Covariant Charges
in Chern–Simons AdS$_3$ Gravity

G. Allemandi, M. Francaviglia, M. Raiteri
Dipartimento di Matematica, Università degli Studi di Torino,
Via Carlo Alberto 10, 10123 Torino, Italy

Abstract

We try to give hereafter an answer to some open questions about the definition of conserved quantities in Chern-Simons theory, with particular reference to Chern-Simons AdS$_3$ Gravity. Our attention is focused on the problem of global covariance and gauge invariance of the variation of Noether charges. A theory which satisfies the principle of covariance on each step of its construction is developed, starting from a gauge invariant Chern–Simons Lagrangian and using a recipe developed in [2] and [27] to calculate the variation of conserved quantities. The problem to give a mathematical well-defined expression for the infinitesimal generators of symmetries is pointed out and it is shown that the generalised Kosmann lift of spacetime vector fields leads to the expected numerical values for the conserved quantities when the solution corresponds to the BTZ black hole. The first law of black holes mechanics for the BTZ solution is then proved and the transition between the variation of conserved quantities in Chern-Simons AdS$_3$ Gravity theory and the variation of conserved quantities in General Relativity is analysed in detail.

Dedicated to Matteo Raiteri, born in the Fifth of October 2002

1 Introduction

Chern-Simons field theories have been widely studied in the past decades as a possible model to analyse the classical and quantum behaviour of the gravitational field. Efforts were focused towards rewriting gravity as a gauge theory with gauge group the Poincaré group or the (anti) de Sitter group. To this purpose, in place of the Hilbert-Einstein Lagrangian, a Chern-Simons Lagrangian was considered in which the gauge potential is a linear combination of the vielbein and the spin connection. This is possible in all odd dimensions and particularly in dimension three, where field equations reproduce exactly the Einstein
field equations; see [1, 3, 4, 15, 42] where it is shown that $2 + 1$ gravity with a negative cosmological constant can be formulated as a Chern-Simons theory (see also [16] for higher dimensional Chern-Simons gravity). In particular it was found in [14] that a particular solution of Chern-Simons theory corresponds to the well-known BTZ black hole [5].

The interest in 3-dimensional Chern-Simons theory as a possible and simpler model to analyse $(2 + 1)$-dimensional gravity was also strengthened by the observation that the thermodynamics of higher dimensional black holes can be understood in terms of the BTZ solution. The BTZ solution provides indeed a model for the geometry of a great amount of black hole solutions relevant to string theory, the geometry of which can be factorized in the product of $BTZ \times M$, where $M$ is a simple manifold; see [30].

According to the renewed interest in Chern–Simons theories a lot of papers dealing with the problem of gauge symmetries and gauge charges for Chern-Simons theories have appeared in the recent years, all addressed to analyse the origin of the gravitational boundary degrees of freedom and, eventually, to understand the statistical mechanical origin of Bekenstein-Hawking entropy via a micro- and grand-canonical calculation (see [3, 4, 10, 15] and references therein).

Motivated by this state of affairs, in the present article we deal with the problem of developing a totally covariant formulation for Chern-Simons conserved quantities, with particular attention to $SL(2, \mathbb{R}) \times SL(2, \mathbb{R}) \simeq SO(2, 2)$ Chern–Simons theory in dimension three (with this gauge choice, Chern–Simons theory is well suited to describe AdS gravity; see e.g. [1, 3, 4, 14, 15, 42]). This issue is tackled, first of all, by framing Chern–Simons theory into the mathematical domain of gauge natural theories, which provide a unifying mathematical arena to describe all classical Lagrangian field theories and, in particular, are fundamental to mathematically analyse the interaction of gravity with fermionic and bosonic matter; see e.g. [22, 23]. The gauge natural approach is essentially based on the Lagrangian formulation of field theories in terms of fiber bundles and the Calculus of Variations on jet bundles. Hence, in order to have a gauge natural formulation of Chern–Simons theory, the first problem we meet is to construct a covariant Chern–Simons Lagrangian, where covariance is referred both to spacetime diffeomorphisms and gauge transformations. Indeed, despite the field equations are covariant for both spacetime diffeomorphisms and gauge transformations, the Chern-Simons Lagrangian is not covariant for gauge transformations. To solve this problem we define a covariant Lagrangian for the theory which differs from the usual Chern-Simons Lagrangian just for the addition of a divergence term. In such a way a covariantized Lagrangian is obtained and, in the meantime, the field equations remain unchanged. The Lagrangian we calculate hereafter is the same “covariantized” Lagrangian obtained in [8] through the use of the “transgression” formula [18]. In the same paper [8] a calculation of conserved quantities based on Noether theorems and the covariant formula for the superpotential of the theory were obtained. Nevertheless, we
shall show in the present paper that the superpotential, even though it is a well-defined mathematical object, in practice it is not suited to calculate conserved quantities. Indeed when conserved quantities are explicitly calculated for the Chern–Simons solution corresponding to the BTZ black hole via the integration of the superpotential on a circle enclosing the horizon, the expected values for mass and angular momentum are not recovered even if the circle tends to spatial infinity (see equation (33) below). This is a rather undesirable result. In fact, even if the same physical solution of a field theory can be obtained from two, or more, different field theories (whose field content is nevertheless equivalent on–shell) it would be reasonable that physical observables depend solely on the given solution, e.g. the black hole, and not on the theory the solution comes from. Roughly speaking, mass, angular momentum and charge of a black hole are physical properties of the black hole itself and they should not change if the same black hole solution is obtained either from the Hilbert Lagrangian of General Relativity or from the Chern–Simons Lagrangian. Hence, the Noether charges which generate conserved quantities have to be somehow tied to the solution under examination rather than to the Lagrangian which generates the equations of motion. Since the Chern–Simons Lagrangian can be mapped into the Hilbert Lagrangian only modulo divergence terms, this basically means that the general formula for the conserved currents we are interested in has to be linked only to the homology class of the Lagrangian (i.e., it has to not depend on additional divergence terms). In this way conserved quantities are related to the equations of motion and do not depend on the representative chosen inside the homology class of Lagrangians.

A step towards the solution of this problem can be found in the papers [32], whereby conserved quantities are obtained from the equations of motion via a cascade equation. Similarly, in the present paper we shall mainly be concerned with the issue of developing a fully covariant approach to conserved quantities in Chern–Simons gravity which could lead to a direct correspondence with charges in General Relativity, thus giving the expected values for mass, angular momentum and entropy for the BTZ solution.

The main problems to be solved are two. The first one deals with the very basic object which has to enter in the definition of conserved quantities. Indeed, we will check that the superpotential does not reproduce the correct numerical values, even if in a first approach to conserved quantities it seems to be the best candidate to describe global charges. The reason of these discrepancies is mainly due to the fact that the naive definition of charges via the superpotential alone does not take into a proper account the role played by boundary terms. Boundary conditions have to be imposed on the dynamical fields, namely Dirichlet or Neumann conditions, specifying which are the intensive or extensive variables or, equivalently, which are the control–response parameters. The choice of boundary conditions basically reflects onto the choice of a background solution the starting solution has to be matched with. Physically speaking, the background fixes the zero level for all conserved quantities ([6, 11, 17, 20, 25]). A way to overcome the problems which come from the background fixing procedure is to define the variation of conserved quantities, as suggested in [6, 25, 27, 32].
In this way a covariant analysis of boundary terms, à la Regge–Teitelboim [38], can be implemented, leading to the covariant ADM formalism for conserved quantities. This is exactly the approach we shall develop also for Chern–Simons theory.

The second problem we are faced with is related to the choice of the symmetry generators. This mathematical problem is by no means trivial and it deserves a careful investigation. Indeed, in a natural theory such as General Relativity, the action of spacetime vector fields on the dynamical fields is unambiguously defined. This means that we know how the fields are Lie dragged along spacetime directions and this is enough to build a mathematical consistent theory of conserved quantities. On the contrary, in a theory with gauge invariance (such as Chern–Simons theory as well as any gauge natural theory; see [34]) there is no canonical way to construct conserved quantities starting from a given spacetime vector field. Indeed, in those theories there is no canonical way to lift the spacetime diffeomorphisms on the configuration space of the theory, i.e. the target space where fields take their values. On the contrary, there may exist different non canonical (but nevertheless global and hence well-defined) ways to perform lifts of spacetime diffeomorphisms. This in turn implies that the transformation rules of the dynamical fields under spacetime diffeomorphisms are not defined until a preferred lift procedure has been somehow selected. For example we shall see that in the Chern–Simons theory there are exactly three distinct dynamical connections. All of them can be equally well used, from a mathematical viewpoint, to define lifted vector fields on the configuration bundle starting from spacetime diffeomorphisms. These vector fields, in their turn, enter into the definition of Lie derivatives of the dynamical fields and they eventually lead to different definitions of conserved quantities. All these definitions provide well-defined and mathematical meaningful expressions. But which of them is also physically meaningful? In this paper we have tried to answer this question by testing all the admissible definitions of conserved quantities with the BTZ black hole solution. The numerical results obtained suggest that the only viable definition of (the variation of) conserved quantities inside the Chern–Simons gravity framework is the one based on the generalized Kosmann lift of spacetime diffeomorphisms [22, 28]. Conserved quantities computed with the generalized Kosmann lift reproduce, in fact, exactly the expected values for the BTZ mass, angular momentum and entropy, while other choices do not lead to meaningful results. Moreover, when we perform the transition from Chern–Simons AdS$_3$ gravity to General Relativity, the formula expressing the (variation of the) conserved quantities in Chern–Simons gravity is mapped exactly into the formula for (the variation of) the conserved quantities in General Relativity found in [6, 11, 27, 33], as one should expect.

The present paper is organized as follows. In Section 2 we define the covariant Lagrangian for $SL(2, \mathbb{R}) \times SL(2, \mathbb{R})$ Chern–Simons theory. In Section 3 we illustrate the geometric framework for the Hamiltonian and symplectic formulation of the theory and we derive a general formula to calculate the variation of conserved quantities. In Section 4 we analyse the problem of defining the lift to the
configuration bundle for an infinitesimal generator of symmetries over spacetime and we define the generalized Kosmann lift. In Section 5 we explicitly calculate the transition of the variation of conserved quantities from Chern-Simons theory to (2 + 1)–dimensional gravity. In the Appendix A are summarized the notations and the formulae entering the calculations of Section 5. In Appendix B the formalism developed throughout the paper is applied to the general anti–de Sitter solution and to the one–particle solution of Chern–Simons theory.

2 The Covariant Chern-Simons Lagrangian

The 3–dimensional Chern–Simons Lagrangian can be written as:

$$L_{CS}(A) = \frac{\kappa}{4\pi} \epsilon_{\mu\nu\rho} \text{Tr} \left( A_\mu d_\nu A_\rho + \frac{2}{3} A_\mu A_\nu A_\rho \right) d^3 x$$

where $\kappa$ is a constant which will be fixed later, while $A_\mu = A^i_\mu J_i$ are the coefficients of the connection 1–form $A = A_\mu dx^\mu$ taking their values in any Lie algebra $\mathfrak{g}$ with generators $J_i$. By fixing $\mathfrak{g} = sl(2, \mathbb{R})$ and choosing the generators

$$J_0 = \frac{1}{2} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad J_1 = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad J_2 = \frac{1}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

we have $[J_i, J_j] = \eta^{jk} \epsilon_{kij} J_l$ and $\text{Tr}(J_i J_j) = 1/2 \eta_{ij}$, with $\eta = \text{diag}(-1, 1, 1)$ and $\epsilon_{012} = 1$. Hence, the Lagrangian (1) can be explicitly written as:

$$L_{CS}(A) = \frac{\kappa}{8\pi} \epsilon_{\mu\nu\rho} \left( \eta_{ij} A^i_\mu d_\nu A^j_\rho + \frac{1}{3} \epsilon_{ijk} A^i_\mu A^j_\nu A^k_\rho \right) d^3 x$$

$$= \frac{\kappa}{16\pi} \epsilon_{\mu\nu\rho} \left( \eta_{ij} F^i_{\mu\nu} A^j_\rho - \frac{1}{3} \epsilon_{ijk} A^i_\mu A^j_\nu A^k_\rho \right) d^3 x$$

where $F^i_{\mu\nu} = d_\mu A^i_\nu - d_\nu A^i_\mu + \epsilon^{ijk} A^j_\mu A^k_\nu$ is the field strength.

We then consider two independent $sl(2, \mathbb{R})$ dynamical connections $A$ and $\bar{A}$, the evolution of which is dictated by the Lagrangian

$$L_{CS}(A, \bar{A}) = L_{CS}(A) - L_{CS}(\bar{A})$$

which is nothing but the difference of two Chern–Simons Lagrangians (1), one for each dynamical connection. Field equations ensuing from (1) are of course:

$$\begin{cases} 
\eta_{ij} \epsilon^{\mu\nu\rho} F^i_{\mu\nu} = 0 \\
\eta_{ij} \epsilon^{\mu\nu\rho} \bar{F}^i_{\mu\nu} = 0
\end{cases}$$

Starting from the fields $A, \bar{A}$ it is then possible (see [1, 3, 14, 22]) to define two new dynamical fields, $e^i$ and $\omega^i$, through the rule:

$$A^i = \omega^i + \frac{1}{l} e^i \quad \bar{A}^i = \omega^i - \frac{1}{l} e^i \quad (l = \text{constant})$$
In terms of the new \((e, w)\) variables field equations (5) become:

\[
R_{ij} = -\frac{1}{l^2} e^i \wedge e^j, \quad T^i = de^i + \omega_i^j \wedge e^j = 0
\] (7)

where \(\omega^i = 1/2 \eta_{jk} \omega^{kl} \) and \(R^i_j = d\omega^i_j + \omega^i_k \wedge \omega^k_j\). Equations (7) are nothing but Einstein’s equations with cosmological constant \(\Lambda = -1/l^2\) written in terms of the triad field \(e^i\) and the torsion–free spin connection \(\omega^i_j\). Moreover the Lagrangian (4), in the new variables, reads as:

\[
L_{CS}(A(w, e), \bar{A}(w, e)) = \frac{k}{8\pi} \epsilon^{\mu\nu\rho} \eta_{ij} A^i_\mu d\nu \bar{A}^j_\rho + \frac{1}{3} \epsilon_{ijk} \bar{A}^i_\mu e^j_\nu e^k_\rho \] (8)

with \(g_{\mu\nu} = \eta_{ij} e^i_\mu e^j_\nu\) and \(R_{\mu\nu} = R^k_{\mu\nu} e^i_\mu e^k_\nu\) being the Ricci tensor of the metric \(g\). Notice, however, that the transition from Chern–Simons theory to General Relativity displays some theoretical undesirable features. Indeed, while Chern–Simons equations of motion are manifestly covariant with respect to spacetime diffeomorphism as well as with respect to gauge transformations, the Chern–Simons Lagrangian (4) is not gauge invariant. If we consider a pure gauge transformation with generator \(\xi^i\) acting on the gauge potential, i.e.

\[
\delta_\xi A^i_\mu = D_\mu \xi^i
\] (9)

from (8) we obtain:

\[
\delta_\xi L_{CS} = d_\mu \left\{ \frac{k}{8\pi} \epsilon^{\mu\nu\rho} \eta_{ij} A^i_\nu d\rho \xi^j \right\}
\] (10)

The fact that (4) is not gauge invariant becomes even more explicit in expression (8) where the non invariant term is ruled out into the divergence part. The simplest way to overcome this drawback is to push the divergence term appearing in the right hand side of (8) into the left hand side, defining in this way a global covariant Chern–Simons Lagrangian \(L_{CS^{\text{cov}}}\):

\[
L_{CS^{\text{cov}}}(A, \bar{A}) = \frac{k}{8\pi} \epsilon^{\mu\nu\rho} \eta_{ij} A^i_\mu d\nu \bar{A}^j_\rho + \frac{1}{3} \epsilon_{ijk} \bar{A}^i_\mu e^j_\nu e^k_\rho \] (11)

or, equivalently:

\[
L_{CS^{\text{cov}}}(A, \bar{A}) = \frac{k}{8\pi} \epsilon^{\mu\nu\rho} \left\{ \eta_{ij} F^i_{\mu\nu} B^j_\rho + \eta_{ij} D_\mu B^i_\nu B^j_\rho + \frac{1}{3} \epsilon_{ijk} B^i_\mu B^j_\nu B^k_\rho \right\} d^3x
\] (12)

where \(D_\mu\) is the covariant derivative with respect to the connection \(A\) and we set
\( B^i_\mu = A^i_\mu - \tilde{A}^i_\mu \). Being \( B^i_\mu \) tensorial, expression (12) transforms as a scalar function under gauge transformations (and as a scalar density under diffeomorphisms). Taking into account definition (6) we now obtain:

\[
L_{\text{CScov}}(A(\omega, e), \tilde{A}(\omega, e)) = \frac{\kappa}{4\pi} \sqrt{g} (g^{\mu\nu} R_{\mu\nu} - 2\Lambda) (13)
\]

meaning that the Chern–Simons Lagrangian (11) is mapped exactly (i.e. without undesirable non invariant boundary terms) into the Hilbert Lagrangian for General Relativity with a negative cosmological constant, provided we set

\[
\kappa = \frac{l}{4G} (14)
\]

being \( G \) the Newton’s constant (and setting \( c = 1 \)). We remark that the Lagrangian (11), or equivalently (12), is the same “covariantized” Lagrangian already obtained in \( \text{[8]} \) through the use of the “transgression” formula established by Chern and Simons (see \( \text{[18]} \)). It was shown in \( \text{[8]} \) that a “covariantization” procedure can be applied to each Chern–Simons Lagrangian in dimension three, independently on the relevant gauge group of the theory.

### 3 Variation of Noether Charges

The approach here proposed to calculate the fundamental parameters of the theory, such as mass, angular momentum and gauge charges, is based on a geometrical Lagrangian formalism for classical field theories. In this framework conserved currents and conserved quantities can be calculated by means of the first and the second Noether theorem as shown in \( \text{[7]} \), \( \text{[24]} \), \( \text{[27]} \). The Hamiltonian formalism for the theory can also be derived identifying the variation of the Hamiltonian with the variation of the Noether current with respect to a vector field transversal to a Cauchy hypersurface in spacetime \( \text{[27]} \). The variation of energy is then naturally defined as the on-shell value of the variation of the Hamiltonian. The advantages which derive in using this approach are related to the fact that all quantities we are going to introduce (Noether currents, Noether charges and symplectic forms) are both covariant and gauge invariant, thereby having a global geometrical interpretation. Physically speaking this means that all formulae retain their validity independently on the observer, i.e. independently on the coordinate system in which formulae can be expressed and independently on the spacetime splitting into space + time. The whole theory is also independent on the addition of divergence terms to the Lagrangian. Hence we have not to care about choosing a representative inside the cohomology class of the Lagrangians. We only have to care that the representative Lagrangian be covariant in order to frame the whole theory of conserved quantities in a well-posed geometric background (this is the ultimate reason why we have chosen the Lagrangian (11) in place of (8): the latter Lagrangian is not gauge invariant!).

Moreover we shall see that in presence of Killing vector fields the variation of the Noether charges, which are naïvely defined through integration on a \((n-2)\)-dimensional surface \( B \) in spacetime, does not change inside the homology class
of 2-dimensional surfaces to which \( B \) belongs (see [27]). Roughly speaking this is the mathematical property which will allow to formulate the first law of black holes mechanics.

In addition, in the framework we shall develop we do not have to impose \textit{a priori} boundary conditions to make the variational principle well defined. Boundary conditions just assume a fundamental role, \textit{a posteriori}, in the formal integration of the variational equation which defines the Noether charges, e.g. the variation of energy. Different boundary conditions on the fields (corresponding to different ways in which the physical system can interact with the outside) can be imposed on the same variational equation leading to different physical interpretations of the results, e.g. internal energy for Dirichlet boundary conditions, free energy for Neumann boundary conditions and so on; see [2, 11, 17, 27, 33].

We shall assume that the reader is already familiar with the geometric language of fiber bundles and with the calculus of variations on jet bundles (see, e.g. [24, 34, 39]). We just recall few notions in order to fix the notation. As it is common use in a geometric approach to field theories the Lagrangian \( L(j^k \varphi) = L(j^k \varphi) \, ds \) is considered as a global horizontal \( m \)-form on the \( k \)-order prolongation of the configuration bundle \( Y \to M \) (which is a fiber bundle over the base manifold \( M \), with \( \dim M = m \)), \( L \) is the Lagrangian density, \( ds \) is the volume form on \( M \) (in a coordinate chart \( ds = dx^1 \wedge \ldots \wedge dx^m \)) while \( \varphi \) are the fields of the theory, considered as sections of the configuration bundle, i.e. \( \varphi : M \to Y \). In the sequel we shall be mainly concerned with first order theories, i.e. theories in which the Lagrangian depends only on the fields together with their first derivatives (for higher order theories we refer the interested reader to [27]). As it is well known, the variation of the Lagrangian, after integration by parts, splits into the sum of two terms, called, respectively, the Euler-Lagrange and the Poincaré-Cartan morphisms:

\[
\delta_X L(j^0 \varphi) = < E(L, j^2 \varphi), X > + d < F(L, j^1 \varphi), X > \tag{15}
\]

where \( X \) is any vertical vector field on the configuration bundle (namely, \( X = \delta \varphi \partial/\partial \varphi \) describes the infinitesimal deformation of the dynamical fields), while \( < , > \) denotes the canonical pairing between differential forms and vector fields. Locally:

\[
< E(L, j^2 \varphi), X > = \left\{ \frac{\partial L}{\partial \varphi^A} - d \mu \frac{\partial L}{\partial (d \mu \varphi^A)} \right\} X^A \, ds \tag{16}
\]

\[
< F(L, j^1 \varphi), X > = \frac{\partial L}{\partial (d \mu \varphi^A)} X^A \, ds_\mu \quad (d s_\mu = \partial_\mu | ds)
\]

where we have collectively labelled the fields with the index \( A = 1, \ldots, n = \dim Y - \dim M \). The Euler-Lagrange morphism selects the critical sections \( \varphi \) (i.e. the physical field solutions) through the field equations \( E(L, j^2 \varphi) = 0 \). According to [24, 50], for any given projectable vector field \( \Xi \) on the manifold \( Y \) projecting onto the vector field \( \xi \) on \( M \) and locally described by:

\[
\Xi = \xi^\mu \partial_\mu + X^A \partial/\partial \varphi^A, \quad \xi = \xi^\mu \partial_\mu \tag{17}
\]
we say that the Lagrangian $L$ admits a 1–parameter group of symmetries generated by the vector field $\Xi$ if it satisfies the following property:

$$\delta_\Xi L(j^1 \varphi) := \langle \delta L, j^1 L \Xi \varphi \rangle = L_\xi L$$

(18)

where $L_\Xi \varphi = T \varphi \circ \xi - \Xi \circ \varphi$ is the geometrically defined Lie derivative of the section $\varphi$ with respect to $\Xi$ and $L_\xi$ denotes the usual Lie derivative of differential forms; see [34, 39]. From equations (15) and (18) it follows that the Noether currents generated by the infinitesimal symmetry $\Xi$ can be written as:

$$E \mathcal{L}(L, \varphi, \Xi) = \langle F(L, j^1 \varphi), L_\Xi \varphi \rangle - i_\xi L(j^1 \varphi)$$

(19)

and satisfy the conservation law $dE \mathcal{L}(L, \varphi, \Xi) = -\langle E \mathcal{L}(L, j^2 \varphi), L_\Xi \varphi \rangle$. The Noether current (19) is then a $(m-1)$-form closed on–shell which can be integrated on any hypersurface $\Sigma$ of spacetime $M$.

The field theories we shall be concerned with from now on are the gauge natural theories (see [20, 22, 28, 34]). In gauge natural theories the configuration bundle $Y$ is associated to a given principal bundle $P \to M$, with Lie group $G$. Moreover each projectable vector field on $P$, which can be locally written as $\Xi_P = \xi^\mu \partial_\mu + \xi^i \rho_i$ (having denoted with $\rho = (g \partial / \partial g)$, $g \in G$, a basis for right invariant vector field on $P$ in a trivialization $(x, g)$ of $P$) canonically induces a vector field (17) on the configuration bundle for which the property (18) holds true. Roughly speaking gauge natural theories are the ones which admit the group $\text{Aut}(P)$ of the automorphisms of a principal bundle $P$ as group of symmetries. For instance, Yang–Mills theories on a dynamical background as well as the covariant Chern-Simons Lagrangian (11) are examples of gauge natural theories. Each vector field $\Xi_P = \xi^\mu \partial_\mu + \xi^i \rho_i$ on the relevant principal bundle $P$ induces on the bundle of connections (the sections of which are the connection 1–forms $A_\mu^i$) the vector field

$$\Xi = \xi^\mu \partial_\mu + \xi^i \frac{\partial}{\partial A_\mu^i}, \quad \Xi_\mu^i = -d_\mu \xi^\nu A_\nu^i - C_{jk}^i A_\mu^j \xi^k - d_\mu \xi^i$$

(20)

Notice that vertical vector fields, i. e. the generators of “pure” gauge transformations, denoted from now on as $\Xi^v$, are the ones for which $\xi^\mu = 0$; in this case $\Xi_\mu^i = -D_\mu \xi^i$. Each vector field (20) satisfies the property (18), thereby inducing a Noether current (19).

Specifically, in the Chern–Simons theory (11), the relevant principal bundle $P$ of the theory is a $SL(2, \mathbb{R})$ principal bundle and the connections $A$ and $\bar{A}$ are two different sections of the associated configuration bundle $Y = J^1 P / SL(2, \mathbb{R})$ which is called the bundle of connections; see [34, 39].

The generalization to gauge natural theories of the second Noether theorem (see [20, 22, 28, 24]) states that in each gauge natural theory the Noether current can be canonically split, through an integration by parts, into two terms:

$$E \mathcal{L}(L, \varphi, \Xi) = \tilde{E} \mathcal{L}(L, \varphi, \Xi) + d[U(L, \varphi, \Xi)]$$

(21)

9
called, respectively, the reduced current $\tilde{\mathcal{E}}$, which identically vanishes on shell since it is proportional to field equations, and the superpotential $U$ of the theory. The reduced current $\tilde{\mathcal{E}}$ is unique while the superpotential $U$ is unique modulo cohomology. It can be uniquely fixed by choosing a specific connection and integrating covariantly (see [26]).

**Example 3.1** The superpotential $U(L_{\text{CScov}}, A, \tilde{A}, \Xi) = 1/2 U^{\mu\nu} d_{\mu\nu}$ written in local coordinates (with $d_{\mu\nu} = \partial_\nu |\partial_\mu | ds$), associated to the Lagrangian (12) and relative to the vector field (20) has been calculated in [8]:

$$U^{\mu\nu} = \frac{k}{8\pi} \epsilon^{\mu\nu\rho} \eta_{ij} B_{\rho} \left\{ \xi^i + \tilde{\xi}^i \right\}$$

where $\Xi_P = \xi^\mu \partial_\mu + \xi^i \rho_i$ denotes now a generic vector field on the $SL(2, \mathbb{R})$ principal bundle $P$ of the theory and

$$\xi^i = \xi^i + A^i_\mu \xi^\mu, \quad \tilde{\xi}^i = \xi^i + \tilde{A}^i_\mu \xi^\mu$$

are the vertical parts of $\Xi_P$ with respect to $A^i_\mu$ and $\tilde{A}^i_\mu$, respectively.

We recall in fact that a vector field $\Xi_P$ on a principal bundle $P$ can be split, once a given connection $A^i_\mu$ has been chosen, into the sum of its horizontal and vertical parts:

$$\Xi_P = \xi^\mu \partial_\mu + \xi^i \rho_i$$

where we have set:

$$\xi^i = \xi^i + A^i_\mu \xi^\mu, \quad \tilde{\xi}^i = \xi^i + \tilde{A}^i_\mu \xi^\mu$$

Notice that both the components $\xi^i$ and $\tilde{\xi}^i$ in (23) transform as vectors under gauge transformations since the non–tensorial character of $\xi^i$ is cured by the non–tensorial character of $A^i_\mu$.

Assuming that the topology of spacetime is diffeomorphic to $\Sigma \times \mathbb{R}$ (see Appendix A), the Noether charge relative to the vector field $\Xi_P$ can be calculated from the formula:

$$Q_B(L_{\text{CScov}}, A, \tilde{A}, \Xi) = \int_B U(L_{\text{CScov}}, A, \tilde{A}, \Xi)$$

where $B \simeq S^1$ is a 1-dimensional surface embedded into $\Sigma$. Now, it would seem physically reasonable to define the energy as the Noether charge relative to a spacetime vector field $\xi$ transverse to $\Sigma$, i.e. by simply setting $\Xi_P = \xi^\mu \partial_\mu$ in formula (22), i.e. $\xi^i = 0$. Nevertheless, as it was already pointed out in [8], this prescription is not admissible from a mathematical viewpoint since the group $\text{Diff}(M)$ of spacetime diffeomorphisms is not a global invariance group for the theory. Indeed, in gauge natural theories the symmetry group is the group $\text{Aut}(P)$ and $\text{Diff}(M)$ is not a canonical and natural subgroup of it. Namely, the splitting of $\text{Aut}(P)$ into $\text{Diff}(M)$ is by no means canonical and it is
meaningful only locally (we shall enter into the details of the matter below). Roughly speaking, this means that, given a spacetime vector field $\xi$, there is no canonical way to define a vector field $\Xi$ on the principal bundle and consequently to define the associated vector field $\xi$ on the configuration bundle $Y$ which enters \((26)\). However, we can define, in a non canonical but nevertheless global way, a lift of spacetime vector fields through a dynamical connection $\omega$. In the theory under examination there are three different dynamical connections: the original connections $A_i^\mu$ and $\bar{A}_i^\mu$ and their combination $\omega_i^\mu = (A_i^\mu + \bar{A}_i^\mu)/2$; accordingly, different lifts can be defined. In \([8]\) the horizontal lift with respect to the connection $\bar{A}$ was considered. These choices correspond to set in (22):

$$\bar{\xi}_j^i(V) = 0, \quad \xi_j^i = -\bar{A}_j^\mu \xi^\mu$$

thus leading to the Noether charge:

$$Q_B(L_{\text{CScov}}, \Xi) = \int_B \frac{\kappa}{8\pi} \eta_{ij} B^i_\rho B^j_\mu \xi^\mu dx^\rho$$

(28)

(other choices will be considered in detail hereafter). We can now check the viability of formula (28) by specifying it for the BTZ solution.

In the $\mathfrak{so}(2, \mathbb{R})$ basis \([3] \) and in the coordinate system $(x^\mu) = (t, \rho, \phi)$ on spacetime, the connections $A = A_i^\mu dx^\mu J_i$ and $\bar{A} = \bar{A}_i^\mu dx^\mu J_i$ corresponding to the (exterior of) the BTZ black hole solution are given by \([3, 14, 15]\):

$$\pm A_i^\mu = \begin{pmatrix} \pm \frac{r_+ + r}{r} \sinh \rho & 0 & \frac{r_+ + r}{r} \sinh \rho \\ 0 & \frac{1}{r_+ + r} \cosh \rho & 0 \\ \frac{r_+ + r}{r} \cosh \rho & 0 \end{pmatrix}$$

(29)

where the $(+)$ is referred to $A$ while $(-)$ is referred to $\bar{A}$. For the triad field $e_i^\mu$ we have:

$$e_i^\mu = \frac{l}{2} \left( (+) A_i^\mu - (-) A_i^\mu \right) = \begin{pmatrix} \frac{r_+}{l} \sinh \rho & 0 & r_- \sinh \rho \\ 0 & l & 0 \\ \frac{r_+}{l} \cosh \rho & 0 & r_+ \cosh \rho \end{pmatrix}$$

(30)

where

$$M = \frac{r_+^2 + r_-^2}{8G l^2}, \quad J = \frac{r_+ r_-}{4G l}$$

are, respectively, the mass and the angular momentum of the black hole. The BTZ metric $g$ of components $g_{\mu\nu} = \eta_{ij} e_i^\mu e_j^\nu$ then becomes:

$$g = -\sinh^2 \rho (r_+/l dt - r_- d\phi)^2 + l^2 dp^2 + \cosh^2 \rho (r_+/l dt - r_+ d\phi)^2$$

(32)

Defining the surface $B$ as the surface of constant $t$ and constant $\rho$, from formula (28) and the solution (29) we obtain:

$$Q_B(L_{\text{CScov}}, (\pm) A, \partial_t) = \frac{1}{l}$$

$$Q_B(L_{\text{CScov}}, (\pm) A, \partial_\phi) = \frac{1}{4\pi} \left( \cosh^2 \rho r_+^2 - \sinh^2 r_-^2 \right)$$

The best known of these ways is the horizontal lift which consists simply in setting $\xi_j^i(V) = 0$ in formula (28).
having denoted with $\hat{\partial}_t$ and $\hat{\partial}_\phi$, respectively, the lift of the spacetime vector fields $\partial_t$ and $\partial_\phi$ through the prescription (27). These results do not agree with the expected physical quantities $M$ and $J$, respectively, even in the limit $\rho \to \infty$; see [5, 9, 10, 14]. This can be seen as an hint that the superpotential or the lift chosen in [8], or both, are not suited in calculating physical observables, at least in the Chern–Simons formulation of $(2 + 1)$ gravity.

Let us then consider again the expression (19) and let us perform the variation of the Noether current $E \langle L, \varphi, \Xi \rangle$ with a (vertical) vector field $X = \delta \varphi \frac{\partial}{\partial \varphi}$. We obtain:

$$\begin{align*}
\delta X E \langle L, \varphi, \Xi \rangle &= \delta X < F(L, j^1 \varphi), L \Xi \varphi > - i_\xi [\delta_X L(j^1 \varphi)] \\
&= \delta X < F(L, j^1 \varphi), L \Xi \varphi > - \mathcal{L}_\xi < F(L, j^1 \varphi), X > \\
&= - i_\xi < \mathcal{E}(L, j^2 \varphi), X > + d[i_\xi < F(L, j^1 \varphi), X >] \\
&= \omega(\varphi, X, L \Xi \varphi) + d(i_\xi < F(L, j^1 \varphi), X >) \\
&= - i_\xi < \mathcal{E}(L, j^2 \varphi), X >
\end{align*}$$

where, in passing from the first to the second equality, we made use of (15) and of the rule $\mathcal{L}_\xi = d i_\xi + i_\xi d$. In [34] we have denoted with $\omega(\varphi, X, L \Xi \varphi)$ the naive symplectic current [13, 31]:

$$\omega(\varphi, X, L \Xi \varphi) = \delta X < F(L, j^1 \varphi), L \Xi \varphi > - \mathcal{L}_\xi < F(L, j^1 \varphi), X >$$

For field theories described through a first order Lagrangian the Poincaré–Cartan morphism $< F(L, j^1 \varphi), X >$ depends on the fields $\varphi$ together with their first derivatives and depends linearly on the components of the vector field $X$ (see [16]). Hence formula (35), by using Leibniz rule, can be rewritten:

$$\begin{align*}
\omega(\varphi, X, L \Xi \varphi) &= < \delta_X F(L, j^1 \varphi), L \Xi \varphi > - < \mathcal{L}_\Xi F(L, j^1 \varphi), X > \\
&+ < F(L, j^1 \varphi), \delta_X (L \Xi \varphi) - L \Xi X >
\end{align*}$$

Moreover, for first order theories, the Poincaré–Cartan morphism, which in practice is given by the derivatives of the Lagrangian with respect to the first derivatives of the fields, is nothing but the mathematical object describing the generalized momenta conjugated to the dynamical fields (see [2]). In trying to establish a correspondence with Classical Mechanics we could say that the first two terms in (36) correspond to the expression $\delta p  \dot{q} - \dot{p} \delta q$ which generates the Hamilton equations of motion through the rule $\delta H = \dot{q} \delta p - \dot{p} \delta q$ (in this analogy the time derivative of Classical Mechanics is replaced, in field theories, by the Lie derivative $\mathcal{L}_\Xi$).

What about the further contribution to (36)? Carrying on the analogy with Classical Mechanics this term would correspond to the expression $p [\delta (d_t q) - d_t (\delta q)]$ which is clearly equal to zero in Classical Mechanics since the time
derivative commute with the variation of the configuration variables. The situation is quite different in field theories. To realize this property let us consider gauge theories (i.e., theories where the dynamical field is the gauge potential \( A^i_\mu \)). In this case we have:

\[
\delta_X (\mathcal{L}_\Xi A^i_\mu) = \delta_X \left\{ \xi^\rho d_\rho A^i_\mu + d_\mu \xi^i + C_{j|h}^i A^j_\mu \xi^h \right\}
\]

\[
= \xi^\rho d_\rho (\delta_X A^i_\mu) + d_\mu \xi^i (\delta_X A^i_\mu) + C_{j|h}^i (\delta_X A^j_\mu) \xi^h
\]

\[
+ d_\mu (\delta_X \xi^i) + C_{j|h}^i A^j_\mu (\delta_X \xi^h)
\]

\[
= \mathcal{L}_\Xi (\delta_X A^i_\mu) + D_\mu (\delta_X \xi^i)
\]

so that:

\[
\delta_X (\mathcal{L}_\Xi A^i_\mu) - \mathcal{L}_\Xi (\delta_X A^i_\mu) = D_\mu (\delta_X \xi^i)
\]

If we consider the case in which the components \( \xi^i \) are built out of the dynamical fields (see, e.g., formula (27)) the variations \( \delta_X \xi^i \) are different from zero and the term (38) does not vanish. However, through an integration by parts procedure, the naive symplectic current (36) for gauge theories splits as:

\[
\omega(\varphi, X, L_{\Xi \varphi}) = \tilde{\omega}(\varphi, X, L_{\Xi \varphi}) + d[\tau(\varphi, X, \Xi)] + f(\mathcal{E}(L, \varphi, X))
\]

where

\[
\tilde{\omega}(\varphi, X, L_{\Xi \varphi}) = <\delta_X F(L, j^1 \varphi), L_{\Xi \varphi}> - <L_{\Xi \varphi} F(L, j^1 \varphi), X>
\]

is the reduced symplectic \((m-1)\)–form and \(f(\mathcal{E}(L, \varphi, X))\) denotes a term proportional to the equations of motion.

Collecting together formulae (34) and (39) we have:

\[
\delta_X \tilde{\mathcal{E}}(L, \varphi, \Xi) = \tilde{\omega}(\varphi, X, L_{\Xi \varphi}) + d[\tau(\varphi, X, \Xi)] + f(\mathcal{E}(L, \varphi, X)) + \text{E.M.}
\]

where E.M means terms proportional to the Euler–Lagrange morphism \( \mathcal{E}(L, j^2 \varphi, X) \) and hence vanishing on–shell. On the other hand, from formula (21) we have

\[
\delta_X \tilde{\mathcal{E}}(L, \varphi, \Xi) = \delta_X \tilde{\mathcal{E}}(L, \varphi, \Xi) + d[\delta_X U(L, \varphi, \Xi)]
\]

Comparing (41) with (42) we finally obtain:

\[
\delta_X \tilde{\mathcal{E}}(L, \varphi, \Xi) + d[\delta_X U(L, \varphi, \Xi) - i_\xi < F(L, \varphi), X > - \tau(\varphi, X, \Xi)] = \tilde{\omega}(\varphi, X, L_{\Xi \varphi}) + \text{E.M.}
\]

\footnote{We stress that a splitting similar to (39) occurs also for higher order natural theories, e.g., General Relativity, even though its origin is quite different (see [27]). Indeed, the divergence term \( \tau \) does not arise from the third term of (36), which is identically vanishing for natural theories, but from an integration by parts applied to the first two contributions. In both cases, i.e., in the first order gauge natural theories considered herein as well as in the higher order natural theories treated in [2, 27], it is the analogy with Classic Mechanics which suggests how to perform the splitting. All the terms in (39) which are not in the form \( \delta p \dot{q} - \dot{p} \delta q \) have to be decomposed, through integrations by parts, under the form \( d[\tau(\varphi, X, \Xi)] + f(\mathcal{E}(L, \varphi, X)) \).}
This formula can be seen as the counterpart in field theories of the variational equation
\[ \delta H = \dot{q} \delta p - \dot{p} \delta q + \left[ \partial_t (\partial L/\partial \dot{q}) - \partial L/\partial q \right] \delta q \]
of Classical Mechanics. This analogy suggests to define the variation \( \delta_X H \) of the Hamiltonian density conjugate to the vector field \( \Xi \) as follows:

\[
\delta_X \left[ H(L, \varphi, \Xi) \right] = \delta_X \tilde{E} \cdot (L, \varphi, \Xi) + d \left[ \delta_X U(L, \varphi, \Xi) - i_\xi < F(L, \varphi) , \mathcal{L}_\Xi \varphi > - \tau(\varphi, X, \Xi) \right]
\]

so that

\[
\delta_X H(L, \varphi, \Xi) = \dot{\varphi}(\varphi, X, \mathcal{L}_\Xi \varphi) + E.M. \tag{45}
\]

Given a Cauchy hypersurface \( \Sigma \) the variation \( \delta_X H \) of the Hamiltonian is simply defined as

\[
\delta_X H = \int_\Sigma \delta_X H.
\]

We remark that the right hand side of the equation (43) does not contain divergence terms at all. This means that the divergence terms \( d \left[ \delta_X U - i_\xi F - \tau \right] \) in (43) exactly cancel out the divergence terms arising in the variation \( \delta_X \tilde{E} \cdot \), i.e.

\[
d \left( \frac{\partial \tilde{E} \cdot}{\partial (d\varphi)} \delta_X \varphi \right) = -d \left[ \delta_X U - i_\xi F - \tau \right] \tag{46}
\]

hence leading to a formula for \( \delta H \) which is divergence–free and which gives rise to the proper Hamilton equations of motion (45).

The definition (44) of the variation of the Hamiltonian density is close in spirit with (and it can be seen as a covariant generalization of) the original idea of Regge and Teitelboim to handle boundary terms: all boundary terms arising in the variation \( \delta H \) of the Hamiltonian are added (with a minus sign) into the definition of the naive Hamiltonian in order to define (the variation of) a new Hamiltonian function endowed with a well defined variational principle and hence suited to be used as the generator of the allowed surface deformations; see [27, 32, 38].

The terms of (44) under the exterior differential are the only ones surviving on shell, i.e. when we consider a field \( \varphi \) which is a solution of field equations (we remind that \( \tilde{E} \) is proportional to field equations) and a variation \( \delta_X \) performed along the space of solutions (i.e. \( \delta_X \tilde{E} = 0 \)). The variation \( \delta_X Q \) of the Noether charges relative to a particular solution \( \varphi \), relative to the surface \( \Sigma \) and relative to the vector field \( \Xi \), are hence defined as the integral on the boundary \( \partial \Sigma \):

\[
\delta_X Q_{\partial \Sigma}(L, \varphi, \Xi) = \int_{\partial \Sigma} \delta_X U(L, \varphi, \Xi) - i_\xi < F(L, \varphi) , X > - \tau(\varphi, X, \Xi) \tag{47}
\]

From equation (45) it then follows that \( \delta_X Q \) satisfies, on–shell, the master equation:

\[
\delta_X Q(L, \varphi, \Xi) = \int_\Sigma \dot{\omega}(\varphi, X, \mathcal{L}_\Xi \varphi) \tag{48}
\]

Remark 3.2 Definition (47) and equation (48) deserve now some further comment.
1– First of all we stress that the (variation of) conserved quantities is not defined only through the (variation of) of the superpotential since two more corrective terms have to be added in the definition. This is one of the reasons why formula (33) leads to a wrong result.

2– The definition (47) do not depend on the representative $L$ chosen inside the homology class $[L]$ of Lagrangians (two Lagrangians $L$ and $L'$ belong to the same class $[L]$ if they differ only for divergence terms, which entails that they give rise to the same equations of motion). This property descends from (46): all the terms $(\delta_X U - i \xi F - \tau)$ which constitute the density of the the variation of the charges can be obtained directly and altogether from the reduced current $\tilde{E}$. Since $\tilde{E}$ is basically a linear combination of field equations with coefficients given by the vector field $\Xi$ (see e.g. [20], [24] for a rigorous proof of this statement) the only mathematical data we need for defining $\delta_X Q$ are the equations of motion and the generator of symmetries $\Xi$. Nevertheless, even though (46) can be used, in practice, as an operative schema to calculate explicitly $\delta_X Q$ (we point out that this is exactly the approach followed by Julia and Silva in [32]), from a theoretical point of view, formula (46) hides the symplectic informations which are instead manifest in (48).

3– Let us then consider formula (48). Recalling definition (40) we see that, if $\Xi$ is a Killing vector for the solution, i.e. $\mathcal{L}_\Xi \varphi = 0$, the reduced symplectic form $\tilde{\omega}$ does vanish. From (48) we obtain then:

$$\delta_X Q(L, \varphi, \Xi) = \int_{\partial \Sigma} \delta_X Q(L, \varphi, \Xi) = 0 \quad (\delta_X Q = \delta_X U - i \xi F - \tau) \quad (49)$$

This latter equation is nothing but the conservation law of conserved quantities. Indeed, let us assume that a metric on spacetime can be built out of the dynamical fields, let us consider a (local) foliation of the $m$–dimensional spacetime $M$ into space + time and let us consider a timelike hypersurface $\mathcal{B}$, namely a world tube in $M$; referring to the example (32) $\mathcal{B}$ would correspond to a surface with constant $\rho$. Let us then denote by $B_{t_1}$ and $B_{t_2}$, respectively, the $(m–2)$–surfaces generated by the intersection of $\mathcal{B}$ with two spacelike hypersurfaces $\Sigma_{t_1}$ and $\Sigma_{t_2}$ at constant times $t_1$ and $t_2$. Since $B_{t_1} \cup B_{t_2}$ defines a boundary in $\mathcal{B}$, from (48) we infer that:

$$\int_{B_{t_1}} \delta_X Q(L, \varphi, \Xi) - \int_{B_{t_2}} \delta_X Q(L, \varphi, \Xi) = 0 \quad (50)$$

so that, if $\mathcal{L}_\Xi \varphi|_\mathcal{B} = 0$, then $\delta_X Q$ is conserved in time.

On the other hand let us consider a portion $D$ of a spacelike hypersurface $\Sigma_{t_0}$ at a given time $t_0$ (for example a generic surface of constant $t$ for the BTZ spacetime (32)) and let us suppose that the (oriented) boundary $\partial \Sigma$ is formed by the disjoint union of two $(m–2)$ surfaces $S$ and $S'$, e.g. two circles of constant $\rho$ in the spacetime the metric of which is defined by (32). Assuming that $\Xi$ is a
Killing vector on $\Sigma$, we have
\[
\int_S \delta_X Q(L, \varphi, \Xi) - \int_{S'} \delta_X Q(L, \varphi, \Xi) = 0 \tag{51}
\]
Referring now to the black hole solution and denoting by $S$ and $S'$, respectively, the spatial infinity and the horizon, formula (51) explains why conserved quantities such as mass and angular momentum, which are naïvely calculated at spatial infinity $S$, are related to properties of the horizon $S'$: this comes from the homological properties of $\delta_X Q$.

4– Formula (47) has a drawback: in fact it provides only the variation $\delta_X Q$ of conserved quantities and the conserved quantities $Q$ are obtained only after a formal integration. Nevertheless the integrability of (47) is not a priori assured. It depends on the boundary conditions $\delta_X \varphi|_{\partial \Sigma}$ we impose. Starting from the same expression, different boundary conditions may lead to different results corresponding to different physical interpretations of conserved quantities. For example the recipe (47) has been proven to give the expected values for the quasilocal energy for Einstein and Einstein-Maxwell theories once the variational equation is solved with Dirichlet boundary conditions: see [2] and [27].

We want hereafter to generalize the above formalism to the case of Chern-Simons theories with applications to Chern-Simons AdS$_3$ gravity and to BTZ black holes. We assume as a Lagrangian for the theory the covariant Lagrangian (12). The superpotential is given by (22) and (23) while the Poincaré–Cartan morphism has been calculated in [8] to be the following:
\[
\langle F(L_{\text{CScov}}, A, \bar{A}) \rangle = \kappa \frac{8}{\pi} \epsilon^{\mu \nu \rho} \eta_{ij} B_j^i \delta_X (A^k \nu + \bar{A}^k \nu) \rangle ds_\mu \tag{52}
\]
Hence, from definition (39) and the property (38) we obtain (compare with (39)):
\[
\omega^\mu (A, \bar{A}, X, \Xi) = \tilde{\omega}^\mu (A, \bar{A}, X, \Xi) + d_\nu \left\{ \tau^\mu \nu (A, \bar{A}, X, \Xi) \right\} - \kappa \frac{8}{\pi} \epsilon^{\mu \nu \rho} \eta_{ij} \delta_X \xi^i \{ F^j \nu + \bar{F}^j \nu \} \tag{53}
\]
with:
\[
\tilde{\omega}^\mu (A, \bar{A}, X, \Xi) = \frac{K}{8 \pi} \epsilon^{\mu \nu \rho} \eta_{ij} \left[ \delta_X B^j \nu \delta (A^i \nu + \bar{A}^i \nu) - \delta X \bar{B}^j \rho \delta_X (A^i \nu + \bar{A}^i \nu) \right] = \frac{K}{2 \pi} \epsilon^{\mu \nu \rho} \eta_{ij} \left[ \delta_X c^j \nu \delta (A^i \nu + \bar{A}^i \nu) - \delta X \bar{c}^j \rho \delta_X \omega^i \nu \right] \tag{54}
\]
and
\[
\tau^\mu \nu (A, \bar{A}, X, \Xi) = \frac{K}{4 \pi} \epsilon^{\mu \nu \rho} \eta_{ij} B^j \rho \delta_X (\xi^i) \tag{55}
\]
We stress that the third term in the right hand side of (53) identically vanishes on shell, so that (53) reproduces exactly the structure (39). We also stress
that the reduced symplectic form \( \tilde{\omega} \) as given by (54) provides us the correct symplectic structure for General Relativity once we have identified \( e^j_\rho \) with the vielbein and \( \omega^i_\nu \) with the spin connection, according to (6). Then \( e^j_\rho \) and \( \omega^i_\nu \) can be recognized as dynamical variables conjugated to each other, i.e. they form a pair of \((q, p)\) variables in the appropriate phase space.

Inserting (22), (52) and (55) into the definition (47) we finally obtain:

\[
\delta X_A^q(L_{CS\text{cov}}, A, \bar{A}, \Xi) = \frac{\kappa}{4\pi} \eta_{ij} \int_B [\xi^i(V) \delta X A^j_\mu - \bar{\xi}^i(V) \delta X \bar{A}^j_\mu] dx^\mu
\]

(56)

with \( \xi^i(V) = \xi^i + A^i_\mu \xi^\mu \), \( \bar{\xi}^i(V) = \xi^i + \bar{A}^i_\mu \xi^\mu \). Notice that the above definition for the variation of the charges in Chern–Simons theory is clearly covariant as well as gauge invariant (indeed \( \delta X A, \delta X \bar{A}, \xi(V) \) and \( \bar{\xi}(V) \) are all gauge vectors).

4 Generalized Kosmann Lift

Once the explicit formula (56) for the variation of charges has been established we are faced with another problem. Our goal is to make use of the same formula (56) in order to calculate different physical quantities, such as energy and angular momentum, for the BTZ black hole. To do that we have to appropriately choose a vector field \( \Xi \) on the configuration bundle \( Y \) the projection \( \xi \) of which on \( M \) is the generator of time translations and angular rotations. As already remarked this is not an obvious choice. Indeed the vector fields \( \Xi \) projecting onto the same spacetime vector field are far from being unique! Given a vector \( \Xi \) all the vectors obtained from it through the addition of a generic vertical vector give in fact rise to the same projected vector. Hence the fundamental problem is to find a physically reasonable mathematical rule to lift up to the configuration bundle a given spacetime vector field. While in natural theories there exists a preferred rule, namely the natural lift (see, e.g. [34]), this is not at all the case in gauge natural theories.

We now try to reformulate the problem in terms of local coordinates. In the gauge natural theory we are analysing, i.e. the theory described by (11), an infinitesimal generator of Lagrangian symmetries is any vector field \( \Xi \) of the kind (20), which is functorially associated to a given vector field:

\[
\Xi_P = \xi^a \partial_a + \xi^t \rho_t
\]

(57)
on the relevant \( SL(2, \mathbb{R}) \) principal bundle of the theory. Given a spacetime vector field \( \xi = \xi^a \partial_a \), the problem to define its lift up to the configuration bundle corresponds to the problem of defining a rule for constructing the components \( \xi^t \) in (57) starting from the components \( \xi^a \), the dynamical fields \( A^t_\mu \), \( \bar{A}^t_\mu \) together with their derivatives. This rule must be mathematically well–defined in the sense that \( \xi^t(V) \) and \( \bar{\xi}^t(V) \) must transform as vectors under gauge transformations. For example, this requirement forbids us to simply set \( \xi^t = 0 \) since this choice in general is not globally defined.
In the sequel we shall consider some of the admissible lifts we are allowed to construct. One of them is known in geometrically oriented literature as the generalized Kosmann lift of vector fields. The Kosmann lift was defined for the first time in [22] in order to establish a connection between the ad hoc definition of Lie derivative of spinor fields given in [35] and the general theory of Lie derivatives on fiber bundles. We shall not enter here into the mathematical details the definition of the Kosmann lift is based on. For this issue we refer the interested reader to [28] where an exhaustive bibliography can also be found. We shall only specialize to the present case the formalism there developed. To this end we just outline that the generalized Kosmann lift we are going to construct. One of them is known in geometrically oriented literature as the latter formula shows explicitly that the components $\xi^h$ and $\bar{\xi}^h$ suggest how different global lifts can be defined. Indeed, in Chern-Simons theory with $g = sl(2, \mathbb{R})$ and three dynamical connections $A, \bar{A}$ and $\bar{A}$, in their turn, are built out only from the components $\xi^a$ and their derivatives. They are independent on any dynamical connection, as it is clearly shown by (60). However using the spin connection induced by the frame $e^a_\mu$:

$$\omega^a_{\mu} = e^a_\alpha (\Sigma^\alpha_{\beta \mu} - \Sigma^\beta_{\alpha \mu} + \Sigma^\alpha_{\mu \beta}) e^\beta_{\mu}, \quad \Sigma^\alpha_{\beta \mu} = e^a_c \partial_{[\beta} e^\beta_{\mu]}$$

(61)

the expression for the Kosmann lift can be rewritten in the following way:

$$\xi^k_{(K)} = e^k_\rho (e^\rho_{\mu} d_{\mu} \xi^\alpha - \xi^\alpha d_{\mu} e^\rho_{\mu}) - \omega^k_{\mu \alpha} e^\alpha_{\mu}$$

(62)

The latter formula shows explicitly that the components $\xi^a_{(V)} = \xi^a_{(K)} + A^a_{\mu} \xi^\mu$ and $\bar{\xi}^a_{(V)} = \bar{\xi}^a_{(K)} + \bar{A}^a_{\mu} \xi^\mu$ correctly transform as gauge vectors. Indeed we have:

$$\xi^a_{(V)} = \frac{1}{2} g^{ab} \epsilon_{bcd} \left\{ e^{\mu \nu \rho \sigma} (\nabla_\mu (e^\rho_{\nu} e^\sigma_{\mu} + (A^\rho_{\mu} - \omega^\rho_{\mu}) \xi^\mu) \right\}$$

$$= \frac{1}{2} g^{ab} \epsilon_{bcd} e^{\mu \nu \rho \sigma} (\nabla_\mu (e^\rho_{\nu} e^\sigma_{\mu} + (A^\rho_{\mu} - \omega^\rho_{\mu}) \xi^\mu)$$

$$\bar{\xi}^a_{(V)} = \frac{1}{2} g^{ab} \epsilon_{bcd} \left\{ e^{\mu \nu \rho \sigma} (\nabla_\mu (e^\rho_{\nu} e^\sigma_{\mu} + (\bar{A}^\rho_{\mu} - \omega^\rho_{\mu}) \xi^\mu) \right\}$$

$$= \frac{1}{2} g^{ab} \epsilon_{bcd} e^{\mu \nu \rho \sigma} (\nabla_\mu (e^\rho_{\nu} e^\sigma_{\mu} - \frac{1}{4} e^\rho_{\mu} \xi^\mu)$$

The relations (59) and (62) between the infinitesimal symmetry generators $\xi^\mu$ and $\xi^a_{(K)}$ suggest how different global lifts can be defined. Indeed, in Chern-Simons theory with $g = sl(2, \mathbb{R})$ and three dynamical connections $A, \bar{A}$ and
\[ \omega = A + \bar{A}, \] we have the possibility to introduce different lifts which are mathematically well defined global lifts. They can be obtained formally from (62) by arbitrarily replacing one of the dynamical connections \( A, \bar{A} \) and \( \omega \) into the expression for the covariant derivative and another one into the last term of (62). For example we can choose the lifts defined by means of:

\[ \xi_k^\mu = e_\mu^h (A) \nabla_\mu (e_\nu^k \xi_\nu) - \omega_\mu^h \xi_\mu \]  

(64)

where we have used the dynamical connections \( A \) and \( \omega \). Interchanging in this way the dynamical connections \( A, \bar{A} \) and \( \omega \) we can define nine different lifts; notice, however, that two of them, namely, the one with \( A, \bar{A} \) and the one with \( \bar{A}, A \), are both identical to the Kosmann lift (62) owing to the splitting (6). These lifts are global as much as the Kosmann lift and there is no mathematical prescription to select one among them. However it can be easily shown that each different lift we can define gives different values for the Lie derivatives of the dynamical fields and also different Noether conserved quantities. The choice we make among them in evaluating the variation of global charges (following the recipe of formula (56)) has therefore to be dictated by pure physical considerations. We then proceed as follows. Evaluating the different possible lifts for the two spacetime vector fields which generate, respectively, time translations and angular rotations for BTZ solution (29), we construct the corresponding variations of conserved quantities according to (56) and we look at the ones reproducing the expected values of mass and angular momentum. That seems to be a physically acceptable criterium to select among the different lifts.

For instance, performing the calculations using the Kosmann lift we obtain the correct values for mass and angular momentum:

\[ Q(L_{CScov}, K(\partial_\xi), B) = \frac{r_+^2 + r_-^2}{8G} = M \]  

(65)

\[ Q(L_{CScov}, K(\partial_\phi), B) = \frac{\tau_+ + \tau_-}{4\omega} = J \]  

(66)

independently on the radius of the circle \( B \) (to be rigorous, the results (65) and (66) are correct modulo a constant of integration which can be viewed as the charge of a background solution and can therefore be set equal to zero fixing, in this way, the zero level for the measurement of the charges). For each one of the other six possible lifts we obtain instead non integrable expressions which, obviously, have no interest here. Just to show one let us consider the lift of formula (64); we obtain then for the conserved quantity associated with the lift of \( \xi = \partial_t \) the following:

\[ \delta Q = \kappa \left\{ r_+ \delta r_+ + r_- \delta r_- - r_- \cosh^2(\rho) \delta r_+ + r_+ \sinh^2(\rho) \delta r_- \right\} \]  

(67)

which is manifestly non integrable since \( \frac{\delta}{\delta r_+} \left( \frac{\delta Q}{\delta r_-} \right) \neq \frac{\delta}{\delta r_-} \left( \frac{\delta Q}{\delta r_+} \right) \).

We stress that the only lift providing the expected results is then the generalized Kosmann lift. This justifies a posteriori the choice of the Kosmann lift to construct the infinitesimal generator of symmetries for our theory.
also stress again that, among all the possible lifts, the Kosmann lift is the only one which does not involve a connection whatsoever (see (60)). In a practical language we could say that, among the different lifts, the generalized Kosmann lift is the “most natural”.

Let us now consider the results (65) and (66). As we already pointed out these numerical values are independent on the radius of the circle $B$ on which integration is performed. This fact is not surprising. Indeed we know, see equation (48) and expression (54), that, on shell, the master formula (56) obeys the equation:

$$\frac{\kappa}{4\pi} \eta_{ij} \int_{\partial \Sigma} [\xi^i_\mu \delta X A^i_\mu - \xi^i_\mu \delta X A^i_\mu] dx^\mu = \int \frac{\kappa}{2\pi^2} e^{\mu\nu\rho} \eta_{ij} [\delta X e^\rho_\mu L_\Xi \omega^\nu_\rho - L_\Xi e^\rho_\mu \delta X \omega^\nu_\rho]$$

(69)

so that the left hand side is a homological invariant iff the right hand side is vanishing. Let us then calculate the Lie derivatives $L_\Xi e^a_\mu$ and $L_\Xi \omega^a_\mu$. From the general formula $L_\Xi A^i_\mu = \xi^i_\rho F^i_{\rho\mu} + D_\mu \xi^i_\nu$ and the splitting (3) we have:

$$L_\Xi e^a_\mu = \xi^a_\rho d_\rho e^a_\mu + d_\mu \xi^a_\rho e^a_\rho + \epsilon^a_{\beta\gamma} e^\beta_\mu e^\gamma_\nu$$

$$L_\Xi \omega^a_\mu = \xi^a_\rho d_\rho \omega^a_\mu + d_\mu \xi^a_\rho \omega^a_\rho + \hat{D}_\mu \xi^a_\nu$$

(70)

When we take $\Xi$ equal to the (generalized) Kosmann lift (60) we obtain:

$$L_{K(\xi)} e^a_\mu = \frac{1}{2} e^{\nu a} L_\xi g_{\nu\mu}$$

$$L_{K(\xi)} \omega^i_{\kappa\mu} = -\frac{1}{2} e^b a^i D_k [L_\xi g_{\nu\mu}] + L_\xi \Gamma^a_{\nu b} e^b_{k} e^i_{\nu}$$

(71)

where $L_\xi$ denotes the usual Lie derivative with respect to the spacetime vector field $\xi$. Notice that

$$L_{K(\xi)} g_{\mu\nu} = \eta_{ij} \left( (L_{K(\xi)} e^i_\rho) e^j_\nu + e^i_\nu (L_{K(\xi)} e^j_\rho) \right) = L_\xi g_{\mu\nu} = \nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu$$

(72)

so that the generalized Kosmann lift reproduces the usual Lie derivative of the metric with respect to diffeomorphisms. Both derivatives (71) are vanishing for the BTZ metric when $\xi$ is either $\partial_t, \partial_\varphi$ or any linear combinations of them with constant coefficients. In all these cases the right hand side of (69) is vanishing so that:

$$\delta X Q (L_{\text{Cov}}, K(\partial_t + \Omega \partial_\varphi), B_1) = \delta X Q (L_{\text{Cov}}, K(\partial_t + \Omega \partial_\varphi), B_2)$$

(73)

3We point out that it is not natural, i.e. functorial, in a rigorous mathematical language, since it does not preserve commutators:

$$[K(\xi), K(\eta)]^{ab} = K([\xi, \eta])^{ab} + \frac{1}{2} e^c_{[a} [\xi, \eta] g^{mu} e_{b]} L_\xi g_{\mu\nu}$$

(68)

Only if at least one of the vector fields $\xi$ or $\eta$ is a conformal Killing vector for the metric then $[K(\xi), K(\eta)] = K([\xi, \eta])$. This is the reason why the generalized Kosmann lift is sometimes referred to as “the quasi–natural” lift in gauge theories; see [28].
where $\Omega$ is a constant and $B_1$ and $B_2$ are two homologic surfaces, i.e. they form the boundary $\partial \Sigma$ of a two dimensional surface $\Sigma$.

The above formula allows us to formulate the first law of black hole dynamics. Indeed if we set $\Omega$ equal to the (constant) angular velocity of the BTZ black hole (see [3]):

$$\Omega = \frac{r_-}{8Glr_+}$$

and $B_1$ equal to any circle of constant radius enclosing the black hole horizon the left hand side of (73) turns out to be:

$$\delta Q\left(K(\partial_t + \Omega \partial_\phi), B_1\right) = \delta Q\left(K(\partial_t), B_1\right) + \Omega \delta Q\left(K(\partial_\phi), B_1\right)$$

where in the last equality we made use of the results (65) and (66). On the other hand if in (73) we take $B_2$ equal to the outer horizon $H$ of the black hole (i.e. $\rho = 0$) we have, after some algebraic calculations:

$$\delta Q\left(K(\partial_t + \Omega \partial_\phi), H\right) = T \delta S$$

where $T = \left(\frac{r_-^2 - r_+^2}{2\pi^2 l_+^2}\right)$ is the temperature for the BTZ black hole and $S = \frac{(2\pi r_+)}{4G}$ is one quarter of the horizon area. Equating (73) and (76) we obtain that the first law of black holes mechanics

$$T \delta S = \delta M + \Omega \delta J$$

holds true also in the domain of a Chern–Simons formulation of $(2 + 1)$ gravity.

5 Transition to General Relativity Variables

In this section we shall show that formula (56), once we specialize it for the Kosmann lift and we make use of the splitting (6), reproduces exactly the formula for the variation of conserved quantities in General Relativity found elsewhere, see [27]. The notations and the basic formulae relative to the ADM foliation of spacetime entering the calculations of this section are summarized in the Appendix A.

For the sake of clarity we recall here the formula (56):

$$\delta_X Q_B(L_{CScov}, A, \bar{A}, \Xi) = \frac{\kappa}{4\pi} \int_B \eta_{ij} [\xi_i^{(V)} \delta A^j - \xi^{(V)}_i \delta \bar{A}_j] dx^\mu, \quad \kappa = \frac{l}{4G}$$

We now insert the splitting (8) for the connections $A$ and $\bar{A}$ in the above expression, and we make use of the expression (63) for $\xi_i^{(V)}$ and $\xi^{(V)}_i$ obtained via the Kosmann lift. This yields

$$\delta_X Q_B(L_{CScov}, A, \bar{A}, K(\xi)) = \frac{\kappa}{4\pi l} \int_B \epsilon_{a ij} \epsilon^{\nu j} \epsilon^\rho \bar{\nabla}_\nu \delta e^a_\mu + \epsilon^a_\nu \delta \omega^{ij}_\mu] dx^\mu$$

(79)
where \( \omega_\mu^\alpha = \frac{1}{4} \epsilon_{\alpha\beta\gamma} \omega_\mu^{\beta\gamma} \). In the following we shall calculate the conserved quantities (79) with respect to the generalized Kosmann lift of a spacetime vector field \( \xi^\mu = Nu^\mu + N^\mu \) (see Appendix A for the notation) and our goal will be to express the above formula in terms of metric quantities adapted to an orthogonal foliation of spacetime.

First of all we remind that, in the notations summarized in the Appendix A, we have:

\[
\int_B \sqrt{g} f_\mu dx^\mu = \int_B f_\mu \epsilon^{\mu\nu\rho} n_\nu u_\rho \sqrt{\sigma} \tag{80}
\]

for any 1–form \( f = \sqrt{g} f_\mu dx^\mu \), being \( \sigma = \det(\sigma_{\mu\nu}) \). By making use of the properties \( \epsilon_{\alpha\beta\gamma} \epsilon_{\mu\nu\rho} = \det(\ell) \epsilon_{\mu\nu\rho} \epsilon_{\alpha\beta\gamma} = \delta^{\alpha\beta\gamma}_{\mu\nu\rho} \), taking into account (76), (77) and the relations \( \epsilon^\alpha \delta e^\alpha_\beta = \delta \sqrt{g}/\sqrt{\sigma} \), \( \sqrt{g} = NV \sqrt{\sigma} \) (see Appendix A) the first term under integration in (79) can be expressed as:

\[
\int_B \epsilon_{\alpha\beta\gamma} \epsilon_{\mu\nu\rho} \nabla_\nu \xi^\rho \delta e^\alpha_\mu dx^\mu = \int_B \nabla_\nu \xi^\rho (u_\mu n_\nu - u_\nu n_\mu) \delta \sqrt{\sigma} \tag{81}
\]

If now we apply Leibniz rule:

\[
u_\mu \nabla_\nu \xi^\rho = \nabla_\mu (u_\nu \xi^\rho) - \xi^\rho \nabla_\mu u_\nu = -\nabla_\mu N - \xi^\rho \nabla_\mu u_\nu \tag{82}
\]

\[
\delta e^\alpha_\beta \xi^\rho = \nabla_\mu (\delta e^\alpha_\beta \xi^\rho) - \delta e^\alpha_\beta \nabla_\mu \xi^\rho = -\xi^\rho \nabla_\nu \delta e^\beta_\nu \tag{83}
\]

and we recall formulae (86) and (88) we get:

\[
\int_B \epsilon_{\alpha\beta\gamma} \epsilon_{\mu\nu\rho} \nabla_\nu \xi^\rho \delta e^\alpha_\mu dx^\mu = \int_B \delta \sqrt{\sigma} [\mu^\nu \Theta_{\mu\nu} + n^\nu \nabla_\mu N - n^\mu \xi^\nu K_{\mu\nu}] = \int_B \delta \sqrt{\sigma} [2N(n_\mu) - 2N^\mu n_\nu K_{\mu\nu}] \tag{84}
\]

where we made use of \( h^\mu_\alpha a_\mu = h^\mu_\alpha \nabla_\mu N/N \).

Calculations are more involved for the second term of (79). We start from the compatibility condition:

\[
D_\gamma e^i_\mu = \nabla_\gamma e^i_\mu + \omega^i_\nu \gamma^\nu e^i_\mu = 0 \quad (\nabla_\gamma e^i_\mu = d_\gamma e^i_\mu - \Gamma^i_\mu \gamma^\nu e^\nu_\rho) \tag{85}
\]

to express the spin connection as \( \omega^i_\mu = -\epsilon^i_\nu \nabla_\mu e^\nu_\iota \). After some algebraic calculations we then obtain:

\[
\int_B \epsilon_{\alpha\beta\gamma} \epsilon^\nu_\iota \delta \omega^i_\mu dx^\mu = \int_B \epsilon^\nu_\iota \delta \omega^i_\mu \nabla_\mu \epsilon^\alpha_\beta \delta_\nu_\iota \epsilon^\beta_\alpha \mu
\]

\[
= 2 \int_B \sqrt{\sigma} [\xi^\delta \delta e^\gamma_\delta n_\beta \nabla_\gamma u^\beta_\delta - N \delta(\nabla_\gamma n_\beta) + N e^\gamma_\delta \delta e^\gamma_\delta \nabla_\gamma n_\beta]
\]

Since \( e^i_\iota \delta e^\gamma_\delta = 2 \delta g_\beta_\gamma \), using formulae (86) together with (87) and (88), we can rewrite:

\[
\int_B \epsilon_{\alpha\beta\gamma} \epsilon^\nu_\iota \delta \omega^i_\mu dx^\mu = 2 \int_B \sqrt{\sigma} [N \delta \kappa - N^\alpha \delta(n_\beta K_\gamma^\beta) + N^2 \kappa^\alpha \delta \sigma_\alpha_\beta] \tag{86}
\]

\[
= \int_B \left\{ 2N \delta(\sqrt{\sigma} K) - 2 \sqrt{\sigma} N^\alpha \delta(n_\beta K_\alpha^\beta) - N \sqrt{\sigma} (K \sigma^\alpha_\beta - K^\alpha_\beta) \delta \sigma_\alpha_\beta \right\}
\]

22
Summing up the two terms \( (84) \) and \( (86) \) and multiplying the result with \( \kappa/4\pi l = 1/16\pi G \), we finally obtain the explicit formula for the variation of the conserved quantity relative to the Kosmann lift of the vector field \( \xi \). It reads:

\[
\delta_X Q_B(L_{CS\text{cov}}, A, \bar{A}, K(\xi)) = \int_B \left\{ N\delta_X E - N^\alpha \delta_X J_\alpha + N \frac{\sqrt{\sigma}}{2} s^\alpha\beta \delta_X \sigma_{\alpha\beta} \right\}
\]

where:

\[
\begin{align*}
E &= \frac{1}{8\pi G} \sqrt{\sigma} K \\
J_\alpha &= \frac{1}{8\pi G} \sqrt{\sigma} \sigma^\mu_\alpha K_\mu^\nu n_\nu \\
s^\mu\nu &= \frac{1}{8\pi G} [(n^\alpha a_\alpha) \sigma^{\mu\nu} - K \sigma^{\mu\nu} + K^{\mu\nu}]
\end{align*}
\]

are, respectively, the quasilocal energy density, the quasilocal angular momentum density and the surface pressure; see \([9, 11]\).

Formula \((87)\) reproduces exactly the formula for the variation of conserved quantities found in \([6, 9, 11, 27, 33]\) for General Relativity. We stress again that the full correspondence between the variation of conserved quantities in the \(SL(2, \mathbb{R})\) Chern–Simons (covariant) theory and \((2 + 1)\) General Relativity essentially depends on the choice of the generalized Kosmann lift, thereby selecting the Kosmann lift, whenever it can be defined (see \([28]\)), as the preferred lift to be considered in the domain of gauge natural theories. This is also in accordance with the results of \([24, 37]\) where the Kosmann lift was used to calculate the superpotential in the tetrad–affine formulation of General Relativity, as well as with the results of \([21]\) where the same lift was considered in the domain of BCEA theories. Nevertheless we again point out that the generalized Kosmann lift is just one among the various possibilities and there does not exist a mathematical reason to select it, while there exists only, a posteriori, a physical justification for its choice; see \([37]\).

### 6 Acknowledgments

We are grateful to A. Borowiec of the University of Warsaw and to L. Fatibene, M. Ferraris and M. Godina of the University of Torino for useful discussions and suggestions on the subject. We mention that this research has been performed under no support from the Italian Ministry of Research.

### A The Orthogonal Foliation of Spacetime

We consider a three dimensional region \( D \subseteq M \) of a Lorentzian three dimensional manifold \((M, g)\) and a foliation of it into spacelike hypersurfaces \( \{\Sigma_t\} \), being \( t \in \mathbb{R} \) the parameter of the foliation. Denoting by \( B_t \) the boundary \( B_t = \partial \Sigma_t \) of each leaf of the foliation the union \( \mathcal{B} = \cup_{t \in \mathbb{R}} B_t \) defines a timelike hypersurface, the unit normal of which we denote by \( n^\mu \). We shall restrict the
attention solely to the case of orthogonal foliations, i.e. foliations for which it holds $u^\mu n_\mu|_B = 0$, having denoted by $u^\mu$ the components of the unit vector field which is everywhere orthogonal to each $\Sigma_t$ (see, e.g. [11, 12]).

On the hypersurface $B$ the spacetime metric $g$ can be decomposed as:

$$g_{\mu\nu} = \sigma_{\mu\nu} + n_\mu n_\nu - u_\mu u_\nu$$

(89)

where $\sigma_{\mu\nu}$ is the metric induced by $g_{\mu\nu}$ on the surface $B_t$. The metric can be also expressed in terms of the triad fields $e^i_\mu$ as $g_{\mu\nu} = \eta_{ij} e^i_\mu e^j_\nu$ where $\eta_{ij} = \text{diag}(-1,1,1)$. In the sequel, on each surface $B_t$ we shall use the following notation:

$$\begin{cases}
  u_\mu = e^0_\mu \\
  n_\mu = e^1_\mu \\
  \sigma_{\mu\nu} = \eta_{ij} e^i_\mu e^j_\nu
\end{cases}$$

(90)

so that $g$ results to be: $g_{\mu\nu} = \sigma_{\mu\nu} + e^1_\mu e^1_\nu - e^0_\mu e^0_\nu$. Obviously we have: $u_\mu u^\mu = e^0_\mu e^0_\mu = -1$, $n_\mu n^\mu = e^1_\mu e^1_\mu = 1$, $u^\mu e^\lambda_a = \delta^0_a$ and $n^\mu e^\lambda_a = \delta^1_a$.

In a system of coordinates $(t, \rho, \varphi)$ adapted to the foliation and for which $B$ is a constant $\rho$ hypersurface, the components of the vectors $u_\mu, n_\mu$ can be expressed as:

$$\begin{cases}
  u_\mu = (-N, 0, 0) \\
  n_\mu = (0, V, 0)
\end{cases}$$

(91)

where $N$ is the ordinary lapse in a foliation–adapted ADM decomposition of the metric while $V$ is called the radial lapse. The vector field $\xi = \partial_t$ is defined, in terms of the ADM lapse and shift, as:

$$\xi^\mu = N u^\mu + N^\mu$$

(92)

and the orthogonal condition $u^\mu n_\mu|_B = 0$ implies $\xi^\mu n_\mu|_B = N^\mu n_\mu|_B = 0$. The variations $\delta_X g$ of the metric with respect to a vertical vector field $X = \partial g_{\mu\nu} \frac{\partial}{\partial g_{\mu\nu}}$ can be written as:

$$\delta_X g_{\mu\nu} = -2 \frac{2}{N} u_\mu u_\nu \delta N - \frac{2}{N} h_{\alpha(\mu} u_{\nu)} \delta N^\alpha + h^a_{(\mu} h^\beta_{\nu)} \delta h_{\alpha\beta}$$

(93)

where $h_{\mu\nu} = g_{\mu\nu} + u_\mu u_\nu$ are the components of the metric on each $\Sigma_t$. Moreover, from (91) we have:

$$\begin{align}
  \delta u_\mu &= \frac{\delta N}{N} u_\mu \\
  \delta n_\mu &= \frac{\delta V}{V} n_\mu
\end{align}$$

(94)

The extrinsic curvature of the generic hypersurface $\Sigma_t$ embedded in $M$ is defined by:

$$K_{\mu\nu} = -h^0_{\mu} \nabla_\alpha u^\alpha$$

(95)

while the extrinsic curvature of the hypersurface $B$ embedded in $M$ results to be:

$$\Theta_{\mu\nu} = -\gamma^0_{\mu} \nabla_\alpha n^\alpha$$

(96)
where $\gamma_{\mu\nu} = g_{\mu\nu} - n_\mu n_\nu$ is the metric on $\mathcal{B}$ induced by $g$. The extrinsic curvature of $B_t$ as a surface embedded in $\Sigma_t$ can be expressed, for each $t \in \mathbb{R}$, as:

$$K_{\mu\nu} = -\sigma_\mu D_\alpha n_\nu$$

(97)

where $D$ is the metric covariant derivative with respect to the metric $h$ of $\Sigma$.

The extrinsic curvatures of $\mathcal{B}$, $\Sigma$ and $B$ can be expressed each one in terms of the others via the relation [11, 12]:

$$\Theta_{\mu\nu} = K_{\mu\nu} + (n^\alpha a_\alpha) u_\mu u_\nu + 2\sigma_{\alpha\mu(u_\nu)} K_{\alpha\beta} n_\beta$$

(98)

where $a$ is the covariant acceleration of the normal $u$, i.e. $a_\mu = u^\rho \nabla_\rho u_\mu$. Denoting by $b$ the covariant acceleration of the normal $n$, i.e. $b_\mu = n^\rho \nabla_\rho n_\mu$, we obtain the two formulæ:

$$\begin{align*}
\nabla_\rho u_\mu &= -K_{\rho\mu} - u_\rho a_\mu \\
\nabla_\rho n_\mu &= -\Theta_{\rho\mu} + n_\rho b_\mu
\end{align*}$$

(99)

### B More Examples

Apart from the BTZ solution, other solutions have been found for Chern-Simons field equations in the framework of Chern-Simons Gravity. We are going to analyse the Chern-Simons anti–de Sitter solution [4] and the single particle solution [19, 36].

#### B.1 The anti–de Sitter solution

Let us consider a three dimensional manifold $M$ with a boundary $\partial M$ which has the topology of a torus. We assume $(\omega, \bar{\omega}, \rho)$ as coordinates over $M$ such that the boundary is located at $\rho = \infty$ and the torus is labelled by complex coordinates $(\omega, \bar{\omega})$. A solution of the $SL(2, \mathbb{C})$ Chern-Simons field equations depending on the coordinates on the torus can be found in [4] and it corresponds to the Euclidean anti–de Sitter solution. In the coordinates chosen a general $SL(2, \mathbb{C})$ connection reads as $A = A^i J_i$ with $A^i = A^i_\omega d\omega + A^i_{\bar{\omega}} d\bar{\omega} + A^i_\rho d\rho$.

Imposing the gauge condition:

$$A_\rho = i J_3$$

(102)

4 These complex coordinates on spacetime are related to the usual spherical coordinates by means of the following expressions:

$$\begin{align*}
\omega &= \phi + it \\
\bar{\omega} &= \phi - it \\
e^\nu &= \nu
\end{align*}$$

(100)

5 In agreement with the notation of [4] the generators of $SL(2, \mathbb{C})$ are:

$$J_1 = \frac{i}{2} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad J_2 = \frac{i}{2} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad J_3 = \frac{i}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

(101)
and the boundary condition \( A_\omega = 0 \) at \( \partial M \), we obtain the following solution
for the Chern-Simons equations \( F_{\mu\nu} = 0 \):

\[
A^i_{\mu} = \begin{pmatrix}
  -e^{\ell t + 4e^{-\rho}L(\omega)G} & 0 & 0 \\
  i\ell e^{\ell t + 4e^{-\rho}L(\omega)G} & 0 & 0 \\
  0 & 0 & i
\end{pmatrix}
\]

(103)
in terms of an arbitrary chiral function \( L(\omega) \). With the same method we obtain
the conjugate solution, by imposing \( \bar{A}_\omega = 0 \) at the boundary:

\[
\bar{A}^i_{\mu} = \begin{pmatrix}
  0 & -e^{\ell t - 4e^{-\rho} \bar{L}(\bar{\omega})G} & 0 \\
  0 & -i\ell e^{\ell t + 4e^{-\rho} \bar{L}(\bar{\omega})G} & 0 \\
  0 & 0 & -i
\end{pmatrix}
\]

(104)
The metric \( g_{\mu\nu} = \eta_{ij} e^i_{\mu} e^j_{\nu} \), where \( e^i_{\mu} = \frac{1}{4}(A^i_{\mu} - \bar{A}^i_{\mu}) \), turns out to be:

\[
ds^2 = 4Gl(\dot{L}d\omega^2 + \dot{\bar{L}}d\bar{\omega}^2) + (l^2 e^{2\rho} + 16G^2 L \bar{L} e^{-2\rho})d\omega d\bar{\omega} + l^2 d\rho^2
\]

(105)
and it is an exact solution of Einstein equations (with negative cosmological
constant \( \Lambda = -1/l^2 \)) depending on two arbitrary functions \( L(\omega), \bar{L}(\bar{\omega}) \).

We perform the calculations to obtain, from (56) with \( \eta = \delta \) and \( \kappa = -l/4G \), the variation of the conserved quantity relative to the generalized
\( SO(3) \) Kosmann lift of the vector field \( \xi = \partial_t = i\left( \frac{\partial}{\partial \omega} - \frac{\partial}{\partial \bar{\omega}} \right) \). The variation of the
“mass” results to be:

\[
\delta_X Q_B(L_{\text{CScov}}, K(\partial_t), A, \bar{A}) = \frac{1}{l} \left\{ \delta_X L(\omega) + \delta_X \bar{L}(\bar{\omega}) \right\}
\]

(106)
which can be integrated to obtain the “mass” for the anti–de Sitter solution:

\[
M := Q_B(L_{\text{CScov}}, K(\partial_t), A, \bar{A}) = \frac{1}{l} \left\{ L(\omega) + \bar{L}(\bar{\omega}) \right\} + \text{const}
\]

(107)
With the generalized Kosmann lift of the vector field \( \xi = \partial_{\phi} = \left( \frac{\partial}{\partial \omega} - \frac{\partial}{\partial \bar{\omega}} \right) \), we obtain:

\[
\delta_X Q_B(L_{\text{CScov}}, K(\partial_{\phi}), A, \bar{A}) = \delta_X L(\omega) - \delta_X \bar{L}(\bar{\omega})
\]

(108)
such that the “angular momentum” results to be:

\[
J := Q_B(L_{\text{CScov}}, K(\partial_{\phi}), A, \bar{A}) = L(\omega) - \bar{L}(\bar{\omega}) + \text{const}
\]

(109)
We remark that we continued to call the quantities (107) and (109), respectively,
the “mass” and “angular momentum” since they correspond exactly to these
quantities when the solution (105) becomes a black hole solution, i.e. when
\( L = L_0, \bar{L} = \bar{L}_0 \) and \( Ml = L_0 + \bar{L}_0, J = L_0 - \bar{L}_0 \).
B.2 The particle solution

In the same \( sl(2, \mathbb{R}) \) basis \([2]\) and in the coordinates system \( (t, \rho, \varphi) \) the exact solution corresponding to a particle source in \((2 + 1)\) dimensional gravity turns out to be:

\[
A^i_\mu = \begin{pmatrix}
\sqrt{\rho^2 + \gamma^2} & 0 & \sqrt{\rho^2 + \gamma^2} \\
0 & \frac{1}{\sqrt{\rho^2 + \gamma^2}} & 0 \\
\rho & 0 & \rho
\end{pmatrix}
\]

(110)

\[
\bar{A}^i_\mu = \begin{pmatrix}
-\sqrt{\rho^2 + \tilde{\gamma}^2} & 0 & \sqrt{\rho^2 + \tilde{\gamma}^2} \\
0 & -\frac{1}{\sqrt{\rho^2 + \tilde{\gamma}^2}} & 0 \\
\rho & 0 & -\rho
\end{pmatrix}
\]

(111)

where \( \gamma = 1 - \alpha, \tilde{\gamma} = 1 - \tilde{\alpha} \) and \( \pi(\alpha + \tilde{\alpha}) \) is the deficit angle of the conical singularity introduced by the particle in the geometry of spacetime (see e.g. \([3, 11]\)).

Using formula \([50]\) it is possible to calculate the conserved quantities relative to the infinitesimal generators of symmetries in spacetime, suitably lifted by means of the generalized Kosmann lift. The variations of mass and angular momentum turn out to be:

\[
\delta X Q_B(L_{CScov}, K(\partial_t), A, \bar{A}) = -\frac{1}{4}(\gamma \delta X \gamma + \tilde{\gamma} \delta X \tilde{\gamma})
\]

\[
\delta X Q_B(L_{CScov}, K(\partial_\varphi), A, \bar{A}) = -(\gamma \delta X \gamma - \tilde{\gamma} \delta X \tilde{\gamma})
\]

(112)

Integrating the above expressions around the conical singularity we obtain for the mass and the angular momentum the following values:

\[
Q_B(L_{CScov}, K(\partial_t), A, \bar{A}) = -\frac{1}{4}(\gamma^2 + \tilde{\gamma}^2) + const = \frac{1}{4}(L^+_0 + L^-_0) + const
\]

\[
Q_B(L_{CScov}, K(\partial_\varphi), A, \bar{A}) = -\frac{1}{4}(\gamma^2 - \tilde{\gamma}^2) + const = L^+_0 - L^-_0 + const
\]

(113)

where we have set \( L^+_0 = -\frac{1}{2}(\gamma^2) \) and \( L^-_0 = -\frac{1}{2}(\tilde{\gamma}^2) \).

These results are in accordance with the values found by Martinec in \([30]\) for the ADM mass and spin of the particle.

References

[1] A.Achucarro, P.K.Townsend, Phys.Lett. B 180 (1986), 89.

[2] G. Allemandi, M. Francaviglia, M. Raiteri, Class. Quantum Grav. 19 (2002), 2633-2655 (gr-qc/0110104).

[3] M. Bañados, T. Broz,M. E. Ortiz, Nucl.Phys. B545 (1999) 340-370, (hep-th/9802076).

[4] M. Bañados, Invited talk at the Second Meeting ”Trends in Theoretical Physics”, held in Buenos Aires, December 1998, (hep-th/9901148).
[5] M. Bañados, C. Teitelboim, J. Zanelli, Phys. Rev. Lett. 69 (1992) 1849; M. Bañados, M. Henneaux, C. Teitelboim, J. Zanelli, Phys. Rev. D48 (1993) 1506.

[6] I. Booth, gr-qc/0105009

[7] A. Borowiec, M. Ferraris and M. Francaviglia, J. Phys. A: Math. Gen. 31 (1998), 8823 (hep-th/9801126).

[8] A. Borowiec, M. Ferraris, M. Francaviglia: A covariant Formalism for Chern–Simons Gravity, in press in J. Phys. A.

[9] J.D. Brown, J. Creighton, R.B. Mann, Phys. Rev. D 50 (10) (1994), 6394.

[10] J. D. Brown, M. Henneaux, Comm. Math. Phys. 104 (1986), 207-226.

[11] J. D. Brown, J. W. York, Phys. Rev. D 47 (4), (1993), 1407; J. D. Brown, J. W. York, Phys. Rev. D 47 (4) (1993), 1420.

[12] J. D. Brown, S.R. Lau, J. W. York, to be published in Annals of Physics (gr-qc/0010024).

[13] G. Burnett, R.M. Wald, Proc. Roy. Soc. Lond., A430 (1990), 56; J. Lee, R.M. Wald, J. Math. Phys 31 (1990), 725.

[14] D. Cangemi, M. Leblanc, R.B. Mann, Phys. Rev. D 48 (8) (1993), 3606.

[15] S. Carlip, Class. Quant. Grav. 15 (1998) 3609-3625, (hep-th/9806026); S. Carlip gr-qc/9503024, S. Carlip gr-qc/9305020.

[16] A. H. Chamseddine and J. Fröhlich, Commun. Math. Phys. 147 (1992), 549; A. H. Chamseddine, Nucl. Phys. B 346 (1990), 213-234; M. Bañados, C. Teitelboim, J. Zanelli, Phys. Rev. D49 (1994) 975-986, gr-qc/9307033.

[17] C.-M. Chen, J. M. Nester, Gravitation & Cosmology 6, (2000), 257 (gr-qc/0001088).

[18] S. S. Chern, Proc. Nat. Acad. Sci. USA 68(4) (1971), 791; S. S. Chern and J. Simons, Proc. Nat. Acad. Sci. USA 68(4) (1971), 791; S. S. Chern and J. Simons, Ann. Math 99 (1974), 48.

[19] S. Deser, R. Jackiw, Ann. Phys.153 (1984), 405–416.

[20] L. Fatibene, M. Ferraris, M. Francaviglia, M. Raiteri, Annals of Phys. 275 (1999), 27.

[21] L. Fatibene, M. Ferraris, M. Francaviglia, M. Raiteri, Phys. Rev. D60, 124012 (1999); L. Fatibene, M. Ferraris, M. Francaviglia, M. Raiteri, Phys. Rev. D60, 124013 (1999).
[22] L. Fatibene, M. Ferraris, M. Francaviglia, M. Godina in: Proc. 6th International Conference on Differential Geometry and its Applications, J. Janyska, I. Kolar, I. Slovak Eds. (Brno, Czech Republic, 1995) 549–558; L. Fatibene, M. Ferraris, M. Francaviglia, M. Godina, Gen. Relativ. Grav. 30 (1998), 1371.

[23] L. Fatibene, M. Ferraris, M. Francaviglia, J. Math. Phys. 38 (1997), 3953 ;

[24] M. Ferraris, M. Francaviglia, in: Mechanics, Analysis and Geometry: 200 Years after Lagrange, Editor: M. Francaviglia, Elsevier Science Publishers B.V., (Amsterdam, 1991) 451.

[25] M. Ferraris and M. Francaviglia, Atti Sem. Mat. Univ. Modena, 37 (1989), 61; M. Ferraris and M. Francaviglia, Gen. Rel. Grav., 22, (9) (1990), 965; M. Ferraris, M. Francaviglia, I. Sinicco, Il Nuovo Cimento, 107B, N. 11, (1992), 1303; M. Ferraris, M. Francaviglia, 7th Italian Conference on General Relativity and Gravitational Physics, Rapallo (Genoa), September 3–6, 1986.

[26] M. Ferraris, M. Francaviglia and O. Robutti, in: Géométrie et Physique, Proceedings of the Journées Relativistes 1985 (Marseille, 1985), 112 – 125; Y. Choquet-Bruhat, B. Coll, R. Kerner, A. Lichnerowicz eds. (Hermann, Paris, 1987).

[27] M. Francaviglia, M. Raiteri, Class. Quantum Grav. 19, (2002), 237 – 258.

[28] M. Godina, P. Matteucci, in press for J. Geom. Phys. (math.DG/0201235).

[29] M. Godina, P. Matteucci, J. Geom. Phys. 39 (2001), 265-275.

[30] S. Hyun hep-th/9704005, K. Sfetsos, K. Skenderis hep-th/9711138

[31] V. Iyer and R. Wald, Phys. Rev. D 50, (1994), 846; R.M. Wald, J. Math. Phys., 31, (1993), 2378.

[32] B. Julia, S. Silva, Class. Quantum Grav., 15, (1998), 2173 [gr–qc/9804029]; B. Julia, S. Silva, Class. Quantum Grav., 17, (2000), 4733 [gr–qc/0005127].

[33] J. Kijowski, Gen. Relativ. Grav. 29, (1997), 307.

[34] I. Kolář, P.W. Michor, J. Slovak, Natural Operations in Differential Geometry, Springer–Verlag, (New York, 1993).

[35] Y. Kosmann, Dérivées de Lie des spineurs, Ann. Mat. Pura Appl. (4), 91, (1972), 317.

[36] E.J. Martinec, hep-th/9809021

[37] P. Matteucci, gr–qc/0201073

[38] T. Regge, C. Teitelboim, Annals of Physics 88, (1974), 286.
[39] D.L. J. Saunders, *The Geometry of Jet Bundles*, Cambridge University Press (Cambridge, 1989).

[40] A. Trautman, in: *Gravitation: An Introduction to Current Research*, L. Witten ed. (Wiley, New York, 1962) 168; A. Trautman, Commun. Math. Phys., 6, (1967), 248.

[41] M. Welling, Nuclear Physics B, 515, (1998), 436–452.

[42] E. Witten, *Nucl. Phys. B* 311 (1988), 46; E. Witten, *Commun. Math. Phys. B* 121 (1989), 351.