A continuous-variable formalism for the Faraday atom–light interface

Julia Stasińska¹, #, Simone Paganelli¹, Carles Rodó¹ and Anna Sanpera², ¹
¹ Grup de Física Teòrica: Informació i Fenòmens Quàntics,
Universitat Autònoma de Barcelona, 08193 Bellaterra (Barcelona), Spain.
² ICREA-Institució Catalana de Recerca i Estudis Avançats.
Lluis Companys 23, 08010 Barcelona, Spain
E-mail: # julsta@ifae.es

Abstract. Quantum interfaces between polarized atomic ensembles and coherent states of light, applied recently to manipulate bipartite and multipartite entanglement, are revisited by means of a continuous-variable formalism. The explicit use of the continuous-variable formalism facilitates significantly the analysis of entanglement between different modes, reducing it to the study of the properties of a final covariance matrix which can be found analytically. Furthermore, it allows to study matter–light interfaces for mixed states, adapting the formalism to the experimental situations in which the initial prepared Gaussian states are, unavoidably, affected by a certain amount of noise. A multipartite scenario, leading to the generation of macroscopic cluster states is presented and analyzed in detail within this formalism.

Keywords: atom–light interface, continuous variables, cluster states

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1. Introduction

A strongly polarized macroscopic atomic ensemble can be described disregarding its individual atomic components and using collective spin variables instead. In this scenario, the interaction of an atomic polarized ensemble with off-resonant linearly polarized light resolves, at a classical level, into a Faraday rotation. The latter refers to the rotation experienced by the polarization of light propagating inside a magnetic medium (polarized atomic ensemble). Moreover, this interaction can also lead to an exchange of quantum fluctuations between matter and light, i.e. a quantum interface. Seminal results exploiting such interface are the generation of spin squeezed states [1], and the entanglement between two macroscopic spatially-separated atomic ensembles [2]. In the first case, squeezing was produced by the interaction of an atomic sample with a squeezed state of light, whereas the second task was achieved by propagating a single laser beam through both samples. The above protocols demand a final measurement on the light (homodyne detection) which projects the state of the atomic ensembles into
a desired final state. This mapping of fluctuations between different physical systems provides a powerful tool to design quantum correlations. The potential applications of such a genuine quantum instrument have just started. Among them the use as a spectroscopic method for strongly correlated ultracold atomic systems \[3\]. Furthermore, if the incident light beam is spatially tailored, the interface provides spatial resolution. This characteristic should permit to directly observe correlations without the need of individual spin addressing \[4\]–\[6\], or even provide a direct measure of the order parameter of some exotic quantum states of matter \[7\]. Conversely, a quantum interface between a strongly correlated system and light should allow to map the correlations of the matter onto the light, leading to the generation of highly non classical states of light.

Here we address matter–light interfaces in the framework of continuous-variable (CV) formalism which strongly facilitates the manipulation and verification of entanglement, both bipartite and multipartite, reducing it to the study of the properties of a final covariance matrix. Such formalism is especially well suited for mixed states since they are treated on an equal footing as pure states. The formalism of continuous variables arises naturally if the initial states of atomic ensembles and light are Gaussian. Then, if the interaction between light and matter is bilinear, the Gaussian character of the subsystems is preserved. Unitary evolutions of such form correspond to symplectic transformations of the covariance matrix of the composite system (matter and light). The final projective measurement on the light, homodyne detection, is also a Gaussian operation and it is easily performed at the level of the final covariance matrix. Therefore, all the necessary tools are at hand.

The paper is organized as follows. In section 2, we briefly state the problem under consideration, starting from the description of the matter and light as continuous-variable systems. Then using the Quantum Non Demolition (QND) effective Hamiltonian of the matter–light interaction we derive the evolution equations (for a detailed derivation of such Hamiltonian the reader is referred to \[8\]). In section 3, we derive explicitly all the steps of the matter–light interface in terms of covariance matrices and symplectic transformations \[9\]–\[10\]. In section 4, we demonstrate the suitability of the formalism by analyzing in detail two examples. The first one corresponds to the well known EPR bipartite entanglement between two macroscopic atomic ensembles, experimentally demonstrated in \[2\]. We consider a more general scenario in which the atomic samples are initially in thermal states with different amount of noise. In such a case entanglement between the atomic samples is produced only above a given value of the matter–light coupling constant. By exploiting different geometries \[11\], the interaction with a second light beam permits to design optimally the final entanglement between the samples and even to delete the entangling action of the first beam. The second case we treat, much more involved, analyzes the generation of cluster states using matter–light interfaces and demonstrates the full power of this formalism in the multipartite scenario. In section 5, we present our conclusions.
2. The Faraday Interaction

The physical system we analyze consists of several atomic ensembles and light beams, the latter playing the role of information carriers between the atomic samples. At a time, only a single light beam interacts with the atomic ensembles. After the interaction, the light beam is measured.

Each atomic ensemble contains a sufficiently large number, \( N_{\text{at}} \), of noninteracting alkali atoms with individual total angular momentum \( \hat{F} \). The ensemble is described by its collective angular momentum \( \hat{J} = (\hat{J}_x, \hat{J}_y, \hat{J}_z) \), where \( \hat{J}_k = \sum_{i=1}^{N_{\text{at}}} \hat{F}_{k,i} \) (\( k = x, y, z \)). All atoms are assumed to be polarized along the \( x \)-direction, which corresponds to preparing them in a particular hyperfine state \( |F, m_F \rangle \). In such a situation, fluctuations in the \( \hat{J}_x \) component of the collective spin are very low, allowing for treating this variable as a classical number \( \hat{J}_x \approx \langle \hat{J}_x \rangle \equiv \hbar J_x = \hbar N_{\text{at}}F \). As a consequence, the quantum character of the collective spin is preserved in the orthogonal spin components, which have a zero mean, but non-zero fluctuations. By appropriate normalization they are made to fulfil the canonical commutation relation,

\[ [\hat{J}_y / \sqrt{\hbar J_x}, \hat{J}_z / \sqrt{\hbar J_x}] = i\hbar. \]

To stress the continuous variable character of the system, we rename the above variables as “position” and “momentum”:

\[
\hat{x}_A = \frac{\hat{J}_y}{\sqrt{\hbar J_x}}, \\
\hat{p}_A = \frac{\hat{J}_z}{\sqrt{\hbar J_x}},
\]

and from now on use only the canonical variables \( \hat{x}_A, \hat{p}_A \) to refer to the atomic sample, where the subindex \( A \) stands for atomic ensemble. Later on when we deal with few ensembles we will use the notation \( \hat{x}_{A,n}, \hat{p}_{A,n} \) to refer to the \( n \)th atomic sample.

On the other hand, the light is taken to be out of resonance from any relevant atomic transition and linearly polarized along, say, the \( x \)-direction. We use the Stokes vector description \( \hat{s} = (\hat{s}_x, \hat{s}_y, \hat{s}_z) \) of light polarization. The components \( \hat{s}_k \) (\( k = x, y, z \)) correspond to the differences between the number of photons (per unit time) with \( x \) and \( y \) linear polarizations, \( \pm \pi/4 \) linear polarizations and the two circular polarizations, i.e.

\[
\begin{align*}
\hat{s}_x &= \frac{\hbar}{2}(\hat{n}_x - \hat{n}_y) = \frac{\hbar}{2}(\hat{a}_x^\dagger \hat{a}_x - \hat{a}_y^\dagger \hat{a}_y), \\
\hat{s}_y &= \frac{\hbar}{2}(\hat{n}_y - \hat{n}_x) = \frac{\hbar}{2}(\hat{a}_x^\dagger \hat{a}_y + \hat{a}_y^\dagger \hat{a}_x), \\
\hat{s}_z &= \frac{\hbar}{2}(\hat{n}_x - \hat{n}_y) = \frac{\hbar}{2\hbar}(\hat{a}_x^\dagger \hat{a}_y - \hat{a}_y^\dagger \hat{a}_x).
\end{align*}
\]  

(2)

These allow for the microscopic description of the interaction with atoms, however, effectively only the following macroscopic observables will be relevant: \( \hat{S}_k = \int_0^T \hat{s}_k(t)dt \), where \( T \) is the duration of the light pulse. So defined operators obey standard angular momentum commutation rules. The assumption of linear polarization along direction \( x \) allows for the approximation \( \hat{S}_x \approx \langle \hat{S}_x \rangle \equiv N_{\text{ph}}\hbar/2 \). Once more, the remaining orthogonal components \( \hat{S}_y \) and \( \hat{S}_z \) are appropriately rescaled in order to make them to
fulfill the canonical commutation rule, \( [\hat{S}_y/\sqrt{\hbar S_x}, \hat{S}_z/\sqrt{\hbar S_x}] = i\hbar \). Straightforwardly, a correspondence equivalent to equation (1) arises:

\[
\hat{x}_L = \frac{\hat{S}_y}{\sqrt{\hbar S_x}}
\]

\[
\hat{p}_L = \frac{\hat{S}_z}{\sqrt{\hbar S_x}}
\]

(3)

which allows to treat the light polarization degrees of freedom on the same footing as the atomic variables. Notice, that while only one light beam is the carrier which entangles the atomic ensembles, a secondary light beam will be normally needed to verify entanglement [8, 11].

In the situation in which a light beam propagates in the YZ plain and passes through a single ensemble at angle \( \alpha \) with respect to direction \( z \), the atom-light interaction can be approximated to the following QND effective Hamiltonian (see [8] for a detailed derivation)

\[
\hat{H}_\text{eff}^{\text{int}}(\alpha) = -\frac{\kappa}{T} \hat{p}_L (\hat{p}_A \cos \alpha + \hat{x}_A \sin \alpha).
\]

(4)

The parameter \( \kappa \) is the coupling constant with the dimension of the inverse of an action. Notice that such Hamiltonian leads to a bilinear coupling between the Stokes operator and the collective atomic spin operators. Evolution can be calculated through the Heisenberg equation for the atoms and using Maxwell–Bloch equation for light, neglecting retardation effect. The variables characterizing the composite system transform according to the following equations ( [8] and references therein):

\[
\hat{x}_A^{\text{out}} = \hat{x}_A^{\text{in}} - \kappa \hat{p}_L^{\text{in}} \cos \alpha,
\]

(5a)

\[
\hat{p}_A^{\text{out}} = \hat{p}_A^{\text{in}} + \kappa \hat{p}_L^{\text{in}} \sin \alpha,
\]

(5b)

\[
\hat{x}_L^{\text{out}} = \hat{x}_L^{\text{in}} - \kappa (\hat{p}_A^{\text{in}} \cos \alpha + \hat{x}_A^{\text{in}} \sin \alpha),
\]

(5c)

\[
\hat{p}_L^{\text{out}} = \hat{p}_L^{\text{in}}.
\]

(5d)

The above equations can be straightforwardly generalized to the case in which a single light beam (\( \hat{x}_L, \hat{p}_L \)) propagates through many samples shining at the \( n \)th sample at a certain angle \( \alpha_n \).

Due to the strong polarization constraint the initial states of the atomic ensembles as well as the one of light can be treated as Gaussian modes. On the other hand, the interaction is a bilinear coupling between the Stokes operator and the collective atomic spin operator, thus, it preserves the Gaussian character of the initial modes and can be interpreted as a Gaussian interaction between two bosonic modes. These facts enable us to tackle the quantum interface within a CV formalism.

3. The matter-light interface in the CV formalism

We start by reviewing the most basic concepts needed to describe Gaussian continuous-variable systems. For further reading, the reader is referred to [12–14] and references...
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therein. For a general quantum system of \(N\) pairs of canonical degrees of freedom (“position” and “momentum”), the commutation relations fulfilled by the canonical coordinates \(\hat{R} = (\hat{x}_1, \hat{p}_1, \ldots, \hat{x}_N, \hat{p}_N)\) can be represented in a matrix form by the symplectic matrix \(\mathcal{J}_N: [\hat{R}_i, \hat{R}_j] = i\hbar (\mathcal{J}_N)_{ij}, i, j = 1, \ldots, 2N,\)

\[
\mathcal{J}_N = \bigoplus_{\mu=1}^{N} \mathcal{J}, \quad \mathcal{J} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.
\] (6)

Gaussian states are, by definition, fully described by the first and second moments of the canonical coordinates. Hence, rather than describing them by their infinite-dimensional density matrix \(\rho\), one can use the Wigner function representation

\[
W(\zeta) = \frac{1}{\pi^N \sqrt{\det \gamma}} \exp \left[ - (\zeta - d)^T \gamma^{-1} (\zeta - d) \right],
\] (7)

which is a function of the first moments through the displacement vector \(d\), and of the second moments through the covariance matrix \(\gamma\), defined as:

\[
d_i = \text{Tr}(\rho \hat{R}_i), \quad \gamma_{ij} = \text{Tr}(\rho \{ \hat{R}_i - d_i, \hat{R}_j - d_j \}).
\] (8)

The variable \(\zeta = (x_1, p_1, \ldots, x_N, p_N)\) is a real phase space vector with probability distribution given by the Wigner function. The covariance matrix corresponding to a quantum state must fulfill the positivity condition

\[
\gamma + i\mathcal{J}_N \geq 0.
\] (9)

In the particular case of a physical system consisting of several atomic ensembles and single light beam the most general covariance matrix takes the form

\[
\gamma = \begin{pmatrix} \gamma^A & C \\ C^T & \gamma^L \end{pmatrix},
\] (10)

where the submatrix \(\gamma^L\) corresponding to light mode, \(\gamma^A\) the atomic ensembles, and \(C\) accounts for the matter light correlations.

If a Gaussian state undergoes a unitary evolution preserving its Gaussian character, which is the case in the physical systems under consideration, then the respective transformation at the level of the covariance matrix is represented by a symplectic matrix \(S\) acting as

\[
\gamma_{\text{out}} = S^T \gamma_{\text{in}} S.
\] (11)

We illustrate how to reconstruct the evolution of the covariance matrix from the propagation equations \((5a)–(5d)\). One notes that the variables describing the system after interaction are expressed as a linear combination of the initial ones. Let us denote by \(K\) the following linear transformation \(K: (\hat{x}^\text{out}_A, \hat{p}^\text{out}_A, \hat{x}^\text{out}_L, \hat{p}^\text{out}_L)^T = K(\hat{x}^\text{in}_A, \hat{p}^\text{in}_A, \hat{x}^\text{in}_L, \hat{p}^\text{in}_L)^T\), which can be straightforwardly obtained from equations \((5a)–(5d)\) and for a single atomic mode reads:

\[
\begin{pmatrix} \hat{x}^\text{out}_A \\ \hat{p}^\text{out}_A \\ \hat{x}^\text{out}_L \\ \hat{p}^\text{out}_L \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & -\kappa \cos \alpha \\ 0 & 1 & 0 & \kappa \sin \alpha \\ -\kappa \sin \alpha & -\kappa \cos \alpha & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \hat{x}^\text{in}_A \\ \hat{p}^\text{in}_A \\ \hat{x}^\text{in}_L \\ \hat{p}^\text{in}_L \end{pmatrix}.
\] (12)
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Since the interaction Hamiltonian is bilinear, the matrix $K$ can be directly applied to a phase space vector $\zeta$ and correspondingly to the covariance matrix, however the sign of the coupling constant $\kappa$ should be changed. This is because the phase space variables undergo the Schrödinger evolution, whereas the quadratures transform according to the Heisenberg picture. Therefore, we define $\tilde{K} = K |_{\kappa \rightarrow (-\kappa)}$, which we apply to the phase space vector and covariance matrix as

$$
\zeta_{\text{out}}^{T} \gamma_{\text{in}}^{-1} \zeta_{\text{out}} = \zeta_{\text{in}}^{T} \tilde{K}^{T} \gamma_{\text{in}}^{-1} \tilde{K} \zeta_{\text{in}} = \zeta_{\text{in}}^{T} (\tilde{K}^{-1} \gamma_{\text{in}} (\tilde{K}^{T})^{-1})^{-1} \zeta_{\text{in}} = \zeta_{\text{in}}^{T} \gamma_{\text{out}}^{-1} \zeta_{\text{in}},
$$

leading to $S = (\tilde{K}^{T})^{-1}$. The above formalism has been explicitly developed for a single sample and a single beam, but it easily generalizes to an arbitrary number of atomic samples and beams, as well as to various geometrical settings.

Finally, the last ingredient essential to describe the matter–light interface at the level of the covariance matrix is the homodyne detection of light. Assuming a zero initial displacement and covariance matrix of the form (10), the measurement of the quadrature $\hat{x}_{L}$ with outcome $\tilde{x}_{L}$ leaves the atomic system in a state described by a covariance matrix

$$
\gamma^{A'} = \gamma^{A} - C(X \gamma^{L} X)^{-1} C^{T},
$$

and displacement

$$
d_{A} = C(X \gamma^{L} X)^{-1}(\tilde{x}_{L}, 0),
$$

where the inverse is understood as an inverse on the support whenever the matrix is not of full rank and $X$ is a diagonal matrix with the same dimension as $\gamma^{L}$ with diagonal entries $(1, 0, 1, 0, \ldots, 1, 0)$.

Inherent to the matter–light interface is the analysis of the remaining atomic samples once the light beam has been measured, in order to check the correlations induced between them. With the CV formalism we have access to the complete information provided by the covariance matrix and displacement vector after interaction. This makes verification of entanglement amenable to covariance matrix criteria (see also [9, 10]).

A structural separability test, which can be only applied when a full covariance matrix is available, is the positive partial transposition (PPT) test [16, 17]. For continuous-variable systems, it corresponds to partial time reversal of the covariance matrix [15], i.e. the change of the sign of the momentum for chosen modes. If the partially time reversed covariance matrix does not fulfill the positivity condition (9), the corresponding state is entangled. This test, however, checks only for the bipartite entanglement. For Gaussian states this criterion is necessary and sufficient for entanglement of $1 \times N$ modes.

Experimentally more convenient separability test is based on variances of collective observables. It was provided for two-mode states in [19] and generalized to many-mode
states in \cite{20}. It states that if an $N$ mode state is separable, then the sum of the variances of the following operators:

\[
\hat{u} = h_1 \hat{x}_1 + \ldots + h_N \hat{x}_N \\
\hat{v} = g_1 \hat{p}_1 + \ldots + g_N \hat{p}_N
\]  

(16)
is bounded from below by a function of the coefficients $h_1, \ldots, h_N, g_1, \ldots, g_N$. Mathematically, the inequality is expressed as

\[
(\Delta \hat{u})^2 + (\Delta \hat{v})^2 \geq f(h_1, \ldots, h_N, g_1, \ldots, g_N)\hbar,
\]  

(17)

where

\[
f(h_1, \ldots, h_N, g_1, \ldots, g_N) = \left| h_l g_l + \sum_{r \in I} h_r g_r \right| + \left| h_m g_m + \sum_{s \in I'} h_s g_s \right|.
\]  

(18)

In the above formula the two modes, $l$ and $m$, are distinguished and the remaining ones are grouped into two disjoined sets $I$ and $I'$. The criterion (17) holds for all bipartite splittings of a state defined by the sets of indices $\{l\} \cup I$ and $\{m\} \cup I'$. For two mode states, the criterion becomes a necessary and sufficient entanglement test, however only after the state is transformed into its standard form by local operations \cite{19}. This local transformations, however, are determined by the form of the covariance matrix. In this sense, the knowledge of the full covariance matrix is essential in order to determine whether the state is entangled. Since in experiment we usually do not have access to this information, we cannot assume that the criterion decides unambiguously about separability.

4. Symplectic and Covariance matrices for bipartite and multipartite entanglement

Let us illustrate the versatility of the formalism in two distinct examples, the first one deals with entanglement of two atomic samples generated and manipulated by interaction and measurement of a light pulse. In the second one, with larger number of atomic samples, we want to show how a cluster state can be generated using the atom–light interface.

4.1. Bipartite entanglement for mixed states: Generation and verification

In the seminal work of Polzik and coworkers \cite{2}, the entanglement between two spatially separated atomic samples was generated in the experimental setup schematically shown in figure 11. In such setup, both light and atomic samples were strongly polarized along the $x$-direction while light propagated along the $z$-direction ($\alpha = 0$). The atomic ensembles were previously addressed with local magnetic fields oriented in opposite directions, enabling to use a single light beam for generation of EPR entanglement and another one for verification, however, between new time-integrated variables. It
is straightforward to generalize the equations of motion (5a)–(5d) to two samples interacting with light according to the Hamiltonian

\[
\hat{H}_{\text{int}}^{(2)} = -\frac{\kappa}{T} \hat{p}_L (\hat{p}_{A,1} + \hat{p}_{A,2}).
\]

(19)

For example, in this case the canonical ”momentum” variable for atoms, \(\hat{p}_{A,n} (n = 1, 2)\), is preserved [compare with equation (5b)], while the canonical ”position” of light after the interaction carries information about the sum of atomic momenta

\[
\hat{x}_{\text{out}}^L = \hat{x}_{\text{in}}^L - \kappa (\hat{p}_{A,1} + \hat{p}_{A,2}).
\]

(20)

Entanglement between the atomic samples is established as soon as the \(\hat{x}_{\text{out}}^L\) component of light is measured. Moreover, it should be emphasized that entanglement is generated independently of the outcome of the measurement, nevertheless the measurement result indicates the displacement of the state.

The setup described above needs some modifications if the individual magnetic field addressing is impossible. The way to overcome this problem was shown in [11], and is summarized in figures 2a and 2b.

Here we give a detailed description of this setup at the level of covariance matrix. Moreover, we further assume that the initial state of the two samples is not a minimum-fluctuation vacuum state as in [8], but a general thermal state. Under such assumptions, the initial state of the composite system is given by the following covariance matrix for atoms and light \(\gamma_{\text{in}} = n_1 1_2^A \oplus n_2 1_2^A \oplus 1_2^L\), where the identity \(1_2\) stands for a single mode and parameters \(n_1, n_2\) are related to temperature by \(n_i = 1/\tanh[\hbar \omega / (2k_B T_i)]\) \((i = 1, 2)\). The symplectic matrix, \(S_{\text{int}}\), describing the interaction of light passing through the samples at zero angle (figure 2a) is given by

\[
S_{\text{int}} = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & \kappa & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & \kappa & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
\kappa & 0 & \kappa & 0 & 0 & 1
\end{pmatrix},
\]

(21)
thus, the covariance matrix after the interaction takes the form expressed in equation (11)

\[
\gamma_{\text{out}} = \begin{pmatrix}
  n_1 + \kappa^2 & 0 & \kappa^2 & 0 & 0 & \kappa \\
  0 & n_1 & 0 & 0 & n_1\kappa & 0 \\
  \kappa^2 & 0 & n_2 + \kappa^2 & 0 & 0 & \kappa \\
  0 & 0 & 0 & n_2 & n_2\kappa & 0 \\
  0 & n_1\kappa & 0 & n_2\kappa & 1 + n_1\kappa^2 + n_2\kappa^2 & 0 \\
  \kappa & 0 & \kappa & 0 & 0 & 1
\end{pmatrix}.
\] (22)

Both modes, representing the samples, are entangled with light, however their reduced state is separable as one can check applying PPT criterion to the covariance matrix of the upper-left block matrix. Entanglement between atomic samples is not produced until one measures a quadrature of light. To demonstrate this, we apply a homodyne measurement of the light mode [see (14) and (15)]. Assuming the measurement outcome \( \tilde{x}_{L,1} \) obtaining the covariance matrix describing the final state of the samples

\[
\gamma_{\text{fin}} = \begin{pmatrix}
  n_1 + \kappa^2 & 0 & \kappa^2 & 0 & 0 & \kappa \\
  0 & \frac{n_1 n_2 \kappa^2 + n_1}{(n_1+n_2)\kappa^2 + 1} & 0 & \frac{n_1 n_2 \kappa^2}{(n_1+n_2)\kappa^2 + 1} & 0 & -\frac{n_1 n_2 \kappa^2}{(n_1+n_2)\kappa^2 + 1} \\
  \kappa^2 & 0 & n_2 + \kappa^2 & 0 & \frac{n_1 n_2 \kappa^2}{(n_1+n_2)\kappa^2 + 1} & 0 \\
  0 & \frac{n_1 n_2 \kappa^2}{(n_1+n_2)\kappa^2 + 1} & 0 & \frac{n_1 n_2 \kappa^2 + n_2}{(n_1+n_2)\kappa^2 + 1} & \frac{n_1 n_2 \kappa^2}{(n_1+n_2)\kappa^2 + 1} & 0 \\
  0 & \frac{n_1 n_2 \kappa^2}{(n_1+n_2)\kappa^2 + 1} & \frac{n_1 n_2 \kappa^2 + n_2}{(n_1+n_2)\kappa^2 + 1} & 0 & \frac{n_1 n_2 \kappa^2}{(n_1+n_2)\kappa^2 + 1} & 0 \\
  \kappa & 0 & \kappa & 0 & 0 & 1
\end{pmatrix},
\] (23)

and the displacement of the final state is

\[
d_{\text{fin}} = \left(0, \frac{\tilde{x}_{L,1}\kappa}{2\kappa^2 + 1}, 0, \frac{\tilde{x}_{L,1}\kappa}{2\kappa^2 + 1} \right).
\] (24)

Notice that the covariance matrix is independent of the measurement outcome, but the latter is present in the displacement vector.
In order to verify that the final state of atomic samples is entangled we use the separability criterion based on the variances of the two commuting variables\[^{19}\]

\[
\left[ \Delta(|\lambda|\hat{p}_{A,1} + \frac{1}{\lambda}\hat{p}_{A,2}) \right]^2 + \left[ \Delta(|\lambda|\hat{x}_{A,1} - \frac{1}{\lambda}\hat{x}_{A,2}) \right]^2 \geq 2\hbar.
\] (25)

We emphasize, however, that due to the fact that in the analyzed setup the measurement possibilities are limited, the only experimentally feasible case is $|\lambda| = 1$. The way to measure such combination of variances was described in detail in\[^{11}\], therefore we will not recall it here. Also the final state is not, in general, in its standard form, therefore, the measurement gives only a sufficient condition for separability.

In the following analysis we restrict to a single inequality involving the collective observables which are squeezed during entanglement generation, i.e. the one for $\lambda = 1$.

From the final covariance matrix\[^{23}\] one can directly compute the variances of the collective atomic spin:

\[
\frac{1}{\hbar} \left[ \Delta(\hat{p}_{A,1} + \hat{p}_{A,2}) \right]^2 = \frac{1}{2} \left( \gamma_{\text{fin},22} + \gamma_{\text{fin},44} + 2\gamma_{\text{fin},24} \right) = \frac{n_1 + n_2}{2(n_1 + n_2)\kappa^2 + 2},
\]

\[
\frac{1}{\hbar} \left[ \Delta(\hat{x}_{A,1} - \hat{x}_{A,2}) \right]^2 = \frac{1}{2} \left( \gamma_{\text{fin},11} + \gamma_{\text{fin},33} + 2\gamma_{\text{fin},13} \right) = \frac{1}{2} (n_1 + n_2).
\] (26)

Substituting the obtained variances into the separability criterion we obtain

\[
\left[ \Delta(\hat{p}_{A,1} + \hat{p}_{A,2}) \right]^2 + \left[ \Delta(\hat{x}_{A,1} - \hat{x}_{A,2}) \right]^2 = \frac{(n_1 + n_2)(n_1 + n_2)\kappa^2 + 2}{2(n_1 + n_2)\kappa^2 + 1},
\] (27)

which violates the bound $2\hbar$ for specific values of $n_1, n_2$, and $\kappa$, i.e. for those that fulfill

\[
\kappa > \frac{\sqrt{2}\sqrt{n_1 + n_2} - 2}{\sqrt{(-n_1 - n_2 + 4)(n_1 + n_2)}}.
\] (28)

The criterion\[^{25}\] for $\lambda = 1$ does not detect all entanglement. We compare the set that is not detected by the inequality to the set of separable states found using the PPT criterion. The respective ranges of parameters $\kappa, n_1, n_2$ are depicted in figures\[^{3}\] and\[^{30}\].

The comparison shows that to generate entanglement between thermal states, much stronger coupling is necessary. Also, the simple criterion based on the variances cannot detect entanglement for $n_1 + n_2 > 4$, even though it is clearly present in the system.

The procedure summarized in figure\[^{2}\] introduces more entanglement in the system squeezing the collective variable $\hat{x}_{A,1} - \hat{x}_{A,2}$. The computation is analogous, therefore will not be repeated. The resulting state is characterized by the following variances:

\[
\frac{1}{\hbar} \left[ \Delta(\hat{p}_{A,1} + \hat{p}_{A,2}) \right]^2 = \frac{1}{\hbar} \left[ \Delta(\hat{x}_{A,1} - \hat{x}_{A,2}) \right]^2 = \frac{n_1 + n_2}{2(n_1 + n_2)\kappa^2 + 2},
\] (29)

The condition for the inequality\[^{25}\] for $\lambda = 1$ to be violated is

\[
\kappa > \sqrt{\frac{n_1 + n_2 - 2}{2n_1 + 2n_2}}.
\] (30)
Figure 3. Comparison of the sets of parameters $\kappa, n_1, n_2$ for which the states are entangled [figures a) and c)], with the ranges of parameters for which they are detected by the spin variance inequality \((25)\) with $\lambda = 1$ [figures b) and d)]. In figures a) and b) we consider the state produced in the setup from figure 2a, whereas in figures c) and d) the one produced in two steps schematically depicted in figures 2a and 2b.

The set of parameters $\kappa, n_1, n_2$ for which the final state is detected by the inequality is depicted in figure 3c. For comparison the set of separable (equivalently PPT) is shown in figure 3d. Again the inequality does not detect all entangled states, however is more efficient than in the first case.

Interestingly enough, our geometrical approach, making the light impinging on each atomic sample at a given angle $\alpha_i$, also opens the possibility of deleting all the entanglement created by the first light beam, by interaction with a second light beam of an appropriate intensity. Notice that the entanglement procedure is intrinsically irreversible because it involves a projective measurement, so coming “deterministically” back to the initial state is not obvious. In [21, 22], a quantum erasing scheme in continuous-variable systems was proposed. The measurement of the meter coordinate entangled with the quantum system causes a back-action. The authors shown that
it is possible to erase the action of the measurement and restore the original state of the system. Here we are interested in deleting the measurement induced entanglement between two atomic samples, exploiting the squeezing and antisqueezing effects produced by the laser beams. Again, a CV formalism greatly simplifies the analysis and indicates in which way the atomic quadratures, squeezed and anti-squeezed by the first light beam, can be restored with the help of a second light pulse.

We begin with the state produced in the first part of the section, represented by the covariance matrix (23). We will perform this part of the analysis only for the initially vacuum state, i.e. \( n_1 = n_2 = 1 \). Evolution due to the interaction with the light beam impinging on each atomic sample at \( \alpha_1 = \alpha_2 = \pi/2 \), schematically represented in figure 2c is reproduced by the following symplectic matrix acting on \( \gamma_{\text{fin}} \oplus \mathbb{I}_2 \):

\[
S_{\text{eraser}} = \begin{pmatrix}
1 & 0 & 0 & 0 & \eta & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & \eta & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & -\eta & 0 & -\eta & 0 & 1
\end{pmatrix}.
\]

(31)

In general, the second light beam may have different properties, hence we use a different coupling constant \( \eta \). After the interaction and measurement of light the state of atomic ensembles is characterized by the following covariance matrix:

\[
\gamma_{2\text{nd}} = \begin{pmatrix}
\frac{\eta^2(2\kappa^2 + 1) + \kappa^2 + 1}{2\eta^2(2\kappa^2 + 1) + 1} & 0 & \frac{\kappa^2 - \eta^2(2\kappa^2 + 1)}{2\eta^2(2\kappa^2 + 1) + 1} & 0 \\
0 & \frac{\eta^2(2\kappa^2 + 1) + \kappa^2 + 1}{2\kappa^2 + 1} & \frac{\kappa^2 - \eta^2(2\kappa^2 + 1)}{2\kappa^2 + 1} & 0 \\
\frac{\kappa^2 - \eta^2(2\kappa^2 + 1)}{2\eta^2(2\kappa^2 + 1) + 1} & 0 & \frac{\eta^2(2\kappa^2 + 1) + \kappa^2 + 1}{2\kappa^2 + 1} & 0 \\
0 & \frac{\kappa^2 - \eta^2(2\kappa^2 + 1)}{2\kappa^2 + 1} & \frac{\eta^2(2\kappa^2 + 1) + \kappa^2 + 1}{2\kappa^2 + 1} & \frac{\kappa^2 - \eta^2(2\kappa^2 + 1)}{2\kappa^2 + 1}
\end{pmatrix}.
\]

(32)

For the specifically adjusted values of \( \kappa \) and \( \eta \), i.e. \( \eta^2 = \kappa^2/(1 + 2\kappa^2) \), the atomic ensembles come back to the initial coherent state, at least at the level of variances. Let us have a look at the displacement vector. Assuming that the outcome of the measurement leading to the generation of entangled state (23) is \( \tilde{x}_{L,1} \) (see equation (24)), and the outcome of the measurement erasing the correlations is \( \tilde{x}_{L,2} \), the total displacement is given by

\[
d_A = \begin{pmatrix}
\sqrt{\frac{\kappa^2}{2\kappa^2 + 1}} \tilde{x}_{L,2}; & \frac{\kappa}{2\kappa^2 + 1} \tilde{x}_{L,1}; & \sqrt{\frac{\kappa^2}{2\kappa^2 + 1}} \tilde{x}_{L,2}; & \frac{\kappa}{2\kappa^2 + 1} \tilde{x}_{L,1}
\end{pmatrix}.
\]

(33)

Hence the final state is equivalent to the initial one only up to the displacement vector.

4.2. Multipartite entanglement: Generation of cluster states using atom-light interfaces.

In [23] a class of \( N \)-qubit quantum states generated in two dimensional arrays of qubits using an Ising-type interaction was presented; these are the so-called cluster states.
Using their scalability properties, Briegel et al. proposed a scheme for a one-way quantum computation [24, 25]. There, two-dimensional cluster states are the entire resource, while computation consists of a sequence of local projective measurements. Furthermore, any two-dimensional cluster state provides a universal quantum computer since it has been proved that any unitary quantum logic network can be efficiently simulated within this scheme. In this sense cluster states, can be regarded as a resource able to generate any type of multi-qubit entanglement by means of two body interactions.

We present here, how the continuous-variable cluster-like states [26, 27] can be generated within the analyzed atom–light interface. We associate the modes of the \( N_s \)-mode system with the vertices of a graph \( G \). The edges between the vertices define the notion of nearest neighbourhood. By \( N_a \) we denote the set of nearest neighbours of vertex \( a \). A cluster is a connected graph. For angular momentum variables, cluster states are defined only asymptotically as those with infinite squeezing in the variables

\[
\hat{p}_a - \sum_{b \in N_a} \hat{x}_b, \quad a \in G.
\]

for all \( a \in G \). We talk about the clusterlike states when squeezing is finite.

In [11] we have shown how to generate a simple cluster-like state between four samples (figure 4a), however in a rotated frame. For each mode we introduce new variables:

\[
\hat{x}'_{A,n} = \frac{1}{\sqrt{2}}(\hat{x}_{A,n} - \hat{p}_{A,n}), \quad \hat{p}'_{A,n} = \frac{1}{\sqrt{2}}(\hat{x}_{A,n} + \hat{p}_{A,n}).
\]

The setup is summarized in figure 4.
The state is generated in four, commuting steps. We will denote an initial state of light and atoms at each step by \( \gamma_{\text{in}}^{(j)} \), a state after interaction by \( \gamma_{\text{out}}^{(j)} \), and the state after the measurement of light by \( \gamma_{\text{fin}}^{(j)} \), \( j = 1, \ldots, 4 \). Note that \( \gamma_{\text{fin}}^{(j)} = \gamma_{\text{fin}}^{(j)} \oplus 1_2 \).

We assume that initially both the atomic samples and light are in a vacuum state \( \gamma_{\text{in}}^{(1)} = 1_{8+2} \). The symplectic matrix (in a rotated frame) reproducing the effect of interaction depicted in figure 4b is

\[
S_I^{(1)} = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & \kappa & 0 \\
0 & 0 & 1 & 0 \\
\kappa & 0 & 0 & 1
\end{pmatrix},
\]

whereas the interaction from figure 4c is reproduced with symplectic matrix

\[
S_I^{(2)} = \begin{pmatrix}
1 & 0 & 0 & -\kappa \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
\kappa & 0 & 0 & 1
\end{pmatrix}.
\]

The other two symplectic matrices can be easily written down basing on \( S_I^{(1)} \) and \( S_I^{(2)} \).

The final state after the sequence of four atom-light interactions followed by the measurement of light is characterized by the covariance matrix

\[
\gamma_{\text{fin}}^{(4)} = \begin{pmatrix}
A & C & E & F \\
C & B & D & E \\
E & D & B & C \\
F & E & C & A
\end{pmatrix},
\]

where \( A, B, C, D, E, \) and \( F \) are the following \( 2 \times 2 \) matrices:

\[
A = \begin{pmatrix}
\frac{5\kappa^6+8\kappa^4+5\kappa^2+1}{5\kappa^4+5\kappa^2+1} & 0 \\
0 & \frac{5\kappa^6+7\kappa^4+5\kappa^2+1}{5\kappa^4+5\kappa^2+1}
\end{pmatrix}, \quad
D = \begin{pmatrix}
0 & \kappa^2(5\kappa^4+5\kappa^2+1) \\
\frac{5\kappa^6+6\kappa^2+2}{5\kappa^4+5\kappa^2+1} & 0
\end{pmatrix},
\]

\[
B = \begin{pmatrix}
\frac{5\kappa^6+7\kappa^4+4\kappa^2+1}{5\kappa^4+5\kappa^2+1} & 0 \\
0 & \frac{10\kappa^6+13\kappa^4+6\kappa^2+1}{5\kappa^4+5\kappa^2+1}
\end{pmatrix}, \quad
E = \begin{pmatrix}
0 & \frac{5\kappa^6+6\kappa^4+2}{5\kappa^4+5\kappa^2+1} \\
\frac{5\kappa^6+5\kappa^4+1}{5\kappa^4+5\kappa^2+1} & 0
\end{pmatrix},
\]

\[
C = \begin{pmatrix}
\kappa^2(5\kappa^4+7\kappa^2+2) & 0 \\
\frac{5\kappa^6+5\kappa^4+1}{5\kappa^4+5\kappa^2+1} & 0
\end{pmatrix}, \quad
F = \begin{pmatrix}
0 & \kappa^2 \frac{5\kappa^6+6\kappa^4+2}{5\kappa^4+5\kappa^2+1} \\
\frac{5\kappa^6+5\kappa^4+1}{5\kappa^4+5\kappa^2+1} & 0
\end{pmatrix}.
\]
In order to check the separability of the produced state, we apply the PPT criterion. The partial time reversal of covariance matrix \( (38) \) with respect to all cuts is negative. Therefore, the state is fully inseparable for all values of \( \kappa \).

Applying a less effective, but experimentally convenient separability test of the form \( (17) \) one detects full inseparability above some threshold value of \( \kappa \). The three inequalities which constitute a necessary separability criterion for this particular cluster state contain variances of the variables squeezed by the interaction and read \( [27] \):

\[
\Delta_1 \equiv [\Delta(\hat{p}^1_1 - \hat{x}^4_2)]^2 + [\Delta(\hat{p}^2_2 - \hat{x}^3_1 - \hat{x}^4_3)]^2 \geq 2\hbar
\]

\[
\Delta_2 \equiv [\Delta(\hat{p}^1_3 - \hat{x}^2_2 - \hat{x}^4_3)]^2 + [\Delta(\hat{p}^2_4 - \hat{x}^2_1 - \hat{x}^4_3)]^2 \geq 2\hbar
\]

\[
\Delta_3 \equiv [\Delta(\hat{p}^2_3 - \hat{x}^1_2)]^2 + [\Delta(\hat{p}^4_4 - \hat{x}^3_3)]^2 \geq 2\hbar.
\]

We determine the variances explicitly from the covariance matrix \( (38) \):

\[
\frac{1}{\hbar} \left[ \Delta(\hat{p}^{(1)} - \hat{x}^{(2)}) \right]^2 = \frac{1}{2} A_{22} + \frac{1}{2} B_{11} - C_{21} = \frac{5\kappa^2 + 2}{10\kappa^4 + 10\kappa^2 + 2},
\]

\[
\frac{1}{\hbar} \left[ \Delta(\hat{p}^{(2)} - \hat{x}^{(1)} - \hat{x}^{(3)}) \right]^2 = \frac{1}{2} B_{22} + \frac{1}{2} A_{11} + \frac{1}{2} B_{11} - C_{12} - D_{21} + E_{11} = \frac{5\kappa^2 + 3}{10\kappa^4 + 10\kappa^2 + 2},
\]

\[
\frac{1}{\hbar} \left[ \Delta(\hat{p}^{(3)} - \hat{x}^{(2)} - \hat{x}^{(4)}) \right]^2 = \frac{1}{2} B_{22} + \frac{1}{2} B_{11} + \frac{1}{2} A_{11} - D_{12} - C_{21} + E_{11} = \frac{5\kappa^2 + 3}{10\kappa^4 + 10\kappa^2 + 2},
\]

\[
\frac{1}{\hbar} \left[ \Delta(\hat{p}^{(4)} - \hat{x}^{(3)}) \right]^2 = \frac{1}{2} A_{22} + \frac{1}{2} B_{11} - C_{12} = \frac{5\kappa^2 + 2}{10\kappa^4 + 10\kappa^2 + 2}.
\]

The dependence of the left-hand sides of inequalities \( (40a) \) on the coupling constant is depicted in figure 5. The inequalities \( (40a) \) and \( (40c) \) are violated for \( \kappa \geq 0.3 \) and the inequality \( (40b) \) is violated for \( \kappa \geq 0.4 \).

5. Summary

We have addressed the problem of manipulation of entanglement between atomic ensembles using Faraday quantum atom–light interface with a continuous-variable formalism. Both the atomic ensembles and light can be prepared in initial Gaussian state and remain Gaussian after all the steps leading to generation of entanglement. This allows us to describe the whole process using the covariance matrix. After a general introduction of the methods we have applied it to derive bounds on the strength of the interaction leading to entanglement between atomic ensembles prepared initially...
in mixed states. Then we have shown how the CV formalism facilitates the analysis of more complex systems. In particular we have addressed the problem of generation of the CV cluster states.

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