On Gauss decomposition of quantum groups

and Jimbo homomorphism

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Abstract

It is shown that the properties of the Gauss decomposition of quantum groups
and the known Jimbo homomorphism permit us to realize these groups as subal-
gebras of well defined algebras constructed from generators of the corresponding
undeformed Lie algebras.

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1. The purpose of this Letter is to show that every quantum group \([\mathfrak{g}]\) from the Cartan list can be considered as a subalgebra of a tensor product of well defined algebras constructed from the corresponding classical Lie algebra. Such a consideration is based on the quantum algebra homomorphism constructed by Jimbo in his ground work \([2]\) and specific properties of the Gauss decomposition of the standard quantized groups of the series \(A_n, B_n, C_n, D_n\) \([5, 6]\). Recall some relevant definitions and results. Below we use the \(R-\)matrix approach to the theory of quantum groups and algebras \([3]\).

According to the \(FRT-\)approach the matrix equation

\[
RT_1T_2 = T_2T_1R
\]  

produces the homogeneous quadratic relations for \(n^2\) generators \(T = (t_{ij}), i, j = 1, \ldots, n\) of a unital associative algebra, a number \(n^2\) by \(n^2\) matrix \(R\) satisfies to the Yang-Baxter equation, \(T_1 = T \otimes I, \ T_2 = I \otimes T\), where \(I\) is a unit matrix. Really a number of independent generators is less than \(n^2\) because of the quantum determinant condition for the quantum groups of the series \(A_n\)

\[
det_q T = \sum_\sigma (-q)^{l(\sigma)} \prod_{i=1}^n T_{\sigma(i)} = 1
\]  

where the sum is over all the permutations \(\sigma\) of the set \((1, 2, \ldots, n)\), and \(l(\sigma)\) is length of \(\sigma\), and the additional condition for the orthogonal and symplectic quantum groups

\[
TCT^tC^{-1} = CT^tC^{-1}T = I
\]  

where \(T^t\) is the matrix transposed to \(T\), and \(C\) is a fixed number matrix \([3]\).

The quadratic relations for generators of the quantum algebra, dual to some quantum group, has the form

\[
R^{(+)}L^{(\pm)}_1L^{(\pm)}_2 = L^{(+)}_2L^{(\pm)}_1R^{(+)}
\]  

\[
R^{(+)}L^{(+)t}L^{(-)t}_2 = L^{(-)t}_2L^{(+)}_1R^{(+)}
\]  

where \(L^{\pm}\) are upper- and lower- triangular matrices of the quantum algebra generators, \(R^{(+)} = PRP\), \(P\) is the transposition operator with the properties \(P^2 = I, PA_1P = A_2\), for any matrix \(A\).

The Gauss decomposition for a quantum group is defined as a transition from the base \(T = (t_{ij})\) of the original generators to the new base \([5, 6]\)

\[
T = T_LT_DT_U
\]  

where \(T_L = (l_{ik})\) and \(T_U = (u_{ik})\) are respectively strictly lower- and upper- triangular matrices, and \(T_D = \text{diag}(d_{kk})\) is a diagonal matrix. Instead \((T_L, T_D, T_U)\) it is often useful
to deal with the base \((T^(-), T^+)\), where \(T^(-) = T_LT_D, T^+ = T_DT_U\). In the last case we identify the diagonal elements \(t_{ii}^(-) = t_{ii}^+\). Below we shall call the generators of the both bases by Gauss generators. The commutation relations for these generators are quadratic too

\[
RT_1^{(\pm)}T_2^{(\pm)} = T_2^{(\pm)}T_1^{(\pm)}R \tag{7}
\]
\[
RdT_1^{(+)}T_2^{(-)} = T_2^{(-)}T_1^{(+)}R_d \tag{8}
\]
\[
RdT_{D1}T_2^{(-)} = T_2^{(-)}T_{D1}R_d \tag{9}
\]
\[
RdT_1^{(+)}T_{D2} = T_{D2}T_1^{(+)}R_d \tag{10}
\]
\[
T_{L1}T_{U2} = T_{U2}T_{L1} \tag{11}
\]

where \(R_d\) is the diagonal part of the \(R\)–matrix. The complete set of the equations for Gauss generators see in \[6\]. Here we remark only that the requirement \[2\] for the quantum special linear groups takes the form \(\prod_{i=1}^n t_{ii}^{(\pm)} = 1\) and the condition \(\[3\]\) for orthogonal and symplectic groups is satisfied by \(T^{(\pm)}\). It is worth noting that the number of independent Gauss generators for each orthogonal or symplectic quantum group exactly corresponds to the number of generators of related classical Lie group. This is because the additional deformed relation entering in the quantum group (and algebra) definition \(\[3\]\) can be resolved explicitly in the Gauss base. Note, as well, that a triangulation procedure \[4\] can leads to a Gauss base with nonquadratic relations (for example, this is true for Jordanian quantum group \(GL_h(2)\) \[3\]).

When the main minors of a \(T\)–matrix are invertible (in particular, the element \(t_{11}\) is invertible) the set of the Gauss generators is equivalent, in algebraic structure, to the set of the original ones. Moreover, in this case we can, in principle, extend the standard comultiplication operation to the Gauss generators by the homomorphism property. Below, however, the problems connected with the Hopf algebra structure of quantum groups with the Gauss base are not considered.

In the work \[4\] the following form decomposition of quantum group generators were introduced

\[
T = M^{(-)}M^{(+)} \tag{12}
\]
\[
[M_1^{(-)}, M_2^{(+)}] = 0. \tag{13}
\]

If \(M^{(-)}, M^{(+)}\) fulfill the \(RTT\)–equations \(\[1\]\) separately then \(T = M^{(-)}M^{(+)}\) fulfills as well and, under evident suggestions \[4\], conversely. Note that above defined \(T^{(\pm)}\) are not isomorphic to FRT’s \(M^{(\pm)}\), but we can to achieve an isomorphism by setting formally \(M^{(-)} = T^{(-)}(\otimes)\mathbb{I}, M^{(+)} = \mathbb{I}(\otimes)T^{(+)}\). The symbol \((\otimes)\) means usual matrix multiplication.
with tensor product of matrix elements. Thus, we get an isomorphism \( M(-)M(+) \cong T(-)T(+) \).

2. The quadratic relations for generators of a quantum algebra \( H \) can be rewritten in the form

\[
R(L_1^{(\pm)})^{-1} (L_2^{(\pm)})^{-1} = (L_2^{(\pm)})^{-1} (L_1^{(\pm)})^{-1} R.
\]

Here we take into account the triangularity of the \( q \)-matrices \( L^{(\pm)} \) and their invertibility. The last is true because in the FRT’s definition of quantum algebras \[3\] there exists the additional requirement for diagonal elements

\[
\text{diag} L^{(\pm)} = 1 I.
\]

For that reason any element of \( (L^{(\pm)})^{-1} \) can be written as a polynomial in algebra generators \( l \) and inverse diagonal generators \( (l_{ii})^{-1} \).

As the additional relations \[3\] has the same form for both orthogonal and symplectic algebras \( L^{(\pm)} \) and quantum groups in the Gauss base \( T^{(\pm)} \), we get an algebra (but not a Hopf algebra) isomorphism

\[
T^{(\pm)} \cong (L^{(\pm)})^{-1}.
\]

Under this isomorphism to the elements \( t^{(\pm)}_{i,i+1}, t^{(-)}_{i+1,i} \) of Borel matrices \( T^{(\pm)} \) correspond the elements \( l^{(\pm)}_{i,i+1}, l^{(-)}_{i+1,i} \) of the matrices \( L^{(\pm)} \):

\[
\begin{align*}
t^{(\pm)}_{i,i+1} &\leftrightarrow - (l^{(\pm)}_{i,i})^{-1} l^{(\pm)}_{i,i+1} (l^{(\pm)}_{i+1,i+1})^{-1}; \\
t^{(-)}_{i+1,i} &\leftrightarrow - (l^{(-)}_{i+1,i+1})^{-1} l^{(-)}_{i+1,i} (l^{(-)}_{i,i})^{-1}.
\end{align*}
\]

(except the algebras of the series \( D_n \), see \[3\]). The elements \( l^{(\pm)}_{i,i+1}, l^{(-)}_{i+1,i} \) are associated with the simple roots of classical algebras through known identification \[3\]. As a consequence we obtain that the elements \( t^{(\pm)}_{i,i+1}, t^{(-)}_{i+1,i} \) satisfy the identities which slightly differ from deformed Serre identities \[1, 2\]. These identities together with the commutation relations between the generators \( t^{(\pm)}_{i,i+1}, t^{(-)}_{i+1,i} \) and the diagonal elements \( t^{(\pm)}_{ii}, t^{(-)}_{ii} \) produce the set of formulas which completely defines the quantum group in Gauss base.

3. Hereafter we restrict our consideration to the quantum groups \( SL_q(n) \), but all the formulas can be easily extended to the orthogonal and symplectic cases.

Let \((H_i, X_i^{(\pm)})\), \((i = 1, ..., n - 1)\) be the Chevalley base for the classical Lie algebra \( sl(n, C) \). The commutation relations of the Borel subalgebra \( b^\pm \) generators and the Serre identities are defined by the algebra simple positive roots \( \alpha_i \):
\[ [H_i, H_j] = 0, \quad [H_i, X_j^{(\pm)}] = \pm (\alpha_i, \alpha_j) X_j^{(\pm)}, \quad i, j = 1, \ldots, n, \quad (15) \]

\[ (ad X_j^{(\pm)})^{1-A_{ij}} (X_j^{(\pm)}) = 0, i \neq j, \quad (16) \]

where \( A \) is the Cartan matrix \( A_{ij} = 2(\alpha_i, \alpha_j)/(\alpha_j, \alpha_j) \).

Let us consider \( n \) linear dependent elements from the Cartan subalgebra of \( sl(n, C) \) defined as

\[ \tilde{H}_i = n^{-1} \left( \sum_{k=i}^{n-1} (n-k) H_k - \sum_{k=1}^{i-1} k H_k \right), \quad i = 1, \ldots, n \quad (17) \]

Such elements were introduced in [3] to establish the relation between the Drinfeld-Jimbo and \( R \)–matrix formulations of the quantum algebra theory. Note, however, that \( \tilde{H}_i \) were regarded as deformed objects in the cited paper.

Introduce the elements \( K_i = e^{\hbar \tilde{H}_i} \) where \( q = e^{\hbar} \) is a deformation parameter. In fact, \( K_i \) are the elements of the maximal commutative subgroup of the complex Lie group \( SL(n, C) \). Thus, we have well defined adjoint action of \( K_i \) on \( sl(n, C) \)

\[ AdK_i(X_j^{(\pm)}) = \begin{cases} 
q \pm 1 X_j^{(\pm)} & \text{if } j = i \\
q \mp 1 X_j^{(\pm)} & \text{if } j = i \mp 1 \\
X_j^{(-)} & \text{if } j \neq i, i \mp 1
\end{cases} \quad (18) \]

The last formulas can be easily proved by direct calculations using (15, 17).

Denote by \( U_i^+ \) an unital algebra generated by \( (K_i^{\pm 1}, X_i^{(+)}) \) and by \( U_i^- \) an unital algebra generated by \( (K_i^{\pm 1}, X_i^{(-)}) \),

\[ U^\pm = U_1^\pm \otimes U_2^\pm \otimes \ldots \otimes U_{n-1}^\pm. \]

In view of \( (13) \) \( U_i^\pm \) is a subalgebra of \( G_H \otimes U^\pm (X_i^\pm) \), where \( G_H \) is the maximal commutative subgroup of \( SL(n, C) \) and \( U^\pm (X_i^\pm) \) is the algebra generated by a single element. Consider, following Jimbo [3], the homomorphism \( \delta^{(\pm)} : U_i^\pm \rightarrow U^\pm \)

\[ \delta^{(\pm)}(K_i) = t_{ii}^{(\pm)} = (K_i^{\pm 1})^{\otimes(n-1)} \quad i = 1, \ldots, n \quad (19) \]

\[ \delta^{(+)}(X_i^{(+)}) = t_{ii+1}^{(+)} = f_i K_i^{\otimes(i-1)} \otimes X_i^{(+) \otimes K_i^{(n-i-1)}} \quad i = 1, \ldots, n-1 \quad (20) \]

\[ \delta^{(-)}(X_i^{(-)}) = t_{i+1,i}^{(-)} = g_i (K_i^{1-})^{\otimes(i-1)} \otimes X_i^{(- \otimes (K_i^{1-})^{\otimes(n-i-1)}} \quad i = 1, \ldots, n-1 \quad (21) \]
In the above formulas $K^\otimes n = K \otimes K \otimes \ldots \otimes K$, ($n$ tensor multipliers); $K^\otimes 1 = K$; $f_i, g_i$ are arbitrary constants. The homomorphism $\delta^{(\pm)}$ slightly differs from the Jimbo’s one \cite{2} in its form but we stress here again that it is a homomorphism of undeformed objects.

Consider the elements $t^{(+)}_{i,i+1}$ defined by (13). In view of (18), they satisfy the following relations

$$t^{(+)}_{i,i+1} t^{(+)}_{j,j+1} = \begin{cases} q^{(\pm)2} t^{(+)}_{j,j+1} t^{(+)}_{i,i+1} & \text{if } j = i \pm 1 \\ t^{(+)}_{i,i+1} t^{(+)}_{j,j+1} & \text{if } j \neq i \pm 1 \end{cases}$$

Put $X_{i}^{(+)} = t^{(+)}_{i,i+1}$. Then the deformed Serre identities, which have the form in our case

$$(X_{i}^{(+)})^2 X_{j}^{(+)} - q^{+1}(q + 1/q)X_{i}^{(+)}X_{j}^{(+)}X_{i}^{(+)} + q^{\pm 2}X_{j}^{(+)}(X_{i}^{(+)})^2 = 0$$

for $j = i \pm 1$, and the commutation relations

$$[X_{i}^{(+)}, X_{j}^{(+)}] = 0 \quad \text{for } j \neq i \pm 1$$

are satisfied by these $t^{(+)}_{i,i+1}$. Therefore, one can uniquely reconstruct the other elements of the $T^{(+)}$ by the formula

$$t^{(+)}_{i,i+k} = \lambda^{1-k} \left( \prod_{l=1}^{k-1} (t^{(+)}_{i+l,i+l})^{-1} \right) [t^{(+)}_{i+1,i+1}, [t^{(+)}_{i+1,i+2}, [\ldots [t^{(+)}_{i+k-2,i+k-1}, t^{(+)}_{i+k-1,i+k}] \ldots]] \right)$$

which easily can be verified by induction. Application of (22) to (19,20) yields

$$t^{(+)}_{i,k+1} = \prod_{j=i}^{k} f_j K_i^\otimes (i-1) \otimes \left( \bigotimes_{l=i}^{k} X_{i}^{(+)} \right) \otimes K_{k+1}^\otimes (n-i-1)$$

(23)

where $\bigotimes_{l=i}^{k} F_l = F_i \otimes F_{i+1} \otimes \ldots \otimes F_k$, $\bigotimes_{l=i}^{i} F_l = F_i$, $k \geq i$. It easy to verify that the constructed elements $T^{(+)} = (t^{(+)}_{i,j})$, $i \leq j$ satisfy $RTT$–equation \cite{1}.

The reconstruction of the elements of $T^{(-)}$ can be carried out in close analogy with the above $T^{(+)}$- case. Note, however, that $\delta^{(\pm)} = \kappa \circ \delta^{(-)}$ where $\kappa$ is the Cartan automorphism

$$H_i \rightarrow -H_i; \quad X_{i}^{(+)} \rightarrow X_{i}^{(-)}.$$

As a result we have immediately

$$t^{(-)}_{k+1,i} = \prod_{j=i}^{k} g_j (K_i^{-1})^\otimes (i-1) \left( \bigotimes_{l=i}^{k} X_{i}^{(-)} \right) \otimes (K_{k+1})^{-1}\otimes (n-i-1).$$

(24)

The above realization of $T^{(\pm)}$ elements permits us accomplish the final step which consists in constructing of the matrix $T = T^{(-)}(\otimes)T^{(+)}$. The latter means that the generators of the quantum group $SL_q(n)$ and all its elements belong to the algebra:
\[ \mathcal{U} = \mathcal{U}^- \otimes \mathcal{U}^+ = \bigotimes_{i=1}^{n-1} \mathcal{U}_i^- \otimes \bigotimes_{j=1}^{n-1} \mathcal{U}_j^+ \]  

(25)

4. To illustrate the above construction let us consider the quantum group \( SL_q(2) \) as an example. The Gauss decomposition for this group is of the form

\[
T = \begin{pmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ l & 1 \end{pmatrix} \begin{pmatrix} A & 0 \\ 0 & A^{-1} \end{pmatrix} \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} A & Au \\ lA & lAu + A^{-1} \end{pmatrix}.
\]

Commutation relations between the Gauss generators \( A, A^{-1}, l, u \) are very simple

\[
Au = quA, \; Al = qlA, \; uA^{-1} = qA^{-1}u, \; lA^{-1} = qA^{-1}l, \; [u, l] = 0.
\]

(26)

According to the above prescription, introduce the matrices \( T^{(\pm)} \):  

\[
T^{(-)} = \begin{pmatrix} A & 0 \\ lA & A^{-1} \end{pmatrix} = \begin{pmatrix} q^{-H/2} & 0 \\ gX^{(-)} & q^{H/2} \end{pmatrix};
\]

\[
T^{(+)} = \begin{pmatrix} A & Au \\ 0 & A^{-1} \end{pmatrix} = \begin{pmatrix} q^{H/2} & fX^{(+)} \\ 0 & q^{-H/2} \end{pmatrix}.
\]

Finally, we have

\[
T = T^{(-)} \otimes T^{(+)} = \begin{pmatrix} q^{-H/2} \otimes q^{H/2} \\ gX^{(-)} \otimes q^{H/2} \end{pmatrix} \begin{pmatrix} q^{-H/2} \otimes X^{(+)} \\ q^{H/2} \otimes q^{-H/2} + fgX^{(-)} \otimes X^{(+)} \end{pmatrix}.
\]

5. Let us make some remarks in conclusion.

- The above construction can be easily generalized to a multiparameter case. For example, for the quantum group \( GL_{p,q}(2) \), defined by the \( R \)-matrix

\[
R_{p,q} = \begin{pmatrix} k & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & k - pq/k & q & 0 \\ 0 & 0 & 0 & k \end{pmatrix},
\]

6
we have the following construction

\[ T = \left( \begin{array}{cc} \frac{k}{p} & 0 \\ c_-X(-) & \frac{k}{q} \\ \end{array} \right) \left( \begin{array}{cc} \frac{k}{q} & \frac{c_+X(+)}{2} \\ 0 & \frac{k}{p} \\ \end{array} \right) \] 

where \( c_{\pm} \) are arbitrary constants. The one-parameter case corresponds to the change \( k \rightarrow q^{1/2}, \ p \rightarrow q^{-1/2} \ q \rightarrow q^{-1/2} \).

- Using any matrix representation of the algebra \( sl(n, C) \), one can obtain a matrix representation for the corresponding quantum group. For instance, using the lowest dimensional representation of \( sl(2) \) by the matrices \( X^+ = \left( \begin{array}{cc} 0 & 1 \\ 0 & 0 \end{array} \right); \ X^- = \left( \begin{array}{cc} 1 & 0 \\ 0 & 0 \end{array} \right); \ H = \left( \begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array} \right) \), one has for \( SL_q(2) \) in view of (27)

\[
\begin{align*}
    t_{11} &= \left( \begin{array}{cccc}
        1 & 0 & 0 & 0 \\
        0 & q & 0 & 0 \\
        0 & 0 & q^{-1} & 0 \\
        0 & 0 & 0 & 1 \\
    \end{array} \right); \\
    t_{12} &= \left( \begin{array}{cccc}
        0 & 0 & 1 & 0 \\
        0 & 0 & 0 & q \\
        0 & 0 & 0 & 0 \\
        0 & 0 & 0 & 0 \\
    \end{array} \right); \\
    t_{21} &= \left( \begin{array}{cccc}
        0 & 0 & 0 & 0 \\
        1 & 0 & 0 & 0 \\
        0 & 0 & 0 & 0 \\
        0 & 0 & q^{-1} & 0 \\
    \end{array} \right); \\
    t_{22} &= \left( \begin{array}{cccc}
        1 & 0 & 0 & 0 \\
        0 & q^{-1} & 1 & 0 \\
        0 & 0 & q & 0 \\
        0 & 0 & 0 & 1 \\
    \end{array} \right).
\end{align*}
\]

- Generalizing the matrix representation of the above subsection we can formally associate with any representation of \( sl(n, C) \) a representation of the corresponding quantum group \( SL_q(n) \). This construction is under investigation.

- We can decrease a number of tensor multipliers in the considered construction if we use the dual algebra \( sl^*(n) \) rather than \( sl(n) \). In the case of \( sl^*(2) \) we have the commutation relations for its generators \( \tilde{H}, \tilde{X}^\pm \) :

\[
[\tilde{H}, \tilde{X}^\pm] = \tilde{X}^\pm, [\tilde{X}^+, \tilde{X}^-] = 0.
\]  

The last commutator suggests to use the Gauss decomposition in the form (1). As a result we have the realization

\[
T = \left( \begin{array}{cc}
    1 & 0 \\
    c_-\tilde{X}^- & 1 \\
\end{array} \right) \left( \begin{array}{cc}
    q^{\tilde{H}} & 0 \\
    0 & q^{-\tilde{H}} \\
\end{array} \right) \left( \begin{array}{cc}
    1 & c_-\tilde{X}^+ \\
    0 & 1 \\
\end{array} \right)
\]

where \( c_{\pm} \) are arbitrary constants.
• From the explicit form of the $SL_q(n)$ generators we can formally obtain a classical limit of $q$-matrix $T$ by direct differentiation of $t_{ij}$ in $h$ and setting $h = 0$. For $SL_q(2)$, choosing the parameters $f = q^{-1} \lambda$, $g = -q \lambda$ \[4\] we get

$$M = dT/dh(h = 0) = \begin{pmatrix} 1/2(1 \otimes H - H \otimes 1) & 2(1 \otimes X^{(+)})(1 - q) \\ -2(X^{(-)} \otimes 1) & -1/2(1 \otimes H - H \otimes 1) \end{pmatrix}.$$  

The elements of $M$-matrix satisfy the commutation relations \[27\].

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