ON A TERMINAL VALUE PROBLEM FOR A SYSTEM OF PARABOLIC EQUATIONS WITH NONLINEAR-NONLOCAL DIFFUSION TERMS

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Abstract. We study a terminal value parabolic system with nonlinear-nonlocal diffusions. Firstly, we consider the issue of existence and ill-posed property of a solution. Then we introduce two regularization methods to solve the system in which the diffusion coefficients are globally Lipschitz or locally Lipschitz under some a priori assumptions on the sought solutions. The existence, uniqueness and regularity of solutions of the regularized problem are obtained. Furthermore, The error estimates show that the approximate solution converges to the exact solution in $L^2$ norm and also in $H^1$ norm.

1. Introduction. Let $T$ be a positive number and $\Omega$ be an open, bounded and connected domain in $\mathbb{R}^d$, $d \geq 1$ with a smooth boundary $\partial \Omega$. We are interested in the following inverse problem: Find $u(x,t), v(x,t)$ for $x \in \Omega, \; t \in [0,T]$, where $u, v$ satisfy the following system

\[
\begin{align*}
\frac{\partial u}{\partial t} - M(\|\nabla u\|_{L^2(\Omega)}(t), \|\nabla v\|_{L^2(\Omega)}(t))\Delta u &= f(x,t), \quad \text{in } \mathcal{D}_T, \\
\frac{\partial v}{\partial t} - N(\|\nabla u\|_{L^2(\Omega)}(t), \|\nabla v\|_{L^2(\Omega)}(t))\Delta v &= g(x,t), \quad \text{in } \mathcal{D}_T, \\
u &= v = 0, \quad \text{on } \partial \mathcal{D}_T, \\
u &= \Phi, \quad v = \Psi, \quad \text{in } \Omega \times \{T\},
\end{align*}
\]

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where \( \mathcal{D}_T := \Omega \times [0, T] \), the function \( \Phi, \Psi \in L^2(\Omega) \) and \( f, g \in L^2(0, T; L^2(\Omega)) \). The operator \(-\Delta\) admits the orthonormal eigenbasis \( \{\theta_p(x)\}_{p \geq 1} \) in \( L^2(\Omega) \), associated with the eigenvalues such that
\[
0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq ... \leq \lambda_p \leq ... \nearrow.
\] (2)

It is well known that for any \( p \in \mathbb{N}^* \), we have
\[
\langle -\Delta u(\cdot, t), \theta_p \rangle_{L^2(\Omega)} = \lambda_p \langle u(\cdot, t), \theta_p \rangle_{L^2(\Omega)}.
\] (3)

Let \( u(x, t) = \sum_{p=1}^{\infty} \langle u(\cdot, t), \theta_p \rangle_{L^2(\Omega)} \theta_p(x) \) be the Fourier series of \( u \) in the space \( L^2(\Omega) \).

In recent years, the nonlocal diffusion problems become an interesting topic. "Kirchhoff's" models type with nonlocal diffusion coefficients have many important applications in nonlinear elasticity, electrorheological fluids, image restoration, biological and ecological systems (see [6, 11, 1, 29, 23, 25, 24] and references therein). The initial value problems (IVPs) for short) for nonlocal diffusion is classical and has been so studied. For example,

- T. Caraballo et al. [10], considered the problem:
\[
\begin{cases}
\frac{\partial u}{\partial t} - M(\|\ell(u)\|_{L^2(H)}) \Delta u = f(u) + g(t), & \text{in } \Omega \times ]0, +\infty[, \\
u = 0, & \text{on } \partial \Omega \times ]0, +\infty[, \\
u = u_\tau, & \text{in } \Omega \times \{\tau\},
\end{cases}
\] (4)

where \( 0 < m \leq M(s) \leq M \), for all \( s \in ]-\infty, \infty[ \) and function \( M \) is locally Lipschitz continuous and depends on the entire population in the domain rather than on the local density, i.e., the evolution of population of species in a container depends on the global population. The authors study the asymptotic behavior of a time-dependent parabolic equation with nonlocal diffusion and nonlinear terms with sublinear growth. In 2018, [13], these same authors also considered the problem (4) by replacing the Laplace operator \( \Delta \) by the \( p \)-laplacian \( \Delta_p \), \( p \geq 2 \). They focussed on the existence of attractors in the phase spaces \( L^2(\Omega) \) and \( L^p(\Omega) \) for time-dependent \( p \)-Laplacian equations with nonlocal diffusion type. Extending the problem (4), Caraballo et al. [11, 12] considered the perturbed nonautonomous nonlocal reaction-diffusion equation
\[
\frac{\partial u}{\partial t} - (1 - \varepsilon) M(\|\ell(u)\|_{L^2(H)}) \Delta u = f(u) + \varepsilon g(t), \quad \text{in } \Omega \times ]0, +\infty[.
\] (5)

Under suitable assumptions, they proved that the family of pullback attractors converges to the corresponding global compact attractor associated with the autonomous nonlocal limit problem when the parameter \( \varepsilon \) goes to zero.

- In [20], M. Gobbino investigated parabolic equations of Kirchhoff type with \((\mathcal{L})\) a self-adjoint linear non-negative operator on a Hilbert space \( H \), namely,
\[
\begin{cases}
\frac{\partial u}{\partial t} + M \left( \left\| (-\mathcal{L})^{\frac{1}{2}} u \right\|^2_H(t) \right) (-\mathcal{L}) u = 0, & \forall t \in ]0, +\infty[, \\
u(0) = u_0 \in H.
\end{cases}
\] (6)

The author proves that the problem (6) admits at least one global solution, is well posed in \( H \), or has no solution under suitable hypotheses on \( u_0 \) and conditions on \( M \). In [18, 19], M. Ghisi and M. Gobbino considered the second order Cauchy
problem
\[
\begin{cases}
    b(t) \frac{\partial u}{\partial t} + M \left( \left\| (-L)^{\frac{1}{2}} u \right\|_{L^2(\Omega)}^2 \right) (-L) u = 0, & \forall t \in ]0, +\infty[, \\
    u(0) = u_0 \in H.
\end{cases}
\tag{7}
\]

The authors proved decay estimates as \( t \to \infty \) for solutions of problem (7). They also showed that their decay rates are optimal in many cases. The interesting results for parabolic equations for Kirchhoff type (nonlocal terms) may be found in [17, 16, 14, 30] and references therein.

- In 2016, R.M.P. Almeida et al. [1] studied existence, uniqueness, long-time behaviour and localization properties of solutions for the non-local system:
\[
\begin{align*}
    \frac{\partial u}{\partial t} - M(\ell_1(u)(t), \ell_2(v)(t)) \Delta u + au^{r-2} u &= f, & \text{in } \mathcal{D}_T, \\
    \frac{\partial v}{\partial t} - N(\ell_1(u)(t), \ell_2(v)(t)) \Delta u + bv^{r-2} v &= g, & \text{in } \mathcal{D}_T, \\
    u = v = 0, & \text{on } \partial \mathcal{D}_T, \\
    u = u_0, & \text{in } \Omega \times \{0\},
\end{align*}
\tag{8}
\]

where \( a, b \) are non-negative constants and \( r > 1, f, g \in L^2(0, T; L^2(\Omega)) \), the diffusion coefficients \( M, N : \mathbb{R}^2 \to \mathbb{R} \) are linear and bounded, \( \ell_i : L^2(\Omega) \to \mathbb{R}, i = 1, 2 \) is a continuous linear form.

- There is also the work of Boulaaras et al. [8, 28] who considered the following problem
\[
\begin{align*}
    \frac{\partial u}{\partial t} - M(\int_{\Omega} |\nabla u|^2 \, dx) \Delta u &= f(u, v), & \text{in } \mathcal{D}_T, \\
    \frac{\partial v}{\partial t} - N(\int_{\Omega} |\nabla v|^2 \, dx) \Delta v &= g(u, v), & \text{in } \mathcal{D}_T, \\
    u = v = 0, & \text{on } \partial \mathcal{D}_T, \\
    u = u_0, & \text{in } \Omega \times \{0\}.
\end{align*}
\tag{9}
\]

For \( f, g \) satisfying suitable conditions, the authors have shown that system (9) has a positive solution.

Up to now, the study of final value problems ((FVPs) for short) for nonlocal diffusion is very scarce. The properties of (FVPs) are very different from those of (IVPs). For (IVPs), we often investigate the existence, blow up, decay, etc. The (FVPs) are not well-posed (discussed in Section 3) whereupon the instability in the sense of Hadamard occurs. Therefore, our purpose is to propose regularized (approximate) problems. Recently, in [33], Tuan et al. considered the following problem:
\[
\frac{\partial u}{\partial t} = M(\|\nabla u\|_{L^2(\Omega)}(t)) \Delta u + f(x, t; u), & \text{in } \mathcal{D}_T, 
\tag{10}
\]
subject to the conditions
\[
\begin{cases}
    u = 0, & \text{on } \partial \mathcal{D}_T, \\
    u = u_T, & \text{in } \Omega \times \{T\}.
\end{cases}
\tag{11}
\]
They used Fourier truncation method and Quasi-reversibility ((QR) for short) to tackle the regularized problems and they presented some stability estimates in $H^1_0$ norm. The results of Tuan et al. for nonlocal problems can be found in [31, 32].

Until now, to our knowledge, there is no work dealing with the system (1) with the nonlocal diffusion terms $M, N$ in the form $M(\|\nabla u\|_{L^2(\Omega)}(t), \|\nabla v\|_{L^2(\Omega)}(t))$ and $N(\|\nabla u\|_{L^2(\Omega)}(t), \|\nabla v\|_{L^2(\Omega)}(t))$. The System (1) is an ill-posed problem, so we need to find suitable methods to construct approximation problems. This problem is not simple due to the fact that the diffusion coefficients $M, N$ contain gradient terms of $u, v$. System (1) can be transformed into a system of nonlinear integral equations. Some classical methods and previous techniques are not applicable to approximate System (1). This case is more difficult for investigation and a new method is required. In this paper, we consider both cases of nonlocal diffusion coefficients that are either global or local Lipschitz functions. For global Lipschitz $M, N$, we apply the (QR) method, for the case of local Lipschitz diffusions, the modified quasi-reversibility method is applied to regularize the System (1).

The plan of the paper is as follows:

Section 2. Preliminaries: abstract framework.

Section 3. Formulations solution of the system and proof of ill-posedness of the System (1) that has a unique mild solution $[u, v] \in [C([0,T]; H^1(\Omega))]^2$.

Section 4. Proof of the existence, uniqueness and regularity of the regularized solutions.

In section 5, we propose the (QR) method and modified (QR) method to regularize System (1) in the two cases of diffusions $M, N$ satisfying either the global Lipschitz (Section 5.1) or local Lipschitz conditions (Section 5.2), respectively. The errors analysis in $L^2(\Omega)$ and $H^1(\Omega)$ have been investigated.

2. Preliminaries. For the Banach space $X$, the norm of $[u, v] \in X \times X$ is defined as

$$\|[u, v]\|_{X^2} = \|u\|_X + \|v\|_X.$$ 

We state the following hypotheses:

(Hyp1): The measurable functions $M, N > 0$, are such that the mappings

$$[\xi, \eta] \mapsto M(\xi, \eta),$$

$$[\xi, \eta] \mapsto N(\xi, \eta),$$

are continuous for $[\xi, \eta] \in \mathbb{R}^2$;

(Hyp2): There exist positive constants $\underline{M}$ and $\overline{M}$ such that

$$\underline{M} \leq M(\xi, \eta) \leq \overline{M}, \quad \text{for all} \quad \xi, \eta \in \mathbb{R},$$

and there exist positive constants $\underline{N}$ and $\overline{N}$ such that

$$\underline{N} \leq N(\xi, \eta) \leq \overline{N}, \quad \text{for all} \quad \xi, \eta \in \mathbb{R};$$

(Hyp3): There exist positive constants $K_M, K_N$ such that $\forall \xi_1, \xi_2, \eta_1, \eta_2 \in \mathbb{R},$

$$|M(\xi_1, \eta_1) - M(\xi_2, \eta_2)| \leq K_M(|\xi_1 - \xi_2| + |\eta_1 - \eta_2|),$$

$$|N(\xi_1, \eta_1) - N(\xi_2, \eta_2)| \leq K_N(|\xi_1 - \xi_2| + |\eta_1 - \eta_2|);$$

(Hyp4): $\Phi, \Psi \in L^2(\Omega)$ represent the exact data, whilst $\Phi^\varepsilon, \Psi^\varepsilon \in L^2(\Omega)$ represent the measured data with noise level $\varepsilon > 0$, such that

$$\|[\Phi^\varepsilon, \Psi^\varepsilon] - [\Phi, \Psi]\|_{L^2(\Omega)^2} \leq \varepsilon.$$
We let \( f^\varepsilon, g^\varepsilon \in L^\infty(0,T;L^2(\Omega)) \) to be the perturbed functions of \( f, g \in L^\infty(0,T;L^2(\Omega)) \) satisfying
\[
∥[f^\varepsilon, g^\varepsilon] - [f, g]∥_{L^\infty(0,T;L^2(\Omega))} ≤ \varepsilon.
\]

We will use the Banach spaces \( C([0,T];X), L^\infty(0,T;X) \) of real measurable functions \( u : \mathbb{R}_T \to X \) (\( X \) is also a Banach space), such that
\[
\|u\|_{C([0,T];X)} = \sup_{t \in [0,T]} \|u(\cdot, t)\|_X < \infty, \quad (12)
\]
and
\[
\|u\|_{L^p(0,T;X)} = \left( \int_0^T \|u(\cdot, t)\|_X^p \, dt \right)^{1/p} < \infty, \quad \forall t \in [0,T],
\]
\[
\|u\|_{L^\infty(0,T;X)} = \text{ess sup}_{t \in [0,T]} \|u(\cdot, t)\|_X < \infty.
\]

The space \( H^1_0(\Omega) \) is equipped with norm and inner product
\[
\|u\|_{H^1_0(\Omega)} = \|\nabla u\|_{L^2(\Omega)} = \left( \int_\Omega |\nabla u(x)|^2 \, dx \right)^{1/2},
\]
\[
\langle u_1, u_2 \rangle_{H^1_0(\Omega)} = \int_\Omega \nabla u_1(x) \cdot \nabla u_2(x) \, dx.
\]
The space
\[
H^q(\Omega) = \left\{ h \in L^2(\Omega) : \sum_{p=1}^\infty \langle h, \theta_p \rangle_{L^2(\Omega)}^2 \lambda_p^q < \infty \right\}, \quad (13)
\]
is equipped with norm
\[
\|h\|_{H^q(\Omega)} = \left( \sum_{p=1}^\infty \langle h, \theta_p \rangle_{L^2(\Omega)}^2 \lambda_p^q \right)^{1/2}. \quad (14)
\]
For some \( \sigma_1, \sigma_2 > 0 \), the Gevrey-type space
\[
\mathcal{G}(\sigma_1 e^{\sigma_2(-\Delta)}) = \left\{ h \in L^2(\Omega) : \sum_{p=1}^\infty \langle h, \theta_p \rangle_{L^2(\Omega)}^2 \lambda_p^{2\sigma_1} e^{2\sigma_2 \lambda_p} < \infty \right\}, \quad (15)
\]
is equipped with the norm
\[
\|h\|_{\mathcal{G}(\sigma_1 e^{\sigma_2(-\Delta)})} = \left( \sum_{p=1}^\infty \langle h, \theta_p \rangle_{L^2(\Omega)}^2 \lambda_p^{2\sigma_1} e^{2\sigma_2 \lambda_p} \right)^{1/2} < \infty. \quad (16)
\]
For \( h \in \mathcal{G}(\sigma_1 e^{\sigma_2(-\Delta)}) \) the Fourier coefficients of \( h \) must decay exponentially as \( p \to \infty \).

3. Mild solution and Ill-posed property. System (1) admits a solution.

**Theorem 3.1.** The System (1) has a unique mild solution \([u, v] \in [C([0,T];H^1(\Omega))]^2\) which satisfies (21)-(22). Furthermore, it is not stable in \( L^2 \)-norm.
Proof. Let \( u(x, t) = \sum_{p=1}^{\infty} u_p(t) \theta_p(x) \) be the Fourier series of \( u \) in \( L^2(\Omega) \). For \( p \geq 1 \), from (1), we can obtain the following ordinary differential equation with given data at \( t = T \):

\[
\begin{aligned}
\frac{d}{dt} u_p(t) + \lambda_p \mathcal{M} (\|\nabla u\|_{L^2(\Omega)}(t), \|\nabla v\|_{L^2(\Omega)}(t)) u_p(t) &= f_p(t), \\
 u_p(T) &= \Phi_p,
\end{aligned}
\]

Integrating (17), we obtain

\[
u_p(t) = \exp \left\{ \lambda_p \int_T^t \mathcal{M} (\|\nabla u\|_{L^2(\Omega)}(s), \|\nabla v\|_{L^2(\Omega)}(s)) \, ds \right\} \Phi_p
- \int_t^T \exp \left\{ \lambda_p \int_s^t \mathcal{M} (\|\nabla u\|_{L^2(\Omega)}(\tau), \|\nabla v\|_{L^2(\Omega)}(\tau)) \, d\tau \right\} f_p(s) \, ds.
\] (18)

Similarly, we obtain

\[
v_p(t) = \exp \left\{ \lambda_p \int_T^t \mathcal{N} (\|\nabla u\|_{L^2(\Omega)}(s), \|\nabla v\|_{L^2(\Omega)}(s)) \, ds \right\} \Psi_p
- \int_t^T \exp \left\{ \lambda_p \int_s^t \mathcal{N} (\|\nabla u\|_{L^2(\Omega)}(\tau), \|\nabla v\|_{L^2(\Omega)}(\tau)) \, d\tau \right\} g_p(s) \, ds.
\] (19)

For convenience, let

\[
E_{(a, b)}^{\lambda_p} \left\{ \mathcal{M} (\|\nabla u\|_{L^2(\Omega)}(t), \|\nabla v\|_{L^2(\Omega)}(t)) \right\}
= \exp \left\{ \lambda_p \int_a^b \mathcal{M} (\|\nabla u\|_{L^2(\Omega)}(t), \|\nabla v\|_{L^2(\Omega)}(t)) \, dt \right\}.\ (20)
\]

Hence, the solution of system (1) is represented by

\[
u(x, t) = \sum_{p=1}^{\infty} \left[ E_{(a, T)}^{\lambda_p} \left\{ \mathcal{M} (\|\nabla u\|_{L^2(\Omega)}(s), \|\nabla v\|_{L^2(\Omega)}(s)) \right\} \Phi_p \right] \theta_p(x)
- \sum_{p=1}^{\infty} \left[ \int_t^T E_{(t, s)}^{\lambda_p} \left\{ \mathcal{M} (\|\nabla u\|_{L^2(\Omega)}(\tau), \|\nabla v\|_{L^2(\Omega)}(\tau)) \right\} f_p(s) \, ds \right] \theta_p(x),
\] (21)

\[
v(x, t) = \sum_{p=1}^{\infty} \left[ E_{(a, T)}^{\lambda_p} \left\{ \mathcal{N} (\|\nabla u\|_{L^2(\Omega)}(s), \|\nabla v\|_{L^2(\Omega)}(s)) \right\} \Psi_p \right] \theta_p(x)
- \sum_{p=1}^{\infty} \left[ \int_t^T E_{(t, s)}^{\lambda_p} \left\{ \mathcal{N} (\|\nabla u\|_{L^2(\Omega)}(\tau), \|\nabla v\|_{L^2(\Omega)}(\tau)) \right\} g_p(s) \, ds \right] \theta_p(x).
\] (22)

For any \( \xi_1, \xi_2 \in C([0, T]; H^1(\Omega)) \), we consider the function

\[
\mathbf{M} : [C([0, T]; H^1(\Omega))]^2 \rightarrow [C([0, T]; H^1(\Omega))]^2,
\]

with

\[
\mathbf{M}(\xi_1, \xi_2)(x, t) = \left[ \mathfrak{M}_1(\xi_1, \xi_2)(x, t); \mathfrak{M}_2(\xi_1, \xi_2)(x, t) \right],
\]
where
\[
M_1(\xi_1, \xi_2)(x, t) := \sum_{p=1}^{\infty} \left[ E_{(t, T)}^{\lambda_p} \left\{ \mathcal{M} \left( \| \nabla \xi_1 \|_{L^2(\Omega)}(s), \| \nabla \xi_2 \|_{L^2(\Omega)}(s) \right) \right\} \theta_p(x) - \int_t^T E_{(t, \tau)}^{\lambda_p} \left\{ \mathcal{M} \left( \| \nabla \xi_1 \|_{L^2(\Omega)}(\tau), \| \nabla \xi_2 \|_{L^2(\Omega)}(\tau) \right) \right\} f_p(s)ds \right] \theta_p(x),
\]
\[
M_2(\xi_1, \xi_2)(x, t) := \sum_{p=1}^{\infty} \left[ E_{(t, T)}^{\lambda_p} \left\{ \mathcal{N} \left( \| \nabla \xi_1 \|_{L^2(\Omega)}(s), \| \nabla \xi_2 \|_{L^2(\Omega)}(s) \right) \right\} \psi_p \theta_p(x) - \int_t^T E_{(t, \tau)}^{\lambda_p} \left\{ \mathcal{N} \left( \| \nabla \xi_1 \|_{L^2(\Omega)}(\tau), \| \nabla \xi_2 \|_{L^2(\Omega)}(\tau) \right) \right\} g_p(s)ds \right] \theta_p(x).
\]

We also define \(M^m, M_1^m, M_2^m\) as follows
\[
M^m(\xi_1, \xi_2) := \underbrace{MM \cdots [M(\xi_1, \xi_2)]}_{m \text{-times}},
\]
and
\[
M_1^m(\xi_1, \xi_2) := \underbrace{M_1 M_1 \cdots [M_1(\xi_1, \xi_2)]}_{m \text{-times}}, \quad M_2^m(\xi_1, \xi_2) := \underbrace{M_2 M_2 \cdots [M_2(\xi_1, \xi_2)]}_{m \text{-times}}.
\]

Now, we show the existence and uniqueness of a solution of the nonlinear equation (18). For \([\xi_1, \xi_2], [\xi_1, \xi_2] \in [C([0, T]; H^1(\Omega))^2\), we shall prove by induction (for any \(m \geq 1\)) that
\[
\|M^m(\xi_1, \xi_2)(\cdot, t) - M^m(\xi_1, \xi_2)(\cdot, t)\|_{H^1(\Omega)}^2 \leq \sqrt{\frac{2K_{\max}^2 C_{\max}^2}{m!}} (T - t)^m \|\xi_1 - \xi_2\|_{C([0, T]; H^1(\Omega))^2}^2.
\]
(23)

For \(m = 1\), using the inequality
\[
|e^a - e^b| \leq |a - b| \max\{e^a, e^b\}, \quad \text{for any} \ a, b \in \mathbb{R},
\]
we obtain
\[
\|M_1(\xi_1, \xi_2)(\cdot, t) - M_1(\xi_1, \xi_2)(\cdot, t)\|_{H^1(\Omega)}^2 \leq \sum_{p=1}^{\infty} \lambda_p \left[ E_{(t, T)}^{\lambda_p} \left\{ \mathcal{M} \left( \| \nabla \xi_1 \|_{L^2(\Omega)}, \| \nabla \xi_2 \|_{L^2(\Omega)} \right) \right\} \right]^2 \|\theta_p\|^2 - \int_t^T E_{(t, \tau)}^{\lambda_p} \left\{ \mathcal{M} \left( \| \nabla \xi_1 \|_{L^2(\Omega)}, \| \nabla \xi_2 \|_{L^2(\Omega)} \right) \right\} f_p(s)ds \right]^2
+ \sum_{p=1}^{\infty} \lambda_p \left[ E_{(t, \tau)}^{\lambda_p} \left\{ \mathcal{M} \left( \| \nabla \xi_1 \|_{L^2(\Omega)}, \| \nabla \xi_2 \|_{L^2(\Omega)} \right) \right\} \right] f_p(s)ds \right]^2
= : M_{11}(\xi_1, \xi_2, \xi_1, \xi_2)(t) + M_{12}(\xi_1, \xi_2, \xi_1, \xi_2)(t).
\]
(25)

First, \(M_{11}(\xi_1, \xi_2, \xi_1, \xi_2)(t)\) is estimated as follows:
\[
|M_{11}(\xi_1, \xi_2, \xi_1, \xi_2)(t)|
\]
way, we obtain

Then, we get inequality (24) and Poincare’s inequality where we have used hypothesis (Hyp3), Parseval’s relation, Hölder’s inequality, the

Similarly,

where we have used hypothesis (Hyp3), Parseval’s relation, Hölder’s inequality, the inequality (24) and Poincare’s inequality \( \| \nabla v \|_{L^2(\Omega)} \leq C_0 \| v \|_{H^1(\Omega)} \). In a similar way, we obtain

Then, we get

where

Similarly,

where
Then, we get
\[
\|M(\xi_1, \xi_2)(\cdot, t) - M(\zeta_1, \zeta_2)(\cdot, t)\|_{H^1(\Omega)}^2 \\
\leq 2 \|M_1(\xi_1, \xi_2)(\cdot, t) - M_1(\zeta_1, \zeta_2)(\cdot, t)\|_{H^1(\Omega)}^2 \\
+ 2 \|M_2(\xi_1, \xi_2)(\cdot, t) - M_2(\zeta_1, \zeta_2)(\cdot, t)\|_{H^1(\Omega)}^2 \\
\leq 4K_{\max}^2 C_0^2 C_1^2 C_{\max, f, g, \phi, \psi}^2 (T - t) \|\xi_1 - \zeta_1\|_{C([0,T];H^1(\Omega))}^2 \\
\] where
\[
K_{\max} = \max\{K_M, K_N\}, \quad C_{\max, f, g, \phi, \psi} = \max\left\{C_{1, \overline{M}, f, \phi}; C_{1, N, g, \psi}\right\}. \quad (28)
\]
Assume that (23) holds for \(m = m_0\). Let us show that (23) holds for \(m = m_0 + 1\). In fact, we have
\[
\|M_1^{m_0+1}(\xi_1, \xi_2)(\cdot, t) - M_1^{m_0+1}(\zeta_1, \zeta_2)(\cdot, t)\|_{H^1(\Omega)}^2 \\
\leq 4K_{\max}^2 C_0^2 C_1^2 C_{\max, f, g, \phi, \psi}^2 \int_t^T \|M_1^{m_0}(\xi_1, \xi_2)(\cdot, s) - M_1^{m_0}(\zeta_1, \zeta_2)(\cdot, s)\|_{H^1(\Omega)}^2 ds \\
\leq \left(\frac{2K_{\max}^2 C_0^2 C_1^2 C_{\max, f, g, \phi, \psi}^2}{m_0!}\right)^{m_0+1} \|\xi_1 - \zeta_1\|_{C([0,T];H^1(\Omega))}^2 \int_t^T (T - s)^{m_0} ds \\
\leq \left(\frac{2K_{\max}^2 C_0^2 C_1^2 C_{\max, f, g, \phi, \psi}^2}{(m_0 + 1)!}\right)^{m_0+1} (T - t)^{m_0+1} \|\xi_1 - \zeta_1\|_{C([0,T];H^1(\Omega))}^2 \]
Using a similar argument, we obtain the estimation for \(M_1^{m_0+1}\). By induction principle, we deduce that (23) holds for all \(m \in \mathbb{N}^*\). Indeed, we have
\[
\|M^m(\xi_1, \xi_2) - M^m(\zeta_1, \zeta_2)\|_{C([0,T];H^1(\Omega))}^2 \\
\leq \left(\frac{2K_{\max}^2 C_0^2 C_1^2 C_{\max, f, g, \phi, \psi}^2}{m!}\right)^m (T - t)^m \|\xi_1 - \zeta_1\|_{C([0,T];H^1(\Omega))}^2 \\
\]
Since
\[
\lim_{m \to \infty} \left(\frac{2K_{\max}^2 C_0^2 C_1^2 C_{\max, f, g, \phi, \psi}^2}{m!}\right)^m = 0,
\]
there exists a positive integer number \(m_0\) such that \(M^{m_0}\) is a contraction. It follows then that the equation \(M^{m_0}(w_1, w_2) = [w_1, w_2]\) has a unique solution \([w_1, w_2] \in C([0,T];H^1(\Omega))\). We claim that \(M(w_1, w_2) = [w_1, w_2]\). In fact, since \(M^{m_0}(w_1, w_2) = (w_1, w_2)\), we know that \(M(M^{m_0}(w_1, w_2)) = M(w_1, w_2)\). This is equivalent to \(M^{m_0}(M(w_1, w_2)) = M(u)\). Hence, \(M(w_1, w_2)\) is a fixed point of \(M^{m_0}\). Moreover, as noted above, \([w_1, w_2]\) is a fixed point of \(M^{m_0}\).
Next, we will show that the solution of System (1) is not stable. For any \( m \in \mathbb{N} \), let us set
\[
\begin{cases}
\tilde{\Phi}_m(x) = \frac{\theta_m(x)}{\lambda_m}, & \text{for all } x \in \Omega, \\
\tilde{\Psi}_m(x) = \frac{\theta_m(x)}{\lambda_m}, & \text{for all } x \in \Omega, \\
f_m(x, t) = e^{-\lambda_m TM} \theta_m(x), & \text{for all } (x, t) \in \Omega \times (0, T), \\
g_m(x, t) = e^{-\lambda_m TN} \theta_m(x), & \text{for all } (x, t) \in \Omega \times (0, T).
\end{cases}
\]

(29)

Let \( \tilde{u}_m, \tilde{v}_m \) satisfy the integral equations
\[
\begin{align*}
\tilde{u}_m(x, t) &= \sum_{p=1}^{\infty} \left[ E_{(t, T)}^{\lambda_p} \left\{ M \left( \| \nabla \tilde{u}_m \|_{L^2(\Omega)}(s), \| \nabla \tilde{v}_m \|_{L^2(\Omega)}(s) \right) \right\} \tilde{\Phi}_m \right] \theta_p(x) \\
&- \sum_{p=1}^{\infty} \left[ \int_{t}^{T} E_{(t, s)}^{\lambda_p} \left\{ M \left( \| \nabla \tilde{u}_m \|_{L^2(\Omega)}(\tau), \| \nabla \tilde{v}_m \|_{L^2(\Omega)}(\tau) \right) \right\} \tilde{f}_m(s) \, ds \right] \theta_p(x), \\
\tilde{v}_m(x, t) &= \sum_{p=1}^{\infty} \left[ E_{(t, T)}^{\lambda_p} \left\{ M \left( \| \nabla \tilde{u}_m \|_{L^2(\Omega)}(s), \| \nabla \tilde{v}_m \|_{L^2(\Omega)}(s) \right) \right\} \tilde{\Psi}_m \right] \theta_p(x) \\
&- \sum_{p=1}^{\infty} \left[ \int_{t}^{T} E_{(t, s)}^{\lambda_p} \left\{ M \left( \| \nabla \tilde{u}_m \|_{L^2(\Omega)}(\tau), \| \nabla \tilde{v}_m \|_{L^2(\Omega)}(\tau) \right) \right\} \tilde{g}_m(s) \, ds \right] \theta_p(x),
\end{align*}
\]

(30)

(31)

where
\[
\begin{align*}
\tilde{\Phi}_m &= \left\langle \tilde{\Phi}_m, \theta_p \right\rangle_{L^2(\Omega)} = \left\langle \tilde{\Phi}_m, \theta_p \right\rangle_{L^2(\Omega)}, \\
\tilde{\Psi}_m &= \left\langle \tilde{\Psi}_m, \theta_p \right\rangle_{L^2(\Omega)}, \\
\tilde{f}_m &= \left\langle \tilde{f}_m, \theta_p \right\rangle_{L^2(\Omega)}, \\
\tilde{g}_m &= \left\langle \tilde{g}_m, \theta_p \right\rangle_{L^2(\Omega)}.
\end{align*}
\]

By an argument analogous to the one above, we conclude that the equation (32) has a unique solution \( \tilde{u}_m \in C([0, T]; H^1(\Omega)) \). Now, noting that \( \| \theta_p \|_{L^2(\Omega)} = 1 \)
\[
\begin{align*}
\| \tilde{u}_m(t) \|_{L^2(\Omega)} &\geq \left\| \sum_{p=1}^{\infty} \left[ E_{(t, T)}^{\lambda_p} \left\{ M \left( \| \nabla \tilde{u}_m \|_{L^2(\Omega)}(s), \| \nabla \tilde{v}_m \|_{L^2(\Omega)}(s) \right) \right\} \right\] \theta_p(t) \right\|_{L^2(\Omega)} \\
&- \left\| \sum_{p=1}^{\infty} \left[ \int_{t}^{T} E_{(t, s)}^{\lambda_p} \left\{ M \left( \| \nabla \tilde{u}_m \|_{L^2(\Omega)}(\tau), \| \nabla \tilde{v}_m \|_{L^2(\Omega)}(\tau) \right) \right\} \right\] \theta_p(t) \right\|_{L^2(\Omega)} \\
&\geq \left\| \sum_{p=1}^{\infty} \left[ \exp \left( \lambda_p \int_{t}^{T} M \, ds \right) \left\langle \frac{\theta_m}{\lambda_m}, \theta_p \right\rangle_{L^2(\Omega)} \right\] \theta_p \right\|_{L^2(\Omega)}.
\end{align*}
\]
Existence - uniqueness - regularity results of the regularized system.

This shows that system (1) ill-posed in the sense of Hadamard in the $E^\beta < M\theta$ for any function $u_0, v_0$. However, we get

$$\lim_{m \to \infty} \|\tilde{u}(m)\|_{C([0,T];L^2(\Omega))} = \lim_{m \to \infty} \|\tilde{v}(m)\|_{L^2(\Omega)} = \lim_{m \to \infty} \frac{1}{\lambda_m} = 0,$$

However,

$$\lim_{m \to \infty} \left(\|\tilde{u}(m)\|_{C([0,T];L^2(\Omega))} + \|\tilde{v}(m)\|_{C([0,T];L^2(\Omega))}\right) = \lim_{m \to \infty} \left(\frac{e^{\lambda_m T M} + e^{\lambda_m T N}}{\lambda_m} - 2T\right) = \infty.$$

This shows that system (1) ill-posed in the sense of Hadamard in the $L^2$-norm. 

4. Existence - uniqueness - regularity results of the regularized system.

Using the quasi-reversibility method, we introduce the following regularized problem

$$\begin{cases}
\partial_t u^\varepsilon_\beta + \mathcal{M}(\|\nabla u^\varepsilon_\beta\|_{L^2(\Omega)}(t), \|\nabla v^\varepsilon_\beta\|_{L^2(\Omega)}(t))\Delta^\beta u^\varepsilon_\beta = f^\varepsilon(x, t), & \text{in } \mathcal{D}_T, \\
\partial_t v^\varepsilon_\beta + N(\|\nabla u^\varepsilon_\beta\|_{L^2(\Omega)}(t), \|\nabla v^\varepsilon_\beta\|_{L^2(\Omega)}(t))\Delta^\beta v^\varepsilon_\beta = g^\varepsilon(x, t), & \text{in } \mathcal{D}_T, \\
u^\varepsilon_\beta = v^\varepsilon_\beta = 0, & \text{on } \partial \mathcal{D}_T, \\
u^\varepsilon_\beta = \Phi^\varepsilon, v^\varepsilon_\beta = \Psi^\varepsilon, & \text{in } \Omega \times \{T\},
\end{cases}
$$

where $\Delta^\beta$ is the linear operator defined by

$$\Delta^\beta w = \sum_{p=1}^\infty \lambda_p^\beta \left< w(\cdot, t), \theta_p \right>_{L^2(\Omega)} \theta_p(x),$$

with

$$\lambda_p^\beta := -\frac{1}{\min \{\overline{M}; N\} T} \log \left(\beta + e^{-\overline{M} T \lambda_p} \right),$$

for any function $w \in L^2(\Omega)$ and the positive constant $\beta := \beta(\varepsilon)$ plays the role of a regularization parameter that satisfies $\beta < 1 - \max \left\{ e^{-\overline{M} T \lambda_p}; e^{-\overline{N} T \lambda_p} \right\}$. Hence, the solution of regularized system (35) is represented by

$$u^\varepsilon_\beta(x, t) = \sum_{p=1}^\infty \left[ E_{(t, T)}^{\beta_p} \left\{ \mathcal{M}(\|\nabla u^\varepsilon_\beta\|_{L^2(\Omega)}(s), \|\nabla v^\varepsilon_\beta\|_{L^2(\Omega)}(s)) \right\} \Phi^\varepsilon \right] \theta_p(x)$$

$$- \sum_{p=1}^\infty \left[ \int_t^T E_{(t, s)}^{\beta_p} \left\{ \mathcal{M}(\|\nabla u^\varepsilon_\beta\|_{L^2(\Omega)}(\tau), \|\nabla v^\varepsilon_\beta\|_{L^2(\Omega)}(\tau)) \right\} f^\varepsilon(s) ds \right] \theta_p(x),$$

(37)
\[ v_2^\alpha(x,t) = \sum_{p=1}^{\infty} \left[ E_{(t,T)}^{\lambda_p} \left\{ N\left( \| \nabla u_2^\alpha \|_{L^2(\Omega)}(s), \| \nabla v_2^\alpha \|_{L^2(\Omega)}(s) \right) \right\} \psi_p^\alpha \right] \theta_p(x) \\
- \sum_{p=1}^{\infty} \left[ \int_t^T E_{(t,s)}^{\lambda_p} \left\{ N\left( \| \nabla u_2^\alpha \|_{L^2(\Omega)}(\tau), \| \nabla v_2^\alpha \|_{L^2(\Omega)}(\tau) \right) \right\} g_p^\alpha(s)ds \right] \theta_p(x). \]

Before stating the results of this section we first define, for \( \beta \in (0,1) \) (which will be assumed from now on) and a function \( w \in C([0,T]; X) \) (\( X \) is Hilbert space), the scaling with \( \beta \) as follows:

\[ S_\beta(X) := \left\{ w \in X : \sup_{t \in [0,T]} \beta^{-\frac{1}{2}} \| w(\cdot,t) \|_X < \infty \right\}, \]

with the norm

\[ \| w \|_{(\beta,X)} = \sup_{t \in [0,T]} \beta^{-\frac{1}{2}} \| w(\cdot,t) \|_X. \]

**Theorem 4.1 (Existence result).** The integral equations (37) and (38) have the solution \( u_2^\alpha, v_2^\alpha \in S_\beta(H^1(\Omega)). \)

For any \( \xi_1, \xi_2 \in S_\beta(H^1(\Omega)) \), we consider the mapping

\[ W : [S_\beta(H^1(\Omega))]^2 \to [S_\beta(H^1(\Omega))]^2, \]

with

\[ W(\xi_1, \xi_2)(x,t) = \left[ \mathcal{W}_1(\xi_1, \xi_2)(x,t); \mathcal{W}_2(\xi_1, \xi_2)(x,t) \right], \]

where

\[ \mathcal{W}_1(\xi_1, \xi_2) = \sum_{p=1}^{\infty} \left[ E_{(t,T)}^{\lambda_p} \left\{ M\left( \| \nabla \xi_1 \|_{L^2(\Omega)}(s), \| \nabla \xi_2 \|_{L^2(\Omega)}(s) \right) \right\} \theta_p(x) \\
- \sum_{p=1}^{\infty} \left[ \int_t^T E_{(t,s)}^{\lambda_p} \left\{ M\left( \| \nabla \xi_1 \|_{L^2(\Omega)}(\tau), \| \nabla \xi_2 \|_{L^2(\Omega)}(\tau) \right) \right\} f_p^\alpha(s)ds \right] \theta_p(x), \]

\[ \mathcal{W}_2(\xi_1, \xi_2) = \sum_{p=1}^{\infty} \left[ E_{(t,T)}^{\lambda_p} \left\{ N\left( \| \nabla \xi_1 \|_{L^2(\Omega)}(s), \| \nabla \xi_2 \|_{L^2(\Omega)}(s) \right) \right\} \psi_p^\alpha \right] \theta_p(x) \\
- \sum_{p=1}^{\infty} \left[ \int_t^T E_{(t,s)}^{\lambda_p} \left\{ N\left( \| \nabla \xi_1 \|_{L^2(\Omega)}(\tau), \| \nabla \xi_2 \|_{L^2(\Omega)}(\tau) \right) \right\} g_p^\alpha(s)ds \right] \theta_p(x). \]

We also define \( W^m, \mathcal{W}_1^m, \mathcal{W}_2^m \) as follows

\[ W^m(\xi_1, \xi_2) := \overbrace{W \cdots W}^{m \text{ times}} \left[ \mathcal{W}(\xi_1, \xi_2) \right], \]

and

\[ \mathcal{W}_1^m(\xi_1, \xi_2) := \overbrace{\mathcal{W}_1 \cdots \mathcal{W}_1(\xi_1, \xi_2)}^{m \text{ times}}, \quad \mathcal{W}_2^m(\xi_1, \xi_2) := \overbrace{\mathcal{W}_2 \cdots \mathcal{W}_2(\xi_1, \xi_2)}^{m \text{ times}}. \]

Now, we show the existence and uniqueness of solutions of the nonlinear equation (18). For \( [\xi_1, \xi_2], [\xi_1, \xi_2] \in [S_\beta(H^1(\Omega))]^2 \), we shall prove by induction (for \( \forall m \geq 1 \)) the estimate

\[ \| W^m(\xi_1, \xi_2) - W^m(\xi_1, \xi_2) \|_{[S_\beta(H^1(\Omega))]^2}. \]
where the constants \( K_{\text{max}}, C_T \) are defined as in Theorem 3.1 and
\[
C_{2,\text{max},f^r,g^r,\Phi^r,\Psi^r} = \max \{ C_{2,\text{max},f^r,\Phi^r}; C_{2,\text{max},g^r,\Psi^r} \},
\]
\[
C_{2,\text{max},f^r,\Phi^r} = \max \left\{ \|f\|_{L^\infty(0,T;H^1(\Omega))}; \|\Phi^r\|_{H^1(\Omega)} \right\},
\]
\[
C_{2,\text{max},g^r,\Psi^r} = \max \left\{ \|g\|_{L^\infty(0,T;H^1(\Omega))}; \|\Psi^r\|_{H^1(\Omega)} \right\}.
\]

For \( m = 1 \), using the inequality (24), we have
\[
\| \mathcal{W}_{11}(\xi_1, \xi_2, \epsilon, \zeta_2)(t) \|_{H^1(\Omega)}^2
= 2 \sum_{p=1}^\infty \lambda_p \left[ E_{\beta,\lambda_p}^\beta \left\{ \mathcal{M} \left( \|\nabla \xi_1\|_{L^2(\Omega)}, \|\nabla \xi_2\|_{L^2(\Omega)} \right) \right\}^2 \right. \\
+ \left. \left( \int_t^T \mathcal{M} \left( \|\nabla \xi_1\|_{L^2(\Omega)}, \|\nabla \xi_2\|_{L^2(\Omega)} \right) - \mathcal{M} \left( \|\nabla \xi_1\|_{L^2(\Omega)}, \|\nabla \xi_2\|_{L^2(\Omega)} \right) \right) \int_t^s |\Phi^r| \, ds \right]^2.
\]

First, \( \mathcal{W}_{11}(\xi_1, \xi_2, \epsilon, \zeta_2)(t) \) is estimated as follows
\[
\| \mathcal{W}_{11}(\xi_1, \xi_2, \epsilon, \zeta_2)(t) \|_p
= 2 \sum_{p=1}^\infty \lambda_p \left[ E_{\beta,\lambda_p}^\beta \left\{ \mathcal{M} \left( \|\nabla \xi_1\|_{L^2(\Omega)}, \|\nabla \xi_2\|_{L^2(\Omega)} \right) \right\}^2 \right. \\
\left. - E_{\beta,\lambda_p}^\beta \left\{ \mathcal{M} \left( \|\nabla \xi_1\|_{L^2(\Omega)}, \|\nabla \xi_2\|_{L^2(\Omega)} \right) \right\}^2 \right] |\Phi^r|_p^2 \\
\leq 2 \sum_{p=1}^\infty \lambda_p^2 \lambda_p^\beta M(T-t) (\lambda_p^\beta)^2 \lambda_p |\Phi^r|_p^2 \\
\leq \left( \int_t^T \mathcal{M} \left( \|\nabla \xi_1\|_{L^2(\Omega)}, \|\nabla \xi_2\|_{L^2(\Omega)} \right) - \mathcal{M} \left( \|\nabla \xi_1\|_{L^2(\Omega)}, \|\nabla \xi_2\|_{L^2(\Omega)} \right) \right)^2 \, ds \right)^2 \\
\leq 2K_M^2 \frac{\log^2(1/\beta)}{\min \left\{ M^2; N^2 \right\} T^2} \beta^{\frac{\beta}{2}} \|\Phi^r\|_{H^1(\Omega)}^2 \left\| \xi_1 - \zeta_2 \right\|_{((\beta,H^1(\Omega)))}^2 \int_t^T \, ds \right)^2 \\
\leq 2K_M^2 \frac{\log^2(1/\beta)}{\min \left\{ M^2; N^2 \right\} T^2} \beta^{\frac{\beta}{2}} \|\Phi^r\|_{H^1(\Omega)}^2 \left\| \xi_1 - \zeta_2 \right\|_{((\beta,H^1(\Omega)))}^2 (T-t),
\]

where we have used hypothesis (Hyp3), Parseval’s relation, Hölder’s inequality, and the inequality (24).

In a similar way, we establish estimation for \( \mathcal{W}_{12}(\xi_1, \xi_2, \epsilon, \zeta_2)(t) \)
\[
\| \mathcal{W}_{12}(\xi_1, \xi_2, \epsilon, \zeta_2)(t) \|_p
\leq 2T \sum_{p=1}^\infty \lambda_p |\lambda_p|^{\beta} \int_t^T \beta^{\frac{\beta}{2}} \int_t^s \mathcal{M} \left( \|\nabla \xi_1\|_{L^2(\Omega)}, \|\nabla \xi_2\|_{L^2(\Omega)} \right)
Then, we get

\[
\beta^{-\frac{2}{3}} \left\| \mathcal{W}(\xi_1, \xi_2)(\cdot, t) - \mathcal{W}(\zeta_1, \zeta_2)(\cdot, t) \right\|_{H^1(\Omega)}^2
\]

\[
\leq 4K_{M_4}^2 \frac{\log^2(1/\beta)}{\min \{M^2, N^2\}} T^2 C_T C_{2, \text{max,} f, \Phi, \Phi} \left\| [\xi_1, \xi_2] - [\zeta_1, \zeta_2] \right\|_{((\beta, H^1(\Omega)))}^2.
\]

whereupon

\[
\left\| \mathcal{W}_1(\xi_1, \xi_2) - \mathcal{W}_1(\zeta_1, \zeta_2) \right\|_{((\beta, H^1(\Omega)))}^2
\]

\[
\leq 4K_{M_4}^2 \frac{\log^2(1/\beta)}{\min \{M^2, N^2\}} T^2 C_T C_{2, \text{max,} f, \Phi, \Phi} \left\| [\xi_1, \xi_2] - [\zeta_1, \zeta_2] \right\|_{((\beta, H^1(\Omega)))}^2.
\]

Similarly,

\[
\left\| \mathcal{W}_2(\xi_1, \xi_2) - \mathcal{W}_2(\zeta_1, \zeta_2) \right\|_{((\beta, H^1(\Omega)))}^2
\]

\[
\leq 4K_{M_4}^2 \frac{\log^2(1/\beta)}{\min \{M^2, N^2\}} T^2 C_T C_{2, \text{max,} g^*, \Phi, \Phi} \left\| [\xi_1, \xi_2] - [\zeta_1, \zeta_2] \right\|_{((\beta, H^1(\Omega)))}^2.
\]

Then, we get

\[
\left\| \mathcal{W}(\xi_1, \xi_2) - \mathcal{W}(\zeta_1, \zeta_2) \right\|_{((\beta, H^1(\Omega)))}^2
\]

\[
\leq 2 \left\| \mathcal{W}_1(\xi_1, \xi_2) - \mathcal{W}_1(\zeta_1, \zeta_2) \right\|_{((\beta, H^1(\Omega)))}^2 + 2 \left\| \mathcal{W}_2(\xi_1, \xi_2) - \mathcal{W}_2(\zeta_1, \zeta_2) \right\|_{((\beta, H^1(\Omega)))}^2
\]

\[
\leq 8K_{M_4}^2 \frac{\log^2(1/\beta)}{\min \{M^2, N^2\}} T^2 C_T C_{2, \text{max,} f^*, g^*, \Phi, \Phi} \left\| [\xi_1, \xi_2] - [\zeta_1, \zeta_2] \right\|_{((\beta, H^1(\Omega)))}^2.
\]
Assume that (39) holds for \( m = m_0 \). Let us show that (23) holds for \( m = m_0 + 1 \). In fact, we have

\[ \beta \frac{2^m}{m!} \left\| \mathcal{W}^{m_0+1}(\xi_1, \xi_2)(\cdot, t) - \mathcal{W}^{m_0+1}(\zeta_1, \zeta_2)(\cdot, t) \right\|_{H^1(\Omega)}^2 \leq 4K^2_M \frac{\log^2(1/\beta)}{\min \left\{ \frac{M^2}{2}, \frac{N^2}{2} \right\} T^2} C^2_2 C^2_2 \max_{\beta, f^*, \Phi^*} \int_t^T \left\| \mathcal{W}^{m_0}(\xi_1, \xi_2)(\cdot, s) - \mathcal{W}^{m_0}(\zeta_1, \zeta_2)(\cdot, s) \right\|_{[(\beta, H^1(\Omega))^2]}^2 ds \]

\[ \leq \left( 4K^2_M \frac{\log^2(1/\beta)}{\min \left\{ \frac{M^2}{2}, \frac{N^2}{2} \right\} T^2} C^2_2 C^2_2 \max_{\beta, f^*, \Phi^*} \right)^{m_0+1} \frac{m_0!}{(m_0 + 1)!} \left( T - t \right)^{m_0+1} \left\| [\xi_1, \xi_2] - [\zeta_1, \zeta_2] \right\|_{[(\beta, H^1(\Omega))^2]}^2. \]

Using a similar argument, we obtain the estimation for \( \mathcal{W}^{2^m+1} \). By induction principle, we deduce that (23) holds for all \( m \in \mathbb{N}^* \). Indeed, we have

\[ \| \mathcal{W}^m(\xi_1, \xi_2) - \mathcal{W}^m(\zeta_1, \zeta_2) \|_{[(\beta, H^1(\Omega))^2]} \leq \frac{\left( 8K^2_{\max} \frac{\log^2(1/\beta)}{\min \left\{ \frac{M^2}{2}, \frac{N^2}{2} \right\} T^2} C^2_2 C^2_2 \max_{\beta, f^*, \Phi^*} \right)^m}{m!} T^m \left\| [\xi_1, \xi_2] - [\zeta_1, \zeta_2] \right\|_{[(\beta, H^1(\Omega))^2]}^2. \]

Since

\[ \lim_{m \to \infty} \left( 2K^2_{\max} \frac{\log^2(1/\beta)}{\min \left\{ \frac{M^2}{2}, \frac{N^2}{2} \right\} T^2} C^2_2 C^2_2 \max_{\beta, f^*, \Phi^*} \right)^m = 0, \]

there exists a positive integer number \( m_0 \) such that \( \mathcal{W}^{m_0} \) is a contraction. It follows that the equation \( \mathcal{W}^{m_0}(w_1, w_2) = [w_1, w_2] \) has a unique solution \( (w_1, w_2) \in S_\beta(H^1(\Omega))^2 \). We claim that \( \mathcal{W}(w_1, w_2) = [w_1, w_2] \). In fact, since \( \mathcal{W}^{m_0}(w_1, w_2) = [w_1, w_2] \), we know that \( \mathcal{W}(\mathcal{W}^{m_0}(w_1, w_2)) = \mathcal{W}(w_1, w_2) \). This is equivalent to \( \mathcal{W}^{m_0}(\mathcal{W}(w_1, w_2)) = \mathcal{W}(w_1, w_2) \). Hence, \( \mathcal{W}(w_1, w_2) \) is a fixed point of \( \mathcal{W}^{m_0} \). Moreover, as noted above, \( [w_1, w_2] \) is a fixed point of \( \mathcal{W}^{m_0} \).

**Theorem 4.2** (uniqueness result). If (Hyp1) – (Hyp3) hold, then the System (37)-(38) has at most a (weak) solution in \( S_\beta(H^1(\Omega))^2 \).
**Proof.** Indeed, if we define $\mathcal{J}_\beta^\varepsilon(x,t) = e^{q(t-T)} \left( u_\beta^\varepsilon(x,t) - \bar{u}_\beta^\varepsilon(x,t) \right)$ for some $q > 0$, it is, by direct computations, easy to obtain that

$$
\partial_t \mathcal{J}_\beta^\varepsilon = q e^{q(t-T)} \left( u_\beta^\varepsilon - \bar{u}_\beta^\varepsilon \right) + e^{q(t-T)} \left( \partial_t u_\beta^\varepsilon - \partial_t \bar{u}_\beta^\varepsilon \right)
$$

$$= q \mathcal{J}_\beta^\varepsilon - \mathcal{M} \left( \| \nabla u_\beta^\varepsilon \|_{L^2(\Omega)}(t), \| \nabla \bar{u}_\beta^\varepsilon \|_{L^2(\Omega)}(t) \right) \Delta^\beta \mathcal{J}_\beta^\varepsilon.
$$

(44)

The action of (44) on $\mathcal{J}_\beta^\varepsilon$ in $\mathcal{S}_\beta(H^1(\Omega))$ gives

$$
\frac{1}{2} \frac{d}{dt} \| \mathcal{J}_\beta^\varepsilon(\cdot,t) \|_{L^2(\Omega)^2}^2
$$

$$= q \| \mathcal{J}_\beta^\varepsilon(\cdot,t) \|_{L^2(\Omega)}^2 - \mathcal{M} \left( \| \nabla u_\beta^\varepsilon \|_{L^2(\Omega)}(t), \| \nabla \bar{u}_\beta^\varepsilon \|_{L^2(\Omega)}(t) \right) \langle \Delta^\beta \mathcal{J}_\beta^\varepsilon(\cdot,t), \mathcal{J}_\beta^\varepsilon(\cdot,t) \rangle_{L^2(\Omega)},
$$

which integrating in time yields

$$
\| \mathcal{J}_\beta^\varepsilon(\cdot,T) \|_{L^2(\Omega)}^2 - \| \mathcal{J}_\beta^\varepsilon(\cdot,t) \|_{L^2(\Omega)}^2
$$

$$= \frac{2}{q} \int_t^T \mathcal{M} \left( \| \nabla u_\beta^\varepsilon \|_{L^2(\Omega)}(s), \| \nabla \bar{u}_\beta^\varepsilon \|_{L^2(\Omega)}(s) \right) \langle \Delta^\beta \mathcal{J}_\beta^\varepsilon(\cdot,s), \mathcal{J}_\beta^\varepsilon(\cdot,s) \rangle_{L^2(\Omega)} ds
$$

(45)

In addition, observe that

$$
\langle \Delta^\beta \mathcal{J}_\beta^\varepsilon, \mathcal{J}_\beta^\varepsilon \rangle_{L^2(\Omega)} = \left\langle \sum_{p=1}^\infty \lambda_p \mathcal{J}_{\beta p}(t), \sum_{p=1}^\infty \mathcal{J}_{\beta p}(t) \right\rangle_{L^2(\Omega)}
$$

$$= \sum_{p=1}^\infty \lambda_p \| \mathcal{J}_{\beta p}(t) \|_{L^2(\Omega)}^2 \leq \frac{1}{\min \{ M, N \}} T \log \left( \frac{1}{\beta} \right) \| \mathcal{J}_\beta(\cdot,t) \|_{L^2(\Omega)}^2,
$$

(46)

where we have used $0 \leq \lambda_p \leq \frac{1}{\min \{ M, N \}} T \log \left( \frac{1}{\beta} \right)$ for $\beta \in [0, 1 - \max \{ e^{-\pi T \lambda \rho}, e^{-\pi T \lambda \rho} \}]$: so combining this with the boundedness of diffusion term given by (Hyp2) then using them in (45) yields

$$
\| \mathcal{J}_\beta(\cdot,T) \|_{L^2(\Omega)} - \| \mathcal{J}_\beta(\cdot,t) \|_{L^2(\Omega)} \geq 2 \left( q - \frac{1}{T} \log \left( \frac{1}{\beta} \right) \right) \int_t^T \| \mathcal{J}_\beta(\cdot,s) \|_{L^2(\Omega)} ds.
$$

Choosing $q \geq \frac{1}{T} \log \left( \frac{1}{\beta} \right)$, it becomes clear that for all $t \in [0, T]$, we have

$$
\| \mathcal{J}_\beta(\cdot,t) \|_{L^2(\Omega)} \leq \| \mathcal{J}_\beta(\cdot,T) \|_{L^2(\Omega)} = 0,
$$

which leads to $u_\beta^\varepsilon \equiv \bar{u}_\beta^\varepsilon$ (similar for $v_\beta^\varepsilon \equiv \bar{v}_\beta^\varepsilon$), this implies the uniqueness of the weak solution $[u_\beta^\varepsilon, v_\beta^\varepsilon]$ in $\mathcal{S}_\beta(H^1(\Omega))^2$. The proof of the theorem is complete. \qed

**Theorem 4.3** (Regularity results). Let $\beta \in [0, 1 - \max \{ e^{-\pi T \lambda \rho}, e^{-\pi T \lambda \rho} \}]$.

- If $\Phi^\varepsilon, \Psi^\varepsilon \in L^2(\Omega)$ and $f^\varepsilon, g^\varepsilon \in \mathcal{S}_\beta(L^2(\Omega))$ then

$$
\left\| [u_\beta^\varepsilon, v_\beta^\varepsilon] \right\|_{((\beta, L^2(\Omega))^2)} \leq \frac{1}{\beta} \left\| [\Phi^\varepsilon, \Psi^\varepsilon] \right\|_{L^2(\Omega)} + T \left\| [f^\varepsilon, g^\varepsilon] \right\|_{((\beta, L^2(\Omega))^2)}.
$$

(47)
• If \( \Phi^c \in G((\Delta)^{1/2}e^{\pi T(-\Delta)}) \), \( \Psi^c \in G((\Delta)^{1/2}e^{\pi T(-\Delta)}) \) and \( f^c \in L^\infty(0, T; G((\Delta)^{3/2}e^{\pi T(-\Delta)})) \), then we get

\[
\| [u^c_{\beta}, v^c_{\beta}] \|_{L^2(\Omega)} \leq \max_{p} \left( M \left( \| \nabla u^c_{\beta} \|_{L^2(\Omega)} (s), \| \nabla v^c_{\beta} \|_{L^2(\Omega)} (s) \right) \right) \frac{\beta}{\pi T} \left( \| \phi^c_p \|_{L^2(\Omega)} \right)^2
\]

(48)

• If \( \Phi^c \in G((\Delta)^{3/2}e^{\pi T(-\Delta)}) \), \( \Psi^c \in G((\Delta)^{3/2}e^{\pi T(-\Delta)}) \) and \( f^c \in L^\infty(0, T; G((\Delta)^{5/2}e^{\pi T(-\Delta)})) \) then we get

\[
\| [u^c_{\beta}, v^c_{\beta}] \|_{H^1(\Omega)} \leq \max_{p} \left( M \left( \| \nabla u^c_{\beta} \|_{L^2(\Omega)} (s), \| \nabla v^c_{\beta} \|_{L^2(\Omega)} (s) \right) \right) \frac{\beta}{\pi T} \left( \| \phi^c_p \|_{L^2(\Omega)} \right)^2
\]

(49)

Proof. The proof of this theorem is divided into some steps. 1st step. Proof of (47). One has

\[
\begin{align*}
\quad u^c_{\beta}(x, t) &= \sum_{p=1}^{\infty} \left( \mathcal{E}_{\pi T}^{M} \left( M \left( \| \nabla u^c_{\beta} \|_{L^2(\Omega)} (s), \| \nabla v^c_{\beta} \|_{L^2(\Omega)} (s) \right) \right) \phi^c_p \right) \theta_p(x) \\
&= :H^\beta_1(x, t) \\
&- \sum_{p=1}^{\infty} \left( \int_{0}^{t} \mathcal{E}_{\pi T}^{M} \left( M \left( \| \nabla u^c_{\beta} \|_{L^2(\Omega)} (\tau), \| \nabla v^c_{\beta} \|_{L^2(\Omega)} (\tau) \right) \right) f^c_p(s) \, ds \right) \theta_p(x) \\
&= :H^\beta_2(x, t)
\end{align*}
\]

(50)

First, we have

\[
\left\| H^\beta_1(\cdot, t) \right\|_{L^2(\Omega)}^2 = \sum_{p=1}^{\infty} \left( \mathcal{E}_{\pi T}^{M} \left( M \left( \| \nabla u^c_{\beta} \|_{L^2(\Omega)} (s), \| \nabla v^c_{\beta} \|_{L^2(\Omega)} (s) \right) \right) \phi^c_p \right)^2 \\
\leq \sum_{p=1}^{\infty} \exp \left\{ -2 \min \left( \frac{1}{M}, \frac{1}{T} \right) \log \left( \beta + e^{\pi T \lambda_p} \right) \frac{\beta}{\pi T} \left( \| \phi^c_p \|_{L^2(\Omega)} \right)^2 \right\} \left( \| \phi^c_p \|_{L^2(\Omega)} \right)^2
\]

(51)

Multiplying by \( \beta^{-2+} \), we have

\[
\beta^{-2+} \left\| H^\beta_1(\cdot, t) \right\|_{L^2(\Omega)}^2 \leq \beta^{-2} \left( \| \phi^c \|_{L^2(\Omega)} \right)^2.
\]

(52)

So, we get

\[
\left\| H^\beta_1(\cdot, t) \right\|_{L^2(\Omega)} \leq \frac{1}{\beta} \left( \| \phi^c \|_{L^2(\Omega)} \right).
\]

(53)
Second, by an argument analogous to the previous one, we get
\[
\left\| \mathcal{H}_2^\beta (\cdot, t) \right\|_{L^2(\Omega)}^2 = \sum_{p=1}^\infty \left[ \int_t^T E_{(t,s)}^{\lambda_p^\beta} \left\{ M \left( \left\| \nabla v_\beta^s \right\|_{L^2(\Omega)} (\tau), \left\| \nabla u_\beta^s \right\|_{L^2(\Omega)} (\tau) \right) \right\} f_\beta^s (s) ds \right]^2
\]
(54)
\[
\leq (T-t) \sum_{p=1}^\infty \int_t^T \exp \left\{ 2 \frac{s-t}{T} \log \left( \frac{1}{\beta} \right) \right\} |f_\beta^s|^2 ds
\]
\[
\leq T \int_t^T \beta^{\frac{2s-2}{s}} \left\| f^s (\cdot, s) \right\|_{L^2(\Omega)}^2 ds.
\]
Multiplying by $\beta^{-2\frac{1}{s}}$, we have
\[
\beta^{-\frac{2}{s}} \left\| \mathcal{H}_2^\beta (\cdot, t) \right\|_{L^2(\Omega)}^2 \leq T \int_t^T \beta^{-\frac{2}{s}} \left\| f^s (\cdot, s) \right\|_{L^2(\Omega)}^2 ds
\]
\[
\leq T \int_t^T \left\| f^s \right\|_{(\beta, L^2(\Omega))}^2 ds \leq T^2 \left\| f^s \right\|_{(\beta, L^2(\Omega))}^2.
\]
(55)
Consequently,
\[
\left\| \mathcal{H}_2^\beta \right\|_{(\beta, L^2(\Omega))} \leq T \left\| f^s \right\|_{(\beta, L^2(\Omega))}.
\]
(56)
Hence, we deduce that
\[
\left\| u_\beta^s \right\|_{(\beta, L^2(\Omega))} \leq \left\| \mathcal{H}_1^\beta \right\|_{(\beta, L^2(\Omega))} + \left\| \mathcal{H}_2^\beta \right\|_{(\beta, L^2(\Omega))} \leq \frac{1}{\beta} \left\| \Phi^s \right\|_{L^2(\Omega)} + T \left\| f^s \right\|_{(\beta, L^2(\Omega))}.
\]
(57)
Similarly, we conclude that
\[
\left\| v_\beta^s \right\|_{(\beta, L^2(\Omega))} \leq \frac{1}{\beta} \left\| \Phi^s \right\|_{L^2(\Omega)} + T \left\| g^s \right\|_{(\beta, L^2(\Omega))}.
\]
(58)
Combining (57) and (58) yields
\[
\left\| u_\beta^s \right\|_{(\beta, L^2(\Omega))} + \left\| v_\beta^s \right\|_{(\beta, L^2(\Omega))}
\]
\[
\leq \frac{1}{\beta} \left( \left\| \Phi^s \right\|_{L^2(\Omega)} + \left\| \Phi^s \right\|_{L^2(\Omega)} \right) + T \left( \left\| f^s \right\|_{(\beta, L^2(\Omega))} + \left\| g^s \right\|_{(\beta, L^2(\Omega))} \right),
\]
(59)
the estimate (49) follows.

2nd step. Proof for (48). One has
\[
\left\| \mathcal{H}_1^\beta (\cdot, t) \right\|_{L^2(\Omega)}^2 = \sum_{p=1}^\infty \left[ E_{(t,T)}^{\lambda_p^\beta} \left\{ M \left( \left\| \nabla v_\beta^s \right\|_{L^2(\Omega)} (s), \left\| \nabla u_\beta^s \right\|_{L^2(\Omega)} (s) \right) \right\} \Phi^s \right]^2
\]
\[
\leq \sum_{p=1}^\infty \exp \left\{ -2 \frac{1}{\min \{ \mathcal{M}, N \} T} \log \left( \beta + e^{-\mathcal{M}T \lambda_p} \right) \right\} \left\| \Phi^s \right\|^2
\]
\[
\leq \sum_{p=1}^\infty e^{2(T-t)\mathcal{M} \lambda_p} \left\| \Phi^s \right\|^2 = \left\| \Phi^s \right\|_{G((-\Delta)^p e^{-\mathcal{M}(-\Delta)})(T)}^2,
\]
(60)
which implies that
\[
\left\| \mathcal{H}_1^\beta (\cdot, t) \right\|_{L^2(\Omega)} \leq \left\| \Phi^s \right\|_{G((-\Delta)^p e^{-\mathcal{M}(-\Delta)})(T)}.
\]
(61)
Using Hölder inequality, it is easily seen that
\[
\left\| \mathcal{H}_2^\beta (\cdot,t) \right\|_{L^2(\Omega)}^2 = \sum_{p=1}^{\infty} \left[ \int_t^T E_{(t,s)}^{\lambda_p^\beta} \left\{ \mathcal{M} \left( \left\| \nabla u_{\beta}^p \right\|_{L^2(\Omega)} (\tau), \left\| \nabla v_{\beta}^p \right\|_{L^2(\Omega)} (\tau) \right) \right\} f_p^\varepsilon (s) ds \right]^2
\leq (T-t) \sum_{p=1}^{\infty} \int_t^T \exp \left\{-2 \frac{1}{\min \{M, N\}} T \log \left( \beta + \exp (-MT\lambda_p) \right) \overline{M} (s - t) \right\} |f_p^\varepsilon|^2 ds
\leq T \sum_{p=1}^{\infty} \int_t^T e^{2(T-t)\overline{M}\lambda_p} \left\| f_c (\cdot,s) \right\|_{L^2(\Omega)}^2 ds
\leq T^2 \left\| f_c \right\|_{L^\infty(0,T; G((\overline{\Delta})^q e^{M\overline{\Delta}}(-\Delta)))}^2.
\]

or,
\[
\left\| \mathcal{H}_2^\beta (\cdot,t) \right\|_{L^2(\Omega)} \leq T \left\| f_c \right\|_{L^\infty(0,T; G(\overline{\Delta})^q e^{M\overline{\Delta}}(-\Delta)))}.
\]

Consequently,
\[
\left\| u_{\beta}^\varepsilon (\cdot,t) \right\|_{L^2(\Omega)} \leq \left\| f_c \right\|_{G((\overline{\Delta})^q e^{M\overline{\Delta}}(-\Delta)))} + T \left\| f_c \right\|_{L^\infty(0,T; G(\overline{\Delta})^q e^{M\overline{\Delta}}(-\Delta)))}.
\]

Similarly as for estimate of \( \left\| v_{\beta}^\varepsilon (\cdot,t) \right\|_{L^2(\Omega)} \), combining this result with (64), we obtain (48).

3rd step. Proof for (49). We first observe that
\[
\left\| \mathcal{H}_1^\beta (\cdot,t) \right\|_{H^1(\Omega)}^2 = \sum_{p=1}^{\infty} \lambda_p^\beta \left[ E_{(t,T)}^{\lambda_p^\beta} \left\{ \mathcal{M} \left( \left\| \nabla u_{\beta}^p \right\|_{L^2(\Omega)} (s), \left\| \nabla v_{\beta}^p \right\|_{L^2(\Omega)} (s) \right) \right\} f_p^\varepsilon \right]^2
\leq \sum_{p=1}^{\infty} \lambda_p^\beta \exp \left\{-2 \frac{1}{\min \{M, N\}} T \log \left( \beta + e^{-MT\lambda_p} \right) \overline{M} (T - t) \right\} |f_p^\varepsilon|^2
\leq \sum_{p=1}^{\infty} \lambda_p^\beta e^{2(T-t)\overline{M}\lambda_p} \left| f_p^\varepsilon \right|^2 = \left\| f_c \right\|_{G((\overline{\Delta})^{q/2} e^{M\overline{\Delta}}(-\Delta)))}^2.
\]

Then, we get
\[
\left\| \mathcal{H}_1^\beta (\cdot,t) \right\|_{H^1(\Omega)} \leq \left\| f_c \right\|_{G((\overline{\Delta})^{q/2} e^{M\overline{\Delta}}(-\Delta)))}.
\]
\[
\leq T \sum_{\mu=1}^{\infty} \lambda_{2} \int_{t}^{T} e^{2(T-t)M \lambda_{\mu}} \left\| f^{\varepsilon}(\cdot, s) \right\|_{L^{2}(\Omega)}^{2} \, ds \\
\leq T^{2} \left\| f^{\varepsilon} \right\|_{L^{\infty}(0, T; \mathbb{G}((-\Delta)^{\gamma/2} e^{M T(-\Delta)})^{2})}^{2},
\]
so, we have
\[
\left\| \mathcal{H}_{2}(\cdot, t) \right\|_{H^{2}(\Omega)} \leq T \left\| f^{\varepsilon} \right\|_{L^{\infty}(0, T; \mathbb{G}((-\Delta)^{\gamma/2} e^{M T(-\Delta)})^{2})},
\]
which combined with (69), leads to (49).
\[\Box\]

5. Regularization results for the System (1).

5.1. The result for globally Lipschitzian $\mathcal{M}, \mathcal{N}$: Quasi-reversibility method.

**Theorem 5.1.** Assume (Hyp1) – (Hyp4) hold, and suppose that the solution $(u, v)$ of the System (1) belongs to $L^{\infty}(0, T; \mathbb{G}((-\Delta)^{\gamma/2} e^{M T(-\Delta)}) \times L^{\infty}(0, T; \mathbb{G}((-\Delta)^{\gamma/2} e^{M T(-\Delta)}))$ and $(u, v) \in [L^{\infty}(0, T; L^{2}(\Omega))]^{2}$. Then for $\beta := \beta(\varepsilon) \in ]0, 1[$ satisfying
\[
\begin{align*}
\lim_{\varepsilon \to 0^{+}} \beta &= 0, \\
\lim_{\varepsilon \to 0^{+}} \varepsilon \beta &= \text{a non-negative real number},
\end{align*}
\]
the error estimate over $L^{2}$-norm is given by
\[
\begin{align*}
\left\| \begin{bmatrix} u_{\beta}^{\varepsilon} \\ v_{\beta}^{\varepsilon} \end{bmatrix} (\cdot, t) - [u, v] (\cdot, t) \right\|_{[L^{2}(\Omega)]^{2}} &
\leq P \beta^{2} + \left\| [\partial_{t} u, \partial_{t} v] \right\|_{[L^{\infty}(0, T; L^{2}(\Omega))]^{2}} \sqrt{\frac{T}{\log \left( \varepsilon^{-1} \right)}},
\end{align*}
\]
for
\[
P = 2\varepsilon \beta^{-1} + \frac{1}{\min\{\lambda_{1}; \lambda_{1}\} \sqrt{T}} \\
\left( \left\| u, v \right\|_{L^{\infty}(0, T; \mathbb{G}((-\Delta)^{\gamma/2} e^{M T(-\Delta)})) \times L^{\infty}(0, T; \mathbb{G}((-\Delta)^{\gamma/2} e^{M T(-\Delta)}))} \right)^{2},
\]
\[
\begin{align*}
D &= \varrho + \varepsilon + \min \left\{ \frac{\lambda_{1}^{2} \mathcal{M}^{2} T^{2}}{8K_{\mathcal{M}}^{2} \left\| u \right\|_{L^{\infty}(0, T; \mathbb{G}((-\Delta)^{\gamma/2} e^{M T(-\Delta)}))}^{2}}, \frac{\lambda_{1}^{2} \mathcal{N}^{2} T^{2}}{8K_{\mathcal{N}}^{2} \left\| v \right\|_{L^{\infty}(0, T; \mathbb{G}((-\Delta)^{\gamma/2} e^{M T(-\Delta)}))}^{2}} \right\} + 1/2 > 0.
\end{align*}
\]
Proof. Let us define $\mathcal{P}_\beta^\varepsilon (x, t) = e^{\Gamma_\beta^\varepsilon (t-T)} (u_\beta^\varepsilon (x, t) - u(x, t))$ for some $\Gamma_\beta^\varepsilon > 0$. From (1) and (35), one deduces that
\[
\partial_t \mathcal{P}_\beta^\varepsilon = \Gamma_\beta^\varepsilon e^{\Gamma_\beta^\varepsilon (t-T)} (u_\beta^\varepsilon - u) + e^{\Gamma_\beta^\varepsilon (t-T)} (\partial_t u_\beta^\varepsilon - \partial_t u)
\]
\[
= \Gamma_\beta^\varepsilon \mathcal{P}_\beta^\varepsilon + e^{\Gamma_\beta^\varepsilon (t-T)} \left[ f^\varepsilon (x, t) - f (x, t) - \mathcal{M}( \|\nabla u_\beta^\varepsilon\|_{L^2(\Omega)} (t), \|\nabla v_\beta^\varepsilon\|_{L^2(\Omega)} (t)) \Delta^\beta u_\beta^\varepsilon \right]
\]
\[
- e^{\Gamma_\beta^\varepsilon (t-T)} \mathcal{M}( \|\nabla u\|_{L^2(\Omega)} (t), \|\nabla v\|_{L^2(\Omega)} (t)) \Delta u.
\]
Therefore, the functional $\mathcal{P}_\beta^\varepsilon$ satisfies
\[
\partial_t \mathcal{P}_\beta^\varepsilon + \mathcal{M}( \|\nabla u_\beta^\varepsilon\|_{L^2(\Omega)} (t), \|\nabla v_\beta^\varepsilon\|_{L^2(\Omega)} (t)) \Delta^\beta \mathcal{P}_\beta^\varepsilon - \Gamma_\beta^\varepsilon \mathcal{P}_\beta^\varepsilon
\]
\[
= e^{\Gamma_\beta^\varepsilon (t-T)} \left[ f^\varepsilon (x, t) - f (x, t) \right]
\]
\[
- e^{\Gamma_\beta^\varepsilon (t-T)} \left[ \mathcal{M}( \|\nabla u_\beta^\varepsilon\|_{L^2(\Omega)} (t), \|\nabla v_\beta^\varepsilon\|_{L^2(\Omega)} (t)) \right]
\]
\[
- e^{\Gamma_\beta^\varepsilon (t-T)} \mathcal{M}( \|\nabla u\|_{L^2(\Omega)} (t), \|\nabla v\|_{L^2(\Omega)} (t)) \Delta u.
\]
Taking the action of (72) on $\mathcal{P}_\beta^\varepsilon$ in $C \left( [0, T]; L^2(\Omega) \right)$, by direct computations one obtains
\[
\frac{1}{2} \frac{d}{dt} \left\| \mathcal{P}_\beta^\varepsilon (\cdot, t) \right\|_{L^2(\Omega)}^2 - \Gamma_\beta^\varepsilon \left\| \mathcal{P}_\beta^\varepsilon (\cdot, t) \right\|_{L^2(\Omega)}^2
\]
\[
= e^{\Gamma_\beta^\varepsilon (t-T)} \left\langle f^\varepsilon (\cdot, t) - f (\cdot, t), \mathcal{P}_\beta^\varepsilon (\cdot, t) \right\rangle_{L^2(\Omega)}
\]
\[
- e^{\Gamma_\beta^\varepsilon (t-T)} \left[ \mathcal{M}( \|\nabla u_\beta^\varepsilon\|_{L^2(\Omega)} (t), \|\nabla v_\beta^\varepsilon\|_{L^2(\Omega)} (t)) \right]
\]
\[
- \mathcal{M}( \|\nabla u\|_{L^2(\Omega)} (t), \|\nabla v\|_{L^2(\Omega)} (t)) \Delta u (\cdot, t)
\]
\[
- e^{\Gamma_\beta^\varepsilon (t-T)} \mathcal{M}( \|\nabla u_\beta^\varepsilon\|_{L^2(\Omega)} (t), \|\nabla v_\beta^\varepsilon\|_{L^2(\Omega)} (t)) \Delta^\beta \mathcal{P}_\beta^\varepsilon (\cdot, t)
\]
\[
- \mathcal{M}( \|\nabla u\|_{L^2(\Omega)} (t), \|\nabla v\|_{L^2(\Omega)} (t)) \Delta u (\cdot, t)
\]
\[
= e^{\Gamma_\beta^\varepsilon (t-T)} \left\langle f^\varepsilon (\cdot, t) - f (\cdot, t), \mathcal{P}_\beta^\varepsilon (\cdot, t) \right\rangle_{L^2(\Omega)}
\]
\[
+ \frac{1}{2} \int_t^T \mathcal{M}( \|\nabla u_\beta^\varepsilon\|_{L^2(\Omega)} (s), \|\nabla v_\beta^\varepsilon\|_{L^2(\Omega)} (s)) \left\langle \Delta^\beta \mathcal{P}_\beta^\varepsilon (\cdot, s), \mathcal{P}_\beta^\varepsilon (\cdot, s) \right\rangle_{L^2(\Omega)} ds
\]
\[
+ 2 \Gamma_\beta^\varepsilon \int_t^T \left\| \mathcal{P}_\beta^\varepsilon (\cdot, s) \right\|_{L^2(\Omega)}^2 ds + J_1 + J_2 + J_3,
\]
which integrated in time yields
\[
\left\| \mathcal{P}_\beta^\varepsilon (\cdot, T) \right\|_{L^2(\Omega)}^2 - \left\| \mathcal{P}_\beta^\varepsilon (\cdot, t) \right\|_{L^2(\Omega)}^2
\]
\[
+ 2 \int_t^T \mathcal{M}( \|\nabla u_\beta^\varepsilon\|_{L^2(\Omega)} (s), \|\nabla v_\beta^\varepsilon\|_{L^2(\Omega)} (s)) \left\langle \Delta^\beta \mathcal{P}_\beta^\varepsilon (\cdot, s), \mathcal{P}_\beta^\varepsilon (\cdot, s) \right\rangle_{L^2(\Omega)} ds
\]
\[
= 2 \Gamma_\beta^\varepsilon \int_t^T \left\| \mathcal{P}_\beta^\varepsilon (\cdot, s) \right\|_{L^2(\Omega)}^2 ds + J_1 + J_2 + J_3,
\]
where $J_i, (i = 1, 2, 3)$ are, respectively, defined by
\[
J_1 := 2 \int_t^T e^{\Gamma_\beta^\varepsilon (s-T)} \left\langle f^\varepsilon (\cdot, s) - f (\cdot, s), \mathcal{P}_\beta^\varepsilon (\cdot, s) \right\rangle_{L^2(\Omega)} ds,
\]
\[
J_2 := -2 \int_t^T e^{\Gamma_\beta^\varepsilon (s-T)} \left[ \mathcal{M}( \|\nabla u_\beta^\varepsilon\|_{L^2(\Omega)} (t), \|\nabla v_\beta^\varepsilon\|_{L^2(\Omega)} (t)) \right]
\]
\[
- e^{\Gamma_\beta^\varepsilon (t-T)} \mathcal{M}( \|\nabla u\|_{L^2(\Omega)} (t), \|\nabla v\|_{L^2(\Omega)} (t)) \Delta u (\cdot, t), \mathcal{P}_\beta^\varepsilon (\cdot, t) \right\rangle_{L^2(\Omega)}
\]
\[
- \mathcal{M}( \|\nabla u\|_{L^2(\Omega)} (t), \|\nabla v\|_{L^2(\Omega)} (t)) \Delta u (\cdot, t)
\]
\[
= e^{\Gamma_\beta^\varepsilon (t-T)} \left\langle f^\varepsilon (\cdot, t) - f (\cdot, t), \mathcal{P}_\beta^\varepsilon (\cdot, t) \right\rangle_{L^2(\Omega)}
\]
\[
+ \frac{1}{2} \int_t^T \mathcal{M}( \|\nabla u_\beta^\varepsilon\|_{L^2(\Omega)} (s), \|\nabla v_\beta^\varepsilon\|_{L^2(\Omega)} (s)) \left\langle \Delta^\beta \mathcal{P}_\beta^\varepsilon (\cdot, s), \mathcal{P}_\beta^\varepsilon (\cdot, s) \right\rangle_{L^2(\Omega)} ds
\]
\[
+ 2 \Gamma_\beta^\varepsilon \int_t^T \left\| \mathcal{P}_\beta^\varepsilon (\cdot, s) \right\|_{L^2(\Omega)}^2 ds + J_1 + J_2 + J_3,
\]
\[
\mathcal{J}_3 := -2 \int_t^T e^{r_\beta(s-T)} \mathcal{M}\left(\|\nabla u_\beta\|_{L^2(\Omega)}(s), \|\nabla v_\beta\|_{L^2(\Omega)}(s)\right) \left\langle \Delta^3 u(\cdot, s) - \Delta u(\cdot, s), \Psi_\beta^* (\cdot, s) \right\rangle_{L^2(\Omega)} \, ds.
\]

For the first term \( \mathcal{J}_1 \), we have the estimate

\[
|\mathcal{J}_1| \leq 2 \left\| f^e - f \right\|_{L^\infty(0,T;L^2(\Omega))} \int_t^T \left\| \Psi_\beta^* (\cdot, s) \right\|_{L^2(\Omega)} \, ds = 2 \varepsilon \int_t^T \left\| \Psi_\beta^* (\cdot, s) \right\|_{L^2(\Omega)} \, ds.
\]

Similarly, we get (using the inequality \( \| y \|_{L^2(\Omega)} \| z \|_{L^2(\Omega)} \leq \frac{1}{2} \| y \|_{L^2(\Omega)}^2 + c \| z \|_{L^2(\Omega)}^2 \))

\[
|\mathcal{J}_2| \leq \frac{2K_M}{\lambda_1 \mathcal{T}^2} \left( \int_t^T \left( \left\| \nabla \Psi_\beta^* (\cdot, s) \right\|_{L^2(\Omega)} + \left\| \nabla \Psi_\beta^* (\cdot, s) \right\|_{L^2(\Omega)} \right) \right) \| \Delta u(\cdot, s) \|_{L^2(\Omega)} \, ds
\]

\[
\leq \frac{2K_M}{\lambda_1 \mathcal{T}^2} \left( \int_t^T \left( \left\| \Psi_\beta^* (\cdot, s) \right\|_{H^1(\Omega)} + \left\| \Psi_\beta^* (\cdot, s) \right\|_{H^1(\Omega)} \right) \right) \| \Delta u(\cdot, s) \|_{L^2(\Omega)} \, ds
\]

\[
\leq \frac{2}{4K_M^2} \int_t^T \left( \left\| \Psi_\beta^* (\cdot, s) \right\|_{H^1(\Omega)}^2 + \left\| \Psi_\beta^* (\cdot, s) \right\|_{H^1(\Omega)}^2 \right) \, ds
\]

\[
+ \frac{\lambda_1^2 \mathcal{T}^2 T^2}{4K_M^2} \int_t^T \left\| \Psi_\beta^* (\cdot, s) \right\|_{L^2(\Omega)}^2 \, ds,
\]

where we have used \((a + b)^2 \leq 2a^2 + 2b^2\) and we have recalled the spectral representation and its consequence driven by the elementary inequality \( a < e^a \), for all \( a > 0 \) that

\[
\Delta u(x,t) = -\sum_{p=1}^\infty \lambda_p u_p(t) \theta_p(x),
\]

\[
\left\| \Delta u(\cdot,t) \right\|_{L^2(\Omega)} \leq \sqrt{\sum_{p=1}^\infty \lambda_p^2 e^{2\mathcal{T} \lambda_p} \left\| u_p(t) \right\|_{L^2(\Omega)}^2} \leq \frac{1}{\lambda_1 \mathcal{T}^2} \left\| u(\cdot,t) \right\|_{H^1(\Omega)}^2 e^{\mathcal{T} \lambda_p (\cdot, \Delta)}.
\]

For \( \mathcal{J}_3 \), applying Hölder’s inequality and using (Hyp2) with the basic inequality \( \log(1 + a) \leq a \), for all \( a > 0 \) and

\[
(\Delta^3 - \Delta) u(x,t) = \sum_{p=1}^\infty (\lambda_p^3 + \lambda_p) u_p(t) \theta_p(x)
\]

\[
= \frac{1}{\mathcal{T}^2} \sum_{p=1}^\infty \log \left( \frac{e^{\mathcal{T} \lambda_p}}{\beta + e^{-\mathcal{T} \lambda_p}} \right) u_p(t) \theta_p(x)
\]

we obtain by Parseval’s relation that

\[
|\mathcal{J}_3| \leq \mathcal{T}^2 \int_t^T \left\| (\Delta^3 - \Delta) u(\cdot, s) \right\|_{L^2(\Omega)}^2 \, ds + \int_t^T \left\| \Psi_\beta^* (\cdot, s) \right\|_{L^2(\Omega)}^2 \, ds
\]

\[
\leq \frac{4K_M^2}{\lambda_1^2 \mathcal{T}^2} \int_t^T \left( \left\| \Psi_\beta^* (\cdot, s) \right\|_{H^1(\Omega)}^2 + \left\| \Psi_\beta^* (\cdot, s) \right\|_{H^1(\Omega)}^2 \right) \, ds
\]

\[
+ \frac{\lambda_1^2 \mathcal{T}^2 T^2}{4K_M^2} \int_t^T \left\| \Psi_\beta^* (\cdot, s) \right\|_{L^2(\Omega)}^2 \, ds.
\]
\[
\leq M^2 \int_t^T \left[ \frac{1}{M^2 T^2} \sum_{p=1}^{\infty} \log^2 \left( \frac{e^{MT\lambda_p}}{\beta + e^{-MT\lambda_p}} \right) |u_p(s)|^2 \right] \, ds + \int_t^T \|\Psi^\varepsilon(\cdot, s)\|_{L^2(\Omega)}^2 \, ds
\]
\[
\leq M^2 \int_t^T \left[ \frac{1}{M^2 T^2} \sum_{p=1}^{\infty} \log^2 \left( \frac{e^{MT\lambda_p} \beta + 1}{e^{2MT\lambda_p}} \right) |u_p(s)|^2 \right] \, ds + \int_t^T \|\Psi^\varepsilon(\cdot, s)\|_{L^2(\Omega)}^2 \, ds
\]
\[
\leq M^2 \int_t^T \left[ \frac{1}{M^2 T^2} \sum_{p=1}^{\infty} \log^2 \left( e^{MT\lambda_p} \beta + 1 \right) |u_p(s)|^2 \right] \, ds + \int_t^T \|\Psi^\varepsilon(\cdot, s)\|_{L^2(\Omega)}^2 \, ds
\]
\[
\leq M^2 \int_t^T \left[ \frac{1}{M^2 T^2} \sum_{p=1}^{\infty} \lambda_p^2 e^{MT\lambda_p} \beta^2 |u_p(s)|^2 \right] \, ds + \int_t^T \|\Psi^\varepsilon(\cdot, s)\|_{L^2(\Omega)}^2 \, ds
\]
\[
\leq \frac{\beta^2}{\lambda^2 T} \|u\|_{L^\infty(0, T; \mathbb{C})}^2 + \int_t^T \|\Psi^\varepsilon(\cdot, s)\|_{L^2(\Omega)}^2 \, ds.
\]

Combining (73)-(76) gives
\[
\|\Psi^\varepsilon(\cdot, T)\|_{L^2(\Omega)}^2 - \|\Psi^\varepsilon(\cdot, t)\|_{L^2(\Omega)}^2
\]
\[
+ 2 \int_t^T \mathcal{M} \left( \|\nabla u^\varepsilon\|_{L^2(\Omega)}(s), \|\nabla u^\varepsilon\|_{L^2(\Omega)}(s) \right) \left( \Delta^\beta \Psi^\varepsilon(\cdot, s), \Psi^\varepsilon(\cdot, s) \right)_{L^2(\Omega)} \, ds
\]
\[
\geq \left[ 2\beta^2 - 2\varepsilon - \frac{\lambda^2 M^2 T^2}{4K_1^2 \|u\|_{L^\infty(0, T; \mathbb{C})}^2 - 1} \right] \int_t^T \|\Psi^\varepsilon(\cdot, s)\|_{L^2(\Omega)}^2 \, ds
\]
\[
- 2 \int_t^T \left( \|\Psi^\varepsilon(\cdot, s)\|_{H^1_0(\Omega)} + \|\nabla \Psi^\varepsilon(\cdot, s)\|_{H^1_0(\Omega)} \right) \, ds
\]
\[
- \frac{\beta^2}{\lambda^2 T} \|u\|_{L^\infty(0, T; \mathbb{C})}^2 \, ds.
\]

It now remains to estimate the following by using (Hyp2) and (46)
\[
\int_t^T \mathcal{M} \left( \|\nabla u^\varepsilon\|_{L^2(\Omega)}(s), \|\nabla u^\varepsilon\|_{L^2(\Omega)}(s) \right) \left( \Delta^\beta \Psi^\varepsilon(\cdot, s), \Psi^\varepsilon(\cdot, s) \right)_{L^2(\Omega)} \, ds
\]
\[
\leq \frac{1}{T} \log \left( \frac{1}{\beta} \right) \int_t^T \|\Psi^\varepsilon(\cdot, s)\|_{L^2(\Omega)}^2 \, ds.
\]

By choosing \( \beta = \frac{1}{\varepsilon} \) and furthermore, noticing the fact that
\[
\|\Psi^\varepsilon(\cdot, T)\|_{L^2(\Omega)} = \|\varepsilon - \Phi\|_{L^2(\Omega)} \leq \varepsilon, \quad \text{and} \quad \|\Psi^\varepsilon(\cdot, t)\|_{L^2(\Omega)} \leq C_0 \|\Psi^\varepsilon(\cdot, t)\|_{H^1_0(\Omega)},
\]
we thus put (78) into (77) to get
\[
\|\Psi^\varepsilon(\cdot, t)\|_{L^2(\Omega)}^2 + (2\varepsilon C_0 - 2) \int_t^T \|\Psi^\varepsilon(\cdot, s)\|_{H^1_0(\Omega)}^2 \, ds - 2 \int_t^T \|\nabla \Psi^\varepsilon(\cdot, s)\|_{H^1_0(\Omega)}^2 \, ds
\]
\[
\leq \varepsilon^2 + \frac{\beta^2}{\lambda^2 T} \|u\|_{L^\infty(0, T; \mathbb{C})}^2 \, ds.
\]
Similarly, we have
\[
\sum (79) \text{ and } (80), \text{ we obtain }
\]
Similarly, we have
\[
\text{which leads to (71).}
\]
0. In fact, we may find
\[
\text{We claim that for every } \varepsilon > 0, \text{ there exists a unique }
\]
\[
\text{which can be written, via the expression of } I_2^e, \text{ as }
\]
\[
\text{where }
\]
\[
\text{which leads to (71).}
\]
Due to the continuity of \( u_t, v_t \), we get for \( \varepsilon \) small enough
\[
\left\| [u^e_{\beta}, v^e_{\beta}] (\cdot, t^e) - [u, v] (\cdot, 0) \right\|_{L^2(\Omega)^2}
\]
\[
\leq \left\| [u^e_{\beta}, v^e_{\beta}] (\cdot, t^e) - [u, v] (\cdot, t^e) \right\|_{L^2(\Omega)^2} + \left\| [u^e_{\beta}, v^e_{\beta}] (\cdot, t^e) - [u, v] (\cdot, 0) \right\|_{L^2(\Omega)^2}
\]
\[
\leq \left\| [u^e_{\beta}, v^e_{\beta}] (\cdot, t^e) - [u, v] (\cdot, t^e) \right\|_{L^2(\Omega)^2} + \int_{t^e}^{t^\varepsilon} \left\| [\partial_s u, \partial_s v] (\cdot, s) \right\|_{L^2(\Omega)^2} \, ds
\]
\[
\leq P\beta^e T e^{D(T-t)} + t^e \left\| [\partial_t u, \partial_t v] \right\|_{L^\infty(0,T; L^2(\Omega)^2)} \text{ for some } t^e \in (0, T).
\]
We claim that for every \( \varepsilon > 0 \), there exists a unique \( t^\varepsilon \in (0, T) \) such that \( \lim_{\varepsilon \to 0^+} t^\varepsilon = 0. \) In fact, we may find
\[
t^\varepsilon = \beta^e T.
Using the inequality $\ln t > -\frac{1}{t}$ for every $t > 0$, we obtain
\[ t^\varepsilon < \left( \frac{T}{\log \left( \frac{1}{\beta} \right)} \right). \]

We are thus led to the following estimate
\[
\left\| \left[ u_\beta, v_\beta \right] \left( \cdot, t^\varepsilon \right) - [u, v] \left( \cdot, 0 \right) \right\|_{L^2(\Omega)}^2
\leq \left( Pe^{DT} + \left\| \left[ \partial_t u, \partial_t v \right] \right\|_{L^\infty(0,T;L^2(\Omega))}^2 \right) \frac{T}{\log \left( \frac{1}{\beta} \right)}.
\]

This completes the proof of the theorem. \( \square \)

5.2. The result for locally Lipschitzian $M, N$: Modified quasi-reversibility method. In section 5.1, instead of the globally Lipschitz condition (Hyp3) of $M, N$, we can impose the locally Lipschitz property in $L^2$-norm (Hyp5): For all $R > 0, \exists K_M(R), K_N(R) : 0 < K_M(R), K_N(R) < \infty, \forall (u_i, v_i) \in \mathbb{B}_R, i = 1, 2, \forall t \in [0, T]$, such that
\[
\left\| M_R \left( \left\| \nabla u \right\|_{L^2(\Omega)}(t), \left\| \nabla v \right\|_{L^2(\Omega)}(t) \right) \right\|_{L^2(\Omega)}
\leq K_M(R) \left\| [u_1, v_1](\cdot, t) - [u_2, v_2](\cdot, t) \right\|_{L^2(\Omega)}^2,
\]
\[
\left\| N_R \left( \left\| \nabla u \right\|_{L^2(\Omega)}(t), \left\| \nabla v \right\|_{L^2(\Omega)}(t) \right) \right\|_{L^2(\Omega)}.
\]
\[
\leq K_N(R) \left\| [u_1, v_1](\cdot, t) - [u_2, v_2](\cdot, t) \right\|_{L^2(\Omega)}^2,
\]
where $\mathbb{B}_R$ is the closed ball in $L^2(\Omega) \times L^2(\Omega)$ of center 0, radius $R$. Then we can use the following sequences of globally Lipschitz functions
\[
M_{R^\varepsilon} \left( \left\| \nabla u \right\|_{L^2(\Omega)}(t), \left\| \nabla v \right\|_{L^2(\Omega)}(t) \right) \quad (83)
\]
\[
:= \left\{ \begin{array}{ll}
M \left( \left\| \nabla u \right\|_{L^2(\Omega)}(t), \left\| \nabla v \right\|_{L^2(\Omega)}(t) \right), & \text{if } \left\| [u, v](\cdot, t) \right\|_{C([0,T],L^2(\Omega))}^2 \leq R^\varepsilon,
M \left( \frac{R^\varepsilon \left\| \nabla u \right\|_{L^2(\Omega)}(t)}{\left\| [u, v](\cdot, t) \right\|_{L^2(\Omega)}^2}, \frac{R^\varepsilon \left\| \nabla v \right\|_{L^2(\Omega)}(t)}{\left\| [u, v](\cdot, t) \right\|_{L^2(\Omega)}^2} \right), & \text{if } \left\| [u, v](\cdot, t) \right\|_{C([0,T],L^2(\Omega))}^2 > R^\varepsilon,
\end{array} \right.
\]
\[
N_{R^\varepsilon} \left( \left\| \nabla u \right\|_{L^2(\Omega)}(t), \left\| \nabla v \right\|_{L^2(\Omega)}(t) \right) \quad (84)
\]
\[
:= \left\{ \begin{array}{ll}
N \left( \left\| \nabla u \right\|_{L^2(\Omega)}(t), \left\| \nabla v \right\|_{L^2(\Omega)}(t) \right), & \text{if } \left\| [u, v](\cdot, t) \right\|_{C([0,T],L^2(\Omega))}^2 \leq R^\varepsilon,
N \left( \frac{R^\varepsilon \left\| \nabla u \right\|_{L^2(\Omega)}(t)}{\left\| [u, v](\cdot, t) \right\|_{L^2(\Omega)}^2}, \frac{R^\varepsilon \left\| \nabla v \right\|_{L^2(\Omega)}(t)}{\left\| [u, v](\cdot, t) \right\|_{L^2(\Omega)}^2} \right), & \text{if } \left\| [u, v](\cdot, t) \right\|_{C([0,T],L^2(\Omega))}^2 > R^\varepsilon,
\end{array} \right.
\]

to approximate $M, N$ and our QR method can be straightforwardly applied. We can easily prove that the reaction function considered in (83) and (84) satisfy the globally Lipschitz condition in $L^2$-norm and the regularized solution for this unstable problem can be found.

Consider the following system
\[
\partial_t U_\beta^\varepsilon = M_{R^\varepsilon} \left( \left\| \nabla U_\beta^\varepsilon \right\|_{L^2(\Omega)}(t), \left\| \nabla U_\beta^\varepsilon \right\|_{L^2(\Omega)}(t) \right) \Delta U_\beta^\varepsilon + L_\beta^\varepsilon U_\beta^\varepsilon + f^\varepsilon(x, t), \quad (85)
\]
\[
\partial_t V_\beta^\varepsilon = N_{R^\varepsilon} \left( \left\| \nabla U_\beta^\varepsilon \right\|_{L^2(\Omega)}(t), \left\| \nabla U_\beta^\varepsilon \right\|_{L^2(\Omega)}(t) \right) \Delta V_\beta^\varepsilon + L_\beta^\varepsilon V_\beta^\varepsilon + g^\varepsilon(x, t), \quad (86)
\]
supplemented with final conditions
\[
U_\beta^\varepsilon(x, T) = \Psi^\varepsilon(x), \quad V_\beta^\varepsilon(x, T) = \Phi^\varepsilon(x), \quad x \in \Omega, \quad (87)
\]
and Dirichlet boundary condition $U^\varepsilon_\beta = V^\varepsilon_\beta = 0$ on $\partial \Omega_T$.
Here $\beta := \beta(\varepsilon) > 0$ is the regularization parameter, satisfying $\beta(\varepsilon) \to 0$ when $\varepsilon \to 0$, and will be chosen later. The operators $L^\varepsilon_\beta, \tilde{L}^\varepsilon_\beta$ are given by

$$
L^\varepsilon_\beta w = \sum_{p=1}^{\infty} \frac{\log \left( 1 + \beta e^{T \tilde{M}_\lambda_p} \right)}{T} \langle w, \theta_p \rangle_{L^2(\Omega)} \theta_p(x),
$$

$$
\tilde{L}^\varepsilon_\beta w = \sum_{p=1}^{\infty} \frac{\log \left( 1 + \beta e^{T \tilde{N}_\lambda_p} \right)}{T} \langle w, \theta_p \rangle_{L^2(\Omega)} \theta_p(x),
$$

where $\tilde{M}, \tilde{N}$ are a positive constants, satisfying $\tilde{M} \geq M, \tilde{N} \geq N$, with $M, N$ given in $(A_2)$. Let us define the following operators

$$
P^\varepsilon_\beta w = L^\varepsilon_\beta w + \tilde{M} \Delta w,
$$

$$
\tilde{P}^\varepsilon_\beta w = \tilde{L}^\varepsilon_\beta w + \tilde{N} \Delta w,
$$

$$
\mathcal{A}_{R^\varepsilon} \left( \|\nabla u\|_{L^2(\Omega)}(t), \|\nabla v\|_{L^2(\Omega)}(t) \right) = M - \mathcal{M}_{R^\varepsilon} \left( \|\nabla u\|_{L^2(\Omega)}(t), \|\nabla v\|_{L^2(\Omega)}(t) \right),
$$

$$
\mathcal{B}_{R^\varepsilon} \left( \|\nabla u\|_{L^2(\Omega)}(t), \|\nabla v\|_{L^2(\Omega)}(t) \right) = N - \mathcal{N}_{R^\varepsilon} \left( \|\nabla u\|_{L^2(\Omega)}(t), \|\nabla v\|_{L^2(\Omega)}(t) \right).
$$

Notice that, from $(A_2) - (A_4)$, for all $t \in [0, T], (u, v), (u_1, v_1), (u_2, v_2) \in [L^2(\Omega)]^2$, we have

$$
0 < \tilde{M} - M \leq \mathcal{A}_{R^\varepsilon} \left( \|\nabla u\|_{L^2(\Omega)}(t), \|\nabla v\|_{L^2(\Omega)}(t) \right) \leq \tilde{M} - M,
$$

$$
0 < \tilde{N} - N \leq \mathcal{B}_{R^\varepsilon} \left( \|\nabla u\|_{L^2(\Omega)}(t), \|\nabla v\|_{L^2(\Omega)}(t) \right) \leq \tilde{N} - N,
$$

$$
\left| \mathcal{A}_{R^\varepsilon} \left( \|\nabla u_1\|_{L^2(\Omega)}(t), \|\nabla v_1\|_{L^2(\Omega)}(t) \right) - \mathcal{A}_{R^\varepsilon} \left( \|\nabla u_2\|_{L^2(\Omega)}(t), \|\nabla v_2\|_{L^2(\Omega)}(t) \right) \right|
$$

$$
= \left| \mathcal{M}_{R^\varepsilon} \left( \|\nabla u_1\|_{L^2(\Omega)}(t), \|\nabla v_1\|_{L^2(\Omega)}(t) \right) \right|
$$

$$
- \left| \mathcal{M}_{R^\varepsilon} \left( \|\nabla u_2\|_{L^2(\Omega)}(t), \|\nabla v_2\|_{L^2(\Omega)}(t) \right) \right|
$$

$$
\leq K_M(R^\varepsilon) \|[u_1, v_1](\cdot, t) - [u_2, v_2](\cdot, t)\|_{L^2(\Omega)^2},
$$

$$
\left| \mathcal{B}_{R^\varepsilon} \left( \|\nabla u_1\|_{L^2(\Omega)}(t), \|\nabla v_1\|_{L^2(\Omega)}(t) \right) - \mathcal{B}_{R^\varepsilon} \left( \|\nabla u_2\|_{L^2(\Omega)}(t), \|\nabla v_2\|_{L^2(\Omega)}(t) \right) \right|
$$

$$
= \left| \mathcal{N}_{R^\varepsilon} \left( \|\nabla u_1\|_{L^2(\Omega)}(t), \|\nabla v_1\|_{L^2(\Omega)}(t) \right) \right|
$$

$$
- \left| \mathcal{N}_{R^\varepsilon} \left( \|\nabla u_2\|_{L^2(\Omega)}(t), \|\nabla v_2\|_{L^2(\Omega)}(t) \right) \right|
$$

$$
\leq K_N(R^\varepsilon) \|[u_1, v_1](\cdot, t) - [u_2, v_2](\cdot, t)\|_{L^2(\Omega)^2}.
$$

The main equations of system (85), (86) can be rewritten as

$$
\partial_t U^\varepsilon_\beta - \mathcal{A}_{R^\varepsilon} \left( \|\nabla U^\varepsilon_\beta\|_{L^2(\Omega)}, \|\nabla U^\varepsilon_\beta\|_{L^2(\Omega)}(t) \right) \Delta U^\varepsilon_\beta = P^\varepsilon_\beta U^\varepsilon_\beta + f^\varepsilon(x, t),
$$

$$
\partial_t V^\varepsilon_\beta - \mathcal{B}_{R^\varepsilon} \left( \|\nabla V^\varepsilon_\beta\|_{L^2(\Omega)}, \|\nabla V^\varepsilon_\beta\|_{L^2(\Omega)}(t) \right) \Delta V^\varepsilon_\beta = \tilde{P}^\varepsilon_\beta V^\varepsilon_\beta + g^\varepsilon(x, t).
$$

To obtain the boundedness of regularized operator, the following technical lemma will play the key role. We shall omit the easy proof of this lemma.
Lemma 5.2. 1. For any \( \epsilon \) is defined as in (70). The solution of the System (1) satisfies
\[
\| L^2_\beta u \|_{L^2(\Omega)} \leq \frac{\beta}{T} \| w \|_{G((-\Delta)^0e^{MT(-\Delta)})},
\]
\[
\| L^2_\beta w \|_{L^2(\Omega)} \leq \frac{\beta}{T} \| w \|_{G((-\Delta)^0e^{MT(-\Delta)})}.
\]

2. For any \( w \in L^2(\Omega) \), it yields
\[
\| P^\beta w \|_{L^2(\Omega)} \leq \frac{1}{T} \log \left( \frac{1}{\beta} \right) \| w \|_{L^2(\Omega)},
\]
\[
\| P^\beta w \|_{L^2(\Omega)} \leq \frac{1}{T} \log \left( \frac{1}{\beta} \right) \| w \|_{L^2(\Omega)}.
\]

5.2.1. \( L^2 \)-estimate.

Theorem 5.3. Suppose that (Hyp1), (Hyp2), (Hyp4), (Hyp5) hold and the constant \( \beta \) is defined as in (70). The solution of the System (1) satisfies
\[
u \in L^2(0,T;G((-\Delta)^0e^{NT(-\Delta)}) \cap L^\infty(0,T;H^1_0(\Omega)) \cap C^1(0,T;L^2(\Omega)) )
\]
\[
v \in L^2(0,T;G((-\Delta)^0e^{NT(-\Delta)}) \cap L^\infty(0,T;H^1_0(\Omega)) \cap C^1(0,T;L^2(\Omega)) )
\]

Denote
\[
E = \max \left\{ \left\| [u,v] \right\|_{L^2(0,T;G((-\Delta)^0e^{MT(-\Delta)}) \times L^2(0,T;G((-\Delta)^0e^{MT(-\Delta)}))};
\left\| [u,v] \right\|_{C^1(0,T;L^2(\Omega))} ; \left\| [u,v] \right\|_{L^\infty(0,T;H^1_0(\Omega))} \right\}.
\]

For \( K_M(\mathbb{R}^\epsilon) \), \( K_N(\mathbb{R}^\epsilon) \) of \( M_{\mathbb{R}^\epsilon}, N_{\mathbb{R}^\epsilon} \), choosing \( \mathbb{R}^\epsilon \), such that for some \( \Lambda > 0 \), yields
\[
K_{\max}(\mathbb{R}^\epsilon) := \max \{ K_M(\mathbb{R}^\epsilon); K_N(\mathbb{R}^\epsilon); 1 \} \leq \frac{1}{\sqrt{T}} \log^{1/2} \left( \log^A \left( \frac{1}{\beta} \right) \right).
\]

Then, the following estimation holds
\[
\left\| \left[ U^\beta_{\epsilon}, V^\beta_{\epsilon} \right](\cdot,t) - [u,v](\cdot,t) \right\|_{L^2(\Omega)^2} \leq \left( 2\epsilon\beta^{-1}\sqrt{T} + 1 + \sqrt{2E/T} \right) \beta^2 \left[ \log \left( \frac{1}{\beta} \right) \right]^{\frac{\Lambda + M + 2M + N}{2M} \epsilon^2 + 2E/T}.
\]

From (93) we imply the stability for \( t \in (0,T) \). Moreover, there exists \( t_\epsilon \in (0,T) : \lim_{\epsilon \to 0} t_\epsilon = 0 \), such that
\[
\left\| \left[ U^\beta_{\epsilon}, V^\beta_{\epsilon} \right](\cdot,t_\epsilon) - [u,v](\cdot,0) \right\|_{L^2(\Omega)^2} \leq \left( \frac{T}{\log \left( \frac{1}{\beta} \right)} \right) \left( 2\epsilon\beta^{-1}\sqrt{T} + 1 + \sqrt{2E/T} \right) \left[ \log \left( \frac{1}{\beta} \right) \right]^{\frac{\Lambda + M + 2M + N}{2M} \epsilon^2 + 2E/T}.
\]

Notice that if we take \( 0 < \Lambda < \frac{M + N}{2M + 2N + 4M} \), then the right hand side of (94) tends to 0, we have the stability at \( t = 0 \).

Proof. Let us define
\[
X^\beta_{\epsilon}(x,t) = e^{\Gamma_{\beta}(t-T)}(U^\beta_{\epsilon} - u)(x,t), \quad Y^\beta_{\epsilon}(x,t) = e^{\Gamma_{\beta}(t-T)}(V^\beta_{\epsilon} - v)(x,t),
\]
where \( \Gamma_\beta > 0 \) is a positive constant that will be chosen later. From (1), (85), with some computations, we have

\[
\frac{d}{dt} \left\langle \mathbf{x}_\beta^\varepsilon(\cdot, t), \Theta \right\rangle_{L^2(\Omega)} - \mathcal{A}_{\mathcal{R}^*} \left( \left\| \nabla U_\beta^\varepsilon \right\|_{L^2(\Omega)}(t), \left\| \nabla U_\beta^\varepsilon \right\|_{L^2(\Omega)}(t) \right) = \mathcal{M}_{\mathcal{R}^*} \left( \left\| \nabla U_\beta^\varepsilon \right\|_{L^2(\Omega)}(t), \left\| \nabla U_\beta^\varepsilon \right\|_{L^2(\Omega)}(t) \right) + \mathcal{S} \left( \left\| \nabla u \right\|_{L^2(\Omega)}(t), \left\| \nabla v \right\|_{L^2(\Omega)}(t) \right) \left( \nabla u(\cdot, t), \nabla \Theta \right)_{L^2(\Omega)}
\]

Taking \( \Theta = \mathbf{x}_\beta^\varepsilon \), and integrating from \( t \) to \( T \), it yields

\[
\left\| \mathbf{x}_\beta^\varepsilon(\cdot, T) \right\|_{L^2(\Omega)}^2 - \left\| \mathbf{x}_\beta^\varepsilon(\cdot, t) \right\|_{L^2(\Omega)}^2 - 2 \int_t^T \int_\Omega \Gamma_\beta \left\| \mathbf{X}_\beta^\varepsilon(\cdot, s) \right\|^2_{L^2(\Omega)} ds
\]

\[
= 2 \int_t^T e^{\gamma_0(s-T)} \left\langle \mathbf{L}_\beta \left( \mathbf{X}_\beta^\varepsilon(\cdot, s) \right), \mathbf{X}_\beta^\varepsilon(\cdot, s) \right\rangle_{L^2(\Omega)} ds + 2 \int_t^T \left\langle \mathbf{P}_\beta \mathbf{X}_\beta^\varepsilon(\cdot, s), \mathbf{X}_\beta^\varepsilon(\cdot, s) \right\rangle_{L^2(\Omega)} ds
\]

\[
- 2 \int_t^T e^{\gamma_0(s-T)} \left[ \mathcal{M}_{\mathcal{R}^*} \left( \left\| \nabla U_\beta^\varepsilon \right\|_{L^2(\Omega)}(s), \left\| \nabla U_\beta^\varepsilon \right\|_{L^2(\Omega)}(s) \right) \right] \left\langle \nabla u(\cdot, s), \nabla \mathbf{X}_\beta^\varepsilon(\cdot, s) \right\rangle_{L^2(\Omega)} ds
\]

Applying Hölder’s inequality and lemma 5.2, we obtain

\[
\left\| \mathbf{L}_\beta \left( \mathbf{X}_\beta^\varepsilon(\cdot, s) \right) \left\|_{L^2(\Omega)} \left\| \mathbf{X}_\beta^\varepsilon(\cdot, s) \right\|_{L^2(\Omega)} ds \leq \frac{2}{T} \int_t^T \left\| u(\cdot, s) \right\|_{G_\beta((-\Delta)\rho e^{-\gamma_0 T}(\cdot))} \left\| \mathbf{X}_\beta^\varepsilon(\cdot, s) \right\|_{L^2(\Omega)} ds
\]

\[
\leq \frac{\beta^2}{T^2} \int_t^T \left\| u(\cdot, s) \right\|_{G_\beta((-\Delta)\rho e^{-\gamma_0 T}(\cdot))}^2 ds + \int_t^T \left\| \mathbf{X}_\beta^\varepsilon(\cdot, s) \right\|_{L^2(\Omega)}^2 ds
\]

\[
\leq \frac{\beta^2}{T^2} \left\| u \right\|_{L^2(\Omega)}^2 G_\beta((-\Delta)\rho e^{-\gamma_0 T}(\cdot)) + \int_t^T \left\| \mathbf{X}_\beta^\varepsilon(\cdot, s) \right\|_{L^2(\Omega)}^2 ds.
\]

For \( \| \| \), using lemma 5.2

\[
\left\| \mathbf{P}_\beta \mathbf{X}_\beta^\varepsilon(\cdot, s) \right\|_{L^2(\Omega)} \leq \frac{2}{T} \log \left( \frac{1}{\beta} \right) \int_t^T \left\| \mathbf{X}_\beta^\varepsilon(\cdot, s) \right\|_{L^2(\Omega)}^2 ds.
\]
Next, we estimate \( \| \| \) (noting that \( e^{f s} < 1 \), for all \( 0 \leq t \leq s \leq T \))

\[
\| \| \leq 2 \int_t^T e^{f s} \left\| \left( f^2 (\cdot, s) - f (\cdot, s), \mathcal{X}_s (\cdot, s) \right) \right\| ds \\
\leq 2 \int_t^T \left\| f^2 (\cdot, s) - f (\cdot, s) \right\| \mathcal{X}_s (\cdot, s) \| L^2(\Omega) \| L^2(\Omega) \| ds \\
\leq \int_t^T \left\| f^2 - f \right\| ^2 L^\infty(0,T;L^2(\Omega)) ds + \int_t^T \| \mathcal{X}_s (\cdot, s) \| ^2 \| L^2(\Omega) \| ds \\
\leq T \epsilon^2 + \int_t^T \| \mathcal{X}_s (\cdot, s) \| ^2 \| L^2(\Omega) \| ds.
\]

Notice that \( R^\epsilon \to \infty \), when \( \epsilon \to 0 \), since \( u, v \in L^\infty(0,T;L^2(\Omega)) \), we can choose a sufficiently small \( \epsilon \), such that for a.e \( (x, t) \in \Omega_T : \| [u, v] \| L^\infty(0,T;L^2(\Omega)) \| < R^\epsilon \), implies that

\[
\mathcal{M} \left( \| \nabla u \| L^2(\Omega) (t), \| \nabla v \| L^2(\Omega) (t) \right) \\
\approx \mathcal{M}_{\mathcal{R}^\epsilon} \left( \| \nabla u \| L^2(\Omega) (t), \| \nabla v \| L^2(\Omega) (t) \right), \text{ a.e in } \mathcal{D}_T.
\]

Using (Hyp5), Hölder’s inequality and Cauchy’s inequality it gives

\[
\| IV \| \\
\leq 2K_M(R^\epsilon) \int_t^T \| (\mathcal{X}_s, \mathcal{Y}_s) (\cdot, s) \| \| L^2(\Omega) \| \| \nabla u (\cdot, s) \| \| L^2(\Omega) \| \| \nabla \mathcal{X}_s (\cdot, s) \| \| L^2(\Omega) \| ds \\
\leq \frac{[K_M(R^\epsilon)]^2 \| u \| ^2 L^\infty(0,T;H^1(\Omega))}{M} \int_t^T \| (\mathcal{X}_s, \mathcal{Y}_s) (\cdot, s) \| \| L^2(\Omega) \| ^2 ds \\
+ 2M \int_t^T \| \nabla \mathcal{X}_s (\cdot, s) \| \| L^2(\Omega) \| ^2 ds.
\]

Combining (96)-(100), and choosing \( \Gamma_\beta = \frac{1}{T} \log(\frac{1}{2}) \), we obtain

\[
\| \mathcal{X}_s (\cdot, t) \| ^2 \| L^2(\Omega) \\
\leq \epsilon^2 (T + 1) + \frac{\beta^2}{T} \| u \| ^2 L^2(\Omega) \| \| + 2 \int_t^T \| \mathcal{X}_s (\cdot, s) \| ^2 \| L^2(\Omega) \| ds \\
+ \frac{[K_M(R^\epsilon)]^2 \| u \| ^2 L^\infty(0,T;H^1(\Omega))}{M} \int_t^T \| (\mathcal{X}_s, \mathcal{Y}_s) (\cdot, s) \| \| L^2(\Omega) \| ^2 ds,
\]

where we have used the hypothesis (Hyp2) in the fact that

\[
\int_t^T A_{\mathcal{R}^\epsilon} \left( \| \nabla u \| \| L^2(\Omega) (s), \| \mathcal{X}_s (\cdot, s) \| \| L^2(\Omega) \| \| \nabla \mathcal{X}_s (\cdot, s) \| \| L^2(\Omega) \| \| ds.
\]

In a similar manner, we obtain the estimate for \( \mathcal{Y}_s \), summing with (101), we have

\[
\| (\mathcal{X}_s, \mathcal{Y}_s) (\cdot, t) \| ^2 \| L^2(\Omega) \| ^2 \leq 2 \| \mathcal{X}_s (\cdot, t) \| ^2 \| L^2(\Omega) \| + 2 \| \mathcal{Y}_s (\cdot, t) \| ^2 \| L^2(\Omega) \|
\leq 4 \epsilon^2 (T + 1) + \frac{\beta^2}{T} \left( \| u \| ^2 L^2(\Omega) \| \| + \| v \| ^2 L^2(\Omega) \| \| \right).
\]
Using the inequality
\[ \frac{2[K_M(R')]{\|u\|^2_{L^\infty(0,T;H^1_0(\Omega))}}} {M} + \frac{2[K_M(R')]{\|v\|^2_{L^\infty(0,T;H^1_0(\Omega))}}} {N} + 4 \]
\[ \int_0^T \|(X_\beta^s, Y_\beta^s)(\cdot, s)\|^2_{L^2(\Omega)} ds \]
\[ \leq 4\varepsilon^2(T + 1) + \frac{2\beta^2}{T^2} E^2 + K_{\max}^2(R') \frac{2M + 2N + 4M N}{M N} E^2 \]
\[ \int_0^T \|(X_\beta^s, Y_\beta^s)(\cdot, s)\|^2_{L^2(\Omega)} ds. \]

Applying Gronwall’s inequality, we arrive at
\[ \left\| (X_\beta^s, Y_\beta^s)(\cdot, t) \right\|^2_{L^2(\Omega)} \leq \left( 4\varepsilon^2(T + 1) + \frac{2\beta^2}{T^2} E^2 \right) \exp \left\{ K_{\max}^2(R') \frac{2M + 2N + 4M N}{M N} E^2(T - t) \right\}, \]
which leads to
\[ \left\| [u, v](\cdot, t) \right\|^2_{L^2(\Omega)} = e^{2T\varepsilon(T-t)} \left\| (X_\beta^s, Y_\beta^s)(\cdot, t) \right\|^2_{L^2(\Omega)} \]
\[ \leq \left( 4\varepsilon^2 \beta^{-2}(T + 1) + \frac{2E^2}{T^2} \right) \beta^\frac{2}{\beta} \left\| [u, v](\cdot, t) \right\|^2_{L^2(\Omega)}, \]
where
\[ T\beta = \frac{1}{T} \log \left( \frac{1}{\beta} \right), \quad K_{\max}(R') \leq \frac{1}{T} \log \left( \log A \left( \frac{1}{\beta} \right) \right). \]

Using the inequality \( \sqrt{a^2 + b^2} \leq a + b \), we can easily imply (93).

Now, for every \( \varepsilon > 0 \) small, let us take the unique solution \( t_\varepsilon \in (0, T) \) of the equation \( t = \beta^T \). Notice that \( \lim_{\varepsilon \to 0} t_\varepsilon = 0 \) and \( t_\varepsilon \leq \frac{T}{\log(\beta)} \). Thus, from (93), we obtain
\[ \left\| [u, v](\cdot, t_\varepsilon) \right\|^2_{L^2(\Omega)} \leq \left\| [u, v](\cdot, 0) \right\|^2_{L^2(\Omega)} + \left\| (u, v)(\cdot, 0) \right\|^2_{L^2(\Omega)} \]
\[ \leq \left( 4\varepsilon^2 \beta^{-1}(T + 1) + \frac{2E^2}{T^2} \right) \beta^\frac{2}{\beta} \left\| [u, v](\cdot, 0) \right\|^2_{L^2(\Omega)} \]
\[ + T\varepsilon \left\| \partial_t u, \partial_t v \right\|_{C([0,T];L^2(\Omega))^2} \]
\[ \leq \sqrt{\frac{T}{\log(\beta)}} \left( 4\varepsilon^2 \beta^{-1}(T + 1) + \frac{2E^2}{T^2} \right) \beta^\frac{2}{\beta} \left\| [u, v](\cdot, 0) \right\|^2_{L^2(\Omega)} \]
\[ + 2E \right). \]

The proof of the theorem is completed. \( \square \)

5.2.2. \( H^1(\Omega) \)-estimate.

**Theorem 5.4.** Suppose that (Hyp1), (Hyp2), (Hyp5) hold and the constants \( \beta, K_{\max}(R') \) are chosen as in Theorem 5.3. The exact solution of system (1) satisfies
\[ u \in L^2(0, T; G((-\Delta)^0, e^{MT(-\Delta)})) \cap L^\infty(0, T; H^2(\Omega)) \cap C^1(0, T; H_0^1(\Omega)), \]
\[ v \in L^2(0, T; G((-\Delta)^0, e^{NT(-\Delta)})) \cap L^\infty(0, T; H^2(\Omega)) \cap C^1(0, T; H_0^1(\Omega)). \]
Denote
\[ E^* = \max \left\{ \left\| [u, v] \right\|_{G((-\Delta)^\alpha_1, \mathbb{R}) \times G((-\Delta)^\alpha_2, \mathbb{R})}, \left\| [u, v] \right\|_{C^1(0,T; H^1_0(\Omega))^2}, \left\| [u, v] \right\|_{L^\infty(0,T; H^2(\Omega))^2} \right\}. \]

The input data error satisfies
\[ \Phi^e, \Psi^e, \Phi, \Psi \in H^1_0(\Omega), \]
\[ \left\| \Phi^e - \Phi \right\|_{H^1_0(\Omega)} + \left\| \Psi^e - \Psi \right\|_{H^1_0(\Omega)} \leq \varepsilon. \]

Then, the following estimation holds
\[ \left\| [U^e_\beta, V^e_\beta] (\cdot, t) - [u, v] (\cdot, t) \right\|_{H^1(\Omega)}^2 \leq \left( \frac{\varepsilon}{\beta} C_{3, M_1, N_1, T} + C_{4, M_1, N_1, T, E, E^*} \right) \beta^2 \left[ \log \left( \frac{1}{\beta} \right) \right] \frac{1}{\beta} + 2E^* \sqrt{\frac{T}{\log \left( \frac{1}{t} \right)}}, \quad (103) \]

We choose the regularization parameter \( \beta (\varepsilon) = \varepsilon \). From this we get the stability for \( t \in (0, T] \). Furthermore, there exists \( t_\varepsilon \in (0, T) : \lim_{\varepsilon \to 0} t_\varepsilon = 0 \), such that
\[ \left\| [U^e_\beta, V^e_\beta] (\cdot, t_\varepsilon) - [u, v] (\cdot, t_\varepsilon) \right\|_{H^1(\Omega)}^2 \leq \left( \frac{\varepsilon}{\beta} C_{3, M_1, N_1, T} + C_{4, M_1, N_1, T, E, E^*} \right) \beta^2 \left[ \log \left( \frac{1}{\beta} \right) \right] \frac{1}{\beta} + 2E^* \sqrt{\frac{T}{\log \left( \frac{1}{t_\varepsilon} \right)}}, \quad (104) \]

or we have the stability at \( t = 0 \). Here \( C_{3, M_1, N_1, T}, C_{4, M_1, N_1, T, E, E^*}, C^2_{5, M_1, N_1, T, E, E^*}, C_0 \) are constants independent of \( t, \varepsilon \).

Proof. As in the previous section, we define
\[ X^e_\beta (x, t) = e^{T^e_\beta (t-T)} (U^e_\beta - u) (x, t), \quad Y^e_\beta (x, t) = e^{T^e_\beta (t-T)} (V^e_\beta - v) (x, t). \]

Since the hypothesis \( u, v \in L^\infty (0, T; H^2(\Omega)), \) yields \( \Delta X^e_\beta, \Delta Y^e_\beta \in L^2 (0, T; L^2(\Omega)), \)

From (95), taking \( \Theta = \lambda_\beta \left( X^e_\beta (\cdot, t), \theta_p (\cdot) \right) \theta_p (x) \) summing from \( p = 1 \) to \( \infty \), and then integrating from \( t \) to \( T \). By some simple calculations, we have that
\[ \begin{align*}
\| \nabla U^e_\beta (\cdot, t) \|_{L^2 (\Omega)}^2 & - \| \nabla U^e_\beta (\cdot, t) \|_{L^2 (\Omega)}^2 - 2T \int_t^T \| \nabla U^e_\beta (\cdot, s) \|_{L^2 (\Omega)}^2 ds \\
& = -2 \int_t^T \mathcal{M}_R \left( \| \nabla U^e_\beta (\cdot, s) \|_{L^2 (\Omega)}^2 - \mathcal{M} (\| \nabla u (\cdot, s) \|, \| \nabla v (\cdot, s) \|) \right) \langle \Delta u (\cdot, s), \Delta X^e_\beta (\cdot, s) \rangle_{L^2 (\Omega)} ds \\
& = -2 \int_t^T e^{T^e_\beta (s-T)} \langle L^e_\beta u (\cdot, s), \Delta X^e_\beta (\cdot, s) \rangle_{L^2 (\Omega)} ds + 2 \int_t^T \langle P^e_\beta \nabla X^e_\beta (\cdot, s), \nabla X^e_\beta (\cdot, s) \rangle_{L^2 (\Omega)} ds \\
& = \| u \|_{H^1 (0, T)}^2 + \int_0^T \langle P^e_\beta u (\cdot, s), \Delta X^e_\beta (\cdot, s) \rangle_{L^2 (\Omega)} ds \\
& = \text{terms 1, 2, 3}.
\end{align*} \]

The above terms make sense since the linearity of \( L^e_\beta, P^e_\beta \), Lipschitz property of \( \mathcal{M}_R^e \), and the fact that \( \Delta X^e_\beta, \Delta Y^e_\beta, \Delta u, \Delta v \in L^2 (0, T; L^2 (\Omega)). \)
Thanks to Lemma 5.2, we have

\[
\|B_1\| \leq 2K_M(R^e) \int_t^T \left\| (X^e_{\beta}, \delta^e_{\beta})(:, s) \right\|_{L^2(\Omega)} \|\Delta u(:, s)\|_{L^2(\Omega)} \|\Delta X^e_{\beta}(\cdot, s)\|_{L^2(\Omega)} \, ds
\]

(106)

Using Hölder’s inequality, Cauchy’s inequality, lemma 5.2, it gives

\[
\|B_2\| \leq 2 \int_t^T \left\| L^e_{\beta}u(:, s) \right\|_{L^2(\Omega)} \|\Delta X^e_{\beta}(\cdot, s)\|_{L^2(\Omega)} \, ds
\]

(107)

\[
\leq \frac{6}{MT^2} \int_t^T \|u(:, s)\|^2_{\Omega} + \frac{2M}{3} \int_t^T \|\Delta X^e_{\beta}(\cdot, s)\|^2_{L^2(\Omega)} \, ds
\]

\[
\leq \frac{6\beta^2(E^e)^2}{MT^2} + \frac{2M}{3} \int_t^T \|\Delta X^e_{\beta}(\cdot, s)\|^2_{L^2(\Omega)} \, ds.
\]

Using Hölder’s inequality, hypothesis (Hyp5), we obtain

\[
|B_3| \leq 2 \int_t^T \left\| P^e_{\beta} \nabla X^e_{\beta}(\cdot, s) \right\|_{L^2(\Omega)} \|\nabla X^e_{\beta}(\cdot, s)\|_{L^2(\Omega)} \, ds
\]

(108)

\[
\leq \frac{2}{T} \log \left( \frac{1}{\beta} \right) \int_t^T \|\nabla X^e_{\beta}(\cdot, s)\|^2_{L^2(\Omega)} \, ds.
\]

Using (Hyp4), we estimate \(B_4\)

\[
|B_4| = 2 \int_t^T e^{\gamma(s-t)} \left\{ f^e(:, s)-f(:, s), \Delta X^e_{\beta}(\cdot, s) \right\}_{L^2(\Omega)} \, ds
\]

(109)

\[
\leq \frac{6}{MT} \int_t^T e^{2\gamma(s-t)} \|f^e(:, s)-f(:, s)\|^2_{L^2(\Omega)} \, ds + \frac{2M}{3} \int_t^T \|\Delta X^e_{\beta}(\cdot, s)\|^2_{L^2(\Omega)} \, ds
\]

\[
\leq \frac{6\gamma^2T}{M} + \frac{2M}{3} \int_t^T \|\Delta X^e_{\beta}(\cdot, s)\|^2_{L^2(\Omega)} \, ds.
\]

Choosing \(T_{\beta} = \frac{T}{\gamma} \log \left( \frac{1}{\beta} \right)\). From (105)-(109), we deduce

\[
\|\nabla X^e_{\beta}(\cdot, t)\|^2_{L^2(\Omega)}
\]

\[
\leq \epsilon^{2} \left( \frac{6T}{M} + 1 \right) + \frac{6\beta^2(E^e)^2}{MT^2} + \frac{6K_M(R^e)^2}{M} \frac{2}{T^2} \int_t^T \left\| (X^e_{\beta}, \delta^e_{\beta})(:, s) \right\|^2_{L^2(\Omega)} \, ds.
\]

In a similar manner, we obtain estimate for \(\delta_{\beta}\). Therefore

\[
\| (\nabla X^e_{\beta}, \nabla \delta^e_{\beta})(:, t) \|^2_{L^2(\Omega)}
\]

\[
\leq 2\epsilon^{2} \left( \frac{6T}{M} + \frac{N}{N} + 2 \right) + \frac{6\beta^2(E^e)^2}{M} \frac{2}{T^2} \frac{M + N}{M} \int_t^T \left\| (X^e_{\beta}, \delta^e_{\beta})(:, s) \right\|^2_{L^2(\Omega)} \, ds.
\]

(110)
6. Conclusions.

In order to increase the significance of the study, numerical results should be presented and discussed illustrating the theoretical findings in terms of accuracy and stability. However, in the case of System (1) with nonlocal diffusion coefficients take place, it goes beyond in this research due to the complications of such ill-posed problems, requiring huge efforts. Therefore, it should be considered carefully in a forthcoming work.

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