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Large values of short character sums

Abstract. In this paper, we prove that for any $A > 0$ there exist infinitely many primes $p$ for which sums of the Legendre symbol modulo $p$ over an interval of length $(\ln p)^A$ can take large values.

§ 1. Introduction

The article [1] contains a variety of propositions on the values of character sums, including the following:

**Theorem 1.** Let $\chi$ be a nonprincipal character to the modulus $q$ and assume that the Riemann hypothesis for the function $L(s, \chi)$ holds. Then for $q \to \infty$ and $\frac{\ln x}{\ln \ln q} \to \infty$ we have

$$\sum_{n \leq x} \chi(n) = o(x)$$

Denote by \((\frac{a}{b})\) the Kronecker-Jacobi symbol and let $D$ be a fundamental discriminant, that is, $D \equiv 1 \pmod{4}$ and squarefree or $4 \mid D$ and $\frac{D}{4} \equiv 2$ or $3 \pmod{4}$ and is a squarefree number. Then the arithmetical function \((\frac{D}{n})\) is a primitive character to the modulus $|D|$ ([5], Theorem 9.13). Therefore, if the generalized Riemann hypothesis is true and $x(D)$ satisfies the condition

$$\frac{x(D)}{(\ln |D|)^A} \to +\infty$$

for any $A$, then the sum of \((\frac{D}{n})\) over the interval $[1, x(D)]$ is $o(x(D))$ as $|D| \to \infty$. In the opposite direction, the following theorem is proved:

**Theorem 2.** For all sufficiently large $q$ and any $A > 0$ there exists a fundamental discriminant $D$ with $q \leq |D| \leq 2q$ such that the inequality

$$\sum_{n \leq x} \left(\frac{D}{n}\right) \gg_A x$$

holds, where $x = (\frac{1}{3} \ln q)^A$.

In this paper we prove a strengthening of the Theorem 2, namely, the following fact:

**Theorem 3.** Let $A \geq 1$ be an arbitrarily large fixed number, $x \geq x_0(A)$ and $y = (\ln x)^A$. Then there exists a prime number $p$ with $x < p \leq 2x$ such that the inequality
\[ \sum_{n \leq y} \left( \frac{n}{p} \right) \gg_A y \]

holds.

**Remark 1.** Let \( A > 0 \) and \( x \) be large enough. Taking \( p \in (x, 2x] \) such that
\[ \sum_{n \leq (\ln x)^A} \left( \frac{n}{p} \right) \gg_A (\ln p)^A \]
and \( G = \mathbb{Z}/p\mathbb{Z}, X = \{1, 2, \ldots, z\}, Y = \{0, z, 2z, \ldots, z(z - 1)\} \subset G \), where \( z = \lfloor (\ln p)^A/2 \rfloor \) we get
\[ \sum_{\substack{x \in X \atop y \in Y}} \left( \frac{x + y}{p} \right) = \sum_{n \leq z^2} \left( \frac{n}{p} \right) \geq c_A \ln p)^A \sim c_A |X||Y| \]
for some positive constant \( c_A \). So, if \( A \) is the set of all quadratic residues in \( \mathbb{Z}/p\mathbb{Z} \), we have
\[ \sum_{\substack{x \in X \atop y \in Y}} \chi_A(x + y) = \frac{1}{2} |X||Y| + \frac{1}{2} \sum_{\substack{x \in X \atop y \in Y}} \left( \frac{x + y}{p} \right) \geq \frac{1 + c_A + o(1)}{2} |X||Y|, \]
where \( \chi_A(\cdot) \) is the characteristic function of the set \( A \). This inequality, together with the Theorem 2 of the paper [4] proves that the set of quadratic residues modulo \( p \) does not behave like a random subset of \( \mathbb{Z}/p\mathbb{Z} \) for infinitely many \( p \).

**§ 2. Lemmas and proof of main theorem**

To prove the Theorem 3, we need some auxiliary lemmas.

**Lemma 1.** For some constants \( c \) and \( c_1 > 0 \) and any nonprincipal character \( \chi \) to the modulus \( q \leq e^{c_1 \sqrt{\ln x}} \) we have
\[ \sum_{p \leq x} \chi(p) \ln p = -\delta \frac{x^\beta}{\beta} + O(xe^{-c_1 \sqrt{\ln x}}), \]
where \( \delta = 1 \) if \( L(s, \chi) \) has a real zero \( \beta \) with \( \beta > 1 - \frac{c}{\ln q} \) and \( \delta = 0 \) otherwise.

**Proof.** See [3], Chapter IX, proof of the Theorem 6.

**Lemma 2.** Let \( \chi_1 \) and \( \chi_2 \) be two different primitive real characters to the moduli \( q_1 \) and \( q_2 \), and suppose that the functions \( L(s, \chi_1) \) and \( L(s, \chi_2) \) have real zeros \( \beta_1 \) and \( \beta_2 \). Then the inequality
\[ \min\{\beta_1, \beta_2\} < 1 - \frac{c_2}{\ln(q_1q_2)} \]
holds for some positive absolute constant \( c_2 \).

**Proof.** See [3], Chapter IX, Theorem 4.
Lemma 3. Let $\chi$ be a real primitive character to the modulus $q$. Then for some absolute constant $c_3 > 0$ we have

$$L(\sigma, \chi) \neq 0 \text{ for } \sigma > 1 - \frac{c_3}{\sqrt{q} \ln^4 q}.$$ 

Proof. See [3], Chapter IX, Theorem 3.

Lemma 4. Let $a \neq 0$ be an integer with $a = bc^2$, where $b$ is squarefree and $c$ is a positive integer. Then there exists a primitive quadratic character $\chi_b$ to the modulus $|b^*|$ such that for any odd prime $p$ the equality

$$\left( \frac{a}{p} \right) = \chi_{0,c}(p) \chi_b(p)$$

holds, where

$$b^* = \begin{cases} 
    b, & \text{if } b \equiv 1 \pmod{4} \\
    4b, & \text{if } b \not\equiv 1 \pmod{4}
\end{cases}$$

and $\chi_{0,c}$ is the principal character to the modulus $c$.

Proof of Lemma 4. Using the multiplicativity of Legendre symbol, we obtain

$$\left( \frac{a}{p} \right) = \left( \frac{b}{p} \right) \left( \frac{c^2}{p} \right).$$

If $p | c$, then $\left( \frac{c^2}{p} \right) = 0$. On the other hand, if $c$ and $p$ are coprime, then $c^2$ is a quadratic residue modulo $p$, so $\left( \frac{c^2}{p} \right) = 1$. Consequently, the equality

$$\left( \frac{c^2}{p} \right) = \chi_{0,c}(p)$$

holds.

Let us now prove that $\left( \frac{b}{p} \right) = \chi_b(p)$ for some primitive character $\chi_b$ to the modulus $|b^*|$. Note that $b^*$ is a fundamental discriminant. Indeed, if $b \equiv 1 \pmod{4}$ then $b^* = b$ is a fundamental discriminant. If, however, $b \not\equiv 1 \pmod{4}$, then due to the fact that $b$ is squarefree we have $b \equiv 2$ or $3 \pmod{4}$, thus $b^* = 4b$ is also a fundamental discriminant. Hence, $\chi_b(n) = \left( \frac{b^*}{n} \right)$ is a primitive quadratic character to the modulus $|b^*|$ (see [5], Theorem 9.13). On the other hand, $\frac{b^*}{n}$ can take only two values: 1 and 4. Therefore, for any odd prime number $p$ we have

$$\chi_b(p) = \left( \frac{b^*}{p} \right) = \left( \frac{b^*/b}{p} \right) \left( \frac{b}{p} \right) = \left( \frac{b}{p} \right),$$

which concludes the proof.

Lemma 5. Let $A \geq 1$, $x \geq x_0(A)$, $M = (\ln x)^{1/3}$, $N = M^{3A} = (\ln x)^A$ and $P(M) = \prod_{q \leq M} q$, where $q$ runs over prime numbers. For any squarefree $c \leq e^{2M}$ we define
Then we have \( r_{A,x}(c) \ll \frac{N \ln N}{M} \) if \( c \nmid P(M) \) and \( r_{A,x}(c) = r_{A,x}(1) \gg_A N \) if \( c \mid P(M) \).

**Proof of Lemma 5.** If \( c \mid P \), then \( c = sr \), where \( s \mid P \) and all the prime factors of \( r \) are greater than \( M \).

Starting with the pair \( (a, d) \) which satisfies the conditions \( ad = cy^2 \), \( a \leq N \) and \( d \mid P(M) \), we construct the integer triple \( (d_1, s_1, z) \) in the following way:

\[
d_1 = \frac{d}{(d,s)}, s_1 = \frac{s}{(s,d)}, z = \frac{y}{d_1}.
\]

Let us show that \( z \) is integer. Indeed, we have \( ad = cy^2 \), therefore \( ad_1 = s_1 ry^2 \), but \( (d_1, s_1 r) = 1 \) as all the prime factors of \( r \) are greater than \( M \) and \( d_1 \) and \( s_1 \) are coprime by definition. Hence, \( y^2 \) is divisible by \( d_1 \). But \( d_1 \) is squarefree, so \( y \) is also divisible by \( d_1 \), consequently \( z \) is integer. On the other hand, \( d_1 s_1 z^2 \leq \frac{N}{r} < \frac{N}{M} \). Indeed,

\[
d_1 s_1 z^2 = \frac{s_1 y^2}{d_1} = \frac{sy^2}{a} = \frac{cy^2}{dr} = \frac{a}{r} \leq \frac{N}{r}.
\]

Furthermore, the pair \( (a, d) \) can be recovered from the triple \( (s_1, d_1, z) \), due to the fact that

\[
a = rs_1 d_1 z^2 \quad \text{and} \quad d = \frac{sd_1}{s_1}.
\]

Note now that for any \( n \leq \frac{N}{M} \) the number of triples \( (s_1, d_1, z) \) such that \( s_1 \) and \( d_1 \) are squarefree and coprime and \( n = s_1 d_1 z^2 \) does not exceed the number of divisors of \( n \). Indeed, from these conditions we deduce that \( s_1 d_1 \) is a squarefree number, so it is equal to the squarefree part of \( n \). Hence, \( z \) is uniquely defined and the number of possible pairs \( (s_1, d_1) \) is equal to the number of divisors of the squarefree part of \( n \). Therefore, \( r_{A,x}(c) \) does not exceed the number of triples satisfying these conditions, which is less than or equal to

\[
\sum_{n \leq \frac{N}{M}} d(n) \ll \frac{N \ln N}{M},
\]

which completes the proof in this case.

In the case when \( c \mid P(M) \), with any pair \( (a, d) \) satisfying the conditions \( 1 \leq a \leq N \), \( d \mid P(M) \), \( ad = cy^2 \) for some \( y \in \mathbb{Z} \) we associate the pair \( \left( a, \frac{cd}{(c,d)^2} \right) \). It is easy to see that \( \frac{cd}{(c,d)^2} \mid P(M) \) and

\[
\frac{acd}{(c,d)^2} = \frac{a^2 y^2}{(c,d)^2} = z^2,
\]

where \( z = \frac{cy}{(c,d)} \) is integer. Conversely, any pair \( (a, d) \) with \( 1 \leq a \leq N \), \( d \mid P(M) \), \( ad = z^2 \) for some \( z \in \mathbb{Z} \) corresponds to the pair \( \left( a, \frac{cd}{(c,d)^2} \right) \). Then \( \frac{cd}{(c,d)^2} \mid P(M) \) and
$$\frac{acd}{(c,d)^2} = \frac{cz^2}{(c,d)^2} = cy^2$$

for some integer $y$, as $z$ is divisible by $d$. These maps are mutually inverse, due to the fact that for $D = \frac{cd}{(c,d)^2}$ we have $(c,D) = \frac{c}{(c,d)}$ and so $\frac{cD}{(c,D)^2} = d$. Therefore $r_{A,x}(1) = r_{A,x}(c)$. Now note that for any $M$–smooth number $a \leq N$ there exists $d \mid P(M)$ with $ad = y^2$ (it suffices to choose the squarefree part of $a$ instead of $d$). Consequently,

$$r_{A,x}(1) \geq \Psi(N, M) = (\rho(3A) + o(1))N \geq \exp(-3A(\ln(3A) + \ln \ln A))N$$

for all large enough $A$, where $\rho$ is the Dickman function (see [2], pp 4-5), which completes the proof of lemma.

**Lemma 6.** Assume that $x > e^4$ and let $\beta$ be a real number with $0 < 1 - \beta \leq \frac{2}{\ln x}$. Then we have

$$x - \frac{(2x)^\beta}{\beta} + \frac{x^\beta}{\beta} \geq \frac{(1 - \beta)x \ln x}{4e^2}.$$ 

**Proof of Lemma 6.** As $\beta < 1$, we have $(2x)^\beta < 2x^\beta$. Thus,

$$x - \frac{(2x)^\beta}{\beta} + \frac{x^\beta}{\beta} > x - \frac{x^\beta}{\beta}.$$ 

On the other hand,

$$x - \frac{x^\beta}{\beta} = \int_\beta^1 \frac{x^s}{s^2}(s \ln x - 1)ds.$$ 

For $\beta \leq s \leq 1$ the inequalities

$$\frac{x^s}{s^2} \geq x^\beta \geq x^{1 - 2/\ln x} = \frac{x}{e^2}$$

hold and

$$s \ln x - 1 \geq \beta \ln x - 1 \geq \ln x \left(1 - \frac{2}{\ln x}\right) - 1 = \ln x - 3 \geq \frac{\ln x}{4},$$

consequently,

$$x - \frac{x^\beta}{\beta} \geq \int_\beta^1 \frac{x \ln x}{4e^2}ds = \frac{(1 - \beta)x \ln x}{4e^2}.$$ 

The lemma is proved.

**Remark 2.** For $1 - \frac{2}{\ln x} \geq \beta \geq \frac{1}{2}$ we have

$$\frac{(2x)^\beta}{\beta} \leq 4x^\beta \leq \frac{4}{e^2}x,$$

hence
\[ x - \frac{(2x)^\beta}{\beta} \geq x \left( 1 - \frac{4}{e^2} \right) \geq \frac{x}{e^2}. \]

Therefore, for all \( \frac{1}{2} \leq \beta < 1 \) and \( x > e^4 \) the inequality
\[ x - \frac{(2x)^\beta}{\beta} + \frac{x^{\beta}}{\beta} \geq \frac{(1-\beta)x}{e^2} \]
holds.

Let us now prove the Theorem 3.

**Proof of Theorem 3.** Fix some \( A \geq 1 \), choose large enough \( x \) and, following the notation of Lemma 5, take \( M = (\ln x)^{1/3} \), \( N = M^{3A} = (\ln x)^A \) and \( P(M) = \prod_{q \leq M} q \). Let us introduce the following notations
\[ w_p(M) = \prod_{q \leq M} \left( 1 + \left( \frac{q}{p} \right) \right) \]
and
\[ S(p, N) = \sum_{n \leq N} \left( \frac{n}{p} \right). \]

It is easy to see that \( w_p(M) \geq 0 \) for any prime \( p \). To prove the Theorem 3 it suffices to show that there exists at least one \( p \in (x, 2x] \) such that \( S(p, N) \gg_A N \).

Consider the following sums:
\[ S_0(x) = \sum_{x < p \leq 2x} w_p(M) \ln p \]
and
\[ S_1(x, A) = \sum_{x < p \leq 2x} w_p(M) S(p, N) \ln p. \]

By the nonnegativity of \( w_p(M) \), it is enough to show that the sum \( S_0(x) \) is positive and
\[ S_1(x, A) \gg_A N S_0(x). \]

We will prove the positivity of \( S_0(x) \) first.

Expanding the brackets in the definition of \( w_p(M) \), we obtain
\[ w_p(M) = 1 + \sum_{d \mid P(M), d > 1} \left( \frac{d}{p} \right). \]

Therefore,
\[ S_0(x) = \sum_{x < p \leq 2x} \ln p + \sum_{d \mid P(M)} \sum_{x < p \leq 2x} \left( \frac{d}{p} \right) \ln p. \]
Let us choose a positive $c_4$ such that $c_4 < 0.1 \min \{c, c_1, c_2, c_3\}$, where $c, c_1, c_2, c_3$ are the absolute constants from lemmas 1, 2 and 3. Then, by the lemmas 1 and 4, for any squarefree integer $1 < f \leq e^{2M}$ there exists a primitive real character $\chi_f$ to the modulus $f^*$ such that

$$\sum_{x < p \leq 2x} \left( \frac{f}{p} \right) \ln p = - \delta_f \frac{(2x)^\beta - x^\beta}{\beta} + O(xe^{-c_4 \sqrt{\ln x}}),$$

(2.1)

where $\delta_f = 1$ if the function $L(s, \chi_f)$ has a real zero $\beta$ with $\beta > 1 - \frac{c_4}{\ln f}$ and $\delta_f = 0$ otherwise.

On the other hand, due to the Lemma 2 there exists at most one squarefree $f \leq e^{2M}$ such that $\delta_f = 1$. In the case when such a number exists, we denote it by $g$ and the corresponding real zero by $\beta_g$. Using lemma 3, for large enough $x$ we obtain the inequality

$$\beta_g \leq 1 - \frac{c_4}{\sqrt{g \ln^2 g}} \leq 1 - \frac{1}{g} \leq 1 - e^{-2M}.$$

Using the equality (2.1) and relations $P(M) \leq e^{2M} \ll e^{0.4c_4 \sqrt{\ln x}}$, we get the following expression for $S_0(x)$:

$$S_0(x) = x - \delta_1 \frac{(2x)^\beta_g - x^\beta_g}{\beta_g} + O(xe^{-0.5c_4 \sqrt{\ln x}}),$$

where $\delta_1 = 1$, if $g$ exists and divides $P(M)$ and $\delta_1 = 0$ otherwise. Due to the inequality $1 - \beta_g \geq 1 - e^{-2M}$ and Lemma 6, we obtain

$$x - \delta_1 \frac{(2x)^\beta_g - x^\beta_g}{\beta_g} \gg x(1 - \beta_g) \gg xe^{-2M} \gg xe^{-0.4c_4 \sqrt{\ln x}}.$$

Consequently,

$$S_0(x) \sim x - \delta_1 \frac{(2x)^\beta_g - x^\beta_g}{\beta_g},$$

and, in particular, $S_0(x) > 0$. Let us now estimate $S_1(A, x)$. Multiplying the expressions for $w_p(M)$ and $S(p, N)$, we get

$$w_p(M)S_p(N) = \sum_{d | P(M)} \sum_{n \leq N} \left( \frac{dn}{p} \right).$$

If $d \mid P(M)$ and $n \leq N$, then $dn \leq e^{2M}$. If $b$ is a squarefree part of $dn$, then for any prime $x < p \leq 2x$ we have $\left( \frac{dn}{p} \right) = \left( \frac{b}{p} \right)$. In view of this, for any prime $x < p \leq 2x$ the equality

$$w_p(M)S_p(N) = \sum_{b \leq e^{2M}} r_{A,x}(b) \left( \frac{b}{p} \right)$$

holds. Taking the sum of this expression over all prime numbers from the interval $(x, 2x]$, we get
\[ S_1(A, x) = \sum_{b \leq e^{2M}} r_{A, x}(b) \sum_{p < x \leq 2x} \left( \frac{b}{p} \right) \ln p = \]

\[ = r_{A, x}(1) \sum_{x < p \leq 2x} \left( \frac{1}{p} \right) + \delta_2 r_{A, x}(g) \sum_{x < p \leq 2x} \left( \frac{g}{p} \right) + O \left( \sum_{b \not\equiv 1 \mod{q}, b \leq e^{2M}} r_{A, x}(b) e^{-c_4 \sqrt{\ln x}} \right) = \]

\[ = x r_{A, x}(1) - \delta_2 \frac{(2x)^{\beta_g} - x^{\beta_g}}{\beta_g} r_{A, x}(g) + O(x e^{-0.5c_4 \sqrt{\ln x}}), \]

where \( \delta_2 = 1 \), if \( g \) exists and \( \delta_2 = 0 \) otherwise.

If \( g \) doesn’t exist, then \( \delta_1 = \delta_2 = 0 \), so

\[ S_1(A, x) = x r_{A, x}(1) + O(x e^{-0.5c_4 \sqrt{\ln x}}) \gg_A N x \]

by the Lemma 5. But we also have \( S_0(A, x) \asymp x \) and so

\[ \frac{S_1(A, x)}{S_0(x)} \gg_A N, \]

which concludes the proof in this case.

If \( g \) exists and does not divide \( P(M) \), then \( \delta_2 = 1 \) and \( \delta_1 = 0 \). Therefore,

\[ S_0(x) \asymp x \]

and

\[ S_1(A, x) = x r_{A, x}(1) - \frac{(2x)^{\beta_g} - x^{\beta_g}}{\beta_g} r_{A, x}(g) + O(x e^{-0.5c_4 \sqrt{\ln x}}). \]

By the Lemma 5, \( r_{A, x}(1) \gg_A N \) and \( r_{A, x}(g) \ll \frac{N \ln N}{M} \), hence

\[ S_1(A, x) = r_{A, x}(1) x \left( 1 + O \left( \frac{\ln N}{M} \right) \right). \]

Consequently,

\[ \frac{S_1(A, x)}{S_0(x)} \gg_A N. \]

It remains to consider the case when \( g \) exists and divides \( P(M) \). Then we have \( \delta_1 = \delta_2 = 1 \), thus

\[ S_0(x) \asymp x - \frac{(2x)^{\beta_g} - x^{\beta_g}}{\beta_g} \]

and

\[ S_1(A, x) = x r_{A, x}(1) - \frac{(2x)^{\beta_g} - x^{\beta_g}}{\beta_g} r_{A, x}(g) + O(x e^{-0.5c_4 \sqrt{\ln x}}). \]

Due to the Lemma 5, \( r_{A, x}(1) = r_{A, x}(g) \gg_A N \). Using Lemma 6, we get
\[
\frac{S_1(A, x)}{S_0(x)} = r_{A,x}(1) + O\left(\frac{e^{-0.5c_4\sqrt{\ln x}}}{1 - \beta_g}\right) = (\rho(3A) + o(1)) N + O\left(e^{-0.1c_4\sqrt{\ln x}}\right) \gg A N,
\]

which completes the proof of our theorem.

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