The interplay between group crossed products, semigroup crossed products and toeplitz algebras

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The interplay between group crossed products, semigroup crossed products and toeplitz algebras

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Abstract. Realization of group crossed products constructed by decomposition, as semigroup crossed products. And connected it to Toeplitz algebra of ordered group quotient to get some preliminaries description for the further study on the structure of Toeplitz algebras of ordered group which is finitely generated.

1. Introduction
The theory of crossed products by semigroups of endomorphisms is a generalization of the theory of crossed products by groups of automorphisms which firstly introduced by Cuntz in [1]. In some years later, Stacey in [2] studied particularly the crossed product on the semigroup of natural numbers (including 0). Based on the ideas of Stacey, [3] discussed their notion of crossed products by semigroups of $C^*$-algebra. In [3], Adji and her collaborators showed the realization of semigroup crossed product as Toeplitz algebras of ordered group. For another example result on semigroup crossed products, we can refer to [4].

Rosjanuardi in [5], has shown the connection between group crossed product and semigroup crossed product. Rosjanuardi used technique of decomposition to semigroup crossed products so that it can be viewed as group crossed products. Moreover, Rosjanuardi and Albania in [6] applied this technique of decomposition to the graph algebras and as their result this graph algebras can be examined as semigroup crossed products. In this study, we adapted the result in [5] and improved to the case where the ordered group is finitely generated. However, as the preliminary study we limited the case to direct sum of three integers. The very first result on this case has already published in [7] but it did not consider its application to Toeplitz algebras.

2. Methods
Let $\Gamma$ be the direct sum of three discrete abelian groups $\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$. Equipped with the lexicographic order then it is a totally ordered group. It can be verified that $\Gamma$ has more than one order ideals, that are $I_1 = \{ (0,0,n) : n \in \mathbb{Z} \}$ and $I_2 = \{ (0,m,n) : m,n \in \mathbb{Z} \}$. Suppose $I = I_1$, results in [7] have lead us to a conclusion that the semigroup crossed products $B_{(\Gamma \setminus I)} \times_{\tau} \Gamma^+$ can be thought as the group crossed products via decomposition technique, that is, in the form of an iterated group crossed products $(B_{(\Gamma \setminus I)}^+ \times_{\tau} (\Gamma \setminus I)^+) \times_\beta I$. Applying the realization of Toeplitz algebras as semigroup crossed
products done in [3], it follows immediately that the iterated crossed products
\( (B(\Gamma I)^+ \times_\tau (\Gamma I)^+) \times_\beta I \) is isomorphic to \( \mathcal{T}(\Gamma I) \times_\beta I \).

3. Results and Discussion
Let \( \Gamma \) be a group and \( A \) be a \( C^* \)-algebra. Define a continuous homomorphism \( \alpha \) of \( \Gamma \) into the group of automorphisms of \( A \). We called \( \alpha \) as an action of \( \Gamma \) on \( A \). The system which consists of a \( C^* \)-algebra \( A \), a group \( \Gamma \), and an action \( \alpha \), write \( (A, \Gamma, \alpha) \), is called as a dynamical system. Through the classes of representation called covariant representation, we will built out a new \( C^* \)-algebra from a dynamical system in the form of crossed products. A covariant representation of \((A, \Gamma, \alpha)\) is a couple of representation \((\pi, V)\), where \( \pi \) is a non-degenerate representation of \( A \) and \( V \) is a unitary representation of \( \Gamma \) such that \( V_s V_t = V_{s+t} \) and this pair satisfies the covariance condition \( \pi(\alpha_t(a)) = V_t \pi(a) V_t^* \) for \( a \in A \) and \( t \in \Gamma \).

3.1. Group Crossed Products
For any dynamical system, we can construct a \( C^* \)-algebra which is a crossed products, a triple \((B, i_A, i_\Gamma)\), consisting of a \( C^* \)-algebra \( B \) equipped with a non-degenerate homomorphism \( i_A : A \to B \) and a homomorphism of \( \Gamma \) into the multiplier algebra of \( B \) \( M(B) \) such that \( i_\Gamma(s)i_\Gamma(t) = i_\Gamma(s+t) \) for all \( s, t \in \Gamma \) and that satisfies the conditions as follows:

1) \( i_A(\alpha_t(a)) = i_\Gamma(t)i_A(a)i_\Gamma(t)^* \) for \( a \in A \) and \( t \in \Gamma \);
2) for any covariant representation of \((A, \Gamma, \alpha)\), there is a non-degenerate representation \( \pi \times V \) of \( B \) such that \( (\pi \times V) \circ i_A = \pi \) and \( \pi \times V \circ i_\Gamma = V \);
3) \( B \) is generated by \( \{i_A(a); a \in A\} \) and \( \{i_\Gamma(x); x \in \Gamma\} \).

We conclude that there is such crossed product and that it is uniquely up to isomorphism.

3.2. Semigroup Crossed Products
Suppose \( \Gamma \) is a totally ordered discrete abelian group, a homomorphism \( \theta \) of \( \Gamma \) into \( \mathbb{R} \) and the semigroup \( \Gamma^+ = \Gamma \cap \theta^{-1}([0,\infty)) \) called as the positive cone of \( \Gamma \). Define an action of \( \Gamma^+ \) on a \( C^* \)-algebra \( A \) by endomorphisms, \( \alpha_x : \Gamma^+ \to \text{End}(A) \) such that \( \alpha_x \) is extendible for all \( x \in \Gamma^+ \). The triple \((A, \Gamma^+, \alpha)\) is called as semigroup dynamical system. Following the group crossed product, the semigroup crossed products constructed from \((A, \Gamma^+, \alpha)\) is a \( C^* \)-algebra where its representations are in correspondence one-to-one with the covariant representation \((\pi, V)\) in which \( \pi \) is a non-degenerate representation of \( A \) and \( V \) is an isometric representation such that \( V_s V_t = V_{s+t} \) and \( \pi(\alpha_t(a)) = V_t \pi(a) V_t^* \) for \( a \in A \) and \( t \in \Gamma \). However, Stacey in [2] emphasized that the semigroup dynamical system may have no covariant representation. On other word, the crossed products is \( \{\{\}\} \). Adji and Rosjaniardi in [8] showed that the guaranty of the existence of crossed product \((A, \Gamma^+, \alpha)\) is on the presence of its non-trivial covariant representation.

We use the notion of crossed products by semigroup of endomorphisms from [3] or [4]. A crossed product for dynamical system \((A, \Gamma^+, \alpha)\) is a triple \((B, i_A, i_\Gamma)\) where \( B \) is a \( C^* \)-algebra, \( i_A : A \to B \) is a non-degenerate homomorphism, and a homomorphism \( i_\Gamma : \Gamma^+ \to M(B) \) of \( \Gamma^+ \) into the multiplier algebra \( M(B) \) of \( B \) such that \( i_\Gamma(s)i_\Gamma(t) = i_\Gamma(s+t) \) for all \( s, t \in \Gamma^+ \), and that satisfies the conditions as follows:

1) \( i_A(\alpha_t(a)) = i_\Gamma(t)i_A(a)i_\Gamma(t)^* \) for \( a \in A \) and \( t \in \Gamma^+ \);
2) for all covariant representation \((\pi, V)\) of \((A, \Gamma^+, \alpha)\), there is a non-degenerate representation \( \pi \times V \) of \( B \) such that \( (\pi \times V) \circ i_A = \pi \) and \( \pi \times V \circ i_\Gamma = V \);
3) \( B \) is generated by elements of the form \( \{i_A(a)i_\Gamma(x); a \in A, x \in \Gamma^+\} \).

Note that, there is a uniquely up to isomorphism and the crossed products will be denoted by \( A \times_\alpha \Gamma^+ \).
3.3. Toeplitz Algebras

3.3.1. Definition. [3] Let $\Gamma$ be a totally ordered group and the positive cone $\Gamma^+$. The family of evaluation maps $\{\epsilon_x: x \in \Gamma\}$ is an orthonormal basis for $L^2(\Gamma^+)$, and by definition, the Hardy space $H^2(\Gamma^+)$ is the closed linear span of $\{\epsilon_x: x \in \Gamma^+\}$. The Toeplitz algebra $\mathcal{T}(\Gamma^+)$ is the $C^*$-algebra of $B(H^2(\Gamma^+))$ generated by Toeplitz operator $T_\phi$ in which $\phi \in \mathcal{C}(\Gamma^+)$, that is, $\phi$ is the operator on $H^2(\Gamma^+)$ defined by $T_\phi(f) = P(\phi f)$ with $P$ denote the projection of $L^2$ onto $H^2$.

3.3.2. The abstract version of Toeplitz algebra. Consider a closed subspace $B_{I^+}$ of $l^\infty(\Gamma)$ spanned by $\{1_x: x \in \Gamma^+\}$, where

$$1_x(y) = \begin{cases} 1, & \text{if } y \geq x, \\ 0, & \text{else} \end{cases}$$

Moreover, this space is a $C^*$-subalgebras of $l^\infty(\Gamma)$. Note that, the automorphisms $\tau_x$ of $l^\infty(\Gamma)$ defined by $\tau_x(f)(y) = f(y-x)$ such that $\tau_x(1_y) = 1_{x+y}$, and thus restriction of $\tau$ to $\Gamma^+$ give a definition for action $\alpha$ of $\Gamma^+$ by endomorphisms of $B_{I^+}$. Adji and his collaborators in [3] have shown that there exists an injective homomorphism of $B_{I^+}$ onto the Toeplitz algebra $\mathcal{T}(\Gamma^+)$, mapping the generator of $B_{I^+}$ onto the operator Toeplitz $T_x$. This semigroup crossed product that we called as the abstract version of Toeplitz algebra.

3.4. Group Crossed Products, Semigroup Crossed Products and Toeplitz Algebras

Let $\Gamma$ be a direct sum of three abelian groups of integers $\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$, by providing an order to $\Gamma$ which is lexicographic order defined in the following:

$$(x_1)_i \leq (y_1)_i \Leftrightarrow x_1 < y_1, \text{ or } x_1 = y_1 \text{ and } (x_i)_i \leq (y_i)_i \text{ for } i = 2, 3,$$

then $\Gamma$ is a totally ordered group.

An order of an abelian group $(G, \leq)$ is a subgroup $G$ such that if $x \in G, y \in I^+$ such that $0 < x \leq y$ then $x \in I$. It can be verified that $\Gamma$ has more than one order ideal which are $I_1 = \{(0, 0, n): n \in \mathbb{Z}\}$ and $I_2 = \{(0, m, n): m, n \in \mathbb{Z}\}$, see [7] for detail. Suppose $I = I_1$, we will consider this order ideal of $\Gamma$ as our first study. Next, we will consider the $C^*$-subalgebras $B_{I^+}$ of $l^\infty(\Gamma \setminus I)$ that is a linear span of $\{1_{[a,b,c]} \in l^\infty(\Gamma \setminus I): [a,b,c] \in (\Gamma \setminus I)^+\}$ in which

$$1_{[a,b,c]}([x,y,z]) = \begin{cases} 1, & \text{if } [x,y,z] \geq [a,b,c]; \\ 0, & \text{else}. \end{cases}$$

Define an action $\alpha$ of $(\Gamma \setminus I)^+$ by endomorphisms on $B_{I^+}$, by $\tau_{[a,b,c]}(1_{[x,y,z]}) = 1_{[a+x,b+y,c+z]}$. The system $(B_{I^+}, (\Gamma \setminus I)^+, \tau)$ has a non-trivial covariant representation which its standard construction discussed in [5], then by applying Proposition 2.2 of [8], the crossed product $(B_{I^+} \rtimes \alpha (\Gamma \setminus I)^+, j_{I \setminus I^+}, I)$ exists.

3.4.1. Lemma. [7, Lemma 4.1] Suppose $I$ is the direct sum $(\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z})_{\text{lex}}, I := \{(0,0,n): n \in \mathbb{Z}\}$ and $\beta$ is an action of $I$ defined by trivial action in the following:

$$\beta_{(a,0,m)}(j_{I \setminus I^+}([x,y,z])) = j_{I \setminus I^+}([x,y,z]).$$

Then the crossed products $(B_{I^+} \rtimes \alpha (\Gamma \setminus I)^+) \rtimes \beta I$ exists.
3.4.2. **Lemma.** [7, Lemma 4.2] Let $\Gamma$ be the direct sum $(\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z})_{lex}$, and $I := \{(0,0,n): n \in \mathbb{Z}\}$. Define an action $\tilde{\tau}$ of $\Gamma^+$ by endomorphism of $B_{(\Gamma^+ I)^+}$ as follows,

$$\tilde{\tau}_{(a,b,c)}(1_{[x,y,z]}) = 1_{[a+x,y+z]}.$$ 

then the crossed product $(B_{(\Gamma^+ I)^+} \times_\tau \Gamma^+, l_B, l_{\Gamma^+})$ exists.

3.4.3. **Theorem.** [7, Theorem 4.3] Suppose $\Gamma$ is the direct sum $(\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z})_{lex}$ and $I := \{(0,0,n): n \in \mathbb{Z}\}$. The $C^*$-algebra $B_{(\Gamma^+ I)^+} \times_\tau (\Gamma^+ I)^+$ is isomorphic to the $C^*$-algebra $(B_{(\Gamma^+ I)^+} \times_\tau (\Gamma^+ I)^+) \times_\beta I$.

4. **Conclusion**

The next corollary shows that we can view the decomposition of semigroup crossed products as resulted in [7, Theorem 4.3] as the group crossed products of Toeplitz algebras of ordered group quotient.

**Corollary.**

Let $\Gamma$ be the direct sum $(\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z})_{lex}$ and $I := \{(0,0,n): n \in \mathbb{Z}\}$. Then the $C^*$-algebra $(B_{(\Gamma^+ I)^+} \times_\tau (\Gamma^+ I)^+) \times_\beta I$ is isomorphic to the crossed products of the dynamical system $(\mathcal{T}(\Gamma^+ I), \beta, I)$, that is $\mathcal{T}(\Gamma^+ I) \times_\beta I$.

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