A NOTE ON THE DOMAIN MAPPING METHOD WITH ROUGH DIFFUSION COEFFICIENTS

M. D. PETERS

ABSTRACT. In this article, we consider elliptic diffusion problems on random domains with non-smooth diffusion coefficients. We start by illustrating the problems that arise from a non-smooth diffusion coefficient by recapitulating the corresponding regularity analysis. Then, we propose an alternative approach to address this problem by means of a perturbation method. Based on the assumption that the diffusion coefficient can be decomposed in a possibly deterministic, analytic part and a rough random perturbation, we derive approximation results in terms of the perturbations amplitude for the approximation of quantities of interest of the solution. Numerical examples are given in order to validate and quantify the theoretical results.

1. Introduction

Often, problems arising in science and engineering can be modeled in terms of boundary value problems. In general, the numerical solution of the latter is well understood if all input parameters are known exactly. In practice, however, this might be a too strong assumption if input parameters are only known up to certain measurement tolerances. In this view, particularly the treatment of uncertainties in the computational domain has become of growing interest, see e.g. [6, 7, 15, 18, 20, 26, 27]. In this article, we consider the elliptic diffusion equation

\[(1) \quad - \text{div} (a(\omega) \nabla u(\omega)) = f \quad \text{in} \quad D(\omega), \quad u(\omega) = 0 \quad \text{on} \quad \partial D(\omega),\]

as a model problem where the underlying domain \(D \subset \mathbb{R}^d\) and the diffusion coefficient \(a(\omega)\) are assumed to be random. This model may be used to account for tolerances in the shape of products fabricated by line production or shapes which stem from inverse problems, like e.g. tomography. The latter has recently been discussed, in the case of a deterministic diffusion coefficient, in [11].

Besides the fictitious domain approach considered in [6], one might essentially distinguish two approaches to deal with uncertain domains: the perturbation method, see e.g. [18] and the references therein, which is suitable to handle small perturbations of the nominal shape and the domain mapping method, see e.g. [27]. The perturbation approach has also successfully been applied to deal with random data, like random diffusion coefficients and random loadings, see e.g. [1][2][6][13][21][22].
In this work, we shall combine the domain mapping method for the numerical treatment of the random domain with the perturbation approach for dealing with the random coefficient. A similar hybrid approach has recently been considered in \[8\] to account for uncertainties induced by the computational domain. There, the domain mapping method is employed to capture the large deformations of the domain, while the perturbation to approach is used to account for the smaller deformations.

Usually, if the solution \( u(x,\omega) \) to (1) provides sufficient regularity, quantities of interest, like expectation and variance, are computed by sophisticated sparse quadrature and quasi-Monte Carlo methods, see e.g. \[5,10,12,24,25,28\]. These methods alleviate the computational burden that comes along with the high dimensionality inherent to this class of problems. The analysis recently published in \[7,15,20\] shows that such regularity results are also available for the solution to (1), if the underlying data, i.e. the diffusion coefficient, the loading and possible data boundary data are analytic functions. In practice, this can be considered a strong limitation. Therefore, our goal in this work is to weaken this requirement.

Exemplarily, we focus here on the diffusion coefficient and emphasize that other data can be handled in a similar way. Assuming an essentially smooth diffusion coefficient, i.e. a diffusion coefficient of the form

\[
a = a_s + \varepsilon a_r, \quad 0 < \varepsilon \ll 1,
\]

where \( a_s \) is an analytic function and \( a_r \) is essentially bounded, i.e. \( \|a_r\|_{L^\infty} \leq 1 \), we will derive approximation results for quantities of interest of the solution \( u \) in terms of the perturbation’s amplitude.

The rest of this article is organized as follows. In Section 2 we introduce some basic definitions and introduce the domain mapping method. Section 3 adapts the regularity results from \[15\] for the situation of an analytic and random diffusion coefficient. The subsequent Section 4 is the main contribution of this article. Here, we consider the case of rough diffusion coefficients in the domain mapping framework and derive approximation results for the perturbation approach under consideration. Afterwards, Section 5 gives a brief overview of the numerical realization of the presented method. Finally, Section 6 provides a numerical example to validate and quantify the theoretical results.

2. Problem formulation

In what follows, let \( D_{ref} \subset \mathbb{R}^d \) for \( d \in \mathbb{N} \) (of special interest are the cases \( d = 2, 3 \)) be a domain with Lipschitz continuous boundary \( \partial D_{ref} \) and let \( (\Omega, \mathcal{F}, \mathbb{P}) \) be a probability space with \( \sigma \)-field \( \mathcal{F} \subset 2^\Omega \) and a complete probability measure \( \mathbb{P} \), i.e. for all \( A \subset B \) and \( B \in \mathcal{F} \) with \( \mathbb{P}[B] = 0 \) it follows \( A \in \mathcal{F} \). We are interested in computing quantities of interest of the solution to the elliptic diffusion problem

\[
-\text{div} \left( a(x,\omega)\nabla u(x,\omega) \right) = f(x) \quad \text{in} \; D(\omega),
\]

\[
u(x,\omega) = 0 \quad \text{on} \; \Gamma(\omega) := \partial D(\omega)
\]
A NOTE ON THE DOMAIN MAPPING METHOD WITH ROUGH DIFFUSION COEFFICIENTS

for $\mathbb{P}$-almost every $\omega \in \Omega$. Note that the case of non-homogeneous Dirichlet data in the domain mapping method can always be reduced to the above homogeneous case, see e.g. [11].

In order to guarantee the well posedness of (2), we assume that all data, i.e. the diffusion coefficient $a$ and the loading $f$ are defined with respect to the hold-all domain

$$ D := \bigcup_{\omega \in \Omega} D(\omega). $$

The diffusion coefficient $a(x, \omega)$ shall be uniformly elliptic, i.e. there exist $\underline{a}, \overline{a} \in (0, \infty)$ such that

$$ \underline{a} \leq \text{ess inf}_{x \in D} a(x, \omega) \leq \text{ess sup}_{x \in D} a(x, \omega) \leq \overline{a} $$

$\mathbb{P}$-almost surely.

We make the crucial assumption that the random variation in the coefficient $a(x, \omega)$ is independent of the random variation in the domain $D(\omega)$. We note that under the same constraint of independence, it is also possible to consider random loadings or even random boundary data. In order to model the random domain, as in [11, 15], we assume the existence of a uniform $C^1$-diffeomorphism $V: \overline{D_{\text{ref}}} \times \Omega \to \mathbb{R}^d$, i.e.

$$ \|V(\omega)\|_{C^1(\overline{D_{\text{ref}}}; \mathbb{R}^d)}, \|V^{-1}(\omega)\|_{C^1(\overline{D_{\text{ref}}}; \mathbb{R}^d)} \leq C_{\text{uni}} $$

for $\mathbb{P}$-almost every $\omega \in \Omega$, such that

$$ D(\omega) = V(D_{\text{ref}}, \omega). $$

Applying the domain mapping approach, the variational formulation reads then:

Find $\hat{u}(\omega) \in H^1_0(D_{\text{ref}})$ such that

$$ \int_{D_{\text{ref}}} A(\omega) \nabla \hat{u}(\omega) \nabla v \, dX = \int_{D_{\text{ref}}} f_{\text{ref}}(\omega) v \, dX \quad \text{for all } v \in H^1_0(D_{\text{ref}}), $$

where

$$ A(X, \omega) := (a \circ V)(X, \omega) \cdot (V'V')^{-1}(X, \omega) \cdot \det V'(X, \omega) $$

and

$$ f_{\text{ref}}(X, \omega) := (f \circ V)(X, \omega) \cdot \det V'(X, \omega). $$

Herein, $V'$ denotes the Jacobian of $V$. In addition, we make use of the convention that $X \in D_{\text{ref}}$ always refers to a material point, while $x \in \mathbb{R}^d$ denotes a spatial point.

Then, there is the following one-to-one correspondence between the solution $u$ to (2) and the solution $\hat{u}$ to (4). It holds

$$ u(x, \omega) = (\hat{u} \circ V^{-1})(x, \omega) \quad \text{and} \quad \hat{u}(X, \omega) = (u \circ V)(X, \omega), $$

see e.g. [15].
3. ANALYTIC DIFFUSION COEFFICIENTS

In this section, we shall briefly recall the essential regularity results for the solution \( \hat{u} \) given that \( a \) is an analytic function and refer to \([7,15]\) for a more comprehensive discussion of this topic.

In order to approximate the solution \( \hat{u} \) numerically, one usually starts from a truncated Karhunen-Loève expansion of the underlying random fields, i.e.

\[
a(x, \omega) = \mathbb{E}[a](x) + \sum_{k=1}^{N} \sqrt{\lambda_k} \psi_k(x) X_k(\omega),
\]

\[
V(X, \omega) = \mathbb{E}[V](X) + \sum_{k=1}^{M} \sqrt{\mu_k} \phi(X) Y_k(\omega)
\]

with families \( \{X_k\}_k \) and \( \{Y_k\}_k \) of uncorrelated and centered random variables. One possibility to compute such a truncated Karhunen-Loève expansion numerically is to employ a pivoted Cholesky decomposition, see e.g. \([14]\) and the references therein.

As equation (6) already indicates, we will assume here that the diffusion coefficient is given in spatial coordinates. In view of the subsequent analysis, this is the more challenging situation. Nevertheless, we emphasize that the presented analysis is also capable of dealing with a diffusion coefficient that is represented in material coordinates, i.e. \( a(X, \omega) \).

Here and in the following, we make the common assumption that the families \( \{X_k\}_k \) and \( \{Y_k\}_k \) are even independent and identically distributed. As a consequence, the two families are particularly independent with respect to each other. After a possible scaling, we have that the range of the random variables is \( \Gamma := [-1,1] \). We further assume that the random variables exhibit densities with respect to the Lebesgue measure, such that the corresponding push-forward measures are given by

\[
\mathbb{P} \circ X^{-1} = \rho_X(y) \, dy \quad \text{and} \quad \mathbb{P} \circ Y^{-1} = \rho_Y(z) \, dz,
\]

respectively. These densities are of product structure due to the independence of the random variables, i.e. \( \rho_X(y) = \rho_{X,1}(y_1) \cdot \rho_{X,N}(y_N) \) and \( \rho_Y(z) = \rho_{Y,1}(z_1) \cdot \rho_{Y,M}(z_M) \), respectively. Moreover, the centeredness yields

\[
\int_{\Gamma} y_k \rho_{X,k}(y_k) \, dy_k = 0 \quad \text{for } k = 1, \ldots, N \quad \text{and} \quad \int_{\Gamma} z_k \rho_{Y,k}(z_k) \, dz_k = 0 \quad \text{for } k = 1, \ldots, M.
\]

Therefore, we can reparametrize the expansions from (6) and write

\[
a(x, y) = \mathbb{E}[a](x) + \sum_{k=1}^{N} \sqrt{\lambda_k} \psi_k(x) y_k, \quad y \in \Gamma^N,
\]

\[
V(X, z) = \mathbb{E}[V](X) + \sum_{k=1}^{M} \sqrt{\mu_k} \phi(X) z_k, \quad z \in \Gamma^M.
\]
This yields the parametrized variational formulation

Find \( \hat{u}(y, z) \in H^1_0(D_{\text{ref}}) \) such that

\[
\int_{D_{\text{ref}}} A(y, z) \nabla \hat{u}(y, z) \nabla v \, dX = \int_{D_{\text{ref}}} f_{\text{ref}}(z) v \, dX \quad \text{for all } v \in H^1_0(D_{\text{ref}}),
\]

(9)

The expectation of \( \hat{u} \) is given by the Bochner-type integral

\[
\mathbb{E}[\hat{u}] (X) := \int_{\Gamma^N} \int_{\Gamma^M} \hat{u}(X, y, z) \rho_X(y) \rho_Y(z) \, dz \, dy
\]

and its variance by

\[
\nabla[\hat{u}] (X) := \int_{\Gamma^N} \int_{\Gamma^M} \left( \hat{u}(X, y, z) - \mathbb{E}[\hat{u}](X) \right)^2 \rho_X(y) \rho_Y(z) \, dz \, dy.
\]

In a similar fashion, we can also compute other quantities of interest, e.g. \( \mathbb{E}[F(\hat{u})] \), where

\[
F : H^1_0(D_{\text{ref}}) \rightarrow \mathbb{R}
\]

is a continuous and linear functional. Note that there exists a version of Fubini's theorem for Bochner integrals. It guarantees that the order of integration in the expressions for the expectation and the variance can be interchanged, see [19].

As we have seen so far, the computation of quantities of interest results in very high dimensional quadrature problems. In order to solve these quadrature problems efficiently, one usually exploits the smoothness of the solution \( \hat{u} \) with respect to the parameters \( y \in \Gamma^N \) and \( z \in \Gamma^M \):

Let all eigenfunctions \( \psi_k \) of the diffusion coefficient \( a \) be analytic, i.e.

\[
\| \partial_x^{\alpha} \psi_k \|_{L^\infty(D)} \leq \alpha! \rho^{-|\alpha|} c_\psi, \quad \alpha \in \mathbb{N}^d,
\]

for \( \rho \in (0, 1) \) and \( c_\psi > 0 \) uniformly in \( k \). Then there holds, cp. [15] Lemma 5,

\[
\| \partial_z^{\alpha} (\psi_k \circ V) \|_{L^\infty(\Gamma^M; L^\infty(D_{\text{ref}}))} \leq |\alpha|! c_\psi \left( \frac{d}{\rho \log 2} \right)^{|\alpha|} \gamma^\alpha \quad \text{for all } \alpha \in \mathbb{N}^M.
\]

Assuming, for the sake of simplicity, that the expectation \( \mathbb{E}[a] \) satisfies the same bound [10], we obtain a similar result for \( \mathbb{E}[a \circ V] \). Therefore, we can compute the derivatives of the transported diffusion coefficient \( a \circ V \) according to

\[
\| \partial_x^{\alpha} \partial_z^{\beta} (a \circ V) \|_{L^\infty(\Gamma^M + N; L^\infty(D_{\text{ref}}))} \leq |\beta|! c_\psi \left( \frac{d}{\rho \log 2} \right)^{|\beta|} \gamma^\beta \left( 1 + \sum_{k=1}^N \sqrt{\lambda_k} \right), \quad \alpha = 0
\]

and

\[
\| \partial_y^{\alpha} \partial_z^{\beta} (a \circ V) \|_{L^\infty(\Gamma^{N+M}; L^\infty(D_{\text{ref}}))} \leq |\beta|! \sqrt{\lambda_k} c_\psi \left( \frac{d}{\rho \log 2} \right)^{|\beta|} \gamma^\beta, \quad \alpha_k = 1.
\]

Since \( a \) is affine with respect to \( y \in \Gamma^N \), all higher order derivatives with respect to any \( y_k \) vanish. Using the crude bound

\[
\| \partial_y^{[\alpha, \beta]} (a \circ V) \|_{L^\infty(\Gamma^{N+M}; L^\infty(D_{\text{ref}}))} \leq (|\alpha| + |\beta|)! C c_\psi^{|\alpha| + |\beta|} \gamma^{[\alpha, \beta]} \]
for some constants $C, c > 0$ and $\gamma := [\gamma_a, \gamma_V]$ with

$$
\gamma_a := \left[ \| \sqrt{\lambda_1 \psi_1} \|_{L^\infty(D)}, \ldots, \| \sqrt{\lambda_N \psi_N} \|_{L^\infty(D)} \right],
$$
$$
\gamma_V := \left[ \| \mu_1 \phi_1 \|_{W^{1,\infty}(D_{	ext{ref}}; \mathbb{R}^d)}, \ldots, \| \mu_M \phi_M \|_{W^{1,\infty}(D_{	ext{ref}}; \mathbb{R}^d)} \right],
$$

it is then easy to show that the derivatives of the transported diffusion coefficient $A(X, y, z)$ exhibit the same behavior, cf. [15]. Therefore, we can conclude in the same way as in [15, Theorem 5] and obtain the following result.

**Theorem 3.1.** Let $\hat{u}(y, z)$ be the solution to (9) and assume that the eigenfunctions $\{\psi_k\}_k$ in the Karhunen-Loève expansion of the diffusion coefficient are analytic in accordance with (10). Then, there exist constants $C, c > 0$ such that

$$
\left\| \frac{\partial^{[\alpha, \beta]} \hat{u}}{\partial [y, z]} \right\|_{L^\infty(\Omega; L^\infty(D; \mathbb{R}^d))} \leq (|\alpha| + |\beta|) C c^{\alpha + |\beta|} |\alpha, \beta| \quad \text{for all } \alpha \in \mathbb{N}^N, \beta \in \mathbb{N}^M.
$$

Regularity estimates of this form allow for sparse quadrature and collocation methods. As we have seen, they come at the cost of high regularity requirements of the underlying data, see also [7, 11, 15, 20]. One possibility to bypass this drawback in the case of small rough perturbations is considered in the following section.

### 4. Rough diffusion coefficients

In this section, we assume that the diffusion coefficient under consideration can be decomposed into an analytic, deterministic part $a_s(x)$ and a non-smooth centered random part $a_r(x, \omega)$ of small magnitude, i.e.

$$
a(x, \omega) = a_s(x) + \varepsilon a_r(x, \omega) \quad \text{for } 0 < \varepsilon \ll 1,
$$

where $\|a_r(x, \omega)\|_{L^\infty(\Omega; L^\infty(D(\omega)))} \leq 1$ and $E[a_r(x)] \equiv 0$. Note that the results presented in this section also remain valid if $a_s$ is also subjected to randomness.

We assume that $a_r$ exhibits a Karhunen-Loève expansion of the form

$$
a_r(x, \omega) = E[a_r](x) + \sum_{k=1}^N \sqrt{\lambda_k} \psi_k(x) X_k(\omega),
$$

where we have only a slow decay of $\sqrt{\lambda_k}$ for $k \to \infty$ due to the roughness of the field. As in the previous section, we parametrize the diffusion coefficient according to

$$
a_r(x, y) = \sum_{k=1}^N \sqrt{\lambda_k} \psi_k(x) y_k, \quad y \in \Gamma^N,
$$

with the corresponding product measure $\rho_X(y) \, dy$.

Next, we decompose the transported diffusion coefficient according to

$$
A(X, y, z) = A_s(X, z) + \varepsilon A_r(X, y, z),
$$

with

$$
A_s(X, z) := (a_s \circ V)(X, z)(V'V')^{-1}(X, z) \det V'(X, z)
$$
and
\[ A_s(X, y, z) := (a_r \circ V)(X, y, z)(V'^\top V')^{-1}(X, z) \det V'(X, z) \]
The independence and the centeredness, cf. [7], of the families \( \{X_k\}_k \) and \( \{Y_k\}_k \) imply
\[
\mathbb{E}[A_r](X) = \int_{\Gamma_N} \int_{\Gamma_M} \sum_{k=1}^N \sqrt{\lambda_k}(\psi_k V)(X, z)y_k(V'^\top V')^{-1}(X, z) \det V'(X, z) \rho_X(y) \rho_Y(z) \, dz \, dy
\]
\[
= \sum_{k=1}^N \sqrt{\lambda_k} \int_{\Gamma_M} (\psi_k V)(X, z)(V'^\top V')^{-1}(X, z) \det V'(X, z) \rho_Y(z) \, dz \int_{\Gamma_N} y_k \rho_X(y) \, dy
\]
\[
= 0.
\]
Therefore, the rough part \( A_r(X, y, z) \) of the transported diffusion coefficient remains centered.

The pivotal idea is now, to linearize the coefficient-to-solution map in a vicinity of the smooth part \( a_s \) of the diffusion coefficient and treat the perturbation \( a_r\) as a small systematic error. To that end, we apply the following result from [13], which has been modified in order to deal with both, a random diffusion coefficient and random loading.

**Lemma 4.1.** Let \( \hat{u}_0(z) \in H^1_0(D_{ref}) \) be the solution to
\[
- \text{div}(A_s(z) \nabla \hat{u}_0(z)) = f_{ref}(z) \text{ in } D_{ref}, \quad \hat{u}_0(z) = 0 \text{ on } \partial D_{ref}.
\]
Then, for \( A(z) := A_s(z) + \varepsilon \delta A \), the mapping
\[
G: L^\infty(D_{ref}; \mathbb{R}^{d \times d}) \to H^1_0(D_{ref}), \quad A(z) \mapsto G(A(z)) := \hat{u}(z)
\]
is Fréchet differentiable, where the derivative \( \delta \hat{u}(z) = \delta \hat{u}[\delta A](z) \in H^1_0(D_{ref}) \) with respect to the direction \( \delta A \in L^\infty(D_{ref}; \mathbb{R}^{d \times d}) \) is given by
\[
- \text{div}(A_s(z) \nabla \delta \hat{u}[\delta A](z)) = \text{div}(\delta A \nabla \hat{u}_0(z)) \text{ in } D_{ref}, \quad \delta \hat{u}[\delta A](z) = 0 \text{ on } \partial D_{ref}.
\]

**Proof.** Let \( \varepsilon > 0 \) be sufficiently small such that \( A_s(z) + \varepsilon \delta A \) is still uniformly elliptic. We consider the diffusion problems
\[
- \text{div}(A_s(z) \nabla \hat{u}_0(z)) = f_{ref}(z) \text{ in } D_{ref}, \quad u(z) = 0 \text{ on } \partial D_{ref}
\]
\[
- \text{div}((A_s(z) + \varepsilon \delta A) \nabla \hat{u}_0(z)) = f_{ref}(z) \text{ in } D_{ref}, \quad u(z) = 0 \text{ on } \partial D_{ref}
\]
Considering the difference of their respective weak formulations yields for \( (\hat{u}_\varepsilon - \hat{u}_0)(z) \in H^1_0(D_{ref}) \) the equation
\[
\int_{D_{ref}} A_s(z) \nabla (\hat{u}_\varepsilon - \hat{u}_0)(z) \nabla v + \varepsilon \delta A \nabla \hat{u}_\varepsilon \nabla v \, dX = 0 \quad \text{for all } v \in H^1_0(D_{ref}).
\]
Hence, the function \( \frac{1}{\varepsilon} (\hat{u}_\varepsilon - \hat{u}_0)(z) \) satisfies the variational formulation
\[
\int_{D_{ref}} A_s(z) \nabla \frac{(\hat{u}_\varepsilon - \hat{u}_0)(z)}{\varepsilon} \nabla v \, dX = - \int_{D_{ref}} \delta A \nabla \hat{u}_\varepsilon(z) \nabla v \, dX \quad \text{for all } v \in H^1_0(D_{ref}).
\]
From the uniform ellipticity of \( A_s(z) \) and the boundedness of \( \delta A \) it is easy to derive that
\[
\| (\hat{u}_\varepsilon - \hat{u}_0)(z) \|_{H^1_0(D_{ref})} \to 0 \quad \text{as } \varepsilon \to 0.
\]
Therefore, we conclude that
\[
\delta u[\delta A](z) := \lim_{\varepsilon \to 0} \frac{(\hat{u}_\varepsilon - \hat{u}_0)(z)}{\varepsilon}
\]
is the Gâteaux derivative of \(\hat{u}(z)\) in direction \(\delta A\). It remains to show that \(\delta u[\delta A](z)\) is also the Frechét derivative. Since \(\delta u[\delta A](z)\) is obviously linear, the Frechét differentiability follows from
\[
\|(\hat{u}_\varepsilon - \hat{u}_0)(z) - \varepsilon \delta u[\delta A](z)\|_{H^1_0(D_{\text{ref}})}
\leq C \sup_{v \in H^1_0(D_{\text{ref}})} \frac{1}{\|v\|_{H^1_0(D_{\text{ref}})}} \int_{D_{\text{ref}}} A_s(z) \nabla (\hat{u}_\varepsilon - \hat{u}_0)(z) - \varepsilon \delta u[\delta A](z) \nabla v \, dX
\]
\[
= C \sup_{v \in H^1_0(D_{\text{ref}})} \frac{1}{\|v\|_{H^1_0(D_{\text{ref}})}} \left( \int_{D_{\text{ref}}} \left[ (A_s(z) + \varepsilon \delta A) \nabla \hat{u}_\varepsilon(z) - A_s(z) \nabla \hat{u}_0(z) \right] \nabla v \, dX - \varepsilon \int_{D_{\text{ref}}} \delta A \nabla \hat{u}_\varepsilon(z) + A_s(z) \nabla \delta u[\delta A](z) \nabla v \, dX \right)
\]
\[
= \varepsilon C \sup_{v \in H^1_0(D_{\text{ref}})} \frac{1}{\|v\|_{H^1_0(D_{\text{ref}})}} \int_{D_{\text{ref}}} \delta A \nabla (\hat{u}_\varepsilon - \hat{u}_0)(z) \nabla v \, dX
\]
\[
\leq \varepsilon C \|\delta A\|_{L^\infty(D_{\text{ref}};\mathbb{R}^{d \times d})} \|(\hat{u}_\varepsilon - \hat{u}_0)(z)\|_{H^1_0(D_{\text{ref}})}.
\]
Herein, the first inequality follows from the uniform ellipticity, i.e. \(C > 0\) is the inverse of the ellipticity constant, while the second equality follows from (15) and (16). \(\square\)

The proof of the previous lemma requires only the boundedness of the perturbation \(\delta A\). Therefore, the following corollary is immediate.

**Corollary 4.2.** The derivative \(\delta \hat{u}(y, z) := \delta \hat{u}[A_r](y, z)\) is given by the diffusion problem
\[
- \text{div} \left( A_s(z) \nabla \delta \hat{u}(y, z) \right) = \text{div} \left( A_r(y, z) \nabla \hat{u}_0(z) \right) \text{ in } D_{\text{ref}}, \delta \hat{u}(y, z) = 0 \text{ on } \partial D_{\text{ref}}.
\]
Moreover, due to the linearity of the Frechét derivative, there holds
\[
\delta \hat{u}(y, z) = \sum_{k=1}^{N} \sqrt{\lambda_k} \delta \hat{u}\left((\psi_k \circ V)(z)\right) (V'^T V')^{-1}(z) \det V'(z) y_k.
\]

As a consequence of Lemma 4.1 and the subsequent corollary, we may expand \(\hat{u}_\varepsilon(y, z)\) into a first order Taylor expansion
\[
\hat{u}_\varepsilon(y, z) = \hat{u}_0(z) + \varepsilon \delta \hat{u}[A_r(y, z)](y, z) + O(\varepsilon^2),
\]
where \(\hat{u}_0(z)\) is the solution to (14). Based on this expansion, the next theorem gives us expansions for the expectation and the variance of \(\hat{u}_\varepsilon\), see also [13].

**Theorem 4.3.** For \(\varepsilon > 0\) sufficiently small, there holds
\[
E[\hat{u}_\varepsilon] = E[\hat{u}_0] + O(\varepsilon^2) \text{ in } H^1_0(D_{\text{ref}}).
\]
Herein, \(\hat{u}_0 \in H^1_0(D_{\text{ref}})\) satisfies the diffusion problem (14). Moreover, there holds
\[
\mathbb{V}[\hat{u}_\varepsilon] = \mathbb{V}[\hat{u}_0] + O(\varepsilon^2) \text{ in } W^{1,1}_0(D_{\text{ref}}).
\]
Proof. To simplify notation, we abbreviate again \( \delta \hat{u}(y, z) := \delta \hat{u}^*[A_r](y, z) \). Applying the Taylor expansion \([13]\), we arrive at

\[
E[\hat{u}_2] = E[\hat{u}_0 + \varepsilon \delta \hat{u} + O(\varepsilon^2)] = E[\hat{u}_0] + \varepsilon E[\delta \hat{u}] + O(\varepsilon^2),
\]

by the linearity of the expectation. Thus, we have to show that \( E[\delta \hat{u}] = 0 \). Employing \([17]\), it holds

\[
E[\delta \hat{u}] = \int_{\Gamma_N} \int_{\Gamma_M} \delta \hat{u}(y, z) \rho_X(y) \rho_Y(z) \, dz \, dy
\]

\[
= \int_{\Gamma_N} \int_{\Gamma_M} \sum_{k=1}^N \sqrt{\lambda_k} \delta \hat{u}[(\psi_k \circ V)(z)](V'^T V')^{-1}(z) \det V'(z) y_k \rho_X(y) \rho_Y(z) \, dz \, dy
\]

\[
= \sum_{k=1}^N \sqrt{\lambda_k} \int_{\Gamma_N} \int_{\Gamma_M} \delta \hat{u}[(\psi_k \circ V)(z)](V'^T V')^{-1}(z) \det V'(z) y_k \rho_X(y) \, dy \int_{\Gamma_N} \rho_Y(z) \, dz
\]

\[
= 0
\]

due to the centeredness of the parameters \( y_k \), see \([7]\).

Employing the result for the expectation, we obtain for the variance

\[
\mathbb{V}[\hat{u}] = \int_{\Gamma_N} \int_{\Gamma_M} (\hat{u}(y, z) - E[\hat{u}])^2 \rho_X(y) \rho_Y(z) \, dz \, dy
\]

\[
= \int_{\Gamma_N} \int_{\Gamma_M} ((\hat{u}_0(z) - E[u_0]) + \varepsilon \delta \hat{u}(y, z) + O(\varepsilon^2))^2 \rho_X(y) \rho_Y(z) \, dz \, dy
\]

\[
= \mathbb{V}[\hat{u}_0] + 2 \int_{\Gamma_N} \int_{\Gamma_M} (\hat{u}_0(z) - E[u_0])(\varepsilon \delta \hat{u}(y, z) + O(\varepsilon^2)) \rho_X(y) \rho_Y(z) \, dz \, dy
\]

\[
+ \varepsilon^2 \int_{\Gamma_N} \int_{\Gamma_M} (\delta \hat{u}(y, z) + O(\varepsilon))^2 \rho_X(y) \rho_Y(z) \, dz \, dy.
\]

Since the last term is bounded with respect to the \( W^{1,1}_0(D_{\text{ref}}) \)-norm, see e.g. \([16]\), it remains to show that the \( \varepsilon \) dependent part in the second term vanishes. Again making use of \([17]\), this can be seen as follows

\[
2 \varepsilon \int_{\Gamma_N} \int_{\Gamma_M} (\hat{u}_0(z) - E[u_0]) \delta \hat{u}(y, z) \rho_X(y) \rho_Y(z) \, dz \, dy
\]

\[
= \int_{\Gamma_N} \int_{\Gamma_M} (\hat{u}_0(z) - E[u_0]) \sum_{k=1}^N \sqrt{\lambda_k} \delta \hat{u}[(\psi_k \circ V)(z)](V'^T V')^{-1}(z) \det V'(z) y_k \rho_X(y) \rho_Y(z) \, dz \, dy
\]

\[
= \sum_{k=1}^N \sqrt{\lambda_k} \int_{\Gamma_N} \int_{\Gamma_M} (\hat{u}_0(z) - E[u_0]) \delta \hat{u}[(\psi_k \circ V)(z)](V'^T V')^{-1}(z) \det V'(z) \rho_Y(z) \, dz \int_{\Gamma_N} y_k \rho_X(y) \, dy
\]

\[
= 0,
\]

where we again exploit the centeredness of the coordinates \( y_k \). This completes the proof.

\( \square \)

Remark 4.4. If the diffusion coefficient is already given in material coordinates, the summands in expression \([13]\) simplify towards

\[
A_s(X, z) := a_s(X)(V'^T V')^{-1}(X, z) \det V'(X, z)
\]
and
\[ A_r(\mathbf{X}, \mathbf{y}, \mathbf{z}) := a_r(\mathbf{X}, \mathbf{y})(\mathbf{V}^T \mathbf{V}')^{-1}(\mathbf{X}, \mathbf{z}) \det \mathbf{V}'(\mathbf{X}, \mathbf{z}). \]

In this case, it is also easy to see that \( E[A_r] = 0 \). Consequently, Theorem 4.3 remains valid in this case. Moreover, it is straightforward to apply the previous derivation to diffusion problems with anisotropic coefficients, i.e. \( a(\mathbf{x}, \omega) \) is a matrix-valued function, cf. [17].

Theorem 4.3 tells us that, given the boundedness in the rough part \( a_r \), we obtain a quadratic approximation of the solution’s expectation and variance in terms of the perturbation’s amplitude. A similar result holds for output functionals of the solution.

**Corollary 4.5.** Let \( F: H^1_0(D_{ref}) \rightarrow \mathbb{R} \) be a continuous and linear functional. Then, for \( \varepsilon > 0 \) sufficiently small, there holds
\[ E[F(\hat{u})] = E[F(\hat{u}_0)] + \mathcal{O}(\varepsilon^2) \quad \text{in} \ H^1_0(D_{ref}). \]

Herein, \( \hat{u}_0 \in H^1_0(D_{ref}) \) satisfies the diffusion problem [14]. Moreover, there holds
\[ \nabla^2[F(\hat{u})] = \nabla^2[F(\hat{u}_0)] + \mathcal{O}(\varepsilon^2) \quad \text{in} \ W^{1,1}_0(D_{ref}). \]

**Proof.** By exploiting the continuity and the linearity of \( F \), the proof can be conducted similarly to the proof of Theorem 4.3. \( \square \)

### 5. Numerical realization of the domain mapping method

![Figure 1](image_url)

**Figure 1.** Construction of parametric finite elements and the action of the random vector field.

In this section, we describe the numerical realization of the domain mapping method and, in particular, the discretization of the random vector field. In addition, we give a brief description how the diffusion coefficient, which is given in spatial coordinates, can be represented such that it is feasible within the domain mapping method. For the sake of completeness, we start by introducing the parametric finite element method which is used in the numerical examples. Nevertheless, we emphasize that the construction presented in Section 5.2 is also viable other finite element discretizations based on nodal basis functions.
5.1. Finite element approximation. For the spatial discretization, we employ linear (iso-) parametric finite elements, see [3][4][23]. To that end, the domain \( D_{ref} \) shall be given by a collection of simplicial smooth patches. More precisely, let \( \triangle \) denote the reference simplex in \( \mathbb{R}^d \). Then, the domain \( D_{ref} \) shall be partitioned into \( K \) patches

\[
D_{ref} = \bigcup_{j=1}^{K} \tau_0,j, \quad \tau_0,j = \kappa_j(\triangle), \quad j = 1, 2, \ldots, K.
\]  

(19)

The intersection \( \tau_0,j \cap \tau_0,j' \), \( j \neq j' \), of any two patches \( \tau_0,j \) and \( \tau_0,j' \) is supposed to be either empty or a common lower dimensional face.

Based on this construction, it is straightforward to introduce a hierarchical mesh on \( D_{ref} \), which is feasible for a geometric multigrid solver: A mesh on level \( \ell \) on \( D_{ref} \) is obtained by regular subdivisions of depth \( \ell \) of the reference simplex into \( 2^\ell \) sub-simplices. Then, by mapping this mesh via the parametrizations \( \kappa_j, j = 1, \ldots, K \), we obtain \( K2^\ell \) elements \( \{\tau_{\ell,i}\}_i \) for \( D_{ref} \), see the first mapping in Figure 1 for a visualization of this situation.

In order to guarantee that the triangulation \( \mathcal{T}_\ell := \{\tau_{\ell,i}\}_i \), on level \( \ell \) results in a regular mesh for \( D_{ref} \), the parametrizations \( \{\kappa_j\}_j \) are supposed to be \( C^0 \) compatible in the following sense: there exists a bijective, affine mapping \( \Xi : \triangle \rightarrow \triangle \) such that for all \( X = \kappa_i(s) \) on a common interface of \( \tau_0,j \) and \( \tau_0,j' \), it holds that \( \kappa_j(s) = (\kappa_{j'} \circ \Xi)(s) \). Thus, the diffeomorphisms \( \kappa_j \) and \( \kappa_{j'} \) coincide at the common interface except for orientation.

Finally, we define the piece wise finite element ansatz functions by lifting the Lagrangian finite elements from \( \triangle \) to the domain \( D_{ref} \) by using the parametrizations \( \kappa_j \). To that end, let \( \Phi_\ell = \{\varphi_{\ell,i} : i \in I_\ell\} \), where \( I_\ell \) is a suitable index set, denote the Lagrangian piece wise linear basis functions on the \( \ell \)-th subdivision \( \triangle_\ell \) of the reference simplex. Then, we obtain the finite element space

\[
V_{\triangle,\ell} = \text{span}\{\varphi_{\ell,j} : j \in I_\ell\} = \{u \in C(\triangle) : u|_\tau \in \Pi_1 \text{ for all } \tau \in \triangle_\ell\}
\]

with \( \dim V_{\triangle,\ell} \approx 2^{\ell d} \) and \( \Pi_1 \) denoting the space of linear polynomials. Continuous basis functions whose support overlaps with several patches are obtained by gluing across patch boundaries, using the \( C^0 \) inter-patch compatibility. This yields a nested sequence of finite element spaces

\[
V_{\text{ref},\ell} := \{v \in C(D_{ref}) : v|_{\kappa_j(\triangle)} = \varphi \circ \kappa_j^{-1}, \varphi \in V_{\triangle,\ell}, \quad j = 1, \ldots, K\} \subset H^1(D_{ref})
\]

with \( \dim V_{\text{ref},\ell} \approx K2^\ell \). It is well known that the spaces \( V_{\text{ref},\ell} \) satisfy the following approximation result. It holds

\[
\inf_{v_\ell \in V_{\text{ref},\ell}} \|u - v_\ell\|_{H^1(D_{ref})} \leq c\ell^2\|u\|_{H^2(D_{ref})}, \quad u \in H^2(D_{ref}),
\]

(20)

for some constant \( c > 0 \). This result remains valid if we replace the parametrizations \( \{\kappa_j\}_j \) by their piece wise affine approximation, i.e. when we replace the curved edges by planar ones, see e.g. [3][4]. In this view, one might also directly start from a polygonal approximation of \( D_{ref} \).
5.2. Discretization of the random vector field. Next, we explain how the random vector field can be approximated numerically. For the sake of a less cumbersome notation, we present the construction only for \( d = 2 \) and emphasize that the construction for \( d = 3 \) can be performed in complete analogy.

Let the random vector field be given by its expectation \( \mathbb{E}[V](X) \) and its covariance function \( \text{Cov}[V](X, X') = [\text{Cov}_{i,j}[V](X, X')]_{i,j=1,2} \). Moreover, let \( \{X_i\}_{i=1}^{n} \subset D_{\text{ref}} \) be the centers of the nodal linear finite element basis \( \{\varphi_1, \ldots, \varphi_n\} \), i.e. \( \varphi_i(X_j) = \delta_{i,j} \). Then, we can represent the expectation by its finite element interpolant

\[
\mathbb{E}[V](X) \approx \sum_{i=1}^{n} \mathbb{E}[V](X_i) \varphi_i(X)
\]

and in complete analogy

\[
\text{Cov}[V](X, X') \approx \sum_{i,j=1}^{n} \text{Cov}[V](X_i, X_j) \varphi_i(X) \varphi_j(X').
\]

Now, in order to determine the Karhunen-Loève expansion of \( V \), we have to solve the operator eigenvalue problem

\[
\int_{D_{\text{ref}}} \text{Cov}[V](X, X') \varphi(X') \, dX' = \mu \varphi(X).
\]

Thus, by replacing \( \text{Cov}[V] \) by its finite element interpolant and testing with respect to the basis functions \( \{[\varphi_i, 0]^T, [0, \varphi_i]^T\}_{i=1}^{n} \), we end up with the generalized algebraic eigenvalue problem

\[
\begin{bmatrix}
M & M \\
M & M
\end{bmatrix}
\begin{bmatrix}
C & M \\
M & M
\end{bmatrix}
v = \mu
\begin{bmatrix}
M & M \\
M & M
\end{bmatrix}
v, \quad v \in \mathbb{R}^{2n}.
\]

Herein,

\[
C := \begin{bmatrix}
[\text{Cov}_{1,1}[V](X_i, X_j)]_{i,j=1}^{n} \\
[\text{Cov}_{2,1}[V](X_i, X_j)]_{i,j=1}^{n}
\end{bmatrix}
\begin{bmatrix}
[\text{Cov}_{1,2}[V](X_i, X_j)]_{i,j=1}^{n} \\
[\text{Cov}_{2,2}[V](X_i, X_j)]_{i,j=1}^{n}
\end{bmatrix} \in \mathbb{R}^{2n \times 2n}
\]

is the covariance function evaluated in all combinations of grid points and

\[
M := [m_{i,j}]_{i,j=1}^{n} \in \mathbb{R}^{n \times n} \quad \text{with} \quad m_{i,j} = \int_{D_{\text{ref}}} \varphi_i \varphi_j \, dX
\]

denotes the finite element mass matrix.

The algebraic eigenvalue problem (21) can now be efficiently solved by means of the pivoted Cholesky decomposition as follows. Let \( C \approx \text{LL}^T \) with \( L \in \mathbb{R}^{2n \times M} \) be the pivoted Cholesky decomposition of \( C \) as described in, e.g. [14]. Then, we approximate the eigenvalue problem (21) by

\[
\begin{bmatrix}
M & M \\
M & M
\end{bmatrix}
\begin{bmatrix}
\text{LL}^T & M \\
M & M
\end{bmatrix}
v = \mu
\begin{bmatrix}
M & M \\
M & M
\end{bmatrix}
v, \quad v \in \mathbb{R}^{2n}.
\]
This eigenvalue problem is equivalent to the usually much smaller eigenvalue problem
\[(23)\]
\[L^\top \begin{bmatrix} M & \tilde{M} \\ \tilde{M} & M \end{bmatrix} L \tilde{v} = \mu \tilde{v}, \quad \tilde{v} \in \mathbb{R}^M.\]

In particular, if \(\tilde{v}_i\) is an eigenvector of (23) with eigenvalue \(\mu_i\), then \(v_i := L \tilde{v}_i\) is an eigenvector of (22) with eigenvalue \(\mu_i\). Moreover, there holds
\[v_i^\top \begin{bmatrix} M & \tilde{M} \\ \tilde{M} & M \end{bmatrix} v_j = \mu_i \delta_{i,j}.\]

**Remark 5.1.** The cost for computing the pivoted Cholesky decomposition is \(O(2nM^2)\) and, since all entries of \(C\) can be computed on the fly without the need of storing the entire matrix \(C\), the storage cost is \(O(2nM)\). Moreover, the small eigenvalue problem (23) can be solved with cost \(O(M^3)\). Thus, since usually \(M \ll n\) the overall cost for computing the Karhunen-Loève expansion of \(V\) by the suggested approach is also \(O(2nM^2)\) in total.

By the presented construction, we obtain a piece wise affine approximation of the random vector field \(V\), see the second mapping in Figure 1. Note that the uniformity condition (3) guarantees that the functional determinant \(\det V'\) has a constant sign, see e.g. [15]. Thus, without loss of generality, we may assume \(\det V' > 0\). For a sufficiently small mesh width \(h > 0\), this property carries over to its piece wise affine approximation. Therefore, each random realization of the piece wise affine approximation maps a mesh onto a mesh by simply moving the mesh points and keeping the topology, i.e. the sets of point indices that make up an element, fixed.

As a consequence, the realizations of the solution to (4) can either be directly computed on the reference domain \(D_{\text{ref}}\) or on its image \(V(D_{\text{ref}}, \omega)\). In particular, there holds a similar approximation result to (20) for the mapped finite element space on \(V(D_{\text{ref}}, \omega)\), see e.g. [2][15]. Finally, we remark that in case that the solution \(u\) has been computed on \(V(D_{\text{ref}}, \omega)\), the corresponding solution \(\hat{u}\) on \(D_{\text{ref}}\) can be easily retrieved by assigning the computed node values of the solution to the respective mesh points in \(D_{\text{ref}}\), cf. (5).

5.3. **Discretization of the diffusion coefficient.** Since the diffusion coefficient is represented in spatial coordinates, we have to provide an efficient means to evaluate it for each particular realization of the domain \(V(D_{\text{ref}}, \omega)\). To that end, we assume that the hold-all \(D\) can be subdivided by a Cartesian grid, for example, the hold all can be chosen as a rectangle for \(d = 2\) and as a cuboid for \(d = 3\). Then, after introducing a uniform grid for \(D\), we can perform the computation of the Karhunen-Loève expansion exactly as for the random vector field.

Now, in order to evaluate \(a(x, \omega)\) at a certain point \(V(D_{\text{ref}}, \omega)\), we only need to retrieve the containing grid cell in \(D\) which is, due to the simple structure of the mesh on \(D\) an \(O(1)\) operation.
6. NUMERICAL RESULTS

In this section, we present a numerical example to validate the presented approach. For the sake of computation times, we consider only an example in two spatial dimensions. The reference domain is given by the unit disc, i.e.

\[ D_{\text{ref}} := \{ X \in \mathbb{R}^2 : \|X\|_2 < 1 \}. \]

The random vector field is represented via its expectation and covariance function according to

\[
\mathbb{E}[V](X) = X, \quad \text{Cov}[V](X, X') = \frac{1}{1000} \begin{bmatrix} 5 \exp\left(-2\|X - X'\|_2^2\right) & \exp(-0.1\|2X - X'\|_2^2) \\ \exp(-0.1\|X - 2X'\|_2^2) & 5 \exp(-0.5\|X - X'\|_2^2) \end{bmatrix}.
\]

Moreover, we assume that the random variables in the corresponding Karhunen-Loève expansion are independent and uniformly distributed on \([\sqrt{3}, \sqrt{3}]\), i.e. they have normalized variance.

We aim at computing the expectation and the variance of the solution to

\[- \text{div} \left( a_\varepsilon(\omega) \nabla u_\varepsilon(\omega) \right) = 1 \quad \text{in} \ D(\omega), \quad u_\varepsilon(\omega) = 0 \quad \text{on} \ \partial D(\omega).
\]

![Figure 2. Expectation (left) and first eigenfunction (right) of the random diffusion coefficient.](image)

Herein, the diffusion coefficient is also defined via its expectation and covariance as

\[
\mathbb{E}[a_\varepsilon](x) = 1 + \frac{x_1^2 - x_2^2}{40}, \quad \text{Cov}[a_\varepsilon](x, x') = \varepsilon^2 \frac{1}{100} \begin{bmatrix} 2 \exp\left(-\|x - x'\|_2^2\right) & +9g(x)g(x') \end{bmatrix}, \quad \varepsilon \geq 0
\]

where

\[ g(x) := \max\{0, 1 - |x_1|\} \cdot \max\{0, 1 - |x_2|\} \]

denotes the tensor product hat-function. Again, we assume that the random variables in the corresponding Karhunen-Loève expansion are independent and uniformly distributed on \([\sqrt{3}, \sqrt{3}]\).
We remark that the tensor product hat function \( g(x) \) is only Lipschitz continuous. Therefore, \( g \circ V \) and, consequently, \( a \circ V \) are non-smooth functions, see the right panel in Figure 2 for a visualization of the first eigenfunction in the diffusion coefficient’s Karhunen-Loève expansion. Consequently, Theorem 3.1 cannot be applied here and we resort to the perturbation approach. To that end, we set

\[
a_\varepsilon(x, \omega) = a_s(x) + \varepsilon a_r(x, \omega) := \mathbb{E}[a](x) + \varepsilon \sum_{k=1}^{N} \sqrt{\lambda_k} \psi_k(x) X_k(\omega),
\]

where the latter refers to the truncated Karhunen-Loève expansion of \( a_\varepsilon(x, \omega) \).

The computation of the finite element approximations and of the Karhunen-Loève expansion for the random vector field are carried out on a mesh for \( D_{\text{ref}} \) with mesh width \( h = 2^{-6} \). The Karhunen-Loève expansion is truncated such that the truncation error is smaller than \( 10^{-2} \), see e.g. [11]. This results in \( M = 52 \) parameters. The Karhunen-Loève expansion of the diffusion coefficient is computed on the hold-all domain \( D = [-2, 2]^2 \), which is discretized by a quadrilateral mesh of size \( h = 4 \cdot 10^{-3} \). The coefficient’s Karhunen-Loève expansion is also truncated such that the error is smaller than \( 10^{-2} \). This results in \( N = 10 \) additional parameters.

Hence, in terms of parameters, we are facing here a 62 dimensional quadrature problem with a non-smooth integrand. This means that we cannot use sophisticated sparse- or quasi-Monte Carlo quadrature methods to obtain suitable reference solutions. Instead, we employ the plain vanilla Monte Carlo quadrature with a huge amount, i.e. \( 10^8 \), of samples. Each reference solution is then calculated by averaging five runs of the Monte Carlo simulation, resulting in \( 5 \cdot 10^8 \) samples in total.

Figure 3 shows visualizations of the expectation for \( \varepsilon = 0.01, 0.25, 0.5, 1 \). It turns out that the expectation looks rather similar in all four cases. This is in contrast to the variance, which is depicted for the same values of \( \varepsilon \) in Figure 4. Here, for \( \varepsilon = 0.01, 0.25 \) the shape of the variance is governed by the anisotropy that is induced by the random vector field, see (24), while for \( \varepsilon = 0.5, 1 \) the shape is governed by the shape of the first eigenfunction of the random diffusion coefficient, see Figure 2.

For, the computation of \( \mathbb{E}[\hat{u}_0] \) and \( \nabla[\hat{u}_0] \), cf. Theorem 4.3, we are in the setting that is considered in Section 3, i.e. all data are analytic functions. Consequently, \( \hat{u}_0 \) is also an analytic function with respect to the parameters \( z \in \Gamma^M \). Hence, we may use the sparse, anisotropic quadrature method based on the Gauss-Legendre points, see [12], to compute \( \mathbb{E}[\hat{u}_0] \) and \( \nabla[\hat{u}_0] \) in an efficient manner.\(^2\)

---

\(^1\) The computations have been carried out on the Euler cluster managed by the Scientific IT Services at the ETH Zurich, see https://sis.id.ethz.ch/hpc, with up to 1400 cores.

\(^2\) The implementation of the sparse grid quadrature is available on https://github.com/muchip/SPQR.
In Theorem 4.3, we have derived a point-wise error estimate, thus, we will measure here, in order to validate this theorem numerically,

\[ \| E[\hat{u}_\varepsilon] - E[\hat{u}_0] \|_{H^1(D_{ref})} \text{ and } \| \nabla[\hat{u}_\varepsilon] - \nabla[\hat{u}_0] \|_{W^{1,1}(D_{ref})}, \]

respectively. We compute these errors for the values \( \varepsilon = 0.03125, 0.0625, 0.125, 0.25, 0.5, 1. \) For \( \varepsilon = 1, \) the maximum possible perturbation is approximately 0.82, i.e. \( \| a_r(x, \omega) \|_{L^\infty(D)} \approx 0.82 \) uniformly in \( \omega \in \Omega. \)

To compute the expectation and the variance of \( \hat{u}_0 \) such that the theoretical rate of \( \varepsilon^2 \) is achieved for all considered values of \( \varepsilon, \) the sparse grid quadrature on level \( q = 11, \) cf. [12], resulting in 6859 quadrature points is sufficient. Figure 5 shows that the theoretical approximation rate of \( \varepsilon^2 \) in terms of the diffusion coefficient’s perturbation’s magnitude is perfectly attained in this example.

7. Conclusion

For the domain mapping method, the presented analysis indicates that smooth data are required in order to derive regularity results for the solution. Thus quadrature methods
A NOTE ON THE DOMAIN MAPPING METHOD WITH ROUGH DIFFUSION COEFFICIENTS

Figure 4. Variance of the solution $\hat{u}$ for $\varepsilon = 0.01, 0.25, 0.5, 1$.

Figure 5. Convergence of the perturbation approach with respect to $\varepsilon$. 
relying on the smoothness of the integrand may not be feasible to compute quantities of interest if the underlying data are non-smooth. In the case of small rough perturbations of the diffusion coefficient, a viable alternative is the combination of the domain mapping method for the randomly deforming domain with the perturbation approach for the diffusion coefficient. To that end, we have derived approximation results based on a first order Taylor expansion of the diffusion problem’s solution with respect to the diffusion coefficient. The approximation results guarantee a quadratic approximation of the solution’s expectation and variance in terms of the perturbations amplitude. The presented numerical example corroborates this result.

References

[1] I. Babuška and P. Chatzipantelidis. On solving elliptic stochastic partial differential equations. *Computer Methods in Applied Mechanics and Engineering*, 191(37-38):4093–4122, 2002.

[2] F. Bonizzoni, F. Nobile, and D. Kressner. Tensor train approximation of moment equations for elliptic equations with lognormal coefficient. *Computer Methods in Applied Mechanics and Engineering*, 308:349–376, 2016.

[3] D. Braess. *Finite Elemente: Theorie, Schnelle Löser und Anwendungen in der Elastizitätstheorie*. Springer, London, 2007.

[4] S. C. Brenner and L. R. Scott. *The Mathematical Theory of Finite Element Methods*. Springer, Berlin, 3rd edition, 2008.

[5] R. Caflisch. Monte Carlo and quasi-Monte Carlo methods. *Acta Numerica*, 7:1–49, 1998.

[6] C. Canuto and T. Kozubek. A fictitious domain approach to the numerical solution of PDEs in stochastic domains. *Numerische Mathematik*, 107(2):257–293, 2007.

[7] J. E. Castrillon-Candas, F. Nobile, and R. F. Tempone. Analytic regularity and collocation approximation for elliptic PDEs with random domain deformations. *Computers & Mathematics with Applications*, 71(6):1173–1197, 2016.

[8] J. E. Castrillon-Candas, F. Nobile, and R. F. Tempone. Hybrid collocation perturbation for pdes with random domains. arXiv preprint arXiv:1703.10040, 2017.

[9] A. Chernov and C. Schwab. First order k-th moment finite element analysis of nonlinear operator equations with stochastic data. *Mathematics of Computation*, 82(284):1859–1888, 2013.

[10] J. Dick, F. Kuo, Q. Le Gia, D. Nuyens, and C. Schwab. Higher order QMC Petrov–Galerkin discretization for affine parametric operator equations with random field inputs. *SIAM Journal on Numerical Analysis*, 52(6):2676–2702, 2014.

[11] R. N. Gantner and M. D. Peters. Higher order quasi-monte carlo for bayesian shape inversion. Preprint 2016-18, Mathematisches Institut, Universität Basel, 2016. (to appear in SIAM/ASA J. Uncertain. Quantif.).

[12] A.-L. Haji-Ali, H. Harbrecht, M. D. Peters, and M. Siebenmorgen. Novel results for the anisotropic sparse grid quadrature. *Journal of Complexity*, 2018.

[13] H. Harbrecht, M. Peters, and M. Siebenmorgen. Combination technique based k-th moment analysis of elliptic problems with random diffusion. *Journal of Computational Physics*, 252:128–141, 2013.
[14] H. Harbrecht, M. Peters, and M. Siebenmorgen. Efficient approximation of random fields for numerical applications. *Numerical Linear Algebra with Applications*, 22(4):596–617, 2015.

[15] H. Harbrecht, M. Peters, and M. Siebenmorgen. Analysis of the domain mapping method for elliptic diffusion problems on random domains. *Numerische Mathematik*, 134(4):823–856, 2016.

[16] H. Harbrecht, M. Peters, and M. Siebenmorgen. Multilevel accelerated quadrature for pdes with log-normally distributed diffusion coefficient. *SIAM/ASA Journal on Uncertainty Quantification*, 4(1):520–551, 2016.

[17] H. Harbrecht, M. D. Peters, and M. Schmidlin. Uncertainty quantification for pdes with anisotropic random diffusion. *SIAM Journal on Numerical Analysis*, 55(2):1002–1023, 2017.

[18] H. Harbrecht, R. Schneider, and C. Schwab. Sparse second moment analysis for elliptic problems in stochastic domains. *Numerische Mathematik*, 109(3):385–414, 2008.

[19] E. Hille and R. S. Phillips. *Functional analysis and semi-groups*, volume 31. American Mathematical Society, Providence, 1957.

[20] R. Hiptmair, L. Scarabosio, C. Schillings, and Ch. Schwab. Large deformation shape uncertainty quantification in acoustic scattering. *Advances in Computational Mathematics*, 2018.

[21] J. B. Keller. Stochastic equations and wave propagation in. *Stochastic processes in mathematical physics and engineering*, 16:145, 1964.

[22] M. Kleiber and T. D. Hien. *The Stochastic Finite Element Method: Basic Perturbation Technique and Computer Implementation*. Wiley, Chichester, 1992.

[23] M. Lenoir. Optimal isoparametric finite elements and error estimates for domains involving curved boundaries. *SIAM Journal on Numerical Analysis*, 23(3):562–580, 1986.

[24] F. Nobile, R. Tempone, and C. Webster. A sparse grid stochastic collocation method for partial differential equations with random input data. *SIAM Journal on Numerical Analysis*, 46(5):2309–2345, 2008.

[25] F. Nobile, R. Tempone, and C. G. Webster. An anisotropic sparse grid stochastic collocation method for partial differential equations with random input data. *SIAM Journal on Numerical Analysis*, 46(5):2411–2442, 2008.

[26] D. M. Tartakovsky and D. Xiu. Stochastic analysis of transport in tubes with rough walls. *Journal of Computational Physics*, 217(1):248–259, 2006.

[27] D. Xiu and D. M. Tartakovsky. Numerical methods for differential equations in random domains. *SIAM Journal on Scientific Computing*, 28(3):1167–1185, 2006.

[28] Jakob Zech and Christoph Schwab. Convergence rates of high dimensional smolyak quadrature. *SAM Research Report*, 2017, 2017.

Michael D. Peters. Department of Biosystems Science and Engineering, ETH Zurich, Switzerland.

E-mail address: michael.peters@bsse.ethz.ch