MARTIN BOUNDARY OF A KILLED RANDOM WALK IN $\mathbb{Z}_+^n$

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Abstract. The Martin compactification is investigated for a $d$-dimensional random walk which is killed when at least one of its coordinates becomes zero or negative. The limits of the Martin kernel are represented in terms of the harmonic functions of the associated induced Markov chains. It is shown that any sequence of points $x_n \in \mathbb{Z}_+^d$ with $\lim_n |x_n| = \infty$ and $\lim_n x_n/|x_n| = q$ is fundamental in the Martin compactification of $\mathbb{Z}_+^d$ if up to the multiplication by constants, the induced Markov chain corresponding to the direction $q$ has a unique positive harmonic function. The full Martin compactification is obtained for Cartesian products of one-dimensional random walks. The methods involve a ratio limit theorem and a large deviation principle for sample paths of scaled processes leading to the logarithmic asymptotics of the Green function.

1. Introduction and main results

In the present paper, we investigate the Martin boundary of a random walk on $\mathbb{Z}^d$ which is killed upon the first time when at least one of its coordinate becomes negative or zero. Such a random walk $(Z(t))$ is a Markov chain on the state space $\mathbb{Z}_+^d \doteq \{ x = (x^1, \ldots, x^d) \in \mathbb{Z}^d : x^i > 0, \ \forall \ i = 1, \ldots, d \}$ with a substochastic transition matrix $(p(x, x') = \mu(x' - x), \ x, x' \in \mathbb{Z}_+^d)$. The Green function of $(Z(t))$ is therefore given by

$$G(x, x') = \sum_{t=0}^{\infty} P_x(X(t) = x') = \sum_{t=0}^{\infty} P_x(S(t) = x', \ \tau > t)$$

where $(S(t))$ is a homogeneous random walk on $\mathbb{Z}^d$ with transition probabilities $p(x, x') = \mu(x' - x)$ and $\tau$ is the first time when the random walk $(S(t))$ exits from $\mathbb{Z}_+^d$.

For a transient discrete time Markov chain on a countable discrete state space $E$ with the Green function $G(x, x')$, the Martin compactification of $E$ is the unique smallest compactification of the discrete set $E$ for which the Martin kernels

$$K(x, \cdot) \doteq G(x, \cdot)/G(x_0, \cdot)$$

extend continuously for all $x \in E$. An explicit description of the Martin compactification is usually a non-trivial problem and the most of the existing results in this domain were obtained for so-called homogeneous random walks, when the transition probabilities of the process are invariant with respect to the translations over the state space $E$ (see the book of Woess [19]). For non-homogeneous Markov chains, there are few examples where the Martin compactification was identified. Cartier [4].

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identified the Martin compactification for random walks on non-homogeneous trees. Doney [6] described the harmonic functions and the Martin boundary of a random walk \((Z(n))\) on \(Z\) killed on the negative half-line \(\{z : z < 0\}\). Alili and Doney [2] extend this result for the corresponding space-time random walk \(S(n) = (Z(n), n)\). For Brownian motion on a half-space, the Martin boundary was obtained in the book of Doob [7]. Kurkova and Malyshev [12] described the Martin boundary for nearest neighbor random walks on \(Z\times N\) and on \(Z^2_+\), with reflected conditions on the boundary. The recent results of Raschel [15] and Kurkova and Raschel [13] identify the Martin compactification for random walks in \(Z^2_+\) with jumps at distance at most 1 and absorbing boundary. All these results use the methods that seem to be unlikely to apply in a more general situation. The methods of Doney [6] and Alili and Doney [2] rely on one-dimensional structure of the process \((Z(n))\). For Brownian motion, in [7], the explicit form of the Green function is used. Kurkova and Malyshev [12], Raschel [15] and Kurkova and Raschel [13] use an analytical method where the geometrical structure of the elliptic curve defined by the jump generating function of the random walk is crucial: this method work only if the corresponding elliptic curve is homeomorphic to the torus. It seems therefore difficult to extend this method for higher dimensions and for arbitrary jumps.

In the present paper we develop large deviation method proposed in [9], where the Martin compactification was identified for a random walks in a half-space \(Z^{d-1} \times Z_+\) killed on the boundary. The main ideas of this method can be summarized as follows:

- The ratio limit theorem allow to identify the limits of the Martin kernel \(K(z, z_n)\) when the logarithmic asymptotic of the Green function for a given sequence \((x_n)\) is zero.
- The logarithmic asymptotics of the Green function were obtained from the sample path large deviation estimates of scaled processes, in terms of the corresponding quasipotential.
- The ratio limit theorem is applied for a twisted Markov process, with an appropriated exponential change of the measure, for which the corresponding logarithmic asymptotic of the Green function is zero.
- The limits of the Martin kernel of the original random walk are obtained from those of twisted random walk by using the inverse change of the measure.

In [10] this method was used to identify Martin compactification for a random walks in a half-space \(Z^{d-1} \times Z_+\) with reflected boundary. The ratio limit theorem and the large deviation estimates for the Green function were combined there with Pascal’s method applied with a suitable renewal equation.

In the recent paper of Ignatiouk and Loree [11] the large deviation method was applied to describe Martin compactification for a random walk in \(Z^2\) killed upon the first exit from \(Z^2_+ = \{x = (x^1, x^2) \in Z^2 : x^1 > 0\}\). The main ideas of this paper are the following: for a sequence of points \(x_n \in Z^2_+\) with \(\lim_n |x_n| = \infty\) and \(\lim_n x_n/|x_n| = q = (q^1, q^2) \in \mathbb{R}^2\) the limits of the Martin kernel were deduced from the those of the corresponding local random walk on \(\{x \in Z^2 : x^i > 0 \text{ for all } i \in \Lambda\}\) with \(\Lambda = \{i \in \{1, 2\} : q^i = 0\}\). Such a local random walk is obtained from the original random walk on \(Z^2_+\) by removing the boundary \(\{x \in Z^2 : x^i = 0 \text{ for } i \in \Lambda\}\), the transition probabilities of the original random walk are then extended on the larger state space \(\{x \in Z^d : x^i > 0 \text{ for all } i \in \Lambda\}\) by homogeneity. For the
Moreover, for \( x \) (native or zero. For a subset \( \Lambda \) of \( \{1, \ldots, d\} \), the first time when the \( i \)-th coordinate of the random walk \((S(t))\) on \( \mathbb{Z}^d \) with transition probabilities \( p(z, z') = \mu(z' - z) \) is irreducible.

For \( i \in \{1, \ldots, d\} \), we denote by \( x^i \) the \( i \)-th coordinate of \( x \in \mathbb{R}^d \). Similarly, \( S^i(t) \) denotes the \( i \)-th coordinate of \( S(t) \) and

\[
\tau_i = \inf\{n \geq 0 : S^i(n) \leq 0\}
\]

is the first time when the \( i \)-th coordinate of the random walk \((S(t))\) becomes negative or zero. For a subset \( \Lambda \) of \( \{1, \ldots, d\} \), we denote \( \Lambda^c \triangleq \{1, \ldots, d\} \setminus \Lambda \) and

\[
\tau_\Lambda = \min_{i \in \Lambda} \tau_i = \inf\{n \geq 0 : S^i(n) \leq 0 \text{ for some } i \in \Lambda\}.
\]

Moreover, for \( x \in \mathbb{R}^d \), we let \( x^\Lambda \triangleq (x^i)_{i \in \Lambda} \in \mathbb{R}^\Lambda \) and similarly \( (S^\Lambda(t)) = (S^i(t))_{i \in \Lambda} \).

For \( \Lambda \neq \emptyset \), the process \((S^\Lambda(t))\) is a random walk on the lattice

\[
\mathbb{Z}^\Lambda \triangleq \{u = (u^i)_{i \in \Lambda} : u^i \in \mathbb{Z}\}
\]

with transition probabilities \( p_\Lambda(u, u') = \mu_\Lambda(u' - u) \) where

\[
\mu_\Lambda(u) = \sum_{x \in \mathbb{Z}^\Lambda} \mu(x), \quad u \in \mathbb{Z}^\Lambda.
\]

A substochastic random walk on \( \mathbb{Z}^\Lambda_\tau = \{u \in \mathbb{Z}^\Lambda : u^i > 0 \text{ for all } i \in \Lambda\} \) with transition matrix \( p_\Lambda(u, u') = \mu_\Lambda(u' - u) \), \( u, u' \in \mathbb{Z}^\Lambda_\tau \) which is identical to \( S^\Lambda(t) \) for \( t < \tau_\Lambda \) and killed at the time \( \tau_\Lambda \) is denoted by \((X^\Lambda(t))\). We call \((X^\Lambda(t))\) the induced Markov chain of the random walk \((S(t))\) corresponding to a given subset \( \Lambda \) of \( \{1, \ldots, d\} \). For \( \Lambda = \{1, \ldots, d\} \) we have therefore \( \tau_\Lambda = \tau \) and \( X^\Lambda(t) = Z(t) \). It is convenient moreover to introduce the induced Markov chain for \( \Lambda = \emptyset \). In the last case, the random walk \((S^\Lambda(t))\) and the induced Markov chain \((X^\Lambda(t))\) are...
Suppose that the conditions (A1)-(A3) are satisfied and let the coordinates of the mean \( M \) be non-negative. Then for any harmonic function \( f > 0 \) of the induced Markov chain \( (X_M(t)) \), the function

\[
h(x) = f(x^{\Lambda(M)}) - \mathbb{E}_x\left(f(S^{\Lambda(M)}(\tau)), \ \tau < \tau_{\Lambda(M)}\right)
\]

is strictly positive and harmonic for \((Z(t))\) and for any fundamental sequence of points \( x_n \in \mathbb{Z}_+^d \) with \( \lim_n |x_n| = \infty \) and \( \lim_n x_n/|x_n| = M/M \) there is a harmonic function \( f > 0 \) of the induced Markov chain \( (X_M(t)) \) such that for any \( x \in \mathbb{Z}_+^d \),

\[
\lim n G(x, x_n)/G(x_0, x_n) = \frac{f(x^{\Lambda(M)}) - \mathbb{E}_x(f(S^{\Lambda(M)}(\tau)), \ \tau < \tau_{\Lambda(M)})}{f(x_0^{\Lambda(M)}) - \mathbb{E}_{x_0}(f(S^{\Lambda(M)}(\tau)), \ \tau < \tau_{\Lambda(M)})}
\]

If moreover a harmonic function \( f > 0 \) of the induced Markov chain \( (X_M(t)) \) is unique to constant multiples, then any sequence \( x_n \in \mathbb{Z}_+^d \) with \( \lim_n |x_n| = \infty \) and \( \lim_n x_n/|x_n| = M/M \) is fundamental for \((Z(t))\) and satisfies (1.5).

Recall that in a particular case, when the coordinates of the mean vector \( M \) are all non-zero (i.e. when \( \Lambda(M) = \emptyset \)) the induced Markov chain \( (X_M(t)) \) is constant and \( \tau_{\Lambda(M)} = \infty \). In this case, the harmonic functions \( (X_M(t)) \) are therefore constant and the function (1.4) is a constant multiple of the function

\[
x \rightarrow \mathbb{P}_x(\tau = \infty).
\]

Using therefore Theorem 1 one gets

**Corollary 1.1.** Suppose that the conditions (A1)-(A3) are satisfied and let the coordinates of the mean \( M \) be strictly positive. Then

- the function \( h(x) = \mathbb{P}_x(\tau = \infty) \) is a strictly positive and harmonic for the Markov chain \((Z(t))\),
any sequence \( x_n \in \mathbb{Z}_+^d \) with \( \lim_n |x_n| = \infty \) and \( \lim_n x_n/|x_n| = M/|M| \) is fundamental for \((Z(t))\) and for any \( x \in Z_+^d \),

\[
\lim_n G(x, x_n)/G(x_0, x_n) = P_x(\tau = \infty)/P_{x_0}(\tau = \infty).
\]

Consider now the case when at least one of the coordinates of the mean vector \( M \) is zero, i.e. when \( \Lambda(M) \neq \emptyset \). In this case, the induced Markov chain \((X_M(t))\) is identical to the homogeneous random walk \((S^\Lambda(M)(t))\) on \( Z^\Lambda(M) \) before the time \( \tau_{\Lambda(M)} \) and killed at the time \( \tau_{\Lambda(M)} \), i.e. when at least one of the coordinates of \((S^\Lambda(M)(t))\) becomes zero or negative. Remark moreover that the mean jump of the random walk \((S^\Lambda(M)(t))\) is equal to zero because

\[
E_0(S^\Lambda(M)(1)) = \sum_{x \in \mathbb{Z}^d} \mu(x) x^\Lambda(M) = M^\Lambda(M) = 0.
\]

Hence, to identify the limiting behavior of the Martin kernel \( G(x, x_n)/G(x_0, x_n) \) for a sequence of points \( x_n \in \mathbb{Z}_+^d \) with \( \lim_n |x_n| = \infty \) and \( \lim_n x_n/|x_n| = M \), one has to identify the positive harmonic functions of a random walk on \( Z^d \) (with \( l = |\Lambda(M)| \) which has zero mean and is killed at the first exit from \( Z_+^d \). Unfortunately, for \( l > 1 \) there are is now general results in this domain. We hope that our paper will motivate the efforts in order to solve such a non-trivial problem.

If \( |\Lambda(M)| = 1 \), i.e. when \( \Lambda(M) = \{i\} \) for some \( 1 \leq i \leq d \), the induced Markov chain \((X_M(t))\) is a homogeneous random walk on \( Z \) killed when hitting the negative half-line \( \{k \in Z : k \leq 0\} \). In this case, the harmonic functions can be described by using the results of Doney [6] (see also Example E 27.3 in Chapter VI of Spitzer [18]). Here, using Theorem [4] one gets

**Proposition 1.1.** Suppose that the conditions (A1)-(A3) are satisfied and let the coordinates of the mean \( M \) be non-negative. Suppose moreover that only one of the coordinates of \( M \) is zero, i.e. \( \Lambda(M) = \{i\} \) for some \( 1 \leq i \leq d \). Then

- the function \( h_M(x) = x^i - E_x(S^i(\tau)) \) is strictly positive and harmonic for \((Z(t))\),
- any sequence \( x_n \in \mathbb{Z}_+^d \) with \( \lim_n |x_n| = \infty \) and \( \lim_n x_n/|x_n| = M/|M| \) is fundamental for \((Z(t))\) and for any \( x \in \mathbb{Z}_+^d \),

\[
\lim_n G(x, x_n)/G(x_0, x_n) = h_M(x)/h_M(x_0).
\]

The proof of this proposition is given in Section 9.

While for \( |\Lambda(M)| > 1 \), the harmonic functions of the induced Markov chain \((X_M(t))\) are not known in general, the results of Picardello and Woess [1] allow us to identify them under the following additional assumption

(A4) \( \mu_{\Lambda(M)}(u) = 0 \) if \( u^i u^j \neq 0 \) for some \( i < j \), \( i, j \in \Lambda(M) \), i.e. only one coordinate of \( X_M(t) \) can change during a transition \( X_M(t) \rightarrow X_M(t + 1) \).

This condition is satisfied for a nearest neighbor random walk and more generally, for Cartesian products of one-dimensional random walks (see [1] [19]). Using Theorem [4] we obtain

**Proposition 1.2.** Under the hypotheses (A1)-(A4), the following assertions hold:
the function
\[ h_M(x) = \prod_{i \in \Lambda(M)} x^i - \mathbb{E}_x \left( \prod_{i \in \Lambda(M)} (S^i(\tau)) \right) \]
is strictly positive and harmonic for the Markov chain \((Z(t))\),
- any sequence \(x_n \in \mathbb{Z}_+^d\) with \(\lim_n |x_n| = \infty\) and \(\lim_n x_n/|x_n| = M/|M|\)
  is fundamental for \((Z(t))\) and for any \(x \in \mathbb{Z}_+^d\),
\[ \lim_n G(x, x_n)/G(x_0, x_n) = h_M(x)/h_M(x_0). \]

The proof of this proposition is given in Section 9.

To identify the limiting behavior of the Martin kernel \(G(x, x_n)/G(x_0, x_n)\) for
a sequence of points \(x_n \in \mathbb{Z}_+^d\) with \(\lim_n |x_n| = \infty\) and \(\lim_n x_n/|x_n| = q\) for an
arbitrary vector
\[ q \in S^d_+ \doteq \{ x \in \mathbb{R}^d : |x| = 1 \text{ and } x^i \geq 0 \text{ for all } i = 1, \ldots, d \} \]
the method of the exponential change of measure is used. Namely, for a given
\(a \in D = \{ a \in \mathbb{R}^d : \varphi(a) \leq 1 \}\) we consider a twisted random walk \((S_a(t))\) on
\(\mathbb{Z}^d\) with transition probabilities \(p_a(x, x') = \mu(x' - x) \exp(a \cdot (x' - x))\). Such
a random walk is stochastic if and only if the point \(a\) belongs to the boundary
\(\partial D = \{ a \in \mathbb{R}^d : \varphi(a) = 1 \}\) of \(D\). For \(a \in \partial D\), we let
\[ \tau^a = \inf \{ t \geq 0 : S^i_a(t) \leq 0 \text{ for some } i = 1, \ldots, d \} \]
and we denote by \((Z_a(t))\) the random walk on \(\mathbb{Z}^d\) which is identical to \((S_a(t))\) before
the time \(\tau^a\) and killed at the time \(\tau^a\). The Green function of the twisted random
walk \((Z_a(t))\) is denoted by \(G_a(x, x')\).

Furthermore, under the assumptions \((A1)-(A4)\), the set \(D\) is compact and strictly
convex, the gradient \(\nabla \varphi(a)\) exists everywhere on \(\mathbb{R}^d\) and does not vanish on the
boundary \(\partial D = \{ a \in \mathbb{R}^d : \varphi(a) = 1 \}\), and the mapping
\[ a \rightarrow q(a) = \nabla \varphi(a)/|\nabla \varphi(a)| \]
determines a homeomorphism from \(\partial D\) to the unit sphere \(S^d = \{ q \in \mathbb{R}^d : |q| = 1 \}\)
(see [8]). We denote by \(q \rightarrow a(q)\) the inverse mapping of \((1.6)\) and we let \(a(q) = a(q/|q|)\) for a non-zero \(q \in \mathbb{R}^d\). According to this notation, \(a(q)\) is the only point
in \(\partial D\) where the vector \(q\) is normal to the convex set \(D\). For \(q \in S^d_+\) and \(a = a(q)\),
the mean of the twisted random walk \((S_{a(q)}(t))\) is given by
\[ M(a(q)) = \sum_{x \in \mathbb{Z}^d} \mu(x) \exp(a(q) \cdot x) x = \nabla \varphi(a)|_{a=a(q)} = |\nabla \varphi(a)|_{a=a(q)} q \]
and consequently,
\[ M(a(q))/|M(a(q))| = q. \]
Since clearly,
\[ G_{a(q)}(x, x') = \exp(a(q) \cdot (x' - x)) G(x, x'), \]
the limiting behavior of the Martin kernel \(G(x, x_n)/G(x_0, x_n)\) when \(x_n \rightarrow \infty\) and
\(x_n/|x_n| \rightarrow q\) can be obtained from Theorem 1 applied for the twisted random walk
\((S_{a(q)}(t))\). For instance, using the equality
\[ P_x(\tau_{a(q)} = \infty) = 1 - P_x(\tau_{a(q)} < \infty) = 1 - \mathbb{E}_x(\exp(a(q) \cdot (S(\tau) - x), \tau < \infty), \]
Corollary 1.2. Suppose that the conditions (A1)-(A3) are satisfied and let the coordinates of \( q \in S^d_+ \) be strictly positive. Then
- the function \( h_q(x) = \exp(a(q) \cdot x) - \mathbb{P}_x(\exp(a(q) \cdot S(t)), \tau < \infty) \) is strictly positive and harmonic for the Markov chain \((Z(t))\),
- any sequence of points \( x_n \in \mathbb{Z}^d_+ \) with \( \lim_n |x_n| = \infty \) and \( \lim_n x_n/|x_n| = q \) is fundamental for \((Z(t))\) and for any \( x \in \mathbb{Z}^d_+ \),
\[
\lim_n G(x, x_n)/G(x_0, x_n) = h_q(x)/h_q(x_0).
\]

Similarly, from Proposition 1.1 it follows

Corollary 1.3. Suppose that the conditions (A1)-(A3) are satisfied and let only one of the coordinates of \( q \in S^d_+ \) be zero : i.e. \( A(q) = \{i\} \) for some \( 1 \leq i \leq d \). Then
- the function \( h_q(x) = x^i \exp(a(q) \cdot x) - \mathbb{E}_x(S^i(\tau) \exp(a(q) \cdot S(\tau)), \tau < \infty) \) is strictly positive and harmonic for \((Z(t))\),
- any sequence \( x_n \in \mathbb{Z}^d_+ \) with \( \lim_n |x_n| = \infty \) and \( \lim_n x_n/|x_n| = q \) is fundamental for \((Z(t))\) and for any \( x \in \mathbb{Z}^d_+ \),
\[
\lim_n G(x, x_n)/G(x_0, x_n) = h_q(x)/h_q(x_0).
\]

Finally, with Proposition 1.2 one gets the full Martin compactification for \((Z(t))\) under the following additional assumption
\[(A4') \ \mu(x) = 0 \text{ if } x^i x^j \neq 0 \text{ for some } 1 \leq i < j \leq d.\]

Corollary 1.4. Under the hypotheses (A1)-(A3) and (A4'), for any \( q \in S^d_+ \),
- the function
\[
h_q(x) = \exp(a(q) \cdot x) \prod_{i \in A(q)} x^i - \mathbb{E}_x \left( \exp(a(q) \cdot S(\tau)) \prod_{i \in A(q)} S^i(\tau), \tau < \infty \right)
\]
is strictly positive and harmonic for \((Z(t))\),
- any sequence of points \( x_n \in \mathbb{Z}^d_+ \) with \( \lim_n |x_n| = \infty \) and \( \lim_n x_n/|x_n| = q \) is fundamental for the Markov chain \((Z(t))\) and for any \( x \in \mathbb{Z}^d_+ \),
\[
\lim_n G(x, x_n)/G(x_0, x_n) = h_q(x)/h_q(x_0)
\]

Our paper is organized as follows. In Section 2 the main ideas of the proof of our main result are given. We introduce there a local random walk \((Z_{M(t)})\) which has asymptotically the same statistical behavior as the Markov chain \((Z(t))\) on \( \mathbb{Z}^d_+ \) far from the boundary \( \cup_{i \in A(M)} \{x^i = 0\} \). For a sequence \( x_n \in \mathbb{Z}^d_+ \) with \( \lim_n |x_n| = \infty \) and \( \lim_n x_n/|x_n| = M/|M| \), the limiting behavior of the Martin kernel \( G(x, x_n)/G(x_0, x_n) \) of the Markov chain \((Z(t))\) is obtained from the limiting behavior of the Martin kernel \( G_{M(t)}(x, x_n)/G_{M(t)}(x_0, x_n) \) of the local process \((Z_{M(t)})\). Our main tools are large deviation estimates of the Green function and a ratio limit theorem for the local process \((Z_{M(t)})\). The large deviation estimates of the Green functions are obtained from the sample path large deviation estimates of the scaled processes \( X_{\varepsilon}(t) \sim \varepsilon Z(t/\varepsilon) \) and \( Z_{M(t)}^{\varepsilon}(t) \sim \varepsilon Z_{M(t)}([t/\varepsilon]) \) in Section 2. Section 3 is devoted to the estimates of the hitting probabilities of
the induced Markov chain \((X_\Lambda(t))\). The ratio limit theorem and its corollaries are given in Sections 1 and 3. Section 7 is devoted to the limiting behavior of the Martin kernel \(G_{\Lambda,M}(x, x_n)/G_{\Lambda,M}(x_0, x_n)\) of the local Markov-additive process \((Z_{\Lambda,M}(t))\) and in Section 8 Theorem 1 is proved. In Section 9 we prove Propositions 1.1 and 1.2.

2. Local Markov-additive processes and the main ideas of the proof

For \(\Lambda \subset \{1, \ldots, d\}\) we introduce a random walk \((Z_\Lambda(t))\) on

\[ Z^{\Lambda, d}_+ = \{ x \in \mathbb{Z}^d : x_i > 0, \forall i \in \Lambda \}, \]

with a substochastic transition matrix \(p(x, x') = \mu(x' - x), x, x' \in \mathbb{Z}^{\Lambda, d}_+\). This random walk is identical to \((S(t))\) for \(t < \tau_{\Lambda}\) and killed at the time \(\tau_{\Lambda}\). Recall that \(\tau_{\Lambda}\) denotes the first time when one of the coordinates of the vector \(S^\Lambda(t) = (S^i(t))_{i \in \Lambda}\) becomes negative or zero and the induced Markov chain \(X^\Lambda(t)\) is killed at the time \(\tau_{\Lambda}\) and identical to \(S^\Lambda(t)\) for \(t < \tau_{\Lambda}\). Another equivalent representation of the induced Markov chain \((X^\Lambda(t))\) is therefore the following :

\[ X^\Lambda(t) = (Z^\Lambda(t), i \in \Lambda) \doteq Z^\Lambda(t). \]

Remark moreover that the transition probabilities of the Markov process \((Z_\Lambda(t))\) are invariant with respect to the translations on \(x\) for all \(x \in \mathbb{Z}^d\) with \(x^\Lambda = 0\):

\[ P_\Lambda(x', x''') = P_\Lambda(x' + x, x''') + x, \quad \forall x', x'' \in Z^{\Lambda, d}_+. \]

Hence, according to the usual terminology, our process \((Z_\Lambda(t))\) is Markov-additive with an additive part \(Z^{\Lambda}_+ (t) = (Z^\Lambda_1(t), i \in \Lambda^c)\) and a Markovian part \(Z^{\Lambda}_-(t) = X^\Lambda(t)\). To simplify the notation we denote \(Y_\Lambda(t) = Z^{\Lambda}_+ (t)\) and we identify \(Z_\Lambda(t)\) with \((X^\Lambda(t), Y_\Lambda(t))\). Similarly, it is convenient to identify the points \(x \in Z^{\Lambda, d}_+\) and \((x^\Lambda, x^\Lambda^c) \in Z^\Lambda_+ \times Z^{\Lambda^c}\).

The random walk \((Z_\Lambda(t))\) is called a local Markov-additive process corresponding to the set \(\Lambda \subset \{1, \ldots, d\}\). Remark that for \(\Lambda = \emptyset\), this is our homogeneous random walk \((S(t))\) on \(\mathbb{Z}^d\), while for \(\Lambda = \{1, \ldots, d\}\), \(Z_\Lambda(t) = Z(t)\) is the random walk with the same transition probabilities as \((S(t))\) on \(\mathbb{Z}^d_+\) and killed upon the first time \(\tau = \min_i \tau_i\) when one of its coordinates becomes negative or zero. The Green function

\[ G(x, x^\tau) = \sum_{n=0}^{\infty} \mathbb{P}_x(S(n) = x^\tau, \tau > n) \]

of the random walk \((Z(t))\) can be represented in terms of the Green function

\[ G_\Lambda(x, x^\tau) = \sum_{n=0}^{\infty} \mathbb{P}_x(Z_\Lambda(n) = x^\tau) = \sum_{n=0}^{\infty} \mathbb{P}_x(S(n) = x^\tau, \tau_\Lambda > n) \]

of the random walk \((Z_\Lambda(t))\) in the following way : for any \(x, x' \in \mathbb{Z}^d_+\),

\begin{equation}
G(x, x') = G_\Lambda(x, x') - \mathbb{E}_x \left( G_\Lambda(S(\tau), x'), \tau < \tau_\Lambda \right)
\end{equation}

The main steps of the proof of Theorem 1 are the following :

Step 1. The transition probabilities of the local random walk \(Z_\Lambda(t) = (X^\Lambda(t), Y_\Lambda(t))\) are invariant with respect to the translations on \(x \in \mathbb{Z}^d\) with \(x^\Lambda = 0\), and hence, a function \(f > 0\) on \(\mathbb{Z}^d_+\) is harmonic for the induced Markov chain \((X^\Lambda(t))\) if and only if the function \(h(x) = f(x^\Lambda)\) is harmonic for the local process \((Z_\Lambda(t))\). For
\( \Lambda = \Lambda(M) \triangleq \{ i \in \{1, \ldots, d\} : M_i = 0 \} \) we prove the following property: if a sequence of points \( x_n \in \mathbb{Z}_+^{\Lambda, d} \) with \( \lim_n |x_n| = \infty \) and \( \lim_n x_n / |x_n| = M / |M| \) is fundamental for the local process \( (Z(t)) \) then

\[
\lim_n G_\Lambda(x, x_n) / G_\Lambda(x_0, x_n) = f(x^\Lambda) / f(x_0^\Lambda), \quad \forall x \in \mathbb{Z}_+^{\Lambda, d},
\]

for some harmonic function \( f > 0 \) of \( (X_M(t)) \). This result is obtained by using the method developed in [9]: a ratio limit theorem is combined with large deviation estimates of the Green function \( G_\Lambda(x, x_n) \). With the large deviation estimates we prove that

\[
\lim_n \frac{1}{|x_n|} \log G_\Lambda(x, x_n) = 0
\]

and next, using the ratio limit theorem of the paper [9] we deduce that

\[
\lim_n G_\Lambda(x, x_n) / G_\Lambda(\hat{x}, x_n) = 1
\]

for all \( x, \hat{x} \in \mathbb{Z}_+^{\Lambda, d} \) with \( x^\Lambda = \hat{x}^\Lambda \). Furthermore, we show that the limit

\[
h(x) = \lim_k G_\Lambda(x, x_{n_k}) / G_\Lambda(x_0, x_{n_k})
\]

is a harmonic function of \( (Z(t)) \) and from \( \text{(2.3)} \) we get the equality \( h(x) = h(\hat{x}) \) for all \( x, \hat{x} \in \mathbb{Z}_+^{\Lambda, d} \) with \( x^\Lambda = \hat{x}^\Lambda \). If the only harmonic functions of the induced Markov chain \( (X^\Lambda(M)(t)) \) are the constant multiples of the function \( h(x) = f(x^\Lambda) \), the above arguments prove that any sequence of points \( x_n \in \mathbb{Z}_+^{\Lambda, d} \) with \( \lim_n |x_n| = \infty \) and \( \lim_n x_n / |x_n| = M / |M| \) is fundamental for the local process \( (Z(t)) \) and satisfies the equality \( \text{(2.2)} \).

Step 2. The renewal equation \( \text{(2.1)} \) is next used to deduce the following result: if a sequence points \( x_n \in \mathbb{Z}_+^d \) with \( \lim_n |x_n| = \infty \) and \( \lim_n x_n / |x_n| = M / |M| \) is fundamental for the local process \( (Z(M)(t)) \) then it is also fundamental for the random walk \( (Z(t)) \) and satisfies the equality \( \text{(1.3)} \) with some harmonic function \( f > 0 \) of \( (X_M(t)) \). The main ideas are here the following: for \( \Lambda = \Lambda(M) \), using \( \text{(2.1)} \) and \( \text{(2.2)} \) one can get the equality

\[
\lim_n G(x, x_n) / G_\Lambda(\hat{x}, x_n) = \left( f(x^\Lambda) - \mathbb{E}_x f(S(\tau)), \tau < \tau_\Lambda \right) / f(\hat{x}^\Lambda)
\]

if one can prove the exchange of the limits

\[
\lim_n \mathbb{E}_x \left( \frac{G_\Lambda(S(\tau), x_n)}{G_\Lambda(\hat{x}, x_n)}, \tau = \tau_\Lambda^c < \tau_\Lambda \right) = \mathbb{E}_x \left( \lim_n \frac{G_\Lambda(S(\tau), x_n)}{G_\Lambda(\hat{x}, x_n)}, \tau = \tau_\Lambda^c < \tau_\Lambda \right)
\]

In a straightforward way, the last equality seems very difficult to obtain: the classical convergence theorems do not work here because there are no known suitable uniform estimates of the Martin kernel \( G_\Lambda(w, x_n) / G_\Lambda(\hat{x}, x_n) \). We first show that the right hand side of \( \text{(2.1)} \) can be decomposed into a main part

\[
G_\Lambda(x, x_n) - \mathbb{E}_x \left( G_\Lambda(S(\tau), x_n), \tau < \tau_\Lambda, |S(\tau)| < \delta |x_n| \right)
\]
and a corresponding negligible part with an arbitrary \( \delta > 0 \), by using large deviation estimates of the Green functions. Next we prove that

\[
\lim_n \mathbb{E}_x \left( \frac{G_\Lambda(S(\tau), x_n)}{G_\Lambda(x', x_n)}, \tau < \tau_\Lambda, |S(\tau)| < \delta|x_n| \right) = \mathbb{E}_x \left( \lim_n \frac{G_\Lambda(S(\tau), x_n)}{G_\Lambda(x', x_n)}, \tau = \tau_{\Lambda^c} < \tau_\Lambda \right)
\]

by using the dominated convergence theorem is used: it is shown that

\[
\mathbb{E}_x (\exp(\varepsilon|Z_\Lambda(\tau)|)), \tau < \tau_\Lambda < \infty
\]

for \( \varepsilon > 0 \) small enough and it is proved that for any \( x' \in \mathbb{Z}_+^d \) and \( \varepsilon > 0 \) there are \( \delta > 0 \) and \( C > 0 \) such that

\[
\mathbf{1}_{\{\delta < |x_n|\}} G_\Lambda(w, x_n)/G_\Lambda(x', x_n) \leq C \exp(\varepsilon|w|), \forall w \in \mathbb{Z}_+^d.
\]

A similar method was earlier used in [11] where the Martin compactification was identified for a random walk on \( \mathbb{Z}^2 \) killed upon the first exit from \( \mathbb{Z}_+^2 \). In our setting, the main steps of the proof are quite similar to those of the paper [11] but the intermediate results are much more delicate to get because of a higher dimension and a more general assumption on the transition probabilities of the process, in a difference of the paper [11], we do not assume the jump generating function \( \Xi \) to be finite everywhere in \( \mathbb{R}^d \) but only in a neighborhood of the set \( D \). Such a more general setting is important in view of the applications to a large class of random walks where the probability of the jumps decreases exponentially when the size jumps tends to infinity.

If the jump generating function \( \Xi \) is finite everywhere in \( \mathbb{R}^d \), any exponential function is integrable with respect of the measure \( \mu \), and the fact that the limit (2.4) is a harmonic function of the local Markov-additive process \( (S_\Lambda(M)(t)) \) is a simple consequence of a rough estimate of the Martin kernel \( G_\Lambda(M) \) by an exponential function \( C(x') \exp(\kappa|x|) \) with a large constant \( \kappa > 0 \). Such a rough estimate easily follows from the Harnack inequality. In our setting, the only exponential functions \( C(x') \exp(\kappa|x|) \) which are integrable with respect to the measure \( \mu \) are those with a small \( \kappa > 0 \) and hence, we need a more careful estimate of the Martin kernel \( G_\Lambda(M) \).

Another point where the arguments of the paper [11] do not work is the proof of the inequality (2.7). The most difficult is here the case when at least one of the coordinates of the mean vector \( M \) is zero. In [11], the proof of (2.7) for such a vector \( M \) heavily relies on the fact that the corresponding induced Markov chain \( (X_M(t)) \) is a recurrent random walk on \( \mathbb{Z} \) killed when hitting the negative half-line \( \{k \in \mathbb{Z} : k < 0\} \). An important property of such a random walk, which is essential for the proof of (2.7) in [11], is that for any \( \varepsilon \in \mathbb{Z} 
\]

\[
P_k(X_M(t) = k + \varepsilon \text{ for some } t > 0) \rightarrow 1 \quad \text{as} \quad k \rightarrow \infty
\]

(see [11]). In a more general case, the induced Markov chain \( (X_M(t)) \) is a random walk on \( \mathbb{Z}_+^d \) having a finite variance, zero mean and killed upon the first exit from \( \mathbb{Z}_+^d \). Remark that for \( \text{Card}(\Lambda(M)) \geq 3 \) a random walk with a finite variance and zero drift on \( \mathbb{Z}_+^d \) is transient and moreover, for \( \text{Card}(\Lambda(M)) \geq 2 \), the property (2.8) fails to hold when \( k \in \mathbb{Z}_+^d \) tends to infinity along one of the axis \( \{x \in \mathbb{Z}_+^d : x_i > 0 \text{ and } x_j = 0 \text{ for } j \neq i\}, i \in \Lambda(M) \). In the present
paper, the inequality (2.7) and the careful desired estimates of the Martin kernel \( G_{\Lambda_M}(x,x_n)/G_{\Lambda_M}(x',x_n) \) are obtained by using a new approach, from the ratio limit theorem and large deviation estimates of the induced Markov chain \((X_M(t))\).

3. LARGE DEVIATION RESULTS

In this section we obtain large deviation estimates for scaled local Markov-additive processes and we deduce from them the large deviation asymptotics of the Green functions. Before to prove the large deviation results we show that the local Markov-additive processes satisfy the following communication condition.

3.1. Communication condition.

**Definition 3.1.** A discrete time Markov chain \((Z(t))\) on a countable state space \(E \subset \mathbb{Z}^d\) is said to satisfy the communication condition on \(E_0 \subset E\) if there exist \(\theta > 0\) and \(C > 0\) such that for any \(x \neq x'\), \(x, x' \in E_0\) there is a sequence of points \(x_0, x_1, \ldots, x_n \in E_0\) with \(x_0 = x, x_n = x'\) and \(n \leq C|x'|\) such that

\[
|x_i - x_{i-1}| \leq C \quad \text{and} \quad P_{x_{i-1}}(Z(1) = x_i) \geq \theta, \quad \forall i = 1, \ldots, n.
\]

**Lemma 3.1.** Under the hypotheses \((A1)\), for any \(\Lambda \in \{1, \ldots, d\}\), the local Markov-additive process \((Z_\Lambda(t))\) satisfies the communication condition on \(Z_\Lambda^{+}\).

**Proof.** To prove this proposition it is sufficient to show that for any unit vector \(e \in \mathbb{Z}^d\) there is a sequence of vectors \(u_{e,1}, \ldots, u_{e,n(e)} \in \text{supp}(\mu) = \{x \in \mathbb{Z}^d : \mu(x) > 0\}\) with \(u_{e,1} + \cdots + u_{e,n(e)} = e\) such that

\[
(3.1) \quad x + u_{e,1} + \cdots + u_{e,k} \in Z_\Lambda^{+}, \quad \forall k = 1, \ldots, n(e), \quad \text{whenever} \ x, x + e \in Z_\Lambda^{+}.
\]

The communication condition will be satisfied then with

\[
\theta = \min_{e} \min_{k} \mu(u_{e,k}) > 0 \quad \text{and} \quad C = \max_{e} \max_{k} \left\{ d n(e), \max_{k} |u_{e,k}| \right\}.
\]

Suppose first that the coordinates \(e^i\) of the unit vector \(e\) are non-negative for all \(i \in \Lambda\) (i.e. either \(e^i = 0\) for all \(i \in \Lambda\) or \(e^i = 1\) for some \(i \in \Lambda\) and \(e^j = 0\) for \(j \neq i\)) and let us consider \(\hat{x} \in Z_\Lambda^{+}\) with \(\hat{x}^i = 1\) for \(i \in \Lambda\) and \(\hat{x}^j = 0\) for \(i \in \Lambda^c\). Then clearly, \(\hat{x} + e \in Z_\Lambda^{+}\) and there are \(u_{e,1}, \ldots, u_{e,n(e)} \in \text{supp}(\mu) = \{x \in \mathbb{Z}^d : \mu(x) > 0\}\) with \(u_{e,1} + \cdots + u_{e,n(e)} = e\) and

\[
\hat{x} + u_{e,1} + \cdots + u_{e,k} \in Z_\Lambda^{+}, \quad \forall k = 1, \ldots, n(e),
\]

because the Markov process \((Z_\Lambda(t))\) is irreducible on \(Z_\Lambda^{+}\). Since for any \(x \in Z_\Lambda^{+}\) and \(i \in \Lambda\), the \(i\)-th coordinate \((x + u_{e,1} + \cdots + u_{e,k})^i\) of the vector \(x + u_{e,1} + \cdots + u_{e,k}\) is greater or equal to the \(i\)-th coordinate \((\hat{x} + u_{e,1} + \cdots + u_{e,k})^i\) of the vector \(\hat{x} + u_{e,1} + \cdots + u_{e,k}\) then we get also (3.1) for all \(x \in Z_\Lambda^{+}\).

Similarly, when a unit vector \(e\) has a negative non-zero coordinate \(e^i = -1\) for some \(i \in \Lambda\) (and consequently, \(e^j = 0\) for \(j \neq i\)), we consider \(\hat{x} \in Z_\Lambda^{+}\) with \(\hat{x}^i = 2, \ \hat{x}^j = 1\) for \(j \in \Lambda, j \neq i\) and \(\hat{x}^j = 0\) for \(j \in \Lambda^c\). For such a point \(\hat{x}\), one has \(\hat{x} + e \in Z_\Lambda^{+}\) and consequently, there is sequence of vectors \(u_{e,1}, \ldots, u_{e,n(e)} \in \text{supp}(\mu)\) with \(u_{e,1} + \cdots + u_{e,n(e)} = e\) and

\[
\hat{x} + u_{e,1} + \cdots + u_{e,k} \in Z_\Lambda^{+}, \quad \forall k = 1, \ldots, n(e).
\]

Moreover, for any point \(x \in Z_\Lambda^{+}\) for which \(x + e \in Z_\Lambda^{+}\), one gets \(x^i \geq \hat{x}^i \geq \hat{x}^j \geq 1 = \hat{x}^j\) for \(j \in \Lambda, j \neq i\). For all \(x \in Z_\Lambda^{+}\) for which \(x + e \in Z_\Lambda^{+}\), the \(i\)-th
coordinate \((x + u_{e,1} + \cdots + u_{e,k})^j\) of the vector \(x + u_{e,1} + \cdots + u_{e,k}\) is therefore greater or equal to the \(i\)-th coordinate \((\tilde{x} + u_{e,1} + \cdots + u_{e,k})^i\) of the vector \(\tilde{x} + u_{e,1} + \cdots + u_{e,k}\) and consequently (3.1) holds.

3.2. Large deviation properties of scaled processes. To formulate the large deviation result we need to introduce the following notations: \(D([0, T], \mathbb{R}^d)\) denotes the set of all right continuous functions with left limits from \([0, T]\) to \(\mathbb{R}^d\) endowed with Skorohod metric (see Billingsley [3]). We let

\[
\mathbb{R}_{+}^{d,d} = \{ x \in \mathbb{R}^d : x_i \geq 0, \forall i \in \Lambda \}.
\]

For \(x \in \mathbb{R}_{+}^{d,d}\) we denote by \([x]\) the nearest lattice point to \(x\) in \(\mathbb{Z}_{+}^{d,d}\). For \(t \in \mathbb{R}_{+}\), \([t]\) denotes the integer part of \(t\).

The following proposition proves the lower large deviation bound for the family \(Z_{\Lambda}^\varepsilon(t) = \varepsilon Z_{\Lambda}([t/\varepsilon])\) in \(D([0, T], \mathbb{R}^d)\) with the rate function

\[
(3.2) \quad I_{[0, T]}(\phi) = \begin{cases} 
\int_0^T (\log \varphi)^*(\dot{\phi}(t)) \, dt, & \text{if } \phi \text{ is absolutely continuous and } \\
+\infty & \phi(t) \in \mathbb{R}_{+}^{d,d} \text{ for all } t \in [0, T],
\end{cases}
\]

where \(\varphi\) is the jump generating function defined by (1.1) and \((\log \varphi)^*\) denotes the convex conjugate of the function \(\log \varphi\) defined by

\[
(\log \varphi)^*(v) = \sup_{a \in \mathbb{R}^d} \left( a \cdot v - \log \varphi(a) \right), \quad v \in \mathbb{R}^d.
\]

Recall that a continuous function \(\phi : [0, T] \rightarrow \mathbb{R}_{+}^{d,d}\) is called absolutely continuous if the derivative \(\dot{\phi}(t)\) exists almost everywhere on \([0, T]\) (with respect to the Lebesgue measure on \([0, T]\)) and for any \(t \in [0, T],
\[
\phi(t) = \phi(0) + \int_0^t \dot{\phi}(s) \, ds.
\]

Proposition 3.1. Under the hypotheses (A1) and (A2), for any \(x \in \mathbb{R}_{+}^{d,d}\), \(T > 0\) and an open set \(O \subset D([0, T], \mathbb{R}^d),
\[
\lim_{\delta \to 0} \liminf_{\varepsilon \to 0} \inf_{x' \in E, |x'-x| < \delta} \varepsilon \log \mathbb{P}_{[x'/\varepsilon]}(Z_{\Lambda}^\varepsilon(\cdot) \in O) \geq - \inf_{\phi \in O, \phi(0) = x} I_{[0, T]}(\phi),
\]

The proof of (3.3) uses the communication condition of Proposition 3.1 together with the lower large deviation bound of Mogulskii’s theorem (see [5]) and is quite similar to the proof of the corresponding lower bound of Proposition 4.1 of the paper [5]. While the hole Mogulskii’s theorem was proved under a more restrictive condition, when the jump generating function (1.1) is finite everywhere on \(\mathbb{R}^d\), for the proof of its lower bound our Assumption (A3) is sufficient (see [5]).

3.3. Large deviation estimates of the Green functions. The large deviation estimates of scaled processes \(Z_{\Lambda}^\varepsilon(t) \doteq \varepsilon Z_{\Lambda}([t/\varepsilon])\) are now used to get the large deviation estimates of the Green functions

\[
G_{\Lambda}(x, x') = \sum_{t=0}^\infty \mathbb{P}_x(Z_{\Lambda}(t) = x') = \sum_{t=0}^\infty \mathbb{P}_x(S(t) = x', \tau_\Lambda > t).
\]
Proposition 3.2. Under the hypotheses (A1)-(A3), for any \( q, q' \in \mathbb{R}^{1,d} \) and any sequences \( x_n, x'_n \in \mathbb{Z}^{1,d}_+ \) and \( \varepsilon_n > 0 \) with \( \lim_n \varepsilon_n = 0 \), \( \lim_n \varepsilon_n x'_n = q' \) and \( \lim_n \varepsilon_n x_n = q \),

\[
\liminf_{n \to \infty} \varepsilon_n \log G_A(x'_n, x_n) \geq - \sup_{a \in D} a \cdot (q - q').
\]

Proof. The proof of this proposition is similar to the proof of Proposition 4.2 of Ignatiouk-Robert [9]. The main arguments of this proof are the following:

For any \( r > 0 \) and \( T > 0 \), using the lower large deviation bound (3.3) with an open set \( \mathcal{O} = \{ \phi : |\phi(T) - q| < r \} \) one gets

\[
\begin{align*}
\liminf_{\delta \to 0} \liminf_{n \to \infty} & \inf_{x' \in \mathbb{Z}^{1,q}_+: |\varepsilon_n x' - q| < \delta} \varepsilon_n \log \sum_{x \in \mathbb{Z}^{1,q}_+: |\varepsilon_n x - q| < r} G_A(x', x) \\
& \geq \liminf_{\delta \to 0} \liminf_{n \to \infty} \inf_{x' \in \mathbb{Z}^{1,q}_+: |\varepsilon_n x' - q| < \delta} \varepsilon_n \log \mathbb{P}_{x'}(|Z^\varepsilon_n(1)| < r) \\
& \geq - \inf_{\phi: \phi(0) = q, \phi(T) = q} \mathcal{I}_0^1(\phi) \\
\end{align*}
\]

Furthermore, by Lemma 3.1, for any \( x \neq x', x, x' \in \mathbb{Z}^{1,d}_+ \) there is a sequence of points \( w_0, w_1, \ldots, w_k \in \mathbb{Z}^{1,d}_+ \) with \( w_0 = x' \), \( w_k = x \) and \( k \leq C|x' - x| \) such that

\[
|w_i - w_{i-1}| \leq C, \quad \mathbb{P}_{w_{i-1}}(Z(1) = w_i) \geq \theta, \quad \forall i = 1, \ldots, k.
\]

From this it follows that for any \( x \neq x', x, x' \in \mathbb{Z}^{1,d}_+ \), there is \( 0 < t \leq C|x - x'| \) such that

\[
\mathbb{P}_{x'}(Z(t) = x) \geq \theta^t \geq \theta^{|x'-x|}
\]

and consequently,

\[
\begin{align*}
G_A(x', x_n) & \geq G_A(x', x) \mathbb{P}_{x'}(Z(t) = x_n) \geq G_A(x', x) \theta^{|x-x_n|} \\
& \geq G_A(x', x) \theta^{|x-q/\varepsilon_n| + C|x-q/\varepsilon_n|}.
\end{align*}
\]

Using moreover the inequality \( \text{Card}\{x \in \mathbb{Z}^d : |x - q/\varepsilon_n| < R\} \leq (2R + 1)^d \) with

\[
R = r/\varepsilon_n
\]

one obtains

\[
G_A(x', x_n) \geq \frac{1}{(1+2r/\varepsilon_n)^d} \sum_{x : |x - q/\varepsilon_n| < r} G_A(x', x)
\]

for all those \( n \in \mathbb{N} \) for which \( |q - q_n| < r \) and consequently, for any \( r > 0 \),

\[
\begin{align*}
\liminf_{\delta \to 0} \liminf_{n \to \infty} & \inf_{x' : |\varepsilon_n x' - q'| < \delta} \varepsilon_n \log G_A(x', x_n) \geq -2Cr \log \theta \\
& + \liminf_{\delta \to 0} \liminf_{n \to \infty} \inf_{x' : |\varepsilon_n x' - q'| < \delta} \varepsilon_n \log \sum_{x : |x - q/\varepsilon_n| < r} G_A(x', x). \\
\end{align*}
\]

Letting at the last inequality \( r \to 0 \) and using (3.3) one gets

\[
\begin{align*}
\liminf_{n \to \infty} \varepsilon_n \log G_A(x'_n, x_n) & \geq - \inf_{T > 0} \inf_{\phi(0) = q', \phi(T) = q} \mathcal{I}_0^1(\phi) \\
\end{align*}
\]

Since for \( \phi(t) = q' + (q - q')t/T \),

\[
\mathcal{I}_0^1(\phi) = \int_0^T (\log \phi)^* (\phi(t)) \, dt = T (\log \phi)^* \left( \frac{q - q'}{T} \right)
\]
and by Theorem 13.5 of Rockafellar \[16\],

\[
\inf_{T > 0} \frac{T(\log \varphi)^*}{T} \left( \frac{q - q'}{T} \right) = \sup_{a : \varphi(a) \leq 1} a \cdot (q - q'),
\]

from the last inequality it follows that

\[
\lim_{n \to \infty} \varepsilon_n \log G_\Lambda(x'_n, x_n) \geq - \inf_{T > 0} T(\log \varphi)^* \left( \frac{q - q'}{T} \right) = - \sup_{a \in D} a \cdot (q - q')
\]

and consequently, (3.4) holds.

A straightforward consequence of this proposition is the following statement.

**Corollary 3.1.** Under the hypotheses (A1)-(A4), for any \( q \neq q', q, q' \in \mathbb{R}_+^{\Lambda,d} \), and any sequences \( x_n, x'_n \in \mathbb{Z}_+^{\Lambda,d} \) and \( \varepsilon_n > 0 \) with \( \lim_n \varepsilon_n = 0 \), \( \lim_n \varepsilon_n x'_n = q' \) and \( \lim_n \varepsilon_n x_n = q \).

\[
\lim_{n \to \infty} \varepsilon_n \log G_\Lambda(x'_n, x_n) \geq -a(q - q') \cdot (q - q')
\]

**Proof.** Indeed, under the hypotheses (A1)-(A4), the point \( a(q - q') \) is the only point on the boundary of the set \( \{ a \in \mathbb{R}_d : \varphi(a) \leq 1 \} \) where the vector \( q - q' \) is normal to the set \( D \) and consequently, the right hand side of (3.4) is equal to the right hand side of (3.6).

Another immediate consequence of Proposition 3.2 is the following statement.

**Corollary 3.2.** Suppose that the conditions (A1)-(A4) are satisfied and let \( M^i \geq 0 \) for all \( i \in \Lambda \). Then for any sequences \( x'_n, x_n \in \mathbb{Z}_+^{\Lambda,d} \) and \( \varepsilon_n > 0 \) with \( \lim_n \varepsilon_n = 0 \), \( \lim_n \varepsilon_n x'_n = 0 \) and \( \lim_n \varepsilon_n x_n = M \).

\[
\lim_{n \to \infty} \varepsilon_n \log G_\Lambda(x'_n, x_n) = 0
\]

**Proof.** To deduce this estimate from (3.6) it is sufficient to notice that for \( q' = 0 \) and \( q = M \), under the hypotheses (A1)-(A4), one has \( a(q - q') = a(M) = 0 \).

## 4. Ratio limit theorem

In this section we identify the limiting behavior of the Martin kernel

\[
G_{\Lambda(M)}(x, x_n)/G_{\Lambda(M)}(\hat{x}, x_n)
\]

when \( \lim_n |x_n| = \infty \) and \( \lim_n x_n/|x_n| = M/|M| \) for \( x, \hat{x} \in \mathbb{Z}_+^{\Lambda,d} \) with \( x^\Lambda = \hat{x}^\Lambda \). To get this results, the large deviation estimates of the Green function \( G_{\Lambda(M)}(x, x_n) \) are combined with the results of the paper \[11\]. Proposition 7.3 of \[11\] applied for the Markov-additive process \( (Z_\Lambda(t)) \) proves the following statement.

**Proposition 4.1.** Suppose that the conditions (A1) - (A3) are satisfied and let \( \mu(0) > 0 \). Suppose moreover that a sequence of points \( x_n \in \mathbb{Z}_+^{\Lambda,d} \) is such that \( \lim_n |x_n| = \infty \) and

\[
\lim_{\delta \to 0} \liminf_{n \to \infty} \inf_{\{ x \in \mathbb{Z}_+^{\Lambda,d} : |x| < \delta |x_n| \}} \frac{1}{|x_n|} \log G_\Lambda(x, x_n) \geq 0.
\]
Then

\[
\lim_{\delta \to 0} \liminf_{n \to \infty} \inf_{x \in \mathbb{Z}^d_+ : |x| < \delta|x_n|} G_\Lambda(x + w, x_n)/G_\Lambda(x, x_n) = \begin{cases} 
\lim_{\delta \to 0} \limsup_{n \to \infty} \sup_{x \in \mathbb{Z}^d_+ : |x| < \delta|x_n|} G_\Lambda(x + w, x_n)/G_\Lambda(x, x_n) = 1
\end{cases}
\]

for all \( x \in \mathbb{Z}^d_+ \) and \( w \in \mathbb{Z}^d \) with \( w^\Lambda = 0 \).

Corollary 3.2 combined with Proposition 4.1 provides the following statement.

**Proposition 4.2.** Suppose that the conditions (A1)-(A3) are satisfied. Then for \( \Lambda = \Lambda(M) \), relations (4.1) hold for any sequence of points \( x_n \in \mathbb{Z}^d_+ \) with \( \lim_n |x_n| = \infty \) and \( \lim_n x_n/|x_n| = M/|M| \).

**Proof.** Under the hypotheses (A1) - (A3), Corollary 3.2 and Proposition 4.1 imply this statement for strongly aperiodic random walk \( (Z_\Lambda(t)) \), i.e. when \( \mu(0) > 0 \). Hence, to prove our proposition we need to show that the last assumption can be omitted. For this we consider a modified substochastic random walk \( (\tilde{Z}_\Lambda(t)) \) on \( \mathbb{Z}^d_+ \) with transition probabilities

\[
P_x(\tilde{Z}_\Lambda(1) = x') = (1 - \varepsilon)\mu(x' - x) + \varepsilon \delta_{x,x'}
\]

where \( 0 < \varepsilon < 1 \) and \( \delta_{x,x'} \) denotes Kronecker’s symbol: \( \delta_{x,x} = 1 \) and \( \delta_{x,x'} = 0 \) for \( x \neq x' \). One can represent the random walk \( (\tilde{Z}_\Lambda(t)) \) in terms of the random walk \( (Z_\Lambda(t)) \) as follows: let \( (\theta_n) \) be a sequence of independent identically distributed Bernoulli random variables with \( P(\theta_n = 1) = 1 - \varepsilon \) which are independent on the random walk \( (Z_\Lambda(t)) \), then letting \( N(n) = \theta_1 + \cdots + \theta_n \), one gets \( \tilde{Z}_\Lambda(n) = Z_\Lambda(N(n)) \) for all \( n \in \mathbb{N} \). For the Green function \( \tilde{G}_\Lambda(x, x') \) of the modified random walk \( (\tilde{Z}_\Lambda(t)) \) one gets therefore

\[
\tilde{G}_\Lambda(x, x') = \sum_{n=0}^{\infty} P_x(Z_\Lambda(N(n)) = x') = \sum_{n=0}^{\infty} \sum_{k=0}^{n} P_x(Z_\Lambda(k) = x') P(N(n) = k)
\]

(4.2) \[
= \sum_{k=0}^{\infty} P_x(Z_\Lambda(k) = x') \sum_{n=k}^{\infty} C_n^{k}(1 - \varepsilon)^{k} \varepsilon^{n-k} = (1 - \theta)^{-1}G_\Lambda(x, x').
\]

For \( \Lambda = \Lambda(M) \), the last equality and Corollary 3.2 applied for the random walk \( (Z_\Lambda(t)) \) show that

\[
\lim_{\delta \to 0} \liminf_{n \to \infty} \inf_{x \in \mathbb{Z}^d_+ : |x| < \delta|x_n|} \frac{1}{|x_n|} \log \tilde{G}_\Lambda(x, x_n) = \lim_{\delta \to 0} \liminf_{n \to \infty} \inf_{x \in \mathbb{Z}^d_+ : |x| < \delta|x_n|} \frac{1}{|x_n|} \log G_\Lambda(x, x_n) \geq 0.
\]

The new random walk \( (\tilde{Z}_\Lambda(t)) \) satisfies therefore the conditions of Proposition 4.1. Hence, for \( \Lambda = \Lambda(M) \) and \( w \in \mathbb{Z}^d_+ \) with \( w^\Lambda = 0 \),

\[
\lim_{\delta \to 0} \liminf_{n \to \infty} \inf_{x \in \mathbb{Z}^d_+ : |x| < \delta|x_n|} \tilde{G}_\Lambda(x + w, x_n)/\tilde{G}_\Lambda(x, x_n) = \lim_{\delta \to 0} \limsup_{n \to \infty} \sup_{x \in \mathbb{Z}^d_+ : |x| < \delta|x_n|} \tilde{G}_\Lambda(x + w, x_n)/\tilde{G}_\Lambda(x, x_n) = 1
\]

and using again the equality (4.2) we get (4.1).
An immediate consequence of Proposition 4.2 is the following statement.

**Corollary 4.1.** Suppose that the conditions (A1) - (A3) are satisfied and let a sequence \( x_n \in \mathbb{Z}^{\Lambda(M),d}_+ \) with \( \lim_n |x_n| = \infty \) and \( \lim_n x_n/|x_n| = M/|M| \) converge to a point of the Martin boundary \( \partial_M(\mathbb{Z}^d_+) \) for the random walk \((Z_{\Lambda(M)}(t))\). Then the limit

\[
(4.3) \quad h(x) = \lim_{n \to \infty} G_{\Lambda(M)}(x, x_n)/G_{\Lambda(M)}(x_0, x_n)
\]

satisfies the equality

\[
(4.4) \quad h(x + w) = h(x) \quad \text{for all } x \in \mathbb{Z}^{\Lambda(M),d}_+ \text{ and } w \in \mathbb{Z}^d \text{ with } w^{\Lambda} = 0.
\]

Remark moreover that under the hypotheses of Corollary 4.1 by Fatou’s lemma, the function \( h \) defined by (4.3) is super-harmonic for the random walk \((Z_{\Lambda(M)}(t))\) and hence, letting for \( u \in \mathbb{Z}^{\Lambda(M)}_+ \),

\[
f(x^{\Lambda(M)}) = h(x) \quad \text{with } x \in \mathbb{Z}^{\Lambda(M),d}_+ \text{ such that } x^{\Lambda(M)} = u
\]

and using (4.3) one gets a positive function \( f \) on \( \mathbb{Z}^{\Lambda(M)}_+ \) satisfying the inequality

\[
\mathbb{E}_u (f(X_M(t))) = \mathbb{E}_x \left( h(Z_{\Lambda(M)}(t)) \right) \leq h(x) = f(u).
\]

The resulting function \( f \) is therefore super-harmonic for the induced Markov chain \((X_M(t))\). Moreover, if the function (4.3) is harmonic for \((Z_{\Lambda(M)}(t))\) then the last inequality holds with the equality and consequently, the function \( f \) is harmonic for the induced Markov chain \((X_M(t))\). The next step of our proof shows that under the hypotheses of Corollary 4.1 the function (4.3) is always harmonic for \((Z_{\Lambda(M)}(t))\). This result would be a simple consequence of dominated convergence theorem and the Harnack inequality if instead of the assumption (A2) we assume that the jump generating function \( \varphi \) is finite everywhere on \( \mathbb{R}^d \); indeed, in this case any exponential function \( \exp(\varepsilon |x|) \) is integrable with respect to the probability measure \( \mu \) and using the Harnack inequality, one can easily show that for any \( n \geq 0 \) and \( x \in \mathbb{Z}^{\Lambda(M),d}_+ \)

\[
G_{\Lambda(M)}(x, x_n)/G_{\Lambda(M)}(x_0, x_n) \leq C \exp(\varepsilon |x|)
\]

with some \( C > 0 \) and \( \varepsilon > 0 \) do not depending on \( x \) and \( n \). In our setting, the exponential functions \( \exp(\varepsilon |x|) \) are integrable only if \( \varepsilon > 0 \) is small enough and hence, we need to get this inequality with a suitable small \( \varepsilon > 0 \). For this we need to estimate hitting probabilities of the induced Markov chain \((X_M(t))\). This is a subject of the following section.

5. Hitting probabilities of the induced Markov chain

Recall that the induced Markov chain \( X_M(t) \doteq X^{\Lambda(M)}(t) \) corresponding to the set \( \Lambda(M) = \{ i \in \{1, \ldots, d\} : M^i = 0 \} \) is a substochastic random walk on \( \mathbb{Z}^{\Lambda(M)}_+ \doteq \{ u = (u^i)_{i \in \Lambda(M)} \in \mathbb{Z}^{\Lambda(M)} : u_i > 0 \text{ for all } i \in \Lambda(M) \} \) with transition probabilities

\[
p_{\Lambda(M)}(u, u') \doteq \mu_{\Lambda(M)}(u' - u) \doteq \sum_{x \in \mathbb{Z}^d : x^{\Lambda(M)} = u' - u} \mu(x),
\]
It is identical to the random walk $S^{\Lambda(M)}(t) = (S^i(t))_{i \in \Lambda(M)}$ on $\mathbb{Z}^{\Lambda(M)}$ before it first exits from $\mathbb{Z}_+^{\Lambda(M)}$. The mean jump of the random walk $S^{\Lambda(M)}(t)$ is equal to zero because according to the definition of the set $\Lambda(M)$,

\begin{equation}
\sum_{u \in \mathbb{Z}^{\Lambda(M)}} \mu_{\Lambda(M)}(u) u = \sum_{x \in \mathbb{Z}^d : x^{\Lambda(M)}} \mu(x) x^{\Lambda(M)} = M^{\Lambda(M)} = 0.
\end{equation}

The main result of this section is the following statement.

**Proposition 5.1.** Under the hypotheses (A1)-(A3), for any $\hat{u} \in \mathbb{Z}_+^{\Lambda(M)}$,

\begin{equation}
\lim_{u \in \mathbb{Z}_+^{\Lambda(M)}, |u| \to \infty} \frac{1}{|u|} \log \mathbb{P}_\hat{u}(X_M(t) = u \text{ for some } t \geq 0) = 0.
\end{equation}

**Proof.** Since clearly $\mathbb{P}_\hat{u}(X_M(t) = u \text{ for some } t > 0) \leq 1$, to prove this proposition it is sufficient to show that for any $\hat{u} \in \mathbb{Z}_+^{\Lambda(M)}$

\begin{equation}
\liminf_{u \in \mathbb{Z}_+^{\Lambda(M)}, |u| \to \infty} \frac{1}{|u|} \log \mathbb{P}_\hat{u}(X_M(t) = u \text{ for some } t \geq 0) \geq 0.
\end{equation}

Moreover, since the Markov process $(X_M(t))$ is irreducible on $\mathbb{Z}_+^{\Lambda(M)}$ it is sufficient to prove [5.2] for $\hat{u} = (\hat{u}^i)_{i \in \Lambda(M)}$ with $\hat{u}^i = 1$ for all $i \in \Lambda(M)$. The proof of this inequality depends on the number of elements $|\Lambda(M)|$ of the set $\Lambda(M)$ and is different in each of the following cases:

- **case 1:** $|\Lambda(M)| = 1$,
- **case 2:** $|\Lambda(M)| \geq 3$,
- **case 3:** $|\Lambda(M)| = 2$.

In the first case, $(S^{\Lambda(M)}(t))$ is a random walk on $\mathbb{Z}$ with zero mean (see [5.1]) and a finite variance because its jump generating function

$$
\varphi_{\Lambda(M)}(\alpha) = \sum_{u \in \mathbb{Z}^{\Lambda(M)}} \mu_{\Lambda(M)}(u) \exp(\alpha \cdot u), \quad \alpha \in \mathbb{R}^{\Lambda(M)}
$$

is finite in a neighborhood of zero in $\mathbb{R}$. In this case, the random walk $(S^{\Lambda(M)}(t))$ is therefore recurrent and for any $u \in \mathbb{Z}$,

\begin{equation}
\mathbb{P}_u(S^{\Lambda(M)}(t) = u + 1 \text{ for some } t > 0) = 1.
\end{equation}

Since the induced Markov chain $(X_M(t))$ is identical to $(S^{\Lambda(M)}(t))$ for $t < \tau_{\Lambda(M)}$ and killed upon the time $\tau_{\Lambda(M)}$, from this it follows that for any $u > 0$,

$$
\mathbb{P}_{u+k}(X_M(t) = u + k + 1 \text{ for some } t > 0) = \mathbb{P}_{u+k}(S^{\Lambda(M)}(t) = u + k + 1 \text{ for some } 0 < t < \tau_{\Lambda(M)}) \to 1 \text{ as } k \to \infty
$$

(for more details, see Lemma 7.2 of the paper [11]). Hence, by strong Markov property applied for a sequence of stopping times $(T_n)$ with $T_0 = 0$ and

$$
T_n = \inf\{t > T_{n-1} : S^{\Lambda(M)}(t) = u + n\}, \text{ for } n \geq 1,
$$

$$
\mathbb{P}_{u+k}(X_M(t) = u + k + 1 \text{ for some } t > 0) = \mathbb{P}_{u+k}(S^{\Lambda(M)}(t) = u + k + 1 \text{ for some } 0 < t < \tau_{\Lambda(M)}) \to 1 \text{ as } k \to \infty
$$
one gets
\[
\liminf_{n \to \infty} \frac{1}{n} \log \mathbb{P}_\hat{u} (X_M(t) = \hat{u} + n \text{ for some } t > 0) \\
\geq \liminf_{n \to \infty} \frac{1}{n} \log \prod_{k=0}^{n-1} \mathbb{P}_{\hat{u} + k} (X_M(t) = \hat{u} + k + 1 \text{ for some } t > 0) \\
= \lim_{n \to \infty} \frac{1}{n} \sum_{k=0}^{n-1} \log \mathbb{P}_{\hat{u} + k} (X_M(t) = \hat{u} + k + 1 \text{ for some } t > 0) \\
= 0
\]

where the last relation follows from (5.3) by Cezaro’s theorem. In the case when \(|\Lambda(M)| = 1\), the inequality (5.2) is therefore verified.

**Case 2.** Suppose now that \(|\Lambda(M)| \geq 3\). Here, to get (5.2) we use large deviation estimates of the Green function of the induced Markov chain \((X_M(t))\)

\[
g_M(u, u') = \sum_{t=0}^{\infty} \mathbb{P}_u (X_M(t) = u'), \quad u, u' \in \mathbb{Z}^+_{\Lambda(M)}
\]

and the equality

\[
g_M(\hat{u}, u) = \mathbb{P}_\hat{u} (X_M(t) = u \text{ for some } t \geq 0) \times g_M(u, u)
\]

In this case, the random walk \((S_{\Lambda(M)}(t))\) is transient, the function \(g_M(u, u)\) is bounded above by the Green function of \((S_{\Lambda(M)}(t))\):

\[
g_M(u, u) = \sum_{t=0}^{\infty} \mathbb{P}_u (X_M(t) = u) = \sum_{t=0}^{\infty} \mathbb{P}_u (S_{\Lambda(M)}(t) = u, \tau_{\Lambda(M)} > t) \\
\leq \sum_{t=0}^{\infty} \mathbb{P}_u (S_{\Lambda(M)}(t) = u) = \sum_{t=0}^{\infty} \mathbb{P}_0 (S_{\Lambda(M)}(t) = 0) < \infty,
\]

and consequently, the left hand side of (5.2) is greater or equal to

\[
\liminf_{u \in \mathbb{Z}^+_{\Lambda(M)}, |u| \to \infty} \frac{1}{|u|} \log g_M(\hat{u}, u).
\]

To get (5.2) it is therefore sufficient to show that for any sequence \(u_n \in \mathbb{Z}^+_{\Lambda(M)}\) with \(\lim_n |u_n| = \infty\),

\[
\liminf_{n} \frac{1}{|u_n|} \log g_M(\hat{u}, u_n) \geq 0.
\]

Moreover, since the set

\[
\mathcal{S} = \{ u \in \mathbb{R}^{\Lambda(M)} : |u| = 1 \text{ and } u^i \geq 0, \text{ for all } i \in \Lambda(M) \}
\]

is compact, it is sufficient to prove that (5.5) holds when \(\lim_n u_n/|u_n| = e\), for each \(e \in \mathcal{S}\). The proof of this inequality uses the large deviation estimates similar to that of Section 3. Namely, denote

\[
\mathbb{R}^+_{\Lambda(M)} = \{ u \in \mathbb{R}^{\Lambda(M)} : u^i \geq 0, \forall i \in \Lambda(M) \}
\]

and let

\[
\varphi_{\Lambda(M)}(\alpha) = \sum_{u \in \mathbb{Z}^+_{\Lambda(M)}} \mu_{\Lambda(M)}(u) \exp(\alpha \cdot u), \quad \alpha \in \mathbb{R}^{\Lambda(M)},
\]
be the jump generating function of the random walk \((S^{\Lambda(M)}(t))\). Then for any \(q \neq q', q, q' \in \mathbb{R}_+^{\Lambda(M)}\) and any sequences \(u'_n, u_n \in \mathbb{R}_+^{\Lambda(M)}\) and \(\varepsilon_n > 0\) with \(\lim_n \varepsilon_n = 0\), \(\lim_n \varepsilon_n u'_n = q'\) and \(\lim_n \varepsilon_n u_n = q\), the same arguments as in the proof of Proposition 5.2 prove that
\[
(5.6) \quad \liminf_{n \to \infty} \varepsilon_n \log g_M(u'_n, u_n) \geq - \sup_{\alpha \in \mathbb{R}^{\Lambda(M)}: \varphi_{\Lambda(M)}(\alpha) \leq 1} a \cdot (q - q').
\]

Moreover, since the mean of the random walk \((S^{\Lambda(M)}(t))\) is zero (see (5.1)), then the set \(\{\alpha \in \mathbb{R}^{\Lambda(M)} : \varphi_{\Lambda(M)}(\alpha) \leq 1\}\) contains an only point zero and consequently, the right hand side of (5.6) is equal to zero. Using therefore (5.6) with \(u'_n = \hat{u}\), \(\varepsilon_n = 1/|u_n|\), \(q' = 0\) and \(q = e \in S\) one gets \((5.5)\) for any sequence of points \(u_n \in \mathbb{Z}_+^{\Lambda(M)}\) with \(\lim_n |u_n| = \infty\) and \(\lim_n u_n/|u_n| = e\).

**Case 3.** Consider now the case when \(|\Lambda(M)| = 2\). Remark that in this case, the same arguments as above prove the inequality \((5.5)\) for any sequence \(u_n \in \mathbb{Z}_+^{\Lambda(M)}\) with \(\lim_n |u_n| = \infty\). However, the right hand side of \((5.4)\) in this case infinite because the random walk \((S^{\Lambda(M)}(t))\) is recurrent and hence, in order to get \((5.2)\) one should be more careful. To prove \((5.2)\) in this case we first notice that for any unit vector \(e \in \mathbb{R}^d\) with \(e^j = 1\) for some \(i \in \Lambda(M)\) and \(e^j = 0\) for \(j \neq i\),
\[
g_M(\hat{u} + ne, \hat{u} + ne) = \sum_{t=0}^{\infty} \mathbb{P}_{\hat{u} + ne}(X_M(t) = \hat{u} + ne)
= \sum_{t=0}^{\infty} \mathbb{P}_{\hat{u} + ne}(S^{\Lambda(M)}(t) = \hat{u} + ne, \tau(\Lambda(M)) > t)
= \sum_{t=0}^{\infty} \mathbb{P}_{\hat{u} + ne}(S^{\Lambda(M)}(t) = \hat{u} + ne, \tau_j > t \text{ for all } j \in \Lambda(M))
\leq \sum_{t=0}^{\infty} \mathbb{P}_{\hat{u} + ne}(S^j(t) = \hat{u}^j, \tau_j > t \text{ for } j \in \Lambda(M) \setminus \{i\})
\leq \sum_{t=0}^{\infty} \mathbb{P}_{\hat{u}}(S^j(t) = \hat{u}^j, \tau_j > t \text{ for } j \in \Lambda(M) \setminus \{i\}).
\]

When \(|\Lambda(M)| = 2\), the right hand side of the above inequality is finite because for \(j \in \Lambda(M) \setminus \{i\}\), the random walk \((S^j(t))\) killed upon the time \(\tau_j\) is transient, and consequently,
\[
g_M(\hat{u} + ne) = \mathbb{P}_{\hat{u}}(X_M(t) = \hat{u} + ne \text{ for some } t > 0) \times g_M(\hat{u} + ne, \hat{u} + ne)
\leq C \mathbb{P}_{\hat{u}}(X_M(t) = \hat{u} + ne \text{ for some } t > 0)
\]
with some \(C > 0\) do not depending on \(n\). The last relation combined with \((5.5)\) proves that for any unit vector \(e \in \mathbb{R}^d\) with \(e^i = 1\) for some \(i \in \Lambda(M)\) and \(e^j = 0\) for \(j \neq i\), the following inequality holds
\[
(5.7) \quad \liminf_{n \to \infty} \frac{1}{n} \log \mathbb{P}_{\hat{u}}(X_M(t) = \hat{u} + ne \text{ for some } t > 0) \geq 0.
\]

For \(\Lambda(M) = \{i, j\}\) with \(1 \leq i < j \leq d\), consider now the unit vectors \(e_i = (e^i_1, e^i_2)\) and \(e_j = (e^j_1, e^j_2)\) in \(\mathbb{Z}_+^{\Lambda(M)}\) with \(e^i_1 = e^j_1 = 1\) and \(e^i_2 = e^j_2 = 0\). Then \(\hat{u} = e_i + e_j\) and for any \(u \in \mathbb{Z}_+^{\Lambda(M)}\),
\[
u = \hat{u} + ne_i + me_j
\]
with some non-negative integers \( n, m \). To prove (5.2) it is sufficient to show that
\[
\liminf_{n + m} \frac{1}{n + m} \log \mathbb{P}_\hat{u} (X_M(t) = \hat{u} + ne_i + me_j \text{ for some } t > 0) \geq 0
\]
when \( n + m \to \infty \). The last relation is a consequence of (5.7). Indeed, for any \( u \in \mathbb{Z}^{\Lambda(M)}_+ \), one has
\[
\mathbb{P}_u (X_M(t) = u + me_j \text{ for some } t > 0) \geq \mathbb{P}_\hat{u} (X_M(t) = \hat{u} + me_j \text{ for some } t > 0).
\]
When applied with \( u = \hat{u} + ne_i \), the last inequality shows that\[
\mathbb{P}_\hat{u} (X_M(t) = \hat{u} + ne_i + me_j \text{ for some } t > 0) \geq \mathbb{P}_\hat{u} (X_M(t) = \hat{u} + me_j \text{ for some } t > 0) \times \mathbb{P}_\hat{u} (X_M(t) = \hat{u} + ne_i \text{ for some } t > 0).
\]
The left hand side of (5.8) is therefore greater or equal to
\[
\liminf_{n + m} \frac{1}{n + m} \log \mathbb{P}_\hat{u} (X_M(t) = \hat{u} + ne_i \text{ for some } t > 0) + \liminf_{n + m} \frac{1}{n + m} \log \mathbb{P}_\hat{u} (X_M(t) = \hat{u} + me_j \text{ for some } t > 0)
\]
and consequently, using (5.7) one gets (5.8). \( \square \)

When combined with the Harnack inequality, the above proposition implies the following statement.

**Corollary 5.1.** Let \( f > 0 \) be a harmonic function of the induced Markov chain \((X_M(t))\). Then under the hypotheses (A1)-(A3),
\[
(5.9) \quad \limsup_{|u| \to \infty} \frac{1}{|u|} \log f(u) \leq 0.
\]

**Proof.** Indeed, by the Harnack inequality (see [19]), for any \( u, \hat{u} \in \mathbb{Z}^{\Lambda(M)}_+ \),
\[
f(u) \leq f(\hat{u})/\mathbb{P}_\hat{u}(X_M(t) = u \text{ for some } t \geq 0)
\]
and hence, (5.2) implies (5.9). \( \square \)

6. **Estimates of the local Martin kernel** \( G_{\Lambda(M)}(x, x_n)/G_{\Lambda(M)}(x_0, x_n) \)

Throughout this section, to simplify the notation, we let \( \Lambda(M) = \Lambda \). Our main result is here the following proposition.

**Proposition 6.1.** Suppose that the conditions (A1)-(A3) are satisfied and let a sequence of points \( x_n \in \mathbb{Z}^{\Lambda,d}_+ \) be such that \( \lim_n |x_n| = \infty \) and \( \lim_n x_n/|x_n| = M/|M| \). Then for any \( \hat{x} \in \mathbb{Z}^{\Lambda,d}_+ \) and \( \varepsilon > 0 \) there are \( N > 0 \), \( \delta > 0 \) and \( C > 0 \) such that
\[
(6.1) \quad 1_{\{|x| \leq \delta |x_n|\}} G_{\Lambda}(x, x_n)/G_{\Lambda}(\hat{x}, x_n) \leq C \exp(\varepsilon |x|)
\]
for all \( n \geq N \) and \( x \in \mathbb{Z}^{\Lambda,d}_+ \).

To prove this proposition we need the following lemmas.
Lemma 6.1. Under the hypotheses of Proposition 6.1 for any \( \varepsilon > 0 \) there are \( N > 0 \) and \( \delta > 0 \) such that

\[
\exp(-\varepsilon|w|) \leq G_A(x + w, x_n)/G_A(x, x_n) \leq \exp(\varepsilon|w|)
\]

for all \( n > N, x \in \mathbb{Z}_+^d \) and \( w \in \mathbb{Z}^d \) with \( w^A = 0 \) satisfying the inequality \( \max\{|w|, |x|\} \leq \delta|x_n| \).

Proof. By Proposition 4.2 for any \( \varepsilon > 0 \) there are \( N > 0 \) and \( \delta > 0 \) such that

\[
\exp(-\varepsilon/d) \leq G_A(x' + e, x_n)/G_A(x', x_n) \leq \exp(\varepsilon/d)
\]

for all \( n > N \) and \( x' \in \mathbb{Z}_+^d \), satisfying the inequality \(|x'| < \delta(d+1)|x_n|\), and any unit vector \( e \in \mathbb{Z}^d \) with \( e^A = 0 \). For \( w \in \mathbb{Z}^d \) with \( w^A = 0 \) and \( |w| = 1 \) the inequalities (6.2) are therefore verified. To get these inequalities for an arbitrary \( w \in \mathbb{Z}^d \) with \( w^A = 0 \), it is sufficient to consider a sequence of unit vectors \( e_1, \ldots, e_k \in \mathbb{Z}^d \) with \( e_i^A = 0 \) and \( k \leq d|w| \) such that \( e_1 + \cdots + e_k = w \) and to apply the inequalities (6.3)

\[
\frac{G_A(x + w, x_n)}{G_A(x, x_n)} = \frac{G_A(x + e_1, x_n)}{G_A(x, x_n)} \prod_{i=1}^{k-1} \frac{G_A(x + e_1 + \cdots + e_i + 1, x_n)}{G_A(x + e_1 + \cdots + e_i, x_n)}
\]

first with \( x' = x \) and \( e = e_1 \) and next with \( x' = x + e_1 + \cdots + e_i \) and \( e = e_{i+1} \) for every \( i = 1, \ldots, k - 1 \). The resulting estimates

\[
\exp(-k\varepsilon/d) \leq G_A(x + w, x_n)/G_A(x, x_n) \leq \exp(k\varepsilon/d)
\]

and the inequality \( k \leq d|w| \) provide (6.2).

Lemma 6.2. Under the hypotheses of Proposition 6.1 for any \( \varepsilon > 0 \) there are \( \delta > 0, N > 0 \) and a positive integer \( k > 0 \) such that for any \( n > N, x \in \mathbb{Z}_+^d \) and a unit vector \( e \in \mathbb{Z}_+^d \) with \( e^A = 0 \),

\[
1_{\{|x| < \delta|x_n|\}}G_A(x + ke, x_n)/G_A(x, x_n) \leq \exp(\varepsilon k).
\]

Proof. To prove this lemma we combine Lemma 6.1 with Proposition 5.1. By Lemma 6.1 for any \( \sigma > 0 \) there are \( N(\sigma) > 0 \) and \( \delta(\sigma) > 0 \) such that

\[
G_A(\hat{x}, x_n) \geq \exp(-\sigma|\hat{x} - \hat{\varepsilon}|)G_A(\hat{x}, x_n)
\]

for all \( n > N(\sigma) \) and \( \hat{x}, \hat{\varepsilon} \in \mathbb{Z}_+^d \) with \( \hat{x}^A = \hat{\varepsilon}^A \) and satisfying the inequality \( \max\{|\hat{x}|, |\hat{x} - \hat{\varepsilon}|\} \leq \delta(\sigma)|z_n| \). We use here the first inequality of (6.2) with \( x = \hat{x} \) and \( w = \hat{x} - \hat{\varepsilon} \). Furthermore, for a vector \( u \in \mathbb{Z}_+^d \) denote

\[
T_u \doteq \inf\{t > 0 : Z_\Lambda(t) = Z_\Lambda(0) + u^\Lambda\}.
\]

Then for any \( R > 0 \),

\[
G_A(x, x_n) \geq \mathbb{E}_x\left(G_A(Z_\Lambda(T_u), x_n) ; T_u < \infty\right) \geq \mathbb{E}_x\left(G_A(Z_\Lambda(T_u), x_n) ; T_u < \infty, |Z_\Lambda(T_u) - x - u| \leq R\right).
\]

The inequality (6.5) applied for the right hand side of the last inequality with \( \hat{x} = Z_\Lambda(T_u) \) and \( \hat{x} = x + u \) shows that

\[
G_A(x, x_n)/G_A(x + u, x_n) \geq \mathbb{E}_x(\exp(-\sigma|Z_\Lambda(T_u) - x - u|) ; T_u < \infty, |Z_\Lambda(T_u) - x - u| \leq R)
\]
for all $n \geq N(\sigma)$ and $R > 0$ satisfying the inequalities $0 < R \leq \delta(\sigma)|x_n|$ and $|x + u| \leq \delta(\sigma)|x_n|$. Moreover, let $\hat{x} \in \mathbb{Z}^+_{\Lambda}$ be such that $\hat{x}^i = 1$ for $i \in \Lambda$ and $\hat{x}^i = 0$ for $i \in \Lambda^c$. Then the right hand side of the above inequality is greater or equal to
\[
\mathbb{E}_\hat{x} \left( \exp(-\sigma|Z_\Lambda(T_u) - \hat{x} - u|); T_u < \infty, |Z_\Lambda(T_u) - \hat{x} - u| \leq R \right)
\]
and consequently,
\[
G_\Lambda(x, x_n) G_\Lambda(x + u, x_n) \geq \mathbb{E}_\hat{x} \left( \exp(-\sigma|Z_\Lambda(T_u) - \hat{x} - u|); T_u < \infty, |Z_\Lambda(T_u) - \hat{x} - u| \leq R \right)
\]
for all $R > 0$, $n \geq N(\sigma)$ and $x, u \in \mathbb{Z}^+_{\Lambda}$ satisfying the inequalities $|x + u| \leq \delta(\sigma)|x_n|$ and $R \leq \delta(\sigma)|x_n|$. Remark now that by monotone convergence theorem, the right hand side of (6.6) tends to $\mathbb{P}_\hat{x}(T_u < \infty)$ as $\sigma \to 0$ and $R \to \infty$. Since clearly,
\[
\mathbb{P}_\hat{x}(T_u < \infty) < 1
\]
from this it follows that for any $u \in \mathbb{Z}^+_{\Lambda}$, there are $\sigma(u) > 0$ and $R(u) > 0$ such that
\[
G_\Lambda(x, x_n) G_\Lambda(x + u, x_n) \geq (\mathbb{P}_\hat{x}(T_u < \infty))^2.
\]
for all $n \geq N(\sigma(u))$ and $x \in \mathbb{Z}^+_{\Lambda}$ satisfying the inequalities $R(u) < \delta(\sigma(u))|x_n|$ and $|x + u| \leq \delta(\sigma(u))|x_n|$.

Recall finally that for $\Lambda = \Lambda(M)$, $Z_\Lambda^*(t) = X_M(t)$ is the induced Markov chain on $\mathbb{Z}^+_{\Lambda}$ corresponding to the set $\Lambda$. Another equivalent definition of the stopping time $T_u$ is therefore the following :
\[
T_u = \inf \{t > 0 : X_M(t) = X_M(0) + u^\Lambda \}.
\]
Hence, by Proposition [6.1]
\[
\frac{1}{k} \log \mathbb{P}_\hat{x}(T_{ke} < \infty) \to 0 \quad \text{as} \quad k \to \infty
\]
and consequently, for any $\varepsilon > 0$ there is a positive integer $k > 0$ such that
\[
\mathbb{P}_\hat{x}(T_{ke} < \infty) \geq \exp(-\varepsilon k/2)
\]
for any unit vector $e \in \mathbb{Z}^+_{\Lambda}$. Letting therefore $\sigma = \sigma(ke)$ and $R = R(ke)$ we conclude that for any $\varepsilon > 0$ there are $k > 0$, $N = N(\sigma) > 0$ and $\delta = \delta(\sigma) > 0$ such that
\[
G_\Lambda(x, x_n) G_\Lambda(x + ke, x_n) \leq \exp(-\varepsilon k)
\]
for any unit vector $e \in \mathbb{Z}^+_{\Lambda}$ and all $n \geq N(\sigma)$ and $x \in \mathbb{Z}^+_{\Lambda}$ satisfying the inequalities $R < \delta|x_n|$ and $|x + ke| \leq \delta|x_n|$. Since $\lim_n |x_n| = +\infty$ the last statement proves Lemma [6.2].

**Lemma 6.3.** Under the hypotheses of Proposition [6.1], for any $\varepsilon > 0$ there are $\delta > 0$, $N > 0$ and a positive integer $k > 0$ such that for any $n > N$, $x \in \mathbb{Z}^+_{\Lambda}$ and $u \in \mathbb{N}^d$ with $u^\Lambda = 0$,
\[
1_{\{\delta|x_n| > \delta|x_n|\}} G_\Lambda(x + ku, x_n) / G_\Lambda(x, x_n) \leq \exp(\varepsilon k|u|).
\]

**Proof.** Indeed, by Lemma [6.2] for any $\varepsilon > 0$ there are $\delta > 0$, $N > 0$ and a positive integer $k > 0$ such that
\[
1_{\{|x| < 2\delta|x_n|\}} G_\Lambda(x + ke, x_n) / G_\Lambda(x, x_n) \leq \exp(\varepsilon k/d)
\]
for any unit vector \( e \in \mathbb{Z}_+^d \) with \( e^\Lambda = 0 \), \( n > N \) and \( x \in \mathbb{Z}_+^d \). Using this inequality at the right hand side of

\[
1_{\{|x|+kn<\delta|x_n|\}} \frac{G_\Lambda(\tilde{x}+kmn, x_n)}{G_\Lambda(\tilde{x}, x_n)} \leq \prod_{j=1}^m 1_{\{|x|+(j-1)n<\delta|x_n|\}} \frac{G_\Lambda(\tilde{x}+kjm, x_n)}{G_\Lambda(\tilde{x}+k(j-1)m, x_n)}
\]

with \( x = \tilde{x} + k(j-1)m \) for every \( j = 1, \ldots, m \) one gets

\[
(6.9) \quad 1_{\{|x|+kn<\delta|x_n|\}} \frac{G_\Lambda(\tilde{x}+kmn, x_n)}{G_\Lambda(\tilde{x}, x_n)} \leq \exp(\varepsilon km/d)
\]

for any unit vector \( e \in \mathbb{Z}_+^d \) with \( e^\Lambda \) = 0, \( n > N \), \( m \geq 0 \) and \( \tilde{x} \in \mathbb{Z}_+^d \). Suppose now that \( \Lambda = \{j_1, \ldots, j_{|\Lambda|}\} \) with \( j_1 < \cdots < j_{|\Lambda|} \) and let \( e_j = (e^{1}_{j}, \ldots, e^{d}_{j}) \) denote the unit vector in \( \mathbb{Z}^d \) with \( e^{i}_{j} = 1 \) and \( e^{i}_{j} = 0 \) for \( i \neq j \). Then for any \( u \in \mathbb{N}^d \) with \( u^\Lambda = 0 \) one has

\[
u = \sum_{j \in \Lambda} u^j e_j \quad \text{with} \quad u^j \in \mathbb{N} \quad \text{for all} \quad j \in \Lambda
\]

and hence, letting for \( l = 0, \ldots, |\Lambda| \)

\[
m_l = u^i \quad \text{and} \quad z_l = x + k \sum_{i \leq l} m_i e_j,
\]

we obtain \( z_0 = x, \ z_{|\Lambda|} = u + ku, \ z_l = z_{l-1} + m_l e_j \) and

\[
|z_{l-1} + km_l| \leq |x| + k \sum_{i \leq l} m_i e_j + km_l \leq |x| + k|u| + km_l \leq 2(|x| + k|u|)
\]

for all \( l = 1, \ldots, |\Lambda| \). From this it follows that

\[
1_{\{|x|+kn<\delta|x_n|\}} \frac{G_\Lambda(x+kn, x_n)}{G_\Lambda(x, x_n)} = 1_{\{|x|+kn<\delta|x_n|\}} \prod_{l=1}^{|\Lambda|} \frac{G_\Lambda(z_{l-1} + km_l e_j, x_n)}{G_\Lambda(z_{l-1}, x_n)}
\]

\[
\leq \prod_{l=1}^{|\Lambda|} 1_{\{|z_{l-1}|+km_l<\delta|x_n|\}} \frac{G_\Lambda(z_{l-1} + km_l e_j, x_n)}{G_\Lambda(z_{l-1}, x_n)}
\]

and consequently, using \((6.9)\) with \( \tilde{x} = z_{l-1} \), \( m = m_l \) and \( e = e_j \) for every \( l = 1, \ldots, |\Lambda| \) we conclude that

\[
1_{\{|x|+kn<\delta|x_n|\}} \frac{G_\Lambda(x+kn, x_n)}{G_\Lambda(x, x_n)} \leq \exp \left( \varepsilon k \sum_{l=1}^{|\Lambda|} m_l/d \right) \leq \exp(\varepsilon k|u|).
\]

for all \( n > N \).

When combined with Lemma \([5.1]\) Lemma \([6.3]\) implies the following estimates.

**Lemma 6.4.** Under the hypotheses of Proposition \([6.1]\), for any \( \varepsilon > 0 \) there are \( \delta > 0, \ N > 0 \) and \( C > 0 \) such that for any \( n > N, \ x \in \mathbb{Z}_+^d \) and \( u \in \mathbb{N}^d \) with \( u^\Lambda = 0 \),

\[
(6.10) \quad 1_{\{|x|+|u|<\delta|x_n|\}} \frac{G_\Lambda(x+u, x_n)}{G_\Lambda(x, x_n)} \leq C \exp(\varepsilon |u|).
\]

**Proof.** Indeed, Lemma \([6.3]\) proves that for any \( \varepsilon > 0 \) there are \( \delta > 0, \ N > 0 \) and \( k \geq 1 \) such that for any \( n > N, \ x \in \mathbb{Z}_+^d \) and \( \tilde{u} \in k \mathbb{N}^d \) with \( \tilde{u}^\Lambda = 0 \), one has

\[
(6.11) \quad 1_{\{|x|+|\tilde{u}|<2\delta|x_n|\}} \frac{G_\Lambda(x+\tilde{u}, x_n)}{G_\Lambda(x, x_n)} \leq \exp(\varepsilon |\tilde{u}|).
\]
Remark now that for any $u \in \mathbb{N}^d$ with $u^{A_c} = 0$ there is $\tilde{u} \in k\mathbb{N}^d$ with $\tilde{u}^{A_c} = 0$ such that
\begin{equation}
\tilde{u}^i < u^i \leq \tilde{u}^i + k \quad \text{for all } i \in A
\end{equation}
and by the Harnack inequality (see [19]),
\[
G_A(x + u, x_n)/G_A(x, x_n) = \frac{G_A(x + u, x_n)}{G_A(x + \tilde{u}, x_n)} \times \frac{G_A(x + \tilde{u}, x_n)}{G_A(x, x_n)} \leq \frac{1}{\mathbb{P}_{x+\tilde{u}}(Z_A(t) = x + u \text{ for some } t \geq 0)} \times \frac{G_A(x + \tilde{u}, x_n)}{G_A(x, x_n)}.
\]
Moreover since by Lemma 6.4, the random walk $(Z_A(t))$ satisfies the communication condition on $\mathbb{Z}_+^{\Lambda_c}$, then there is $0 < \theta < 1$ such that
\[
\mathbb{P}_{x+\tilde{u}}(Z_A(t) = x + u \text{ for some } t \geq 0) \geq \theta^{\left|u-\tilde{u}\right|} \geq \theta^{kd}
\]
and consequently,
\[
G_A(x + u, x_n)/G_A(x, x_n) \leq \theta^{-kd}G_A(x + \tilde{u}, x_n)/G_A(x, x_n).
\]
The last inequality combined with (6.11) and (6.12) proves (6.10) for $n > N$ large enough with $C = \exp(\varepsilon kd)\theta^{-kd}$. \hfill \square

**Proof of Proposition 6.4.** Since the Markov chain $(Z_A(t))$ is irreducible on $\mathbb{Z}_+^{\Lambda_c}$ it is sufficient to prove this proposition for $\hat{x} \in \mathbb{Z}_+^{\Lambda_c}$ with $\hat{x}^i = 1$ for $i \in A$ and $\hat{x}^i = 0$ for $i \in A_c$. For this we combine Lemma 6.1 with Lemma 6.4. For a given $\hat{x} \in \mathbb{Z}_+^{\Lambda_c}$, Lemma 6.1 proves that for any $\varepsilon > 0$ there $N_1(\varepsilon) > 0$ and $\delta_1(\varepsilon) > 0$ such that
\begin{equation}
G_A(\hat{x} + w, x_n)/G_A(\hat{x}, x_n) \leq \exp(\varepsilon|w|)
\end{equation}
for all $n > N_1(\varepsilon)$ and $\hat{x}, w \in \mathbb{Z}_+^{\Lambda_c}$ with $w^{A_c} = 0$ such that $|w| \leq \delta_1(\varepsilon)|x_n|$. While by Lemma 6.4 for any $\varepsilon > 0$ there are $\delta_2(\varepsilon) > 0$, $N_2(\varepsilon) > 0$ and $C(\varepsilon) > 0$ such that
\begin{equation}
G_A(\hat{x} + u, x_n)/G_A(\hat{x}, x_n) \leq C(\varepsilon)\exp(\varepsilon|u|).
\end{equation}
for any $n > N_2(\varepsilon)$, $x \in \mathbb{Z}_+^{\Lambda_c}$ and $u \in \mathbb{N}^d$ with $u^{A_c} = 0$ satisfying the inequality $|\hat{x}| + |u| < \delta_2(\varepsilon)|x_n|$. Letting for $x \in \mathbb{Z}_+^{\Lambda_c}$,
\[
w = \sum_{i \in \Lambda_c} x^i e_i \quad \text{and} \quad u = \sum_{i \in \Lambda} (x^i - 1)e_i
\]
one gets $u \in \mathbb{N}^d$ with $u^{A_c} = 0$ and $w \in \mathbb{Z}_+^{\Lambda_c}$ with $w^{A_c} = 0$ such that
\[
x = \hat{x} + w + u \quad \text{and} \quad \max\{|u|, |w|\} \leq |x - \hat{x}| \leq |x| + d.
\]
Hence, using the inequalities (6.13) and (6.14) with $\tilde{x} = \hat{x} + w$ we obtain
\[
G_A(x, x_n)/G_A(\hat{x}, x_n) = G_A(x, x_n)/G_A(\tilde{x}, x_n) \times G_A(\tilde{x}, x_n)/G_A(\hat{x}, x_n) \leq C(\varepsilon)\exp(\varepsilon|w| + \varepsilon|u|) \leq C(\varepsilon)\exp(2\varepsilon|x| + 2d)
\]
for all $n > \max\{N_1(\varepsilon), N_2(\varepsilon)\}$ and $x \in \mathbb{Z}_+^{\Lambda_c}$ satisfying the inequality $|x| + d \leq \delta_1(\varepsilon)|x_n|$ and $2|x| + 3d < \delta_2(\varepsilon)|x_n|$. Since $\lim_n |x_n| = \infty$, the last relations prove (6.11) with $C = C(\varepsilon/2)\exp(2d)$ and $0 < \delta < \min\{\delta_1(\varepsilon/2), \delta_2(\varepsilon/2)/2\}$ for $n$ large enough.
7. Limiting behavior of the local Martin kernel

The estimates obtained in the previous section are now combined with the results of Section 6 in order to investigate the limiting behavior of the Martin kernel $G_{M}(x, x_{n})/G_{M}(x_{0}, x_{n})$ when $|x_{n}| \to \infty$ and $x_{n}/|x_{n}| \to M/|M|$. To simplify the notations, we denote throughout this section $\Lambda = \Lambda(M)$. Our main result is here the following proposition.

**Proposition 7.1.** Suppose that the conditions (A1) - (A3) are satisfied and let a sequence of points $x_{n} \in \mathbb{Z}_{+}^{d}$ with $\lim_{n} |x_{n}| = +\infty$ and $\lim_{n} x_{n}/|x_{n}| = M/|M|$ be fundamental for the Markov process $(Z_{\Lambda}(t))$. Then there is a harmonic function $f > 0$ of the induced Markov chain $(X_{\Lambda}(t))$ such that for any $x \in \mathbb{Z}_{+}^{d}$,

$$
\lim_{n \to \infty} G_{\Lambda}(x, x_{n})/G_{\Lambda}(x_{0}, x_{n}) = f(x^{\Lambda})/f(x_{0}^{\Lambda}).
$$

(7.1)

As a consequence of this result one gets

**Corollary 7.1.** Suppose that the conditions (A1) - (A3) are satisfied and let the only non-negative harmonic functions of the induced Markov chain $(X_{\Lambda}(t))$ be the constant multiples of $f > 0$. Then any sequence $x_{n} \in \mathbb{Z}_{+}^{d}$ with $\lim_{n} |x_{n}| = +\infty$ and $\lim_{n} x_{n}/|x_{n}| = M/|M|$ is fundamental for the Markov process $(Z_{\Lambda}(t))$ and the equality (7.1) holds with a given function $f$.

**Proof.** Indeed, if a sequence of points $x_{n} \in \mathbb{Z}_{+}^{d}$ is such that $\lim_{n} |x_{n}| = +\infty$ and $\lim_{n} x_{n}/|x_{n}| = M/|M|$, then by Proposition 7.1 for any fundamental subsequence $(x_{n_{k}})$, the sequence of functions $G_{\Lambda}(:, x_{n_{k}})/G_{\Lambda}(x_{0}, x_{n_{k}})$ converges point-wise to the same limit $f(\cdot)/f(x_{0})$. By compactness, the sequence $(x_{n})$ is therefore fundamental itself and satisfies the equality (7.1). □

The proof of Proposition 7.1 uses the following preliminary results.

**Lemma 7.1.** Under the hypotheses (A2),

$$
\lim_{r \to \infty} \sup_{r} \frac{1}{r} \log \sum_{x \in \mathbb{Z}_{+}^{d} : |x| \geq r} \mu(x) < 0.
$$

(7.2)

**Proof.** Indeed, under the hypotheses (A2), there is $\theta > 0$ such that

$$
C_{\theta} := \sup_{a \in \mathbb{Z}_{+}^{d} : |a| \leq \theta} \varphi(a) < \infty
$$

and hence, for any unit vector $e \in \mathbb{Z}_{+}^{d}$,

$$
C_{\theta} \geq \varphi(\theta e) = \sum_{x : e \cdot x \geq r} \mu(x) \exp(\theta e \cdot x) \geq \exp(\theta r) \sum_{x : e \cdot x \geq r} \mu(x).
$$

From the last inequality it follows that

$$
\sum_{x \in \mathbb{Z}_{+}^{d} : |x| \geq r} \mu(x) \leq 2d \max_{e \in \mathbb{Z}_{+}^{d} : |e| = 1} \sum_{x \in \mathbb{Z}_{+}^{d} : e \cdot x \geq r} \mu(x) \leq 2dC_{\theta} \exp(-\theta r)
$$

and consequently, (7.2) holds. □

Using this lemma together with Corollary 6.2, we obtain
Lemma 7.2. Suppose that the conditions (A1)-(A3) are satisfied and let a sequence \( x_n \in \mathbb{Z}_+^{\Lambda,d} \) be such that \( \lim_n |x_n| = +\infty \) and \( \lim_n x_n/|x_n| = M/|M| \). Then for any \( \delta > 0 \) and \( x \in \mathbb{Z}_+^{\Lambda,d} \),

\[
(7.3) \quad \lim_{n \to \infty} \frac{1}{|x_n|} \sum_{z \in \mathbb{Z}_+^{\Lambda,d}:|z| \geq \delta |x_n|} \mu(z-x) G_\Lambda(z,x_n) = 0.
\]

Proof. Indeed, for any \( n \in \mathbb{N} \) and \( z \in \mathbb{Z}_+^{\Lambda,d} \),

\[
G_\Lambda(z,x_n) \leq G_\Lambda(x_n,x_n) = \sum_{t=0}^{\infty} p_{x_n}(S(t) = x_n, \tau_\Lambda > t)
\]

\[
\leq \sum_{t=0}^{\infty} p_{x_n}(S(t) = x_n) = \sum_{t=0}^{\infty} p_0(S(t) = 0) < \infty,
\]

and hence, by Lemma 7.2,

\[
\limsup_{n \to \infty} \frac{1}{|x_n|} \log \sum_{z \in \mathbb{Z}_+^{\Lambda,d}:|z| \geq \delta |x_n|} \mu(z-x) G_\Lambda(M,z) = 0.
\]

The last inequality proves (7.3) because by Corollary 3.2,

\[
\limsup_{n \to \infty} \frac{1}{|x_n|} \log G_\Lambda(M,x,x_n) = 0.
\]

Lemma 7.2 combined with Proposition 6.1 implies the following property.

Lemma 7.3. Suppose that the conditions (A1)-(A3) are satisfied and let a sequence of points \( x_n \in \mathbb{Z}_+^{\Lambda,d} \) with \( \lim_n |x_n| = +\infty \) and \( \lim_n x_n/|x_n| = M/|M| \) be fundamental for the Markov process \( (Z_\Lambda(t)) \). Then the limit

\[
(7.4) \quad h(x) = \lim_{n \to \infty} G_\Lambda(x,x_n)/G_\Lambda(x_0,x_n)
\]

is a harmonic function of \( (Z_\Lambda(t)) \).

Proof. To prove this statement one has to show that

\[
\lim_{n \to \infty} G_\Lambda(x,x_n)/G_\Lambda(x_0,x_n) = \sum_{z \in \mathbb{Z}_+^{\Lambda,d}} \mu(z-x) \lim_{n \to \infty} G_\Lambda(z,x_n)/G_\Lambda(x_0,x_n)
\]

Since for \( x \neq x_n \),

\[
G_\Lambda(x,x_n) = \sum_{z \in \mathbb{Z}_+^{\Lambda,d}} \mu(z-x) G_\Lambda(z,x_n)
\]

and by Lemma 7.2 for any \( \delta > 0 \),

\[
\lim_{n \to \infty} \frac{1}{|x_n|} \sum_{z \in \mathbb{Z}_+^{\Lambda,d}:|z| \geq \delta |x_n|} \mu(z-x) G_\Lambda(z,x_n) = 0,
\]
it is sufficient to show that for some \( \delta > 0 \),

\[
\lim_{n \to \infty} \sum_{z \in \mathbb{Z}_+^d, |z| \leq \delta |x_n|} \mu(z - x) \frac{G_A(z, x_n)}{G_A(x_0, x_n)} = \sum_{z \in \mathbb{Z}_+^d} \mu(z - x) \lim_{n \to \infty} \frac{G_A(z, x_n)}{G_A(x_0, x_n)}
\]

The proof of the last relation uses Proposition \[6.1\] and the dominated convergence theorem. By dominated convergence theorem, (7.5) holds if there exists a positive and \( \mu \)-integrable function \( C(z) \) on \( \mathbb{Z}_+^d \) such that

\[
1_{|z| \leq \delta |x_n|} \frac{G_A(z, x_n)}{G_A(x_0, x_n)} \leq C(z) \quad \text{for all } z \in \mathbb{Z}_+^d.
\]

Because of the assumption (A2), an exponential function \( C(z) = C \exp(\varepsilon |z|) \) is \( \mu \)-integrable for any \( \varepsilon > 0 \) is small enough, and Proposition \[6.1\] proves that for any \( \varepsilon > 0 \) there are \( C > 0 \) and \( \delta > 0 \) for which (7.6) also holds with \( C(z) = C \exp(\varepsilon |z|) \).

**Proof of Proposition \[7.1\]**. This proposition is a consequence of Lemma \[7.3\] and Corollary \[4.1\]. Indeed, let a sequence of points \( x_n \in \mathbb{Z}_+^d \) with \( \lim_n |x_n| = +\infty \) and \( \lim_n x_n/|x_n| = M/|M| \) be fundamental for the Markov chain \((Z_A(t))\) and let

\[
h(x) = \lim_{n \to \infty} G_A(x, x_n)/G_A(x_0, x_n).
\]

Then by irreducibility,

\[
h(x) = \lim_{n \to \infty} \frac{G_A(x, x_n)}{G_A(x_0, x_n)} \geq \mathbb{P}_x(Z_A(t) = x_0, \text{ for some } t \geq 0) > 0
\]

and by Lemma \[7.3\] the function \( h \) harmonic for \((Z_A(t))\). Moreover, by Corollary \[4.1\] \( h(x + w) = h(x) \) for all \( x \in \mathbb{Z}_+^d \) and \( w \in \mathbb{Z}^d \) with \( w^\Lambda = 0 \) and consequently, the function \( h(x) = h(x^1, \ldots, x^n) \) does not depend on \( x^i \) for \( i \notin \Lambda \). Letting therefore

\[
f(x^\Lambda) \doteq h(x)
\]

one gets a function \( f > 0 \) on \( \mathbb{Z}_+^d \) satisfying \[7.1\] with \( f(x_0^\Lambda) = h(x_0) = 1 \) and such that

\[
\mathbb{E}_x(f(X_M(1))) = \mathbb{E}_x(f(Z_A^\Lambda(1))) = \mathbb{E}_x(h(Z_A(1))) = h(x) = f(x^\Lambda).
\]

The last relation shows that the function \( f \) is harmonic for the induced Markov chain \((X_M(t))\).

\[\square\]

**8. From local to the original process: proof of Theorem \[1\]**

The results of the previous sections are now used to obtain the asymptotic behavior of the Martin kernel of the original Markov chain \((Z(t))\).

**8.1. Principal part of the renewal equation.** The first result of this section proves that for a sequence of points \( x_n \in \mathbb{Z}_+^d \) with \( \lim_n |x_n| = \infty \) and \( \lim_n x_n/|x_n| = M/|M| \) the right hand side of the renewal equation \[2.1\] with \( x' = x_n \) and \( \Lambda = \Lambda(M) \) can be decomposed into the main part

\[
\Xi_{\delta, \Lambda}(x, x_n) = G_A(x, x_n) - \mathbb{E}_x(G_A(S(\tau), x_n), \tau < \tau_\Lambda, |S(\tau)| < \delta |x_n|)
\]

and the corresponding negligible part

\[
\Xi_{\delta, \Lambda}(x, x_n) - G(x, x_n) = \mathbb{E}_x(G_A(S(\tau), x_n), \tau < \tau_\Lambda, |S(\tau)| \geq \delta |x_n|)
\]

with an arbitrary \( \delta > 0 \). This is a subject of the following proposition.
Proposition 8.1. Suppose that the conditions (A1)-(A3) are satisfied and let the coordinates of the mean $M$ be non-negative. Suppose moreover that a sequence of points $x_n \in \mathbb{Z}_+^d$ is such that $\lim_n |x_n| = \infty$ and $\lim_n x_n/|x_n| = M/|M|$. Then for $\Lambda = \Lambda(M)$ and any $\delta > 0$,

$$\lim_{n \to \infty} \mathbb{E}_{\Lambda}(x, x_n)/G(x, x_n) = 1$$

Proof. Indeed, for any $a \in D$, the exponential function $f(x) = \exp(a \cdot x)$ is super-harmonic for the random walk $(S(t))$ and hence, by the Harnack inequality (see Theorem 19),

$$\mathbb{P}_x(S(t) = x' \text{ for some } t \geq 0) \leq \exp(a \cdot (x - x')) \quad \forall x, x' \in \mathbb{Z}^d.$$

the Green function

$$G_S(x, x') = \sum_{t=0}^{\infty} \mathbb{P}_x(S(t) = x')$$

of the random walk $(S(t))$ satisfies therefore the inequality

$$G_S(x, x') = G_S(x', x') \mathbb{P}_x(S(t) = x' \text{ for some } t \geq 0) \leq G_S(x', x') \exp(a \cdot (x - x')) = G_S(0, 0) \exp(a \cdot (x - x')).$$

Since clearly, $G_{\Lambda}(x, x') \leq G_S(x, x')$ for all $x, x' \in \mathbb{Z}_+^d$, the last relation proves (8.2) with $C = G_S(0, 0)$. \hfill \square

Recall that for a non-zero vector $q \in \mathbb{R}^d$, the only point in $D$ where the linear function $a \to a \cdot q$ achieves its maximum over the set $D$ is denoted $a(q)$. For $q = 0$ it is convenient to let $a(0) = 0 \in D$. Then

$$\max_{a \in D} a \cdot q = a(q) \cdot q$$

for any $q \in \mathbb{R}^d$ and from Lemma 8.1 it follows

Corollary 8.1. Under the hypotheses (A1)-(A3), there is $C > 0$ such that for any $\Lambda \subset \{1, \ldots, d\}$ and $x, x' \in \mathbb{Z}_+^\Lambda$,

$$G_{\Lambda}(x, x') \leq C \exp(-a(x' - x) \cdot (x' - x)).$$

Lemma 8.2. Suppose that the conditions (A1)-(A3) are satisfied and let a sequence $x_n \in \mathbb{Z}_+^d$ be such that $\lim_n |x_n| = +\infty$. Then there is $R > 0$ such that for any $x \in \mathbb{Z}_+^d$,

$$\limsup_{n \to \infty} \frac{1}{|x_n|} \log \sum_{z \in \mathbb{Z}_+^d, |z| \geq R |x_n|} \exp(-a(z-x) \cdot (z-x) - a(x_n-z) \cdot (x_n-z)) < 0$$

Proof. Indeed, for any $n \in \mathbb{N}$ and $x, z \in \mathbb{Z}_+^d$

$$a(z-x) \cdot (z-x) + a(x_n-z) \cdot (x_n-z) = \sup_{a \in D} a \cdot (z-x) + \sup_{a \in D} a \cdot (x_n-z) \geq a(z) \cdot (z-x) + a(-z) \cdot (x_n-z) \geq a(z) \cdot z + a(-z) \cdot (-z) - (|x| + |x_n|) \sup_{a \in D} |a|$$
and consequently, the left hand side of (8.3) does not exceed
\[ \limsup_{n \to \infty} \frac{1}{|x_n|} \log \sum_{z \in \mathbb{Z}^d : |z| \geq R|x_n|} \exp(-a(z) \cdot z - a(-z) \cdot (-z) + (|x| + |x_n|) \sup_{a \in D} |a|) \]
\[ = \sup_{a \in D} |a| + \limsup_{n \to \infty} \frac{1}{|x_n|} \log \sum_{z \in \mathbb{Z}^d : |z| \geq R|x_n|} \exp(-a(z) \cdot z - a(-z) \cdot (-z)). \]

To prove Lemma 8.2, it is therefore sufficient to show that for \( R > 0 \) large enough,
\[ (8.4) \limsup_{n \to \infty} \frac{1}{|x_n|} \log \sum_{z \in \mathbb{Z}^d : |z| \geq R|x_n|} \exp(-a(z) \cdot z - a(-z) \cdot (-z)) < -\sup_{a \in D} |a|. \]

For this we have to investigate the function
\[ \lambda(q) = a(q) \cdot q + a(-q) \cdot (-q) = \sup_{a \in D} a \cdot q + \sup_{a \in D} a \cdot (-q). \]

As a supremum of a collection of convex functions \( q \to a \cdot q - a' \cdot q \) over a compact set \( (a, a') \in D \times D \), this function is finite, convex and therefore continuous on \( \mathbb{R}^d \). Moreover, recall that under the hypotheses (A1)-(A3), the mapping \( q \to a(q) \) determines a homeomorphism from the unit sphere \( S^d = \{ q \in \mathbb{R}^d : |q| = 1 \} \) to the boundary \( \partial D \) of the set \( D \) and for any non-zero \( q \in \mathbb{R}^d \), the point \( a(q) \equiv a(q/|q|) \) is the only point in \( D \) where the supremum of the linear function \( a \to a \cdot q \) over \( a \in D \) is attained. Hence, for any \( q \neq 0 \), one has \( a(-q) \neq a(q) \),
\[ \lambda(q) = |q| \lambda(q/|q|), \]
and
\[ \lambda(q) = a(q) \cdot q + \sup_{a \in D} a \cdot (-q) > a(q) \cdot q + a(q) \cdot (-q) = 0, \]
from which it follows that
\[ \lambda(q) \geq \lambda^* |q| \]
with
\[ \lambda^* \equiv \inf_{q \in \mathbb{R}^d : |q| = 1} a(q) \cdot q + a(-q) \cdot (-q) > 0. \]
The last relations show that the left hand side of (8.4) does not exceed
\[ \limsup_{n \to \infty} \frac{1}{|x_n|} \log \sum_{z \in \mathbb{Z}^d : |z| \geq R|x_n|} \exp(-\lambda^* |z|) \]
\[ \leq \limsup_{n \to \infty} \frac{1}{|x_n|} \log \left( \exp(-\lambda^* R|x_n|/2) \sum_{z \in \mathbb{Z}^d : |z| \geq R|x_n|} \exp(-\lambda^* |z|/2) \right) \]
\[ \leq -\lambda^* R/2 + \limsup_{n \to \infty} \frac{1}{|x_n|} \log \sum_{z \in \mathbb{Z}^d : |z| \geq R|x_n|} \exp(-\lambda^* |z|/2) \]
where
\[ \limsup_{n \to \infty} \frac{1}{|x_n|} \log \sum_{z \in \mathbb{Z}^d : |z| \geq R|x_n|} \exp(-\lambda^* |z|/2) \leq \limsup_{n \to \infty} \frac{1}{|x_n|} \log \sum_{z \in \mathbb{Z}^d} \exp(-\lambda^* |z|/2) \]
\[ \leq 0 \]
because for \( \lambda^* > 0 \), the series
\[ \sum_{z \in \mathbb{Z}^d} \exp(-\lambda^* |z|/2) \]
converge. The inequality \(4.8\) holds therefore for \(R > 2\sup_{a \in D} |a|/\lambda^\ast\).

\textbf{Lemma 8.3.} Suppose that the conditions (A1)-(A3) are satisfied and let a compact set \(V \subset \mathbb{R}^d\) be such that \(V \cap \{cM : c \geq 0\} = \emptyset\). Then for any \(x \in \mathbb{Z}^d\) and any sequence \(x_n \in \mathbb{Z}^d\) with \(\lim_n |x_n| = +\infty\) and \(\lim_n x_n/|x_n| = M/|M|\), one has

\[
\limsup_{n \to \infty} \frac{1}{|x_n|} \log \sum_{z \in \mathbb{Z}^d : z \in |x_n|V} \exp(-a(z-x) \cdot (z-x) - a(x_n-z) \cdot (x_n-z)) < 0
\]

\[\text{Proof. Indeed, let a compact set } V \subset \mathbb{R}^d \text{ be such that } V \cap \{cM : c \geq 0\} = \emptyset. \text{ Then letting } c = \sup_{a \in D} |a|, \text{ for any } x \in \mathbb{Z}^d, n \in \mathbb{N} \text{ and } z \in \mathbb{Z}^d \cap (|x_n|V) \text{ one gets}
\]

\[
\sup_{a \in D} a \cdot (x_n - z) = \sup_{a \in D} \left( a \cdot \left( x_n - \frac{|x_n|}{|M|} M \right) + a \cdot \left( \frac{|x_n|}{|M|} - z \right) \right) 
\]

\[
\geq -c \left| x_n - \frac{|x_n|}{|M|} M \right| + \sup_{a \in D} a \cdot \left( \frac{|x_n|}{|M|} - z \right)
\]

and

\[
\sup_{a \in D} a \cdot (z-x) \geq a(z) \cdot (z-x) \geq \sup_{a \in D} a \cdot z - c|x|
\]

from which it follows that

\[
a(z-x) \cdot (z-x) + a(x_n-z) \cdot (x_n-z) = \sup_{a \in D} a \cdot (z-x) + \sup_{a \in D} a \cdot (x_n-z) \geq \sup_{a \in D} a \cdot z + \sup_{a \in D} a \cdot \left( \frac{|x_n|}{|M|} M - z \right) - c|x| - c \left| x_n - \frac{|x_n|}{|M|} M \right|
\]

and consequently,

\[
\limsup_{n \to \infty} \frac{1}{|x_n|} \log \sum_{z \in \mathbb{Z}^d : z \in |x_n|V} \exp\left(-a(z-x) \cdot (z-x) - a(x_n-z) \cdot (x_n-z)\right)
\]

\[
\leq \limsup_{n \to \infty} \frac{1}{|x_n|} \log \sum_{z \in \mathbb{Z}^d : z \in |x_n|V} \exp\left(-\sup_{a \in D} a \cdot z - \sup_{a \in D} a \cdot \left( \frac{|x_n|}{|M|} M - z \right)\right)
\]

\[
+ c \lim_{n \to \infty} \left( \frac{|x|}{|x_n|} + \frac{|x_n|}{|x|} \right)
\]

\[
\geq \sup_{a \in D} a \cdot z + \sup_{a \in D} a \cdot \left( \frac{|x_n|}{|M|} M - z \right)
\]

\[
\text{where the last relation holds because } |x_n| \to \infty \text{ and } x_n/|x_n| \to M/|M| \text{ as } n \to \infty.
\]

Moreover, letting

\[
\lambda_M(q) = \sup_{a \in D} a \cdot q + \sup_{a \in D} a \cdot \left( \frac{M}{|M|} - q \right)
\]

one gets

\[
\sup_{a \in D} a \cdot z + \sup_{a \in D} a \cdot \left( \frac{|x_n|}{|M|} M - z \right) = |x_n| \left( \sup_{a \in D} a \cdot \frac{z}{|x_n|} + \sup_{a \in D} a \cdot \left( \frac{M}{|M|} - \frac{z}{|x_n|} \right) \right)
\]

\[
= |x_n| \lambda_M \left( \frac{z}{|x_n|} \right) \geq |x_n| \inf_{q \in V} \lambda_M(q)
\]
for all \( z \in \mathbb{Z}^d \cap (|x_n| V) \). The last inequality combined with (8.6) shows that the left hand side of (8.5) does not exceed
\[
- \inf_{q \in V} \lambda_M (q) + \limsup_{n \to \infty} \frac{1}{|x_n|} \log \text{Card}(\{ z \in \mathbb{Z}^d : z \in |x_n| V \}).
\]
Since for a compact set \( V \subset \mathbb{R}^d \), the number of points \( \text{Card}(\{ z \in \mathbb{Z}^d : z \in rV \}) \) of the set \( \{ z \in \mathbb{Z}^d : z \in rV \} \) tends to infinity polynomially with respect to \( r \) as \( r \to \infty \), we conclude that
\[
\limsup_{n \to \infty} \frac{1}{|x_n|} \log \text{Card}(\{ z \in \mathbb{Z}^d : z \in |x_n| V \}) = 0
\]
and
\[
\limsup_{n \to \infty} \frac{1}{|x_n|} \log \sum_{z \in \mathbb{Z}^d : z \in |x_n| V} \exp (-a(z - x) \cdot (z - x) - a(x_n - z) \cdot (x_n - z)) \leq - \inf_{q \in V} \lambda_M (q).
\]
(8.7)
To complete the proof of (8.5) it is now sufficient to show that
\[
\inf_{q \in V} \lambda_M (q) > 0.
\]
(8.8)
For this we investigate the function \( \lambda_M (\cdot) \). As a supremum of a collection of convex functions
\[
q \to a \cdot q + a' \cdot \left( \frac{M}{|M|} - q \right)
\]
over the compact set \( (a, a') \in D \times D \), this function is finite, convex and therefore continuous on \( \mathbb{R}^d \). Moreover, recall that the mapping \( q \to a(q) \) determines a homeomorphism from the unit sphere \( S^d = \{ q \in \mathbb{R}^d : |q| = 1 \} \) to the boundary \( \partial D \) of the set \( D \) and for any non-zero \( q \in \mathbb{R}^d \), the point \( a(q) = a(q/|q|) \) is the only point in \( D \) where the supremum of the linear function \( a \to a \cdot q \) over \( a \in D \) is attained. Since \( V \cap \{ cM : c \geq 0 \} = \emptyset \), from this it follows that for any \( q \in V \),
\[
a(M) \neq a(q), \quad a(M) \neq a \left( \frac{M}{|M|} - q \right)
\]
and
\[
\lambda_M (q) = \sup_{a \in D} a \cdot q + \sup_{a \in D} a \cdot \left( \frac{M}{|M|} - q \right)
\]
\[
> a(M) \cdot q + a(M) \cdot \left( \frac{M}{|M|} - q \right) = a(M) \cdot \frac{M}{|M|}.
\]
Since according to the definition of the mapping \( q \to a(q) \) (see Section I)
\[
a(M) = a(\nabla \varphi(0)) = 0
\]
this proves that \( \lambda_M (q) > 0 \) for all \( q \in V \) and consequently (8.8) holds. The inequality (8.8) combined with (8.7) provides (8.5). □
Proof of Proposition 8.1. Remark first of all that by Corollary 8.1, for any \( \delta > 0 \),
\[
0 < \Xi_{\delta, A}(x, x_n) - G(x, x_n) = \sum_{w \in \mathbb{Z}_+^d \setminus \mathbb{Z}_+^d: |w| \geq \delta |x_n|} \mathbb{P}_x(S(\tau) = w, \tau < \tau_A) G_A(w, x_n)
\]
\[
\leq \sum_{w \in \mathbb{Z}_+^d \setminus \mathbb{Z}_+^d: |w| \geq \delta |x_n|} C^2 \exp \left( -a(z - x) \cdot (z - x) - a(x_n - z) \cdot (x_n - z) \right)
\]
and recall that by Proposition 6.1,
\[
\liminf_{n \to \infty} \frac{1}{|x_n|} \log G(x, x_n) = 0.
\]
To prove Proposition 8.1 it is therefore sufficient to show that
\[
(8.9) \limsup_{n \to \infty} \frac{1}{|x_n|} \log \left( \sum_{z \in \mathbb{Z}_+^d \setminus \mathbb{Z}_+^d: |z| \geq \delta |x_n|} \exp \left( -a(z - x) \cdot (z - x) - a(x_n - z) \cdot (x_n - z) \right) \right) < 0.
\]
For this we use Lemma 8.2 with \( R > 0 \) large enough, Lemma 8.3 with a compact set
\[
V = \{ q \in \mathbb{R}^d : \delta \leq |q| \leq R \text{ and } q^i \leq 0 \text{ for all } i \in \Lambda^c(M) \}
\]
and Lemma 1.2.15 of Dembo and Zeitouni [5]. Indeed, denote for \( 0 < \delta < R \),
\[
\Sigma_{\delta, R}^1(x, x_n) \doteq \sum_{z \in \mathbb{Z}_+^d \setminus \mathbb{Z}_+^d: \delta |x_n| \leq |z| \leq R |x_n|} \exp \left( -a(z - x) \cdot (z - x) - a(x_n - z) \cdot (x_n - z) \right)
\]
and let
\[
\Sigma_{R}^2(x, x_n) \doteq \sum_{z \in \mathbb{Z}_+^d: |z| > R |x_n|} \exp \left( -a(z - x) \cdot (z - x) - a(x_n - z) \cdot (x_n - z) \right).
\]
Then by Lemma 1.2.15 of Dembo and Zeitouni [5], the left hand side of (8.9) is equal to
\[
\limsup_{n \to \infty} \frac{1}{|x_n|} \log \left( \Sigma_{\delta, R}^1(x, x_n) + \Sigma_{R}^2(x, x_n) \right)
\]
\[
= \max \left\{ \limsup_{n \to \infty} \frac{1}{|x_n|} \log \Sigma_{\delta, R}^1(x, x_n), \limsup_{n \to \infty} \frac{1}{|x_n|} \log \Sigma_{R}^2(x, x_n) \right\},
\]
where by Lemma 8.3, for any \( 0 < \delta < R \),
\[
\limsup_{n \to \infty} \frac{1}{|x_n|} \log \Sigma_{\delta, R}^2(x, x_n) < 0
\]
and by Lemma 8.2
\[
\limsup_{n \to \infty} \frac{1}{|x_n|} \log \Sigma_{R}^2(x, x_n) < 0
\]
if \( R > 0 \) is large enough. Using these relations with \( R > 0 \) large enough and \( 0 < \delta < R \) one gets therefore (8.9).
8.2. Generating functions of hitting probabilities.

**Proposition 8.2.** Suppose that the conditions (A1)-(A3) are satisfied, the coordinates of the random walk are non-negative and let $M = M(M)$. Then there is $\varepsilon_0 > 0$ such that for any $0 < \varepsilon < \varepsilon_0$,

$$E_x(|S(\tau)|), \tau < \tau_A < \infty, \quad \forall x \in \mathbb{Z}_+^d,$$

and for any $x^A \in \mathbb{Z}_+^d$,

$$E_x(|S^A(\tau)|), \tau < \tau_A \to 0 \quad \text{as} \quad \min_{i \in \Lambda} |x^i| \to \infty.$$

The proof of this proposition uses the following lemmas

**Lemma 8.4.** Under the hypotheses (A1)-(A3), for any $a \in D$ and $x \in \mathbb{Z}_+^d$,

$$E_x(x \cdot S(\tau)), \tau < \tau_A \leq \exp(a \cdot x)$$

**Proof.** To get this inequality it is sufficient to notice that for any $a \in D$, the exponential function $a \rightarrow \exp(a \cdot x)$ is super-harmonic for the Random walk $(S(t))$ and the quantity

$$E_x(x \cdot S(\tau)), \tau < \tau_A \times \exp(-a \cdot x)$$

is equal to the probability that the twisted substochastic random walk $(\tilde{S}(t))$ having transition probabilities $\tilde{p}(x, x') = p(x, x') \exp(a \cdot (a' - a))$ exits from $\mathbb{Z}_+^d$ before the first time when at least one of its coordinates $(S^i(\tau))$ with $i \in \Lambda$ becomes negative or zero.

Let $e_i = (e_i^1, \cdots, e_i^d)$ denote the unit vector in $\mathbb{R}^d$ with $e_i^j = 1$ and $e_i^j = 0$ for $j \neq i$.

**Lemma 8.5.** Under the hypotheses of Proposition 8.2, for any $\Lambda' \subset \{1, \ldots, d\}$ such that $\Lambda(M) \subset \Lambda' \neq \{1, \ldots, d\}$, there are $\delta > 0$ and $\sigma > 0$ for which the points

$$\tilde{a}_{\Lambda'} = -\delta \sum_{i \in \Lambda'} M^i e_i + \sigma \sum_{i \in \Lambda(M)} e_i$$

and

$$\tilde{a}_{\Lambda'} = -\delta \sum_{i \in \Lambda'} M^i e_i + \sigma \sum_{i \in \Lambda'} e_i$$

belong to the set $D$.

**Proof.** Indeed, recall that $\Lambda(M) \doteq \{i \in \{1, \ldots, d\} : M^i = 0\}$. Hence, for $i \notin \Lambda(M)$, under the hypotheses of Proposition 8.2,

$$M^i \doteq \left(\nabla \varphi(0)\right)^i \doteq \frac{\partial}{\partial a^i} \varphi(a)\bigg|_{a=0} > 0$$

and consequently, for some $\delta > 0$ small enough and any $\Lambda' \subset \{1, \ldots, d\}$ such that $\Lambda(M) \subset \Lambda' \neq \{1, \ldots, d\}$,

$$\varphi\left(-\delta \sum_{i \in \Lambda'} M^i e_i\right) < \varphi(0) = 1$$

The point

$$a_{\Lambda'} \doteq -\delta \sum_{i \in \Lambda'} M^i e_i$$

belongs therefore to the interior of the set $D$ and consequently there is $\sigma > 0$ for which the points

$$\tilde{a}_{\Lambda'} = a_{\Lambda'} + \sigma \sum_{i \in \Lambda(M)} e_i$$

and

$$\tilde{a}_{\Lambda'} = a_{\Lambda'} + \sigma \sum_{i \in \Lambda'} e_i$$

also belong to the interior of the set $D$. \qed
Proof of Proposition \[8.2\] Recall that \( \tau_i \) denotes the first time when the \( i \)-th coordinate of the random walk \( (S(t)) \) becomes negative or zero,

\[
\tau \doteq \min_{i=1,\ldots,d} \tau_i, \quad \text{and} \quad \tau_{\Lambda} \doteq \min_{i \in \Lambda} \tau_i
\]

Hence, on the event \( \{ \tau < \tau_{\Lambda} \} \), one has \( S^i(\tau) > 0 \) for all \( i \in \Lambda \) and \( S^i(\tau) \leq 0 \) for some \( i \notin \Lambda \). Letting \( \Lambda_+(x) = \{ i \in \{1,\ldots,d \} : x^i > 0 \} \), we get therefore

\[
\mathbb{E}_x(\exp(\varepsilon|S(\tau)|), \tau < \tau_{\Lambda}) = \sum_{\begin{smallmatrix} \Lambda' \subseteq \Lambda \\ \Lambda' \neq \{1,\ldots,d \} \end{smallmatrix}} \mathbb{E}_x(\exp(\varepsilon|S(\tau)|), \Lambda_+(S(\tau)) = \Lambda', \tau < \tau_{\Lambda}).
\]

Furthermore, by Lemma \[8.5\] for \( \Lambda = \Lambda(M) \) and any \( \Lambda' \subset \{1,\ldots,d \} \) such that \( \Lambda \subset \Lambda' \neq \{1,\ldots,d \} \) there are \( \delta > 0 \) and \( \sigma > 0 \) for which

\[
\hat{a}_{\Lambda'} = -\delta \sum_{i \in \Lambda'} M^i e_i + \sigma \sum_{i \in \Lambda'} e_i \in D.
\]

For such a point \( \hat{a}_{\Lambda'} = (\hat{a}_1^\Lambda', \ldots, \hat{a}_d^\Lambda') \), on the event \( \{ \Lambda_+(S(\tau)) = \Lambda' \} \),

\[
\hat{a}_{\Lambda'} \cdot S(\tau) = -\delta \sum_{i \in \Lambda'} M^i S^i(\tau) + \sigma \sum_{i \in \Lambda'} S^i(\tau)
\]

\[
= \delta \sum_{i \in \Lambda''} M^i |S^i(\tau)| + \sigma \sum_{i \in \Lambda'} |S^i(\tau)| \geq |S(\tau)| \min_{\sigma, i \in \Lambda'} M^i.
\]

Using this inequality at the right hand side of \[8.10\] with

\[
0 < \varepsilon < \min \{ \sigma, \delta \min_{i \in \Lambda'} M^i \}
\]

we obtain

\[
\mathbb{E}_x(\exp(\varepsilon|S(\tau)|), \tau < \tau_{\Lambda}) \leq \sum_{\begin{smallmatrix} \Lambda' \subseteq \Lambda \\ \Lambda' \neq \{1,\ldots,d \} \end{smallmatrix}} \mathbb{E}_x(\exp(\hat{a}_{\Lambda'} \cdot S(\tau)), \Lambda_+(S(\tau)) = \Lambda', \tau < \tau_{\Lambda})
\]

where by Lemma \[8.4\]

\[
\mathbb{E}_x(\exp(\hat{a}_{\Lambda'} \cdot S(\tau)), \Lambda_+(S(\tau)) = \Lambda', \tau < \tau_{\Lambda}) \leq \exp(\hat{a}_{\Lambda'} \cdot x)
\]

and consequently,

\[
\mathbb{E}_x(\exp(\varepsilon|S(\tau)|), \tau < \tau_{\Lambda}) \leq \sum_{\begin{smallmatrix} \Lambda' \subseteq \Lambda \\ \Lambda' \neq \{1,\ldots,d \} \end{smallmatrix}} \exp(\hat{a}_{\Lambda'} \cdot x) < \infty.
\]

The first assertion of Proposition \[8.2\] is therefore proved. To prove the second assertion of this proposition we use again Lemmas \[8.4\] and \[8.5\] but with the points

\[
\tilde{a}_{\Lambda'} = -\delta \sum_{i \in \Lambda''} M^i e_i + \sigma \sum_{i \in \Lambda(M)} e_i
\]

The same arguments as above shows that on the event \( \{ \Lambda \subset \Lambda_+(S(\tau)) = \Lambda' \} \),

\[
\tilde{a}_{\Lambda'} \cdot S(\tau) = -\delta \sum_{i \in \Lambda''} M^i S^i(\tau) + \sigma \sum_{i \in \Lambda} S^i(\tau)
\]

\[
= \delta \sum_{i \in \Lambda''} M^i |S^i(\tau)| + \sigma \sum_{i \in \Lambda} |S^i(\tau)| \geq \sigma \sum_{i \in \Lambda} |S^i(\tau)| = \sigma |S^\Lambda(\tau)|
\]
and consequently, for $0 < \varepsilon \leq \sigma$,
\[
E_x(\exp(\varepsilon |S^\Lambda(\tau)|), \tau < \tau_\Lambda) \leq \sum_{\Lambda' : \Lambda \subseteq \Lambda', \Lambda' \neq \{1, \ldots, d\}} E_x(\exp(\tilde{a}_{\Lambda'} \cdot S(\tau)), \Lambda_+(S(\tau)) = \Lambda', \tau < \tau_\Lambda)
\leq \sum_{\Lambda' : \Lambda \subseteq \Lambda', \Lambda' \neq \{1, \ldots, d\}} \exp(\tilde{a}_{\Lambda'} \cdot x) < \infty.
\]

Since for any $x^\Lambda \in \mathbb{Z}_+^d$, and $\Lambda' \subset \{1, \ldots, d\}$ such that $\Lambda \subset \Lambda' \neq \{1, \ldots, d\}$,
\[
\tilde{a}_{\Lambda'} \cdot x = -\delta \sum_{i \in \Lambda^c} M^i x^i + \sigma \sum_{i \in \Lambda} x^i \to -\infty \quad \text{as} \quad \min_{i \in \Lambda^c} x^i \to \infty,
\]
the last inequality proves the second assertion of Proposition 8.2. \hfill \Box

### 8.3. Harmonic functions

When combined with Corollary 5.11, Proposition 8.2 implies the following statement.

**Proposition 8.3.** Suppose that the conditions (A1)-(A3) are satisfied, the coordinates of the vector $M$ are non-negative and let $\Lambda = \Lambda(M)$. Then for any harmonic function $f > 0$ of the induced Markov chain $(X_M(t))$, the function
\[
(8.11) \quad h(x) = f(x^\Lambda) - E_x(f(S^\Lambda(\tau)), \tau < \tau_\Lambda)
\]
is finite, strictly positive and harmonic for the Markov chain $(Z(t))$.

**Proof.** If $f > 0$ is a harmonic function of the induced Markov chain $(X_M(t))$ then the function $x \to f(x^\Lambda)$ is harmonic for the local Markov process $Z_\Lambda(t) = Z_{\Lambda(M)}(t) = (X_M(t), Y_M(t))$ because according to the definition of the induced Markov chain $(X_M(t))$ (see Section 2),
\[
X_M(t) = Z_\Lambda^\Lambda(t), \quad \forall t \geq 0.
\]

Since the local Markov chain $Z_\Lambda(t)$ is identical to the random walk $S(t)$ for $t < \tau_\Lambda$ and is killed upon the time $\tau_\Lambda$, from this it follows that
\[
f(x^\Lambda) \geq E_x(f(Z_\Lambda^\Lambda(\tau_{\Lambda'}))), \quad \tau_{\Lambda'} < \tau_\Lambda = E_x(f(S^\Lambda(\tau)), \tau < \tau_\Lambda)
\]
where the last equality holds because $\tau = \min\{\tau_\Lambda, \tau_{\Lambda'}\}$. The function $(8.11)$ is therefore finite and non-negative. Furthermore, for any $x \in \mathbb{Z}_+^d$, using again the fact that the function $x \to f(x^\Lambda)$ is harmonic for $(Z_\Lambda(t))$, one gets
\[
f(x^\Lambda) = E_x(f(Z_\Lambda^\Lambda(1))) = E_x(f(Z_\Lambda^\Lambda(1)), \tau = 1) + E_x(f(Z_\Lambda^\Lambda(1)), \tau > 1) = E_x(f(S^\Lambda(1)), \tau = 1 < \tau_\Lambda) + E_x(f(S^\Lambda(1)), \tau > 1).
\]

Since moreover,
\[
E_x(f(S^\Lambda(\tau)), \tau < \tau_\Lambda) = E_x(f(S^\Lambda(\tau)), \tau = 1 < \tau_\Lambda) + E_x(f(S^\Lambda(\tau)), 1 < \tau < \tau_\Lambda)
\]
then for any $x \in \mathbb{Z}_+^d$, 
\[ h(x) = f(x^\Lambda) - \mathbb{E}_x(f(S^\Lambda(\tau)), \tau < \tau_A) \]
\[ = \mathbb{E}_x(f(S^\Lambda(1)), \tau > 1) - \mathbb{E}_x(f(S^\Lambda(\tau)), 1 < \tau < \tau_A) \]
\[ = \sum_{w \in \mathbb{Z}_+^d} \mu(w - x)f(w^\Lambda) - \sum_{w \in \mathbb{Z}_+^d} \mu(w - x)\mathbb{E}_w(f(S^\Lambda(\tau)), \tau < \tau_A) \]
\[ = \sum_{w \in \mathbb{Z}_+^d} \mu(w - x)h(w) \]

and consequently, the function $h$ is harmonic for the Markov chain $(Z(t))$. Now, to complete the proof of this proposition we have to show that the function $h$ is strictly positive on $\mathbb{Z}_+^d$. For this we use Propositions 8.2 and Corollary 5.1. Recall that by Corollary 5.1,
\[ \limsup_{|u| \to \infty} \frac{1}{|u|} \log f(u) \leq 0. \]
Hence, for any $\varepsilon > 0$ there is $C > 0$ such that 
\[ 0 < f(u) \leq C \exp(|\varepsilon| |u|), \quad \forall u \in \mathbb{Z}_+^d \]
and consequently, for any $x \in \mathbb{Z}_+^d$,
\[ h(x) \geq f(x^\Lambda) - C \mathbb{E}_x(\exp(\varepsilon |S^\Lambda(\tau)|)), \tau < \tau_A). \]

Since by Proposition 8.2 for $\varepsilon > 0$ small enough and any $x^\Lambda \in \mathbb{Z}_+^d$,
\[ \mathbb{E}_x(\exp(\varepsilon |S^\Lambda(\tau)|)), \tau < \tau_A) \to 0 \quad \text{as} \quad \min_{i \in \Lambda} x^i \to +\infty, \]

from this it follows that there is $\tilde{x} \in \mathbb{Z}_+^d$ (with a large $\min_{i \in \Lambda} \tilde{x}^i$) for which $h(\tilde{x}) > 0$. Using finally the Harnack inequality
\[ h(x) \geq h(\tilde{x}) \mathbb{P}_x(Z(t) = \tilde{x} \quad \text{for some} \quad t \geq 0) \]
and the fact that under the hypotheses (A2), the Markov chain $(Z(t))$ is irreducible on $\mathbb{Z}_+^d$, we conclude that $h(x) > 0$ for all $x \in \mathbb{Z}_+^d$. \(\square\)

8.4. Proof of Theorem 1. We are ready now to complete the proof of Theorem 1. Indeed, suppose that the conditions (A1)-(A3) are satisfied and the coordinates of the mean vector $M$ are non-negative. Then by Proposition 8.3 for any harmonic function $f > 0$ of the induced Markov chain $(X_M(t))$, the function
\[ h(x) = f(x^\Lambda) - \mathbb{E}_x(f(S^\Lambda(\tau)), \tau < \tau_A) \]
with $\Lambda = \Lambda(M) \doteq \{i : M^i = 0\}$ is finite, strictly positive and harmonic for the Markov chain $(Z(t))$. Hence, the first assertion of Theorem 1 is already proved.

Suppose now that the sequence of points $z_n \in \mathbb{Z}_+^d$ with $\lim_{n \to \infty} |z_n| = \infty$ and $\lim_{n \to \infty} z_n/|z_n| = M/|M|$ is fundamental for the local random walk $(Z_\Lambda(t))$ with $\Lambda = \Lambda(M) \doteq \{i : M^i = 0\}$. Then by Proposition 7.1 there is a harmonic function $f > 0$ of the induce Markov chain $(X_M(t))$ such that
\[ \lim_{n \to \infty} G_\Lambda(x, z_n)/G_\Lambda(x_0, z_n) = f(x^\Lambda)/f(x_0^\Lambda), \quad \forall x \in \mathbb{Z}_+^d \]

\[ \text{Corollary 5.1} \]
Whenever the last limit exists. By Proposition 6.1, for any $n \in \mathbb{Z}_+^d$,
\begin{equation}
(8.14) \quad \lim_{n \to \infty} G(x, z_n)/G(x_0, z_n) = \frac{1}{f(x_0^M)} \left( f(x^M) - \mathbb{E}_x \left( f(S^M(\tau), \tau < \tau_N) \right) \right).
\end{equation}

By Proposition 8.1, the right hand side of the above renewal equation can be decomposed into a main part
\begin{equation}
\Xi_{\delta,N}(x, z_n) \triangleq G(x, z_n) - \mathbb{E}_x \left( G(S^M(\tau), z_n), \tau < \tau_N, |S(\tau)| < \delta |z_n| \right)
\end{equation}
and the corresponding negligible part $\Xi_{\delta,N}(x, z_n) - G(x, z_n)$ so that for any $\delta > 0$ and $x \in \mathbb{Z}_+^d$,
\begin{equation}
\lim_{n \to \infty} \Xi_{\delta,N}(x, z_n)/G(x, z_n) = 1.
\end{equation}

From this it follows that
\begin{equation}
\lim_{n \to \infty} G(x, z_n)/G(x_0, z_n) = \lim_{n \to \infty} \Xi_{\delta,N}(x, z_n)/G(x_0, z_n)
\end{equation}
whenever the last limit exists. By Proposition 6.1 for any $\varepsilon > 0$ there are $N > 0$, $C > 0$ and $\delta > 0$ such that for any $n \geq N$,
\begin{equation}
(8.15) \quad 1_{|w| < \delta |z_n|} G(w, z_n)/G(x_0, z_n) \leq C \exp(\varepsilon |w|), \quad \forall w \in \mathbb{Z}_+^d.
\end{equation}

Since by Proposition 8.3 for $\varepsilon > 0$ small enough,
\begin{equation}
\mathbb{E}_x(\exp(\varepsilon |S(\tau)|), \tau < \tau_N) < \infty,
\end{equation}
using dominated convergence theorem from (8.12) and (8.15) it follows that for some $\delta > 0$, the limit at the right hand side of (8.14) exists and is equal to the right hand side of (8.13). Relation (8.13) is therefore proved. Since by Proposition 8.3 the right hand side of (8.13) is non-zero, we conclude that
\begin{equation}
\lim_{n \to \infty} G(x, z_n)/G(x_0, z_n) = \frac{f(x^M) - \mathbb{E}_x \left( f(S^M(\tau), \tau < \tau_N) \right)}{f(x_0^M) - \mathbb{E}_x \left( f(S^M(\tau), \tau < \tau_N) \right)}, \quad \forall x \in \mathbb{Z}_+^d.
\end{equation}

Any fundamental sequence $z_n \in \mathbb{Z}_+^d$ of the local random walk $(Z_{(M)}(t))$, with $\lim_n |z_n| = \infty$ and $\lim_n z_n/|z_n| = M/|M|$, is therefore fundamental for the random walk $(Z(t))$.

Consider now a sequence $x_n \in \mathbb{Z}_+^d$ with $\lim_n |x_n| = \infty$ and $\lim_n x_n/|x_n| = M/|M|$ which is fundamental for the random walk $(Z(t))$ and let
\begin{equation}
(8.17) \quad h(x) = \lim_{n \to \infty} \frac{G(x, x_n)}{G(x_0, x_n)}, \quad \forall x \in \mathbb{Z}_+^d.
\end{equation}

Then by compactness, there is a subsequence $(x_{n_k})$ which is also fundamental for the local random walk $(Z_{(M)}(t))$ and consequently, there exist a harmonic function $f > 0$ of the induced Markov chain $(X_M(t))$ such that
\begin{equation}
\lim_{k \to \infty} G(x, x_{n_k})/G(x_0, x_{n_k}) = \frac{f(x^M) - \mathbb{E}_x \left( f(S^M(\tau), \tau < \tau_N) \right)}{f(x_0^M) - \mathbb{E}_x \left( f(S^M(\tau), \tau < \tau_N) \right)}, \quad \forall x \in \mathbb{Z}_+^d
\end{equation}
(we use here (8.10) with $z_k = x_{n_k}$). Comparison of the last relation with (8.17) shows that
\begin{equation}
(8.18) \quad h(x) = \frac{f(x^M) - \mathbb{E}_x \left( f(S^M(\tau), \tau < \tau_N) \right)}{f(x_0^M) - \mathbb{E}_x \left( f(S^M(\tau), \tau < \tau_N) \right)}, \quad \forall x \in \mathbb{Z}_+^d.
\end{equation}
and consequently,
\[
(8.18) \quad \lim_{n \to \infty} \frac{G(x, x_n)}{G(x_0, x_n)} = \frac{f(x^A) - E_x(f(S^A(\tau), \tau < \tau_\Lambda))}{f(x_n^A) - E_{x_n}(f(S^A(\tau), \tau < \tau_\Lambda))}, \quad \forall x \in \mathbb{Z}_d^d.
\]

The second assertion of Theorem 1 is therefore also proved.

Suppose finally that a harmonic function \( f > 0 \) of the induced Markov chain \((X_M(t))\) is unique to constant multiples and let a sequence \( x_n \in \mathbb{Z}_d^d \) be such that \( \lim_n |x_n| = \infty \) and \( \lim_n x_n/|x_n| = M/M \). Then for any subsequence \( z_k = x_{n_k} \) which is fundamental for the Markov chain \((Z(t))\), one has
\[
\lim_{n \to \infty} \frac{G(x, x_{n_k})}{G(x_0, x_{n_k})} = \frac{f(x^A) - E_x(f(S^A(\tau), \tau < \tau_\Lambda))}{f(x_{n_k}^A) - E_{x_{n_k}}(f(S^A(\tau), \tau < \tau_\Lambda))}, \quad \forall x \in \mathbb{Z}_d^d.
\]
with the same function \( f \). By compactness, from this it follows that the sequence \((x_n)\) is fundamental itself and satisfies (8.18). Theorem 1 is therefore proved.

9. Proofs of Proposition 1.1 and Proposition 1.2

Remark first of all that Proposition 1.1 is a particular case of Proposition 1.2 when the set \( \Lambda(M) = \{ i : M^i = 0 \} \) contains only one point. The proof of Proposition 1.1 is therefore the same and even simpler than the proof of Proposition 1.2. To prove Proposition 1.2 we use the results of of Picardello and Woess [1]. Before formulating these results we recall some useful properties of minimal \( t \)-harmonic functions and the convergence norm of transition kernels.

9.1. Convergence norm and \( t \)-harmonic functions.

**Definition 9.1.** Let \( P = (p(x, x'), x, x' \in E) \) be a transition kernel of a time-homogeneous, irreducible Markov chains \( \xi = (\xi(t)) \) on a countable, discrete state spaces \( E \).

1. For \( t > 0 \), a positive function \( f : E \to \mathbb{R}_+ \) is said to be \( t \)-harmonic for \( P \) if it satisfies the equality \( Pf = tf \) (i.e. if it is an eigenvector of the transition operator \( P \) with respect to the eigenvalue \( t \)).
2. A \( t \)-harmonic function \( f > 0 \) is said to be minimal if for any \( t \)-harmonic function \( \tilde{f} > 0 \) the inequality \( \tilde{f} \leq f \) implies the equality \( \tilde{f} = cf \) with some \( c > 0 \).
3. The convergence norm \( \rho(P) \) of \( P \) is defined by
\[
\rho(P) = \lim_{n \to \infty} \sup_{x} \left( P_x(\xi(n) = x') \right)^{1/n}.
\]

By irreducibility, \( \rho(P) \) does not depend on \( x, x' \in E \) (see Seneta [17]).

By Perron-Frobenius theorem (see [17]), for finite state space \( E \), the quantity \( \rho(P) \) is equal to the maximal real eigenvalue of the matrix \( P \). When the state space \( E \) is infinite (and countable), Theorem 6.3 of Seneta [17] gives another equivalent representation of the convergence norm of an irreducible transition kernel \( P \):
\[
(9.1) \quad \rho(P) = \sup_{K \subseteq E} \rho_K(P)
\]
where the supremum is taken over all finite subsets \( K \subseteq E \), and for any finite set \( K \subseteq E \), \( \rho_K(P) \) is the maximal real eigenvalue of the truncated transition matrix \( (p(x, x'), x, x' \in K) \).
Recall finally that for \( t > 0 \), the set of \( t \)-harmonic functions of an irreducible Markov kernel \( P \) on a countable state space \( E \) is nonvoid only if \( t \geq \rho(P) \), see Pruitt [14]. For \( t = 1 \), the \( t \)-harmonic functions are called harmonic.

Consider now a probability measure \( \nu \) on \( \mathbb{Z} \), let \( (\xi(t)) \) be a random walk on \( \mathbb{Z} \) with transition probabilities \( p(k, k') = \nu(k' - k) \) and let \( T \) denote the first time when the random walk \( (\xi(t)) \) becomes negative or zero:

\[
T = \inf\{ n \geq 0 : \xi(n) \leq 0 \}.
\]

To prove Propositions 1.1 and 1.2 we need to identify the convergence norm and the harmonic functions of the substochastic Markov kernel

\[
P_+ = (p(k, k') = \nu(k' - k), \, k, k' > 0)
\]

on \( \mathbb{Z}_+ = \{ k \in \mathbb{Z} : k > 0 \} \). The assumptions we need on the measure \( \nu \) are the following

(B1) the substochastic matrix \( P_+ = (p(k, k') = \nu(k' - k), \, k, k' > 0) \) is irreducible,

(B2) \( \sum_{k \in \mathbb{Z}} \nu(k) = 0 \) and

(B3) for some \( \varepsilon > 0 \),

\[
\sum_{k \in \mathbb{Z}} \nu(k) \exp(\varepsilon|k|) < \infty.
\]

**Proposition 9.1.** Under the hypotheses (B1)-(B3), \( \rho(P_+) = 1 \) and the only positive harmonic functions of \( P_+ \) are the constant multiples of

\[
f(k) = k - \mathbb{E}_k(\xi(T)), \quad k \in \mathbb{Z}_+.
\]

**Proof.** The inequality \( \rho(P_+) \leq 1 \) is clearly satisfied because the matrix \( P_+ \) is substochastic. To prove that \( \rho(P_+) \geq 1 \) we consider a transition kernel

\[
P = \nu(k' - k), \, k, k' \in \mathbb{Z}
\]

of the homogeneous random walk \( (\xi(t)) \). Under the hypotheses (H1)-(H2), this is an irreducible random walk on \( \mathbb{Z} \) with zero mean and a finite variance. The random walk \( (\xi(t)) \) is therefore recurrent and consequently, \( \rho(P) = 1 \). Moreover, using (9.1) one gets

\[
\rho(P) = \sup_{K \subset \mathbb{Z}} \rho_K(P) \quad \text{and} \quad \rho(P_+) = \sup_{K \subset \mathbb{Z}_+} \rho_K(P)
\]

where the supremums are taken over finite sets \( K \) and \( \rho_K(P) \) is the maximal real eigenvalue of the truncated matrix \( (\nu(k' - k), \, k, k' \in K) \). Since the components of the matrix \( P \) are invariant with respect to the translations on \( k \in \mathbb{Z} \),

\[
\rho_K(P) = \rho_{k+K}(P), \quad \forall k \in \mathbb{Z}
\]

for any finite set \( K \subset \mathbb{Z} \). Hence, the right hand sides of the equalities (9.2) are equal to each other and consequently, \( \rho(P_+) = \rho(P) = 1 \). The first assertion of Proposition 9.1 is therefore proved.

When \( \nu(0) > 0 \), the second assertion follows from Theorem 1 of Doney [6] (see also Example E 27.3 in Chapter VI of Spitzer [18]) and is proved in Lemma 5.3 of the paper [9]. To prove the second assertion for a probability measure \( \nu \) with \( \nu(0) = 0 \), it is sufficient to notice that the transition kernel \( P_+ \) has the same harmonic functions as the modified transition kernel \( \tilde{P}_+ = (\tilde{p}(k, k') = \tilde{\nu}(k' - k), \, k, k' \in \mathbb{Z}_+) \) with

\[
\tilde{\nu}(k) = \begin{cases} (1 - \theta)\nu(k) & \text{if } k \neq 0, \\ \theta & \text{for } k = 0, \end{cases}
\]

where \( 0 < \theta < 1 \).
and that the modified random walk \((\tilde{\xi}(t))\) with transition probabilities \(\tilde{p}(k, k') = \tilde{v}(k' - k)\) and \(\tilde{T} = \inf\{n \geq 0 : \xi(t) \leq 0\}\) satisfy the equality \(E_k(\tilde{\xi}(\tilde{T})) = E_k(\xi(T))\) for all \(k \in \mathbb{Z}_+\).

9.2. Cartesian products of Markov chains. Let \(P = (P(u, u'), u, u' \in E_1)\) and \(Q = (Q(u, u'), u, u' \in E_2)\) be two transition kernels of two time-homogeneous Markov chains \(\xi(t)\) and \(\eta(t)\) on countable, discrete state spaces \(E_1\) and \(E_2\) respectively. A Markov chain on the Cartesian product \(E_1 \times E_2\) having transition kernel \(R_a = \alpha P \otimes 1_{E_2} + (1 - \alpha) 1_{E_1} \otimes Q\) is called a Cartesian product of Markov chains \(\xi\) and \(\eta\). Transition probabilities of such a Markov chain are given by

\[
R_a((u^1, u^2), (v^1, v^2)) = \begin{cases} 
\alpha P(u^1, v^1) & \text{if } u^2 = v^2, \\
(1 - \alpha) Q(u^2, v^2) & \text{if } u^1 = v^1, \\
0 & \text{otherwise.}
\end{cases}
\]

The kernel \(R_a = (R_a(u, v), u, v \in E \times E_2)\) is a Cartesian product of transition kernels \(P\) and \(Q\). By Lemma 3.1 of Picardello and Woess [1], the convergence norm of the Cartesian product \(R_a\) is related to those of \(P\) and \(Q\) as follows

\[
\rho(R_a) = \alpha \rho(P) + (1 - \alpha) \rho(Q)
\]

and it is clear that for any \(r \geq \rho(P)\) and \(s \geq \rho(Q)\), if \(f > 0\) is a \(r\)-harmonic function of \(P\) and \(g > 0\) is a \(s\)-harmonic function of \(Q\) then the function

\[
h(u^1, u^2) = f \otimes g \ (u^1, u^2) \equiv f(u^1)g(u^2), \quad (u^1, u^2) \in E_1 \times E_2
\]

is a \(t\)-harmonic function of \(R_a\) with \(t = \alpha r + (1 - \alpha)s\). Conversely, for minimal \(t\)-harmonic functions of the Cartesian product \(R_a\), Theorem 3.2 of Picardello and Woess [1] proves the following property.

**Theorem [Picardello and Woess [1]].** If the transition kernels \(P\) and \(Q\) are stochastic and irreducible respectively on \(E_1\) and \(E_2\), then for any \(0 < a < 1\) and \(t \geq \rho(R_a)\), every minimal \(t\)-harmonic function \(h > 0\) of the Cartesian product \(R_a\) is of the form \(f \otimes g\) with some minimal \(r\)-harmonic function \(f > 0\) of \(P\), a minimal \(s\)-harmonic function \(g > 0\) of \(Q\) and \(r \geq \rho(P)\), \(s \geq \rho(Q)\) satisfying the equality \(ar + (1 - a)s = t\).

In a particular case, when \(\rho(P) = \rho(Q) = 1\), for minimal harmonic (i.e. \(t\)-harmonic with \(t = 1\)) functions, this result implies the following statement.

**Corollary 9.1.** Suppose that the transition kernels \(P\) and \(Q\) are stochastic and irreducible respectively on \(E_1\) and \(E_2\) and let \(\rho(P) = \rho(Q) = 1\). Then for any \(0 < a < 1\), every minimal harmonic function \(h > 0\) of \(R_a\) is of the form \(f \otimes g\) with some minimal harmonic functions \(f > 0\) and \(g > 0\) of \(P\) and \(Q\) respectively.

For sub-stochastic transition kernels, one gets therefore

**Proposition 9.2.** Suppose that the sub-stochastic transition kernels \(P\) and \(Q\) are irreducible respectively on \(E_1\) and \(E_2\) and let \(\rho(P) = \rho(Q) = 1\). Suppose moreover that every positive harmonic function of \(P\) is a constant multiple of \(f > 0\) and every positive harmonic function of \(Q\) is a constant multiple of \(g > 0\). Then for any \(0 < a < 1\), every positive harmonic function of \(R_a\) is a constant multiple of the function \(f \otimes g\).
Proof. It is sufficient to apply the above statement for twisted transition kernels \( \tilde{P} = (\tilde{P}(u, u'), u, u' \in \mathbb{E}_1) \) and \( \tilde{Q} = (\tilde{Q}(u, u'), u, u' \in \mathbb{E}_2) \) defined by
\[
\tilde{P}(u, u') = P(u, u')f(u')/f(u), \quad \text{for } u, u' \in \mathbb{E}_1
\]
and
\[
\tilde{Q}(u, u') = Q(u, u')g(u')/g(u), \quad \text{for } u, u' \in \mathbb{E}_2
\]
Under the hypotheses of Proposition 9.2 these transition kernels are clearly stochastic and irreducible, the only positive harmonic functions of \( \tilde{P} \) (resp. \( \tilde{Q} \)) are constant and
\[
\rho(\tilde{P}) = \rho(P) = \rho(Q) = \rho(\tilde{Q}) = 1.
\]
By Corollary 9.1 from this it follows that every minimal harmonic function of the Cartesian product \( \tilde{R}_a = a \tilde{P} \otimes 1_{\mathbb{E}_2} + (1 - a) 1_{\mathbb{E}_1} \times \tilde{Q} \) is also constant. To compete the proof of our proposition it is now sufficient to compare the transitions kernels \( \tilde{R}_u = (\tilde{R}_u(u, v), u, v \in E \times \mathbb{E}_2) \) and \( \tilde{R}_u = (\tilde{R}_u(u, v), u, v \in E \times \mathbb{E}_2) \). Since clearly
\[
\tilde{R}_u(u, v) = R_u(u, v) f \otimes g(v)/f \otimes g(u), \quad \forall u, v \in E_1 \times E_2,
\]
this proves that the only minimal harmonic functions (and consequently, also the only positive harmonic functions) of \( \tilde{R}_u \) are the constant multiples of \( f \otimes g \). \( \square \)

For \( a = (a_1, \ldots, a_m) \in [0, 1]^m \) with \( a_1 + \cdots + a_m = 1 \), a Cartesian product \( R_a \) of \( m \) transition kernels \( P_1, \ldots, P_m \) of time-homogeneous Markov chains \( \xi_1(t), \ldots, \xi_m(t) \) on countable, discrete state spaces \( E_1, \ldots, E_m \), is defined by
\[
R_a = a_1P_1 \otimes 1_{E_2} \otimes \cdots \otimes 1_{E_m} + a_21_{E_1} \otimes P_2 \otimes 1_{E_3} \otimes \cdots 1_{E_m} + \cdots + a_m1_{E_1} \otimes \cdots \otimes 1_{E_{m-1}} \otimes P_m.
\]
By induction with respect to \( m \) and using (9.2), from Proposition 9.2 it follows

Corollary 9.2. Suppose that the sub-stochastic transition kernels \( P_1, \ldots, P_m \) are irreducible respectively on \( E_1, \ldots, E_k \) and let \( \rho(P_i) = 1 \) for all \( i = 1, \ldots, m \). Suppose moreover that for any \( i = 1, \ldots, m \), every positive harmonic function of \( P_i \) is a constant multiple of \( f_i > 0 \). Then for any \( a = (a_1, \ldots, a_m) \in [0, 1]^m \) with \( a_1 + \cdots + a_m = 1 \), every positive harmonic function of \( R_a \) is a constant multiple of the function \( f_1 \otimes \cdots \otimes f_m \).

9.3. Application for a homogeneous random walk on \( \mathbb{Z}^m \). Let \( \xi(t) \) be a homogeneous random walk on \( \mathbb{Z}^m \) with transition probabilities
\[
P(u, u') = \nu(u' - u), \quad u, u' \in \mathbb{Z}^m
\]
and let \( \tau_i \) denote the first time when the \( i \)-th coordinate \( \xi^i(t) \) of \( \xi(t) \) becomes negative or zero. Suppose moreover that the probability measure \( \nu \) on \( \mathbb{Z}^m \) satisfies the following conditions.

(C1) For \( \mathbb{Z}_+^m = \{u \in \mathbb{Z}^m : u_i > 0, \ \forall i = 1, \ldots, m\} \) the sub-stochastic matrix \( P(u, u') = \nu(u' - u), \ u, u' \in \mathbb{Z}_+^m \) is irreducible.

(C2) The function
\[
\varphi_\xi(\alpha) = \sum_{u \in \mathbb{Z}^m} \nu(u) \exp(\alpha \cdot u)
\]
is finite in a neighborhood of zero in \( \mathbb{R}^m \).

(C3) \( \sum_{u \in \mathbb{Z}^m} u\nu(u) = 0 \).

(C4) \( \nu(u) = 0 \) if \( u_iu_j \neq 0 \) for some \( 1 \leq i < j \leq m \).
Under the hypotheses (C1)-(C4), for every $1 \leq i \leq m$, the $i$-th coordinate $(\xi^i(t))$ of $(\xi(t))$ is a recurrent random walk on $\mathbb{Z}$ and consequently, almost surely $\tau_i < \infty$. If the hypotheses (C1)-(C4) are satisfied and moreover \( \nu(0) = 0 \) then the transition kernel $P_+$ is a Cartesian product of $m$ irreducible transition kernels $P_i = (P_i(u, u')) = \nu_i(k' - k), k, k' \in \mathbb{Z}_+$:

\[
P_+ = a_1 P_1 \otimes 1_{\mathbb{Z}_+} \otimes \cdots \otimes 1_{\mathbb{Z}_+} + a_2 1_{\mathbb{Z}_+} \otimes P_2 \otimes 1_{\mathbb{Z}_+} \cdots + a_m 1_{\mathbb{Z}_+} \otimes \cdots \otimes 1_{\mathbb{Z}_+} \otimes P_m
\]

with

\[
a_i = \sum_{k \in \mathbb{Z}} \nu(ke_i)
\]

and

\[
\nu_i(k) = \frac{1}{a_i} \nu(ke_i), \quad k \in \mathbb{Z}
\]

where $e_i = (e_i^1, \ldots, e_i^m)$ denotes the unit vector in $\mathbb{Z}^m$ with $e_i^j = 1$ and $e_i^j = 0$ for $j \neq i$. Because of the assumptions (C1)-(C3), the probability measures $\nu_1, \ldots, \nu_m$ satisfy the conditions (B1)-(B3) of Proposition 9.1 and hence, using Corollary 9.2 one gets

**Corollary 9.3.** If the conditions (C1)-(C4) are satisfied and $\nu(0) = 0$ then the only positive harmonic functions of the transition kernel $P_+$ are the constant multiples of the function

\[
f(u) = \prod_{i=1}^m (u^i - \mathbb{E}_u(\eta_i(T_i)))
\]

where $(\eta_i(t))$ is a random walk on $\mathbb{Z}$ with transition probabilities $p_i(k, k') = \nu_i(k' - k)$ and $T_i = \inf\{n \geq 0 : \eta_i(n) \leq 0\}$.

The main result of this section is the following statement.

**Proposition 9.3.** Under the hypotheses (C1)-(C4), the only positive harmonic functions of the substochastic kernel $P_+$ are the constant multiples of the function

\[
f(u) = \prod_{i=1}^m u^i - \mathbb{E}_u \left( \prod_{i=1}^m \xi^i \left( \min_{1 \leq i \leq m} \tau_i \right) \right), \quad u \in \mathbb{Z}_+^m
\]

Corollary 9.3 proves this proposition when $\nu(0) = 0$ and the coordinates $(\xi^i(t))$ are lower-semicontinuous, i.e. if $\nu(u) = 0$ whenever $u^i < -1$ for some $1 \leq i \leq m$. Indeed, in this case, almost surely, $\xi^i(\tau_i) = \eta^i(T_i) = 0$ and consequently, the right hand sides of (9.8) and (9.9) are equal to $u^1 \times \cdots \times u^m$. To prove Proposition 9.3 in a general case, we need the following preliminary results. As above, for a given $\Lambda \subset \{1, \ldots, m\}$, we denote

\[
\tau_\Lambda = \min_{i \in \Lambda} \tau_i
\]

**Lemma 9.1.** Under the hypotheses (C1)-(C4), for any $\varepsilon > 0$ there is $\delta_\varepsilon > 0$ such that for any $0 < \delta < \delta_\varepsilon$ and any non-empty subset $\Lambda \subset \{1, \ldots, m\}$,

\[
\mathbb{E}_u \left( \exp \left( -\varepsilon \tau_\Lambda + \delta \sum_{i=1}^m |\xi^i(\tau_\Lambda)| \right) \right) < \infty.
\]
Proof. To prove this lemma we first notice that for any \(\delta > 0\),
\[
\exp \left( \delta \sum_{i=1}^{m} |\xi^i(\tau_\Lambda)| \right) \leq \sup_{\alpha = (\alpha^1, \ldots, \alpha^m) \in \mathbb{R}^m: |\alpha^i| = \delta, \forall i = 1, \ldots, m} \exp (\alpha \cdot \xi(\tau_\Lambda))
\]
\[
\leq \sum_{\alpha = (\alpha^1, \ldots, \alpha^m) \in \mathbb{R}^m: |\alpha^i| = \delta, \forall i = 1, \ldots, m} \exp (\alpha \cdot \xi(\tau_\Lambda))
\]
from which it follows that
\[
\mathbb{E}_u \left( \exp (-\varepsilon \tau_\Lambda + \delta \sum_{i=1}^{m} |\xi^i(\tau_\Lambda)|) \right) \leq \sum_{\alpha = (\alpha^1, \ldots, \alpha^m) \in \mathbb{R}^m: |\alpha^i| = \delta, \forall i = 1, \ldots, m} \mathbb{E}_u \left( \exp (-\varepsilon \tau_\Lambda + \alpha \cdot \xi(\tau_\Lambda)) \right)
\]
\[
\leq 2^m \sup_{\alpha \in \mathbb{R}^m: |\alpha| \leq m\delta} \mathbb{E}_u \left( \exp (-\varepsilon \tau_\Lambda + \alpha \cdot \xi(\tau_\Lambda)) \right)
\]
where
\[
\mathbb{E}_u \left( \exp (-\varepsilon \tau_\Lambda + \alpha \cdot \xi(\tau_\Lambda)) \right) = \sum_{n=0}^{\infty} \exp (-\varepsilon n) \mathbb{E}_u \left( \exp (\alpha \cdot \xi(n)) \right)
\]
\[
= \exp (\alpha \cdot u) \sum_{n=0}^{\infty} \exp (-\varepsilon n) \varphi_\xi(\alpha)^n = \frac{\exp (\alpha \cdot u)}{1 - \exp (-\varepsilon) \varphi_\xi(\alpha)}
\]
whenever \(\varphi_\xi(\alpha) < \exp(\varepsilon)\). Since the jump generating function \(\varphi_\xi\) is continuous and
\(\varphi_\xi(0) = 1\), from this it follows that the left hand side of (9.11) is finite whenever \(\delta > 0\) is small enough. Lemma [9.1] is therefore proved. \(\square\)

Lemma 9.2. Suppose that the conditions (C1)-(C4) are satisfied and let \(\nu(0) = 0\). Then for any \(u \in \mathbb{Z}_+^m\),
\[
(9.11) \quad \mathbb{E}_u \left( \xi^m(\tau_m) \prod_{1 \leq i \leq m-1} \xi^i \left( \min_{1 \leq i \leq m-1} \tau_i \right) \right)
\]
\[
= \mathbb{E}_u(\xi^m(\tau_m)) \times \mathbb{E}_u \left( \prod_{1 \leq i \leq m-1} \xi^i \left( \min_{1 \leq i \leq m-1} \tau_i \right) \right).
\]

Proof. To prove this lemma we consider a continuous-time version of the random walk \((\xi(t))\) defined by
\[
\dot{\xi}(t) = \xi(\mathcal{N}(t)), \quad t \in [0, +\infty[,
\]
where \(\mathcal{N}(t)\) is the Poisson process on \([0, +\infty[\) with rate 1. Because of the assumption (C4), the coordinates \((\dot{\xi}^1(t)), \ldots, (\dot{\xi}^m(t))\) of such a random walk \((\dot{\xi}(t))\) perform independent random walks on \(\mathbb{Z}\) for any \(t \geq 0\), \(\mathcal{N}(t) = \mathcal{N}_1(a_1 t) + \cdots + \mathcal{N}_m(a_m t)\) and
\[
\dot{\xi}^i(t) = \eta_i(\mathcal{N}_i(a_i t))
\]
and for any non-empty subset \( \Lambda \) equalities are well defined, because clearly, Remark first that by Lemma 9.1, the left and the right hand side s of these equalities are well defined, because clearly, \( E_1 \) (9.13) \( E \)

Lemma 9.3. Suppose that the conditions (C1)-(C4) are satisfied and let \( \nu(0) = 0 \). Then for any \( 1 \leq i \leq m \) and \( k \in \mathbb{Z}_+ \),

\[
E_k(\xi(\tau_i)) = E_k(\eta^i(T_i))
\]

and for any non-empty subset \( \Lambda \subset \{1, \ldots, m\} \), \( \varepsilon > 0 \) and \( u = (u^1, \ldots, u^m) \in \mathbb{Z}_+^m \),

\[
E_u\left(\exp(-\varepsilon \tau_\Lambda) \prod_{i=1}^m \xi_i^i(\tau_\Lambda)\right) = E_u\left(\exp(-\varepsilon \tau_\Lambda) \prod_{i \in \Lambda} \xi_i^i(\tau_\Lambda)\right) \times \prod_{i \notin \Lambda} u^i,
\]

Proof. Remark first that by Lemma 9.1, the left and the right hand sides of these equalities are well defined, because clearly,

\[
\left| \xi_i^i(\tau_\Lambda) \right| \leq \frac{1}{\delta} \exp(\delta \left| \xi_i^i(\tau_\Lambda) \right|), \quad \forall \delta > 0.
\]

Consider now a sequence of independent identically distributed random variables \( (\theta_n) \) taking the values in the set \( \{1, \ldots, m\} \) with \( \mathbb{P}(\theta_n = i) = a_i \) for all \( 1 \leq i \leq m \) with the quantities \( a_i \) defined by (9.6). Denote \( N_i(t) = 1_{\{\theta_i = i\}} + \cdots + 1_{\{\theta_i = i\}} \) and let \( (\eta_i(t), \ldots, \eta_m(t)) \) be independent random walks on \( \mathbb{Z} \) which are independent on the sequence \( (\theta_n) \) and have transition probabilities \( p_i(k, k') = \nu_i(k' - k) \) defined by (9.7) respectively for \( 1 \leq i \leq m \). Then our random walk \( \xi(t) = (\xi^1(t), \ldots, \xi^m(t)) \) can be represented in the following way : \( \xi^i(t) = \eta^i(N_i(t)) \) for any \( i \in \{1, \ldots, m\} \) and \( t \in \mathbb{N} \). According to this representation, the stopping times \( \tau_i = \inf\{n \geq 0 : \xi_t^i(n) \leq 0\} \) and \( T_i = \inf\{n \geq 0 : \eta^i(n) \leq 0\} \) are related as follows :

\[
T_i = N_i(\tau_i) \quad \text{and} \quad \tau_i = \inf\{n \geq 0 : N_i(n) = T_i\}.
\]

Since clearly, \( \xi^i(\tau_i) = \eta^i(T_i) \), one gets therefore (9.12). Furthermore, for given \( \tau_\Lambda \) and \( N_1(\tau_\Lambda), \ldots, N_m(\tau_\Lambda) \), the random vectors \( (\xi^i(\tau_\Lambda) = \eta^i(N_i(\tau_\Lambda)), i \in \Lambda) \) and \( (\xi^i(\tau_\Lambda) = \eta^i(N_i(\tau_\Lambda)), i \notin \Lambda) \) are conditionally independent and the conditional expectation

\[
E_u\left(\prod_{i \in \{1, \ldots, m\} \setminus \Lambda} \xi_i^i(\tau_\Lambda) \mid \tau_\Lambda, N_1(\tau_\Lambda), \ldots, N_m(\tau_\Lambda)\right)
\]

is equal to

\[
E_u\left(\prod_{i \in \{1, \ldots, m\} \setminus \Lambda} \eta^i(N_i(\tau_\Lambda)) \mid \tau_\Lambda, N_1(\tau_\Lambda), \ldots, N_m(\tau_\Lambda)\right) = \prod_{i \in \{1, \ldots, m\} \setminus \Lambda} E_{\eta^i} \left( \eta^i(N_i(\tau_\Lambda)) \mid N_i(\tau_\Lambda) \right).
\]
Moreover, because of the assumption (C3), \( E_{u^i} \left( \eta^i (N_t (\tau_\Lambda)) \mid N_t (\tau_\Lambda) \right) = u^i \) for any \( i \notin \Lambda \). The left hand side of (9.13) is therefore equal to

\[
E_u \left( \exp(-\varepsilon \tau_\Lambda) \prod_{i \in \Lambda} \xi^i (\tau_\Lambda) \right) \prod_{i \in \{1, \ldots, m \} \setminus \Lambda} u^i.
\]

and consequently, (9.13) holds. \( \square \)

By strong Markov property, from (9.13) it follows

**Corollary 9.4.** Under the hypotheses of Lemma 9.3 for any non-empty subset \( \Lambda \subset \{1, \ldots, m\} \), \( \varepsilon > 0 \) and \( u = (u^1, \ldots, u^m) \in \mathbb{Z}_+^m \),

\[
(9.14) \quad E_u \left( \exp(-\varepsilon \tau_\Lambda) \prod_{i \in \Lambda} \xi^i (\tau_\Lambda) \right) = \prod_{i \in \Lambda} \xi^i (\tau_{\{1, \ldots, m \} \setminus \Lambda} < \tau_\Lambda).
\]

We are ready now to get another equivalent representation of the function (9.8).

**Lemma 9.4.** If the conditions (C1)-(C4) are satisfied and \( \nu(0) = 0 \), then for any \( u = (u^1, \ldots, u^m) \in \mathbb{Z}_+^m \),

\[
(9.15) \quad \prod_{i=1}^m \left( u^i - E_{u^i} (\eta_i (T_\tau)) \right) = \prod_{i=1}^m \left( u^i - E_{u^i} (\xi^i (\tau_i)) \right) = \prod_{i=1}^m \left( u^i - E_u \left( \prod_{1 \leq j \leq m} \xi^i \left( \min_{1 \leq j \leq m} \tau_j \right) \right) \right)
\]

Proof. The first equality of (9.15) follows from (9.12). To prove the second equality, it is convenient to use the induction with respect to \( m \). For \( m = 1 \), this equality is trivial. Suppose now that (9.15) holds for some \( m = k \geq 1 \). Then

\[
\prod_{i=1}^{k+1} \left( u^i - E_{u^i} (\xi^i (\tau_i)) \right) = \left( \prod_{i=1}^k u^i - E_u \left( \prod_{1 \leq j \leq k} \xi^i \left( \min_{1 \leq j \leq k} \tau_j \right) \right) \right)
\]

\[
\times \left( u^{k+1} - E_{u^{k+1}} (\xi^{k+1} (\tau_{k+1})) \right)
\]

and hence, to get (9.15) for \( m = k + 1 \), it is sufficient to show that

\[
(9.16) \quad E_u \left( \prod_{i=1}^{k+1} \xi^i \left( \min_{1 \leq j \leq k+1} \tau_j \right) \right) = u^{k+1} \times E_u \left( \prod_{i=1}^k \xi^i \left( \min_{1 \leq j \leq k} \tau_j \right) \right)
\]

\[
+ E_{u^{k+1}} (\xi^{k+1} (\tau_{k+1})) \times \prod_{i=1}^k u^i - E_{u^{k+1}} (\xi^{k+1} (\tau_{k+1})) \times E_u \left( \prod_{1 \leq j \leq k} \xi^i \left( \min_{1 \leq j \leq k} \tau_j \right) \right)
\]

To prove this equality let us notice that because of the assumption (C4), almost surely \( \tau_i \neq \tau_j \) for all \( 1 \leq i < j \leq k+1 \). Hence, almost surely, only one of the coordinates \( \xi^i (\min_{1 \leq j \leq k+1} \tau_j) \), \( \ldots, \xi^{k+1} (\min_{1 \leq j \leq k+1} \tau_j) \) is negative or zero, and consequently,

\[
\prod_{i=1}^{k+1} \xi^i \left( \min_{1 \leq j \leq k+1} \tau_j \right) \leq 0.
\]
The left hand side of (9.16) is therefore negative or zero. While we do not yet know whether the left hand side of (9.16) finite or equal to $-\infty$, the quantities

$$E_u \left( \exp \left( -\varepsilon \min_{1 \leq j \leq k+1} \tau_j \right) \prod_{i=1}^{k+1} \xi^i \left( \min_{1 \leq j \leq k+1} \tau_j \right) \right)$$

are well defined for any $\varepsilon > 0$ by Lemma 9.1 and by monotone convergence theorem, (9.17)

$$E_u \left( \prod_{i=1}^{k+1} \xi^i \left( \min_{1 \leq j \leq k+1} \tau_j \right) \right) = \lim_{\varepsilon \to 0+} E_u \left( \exp \left( -\varepsilon \min_{1 \leq j \leq k+1} \tau_j \right) \prod_{i=1}^{k+1} \xi^i \left( \min_{1 \leq j \leq k+1} \tau_j \right) \right).$$

Furthermore, using again the fact that almost surely $\tau_i \neq \tau_{k+1}$ for all $i = 1, \ldots, k$, one gets

$$E_u \left( \exp \left( -\varepsilon \min_{1 \leq j \leq k+1} \tau_j \right) \prod_{i=1}^{k+1} \xi^i \left( \min_{1 \leq j \leq k+1} \tau_j \right) \right)$$

$$= E_u \left( \exp \left( -\varepsilon \min_{1 \leq j \leq k} \tau_j \right) \prod_{i=1}^{k+1} \xi^i \left( \min_{1 \leq j \leq k} \tau_j \right), \min_{1 \leq j \leq k} \tau_j < \tau_{k+1} \right)$$

$$+ E_u \left( \exp \left( -\varepsilon \tau_{k+1} \right) \prod_{i=1}^{k+1} \xi^i \left( \tau_{k+1} \right), \min_{1 \leq j \leq k} \tau_j > \tau_{k+1} \right)$$

$$= E_u \left( \exp \left( -\varepsilon \min_{1 \leq j \leq k} \tau_j \right) \prod_{i=1}^{k+1} \xi^i \left( \min_{1 \leq j \leq k} \tau_j \right) \right) + E_u \left( \exp \left( -\varepsilon \tau_{k+1} \right) \prod_{i=1}^{k+1} \xi^i \left( \tau_{k+1} \right) \right)$$

$$- E_u \left( \exp \left( -\varepsilon \min_{1 \leq j \leq k} \tau_j \right) \prod_{i=1}^{k+1} \xi^i \left( \min_{1 \leq j \leq k} \tau_j \right), \min_{1 \leq j \leq k} \tau_j > \tau_{k+1} \right)$$

$$- E_u \left( \exp \left( -\varepsilon \tau_{k+1} \right) \prod_{i=1}^{k+1} \xi^i \left( \tau_{k+1} \right), \min_{1 \leq j \leq k} \tau_j < \tau_{k+1} \right).$$

For the right hand side of the last equality, Lemma 9.3 applied with $\Lambda = \{1, \ldots, k\}$ proves that the first term if is equal to

$$u^{k+1} \times E_u \left( \exp \left( -\varepsilon \min_{1 \leq j \leq k} \tau_j \right) \prod_{i=1}^{k} \xi^i \left( \min_{1 \leq j \leq k} \tau_j \right) \right),$$

again by Lemma 9.3 but now with $\Lambda = \{t_{k+1}\}$ the second term is equal to

$$E_u \left( \exp \left( -\varepsilon \tau_{k+1} \right) \xi^{k+1} \left( \tau_{k+1} \right) \right) \times \prod_{i=1}^{k} u^i,$$

Corollary 9.4 applied with $\Lambda = \{1, \ldots, k\}$ shows that the third term is equal to

$$E_u \left( \exp \left( -\varepsilon \min_{1 \leq j \leq k} \tau_j \right) \xi^{k+1} \left( \tau_{k+1} \right) \times \prod_{i=1}^{k} \xi^i \left( \min_{1 \leq j \leq k} \tau_j \right), \min_{1 \leq j \leq k} \tau_j > \tau_{k+1} \right)$$

and again by Corollary 9.4 but now with $\Lambda = \{t_{k+1}\}$, the fourth term is equal to

$$E_u \left( \exp \left( -\varepsilon \tau_{k+1} \right) \xi^{k+1} \left( \tau_{k+1} \right) \times \prod_{i=1}^{k} \xi^i \left( \min_{1 \leq j \leq k} \tau_j \right), \min_{1 \leq j \leq k} \tau_j > \tau_{k+1} \right).$$
Since the sum of the last two terms is equal to
\[ \mathbb{E}_u \left( \exp \left( -\varepsilon \min_{1 \leq j \leq k} \tau_j \right) \xi^{k+1}(\tau_{k+1}) \times \prod_{i=1}^{k} \xi^{\min_{1 \leq j \leq k} \tau_j} \right) \]
letting \( \varepsilon \to 0 \) and using monotone convergence theorem one gets
\[ \mathbb{E}_u \left( \prod_{i=1}^{k+1} \xi^{\min_{1 \leq j \leq k+1} \tau_j} \right) = u^{k+1} \times \mathbb{E}_u \left( \prod_{i=1}^{k} \xi^{\min_{1 \leq j \leq k} \tau_j} \right) \]
\[ + \mathbb{E}_{u+1}(\xi^{k+1}(\tau_{k+1})) \times \prod_{i=1}^{k} u^i - \times \mathbb{E}_u \left( \xi^{k+1}(\tau_{k+1}) \times \prod_{i=1}^{k} \xi^{\min_{1 \leq j \leq k} \tau_j} \right). \]
The last relation combined with Lemma 9.2 proves (9.15).

\[ \square \]

**Proof of Proposition 9.3.** Lemma 9.3 and Corollary 9.3 prove this proposition in the case where \( \nu(0) = 0 \). Suppose now that \( \nu(0) \neq 0 \) and let us notice that a positive function \( f > 0 \) on \( \mathbb{Z}_d^m \) is harmonic for the transition kernel \( \tilde{P}_+ \) if and only if it is harmonic for the new transition kernel \( \tilde{P}_+ = (\tilde{P}(u, u') = \tilde{\nu}(u' - u), u, u' \in \mathbb{Z}_d^m) \) with
\[ \tilde{\nu}(u) = \begin{cases} (1 - \nu(0))^{-1} \nu(u) & \text{if } u \neq 0, \\ 0 & \text{if } u = 0. \end{cases} \]

If the original probability measure \( \nu \) satisfies the conditions (C1)-(C4), the modified probability measure also satisfies the conditions (C1)-(C4). Since \( \tilde{\nu}(0) = 0 \), the only positive harmonic functions of the modified transition kernel \( \tilde{P}_+ \), as well as of the original transition kernel \( P_+ \), are therefore the constant multiples of the function
\[ f(u) = \prod_{i=1}^{m} u^i - \mathbb{E}_u \left( \prod_{i=1}^{m} \tilde{\xi}^{\min_{1 \leq i \leq m} \tau_i} \right), \quad u \in \mathbb{Z}_d^m \]
where \( \tilde{\xi}(t) = (\tilde{\xi}_1(t), \ldots, \tilde{\xi}_m(t)) \) is the new random walk on \( \mathbb{Z}_d^m \) with transition probabilities \( \tilde{\rho}(u, u') = \tilde{\nu}(u' - u) \) and \( \tilde{\tau}_i = \inf \{ t \geq 0 : \tilde{\xi}_i(t) \leq 0 \} \) for \( 1 \leq i \leq m \). Consider finally a sequence of independent identically distributed Bernoulli random variables \( (\theta_n) \) with \( \mathbb{P}(\theta_n = 1) = 1 - \nu(0) \). The random walk \( (\tilde{\xi}(t)) \) can be represented in terms of the random walk \( (\xi(t)) \) as follows: if the random walk \( (\xi(t)) \) and the sequence \( (\theta_n) \) are independent, then letting \( N(n) = \theta_1 + \cdots + \theta_n \) one gets \( \xi(n) = \tilde{\xi}(N(n)) \). With such a representation, almost surely,
\[ \prod_{i=1}^{m} \tilde{\xi}^{\min_{1 \leq i \leq m} \tau_i} = \prod_{i=1}^{m} \xi^{\min_{1 \leq i \leq m} \tau_i} \]
and consequently, the right hand side of (9.18) is equal to the right hand side of (9.9). Proposition 9.3 is therefore proved.

9.4. **Proofs of Proposition 1.1 and Proposition 1.2.** Recall that for \( \Lambda = \Lambda(M) = \{ i : M^i = 0 \} \), the random process \( S^\Lambda(t) = (S^\Lambda(t))_{t \in \Lambda} \) is a homogeneous random walk on \( \mathbb{Z}_d^\Lambda \) with transition probabilities
\[ \mathbb{P}_u \left( S^\Lambda = u' \right) = \mu_\Lambda(u' - u) = \sum_{x \in \mathbb{Z}_d^\Lambda : x^\Lambda = u' - u} \mu(x), \quad u, u' \in \mathbb{Z}_d^\Lambda. \]
and zero mean
\[ E_0(S^A(1)) = \sum_{u \in Z^d} \mu_A(u)u = \sum_{x \in Z^d} \mu(x)x^A = M^A = 0. \]

The induced Markov chain \((X_M(t))\) is identical to the random walk \(S^{A(M)}(t)\) for \(t < \tau_{A(M)} = \inf\{t \geq 0 : S^i(t) \leq 0 \text{ for some } i \in A(M)\}\) and killed at the time \(\tau_{A(M)}\). The transition kernel of the Markov chain \((X_M(t))\) is
\[ P_+ = \left(p(u, u') = \mu_{A(M)}(u' - u), u, u' \in Z^{A(M)}\right). \]

Under the hypotheses (A1)-(A4), the probability measure \(\mu_{A(M)}\) satisfies the conditions (C1)-(C4) of Proposition 9.3 and consequently, every minimal harmonic function \(f > 0\) of \((X_M(t))\) is a constant multiple of the function
\[ f(u) = \prod_{i \in A(M)} u^i - E_u \left(\prod_{i \in A(M)} S^i(\tau_{A(M)})\right). \]

By Theorem 1, from this it follows that every sequence \(x_n \in Z^d_+\) with \(\lim_n |x_n| = \infty\) and \(\lim_n x_n/|x_n| = M\) is fundamental for the Markov chain \((Z(t))\) and for any \(x \in Z^d_+\),
\[ \lim_n G(x, x_n)/G(x_0, x_n) = h(x)/h(x_0) \]
with
\[ h(x) = f(x^{A(M)}) - E_x \left(f(S^{A(M)}(\tau)), \tau < \tau_{A(M)}\right). \]

Remark finally that for any \(x \in Z^d_+\),
\[ h(x) = \prod_{i \in A(M)} x^i - E_x \left(\prod_{i \in A(M)} S^i(\tau_{A(M)})\right) - E_x \left(\prod_{i \in A(M)} S^i(\tau), \tau < \tau_{A(M)}\right) \]
\[ + E_x \left(\prod_{i \in A(M)} S^{A(M)}(\tau)\right) \left(\prod_{i \in A(M)} S^i(\tau_{A(M)}), \tau < \tau_{A(M)}\right). \]

Since by strong Markov property, the last term is equal to
\[ E_x \left(\prod_{i \in A(M)} S^i(\tau_{A(M)}), \tau < \tau_{A(M)}\right), \]
we conclude that for any \(x \in Z^d_+\),
\[ h(x) = \prod_{i \in A(M)} x^i - E_x \left(\prod_{i \in A(M)} S^i(\tau_{A(M)}), \tau \geq \tau_{A(M)}\right) \]
\[ - E_x \left(\prod_{i \in A(M)} S^i(\tau), \tau < \tau_{A(M)}\right) \]
\[ = \prod_{i \in A(M)} x^i - E_x \left(\prod_{i \in A(M)} S^i(\tau), \tau < \infty\right). \]
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