COMPARISON BOUNDS FOR PERTURBED SCHRÖDINGER OPERATORS WITH SINGLE-WELL POTENTIALS

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Abstract. We prove bounds on the sum of the differences between the eigenvalues of a Schrödinger operator and its perturbation. Our results hold for operators in one dimension with single-well potentials. We rely on a variation of the well-known factorisation method. In the Pöschl-Teller and Coulomb cases we are able to use the explicit factorisations to establish improved bounds.

1. Introduction

Consider the self-adjoint Schrödinger operator $H_V = H_0 - V = -\Delta - V$ on $L^2(\mathbb{R}^d)$. When $V$ has suitable decay and a non-trivial positive part then it may have finite or infinitely many negative eigenvalues $\{E_k(H_V)\}_{k=1}$. The celebrated Lieb-Thirring inequality [LT76] states that

$$\sum_k |E_k(H_V)|^\gamma \leq L_{\gamma,d} \int_{\mathbb{R}^d} V(x)^{\gamma + \frac{d}{2}} \, dx \tag{1.1}$$

with $\gamma \geq 1/2$ if $d = 1$, $\gamma > 0$ if $d = 2$ and $\gamma \geq 0$ for $d \geq 3$, where we use $V_+ = (|V| + V)/2$. The endpoint cases for $d = 1$ and $d \geq 3$ were settled in [Wei96] and [Cwi77, Lie76, Roz72] respectively.

Substantial progress has been made in determining the sharp constants in (1.1), most notably in [HLT02] and [LW00] (see [Sch22] or [Fra20] for a review). In the latter, Laptev and Weidl established that $L_{\gamma,d} = L_{\gamma,d}^{cl}$ for all $\gamma \geq 3/2$ and $d \geq 1$, where

$$L_{\gamma,d}^{cl} = \frac{\Gamma(\gamma + 1)}{(4\pi)^{d/2}\Gamma(\gamma + 1 + \frac{d}{2})}$$

is the semiclassical constant. Their argument relied on proving the Lieb-Thirring bound in the case $\gamma = 3/2$, $d = 1$ for operator-valued potentials with optimal constant $L_{1,3/2}^{cl} = 3/16$. The scalar result followed by a lifting argument to higher dimensions and to $\gamma \geq 3/2$ by a standard lifting argument of Aizenman and Lieb [AL78].

Shortly after, a simple proof of their results was found by Benguria and Loss in [BL00]. Their argument relied on a factorisation of one-dimensional Schrödinger operators according to so-called creation and annihilation operators (which we detail below).

This factorisation scheme, sometimes known as the commutation method, has been used by many authors. In particular, to determine the eigenvalues of $H_V$ for explicit
V (see examples in Section 4 and in [FLW22]). Its first known application to trace inequalities was by Schmincke in [Sch78] where the author derived the lower bound
\[
\sum_k \sqrt{|E_k(H_V)|} \geq \frac{1}{4} \int_{\mathbb{R}^d} V(x) \, dx.
\]
Following its use by Benguria and Loss for Lieb-Thirring inequalities it has been employed to obtain analogous bounds for the Robin boundary case on the half-line by Exner, Laptev and Usman in [ELU14]. This was improved by Schimmer in [Sch19] by using an alteration of this idea, known as the double commutation method.

Most recently, in [Lap21] the author used a variation of the factorisation method to derive the Hardy-Lieb-Thirring inequality, with a conjectured sharp constant. That is, the inequalities found by Ekholm and Frank in [EF08] which are improvements of the standard bounds (1.1) for operators \( H_V \) above where \( H_0 = -\Delta - V_0 \) on \( L^2(\mathbb{R}^d) \) with Dirichlet boundary conditions (see also [EF06]).

The results we present here can be seen as a generalisation of Laptev’s work. We use the factorisation method to derive eigenvalue comparison inequalities between \( H_0 = -\Delta - V_0 \) and the perturbed operator \( H_V = H_0 - V \) in one dimension. Results for perturbations of this type already exist. Most notably, a result by Frank, Simon and Weidl was produced in [FSW08] in which they derived
\[
\sum_k |E_k(H_V)|^\gamma \leq \beta(V_0)^{\gamma+\frac{d}{2}} L_{\gamma,d} \int_{\mathbb{R}^d} V(x)^{(\gamma+\frac{d}{2})} \, dx
\]
for \( \gamma, d \) and \( L_{\gamma,d} \) as in (1.1), where \( H_0 \) has positive spectrum and a ‘regular ground state’. The latter includes those with potentials of the form \( V_0 = \Delta u/u \) where \( u \) is some strictly positive function, in which case one has \( \beta(V_0) = (\sup(u)/\inf(u))^2 \).

Our main result is an upper bound on the sum of the gap between the first eigenvalue of the Schrödinger operator \( H_0 \) with a single-well potential \( V_0 \) and the eigenvalues of its perturbation by some potential \( V \). By single-well we mean a function that is non-decreasing up to some point, and non-increasing thereafter.

**Theorem 1.** Let \( V_0 \) be positive and single-well with \((1 + |x|)V_0 \in L^1(\mathbb{R}) \) and let \( V_+ \in L^2(\mathbb{R}) \). Then the following holds
\[
\sum_k \left( \sqrt{|E_k(H_V)|} - \sqrt{|E_k(H_0)|} \right) \leq \frac{3}{16} \int_{\mathbb{R}} V(x)^2 \, dx. \tag{1.2}
\]

The inequality is optimal in the sense that as \( V_0 \) is taken to zero what remains is precisely the sharp Lieb-Thirring inequality. Though the bound differs to that in [FSW08], it can be seen as a similar estimate for perturbed operators where the correction appears on the left-hand side and for which we have explicit sharp constant. We shall see that a lifting in \( \gamma \) is possible, with extra cost.

As a consequence of our proof we find the same result on the half-line for the Schrödinger operators \( H_{0,\sigma} = -\frac{d^2}{dx^2} - V_0 \) and \( H_{V,\sigma} = H_0 - V \) on \( L^2(\mathbb{R}_+) \) with Robin boundary conditions \( \varphi'(0) - \sigma \varphi(0) = 0, \ \sigma \geq 0 \).
Theorem 2. Let $V_0$ be positive and single-well with $V_0 \in L^1(\mathbb{R}_+)$ and let $V_+ \in L^2(\mathbb{R}_+)$. Then the following holds

$$\sum_k \left( \sqrt{|E_k(H_{V,\sigma})|} - \sqrt{|E_k(H_0)|} \right)^3 \leq \frac{3}{8} \int_{\mathbb{R}_+} V(r)^2_+ dr. \quad (1.3)$$

The inequality we obtain through our factorisation method has a more general form than theorems 1 and 2. In both cases, it’s the log-concavity of the ground state of $H_0$ and $H_{0,\sigma}$ for single-well $V_0$ that enables us to derive these results. Where we have an explicit factorisation for $H_0$ we are able to show something stronger and improve (1.2) and (1.3). The following two results show that this is true for the Pöschl-Teller and Coulomb potentials, both of which are single well. Whether this improvement holds for single-well $V_0$ or another class of potentials is left as an interesting open question.

Theorem 3. Let $V_0(x) = \nu(\nu + 1) \sech^2(x)$ with $\nu > 0$ and let $V_+ \in L^2(\mathbb{R})$. Then the following holds

$$\sum_{k=1}^{[\nu]} \left( \sqrt{|E_k(H_{V})|} - \sqrt{|E_k(H_0)|} \right)^3 + \sum_{k=[\nu]+1}^\infty |E_k(H_{V})|^{3/2} \leq \frac{3}{16} \int_{\mathbb{R}} V(x)^2_+ dx$$

where $E_k(H_0) = -((\nu - k + 1)^2$ are the eigenvalues of $H_0$. Furthermore, under the same conditions on $V_0$,

$$\sum_{k=1}^{[\nu]} (|E_k(H_{V})| - 2|E_k(H_0)|)^\gamma_+ \leq 2\sqrt{2}L^cl_{\gamma,1} \int_{\mathbb{R}} V(x)^{\gamma + 1/2}_+ dx \quad (1.4)$$

holds for all $\gamma \geq 3/2$.

We see two improvements in this case. The first inequality contains all discrete spectral data from $H_V$ and in the second we see that an Aizenman-Lieb lifting argument works. The same result holds for the Coulomb potential, generalising the Hardy-Lieb-Thirring bound and the proof of Laptev in [Lap21]. We identify $\sigma = \infty$ for the Robin operators with Dirichlet boundary conditions.

Theorem 4. Let $V_0(x) = \kappa/r - \nu(\nu + 1)/r^2$ with $\nu \geq -1/2$, $\kappa > 0$ and let $V_+ \in L^2(\mathbb{R}_+)$. Then the following holds

$$\sum_{k=1} \left( \sqrt{|E_k(H_{V,\sigma})|} - \sqrt{|E_k(H_0,\sigma)|} \right)^3 \leq \frac{3}{8} \int_{\mathbb{R}_+} V(r)^2_+ dr$$

where $E_k(H_{0,\sigma}) = -\kappa^2/4(\nu + k)^2$ are the eigenvalues of $H_{0,\sigma}$. Furthermore, under the same conditions on $V_0$,

$$\sum_{k=1} (|E_k(H_{V,\sigma})| - 2|E_k(H_0,\sigma)|)^\gamma_+ \leq 4\sqrt{2}L^cl_{\gamma,1} \int_{\mathbb{R}_+} V(r)^{\gamma + 1/2}_+ dr \quad (1.5)$$

holds for all $\gamma \geq 3/2$. 
In what proceeds we begin by introducing the factorisation scheme for the perturbed Schrödinger operator $H_V$. We prove a technical proposition which will be the general inequality from which we derive theorems 1-4. In Section 3 we prove that the single-well property for $V_0$ is sufficient to apply our bound and then proceed to prove theorems 1 and 2 for the stated restrictions on $V_0$. Finally, in Section 3 we present the explicit factorisations of the Pöschl-Teller and Coulomb operators and formulate the improved results in these cases.

Remark 5. The bounds (1.2), (1.3), (1.4) and (1.5) suggest that the perturbation $V$ needs to be large relative to $V_0$ in order to contain a bound close of $H_V$. In particular, if $V_0$ is small enough that it only produces one eigenvalue then

$$\left(\sqrt{|E_1(H_V)|} - \sqrt{|E_1(H_0)|}\right)^3 + \sum_{k=2}^n |E_k(H_V)|^{3/2} \leq \frac{3}{16} \int_\mathbb{R} V(x)^2 \, dx.$$

2. Factorisation scheme for $H_V$

We start with the assumption that $V_0, V \in C_0^\infty(\mathbb{R})$ and $V_0, V \geq 0$. The corresponding operators $H_0 = -\frac{d^2}{dx^2} - V_0$ and $H_V = H_0 - V$ on $L^2(\mathbb{R})$ have negative eigenvalues which we denote by $\{E_k(H_0)\}_{k=1}^M$ and $\{E_k(H_V)\}_{k=1}^N$ in increasing order (it is clear that $N \geq M$). For the sake of brevity, we will denote $\mu_k = -E_k(H_0)$ and $\lambda_k = -E_k(H_V)$, unless we wish to emphasise the underlying operator.

The factorisation scheme for $H_0$ goes as follows: let $\varphi_1$ be the first eigenfunction of $H_0$ corresponding to $-\mu_1$. It is well-known that $\varphi_1$ can be taken to be positive without zeros. Consider the first order operator $D_1 = -\frac{d}{dx} - g_1$, where $g_1(x) = \varphi_1'(x)/\varphi_1(x)$. We see that $H_0$ factorises by $D_1$, in the sense that

$$D_1^* D_1 = H_0 + \mu_1,$$
$$D_1 D_1^* = H_0 - 2g_1^2 + \mu_1.$$

This follows by expanding with $D_1^* = -\frac{d}{dx} - g_1$ and using that $g_1$ solves the Riccati equation

$$g_1' = \mu_1 - V_0 - g_1^2$$

(2.2)
deduced directly from its definition. We note that the relations (2.1) can be seen precisely by looking on the associated quadratic forms, where it holds for all functions in the form core $C_0^\infty(\mathbb{R})$.

As a consequence it follows that $D_1^* D_1$ has discrete spectrum consisting of the eigenvalues $\{-\mu_k + \mu_1\}_{k=1}^M$. It is known that $D_1^* D_1$ and $D_1 D_1^*$ have the same non-zero discrete spectrum. Thus, if zero is not an eigenvalue of $D_1 D_1^*$ then we define the operator

$$H_0^{(2)} = H_0 - 2g_1^2 = -\frac{d^2}{dx^2} - V_0^{(2)}$$

and deduce that it has eigenvalues $\{-\mu_k\}_{k=2}^M$. We call this the lifted operator, with lifted potential $V_0^{(2)} = V_0 + 2g_1^2$. It can be deduced that zero isn’t an eigenvalue of
The discrete spectrum of $H$ holds for these objects. We repeat this procedure until we exhaust

denote the eigenfunctions of $H$ of $Q$ derived from solving for $\varphi$ from (2.4) we observe that $Q$ where $g$

Again, we use that $Q$ with $g$

For the perturbed operator $H_V$ we proceed with the same argument. Consider its lowest eigenvalue $-\lambda_1$ and corresponding eigenfunction $\psi_1$ and define the operator $Q_1 = D_1 - f_1$, where $D_1$ is as before and $f_1 = D_1 \psi_1/\psi_1$. It follows that

$$Q_1^*Q_1 = H_V + \lambda_1, \quad Q_1Q_1^* = H_V + \lambda_1 - 2f'_1 - 2g'_1$$

where we use that $f_1$ solves the Riccati-type equation

$$f'_1 = \lambda_1 - \mu_1 - V - f_1^2 - 2g_1f_1. \quad (2.5)$$

From the behaviour of $\psi_1$ outside the support of both potentials, and using (2.3), we have

$$f_1(x) = \begin{cases} -\sqrt{\mu_1} + \sqrt{\mu_1}, & \text{as } x \to \infty, \\ \sqrt{\mu_1}, & \text{as } x \to -\infty. \end{cases}$$

Again, we use that $Q_1^*Q_1$ and $Q_1Q_1^*$ have the same non-zero discrete spectrum and from (2.4) we observe that $Q_1^*Q_1$ has eigenvalues given by $\{-\lambda_k + \lambda_1\}_{k=1}^N$. We suppose that $0$ is an eigenvalue of $Q_1Q_1^*$. If so, then there would exist a nontrivial $\psi \in L^2(\mathbb{R})$ with $Q_1Q_1^*\psi = 0$ which would imply that $Q_1^*\psi = 0$ and thus $\psi$ solves

$$-\psi' - g_1 \psi - f_1 \psi = 0.$$ 

However, outside the supports of $V_0$ and $V$ this would result in the asymptotic behaviour $\psi(x) \sim \exp \left( \pm \sqrt{\lambda_k} x \right)$ as $x \to \pm \infty$, thus $\psi \notin L^2(\mathbb{R})$ and $0$ is not an eigenvalue of $Q_1Q_1^*$. We conclude from (2.4) that the new Schrödinger operator defined as

$$H_V^{(2)} = H_V - 2f'_1 - 2g'_1 = H_0^{(2)} - (V + 2f'_1) = H_0^{(2)} - V^{(2)} \text{ in } L^2(\mathbb{R})$$

has lifted discrete spectrum $\{-\lambda_k\}_{k=2}^N$.

We repeat this process. We use the ground state of the operators $H_0^{(2)}$ and $H_V^{(2)}$ to factorise again and so on. Explicitly, we obtain a sequence of lifted operators $H_0^{(k)}$ and $H_V^{(k)}$ with ground states $u_k = (\prod_{j=1}^{k-1} D_j) \varphi_k$ and $v_k = (\prod_{j=1}^{k-1} (D_j - f_j)) \psi_k$, respectively; where $g_j = u_j'/u_j$, $D_j = \frac{d}{dx} - g_j$, $f_j = D_j v_j / v_j$, $Q_j = D_j - f_j$ and where $\varphi_k$ and $\psi_k$ denote the eigenfunctions of $H_0$ and $H_V$, respectively. Analogous asymptotics and Riccati equations hold for these objects. We repeat this procedure until we exhaust the discrete spectrum of $H_0$.

Following [BL00], we form a trivial integral inequality and use the Riccati equations, (2.5), satisfied by $f_k$ and the asymptotic properties of $f_k$ and $g_k$ to obtain eigenvalue.
bounds. Consider for \( k \leq M \)

\[
0 \leq \int_{\mathbb{R}} \left( V^{(k+1)} \right)^2 \, dx = \int_{\mathbb{R}} \left( V^{(k)} + 2f_k \right)^2 \, dx
\]

\[
= \int_{\mathbb{R}} (V^{(k)})^2 \, dx + 4 \int_{\mathbb{R}} f_k (V^{(k)} + f_k) \, dx,
\]

\[
= \int_{\mathbb{R}} (V^{(k)})^2 \, dx + 4 \int_{\mathbb{R}} f_k^\prime (\lambda_k - \mu_k - f_k^2 - 2g_k f_k) \, dx,
\]

\[
= \int_{\mathbb{R}} (V^{(k)})^2 \, dx + 4f_k (\lambda_k - \mu_k) \bigg|_{-\infty}^{\infty} - \frac{4}{3} f_k^3 \bigg|_{-\infty}^{\infty}
\]

\[
- 4f_k^2 g_k \bigg|_{-\infty}^{\infty} + 4 \int_{\mathbb{R}} g_k^\prime f_k^2 \, dx
\]

from which the asymptotics give us explicit terms, we find

\[
\frac{8}{3} \left( 2\lambda_k^3 - 3\lambda_k \sqrt{\mu_k} + \mu_k^\frac{3}{2} \right) \leq \int_{\mathbb{R}} (V^{(k)})^2 \, dx + 4 \int_{\mathbb{R}} g_k^\prime f_k^2 \, dx.
\]

Using this inductively, we conclude that

\[
\sum_{k=1}^{M} \left( \lambda_k^\frac{3}{2} - \frac{3}{2} \lambda_k \sqrt{\mu_k} + \frac{\mu_k^\frac{3}{2}}{2} \right) \leq \frac{3}{16} \int_{\mathbb{R}} V^2 \, dx + \frac{3}{4} \sum_{k=1}^{N} \int_{\mathbb{R}} g_k^\prime f_k^2 \, dx.
\]

After we have exhausted all the \( M \) eigenvalues of \( H_0 \) we have the operator

\[
H^{(M+1)}_V = -\frac{d^2}{dx^2} - V^{(M+1)} - V^{(M+1)}.
\]

Since \( H_0^{(M+1)} \) has no negative eigenvalues, it follows that \( V^{(M+1)} \leq 0 \) almost everywhere. Therefore, we remove \( V^{(M+1)} \) at this step and by the variational principle we conclude that \( |E_k(H^{(M+1)}_V)| \leq |E_k(-\frac{d^2}{dx^2} - V^{(M+1)})| \) for all \( k \). We directly apply the method of [BL00] to this new operator and find that

\[
\sum_{k=M+1}^{N} |E_k(H_V)|^{\frac{3}{2}} \leq \sum_{k=1}^{N} |E_k(H)|^{\frac{3}{2}} \leq \frac{3}{16} \int_{\mathbb{R}} (V^{(M+1)})^2 \, dx.
\]

Combining with the inductive scheme above, this allows us to include the extra eigenvalues of \( H_V \) in our estimate, which we state in the proposition below.

Finally, we note that the restrictions on \( V_0 \) and \( V \) can be loosened. All we require from \( V_0 \) and \( V \) is that the asymptotics used above hold true. In particular, we require that the ground states of the lifted operators \( H_0^{(k)} \) and \( H_V^{(k)} \), \( u_k \) and \( v_k \) given above, satisfy

\[
u_k(x) \sim e^{\pm \sqrt{\lambda_k} x}, \quad \text{and}
\]

\[
v_k(x) \sim e^{\mp \sqrt{\lambda_k} x}
\]

as \( x \to \pm \infty \), for all \( k \). In this case there may be finite or infinitely many eigenvalues (accumulating at zero).
**Proposition 6.** Consider the Schrödinger operators $H_0$ and $H_V$ in $L^2(\mathbb{R})$ with negative eigenvalues $\{-\mu_k\}_{k=1}^M$ and $\{-\lambda_k\}_{k=1}^N$ respectively such that the lifted eigenfunctions satisfy (2.6). Then for any $K \leq N$ it follows that

$$
\sum_{k=1}^K \left[ (\sqrt{\lambda_k} - \sqrt{\mu_k})^3 + \frac{3}{2} \sqrt{\mu_k} \left( \sqrt{\lambda_k} - \sqrt{\mu_k} \right)^2 \right] \leq \frac{3}{16} \int_{\mathbb{R}} V^2 \, dx + \frac{3}{4} \sum_{k=1}^K \int_{\mathbb{R}} g_k^2 f_k^2 \, dx
$$

with $g_k$ and $f_k$ as above and where we take $\mu_k = 0, g_k \equiv 0$ for $k > M$.

We make a remark about the case for the operators with Robin boundary conditions.

**Remark 7.** The operators $H_{0,\sigma}$ and $H_{V,\sigma}$ in $L^2(\mathbb{R}_+)$ with boundary conditions $\varphi'(0) - \sigma \varphi(0) = 0$ factorise similarly. In particular, (2.1) and (2.4) hold with $g_l$ and $f_1$ analogously defined according to the ground states. For compactly supported potentials the same asymptotic behaviour holds as $x \to \infty$ with $g_l(0) = \sigma$ and $f_1(0) = 0$ at the boundary. Thus, in particular, we find

$$
\left( \sqrt{|E_1(H_{V,\sigma})|} - \sqrt{|E_1(H_{0,\sigma})|} \right)^3 \leq \frac{3}{8} \int_{\mathbb{R}_+} V^2 \, dx + \frac{3}{2} \int_{\mathbb{R}_+} g_l^2 f_1^2 \, dx
$$

(2.7)

which holds for more general $V_0, V$ as in the above proposition (satisfying the condition (2.6) at $+\infty$). In this case an important distinction occurs after factorisation. For the lifted operators one obtains the Schrödinger operators $H_{0,\sigma}^{(k)}$ and $H_{V,\sigma}^{(k)}$ defined as before but now with Dirichlet conditions at 0. Later we will see that this can be circumnavigated with a variational argument.

The next two sections are largely concerned with removing the last term in the above proposition. A sufficient condition is that $g_k'$ is negative almost everywhere; that is, the ground states of the operators $H_0^{(k)}$ are log-concave. This is equivalent to $H_0^{(k+1)} \geq H_0^{(k)}$ in the form sense for all $k$. An ordering of this sort seems like a reasonable assumption, given that the energies are lifted from $k$ to $k + 1$. In what follows, we are able to show this for $k = 1$ when $V_0$ is single well and for all $k$ for specific examples. However, this stronger property does not generally hold, see Remark 8.

Finally, we note that we will only consider restrictions on $V_0$. The removal of the last term may be possible in more generality, which we leave as an open problem.

**Remark 8.** Consider the case where $V_0$ is a double-well potential, it can be seen that the corresponding ground state is also double-well, hence not quasi-concave and thus not log-concave. An analytic proof of this can be found for the hyperbolic double-well potential in [Dow13].

3. Proof of theorems 1 and 2

Log-concavity of the ground state has been explored by many authors. On a bounded domain, for the Dirichlet and Neumann Laplacians this is a simple fact (for the Robin case there are exceptions, see [ACH20]). For Schrödinger operators on
a bounded domain, Brescamp and Lieb showed as a consequence of a functional inequality in [BL02] that this holds for concave potentials. An application, and simpler proof, of this fact was presented in [SWYY85] where the authors used it to derive a lower bound on the difference of the first two eigenvalues, the ‘fundamental gap’. The single-well condition arose with the same application in the paper of Ashbaugh and Benguria [AB89], though log-concavity was not explicitly shown or used.

Works by Baumgartner, Grosse and Martin have shown that log-concavity holds for ground state of operators on $L^2(\mathbb{R})$ of the form $H_V$ where $V$ is ‘$V_0$–concave’, see [Bau90]. Originally this was shown for Hardy-type $V_0$ in [BGM86], a proof which was later simplified in [AB88]. For our application we only need to look on $H_0$.

To prove Theorem 1 we apply Proposition 6 whilst removing the last term in the bound. Log-concavity is sufficient for the latter point, but the concavity condition on $V_0$ of Brescamp-Lieb causes problems with the asymptotics required (2.6). Thus, we look at $V_0$ that are single well and prove log-concavity in this case. The remainder of the proof relies on a variational argument.

**Lemma 9.** Suppose that $(1 + |x|)V_0 \in L^1(\mathbb{R})$ and that $V_0$ is positive and single-well, then the eigenfunction $\varphi$ corresponding to the lowest eigenvalue $-\mu$ of $H_0$ on $L^2(\mathbb{R})$ is log-concave and $\varphi(x) \sim e^{\mp \sqrt{\mu} x}$ as $x \to \pm \infty$.

**Proof.** The asymptotic decay of the ground state follows from $(1 + |x|)V_0 \in L^1(\mathbb{R})$ by considering the Jost functions (see [CSN89]). With this in hand, we define the log-derivative of the ground state as $g = \varphi'/\varphi$. It is weakly differentiable (hence continuous), obeys the Riccati equation (2.2) and tends to $\mp \sqrt{\mu}$ as $x \to \pm \infty$.

Suppose that $g' > 0$ (almost everywhere) on some open interval $I_1$. If there is no disjoint open interval where $g' < 0$ beforehand then the asymptotics tell us that $g > \sqrt{\mu}$ on $I_1$ and

$$V_0(x) = \mu - g^2(x) - g'(x) - g(x) < -g'(x) < 0 \text{ on } I_1$$

using the Riccati equation, which contradicts the positivity of $V_0$. Hence, there exists a open interval $I_0$ on which $g' < 0$. Furthermore, we can assume that $g > -\sqrt{\mu}$ on $I_1$, otherwise one can find that $V_0 < 0$ as before and obtain a contradiction. Therefore, using the behaviour of $g$ as $x \to \infty$ there must exist another disjoint open interval $I_2$ after, on which $g' < 0$.

We deduce the existence of three points $x_0 < x_1 < x_2$ with $x_k \in I_k$, $k = 1, 2, 3$ for which $g(x_1) = g(x_2) = g(x_2) = c$. Using the Riccati equation and the respective sign of $g'$ we find

$$V_0(x_1) > \mu - c^2,$$  \hspace{1cm} (3.1)
$$V_0(x_2) < \mu - c^2,$$  \hspace{1cm} (3.2)
$$V_0(x_3) > \mu - c^2.$$  \hspace{1cm} (3.3)
If $V_0$ is continuous we are finished as this contradicts the single-well property. Otherwise, by continuity of $g$, the inequalities (3.1), (3.2) and (3.3) extend a.e. to neighbourhoods about $x_1$, $x_2$ and $x_3$ respectively. We conclude that $g' \leq 0$.

Assume that $V_0$ satisfies the conditions of the lemma and that $V \in C_0^\infty(\mathbb{R})$ and $V \geq 0$. We apply Proposition 6 with $K = 1$ for a bound on the first eigenvalues, removing the last term owing to the negativity of $g_1$. To obtain the full statement we use a variational argument.

As a consequence of Lemma 9, the lifted operator $H_{V}^{(2)}$ satisfies

$$H_{V}^{(2)} \geq -\frac{d^2}{dx^2} - V_0 - V^{(2)} = H_{V}^{(2)}$$

in the quadratic form sense. We look on the operator $H_{V}^{(2)}$ and denote it by $H^{(2)}$, thus we have $|E_2(H_{V})| = |E_1(H_{V}^{(2)})| \leq |E_1(H^{(2)})|$. We lift $H^{(2)} = H_{V}^{(2)}$ again, according to Section 2, dispose of $g_1$ and define the lifted operator

$$H^{(3)} = H_{V}^{(2)} \geq -\frac{d^2}{dx^2} - V_0 - (V^{(2)})^{(2)} = H_{V}^{(2)}.$$  

Repeating this, we obtain a sequence of operators $H^{(k)} = H_{V_k}$ with $V_k = ((V^{(2)})^{(2)})^{(2)}$, iterated $k - 1$ times, and such that $|E_k(H_{V})| \leq |E_1(H^{(k)})|$. Once $|E_k(H_{V})| \leq |E_1(H_0)|$ then we stop. Applying the inductive integral method for this sequence of potentials, as in Proposition 6, we form

$$\sum_{k} \left[\sqrt{|E_k(H_{V})|} - \sqrt{|E_1(H_0)|}\right]^3 \leq \sum_{k} \left(\sqrt{|E_1(H^{(k)})|} - \sqrt{|E_1(H_0)|}\right)^3 \leq \frac{3}{16} \int_{\mathbb{R}} V^2 \, dx.$$  

We finish the proof by noting that a standard approximation and variational argument can be applied to extend (3.4) to more general $V$.

Finally, we note that on the half-line for the operator $H_{0,\sigma}$ with $\sigma \geq 0$, an analogous version of Lemma 9 holds by near-identical proof. In this case the asymptotic requirements for (2.7) at infinity are satisfied when $V_0 \in L^1(\mathbb{R}_+)$, which follows by work in [DU11]. From Remark 7, the bound in Theorem 2 follows by the same variational idea as above, where at each step we also use that $H_{V,\sigma} \leq H_{V,\infty}$.

Remark 10. Whether the single-well property can be used to remove the error term in Proposition 6 entirely (for any $K$) is left open. However, clearly an improvement of (3.3) can be found if we know that more than one of the lifted ground states are log-concave.

4. Proof of Theorems 3 and 4

Finally, we apply our result from Section 2 to two well-known examples of $H_0$. For the Pöschl-Teller potential, this will be a direct result of the explicit factorisation and application of Proposition 6. For the Coulomb potential we will need to be careful.
about the Dirichlet boundary conditions. In both cases we will see that we can obtain superior estimates to those in theorems 1 and 2.

4.1. Pöschl-Teller Potential. Consider the case of the potential

\[ V_0(x) = \nu(\nu + 1) \text{sech}^2(x) \]

with \( \nu > 0 \). It has an explicit factorisation according to the first order operator \( D = \frac{d^2}{dx^2} + \nu \tanh(x) \). Following the factorisation method (outlined in Section 2), its lifted potential is given by

\[ V_0^{(2)}(x) = \nu(\nu - 1) \text{sech}^2(x), \]

thus we can see that \( H_0^{(2)} \) has the same shaped potential as \( H_0 \), with a change of parameter. It follows that

\[ V_0^{(k)}(x) = (\nu - k + 1)(\nu - k + 2) \text{sech}^2(x). \]

Using this scheme we can explicitly compute the negative eigenvalues to be

\[ E_k(H_0) = \left(\nu - k + 1\right)^2, \]

\( k = 1, \ldots, \lfloor \nu \rfloor \) as well as the corresponding eigenfunctions, see [FLW22]. Each lifted operator has ground state given by \( u_k = \cosh^{-\nu+k}(x) \) with log-derivative given by \( g_k = -(\nu - k + 1) \tanh(x) \). Thus \( V_0 \) satisfies the requirements of Proposition 4, and at each stage its lifted ground state is log-concave. If \( V \in C_0^\infty(\mathbb{R}) \) then \( V_0 \) also satisfies the conditions (2.6). Thus, the conclusion that

\[ \sum_{k=1}^{\lfloor \nu \rfloor} \left( \sqrt{|E_k(H_0)|} - (\nu - k + 1) \right)^3 + \sum_{k=\lfloor \nu \rfloor + 1} |E_k(H_0)|^{3/2} \leq \frac{3}{16} \int V(x)^2_+ \, dx. \]

Where the standard approximation and variational arguments extend this to more general \( V \).

The second bound (1.4) is just obtained from the full statement of the proposition by halving \( |E_k(H_0)| \) to obtain a bound on \( \sum_k (E_k(H_0)/2 - E_k(H_0))^{3/2} \) and using the standard Aizenman-Lieb argument of [AL78].

4.2. Coulomb potential. Now consider the case of the operator with Coulomb potential on the positive real line with Dirichlet boundary condition at 0. That is,

\[ H_{0,\infty} = -\frac{d^2}{dr^2} + \frac{\nu(\nu + 1)}{r^2} - \frac{\kappa}{r} \text{ in } L^2(\mathbb{R}_+), \]

where \( \nu > -1/2 \) and \( \kappa > 0 \). The explicit factorisation is given according to the first order operator \( D = \frac{d}{dr} - \frac{(\nu+1)}{r} + \frac{\kappa}{2(\nu+1)}, \) from which the lifted potential is found to be

\[ V_0^{(2)}(r) = -\frac{(\nu + 1)(\nu + 2)}{r^2} + \frac{\kappa}{r}. \]

We see that \( H_{0,\infty}^{(2)} \) has the same shaped potential with different parameters. Iterating this, we find that

\[ V_0^{(k)}(r) = -\frac{(\nu + k - 1)(\nu + k)}{r^2} + \frac{\kappa}{r}. \]
This scheme can be used to explicitly calculate the negative eigenvalues as $E_k(H_{0,x}) = -\kappa^2/4(\nu + k)^2$, $k \in \mathbb{N}$ and the associated eigenfunctions, see [FLW22]. The lifted ground states are given by $u_k(r) = r^{\nu+1}e^{-r\sqrt{|E_k(H_{0,x})|}}$, these are log-concave with $g_k' = -(\nu + k)/r^2$. Though these $u_k$ do not obey the asymptotic requirements of Remark 7 we still have that $g_k \to -\sqrt{|E_k(H_{0,x})|}$ at infinity. However, we cannot directly apply the ideas of the Robin case since the value of $g_k$ at 0 is potentially problematic. We treat this with care, following [Lap21].

We revert to the scheme in Section 2 and denote by $\lambda_k = -E_k(H_{V,x})$. Assume that $V \in C_0^\infty(\mathbb{R}_\pm)$ and consider the behaviour of the ground state of $H_{V,x}, \psi_1$, outside of the support of $V$. Then $\psi_1$ satisfies

$$-\psi_1''(r) + \left(\frac{v(v+1)}{r^2} - \frac{\kappa}{r}\right)\psi_1(r) = -\lambda_1 \psi_1(r).$$

The solution can be found in terms of the Whittaker functions. Using the asymptotic expansion of these it follows that $f_1$ (defined as in Section 2) satisfies

$$f_1(r) = \begin{cases} \frac{r(-\kappa + 2\sqrt{\Lambda_1}\nu + 2\sqrt{\Lambda_1}) + 2(\nu+1)^2(2\nu+3)}{4(\nu+1)^2(2\nu+3)} + O(r^2), & \text{as } r \to 0 \\ -\sqrt{\Lambda_1} + \frac{\kappa}{2r\sqrt{\Lambda_1}} + O(r^{-2}), & \text{as } r \to \infty. \end{cases}$$

We use this in the inductive method of Section 2 with $g_1' < 0$. Together with the fact that $H_{0,x}$ lifts to another Coulomb operator we can check asymptotic requirements are met for all lifted potentials. Thus the following inequality holds,

$$\sum_{k=1}^{\infty} \left(\sqrt{|E_k(H_V)|} - \frac{\kappa}{2(\nu + k)}\right)^3 \leq \frac{3}{8} \int_{\mathbb{R}_+} V(r)^2 \, dr$$

which extends to a larger class of $V$ as before.

Remark 11. Both examples in this section are known as shape-invariant potentials; that is, at each step of factorisation the new potentials have the same shape but differ by some parameter. Such potentials are fully factorisable, which in these cases allows us to apply the full version of Proposition 6. See [GM08] for further examples of shape-invariant potentials.

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