Density of kinks after a quench: When symmetry breaks, how big are the pieces?

Pablo Laguna\(^{(1,2)}\) and Wojciech Hubert Zurek\(^{(1)}\)

\(^{(1)}\) Theoretical Astrophysics, T-6, MS B288
Los Alamos National Laboratory, Los Alamos, NM 87545, USA
\(^{(2)}\) Department of Astronomy & Astrophysics and Center for Gravitational Physics & Geometry
Penn State University, University Park, PA 16802, USA
(March 24, 2022)

Numerical study of order parameter dynamics in the course of second order (Landau-Ginzburg) symmetry breaking transitions shows that the density of topological defects, kinks, is proportional to the fourth root of the rate of the quench. This confirms the more general theory of domain-size evolution in the course of symmetry breaking transformations proposed by one of us \(^{[1]}\). Using these ideas, it is possible to compute the density of topological defects from the quench timescale and from the equilibrium scaling of the correlation length and relaxation time near the critical point.

\[ \langle |\delta \varphi(x, t)| \rangle \sim \exp(-|\Delta|/\xi) \]

Below \(T_C\), the spatial order is established on scales much larger than \(\xi\); however, the scale over which the healing occurs —for example, from the “wounds” inflicted by topological defects—is characterized by \(\xi\). Similarly, \(\tau\) characterizes the time required for the order parameter to relax to its equilibrium value. During the phase transition, in the immediate vicinity of the critical temperature, the motion will be often overdamped (that is, dominated by the first time derivative of the order parameter) and \(\tau = \tau_0/|\epsilon|\).

The estimate of the density of defects put forward in Ref. \(^{[4]}\) is based on a linear quench,

\[ \epsilon = t/Q \]

which is expected to be a suitable approximation in the neighborhood of \(T_C\). In \(^{[4]}\), \(Q\) is the quench timescale.

\[ \xi = \xi_0/|\epsilon\|^\nu \]
and \( t \) the time before \((t < 0)\) and after \((t > 0)\) the transition \((\epsilon = 0)\). When the critical temperature is crossed at the fixed rate given by Eq. \((3)\), there will come a moment when the order parameter —because of the critical slowing down implied by Eq. \((3)\)— will be simply unable to adjust its actual correlation length to the equilibrium \( \xi \) given by Eq. \((1)\). This will occur when the time remaining until the transition, \( \tilde{t} \), equals the relevant dynamical relaxation time \( \tau(\tilde{t}) \), that is, when

\[
\tau(\tilde{t}) = \frac{\tau_0}{|\tilde{\eta}|/\tau_Q} = \tilde{t} .
\]  

(5)

This equation yields a relative temperature

\[
\dot{\epsilon} = \frac{\tilde{t}}{\tau_Q} = \left( \frac{\tau_0}{\tau_Q} \right)^{\frac{1}{\mu}},
\]  

(6)

at which the evolution of the order parameter will cease to follow equilibrium. At this point, the correlation length of \( \varphi \) will cease to diverge in accord with Eq. \((1)\), as the phase transition region is traversed. Instead, \( \xi \) will reach a value approximately given by

\[
\dot{\xi} = \frac{\xi_0}{|\tilde{\eta}|^{\mu}} = \left( \frac{\tau_0}{\tau_Q} \right)^{\frac{1}{\mu}}.
\]

(7)

Regions more distant than \( \dot{\xi} \) will be forced to select the new vacuum independently.

An essentially equivalent estimate of the characteristic domain size is obtained from the “causal horizon” idea, but only when the appropriate velocity of the signal propagation on \( \varphi \) is adopted \([\ref{1}] [\ref{3}]\). That velocity is of the order of \( v = \xi/\tau \) for the dynamical processes correlating different regions of the order parameter. Thus, the size of the corresponding causal horizon size (and, therefore, the size of the domain) will be no more than \( h(\tilde{t}) = v(\tilde{t}) \tilde{t} \approx \dot{\xi} \).

In any case, the size of independently selected domains of the new vacuum will be given, to the leading order, by \( \dot{\xi} \), which will also determine the typical separation of topological defects and, therefore, the initial defect density \( n \). For instance, for monopoles \( n \approx 1/(f\tilde{\xi}^D) \), with \( D \) the space dimension and \( f \sim O(1) \) a factor presumably somewhat larger than unity. \( f \) takes into account the possibility that the independent choices of the vacuum may be similar anyway, and that the correlations will grow slowly following the instant \( \tilde{t} \) due to diffusion, etc.; the key prediction of the theory of \([\ref{1}]\) is in any case the scaling, as is usually the case in the critical phenomena.

The aim of the work in this Letter is to test this theory with a computer experiment. The immediate motivation of our research comes from recent superfluid experiments \([\ref{1}] [\ref{2}] [\ref{4}]\). These experiments follow the original suggestion in Ref. \([\ref{1}]\) and appear to support the estimate of defect formation based on \( \dot{\xi} \). While the quench-generated density of vortex lines is somewhat uncertain, it is, nonetheless, in accord with the theory summed up above but in conflict with the ideas based on activation and Ginzburg temperature \([\ref{12}]\). However, these experiments were, at least so far, unable to vary the quench time \( \tau_Q \). Thus, it was not possible to systematically test the key predictions of Ref. \([\ref{1}]\), namely the power law dependence of the size of the fragments of the broken symmetry vacuum on the quench rate \( \tau_Q^{-1} \) given by Eq. \((7)\), and the complementary dependence of the initial number of defects.

To investigate this issue, we considered the numerical evolution of a one-dimensional system \([\ref{13}]\) for a real field \( \varphi \) according to the equation of motion derived from the Landau-Ginzburg potential \( V(\varphi) = (\varphi^2 - 2\epsilon\varphi^2 + 1)/8 \). The system is assumed to be in contact with a thermal reservoir. Thus, it obeys the Langevin equation

\[
\ddot{\varphi} + \eta\dot{\varphi} - \partial_{xx}\varphi + \partial_x V(\varphi) = \vartheta .
\]

(8)

The noise term \( \vartheta \) has correlation properties

\[
\langle \vartheta(x,t), \vartheta(x',t') \rangle = 2\eta\theta\delta(x'-x)\delta(t'-t) .
\]

(9)

In Eq. \((9)\), \( \eta \) is the overall damping constant which also helps characterize the amplitude of the noise through Eq. \((8)\). The parameter \( \epsilon \) measures the distance from the phase transition and varies according to Eq. \((4)\); that is, \( \epsilon = \min(1, t/\tau_Q) \). \( \theta \) describes the temperature of the reservoir and is kept constant throughout this work. This separate parametrization of \( \epsilon \) and \( \theta \) was adopted to correspond to the situation in superfluid He \( ^4 \), where the symmetry breaking can be induced by the change of the pressure and occurs with inconsequential adjustments of the absolute temperature \([\ref{13}]\).

The second order time evolution allows us in principle to make contact with cosmology, but, in the regime considered here, the evolution is dominated by the dissipative term \( \eta\dot{\varphi} \). Hence, in effect, we are dealing with the time dependent Landau-Ginzburg equation.

We investigate the creation of kinks as a function of \( \tau_Q \) by starting at some \( \epsilon < 0 \) suitably above the transition, and then gradually adjusting the shape of the potential in accord with Eq. \((3)\). Figure \( \ref{fig:2} \) shows a sequence of “snapshots” of \( \varphi \) obtained in the course of such a quench. When \( \epsilon < 0 \), \( \varphi \) fluctuates around its expectation value \( \langle \varphi \rangle = 0 \). The same situation initially persists for slightly positive \( \epsilon \). However, further below \( T_C \), \( \varphi \) settles locally around one of the two alternatives: \( \langle \varphi \rangle = \pm\sqrt{\epsilon} \). Moreover, local choices of one of these two alternatives cannot be easily undone once a certain symmetry breaking is selected, unless the kinks are separated by distances no larger than \( \dot{\xi} \) (see Fig. \( \ref{fig:3} \)).

To test the predictions of defect density in Ref. \([\ref{1}]\), we note again that, for a sizable \( \eta \) and sufficiently small \( \epsilon \), the damping term \( \eta\dot{\varphi} \) in Eq. \((8)\) is bound to dominate. Such overdamped evolution will take place whenever \( \eta^2 > |\epsilon| \) \([\ref{14}]\). In our case, \( \eta > 1 \) and \( |\epsilon| < 1 \). The characteristic relaxation time is then
given by $\tau = \eta/|\epsilon|$. Consequently, $\mu = 1$ in Eq. (2), and
\begin{equation}
\dot{\xi} = \sqrt{\eta/\tau_Q}.
\end{equation}

The corresponding value of the relative temperature immediately yields, $\dot{\epsilon} = \dot{\xi}/\tau_Q = \sqrt{\eta/\tau_Q}$. It follows that
\begin{equation}
\dot{\xi} = \xi_0 (\tau_Q/\eta)^{1/4},
\end{equation}

where we have adopted $\nu = 1/2$ in accord with the Landau-Ginzburg theory and in agreement with the critical exponents inferred from the behavior of the healing length in Fig. 1. Eqs. (10) and (11) are expected to be applicable as long as the condition $\eta^2 > |\epsilon|$ holds at $\dot{\epsilon}$, which in turn implies $\eta/\tau_Q > 1$.

Figure 3 is the principal result of our paper. It illustrates the density of kinks obtained in a sequence of quenches with $\eta = 1$ for various $\tau_Q$ values. For each $\tau_Q$, the phase transition was simulated 15 times, starting at $\epsilon = -1$ (except for the shortest and longest $\tau_Q$, which were initiated at $\epsilon = -10/\sqrt{\tau_Q}$). Our computational domain had periodic boundary conditions. Production runs were carried out with resolution of 16,384 gridpoints. Convergence was checked by comparing results at different resolutions. The “physical” size of the computational ring in the production runs was 2,048 units. At this scale, the ring is large enough, so boundary effects are avoided.

The number of kinks produced by the quench was obtained by counting the number of zeros of $\varphi$. Above and immediately below $T_C$, there is a significant number of zeros which have little to do with the kinks (see Fig. 3). However, as the quench proceeds, the number of zeros quickly evolves towards an “asymptotic” value (see Fig. 1). This change of the density of zeros and its eventual stabilization is associated with the obvious change of the character of $\varphi$ and with the appearance of the clearly defined kinks. By then, the number of kinks is nearly constant in the runs with long $\tau_Q$, and, correspondingly, their kink density is low. Even in the runs with the smallest $\tau_Q$, there is still a clear break between the post-quench rates of the disappearance of zeros and the long-time, relatively small rate at which the kinks annihilate.

In the regime here investigated, the theoretical scaling relation for the number density of kinks, $n \approx (\eta/\tau_Q)^{\frac{4}{7}} \approx 1/\xi$ appears to be well satisfied, with $f \sim 8$. It is interesting to note that a similar value of $f$ has been deduced from the experiment in Ref. [1]. Experimentally we find $n = (8.7 \times 10^{-2} \pm 1.0 \times 10^{-3})\xi_Q^{0.29 \pm 0.02}$, when the kinks are counted at approximately the same $t/\tau_Q$ value. A similar scaling is also obtained for kinks counted at equal $t$ times after the quench.

In summary, our numerical experiment appears to provide a strong confirmation of the theoretical predictions given by one of us [1]. As expected, the density of topological defects is somewhat less than the inverse of $\xi$ but of the right order of magnitude. Most importantly, the scaling of the kink density with $\tau_Q$ follows closely theoretical expectations. These results are also supported by the dependence of the number of kinks on the damping parameter $\eta$, as well as by the preliminary results of the computer experiments involving complex order parameter and/or more than one dimensional space [13].

We thank S. Habib for helpful discussions. This work was partially supported by NSF grants PHY 93-09834, 93-57219 (NYI) to P.L. and NASA HPCC to W.H.Z.

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[12] Kibble has also suggested [1] that thermal activation in the correlation-sized volumes, which is thought to occur below $T_C$ but above the Ginzburg temperature $T_G$, will play a crucial role in determining the density of the defects. This would lead to the separation of defects of the order of the correlation length at $T_G$. Much of the past cosmological literature was based on this assumption, which differs from the non-equilibrium paradigm of Ref. [1]. It now appears that, at least in the cases investigated to date, the association between the $T_G$ and the density of defects has been not only ruled out by superfluid experiments [12] but also—as we show in this paper—by numerical simulations.
[13] In one dimension, there are no true phase transitions at a finite temperature (see, e.g. N. Goldenfeld, Lectures on Phase Transitions and the Renormalization Group, Addison Wesley, Reading, Massachusetts 1992). Neverthe-
less, symmetry breaking can occur (although not with an infinite range order) which will suffice for our purposes.

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FIG. 1. Characteristic equilibrium correlation (or healing) length, $\xi = \xi_0/|\epsilon|^\nu$, for the system under investigation. The best fitting yields $\xi_0 = 1.38 \pm 0.06$, $\nu = 0.41 \pm 0.03$ ($\chi^2 = 1.2$) above $T_C$, and $\xi_0 = 1.02 \pm 0.04$, $\nu = 0.48 \pm 0.02$ ($\chi^2 = 3.7$) below $T_C$, close to the Landau-Ginzburg exponent of $\nu = 1/2$.

FIG. 2. Snapshots of $\phi$ during kink formation with a quench timescale of $\tau_Q = 64$ and damping parameter $\eta = 1$. The figures, from top to bottom, correspond to $t = -80.0, 7.5, 32.5, 45.0$ and 333.0, respectively.

FIG. 3. Number of defects as a function of quench timescale. The straight line is the best fit to $n \propto \tau_Q^a$ with $a = 0.28 \pm 0.02$ ($\chi^2 = 1.96$). This exponent compares favorably with the theoretical prediction of $1/4$ based on the theory in Ref. [1].

FIG. 4. Average number of zeros as a function of time in units of $\tau_q$; from top to bottom $\tau_Q = 4, 8, ..., 2048, 4096$. The number of kinks used in Fig. 3 were obtained at $t/\tau_Q \sim 4$, except in the large, computationally expensive, $\tau_Q$ cases where an extrapolated value was used.