CONSERVATION LAWS, SYMMETRIES, AND LINE SOLITON SOLUTIONS OF GENERALIZED KP AND BOUSSINESQ EQUATIONS WITH $p$-POWER NONLINEARITIES IN TWO DIMENSIONS

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Nonlinear generalizations of integrable equations in one dimension, such as the Korteweg–de Vries and Boussinesq equations with $p$-power nonlinearities, arise in many physical applications and are interesting from the analytic standpoint because of their critical behavior. We study analogous nonlinear $p$-power generalizations of the integrable Kadomtsev–Petviashvili and Boussinesq equations in two dimensions. For all $p \neq 0$, we present a Hamiltonian formulation of these two generalized equations. We derive all Lie symmetries including those that exist for special powers $p \neq 0$. We use Noether’s theorem to obtain conservation laws arising from the variational Lie symmetries. Finally, we obtain explicit line soliton solutions for all powers $p > 0$ and discuss some of their properties.

Keywords: line soliton, conservation law, Kadomtsev–Petviashvili equation

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1. Introduction

The Kadomtsev–Petviashvili (KP) equation

$$(u_t + \alpha u u_x + u_{xxx})_x + \beta u_{yy} = 0 \quad (1)$$

and its modified version

$$(u_t + \alpha u^2 u_x \pm \sqrt{2\alpha|\beta|} u_x \partial_x^{-1} u_y + u_{xxx})_x + \beta u_{yy} = 0 \quad (2)$$

are integrable systems in $2+1$ dimensions. These equations arise in important physical applications, most notably as models for shallow water waves [1], ion acoustic waves [2], and magnetic excitations in thin films [3], and they have line soliton and lump solutions [4]–[8]. Both equations have a rich mathematical structure. In particular, their dimensional reductions yield many physically relevant integrable systems in 1+1 dimensions.

Nonlinear generalizations of $(1+1)$-dimensional integrable systems with $p$-power nonlinearities have been extensively studied. The best known example is the generalized Korteweg-de Vries (KdV) equation $u_t + u^p u + u_{xxx} = 0$ with $p \neq 0$, which becomes the ordinary KdV equation with $p = 1$ and the modified KdV
equation with \( p = 2 \). The KdV equation is a model of unidirectional shallow water waves, and the modified KdV equation is a model of unidirectional ion acoustic waves. For a general power \( p \), the generalized KdV equation has solitary wave solutions

\[
u = \left( \frac{(p + 1)(p + 2)}{2} \right)^{1/p} \text{sech}^{2/p} \left( \frac{p}{2} \sqrt{c(x - ct)} \right),
\]

whose interactions depend sensitively on the value of \( p \). This sensitivity can be understood by considering the scaling properties of the conserved mass and energy

\[
\mathcal{M}[\nu] = \int_{-\infty}^{\infty} \nu^2 \, dx, \quad \mathcal{E}[\nu] = \int_{-\infty}^{\infty} \left( \frac{1}{2} \nu_x^2 - \frac{1}{(p + 1)(p + 2)} \nu^{p+2} \right) \, dx
\]

for initial-value solutions \( \nu(x, t) \). Under the scaling symmetry \( x \to \lambda x, \ t \to \lambda^3 t, \ \nu \to \lambda^{-2/p} \nu \), the mass is scaling-invariant for \( p = 4 \), and the energy has a negative scaling weight for any \( p > 0 \). These properties can be used to show [9] that solitary waves are stable under arbitrary perturbations with respect to the \( H^1 \) norm, \( \| \nu \|_{H^1} = \int_{-\infty}^{\infty} (\nu^2 + \nu_x^2) \, dx \), if \( p \) is smaller than the critical power \( p^* = 4 \). This stability extends to the critical case \( p = 4 \) for solitary waves of sufficiently small energy.

There has been much less study of \( p \)-power generalizations of the KP and modified KP equations themselves. In particular, the generalized KP (gKP) equation

\[
\nu_t + \alpha \nu^p \nu_x + \nu_{xxx} + \beta \nu_{yyy} = 0, \quad p \neq 0,
\]

is a natural (2+1)-dimensional analogue of the gKdV equation. An interesting variant is given by

\[
\nu_{tt} = \nu_{xx} + \alpha (\nu^{p+1})_{xx} + \gamma \nu_{xxxx} + \beta \nu_{yy}, \quad p \neq 0,
\]

which is a (2+1)-dimensional generalized Boussinesq (2D gB) equation. Its reduction to 1+1 dimensions gives a \( p \)-power generalization of the ordinary Boussinesq equation

\[
\nu_{tt} = \nu_{xx} + \alpha (\nu^2)_{xx} + \gamma \nu_{xxxx}.
\]

This equation is an integrable system that can be obtained by a dimensional reduction of the KP equation, and it is a physically relevant model of bidirectional shallow water waves. One very interesting feature of the Boussinesq equation with \( \gamma < 0 \) is that its solitary wave solutions have an instability that can lead to formation of a singularity in a finite time [10], [11]. This behavior contrasts with the typical stability of solitons in integrable systems.

Our purpose here is to determine the conservation laws, symmetries, and line soliton solutions admitted by gKP equation (3) and 2D gB equation (4) in 2+1 dimensions for all nonlinearity powers \( p > 0 \).

In Sec. 2, we formulate both the gKP and the 2D gB equations for arbitrary \( p \neq 0 \) as Hamiltonian systems by introducing potentials. In Sec. 3, we then obtain all Lie point symmetries and find the class of variational point symmetries. In Sec. 4, we use Noether’s theorem in the Hamiltonian setting to derive the corresponding conservation laws. These results include any point symmetries and any Noether conservation laws that arise for special powers \( p \neq 0 \).

In Sec. 5, we consider line solitons \( \nu = U(x + \mu y - \nu t) \) for both the gKP and the 2D gB equations for all \( p > 0 \), where the parameters \( \mu \) and \( \nu \) determine the direction and the speed of the line soliton. Using the conservation laws obtained for the gKP and 2D gB equations, we directly reduce the resulting fourth-order nonlinear differential equations for \( U \) to first-order separable differential equations. We then integrate these
separable differential equations to obtain an explicit form of the line soliton solutions for all \( p > 0 \) with \( \mu \) and \( c \) arbitrary except for satisfying a kinematic inequality. We discuss some properties of these solutions. In Sec. 6, we conclude with a few remarks.

In the case \( p = 2 \), a few conservation laws of gKP equation (3) and 2D gB equation (4) were previously found in [12], [13]. Some solutions of the gKP equation in three dimensions were obtained in [14] without any discussion of their properties. A general treatment of symmetries, conservation laws, and their applications to differential equations can be found in [15]–[17].

### 2. Hamiltonian structure

It is useful to begin by noting that the parameters in gKP equation (3) transform under scalings
\[
 t \rightarrow \lambda^3 t, \quad x \rightarrow \lambda x, \quad y \rightarrow \lambda^a y, \quad u \rightarrow \lambda^b u \quad \text{as} \quad \alpha \rightarrow \lambda^{-pb-2} \alpha \quad \text{and} \quad \beta \rightarrow \lambda^{2a-4} \beta.
\]
Hence, we can set \( \alpha = 1 \) and \( \beta = \pm 1 \) without loss of generality. We can likewise set \( \alpha = 1 \), \( \beta = \pm 1 \), and \( \gamma = \pm 1 \) in 2D gB equation (4). Therefore, we hereafter consider the gKP and gB equations in the respective forms
\[
 (u_t + u^p u_x + u_{xxx})_x + \sigma^2 u_{yy} = 0 \tag{6}
\]
and
\[
 u_{tt} = u_{xx} + (u^{p+1})_{xx} \pm u_{xxxx} + \sigma^2 u_{yy}, \tag{7}
\]
where \( p \neq 0 \) and \( \sigma^2 = \pm 1 \). Both the gKP equation and the 2D gB equation can be expressed as Hamiltonian evolution equations. The simplest way to derive this formulation is to first consider a Lagrangian form of the equations.

For the KP equation, a Lagrangian is known [18] using a potential \( v(x, y, t) \) given by
\[
 u = v_x. \tag{8}
\]
In terms of this potential, the gKP equation becomes
\[
 0 = u_{tx} + v_x^p v_{xx} + v_{xxxx} + \sigma^2 v_{yy} = \frac{\delta L}{\delta v}, \tag{9}
\]
where \( L \) is the Lagrangian
\[
 L = -\frac{1}{2} v_t v_x + \frac{1}{2} v_x^2 - \frac{1}{(p + 1)(p + 2)} v_x^{p+2} - \frac{1}{2} \sigma^2 v_y^2. \tag{10}
\]
The associated Hamiltonian formulation is then given by
\[
 v_t = -D_x^{-1} \left( \frac{\delta H}{\delta v} \right), \tag{11}
\]
where
\[
 H = \int_{y^2} \left( \frac{1}{2} v_x^2 - \frac{1}{(p + 1)(p + 2)} v_x^{p+2} - \frac{1}{2} \sigma^2 v_y^2 \right) dx \, dy \tag{12}
\]
is the Hamiltonian functional and \( D_x^{-1} \) is a Hamiltonian operator [15]. This formulation can be equivalently expressed in terms of \( u \) by using the variational derivative identity \( \delta H / \delta v = -D_x (\delta H / \delta u) \), which yields
\[
 u_t = D_x \left( \frac{\delta H}{\delta u} \right), \tag{13}
\]
where the Hamiltonian density is now a nonlocal expression in terms of $u$. We note that if we set $\sigma^2 = 0$, then we obtain the standard Hamiltonian formulation of the gKdV equation [19].

A similar Lagrangian formulation can be obtained for the 2D gB equation starting from the same potential (8). The corresponding form of the 2D gB equation is given by

$$0 = v_{tt} - v_{xx} - (v_x^{p+1})_x \mp v_{xxxx} - \sigma^2 v_{yy} = \frac{\delta L}{\partial v},$$

(14)

where

$$L = -\frac{1}{2}v_t^2 + \frac{1}{2}v_x^2 + \frac{1}{p+2}v_x^{p+2} + \frac{1}{2}v_{xx}^2 + \frac{1}{2}\sigma^2 v_y^2$$

is the Lagrangian. To obtain the associated Hamiltonian formulation, we must convert this formulation into an equivalent first-order evolution system

$$v_t = w, \quad w_t = v_{xx} + (v_x^{p+1})_x \pm v_{xxxx} + \sigma^2 v_{yy}. \quad (16)$$

The associated Hamiltonian formulation for this system is then given by

$$\begin{pmatrix} v \\ w \end{pmatrix}_t = J \begin{pmatrix} \delta H/\delta v \\ \delta H/\delta w \end{pmatrix}, \quad J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

(17)

where

$$H = \int_{\mathbb{R}^2} \left( \frac{1}{2}w^2 + \frac{1}{2}v_x^2 + \frac{1}{p+2}v_x^{p+2} + \frac{1}{2}v_{xx}^2 + \frac{1}{2}\sigma^2 v_y^2 \right) dx \, dy$$

(18)

is the Hamiltonian functional and $J$ plays the role of a Hamiltonian operator [15]. An equivalent formulation in terms of $u$ arises from the relation $u_t = v_{tx} = w_x$ combined with the previous variational derivative identity $\delta H/\delta v = -D_x(\delta H/\delta u)$. This yields

$$\begin{pmatrix} v \\ w \end{pmatrix}_t = D \begin{pmatrix} \delta H/\delta u \\ \delta H/\delta w \end{pmatrix}, \quad D = \begin{pmatrix} 0 & D_x \\ D_x & 0 \end{pmatrix},$$

(19)

where $D$ is a Hamiltonian operator and the Hamiltonian density is now a nonlocal expression in terms of $u$ and $w$. We note that if we set $\sigma^2 = 0$, then we obtain a Hamiltonian formulation of the 1+1 gB equation, which in the case $p = 1$ coincides with the first Hamiltonian structure [15] of ordinary Boussinesq equation (5).

These Hamiltonian formulations motivate studying the symmetries and conservation laws of the gKP and 2D gB equations in their respective potential forms (9) and (14).

3. Symmetries

Symmetries are a basic structure of nonlinear evolution equations. They yield transformation groups under which the set of solutions of a given evolution equation is mapped into itself and can thus be used to find group-invariant solutions.

We first consider point symmetries $(x, y, t, v) \rightarrow (\tilde{x}, \tilde{y}, \tilde{t}, \tilde{v})$, where $\tilde{x}, \tilde{y}, \tilde{t},$ and $\tilde{v}$ are functions of $x, y, t,$ and $v$. For gKP equation (9) and 2D gB equation (14), an infinitesimal point symmetry consists of a generator

$$X = \xi^x(x, y, t, v)\partial_x + \xi^y(x, y, t, v)\partial_y + \tau(x, y, t, v)\partial_t + \eta(x, y, t, v)\partial_v,$$
whose prolongation leaves the corresponding equation invariant. Every point symmetry can be expressed in an equivalent characteristic form

\[ \hat{X} = P \partial_v, \quad P = \eta(x, y, t, v) - \xi^x(x, y, t, v)v_x - \xi^v(x, y, t, v)v_y - \tau(x, y, t, v)v_1, \]  

(21)

which acts only on \( v \). The function \( P \) is called the symmetry characteristic.

The invariance of the gKP equation in potential form (9) is expressed by the condition

\[ 0 = D_x(D_1P + v_v^pD_xP + D_3^2P) + \sigma^2 D_y^2P, \]  

(22)

which holds for all solutions \( v(x, y, t) \) of the gKP potential equation. Similarly, the invariance of the 2D gB equation in potential form (14) is given by

\[ 0 = D_x^2P - D_x^2(P + (p + 1)v_v^pD_xP \pm D_3^2P) - \sigma^2 D_y^2P, \]  

(23)

which holds for all solutions \( v(x, y, t) \) of the 2D gB potential equation. Each of these invariance conditions splits with respect to \( x \) derivatives and \( y \) derivatives of \( v \), yielding an overdetermined system of equations for \( \eta(x, y, t, v) \), \( \xi^x(x, y, t, v) \), \( \xi^v(x, y, t, v) \), and \( \tau(x, y, t, v) \), along with \( p \), subject to the classification condition \( p \neq 0 \). It is straightforward to set up and solve these two determining systems using Maple. In particular, the Maple package \texttt{rifsimp} can be used to obtain a complete classification of all solution cases.

We now summarize the results.

\textbf{Theorem 3.1.} The following statements hold:

1. The point symmetries admitted by generalized KP potential equation (9) for arbitrary \( p \neq 0 \) are generated by

\[ \tau_1 = 0, \quad \xi^x_1 = 1, \quad \xi^y_1 = 0, \quad \eta_1 = 0, \quad (24a) \]

\[ \tau_2 = 0, \quad \xi^x_2 = 0, \quad \xi^y_2 = 1, \quad \eta_2 = 0, \quad (24b) \]

\[ \tau_3 = 1, \quad \xi^x_3 = 0, \quad \xi^y_3 = 0, \quad \eta_3 = 0, \quad (24c) \]

\[ \tau_4 = 3t, \quad \xi^x_4 = x, \quad \xi^y_4 = 2y, \quad \eta_4 = (1 - 2/p)v, \quad (24d) \]

\[ \tau_5 = 0, \quad \xi^x_5 = y, \quad \xi^y_5 = -2\sigma^2 t, \quad \eta_5 = 0, \quad (24e) \]

\[ \tau_6 = 0, \quad \xi^x_6 = 0, \quad \xi^y_6 = 0, \quad \eta_6 = f_1(t) + f_2(t)y. \quad (24f) \]

Symmetries (24a)–(24c) are translations, symmetry (24d) is a scaling, and symmetry (24e) is a rotation in the plane \((y, t + \sigma^2 x)\) combined with a boost in the plane \((y, t - \sigma^2 x)\). Symmetry (24f) is a linear combination of two infinite-dimensional families corresponding to the general solution of \( P_{yy} = 0 \) for \( P(y, t) \).

2. Additional point symmetries are admitted by generalized KP potential equation (9) only if \( p = 1 \):

\[ \tau_7 = f_3(t), \]

\[ \xi^x_7 = -\frac{1}{6}\sigma^2 f_3''(t)y^2 + \frac{1}{2}\sigma^2 f_3'(t)y + \frac{1}{3}f_3(t)x + f_5(t), \quad \xi^y_7 = \frac{2}{3}f_3'(t)y + f_4(t), \]

\[ \eta_7 = \frac{1}{72}f_3'''(t)y^4 + \frac{1}{12}\sigma^2 f_3''(t)y^3 - \frac{1}{6}\sigma^2 f_3''(t)y^2x - \frac{1}{2}\sigma^2 f_3''(t)y^2 - \frac{1}{2}f_3'(t)y + \]

\[ + \frac{1}{6}f_3'(t)x^2 + f_5(t)x - \frac{1}{3}f_3(t)v. \]  

(25)
This symmetry is a linear combination of three infinite-dimensional families and includes the case $p = 1$ of scaling symmetry (24d) for $f_3(t) = 3t$ and $f_4(t) = f_5(t) = 0$.

The corresponding symmetry transformation groups for arbitrary $p \neq 0$ are respectively given by

\begin{align}
(x, y, t, v) &\rightarrow (x + \epsilon, y, t, v), \\
(x, y, t, v) &\rightarrow (x, y + \epsilon, t, v), \\
(x, y, t, v) &\rightarrow (x, y, t + \epsilon, v), \\
(x, y, t, v) &\rightarrow (e^{\epsilon}x, e^{\epsilon^2}y, e^{3\epsilon}t, e^{(1-2/p)\epsilon}v), \\
(x, y, t, v) &\rightarrow (x + \epsilon y - \sigma^2 e^t, y - 2\epsilon \sigma^2 t, t, v), \\
(x, y, t, v) &\rightarrow (x, y, t, f_1(t)\epsilon + f_2(t)e\epsilon v + v),
\end{align}

where $\epsilon \in \mathbb{R}$ is the group parameter. For $p = 1$, the symmetry transformation group generated by the three infinite families (25) was discussed in [20].

**Theorem 3.2.** The following statements hold:

1. The point symmetries admitted by 2D gB potential equation (14) for arbitrary $p \neq 0$ are generated by

\begin{align}
\tau_1 &= 0, \quad \xi_1^x = 1, \quad \xi_1^y = 0, \quad \eta_1 = 0, \\
\tau_2 &= 0, \quad \xi_2^x = 0, \quad \xi_2^y = 1, \quad \eta_2 = 0, \\
\tau_3 &= 1, \quad \xi_3^x = 0, \quad \xi_3^y = 0, \quad \eta_3 = 0, \\
\tau_4 &= y, \quad \xi_4^x = 0, \quad \xi_4^y = \sigma^2 t, \quad \eta_4 = 0, \\
\tau_5 &= 0, \quad \xi_5^x = 0, \quad \xi_5^y = 0, \quad \eta_5 = f_1(y + \sigma t) + f_2(y - \sigma t).
\end{align}

Symmetries (27a)-(27c) are translations, and symmetry (27d) is a boost in the plane $(y, t)$. Symmetry (27!c) is a linear combination of two infinite-dimensional families corresponding to the general solution of $P_{tt} - \sigma^2 P_{yy} = 0$ for $P(y, t)$.

2. Additional point symmetries are admitted by 2D gB potential equation (14) only if $p = 1$:

\begin{equation}
\tau_6 = 2t, \quad \xi_6^x = x, \quad \xi_6^y = 2y, \quad \eta_6 = -(v + x).
\end{equation}

This symmetry is a scaling combined with a shift in $v$.

The corresponding symmetry transformation groups for arbitrary $p \neq 0$ are respectively given by

\begin{align}
(x, y, t, v) &\rightarrow (x + \epsilon, y, t, v), \\
(x, y, t, v) &\rightarrow (x, y + \epsilon, t, v), \\
(x, y, t, v) &\rightarrow (x, y, t + \epsilon, v), \\
(x, y, t, v) &\rightarrow (x, \cosh(\epsilon) y + \sinh(\epsilon) t, \cosh(\epsilon) t + \sinh(\epsilon) t, v), \\
(x, y, t, v) &\rightarrow (x, y, t, f_1(t)\epsilon + f_2(t)e\epsilon v + v),
\end{align}

where $\epsilon \in \mathbb{R}$ is the group parameter. The symmetry transformation group generated by scaling shift (28) in the case $p = 1$ is given by

\begin{equation}
(x, y, t, v) \rightarrow (e^\epsilon x, e^{2\epsilon} y, e^{2\epsilon} t, e^{-\epsilon} v - \frac{1}{2} e^\epsilon x).
\end{equation}
**Variational symmetries.** We next determine which of the symmetries in Theorems 3.1 and 3.2 are variational.

We recall that an infinitesimal symmetry \( \hat{X} = P \partial_v \) is a variational symmetry if it leaves any given Lagrangian \( L \) invariant up to a total divergence,

\[
\hat{X}L = D_t \Psi^t + D_x \Psi^x + D_y \Psi^y,
\]

where \( \Psi^t, \Psi^x, \) and \( \Psi^y \) are functions of \( t, x, y, v, \) and derivatives of \( v \). This invariance condition is typically checked by first computing \( \hat{X}L \) and then attempting to use integration by parts to bring this expression into the form of a total divergence. A much more efficient method can be used based on the Euler operator (i.e., the variational derivative)

\[
E_v = \partial_v - D_t \partial_{v_t} - D_x \partial_{v_x} - D_y \partial_{v_y} + D^2_t \partial_{v_{tt}} + D^2_x \partial_{v_{xx}} + D^2_y \partial_{v_{yy}} + 
+ D_t D_x \partial_{v_{tx}} + D_t D_y \partial_{v_{ty}} + D_x D_y \partial_{v_{xy}} + \ldots
\]

This operator has the property that it annihilates a function of \( t, x, y, v, \) and derivatives of \( v \) iff the function is equal to a total divergence \( D_t \Psi^t + D_x \Psi^x + D_y \Psi^y \). Consequently, variational symmetry condition (31) can be formulated as

\[
E_v(\hat{X}L) = 0.
\]

Moreover, this condition can be further simplified [15], [21] using the variational identity

\[
\hat{X}L = E_v(L) \hat{X}v + D_t \Phi^t + D_x \Phi^x + D_y \Phi^y,
\]

which is derived by integration by parts, where \( E_v(L) \) is the variational derivative of \( L \). Combining identity (34) and Euler-operator equation (33), we see that variational symmetry condition (31) becomes

\[
E_v(PE_v(L)) = 0.
\]

This equation involves only the symmetry characteristic \( P \) and the expression for the left-hand side of Euler–Lagrange equations \( E_v(L) = 0 \) given by \( L \) (which is unchanged if any total divergence is added to \( L \)).

We now apply variational symmetry condition (35) to the point symmetries admitted by gKP equation (9) and 2D gB equation (14), where \( L \) is the respective Lagrangian (10) and (15). A straightforward computation gives the following results.

**Proposition 3.1.** The variational point symmetries admitted by gKP potential equation (9) are generated by translation symmetries (24a)–(24c), rotation-boost symmetry (24e), and infinite symmetry families (24f) and (25). Scaling symmetry (24d) is variational if the nonlinearity power is \( p = 1 \), which coincides with the case \( f_3(t) = 3t, f_4(t) = f_5(t) = 0 \) in infinite family (25).

**Proposition 3.2.** The variational point symmetries admitted by 2D gB potential equation (14) are generated by translation symmetries (27a)–(27c), boost symmetry (27d), and infinite symmetry families (27e). Scaling shift symmetry (28) is not variational for any nonlinearity power \( p \).
4. Conservation laws

Conservation laws are fundamentally important for nonlinear evolution equations because they provide physical, conserved quantities and also conserved norms for all solutions.

For gKP equation (9) and 2D gB equation (14), a local conservation law is a continuity equation

\[ D_t T + D_x X + D_y Y = 0 \]  

(36)

holding for all solutions \( v(x, y, t) \) of the corresponding equations, where \( T \) is the conserved density and \( (X, Y) \) is the spatial flux, which are functions of \( t, x, y, v \), and derivatives of \( v \). If solutions \( v(x, y, t) \) are considered in a given spatial domain \( \Omega \subseteq \mathbb{R}^2 \), then every conservation law yields a corresponding conserved integral

\[ C[v] = \int_{\Omega} T \, dx \, dy \]  

(37)

satisfying the global balance equation

\[ \frac{d}{dt} C[v] = -\int_{\partial\Omega} (X, Y) \cdot \hat{n} \, ds, \]  

(38)

where \( \hat{n} \) is the unit outward normal vector of the domain boundary curve \( \partial\Omega \) and \( ds \) is the arc length on this curve. Global balance equation (37) has the physical meaning that the rate of change of \( C[v] \) on the spatial domain is balanced by the net outward flux through the boundary of the domain.

A conservation law is locally trivial if for all solutions \( v(x, y, t) \) in \( \Omega \), the conserved density \( T \) reduces to a spatial divergence \( D_x \Psi_x + D_y \Psi_y \), and the spatial flux \( (X, Y) \) reduces to a time derivative \( -D_t(\Psi_x, \Psi_y) \), because global balance equation (38) then becomes an identity. Similarly, two conservation laws are locally equivalent if they differ by a locally trivial conservation law for all solutions \( v(x, y, t) \) in \( \Omega \). We are interested only in locally nontrivial conservation laws.

Any nontrivial conservation law (36) can be expressed in an equivalent characteristic form [15], [16], [21] given by a divergence identity holding outside the space of solutions \( v(x, y, t) \). For the gKP and 2D gB equations, conservation laws have the respective characteristic forms

\[ D_t \tilde{T} + D_x \tilde{X} + D_y \tilde{Y} = (v_{tx} + v_x^p v_{xx} + v_{xxxx} + \sigma^2 v_{yy})Q \]  

(39)

and

\[ D_t \tilde{T} + D_x \tilde{X} + D_y \tilde{Y} = (v_{tt} - v_{xx} - (v_x^{p+1})_x + v_{xxxx} - \sigma^2 v_{yy})Q, \]  

(40)

where \( Q, \tilde{T}, \tilde{X}, \) and \( \tilde{Y} \) are functions of \( t, x, y, v \), and derivatives of \( v \) where the conserved density \( \tilde{T} \) and the spatial flux \( (\tilde{X}, \tilde{Y}) \) reduce to \( T \) and \( (X, Y) \) when restricted to all solutions \( v(x, y, t) \) of the respective equations. These divergence identities are called the characteristic equation for the conservation law, and the function \( Q \) is called the conservation law multiplier. In particular, \( Q \) has the property that it is nonsingular if evaluated on any solution \( v(x, y, t) \). Consequently, we note that the characteristic equation of a conservation law is locally equivalent to the conservation law itself.

Because the gKP and 2D gB equations each have a Lagrangian formulation, Noether’s theorem [16], [21] shows that for both equations, there is a one-to-one correspondence between locally nontrivial conservation laws (up to equivalence) and variational symmetries. Specifically, this correspondence can stated simply as \( Q = P \), where \( Q \) is a conservation law multiplier and \( P \) is a variational symmetry characteristic. The conserved density \( \tilde{T} \) and the spatial flux \( (\tilde{X}, \tilde{Y}) \) corresponding to a given variational symmetry characteristic \( P \) can be straightforwardly obtained by several methods [16], [21], [22], starting from characteristic equations (39) and (40).
For any given variational symmetry characteristic \( P \), we can apply an iterated integration process [22] directly to the terms in the expression \( PE_v(L) \), which yields \( \tilde{T} \), \( \tilde{X} \), and \( \tilde{Y} \). This method can sometimes be lengthy or awkward, depending on the complexity of the expression \( PE_v(L) \). A more systematic method [16] is to split the characteristic equation \( D_t \tilde{T} + D_x \tilde{X} + D_y \tilde{Y} = PE_v(L) \) with respect to \( v \) and its derivatives and thus obtain an overdetermined linear system that can be solved for \( \tilde{T} \), \( \tilde{X} \), and \( \tilde{Y} \). An alternative, more direct method is to use a homotopy integral formula that inverts the Euler operator in the variational symmetry equation \( E_v(PE_v(L)) = 0 \). The simplest version of this formula appeared in [16], [21], and a more complicated general version was given in [15].

We now summarize the conservation laws arising from the variational point symmetries obtained in Propositions 3.1 and 3.2.

**Theorem 4.1.** The following statements hold:

1. The conservation laws corresponding to the variational point symmetries admitted by gKP potential equation (9) for arbitrary \( p \neq 0 \) are given by

\[
T_1 = \frac{1}{2} v_{xx}^2 - \frac{\sigma}{2} \frac{v_y^2}{v_x} - \frac{1}{(p+1)(p+2)} v_{x}^{p+2},
\]

\[
X_1 = v_1 v_{xxx} - v_{xx} v_{xx} + \frac{1}{p+1} v_x^{p+1} v_t + \frac{1}{2} v_{x}^2,
\]

\[
Y_1 = \sigma^2 v_t v_y,
\]

\[
T_2 = \frac{1}{2} v_x^2,
\]

\[
X_2 = v_x v_{xxx} - \frac{1}{2} v_{xx}^2 + \frac{1}{p+2} v_x^{p+2} - \frac{\sigma^2}{2} v_y^2,
\]

\[
Y_2 = \sigma^2 v_x v_y,
\]

\[
T_3 = \frac{1}{2} v_x v_y,
\]

\[
X_3 = v_y v_{xxx} - v_{xx} v_{xy} + \frac{1}{p+1} v_y v_x^{p+1} + \frac{1}{2} v_t v_y,
\]

\[
Y_3 = \frac{1}{2} v_{xx}^2 + \frac{\sigma^2}{2} v_y^2 - \frac{1}{(p+1)(p+2)} v_x^{p+2} - \frac{1}{2} v_t v_x,
\]

\[
T_4 = \frac{1}{2} (v_{xy}^2) - \sigma^2 t v_y v_x,
\]

\[
X_4 = (v_{xy} - 2 \sigma^2 t v_y) v_{xxx} - \frac{1}{2} (v_y^2 + 2 \sigma^2 t v_y v_x) - \frac{\sigma^2}{2} v_y^2 - \sigma^2 t v_y v_t - \frac{1}{p+1} t v_y v_x^{p+1} + \frac{1}{p+2} v_y^{p+2},
\]

\[
Y_4 = - \sigma^2 t v_x^{p+2} - 2 \sigma^2 v_x t v_x + \sigma^2 v_y v_x + \sigma^2 t v_t v_x + 2 \sigma^2 \frac{t}{(p+1)(p+2)} v_x^{p+2},
\]

\[
T_5 = 0,
\]

\[
X_5 = \left( \frac{1}{p+1} v_x^{p+1} + v_{xxx} + v_t \right) (f_1(t) y + f_2(t)),
\]

\[
Y_5 = \sigma^2 ((v_y - v) f_1(t) + f_2(t) v_y).
\]
2. The only additional conservation laws corresponding to variational point symmetries admitted by gKP potential equation (9) arise for $p = 1$:

$$T_6 = \frac{1}{2} f_3(t) v_x^2 + f_3(t) v,$$

$$X_6 = \sigma^2 f_3(t) v_x + \frac{1}{2} f_3'''(t)y^2 - f_3'(t)x v_{xxx} - \frac{1}{2} f_3(t) v_x^2 + f_3'(t) v_{xx} + \frac{1}{3} f_3(t) v_x^3 +$$

$$+ \left( \frac{\sigma^2}{4} y^2 f_3'''(t) - \frac{1}{2} f_3'(t)x \right) v_x^2 - \frac{\sigma^2}{2} f_3(t) v_y^2 + \left( \frac{\sigma^2}{2} f_3''(t)y^2 - f_3'(t)x \right) v_t, \quad (46)$$

$$Y_6 = \sigma^2 f_3(t) v_x v_y + \left( \frac{1}{2} f_3''(t)y^2 - \sigma^2 f_3'(t)x \right) v_y - f_3''(t)yv,$$

$$T_7 = -\frac{\sigma^2}{4} f_4'(t) y v_x^2 + \frac{1}{2} f_4(t) v_y v_x - \frac{\sigma^2}{2} f_4''(t) yv,$$

$$X_7 = \left( \frac{\sigma^2}{2} f_4'(t) y v_x + f_4(t) v_y - \frac{1}{12} f_4'''(t)y^3 + \frac{\sigma^2}{2} f_4''(t) y v_x \right) v_{xxx} +$$

$$+ \frac{\sigma^2}{4} f_4'(t) y v_x^2 - \left( f_4(t) v_{xy} + \frac{\sigma^2}{2} f_4''(t) y v_x \right) v_x -$$

$$- \frac{\sigma^2}{6} f_4'(t) y v_{xx}^3 + \left( \frac{1}{2} f_4(t) v_y - \frac{1}{24} f_4''(t)y^3 + \frac{\sigma^2}{4} f_4''(t) y v_x \right) v_x^2 +$$

$$+ \frac{1}{4} f_4'(t) y v_{xx}^2 + \frac{1}{2} f_4(t) v_t v_y + \left( -\frac{1}{12} f_4'''(t) y^3 + \frac{\sigma^2}{2} f_4''(t)v \right) v_t, \quad (47)$$

$$Y_7 = \frac{1}{2} f_4(t) v_x^2 - \frac{1}{6} f_4(t) v_x^3 - \left( \frac{1}{2} f_4'(t) y v_x + \frac{1}{2} f_4(t) v_t \right) v_x + \frac{\sigma^2}{2} f_4(t) v_y^2 +$$

$$+ \left( -\frac{\sigma^2}{12} f_4'''(t)y^3 + \frac{1}{2} f_4''(t) y v_x \right) v_y + \left( \frac{\sigma^2}{4} f_4'''(t) y^2 - \frac{1}{2} f_4'(t)x \right) v,$$

$$T_8 = \frac{1}{2} f_3(t) v_x^2 - \frac{1}{6} f_3(t) v_x^3 + \frac{1}{12} \left( 2 f_3'(t)x - \sigma^2 f_3''(t)y^2 \right) v_x^2 + \frac{1}{3} f_3(t) y v_y v_x -$$

$$- \frac{\sigma^2}{2} f_3(t) v_x^2 + \frac{1}{6} \left( 2 f_3'''(t)x - \sigma^2 f_3''(t)y^2 \right) v,$$

$$X_8 = \left( \left( -\frac{\sigma^2}{6} f_3'''(t)y^2 + \frac{1}{3} f_3''(t)x \right) v_x + \frac{2}{3} f_3'(t) y v_y + f_3(t) v_t + \frac{1}{3} f_3'(t)v -$$

$$- \frac{1}{72} f_3'''(t)y^4 + \frac{\sigma^2}{6} f_3'''(t) y^2 x - \frac{1}{6} f_3'(t)x^2 \right) v_{xxx} +$$

$$+ \left( \frac{\sigma^2}{12} f_3''(t)y^2 - \frac{1}{6} f_3'(t)x \right) v_{xx}^2 +$$

$$+ \left( -\frac{2}{3} f_3'(t) y v_{xy} - \frac{2}{3} f_3'(t)x - \frac{\sigma^2}{6} f_3'''(t)y^2 + \frac{1}{3} f_3''(t)x - f_3(t) v_{xx} \right) v_{xx} +$$

$$+ \left( -\frac{\sigma^2}{18} f_3''(t)y^2 + \frac{1}{9} f_3'(t)x \right) v_x^3 +$$

$$+ \left( \frac{1}{3} f_3'(t) y v_y + \frac{1}{2} f_3(t) v_t + \frac{1}{6} f_3'(t)v - \right), \quad (48)$$
The conservation laws corresponding to the variational point symmetries admitted by Theorem 4.2.

The following statements hold (14) for arbitrary $p = 1$ if $v = 1$,

\begin{align}
- & \frac{1}{144} f_3'''(t) y^4 + \frac{\sigma^2}{12} f_3''(t) y^2 x - \frac{1}{12} f_3''(t) x^2 \right) v_x^2 - \\
- & \frac{1}{3} f_3'(t) v_x + \left( \frac{1}{12} f_3''(t) y^2 - \frac{\sigma^2}{6} f_3'(t) x \right) v_y^2 + \frac{1}{3} f_3'(t) y v_x v_y + \frac{1}{2} f_3'(t) v_x^2 + \\
+ & \left( \frac{1}{3} f_3'(t) v - \frac{1}{72} f_3'''(t) y^4 + \frac{\sigma^2}{6} f_3''(t) y^2 x - \frac{1}{6} f_3''(t) x^2 \right) v_t.
\end{align}

$Y_8 = \frac{1}{3} f_3'(t) y v_x^3 - \frac{1}{9} f_3'(t) y v_x^3 +$

\[ \left( \left( -\frac{1}{6} f_3'(t) y^2 + \frac{\sigma^2}{3} f_3'(t) x \right) v_y - \frac{1}{3} f_3'(t) y v_t \right) v_x + \frac{\sigma^2}{3} f_3'(t) y v_y + \\
+ \left( \sigma^2 f_3'(t) v_t + \frac{\sigma^2}{3} f_3'(t) v - \frac{1}{72} \sigma^2 f_3'''(t) y^4 + \frac{1}{6} f_3''(t) y^2 x - \frac{\sigma^2}{6} f_3''(t) x^2 \right) v_y + \\
+ \left( \frac{\sigma^2}{18} f_3'''(t) y^3 - \frac{1}{3} f_3'''(t) y x \right) v. \]

**Theorem 4.2.** The following statements hold:

1. The conservation laws corresponding to the variational point symmetries admitted by 2D gB potential equation (14) for arbitrary $p \neq 0$ are given by

\begin{align}
T_1 &= \frac{1}{2} (v_x^2 + v_{xx}^2 + v_z^2 + \sigma^2 v_y^2) - \frac{1}{p+2} v_x^{p+2}, \\
X_1 &= -(v_{x}^{p+1} + v_{xxx} + v_x) v_t \pm v_{x} v_{xx}, \\
Y_1 &= -\sigma^2 v_x v_y, \\
T_2 &= v_x v_t, \\
X_2 &= \frac{p+1}{p+2} v_x^{p+2} \pm \left( \frac{v_x v_{xxx} - \frac{1}{2} v_x^2}{v_x} + \frac{\sigma^2}{2} v_y^2 - \frac{1}{2} v_t^2 - \frac{1}{2} x^2 \right), \\
Y_2 &= -\sigma^2 v_x v_y, \\
T_3 &= v_y v_t, \\
X_3 &= -(v_{x}^{p+1} + v_{xxx} + v_x) v_y \pm v_{xy} v_{xx}, \\
Y_3 &= \frac{1}{p+2} v_x^{p+2} + \frac{1}{2} (v_{x}^2 + v_x^2 + \sigma^2 v_y^2 - v_t^2), \\
T_4 &= \frac{1}{2} y (v_x^2 + v_{xx}^2 + v_x + \sigma^2 v_y^2) + \sigma^2 v_x v_t + \frac{1}{p+2} y v_x^{p+2}, \\
X_4 &= -(y \sigma^2 v_y + y v_t) (\pm v_{xxx} + v_x + v_{x}^{p+1}) \pm (\sigma^2 v_{xy} + y v_t v_x) v_{xx}, \\
Y_4 &= \frac{1}{p+2} t v_x^{p+2} - \sigma^2 \left( \frac{1}{2} (\pm v_x^2 + \sigma^2 v_y^2 + v_t - v_x^2) t + y v_t v_y \right), \\
T_5 &= (f_1(y + \sigma t) + f_2(y - \sigma t)) v_t - \sigma (f_1(y + \sigma t) - f_2(y - \sigma t)) v, \\
X_5 &= -(f_1(y + \sigma t) + f_2(y - \sigma t)) (\pm v_{xxx} + v_x + v_{x}^{p+1}), \\
Y_5 &= \sigma^2 (f_1(y + \sigma t) + f_2(y - \sigma t)) v - (f_1(y + \sigma t) + f_2(y - \sigma t)) v_y. 
\end{align}
2. No additional conservation laws corresponding to variational point symmetries admitted by 2D gB potential equation (14) arise for any \( p \neq 0 \).

4.1. Conserved quantities of the gKP equation. For the gKP potential equation, conservation law (41) arises from time-translation symmetry (24a) and yields the energy quantity

\[
E[v] = \int_\Omega \left( \frac{1}{2} v_{xx}^2 - \frac{\sigma^2}{2} v_y^2 - \frac{1}{p+1}(p+2) v^{p+2}_x \right) dx \, dy. \tag{54}
\]

Conservation laws (42) and (43) arise from space-translation symmetries (24b) and (24c), both of which yield momentum quantities

\[
\mathcal{P}_x[v] = \int_\Omega \frac{1}{2} v_x^2 \, dx \, dy, \tag{55}
\]

\[
\mathcal{P}_y[v] = \int_\Omega v_x v_y \, dx \, dy. \tag{56}
\]

Conservation law (44) arises from rotation-boost symmetry (24d) and yields an analogous momentum quantity

\[
Q[v] = \int_\Omega \left( \frac{1}{2} y v_x^2 - \sigma^2 t v_y v_x \right) dx \, dy. \tag{57}
\]

In addition to these conserved quantities, there are two spatial flux quantities that arise from conservation law (45) and describe conserved topological charges,

\[
\mathcal{F}_1[v] = \int_{\partial \Omega} \left( \frac{1}{p+1} v_x^{p+1} + v_{xxx} + v_t, \sigma^2 v_y \right) \cdot \hat{n} \, ds = 0, \tag{58}
\]

\[
\mathcal{F}_2[v] = \int_{\partial \Omega} \left( y \left( \frac{1}{p+1} v_x^{p+1} + v_{xxx} + v_t \right), \sigma^2 (y v_y - v) \right) \cdot \hat{n} \, ds = 0.
\]

All of these conserved quantities (54)–(58) hold for \( p \neq 0 \). They are homogeneous under scaling symmetry (26d). In particular, they have the respective scaling weights

\[
w_\varepsilon = 1 - \frac{4}{p}, \quad w_{\mathcal{P}_x} = 3 - \frac{4}{p}, \quad w_{\mathcal{P}_y} = 2 - \frac{4}{p}, \quad w_Q = 5 - \frac{4}{p},
\]

and

\[
w_{\mathcal{F}_1} = -\frac{2}{p}, \quad w_{\mathcal{F}_2} = 2 - \frac{2}{p},
\]

as defined by \( C[v] \rightarrow e^{w r} C[v] \).

Finally, the three infinite families of conservation laws (46)–(48), which hold only for \( p = 1 \), yield two conserved dilatational momentum quantities,

\[
\tilde{\mathcal{P}}_x[v] = \int_\Omega \left( \frac{1}{2} f_5(t) v_x^2 + f'_5(t) v \right) dx \, dy,
\]

\[
\tilde{\mathcal{P}}_y[v] = \int_\Omega \left( \frac{1}{2} f_4(t) v_y v_x - \frac{\sigma^2}{4} f'_4(t) y v_y^2 - \frac{\sigma^2}{2} f''_4(t) y v \right) dx \, dy,
\]

and a conserved dilatational energy quantity,

\[
\tilde{E}[v] = \int_\Omega \left( \frac{1}{2} f_3(t) v_x^2 - \sigma^2 v_y^2 - \frac{1}{3} v_x^3 \right) + \frac{1}{3} f'_3(t) \left( \frac{1}{2} v_x v_y + y v_y v_x \right) + \frac{1}{3} f''_3(t) \left( x v - \frac{1}{4} y^2 v_x^2 \right) + \frac{\sigma^2}{6} f'''_3(t) y^2 v \right) dx \, dy. \tag{60}
\]
All of them are not scaling homogeneous unless the functions $f_3$, $f_4$, and $f_5$ are chosen to be monomials in $t$.

Conserved energies (54) and (60), conserved momenta (55)–(57), (59), and topological charges (58) each exhibit an essential dependence on the potential $v$. Consequently, the corresponding conservation laws are nonlocal in terms of $u$.

The only conservation law that is local in terms of $u$ is the $x$ momentum, whose corresponding conserved quantity (55) describes the mass of $u$:

$$\mathcal{M}[u] = \int_{\Omega} u^2 \, dx \, dy = 2P^x[v],$$  \hspace{1cm} (61)

This quantity is scaling invariant iff $p = 4/3$. In comparison, the energy is scaling invariant iff $p = 4$.

### 4.2. Conserved quantities of the 2D gB equation.

Similarly, for the gB potential equation, conservation law (49) yields the energy quantity

$$\mathcal{E}[v] = \int_{\Omega} \left( \frac{1}{2} v^2_t + v^2_x + v^2_y + \sigma^2 v^2_y + \frac{1}{p + 2} y v^{p+2} \right) dx \, dy,$$  \hspace{1cm} (62)

which arises from time-translation symmetry (27a). Conservation laws (50) and (51) yield the momentum quantities

$$P^x[v] = \int_{\Omega} v_x v_t \, dx \, dy,$$  \hspace{1cm} (63)

$$P^y[v] = \int_{\Omega} v_y v_t \, dx \, dy,$$  \hspace{1cm} (64)

which arise from space-translation symmetries (27b) and (27c). Conservation law (52) yields the boost-momentum quantity

$$Q[v] = \int_{\Omega} \left( \frac{1}{2} y(v^2_t + v^2_x + v^2_y + \sigma^2 v^2_y) + \sigma^2 t v_y v_t + \frac{1}{p + 2} y v^{p+2} \right) dx \, dy,$$  \hspace{1cm} (65)

arising from boost symmetry (27d). The infinite family of conservation laws (53) yield a corresponding infinite family of quantities

$$T[v] = \int_{\Omega} \left( (f_1(y + \sigma t) + f_2(y - \sigma t)) v_t - \sigma(f'_1(y + \sigma t) - f'_2(y - \sigma t)) v_t \right) dx \, dy,$$  \hspace{1cm} (66)

which are counterparts of the transverse momentum quantities known for the linear wave equation $v_{tt} - \sigma^2 v_{yy} = 0$. In particular, if we set $f_1 = f_2 = 1/2$, then the conserved quantity

$$T[v] = \int_{\Omega} v_t \, dx \, dy = \dot{A}[v]$$

is the time derivative of the net amplitude

$$A[v] = \int_{\Omega} v \, dx \, dy.$$  

This implies $\dot{A}[v] = 0$ and hence $A[v] = t \dot{A}[v] \big|_{t=0} + A[v] \big|_{t=0}$. Therefore, the net amplitude has the same motion as a free particle. If we next set $f_1 = (y + \sigma t)/2$ and $f_2 = (y - \sigma t)/2$, then we obtain the conserved quantity

$$T[v] = \int_{\Omega} y v_t \, dx \, dy = \dot{A}_1[v],$$
which is the time derivative of the first $y$ moment of the net amplitude,

$$A_1[v] = \int_\Omega y v \, dx \, dy.$$  

This implies $\dot{A}_1[v] = 0$ and hence $A_1[v] = t\dot{A}_1[v]|_{t=0} + A_1[v]|_{t=0}$. If we set $f_1 = (y + \sigma t)^2/2$ and $f_2 = (y - \sigma t)^2/2$, then the resulting conserved quantity is

$$T[v] = \int_\Omega ((y^2 + \sigma^2 t^2) v_t - 2\sigma^2 t v) \, dx \, dy = \dot{A}_2[v] + \sigma^2 t^2 A[v] - 2\sigma^2 t A[v],$$

where $A_2[v] = \int_\Omega y^2 v \, dx \, dy$ is the second $y$ moment of the net amplitude. Hence, we obtain $\dot{A}_2[v] = 2\sigma^2 A[v]$, which yields

$$A_2[v] = \frac{1}{3} \sigma^2 t^3 \dot{A}[v]|_{t=0} + \sigma^2 t^2 A[v]|_{t=0} + t \dot{A}_2[v]|_{t=0} + A_2[v]|_{t=0}.$$  

Similar expressions arise for the higher $y$ moments $A_n[v] = \int_\Omega y^n v \, dx \, dy$, $n = 3, 4, \ldots$.

Conserved quantities (62)–(66) hold for $p \neq 0$. They have no scaling properties unless $p = 1$, because this is the only nonlinearity power for which scaling symmetry (28) exists. This scaling symmetry contains a shift on $v$ and generates transformation group (30). The derivatives of $v$ can be shown to transform in a scaling homogeneous manner except for $v_x \to e^{-2\epsilon} v_x - 1/2$, which contains a shift.

To determine how the conserved quantities transform, we can apply a general result [23] that relates the symmetry action on conservation laws to the commutators of the scaling symmetry and the variational symmetries that generate the conservation laws. We find that energy (62), $y$ momentum (64), and boost momentum (65) are each scaling homogeneous modulo the addition of a locally trivial conserved quantity. In particular, they have the respective scaling weights

$$w_{\mathcal{E}} = w_{\mathcal{P}_y} = w_{\mathcal{P}_0} = -3, \quad w_{\mathcal{Q}} = -1,$$

as defined by $C[v] \to e^{w v} C[v]$. In contrast, we find that $x$ momentum (63) is only scaling homogeneous modulo the addition of the conserved transverse momentum quantity $\dot{A}[v]$ and also a locally trivial conserved quantity. Moreover, the infinite family of quantities (66) is not scaling homogeneous unless the functions $f_1$ and $f_2$ are chosen to be monomials in $y \pm \sigma t$.

Finally, all conserved quantities (62)–(66) depend essentially on the potential $v$, and the corresponding conservation laws are hence nonlocal in terms of $u$.

5. Line soliton solutions

A line soliton is a solitary traveling wave

$$u = U(x + \mu y - \nu t)$$

in two dimensions, where the parameters $\mu$ and $\nu$ determine the direction and the speed of the wave. A more geometric form for a line soliton is given by writing $x + \mu y = (x, y) \cdot \hat{k}$, where $\hat{k} = (1, \mu)$ is a constant vector in the plane $(x, y)$. The traveling-wave variable can then be expressed as

$$\xi = x + \mu y - \nu t = |\hat{k}|((\hat{k} \cdot (x, y)) - ct),$$

where the unit vector $\hat{k} = (\cos \theta, \sin \theta)$ with $\tan \theta = \mu$ and the constant $c = \nu/|\hat{k}|$, with $|\hat{k}|^2 = 1 + \mu^2$ give the respective propagation direction and speed of the line soliton. Here, the direction satisfies $-\pi/2 \leq \theta \leq \pi/2$. 

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and the speed can be \( c > 0 \) or \( c < 0 \). We note that line soliton (68) is invariant under the two-parameter group of translations \( t \rightarrow t + \epsilon_1, \ x \rightarrow x + \epsilon_1 \nu - \epsilon_2 \mu, \ y \rightarrow y + \epsilon_2 \) with \( \epsilon_1, \epsilon_2 \in \mathbb{R} \).

We now derive explicit line soliton solutions (68) for gKP equation (3) and 2D gB equation (4). It is convenient to use the coordinate form of the traveling-wave variable \( \xi = x + \mu y - \nu t \) for this derivation.

Substituting line soliton expression (68) in both the gKP and the 2D gB equations in potential form yields a nonlinear third-order ODE. To reduce these respective ODEs to separable ODEs that can be integrated, we need two functionally independent first integrals. The necessary first integrals can be obtained from the conservation laws in Theorems 4.1 and 4.2 by the following steps.

We assume that a conservation law \( D_t T + D_x X + D_y Y = 0 \) does not contain the variables \( t, x, \) and \( y \) explicitly. The conservation law then yields a first integral of the line soliton ODE by the reductions

\[
D_t|_{u=U(\xi)} = -\nu \frac{d}{d\xi}, \quad D_x|_{u=U(\xi)} = \frac{d}{d\xi}, \quad D_y|_{u=U(\xi)} = \mu \frac{d}{d\xi}.
\]

This gives

\[
\frac{d}{d\xi} ((X + \mu Y - \nu T)|_{u=U(\xi)}) = 0. \tag{71}
\]

The first integral is therefore

\[ X + \mu Y - \nu T = C = \text{const}, \tag{72}\]

which has the physical meaning of the spatial flux of the conserved quantity \( \int_{\Omega} T \, dx \, dy \) in a reference frame moving with speed \( c \) in the direction \( k \) in the plane \( (x, y) \) (namely, the rest frame of the line soliton).

5.1. The gKP line solitons. We begin with gKP potential equation (9). Using relation (8) for \( u \) in terms of the potential \( v \) and substituting line soliton expression (68), we obtain the nonlinear third-order ODE

\[ (\sigma^2 \mu^2 - \nu)U'' + U^p U' + U''' = 0 \tag{73}\]

for \( U(\xi) \). From Theorem 4.1, we see that the gKP conservation laws that do not explicitly contain \( t, x, \) and \( y \) are energy (41), \( x \) and \( y \) momenta (42) and (43), and spatial flux (45) with \( f_2 = 1 \) and \( f_1 = 0 \). Applying first integral formula (72) to these four conservation laws, we find that the energy and \( y \) momentum up to a constant proportionality factor yield the same first integral as the \( x \) momentum,

\[ C_1 = UU'' + \frac{1}{2}(\sigma^2 \mu^2 - \nu)U^2 - \frac{1}{2}U'^2 + \frac{1}{p + 2}U^{p+2}, \tag{74}\]

and the spatial flux yields an independent first integral,

\[ C_2 = U'' + (\sigma^2 \mu^2 - \nu)U + \frac{1}{p + 1}U^{p+1}. \tag{75}\]

Because we are interested in a solitary wave solution, we impose the asymptotic conditions \( U, U', U'' \rightarrow 0 \) as \( |\xi| \rightarrow \infty \). This requires \( C_1 = C_2 = 0 \). Combining the two first integrals (74) and (75), we then obtain the first-order separable ODE

\[ U'^2 = (\nu - \mu^2 \sigma^2)U^2 - \frac{2}{(p + 2)(p + 1)}U^{p+2}, \tag{76}\]

which can be straightforwardly integrated. Its general solution up to a shift in \( \xi \) is given by

\[ U(\xi) = (2(p + 2)(p + 1)\kappa)^{1/p} \text{sech} \left( \frac{1}{2}p(p + 1)(p + 2)\sqrt{\kappa} \xi \right)^{2/p}, \quad \kappa = \nu - \mu^2 \sigma^2, \tag{77}\]
which is smooth and satisfies the desired decay conditions if \( \kappa > 0 \) and \( p > 0 \). This is the general line soliton solution of gKP equation (3) for all \( p > 0 \). It coincides with the well-known line soliton solution [4] for the ordinary KP equation (1) if \( p = 1 \).

We next discuss a few properties of solution (77).

First, the parameters \( \mu = \tan \theta \) and \( \nu = c\sqrt{1 + \mu^2} \) give the angle \( \theta \) of the direction of motion of the line soliton with respect to the \( x \) axis and the speed \( c \) of the line soliton along its direction of motion. These two parameters must satisfy the kinematic condition \( \nu > \text{sgn}(\sigma^2)\mu \), where we recall that \( \sigma^2 = \pm 1 \). Because \( c \) must be finite, we conclude that the line soliton can propagate in any direction except \( \theta = \pi/2 \) (namely, parallel to the \( y \) axis). In the case \( \sigma^2 = -1 \), the kinematic condition is satisfied for all positive speeds \( c > 0 \) and also for negative speeds \( 0 < c < -\mu^2/\sqrt{1 + \mu^2} \). In contrast, in the case \( \sigma^2 = 1 \), the kinematic condition requires that the speed be greater than \( c > \mu^2/\sqrt{1 + \mu^2} \) and hence must be positive. These kinematic properties are independent of the nonlinearity power.

Last, the physical conserved quantities for the line soliton are given by the \( x \) and \( y \) momenta densities
\[
\left( \frac{q}{2} \right)^{1+2/p} \text{sech}\left( \frac{pq}{2} \sqrt{\kappa \xi} \right)^{4/p} \text{sech}\left( \frac{pq}{2} \sqrt{\kappa \xi} \right)^{4/p}, \quad \mu \left( \frac{q}{2} \right)^{1+2/p} \text{sech}\left( \frac{pq}{2} \sqrt{\kappa \xi} \right)^{4/p}
\]
and the energy density
\[
\left( \frac{q}{2} \right)^{1+2/p} \text{sech}\left( \frac{pq}{2} \sqrt{\kappa \xi} \right)^{4/p} \left( \left( \frac{q}{2} \right)^{1+2/p} \text{sech}\left( \frac{pq}{2} \sqrt{\kappa \xi} \right)^{4/p} \right),
\]
where \( q = (p+1)(p+2) \). These densities are obtained from conservation law expressions (41)–(43) evaluated for line soliton solution (77).

5.2. The 2D gB line solitons. We apply the preceding derivation to 2D gB potential equation (14).

The nonlinear third-order ODE for line soliton (68) is given by
\[
(1 + \sigma^2 \mu^2 - \nu^2)U'' + (p + 1)U^p U' \pm U''' = 0. \quad (78)
\]
From Theorem 4.2, we see that the 2D gB conservation laws that do not explicitly contain \( t \), \( x \), and \( y \) are energy (49), \( x \) and \( y \) momenta (50), (51), and transverse momentum (53) with \( f_1 = f_2 = 1 \). First integral formula (72) then yields
\[
C_1 = \pm U'' + (1 + \mu^2 \sigma^2 - \nu^2)U + U^{p+1}, \quad (79)
\]
which arises from the transverse momentum, and also
\[
C_2 = \pm (UU'' - \frac{1}{2}U'^2) + \frac{1}{2}(1 + \mu^2 \sigma^2 - \nu^2)U^2 + \frac{p+1}{p+2}U^{p+2}, \quad (80)
\]
which arises from the energy and also from both the \( x \) and the \( y \) momenta. These two first integrals are functionally independent. Because we are interested in a solitary wave solution, we impose the asymptotic conditions \( U, U', U'' \rightarrow 0 \) as \( |\xi| \rightarrow \infty \), which require \( C_1 = C_2 = 0 \). The two first integrals (79) and (80) can then be combined to obtain the first-order separable ODE
\[
U'^2 = \pm \left( (\nu^2 - \mu^2 \sigma^2 - 1)U^2 - \frac{2}{p+2}U^{p+2} \right), \quad (81)
\]
which is straightforward to integrate. Its general solution up to a shift in \( \xi \) is given by
\[
U(\xi) = \left( \frac{(p+2)}{2} \right)^{1/p} \text{sech}\left( \frac{p(p+2)}{2} \sqrt{\pm \kappa} \xi \right)^{2/p}, \quad \kappa = \nu^2 - \mu^2 \sigma^2 - 1. \quad (82)
\]
This solution is smooth and satisfies the desired decay conditions if $\kappa > 0$ and $\rho > 0$ or $\kappa < 0$ and $1/\rho$ is a rational number with an odd denominator. Hence, for $\kappa > 0$, solution (82) of the 2D gB equation with the plus sign in the radicand describes a line soliton for all $\rho > 0$, and for $\kappa < 0$, solution (82) with the minus sign in the radicand describes a line soliton if $1/\rho$ is a rational number with an odd denominator.

We next discuss a few properties of solution (82). We recall that $\mu$ and $\nu$ give the angle $\theta = \arctan \mu$ of the direction of motion of the line soliton with respect to the $x$ axis and the speed $c = \nu / \sqrt{1 + \mu^2}$ of the line soliton. Because $c$ must be finite, we note that the direction of motion cannot be parallel to the $y$ axis.

First, in the case with the plus sign in the radicand in (82), the parameters must satisfy the kinematic condition $\nu^2 > \text{sgn}(\sigma^2) \mu^2 + 1$. This implies that $|c| > 1$ if $\sigma^2 = 1$ and $|c| > \sqrt{(1 - \mu^2)/(1 + \mu^2)}$ if $\sigma^2 = -1$. Hence, the direction of motion is within an angle of $\pi/4$ with respect to the $x$ axis if $\sigma^2 = -1$, and there is a minimum speed $|c| > \sqrt{(1 - \mu^2)/(1 + \mu^2)} > 0$.

Second, in the case with the minus sign, the soliton parameters $\mu$ and $\nu$ must satisfy the opposite kinematic condition $\nu^2 < \text{sgn}(\sigma^2) \mu^2 + 1$. Consequently, in the case $\sigma^2 = 1$, there is a maximum speed $|c| < 1$. In the case $\sigma^2 = -1$, the direction of motion is confined within an angle of $\pi/4$ with respect to the $x$ axis, and the speed satisfies $|c| < \sqrt{(1 - \mu^2)/(1 + \mu^2)}$. These kinematic properties are independent of the nonlinearity power.

Last, the physical conserved quantities for the line soliton are given by the $x$ and $y$ momenta densities,

\[-\nu \left( \frac{q}{2\mu} \right)^{1/\rho} \text{sech} \left( \frac{pq}{2} \sqrt{\pm \kappa \xi} \right)^{4/\rho}, \quad -\mu \nu \left( \frac{q}{2\kappa} \right)^{1+2/\rho} \text{sech} \left( \frac{pq}{2} \sqrt{\pm \kappa \xi} \right)^{4/\rho},\]

and the energy density

\[-\left( \frac{q}{2\kappa} \right)^{1+2/\rho} \text{sech} \left( \frac{pq}{2} \sqrt{\pm \kappa \xi} \right)^{4/\rho} \left( \left( q + \frac{1}{q} \right) \tanh \left( \frac{pq}{2} \sqrt{\pm \kappa \xi} \right)^2 - \frac{2q^2}{\kappa} \right),\]

where $q = p + 2$. These densities are obtained from conservation law expressions (49)–(51) evaluated for line soliton solution (82).

6. Concluding remarks

For gKP equation (3) and 2D gB equation (4) with $p$-power nonlinearities, we have derived all conservation laws that arise from variational point symmetries and used them to obtain the general line soliton solutions of these two equations for all powers $p > 0$. We also discussed some basic kinematic properties of the line solitons.

Our results can be further used as a starting point to investigate the stability of the line soliton solutions for the gKP equation. It is known [2], [24] that line solitons of the KP equation ($p = 1$) are stable in the case $\sigma^2 = 1$ and unstable in the opposite case $\sigma^2 = -1$. It would be interesting to determine the stability of the line solitons for higher powers $p \geq 2$. Compared with the KdV equation, the critical power for the conserved mass in two dimensions is $p = 4/3$, while the critical power for the conserved energy is $p = 4$.

In addition to line soliton solutions, the KP equation has lump solutions, which are rational functions and involve several parameters. No lump solutions are known for the modified KP equation, and an interesting question is to see if a similar result holds for the gKP equation for all $p > 1$. The stability of line solitons and the existence of lump solutions can be similarly investigated for the 2D gB equation.

As another direction of investigation, we mention a recent study [25] of solutions of a combined KP and modified KP equation

\[ (u_t + 6\alpha u u_x + 6\beta u^2 u_x + u_{xxx})_x + u_{yy} = 0 \]
and also a combined 2D Boussinesq and modified 2D Boussinesq equation

\[ u_{tt} = u_{xx} + (6\alpha uu_x + 6\beta u^2u_x + u_{xxx})x + u_{yy}. \]

A natural problem is to seek line solitons, lump solutions, and other types of solitary waves and also conservation laws of the fully generalized KP and 2D Boussinesq equations

\[ (u_t + f(u)u_x + u_{xxx})x + \sigma^2 u_{yy} = 0, \quad u_{tt} = u_{xx} + (f(u)u_x + u_{xxx})x + \sigma^2 u_{yy}. \]

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