Rescuing constraints on modified gravity through relativistic distortions in large-scale structure

Sveva Castello, Nastassia Grimm, and Camille Bonvin
Département de Physique Théorique and Center for Astroparticle Physics,
Université de Genève, Quai E. Ansermet 24, CH-1211 Genève 4, Switzerland
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The distribution of galaxies provides an ideal laboratory to test for deviations from General Relativity. In particular, redshift-space distortions are commonly used to constrain modifications to the Poisson equation, which governs the strength of dark matter clustering. Here, we show that these constraints rely on the validity of the weak equivalence principle, which has never been tested for the dark matter component. Relaxing this restrictive assumption leads to modifications in the growth of structure that are fully degenerate with modifications induced by the Poisson equation. This in turns strongly degrades the constraining power of redshift-space distortions. Such degeneracies can however be broken and tight constraints on modified gravity can be recovered using relativistic distortions in the galaxy distribution, which will be observable by the coming generation of large-scale structure surveys.

INTRODUCTION

One of the main goals of large-scale structure surveys is to determine whether the laws of gravity at cosmological scales are consistent with General Relativity (GR). This is motivated by the fact that modified gravity theories are able to explain the observed accelerated expansion of the Universe at late time without a cosmological constant or a dark energy component (see e.g. [1, 2] for reviews), therefore providing a viable alternative to the standard ΛCDM cosmological model.

Various theoretical frameworks have been developed in recent years to test deviations from GR. A simple and generic approach (see e.g. [3]) consists in modifying the Poisson equation and the relation between the two gravitational potentials describing the geometry of the Universe [4] by two unknown functions, µ and η:

\[ k^2\Psi = -4\pi G a^2 \mu (z,k) \delta \rho_m, \]  
\[ \Phi = \eta (z,k) \Psi. \]  

Redshift-space distortions (RSD) have been used to place constraints on µ [4], while gravitational lensing (cosmic shear or CMB lensing) is sensitive to the combination \( \Sigma = \mu(1 + \eta)/2 \) [5, 6]. The current bounds are in agreement with GR, \( \mu = \eta = 1 \) [7]. Future lensing and spectroscopic surveys, like Euclid, DESI and the SKA, are expected to measure these functions at the level of \( 10^{-3} \) \( - 10^{-2} \) [7,9].

In this Letter, we show that these constraints suffer from one important limitation: they are derived assuming that the theory of gravity preserves the weak equivalence principle (WEP). The WEP has been validated up to great precision for the particles of the Standard Model (see e.g. [10]), but it has never been tested for the unknown dark matter component. Since the velocity of galaxies is largely driven by their dark matter halos, it is legitimate to question this assumption. A violation of the WEP arises in various modified gravity theories, such as models interacting dark matter and dark energy [11, 12], or when dark matter (contrary to baryonic matter) is non-minimally coupled to the space-time geometry [13, 14]. A breaking of the WEP can also emerge from violations of Lorentz invariance, screened modifications of GR [15], and violations of the strong equivalence principle (Nordtvedt effect [16]) if a large fraction of dark matter were made of compact objects [17].

The aim of our work is first to study how the constraints on deviations from GR change if we allow for a breaking of the WEP for dark matter. In particular, we will show that the RSD are able to place tight bounds on µ only provided that the WEP is valid. When dropping this restrictive assumption, µ becomes completely degenerate with one of the parameters encoding deviations from the WEP and cannot be constrained on its own anymore. However, the situation radically changes when adding a relativistic observable into the game: gravitational redshift, which can be targeted through cross-correlations of two populations of galaxies [18, 26]. This effect is directly sensitive to the time component of the metric, Ψ, and is therefore uniquely adapted to constrain deviations from the WEP [27, 29]. Here, we demonstrate that including this new observable in the analysis efficiently breaks the degeneracy among the gravity modifications.

CONSTRAINING MODIFIED GRAVITY WITH REDSHIFT-SPACE DISTORTIONS

RSD surveys map the distribution of galaxies and provide measurements of the galaxy number counts fluctua-

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1 We use the perturbed Friedmann metric: \( ds^2 = a^2[-(1 + 2\Psi)dt^2 + (1 - 2\Phi)dx^2], \) where \( \tau \) denotes conformal time.
tions
\[ \Delta(n, z) = b \delta - \frac{1}{H} \partial_r(V \cdot n), \]
where \( N \) is the number of galaxies per pixel detected in direction \( n \) and at redshift \( z \), and \( \bar{N} \) denotes the average number of galaxies per pixel. In the linear regime, the observable \( \Delta \) is dominated by two contributions, [30]

\[ \Delta(n, z) = b \delta - \frac{1}{H} \partial_r(V \cdot n). \]

The first term encodes the fluctuations in the matter density (\( b \) denotes linear bias), while the second term contains the well-known redshift-space distortions, which depend on the galaxy peculiar velocity \( V \). The parameter \( H \) denotes the Hubble parameter in conformal time and \( r \) is the comoving distance to the galaxies.

The standard way of efficiently extracting information from \( \Delta(n, z) \) is to measure its two-point correlation function \( \xi \equiv \langle \Delta(z, n) \Delta(z', n') \rangle \). Because of statistical isotropy, the correlation function can be expanded in Legendre polynomials and, at linear order, it only contains three multipoles: a monopole, quadrupole and hexadecapole. Assuming that there is no exchange of energy between dark matter and the other constituents (such that the continuity equation can be used), these multipoles can be written as

\[ \xi_0(z, d) = \int \frac{dk}{2\pi^2} \frac{P_\delta^0(k, z)}{\sigma_\delta^0(z)} j_0(kd), \]

[54]

where \( f(z) \equiv \frac{\ln(\delta)}{\ln(a)} \) is the growth rate of structure, \( b(z) \equiv b(z)\sigma_\delta(z), \( \bar{f}(z) \equiv f(z)\sigma_\delta(z) \) and

\[ \mu_\ell(z, d) = \int \frac{dk}{2\pi^2} \frac{P_\delta(k, z)}{\sigma_\delta^2(z)} j_\ell(kd). \]

Here, we have introduced a redshift \( z_* \), chosen to be well in the matter-dominated era (before cosmic acceleration started), and we assume that GR is recovered at this redshift. The functions \( \mu_\ell(z, d) \) are therefore fully determined by early-Universe physics, and tightly constrained by CMB observations [6]. The impact of different theories of gravity is then completely encoded in the amplitude \( \sigma_\delta(z) \) and the growth rate \( f(z) \).

Combining measurements of the three multipoles, it is possible to directly measure the two quantities \( b(z) \) and \( \bar{f}(z) \) as a function of redshift [4, 8]. These model-independent measurements can then be compared with theoretical predictions for any given model of gravity, and used to place constraints on the parameters space, possibly excluding specific models. This method has for example been used in SDSS [4] to constrain the parameter \( \mu \), which encodes modifications to Poisson equation [1]. Measurements of \( \bar{f} \) are translated into constraints on \( \mu \) by combining Eq. (1) with the continuity and Euler equation at sub-horizon scales \( (k \gg H) \), yielding an evolution equation for the density, see e.g. [3]

\[ \delta'' + \left( 1 + \frac{H'}{H} + \Theta \right) \delta' - \frac{3}{2} \frac{\Omega_{m,0}}{a} \left( \frac{H_0}{H} \right)^2 \mu \delta = 0, \]

[56]

where a prime denotes derivatives with respect to \( \ln a \). For a given function \( \mu \), this equation can be numerically solved to determine \( f \) and \( \sigma_\delta \) and consequently \( \bar{f} \).

The situation drastically changes when relaxing the assumption that dark matter obeys the WEP. The Euler equation for the dark matter velocity is modified with a source term

\[ V' + V - \frac{k}{H} \Psi = E^{\text{break}}, \]

[8]

where \( V \) denotes the velocity potential in Fourier space. The exact functional form of \( E^{\text{break}} \) depends on the mechanism responsible for the violation of the equivalence principle. In [27], it was shown that, if dark matter is non-minimally coupled to a new degree of freedom propagating gravity such as a scalar or vector field, one generically obtains

\[ E^{\text{break}} = -\Theta(z)V + \frac{k}{H} \Gamma(z)\Psi. \]

The term proportional to \( \Theta \) is a friction term describing the impact of the new degree of freedom on the velocity evolution, whereas the term proportional to \( \Gamma \) encodes a fifth force acting on dark matter. For simplicity, we assume in the following that the growth and velocity of galaxies are driven by that of dark matter (see Appendix A for a discussion on the impact of a fraction of baryons). Combining Eq. (8) with Eq. (1) and the continuity equation, we obtain the following modified evolution equation for the dark matter density

\[ \delta'' + \left( 1 + \frac{H'}{H} + \Theta \right) \delta' - \frac{3}{2} \frac{\Omega_{m,0}}{a} \left( \frac{H_0}{H} \right)^2 \mu (\Gamma + 1) \delta = 0. \]

[10]

It is clear from this equation that there is a complete degeneracy between \( \Gamma \) and \( \mu \). This reflects the fact that the clustering of dark matter can be enhanced in two ways: either by adding a fifth force directly acting on dark matter (positive \( \Gamma \)), or by increasing the depth of the gravitational potential associated to a given density $^{2}$

Note that Eq. (6) assumes that \( \mu \) is independent of \( k \), which is the case for some classes of modified gravity models [32, 33]. If \( \mu \) depends on \( k \), the functions \( \sigma_\delta \) and \( f \) become scale-dependent and have to be taken inside the integrals over \( k \).
distribution ($\mu > 1$), which in turn increases the infall and clustering of dark matter. In addition, we expect from Eq. (10) a further degeneracy between $\mu (\Gamma + 1)$ and the parameter $\Theta$, which tends to slow down dark matter clustering through friction.

Eq. (10) can be solved numerically for given $\mu$, $\Theta$ and $\Gamma$. It is common to assume that the modifications evolve proportionally to the background evolution of dark energy $^{32-34}$, i.e. that they become relevant only during the phase of accelerated expansion of the Universe, such that

\[ \mu(z) = 1 + \mu_0 \Omega_A(z)/\Omega_{\Lambda,0}, \tag{11} \]

\[ \Theta(z) = \Theta_0 \Omega_A(z)/\Omega_{\Lambda,0} \quad \text{and} \quad \Gamma(z) = \Gamma_0 \Omega_A(z)/\Omega_{\Lambda,0}. \]

**IMPACT ON CURRENT AND FUTURE CONSTRAINTS ON $\mu$**

We start by studying how the current constraints on $\mu_0$ are degraded if the WEP is not enforced. We use the measurements of $\tilde{f}_i = f(z_i)\sigma_8(z_i)$ from BOSS and eBOSS in 6 redshift bins (see Table 3 in [4]). When the WEP is enforced, we can directly translate the constraints on $\tilde{f}_i$ into constraints on $\mu_0$ using the Fisher formalism, yielding $\sigma_{\mu_0} = 0.21$\(^3\). On the other hand, when dropping the assumption that the WEP is valid, the full degeneracy between $\mu_0$ and $\Gamma_0$ implies that only the sum $\mu_0 + \Gamma_0$ (and not $\mu_0$ alone) can be constrained by RSD measurements. We therefore calculate a Fisher matrix for the parameter space $\{\mu_0 + \Gamma_0, \Theta_0\}$. The joint constraints are shown in Fig. 1. We see a strong degeneracy between $\mu_0 + \Gamma_0$ and $\Theta_0$, which is due to the fact that $\Theta_0$ slows down the growth of structure, while $\mu_0 + \Gamma_0$ accelerates it.

\[^3\text{This is comparable to the value $\sigma_{\mu_0} = 0.25$ obtained in [4] when combining RSD and weak lensing.}\]

As a consequence, the marginalised constraint on $\mu_0 + \Gamma_0$ is very large: $\sigma_{\mu_0 + \Gamma_0} = 6.05$, i.e. 30 times larger than the original constraints on $\mu_0$.

We also perform Fisher forecasts for the upcoming generation of cosmological surveys, assuming GR as the fiducial model. We consider two catalogues: the Bright Galaxy Sample (BGS) of DESI, which will observe 10 million galaxies up to $z = 0.5$, and the SKA phase 2, which will observe close to a billion galaxies up to $z = 2$.

The survey specifications (number density and volume) are taken from $^{35}$ and $^{8}$, respectively, and the fiducial cosmology is fixed to the latest Planck values $^{3}$. We choose $z_s = 10$ and fix the minimum separation $d_{\text{min}} = 20 \text{Mpc}/h$, such that non-linear effects are negligible $^{25}$. We let the bias evolve according to the fitting functions given in $^{35}$: $b_{\text{BGS}} = b_0 \delta(0)/\delta(z)$ for DESI, involving one free parameter with fiducial value $b_0 = 1.34$; and $b_{\text{SKA}} = b_1 \exp(b_2 z)$ for SKA2, involving two free parameters with fiducial values $b_1 = 0.554$ and $b_2 = 0.783$. We include shot noise and cosmic variance in the variance of the multipoles (see Appendix C of [27]) and account for cross-correlations between different multipoles.

Our results are summarized in Table I, again showing a significant degradation of the constraints when allowing for a violation of the WEP. Hence, RSD measurements provide tight constraints on $\mu_0$ as in [4] only when ignoring a wide range of modified gravity models $^{11-17}$. We also underline that we have assumed $\mu_0$, $\Gamma_0$ and $\Theta_0$ to be scale-independent. We have checked that, in the specific case where $\mu$ depends on $k$ with a known scaling $^{32}$, while $\Theta$ and $\Gamma$ remain scale-independent, the degeneracy is immediately broken. On the other hand, if the scale-dependence is unknown, or if $\mu$ and $\Gamma$ share the same scale-dependence, the degeneracy persists.

**DEUS EX MACHINA: RELATIVISTIC EFFECTS**

We now demonstrate that by extending RSD analyses we can recover tight constraints on modified gravity, and in particular simultaneously constrain $\mu_0$ and a violation of the WEP. It was shown in the last decade that the observed galaxy number counts fluctuations are not only described by a density term and RSD as in Eq. (4), but that various relativistic distortions also give a contribution $^{36-38}$. These effects have a negligible impact on the even multipoles $^{39}$, but some of them have the particu-

![Fig. 1](image-url)
The dipole arises from the following corrections to $\Delta$:

$$
\Delta^{\text{rel}} = \frac{1}{\mathcal{H}} \partial_r \Psi + \frac{1}{\mathcal{H}} \mathbf{\dot{V}} \cdot \mathbf{n} + \left( 1 - 5s + \frac{5s - 2}{H^r} - \mathcal{H}^r \right) \mathbf{V} \cdot \mathbf{n},
$$

where a dot denotes derivatives with respect to conformal time, $s$ is the magnification bias and $f^{\text{rel}}$ is the evolution bias. These quantities can be directly measured for a given population of galaxies (see Appendix B). The first term in Eq. (12) encodes the contribution from gravitational redshift, which changes the apparent size of a redshift bin located inside a gravitational potential, while the second and third terms are Doppler effects.

In order to measure the dipole, two distinct populations of galaxies are needed, e.g. a bright population, $B$, and faint population, $F$. The dipole $\xi_1 = \xi_1(z,d)$ in the cross-correlation can be written in the following way using Eqs. (1) and (9):

$$
\xi_1 = \frac{\mathcal{H}}{H_0} \nu_1(d,z_a) \left[ 5f \left( \bar{b}_B s_F - \bar{b}_F s_B \right) \left( 1 - \frac{1}{r H} \right) + 3f^2 \Delta s \left( 1 - \frac{2}{r H} \right) + \frac{\Delta \hat{b}}{r H} \right] \frac{\sqrt{\sigma_0^2(z) \Gamma}}{\sigma_{\lambda_1}} - \frac{2}{5} \Delta \hat{b} \frac{d}{r} \mu_2(d,z_a),
$$

which is essential to break the degeneracy between the parameters.

The last term in Eq. (13) proportional to $\mu_2$ arises from the wide-angle contribution [19]. We note that $\mu$, $\Theta$ and $\Gamma$ enter the dipole in two ways: first, through their impact on $f$ (as in the even multipoles), but also directly through the terms in the last line of Eq. (13). These terms are due to the gravitational redshift contribution $\partial_r \Psi$ in Eq. (12), which is expected to be robustly detected when adding the relativistic dipole to RSD measurements.

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$$
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$$

where a dot denotes derivatives with respect to conformal time, $s$ is the magnification bias and $f^{\text{rel}}$ is the evolution bias. These quantities can be directly measured for a given population of galaxies (see Appendix B). The first term in Eq. (12) encodes the contribution from gravitational redshift, which changes the apparent size of a redshift bin located inside a gravitational potential, while the second and third terms are Doppler effects.

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$$

The marginalised constraints are presented in Table I and the joint constraints for SKA2 are plotted in Fig. 2. The constraints on $\sigma_{\mu_0 + \Gamma_0}$ are only marginally improved by adding the dipole. However, the dipole gives a decisive contribution by breaking the degeneracies, allowing us to constrain the three parameters individually. Since we are introducing two additional parameters, we expect that the bounds on $\mu_0$ are not as tight as in the RSD-only analyses restricted to the validity of the WEP. Nevertheless, when diagonalizing the Fisher matrix, we always identify a combination of parameters $\lambda_1$ with even tighter constraints than in the first line of Table I. The exact combination slightly varies depending on

\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|c|}
\hline
& DESI & SKA2 & SKA2 with baryons \\
\hline
$\sigma_{\mu_0 + \Gamma_0}$ & 0.41 & 0.067 & 0.083 \\
$\sigma_{\mu_0}$ & 1.77 & 0.147 & 0.147 \\
$\sigma_{\Gamma_0}$ & 1.79 & 0.162 & 0.190 \\
$\sigma_{\Theta_0}$ & 0.50 & 0.085 & 0.100 \\
$\sigma_{\lambda_1}$ & 0.01 & 0.002 & 0.003 \\
\hline
\end{tabular}
\caption{Forecasted constraints on $\mu_0 + \Gamma_0$, the individual parameters $\{\mu_0, \Gamma_0, \Theta_0\}$ and the best-measured eigenvector $\lambda_1$ when adding the relativistic dipole to RSD measurements.}
\end{table}
on the specifications: for the baseline SKA2 analysis, we have \( \lambda_1 = 0.62 \mu_0 + 0.62 \Gamma_0 - 0.49 \Theta_0 \), with \( \sigma_{\lambda_1} = 0.002 \). This indicates that the inclusion of the relativistic dipole yields extremely stringent constraints on deviations from GR, with no restriction to a particular class of models.

Finally, we have checked that the analysis holds if we add a fraction of baryons (obeying the WEP) to the density and velocity evolution (see Appendix A for details). The results are presented in Table II and show that the constraints on \( \mu_0 \), \( \Gamma_0 \) and \( \Theta_0 \) are only marginally affected.

**CONCLUSION**

The main goal of large-scale structure surveys is to test the laws of gravity, in order to determine whether the accelerated expansion of the Universe is due to dark energy or to modifications of gravity at cosmological distances. RSD measurements of the growth of structure are highly sensitive to deviations from GR, since the growth rate directly probes the dynamical evolution of dark matter clustering. As such, RSD are a key observable to constrain deviations in the Poisson equation, encoded in the parameter \( \mu \).

In this Letter, we have shown that the clustering of dark matter is also affected by violations of the WEP. In particular, if dark matter is sensitive to a fifth force mediated by a new gravitational degree of freedom, its clustering is enhanced. On the other hand, if the new degree of freedom generates friction on dark matter, its clustering is suppressed. Since RSD are only sensitive to the growth rate, they cannot distinguish between these different effects. Therefore, constraints on the parameter \( \mu \) are completely spoiled by allowing for violations of the WEP for dark matter.

Luckily, as in all good plays, a *Deus ex machina* rescues the situation: relativistic distortions in the galaxy number counts, which break the degeneracy between \( \mu \) and the parameters governing violations of the WEP. Our study shows that by combining RSD with a measurement of the dipole generated by these distortions, we will be able to recover tight constraints on all parameters with the coming generation of surveys.

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**APPENDIX A: IMPACT OF A FRACTION OF BARYONS**

In our analysis, we have assumed that the velocity of galaxies is completely determined by the velocity of the dark matter halos. Here we repeat the analysis with the SKA2 survey specifications, including a baryonic contribution that obeys the WEP (see also [29]). We write the total density and velocity as \( \delta = x \delta_{dm} + (1 - x) \delta_b \) and \( V = x V_{dm} + (1 - x) V_b \), where \( x \) is the fraction of dark matter inside a galaxy: \( x = \rho_{dm}/\rho_m \). This leads to a system of coupled differential equations

\[
\frac{\delta_{dm}}{\delta t} + H(1 + \Theta) \delta_{dm} - \frac{3}{2} \rho_m \Omega_m(z) \mu(1 + \Gamma) [x \delta_{dm} + (1 - x) \delta_b] = 0,
\]

\[
\frac{\delta_b}{\delta t} + H \delta_b - \frac{3}{2} \rho_m \Omega_m(z) \mu [x \delta_{dm} + (1 - x) \delta_b] = 0,
\]

which can be solved numerically using \( \delta_{dm} = \delta_b \) as initial condition at \( z_* \), when violations of the WEP are negligible, see Eq. (11).

We set \( x = 0.85 \), which is a typical value for individual massive galaxies [12] and also roughly corresponds the cosmic average [6]. The results are presented in Table II showing a small degradation of the constraints on \( \Theta_0 \) and \( \Gamma_0 \) of roughly 18%. On the other hand, the bounds on \( \mu_0 \) are not significantly affected.

**APPENDIX B: MAGNIFICATION AND EVOLUTION BIASES**

Galaxy surveys are usually flux limited, i.e. they detect only galaxies with a flux above a given threshold \( F_c \). This generates additional fluctuations in the galaxy number counts, which have been calculated e.g. in [19] [38]. Here we derive how this effect impacts two populations of galaxies.

We denote by \( N_B(z, n) \) the number of bright galaxies per pixel, i.e. the number of galaxies above a chosen flux limit \( F_{cut} \):

\[
N_B(z, n) \equiv N(z, n, F \geq F_{cut}).
\]

Since light propagation is affected by inhomogeneities, \( F_{cut} \) corresponds to a different luminosity threshold in different directions: \( L_{cut}(z, n) = L_{cut}(z) + \delta L_{cut}(z, n) \). Here, \( L_{cut} \) denotes the luminosity threshold associated to \( F_{cut} \) in a homogeneous Universe, and \( \delta L_{cut} \) is the departure from this average due to fluctuations. We obtain

\[
N_B(z, n) = N(z, n, L \geq \bar{L}_{cut}(z) + \delta L_{cut}(z, n)) \approx N(z, n, L \geq L_{cut}(z)) - \frac{5}{2} s(z, L_{cut}) \frac{\delta L_{cut}}{L_{cut}} \, ,
\]

where

\[
s(z, L_{cut}) \equiv \frac{2 \delta}{\partial L_{cut}} N(z, L \geq L_{cut}) .
\]
On the other hand, the faint galaxies have a flux smaller than $F_{\text{cut}}$, but larger than the flux threshold of the survey $F_s$:

$$N_F(z, \mathbf{n}, F_{\text{cut}} > F \geq F_s) = N(z, \mathbf{n}, F \geq F_s) - N(z, \mathbf{n}, F \geq F_{\text{cut}}) 
\simeq N(z, \mathbf{n}, L_{\text{cut}} > L \geq L_s) 
- \frac{5}{2} s(z, L_s) \frac{\delta L_s}{L_s} + \frac{5}{2} s(z, L_{\text{cut}}) \frac{\delta L_{\text{cut}}}{L_{\text{cut}}},$$

where

$$s(z, L_s) \equiv \frac{2}{5} \frac{\partial}{\partial L_s} N(z, L \geq L_s).$$

For a fixed flux, the fluctuations in luminosity are directly related to the fluctuations of the luminosity distance, which have been calculated in [39, 44]:

$$\frac{\delta L_{\text{cut}}}{L_{\text{cut}}} = \frac{\delta L_s}{L_s} = \frac{2}{5} \frac{\delta d_L(z, \mathbf{n})}{d_L(z)}. \quad (22)$$

Here we are only interested in the terms contributing to the dipole, i.e., those proportional to the peculiar velocity. Inserting Eq. (22) into (18) and (20) we find that the flux thresholds generate fluctuations in $\Delta$ for the bright and faint populations of the form

$$\Delta_{\text{mag}} = -5 s_{B,F}(z) \left(1 - \frac{1}{rH}\right) \mathbf{V} \cdot \mathbf{n}, \quad (23)$$

where

$$s_B(z) \equiv s(z, L_{\text{cut}}), \quad s_F(z) \equiv s(z, L_s) - s(z, L_{\text{cut}}). \quad (24, 25)$$

To calculate $s_B(z)$ and $s_F(z)$ for SKA2 we use the fitting function for $s(z, L)$ given in [35], with a flux sensitivity limit $F_s$ of 5 $\mu$Jy. We then choose the flux cut $F_{\text{cut}}$ in each redshift bin such that we have the same number of bright and faint galaxies. For DESI, we use the model developed in [39] for the magnification bias of the BGS with a magnitude limit $m_s = 19.5$, and we set the magnitude cut by again imposing the same number of bright and faint galaxies. The resulting magnification bias functions for the bright and faint populations of both surveys are shown in Fig. 3.

The number counts fluctuations $\Delta_{\text{rel}}$ also depend on the evolution bias $f_{\text{evol}}$, see Eq. (12). The evolution bias describes the evolution of the galaxy population with time, while taking the selection function into account [38]. For the forecasts, we use $f_{\text{evol}}^B = f_{\text{evol}}^F = 0$. Once data will be available, the magnification bias and the evolution bias can be measured from the average number of galaxies.
