THE $h$-VECTOR OF A LADDER DETERMINANTAL RING COGENERATED BY $2 \times 2$ MINORS IS LOG-CONCAVE

MARTIN RUBEY

Abstract. We show that the $h$-vector of a ladder determinantal ring cogenerated by $M = [u_1 \mid v_1]$ is log-concave. Thus we prove an instance of a conjecture of Stanley, resp. Conca and Herzog.

In honour of Miriam Rubey, at the occasion of her second birthday

1. Introduction

Definition 1.1. A sequence of real numbers $a_1, a_2, \ldots, a_n$ is logarithmically concave, for short log-concave, if $a_{i-1}a_{i+1} \leq a_i^2$ for $i \in \{2, 3, \ldots, n-1\}$.

Numerous sequences arising in combinatorics and algebra have, or seem to have this property. In the paper [13] written in 1989, Richard Stanley collected various results on this topic. (For an update see [3].) There he also stated the following conjecture:

Conjecture 1.2. Let $R = R_0 \oplus R_1 \oplus \ldots$ be a graded (Noetherian) Cohen-Macaulay (or perhaps Gorenstein) domain over a field $K = R_0$, which is generated by $R_1$ and has Krull dimension $d$. Let $H(R, m) = \dim_K R_m$ be the Hilbert function of $R$ and write

$$\sum_{m \geq 0} H(R, m)x^m = (1 - x)^{-d} \sum_{i=0}^{s} h_ix^i.$$ 

Then the sequence $h_0, h_1, \ldots, h_s$ is log-concave.

The sequence $h_0, h_1, \ldots, h_s$ is called the $h$-vector of the ring. Originally the question was to decide whether a given sequence can arise as the $h$-vector of some ring. In this sense the validity of the conjecture would imply that log-concavity was a necessary condition on the $h$-vector.

It is now known however [12, 3] that Stanley’s conjecture is not true in general. Several natural weakenings have been considered, but are still open. For example, Aldo Conca and Jürgen Herzog conjectured that the $h$-vector would be log-concave for the special case where $R$ is a ladder determinantal ring. (Note that ladder determinantal rings are Cohen-Macaulay, as was shown in [8 Corollary 4.10], but not necessarily Gorenstein.) We will prove the conjecture of Conca and Herzog in the simplest case, i.e., where $R$ is a ladder determinantal ring cogenerated by $2 \times 2$ minors, see Corollary 4.6.

In the case of ladder determinantal rings the $h$-vector has a nice combinatorial interpretation. This follows from work of Abhyankar and Kulkarni [1, 2, 10, 11], Bruns, Conca, Herzog, and Trung [4, 5, 6, 8]. In the following paragraphs, which are taken almost verbatim from [9], we will explain these matters.
2. Ladders, ladder determinantal rings and non-intersecting lattice paths

First we have to introduce the notion of a ladder:

**Definition 2.1.** Let $X = (x_{i,j})_{0 \leq i \leq b, 0 \leq j \leq a}$ be a $(b+1) \times (a+1)$ matrix of indeterminates. Let $Y = (y_{i,j})_{0 \leq i \leq b, 0 \leq j \leq a}$ be another matrix of the same dimensions, with the property that $y_{i,j} \in \{0, x_{i,j}\}$, and if $y_{i,j} = x_{i,j}$ and $y_{i',j'} = x_{i',j'}$, where $i \leq i'$ and $j \leq j'$, then $y_{r,s} = x_{r,s}$ for all $r$ and $s$ with $i \leq r \leq i'$ and $j \leq s \leq j'$. Such a matrix $Y$ is called a ladder.

A ladder region $L$ is a subset of $\mathbb{Z}^2$ with the property that if $(i,j)$ and $(i',j') \in L$, $i \leq i'$ and $j \geq j'$ then $(r,s) \in L$ for all $r \in \{i,i+1, \ldots, i'\}$ and $s \in \{j',j'+1, \ldots, j\}$. Clearly, a ladder region can be described by two weakly increasing functions $L$ and $\overline{L}$, such that $L$ is exactly the set of points $\{(i,j) ; L(i) \leq j \leq \overline{L}(i)\}$.

We associate with $Y$ a ladder region $L \subset \mathbb{Z}^2$ via $(j,b-i) \in L$ if and only if $y_{i,j} = x_{i,j}$.

In Figure 1a an example of a ladder with $a = 8$ and $b = 9$ is shown, the corresponding ladder region is shown in Figure 1b.

Now we can define the ring we are dealing with:

**Definition 2.2.** Given a $(b+1) \times (a+1)$ matrix $Y$ which is a ladder, fix a “bivector” $M = [u_1, u_2, \ldots, u_n \mid v_1, v_2, \ldots, v_n]$ of integers with $1 \leq u_1 < u_2 < \cdots < u_n \leq b+1$ and $1 \leq v_1 < v_2 < \cdots < v_n \leq a+1$. By convention we set $u_{n+1} = b+2$ and $v_{n+1} = a+2$.

Let $K[Y]$ denote the ring of all polynomials over some field $K$ in the $y_{i,j}$’s, where $0 \leq i \leq b$ and $0 \leq j \leq a$. Furthermore, let $I_M(Y)$ be the ideal in $K[Y]$ that is generated by those $t \times t$ minors of $Y$ that contain only nonzero entries, whose rows form a subset of the last $u_t-1$ rows or whose columns form a subset of the last $v_t-1$ columns, $t \in \{1,2,\ldots,n+1\}$. Thus, for $t = n+1$ the rows and columns of minors are unrestricted.

The ideal $I_M(Y)$ is called a ladder determinantal ideal generated by the minors defined by $M$. We call $R_M(Y) = K[Y]/I_M(Y)$ the ladder determinantal ring cogenerated by the minors defined by $M$, or, in abuse of language, the ladder determinantal ring cogenerated by $M$.

Note that we could restrict ourselves to the case $u_1 = v_1 = 1$, because all the elements of $Y$ that are in one of the last $u_1-1$ rows or in one of the last $v_1-1$ columns are in the ideal.

Next, we introduce the combinatorial objects that will accompany us throughout the rest of this paper:

**Definition 2.3.** A two-rowed array of length $k$ is a pair of strictly increasing sequences of integers, both of length $k$. A two-rowed array $T = (v_1, v_2 \ldots v_k)$ is bounded by $A = (A_1, A_2)$ and $E = (E_1, E_2)$, if $A_1 \leq a_1 < a_2 < \cdots < a_k \leq E_1 - 1$ and $A_2 + 1 \leq b_1 < b_2 < \cdots < b_k \leq E_2$.

Given any subset $L$ of $\mathbb{Z}^2$, we say that the two-rowed array $T$ is in $L$, if $(a_i, b_i) \in L$ for $i \in \{1,2,\ldots,k\}$. By $T^L_k(A \rightarrow E)$ we will denote the set of two-rowed arrays of length $k$, bounded by $A$ and $E$ which are in $L$. The total length of a family of two-rowed arrays is just the sum of the lengths of its members.
Let $T_1 = (a_1, a_2, \ldots, a_k)$ and $T_2 = (b_1, b_2, \ldots, b_l)$ be two-rowed arrays bounded by $A^{(1)} = (A_1^{(1)}, A_2^{(1)})$ and $E^{(1)} = (E_1^{(1)}, E_2^{(1)})$ and $A^{(2)} = (A_1^{(2)}, A_2^{(2)})$ and $E^{(2)} = (E_1^{(2)}, E_2^{(2)})$ respectively. Set $a_{k+1} = E_1^{(1)}$ and $b_0 = A_2^{(1)}$. We say that $T_1$ and $T_2$ intersect if there are indices $I$ and $J$ such that

$$(\times) \quad \begin{align*}
    x_I &\leq a_I \\
    b_{J-1} &\leq y_J
\end{align*}$$

where $1 \leq I \leq k+1$ and $1 \leq J \leq l$. A family of two-rowed arrays is non-intersecting if no two arrays in it intersect.

Note that a two-rowed array in $T_k^b(A \rightarrow E)$ can be visualized by a lattice path with east and north steps, that starts in $A$ and terminates in $E$ and has exactly $k$ north-east turns which are all in $L$: Each pair $(a_i, b_i)$ of a two-rowed array $(a_1, \ldots, a_k)$ then corresponds to a north-east turn of the lattice path. It is easy to see that Condition $(\times)$ holds if and only if the lattice paths corresponding to $T_1$ and $T_2$ intersect.

For an example see Figure 1c, where the three two-rowed arrays

$$T^{(1)} = \begin{pmatrix} 2 & 3 \\ 6 & 7 \end{pmatrix}, \quad T^{(2)} = \begin{pmatrix} 3 & 5 \\ 4 & 6 \end{pmatrix}, \quad \text{and} \quad T^{(3)} = \begin{pmatrix} 2 & 4 & 6 \\ 1 & 3 & 4 \end{pmatrix}$$

bounded by $A^{(1)} = (0, 3)$, $A^{(2)} = (0, 2)$, $A^{(3)} = (0, 0)$ and $E^{(1)} = (5, 9)$, $E^{(2)} = (7, 9)$, $E^{(3)} = (8, 9)$ are shown as lattice paths. The points of the ladder-region $L$ are drawn as small dots, the circles indicate the start- and endpoints and the big dots indicate the north-east turns.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{ladder_region.png}
\caption{A ladder with $a = 8$ and $b = 9$, the corresponding ladder region, and a triple of non-intersecting lattice paths in this ladder.}
\end{figure}

3. A COMBINATORIAL INTERPRETATION OF THE $h$-VECTOR OF A LADDER DETERMINANTAL RING

We are now ready to state the theorem which reveals the combinatorial nature of the $h$-vector of $R_M(Y) = K[Y]/I_M(Y)$, the ladder determinantal ring cogenerated by $M$. 

Theorem 3.1. Let $Y = (y_{i,j})_{0 \leq i < b, \ 0 \leq j < a}$ be a ladder and let $M = [u_1, u_2, \ldots , u_n \mid v_1, v_2, \ldots , v_n]$ be a bivector of integers with $1 \leq u_1 < u_2 < \cdots < u_n \leq a + 1$ and $1 \leq v_1 < v_2 < \cdots < v_n \leq b + 1$. For $i \in \{1, 2, \ldots , n\}$ let

$$A^{(i)} = (0, u_{n+1-i} - 1)$$

$$E^{(i)} = (a - v_{n+1-i} + 1, b).$$

Let $L^{(1)} = L$ be the ladder region associated with $Y$ and for $i \in \{1, 2, \ldots , n - 1\}$ let

$$L^{(i)} = \{(x, y) \in L^{(i+1)} : x \leq E^{(i)}, y \geq A^{(i)} \text{ and } (x+1, y-1) \in L^{(i+1)}\}.$$

Finally, for $i \in \{1, 2, \ldots , n\}$ let

$$B^{(i)} = \{(x, y) \in L^{(i)} : (x+1, y-1) \notin L^{(i)}\}.$$

and let $d$ be the cardinality of $\bigcup_{i=1}^{n} B^{(i)}$.

Then, under the assumption that all of the points $A^{(i)}$ and $E^{(i)}$, $i \in \{1, 2, \ldots , n\}$, lie inside the ladder region $L$, the Hilbert series of the ladder determinantal ring $R_M(Y) = K[Y] / I_M(Y)$ equals

$$\sum_{\ell \geq 0} \dim_K R_M(Y)_\ell z^\ell = \sum_{\ell \geq 0} \left| T^L_\ell(A \mapsto E) \right| \frac{z^\ell}{(1-z)^d}.$$

Here, $R_M(Y)_\ell$ denotes the homogeneous component of degree $\ell$ in $R_M(Y)$ and $\left| T^L_\ell(A \mapsto E) \right|$ is the number of non-intersecting families of two-rowed arrays with total length $\ell$, such that the $i$th two-rowed array is bounded by $A^{(i)}$ and $E^{(i)}$ and is in $L^{(i)} \setminus B^{(i)}$ for $i \in \{1, 2, \ldots , n\}$.

The sets $B^{(i)}$, $i \in \{1, 2, \ldots , n\}$ can be visualized as being the lower-right boundary of $L^{(i)}$. Viewed as a path, there are exactly $E^{(i)}_1 - A^{(i)}_1 + E^{(i)}_2 - A^{(i)}_2 + 1$ lattice points on $B^{(i)}$, but not all of them are necessarily in $L$. However, if $L$ is an upper ladder, that is, $(a,0) \in L$, then this must be the case and we have

$$d = \sum_{i=1}^{n} \left( E^{(i)}_1 - A^{(i)}_1 + E^{(i)}_2 - A^{(i)}_2 + 1 \right)$$

$$= \sum_{i=1}^{n} (a - v_{n+1-i} + 1 + b - u_{n+1-i} + 1 + 1)$$

$$= n(a + b + 3) - \sum_{i=1}^{n} (u_i + v_i),$$

as in [9].

In Figure 2a, an example for a ladder region $L$ with $a = 8$ and $b = 9$ is given. The small dots represent elements of $L$, the circles on the left and on the top of $L$ represent the points $A^{(i)}$ and $E^{(i)}$, $i \in \{1, 2, 3\}$ that are specified by the minor $M = [1, 3, 4 \mid 1, 2, 4]$. The dotted lines indicate the lower boundary of $L^{(i)}$. Note that the point $(4,9)$ is not an element of $L$. Therefore, in this example we have

$$d = n(a + b + 3) - \sum_{i=1}^{n} (u_i + v_i) - 1 = 44.$$
a. a ladder region with $a = 8$ and $b = 9$

b. a 10 dimensional face of $\Delta_{[1,3,4][1,2,4]}(Y)$

**Figure 2.**

**Figure 3.** Constructing a family of non-intersecting lattice paths, such that the $i^{th}$ path stays above $L^{(i)}$, $i \in \{1,2,3\}$

**Figure 4.** The corresponding family of non-intersecting lattice paths, where the $i^{th}$ path has north-east turns only in $L^{(i)}$ for $i \in \{1,2,3\}$
Proof: We will use results of Jürgen Herzog and Ngô Viêt Trung. In Section 4 of [8], ladder determinantal rings are introduced and investigated.

We equip the indeterminates \( x_{i,j}, i \in \{0,1,\ldots,b\} \) and \( j \in \{0,1,\ldots,a\} \) with the following partial order:

\[
x_{i,j} \leq x_{i',j'} \text{ if } i \geq i' \text{ and } j \leq j'.
\]

A \( t \)-antichain in this partial order is a family of elements \( x_{r_1,s_1}, x_{r_2,s_2}, \ldots, x_{r_t,s_t} \) such that \( r_1 < r_2 < \cdots < r_t \) and \( s_1 < s_2 < \cdots < s_t \). Thus, a \( t \)-antichain corresponds to a sequence \( (s_1, b - r_1), (s_2, b - r_2), \ldots, (s_t, b - r_t) \) of \( t \) points in the ladder region associated with \( Y \), where each point lies strictly south-east of the previous ones.

Let \( D_t \) be the union of the last \( u_t - 1 \) rows and the last \( r_t - 1 \) columns of \( Y \). Let \( \Delta_M(Y) \) be the simplicial complex whose \( k \)-dimensional faces are subsets of elements of \( Y \) of cardinality \( k + 1 \) which do not contain a \( t \)-antichain in \( D_t \) for \( t \in \{1,2,\ldots,n + 1\} \). Let \( f_k \) be the number of \( k \)-dimensional faces of \( \Delta_M(Y) \) for \( k \geq 0 \). Then, Corollary 4.3 of [8] states, that

\[
\dim_K R_M(Y)_k = \sum_{k \geq 0} \binom{\ell - 1}{k} f_k.
\]

In the following, we will find an expression for the numbers \( f_k \) involving certain families of non-intersecting lattice paths.

In Figure 2b, a 10-dimensional face of \( \Delta_{[1,3,4][1,2,4]}(Y) \) is shown, the elements of the face are indicated by bold dots. We will describe a modification of Viennot’s ‘light and shadow procedure’ (with the sun in the top-left corner) that produces a family of \( \ell \)-non-intersecting lattice paths such that the \( i \)-th path runs from \( A^{(i)} = (0, u_{n+1-i}) \) to \( E^{(i)} = (a - v_{n+1-i}, b) \) and has north-east turns only in \( L^{(i)} \), for \( i \in \{1,2,\ldots,n\} \).

Imagine a sun in the top-left corner of the ladder region and a wall along the lower-right border \( B^{(1)} \) of \( L^{(1)} \). Then each lattice point \( (r,s) \) that is either in \( B^{(1)} \) or corresponds to an element \( x_{s,b-r} \) of the face casts a ‘shadow’ \( \{ (x,y) : x \geq r, y \leq s \} \).

The first path starts at \( A^{(1)} \), goes along the north-east border of this shadow and terminates in \( E^{(1)} \). In the left-most diagram of Figure 3 this is accomplished for the face shown in Figure 2b.

In the next step, we remove the wall on \( B^{(1)} \) and all the elements of the face which correspond to lattice points lying on the first path. Then the procedure is iterated. See Figure 3 for an example. Let \( P \) be the resulting family of non-intersecting lattice paths.

Now, for each \( i \in \{1,2,\ldots,n\} \), we remove all elements of the face except those which correspond to north-east turns of the \( i \)-th path and do not lie on \( B^{(i)} \). In the example, \((5,8)\) is a north-east turn of the second path but lies on \( B^{(2)} \), therefore the corresponding element \( x_{1,5} \) of the face is removed. On the other hand, \((4,5)\) lies on \( B^{(3)} \), but is a nort-east turn of the third path, so the corresponding element \( x_{4,4} \) of the face is kept.

This set of north-east turns defines another family of non-intersecting lattice paths \( P' \) that has the property that the \( i \)-th path has north-east turns only in \( L^{(i)} \) for \( i \in \{1,2,\ldots,n\} \).

We now want to count the number of faces of \( \Delta_M(Y) \) that reduce under ‘light and shadow’ to a given family of lattice paths \( P' \) with this property. Clearly, \( P' \) can be translated into a family \( P \) of non-intersecting lattice paths such that the \( i \)-th
path does not go below \( B^{(i)} \) for \( i \in \{1, 2, \ldots, n\} \). Note that the number of lattice points on such a family \( P \) of paths is always equal to \( d \), independently of the given face. Thus, if \( m \) is the number of north-east turns of \( P' \), there are

\[
\binom{d - m}{k + 1 - m}
\]

families of non-intersecting lattice paths \( P \) that reduce to \( P' \).

Hence, \( f_k = \binom{d - m}{k + 1 - m} \left| T^L_k(A \mapsto E) \right| \) and we obtain

\[
\sum_{\ell \geq 0} \dim_K R_M(Y)_\ell z^\ell = \sum_{\ell \geq 0} \left( \sum_{k \geq 0} \binom{\ell - 1}{k} f_k \right) z^\ell
\]

\[
= \sum_{m \geq 0} \left| T^L_m(A \mapsto E) \right| \sum_{\ell \geq 0} \sum_{k \geq 0} \binom{\ell - 1}{k} \binom{d - m}{k + 1 - m} \left| T^L_m(A \mapsto E) \right| z^\ell
\]

and if we sum the inner sum by means of the Vandermonde summation (see for example [7], Section 5.1, (5.27)),

\[
\sum_{\ell \geq 0} \dim_K R_M(Y)_\ell z^\ell = \sum_{m \geq 0} \left| T^L_m(A \mapsto E) \right| \sum_{\ell \geq 0} z^\ell \binom{d + \ell - m - 1}{d - 1}
\]

\[
= \frac{\sum_{m \geq 0} \left| T^L_m(A \mapsto E) \right| z^m}{(1 - z)^d}.
\]

\[
\square
\]

4. Log-concavity of the \( h \)-vector in the case \( M = [u_1 \mid v_1] \)

In this paper we will settle Stanley’s conjecture when \( R \) is a ladder determinantal ring cogenerated by \( M \), where \( M \) is just a pair of integers, i.e., \( n = 1 \). We want to stress, however, that data strongly suggest that Conca and Herzog’s conjecture is also true for arbitrary \( n \).

By the preceding theorem, in the case we are going to tackle, the sum \( \sum_{i=0}^n h_i x^i \) that appears in the conjecture is the generating function \( \sum_{k \geq 0} \left| T^L_k(A \mapsto E) \right| z^k \) of two-rowed arrays bounded by \( A \) and \( E \) which are in the ladder region \( L \).

As the bounds \( A \) and \( E \) will not be of any significance throughout the rest of this paper, we will abbreviate \( T^L_k(A \mapsto E) \) to \( T^L_k \). We will show that the \( h \)-vector is log-concave by constructing an injection from \( T^L_{k+1} \times T^L_{k-1} \) into \( T^L_k \times T^L_k \). This injection will involve some cut and paste operations that we now define:

**Definition 4.1.** Let \( A \) and \( X \) be two strictly increasing sequences of integers, such that the length of \( X \) is the length of \( A \) minus two, i.e., \( A = (a_1, a_2, \ldots, a_{k+1}) \) and \( X = (x_1, x_2, \ldots, x_{k-1}) \) for some \( k \geq 1 \). A cutting point of \( A \) and \( X \) is an index \( l \in \{1, 2, \ldots, k\} \) such that

\[
(*) \quad a_l < x_l,
\]

and

\[
x_{l-1} < a_{l+1},
\]

where we require the inequalities to be satisfied only if all variables are defined. Hence, 1 is a cutting point if \( a_1 < x_1 \), and \( k \) is a cutting point if \( x_{k-1} < a_{k+1} \).
The image of $A$ and $X$ obtained by cutting at $l$ is
\[
\begin{array}{cccccc}
a_1 & a_2 & \ldots & a_{l-1} & a_l & x_l & x_{l+1} & \ldots & x_{k-1} \\
x_1 & x_2 & \ldots & x_{l-1} & a_{l+1} & a_{l+2} & \ldots & \ldots & a_{k+1}
\end{array}
\]

Note that both the resulting sequences have length $k$.

**Lemma 4.2.** Let $A = (a_1, a_2, \ldots, a_{k+1})$ and $X = (x_1, x_2, \ldots, x_{k-1})$ be strictly increasing sequences of integers, such that the length of $X$ is the length of $A$ minus two. Then there exists at least one cutting point of $A$ and $X$.

**Proof.** If $a_l \geq x_l$ for $l \in \{1, 2, \ldots, k-1\}$ then $a_{k+1} > a_{k-1} \geq x_{k-1}$ and $k$ is a cutting point. Otherwise, let $l$ be minimal such that $a_l < x_l$. If $l = 1$ then 1 is a cutting point. Otherwise, because of the minimality of $l$, we have $a_{l+1} > a_{l-1} \geq x_{l-1}$, thus $l$ is a cutting point. \qed

**Definition 4.3.** Let $T = (T_1, T_2) \in T_{k+1} \times T_{k-1}$ be a pair of two-rowed arrays. Then a top cutting point of $T$ is a cutting point of the top rows of $T_1$ and $T_2$ and a bottom cutting point of $T$ is a cutting point of the bottom rows of $T_1$ and $T_2$.

A pair $(l, m)$, where $l, m \in \{1, 2, \ldots, k\}$, such that $l$ is a top cutting point and $m$ is a bottom cutting point of $T_1$ and $T_2$ is a cutting point of $T$. Cutting the top rows of $T$ at $l$ and the bottom rows at $m$ we obtain the image of $T$. Note that both of the two-rowed arrays in the image have length $k$. More pictorially, if $l < m$,
\[
\begin{array}{cccccc}
a_1 & \ldots & a_l & x_l & \ldots & x_{m-1} & x_{m} & \ldots & x_{k-1} \\
b_1 & \ldots & b_{l+1} & b_m & \ldots & b_{m-1} & b_{m} & \ldots & b_{k+1}
\end{array}
\]

and similarly if $l \geq m$.

For $T = (T_1, T_2) \in T_{k+1}^L \times T_{k-1}^L$, the pair $(l, m)$ is an allowed cutting point of $T$, if both of the two-rowed arrays in the obtained image are in $L$.

In Lemma 5.1 we will prove that every pair of two-rowed arrays in $T_{k+1}^L \times T_{k-1}^L$ has at least one allowed cutting point. This motivates the following definition:

**Definition 4.4.** Let $T = (T_1, T_2) \in T_{k+1}^L \times T_{k-1}^L$ a pair of two-rowed arrays as before. Consider all allowed cutting points $(l, m)$ of $T$. Select those with $|l - m|$ minimal. Among those, let $(l, m)$ be the pair which comes first in the lexicographic order. Then we call $(l, m)$ the optimal cutting point of $T$.

Now we are ready to state our main theorem, which implies that Stanley’s conjecture is true, when $R$ is a ladder determinantal ring cogenerated by a pair of integers $M$:

**Theorem 4.5.** Let $L$ be a ladder region. Let $T \in T_{k+1}^L \times T_{k-1}^L$. Define $I(T)$ to be the pair of two-rowed arrays obtained by cutting $T$ at its optimal cutting point. Then $I$ is well-defined and an injection from $T_{k+1}^L \times T_{k-1}^L$ into $T^L \times T^L$.

**Corollary 4.6.** The $h$-vector of the ladder determinantal ring cogenerated by $M = [u_1 | v_1]$ is log-concave.

**Proof of the corollary.** By Theorem 3.3, the $h$-vector of this ring is equal to the generating function $\sum_{k \geq 0} |T_k^L(A \rightarrow E)| z^k$ of two-rowed arrays bounded by $A = (0, u_1 - 1)$ and $E = (a - v_1 + 1, b)$ which are in the ladder region $L$. By the
preceding theorem, there is an injection from $\mathcal{T}^L_{k+1}(A \mapsto E) \times \mathcal{T}^L_{k-1}(A \mapsto E)$ into $\mathcal{T}^L_{k}(A \mapsto E) \times \mathcal{T}^L_{k}(A \mapsto E)$, thus

$$|\mathcal{T}^L_{k+1}(A \mapsto E)| \cdot |\mathcal{T}^L_{k-1}(A \mapsto E)| \leq |\mathcal{T}^L_{k}(A \mapsto E)|^2.$$ 

\[ \Box \]

We will split the proof of Theorem 4.5 in two parts. In Section 5, we show that the mapping $I$ is well-defined, that is, for any pair of two-rowed arrays $T \in \mathcal{T}^L_{k+1} \times \mathcal{T}^L_{k-1}$ there is an allowed cutting point. Finally, in Section 5, we show that $I$ is indeed an injection.

5. The mapping $I$ is well-defined

**Lemma 5.1.** Let $L$ be a ladder region. Then for every pair of two-rowed arrays in $\mathcal{T}^L_{k+1} \times \mathcal{T}^L_{k-1}$ there is an allowed cutting point $(l, m)$.

For the proof of this lemma, we have to introduce some more notation: Let $(T_1, T_2) \in \mathcal{T}^L_{k+1} \times \mathcal{T}^L_{k-1}$ with $T_1 = \left( a_1, a_2, \ldots, a_{k+1} \right)$ and $T_2 = \left( b_1, b_2, \ldots, b_{k+1} \right)$. We say that Inequality $\lceil$top$\rceil$ holds for an interval $[c, d]$ if

$$(\text{top}) \quad L(a_j) \geq y_{j-1},$$

for $j \in [c, d]$. Inequality $\lceil$top$\rceil$ holds for an interval $[c, d]$ if

$$(\text{top}) \quad L(a_j) \leq y_{j-1},$$

for $j \in [c, d]$. Similarly, Inequality $\lceil$bottom$\rceil$ holds for an interval $[c, d]$ if

$$(\text{bottom}) \quad L(x_{j-1}) \geq b_{j},$$

for $j \in [c, d]$. Inequality $\lceil$bottom$\rceil$ holds for an interval $[c, d]$ if

$$(\text{bottom}) \quad L(x_{j-1}) \leq b_{j},$$

for $j \in [c, d]$, where $L$ and $\overline{L}$ are as in Definition 2.1. We say that any of these inequalities holds for a cutting point $(l, m)$ if it holds for the interval $[l+1, m]$ if $l < m$ and for the interval $[m+1, l]$ if $m < l$. Clearly, a cutting point $(l, m)$ is allowed if and only if all of these inequalities hold for it.

Most of the work is done by the following lemma:

**Lemma 5.2.** Let $T = (T_1, T_2) \in \mathcal{T}^L_{k+1} \times \mathcal{T}^L_{k-1}$, $T_1 = \left( a_1, a_2, \ldots, a_{k+1} \right)$ and $T_2 = \left( b_1, b_2, \ldots, b_{k+1} \right)$. Let $\mathcal{L}$ and $\overline{L}$ be top cutting points, such that there is no top cutting point in the closed interval $[\mathcal{L}+1, \overline{L}-1]$. Similarly, let $\mathcal{m}$ and $\overline{m}$ be bottom cutting points, such that there is no bottom cutting point in the closed interval $[\mathcal{m}+1, \overline{m}-1]$. Then for both of the intervals $[\mathcal{L}+1, \overline{L}]$ and $[\mathcal{m}+1, \overline{m}]$,

- either $(\text{top})$ or $(\text{bottom})$ hold,
- either $(\text{top})$ or $(\text{bottom})$ hold,
- either $(\text{top})$ or $(\text{bottom})$ hold,
- either $(\text{bottom})$ or $(\text{bottom})$ hold.

Let $l_{\text{min}}, l_{\text{max}}, m_{\text{min}}$ and $m_{\text{max}}$ be the minimal and maximal top and bottom cutting points. Then we have

- $(\text{top})$ and $(\text{bottom})$ hold for $[2, \max(l_{\text{min}}, m_{\text{min}})]$ and
- $(\text{top})$ and $(\text{bottom})$ hold for $[\min(l_{\text{max}}, m_{\text{max}}), k]$. 


Proof. Suppose that \( \text{top} \) does not hold for the interval \([l+1, \bar{l}]\). We claim that in that case there is an index \( j \in [l+1, \bar{l} - 1] \) such that \( a_j < x_j \); For, by hypothesis there is an index \( i \in [l+1, \bar{l}] \) such that \( \mathcal{L}(a_i) < y_{i-1} \). We have \( \mathcal{L}(a_i) < y_{i-1} \leq L(x_{i-1}) \) and because \( L \) is a weakly increasing function, \( a_i < x_{i-1} \). It follows that \( a_{i-1} < a_i < x_{i-1} < x_i \). Thus, if \( i = \bar{l} \) we choose \( j = i - 1 \), otherwise \( j = i \).

The same statement is true if \( \text{bottom} \) does not hold for the interval \([l+1, \bar{l}]\): In this case there must be an index \( i \in [l+1, \bar{l}] \) such that \( \mathcal{L}(x_{i-1}) > b_i \). We conclude that \( L(a_i) \leq b_i \leq L(x_{i-1}) \) and thus \( a_i < x_{i-1} \).

Next, we will use induction to prove that
\[
(**) \\
\quad a_l < x_l \\
\quad \text{and} \quad a_{l+1} \leq x_{l+1}
\]
for \( l \in [l+1, \bar{l} - 1] \). We will first do an induction on \( l \) to establish the claim for \( l \in [j, \bar{l} - 1] \).

We start the induction at \( l = j \): Above we already found that \( a_j < x_j \). Therefore we must have \( a_{j+1} \leq x_{j-1} \), because otherwise \( j \) would satisfy \( \text{top} \) and hence were a top cutting point.

Now suppose that \( \text{top} \) holds for a particular \( l < \bar{l} - 1 \). Then \( a_{l+1} \leq x_{l-1} < x_{l+1} \), and, because there is no top cutting point at \( l+1 \), we have \( a_{l+2} \leq x_{l} \).

Similarly, to establish \( \text{bottom} \) for \( l \in [l+1, j] \) we do a reverse induction on \( l \). Suppose that \( \text{bottom} \) holds for a particular \( l > \bar{l} + 1 \). Then \( a_{l-1} < a_{l+1} \leq x_{l-1} \), and, because there is no top cutting point at \( l-1 \), we have \( a_l \leq x_{l-2} \).

Thus we obtain
\[
\mathcal{L}(x_{l-1}) \geq \mathcal{L}(x_{l-2}) \geq \mathcal{L}(a_l) \geq b_l, \quad \text{and} \\
\mathcal{L}(x_{l-1}) \geq \mathcal{L}(a_{l+1}) \geq b_{l+1} \geq b_l,
\]
which means that \( \text{bottom} \) holds for the interval \([l+1, \bar{l}]\).

Furthermore,
\[
\mathcal{L}(a_{l+1}) \leq \mathcal{L}(a_{l+2}) \leq \mathcal{L}(x_{l}) \leq y_{l}, \quad \text{and} \\
\mathcal{L}(a_l) \leq \mathcal{L}(x_{l-2}) \leq y_{l-2} \leq y_{l-1},
\]
which means that \( \text{top} \) holds for the interval \([l+1, \bar{l}]\).

Next we show that \( \text{top} \) and \( \text{bottom} \) hold for the interval \([2, l_{\min}]\): Assume that either of these inequalities does not hold for the interval \([2, l_{\min}]\) and that \([2, l_{\min}]\) does not contain a top cutting point except \( l_{\min} \). Then the above reverse induction implies that \( a_1 \leq a_3 < x_1 \), which means that \( 1 \) is a top cutting point. Thus, \( l_{\min} = 1 \) and the interval \([2, l_{\min}]\) is empty.

The other assertions are shown in a completely analogous fashion. \( \square \)

We are now ready to establish Lemma 5.1.

Proof of Lemma 5.1 Let \( T = (T_1, T_2) \in \mathcal{K}_{k+1}^L \times \mathcal{K}_{k-1}^L \). By Lemma 1.2 there is at least one cutting point \((l, m)\) of \( T \). Let \( l_{\min}, l_{\max}, m_{\min} \) and \( m_{\max} \) be the minimal and maximal top and bottom cutting points of \( T \) as before.

If there is an index \( j \) which is a top and a bottom cutting point of \( T \), then trivially \( -(j, j) \) is an allowed cutting point. Otherwise, we have to show that there is a cutting point \((l, m)\) for which \( \text{top}, \text{top}, \text{bottom}, \text{bottom} \) hold. Suppose that this is not the case.
For the inductive proof which follows, we have to introduce a convenient indexing scheme for the sequence of top and bottom cutting points. Let

\[ m_{i,0} = \max \{ m : m < l_{\min} \text{ and } m \text{ is a bottom cutting point} \}, \]

\[ m_{i,0} = \max \{ m : m < l_{i-1,1} \text{ and } m \text{ is a bottom cutting point} \} \] for \( i > 1, \)

and \( l_{i,0} = \max \{ l : l < m_{i,1} \text{ and } l \text{ is a top cutting point} \} \] for \( i \geq 1, \)

where \( m_{i,j+1} \) is the bottom cutting point directly after \( m_{i,j} \), and \( l_{i,j+1} \) is the top cutting point directly after \( l_{i,j} \). Furthermore, we set \( l_{0,1} = l_{\min}. \)

More pictorially, we have the following sequence of top and bottom cutting points for \( i \geq 1: \)

\[ \cdots < m_{i,0} < l_{i-1,1} < l_{i-1,2} < \cdots < l_{i,0} < m_{i,1} < m_{i,2} < \cdots < m_{i+1,0} < \cdots \]

If \( m_{\min} > l_{\min} \), then \( m_{i,1} \) does not exist, of course. Note that there are no bottom cutting points between \( l_{i,1} \) and \( l_{i+1,0} \), and there are no top cutting points between \( m_{i,1} \) and \( m_{i+1,0}. \)

Suppose first that \( m_{\min} < l_{\min}. \) By induction on \( i, \) we will show that \( \text{(top)} \) and \( \text{(bottom)} \) hold for the cutting points \( (l_{i-1,1}, m_{i,0}) \), where \( i \geq 1. \) By Lemma 5.2 we know that \( \text{(top)} \) and \( \text{(bottom)} \) are satisfied for the cutting point \( (l_{\min}, m_{i,0}) \), because \( [m_{i,0} + 1, l_{\min}] \subseteq [2, l_{\min}]. \) It remains to perform the induction step, which we will divide into five simple steps.

**Step 1.** \( \text{(top)} \) and \( \text{(bottom)} \) hold for the interval \([m_{i,0} + 1, l_{i-1,1}]. \) This is just a restatement of the induction hypothesis, i.e., that \( \text{(top)} \) and \( \text{(bottom)} \) hold for the cutting point \((l_{i-1,1}, m_{i,0}). \)

**Step 2.** Either \( \text{(bottom)} \) or \( \text{(top)} \) does not hold for the interval \([m_{i,0} + 1, m_{i,1}]. \) Because of Step 1, not both of \( \text{(bottom)} \) and \( \text{(top)} \) can hold for \((l_{i-1,1}, m_{i,0}) \), lest this was an allowed cutting point. Thus either \( \text{(bottom)} \) or \( \text{(top)} \) does not hold for \([m_{i,0} + 1, l_{i-1,0} + 1]. \) This interval is contained in \([m_{i,0} + 1, m_{i,1}], \) thus the inequalities \( \text{(bottom)} \) and \( \text{(top)} \) cannot hold on this interval either.

**Step 3.** \( \text{(top)} \) and \( \text{(bottom)} \) hold for \([l_{i,0} + 1, m_{i,1}]. \) Suppose that \( \text{(bottom)} \) does not hold for \([m_{i,0} + 1, m_{i,1}]. \) Then, by Lemma 5.2 we obtain that \( \text{(top)} \) and \( \text{(bottom)} \) hold for \([m_{i,0} + 1, m_{i,1}], \) because this interval contains no bottom cutting points except \( m_{i,1}. \) The same is true, if \( \text{(top)} \) does not hold for \([m_{i,0} + 1, m_{i,1}]. \) Because \([l_{i,0} + 1, m_{i,1}]. \) is a subset of this interval, \( \text{(top)} \) and \( \text{(bottom)} \) hold for the cutting point \((l_{i,0}, m_{i,1}), \) or, equivalently, for the interval \([l_{i,0} + 1, m_{i,1}]. \)

**Step 4.** Either \( \text{(bottom)} \) or \( \text{(top)} \) does not hold for \([l_{i,0} + 1, l_{i,1}]. \) Because of Step 3, not both of \( \text{(bottom)} \) and \( \text{(top)} \) can hold for the cutting point \((l_{i,0}, m_{i,1}), \) nor for the greater interval \([l_{i,0} + 1, l_{i,1}]. \)

**Step 5.** \( \text{(top)} \) and \( \text{(bottom)} \) hold for \([m_{i+1,0} + 1, l_{i,1}]. \) The interval \([l_{i,0} + 1, l_{i,1}] \) does not contain a top cutting point except \( l_{i,1} \), thus by Lemma 5.2 and Step 4 we see that \( \text{(top)} \) and \( \text{(bottom)} \) hold. Finally, because \([m_{i+1,0} + 1, l_{i,1}] \subseteq [l_{i,0} + 1, l_{i,1}] \), \( \text{(top)} \) and \( \text{(bottom)} \) hold for the cutting point \((l_{i,1}, m_{i+1,0}). \)

If \( l_{\max} > m_{\max} \), then we encounter a contradiction: Let \( r \) be such that \( m_{r,0} = m_{\max}. \) We have just shown that \( \text{(top)} \) and \( \text{(bottom)} \) hold for the cutting point \((l_{r-1,1}, m_{r,0}). \) Furthermore, by Lemma 5.2, \( \text{(bottom)} \) and \( \text{(top)} \) hold for \([m_{r,0}, k] \)
and thus also for \((l_{r-1}, m_r, 0)\). Hence, this would be an allowed cutting point, contradicting our hypothesis.

If \(l_{max} \leq m_{max}\), let \(r\) be such that \(l_{r, 0} = l_{max}\). By the induction (Step 3) we find that \(top_{l, m, T}\) and \(bottom_{l, m, T}\) hold for the cutting point \((l_{r, 0}, m_{r, 1})\). Again, because of Lemma 5.2, we know that \(top_{l, m, T}\) and \(bottom_{l, m, T}\) holds for \([l_{r, 0}, k]\) and thus also for \((l_{r, 0}, m_{r, 1})\). Hence, we had an allowed cutting point in this case also.

The case that \(m_1 > t_1\) is completely analogous. \(\square\)

6. THE MAPPING I IS AN INJECTION

Lemma 6.1. The mapping \(I\) defined above is an injection.

Proof. Suppose that \(I(T) = I(T')\) for \(T = (T_1, T_2)\) and \(T' = (T'_1, T'_2)\), such that \(T\) and \(T'\) are elements of \(L_{k+1} \times L_{k-1}\). Let \((l, m)\) be the optimal cutting point of \(T\), and let \((l', m')\) be the optimal cutting point of \(T'\).

Observe that we can assume \(\min(l, m, l', m') = 1\), because the elements of \(T\) and \(T'\) with index less than or equal to this minimum retain their position in \(I(T)\). Likewise, we can assume that \(\max(l, m, l', m') = k\).

Furthermore, we can assume that \(l \leq l'\), otherwise we exchange the meaning of \(T\) and \(T'\). Thus, we have to consider the following twelve situations:

1. \(1 = l \leq l' \leq m \leq m' = k\)
2. \(1 = l \leq l' \leq m' \leq m = k\)
3. \(1 = l \leq m \leq l' \leq m' = k\)
4. \(1 = l \leq m \leq m' \leq l' = k\)
5. \(1 = l \leq m' \leq l' \leq m = k\)
6. \(1 = l \leq m' \leq m \leq l' = k\)
7. \(1 = m \leq l \leq l' \leq m' = k\)
8. \(1 = m \leq l \leq m' \leq l' = k\)
9. \(1 = m \leq m' \leq l \leq l' = k\)
10. \(1 = m' \leq l \leq l' \leq m = k\)
11. \(1 = m' \leq l \leq m \leq l' = k\)
12. \(1 = m' \leq m \leq l \leq l' = k\)

We shall divide these twelve cases into two portions according to whether \(l \leq m\) or not.

A: \(1 \leq m\). In the Cases (1)–(6), (10) and (11) we have \(l \leq m\), thus the pair of two-rowed arrays \(T = (T_1, T_2) \in L_{k+1} \times L_{k-1}\) looks like

\[
\begin{array}{c|cccc}
| a_1 | a_2 | \ldots | a_{l+1} | a_{k+1} \\
\hline
| b_1 | \ldots | b_m | b_{m+1} | \ldots | b_{k+1} \\
\end{array}
\]

\[
\begin{array}{ccc|c|cc}
| x_1 | \ldots | x_{l-1} | x_l | \ldots | x_{k-1} \\
\hline
| y_1 | \ldots | y_{m-1} | y_m | \ldots | y_{k-1} \\
\end{array}
\]

Cutting at \((l, m)\) we obtain \(I(T) \in L_k \times L_k\):

\[
\begin{array}{c|c|ccc}
| a_1 | a_2 | \ldots | a_{l+1} | a_{k+1} \\
\hline
| b_1 | \ldots | b_m | y_m | \ldots | y_{k-1} \\
\end{array}
\]

\[
\begin{array}{c|c|ccc|c|ccc}
| x_1 | \ldots | x_{l-1} | x_l | \ldots | x_{k-1} \\
\hline
| y_1 | \ldots | y_{m-1} | b_{m+1} | \ldots | b_{k+1} \\
\end{array}
\]

If \(l = 1\), then the top row of the second array in \(I(T)\) is \((a_2, a_3, \ldots, a_{k+1})\), if \(m = k\), then the bottom row of the first array in \(I(T)\) is \((b_1, b_2, \ldots, b_k)\).
Case (1), $1 \leq l' \leq m \leq m' = k$. Given that $I(T) = I(T')$, the pair $T'$ can be expressed in terms of the entries of $T$ as follows:

$$
\begin{array}{|c|c|c|c|c|c|c|c|c|}
\hline
a_1 & x_1 & \ldots & x_{\nu-1} & \vdots & a_{\nu+1} & \ldots & a_{k+1} \\
\hline
b_1 & \ldots & \ldots & \ldots & \ldots & b_m & \ldots & y_{k-1} & b_{k+1} \\
\hline
a_2 & \ldots & a_\nu & \vdots & x_\nu & \ldots & x_{k-1} \\
\hline
y_1 & \ldots & \ldots & \ldots & \ldots & y_{m-1} & \ldots & b_{m+1} & \vdots \\
\hline
\end{array}
$$

The vertical dots indicate the cut $(l', m')$ which would result in $I(T')$. We show that the cutting point $(l, m) = (1, m)$, indicated above by the vertical lines, is in fact an allowed cutting point for $T'$: Cutting at $(1, m)$ yields

$$
\begin{array}{|c|c|c|c|c|c|c|c|c|}
\hline
a_1 & a_2 & \ldots & a_\nu & \vdots & x_\nu & \ldots & x_{m-1} & \ldots & x_{k-1} \\
\hline
b_1 & \ldots \ldots & a_{\nu+1} & \ldots & a_m & \ldots & a_{k+1} \\
\hline
x_1 & \ldots & x_{\nu-1} & \vdots & a_{\nu+1} & \ldots & b_m & \ldots & b_k & \vdots \\
\hline
y_1 & \ldots \ldots & y_{m-1} & \ldots & y_m & \ldots & y_{k-1} & y_{k+1} \\
\hline
\end{array}
$$

Note, that this is the same pair of two-rowed arrays we obtain by cutting $T$ at $(l', m')$. We have to check that the pair of two-rowed arrays $\tilde{T}$ is in the ladder region.

Clearly,

$$(a_2, b_2), (a_3, b_3), \ldots, (a_\nu, b_\nu)$$

and

$$(x_1, y_1), (a_2, b_2), \ldots, (x_{\nu-1}, y_{\nu-1})$$

are in the ladder region, because these pairs appear also in $T$. Furthermore, the pairs

$$(x_\nu, b_{\nu+1}), (x_{\nu+1}, b_{\nu+2}), \ldots (x_{m-1}, b_m)$$

and

$$(a_{\nu+1}, y_\nu), (a_{\nu+2}, y_{\nu+1}), \ldots (a_m, y_{m-1})$$

appear in $I(T)$ and are therefore in the ladder region, too. All the other pairs, i.e.,

$$(a_1, y_1), (x_m, b_{m+1}), (x_{m+1}, b_{m+2}), \ldots (x_{k-1}, b_k), (a_{m+1}, y_{m+1}), (a_{m+2}, y_{m+1}), \ldots (a_k, y_{k-1})$$

are unaffected by the cut and appear in $T'$. Thus we have that $(l, m)$ and $(l', m')$ are allowed cuts for $T$ and $T'$. We required that $(l, m)$ is optimal for $T$ and that $(l', m')$ is optimal for $T'$, therefore we must have $l = l'$ and $m = m'$.

In all the other cases the reasoning is very similar. Thus we only print the pairs of two-rowed arrays $T'$ and $\tilde{T}$ and leave it to the reader to check that $\tilde{T}$ is in the ladder region.
Case (2), \(1 \leq l' \leq m' \leq m = k\). The pair \(T'\) can be expressed in terms of the entries of \(T\) as follows:

\[
\begin{array}{c|c c c c c c c c c c c c c c c c}
\hline
a_1 | x_1 & \cdots & x_{l' - 1} & \vdash & a_{l' + 1} & \cdots & & \hline
b_1 & \cdots & b_{m'} & \vdash & y_{m'} & \cdots & y_{k-1} & \vdash & b_{k+1} \\
\hline
a_2 & x_{l'} & \cdots & x_{k-1} \\
y_1 & \cdots & y_{m'-1} & \vdash & b_{m'+1} & \cdots & b_k \\
\hline
\end{array}
\]

Cutting at \((l, m)\) yields

\[
\begin{array}{c|c c c c c c c c c c c c c c c c}
\hline
a_1 | a_2 & \cdots & a_{l'} & \vdash & x_{l'} & \cdots & x_{m'-1} & \cdots & x_{k-1} \\
b_1 & \cdots & b_{m'} & \vdash & y_{m'} & \cdots & y_{k-1} \\
y_1 & \cdots & y_{m-1} & \vdash & b_{m'+1} & \cdots & b_k \\
\hline
\end{array}
\]

Case (3), \(1 \leq m \leq l' \leq m' = k\). The pair \(T'\) can be expressed in terms of the entries of \(T\) as follows:

\[
\begin{array}{c|c c c c c c c c c c c c c c c c}
\hline
a_1 | x_1 & \cdots & x_{l'-1} & \vdash & a_{l'+1} & \cdots & a_{k+1} \\
b_1 & \cdots & b_m & \vdash & y_m & \cdots & y_{k-1} & \vdash & b_{k+1} \\
\hline
a_2 & \vdash & x_{l'} & \cdots & x_{k-1} \\
y_1 & \vdash & y_{m-1} & \vdash & b_{m'+1} & \cdots & b_k \\
\hline
\end{array}
\]

Cutting at \((l, m)\) yields

\[
\begin{array}{c|c c c c c c c c c c c c c c c c}
\hline
a_1 | a_2 & \cdots & a_{l'} & \vdash & x_{l'} & \cdots & x_{k-1} \\
b_1 & \vdash & b_m & \vdash & b_{m'+1} & \cdots & b_k \\
\hline
x_{l'-1} & \vdash & a_{l'+1} & \cdots & a_{k+1} \\
y_1 & \vdash & y_{m-1} & \vdash & y_m & \cdots & y_{k-1} & \vdash & b_{k+1} \\
\hline
\end{array}
\]

Case (4), \(1 \leq m \leq m' \leq l' = k\). The pair \(T'\) can be expressed in terms of the entries of \(T\) as follows:

\[
\begin{array}{c|c c c c c c c c c c c c c c c c}
\hline
a_1 | x_1 & \cdots & x_{k-1} & \vdash & a_{k+1} \\
b_1 & \cdots & b_m & \vdash & y_m & \cdots & y_{m'-1} & \vdash & b_{m'+1} & \cdots & b_{k+1} \\
\hline
a_2 & \vdash & \cdots & a_{k+1} \\
y_1 & \vdash & \cdots & b_{m'+1} & \vdash & \cdots & y_{k-1} \\
\hline
\end{array}
\]

Cutting at \((l, m)\) yields

\[
\begin{array}{c|c c c c c c c c c c c c c c c c}
\hline
a_1 | a_2 & \cdots & a_{l'} & \vdash & x_{l'} & \cdots & x_{k-1} \\
b_1 & \vdash & b_m & \vdash & b_{m'+1} & \cdots & b_k \\
\hline
x_{k-1} & \vdash & a_{k+1} \\
y_1 & \vdash & y_{m-1} & \vdash & y_m & \cdots & y_{m'-1} & \vdash & b_{m'+1} & \cdots & b_{k+1} \\
\hline
\end{array}
\]
Case (5), $1 \leq m' \leq l' \leq m = k$. The pair $T'$ can be expressed in terms of the entries of $T$ as follows:

$$
\begin{align*}
(T') & \quad \begin{array}{c|ccc|ccc}
a_1 & x_1 & \cdots & x_{l'-1} & a_{l'+1} & \cdots & a_{k+1} \\
b_1 & & & b_{m'} & & & \\
\end{array} \\
& \quad \begin{array}{c|ccc|ccc}
a_2 & \cdots & a_{l'} & x_{l'} & x_{k-1} \\
y_1 & \cdots & y_{m'-1} & b_{m'+1} & \cdots & b_k \\
\end{array}
\end{align*}
$$

Cutting at $(l, m)$ yields

$$
\begin{align*}
(T') & \quad \begin{array}{c|ccc|ccc}
a_1 & a_2 & \cdots & a_{l'} & x_{l'} & x_{k-1} \\
b_1 & & & b_{m'} & y_{m'} & y_{k-1} \\
\end{array} \\
& \quad \begin{array}{c|ccc|ccc}
a_2 & \cdots & y_{m'-1} & b_{m'+1} & \cdots & b_k \\
y_1 & \cdots & y_{m'-1} & b_{m'+1} & \cdots & b_k \\
\end{array}
\end{align*}
$$

Case (6), $1 \leq m' \leq m \leq l' = k$. The pair $T'$ can be expressed in terms of the entries of $T$ as follows:

$$
\begin{align*}
(T') & \quad \begin{array}{c|ccc|c}
a_1 & x_1 & \cdots & x_{k-1} & a_{k+1} \\
b_1 & b_{m'} & y_{m'} & \cdots & y_{m-1} \\
\end{array} \\
& \quad \begin{array}{c|c}
a_2 & a_k \\
y_1 & \cdots & y_{m'-1} \\
\end{array}
\end{align*}
$$

Cutting at $(l, m)$ yields

$$
\begin{align*}
(T') & \quad \begin{array}{c|ccc|c}
a_1 & a_2 & \cdots & x_{k-1} & a_{k+1} \\
b_1 & b_{m'} & y_{m'} & \cdots & y_{m-1} \\
\end{array} \\
& \quad \begin{array}{c|c}
a_2 & a_k \\
y_1 & \cdots & y_{m'-1} \\
\end{array}
\end{align*}
$$

Case (10), $1 \leq m' \leq l' \leq m = k$. The pair $T'$ can be expressed in terms of the entries of $T$ as follows:

$$
\begin{align*}
(T') & \quad \begin{array}{c|cccc|c}
a_1 & \cdots & a_l & x_l & \cdots & x_{l'-1} & a_{l'+1} & \cdots & a_{k+1} \\
b_1 & y_1 & \cdots & y_{l-1} & a_{l+1} & a_{l'} & x_{l'} & \cdots & x_{k-1} \\
\end{array} \\
& \quad \begin{array}{c|c}
b_2 & b_k \\
\end{array}
\end{align*}
$$

Cutting at $(l, m)$ yields

$$
\begin{align*}
(T') & \quad \begin{array}{c|cccc|c}
a_1 & \cdots & a_l & a_{l+1} & \cdots & a_{l'} & x_{l'} & \cdots & x_{k-1} \\
b_1 & y_1 & \cdots & y_{l-1} & x_l & \cdots & x_{l'-1} & a_{l'+1} & \cdots & a_{k+1} \\
\end{array} \\
& \quad \begin{array}{c|c}
b_2 & b_k \\
\end{array}
\end{align*}
$$
Case (11), $1 = m' \leq l \leq m \leq l' = k$. The pair $T'$ can be expressed in terms of the entries of $T$ as follows:

$$(T')$$

\[
\begin{array}{cccccc}
  a_1 & \ldots & a_l & x_l & \ldots & x_{k-1} & b_{k+1} \\
  b_1 \mid y_1 & \ldots & y_{m-1} & b_{m+1} & \ldots & b_{k+1} \\
  x_1 \ldots x_{l-1} & \mid a_{l+1} & \ldots & x_{k-1} & \mid a_{k+1} \\
  b_2 & \ldots & b_m & y_m & \ldots & y_{k-1} \\
\end{array}
\]

Cutting at $(l, m)$ yields

$$(\tilde{T})$$

\[
\begin{array}{cccccc}
  a_1 & \ldots & a_l & a_l & \ldots & a_k \\
  b_1 \mid y_1 & \ldots & y_{m-1} & y_m & \ldots & y_{k-1} \\
  x_1 \ldots x_{l-1} & x_l & \ldots & x_{k-1} & \mid a_{k+1} \\
  b_2 & \ldots & b_m & b_{m+1} & \ldots & b_{k+1} \\
\end{array}
\]

B: $m \leq l$. In the Cases (7)–(9) and (12) we have $m \leq l$, thus the pair of two-rowed arrays $T = (T_1, T_2) \in T_{k+1}^L \times T_{k-1}^L$ looks like

\[
\begin{array}{cccc}
  a_1 & \ldots & a_l & a_l \mid a_{l+1} \ldots a_{k+1} \\
  b_1 \mid y_1 & \ldots & y_{m-1} & y_m \ldots y_{k-1} \\
  x_1 \ldots x_{l-1} & x_l & \ldots & x_{k-1} \\
  y_1 \ldots y_{m-1} & y_m & \ldots & y_{k-1} \\
\end{array}
\]

Cutting at $(l, m)$ we obtain $I(T) \in T_{k+1}^L \times T_{k-1}^L$:

\[
\begin{array}{cccc}
  a_1 & \ldots & a_l & a_l \mid x_l \ldots x_{k-1} \\
  b_1 \mid y_1 & \ldots & b_m & y_m \ldots y_{k-1} \\
  x_1 \ldots x_{l-1} & a_{l+1} \ldots a_{k+1} \\
  y_1 \ldots y_{m-1} & b_{m+1} & \ldots & b_{k+1} \\
\end{array}
\]

Case (7), $1 = m \leq l \leq l' \leq m' = k$. The pair $T'$ can be expressed in terms of the entries of $T$ as follows:

$$(T')$$

\[
\begin{array}{cccccc}
  a_1 & \ldots & a_l & x_l & \ldots & x_{v'-1} & a_{v'+1} \ldots a_{k+1} \\
  b_1 \mid y_1 & \ldots & y_{m-1} & y_m \ldots y_{k-1} & b_{k+1} \\
  x_1 \ldots x_{l-1} & a_{l+1} & \ldots & a_{v'} & x_{v'} \ldots x_{k-1} & \mid b_k \mid b_k \\
  b_2 & \ldots & b_m & b_{m+1} & \ldots & b_{k+1} \\
\end{array}
\]

Cutting at $(l, m)$ yields

$$(\tilde{T})$$

\[
\begin{array}{cccccc}
  a_1 & \ldots & a_l & a_l \mid a_{l+1} \ldots a_{v'} \mid x_{v'} \ldots x_{k-1} & b_{k+1} \\
  b_1 \mid b_2 & \ldots & b_m & y_m \ldots y_{k-1} & b_{k+1} \\
  x_1 \ldots x_{l-1} & x_{l} & \ldots & x_{v'-1} & a_{v'+1} \ldots a_{k+1} \mid y_1 \mid y_1 \\
  y_1 \ldots y_{m-1} & y_{m+1} & \ldots & y_{k-1} & b_{k+1} \\
\end{array}
\]

Case (8), \(1 = m \leq l \leq m' \leq l' = k\). The pair \(T'\) can be expressed in terms of the entries of \(T\) as follows:

\[
\begin{array}{cccc}
 a_1 & \ldots & a_l & \mid x_l \ldots x_{k-1} & \mid a_{k+1} \\
 b_1 & \mid y_1 & \ldots & y_{m'-1} & \mid b_{m'+1} \\
 \end{array}
\]

\[
\begin{array}{cccc}
 x_1 & \ldots & x_{l-1} & \mid a_{l+1} & \ldots & a_k & \mid b_{k+1} \\
 b_2 & \ldots & b_{m'} & \mid y_{m'} & \ldots & y_{k-1} \\
 \end{array}
\]

Cutting at \((l, m)\) yields

\[
\begin{array}{cccc}
 a_1 & \ldots & a_l & \mid a_{l+1} & \ldots & a_k & \mid b_{k+1} \\
 b_1 & \mid b_2 & \ldots & b_{m'} & \mid y_{m'} & \ldots & y_{k-1} \\
 \end{array}
\]

\[
\begin{array}{cccc}
 x_1 & \ldots & x_{l-1} & \mid x_l & \ldots & x_{k-1} & \mid a_{k+1} \\
 y_1 & \ldots & y_{m'-1} & \mid b_{m'+1} & \ldots & b_{k+1} \\
 \end{array}
\]

Case (9), \(1 = m \leq m' \leq l \leq l' = k\). The pair \(T'\) can be expressed in terms of the entries of \(T\) as follows:

\[
\begin{array}{cccc}
 a_1 & \ldots & a_l & \mid x_l \ldots x_{k-1} & \mid a_{k+1} \\
 b_1 & \mid y_1 & \ldots & y_{m'-1} & \mid b_{m'+1} \\
 \end{array}
\]

\[
\begin{array}{cccc}
 x_1 & \ldots & x_{l-1} & \mid a_{l+1} & \ldots & a_k & \mid b_{k+1} \\
 b_2 & \ldots & b_{m'} & \mid y_{m'} & \ldots & y_{k-1} \\
 \end{array}
\]

Cutting at \((l, m)\) yields

\[
\begin{array}{cccc}
 a_1 & \ldots & a_l & \mid a_{l+1} & \ldots & a_k & \mid y_{k-1} \\
 b_1 & \mid b_2 & \ldots & b_{m'} & \mid y_{m'} & \ldots & y_{k-1} \\
 \end{array}
\]

\[
\begin{array}{cccc}
 x_1 & \ldots & x_{l-1} & \mid x_l & \ldots & x_{k-1} & \mid a_{k+1} \\
 y_1 & \ldots & y_{m'-1} & \mid b_{m'+1} & \ldots & b_{k+1} \\
 \end{array}
\]

Case (12), \(1 = m' \leq m \leq l \leq l' = k\). The pair \(T'\) can be expressed in terms of the entries of \(T\) as follows:

\[
\begin{array}{cccc}
 a_1 & \ldots & a_l & \mid x_l \ldots x_{k-1} & \mid a_{k+1} \\
 b_1 & \mid y_1 & \ldots & y_{m-1} & \mid b_{m+1} \\
 \end{array}
\]

\[
\begin{array}{cccc}
 x_1 & \ldots & x_{l-1} & \mid a_{l+1} & \ldots & a_k & \mid b_{k+1} \\
 b_2 & \ldots & b_m & \mid y_m & \ldots & y_{k-1} \\
 \end{array}
\]

Cutting at \((l, m)\) yields

\[
\begin{array}{cccc}
 a_1 & \ldots & a_l & \mid a_{l+1} & \ldots & a_k & \mid y_{k-1} \\
 b_1 & \mid b_2 & \ldots & b_m & \mid y_m & \ldots & y_{k-1} \\
 \end{array}
\]

\[
\begin{array}{cccc}
 x_1 & \ldots & x_{l-1} & \mid x_l & \ldots & x_{k-1} & \mid a_{k+1} \\
 b_2 & \ldots & b_m & \mid b_{m+1} & \ldots & b_{k+1} \\
 \end{array}
\]

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