NOETHER-LEFSCHETZ THEORY
AND THE YAU-ZASLOW CONJECTURE

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0. Introduction

0.1. Yau-Zaslow conjecture. Let \( S \) be a nonsingular projective \( K3 \) surface, and let
\[ \beta \in \text{Pic}(S) = H^2(S, \mathbb{Z}) \cap H^{1,1}(S, \mathbb{C}) \]
be a nonzero effective curve class. The moduli space \( \overline{M}_0(S, \beta) \) of genus 0 stable maps (with no marked points) has the expected dimension
\[ \dim_{\text{vir}} \left( \overline{M}_0(S, \beta) \right) = \int_{\beta} c_1(S) + \dim_{\mathbb{C}}(S) - 3 = -1. \]
Hence, the virtual class \( \left[ \overline{M}_0(S, \beta) \right]_{\text{vir}} \) vanishes, and the standard Gromov-Witten theory is trivial.

Curve counting on \( K3 \) surfaces is captured instead by the reduced Gromov-Witten theory constructed first via the twistor family in [6]. An algebraic construction following [1, 2] is given in [31]. Since the reduced class
\[ \left[ \overline{M}_0(S, \beta) \right]_{\text{red}} \in H_0(\overline{M}_0(S, \beta), \mathbb{Q}) \]
has dimension 0, the reduced Gromov-Witten integrals of \( S \),
\[ R_{0, \beta}(S) = \int_{\left[ \overline{M}_0(S, \beta) \right]_{\text{red}}} 1 \in \mathbb{Q}, \]
are well-defined. For deformations of \( S \) for which \( \beta \) remains a \((1, 1)\)-class, the integrals (1) are invariant.

The second cohomology of \( S \) is a rank 22 lattice with intersection form
\[ H^2(S, \mathbb{Z}) \cong U \oplus U \oplus U \oplus E_8(-1) \oplus E_8(-1), \]
where
\[ U = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \]
and
\[ E_8(-1) = \begin{pmatrix} -2 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & -2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & -2 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -2 \end{pmatrix} \]
is the (negative) Cartan matrix. The intersection form (2) is even.

The divisibility \( m(\beta) \) is the maximal positive integer dividing the lattice element \( \beta \in H^2(S, \mathbb{Z}) \). If the divisibility is 1, \( \beta \) is primitive. Elements with equal divisibility and norm are equivalent up to orthogonal transformations of \( H^2(S, \mathbb{Z}) \). By straightforward deformation arguments using the Torelli theorem for \( K3 \) surfaces, \( R_{0, \beta}(S) \) depends, for effective classes, only on the divisibility \( m(\beta) \) and the norm \( \langle \beta, \beta \rangle \). We will omit the argument \( S \) in the notation.

The genus 0 BPS counts associated to \( K3 \) surfaces have the following definition. Let \( \alpha \in \text{Pic}(S) \) be a nonzero class which is both effective and primitive. The
Gromov-Witten potential $F_\alpha(v)$ for classes proportional to $\alpha$ is

$$F_\alpha = \sum_{m > 0} R_{0,m\alpha} v^{m\alpha}. $$

The BPS counts $r_{0,m\alpha}$ are uniquely defined via the Aspinwall-Morrison formula,

$$F_\alpha = \sum_{m > 0} r_{0,m\alpha} \sum_{d > 0} \frac{z^{d\alpha}}{d^3}, $$

for both primitive and divisible classes.

The Yau-Zaslow conjecture [36] predicts the values of the genus 0 BPS counts for the reduced Gromov-Witten theory of $K3$ surfaces. We interpret the conjecture in two parts.

**Conjecture 1.** The BPS count $r_{0,\beta}$ depends upon $\beta$ only through the norm $\langle \beta, \beta \rangle$.

Conjecture 1 is rather surprising from the point of view of Gromov-Witten theory since $R_{0,\beta}$ certainly depends upon the divisibility of $\beta$. Let $r_{0,m,h}$ denote the genus 0 BPS count associated to a class $\beta$ of divisibility $m$ satisfying $\langle \beta, \beta \rangle = 2h - 2$.

Assuming Conjecture 1 holds, we define

$$r_{0,h} = r_{0,m,h},$$

independent of $m$.

**Conjecture 2.** The BPS counts $r_{0,h}$ are uniquely determined by

$$\sum_{h \geq 0} r_{0,h} q^h = \prod_{n=1}^{\infty} (1 - q^n)^{-24}. $$

Conjecture 2 can be written in terms of the Dedekind $\eta$ function

$$\sum_{h \geq 0} r_{0,h} q^{h-1} = \eta(\tau)^{-24}, $$

where $q = e^{2\pi i \tau}$.

The conjectures have been previously proven in very few cases. A proof of the Yau-Zaslow formula for primitive classes $\beta$ via Euler characteristics of compactified Jacobians following [36] can be found in [3, 7, 11]. The Yau-Zaslow formula was proven via Gromov-Witten theory for primitive classes $\beta$ by Bryan and Leung [6]. An early calculation by Gathmann [13] for a class $\beta$ of divisibility 2 was important for the correct formulation of the conjectures. Conjectures 1 and 2 have been proven in the divisibility 2 case by Lee and Leung [26] and Wu [35]. The main result of the paper is a proof of Conjectures 1 and 2 in all cases.

**Theorem 1.** The Yau-Zaslow conjecture holds for all nonzero effective classes $\beta \in \text{Pic}(S)$ on a $K3$ surface $S$.  

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1. Independence of $m$ holds when $2m^2$ divides $2h - 2$. Otherwise, no such class $\beta$ exists and $r_{0,m,h}$ is defined to vanish.
0.2. Noether-Lefschetz theory.

0.2.1. Lattice polarization. Let $S$ be a $K3$ surface. A primitive class $L \in \text{Pic}(S)$ is a quasi-polarization if

$$\langle L, L \rangle > 0 \quad \text{and} \quad \langle L, [C] \rangle \geq 0$$

for every curve $C \subset S$. A sufficiently high tensor power $L^n$ of a quasi-polarization is base point free and determines a birational morphism

$$S \to \tilde{S}$$

contracting A-D-E configurations of $(-2)$-curves on $S$. Hence, every quasi-polarized $K3$ surface is algebraic.

Let $\Lambda$ be a fixed rank $r$ primitive embedding $\Lambda \subset U \oplus U \oplus E_8(-1) \oplus E_8(-1)$ with signature $(1, r - 1)$, and let $v_1, \ldots, v_r \in \Lambda$ be an integral basis. The discriminant is

$$\Delta(\Lambda) = (-1)^{r-1} \det \begin{pmatrix} \langle v_1, v_1 \rangle & \cdots & \langle v_1, v_r \rangle \\ \vdots & \ddots & \vdots \\ \langle v_r, v_1 \rangle & \cdots & \langle v_r, v_r \rangle \end{pmatrix}.$$ 

The sign is chosen so that $\Delta(\Lambda) > 0$.

A $\Lambda$-polarization of a $K3$ surface $S$ is a primitive embedding

$$j : \Lambda \to \text{Pic}(S)$$

satisfying two properties:

(i) the lattice pairs $\Lambda \subset U^3 \oplus E_8(-1)^2$ and $\Lambda \subset H^2(S, \mathbb{Z})$ are isomorphic via an isometry which restricts to the identity on $\Lambda$,

(ii) $\text{Im}(j)$ contains a quasi-polarization.

By (ii), every $\Lambda$-polarized $K3$ surface is algebraic.

The period domain $M$ of Hodge structures of type $(1, 20, 1)$ on the lattice $U^3 \oplus E_8(-1)^2$ is an analytic open set of the 20-dimensional nonsingular isotropic quadric $Q$.

$$M \subset Q \subset \mathbb{P}(U^3 \oplus E_8(-1)^2 \otimes \mathbb{C})$$

Let $M_\Lambda \subset M$ be the locus of vectors orthogonal to the entire sublattice $\Lambda \subset U^3 \oplus E_8(-1)^2$.

Let $\Gamma$ be the isometry group of the lattice $U^3 \oplus E_8(-1)^2$, and let

$$\Gamma_\Lambda \subset \Gamma$$

be the subgroup restricting to the identity on $\Lambda$. By global Torelli, the moduli space $\mathcal{M}_\Lambda$ of $\Lambda$-polarized $K3$ surfaces is the quotient

$$\mathcal{M}_\Lambda = M_\Lambda / \Gamma_\Lambda.$$ 

We refer the reader to [10] for a detailed discussion.

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2An embedding of lattices is primitive if the quotient is torsion free.
0.2.2. Families. Let $X$ be a compact 3-dimensional complex manifold equipped with holomorphic line bundles

$$L_1, \ldots, L_r \to X$$

and a holomorphic map

$$\pi : X \to C$$

to a nonsingular complete curve.

The tuple $(X, L_1, \ldots, L_r, \pi)$ is a 1-parameter family of nonsingular $\Lambda$-polarized K3 surfaces if

(i) the fibers $(X_\xi, L_{1,\xi}, \ldots, L_{r,\xi})$ are $\Lambda$-polarized $K3$ surfaces via

$$v_i \mapsto L_{i,\xi}$$

for every $\xi \in C$;

(ii) there exists a $\lambda^\pi \in \Lambda$ which is a quasi-polarization of all fibers of $\pi$ simultaneously.

The family $\pi$ yields a morphism,

$$\iota_\pi : C \to \mathcal{M}_\Lambda,$$

to the moduli space of $\Lambda$-polarized $K3$ surfaces.

Let $\lambda^\pi = \lambda^\pi_1 v_1 + \cdots + \lambda^\pi_r v_r$. A vector $(d_1, \ldots, d_r)$ of integers is positive if

$$\sum_{i=1}^r \lambda^\pi_i d_i > 0.$$ 

If $\beta \in \text{Pic}(X_\xi)$ has intersection numbers

$$d_i = \langle L_{i,\xi}, \beta \rangle,$$

then $\beta$ has positive degree with respect to the quasi-polarization if and only if $(d_1, \ldots, d_r)$ is positive.

0.2.3. Noether-Lefschetz divisors. Noether-Lefschetz numbers are defined in [31] by the intersection of $\iota_\pi(C)$ with Noether-Lefschetz divisors in $\mathcal{M}_\Lambda$. We briefly review the definition of the Noether-Lefschetz divisors.

Let $(L, \iota)$ be a rank $r+1$ lattice $L$ with an even symmetric bilinear form $\langle , \rangle$ and a primitive embedding

$$\iota : \Lambda \to L.$$ 

Two data sets $(L, \iota)$ and $(L', \iota')$ are isomorphic if there is an isometry which restricts to the identity on $\Lambda$. The first invariant of the data $(L, \iota)$ is the discriminant $\Delta \in \mathbb{Z}$ of $L$.

An additional invariant of $(L, \iota)$ can be obtained by considering any vector $v \in L$ for which $v \cdot \iota(\Lambda) \neq 0$.

(5) $$L = \iota(\Lambda) \oplus \mathbb{Z}v.$$ 

The pairing

$$\langle v, \cdot \rangle : \Lambda \to \mathbb{Z}$$

determines an element of $\delta_v \in \Lambda^*$. Let $G = \Lambda^*/\Lambda$ be the quotient defined via the injection $\Lambda \to \Lambda^*$ obtained from the pairing $\langle , \rangle$ on $\Lambda$. The group $G$ is abelian of order equal to the discriminant $\Delta(\Lambda)$. The image

$$\delta \in G/\pm$$

\[3\] Here, $\oplus$ is used just for the additive structure (not the orthogonal direct sum).
of $\delta_v$ is easily seen to be independent of $v$ satisfying (5). The invariant $\delta$ is the coset of $(\mathbb{L}, \iota)$.

By elementary arguments, two data sets $(\mathbb{L}, \iota)$ and $(\mathbb{L}', \iota')$ of rank $r+1$ are isomorphic if and only if the discriminants and cosets are equal.

Let $v_1, \ldots, v_r$ be an integral basis of $\Lambda$ as before. The pairing of $\mathbb{L}$ with respect to an extended basis $v_1, \ldots, v_r, v$ is encoded in the matrix

$$L_{h, d_1, \ldots, d_r} = \begin{pmatrix}
\langle v_1, v_1 \rangle & \cdots & \langle v_1, v_r \rangle & d_1 \\
\vdots & \ddots & \vdots & \vdots \\
\langle v_r, v_1 \rangle & \cdots & \langle v_r, v_r \rangle & d_r \\
d_1 & \cdots & d_r & 2h - 2
\end{pmatrix}.$$ 

The discriminant is

$$\Delta(h, d_1, \ldots, d_r) = (-1)^r \det(L_{h, d_1, \ldots, d_r}).$$

The coset $\delta(h, d_1, \ldots, d_r)$ is represented by the functional $v_i \mapsto d_i$.

The Noether-Lefschetz divisor $P_{\Delta, \delta} \subset \mathcal{M}_\Lambda$ is the closure of the locus of $\Lambda$-polarized $K3$ surfaces $S$ for which $(\text{Pic}(S), j)$ has rank $r+1$, discriminant $\Delta$, and coset $\delta$. By the Hodge index theorem, $P_{\Delta, \delta}$ is empty unless $\Delta > 0$.

Let $h, d_1, \ldots, d_r$ determine a positive discriminant $\Delta(h, d_1, \ldots, d_r) > 0$.

The Noether-Lefschetz divisor $D_{h, (d_1, \ldots, d_r)} \subset \mathcal{M}_\Lambda$ is defined by the weighted sum

$$D_{h, (d_1, \ldots, d_r)} = \sum_{\Delta, \delta} m(h, d_1, \ldots, d_r | \Delta, \delta) \cdot [P_{\Delta, \delta}],$$

where the multiplicity $m(h, d_1, \ldots, d_r | \Delta, \delta)$ is the number of elements $\beta$ of the lattice $(\mathbb{L}, \iota)$ of type $(\Delta, \delta)$ satisfying

$$\langle \beta, \beta \rangle = 2h - 2, \quad \langle \beta, v_i \rangle = d_i.$$ 

If the multiplicity is nonzero, then $\Delta | \Delta(h, d_1, \ldots, d_r)$, so only finitely many divisors appear in the above sum.

If $\Delta(h, d_1, \ldots, d_r) = 0$, the divisor $D_{h, (d_1, \ldots, d_r)}$ has an alternate definition. The tautological line bundle $\mathcal{O}(-1)$ is $\Gamma$-equivariant on the period domain $\mathcal{M}_\Lambda$ and descends to the Hodge line bundle

$$\mathcal{K} \to \mathcal{M}_\Lambda.$$ 

We define $D_{h, (d_1, \ldots, d_r)} = \mathcal{K}^*$. See [31] for an alternate view of degenerate intersection.

If $\Delta(h, d_1, \ldots, d_r) < 0$, the divisor $D_{h, (d_1, \ldots, d_r)}$ on $\mathcal{M}_\Lambda$ is defined to vanish by the Hodge index theorem.

0.2.4. Noether-Lefschetz numbers. Let $\Lambda$ be a lattice of discriminant $l = \Delta(\Lambda)$, and let $(X, L_1, \ldots, L_r, \pi)$ be a 1-parameter family of $\Lambda$-polarized $K3$ surfaces. The Noether-Lefschetz number $NL_{h, (d_1, \ldots, d_r)}^r$ is the classical intersection product

$$NL_{h, (d_1, \ldots, d_r)}(7) = \int_C \iota^* [D_{h, (d_1, \ldots, d_r)}].$$
Let $\text{Mp}_2(\mathbb{Z})$ be the metaplectic double cover of $SL_2(\mathbb{Z})$. There is a canonical representation $[4]$ associated to $\Lambda$, $\rho_\Lambda^*: \text{Mp}_2(\mathbb{Z}) \to \text{End}(\mathbb{C}[G])$.

The full set of Noether-Lefschetz numbers $NL^\pi_{h,d_1,\ldots,d_r}$ defines a vector-valued modular form

$$\Phi^\pi(q) = \sum_{\gamma \in G} \Phi^\pi_\gamma(q) v_\gamma \in \mathbb{C}[[q^{\frac{1}{2}}]] \otimes \mathbb{C}[G],$$

of weight $\frac{22-r}{2}$ and type $\rho_\Lambda^*$ by results[4] of Borcherds and Kudla-Millson [4][25]. The Noether-Lefschetz numbers are the coefficients[5] of the components of $\Phi^\pi$,

$$NL^\pi_{h,(d_1,\ldots,d_r)} = \Phi^\pi_\gamma \left[ \frac{\Delta(h,d_1,\ldots,d_r)}{2l} \right],$$

where $\delta(h,d_1,\ldots,d_r) = \pm \gamma$. The modular form results significantly constrain the Noether-Lefschetz numbers.

0.2.5. Refinements. If $d_1,\ldots,d_r$ do not simultaneously vanish, refined Noether-Lefschetz divisors are defined. If $\Delta(h,d_1,\ldots,d_r) > 0,$

$$D_{m,h,(d_1,\ldots,d_r)} \subset D_{h,(d_1,\ldots,d_r)}$$

is defined by requiring the class $\beta \in \text{Pic}(S)$ to satisfy (6) and have divisibility $m > 0$. If $\Delta(h,d_1,\ldots,d_r) = 0$, then

$$D_{m,h,(d_1,\ldots,d_r)} = D_{h,(d_1,\ldots,d_r)}$$

if $m > 0$ is the greatest common divisor of $d_1,\ldots,d_r$ and 0 otherwise.

Refined Noether-Lefschetz numbers are defined by

$$(8) \quad NL^\pi_{m,h,(d_1,\ldots,d_r)} = \int_C t^*_\pi[D_{m,h,(d_1,\ldots,d_r)}].$$

In Section 2.3 the full set of Noether-Lefschetz numbers $NL^\pi_{h,(d_1,\ldots,d_r)}$ is easily shown to determine the refined numbers $NL^\pi_{m,h,(d_1,\ldots,d_r)}$.

0.3. Three theories. The main geometric idea in the proof is the relationship of three theories associated to a 1-parameter family

$$\pi: X \to C$$

of $\Lambda$-polarized $K3$ surfaces:

(i) the Noether-Lefschetz numbers of $\pi$,

(ii) the genus 0 Gromov-Witten invariants of $X$,

(iii) the genus 0 reduced Gromov-Witten invariants of the K3 fibers.

The Noether-Lefschetz numbers (i) are classical intersection products while the Gromov-Witten invariants (ii)-(iii) are quantum in origin. For (ii), we view the theory in terms of the Gopakumar-Vafa invariants[16][17].

Let $n^X_{0,(d_1,\ldots,d_r)}$ denote the Gopakumar-Vafa invariant of $X$ in genus 0 for $\pi$-vertical curve classes of degrees $d_1,\ldots,d_r$ with respect to the line bundles $L_1,\ldots,L_r$. Let $r^X_{0,m,h}$ denote the reduced K3 invariant defined in Section 0.3. The following

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While the results of the papers [4][25] have considerable overlap, we will follow the point of view of Borcherds.

If $f$ is a series in $q$, $f[k]$ denotes the coefficient of $q^k$.

A review of the definitions can be found in Section 2.3.
result is proven\footnote{The result of \cite{31} is stated in the rank $r = 1$ case, but the argument is identical for arbitrary $r$.} in \cite{31} by a comparison of the reduced and usual deformation theories of maps of curves to the K3 fibers of $\pi$.

\textbf{Theorem 2.} For degrees $(d_1, \ldots, d_r)$ positive with respect to the quasi-polarization $\lambda^\pi$,
\[
n^\pi_{0, (d_1, \ldots, d_r)} = \sum_{h=0}^{\infty} \sum_{m=1}^{\infty} r_{0,m,h} \cdot NL^\pi_{m,h,(d_1, \ldots, d_r)}.
\]

\section{Proof of Theorem 1} The STU model described in Section 1 is a special family of rank 2 lattice polarized K3 surfaces
\[
\pi_{STU} : X_{STU} \to \mathbb{P}^1.
\]
The fibered K3 surfaces of the STU model are themselves elliptically fibered. The proof of Theorem 1 proceeds in four basic steps:

(i) The modular form \cite{4, 25} determining the intersections of the base $\mathbb{P}^1$ with the Noether-Lefschetz divisors is calculated. For the STU model, the modular form has vector dimension 1 and is proportional to the product $E_4 E_6$ of Eisenstein series.

(ii) Theorem 2 is used to show the 3-fold BPS counts $n^\pi_{0, (d_1, d_2)}$ then determine all the reduced K3 invariants $r_{0,m,h}$. Strong use is made of the rank 2 lattice of the STU model.

(iii) The BPS counts $n^\pi_{0, (d_1, d_2)}$ are calculated via mirror symmetry. Since the STU model is realized as a Calabi-Yau complete intersection in a nonsingular toric variety, the genus 0 Gromov-Witten invariants are obtained after proven mirror transformations from hypergeometric series. The Klemm-Lerche-Mayr identity, proven in Section 3, shows that the invariants $n^\pi_{0, (d_1, d_2)}$ are themselves related to modular forms.

(iv) Theorem 1 then follows from the Harvey-Moore identity which simultaneously relates the modular structures of
\[
n^\pi_{0, (d_1, d_2)}, \quad r_{0,m,h}, \quad NL^\pi_{m,h,(d_1, d_2)}
\]
in the form specified by Theorem 2. D. Zagier’s proof of the Harvey-Moore identity is presented in Section 4.

The strategy of proof is special to genus 0. Much less is known in higher genus. The Katz-Klemm-Vafa conjecture \cite{21, 31} for the integral\footnote{The integrand $\lambda_g$ is the top Chern class of the Hodge bundle on $\overline{M}_g(X, \beta)$.}.
\[
\int_{[\overline{M}_g(S, \beta)]^{red}} (-1)^g \lambda_g
\]
is a particular generalization of the Yau-Zaslow formula to higher genera. The KKV formula does not yet appear easily approachable in Gromov-Witten theory\footnote{For $g = 1$, the KKV formula follows for all classes on K3 surfaces from the Yau-Zaslow formula via the boundary relation for $\lambda_1$.}.
1. The STU model

1.1. Overview. The STU model is a particular nonsingular projective Calabi-Yau 3-fold $X$ equipped with a fibration

$$\pi : X \to \mathbb{P}^1.$$  

Except for 528 points $\xi \in \mathbb{P}^1$, the fibers $X_\xi = \pi^{-1}(\xi)$ are nonsingular elliptically fibered $K3$ surfaces. The 528 singular fibers $X_\xi$ have exactly 1 ordinary double point singularity each.

The 3-fold $X$ is constructed as an anticanonical section of a nonsingular projective toric 4-fold $Y$. The Picard rank of $Y$ is 6. The fibration (9) is obtained from a nonsingular toric fibration

$$\pi^Y : Y \to \mathbb{P}^1.$$  

The image of

$$\text{Pic}(Y) \to \text{Pic}(X_\xi)$$

determines a rank 2 sublattice of each fiber $\text{Pic}(X_\xi)$ with intersection form

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$  

The toric data describing the construction of $X \subset Y$ and the fibration structure are explained here.

1.2. Toric varieties. Let $N$ be a lattice of rank $d$,  

$$N \cong \mathbb{Z}^d.$$  

A fan $\Sigma$ in $N$ is a collection of strongly convex rational polyhedral cones containing all faces and intersections. A toric variety $V_\Sigma$ is canonically associated to $\Sigma$. The variety $V_\Sigma$ is complete of dimension $d$ if the support of $\Sigma$ covers $N \otimes_{\mathbb{Z}} \mathbb{R}$. If all cones are simplicial and if all maximal cones are generated by a lattice basis, then $V_\Sigma$ is nonsingular. See [8, 12, 32] for the basic properties of toric varieties.

Let $\Sigma$ be a fan corresponding to a nonsingular complete toric variety. A 1-dimensional cone of $\Sigma$ is a ray with a unique primitive vector. Let $\Sigma^{(1)}$ denote the set of 1-dimensional cones of $\Sigma$ indexed by their primitive vectors

$$\{\rho_1, \ldots, \rho_n\}.$$  

Let $r^1, \ldots, r^\ell$ be a basis over the integers of the module of relations among the vectors (10). We write the $j^{\text{th}}$ relation as

$$r^1_j \rho_1 + \ldots + r^n_j \rho_n = 0.$$  

Define a torus

$$(\mathbb{C}^*)^\ell \cong \prod_{j=1}^{\ell} \mathbb{C}^*_j$$

with factors indexed by the relations.

---

10 The model has been studied in physics since the 1980s. The letter $S$ stands for the dilaton and $T$ and $U$ label the torus moduli in the heterotic string. The STU model was an important example for the duality between type IIA and heterotic strings formulated in [29]. The ideas developed in [18] [19] [23] [24] [30] about the STU model play an important role in our paper.
A simple description of $V_\Sigma$ is obtained via a quotient construction. Let $\{z_i\}_{1 \leq i \leq n}$ be coordinates on $\mathbb{C}^n$ corresponding to the primitives $\rho_i$ of the rays in $\Sigma^{(1)}$. An action of $\mathbb{C}^*_j$ on $\mathbb{C}^n$ is defined by

$$\lambda_j \cdot (z_1, \ldots, z_n) = (\lambda_j^{r_j^1} z_1, \ldots, \lambda_j^{r_j^n} z_n), \quad \lambda_j \in \mathbb{C}^*_j.$$  

(11)

In order to obtain a well-behaved quotient for the induced $(\mathbb{C}^*)^\ell$-action on $\mathbb{C}^n$, an exceptional set $Z(\Sigma) \subset \mathbb{C}^n$ consisting of a finite union of linear subspaces is excluded. The linear space defined by $\{z_i = 0 | i \in I\}$ is contained in $Z(\Sigma)$ if there is no single cone in $\Sigma$ containing all of the primitives $\{\rho_i\}_{i \in I}$. After removing $Z(\Sigma)$, the quotient

$$V_\Sigma = \left(\mathbb{C}^n \setminus Z(\Sigma)\right) / (\mathbb{C}^*)^\ell$$  

(12)

yields the toric variety associated to $\Sigma$.

Since $\ell = n - d$, the complex dimension of the quotient $V_\Sigma$ equals the rank $d$ of the lattice $N$. The variety $V_\Sigma$ is equipped with the action of the quotient torus $T = (\mathbb{C}^*)^n / (\mathbb{C}^*)^\ell$.

The rank of Pic($V_\Sigma$) is $\ell$. The primitives $\rho_i$ are in 1–to–1 correspondence with the $T$-invariant divisors $D_i$ on $V_\Sigma$ defined by

$$D_i = \{z_i = 0\} \subset V_\Sigma.$$  

(13)

Conversely, the homogeneous coordinate $z_i$ is a section of the line bundle $\mathcal{O}(D_i)$.

The anticanonical divisor class of $V_\Sigma$ is determined by

$$-K_{V_\Sigma} = \sum_{i=1}^n D_i.$$  

(14)

1.3. The toric 4-fold $Y$. The fan $\Sigma$ in $\mathbb{Z}^4$ defining the toric 4-fold $Y$ has 10 rays with primitive elements:

$$\begin{align*}
\rho_1 &= (1, 0, 2, 3), & \rho_2 &= (-1, 0, 2, 3), \\
\rho_3 &= (0, 1, 2, 3), & \rho_4 &= (0, -1, 2, 3), \\
\rho_5 &= (0, 0, 2, 3), & \rho_6 &= (0, 0, -1, 0), & \rho_7 &= (0, 0, 0, -1), \\
\rho_8 &= (0, 0, 1, 2), & \rho_9 &= (0, 0, 0, 1), & \rho_{10} &= (0, 0, 1, 1).
\end{align*}$$

The full fan $\Sigma$ is obtained from the convex hull of the 10 primitives. By explicitly checking each of the 24 dimension-4 cones, $Y$ is seen to be a complete nonsingular toric 4-fold.

Generators $r^1, \ldots, r^6$ of the rank 6 module of relations among the primitives can be taken to be

$$\begin{align*}
\rho_1 + \rho_2 + 4\rho_6 + 6\rho_7 &= 0, \\
\rho_3 + \rho_4 + 4\rho_6 + 6\rho_7 &= 0, \\
\rho_5 + 2\rho_6 + 3\rho_7 &= 0, \\
\rho_6 + 2\rho_7 + \rho_8 &= 0, \\
\rho_6 + \rho_7 + \rho_9 &= 0, \\
\rho_6 + \rho_7 + \rho_{10} &= 0.
\end{align*}$$
By the identification (11) of $-K_Y$, the product $\prod_{i=1}^{10} z_i$ defines an anticanonical section. Hence, every product
\[ \prod_{i=1}^{10} z_i^{m_i}, \quad m_i \geq 0, \]
which is homogeneous of degree $\sum_{i=1}^{10} r_i^{m_i}$ with respect to the action (11) of $\mathbb{C}^*$ also defines an anticanonical section. Hence,
\begin{align*}
(15) \quad &z_1^{12} z_4^{12} z_5^{6} z_8^{4} z_9^{2} z_{10}^{3}, \\
&z_2^{12} z_4^{12} z_5^{6} z_8^{4} z_9^{2} z_{10}^{3}, \\
&z_3^{12} z_5^{12} z_5^{6} z_8^{4} z_9^{2} z_{10}^{3}, \\
&z_4^{12} z_5^{12} z_5^{6} z_8^{4} z_9^{2} z_{10}^{3}, \\
&z_6^{3} z_8^{2} z_9^{2}, \\
&z_7^{2} z_{10}^{2} \\
\end{align*}
are all sections of $-K_Y$.

From the definitions, we find that $Z(\Sigma)$ consists of the union of the following 11 linear spaces of dimension 2 in $\mathbb{C}^4$:
\begin{align*}
I_1 &= \{1, 2\}, \quad I_2 = \{3, 4\}, \quad I_3 = \{5, 6\}, \quad I_4 = \{5, 7\}, \\
I_5 &= \{5, 9\}, \quad I_6 = \{6, 8\}, \quad I_7 = \{6, 10\}, \quad I_8 = \{7, 8\}, \\
I_9 &= \{7, 9\}, \quad I_{10} = \{8, 10\}, \quad I_{11} = \{9, 10\}.
\end{align*}

Recall, $I_k$ indexes the coordinates which vanish.

A simple verification shows that the 6 sections (15) of $-K_Y$ do not have a common zero on the prequotient $\mathbb{C}^n \setminus Z(\Sigma)$. Hence, $-K_Y$ is generated by global sections on $Y$. A hypersurface
\[ X \subset Y \]
defined by a generic section of $-K_Y$ is nonsingular by Bertini’s Theorem. By adjunction, $X$ is Calabi-Yau.

1.4. Fibrations. The toric variety $Y$ admits two obvious fibrations
\[ \pi^Y : Y \to \mathbb{P}^1, \quad \mu^Y : \to \mathbb{P}^1 \]
given in homogeneous coordinates by
\[ \pi^Y(z_1, \ldots, z_{10}) = [z_1, z_2], \quad \mu^Y(z_1, \ldots, z_{10}) = [z_3, z_4]. \]
Since $Z(\Sigma)$ contains the linear spaces
\[ I_1 = \{1, 2\}, \quad I_2 = \{3, 4\}, \]
both $\pi^Y$ and $\mu^Y$ are well-defined.

Consider first $\pi^Y$. The fibers of $\pi^Y$ are nonsingular complete toric 3-folds defined by the fan in
\[ \mathbb{Z}^3 \subset \mathbb{Z}^4, \quad (c_1, c_2, c_3) \mapsto (0, c_1, c_2, c_3) \]
determined by the primitives $\rho_3, \ldots, \rho_{10}$.

Let $X$ be obtained from a generic section of $-K_Y$. Let
\[ \pi : X \to \mathbb{P}^1 \]
be the restriction $\pi^Y|_X$.

**Proposition 1.** Except for 528 points $\xi \in \mathbb{P}^1$, the fibers
\[ X_\xi = \pi^{-1}(\xi) \]
are nonsingular elliptically fibered $K3$ surfaces. The 528 singular fibers $X_\xi$ each have exactly 1 ordinary double point singularity.
Proof: Let \( P_{k,k}(z_1, z_2|z_3, z_4) \) denote a bihomogeneous polynomial of degree \( k \) in \((z_1, z_2)\) and degree \( k \) in \((z_3, z_4)\). Let
\[
F = P_{12,12}(z_1, z_2|z_3, z_4), \quad G = P_{8,8}(z_1, z_2|z_3, z_4), \quad H = P_{4,4}(z_1, z_2|z_3, z_4)
\]
be bihomogeneous polynomials. Then
\[
F = 6z_5^6 4z_8^2z_{10}^2, \quad G = 3z_6^3z_8z_9z_{10}, \quad H = 2z_5^2z_6^2z_8^2z_{10}, \quad z_6^3z_8z_9^2, \quad z_7^2z_{10}
\]
all determine sections of \(-K_Y\).

Let \( X \) be defined by a generic linear combination of the sections \((17)\). Since the base point free system \((15)\) is contained in \((17)\), \( X \) is nonsingular. We will prove that all the fibers \( X_\ell \) of \( X \) are nonsingular, except for finitely many with exactly 1 ordinary double point each, by an explicit study of the equations.

Since \( I_7 = \{6,10\} \), \( I_9 = \{8,10\} \), and \( I_{11} = \{9,10\} \) are in \( Z(\Sigma) \), we easily see that \( X \cap D_{10} = \emptyset \) if the coefficient of \( z_6^3z_8z_9^2 \) is nonzero. Similarly,
\[
X \cap D_8 = \emptyset, \quad X \cap D_9 = \emptyset.
\]
Hence, using the last 3 factors of the torus \((\mathbb{C}^*)^\ell\), the coordinates \( z_8, z_9, \) and \( z_{10} \) can all be set to 1. The equation for \( X \) simplifies to
\[
F = 6z_5^6 + Gz_6^3z_9 + H = z_6^3z_8^2z_8^2 + \alpha z_6^3 + \beta z_7^2.
\]

The coordinates \( z_1 \) and \( z_2 \) do not simultaneously vanish on \( Y \). There are two charts to consider. By symmetry, the analysis on each is identical, so we assume \( z_1 \neq 0 \). Using the first factor of \((\mathbb{C}^*)^\ell\), we set \( z_1 = 1 \). By the same reasoning, we set \( z_3 = 1 \) using the second factor of \((\mathbb{C}^*)^\ell\). Since \( I_3 = \{5,6\} \) and \( I_4 = \{5,7\} \) are in \( Z(\Sigma) \), either \( z_6 \neq 0 \) or both \( z_6 \) and \( z_7 \) do not vanish.

Case \( z_5 \neq 0 \). Using the third factor of \((\mathbb{C}^*)^\ell\) to set \( z_5 = 1 \), we obtain the equation
\[
F(1, z_2|1, z_4) + H(1, z_2|1, z_4)z_6 + G(1, z_2|1, z_4)z_6^2 + \alpha z_6^3 + \beta z_7^2
\]
in \( \mathbb{C}^4 \) with coordinates \( z_2, z_4, z_6, z_7 \). The map \( \pi \) is given by the \( z_2 \) coordinate. The partial derivative of \((18)\) with respect to \( z_7 \) is \( 2\beta z_7 \). Hence, if \( \beta \neq 0 \), all singularities of \( \pi \) occur when \( z_7 = 0 \).

We need only analyze the reduced dimension case
\[
F(1, z_2|1, z_4) + H(1, z_2|1, z_4)z_6 + G(1, z_2|1, z_4)z_6^2 + z_6^3
\]
with coordinates \( z_2, z_4, z_6 \). Here, \( \alpha \) has been set to 1 by scaling the equation. We must show that all the fibers of \( \pi \) are nonsingular curves except for finitely many with simple nodes. We view equation \((19)\) as defining a 1-parameter family of paths \( \gamma_{z_2}(z_4) \) in the space
\[
C = \{ \gamma_0 + \gamma_1 z_6 + \gamma_2 z_6^2 + z_6^3 \mid \gamma_0, \gamma_1, \gamma_2 \in \mathbb{C} \}
\]
of cubic polynomials in the variable \( z_6 \). The coordinate of the path is \( z_4 \). The variable \( z_2 \) indexes the family of paths.

Let \( \Delta \subset C \) be the codimension 1 discriminant locus of cubics with double roots. The discriminant is irreducible with cuspidal singularities in codimension 2 in \( C \). The possible singularities of the fiber \( \pi^{-1}(\lambda) \) occur only when the path \( \gamma_{z_2}(z_4) \) intersects \( \Delta \). The fiber \( \pi^{-1}(\lambda) \) is nonsingular over such an intersection point if either

(i) \( \gamma_\lambda \) is transverse to \( \Delta \) at a nonsingular point of \( \Delta \).

(ii) \( \gamma_\lambda \) is transverse to the codimension 1 tangent cone of a singular point of \( \Delta \).
The fiber $\pi^{-1}(\lambda)$ has a simple node over an intersection point of the path $\gamma(\lambda)$ with $\Delta$ if

(iii) $\gamma$ is tangent to $\Delta$ at a nonsingular point of $\Delta$.

The above are all the possibilities which can occur in a generic 1-parameter family of paths in the space of cubic equations. Possibility (iii) can happen only for finitely many $\lambda$ and just once for each such $\lambda$.

Case $z_6 \neq 0$ and $z_7 \neq 0$. Using the third factor of $(\mathbb{C}^*)^5$ to set $z_6 = 1$, we obtain the equation

$$F(1, z_2|1, z_4) + H(1, z_2|1, z_4) + G(1, z_2|1, z_4) + \alpha + \beta z_7^2$$

in $\mathbb{C}^4$ with coordinates $z_2, z_4, z_5, z_7$. The partial derivative of (20) with respect to $z_7$ is not 0 for $z_7 \neq 0$. Hence, there are no singular fibers of $\pi$ on the chart.

We have proven that all the fibers $X_\xi$ of $\pi$ are nonsingular except for finitely many with exactly 1 ordinary double point each. Let $X_\xi$ be a nonsingular fiber. Let

$$\mu : X \to \mathbb{P}^1$$

be the restriction $\mu|_X$. The fibers of the product

$$(\pi, \mu) : X \to \mathbb{P}^1 \times \mathbb{P}^1$$

are easily seen to be anticanonical sections of the nonsingular toric surface $W$ with fan in $\mathbb{Z}^2$ determined by the primitives $\rho_5, \ldots, \rho_{10}$. These anticanonical sections are elliptic curves. Since $X_\xi$ has trivial canonical bundle by adjunction and the map

$$\mu : X_\xi \to \mathbb{P}^1$$

is dominant with elliptic fibers, we conclude that $X_\xi$ is an elliptically fibered $K3$ surface.

The Euler characteristic of $X$ can be calculated by toric intersection in $Y$,

$$\chi_{\text{top}}(X) = -480.$$ 

The Euler characteristic of a nonsingular $K3$ fibration over $\mathbb{P}^1$ is 48. Since each fiber singularity reduces the Euler characteristic by 1, we conclude that $\pi$ has exactly 528 singular fibers. \(\square\)

For emphasis, we will sometimes denote the STU model by

$$\pi^{\text{STU}} : X^{\text{STU}} \to \mathbb{P}^1.$$ 

1.5. Divisor restrictions. The divisors $D_1, D_2, D_3, D_9$, and $D_{10}$ have already been shown to restrict to the trivial class in Pic($X_\xi$). The divisors $D_3$ and $D_4$ restrict to the fiber class $F \in \text{Pic}(X_\xi)$ of the elliptic fibration

$$\mu : X_\xi \to \mathbb{P}^1.$$ 

Certainly $F^2 = 0$. Let $S \in \text{Pic}(X_\xi)$ denote the restriction of $D_5$. Toric calculations yield the products

$$F \cdot S = 1, \quad S \cdot S = -2.$$

---

1\ A cusp of $\pi^{-1}(\lambda)$ occurs, for example, when the path has contact order 3 at a nonsingular point of the discriminant.

1\ Since the product $(\mu^Y, \mu^Y) : Y \to \mathbb{P}^1 \times \mathbb{P}^1$ has fibers isomorphic to the nonsingular complete (hence projective) toric surface $W$, the 4-fold $Y$ is projective.
Hence, $S$ may be viewed as the section class of the elliptic fibration \[21\]. The divisors $D_6$ and $D_7$ restrict to classes in the rank 2 lattice generated by $F$ and $S$.

The restriction of $\text{Pic}(Y)$ to each fiber $X_\xi$ is a rank 2 lattice generated by $F$ and $S$ with intersection form \[
\begin{pmatrix}
0 & 1 \\
1 & -2
\end{pmatrix}.
\]

We may also choose generators $L_1 = F$ and $L_2 = F + S$ with intersection form \[
\Lambda = \begin{pmatrix}
0 & 1 \\
1 & 0
\end{pmatrix}.
\]

1.6. 1-parameter families. Let $X$ be a compact 3-dimensional complex manifold equipped with two holomorphic line bundles $L_1, L_2 \to X$ and a holomorphic map $\pi : X \to C$ to a nonsingular complete curve.

The data $(X, L_1, L_2, \pi)$ determine a family of $\Lambda$-polarized $K3$ surfaces if the fibers $(X_\xi, L_{1, \xi}, L_{2, \xi})$ are $K3$ surfaces with intersection form \[
\begin{pmatrix}
L_{1, \xi} \cdot L_{1, \xi} & L_{2, \xi} \cdot L_{1, \xi} \\
L_{1, \xi} \cdot L_{2, \xi} & L_{2, \xi} \cdot L_{2, \xi}
\end{pmatrix} = \begin{pmatrix}
0 & 1 \\
1 & 0
\end{pmatrix}
\] and there exists a simultaneous quasi-polarization. The 1-parameter family $(X, L_1, L_2, \pi)$ yields a morphism, $\iota_\pi : C \to \mathcal{M}_\Lambda$, to the moduli space of $\Lambda$-polarized $K3$ surfaces.

The construction $(X^{STU}, L_1, L_2, \pi^{STU})$ of the STU model in Sections 1.3-1.5 is almost a 1-parameter family of $\Lambda$-polarized $K3$ surfaces. The only failing is the 528 singular fibers of $\pi^{STU}$. Let $\epsilon : C \to \mathbb{P}^1$ be a hyperelliptic curve branched over the 528 points of $\mathbb{P}^1$ corresponding to the singular fibers of $\pi$. The family $\epsilon^* (X^{STU}) \to C$ has 3-fold double point singularities over the 528 nodes of the fibers of the original family. Let $\overline{\pi}^{STU} : \overline{X}^{STU} \to C$ be obtained from a small resolution $\overline{X}^{STU} \to \epsilon^*(X^{STU})$. Let $L_i \to \overline{X}^{STU}$ be the pull-back of $L_i$ by $\epsilon$. The data $\overline{X}^{STU}, L_1, L_2, \overline{\pi}^{STU}$ determine a 1-parameter family of $\Lambda$-polarized $K3$ surfaces; see Section 5.3 of \[31\]. The simultaneous quasi-polarization is obtained from the projectivity of $X^{STU}$. 

1.7. Gromov-Witten invariants. Since $X^{STU}$ is defined by an anticanonical section in a semi-positive nonsingular toric variety $Y$, the genus 0 Gromov-Witten invariants have been proven by Givental [14, 15, 29, 33] to be related by mirror transformation to hypergeometric solutions of the Picard-Fuchs equations of the Batyrev-Borisov mirror. By Section 5.3 of [31], the Gromov-Witten invariants of $\tilde{X}^{STU}$ are exactly twice the Gromov-Witten invariants of $X^{STU}$ for curve classes in the fibers.

2. Noether-Lefschetz numbers and reduced $K3$ invariants

2.1. Refined Noether-Lefschetz numbers. Following the notation of Section 0.2, let $\Lambda \subset U \oplus U \oplus U \oplus E_8(-1) \oplus E_8(-1)$ be primitively embedded with signature $(1, r - 1)$ and integral basis $v_1, \ldots, v_r$. Let $(X, L_1, \ldots, L_r, \pi)$ be a 1-parameter family of $\Lambda$-polarized $K3$ surfaces. Let $d_1, \ldots, d_r$ be integers which do not all vanish.

Lemma 1. The Noether-Lefschetz numbers $NL^\pi_{h,(d_1,\ldots,d_r)}$ completely determine the refinements $NL^\pi_{m,h,(d_1,\ldots,d_r)}$.

Proof. By definition, the refined Noether-Lefschetz numbers satisfy two elementary identities. The first is

$$NL^\pi_{h,(d_1,\ldots,d_r)} = \sum_{m=1}^{\infty} NL^\pi_{m,h,(d_1,\ldots,d_r)}.$$ 

If $m$ does not divide all $d_i$, then $NL^\pi_{m,h,(d_1,\ldots,d_r)}$ vanishes. If $m$ divides all $d_i$, then a second identity holds:

$$NL^\pi_{m,h,(d_1,\ldots,d_r)} = NL^\pi_{1,h',(d_1/m,\ldots,d_r/m)},$$

where $2h - 2 = m^2(2h' - 2)$.

If $\Delta(h, d_1, \ldots, d_r) = 0$, the refined number $NL^\pi_{m,h,(d_1,\ldots,d_r)}$ vanishes by definition unless $m$ is the GCD of $(d_1, \ldots, d_r)$. In the latter case,

$$NL^\pi_{h,(d_1,\ldots,d_r)} = NL^\pi_{m,h,(d_1,\ldots,d_r)}.$$ 

Hence the lemma is trivial in the $\Delta(h, d_1, \ldots, d_r) = 0$ case.

If $\Delta(h, d_1, \ldots, d_r) > 0$, we prove the lemma by induction on $\Delta$. The second identity reduces us to the case where $m = 1$. The first identity determines the $m = 1$ case in terms of the Noether-Lefschetz number $NL_{h,(d_1,\ldots,d_r)}$ and refined numbers with

$$\Delta(h', d'_1, \ldots, d'_r) < \Delta(h, d_1, \ldots, d_r).$$

□

2.2. STU model. The resolved version of the STU model

$$\pi^{STU} : \tilde{X}^{STU} \to C$$

is lattice polarized with respect to

$$\Lambda = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$ 

The application of the results of [3, 25] to the STU model is extremely simple. Since the lattice $\Lambda$ is unimodular, the corresponding representation $\rho^*_\Lambda$ is 1-dimensional.
and, in fact, is the trivial representation of $M_{p_4}(\mathbb{Z})$. The Noether-Lefschetz degrees are thus encoded by a scalar modular form of weight $\frac{22 - r}{2} = 10$. The space of such forms is well known to be of dimension 1 and spanned by the product of the Eisenstein series \[ E_{10}(q) = E_4(q)E_6(q) = 1 - 264 \sum_{n \geq 1} \sigma_9(n)q^n. \]

Hence, a single Noether-Lefschetz calculation determines the full series.

Lemma 2. $NL_{\tilde{\pi}_{(0,0),0}} = 1056$.

Proof. By Proposition 1, the STU model

$$\pi^{STU}: X^{STU} \to \mathbb{P}^1$$

has 528 nodal fibers. Let $S$ be a fiber of the resolved family $\tilde{\pi}^{STU}$ lying over a singular fiber of $\pi$. The Picard lattice of $S$ certainly contains

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

spanned by $L_1, L_2$, and the $(-2)$-curve $E$ of the small resolution. Let

$$\tilde{\iota}: C \to \mathcal{M}_A$$

be the map to moduli. Since a class $\beta$ satisfying

$$\langle \beta, \beta \rangle = -2$$

on a $K3$ surface is either effective or anti-effective, the set-theoretic intersections of $\tilde{\iota}$ with $D_{0,(0,0)}$ correspond to fibers of $\tilde{\pi}$, where $L_1$ and $L_2$ do not generate an ample class, precisely, the 528 fibers of $\tilde{\pi}$ lying over the singular fibers of $\pi$.

The divisor $D_{0,(0,0)}$ has multiplicity exactly 2 at the 528 intersections with $\tilde{\iota}$ since $E$ and $-E$ are the only $-2$ classes orthogonal to $L_1$ and $L_2$. Finally, since $E$ has normal bundle $(-1,-1)$ in $X^{STU}$, the curve $\tilde{\iota}$ is transverse to the reduced divisor $\frac{1}{2}D_{0,(0,0)}$ at the 528 intersections. We conclude that $NL_{\tilde{\pi}_{(0,0),0}} = 528 \cdot 2 = 1056$. \(\square\)

Proposition 2. The Noether-Lefschetz degrees of the resolved STU model are given by the equation

$$NL_{\tilde{\pi}_{h,(d_1,d_2)}} = -4E_4(q)E_6(q) \left[ \frac{\Delta(h,d_1,d_2)}{2} \right].$$

---

13 The Eisenstein series $E_{2k}$ is the modular form defined by the equation

$$-\frac{B_{2k}}{4k} E_{2k}(q) = -\frac{B_{2k}}{4k} + \sum_{n \geq 1} \sigma_{2k-1}(n)q^n,$$

where $B_{2n}$ is the $2n^{th}$ Bernoulli number and $\sigma_n(k)$ is the sum of the $k^{th}$ powers of the divisors of $n$,

$$\sigma_k(n) = \sum_{i|n} i^k.$$
2.3. BPS states. Let \((\bar{X}^{\text{STU}}, \bar{L}_1, \bar{L}_2, \bar{\pi}^{\text{STU}})\) be the \(\Lambda\)-polarized STU model. The vertical classes are the kernel of the push-forward map by \(\bar{\pi}\),

\[
0 \to H_2(\bar{X}, \mathbb{Z}) \bar{\pi} \to H_2(\bar{X}, \mathbb{Z}) \to H_2(C, \mathbb{Z}) \to 0.
\]

While \(\bar{X}\) need not be a projective variety, \(\bar{X}\) carries a \((1,1)\)-form \(\omega_K\) which is Kähler on the \(K3\) fibers of \(\bar{\pi}\). The existence of a fiberwise Kähler form is sufficient to define the Gromov-Witten theory for vertical classes,

\[
0 \neq \gamma \in H_2(\bar{X}, \mathbb{Z}) \bar{\pi}.
\]

The fiberwise Kähler form \(\omega_K\) is obtained by a small perturbation of the quasi-Kähler form obtained from the quasi-polarization. The associated Gromov-Witten theory is independent of the perturbation used.

Let \(\bar{M}_0(\bar{X}, \gamma)\) be the moduli space of stable maps from connected genus 0 curves to \(\bar{X}\). Gromov-Witten theory is defined by integration against the virtual class,

\[
N_{0,\gamma}^{\bar{X}} = \int_{\bar{M}_0(\bar{X}, \gamma)_{\text{vir}}} 1.
\]

The expected dimension of the moduli space is 0.

The genus 0 Gromov-Witten potential \(F^{\bar{X}}(v)\) for nonzero vertical classes is the series

\[
F^{\bar{X}} = \sum_{\gamma \neq 0 \in H_2(\bar{X}, \mathbb{Z})} N_{0,\gamma}^{\bar{X}} v^\gamma,
\]

where \(v\) is the curve class variable. The BPS counts \(n_{0,\gamma}^{\bar{X}}\) of Gopakumar and Vafa are uniquely defined by the following equation:

\[
F^{\bar{X}} = \sum_{\gamma \neq 0 \in H_2(\bar{X}, \mathbb{Z})} n_{0,\gamma}^{\bar{X}} \sum_{d > 0} \frac{v^{d\gamma}}{d^3}.
\]

Conjecturally, the invariants \(n_{0,\gamma}^{\bar{X}}\) are integral and obtained from the cohomology of an as yet unspecified moduli space of sheaves on \(\bar{X}\). We do not assume that the conjectural properties hold.

Using the \(\Lambda\)-polarization, we define the BPS counts

\[
n_{0, (d_1, d_2)}^{\bar{X}} = \sum_{\gamma \in H_2(\bar{X}, \mathbb{Z})^\alpha, \int_{\bar{\pi}} L_i = d_i} n_{0,\gamma}^{\bar{X}}
\]

when \(d_1\) and \(d_2\) are not both 0.

The original STU model,

\[
\pi^{\text{STU}} : X^{\text{STU}} \to \mathbb{P}^1,
\]

with 528 singular fibers is a nonsingular, projective, Calabi-Yau 3-fold. Hence the Gromov-Witten invariants are well-defined. Let \(n_{0, (d_1, d_2)}^{X}\) denote the fiberwise Gopakumar-Vafa invariant with degrees \(d_i\) measured by \(L_i\). By the argument of Section 1.7

\[
n_{0, (d_1, d_2)}^{\bar{X}} = 2n_{0, (d_1, d_2)}^{X}
\]

when \(d_1\) and \(d_2\) are not both 0.
2.4. Invertibility of constraints. Let \( \mathcal{P} \subset \mathbb{Z}^2 \) be the set of pairs
\[
\mathcal{P} = \{ (d_1, d_2) \neq (0, 0) \mid d_1 \geq 0, \ d_1 \geq -d_2 \}.
\]
Pairs \((d_1, d_2) \in \mathcal{P}\) are certainly positive with respect to any quasi-polarization for \(\tilde{\pi}^{\text{STU}}\) since such \((d_1, d_2)\) can be realized by linear combinations of the effective classes \(F\) and \(S\).

Theorem 2 applied to the resolved STU model yields the equation
\[
(25) \quad \tilde{n}_{0,(d_1,d_2)}^\pi = \sum_{h=0}^{\infty} \sum_{m=1}^{\infty} r_{0,m,h} \cdot NL_{m,h,(d_1,d_2)}^{\tilde{\pi}}
\]
for \((d_1, d_2) \in \mathcal{P}\). The BPS states on the left side will be computed by mirror symmetry in Section 3. The refined Noether-Lefschetz degrees are determined by Lemma 1 and Proposition 2. Consequently, equation (25) provides constraints on the reduced K3 invariants \(r_{0,m,h}\).

The integrals \(r_{0,m,h}\) are very simple in case \(h \leq 0\). By Lemma 2 of [31], \(r_{0,m,h} = 0\) for \(h < 0\),
\[
r_{0,1,0} = 1,
\]
and \(r_{0,m,0} = 0\) otherwise.

**Proposition 3.** The set of integrals \(\{r_{0,m,h}\}_{m \geq 1, h > 0}\) is uniquely determined by the set of constraints (25) for \((d_1 \geq 0, \ d_2 > 0)\) and the integrals \(r_{0,m,h} \leq 0\).

**Proof.** A certain subset of the linear equations with \(d_2 > 0\) will be shown to be upper triangular in the variables \(r_{0,m,h}\). Picard rank 2 is crucial for the argument.

Let us fix in advance the values of \(m \geq 1\) and \(h > 0\). We proceed by induction on \(m\) assuming the reduced invariants \(r_{0,m',h}\) have already been determined for all \(m' < m\). The assumption is vacuous when \(m = 1\). We can also assume that \(r_{0,m,h'}\) has been determined inductively for \(h' < h\). If \(2h - 2\) is not divisible by \(2m^2\), then we have \(r_{0,m,h} = 0\), so we can further assume that
\[
2h - 2 = m^2(2s - 2)
\]
for an integer \(s > 0\).

Consider equation (25) for \((d_1, d_2) = (m(s - 1), m)\). Certainly
\[
NL_{m',h',(m(s-1),m)}^{\tilde{\pi}} = 0
\]
unless \(m'\) divides \(m\). By the Hodge index theorem, we must have
\[
(26) \quad \Delta(h', m(s-1), m) = 2 - 2h' + m^2(2s - 2) \geq 0
\]
if \(NL_{m,h',(m(s-1),m)}^{\tilde{\pi}} \neq 0\). Inequality (26) implies that \(h' \leq h\).

Therefore, the constraint (25) takes the form
\[
\tilde{n}_{0,(m(s-1),m)} = r_{0,m,h} NL_{m,h,(m(s-1),m)}^{\tilde{\pi}} + \ldots
\]
where the dots represent terms involving \(r_{0,m',h'}\) with either
\[
m' < m \quad \text{or} \quad m' = m, \ h' < h.
\]
The leading coefficient is given by
\[
NL_{m,h,(m(s-1),m)}^{\tilde{\pi}} = NL_{h,(m(s-1),m)}^{\tilde{\pi}} = -4.
\]
As the system is upper-triangular, we can invert to solve for \(r_{0,m,h}\). \(\square\)
2.5. Proof of the Yau-Zaslow conjecture. By Proposition 3, we need only show that the answer for \( r_{0,m,h} \) predicted by the Yau-Zaslow conjecture satisfies the constraints (25) for all pairs \((d_1 \geq 0, \ d_2 > 0)\).

Let \( X^{STU} \) be the original Calabi-Yau 3-fold of the STU model. Let

\[
D^3_2 F^X = \sum_{(d_1, d_2) \in \mathcal{P}} d_2^3 N^X_{0,(d_1, d_2)} q_1^{d_1} q_2^{d_2}
\]

be the third derivative\(^{14}\) of the genus 0 Gromov-Witten series for \( \pi \)-vertical classes in \( \mathcal{P} \).

We can calculate \( D^3_2 F^X \) by the constraint (25) assuming the validity of the Yau-Zaslow conjecture,

\[
D^3_2 F^X = \sum_{(d_1, d_2) \in \mathcal{P}} d_2^2 c(d_1, d_2) \frac{q_1^{d_1} q_2^{d_2}}{1 - q_1^{d_1} q_2^{d_2}},
\]

where \( c(k, l) \) is the coefficient of \( q^{kl} \) in

\[-2 \frac{E_4(q) E_6(q)}{\eta^{24}(q)}.\]

**Proposition 4.** The Yau-Zaslow conjecture is implied by the identity

\[
\sum_{(d_1, d_2) \in \mathcal{P}} d_2^3 N^X_{0,(d_1, d_2)} q_1^{d_1} q_2^{d_2} = \sum_{(d_1, d_2) \in \mathcal{P}} d_2^3 c(d_1, d_2) \frac{q_1^{d_1} q_2^{d_2}}{1 - q_1^{d_1} q_2^{d_2}}.
\]

**Proof.** The \( q_1^{d_1} q_2^{d_2} \) coefficient of the above identity is simply \( d_2^3 \) times the constraint (25). Since we only require the constraints in case

\((d_1 \geq 0, \ d_2 > 0) \in \mathcal{P},\)

the identity implies all the constraints we need.

The remainder of the paper is devoted to the proof of Proposition 4. The genus 0 Gromov-Witten invariants of \( X \) are related, after a mirror transformation, to hypergeometric solutions of the associated Picard-Fuchs system of differential equations. Hence, Proposition 4 amounts to a subtle identity among special functions.

3. Mirror transform

3.1. Picard-Fuchs. Let \( \pi : X \to \mathbb{P}^1 \) be the STU model. Let

\[\delta_0 \in H^*(X, \mathbb{C})\]

denote the identity class. A basis of \( H^2(X, \mathbb{C}) \) is obtained from the restriction of the toric divisors of \( Y \) discussed in Section 1.5.\(^{15}\)

\[\delta_1 = 2D_1 + 2D_3 + D_5, \ \delta_2 = D_3, \ \delta_3 = D_1.\]

Recall, \( \delta_3 \) vanishes on the fibers of \( \pi \). Let \( \{ \delta_j \} \) be a full basis of \( H^*(X, \mathbb{C}) \) extending the above selections.

Let \( u_1, u_2, u_3 \) be the canonical coordinates for the mirror family with respect to the divisor basis \( \delta_1, \delta_2, \delta_3 \). Let

\[\theta_i = u_i \frac{\partial}{\partial u_i}.\]

\(^{14}\) \( D_2 = q_2 \frac{d}{dq_2} \)

\(^{15}\)
The Picard-Fuchs system associated to the mirror of $X^{STUV}$ is:

$$L_1 = \theta_1 (\theta_1 - 2 \theta_2 - 2 \theta_3) - 12 (6 \theta_1 - 5) (6 \theta_1 - 1) u_1,$$

$$L_2 = \theta_2^2 - (2 \theta_2 + 2 \theta_3 - \theta_1 - 2) (2 \theta_2 + 2 \theta_3 - \theta_1 - 1) u_2,$$

$$L_3 = \theta_3^2 - (2 \theta_2 + 2 \theta_3 - \theta_1 - 2) (2 \theta_2 + 2 \theta_3 - \theta_1 - 1) u_3. \tag{29}$$

The system is obtained canonically from the Batyrev-Borisov construction; see [9] for the formalism.

3.2. Solutions. A fundamental solution to the Picard-Fuchs system can be written in terms of GKZ hypergeometric series,

$$\varpi \in H^*(X, \mathbb{C}) \otimes \mathbb{C}[\log(u_1), \log(u_2), \log(u_3)\lbrack \rbrack u_1, u_2, u_3]. \tag{30}$$

Let $\varpi(u, \delta_j)$ be the corresponding coefficient of (30). Then

$$L_i \varpi(u, \delta_j) = 0.$$

The standard normalization of $\varpi$ satisfies two important properties:

(i) The $\delta_0$ coefficient is the unique solution

$$\varpi(u, \delta_0) = 1 + O(u)$$

holomorphic at $u = 0$.

(ii) For $1 \leq i \leq 3$,

$$\varpi(u, \delta_i) = \frac{\varpi(u, \delta_0)}{2\pi i} \log(u_i) + O(u)$$

are the logarithmic solutions.

Let $T_1, T_2, T_3$ be coordinates on $H^2(X, \mathbb{C})$ with respect to the basis $\delta$. The mirror transformation is defined by

$$T_i = \frac{\varpi(u, \delta_i)}{\varpi(u, \delta_0)} = \frac{1}{2\pi i} \log(u_i) + O(u)$$

for $1 \leq i \leq 3$.

The mirror transformation relates the genus 0 Gromov-Witten theory of $X$ to the Picard-Fuchs system for the mirror family. For anticanonical hypersurfaces in toric varieties, a proof is given in [15].

3.3. Mirror transform for $q_3 = 0$. We introduce two modular parameters

$$\tau_1 = T_1, \quad \tau_2 = T_1 + T_2. \tag{31}$$

For $i = 1$ and 2, let

$$\hat{q}_i = \exp(2\pi i \tau_i),$$

and let $q_3 = \exp(2\pi i T_3)$.

Our first step is to find a modular expression for the mirror map and the period $\varpi(u, \delta_0)$ to leading order in $q_3$. We prove two formulas discovered by Klemm, Lerche, and Mayr in [24].

Lemma 3. We have

$$u_1 = \frac{2(j(\hat{q}_1) + j(\hat{q}_2) - \mu)}{j(\hat{q}_1)j(\hat{q}_2) + \sqrt{j(\hat{q}_1)(j(\hat{q}_1) - \mu)j(\hat{q}_2)(j(\hat{q}_2) - \mu)}} + O(q_3),$$

$$u_2 = \frac{(j(\hat{q}_1)j(\hat{q}_2) + \sqrt{j(\hat{q}_1)(j(\hat{q}_1) - \mu)j(\hat{q}_2)(j(\hat{q}_2) - \mu))^2}}{4j(\hat{q}_1)j(\hat{q}_2)(j(\hat{q}_1) + j(\hat{q}_2) - \mu)^2} + O(q_3).$$
where $\mu = 1728$ and

$$j(q) = \frac{E_4^3}{\eta^{24}} = \frac{1}{q} + 744 + 196884q + O(q^2)$$

is the normalized $j$ function.

**Lemma 4.** \( \lim_{q_3 \to 0} \varpi(u, \delta_0) = E_4(\hat{q}_1)^{1/2} E_4(\hat{q}_2)^{1/2} \).

**Proof.** We prove Lemmas 3 and 4 together. The first step is to perform the following change of variables:

$$u_1 = z_1, \quad u_2 = \frac{z_2}{2} \left(1 + \sqrt{1 - 4z_3}\right), \quad u_3 = \frac{z_2}{2} \left(1 - \sqrt{1 - 4z_3}\right),$$

with the inverse change

$$z_1 = u_1, \quad z_2 = u_2 + u_3, \quad z_3 = \frac{u_2 u_3}{(u_2 + u_3)^2}.$$ 

In the new variables, the limit $u_3 \to 0$ becomes the limit $z_3 \to 0$.

The statement of Lemma 3 in the variables $z_i$ remains unchanged to first order in $q_3$. We will prove

$$z_1 = \frac{2 \left(j(\hat{q}_1) + j(\hat{q}_2) - \mu\right)}{j(\hat{q}_1) j(\hat{q}_2) + \sqrt{j(\hat{q}_1) (j(\hat{q}_1) - \mu) \sqrt{j(\hat{q}_2) (j(\hat{q}_2) - \mu)}}} + O(q_3),$$

$$z_2 = \frac{(j(\hat{q}_1) j(\hat{q}_2) + \sqrt{j(\hat{q}_1) (j(\hat{q}_1) - \mu) \sqrt{j(\hat{q}_2) (j(\hat{q}_2) - \mu)}})^2}{4 j(\hat{q}_1) j(\hat{q}_2) (j(\hat{q}_1) + j(\hat{q}_2) - \mu)^2} + O(q_3).$$

The Picard-Fuchs differential operators (20) can be rewritten as

$$\mathcal{L}_1'(z) = \mathcal{L}_1(u),$$

$$z_2 \sqrt{1 - 4z_3} \mathcal{L}_2'(z) = \mathcal{L}_2(u) - \mathcal{L}_3(u),$$

$$z_2 \sqrt{1 - 4z_3} \mathcal{L}_3'(z) = u_3 \mathcal{L}_2(u) - u_2 \mathcal{L}_3(u),$$

with

$$\mathcal{L}_1' = \theta_1 (\theta_1 - 2 \theta_2) - 12 (6 \theta_1 - 5) (6 \theta_1 - 1) z_1,$$

$$\mathcal{L}_2' = \theta_2 (\theta_2 - 2 \theta_3) - (2 \theta_2 - \theta_1 - 2) (2 \theta_2 - \theta_1 - 1) z_2,$$

$$\mathcal{L}_3' = \theta_3^2 - (2 \theta_3 - \theta_2 - 2) (2 \theta_3 - \theta_2 - 1) z_3,$$

where now $\theta_i = z_i \frac{d}{dz_i}$. Since $\mathcal{L}_3'(z) \to 0$ in the limit $z_3 \to 0$, we need only focus on $\mathcal{L}_1'(z)$ and $\mathcal{L}_2'(z)$.

Next, we transform $\mathcal{L}_1'(z)$ and $\mathcal{L}_2'(z)$ to new variables $y_1, y_2, y_3$ via the change

$$z_1 = \frac{2 (y_1 + y_2 - \mu)}{y_1 y_2 + \sqrt{y_1 (y_1 - \mu) y_2 (y_2 - \mu)}},$$

$$z_2 = \frac{(y_1 y_2 + \sqrt{y_1 (y_1 - \mu) y_2 (y_2 - \mu)})^2}{4 y_1 y_2 (y_1 + y_2 - \mu)^2},$$

$$z_3 = y_3.$$
We obtain
\[ L''_1 = y_1^2 y_2 (y_1 - \mu) \partial^2 y_1 + y_1 y_2 (y_1 - \frac{\mu}{2}) \partial y_1 - y_1 y_2 (y_2 - \mu) \partial^2 y_2 \]
\[ - y_1 y_2 (y_2 - \frac{\mu}{2}) \partial y_2 + 60 (y_1 - y_2), \]
\[ L''_2 = -y_1^2 (y_1 - \mu) \partial^2 y_1 + y_1 (\frac{\mu}{2} - y_1) \partial y_1 + y_2^2 (y_2 - \mu) \partial^2 y_2 + y_2 (y_2 - \frac{\mu}{2}) \partial y_2 \]
\[ - 2y_1 y_3 (y_1 - \mu) \partial y_1 \partial y_3 + 2y_2 y_3 (y_2 - \mu) \partial y_2 \partial y_3. \]
In the limit \( y_3 \to 0 \), the second line on the right for \( L''_2 \) vanishes. We can combine \( L''_1 \) and \( L''_2 \) to obtain the following simple forms:
\[ L''_1 + y_1 \lim_{y_3 \to 0} L''_2 = (y_1 - y_2) \left( 60 - \left( y_1 - \frac{\mu}{2} \right) y_1 \partial y_1 - (y_1 - \mu) y_1^2 \partial^2 y_1 \right), \]
\[ L''_1 + y_2 \lim_{y_3 \to 0} L''_2 = (y_1 - y_2) \left( 60 - \left( y_2 - \frac{\mu}{2} \right) y_2 \partial y_2 - (y_2 - \mu) y_2^2 \partial^2 y_2 \right). \]

The solution \( \varpi(y, \delta_0)_{y_3=0} \) therefore satisfies the differential equation
\[ \mathcal{L} = (y - \mu) y^2 \partial^2 y + \left( y - \frac{\mu}{2} \right) y \partial y - 60 \]
in both \( y_1 \) and \( y_2 \).

Changing (33) to the variable \( t = \frac{1728}{y} \) yields
\[ \mathcal{L} = t (1 - t) \partial^2 t + (1 - \frac{3}{2} t) \partial t - \frac{5}{144}, \]
which by comparing with the general hypergeometric differential operator
\[ \mathcal{L} = t (1 - t) \partial^2 t + (c - (1 + a + b)t) \partial t - ab \]
is identified with the system
\[ _2F_1(a, b; c; t) = _2F_1\left( \frac{1}{12}, \frac{5}{12}; 1; t(\tau) \right). \]

According to the results of Klein and Fricke as reviewed in [37], we have a unique (up to scaling) solution \( g_0 \) to (33) locally analytic at \( y = \infty \). The solution can be written as
\[ g_0(j(\tau)) = (E_4)^{\frac{1}{4}}(\tau), \quad y(\tau) = j(\tau). \]
Moreover, the inverse is
\[ \tau(y) = \frac{g_1(y)}{2\pi i g_0(y)}, \]
where \( g_1 \) is a logarithmic solution at \( y = \infty \) of \( \mathcal{L} \), unique up to normalization and addition of \( g_0 \).

Transformation of the solution \( \varpi(u, \delta_0) \) is seen to be analytic in a neighborhood of \( t_1 = t_2 = 0 \). We conclude that
\[ \varpi(u, \delta_0)_{u_3=0} = E_4^{\frac{1}{4}}(\tau_1) E_4^{\frac{1}{4}}(\tau_2). \]
By comparing the first few coefficients of the actual solutions \( \varpi(u, \delta_i) \) in the \( u_3 \to 0 \) limit, we can uniquely identify
\[ \tau_1(u) = T_1(u), \quad \tau_2(u) = T_1(u) + T_2(u). \]
Hence, Lemma 4 is established. Lemma 3 is proven by transforming back to the \( u_1 \) and \( u_2 \) variables. \( \square \)
Restricted to a K3 fiber of \( \pi: X \to \mathbb{P}^1 \), we have
\[
\delta_1 = 2F + S, \quad \delta_2 = F.
\]
The coordinates \( 2\pi i\tau_1 \) and \( 2\pi i\tau_2 \) correspond to the divisor basis
\[
L_2 = F + S, \quad L_1 = F
\]
of the K3 fiber. Since the variables \( q_1 \) and \( q_2 \) of Section 2 measure degrees against
\( L_1 \) and \( L_2 \), we see that
\[
\tilde{q}_1 = q_2 \quad \text{and} \quad \tilde{q}_2 = q_1
\]
for the fiber geometry.

3.4. B-model. The mirror transformation results of Section 3.3 together with a B-model calculation of the periods will be used to prove the following result discovered by Klemm, Mayr, and Lerche [24].

**Proposition 5.** We have
\[
2 + \sum_{(d_1, d_2) \in \mathcal{P}} d_1^3 N_{0, (d_1, d_2)} q_1^{d_1} q_2^{d_2} = 2 \frac{E_4(q_1) E_6(q_1)}{\eta^{24}(q_1)} \frac{E_4(q_2)}{j(q_1) - j(q_2)}.
\]
The left side of Proposition 5 is the left side of Proposition 3 with an added degree 0 constant 2.

**Proof.** We will use following universal expression for the Gromov-Witten invariants of \( X \) in terms of the periods of the mirror:
\[
2 + \sum_{(d_1, d_2) \in \mathcal{P}} d_1^3 N_{0, (d_1, d_2)} q_1^{d_1} q_2^{d_2} = \lim_{q_3 \to 0} \frac{1}{\omega(u(T), \delta_0)^2} \sum_{i, j, k=1}^3 \frac{\partial u_i \partial u_j \partial u_k}{\partial \tau_1 \partial \tau_1 \partial \tau_1} Y_{i, j, k}(u(T)),
\]
where the \( Y_{i, j, k} \) are the Yukawa couplings of the mirror family; see [9] [24].
The periods \( Y_{i, j, k} \) can be explicitly computed via Griffith transversality [24] and greatly simplify in the \( q_3 \to 0 \) limit. We tabulate the results below:

- \( Y_{111} = \frac{8(1 - \tilde{u}_1)}{\tilde{u}_1 \Delta_1} \),
- \( Y_{112} = \frac{2(1 - \tilde{u}_1)^2 + \tilde{u}_1^2 (\tilde{u}_2 - \tilde{u}_3)}{\tilde{u}_1^2 \tilde{u}_2 \Delta_1} \),
- \( Y_{113} = \frac{2(1 - \tilde{u}_1)^2 + \tilde{u}_1^2 (\tilde{u}_3 - \tilde{u}_2)}{\tilde{u}_1^2 \tilde{u}_3 \Delta_1} \),
- \( Y_{122} = \frac{2\tilde{u}_1 (1 - \tilde{u}_1)}{\tilde{u}_2 \Delta_1} \),
- \( Y_{123} = \frac{(1 - \tilde{u}_1) ((1 - \tilde{u}_1)^2 - (\tilde{u}_2 + \tilde{u}_3) \tilde{u}_1^2)}{\tilde{u}_1 \tilde{u}_2 \tilde{u}_3 \Delta_1} \).

Here, we have introduced the variables
\[
\tilde{u}_1 = 432 u_1, \quad \tilde{u}_2 = 4 u_2, \quad \tilde{u}_3 = 4 u_3
\]
and the discriminant loci
\begin{equation}
\Delta_1 = (1 - \tilde{u}_1)^4 - 2(\tilde{u}_2 + \tilde{u}_3)\tilde{u}_1^2(1 - \tilde{u}_1)^2 + (\tilde{u}_2 - \tilde{u}_3)^2\tilde{u}_1^4,
\end{equation}
\begin{equation}
\Delta_2 = (1 - \tilde{u}_2 - \tilde{u}_3)^2 - 4\tilde{u}_2\tilde{u}_3.
\end{equation}

The quantities $A_2$ and $A_3$ are defined by
\begin{equation}
A_2 = (1 + \tilde{u}_2 - \tilde{u}_3) (1 - \tilde{u}_1)^2 + \tilde{u}_1^2 (1 - \tilde{u}_3 - 3 \tilde{u}_2) (\tilde{u}_2 - \tilde{u}_3),
A_3 = (1 + \tilde{u}_3 - \tilde{u}_2) (1 - \tilde{u}_1)^2 + \tilde{u}_1^2 (1 - \tilde{u}_2 - 3 \tilde{u}_3) (\tilde{u}_3 - \tilde{u}_2).
\end{equation}

The normalizations of the Yukawa couplings $Y_{i,j,k}$ are fixed by the classical intersections.

The leading behavior of the mirror map for $u_1, u_2$ is obtained by rewriting Lemma 3 in terms of $E_4(\tau_i)$ and $E_6(\tau_i)$ as
\begin{equation}
u_1 = \frac{1}{864} \left( 1 - \frac{E_6(\tau_i) E_6(\tau_j)}{E_4(\tau_i)^{\frac{3}{2}} E_4(\tau_j)^{\frac{3}{2}}} \right) + O(q_3),
\end{equation}
\begin{equation}
u_2 = \frac{E_4(\tau_i)^3 - E_6(\tau_i)^2}{4 \left( E_4(\tau_i)^{\frac{3}{2}} E_4(\tau_j)^{\frac{3}{2}} - E_6(\tau_i) E_6(\tau_j) \right)^2} + O(q_3).
\end{equation}

Denote the leading behavior of the last mirror map by
\begin{equation}
u_3 = q_3 f_3(\hat{q}_1, \hat{q}_2) + O(q_3^2).
\end{equation}

The derivatives of the mirror maps with respect to $T_2$ are easily evaluated using the standard identities
\begin{align*}
q \frac{d}{dq} E_2 &= \frac{1}{12}(E_2^2 - E_4), \\
q \frac{d}{dq} E_4 &= \frac{1}{2}(E_2 E_4 - E_6), \\
q \frac{d}{dq} E_6 &= \frac{1}{2}(E_2 E_6 - E_4^2), \\
q \frac{d}{dq} j &= -j \frac{E_6}{E_4}.
\end{align*}

We find, to leading order in $q_3$,
\begin{align*}
\frac{\partial \nu_1}{\partial \tau_1} &= \frac{E_6(\tau_2) \left( E_4(\tau_1)^3 - E_6(\tau_1)^2 \right)}{1728 E_4(\tau_2)^{\frac{3}{2}} E_4(\tau_1)^{\frac{3}{2}}}, \\
\frac{\partial \nu_2}{\partial \tau_2} &= \sqrt{E_4(\tau_1) \left( E_4(\tau_2)^3 - E_6(\tau_2)^2 \right) \left( -\left( E_4(\tau_1)^{\frac{3}{2}} E_6(\tau_2) \right) + E_6(\tau_1) \left( E_4(\tau_1)^3 - E_6(\tau_1)^2 \right) \right)} \\
&\quad \cdot \frac{4 \left( E_4(\tau_1)^{\frac{3}{2}} E_4(\tau_2)^{\frac{3}{2}} - E_6(\tau_2) E_6(\tau_1) \right)}{4 \left( E_4(\tau_1)^{\frac{3}{2}} E_4(\tau_2)^{\frac{3}{2}} - E_6(\tau_2) E_6(\tau_1) \right)^2}.
\end{align*}

The derivative $\frac{\partial \nu_3}{\partial \tau_1}$ can be written to this order as
\begin{equation}
\frac{\partial \nu_3}{\partial \tau_1} = \frac{\nu_3}{f_3(\hat{q}_1, \hat{q}_2)} \frac{\partial}{\partial \tau_1} f_3(\hat{q}_1, \hat{q}_2) + O(u_3^2).
\end{equation}

There are many simplifications in the limit $u_3 \to 0$. First the triple couplings
\begin{align*}
Y_{133}, & \quad Y_{233}, \quad Y_{333}\n\end{align*}
do not have enough inverse powers of \( u_3 \) and therefore do not contribute by the vanishing \[38\]. Second, the surviving \( Y_{i,j,k} \) simplify in the limit. We evaluate

\[
\lim_{q_3 \to 0} \frac{1}{\varpi(u(T),\vartheta_0)^3} \sum_{i,j,k=1}^3 \frac{\partial u_i}{\partial \tau_1} \frac{\partial u_j}{\partial \tau_1} \frac{\partial u_k}{\partial \tau_1} Y_{i,j,k}(u(T)) = -2 \frac{E_4(\tau_2) E_4(\tau_1) E_6(\tau_2) (E_4(\tau_1)^3 - E_6(\tau_1)^2)}{E_4(\tau_2)^3 E_6(\tau_1)^2 - E_4(\tau_1)^3 E_6(\tau_2)^2}.
\]

The possible linear dependence on \( f_3(\hat{q}_1,\hat{q}_2) \) drops out as claimed in \[24\]. Using the standard identities

\[
j = \frac{E_4^3}{\eta^{24}}, \quad \eta^{24} = E_4^3 - E_6^2,
\]

we obtain the right side of Proposition \[5\].

\[
\square
\]

4. The Harvey-Moore identity

4.1. Proof of Proposition [4]. After evaluating the left side via Proposition [5] and dividing by 2, Proposition [4] amounts to a modular form identity. Let

\[
f(\tau) = \frac{E_4(\tau) E_6(\tau)}{\eta(\tau)^{24}} = \sum_{n=-1}^{\infty} c(n) q^n,
\]

where \( q = \exp(2\pi i \tau) \). Then, we must prove

\[
\frac{f(\tau_1) E_4(\tau_2)}{j(\tau_1) - j(\tau_2)} = \frac{q_1}{q_1 - q_2} + E_4(\tau_2) - \sum_{d,k,l>0} \ell^3 c(kl) q_1^{kd} q_2^{ld}.
\]

Equation \[40\] is the Harvey-Moore identity conjectured in \[18\].

4.2. Zagier’s proof of the Harvey-Moore identity. The Harvey-Moore identity implies Proposition [4] and concludes the proof of the Yau-Zaslow conjecture. We present here Zagier’s argument from \[38\].

Let \( S_k \subset M_k \subset M^!_k \) denote the spaces of cusp forms, modular forms, and weakly holomorphic\[15\] modular forms for \( \Gamma = \text{SL}(2,\mathbb{Z}) \). Certainly

\[
f(\tau) \in M^!_{1,2}.
\]

For each \( n \geq 0, \) there is a unique function \( F_n \in M^!_4 \) satisfying

\[
F_n(\tau) = q^{-n} + O(q)
\]
as \( j(\tau) \to \infty \). Uniqueness follows from the vanishing of \( S_4 \). Existence follows by writing \( F_n(\tau) \) as \( E_4(\tau) \) times a polynomial in \( j(\tau) \).

\[
F_0 = E_4, \quad F_1 = E_4(j - 984), \quad F_2 = E_4(j^2 - 1728j + 393768) \ldots
\]

We draw several consequences:

(i) \( F_1|T_n = n^3 F_n \) for all \( n \geq 1 \), where \( T_n \) is the \( n \)th Hecke operator in weight 4. Indeed, \( T_n \) sends \( M^!_4 \) to itself and, by standard formulas for the action of \( T_n \) on Fourier expansions, \( T_n \) sends \( q^{-1} + O(q) \) to \( n^3 q^{-n} + O(q) \).

\[15\]Holomorphic except for a possible pole at infinity.
(ii) $F_1 = -f'''$, where the prime denotes differentiation by

$$
\frac{1}{2\pi i} \frac{d}{d\tau} = q \frac{d}{dq}.
$$

We see that $f'''$ lies in $M_{4}^{k}$ by the $k = 4$ case of Bol’s identity,

$$
\frac{d^{k-1}}{d\tau^{k-1}} (f_{|2-k'}) = \left( \frac{d^{k-1} f}{d\tau^{k-1}} \right) |_{k' \gamma} \quad \forall \gamma \in \Gamma.
$$

Since the Fourier expansion of $f'''$ begins as $-q^{-1} + \mathcal{O}(q)$, the claim is proven.

(iii) For $I(\tau_1) > \max_{\gamma \in \Gamma} I(\gamma \tau_2)$,

$$
\frac{f(\tau_1) E_4(\tau_2)}{j(\tau_1) - j(\tau_2)} = \sum_{n=0}^{\infty} F_n(\tau_2) q_n.
$$

Let $L(\tau_1, \tau_2)$ denote the left side of (4.12). We see that $L(\tau_1, \tau_2)$ is a meromorphic modular form in $\tau_2$ with a simple pole of residue $-\frac{1}{2\pi i}$ at $\tau_2 = \tau_1$ (since $j' = -E_2 E_6 / \eta^2$) and no poles outside $\Gamma_1$. Moreover, $L(\tau_1, \tau_2)$ tends to 0 as $\mathcal{J}(\tau_2) \to \infty$. These properties characterize $L(\tau_1, \tau_2)$ uniquely and show that the $n$th Fourier coefficient with respect to $\tau_1$ for $\mathcal{J}(\tau_1) \to \infty$ has the properties characterizing $F_n(\tau_2)$.

Combining (i) and (ii) with the formula for the action of $T_n$ on Fourier expansions, we obtain

(41)  \[ F_n(\tau) = (-n^{-3} f''') | T_n = n^{-3} \left( q^{-1} - \sum_{m=1}^{\infty} m^3 c(m) q^m \right) | T_n \]

$$
= \left( q^{-n} - \sum_{k,d>0, k,d|n} \ell^3 c(k\ell) q^{\ell d} \right)
$$

for $n > 0$. The Harvey-Moore identity follows from (41) and (iii) together with the equality $F_0 = E_4$.

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