A ONE-RELATOR GROUP WITH LONG LOWER CENTRAL SERIES

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Abstract. A one-relator group with lower central series of length $\omega^2$ is constructed. This answers a problem of G. Baumslag.

1. Introduction

For a group $G$, the lower central series are defined inductively as follows:

$$\gamma_1(G) = G,$$
$$\gamma_{\alpha+1}(G) = [\gamma_\alpha(G), G]$$

and $\gamma_\tau(G) = \bigcap_{\alpha<\tau} \gamma_\alpha(G)$ for a limit ordinal $\tau$. The smallest ordinal $\alpha$, such that $\gamma_\alpha(G) = \gamma_{\alpha+1}(G)$ is called the length of the lower central series of $G$. First examples of finitely presented groups with the lower central series of length greater than $\omega$ were constructed by J. Levine [3]. T. Cochran and K. Orr constructed examples of 3-manifold groups with lower central series of length greater than $\omega$ in [2].

G. Baumslag asked the following question ([1], Problem 10): Is the lower central series of a 1-relator group of length at most $\omega^n$ for some finite ordinal $n$?

Consider the following one-relator group

$$G = \langle a, b \mid a^{b^2} = aa^{3b} \rangle$$

This group is a semidirect product $F_2 \rtimes \mathbb{Z}$, where a generator of the cyclic group acts on $F_2 = F(x, y)$ as the automorphism

$$x \mapsto y,$$
$$y \mapsto xy^3.$$ 

The main result is the following

**Theorem 1.** The group $G$ has the lower central series of length $\omega^2$.

The group $G$ provides an example of a one-relator group with length greater than $\omega n$ for any finite $n$, thus answering the above problem of G. Baumslag.

2. Proof of theorem 1

Let $\Gamma$ be a group and $M$ a $\mathbb{Z}[\Gamma]$-module. $M$ is called residually nilpotent if

$$\cap_k M \Delta^k(\Gamma) = 0,$$

where $\Delta^k(\Gamma)$ is the $k$th power of the augmentation ideal $\Delta(\Gamma) := \ker(\mathbb{Z}[\Gamma] \to \mathbb{Z})$.

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Lemma 2. Let $M$ be a finitely generated free abelian group of rank $m \geq 1$ and an infinite cyclic group with generator $t$ acts on $M$ as a matrix $A \in GL_m(\mathbb{Z})$. Suppose that the product of any collection of eigenvalues of $A - Id$ is not $\pm 1$. Then the module $M$ is residually nilpotent $\mathbb{Z}[[t]]$-module.

Proof. Let $\Delta$ be the augmentation ideal of $\mathbb{Z}[[t]]$. Observe that, for any $n \geq 1$, the power $\Delta^n$ is the ideal generated by $(1-t)^n$. Consider the submodule $N := M\Delta^n = \bigcap_{k \geq 1} M\Delta^k$ and assume that $N \neq 0$. The Krull intersection theorem implies that $NI = N$, hence $N(1-t) = N$ and the restriction of $(1-t)$ on $N$ defines a bijection on $N$. Hence $\det(A - Id) = \pm 1 = \beta_1 \ldots \beta_k$, where $(\beta_1, \ldots, \beta_k)$ is some collection of eigenvalues of $A - Id$. This gives a needed contradiction. Hence $N = 0$. \hfill $\Box$

Now we prove theorem 1. As usual, if $x, y, a_1, \ldots, a_{k+1}$ are elements of a group $G$ we set $[x, y] = x^{-1}y^{-1}xy$, $x^y = y^{-1}xy$ and define
\[ [a_1, a_2, \ldots, a_{k+1}] = [[a_1, \ldots, a_k], a_{k+1}] \quad (k > 1). \]
For $n \geq 1$, define $T_n$ as a normal closure in $G$ of all elements of the form $[x_1, \ldots, x_n]$, where all $x_i$'s are either $a$ or $a^b$. We have
\[ H_1 = \langle a \rangle^G = F(a, a^b), \quad H_2 = \langle [a, a^b] \rangle^G, \quad H_3 = \langle [a, a^b, a], [a, a^b, a^b] \rangle^G \]

Therefore, the element $[a, a^b]$ is a generalized 3-torsion element. On the other hand, taking the commutator with $a$ of the both sides of the relation, we get
\[ [a^b, a] = [a, a^b]a^3b^{-1} \]

Therefore $[a, a^b]$ is a generalized 2-torsion element. This implies that $[a, a^b] \in \gamma_2(G)$. Since $H$ is residually nilpotent, we conclude that $\gamma_3(H) = \langle [a, a^b] \rangle^G$.
Define the group $T$ is residually nilpotent ± a product of elements of the form integers of the form $\Delta$ is the augmentation ideal of their conjugates in the sense of quadratic irrationality also is ± $U_l, s$ ω $G/\gamma$ statement for $n$ is trivial in $T$.

Consider the quotient the conjugate quadratic irrationality to this point that the group $G$ has length greater than $\omega$. Indeed, since $H_2(H) = \mathbb{Z}/2$ and $H_2(G) = 0$, we get

$$H_2(H) = H_2(G/\omega(G)) \simeq \omega(G)/\omega+1(G) \simeq \mathbb{Z}/2$$

and

$$[a, a^b] \in \omega(G) \setminus \omega+1(G), \quad [a, a^b]^2 \in \omega+1(G).$$

Now suppose that the statement (1) holds for a given $n$. Here we prove this statement for $n + 1$.

We have an inclusion

$$\gamma_{n+1}(G) \subseteq [H_{n+1}, G, \ldots, G], \quad k \geq 1$$

Define the group $T := G/H_{n+2}$. By the assumption of induction, the term $\gamma_{n+1}(T)$ lies in the quotient $H_{n+1}/H_{n+2}$, the term $\gamma_{n+k}(T)$ is a subgroup of the quotient $[H_{n+1}, G, \ldots, G]H_{n+2}/H_{n+2}$ and can be presented as a product of conjugates of the elements

$$[a, b, \ldots, b].H_{n+2}, \quad \alpha \in H_{n+1}.$$ 

Consider the quotient $H_{n+1}/H_{n+2}$ as a $\mathbb{Z}[(b)]$-module. If the module $H_{n+1}/H_{n+2}$ is residually nilpotent $\mathbb{Z}[(b)]$-module, i.e.

$$0 = \cap_k (H_{n+1}/H_{n+2}) \Delta^k,$$

where $\Delta$ is the augmentation ideal of $\mathbb{Z}[(b)]$, then the intersection

$$\cap_k ([H_{n+1}, G, \ldots, G]H_{n+2})/H_{n+2}$$

is trivial in $T$ and the statement (1) will follow for $n + 1$. Denote $M := H_1/H_2$.

Since $H_1$ is a free group and $H_1 = \gamma_1(H_1)$, $i \geq 1$, the quotient $H_{n+1}/H_{n+2}$ is the $n + 1$-st Lie power $L^{n+1}(M)$ by Magnus-Witt theorem (see [3, 8]). Lie powers are subgroups of tensor powers $L^{n+1}(M) \subseteq \otimes^{n+1}(M)$ and therefore,

$$\cap_k L^{n+1}(M) \Delta^k \subseteq \cap_k \otimes^{n+1}(M) \Delta^k.$$ 

Here we consider the tensor powers of $M$ as $\mathbb{Z}[(b)]$-modules with the diagonal action of $\langle b \rangle$.

For given matrices $A$ and $B$, the eigenvalues of their tensor product $A \otimes B$ consist of all possible products of the eigenvalues of $A$ and $B$. The eigenvalues of the matrix $U$ are $\alpha_1 = \frac{3 + \sqrt{13}}{2}, \alpha_2 = \frac{3 - \sqrt{13}}{2}$. Hence, the eigenvalues of the tensor power $U^{\otimes n+1}$ are $\pm \alpha_1^k$ and $\pm \alpha_2^k$ for some $l, s \geq 1$. Suppose that one can present $\pm 1$ as a product of elements of the form $\pm \alpha_1^k + 1$ and $\pm \alpha_2^k + 1$. Then the product of their conjugates in the sense of quadratic irrationality also is $\pm 1$. Since $\alpha_1^k + 1$ is the conjugate quadratic irrationality to $\alpha_1^k$, one can present $\pm 1$ as a product of integers of the form $(\alpha_1^l - 1)(\alpha_2^l - 1)$ and $(\alpha_1^s - 1)(\alpha_2^s - 1)$

for different $l, s \geq 1$. Since $(\alpha_1 - 1)(\alpha_2 - 1) = -3$ divides $(\alpha_1^l - 1)(\alpha_2^l - 1)$ for any $l \geq 1$, the absolute values of the terms $(\alpha_1^l - 1)(\alpha_2^l - 1)$ are greater than 1. For an odd $s$, the product $(\alpha_1^s + 1)(\alpha_2^s + 1)$ divides $(\alpha_1 + 1)(\alpha_2 + 1) = 3$. For an even $s$,
the terms \((\alpha_1^2 + 1)\) and \((\alpha_2^2 + 1)\) are greater than 1. We get a contradiction with assumption that one can present \(\pm 1\) as a product of elements of the form \(\pm \alpha_1 \pm 1\) and \(\pm \alpha_2 \pm 1\).

We conclude that, for any \(n\), the matrix \(U^{\otimes n+1} - \text{Id}\) satisfies the conditions of lemma 2 and therefore the \((n + 1)\)st tensor power \(M^{\otimes n+1}\) is a residually nilpotent \(\mathbb{Z}[[b]]\)-module. This implies that the Lie power \(L^{n+1}(M)\) also is a residually nilpotent \(\mathbb{Z}[[b]]\)-module and the inductive step is done,

\[
\gamma_{(n+1)\omega}(G) \subseteq H_{n+2},
\]

proving \((1)\) for all \(n\).

Since \(H_1\) is a free group, it is residually nilpotent, and we get

\[
\gamma_{\omega^2}(G) = \cap_n \gamma_{n\omega}(G) \subseteq \cap_n H_{n+1} = \cap_n \gamma_{n+1}(H_1) = 1.
\]

Since \([a^k, a] \in \gamma_{\omega}(G)\), we have

\[
1 \neq [[a^k, a], a, [a^k, a], \ldots, [a^k, a]] \in \gamma_{(k+1)\omega}(G), \; k \geq 1.
\]

This element clearly is non-trivial, since it is a basic commutator in the basic elements \(a, a^b\) of the free group \(H_1 = F(a, a^b)\). Hence, the lower central series length of \(G\) is \(\omega^2\) and theorem 1 is proved.

**Remark 3.** Observe that one can easily use the proof of theorem 1 to prove the following more general result. Let \(F_n\) be a free group and an element \(t\) acts on \(F_n\) as an automorphism \(\phi : F_n \to F_n\). Define \(\Phi = F_n \rtimes \langle t \rangle\). Suppose that any tensor power \((F_{ab})^\otimes k\) \((k \geq 1)\) is a residually nilpotent \(\mathbb{Z}[[t]]\)-module. Then \(\gamma_{\omega^2}(\Phi) = 1\).

In particular, if \(\Phi\) is not residually nilpotent, then the lower central series length of \(\Phi\) is exactly \(\omega^2\).

We add the next remark here due to its applications in the theory of localizations \([5]\).

**Remark 4.** The 2\(^{th}\) lower central series term of \(G\) can be described as follows

\[
\gamma_{2\omega}(G) = \langle [a, a^b, a], [a, a^b, a^b] \rangle^G
\]

We proved before that there is an inclusion \(\gamma_{2\omega}(G) \subseteq H_3\). To see the converse inclusion, observe that \(a\) and \(a^b\) are generalized 3-torsion elements, hence

\[
[a, a^b, a]^{3^l}, [a, a^b, a^b]^{3^l} \subseteq \gamma_{\omega+l}(G), \; l \geq 1.
\]

However, the same is true for powers of 2, due to \((1)\). Therefore, \(H_3 \subseteq \gamma_{2\omega}(G)\) and description \((5)\) follows.

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