NOTE ON THE KATO PROPERTY OF SECTORIAL FORMS

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Abstract. We characterise the Kato property of a sectorial form $a$, defined on a Hilbert space $V$, with respect to a larger Hilbert space $H$ in terms of two bounded, selfadjoint operators $T$ and $Q$ determined by the imaginary part of $a$ and the embedding of $V$ into $H$, respectively. As a consequence, we show that if a bounded selfadjoint operator $T$ on a Hilbert space $V$ is in the Schatten class $S_p(V)$ ($p \geq 1$), then the associated form $a_T(\cdot, \cdot) := \langle (I + iT)\cdot, \cdot \rangle_V$ has the Kato property with respect to every Hilbert space $H$ into which $V$ is densely and continuously embedded. This result is in a sense sharp. Another result says that if $T$ and $Q$ commute then the form $a_T$ has the Kato property.

1. Introduction and preliminaries

Let $a : V \times V \to \mathbb{C}$ be a bounded, sectorial, coercive, sesquilinear form on a complex Hilbert space $V$, which is densely and continuously embedded into a second Hilbert space $H$. Then $a$ induces a sectorial, invertible operator $L_H$ on $H$, and Kato’s square root problem is to know whether the domain of $L_H^\frac{1}{2}$ is equal to the form domain $V$. If this is the case, then we say that the couple $(a, H)$ has the Kato property. In this short note we characterise the Kato property of $(a, H)$ in terms of two bounded, selfadjoint operators $T, Q \in \mathcal{L}(V)$ determined by the imaginary part of $a$ and by the embedding of $V$ into $H$, respectively. We show that the Kato property of $(a, H)$ is equivalent to the similarity of $Q(I + iT)^{-1}$ to an accretive operator, or to the similarity of $(I + Q + iT)(I - Q + iT)^{-1}$ to a contraction; see Theorem 2.1. The established link to different characterisations known in the literature provides an interesting connection between a variety of techniques and results mainly from operator theory of bounded operators, harmonic analysis, interpolation theory, or abstract evolution equations.

In particular, we show that if a bounded, selfadjoint operator $T$ on a Hilbert space $V$ is in the Schatten class $S_p(V)$ for some $p \geq 1$, then the associated form $a_T(\cdot, \cdot) := \langle (I + iT)\cdot, \cdot \rangle_V$ has the Kato property with respect to every Hilbert space $H$ into which $V$ is densely and continuously embedded; see Corollary 3.2. This result is in a sense sharp; see Proposition 4.1.

On the other hand, if $a$ is an arbitrary bounded, sectorial, coercive form on $V$, then for every nonnegative, injective operator $Q \in \mathcal{L}(V)$ which is of the form $I + P$ with $P \in S_p(V)$ ($p \geq 1$), the pair $(a, H_Q)$ has the Kato property, where $H_Q$ is the completion of $V$ with respect to the inner product $\langle Q, \cdot, \cdot \rangle_V$; see Corollary 3.3. Another straightforward consequence of Theorem 2.1 says that for every pair $(T, Q)$ of selfadjoint, commuting operators (with $Q$ being nonnegative and injective), the form $a_T$ has the Kato property with respect to $H_Q$; see Corollary 2.2.

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We conclude this introduction with some preliminaries.

1.1. **Forms.** Let \( a \) be a bounded, sesquilinear form on a complex Hilbert space \( V \). Denote by \( a^* \) the adjoint form of \( a \), that is, \( a^*(u,v) := \overline{a(v,u)} \) for every \( u, v \in V \). Then we call

\[
\begin{align*}
\mathfrak{s} &:= \text{Re} a := (a + a^*)/2 \quad \text{and} \\
\mathfrak{t} &:= \text{Im} a := (a - a^*)/2i
\end{align*}
\]

the real part and the imaginary part of \( a \), respectively. Note that \( \mathfrak{s} = \text{Re} a \) and \( \mathfrak{t} = \text{Im} a \) are symmetric forms on \( V \) and \( a = \mathfrak{s} + i\mathfrak{t} \). Throughout the following, we assume that \( a \) is coercive in the sense that \( \text{Re} a(u,u) \geq \eta \|u\|^2_V \) for some \( \eta > 0 \) and every \( u \in V \). This means that \( \mathfrak{s} = \text{Re} a \) is an equivalent inner product on \( V \), and for simplicity we assume that \( \mathfrak{s} \) is equal to the inner product on \( V \): \( \mathfrak{s}(u,v) = \langle u,v \rangle_V \) \((u, v \in V)\). We shall also assume that \( a \) is sectorial, that is, there exists \( \beta \geq 0 \) such that

\[
|\text{Im} a(u,u)| \leq \beta \text{Re} a(u,u), \quad u \in V.
\]

Let \( H \) be a second Hilbert space such that \( V \) is densely and continuously embedded into \( H \), that is, there exists a bounded, injective, linear operator \( j : V \to H \) with dense range. In the sequel we identify \( V \) with \( j(V) \). The embedding \( j \) induces a bounded, linear embedding \( j' : H \to V' \) (where \( V' \) is the space of bounded, antilinear functionals on \( V \)) given by

\[
j'(u) := \langle u, \cdot \rangle_H, \quad u \in H.
\]

Then we have the following picture:

\[
V \xrightarrow{j} H \xrightarrow{j'} V' \text{ and } V \xrightarrow{j'j} V'.
\]

We write also \( J := j'j \) for the linear embedding of \( V \) into the dual space \( V' \). As usual, \( V' \) is equipped with the inner product \( \langle u, v \rangle_{V'} := \langle I_V u, I_V v \rangle_V \), where \( I_V : V' \to V \) is the Riesz isomorphism.

1.2. **Bounded operators associated with the pair \((a,H)\).** Let \((a,H)\) be given as above. We define two associated bounded, linear operators on \( V \). In fact, by the Riesz-Frèchet representation theorem, there exist two unique selfadjoint operators \( T = T_a \), \( Q = Q_H \in \mathcal{L}(V) \), such that

\[
t(u,v) = \langle Tu, v \rangle_V \quad \text{and} \quad \langle u,v \rangle_H = \langle Qu, v \rangle_V \quad \text{for every } u, v \in V,
\]

and hence, by recalling our convention that \( \mathfrak{s} = \langle \cdot, \cdot \rangle_V \),

\[
a(u,v) = \langle (I + iT)u, v \rangle_V \quad \text{for every } u, v \in V.
\]

Moreover, since \( \langle \cdot, \cdot \rangle_H \) is an inner product, \( Q \) is nonnegative and injective. In fact, \( Q = j'Tj \), where \( j' : H \to V \) is the Hilbert space adjoint of \( j \).

Conversely, every selfadjoint operator \( T \in \mathcal{L}(V) \) induces via the equality (1.3) a bounded, sesquilinear, sectorial form \( a \) on \( V \) for which \( \text{Re} a \) coincides with the inner product \( \langle \cdot, \cdot \rangle_V \), and for which \( \text{Im} a \) is represented by \( T \). Similarly, every nonnegative, injective operator \( Q \in \mathcal{L}(V) \) induces via the equality (1.2) an inner product \( \langle \cdot, \cdot \rangle_H := \langle Q_{\cdot}, \cdot \rangle_V \) on \( V \), and thus, by taking the completion, a Hilbert space \( H_Q \) into which \( V \) is densely and continuously embedded.

We say that the pair of operators \((T,Q)\) is associated with the pair \((a,H)\), or, conversely, the pair \((a,H)\) is associated with the pair \((T,Q)\).
1.3. **Unbounded operators associated with the pair** \((a, H)\). Given a pair \((a, H)\) as above, we define also associated closed, linear operators on \(H\) and \(V'\).

First, we denote by \(L_H := L_{a,H}\) the, in general, unbounded operator on \(H\) given by
\[
\mathcal{D}(L_H) := \{ u \in j(V) : \exists f \in H \forall v \in V : a(j^{-1}u, v) = \langle f, jv \rangle_H \},
\]
\[
L_H u := f.
\]
Second, we denote by \(L_{V'} := L_{a,V'}\), the operator on \(V'\) which is given by
\[
\mathcal{D}(L_{V'}) := (j')^{-1}(V) = J(V),
\]
\[
L_{V'} u := a(J^{-1}u, \cdot).
\]
In a similar way we define the operators \(L_{s,H}\) and \(L_{s,V'}\) associated with the real part \(s = \text{Re} a\).

Recall that a closed, linear operator \((A, \mathcal{D}(A))\) on a Banach space \(X\) is called **sectorial of angle** \(\theta \in (0, \pi)\) if
\[
\sigma(A) \subseteq \Sigma_\theta := \{ z \in \mathbb{C} : |\text{arg} z| \leq \theta \},
\]
and if for every \(\theta' \in (\theta, \pi)\) one has
\[
\sup_{z \notin \Sigma_{\theta'}} \|zR(z, A)\| < \infty.
\]
We simply say that \(A\) is **sectorial** if it is sectorial for some angle \(\theta \in (0, \pi)\). The **numerical range** of a closed, linear operator \((A, \mathcal{D}(A))\) on a Hilbert space \(H\) is the set
\[
W(A) := \{ \langle Au, u \rangle_H : u \in \mathcal{D}(A), \|u\|_H = 1 \}.
\]
The operator \(A\) is said to be \(\theta\)-**accretive** for \(\theta \in (0, \pi)\), if \(W(A) \subseteq \Sigma_\theta\), that is, if
\[
|\text{arg} \langle Au, u \rangle_H| \leq \theta \text{ for every } u \in \mathcal{D}(A).
\]
If \(\theta = \frac{\pi}{2}\), that is, \(\text{Re} \langle Au, u \rangle_H \geq 0\) for every \(u \in \mathcal{D}(A)\), we say that \(A\) is **accretive**.

Both operators \(L_H\) and \(L_{V'}\) defined above are sectorial for some angle \(\theta \in (0, \frac{\pi}{2})\). Since \(a\) is assumed to be coercive, we have \(0 \in \rho(L_H)\) and \(0 \in \rho(L_{V'})\), that is, both \(L_H\) and \(L_{V'}\) are isomorphisms from their respective domains onto \(H\) and \(V'\), respectively; see e.g. [14, Theorem 2.1, p. 58].

It is easy to check that the numerical range of \(L_H\) is contained in the sector \(\Sigma_\theta\) with \(\theta = \arctan \beta\) and in particular \(L_H\) is \(\theta\)-accretive. As a consequence, by [8, Theorem 11.13], \(L_H\) admits a bounded \(H^\infty\) functional calculus. We refer the reader to [8] or [4] for the background on fractional powers and \(H^\infty\) functional calculus of sectorial operators.

2. **Characterisation of the Kato property**

Let \((a, H)\) be as above, that is, \(a\) is a bounded, sectorial, coercive, sesquilinear form on a Hilbert space \(V\) which embeds densely and continuously into a second Hilbert space \(H\). Let \(L_H := L_{a,H}\) be defined as above. We say that the couple \((a, H)\) has **Kato’s property** if \(\mathcal{D}(L_H^{1/2}) = V\). By the Closed Graph Theorem, if \((a, H)\) has the Kato property, then the norms \(\|L_H^{1/2} \cdot \|_H\) and \(\| \cdot \|_V\) are equivalent on \(V\). According to Kato [6] and Lions [10], the coincidence of any two of the spaces \(\mathcal{D}(L_H^{1/2})\), \(\mathcal{D}(L_H^{1/2})\) and \(V\) implies the coincidence of the three.

Moreover, by Subsection 1.2, it is natural to say that a pair \((T, Q)\) of selfadjoint, bounded operators on a Hilbert space \(V\), with \(Q\) being nonnegative and injective, has the
Kato property, if the associated pair \((a_T, H_Q)\) has the Kato property, where \(a_T(u, v) = \langle (I + iT)u, v \rangle_V (u, v \in V)\) and \(H_Q\) is the completion of \((V, \langle \cdot, \cdot \rangle_V)\).

The main result of this section is the following characterisation of the Kato property of \((a, H)\) in terms of the associated pair of bounded operators \((T, Q)\).

**Theorem 2.1.** Let \((T, Q)\) be the pair of operators associated with \((a, H)\) as above. Then the following assertions are equivalent:

(i) \((a, H)\) has the Kato property

(ii) There exists a positive operator \(S\) on \(V\) such that

\[ \langle QS(I + iT)u, u \rangle_V \in \Sigma_\theta \text{ for every } u \in V \text{ and some } \theta < \frac{\pi}{2}, \]

(iii) There exists a positive operator \(S\) on \(V\) such that

\[ \langle S(I - iT)^{-1}Qu, u \rangle_V \in \Sigma_\theta \text{ for every } u \in V \text{ and some } \theta < \frac{\pi}{2}, \]

that is, \((I - iT)^{-1}Q\) is similar to a \(\theta\)-accretive operator, or, equivalently, the operator \((I - iT)^{-1}Q\) has a bounded \(H^\infty(\Sigma_\theta)\) functional calculus.

(iv) There exists a positive operator \(S\) on \(V\) such that

\[ \langle SQ(I + iT)^{-1}u, u \rangle_V \in \Sigma_\theta \text{ for every } u \in V \text{ and some } \theta < \frac{\pi}{2}, \]

that is, \(Q(I + iT)^{-1}\) is similar to a \(\theta\)-accretive operator, or, equivalently, the operator \(Q(I + iT)^{-1}\) has a bounded \(H^\infty(\Sigma_\theta)\) functional calculus.

(v) The operator \((I - Q + iT)(I + Q + iT)^{-1}\) is polynomially bounded.

(vi) The operator \((I - Q + iT)(I + Q + iT)^{-1}\) is similar to a contraction.

Recall that, if \(T, Q \in \mathcal{L}(V)\) are selfadjoint operators, then \(QT\) is selfadjoint if and only if \(T\) and \(Q\) commute, or if and only if \(\langle QTu, u \rangle_V \in \mathbb{R}\) for every \(u \in V\). Therefore, the above Theorem 2.1(ii') gives the following sufficient condition for \((a, H)\) to have the Kato property.

**Corollary 2.2.** If \(T\) and \(Q\) commute, then \((a, H)\) has the Kato property.

We start with auxiliary results on the operators appearing in Theorem 2.1.

**Lemma 2.3.** Let \(T\) and \(Q\) be selfadjoint, bounded operators on a Hilbert space \(V\). Assume that \(Q\) is nonnegative. Then, the operator \(A := Q(I + iT)^{-1} \in \mathcal{L}(V)\) is sectorial of angle \(\theta < \frac{\pi}{2}\).

**Proof.** By a standard argument based on the Neumann series extension it is sufficient to show that \(\sup_{\Re z \leq 0} \|zR(z, A)\| < \infty\). Note that for every \(z \in \mathbb{C}\) with \(\Re z \leq 0\) and every \(u \in V\) we have

\[ ((z + izT - Q)u, u)_V = (\Re z \|u\|_V^2 - \Im z \langle Tu, u \rangle_V - \langle Qu, u \rangle) + i (\Im z \|u\|_V^2 + \Re z \langle Tu, u \rangle_V), \]

and therefore

\[ |((z + izT - Q)u, u)_V|^2 = |z|^2 \|u\|_V^4 - 2 \Re z \|u\|_V^2 \langle Qu, u \rangle_V + \langle z^2 (Tu, u)_V^2 + 2 \Im z \langle Tu, u \rangle_V \langle Qu, u \rangle_V + \langle Qu, u \rangle^2 \]

\[ \geq |z|^2 \|u\|_V^4, \]
or, by the Cauchy-Schwarz inequality,
\[ \| (z + izT - Q)u \|_{V} \geq |z| \| u \|_{V}. \]
This inequality implies that \( z + izT - Q \) is injective and has closed range. A duality argument, using similar estimates as above, shows that \( z + izT - Q \) has dense range, and therefore \( z + izT - Q \) is invertible for every \( z \in \mathbb{C} \) with \( \text{Re} \ z \leq 0 \). Moreover, the above inequality shows that
\[ \sup_{\text{Re} \ z \leq 0} \| z(z + izT - Q)^{-1} \| \leq 1. \]
As a consequence, \( z - A = (z + izT - Q)(I + iT)^{-1} \) is invertible for every \( z \in \mathbb{C} \) with \( \text{Re} \ z \leq 0 \) and
\[ \sup_{\text{Re} \ z \leq 0} \| zR(z, A)\| = \sup_{\text{Re} \ z \leq 0} \| (I + iT) z(z + izT - Q)^{-1} \| \leq \| I + iT \|. \]

Let \( A \in \mathcal{L}(V) \) be a bounded, sectorial operator of angle \( \theta \in (0, \frac{\pi}{2}) \), and let \( C := (I - A)(I + A)^{-1} \) be its Cayley transform. Then the equality
\[ (z - 1)(z - C)^{-1} = \frac{z - 1}{z + 1} \left( \frac{z - 1}{z + 1} + A \right)^{-1} (I + A) \]
shows that \( C \) is a Ritt operator, that is, \( \sigma(C) \subseteq D \cup \{1\} \) (where \( D \subset \mathbb{C} \) is the open unit disk) and
\[ \sup_{|z| > 1} \| (z - 1)R(z, C) \| < \infty. \]
From this and the preceding lemma, we obtain the following statement.

**Lemma 2.4.** The Cayley transform
\[ C = (I - A)(I + A)^{-1} = (I - Q + iT)(I + Q + iT)^{-1} \]
of the operator \( A = Q(I + iT)^{-1} \) is a Ritt operator.

Recall that a bounded operator \( C \) on a Hilbert space \( V \) is a Ritt operator if and only if it is power bounded and \( \sup_{n \in \mathbb{N}} \| C^{n} - C^{n+1} \| < \infty \); see [12]. Furthermore, a bounded operator \( C \) on a Hilbert space is polynomially bounded if there exists a constant \( M \geq 0 \) such that for every polynomial \( p \) one has
\[ \| p(C) \| \leq M \sup_{|z| \leq 1} |p(z)|. \]

The proof of Theorem 2.1 is a consequence of the characterisation of the Kato property by means of the boundedness of the \( H^{\infty} \) functional calculus for the operator \( L_{V'} = L_{a,V'} \) given by Arendt in [11] Theorem 5.5.2, p.45.

**Lemma 2.5.** Let \( L_{V'} = L_{a,V'} \) be the operator associated with \( (a, H) \) as above. Then the following assertions are equivalent:

(i) \( (a, H) \) has the Kato property.

(ii) \( L_{V'} \) has a bounded \( H^{\infty} \) functional calculus.

Moreover, if (i) or (ii) holds, then \( L_{V'} \) has a bounded \( H^{\infty}(\Sigma_{\theta}) \) functional calculus for every \( \theta > \arctan \beta \) with \( \beta \) as in (1.1).

For the convenience of the reader we recall the proof of this result using our notation and with slight modifications.
Proof. First of all, note that the operator \( L_H \) can be expressed as the operator \( j'^{-1}L_{V'}j' \) with domain \( \{ u \in H : j'u \in \mathcal{D}(L_{V'}) \} \) and \( L_{V'}j'u \in j'(H) \). Then \((\lambda - L_H)^{-1} = j'^{-1}(\lambda - L_{V'})^{-1}j'\) for every \( \lambda \notin \Sigma_0 \), and by the definition of the square roots via contour integrals,
\[
L_H^{\frac{1}{2}} = j'^{-1}L_{V'}^{\frac{1}{2}}j'.
\]

(i)⇒(ii) Therefore, if \( \mathcal{D}(L_H^{\frac{1}{2}}) = \mathcal{R}(L_H^{\frac{1}{2}}) = j(V) \), then
\[
\mathcal{D}(L_{V'}^{\frac{1}{2}}) := L_{V'}^{\frac{1}{2}}(V') = L_{V'}^{\frac{1}{2}}(j'(V)) = L_{V'}^{\frac{1}{2}}jL_{V'}^{\frac{1}{2}}(H) = L_{V'}^{\frac{1}{2}}j(\lambda - L_{V'})^{-1}j' = j'(H),
\]
where the last equality follows from \( L_{V'}^{\frac{1}{2}}, L_{V'}^{\frac{1}{2}} = L_{V'} \); see e.g. \cite{8} Theorem 15.15, p.289]. By \cite{14} Corollary 2.3, p.113], \( j'(H) = [V', \mathcal{D}(L_{a,V'})]_{\frac{1}{2}} \), where on \( \mathcal{D}(L_{a,V'}) = J(V) \) we consider the graph norm of \( L_{a,V'} \) that is, \( \|L_{a,V'} \cdot \|_{V'} + \| \cdot \|_{V'} \). Since for \( v \in V \) we have
\[
I_V L_{V'} Jv = (I + iT)v \quad \text{and} \quad I_V L_{a,V'} Jv = v,
\]
hence
\[
\|L_{V'} Jv\|_{V'} = \|(I + iT)v\|_V \quad \text{and} \quad \|L_{V'} Jv\|_{V'} = \|v\|_V.
\]
Consequently, the invertibility of \( I + iT \) implies that the graph norm of \( L_{a,V'} \) is equivalent to the graph norm of \( L_{V'} = L_{a,V'} \) on \( \mathcal{D}(L_{V'}) = J(V) \). Therefore, we get that
\[
[V', \mathcal{D}(L_{V'})]_{\frac{1}{2}} = \mathcal{D}(L_{V'}^{\frac{1}{2}}).
\]
Hence, by \cite{14} Theorem 16.3, p.532], \( L_{V'} \) has a bounded \( H^\infty \) functional calculus.

(ii)⇒(i) On the other hand, if \( L_{V'} \) has a bounded \( H^\infty \) functional calculus, then as above we get \( \mathcal{D}(L_{V'}^{\frac{1}{2}}) = [V', \mathcal{D}(L_{V'})]_{\frac{1}{2}} = j'(H) \). Therefore,
\[
\mathcal{D}(L_H^{\frac{1}{2}}) := L_H^{\frac{1}{2}}j^{-1}(j'(H)) = L_H^{\frac{1}{2}}j^{-1}L_{V'}^{\frac{1}{2}}(V')
\]
\[
= j^{-1}j'L_H^{\frac{1}{2}}j^{-1}L_{V'}^{\frac{1}{2}}(V') = j^{-1}L_{V'}^{\frac{1}{2}}L_{V'}^{\frac{1}{2}}(V') = j^{-1}j'(V)
\]
\[
= j(V).
\]

For the last statement about the angle of the \( H^\infty \) functional calculus first note, that \( \mathcal{D}(L_{V'}^{\frac{1}{2}}) = [V', \mathcal{D}(L_{V'})]_{\frac{1}{2}} = j'(H) \) yields
\[
L_H = j^{-1}L_{V'}^{\frac{1}{2}}L_{V'}^{\frac{1}{2}}j'j'.
\]
Moreover, by the Closed Graph Theorem, the operator \( j' \) is an isomorphism from \( H \) onto \( \mathcal{D}(L_{V'}^{\frac{1}{2}}) = j'(H) \) equipped with the graph norm. Since the operator \( L_H \) is \( (\text{arctan} \, \beta) \)-accretive, therefore it is sectorial of angle \( \text{arctan} \, \beta \), and consequently the operator \( L_{V'} \), too. Finally, for example, by \cite{14} Theorem 16.3, p.532] (cf. \cite{14} Remark 16.2, p.536), \( L_{V'} \) has a bounded \( H^\infty \) functional calculus in any sectorial domain \( \Sigma_0 \) with \( \theta > \text{arctan} \, \beta \). This completes the proof. \( \Box \)

Proof of Theorem 2.7 Assume that \( (a,H) \) has the Kato property. By Lemma 2.3, \( L_{V'} \) has a bounded \( H^\infty (\Sigma_0) \) functional calculus for every \( \theta > \text{arctan} \, \beta \). Fix \( \theta \in (\text{arctan} \, \beta, \frac{\pi}{2}) \). By the characterisation of the boundedness of the \( H^\infty \) functional calculus, \cite{8} Theorem 11.13, p.229], \( L_{V'} \) is \( \theta \)-accretive with respect to an equivalent inner product \( \langle \cdot , \cdot \rangle_\theta \) on \( V' \). Let \( S \in \mathcal{L}(V') \) be the positive operator such that \( \langle \cdot , \cdot \rangle_\theta = \langle S \cdot , \cdot \rangle_{V'} \). Then \( S := I_V S I_{V'}^{-1} \in \mathcal{L}(V) \) is a positive operator on \( V \).
First, note that $I_Lv.Jv = (I + iT)v$ and $Iv.Jv = Qv$ for every $v \in V$. Then,

$$
(L_Lv.Jv)_\theta = \langle S_Lv.Jv.v \rangle_V,
$$

$$
= \langle I_Lv.SI_Lv.(1) \rangle_V
$$

$$
= \langle I_Lv.Jv.(1) \rangle_V
$$

for every $v \in V$. Therefore, the operator $QS(I + iT)$ is $\theta$-accretive with respect to $\langle \cdot, \cdot \rangle_V$. Therefore, (i) $\Rightarrow$ (ii$'$). The implication (ii$'$) $\Rightarrow$ (i) follows from a similar argument.

The equivalences (ii) $\iff$ (iii) $\iff$ (iv) follow from the following chain of equivalences which holds for every positive operator $S \in L(V)$ and $\theta \in (0, \frac{\pi}{2}]$:

$QS(I + iT)$ is $\theta$-accretive

$\iff \forall u \in V : \langle QS(I + iT)u, u \rangle \in \Sigma_\theta$

$\iff \forall u \in V : \langle QSu, (I + iT)^{-1}u \rangle \in \Sigma_\theta$

$\iff \forall u \in V : \langle u, S^{\frac{1}{2}}Q(I + iT)^{-1}S^{-\frac{1}{2}}u \rangle \in \Sigma_\theta$

$\iff S^{\frac{1}{2}}(I - iT)^{-1}QS^{-\frac{1}{2}}$ is $\theta$-accretive

$\iff S^{\frac{1}{2}}Q(I + iT)^{-1}S^{-\frac{1}{2}}$ is $\theta$-accretive.

For (iv) $\iff$ (v), set $A := Q(I - iT)^{-1}$, and note that its Cayley transform is given by

$$
C := \phi(A) = (I - Q + iT)(I + Q + iT)^{-1},
$$

where $\phi$ is the conformal map $\phi(z) := (1 - z)(1 + z)^{-1}$ from $\Sigma_{\frac{\pi}{2}}$ onto $\{ |z| < 1 \}$. Moreover, for every polynomial $p$ we have

$$
p(C) = (p \circ \phi)(A) \quad \text{and} \quad \sup_{z \in \Sigma_\theta} |(p \circ \phi)(z)| \leq \sup_{|z| < 1} |p(z)|.
$$

Therefore, the boundedness of the $H^\infty(\Sigma_\theta)$ functional calculus of $A$ with $\theta \leq \frac{\pi}{2}$ yields the polynomial boundedness of its Cayley transform $C$. For the converse, by Runge’s theorem, it is easy to see that $A$ has a bounded $R(\Sigma_{\frac{\pi}{2}})$ functional calculus; here $R(\Sigma_{\frac{\pi}{2}})$ stands for the algebra of rational functions with poles outside $\Sigma_{\frac{\pi}{2}}$. Then, the boundedness of the $H^\infty(\Sigma_{\frac{\pi}{2}})$ functional calculus follows again by an approximation argument and McIntosh’s convergence theorem [11 Section 5, Theorem]; see also [4 Proposition 3.13, p.66]. Since $A$ is $\theta$-sectorial for some $\theta < \frac{\pi}{2}$, [8 Theorem 11.13] gives (iv).

Finally, for (v) $\Rightarrow$ (vi), since the Cayley transform $C = \phi(A)$ is a Ritt operator, see Lemma [2,4] by [9 Theorem 5.1], it is similar to a contraction. The converse is a consequence of the von Neumann inequality. □

Remark 2.6. (a) By [8 Theorem 11.13 H7]) one can show that if $(a, H)$ has the Kato property, then (iv) in Theorem [2,4] holds with

$$
S := \int_{\Sigma_{\sigma - \theta}} A^*e^{zA}Ae^{zA} \, dz = \int_{\Sigma_{\sigma - \theta}} |Ae^{zA}|^2 \, dz,
$$

where $A := Q(I - iT)^{-1}$ and the integral exists in the weak operator topology.

(b) Note that in the case when the operator $Q$ is invertible on $V$, or equivalently, the inner products on $H$ and $V$ are equivalent, then $(a, H)$ has the Kato property simply
because $L_H \in \mathcal{L}(H) = \mathcal{L}(V)$. It should be pointed out, that in this case the similarity to a contraction of the operator

$$C = (I - Q + iT)(I + Q + iT)^{-1} = (T - i(I - Q))(T - i(I + Q))^{-1},$$

which is stated in Theorem 2.1 (vi), can be proved in a straightforward way. Indeed, in [3] Theorem 1] Fan proved that an operator $C \in \mathcal{L}(V)$ with $1 \in \rho(C)$ is similar to an accretive operator on $V$.

$$\quad$$

3. Kato property and triangular operators

Recall that a bounded operator $\Delta$ on a Hilbert space $V$ is triangular if there exists a constant $M \geq 0$ such that

$$\sum_{j=1}^{n} |a_j| \leq M \sup_{|a_j|=1} \left| \sum_{j=1}^{n} a_j v_j \right|,$$

for every $n \in \mathbb{N}$ and every $u_1, \ldots, u_n, v_1, \ldots, v_n \in V$. By a theorem of Kalton [5, Theorem 5.5], an operator $\Delta$ on $V$ is triangular if and only if $\sum_{n=1}^{\infty} \frac{\|\Delta^n\|}{n+1} < \infty$, where $(s_n(\Delta))_{n \in \mathbb{N}}$ is the sequence of singular values of $\Delta$. Therefore, the Schatten-von Neumann classes are included in the class of triangular operators. We refer the reader to [5] Section 5 for basic properties of triangular operators. One interest in the class of triangular operators stems from the following perturbation theorem by Kalton [5, Theorem 7.7].

**Lemma 3.1.** Let $A$ and $B$ be two sectorial operators on a Hilbert space $H$. Assume that $B$ has a bounded $H^{\infty}$ functional calculus, and that $A = (I + \Delta)B$ for some triangular operator $\Delta$. Then $A$ has a bounded $H^{\infty}$ functional calculus, too.

Combining this result with Theorem 2.1 we show that the Kato property of $(a, H)$ is preserved under certain triangular perturbations of the imaginary part of $a$, and in particular, that for every bounded, selfadjoint operators $T$ and $Q$ on a Hilbert space $V$ such that $T$ is triangular and $Q$ is nonnegative and injective, the pair $(T, Q)$ has the Kato property, that is, $Q(I - iT)^{-1}$ is similar to an accretive operator on $V$.

**Corollary 3.2.** Let $a$ and $b$ be two sectorial forms on $V$ with the same real parts, that is, $\text{Re}a = \text{Re}b$. Let the imaginary parts $t_a$ and $t_b$ of $a$ and $b$ be determined by selfadjoint operators $T_a$, $T_b \in \mathcal{L}(V)$, respectively. Assume that $(b, H)$ has the Kato property, and that $T_a - T_b$ is a triangular operator. Then $(a, H)$ has the Kato property, too.

In particular, if $T_a$ is a triangular operator, then $(a, H)$ has the Kato property for every Hilbert space $H$ into which $V$ is densely and continuously embedded.

**Proof of Corollary 3.2** Note that by the second resolvent equation we get

$$\quad$$

Therefore, since the operator $i(I - iT_a)^{-1}(T_a - T_b)$ is triangular, the claim follows from Lemma 2.3, Lemma 3.1 and Theorem 2.3 (iii).
Alternatively, note that Arendt’s result, Lemma 2.5, used in the proof of Theorem 2.1 can be directly applied to get Corollary 3.2. Indeed, set \( \Delta := L_{a,V'}L_{b,V}^{-1} - I \). By our assumption, \( L_{b,V} \) admits a bounded \( H^\infty \) functional calculus. We also recall that both \( L_{a,V'} \) and \( L_{b,V'} \) are sectorial operators. By Lemma 3.1, it is thus sufficient to show that the operator \( \Delta \) is triangular. Since Re \( a = \text{Re } b \), we get \( \Delta = i (L_{\text{im } a,V'} - L_{\text{im } b,V'}) L_{b,V}^{-1} - I \). Fix \( u \) and \( v \) in \( V' \). Then
\[
\langle \Delta u, v \rangle_{V'} = \langle I_V \Delta u, I_V v \rangle_{V'}
\]
\[
= i \langle (L_{\text{im } a,V'} - L_{\text{im } b,V'}) L_{b,V}^{-1} u, I_V v \rangle
\]
\[
= i \text{Im } a ((L_{b,V}J)^{-1}u, I_V v) - i \text{Im } b ((L_{b,V}J)^{-1}u, I_V v)
\]
\[
= i (L_{b,V}J)^{-1}u, (T_a - T_b) I_V v \rangle_{V'}.
\]
Since \( L_{b,V}J \) is an isomorphism from \( V \) onto \( V' \), the triangularity of \( \Delta \) is equivalent to the triangularity of \( T_a - T_b \).

For the proof of the second statement, it is sufficient to apply the one just proved for a symmetric form, that is, \( b \) with \( \text{Im } b = 0 \). □

In an analogous way, by combining Lemma 3.1 with Theorem 2.1 (iii), we get the following perturbation result for the real parts of forms.

**Corollary 3.3.** Let \( a \) and \( b \) be two sectorial forms on a space \( V \) with the same imaginary parts, that is, \( t_a = t_b \), and equivalent real parts \( s_a \) and \( s_b \). Let \( S \in \mathcal{L}(V) \) be such that \( s_a (Su, v) = s_b (u, v) \).

If \( S - I \) is triangular and \((b, H)\) has the Kato property, then \((a, H)\) has the Kato property, too.

**Proof.** According to Theorem 2.1 (iii), if \((H, \langle \cdot, \cdot \rangle)\) is a Hilbert space such that \((b, H)\) has the Kato property, then the operator \((I - iT_b)^{-1}Q_b\), where \( Q_b \) is a nonnegative, injective operator on \((V, q_b)\) with \( \langle u, v \rangle = q_b (Q_b u, v) \) \((u, v \in V)\), has a bounded \( H^\infty \) functional calculus. The corresponding operator \( Q_a \) for the form \( a \) is equal to \( SQ_b \) and \( T_a = ST_b \).

Hence,
\[
(I - iT_b)^{-1}Q_a = (I - iST_b)^{-1}SQ_b.
\]
Then, since \((iT_b - iST_b)(I - iT_b)^{-1}Q_b\) is triangular and
\[
(I - iST_b)^{-1}Q_b - (I - iT_b)^{-1}Q_b = (I - iST_b)^{-1}(iT_b - iST_b)(I - iT_b)^{-1}Q_b
\]
the operator \((I - iST_b)^{-1}Q_b\) has a bounded \( H^\infty \) functional calculus. Moreover, note that
\[
(I - iST_b)^{-1}SQ_b - (I - iST_b)^{-1}Q_b = (I - iST_b)^{-1}(S - I)Q_b
\]
\[
= \Delta (I - iST_b)^{-1}Q_b,
\]
where
\[
\Delta = (I - iST_b)^{-1}(S - I)(I - iST_b)
\]
is triangular. Therefore, again by Lemma 3.1, \((I - iST_b)^{-1}SQ_b\) has a bounded \( H^\infty \) functional calculus, which completes the proof. □

Finally, for the sake of completeness, we state a perturbation result for the operator \( Q \) generating the Hilbert space \( H \) in \((a, H)\). Its proof follows directly from Lemma 3.1 and Theorem 2.1 (iv).
Corollary 3.4. Assume that \((a, H)\) has the Kato property. Let \(Q\) be the nonnegative, injective operator on \(V\) associated with \(H\). Then, \((a, H_Q)\) has the Kato property for every Hilbert space \(H_Q\) with \(Q\) being a triangular perturbation of \(Q\), that is, \(\hat{Q} = (I + \Delta)Q\) for some triangular operator \(\Delta\).

4. Optimality of Corollary 3.2

Below, we show that the class of triangular operators is, in a sense, the largest subclass of compact operators for which Corollary 3.2 holds. Recall that, by [5] Theorem 5.5, a compact operator \(T\) is not triangular, if \(\sum_{n\in\mathbb{N}} \frac{s_n(T)}{n} = \infty\), where \(s_n(T)\) \((n\in\mathbb{N})\) stands for the \(n\)-th singular value of \(T\).

Proposition 4.1. Let \((a_n)_{n\in\mathbb{N}}\) be a nonincreasing sequence of positive numbers with \(\sum_{n\in\mathbb{N}} \frac{a_n}{n} = \infty\). Then, there exists a sectorial form \(a\) such that the singular values \((s_n(T_a))_{n\in\mathbb{N}}\) determined by the imaginary part of \(a\) satisfy \(s_n(T) \leq a_n\) \((n\in\mathbb{N})\), but not for every Hilbert space \(H\) for which \(V\) is densely and continuously embedded in \(H\), the couple \((a, H)\) has the Kato property.

Equivalently, there exist a selfadjoint, compact operator \(T\) on a Hilbert space \(V\) with \(s_n(T) \leq a_n\) \((n\in\mathbb{N})\), and a nonnegative, injective operator \(Q\) on \(V\) such that \(Q(I + iT)^{-1}\) is not similar to an accretive operator.

In order to construct an example we adapt two related results from [5] and [2]. Recall that, in [2], the sesquilinear form \(a\) on a Hilbert space \(H\) is expressed as

\[(4.1) \quad a(u, v) = \langle ASu, Sv \rangle_H, \quad u, v \in V := \mathcal{D}(S),\]

where \(S\) is a positive selfadjoint (not necessarily bounded) operator on \(H\), and \(A\) is a bounded invertible \(\theta\)-accretive operator on \(H\) for some \(\theta < \pi/2\). (Here, we call the selfadjoint operator \(S\) on \(H\) positive if \(\langle Su, u \rangle_H > 0\) for all \(u \in \mathcal{D}(A) \setminus \{0\}\).) Then, following Kato's terminology [5], \(a\) is a regular accretive form in \(H\). The operator \(I_a, H\) associated with the form \(a\) on \(H\) is given by \(SAS\). Note that \(s = \text{Re} a\) is an equivalent inner product to \(\langle S^*, S^* \rangle_H\), and in order to put it in our setting, we additionally assume that \(0 \in \rho(S)\). Then \(s\) is a complete inner product on \(V := \mathcal{D}(S)\). In fact, since \(S\) is selfadjoint, to get the completeness of this inner product, it is sufficient that \(S\) is injective and has closed range.

For the convenience of the reader we restate two auxiliary results from [5] and [2].

Lemma 4.2 ([5, Theorem 8.3]). Let \(H\) be a separable Hilbert space and let \((e_n)_{n \in \mathbb{N}}\) be an orthonormal basis. Let \(S\) be the sectorial operator defined by \(Se_n = 2^ne_n\) \((n \in \mathbb{N})\) with \(\mathcal{D}(S) := \{x \in H : \sum_{n=1}^{\infty} 2^{2n}|\langle x, e_n \rangle_H|^2 < \infty\}\). Suppose \(K \in L(H)\) is a non-triangular compact operator. Then, there exist bounded operators \(U\) and \(W\) on \(H\) such that for every \(m \in \mathbb{N}\), the operator \((I + 2^{-m} WKU)S\) fails to have a bounded \(H^\infty\) functional calculus.

Lemma 4.3 ([2, Theorem 10.1]). Let \(A, S, a\) have the properties specified above. Then \((a, H)\) has the Kato property if and only if the operator \(AS\) has a bounded \(H^\infty\) functional calculus.

Lemma 4.4. Let \(A, S, a\) have the properties specified above. Let \(T\) and \(Q\) be the operators associated with \(a\) and \(H\).

(i) The operator \(T \in L(V)\) is compact if and only if the operator \(\text{Im} A \in L(H)\) is compact. Then, \(s_n(T) \simeq s_n(\text{Im} A)\) \((n \in \mathbb{N})\), that is, there exists \(c > 0\) such that \(c^{-1}s_n(T) \leq s_n(\text{Im} A) \leq cs_n(T)\) for every \(n \in \mathbb{N}\).
(ii) The operator $Q \in \mathcal{L}(V)$ is compact if and only if the embedding of $V$ into $H$ is compact, if and only if $S^{-1} \in \mathcal{L}(H)$ is compact. Then, $s_n(Q) \simeq s_n(S^{-1}) \simeq s_n(j)$ ($n \in \mathbb{N}$) where $j$ denotes the canonical embedding of $V$ into $H$.

Proof. First, note that the operators $T$ and $Q$ are of the form:

$$T = S^{-1}(\text{Re } A)^{-1}\text{Im } A S$$

and

$$Q = S^{-1}(\text{Re } A)^{-1}S^{-1},$$

where $\text{Re } A$ and $\text{Im } A$ denote the real and the imaginary part of $A$, and $S^{-1}$ is the restriction of $S^{-1} \in \mathcal{L}(H)$ to $V$, considered as an operator in $\mathcal{L}(V,H)$. These expressions give the first statements in (i) and (ii). The second assertion in (i) follows in a straightforward from, e.g., [13, Theorem 7.7, p. 171].

To prove the corresponding one of (ii), assume that $S^{-1}$ is compact with spectrum $\sigma(S^{-1}) = \{\mu_n\}_{n \in \mathbb{N}}$ where $\mu_n \to 0^+$ as $n \to \infty$. Therefore, there exists an orthonormal system $\{e_n\}_{n \in \mathbb{N}}$ in $H$ such that $Sh = \sum_n \mu_n h e_n$ for $h \in \mathcal{D}(S) = \{h \in H : \sum \mu_n^2 |(h,e_n)|^2 < \infty \}$. Let $C : V_* \to H$, $Cu := S^{-1}u$, $u \in \mathcal{D}(S)$, where $V_*$ denote the Hilbert space $(\mathcal{D}(S), (S^* S)_{H})$. Of course, $C \in \mathcal{L}(V_*, H)$ and $C^* C \in \mathcal{L}(V_*)$ are compact. Moreover, note that

$$C^* Cu = \sum_n \mu_n^2 \langle u, g_n \rangle_{V_*} g_n, \quad u \in V_*,$$

where $g_n := \mu_n^{-2} e_n$ ($n \in \mathbb{N}$) is an orthonormal basis for $V_*$. Thus, the singular values of $C$ are given by $s_n(C) = \mu_n$, $n \in \mathbb{N}$.

Now, let $I_S$ denote the identity map on $\mathcal{D}(S)$ considered as an operator from $V$ onto $V_*$. Therefore, we have $S^{-1} = CI_S$ and, by [13, Theorem 7.1, p. 171], we get

$$s_n(Q) \leq \|S^{-1}(\text{Re } A)^{-1}\|_{\mathcal{L}(H)} s_n(CI_S) = \|S^{-1}(\text{Re } A)^{-1}\|_{\mathcal{L}(H)} \|I_S\|_{\mathcal{L}(V_*,V)} s_n(C^*)$$

and

$$s_n(C) = s_n((\text{Re } A)SQI_S^{-1}) \leq \|S^{-1}(\text{Re } A)\|_{\mathcal{L}(V,H)} \|QI_S^{-1}\|_{\mathcal{L}(V_*,V)} s_n(Q).$$

Finally, note that $s_n(S^{-1})$ is equal to the $n$-th singular value of the embedding of $V_*$ into $H$. This completes the proof. \hfill \Box

Proof of Proposition 4.2. Suppose that $H$, $S$, $K$, $U$, $W$ have the properties specified above in Lemma 4.1. Fix $m \in \mathbb{N}$ such that the numerical range of the operator $A := I + 2^{-m} W K U$ is contained in $\{ |z| < 1 \}$. Then, by Lemma 4.3 the couple $(a, H)$ does not have the Kato property, where the form $a$ corresponds to the operators $A$ and $S$ as above. Moreover, by Lemma 4.3(i),

$s_n(T) \simeq s_n(\text{Im } A) \simeq s_n((2^{-m} W K U - U^* K^* W^*)) \leq s_n(K) \quad (n \in \mathbb{N}).$

This completes the proof. \hfill \Box
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