Dirichlet boundary value problems for uniformly elliptic equations in modified local generalized Sobolev–Morrey spaces

Vagif S. Guliyev\textsuperscript{1,2}, Tahir S. Gadjiev\textsuperscript{1} and Shahla Galandarova\textsuperscript{1}

\textsuperscript{1}Institute of Mathematics and Mechanics of NAS of Azerbaijan, AZ1141 Baku, Azerbaijan
\textsuperscript{2}S. M. Nikolskii Institute of Mathematics at RUDN University, Moscow, Russia, 117198

Received 3 August 2017, appeared XX October 2017
Communicated by Maria Alessandra Ragusa

Abstract. In this paper, we study the boundedness of the sublinear operators, generated by Calderón–Zygmund operators in local generalized Morrey spaces. By using these results we prove the solvability of the Dirichlet boundary value problem for a polyharmonic equation in modified local generalized Sobolev–Morrey spaces. We obtain a priori estimates for the solutions of the Dirichlet boundary value problems for the uniformly elliptic equations in modified local generalized Sobolev–Morrey spaces defined on bounded smooth domains.

Keywords: solvability of Dirichlet problem, modified local generalized Sobolev–Morrey spaces, a priori estimates, uniformly elliptic equations.

2010 Mathematics Subject Classification: 35J30, 35J40, 42B20, 42B25.

1 Introduction

The classical Morrey spaces $L_{p,\lambda}$ are originally introduced in order to study the local behavior of solutions to elliptic partial differential equations. In fact, the better inclusion between the Morrey and Hölder spaces permits to obtain regularity of the solution to elliptic boundary value problems. For the properties and applications of the classical Morrey spaces we refer the readers to [30,34].

In [8] Chiarenza and Frasca showed boundedness of the Hardy–Littlewood maximal operator in $L_{p,\lambda}(\mathbb{R}^n)$ that allows them to prove continuity in these spaces of some classical integral operators. The results in [8] allow us to study the regularity of the solutions of elliptic/parabolic equations and systems in $L_{p,\lambda}$ (see [9, 11, 12, 33, 35–37] and the references therein). In [31] Mizuhara extended the Morrey’s concept of integral average over a ball with a certain growth, taking a weight function $\varphi(x,r) : \mathbb{R}^n \times \mathbb{R}_+ \rightarrow \mathbb{R}_+$ instead of $r^\lambda$. So he put the beginning of the study of the generalized Morrey spaces $M_{p,\varphi}$, $p > 1$ with $\varphi$ belonging to various classes of weight functions. In [32] Nakai proved boundedness of the maximal and

\textsuperscript{2}Corresponding author. Email: vagif@guliyev.com
Calderón–Zygmund operators in $M_{p, \varphi}$ imposing suitable integral and doubling conditions on $\varphi$. Taking a weight $w(x, t) = \varphi(x, t)^p t^\mu$ the conditions of Mizuhara–Nakai become
\[
\int_r^\infty \varphi(x, \tau) \frac{d\tau}{\tau} \leq C \varphi(x, r)^p, \quad C^{-1} \leq \frac{\varphi(x, t)}{\varphi(x, r)} \leq C, \quad \forall r \leq t \leq 2r,
\]
where the constants do not depend on $t$, $r$ and $x \in \mathbb{R}^n$.

In series of works, the first author studies the continuity in generalized Morrey spaces of sublinear operators generated by various integral operators as Calderón–Zygmund, Riesz and others (see [4,21,23]). The following theorem obtained in [21,23] extends the results of Nakai to the generalized Morrey spaces with weight $w(x, t) = \varphi(x, t)t^\mu$ (for the definition of the spaces see Section 2).

**Theorem A** ([23, Theorem 6.2]). Let $1 \leq p < \infty$ and $(\varphi_1, \varphi_2)$ satisfy the condition
\[
\int_r^\infty \varphi_1(x, \tau) \frac{d\tau}{\tau} \leq C \varphi_2(x, r), \quad (1.1)
\]
where $C$ does not depend on $x$ and $r$. Then the Calderón–Zygmund operators are bound from $M_{p, \varphi_1}(\mathbb{R}^n)$ to $M_{p, \varphi_2}(\mathbb{R}^n)$ for $p > 1$ and from $M_{1, \varphi_1}(\mathbb{R}^n)$ to the weak space $WM_{p, \varphi_2}(\mathbb{R}^n)$.

This result is extended on spaces with weaker condition on the weight pair $(\varphi_1, \varphi_2)$ (see [4]). A further development of the generalized Morrey spaces can be found in the works [4,24] and the references therein. In [4,24], Guliyev et al. obtained a weaker than (1.1) condition on the pair $(\varphi_1, \varphi_2)$ which is optimal and ensure the boundedness of the classical integral operators from $M_{p, \varphi_1}(\mathbb{R}^n)$ to $M_{p, \varphi_2}(\mathbb{R}^n)$. Precisely, if
\[
\int_r^\infty \text{ess sup}_{t+s \leq \infty} \varphi_1(x, s) s^\frac{n}{r} dt \leq C \varphi_2(x, r), \quad (1.2)
\]
then the Calderón–Zygmund operators are bound from $M_{p, \varphi_1}(\mathbb{R}^n)$ to $M_{p, \varphi_2}(\mathbb{R}^n)$ for $p > 1$ and from $M_{1, \varphi_1}(\mathbb{R}^n)$ to the weak space $WM_{p, \varphi_2}(\mathbb{R}^n)$.

We use this integral inequality to obtain the Calderón–Zygmund type estimate for the $M_{p, \varphi}$-regularity of the solution. These results allow us to study the regularity of the solutions of various linear elliptic and parabolic boundary value problems in $M_{p, \varphi}$ (see [27,28,38]).

Later these results are extended on the local generalized Morrey spaces, which is obtained the boundedness of the Calderón–Zygmund operators from one local generalized Morrey space $LM_{p, \varphi_1}^{x_0}(\mathbb{R}^n)$ to another $LM_{p, \varphi_2}^{x_0}(\mathbb{R}^n)$, $x_0 \in \mathbb{R}^n$ (see [25,26]), if the pair functions $(\varphi_1, \varphi_2)$ satisfy the following condition
\[
\int_r^\infty \text{ess sup}_{t+s \leq \infty} \varphi_1(x_0, s) s^\frac{n}{r} dt \leq C \varphi_2(x_0, r), \quad (1.3)
\]
where $C$ does not depend on $r$.

In this paper we study the boundedness of the sublinear operators, generated by Calderón–Zygmund operators in local generalized Morrey spaces. By using these results we obtain the regularity of the solutions of higher order uniformly elliptic boundary value problem in modified local generalized Sobolev–Morrey spaces defined on bounded smooth domains.

The paper is organized as follows. In Section 2 we give some definitions and some estimates of the Green function and the Poisson kernels. In Section 3 we prove the boundedness of the sublinear operators, generated by Calderón–Zygmund operators in the local generalized...
Morrey spaces. Further, we obtain the regularity estimates for the solvability of the Dirichlet boundary value problem for polyharmonic equation in modified local generalized Sobolev–Morrey spaces. In Section 4 we prove a priori estimates for the solutions of the Dirichlet boundary value problems for the uniformly elliptic equations in modified local generalized Sobolev–Morrey spaces defined on bounded smooth domains.

By $A \lesssim B$ we mean that $A \leq CB$ with some positive constant $C$ independent of appropriate quantities. If $A \lesssim B$ and $B \lesssim A$, we write $A \approx B$ and say that $A$ and $B$ are equivalent.

2 Definitions and statement of the problem

**Definition 2.1.** Let $\varphi : \Omega \times \mathbb{R}^+ \to \mathbb{R}^+$ be a measurable function and $1 \leq p < \infty$. For any domain $\Omega$ the generalized Morrey space $M_{p,\varphi}(\Omega)$ (the weak generalized Morrey space $WM_{p,\varphi}(\Omega)$) consists of all $f \in L_p^\text{loc}(\Omega)$ such that

$$
\|f\|_{M_{p,\varphi}(\Omega)} = \sup_{x \in \Omega, 0 < r < d} \frac{1}{\varphi(x,r)} \frac{1}{|B(x,r)|^{1/p}} \|f\|_{L_p(\Omega(x,r))} < \infty,
$$

$$
\left( \|f\|_{WM_{p,\varphi}(\Omega)} = \sup_{x \in \Omega, 0 < r < d} \frac{1}{\varphi(x,r)} \frac{1}{|B(x,r)|^{1/p}} \|f\|_{W^{1,p}_\text{loc}(\Omega(x,r))} < \infty \right)
$$

where $d = \sup_{x,y \in \Omega} |x - y|$, $B(x,r) = \{y \in \mathbb{R}^n : |x - y| < r \}$ and $\Omega(x,r) = \Omega \cap B(x,r)$.

In the case $\varphi(x,r) = r^{\lambda/n}$, $M_{p,\varphi} = L_{p,\lambda}$, where $0 < \lambda < n$. If $\lambda = 0$, then $L_{p,0}(\mathbb{R}^n) = L_p(\mathbb{R}^n)$, if $\lambda = n$, then $L_{p,n}(\mathbb{R}^n) = L_\infty(\mathbb{R}^n)$. In the case $\lambda < 0$ or $\lambda > n$, $L_{p,\lambda}(\mathbb{R}^n) = \Theta$, where $\Theta$ is the set of all functions equivalent to 0 on $\mathbb{R}^n$.

**Definition 2.2.** Let $\varphi(x,r)$ be a positive measurable function on $\Omega \times (0,d)$ and $1 \leq p < \infty$. Fixed $x_0 \in \Omega$, we denote by $LM_{p,\varphi}^{\{x_0\}}(\Omega)$ ($WLM_{p,\varphi}^{\{x_0\}}(\Omega)$) the local generalized Morrey space (the weak local generalized Morrey space), the space of all functions $f \in L_p^\text{loc}(\Omega)$ with finite quasinorm

$$
\|f\|_{LM_{p,\varphi}^{\{x_0\}}(\Omega)} = \sup_{0 < r < d} \frac{1}{\varphi(x_0,r)} \frac{1}{|B(x_0,r)|^{1/p}} \|f\|_{L_p(\Omega(x_0,r))},
$$

$$
\left( \|f\|_{WLM_{p,\varphi}^{\{x_0\}}(\Omega)} = \sup_{0 < r < d} \frac{1}{\varphi(x_0,r)} \frac{1}{|B(x_0,r)|^{1/p}} \|f\|_{W^{1,p}_\text{loc}(\Omega(x_0,r))} \right)
$$

**Definition 2.3.** Let $\varphi(x,r)$ be a positive measurable function on $\Omega \times (0,d)$ and $1 \leq p < \infty$. We denote by $\tilde{M}_{p,\varphi}(\Omega)$ ($\tilde{M}_{p,\varphi}(\Omega)$) the modified generalized Morrey space (the modified weak generalized Morrey space), the space of all functions $f \in L_p(\Omega)$ with finite norm

$$
\|f\|_{\tilde{M}_{p,\varphi}(\Omega)} = \|f\|_{M_{p,\varphi}(\Omega)} + \|f\|_{L_p(\Omega)}
$$

$$
\left( \|f\|_{W\tilde{M}_{p,\varphi}(\Omega)} = \|f\|_{WM_{p,\varphi}(\Omega)} + \|f\|_{W_p(\Omega)} \right).
$$

**Definition 2.4.** Let $\varphi(x,r)$ be a positive measurable function on $\Omega \times (0,d)$ and $1 \leq p < \infty$. Fixed $x_0 \in \Omega$, we denote by $\tilde{LM}_{p,\varphi}^{\{x_0\}}(\Omega)$ ($\tilde{LM}_{p,\varphi}^{\{x_0\}}(\Omega)$) the modified local generalized Morrey space (the modified weak local generalized Morrey space), the space of all functions $f \in L_p(\Omega)$
with finite norm
\[
\|f\|_{L^{p,q}_{m}(\Omega)} = \|f\|_{LM^{\{x_0\}}_{p,q}^{(m)}(\Omega)} + \|f\|_{L^{p}(\Omega)}
\]
\[
\|f\|_{W^{2m}_{p,q}(\Omega)} = \|f\|_{ WM^{\{x_0\}}_{p,q}^{(2m)}(\Omega)} + \|f\|_{W^{2}_{p}(\Omega)}.
\]

Definition 2.5. The modified generalized Sobolev–Morrey space \(W_{p,q}^{2m}(\Omega)\) consist of all functions \(u \in W_{p}^{2m}(\Omega)\) with distributional derivatives \(D^s u \in LM_{p,q}^{\{x_0\}}(\Omega)\), \(0 \leq |s| \leq 2m\), endowed with the norm
\[
\|u\|_{W_{p,q}^{2m}(\Omega)} = \sum_{0 \leq |s| \leq 2m} \|D^s u\|_{LM_{p,q}^{\{x_0\}}(\Omega)}.
\]

The modified local generalized Sobolev–Morrey space \(W_{p,q}^{2m,\{x_0\}}(\Omega)\) consist of all functions \(u \in W_{p}^{2m}(\Omega)\) with distributional derivatives \(D^s u \in \tilde{LM}^{\{x_0\}}_{p,q}(\Omega)\), \(0 \leq |s| \leq 2m\), endowed with the norm
\[
\|u\|_{W_{p,q}^{2m,\{x_0\}}(\Omega)} = \sum_{0 \leq |s| \leq 2m} \|D^s u\|_{LM_{p,q}^{\{x_0\}}(\Omega)}.
\]

The space \(W_{p,q}^{2m,\{x_0\}}(\Omega) \cap \tilde{W}^{1}_{p}(\Omega)\) consists of all functions \(u \in \tilde{W}^{1}_{p}(\Omega)\) with \(D^s u \in LM_{p,q}^{\{x_0\}}(\Omega)\), \(0 \leq |s| \leq 2m\) and is endowed by the same norm. Recall that \(\tilde{W}^{1}_{p}(\Omega)\) is the closure of \(C_0^\infty(\Omega)\) with respect to the norm in \(W_{1,p}(\Omega)\).

At first we consider the Dirichlet boundary value problem for polyharmonic equation
\[
\begin{aligned}
(-\Delta)^m u &= f & \text{in } \Omega, \\
u &= \frac{\partial u}{\partial n} = \cdots = \frac{\partial^{m-1} u}{\partial n^{m-1}} = g & \text{on } \partial \Omega,
\end{aligned}
\tag{2.1}
\]
where \(\Omega \subset \mathbb{R}^n\), \(n \geq 2\) is a bounded domain with sufficiently smooth boundary.

For the solutions of the problem (2.1) we give some estimates for the Green function and the Poisson kernels. Later we obtain a priori estimates for solvability of problem (2.1) in the local generalized Morrey spaces.

Let \(G_m(x,y)\) be the Green function and \(K_j(x,y)\), \(j = 0, m - 1\) the Poisson kernels of problem (2.1). Then the solution of problem (2.1) can be written as
\[
u(x) = \int_{\Omega} G_m(x,y)f(y)dy + \sum_{j=0}^{m-1} \int_{\partial \Omega} K_j(x,y)g(y)d\sigma_y
\]
for correspondingly \(f\) and \(g\). For example, when \(m = 2\) and \(n = 2\) we will used that there is a constant \(C(\Omega)\) such that
\[
|G_2(x,y)| \leq C(\Omega)d(x)d(y) \min \left\{ 1, \frac{d(x)d(y)}{|x-y|^2} \right\},
\tag{2.2}
\]
which was proved in [29], where \(d\) is the distance of \(x\) to the boundary \(\partial \Omega\)
\[
d(x) = \inf_{\bar{x} \in \partial \Omega} |x - \bar{x}|.
\tag{2.3}
\]

However, we would like to mention that for \(G_m\) and \(K_j\) estimates are the optimal tools for deriving regularity results in spaces that involve to behavior at the boundary. Coming back
to the $m = n = 2$ it follows from (2.2) that the solution of problem (2.1) satisfies the following estimates for appropriate $f$ at $g = 0$

$$
\|ud^{-2}\|_{L^\infty(\Omega)} \leq C(\Omega) \|f\|_{L^1(\Omega)},
$$
$$
\|u\|_{L^\infty(\Omega)} \leq C(\Omega) \|f d^2\|_{L^1(\Omega)}.
$$

We also derive estimates for derivative of kernels. We will focus on estimate that contain growth rates near the boundary. These estimates are optimal. Indeed, when we consider $G_m(x,y)$ for $\Omega = B(x,r)$ a ball in $\mathbb{R}^n$ the growth rates near the boundary are sharp (see [18]). For $m = 1$ or $m \geq 2$ and $\Omega = B(x,r)$ it is known that the Green function is positive and can even be estimated from below by a positive function with the same singular behavior (see [19]). Let us remind that for $m \geq 2$ the Green function in general is not positive. For general domains the optimal behavior in absolute values is captured in our estimates. Sharp estimates for $K_{m-1}$ and $K_{m-2}$ in the case of a ball can be found in [20]. In [5] Barbatis considered the pointwise estimates for the Green function of higher order parabolic problems on domains and derived pointwise estimates for the kernel. For higher order parabolic systems the classical estimates obtained by Eidelman [17] were not considered in domains with boundary. For a survey on spectral theory of higher order elliptic operators, including some estimates for the corresponding kernels, we refer to [14].

Let $G$ a function on $\Omega \times \Omega$ and $\alpha, \beta \in \mathbb{N}^n$. Derivatives of $G$ are denoted by

$$
D_\alpha^\beta D_\alpha^\beta f(x,y) = \frac{\partial|\alpha|}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}} \frac{\partial|\beta|}{\partial y_1^{\beta_1} \cdots \partial y_n^{\beta_n}} G(x,y),
$$

where $|\alpha| = \sum_{k=1}^n \alpha_k$, $|\beta| = \sum_{k=1}^n \beta_k$.

For completeness we will give some estimates for $G_m(x,y)$ and $K_j(x,y)$ depending on the distance to the boundary and auxiliary results with proof. We will do by estimating the $j$-th derivative through an integration of the $(j+1)$-th derivative along a path to the boundary. The dependence on the distance to the boundary $d(x)$ will appear closing a path which length is proportional to $d(x)$. The path will be constructed in Lemma 2.10.

**Theorem 2.6 ([15, 29]).** Let $G_m(x,y)$ be the Green function of problem (2.1). Then for every $x, y \in \Omega$ the following estimates hold:

1. if $2m - n > 0$, then

$$
|G_m(x,y)| \leq d^{m-\frac{2}{n}}(x) \cdot d^{m-\frac{2}{n}}(y) \min \left\{ 1, \frac{d(x) d(y)}{|x - y|^2} \right\}^\frac{2}{n};
$$

2. if $2m - n = 0$, then

$$
|G_m(x,y)| \leq \log \left( 1 + \min \left\{ 1, \frac{d(x) d(y)}{|x - y|^2} \right\} \right) ;
$$

3. if $2m - n < 0$, then

$$
|G_m(x,y)| \leq |x - y|^{2m-n} \min \left\{ 1, \frac{d(x) d(y)}{|x - y|^2} \right\}^m.
$$

**Theorem 2.7 ([15, 29]).** Let $K_j(x,y)$, $j = 0, m - 1$ be the Poisson kernels of problem (2.1). Then for every $x \in \Omega, y \in \partial \Omega$ the following estimates hold:

$$
|K_j(x,y)| \leq \frac{d^m(x)}{|x - y|^{n - j + m - 1}}.
$$
Remark 2.8. If \( n - 1 < j \leq m - 1 \), then from (2.4) we get the inequality
\[
|K_j(x, y)| \leq d^{1+j-n}(x)
\] (2.5)
on \( \Omega \times \partial \Omega \).

Remark 2.9. The estimates in Theorem 2.7 hold for any uniformly elliptic operator of order \( 2m \).

In [19] the estimates in Theorem 2.6 are given for the case that \( \Omega = B(x, r) \) in \( \mathbb{R}^n \). In there the authors use an explicit formula for the Green’s function, given in [6].

For general domains one cannot expect an explicit formula for the Green’s functions and the Poisson kernels. We will use the estimates for \( G_m(x, y) \) and \( K_j(x, y) \) given in [29]. In [29] for sufficiently regular domains \( \Omega \) some estimates for the Green’s function and Poisson kernels was proved.

The following lemma is valid.

Lemma 2.10. Let \( x \in \Omega \) and \( y \in \overline{\Omega} \). There exists a curve \( \gamma^y_x : [0, 1] \rightarrow \Omega \) with \( \gamma^y_x(0) = x, \gamma^y_x(1) \in \partial \Omega \) and
1. \( |\gamma^y_x(t) - y| \geq \frac{1}{2}|x - y| \) for every \( t \in [0, 1] \), \( \quad \) (2.6)
2. \( l \leq (1 + \pi)d(x), \quad \) where \( l \) is the length of \( \gamma^y_x \). \( \quad \) (2.7)
Moreover, if \( \tilde{\gamma}^y_x : [0, l] \rightarrow \overline{\Omega} \) is the parametrization by arc length of \( \gamma^y_x \), then the following inequalities hold
3. \( \frac{1}{5}s \leq |x - \tilde{\gamma}^y_x(s)| \leq s \) for \( s \in [0, l] \). \( \quad \) (2.8)

We proceed with the proof of Theorem 2.6 and start from the estimates in [29] of the \( m \)-th derivative of \( G_m(x, y) \).

Integrating this function along the path \( \gamma^y_x \) of Lemma 2.10. We find the estimates of the \((m-1)\)-th derivative of \( G_m(x, y) \) in terms of the distance to the boundary. Iterating the procedure \( m \) times we find the results as stated in Theorem 2.6.

We use some auxiliary results which can easy obtain from [29]. From these results we get the following theorem.

Theorem 2.11 ([15, 29]). Let \( G_m(x, y) \) be the Green’s function of problem (2.1), \( k \in \mathbb{N}^n \). Then for every \( x, y \in \Omega \), the following estimates hold.

1. For \( |k| \geq m \): if \( 2m - n - |k| < 0 \), then
\[
\left|D^k_x G_m(x, y)\right| \leq C|x - y|^{2m-n-|k|} \min \left\{ 1, \frac{d(y)}{|x - y|} \right\}^m;
\]
if \( 2m - n - |k| = 0 \), then
\[
\left|D^k_x G_m(x, y)\right| \leq C \log \left( 1 + \frac{d^m(y)}{|x - y|^m} \right) \approx \log \left( 2 + \frac{d(y)}{|x - y|} \right) \min \left\{ 1, \frac{d(y)}{|x - y|} \right\}^m;
\] (2.9)
if \( 2m - n - |k| > 0 \), then
\[
\left|D^k_x G_m(x, y)\right| \leq C d^{2m-n-|k|}(y) \min \left\{ 1, \frac{d(y)}{|x - y|} \right\}^{n+|k|-m}.
\]
2. For $|k| < m$: if $2m - n - |k| < 0$, then

$$
|D^k_m G_m(x, y)| \leq C |x - y|^{2m - n - |k|} \min \left\{ 1, \frac{d(x)}{|x - y|} \right\}^{m - |k|} \min \left\{ 1, \frac{d(y)}{|x - y|} \right\}^m.
$$

If $2m - n - |k| = 0$, then

$$
|D^k_m G_m(x, y)| \leq C \log \left( 1 + \frac{d_m(y)d_{m-|k|}(x)}{|x - y|^{2m - |k|}} \right)
\approx \log \left( 2 + \frac{d(y)}{|x - y|} \right) \min \left\{ 1, \frac{d(y)}{|x - y|} \right\}^m \min \left\{ 1, \frac{d(x)}{|x - y|} \right\}^{m - |k|}.
\tag{2.10}
$$

If $2m - n - |k| > 0$, and moreover

a) $m - \frac{n}{2} \leq |k|$, then

$$
|D^k_m G_m(x, y)| \leq C d^{2m - n - |k|}(y) \min \left\{ 1, \frac{d(x)}{|x - y|} \right\}^{m - |k|} \min \left\{ 1, \frac{d(y)}{|x - y|} \right\}^{n - m + |k|};
$$

b) $|k| < m - \frac{n}{2}$, then

$$
|D^k_m G_m(x, y)| \leq C d(y)^{m - \frac{n}{2}} d^{2m - n - |k|}(x) \min \left\{ 1, \frac{d(x)d(y)}{|x - y|^2} \right\}^\frac{n}{2}.
$$

Proof. Let $x, y \in \Omega$. We use the estimates derivatives of $G_m(x, y)$ from [29]. The estimates for the lower order derivatives of $G_m(x, y)$ will be obtained by integrating the higher order derivatives along the path $\gamma^y_z$ from Lemma 2.10. This lemma corresponds to one of the integration steps. For example, with $\alpha, \beta \in \mathbb{N}^n$ and if $\bar{x} \in \partial\Omega$ the endpoint of $\gamma^y_z$, then we find

$$
D^\alpha_x D^\beta_y G_m(x, y) = D^\alpha_x D^\beta_y G_m(\bar{x}, y) + \int_{\gamma^y_z} \nabla_z D^\alpha_x D^\beta_y G_m(z, y) dz.
\tag{2.11}
$$

If $|\alpha| \leq m - 1$, then the first term on the right hand side of (2.11) equals to zero and we get

$$
|D^\alpha_x D^\beta_y G_m(x, y)| \leq \int_0^1 \left| \nabla_x D^\alpha_x D^\beta_y G_m(\gamma^y_z(s), y) \right| ds.
\tag{2.12}
$$

If $|\beta| \leq m - 1$, then similarly by integrating with respect to $y$ we find

$$
|D^\alpha_x D^\beta_y G_m(x, y)| \leq \int_0^1 \left| \nabla_y D^\beta_y D^\alpha_x G_m(x, \gamma^y_z(s)) \right| ds.
\tag{2.13}
$$

We distinguish the cases as in the statement of the theorem.

Case 1. Let $|k| = r \geq m$ and $\beta \in \mathbb{N}^n$ with $|\beta| = m - 1$. Then from $k = \alpha$ and using the estimates from [29], we get

$$
|D^\alpha_x D^\beta_y G_m(x, y)| \leq |x - y|^{m - n - r}.
$$

Case 2. Let $|k| = r < m$. Also we using the estimates for $|D^\beta_y D^\alpha_x D^k_m G_m(x, y)|$ from [29] and then integrates $m$ times with respect of $y$ and $m - r$ times with respect to $x$.

Thus the theorem is proved. \qed
The proof of Theorem 2.7. The method of proof is similar to the one used in Theorem 2.6. A difference is that in this case there is no symmetry between $x$ and $y$. The following lemma, that corresponds to one integration step is as follows.

**Lemma 2.12.** Let $\nu, k \in \mathbb{N}$ with $k \geq 2$. If

$$|\nabla_{x}H(x, y)| \lesssim |x - y|^{-k}d^{\nu}(x)$$

for $x \in \Omega$, $y \in \partial\Omega$ and $H(\bar{x}, y) = 0$ for every $\bar{x} \in \partial\Omega$ with $\bar{x} \neq y$, then the following inequality holds

$$|H(x, y)| \lesssim |x - y|^{-k}d^{\nu+1}(x)$$

for $x \in \Omega$, $y \in \partial\Omega$.

If we use previous auxiliary results, then we can easily prove Lemma 2.12.

The Lemma 2.12 allow us to prove the following theorem for which Theorem 2.7 is a special case.

**Theorem 2.13.** ([15, 29]) Let $K_{j}(x, y)$, $j = 0, m - 1$ be the Poisson kernels of problem (2.1) and $\alpha \in \mathbb{N}^{n}$ with $|\alpha| \leq m - 1$. Then the following estimate

$$|D_{x}^{\alpha}K_{j}(x, y)| \lesssim \frac{d^{m-|\alpha|}(x)}{|x - y|^{n-j+m-1}}$$

holds for $x \in \Omega$ and $y \in \partial\Omega$.

**Remark 2.14.** The estimates of $D_{x}^{\alpha}K_{j}(x, y)$ for $|\alpha| \geq m$ can be found from [29]. Following estimate is valid

$$|D_{x}^{\alpha}K_{j}(x, y)| \lesssim |x - y|^{-n+j-|\alpha|+1}.$$

3 Sublinear operators, generated by Calderón–Zygmund operators in local generalized Morrey spaces

Let $\Omega$ be an open bounded subset of $\mathbb{R}^{n}$. Suppose that $T$ represents a linear or a sublinear operator, which satisfies that for any $f \in L_{1}(\Omega)$

$$|Tf(x)| \leq c_{0} \int_{\Omega} \frac{|f(y)|dy}{|x - y|^{n}}, \quad x \notin \text{supp}(f),$$

where $c_{0}$ is independent of $f$ and $x$.

The following local estimates for the sublinear operator satisfying condition (3.1) are valid.

**Lemma 3.1.** Let $1 \leq p < \infty$, $\Omega$ be an open bounded subset of $\mathbb{R}^{n}$, $x_{0} \in \Omega$, $0 < r \leq d$, $d = \sup_{x, y \in \Omega} |x - y| < \infty$. Let also $T$ be a sublinear operator satisfying condition (3.1), and bounded from $L_{p}(\Omega)$ to $WL_{p}(\Omega)$, and bounded on $L_{p}(\Omega)$ for $p > 1$.

(i) Then the inequality

$$\|Tf\|_{WL_{p}(\Omega(x_{0}, r))} \lesssim r^{\frac{n}{p-1}} \int_{r}^{d} t^{\frac{n}{p}-1} \|f\|_{L_{p}(\Omega(x_{0}, t))} dt + r^{\frac{n}{p}} \|f\|_{L_{p}(\Omega)}$$

holds for any $\Omega(x_{0}, r)$ and for any $f \in L_{p}(\Omega)$.  

Electronic copy available at: https://ssrn.com/abstract=3298593
(ii) Moreover, for \( p > 1 \) the inequality
\[
\| T f \|_{L_p(\Omega(x_0,r))} \lesssim r^{\frac{n}{p}} \int_r^d t^{-\frac{n}{p}-1} \| f \|_{L_p(\Omega(x_0,t))} \, dt + r^n \| f \|_{L_p(\Omega)}
\] (3.3)
holds for any \( \Omega(x_0,r) \) and for any \( f \in L_p(\Omega) \).

Proof. Let \( 1 \leq p < \infty \). Since
\[
\int_r^d t^{-\frac{n}{p}-1} \| f \|_{L_p(\Omega(x_0,t))} \, dt \geq r^{\frac{n}{p}} \int_r^d t^{-\frac{n}{p}-1} \, dt
\] \( \approx \| f \|_{L_p(\Omega(x_0,r))} (d^{\frac{n}{p}} - r^{\frac{n}{p}}) \), \( r \in (0,d) \),
we get that
\[
\| f \|_{L_p(\Omega(x_0,r))} \lesssim r^{\frac{n}{p}} \int_r^d t^{-\frac{n}{p}-1} \| f \|_{L_p(\Omega(x_0,t))} \, dt + r^n \| f \|_{L_p(\Omega)}, \quad r \in (0,d).
\] (3.4)

(i). Assume that \( 1 \leq p < \infty \). Let \( r \in (0,d/2) \). We write \( f = f_1 + f_2 \) with \( f_1 = f \chi_{\Omega(x_0,2r)} \) and \( f_2 = f \chi_{\Omega \setminus (x_0,2r)} \). Taking into account the linearity of \( T \), we have
\[
\| T f \|_{WL_p(\Omega(x_0,r))} \leq \| T f_1 \|_{WL_p(\Omega(x_0,r))} + \| T f_2 \|_{WL_p(\Omega(x_0,r))}.
\] (3.5)

Since \( f_1 \in L_p(\Omega) \), in view of (3.4), the boundedness of \( T \) from \( L_p(\Omega) \) to \( WL_p(\Omega) \) implies that
\[
\| T f_1 \|_{WL_p(\Omega(x_0,r))} \leq \| T f_1 \|_{WL_p(\Omega)} \lesssim \| f_1 \|_{L_p(\Omega)} \approx \| f \|_{L_p(\Omega(x_0,2r))}
\] \( \lesssim r^{\frac{n}{p}} \int_r^d t^{-\frac{n}{p}-1} \| f \|_{L_p(\Omega(x_0,t))} \, dt + r^n \| f \|_{L_p(\Omega)}, \) (3.6)

where the constant is independent of \( f, x_0 \) and \( r \).

We have
\[
| T f_2(x) | \leq \int_{\Omega \setminus (x_0,2r)} \frac{|f(y)| \, dy}{|x - y|^{n-1}}, \quad x \in \Omega(x_0,r).
\]
It’s clear that \( x \in \Omega(x_0,r), y \in \Omega \setminus (x_0,2r) \) implies \( (1/2)|x_0 - y| \leq |x - y| < (3/2)|x_0 - y| \). Therefore we obtain that
\[
\| T f_2 \|_{L_p(\Omega(x_0,r))} \approx r^{\frac{n}{p}} \int_{\Omega \setminus (x_0,2r)} \frac{|f(y)| \, dy}{|x_0 - y|^{n-1}}.
\]

By Fubini’s theorem, we get that
\[
\int_{\Omega \setminus (x_0,2r)} \frac{|f(y)| \, dy}{|x_0 - y|^{n-1}} \approx \int_{\Omega \setminus (x_0,2r)} |f(y)| \left( 1 + \int_{|x_0 - y|}^{\infty} \frac{ds}{s^{n-1}} \right) \, dy
\] \( = \int_{\Omega \setminus (x_0,2r)} |f(y)| \, dy + \int_{\Omega \setminus (x_0,2r)} |f(y)| \left( \int_{|x_0 - y|}^{\infty} \frac{ds}{s^{n-1}} \right) \, dy
\] \( = \int_{\Omega \setminus (x_0,2r)} |f(y)| \, dy + \int_{2r}^{d} \left( \int_{|x_0 - y|}^{s} |f(y)| \, dy \right) \frac{ds}{s^{n-1}}
\] \( \leq \int_{\Omega \setminus (x_0,2r)} |f(y)| \, dy + \int_{2r}^{d} \left( \int_{\Omega(x_0,s)} |f(y)| \, dy \right) \frac{ds}{s^{n-1}}
\]

Applying Hölder’s inequality, we arrive at
\[
\int_{\Omega \setminus (x_0,2r)} |f(y)| \, dy \lesssim \| f \|_{L_p(\Omega)} + \int_{2r}^{d} s^{-\frac{n}{p}-1} \| f \|_{L_p(\Omega(x_0,s))} \, ds.
\]
Thus the inequality
\begin{equation}
\|Tf_2\|_{L_p(\Omega(x_0,t))} \lesssim r^\frac{\gamma}{p} \int_r^d s^{-\frac{\gamma}{p}-1} \|f\|_{L_p(\Omega(x_0,s))} \, ds + r^\frac{\gamma}{p} \|f\|_{L_p(\Omega)} \tag{3.7}
\end{equation}
holds for all $r \in (0, d/2)$.

On the other hand, since
\[
\|Tf_2\|_{W_{L_p}(\Omega(x_0,t))} \leq \|Tf_2\|_{L_p(\Omega(x_0,t))}
\]
using (3.7), we get that
\begin{equation}
\|Tf_2\|_{W_{L_p}(\Omega(x_0,t))} \lesssim r^\frac{\gamma}{p} \int_r^d s^{-\frac{\gamma}{p}-1} \|f\|_{L_p(\Omega(x_0,s))} \, ds + r^\frac{\gamma}{p} \|f\|_{L_p(\Omega)} \tag{3.8}
\end{equation}
holds true for all $r \in (0, d/2)$.

Finally, combining (3.6) and (3.8), we obtain that
\[
\|Tf\|_{W_{L_p}(\Omega(x_0,t))} \lesssim r^\frac{\gamma}{p} \int_r^d s^{-\frac{\gamma}{p}-1} \|f\|_{L_p(\Omega(x_0,s))} \, ds + r^\frac{\gamma}{p} \|f\|_{L_p(\Omega)}
\]
holds for all $r \in (0, d/2)$ with a constant independent of $f$, $x_0$ and $r$.

Let now $r \in [d/2, d)$. Then, using $(L_p(\Omega), W_{L_p}(\Omega))$-boundedness of $T$, we obtain
\[
\|Tf\|_{W_{L_p}(\Omega(x_0,t))} \lesssim \|f\|_{L_p(\Omega)} \approx r^\frac{\gamma}{p} \|f\|_{L_p(\Omega)},
\]
and, inequality (3.2) holds.

(ii). Assume that $1 < p < \infty$. Let again $r \in (0, d/2)$. We write $f = f_1 + f_2$ with $f_1 = f\chi_{\Omega(x_0,2r)}$ and $f_2 = f\chi_{\Omega(x_0,2r)}$. Taking into account the linearity of $T$, we have
\begin{equation}
\|Tf\|_{L_p(\Omega(x_0,t))} \leq \|Tf_1\|_{L_p(\Omega(x_0,t))} + \|Tf_2\|_{L_p(\Omega(x_0,t))}. \tag{3.9}
\end{equation}
Since $f_1 \in L_p(\Omega)$, in view of (3.4), the boundedness of $T$ on $L_p(\Omega)$ implies that
\[
\|Tf_1\|_{L_p(\Omega(x_0,t))} \leq \|Tf_1\|_{L_p(\Omega)} \lesssim \|f_1\|_{L_p(\Omega)} \approx \|f\|_{L_p(\Omega(x_0,2r))}
\]
\begin{equation}
\lesssim r^\frac{\gamma}{p} \int_r^d t^{-\frac{\gamma}{p}-1} \|f\|_{L_p(\Omega(x_0,t))} \, dt + r^\frac{\gamma}{p} \|f\|_{L_p(\Omega)}, \tag{3.10}
\end{equation}
where the constant is independent of $f$, $x_0$ and $r$.

Combining (3.9), (3.10) and (3.7), we get inequality (3.3) holds for all $r \in (0, d/2)$ with a constant independent of $f$, $x_0$ and $r$.

If $r \in [d/2, d)$, then, using the boundedness of $T$ on $L_p(\Omega)$, we obtain that
\[
\|Tf\|_{L_p(\Omega(x_0,t))} \leq \|Tf\|_{L_p(\Omega)} \lesssim \|f\|_{L_p(\Omega)} \approx r^\frac{\gamma}{p} \|f\|_{L_p(\Omega)},
\]
and, inequality (3.3) holds. \hfill \square

Now we are going to use the following statement on the boundedness of the weighted Hardy operator
\[
H_w^* g(t) := \int_t^d g(s) w(s) \, ds, \quad 0 < t \leq d < \infty,
\]
where $w$ is a fixed function non-negative and measurable on $(0, d)$.

The following theorem was proved in [25].

Theorem 3.2. Let \( v_1, v_2 \) and \( w \) be positive almost everywhere and measurable functions on \((0, d)\). The inequality
\[
\text{ess sup}_{0<t<d} v_2(t) H_w g(t) \leq C \text{ess sup}_{0<t<d} v_1(t) g(t)
\]  
(3.11)
holds for some \( C > 0 \) for all non-negative and non-decreasing \( g \) on \((0, d)\) if and only if
\[
B := \text{ess sup}_{0<t<d} v_2(t) \int_t^d \frac{w(s)ds}{\text{ess sup}_{s<t<d} v_1(t)} < \infty.
\]
(3.12)
Moreover, if \( C^* \) is the minimal value of \( C \) in (3.11), then \( C^* = B \).

Remark 3.3. In (3.11) and (3.12) it is assumed that \( \frac{1}{\infty} = 0 \) and \( 0 \cdot \infty = 0 \).

Theorem 3.4. Let \( 1 \leq p < \infty \), \( \Omega \) be an open bounded subset of \( \mathbb{R}^n \), \( x_0 \in \Omega \), and \( (\varphi_1, \varphi_2) \) satisfy the condition
\[
\int_r^d \text{ess inf}_{t<\tau<\infty} \varphi_1(x_0, \tau) \tau^{\frac{p}{p+1}} dt \leq C \varphi_2(x_0, r),
\]
(3.13)
where \( C \) does not depend on \( r \). Let also \( T \) be a sublinear operator satisfying condition (3.1), and bounded from \( L_p(\Omega) \) to \( \text{WL}_p(\Omega) \), and bounded on \( L_p(\Omega) \) for \( p > 1 \). Then there exists \( c = c(p, \varphi_1, \varphi_2, n) > 0 \) such that
\[
\|Tf\|_{\text{WL}^{r_0}_{p,2}}(\Omega) \leq c \|f\|_{\text{LM}^{r_0}_{p,1}}(\Omega),
\]
Moreover, for \( p > 1 \) there exists \( c = c(p, \varphi_1, \varphi_2, n) > 0 \) such that
\[
\|Tf\|_{\text{LM}^{r_0}_{p,2}}(\Omega) \leq c \|f\|_{\text{LM}^{r_0}_{p,1}}(\Omega).
\]

Proof. By Theorem 3.2 and Lemma 3.1 with \( v_2(r) = \varphi_2(x_0, r)^{-1} \), \( v_1(r) = \varphi_1(x_0, r)^{-1} r^{-\frac{n}{p}} \) and \( w(r) = r^{-\frac{n}{p}} \) we have
\[
\|Tf\|_{\text{WL}^{r_0}_{p,2}}(\Omega) \leq \sup_{0<r<d} \varphi_1(x_0, r)^{-1} \int_r^d \|f\|_{\text{WL}_p(\Omega(x_0,t))} \frac{dt}{t^\frac{1}{p+1}} + \|Tf\|_{L_p(\Omega)}
\]
\[
\leq \sup_{0<r<d} \varphi_1(x_0, r)^{-1} r^{-\frac{n}{p}} \|f\|_{L_p(\Omega(x_0,r))} + \|f\|_{L_p(\Omega)}
\]
\[
= \|f\|_{\text{LM}^{r_0}_{p,1}}(\Omega) + \|f\|_{L_p(\Omega)} = \|f\|_{\text{LM}^{r_0}_{p,2}}(\Omega),
\]
and for \( 1 \leq p < \infty \)
\[
\|Tf\|_{\text{LM}^{r_0}_{p,2}}(\Omega) \leq \sup_{0<r<d} \varphi_1(x_0, r)^{-1} \int_r^d \|f\|_{L_p(\Omega(x_0,t))} \frac{dt}{t^\frac{1}{p+1}} + \|Tf\|_{L_p(\Omega)}
\]
\[
\leq \sup_{0<r<d} \varphi_1(x_0, r)^{-1} r^{-\frac{n}{p}} \|f\|_{L_p(\Omega(x_0,r))} + \|f\|_{L_p(\Omega)}
\]
\[
= \|f\|_{\text{LM}^{r_0}_{p,1}}(\Omega) + \|f\|_{L_p(\Omega)} = \|f\|_{\text{LM}^{r_0}_{p,2}}(\Omega).
\]

From Theorem 3.4 we get the following corollary.
Let \( 1 \leq p < \infty \), \( \Omega \) be an open bounded subset of \( \mathbb{R}^n \), \( x_0 \in \Omega \), and \((\varphi_1, \varphi_2)\) satisfy the condition

\[
\int_r^d \text{ess inf}_{t \leq \tau < \infty} \varphi_1(x, \tau) \frac{\tau^{n/2}}{t^{n+1}} \, dt \leq C \varphi_2(x, r), 
\]

(3.14)

where \( C \) does not depend on \( x \) and \( r \). Let also \( T \) be a sublinear operator satisfying condition (3.1), and bounded from \( L_p(\Omega) \) to \( WL_p(\Omega) \), and bounded on \( L_p(\Omega) \) for \( p > 1 \). Then there exists \( c = c(p, \varphi_1, \varphi_2, n) > 0 \) such that

\[
\|Tf\|_{W M_{p,\varphi_1}(\Omega)} \leq c\|f\|_{\tilde{M}_{p,\varphi_2}(\Omega)}.
\]

Moreover, for \( p > 1 \) there exists \( c = c(p, \varphi_1, \varphi_2, n) > 0 \) such that

\[
\|Tf\|_{\tilde{M}_{p,\varphi_2}(\Omega)} \leq c\|f\|_{\tilde{M}_{p,\varphi_2}(\Omega)}.
\]

4 Dirichlet boundary value problem for polyharmonic equation in modified local generalized Sobolev–Morrey spaces

Now we will derive regularity estimates for solution of problem (2.1) when \( g = 0 \)

\[
\begin{cases}
(-\Delta)^m u = f & \text{in } \Omega, \\
\frac{\partial u}{\partial \nu} = 0 & \text{on } \Omega,
\end{cases}
\]

(4.1)

where \( 0 \leq k \leq m - 1 \), \( \Omega \subset \mathbb{R}^n \) is bounded.

We get the estimates of solution problem (4.1) in modified local generalized Sobolev–Morrey spaces

\[
\|u\|_{W_{p,\varphi_2}^m(\Omega)} \lesssim \|f\|_{\tilde{M}_{p,\varphi_2}(\Omega)}.
\]

Note that

\[
Kf(x) = \lim_{\epsilon \to 0} \int_{|x-y| > \epsilon} \sum_{|\alpha| = 2m} D^\alpha x G_m(x-y)f(y)dy
\]

is the Calderón–Zygmund operator. Here and later we take, that function \( f \) define in \( \mathbb{R}^n \), also this function is continuity extended to exterior of domain \( \Omega \) with zero. The function \( D^\alpha x G_m(x, y) \in C^\infty(\mathbb{R}^n \setminus \{0\}) \) and this function is homogeneous of order \( m - \). Hence consequence, that \( D^\alpha x G_m(x, y) \) homogeneous of order \( 2m - n \) and tends to zero on unit sphere (see [15]). Then from general theory giving in [7] consequence that \( K \) bounded operator on \( L_p(\mathbb{R}^n) \) for \( 1 < p < \infty \). Moreover, maximal singularity operator

\[
\tilde{K}f(x) = \sup_{\epsilon > 0} \int_{|x-y| > \epsilon} \sum_{|\alpha| = 2m} D^\alpha x G_m(x-y)f(y)dy
\]

also a bounded on \( L_p(\mathbb{R}^n) \) for \( 1 < p < \infty \).

From Theorem 3.4 we get the following corollary.

Corollary 4.1. Let \( 1 < p < \infty \), \( \Omega \) be an open bounded subset of \( \mathbb{R}^n \), \( x_0 \in \Omega \), and \((\varphi_1, \varphi_2)\) satisfy the condition (3.13). Then operators \( M \) and \( K \) are bounded from \( \tilde{M}_{p,\varphi_1}(\Omega) \) to \( \tilde{M}_{p,\varphi_2}(\Omega) \).

From Corollary 3.5 we get the following.

Corollary 4.2. Let \( 1 < p < \infty \), \( \Omega \) be an open bounded subset of \( \mathbb{R}^n \), and \((\varphi_1, \varphi_2)\) satisfy the condition (3.14). Then operators \( M \) and \( K \) are bounded from \( M_{p,\varphi_1}(\Omega) \) to \( M_{p,\varphi_2}(\Omega) \).

Electronic copy available at: https://ssrn.com/abstract=3298593
Theorem 4.3. Let \( 1 < p < \infty, x_0 \in \Omega, \Omega \subset \mathbb{R}^n \) be a bounded domain with \( \partial \Omega \subset C^2 \), and \((\varphi_1, \varphi_2)\) satisfy the condition (3.13). Let also \( f \in \tilde{LM}^{(x_0)}_{\varphi_1}(\Omega) \) and function \( u \) is a solution of problem (4.1). Then there is exist constant \( C \) which dependent only at \( n, \varphi \) and \( \Omega \) such that
\[
\|u\|_{W^{2m}(x_0)} \leq C\|f\|_{\tilde{LM}^{(x_0)}(\Omega)}.
\] (4.2)

Proof. The proved consequence from the above estimates of the Green’s function from [27]:
the following inequalities
\[
|u(x)| + \left| \sum_{|\alpha| \leq m} D^\alpha u(x) \right| \lesssim Mf(x), \quad (4.3)
\]
\[
\left| \sum_{|\alpha| \leq 2m} D^\alpha u(x) \right| \lesssim \tilde{K} f(x) + Mf(x) + |f(x)|. \quad (4.4)
\]
hold uniformly for any \( x \in \Omega \).

With similarly ideas can be proved estimates
\[
|u(x)| + \left| \sum_{|\alpha| \leq m} D^\alpha u(x) \right| \lesssim Mf(x), \quad (4.5)
\]
\[
\left| \sum_{|\alpha| \leq 2m} D^\alpha u(x) \right| \lesssim \tilde{K} f(x) + Mf(x) + |f(x)|. \quad (4.6)
\]

Now we passing to prove of Theorem 4.3. From Corollary 4.1 imply that the operators \( M \) and \( \tilde{K} \) are bounded in \( \tilde{LM}^{(x_0)}_{\varphi_1}(\Omega) \). Therefore statement of Theorem 4.3 and estimate (4.2) the immediately consequence from inequalities (4.5), (4.6) and Corollary 4.1.

Theorem 4.3 is proved.

From inequalities (4.5), (4.6) and Corollary 4.2 we get the following corollary.

Corollary 4.4. Let \( 1 < p < \infty, \Omega \subset \mathbb{R}^n \) be a bounded domain with \( \partial \Omega \subset C^2 \), and \((\varphi_1, \varphi_2)\) satisfy the condition (3.14). Let also \( f \in \tilde{M}^{(x_0)}_{\varphi_1}(\Omega) \) and function \( u \) is a solution of problem (4.1). Then there is exist constant \( C \) which dependent only at \( n, \varphi \) and \( \Omega \) such that
\[
\|u\|_{W^{2m}(x_0)} \leq C\|f\|_{\tilde{M}^{(x_0)}(\Omega)}.
\]

5 Estimates of solutions any higher order uniformly elliptic equation with smooth coefficients in modified local generalized Sobolev–Morrey spaces

Consider the boundary value problem
\[
\begin{aligned}
L u &= f \quad \text{in } \Omega \\
B_j u &= \varphi_j \quad \text{on } \partial \Omega
\end{aligned}
\] (5.1)

for \( j = 0, \ldots, m - 1 \). The following assumptions hold.

1. The operator
\[
L u = \sum_{|\alpha| \leq 2m} a_{\alpha,j}(x) D^\alpha u
\]
is uniformly elliptic: there exists a constant $\gamma > 0$, such that

$$\gamma^{-1} |\xi|^2 \leq \sum_{\alpha, j} a_{\alpha, j}(x) \xi_{\alpha} \xi_{j} \leq \gamma |\xi|^2, \quad \text{a.e. } x \in \Omega, \forall \xi \in \mathbb{R}^n$$

$$a_{\alpha, j}(x) = a_{j, \alpha}(x)$$

2. The boundary operators

$$B_j = \sum_{|\beta| \leq m_j} b_{j, \beta} D^\beta, \quad \text{for } j = 0, m - 1$$

satisfy the complementing condition relative to $L$ (see the complementing condition on page 663 of [17]).

3. Let $l_1 > \max_j(2m - m_j)$ and $l_0 = \max_j(2m - m_j)$. The coefficients $a_{\alpha, j}$ belong to $C^{l_1 + 1}(\Omega)$ and $b_{j, \beta}$ belong to $C^{l_1 + 1}(\partial \Omega)$.

4. The boundary $\partial \Omega$ is $C^{l_1 + 2m + 1}$.

5. $f \in L^m_{p, \varphi}(\Omega)$ with $1 < p < \infty$ and $\varphi : \Omega \times \mathbb{R}^+ \to \mathbb{R}^+$ measurable.

**Theorem 5.1.** Let us consider the boundary value problem (5.1) and satisfy conditions 1–5 and also condition of Theorem 4.3. Then there is exist constant $C$ which dependent only at $n, \varphi$ and $\Omega$ such that

$$\|u\|_{W^{2m}_{p, \varphi}} \leq C \|f\|_{\overline{L^m_{p, \varphi}}(\Omega)}.$$  \hspace{1cm} (5.2)

Theorem 5.1 similarly ideas of Theorem 4.3 is proved.

For this it will be enough to consider the Krasovsky work [29]. We will recall the theorem in [29] which gives the estimates of the Green's function and the Poisson kernels. The proved consequence from this estimates. As in proof of Theorem 4.3 we use estimates (4.5), (4.6) and Corollary 4.1. Therefore statement of theorem and estimate (5.2) the immediately consequence from inequalities (4.5), (4.6). Theorem 5.1 is proved.

From inequalities (4.5), (4.6) and Corollary 4.2 we get the following corollary.

**Corollary 5.2.** Let us consider the boundary value problem (5.1) and satisfy conditions 1–5 and also condition of Corollary 4.4. Then there is exist constant $C$ which dependent only at $n, \varphi$ and $\Omega$ such that

$$\|u\|_{W^{2m}_{p, \varphi}} \leq C \|f\|_{\overline{L^m_{p, \varphi}}(\Omega)}.$$  \hspace{1cm} (5.2)

**Acknowledgements**

We thank the referee(s) for careful reading the paper and useful comments.

The research of V. S. Guliyev was partially supported by the Ministry of Education and Science of the Russian Federation (the Agreement number No. 02.a03.21.0008) and by the grant of Presidium of Azerbaijan National Academy of Science.
References

[1] D. R. Adams, A note on Riesz potentials, Duke Math. 42(1975), No. 4, 765–778. MR0458158

[2] S. Agmon, Lectures on elliptic boundary value problems, Prepared for publication by B. Frank Jones, Jr. with the assistance of George W. Batten, Jr. Van Nostrand Mathematical Studies, No. 2 D. Van Nostrand Co., Inc., Princeton, N.J.–Toronto–London 1965. MR0178246

[3] S. Agmon, A. Douglis, L. Nirenberg, Estimates near the boundary for solutions of elliptic partial differential equations satisfying general boundary conditions. I., Comm. Pure Appl. Math. 12(1959), 623–727. MR0125307; https://doi.org/10.1002/cpa.3160120405

[4] A. Akbulut, V.S. Guliyev, R. Mustafayev, On the boundedness of the maximal operator and singular integral operators in generalized Morrey spaces, Math. Bohem. 137(2012), No. 1, 27–43. MR2978444

[5] G. Barbatis, Sharp heat kernel bounds and Finsler-type metrics, Quart. J. Math. Oxford Ser. (2) 49(1998), No. 195, 261–277. MR1645623; https://doi.org/10.1093/qjmath/49.195.261

[6] T. Boggio, Sulle funzioni di Green d’ordine $m$, Rend. Circ. Mat. Palermo 20(1905), 97–135.

[7] A. P. Calderón, A. Zygmund, On the existence of certain singular integrals, Acta Math. 88(1952), No. 1, 85–139. MR0052553; https://doi.org/10.1007/BF02392130

[8] F. Chiarenza, M. Frasca, Morrey spaces and Hardy–Littlewood maximal function, Rend. Mat. Appl. 7(1987), No. 3–4, 273–279. MR0985999

[9] F. Chiarenza, M. Frasca, P. Longo, $W_{2,p}$-solvability of Dirichlet problem for nondivergence elliptic equations with VMO coefficients, Trans. Amer. Math. Soc. 336(1993), No. 2, 841–853. MR1088476; https://doi.org/10.2307/2154379

[10] P. Clement, G. Sweers, Uniform anti-maximum principles, J. Differential Equations, 164(2000), No. 1, 118–154. MR1761420; https://doi.org/10.1006/jdeq.1999.3745

[11] G. Di Fazio, D. K. Palagachev, M. A. Ragusa, Global Morrey regularity of strong solutions to the Dirichlet problem for elliptic equations with discontinuous coefficients, J. Funct. Anal. 166(1999), No. 2, 179–196. MR1707751; https://doi.org/10.1006/jfan.1999.3425

[12] D. Fan, S. Lu, D. Yang, Regularity in Morrey spaces of strong solutions to nondivergence elliptic equations with VMO coefficients, Georgian Math. J. 5(1998), No. 5, 425–440. MR1643604; https://doi.org/10.1023/B:GEOR.000008114.52420.af

[13] M. Fila, P. Souplet, F. Weissler, Linear and nonlinear heat equations in $L^1$ spaces and universal bounds for global solutions, Math. Ann. 320 (2001), 87–113. MR1835063; https://doi.org/10.1007/PL00004471

[14] E.B. Davies, $L_p$ spectral theory of higher-order elliptic differential operators, Bull. London Math. Soc. 29(1997), No. 5, 513–546. MR1458713; https://doi.org/10.1112/S002460939700324X
[15] A. Dall’Acqua, G. Sweers, Estimates for Green function and Poisson kernels of higher order Dirichlet boundary value problems, J. Differential Equations 205 (2004), No. 2, 466–487. MR2092867; https://doi.org/10.1016/j.jde.2004.06.004

[16] R. G. Duran, M. Sanmartino, M. Toschi, Weighted a priori estimates for Poisson equation, Indiana Univ. Math. J. 57 (2008), No. 7, 3463–3478. MR2492240; https://doi.org/10.1512/iumj.2008.57.3427

[17] S. D. Eidel’man, Parabolic systems, London North-Holland Publishing Co., Amsterdam–London; Wolters-Noordhoff Publishing, Groningen, 1969. MR0252806

[18] H.-Ch. Grunau, G. Sweers, Sharp estimates for iterated Green functions, Proc. Roy. Soc. Edinburgh Sect. A 132 (2002), No. 1, 91–120. MR1884473; https://doi.org/10.1017/S0308210500001542

[19] H.-Ch. Grunau, G. Sweers, Positivity for equations involving polyharmonic operators with Dirichlet boundary conditions, Math. Ann. 307 (1997), No. 4, 589–626. MR1464133; https://doi.org/10.1007/s002080000052

[20] H.-Ch. Grunau, G. Sweers, The role of positive boundary data in generalized clamped plate equations, Z. Angew. Math. Phys. 49 (1998), No. 3, 420–435. MR1629545; https://doi.org/10.1007/s000000050100

[21] V. S. Guliyev, Integral operators on function spaces on the homogeneous groups and on domains in G (in Russian), Doctor’s degree dissertation, Moscow, Mat. Inst. Steklov, 1994.

[22] V. S. Guliyev, Function spaces, integral operators and two weighted inequalities on homogeneous groups. Some applications (in Russian), Baku, 1999.

[23] V. S. Guliyev, Boundedness of the maximal, potential and singular operators in the generalized Morrey spaces, J. Inequal. Appl. 2009, Art. ID 503948, 20 pp. MR2579556; https://doi.org/10.1155/2009/503948

[24] V. S. Guliyev, S. S. Aliyev, T. Karaman, P. Shukurov, Boundedness of sublinear operators and commutators on generalized Morrey spaces, Integral Equations Operator Theory, 71 (2011), No. 3, 327–355. MR2852191; https://doi.org/10.1007/s00020-011-1904-1

[25] V. S. Guliyev, Generalized local Morrey spaces and fractional integral operators with rough kernel, J. Math. Sci. (N. Y.) 193 (2013), No. 2, 211–227. MR3077185; https://doi.org/10.1007/s10958-013-1448-9

[26] V. S. Guliyev, Local generalized Morrey spaces and singular integrals with rough kernel, Azerb. J. Math. 3 (2013), No. 2, 79–94. MR3084931

[27] V. S. Guliyev, L. Softova, Global regularity in generalized Morrey spaces of solutions to nondivergence elliptic equations with VMO coefficients, Potential Anal. 38 (2013), No. 3, 843–862. MR3034602; https://doi.org/10.1007/s11118-012-9299-4

[28] V. Guliyev, L. Softova, Generalized Morrey regularity for parabolic equations with discontinuity data. Proc. Edinb. Math. Soc. (2) 58 (2015), No. 1, 199–218. MR3333985; https://doi.org/10.1017/S0013091513000758
[29] J. P. Krasovsky, Isolation of the singularity in Green’s function (in Russian), Izv. Akad. Nauk SSSR Ser. Mat. 31(1967), 977–1010. MR0223740

[30] C. B. Morrey, On the solutions of quasi-linear elliptic partial differential equations, Trans. Amer. Math. Soc. 43(1938), No. 1, 126–166. MR1501936; https://doi.org/10.2307/1989904

[31] T. Mizuhara, Boundedness of some classical operators on generalized Morrey spaces, in: Harmonic analysis (Sendai, 1990) ICM-90 Satell. Conf. Proc., pp. 183–189. MR1261439

[32] E. Nakai, Hardy–Littlewood maximal operator, singular integral operators and the Riesz potentials on generalized Morrey spaces, Math Nachr. 166(1994), 95–103. MR1273325; https://doi.org/10.1002/mana.19941660108

[33] D. K. Palagachev, M. A. Ragusa, L. G. Softova, Cauchy–Dirichlet problem in Morrey spaces for parabolic equations with discontinuous coefficients, Boll. Unione Mat. Ital. Sez. B Artic. Ric. Mat. (8) 6(2003), No. 3, 667–683. MR2014826

[34] J. Peetre, On the theory $L_{p,\lambda}$, J. Funct. Anal. 4(1969), 71–87. MR0241965

[35] A. Scapellato, On some qualitative results for the solution to a Dirichlet problem in local generalized Morrey spaces, in: S. Sivasundaram (Ed.), AIP Conference Proceedings Vol. 1798, Article Number: UNSP 020138, 2017. https://doi.org/10.1063/1.4972730

[36] A. Scapellato, Some properties of integral operators on generalized Morrey spaces, in: T. E. Simos, Ch. Tsitouras (Eds.), AIP Conference Proceedings, Vol. 1863, 21, Art. no. 510004, 2017. https://doi.org/10.1063/1.4992662

[37] L. G. Softova, Singular integral operators in Morrey spaces and interior regularity of solutions to systems of linear PDEs, J. Global Optim. 40(2008), 427–442. MR2373570; https://doi.org/10.1007/s10898-007-9213-6

[38] L. G. Softova, The Dirichlet problem for elliptic equations with VMO coefficients in generalized Morrey spaces, in: Advances in harmonic analysis and operator theory, Operator Theory: Advances and Applications, Vol. 229, Springer, 2013, pp. 371–386. MR3060425; https://doi.org/10.1007/978-3-0348-0516-2_21