Strongly localized dark modes in binary discrete media with cubic-quintic nonlinearity within the anti-continuum limit

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Abstract. The existence of dark strongly localized modes of binary discrete media with cubic-quintic nonlinearity is numerically demonstrated by solving the relevant discrete nonlinear Schrödinger equations. In the model, the coupling coefficients between adjacent sites are set to be relatively small representing the anti-continuum limit. In addition, approximated analytical solutions for vectorial solitons with various topologies are derived. Stability analysis of the localized states was performed using the standard linearized eigenfrequency problem. The prediction from the stability analysis are furthermore verified by direct numerical integrations.

1. Introduction
The study of localized solutions in discrete system has gain central attraction for nearly three decades, and this was motivated by its profound impacts on various branches of science, mainly solid-state physics and nonlinear optics [3, 6, 10–13]. The stability analysis and evolution of localized states frequently described by discrete nonlinear Schrödinger equation (DNLSE). Perhaps this was first initiated by experimental work on the fabricated AlGaAs waveguide arrays reported in [4], followed by the mathematical model presented in [5]. The constitution of fundamental dark discrete soliton is found in [9] on the anomalous diffraction region of AlGaAs waveguide arrays related with Kerr-type cubic nonlinearity. This was known as the effect of defocusing nonlinearity in DNLSE system. DNLSE also describing other areas in physics, i.e. Bose-Einstein condensates (BEC) and optically induced lattice [7].

In systems with more than one component, extra interaction mechanism that rise up can strongly affecting the conditions of the existence of solitons. Localized vectorial modes form a species of composite solutions that consist of two or more mutually self-trap components in a nonlinear medium. Significantly, if not all components exist at the same time, such composite systems could not be observed. In addition, the existence of dark solitons requires modulational stability of the background. Strongly localized, dark vectorial modes in discrete cubic media described by DNLSE’s where shown to exist, along with stability region [8]. As far as discrete binary cubic-quintic (CQ) media being concerned, only modulational instability (MI) of the plane-wave solutions has been reported, where for the same set of initial parameters i.e. same amplitudes and inter-component coupling constants; quintic nonlinear yields fundamentally
stronger effect than cubic nonlinearity [2], which provides some information on the stability region of dark solitons but their existence is not yet verified. Thus, this paper examines the existence of dark strongly localized modes (SLM’s) in binary discrete media with CQ nonlinearity.

This paper is organized as follows: in Section 2, we introduced the mathematical models representing binary discrete media with CQ nonlinearity with its conserved quantities, in Section 3, the existence and linear stability analysis of the solutions are discussed. Analytic expressions for vectorial dark SLM’s of different topologies are derived in Section 3 too.

2. The Mathematical Model

Binary discrete media with cubic-quintic nonlinearity can be described by two-component CQDNLSE:

\[\begin{align*}
\frac{dA_n}{dz} &= -c_a (A_{n+1} + A_{n-1}) - \lambda (|A_n|^2 + |B_n|^2) A_n \\
&\quad - \gamma (|A_n|^4 + 2\alpha |A_n|^2 |B_n|^2 + \alpha |B_n|^4) A_n, \\
\frac{dB_n}{dz} &= -c_b (B_{n+1} + B_{n-1}) - \lambda (|B_n|^2 + |A_n|^2) B_n \\
&\quad - \gamma (|B_n|^4 + 2\alpha |B_n|^2 |A_n|^2 + \alpha |A_n|^4) B_n,
\end{align*}\]

where \(A_n(z), B_n(z)\) are the complex-valued wave functions at site \(n\); \(c_a, c_b\) denote the coupling constants between adjacent sites; variable \(z\) denotes propagation coordinate in optics or time in BEC application; \(\lambda\) and \(\gamma\) are the cubic and quintic nonlinear coefficients respectively; and \(\beta\) and \(\alpha\) denote the cubic and quintic inter-component nonlinear coupling or cross-phase modulation (XPM) coefficients respectively. Here, the self-phase modulation (SPM) coefficients are rescaled to one. Although \(\lambda\) and \(\gamma\) also can be rescaled to one, these coefficients are preserved for convenience to examine the effects of only cubic (\(\lambda \neq 0, \gamma = 0\)), only quintic (\(\lambda = 0, \gamma \neq 0\)), or weighted contribution of both coefficients (\(\lambda \neq 0, \gamma \neq 0\)) to overall MI [2]. Setting \(\gamma = 0\) and \(\beta = 1/\lambda\) transforms Eqs. (1)-(2) into the coupled DNLSE studied in [8], while by posing \(\beta = \alpha = 0\) Eqs. (1)-(2) become uncoupled and turned into two one-dimensional CQDNLSE’s reported in [1,14].

Eqs. (1)-(2) are derived from the Hamiltonian

\[H = H_a + H_b + H_{int}\]

through

\[i \frac{dA_n}{dz} = -\frac{\delta H}{\delta A_n^*}, \quad i \frac{dB_n}{dz} = -\frac{\delta H}{\delta B_n^*},\]

where

\[\begin{align*}
H_a &= \sum_n \left[ c_a (A_n^* A_{n+1} + A_{n-1} A_n^*) + \frac{\lambda}{2} |A_n|^4 + \frac{\gamma}{3} |A_n|^6 \right], \\
H_b &= \sum_n \left[ c_b (B_n^* B_{n+1} + B_{n-1} B_n^*) + \frac{\lambda}{2} |B_n|^4 + \frac{\gamma}{3} |B_n|^6 \right], \\
H_{int} &= \sum_n \left[ \beta |A_n|^2 |B_n|^2 + \alpha \gamma (|A_n|^4 |B_n|^2 + |A_n|^2 |B_n|^4) \right],
\end{align*}\]

Eqs. (1)-(2) have two conserved quantities: the Hamiltonian \(H\) and the total excitation norm \(N = \sum_n (|A_n|^2 + |B_n|^2)\).
3. The Existence and Stability of Dark Strongly Localized Modes

The existence of dark SLM’s can be investigated by setting \( A_n = \alpha_n \exp(ik_n z) \), \( B_n = \beta_n \exp(ik_n z) \) and substitute into Eqs. (1)-(2) one obtains the steady-state equations for real amplitudes \( \alpha_n \) and \( \beta_n \):

\[
-k_a \alpha_n + c_a \left( \alpha_{n+1} + \alpha_{n-1} \right) + \lambda \left( \alpha_n^2 + \beta_n^2 \right) \alpha_n + \gamma \left( \alpha_n^4 + 2\alpha_n^2 \beta_n^2 + \alpha_n^2 \beta_n^4 \right) \alpha_n = 0, \tag{8}
\]

\[
-k_b \beta_n + c_b \left( \beta_{n+1} + \beta_{n-1} \right) + \lambda \left( \beta_n^2 + \alpha_n^2 \right) \beta_n + \gamma \left( \beta_n^4 + 2\beta_n^2 \alpha_n^2 + \alpha_n^4 \right) \beta_n = 0. \tag{9}
\]

The exact solutions of steady-state equations (8)-(9) give rise stationary excitations in each site of a lattice that yet constitute dark SLM’s of different topologies. Topologies considered in this paper are restricted to the kink-like kinds, as what have been done in [8] for discrete cubic media.

Numerical stability analysis was performed by linearization using the ansatz

\[
A_n = \exp(ik_n z) \{ \alpha_n + \delta [c_n \exp(-i\omega z) + d_n \exp(i\omega z)] \},
\]

\[
B_n = \exp(ik_n z) \{ \beta_n + \delta [f_n \exp(-i\omega z) + g_n \exp(i\omega z)] \},
\]

where \( \omega \)'s denote the eigenfrequencies and \( \delta \ll 1 \). Substituting the ansatz into Eqs. (1)-(2) yields the (linearization) eigenfrequency problem at \( O(\delta) \):

\[
\omega \begin{pmatrix} c_k \\ d_k \\ f_k \\ g_k \end{pmatrix} = \begin{pmatrix} \partial F_{a,i} / \partial \alpha_j & \partial F_{a,i} / \partial \alpha_j^* & \partial F_{a,i} / \partial \beta_j & \partial F_{a,i} / \partial \beta_j^* \\ \partial F_{b,i} / \partial \alpha_j & \partial F_{b,i} / \partial \alpha_j^* & \partial F_{b,i} / \partial \beta_j & \partial F_{b,i} / \partial \beta_j^* \\ \partial F_{b,i} / \partial \alpha_j & \partial F_{b,i} / \partial \alpha_j^* & \partial F_{b,i} / \partial \beta_j & \partial F_{b,i} / \partial \beta_j^* \\ \partial F_{b,i} / \partial \alpha_j & \partial F_{b,i} / \partial \alpha_j^* & \partial F_{b,i} / \partial \beta_j & \partial F_{b,i} / \partial \beta_j^* \end{pmatrix} \begin{pmatrix} c_k \\ d_k \\ f_k \\ g_k \end{pmatrix},
\]

where

\[
F_{a,i} = c_a \left( \alpha_{i+1} + \alpha_{i-1} \right) + \lambda (|\alpha_i|^2 + |\beta_i|^2) \alpha_i + \gamma \left( \alpha_i^4 + 2\alpha_i^2 |\beta_i|^2 + |\alpha_i|^4 \right) \alpha_i - k_a \alpha_i,
\]

\[
F_{b,i} = c_b \left( \beta_{i+1} + \beta_{i-1} \right) + \lambda (|\beta_i|^2 + |\alpha_i|^2) \beta_i + \gamma \left( \beta_i^4 + 2\beta_i^2 |\alpha_i|^2 + |\beta_i|^4 \right) \beta_i - k_b \beta_i.
\]
The existence of imaginary parts of the eigenfrequencies $\omega$’s in the eigenfrequency spectrum denotes the instability of the solutions.

The space-time evolution of the solutions can be achieved by direct numerical integrations of Eqs. (1)-(2). The initial conditions of the model are taken from the exact solutions of Eqs. (8)-(9) with additional random, uniformly distributed perturbation of amplitude $10^{-4}$. We utilize the fourth-order Runge-Kutta procedure to perform the direct numerical integration.

3.1. Odd dark SLM’s

These modes are defined as every site has nonzero excitation except the central site (see Fig. 1a), and have the solution in the form $\alpha_n = A(\ldots, 1, 1, 1, 0, a_1, 1, 1, 1, \ldots)$ and $\beta_n = B(\ldots, 1, 1, b_1, 0, b_1, 1, 1, 1, \ldots)$. Substituting the above ansatz into Eqs. (8)-(9) straightforwardly we obtain the expressions for the dispersion relation

$$k_a = 2c_a + \lambda(A^2 + \beta B^2) + \gamma(A^4 + 2\alpha A^2 B^2 + \alpha B^4)$$

$$k_b = 2c_b + \lambda(B^2 + \beta A^2) + \gamma(B^4 + 2\alpha B^2 A^2 + \alpha A^4)$$

and the secondary excitations

$$a_1 = 1 - \zeta, \quad b_1 = 1 - \xi, \quad \zeta, \xi \ll 1$$

where

$$\zeta = \frac{\lambda(c_a - \beta c_b) + 2\gamma\left[\alpha A^2 (c_a - c_b) - B^2 (\alpha c_a - c_b)\right]}{2A^2 \sigma}$$

$$\xi = \frac{\lambda(c_b - \beta c_a) + 2\gamma\left[\alpha B^2 (c_a - c_b) - A^2 (\alpha c_a - c_b)\right]}{2B^2 \sigma}$$

$$\sigma = \lambda^2 (\beta^2 - 1) + 4\gamma^2 (\alpha - 1) \left[\alpha (A^2 + B^2) + (\alpha^2 + 1) A^2 B^2\right] + 2\lambda\gamma [\alpha (2\beta - 1) - 1] (A^2 + B^2).$$

It is worth to note that the choice of the value of peak amplitudes $A$ and $B$ is arbitrary. The approximation $\zeta, \xi \ll 1$ is concordant with the subject of this paper, i.e. strongly localized modes. These derivations are restricted to the first-order approximations primarily because of the interest on the physical aspects of the problem, thus deviations from the background of the order $\zeta^2, \xi^2$ and smaller are neglected. Examples of odd dark modes are displayed in Fig. 2.

In Fig. 2a, the absence of imaginary parts of the eigenfrequencies $\omega$ in the eigenfrequency spectrum (second top left panel) denotes that the solutions are stable at $c = -0.1$ (from hereby, we fix $c_a = c_b \equiv c$ to minimize the number of parameters). Meanwhile, the numerical stability analysis (second top right panel) showing that the solutions are said to be linearly stable for all $c$ up to $c = -0.1$, where the dependence of the lowest squared eigenfrequencies $\omega^2$ (solid line) is reported, as a function of $c$, to be always non-negative. Results from the direct numerical integrations of Eqs. (1)-(2) are found to be in full agreement with the stability analysis above. The deviation of the solutions from the exact ones are very small.

In contrast, in Fig. 2b, the eigenfrequency spectrum exhibits the presence of imaginary part of the eigenfrequencies $\omega$, while the numerical stability analysis shows that the lowest squared eigenfrequencies $\omega^2$ is always negative up to $c = -0.1$. These results indicate the instability of the solutions. Direct numerical integrations of the model depicts that the instability of the solutions gives rise to the localized states after transient, and the deviations of the solutions from the exacts’ are large.
Figure 2: First row panels. Exact odd dark SLM’s of Eqs. (1)-(2) of components $A_n$ and $B_n$. Second row panels. Eigenfrequency spectrum (left panel) and the numerical stability analysis as functions of $c$ (right panel) corresponds to the SLM’s depicted in the top panels. Third row panels. Space-time evolution of $|A_n|^2$ and $|B_n|^2$, respectively. Bottom row panels Deviation of respective components from their exact solutions. Parameters: $A = 1.01$, $B = 1$, $c_a = c_b = c = -0.1$, $\lambda = \gamma = 1$, $\alpha = 1$, and (a) $\beta = 0.9$, (b) $\beta = 1.1$.

3.2. Even dark SLM’s

These modes are centered between two adjacent lattice sites (refer Fig. 1b), and have the ansatz of the form $\alpha_n = A(\ldots, 1, 1, 1, \alpha_2, \alpha_1, \alpha_2, 1, 1, \ldots)$ and $\beta_n = B(\ldots, 1, 1, 1, \beta_2, \beta_1, \beta_1, \beta_2, 1, 1, \ldots)$ where $\alpha_2$ and $\beta_2$ are assumed to be near to unity as a result of the strong localization. Substituting the above ansatz into Eqs. (8)-(9) one can obtain the dispersion relations same as in Eqs. (10)-(11) and the excitations

$$
a_2 = 1 - \frac{c_a \left[ \lambda \beta - 2 \gamma \alpha (B^2 + A^2) \right]}{2A^2\sigma} (1 - b_1) - \frac{c_a \left[ \lambda - 4 \gamma (A^2 + \alpha B^2) \right]}{2A^2\sigma} (1 - a_1),$$

$$b_2 = 1 - \frac{c_a \left[ \lambda \beta - 2 \gamma \alpha (A^2 + B^2) \right]}{2B^2\sigma} (1 - a_1) - \frac{c_b \left[ \lambda - 4 \gamma (\alpha A^2 + B^2) \right]}{2B^2\sigma} (1 - b_1),$$

where

$$\sigma = 8\lambda^2 A^2 B^2 (\alpha^2 + 2) + 4\lambda \gamma / \alpha B^2 - 4\gamma^2 \alpha (\alpha - 4) (A^4 + B^4) - 8\lambda \gamma (\alpha + 1) (A^2 + B^2) - \lambda^2 \beta^2.$$
Here, \( a_1 \) and \( b_1 \) can be determined by solving the following system of nonlinear equations
\[
\begin{align*}
\{ & A^2 \left[ \lambda + \gamma A^2(a_1^2 + 1) \right] (a_1^2 - 1) + B^2 \left[ \lambda \beta + \gamma \alpha B^2(b_1^2 + 1) \right] (b_1^2 - 1) \\
& + 2\gamma \alpha A^2 B^2 (a_1^2 b_1^2 - 1) \} a_1 + c_a (1 - 3a_1) = 0, \\
\{ & B^2 \left[ \lambda + \gamma B^2(b_1^2 + 1) \right] (b_1^2 - 1) + A^2 \left[ \lambda \beta + \gamma \alpha A^2(a_1^2 + 1) \right] (a_1^2 - 1) \\
& + 2\gamma \alpha B^2 A^2 (b_1^2 a_1^2 - 1) \} b_1 + c_b (1 - 3b_1) = 0.
\end{align*}
\]
(15)
(16)

For some particular cases Eqs. (15)-(16) can be substantially reduced. Suppose that \(|a_1|, |b_1| \ll 1\), from Eqs. (13)-(16) one can obtain
\[
\begin{align*}
a_1 &= \frac{c_a}{k_a - 2c_a}, \quad b_1 = \frac{c_b}{k_b - 2c_b}, \\
a_2 &= 1 - \frac{c_b \left[ \lambda \beta - 2\gamma \alpha (B^2 + A^2) \right] - c_a \left[ \lambda - 4\gamma (A^2 + \alpha B^2) \right]}{2A^2 \sigma}, \\
b_2 &= 1 - \frac{c_a \left[ \lambda \beta - 2\gamma \alpha (A^2 + B^2) \right] - c_b \left[ \lambda - 4\gamma (B^2 + \alpha A^2) \right]}{2B^2 \sigma}.
\end{align*}
\]

The evolutions of stable and unstable propagations are displayed in Fig. 3. The solutions exhibit the same behaviours discussed in the odd dark modes case. Numerous simulations on odd and even dark SLM’s are found in excellent agreement with the background stability studied in [2].

3.3. Symbiotic dark–anti-dark SLM’s
Beside the coupled CDNLS, our model (1)-(2) also exhibit the special attribute of supporting symbiotic localized modes, i.e. modes with the combination of different topologies. A combination of dark and anti-dark localized modes (refer Fig. 1c) is described by Eqs. (15)-(16). This particular form of SLM’s is entirely caused by the vectorial nature of the interaction between components which does not present in the continuum limit or in the scalar discrete system [8]. Nevertheless, if the solutions of Eqs. (15)-(16) correspond to the structure of this mode are found, it is not a guarantee that dark–anti-dark modes are exist across the continuation from anti-continuum limit.

However, the modulational stability of the background — although it is essential — is insufficient to probe the existence of stable solitons. In Fig. 4, the parameters used are the ones with stable background in [2]. In Fig. 4a, linear stability analysis predicts that the solutions are stable up to \( c = -0.1 \), and direct numerical integrations confirms the prediction. Meanwhile in Fig. 4b, the presence of two pairs of imaginary parts of the eigenfrequencies \( \omega \) in the eigenfrequency spectrum illustrating the weak oscillatory instability of the solutions at \( c = -0.1 \), which is verified by direct numerical integrations of the model.

3.4. Symbiotic bright–dark SLM’s
Another kind of symbiotic modes is the combination of a bright \( A \)-component and a dark \( B \)-component. The ansatz \( \alpha_n = A(\ldots, 0, 0, 0, a_1, 1, a_1, 0, 0, 0, \ldots) \) and \( \beta_n = B(\ldots, 1, 1, 1, b_1, 0, b_1, 1, 1, 1, \ldots) \) allow us to recognize an odd feature of bright–dark modes sketched in Fig. 1d. The secondary excitations \( a_1 \) and \( b_1 \) can be found from the related set of algebraic equations as
\[
\begin{align*}
a_1 &= \frac{c_a}{\lambda (A^2 - \beta B^2) + \gamma (A^4 - \alpha B^4)}, \\
b_1 &= 1 - \xi, \quad \xi = \frac{-c_b}{2(\lambda - 2\gamma B^2)B^2} \ll 1,
\end{align*}
\]
(17)
(18)
Figure 3: Same as in Fig. 2 but for even dark modes. All parameters are fixed as in Fig. 2.

and the corresponding dispersion relations as

\[ k_a = (\lambda + \gamma A^2)A^2, \quad k_b = 2c_b + (\lambda + \gamma B^2)B^2. \]  

(19)

In Fig. 5a, eigenfrequency spectrum shows purely real parts of eigenfrequencies \( \omega \), while the numerical linear analysis indicating that the solutions should be stable up to \( c = -0.1 \). Direct numerical integrations illustrate that the solutions propagate with stability across transient.

The same kind of solutions can arise in the even case \( \alpha_n = A(\ldots, 0, 0, 0, a, 1, s_a,a, 0, 0, \ldots) \) and \( \beta_n = B(\ldots, 1, 1, 1, b_2, b_1, b_1, b_2, 1, 1, \ldots) \), where \( s_a = \pm 1 \), with dispersion relations

\[ k_a = c_a s_a + (\lambda + \gamma A^2)A^2, \quad k_b = 2c_b + (\lambda + \gamma A^2)B^2 \]  

(20)

and the excitations

\[ a = \frac{c_a}{(\lambda + \gamma A^2)A^2 - (\lambda\beta + \gamma\alpha B^2)B^2}, \quad b_1 = \frac{c_b}{(\lambda + \gamma B^2)B^2 - (\lambda\beta + \gamma\alpha A^2)A^2}, \]  

\[ b_2 = 1 - \xi, \quad \xi = -\frac{c_b(1 - b_1)}{2(\lambda + 2\gamma B^2)B^2} \ll 1. \]  

(21)  

(22)

In Fig. 5b, a pair of imaginary parts of eigenfrequencies \( \omega \) arise in the eigenfrequency spectrum due to the phase invariance, denoting strong exponential instability of the solutions. Negative
values of the lowest squared eigenfrequencies $\omega^2$ illustrate the instability of the solution up to $c = -0.1$. These predictions are verified by direct numerical integrations, where the effect of instability is stronger on bright $A$-component than dark $B$-component. The background stability is again inadequate condition for the existence of stable localized modes.

4. Conclusion
In this paper we have demonstrated the existence of dark strongly localized modes in binary discrete media with cubic and quintic nonlinearities modelled by the coupled CQDNLSE. The stability of these modes has been probed both by linear spectral analysis and direct numerical integrations. We found that stability of odd and even dark modes confirms the results on modulational stability region identified in [2], while the symbiotic modes are not restrained by it. For odd and even dark modes, instability gives rise to localized states. Meanwhile, the existence of symbiotic dark–anti-dark SLM’s are restricted to a small range of parameters. For the symbiotic bright-dark SLM’s, even bright component are found to be unstable earlier that dark component.
Figure 5: Same as in Fig. 2 but for symbiotic bright-dark modes. Parameters: $A = 1.01$, $B = 1$, $c_a = c_b = c = -0.1$, $\lambda = \gamma = 1$, and $\beta = -3$, $\alpha = 2$, (a) odd modes and (b) even modes with $s_a = -1$.

Acknowledgment
The authors would like to acknowledge supports from the Ministry of Higher Education, Malaysia under research grant no. FRGS 144870-169605.

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