Dark matter in the $SO(5) \times U(1)$ gauge-Higgs unification

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In the $SO(5) \times U(1)$ gauge-Higgs unification, the lightest neutral component of $n_F$ $SO(5)$-spinor fermions (dark fermions), which are relevant as they have the observed unstable Higgs boson, becomes the dark matter of the universe. We show that the relic abundance of the dark matter determined by WMAP and Planck data is reproduced, below the bound placed by the direct-detection experiment by LUX, by a model with one light and three heavier ($n_F = 4$) dark fermions with the lightest one with a mass from 2.3 TeV to 3.1 TeV. The corresponding Aharonov–Bohm phase $\theta_H$ in the fifth dimension ranges from 0.097 to 0.074. The case of $n_F = 3$ ($n_F = 5, 6$) dark fermions yields a relic abundance smaller (larger) than the observed limit.

Subject Index B40, B43, B71, B73

1. Introduction

The Higgs boson with a mass around 125.5 GeV was discovered at LHC [1,2]. It is not clear, however, whether or not the particle discovered is precisely the Higgs boson specified in the standard model (SM). Physics beyond the standard model may be hiding, showing up at the upgraded LHC. Couplings of the Higgs boson to other particles may slightly deviate from those in the SM, and new particles may be produced, say, in the 4–7 TeV range. The SM lacks a principle governing the dynamics of the Higgs boson. Further, the SM provides no clue to explaining the dark matter (DM) in the universe.

In the gauge-Higgs unification (GHU), the Higgs boson is unified with gauge bosons. The 4D Higgs boson appears as a part of the extra-dimensional component of gauge fields so that its dynamics is governed by the gauge principle [3–10]. It has been shown that, in the $SO(5) \times U(1)$ GHU in the Randall–Sundrum warped space, the low-energy physics appears almost the same as that in the SM, consistent with all LHC data [11–18]. Contributions of Kaluza–Klein (KK) excited modes to the $H \rightarrow \gamma \gamma$ decay, e.g., turn out very small [17]. Higgs couplings to gauge bosons, quarks, and leptons at the tree level are suppressed by a common factor $\cos \theta_H$ where $\theta_H$ is the Aharonov–Bohm (AB) phase in the extra dimension [19–24]. All of the precision measurements, the tree-unitary constraint, and the $Z'$ search indicate that $\theta_H < 0.2$ [11,25]. The $SO(5) \times U(1)$ GHU predicts new structure at higher energies. The masses of the first KK modes of $Z$ and $\gamma$ are predicted to be 3–7 TeV for $\theta_H = 0.1–0.2$. The Higgs cubic and quartic self-coupling should be smaller than those in the SM by 10%–20% [18]. Many other signals of GHU have been investigated [26–38].
Another important issue is the dark matter [39]. Supersymmetric theory, the leading model of physics beyond the SM, predicts the lightest supersymmetric particle as a dark matter candidate [40,41]. The lightest KK particle in universal extra-dimension (UED) models [42–47], the lightest T-odd particle in the little Higgs models [48–50], a fermionic composite state in the composite Higgs models [51–53], and axions [54–58] can be identified as dark matter. In the Higgs portal scenario the Higgs boson couples to dark matter in the hidden sector [59–63], and the dynamical dark matter scenario has been proposed [64,65]. Is there a dark matter candidate in the SO(5) × U(1) gauge-Higgs unification model? Can it explain the relic abundance reduced from the WMAP/Planck data and other observations, within the constraints from direct-detection searches? A few scenarios for dark matter in GHU have been proposed [66–69]. In this paper we would like to show that the realistic SO(5) × U(1) gauge-Higgs unification model contains a natural candidate for dark matter.

In the minimal SO(5) × U(1) gauge-Higgs unification model, in which only quark–lepton vector multiplets and associated brane fermions are introduced in the fermion sector, the effective potential is minimized at $\theta_H = \frac{1}{2} \pi$, which in turn implies that the Higgs boson becomes stable, contradicting the observation [13,15,68]. To have an unstable Higgs boson, it is necessary to introduce fermion multiplets in the spinor representation of SO(5) that do not appear at low energies [17]. Indeed, the presence of these fermions, with the gauge fields and top quark multiplet, naturally leads to $0 < \theta_H < \frac{1}{2} \pi$, yielding predictions consistent with the observation. One remarkable property is that, independent of the details of these SO(5)-spinor fermions, there appear universality relations among $\theta_H$, the masses of KK $Z$/photon, and the Higgs self couplings.

We show that the lightest neutral component of the SO(5)-spinor fermions is absolutely stable, and becomes the dark matter of the universe. For this reason the SO(5)-spinor fermion is called a dark fermion in the present paper. It is heavy, with a mass around 2–4 TeV, but its couplings to the Higgs boson are small. From its relic abundance, the number and structure of the dark fermion multiplets are inferred. It is curious that the Higgs dynamics is intimately related to the dark matter in the gauge-Higgs unification.

The paper is organized as follows. In Sect. 2 the SO(5) × U(1) model is introduced. In Sect. 3 it is shown that the neutral components of dark fermions become the dark matter, and the relic abundance is evaluated. In Sect. 4 the spin-independent cross section of the dark matter candidate with nucleons is evaluated, and the compatibility with the constraint coming from the direct-detection experiments, XENON100 and LUX [70,71], is examined. It will be found that the model with $n_F = 4$ non-degenerate dark fermions, with the lightest one of a mass 2.3–3.1 TeV, explains the relic abundance of the dark matter determined from the WMAP/Planck data below the bound placed by the direct-detection observation of LUX. Section 5 is devoted to the conclusion and discussions. In the appendixes, the wave functions and couplings of dark fermions and relevant gauge bosons are summarized.

### 2. Model

The model of the SO(5) × U(1) GHU is defined in the Randall–Sundrum (RS) warped space with a metric

$$ds^2 = G_{MN} dx^M dx^N = e^{-2\sigma(y)} \eta_{\mu\nu} dx^\mu dx^\nu + dy^2,$$

where $\eta_{\mu\nu} = \text{diag}(-1, 1, 1, 1)$, $\sigma(y) = \sigma(y + 2L) = \sigma(-y)$, and $\sigma(y) = k|y|$ for $|y| \leq L$. The Planck and TeV branes are located at $y = 0$ and $y = L$, respectively. The bulk region $0 < y < L$ is
The quark–lepton multiplets $A_M$, $B_M$, quark–lepton multiplets $\Psi_a$, SO(5)–spinor fermions (dark fermions) $\Psi_F$, brane fermions $\hat{x}_{aR}$, and a brane scalar $\hat{\Phi}$ [14,17]. The bulk part of the action is given by

$$S_{\text{bulk}} = \int d^5x \sqrt{-G} \left[ -\text{tr} \left( \frac{1}{4} F^{(A)MN} F^{(A)}_{MN} + \frac{1}{2\xi} \left( f_{gf}^{(A)} \right)^2 + \mathcal{L}_{\text{gh}}^{(A)} \right) ight. \\
- \left( \frac{1}{4} F^{(B)MN} F^{(B)}_{MN} + \frac{1}{2\xi} \left( f_{gf}^{(B)} \right)^2 + \mathcal{L}_{\text{gh}}^{(B)} \right) \\
+ \sum_a \bar{\Psi}_a \mathcal{D}(c_a) \Psi_a + \sum_{i=1}^{n_F} \bar{\Psi}_F_i \mathcal{D}(c_F_i) \Psi_F_i \right].$$

$$\mathcal{D}(c) = \Gamma^A e_A^M \left( \partial_M + \frac{1}{8} \omega_{MBC} \left[ \Gamma^B, \Gamma^C \right] - ig_A A_M - ig_B Q_X B_M \right) - c\sigma'(y). \tag{2.2}$$

The gauge fixing and ghost terms are denoted as functionals with subscripts $gf$ and $gh$, respectively. $F_{MN}^{(A)} = \partial_M A_N - \partial_N A_M - ig_A [A_M, A_N]$, and $F_{MN}^{(B)} = \partial_M B_N - \partial_N B_M$. The color SU(3)$_C$ gluon fields and their interactions have been suppressed in the present paper. The SO(5) gauge fields $A_M$ are decomposed as

$$A_M = \sum_{aL=1}^3 A_M^{aL} T^{aL} + \sum_{aR=1}^3 A_M^{aR} T^{aR} + \sum_{\hat{a}=1}^4 A_M^{\hat{a}} \hat{T}^{\hat{a}}, \tag{2.3}$$

where $T^{al,aR}(a_L, a_R = 1, 2, 3)$ and $\hat{T}(\hat{a} = 1, 2, 3, 4)$ are the generators of SO(4) $\simeq$ SU(2)$_L \times$ SU(2)$_R$ and SO(5)/SO(4), respectively.

In the fermion part $\bar{\Psi} = i \Psi \gamma^0$ and $\Gamma^M$ matrices are given by

$$\Gamma^\mu = \gamma^\mu = \begin{pmatrix} \bar{\sigma}^\mu & \sigma^\mu \end{pmatrix}, \quad \Gamma^5 = \gamma^5 = \begin{pmatrix} 1 \\ -1 \end{pmatrix},$$

$$\sigma^\mu = (1, \bar{\sigma}), \quad \bar{\sigma}^\mu = (-1, \bar{\sigma}). \tag{2.4}$$

The quark–lepton multiplets $\Psi_a$ are introduced in the vector representation of SO(5). In contrast, $n_F$ dark fermions $\Psi_F$ are introduced in the spinor representation. The $c$ term in Eq. (2.2) gives a bulk kink mass, where $\sigma'(y) = k \epsilon(y)$ is a periodic step function with a magnitude $k$. The dimensionless parameter $c$ plays an important role in controlling the profiles of fermion wave functions.

The orbifold boundary conditions at $y_0 = 0$ and $y_1 = L$ are given by

$$\begin{pmatrix} A_\mu \\ A_y \end{pmatrix}(x, y_j - y) = P_{\text{vec}} \begin{pmatrix} A_\mu \\ -A_y \end{pmatrix}(x, y_j + y) P_{\text{vec}}^{-1},$$

$$\begin{pmatrix} B_\mu \\ B_y \end{pmatrix}(x, y_j - y) = \begin{pmatrix} B_\mu \\ -B_y \end{pmatrix}(x, y_j + y),$$

$$\Psi_a(x, y_j - y) = P_{\text{vec}} \Gamma^5 \Psi_a(x, y_j + y),$$

$$\Psi_F(x, y_j - y) = \eta_F (-1)^j P_{sp} \Gamma^5 \Psi_F(x, y_j + y), \quad \eta_F = \pm 1,$$

$$P_{\text{vec}} = \text{diag} (-1, -1, -1, -1, +1), \quad P_{sp} = \text{diag} (+1, +1, +1, -1). \tag{2.5}$$
The SO(5) $\times$ U(1)$_X$ symmetry is reduced to SO(4) $\times$ U(1)$_X$ $\simeq$ SU(2)$_L$ $\times$ SU(2)$_R$ $\times$ U(1)$_X$ by the orbifold boundary conditions. Various orbifold boundary conditions fall into a finite number of equivalence classes of boundary conditions [72,73]. The physical symmetry of the true vacuum in each equivalence class of boundary conditions is dynamically determined at the quantum level by the Hosotani mechanism. Recently, dynamics for selecting boundary conditions has been proposed as well [74]. The Hosotani mechanism has been explored and established, not only in perturbation theory, but also on the lattice nonperturbatively [75].

The brane action $S_{\text{brane}}$ contains brane fermions $\bar{\psi}_{aR}(x)$, a brane scalar $\hat{\Phi}(x)$, $A_{\mu}(x, y = 0)$, and $\Psi_d(x, y = 0)$. It manifestly preserves gauge invariance in SO(4) $\times$ U(1)$_X$. $\Phi$ develops a non-vanishing expectation value $\langle \Phi \rangle \gg m_{\text{KK}}$, which results in spontaneous breaking of SO(4) $\times$ U(1)$_X$ into SU(2)$_L$ $\times$ U(1)$_Y$ and makes all exotic fermions heavy.

The 4D Higgs field, which is a bidoublet in SU(2)$_L$ $\times$ SU(2)$_R$, appears as a zero mode in the SO(5)/SO(4) part of the fifth-dimensional component of the vector potential $A_i^a(x, y)$ with custodial symmetry [11,76,77]. Without loss of generality one can set $\langle A_i^a \rangle \propto \delta^{a4}$ when the electroweak symmetry is spontaneously broken. The zero modes of $A_i^a (a = 1, 2, 3)$ are absorbed by $W$ and $Z$ bosons. The 4D neutral Higgs field $H(x)$ is a fluctuation mode of the Wilson line phase $\theta_H$, which is an Aharonov–Bohm phase in the fifth dimension:

$$A_i^4(x, y) = \{ \theta_H f_H + H(x) \} u_H(y) + \cdots,$$

$$\exp \left\{ -i \frac{\theta_H}{2} \cdot 2\sqrt{2} \hat{T}^4 \right\} = \exp \left\{ i g_A \int_0^L dy \langle A_4 \rangle \right\}.$$

$$f_H = \frac{2}{g_A} \sqrt{\frac{k}{z_L^2 - 1}} = \frac{2}{g_w} \sqrt{\frac{k}{L(z_L^2 - 1)}}. \quad (2.6)$$

Here the wave function of the 4D Higgs boson is given by $u_H(y) = [2k/(z_L^2 - 1)]^{1/2} e^{2ky}$ for $0 \leq y \leq L$ and $u_H(-y) = u_H(y) = u_H(y + 2L)$. $g_w = g_A/\sqrt{L}$ is the dimensionless 4D SU(2)$_L$ coupling.

For each generation, two vector multiplets $\Psi_1$ and $\Psi_2$ for quarks and two vector multiplets $\Psi_3$ and $\Psi_4$ for leptons are introduced. In contrast, the dark fermion $\Psi_{F_i}$ belongs to the spinor representation of SO(5), having four components:

$$\Psi_{F_i} = \left( \begin{array}{c} \psi_{i1}^j \\ \psi_{i2}^j \\ \psi_{r1}^i \\ \psi_{r2}^i \end{array} \right). \quad (2.7)$$

$\psi_j^i$ and $\psi_j^i$ are SU(2)$_L$ and SU(2)$_R$ doublets, respectively. They mix with each other for $\theta_H \neq 0$. The electric charge is given by $Q_{\text{EM}} = T^3_L + T^3_R + Q_X$. We take $Q_X = \frac{1}{2}$ for $\Psi_{F_i}$ so that it contains charge 1 and 0 components.

The KK decomposition of the $\Psi_{F_i}$ fields is summarized in Appendix B. With the boundary condition (2.5) $\Psi_{F_i}(x, z)$ does not have zero modes, and is expanded in the KK modes $F^{(n)}_{i1}(x)$ and $F^{(n)}_{i2}(x) (n = 1, 2, 3, \ldots)$ as in (B1). The mass spectrum is determined by (B7). With $\eta_{F_i} = +1$ in the boundary condition for $\Psi_{F_i}$ in (2.5) and for small $\theta_H$ the odd KK number modes $F^{(n)}_{i1}$, $F^{(n)}_{i2}$ ($n$: odd) are mostly SU(2)$_R$ doublets, containing some SU(2)$_L$ components. The even KK number modes $F^{(n)}_{i1}$, $F^{(n)}_{i2}$ ($n$: even) are mostly SU(2)$_L$ doublets. Consequently, the first KK modes $F^{(1)}_{i1}$, $F^{(1)}_{i2}$ couple to the SU(2)$_L$ gauge bosons ($W$ and $Z$) very weakly. On the other hand, with
The relic density of the dark fermion. We mostly follow the arguments in Refs. [39,42,78]. The Boltzmann equation for $F^0_i$ is given by

$$
\frac{dn_{(F^0_i)}}{dt} = -3Hn_{(F^0_i)} + \sum_{X,X'} \left[ \sigma(F^0_i \rightarrow XX')v \left(n_{(F^0_i)n_{(F^0_i)}} - n_{(\overline{F^0_i})n_{(\overline{F^0_i})}} \right) \right] \\
- \sum_{X,X'} \left[ \sigma(F^{-}_i \rightarrow XX')v \left(n_{(F^{-}_i)n_{(F^{-}_i)}} - n_{(\overline{F^{-}_i})n_{(\overline{F^{-}_i})}} \right) \right] \\
- \sum_j \left[ \sigma(F^0_j \rightarrow F^+_jF^-_j)v \left(n_{(F^0_j)n_{(F^0_j)}} - n_{(\overline{F^0_j})n_{(\overline{F^0_j})}} \right) \right] \\
- \sum_{X,X'} \left[ \sigma(F^0_iX \rightarrow F^+_iX')v_{(F^0_i)n_{(X)}} - \sigma(v(F^+_iX' \rightarrow F^0_iX)n_{(F^0_i)n_{(X)}}) \right]. \tag{3.1}
$$

Similar relations are obtained for $\overline{F}^0_i$ and $F^\pm_i$. Here $H$ is the Hubble constant, $n_{(F)}$ denotes the number density of $F$, and $X$ represents an SM field. The number density of $F$ in thermal equilibrium is given by $n_{eq}(F) = g_{x}(m_{x}T/2\pi^{3/2})\exp(-m_{x}/T)$ where $g_{x}$ and $m_{x}$ are the number of the degrees of freedom and mass of $x$, respectively. If $F^\pm$ is heavier than $F^0$, a term describing $F^+ \rightarrow F^0 f f'$ decay should be added on the right-hand side of (3.1):

$$
- \left(n_{(F^+_i)} - n_{(\overline{F}^0_i)}\right)\Gamma(F^+_i \rightarrow F^0_i f f'), \tag{3.2}
$$

where $f, f'$ are fermions in the SM and $\Gamma$ denotes a decay width.

The effective interactions relevant to annihilations of dark fermions are given by

$$
\mathcal{L}_{eff} \supset Z_{\mu} \left\{ \sum_{i=1}^{n_F} \overline{F}^0_i \gamma \mu \frac{g_w}{\cos \theta_W} (V_F + \gamma_5 A_F) F^0_i + \sum_{i=1}^{n_F} \overline{F}^+_i \gamma \mu \frac{g_w}{\cos \theta_W} (V_{F^+} + \gamma_5 A_{F^+}) F^+_i \\
+ \sum_{f} \bar{f} \gamma \mu \frac{g_w}{\cos \theta_W} (v_f + \gamma_5 a_f) f \right\}
$$

3. Relic density

By considering annihilations and decays of dark fermions in the early universe, one can evaluate the relic density of the dark fermion. We mostly follow the arguments in Refs. [39,42,78].
At the quantum level, the masses of \( F^\pm \) and \( F^0 \) receive finite corrections \( \delta m_{F^\pm} \) and \( \delta m_{F^0} \), respectively, and the degeneracy is lifted by one-loop corrections involving the photon and KK photons, which appear only in \( \delta m_{F^\pm} \), as depicted in Fig. 1. The mass difference between \( F^\pm \) and \( F^0 \), \( \delta m_{F^\pm} - \delta m_{F^0} \), can be evaluated in an analogous way to the universal extra dimension [79], and in the case of the warped extra dimension it is estimated by

\[
\delta m_{F^\pm} - \delta m_{F^0} \sim m_F \frac{\alpha_{\text{EM}}}{4\pi} \cdot K,
\]

where \( \alpha_{\text{EM}} \) is the electromagnetic fine-structure constant. In UED \( K = \ln(\Lambda^2/\mu^2) \) where \( \Lambda \) and \( \mu \) is the cut-off scale and a renormalization scale, respectively, and \( \Lambda/\mu \sim 10 \). In the RS spacetime only the first few KK excited states of each fields enter the quantum corrections. In particular, the

\[
\sum_{V=\text{string}} V^\mu \left\{ \sum_{i=1}^{n_F} \tilde{F}_i^0 \gamma^\mu g_w \left( V_F^{(V)} + \gamma_S A_F^{(V)} \right) F_i^0 + \sum_{i=1}^{n_F} \tilde{F}_i^+ \gamma^\mu g_w \left( V_{F^+}^{(V)} + \gamma_S A_{F^+}^{(V)} \right) F_i^+ \right\}
+ \sum_{f} \bar{f} \gamma^\mu g_w \left( v_f^{(V)} + \gamma_S a_f^{(V)} \right) f
\]

and by charged currents in Eq. (3.4). Here \( H \) denotes the Higgs boson, and \( f \) refers to a fermion in the SM (quarks, leptons, and neutrinos).

For the decays (3.2) the corresponding interaction terms in the effective Lagrangian are

\[
\mathcal{L}_{\text{eff}} \supset \sum_{V=W, W^{(1)}, W^{(1)}_R} V^\mu \frac{g_w}{\sqrt{2}} \left\{ \sum_{i=1}^{n_F} \tilde{F}_i^0 \gamma^\mu \left( V_F^{(V)} + \gamma_S A_F^{(V)} \right) F_i^0 + \sum_{f} \bar{f} \gamma^\mu \left( v_f^{(V)} + \gamma_S a_f^{(V)} \right) f \right\} + \text{(h.c.)},
\]

where \( f \) and \( f' \) refer to up-type quark (neutrino) and down-type quark (charged lepton), respectively. A CKM-like mixing matrix \( U^{(V)\text{CKM}} \) is a unit matrix for leptons and is assumed to approximately coincide with the CKM matrix for \( V = W \). For the spinor fermion \( F \), the right- and left-handed couplings \( g^V_{F/R/L} \equiv g_w \left( V_F^{(V)} \pm A_F^{(V)} \right)/2 \) are given in Appendix C.22, and for the SM fermions the couplings can be found in Ref. [18]. In particular, the \( W_R \) boson is found to have no couplings to the SM fermions.

3.1. Decays vs conversions of charged dark fermions

At the quantum level, the masses of \( F^\pm \) and \( F^0 \) receive finite corrections \( \delta m_{F^\pm} \) and \( \delta m_{F^0} \), respectively, and the degeneracy is lifted by one-loop corrections involving the photon and KK photons, which appear only in \( \delta m_{F^\pm} \), as depicted in Fig. 1. The mass difference between \( F^\pm \) and \( F^0 \), \( \delta m_{F^\pm} - \delta m_{F^0} \), can be evaluated in an analogous way to the universal extra dimension [79], and in the case of the warped extra dimension it is estimated by

\[
\delta m_{F^\pm} - \delta m_{F^0} \sim m_F \frac{\alpha_{\text{EM}}}{4\pi} \cdot K,
\]

where \( \alpha_{\text{EM}} \) is the electromagnetic fine-structure constant. In UED K = \( \ln(\Lambda^2/\mu^2) \) where \( \Lambda \) and \( \mu \) is the cut-off scale and a renormalization scale, respectively, and \( \Lambda/\mu \sim 10 \). In the RS spacetime only the first few KK excited states of each fields enter the quantum corrections. In particular, the
Fig. 1. Diagrams contributing to the fermion mass difference $\Delta m_F = \delta m_{F^+} - \delta m_{F^0}$.

Fig. 2. Charged dark fermion decay. $\ell^-$ and $\bar{\nu}$ can be replaced with down-type quarks and up-type anti-quarks, respectively.

coupling of right-handed $F^\pm(1)$ to $\gamma^{(1)}$ is several times as large as the electromagnetic coupling. It follows that $K \sim \mathcal{O}(10)$. Similarly, quantum corrections due to higher KK modes to the gauge couplings also become small, and a large cut-off scale is allowed [80,81].

A charged dark fermion decays to a neutral dark fermion and a charged vector boson, and hence to charged leptons and neutrinos, or light down-type quarks and up-type antiquarks (see Fig. 2).

In the SO(5) × U(1) GHU model, we have three charged vector bosons at low energies: $W$, the first KK excited state of $W$, and the lowest KK mode of the $W_R$ boson. A charged dark fermion $F^+$ decays to $F^0$ mainly by emitting a $W$ boson, because $W^{(1)}$ is heavy and interacts with $F^+$ and $F^0$ very weakly, and $W_R^{(1)}$ cannot decay to the SM fermions. If the mass difference between the charged and neutral dark fermions, $\Delta m_F \equiv m_{F^+} - m_{F^0} \simeq \delta m_{F^+} - \delta m_{F^0}$, is much smaller than $m_W$, the decay rate is given by

$$
\Gamma(F^- \rightarrow F^0 \ell \bar{\nu})
= \frac{G_F^2}{192\pi^3} m_F^{-5} \left[ \left( \frac{g_{W}'}{g_{W}^{L/R}} \right)^2 + \left( \frac{g_{W}'}{g_{W}^{L/R}} \right)^2 \right] \left[ \left( \frac{\epsilon_W}{g_{F}^{L/R}} \right)^2 + \left( \frac{\epsilon_W}{g_{F}^{L/R}} \right)^2 \right] \left[ f \left( \frac{m_{F^0}^2}{m_{F^-}^2} \right) - 4 g_{F}^{W} g_{F}^{W} \left( \frac{m_{F^0}^2}{m_{F^-}^2} \right) \right]
= \frac{G_F^2}{192\pi^3} \Delta m_F^{-5} \left[ \left( \frac{\epsilon_W}{g_{F}^{L/R}} \right)^2 + \left( \frac{\epsilon_W}{g_{F}^{L/R}} \right)^2 \right] \left[ \frac{64}{5} \left( \frac{\epsilon_W}{g_{F}^{L/R}} \right)^2 + \left( \frac{\epsilon_W}{g_{F}^{L/R}} \right)^2 - g_{F}^{W} g_{F}^{W} \right] + \mathcal{O} \left( \frac{\Delta m_F}{m_F^6} \right),
$$

(3.6)

where $g_{f/L/R}^{V} \equiv g_{f/L/R}^{V} / g_{W}$, and

$$
f(x) = 1 - 8x + 8x^3 - x^4 - 12x^2 \ln x,
g(x) = 1 + 9x - 9x^2 - x^3 + 6x(1 + x) \ln x.
$$

(3.7)

In the second equality in (3.6), we have assumed $\Delta m_F \ll m_{F^+}, m_{F^0}$ and have invoked approximations

$$
f(1 - x)^2 = \frac{64}{5} x^5 - \frac{96}{5} x^6 + \mathcal{O}(x^7),
g((1 - x)^2) = \frac{16}{5} x^5 - \frac{8}{5} x^6 + \mathcal{O}(x^7).
$$

(3.8)
The annihilation processes and corresponding diagrams of the dark fermions are tabulated in Table 1 and Fig. 4. We note that the masses of the first excited states of SM fermions (bosons) are about 105 MeV. In order that the $F^\pm$ lifetime is much shorter than the typical timescale of the weakly interacting massive particle (WIMP) DM formation, i.e. $\tau_{F^\pm} \ll 10^{-10}$ sec, the mass difference of the dark fermions must be of the order of 10 GeV or larger. The mass difference (3.5) will satisfy this condition for $m_F \gtrsim 2$ TeV with $K \sim O(10)$. Hereafter we assume that these conditions are satisfied and $F^\pm$ decays sufficiently quickly. We also note that if charged fermions $F^\pm$ do not decay sufficiently fast, they would remain after the DM freeze-out and would subsequently decay to $F^0$, doubling the relic DM density.

In the right-hand side of the Boltzmann equation (3.1), the last two terms correspond to the $F^0 \leftrightarrow F^+$ conversion depicted in Fig. 3.

The process depicted as (a) in Fig. 3, in particular $F^+ F^-$ pair production through this process, is kinematically allowed since $m_F \gg \Delta m_F$. Although the process depicted as (b) in Fig. 3 is suppressed by the small $F \bar{F} W W$ coupling, which is of the order of $10^{-3}$, this conversion process can dominate due to the large ratio of $n_{(F^\pm)}^{eq}/n_{(F^0)}^{eq} \sim (T/m_F)^{3/2} \exp(m_F/T) \sim 10^{10}$ for $T/m_F \sim 30$ [78].

Thus we have $n_{(F^\pm)}^{eq} \sim n_{(F^0)}^{eq}$ before freeze-out, and after freeze-out the $F^+$ decay to $F^0$. The relic density of the dark fermion in the present universe is given by the sum of the charged and neutral dark fermions at freeze-out. In the following we calculate the number density of all dark fermions.

### 3.2. Pair annihilations and relic density of dark fermions

The annihilation processes and corresponding diagrams of the dark fermions are tabulated in Table 1 and Fig. 4. We note that the masses of the first excited states of SM fermions (bosons) are about $m_{KK}$ (0.8$m_{KK}$). The mass of dark fermions is smaller than half of $m_{KK}$, so that the the final states of the annihilation of dark fermions involve only SM particles.

We consider the case where $\theta_H$ is small ($z_L \lesssim 10^5$ or $\theta_H \lesssim 0.15$). In such a case, the dark fermion is heavy and some of the annihilation amplitudes are processes that are suppressed by $\sin^2 \theta_H$. We find that, for most of the processes, the annihilation cross sections are too small to explain the current relic density. In particular, we find that $\bar{F} F W$, $\bar{F} F Z$, $\bar{F} F Z^{(n)}$, and $Z_R^{(n)} W W$ couplings are suppressed by the $\sin^2 \theta_H$ factor (see Appendixes C and D). One finds that process (a-i) is suppressed by the small Higgs–Yukawa couplings of $F \bar{F}$ and processes (a-ii) with $V = Z$ and $Z^{(n)}$ are suppressed by the small $Z^{(n)} F \bar{F}$ couplings. Processes (a-iii) and (a-iv) are suppressed by the
Table 1. Pair annihilation processes of dark fermions ($F = F^0, \bar{F}^0, F^+, \bar{F}^+$). (a-i)–(a-v) are annihilation processes of neutral and charged dark fermions, whereas (ac-i)–(ac-iv) are those of charged dark fermions. (co-i)–(co-v) are for co-annihilation of the neutral and charged dark fermions. In the intermediate states, ‘$n$’ denotes the KK excitation level ($n \neq 0$). In the final states, $q, l,$ and $\nu$ denotes quarks, charged leptons, and neutrinos in the SM. Corresponding diagrams are shown in Fig. 4.

| Process | Diagrams |
|---------|----------|
| (a-i) $F \bar{F} \rightarrow (S = H, H^{(n)}) \rightarrow q\bar{q}, l\bar{l}$ | (a) |
| (a-ii) $F \bar{F} \rightarrow (V = Z, Z^{(n)}, Z_R^{(n)}) \rightarrow q\bar{q}, l\bar{l}, \nu\bar{\nu}$ | (b) |
| (a-iii) $F \bar{F} \rightarrow ZZ, t$- and $u$-channels | (c), (d) |
| (a-iv) $F \bar{F} \rightarrow W^+W^- t$-channel | (c) |
| (a-v) $F \bar{F} \rightarrow (V = Z, Z^{(n)}, Z_R^{(n)}) \rightarrow W^+W^-$ | (e) |
| (ac-i) $F^+ \bar{F}^0 \rightarrow \gamma\gamma$, $t$- and $u$-channels | (c), (d) |
| (ac-ii) $F^+ \bar{F}^0 \rightarrow Z\gamma$, $t$- and $u$-channels | (c) (d) |
| (ac-iii) $F^+ \bar{F}^0 \rightarrow (V = \gamma, \gamma^{(n)}) \rightarrow q\bar{q}, l\bar{l}$ | (b) |
| (ac-iv) $F^+ \bar{F}^0 \rightarrow (V = \gamma, \gamma^{(n)}) \rightarrow W^+W^-$ | (e) |
| (co-i) $F^+ \bar{F}^0 \rightarrow (V = W^+, W^{+(n)}, W_R^{+(n)}) \rightarrow q\bar{q}', \nu\bar{l}$ | (b) |
| (co-ii) $F^+ \bar{F}^0 \rightarrow (V = W^+, W^{+(n)}, W_R^{+(n)}) \rightarrow W^+Z$ | (e) |
| (co-iii) $F^+ \bar{F}^0 \rightarrow (V = W^+, W^{+(n)}, W_R^{+(n)}) \rightarrow W^+\gamma$ | (e) |
| (co-iv) $F^+ \bar{F}^0 \rightarrow W^+Z, t$- and $u$-channels | (c), (d) |
| (co-v) $F^+ \bar{F}^0 \rightarrow W^+\gamma, t$- and $u$-channels | (c), (d) |

Fig. 4. $F\bar{F}$ annihilation diagrams. (a) s-channel annihilation to a fermion pair through the Higgs boson; (b) s-channel, to fermions through a vector boson; (c),(d) t- and u-channel annihilations to two vector bosons; (e) s-channel annihilation to two vector bosons.

small $W^- F^+ \bar{F}$ and $ZF\bar{F}$ couplings. All processes of (a-v) are suppressed by the small $Z^{(n)} F\bar{F}$ coupling and small $Z_R^{(n)} W^+ W^-$ couplings. Thus one finds that only process (a-ii) with $V = Z_R^{(1)}$ is unsuppressed and could be enhanced by both the Breit–Wigner resonance [45] of $Z_R^{(1)}$ and the large right-handed couplings of $Z_R^{(1)}$ to quarks and leptons.
For the annihilation of charged dark fermions, we see that process (ac-i) is not suppressed by couplings. However, the annihilation cross section

$$\sigma(F_i^+F_i^- \rightarrow \gamma\gamma) \cdot v = \frac{e^4}{8\pi m_F^2} + O(v^2),$$

(3.10)

where $v$ is the relative velocity of initial particles, is numerically small and negligible with $m_F \gtrsim 2\text{ TeV}$. Process (ac-ii) is suppressed by $F\tilde{F}Z$ couplings. The cross section in process (ac-iii) with $V = \gamma$ is estimated as

$$\sum_f \sigma(F_i^+F_i^- \rightarrow \gamma \rightarrow f\bar{f}) \cdot v = 8 \cdot \frac{e^4}{16\pi m_F^2} + O(v^2).$$

(3.11)

Process (ac-iii) with $V = \gamma^{(1)}$ can be enhanced by both large right-handed coupling of fermions and Breit–Wigner resonances. Process (ac-iv) is suppressed by the small $\gamma^{(1)}W^+W^-$ coupling.

As for co-annihilation, we have tabulated possible processes in Table 1. We find that process (co-i) with $V = W^+, W^{+(n)}$ is suppressed by small $F^+\tilde{F}^0W^-$ couplings and process (co-i) with $V = W_R$ is forbidden because of vanishing $W_R\tilde{f}f$ couplings. Process (co-ii) with $V = W, W^{(n)}$ is suppressed by small $\tilde{F}FW$ couplings, and (co-ii) with $V = W_R$ is suppressed by the small $W_R - W - Z$ couplings. Process (co-iii) with $V = W, W^{+(n)}$ is suppressed by small $F\tilde{F}W^{(n)}$ couplings. Process (co-iii) with $V = W_R$ is forbidden by the vanishing $W_RW\gamma$ coupling, which ensures the orthonormality of the KK gauge bosons. Processes (co-iv) and (co-v) are suppressed by small $F\tilde{F}Z$ and $F^+\tilde{F}^0W^-$ couplings. Hence we found that all of the co-annihilation processes are either vanishing or strongly suppressed.

Thus we find that the relevant processes for dark fermion annihilation are the following s-channel processes:

$$\begin{align*}
F^0\tilde{F}^0 & \rightarrow Z_R^{(1)} \rightarrow q\bar{q}, \bar{L}, \nu\bar{\nu}, \\
F^+F^- & \rightarrow \gamma, \gamma^{(1)} \rightarrow q\bar{q}, \bar{L}, \\
F^+F^- & \rightarrow Z_R^{(1)} \rightarrow q\bar{q}, \bar{L}, \nu\bar{\nu},
\end{align*}$$

(3.12)

and all other annihilation and co-annihilation processes are negligible.

In the following, we calculate the relic density of the dark fermions using annihilation cross sections of the processes given in (3.12). For charged dark fermions, the annihilation cross section of $F_i^+F_i^-$ to the SM fermions is given by

$$\sum_f \sigma(F_i^+F_i^- \rightarrow \{\gamma, \gamma^{(1)}, Z_R^{(1)}\} \rightarrow \bar{f}f)$$

$$= 8 \cdot \frac{e^4}{16\pi\beta}s^2 \left(s + 4m_F^2 + \frac{1}{3}\beta^2 \right)$$

$$+ \frac{1}{64\pi\beta} \left[ \frac{s}{Z_R^{(1)}Z_R^{(1)} + m^2} \right] g_w^4 \left( \sum_f \left[ \left( g_{f\bar{L}}^{(1)} \right)^2 + \left( g_{fR}^{(1)} \right)^2 \right] \right)$$

$$\times \left\{ 1 + \frac{\beta^2}{3} \left[ \left( g_{Z_R^{(1)}F^+L}^{(1)} \right)^2 + \left( g_{Z_R^{(1)}F^+R}^{(1)} \right)^2 \right] + \frac{8m_F^2}{s} g_{Z_R^{(1)}F^+L} g_{Z_R^{(1)}F^+R} \right\}$$

$$+ \frac{s}{(s - m_F^2)^2 + m^2} \left[ \sum_f \left[ \left( g_{f\bar{L}}^{(1)} \right)^2 + \left( g_{fR}^{(1)} \right)^2 \right] \right].$$
\[ \times \left\{ \left( 1 + \frac{\beta^2}{3} \right) \left[ \frac{1}{g_{F+L}} \right] + \frac{m_{F+}^2}{s} g_{F+L} g_{F+R} \right\} + 2 \cdot \frac{s - m_{Z_R}^2}{\left[ \left( s - m_{Z_R}^2 \right)^2 + m_{Z_R}^2 \right] \left[ \left( s - m_{\gamma}^2 \right)^2 + m_{\gamma}^2 \right]} \cdot \frac{s}{s - m_{Z_R}^2} \right\} \]

\[ \times g_{\gamma}^2 e^2 \left( \sum_f \left[ \frac{Z_R^{\gamma}(f)}{g_{F+L} g_{F+R} + g_{F-+L} g_{F-+R}} \right] \right) \left( 1 + \frac{\beta^2}{3} \right) \left[ \frac{Z_R^{\gamma}(f)}{g_{F+L} g_{F+R} + g_{F-+L} g_{F-+R}} \right], \]  

(3.13)

where \( g_{F+/F+} = V_f^{(V)} + A_f^{(V)} + \delta g_{F+/F+} \equiv v_f^{(V)} \pm A_f^{(V)} (V = Z_R^{\gamma}, \gamma) \), and the couplings are summarized in Sect. 5. \( \beta \equiv \sqrt{1 - 4m_F^2/s} \) and \( s \) is the invariant mass of \( F \bar{F} \). We have neglected the \( \gamma - \gamma(1) \) and \( \gamma - Z_R^{(1)} \) interference terms. The \( f_i f_i \) annihilation cross section \( \sum_f \sigma(F_i F_i \to Z_R^{(1)} \to f f) \) is obtained from (3.13) by replacing \( f_i f_i \) with \( f_i f_i \equiv V_f^{(V)} \pm A_f^{(V)} \) and ignoring the \( e^2 \) and \( e^4 \) terms. \( \Gamma_{Z_R^{(1)}} \) and \( \Gamma_{\gamma(1)} \) are the total decay rate of \( Z_R^{(1)} \) and \( \gamma(1) \) bosons, and \( \Gamma \) is estimated to be

\[ \Gamma_{\gamma(1)} = \sum_f N_{c,f} m_{Z_R^{(1)}} g_{\gamma}^2 \left( \frac{Z_R^{(1)}(f)}{g_{F+L} g_{F+R}, m_{\gamma}^2/m_{Z_R^{(1)}}} \right) \]

\[ + \sum_f m_{Z_R^{(1)}} g_{\gamma}^2 \left[ \gamma \left( g_{F+L} g_{F+R}, m_{F+}^2/m_{Z_R^{(1)}} \right) + \gamma \left( g_{F-+L} g_{F-+R}, m_{F-}^2/m_{Z_R^{(1)}} \right) \right] \right\}, \]

(3.14)

\( \Gamma_{\gamma(1)} \) is obtained in an analogous way. Here \( N_{c,f} = 3 \) (1) when \( f \) is a quark (charged lepton or neutrino). \( m_f \) is the mass of the SM fermion. We note that the \( F^\pm \) contributions in (3.14) are rather large.

Let \( n_0 [n_+] \) be the number density of \( F_i^0 \) and \( \bar{F}_i^0 \) \( (F_i^+ \text{ and } F_i^-) \) \( (i = 1, \ldots, n_F) \), and \( \sigma_0 (\sigma_+) \) be the annihilation cross section of \( F_i^0 \) \( (F_i^+) \). Then the evolution of the total number density of the DM is given by \( n \equiv 2n_F(n_0 + n_+) \), and the time evolution of \( n \) is governed by the Boltzmann equation

\[ \frac{dn}{dt} = -3Hn - 2n_F \langle \sigma_0 v \rangle \left( n_0^2 - n_0^2 \right) - 2n_F \langle \sigma_+ v \rangle \left( n_+^2 - n_+^2 \right), \]  

(3.15)

where \( n_0^{+} + n_0^{-} \) is the number density in thermal equilibrium and is approximated by \( n_0^{+} \approx g_0^{+}/(m_{F/0} + T/2\pi)^{3/2} \exp(-m_{F/0} + T) \) with \( g_0^{+} = 2 \) being the number of degrees of freedom of \( F_i^0 \) and \( F_i^+ \). Using the relations \( n_0^{+}/n_0^{-} = n/n_E \) and \( n_0^{-}/n_0^{-} = n_+^{-}/n_+^{-} = 1/4n_F \), we obtain

\[ \frac{dn}{dt} = -3Hn - \langle \sigma v \rangle \left( n^2 - n_0^2 \right), \]

\[ \sigma v \equiv \frac{\sigma_0 v + \sigma_+ v}{8n_F}. \]  

(3.16)

We introduce \( Y_{(eq)} \equiv n_{(eq)} / S \) where \( S = 2\pi^2 g_* T^3 / 45 \) is the entropy density. \( g_* \) is the degree of freedom at the freeze-out temperature \( T_F \) and we take \( g_* = 92 \). Conservation of entropy per co-moving volume, \( SdV = \text{const} \) \( (a_{SF} \) is the scale factor of the expanding universe), reads
\( dn/dt + 3Hn = SdY/dt \). The Hubble constant is given by \( H^2 = 4\pi^2 g_* T^4/(45M_{Pl}^2) \) and \( t = 1/2H \) in the radiation-dominant era. \( M_{Pl} \) is the Planck mass. Hence we rewrite the Boltzmann equation as

\[
\frac{dY}{dx} = \frac{\langle \sigma v \rangle}{H} \frac{1}{x} S\left( Y^2 - Y_{eq}^2 \right),
\]

where \( x \equiv T/m_F \) and \( T \) is the temperature of the universe. \( \langle \sigma v \rangle = \langle \sigma v \rangle(x) \) is the thermal-averaged cross section discussed later. \( n_{eq} \) is the density in thermal equilibrium, and becomes

\[
n_{eq} = g_{\text{eff}} \left( \frac{m_FT}{2\pi} \right)^{3/2} e^{-m_F/T}
\]

\( (g_{\text{eff}} = 2 \cdot 4n_F \) is the degree of freedom of the dark fermions) in the non-relativistic limit. Defining \( \Delta \equiv Y - Y_{eq} \) and \( \Delta' \equiv d\Delta/dx, \) \( Y_{eq} \equiv dY_{eq}/dx, \) we rewrite (3.17) as

\[
\Delta' = -Y_{eq} + f(x) (2Y_{eq} + \Delta), \quad f(x) = \sqrt{\frac{\pi g_*}{45}} m_F M_{Pl} \langle \sigma v \rangle,
\]

which is written at early times \( (x \gg x_f \equiv T_f/m_F, |\Delta'| \ll |Y_{eq}'|) \) as

\[
\Delta = \frac{Y_{eq}'}{f(x)(2Y_{eq} + \Delta)}.
\]

At late times \( (T \ll T_f), Y_{eq} \ll Y \sim \Delta \) and \( |\Delta'| \gg |Y_{eq}'|, \) hence (3.19) reads

\[
\Delta^{-2} \Delta' = f(x).
\]

Integrating (3.21) with \( x \) from zero to \( x_f \equiv T_f/m_F, \) we obtain

\[
Y_0^{-1} \simeq \Delta_0^{-1} = \int_0^{x_f} f(x)dx = \sqrt{\frac{\pi g_*}{45}} M_{Pl} m_F J_f, \quad J_f = \int_0^{x_f} \langle \sigma v \rangle(x)dx,
\]

where we have used \( \Delta(x_f) \gg \Delta(x = 0). \) Thus the relic density of the dark fermions at the present time is given by

\[
\Omega_{DM} h^2 = \frac{\rho_{DM}}{\rho_c} h^2 = \frac{m_F S_0 Y_0 h^2}{\rho_c} = \frac{1.04 \times 10^9}{M_{Pl}} \frac{1}{\sqrt{8\pi J_f}},
\]

where \( \rho_{DM} = m_F S_0 Y_0 \) and \( \rho_c = 3H_0^2 M_{Pl}^2/8\pi = 1.054 \times 10^{-5} \text{ GeV cm}^{-3} \) have been used. \( S_0 = 2889.2 \text{ cm}^{-3} \) is the entropy density of the present universe.

The freeze-out temperature is determined by solving the condition

\[
\Delta(x_f) = c Y_{eq}(x_f),
\]

with \( \Delta \) in the early time. \( c \) is an numerical factor of order unity and is determined by matching the late-time and early-time solutions. Hereafter we take \( c = 1/2. \) Equation (3.24) with (3.20) gives the following transcendental relation:

\[
x_f^{-1} = \ln \left( c(c + 2) \sqrt{\frac{45 g_{\text{eff}} m_F M_{Pl} x_f^{1/2} \langle \sigma v \rangle}{2\pi^3 g_s^{1/2}}} \right),
\]

which can be solved by numerical iteration.
Table 2. \(\theta_H, c_{\text{top}}, c_F,\) and \(m_F\) for \(z_L\) and \(n_F = 3, 4, 5,\) and \(6,\) in the case where dark fermions are degenerate.

| \(n_F\) | \(z_L\) | \(\theta_H\) | \(c_{\text{top}}\) | \(c_F\) | \(m_F\) [TeV] |
|--------|--------|-------------|----------------|-------|-----------|
| 3      | \(10^8\) | 0.360       | 0.357          | 0.385 | 0.670     |
|        | \(10^6\) | 0.177       | 0.296          | 0.309 | 1.54      |
|        | \(10^5\) | 0.117       | 0.227          | 0.235 | 2.54      |
|        | \(2 \times 10^4\) | 0.0859 | 0.137          | 0.127 | 3.88      |
| 4      | \(10^8\) | 0.355       | 0.357          | 0.423 | 0.567     |
|        | \(10^6\) | 0.174       | 0.292          | 0.374 | 1.27      |
|        | \(10^5\) | 0.115       | 0.227          | 0.332 | 2.03      |
|        | \(3 \times 10^4\) | 0.0917 | 0.168          | 0.299 | 2.66      |
|        | \(10^4\) | 0.0737      | 0.0366         | 0.256 | 3.46      |
| 6      | \(10^8\) | 0.348       | 0.356          | 0.461 | 0.455     |
|        | \(10^6\) | 0.171       | 0.292          | 0.434 | 1.00      |
|        | \(10^5\) | 0.113       | 0.227          | 0.414 | 1.57      |
|        | \(10^4\) | 0.0724      | 0.0365         | 0.379 | 2.57      |

The precise form of the velocity-averaged cross section \(\langle \sigma v \rangle\) is given in Ref. [82]. When \(\sigma v\) is expanded in \(v^2\) as

\[
\sigma v = a + b v^2 + \cdots = a + b \left( (s - 4m_F^2)/m_F^2 \right) + \cdots, \tag{3.26}
\]

we obtain

\[
\langle \sigma v \rangle = 4\pi \left( \frac{m_F}{4\pi T} \right)^{3/2} \int_0^\infty dv v^2 e^{-m_F v^2 / 4T} \sigma v
= a + 6bT/m_F + \cdots. \tag{3.27}
\]

In the present case \(x_f \sim 1/30\) and therefore only the first term in the \(v^2\) expansion in Eq. (3.26) is kept in the following analysis.

### 3.3. Relic density of degenerate dark fermions

First we consider the case in which all dark fermions are degenerate. In the numerical study of this paper, we have adopted \(\alpha_{\text{EM}} = e^2/4\pi = 1/128, \sin^2 \theta_W = 0.2312, m_Z = 91.1876\) GeV, and \(m_{\text{top}} = 171.17\) GeV [83]. In Table 2, we have summarized the values of \(\theta_H\), the bulk mass parameters of the top quark \(c_{\text{top}}\) and the dark fermion \(c_F\), and mass of the dark fermion \(m_F\) for particular values of \((z_L, n_F)\). \(\theta_H, c_{\text{top}},\) and \(c_F\) are chosen so that we obtain a 126 GeV Higgs mass [17,18].

In Fig. 5 the relic density of the dark fermions for \(n_F = 3, 4, 5,\) and \(6,\) is plotted. In the plot, the best value [68% confidence level (CL) limits] of the relic density of the cold dark matter observed by Planck [84],

\[
\Omega_{\text{CDM}} h^2 = 0.11805 \quad [0.1186 \pm 0.0031], \tag{3.28}
\]

is also shown. Here Hubble’s expansion rate \(H_0 = 100h\) km s\(^{-1}\) Mpc\(^{-1}\), 100\(h = 67.11\) [67.4 \(\pm\) 1.4].

In our previous work [18], we have constrained \(z_L\) by \(z_L \lesssim 10^6\) because no evidence of the neutral boson resonances in LHC has been seen. For \(z_L \lesssim 10^6\), we found that no parameter regions can explain the current DM density. For \(n_F = 3\) we obtain \(\Omega_{\text{DM}} h^2 \lesssim 0.08\) for any value of \(z_L\). For \(n_F = 4\) and \(z_L \leq 10^6\), we have \(\Omega_{\text{DM}} h^2 \gtrsim 0.2\). For \(n_F = 5\) and \(6,\) the predicted densities are larger than the limit on the closure universe.
Fig. 5. Relic density of neutral dark fermions in the case of $n_f$ degenerate dark fermion multiplets ($n_F = 3, 4, 5, 6$). Data points are, from right to left, $z_L = 10^4 (2 \times 10^4)$ to $10^5$ with steps of $10^4, 10^5, 10^6, 10^7$, and $10^8$ for $n_F = 4, 5, 6$ ($n_F = 3$). The current observed limit of $\Omega_{\text{DM}} h^2$ and the lower bound of the overclosure of the universe are indicated as horizontal lines.

Fig. 6. $F_0^0$ decay to $F_0^0$ by emitting one $Z$ boson or two $W$ bosons.

We remark that for $n_F = 3$ the relic density becomes very small at $z_L \sim 3 \times 10^4$ due to the fact that the masses of the first KK vector bosons are very close to twice the mass of the dark fermions, and enhancement due to the Breit–Wigner resonance happens. A similar mechanism occurs in some of the universal extra-dimension models [43,44,46,47].

3.4. Current mixing

So far it has been supposed that $n_F$ multiplets of SO(5)-spinor fermions $\Psi_F$, are degenerate. There is an intriguing scenario in which some of them are heavier than others, only the lightest $F_i^{0(1)}$ becoming dark matter. A typical mass of $F_i^{0(1)}$ is 1–3 TeV. We show that the mass difference of $O(200)$ GeV and small mixing could fulfill this job.

Let us denote the lightest particles of heavy and light SO(5)-spinor fermions by $(F^+_h, F^0_h)$ and $(F^+_l, F^0_l)$, respectively. Charged $F^+_l$ and $F^+_h$ are heavier than the corresponding neutral ones, and are supposed to decay sufficiently fast. $F^0_h$ also needs to decay sufficiently fast in order for the scenario to work. $F^0_h$ can decay either as $\rightarrow F^0_l + Z$ or as $\rightarrow F^+_l + W^- \rightarrow F^0_l + W^+ + W^-$, as shown in Fig. 6. For this process the off-diagonal neutral or charged current is necessary. We examine in this subsection how the off-diagonal currents are generated.

To be concrete, let us suppose that there are only two SO(5)-spinor fermion multiplets, $\Psi_{F_h}$ and $\Psi_{F_l}$, which are gauge eigenstates. We suppose that $\Psi_{F_l}$ obeys the boundary condition $\eta_{F_l} = +1$ in
whereas $\Psi_{F_h}$ satisfies the flipped boundary condition $\eta_{F_h} = -1$. It is easy to confirm that their KK spectrum is given by \((B7)\) for both $\Psi_{F_h}$ and $\Psi_{F_i}$. The lowest mode $(F_h^{+1}, F_h^{0(1)})$ is mostly an SU(2)$_L$ doublet, whereas $(F_i^{+1}, F_i^{0(1)})$ is mostly an SU(2)$_R$ doublet.

Let us denote the gauge (mass) eigenstates of the lightest modes of $\Psi_{F_h}$, $\Psi_{F_i}$ by $\hat{\tilde{F}}_h^{+}, \hat{\tilde{F}}_h^{0}, \hat{\tilde{F}}_i^{+}, \hat{\tilde{F}}_i^{0}$ $(F_h^{+}, F_h^{0}, F_i^{+}, F_i^{0})$. The most general form of bulk mass terms for $\Psi_{F_h}$ and $\Psi_{F_i}$ is

$$\mathcal{L}_{\phi}^{\text{4D mass}} = -\sigma' \left\{ c_{F_h} \tilde{\Psi}_{F_h} \Psi_{F_h} + c_{F_i} \tilde{\Psi}_{F_i} \Psi_{F_i} \right\} - \tilde{\Delta} \left\{ \tilde{\Psi}_{F_h} \Psi_{F_i} + \tilde{\Psi}_{F_i} \Psi_{F_h} \right\}. \quad (3.29)$$

We note that $\tilde{\Psi}_{F_h} \Psi_{F_h}$ and $\tilde{\Psi}_{F_i} \Psi_{F_i}$ are odd under parity $y \rightarrow -y$, whereas $\tilde{\Psi}_{F_h} \Psi_{F_i}$ is even. The $\tilde{\Delta}$ term induces mass mixing among $\hat{\tilde{F}}_h^{+}$ and $\hat{\tilde{F}}_i^{+}$, and among $\hat{\tilde{F}}_h^{0}$ and $\hat{\tilde{F}}_i^{0}$. $c_{F_h}$ and $c_{F_i}$ generate masses $\hat{\tilde{m}}_h$ and $\hat{\tilde{m}}_i$ for $(\hat{\tilde{F}}_h^{+}, \hat{\tilde{F}}_h^{0})$ and $(\hat{\tilde{F}}_i^{+}, \hat{\tilde{F}}_i^{0})$. We suppose that $c_{F_h} < c_{F_i}$ so that $\hat{\tilde{m}}_h > \hat{\tilde{m}}_i$. As described in Sect. 3.1, charged states acquire radiative corrections (3.5), a $\hat{\tilde{m}}_h$ ($\hat{\tilde{m}}_i$) for $\hat{\tilde{F}}_h^{+}$ ($\hat{\tilde{F}}_i^{+}$) where $a$ is $O(10^{-3} \rightarrow 10^{-2})$.

Hence the mass matrices are given by

$$\mathcal{L}_{\phi}^{\text{4D mass}} = -\left( \hat{\tilde{F}}_h^{+}, \hat{\tilde{F}}_i^{+} \right) \mathcal{M}_+ \left( \hat{\tilde{F}}_h^{+}, \hat{\tilde{F}}_i^{+} \right) - \left( \hat{\tilde{F}}_h^{0}, \hat{\tilde{F}}_i^{0} \right) \mathcal{M}_0 \left( \hat{\tilde{F}}_h^{0}, \hat{\tilde{F}}_i^{0} \right), \quad (3.30)$$

We suppose that $\Delta \ll \hat{\tilde{m}}_h, \hat{\tilde{m}}_i$. We diagonalize the two matrices to obtain

$$\mathcal{L}_{\phi}^{\text{4D mass}} = -m_{F_h} \hat{\tilde{F}}_h^{+} \hat{\tilde{F}}_h^{+} - m_{F_i} \hat{\tilde{F}}_i^{+} \hat{\tilde{F}}_i^{+} - m_{F_h} \hat{\tilde{F}}_h^{0} \hat{\tilde{F}}_h^{0} - m_{F_i} \hat{\tilde{F}}_i^{0} \hat{\tilde{F}}_i^{0}.$$

$$\begin{align*}
\left( \begin{array}{c} F_h^{+} \\ F_i^{+} \end{array} \right) &= V \left( \frac{1}{2} \alpha_+ \right) \left( \begin{array}{c} \hat{\tilde{F}}_h^{+} \\ \hat{\tilde{F}}_i^{+} \end{array} \right), \\
\left( \begin{array}{c} F_h^{0} \\ F_i^{0} \end{array} \right) &= V \left( \frac{1}{2} \alpha_0 \right) \left( \begin{array}{c} \hat{\tilde{F}}_h^{0} \\ \hat{\tilde{F}}_i^{0} \end{array} \right), \\
\left( \begin{array}{c} m_{F_h}^{+} \\ m_{F_i}^{+} \end{array} \right) &= \frac{1}{2} (1 + a) (\hat{\tilde{m}}_h + \hat{\tilde{m}}_i) \pm \sqrt{\frac{1}{4} (1 + a)^2 (\hat{\tilde{m}}_h - \hat{\tilde{m}}_i)^2 + \Delta^2}, \\
V(\alpha) &= \left( \begin{array}{cc} \cos \alpha & \sin \alpha \\ -\sin \alpha & \cos \alpha \end{array} \right), \quad \tan \alpha_+ = \frac{2\Delta}{(1 + a)(\hat{\tilde{m}}_h - \hat{\tilde{m}}_i)}. \quad (3.31)
\end{align*}$$

The masses $(m_{F_h}^{+}, m_{F_i}^{+})$ and angle $\alpha_0$ are obtained from $(m_{F_h}^{+}, m_{F_i}^{+})$ and $\alpha_+$ by taking $a \rightarrow 0$.

The couplings to $Z$ (the neutral currents) are given originally by

$$Z_{\mu} \left[ \sum_{F_j = F_h^{+}, F_i^{+}, F_h^{0}, F_i^{0}} \left\{ g^Z_{F_j L} \hat{\tilde{F}}_{jL} Y_{\mu} \hat{\tilde{F}}_{jL} + g^Z_{F_j R} \hat{\tilde{F}}_{jR} Y_{\mu} \hat{\tilde{F}}_{jR} \right\} \right]. \quad (3.32)$$

Similarly, the couplings to $W$ (the charged currents) are given by

$$W_{\mu} \left[ \sum_{j = h, i} \left\{ g^W_{F_j L} \hat{\tilde{F}}_{jL} Y_{\mu} \hat{\tilde{F}}_{jL} + g^W_{F_j R} \hat{\tilde{F}}_{jR} Y_{\mu} \hat{\tilde{F}}_{jR} \right\} \right] + (\text{h.c.}). \quad (3.33)$$

We recall that $(F_h^{+}, F_h^{0})$ is mostly an SU(2)$_L$ doublet, whereas $(F_i^{+}, F_i^{0})$ is mostly an SU(2)$_R$ doublet with the boundary conditions imposed on $\Psi_{F_h}$ and $\Psi_{F_i}$. Therefore $g^Z_{F_h L} \gg g^W_{F_h L}$ and $g^W_{F_i L} \gg g^W_{F_i L}$. 

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etc. In terms of mass eigenstates the neutral current becomes

\[
\left( \bar{F}_{hL}^0, \bar{F}_{IL}^0 \right) \left\{ \frac{g_{F^0_{hL}}^Z + g_{F^0_{IL}}^Z}{2} + \frac{g_{F^0_{hL}}^Z - g_{F^0_{IL}}^Z}{2} U(\alpha_0) \right\} \gamma^\mu \left( F_{hL}^0 / F_{IL}^0 \right) \\
+ \left( \bar{F}_{hL}^+, \bar{F}_{IL}^+ \right) \left\{ \frac{g_{F^+_{hL}}^Z + g_{F^+_{IL}}^Z}{2} + \frac{g_{F^+_{hL}}^Z - g_{F^+_{IL}}^Z}{2} U(\alpha_+) \right\} \gamma^\mu \left( F_{hL}^+ / F_{IL}^+ \right) + (L \to R),
\]

where

\[
U(\alpha) = \begin{pmatrix} \cos \alpha & -\sin \alpha \\ -\sin \alpha & \cos \alpha \end{pmatrix}.
\]

The charged current is

\[
\left( \bar{F}_{hL}^+, \bar{F}_{IL}^+ \right) \left\{ \frac{g_{F^+_h L}^W + g_{F^+_I L}^W}{2} V \left( \frac{\alpha_+ - \alpha_0}{2} \right) + \frac{g_{F^0_{h L}}^W - g_{F^0_{I L}}^W}{2} U \left( \frac{\alpha_+ + \alpha_0}{2} \right) \right\} \gamma^\mu \left( F_{hL}^0 / F_{IL}^0 \right) + (L \to R).
\]

We recognize that off-diagonal neutral and charged currents are generated for dark fermions obeying the distinct boundary conditions.

For small \( \theta_H \), heavy dark fermions have much larger couplings to \( W \) and \( Z \) than light dark fermions. Let us suppose that \( \Delta \ll \tilde{m}_h - \tilde{m}_l \) so that \( \frac{1}{2} \alpha_0 \sim \Delta / (\tilde{m}_h - \tilde{m}_l) \ll 1 \) and \( \alpha_+ \sim \alpha_0 / (1 + a) \ll 1 \). The \( Z \) coupling of \( F_{hL}^0 \) is \( g_{F^0_{hL}}^Z \) approximately \( g_{F^0_{hL}}^Z + g_{F^0_{hL}}^Z (\frac{1}{2} \alpha_0)^2 \). We assume that \( (\frac{1}{2} \alpha_0)^2 \ll \text{sup} \left( |g_{F^0_{hL}}^Z / g_{F^0_{hL}}^Z|, |g_{F^0_{hL}}^Z / g_{F^0_{hL}}^Z| \right) \) so that the estimate of the cross section for the direct-detection experiments discussed in the next section remains valid.

The couplings for \( F_{hL}^0 \to F_{hL}^0 + Z \) and for \( F_{hL}^0 \to F_{hL}^+ + W \) are approximately \( -\frac{1}{2} g_{F^0_{hL}}^Z \alpha_0 \) and \( -\frac{1}{2} g_{F^0_{hL}}^W \alpha_0 \), respectively. With a moderate \( \frac{1}{2} \alpha_0 \sim \frac{1}{2} \text{sup} \left( |g_{F^0_{hL}}^Z / g_{F^0_{hL}}^Z|^{1/2}, |g_{F^0_{hL}}^Z / g_{F^0_{hL}}^Z|^{1/2} \right) \), \( F_{hL}^0 \) decays sufficiently fast. Only the light dark fermion \( F_{IL}^0 \) becomes a candidate for dark matter.

### 3.5. Relic density of non-degenerate dark fermions

Let us examine the case with non-degenerate dark fermions. We separate the \( n_F \) dark fermions \( \{F_i^+ \} \) into \( \bar{n}_F \) light fermions \( \{F_i^+ \} \) with bulk mass \( c_{F_i} \) and \( n_F \) heavy fermions \( \{F_h, F_h \} \) with \( c_{F_h} \). Here \( \Delta c_F = c_F - c_{F_h} > 0 \). \( c_F \) and \( c_{F_h} \) are chosen so as to keep the values of \( \theta_H \) and \( m_H \) unchanged. In Table 3, the values of \( c_F \) become \( \Delta c_F \), and the corresponding fermion masses are tabulated. The changes in the couplings of \( \bar{n}_F \) light fermions to vector bosons from those in the degenerate case are found to be small.

At the temperature \( T \gtrsim m_{F_h} - m_{F_1} \), the heavy–light conversion process depicted in Fig. 7 dominates, and both \( F_h \) and \( F_l \) obey the Boltzmann distribution. When \( m_{F_h} - m_{F_1} \gtrsim T_f = \mathcal{O}(100 \text{ GeV}) \), the number density of \( F_h \) becomes much smaller than that of \( F_1 \).

In contrast to \( F_1 \), \( F_h \) obeys the boundary condition \( \eta_{F_h} = -1 \) and its couplings to \( W \) and \( Z \) are not suppressed, whereas its coupling to \( Z_R \) is suppressed. Thus the dominant annihilation processes of \( F_h \) are s-channel processes of \( F \bar{F} \) annihilation to the SM fermions through \( Z^{(1)} \) and \( \gamma^{(1)} \) [(a–ii) with \( V = Z^{(1)} \) and (ac–iv) with \( V = \gamma^{(1)} \) in Table 1] and co-annihilation through \( W^{(1)} \) [(co–i) with...
The dominant processes of $F_h \leftrightarrow F_l$ conversion. $f$ and $f'$ are the SM fermions. $\alpha$ denotes suppression of the $FFW$, $FFZ$ vertex factor by mixing.

\[ V = W^{(1)} \] in Table 1. The time evolutions of the total dark fermion density are given by

\[
\frac{dn}{dt} = -3Hn - 2n_F^{\text{light}}(\sigma_{l0}v)\left(n_{l0}^2 - n_{l0,\text{eq}}^2\right) - 2n_F^{\text{light}}(\sigma_{l+}v)\left(n_{l+}^2 - n_{l+\text{,eq}}^2\right) - 2n_F^{\text{heavy}}(\sigma_{h0}v)\left(n_{h0}^2 - n_{h0,\text{eq}}^2\right) - 2n_F^{\text{heavy}}(\sigma_{h+}v)\left(n_{h+}^2 - n_{h+,\text{eq}}^2\right) - 4n_F^{\text{heavy}}(\sigma_{hc}v)\left(n_{h0}n_{h+} - n_{h0,\text{eq}}n_{h+,\text{eq}}\right) = -3Hn - 2n_F^{\text{light}}\left(\frac{n_{\text{eq}}}{n_{\text{eq}}}\right)^2 \left[\langle \sigma_{l0}v \rangle + \langle \sigma_{l+}v \rangle\right] (n^2 - n_{\text{eq}}^2) - 2n_F^{\text{heavy}}\left(\frac{n_{h\text{eq}}}{n_{\text{eq}}}\right)^2 \left[\langle \sigma_{h0}v \rangle + \langle \sigma_{h+}v \rangle + 2\langle \sigma_{hc}v \rangle\right] (n^2 - n_{\text{eq}}^2) \equiv -3Hn - \langle \sigma_{\text{ND}}^{\text{eff}}v \rangle (n^2 - n_{\text{eq}}^2),
\]

where $n_{w0}$ and $n_{w+}$ ($w = h, l$) are the number densities of $F_{w,i}^0$, $F_{w,i}^+$ ($i = 1, \ldots , n_F^{\text{light}}$ for $w = l$, and $1, \ldots , n_F^{\text{heavy}}$ for $w = h$), respectively. $\sigma_{w0}$, $\sigma_{w+}$, and $\sigma_{hc}$ are the cross sections of $F_{w,i}^{00}$, $F_{w,i}^{0+}$, $F_{w,i}^{+0}$ annihilations and $F_{h,i}^{0+}$ co-annihilation, respectively. We have also used

\[
\frac{n_{w0}/+}{n_{\text{eq}}} \simeq \frac{n_{w0}/+\text{,eq}}{n_{\text{eq}}} , \quad n_{w0}\simeq n_{w0\text{,eq}}\equiv n_{w+\text{,eq}} , \quad w = h, l.
\]
The number densities in thermal equilibrium are given by

\[
\begin{align*}
n_{n}^{\text{eq}} &= \frac{1}{4n_{F}^{\text{light}} + 4n_{F}^{\text{heavy}}(1 + \eta)^{3/2}\exp(-\eta/x)}, \\
n_{h}^{\text{eq}} &= \frac{(1 + \eta)^{3/2}\exp(-\eta/x)}{4n_{F}^{\text{light}} + 4n_{F}^{\text{heavy}}(1 + \eta)^{3/2}\exp(-\eta/x)}, \quad \eta \equiv \frac{m_{F_{h}} - m_{F_{l}}}{m_{F_{l}}},
\end{align*}
\] (3.39)

and \(g_{\text{eff}}\) in (3.18) will be replaced with

\[
g_{\text{eff}}^{\text{ND}} = 2 \cdot 4n_{F}^{\text{light}} + 2 \cdot 4n_{F}^{\text{heavy}}(1 + \eta)^{3/2}\exp(-\eta/x).
\] (3.40)

When \(\eta/x \gg 1\), the Boltzmann equation (3.37) with (3.39) can be approximated by

\[
\frac{dn}{dt} = -3Hn_{\text{eff}}^{\text{ND}} v \bigg|_{\eta \to \infty} \left( n^{2} - n_{\text{eq}}^{2} \right),
\]

\[
\sigma_{\text{eff}}^{\text{ND}} v \bigg|_{\eta \to \infty} = \frac{1}{8n_{F}^{\text{light}}} \left[ \sigma_{00} v + \sigma_{0+} v \right],
\] (3.41)

and \(g_{\text{eff}}^{\text{ND}} \bigg|_{\eta \to \infty} = 2 \cdot 4n_{F}^{\text{light}}\). With this approximation, one can calculate the relic density of the dark fermion by following the procedure described in Sect. 3.2. The effective cross section, and therefore \(J_{f}\) in (3.22), is enhanced by a factor \(\sigma_{\text{eff}}^{\text{ND}} v \bigg|_{\eta \to \infty}/\sigma_{\text{eff}} v \simeq n_{F}/n_{F}^{\text{light}}\), which results in the reduction of the relic density by a factor \(n_{F}^{\text{light}}/n_{F}\), as seen from (3.23). If \(\eta\) is not so large, the approximation (3.41) is not valid any more. In particular, for \(\eta \sim 0\) the Boltzmann equation (3.37) become almost identical to (3.16), and the relic density will be increased up to that in the degenerate case. Effects of small \(\eta\) on \(\Omega_{\text{DM}} h^{2}\) (3.23) mainly appear in the change of the value of \(J_{f}\) (or \(\langle \sigma_{\text{eff}} v \rangle\)). Numerically we find that \(J_{f}\) determined from (3.41) well approximates \(J_{f}\) determined from (3.37) with (3.39) at \(\mathcal{O}(5\%)\) accuracy when \(\eta \gtrsim 0.10\) for \(x = x_{f} \simeq 1/30\) and \(\sigma_{00}/\sigma_{0+} \sim \sigma_{00}/\sigma_{0+}\).

We note that, in the cross section (3.13), the total decay width of \(Z_{R}^{(1)}\) (3.14) can be modified so that it consists of \(n_{F}^{\text{light}} F_{l}\) and \(n_{F}^{\text{heavy}} F_{h}\) partial decay widths, as \(Z_{R} F_{l} \bar{F}_{l}\) and \(Z_{R} F_{h} \bar{F}_{h}\) couplings are not the same. The total decay width of \(\gamma^{(1)}\) does not change so much, since \(\gamma^{(1)} F \bar{F}\) couplings are invariant under the exchange \(\text{SU}(2)_{L} \leftrightarrow \text{SU}(2)_{R}\). Numerically we find that the change in the cross section (3.13) induced from the change in decay widths amounts only to a few percent.

From Table 3, we see that for \(\Delta c_{F} \gtrsim 0.04\) the condition \(\eta \gtrsim 0.1\) is satisfied and the cross section formula (3.41) is valid. In Fig. 8 we have plotted the relic density of the dark fermion determined from the Boltzmann equation (3.41) for \(\Delta c_{F} = 0.04\) and 0.06 in the case of \(n_{F} = 4\) with \((n_{F}^{\text{light}}, n_{F}^{\text{heavy}}) = (1, 3)\). For \(\Delta c_{F} < 0.04\), the approximated formula (3.41) is no longer valid, and the relic density can be much larger than those for \(\Delta c_{F} \gtrsim 0.04\). By inter-/extrapolating the \(\Omega_{\text{DM}} h^{2}\) with respect to \(\Delta c_{F}\) and \(z_{L}\), we plot the parameter region \((\Delta c_{F}, z_{L})\) allowed by the experimental limit on the current relic density in Fig. 9. It is seen that the observed current relic density is obtained when \(10^{4} \lesssim z_{L} \lesssim 10^{6} (0.07 \lesssim \theta_{H} \lesssim 0.17)\) in the range \(0.04 \lesssim \Delta c_{F} \lesssim 0.07\). The mass of the dark fermion \(m_{\text{DM}}\) varies within the range of \((1000, 3100)\) GeV. For \(n_{F} = 5, 6\) and \(n_{F} = 4\) with \((n_{F}^{\text{light}}, n_{F}^{\text{heavy}}) = (2, 2), (3, 1),\) we find no parameter region that explains the current DM density.

In the numerical study we have used an approximation explained in Sect. 3.2. In the case in which the Breit–Wigner resonance enhances the DM relic density, a more rigorous treatment may be required [78]. In the case under consideration, the effect of the enhancement is found to be mild. Quantitatively, in the notation of Ref. [78] we obtain \(\epsilon = (\Gamma_{V}/m_{V})^{2} = \mathcal{O}(0.005)\)
Fig. 8. Relic density of the dark fermion versus $m_{\text{DM}} = m_{F_l}$ for $n_F = 4$ ($n_F^{\text{light}} = 1$, $n_F^{\text{heavy}} = 3$). Thick-solid and thick-dotted lines are $\Delta c_F = c_F - c_{F_h} = 0.06$ and 0.04, respectively. Data points are, from right to left, $z_L = 10^4$ to $10^5$ with an interval $10^4$, $3 \times 10^5$, and $10^6$. Horizontal lines around $\Omega_{\text{DM}} h^2 \sim 0.12$ show the observed 68% confidence level (CL) limit of the relic density of the cold dark matter.

Fig. 9. Parameter region ($\Delta c_F, z_L$) allowed by the limits of relic density. Inner and outer colored regions are allowed with the 68% CL limit and twice the 68% CL limit $\Omega_{\text{DM}} h^2 \subset [0.1186 \pm 2 \times 0.0031]$, respectively. The mass of the dark fermion $m_{F_l}$ and a mass ratio $\eta \equiv (m_{F_h} - m_{F_l}) / m_{F_l}$ are also indicated as solid and dashed lines, respectively.

Before closing this section, we make a few comments. First we comment on the effect of dark fermions on the electroweak precision parameters [85,86], in particular on the $S$ parameter. Since the dark fermions have vector-like couplings to the $Z$ boson, the contribution to the $S$ parameter from an SU(2) doublet $\{F^+, F^0\}$ is estimated to be

$$
\Delta(\alpha_{\text{EM}} S) \simeq 4 s_w^2 c_w^2 \Pi' (0) \sum_{F = F^+, F^0} \left( g_{FV}^Z \right)^2 - c_w^2 - s_w^2 \frac{g_{FV}^Z Q_F e - Q^2 e^2}{c_w s_w},
$$

$$
c_w \equiv \cos \theta_W, \quad s_w \equiv \sin \theta_W.
$$

(3.42)
where $g_{ZV}^2 \equiv (g_{ZL}^2 + g_{ZR}^2)/2$ and $Q_F$ are the vector coupling to $Z$ and the electric charge of $F$, respectively. $\Pi(p^2)$ is the vacuum polarization function, which is induced by the one-loop fermion with vector-type coupling. Numerically we find that in both cases of $F_l (\eta_{F_l} = +1)$ and $F_h (\eta_{F_h} = -1)$ the sum of the right-hand side in (3.42) vanishes accurately. Hence there are no sizable corrections of the $S$ parameter from dark fermions.

Secondly, as a stabilization mechanism for the branes, one can introduce some dynamical model à la Goldberger–Wise [87]. In such a case the phase transition of the radion field may alter the thermal history of the universe drastically [88]. Here we have supposed that the critical temperature of the radion phase transition, $T_\phi$, is much higher than the freeze-out temperature of the dark fermions, e.g., $T_\phi \gg T_f \sim 100 \text{ GeV}$.

4. Direct detection

In this section, we analyze the elastic scattering of the dark fermion ($F^0$) off a nucleus [40,41,89] and examine the constraint coming from direct-detection experiments [70,71]. The dominant and sub-dominant processes are shown in Fig. 10. The dominant process of the $F^0$–nucleus scattering turns out to be $Z$ boson exchange, though the $Z$–$F^0$ coupling is very small. The $Z_R^{(1)}$–$F^0$ coupling is larger, but $Z_R^{(1)}$ is heavy. Subdominant are the processes of $Z_R^{(2)}$ and Higgs exchange. Contributions from other processes are negligible.

In the scattering of $F^0$ on nuclei with large mass number $A$, scalar and vector interactions dominate for the spin-independent cross section. Therefore the effective Lagrangian at low energies is given by

$$\mathcal{L}_{\text{int}} \simeq \sum_q \left\{ -\left( \frac{g_w v_q}{m_Z \cos \frac{\theta_W}{2}} V_F + \frac{g_w^2 v_q (Z_R^{(1)})}{m_Z^2} V_F (Z_R^{(1)}) \right) \bar{q} \gamma^0 q F^0 \tilde{F}^0 F^0 + \frac{y_q Y_F}{m_H^2} \bar{q} q F^0 \right\}.$$ (4.1)

To evaluate the scattering amplitude by the Higgs exchange, we need to estimate the nucleon matrix element

$$(N|\bar{q} q|N) = m_N f_{Tq}^{(N)},$$ (4.2)

where $N = p, n$. For heavy quarks ($Q = c, b, t$) one has

$$f_{TQ}^{(N)} = \frac{2}{27} \left( 1 - \sum_{q = u, d, s} f_{Tq}^{(N)} \right).$$ (4.3)

In the GHU model, quark couplings satisfy $v_q |_{\text{GHU}} \simeq v_q |_{\text{SM}}$ and

$$y_q |_{\text{GHU}} \simeq y_q |_{\text{SM}} \cos \theta_H = \frac{g_w}{2m_W} v_q \cos \theta_H.$$ (4.4)

---

Fig. 10. Dominant and subdominant processes of $F^0$–nucleus scattering.
Table 4. $F^0$ mass $m_F$ and the spin-independent cross section $\sigma_N$ of the $F^0$–nucleon scattering for $n_F = 4, 5, 6$ degenerate dark fermions.

| $z_L$ | $\theta_H$ | $m_F$ (TeV) | $\sigma_N$ (cm$^2$) |
|-------|------------|-------------|---------------------|
| 10$^3$ | 0.115      | 2.03        | 5.33 x 10$^{-44}$   |
| 5 x 10$^4$ | 0.101      | 2.36        | 3.78 x 10$^{-44}$   |
| 3 x 10$^4$ | 0.092      | 2.66        | 2.99 x 10$^{-44}$   |
| 2 x 10$^4$ | 0.085      | 2.92        | 2.53 x 10$^{-44}$   |
| 10$^4$   | 0.074      | 3.46        | 2.03 x 10$^{-44}$   |
| 10$^5$   | 0.114      | 1.75        | 3.67 x 10$^{-44}$   |
| 10$^4$   | 0.073      | 2.91        | 1.01 x 10$^{-44}$   |
| 10$^4$   | 0.113      | 1.57        | 2.96 x 10$^{-44}$   |
| 10$^4$   | 0.072      | 2.56        | 0.72 x 10$^{-44}$   |

The spin-independent cross section of the $F^0$–nucleon elastic scattering becomes

$$\sigma_0 \equiv \int_0^{4M_r^2 v^2} \frac{d\sigma}{d|q|^2} \left| \frac{d|q|^2}{d|q|^2} \right| dq$$

$$= \frac{M_r^2}{\pi} \left\{ Z(b_p + f_p) + (A - Z)(b_n + f_n) \right\}^2,$$

(4.5)

where $M_r$ is the $F^0$–nucleus reduced mass and $Z (A)$ is the atomic (mass) number of the nucleus.

The spin-independent cross section of the $F^0$–nucleon elastic scattering $\sigma_N$ can be written as

$$\sigma_N \equiv \frac{1}{A^2 M_r^2} \sigma_0,$$

(4.7)

where $m_r$ is the $F^0$–nucleon reduced mass.

The $F^0$–nucleon cross sections $\sigma_N$ are shown in Tables 4 and 5 and Fig. 11. In the numerical evaluation we have employed the values given in Ref. [41]:

$$f_{Tq}^{(p)} = 0.020, \quad f_{Tq}^{(n)} = 0.026, \quad f_{Tq}^{(p)} = 0.118,$$

$$f_{Tq}^{(p)} = 0.014, \quad f_{Tq}^{(n)} = 0.036, \quad f_{Tq}^{(n)} = 0.118.$$

(4.8)
Recent lattice simulations show smaller values for \( f_{TS}^{(N)} \) [90–99], which yield slightly smaller cross sections than those described below.

In the previous section we have seen that, when all \( n_F \) dark fermions are degenerate, there are no parameter regions that reproduce the observed value of the relic DM density. It was shown that the observed DM density can be obtained when there are \( n_F^{\text{light}} \) light dark fermions and \( n_F^{\text{heavy}} \) heavy dark fermions of opposite \( \eta_F \) in the boundary conditions. In particular, for the parameter set of \((n_F^{\text{light}}, n_F^{\text{heavy}}) = (1, 3), \) the region \( 0.04 \lesssim \Delta c_F \lesssim 0.07, z_L \lesssim 10^6 \) successfully explains the relic...
abundance, as shown in Fig. 9. The allowed band region in Fig. 9 is mapped in Fig. 11 for the spin-independent cross section for the $F^0$–nucleon elastic scattering. The purple and light purple bands there represent the regions allowed by the limit of the relic abundance of DM at the 68% CL and by twice that, respectively. It is seen that the band region from $z_L = 10^4$ to $4 \times 10^4$ is allowed by the direct-detection experiments of LUX [71] and XENON100 [70]. In the allowed region the dark fermion mass ranges from 3.1 TeV to 2.3 TeV, whereas the AB phase $\theta_H$ ranges from 0.074 to 0.097.

The mass of $Z'$ bosons (the lowest $Z_R$ boson and the first KK modes $Z^{(1)}$ and $\gamma^{(1)}$) ranges from 8 TeV to 6.5 TeV. For reference we have added, in Fig. 11, the expected limit from the 300 live-days result of the LUX experiment. The XENON 1T experiment is expected to give a limit one order of magnitude smaller than that of the LUX 300 live-days experiment in the cross section.

We remark that the $n_F = 3$ case predicts too small relic densities, as shown in Fig. 8. This implies that the dark fermions in the GHU model account for only a fraction of the dark matter of the universe, and the model is not excluded by direct-detection experiments.

5. Conclusion and discussions

In the present paper we have given a detailed analysis of DM in GHU. In the SO(5) × U(1) GHU, the observed unstable Higgs boson is realized by introducing SO(5)-spinor fermions. Spinor fermions do not directly interact with SO(5)-vector fermions that contain the SM quarks and leptons. Therefore the total spinor–fermion number is conserved and the lightest one can remain as dark matter in the current universe. Such fermions are referred to as “dark fermions”.

In Sect. 3 we have evaluated the relic density of the dark fermions. Although charged and neutral dark fermions are degenerate at tree level, charged fermions become heavier than neutral ones through loop effects so that the charged dark fermions decay into neutral ones much earlier than they cooled down at their freeze-out temperature. We found that, among the various annihilation processes of dark fermions, dominant ones are those in which a dark fermion and its antiparticle annihilate into the SM fermions mediated by the lowest KK $Z_R$ boson and the first KK photon. We have also evaluated the annihilation cross section and obtained the relic densities of the dark fermions in the current universe for various values of $n_F$ and $z_L$. The results depend sensitively on the number of dark fermions $n_F$. When all neutral dark fermions are degenerate, no solution has been found that explains the observed value of the relic density of dark matter and is consistent with the limit from direct-detection experiments. For $n_F = 3$ the relic density becomes much smaller than the bound, because twice the mass of the dark fermion is close to the mass of the $Z_R$ boson and the annihilation is enhanced by the resonance. For $n_F = 4, 5$, and 6 the relic density becomes larger than the bound.

We have considered the case in which $n_F$ dark fermions consist of $n_F^\text{light}$ lighter fermions and $n_F^\text{heavy}$ heavier fermions. They are mixed with each other through the bulk mass terms that can be introduced when lighter and heavier fermions have opposite signs of $\eta_F$ in the boundary conditions under reflections at the TeV and Planck branes. When the mass difference of these fermions is sufficiently large (more than $\mathcal{O}(100 \text{ GeV})$), heavier ones decay quickly to lighter ones and the effective number of species of the dark fermions can be reduced from $n_F$ to $n_F^\text{light}$. Accordingly, the relic density reduces to $n_F^\text{light} / n_F$ of that in the degenerate case. For $n_F = 4$ it is found that one can obtain the relic density consistent with the experimental bound for $10^4 \lesssim z_L \lesssim 10^6$, $0.04 \lesssim \Delta c_F \lesssim 0.07$ when $\left(n_F^\text{light}, n_F^\text{heavy}\right) = (1, 3)$. In the cases of $\left(n_F^\text{light}, n_F^\text{heavy}\right) = (2, 2), (3, 1)$ and of $n_F = 5$ and $n_F = 6$, no solution has been found. We comment that there are no sizable corrections to the $S$ parameter from the dark-fermion loops.
In Sect. 4, we calculated the scattering cross section of the dark fermions with nucleons. The dark fermions have very small Higgs–Yukawa and $Z$-boson couplings, both of which are suppressed by powers of $\sin \theta_H$. We evaluated the spin-independent cross sections and compared them with the experimental bound obtained in the recent experiments of WIMP direct detection [70,71]. Combined with the constraint from the relic density, we showed that the region $10^4 \lesssim z_L \lesssim 4 \times 10^4$ for $(n_L^{\text{light}}, n_L^{\text{heavy}}) = (1, 3)$ is viable. The corresponding mass of the dark matter candidate (dark fermions) ranges from 3.1 TeV to 2.3 TeV, whereas the AB phase $\theta_H$ ranges from 0.074 to 0.097.

The mass of $Z'$ bosons ranges from 8 TeV to 6.5 TeV.

The $n_F = 4$ model with one light and three heavy dark fermions with opposite boundary conditions is consistent with the current direct-detection experiments. Such dark fermions should be detected in direct-detection experiments in the near future. For $n_F = 3$, our model cannot explain the current DM density. In this case the current DM density should be accounted for by dark matter generated by another mechanism, such as axion DM [57] and dynamical dark matter [64,65]. In this case DM in the GHU model may or may not be detected, depending on the property of the dominant dark matter components.

The gauge-Higgs unification scenario is viable and promising. The $SO(5) \times U(1)$ GHU predicts new $Z'$ bosons in the 6.5–8 TeV region and deviation of the self-couplings of the Higgs boson from the SM, which can be explored and checked at the upgraded LHC and ILC experiments. We stress again that the model naturally contains the dark matter candidate (dark fermions) in the mass range 2.3–3.1 TeV. The mass and cross section of the dark fermions are within the reach of ongoing and future experiments, and the allowed parameter region of this model can be explored with future collider experiments [18]. Pinning down its mass fixes the value of $\theta_H$, which further yields more predictions of GHU in collider experiments.

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Appendix A. $SO(5)$ generators and base functions

SO(5) generators in the spinorial representation are defined as

$$T^a_L = \frac{1}{2} \left( \sigma^a \right), \quad T^a_R = \frac{1}{2} \left( \sigma^a \right), \quad \hat{T}^a = \frac{1}{2 \sqrt{2}} \left( -i \sigma^a \right), \quad \hat{T}^4 = \frac{1}{2 \sqrt{2}} \left( I \right),$$

and $\text{Tr}[T^\alpha, T^\beta] = 8 \delta^{\alpha \beta}$ holds.

The mode functions for the KK towers are expressed in terms of Bessel functions. For gauge fields we define

$$C(z; \lambda) = \frac{\pi}{2} \lambda z F_{1,0}(\lambda z, \lambda z_L), \quad C'(z; \lambda) = \frac{\pi}{2} \lambda^2 z F_{0,0}(\lambda z, \lambda z_L),$$

$$S(z; \lambda) = -\frac{\pi}{2} \lambda z F_{1,1}(\lambda z, \lambda z_L), \quad S'(z; \lambda) = -\frac{\pi}{2} \lambda^2 z F_{0,1}(\lambda z, \lambda z_L).$$
\begin{equation}
\hat{S}(z; \lambda) = \frac{C(1; \lambda)}{S(1; \lambda)} S(z; \lambda),
\end{equation}

\begin{equation}
F_{\alpha, \beta}(u, v) = J_\alpha(u)Y_\beta(v) - Y_\alpha(u)J_\beta(v).
\end{equation}

These functions satisfy
\begin{align*}
C(z_L; \lambda) &= z_L, \quad C'(z_L; \lambda) = 0, \quad S(z_L; \lambda) = 0, \quad S'(z_L; \lambda) = \lambda, \\
CS' - SC' &= \lambda z.
\end{align*}

For fermions with a bulk mass parameter \( c \) we define
\begin{align*}
\begin{pmatrix}
C_L \\
S_L
\end{pmatrix}(z; \lambda, c) &= \pm \frac{\pi}{2} \sqrt{zz_L} F_{c+\frac{1}{2}, c+\frac{1}{2}} (\lambda z, \lambda z_L), \\
\begin{pmatrix}
C_R \\
S_R
\end{pmatrix}(z; \lambda, c) &= \mp \frac{\pi}{2} \sqrt{zz_L} F_{c-\frac{1}{2}, c+\frac{1}{2}} (\lambda z, \lambda z_L).
\end{align*}

They satisfy
\begin{align*}
D_+ (c) \begin{pmatrix}
C_L \\
S_L
\end{pmatrix} &= \lambda \begin{pmatrix}
S_R \\
C_R
\end{pmatrix}, \\
D_- (c) \begin{pmatrix}
C_R \\
S_R
\end{pmatrix} &= \lambda \begin{pmatrix}
S_L \\
C_L
\end{pmatrix}, \\
D_\pm (c) &= \pm \frac{d}{dz} + \frac{c}{z},
\end{align*}

and
\begin{equation}
C_L C_R - S_L S_R = 1.
\end{equation}

**Appendix B. Wave functions of dark fermions**

The dark fermion \( \Phi_{Fi} \) is introduced in the spinorial representation of SO(5). With the charge assignment of \( Q_E = T^3_L + T^3_R + Q_X \) and \( Q_X = \frac{1}{2} \), \( \Phi_{Fi}(x, z) \) is decomposed into KK modes \( F_{i}^{(n)}(x) \) and \( F_{i}^{0(n)}(x) \) \((n = 1, 2, 3, \ldots)\) in the twisted gauge in which \( \langle A_z \rangle \) vanishes:

\begin{equation}
\psi_{Fi} = \psi_{Fi,R} + \psi_{Fi,L} = \sum_{n} \psi_{Fi}^{(n)}, \quad \psi_{Fi} = \psi_{Fi,R}^{(n)} + \psi_{Fi,L}^{(n)},
\end{equation}

\begin{equation}
\gamma^5 \begin{pmatrix}
\psi_{Fi,R} \\
\psi_{Fi,L}
\end{pmatrix} = \begin{pmatrix}
+ \psi_{Fi,R} \\
- \psi_{Fi,L}
\end{pmatrix},
\end{equation}

\begin{align*}
\psi_{Fi,R}^{(n)}(x, z) &= \sqrt{k z^2} \begin{pmatrix} F_{i,R}^{(n)}(x) \\
0 \\
0 \\
0 \end{pmatrix}, \\
\psi_{Fi,L}^{(n)}(x, z) &= \sqrt{k z^2} \begin{pmatrix} F_{i,L}^{(n)}(x) \\
0 \\
0 \\
0 \end{pmatrix}.
\end{align*}

Here the suffixes \( l \) and \( r \) refer to the two SU(2) of SO(4) = SU(2) \( _L \) \( \times \) SU(2) \( _R \) \( \subset \) SO(5).
\( \Psi_F \) in the twisted gauge satisfies a free Dirac equation. The left- and right-handed components of \( \Psi_F = z^{-2} \Psi_F \) satisfy

\[
\sigma \cdot \partial \Psi_{F,i,L} = k D_- (c) \Psi_{F,i,R}, \\
\bar{\sigma} \cdot \partial \Psi_{F,i,R} = k D_+ (c) \Psi_{F,i,L}.
\] (B2)

Let us denote the SU(2) \(_L\) (SU(2) \(_R\)) component of \( \Psi_{F,i,L} \) by \( \Psi_{F,i,L} \) (\( \Psi_{F,i,R} \)), etc. The boundary condition for \( \Psi_{F,i} \) with \( \eta_{F,i} = +1 \) in (2.5) is transformed in the twisted gauge in the conformal coordinates to

\[
\cos \frac{1}{2} \theta_H \Psi_{F,i,L}(1) - i \sin \frac{1}{2} \theta_H \Psi_{F,i,R}(1) = 0, \\
- \sin \frac{1}{2} \theta_H \Psi_{F,i,L}(1) + \cos \frac{1}{2} \theta_H \Psi_{F,i,R}(1) = 0, \\
\cos \frac{1}{2} \theta_H D_- \Psi_{F,i,L}(1) - i \sin \frac{1}{2} \theta_H D_- \Psi_{F,i,R}(1) = 0, \\
- \sin \frac{1}{2} \theta_H D_+ \Psi_{F,i,L}(1) + \cos \frac{1}{2} \theta_H D_+ \Psi_{F,i,R}(1) = 0,
\] (B3)

\[
\tilde{\Psi}_{F,i,L}(z_L) = 0, \quad D_- \tilde{\Psi}_{F,i,L}(z_L) = 0, \\
D_- \tilde{\Psi}_{F,i,R}(z_L) = 0, \quad \tilde{\Psi}_{F,i,R}(z_L) = 0.
\] (B4)

By making use of (B4), the eigenmodes can be written as

\[
\begin{pmatrix}
\tilde{\Psi}_{F,i,L}(z) \\
\tilde{\Psi}_{F,i,R}(z)
\end{pmatrix} =
\begin{pmatrix}
A_1 C_L(z; \lambda, c_{F_i}) \\
B_1 S_L(z; \lambda, c_{F_i})
\end{pmatrix},
\begin{pmatrix}
\tilde{\Psi}_{F,i,L}(z) \\
\tilde{\Psi}_{F,i,R}(z)
\end{pmatrix} =
\begin{pmatrix}
A_2 S_R(z; \lambda, c_{F_i}) \\
B_2 C_R(z; \lambda, c_{F_i})
\end{pmatrix}.
\] (B5)

Then (B3) leads to

\[
M \begin{pmatrix} A_1 \\ B_1 \end{pmatrix} = M \begin{pmatrix} A_2 \\ B_2 \end{pmatrix} = 0,
\]

\[
M = \begin{pmatrix}
\cos \frac{1}{2} \theta_H C_L(1) & -i \sin \frac{1}{2} \theta_H S_L(1) \\
-i \sin \frac{1}{2} \theta_H S_R(1) & \cos \frac{1}{2} \theta_H C_R(1)
\end{pmatrix}.
\] (B6)

where \( C_L(z) = C_L(z; \lambda, c_{F_i}), S_R(z) = S_R(z; \lambda, c_{F_i}) \), etc.

The mass spectrum \( \{ m_{F,i,n} = k \lambda_{i,n} \} \) is determined by \( \det M = 0 \), or by

\[
C_L(1; \lambda_{i,n}, c_{F_i}) C_R(1; \lambda_{i,n}, c_{F_i}) - \sin^2 \frac{\theta_H}{2} = 0.
\] (B7)

The corresponding wave functions are given by

\[
\begin{pmatrix}
\ell_i^{(n)}_{i,L}(z) \\
\ell_i^{(n)}_{i,R}(z)
\end{pmatrix} =
\begin{pmatrix}
\cos \frac{1}{2} \theta_H C_L(1) \\
\cos \frac{1}{2} \theta_H C_R(1)
\end{pmatrix},
\begin{pmatrix}
\ell_i^{(n)}_{i,L}(z) \\
\ell_i^{(n)}_{i,R}(z)
\end{pmatrix} =
\begin{pmatrix}
\ell_i^{(n)}_{i,L}(z) \\
\ell_i^{(n)}_{i,R}(z)
\end{pmatrix} =
\begin{pmatrix}
\cos \frac{1}{2} \theta_H C_L(1) \\
\cos \frac{1}{2} \theta_H C_R(1)
\end{pmatrix},
\] (B8)

with \( \lambda = \lambda_{i,n} \). The normalization factors \( r^{(n)}_i \) and \( r'^{(n)}_i \) are determined by the condition

\[
\int_1^{z_L} dz \left\{ |\ell_i^{(n)}_{i,L}|^2 + |\ell_i^{(n)}_{i,R}|^2 \right\} = \int_1^{z_L} dz \left\{ |\ell_i^{(n)}_{i,L}|^2 + |\ell_i^{(n)}_{i,R}|^2 \right\} = 1.
\] (B9)
to be

\[ r_i^{(n)} = \int_1^{z_L} dz \left[ \sin^2 \frac{1}{2} \theta_H S_L(z^2) + \cos^2 \frac{1}{2} \theta_H C_L(z^2) \right] \]

\[ = \int_1^{z_L} dz \left[ \sin^2 \frac{1}{2} \theta_H S_L(z^2) + \cos^2 \frac{1}{2} \theta_H C_L(z^2) \right], \]

\[ r_i^{(n)} = \int_1^{z_L} dz \left[ \cos^2 \frac{1}{2} \theta_H C_R(z^2) + \sin^2 \frac{1}{2} \theta_H S_R(z^2) \right] \]

\[ = \int_1^{z_L} dz \left[ \cos^2 \frac{1}{2} \theta_H C_R(z^2) + \sin^2 \frac{1}{2} \theta_H S_R(z^2) \right]. \quad (B10) \]

One comment is in order about the \( \theta_H \to 0 \) limit of the wave functions. For \( \theta_H = 0 \) the spectrum (B7) is determined by either \( C_R(1) = C_R(1; \lambda_i, 2n, c_{F_i}) = 0 \) or \( C_L(1) = C_L(1; \lambda_i, 2n, c_{F_i}) = 0 \) \( (n = 1, 2, 3, \ldots) \) where the eigenvalues have been ordered as \( 0 < \lambda_i, 1 < \lambda_i, 2 < \lambda_i, 3 < \ldots \). The case \( C_R(1) = 0 \) corresponds to excitations of the \( SU(2)_R \) doublet component, \( C_L(1) = 0 \) to excitations of the \( SU(2)_L \) doublet component. For \( C_R(1) = 0 (C_L(1) = 0), r_i^{(n)}/sin^2 \frac{1}{2} \theta_H \neq 0 (r_i^{(n)}/sin^2 \frac{1}{2} \theta_H \neq 0) \) at \( \theta_H = 0 \).

In the boundary condition for \( \Psi_{F_i} \), one could adopt \( \eta_{F_i} = -1 \) in (2.5). In the case of non-degenerate dark fermions, the heavy dark fermion multiplet satisfies this flipped boundary condition. In this case the corresponding wave functions and Kaluza–Klein masses are obtained from the above formulas by the replacement

\[ c_H \leftrightarrow is_H, \quad C_L \leftrightarrow S_L, \quad S_R \leftrightarrow C_R. \quad (B11) \]

The spectrum is determined by the same equation as in (B7). The lowest mode mostly becomes an \( SU(2)_L \) doublet for small \( \theta_H \).

**Appendix C.** Gauge and Higgs couplings of dark fermions

**C.1. Couplings to the Higgs boson**

Couplings to the Higgs boson are read from the gauge interaction

\[ \int_1^{z_L} dz \sqrt{G} \epsilon^a \Psi_F (g_A A_z + Q \bar{e} B_z) i \gamma^8 \Psi_F , \quad \sqrt{G} \epsilon^a g_A = L G_w / k^4 \epsilon^{-4}, \quad (C1) \]

where

\[ A_z(x, z) = \hat{H} + \sum_{a=1}^3 \hat{G}^a + \sum_{a=1}^3 \hat{D}^a, \]

\[ \hat{H} = \sum_{n} H^{(n)}(x) u_{H^{(n)}} T^4, \]

\[ \hat{G}^a = \sum_{n} G^{a(n)}(x) \left[ u_{G^{(n)}} T^4 + T^4 \right], \]

\[ \hat{D}^a = \sum_{n} D^{a(n)}(x) \left[ u_{D^{(n)}} T^4 - T^4 \right] + \hat{u}_{D^{(n)}} T^4, \]

\[ B_z(x, z) = \sum_{n} B^{(n)}(x) u_{B^{(n)}}(z), \quad (C2) \]
Table C1. The Higgs–Yukawa couplings $y_{F_i^{(n)}}$ in (C6) in the case of degenerate dark fermions with the parameters specified in Table 2.

| $n_F$ | $z_L$  | $y_{F_i^{(n)}}$ |
|-------|--------|-----------------|
| 3     | $10^8$ | -0.106          |
|       | $10^6$ | -0.071          |
|       | $10^5$ | -0.064          |
|       | $2 \times 10^4$ | -0.089 |
| 4     | $10^8$ | -0.082          |
|       | $10^6$ | -0.049          |
|       | $10^5$ | -0.038          |
|       | $3 \times 10^4$ | -0.034 |
|       | $10^4$ | -0.033          |
| 6     | $10^8$ | -0.060          |
|       | $10^6$ | -0.034          |
|       | $10^5$ | -0.024          |
|       | $10^4$ | -0.017          |

$G^{a(n)}$, $D^{a(n)}$, and $B^{(n)}$ are Nambu–Goldstone bosons and only the $\hat{H}$ is the tower of the physical scalar particles. Hereafter we consider only Higgs couplings. The Higgs wave functions are given by

$$u_{H^{(0)}}(z) = \sqrt{\frac{2}{k(z^2_L - 1)}} z,$$  \hspace{1cm} (C3)

for the zero-mode Higgs boson, and

$$u_{H^{(n)}(z)} = \frac{1}{\sqrt{r_{H^{(n)}}}} S'(z; \lambda_{H^{(n)}}), \quad r_{H^{(n)}} = \int_1^{z_L} \frac{kdz}{z} S'(z; \lambda_{H^{(n)}})^2,$$ \hspace{1cm} (C4)

for KK excitations ($n \geq 1$). Here $S(1; \lambda_{H^{(n)}}) = 0$ is satisfied. The building block for the $H \bar{F}^{(n)}F^{(n)}$ Yukawa coupling is given by

$$\Psi_F \gamma^\mu (g_A A^\mu + Q_X g_B B^\mu) \Psi_F = i \frac{kz^4}{2 \sqrt{2} f_j^{(n)}} \sin \frac{\theta_H}{2} \cos \frac{\theta_H}{2} S_L(1) C_L(1) \left[ F_j^{(n)} F_j^{(n)} - \bar{F}_j^{(n)} F_j^{(n)} \right],$$ \hspace{1cm} (C5)

where $C_L(z)C_R(z) - S_L(z)S_R(z) = 1$ has been used. Hence the Higgs–Yukawa coupling in the 4D Lagrangian, $\mathcal{L}_{4D} \supset y_{F_i^{(n)}} H^{(0)} \bar{F}_i^{(n)} F_i^{(n)}$, is given by

$$y_{F_i^{(n)}} = \frac{g_w}{4} \frac{1}{r_i^{(n)}} \sqrt{k L (z^2_L - 1)} \sin \frac{\theta_H}{2} \cos \frac{\theta_H}{2} S_L(1; \lambda_i, c_F) C_L(1; \lambda_i, c_F).$$ \hspace{1cm} (C6)

In Tables C1 and C2, we have summarized the Higgs–Yukawa couplings of $F$. In Table C2 the couplings in non-degenerate cases are summarized.

C.2. Couplings to vector bosons

Couplings to the vector bosons are read off from the gauge interaction in the 5D action

$$\int_0^{z_L} dz \sqrt{G} \gamma^\mu \psi \gamma^\mu (g_A A^\mu + Q_X g_B B^\mu) \psi = \frac{g_w \sqrt{L}}{k z^4} S_\mu,$$ \hspace{1cm} (C7)
where \( A_\mu(x, z) \) and \( B_\mu(x, z) \) decompose to the Kaluza–Klein towers

\[
A_\mu(x, z) = \hat{W}_\mu^- + \hat{W}_\mu^+ + \hat{Z}_\mu^A + \hat{A}_\mu^\gamma + \hat{W}_{R\mu}^- + \hat{W}_{R\mu}^+ + \hat{Z}_{R\mu}^A + \hat{A}_{R\mu}^\gamma, \\
B_\mu(x, z) = \hat{Z}_\mu^B + \hat{A}_\mu^\gamma + \hat{Z}_{R\mu}^B.
\]

The gauge couplings in (C7) consist of

\[
\tilde{\Psi}_F \gamma^\mu \left( g_A \hat{\Psi}_\mu \right) \Psi_F, \quad \text{for } V = W, W_R, A^4, \\
\tilde{\Psi}_F \gamma^\mu \left( g_A \hat{\Psi}_\mu^A + Q_{XGB} \hat{\Psi}_\mu^B \right) \Psi_F, \quad \text{for } V = A^\gamma, Z, Z_R.
\]

Each tower is decomposed to KK modes. For \( W, W_R, \) and \( A^4 \) bosons,

\[
\hat{W}_\mu^\pm = \sum_n W_{\mu}^{\pm(n)}(x) \left\{ h_{W_{\mu}}^L \frac{T^1 \pm iT^2}{\sqrt{2}} + h_{W_{\mu}}^R \frac{T^1 \pm iT^2}{\sqrt{2}} + \hat{h}_{W_{\mu}} \frac{T^1 \pm iT^2}{\sqrt{2}} \right\}, \\
\hat{W}_{R\mu}^\pm = \sum_n W_{R\mu}^{\pm(n)}(x) \left\{ h_{W_{R\mu}}^L \frac{T^1 \pm iT^2}{\sqrt{2}} + h_{W_{R\mu}}^R \frac{T^1 \pm iT^2}{\sqrt{2}} + \hat{h}_{W_{R\mu}} \frac{T^1 \pm iT^2}{\sqrt{2}} \right\}, \\
\hat{A}_{\mu}^4 = \sum_n A_{\mu}^{4(n)}(x) h_{A^4} \hat{T}^4,
\]

where \( \hat{W}_\mu^\pm = (\hat{W}_\mu^1 \pm i \hat{W}_\mu^2) / \sqrt{2} \), etc., whereas, for \( A^\gamma, Z, \) and \( Z_R \) bosons,

\[
\left( \hat{A}_\mu^\gamma, \hat{A}_\mu^\gamma \right) = \sum_n A_{\mu}^{\gamma(n)}(x) \left( h_{\gamma R}^L T^3_L + h_{\gamma R}^R T^3_R + h_{\gamma}^L T^3_L + h_{\gamma}^R T^3_R \right), \\
\left( \hat{Z}_\mu^A, \hat{Z}_\mu^A \right) = \sum_n Z_{\mu}^{A(n)}(x) \left( h_{Z_R}^L T^3_L + h_{Z_R}^R T^3_R + h_{Z}^L T^3_L + h_{Z}^R T^3_R \right), \\
\left( \hat{Z}_{R\mu}^A, \hat{Z}_{R\mu}^A \right) = \sum_n Z_{R\mu}^{A(n)}(x) \left( h_{Z_R}^L T^3_L + h_{Z_R}^R T^3_R + h_{Z}^L T^3_L + h_{Z}^R T^3_R \right).
\]

Here \( n = 0, 1, 2, \ldots (1, 2, \ldots) \) for \( A^\gamma, W_\mu, \) and \( Z_\mu \) (\( W_{R\mu}, Z_{R\mu}, \) and \( A^4 \)). \( A_{\mu}^{\gamma(0)}(x), W_{\mu}^{(0)}, \) and \( Z_\mu^{(0)} \) correspond to the photon, \( W, \) and \( Z \) bosons, respectively.

\[\text{Table C2.}\] The Higgs–Yukawa couplings \( y_{F_{i}^{(l)}}^{l} \) of the light dark fermion in (C6) in the case of non-degenerate \( (n^l_F, n^h_F) = (1, 3) \) dark fermions with the parameters specified in Table 3.

| \( \Delta c_F \) | \( z_L \) | \( y_{F_{i}^{(l)}}^{l} \) |
|---|---|---|
| 0.04 | \( 10^b \) | -0.042 |
| \( 10^3 \) | -0.033 |
| \( 3 \times 10^4 \) | -0.029 |
| \( 10^4 \) | -0.028 |
| 0.06 | \( 10^b \) | -0.038 |
| \( 10^5 \) | -0.030 |
| \( 3 \times 10^4 \) | -0.027 |
| \( 10^4 \) | -0.026 |
C.2.1. Couplings to $\gamma^{(n)}$, $Z^{(n)}$, $Z_R^{(n)}$, and $A^4$. Here we summarize the dark fermion couplings to the neutral vector bosons. It would be useful to collect the building blocks for the couplings. For the KK fermions $\Psi_F^{(n)}$ and $\Psi_F^{(m)}$, we have

\[
\Psi_F^{(n)T} \gamma^\mu \Psi_F^{(m)} = \frac{k^4 z^n}{2} e^{(n)F_L} e^{(m)F_R} \left[ F_L^{(n)} \gamma^\mu F_L^{(m)} - F_L^{(0)} \gamma^\mu F_L^{(0)} \right] + (F_L, f_{IL} \to F_R, f_{IR}),
\]

\[
\Psi_F^{(n)T} \gamma^\mu \Psi_F^{(m)} = \frac{k^4 z^n}{2} f^{(n)F_L} f^{(m)F_R} \left[ F_L^{(n)} \gamma^\mu F_L^{(m)} - F_L^{(0)} \gamma^\mu F_L^{(0)} \right] + (F_L, f_{IL} \to F_R, f_{IR}),
\]

\[
\Psi_F^{(n)T} \gamma^\mu \Psi_F^{(m)} = k^4 \frac{z^n}{2} \left[ f^{(n)F_L} f^{(m)F_R} - f^{(n)F_R} f^{(m)F_L} \right] \left[ F_L^{(n)} \gamma^\mu F_L^{(m)} - F_L^{(0)} \gamma^\mu F_L^{(0)} \right] + (F_L, f_{IL} \to F_R, f_{IR}),
\]

\[
\Psi_F^{(n)T} \gamma^\mu \Psi_F^{(m)} = k^4 \left[ f^{(n)F_L} f^{(m)F_R} + f^{(n)F_R} f^{(m)F_L} \right] \left[ F_L^{(n)} \gamma^\mu F_L^{(m)} + F_L^{(0)} \gamma^\mu F_L^{(0)} \right] + (F_L, f_{IL} \to F_R, f_{IR}, f_{IR}).
\]

(C12)

In the following we summarize the couplings in the case of $n = m = 1$.

Electromagnetic photon $\gamma = \gamma^{(0)}$. For the photon $A_{\mu}^{(0)}$, the wave functions are given by

\[
h_{\gamma^{(0)}}^{L} = h_{\gamma^{(0)}}^{R} = \frac{1}{\sqrt{1 + s^2 \phi}} s_{\phi}, \quad h_{\gamma^{(0)}}^{B} = \frac{1}{\sqrt{1 + s^2 \phi}} c_{\phi},
\]

(C13)

where $c_{\phi}$ and $s_{\phi}$, given by

\[
c_{\phi} \equiv \cos \phi = \frac{g_A}{\sqrt{g_A^2 + g_B^2}}, \quad s_{\phi} \equiv \sin \phi = \frac{g_B}{\sqrt{g_A^2 + g_B^2}},
\]

(C14)

parameterize the mixing of $A_M$ and $B_M$, and are related to the Weinberg angle $\theta_W$ by $\sin \phi = \tan \theta_W$. The couplings between dark fermions and the photon can be read from

\[
\int_{1}^{z_L} dz \sqrt{G} e^{\mu} \Psi_F^{(n)} [g_{AA_{\mu}^{(0)}} + Q_X g_{BB_{\mu}^{(0)}}] \gamma^\mu \Psi_F^{(m)} = e A_{\mu}^{(0)}(x) \int_{1}^{z_L} dz \left[ f^{(n)F_L} f^{(m)F_R} + f^{(n)F_R} f^{(m)F_L} \right] \times \left[ \left( Q_X + \frac{1}{2} \right) F_L^{(n)} + \left( Q_X - \frac{1}{2} \right) F_L^{(0)} \right] + (L \to R)
\]

\[
= e A_{\mu}^{(0)}(x) \delta_{n,m} \left( Q_X + \frac{1}{2} \right) F_L^{(n)} + \left( Q_X - \frac{1}{2} \right) F_L^{(0)} \right) \right) \},
\]

where the orthogonality condition (B9) has been used. $F^+ (F^0)$ has electric charge $Q_X + \frac{1}{2}$ ($Q_X - \frac{1}{2}$). The Kaluza–Klein level for fermions is preserved.

KK photons. The wave functions for the KK photons $\gamma^{(n)} (n \geq 1)$ are given by

\[
\left( \begin{array}{c}
\frac{h_{\gamma^{(n)}}^{L}}{h_{\gamma^{(n)}}^{R}} \\
\frac{h_{\gamma^{(n)}}^{B}}
\end{array} \right) = \frac{1}{\sqrt{1 + s^2 \phi}} \frac{1}{\sqrt{1 + s^2 \phi}} \left( s_{\phi} \right) C(z), \quad r_{\gamma^{(n)}} = \int_{1}^{z_L} dz C(z)^2.
\]

(C16)
Table C3. The mass and left- and right-handed couplings to $F^+$ in (C17) in units of electromagnetic coupling $e$ of the first KK photon in the case of degenerate dark fermions with the parameters specified in Table 2.

| $n_F$ | $z_L$ | $m_{\gamma^{(1)}}$ [TeV] | $g_{F^+ L}^{\gamma^{(1)}}$ | $g_{F^+ R}^{\gamma^{(1)}}$ |
|------|------|------------------|-----------------|-----------------|
| 3    | $10^8$ | 2.42             | 0.19            | 4.16            |
|      | $10^6$ | 4.26             | 0.28            | 3.61            |
|      | $10^5$ | 5.92             | 0.38            | 3.31            |
|      | $2 \times 10^4$ | 7.55             | 0.52            | 3.09            |
| 4    | $10^8$ | 2.46             | 0.06            | 4.15            |
|      | $10^6$ | 4.32             | 0.11            | 3.59            |
|      | $10^5$ | 6.00             | 0.15            | 3.28            |
|      | $3 \times 10^4$ | 7.19             | 0.17            | 3.10            |
|      | $10^4$ | 8.52             | 0.21            | 2.93            |
| 6    | $10^8$ | 2.50             | $-0.06$         | 4.14            |
|      | $10^6$ | 4.40             | $-0.05$         | 3.58            |
|      | $10^5$ | 6.12             | $-0.04$         | 3.26            |
|      | $10^4$ | 8.68             | $-0.03$         | 2.90            |

Table C4. The left- and right-handed couplings to the light $F^+_1$ in (C17) in units of electromagnetic coupling $e$ of the first KK photon in the case of non-degenerate ($n_F^{\mathrm{light}}, n_F^{\mathrm{heavy}} = (1, 3)$) dark fermions with the parameters specified in Table 3.

| $\Delta c_F$ | $z_L$ | $g_{F^+_L}^{\gamma^{(1)}}$ | $g_{F^+_R}^{\gamma^{(1)}}$ |
|--------------|------|------------------|-----------------|
| 0.04         | $10^6$ | 0.03             | 3.58            |
|              | $10^5$ | 0.08             | 3.27            |
|              | $3 \times 10^4$ | 0.11             | 3.09            |
|              | $10^4$ | 0.16             | 2.92            |
| 0.06         | $10^6$ | 0.01             | 3.58            |
|              | $10^5$ | 0.04             | 3.26            |
|              | $3 \times 10^4$ | 0.08             | 3.09            |
|              | $10^4$ | 0.13             | 2.92            |

where $C(z) = C(z; \lambda_{\gamma^{(n)}})$ and $\lambda_{\gamma^{(n)}}$ satisfy $C'(1; \lambda_{\gamma^{(n)}}) = 0$. Hence the couplings are given by

$$
\sum_{c=\pm, 0} \gamma^{(n)}_{\mu} \left[ g_{F^+ L}^{\gamma^{(n)}} \tilde{F}_{L \gamma}^{c \mu} F_{L}^{c} + g_{F^+ R}^{\gamma^{(n)}} \tilde{F}_{R \gamma}^{c \mu} F_{R}^{c} \right]
= \gamma^{(n)}_{\mu} (x) \left( (Q x + \frac{1}{2}) \tilde{F}_{L \gamma}^{+ \mu} F_{L}^{+} + (Q x - \frac{1}{2}) \tilde{F}_{L \gamma}^{0 \mu} F_{L}^{0} \right)
\times \frac{e \sqrt{L}}{\sqrt{F_{\gamma^{(n)}}}} \int_{z_L}^{z_1} dz \left[ |f_{L L}|^2 + |f_{L R}|^2 \right] + (L \to R).
$$

(C17)

Note that the couplings are left–right asymmetric, i.e., $g_{F^+ \gamma^{(n)}}^L \neq g_{F^+ \gamma^{(n)}}^R$ for $n \geq 1$. In Tables C3 and C4, the $\gamma^{(1)} F^+ F^-$ couplings are tabulated.
The wave functions of the $Z$ tower are given by

$$
\begin{pmatrix}
    h^L_{Z(n)} \\
    h^R_{Z(n)} \\
    \hat{h}_{Z(n)}
\end{pmatrix}
= \frac{1}{\sqrt{1 + s^2_\phi}} \frac{1}{\sqrt{T_{Z(n)}}}
\begin{pmatrix}
    \frac{c^2_\phi + (1 + s^2_\phi) \cos \theta_H}{\sqrt{2}} C(z) \\
    \frac{c^2_\phi - (1 + s^2_\phi) \cos \theta_H}{\sqrt{2}} C(z) \\
    - (1 + s^2_\phi) \sin \theta_H \hat{S}(z)
\end{pmatrix},
$$

where $C(z) \equiv C(z; \lambda_{Z(n)})$, $\hat{S}(z) \equiv \hat{S}(z; \lambda_{Z(n)})$, and $\lambda_{Z(n)}$ satisfy

$$2S(z; \lambda_{Z(n)})C'(z; \lambda_{Z(n)}) + (1 + s^2_\phi)\lambda_{Z(n)} \sin^2 \theta_H = 0. \tag{C19}$$

The smallest positive root $\lambda_{Z(0)}$ is related to the $Z$-boson mass by $m_Z = k \cdot \lambda_{Z(0)}$. In terms of these the couplings of $F$ to the $Z(n)$ boson are given by

$$\mathcal{L}_{4D} \supset Z^{(n)}_\mu \sum_{c=\mp} \left[ g_{F^c} F^c_{L} \bar{F}^c_{L} \gamma^\mu F^c_{L} + g_{F^c} F^c_{R} \bar{F}^c_{R} \gamma^\mu F^c_{R} \right]$$

$$= \frac{g_{V}}{\sqrt{2} \cos \theta_W} Z^{(n)}_\mu \sum_{c=\mp} \bar{F}^c_{L} \gamma^\mu F^c_{L} \int_1^{z_L} dz \left[ \frac{1}{2} \left[ |f_{IL}|^2 + |f_{LR}|^2 \right] C(z) \right]$$

$$+ \cos \theta_H C(z) \left[ |f_{IL}|^2 - |f_{LR}|^2 \right] - i \sin \theta_H \hat{S}(z) \left[ f_{IL}^* f_{IL} - f_{LR}^* f_{LR} \right]$$

$$- (Q_X + I^{(c)}_3) \sin^2 \theta_W \cdot 2C(z) \left[ |f_{IL}|^2 + |f_{LR}|^2 \right] + (L \rightarrow R), \tag{C20}$$

where $I^{(c)}_3 = \frac{1}{2} (\mp \frac{i}{2})$ for $c = \pm (0)$. We note that if the $F$ obey the boundary condition (b.c.) $\eta_F = +1$ the $Z^{(n)}$ coupling to a fermion $F^0$ with $Q_{EM} = Q_X + I^{(0)}_3 = 0$ is suppressed by $\sin^2 (\theta_H / 2)$, because $f_{IL} \propto \sin (\theta_H / 2)$. We have summarized the $ZF \bar{F}$ couplings in Tables C5, C6, and C7, and the $Z^{(1)} F \bar{F}$ couplings in Tables C8 and C9.

The wave functions of the $Z_R$ tower are given by

$$\begin{pmatrix}
    h^L_{Z_R(n)} \\
    h^R_{Z_R(n)} \\
    \hat{h}_{Z_R(n)}
\end{pmatrix}
= \frac{1}{\sqrt{1 + (1 + 2t^2_\phi) \cos^2 \theta_H}} \frac{1}{\sqrt{T_{Z_R(n)}}}
\begin{pmatrix}
    \frac{1 - \cos \theta_H}{\sqrt{2}} C(z) \\
    \frac{-1 - \cos \theta_H}{\sqrt{2}} C(z) \\
    \sqrt{2} t^2_\phi \cos \theta_H \hat{S}(z)
\end{pmatrix},
$$

where $C(z) = C(z; \lambda_{Z_R(n)})$ and $\lambda_{Z_R(n)}$ satisfy $C(1; \lambda_{Z_R(n)}) = 0$. Hence the $Z_R^{(n)} \bar{F} F$ couplings are given by

$$\mathcal{L}_{4D} \supset Z_{R(n)}^\mu \sum_{c=\mp} \left[ g_{F^c} F^c_{L} \bar{F}^c_{L} \gamma^\mu F^c_{L} + g_{F^c} F^c_{R} \bar{F}^c_{R} \gamma^\mu F^c_{R} \right]$$
\[
Z_{R}^{(n)} = \frac{g_{w} \sqrt{L}}{\sqrt{2} \left[ 1 + \frac{\cos \theta_{W}}{\cos 2\theta_{W}} \sqrt{F_{R}^{(n)}} \right]} \sum_{c=+,-,0} \bar{F}_{L}^{c} Y^{c} F_{L}^{c} \int_{z_{L}}^{z_{L}} dz C(z)
\]
\[\times \left[ I_{3}^{(c)} \left\{ - \cos \theta_{W} \left[ |f_{L}|^2 + |f_{R}|^2 \right] + \left[ |f_{L}|^2 - |f_{R}|^2 \right] \right\} + 2 Q_{X} \frac{\sin^2 \theta_{W}}{\cos 2\theta_{W}} \cos \theta_{W} \left[ |f_{L}|^2 + |f_{R}|^2 \right] \right\} + (L \rightarrow R). \tag{C22}
\]

**Table C5.** The left- and right-handed couplings in units of \(g_{w}\) of \(F\) to the \(Z\) boson in (C20) with b.c. \(\eta_{F} = +1\) in the case of degenerate dark fermions with the parameters specified in Table 2.

| \(n_{F}\) | \(z_{L}\) | \(g_{F+L}^{Z}\) | \(g_{F-R}^{Z}\) | \(g_{F+L}^{Z} \times 10^{4}\) | \(g_{F-R}^{Z} \times 10^{4}\) |
|---|---|---|---|---|---|
| 3 | \(10^{8}\) | -0.260 | -0.242 | -40.1 | -227.3 |
| | \(10^{6}\) | -0.261 | -0.257 | -21.8 | -69.6 |
| | \(10^{5}\) | -0.262 | -0.260 | -19.7 | -42.7 |
| | \(2 \times 10^{4}\) | -0.259 | -0.258 | -41.3 | -58.4 |
| 4 | \(10^{8}\) | -0.261 | -0.244 | -25.2 | -204.9 |
| | \(10^{6}\) | -0.263 | -0.258 | -11.4 | -55.9 |
| | \(10^{5}\) | -0.263 | -0.261 | -7.6 | -27.7 |
| | \(3 \times 10^{4}\) | -0.263 | -0.262 | -6.7 | -19.7 |
| | \(10^{4}\) | -0.263 | -0.262 | -6.5 | -15.4 |
| 6 | \(10^{8}\) | -0.263 | -0.246 | -14.2 | -186.0 |
| | \(10^{6}\) | -0.263 | -0.259 | -5.8 | -47.7 |
| | \(10^{5}\) | -0.263 | -0.262 | -3.4 | -21.9 |
| | \(10^{4}\) | -0.264 | -0.263 | -2.1 | -9.8 |

**Table C6.** The left- and right-handed couplings in units of \(g_{w}\) of \(F\) to the \(Z\) boson in (C20) with b.c. \(\eta_{F} = -1\) in the case of degenerate dark fermions with the parameters specified in Table 2.

| \(n_{F}\) | \(z_{L}\) | \(g_{F+L}^{Z}\) | \(g_{F-R}^{Z}\) | \(g_{F+L}^{Z} \times 10^{4}\) | \(g_{F-R}^{Z} \times 10^{4}\) |
|---|---|---|---|---|---|
| 4 | \(10^{8}\) | 0.304 | 0.287 | -0.569 | -0.552 |
| | \(10^{6}\) | 0.306 | 0.301 | -0.569 | -0.565 |
| | \(10^{5}\) | 0.306 | 0.305 | -0.570 | -0.569 |

**Table C7.** The left- and right-handed couplings in units of \(g_{w}\) of \(F\) to the \(Z\) boson in (C20) with b.c. \(\eta_{F} = +1\) in the case of non-degenerate \((n_{F}^{light}, n_{F}^{heavy}) = (1, 3)\) dark fermions with the parameters specified in Table 3.

| \(\Delta c_{F}\) | \(z_{L}\) | \(g_{F+L}^{Z}\) | \(g_{F-R}^{Z}\) | \(g_{F+L}^{Z} \times 10^{4}\) | \(g_{F-R}^{Z} \times 10^{4}\) |
|---|---|---|---|---|---|
| 0.04 | \(10^{6}\) | -0.263 | -0.259 | -8.4 | -52.2 |
| | \(10^{5}\) | -0.263 | -0.261 | -5.9 | -25.5 |
| | \(3 \times 10^{4}\) | -0.263 | -0.262 | -5.1 | -17.8 |
| | \(10^{4}\) | -0.263 | -0.262 | -5.0 | -13.5 |
| 0.06 | \(10^{6}\) | -0.263 | -0.259 | -7.2 | -50.8 |
| | \(10^{5}\) | -0.263 | -0.261 | -5.1 | -24.6 |
| | \(3 \times 10^{4}\) | -0.263 | -0.262 | -4.5 | -17.0 |
| | \(10^{4}\) | -0.263 | -0.263 | -4.4 | -12.8 |
We note that, unlike the case of the $F$ to the first KK $Z$ boson $Z^{(1)}$ in (C20) with b.c. $\eta_F = +1$ in the case of degenerate dark fermions with the parameters specified in Table 2.

| $n_F$ | $z_L$ | $m_{Z^{(1)}}$ [TeV] | $g_{F+L}^{Z^{(1)}}$ | $g_{F+R}^{Z^{(1)}}$ | $g_{F-R}^{Z^{(1)}}$ | $g_{F-R}^{Z^{(1)}}$ |
|-------|-------|----------------------|---------------------|---------------------|---------------------|---------------------|
| 3     | $10^6$| 2.42                 | $-0.02$             | $-1.07$             | $-0.04$             | $-0.08$             |
|       | $10^6$| 4.25                 | $-0.06$             | $-0.95$             | $-0.02$             | $-0.02$             |
|       | $10^6$| 5.92                 | $-0.09$             | $-0.87$             | $-0.01$             | $-0.01$             |
| 2 × $10^4$ | 7.54  | $-0.12$             | $-0.81$             | $-0.02$             | $-0.02$             | $-0.00$             |
| 4     | $10^6$| 2.45                 | 0.00                | $-1.06$             | $-0.02$             | $-0.08$             |
|       | $10^6$| 4.32                 | $-0.02$             | $-0.94$             | $-0.01$             | $-0.02$             |
|       | $10^6$| 6.00                 | $-0.03$             | $-0.86$             | $-0.01$             | $-0.01$             |
|       | $10^4$| 8.52                 | $-0.05$             | $-0.77$             | $-0.00$             | $-0.00$             |
| 6     | $10^6$| 2.50                 | 0.02                | $-1.06$             | $-0.01$             | $-0.07$             |
|       | $10^6$| 4.40                 | 0.02                | $-0.94$             | $-0.00$             | $-0.01$             |
|       | $10^5$| 6.13                 | 0.01                | $-0.86$             | $-0.00$             | $-0.01$             |
|       | $10^4$| 8.68                 | 0.01                | $-0.77$             | $-0.00$             | $-0.00$             |

In Tables C10, C11, and C12, we have summarized the $Z_R \bar{F} \bar{F}$ couplings.

$A^4$ boson. Diagonal $\bar{F}^{(n)} F^{(n)} A^4$ couplings vanish, because one finds, for the left-hand couplings,

$$
\bar{\psi}_{FL}^{(n)} \gamma^\mu \tilde{T}^4 \psi_{FL}^{(n)} = k z \frac{4}{2\sqrt{2}} \tilde{z}_{L}^{(n)} \gamma^\mu \tilde{f}_{R}^{(n)} \left( f_{R}^{(n)} f_{R}^{(n)} \right) \left( \begin{array}{c} 1 \\ 2 \end{array} \right) \left( \begin{array}{c} f_{R}^{(n)} \\ f_{R}^{(n)} \end{array} \right),
$$

with

$$
\propto \left\{ S_L(1, \lambda_m) C_L(z, \lambda_m) C_L(1, \lambda_n) S_L(z, \lambda_n) - (\lambda_m \leftrightarrow \lambda_n) \right\},
$$

and a similar relation for right-handed couplings.

C.2.2. Couplings to $W$ and $W_R$ bosons. The building blocks for $W \bar{F} \bar{F}$ and $W_R \bar{F} \bar{F}$ couplings are

$$
\bar{\psi}_{FL}^{(n)} W^+ \gamma^\mu \psi_{FL}^{(n)} = \frac{k z_4}{2} \tilde{f}_{L}^{(n)} \tilde{f}_{L}^{(n)} \left[ \tilde{F}_{L}^{+(n)} \gamma^\mu \tilde{F}_{L}^{0(m)} - \tilde{F}_{L}^{+(n)} \gamma^\mu \tilde{F}_{L}^{0(m)} \right] + (F_L, f_{L} \rightarrow F_R, f_{L}),
$$

$$
\bar{\psi}_{FL}^{(n)} W^+ \gamma^\mu \psi_{FL}^{(n)} = \frac{k z_4}{2} \tilde{f}_{R}^{(n)} \tilde{f}_{R}^{(n)} \left[ \tilde{F}_{L}^{+(n)} \gamma^\mu \tilde{F}_{L}^{0(m)} - \tilde{F}_{L}^{+(n)} \gamma^\mu \tilde{F}_{L}^{0(m)} \right] + (F_L, f_{L} \rightarrow F_R, f_{L}).
$$
\[ \tilde{\Psi}_F^{(n)} T^\dagger \gamma^\mu \tilde{\Psi}_F^{(m)} = \frac{k z_L^4}{2i} \left[ f_{(n)L} f_{(m)L} - f_{(n)L} f_{(m)L} \right] \left[ i \tilde{F}_L^{+(n)} \gamma^\mu F_L^{0(m)} - \tilde{F}_L^{+(m)} \gamma^\mu F_L^{0(n)} \right] + (F_L, f_{IL}, f_{JR}, f_{FR}). \] (C24)

In the following, we summarize the \( W^{-F^{0(1)}F^{+(1)}} \) and \( W^{-F^{0(1)}F^{+(1)}} \) \( (m = n = 1) \) couplings.

**Table C10.** The left- and right-handed couplings in units of \( g_w \) of \( F \) to \( Z_R^{(1)} \) in (C22) with b.c. \( \eta_F = +1 \) in the case of degenerate dark fermions with the parameters specified in Table 2.

| \( n_F \) | \( z_L \) | \( m_{Z_R^{(1)}} \) [TeV] | \( g_{Z_R^{(1)}F^+L} \) | \( g_{Z_R^{(1)}F^+R} \) | \( g_{Z_R^{(1)}F^0L} \) | \( g_{Z_R^{(1)}F^0R} \) |
|---------|---------|----------------|----------------|----------------|----------------|----------------|
| 3       | \( 10^8 \) | 2.34          | -0.09          | -1.05          | 0.25           | 2.55           |
|         | \( 10^6 \) | 4.06          | -0.13          | -0.90          | 0.34           | 2.23           |
|         | \( 10^5 \) | 5.59          | -0.16          | -0.82          | 0.42           | 2.06           |
|         | \( 2 \times 10^4 \) | 7.05          | -0.20          | -0.77          | 0.51           | 1.93           |
| 4       | \( 10^8 \) | 2.37          | -0.07          | -1.05          | 0.18           | 2.54           |
|         | \( 10^6 \) | 4.12          | -0.10          | -0.89          | 0.24           | 2.22           |
|         | \( 10^5 \) | 5.70          | -0.11          | -0.82          | 0.29           | 2.04           |
|         | \( 3 \times 10^4 \) | 6.74          | -0.12          | -0.78          | 0.32           | 1.94           |
|         | \( 10^4 \) | 7.92          | -0.14          | -0.73          | 0.35           | 1.84           |
| 6       | \( 10^8 \) | 2.42          | -0.04          | -1.05          | 0.12           | 2.54           |
|         | \( 10^6 \) | 4.20          | -0.06          | -0.89          | 0.16           | 2.21           |
|         | \( 10^5 \) | 5.78          | -0.07          | -0.81          | 0.18           | 2.03           |
|         | \( 10^4 \) | 8.11          | -0.08          | -0.73          | 0.21           | 1.83           |

**Table C11.** The left- and right-handed couplings in units of \( g_w \) of \( F \) to \( Z_R^{(1)} \) in (C22) with b.c. \( \eta_F = -1 \) in the case of degenerate dark fermions with the parameters specified in Table 2.

| \( n_F \) | \( z_L \) | \( g_{Z_R^{(1)}F^+L} \) | \( g_{Z_R^{(1)}F^+R} \) | \( g_{Z_R^{(1)}F^0L} \) | \( g_{Z_R^{(1)}F^0R} \) |
|---------|---------|----------------|----------------|----------------|----------------|
| 4       | \( 10^8 \) | 0.05           | 0.80           | 0.07           | 0.69           |
|         | \( 10^6 \) | 0.07           | 0.68           | 0.08           | 0.65           |
|         | \( 10^5 \) | 0.08           | 0.62           | 0.09           | 0.61           |
|         | \( 3 \times 10^4 \) | 0.09           | 0.58           | 0.10           | 0.58           |
|         | \( 10^4 \) | 0.11           | 0.55           | 0.11           | 0.55           |

**Table C12.** The left- and right-handed couplings in units of \( g_w \) of \( F \) to \( Z_R^{(1)} \) in (C22) with b.c. \( \eta_F = +1 \) in the case of non-degenerate \( (\eta_F^{light}, \eta_F^{heavy}) = (1, 3) \) dark fermions with the parameters specified in Table 3.

| \( \Delta c_F \) | \( z_L \) | \( g_{Z_R^{(1)}F^+L} \) | \( g_{Z_R^{(1)}F^+R} \) | \( g_{Z_R^{(1)}F^0L} \) | \( g_{Z_R^{(1)}F^0R} \) |
|-------|---------|----------------|----------------|----------------|----------------|
| 0.04  | \( 10^6 \) | -0.08          | -0.89          | 0.20           | 2.22           |
|       | \( 10^5 \) | -0.10          | -0.81          | 0.25           | 2.03           |
|       | \( 3 \times 10^4 \) | -0.11          | -0.77          | 0.28           | 1.93           |
|       | \( 10^4 \) | -0.13          | -0.73          | 0.32           | 1.84           |
| 0.06  | \( 10^6 \) | -0.07          | -0.89          | 0.18           | 2.21           |
|       | \( 10^5 \) | -0.09          | -0.81          | 0.23           | 2.03           |
|       | \( 3 \times 10^4 \) | -0.10          | -0.77          | 0.26           | 1.93           |
|       | \( 10^4 \) | -0.12          | -0.73          | 0.30           | 1.83           |
\( W \) boson. The wave functions of the \( W \) tower are

\[
\begin{pmatrix}
  h_W^{L(n)} \\
  h_W^{R(n)} \\
  h_W^{R(n)}
\end{pmatrix} = \frac{1}{\sqrt{r_W^{(n)}}} \begin{pmatrix}
  1+\cos^2 \theta_C C(z) \\
  \frac{1}{\sqrt{2}} C(z) \\
  -\sin \theta_C \hat{S}(z)
\end{pmatrix},
\]

where

\[
r_W^{(n)} = \int_1^{z_L} \frac{dz}{k_z} \left\{ (1+\cos^2 \theta_H)C(z)^2 + \sin^2 \theta_H \hat{S}(z)^2 \right\},
\] (C25)

where \( C(z) = C(z; \lambda_{W^{(n)}}), \hat{S}(z) = \hat{S}(z; \lambda_{W^{(n)}}) \) and \( \lambda_{W^{(n)}} \) satisfies

\[
2S(z; \lambda_{W^{(n)}})C(1; \lambda_{W^{(n)}}) + \lambda_{W^{(n)}} \sin^2 \theta_H = 0.
\] (C26)

\( W^{(0)} \) is the \( W \) boson whose mass is given by \( m_W = k \cdot \lambda_{W^{(0)}} \). The couplings

\[
\mathcal{L}_{4D} \supset W_{\mu}^{- (n)} \left[ g_{FL}^{W(n)} f_L^{0 \gamma^\mu} F_L^+ + g_{FR}^{W(n)} f_R^{0 \gamma^\mu} F_R^+ \right] + \text{(h.c.)}
\]

are given by

\[
g_{FL}^{W(n)} = \frac{g_w}{\sqrt{2}} \frac{\sqrt{L}}{\sqrt{r_W^{(n)}}} \int_1^{z_L} dz \left\{ C(z)(1+\cos \theta_H) |f_{1L}|^2 + (1-\cos \theta_H) |f_{1R}|^2 \right\}
-
\sin \theta_H \hat{S}(z)i \left\{ f_{1L}^* f_{1R} - f_{1R}^* f_{1L} \right\},
\] (C27)

and \( g_{FR}^{W(n)} \) is obtained by replacements \( f_{1L} \rightarrow f_{1R} \). We note that for the dark fermion obeying b.c. \( \eta_F = +1 \) these couplings are suppressed by \( \sin^2(\theta_H/2) \), because \( f_{1L} \propto \sin(\theta_H/2) \). The \( W F \) and \( W^{(1)} F \) couplings are summarized in Tables C13 and C14.

\( W_R \) boson. The wave functions of the \( W_R \) tower are given by

\[
\begin{pmatrix}
  h_W^{L(n)} \\
  h_W^{R(n)} \\
  h_W^{R(n)}
\end{pmatrix} = \frac{1}{\sqrt{1+\cos^2 \theta_H}} \frac{1}{\sqrt{r_W^{(n)}}} \begin{pmatrix}
  -\cos \theta_H + \frac{1}{\sqrt{2}} \\
  \frac{1}{\sqrt{2}} \\
  -\frac{1-\cos \theta_H}{\sqrt{2}}
\end{pmatrix} C(z),
\]

where \( C(z) = C(z; \lambda_{W^{(n)}}) \) and \( \lambda_{W^{(n)}} \) is defined by \( C(1; \lambda_{W^{(n)}}) = 0 \). In an analogous way to the \( W \) boson, we obtain the couplings

\[
\mathcal{L}_{4D} \supset W_{\mu}^{R (n)} \left[ g_{FL}^{W(n)} f_L^{0 \gamma^\mu} F_L^+ + g_{FR}^{W(n)} f_R^{0 \gamma^\mu} F_R^+ \right] + \text{(h.c.)},
\]

\[
g_{FL}^{W(n)} = \frac{g_w}{\sqrt{2}} \frac{\sqrt{L}}{\sqrt{r_W^{(n)}}} \sqrt{1+\cos^2 \theta_H}
\times \int_1^{z_L} dz C(z) \left\{ (1-\cos \theta_H) |f_{1L}|^2 + (1-\cos \theta_H) |f_{1R}|^2 \right\},
\] (C29)

and \( g_{FR}^{W(n)} \) is obtained by replacing \( f_{1L} \) with \( f_{1R} \). The \( W^{(1)} F \) couplings are summarized in Tables C13 and C14.
Table C13. The left- and right-handed couplings $F_0 F^+ V^-$ (in units of $g_w/\sqrt{2}$) of $F$ to a charged vector boson $V^-$ ($V = W$, $W^{(i)}$, and $W_R^{(i)}$) in (C27) with b.c. $\eta_F = +1$ in the case of degenerate dark fermions with the parameters specified in Table 2.

| $n_F$ | $z_L$ | $g_{s_{FL}}^W \times 10^3$ | $g_{s_{FR}}^W \times 10^3$ | $m_{W^{(i)}} [TeV]$ | $g_{s_{FL}}^{W^{(i)}} \times 10^3$ | $g_{s_{FR}}^{W^{(i)}} \times 10^3$ | $m_{W_R^{(i)}} [TeV]$ | $w_{s_{FL}}^W$ | $w_{s_{FR}}^W$ |
|-------|-------|--------------------------|--------------------------|-----------------|--------------------------|--------------------------|-----------------|----------------|----------------|
| 3     | $10^8$ | 7.0                      | 39.8                     | 2.42            | 61.9                      | 136                      | 2.34            | -0.41          | -3.11          |
| 10$^6$| 4.4    | 35.9                     | 40.2                     | 132.2           | 2.37                      | -0.30                    | -3.10           |
| 2$\times10^4$ | 2.0    | 9.7                      | 15.0                     | 26.8            | 4.12                      | -0.41                    | -2.65           |
| 10$^6$| 0.4    | 1.7                      | 23.4                     | 127.3           | 2.42                      | -0.19                    | -3.10           |
| 10$^4$| 1.0    | 8.4                      | 8.0                      | 25.9            | 4.20                      | -0.26                    | -2.64           |
| 10$^4$| 0.4    | 1.7                      | 2.3                      | 3.6             | 8.07                      | -0.36                    | -2.16           |

Table C14. The left- and right-handed couplings $F_0 F^+ V^-$ (in units of $g_w/\sqrt{2}$) of $F$ to a charged vector boson $V^-$ ($V = W$, $W^{(i)}$, and $W_R^{(i)}$) in (C27) with b.c. $\eta_F = -1$ in the case of degenerate dark fermions with the parameters specified in Table 2.

| $n_F$ | $z_L$ | $w_{s_{FL}}^W$ | $w_{s_{FR}}^W$ | $w_{s_{FL}}^{W^{(i)}}$ | $w_{s_{FR}}^{W^{(i)}}$ | $w_{s_{FL}}^{W_R^{(i)}}$ | $w_{s_{FR}}^{W_R^{(i)}}$ |
|-------|-------|----------------|----------------|------------------------|------------------------|--------------------------|--------------------------|
| 4     | $10^8$| 0.997         | 0.966         | 0.04                   | -0.019                 | 0.099                    |                         |
| 10$^6$| 0.998 | 0.991         | 0.10          | 3.59                   | -0.008                 | 0.020                    |                         |
| 10$^4$| 0.999 | 0.998         | 0.21          | 2.93                   | -0.004                 | 0.003                    |                         |

Appendix D. $V W^+ W^-$ vector-boson couplings

In terms of the wave functions for the $W$ boson and other neutral vector bosons $V = Z, Z_R, A^\gamma, A^\tilde{\gamma}$, one can read the $V W^+ W^-$ couplings from the relation

$$g_A \int_1^{z_L} \frac{dz}{k z} \partial_\mu \hat{W}_\rho \hat{W}_\sigma (x, z) = \sum_{n, r, s} g_{V^{(n)} W^{(r)} W^{-(s)}} \left(\partial_\mu \hat{V}_\nu^{(n)}\right) W^{(r)} W^{-(s)} (x). \tag{D1}$$

Hereafter we summarize the formulas for $V^{(n)} W^+ W^-$ couplings. Numerically computed values of the $V W^+ W^-$ ($V = Z, Z^{(1)}, Z_R^{(1)}$, and $\gamma^{(1)}$) couplings are summarized in Table D1. These couplings depend sensitively on $z_L$ and $\theta_H$, but very weakly on $n_F$, thanks to the universality relations in the model [17,18].

$\gamma^{(n)} W^+ W^-$ coupling. The $\gamma^{(n)} W^+ W^-$ coupling is given by

$$g_{\gamma^{(n)} W W} = g_w \sqrt{L} \int_1^{z_L} \frac{dz}{k z} \left( h_{\gamma^{(n)}}^L \left( \hat{h}_W^L \right)^2 + \left( \hat{h}_W^R \right)^2 \right) + h_{\gamma^{(n)}}^R \left( \left( \hat{h}_W^R \right)^2 + \left( \hat{h}_W^L \right)^2 \right). \tag{D2}$$

In particular, for the photon $\gamma = \gamma^{(0)}$ we obtain

$$g_{\gamma W W} = e \quad \text{(electromagnetic coupling)}. \tag{D3}$$

and for KK excited photons ($n \neq 0$) we have

$$g_{\gamma^{(n)} W W} = e \sqrt{L} \int_1^{z_L} \frac{dz}{k z} \sqrt{\gamma^{(n)}} \left[ \left( h_{\gamma^{(n)}}^L \right)^2 + \left( h_{\gamma^{(n)}}^R \right)^2 + \left( \hat{h}_W \right)^2 \right]. \tag{D4}$$
Table D1. Triple vector-boson couplings $V W^+ W^-$ with $V = Z, Z^{(1)}, Z_R^{(1)}$ (D5), (D6) in units of $g_w$ and $\gamma^{(1)} W^+ W^-$ in units of electric charge $e$.

| $n_F$ | $z_L$ | $g_{WWZ}$ | $g_{WWZ^{(1)}} \times 10^2$ | $g_{WWZ_R^{(1)}} \times 10^2$ | $g_{WW(\gamma)} \times 10^2$ |
|------|------|-----------|-----------------|-----------------|-----------------|
| 4    | 10^8 | 0.811     | 1.506           | 0.391           | -0.417          |
|      | 10^6 | 0.861     | 0.459           | 0.114           | -0.115          |
|      | 10^5 | 0.870     | 0.225           | 0.055           | -0.054          |
|      | 10^4 | 0.874     | 0.105           | 0.025           | -0.024          |

$Z^{(n)} W^+ W^-$ coupling.

$$g_{Z^{(n)}WW} = g_w \sqrt{L} \int_{z_L}^{z_R} \frac{dz}{k_z} \left[ h_{Z^{(n)}}^{L} \left[ (h_{W}^{L})^2 + (\hat{h}_{W}^{L})^2 \right] + h_{Z^{(n)}}^{R} \left[ (h_{W}^{R})^2 + (\hat{h}_{W}^{R})^2 \right] \right] + \hat{h}_{Z^{(n)}} (h_{W}^{L} + h_{W}^{R}) \hat{h}_{W}.$$  \hspace{2cm} (D5)

$Z_R^{(n)} W^+ W^-$ coupling.

$$g_{Z_R^{(n)}WW} = g_w \sqrt{L} \int_{z_L}^{z_R} \frac{dz}{k_z} \left[ h_{Z_R^{(n)}}^{L} \left[ (h_{W}^{L})^2 + (\hat{h}_{W}^{L})^2 \right] + h_{Z_R^{(n)}}^{R} \left[ (h_{W}^{R})^2 + (\hat{h}_{W}^{R})^2 \right] \right]. \hspace{2cm} (D6)$$

We note that this coupling is suppressed by $\sin^2 \theta_H$ because

$$\hat{h}_{Z_R^{(n)}}, \hat{h}_W \propto \sin^2(\theta_H/2), \hspace{0.5cm} \hat{h}_{W} \propto \sin \theta_H.$$

$A^{(n)} W^+ W^-$ coupling. $A^{(n)} W^+(r) W^-(s)$ coupling vanishes when $r = s$. In particular, for $r = s = 0$ we obtain

$$g_{A^{(n)}WW} = 0. \hspace{2cm} (D7)$$

$W_R^{(n)} W^- Z$ coupling.

$$g_{W_R^{(n)}WZ} = g_w \sqrt{L} \int_{z_L}^{z_R} \frac{dz}{k_z} \left[ h_{W_R^{(n)}}^{L} h_{W}^{L} h_{Z}^{L} + h_{W_R^{(n)}}^{R} h_{W}^{R} h_{Z}^{R} + \frac{1}{2} (h_{W_R^{(n)}}^{L} + h_{W_R^{(n)}}^{R}) \hat{h}_{W}^{L} \hat{h}_{Z}^{L} \right]. \hspace{2cm} (D8)$$

This coupling is suppressed by $\sin^2(\theta_H/2)$ because

$$\hat{h}_{W_R^{(n)}}, \hat{h}_{W_R^{(n)}}, \hat{h}_{W_R^{(n)}}, \hat{h}_{W_R^{(n)}}, \hat{h}_{W_R^{(n)}} \propto \sin^2(\theta_H/2).$$

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