An Analysis of Closed-Loop Stability for Linear Model Predictive Control Based on Time-Distributed Optimization

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Abstract—Time-distributed Optimization (TDO) is an approach for reducing the computational burden of Model Predictive Control (MPC). When using TDO, optimization iterations are distributed over time by maintaining a running solution estimate and updating it at each sampling instant. In this paper, the specific case of using TDO in linear-quadratic MPC subject to input constraints is studied, and analytic expressions for the system gains and the sufficient amount of iterations required for asymptotic stability are derived. Further, it is shown that the closed-loop stability of TDO-based MPC can be guaranteed using multiple mechanisms including increasing the number of solver iterations, preconditioning the optimal control problem, adjusting the MPC cost matrices, and reducing the length of the receding horizon. These results in a linear system setting also provide insights and guidelines that could be more broadly applicable, e.g., to nonlinear MPC.

I. INTRODUCTION

Model Predictive Control (MPC) is a feedback strategy that generates inputs by solving an Optimal Control Problem (OCP) over a finite receding horizon [1]. To implement MPC, the solution of the OCP must be computed within the sampling period of the controller; this may not always be feasible for systems with limited computing power, fast sampling rates, and/or highly nonlinear dynamics.

An approach to reduce the computational footprint of MPC is to employ an iterative optimization method and truncate it in a manner that maintains important characteristics of the closed-loop system, such as stability and constraint satisfaction, resulting in a suboptimal MPC feedback law. One approach to suboptimal MPC is to use Time-distributed Optimization (TDO), which can be considered to be a generalization of the popular real-time iteration (RTI) scheme [2]. When implemented using TDO, the controller maintains a running solution estimate which is improved at each sampling instant using a finite number of optimizer iterations. The closed-loop behavior of TDO-based MPC (TDO-MPC) is described by the interconnection of two dynamical systems, as shown in Figure 1.

Fig. 1. Optimal MPC is a static feedback law. Time-distributed MPC is a compensator with a solution estimate \( \nu \) as its internal state and dynamics defined by \( \nu \) iterations of an optimization algorithm denoted by \( T^{\nu} \).

In a previous paper [3], we studied the stability and robustness of a general TDO-MPC formulation with any locally Input-to-state Stable (ISS) MPC feedback law and any optimization algorithm with at least q-linear convergence. The coupled plant-optimizer system was analyzed using the ISS framework and sufficient conditions for asymptotic stability and constraint satisfaction were provided. In this paper, we focus on the specific case of linear-quadratic MPC subject to input constraints to derive analytic expressions for the ISS gains of the plant, optimizer, and closed-loop system. Using these expressions, we investigate specific mechanics for ensuring the stability of TDO-MPC from an analytical and numerical perspective.

The stability, performance, and robustness of various suboptimal MPC methods are studied in the literature. Conditions under which a feasible suboptimal MPC law is stabilizing for constrained discrete-time nonlinear systems are derived in [4]. Suboptimal MPC of unconstrained discrete-time nonlinear systems is studied in [5], while suboptimal MPC of input constrained continuous-time systems is studied in [6]. The robustness of optimal and suboptimal MPC is studied in [7]. The stability of the RTI scheme for unconstrained nonlinear systems is analyzed in [2]. Our previous work [3] extends this analysis by considering both state and control constraints. Alternative control schemes which use continuous-time representations of the optimizer dynamics are investigated in [8] and [9]. For a more complete discussion and comparison of TDO-MPC and suboptimal MPC, see [3].

Deeper insight can be obtained by studying specific TDO-MPC schemes. The stability of suboptimal continuous time nonlinear MPC subject to input constraints is studied in [6] under the assumption of a linearly convergent optimization algorithm; a specific fixed-point method satisfying these assumptions is proposed in [10]. A suboptimal proximal gradient method for linear-quadratic MPC (LQMP) subject to input constraints is proposed in [11] and proposes a Bryden-based projection technique to ensure satisfaction of a terminal state constraint and thus stability. A method for suboptimal LQMP with state and control constraints using a dual accelerated gradient projection is proposed in [12] which tightens constraints based on a pre-specified degree of suboptimality to ensure stability. Input constrained LQMP, implemented using a primal accelerated gradient method, is considered in [13] and bounds on the number of iterations needed to achieve a pre-specified level of suboptimality are derived. However, [13] does not provide a mechanism for choosing the suboptimality tolerance so as to ensure stability.

Existing literature has established conditions for the stability of various TDO-MPC methods, however these studies only consider techniques for enforcing terminal constraints or increasing the number of iterations (or decreasing the size of the discretization step for continuous-time methods) as a mechanism for achieving stability. This paper presents a detailed systems theoretic analysis of the system in Figure 1 to identify and analyze multiple mechanisms for ensuring stability of the closed-loop TDO-MPC system. For analytical tractability, we focus on input constrained LQMP using primal gradient-based optimization methods and derive computable analytic expressions for the MPC and optimizer gains, a sufficient condition for asymptotic stability, and a corresponding iteration bound. Using these expression, we identify several approaches for ensuring asymptotic stability, namely increasing the number of solver...
iterations, preconditioning the OCP, tuning the cost function, and reducing the prediction horizon. Existing literature on similar problems has only considered how some of these factors affect suboptimality, and not stability [13], or has only considered increasing the amount of iterations as a means of achieving stability [12]. Additionally, the iteration bound in this paper is independent of any (arbitrary) pre-specified degree of suboptimality, unlike in [13] and [12].

Using two numerical examples, we show that the iteration bound for asymptotic stability is comparable to the iteration bound for suboptimality computed in [13] for a stable system, and exhibits the same trends as the amount of iterations needed to stabilize an unstable system in simulation. Although we establish these results in a particular setting, they provide insight and guidelines on how stability can be ensured more generally, including in nonlinear settings.

**Notation:** The normal cone mapping of a closed, convex set $C$ is defined as follows:

$$N_C(v) = \begin{cases} \{ y \mid y^T (w - v) \leq 0, \forall w \in C \}, & \text{if } v \in C, \\ \emptyset, & \text{else.} \end{cases}$$

The radius of $C$, a closed neighbourhood of the origin, is defined as $\text{rad}(C) = \max r,$ s.t. $|x| \leq r \Rightarrow x \in C$. If $A \in \mathbb{R}^{m \times n}$, $B \in \mathbb{R}^{p \times q}$, then $A \otimes B \in \mathbb{R}^{mp \times nq}$ denotes the Kronecker product. Let $(\mathcal{S}_+^n, \mathcal{S}_0^+) \in \mathbb{R}^{n \times n}$ denote the set of symmetric $n \times n$ positive (definite, semidefinite) matrices. Given $x \in \mathbb{R}^n$ and $w \in \mathbb{S}_0^+$, the $W$-norm of $x$ is $\| x \|_W = \sqrt{x^T W x}$. If $A \in \mathbb{R}^{n \times n}$, $W \in \mathbb{S}_0^+$ and $V \in \mathbb{S}_0^+$ then the induced norm is $\| A \|_{W,V} = \| \sqrt{V} A \sqrt{W} \|_2$. Given $M \in \mathbb{S}_0^n$, we use $\lambda^\nu_W(M)$ and $\lambda^\nu_V(M)$ respectively to denote the minimum and maximum eigenvalues of $\sqrt{V}^{-1} W M \sqrt{W}^{-1}$; these satisfy $\lambda^\nu_W(M) \| x \|_W^2 \leq \| x \|_M^2 \leq \lambda^\nu_V(M) \| x \|_W^2$. The condition number of $M \in \mathbb{S}_0^n$ is $\kappa(M) = \lambda^\nu(M)/\lambda^\nu(M)$. If the subscript is omitted then it is understood that $W = I$. Our use of comparison functions, e.g., class $K, K_L, \text{ or } L$ functions follows [14]. We also make extensive use of Input-to-State Stability (ISS) analysis tools such as asymptotic gains, see [15, 10] for more details, and use $\limsup$ as shorthand for $\limsup$.

**II. PROBLEM SETTING**

Consider a Linear Time Invariant (LTI) system

$$x_{k+1} = Ax_k + Bu_k, \quad (1)$$

where $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times m}$, $x \in \mathbb{R}^n$ is the state, and $u \in \mathbb{R}^m$ is the control input. The control objective is stabilize the origin [1] while enforcing the input constraint $u_k \in U$, $\forall k > 0$.

We will approach the problem using MPC. To do so, consider the following Parameterized Optimal Control Problem (POCP)

$$\min_{\xi, \nu} \| x_k \|_2^2 + \sum_{i=0}^{N-1} \| x_i \|_Q^2 + \| u_i \|_2^2 \quad (2a)$$

s.t. $\xi_{i+1} = Ax_i + Bu_i$, $i = 0, \ldots, N-1$, $\xi_0 = x$, $\mu_1 \in U$, $i = 0, \ldots, N-1$, $\nu \in \mathcal{V}$

where $N > 0$ is the horizon length, $Q \in \mathbb{R}^{n \times n}$, $R \in \mathbb{R}^{m \times m}$ and $P \in \mathbb{R}^{n \times n}$ are weighting matrices, $x$ is the parameter/measured state, $\nu = (\mu_0, \ldots, \mu_{N-1})$, and $\xi = (\xi_0, \ldots, \xi_N)$. The MPC feedback law (see, e.g., [17] for [1]) is

$$u = K(x) = \Xi \nu^*(x), \quad (3)$$

where $\Xi = [1 \ 0 \ \ldots \ 0]^T \otimes I_m$ selects $\mu_0$ from $\nu$ and $\nu^*(x)$ denotes the global solution of [2] for the parameter value $x$.

We make the following assumptions to ensure that [3] can be used to construct a stabilizing feedback law for [1].

**Assumption 1.** System [1] is stabilizable and the matrices in [2] satisfy $R \in \mathbb{S}_+^m$, $Q \in \mathbb{S}_+^n$, and $P = Q + A^T PA - (A^T PB)(R + B^T PB)^{-1}(B^T PA) \in \mathbb{S}_+^n$.

**Assumption 2.** The constraint set $U \subseteq \mathbb{R}^m$ is closed, convex, and contains the origin in its interior.

Remark 1. The assumption $Q > 0$ can be replaced with the weaker condition $Q \succeq 0$ and $(A, Q)$ observable. However, the assumption $Q > 0$ lends itself to a tighter ISS gain.

Oftentimes, not enough computational resources are available to solve [2] at each iteration. Instead, we perform a finite number $\ell \in (0, \infty)$ of iterations at each sampling instant and warmstart the optimization algorithm using the estimate from the previous sampling instant. This leads to a coupled plant-optimizer system

$$\begin{align*}
x_{k+1} &= Ax_k + Bu_k, \quad (4a) \\
\nu_k &= T^\ell(\nu_{k-1}, x_k), \quad (4b) \end{align*}$$

where $\nu$ is a running solution estimate and $T^\ell$ represents $\ell$ iterations of an optimization algorithm[1]. In this paper, $T^\ell$ will represent both accelerated and non-accelerated projected gradient methods.

In a recent paper [3], we analyzed a generalized version of [4] using Input-to-State Stability (ISS) and small-gain tools. However, since the setting was fairly general, the paper [3] was limited to existence type proofs. In this paper, we consider an analytically tractable special case and derive quantitative expressions for all the quantities of interest. Our goal is to analyze these expressions to better understand what factors that influence the properties of [4].

**III. OPTIMIZATION STRATEGY**

The OCP [2] can be rewritten using the state transition equation

$$\xi = \dot{B} \nu + \dot{A} x, \quad (5)$$

where

$$\dot{B} = \begin{bmatrix} 0 & 0 & 0 \\
B & 0 & 0 \\
\vdots & \ddots & \vdots \\
A^{N-1} B & \ldots & A^N \\
\end{bmatrix}, \quad \text{and } \dot{A} = \begin{bmatrix} I \\
A \\
\vdots \\
A^N \\
\end{bmatrix}. \quad (6)$$

Substituting [5] into [2a] leads to the condensed POCP

$$\min_{\nu \in \mathcal{V}} f(\nu, x) = \nu^T \Lambda \nu + 2\nu^T G x + x^T W x, \quad (7)$$

where $\mathcal{V} = \mathcal{U}^N$ and the cost matrices are $H = \dot{B}^T \dot{B} + (I_N \otimes R)$, $G = \dot{B}^T \dot{H} \Lambda$, and $W = Q + \dot{A}^T \dot{H} A$, with $\dot{H} = (I_N \otimes Q) 0 \ 0 \ P$. Note that $H > 0$ for $R > 0$. The following lemma, whose proof can be found in the appendix, characterizes $W$.

**Lemma 1.** The matrix $W$ in [7] satisfies $W \succeq P \succ 0$ if: i) $Q \succeq 0$, ii) $(A, Q)$ are observable, iii) $R > 0$, and iv) $P$ satisfies the Riccati equation in Assumption [1].

Since [7] is strongly convex, the following Variational Inequality (VI) is necessary and sufficient for optimality [18],

$$H \nu + G x + \nabla \nu(\nu) \succeq 0, \quad (8)$$

and the solution mapping

$$S(x) = \{ \nu \mid H \nu + G x + \nabla \nu(\nu) \succeq 0 \}, \quad (9)$$

1The operator $T^\ell$ is formally defined in [11].
is a function. A common approach for solving [9] is with an iterative optimization algorithm. A single iteration of a typical optimization algorithm can be represented by

\[ \nu_{i+1} = T(\nu_i, x), \]  

where \( T : \mathbb{R}^{N_m} \times \mathbb{R}^n \rightarrow \mathbb{R}^{N_m} \). Performing multiple iterations leads to the following recursive definition for the optimization operator \( T^\ell \),

\[ T^\ell(\nu, x) = T(T^{\ell-1}(\nu, x), x), \]

where \( \nu \in \mathbb{R}^{N_m} \) is the solution estimate, \( x \) is the parameter, and \( T^\ell(\nu, x) := \nu \). In the following subsections we present two possible choices for \( T \).

A. Projected Gradient Method

One method for solving [10] is the projected gradient method (PGM)

\[ \nu_{i+1} = \Pi_V[\nu_i - \alpha \nabla f(\nu_i, x)], \]

where \( \alpha = 2/(\lambda^+(H) + \lambda^-(H)) \) and \( \Pi_V \) denotes Euclidean projection onto \( V \). This particular choice of the step size \( \alpha \) maximizes the convergence rate of the algorithm. The PGM can be interpreted as a fixed point iteration applied to (8). The following theorem addresses its convergence rate.

**Theorem 1.** (Lemma 3.1) Let \( T \) represent the PGM (11), pick any \( x \in \mathbb{R}^n \) and suppose Assumptions 2 and hold. Then, for any \( \nu \in \mathbb{R}^{N_m} \),

\[ \|T^\ell(\nu, x) - S(x)\|_2 \leq \eta^\ell \|\nu - S(x)\|_2, \]

where \( \eta = (\kappa - 1)/(\kappa + 1) \) and \( \kappa = \lambda^+(H)/\lambda^-(H) \).

B. Accelerated Projected Gradient Method

Another option for defining \( T \) is the accelerated projected gradient method [20]. This method uses Nesterov acceleration [21], to achieve higher convergence rates. The method is summarized in Algorithm 1 and the following theorem summarizes the convergence properties of the APGM for strongly convex problems.

**Algorithm 1 Accelerated Projected Gradient Method**

**Input:** \( \nu \in V, x, \ell > 0 \)

**Output:** \( \nu^+ = \nu_\ell \)

1. \( m = 2\lambda^-(H), L = 2\lambda^+(H), \kappa = L/m \)
2. \( \theta_0 = 1, \theta_{-1} = 0, z_0 = \nu \)
3. for \( k = 0, \ldots, \ell - 1 \) do
   4. \( y_k = y_{k-1} + \frac{\theta_{k+1}}{\theta_k} (z_k - y_{k-1}), \gamma_k = \theta_{k+1} L \)
   5. \( v_k = \Pi_V[y_k - L^{-1} \nabla f(y_k)] \)
   6. \( z_{k+1} = v_k + \gamma_k (v_k - y_k) \)
   7. \( \theta_{k+1} = \frac{\zeta_k}{\gamma_k} \left( \sqrt{1 + 4\theta_k^2/\zeta_k^2} - 1 \right), \zeta_k = \theta_k^2 - \kappa^{-1} \)
8. end for

**Theorem 2.** (Lemma 22, 23). Let \( T^\ell \) represent Algorithm 1, pick any \( x \in \mathbb{R}^n \), and suppose Assumptions 2 and hold. Then for any \( \nu \in V \), \( \nu^+ = T^\ell(\nu, x) \) satisfies

\[ f(\nu^+, x) - f(\nu^+, x) \leq \lambda^+(H) \left( 1 - \frac{1}{\sqrt{\kappa(H)}} \right)^{\ell-1} \|\nu_0 - \nu^+\|_2^2. \]

We can also exploit strong convexity to show reduction of the error.

**Lemma 2.** Suppose the conditions of Theorem 2 hold and let \( \nu^+ = T^\ell(\nu, x) \). Then

\[ \|T^\ell(\nu, x) - S(x)\|_H \leq \eta_\ell(\ell) \|\nu - S(x)\|_H, \]

where \( \eta_\ell(\ell) = \sqrt{\kappa(H)} \left( 1 - \frac{1}{\sqrt{\kappa(H)}} \right)^{\ell-1} \) and \( \eta_\ell \in \mathcal{L} \).

**Proof.** Since \( S(x) \) minimizes [11], which is a strongly convex function in \( \nu \) for any \( x \), we have that

\[ f(\nu^+, x) \geq f(S(x), x) + \|\nu^+ - S(x)\|^2_H \forall \nu^+ \in V. \]

Combining (13) with \( \lambda^+(H) \|\nu^+ - S(x)\|_2^2 \leq \|\nu^+ - S(x)\|^2_H \) and Theorem 2 yields

\[ \|\nu^+ - S(x)\|_H^2 \leq \kappa(H) \left( 1 - \sqrt{\kappa(H)} \right)^{-1} \|\nu - S(x)\|_H^2. \]

Taking the square root of both sides completes the proof.

IV. Stability Analysis of the Coupled System

In this section, we analyze [12] using ISS tools. Our goal is to derive verifiable conditions under which the real-time implementation of MPC leads to an asymptotically stable closed-loop system.

A. Properties of the Solution Mapping

We begin with an analysis of the OCP solution mapping, since both subsystems depend strongly on its Lipschitz constant.

**Theorem 3.** Under Assumption 2 the solution mapping \( \Pi_V \) satisfies

\[ \lambda^-(H) \|S(x) - S(y)\|_2 \leq \|S(x) - S(y)\|_H \leq \|\tilde{G}\|_2 \|x - y\|_\rho, \]

and \( S(0) = 0 \), for all \( x, y \in \mathbb{R}^n \) where \( \tilde{G} = \sqrt{\kappa^{-1}} G \sqrt{\kappa^{-1}} \).

**Proof.** Assumption 2 implies \( H > 0 \). Therefore, [S] is a strongly monotone operator in \( \nu \) implying that \( S \) is a function. Moreover, \( \nu(x) = (0, 0) \) satisfies [S] so that \( S(0) = 0 \). To derive the Lipschitz constant, pick any \( x, y \in \mathbb{R}^n \) and let \( v = S(x), w = S(y) \). Due to the properties of the normal cone mapping

\[ Hv + Gx + N_V(v) \ni 0 \Leftrightarrow \langle Hv + Gx, v' - v \rangle \geq 0 \forall v', v \in V, \]

and \( x \in \mathbb{R}^n \). Applying this property to \( Hv + Gx + N_V(v) \ni 0 \) and \( Hw + GY + N_V(w) \ni 0 \), with \( w' = v \) and \( v' = w \), leads to

\[ \langle Hv + Gx, w - v \rangle \geq 0, \text{ and } \langle Hw + GY, v - w \rangle \geq 0. \]

Combining these inequalities yields

\[ \langle Hv + Gx, v - w \rangle \leq 0 \leq \langle Hw + GY, v - w \rangle, \]

\[ \Rightarrow \langle H(v - w), v - w \rangle \leq \langle G(y - x), v - w \rangle. \]
Finally, recognizing the left hand side of (17) as $\|v - w\|^2_H$ and applying the Cauchy-Schwarz inequality leads to
\[
\|v - w\|^2_H \leq (v - w)^T G (y - x), \tag{18a}
\]
\[
\leq \sqrt{T} (v - w)^T G \sqrt{T} (y - x), \tag{18b}
\]
\[
\leq \|G\|_2 \|v - w\|_H \|x - y\|_P, \tag{18c}
\]
which implies the right hand inequality in Theorem 3. Combining (18c) with $\lambda^-(H)\|v - w\|^2_H \leq \|v - w\|^2_H$ yields the left hand inequality and completes the proof. \hfill \square

### B. ISS Gain of Optimal MPC

Having detailed the properties of the solution mapping, it is now possible to derive an asymptotic gain for the closed-loop system
\[
x_{k+1} = Ax_k + B(K(x_k) + d_k), \tag{19}
\]
where the disturbance $d_k \in \mathbb{R}^m$ represents suboptimality due to incomplete optimization. We use the optimal cost of (17)
\[
V(x) = \|S(x)\|^2_H + 2S(x)^T G x + \|x\|^2_W, \tag{20}
\]
which coincides with the optimal cost of (2), as our candidate ISS-Lyapunov function. To streamline the remainder of our discussion, we introduce a few technical results.

**Lemma 3.** Under Assumptions 2 and 2, the variation of (20) satisfies the following bound,
\[
V(x + \delta) - V(x) \leq \beta^2(2\|x\|_P + \|\delta\|_P)\|\delta\|_P, \tag{21}
\]
for all $x, \delta \in \mathbb{R}^n$, with $\beta^2 = (\lambda^*_P(W) + 2\|G\|_2^2, \lambda^*_P(W) > 1$, and $G$ is defined in Theorem 3.

**Proof.** See appendix. \hfill \square

**Corollary 2.** Under Assumptions 2 and 2 the optimal cost function (20) satisfies the following
\[
\|x\|^2 \leq V(x) \leq \beta^2\|x\|^2. \tag{22}
\]

To establish an ISS bound for the ideal closed-loop system, we introduce the following backwards reachable set. Let $\chi(k, x, u)$ denote the solution of $x_{k+1} = Ax_k + Bu_k$ at time-step $k$ with an initial condition $x_0 = x$ and an input sequence $u = \{u_i\}_{i=0}^\infty$. Then, define
\[
\Gamma_N = \{ x \mid \exists u \in U \text{ st. } \chi(N, x, u) \in O_\infty \} \subseteq \mathbb{R}^n, \tag{23}
\]
where $O_\infty \subseteq \mathbb{R}^n$ is the maximal constraint admissible set for the system $x_{k+1} = (A - BK)x_k$ and $K = (R + B^T P B)^{-1}(B^T P A)$ is the linear quadratic regulator (LQR) gain.

**Assumption 3.** The initial condition satisfies $x_0 \in \Gamma_N$.

With these results in place, we now prove the following:

**Theorem 4.** Let Assumptions 2 and 2 hold, then, given suitable restrictions $\{d_k\} \subseteq D$ on the disturbance, the ideal closed-loop system (19) is ISS and satisfies $\kappa(x_k) \in U$, $\forall k \geq 0$. Moreover, its asymptotic gain $\gamma_1$ satisfies
\[
\lim_{k \to \infty} \|x_k\|_P \leq \gamma_1 \lim_{k \to \infty} \|Bd_k\|_P, \tag{24}
\]
where
\[
\gamma_1 = \frac{\beta^4}{\lambda^*_P(Q)} \left(1 + \sqrt{1 + \frac{\lambda^*_P(Q)}{\beta^4}}\right), \tag{25}
\]
and $D = \{d \mid \gamma_1 \|Bd\|_P \leq \lambda^-(P)\text{rad}(\Gamma_N)\}$.

**Proof.** Let
\[
x^+ = Ax + BK(x), \tag{26}
\]
\[
x^+ = Ax + B(K(x) + d), \tag{27}
\]
denote the ideal and disturbed one-step variations. Consider the ISS-Lyapunov Candidate Function (20) and let $x \in \Gamma_N$. As detailed in (17), the ideal one-step variation satisfies
\[
V(x^+) - V(x) \leq -\|x\|^2_Q, \tag{28}
\]
due to the choice of the terminal cost matrix $P$ and the restriction $x \in \Gamma_N$. Thus, it follows from Lemma 3 and (28) that
\[
V(x^+) - V(x) = [V(x^+) - V(x^0)] + [V(x^+) - V(x)], \leq \left[\beta^2\|Bd\|^2_P + 2\beta^2\|Bd\|_P\|x^+\|_P\right] - \|x\|^2_Q. \tag{29}
\]
Due to Corollary 2, $\|x^+\|^2 \leq V(x^+) < V(x) \leq \beta\|x\|^2_P$, thus
\[
V(x^+) - V(x) \leq \beta^2\|Bd\|^2_P + 2\beta^2\|Bd\|_P\|x\|_P - \lambda^*_P(Q)\|x\|^2_P. \tag{30}
\]
We now note that
\[
\|x\|_P \geq \frac{\gamma_1}{\beta} \|Bd\|_P \Rightarrow V(x^+) \leq V(x). \tag{31}
\]
By taking into account the upper bound in (22), we note that the set $\Omega = \{ x \mid V(x) \leq \gamma^2\|Bd\|^2_P \}$ is forward-invariant. Following from the lower bound in (22), $x^+_k \in \Omega$ implies $\|x_k\|^2 \leq \gamma_1\|Bd\|_P$. This is sufficient to obtain the stated ISS gain, see [25, Remark 3.3], as long as $\Omega \subseteq \Gamma_N$. Thus, we introduce the restriction $d_k \in D$, with $D = \{d \mid \gamma_1\|Bd\|_P \leq \lambda^-(P)\text{rad}(\Gamma_N)\}$.

**C. ISS Gain of the PGM methods**

Having shown that the closed-loop system is ISS under the ideal MPC feedback law, we now show that the optimizer dynamics (11) are ISS when $T$ is defined using the PGM. Moreover, we derive a computable expression for the corresponding asymptotic gain.

**Theorem 5.** Consider the optimizer dynamics $\nu^+ = T^f(\nu, x)$, that represents the PGM (12) and $T^f$ is defined in (11). Under Assumptions 2 and 2 the error signal, $e = \nu - S(x)$, is ISS with respect to the state update $\Delta x = x^+ - x$ and satisfies
\[
\lim_{k \to \infty} \|e_k\|_2 \leq \gamma_2(\ell) \lim_{k \to \infty} \|\Delta x_k\|_P, \tag{32}
\]
where $\gamma_2(\ell) = b \eta^f/(1 - \eta^f), \gamma_2 \in L, b = \|\dot{G}\|_2/\sqrt{\lambda^-(H)}$, and $\dot{G}, \eta$ are as defined in Theorems 3 and 2, respectively.

**Proof.** Combining Theorems 3 and 2 with the triangle inequality yields
\[
\|\nu^+ - S(x^+)\|_2 \leq \eta\|\nu - S(x^+)\|_2 \tag{33a}
\]
\[
\leq \eta\|\nu - S(x) + [S(x) - S(x^+)]\|_2 \tag{33b}
\]
\[
\leq \eta\|\nu - S(x)\|_2 + \eta\|S(x^+) - S(x)\|_2 \tag{33c}
\]
\[
\leq \eta\|\nu - S(x)\|_2 + \eta\|b\|\|x^+ - x\|_P, \tag{33d}
\]
where $b = \|\dot{G}\|_2/\sqrt{\lambda^-(H)}$ is the Lipschitz constant for $S$ derived in Theorem 3 and $\eta < 1$ is the convergence rate from Theorem 3. Since $\eta < 1$, it follows from [15, Example 3.4] that the error signal $e = \nu - S(x)$ is ISS with respect to the input $\Delta x = x^+ - x$ and satisfies
\[
\lim_{k \to \infty} \|e_k\|_2 \leq \gamma_2 \lim_{k \to \infty} \|\Delta x_k\|_2, \tag{34}
\]
with the asymptotic gain $\gamma_2 = \eta b/(1 - \eta^f)$. \hfill \square
D. ISS gain of the APGM method

Next, we show that (11) is ISS when $\mathcal{T}$ represents the APGM dynamics defined in Algorithm 1.

**Theorem 6.** Consider the optimizer dynamics $\nu^+ = \mathcal{T}^\ell(\nu, x)$, where $\mathcal{T}^\ell$ represents Algorithm 2. Under Assumptions 1 and 2, the error signal, $e = \nu - S(x)$ is ISS with respect to the state update $\Delta x = x^+ - x$ provided that $\ell > \bar{\ell}$ where $\ell$ is defined in Corollary 7.

Moreover, the optimizer system satisfies the asymptotic bound

$$\liminf_{k \to \infty} \|e_k\|_H \leq \frac{\gamma_2^2(\ell)}{\ell} \liminf_{k \to \infty} \|\Delta x_k\|_p,$$

with ISS gain $\gamma_2^2(\ell) = \eta_\ell(\ell)(\hat{G})^2/\ell(1 - \eta_\ell(\ell))$, where $\eta_\ell \in \mathcal{L}$ and $\hat{G}$ are defined in Lemma 2 and Theorem 1 respectively.

**Proof.** The proof is similar to that of Theorem 3. Applying Theorem 1 and Lemma 2 and following the same steps as in (11) using the $H$-norm in place of the 2-norm, yields

$$\rho + S(x^+)|h| \leq \eta_\ell(\ell)|\nu - S(x)|_H + \eta_\ell(\ell)(\hat{G})^2 \|x^+ - x\|_p.$$  

Invoking Corollary 4 we have $\eta_\ell(\ell) < 1$ for all $\ell > \bar{\ell}$. Thus the error signal $e = \nu - S(x)$ is ISS with respect to $\Delta x$, provided $\ell > \bar{\ell}$.

Following [15] Example 3.4) we derive the asymptotic bound

$$\liminf_{k \to \infty} \|e_k\|_H \leq \frac{\gamma_2^2(\ell)}{\ell} \liminf_{k \to \infty} \|\Delta x_k\|_p,$$

where the ISS gain is $\gamma_2^2(\ell) = \eta_\ell(\ell)(\hat{G})^2/\ell(1 - \eta_\ell(\ell))$. \hfill $\square$

E. Stability of the Interconnection

Having characterized the ISS properties of both the MPC and the optimizer, we now consider the interconnected system (4). The following theorem identifies sufficient conditions under which (4) is asymptotically stable when $\mathcal{T}$ is defined using the APGM.

**Theorem 7.** Suppose Assumptions 1 and 2 hold. Then, the closed-loop system (4) is asymptotically stable if $\zeta_1 \gamma_1 \gamma_2(\ell) < 1$, where $\zeta = 2\sqrt{\gamma_1 B \Xi}_2$, $\Xi$ is defined in (3), and $\gamma_1$, $\gamma_2$ are defined in Theorems 4 and 5.

**Proof.** To begin, note that

$$\liminf_{k \to \infty} \|\Delta x_k\|_p = \liminf_{k \to \infty} \|x_{k+1} - x_k\|_p,$$

$$\leq \liminf_{k \to \infty} \|x_{k+1} - x_k\|_p + \liminf_{k \to \infty} \|x_k\|_p = 2 \liminf_{k \to \infty} \|x_k\|_p$$

Thus, as a result of Theorem 4

$$\liminf_{k \to \infty} \|\Delta x_k\|_p \leq 2\gamma_1 \liminf_{k \to \infty} \|Bd_k\|_p,$$

provided the restriction $\gamma_1 \|Bd_k\|_p \in \Gamma_N$ holds for all $k \geq 0$. By specializing the input disturbance to $d_k = \Xi(\nu_k - S(x_k))$, it follows from the definitions of the $P$ norm that

$$\|Bd\|_p \leq \sqrt{\gamma_1 B \Xi}_2 \|\nu_k - S(x_k)\|_2.$$  

As a result, the MPC subsystem is not only ISS (as stated in Theorem 3), but it also satisfies

$$\liminf_{k \to \infty} \|\Delta x_k\|_p \leq \zeta_1 \liminf_{k \to \infty} \|e_k\|_2,$$

where $e = \nu - S(x)$, and $\zeta = 2\sqrt{\gamma_1 B \Xi}_2$. Moreover, the PGM subsystem in (4) is ISS with gain $\gamma_2$ by Theorem 5 and satisfies

$$\liminf_{k \to \infty} \|e_k\|_2 \leq \gamma_2(\ell) \liminf_{k \to \infty} \|\Delta x_k\|_p.$$

Combining (37) and (38) we conclude that

$$\liminf_{k \to \infty} \|e_k\|_2 \leq \zeta_1 \gamma_2(\ell) \liminf_{k \to \infty} \|e_k\|_2,$$

therefore, by the Small Gain Theorem 15, the interconnected system is asymptotically stable if $\zeta_1 \gamma_1 \gamma_2(\ell) < 1$ and

$$\gamma_1 \|Bd_k\|_p \leq \lambda(\gamma_1 \rho)\angle(\Gamma_N), \forall k \geq 0.$$  

As detailed in [26], there exists some $R \subseteq \Gamma_N \times R^{N_m} \neq \emptyset$ such that, if $(x_0, u_0) \in R$, then (40) is satisfied $\forall k \geq 0$. \hfill $\square$

**Corollary 3.** Under Assumptions 1 and 2 the PGM based closed-loop system is asymptotically stable if

$$\ell > \ell^* = \frac{\log(\zeta_1 b + 1)}{\log(\zeta_\ell)},$$

where $\zeta$, $\gamma_1$, $b$, and $\eta$ are defined in Theorems 4, 5, and 6. Moreover, since $\eta < 1$ and the other constants are finite, $\ell^* > 0$.

The following theorem mirrors Theorem 7 when $\mathcal{T}$ is defined using the APGM instead of the PGM.

**Theorem 8.** Suppose Assumptions 1 and 2 hold and let $e = \nu - S(x)$. Then, if $\mathcal{T}$ represents Algorithm 2, the corresponding closed-loop system (4) is asymptotically stable if $\zeta_1 \gamma_1 \gamma_2(\ell) < 1$ and $\ell > \bar{\ell}$, where $\zeta_\ell = 2\sqrt{\gamma_1 B \Xi}_2\angle(\Gamma_N), \ell$ is defined in Corollary 7 and $\gamma_1$, $\gamma_2$ are defined in Theorems 4 and 6.

**Proof.** The proof is nearly identical to that of Theorem 7. Simply replace (30) with $\|Bd\|_p \leq \sqrt{\gamma_1 B \Xi}_2\angle(\Gamma_N)$, and $\ell$ as defined in Corollary 7 and Theorem 6 (40).

The resulting small gain condition is

$$\liminf_{k \to \infty} \|e_k\|_H \leq \zeta_\ell \gamma_1 \gamma_2^2(\ell) \liminf_{k \to \infty} \|e_k\|_H,$$

and the restrictions to Theorems 4 and 6 are (40) and $\ell > \bar{\ell}$. \hfill $\square$

**Corollary 4.** Under Assumptions 1 and 2 the APGM based closed-loop system is asymptotically stable if $\ell > \max(\ell_0^*, \ell^*)$ where

$$\ell_0^* = 1 - 2\log((\hat{G})^2 \zeta_\ell \gamma_1 + 1) + \log(1 - 1/\sqrt{\gamma(H)})$$

and $\zeta_\ell$, $\gamma_1$, and $\hat{G}$, are defined in Theorems 4, 6, and 8.

V. DISCUSSION AND NUMERICAL EXAMPLES

Theorems 7 and 8 provide sufficient conditions for the stability of TDO-MPC under fairly strict assumptions (LTI system, convex input constraints, no state constraints, projected gradient-type solvers). Although less general than existing literature, (e.g. [2], [3]), the proposed setting provides new insight on possible mechanisms that can be leveraged to ensure convergence. To guide the discussion, we will consider two benchmark systems: one stable and one unstable.

**Jones System:** For the purpose of direct comparison with existing literature, we consider the stable system addressed in [13], i.e.

$$x^+ = \begin{bmatrix}
0.7 & -0.1 & 0 & 0 \\
0.2 & -0.5 & 0 & 0 \\
0 & 0.1 & 0 & 1 \\
0.5 & 0 & 0.5 & 0
\end{bmatrix} x + \begin{bmatrix}
0 & 0.1 \\
0.1 & 1 \\
0.1 & 0 \\
0 & 0
\end{bmatrix} u,$$

subject to the initial conditions $x_0 = [10 \ 10 \ 10 \ 10]^T$, input constraints $U = [-1, 1]$, and state cost with $Q = I$. Unless otherwise specified, the nominal values for the horizon length, input weighting matrix, and solver iterations are $N = 5$, $R = I$, $\ell = 10$.

**Inverted Pendulum:** We use a model of an inverted pendulum on a cart to investigate the closed-loop behavior of TDO-MPC for an unstable system. The equations of motion are

$$\frac{4}{3} ml^2 \ddot{\theta} - ml \ddot{y} = mlg \theta$$

$$(M + m) \ddot{y} - ml \ddot{\theta} = -bg + F,$$
where $y$ is the position of the cart, $\theta$ is the angle of the pendulum, $g = 9.81 \text{ m/s}^2$ is the gravitational constant, $M = 1 \text{ kg}$ is the mass of the cart, $m = 0.1 \text{ kg}$, $b = 0.1 \text{ Ns/m}$, and $l = 1 \text{ m}$ are the mass, damping coefficient, and length of the pendulum respectively. The states and control inputs are

$$x = [y \dot{y} \theta \dot{\theta}]^T, \quad u = F.$$  \hspace{1cm} (45)

The angle $\theta = 0$ corresponds to the upright position and the linear model is generated by linearization about the origin. Given the initial state $x_0 = [2 0 0 0]^T$, the control objective is to drive the system to the origin under constraints $U = [-1, 1]$. The control law is implemented using a sampling period of $\tau = 0.2 \text{ s}$ and state weighting matrix $Q = I$. Unless otherwise specified, the nominal values for the horizon length, input weighting matrix, PGM solver iterations, and APGM solver iterations are $N = 7$, $R = I$, $\ell_{\text{PGM}} = 10^5$, and $\ell_{\text{APGM}} = 10^4$.

The following subsections describe a few different mechanisms for ensuring the closed-loop stability of TDO-MPC and the advantages and disadvantages of each.

A. Increase solver iterations

As detailed in, e.g. [1], [12], [13], the obvious way to stabilize TDO-MPC is to increase the number of optimizer iterations per timestep. Indeed, as the ISS gain of PGM is $\gamma_2(\ell) = \frac{bn}{(1 - \eta^n)}$, with $\eta < 1$, it follows that $\gamma_2 \to 0$ monotonically as $\ell \to \infty$. Similarly, for $\ell > \ell$, the asymptotic gain of APGM (23) is such that $\gamma_2(\ell) \to 0$ monotonically as $\ell \to \infty$. Figures 2 and 3 illustrate how increasing $\ell$ can help achieve stability.

The main limitation with this option is that computation time is generally proportional to the number of iterations. As such, $\ell$ is effectively upper-bounded by the real-time requirements of the application.

B. Use preconditioning

Another option is to improve the condition number $\kappa$ to decrease the ISS gain of the optimizer. Indeed, since the convergence rate of PGM and APGM are proportional to $1 - \kappa^{-1}$ and $1 - \sqrt{\kappa^{-1}}$, respectively, it follows that $\eta \to 0$ and $\eta_{\text{APG}}(\ell) \to 0$ monotonically as $\kappa \to 1$. Further, the APGM is more suited to ill-conditions problems since its convergence rate depends on the square-root of the condition number. Figures 2 and 3 illustrate how preconditioning affects stability.

For the OCP in (7), an explicit preconditioning process can be performed by defining the preconditioned OCP

$$\min_{\nu \in \mathcal{V}} \quad f(\nu, x) = \hat{\nu}^T \hat{D}^T \hat{H} \hat{D} \nu + 2\hat{\nu}^T \hat{D}^T \hat{G} \dot{x} + x^T \hat{W} x,$$  \hspace{1cm} (46)

with $D \in \mathbb{S}_+^{b+}$, $\hat{Y} = D^{-1} Y$ and $\hat{\nu} = D^{-1} \nu$, such that $\kappa(\hat{D} \hat{H} \hat{D}) < \kappa(H)$. If $D$ is diagonal, the projection onto the transformed constraint set $\mathcal{V}$ remains simple. The optimal diagonal preconditioner $D$ can be computed by solving an offline convex semi-definite programming problem, see [13] Section V-C. Since preconditioning always improves convergence, we will only consider the preconditioned variants of each algorithm for the remainder of the examples.

Although there is a limit to how much $\kappa$ can be reduced, there is virtually no drawback to diagonal preconditioning strategies. When appropriate, pre-stabilization of (4) is another effective tool, as is non-diagonal preconditioning. These methods lead to polyhedral constraint sets that cannot be easily projected onto. As such the use of dual methods such as GPAD [27] for TDO is a promising direction for future work.

C. Tune the cost function

A third mechanism for stabilizing TDO-MPC is to adjust the weighting matrices in the OCP (4). Indeed, for a fixed $Q$, the value of $R$ impacts $\gamma_1$ and $\gamma_2$ through a variety of different, and sometimes opposing, mechanisms:

- **Condition Number:** Since $H = \hat{B}^T \hat{H} \hat{B} + (I_N \otimes R)$, increasing $\lambda^- (R)$ will reduce $\lambda^+ (H)$, which reduces $\gamma_2$;

- **Feedback Gain:** Since $R$ penalizes the control effort, increasing $\lambda^- (R)$ decreases the Lipschitz constant of the solution mapping $S(x) \leq \|G\| \|x\|$;

- **Closed-loop Cost:** Given $\lambda^- (R) \to \infty$ the optimal cost of the closed-loop system tends to the cost of the open-loop system subject to $u = 0$. Thus, the matrix $W$ in (7) satisfies $W \preceq U$, where $U$ satisfies the Lyapunov equation $U = Q + A^T U A$. If $A$ is Schur, $U$ exists and $W$ is bounded. Otherwise, the closed-loop cost $W$ will grow unbounded.

Since the MPC gain satisfies $\gamma_1 \propto (\|G\| + \lambda^+_R (W))^2$, the dependency between $\gamma_1$ and $\lambda^- (R)$ is not monotonic and is contingent on the properties of the state matrix $A$. For both stable and unstable systems, increasing $\lambda^- (R)$ reduces $\|G\|$. However for unstable systems, the dependency on $\lambda^+_R (W)$ will eventually lead to an increase in the MPC gain $\gamma_1$. Figures 4 and 5 illustrate the effects of $R$ on the benchmark systems.

Even in the case of a stable system, it may not be desirable to select an arbitrarily large input penalty because it typically leads to longer response times. As such, the choice of $R$ is subject to a trade-off between the stability and performance of TDO-MPC.

D. Decrease the prediction horizon

The final option for stabilizing TDO-MPC is to decrease the horizon length $N$. Although this solution is counter-intuitive, equations

$^1$Instability in (4) leads to ill-conditioning of $H$. 

$^2$Instability in (7) grows unbounded.
in appendix show that reducing $N$ will reduce $\lambda_p(W)$. This will decrease $\gamma_1$ as detailed in the previous subsection. Moreover, it follows from \((6)\) and $H = \tilde{B}^T H \tilde{B} + (I_N \otimes R)$, that reducing $N$ also improves the condition number $\kappa(H)$, thus decreasing $\gamma_2$. Figures \(6\) \(7\) illustrate how reducing $N$ can help stabilize TDO-MPC.

The main drawback of this option is that our analysis relies on Assumption \(3\). Since reducing $N$ reduces the set of initial conditions for which TDO-MPC is applicable, the prediction horizon is effectively lower-bounded by the recursive feasibility condition.

### VI. Comparisons

For stable systems, the iteration bounds $\ell^*$ presented in Figure \(6\) falls between a range of 4 to 10 iterations for both the PGM and APGM. This is consistent with the range reported in \([13\) Figure 2d]. Moreover, rather than guaranteeing an arbitrary suboptimality bound, our analysis directly certifies closed-loop stability. The PGM gives rise to better bounds in this case since it is a descent method and the Jones system is relatively well conditioned.

For unstable systems, Figure \(8\) shows that the overall trend obtained for $\ell^*$ is consistent with simulation results. Although these values are too conservative for certification purposes, our analysis provides useful guidelines for the design and tuning of TDO-MPC.

### VII. Conclusion

This paper analyzed the closed-loop properties of a class of TDO-MPC and provided detailed guidelines on how to ensure asymptotic stability using a variety of mechanisms, namely: increasing the number of solver iterations, using preconditioning techniques, tuning the weighting matrices in the cost, and decreasing the prediction horizon. Future work will focus on tightening the bound for unstable systems, investigating the use of dual methods in TDO, and proving that the proposed guidelines are applicable to more general TDO-MPC settings.
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APPENDIX

A. PROOF OF LEMMA 1

First, define

\[ W_1 = \sum_{k=0}^{N-1} (A^k)T Q(A^k), \quad W_2 = (A^N)^T P(A^N) \]  

so that \( W = W_1 + W_2 \). By Assumption 1, \( A^T PA = P - Q + (A^T PB)(R + B^T PB)^{-1}(B^T PA) \).

Using (48) to express \( (A^k)^T P(A^k) \) and summing up from \( k = 1 \) to \( N - 1 \) yields

\[ W = P + \sum_{k=1}^{N} (A^k)^T PB(R + B^T PB)^{-1}B^T P(A^k). \]

Since the assumptions of Lemma 1 imply \( P > 0 \) the result follows.

B. PROOF OF LEMMA 3

(i) From the triangle inequality,

\[ \|x + \delta\|^2_P \leq \|x\|^2_P + \|\delta\|^2_P. \]

Combining this with the properties of the matrix norm we obtain,

\[ \|x + \delta\|^2_P \leq \lambda_P^+(W)(2\|x\|_P\|\delta\|_P + \|\delta\|^2_P). \]

(ii) Using the properties of norms we have that

\[ \|S(x + \delta)\|^2_H \leq \|S(x + \delta) - S(x)\|^2_H + \|S(x + \delta) - S(x)\|_H. \]

\[ \leq \|\Delta S\|^2_H + 2\|\Delta S\|_H\|\Delta S\|_H + \|\Delta S\|^2_H, \]

where \( \Delta S = S(x + \delta) - S(x) \). Combining this with Theorem 3 we obtain that

\[ \|S(x + \delta)\|^2_H - \|S(x)\|^2_H \leq \|\Delta S\|^2_H + 2\|\Delta S\|_H\|\Delta S\|_H. \]

(iii) Recall that \( \hat{G} = \sqrt{H}^T G \sqrt{P}^{-1} \), so that

\[ (S(x + \delta) - S(x))^T Gx = \sqrt{H}(S(x + \delta) - S(x))^T \hat{G} \sqrt{P} x. \]

\[ \|S(x + \delta) - S(x)^T Gx\| \leq \|\hat{G}\|_2\|x\|_P + \|\Delta S\|_H\|\Delta S\|_H + \|\Delta S\|^2_H, \]

where the second inequality follows from Theorem 3.

(iv) It follows from the definition of \( \hat{G} \) that

\[ S(x + \delta)^T G\delta = (\sqrt{H}(S(x + \delta) - S(x))^T \hat{G} \sqrt{P} \delta. \]

Thus, using Theorem 3 and that \( S(0) = 0 \), we obtain that

\[ \|S(x + \delta)^T G\delta\| \leq \|\hat{G}\|_2\|S(x + \delta) - S(0)\|_H\|\delta\|_P, \]

\[ \leq \|\hat{G}\|_2\|x + \delta\|_P\|\delta\|_P, \]

\[ \leq \|\hat{G}\|_2\|x\|_P + \|\Delta S\|_H\|\Delta S\|_H\|\delta\|_P. \]

Combining (i)-(iv) and collecting like terms completes the proof.