Abstract. We review recent works that relate entanglement of random vectors to their localization properties. In particular, the linear entropy is related by a simple expression to the inverse participation ratio, while next orders of the entropy of entanglement contain information about e.g. the multifractal exponents. Numerical simulations show that these results can account for the entanglement present in wavefunctions of physical systems.

Key words: Quantum information; Entanglement; Random vectors; Localization; Multifractals

1. Introduction

Quantum mechanics has always seemed puzzling since its first construction in the first half of the twentieth century. Many properties are different from the world of classical physics in which our intuition is built. The development of quantum information science in the last decades has exemplified this aspect. Indeed, it was realized that it is in principle possible to exploit the features of quantum mechanics to treat information in a different way from what a classical computer would do. In this context, the specific properties of quantum mechanics are put forward as new resources which enable to treat information in completely new ways.

One of the most peculiar properties of quantum mechanics is entanglement, that is the possibility to construct quantum states of several subsystems that cannot be factorized into a product of individual states of each subsystem. Such entangled states are the most common in quantum mechanics, and they display correlations which cannot be seen in a classical world, exemplified by e.g. the Einstein–Podolsky–Rosen “paradox.” Entanglement is also a resource for quantum information (see Nielsen and Chuang 2000 and references therein), and has been widely studied as such in the past few years.

Despite intensive work, entanglement remains a somewhat mysterious property of physical systems. The structure of entanglement of systems even...
with small numbers of particles is hard to characterize. Even properly measuring the entanglement present in a system is difficult for mixed states. This is all the more important since recent results have shown that (at least for pure states) if a process creates a sufficiently low level of entanglement, it can be simulated efficiently by a classical computer (Jozsa and Linden 2003; Vidal 2003). This gives a limit on the speedup over classical computation a quantum computer can achieve, and also gives rise to interesting proposals for building classical algorithms simulating weakly entangled quantum systems (Verstraete et al. 2004).

In this paper, we review recent results we obtained (details can be found in Giraud et al. 2007, 2009), which concern the relationship of entanglement to localization properties of a quantum state. Our strategy is to consider $n$-qubit systems, and to study entanglement of quantum states relative to their localization properties in the $2^n$-dimensional Hilbert space in the computational basis. We obtain analytical results for random states, that is ensemble of quantum states sharing some properties. Such random states have been recently studied in the literature. They are interesting in themselves, since it has been shown for example in quantum information that they are useful in various quantum protocols (Harrow et al. 2004; Hayden et al. 2004; Bennett et al. 2005; Cappellaro et al. 2005). This motivated a recent activity in the quantum information community to try and produce efficiently such random vectors or random operators through quantum algorithms (Emerson et al. 2003; Weinstein and Hellberg 2005), and to characterize their entanglement properties (Scott 2004; Sommers and Zyczkowski 2004; Giraud 2007a, b; Zidaric 2007; Zidaric et al. 2007; Facchi et al. 2008). In addition to their intrinsic usefulness, random states are important since they can describe typical states of a “complex” system. For example, it has been known for some times now that random vectors built from Random Matrix Theory (RMT) can describe faithfully the properties of quantum Hamiltonian systems whose classical limit is chaotic, and more generally of many complex quantum systems (Giannoni et al. 1991). Such random vectors are ergodic, and the entanglement they contain has been calculated some time ago (Lubkin 1978; Page 1993). However, in many quantum systems, the wavefunctions are not ergodic but localized. This can correspond to electrons in a disordered potential, which are exponentially localized due to Anderson localization. It can also be seen in many-body interacting systems, where the presence of a moderate interaction can lead to states partially localized in energy. Some systems are in a well-defined sense neither ergodic neither localized: they correspond to e.g. states at the Anderson transition between localized and delocalized states, and can show multifractal properties (Mirlin 2000; Evers and Mirlin 2007).