Design of uncertainty and disturbance estimator based tracking control for fuzzy switched systems

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Abstract
In this paper, an uncertainty and disturbance estimator (UDE)-based control is employed to ensure the finite-time tracking and disturbance rejection performance for a class of Takagi–Sugeno fuzzy switched systems including additive time-varying delays, unknown time-varying uncertainties and disturbances. To be precise, the robust tracking performance is achieved by estimating unknown uncertainties and disturbances with the aid of low-pass filter by means of appropriate bandwidth choice of the desired filter. Moreover, a new delay-dependent finite-time stability criterion is derived by employing fuzzy-dependent Lyapunov–Krasovskii functional, extended Wirtinger’s integral inequality and average dwell-time approach to enforce the addressed system to track the given reference system within a finite period of time. Finally, the developed theoretical results are ensured by presenting the numerical examples with simulation results.

1 | INTRODUCTION

In recent years, switched systems have received considerable attention due to their wide applications in several real processes such as network communications, mechanical systems, aircraft systems, electric circuits, traffic control, power systems and mobile robots. In particular, switched systems consists of many subsystems wherein the coordination between the subsystem takes place with respect to the desired switching rule and also it generally exhibits switching in structure, criterion and parameters due to the abrupt changes in environment which are indispensable [1–5]. More specifically, the switching rule indicates the activation of individual subsystem at each instant of time to explore the dynamics of the system in distinct modes. Therefore, the stability analysis of various kinds of switched systems through the average-dwell time approach has received much attention among many researchers [6–9]. For instance, the authors in [6] developed a robust dynamic output feedback based switching-type control law for a class of linear uncertain time-varying systems based on Lyapunov theory. Cheng et al. [7] presented an event-triggered algorithm for fuzzy Markovian jump systems based on asynchronous operation approach and switching policies. Besides, the robust control protocol for switched systems with parametric uncertainties and input saturation is proposed in [9] by designing some nested ellipsoid invariant sets that make the region of attraction as large as possible by maximising the outmost ellipsoid.

In addition, most of the real world problems are mathematically modelled as a complex nonlinear systems in which nonlinearities induce complications while dealing with the stability and stabilisation analysis [10–15]. To endure this complication, the nonlinear system is decomposed into several linear systems with different weights through the Takagi–Sugeno (TS) fuzzy model approach [16–27]. On the other hand, most of the practical models are significantly affected by unpredictable uncertainties and unmeasurable exogenous disturbances, which lead to poor system performance and may cause instability [28–35]. However the estimated values of the uncertainties and disturbances are required to compensate the disturbance attenuation level of the nonlinear control systems [36–38]. But in practice the unpredictable changes are difficult to measure and sometimes impossible. To conquer the above stated difficulties, an effective and quickly estimable uncertainty and disturbance estimator (UDE)-based controller scheme is implemented which has been widely applied for many dynamical systems in recent years (see [39–43] and the references therein). It should be noted that the
UDE-based controller perfectly estimates the unknown disturbances, nonlinearities and uncertainties through insertion of a low-pass filter with appropriate bandwidth [41]. More specifically, tracking controller is imperative and able to ensure the better performance of system dynamics and possibly track the desired trajectories even in the presence of lumped disturbances [44, 45].

It is well known that the trajectories depend on past and present states of the system and if the needs of both present and past states are ignored, then it may distort the desired system performance. In general, time-varying delay usually appeared in real-world problems and it can cause instability, oscillations and performance degradation [46–49]. Since the measured signals are transmitted from one spot to another might encounter delays from different parts of the systems with different properties. Therefore, remarkable research works among the research community are carried out on dynamic control systems with additive time-varying delay [50, 51]. It should be noted that the finite-time tracking performance will be more effective than the exponential tracking performance for nonlinear control systems [52–56]. It is worth mentioning that the finite-time tracking controller ensures the system states to track the desirable signals within a finite period of time even in the presence of noises. However, to the authors’ knowledge, no work has been available yet for the UDE-based finite-time tracking control design for the TS fuzzy switched system including additive time-varying delays, unpredictable uncertainties and unknown external disturbances which motivates this study. Specifically, in this article new constraints are developed to attain the less convergence rate of the error dynamics and also to reject the disturbances via a perfect estimation technique within a finite period of time by employing a new fuzzy-dependent Lyapunov–Krasovskii functional, extended Wirtinger’s integral inequality and average dwell-time approach. The main contribution and novelties proposed in this article are summarized as follows:

1. A novel UDE-based controller is designed to ensure the finite-time tracking performance of TS fuzzy switched systems with additive time-varying delays, unpredictable uncertainties and unknown external disturbances.
2. A new fuzzy switched reference model with additive time-varying delays, uniformly bounded and piecewise continuous reference input is considered for the addressed system.
3. In order to estimate unpredictable uncertainties and unknown disturbances, a UDE-based approach is employed via fuzzy switched error dynamics and the desired convergence matrices are determined.
4. Precisely, the proposed controller can overcome the disturbance effect and relax general assumptions of the system.

Notations: The notation $X \succ (\succeq, \prec, \preceq) 0$ represents positive definite (semi-positive definite, negative-definite, semi-negative definite) matrix. $\mathbb{Z}$ denotes the set of all positive integers. $\mathbb{R}^n$ and $\mathbb{R}^+$, respectively represent the set of all $n \times n$ real-valued matrices and positive real numbers. $(.)^T$ symbolise the transpose of the matrix. $(.)^+$ denotes the pseudo inverse of any matrix $A$. $L_2[0, \infty)$ implies the space of all square integrable functions.

### PROBLEM FORMULATION

Consider a class of switched systems with time-varying uncertainties, additive time-varying delays and external disturbances. The dynamics of the considered system are represented as a TS fuzzy based differential equations with $r_{\sigma(t)}$ fuzzy rules. The description of $\ell$-th fuzzy rule is as follows:

Plant rule $R^e_{\sigma(t)}$: IF $\bar{\eta}_1(t)$ is $\bar{\xi}_{\sigma(t)}(i)$, $\bar{\eta}_2(t)$ is $\bar{\xi}_{\sigma(t)}(j)$... and $\bar{\eta}_r(t)$ is $\bar{\xi}_{\sigma(t)}(k)$ THEN

$$
\dot{x}(t) = (A_{\sigma(t)} + \Delta A_{\sigma(t)}(t))x(t) + (A_{\sigma(t)}g(t) + \Delta A_{\sigma(t)}g(t))
\times w(t) + B_{\sigma(t)}u_{\sigma(t)}(t) + C_{\sigma(t)}w(t),
$$

where $R^e_{\sigma(t)} = \{1, \ldots, r_{\sigma(t)}\}$ describes the $\ell$-th fuzzy rule and $r_{\sigma(t)} \in \mathbb{Z}$ represents the number of IF-THEN rules in the $\sigma$-th switched subsystem, wherein piecewise constant function $\sigma(t) : \mathbb{R}^+ \rightarrow \mathcal{S}$ represents the switching law in which $\mathcal{S} = \{1, \ldots, \mathcal{S}\}$ and $\mathcal{S}$ denotes overall modes in switched system; $\bar{\xi}_{\sigma(t)}$ denotes the fuzzy set of $\ell$-th rule; $\bar{\eta}_i(t)$ = $\{\bar{\eta}_1(t), \bar{\eta}_2(t), \ldots, \bar{\eta}_r(t)\}$ $i$ $\in$ $\{1, \ldots, k\}$ and $k$ is the positive integer) is the premise variables; $x(t) \in \mathbb{R}^n$ denotes the state vector, $u_{\sigma(t)}(t) \in \mathbb{R}^{n_u}$ is the control input vector, $w(t) \in \mathbb{R}^m$ is the external disturbance which belongs to $L_2[0, \infty)$, $A_{\sigma(t)}$, $A_{\sigma(t)}g(t)$, $B_{\sigma(t)}$ and $C_{\sigma(t)}$ are constant matrices with appropriate dimensions of $\mathcal{S}$-subsystem; $\Delta A_{\sigma(t)}g(t)$ and $\Delta A_{\sigma(t)}g(t)$ represent the unknown time-varying uncertainties of $\mathcal{S}$-subsystem; $d(t)$ and $\bar{d}(t)$ are the time-varying delays, which satisfy the conditions $0 \leq d(t) \leq d_1(t) \leq \bar{d}$, $0 \leq \bar{d} \leq \bar{d}_2$, and $d(t)$ is $\bar{d}_1$, and $\bar{d}_2$, $\bar{d}_1$ and $\bar{d}_2$ are known positive scalars. Let $\bar{d}(t) = d_1(t) + d_2(t)$ and $\bar{d} = \bar{d}_1 + \bar{d}_2$.

Based on fuzzy blending, the $\varrho$-th fuzzy switched system (1) can be described by

$$
\dot{x}(t) = \sum_{i=1}^{r_{\varrho(t)}} \Psi_{\varrho(i)}(\bar{\eta}(t)) \left[ (A_{\varrho(i)}g(t) + \Delta A_{\varrho(i)}g(t))x(t) + (A_{\varrho(i)}g(t) + \Delta A_{\varrho(i)}g(t)) \right]
\times w(t) + B_{\varrho(i)}u_{\varrho(i)}(t) + C_{\varrho(i)}w(t),
$$

where $\Psi_{\varrho(i)}(\bar{\eta}(t)) = \int_{\varrho(t)}^{\varrho(t)} \mu_{\varrho(i)}(\bar{\eta}(t)) \sum_{i=1}^{r_{\varrho(i)}} \Psi_{\varrho(i)}(\bar{\eta}(t))$ is the fuzzy weight function in which $\mu_{\varrho(i)}(\bar{\eta}(t))$ = $\int_{\varrho(t)}^{\varrho(t)} \xi_{\varrho(i)}(\bar{\eta}(t))$, $\sum_{i=1}^{r_{\varrho(i)}} \mu_{\varrho(i)}(\bar{\eta}(t)) = 1$, $\forall t > 0$ and $\mu_{\varrho(\varrho(i))} \geq 0$. Then, we obtain $\Psi_{\varrho(i)}(\bar{\eta}(t)) \geq 0$ and $\sum_{i=1}^{r_{\varrho(i)}} \Psi_{\varrho(i)}(\bar{\eta}(t)) = 1$, $\forall t > 0$. Thereafter, for notation simplicity, let $\Delta A_{\varrho(i)}g(t) = \sum_{i=1}^{r_{\varrho(i)}} \Psi_{\varrho(i)}(\bar{\eta}(t))A_{\varrho(i)}g(t) - A_{\varrho(i)}g(t)$, $\Delta A_{\varrho(i)}g(t) = \sum_{i=1}^{r_{\varrho(i)}} \Psi_{\varrho(i)}(\bar{\eta}(t))A_{\varrho(i)}g(t) - A_{\varrho(i)}g(t)$, $\Delta A_{\varrho(i)}g(t) = \sum_{i=1}^{r_{\varrho(i)}} \Psi_{\varrho(i)}(\bar{\eta}(t))A_{\varrho(i)}g(t) - A_{\varrho(i)}g(t)$, $B_{\varrho(i)} = \sum_{i=1}^{r_{\varrho(i)}} \Psi_{\varrho(i)}(\bar{\eta}(t))B_{\varrho(i)}$ and $C_{\varrho(i)} = \sum_{i=1}^{r_{\varrho(i)}} \Psi_{\varrho(i)}(\bar{\eta}(t))C_{\varrho(i)}$. Finally, the above equation can be rewritten as

$$
\dot{x}(t) = (A_{\varrho(i)}g(t) + \Delta A_{\varrho(i)}g(t))x(t) + (A_{\varrho(i)}g(t) + \Delta A_{\varrho(i)}g(t)) \times (d_1(t) - d_2(t))
\times w(t) + B_{\varrho(i)}u_{\varrho(i)}(t) + C_{\varrho(i)}w(t).
$$

(3)

To develop the UDE-based tracking controller, the TS fuzzy reference model is considered for the $\varrho$-th subsystem (1) in the
following form:

\[
\dot{x}_n(t) = \sum_{r=1}^\infty \psi_{r}(\eta(t)) [A_{spslt} x_n(t) + A_{dpslt} x_n(t) - d_1(t) - d_2(t)] + B_{spslt} r(t),
\]

(4)

where \(x_n(t)\) and \(r(t)\) denote reference state and reference input, respectively. For notational simplicity, the coefficient matrices of the \(r\)-th reference subsystem are taken as \(A_{spslt} = \sum_{r=1}^\infty \psi_{r}(\eta(t)) A_{spslt}, A_{dpslt} = \sum_{r=1}^\infty \psi_{r}(\eta(t)) A_{dpslt}\) and \(B_{spslt} = \sum_{r=1}^\infty \psi_{r}(\eta(t)) B_{spslt}\).

The state feedback controller \(u_p(t)\) will be determined to track the desired reference trajectories with the aid of known dynamics of the addressed system, which forces that the error vector \(e(t) = x_n(t) - x(t)\) converges to zero. Consider the closed-loop error dynamics for the TS fuzzy switched system (1) in the following form:

\[
\dot{e}(t) = (\hat{A}_{spslt} + \hat{K}_{gy}) e(t) + \hat{A}_{dpslt} e(t) - d_1(t) - d_2(t),
\]

(5)

where \(\hat{K}_{gy} = \sum_{r=1}^\infty \psi_{r}(\eta(t)) K_{gy}\) are the feedback gain matrices of the \(r\)-th error subsystem, which to be determined. Especially, the control gain matrices \(K_{gy}\) are chosen based on the pole placement strategy to guarantee the stability of the error dynamics.

From the equations (3)--(5), we obtain

\[
\dot{\bar{y}}_{p} = \hat{A}_{spslt} x_n(t) + \hat{A}_{dpslt} (x(t) - d_1(t) - d_2(t)) + \hat{B}_{spslt} r(t) \times (t) - \hat{A}_{spslt} x_n(t) - \hat{A}_{dpslt} x(t) - d_1(t) - d_2(t))
\]

(6)

where the collection of unknown uncertainties and unpredictable disturbances is given by

\[
u_{p}(t) = \Delta \hat{A}_{spslt} x_n(t) + \Delta \hat{A}_{dpslt} x(t) - d_1(t) - d_2(t)) + \hat{C}_{p} w(t).
\]

(7)

The above lumped disturbances are equivalently reconstructed with respect to the known system dynamics and control signals. Now, the reformulated disturbance estimation can be represented as

\[
u_{p}(t) = \dot{v}(t) = \hat{A}_{spslt} x_n(t) - \hat{A}_{dpslt} x(t) - d_1(t) - d_2(t)) - \hat{B}_{spslt} u_{p}(t).
\]

(8)

However, the above control law consists of state derivative and to conquer this issue, a filtration process is carried out. Finally, the unknown lumped disturbances can be estimated as

\[
\nu_{p}(t) = \nu_{p}(t) * q_{1}(f)
\]

\[
= \{\dot{v}(t) \times (t) - \hat{A}_{spslt} x_n(t) - \hat{A}_{dpslt} x(t) - d_1(t) - d_2(t))
\]

(9)

\[
- \hat{B}_{spslt} u_{p}(t) * q_{1}(f),
\]

where "*" denotes the convolution operator; \(q_{1}(f)\) is the impulse response of the filter \(Q_{f}(f)\). Now, the UDE-based control strategy in (6) is reconstructed via replacing the unknown lumped disturbance with \(u_{p} u_{p}\), in (9), we get

\[
\hat{B}_{spslt} u_{p}(t) = \hat{A}_{spslt} x_n(t) + \hat{A}_{dpslt} x(t) - d_1(t) - d_2(t) + \hat{B}_{spslt} r(t)
\]

\[
\times (t) - \hat{A}_{spslt} x_n(t) - \hat{A}_{dpslt} x(t) - d_1(t) - d_2(t)) - K_{gy}
\]

\[
\times e(t) - [\hat{A}_{spslt} x_n(t) - \hat{A}_{dpslt} x(t) - d_1(t) - d_2(t))
\]

(10)

Now, by taking inverse Laplace transform on both sides of the above equation, we obtain

\[
\hat{B}_{spslt} u_{p}(t) = \hat{B}_{spslt} + \hat{B}_{spslt} r(t) - K_{gy} e(t) - \hat{A}_{spslt} x(t)
\]

\[
\times (t) - d_1(t) - d_2(t)) - \hat{A}_{spslt} x(t)
\]

\[
\hat{A}_{spslt} x_n(t) - \hat{A}_{dpslt} x(t) - d_1(t) - d_2(t)) - \hat{B}_{spslt} u_{p}(t) * q_{1}(f).
\]

(11)

where \(\hat{B}_{spslt} = (\hat{B}_{spslt})^{+} [\hat{L}^{-1}\{\frac{1}{1-Q_{f}(f)}\} \ast [\hat{A}_{spslt} x(t) + \hat{A}_{dpslt} x(t) - d_1(t) - d_2(t)) + \hat{B}_{spslt} r(t) - K_{gy} e(t)] ] - \hat{K}_{gy} e(t)]\), UDE = \((\hat{B}_{spslt})^{+} [\hat{A}_{spslt} x(t) - \hat{A}_{dpslt} x(t) - d_1(t) - d_2(t)) - \hat{A}_{spslt} x_n(t) - \hat{A}_{dpslt} x(t) - d_1(t) - d_2(t)) - \hat{B}_{spslt} u_{p}(t) * q_{1}(f)\) and error feedback = \((\hat{B}_{spslt})^{+} [L^{-1}\{\frac{1}{1-Q_{f}(f)}\} \ast [\hat{K}_{gy} e(t)]\).

The main objective of the proposed UDE-based control scheme is that it should increase the tracking error convergence rate with respect to the bandwidth \(\delta\). It should be mentioned that the strong disturbance restrain capacity is retained by means of appropriate bandwidth selection of the desired low-pass filter. Thus the filter design in control algorithm is imperative and it can be defined in the following form:

\[
Q_{f}(f) = \frac{1}{\delta^{2} + 1}.
\]

(12)

By imposing the above low-pass filter, the proposed control law (10) can be rewritten as

\[
\hat{B}_{spslt} u_{p}(t) = (\hat{B}_{spslt})^{+} [\hat{A}_{spslt} x_n(t) + \hat{A}_{dpslt} x(t) - d_1(t) - d_2(t))
\]

\[
+ \hat{B}_{spslt} r(t) - \hat{A}_{spslt} x_n(t) - \hat{A}_{dpslt} x(t) - d_1(t) - d_2(t))
\]

\[
+ \delta \left[ \left\{ \hat{A}_{spslt} + \hat{K}_{gy} e(t) \right\} - \hat{A}_{spslt} e(t) - d_1(t) - d_2(t))
\]

(13)

\[
- \hat{A}_{spslt} + \hat{K}_{gy} \int_{0}^{\infty} e(\tau) d\tau - \hat{A}_{dpslt} \int_{0}^{\infty} e(t) - d(t) d\tau \right\}
\]
Remark 1. It is worthy to mention that the most of the exiting control approaches require some assumptions on unpredictable aspects [28–33] to obtain the required result. More precisely, in active disturbance rejection control (ADRC) approach [28], the disturbance attenuation is accomplished with the aid of extended state observer and nonlinear feedback. Unfortunately, the boundedness of the derivative of the disturbance cannot be handled by the ADRC design. In particular, the UDE based control strategy is proposed by relaxing this assumption on the disturbance and only requires the bandwidth of the disturbance for the filter design. Besides, the equivalent-input-disturbance-based control approach [31] requires that the disturbance should be applied or interpreted to the input channel, whereas the proposed UDE-based control does not have this restriction. The chattering phenomenon may occur in the sliding mode control scheme (SMCS) due to the existence of signum function in the control design. Precisely, the simulation results in [32] demonstrate that disturbance observer based control approach (DOBCA) is better than sliding mode control for tracking control problem of robotic manipulators. Notably, the basic idea of SMCS is to use switching functions with gains larger enough to cover the uncertainties. Moreover, in DOBCA [33], the nominal plant model is incorporated in taking the inverse of the plant model. But the UDE does not, so the UDE-based control is applicable to a wider class of systems and is easy to be implemented and tuned while bringing very good robust performance.

Figure 1 presents the block diagram of the proposed controller structure. Furthermore, to obtain the stability criterion for the system (5), the following lemma and definitions are indispensable:

Lemma 1 ([46]). For all continuously differentiable function \( \lambda : [x, y] \rightarrow \mathbb{R}^k \) and a positive definite matrix \( \mathbf{Z} \), the inequality \( \int_{t_1}^{t_2} \lambda^T(s)\mathbf{Z}\lambda(s)ds \leq \frac{1}{y-x}Y^T\Theta Y \) holds, where

\[
Y^T = \begin{bmatrix}
\int_{x}^{y} \lambda^T(s)ds & \int_{x}^{y} \lambda^T(s)du & \int_{x}^{y} du & \int_{x}^{y} dv \\
\int_{x}^{y} \lambda du & \int_{x}^{y} du & \int_{x}^{y} dv & \int_{x}^{y} dv \\
\int_{x}^{y} dv & \int_{x}^{y} dv & \int_{x}^{y} dv & \int_{x}^{y} dv \\
\end{bmatrix}
\]

and

\[
\Theta = \begin{bmatrix}
\frac{1}{y-x}Z & \frac{1}{y-x}Z & \frac{1}{y-x}Z & \frac{1}{y-x}Z \\
\frac{1}{y-x}Z & \frac{1}{y-x}Z & \frac{1}{y-x}Z & \frac{1}{y-x}Z \\
\frac{1}{y-x}Z & \frac{1}{y-x}Z & \frac{1}{y-x}Z & \frac{1}{y-x}Z \\
\frac{1}{y-x}Z & \frac{1}{y-x}Z & \frac{1}{y-x}Z & \frac{1}{y-x}Z \\
\end{bmatrix}
\]

Definition 1 ([23]). If \( \mathcal{N}_\sigma(t_1, t_2) \leq \mathcal{N}_0 + \frac{t_2 - t_1}{\tau} \), where \( \mathcal{N}_\sigma \) represents the switching numbers of \( \sigma(t) \), holds for each \( t_2 > t_1 \geq 0 \), chatter bound \( \mathcal{N}_0 \geq 1 \), and \( \tau > 0 \), then \( \tau^* \) is called the average dwell-time.
Theorem 1. Consider the closed-loop error system (5) with the known real constant bandwidth of the lumped disturbance. For given positive scalars $\alpha, \bar{r}_1, \bar{t}_2, \mu$ and $\bar{\beta}_f$ ($\ell = 1, \ldots, r - 1$), the system (5) is finite-time stable with respect to $(\varepsilon_1, \varepsilon_2, \bar{a}_1, \bar{d}_2, \mathcal{M})$, if there exist real symmetric matrices $R_{\phi_0} > 0$ ($k = 1, \ldots, 5$), $R_{\phi_f} > 0$, such that the following inequalities hold for $1 \leq \ell < j \leq r$ and $\phi, \tilde{\phi} \in \{1, \ldots, \bar{\sigma}\}$:

$$R_{\phi_f} > R_{\phi_0}, \quad (j = 1, \ldots, r - 1)$$

$$\Pi_{a/\ell}^{a/\ell} < 0, \quad (n, s, f, \ell = 1, \ldots, r)$$

$$\Pi_{a/\ell}^{a/\ell} + \Pi_{a/\ell}^{a/\ell} < 0, \quad (n, s, f, j = 1, \ldots, r), \ell \neq j$$

$$R_{\phi_0} < \mu R_{\phi_f}, R_{\phi_f} < \mu R_{\phi_0},$$

$$\eta_{\varepsilon_1} < \lambda_1 e^{T} \varepsilon_2,$$

where

$$\Pi_{a/1}^{a/1} = A_{\phi_0} R_{\phi_0} + R_{\phi_0} A_{\phi_0}^T + Y_{\phi_0} + (Y_{\phi_0}(t))^T + \alpha R_{\phi_f} + \alpha R_{\phi_0} + \alpha \Phi_{\phi_0} + \sum_{i=1}^{\varepsilon_1} \phi_i (R_{\phi_f} - R_{\phi_0}), \quad \Pi_{a/2}^{a/2} = R_{\phi_0} A_{\phi_0} \Phi_{\phi_0},$$

$$\Pi_{a/2}^{a/2} = \Pi_{a/2}^{a/2} - (1 - \bar{r}_1) \exp(-\alpha \bar{a}) (R_{\phi_f} - R_{\phi_0}), \quad \Pi_{a/3}^{a/3} = \Pi_{a/3}^{a/3} - \Pi_{a/3}^{a/3} = -9 \exp(-\alpha \bar{d})$$

$$R_{\phi_0}, \quad \Pi_{a/4}^{a/4} = \frac{192 \exp(-\alpha \bar{a}) R_{\phi_0}}{2\mu}, \quad \Pi_{a/5}^{a/5} = \frac{60 \exp(-\alpha \bar{d}) R_{\phi_0}}{2\mu},$$

$$\Pi_{a/6}^{a/6} = \frac{-2 \exp(-\alpha \bar{a}) R_{\phi_0}}{\bar{\beta}_f - \mu \Phi_{\phi_0}}, \quad \Pi_{a/7}^{a/7} = \frac{-2 \exp(-\alpha \bar{d}) R_{\phi_0}}{\bar{\beta}_f - \mu \Phi_{\phi_0}}, \quad \Pi_{a/7}^{a/7} = \frac{-2 \exp(-\alpha \bar{a}) R_{\phi_0}}{\bar{\beta}_f - \mu \Phi_{\phi_0}},$$

$$\eta_{\varepsilon_1} < \lambda_1 e^{T} \varepsilon_2,$$

where $\bar{a} = \bar{a}_1 + \bar{d}_2, \bar{f} = \bar{t}_1 + \bar{t}_2, R_{\phi_0}(t) = M^T R_{\phi_0}(t) M^{-1}, \bar{r}_1 = 1, \ldots, 5$,

$$w = \max_{\phi \in \mathcal{S}_+} \lambda_{\max}(R_{\phi_0}(t)) + \exp(\alpha \bar{a}_1) \lambda_{\max}(R_{\phi_0}(t)) + \alpha \bar{d}, \quad \tilde{\phi}_i = 1, \ldots, 5$$

The proof of this theorem involves the use of Lyapunov-Krasovskii functional and the construction of a suitable Lyapunov function candidate. The error system (5) is finite-time stable if and only if there exists a positive constant $\xi_0$ such that

$$\zeta_x > \xi_0 = \frac{\mathcal{X} \ln \mu}{\ln \frac{\tilde{\xi}_0}{\xi_0} + \alpha \mathcal{X}},$$

where $\mu > 1$. Moreover, the error feedback control gain matrices are computed by

$$K_{\phi_f} = Y_{\phi_0} R_{\phi_f}^{-1}.$$

Proof. Here the finite-time stability constraints are established by considering the following Lyapunov-Krasovskii functional candidate for the closed-loop error system (5):

$$V_{\phi_f}(t) = \sum_{j=1}^{3} V_{\phi_j}(t),$$

where

$$V_{\phi_f}(t) = e^T(t) R_{\phi_f}(t) e(t),$$

$$V_{\phi_0}(t) = \int_{-d_1(t)-d_2(t)}^{t} \exp(\alpha \bar{a}) e^T(s) R_{\phi_0} e(s) ds$$

$$+ \int_{-d_1(t)}^{t} \exp(\alpha \bar{a}) e^T(s) R_{\phi_0} e(s) ds$$

$$+ \int_{-d_2(t)}^{t} \exp(\alpha \bar{a}) e^T(s) R_{\phi_0} e(s) ds$$

with $R_{\phi_f}(t) = \sum_{j=1}^{3} \Psi_{\phi_0}(t) R_{\phi_f},$ here $R_{\phi_f}, R_{\phi_0}, \bar{r}_1 \in \{1, \ldots, 5\}$, are positive-definite matrices for the $\phi_i$-th subsystem.

The time derivative of the considered Lyapunov-Krasovskii functional (20) along the trajectories of the error system (5) is given by

$$\dot{V}_{\phi_f}(t) = 2 e^T(t) R_{\phi_f}(t) e(t) + e^T(t) \dot{R}_{\phi_f}(t) e(t),$$

$$= 2(\bar{a} e^T(t) R_{\phi_f}(t) e(t) + \bar{d} e^T(t) \dot{R}_{\phi_f}(t) e(t) - \bar{r} \exp(-\alpha \bar{d}))$$

$$\times R_{\phi_f}(t) e(t) + e^T(t) R_{\phi_f}(t) e(t) - \alpha V_{\phi_f}(t),$$

$$\dot{V}_{\phi_0}(t) \leq e^T(t) (R_{\phi_0}(t) + R_{\phi_0}(t)) e(t) - (1 - \bar{r}) \exp(-\alpha \bar{d})$$
\[ V_\phi(t) \leq (\hat{d})^2 t^2 (t) R_{\phi\phi} e(t) - \hat{d} \]
\[ \times \int_{t-\hat{d}(t)}^t \exp\{-\alpha \hat{d}\} e^T(t) R_{\phi\phi} e(t) ds \]
\[ - \alpha V_\phi(t), \quad (23) \]

where \( \hat{d}(t) = d_1(t) + d_2(t), \quad \hat{d} = \hat{d}_1 + \hat{d}_2, \quad \bar{\epsilon} = \bar{\epsilon}_1 + \bar{\epsilon}_2. \) The reformulated form of the integral term in the above inequality (23) by employing Lemma 1 is given by

\[ -\hat{d} \int_{t-\hat{d}(t)}^t \exp\{-\alpha \hat{d}\} e^T(t) R_{\phi\phi} e(t) ds \]
\[ \leq \xi^T(t) \exp\{-\alpha \hat{d}\} \begin{bmatrix} -9 R_{\phi\phi} & \frac{36}{d1} R_{\phi\phi} & \frac{60}{d2} R_{\phi\phi} \\ \ast & \frac{-192}{d^2} R_{\phi\phi} & \frac{360}{d^3} R_{\phi\phi} \\ \ast & \ast & \frac{-720}{d^4} R_{\phi\phi} \end{bmatrix} \xi(t), \quad (24) \]

where \( \xi^T(t) = \int_{t-\hat{d}(t)}^t e^T(t) ds \int_{t-\hat{d}(t)}^t e^T(t) ds \int_{t-\hat{d}(t)}^t e^T(t) ds \int_{t-\hat{d}(t)}^t e^T(t) ds \int_{t-\hat{d}(t)}^t e^T(t) ds \int_{t-\hat{d}(t)}^t e^T(t) ds. \)

Moreover, with the aid of fuzzy weighting function property and (13), it is easy to obtain that

\[ \hat{R}_\phi(t) = \sum_{\ell=1}^r \Psi_{\phi\ell}(\eta(t)) R_{\phi\phi} = \sum_{\ell=1}^r \Psi_{\phi\ell}(\eta(t)) (R_{\phi\phi} - R_{\phi\phi}) \]
\[ \leq \sum_{\ell=1}^r \hat{\beta}_{\phi\ell} (R_{\phi\phi} - R_{\phi\phi}). \quad (25) \]

Now, by combining the inequalities (21)–(25), we get

\[ V_\phi(t) + \alpha V_\phi(t) \leq \sum_{s=1}^r \sum_{j=1}^r \sum_{t=1}^r \sum_{s=1}^r \sum_{j=1}^r \sum_{t=1}^r \Psi_{\phi\phi}(\eta(t - \hat{d}(t))) \]
\[ \times \Psi_{\phi\phi}(\eta(t - d_1(t))) \Psi_{\phi\phi}(\eta(t - d_2(t))) \]
\[ \times \Psi_{\phi\ell}(\eta(t)) \Psi_{\phi\ell}(\eta(t)) \xi^T(t) \xi(t), \quad (26) \]

Integrate the above inequality from \( t_a \) to \( t \), we obtain

\[ V_\phi(t) \leq \exp\{-\alpha(t - t_a)\} V_\phi(t_a). \quad (27) \]

Then, from the inequality (17), we can have

\[ V_\phi(t_a) \leq \mu V_\phi(t_a^-) \quad (28) \]

where \( \mu > 1. \) Now, from (26) and (27), we get

\[ V_\phi(t) \leq \mu \exp\{-\alpha(t - t_a)\} V_\phi(t_a^-) \]
\[ \leq \mu^2 \exp\{-\alpha(t - t_{a-1})\} V_\phi(t_{a-1}^-) \]
\[ \leq \cdots \leq \mu^N(t, 0) \exp\{-\alpha \Xi\} V_\phi(0), \quad (29) \]

where \( \mathcal{N}_c(0, \Xi) \leq (t_f - t_a) t_a \) is the number of switching of \( \phi\)-th subsystem over the interval \([0, \Xi]\), here \( \Xi \) is the average dwell-time. Let \( R_{\phi\phi} = \mathcal{M}^T R_{\phi\phi} \mathcal{M} \), \( i = 1, \ldots, 5 \), \( R_{\phi\phi} = \mathcal{M}^T R_{\phi\phi} \mathcal{M} \) and from the above inequality (28), it follows that

\[ V_\phi(t) \geq e^T(t) R_{\phi\phi} e(t) \geq \min_{\phi \in \Lambda_v} \{\lambda_{\min}(R_{\phi\phi}(t))\} e^T(t) \mathcal{M} e(t). \quad (30) \]

On the other hand, we have

\[ V(0) = e^T(0) R_{\phi\phi}(0) e(0) + \int_{t_0}^{t_f} \exp[\alpha \ell] e^T(t) R_{\phi\phi}(\varepsilon) e(t) d\varepsilon \]
\[ + \int_{t_f}^{0} \exp[\alpha \ell] e^T(t) R_{\phi\phi}(\varepsilon) e(t) d\varepsilon \]
\[
\lambda_1 e^T(t)M e(t) \leq \exp\left(-\alpha - \frac{\ln \mu}{\frac{1}{2}e}\right)\lambda_1 e_1,
\]

where \(\alpha\) is same as defined in theorem statement. By combining the equations (28)–(30), we get

\[
\lambda_1 e^T(t)M e(t) \leq \exp\left(-\alpha - \frac{\ln \mu}{\frac{1}{2}e}\right)\lambda_1 e_1.
\]

If the conditions (18) and (19) hold, we get \(e^T(t)M e(t) < e_2\). Hence, the closed-loop error system in (5) is finite-time stable with respect to \((e_1, e_2, \mathcal{K}, \delta_1, \delta_2, \mathcal{M})\). This completes the proof.

Supposing that the time-varying delay is not considered in the system (1), then the desired finite-time tracking performance of the delay-free system of (1) is achieved by ensuring the finite-time stability of the error system without delay term. Hence, the conditions to ensure finite-time stability of the concerned delay-free system with the aid of UDE-based switched fuzzy controller is designed in the following corollary.

**Corollary 1.** Let us assume the closed-loop error system (5) without the delay terms and with known bandwidth value of the lumped disturbance. For given positive scalars \(\alpha, \mu\) and \(\beta_2\) \((\ell = 1, \ldots, r - 1)\), the error system (5) in the absence of the delays is finite-time stable with respect to \((e_1, e_2, \mathcal{K}, \mathcal{M})\), if there exist real symmetric matrices \(R_{\psi \ell} > 0\) such that the following inequalities hold for \(1 \leq \ell < j \leq r\) and \(\psi, \tilde{\psi} \in \{1, \ldots, s\}\):

\[
R_{\psi j} > R_{\psi i}, \quad (j = 1, \ldots, r - 1)
\]

\[
\Pi_{\ell \ell} < 0, \quad (\ell = 1, \ldots, r)
\]

\[
\Pi_{\ell j} + \Pi_{jj} < 0, \quad (\ell, j = 1, \ldots, r), \ell \neq j
\]

\[
R_{\psi \ell} < \mu R_{\psi \ell},
\]

where \(\psi, \tilde{\psi} \in \{1, \ldots, s\}\).

**Remark 2.** The proposed UDE-based control scheme consists of three components such as state feedback, error feedback and lumped disturbance estimation. Thus the controller is designed with respect to the known dynamics of the system and reference state. In addition to that, the unknown lumped disturbances are effectively estimated through the low-pass filter. Therefore, the three components effectively overcome the effects of undesirable changes in system and makes the error system to be converged. Further, the UDE-based controller can be used to effectively track both periodic and aperiodic reference signals of the addressed system.

**Remark 3.** The filter design chosen in this study is imperative to attenuate the undesirable disturbance and uncertain properties in system trajectories. It should be noted that the signals are estimated by filter process wherein the input signals are amplified without shifting the phase value and also the signals beyond the bandwidth value are blocked. Therefore, the signals are well estimated with more accuracy through the proposed controller.

**Remark 4.** It should be pointed out that the fuzzy-dependent Lyapunov–Krasovskii functional effectively reduced the conservativeness in the stability analysis rather than the quadratic Lyapunov–Krasovskii functional. Specifically, fuzzy-dependent Lyapunov–Krasovskii functional is the weighted sum of the quadratic Lyapunov–Krasovskii functional. It is noted that the same set of premise membership functions and the same number of rules are shared in both the fuzzy model and fuzzy controller due to the implementation of parallel distribution compensation concept. In particular, parallel distribution compensation based results have some limitations on design flexibility, computational demand and robustness property [16]. To overcome such issues, membership-function-dependent technique is considered in [17, 18], which can unify the three categories such as perfectly, partially and imperfectly matched premises for the stability analysis of a fuzzy model. In our future work, we have planned to study the stabilisation for the...
switched fuzzy systems with imperfectly matched premises by using the membership-function-dependent technique.

4 | NUMERICAL VALIDATION

In this section, the proposed UDE-based fuzzy switched controller performance for the addressed system is illustrated with simulation results. To be precise, the effectiveness of the developed controller to achieve finite-time tracking against unknown time-varying uncertainties, additive time-varying delays and unknown disturbances is clearly examined in first example. In the next example, mass-spring-damping system is presented to show the applicability of the designed control scheme.

Example 1: Consider the TS fuzzy switched system (3) consisting of two switched subsystems with two fuzzy rules. The corresponding state parameters are chosen as

Subsystem 1:

\[
A_{11} = \begin{bmatrix} -1 & -2 \\ 1 & 1 \end{bmatrix}, \quad A_{12} = \begin{bmatrix} -1 & 2 \\ -1 & 1 \end{bmatrix}, \quad A_{d11} = \begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix}, \quad A_{d12} = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}, \quad B_{11} = \begin{bmatrix} 0.4 & 0.6 \\ 0.8 & 1 \end{bmatrix}, \quad B_{12} = \begin{bmatrix} 0.3 & 0.5 \\ 0.7 & 0.9 \end{bmatrix},
\]

\[
\Delta A_{11}(t) = \begin{bmatrix} 0.02 \cos(t) & -0.05 \sin(t) \\ 0.07 \cos(0.1t) & -0.03 \sin(t) \end{bmatrix}, \quad C_{11} = \begin{bmatrix} 2 \\ -1 \end{bmatrix},
\]

\[
\Delta A_{12}(t) = \begin{bmatrix} -0.04 \sin(t) & 0.05 \sin(t) \\ -0.07 \sin(t) & 0.03 \sin(t) \end{bmatrix}, \quad C_{12} = \begin{bmatrix} 1 \\ -2 \end{bmatrix},
\]

\[
\Delta A_{d11}(t) = \begin{bmatrix} 0.1 \sin^2(0.5t) & 0.2 \cos^2(0.3t) \\ 0.1 \cos^2(2t) & 0.2 \cos^2(0.8t) \end{bmatrix},
\]

\[
\Delta A_{d12}(t) = \begin{bmatrix} 0.1 \cos(0.3t) & 0.3 \cos(0.6t) \\ -0.2 \sin(0.9t) & 0.1 \cos(0.2t) \end{bmatrix}.
\]

Subsystem 2:

\[
A_{21} = \begin{bmatrix} 2 & -1 \\ 1 & -1 \end{bmatrix}, \quad A_{22} = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}, \quad A_{d21} = \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix}, \quad A_{d22} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}, \quad B_{21} = \begin{bmatrix} 0.5 & 0.4 \\ 0.5 & 0.6 \end{bmatrix}, \quad B_{22} = \begin{bmatrix} 0.6 & 0.3 \\ 0.7 & 0.9 \end{bmatrix},
\]

\[
\Delta A_{21}(t) = \begin{bmatrix} -0.1 \sin(0.5t) & 0.4 \cos(0.7t) \\ -0.1 \sin(0.9t) & 0.2 \cos(0.8t) \end{bmatrix}, \quad C_{21} = \begin{bmatrix} 2 \end{bmatrix},
\]

\[
\Delta A_{22}(t) = \begin{bmatrix} \sin(0.1t) \cos(0.2t) & 0.2 \sin(0.3t) \cos(0.4t) \\ \sin(0.5t) \cos(0.6t) & 0.2 \sin(0.7t) \cos(0.8t) \end{bmatrix},
\]

\[
C_{22} = \begin{bmatrix} -2 & -3 \end{bmatrix}, \quad \Delta A_{d21}(t) = \begin{bmatrix} -0.1 \cos^2(t) & \sin^2(t) \\ -0.2 \cos^2(2t) & \sin^2(2t) \end{bmatrix},
\]

\[
\Delta A_{d22}(t) = \begin{bmatrix} 0.3 \sin(3t) & 0.4 \cos(4t) \\ 0.5 \sin(5t) & 0.6 \cos(6t) \end{bmatrix}.
\]

\[\Delta \bar{A}_{d22}(t) = \begin{bmatrix} 0.3 \sin(3t) & 0.4 \cos(4t) \\ 0.5 \sin(5t) & 0.6 \cos(6t) \end{bmatrix}.
\]

Now the reference model is considered with two switched subsystems and two fuzzy rules as follows:

Subsystem 1:

\[A_{m11} = \begin{bmatrix} 0.5 & 0.1 \\ 0.6 & -0.2 \end{bmatrix}, \quad A_{m12} = \begin{bmatrix} 0.2 & 0.1 \\ -2 & -1 \end{bmatrix}, \quad A_{m21} = \begin{bmatrix} -1 & 0.6 \\ 0.4 & -1 \end{bmatrix}, \quad A_{m22} = \begin{bmatrix} 0 & -0.1 \\ 0 & -0.1 \end{bmatrix}, \quad B_{m11} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}, \quad B_{m12} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}.
\]

Subsystem 2:

\[A_{m21} = \begin{bmatrix} 0.5 & 0.1 \\ 0.6 & -0.2 \end{bmatrix}, \quad A_{m22} = \begin{bmatrix} -0.1 & 0.1 \\ -3 & -5 \end{bmatrix}, \quad A_{m21} = \begin{bmatrix} -1 & 0.0 \\ 0 & -0.2 \end{bmatrix}, \quad B_{m21} = \begin{bmatrix} 1 \\ -4 \end{bmatrix}, \quad B_{m22} = \begin{bmatrix} -1 \\ -3 \end{bmatrix}.
\]

Moreover, the uniformly bounded reference input is taken as

\[r(t) = \begin{cases} \sin(2t) \sin(3t), & 1 < t < 5 \\ \sin(4t) \cos(2t), & 8 < t \leq 10 \\ \sin(2t) \sin(5t), & \text{otherwise} \end{cases}
\]

The additive time-varying delays are taken as \(d_1(t) = 0.2 + 0.3 \sin(0.1t)\) and \(d_2(t) = 0.5 + 0.2 \cos(t)\) and with \(\beta_1 = 0.5, \beta_2 = 0.7, \epsilon_1 = 0.5, \epsilon_2 = 0.2, \varphi_1 = 0.3\) and \(\varphi_2 = 0.2\). For simulation purposes, the fuzzy weighting functions are chosen as \(\Psi_{11}(x_1(t)) = \Psi_{21}(x_1(t)) = \frac{\sin^2(x_1(t))}{5}\) and \(\Psi_{12}(x_1(t)) = \Psi_{22}(x_1(t)) = \frac{3 \sin(x_1(t))}{5}\). Also the external disturbances are chosen as

\[w(t) = \begin{cases} \sin(2t) \sin(3t), & 1 < t < 2 \\ 1.6 \sin(\pi t), & 3 < t < 5 \\ 4, & 6 < t < 8 \\ 5 \cos(5t), & \text{otherwise} \end{cases}
\]

Now, the derived constraints in Theorem 1 are solved with respect to the parameters \(\beta_1 = 0.6, \alpha = 0.2, \mu = 1.3, \epsilon_1 = 0.6, \epsilon_2 = 2, \varphi = 20\) and \(\mathcal{M} = I\), we obtain the associated gain matrices as

\[K_{11} = \begin{bmatrix} -2.7040 & -0.0616 \\ 0.0207 & 4.1806 \end{bmatrix}, \quad K_{12} = \begin{bmatrix} -8.0588 & -1.7625 \\ -1.8827 & -12.4385 \end{bmatrix}.
\]
The tracking performance of the state trajectories to their desired reference trajectories are depicted in Figure 2 with the initial conditions for the considered system and reference model as $[0.1 \ 0.3]^T$ and $[0 \ -0.2]^T$ respectively. In Figure 2, the tracking performance of the state dynamics through the developed controller (11) with and without the estimated value of the lumped disturbances demonstrates the importance of incorporating the UDE estimation values in controller design. The unpredictable uncertainties and unmeasurable disturbances in the addressed system (1) are effectively handled by the estimated lumped disturbances via designing a low-pass filter with the desired bandwidth value $\delta = 0.001$, which is clearly observed from Figure 3, wherein $u_{\text{q1}}$ represents the unknown lumped disturbance and their corresponding estimation is represented as $u_{\text{q1e}}$. Moreover to show the impact and importance of low-pass filter in the error dynamics, the state responses of the error system for distinct bandwidth values are plotted in Figure 4. Figure 4 exhibits that the appropriate bandwidth selection is significant to obtain better tracking performance. It is noted that if the bandwidth value decreases, then the convergence rate of the error state increases. Figure 5 shows the corresponding UDE-based fuzzy switched control responses and the switching mode of the considered system with respect to the consistent average dwell-time $\zeta^* = 1.3\varepsilon$. The error state trajectories are presented in Figure 6 which shows that the state trajectories are restricted within the prescribed bound value of $\varepsilon_2$. It is clearly revealed that the considered system attains the desired tracking performance within a finite period of time. In addition, the tracking performance of the system (1) for distinct
reference inputs such as sinusoidal signal, step signal, aperiodic signal and sawtooth signal are presented in Figure 7. Specifically Figure 7 illustrates the effective tracking performance of the system with minimum tracking error for different kinds of reference inputs, which ensures the robustness of the developed controller. From the simulation results, it is seen that the developed control strategy more effectively handles the unknown lumped disturbances with undesirable changes and also ensures the outstanding tracking performance for various reference inputs.

**Example 2:** Consider the well-known mass-spring-damper system to verify the applicability of the proposed UDE-based switched fuzzy controller. Based on Newton’s law, the second order differential equation of the mass-spring-damper model is expressed in the following form:

\[ m\ddot{x} + F_j + F_i = u(t), \]

where the mass of the spring is denoted as \( m \); the friction and restoring force with nonlinear nature are represented as \( F_j \) and \( F_i \), respectively and the term \( u(t) \) denotes the control input. As in [27], we modelled the considered system as switched fuzzy system (1) without time delays and uncertainties. The associated state matrices are

\[
A_{11} = \begin{bmatrix} 0 & 1 \\ -0.02 & 0 \end{bmatrix}, \quad A_{12} = \begin{bmatrix} 0 & 1 \\ -0.02 & -0.225 \end{bmatrix}, \\
A_{21} = \begin{bmatrix} 0 & 1 \\ -1.5275 & 0 \end{bmatrix}, \quad A_{22} = \begin{bmatrix} 0 & 1 \\ -1.5275 & -0.225 \end{bmatrix},
\]

\[
B_{11} = B_{12} = B_{21} = B_{22} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.
\]

Now, the membership terms are obtained as \( \Psi_{11}(x_1(t)) = 1 - x_1^2(t)/2.25; \quad \Psi_{21}(x_2(t)) = 1 - x_2^2(t)/2.25; \quad \Psi_{12}(x_1(t)) = x_1^2(t)/2.25 \) and \( \Psi_{22}(x_2(t)) = x_2^2(t)/2.25 \). For simulation purposes, we consider the external disturbance and its associated matrices as

\[
w(t) = 5\sin(0.5t); \quad C_{11} = \begin{bmatrix} 0 \\ 0.5 \end{bmatrix}; \quad C_{12} = \begin{bmatrix} 0 \\ 0.6 \end{bmatrix};
\]

\[
C_{21} = \begin{bmatrix} 0 \\ 0.1 \end{bmatrix}; \quad C_{22} = \begin{bmatrix} 0 \\ 0.2 \end{bmatrix}.
\]

The corresponding matrices of the reference model that consists of two switched subsystems and two fuzzy rules which are selected as follows:

\[
A_{m_{11}} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}, \quad A_{m_{12}} = \begin{bmatrix} 0 & 0.8 \\ -1.5 & -2.6 \end{bmatrix}, \quad A_{m_{21}} = \begin{bmatrix} 0 & 1.5 \\ -0.2 & -0.3 \end{bmatrix},
\]

\[
A_{m_{22}} = \begin{bmatrix} 0.4 & 1.2 \\ 0.5 & 0.8 \end{bmatrix}, \quad B_{m_{11}} = B_{m_{12}} = B_{m_{21}} = B_{m_{22}} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.
\]

**FIGURE 7** Tracking performance of the states to the reference trajectories for different reference inputs
In addition, the uniformly bounded reference input is taken as \( r(t) = 8 \sin(t) \cos(t) \). Based on the above stated parameter values, the corresponding gain matrices are obtained via solving the LMIs proposed in Corollary 1 by resorting into the MATLAB LMI toolbox. The obtained gain matrices are

\[
K_{11} = \begin{bmatrix} -0.3838 & -0.1767 \\ -0.5046 & 1.8729 \end{bmatrix}; \quad K_{12} = \begin{bmatrix} -0.4448 & -0.1577 \\ -0.6072 & 2.0508 \end{bmatrix};
\]

\[
K_{21} = \begin{bmatrix} -0.4122 & -0.6708 \\ -1.9203 & -0.4134 \end{bmatrix}; \quad K_{22} = \begin{bmatrix} -0.8687 & -0.7740 \\ -2.6003 & -1.7647 \end{bmatrix}.
\]

Based on the obtained gain matrices and considered parameters, the simulation results of the addressed mass-spring-damper system are presented in Figures 8–11 to show the applicability and potential of the developed controller under the initial conditions \( x(0) = \begin{bmatrix} 0.5 & -0.2 \end{bmatrix}^T \) and \( \dot{x}(0) = \begin{bmatrix} -0.2 & 0.1 \end{bmatrix}^T \). To be precise, Figure 8 depicts the tracking performance of the state trajectories to the reference trajectories via the proposed controller. Moreover, the proposed controller effectively eliminates and estimates the lumped disturbance which is validated by presenting the lumped disturbance estimation performance of the considered system in Figure 9. Figure 10 shows the performance of the proposed controller and the switching mode of the system. Further, the error trajectories are converged within the optimal bound value of \( \epsilon_2 \), which is depicted in Figure 11. Therefore, the developed UDE-based control strategy guarantees the finite-time stability of the error system with respect to \( (1, 5, 10, I) \) and also ensures the perfect estimation of the lumped disturbances.

Example 3: For the purpose of showing the advantage and inherent potential of the proposed control scheme among the existing literature, a comparative study is provided in this numerical example. In this connection, the duffing forced-oscillation system is considered to verify our proposed method. The dynamic model of the duffing forced-oscillation system is represented by the following differential equations [44]:

\[
\begin{cases}
\dot{x}_1(t) = x_2(t), \\
\dot{x}_2(t) = -x_1^3(t) - 0.1x_2(t) + 12\cos(t) + w(t). 
\end{cases}
\]

Now, the state space representation of the above equation is considered in delay-free form of equation (1) with two fuzzy rules and one switched subsystem, the associated system matrices are given as follows:

\[
A_{11} = \begin{bmatrix} 0 & 1 \\ 0 & -0.1 \end{bmatrix}, \quad A_{12} = \begin{bmatrix} 0 & 1 \\ -25 & -0.1 \end{bmatrix}, \\
B_{11} = B_{12} = C_{11} = C_{12} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad w(t) = 12\cos(t).
\]

Moreover, the membership functions of the considered model are obtained as \( \psi_{11}(x_1(t)) = 1 - x_1^2(t)/25 \) and \( \psi_{12} = x_1^2(t)/25 \). In order to compare our approach with the controller designed in [44], the reference model is considered as same as in [44]. For simulation purpose, we consider the band-
width value of the designed low-pass filter as $\mathcal{F} = 0.05$, $\epsilon_1 = 1$, $\epsilon_2 = 4$, $\mathcal{T} = 20$, $\mathcal{M} = I$ and the initial conditions as $x(0) = [2 -1]^T$; $x_m(t) = [-0.51]^T$. Now, the LMI-based conditions given in Corollary 1 are solved with respect to the above considered parameters, we obtain the gain matrix $K = [-0.511.5]$. In accordance with the $H_\infty$ tracking controller designed in [44] and proposed UDE-based control law in (13), the tracking performance between the system and reference state trajectories are presented in Figure 12. Further, the tracking error response under both the UDE-based and $H_\infty$ approach is plotted in Figure 13. It is clearly depicted from the Figure 13 that the proposed approach ensures the better tracking performance than the approach followed in [44]. Hence, it is strongly concluded that the proposed UDE-based controller provides better tracking performance than the approach in [44].

**Remark 5.** Notably, by considering suitable Lyapunov–Krasovskii functional and employing advanced integral inequalities like extended Wirtinger integral inequality, we may reduce conservatism. Despite the advantages, the computational complexity may increases since the stability conditions derived in Theorem 1 are primarily depends on several parameters such as $\alpha, \beta, \mu, \beta_\ell (\ell = 1, \ldots, r - 1), \epsilon_1, \epsilon_2, \mathcal{T}, \delta_l, \gamma_l$ and $\mathcal{M}$. Precisely, these parameters are selected on the basis of trial

and error approach to ensure the feasibility of the LMI based conditions. Besides, the selection of bandwidth values plays a major role in guaranteeing the stability of the error system without adjusting control gain matrices, which is validated via the simulation result provided in Figure 4. Therefore, the proposed control scheme has less computational complexity than the existing methods.

## 5 CONCLUSIONS

The robust finite-time tracking control problem for a class of TS fuzzy switched systems subject to unpredictable uncertainties and unknown disturbances is examined in this paper via UDE-based approach. By constructing a fuzzy-dependent Lyapunov–Krasovskii functional, a set of delay-dependent stability constraints is derived by employing the extended Wirtinger’s integral inequality to ensure the finite-time stability of the error dynamics and also to estimate and compensate the unpredictable dynamics of the addressed system. The significance of the proposed theoretical result is revealed via the simulation results of numerical examples. Specifically, the mass-spring-damper model is addressed to verify the applicability of the developed theoretical result. Moreover, the existence of small amount of input delay affects the filter measurement and control performance. Thus, a UDE-based truncated predictive control design is needed for higher-order fuzzy switched systems, which will be the topic of our future research work.

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