A generalized Vitali set from nonextensive statistics

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Abstract We address a generalization of the Vitali set through a deformed translational property that stems from a generalized algebra derived from the nonextensive statistics. The generalization is based on the so-called $q$-addition $x \oplus_q y = x + y + (1-q)xy$ for rational values of $q$, where the ordinary formalism is recovered when the control parameter $q \to 1$. The generalized Vitali set is non-measurable for all rational parameter $\frac{1}{2} < q \leq 1$, but in the limit $q \to \frac{1}{2}$ the non-measurability cannot be guaranteed. Furthermore, assuming measurability when $q \to \frac{1}{2}$, then this must be positive. Monotonicity, $\sigma$-additivity, $\sigma$-finiteness, and translational invariance are generalized according to the structure of the $q$-addition and of the $q$-integral.

Keywords Vitali set · nonextensive algebra · Axiom of Choice · measure theory

1 Introduction

The Vitali set represents one of the most relevant objects in mathematics. It concerns the measurability problem and uses the Axiom of Choice (AC) for its construction. It constitutes the first elementary example of non-measurable set (in the sense of the Lebesgue measure) of real numbers, found originally by Giuseppe Vitali [1]. Indeed, there is no an unique Vitali set but rather an uncountable family of them depending on the choice function given by the AC. The Vitali set gave a big boost in the foundations of maths and logic, for instance, by inspiring implicitly Zermelo-Fraenkel models [2] that do not require the AC. One of the intuitive requirements employed for the Vitali construction is the invariance of the Lebesgue measure under translations,
thus presupposing a sum operation between the set to be translated and the value of the translation.

On the other hand, accumulative evidence in multiple phenomena like non-ergodicity, long-range systems, etc. motivated a generalization of the statistical mechanics for addressing these issues in a appropriate way than the traditional descriptions [3,4]. Parallel to this progress, several generalized algebras were developed from the mathematical background of the nonextensive statistics [5,6,7,9]. In these structures the usual operations of the real numbers are generalized in terms a control parameter, usually denoted by $q$ within this context.

The goal of this letter is to provide a generalization of the Vitali set, by making use of the generalized $q$-addition [6] and the $q$-integral as a nonextensive version of the Lebesgue integral as well as the AC, that recover their usual definitions in the limit $q \to 1$ and whose non-measurability cannot be demonstrated when $q \to \frac{1}{2}$.

The letter is organized as follows. In Section 2 we briefly review some generalized operations of the nonextensive algebra that will be used. The generalized Vitali set is built in Section 3 by employing the AC and the $q$-sum. We prove its monotonicity, $\sigma$-additivity, $\sigma$-finiteness and $q$-translation invariance. Non-measurability is shown for $\frac{1}{2} < q \leq 1$. Particularly, non-measurability is not guaranteed for $q \to \frac{1}{2}$ and if the generalized Vitali set is assumed to be measurable for $q \to \frac{1}{2}$, then its measure must be positive. Here we also elaborates further remarks about the proposed generalized Vitali set and the measure theory and the AC. Finally, in Section 4 some conclusions and perspectives are outlined.

2 Brief review on basic concepts

We present the minimal notions and concepts to be used in this work.

2.1 Generalized addition, derivative and integral operations

Based on a original motivation by the Tsallis entropy $S_q$, with $q \in \mathbb{R}$ the control parameter, $k$ a dimensional positive constant and $(p_1, \ldots, p_W)$ a discretized probability distribution,

$$S_q[[p_i]] = k \sum_{i=1}^{W} p_i^q - 1 \quad \frac{1}{1-q},$$  \hfill (1)

generalizations of the logarithm and the exponential, the so-called $q$-logarithm and $q$-exponential are defined by [10]

$$\ln_q x = \frac{x^{1-\frac{1}{q}}-1}{1-\frac{1}{q}} \quad (x > 0),$$

$$e_q(x) = [1 + (1-q)x]^{\frac{1}{1-q}} \quad (x \in \mathbb{R}),$$  \hfill (2)
where \([A]_+ = \max\{A, 0\}\). From (2) it follows

\[
\ln_q(xy) = \ln_q x + \ln_q y + (1 - q) \ln_q x \ln_q y, \quad c_q(x)c_q(y) = c_q(x + y + (1 - q)xy). \tag{3}
\]

From these relations, a generalization of the traditional operations of the real numbers (called nonextensive algebra), deformed by the control parameter \(q\), is defined \([5,6]\):

\[
x \oplus_q y = x + y + (1 - q)xy, \\
x \ominus_q y = x - y \frac{1}{1 + (1 - q)x} \quad (y \neq \frac{1}{q - 1}), \\
x \otimes_q y = \left[x^{1-q} - y^{1-q} + 1\right]_+ \quad (x, y > 0), \\
x \oslash_q y = \left[x^{1-q} - y^{1-q} + 1\right]_+ \quad (x, y > 0), \tag{4}
\]

that are the so-called \(q\)-sum, \(q\)-difference, \(q\)-multiplication, and \(q\)-division.

A deformed differential is defined through the \(q\)-difference \([6]\):

\[
d_q x := \lim_{x' \to x} \frac{x' \ominus_q x}{dx} = \frac{1}{1 + (1 - q)x}. \tag{5}
\]

The definition of a deformed number \([11,12]\)

\[
x_q := \ln\left(c_q(x)\right) = \frac{1}{1 - q} \ln(1 + (1 - q)x) \tag{6}
\]

allows the identity \(dx_q = d_q x\), i.e., the differential of the deformed \(q\)-variable is equal to the \(q\)-differential of an ordinary variable (see \([12]\)). The \(q\)-derivative is defined as

\[
(D_q f)(x) = \frac{df}{d_q x} = (1 + (1 - q)x) \frac{df}{dx}, \tag{7}
\]

and, consistently, the \(q\)-integral

\[
(I_q(f))(x) = \int f(x) d_q x. \tag{8}
\]

Deformed \(q\)-numbers and deformed \(q\)-sum are related through

\[
(x \oplus_q y)_q = x_q + y_q. \tag{9}
\]
2.2 The Vitali set

We review the standard construction of the Vitali set. We first recall the AC in its traditional form.

**Axiom of Choice (AC):** for every family \((B_i)_{i \in I}\) of nonempty sets there exists a set composed by \((x_i)_{i \in I}\) with \(x_i \in B_i\) for all \(I\).

In this way, given a family of nonempty sets, the AC guarantees the existence of a set whose elements belong to each member of the family, by extracting exactly one element of each one of them. The AC lies on the foundations of the mathematics since it constitutes a tool for demonstrating the existence of important notions, like the existence of a basis for all vectorial space, or the non-measurable Vitali set, as described below.

For two arbitrary numbers \(x, y\) in \([0, 1]\) an specific relation in \([0, 1]\), denoted by \(x \sim y\), can be defined by

\[
x \sim y \iff x - y \in \mathbb{Q}
\]

being \(\mathbb{Q}\) the set of rational numbers. Since \(\sim\) is an equivalence relation\(^1\) by applying the AC to the family of equivalence classes a set \(V\) is obtained, containing exactly one representative of each equivalence class.

Let us assume \(V\) measurable with \(\mu(A) = \int_A dx\), the Lebesgue measure (the usual integral) on the measurable sets of \(\mathbb{R}\). Since \(\mathbb{Q}\) is numerable then in particular \(\mathbb{Q} \cap [-1, 1]\) is also numerable. Thus, \(\mathbb{Q} \cap [-1, 1]\) can be enumerated by a sequence \(\{r_k\}_{k \in \mathbb{N}}\).

If \(x \in [0, 1]\), by the construction of \(V\), there exists \(v \in V\) and \(r_k \in [-1, 1]\) with \(x = v + r_k\). So, \([0, 1] \subseteq \bigcup_k V + r_k\). Also, if \(z \in V + r_k \cap V + r_l\) we have \(z = v + r_k = v' + r_l\) and \(v - v' = r_k - r_l \in \mathbb{Q}\) so \(v = v'\) and \(r_k = r_l\). This shows that the translated sets \(V + r_k\) are pairwise disjoint. Finally, \(\bigcup_k V + r_k \in [-1, 2]\) by construction. Therefore,

\[
[0, 1] \subseteq \bigcup_k V + r_k \subseteq [-1, 2]. \tag{11}
\]

Since \(\mu\) is monotous and satisfies \(\sigma\)-additivity, Eq. (11) implies

\[
1 \leq \sum_{k=1}^{\infty} \mu(V + r_k) \leq 3, \tag{12}
\]

where \(\mu([0, 1]) = 1\) and \(\mu([-1, 2]) = 3\). Now by the translational invariance of \(\mu\) is \(\mu(V + r_k) = \mu(V)\) for all \(k\) and therefore by inserting this in (12)

\[
1 \leq \sum_{k=1}^{\infty} \mu(V) \leq 3.
\]

Due to these inequalities it is clear that \(\mu(V)\) cannot be infinite. If \(\mu(V)\) has a finite value the series results infinite, which is a contradiction. Hence, a value

\(^1\) \(\sim\) is an equivalence relation if satisfies reflexivity \((x \sim x)\), symmetry \((x \sim y \rightarrow y \sim x)\), and transitivity \((x \sim y, y \sim z \rightarrow x \sim z)\).
for \( \mu(V) \) cannot be defined. Thus, in this standard demonstration the elements used were: the AC, the monotonicity, the \( \sigma \)-additivity and the translational invariance of the Lebesgue measure. Finally, this forces to admit the existence of non-measurable sets in \( \mathbb{R} \).

### 3 Generalizing the Vitali set

The \( q \)-algebra and \( q \)-calculus are used in the present generalization of the Vitali set. All the formalism recovers the standard one for \( q \to 1 \). Our strategy is to generalize the relation \( \text{[10]} \):

\[
x \sim_q y \iff x = y \oplus_q r \quad \text{with} \quad r \in \mathbb{Q}, \quad q \in \mathbb{Q}
\]

The restriction \( q \in \mathbb{Q} \) is to guarantee that \( \sim_q \) is an equivalence relation.

From \( \text{[13]} \), \( \sim_q \) results an equivalence relation: Reflexivity: \( x \sim_q x \) since using \( \text{[4]} \), it results \( x = x \oplus_0 0 \). Symmetry: If \( x \sim_q y \) it follows that \( x = y + r + (1-q)y r \) with \( r \in \mathbb{Q} \). So \( y = \frac{x - r}{1+(1-q)r} = \frac{x}{1+(1-q)r} + \frac{r}{1+(1-q)r} \in \mathbb{Q} \), and \( y \sim_q x \).

Transitivity: If \( x \sim_q y \) and \( y \sim_q z \) we have \( x = y \oplus_q r_1 \) and \( y = z \oplus_q r_2 \). By the associativity of the \( q \)-sum \( x = (z \oplus_q r_2) \oplus_q r_1 = z \oplus_q (r_2 \oplus_q r_1) \) with \( r_2 \oplus_q r_1 \in \mathbb{Q} \). This implies \( x \sim_q z \).

Now we can proceed with the Vitali construction in the usual form, by applying the AC to the equivalence classes of \( \sim_q \) in \([0,1]\), denoted by \( V_q \). Some properties of \( V_q \) can be summarized.

**Lemma 1** Let \( q \) be such that \( 0 \leq q \leq 1 \). If \( \{r_k\} \) is an enumeration of \( \mathbb{Q} \cap [-1,1] \) the following properties are satisfied:

(i) The \( q \)-translated sets \( V_q \oplus_q r_k \) are disjoint pairwise.

(ii) \( [0,1] \subseteq \bigcup_k V_q \oplus_q r_k \subseteq [-2,3] \).

**Proof** (i) : If \( z \in V_q \oplus_q r_k \cap V_q \oplus_q r_l \) we obtain \( z = v \oplus_q r_k = v' \oplus_q r_l \). So \( v = v' \oplus_q \left( \frac{r_l - r_k}{1+(1-q)r_k} \right) \in \mathbb{Q} \), so \( v = v' \) and \( r_k = r_l \).

(ii) : If \( x \in [0,1] \) there exists \( v \in V_q \subseteq [0,1] \) and \( r \in \mathbb{Q} \) with \( x = v \oplus_q r \). Since \( 0 \leq x, v \leq 1 \) and \( 0 \leq 1 - q \leq 1 \) this implies that \( -1 \leq r = \frac{x - v}{1+(1-q)v} \leq 1 \) with \( r \in \mathbb{Q} \). Thus, \( [0,1] \subseteq \bigcup_k V_q \oplus_q r_k \) with \( r_k \in \mathbb{Q} \cap [-1,1] \).

Moreover, if \( z \in V_q \oplus r_k \) then \( z = v + (1+(1-q)v)r_k \) with \( 0 \leq v \leq 1 \), \( -1 \leq r_k \leq 1 \) and \( 0 \leq 1 - q \leq 1 \). Joining these inequalities, it results \( -2 \leq v(1+(1-q)r_k) + r_k \leq 3 \). Hence, \( V_q \oplus_q r_k \subseteq [-2,3] \). Thus, \( \bigcup_k V_q \oplus_q r_k \subseteq [-2,3] \).

Before analyzing the measurability of \( V_q \), we need to list some properties of the \( q \)-integral.

**Lemma 2** (Monotonicity, \( \sigma \)-additivity, \( \sigma \)-finiteness and \( q \)-invariance translatable of the \( q \)-integral)

Considering the measure \( \mu_q(A) = \int_A d_q x \) on the \( q \)-algebra \( \Sigma_q \) of subsets of real numbers contained in \( (\frac{1}{1-q}, +\infty) \) some properties are satisfied.

\[ ^* \text{Here we are considering} \ (\frac{1}{1-q}, +\infty) \text{is the universal set, so the complement of a subset} \ A \text{is understood to be} \ (\frac{1}{1-q}, +\infty) - A. \]
(a) $\mu_q$ is monotone, $\sigma$-additive and $\sigma$-finite.

(b) $\mu_q(A \oplus v) = \mu_q(A) \forall A \in \Sigma_q$ and $v \in \left(\frac{-1}{1-q}, +\infty\right)$.

(c) $\mu_q(\alpha A) = \alpha \mu_{1-(1-q)\alpha}(A) \forall \alpha > 0$.

Proof (a): The monotonicity and $\sigma$-additivity is a consequence of the properties of the Lebesgue integral and of the definition of $\mu_q$ given by (8). To see the $\sigma$-finiteness it is sufficient to notice $\left(\frac{-1}{1-q}, +\infty\right) = \bigcup_{n=1}^{\infty} A_n \cup \bigcup_{k=1}^{\infty} B_k$ with $A_n = \left[\frac{-1}{1-q} + \frac{1}{n+1}, \frac{-1}{1-q} + \frac{1}{n}\right]$ for all $n \in \mathbb{N}$ and $B_k = \left[\frac{-1}{1-q} + k, \frac{1}{1-q} + k + 1\right]$ for all $k \in \mathbb{N}$ where $\mu_q(A) = \mu_q(B_k) = \frac{1}{1-q} \ln (1 + (1-q)k) < \infty \forall i \in \mathbb{N}$.

(b): Since $\Sigma_q$ is a $\sigma$-algebra and by the $\sigma$-additivity it is sufficient to prove for intervals $[x_1, x_2]$. Since $0 \leq 1 - q$ we have $[x_1, x_2] \oplus v = [x_1 \oplus v, x_2 \oplus v]$ for $v \in \left(\frac{-1}{1-q}, +\infty\right)$. Then, by the definition of $\mu_q$ we obtain $\mu_q([x_1, x_2] \oplus v) = \int_{x_1 \oplus v}^{x_2 \oplus v} d\mu = \int_{x_1 \oplus v}^{x_2 \oplus v} \frac{dx}{1+(1-q)x} = (x_2 \oplus v)q - (x_1 \oplus v)q = (x_2q + vq - (x_1q + vq) = (x_2)q - (x_1)q = \mu_q([x_1, x_2])$, where we have used that $(x \oplus y)q = xq + yq$ for all pair of real numbers $x, y$.

(c): Let us show it for an arbitrary interval $[x_1, x_2]$. Since $\alpha [x_1, x_2] = [\alpha x_1, \alpha x_2]$ then $\mu_q(\alpha [x_1, x_2]) = \int_{\alpha x_1}^{\alpha x_2} \frac{dx}{1+(1-q)x} = \frac{1}{1-q} \ln \left(\frac{1+(1-q)x_2}{1+(1-q)x_1}\right)$ that can be written as $\mu_q(\alpha [x_1, x_2]) = \frac{1}{1-q} \ln \left(\frac{1+(1-q)x_2}{1+(1-q)x_1}\right)$ with $q' = 1 - (1-q)\alpha$. This completes the proof.

With the help of Lemmas (1) and (2) it is straightforward to study the measurability of $V_q$. Thus, we arrive to our main contribution of the present work.

Theorem 1 (Non-measurability of $V_q$ for $\frac{1}{2} < q \leq 1$)

The generalized Vitali set $V_q$ results non-measurable for $\frac{1}{2} < q \leq 1$. However, when $q \to \frac{1}{2}$ the non-measurability cannot be guaranteed. Furthermore, if $V_{\frac{1}{2}}$ is measurable then $\mu_{\frac{1}{2}}(V_{\frac{1}{2}}) > 0$.

Proof: From the monotonicity of $\mu_q$, and due to Lemma (1) (ii)

$$\mu_q([0, 1]) \leq \mu_q\left(\bigcup_{k} V_q \oplus r_k\right) \leq \mu_q([-2, 3]),$$

which can be recasted from the $\sigma$-additivity of $\mu_q$ (Lemma (2) (b)), as

$$\mu_q([0, 1]) \leq \sum_{k=1}^{\infty} \mu_q(V_q \oplus r_k) \leq \mu_q([-2, 3]).$$

Now, since the sequence $\{r_k\}$ is in $[-1, 1]$, it must be $\{r_k\} \subseteq \left(\frac{-1}{1-q}, +\infty\right)$ (because $\frac{-1}{1-q} < -1$ for $\frac{1}{2} < q \leq 1$). Hence, if we apply the $q$-invariance translational of $\mu_q$ (Lemma (2) (b)) to the last equation we obtain

$$\frac{1}{1-q} \ln (2-q) \leq \mu_q(V_q) \left(\sum_{k=1}^{\infty} 1\right) \leq \frac{1}{1-q} \ln \left(\frac{4 - 3q}{2q - 1}\right),$$

(14)
valid for all $\frac{1}{2} < q \leq 1$, where $\mu_q([0, 1]) = \frac{1}{1-q} \ln(2 - q)$ and $\mu_q([-2, 3]) = \frac{1}{1-q} \ln \left( \frac{4-3q}{2q-1} \right)$.

If $\mu_q(V_q)$ is finite (and non-zero) or infinite from (14) we have $\frac{1}{1-q} \ln \left( \frac{4-3q}{2q-1} \right) = \infty$, which is a contradiction since $\frac{1}{2} < q \leq 1$. Neither $\mu_q(V_q)$ can be zero since $\frac{1}{1-q} \ln(2 - q) > 0$. Thus, $V_q$ cannot be measurable for $\frac{1}{2} < q \leq 1$.

On the other hand, taking the limit $q \to \frac{1}{2}$ in (14), we have $\frac{1}{1-q} \ln(2 - q)$ tends to $3 \ln(\frac{4}{3})$ and $\frac{1}{1-q} \ln \left( \frac{4-3q}{2q-1} \right) \to +\infty$, so

$$3 \ln \left( \frac{4}{3} \right) \leq \mu_{\frac{1}{2}}(V_{\frac{1}{2}}) \left( \sum_{k=1}^{\infty} 1 \right) \leq +\infty,$$

(15)

which does not lead to any contradiction. Therefore, from (15) we cannot conclude the non-measurability of $V_{\frac{1}{2}}$. Finally, assuming $V_{\frac{1}{2}}$ measurable and using (15), then $\mu_{\frac{1}{2}}(V_{\frac{1}{2}})$ must be positive. This completes the proof.

3.1 A discussion concerning the measure theory

By using the AC for demonstrating theorems, some kind of counterintuitive situations can appear, many of them, concerning geometrical notions: non-measurable sets as the Vitali one, Banach-Tarski theorem [14], etc. In a general way, these results force to make one of the subsequent concessions in order to have a fair definition of volume:

(A) The volume of a set changes when it is rotated or translated.
(B) The volume of the union of two disjoint sets is not equal to the sum of their volumes.
(C) There are non-measurable sets.
(D) ZFC axioms (Zermelo-Fraenkel set theory with the AC) could be altered.

The measure theory chooses (C), maintaining invariance (against rotations and translations), additivity and the ZFC axioms as logical and intuitive requisites for a reasonable notion of volume. Looking at the construction of the standard Vitali set $V$, the AC is the fundamental element to conclude the non-measurability.

For the generalized Vitali set $V_q$ occurs the same within the range of values $\frac{1}{2} < q \leq 1$, where the translation and the integral are the nonextensive ones ($q$-translation and $q$-integral). However, when $q = \frac{1}{2}$, from the monotonicity, the $\sigma$-additivity, the invariance under $\frac{1}{2}$-translations of $\mu_{\frac{1}{2}}$ along with the AC, the non-measurability of $V_{\frac{1}{2}}$ cannot be obtained, as it is established by the Theorem 1. Even more, if $V_{\frac{1}{2}}$ is measurable, then it must have a positive measure. Therefore, in the construction of the generalized Vitali set, the crucial element to analyze the measurability results to be the non-additive index $q$ rather than the AC, contrarily to be expected. Thus, the role played by the $q$-algebra turns out relevant in the context of a measure theory where the
Table 1 Some characteristics of the standard measure and its nonextensive version for the range of values $\frac{1}{2} \leq q \leq 1$. The novel fact is that, for $q = \frac{1}{2}$ the non-measurability of $V_\frac{1}{2}$ is not followed, even though the AC is used.

| Properties                        | standard measure $\mu$ | nonextensive measure $\mu_q$ |
|-----------------------------------|------------------------|-----------------------------|
| monotonicity                      | YES                    | YES                         |
| $\sigma$-additivity               | YES                    | YES                         |
| translational invariance          | YES                    | NO ($q \neq 1$)             |
| $q$-translational invariance      | NO ($q \neq 1$)        | YES                         |
| $\sigma$-finiteness               | YES                    | YES; $\infty$ for $q = \frac{1}{2}$ |
| ZFC axioms                        | YES                    | YES                         |
| Vitali non-measurability          | YES                    | YES ($q \neq \frac{1}{2}$) |

translation and the integral are replaced by their nonextensive versions. For $q = \frac{1}{2}$, the conditions $(A)$-$(D)$ can be adapted into the corresponding ones, compatible with a measure theory provided with the $q$-addition and the $q$-integral of the $q$-algebra:

$(A')$ The $q$-volume of a set is invariant against $q$-translations.

$(B')$ The $q$-volume of the union of two disjoint sets is equal to the sum of their $q$-volumes.

$(C')$ There are non-measurable sets for $\frac{1}{2} < q \leq 1$, except probably for $q = \frac{1}{2}$.

$(D')$ ZFC axioms could be not altered for $q = \frac{1}{2}$ in a nonextensive measure theory.

We summarize our discussion in Table 1.

4 Conclusions

We have presented an extension of the Vitali construction by making use of the $q$-sum, instead of standard sum, in the context of generalized algebraic operations on the real numbers inherited by nonextensive statistics, and where the standard ones are recovered when $q \rightarrow 1$ as a special case. The traditional proof of the non-measurability remains valid only within the range of the rational values $\frac{1}{2} < q \leq 1$. For $q = \frac{1}{2}$ we have showed that the AC is not sufficient to determine the non-measurability of the generalized Vitali set $V_\frac{1}{2}$.

In this manner, a deformation of the algebraic structure of the operations may not lead to the counterintuitive results of the measure theory and geometry as the non-measurable sets or the Banach-Tarski theorem.

A reexamination of other geometrical constructions (as the Vitali set) by employing deformed operations (as the provided by the $q$-algebra) are desirable to be explored in order to study the interplay between the axioms used to formalize geometrical and physical notions. This future proposal could play a complementary role with others in the literature as the Solovey’s model [18], where the existence of non-measurable sets for the Lebesgue measure is not provable within ZF set theory without the AC.

3 Here we refer to the measure theory provided with the measure $\mu_q$. 
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