Simple, Robust and Optimal Ranking from Pairwise Comparisons

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Abstract

We consider data in the form of pairwise comparisons of \(n\) items, with the goal of precisely identifying the top \(k\) items for some value of \(k < n\), or alternatively, recovering a ranking of all the items. We analyze the Copeland counting algorithm that ranks the items in order of the number of pairwise comparisons won, and show it has three attractive features: (a) its computational efficiency leads to speed-ups of several orders of magnitude in computation time as compared to prior work; (b) it is robust in that theoretical guarantees impose no conditions on the underlying matrix of pairwise-comparison probabilities, in contrast to some prior work that applies only to the BTL parametric model; and (c) it is an optimal method up to constant factors, meaning that it achieves the information-theoretic limits for recovering the top \(k\)-subset. We extend our results to obtain sharp guarantees for approximate recovery under the Hamming distortion metric, and more generally, to any arbitrary error requirement that satisfies a simple and natural monotonicity condition.

1 Introduction

Ranking problems involve a collection of \(n\) items, and some unknown underlying total ordering of these items. In many applications, one may observe (noisy) comparisons between various pairs of items. Examples include matches between football teams in tournament play; consumer’s preference ratings in marketing; and certain types of voting systems in politics. Given a set of such noisy comparisons between items, it is often of interest to find the true underlying ordering of all \(n\) items, or alternatively, given some given positive integer \(k < n\), to find the subset of \(k\) most highly rated items. These two problems are the focus of this paper.

There is a substantial literature on the problem of finding approximate rankings based on noisy pairwise comparisons. A number of papers (e.g., [KMS07, BM08, Eri13]) consider models in which the probability of a pairwise comparison agreeing with the underlying order is identical across all pairs. These results break down when for one or more pairs, the probability of agreeing with the underlying ranking is either comes close to or is exactly equal to \(\frac{1}{2}\). Another set of papers [Hum04, NOS12, HOX14, SPX14, SBB+16] work using parametric models of pairwise comparisons, and address the problem of recovering the parameters associated to every individual item. A more recent line of work [Cha14, SBGW16, SBW16] studies a more general class of models based on the notion of strong stochastic transitivity (SST), and derives conditions on recovering the pairwise comparison probabilities themselves. However, it remains unclear whether or not these results can directly extend to tight bounds.
for the problem of recovery of the top $k$ items. The works \cite{JS08, MGC11, AS12, DIS15} consider mixture models, in which every pairwise comparison is associated to a certain individual making the comparison, and it is assumed that the preferences across individuals can be described by a low-dimensional model.

Most related to our work are the papers \cite{WJJ13, RA14, RGLA15, CS15}, which we discuss in more detail here. Wauthier et al. \cite{WJJ13} analyze a weighted counting algorithm to recover approximate rankings; their analysis applies to a specific model in which the pairwise comparison between any pair of items remains faithful to their relative positions in the true ranking with a probability common across all pairs. They consider recovery of an approximate ranking (under Kendall’s tau and maximum displacement metrics), but do not provide results on exact recovery. As the analysis of this paper shows, their bounds are quite loose: their results are tight only when there are a total of at least $\Theta(n^2)$ comparisons. The pair of papers \cite{RA14, RGLA15} by Rajkumar et al. consider ranking under several models and several metrics. In the part that is common with our setting, they show that the counting algorithm is consistent in terms of recovering the full ranking, which automatically implies consistency in exactly recovering the top $k$ items. They obtain upper bounds on the sample complexity in terms of a separation threshold that is identical to a parameter $\Delta_k$ defined subsequently in this paper (see Section 3). However, as our analysis shows, their bounds are loose by at least an order of magnitude. They also assume a certain high-SNR condition on the probabilities, an assumption that is not imposed in our analysis.

Finally, in very recent work on this problem, Chen and Suh \cite{CS15} proposed an algorithm called the Spectral MLE for exact recovery of the top $k$ items. They showed that, if the pairwise observations are assumed to drawn according to the Bradley-Terry-Luce (BTL) parametric model \cite{BT52, Luc59}, the Spectral MLE algorithm recovers the $k$ items correctly with high probability under certain regularity conditions. In addition, they also show, via matching lower bounds, that their regularity conditions are tight up to constant factors. While these guarantees are attractive, it is natural to ask how such an algorithm behaves when the data is not drawn from the BTL model. In real-world instances of pairwise ranking data, it is often found that parametric models, such as the BTL model and its variants, fail to provide accurate fits (for instance, see the papers \cite{DM59, ML65, Tve72, BW97} and references therein).

With this context, the main contribution of this paper is to analyze a classical counting-based method for ranking, often called the Copeland method \cite{Cop51}, and to show that it is simple, optimal and robust. Our analysis does not require that the data-generating mechanism follow either the BTL or other parametric assumptions, nor other regularity conditions such as stochastic transitivity. We show that the Copeland counting algorithm has the following properties:

- Simplicity: The algorithm is simple, as it just orders the items by the number of pairwise comparisons won. As we will subsequently see, the execution time of this counting algorithm is several orders of magnitude lower as compared to prior work.

- Optimality: We derive conditions under which the counting algorithm achieves the stated goals, and by means of matching information-theoretic lower bounds, show that these conditions are tight.

- Robustness: The guarantees that we prove do not require any assumptions on the pairwise-comparison probabilities, and the counting algorithm performs well for various classes of
data sets. In contrast, we find that the spectral MLE algorithm performs poorly when the data is not drawn from the BTL model.

In doing so, we consider three different instantiations of the problem of set-based recovery: (i) Recovering the top \(k\) items perfectly; (ii) Recovering the top \(k\) items allowing for a certain Hamming error tolerance; and (iii) a more general recovery problem for set families that satisfy a natural “set-monotonicity” condition. In order to tackle this third problem, we introduce a general framework that allows us to treat a variety of problems in the literature in an unified manner.

The remainder of this paper is organized as follows. We begin in Section 2 with background and a more precise formulation of the problem. Section 3 presents our main theoretical results on top-\(k\) recovery under various requirements. Section 4 provides the results of experiments on both simulated and real-world data sets. We provide all proofs in Section 5. The paper concludes with a discussion in Section 6.

2 Background and problem formulation

In this section, we provide a more formal statement of the problem along with background on various types of ranking models.

2.1 Problem statement

Given an integer \(n \geq 2\), we consider a collection of \(n\) items, indexed by the set \([n] := \{1, \ldots, n\}\). For each pair \(i \neq j\), we let \(M_{ij}\) denote the probability that item \(i\) wins the comparison with item \(j\). We assume that that each comparison necessarily results in one winner, meaning that

\[
M_{ij} + M_{ji} = 1, \quad \text{and} \quad M_{ii} = \frac{1}{2},
\]

where we set the diagonal for concreteness.

For any item \(i \in [n]\), we define an associated score \(\tau_i\) as

\[
\tau_i := \frac{1}{n} \sum_{j=1}^{n} M_{ij}.
\]

In words, the score \(\tau_i\) of any item \(i \in [n]\) corresponds to the probability that item \(i\) beats an item chosen uniformly at random from all \(n\) items.

Given a set of noisy pairwise comparisons, our goals are (a) to recover the \(k\) items with the maximum values of their scores; and (b) to recover the full ordering of all the items as defined by the score vector. The notion of ranking items via their scores (2) generalizes the explicit rankings under popular models in the literature. Indeed, as we discuss shortly, most models of pairwise comparisons considered in the literature either implicitly or explicitly assume that the items are ranked according to their scores. Note that neither the scores \(\{\tau_i\}_{i \in [n]}\) nor the matrix \(M := \{M_{ij}\}_{i,j \in [n]}\) of probabilities are assumed to be known.

More concretely, we consider a random-design observation model defined as follows. Each pair is associated with a random number of noisy comparisons, following a binomial distribution with parameters \((r, p)\), where \(r \geq 1\) is the number of trials and \(p \in (0, 1]\) is the probability of making a comparison on any given trial. Thus, each pair \((i, j)\) is associated with a binomial random variable with parameters \((r, p)\) that governs the number of comparisons between the
pair of items. We assume that the observation sequences for different pairs are independent. Note that in the special case $p = 1$, this random binomial model reduces to the case in which we observe exactly $r$ observations of each pair; in the special case $r = 1$, the set of pairs compared form an $(n, p)$ Erdős-Rényi random graph.

In this paper, we begin in Section 3.1 by analyzing the problem of exact recovery. More precisely, for a given matrix $M$ of pairwise probabilities, suppose that we let $S_k^*$ denote the (unknown) set of $k$ items with the largest values of their respective scores, assumed to be unique for concreteness.

Given noisy observations specified by the pairwise probabilities $M$, our goal is to establish conditions under which there exists some algorithm $\hat{S}_k$ that identifies $k$ items based on the outcomes of various comparisons such that the probability $\mathbb{P}_M(\hat{S}_k = S_k^*)$ is very close to one.

In the case of recovering the full ranking, our goal is to identify conditions that ensure that the probability $\mathbb{P}_M(\bigcap_{k \in [n]} (\hat{S}_k = S_k^*))$ is close to one.

In Section 3.2, we consider the problem of recovering a set of $k$ items that approximates $S_k^*$ with a minimal Hamming error. For any two subsets of $[n]$, we define their Hamming distance $D_H$, also referred to as their Hamming error, to be the number of items that belong to exactly one of the two sets—that is

$$D_H(A, B) = \text{card}\left(\{A \cup B\} \setminus \{A \cap B\}\right). \quad (3)$$

For a given user-defined tolerance parameter $h \geq 0$, we derive conditions that ensure that $D_H(\hat{S}_k, S_k^*) \leq 2h$ with high probability.

Finally, we generalize our results to the problem of satisfying any a general class of requirements on set families. These requirements are specified in terms of which $k$-sized subsets of the items are allowed, and is required to satisfy only one natural condition, that of set-monotonicity, meaning that replacing an item in an allowed set with a higher rank item should also be allowed. See Section 3.3 for more details on this general framework.

### 2.2 A range of pairwise comparison models

To be clear, our work makes no assumptions on the form of the pairwise comparison probabilities. However, so as to put our work in context of the literature, let us briefly review some standard models used for pairwise comparison data.

**Parametric models:** A broad class of parametric models, including the Bradley-Terry-Luce (BTL) model as a special case [BT52, Luc59], are based on assuming the existence of “quality” parameter $w_i \in \mathbb{R}$ for each item $i$, and requiring that the probability of an item beating another is a specific function of the difference between their values. In the BTL model, the probability $M_{ij}$ that $i$ beats $j$ is given by the logistic model

$$M_{ij} = \frac{1}{1 + e^{-(w_i - w_j)}}. \quad (4a)$$

More generally, parametric models assume that the pairwise comparison probabilities take the form

$$M_{ij} = F(w_i - w_j), \quad (4b)$$

where $F : \mathbb{R} \rightarrow [0, 1]$ is some strictly increasing cumulative distribution function.
By construction, any parametric model has the following property: if \( w_i > w_j \) for some pair of items \((i, j)\), then we are also guaranteed that \( M_{i\ell} > M_{j\ell} \) for every item \( \ell \). As a consequence, we are guaranteed that \( \tau_i > \tau_j \), which implies that ordering of the items in terms of their quality vector \( w \in \mathbb{R}^n \) is identical to their ordering in terms of the score vector \( \tau \in \mathbb{R}^n \). Consequently, if the data is actually drawn from a parametric model, then recovering the top \( k \) items according to their scores is the same as recovering the top \( k \) items according their respective quality parameters.

**Strong Stochastic Transitivity (SST) class:** The class of strong stochastic transitivity (SST) models is a superset of parametric models [SBGW16]. It does not assume the existence of a quality vector, nor does it assume any specific form of the probabilities as in equation (4a). Instead, the SST class is defined by assuming the existence of a total ordering of the \( n \) items, and imposing the inequality constraints \( M_{i\ell} \geq M_{j\ell} \) for every pair of items \((i, j)\) where \( i \) is ranked above \( j \) in the ordering, and every item \( \ell \). One can verify that an ordering by the scores \( \{\tau_i\}_{i \in [n]} \) of the items lead to an ordering of the items that is consistent with that defined by the SST class.

Thus, we see that in a broad class of models for pairwise ranking, the total ordering defined by the score vectors (2) coincides with the underlying ordering used to define the models. In this paper, we analyze the performance of a counting algorithm, without imposing any modeling conditions on the family of pairwise probabilities. The next three sections establish theoretical guarantees on the recovery of the top \( k \) items under various requirements.

### 2.3 Copeland counting algorithm

The analysis of this paper focuses on a simple counting-based algorithm, often called the Copeland method [Cop51]. It can be also be viewed as a special case of the Borda count method [dB81], which applies more generally to observations that consist of rankings of two or more items. Here we describe how this method applies to the random-design observation model introduced earlier.

More precisely, for each distinct \( i, j \in [n] \) and every integer \( \ell \in [r] \), let \( Y_{ij}^\ell \in \{-1, 0, +1\} \) represent the outcome of the \( \ell^{th} \) comparison between the pair \( i \) and \( j \), defined as

\[
Y_{ij}^\ell = \begin{cases} 
0 & \text{no comparison between } (i,j) \text{ in trial } \ell \\
+1 & \text{if comparison is made and item } i \text{ beats } j \\
-1 & \text{if comparison is made and item } j \text{ beats } i.
\end{cases}
\]

Note that this definition ensures that \( Y_{ij}^\ell = -Y_{ji}^\ell \). For \( i \in [n] \), the quantity

\[
N_i := \sum_{j \in [n]} \sum_{\ell \in [r]} 1\{Y_{ij}^\ell = 1\}
\]

corresponds to the number of pairwise comparisons won by item \( i \). Here we use \( 1\{\cdot\} \) to denote the indicator function that takes the value 1 if its argument is true, and the value 0 otherwise. For each integer \( k \), the vector \( \{N_i\}_{i=1}^n \) of number of pairwise wins defines a \( k \)-sized subset

\[
\tilde{S}_k = \left\{ i \in [n] \mid N_i \text{ is among the } k \text{ highest number of pairwise wins} \right\},
\]

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corresponding to the set of \( k \) items with the largest values of \( N_i \). Otherwise stated, the set \( \tilde{S}_k \) corresponds to the rank statistics of the top \( k \)-items in the pairwise win ordering. (If there are any ties, we resolve them by choosing the indices with the smallest value of \( i \).)

3 Main results

In this section, we present our main theoretical results on top-\( k \)-recovery under the three settings described earlier. Note that the three settings are ordered in terms of increasing generality, with the advantage that the least general setting leads to the simplest form of theoretical claim.

3.1 Thresholds for exact recovery of the top \( k \) items

We begin with the goal of exactly recovering the \( k \) top-ranked items. As one might expect, the difficulty of this problem turns out to depend on the degree of separation between the top \( k \) items and the remaining items. More precisely, let us use \((k)\) and \((k+1)\) to denote the indices of the items that are ranked \( k^{th} \) and \((k+1)^{th} \) respectively. With this notation, the \( k \)-separation threshold \( \Delta_k \) is given by

\[
\Delta_k := \tau(k) - \tau(k+1) = \frac{1}{n} \sum_{i=1}^{n} M_{(k)i} - \frac{1}{n} \sum_{i=1}^{n} M_{(k+1)i}
\]

(8)

In words, the quantity \( \Delta_k \) is the difference in the probability of item \((k)\) beating another item chosen uniformly at random, versus the same probability for item \((k+1)\).

As shown by the following theorem, success or failure in recovering the top \( k \) entries is determined by the size of \( \Delta_k \) relative to the number of items \( n \), observation probability \( p \) and number of repetitions \( r \). In particular, consider the family of matrices

\[
\mathcal{F}_k(\alpha; n, p, r) := \left\{ M \in [0, 1]^{n \times n} \mid M + M^T = 11^T, \text{ and } \Delta_k \geq \alpha \sqrt{\frac{\log n}{npr}} \right\}
\]

(9)

To simplify notation, we often adopt \( \mathcal{F}_k(\alpha) \) as a convenient shorthand for this set, where its dependence on \((n, p, r)\) should be understood implicitly.

With this notation, the achievable result in part (a) of the following theorem is based on the estimator that returns the set \( \tilde{S}_k \) of the the \( k \) items defined by the number of pairwise comparisons won, as defined in equation (7). On the other hand, the lower bound in part (b) applies to any estimator, meaning any measurable function of the observations.

Theorem 1. (a) For any \( \alpha \geq 8 \), the maximum pairwise win estimator \( \tilde{S}_k \) from equation (7) satisfies

\[
\sup_{M \in \mathcal{F}_k(\alpha)} \mathbb{P}_{M}[\tilde{S}_k \neq S^*_k] \leq \frac{1}{n^{1/4}}.
\]

(10a)

(b) Conversely, suppose that \( n \geq 7 \) and \( p \geq \frac{\log n}{2nr} \). Then for any \( \alpha \leq \frac{1}{7} \), the error probability of any estimator \( \tilde{S}_k \) is lower bounded as

\[
\sup_{M \in \mathcal{F}_k(\alpha)} \mathbb{P}_{M}[\tilde{S}_k \neq S^*_k] \geq \frac{1}{7}.
\]

(10b)
Remarks: First, it is important to note that the negative result in part (b) holds even if the supremum is further restricted to a particular parametric sub-class of \( F_k(\alpha) \), such as the pairwise comparison matrices generated by the BTL model, or by the SST model. Our proof of the lower bound for exact recovery is based on a generalization of a construction introduced by Chen and Suh [CS15], one adapted to the general definition \( \Delta_k \) of the separation threshold.

Second, we note that in the regime \( p < \frac{\log n}{2nr} \), standard results from random graph theory [ER60] can be used to show that there are at least \( \sqrt{n} \) items (in expectation) that are never compared to any other item. Of course, estimating the rank is impossible in this pathological case, so we omit it from consideration.

Third, the two parts of the theorem in conjunction show that the counting algorithm is essentially optimal. The only room for improvement is in the difference between the value 8 of \( \alpha \) in the achievable result, and the value \( \frac{1}{7} \) in the lower bound.

Theorem 1 can also be used to derive guarantees for recovery of other functions of the underlying ranking. Here we consider the problem of identifying the ranking of all \( n \) items, say denoted by the permutation \( \pi^* \). In this case, we require that each of the separations \( \{\Delta_j\}_{j=1}^{n-1} \) are suitably lower bounded: more precisely, we study models \( M \) that belong to the intersection \( \bigcap_{j=1}^{n-1} F_j(\gamma) \).

**Corollary 1.** Let \( \tilde{\pi} \) be the permutation of the items specified by the number of pairwise comparisons won. Then for any \( \alpha \geq 8 \), we have

\[
\sup_{M \in \bigcap_{j=1}^{n-1} F_j(\alpha)} \mathbb{P}_M[\tilde{\pi} \neq \pi^*] \leq \frac{1}{n^{13}}.
\]

Moreover, the separation condition on \( \{\Delta_j\}_{j=1}^{n-1} \) that defines the set \( \bigcap_{j=1}^{n-1} F_j(\alpha) \) is unimprovable beyond constant factors.

This corollary follows from the equivalence between correct recovery of the ranking and recovering the top \( k \) items for every value of \( k \in [n] \).

**Detailed comparison to related work:** In the remainder of this subsection, we make a detailed comparison to the related works [WJJ13, RA14, RGLA15, CS15] that we briefly discussed earlier in Section 1.

Wauthier et al. [WJJ13] analyze a weighted counting algorithm for approximate recovery of rankings; they work under a model in which \( M_{ij} = \frac{1}{2} + \gamma \) whenever item \( i \) is ranked above item \( j \) in an assumed underlying ordering. Here the parameter \( \gamma \in (0, \frac{1}{2}] \) is independent of \( (i,j) \), and as a consequence, the best ranked item is assumed to be as likely to meet the worst item as it is to beat the second ranked item, for instance. They analyze approximate ranking under Kendall tau and maximum displacement metrics. In order to have a displacement upper bounded by some \( \delta > 0 \), their bounds require the order of \( \frac{n}{\delta^{5/2}} \) pairwise comparisons. In comparison, our model is more general in that we do not impose the \( \gamma \)-condition on the pairwise probabilities. When specialized to the \( \gamma \)-model, the quantities \( \{\Delta_j\}_{j=1}^{n-1} \) in our analysis takes the form \( \Delta_j = \frac{2\gamma}{n} \), and Corollary 1 shows that \( \frac{n \log n}{\min_{j \in [n]} \Delta_j} = \frac{n^2 \log n}{\gamma^2} \) observations are sufficient to recover the exact total ordering. Thus, for any constant \( \delta \), Corollary 1 guarantees recover with a multiplicative factor of order \( \frac{n^2}{\log n} \) smaller than that established by Wauthier et al. [WJJ13].
The pair of papers [RA14, RGLA15] by Rajkumar et al. consider ranking under several models and several metrics. For the subset of their models common with our setting—namely, Bradley-Terry-Luce and the so-called low noise models—they show that the counting algorithm is consistent in terms of recovering the full ranking or the top subset of items. The guarantees are obtained under a low-noise assumption: namely, that the probability of any item $i$ beating $j$ is at least $\frac{1}{2} + \gamma$ whenever item $i$ is ranked higher than item $j$ in an assumed underlying ordering. Their guarantees are based on a sample size of at least $\log \frac{n}{\gamma^2 \mu^2}$, where $\mu$ is a parameter lower bounded as $\mu \geq \frac{1}{n^2}$. Once again, our setting allows for the parameter $\gamma$ to be arbitrarily close to zero, and furthermore as one can see from the discussion above, our bounds are much stronger. Moreover, while Rajkumar et al. focus on upper bounds alone, we also prove matching lower bounds on sample complexity showing that our results are unimprovable beyond constant factors. It should be noted that Rajkumar et al. also provide results for other types of ranking problems that lie outside the class of models treated in the current paper.

Most recently, Chen and Suh [CS15] show that if the pairwise observations are assumed to draw according to the Bradley-Terry-Luce (BTL) parametric model (4a), then their proposed Spectral MLE algorithm recovers the $k$ items correctly with high probability when a certain separation condition on the parameters $\{w_i\}_{i=1}^n$ of the BTL model is satisfied. In addition, they also show, via matching lower bounds, that this separation condition are tight up to constant factors. In real-world instances of pairwise ranking data, it is often found that parametric models, such as the BTL model and its variants, fail to provide accurate fits [DM59, ML65, Tve72, BW97]. Our results make no such assumptions on the noise, and furthermore, our notion of the ordering of the items in terms of their scores (2) strictly generalizes the notion of the ordering with respect to the BTL parameters. In empirical evaluations presented subsequently, we see that the counting algorithm is significantly more robust to various kinds of noise, and takes several orders of magnitude lesser time to compute.

Finally, in addition to the notion of exact recovery considered so far, in the next two subsections we also derive tight guarantees for the Hamming error metric and more general metrics inspired by the requirements of many relevant applications [IBS08, MTW05, BO03, MAEA05, KS06, FLN03].

### 3.2 Approximate recovery under Hamming error

In the previous section, we analyzed performance in terms of exactly recovering the $k$-ranked subset. Although exact recovery is suitable for some applications (e.g., a setting with high stakes, in which any single error has a large price), there are other settings in which it may be acceptable to return a subset that is “close” to the correct $k$-ranked subset. In this section, we analyze this problem of approximate recovery when closeness is measured under the Hamming error. More precisely, for a given threshold $h \in [0, k)$, suppose that our goal is to output a set $k$-sized set $\hat{S}_k$ such that its Hamming distance to the set $S_k^*$ of the true top $k$ items, as defined in equation (3), is bounded as

$$D_H(\hat{S}_k, S_k^*) \leq 2h.$$  \hspace{1cm} (11)

Our goal is to establish conditions under which it is possible (or impossible) to return an estimate $\hat{S}_k$ satisfying the bound (11) with high probability.\footnote{The requirement $h < k$ is sensible because if $h \geq k$, the problem is trivial: any two $k$-sized sets $\hat{S}_k$ and $S_k^*$ satisfy the bound $D_H(\hat{S}_k, S_k^*) \leq 2k \leq 2h$.}
As before, we use \((1), \ldots, (n)\) to denote the permutation of the \(n\) items in decreasing order of their scores. With this notation, the following quantity plays a central role in our analysis:

\[
\Delta_{k,h} := \tau(k-h) - \tau(k+h+1).
\]  

(12a)

Observe that \(\Delta_{k,h}\) is a generalization of the quantity \(\Delta_k\) defined previously in equation (8); more precisely, the quantity \(\Delta_k\) corresponds to \(\Delta_{k,h}\) with \(h = 0\). We then define a generalization of the family \(\mathcal{F}_k(\alpha; n, p, r)\), namely

\[
\mathcal{F}_{k,h}(\alpha; n, p, r) := \{ M \in [0,1]^{n \times n} \mid M + M^T = 11^T, \text{ and } \Delta_{k,h} \geq \alpha \sqrt{\log n / npr} \}. \]

(12b)

As before, we frequently adopt the shorthand \(\mathcal{F}_{k,h}(\alpha)\), with the dependence on \((n, p, r)\) being understood implicitly.

**Theorem 2.** (a) For any \(\alpha \geq 8\), the maximum pairwise win set \(\tilde{S}_k\) satisfies

\[
\sup_{M \in \mathcal{F}_{k,h}(\alpha)} \mathbb{P}_M[D_H(\tilde{S}_k, S_k^*) > 2h] \leq \frac{1}{n^{14}}. \quad (13a)
\]

(b) Conversely, in the regime \(p \geq \frac{\log n}{2np}\) and for given constants \(\nu_1, \nu_2 \in (0,1)\), suppose that \(2h \leq \frac{1}{1+\nu_2} \min\{n^{1-\nu_1}, k, n-k\}\). Then for any \(\alpha \leq \frac{\sqrt{\nu_1 \nu_2}}{14}\), any estimator \(\hat{S}_k\) has error at least

\[
\sup_{M \in \mathcal{F}_{k,h}(\alpha)} \mathbb{P}_M[D_H(\hat{S}_k, S_k^*) > 2h] \geq \frac{1}{7}, \quad (13b)
\]

for all \(n\) larger than a constant \(c(\nu_1, \nu_2)\).

This result is similar to that of Theorem 1 except that the relaxation of the exact recovery condition allows for a less constrained definition of the separation threshold \(\Delta_{k,h}\). As with Theorem 1, the lower bound in part (b) applies even if probability matrix \(M\) is restricted to lie in a parametric model (such as the BTL model), or the more general SST class. The counting algorithm is thus optimal for estimation under the relaxed Hamming metric as well.

Finally, it is worth making a few comments about the constants appearing in these claims. We can weaken the lower bound on \(\Delta_k\) required in Theorem 2(a) at the expense of a lower probability of success; for instance, if we instead require that \(\alpha \geq 4\), then the probability of error is guaranteed to be at most \(n^{-2}\). Subsequently in the paper, we provide the results of simulations with \(n = 500\) items and \(\alpha = 4\). On the other hand, in Theorem 2(b), if we impose the stronger upper bound \(\alpha = \mathcal{O}(1/\sqrt{h \log n})\), then we can remove the condition \(h \leq n^{1-\nu_1}\).

### 3.3 An abstract form of \(k\)-set recovery

In earlier sections, we investigated recovery of the top \(k\) items either exactly or under a Hamming error. Exact recovery may be quite strict for certain applications, whereas the property of Hamming error allowing for a few of the top \(k\) items to be replaced by arbitrary items may be undesirable. Indeed, many applications have requirements that go beyond these metrics; for instance, see the papers [IBS08, MTW05, BO03, MAEA05, KS06, FLN03] and
references therein for some examples. In this section, we generalize the notion of exact or Hamming-error recovery in order to accommodate a fairly general class of requirements.

Both the exact and approximate Hamming recovery settings require the estimator to output a set of \( k \) items that are either exactly or approximately equal the true set of top \( k \) items. When is the estimate deemed successful? One way to think about the problem is as follows. The specified requirement of exact or approximate Hamming recovery is associated to a set of \( k \)-sized subsets of the \( n \) possible ranks. The estimator is deemed successful if the true ranks of the chosen \( k \) items equals one of these subsets. In our notion of generalized recovery, we refer to such sets as allowed sets. For example, in the case \( k = 3 \), we might say that the set \( \{1, 4, 10\} \) is allowed, meaning that an output consisting of the “first”, “fourth” and “tenth” ranked items is considered correct.

In more generality, let \( \mathcal{S} \) denote a family of \( k \)-sized subsets of \([n]\), which we refer to as family of allowed sets. Notice that any allowed set is defined by the positions of the items in the true ordering and not the items themselves. Once some true underlying ordering of the \( n \) items is fixed, each element of the family \( \mathcal{S} \) then specifies a set of the items themselves. We use these two interpretations depending on the context — the definition in terms of positions to specify the requirements, and the definition in terms of the items to evaluate an estimator for a given underlying probability matrix \( M \).

We let \( \mathcal{S}^\dagger_k \) denote a \( k \)-set estimate, meaning a function that given a set of observations as input, returns a \( k \)-sized subset of \([n]\) as output.

**Definition 1** (\( \mathcal{S} \)-respecting estimators). For any family \( \mathcal{S} \) of allowed sets, a \( k \)-set estimate \( \mathcal{S}^\dagger_k \) respects its structure if the set of \( k \) positions of the items in \( \mathcal{S}^\dagger_k \) belongs to the set family \( \mathcal{S} \).

Our goal is to determine conditions on the set family \( \mathcal{S} \) under which there exist estimators \( \mathcal{S}^\dagger_k \) that respect its structure. In order to illustrate this definition, let us return to the examples treated thus far:

**Example 1** (Exact and approximate Hamming recovery). The requirement of exact recovery of the top \( k \) items has \( \mathcal{S} \) consisting of exactly one set, the set of the top \( k \) positions \( \mathcal{S} = \{[k]\} \). In the case of recovery with a Hamming error at most \( 2h \), the set \( \mathcal{S} \) of all allowed sets consists all \( k \)-sized subsets of \([n]\) that contain at least \((k - h)\) positions in the top \( k \) positions. For instance, in the case \( h = 1 \), \( k = 2 \) and \( n = 4 \), we have

\[
\mathcal{S} = \\{\{1, 2\}, \{1, 3\}, \{1, 4\}, \{2, 3\}, \{2, 4\}\}.
\]

Apart from these two requirements, there are several other requirements for top-\( k \) recovery popular in the literature \[CCF+01\] [FLN03] [BO03] [MTW05] [MAEA05] [KS06] [IBS08]. Let us illustrate them with another example:

**Example 2.** Let \( \pi^* : [n] \to [n] \) denote the true underlying ordering of the \( n \) items. The following are four popular requirements on the set \( \mathcal{S}^\dagger_k \) for top-\( k \) identification, with respect to the true permutation \( \pi^* \), for a pre-specified parameter \( \epsilon \geq 0 \).

(i) All items in the set \( \mathcal{S}^\dagger_k \) must be contained contained within the top \((1 + \epsilon)k\) entries:

\[
\max_{i \in \mathcal{S}^\dagger_k} \pi^*(i) \leq (1 + \epsilon)k. \quad (14a)
\]

\footnote{In case of two or more items with identical scores, the choice of any of these items is considered valid.}
(ii) The rank of any item in the set $S_k^\dagger$ must lie within a multiplicative factor $(1 + \epsilon)$ of the rank of any item not in the set $S_k^\dagger$:  
$$\max_{i \in S_k^\dagger} \pi^*(i) \leq (1 + \epsilon) \min_{j \in [n] \setminus S_k^\dagger} \pi(j).$$  
(14b)

(iii) The rank of any item in the set $S_k^\dagger$ must lie within an additive factor $\epsilon$ of the rank of any item not in the set $S_k^\dagger$:  
$$\max_{i \in S_k^\dagger} \pi^*(i) \leq \min_{j \in [n] \setminus S_k^\dagger} \pi^*(j) + \epsilon.$$  
(14c)

(iv) The sum of the ranks of the items in the set $S_k^\dagger$ must be contained within a factor $(1 + \epsilon)$ of the sums of ranks of the top $k$ entries:  
$$\sum_{i \in S_k^\dagger} \pi^*(i) \leq (1 + \epsilon) \frac{1}{2} k(k + 1).$$  
(14d)

Note that each of these requirements reduces to the exact recovery requirement when $\epsilon = 0$. Moreover, each of these requirements can be rephrased in terms of families of allowed sets. For instance, if we focus on requirement (i), then any $k$-sized subset of the top $(1 + \epsilon)k$ positions is an allowable set.

In this paper, we derive conditions that govern $k$-set recovery for allowable set systems that satisfy a natural “monotonicity” condition. Informally, the monotonicity condition requires that the set of $k$ items resulting from replacing an item in an allowed set with a higher ranked item must also be an allowed set. More precisely, for any set $\{t_1, \ldots, t_k\} \subseteq [n]$, let $\Lambda(\{t_1, \ldots, t_k\}) \subseteq 2^n$ be the set defined by all of its monotone transformations—that is  
$$\Lambda(\{t_1, \ldots, t_k\}) := \{\{t_1', \ldots, t_k'\} \subseteq [n] \mid t_j' \leq t_j \text{ for every } j \in [k]\}.$$  

Using this notation, we have the following:

**Definition 2** (Monotonic set systems). The set $\mathfrak{S}$ of allowed sets is a monotonic set system if  
$$\Lambda(T) \subseteq \mathfrak{S} \text{ for every } T \in \mathfrak{S}.$$  
(15)

One can verify that condition (15) is satisfied by the settings of exact and Hamming-error recovery, as discussed in Example 1. The condition is also satisfied by all four requirements discussed in Example 2.

The following theorem establishes conditions under which one can (or cannot) produce an estimator that respects an allowable set requirement. In order to state it, recall the score $\tau_i := \frac{1}{n} \sum_{j=1}^n M_{ij}$, as previously defined in equation (2) for each $i \in [n]$. For notational convenience, we also define $\tau_i := -\infty$ for every $i > n$. Consider any monotonic family of allowed sets $\mathfrak{S}$, and for some integer $\beta \geq 1$, let $T^1, \ldots, T^{\beta} \in \mathfrak{S}$ such that $\mathfrak{S} = \bigcup_{b \in [\beta]} \Lambda(T^b)$. For every $b \in [\beta]$, let $t^b_1 < \cdots < t^b_k$ denote the entries of $T^b$. We then define the critical threshold based on the scores:  
$$\Delta_{\mathfrak{S}} := \max_{b \in [\beta]} \min_{j \in [k]} (\tau(j) - \tau(k+t^b_j - j + 1)).$$  
(16)
The term $\Delta_{\mathcal{S}}$ is a further generalization of the quantities $\Delta_k$ and $\Delta_{k,h}$ defined in earlier sections. We also define a generalization $\mathcal{F}_{\mathcal{S}}(\cdot)$ of the families $\mathcal{F}_k(\cdot)$ and $\mathcal{F}_{k,h}(\cdot)$ as

$$
\mathcal{F}_{\mathcal{S}}(\alpha; n, p, r) := \left\{ M \in [0, 1]^{n \times n} \mid M + M^T = 11^T \text{ and } \Delta_{\mathcal{S}} \geq \alpha \sqrt{\log n / npr} \right\}.
$$

As before, we use the shorthand $\mathcal{F}_{\mathcal{S}}(\alpha)$, with the dependence on $(n, p, r)$ being understood implicitly.

**Theorem 3.** Consider any allowable set requirement specified by a monotonic set class $\mathcal{S}$.

(a) For any $\alpha \geq 8$, the maximum pairwise win set $\tilde{S}_k$ satisfies

$$
\sup_{M \in \mathcal{F}_{\mathcal{S}}(\alpha)} \mathbb{P}_M [\tilde{S}_k \notin \mathcal{S}] \leq \frac{1}{n^{1/2}}.
$$

(b) Conversely, in the regime $p \geq \frac{\log n}{2npr}$, and for given constants $\mu_1 \in (0, 1), \mu_2 \in (\frac{3}{4}, 1]$, suppose that $\max_{b \in [\beta]} b_{(\mu_2 k)} \leq \frac{n}{2}$ and $8(1 - \mu_2)k \leq n^{1-\mu_1}$. Then for any $\alpha$ smaller than a constant $c_u(\mu_1, \mu_2) > 0$, any estimator $\hat{S}_k$ has error at least

$$
\sup_{M \in \mathcal{F}_{\mathcal{S}}(\alpha)} \mathbb{P}_M [\hat{S}_k \notin \mathcal{S}] \geq \frac{1}{15},
$$

for all $n$ larger than a constant $c_0(\mu_1, \mu_2)$.

A few remarks on the lower bound are in order. First, the lower bound continues to hold even if the probability matrix $M$ is restricted to follow a parametric model such as BTL or restricted to lie in the SST class. Second, in terms of the threshold for $\alpha$, the lower bound holds with $c_u(\mu_1, \mu_2) = \frac{1}{15} \sqrt{\mu_1 \min \left\{ \frac{1}{4(1-\mu_2)^2} - 1, \frac{1}{2} \right\}}$. Third, it is worth noting that one must necessarily impose some conditions for the lower bound, along the lines of those required in Theorem 3(b) for the allowable sets to be “interesting” enough.

As a concrete illustration, consider the requirement defined by the parameters $b = 1$, $k = 1$ and $\mathcal{S} = \Lambda(\{n - \sqrt{n}\})$. For $\mu_1 = \mu_2 = \frac{9}{10}$, this requirement satisfies the condition $8(1 - \mu_2)k \leq n^{1-\mu_1}$ but violates the condition $t_{(\mu_2 k)} \leq \frac{9}{2}$. Now, a selection of $k = 1$ item made uniformly at random (independent of the data) satisfies this allowable set requirement with probability $1 - \frac{1}{\sqrt{n}}$. Given the success of such a random selection algorithm in this parameter regime, we see that the lower bounds therefore cannot be universal, but must require some conditions on the allowable sets.

### 4 Simulations and experiments

In this section, we empirically evaluate the performance of the counting algorithm and compare it with the Spectral MLE algorithm via simulations on synthetic data, as well as experiments using datasets from the Amazon Mechanical Turk crowdsourcing platform.
Figure 1. Simulation results comparing Spectral MLE and the counting algorithm in terms of error rates for exact recovery of the top $k$ items, and computation time. (a) Histogram of fraction of instances where the algorithm failed to recover the $k$ items correctly, with each bar being the average value across 50 trials. The counting algorithm has 0% error across all problems, while the spectral MLE is accurate for parametric models (BTL, Thurstone), but increasingly inaccurate for other models. (b) Histogram plots of the maximum computation time taken by the counting algorithm and the minimum computation time taken by Spectral MLE across all trials. Even though this maximum-to-minimum comparison is unfair to the counting algorithm, it involves five or more orders of magnitude less computation.

4.1 Simulated data

We begin with simulations using synthetically generated data with $n = 500$ items and observation probability $p = 1$, and with pairwise comparison models ranging over six possible types. Panel (a) in Figure 1 provides a histogram plot of the associated error rates (with a bar for each one of these six models) in recovering the $k = n/4 = 125$ items for the counting algorithm versus the Spectral MLE algorithm. Each bar corresponds to the average over 50 trials. Panel (b) compares the CPU times of the two algorithms. The value of $\alpha$ (and in turn, the value of $r$) in the first five models is as derived in Section 3.1. In more detail, the six model types are given by:

(I) Bradley-Terry-Luce (BTL) model: Recall that the theoretical guarantees for the Spectral MLE algorithm [CS15] are applicable to data that is generated from the BTL model [4a], and as guaranteed, the Spectral MLE algorithm gives a 100% accuracy under this model. The counting algorithm also obtains a 100% accuracy, but importantly, the counting algorithm requires a computational time that is five orders of magnitude lower than that of Spectral MLE.

(II) Thurstone model: The Thurstone model [Thu27] is another parametric model, with the function $F$ in equation (4b) set as the cumulative distribution function of the standard Gaussian distribution. Both Spectral MLE and the counting algorithm gave 100% accuracy under this model.

(III) BTL model with one (non-transitive) outlier: This model is identical to BTL, with one modification. Comparisons among $(n - 1)$ of the items follow the BTL model as before, but the remaining item always beats the first $\frac{n}{4}$ items and always loses to each of the other items. We see that the counting algorithm continues to achieve an accuracy of 100%
as guaranteed by Theorem 1. The departure from the BTL model however prevents the Spectral MLE algorithm from identifying the top $k$ items.

(IV) **Strong stochastic transitivity (SST) model:** We simulate the “independent diagonals” construction of [SBGW16] in the SST class. Spectral MLE is often unsuccessful in recovering the top $k$ items, while the counting algorithm always succeeds.

(V) **Mixture of BTL models:** Consider two sets of people with opposing preferences. The first set of people have a certain ordering of the items in their mind and their preferences follow a BTL model under this ordering. The second set of people have the opposite ordering, and their preferences also follow a BTL model under this opposite ordering. The overall preference probabilities is a mixture between these two sets of people. In the simulations, we observe that the counting algorithm is always successful while the Spectral MLE method often fails.

(VI) **BTL with violation of separation condition:** We simulate the BTL model, but with a choice of parameter $r$ small enough that the value of $\alpha$ is about one-tenth of its recommended value in Section 3.1. We observe that the counting algorithm incurs lower errors than the Spectral MLE algorithm, thereby demonstrating its robustness.

To summarize, the performance of the two algorithms can be contrasted in the following way. When our stated lower bounds on $\alpha$ are satisfied, then consistent with our theoretical claims, the Copeland counting algorithm succeeds irrespective of the form of the pairwise probability distributions. The Spectral MLE algorithm performs well when the pairwise comparison probabilities are faithful to parametric models, but is often unsuccessful otherwise. Even when the condition on $\alpha$ is violated, the performance of the counting algorithm remains superior to that of the Spectral MLE algorithm, thereby demonstrating its robustness.

3. In terms of computational complexity, for every instance we simulated, the counting algorithm took several orders of magnitude less time as compared to Spectral MLE.

4.2 **Experiments on data from Amazon Mechanical Turk**

In this section, we describe experiments on real world datasets collected from the Amazon Mechanical Turk (mturk.com) commercial crowdsourcing platform.

4.2.1 **Data**

In order to evaluate the accuracy of the algorithms under consideration, we require datasets consisting of pairwise comparisons in which the questions can be associated with an objective and verifiable ground truth. To this end, we used the “cardinal versus ordinal” dataset from our past work [SBB+16]; three of the experiments performed in that paper are suitable for the evaluations here—namely, ones in which each question has a ground truth, and the pairs of items are chosen uniformly at random. The three experiments tested the workers’ general knowledge, audio, and visual understanding, and the respective tasks involved: (i) identifying the pair of cities with a greater geographical distance, (ii) identifying the higher frequency key of a piano, and (iii) identifying spelling mistakes in a paragraph of text. The number of items $n$ in the three experiments were 16, 10 and 8 respectively. The total number of pairwise comparisons were 408, 265 and 184 respectively. The fraction of pairwise comparisons whose

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3 Note that part (b) of Theorem 1 is a minimax converse meaning that it appeals to the worst case scenario.
Figure 2. Evaluation of Spectral MLE and the counting algorithm on three datasets from Amazon Mechanical Turk in terms of the error rates for top $k$-subset recovery. The three panels plot the Hamming error when recovering the top $k$ items in the three datasets when a $q^{th}$ fraction of the total data is used, for various values of subsampling probability $q \in (0, 1]$. The counting algorithm consistently outperforms the Spectral MLE algorithm.

outcomes were incorrect (as compared to the ground truth) in the raw data are 17%, 20% and 40% respectively.

4.2.2 Results

We compared the performance of the counting algorithm with that of the Spectral MLE algorithm. For each value of a “subsampling probability” $q \in \{0.1, 0.2, \ldots, 1.0\}$, we subsampled a fraction $q$ of the data and executed both algorithms on this subsampled data. We evaluated the performance of the algorithms on their ability to recover the top $k = \lceil \frac{n}{4} \rceil$ items under the Hamming error metric.

Figure 2 shows the results of the experiments. Each point in the plots is an average across 100 trials. Observe that the counting algorithm consistently outperforms Spectral MLE. (We think that the erratic fluctuations in the spelling mistakes data are a consequence of a high noise and a relatively small problem size.) Moreover, the Spectral MLE algorithm required about 5 orders of magnitude more computation time (not shown in the figure) as compared to counting. Thus the counting algorithm performs well on simulated as well as real data. It outperforms Spectral MLE not only when the number of items is large (as in the simulations) but also when the problem sizes are small as seen in these experiments.

5 Proofs

We now turn to the proofs of our main results. We continue to use the notation $[i]$ to denote the set $\{1, \ldots, i\}$ for any integer $i \geq 1$. We ignore floor and ceiling conditions unless critical to the proof.

Our lower bounds are based on a standard form of Fano’s inequality [CT12, Tsy08] for lower bounding the probability of error in an $L$-ary hypothesis testing problem. We state a version here for future reference. For some integer $L \geq 2$, fix some collection of distributions $\{P_1, \ldots, P_L\}$. Suppose that we observe a random variable $Y$ that is obtained by first sampling an index $A$ uniformly at random from $[L] = \{1, \ldots, L\}$, and then drawing $Y \sim P_A$. (As a result, the variable $Y$ is marginally distributed according to the mixture distribution $P = \frac{1}{L} \sum_{a=1}^{L} P_a$.) Given the observation $Y$, our goal is to “decode” the value of $A$, correspond-
ing to the index of the underlying mixture component. Using $Y$ to denote the sample space associated with the observation $Y$, Fano’s inequality asserts that any test function $\phi: Y \to [L]$ for this problem has error probability lower bounded as

$$\mathbb{P}[\phi(Y) \neq A] \geq 1 - \frac{I(Y; A) + \log 2}{\log L},$$

where $I(Y; A)$ denotes the mutual information between $Y$ and $A$. A standard convexity argument for the mutual information yields the weaker bound

$$\mathbb{P}[\phi(Y) \neq A] \geq 1 - \frac{\max_{a, b \in [L]} D_{KL}(P^a \| P^b) + \log 2}{\log L}, \tag{19}$$

We make use of this weakened form of Fano’s inequality in several proofs.

5.1 Proof of Theorem 1

We begin with the proof of Theorem 1, dividing our argument into two parts.

5.1.1 Proof of part (a)

For any pair of items $(i, j)$, let us encode the outcomes of the $r$ trials by an i.i.d. sequence $V^{(\ell)}_{ij} = [X^{(\ell)}_{ij} \ X^{(\ell)}_{ji}]^T$ of random vectors, indexed by $\ell \in [r]$. Each random vector follows the distribution

$$\mathbb{P}[X^{(\ell)}_{ij}, X^{(\ell)}_{ji}] = \begin{cases} 1 - p & \text{if } (x^{(\ell)}_{ij}, x^{(\ell)}_{ji}) = (0, 0) \\ pM_{ij} & \text{if } (x^{(\ell)}_{ij}, x^{(\ell)}_{ji}) = (1, 0) \\ p(1 - M_{ij}) & \text{if } (x^{(\ell)}_{ij}, x^{(\ell)}_{ji}) = (0, 1) \\ 0 & \text{otherwise.} \end{cases}$$

With this encoding, the variable $W_a := \sum_{\ell \in [r]} \sum_{z \in [n] \setminus \{a\}} X^{(r)}_{az}$ encodes the number of wins for item $a$.

Consider any item $a \in S^*_k$ which ranks among the top $k$ in the true underlying ordering, and any item $b \in [n] \setminus S^*_k$ which ranks outside the top $k$. We claim that with high probability, item $a$ will win more pairwise comparisons than item $b$. More precisely, let $E_{ba}$ denote the event that item $b$ wins at least as many pairwise comparisons than $a$. We claim that

$$\mathbb{P}(E_{ba}) \leq \exp \left( -\frac{\frac{1}{4} (rpm\Delta_k)^2}{rpm(2 - \Delta_k) + \frac{2}{3} rpm\Delta_k} \right) \leq \frac{1}{n^{16}}. \tag{20}$$

Given this bound, the probability that the counting algorithm will rank item $b$ above $a$ is no more than $n^{-16}$. Applying the union bound over all pairs of items $a \in S^*_k$ and $b \in [n] \setminus S^*_k$ yields $\mathbb{P}[\hat{S}_k \neq S^*_k] \leq n^{-14}$ as claimed.

We note that inequality (ii) in equation (20) follows from inequality (i) combined with the condition on $\Delta_k$ that arises by setting $\alpha \geq 8$ as assumed in the hypothesis of the theorem. Thus, it remains to prove inequality (i) in equation (20). By definition of $E_{ba}$, we have

$$\mathbb{P}(E_{ba}) = \mathbb{P} \left( \sum_{\ell \in [r]} \sum_{z \in [n] \setminus \{b\}} X^{(\ell)}_{bz} - \sum_{\ell \in [r]} \sum_{z \in [n] \setminus \{a\}} X^{(\ell)}_{az} \geq 0 \right). \tag{21}$$
It is convenient to recenter the random variables. For every \( \ell \in [r] \) and \( z \in [n] \setminus \{a, b\} \), define the zero-mean random variables
\[
\overline{X}_{aa}^{(\ell)} = X_{aa}^{(\ell)} - \mathbb{E}[X_{aa}^{(\ell)}] = X_{aa}^{(\ell)} - pM_{az} \quad \text{and} \quad \overline{X}_{bb}^{(\ell)} = X_{bb}^{(\ell)} - \mathbb{E}[X_{bb}^{(\ell)}] = X_{bb}^{(\ell)} - pM_{bz}.
\]
Also, let
\[
\overline{X}_{ab}^{(\ell)} = (X_{ab}^{(\ell)} - X_{ba}^{(\ell)}) - \mathbb{E}[X_{ab}^{(\ell)} - X_{ba}^{(\ell)}] = (X_{ab}^{(\ell)} - X_{ba}^{(\ell)}) - (pM_{ab} - pM_{ba}).
\]
We then have
\[
\mathbb{P}(\mathcal{E}_{ba}) = \mathbb{P}\left( \sum_{\ell \in [r]} \left( \sum_{z \in [n] \setminus \{a, b\}} \overline{X}_{bz}^{(\ell)} - \sum_{z \in [n] \setminus \{a, b\}} \overline{X}_{az}^{(\ell)} - \overline{X}_{ab}^{(\ell)} \right) \geq rp \sum_{z \in [n]} (M_{az} - M_{bz}) \right).
\]
Since \( a \in S_{k}^{*} \) and \( b \in [n] \setminus S_{k}^{*} \), from the definition of \( \Delta_{k} \), we have \( n\Delta_{k} \leq \sum_{z \in [n]} (M_{az} - M_{bz}) \), and consequently
\[
\mathbb{P}(\mathcal{E}_{ba}) \leq \mathbb{P}\left( \sum_{\ell \in [r]} \left( \sum_{z \in [n] \setminus \{a, b\}} \overline{X}_{bz}^{(\ell)} - \sum_{z \in [n] \setminus \{a, b\}} \overline{X}_{az}^{(\ell)} - \overline{X}_{ab}^{(\ell)} \right) \geq rpn\Delta_{k} \right). \tag{22}
\]
By construction, all the random variables in the above inequality are zero-mean, mutually independent, and bounded in absolute value by 2. These properties alone would allow us to obtain a tail bound by Hoeffding’s inequality; however, in order to obtain the stated result \([20]\), we need the more refined result afforded by Bernstein’s inequality (e.g., \([BLM13]\)). In order to derive a bound of Bernstein type, the only remaining step is to bound the second moments of the random variables at hand. Some straightforward calculations yield
\[
\mathbb{E}[(-\overline{X}_{az}^{(\ell)})^{2}] \leq pM_{az}, \quad \mathbb{E}[(\overline{X}_{bz}^{(\ell)})^{2}] \leq pM_{bz}, \quad \text{and} \quad \mathbb{E}[(\overline{X}_{ab}^{(\ell)})^{2}] \leq pM_{ab} + pM_{ba}.
\]
It follows that
\[
\sum_{z \in [n] \setminus \{a, b\}} \mathbb{E}[(-\overline{X}_{az}^{(\ell)})^{2}] + \sum_{z \in [n] \setminus \{a, b\}} \mathbb{E}[(\overline{X}_{bz}^{(\ell)})^{2}] + \mathbb{E}[(\overline{X}_{ab}^{(\ell)})^{2}] \leq p \left( \sum_{z \in [n] \setminus \{a, b\}} (M_{az} + M_{bz}) + M_{ab} + M_{ba} \right)
\]
\[
\leq p \left( 2 \sum_{z \in [n]} M_{az} - n\Delta_{k} \right) \leq pn(2 - \Delta_{k}),
\]
where the inequality (iii) follows from the definition of \( \Delta_{k} \), and step (iv) follows because \( M_{az} \leq 1 \) for every \( z \) and \( M_{aa} = \frac{1}{2} \). Applying the Bernstein inequality now yields the stated bound \([20](i)\).
5.1.2 Proof of part (b)

The symmetry of the problem allows us to assume, without loss of generality, that $k \leq \frac{n}{2}$. We prove a lower bound by first constructing an ensemble of $n - k + 1$ different problems, and considering the problem of distinguishing between them. For each $a \in \{k - 1, k, \ldots, n\}$, let us define the $k$-sized subset $S^*[a] := \{1, \ldots, k - 1\} \cup \{a\}$, and the associated matrix of pairwise probabilities

$$M^a_{ij} :=
\begin{cases}
\frac{1}{2} & \text{if } i, j \in S^*[a], \text{ or } i, j \notin S^*[a] \\
\frac{1}{2} + \delta & \text{if } i \in S^*[a] \text{ and } j \notin S^*[a] \\
\frac{1}{2} - \delta & \text{if } i \notin S^*[a] \text{ and } j \in S^*[a],
\end{cases}
$$

where $\delta \in (0, \frac{1}{2})$ is a parameter to be chosen. We use $P^a$ to denote probabilities taken under pairwise comparisons drawn according to the model $M^a$.

One can verify that the construction above falls in the intersection of parametric models and the SST model. In the parametric case, this construction amounts to having the parameters associated to every item in $S^*_k$ to have the same value, and those associated to every item in $[n] \setminus S^*_k$ to have the same value. Also observe that for every such distribution $P^a$, the associated $k$-separation threshold $\Delta_k = \delta$.

Any given set of observations can be described by the collection of random variables $Y = \{Y^{(\ell)}_{ij}, j > i \in [n], \ell \in [r]\}$. When the true underlying model is $P^a$, the random variable $Y^{(\ell)}_{ij}$ follows the distribution

$$Y^{(\ell)}_{ij} =
\begin{cases}
0 & \text{with probability } 1 - p \\
i & \text{with probability } pM^a_{ij} \\
j & \text{with probability } p(1 - M^a_{ij}).
\end{cases}
$$

The random variables $\{Y^{(\ell)}_{ij}\}_{i,j \in [n], i < j, \ell \in [r]}$ are mutually independent, and the distribution $P^a$ is a product distribution across pairs $\{i > j\}$ and repetitions $\ell \in [r]$.

Let $A \in \{k, \ldots, n\}$ follow a uniform distribution over the index set, and suppose that given $A = a$, our observations $Y$ has components drawn according to the model $P^a$. Consequently, the marginal distribution of $Y$ is the mixture distribution $\frac{1}{L} \sum_{a=1}^{L} P^a$ over all $L = n - k + 1$ models. Based on observing $Y$, our goal is to recover the correct index $A = a$ of the underlying model, which is equivalent to recovering the planted subset $S^*[a]$. We use the Fano bound [19] to lower bound the error bound associated with any test for this problem. In order to apply Fano’s inequality, the following result provides control over the Kullback-Leibler divergence between any pair of probabilities involved.

**Lemma 1.** For any distinct pair $a, b \in \{k, \ldots, n\}$, we have

$$D_{KL}(P^a || P^b) \leq \frac{2npn}{45\delta^2 - 1}.
$$

See the end of this section for the proof of this claim.

Given this bound on the Kullback-Leibler divergence, Fano’s inequality [19] implies that any estimator $\phi$ of $A$ has error probability lower bounded as

$$P[\phi(Y) \neq A] \geq 1 - \frac{2npn}{45\delta^2 - 1} + \log 2 \geq \frac{1}{7}.
$$
Here the final inequality holds whenever $\delta \leq \frac{1}{7} \sqrt{\frac{\log n}{np r}}$, $p \geq \frac{\log n}{2 pr}$, $n \geq 7$ and $k \leq \frac{n}{2}$. The condition $p \geq \frac{\log n}{2 pr}$ also ensures that $\delta < \frac{1}{2}$ thereby ensuring that our construction is valid. It only remains to prove Lemma 1.

5.1.3 Proof of Lemma 1

Since the distributions $\mathbb{P}^a$ and $\mathbb{P}^b$ are formed by components that are independent across edges $i > j$ and repetitions $\ell \in [r]$, we have

$$D_{KL}(\mathbb{P}^a \| \mathbb{P}^b) = \sum_{\ell \in [r]} \sum_{1 \leq i < j \leq n} D_{KL}(\mathbb{P}^a(X_{ij}^{(\ell)}) \| \mathbb{P}^b(X_{ij}^{(\ell)})) = r \sum_{1 \leq i < j \leq n} D_{KL}(\mathbb{P}^a(X_{ij}^{(1)}) \| \mathbb{P}^b(X_{ij}^{(1)})),$$

where the second equality follows since the $r$ trials are all independent and identically distributed.

We now evaluate each individual term in right hand side of the above equation. Consider any $i, j \in [n]$. We divide our analysis into three disjoint cases:

Case I: Suppose that $i, j \in [n] \setminus \{a, b\}$. The distribution of $X_{ij}^{(1)}$ is identical across the distributions $\mathbb{P}^a$ and $\mathbb{P}^b$. As a result, we find that

$$D_{KL}(\mathbb{P}^a(X_{ij}^{(1)}) \| \mathbb{P}^b(X_{ij}^{(1)})) = 0.$$

Case II: Suppose that $i = a$, $j \in [n] \setminus \{a, b\}$ or $i = b$, $j \in [n] \setminus \{a, b\}$. We then have

$$D_{KL}(\mathbb{P}^a(X_{ij}^{(1)}) \| \mathbb{P}^b(X_{ij}^{(1)})) \leq p \frac{(2\delta)^2}{(\frac{1}{2} - \delta)(\frac{1}{2} + \delta)}.$$

Case III: Suppose that $i = a$, $j = b$. We then have

$$D_{KL}(\mathbb{P}^a(X_{ij}^{(1)}) \| \mathbb{P}^b(X_{ij}^{(1)})) \leq p \frac{(2\delta)^2}{(\frac{1}{2} - \delta)(\frac{1}{2} + \delta)}.$$

Combining the bounds from all three cases, we find that the KL divergence is upper bounded as

$$\frac{1}{r} D_{KL}(\mathbb{P}^a \| \mathbb{P}^b) \leq 2(n - 2)p \frac{\delta^2}{(\frac{1}{2} - \delta)(\frac{1}{2} + \delta)} + p \frac{(2\delta)^2}{(\frac{1}{2} - \delta)(\frac{1}{2} + \delta)}.$$

Some simple algebraic manipulations yield the claimed result.

5.2 Proof of Corollary 1

We now turn to the proof of Corollary 1. Beginning with the claim of sufficiency, it is easy to see that the ranking is correctly recovered whenever the top $k$ items are correctly recovered for every value of $k \in [n]$. Consequently, one can apply the union bound to (10a) over all values of $k \in [n]$ and this gives the desired upper bound.

Now turning to the claim of necessity, we first introduce some notation to aid in subsequent discussion. Defining the parameter $\Delta_0 := \min_{j \in [n-1]} (\tau(j) - \tau(j+1))$, we have shown that the lower bound

$$\Delta_0 \geq 8 \sqrt{\frac{\log n}{n pr}}.$$
is sufficient to guarantee exact recovery of the full ranking. Further, one must also have
\[
\Delta_0 \leq \frac{1}{n-1} \sum_{j=1}^{n-1} (\tau(j) - \tau(j+1)) = \frac{1}{n-1} (\tau(1) - \tau(n)) \leq \frac{1}{n-1}.
\]

Here we show that these two requirements are tight up to constant factors, meaning that for any value of \(\Delta_0\) satisfying \(\Delta_0 \leq \frac{1}{9} \sqrt{\frac{\log n}{npr}}\) and \(\Delta_0 \leq \frac{1}{9} \sqrt{\frac{\log n}{n-1}}\), there are instances where recovery of the underlying ranking fails with probability at least \(\frac{1}{70}\) for any estimator.

Consider the following ensemble of \((n-1)\) different problems, indexed by \(a \in [n-1]\). For every value of \(a \in [n-1]\), define a permutation \(\pi^a\) of the \(n\) items as
\[
\pi^a(i) = \begin{cases} 
  i + 1 & \text{if } i = a \\
  i - 1 & \text{if } i = a + 1 \\
  i & \text{otherwise.}
\end{cases}
\]

In words, the permutation \(\pi^a\) equals the identity permutation except for the swapping of items \(a\) and \((a+1)\). Define an associated matrix of pairwise-comparison probabilities \(M^a\) as
\[
M^a_{ij} = \frac{1}{2} - (\pi^a(i) - \pi^a(j)) \Delta_0,
\]
and \(M^a_{ji} = 1 - M^a_{ij}\). Let \(P^a\) denote the probabilities taken under pairwise comparisons drawn according to the model \(M^a\). The condition \(\Delta_0 \leq \frac{1}{9} \sqrt{\frac{\log n}{npr}}\) ensures that this construction is a valid probability distribution. One can then compute that under distribution \(P^a\), the score \(\tau^a_i\) of any item \(i\) equals
\[
\tau^a_i = \frac{1}{2} - (\pi^a(i) - \frac{n+1}{2}) \Delta_0.
\]

One can also verify that for any \(a \in [n-1]\), and any \(i \in [n-1]\), we have
\[
\tau^a_{\pi^a(i)} - \tau^a_{\pi^a(i+1)} = \Delta_0,
\]
where we have used the fact that \(\pi^a(\pi^a(i)) = i\). The requirement imposed by the hypothesis is thus satisfied.

We now use Fano’s inequality \(^{[19]}\) obtain the claimed lower bound. In order to apply this result, we first obtain an upper bound on the Kullback-Leibler divergence between the probability distributions of the observed data under any pair of problems constructed above.

**Lemma 2.** For any distinct pair \(a, b \in [n-1]\), we have
\[
D_{KL}(P^a \| P^b) \leq 50npr \Delta_0^2.
\]

See the end of this section for the proof of this claim.

Given this bound on the Kullback-Leibler divergence, the Fano bound \(^{[19]}\) implies that any method \(\phi\) for identifying the true ranking has error probability
\[
P[\phi(Y) \neq A] \geq 1 - \frac{50npr \Delta_0^2 + \log 2}{\log(n-1)} \geq \frac{1}{70},
\]
where the final inequality holds whenever \(\Delta_0 \leq \frac{1}{9} \sqrt{\frac{\log n}{npr}}\) and \(n \geq 9\).

The only remaining detail is the proof of Lemma 2.
5.2.1 Proof of Lemma 2

Since the distributions $P^a$ and $P^b$ are formed by components that are independent across edges $i > j$ and repetitions $\ell \in [r]$, we have

$$D_{\text{KL}}(P^a \| P^b) = \sum_{\ell \in [r]} \sum_{1 \leq i < j \leq n} D_{\text{KL}}(P^a(X_{ij}^{(\ell)}) \| P^b(X_{ij}^{(\ell)})) = r \sum_{1 \leq i < j \leq n} D_{\text{KL}}(P^a(X_{ij}^{(1)}) \| P^b(X_{ij}^{(1)})),$$

where the second equality follows since the $r$ trials are all independent and identically distributed.

We now evaluate each individual term in right hand side of the above equation. Consider any $i, j \in [n]$. We divide our analysis into three disjoint cases:

Case I: Suppose that $i, j \in [n] \{a, a + 1, b, b + 1\}$. The distribution of $X_{ij}^{(1)}$ is identical across the distributions $P^a$ and $P^b$. As a result, we find that

$$D_{\text{KL}}(P^a(X_{ij}^{(1)}) \| P^b(X_{ij}^{(1)})) = 0.$$

Case II: Alternatively, suppose $i \in \{a, a + 1, b, b + 1\}$ and $j \in [n] \{a, a + 1, b, b + 1\}$ or if $j \in \{a, a + 1, b, b + 1\}$ and $i \in [n] \{a, a + 1, b, b + 1\}$. Then we have

$$D_{\text{KL}}(P^a(X_{ij}^{(1)}) \| P^b(X_{ij}^{(1)})) \leq 5p\Delta_0^2,$$

where we have used the fact that $P^a(X_{ij}^{(1)})$ and $P^b(X_{ij}^{(1)})$ both take values in $[\frac{7}{18}, \frac{11}{18}]$ since $\Delta_0 \leq \frac{1}{9n-1}$.

Case III: Otherwise, suppose that both $i, j \in \{a, a + 1, b, b + 1\}$. Then we have

$$D_{\text{KL}}(P^a(X_{ij}^{(1)}) \| P^b(X_{ij}^{(1)})) \leq 20p\Delta_0^2.$$

Combining the bounds from the three cases, we find that the KL divergence is upper bounded as

$$\frac{1}{r} D_{\text{KL}}(P^a \| P^b) \leq 40(n - 4)p\Delta_0^2 + 240p\Delta_0^2 \leq 50np\Delta_0^2,$$

where we have used the assumption $n \geq 9$ to obtain the final inequality.

5.3 Proof of Theorem 2

We now turn to the proof of Theorem 2 beginning with part (a).

5.3.1 Proof of part (a)

Without loss of generality, we can assume that the true underlying ranking is the identity ranking, that is, item $i$ is ranked at position $i$ for every $i \in [n]$. Given the the lower bound $\alpha \geq 8$ is satisfied, Theorem 1 ensures that with probability at least $1 - n^{-16}$, the counting estimator $\hat{S}_k$ ranks every item in $\{1, \ldots, k - h\}$ higher than every item in the set $\{k + h + 1, \ldots, n\}$.

Thus, we are guaranteed that either $\hat{S}_k \subseteq [k + h]$ and/or $[k - h] \subseteq \hat{S}_k$. One can verify either case leads to $|\hat{S}_k \cap [k]| \geq k - h$, thereby proving the claimed result.
5.3.2 Proof of part (b)

We assume without loss of generality that $k \leq \frac{n}{2}$. (Otherwise, one can equivalently study the problem of recovering the last $k$ items.) Since the case $h = 0$ is already covered by Theorem 1(b), we may also assume that $h \geq 1$.

The proof involves construction of $L \geq 1$ sets of probability matrices $\{M^a\}_{a \in [L]}$ of the pairwise comparisons with the following two properties:

(i) For every $a \in [L]$, let $S_k^a \subseteq [n]$ denote the set of the top $k$ items under the $a^{th}$ set of distributions. Then for every $k$-sized set $S \in [n],$

$$\sum_{a=1}^{L} 1\{D_H(S, S_k^a) \leq 2h\} \leq 1.$$  

(ii) If the underlying distribution $a$ is chosen uniformly at random from this set of $L$ distributions, then any estimator that attempts to identify the underlying distribution $a \in [L]$ errs with probability at least $\frac{1}{7}$.

Now consider any estimator $\hat{S}_k$ for identifying the top $k$ items $S_k^\star$. Given property (i), whenever the estimator is successful under the Hamming error requirement $D_H(\hat{S}_k, S_k^\star) \leq 2h$, it must be able to uniquely identify the index $a \in [L]$ of the underlying distribution of pairwise comparison probabilities. However, property (ii) mandates that any estimator for identifying the underlying distribution errs with a probability at least $\frac{1}{7}$. Assuming that such sets of probability distributions satisfying these two properties exist, putting these results together yields the claimed result.

We now proceed to construct probability distributions satisfying the two aforementioned properties. Consider any positive number $\Delta_0$ satisfying the upper bound

$$\Delta_0 \leq \frac{1}{14} \sqrt{\nu_1 \nu_2 \log \frac{n}{npr}}. \quad (24)$$

The $L$ matrices $\{M^a\}_{a \in [L]}$ of probability distributions we construct differ only in a permutation of their rows and columns, and modulo this permutation, have identical values. In other words, these $L$ distributions differ only in the identities of the $n$ items and the values of the pairwise-comparison probabilities $M^a_{(i)(j)}$ among the ordered sequence of the $n$ items are identical across all distributions $a \in [L]$.

For any ordering $(1), \ldots, (n)$ of the $n$ items, for every $a \in [L]$, set

$$M^a_{(i)(j)} = \begin{cases} \frac{1}{2} + \Delta_0 & \text{if } i \in [k] \text{ and } j \notin [k] \\ \frac{1}{2} - \Delta_0 & \text{if } i \notin [k] \text{ and } j \in [k] \\ \frac{1}{2} & \text{otherwise.} \end{cases} \quad (25)$$

Note that the upper bound $[24]$ on $\Delta_0$, coupled with the assumption $p \geq \sqrt{\frac{\log n}{2npr}}$, ensures that $\Delta_0 < \frac{1}{2}$ and hence that our definition $[25]$ leads to a valid set of probabilities. Given this construction, the scores of the $n$ items are $\tau(1) = \cdots = \tau(k) = \tau(k+1) + \Delta_0 = \cdots = \tau(n) + \Delta_0$. The bound $[24]$ ensures that the condition $\alpha \leq \frac{\sqrt{\nu_1 \nu_2}}{14}$ required by the hypothesis of the theorem is satisfied.
It remains to specify the ordering of the $n$ items in each set of probability distributions. This specification relies on the following lemma, that in turn uses a coding-theoretic result due to Levenshtein [Lev71]. It applies in the regime $2h \leq \frac{1}{1+\nu_2} \min \{ n^{1-\nu_1}, k, n-k \}$ for some constants $\nu_1 \in (0,1)$ and $\nu_2 \in (0,1)$, and when $n$ is larger than a $(\nu_1, \nu_2)$-dependent constant.

**Lemma 3.** Under the previously given conditions, there exists a subset $\{ b^1, \ldots, b^L \} \subseteq \{ 0,1 \}^{n/2}$ with cardinality $L \geq e^{\frac{9}{2} \nu_1 \nu_2 h \log n}$, and such that

$$D_H(b^j, 0) = 2(1 + \nu_2)h, \quad \text{and} \quad D_H(b^j, b^k) > 4h \quad \text{for all } j \neq k \in [L].$$

We prove this lemma at the end of this section. Given this lemma, we now complete the proof of the theorem. Map the $\frac{n}{2}$ items $\{ \frac{n}{2} + 1, \ldots, n \}$ to the $\frac{n}{2}$ bits in each of the strings given by Lemma 3. For each $\ell \in [e^{\frac{9}{2} \nu_1 \nu_2 h \log n}]$, let $B_\ell$ denote the $2(1 + \nu_2)h$-sized subset of $\{ \frac{n}{2} + 1, \ldots, n \}$ corresponding to the $2(1 + \nu_2)h$ positions equalling 1 in the $\ell$th string. Also define sets $A_\ell = \{ 1, \ldots, k - 2(1 + \nu_2)h \}$ and $C_\ell = [n] \setminus (A_\ell \cup B_\ell)$. We note that this construction is valid since $2h \leq \frac{1}{1+\nu_2} k$.

We now construct $L = e^{\frac{9}{2} \nu_1 \nu_2 h \log n}$ sets of pairwise comparison probability distributions $M^1, \ldots, M^L$ and show that these sets satisfy the two required properties. As mentioned earlier, each matrix of comparison-probabilities $M^\ell$ takes values as given in (25), but differs in the underlying ordering of the $n$ items. In particular, associate the set $\ell \in [L]$ of distributions to any ordering of the $n$ items that ranks every item in $A_\ell$ higher than every item in $B_\ell$, and every item in $B_\ell$ in turn higher than every item in $C_\ell$. Then for any $\ell$, the set of top $k$ items is given by $A_\ell \cup B_\ell$. From the guarantees provided by Lemma 3 for any distinct $\ell, m \in [L]$, we have $D_H(A_\ell \cup B_\ell, A_m \cup B_m) \geq 4h + 1$. This construction consequently satisfies the first required property.

We now show that the construction also satisfies the second property: namely, it is difficult to identify the true index. We do so using Fano’s inequality (19), for which we denote the probability distribution of the observations due to any matrix $M^\ell, \ell \in [L]$, as $\mathbb{P}^\ell$.

We first derive an upper bound on the Kullback-Leibler divergence between any two distributions $\mathbb{P}^\ell$ and $\mathbb{P}^m$ of the observations. Observe that $\mathbb{P}^\ell(i \succ j) \neq \mathbb{P}^m(i \succ j)$ only if $i \in B_\ell \cup B_m$ or $j \in B_\ell \cup B_m$. In this case, we have $D_{KL}(\mathbb{P}^\ell(i \succ j) \| \mathbb{P}^m(i \succ j)) \leq \frac{4\Delta_0^2}{\frac{1}{4} - \Delta_0^2}$. Since both sets $B_\ell$ and $B_m$ have a cardinality of $2(1 + \nu_2)h$, aggregating over all possible observations across all pairs, we obtain that

$$D_{KL}(\mathbb{P}^\ell \| \mathbb{P}^m) \leq 4(1 + \nu_2)hnpr \frac{4\Delta_0^2}{\frac{1}{4} - \Delta_0^2}. \quad \text{(26)}$$

In the regime $p \geq \frac{\log n}{2nr}$ and $\Delta_0 \leq \frac{1}{14} \sqrt{\frac{\nu_1 \nu_2 \log n}{npr}}$, we have $\Delta_0 \leq \frac{1}{14\sqrt{2}}$. Substituting the inequality $\Delta_0 \leq \frac{1}{14} \sqrt{\frac{\nu_1 \nu_2 \log n}{npr}}$ in the numerator and $\frac{1}{4} - \Delta_0^2 \geq \frac{1}{4} - \left( \frac{1}{14\sqrt{2}} \right)^2$ in the denominator of the right hand side of the bound (26), we find that

$$D_{KL}(\mathbb{P}^\ell \| \mathbb{P}^m) \leq \frac{3}{4} \nu_1 \nu_2 h \log n.$$ 

Now suppose that we drawn $Y$ from some distribution chosen uniformly at random from $\{\mathbb{P}^1, \ldots, \mathbb{P}^L\}$. Applying Fano’s inequality (19) ensures that any test $\phi$ for estimating the index $A$ of the chosen distribution must have error probability lower bounded as

$$\mathbb{P}[\phi(Y) \neq A] \geq \left( 1 - \frac{\frac{3}{4} \nu_1 \nu_2 h \log n + \log 2}{\frac{9}{16} \nu_1 \nu_2 h \log n} \right) \geq \frac{1}{7}.$$ 

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Here the final inequality holds as long as $n$ is larger than some universal constant.

### 5.3.3 Proof of Lemma 3

We divide the proof into two cases depending on the value of $h$.

**Case I:** $h \geq \frac{1}{2
u_1\nu_2}$. Let $L$ denote the number of binary strings of length $m_0$ such that each has a Hamming weight $w_0$ and pair has a Hamming distance at least $d_0$. It is known [Lev71, JV04] that $L$ can be lower bounded as:

$$L \geq \frac{\binom{m_0}{w_0}}{\sum_{i=0}^{\lfloor \frac{d_0-1}{2} \rfloor} \binom{m_0}{w_0-i} \binom{m_0-w_0}{j}} \geq \frac{\binom{m_0}{w_0}}{\frac{d_0+1}{2} \left( \frac{e w_0}{\min\{d_0,w_0\}/2} \right)^{\min\{d_0,w_0\}/2} \left( \frac{e m_0}{\min\{d_0,m_0\}/2} \right)^{\min\{d_0,m_0\}/2}}.$$

Note that for the setting at hand, we have $m_0 = \frac{n}{2}$, $w_0 = 2(1 + \nu_2)h$ and $d_0 = 4h + 1$. Since $\nu_1 \in (0, 1)$ and $\nu_2 \in (0, 1)$, we have the chain of inequalities

$$w_0 < d_0 \leq 4n^{1-\nu_1} \frac{n}{2} = m_0,$$

where the inequality (i) holds when $n$ is large enough. These relations allow for the simplification:

$$\log L \geq \log \left\{ \frac{\binom{m_0}{w_0}}{\frac{d_0+1}{2} \left( \frac{e w_0}{\min\{d_0,w_0\}/2} \right)^{\min\{d_0,w_0\}/2} \left( \frac{e m_0}{\min\{d_0,m_0\}/2} \right)^{\min\{d_0,m_0\}/2}} \right\}$$

$$= (w_0 - d_0/2) \log m_0 - w_0 \log w_0 + \frac{d_0}{2} \log d_0 - \frac{d_0 + w_0}{2} \log(2e) - \log((d_0 + 1)/2).$$

Substituting the values of $w_0$, $d_0$ and $m_0$ and then simplifying yields

$$\log L \geq (2\nu_2 h - \frac{1}{2}) \log n - 2(1 + \nu_2)h \log(2(1 + \nu_2)h) + (2h + \frac{1}{2}) \log(4h + 1)$$

$$- (((3 + \nu_2)h) + \frac{1}{2}) \log(2e) - \log(2h + 1)$$

$$\geq (2\nu_2 h - \frac{1}{2}) \log n - 2\nu_2 h \log(2(1 + \nu_2)h) - c_1' h,$$

where $c_1'$ is a constant whose value depends only on $(\nu_1, \nu_2)$. In the regime $\frac{1}{\nu_1\nu_2} \leq 2h \leq \frac{n^{1-\nu_1}}{1+\nu_2}$, some algebraic manipulations then yield

$$\log L \geq (2\nu_1\nu_2 h - \frac{1}{2}) \log n - c_1' h \geq \nu_1\nu_2 h (\log n - \log 2 - c_1') \geq \frac{9}{10} \nu_1\nu_2 h \log n,$$

where the final inequality holds when $n$ is large enough.

**Case II:** $h < \frac{1}{2\nu_1\nu_2}$. Consider a partition of the $\frac{n}{2}$ bits into $\frac{n}{4(1+\nu_2)h}$ sets of size $2(1 + \nu_2)h$ each. Define an associated set of $\frac{n}{4(1+\nu_2)h}$ sets of binary strings, each of length $\frac{n}{2}$, with the $i^{th}$ string having ones in the positions corresponding to the $i^{th}$ set in the partition and zeros elsewhere. Then each of these strings have a Hamming weight of $2(1 + \nu_2)h$, and every pair has a Hamming distance at least $4(1 + \nu_2)h > 4h$. The total number of such strings equals

$$\exp \left\{ \log \frac{n}{4(1 + \nu_2)h} \right\} \geq \exp \left\{ \log n - \log \left( \frac{2(1 + \nu_2)}{\nu_1\nu_2} \right) \right\} \geq \exp \left\{ \frac{9}{10} \log n \right\} \exp \left\{ 1.8\nu_1\nu_2 h \log n \right\},$$

where the inequalities (i) and (iii) are a result of operating in the regime $h < \frac{1}{2\nu_1\nu_2}$ and the inequality (ii) assumes that $n$ is greater than a $(\nu_1, \nu_2)$-dependent constant.
5.4 Proof of Theorem 3
We now turn to the proof of Theorem 3.

5.4.1 Proof of part (a)
For every $i \in [n]$, let $(i)$ denote the item ranked $i$ according to their latent scores, as defined in equation (2). Recall from the proof of Theorem 1 that for any $u < v \in [n]$, the condition

$$\tau(u) - \tau(v) \geq 8 \sqrt{\frac{\log n}{npr}}$$

ensures that with probability at least $1 - n^{-14}$, every item in the set $\{(1), \ldots, (u)\}$ wins more comparisons than every item in the set $\{(v), \ldots, (n)\}$. Consequently, if the set $\tilde{\mathcal{S}}_k$ contains any item in $\{(v), \ldots, (n)\}$, then it must contain the entire set $\{(1), \ldots, (u)\}$. In other words, at least one of the following must be true: either $\{(1), \ldots, (u)\} \subseteq \tilde{\mathcal{S}}_k$ or $\tilde{\mathcal{S}}_k \subseteq \{(1), \ldots, (v-1)\}$. Consequently, in the regime $v = k + t - u + 1$ for any $1 \leq u \leq k$ and $u \leq t \leq n$, we have that

$$|\tilde{\mathcal{S}}_k \cap \{(1), \ldots, (t)\}| \geq u.$$  \hspace{1cm} (27)

Now consider any $b \in [\beta]$ that satisfies the condition

$$\min_{j \in [k]} (\tau(j) - \tau(k+tb_j-j+1)) \geq 8 \sqrt{\frac{\log n}{npr}}.$$  

For any $j \in [k]$, setting $u = j$ and $v = (k+tb_j-j+1)$ in (27), and applying the union bound over all values of $j \in [k]$ yields that

$$|\tilde{\mathcal{S}}_k \cap \{(1), \ldots, (tb_j)\}| \geq j \quad \text{for every } j \in [k],$$

with probability at least $1 - n^{-13}$. Consequently, we have that

$$\Pr(\tilde{\mathcal{S}}_k \in \Lambda(T_b)) \geq 1 - n^{-13},$$

completing the proof of the claim.

5.4.2 Proof of part (b)
In the regime $\frac{tb_k}{\mu_2k} \leq \frac{n}{2}$ for every $b \in [\beta]$, it suffices to show that any estimator $\tilde{\mathcal{S}}_k$ will incur an error lower bounded as

$$\Pr(|\tilde{\mathcal{S}}_k \cap \{(1), \ldots, (n/2)\}| < \mu_2k) \geq \frac{1}{15},$$

where $(i)$ denotes the item ranked $i$ according to their latent scores according to equation (2).

Our proof relies on the result and proof of the Hamming error case analyzed in Theorem 2(b). To this end, let us set the parameter $h$ of Theorem 2(b) as $h = 2(1 - \mu_2)k$. We claim that this value of $h$ lies in the regime $h \leq \frac{1}{2(1+\nu_2)} \min\{k, n-k, n^{1-\nu_1}\}$ for some values $\nu_1 \in (0, 1)$ and $\nu_2 \in (0, 1)$, as required by Theorem 2(b). This claim follows from the fact that

$$h = 2(1 - \mu_2)k \leq \frac{1}{2(1+\nu_2)}k,$$
for $\nu_2 = \min \left\{ \frac{1}{4(1-\mu_2)} - 1, \frac{1}{2} \right\} \in (0,1)$. Furthermore,

$$h = 2(1 - \mu_2)k \leq \frac{n^{1-\mu_1}}{4} \leq \frac{1}{2(1+\nu_2)} n^{1-\nu_1}$$

for $\nu_1 = \frac{9}{10} \mu_1 \in (0,1)$, where $(i)$ is a result of our assumption $8(1 - \mu_2)k \leq n^{1-\mu_1}$ and $(ii)$ holds when $n$ is large enough. This assumption also implies that $k \leq n - k$ for a large enough value of $n$. We have now verified operation in the regime required by Theorem 2(b).

The construction in the proof of Theorem 2 is based on setting

$$\tau(1) = \cdots \tau(k) = \tau(k+1) + \Delta_0 = \cdots = \tau(n) + \Delta_0,$$

for any real number $\Delta_0$ in the interval $\left(0, \frac{1}{14} \sqrt{\frac{\mu_1 \nu_2 \log n}{npr}} \right]$. This condition is also satisfied in our construction due to the assumed upper bound $\alpha \leq \frac{1}{15} \sqrt{\mu_1 \min \left\{ \frac{1}{4(1-\mu_2)} - 1, \frac{1}{2} \right\}}$. Consequently, the result of Theorem 2(b) implies that in this setting, any estimator $\hat{S}_k$ will incur a Hamming error greater than $h = 2(1 - \nu_2)k$ with probability at least $\frac{1}{7}$, or equivalently,

$$\mathbb{P}(||S_k \cap \{(1), \ldots, (k)\}| < (2\nu_2 - 1)k) \geq \frac{1}{7}.$$

Under this event, the estimator $\hat{S}_k$ contains at most $(2\nu_2 - 1)k - 1$ items from the set of top $k$ items. In order to ensure it gets at least $\nu_2k$ items from $\{(1), \ldots, (n/2)\}$, the remaining $2(1 - \mu_2)k + 1$ chosen items must have at least $(1 - \mu_2)k + 1$ items from $\{(k+1), \ldots, (n/2)\}$.

However, in the construction, items $(k+1), \ldots, (n)$ are indistinguishable from each other, and hence by symmetry these $2(1 - \mu_2)k + 1$ chosen items must contain at least $(1 - \mu_2)k + 1$ items from the set $\{(n/2 + 1), \ldots, (n)\}$ with probability at least $\frac{1}{7}$. Putting these arguments together, we obtain that under this construction, any estimator $\hat{S}_k$ has error probability lower bounded as

$$\mathbb{P}(||S_k \cap \{(1), \ldots, (n/2)\}| < \nu_2k) \geq \frac{1}{14}. \quad (28)$$

It remains to deal with a subtle technicality. The construction above involves items $(k+1), \ldots, (n)$ with identical scores. Recall that in the definition of the user-defined requirement, in case of multiple items with identical scores, we considered the choice of either of such items as valid. The following lemma helps overcome this issue. In order to state the lemma, we define $\|M\|_\infty := \max_{(i,j) \in [n]^2} |M_{ij}|$ for a matrix $M \in \mathbb{R}^{n \times n}$.

**Lemma 4.** Consider any two $(n \times n)$ matrices $M^a$ and $M^b$ of pairwise probabilities such that

$$\|M^a - M^b\|_\infty \leq \epsilon, \quad \|M^a\|_\infty \geq \epsilon, \text{ and } \|M^b\|_\infty \geq \epsilon \quad (29a)$$

for some $\epsilon \in [0,1]$. Then for any $k$-sized sets of items $T_1, \ldots, T_\beta \subseteq [n]$, and any estimator $\hat{S}_k$, we have

$$|\mathbb{P}_{M^a}(\hat{S}_k \in \{T_1, \ldots, T_\beta\}) - \mathbb{P}_{M^b}(\hat{S}_k \in \{T_1, \ldots, T_\beta\})| \leq 6^{n^2r} \epsilon. \quad (29b)$$
See Section 5.4.3 for the proof of this claim.

Now consider an \((n \times n)\) pairwise probability matrix \(M'\) whose entries takes values

\[
M'_{(i)(j)} = \begin{cases} 
\frac{1}{2} + \Delta_0 + \epsilon & \text{if } i \in [k] \text{ and } j \in [n] \setminus [n/2] \\
\frac{1}{2} + \Delta_0 & \text{if } i \in [k] \text{ and } j \in [n/2] \setminus [k] \\
\frac{1}{2} + \epsilon & \text{if } i \in [n/2] \setminus [k] \text{ and } j \in [n] \setminus [n/2] \\
\frac{1}{2} & \text{otherwise,}
\end{cases}
\]

and \(M'_{ii} = 1 - M'_{ij}\), whenever \(i \leq j\). Set \(\epsilon = 7^{-n^2r}\).

One can verify that under the probability matrix \(M'\), the scores of the \(n\) items satisfy the relations

\[
\tau(1) = \cdots = \tau(k) = \tau(k+1) + \Delta_0 = \cdots = \tau(n/2) + \Delta_0 = \tau(n/2+1) + \Delta_0 + \epsilon = \cdots = \tau(n) + \Delta_0 + \epsilon.
\]

The set of items \(\{(1), \ldots, (n/2)\}\) are thus explicitly distinguished from the items \(\{(n/2 + 1), \ldots, (n)\}\). We now call upon Lemma 4 with \(M^a = M'\), and \(M^b\) as the matrix of probabilities constructed in the proof of Theorem 2, where both sets have the same ordering of the items. This assignment is valid given that \(\Delta_0 < \frac{1}{3}\) and \(\epsilon = 7^{-n^2r}\). Lemma 4 then implies that any estimator that is \(\mathcal{S}\)-respecting with probability at least \(1 - \frac{1}{15}\) under \(M^b\) must also be \(\mathcal{S}\)-respecting with probability at least \(1 - \frac{1}{15}\) under \(M^a\). But by equation (28), the latter condition is impossible, which implies our claimed lower bound.

### 5.4.3 Proof of Lemma 4

Let \(P^a\) and \(P^b\) denote the probabilities induced by the matrices \(M^a\) and \(M^b\) respectively. Consider any fixed observation \(Y_1 \subseteq \{0, 1, \phi\}^{r(n \times n)}\), where \(\phi\) denotes the absence of an observation. Given the bounds (29a), some algebra leads to

\[
| P^a(Y = Y_1) - P^b(Y = Y_1) | \leq 2^{n^2r} \epsilon, \tag{30}
\]

where \(P^a(Y = Y_1)\) and \(P^b(Y = Y_1)\) denote the probabilities of observing \(Y_1\) under \(P^a\) and \(P^b\), respectively.

Now consider any estimator \(\hat{S}_k\), which is permitted to be randomized. Let \(L \leq 3^{n^2r}\) denote the total number of possible values of the observation \(Y\), and let \(\{Y_1, \ldots, Y_L\} = \{0, 1, \phi\}^{r(n \times n)}\) denote the set of all possible valid values of the observation. For each \(i \in [L]\), let \(q_i \in [0, 1]\) denote the probability that the estimator \(\hat{S}_k\) succeeds in satisfying the given requirement when the data observed equals \(Y_i\). (Recall that the given requirement is in terms of the actual items and not their positions.) Then we have

\[
| P^1(\hat{S}_k \in \{T_1, \ldots, T_\beta\}) - P^2(\hat{S}_k \in \{T_1, \ldots, T_\beta\}) | = \left| \sum_{i=1}^L P^1(Y = Y_i)q_i - \sum_{i=1}^L P^2(Y = Y_i)q_i \right| \\
\leq \sum_{i=1}^L \left| P^1(Y = Y_i) - P^2(Y = Y_i) \right| q_i \\
\leq \sum_{i=1}^L 2^{n^2r} \epsilon q_i \leq 6^{n^2r} \epsilon,
\]

as claimed, where step (i) follows from our earlier bound (30) and step (ii) uses the bound \(L \leq 3^{n^2r}\).
6 Discussion

In this paper, we analyzed the problem of recovering the $k$ most highly ranked items based on observing noisy comparisons. We proved that an algorithm that simply selects the items that win the maximum number of comparisons is, up to constant factors, an information-theoretically optimal procedure. Our results also extend to recovering the entire ranking of the items as a simple corollary. In empirical evaluations, this algorithm takes several orders of magnitude lower computation time while providing higher accuracy as compared to prior work. The results of this paper thus underscore the philosophy of Occam’s razor that the simplest answer is often correct.

There are number of open questions suggested by our work. The observation model considered here is based on a random number of observations for all pairs of comparisons. It would be interesting to extend our results to cases in which only specific subsets of pairs are observed. Moreover, we considered a random design setting where we do not have any control over which pairs are compared. The notion of allowable sets introduced in this paper apply to recovery of $k$-sized subsets of the items; such a formulation and associated results may apply to recovery of partial or total orderings of the items. A parallel line of literature (e.g., [KK13 BFSC+13 JKD15]) studies settings in which the pairs to be compared can be chosen sequentially in a data-dependent manner, but to the best of our knowledge, this line of literature considers only the metric of exact recovery of the top $k$ items. It is of interest to investigate the Hamming and allowable set recovery problems in such an active setting.

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