TAME PAIRS, DEFINABLE TYPES AND PRO-DEFINABILITY

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Abstract. We show (strict) pro-definability of spaces of definable types in various classical first order theories, including o-minimal expansions of divisible abelian groups, Presburger arithmetic, \( p \)-adically closed fields, real closed and algebraically closed valued fields and closed ordered differential fields. As a particular case, we recover a result of Hrushovski and Loeser about the strict pro-definability of the stable completion of a definable set in algebraically closed valued fields. Furthermore, we prove strict pro-definability of some other distinguished subspaces of the type spaces, which could be viewed as model-theoretic analogue of Huber’s analytification.

Our general strategy is to study the class of stably embedded pairs of models of the above mentioned theories. We show that such classes are elementary in the language of pairs and provide axiomatizations for some of their completions. In the o-minimal setting, our approach provides an alternative axiomatization for the theory of tame pairs defined by Lewemberg and van den Dries (also axiomatized by Pillay).

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1. Introduction

In [11], building on the model theory of algebraically closed valued fields (ACVF), Hrushovski and Loeser developed a theory which provides a model-theoretic account of the Berkovich analytification of algebraic varieties. Informally, given a complete non-archimedean rank 1 valued field \( k \) and an algebraic variety \( X \) over \( k \), they showed how the space of generically stable types on \( X \) over a large algebraically closed valued field extending \( k \), gives a model-theoretic avatar of the analytification \( X^{an} \) of \( X \). Most notably, their association allowed them to obtain results concerning the homotopy type of quasi-projective varieties which were only known under strong algebro-geometric hypothesis on \( X \).

One of the difficulties to study Berkovich spaces from a model-theoretic point of view is that such spaces do not seem to generally have (in ACVF) the structure of a definable set –where

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usual model-theoretic techniques can be applied– but rather canonically the structure of space on types. Part of the novelty of Hrushovski-Loeser’s work lies on the fact that their spaces can be equipped with the structure of a strict pro-definable set, which granted them back the use of different classical model-theoretic tools. It is thus tempting to ask if such a structural result holds for other distinguished subsets of definable types and even for other first-order theories. In this article we aim to give a positive answer to this question. In particular, we will show pro-definability and strict pro-definability of various spaces of definable types in different classical first order structures, including o-minimal expansion of groups, $p$-adically closed fields, real closed valued fields and algebraically closed valued fields. As a particular case, we obtain a new proof of the strict pro-definability of Hrushovski-Loeser spaces.

In a sequel, we will further explore structural properties of some of the above mentioned spaces, and compare them to known spaces of geometric interest. Motivated by Hrushovski and Loeser’s work, we wish to show that there are spaces of definable types that can mimic Huber’s “adification” of an algebraic variety in a similar way the space of generically stable types mimics Berkovich’s analytification of an algebraic variety. We hope this will provide the first steps towards a model theory of adic spaces. In the same spirit, working in real closed valued fields, there are spaces of definable types which can be seen as the model-theoretic counterpart of the analytification of semi-algebraic sets as recently defined by Jell, Scheiderer and Yu in [12]. This is also very much related to the work of Ealy, Haskell and Marikova in [8] on residue field domination in real closed valued fields. This article lays the foundation for a model-theoretic study of such spaces.

Our approach is based on the model theory of pairs. Besides providing the desired implications on pro-definability and strict pro-definability, some of our results concerning pairs are interesting on their own. In particular, we give axiomatizations for different completions of the the theory of elementary pairs of models of $T$, for $T$ either the theory of an o-minimal expansion of a group (admitting quantifier elimination), the theory of algebraically closed valued fields, or the theory of real closed valued fields. In the o-minimal setting, we obtain (as a particular case) an alternative axiomatization of the so called “tame pairs” introduced by Lewemberg and van den Dries in [24] (and also axiomatized by Pillay in [17]).

2. Main results

All definitions will be given in detail in Section 3. Let us formally state the main results of the article.

The first link between pro-definability and model theory of pairs is the following result:

**Theorem** (Later Theorem 4.3.1). Let $T$ be an $L$-theory such that

1. the theory of stably embedded pairs of models of $T$ is elementary in the language of pairs $L_P$;
2. for every small model $M \models T$ and every finite tuple $a \in U$, acl$(Ma)$ is a model of $T$;
3. $T$ is model complete.

Then $T$ has uniform definability of types. In particular, given a definable set $X$, the space of definable types concentrating on $X$ is pro-definable in $L^eq$.

As a corollary we obtain:

**Theorem** (Later Corollary 4.3.3 and Remark 4.3.2). Let $T$ be one of the following theories: a stable theory; an o-minimal theory satisfying properties (2) and (3) of the above theorem; Presburger arithmetic; the theory of $p$-adically closed fields; the theory of real closed and
algebraically closed valued fields; or the theory of closed ordered differential fields. Then, for every definable set $X$, the space $S^\text{def}_X$ of definable types concentrating on $X$ is pro-definable in $\mathcal{L}^\text{eq}$.

Our strategy to proving the previous result requires to show, for all the above listed theories, that their associated theory of stably embedded pairs is elementary in the language of pairs. In most cases this follows by well-known theorems such as the Marker-Steinhorn theorem in o-minimal theories. In the Appendix we gather the remaining proofs to showing all theories above listed have indeed such a property.

To pass from pro-definability to strict pro-definability (i.e., requiring in addition surjective definable transition maps) more work is needed. We will prove strict pro-definability for the associated spaces of theories $T$ we are mostly interested in, namely, for some o-minimal expansions of groups (here admitting quantifier elimination) and both for real closed and algebraically closed valued fields. The result is achieved by axiomatizing certain completions of the theory of stably embedded pairs of models of $T$, which (following van den Dries) we call tame pairs and denote them by $T_{\text{tame}}(\Delta)$ (introduced in Section 7). Informally, $\Delta$ is a set of formulas controlling which definable 1-types over the small model are realized in the extension (that is, given a model $(N,M)$ of $T_{\text{tame}}(\Delta)$, which definable 1-types over $M$ are realized in $N$). Our approach to show strict-pro-definability relies on a relative quantifier elimination result for tame pairs and a construction of special models of such pairs. Both results are interest on its own. Let us state the main two theorems:

**Theorem (Later Theorem 7.2.2).** Let $T$ be one of the following $\mathcal{L}$-theories:

1. an o-minimal theory having quantifier elimination;
2. a completion of the theory of algebraically closed valued fields;
3. the theory of real closed valued fields.

Then, if consistent, the theory $T_{\text{tame}}(\Delta)$ is complete. Furthermore, for every $\mathcal{L}_P$-formula $\psi(x)$ there is an $\mathcal{L}$-formula $\varphi(x)$ such that

$$T_{\text{tame}}(\Delta) \models (\forall x)(P(x) \rightarrow (\psi(x) \leftrightarrow \varphi(x))).$$

As a corollary, we obtained the following.

**Theorem (Later Corollary 5.1.5).** Let $X$ be a definable set definable in one of theories in the theorem above, then

- $S^\text{def}_X$, the set of definable types on $X$,
- $\hat{X}$, the set of bounded definable types on $X$,
- $\hat{X}$, the set of definable types on $X$ that are orthogonal to $\Gamma$, when $T$ is RCVF or ACVF,

are strict pro-definable.

When $T$ is a completion of ACVF and $X$ is a variety defined over a field $K$, the space $\hat{X}$ is the model-theoretic analog of Huber’s analytification of $X$. As stated in the introduction, the properties of such spaces will be developed in a sequel.

The techniques employed to show the previous theorems cannot be used to prove analogue statements for Presburger arithmetic, the theory of $p$-adically closed fields or the theory of closed ordered differential fields (CODF). Following an idea suggested by Martin Hils, we show however that, for Presburger Arithmetic, the space $S^\text{def}_X$ is also strict pro-definable (see later Theorem 5.2.1). We strongly believe the same holds both for $p$-adically closed fields and for CODF, but we left these two cases out of the present article.
The following two questions remain open.

**Question.** Can one characterize NIP theories (or dp-minimal theories) having uniform definability of types?

**Question.** Is there a characterization of NIP theories for which, if the spaces of definable types are pro-definable, then they are also strict pro-definable?

The article is laid out as follows. In Section [3] we provide the needed model-theoretic background and recall definitions and basic properties of pro-definability. Stably embedded pairs are introduced and studied in Section [4] where Theorem 4.3.1 and Corollary 4.3.4 are proved. We use in this section results gathered in the Appendix which guarantee that the class of stably embedded pairs is an elementary class for some classical theories such as Presburger arithmetic, the theory of $p$-adically closed fields and the theory or real closed valued fields. All results on strict pro-definability are presented in Section [5]. Finally, in Sections [6] and [7] tame pairs are introduced and Theorem 7.2.2 is proved.

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3. Preliminaries and notation

3.1. Model theoretic background. We will use standard model-theoretic notation with the following specific conventions. We refer the reader to [21] for a general introduction to model theory.

Let $L$ be a first order language (possibly multi-sorted) and $T$ be a complete $L$-theory. We let $F(L)$ denote the set of $L$-formulas. For a tuple of variables $x$, we let $F_x(L)$ denote the set of $L$-formulas which have the tuple $x$ among their free variables. We let $\ell(x)$ denote the length of the tuple $x$. The sorts of $L$ are denoted by bold letters $S$. Given a variable $x$ with $\ell(x) = 1$, we let $S_x$ denote the sort where $x$ ranges. If $x = (x_1, \ldots, x_m)$, we let $S_x$ denote the product of sorts $S_{x_1} \times \cdots \times S_{x_m}$. For tuples $x = (x_1, \ldots, x_m)$ and $y = (y_1, \ldots, y_m)$, we will often abuse of notation and write $x = y$ as an abbreviation for $\bigwedge_{i=1}^{m} x_i = y_i$.

Let $M$ be a model of $T$ and $S$ be a sort of $L$. We let $S(M)$ denote the set of elements of $M$ which are of sort $S$. For notational consistency, even if $L$ is one-sorted and $M$ is a model of $T$, we write $S_x(M)$ for $M^x$. By a subset of $M$ we mean a subset of the union of all $S(M)$, where $S$ is any sort in $L$. Let $C$ be a subset of $M$. The language $L(C)$ is the language $L$ together with constant symbols for every element in $C$. For a tuple of variables $x$, the set of types over $C$ in the variables $x$ (i.e., the Stone dual of the boolean algebra of $L(C)$-definable subsets of $S_x$) is denoted $S^T_x(C)$. When no ambiguity arises we also write $S_x(C)$. By default, we equip $S_x(C)$ (and any subset) with the logic topology. In cases where a set of types has a geometric content, such sets are also equipped with a different topology. When no confusion arises, we will speak of spaces of types without specifying the topology. Given an $L(C)$-definable subset $X \subseteq S_x$, we say that a type $p \in S_x(C)$ concentrates on $X$ if $p$ contains a formula defining $X$. We denote by $S_X(C)$ the subset of $S_x(C)$ consisting of those types that concentrate on $X$. 
We let \( \mathcal{U} \) be a monster model of \( T \). A subset \( A \subseteq \mathcal{U} \) is said to be small if it is of cardinality smaller than \( |\mathcal{U}| \). A type \( p(x) \) is a global type if \( p \in S_x(\mathcal{U}) \). We will often identify a sort \( S \) with the set of points \( S(\mathcal{U}) \).

Let \( C \) be a small subset of \( \mathcal{U} \). Given an ordered tuple \( x = (x_i)_{i \in I} \) of variables (possibly infinite), a subset \( X \) of \( S_x = \prod_{i \in I} S_{x_i} \) is \( C \)-definable (following Shelah’s terminology) if there is a small collection \( \Theta \) of \( L(C) \)-formulas \( \varphi(x) \) (where only finitely many \( x_i \) occur in each formula) such that \( X = \{ a \in S_x \mid \mathcal{U} \models \varphi(a), \varphi \in \Theta \} \). When the tuple of variables is finite, we also say \( X \) is \( \infty \)-definable.

We let \( \text{dcl} \) and \( \text{acl} \) denote the usual definable and algebraic closure operators (in the model-theoretic sense). Given a tuple \( x \) of length \( \ell(x) = n \), a \( C \)-definable set \( X \subseteq S_x(\mathcal{U}) \) and a subset \( A \) of \( \mathcal{U} \) we let \( X(A) := X \cap \text{dcl}(A)^0 \).

Let \( S \) be a sort and let \( L^S \) be the restriction of \( L \) to \( S \). The sort \( S \) is called dominant if every other \( L \)-sort \( S' \) is a \( L^S \)-sort (that is, an imaginary \( L^S \)-sort). When \( L \) contains a dominant sort \( D \), and \( S \) is an \( L \)-sort such that \( S = D^n/E \) for an \( L|D \)-definable equivalence relation, we assume that \( L \) includes the quotient map \( D^n \rightarrow S \).

### 3.2. Some classical theories

In this section we provide a list of the main classical NIP unstable theories we will be interested in. We simply recall their language and main properties and refer the reader to classical references.

1. Let \( L_{\text{og}} := (+, -, \leq, 0) \) be the language of ordered groups. The \( L_{\text{og}} \)-theory of divisible ordered abelian groups will be denoted DOAG. This theory has quantifier elimination and is o-minimal (see [14], Chapter 3).

2. Let \( L_{\text{or}} := (+, -, \cdot, 0, 1, <) \) denote the language of ordered rings. The \( L_{\text{or}} \)-theory of real closed fields is denoted RCF. It is an o-minimal theory with quantifier elimination.

3. Let \( L_{\text{Pres}} \) be the extension of \( L_{\text{og}} \) by adding a binary relation symbol \( \equiv_n \) for each positive integer \( n \). The \( L_{\text{Pres}} \)-theory of \((\mathbb{Z}, +, -, 0, \leq, (\equiv_n)_{n \geq 1}) \) where \( \equiv_n \) is interpreted as equivalence modulo \( n \), is called Presburger Arithmetic. It has quantifier elimination.

4. Let \( L_{\text{Mac}} \) denote Macintyre’s language, that is, the extension of the language of rings \( L_r \) by adding unary predicates \( P_n \) for each integer \( n \geq 2 \). In \( \mathbb{Q}_p \), each predicate \( P_n \) is interpreted as the subset of \( n^{th} \)-powers. The \( L_{\text{Mac}} \)-theory of \( \mathbb{Q}_p \) is the theory of \( p \)-adically closed fields \( p\text{CF} \). It has quantifier elimination. A similar result holds for finite extensions of \( \mathbb{Q}_p \) by adding finitely many constants to \( L_{\text{Mac}} \) as in [9] Theorem 5.6). The corresponding theories are denoted by \( p\text{CF}_d \), where \( d \) denotes the \( p \)-rank of the extension (see [9] Section 2).

5. Let \( L_{\delta, <} \) be the extension of ordered differential rings, namely, the extension of \( L_{\text{or}} \) by a new unary function symbol \( \delta \) interpreted as a derivation. The \( L_{\delta} \)-theory of closed ordered differential fields CODF introduced in [20] by Singer is complete and admits quantifier elimination (see also [18]).

6. Let \( (K, v) \) be a valued field. The one-sorted language of valued fields \( L_{\text{div}} \) is the extension of \( L_r \) with a binary predicate \( \text{div} \) interpreted in \( (K, v) \) as \( \text{div}(x, y) \) if and only if \( v(x) \leq v(y) \). We let \( v: K \rightarrow \Gamma_\infty(K) \) denote the valuation, \( \Gamma(K), +, <, 0 \) the value group (in additive notation), \( \Gamma_\infty \) the set \( \Gamma(K) \cup \{ \infty \} \), \( R_K \) the valuation ring, \( M_K \) the maximal ideal, \( k_K \) the residue field and \( \text{resr}: R_K \rightarrow k_K \) the residue map. The three-sorted language \( L_{\Gamma, k} \) corresponds to \( (K, L_r), (\Gamma, L_{\text{og}}) \) (where \( L_{\text{og}, \infty} \) is the language \( L_{\text{og}} \) together with a constant symbol \( \infty \)) and \( (k, L_r) \), together with symbols for the valuation and the residue map. Given \( a \in K \) and
γ ∈ Γ∞, the closed ball centered at a of radius γ corresponds to the set
\[ B(a, γ) := \{ x ∈ K \mid v(x - a) ≥ γ \} \]
and when γ ∈ ΓK, the open ball centered at a of radius γ corresponds to the set
\[ B^o(a, γ) := \{ x ∈ K \mid v(x - a) > γ \} \]
By a ball we mean a set which is either a closed ball or an open ball. Given \( a, b ∈ K \), We will often use the notation \( B(a, b) \) (resp. \( B^o(a, b) \)) as a short hand to denote the ball \( B(a, v(a - b)) \) (resp. \( B^o(a, v(a - b)) \)). A Swiss cheese is a set of the form \( B \setminus (\bigcup_{i=1}^m B_i) \) where \( B \) is a ball and each \( B_i \) is a ball strictly contained in \( B \) such that \( B_i \cap B_j = \emptyset \) for \( i ≠ j \). The ball \( B \) is called the positive ball of the Swiss cheese.

Suppose \((K, v)\) is algebraically closed and non-trivially valued. For \( \mathcal{L} \) either \( \mathcal{L}_{\text{div}} \) or \( \mathcal{L}_{Γ,K} \), we let \( \text{ACVF}_{p,q}(\mathcal{L}) \) denote the \( \mathcal{L} \)-theory of \((K, v)\). The theory \( \text{ACVF}_{p,q}(\mathcal{L}) \) has quantifier elimination.

(7) Let \( \mathcal{L}_{\text{div}}^≤ \) be the extension of \( \mathcal{L}_{\text{div}} \) with the binary relation \( ≤ \). Let \( \mathcal{L}_{Γ,K}^≤ \) be the extension of \( \mathcal{L}_{Γ,K} \) where \( \mathcal{L}_≤ \) is extended by \( \mathcal{L}_{\text{or}} \) in both the value field sort and the residue field sort. Let \((K, v, ≤)\) be a real closed valued field. For \( \mathcal{L} \) either \( \mathcal{L}_{\text{div}}^≤ \) or \( \mathcal{L}_{Γ,K}^≤ \), we let \( \text{RCVF}(\mathcal{L}) \) be the \( \mathcal{L} \)-theory of \((K, v, ≤)\). The theory \( \text{RCVF}(\mathcal{L}) \) has quantifier elimination.

3.3. Pro/Ind-definability.

3.3.1. Pro-definable sets. Let \((I, ≤)\) be a small upwards directed partially ordered set and \( C \) be a small subset of \( \mathcal{U} \). A C-definable projective system is a collection \((X_i, f_{ij})\) such that

(1) for every \( i ∈ I \), \( X_i \) is a C-definable set;
(2) for every \( i, j ∈ I \) such that \( i ≥ j \); \( f_{ij} : X_i \to X_j \) is C-definable;
(3) \( f_{ii} \) is the identity on \( X_i \) and \( f_{ik} = f_{jk} \circ f_{ij} \) for all \( i ≥ j ≥ k \).

A pro-C-definable set \( X \) is the projective limit of a C-definable projective system \((X_i, f_{ij})\)
\[ X := \lim_{i∈I} X_i. \]

We say that \( X \) is pro-definable if it is pro-C-definable for some small set of parameters \( C \). Pro-definable sets can also be seen as *-definable sets. By a result of Kamensky \[13\], we may identify \( X \) and \( X(\mathcal{U}) \).

3.3.2. Pro-definable morphisms. Let \( X = \lim_{i∈I} X_i \) and \( Y = \lim_{j∈J} Y_j \) be two pro-C-definable sets with associated C-definable projective systems \((X_i, f_{ij})\) and \((Y_j, g_{jj'})\). A pro-C-definable morphism is a map \( φ : X → Y \) together with a function \( d : J → I \) and a family of C-definable functions \( \{ φ_{ij} : X_i → Y_j \mid i ≥ d(j) \} \) such that, for all \( j ≥ j' \) in \( J \) and all \( i ≥ i' \) in \( I \) with \( i ≥ d(j) \) and \( i' ≥ d(j') \), the following diagram commutes
\[ X \xrightarrow{π_i} X_i \xrightarrow{f_{ij}} X_j \xrightarrow{π_j} Y \]
\[ Y \xrightarrow{φ_j} Y_j \xrightarrow{g_{jj'}} Y_{j'}, \]

where \( π_i \) and \( π_j \) denote the canonical projections \( π_i : X → X_i \) and \( π_j : Y → Y_j \).
3.3.3. **Strict pro-definable.** Let $X$ be a pro-$C$-definable set. We say $X$ is *strict pro-definable* if it can be represented by (i.e., be in pro-definable bijection with) a projective system $(X_i, f_{ij})$ where the transition maps $f_{ij}$ are surjective. Equivalently, $X$ is strict pro-definable if the projection $\pi_i(X)$ is $C$-definable for every $i \in I$.

Viewing pro-definable sets as $\ast$-definable sets in the sense of Shelah, it is worth noting that strict-pro-definable sets resemble more of definable sets than type definable sets. In general, the taking points functor induces an equivalence of category when the model of choice is sufficiently saturated. But for strict-pro-definable sets, restrictions on degrees of saturation can be removed.

3.3.4. **Ind-definable sets.** As the dual for pro-definable sets, we have ind-definable sets. We will only use this notion in Section 5.

Let $(I, \leq)$ be a small downwards directed partially ordered set and $C$ be a small subset of $U$.

A $C$-definable directed system is a collection $(X_i, f_{ij})$ such that

1. For every $i \in I$, $X_i$ is a $C$-definable set;
2. For every $i, j \in I$ such that $i \leq j$, $f_{ij} : X_i \to X_j$ is $C$-definable;
3. $f_{ii}$ is the identity on $X_i$ and $f_{ik} = f_{jk} \circ f_{ij}$ for all $i \leq j \leq k$.

A *ind-$C$-definable set* $X$ is the direct limit of a $C$-definable directed system $(X_i, f_{ij})$

$$X := \lim_{\rightarrow i \in I} X_i.$$  

We say that $X$ is *ind-definable* if it is ind-$C$-definable for some small set of parameters $C$. In particular, sets defined by a small union of definable sets are ind-definable.

3.3.5. **Ind-definable morphisms.** Let $X = \lim_{\rightarrow i \in I} X_i$ and $Y = \lim_{\rightarrow j \in J} Y_j$ be two ind-$C$-definable sets with associated $C$-definable directed systems $(X_i, f_{ij})$ and $(Y_j, g_{jj'})$. A *ind-$C$-definable morphism* is a map $\varphi : X \to Y$ together with a function $d : I \to J$ and a family of $C$-definable functions $\{\varphi_{ij} : X_i \to Y_j \mid j \geq d(i)\}$ such that, for all $j \geq j'$ in $J$ and all $i \geq i'$ in $I$ with $j \geq d(i)$ and $j' \geq d(i')$, the following diagram commutes

$$
\begin{array}{ccc}
X & \xleftarrow{\iota_i} & X_i \\
\downarrow{\varphi} & & \downarrow{\varphi_{ij}} \\
Y & \xleftarrow{\iota_j} & Y_j
\end{array}
\quad
\begin{array}{ccc}
X' & \xleftarrow{\iota'_i} & X'_{i'} \\
\downarrow{\varphi_{ij}} & & \downarrow{\varphi_{ij'}} \\
Y' & \xleftarrow{\iota'_j} & Y'_{j'}
\end{array}
$$

where $\iota_i$ and $\iota_j$ denote the canonical inclusions $\iota_i : X_i \to X$ and $\iota_j : Y_j \to Y$.

3.4. **Definable types and the definable completion of a definable set.**

3.4.1. **Definable types.** Let $A$ and $B$ be arbitrary subsets of $U$. A type $p(x) \in S_x(B)$ is *$A$-definable* (or definable over $A$) if for every $L$-formula $\varphi(x, y)$ there is an $L(A)$-formula $d_p(\varphi)(y)$ such that for every $c \in S_y(B)$

(E1) \[ \varphi(x, c) \in p(x) \iff U \models d_p(\varphi)(c). \]

The map $d_p : \mathcal{F}(L) \to \mathcal{F}(L(A))$ is called a *scheme of definition* for $p$. We say $p \in S(B)$ is *definable* if it is $B$-definable.

3.4.2. **Remark.** If $p \in S_x(B)$ is $A$-definable via some scheme of definition $d_p$, then it is also $A_0$-definable via $d_p$ for some subset $A_0 \subseteq A$ with $A_0$ of cardinality $|\mathcal{F}(L)|$.

The following lemma is left to the reader.

3.4.3. **Lemma.** Let $A, B$ be small sets. Then
(1) if \( tp(a_1/B) \) is \( A \)-definable and \( tp(a_2/Ba_1) \) is \( Aa_1 \)-definable, then \( tp(a_1,a_2/B) \) is \( A \)-definable.

(2) if \( tp(a_1/B) \) is \( A \)-definable and \( a_2 \in acl(Aa_1) \), then \( tp(a_2/B) \) is \( A \)-definable. \( \square \)

3.4.7. Remark. 3.4.7. To see it coincides with \( f \) on \( A \)-simple points, let \( M \) be a small model containing \( C \), let \( a \in X(M) \) and \( b \in Y(M) \) be such that \( f(a) = b \). Then we have that for all \( c \in S_w \)

\[
\begin{align*}
U \models \varphi(b,c) & \iff U \models \varphi(f(a),c) & \iff U \models d_p(\varphi(f(x),c))(c),
\end{align*}
\]

which shows that \( tp(b,U) = f_{\text{def}}(tp(a/U)) \). \( \square \)

3.4.8. Remark. Note that if \( f \) is injective/surjective, so is \( f_{\text{def}} \).

3.4.9. Tensor product. Let \( x \) and \( y \) be disjoint tuples of variables. For \( p \in S_x(U) \) and \( q \in S_y(U) \), assume further that \( p \) is definable. Then the tensor product \( p \otimes q \in S_{x,y}(U) \) is defined by

\[
p \otimes q = tp(a,b/U) \iff b \models q \text{ and } a \models p|U \cup \{b\}.
\]

The tensor product is associative but not necessarily commutative.
3.4.9. Generically stable types. Suppose \( T \) is NIP. A definable type \( p \in S_x(U) \) is generically stable if and only if \( p(x) \otimes p(x') = p(x') \otimes p(x) \), where \( x' \) is a disjoint copy of the tuple of variables \( x \). See [19, Section 2.2.2] for a detailed treatment.

3.5. The definable completion as a pro-definable set. As stated in the introduction, we would like to show the set \( S^\text{def}_X \) is pro-definable for \( X \) a definable set. Let us explain what this means. The set \( S^\text{def}_X \) is pro-definable if there is a functorial bijection \( h^X \) between \( S^\text{def}_X \) and a pro-definable set \( D_X \), where by functorial we mean it satisfies the following two properties:

(1) For every \( M \models T \) over which \( X \) and \( D \) are defined, there is a bijection \( h^X_M : S^\text{def}_X(M) \to D_X(M) \). Furthermore, if \( M \preceq N \), then the following diagram commutes

\[
\begin{array}{ccc}
S^\text{def}_X(M) & \xrightarrow{h^X_M} & D_X(M) \\
\downarrow \iota & & \downarrow i \\
S^\text{def}_X(N) & \xrightarrow{h^X_N} & D_X(N)
\end{array}
\]

where the \( \iota \) and \( i \) are the canonical inclusion maps respectively.

(2) If \( f : X \to Y \) is a definable map, there is a pro-definable morphism \( f' : D_X \to D_Y \) (in the sense of \([3.3.2]\)) such that for every model \( M \) of \( T \) over which \( f, D, X \) and \( Y \) are defined, \( f'_M : D_X(M) \to D_Y(M) \) corresponds to the unique map making the following diagram commute

\[
\begin{array}{ccc}
S^\text{def}_X(M) & \xrightarrow{h^X_M} & D_X(M) \\
\downarrow f^\text{def} & & \downarrow f'_M \\
S^\text{def}_Y(M) & \xrightarrow{h^Y_M} & D_Y(M)
\end{array}
\]

where \( f^\text{def} \) is as in Lemma \([3.4.6]\).

3.6. Other completions.

3.6.1. Bounded types. Let \( T \) be an \( o \)-minimal theory and \( M \) be a model of \( T \). Given an elementary extension \( M \preceq N \), we say that \( N \) is bounded by \( M \) if for every \( b \in N \), there are \( c_1, c_2 \in M \) such that \( c_1 < b < c_2 \). Let \( A \) be a small subset of \( U \) and \( X \) be a definable set. A type \( p \in S^\text{def}_X(A) \) is bounded if for any small model \( M \) containing \( A \) and every realization \( a \models p|M \), there is an elementary extension \( M \preceq N \) with \( a \models N^{\ell(x)} \) and such that \( N \) is bounded by \( M \).

Let \( T \) be either RCVF or a completion of ACVF. Let \( A \) be a small subset of \( U \). A type \( p \in S^\text{def}_X(A) \) is bounded if for any small model \( M \) containing \( A \) and every realization \( a \models p|M \), \( \Gamma(\text{acl}(Ma)) \) is bounded by \( \Gamma(M) \).

3.6.2. The bounded completion. Let \( T \) be either an \( o \)-minimal theory, a completion of ACVF or RCVF. Let \( A \) be a small subset of \( U \) and \( X \) be a definable set. The bounded completion of \( X \) over \( A \), denoted \( \bar{X}(A) \), is the set of bounded global \( A \)-definable types.

3.6.3. Types orthogonal to \( \Gamma \). Let \( T \) be a complete theory extending the theory of valued fields (e.g., RCVF or a completion of ACVF). Let \( A \) be a small subset of \( U \). A type \( p \in S^\text{def}_X(A) \) is said to be orthogonal to \( \Gamma \) if for every model \( M \) containing \( A \) and every realization \( a \models p|M \), \( \Gamma(M) = \Gamma(\text{acl}(Ma)) \).
3.6.4. The orthogonal completion. Let $T$ be either a completion of ACVF or RCVF. Given a definable set $X$, the orthogonal completion of $X$, denoted by $\tilde{X}(A)$, is the set of global $A$-definable types concentrating on $X$ which are orthogonal to $\Gamma$.

By [11] Proposition 2.9.1, when $T$ is a completion of ACVF, the set $\tilde{X}(A)$ also corresponds to the set of definable types which are generically stable and is also called the stable completion of $X$. Note that if $T$ is RCVF every generically stable type is a realized type.

3.6.5. Remark.

(1) If $T$ is an o-minimal expansion of the theory of real closed field, then every bounded definable type is a realized type. However, it is not the case for general o-minimal theories. For example, the type $0^+$ in DOAG is one such type.

(2) Let $T$ be one of the theories listed in Definition 3.6.2 and $M$ be a model of $T$. If $\text{tp}(a/M)$ is bounded, then $\text{tp}(b/M)$ is bounded for every $b \in \text{acl}(M,a)$.

(3) Let $T$ be either a completion of ACVF or RCVF, and let $M$ be a model of $T$. If $\text{tp}(a/M)$ is orthogonal to $\Gamma$, then $\text{tp}(b/M)$ is orthogonal to $\Gamma$ for every $b \in \text{acl}(M,a)$.

3.6.6. Remark. Let $T$ be one of the theories listed in Definition 3.6.2. If $p \in S_x(M)$ is a bounded definable type (resp. orthogonal to $\Gamma$) its canonical extension $p[U] \in S_x(U)$ is also bounded (resp. orthogonal to $\Gamma$). Moreover, if $f : X \to Y$ is a definable function, then the restriction of $f^{\text{def}} : S_X^{\text{def}} \to S_Y^{\text{def}}$ to $\tilde{X}$ (resp. to $\tilde{Y}$) defines a function $\tilde{f} : \tilde{X} \to \tilde{Y}$ (resp. $\tilde{f} : \tilde{X} \to \tilde{Y}$).

In spite of Remark 3.6.6 saying that $\tilde{X}$ (resp. $\tilde{Y}$) is pro-definable means that it is in functorial bijection with a pro-definable set as in Section 3.5.

3.6.7. Geometric interpretation. For $T$ a completion of ACVF, let $V$ be a variety over some valued field $F$. In [11], $V$ is introduced as a model-theoretic analogue of the Berkovich analytification $V^{\text{an}}$ of $V$. Similarly, our aim is to view $\tilde{V}$ as a model-theoretic analogue of the Huber analytification of $V$. When $T$ is RCVF, both $\tilde{V}$ and $\tilde{V}$ are good candidates to be the model-theoretic counterparts of the analytification of semi-algebraic sets defined by defined by Jell, Scheiderer and Yu in [12]. The set $\tilde{V}$ also corresponds to the set of residue field dominated types as defined by Ealy, Haskell and Mariková in [8]. Finally, $S_X^{\text{def}}$ can be viewed as a model-theoretic analogue of the “space of valuations on $V$”. We will present more structural results concerning the set $\tilde{V}$ in a sequel of this article.

3.7. Uniform definability of types. A uniform scheme of definition for $S_X^{\text{def}}$ over $A$ is a pair of maps $(d, c)$

$$d : \mathcal{F}_x(L) \to \mathcal{F}(L(A))$$

$$\varphi(x, y) \mapsto d(\varphi)(y, z_{\varphi})$$

and

$$c : S_X^{\text{def}} \times \mathcal{F}_x(L) \to S_{x^{\varphi}}(A)$$

$$p, \varphi \mapsto c(p, \varphi)$$

such that for every $p \in S_x^{\text{def}}$ and every $L$-formula $\varphi(x, y)$, the map $\varphi(x, y) \mapsto d(\varphi)(y, c(p, \varphi))$ is a scheme of definition for $p$. We say that $S_X^{\text{def}}$ is uniformly definable if it admits a uniform scheme of definition over some small set. The theory $T$ has uniform definability of types if for every finite tuple of variables $x$, $S_x^{\text{def}}$ is uniformly definable. Equivalently, $T$ has uniform definability of types if for every definable set $X$, $S_X^{\text{def}}$ is uniformly definable. When $T$ has a dominant sort $D$, $T$ has uniform definability of types if for every $n \geq 1$, $S^{\text{def}}_D$ is uniformly definable. The following is a routine coding exercise.
3.7.1. **Lemma.** Suppose that for every $L$-formula $\varphi(x,y)$ there are finitely many formulas $\psi_1(y,z_1), \ldots, \psi_n(y,z_n)$ such that for every $p \in S^\text{def}_X$ there are $i \in \{1, \ldots, n\}$ and $c \in S_{z_i}$ such that for all $b \in S_y$

$$\varphi(x,b) \in p(x) \iff \mathcal{U} \models \psi_i(b,c).$$

Then $S^\text{def}_X$ is uniformly definable.

3.8. **Pro-definability of $S^\text{def}_X$ via uniform definability.** The proofs of the following results are extracted *mutatis-mutandis* from [II]. We include them for the reader’s convenience.

Let $X \subseteq S_x$ be a definable set. Suppose that $S^\text{def}_X$ is uniformly definable and let $d = (d, c)$ be a uniform scheme of definition of $S^\text{def}_X$. Given an $L$-formula $\varphi(x,y)$, the tuple of variables $z_\varphi$ is given by $d(\varphi)(y, z_\varphi)$. We associate to this data a map $\tau_d$ defined by

$$(E2) \quad \tau_d: S^\text{def}_X \to \prod_{\varphi \in F_\varphi(L)} S_{z_\varphi} \quad p \mapsto (c(p, \varphi))_{\varphi \in F_\varphi(L)},$$

3.8.1. **Remark.** Let $d = (d, c)$ be a uniform scheme of definition of $S^\text{def}_X$. Note that to show that set $S^\text{def}_X$ is pro-definable it suffices to show that $\tau_d(S^\text{def}_X)$ is $*$-definable (note that $\tau_d$ is injective). Similarly, to show that $S^\text{def}_X$ is strict pro-definable, it suffices to show that each projection $\pi_\varphi: \tau_d(S^\text{def}_X) \to S_{z_\varphi}$ is definable. It is not difficult to show all functoriality properties from Section 3.3 are satisfied.

3.8.2. **Proposition.** Suppose that $S^\text{def}_X$ is uniformly definable. Then $S^\text{def}_X$ is pro-definable in $L^\text{eq}$. In particular, if $T$ has elimination of imaginaries, then $S^\text{def}_X$ is pro-definable.

**Proof.** Fix a uniform scheme of definition $d = (d, c)$ of $S^\text{def}_X$. Without loss of generality we may suppose that $d$ factors through conjunctions, that is, $d(\varphi \land \psi) = d(\varphi) \land d(\psi)$. By elimination of imaginaries, we may further suppose that

$$((\forall z_\varphi)(\forall z'_\varphi)(\forall y)[(d(\varphi)(y, z_\varphi) \leftrightarrow d(\varphi)(y, z_\varphi')) \to z_\varphi = z'_\varphi].$$

Thus, $c(p, \varphi)$ is the canonical parameter of the set defined by the formula $d(\varphi)(y, c(p, \varphi))$ (note that this implies that $d(\varphi)(y, z_\varphi)$ is now an $L^\text{eq}$-formula). Consider the following set of formulas $\Theta$ containing, for each $L$-formula $\varphi(x,y)$, the formula $\theta_\varphi(z_\varphi)$ given by

$$\theta_\varphi(z_\varphi) := (\forall y)(\exists x)(x \in X \land (\varphi(x,y) \leftrightarrow d(\varphi)(y, z_\varphi))).$$

Let $\varphi_1(x,y_1), \ldots, \varphi_m(x,y_m)$ be $L$-formulas, $y$ be the tuple $(y_1, \ldots, y_m)$ and $\varphi(x,y)$ denote the conjunction $\bigwedge_{i=1}^m \varphi_i(x, y_i)$. Since $d$ factors through conjunctions

$$(E3) \quad \models ((\forall z_\varphi)(\forall z'_{\varphi_1}) \cdots (\forall z'_{\varphi_m})(\forall y)[(d(\varphi)(y, z_\varphi) \leftrightarrow \bigwedge_{i=1}^m d(\varphi_i)(y, z_{\varphi_i}))].$$

We claim that

$$\tau_d(S^\text{def}_X) = \{(c_\varphi)_{\varphi} \in \prod_{\varphi \in F_\varphi(L)} S_\varphi \mid (c_\varphi)_{\varphi} \models \Theta\}.$$ 

From left-to-right, let $p \in S^\text{def}_X$ and $\theta_\varphi(z_\varphi)$ be a formula in $\Theta$. We have that

$$\tau_d(p) \models \theta_\varphi(z_\varphi) \iff (c(p, \varphi))_{\varphi} \models \theta_\varphi(z_\varphi)$$

$$\iff \theta_\varphi(c(p, \varphi))$$

$$\iff ((\forall y)(\exists x)(x \in X \land (\varphi(x,y) \leftrightarrow d(\varphi)(y, c(p, \varphi))))),$$

and the last formula holds since for every $y$ any realization of $p$ satisfies such formula.
To show the right-to-left inclusion, let \((c_\varphi)_\varphi\) be such that \((c_\varphi)_\varphi \models \Theta\). Consider the set of formulas

\[ p(x) := \{ x \in X \} \cup \{ \varphi(x, b) \mid U \models d(\varphi)(b, c_\varphi) \}. \]

Let us show that \(p(x)\) is an element of \(S_X(U)\). Once we show \(p(x)\) is consistent, that \(p \in S_X^{\text{def}}\) follows by definition. Let \(\varphi_1(x, b_1), \ldots, \varphi_m(x, b_m)\) be formulas in \(p(x)\). Let \(y = (y_1, \ldots, y_m)\) and \(\varphi(x, y)\) be the conjunction \(\bigwedge_{i=1}^m \varphi_i(x, y_i)\). Letting \(b := (b_1, \ldots, b_n)\), by the definition of \(p(x)\) and (3.13) we have that the formula \(\varphi(x, b)\) is also in \(p(x)\). Moreover, since \((c_\varphi)_\varphi \models \theta_\varphi\) we have in particular that

\[ \models (\exists x)(x \in X \land (\varphi(x, b) \leftrightarrow d(\varphi)(b, c_\varphi))). \]

Finally, since \(\varphi(x, b) \in p(x)\), we must have that \(\models d(\varphi)(b, c_\varphi)\), which shows there is an element satisfying \(\varphi(x, b)\). By compactness, \(p(x)\) is consistent. The result now follows by Remark 3.8.1.

\[ \square \]

4. Stably embedded pairs

4.1. Elementary pairs. Suppose \(L\) is a one-sorted language. Let \(L_P\) be a language extending \(L\) by a new unary predicate \(P\). We denote an \(L_P\)-structure as a pair \((N, A)\) where \(N\) is an \(L\)-structure and \(A \subseteq N\) corresponds to the interpretation of \(P\). Given a complete \(L\)-theory \(T\), the \(L_P\)-theory of elementary pairs of models of \(T\), denoted \(T_P\), is the theory of the class of \(L_P\)-structures \((N, M)\) where \(M \prec N \models T\). A standard argument shows \(T_P\) that is indeed an \(L_P\)-theory.

Given a tuple \(x = (x_1, \ldots, x_m)\), we abuse of notation and write \(P(x)\) as an abbreviation for \(\bigwedge_{i=1}^m P(x_i)\).

When \(L\) is multi-sorted we let \(L_P\) denote the language which extends \(L\) by a new unary predicate \(P_S\) for every \(L\)-sort \(S\). Analogously, an \(L_P\)-structure \(N\) is a model of \(T_P\) if the collection of subsets \(P_S(N)\) forms an elementary \(L\)-substructure of \(N\). We will also denote any such a structure as a pair \((N, M)\) where \(M \prec N \models T\) and for every \(L\)-sort \(S\), \(P_S(N) = S(M)\).

When \(L\) has a dominant sort \(D\), adding a single predicate \(P\) for \(P_D\) suffices. Indeed, given an \(L\)-sort \(S\), the predicate \(P_S\) is \(L_P\)-definable using the predicate \(P_D\). We leave here the details to the reader. When \(L\) has a dominant sort \(D\), we will thus abuse of notation and often identify an \(L\)-elementary pair \((N, M)\) with the corresponding \(L_{|D}\)-elementary pair \((D(N), D(M))\) (and vice versa).

4.2. Stably embedded pairs.

4.2.1. Definition. Let \(M \prec N\) be an elementary extension of models of \(T\). The extension is called stably embedded if for every \(L(N)\)-definable subset \(X \subseteq N^m\), the set \(X \cap M^m\) is \(L(M)\)-definable.

4.2.2. The class of stably embedded models of \(T\), denoted \(SE(T)\), is the class of \(L_P\)-structures \((N, M)\) such that \(M \prec N \models T\) and the extension \(M \prec N\) is stably embedded. Given \(T\) such that \(SE(T)\) is \(L_P\)-elementary (i.e. it is the class of models of a certain \(L_P\)-theory), we let \(T_P^w\) be the \(L_P\)-theory \(Th(SE(T))\) and call it the theory of stably embedded pairs of models of \(T\). The following lemma is a standard exercise.

4.2.3. Lemma. An elementary extension \(M \prec N\) is stably embedded if and only if for every tuple \(a\) in \(N\), the type \(tp(a/M)\) is definable.

\[ \square \]

4.2.4. Theorem. Let \(T\) be one of the following theories
(i) a complete stable theory;
(ii) a complete o-minimal theory;
(iii) Presburger arithmetic, $pCF_d$, CODF, ACVF$_{p,q}$ or RCVF (in any of the languages mentioned in points (6)-(7) of Section 3.2).

Then, the class $\mathcal{SE}(T)$ is an elementary class in $\mathcal{L}_p$.

**Proof.** Suppose $(N, M)$ is an $\mathcal{L}_p$-model of $T_P$, so $M \prec N$.

If $T$ is a complete stable theory then $\mathcal{SE}(T)$ is the class of models of $T_P$ (see [19 Theorem 2.60]).

If $T$ is a complete o-minimal theory, this follows by the Marker-Steinhorn theorem [15]. Indeed we have that $(M, N)$ is stably embedded if and only if $M$ is Dedekind complete in $N$, which is an $\mathcal{L}_p$-elementary property. Similarly, for CODF, by [3 Proposition 3.6], $(N, M)$ is stably embedded if and only if $M$ is Dedekind complete in $N$. For ACVF$_{p,q}(\mathcal{L}_{\Gamma,k})$ the result follows by [6 Theorem 1.9]. For Presburger arithmetic, the result corresponds to Corollary A.1.3. The argument is entirely based on an unpublished note by G. Conant and S. Vojdani in [5]. For real closed valued fields and $p$-adically closed valued fields, the result correspond respectively to Corollaries A.2.4 and A.3.4 whose proof are presented in the Appendix. □

4.2.5. **Question.** Is there a natural characterization of the class of complete NIP theories $T$ for which $\mathcal{SE}(T)$ is $\mathcal{L}_p$-elementary?

4.3. **Uniform definability via stably embedded pairs.** In this section we show the main relation between stably embedded pairs and pro-definability which is gathered in Theorem 4.3.1 and Corollary 4.3.3.

4.3.1. **Theorem.** Let $T$ be such that

(1) $\mathcal{SE}(T)$ is $\mathcal{L}_p$-elementary;
(2) for every small model $M \models T$ and every finite tuple $a \in U$, $acl(Ma)$ is a model of $T$;
(3) $T$ is model complete.

Then $T$ has uniform definability of types. In particular, $S^\text{def}_X$ is pro-definable in $\mathcal{L}^{eq}$ for every definable set $X$.

**Proof.** Fix a model $M$ of $T$ an a $\mathcal{L}$-formula $\varphi(x, y)$. Let $(\psi_i(y, z_i))_{i \in I}$ be an enumeration of $\mathcal{F}_y(\mathcal{L})$. Suppose for a contradiction that no formula $\psi_i(x, z_i)$ provides a uniform definition for $\varphi$ over $M$. This implies, by Lemma 3.7.1, that for every finite subset $J \subseteq I$ there is a definable type $q_J(x) \in S(M)$ such that $\varphi$ cannot be defined in $q_J$ by any of the formulas in $J$. Consider for every $i \in I$ the $\mathcal{L}_p$-formula $\theta_i(x)$

$$\left(\forall z_i \in P(\exists y \in P)(\neg(\varphi(x, y) \leftrightarrow \psi_i(y, z_i)))\right).$$

Let $\Sigma(x) := \{\theta_i(x) \mid i \in I\} \cup T^se_x$. Let us show that $\Sigma(x)$ is consistent. Let $\Sigma_0$ be a finite subset of $\Sigma$ and let $J := \{i \in I \mid \theta_i(x) \in \Sigma_0\}$. By assumption, there is some definable type $q = q_J(x) \in S(M)$ such that $\varphi$ is not defined in $q$ by any of the formulas $\psi_i$ with $i \in J$. Let $a$ be a realization of $q$ in some elementary extension of $M$. By the choice of $a$ we have that $\theta_i(a)$ holds for all $i \in J$. By assumptions (2) and (3), $M \prec N := acl(Ma)$. By Lemmas 3.4.3 and 4.2.3 the pair $(N, M)$ is stably embedded, which shows that $\Sigma_0$ is consistent. Thus, by compactness, $\Sigma$ is consistent. Let $(N', M')$ be a model of $T^se_a$ and $a \in N'$ be a realization of $\Sigma$. By the definition of $\Sigma$, the type $tp(a/M')$ is not definable, which contradicts the fact that the pair $(N', M')$ is stably embedded (by Lemma 4.2.3). □

4.3.2. **Remark.** Note that if $T$ is stable, the above proof gives another way of proving uniform definability of types by taking $N$ to be any model containing $M$ and $a$. In particular, if $T$ is stable, then $S^\text{def}_X$ is pro-definable in $\mathcal{L}^{eq}$ by Theorem 3.8.2.
4.3.3. **Corollary.** All theories from Section 3.2 have uniform definability of types.

*Proof.* It suffices to show that conditions (1)-(4) from Theorem 4.3.1 hold for $T$. Condition (1) is straightforward in each case. That condition (2) holds is part of the content of Theorem 4.2.4. All theories have quantifier elimination, so (4) is also guaranteed. Condition (3) follows by standard arguments using quantifier elimination in each of these theories (for pCF$_d$ see [22]).

4.3.4. **Corollary.** Let $T$ be any theory listed in Section 3.2. Then for every $\mathcal{L}(\mathcal{U})$-definable set $X$, the definable completion $S^X_{\text{def}}$ is a pro-definable set in $\mathcal{L}_{\text{eq}}$. If $T$ has elimination of imaginaries, then $S^X_{\text{def}}$ is pro-definable.

*Proof.* This follows by Corollary 4.3.3 and Proposition 3.8.2.

## 5. Strict pro-definability

Let $T$ be an $\mathcal{L}$-theory with uniform definability of types and assume it eliminates imaginaries. Let $d = (d, c)$ be a uniform scheme of definition and suppose $X$ is a subset of $S_x$. By elimination of imaginaries we may assume that $d(\varphi)(y, z)$ is such that

\[(\forall z)(\forall z')(\forall y)[(d(\varphi)(y, z) \leftrightarrow d(\varphi)(y, z')) \rightarrow z = z'].\]

In this situation, by Proposition 3.8.2, the set $S^X_{\text{def}}$ is pro-definable. It is also strict pro-definable provided that, for every model $M$ of $T$ and every $\mathcal{L}$-formula $\varphi(x, y)$, the projection

\[
\pi(\varphi)(S^X_{\text{def}}(M)) := \{c \in S_x(M) \mid \exists p \in S^X_{\text{def}}(M) \ c = c(p, \varphi)\}
\]

\[
= \{c \in S_x(M) \mid \exists p \in S^X_{\text{def}}(M) \forall b \in S_y(M)(\varphi(x, b) \in p \iff M \models d(\varphi)(b, c))\}
\]

is $\mathcal{L}(M)$-definable. As noticed by Hrushovski and Loeser (see [11, Theorem 3.1.1]), it suffices to show that this projection is ind-definable. Indeed, the pro-definability of $S^X_{\text{def}}$ implies that $\pi(\varphi)(S^X_{\text{def}}(M))$ is $\omega$-definable, and thus if it is also ind-definable, a standard compactness argument implies it is definable.

Similarly, when $T$ is an o-minimal theory, RCVF or a completion of ACVF, one can show strict pro-definability of $\tilde{X}$ and $\tilde{X}$ by showing that the corresponding sets

\[
\pi(\varphi)(\tilde{X}(M)) := \{c \in S_x(M) \mid \exists p \in \tilde{X}(M) \ c = c(p, \varphi)\}
\]

\[
\pi(\varphi)(\tilde{X}(M)) := \{c \in S_x(M) \mid \exists p \in \tilde{X}(M) \ c = c(p, \varphi)\}
\]

are $\mathcal{L}(M)$-definable. Note that we do not require to previously show pro-definability of $\tilde{X}$ or $\tilde{X}$: if the projections are definable, the set $\tau_d(\tilde{X})$ (resp. $\tau_d(\tilde{X})$) as defined in Section 3.8 corresponds to the $*$-definable set formed by all these projections, so pro-definability is granted (see Remark 3.8.1).

5.0.1. **Remark.** Note that to show strict pro-definability of $S^X_{\text{def}}$ for every definable set $X$ it suffices to show it for $S^X_d$ for all finite tuple of variables $x$. In addition, one may suppose without loss of generality that $d$ commutes with boolean combinations, that is, $d(\varphi \land \psi) = d(\varphi) \land d(\psi)$ and $d(\neg \varphi) = \neg d(\varphi)$. In this case we also have that for $\mathcal{L}$-formulas $\varphi(x, y)$ and $\psi(x, y)$ with $d(\varphi)(y, z)$

\[
\pi(\varphi \land \psi)(S^X_{\text{def}}(M)) = \pi(\varphi)(S^X_{\text{def}}(M)) \cap \pi(\psi)(S^X_{\text{def}}(M))\]

\[
\pi(\neg \varphi)(S^X_{\text{def}}(M)) = S_x(M) \setminus \pi(\varphi)(S^X_{\text{def}}(M)).\]

When $T$ is o-minimal, RCVF or a completion of ACVF, we will show pro-definability of $S^X_{\text{def}}$ (resp. $\tilde{X}$ and $\tilde{X}$) with an extra detour argument. We will first show that for all such theories, the corresponding projection is $\mathcal{L}_{p'}$-definable in a particular stably embedded pair
(N, M). Then, we will further show that such an $\mathcal{L}_P$-definable set is also $\mathcal{L}$-definable. We introduce such stably embedded pairs in the following section.

5.1. **Special pairs.** Let $T$ be an $\mathcal{L}$-theory. We will define particular stably embedded pairs having a certain saturation property with respect to definable types and 1-types. We will later (in Section 7) provide an axiomatization of their $\mathcal{L}_P$-theory.

5.1.1. **Definition.** An elementary pair $(N, M)$ is a **special pair** if for all finite tuples of variables $x$

(i) for every $p \in S^\text{def}_x(M)$ there is $a \in N$ such that $a \models p$,

(ii) for every tuple $a$ in $S_x(N)$, the type $tp(a/M) \in S^\text{def}_x(M)$,

and, in addition, when there is one variable, for every $p \in S^\text{def}_x(M)$ and every finite tuple $b \in N$, there is $a \in N$ such that $a \models p|Mb$.

5.1.2. **Definition.** Let $T$ be either an o-minimal theory, RCVF or a completion of ACVF. A pair of $M < N$ of models of $T$ is a **b-special pair** (resp. in the latter two theories, an **o-special pair**) if for every definable set $X$

(i) for every $p \in \tilde{X}(M)$ (resp. $\tilde{X}(M)$) there is $a \in X(N)$ such that $a \models p$,

(ii) for every tuple $a$ in $X(N)$, the type $tp(a/M) \in \tilde{X}(M)$ (resp. $tp(a/M) \in \tilde{X}(M)$),

and, in addition, when $x$ is one variable, for every $p \in \tilde{X}(M)$ (resp. $\tilde{X}(M)$) and every finite tuple $b \in N$, there is $a \in N$ such that $a \models p|Mb$.

Note that special pairs (resp. b-special and o-special pairs) are stably embedded by Lemma 4.2.3. Consider the following property of $T$:

(P) For every small subset $A \subset U$ and a small model $M \subset U \acl(MA)$ is a model of $T$.

Note that almost all theories in Section 3.2 satisfy (P).

5.1.3. **Lemma.** Let $T$ be an $\mathcal{L}$-theory having property (P) and $M$ be a model of $T$. Then there is an elementary extension $N$ of $M$ such that $(N, M)$ is a special pair. If $T$ is either an o-minimal expansion of DOAG with quantifier elimination, RCVF or a completion of ACVF, then there is an elementary extension $N$ of $M$ such that $(N, M)$ is a b-special pair (resp. o-special pair for the latter two theories).

**Proof.** Let $\{p_\alpha\}_{\alpha<\kappa}$ be an enumeration of all $M$-definable types and $\{q_\beta\}_{\beta<\lambda}$ be an enumeration of $M$-definable 1-types. We will construct the model $N$ via an elementary chain. Let $M_0 = M$ and assume $M_\alpha$ has been constructed. Take $a \models p_\alpha|M_\alpha$ and let $M_{\alpha+1,0} = \acl(M,a)$ (which is a model by Property (P)). Assume $M_{\alpha+1,\beta}$ has been constructed and take $a \models q_\beta|M_{\alpha+1,\beta}$, and let $M_{\alpha+1,\beta+1} = \acl(M_{\alpha+1,\beta}, a)$. Let $M_{\alpha+1} = \bigcup_{\beta<\lambda} M_{\alpha+1,\beta}$. By Lemma 4.2.3, $M_\alpha \subseteq M_{\alpha+1}$ is stably embedded. Let $N = \bigcup_{\alpha<\kappa} M_\alpha$. By construction, the pair $(N, M)$ is a special pair.

To construct a b-special pair (resp. an o-special pair) one follows the same construction taking $\{p_\alpha\}_{\alpha<\kappa}$ to be an enumeration of all bounded definable types (resp. definable types orthogonal to $\Gamma$) and $\{q_\beta\}_{\beta<\lambda}$ be an enumeration of all bounded $M$-definable (resp. orthogonal to $\Gamma$) 1-types. Remark 3.6.5 guarantees that the type of any tuple in $M_{\alpha+1}$ is also bounded (resp. orthogonal to $\Gamma$).

Let $(N, M)$ be a special pair and suppose $T$ has uniform definability of types. If $(d, c)$ is a uniform scheme of definition, we have

$$\pi_\varphi(S^\text{def}_x(M)) = \{ c \in S_x(M) \mid \exists a \in S_x(N) \forall b \in S_y(M) (\varphi(a, b) \leftrightarrow d(\varphi)(c, b))\}.$$
and it is easy to see that \( \pi_{\varphi}(S_x^{\text{def}}(M)) \) is therefore an \( \mathcal{L}_P \)-definable subset of \((N, M)\). Similarly if \((N, M)\) is a \(b\)-special pair (resp. an \(o\)-special pair) the projection \(\pi_{\varphi}(S_x(M))\) (resp. \(\pi_{\varphi}(S_x(M))\)) are \(\mathcal{L}_P\)-definable in the pair. To obtain strict pro-definability of \(S_x^{\text{def}}(M)\) (resp. \(S_x^{\text{def}}(M)\)) it remains thus to show that such an \(\mathcal{L}_P\)-definable subset of \(M\) is \(\mathcal{L}\)-definable. This follows from the following stronger result.

5.1.4. Theorem. Let \(T\) be either an \(o\)-minimal expansion of the theory DOAG with QE, a completion of ACVF or RCVF. Let \((N, M)\) be a special pair (resp. \(b\)-special pair, or \(o\)-special pair). Then every \(\mathcal{L}_P\)-definable subset of \(M\) is \(\mathcal{L}\)-definable.

The proof of Theorem 5.1.4 will be given at the end of Section 7 where we will further provide an axiomatization of the \(\mathcal{L}_P\)-theory of special pairs (resp. \(b\)-special and \(o\)-special pairs). As a corollary we obtain:

5.1.5. Corollary. Let \(T\) be either an \(o\)-minimal expansion of the theory DOAG with quantifier elimination, a completion of ACVF or RCVF. Let \(X\) be a definable set. Then \(S_X^{\text{def}}\) and \(\bar{X}\) are strict pro-definable. If \(T\) is one of the two latter theories, \(\bar{X}\) is also strict pro-definable.

Proof. Since all theories \(T\) in the statement have property \([P]\), the result follows from Lemma 5.1.3 and Theorem 5.1.4. \(\square\)

In the case when \(T\) is a completion of ACVF, strict pro-definability of \(\bar{X}\) was proved by Hrushovski and Loeser [11, Theorem 3.1.1].

As mentioned in the introduction, the strategy to show Theorem 5.1.4 cannot be used to prove analogue statements for Presburger arithmetic, the theory of \(p\)-adically closed fields or CODF. We will explain this in Section 8. In the next section we show nevertheless that for Presburger Arithmetic the space \(S_x^{\text{def}}\) is also strict pro-definable.

5.2. Strict pro-definability in Presburger arithmetic. Let \(T\) be Presburger arithmetic and \(M\) be a small model of \(T\). In this case \(\text{acl} = \text{dcl}\). Note that a set \(\{a_1, \ldots, a_k\} \subseteq \mathcal{U}\) is \(\text{acl}\)-independent over \(M\) if

\[
\sum_{i=1}^{k} n_i a_i + u \neq 0 \quad \text{for all } n_i \in \mathbb{Z} \text{ and all } u \in M.
\]

Given a tuple \(b = (b_1, \ldots, b_k)\) and \(c \in M\), we will write \(b < c\) as an abbreviation for \(\bigwedge_{i=1}^{k} b_i < c\) (and similarly for \(c < b\)).

5.2.1. Theorem. The space \(S_x^{\text{def}}(M)\) is strict pro-definable in \(\mathcal{L}_{\text{Pres}}\) for every tuple of variables \(x\).

Proof. Let \(x = (x_1, \ldots, x_n)\) and \((d, c)\) be a uniform scheme of definition. We show that, given a formula \(\varphi(x, y)\) with \(y = (y_1, \ldots, y_m)\), the projection \(\pi_{\varphi}(S_x(M))\) is ind-definable. We let \(I := \{1, \ldots, n\}\) and \(J = \{1, \ldots, m\}\). By quantifier elimination and Remark 5.0.1 we may suppose the formula \(\varphi(x, y)\) is one of the following

(E4) \[
\sum_{i \in I} n_i x_i + \sum_{j} m_{j \in J} y_j + k = 0
\]

(E5) \[
\sum_{i \in I} n_i x_i + \sum_{j \in J} m_j y_j + k \square 0
\]

(E6) \[
\sum_{i \in I} n_i x_i + \sum_{j \in J} m_j y_j + k \equiv_m \ell
\]
where \( \square \in \{ <, > \}, n_i, m_j, k \in \mathbb{Z}, m \in \mathbb{N}_{>1} \) and \( 0 \leq \ell < m \). Without loss of generality we may further suppose that there is at least some \( i \in I \) such that \( n_i \neq 0 \). We proceed by induction on the length \( n \) of \( x \), so suppose \( n = 1 \). Given \( p \in S_x^{\text{def}}(M) \) and \( a \in U \) a realization of \( p \), by Theorem \( \text{[A.1.1]} \) there are only three possibilities: either \( a \in M, a < M \) or \( M < a \). We split in cases with respect to the form of \( \varphi \).

**Case 1:** Suppose \( \varphi \) is as in \( \text{[E.4]} \). Without loss of generality we may suppose that for \( z = (z_1, z_2) \), the formula \( d(\varphi)(y, z) \) is

\[
z_1 = 0 \land n_1 z_2 + \sum_{j \in J} m_j y_i + k = 0
\]

where \( z \) ranges over the \( \emptyset \)-definable set \( \{0\} \times M \cup \{(1,0)\} \) (note there is a bijection between this set and the equivalent classes of \( S_x \) defining the same set \( d(\varphi)(M, z) \)). If \( p \in S_x^{\text{def}}(M) \) and \( p = \text{tp}(a/M) \) for some \( a \in M \), then \( c(p, \varphi) = (0, a) \). If \( p \) is not a realized type, then \( \varphi(x, b) \notin p \) for all \( b \in S_y(M) \), and \( c(p, \varphi) = (1, 0) \). Therefore, \( \{0\} \times M \cup \{(1,0)\} = \pi_\varphi(S_x(M)) \), which shows the projection is definable.

**Case 2:** Suppose \( \varphi \) is as in \( \text{[E.5]} \). Assume without loss of generality that \( n_1 > 0 \) and \( \square \) is \( < \). As in Case 1, we may suppose that for \( z = (z_1, z_2) \), the formula \( d(\varphi)(y, z) \) is

\[
(z_1 = z_2 = 0) \lor (z_1 = 1 \land n z_2 + \sum_{j \in J} m_j y_i + k < 0),
\]

where \( z \) ranges over the \( \emptyset \)-definable set \( \{(0,0)\} \cup \{1\} \times M \cup \{(0,1)\} \). Let \( p \in S_x^{\text{def}}(M) \) and suppose \( p = \text{tp}(a/M) \). If \( a \in M \), then \( c(p, \varphi) = (1, a) \). If \( a < M \), then \( \varphi(a, b) \) holds for all \( b \in S_y(M) \), so \( c(p, \varphi) = (0, 0) \). Finally, if \( M < a \), then \( \varphi(a, b) \) does not hold for any \( b \in S_y(M) \) and \( c(p, \varphi) = (0, 1) \). Therefore, \( \{0\} \times M \cup \{(1,0)\} = \pi_\varphi(S_x(M)) \), so the projection is definable.

**Case 3:** Suppose \( \varphi \) is as in \( \text{[E.6]} \). Let us show that \( \pi_\varphi(S_x(M)) \) is defined by the formula \( \theta(z) \)

\[
(\exists y) d(\varphi)(y, z) \land (\forall y) (\forall y') [(d(\varphi)(y, z) \land d(\varphi)(y', c)) \rightarrow \sum_j m_j y_i \equiv_m \sum_j m_j y'_i)].
\]

If \( c \in \pi_\varphi(S_x(M)) \), then it is easy to see that \( \theta(c) \) holds. So suppose \( \theta(c) \) holds and let \( b \in S_y(M) \) be such that \( d(\varphi)(b, c) \) holds. Then the set of formulas

\[
\{ a < x \mid a \in M \} \cup \{ n_1 x + \sum_{j \in J} m_j b_j + k \equiv_m \ell \}
\]

is consistent. Any type \( p(x) \) containing it is definable by Theorem \( \text{[A.1.1]} \) and \( c = c(p, \varphi) \).

This completes the proof for \( n = 1 \).

Now suppose the result holds for all formulas \( \psi(x', y) \) where \( x' \) has less than \( n \) variables. Let \( u \) be a new single variable. Let \( D \) be the set of tuples \( d = (d_1, \ldots, d_n) \in \mathbb{Z}^n \) such that at least some \( d_i \neq 0 \). For \( d \in D \), let \( i_d \) be minimal such that \( d_{i_d} \neq 0 \), and \( x_d \) be the tuple of variables \( x \) without the variable \( x_{i_d} \). Let \( \psi_d(x_d, u, y) \) be the formula

\[
(\forall x_{i_d})(\sum_i d_i x_i + u = 0 \rightarrow \varphi(x, y)).
\]

We split again in three cases depending on the form of \( \varphi \):

**Case 1:** Suppose \( \varphi \) is as in \( \text{[E.4]} \). For \( d \in D \), let \( w_d \) be the tuple of variables appearing in \( d(\psi_d)(u, y, w_d) \) the definition of \( \psi_d \). Let \( \theta_d(z) \) be the formula

\[
(\exists w_d \in \pi_{\psi_d}(S_{x_d}))(\exists u)(\forall y)(d(\psi_d)(u, y, w_d) \leftrightarrow d(\varphi)(y, z)),
\]
and \( \theta_0(z) \) be the formula \((\forall y) -d(\varphi)(y, z)\). We claim that
\[
\pi_\varphi(S_x(M)) = \theta_0(M) \cup \bigcup_{d \in D} \theta_d(M).
\]
From left-to-write let \( c = c(p, \varphi) \in \pi_\varphi(S_x(M)) \). If some (every) realization of \( p \) is acl-independent over \( M \), then \( \varphi(x, b) \notin p \) for all \( b \in S_y(M) \), hence \( \theta_0(c) \) holds. Suppose that there is a realization \( a \models p \) which is acl-dependent over \( M \). Therefore, there is \( d \in D \) and \( e \in M \) such that the formula \( \sum_i d_i x_i + e = 0 \) is in \( p \). Let \( q(x_d) \) be the restriction of \( p \) to the variables \( x_d \) and \( s = c(q, \psi_d) \in \pi_{\psi_d}(S_{x_d}(M)) \), which exists by induction. By the choice of \( q \),
\[
(\forall y)(d(\psi_d)(e, y, s) \leftrightarrow d(\varphi)(y, c)),
\]
which shows that \( \theta_d(c) \) holds.

From right-to-left, suppose first \( \theta_0(c) \) holds. Then the tensor \( n^{th} \)-power \( p \) of any non-realized definable 1-type \( q \) satisfies that \( c(p, \varphi) = c \). Suppose now that \( \theta_d(c) \) holds. Then there are \( s \in \pi_{\psi_d}(S_{x_d}(M)) \) and \( e \in M \) such that \( (E7) \) holds. Let \( q \in S_{x_d}^{\text{def}} \) be such that \( s = c(q, \psi_d) \). Let \( p(x) \) be the type determined by
\[
q(x_d) \cup \sum_i d_i x_i + e = 0.
\]
The reader may check that \( c = c(p, \varphi) \), hence \( c \in \pi_\varphi(S_x(M)) \).

Case 2: Suppose \( \varphi \) is as in \((E5)\) and without loss of generality that \( \Box \) is \(< \). Let \( \theta_0(z) \) be as in Case 1 and \( \theta_1(z) \) be the formula \((\forall y)d(\varphi)(y, z)\). Similarly as in the previous case, we have
\[
\pi_\varphi(S_x(M)) = \theta_0(M) \cup \theta_1(M) \cup \bigcup_{d \in D} \theta_d(M).
\]
The proof follows the same steps as in the previous case using the fact that if \( p(x) \in S_{x}^{\text{def}}(M) \) has some (every) acl-independent over \( M \) realization, then
\[
\varphi(x, b) \in p \text{ for all } b \in S_y(M) \iff \varphi(x, b) \in p \text{ for some } b \in S_y(M).
\]
This shows that if \( c = c(p, \varphi) \) and \( p \) is as above, then either \( \theta_0(c) \) or \( \theta_1(c) \) hold. The case where all realization of \( p \) are acl-dependent over \( M \) is handled exactly as in Case 1.

Case 3: Suppose \( \varphi \) is as in \((E6)\). Here the same formula \( \theta(z) \) used when \( n = 1 \) defines \( \pi_\varphi(S_x(M)) \). The argument is analogous. \( \Box \)

6. Classifiable types

We work over a complete \( \mathcal{L} \)-theory \( T \). Let \( M \prec N \) be an elementary extension and let \( p \in S_x(M) \) be a definable \( \mathcal{L} \)-type realized in \( N \). We let \( p(N) \) denote the set of realizations of \( p \) in \( N \), that is \( \{ a \in S_x(N) \mid tp(a/M) = p \} \).

Through Sections 6.1 to 6.2 we let \( T_p^\prime \) be an \( \mathcal{L}_p \)-theory extending \( T_p \) and \( (N, M) \) be a model of \( T_p^\prime \).

6.1. Relative isolated types. For \( A \subseteq M \), a definable \( \mathcal{L} \)-type \( p \in S_x(M) \) is \( T_p^\prime \)-isolated over \( A \) if there is an \( \mathcal{L}_p(A) \)-formula \( \varphi(x) \) such that, for every \( \mathcal{L}_p \)-elementary extension \( (N', M') \) of \((N, M)\), \( \varphi(N') = (p|M')(N') \). In particular, \( \varphi(N) = p(N) \). In this situation, we say that \( p \) is \( T_p^\prime \)-isolated by \( \varphi(x) \). Given a model \( M \) of \( T \) and a definable \( \mathcal{L} \)-type \( p \in S_x(M) \), we say \( p \) is \( T_p^\prime \)-isolated if there is an elementary extension \( N \) of \( M \) such that \( (N, M) \models T_p^\prime \) and \( p \) is \( T_p^\prime \)-isolated over \( M \).

Let \( T_p^{\prime\prime} \) be an \( \mathcal{L}_p \) theory extending \( T_p^\prime \) and \( (N, M) \) a model of \( T_p^{\prime\prime} \). If a definable \( \mathcal{L} \)-type \( p \in S_x(M) \) is \( T_p^\prime \)-isolated then it is also \( T_p^{\prime\prime} \)-isolated.
6.2. **Classifying formulas and types.** An $\mathcal{L}_p$-formula $\delta(x,y)$ is a $T'_p$-classifying formula if for every pair $(N,M) \models T'_p$ and every $c \in S_y(M)$ there is a definable $\mathcal{L}$-type $q_c \in S_x(M)$ which is $T'_p$-isolated by $\delta(x,c)$. Note that $y$ can be taken to be the empty tuple, in which case, for every pair $(N,M) \models T'_p$ there is a definable $\mathcal{L}$-type $q \in S_x(M)$ which is $T'_p$-isolated by $\delta(x)$.

For $A \subseteq M$ we say that an definable $\mathcal{L}$-type $p \in S_x(M)$ is $T'_p$-classifiable over $A$ if there is an $\mathcal{L}_p$-classifying formula $\delta(x,y)$ and $a \in S_y(A)$ such that $p$ is $T'_p$-isolated by $\delta(x,a)$. We say $p$ is $T'_p$-classifiable if it is $T'_p$-classifiable over $A$ for some $A \subseteq M$.

6.2.1. **Lemma.** Fix a tuple of variables $x$ and suppose that for every model $(N,M)$ of $T^{se}_p$ and every $a \in S_x(N)$, the $\mathcal{L}$-type $tp(a/M)$ is $T^{se}_p$-classifiable. Then there is a finite set of $T^{se}_p$-classifying formulas $\Delta$ such that for every model $(N,M)$ of $T^{se}_p$ and every $a \in S_x(N)$, $tp(a/M)$ is $T'_p$-classified by a formula $\delta(x,y) \in \Delta$.

**Proof.** Let $\{\delta_j(x,y_j) \mid j \in J\}$ be the set of all $T^{se}_p$-classifying formulas. Suppose for a contradiction the result is false. Then, for every finite subset $I \subseteq J$, there is model $(N_I,M_I)$ of $T^{se}_p$ and $a_I \in S_x(N_I)$ such that $tp(a/M_I)$ is not classified by $\delta_i(x,y_i)$ for all $i \in I$. By compactness, this shows that the set of formulas

$$\Theta(x) := \{ (\forall y_j)(\neg \delta_j(x,y_j) \mid j \in J) \cup T^{se}_p \}$$

is consistent, which contradicts the assumption. \qed

6.3. **Examples.**

6.3.1. **Realized types.** Fix a tuple of variables $x$. The simplest example of a $T_p$-classifiable formula is the formula $x = y \land P(y)$ where $y$ is a tuple of variables such that $S_x = S_y$. It is easy to see that such formulas $T_p$-classify realized types over the smaller model. This also shows that realized types over the small model are always $T_p$-classifiable.

6.3.2. **Strongly minimal theories.** Let $T$ be a strongly minimal theory. Definable 1-types over a model $M$ are then either realized or the “generic type”, that is, the type determined by containing the formulas $\{x \neq a \mid a \in M\}$. This type is $T_p$-classifiable by the formula $\delta(x)$

$$(\forall y)(P(y) \rightarrow x \neq y).$$

6.3.3. **O-minimal theories.** Let $T$ be an o-minimal $\mathcal{L}$-theory and assume without loss of generality $\mathcal{L}$ contains a symbol for the linear order $\{<\}$. Let us show that all definable 1-types over models of $T$ are $T_p$-classifiable. Recall that for a model $M$ of $T$, a 1-type is completely determined by its reduct to the language $\{<\}$. In addition, there are 4 kinds of non-realized definable 1-types in $S_x(M)$, $x$ being a single variable:
The type at $\infty$ determined by containing the set of $L(M)$-formulas $\{a < x \mid a \in M\}$.

The type at $-\infty$ determined by containing the set of $L(M)$-formulas $\{x < a \mid a \in M\}$.

For $c \in M$, the right infinitesimal of $c$ determined by containing the set of $L(M)$-formulas $\{c < x \} \cup \{x < a \mid a \in M, c < a\}$.

For $c \in M$, the left infinitesimal of $c$ determined by containing the set of $L(M)$-formulas $\{x < c\} \cup \{a < x \mid a \in M, a < c\}$.

Table 1: Non-realized definable 1-types models

O-minimality implies that the previous list contains all possible non-realized 1-types over a model. We let the reader verify that the following $L_P$-formulas provide their $T_P$-classifying formulas:

- $\delta_{+\infty}(x) : (\forall w)(P(w) \rightarrow w < x)$ $T_P$-classifies the type $p_{+\infty}$
- $\delta_{-\infty}(x) : (\forall w)(P(w) \rightarrow w < y)$ $T_P$-classifies the type $p_{-\infty}$
- $\delta_{+(x,z)} : P(z) \land z < x \land (\forall w)((P(w) \land z < w) \rightarrow x < w)$ $T_P$-classifies types $p_{c+}$
- $\delta_{-(x,z)} : P(z) \land x < z \land (\forall w)((P(w) \land w < z) \rightarrow w < x)$ $T_P$-classifies types $p_{c-}$

Table 2: Classifying formulas of non-realized definable 1-types over models

6.3.4. C-minimal expansions of valued groups. With a bit of work, the following content can be extended to general C-minimal structures. For simplicity, we will work over a C-minimal $L$-expansion of a valued group. Recall that such an expansion is C-minimal if every for every elementarily structure $N \equiv M$, every $L(N)$-definable subset $X \subseteq N$ is a finite Boolean combination of balls (closed and open). Note that by C-minimality, given an $L$-formula $\varphi(x,y)$ with $\ell(x) = 1$, there is a natural number $n_\varphi$ such that for all $a \in S_y(M)$, $\varphi(x,a)$ is defined by a Boolean combination of at most $n_\varphi$ balls.

Recall that given $a, b \in M$, We will use the notation $B(a,b)$ (resp. $B^o(a,b)$) as a short hand to denote the ball $B(a,v(a-b))$ (resp. $B^o(a,v(a-b))$). We can classify non-realized definable 1-types in the following families.
for \( a, b \in M \) with \( a \neq b \) and \( B := B(a, b) \), the
generic type of \( B \)
| \( p_B \) | contains a formula for \( B \) but for no proper sub-ball |
---|---|
for \( a, b \in M \) with \( a \neq b \) and \( B := B(a, b) \), the
generic type of \( M \setminus B \)
| \( p_{B^c} \) | contains a formula for \( M \setminus B \) but no formula defining \( M \setminus B' \)
for \( a, b \in M \) with \( a \neq b \) and \( B^o := B^o(a, b) \) or
\( B^o = M \), the generic type of \( B^o \)
| \( p_{B^o} \) | contains a formula for \( B^o \) but for no proper sub-ball |
the generic type of \( M \)
| \( p_M \) | contains no formula defining a closed ball |

Table 3: Non-realized definable 1-types over a model

\( C \)-minimality implies that the previous list contains all possible non-realized 1-types over a model. Note that the generic type of \( M \) can also be seen as the generic type of an open ball with radius \(-\infty\).

We let the reader verify that the following \( L_P \)-formulas provide \( T_P \)-classifying formulas for the 1-types above listed:

\[
\delta_B(x, z_1, z_2) : \begin{cases} 
  z_1 \neq z_2 \land P(z_1) \land P(z_2) \land x \in B(z_1, z_2) \land \\
  (\forall w)((P(w) \land w \in B(z_1, z_2)) \rightarrow B(w, x) = B(z_1, z_2))
\end{cases}
\]

| \( \delta_B(x, z_1, z_2) \) | \( p_B \) |
---|---|
\( \delta_{B^c}(x, z_1, z_2) : \begin{cases} 
  z_1 \neq z_2 \land P(z_1) \land P(z_2) \land x \in M \setminus B(z_1, z_2) \land \\
  (\forall w)((P(w) \land w \in B(z_1, z_2)) \rightarrow x \in B(z_1, w))
\end{cases} \)
| \( p_{B^c} \) |
\( \delta_{B^o}(x, z_1, z_2) : \begin{cases} 
  z_1 \neq z_2 \land P(z_1) \land P(z_2) \land x \in B^o(z_1, z_2) \land \\
  (\forall w)((P(w) \land w \in B(z_1, z_2)) \rightarrow x \notin B^o(z_1, w))
\end{cases} \)
| \( p_{B^o} \) |
\( \delta_M(x) : (\forall w)(P(w) \rightarrow v(x) < v(w)) \)
| \( p_M \) |

Table 4: Classifying formulas of non-realized definable 1-types over models

6.3.5. **Real closed valued fields.** Let \( T \) be an expansion of RCVF. We say that expansion is
\( L_{\text{div}, <} \)-minimal if for every model \( M \models T \), every definable subset \( X \subseteq M \) is a finite union of
balls and intervals. The main consequence of such a property is that 1-types are determined
by formulas defining intervals and balls and are thus easy to classify. Given a model \( M \) of
\( T \), a subset \( A \subseteq M \) and \( a \in U \), we write \( A < a \) as a shortcut for “\( x < a \) for all \( x \in A \)”. By \( L_{\text{div}, <} \)-minimality, every non-realized definable 1-type over a model \( M \) of \( T \) is either \( p_{+\infty} \),
\( p_{-\infty} \), \( p_{c+} \), \( p_{c-} \) (as defined in 6.3.3) or one of the following list:
for a ball $B$, open or closed & type & is determined by \\
| the right generic type of $B$ & $p_{B^+}$ & $x \in B$ and $B \cap M < x$ \\
| the left generic type of $B$ & $p_{B^-}$ & $x \in B$ and $x < B \cap M$ \\
| the right generic type of $M \setminus B$ & $p_{B^+_\infty}$ & $B < x$ and $x < a$ for every $a \in M$ such that $B < a$ \\
| the left generic type of $M \setminus B$ & $p_{B^-\infty}$ & $x < B$ and $a < x$ for every $a \in M$ such that $a < B$

Table 5: Remaining non-realized definable 1-types over a model

We let the reader verify that the following $\mathcal{L}_P$-formulas provide $T_P$-classifying formulas for the above listed generic types:

| $T_P$-classifying formula | types classified |
|---------------------------|------------------|
| $\delta_{B^+}(x, z_1, z_2)$ : $\delta_{B^-}(x, z_1, z_2)$ : $\delta_{B^+_\infty}(x, z_1, z_2)$ : $\delta_{B^-\infty}(x, z_1, z_2)$ : |
| $\{z_1 \not= z_2 \land P(z_1) \land P(z_2) \land x \in B(z_1, z_2) \land$ & $p_{B^+}$ \\
| $(\forall w)((P(w) \land w \in B(z_1, z_2)) \rightarrow w < x)$ & $p_{B^-}$ \\
| $\{z_1 \not= z_2 \land P(z_1) \land P(z_2) \land x \in B(z_1, z_2) \land$ & $p_{B^+_\infty}$ \\
| $(\forall w)((P(w) \land w \in B(z_1, z_2)) \rightarrow x < w)$ & $p_{B^-\infty}$ \\
| $\{z_1 \not= z_2 \land P(z_1) \land P(z_2) \land x \in B(z_1, z_2) \land$ & $p_{B^+_\infty}$ \\
| $(\forall w)((P(w) \land w \in B(z_1, z_2)) \rightarrow w < x)$ & $p_{B^-\infty}$ \\
| $\{z_1 \not= z_2 \land P(z_1) \land P(z_2) \land x \in B(z_1, z_2) \land$ & $p_{B^+_\infty}$ \\
| $(\forall w)((P(w) \land w \in B(z_1, z_2)) \rightarrow x < w)$ & $p_{B^-\infty}$ \\
| $\{z_1 \not= z_2 \land P(z_1) \land P(z_2) \land x \in B(z_1, z_2) \land$ & $p_{B^+_\infty}$ \\
| $(\forall w)((P(w) \land w \in B(z_1, z_2)) \rightarrow w < x)$ & $p_{B^-\infty}$ \\
| $\{z_1 \not= z_2 \land P(z_1) \land P(z_2) \land x \in B(z_1, z_2) \land$ & $p_{B^+_\infty}$ \\
| $(\forall w)((P(w) \land w \in B(z_1, z_2)) \rightarrow x < w)$ & $p_{B^-\infty}$ \\
| $\{z_1 \not= z_2 \land P(z_1) \land P(z_2) \land x \in B(z_1, z_2) \land$ & $p_{B^+_\infty}$ \\
| $(\forall w)((P(w) \land w \in B(z_1, z_2)) \rightarrow w < x)$ & $p_{B^-\infty}$ |

Table 6: Classifying formulas of non-realized definable 1-types over models

6.3.6. Remark. Let $T$ be as in either Section 6.3.3, 6.3.4 or 6.3.5 and let $(N, M)$ be a model of $T_P^{\omega}$. In each case, the minimality condition imposed implies that every element $a \in N \setminus M$ lies in the projection of a set defined by a classifying formula. For example, if $T$ is o-minimal, $a$ must satisfy one of the following formulas: $\forall x \delta_{+\infty}(x), \delta_{-\infty}(x), (\exists y)\delta_{+}(x, y)$ or $(\exists y)\delta_{-}(x, y)$.

6.4. Non-examples. In this section we show examples of theories which, despite having uniform definability of types, do not have the property that all their definable 1-types are $T_P^{\omega}$-classifiable. The formal content is giving in the following proposition which should be contrasted with Remark 6.3.6

6.4.1. Proposition. Let $T$ be one of the following theories

(1) Presburger arithmetic
(2) $p\text{CF}_d$
Then, there is a model \((N,M)\) of \(T^\text{se}_P\) and \(a \in N\) with \(\ell(a) = 1\) such that \(tp(a/M)\) is not \(T^\text{se}_P\)-classifiable.

**Proof.** Suppose not. Then, by Lemma 6.2.1 there is a finite subset \(\Delta\) of \(T^\text{se}_P\)-formulas classifying formulas such that for every model \((N,M)\) of \(T^\text{se}_P\) and every \(a \in N\), \(tp(a/M)\) is \(T^\text{se}_P\)-classified by a formula in \(\Delta\). In particular, this implies that

\((*)\) for every model \((N,M)\) of \(T^\text{se}_P\) there are at most \(|M|\) definable 1-types realized in \(N\).

We provide a contradiction for each theory:

1. **Suppose \(T\) is Presburger arithmetic.** Let \(N\) be an \(\aleph_1\)-saturated elementary extension of \(\mathbb{Z}\). By Theorem [A.1.1] \((N,\mathbb{Z})\) is a model of \(T^\text{se}_P\). However, there are \(2^{|\mathbb{N}|}\) definable 1-types over \(\mathbb{Z}\) realized in \(N\). Indeed, for every subset \(A\) of the prime numbers, there is a 1-type \(q_A(x) \in S_x(\mathbb{Z})\) containing the formulas

\[
\{m < x \mid m \in \mathbb{Z}\} \cup \{x \equiv_p 0 \mid p \in A\} \cup \{\neg(x \equiv_p 0) \mid p \notin A\},
\]

which contradicts \((*)\).

2. **Suppose \(T\) is \(p\text{CF}\) and let \(M\) be a countable model of \(T\) with \(\Gamma_M = \mathbb{Z}\).** For each subset \(A\) of \(\mathbb{N}\), let \(q_A \in S_x(M)\) be the type containing the set of formulas

\[
\{m < v(x) \mid m \in \mathbb{Z}\} \cup \{P_n(x) \mid n \in A\} \cup \{\neg P_n(x) \mid n \notin A\}.
\]

If consistent, the type \(q_A\) is definable by a short argument from Theorem [A.3.3]. Let \(\{q_\alpha \mid \alpha < 2^{|\mathbb{N}|}\}\) be an enumeration of consistent types \(q_A\) where \(A\) ranges over all subsets of prime numbers (it is not difficult to see there are uncountably many). By Lemma [5.1.3] let \((N,M)\) be a special pair. By definition \((N,M)\) is a model of \(T^\text{se}_P\). However, \(N\) realizes \(2^{|\mathbb{N}|}\) definable 1-types over \(M\), which contradicts \((*)\).

3. **Suppose \(T\) is CODF and \(M\) is a countable model of CODF.** The reduct of \(M\) to \(\mathcal{L}_{\text{or}}\) is a real closed field. Consider

\[
p^i = \begin{cases} p_\infty(x) & \text{if } i = 0 \\ p_{-\infty}(x) & \text{if } i = 1, \end{cases}
\]

where \(p_\infty(x)\) and \(p_{-\infty}(x)\) denote elements of \(S^\mathcal{L}_{\text{or}}(M)\), that is types over \(M\) as a model of RCF (as defined in Section [6.3.3]). For an \(n\)-sequence \(s\) of zeros and ones, we let \(p_s(x_0,\ldots,x_{n-1})\) by defined as the following tensor product

\[
p_s = \bigotimes_{i=0}^{n-1} p^{s_i}.
\]

Let \(s \in 2^\omega\) be a countable sequence. We denote by \(s_{<n}\) the initial segment \((s_0,\ldots,s_n)\) of \(s\). Let \(p_s(x) \in S_x(M)\) be such that for every \(n < \omega\)

\[
p_s(x) \models p_{s_n}(x,\delta(x),\delta^2(x),\ldots,\delta^n(x)).
\]

By [4] Lemma 2.10, the type \(p_s(x)\) is definable. Let \(\{p_\alpha \mid \alpha < 2^\omega\}\) be an enumeration of the types \(p_s\) for \(s \in 2^\omega\). Again, by Lemma [5.1.3] let \((N,M)\) be a special pair. By definition \((N,M)\) is a model of \(T^\text{se}_P\). But \(N\) realizes \(2^{|\mathbb{N}|}\) definable 1-types over \(M\), which contradicts \((*)\). \(\square\)
6.4.2. Comment. The previous examples suggest a possible (hopefully interesting) model-theoretic dividing line: say a complete $\mathcal{L}$-theory $T$ is \textit{definably bounded} if for every model $M$ of $T$, the space of definable 1-types over $M$ has cardinality $|M|$. Examples include all theories of finite structures and the theories described in Sections 6.3.3, 6.3.4 and 6.3.5. On the other hand, the proof of the above proposition shows that Presburger arithmetic, $p\text{CF}$ and CODF are not definably bounded. While both algebraically and differentially closed fields are stable and definably bounded, the theory of $(\mathbb{Z}, +)$ provides an example of a stable non definably bounded theory. It is not difficult to see, using Lemma 6.2.1 that if all 1-types over models of $T$ are classifiable, then $T$ is definably bounded. We do not know if the converse holds.

7. Tame pairs

Let $T$ be one of the following theories:

(1) an o-minimal theory having quantifier elimination;
(2) a completion of ACVF;
(3) RCVF.

In this section we provide axiomatizations for certain completions of $T^{\text{se}}_P$, denoted $T_{\text{tame}}(\Delta)$, which are controlled by a given set of classifying formulas $\Delta$. One could also try to frame this section in an abstract setting, for example, by letting $T$ be a complete VC-minimal theory satisfying further model-theoretic assumptions (see [10] for a definition of VC-minimality). We will leave this for further considerations. However, let us list the main properties of $T$ shared by the theories in the above list that will be crucially used in this section:

(P1) $T$ has quantifier elimination
(P2) $T^{\text{se}}_P$ exists
(P3) $T$ has uniform definability of types
(P4) all 1-definable types over models of $T$ are $T^{\text{se}}_P$-classifiable.

If $T$ is as in (1)-(3) above (P1) is clearly satisfied, Theorem 4.2.4 implies (P2), Corollary 4.3.3 yields (P3) and the results in Sections 6.3.3, 6.3.4 and 6.3.5 ensure (P4).

For the rest of the section we let $\Phi$ denote the set of $\mathcal{L}_P$-classifying formulas of non-realized definable 1-types, and $\Delta$ be a non-empty subset of $\Phi$.

7.1. The theory $T_{\text{tame}}(\Delta)$. The theory $T_{\text{tame}}(\Delta)$ is defined as the $\mathcal{L}_P$-theory containing the following axioms with respect to a uniform scheme of definition $d = (d, c)$ for $T$:

(T1) for every formula $\varphi(x, y)$ with both $x, y$ tuples of variables, the axiom
$$(\forall x)(\exists z_\varphi)[P(z_\varphi) \land (\forall y)(P(y) \rightarrow (\varphi(x, y) \leftrightarrow d(\varphi)(y, z_\varphi))];$$

(T2) for $\ell(x) = 1$,
$$(\forall x)[\neg P(x) \rightarrow \bigvee_{\delta \in \Delta}(\exists z)(P(z) \land \delta(x, z))],$$

where $\bigvee$ stands for exclusive disjunction.

(T3) for $z$ a tuple of variables and $\delta(x, z) \in \Delta$, for every $\mathcal{L}$-formula $\varphi(x, y)$ the axiom
$$(\forall y)(\exists z)[(P(z) \land \theta^\varphi_\delta(y, z)) \rightarrow (\exists x)(\varphi(x, y) \land \delta(x, z))],$$

where $\theta^\varphi_\delta(y, z)$ is an $\mathcal{L}_P$-formula depending on both $\delta(x, z)$ and $\varphi(x, y)$ (that will be defined for each $T$ respectively in Sections 7.1.1, 7.1.2 and 7.1.3).
Note that axiom (T1) implies that any model of $T_{\text{tame}}(\Delta)$ is a model of $T^\infty_P$. Axioms from (T3) will guarantee a certain extension property in a suitable back-and-forth system, as we will see in Theorem 7.2.2.

Let us now specify specify the form of $\theta^\varphi_\delta(y, z)$ for each $T$ and each possible $\delta \in \Phi$.

7.1.1. $\theta^\varphi_\delta$ for $T$ o-minimal. Let $T$ be a complete o-minimal $\mathcal{L}$-theory having quantifier elimination. Given an $\mathcal{L}$-formula $\varphi(x, y)$, the formula $\theta^\varphi_\delta(y, z)$ is defined by

- if $\delta := \delta_+^\infty(x)$, the formula $\theta^\varphi_\delta(y)$ is
  \[(\exists x)(\varphi(x, y) \land w < x))\];
- if $\delta := \delta_-^\infty(x)$, the formula $\theta^\varphi_\delta(y)$ is
  \[(\exists x)(\varphi(x, y) \land x < w))\];
- if $\delta := \delta_+(x, z)$, the formula $\theta^\varphi_\delta(y, z)$ is
  \[(\forall w)((P(w) \land z < w) \rightarrow (\exists x)\varphi(x, z) \land z < x < w))];
- if $\delta := \delta_-(x, z)$, the formula $\theta^\varphi_\delta(y, z)$ is
  \[(\forall w)((P(w) \land w < z) \rightarrow (\exists x)\varphi(x, z) \land w < x < z))].

7.1.2. $\theta^\varphi_\delta$ for a completion of ACVF. Let $T$ be a completion of ACVF. Given an $\mathcal{L}$-formula $\varphi(x, y)$, the formula $\theta^\varphi_\delta(y, z)$ is defined by

- if $\delta := \delta_B(x, z_1, z_2)$ and $n_\varphi$ is defined as in Section [6.3.3], the formula $\theta^\varphi_\delta(y, z)$ is
  \[(\forall w_1) \cdots (\forall w_{2n_\varphi})(\exists x)(\xi_1(w_1, \ldots, w_{2n_\varphi}, z_1, z_2) \rightarrow \xi_2(x, y, w_1, \ldots, w_{n_\varphi}, z_1, z_2)),\]
  with
  \[
  \xi_1 := \bigwedge_{i=1}^{2n_\varphi} P(w_i) \land \bigwedge_{i=1}^{n_\varphi} B^\varphi(w_i, w_{n_\varphi+i}) \subseteq B(z_1, z_2)
  \]
  \[
  \xi_2 := \varphi(x, y) \land x \in B(z_1, z_2) \setminus \bigcup_{i=1}^{n_\varphi} B^\varphi(w_i, w_{n_\varphi+i}).
  \]
- if $\delta := \delta_B^c(x, z_1, z_2)$, the formula $\theta^\varphi_\delta(y, z)$ is
  \[(\forall w_1)(\forall w_2)(\exists x)(\xi_1(w_1, w_2, z_1, z_2) \rightarrow \xi_2(x, y, w_1, w_2, z_1, z_2)),\]
  with
  \[
  \xi_1 := P(w_1) \land \bigwedge_{i=1}^{n_\varphi} B^\varphi(z_1, z_2) \subseteq B^\varphi(w_1, w_2)
  \]
  \[
  \xi_2 := \varphi(x, y) \land x \in B^\varphi(w_1, w_2) \setminus B^\varphi(z_1, z_2).
  \]
- if $\delta := \delta_B^o(x, z_1, z_2)$ and $n_\varphi$ is defined as in Section [6.3.4], the formula $\theta^\varphi_\delta(y, z)$ is
  \[(\forall w_1) \cdots (\forall w_{2n_\varphi})(\exists x)(\xi_1(w_1, \ldots, w_{2n_\varphi}, z_1, z_2) \rightarrow \xi_2(x, y, w_1, \ldots, w_{n_\varphi}, z_1, z_2)),\]
  with
  \[
  \xi_1 := \bigwedge_{i=1}^{2n_\varphi} P(w_i) \land \bigwedge_{i=1}^{n_\varphi} B(w_i, w_{n_\varphi+i}) \subseteq B^\varphi(z_1, z_2)
  \]
  \[
  \xi_2 := \varphi(x, y) \land x \in B^\varphi(z_1, z_2) \setminus \bigcup_{i=1}^{n_\varphi} B(w_i, w_{n_\varphi+i}).
  \]
- if $\delta := \delta_M(x)$, the formula $\theta^\varphi_\delta(y)$ is
  \[(\forall w)(\exists x)(P(w) \rightarrow (\varphi(x, y) \land v(x) < v(w)))\]
7.1.3. $\theta_\delta^\psi$ for RCVF. Let $T$ be RCVF. Abusing notation, given a definable subset $B$ of a model $M$ of $T$ and an element $b \in M$, we write $B < b$ as an abbreviation for the formula $(\forall u)(u \in B \to u < b)$. Given an $L$-formula $\varphi(x, y)$, the formula $\theta_\delta^\psi(y, z)$ is defined by

- if $\delta \in \Delta$ is either $\delta_{-\infty}(x)$, $\delta_{+\infty}(x)$, $\delta_{+}(x, z)$ or $\delta_{-}(x, z)$, the formula $\theta_\delta^\psi$ is defined as in Section 7.1.1
- if $\delta := \delta_{B_+}(x, z_1, z_2) \in \Delta$, the formula $\theta_\delta^\psi(y, z)$ is

$$\forall u \exists x [(P(u) \land w \in B(z_1, z_2)) \to (\varphi(x, y) \land x \in B(z_1, w_2) \land w < x)];$$

- if $\delta := \delta_{B_-}(x, z_1, z_2) \in \Delta$, the formula $\theta_\delta^\psi(y, z)$ is

$$\forall u \exists x [(P(u) \land w \in B(z_1, z_2)) \to (\varphi(x, y) \land x \in B(z_1, w_2) \land w < x)];$$

- $\delta := \delta_{B_+}(x, z_1, z_2) \in \Delta$, the formula $\theta_\delta^\psi(y, z)$ is

$$\forall u \exists x [(P(u) \land B(z_1, z_2) < w) \to (\varphi(x, y) \land B(z_1, z_2) < x < w)];$$

- $\delta := \delta_{B_-}(x, z_1, z_2) \in \Delta$, the formula $\theta_\delta^\psi(y, z)$ is

$$\forall u \exists x [(P(u) \land w < B(z_1, z_2)) \to (\varphi(x, y) \land w < x < B(z_1, z_2))].$$

7.2. Completeness results. In this section we show that, if consistent, the theory $T_{tame}(\Delta)$ is a completion of $T_P^\psi$ (Theorem 7.2.2). Our proof is inspired by Pillay’s [17, Theorem 2.3], which is a particular case of our result.

Let $T_{tame}$ denote the theory $T_{tame}(\Phi)$. In most of the examples in Section 3.2 $T_{tame}$ will always be consistent. However, it is worth observing that not every choice of $\Delta$ will define a consistent theory. In Section 7.3 we will provide examples of subsets $\Delta$ which induce consistent theories $T_{tame}(\Delta)$. We will also compare how some of these theories corresponds to previous known completions of $T_P$.

The following lemma will be needed in the sequel:

7.2.1. Lemma. Let $M, N$ be two $L$-structures and $f : A \subseteq M \to N$ be a partial isomorphism. Then there is a partial isomorphism $g : acl(A) \to N$ extending $f$.

Proof. By quantifier elimination we may suppose that $f$ is elementary. Suppose $b \in acl(A)$ and let $\varphi(x, a)$ be an $acl(A)$-formula such that $\varphi(M) = \{b_1, \ldots, b_n\}$ with $b = b_1$ and $n$ minimal with such property. Since $f$ is elementary, let $b_1', \ldots, b_n' \in N$ be such that $\varphi(N) = \{b_1', \ldots, b_n'\}$. We claim that the function $g : A \cup \{b_1', \ldots, b_n'\} \to N$ defined by

$$g(x) := \begin{cases} f(a) & \text{if } x \in A \\ b_i' & \text{if } a = b_i \end{cases}$$

is a partial isomorphism. Let $\psi(x_1, \ldots, x_n, y, z)$ be a quantifier free $L$-formula and suppose that $\psi(b_1, \ldots, b_n, a, c)$ holds in $M$, where $a, c$ are tuples from $A$. The minimality of $n$ implies that for every permutation $\mu \in S(n)$, $\psi(b_{\mu(1)}, \ldots, b_{\mu(n)}, a, c)$ also holds in $M$. Then we have
that

\[ M \models (\exists x_1) \cdots (\exists x_n)(\bigwedge_{i \neq j} x_i \neq x_j \land \varphi(x_i, a) \land \bigwedge_{\mu \in S(n)} \psi(x_{\mu(1)}, \ldots, x_{\mu(n)}, a, c)) \]

\[ \iff N \models (\exists x_1) \cdots (\exists x_n)(\bigwedge_{i \neq j} x_i \neq x_j \land \varphi(x_i, a) \land \bigwedge_{\mu \in S(n)} \psi(x_{\mu(1)}, \ldots, x_{\mu(n)}, f(a), f(c))) \]

\[ \iff N \models \bigwedge_{\mu \in S(n)} \psi(b_{\mu(1)}, \ldots, b_{\mu(n)}, f(a), f(c)) \]

\[ \iff N \models \psi(b_1, \ldots, b_n, g(a), g(c)) \]

which shows that \( g \) is a partial isomorphism. \( \square \)

7.2.2. Theorem. Let \( T \) be any of (1)-(3) as in the beginning of the section. Then, if consistent, the theory \( T_{\text{tame}}(\Delta) \) is complete. Furthermore, for every \( \mathcal{L}_P \)-formula \( \psi(x) \) there is an \( \mathcal{L} \)-formula \( \varphi(x) \) such that

\[ T_{\text{tame}}(\Delta) \models (\forall x)(P(x) \rightarrow (\psi(x) \leftrightarrow \varphi(x))). \]

Proof. We will show that \( T_{\text{tame}}(\Delta) \) is complete by a back-and-forth argument. Let \( (N, M) \) and \( (N', M') \) be two \( |\mathcal{L}|^+ \)-saturated models of \( T_{\text{tame}}(\Delta) \). We will use acl to denote the algebraic closure in \( \mathcal{L} \). Suppose that definable \( \mathcal{L} \)-types are defined via a scheme of definition \( (d, c) \).

Consider the following set \( \Sigma \) of partial maps between \( (N, M) \) and \( (N', M') \) defined by the following conditions: for some \( n \in \mathbb{N} \) there are \( n \)-tuples \( a \) in \( N \), \( a' \) in \( N' \), and subsets \( C \subseteq M \), \( C' \subseteq M' \) such that

(i) the tuple \( a \) (resp. \( a' \)) is algebraically independent over \( M \) in \( \mathcal{L} \) (resp. over \( M' \) in \( \mathcal{L} \));
(ii) both \( C \) and \( C' \) have cardinality at most \( |L| \);
(iii) \( \sigma \) takes \( a \) to \( a' \) and \( C \) onto \( C' \);
(iv) \( \text{dom}(\sigma) = \text{acl}(aC) \) and \( \text{range}(\sigma) = \text{acl}(a'C') \);
(v) \( \sigma \) is a partial \( \mathcal{L} \)-isomorphism, and \( (N, M) \models P(x) \) if and only if \( (N', M') \models P(\sigma(x)) \)

for all \( x \in \text{dom}(\sigma) \).

Note that every \( \sigma \in \Sigma \) is an \( \mathcal{L}_P \)-partial isomorphism. Moreover, by the quantifier elimination assumption on \( T \) (property (P1)), every \( \mathcal{L} \)-partial isomorphism is an \( \mathcal{L} \)-partial elementary map. We will show that \( \Sigma \) is an back-and-forth system. Observe in addition that since \( \Sigma \) contains every \( \mathcal{L} \)-elementary map between \( M \) and \( M' \), the “furthermore” part of the statement follows directly by compactness.

It is easy to see that \( \Sigma \) is non-empty. Let \( \sigma \in \Sigma \), \( a, a' \) and \( C, C' \) be as in conditions (i) – (v). As the situation is symmetric, it suffices to show show the “forth” condition, that is, for \( b \in N \) we need to find some \( \rho \in \Sigma \) extending \( \sigma \) and such that \( b \in \text{dom}(\rho) \). We split in cases:

Case 1: Suppose \( b \in \text{acl}(aC) \). Then \( b \in \text{dom}(\sigma) \) by condition (iv).

Case 2: Suppose \( b \in M \setminus \text{acl}(aC) \). Consider the set of formulas

\[ q(x) := \{ \varphi(x, a', c') \mid N \models \varphi(x, a, c) \in \text{tp}_{\mathcal{L}}(b/aC) \} \cup \{ P(x) \}. \]

Let us show that \( q(x) \) is consistent. Since \( q(x) \) is closed under conjunction, it suffices to show that \( \varphi(x, a', c') \land P(x) \) is consistent for \( \varphi \in \text{tp}_{\mathcal{L}}(b/aC) \). For \( \varphi \) such a formula, we have \( N \models (\exists x)\varphi(x, a, c) \) and therefore \( N' \models (\exists x)\varphi(x, a', c') \). Since \( M' \prec N' \), we also have that \( M' \models (\exists x)\varphi(x, a', c') \) which shows that \( (N', M') \models (\exists x)(\varphi(x, a', b') \land P(x)) \). By saturation,
let $b'$ realize $q(x)$. Then the map $\mu := \sigma \cup \{(b, b')\}$ is a partial $\mathcal{L}_P$-isomorphism with domain $abC$ extending $\sigma$. By Lemma 7.2.1, we can extend $\mu$ to $acl(abC)$, which completes this case.

Case 3: Suppose $b \in acl(aM) \setminus M$. Let $c$ be a tuple from $M$ such that $b \in acl(a_c)$. By iterating Case 2, we may suppose that $c$ is a tuple from $C$ and therefore by property (iv) that $b \in dom(\sigma)$.

Case 4: Suppose $b \notin acl(aM)$ so that $ab$ is algebraically independent over $M$. By (T2), the type $tp_L(b/M)$ is definable and $\mathcal{L}_P$-classified by a formula $\delta \in \Delta$. We proceed by cases, depending now both on $T$ and the form of $\delta$.

Case 4.1 Suppose $T$ is o-minimal. Let us show the case where $\delta$ corresponds to $\delta_+(x, y)$. The proof of the remaining cases, being very similar, is left to the reader. Let $c_0 \in M$ be such that $\delta_1(b, c_0)$ holds. Therefore, $tp_L(b/M)$ corresponds to the type $p_{c_0} \in S_x(M)$. By the previous cases, we may suppose $c_0$ is in $C$. Consider the set of $\mathcal{L}_P$-formulas $\Theta(x)$

$$\Theta(x) := \{ \varphi(x, e) \mid \varphi(x, e) \in tp_L(b/aC) \} \cup \{ \delta_+(x, c_0) \}. $$

Let us show $\Theta$ is consistent. As $tp_L(b/aC)$ is closed under conjunction, it suffices to show that for any $\varphi(x, e) \in tp_L(b/aC)$

$$(N, M) \models (\exists x)(\varphi(x, e')) \land \delta_+(x, c_0') \land (N, M) \models (\exists x)(\varphi(x, e) \land c_0' < x < e').$$

Suppose it is false and let $c_1 \in M'$ be such that

$$(N', M') \models c_0' < c_1 \land (\exists x)(\varphi(x, e') \land c_0' < x < c_1').$$

By the previous cases, we may suppose $c_1 \in C'$ and let $c_1 := \sigma^{-1}(c_1')$. Therefore, since $c$ is $\mathcal{L}$-elementary, we have

$$(N, M) \models c_0 < c_1 \land (\exists x)(\varphi(x, e) \land c_0 < x < c_1),$$

which contradicts that $\varphi(b, e)$ and $c_0 < b < c_1$ hold. This shows the consistency of $\Theta(x)$. By saturation, let $b' \in N'$ be any element realizing $\Theta(x)$. Then the map $\mu := \sigma \cup \{(b, b')\}$ is a partial $\mathcal{L}_P$-isomorphism with domain $abC$ extending $\sigma$. We conclude once more by Lemma 7.2.1.

Case 4.2 Suppose $T$ is a completion of ACVF. Let us show the case where $\delta$ corresponds to $\delta_B(x, z_1, z_2)$ and leave the remaining cases to the reader. Let $c_1, c_2 \in M$ be such that $\delta_B(b, c_1, c_2)$ holds. Therefore, $tp_L(b/M)$ corresponds to the generic type of a closed ball $p_B \in S_x(M)$ for $B = B(c_1, c_2)$. By Case 1, we may suppose $c_1$ and $c_2$ are in $C$. Consider the set of $\mathcal{L}_P$-formulas $\Theta(x)$

$$\Theta(x) := \{ \varphi(x, e) \mid \varphi(x, e) \in tp_L(b/aC) \} \cup \{ \delta_B(x, c_1, c_2) \}. $$

As in the previous case, the consistency of $\Theta$ follows by showing

$$(N', M') \models (\exists x)(\varphi(x, e') \land \delta_B(x, c_1', c_2'),$$

for every $\varphi(x, e) \in tp_L(b/aC)$, where $e' = \sigma(e)$. By (T3), it suffices to show that $(N', M') \models \theta^*_\delta(e', c_1', c_2')$ which corresponds in this case to

$$(N', M') \models (\forall w_1) \cdots (\forall w_{2n_e})(\exists x)(\xi_1(w_1, \ldots, w_{2n_e}, c_1', c_2') \rightarrow \xi_2(x, e', w_1, \ldots, w_{n_e}, c_1', c_2')).$$
with
\[ \xi_1 := \bigwedge_{i=1}^{2n_\varphi} P(w_i) \land \bigwedge_{i=1}^{n_\varphi} B^\varphi(w_i, w_{n_\varphi+i}) \subseteq B(c'_1, c'_2) \]
\[ \xi_2 := \varphi(x, e') \land x \in B(c'_1, c'_2) \setminus \bigcup_{i=1}^{n_\varphi} B^\varphi(w_i, w_{n_\varphi+i}). \]

Suppose for a contradiction \((N', M') \models -\theta^\varphi_\delta(e', c'_1, c'_2)\). Then there are \(e'_1, \ldots, e'_{2n_\varphi} \in M'\) such that the open ball \(B^\varphi(e'_i, e'_{n_\varphi+i})\) is contained in \(B(c'_1, c'_2)\) for each \(1 \leq i \leq n_\varphi\), but no \(x \in N'\) satisfies
\[ \varphi(x, e') \land x \in B(c'_1, c'_2) \setminus \bigcup_{i=1}^{n_\varphi} B^\varphi(e'_i, e'_{n_\varphi+i}). \]

Now, by the previous cases, we may extend \(\sigma\) such that \(e'_1, \ldots, e'_{2n_\varphi} \in C'\). But then, this implies
\[(N, M) \models (\forall x)(\xi_1(e_1, \ldots, e_{2n_\varphi}, c_1, c_2) \land -\xi_2(x, e_1, \ldots, e_{n_\varphi}, c_1, c_2)).\]

Since \(b \models p_B\) and \(\varphi(b, e)\) holds by assumption, \(b\) satisfies the above formula, yielding a contradiction. This shows \(\Theta(x)\) is consistent and the proof is finished as in Case 4.1.

**Case 4.3** Suppose \(T\) is RCVF. We only show the case where \(\delta\) corresponds to \(\delta_{B^+}(x, z_1, z_2)\). Let \(c_1, c_2 \in M\) be such that \(\delta_{B^+}(b, c_1, c_2)\) holds. Therefore, \(tp_L(b/M)\) corresponds to the right generic type \(p_{B^+} \in S_x(M)\) where \(B\) is the closed ball \(B(c_1, c_2)\). By the previous cases, we may suppose \(c_1\) and \(c_2\) are in \(C\). We follow the same strategy and show that the following set of \(L_p\)-formulas \(\Theta(x)\) is consistent:
\[ \Theta(x) := \{ \varphi(x, e') \mid \varphi(x, e) \in tp_L(b/aC) \} \cup \{ \delta_B(x, c'_1, c'_2) \}. \]

As in the previous cases by (T3), it suffices to show that \((N', M') \models \theta^\varphi_\delta(e', c'_1, c'_2)\), which in this case corresponds to
\[(N', M') \models (\forall w)(\exists x)((P(w) \land w \in B(c'_1, c'_2)) \rightarrow (\varphi(x, e') \land x \in B(c'_1, c'_2) \land w < x)).\]

Assuming for a contradiction this is not the case, let \(e' \in M'\) be such that
\[(N', M') \models (\forall x)[(e' \in B(c'_1, c'_2) \land -\varphi(x, e') \land x \in B(c'_1, c'_2) \land e' < x)]].\]

By the previous cases, we may suppose \(e' \in C\). Then we obtain that for \(c = \sigma^{-1}(e')\)
\[(N, M) \models (\forall x)[(c \in B(c_1, c_2) \land -\varphi(x, e) \land x \in B(c_1, c_2) \land c < x))],\]

which contradicts that \(b \models p_{B^+}\) and \(\varphi(b, e)\) holds by assumption. As before, this shows \(\Theta(x)\) is consistent and the result is obtained as in Case 4.1.

### 7.3. Consistency of \(T_{tame}(\Delta)\)

In this section, we prove the consistency of \(T_{tame}(\Delta)\) for some interesting examples \(T\) and \(\Delta\):

- \(T\) is a o-minimal expansion of the theory DOAG.
- \(T\) is a completion of ACVF.
- \(T\) is RCVF.

We will show that special pairs (resp. \(b\)-special pairs and \(o\)-special pairs) as defined in Section 5.1 are models of some of the theories \(T_{tame}(\Delta)\).
7.3.1. *O-minimal expansions of divisible ordered abelian groups.* Let us assume $T$ is an o-minimal expansion of the theory of divisible ordered abelian groups.

7.3.2. **Proposition.** Let $(N, M)$ be a special pair. Then $(N, M) \models T_{tame}$.

**Proof.** We proceed to verify (T1) to (T3) in the axiomatization of $T_{tame}(\Delta)$ where $\Delta$ consists of all definable 1-types. Scheme axioms (T1) and (T2) follow easily by the fact that the pair is stably embedded and $T$ has uniform definability of types. It remains to check axiom scheme (T3).

Fix a formula $\varphi(x, y)$ and consider the axiom,

$$((\forall y)((\forall w)(P(w) \rightarrow (\exists x)(\varphi(x, y) \land w < x)) \rightarrow (\exists x)(\varphi(x, y) \land \delta_{+\infty}(x)))$$

Let $a \in S_y(N)$ and $\varphi(x, y)$ be an $\mathcal{L}$-formula, where $x$ is a variable for the home sort. We need to show, if $\varphi(x, a)$ is consistent with $b < x$ for every $b \in M$, there is some $c \in N$ such that $\varphi(c, a)$ and $c > b$ for every $b \in M$. By o-minimality, $\varphi(N, a)$ is a finite union of intervals and points in $N$, let $d$ be the least upper bound of $\varphi(N, a)$ in $N$, if $d$ is $\infty$, the axiom will be satisfied. So it suffices to consider the case when $d \in N$. If $d \in \varphi(N, a)$, then $d$ will be the candidate for the above and we are done in this case. So we may assume that $d \notin \varphi(N, a)$ and hence, we may assume that $\varphi(N, a)$ is of the form $(d', d)$ for some $d', d \in N$. By the definition of special pair, let $e \in N$ be such that $e \models p_0 - |M d'\rangle$. Then $d + e \in (d', d)$ and $d + e > M$. So we verified the above axiom. The axiom corresponding to $\delta_{-\infty}(x)$ can be verified similarly.

Now, let’s consider the following axiom,

$$((\forall y)(\forall z)((P(z) \land (\forall w)((P(w) \land z < w) \rightarrow (\exists x)(\varphi(x, z) \land z < x < w))) \rightarrow (\exists x)(\varphi(x, y) \land \delta_{+}(x, z)))$$

So we need to show the following: “Let $a \in N$ and $b \in M$ be given, and for $\varphi(x, y)$ a $\mathcal{L}$-formula, where $x$ is a variable for the home sort. If $\varphi(x, a)$ is consistent with $b < x < c$ for any $c \in M$, then there is a $d \in N$ such that $\varphi(d, a)$ and $b < d < c$ for every $c \in M$, $c > b$.” By o-minimality, we know that $\varphi(x, a)$ is a finite union of intervals and points in $N$. If $\varphi(N, a)$ is consistent with $b < x < c$ for any $c \in M$, then we may assume that $\varphi(N, a) > b$. Let $d$ be the infimum of $\varphi(N, a)$ in $N$. If $d \in \varphi(N, a)$, we are done. Hence we may assume that $\varphi(N, a)$ is of the form $(d, d')$ for some $d, d' \in N$. By the definition of special pair, let $e \in N$ be such that $e \models p_0 - |M d'\rangle$. Hence $d + e \in \varphi(N, a)$ and $d + e \models p_b + |M$. The axiom corresponding to $\delta_{-}(x, z)$ can be verified similarly. Hence we have verified that $(N, M) \models T_{tame}$. \hfill \Box

7.3.3. **Proposition.** Let $\Delta$ contain the classifying formulas $\delta_{+}(x, z)$ and $\delta_{-}(x, z)$. Let $M$ be a model of $T$ and $(N, M)$ be an $\omega$-special pair. Then $(N, M) \models T_{tame}(\Delta)$.

**Proof.** The verification of axioms (T1) and (T2) follows directly from the definition of $b$-special pair. Axiom scheme (T3) follows from the same argument as in Proposition 7.3.2 (note that all auxiliary types used in the proof are bounded). \hfill \Box

7.3.4. **Remark.** In the case when $T$ is the theory of the real closed fields, $T_{tame}$ is equivalent to the theory of pairs in [24] and [23]. If $T$ is the theory of DOAG, we have that $T_{tame}$ is equivalent to the theory axiomatized in [17]. Note, however, the theory of pairs corresponding to the bounded types is a new object that has not been described in the above mentioned works.

As remarked previously, when $T$ expands real closed fields and $\Delta$ corresponds to bounded types, the theory $T_{tame}(\Delta)$ will be equivalent to the $L_{\mathcal{P}}$-theory in which for every model $M$, it holds that $P(M) = M$, given that there are no non-realized bounded types.
7.3.5. Algebraically closed valued fields. Let $T$ denote a completion of ACVF.

7.3.6. Proposition. Let $M$ be a model of $T$ and $(N, M)$ be a special pair. Then $(N, M) \models T_{tame}$.

Proof. The axiom scheme (T1) follows from the fact that the pair is stably embedded and $T$ has uniform definability of types. Axiom (T2) is satisfied by the fact that we have realizations of all definable types, hence all definable 1-types. It remains to check the axiom scheme (T3). Recall that for a formula $\varphi(x, y)$ and $\delta(y, z)$ a classifying formula, such axioms are of the form

$$(\forall y)(\forall z)((P(z) \land \theta^\varphi_\delta(y, z)) \rightarrow (\exists x)(\varphi(x, y) \land \delta(x, z)),$$

where $\theta^\varphi_\delta(y, z)$ is as defined in Section 7.3.2. We will verify this axioms case by case. Let us first look at the case where $\delta := \delta_B(x, z_1, z_2)$. For $n_\varphi$ as defined in Section 6.3.4, the formula $\theta^\varphi_\delta(y, z)$ is

$$(\forall w_1) \cdots (\forall w_{2n_\varphi})(\exists x)(\xi_1(w_1, \ldots, w_{2n_\varphi}, z_1, z_2) \rightarrow \xi_2(x, y, w_1, \ldots, w_{n_\varphi}, z_1, z_2),$$

with

$$\xi_1 := \bigwedge_{i=1}^{2n_\varphi} P(w_i) \land \bigwedge_{i=1}^{n_\varphi} B^\circ(w_i, w_{n_\varphi+i}) \subseteq B(z_1, z_2)$$

$$\xi_2 := \varphi(x, y) \land x \in B(z_1, z_2) \setminus \bigcup_{i=1}^{n_\varphi} B^\circ(w_i, w_{n_\varphi+i}).$$

Let $a \in S_y(N)$ and $z_1, z_2 \in M$ be given. We need to show that, if $\varphi(x, a)$ is consistent with $x \in B(z_1, z_2) \setminus \bigcup_{i=1}^{n_\varphi} B^\circ(w_i, w_{n_\varphi+i})$ for any $w_i \in M$, then there is a $b \in \varphi(x, a)$ such that $b \in B(z_1, z_2)$ and $b \notin B^\circ(w_1, w_2)$ for any $w_1, w_2 \in M$ with $B^\circ(w_1, w_2) \subseteq B(z_1, z_2)$. By the definition of special pair, let $b \in N$ be a realization of the generic type of the ball $B(z_1, z_2)$ over $M, a, z_1, z_2$. By C-minimality, $\varphi(N, a)$ is a disjoint union of at most $n_\varphi$-many Swiss cheeses with centers and radii in $N$. By taking their intersection with $B(z_1, z_2)$, we may assume all Swiss cheeses to be contained in $B(z_1, z_2)$. Then we need to verify that $\varphi(N, a)$ is not covered by the set

$$\bigcup_{w_1 \in B(z_1, z_2), w_1 \in M} B^\circ(w_1, z_2).$$

If every positive ball $B_i$ is contained in some $B^\circ(w_i, z_2)$, then we have a contradiction to the hypothesis. Hence there is a positive ball $B_i$ in the Swiss cheese decomposition of $\varphi(N, a)$, such that $B_i$ is not contained in $B^\circ(w_1, z_2)$ for any $w_1 \in M$ and $w_1 \in B(z_1, z_2)$. Then, either $B_i$ is contained in an open ball $B^\circ(b, z_2)$ that is disjoint from $B^\circ(w, v(z_1 - z_2))$ for $w \in M$ and $w \in B(z_1, z_2)$, or $B_i = B(z_1, z_2)$. In the first case, we see that any element in the Swiss cheese corresponding to $B_i$ will realize the generic type of $B(z_1, z_2)$ over $M$. In the 2nd case, we see that $B(z_1, z_2)$ is not a union of balls $B^\circ(w, v(z_1 - z_2))$ for finitely many $B^\circ(w', v(z_1 - z_2))$ where $w' \in B(z_1, z_2)$. Hence we have a point in the corresponding Swiss cheese that satisfies the generic type of $B(z_1, z_2)$ over $M$.

Now, we look at the axiom corresponding to the case where $\delta := \delta_B(x, z_1, z_2)$. For and $n_\varphi$ defined as in Section 6.3.4, the formula $\theta^\varphi_\delta(y, z)$ is

$$(\forall w_1) \cdots (\forall w_{2n_\varphi})(\exists x)(\xi_1(w_1, \ldots, w_{2n_\varphi}, z_1, z_2) \rightarrow \xi_2(x, y, w_1, \ldots, w_{n_\varphi}, z_1, z_2),$$
with

\[
\xi_1 := \bigwedge_{i=1}^{2n} P(w_i) \land \bigwedge_{i=1}^{n_u} B(w_i, w_{n_u+i}) \subseteq B^\circ(z_1, z_2)
\]

\[
\xi_2 := \varphi(x, y) \land x \in B^\circ(z_1, z_2) \setminus \bigcup_{i=1}^{n_u} B(w_i, w_{n_u+i}).
\]

Let \( a \in N \) and \( z_1, z_2 \in M \) be given. We wish to show that if \( \varphi(N, a) \) is consistent with \( x \in B^\circ(z_1, z_2) \setminus \bigcup_{i=1}^{n_u} B(w_i, w_{n_u+i}) \) with \( B(w_i, w_{n_u+i}) \subseteq B^\circ(z_1, z_2) \) and \( w_i \in M \), there is a \( b \in \varphi(N, a) \) such that \( b \in B^\circ(z_1, z_2) \) and \( b \notin B(w_1, w_2) \) for any \( w_1, w_2 \in M \) with \( B(w_1, w_2) \subseteq B^\circ(z_1, z_2) \). By the definition of special pair, let \( b \in N \) be an element realizing the generic type \( p_{B^\circ}|\text{acl}(M, a) \) for \( B^\circ = B^\circ(z_1, z_2) \). By \( C \)-minimality, we have that \( \varphi(N, a) \) is a union of at most \( n_x \)-many Swiss cheeses with radii and center in \( \text{acl}(M, a) \). By taking an intersection with \( B^\circ(z_1, z_2) \), we may assume that all of the Swiss cheeses are contained in it. If each positive ball \( B_i \) contains some \( B(w_i, w_{i'}) \) where \( B(w_i, w_{i'}) \subseteq B^\circ(z_1, z_2) \) and \( w_i, w_i' \in M \), we have a contradiction to the hypothesis. Hence we may assume one of the \( B_i \)'s is not contained in any of \( B(w_i, w_{i'}) \subseteq B^\circ(z_1, z_2) \) with \( w_i, w_i' \in M \). If \( B_i \) contains no point in \( M \), we are done in this case. Otherwise, \( B_i \) will contain a ball of radius \( \gamma \) where \( \gamma = |p_{(z_1, z_2)} + |M. \) Then the Swiss cheese corresponding to \( B_i \) contains a point \( c \) such that for each \( z \in B^\circ(z_1, z_2), z \in M, v(c - z) = p_{(w(z_1, z_2))} + |M. \) Then this point \( c \) is the point we needed. The case for \( \delta_{B^\circ}(x, z_1, z_2) \) can be treated similarly.

The last cases for types corresponding to balls with valuative radius \(+\infty\) or \(-\infty\) can be treated in a similar fashion as in the \( o \)-minimal case.

\begin{proof}

7.3.7. **Proposition.** Let \( T \) be a completion of \( \text{ACVF} \), let \((N, M)\) be a \( b \)-special pair, then \((N, M) \models T_{\text{tame}}(\Delta)\), where \( \Delta \) consists of classifying formulas for bounded 1-types.

\end{proof}

\begin{proof}

The verification of both \( (T1) \) and \( (T3) \) follows the same argument as in the proof of Proposition 7.3.6. Axiom \( (T2) \) is satisfied by the definition of \( b \)-special pair: for every tuple \( a \in N, \text{tp}(a/M) \) is bounded and for every bounded 1-type \( p \) over \( M \), there is a realization of \( p \) in \( N \).

\end{proof}

7.3.8. **Proposition.** Let \( T \) be a completion of \( \text{ACVF} \), let \((N, M)\) be an \( o \)-special pair, then \((N, M) \models T_{\text{tame}}(\Delta)\), where \( \Delta \) consists of classifying formulas for 1-types that are orthogonal to \( \Gamma \).

\begin{proof}

The verification of both \( (T1) \) and \( (T3) \) follows the same argument as in the proof of Proposition 7.3.6. Axiom \( (T2) \) is satisfied by the definition of \( o \)-special pair: for every tuple \( a \in N, \text{tp}(a/M) \) is orthogonal to \( \Gamma \) and for every definable 1-type \( p \) over \( M \) which is orthogonal to \( \Gamma \), there is a realization of \( p \) in \( N \).

\end{proof}

7.3.9. **Real closed valued fields.** Let \( T \) be \( \text{RCVF} \).

7.3.10. **Proposition.** Let \( M \) be a real closed valued field and \((N, M)\) be a special pair. Then \((N, M) \models T_{\text{tame}}\).

\begin{proof}

The axiom scheme \( (T1) \) follows from the fact that the pair is stably embedded and \( T \) has uniform definability of types. It also satisfies axiom \( (T2) \) by definition of special pair. It remains to show \( (T3) \). Let \( \varphi(x, y) \) be an \( \mathcal{L} \)-formula. We only show the satisfiability of the axiom

\[
(\forall y)(\forall z)[(P(z) \land \theta^\circ_\delta(y, z)) \rightarrow (\exists x)(\varphi(x, y) \land \delta(x, z))],
\]

\end{proof}
for $z = (z_1, z_2)$, $\delta = \delta_B(x, z_1, z_2)$ and the corresponding $\theta_\delta(y, z)$ which is of the form

$$\forall w(\exists x)(P(w) \land w \in B(z_1, z_2)) \rightarrow (\phi(x, y) \land x \in B(z_1, z_2) \land w < x)].$$

The remaining cases are similar and left to the reader. Fix $a \in S_p(N)$ and $c_1, c_2 \in M$ such that $\theta_\delta(a, c_1, c_2)$ holds. We need to show that there is some $b \in N$ such that $\phi(b, a)$ holds and $b$ realizes the right generic type $p_{B+}$ over $M$, where $B$ denotes the ball $B(c_1, c_2) \cap M$. By quantifier elimination, the definable set defined by $\phi(x, a)$ is a finite union of balls and intervals. Since $\theta_\delta(a, c_1, c_2)$ holds, there is $d \in B(c_1, c_2)$ such that $C = (d, \infty) \cap B(c_1, c_2)$ is contained in $\phi(x, a)$. By the properties of special pairs, let $b \in N$ be a realization of $p_{B+}|\text{Mad}$. Then, $d < b$ and since $b \in B(c_1, c_2)$ we must have that $\phi(b, a)$ holds. Clearly, $b$ realizes $p_{B+}|M$. \hfill \square

The proofs of the following two propositions follow a very similar argument and will be omitted.

7.3.11. **Proposition.** Let $T$ be RCVF, let $(N, M)$ be a $b$-special pair. Then $(N, M) \models T_{tame}(\Delta)$, where $\Delta$ consists of classifying formulas for bounded 1-types.

7.3.12. **Proposition.** Let $T$ be RCVF, let $(N, M)$ be a $a$-special pair. Then $(N, M) \models T_{tame}(\Delta)$, where $\Delta$ consists of classifying formulas for 1-types that are orthogonal to $\Gamma$.

7.3.13. **Remark.** In the spirit of [11], the notion of bounded types can be viewed as an orthogonality condition. Let $p$ be the $\emptyset$-definable type on $\Gamma_{\infty}$, which states that it is greater than anything in $\Gamma$ but less than $\infty$, and in this language, bounded is the same as orthogonal to $p$. We thank Silvain Rideau for pointing this out.

We finish with the proof of Theorem 5.1.4.

**Proof of Theorem 5.1.4.** Let $T$ be either an $o$-minimal expansion of the theory DOAG, a completion of ACVF or RCVF. Let $(N, M)$ be a special pair (resp. $b$-special pair, or $a$-special pair). Let $X$ be an $L$-definable subset of $M$. Then by Propositions 7.3.2, 7.3.6 and 7.3.10 (resp. Propositions 7.3.3, 7.3.4, 7.3.11, 7.3.8, 7.3.12), $(N, M)$ is a model of $T_{tame}$ (resp. $T_{tame}(\Delta)$ for the corresponding set $\Delta$). Thus, by the last part of Theorem 7.2.2, $X$ is $L$-definable. \hfill \square

**APPENDIX A. CHARACTERIZATION OF STABLY EMBEDDED PAIRS**

In this appendix we gather the proof of the remaining cases of Theorem 4.2.4. Section A.1, which is devoted to Presburger arithmetic, is entirely based on a fragment of unpublished notes by G. Conant and S. Vojdani in [5]. In Sections A.2 and A.3 the result is shown for real closed valued fields and $p$-adically closed valued fields, respectively.

A.1. **Presburger arithmetic.** Let $T$ be the theory of Presburger arithmetic as introduced in [3.2]. Recall $T$ has quantifier elimination and definable Skolem functions. Let $M$ be a model. It is not difficult to show that every non-empty definable subset of $X \subseteq M$ which is bounded from below (resp. from above) has a minimal (resp. maximal) element. Given a subset $A \subseteq M$, $\text{dcl}(A)$ is also a model of $T$. Given an elementary extension $M \prec N$ and $a = (a_1, \ldots, a_n) \in N^n$, we let $M(a)$ denote the model $\text{dcl}(M \cup \{a_1, \ldots, a_n\})$. The extension $M \prec N$ is an end-extension if for all $x \in N \setminus M$, either $x > a$ for all $a \in M$, or $x < a$ for all $a \in M$. A type $p(x)$ with $x = (x_1, \ldots, x_n)$ is said to be algebraic over $M$ if it contains a formula of the form $rx_n = \sum_{i=1}^{n-1} s_ix_i + b$ with $r, s_i \in \mathbb{Z}$ and $b \in M$. Note that for $n = 1$, being algebraic over $M$ is equivalent to be realized in $M$.

We have the following characterization of definable types over models in $T$. 

A.1.1. Theorem. Let $M$ be a model of Presburger arithmetic. A type $p \in S_n(M)$ is definable if and only if for every realization $a \models p$, $M \prec M(a)$ is an end-extension.

Proof. Suppose $p$ is definable and for a contradiction that there is $a = (a_1, \ldots, a_n) \in \mathcal{U}^n$ such that $a \models p$ and $M \prec M(a)$ is not an end-extension. Then, there is some $b \in M(a) \setminus M$ and $m_1, m_2 \in M$ such that $m_1 < b < m_2$. By definition, $b \in \text{dcl}(M, a)$, so there are $r \in \mathbb{Z}_{>0}$ and $s_0, s_1, \ldots, s_n \in \mathbb{Z}$ such that

$$rb = s_0 + \sum_{i=1}^{n} s_i a_i.$$  

Note that $c := \sum_{i=1}^{n} s_i a_i \notin M$. Moreover, $rm_1 - s_0 < c < rm_2 - s_0$. Consider the set $X := \{X \in M \mid c < x\}$. Since $p$ is definable, $X$ is $M$-definable. Moreover, $X$ is bounded below by $rm_1 - s_0$ and contains $rm_2 - s_0$, hence it is non-empty. Let $d$ be its minimal element. Then we have that $d - 1 < c < d$, which contradicts that $\mathcal{U}$ is a model.

For the converse, suppose that $M \prec M(a)$ is an end-extension for all $a \in \mathcal{U}^n$ such that $a \models p$. We proceed by induction on $n$.

Case 1: Suppose the result has been proved for $0 < k < n$ and that $p$ is algebraic over $M$. Then the reduct of $p$ to the variables $x_1, \ldots, x_{n-1}$ determines $p$, and the result follows by induction. Note that if $n = 1$, the conditions implies that $p$ is realized in $M$, and is in particular definable.

Case 2: Suppose the result has been proved for $0 < k < n$ and that $p$ is not algebraic over $M$. Let us denote $(\mathbb{Z}^n)^* := \mathbb{Z}^n \setminus \{0, \ldots, 0\}$. By quantifier elimination, it is enough to provide a definition for the following atomic formulas:

$$\varphi_1(x, y) : \sum_{i=1}^{n} s_i x_i = t(y)$$
$$\varphi_2(x, y) : \sum_{i=1}^{n} s_i x_i < t(y)$$
$$\varphi_3(x, y) : \sum_{i=1}^{n} s_i x_i \equiv_m t(y),$$

where $s = (s_1, \ldots, s_n) \in (\mathbb{Z}^n)^*$, $y = (y_1, \ldots, y_t)$, $t(y)$ is an $\mathcal{L}_{\text{Pres}}$-term and $m \in \mathbb{Z}_{>0}$. By assumption, for every $s \in (\mathbb{Z}^n)^*$ either $p$ implies that $\sum_{i=1}^{n} s_i x_i > M$ or $p$ implies that $\sum_{i=1}^{n} s_i x_i < M$. For each $s \in (\mathbb{Z}^n)^*$ and $m \in \mathbb{Z}_{>0}$, let $\alpha(s, m) \in \{0, \ldots, m-1\}$ be such that

$$\sum_{i=1}^{n} s_i x_i \equiv_m \alpha(s, m).$$

We let the reader check that the following formulas provide definitions for $\varphi_i(x, y)$ with $i = 1, 2, 3$:

$$d_p(\varphi_1)(y) : y_1 \neq y_1$$
$$d_p(\varphi_2)(y) : \begin{cases} y_1 \neq y_1 & \sum_{i=1}^{n} s_i a_i > M \\ y_1 = y_1 & \sum_{i=1}^{n} s_i a_i < M \end{cases}$$
$$d_p(\varphi_3)(y) : t(y) \equiv_m \alpha(s, m).$$

\[\Box\]

A.1.2. Corollary. An elementary pair $(N, M)$ of models of Presburger arithmetic is stably embedded if and only if it is an end-extension.

Proof. Follows from Lemma A.2.3 and Theorem A.1.1

\[\Box\]

A.1.3. Corollary. The class $\mathcal{SE}(T)$ is an elementary class in $\mathcal{L}_p$. 

\[\Box\]
A.2. **Real closed valued fields.** Recall that for a valued field \((K,v)\) we let \(\Gamma_K\) denote the value group and \(k_K\) the residue field. Given a valued field extension \((K \subseteq L,v)\) and a subset \(A := \{a_1, \ldots, a_n\} \subseteq L\), we say that \(A\) is \(K\)-valuation independent if for every \(K\)-linear combination \(\sum_{i=1}^{n} c_i a_i \) with \(c_i \in K\),
\[
v \left( \sum_{i=1}^{n} c_i a_i \right) = \min_i (v(c_i a_i)).
\]

The extension \(L|K\) is called vs-defectless\(^1\) if every finitely generated \(K\)-vector subspace \(V\) of \(L\) admits a \(K\)-valuation basis, that is, a \(K\)-valuation independent set which spans \(V\) over \(K\). See [2, 7] for more on vs-defectless extensions.

Let \(M\) be a structure and \(S\) be an imaginary sort in \(M^{eq}\). We let \(\mathcal{L}_S\) be the language having a predicate for every \(\emptyset\)-definable subset \(R \subseteq S^n\). The structure \((S,\mathcal{L}_S)\) in which every \(R\) has the natural interpretation is called the induced structure on \(S\).

**A.2.1. Lemma.** Let \(T\) be an \(\mathcal{L}\)-complete theory and \(M \prec N\) be a stably embedded pair of models of \(T\). Let \(S\) be an imaginary sort in \(T^{eq}\). Consider \(S\) with all the induced structure from \(\mathcal{L}\), then the pair \(S(M) \prec S(N)\) is stably embedded in \(\mathcal{L}_S\).

**A.2.2. Corollary.** Let \((K \prec L,v)\) be a stably embedded pair of either real closed valued fields in \(\mathcal{L}_{\text{div}}^<\) or \(p\)-adically closed fields in \(\mathcal{L}_{\text{Mac}}\). Then, the pairs \(\Gamma_K \prec \Gamma_L\) and \(k_K \prec k_L\) are stably embedded in their respective induced structure languages.

**A.2.3. Theorem.** Let \(K \prec L\) be two real closed valued fields. The following are equivalent

1. the pair \(K \prec L\) is stably embedded in \(\mathcal{L}_{\text{div}}^<\),
2. the valued field extension \(L|K\) is vs-defectless, the pairs \(\Gamma_K \prec \Gamma_L\) and \(k_K \prec k_L\) are stably embedded in \(\mathcal{L}_{\text{log}}\) and \(\mathcal{L}_{\text{or}}\) respectively.

**Proof.** Let \((K \prec L,v)\) be a pair of real closed valued fields.

(1) \(\Rightarrow\) (2): That the pairs \(\Gamma_K \prec \Gamma_L\) and \(k_K \prec k_L\) are stably embedded follows by Corollary [2.2.2] That the extension is vs-defectless follows word for word as for algebraically closed valued fields in [6, Theorem 1.9].

(2) \(\Rightarrow\) (1): Let \(X \subseteq L^m\) be an \(\mathcal{L}_{\text{div}}^<\)-definable set over \(L\). We need to show that \(X \cap K^m\) is \(\mathcal{L}_{\text{div}}^<\)-definable over \(K\). By quantifier elimination, we may suppose that \(X\) is defined by one of the following formulas

(i) \(v(P(x)) \square v(Q(x))\) with \(\square\) either \(\leqslant\) or \(<\),

(ii) \(0 < P(x)\),

where \(P, Q \in L[X]\) with \(X = (X_1, \ldots, X_m)\). When \(X\) is defined by a formula as in (i), one can proceed as in [6, Theorem 1.9], so it remains to show the result for (ii).

Since the extension \(L|K\) is vs-defectless, there is a \(K\)-valuation independent set \(A := \{a_1, \ldots, a_n\} \subseteq L\) and polynomials \(P_i \in K[X]\) with \(i \in I := \{1, \ldots, n\}\) such that \(P(x) = \sum_{i \in I} a_i P_i(x)\). By [2, Lemma 2.23], we can further suppose that for every \(i, j \in I\), if \(v(a_i)\) and \(v(a_j)\) lie in the same coset modulo \(\Gamma_K\), then \(v(a_i) = v(a_j)\). Moreover, at the expense of multiplying \(P_i\) by \(-1\), we can suppose that \(a_i > 0\) for all \(i \in I\).

For each \(\emptyset \neq J \subseteq I\), let \(A_J\) be the set

\[ A_J := \{x \in K^m \mid v(\sum_{i \in I} a_i P_i(x)) = v(a_j P_j(x))\} \text{ if and only if } j \in J \} \]

\(^1\)This is the same as separated as in [1] and [7].
By case (i), we may suppose $A_J$ is definable over $K$. Further, since $A$ is $K$-valuation independent, the sets $A_J$ cover $K^m$ when $J$ varies over all non-empty subsets of $I$. Therefore, it suffices to show that $X \cap A_J$ is definable over $K$ for every $J \subseteq I$. Let us first show how to reduce to the case where $J = I$. If $J \neq I$, then for all $x \in A_J$ we have

$$0 < P(x) \Leftrightarrow 0 < \sum_{i \in J} a_i P_i(x) + \sum_{i \notin J} a_i P_i(x) \Leftrightarrow 0 < \sum_{i \in J} a_i P_i(x),$$

and thus we obtain an equivalent formula where for all $i \in J$, $v(a_i P_i(x))$ is the same. Therefore without loss of generality it suffices to show the case $J = I$. Now, since for all $x \in A_I$, $v(a_i P_i(x)) = v(a_j P_j(x))$ for all $i, j \in I$, $v(a_i)$ and $v(a_j)$ are in the same coset modulo $\Gamma_K$, and hence $v(a_i) = v(a_j)$ for all $i, j \in I$. Also $v(P_i(x)) = v(P_j(x))$ for all $i, j \in I$. Multiplying by a suitable constant $c \in L$, we may suppose that $v(a_i) = 1$ for all $i \in I$. Similarly, multiplying by a suitable constant $c' \in K$, we may suppose that $v(P_i(x)) = 1$ for all $x \in A_I$. We conclude by noting that in this situation, for all $x \in A_I$

$$0 < P(x) \Leftrightarrow 0 < \text{res}(\sum_{i=1}^n a_i P_i(x)) \Leftrightarrow 0 < \sum_{i=1}^n \text{res}(a_i) \text{res}(P_i(x)).$$

Since $k_K$ is stably embedded in $k_L$, the set $\{y \in k^n_K \mid 0 < \sum_{i=1}^n \text{res}(a_i)y_i\}$ is definable over $k_K$. Lifting the parameters, we obtain that $X \cap A_I$ is definable over $K$.

\[\square\]

A.2.4. Corollary. The class $\mathcal{S}(RCVF)$ is an elementary class in $\mathcal{L}_p$ for $\mathcal{L}$ either $\mathcal{L}^\leq_{\text{div}}$ or $\mathcal{L}_{\Gamma, k}^\leq$.

A.3. $p$-adically closed fields. We start by the following lemma.

A.3.1. Lemma. Let $(K \subseteq L,v)$ be a valued field extension. Suppose that every $y \in L$ is of the form $y = x + a$ with $a \in K$ and $x \in L$ such that $|v(x)| > \Gamma_K$. Then the extension is $v$-vs-defectless.

Proof. Let $V \subseteq L$ be a $K$-vector space of dimension $n$. Let us show that $V$ contains elements $\{x_1, \ldots, x_n\}$ such that each $v(x_i)$ lies in a different $\Gamma_K$-coset. By [9, Lemma 3.2.2], this implies that $\{x_1, \ldots, x_n\}$ is a $K$-valuation basis for $V$. We proceed by induction on $n$. Let $\{y_1, \ldots, y_n\}$ be a basis for $V$. For $n = 1$ the result is trivial (take $x_1 = y_1$). Suppose the result holds for all $K$-vector spaces of dimension smaller than $n$. Then, by induction, the $K$-vector space generated by $\{y_1, \ldots, y_{n-1}\}$ contains elements $\{x_1, \ldots, x_{n-1}\}$ such that each $v(x_i)$ lies in a different $\Gamma_K$-coset. Without loss of generality we may assume

$$v(x_{n-1}) > v(x_{n-2}) + \Gamma_K > \cdots > v(x_1) + \Gamma_K.$$

If either $v(y_n) > v(x_{n-1}) + \Gamma_K$, $v(x_1) > v(y_n) + \Gamma_K$, or

$$v(x_{m+1}) > v(y_n) + \Gamma_K > v(x_m) + \Gamma_K,$$

for some $1 \leq m < n - 1$, then we are done by setting $x_n := y_n$. Otherwise, there are $c \in K$ and $1 \leq m \leq n - 1$ such that $v(c y_n) = v(x_m)$. By assumption, let $a \in K$ be such that $c y_n = x + a$ with $v(x) > \Gamma_K$. Therefore, for $b := c y_n - a x_m$ satisfies $v(b) > v(x_m) + \Gamma_K$. If $v(b)$ is in a different $\Gamma_K$-coset than every $v(x_i)$ for $i \in \{1, \ldots, n-1\}$, we are done by setting $x_n := b$. Otherwise, there are $c' \in K$ and $m' > m$ such that $v(c'b) = v(x_{m'})$. Following the same procedure, there is $a' \in K$ such that

$$v(c'b - a'x_{m'}) > v(x_{m'}) + \Gamma_K.$$
Again, for \( b' := c'b - a'x_{m'} \), if \( v(b') \) lies in a different \( \Gamma_K \)-coset than every \( x_i \) for \( i \in \{1, \ldots, n-1\} \), we are done. Otherwise, there must be \( e'' \in K \) such that \( v(e''b') = v(x_{m''}) \) for \( m'' \in \{1, \ldots, n-1\} \) such that either \( m'' > m' \). Iterating this argument at most \( n - m \) times, one finds a \( K \)-linear combination \( x_n := a_n y_n + \sum_{i=1}^{n-1} a_i x_i \) such that \( v(x_n) \) is in a different \( \Gamma_K \)-coset than every \( v(x_i) \) for \( i \in \{1, \ldots, n-1\} \). \( \square \)

Let \((K \subseteq L, v)\) be a valued field extension. Let \( G \) be the convex hull of \( \Gamma_K \) in \( \Gamma_L \) and \( w \) be the valuation on \( L \) obtained by composing \( v \) with the canonical quotient map \( \Gamma_L \to \Gamma_L/G \). Let us denote \( k^w_K \) and \( k^w_L \) the residue fields of \((K, w)\) and \((L, w)\). As \( w \) is trivial on \( K, K \cong k^w_K \).

An element \( a \in L \) is \textit{limit over} \( K \) if the extension \( K(a)/K \) is an immediate extension. We let the reader check that if \( K \) is a \( p \)-adically closed valued field and \( a \) is limit over \( K \), then the type \( tp(a/K) \) is not definable.

A.3.2. Theorem ([1] Part (a) of the main Theorem]). Suppose \((K \prec L, v)\) is a valued field extension of Henselian valued fields of characteristic 0 and let \( w \) be as above. If the canonical embedding \( k^w_K \to k^w_L \) is an isomorphism, then \( K \prec L \) is stably embedded in \( L_{\text{div}} \).

A.3.3. Theorem. Let \( K \prec L \) be two \( p \)-adically closed valued fields. The following are equivalent

1. the pair \( K \prec L \) is stably embedded in \( L_{t} \) (resp. \( L_{\text{Mac}} \)),
2. the valued field extension \( L|K \) is \( vs \)-defectless and the pair \( \Gamma_K \prec \Gamma_L \) is stably embedded in \( L_{\text{Pres}} \).

Proof. (1) \( \Rightarrow \) (2): That \( \Gamma_K \prec \Gamma_L \) is stably embedded follows again by Corollary [A.2.2]. It remains to show that the extension is \( vs \)-defectless. By Lemma [A.3.1], it suffices to show that every element \( y \in L \) is of the form \( x + a \) for \( a \in K \) and \( x \in L \) such that \( |v(x)| > \Gamma_K \). Every element in \( y \in K \) is of such form taking \( x = 0 \), so we may suppose \( y \in L \setminus K \). If \( |v(y)| > \Gamma_K \) take \( a = 0 \). Otherwise, since \( \Gamma_L \) is an end extension of \( \Gamma_K \) by Corollary [A.1.2] we must have \( v(y) \in v(K) \). Suppose there is no \( a \in K \) such that \( |v(y-a)| > \Gamma_K \). Thus, for every \( a \in K, v(y-a) \in \Gamma_K \). But this implies that \( y \) is a limit over \( K \), which contradicts that \( K \) is stably embedded in \( L \). This shows the extension is \( vs \)-defectless.

(2) \( \Rightarrow \) (1): Since the pair is \( vs \)-defectless, there are no limit points in \( L \) over \( K \). Moreover, since \( \Gamma_K \prec \Gamma_L \) is stably embedded, by Corollary [A.1.2] it is an end extension of \( \Gamma_K \). The same argument as in the previous implication shows that every element \( y \in L \) is of the form \( x + a \) for \( a \in K \) and \( x \in L \) such that \( |v(x)| > \Gamma_K \). In particular, the convex hull of \( \Gamma_K \) in \( \Gamma_L \) is \( \Gamma_K \). Let us show that \( k^w_L \) is isomorphic to \( K \). For all \( y \in L \setminus K \) such that \( w(a) = 0 \), there is a unique \( a \in K \) such that \( v(y-a) > \Gamma_K \). Therefore \( \text{res}_w(y) = a \), which shows that \( k^w_L \) is in bijection with \( K \). The result now follows from [A.3.2]. \( \square \)

A.3.4. Corollary. The class \( SE(pCF) \) is an elementary class in \( L_P \) for \( L \) either \( L_{t} \) or \( L_{\text{Mac}} \).

We end the article with the following question.

A.3.5. Question. Is the class of stably embedded henselian valued fields \( L_P \)-elementary?

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