Conformally Invariant Green Functions of Current and Energy-Momentum Tensor in Spaces of Even Dimension $D \geq 4$.

E.S. Fradkin
Institute for Advanced Study, Princeton, NJ08540, USA
and P.N. Lebedev Physical Institute,
Leninsky Prospect, 53, 117924, Moscow, Russia.

M.Ya. Palchik
Inst. of Automation and Electrometry, Novosibirsk 630090, Russia

Abstract

We study the conformally invariant quantum field theory in spaces of even dimension $D \geq 4$. The conformal transformations of current $j_\mu$ and energy-momentum tensor $T_{\mu\nu}$ are examined. It is shown that the set of conformal transformations of particular kind corresponds to the canonical (unlike anomalous) dimensions $l_j = D - 1$ and $l_T = D$ of those fields. These transformations cannot be derived by a smooth transition from anomalous dimensions. The structure of representations of the conformal group, which correspond to these canonical dimensions, is analyzed, and new expressions for the propagators $\langle j_\mu j_\nu \rangle$ and $\langle T_{\mu\nu} T_{\rho\sigma} \rangle$ are derived. The latter expressions have integrable singularities. It is shown that both propagators satisfy non-trivial Ward identities. The higher Green functions of the fields $j_\mu$ and $T_{\mu\nu}$ are considered. The conformal QED and linear conformal gravity are discussed. We obtain the expressions for invariant propagators of electromagnetic and gravitational fields. The integrations over internal photon and graviton lines are performed. The integrals are shown to be conformally invariant and convergent, provided that the new expressions for the propagators are used.
1 Introduction

A family of exactly solvable conformal models in $D$ dimensions is studied in [1–5]. Notwithstanding that the conformal group in $D \geq 3$ is finite-parametric, these models are in many respects similar to two-dimensional conformal theories, and coincide in $D = 2$ with minimal models [6–8]. The approach presented in [1–2] is based on conformally invariant [3] Ward identities for the conserved current $j_\mu$ and the energy-momentum tensor $T_{\mu\nu}$. In $D$-dimensional case the latter fields have the canonical dimensions

$$l_j = D - 1, \quad l_T = D.$$ (1.1)

The conformal symmetry is supposed to be exact, and the energy-momentum tensor to be traceless: $T_{\mu\nu} = 0$.

The conformally invariant propagators

$$\langle j_\mu(x_1)j_\nu(x_2) \rangle, \quad \langle T_{\mu\nu}(x_1)T_{\rho\sigma}(x_2) \rangle$$ (1.2)

are known to be ill-defined for even $D \geq 4$ due to formally non-integrable factors $(x_{12}^2)^{-D+1}$, $(x_{12}^2)^{-D}$. This leads to substantial difficulties when one attempts to include gauge interactions in a conformally-invariant fashion. It is one of the lasting problems in $D \geq 4$ conformal theory. The present article is aimed to impart a detailed analysis and the solution of this problem.

We shall show that for even $D \geq 4$ the conformally invariant propagators [1.2] satisfy non-trivial Ward identities regardlessly to the choice of regularization for their transversal parts. The r.h. sides of the Ward identities are non-zero due to contributions of equal-time commutators between the components of current and energy-momentum tensor. In conformal theory, each commutator is expressed through the central charge $C_j$ or $C_T$.

As shown in [1] (see also [2,3]), the $D$-dimensional analogues of the central charge may be c-number or, as well, operator valued. They arise in operator product expansions

$$j_\mu(x)j_\nu(0) = [C_j] + [P_j(x)] + \ldots,$$

$$T_{\mu\nu}(x)T_{\rho\sigma}(0) = [C_T] + [P_T(x)] + \ldots,$$

where $P_j(x)$ and $P_T(x)$ are the scalar conformal fields of the dimension $d_P = d_P^T = D - 2$.

The operator analogues of the central charge, the fields $P_j(x)$ and $P_T(x)$, contribute to the Ward identities for higher Green functions $\langle j_\mu j_\nu \ldots \rangle$ and $\langle T_{\mu\nu} T_{\rho\sigma} \ldots \rangle$, see [3] for more details.

\footnote{We stress that the conformal symmetry remains exact in this approach. Another approach was discussed lately in [9,10]. These works assume a special redefinition of conformal propagators [1.2] which breaks the conformal symmetry. The power factors are replaced by regularized expressions. After that, the breakdown of the symmetry is shown to result in conformal anomalies. Note that such effects might also be considered in our approach, if the symmetry were to be broken. However, it falls beyond the focus of our discussion.}
The propagators, as well as higher Green functions, may be chosen to have no longitudinal parts. Let us remind that the expectation values of $T$-ordered products of the fields are defined modulo quasilocal terms. One may employ this arbitrariness to pass to transversal propagators. However, such a redefinition of the propagators breaks the conformal symmetry. In the section 2 we show that longitudinal parts of Euclidean propagators are uniquely fixed by conformal symmetry.

As an example, let us start with two-dimensional conformal models. The problems due to divergencies are absent in this case. Still, the Ward identities are non-trivial, as in the case of $D \geq 4$. The conformally invariant propagator of the current

$$\langle j_\mu(x) j_\nu(0) \rangle|_{D=2} = -\frac{1}{8\pi} C_j \left( \delta_{\mu\nu} - 2\frac{x_\mu x_\nu}{x^2} \right) \frac{1}{x^2} = -\frac{1}{4\pi} C_j \partial_\mu \partial_\nu \ln x^2$$

satisfies the following Ward identity:

$$\partial_\mu \langle j_\mu(x) j_\nu(0) \rangle = C_j \partial_\nu \delta(x).$$

Consider the traceless energy-momentum tensor. The conformally invariant propagator

$$\langle T_{\mu\nu}(x) T_{\rho\sigma}(0) \rangle|_{D=2} = \frac{1}{8\pi} C_{T} \left[ g_{\mu\rho}(x) g_{\nu\sigma}(x) + g_{\mu\sigma}(x) g_{\nu\rho}(x) - \delta_{\mu\rho} \delta_{\nu\sigma} \right] \frac{1}{(x^2)^{2+\epsilon}} \bigg|_{\epsilon=0},$$

where $g_{\mu\nu}(x) = \delta_{\mu\nu} - 2\frac{x_\mu x_\nu}{x^2}$, is finite when $\epsilon \to 0$. Using the identity for $D = 2$

$$\delta_{\mu\rho} \partial_\mu \partial_\nu + \delta_{\mu\nu} \partial_\mu \partial_\rho + \delta_{\nu\rho} \partial_\mu \partial_\sigma + \delta_{\nu\sigma} \partial_\mu \partial_\rho = 2 (\delta_{\mu\rho} \partial_\sigma + \delta_{\mu\sigma} \partial_\rho + (\delta_{\mu\rho} \delta_{\nu\sigma} + \delta_{\mu\sigma} \delta_{\nu\rho} - \delta_{\mu\nu} \delta_{\rho\sigma}) \square),$$

one can bring it to the form

$$\langle T_{\mu\nu}(x) T_{\rho\sigma}(0) \rangle|_{D=2} = -\frac{C_{T}}{24} \left[ (\partial_\mu \partial_\nu - \frac{1}{2} \delta_{\mu\nu} \square) \left( \partial_\rho \partial_\sigma - \frac{1}{2} \delta_{\rho\sigma} \square \right) ight]$$

$$- \frac{1}{8} (\delta_{\mu\rho} \delta_{\nu\sigma} + \delta_{\mu\sigma} \delta_{\nu\rho} - \delta_{\mu\nu} \delta_{\rho\sigma} \square)^2 \frac{1}{\square} \delta(x),$$

where $\frac{1}{\square} \delta(x) = -\frac{1}{4\pi} \ln x^2$. The Ward identity may be derived directly from conducting the calculations for $\epsilon \neq 0$ and then passing to the limit [11] $\lim_{\epsilon \to 0} \left[ \epsilon \ln \frac{1}{(x^2)^{1+\epsilon}} \right] = \pi \delta(x)$. As the result one obtains

$$\partial_\nu \langle T_{\mu\nu}(x) T_{\rho\sigma}(0) \rangle|_{D=2} = -\frac{C_{T}}{24} \left[ (\partial_\mu \partial_\nu - \frac{1}{2} \delta_{\mu\nu} \square) \left( \partial_\rho \partial_\sigma - \frac{1}{2} \delta_{\rho\sigma} \square \right) ight]$$

$$- \frac{1}{4} \delta_{\rho\sigma} \partial_\mu \square] \delta(x).$$
This equation is a particular case of the Ward identity to be found in the next section for the case of any even $D$. Passing in \( \left[1.5, 1.8\right] \) to complex variables

\[
x^\pm = x_1 \pm ix_2, \quad \partial^\pm = \frac{1}{2}(\partial_1 \mp i\partial_2), \quad x^\pm \to z, \quad x^- \to \bar{z}, \quad \partial_+ \to \partial_z, \quad \partial_- \to \partial_{\bar{z}}
\]

\[
T_{zz} \sim \frac{1}{2}(T_{11} - T_{22} - iT_{12} - iT_{21}), \quad T_{\bar{z}\bar{z}} \sim \frac{1}{2}(T_{11} + T_{22})
\]

we find well-known results

\[
\langle T_{zz}(z)T_{zz}(0) \rangle = \frac{1}{6}\pi CT_3(\partial_z)^4(\partial_z\partial_{\bar{z}})^{-1}\delta^{(2)}(z, \bar{z}) \sim \frac{1}{z^4},
\]

\[
\partial_z \langle T_{zz}(z)T_{zz}(0) \rangle = \frac{\pi CT_3}{6}(\partial_z)^3\delta^{(2)}(z, \bar{z}).
\]

The conformally invariant expression \(1.7\) differs from the transversal propagator

\[
\sim (\partial_\mu \partial_\nu - \delta_{\mu\nu} \square)(\partial_\rho \partial_\sigma - \delta_{\rho\sigma} \square) \ln x^2
\]

by quasilocal terms. These terms are present in \(1.7\) and may be eliminated by a redefinition of the $T$-ordered product of $T_{\mu\nu}$ components. However the latter will lead to a breakdown of conformal invariance, and to non-zero trace of the energy-momentum tensor.

The expression for the propagator $\langle TT \rangle$ for $D > 2$ results from \(1.3\) after the change $(x^2)^{-2-\epsilon} \to (x^2)^{-D-\epsilon}$. It is divergent in the limit $\epsilon \to 0$ for even $D \geq 4$, since $(x^2)^{-D-\epsilon} \mid_{\epsilon \to 0} \sim (1/\epsilon)^{D/2}\delta(x)$. Similarly, the expression for the propagator $\langle jj \rangle$ for $D > 2$ results from \(1.3\) after the change $(x^2)^{-1} \to (x^2)^{-D+1+\epsilon}$, and also diverges in the limit $\epsilon \to 0$. Note that in two-dimensional case these divergences are cancelled; it can be seen from the differential representations \(1.9,1.3\). We shall show in sections 3–5 that for $D \geq 4$ the latter divergences are formal and contribute neither to Feynman graphs nor to any other contractions available in conformal theory. The presence of the above divergences is caused by peculiar properties of conformal transformations for the fields of dimensions \(1.1\). We shall demonstrate that the conformal transformations of the fields, as well as invariant averages of the fields with anomalous dimensions

\[
l = l_j + \epsilon, \quad l = l_T + \epsilon,
\]

can not be analytically continued to the values at $\epsilon = 0$ because the representations of the conformal group have singular properties at these points. These representations belong to a series of exceptional integer points [12,13] and must be reconstructed completely. This is done in sections 3,4. Conformal transformations of the fields $j_\mu$, $T_{\mu\nu}$ for $\epsilon = 0$ will be shown to have different structures in longitudinal and transversal sectors. Correspondingly, the two types of invariant kernels contributing to the propagators \(1.2\) will be shown to exist. The latter leads to a specific structure of the conformal fields $j_\mu(x)$ and $T_{\mu\nu}(x)$. Two mutually orthogonal sectors of the Hilbert
space, having different physical meanings [1,2] (see also Sec.6), correspond to each of these fields. The elements of the first sector are the equivalence classes, each including a whole set of states. The transformations inside an equivalence class do not alter physical results. We shall show that the above divergences may be eliminated using the transformations inside equivalence classes and hence do not affect the results, see Sec. 3,4.

In sections 5,6 the conformal QED and the linear conformal gravity are considered. The problem of how to single out the contribution due to gauge interactions into the conformally invariant Gauge functions

\[ \langle j_\mu \ldots \rangle, \quad \langle T_{\mu\nu} \ldots \rangle, \quad (1.11) \]

where dots stand for any set of fields, will also be discussed. The above Green functions can be found [1,2] from Ward identities up to transversal conformal parts solely caused by gauge interactions. The framework developed here allows to single out this contribution uniquely, see also [4,5]. The Green functions [1.11] are calculated from Ward identities, their remaining parts describe a theory of direct (non-gauge) interaction. As a result, in a theory without gauge interactions the Green functions [1.11] are uniquely determined by Ward identities. In our opinion, this result appears to be sufficiently promising, since it allows one to derive [1,2] the $D$-dimensional analogues of minimal models. The additional conditions imposed on the functions [1.11] which ensure the absence of contributions due to gauge interactions, as well as the number of models in $D > 2$ and the transition to $D = 2$ theory in this approach, are dealt with in [4,5]. Here we present a more detailed justification of these conditions (section 6).

Let us remark that for the sake of completeness the section 3 contains a brief description of representations of the conformal group in integer points, to the length required for understanding the sections 4–6. At the first reading, section 3 may be skipped. A comprehensive analysis of representations regarding conformal gauge theories is given in [1] (see also [14–17]).

2 Ward Identities for the Propagators $\langle j_\mu j_\nu \rangle$ and $\langle T_{\mu\nu} T_{\rho\sigma} \rangle$

2.1 Longitudinal Part of the Propagator of the Current

Let $j_\mu^l(x)$ be a conformal quantum field of anomalous dimension $l$ in $D$-dimensional Euclidean space. The transformation of conformal inversion

\[ x_\mu \rightarrow Rx_\mu = \frac{x_\mu}{x^2} \quad (2.1) \]

induces the following transformation of the field $j_\mu^l$:

\[ j_\mu^l(x) \rightarrow R, \quad \frac{1}{(x^2)^l}g_{\mu\nu}(x)j_\nu(Rx), \quad (2.2) \]
where
\[ g_{\mu\nu}(x) = \delta_{\mu\nu} - \frac{2x_\mu x_\nu}{x^2}. \] (2.3)

The invariant propagator is determined from the condition
\[ \langle j^i_\mu(x_1) j^j_\nu(x_2) \rangle = \frac{1}{(x_1^2)^l} \frac{1}{(x_2^2)^l} g_{\mu\rho}(x_1) g_{\nu\sigma}(x_2) \langle j^\rho_{(Rx_1)} j^\sigma_{(Rx_2)} \rangle. \] (2.4)

The solution has the form
\[ \langle j^i_\mu(x_1) j^j_\nu(x_2) \rangle = \tilde{C}_j \frac{1}{(x_{12}^2)^l} g_{\mu\nu}(x_{12}), \] (2.5)
where \( \tilde{C}_j \) is some constant.

Consider the behaviour of this propagator near the canonical dimension point
\[ l = l_j + \epsilon = D - 1 + \epsilon. \] (2.6)

All its contractions of the type
\[ \int dx_1 dx_2 A^{D-l}_\mu(x_1) \langle j^i_\mu(x_1) j^j_\nu(x_2) \rangle A^{D-l}_\nu(x_2) \bigg|_{\epsilon \to 0}, \] (2.7)
where \( A^{D-l}_\mu \) is a conformal field of dimension \( D - l \), diverge in the limit of \( \epsilon = 0 \). The term leading in \( \epsilon \) is calculated using the relations [11]
\[ \frac{1}{(x^2)^{D/2+k+\epsilon}} \bigg|_{\epsilon \to 0} \sim \frac{1}{\epsilon} \frac{\pi^{D/2} 4^{-k}}{\Gamma \left( \frac{D}{2} + k \right) \Gamma (k + 1)} \Box^k \delta(x). \] (2.8)

Let us rewrite the propagator as
\[ \Delta^\epsilon_{\mu\nu}(x_{12}) = \langle j^i_\mu(x_1) j^j_\nu(x_2) \rangle = \frac{\tilde{C}_j}{2(D - 1 + \epsilon)(D - 2 + \epsilon)} \]
\[ \times \left[ (\delta_{\mu\nu} - \partial_\mu \partial_\nu) \frac{1}{(x_{12}^2)^{D-2+\epsilon}} - \frac{\epsilon}{D - 2 + 2\epsilon} \delta_{\mu\nu} \right]. \] (2.9)

Hence it is clear that the divergent part of the propagator is transversal.

Note that in the limit \( \epsilon = 0 \) the field \( A^{D-l}_\mu \) coincides with the electromagnetic field. A conformally invariant regularization \( (l_A = 1 \to l'_A = 1 - \epsilon) \) of the field \( A_\mu(x) \) will be treated in section 5. The contractions (2.7) in the longitudinal sector
\[ A^{l}_{\mu}(x) \big|_{\epsilon = 0} = \partial_\mu \varphi, \quad A^{tr}_{\mu}(x) \big|_{\epsilon = 0} = 0 \] (2.10)
are finite in virtue of (2.9). As will be shown in sections 4,5, a new expression for the propagator \( \langle j_\mu j_\nu \rangle \) appears in the transversal sector, while the kernel (2.9) may participate in the longitudinal sector only. Below in this section we consider a theory
without electromagnetic interaction. By definition, only the contractions \( \mathbf{2.7} \) with longitudinal fields \( \mathbf{2.10} \)

\[
\int dx_1 dx_2 A^{\text{long}}_\mu(x_1) \Delta^\epsilon_{\mu\nu}(x_{12}) A^{\text{long}}_\nu(x_2) \bigg|_{\epsilon=0}
\]

are present in that theory. These contractions are transversal because the divergent part of the kernel \( \mathbf{2.9} \) does not contribute to them.

Consider a Ward identity for the propagator \( \langle j_\mu j_\nu \rangle \). Define the l.h.s. of the identity by the relation

\[
\partial_\mu \langle j_\mu(x_1) j_\nu(x_2) \rangle = \lim_{\epsilon \to 0} \partial_\mu \langle j^l_\mu(x_1) j^l_\nu(x_2) \rangle.
\]

From \( \mathbf{2.5} \) one has

\[
\partial_\mu \langle j^l_\mu(x_1) j^l_\nu(x_2) \rangle = -\frac{\epsilon \tilde{C}_j}{(D-1+\epsilon)} \frac{1}{(x_{12})^{D-1+\epsilon}}.
\]

Using \( \mathbf{2.8} \) for \( k = \frac{D-2}{2} \)

\[
\lim_{\epsilon \to 0} \epsilon \frac{1}{(x_{12})^{D-1+\epsilon}} = 4^{-\frac{D-2}{2}} \pi^{D/2} \frac{\Gamma \left( \frac{D}{2} \right)}{\Gamma (D-1) \Gamma \left( \frac{D}{2} \right)} \frac{D}{2} \delta(x),
\]

we find

\[
\partial_\mu \langle j_\mu(x_1) j_\nu(x_2) \rangle = C_j \partial_\nu \frac{D}{2} \delta(x_{12}),
\]

where

\[
C_j = -4^{-\frac{D-2}{2}} \pi^{D/2} \frac{\Gamma \left( \frac{D}{2} \right)}{\Gamma (D)} \tilde{C}_j.
\]

One easily checks that this identity is conformally invariant. The l.h.s of \( \mathbf{2.12} \) transforms as a Euclidean average

\[
\langle j(x_1) j_\nu(x_2) \rangle \neq 0,
\]

where \( j(x) = \partial_\mu j_\mu(x) \) is a conformal scalar of dimension \( D \):

\[
j(x) \to \frac{1}{(x^2)^D} j(Rx).
\]

This transformation law is proved with the help of identities

\[
\partial^2_\mu = \frac{1}{x^2} g_{\mu\nu}(x) \partial^R_{\mu}, \quad g_{\mu\nu}(x) g_{\mu\nu}(x) = \delta_{\mu\nu}, \quad \partial_\mu \left[ \frac{1}{(x^2)^{D-1}} g_{\mu\nu}(x) \right] = 0.
\]

Note that the conformally invariant average including scalar and vector vanishes in the general case

\[
\langle j^l_\mu(x_1) j^l_\nu(x_2) \rangle = 0, \quad \text{if } l_1 \neq D \text{ and } l_2 \neq D - 1.
\]
for any dimension save
\[ l_1 = D, \quad l_2 = D - 1, \]
the latter are exceptional ones. The r.h.s. of the identity 2.12 also transforms like the
quantity 2.13 — this may be checked with the help of relation
\[
\delta(Rx_{12}) = \left(\frac{x_{12}^2}{x_1^2}\right)^D \delta(x_{12}). \tag{2.15}
\]
The invariance of the identity 2.12 is proved.
Thus, the expansion in \( \epsilon \) of the regularized propagator \( \Delta^\epsilon_{\mu\nu} \) has a finite longitudinal
term
\[
\Delta^\text{long}_{\mu\nu}(x_{12}) = \langle j_\mu(x_1) j_\nu(x_2) \rangle^\text{long} = C_j \partial_\mu \partial_\nu \square \frac{\partial}{\partial x_{12}^2} \delta(x_{12}). \tag{2.16}
\]
In the discussions below, the following form of the regularized propagator is useful
\[
\Delta^\epsilon_{\mu\nu} = \frac{\tilde{C}_j}{2(D - 1 + \epsilon)(D - 2 + 2\epsilon)} \left( \delta_{\mu\nu} \square - \partial_\mu \partial_\nu \right) \frac{1}{(x_{12}^2)^{D - 2 + \epsilon}}
+ C_j \partial_\mu \partial_\nu \square \frac{\partial}{\partial x_{12}^2} \delta(x_{12}) + O(\epsilon). \tag{2.17}
\]
Note that in the derivation of the regularized expression 2.17 in some publications
the finite longitudinal term was missing, and the propagator was identified with the
regularized transversal part of this expression.

### 2.2 Transversal Part of the Propagator of the Energy-Momentum Tensor

Let \( T_{\mu\nu}^l(x) \) be a traceless symmetric tensor of dimension \( l \). Under conformal inversion
it transforms as
\[
T_{\mu\nu}^l(x) \overset{R}{\longrightarrow} \frac{1}{(x_{12}^2)^l} g_{\mu\rho}(x) g_{\nu\sigma}(x) T_{\rho\sigma}^l(Rx). \tag{2.18}
\]
The conformal propagator is determined by the condition of invariance
\[
\langle T_{\mu\nu}^l(x_1) T_{\rho\sigma}^l(x_2) \rangle = \frac{1}{(x_{12}^2)^l} g_{\mu\alpha}(x_1) g_{\nu\beta}(x_1) g_{\rho\lambda}(x_2) g_{\sigma\tau}(x_2)
\times \langle T_{\alpha\beta}^l(Rx_1) T_{\lambda\tau}^l(Rx_2) \rangle \tag{2.19}
\]
and has the form
\[
\langle T_{\mu\nu}^l(x_1) T_{\rho\sigma}^l(x_2) \rangle = \tilde{C}_T \left[ g_{\mu\rho}(x_{12}) g_{\nu\sigma}(x_{12}) + g_{\mu\sigma}(x_{12}) g_{\nu\rho}(x_{12}) - \frac{2}{D} \delta_{\mu\nu} \delta_{\rho\sigma} \right] \frac{1}{(x_{12}^2)^l}, \tag{2.20}
\]
where \( \tilde{C}_T \) is some constant. Consider its behaviour near the point of canonical
dimension \( l_T \):
\[
l = l_T + \epsilon = D + \epsilon. \tag{2.21}
\]
The contractions

\[ \int dx_1 dx_2 \ h_{\mu\nu}^{D-l}(x_1) \langle T_{\mu\nu}^l(x_1) T_{\rho\sigma}^l(x_2) \rangle h_{\rho\sigma}^{D-l}(x_2) \bigg|_{\epsilon \to 0} \]  

are divergent in the limit of \( \epsilon = 0 \) if \( h_{\mu\nu}^{\epsilon}(x) \neq 0 \). One can check that the singular in \( \epsilon \) part of this propagator is transversal and has the form:

\[ \Delta_{\mu\nu}(x_{12}) \sim \tilde{C}_T H_{\mu\nu}^{\epsilon} \left( \frac{\partial}{\partial x} \right) \frac{1}{(x_{12}^2)^{D-2+\epsilon}} + O(1), \]  

where \( H_{\mu\nu}^{\epsilon} \) is the transversal (in each index) differential operator

\[ \partial_\mu H_{\mu\nu\rho\sigma}^{\epsilon} \left( \frac{\partial}{\partial x} \right) = 0, \quad H_{\mu\nu\rho\sigma}^{\epsilon} = H_{\rho\sigma\mu\nu}^{\epsilon}, \]

\[ H_{\mu\nu\rho\sigma}^{\epsilon} \left( \frac{\partial}{\partial x} \right) = \left\{ \frac{D-2}{D-1} \partial_\mu \partial_\nu \partial_\rho \partial_\sigma - \frac{1}{2} (\delta_{\mu\rho} \partial_\nu \partial_\sigma + \delta_{\mu\sigma} \partial_\nu \partial_\rho)
+ \delta_{\nu\rho} \partial_\mu \partial_\sigma + \delta_{\nu\sigma} \partial_\mu \partial_\rho) \square - \frac{1}{(D-1)} (\delta_{\mu\rho} \partial_\nu \partial_\sigma + \delta_{\rho\sigma} \partial_\mu \partial_\nu) \square
+ \frac{1}{2} (\delta_{\mu\rho} \delta_{\nu\sigma} + \delta_{\mu\sigma} \delta_{\nu\rho}) \square^2 - \frac{1}{(D-1)} \delta_{\mu\nu} \delta_{\rho\sigma} \square^2 \right\}. \]  

Thus the divergent part of the contraction 2.22 is caused by a contribution of the transversal component of the field \( h_{\mu\nu}(x) \) when \( \epsilon \to 0 \).

In the limit \( \epsilon = 0 \) the field \( h_{\mu\nu}(x) \) coincides with the traceless part of metric tensor, which has a zero dimension: \( l_h = D - l_T = 0 \). In section 5 we describe a conformally invariant regularization of the field \( h_{\mu\nu} \) \( l_h \to l_h^\epsilon = -\epsilon \). In the longitudinal sector consisting of the fields

\[ h_{\mu\nu}^{\long}(x) \bigg|_{\epsilon = 0} = \partial_\mu h_\nu(x) + \partial_\nu h_\mu(x) - \frac{2}{D} \delta_{\mu\nu} \partial_\lambda h_\lambda(x), \quad h_{\mu\nu}^{\long}(x) \bigg|_{\epsilon = 0} = 0 \]

we get the contractions which are finite due to 2.23

\[ \int dx_1 dx_2 \ h_{\mu\nu}^{\long}(x_1) \langle T_{\mu\nu}^l(x_1) T_{\rho\sigma}^l(x_2) \rangle h_{\rho\sigma}^{\long}(x_2). \]  

In the theory with no gravitational interaction only the contractions 2.26 may appear. We show in sections 4–5 that in the transversal sector of the conformal gravity a new expression for the propagator \( \langle T_{\mu\nu} T_{\rho\sigma} \rangle \) arises. This expression is free of divergences at \( \epsilon = 0 \).

Consider the Ward identity for the propagator. By definition, we set

\[ \partial_\nu \langle T_{\mu\nu}(x_1) T_{\rho\sigma}(x_2) \rangle = \lim_{\epsilon \to 0} \partial_\nu \langle T_{\mu\nu}^l(x_1) T_{\rho\sigma}^l(x_2) \rangle. \]
The divergent part present in 2.23 vanishes after taking the derivative. As the result we have from 2.20
\[ \partial_\nu \langle T^l_\mu(x_1)T^l_\rho(x_2) \rangle = \frac{\epsilon \tilde{C}_T}{(D - 1 + \epsilon)(D + 1 + \epsilon)} \left[ \partial_\mu \partial_\rho \partial_\sigma \right. \\
- \frac{(D - 1 + \epsilon)}{2(D + 2 \epsilon)} (\delta_{\mu \rho} \partial_\sigma + \delta_{\mu \sigma} \partial_\rho) \Box - \frac{1 + \epsilon}{D(D + \epsilon)} \delta_{\rho \sigma} \partial_\mu \Box \left. \right] \frac{1}{(x_{12})^{D-1+\epsilon}}. \] (2.27)

Using the relation 2.18 for \( k = \frac{D-2}{2} \) one gets
\[ \partial_\nu \langle T_\mu T_\rho(x_1)T_\rho(x_2) \rangle = C_T \left[ \partial_\mu \partial_\rho \partial_\sigma - \frac{D - 1}{2D} (\delta_{\mu \rho} \partial_\sigma + \delta_{\mu \sigma} \partial_\rho) \right. \\
- \frac{1}{D^2} \delta_{\rho \sigma} \partial_\mu \Box \left. \right] \frac{D-2}{2} \delta(x_{12}), \] (2.28)
where
\[ C_T = -\tilde{C}_T \pi^{D/2} 4^{-(D-2)} \left[ \Gamma(D+2) \Gamma \left( \frac{D}{2} \right) \right]^{-1}. \]

For \( D = 2 \) this identity becomes naturally 1.8. It is not hard to show that the identity 2.28 is conformally invariant. The l.h.s. transforms as a conformally invariant average of a vector and a tensor:
\[ \langle T_\mu(x_1)T_\rho(x_2) \rangle \xrightarrow{R} \frac{1}{(x_1^2)^{D+1}} \frac{1}{(x_2^2)^D} g_{\mu\nu}(x_1)g_{\rho\alpha}(x_2)g_{\sigma\beta}(x_2) \langle T_\nu(Rx_1)T_\alpha(Rx_2) \rangle, \] (2.29)
where \( T_\mu(x) = \partial_\nu T_{\mu\nu}(x) \) is a conformal vector of dimension \( d + 1 \). One easily checks that for \( l = D \) the quantity \( \partial_\nu T_{\mu\nu}(x) \) is indeed the conformal vector
\[ \partial_\nu T_{\mu\nu}(x) \xrightarrow{R} \frac{1}{(x^2)^{D+1}} g_{\mu\rho}(x) \partial_{\nu}^{Rz} T_{\rho\nu}(Rx). \]

Notice that in the general case the conformally invariant average of a vector and a tensor is zero
\[ \langle T^l_\mu(x_1)T^l_\rho(x_2) \rangle, \text{ if } l_1 \neq D + 1 \text{ and } l_2 \neq D, \]
for any dimension except
\[ l_1 = D + 1, \quad l_2 = D. \]
These dimensions are exceptional, and \( \langle T_\mu T_\rho \rangle \neq 0 \). The r.h.s. of the identity 2.28 also transforms by the law 2.24. This is checked with the help of 2.15. The invariance of the identity is proved.

From the discussion above follows that the expansion 2.23 includes a finite longitudinal term of the type
\[ \Delta_{\mu\rho\sigma}(x_{12}) = \langle T_{\mu\nu}(x_1)T_{\rho\sigma}(x_2) \rangle^{\text{long}} = \partial_\mu \Delta_{\nu\rho\sigma}(x_{12}) + \partial_\nu \Delta_{\mu\rho\sigma}(x_{12}) - \frac{2}{D} \delta_{\mu\rho} \partial_\lambda \Delta_{\lambda\rho\sigma}(x_{12}). \] (2.30)
One finds from the Ward identity [2.28]:
\[
\Delta_{\mu\rho\sigma}(x_{12}) = C_T \left\{ \frac{(2D^2 - 3D + 2)}{2D(D-1)} \partial_{\mu} \partial_{\rho} \partial_{\sigma} \\
- \frac{D-1}{2D} (\delta_{\mu\rho} \partial_{\sigma} + \delta_{\mu\sigma} \partial_{\rho}) \Box - \frac{1}{2D(D-1)} \delta_{\rho\sigma} \partial_{\mu} \Box \right\} \Box^{\mu+} \delta(x_{12}).
\] (2.31)

Rewrite the expansion 2.23 in the form
\[
\Delta^\epsilon_{\mu\nu\rho\sigma}(x_{12}) = \langle T^l_{\mu\nu}(x_1) T^l_{\rho\sigma}(x_2) \rangle \bigg|_{\epsilon \to 0} \simeq \Delta^\epsilon_{\mu\nu\rho\sigma}(x_{12}) + \Delta^\text{long}_{\mu\nu\rho\sigma}(x_{12}) + O(\epsilon),
\] (2.32)
where
\[
\Delta^\epsilon_{\mu\nu\rho\sigma}(x_{12}) \sim \tilde{C}_T H^\text{tr}_{\mu\nu\rho\sigma} \left( \frac{\partial}{\partial x} \right) \frac{1}{(x_{12}^2)^{D-2+\epsilon}}.
\]

In a number of works dealing with the expansion 2.32, the finite longitudinal term 2.30 is missing.

For what follows it is helpful to introduce projection operators [1] selecting out longitudinal and transversal sectors. Let us start with the longitudinal projector. We put
\[
P^\text{long}_{\mu\nu\rho\sigma} \left( \frac{\partial}{\partial x} \right) = \partial_{\mu} P^\text{tr}_{\nu\rho\sigma} \left( \frac{\partial}{\partial x} \right) + \partial_{\nu} P^\text{tr}_{\mu\rho\sigma} \left( \frac{\partial}{\partial x} \right) - \frac{2}{D} \delta_{\mu\nu} \partial_{\lambda} P^\text{tr}_{\lambda\rho\sigma} \left( \frac{\partial}{\partial x} \right)
\] (2.34)

It is not hard to check that the operator
\[
P^\text{long}_{\mu\nu\rho\sigma} \partial_{\lambda} = P^\text{long}_{\mu\nu\rho\sigma} P^\text{long}_{\rho\sigma\lambda} = P^\text{long}_{\mu\nu\sigma}.
\]

The transversal projector is defined as
\[
P^\text{tr}_{\mu\nu\rho\sigma} \left( \frac{\partial}{\partial x} \right) = I_{\mu\nu\rho\sigma} - P^\text{long}_{\mu\nu\rho\sigma} \left( \frac{\partial}{\partial x} \right),
\] (2.35)
where
\[
I_{\mu\nu\rho\sigma} = \frac{1}{2} \left( \delta_{\mu\rho} \delta_{\nu\sigma} + \delta_{\mu\sigma} \delta_{\nu\rho} - \frac{2}{D} \delta_{\mu\nu} \delta_{\rho\sigma} \right).
\]

One can show that this operator is related to the operator $H^\text{tr}_{\mu\nu\rho\sigma}$ introduced above by the equality
\[
P^\text{tr}_{\mu\nu\rho\sigma} \left( \frac{\partial}{\partial x} \right) = \frac{1}{\Box^2} H^\text{tr}_{\mu\nu\rho\sigma} \left( \frac{\partial}{\partial x} \right),
\] (2.36)
where $H^\text{tr}$ is given by the expression 2.24.
3  The Transformations of Conformal Group in Integer Points

In this section we demonstrate that the divergences of the expressions 2.17 and 2.32 at \( \epsilon = 0 \) are caused by the specific structure of representations of the conformal group, which are given by the transformation laws 2.2 and 2.18 for \( l = l_j \) and \( l = l_T \). Let us remind that the propagators which are invariant under the transformations of a certain symmetry group may be found as the kernels of invariant bilinear forms in the representation space. For example, the propagator of the irreducible conformal field \( j^l_{\mu} \) may be introduced as the kernel of the form

\[
(A^{D-l}, A^{D-l}) = \int dx_1 dx_2 A^{D-l}_\mu(x_1) \Delta^l_{\mu\nu}(x_{12}) A^{D-l}_\nu(x_2),
\]

where \( A^{D-l}_\mu \) are conformal fields of dimension \( \tilde{l} = D - l \). (We also remind that representations with dimensions \( l \) and \( \tilde{l} = D - l \) are equivalent). Taking into account that \( d^Dx = (x^2)^D d^D(Rx) \), one obtains the expression 2.4 for the kernel \( \Delta^l_{\mu\nu}(x) \), so that the kernel may be identified with the invariant propagator of the field \( j^l_{\mu} \):

\[
\langle j^l_{\mu}(x_1) j^l_{\nu}(x_2) \rangle \sim \Delta^l_{\mu\nu}(x_{12}).
\]

The latter is valid for either irreducible representations or their direct sums, since corresponding kernels are non-degenerate.

It is known that \([10,11]\) the representations of the conformal group defined by the transformation laws 2.2 and 2.18 are irreducible for all values of \( l \) except the points

\[
l_j = D - 1, \ l_A = D - l_j = 1 \quad \text{and} \quad l_T = D, \ l_h = D - l_T = 0,
\]

which belong to a series of exceptional integer points. The representations in these points are undecomposable, while the invariant kernels are degenerate. To derive invariant propagators of the fields with such dimensions, it is primarily necessary to construct irreducible representations and their direct sums. After that, the propagators may be determined from the relations of the type 3.2. Let us demonstrate this on an example of the current.

3.1  Irreducible Representations for the Fields \( A_\mu \) and \( j_\mu \)

The propagator of the current

\[
\Delta_{\mu\nu}(x_{12}) = \langle j_\mu(x_1) j_\nu(x_2) \rangle
\]

is the kernel of the invariant contraction

\[
(A, A) = \int dx_1 dx_2 A_\mu(x_1) \langle j_\mu(x_1) j_\nu(x_2) \rangle A_\nu(x_2),
\]
where $A_{\mu}(x)$ is the electromagnetic potential. Consider the transformation law

$$A_{\mu}(x) \xrightarrow{R} A'_{\mu}(x) = U_R^A A_{\mu}(x) = \frac{1}{x^2} g_{\mu\nu}(x) A_{\mu}(Rx). \quad (3.6)$$

This transformation law leads to a number of difficulties. The contraction $3.5$ is invariant if the propagator $3.4$ satisfies the equation $2.4$ after the formal transition $\epsilon = l - D + 1 \to 0$ is performed in the latter equation. Also, we know that the solution $2.5$ of this equation for $\epsilon \neq 0$ does not admit such a transition to this limit for even $D \geq 4$. Note however, that the equation $2.4$ for $l = D - 1$ admits (for even $D \geq 4$) a special solution which cannot be obtained as a result of transition to the limit $\epsilon \to 0$. This solution is transversal:

$$\Delta^{\text{tr}}_{\mu\nu}(x_{12}) = \langle j_\mu(x_1) j_\nu(x_2) \rangle^{\text{tr}} \sim (\partial_\mu \partial_\nu - \delta_{\mu\nu} \Box) \frac{\Delta^{D-4}}{x^2} \delta(x_{12}). \quad (3.7)$$

Another aspect of this problem consists in the fact that the conformal propagator of the field $A_{\mu}(x)$, defined by the equation $2.4$ for $l = l_A = 1$, turns out to be longitudinal

$$D^{\text{long}}_{\mu\nu}(x_{12}) = \langle A_\mu(x_1) A_\nu(x_2) \rangle^{\text{long}} \sim \frac{1}{x^2_{12}} g_{\mu\nu}(x_{12}) = \frac{1}{2} \partial_\mu \partial_\nu \ln x^2_{12}. \quad (3.8)$$

Note that the propagator of the regularized field $A^{D-l}_{\mu}(x)$ has a transversal component of the order $\epsilon$

$$D^{\epsilon}_{\mu\nu}(x_{12}) \bigg|_{\epsilon \to 0} = \langle A^{D-l}_{\mu}(x_1) A^{D-l}_{\nu}(x_2) \rangle \bigg|_{\epsilon \to 0} = D^{\text{long}}_{\mu\nu}(x_{12}) + \epsilon D^{\text{tr}}_{\mu\nu}(x_{12}) + \ldots, \quad (3.9)$$

where $D^{\epsilon}_{\mu\nu}(x_{12}) \sim (\delta_{\mu\nu} \Box - \partial_\mu \partial_\nu) \ln x^2$. Comparing that with the leading terms of the expansion $2.17$

$$\Delta^{\epsilon}_{\mu\nu}(x_{12}) \bigg|_{\epsilon \to 0} \simeq \frac{1}{\epsilon} \Delta^{\text{tr}}_{\mu\nu}(x_{12}) + \Delta^{\text{long}}_{\mu\nu}(x_{12}) + \ldots, \quad (3.10)$$

where $\Delta^{\epsilon}_{\mu\nu}$ is given by the expression $3.7$ and $\Delta^{\text{long}}_{\mu\nu}$ — by the expression $2.10$, one may conclude that the resolving of the $0 \times \infty$ ambiguity should occur in the transversal sector of the contraction $3.3$. This elucidates the cause of famous difficulties in attempts to analyze conformal gauge theories: the propagator of the gauge field, formally defined at $\epsilon = 0$, looks like longitudinal, and the theory seems to be empty.

To inspect the problem, consider an undecomposable representation $Q_A$ defined by the transformation $3.8$ on the space $M_A$ of all fields $A_{\mu}(x)$. The space $M_A$ has an invariant subspace $M^{\text{long}}_A$

$$M^{\text{long}}_A \subset M_A$$

consisting from longitudinal fields

$$A_{\mu}^{\text{long}} \subset M^{\text{long}}_A, \quad A_{\mu}^{\text{long}}(x) = \partial_\mu \varphi(x).$$
Indeed, the result of conformal transformation of a longitudinal field is also a longitudinal field:

\[ A^\text{long}_\mu(x) \xrightarrow{R} \frac{1}{x^2} g_{\mu\nu}(x) A^\text{long}_\nu(Rx) = \frac{1}{x^2} g_{\mu\nu}(x) \partial^R_{\nu} \varphi(Rx) = \partial^R_{\mu} \varphi(Rx), \]

i.e.,

\[ A^\text{long}_\mu(x) \xrightarrow{R} A^\prime_\mu \text{long}(x) = \partial^R_{\mu} \varphi(x) \in M^\text{long}_A, \]

where \( \varphi'(x) = \varphi(Rx) \). It is essential that the complement of the subspace \( M^\text{long}_A \) up to the total space \( M_A \) is not invariant with respect to transformation \( R \). In particular, the subspace of transversal field is non-invariant:

\[ A_\mu^\text{tr}(x) \xrightarrow{R} \tilde{A}_\mu(x) = \frac{1}{x^2} g_{\mu\nu}(x) A^\text{tr}_\nu(Rx), \quad \partial^R_{\mu} \tilde{A}_\mu(x) \neq 0. \]

This feature is characteristic for any undecomposable representations.

In the case of the current \( j^\mu_\mu(x) \) the undecomposable representation is given by the transformation law

\[ j^\mu_\mu(x) \xrightarrow{R} j'^\mu_\mu(x) = U^j_R j_\mu(x) = \frac{1}{(x^2)^{D-1}} g_{\nu\mu}(x) j^\nu(Rx). \quad (3.11) \]

Denote the space of representation \( Q_j \) as \( M_j \). It consists of all fields \( j^\mu_\mu(x) \). Transversal fields compose an invariant subspace \( M^\text{tr}_j \):

\[ j^\text{tr}_\mu \in M^\text{tr}_j \subset M_j. \]

Indeed, after the transformation of a transversal field \( j^\text{tr}_\mu(x) \) a new transversal field results:

\[ j^\text{tr}_\mu(x) = \frac{1}{(x^2)^{D-1}} g_{\nu\mu}(x) j^\text{tr}_\nu(Rx); \quad \partial^R_{\mu} j^\text{tr}_\mu(x) = 0 \text{ if } \partial^R_{\mu} j^\text{tr}_\mu(x) = 0. \]

To check that, the relation \( \partial^R_{\mu} \left[ \frac{1}{(x^2)^{D-1}} g_{\mu\nu}(x) \right] = 0 \) is used. A complement of \( M^\text{tr}_j \) to the total space \( M_j \) is non-invariant. Thus the transformation law \( R \) does not define a representation on the space of longitudinal currents \( j^\text{long}_\mu = \partial^R_{\mu} j(Rx) \).

Now consider invariant bilinear forms for even \( D \geq 4 \). On the space \( M_A \) exists a unique finite form

\[ \{ A, A \}_0 = \int dx_1 dx_2 A_\mu(x_1) \Delta^\text{tr}_{\mu\nu}(x_{12}) A_\nu(x_2), \quad (3.12) \]

where \( \Delta^\text{tr}_{\mu\nu} \) is the exceptional kernel \( [3, 4] \). This form is degenerate on the invariant subspace \( M^\text{long}_A \)

\[ \{ A^\text{long}, A^\text{long} \}_0 = 0. \quad (3.13) \]
Analogously, the only invariant form on the space $M_j$ has the form

$$\{j, j\}_0 = \int dx_1 dx_2 j_\mu(x_1) D_{\mu\nu}^{long}(x_12) j_\nu(x_2). \quad (3.14)$$

It is degenerate on the invariant subspace $M_j^{tr}$:

$$\{j^{\text{tr}}, j^{\text{tr}}\}_0 = 0. \quad (3.15)$$

To obtain non-degenerate forms one should consider irreducible representations instead of undecomposable representations $Q_A$ and $Q_j$. To each undecomposable representation $Q$ defined on the space $M$ one can attach a pair of irreducible representations. One of them acts on an invariant subspace $M_0 \subset M$, namely, on the subspace where the invariant form is degenerate, see $3.12$–$3.15$. Denote this representation as $Q_0$. It is defined by the initial transformation law. The other irreducible representation is established by the same transformation law on the quotient space

$$\tilde{M} = M/M_0.$$ 

Denote this representation as $\tilde{Q}$. The elements of the space are equivalence classes.

In the case of the potential $A_\mu$ the quotient space

$$\tilde{M}_A = M/M^{\text{long}}$$

consists of equivalence classes $[A_\mu]$, each class including all fields with a fixed transversal component. Any two fields $A_\mu$ and $A_\mu + \partial_\mu \varphi$, where $\varphi$ is an arbitrary function, are the representatives of the same class $[A]$. The transformation law $3.6$ defines an irreducible representation on such classes:

$$\tilde{Q}_A : \quad [A_\mu] \xrightarrow{R} [A_\mu'] . \quad (3.16)$$

The form $3.12$ gives the scalar product on these classes. One may choose transversal representatives in each class — it does not effect the value of the form:

$$\int dx_1 dx_2 A_\mu(x_1) \Delta^{tr}_{\mu\nu}(x_12) A_\nu(x_2) = \int dx_1 dx_2 A'_{\mu}(x_1) \Delta^{tr}_{\mu\nu}(x_12) A'^{tr}_\nu(x_2). \quad (3.17)$$

It follows that the form $\{A, A\}_0$ is non-degenerate when exploited on the quotient space $\tilde{M}_A$. The other irreducible representation, of the $Q_0$ type, is given by the transformation law $3.6$ on an invariant subspace $M^{\text{long}}_A$:

$$A^{\text{long}}_\mu(x) \xrightarrow{R} A'^{\text{long}}_\mu(x).$$

Denote this representation as $Q^{\text{long}}_A$, and consider its invariant scalar product. The latter may be defined by the form

$$\{A^{\text{long}}, A'^{\text{long}}\}_1 = \int dx_1 dx_2 A^{\text{long}}_\mu(x_1) \Delta^{\text{long}}_{\mu\nu}(x_12) A'^{\text{long}}_\nu(x_2), \quad (3.18)$$
where $\Delta_{\mu\nu}^{\text{long}}$ is the longitudinal kernel $2.16$. This form is invariant only on the subspace of longitudinal fields. Its expression may be chosen to be explicitly invariant using the singular kernel $2.17$:

$$
\int dx_1 dx_2 A_{\mu}(x_1) \Delta_{\mu\nu}^{\text{long}}(x_1; x_2) A_{\nu}(x_2) = \int dx_1 dx_2 \tilde{A}_{\mu}(x_1) \Delta_{\mu\nu}^{\epsilon}(x_1; x_2) \tilde{A}_{\nu}(x_2) \bigg|_{\epsilon=0}.
$$

Thus we get the pair of irreducible representations

$$
\tilde{Q}_A \quad \text{and} \quad Q_A^{\text{long}}
$$

acting on the spaces $\tilde{M}_A$ and $M_A^{\text{long}}$. Invariant scalar products are defined on both of these spaces. Accordingly, there is the pair of invariant kernels

$$
\Delta^{\text{tr}}_{\mu\nu}(x_1; x_2) \quad \text{and} \quad \Delta^{\epsilon}_{\mu\nu}(x_1; x_2).
$$

The former one is non-degenerate on the space $\tilde{M}_A$, while the latter is non-degenerate (and finite) on the space $M_A^{\text{long}}$. Now let us consider a direct sum of irreducible representations

$$
\tilde{Q}_A \oplus Q_A^{\text{long}}
$$

defined on the direct sum of spaces

$$
\tilde{M}_A \oplus M_A^{\text{long}}.
$$

As will be shown in the next section, having chosen a suitable realization of conformal transformations one would be able to construct an invariant scalar product on the space $3.23$

$$(A, A) = \{A_{\text{tr}}, A_{\text{tr}}\}_0 + \{A_{\text{long}}, A_{\text{long}}\}_1,
$$

which would define a new propagator of the current.

In an analogous manner one can explore the pair of irreducible representations

$$
Q_j^{\text{tr}} \quad \text{and} \quad \tilde{Q}_j,
$$

associated with an undecomposable representation $Q_j$. The first one, $Q_j^{\text{tr}}$, is given by the transformation law $3.11$ on the invariant space $M_j^{\text{tr}}$:

$$
Q_j^{\text{tr}} : \quad j_{\mu}^{\text{tr}}(x) \xrightarrow{R} j_{\mu}^{\text{tr}}.
$$

The second irreducible representation is given by the law $3.11$ on the quotient space $\tilde{M}_j = M_j/M_j^{\text{tr}}$ which consists of equivalence classes $[j_{\mu}]$:

$$
\tilde{Q}_j : \quad [j_{\mu}] \xrightarrow{R} [j_{\mu}^{\text{tr}}].
$$

Each class contains all fields with a fixed longitudinal component. Any two fields $j_{\mu}$ and $j_{\mu}^{\text{tr}} + \tilde{j}_{\mu}$, where $\tilde{j}_{\mu}$ is a transversal field, belong to the same class. The invariant
form 3.14 defines a scalar product on these equivalence classes. In what follows in each class we select a longitudinal representative. The value of the form will be the same after that:

\[
\int dx_1 dx_2 j_{\mu}(x_1)D^\text{long}_{\mu\nu}(x_{12})j_{\nu}(x_2) = \int dx_1 dx_2 j^\text{long}_{\mu}(x_1)D^\text{long}_{\mu\nu}(x_{12})j^\text{long}_{\nu}(x_2).
\] (3.28)

Consider the invariant scalar product for the representation \( Q_j^\text{tr} \). The latter may be defined by the form

\[
\{j^\text{tr}_j, j^\text{tr}_j\}_1 = \int dx_1 dx_2 j^\text{tr}_\mu(x_1)D^\text{tr}_{\mu\nu}(x_{12})j^\text{tr}_\nu(x_2),
\] (3.29)

where

\[
D^\text{tr}_{\mu\nu} \sim (\delta_{\mu\nu} - \partial_{\mu}\partial_{\nu}) \ln x_{12}^2.
\] (3.30)

Though the kernel \( D^\text{tr}_{\mu\nu} \) is not invariant under conformal inversion

\[
D^\text{tr}_{\mu\nu}(x_{12}) = \frac{1}{x_1^2 x_2^2} g_{\mu\rho}(x_1)g_{\nu\sigma}(x_2)D^\text{tr}_{\mu\nu}(Rx_1 - Rx_2)
- \frac{1}{x_1} g_{\mu\nu}(x_1)\partial_{\eta}^{Rx_1} \left( \frac{1}{2} \ln x_{12}^2 \partial_\eta \ln x_1^2 \right)
- \frac{1}{x_2} g_{\mu\nu}(x_2)\partial_{\eta}^{Rx_2} \left( \frac{1}{2} \ln x_{12}^2 \partial_\eta \ln x_2^2 \right)
- \frac{1}{x_1^2 x_2^2} g_{\mu\nu}(x_{12})\partial_{\eta}^{Rx_1} \partial_{\sigma}^{Rx_2} \left( \ln x_1^2 \ln x_2^2 \right),
\]

the form 3.29 remains invariant as long as considered on transversal fields. Note that the form 3.29 may be written as an explicitly invariant expression if one introduces the invariant kernel which is singular when \( \epsilon \to 0 \):

\[
D^\epsilon_{\mu\nu}(x_{12}) \sim \frac{1}{\epsilon} \frac{1}{(x_1 \ln x_2)^{1-\epsilon}} g_{\mu\nu}(x_{12}) = \frac{1}{\epsilon} D^\text{long}_{\mu\nu}(x_{12}) + D^\text{tr}_{\mu\nu}(x_{12}) + \ldots.
\] (3.31)

The contractions of this kernel with transversal fields are finite in the limit \( \epsilon = 0 \). Now rewrite 3.29 in the form

\[
\{j^\text{tr}_j, j^\text{tr}_j\}_1 = \int dx_1 dx_2 j^\text{tr}_\mu(x_1)D^\epsilon_{\mu\nu}(x_{12})j^\text{tr}_\nu(x_2)\bigg|_{\epsilon = 0}.
\] (3.32)

Here we assume the fields \( j^\text{tr}_\mu \) are subjected to conformally invariant regularization (see sections 4,5):

\[
\partial_{\mu}j^\text{tr}_\mu(x)|_{\epsilon = 0} = 0.
\]

Resultantly, we have a pair of invariant kernels (in accordance with 3.21)

\[
D^\epsilon_{\mu\nu}(x_{12}) \quad \text{and} \quad D^\text{long}_{\mu\nu}(x_{12}).
\] (3.33)

The first kernel is non-degenerate and finite on the space \( \tilde{M}_A \), and the second — on the subspace \( M^\text{long}_A \). Analogously to 3.22 consider a direct sum of representations 3.21:

\[
Q^\text{tr}_j \oplus \tilde{Q}_j,
\] (3.34)
defined on the direct sum of spaces
\[ M_j^\text{tr} \oplus \tilde{M}_j. \] (3.35)

In the next section we derive a non-degenerate propagator \( \langle A_\mu A_\nu \rangle \) defining the latter as a kernel of the invariant product
\[ (j, j) = \left\{ \begin{array}{ll} j^{\text{tr}}, j^{\text{tr}} \right\} & \text{if } j \leq D - 4, \\
\left\{ j^{\text{long}}, j^{\text{long}} \right\} & \text{if } j > D - 4,
\end{array} \] (3.36)
in a specific realization of conformal transformations.

### 3.2 Irreducible Representations for the Fields \( h_{\mu\nu} \) and \( T_{\mu\nu} \)

Define a propagator of the energy-momentum tensor
\[ \Delta_{\mu\nu\rho\sigma}(x_{12}) = \langle T_{\mu\nu}(x_1)T_{\rho\sigma}(x_2) \rangle \] (3.37)
as the kernel of the invariant form
\[ (h, h) = \int dx_1 \ dx_2 \ h_{\mu\nu}(x_1) \langle T_{\mu\nu}(x_1)T_{\rho\sigma}(x_2) \rangle h_{\rho\sigma}(x_2), \] (3.38)
where \( h_{\mu\nu} \) is the traceless part of the metric tensor. Consider the transformation law
\[ h_{\mu\nu}(x) \xrightarrow{R} h'_{\mu\nu}(x) = U^h_R h_{\mu\nu}(x) = g_{\mu\rho}(x)g_{\nu\sigma}(x)h_{\rho\sigma}(Rx). \] (3.39)

If the form 3.38 were invariant with respect to this law, the propagator 3.37 would satisfy the equation 2.19 for \( l = D \). We know that the solution of the equation 2.19 does not allow a limiting transition \( l \to D \) for even \( D \geq 4 \). Nevertheless, there exists an exceptional solution
\[ \Delta^{\text{tr}}_{\mu\nu\rho\sigma}(x_{12}) = \langle T_{\mu\nu}(x_1)T_{\rho\sigma}(x_2) \rangle^{\text{tr}} \sim H^{\text{tr}}_{\mu\nu\rho\sigma} \left( \frac{\partial}{\partial x} \right) \square^{D-4} \delta(x_{12}), \] (3.40)
where \( H^{\text{tr}}_{\mu\nu\rho\sigma} \) is the differential operator 2.24. This solution cannot be derived from the general solution 2.20 as the limit \( l \to D \). Therefore, the propagator of the field \( T_{\mu\nu}(x) \), which is invariant under the transformation
\[ T_{\mu\nu}(x) \xrightarrow{R} T'_{\mu\nu}(x) = U^T_R T_{\mu\nu}(x) = \frac{1}{(x^2)^D} g_{\mu\rho}(x)g_{\nu\sigma}(x)T_{\rho\sigma}(Rx), \] (3.41)
is transversal. On the other hand, the propagator of the field \( h_{\mu\nu}(x) \), being invariant under 3.39, is longitudinal. The solution of the equation 2.19 for \( l = 0 \) reads:
\[ D^{\text{long}}_{\mu\nu\rho\sigma}(x_{12}) = \langle h_{\mu\nu}(x_1)h_{\rho\sigma}(x_2) \rangle^{\text{long}} \sim \left[ g_{\mu\nu}(x_{12})g_{\rho\sigma}(x_{12}) + g_{\mu\sigma}(x_{12})g_{\nu\rho}(x_{12}) - \frac{2}{D} \delta_{\mu\nu}\delta_{\rho\sigma} \right] 
\approx \partial_\mu D_{\nu\rho\sigma}(x_{12}) + \partial_\nu D_{\mu\rho\sigma}(x_{12}) - \frac{2}{D} \delta_{\mu\nu}\partial_\lambda D_{\lambda\rho\sigma}(x_{12}), \] (3.42)
where

\[ D_{\mu,\rho\sigma}(x) = \frac{1}{2} \left[ x_\mu g_{\rho\sigma}(x) + x_\sigma g_{\mu\rho}(x) + \frac{2}{D^2} \delta_{\rho\sigma} x_{\mu} \right]. \]

Hence it follows that similar to the case of current (see 3.9, 3.10 and below), the transversal sector of the contraction 3.38 exhibits an ambiguity \( 0 \times \infty \).

An inspection of undecomposable representations \( Q_h \) and \( Q_T \) defined by the transformation laws 3.39 and 3.41 is a literal recital of the analysis of representations \( Q_A \) and \( Q_j \) studied above. So, we shall restrict ourselves to a brief discourse. Look at the representation \( Q_{\text{h}} \) defined by the law 3.39 in the space of fields \( h_{\mu\nu} \). This space has an invariant subspace \( M_{\text{h}} \), consisting of longitudinal fields

\[ h_{\mu\nu}^{\text{long}}(x) = \partial_{\mu} h_{\nu}(x) + \partial_{\nu} h_{\mu}(x) - \frac{2}{D^2} \delta_{\mu\nu} \partial_{\lambda} h_{\lambda}(x). \]  

(3.43)

The transformation law 3.39 defines the irreducible representation \( \tilde{Q}_{\text{h}} \) in the above space. The form 3.38 with the kernel 3.40 vanishes on \( M_{\text{h}}^{\text{long}} \). An invariant scalar product may be defined by the form 2.26, the latter will be denoted as \( \{ h_{\text{long}}, h_{\text{long}} \} \).

The second irreducible representation, denoted as \( \tilde{Q}_{\text{h}} \), is given by the law 3.39 in the quotient space

\[ \tilde{M}_{\text{h}} = M_{\text{h}} / M_{\text{h}}^{\text{long}}. \]

The elements of this space are equivalence classes, each including all fields with a fixed transversal component. Two fields \( h_{\mu\nu} \) and \( h_{\mu\nu} + h_{\mu\nu}^{\text{long}} \) belong to the same class. The invariant scalar product on \( \tilde{M}_{\text{h}} \) is given by the form

\[ \{ h, h \}_0 = \int dx_1 dx_2 h_{\mu\nu}(x_1) \Delta_{tr}^{\mu\nu\rho\sigma}(x_12) h_{\rho\sigma}(x_2), \]  

(3.44)

where \( \Delta_{tr}^{\mu\nu\rho\sigma} \) is the transversal kernel 3.40. Thus one has a pair of irreducible representations

\[ \tilde{Q}_{\text{h}} \quad \text{and} \quad Q_{\text{h}}^{\text{long}}, \]  

(3.45)

associated with the pair of invariant kernels

\[ \Delta_{tr}^{\mu\nu\rho\sigma}(x_12) \quad \text{and} \quad \Delta_{\epsilon}^{\mu\nu\rho\sigma}(x_12). \]  

(3.46)

The second kernel is given by the expression 2.32 formally divergent at \( \epsilon = 0 \), but gives rise to the form \( \{ , \}_1 \) which is finite on \( M_{\text{h}}^{\text{long}} \). Examine a direct sum of representations

\[ \tilde{Q} \oplus Q_{\text{h}}^{\text{long}}. \]  

(3.47)

An invariant scalar product may be postulated as a sum of forms

\[ (h, h) = \{ h_{\text{tr}}^{\text{tr}}, h_{\text{tr}}^{\text{tr}} \}_0 + \{ h_{\text{long}}^{\text{long}}, h_{\text{long}}^{\text{long}} \}_1. \]  

(3.48)

In the next section we study a realization of representation \( \tilde{Q} \) on the space of transversal fields \( h_{\mu\nu}^{\text{tr}} \) and obtain the expression for the propagator \( \langle T_{\mu\nu} T_{\rho\sigma} \rangle \), given by the formula 3.38.
An undecomposable representation $Q_T$ given by the law 3.41 may be inspected in a similar manner. The space $M_T$ of the representation $Q_T$ has an invariant subspace $M_T^{\text{tr}} \subset M_T$.

Indeed, the transformation of a transversal field $T_{\mu\nu}^{\text{tr}}$ results in a transversal field $T_{\mu\nu}^{\text{tr}}$:

$$\partial_\mu T_{\mu\nu}^{\text{tr}}(x) = \partial_\mu \left\{ \frac{1}{(x^2)^2} T_{\mu\nu} T_{\rho\sigma}(Rx) \right\} = 0.$$  

To check that, one uses the equality $\partial_\mu \left[ \frac{1}{(x^2)^2} T_{\mu\nu} T_{\rho\sigma}(x) \right] = 0$ and the fact that the tensor $T_{\mu\nu}(x)$ is traceless. The transformation law 3.41 defines an irreducible representation $Q_T^{\text{tr}}$ on the space $M_T^{\text{tr}}$. An invariant scalar product on that space may be defined by the form

$$\{ T^{\text{tr}}_\mu, T^{\text{tr}}_\nu \}_{\epsilon=0} = \int dx_1 dx_2 T^{\text{tr}}_{\mu\nu}(x_1) D^{\epsilon}_{\mu\rho\sigma}(x_1) T^{\text{tr}}_{\rho\sigma}(x_2) \bigg|_{\epsilon=0}, \quad (3.49)$$

where

$$D^{\epsilon}_{\mu\rho\sigma}(x) \sim \frac{1}{\epsilon (x^2)^\epsilon} \left[ g_{\mu\nu}(x) g_{\rho\sigma}(x) + g_{\mu\sigma}(x) g_{\nu\rho}(x) - \frac{2}{D} \delta_{\mu\nu} \delta_{\rho\sigma} \right]. \quad (3.50)$$

Though this kernel is formally divergent, the contractions 3.49 are always finite due to 3.42. A conformally invariant regularization of transversal fields $T_{\mu\nu}^{\text{tr}}(x)$ in 3.49 is detailed in section 5. Note that the form 3.49 can be defined using the formally non-invariant kernel $D^{\epsilon}_{\mu\rho\sigma}$ (similar to 3.29):

$$\int dx_1 dx_2 T^{\text{tr}}_{\mu\nu}(x_1) D^{\epsilon}_{\mu\rho\sigma}(x_1) T^{\text{tr}}_{\rho\sigma}(x_2) \bigg|_{\epsilon=0} = \int dx_1 dx_2 T^{\text{tr}}_{\mu\nu}(x_1) P^{\epsilon}_{\mu\rho\sigma} T^{\text{tr}}_{\rho\sigma}(x_2) \bigg|_{\epsilon=0}, \quad (3.51)$$

where

$$D^{\text{tr}}_{\mu\rho\sigma}(x) \sim P^{\text{tr}}_{\mu\rho\sigma} \left( \frac{\partial}{\partial x} \right) \ln x^2, \quad (3.52)$$

and $P \left( \frac{\partial}{\partial x} \right)$ is the projection operator 2.36. The second irreducible representation, denoted as $\tilde{Q}_T$, acts in the quotient space (similar to 3.27)

$$\tilde{M}_T = M_T / M_T^{\text{tr}}.$$  

The invariant scalar product on $\tilde{M}_T$ is defined by the form

$$\{ T, T \}_{0} = \int dx_1 dx_2 T_{\mu\nu}(x_1) D^{\text{long}}_{\mu\rho\sigma}(x_1) T_{\rho\sigma}(x_2). \quad (3.53)$$

Choosing a longitudinal representative in each equivalence class $\tilde{M}_T$

$$T^{\text{long}}_{\mu\nu}(x) = \partial_\mu T_\nu(x) + \partial_\nu T_\mu(x) - \frac{2}{D} \delta_{\mu\nu} \partial_\lambda T_\lambda(x), \quad (3.54)$$
one can rewrite (3.53) in the form (see (3.28)):

$$\{T, T\}_0 = \int dx_1 dx_2 T^\text{long}_{\mu\nu}(x_1) D^\text{long}_{\mu\nu\rho\sigma}(x_{12}) T^\text{long}_{\rho\sigma}(x_2).$$  (3.55)

Thus one has a pair of irreducible representations

$$Q^\text{tr}_T \quad \text{and} \quad \tilde{Q}_T,$$  (3.56)

associated with the pair of invariant forms

$$\{T^\text{tr}, T^\text{tr}\}_1 \quad \text{and} \quad \{T^\text{long}, T^\text{long}\}_0.$$  (3.57)

In the next section we examine the field $T_{\mu\nu}(x)$ which transforms by the irreducible representation

$$Q^\text{tr}_T \oplus \tilde{Q}_T.$$  (3.58)

Choosing a special realization of the representation $\tilde{Q}_T$, we shall introduce an invariant propagator $\langle h_{\mu\nu} h_{\rho\sigma} \rangle$ as the kernel of the form

$$(T, T) = \{T^\text{tr}, T^\text{tr}\}_1 + \{T^\text{long}, T^\text{long}\}_2 = \int dx_1 dx_2 T_{\mu\nu}(x_1) \langle h_{\mu\nu}(x_1) h_{\rho\sigma}(x_2) \rangle T_{\rho\sigma}(x_2).$$  (3.59)

4 New Conformal Transformations and Conformal Propagators $\langle j_\mu j_\nu \rangle, \langle T_{\mu\nu} T_{\rho\sigma} \rangle$

Consider the irreducible representation $\tilde{Q}_A$, see (3.16). The transformation law (3.6) defines it on the space $\tilde{M}_A$ of equivalence classes. Let us present a new realization of this representation on the space of transversal fields. Of course, the transformation law of the field $A_\mu(x)$ will be different in this realization.

First, in each equivalence class $[A_\mu] \subset \tilde{M}_A$ we pass to a transversal representative $A^\text{tr}_\mu \subset [A_\mu]$:

$$A_\mu(x) \to A^\text{tr}_\mu(x) = P^\text{tr} A_\mu(x) = \left( \delta_{\mu\nu} - \frac{\partial_{\mu} \partial_{\nu}}{\Box} \right) A_\nu(x),$$  (4.1)

$$A_\mu, A^\text{tr}_\mu(x) \subset [A_\mu].$$

The transformation (3.6) converts the class $[A_\mu]$ into a new class $[A'_\mu]$, see (3.16). Under that, a transversal representative $A^\text{tr}_\mu$ transforms into a certain (non-transversal) field $A'_\mu \subset [A'_\mu]$

$$A'_\mu(x) = U^A R A^\text{tr}_\mu(x) = \frac{1}{x^2} g_{\mu\nu}(x) A^\text{tr}_\nu(Rx).$$  (4.2)
In the new class \([A'_\mu]\) we pass to a transversal representative \(A^{\text{tr}}_\mu\)
\[
A^{\text{tr}}_\mu(x) = P^{\text{tr}}A'_\mu(x) = \left(\delta_{\mu\nu} - \frac{\partial_{\mu} \partial_{\nu}}{\Box}\right)A'_\nu(x).
\]

The sequence of three transformation 4.1–4.3 defines a realization of the representation \(\tilde{Q}_A\) on transversal fields. Taking into account that longitudinal fields \(A^{\text{long}}_\mu\) transform by the irreducible representation \(Q^{\text{long}}_A\), one can now define a reducible representation \(\tilde{Q}_A \oplus Q^{\text{long}}_A\) on the total space of fields \(A_\mu(x)\) as follows. Decompose the field \(A_\mu\) into the sum of transversal and longitudinal components
\[
A_\mu(x) = A^{\text{tr}}_\mu(x) + A^{\text{long}}_\mu(x) = P^{\text{tr}}A_\mu(x) + P^{\text{long}}A_\mu(x).
\]

With the first term we associate the representation \(\tilde{Q}_A\), and with the second — \(Q^{\text{long}}_A\).

To proceed, introduce the operator
\[
V^A_{R} = P^{\text{tr}}U^A_R P^{\text{tr}} + U^A_R P^{\text{long}},
\]

where \(U^A_R\) is given by 3.6. The transformation law
\[
A_\mu(x) \xrightarrow{R} A'_\mu(x) = V_RA_\mu(x)
\]
defines the reducible representation \(\tilde{Q}_A \oplus Q^{\text{long}}_A\). An expanded form of the transformation 4.5 is:
\[
V_RA_\mu(x) = \left(\delta_{\mu\nu} - \frac{\partial_{\mu} \partial_{\nu}}{\Box}\right) \frac{1}{x^2} g_{\nu\rho}(x) \left(\delta_{\rho\sigma} - \frac{\partial_{\rho} \partial_{\sigma}}{\Box}ight) A_\sigma(Rx)
+ \frac{1}{x^2} g_{\nu\rho}(x) \frac{\partial^R_{\nu} \partial^R_{\rho}}{\Box_{Rx}} A_\rho(Rx)
= \frac{1}{x^2} g_{\mu\nu}(x) A_\nu(Rx) - \frac{\partial^R_{\mu} \partial^R_{\nu}}{\Box_{Rx}} A_\nu(Rx) + \frac{1}{x^2} g_{\mu\rho}(x) \frac{\partial^R_{\rho} \partial^R_{\nu}}{\Box_{Rx}} A_\rho(Rx).
\]

Evidently, each conformal transformation \(g: x_\mu \to gx_\mu\) of the field \(A_\mu(x)\) in the new realization may be put into a form 4.7
\[
A_\mu(x) \xrightarrow{g} V_g A_\mu(x) = P^{\text{tr}}U_g P^{\text{tr}}A_\mu(x) + U_g P^{\text{long}}A_\mu(x),
\]

where \(U_g\) is the transformation operator of the field \(A_\mu(x)\) in a conventional realization. One easily checks that the transformation 4.7 satisfies the group law:
\[
V_{g_2}V_{g_1}A_\mu(x) = V_{g_1g_2}A_\mu(x), \quad \text{if} \quad U_{g_2}U_{g_1}A_\mu(x) = U_{g_1g_2}A_\mu(x).
\]

The check employs an invariance of longitudinal sector (the subspace \(M_\mu^{\text{long}}\)) with respect to transformations \(U_g\) in conventional realization: \(P^{\text{tr}}U_g P^{\text{long}}A_\mu(x) = 0\).
The realization of representation \( Q_j^{tr} \oplus \tilde{Q}_j \), see \([3.33, 3.36]\), on the space of fields \( j_\mu(x) \) may be deduced in an analogous manner. It amounts to introducing the representation \( \tilde{Q}_j \), see \([3.27]\) and below, on the space of fields \( j_\mu^{long}(x) \). As above, we choose a longitudinal representative in each equivalence class \([j_\mu]\) and consider a sequence of transformations like \([4.1, 4.3]\):

\[
j_\mu(x) \rightarrow j_\mu(x) \rightarrow \frac{\partial_{\mu} \partial_{\nu} j_\nu(x)}{\Box} \frac{1}{(x^2)^{D-1}} g_{\nu \rho}(x) \frac{\partial_{\rho} \partial_{\sigma} j_\sigma(x)}{\Box}.
\]

Using the invariance of transversal sector (the subspace \( M_j^{tr} \)) of the fields \( j_\mu(x) \) under the transformation \([3.11]\)

\[
P^{long} U_R^{j} P^{tr} j_\mu(x) = 0,
\] (4.10)

we arrive at the following realization of the representation \( Q_j^{tr} \oplus \tilde{Q}_j \):

\[
j_\mu(x) \xrightarrow{R} V_R^j j_\mu(x) = U_R^j P^{tr} j_\mu(x) + P^{long} U_R^j P^{long} j_\mu(x),
\] (4.11)

or, more explicitly

\[
V_R^j j_\mu(x) = \frac{1}{(x^2)^{D-1}} g_{\mu \nu}(x) \left( \delta_{\nu \rho} - \frac{\partial_{\nu} \partial_{\rho}}{\Box} \right) j_\rho(R x) + \frac{\partial_{\mu} \partial_{\nu}}{\Box} \frac{1}{(x^2)^{D-1}} g_{\nu \rho}(x) \frac{\partial_{\rho} \partial_{\sigma} j_\sigma(x)}{\Box}.
\]

Here the first term corresponds to the representation \( Q_j^{tr} \), while the second — to the representation \( \tilde{Q}_j \). The group law is checked in the same way as in the case of the field \( A_\mu(x) \), see \([3.8]\). On account of \([4.10]\), we have:

\[
V_R^j V_R^j j_\mu(x) = j_\mu(x), \quad \text{since} \quad U_R^j U_R^j j_\mu(x) = j_\mu(x).
\]

Thus, we have formulated a new realization of conformal transformations of the fields \( A_\mu \) and \( j_\mu \). Now consider invariant (in the new sense) propagators of these fields. The conditions of invariance have the form

\[
\langle V_R^A A_\mu(x_1) V_R^A A_\nu(x_2) \rangle = \langle A_\mu(x_1) A_\nu(x_2) \rangle,
\] (4.12)

\[
\langle V_R^j j_\mu(x_1) V_R^j j_\nu(x_2) \rangle = \langle j_\mu(x_1) j_\nu(x_2) \rangle.
\] (4.13)

The solution of these equations has the form \([1.14–16]\):

\[
D_{\mu \nu}(x_{12}) = \langle A_\mu(x_1) A_\nu(x_2) \rangle = g_A \left( \delta_{\mu \nu} - \frac{\partial_{\mu} \partial_{\nu}}{\Box} \right) \frac{1}{x_{12}^4} + \eta \partial_\mu \partial_\nu \ln x_{12},
\] (4.14)

\[
\Delta_{\mu \nu}(x_{12}) = \langle j_\mu(x_1) j_\nu(x_2) \rangle = g_j \left( \delta_{\mu \nu} - \frac{\partial_{\mu} \partial_{\nu}}{\Box} \right) \Box^{-1} \delta(x_{12}) + C_j \partial_\mu \partial_\nu \Box^{-1} \delta(x_{12}),
\] (4.15)
where \( g_A, \eta, g_j \) are constants. The coefficient \( C_j \) in the second term of (4.13) is chosen from the Ward identity (2.12). The propagators (4.14) and (4.15) are the kernels of conformally invariant (in the new sense) contractions (3.24) and (3.35), see also (3.5).

Consider the fields \( h_{\mu\nu}(x) \) and \( T_{\mu\nu}(x) \). Repeating the derivation of transformations (4.4) and (4.11) literally, we get the new conformal transformations in the following form [1], see also [17]:

\[
\begin{align*}
  h_{\mu\nu}(x) &\rightarrow V_{R}^{h} h_{\mu\nu}(x) = P_{R}^{tr} U_{R}^{h} P_{R}^{tr} h_{\mu\nu}(x) + U_{R}^{h} P_{R}^{long} h_{\mu\nu}(x), \\
  T_{\mu\nu}(x) &\rightarrow V_{R}^{T} T_{\mu\nu}(x) = U_{R}^{T} P_{R}^{tr} T_{\mu\nu}(x) + P_{R}^{long} U_{R}^{T} P_{R}^{long} T_{\mu\nu}(x),
\end{align*}
\]

(4.16) (4.17)

where \( U_{R}^{h} \) and \( U_{R}^{T} \) are given by the relations (3.39) and (3.41), and the projection operators \( P_{R}^{tr}, P_{R}^{long} \) are written down in (2.30) and (2.34). These transformations define reducible representations of the conformal group, see (3.43), (3.45) and (3.56), (3.59):

\[
\hat{Q}_{h} \oplus Q_{h}^{long} \quad \text{and} \quad Q_{T}^{tr} \oplus \hat{Q}_{T}.
\]

(4.18)

In the derivation of (4.16), (4.17) we have used the invariance of longitudinal sector of the field \( h_{\mu\nu}(x) \) under the action of \( U_{R}^{h} \), and the invariance of transversal sector of the field \( T_{\mu\nu}(x) \) under the action of \( U_{R}^{T} \) (the subspaces \( M_{h}^{long} \) and \( M_{T}^{long} \))

\[
P_{R}^{tr} U_{R}^{h} P_{R}^{long} h_{\mu\nu}(x) = 0, \quad P_{R}^{long} U_{R}^{T} P_{R}^{tr} T_{\mu\nu}(x) = 0.
\]

The conformally invariant (in the new sense) propagators of the fields \( h_{\mu\nu}(x) \) and \( T_{\mu\nu}(x) \) are found from the conditions of invariance, analogous to (4.12), (4.13). They have the form (see also [1,17] and references therein):

\[
\begin{align*}
  D_{\mu\rho\sigma}(x_{12}) &= \langle h_{\mu\nu}(x_{1}) h_{\rho\nu}(x_{2}) \rangle \\
  &= g_{h} P_{\mu\rho\sigma}^{tr} \left( \frac{\partial}{\partial x} \right) \ln x_{12}^{2} + \eta P_{\mu\rho\sigma}^{long} \left( \frac{\partial}{\partial x} \right) \ln x_{12}^{2},
\end{align*}
\]

\[
\Delta_{\mu\rho\sigma}(x_{12}) = \langle T_{\mu\nu}(x_{1}) T_{\rho\nu}(x_{2}) \rangle
\]

\[
\begin{align*}
  &= g_{T} H_{\mu\rho\sigma}^{tr} \left( \frac{\partial}{\partial x} \right) \Box^{D-4} \delta(x_{12}) + C_{T} H_{\mu\rho\sigma}^{long} \left( \frac{\partial}{\partial x} \right) \Box^{D-4} \delta(x_{12}).
\end{align*}
\]

(4.19) (4.20)

Here \( g_{h}, \eta \) and \( g_{T} \) are constants, the projection operators \( P_{\mu\rho\sigma}^{tr} \) and \( P_{\mu\rho\sigma}^{long} \) are given in (2.36), (2.34), the operator \( H_{\mu\rho\sigma}^{tr} \) is defined in (2.24)

\[
H_{\mu\rho\sigma}^{long} \left( \frac{\partial}{\partial x} \right) = \partial_{\mu} H_{\nu\rho\sigma} \left( \frac{\partial}{\partial x} \right) + \partial_{\nu} H_{\mu\rho\sigma} \left( \frac{\partial}{\partial x} \right) - \frac{2}{D} \delta_{\mu} \partial_{\lambda} H_{\lambda\rho\sigma} \left( \frac{\partial}{\partial x} \right),
\]

(4.21)

where

\[
H_{\mu\rho\sigma} \left( \frac{\partial}{\partial x} \right) = \frac{2D^{2} - 3D + 2}{2D(D - 1)} \partial_{\mu} \partial_{\rho} \partial_{\sigma} - \frac{D - 1}{2D} (\delta_{\mu\rho} \partial_{\sigma} + \delta_{\mu\sigma} \partial_{\rho}) \Box - \frac{1}{2D(D - 1)} \delta_{\rho\sigma} \partial_{\mu} \Box.
\]

(4.22)

The coefficient in the second term is chosen in accordance with the Ward identity (2.28).
5  Equivalence Conditions and Higher Green Functions. The Propagators $\langle jj \rangle$ and $\langle TT \rangle$ on Internal Lines

Let us consider higher Green functions which include the field $A_\mu(x)$ or $j_\mu(x)$ together with some different fields:

$$G^A_\mu(x,\ldots) = \langle A_\mu(x)\Phi_1(x_1)\ldots \rangle, \quad G^j_\mu(x,\ldots) = \langle j_\mu(x)\Phi_1(x_1)\ldots \rangle,$$

(5.1)

where the dots stand for a certain set of fields $\Phi_i, i = 1,2,\ldots$, of arbitrary tensor structure. One can show [1] that the two types of such conformally invariant Green functions exist. The first type includes the Green functions associated to irreducible representations $\tilde{\mathcal{Q}}_A$ and $\tilde{\mathcal{Q}}_j^r$,

$$G^A_{1\mu}(x,\ldots) = \langle A_\mu(x)\ldots \rangle_1, \quad G^j_{1\mu}(x,\ldots) = \langle j_\mu(x)\ldots \rangle_1,$$

(5.2)

while to the second type belong the Green functions associated to representations $\mathcal{Q}_{A^\long}$ and $\tilde{\mathcal{Q}}_j$:

$$G^A_{2\mu}(x,\ldots) = \langle A_\mu(x)\ldots \rangle_2, \quad G^j_{2\mu}(x,\ldots) = \langle j_\mu(x)\ldots \rangle_2.$$

(5.3)

The Green functions $G^j_1$ and $G^j_2$ are associated with representations which act on invariant subspaces $M_j^\long$ and $M_A^\long$, see section 3. So, the functions $G^j_1$ are transversal, while $G^j_2$ are longitudinal:

$$\partial^\mu G^j_{1\mu}(x,\ldots) = 0, \quad G^A_{2\mu}(x,\ldots) = \partial^\mu G(x,\ldots).$$

(5.4)

Each of the Green functions [5.2,5.3] may be represented in terms of partial wave expansion [1], i.e., expressed through an infinite set of three-point functions of tensor fields $\Phi^l_i(x) = \Phi^l_{\mu_1,\ldots,\mu_s}(x)$ of dimension $l$, where $l, s$ run through all possible values. To prove [5.4] and other related statements, it is sufficient to demonstrate their validity for three-point functions.

Consider the functions [5.1] containing scalar fields. Introduce the invariant three-point functions

$$C^{l,s}_{\mu}(x_1x_2x_3) = \langle \Phi^l_{\mu_1,\ldots,\mu_s}(x_1)\varphi(x_2)j_\mu(x_3) \rangle, \quad B^{l,s}_{\mu}(x_1x_2x_3) = \langle \Phi^l_{\mu_1,\ldots,\mu_s}(x_1)\varphi(x_2)A_\mu(x_3) \rangle,$$

(5.5)

where $\varphi(x)$ is a scalar field of dimension $d$. Both $C$- and $B$-functions fall into two types. Denote the functions of the 1st type as $C^{l,s}_{1\mu}, B^{l,s}_{1\mu}$, and of the 2nd — as $C^{l,s}_{2\mu}, B^{l,s}_{2\mu}$. The explicit expressions for these functions are found in [1,2]. In particular, the functions $C^{l,s}_{1\mu}$ and $B^{l,s}_{2\mu}$, which satisfy [5.4], have the form:

$$C^{l,s}_{1\mu}(x_1x_2x_3) \sim \left\{ \lambda^{x_3}(x_2x_1)\lambda^{x_1}(x_2x_3) \right\},$$

$$+ \frac{(l-d)}{s(D-2-l+d+s)} \frac{1}{x_{13}^{2l}} \left[ \sum_{k=1}^{s} g_{\mu_1\mu_2}(x_{13})\lambda^{x_1\mu_1,\ldots,\mu_s}(x_2x_3) - \text{traces} \right]$$

$$\times \left( x_{12}^{2l-d-s-D+2} x_{13}^{-2l} x_{23}^{2l-d-s-D+2} \right),$$

(5.6)
where
\[ \lambda_{\mu}^{x_1}(x_1x_2) = \frac{(x_{13})_{\mu}}{x_{13}} - \frac{(x_{23})_{\mu}}{x_{23}}, \quad \lambda_{\mu_1...\mu_s}^{x_1}(x_2x_3) = \lambda_{\mu_1}^{x_1}(x_2x_3) \ldots \lambda_{\mu_s}^{x_1}(x_2x_3) - \text{traces}, \]
\[ \hat{\mu}_k \text{ means that the index } \mu_k \text{ is omitted}, \]
\[ B_{2\mu}^{l,s}(x_1x_2x_3) \sim \left\{ \lambda_{\mu}^{x_3}(x_2x_1)\lambda_{\mu_1...\mu_s}^{x_1}(x_2x_3) + \frac{1}{(l-d-s)x_{13}} \sum_{k=1}^{s} g_{\mu\mu_k}(x_{13})\lambda_{\mu_1...\hat{\mu}_k...\mu_s}^{x_1}(x_2x_3) - \text{traces} \right\} \]
\[ \times \left( \frac{x_{12}^{l+d-s}}{2} \frac{x_{23}^{l-d-s}}{2} \right). \] (5.7)

One can check that these functions satisfy the conditions 5.4
\[ \partial^{x_3} C_{l\mu}^{l,s}(x_1x_2x_3) = 0, \quad \text{for all } l, s \geq 1, \] (5.8)
\[ B_{2\mu}^{l,s}(x_1x_2x_3) \sim \partial^{x_3} \left[ \lambda_{\mu_1...\mu_s}^{x_1}(x_2x_3)(x_{12}^{l+d-s} \left( \frac{x_{23}}{x_{13}} \right)^{l+d-s} \right]. \] (5.9)

Finally, the functions \( C_{l\mu}^{l,s} \) and \( B_{\mu}^{l,s} \) may be derived from the expressions 5.6, 5.7, if arbitrary coefficients are inserted in front of the second terms in braces to each of these expressions. Note that for \( s = 0 \) no transversal function \( C_{1\mu}^{l,s} \) exists; each of the functions \( C_{l\mu}^{l,s} \) and \( B_{\mu}^{l,s} \) has only one term.

For the Green functions \( \langle A_{\mu} \ldots \rangle \) and \( \langle j_{\mu} \ldots \rangle \), which are invariant under the new transformations 4.7 and 4.11, the general equations may now be easily written down. On account of 5.8, 5.9 (for \( s \geq 1 \)) one has for the three-point functions:
\[ B_{\mu}^{l,s}(x_1x_2x_3) = \langle \Phi_{\mu_1...\mu_s}^{l,s}(x_1)\varphi(x_2)A_{\mu}(x_3) \rangle \]
\[ = \left( \delta_{\mu\nu} - \partial_{\mu}^{x_3} \partial_{\nu}^{x_3} \right) B_{1\nu}^{l,s}(x_1x_2x_3) + B_{2\mu}^{l,s}(x_1x_2x_3) \] (5.10)
\[ C_{\mu}^{l,s}(x_1x_2x_3) = \langle \Phi_{\mu_1...\mu_s}^{l,s}(x_1)\varphi(x_2)j_{\mu}(x_3) \rangle \]
\[ = C_{1\mu}^{l,s}(x_1x_2x_3) + \partial_{\nu}^{x_3} \partial_{\mu}^{x_3} C_{2\mu}^{l,s}(x_1x_2x_3). \] (5.11)

For the higher Green functions we obviously get
\[ G_{\mu}^{A}(x, \ldots) = \langle A_{\mu}(x) \ldots \rangle = \left( \delta_{\mu\nu} - \partial_{\mu}^{a} \partial_{\nu}^{a} \right) G_{1\nu}^{A}(x, \ldots) + \partial_{\nu}^{a} G(x, \ldots), \] (5.12)
\[ G_{\mu}^{j}(x, \ldots) = \langle j_{\mu}(x) \ldots \rangle = G_{1\mu}^{j}(x, \ldots) + \partial_{\nu}^{a} G_{2\nu}^{j}(x, \ldots). \] (5.13)
A remarkable feature of conformal theory is the existence of relations between Green functions of the current and those of the potential. These relations are caused by the structure of representations of the conformal group, and are equivalent to Maxwell equations. Let us discuss it in some detail.

The irreducible representations studied in section 3 are mutually related by equivalence conditions [12,13]

\[ \tilde{Q}_A \sim Q^j, \quad \tilde{Q}_A^{\text{long}} \sim \tilde{Q}_j. \]  

(5.14)

The invariant kernels [3.21] and [3.33] are the kernels of intertwining operators which relate the fields \( A_\mu \) and \( j_\mu \). The first condition [5.14] links the Green functions [5.2] and the second — the functions [5.3]. Using the kernels [3.21], we have:

\[
G^{ij}_{A}(x, \ldots) = \int dy \Delta^{tr}_{\mu\nu}(x-y)G^{A}_{1\nu}(y, \ldots),
\]

(5.15)

\[
G^{ij}_{2\mu}(x, \ldots) = \int dy \Delta^{\epsilon}_{\mu\nu}(x-y)G^{A,\epsilon}_{2\nu}(y, \ldots)_{\epsilon=0},
\]

(5.16)

where \( \Delta^{tr}_{\mu\nu} \) is the kernel [2.17], singular at \( \epsilon = 0 \), and \( G^{A,\epsilon}_{2\nu} \) is a regularized (see below) longitudinal function. These relations may be also rewritten with the help of the kernels [3.33]:

\[
G^{A}_{1\mu}(x, \ldots) = \int dy D^{\epsilon}_{\mu\nu}(x-y)G^{ij\epsilon}_{1\nu}(y, \ldots)_{\epsilon=0},
\]

(5.17)

\[
G^{A}_{2\mu}(x, \ldots) = \int dy D^{\epsilon}_{\mu\nu}(x-y)G^{A,\epsilon}_{2\nu}(y, \ldots),
\]

(5.18)

where \( D^{\epsilon}_{\mu\nu} \) is the kernel [3.31], singular at \( \epsilon = 0 \), and \( G^{ij\epsilon}_{1\nu} \) is a regularized (see below) transversal function, which satisfies [5.4]. Substituting the explicit expression for the kernel \( \Delta^{tr}_{\mu\nu} \) into Eq.5.15, we get for \( D = 4 \) the Maxwell equations for the Green functions. Hence, the Maxwell equations express the equivalence conditions for representations of the conformal group [15,16].

As already mentioned, the higher Green functions can be represented in the form of conformal partial wave expansions. To prove the relations 5.15–5.18, it is sufficient to demonstrate their validity for the invariant three-point functions

\[
C^{l,s}_{1\mu}(x_1x_2x_3) = \int dx_4 \Delta^{tr}_{\mu\nu}(x_{34})B^{l,s}_{1\nu}(x_1x_2x_4),
\]

(5.19)

\[
C^{l,s}_{2\mu}(x_1x_2x_3) = \int dx_4 \Delta^{\epsilon}_{\mu\nu}(x_{34})B^{l,s\epsilon}_{2\nu}(x_1x_2x_4)_{\epsilon=0},
\]

(5.20)

\[
B^{l,s}_{1\mu}(x_1x_2x_3) = \int dx_4 D^{\epsilon}_{\mu\nu}(x_{34})C^{l,s\epsilon}_{1\nu}(x_1x_2x_4)_{\epsilon=0},
\]

(5.21)

\[
B^{l,s}_{2\mu}(x_1x_2x_3) = \int dx_4 D^{\text{long}}_{\mu\nu}(x_{34})C^{l,s}_{2\nu}(x_1x_2x_4).
\]

(5.22)

The calculation of these integrals is reviewed in [1,2,4]. The conformal regularization is used in 5.20 and 5.21. The regularized expressions may be derived by the substitution
of anomalous dimensions \( l'_j = D - 1 + \epsilon \), \( l'_A = 1 - \epsilon \) in place of the canonical ones \( l_j = D - 1, l_A = 1 \). For the latter, it is sufficient to introduce the factors \((x_{12}^2)^{\epsilon/2}(x_{13}^2 x_{23}^2)^{-\epsilon/2}\) and \((x_{12}^2)^{-\epsilon/2}(x_{13}^2 x_{23}^2)^{\epsilon/2}\) into the expressions \[5.6\] and \[5.7\] correspondingly. Acting analogously, one can derive the regularized expressions for the higher Green functions: one just needs to represent them as conformal partial wave expansions in regularized three-point functions.

Let us remark that the equivalence conditions \[5.20\] and \[5.21\] may be rewritten without making use of regularized functions and kernels. Indeed, it is sufficient to apply the realization obtained in the previous section. We get instead of \[5.20\] and \[5.21\]

\[
\frac{\partial^{x_3} \partial^{x_3}}{\Box_{x_3}} C_{2\nu}^{l,s}(x_1 x_2 x_3) = \int dx_4 \Delta_{\mu\nu}^{\text{long}}(x_{34}) B_{2\nu}^{l,s}(x_1 x_2 x_4),
\]

\[
\left( \delta_{\mu\nu} - \frac{\partial^{x_3} \partial^{x_3}}{\Box_{x_3}} \right) B_{1\nu}^{l,s}(x_1 x_2 x_3) = \int dx_4 D_{\mu\nu}^{\text{tr}}(x_{34}) C_{1\nu}^{l,s}(x_1 x_2 x_4).
\]

Consider the conformally invariant integrals over internal photon (or current) line

\[
\int dx \, dy \, G^A_\mu(x, \ldots) \Delta_{\mu\nu}(x - y) G^A_\nu(y, \ldots) = \int dx \, dy \, G^i_\mu(x, \ldots) D_{\mu\nu}(x - y) G^i_\nu(y, \ldots),
\]

where \(G^A_\mu\) and \(G^i_\mu\) are the Green functions \[5.12\] and \[5.13\]; \(\Delta_{\mu\nu}\) and \(D_{\mu\nu}\) are the kernels \[4.15\] and \[4.14\]. Such integrals are encountered in conformally invariant skeleton theory, as well as in exactly solvable \(D\)-dimensional models considered in \([1–5]\). The conformal invariance of the integrals \[5.25\] is clear from the analysis of section 3. They represent invariant contractions \((A, A)\) and \((j, j)\), see \[3.36\] and \[3.24\]. Taking into account \[3.17\] and \[3.13\] and also \[3.28\] and \[3.32\], the integrals \[5.25\] can be rewritten through the regularized expressions which are conformally invariant in a usual sense, i.e., with respect to transformations \[3.6\] and \[3.11\]:

\[
\int G^A_\mu \Delta^i_{\mu\nu} G^A_\nu = \int dx \, dy \, G^A_\mu(x, \ldots) \Delta^i_{\mu\nu}(x - y) G^A_\nu(y, \ldots)
\]

\[
+ \int dx \, dy \, G^{A,\epsilon}_\mu(x, \ldots) \Delta^i_{\mu\nu}(x - y) G^{A,\epsilon}_\nu(y, \ldots)\bigg|_{\epsilon=0},
\]

where \(\Delta^i_{\mu\nu}\) are the kernels \[3.21\], see \[3.7\] and \[2.17\], \(G^A_{2\mu}\) is longitudinal,

\[
\int G^i_\mu D^i_{\mu\nu} G^i_\nu = \int dx \, dy \, G^{i,\epsilon}_\mu(x, \ldots) D^i_{\mu\nu}(x - y) G^{i,\epsilon}_\nu(y, \ldots)\bigg|_{\epsilon=0}
\]

\[
+ \int dx \, dy \, G^{i}_\mu(x, \ldots) D^{\text{long}}_{\mu\nu}(x - y) G^{i}_\nu(y, \ldots),
\]

where \(D^i_{\mu\nu}\) and \(D^{\text{long}}_{\mu\nu}\) are the kernels \[3.33\], see \[3.31\] and \[3.8\]. \(G^i_\mu\) is transversal. The left-hand sides of \[5.27\] and \[5.28\] are equal (under the suitable renormalization of the
propagator $D^A_{\mu\nu}$) and may be expressed through invariant kernels non-singular at $\epsilon = 0$

\[
\int G^A_{\mu} \Delta^A_{\mu\nu} G^A_{\nu} = \int G^j_{\mu} D^A_{\mu\nu} G^j_{\nu} = \int dx dy G^A_{1\mu}(x, \ldots) \Delta^A_{\mu\nu}(x - y) G^A_{1\nu}(y, \ldots) + \int dx dy G^j_{2\mu}(x, \ldots) D^A_{\mu\nu}(x - y) G^j_{2\nu}(y, \ldots).
\]  

(5.29)

All the conclusions concerning the relations 5.25–5.29 are readily transferred to the case of invariant functions $B^{l,s}_{1\mu,1}$, $B^{l,s}_{2\mu,1}$ and $C^{l,s}_{1\mu,1}$, $C^{l,s}_{2\mu,1}$ for $s \geq 1$.

Consider the fields $h_{\mu\nu}$ and $T_{\mu\nu}$. Identically to the case of the fields $A_{\mu}$, $j_{\mu}$, the irreducible representations 3.45 and 3.56 are pairwise equivalent [12,13]:

\[
\tilde{Q}_h \sim Q^T_{tr}, \quad Q^T_{long} \sim \tilde{Q}_T.
\]  

(5.30)

Accordingly, there are two types of invariant higher Green functions:

\[
G^h_{1\mu\nu}(x, \ldots) = \langle h_{\mu\nu}(x) \ldots \rangle, \quad G^T_{1\mu\nu}(x, \ldots) = \langle T_{\mu\nu}(x) \ldots \rangle,
\]  

(5.31)

where the dots stand for any sets of fields. The Green functions of the first type transform by irreducible representations $\tilde{Q}_h$ and $Q^T_{tr}$:

\[
G^h_{1\mu\nu}(x, \ldots) = \langle h_{\mu\nu}(x) \ldots \rangle_1, \quad G^T_{1\mu\nu}(x, \ldots) = \langle T_{\mu\nu}(x) \ldots \rangle_1,
\]  

(5.32)

while those of the second type — by representations $Q^h_{long}$, $\tilde{Q}_T$:

\[
G^h_{2\mu\nu}(x, \ldots) = \langle h_{\mu\nu}(x) \ldots \rangle_2, \quad G^T_{2\mu\nu}(x, \ldots) = \langle T_{\mu\nu}(x) \ldots \rangle_2.
\]  

(5.33)

The function $G^T_{1\mu\nu}$ is transversal,

\[
\partial_\mu G^T_{1\mu\nu}(x, \ldots) = 0,
\]  

(5.34)

while the function $G^h_{2\mu\nu}$ is longitudinal

\[
G^h_{2\mu\nu}(x, \ldots) = \partial_\mu G^h_{\nu}(x, \ldots) + \partial_\nu G^h_{\mu}(x, \ldots) - \frac{2}{D} \delta_{\mu\nu} \partial_\lambda G^h_{\lambda}(x, \ldots),
\]  

(5.35)

where $G^h_{\mu\nu}(x, \ldots)$ is an invariant function of the vector field $h_{\mu}(x)$, see 3.43. For higher Green functions, which are invariant under the new transformations 4.7 and 4.11, the general equations may now be written using 5.35,5.36:

\[
G^h_{\mu\nu}(x, \ldots) = P^{tr}_{\mu\rho\sigma} \left( \frac{\partial}{\partial x} \right) G^h_{1\rho\sigma}(x, \ldots) + G^h_{2\mu\nu}(x, \ldots),
\]  

(5.36)

\[
G^T_{\mu\nu}(x, \ldots) = G^T_{1\mu\nu}(x, \ldots) + P^{long}_{\mu\rho\sigma} \left( \frac{\partial}{\partial x} \right) G^T_{2\rho\sigma}(x, \ldots),
\]  

(5.37)
where $P_{\text{tr}}$ and $P_{\text{long}}$ are the projection operators \([2.36, 2.34]\). $G_{1,2\mu\nu}^h$ and $G_{1,2\mu\nu}^T$ are invariant under the old transformations \([3.39, 3.41]\).

The Green functions \([5.31]\) may be expanded into infinite sets of invariant three-point functions

$$
B_{\mu\nu}^{l,s}(x_1 x_2 x_3) = \langle \Phi_{\mu_1 \ldots \mu_s}(x_1) \varphi(x_2) h_{\mu\nu}(x_3) \rangle, \quad C_{\mu\nu}^{l,s}(x_1 x_2 x_3) = \langle \Phi_{\mu_1 \ldots \mu_s}(x_1) \varphi(x_2) T_{\mu\nu}(x_3) \rangle.
$$

The explicit expressions for these functions may be found in \([1,2,5]\). Exactly as in the case of the current, we have two types of functions \([5.38]\) (for $s \geq 2$):

$$
B_{1\mu\nu}^{l,s}(x_1 x_2 x_3), C_{1\mu\nu}^{l,s}(x_1 x_2 x_3) \quad \text{and} \quad B_{2\mu\nu}^{l,s}(x_1 x_2 x_3), C_{2\mu\nu}^{l,s}(x_1 x_2 x_3),
$$

which correspond to the representations

$$
\hat{Q}_h, Q_T^{\text{tr}} \quad \text{and} \quad Q_h^{\text{long}}, \hat{Q}_T.
$$

Only one type survives when $s = 0, 1 \ [1,2,5]$, namely, $B_{2\mu\nu}^{l,s}$ and $C_{2\mu\nu}^{l,s}$. The function $C_{1\mu\nu}^{l,s}$ is transversal, while $B_{2\mu\nu}^{l,s}$ is longitudinal, see \([5.34, 5.35]\).

Consider the equivalence conditions \([5.30]\). They mutually relate each pair of functions \([5.32, 5.33]\) as well as three-point functions \([5.39]\). These relations may be written in terms of kernels \([5.40, 5.42, 5.50]\):

$$
G_{1\mu\nu}^T(x, \ldots) = \int dy \Delta_{\mu\rho\sigma}(x - y) G_{1\rho\sigma}^h(y, \ldots)
$$

$$
G_{2\mu\nu}^T(x, \ldots) = \int dy \Delta_{\mu\rho\sigma}(x - y) G_{2\rho\sigma}^{h,\epsilon}(y, \ldots)_{\epsilon = 0},
$$

where $\Delta_{\mu\rho\sigma}^{\text{tr}}$ is the invariant transversal kernel \([3.40]\) and $\Delta_{\mu\rho\sigma}^{\epsilon}$ is the kernel \([2.32]\) singular in the $\epsilon = 0$ limit. Let us remind that $G_{2\rho\sigma}^{h,\epsilon}$ is longitudinal for $\epsilon = 0$, see \([5.33]\), so that the integral \([5.41]\) is finite. The invariant regularization of the function $G_{2\rho\sigma}^h$ is provided via substitution of anomalous dimension $l_h^1 = -\epsilon$ in place of canonical one $l_h = 0$, as described above for the case of potential $A_{\mu}$.

It is shown in \([1]\), see also \([17]\), that the relation \([5.40]\) for $D = 4$ is equivalent to the equations of the linear conformal gravity.

The equalities \([5.40, 5.41]\) may be inverted and brought to the form

$$
G_{1\mu\nu}^h(x, \ldots) = \int dy D_{\mu\rho\sigma}^\epsilon(x - y) G_{1\rho\sigma}^T(y, \ldots)_{\epsilon = 0}
$$

$$
G_{2\mu\nu}^h(x, \ldots) = \int dy D_{\mu\rho\sigma}^{\text{long}}(x - y) G_{2\rho\sigma}^T(y, \ldots),
$$

where $D_{\mu\rho\sigma}^\epsilon$ is the ($\epsilon = 0$)-singular kernel \([3.50]\) and $D_{\mu\rho\sigma}^{\text{long}}$ is the longitudinal kernel \([3.42]\). The integral in \([5.42]\) is finite since the function $G_{1\rho\sigma}^{T,\epsilon}$ is transversal for $\epsilon = 0$, and the leading term $\sim 1/\epsilon$ in the expansion of $D_{\mu\rho\sigma}^\epsilon$ is longitudinal. Note that the
relations [5.41] and [5.42] may be rewritten using non-singular kernels if the realization of representations \( \tilde{Q}_h \) and \( \tilde{Q}_T \) from section 4 is utilized.

\[
P_{\mu\nu\rho\sigma}^{\text{long}} \left( \frac{\partial}{\partial x} \right) G_{2\rho\sigma}^T(x, \ldots) = \int dy \Delta_{\mu\nu\rho\sigma}^{\text{long}}(x - y) G_{2\rho\sigma}^h(y, \ldots) \quad (5.44)
\]

\[
P_{\mu\nu\rho\sigma}^{\text{tr}} \left( \frac{\partial}{\partial x} \right) G_{1\rho\sigma}^h(x, \ldots) = \int dy D_{\mu\nu\rho\sigma}^{\text{tr}}(x - y) G_{2\rho\sigma}^T(y, \ldots), \quad (5.45)
\]

where \( \Delta_{\mu\nu\rho\sigma}^{\text{long}} \) and \( D_{\mu\nu\rho\sigma}^{\text{tr}} \) are the finite kernels [2.30] and [3.52].

The invariant three-point functions [5.39] are also related by equivalence conditions. The relations of the type [5.40]–[5.45] for these functions may be proved by direct calculations. The technical details of the such calculations may be found in [1], see also [2,5].

It remains to discuss invariant integrals over internal lines corresponding to the fields \( h_{\mu\nu} \) and \( T_{\mu\nu} \). A literal quotation of the arguments concerning the derivation of equations [5.25]–[5.29] in the case of the fields \( A_\mu, j_\mu \) is sufficient, and will not be reproduced here. The only relevant comment is that the properties of invariant forms [3.38],[3.48],[3.59] and the expressions [4.19],[4.20] for the propagators \( \langle hh \rangle \) and \( \langle TT \rangle \) are applied in this case.

### 6 Irreducible Components of the Current and the Energy-Momentum Tensor

The two types of Green functions for the current were analyzed in the previous section. These functions correspond to the pair of irreducible representations \( Q_{j}^{\text{tr}} \) and \( \tilde{Q}_j \). It proves helpful to introduce two fields: \( j_{\mu}^{\text{tr}}(x) \) and \( \tilde{j}_\mu(x) \). The total Euclidean current \( j_\mu(x) \) transforms by the irreducible representation \( Q_{j}^{\text{tr}} \oplus \tilde{Q}_j \) and may be written as

\[
j_\mu(x) = j_{\mu}^{\text{tr}}(x) + \tilde{j}_\mu(x).
\]

Accordingly, one has a pair of propagators:

\[
\langle j_{\mu}^{\text{tr}}(x_1)j_{\nu}^{\text{tr}}(x_2) \rangle \quad \text{and} \quad \langle \tilde{j}_\mu(x_1)\tilde{j}_\nu(x_2) \rangle. \quad (6.2)
\]

All the Green functions \( \langle j^{\text{tr}}_\mu \ldots \rangle \) of the current \( j^{\text{tr}}_\mu \) are transversal, and its propagator has the form

\[
\Delta_{\mu\nu}^{\text{tr}}(x_{12}) = \langle j_{\mu}^{\text{tr}}(x_1)j_{\nu}^{\text{tr}}(x_2) \rangle \sim (\delta_{\mu\nu} - \partial_\mu \partial_\nu) \Box^{D-4} \delta(x_{12}). \quad (6.3)
\]

The Green functions of the current \( \tilde{j}_\mu \) depend on the choice of realization of the representation \( \tilde{Q}_j \). Choosing different representatives in the equivalence class, see sections 3,4, one can obtain different realizations of the Green functions \( \langle \tilde{j}_\mu \ldots \rangle \). In
section 4 we have examined a special realization of the representation $Q_j$, where the current $j_\mu$ is longitudinal. The propagator of the current in this realization takes the form

$$\langle j_\mu(x_1)j_\nu(x_2) \rangle = C_j \partial_\mu \partial_\nu \frac{D^4}{D^2} \delta(x_{12}).$$  \hspace{1cm} (6.4)

Note that the relevant conformal transformations are non-local and may differ from the transformations of the current $j_\mu$, see 4.11. The propagator of the total current equals to the sum of terms 6.3 and 6.4. The other realization of the representation $\tilde{Q}_j$ has also been studied; the transformations in that one are local and coincide with the transformations of the current $j_\mu$. In the latter realization the propagator of the current demands a regularization (section 2):

$$\langle \tilde{j}_\mu(x_1)\tilde{j}_\nu(x_2) \rangle = \Delta_{\mu\nu}(x_{12}) = C_j \partial_\mu \partial_\nu \frac{1}{x_{12}^{2+\epsilon}} + O(\epsilon).$$  \hspace{1cm} (6.5)

However this is the equivalence class $[j_\mu]$ and not the current $\tilde{j}_\mu$ by itself that has the physical meaning. The framework of conformal theory is constructed in such a manner that the transformations inside an equivalence class do not effect the values of the conformally invariant contractions 5.25,5.26. The latter ones may, in particular, be brought to the form 5.27,5.28 or 5.29. This property of contractions was discussed in section 3 in connection with the specification of invariant forms in the spaces of representations $\tilde{Q}_j$, $Q_{A_{\text{long}}}^A$, see 3.11 and 3.28. Hence one is free use either realization: 6.4 or 6.5, depending on what is actually needed. The transversal component in 6.5 is divergent at $\epsilon = 0$ and does not contribute to conformally invariant graphs. However, the realization 6.5 is more convenient for technical purposes, as demonstrated in [2–5], see [1] for more details.

All the above is also valid for the field $A_\mu(x)$, which transforms by the irreducible representation $\tilde{Q}_A \oplus Q_{A_{\text{long}}}^A$. Introducing irreducible fields $\tilde{A}_\mu(x)$ and $A_{\text{long}}^\mu(x)$, rewrite the field $A_\mu(x)$ as the sum

$$A_\mu(x) = \tilde{A}_\mu(x) + A_{\text{long}}^\mu(x).$$  \hspace{1cm} (6.6)

All the Green functions of the field $A_{\text{long}}^\mu$ are longitudinal. Its propagator has the form

$$\langle A_{\text{long}}^\mu(x_1)A_{\text{long}}^\nu(x_2) \rangle = D_{\mu\nu}^\text{long}(x_{12}) \sim \partial_\mu \partial_\nu \ln x_{12}^2.$$  \hspace{1cm} (6.7)

The Green functions of the field $\tilde{A}_\mu$ depend on the choice of realization of the representation $\tilde{Q}_A$. Different realizations correspond to different representatives of the equivalence class $[A_\mu]$. In section 4 we discussed a realization in which the propagator $\langle \tilde{A}_\mu \tilde{A}_\nu \rangle$ is transversal:

$$\langle \tilde{A}_\mu(x_1)\tilde{A}_\nu(x_2) \rangle \sim (\delta_{\mu\nu} - \partial_\mu \partial_\nu) \ln x_{12}^2.$$  \hspace{1cm} (6.8)

The conformal transformation in this realization are non-local and differ from the transformations of longitudinal field $A_{\text{long}}^\mu$, see 1.4,1.6. Another realization, in which
the conformal transformations are local and coincide with the transformations of the field \( A_{\mu}^{\text{long}} \), is also studied in section 3. In the latter case, the propagator \( \langle \tilde{A}_\mu \tilde{A}_\nu \rangle \) demands a regularization:

\[
\langle \tilde{A}_\mu(x_1)\tilde{A}_\nu(x_2) \rangle = D^\epsilon_{\mu\nu}(x_{12}) \sim \frac{1}{\epsilon (x_{12})^{1+\epsilon}} g_{\mu\nu}(x_{12}).
\]

Unlike \[5.8\], it has a longitudinal component which is singular at \( \epsilon = 0 \) and does not contribute to conformally invariant graphs, see \[5.24,5.28\] and \[5.29\]. Technically, the realization \[6.9\] is more useful.

Thus, henceforth to the end of this paper we use the realization \[6.5\] and \[6.9\]. The equivalence conditions for the representations \[5.14\] may be written in the form of operator relations [1]: in the transversal sector

\[
\begin{align*}
  j^\text{tr}_\mu(x) &= \int dy \Delta^\epsilon_{\mu\nu}(x-y) \tilde{A}_\nu(y), \quad \tilde{A}_\mu(x) = \int dy D^\epsilon_{\mu\nu}(x-y) j^\text{tr}_\nu(y) \Big|_{\epsilon=0}, \quad (6.10) \\
  \tilde{g}_\mu(x) &= \int dy \Delta^\epsilon_{\mu\nu}(x-y) A_{\nu,\epsilon}^{\text{long}}(y) \Big|_{\epsilon=0}, \quad A_{\mu,\epsilon}^{\text{long}}(x) = \int dy D_{\mu\nu}^{\text{long}}(x-y) \tilde{g}_\nu(y). \quad (6.11)
\end{align*}
\]

Here the relations between Euclidean quantum fields are considered as a condensed form of analogous relations for all the Green functions, see \[5.15,5.18\]. The regularized fields \( A_{\mu,\epsilon}^{\text{long}} \) and \( j_{\mu,\epsilon}^{\text{tr}} \) signify that the regularization has been introduced to the Green functions, see section 5. We remind that the equations \[6.10\] in \( D = 4 \) are equivalent to the Maxwell equations [1].

The physical meaning of expansions of Euclidean fields \( j_\mu \) and \( A_\mu \) into pairs of irreducible components may be commented in the following manner. Consider the Euclidean fields \( j_\mu(x) \) and \( T_{\mu\nu}(x) \). The Green functions \( \langle j_\mu \ldots \rangle \) and \( \langle T_{\mu\nu} \ldots \rangle \) are the Euclidean analogues of \( T \)-ordered vacuum expectation values in Minkowski space. Here we treat the Euclidean fields \( j_\mu(x) \) and \( T_{\mu\nu}(x) \) as the symbolic notation for the complete sets of Green functions \( \langle j_\mu \ldots \rangle \) and \( \langle T_{\mu\nu} \ldots \rangle \). Correspondingly, the derivatives of the Euclidean fields \( \partial_{\mu} j_\nu(x) \) and \( \partial_{\mu} T_{\mu\nu}(x) \) denote the derivatives of Green functions \( \partial_\mu \langle j_\nu \ldots \rangle \) and \( \partial_\mu \langle T_{\mu\nu} \ldots \rangle \). Calculating these derivatives, one encounters the two types of terms of different nature. Consider those terms on an example of the conserved current in Minkowski space. One gets for the propagator of the current:

\[
\partial_\mu \langle 0 | T \{ j_\mu(x) j_\nu(0) \} | 0 \rangle = \delta(x_0) \langle 0 | [j_0(x), j_\nu(0)] | 0 \rangle + \langle 0 | T \{ \partial_\mu j_\mu(x) j_\nu(0) \} | 0 \rangle.
\]

The second term vanishes due to the conservation law

\[
\partial_\mu j_\mu^{\text{Mink}}(x) = 0.
\]

To ensure the covariance of the \( T \)-ordered average, one should add quasilocal terms to the first term of the expression. The form of the total contribution of these terms and the commutator is imposed by conformal invariance, and reads

\[
\partial_\mu \langle 0 | T \{ j_\mu(x) j_\nu(0) \} | 0 \rangle = C_\mu \partial_\nu \Box^{-\frac{D-2}{2}} \delta(x).
\]
In Euclidean conformal theory, one associates the above pair of contributions with the irreducible components of the Euclidean current \( j_\mu(x) \), so that

\[
\partial_{\mu} j_\mu(x) = \partial_{\mu} \tilde{j}_\mu(x), \quad \partial_{\mu} j^\text{tr}_\mu(x) = 0.
\]

In particular, the total propagator of the current may be represented as:

\[
\langle j_\mu(x_1) j_\nu(x_2) \rangle = \langle j_\mu(x_1) \tilde{j}_\nu(x_2) \rangle + \langle j^\text{tr}_\mu(x_1) j^\text{tr}_\nu(x_2) \rangle,
\]

\[
\partial_{\mu} j_\mu(x_1) j_\nu(x_2) = \partial_{\mu} \langle j_\mu(x_1) \tilde{j}_\nu(x_2) \rangle = C_{\text{J}} \partial_{\nu} \frac{\partial^2}{\partial x_{12}^2} \delta(x_{12}),
\]

\[
\partial_{\mu} j^\text{tr}_\mu(x_1) j^\text{tr}_\nu(x_2) = 0.
\]

Thus, the two irreducible components \( \tilde{j}_\mu \) and \( j^\text{tr}_\mu \) have different physical meaning, and hence the different group-theoretic structure.

Only the current \( j^\text{tr}_\mu(x) \), but not \( \tilde{j}_\mu(x) \), induces a non-trivial contribution to the electromagnetic interaction. The Green functions of the current \( \tilde{j}_\mu \) satisfy non-trivial Ward identities and contain the information on the (postulated) commutation relations of the total current:

\[
[j_0(x), j_k(0)]_{x^0=0}, [j_0(x), \varphi(0)]_{x^0=0}, \ldots.
\]

As shown in [1,2], all the Green functions \( \langle j_\mu \ldots \rangle \) are uniquely determined by the condition of conformal invariance and by the Ward identities. Resultantly, the operator product expansions \( \tilde{j}_\mu(x)\varphi(0) \) have the form [1,2]

\[
\tilde{j}_\mu(x)\varphi(0) = \sum_s [P_s],
\]

where \( P_s \) are the tensor fields of rank \( s \) and dimension \( d + s \). This result does not depend on the type of interaction and is intrinsically due by conformal symmetry and the contributions of equal-time commutators. The dynamical models are defined [1,2] by the definition of commutators \( [j_\mu(x), j_\nu(0)]_{x^0=0} \). In Euclidean version of the theory, this commutator is determined by the type of the operator product expansion \( \tilde{j}_\mu(x)\tilde{j}_\nu(0) \). The expansion \( \tilde{j}_\mu(x)\tilde{j}_\nu(0) = [C_j] + [P_j] + \ldots \) was considered in [1,2], while in [4] the models with \( C_j \neq 0, P_j(x) = 0 \) were examined.

One should remark that since the current \( \tilde{j}_\mu(x) \) arises as a representative of an equivalence class \( \{ \tilde{j}_\mu \} \subset \tilde{M}_j = M_j / M^\text{tr}_j \), the transversal parts of the Green functions \( \langle j \ldots \rangle \) may be redefined by performing a different choice of representatives. Particularly, in the non-local realization of conformal transformations, considered above, these Green functions are longitudinal. This realization is useful for conformal QED. However, it is essential that the local realization of conformal transformations of the current \( \tilde{j}_\mu \), is needed for the analysis of the operator product expansions \( \tilde{j}_\mu\varphi \) and \( \tilde{j}_\mu\tilde{j}_\nu \). In a local realization, the Green functions \( \langle \tilde{j}_\mu \ldots \rangle \) have quite definite transversal parts, which do not contribute to the electromagnetic interaction since an irreducible component \( \tilde{j}_\mu \) of the total current only appears in contractions of the type \( \int dx \tilde{j}_\mu(x)A_\mu^\text{long}(x) \).
The interaction with the irreducible field $\tilde{A}_\mu$ is caused by the component $j^{ir}_\mu$ of the total current, and has the form $\int dx j^{ir}_\mu(x)\tilde{A}_\mu(x)$.

Let us consider the structure of the Hilbert space of conformal theory more comprehensively. As shown in [1], see also [2], the two types of conformal currents

$$\tilde{j}_\mu(x) \quad \text{and} \quad j^{ir}_\mu(x)$$

(6.12)

may be associated with the two mutually orthogonal sectors of the Hilbert space

$$\tilde{H} \oplus H_0,$$

(6.13)

where $\tilde{H}$ is generated by the states

$$\tilde{j}_\mu(x_1)\varphi(x_2) \mid 0 \rangle, \tilde{j}_\mu(x_1)\tilde{j}_\nu(x_2)\varphi(x_3) \mid 0 \rangle, \ldots,$$

(6.14)

and $H_0$ includes analogous states of the current $j^{ir}_\mu(x)$. The orthogonality of the spaces $\tilde{H}$ and $H_0$ means the vanishing of the Green functions

$$\langle j^{ir}_\mu(x_1)\tilde{j}_\nu(x_2) \rangle = 0, \quad \langle \varphi(x_1)j^{ir}_\mu(x_2)\tilde{j}_\nu(x_3)\varphi^\dagger \rangle = 0.$$  

(6.15)

Due to the equivalence condition 6.10 the subspace $H_0$ contains nothing but electromagnetic degrees of freedom. The arising of non-zero current $j^{ir}_\mu$ necessarily begets the electromagnetic interaction. If we are engaged in the analysis of non-electromagnetic interaction, i.e., are interested in the states of subspace $\tilde{H}$, the problem of separation of the current $\tilde{j}_\mu$ from the total current 6.1 emerges.

The latter leads to the following situation. The conformal symmetry arises as a non-perturbative effect. The conclusions of conformal theory cannot have any analogues in perturbation theory. Moreover, the conformal symmetry may occur in a special class of models, not necessarily lagrangean. Given the structure of the Hilbert space described above, the original presence of gauge interaction in conformal models would be the most natural conjecture. Under that, both irreducible components 6.12 contribute to the total current. A “true” conformal theory must include the complete current 6.1, and hence, due to 6.10 the gauge field $\tilde{A}_\mu$ as well. If one is going to examine an approximate model without gauge interactions, the solution is to be looked for in the restricted class of Green functions $\langle j_\mu \ldots \rangle$, those corresponding to the irreducible representation $\tilde{Q}_j$. This leads to certain restrictions on higher Green functions which guarantee the irreducibility:

$$j_\mu(x) = \tilde{j}_\mu(x), \quad j^{ir}_\mu(x) = 0 \quad \text{on} \quad \tilde{H}.$$  

(6.16)

Such restrictions were studied to a fair extent in [1,2], see also [4].

These works deal with the class of non-gauge models in $D$-dimensional space, which are analogous to two-dimensional conformal models. The irreducibility condition for the current 6.16 is written in the following form [1,2,4]

$$\int dy dz B^{ls}_{1\mu}(xyz)\langle j_\mu(z)\varphi(y) \ldots \rangle = 0 \quad \text{for all } l, s,$$

(6.17)
where $B_{1,\mu}^s$ are the invariant three-point functions introduced in section 5. A theory supplied with such a condition is non-trivial if the operator product expansion $j_\mu(x_1) j_\nu(x_2)$, where $j_\mu(x) = \tilde{j}_\mu(x)$ is the irreducible current, includes the anomalous terms $[C_j]$ and $[P_j]$, see Introduction. The conditions 6.17 allow one to calculate all the Green functions of the current from anomalous [3] Ward identities.

All the above is easily generalized to the case of the energy-momentum tensor and the metric field. One introduces a pair of tensor fields

$$T_{\mu\nu}(x) \quad \text{and} \quad \tilde{T}_{\mu\nu}(x), \quad (6.18)$$

transforming by irreducible representations $Q_{T}^\text{tr}$ and $\tilde{Q}_T$. All the Green functions of the field $T_{\mu\nu}^\text{tr}$ are transversal, and the propagator is given by the expression 3.40. The Green functions of the field $\tilde{T}_{\mu\nu}$ solely depend on the choice of realization of the representation $\tilde{Q}_T$. In the non-local realization of the section 4, the propagator $\langle \tilde{T}\tilde{T} \rangle$ has the form 2.30.

The total propagator of the field

$$T_{\mu\nu}(x) = T_{\mu\nu}^\text{tr}(x) + \tilde{T}_{\mu\nu}(x) \quad (6.19)$$

in this realization reads

$$\langle T_{\mu\nu}(x_1) T_{\rho\sigma}(x_2) \rangle = \Delta_{\mu\nu\rho\sigma}^\text{tr}(x_{12}) + \Delta_{\mu\nu\rho\sigma}^\text{long}(x_{12}) \quad (6.20)$$

and is given by Eq.4.19. In the local realization of the representation $\tilde{Q}_T$ the propagator of the field $T_{\mu\nu}$ is given by the regularized expression 2.32. Identically to the case of electromagnetic interaction, the conformally invariant graphs do not depend on the choice of realization of the representation $\tilde{Q}_T$. This is obvious from the inspection of invariant forms on the spaces of representations $\tilde{Q}_T, Q_h^\text{long}$, see section 3. (The choice of representatives in the equivalence classes $[T_{\mu\nu}]$ and $[h_{\mu\nu}]$ does not alter the values of the forms.) Analogously, the metric tensor

$$h_{\mu\nu}(x) = \tilde{h}_{\mu\nu}(x) + h_{\mu\nu}^\text{long}(x) \quad (6.21)$$

also transforms by the reducible representation $\tilde{Q}_h \oplus Q_h^\text{long}$. The propagator of the irreducible field $h_{\mu\nu}^\text{long}$ is longitudinal, see 3.42, while the propagator of the field $\tilde{h}_{\mu\nu}$ has a transversal part. In the realization of section 4 the total propagator of the field $h_{\mu\nu}$ is given by the expression, see 4.19

$$\langle h_{\mu\nu}(x_1) h_{\rho\sigma}(x_2) \rangle = D_{\mu\nu\rho\sigma}^\text{tr}(x_{12}) + D_{\mu\nu\rho\sigma}^\text{long}(x_{12}). \quad (6.22)$$

In the local realization of the representation $\tilde{Q}_h$ the propagator of the field $\tilde{h}_{\mu\nu}$ is given by the regularized expression 3.50.

Consider the equivalence conditions for the representations 5.30. In the operator notation they read:

$$T_{\mu\nu}^\text{tr}(x) = \int dy \Delta_{\mu\nu\rho\sigma}^\text{tr}(x-y) \tilde{h}_{\rho\sigma}(y), \quad \tilde{T}_{\mu\nu}(x) = \int dy D_{\mu\nu\rho\sigma}(x-y) T_{\rho\sigma}^\text{tr}(y) \quad (6.23)$$

$$\tilde{T}_{\mu\nu}(x) = \int dy \Delta_{\mu\nu\rho\sigma}^\epsilon(x-y) \tilde{h}_{\rho\sigma}^\text{long}(y) \bigg|_{\epsilon=0}, \quad h_{\mu\nu}^\text{long}(x) = \int dy D_{\mu\nu\rho\sigma}^\text{long}(x-y) \tilde{T}_{\rho\sigma}(y). \quad (6.24)$$
The kernels $\Delta^{\mu\nu}_{\rho\sigma}$ and $D_{\mu\nu\rho\sigma}$ are given by the expressions (3.40) and (3.50), while
the kernels $\Delta^{\epsilon}_{\mu\nu\rho\sigma}$ and $D_{\mu\nu\rho\sigma}^{\long}$ — by the expressions (2.30) and (3.42).

The regularization of the fields $T_{\rho\sigma}^{\mu}$ and $h_{\rho\sigma}^{\long}$ means the regularization of the
Green functions, see section 5. Let us remind that when $D = 4$, the equations 6.23
are equivalent to the equations of linear conformal gravity.

The states of the Hilbert space generated by the fields $6.18$ also form a pair of
orthogonal subspaces $H_0$ and $\tilde{H}$. Due to the equivalence conditions 6.23 the subspace
$H_0$ contains solely gravitational degrees of freedom. The field $T^{\mu\nu}$ necessarily begets
a gravitational interaction. Recalling the arguments presented above, we conclude
that the theory without gravitational interaction must include the irreducible tensor
$\tilde{T}_{\mu\nu}$ only, and satisfy the condition

$$T_{\mu\nu}(x) = \tilde{T}_{\mu\nu}(x), \quad T^{\mu\nu}(x) = 0 \quad \text{on } \tilde{H}. \quad (6.25)$$

As shown in [1,2], see also [5], these conditions may be set up as the following conditions on higher Green functions

$$\int dy \, dz \, B^{l,s}_{\mu\nu}(xyz) \langle T_{\mu\nu}(z) \varphi(y) \ldots \rangle = 0 \quad \text{for all } l, s, \quad (6.26)$$

where $B^{l,s}_{\mu\nu}$ are the invariant functions discussed in section 5. A theory supplied
with such a condition is non-trivial if the operator product expansion $T_{\mu\nu}(x_1)T_{\rho\sigma}(x_2)$, where $T_{\mu\nu}(x) = \tilde{T}_{\mu\nu}(x)$ is the irreducible field, includes the anomalous terms $[C_T]$ and $[P_T]$, see Introduction.

Thus the Green functions of the irreducible fields

$$\langle \tilde{j}_{\mu} \varphi \ldots \rangle, \quad \langle \tilde{T}_{\mu\nu} \varphi \ldots \rangle$$

satisfy the conditions 6.17 and 6.26 and are uniquely determined by anomalous
Ward identities for any space dimension (either even or odd [1,2,5]). Such a theory
acquires analogues of null-vectors, each defining an exactly solvable model [1,2,4,5].
Note that the current and the energy-momentum tensor of two-dimensional theory
are analogous to the fields $\tilde{j}_{\mu}$ and $\tilde{T}_{\mu\nu}$, while the fields $j^{\mu\nu}$ and $T^{\mu\nu}$, have no analogues
when $D = 2$. The propagator of the energy-momentum tensor in $D = 2$ has the form [1,7]. Using the identity 1.1 one can bring the propagator to the form which follows from 2.30,2.31 in $D = 2$:

$$\langle T_{\mu\nu}(x_1)T_{\rho\sigma}(x_2) \rangle_{D=2} = \partial_\mu \Delta_{\nu\rho\sigma}(x_{12}) + \partial_\nu \Delta_{\mu\rho\sigma}(x_{12}) - \delta_{\mu\nu} \partial_\lambda \Delta_{\lambda\rho\sigma}(x_{12}),$$

where

$$\Delta_{\mu\rho\sigma}\big|_{D=2} \sim [4\partial_\mu \partial_\rho \partial_\sigma - (\partial_\mu \partial_\rho + \partial_\mu \partial_\sigma)] - \delta_{\rho\sigma} \partial_\mu \Box \ln x^2.$$

Therefore an irreducible representation (of the 6-parametric conformal group in $D = 2$) corresponding to it, is analogous to the representation $\tilde{Q}$ in $D$-dimensional theory.
The theory defined by the conditions 6.25 and 6.26 is a natural candidate to the
generalization of two-dimensional conformal theory.
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Bibliography

1. E.S.Fradkin and M.Ya.Palchik, Conformal Quantum Field Theory in $D$-Dimensions, Kluwer Acad. Publ., 1996
2. E.S.Fradkin and M.Ya.Palchik, Ann. of Phys. 249(1996)44.
3. E.S.Fradkin, M.Ya.Palchik, and V.N.Zaikin, Phys.Rev. D53(1996)7345.
4. E.S.Fradkin and M.Ya.Palchik, preprint, 1C/96/21, Triest, 1996.
5. E.S.Fradkin and M.Ya.Palchik, preprint, 1C/96/22, Triest, 1996.
6. A.A.Belavin, A.M.Polyakov and A.B.Zamolodchikov, Nucl.Phys. B241(1984)333.
7. D.Friden, Z.Qiu and S.Shenker, Phys.Rev.Lett. 52(1984)1575, Phys.Lett. 151B(1985)37.
8. V.G.Knizhnik and A.B.Zamolodchikov, Nucl.Phys. B247(1984)33.
9. H. Osborn and A. C. Petkou, Ann. Phys. (N.Y.) 231 (1994) 311.
10. J. Erdmenger and H. Osborn, Nucl.Phys., B483 (1997) 431.
11. I.M.Gel’fand and G.E.Shilov, Generalized Functions, v.1 (Academic Press, New York, 1964).
12. V.K.Dobrev, G.Mack, V.B.Petkova, S.G.Petrova and I.T.Todorov, Lecture Notes in Physics, v.63 (Springer-Verlag, 1977).
13. A.U.Klimyk and A.M.Gavrilik, Matrics Elements and Klebsh-Gordan Coefficients of Group Representations (Naukova Dumka, Kiev, 1979).
14. M.Ya.Palchik, J.Phys. 16(1983)1523.
15. A.A.Kozhevnikov, M.Ya.Palchik and A.A.Pomeranskii, Yadernaya Fizika, 37(1983)481.
16. E.S.Fradkin, A.A.Kozhevnikov, M.Ya.Palchik and A.A.Pomeranskii, Comm. Math. Phys. 91(1983)529.
17. E.S.Fradkin and M.Ya.Palchik, Class.Quantum Grav. 1(1984)131.