Anisotropic superconductivity mediated by phonons in layered compounds with weak screening effects

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Anisotropic pairing interactions mediated by phonons are examined in layer systems. It is shown that the screening effects become weaker when the layer spacing increases. Then the anisotropic components of the pairing interactions increase with the screening length since the momentum dependence changes. As a result, various types of anisotropic superconductivity occur depending on the parameter region. For example, \( p \)-wave superconductivity occurs when the short-range part of Coulomb repulsion is strong and the layer spacing is large. Two kinds of inter-layer pairing may occur when the layer spacing is not too large. Although the phonon contribution to the \( d \)-wave pairing interaction is weaker than the \( p \)-wave interaction, it increases with the layer spacing. Relevance of the present results to organic superconductors, high-\( T_c \) cuprates, and \( \text{Sr}_2\text{RuO}_4 \) is discussed.

I. INTRODUCTION

Anisotropy of superconducting order parameter and the mechanism of pairing interactions in layered superconductors are recent subjects of much interest. In particular, high-\( T_c \) cuprate superconductors, organic superconductors, and \( \text{Sr}_2\text{RuO}_4 \) compound have been studied by many authors.

There are some evidences that the order parameter has line nodes on Fermi-surface in high-\( T_c \) cuprates. For example, an experiment and a theory on Josephson junction gave an evidence of “\( d \)-wave” order parameter in a cuprate superconductor \[1\]. Linear temperature dependence of the penetration depth was observed at low temperatures \[2\].

On the other hand, superconductivity in \( \text{Sr}_2\text{RuO}_4 \) is considered to be due to spin triplet pairing according to the results of a Knight shift measurement \[3\] and a \( \mu \)SR experiment \[4\]. In the organic superconductor (TMTSF)\(_2\)PF\(_6\), a Knight shift measurement seems to support spin triplet pairing \[5\].

On mechanism of the high transition temperature of the cuprates, pairing interactions of magnetic origin, such as exchange of spin fluctuations \[6\] and a super-exchange interaction between nearest neighbor spins, have been discussed by many authors, because of proximity to the antiferromagnetic phase. However, experimental results of isotope effect suggest that there are contributions to the superconductivity from phonon-mediated interactions in many high-\( T_c \) cuprates \[7,8\]. Absolute values of shifts of \( T_c \) are very large (0.2K \~ 0.7K), but isotope effect exponents \( \alpha \) are small because of the high transition temperature.

Abrikosov proposed a theory based on weak screening of Coulomb interactions and phonon-mediated pairing interactions in which anisotropic \( s \)-wave order parameter was obtained \[9\]. In the presence of on-site Coulomb repulsion, extremely anisotropic \( s \)-wave order parameter with nodes was obtained \[10\]. Bouvier and Bok also calculated an order parameter explicitly, and obtained anisotropic \( s \)-wave in the same model \[11\]. Recently, it has been shown that \( d \)-wave superconductivity is reproduced in a similar model with antiferromagnetic fluctuations \[12,13\].

We proposed in our previous paper \[14\] that triplet pairing superconductivity can be induced by phonon-mediated interactions in ferromagnetic compounds, where singlet pairing is suppressed by Pauli paramagnetic effect.

The origin of the anisotropic components of pairing interactions mediated by phonons is briefly explained as follows. The screening effect limits electron-ion interactions within a range of the order of the screening length. Since the pairing interactions mediated by phonons are obtained by a second order perturbation of the electron-ion interactions, they also have a range of the same order. For example, the screening effect is taken into account as vertex corrections within diagramatic technique \[15\]. When the screening length increases, the interactions are more localized in the momentum space. Hence, the anisotropic components of the interactions increase with the screening length.

In this paper, we examine layered superconductors with the phonon-mediated pairing interactions, extending it to systems with large layer spacing. The layered structure modifies the screening length and the pairing interactions significantly. It is shown that anisotropic components of the pairing interactions increase with the layer spacing. We argue from this result that some aspects of the layered superconductors, such as \( \text{Sr}_2\text{RuO}_4 \), organics, and cuprates, can be explained in the present model.

We also study an effect of anisotropy of density of states in square lattice systems. Although the effect of the anisotropy must be most remarkable when the Fermi-surface is near the van Hove singularities, we consider a system not necessarily near the van Hove singularity but a system with the density of states anisotropy within...
the layers. Enhancement of the transition temperature due to the van Hove singularities has been discussed by many authors. In particular, it has often been discussed that the effect is efficient especially for the $d_{x^2-y^2}$ pairing \[23, 25, 24, 21\]. In this paper, we concentrate on the anisotropy but do not discuss the value of the density of states. It is shown that superconductivity is enhanced by the anisotropy for pairing of $d_{x^2-y^2}$ symmetry, but not for those of $d_{xy}$, $p_x$, and $p_y$ symmetries.

In section II, we define the model of the pairing interactions mediated by phonons. We derive expressions of the coupling constants for various types of anisotropic superconductivity. In section III, we examine the dependence on the layer spacing of the screening length and the pairing interactions. In section IV, we consider a situation in which inter-layer coupling is of the order of intra-layer coupling. In section V, we examine an effect of the anisotropy in the electron dispersion in square lattice systems. Section VI is devoted to discussion and summary.

II. SCREENING EFFECT AND PAIRING INTERACTIONS

First, we introduce a model of pairing interactions. Abrikosov examined an effective pairing interaction mediated by phonons of the form

$$V(q) = g(q_s^2/q^2 + q_s^2) \frac{[\omega(q)]^2}{(\xi_k - \xi_{k+q})^2 - [\omega(q)]^2},$$

(1)

with $q = |q|$ and $q_s = l_s^{-1}$, where $l_s$ denotes the screening length \[17, 18\]. A similar form corresponding to $n = 1$ is obtained by taking into account the screening effect in electron-phonon interactions as explained in a text book \[23\]. If we put $n = 1$ for simplicity and $\xi_k - \xi_{k+q} = 0$ for the electrons near the Fermi-surface in eq. (1) according to Abrikosov \[17\], we obtain a simplified form

$$V(q) = -\frac{g}{2} \frac{q_s^2}{q^2 + q_s^2}. \quad (2)$$

We examined this model in three dimensions in our previous paper \[22\].

In this paper, we consider the layer systems, and define lattice constants $a$ within the layers and $b$ between the layers. We take $x$ and $y$-axes in the direction of the lattice vectors within the layers, and $z$-axis perpendicular to the layers. In eq. (2), the range of the interaction is $l_z/b$ layers in the $z$-direction, while it is $l_s/a$ sites in the layers. Thus, the anisotropy due to the layered structure is partially taken into account for the difference of the lattice constants $b$ and $a$. This model is qualitatively appropriate for long wave length such as $\lambda \sim q^{-1} \gg a, b, l_s$. However, when $b \gg a$, the discrete layered structure in the inter-layer direction must be taken into account for shorter wave length $\lambda \sim q^{-1} \sim b$. Therefore we extend eq. (2) in the form

$$V(q) = -\frac{g}{2} \frac{q_s^2}{|q||^2 + q_s^2} - \frac{g'}{2} \frac{q_s^2}{|q||^2 + q_s^2} \cos q_z b \quad (3)$$

for layer systems, where $q_0$ is the momentum element in the layers. Here we have truncated the interaction at the nearest layers. The first and second terms correspond to the intra-layer and inter-layer interactions, respectively. The parameter $q_s$ is the inverse of the range of the interactions between electrons on the nearest layers in the $x$ and $y$-directions.

The gap equation of superconductivity is written as

$$\Delta(k) = -\frac{1}{\sqrt{N}} \sum_{k'} V(k-k') W(k') \Delta(k'), \quad (4)$$

where

$$W(k') = \tanh \frac{E(k')}{2T} \quad (5)$$

with $E(k) = \sqrt{\xi_k^2 + |\Delta(k)|^2}$ and $N$ the number of lattice sites.

We put the gap function

$$\Delta(k) = \Delta_{||}(k_{||}) \eta(k_z), \quad (6)$$

where $k_{||} = (k_x, k_y)$ and $\eta(k_z)$ is a normalized function of the momentum component $k_z$. From eq. (3), the solution of the gap equation (4) at $T = T_c$ has a form with $\eta(k_z) = 1$, $\sqrt{2} \cos k_z b$, or $\sqrt{2} \sin k_z b$. Then eq. (4) is written as

$$\Delta_{||}(k_{||}) = -\frac{1}{\sqrt{N}} \sum_{k'_||} V(k_{||}, k'_||) W(k'_||) \Delta_{||}(k'_||), \quad (7)$$

where $N_{||}$ denotes the number of sites in a layer, and $V(k_{||}, k'_||)$ denotes the averaged pairing interaction defined by

$$V(k_{||}, k'_||) \equiv \frac{b^2}{(2\pi)^2} \int_{-\pi/b}^{\pi/b} dk_z \int_{-\pi/b}^{\pi/b} dk'_z \eta(k_z) V(k, k') \eta(k'_z). \quad (8)$$

Here we assume that the dispersion in the $z$-direction can be neglected in $\xi_k$ in the gap equation.

We consider cylindrically symmetric Fermi-surface from now on. Hence we put $|k_{||}| = |k'_|| = k_F$ in the pairing interactions eq. (8) and obtain

$$V(\varphi - \varphi') \equiv V(k_{||}, k'_||) = -\frac{g}{\alpha - \cos(\varphi - \varphi')}, \quad (9)$$

with

$$\alpha = 1 + \frac{q_s^2}{2k_F^2}. \quad (10)$$

for $\eta(k_z) = 1$. On the other hand, for the order parameters with $\eta(k_z) = \sqrt{2} \cos k_z b$ and $\eta(k_z) = \sqrt{2} \sin k_z b$, the
expression for $V(\varphi - \varphi')$ is obtained by replacing $g$ and $\alpha$ with $g'$ and $\alpha' = 1 + q' L^2 / 2k_F^2$, respectively, in eq. (9).

We expand the averaged interaction $V(\varphi - \varphi')$ as

$$\begin{align*}
V(\varphi - \varphi') &= \sum_{m=0}^{\infty} V_m n_m \gamma_m(\varphi - \varphi') \\
&= \sum_{m=0}^{\infty} V_m (\gamma_m(\varphi) \gamma_m(\varphi') + \bar{\gamma}_m(\varphi) \bar{\gamma}_m(\varphi')),
\end{align*}$$

and the gap function $\Delta_{\parallel}(\varphi) = \Delta_{\parallel}(\mathbf{k}_\parallel)$ as

$$\Delta_{\parallel}(\varphi) = \sum_{m=0}^{\infty} [\Delta_m \gamma_m(\varphi) + \bar{\Delta}_m \bar{\gamma}_m(\varphi)],$$

where

$$\begin{align*}
\gamma_m(\varphi) &= n_m \cos(m\varphi) \\
\bar{\gamma}_m(\varphi) &= n_m \sin(m\varphi),
\end{align*}$$

with normalization factors

$$n_m = \begin{cases} 1 & \text{for } m = 0 \\ \sqrt{2} & \text{for } m \neq 0. \end{cases}$$

The expansion factor $V_m$ is calculated by

$$V_m = \frac{1}{n_m} \int_0^{2\pi} \frac{d\theta}{2\pi} \gamma_m(\theta) V(\theta).$$

It is easy to perform the integration in eq. (15). For $\eta(k_z) = 1$, we obtain dimensionless coupling constants

$$\lambda_m = gN(0) \left[ \frac{\alpha - 1}{\alpha + 1} [\alpha - \sqrt{\alpha^2 - 1}]^m. \right.$$

Then the superconducting transition temperature $T_c$ is obtained by

$$T_c = 1.13 \omega_D e^{-1/\lambda_m},$$

with $\lambda_m = -V_m N(0)$ from eq. (4), where $N(0)$ is the density of states per site of a given spin.

For $\eta(k_z) = \sqrt{2} \cos k_z b$ and $\sqrt{2} \sin k_z b$ we obtain a similar dimensionless coupling constant as

$$\lambda'_m = \frac{1}{2} \left[ \frac{\alpha' - 1}{\alpha' + 1} [\alpha' - \sqrt{\alpha'^2 - 1}]^m, \right.$$

for nearest neighbor layer pairings. The expression for $T_c$ is the same as eq. (17).

Here, we note that a contribution from the short-range part of the Coulomb repulsion must be subtracted from $\lambda_0$ obtained above. For example in the tight binding model, the on-site Coulomb energy is estimated by

$$U = \int \int d^3r \ d^3r' |w(r)|^2 \frac{e^2}{4\pi\epsilon_0 |r - r'|} |w(r')|^2,$$

where $w(r)$ is the Wannier function. It is obvious that the energy $U$ is not included in our interaction energy eq. (3), since eq. (19) depends on the profile of the Wannier function. Equation (3) describes the behaviors of pairing interactions of longer wave length, while the energy $U$ in eq. (19) is characterized by the local states of electrons on each lattice site.

Therefore we must consider the on-site Coulomb repulsion in addition to the pairing interaction of eq. (3). However, it reduces only the intra-layer $s$-wave pairing interaction but not the other anisotropic pairing interactions because of the symmetry. We define a parameter $U$ so that the $s$-wave interaction $\lambda_0$ is reduced by $\tilde{U}N(0)$. The value of the parameter $U$ is not equal to $U$, because the retardation and spin fluctuation effects should be taken into account. We consider $\tilde{U}$ as a given parameter without estimating it microscopically.

### III. Dependence on the Layer Spacing of the Anisotropic Pairing Interaction

In this section, we calculate anisotropic components of the effective pairing interactions as functions of the layer spacing $b$. We concentrate on the case of intra-layer pairing $\eta(k_z) = 1$ for a while.

The squared inverse of the screening length is

$$q_s^2 = \frac{e^2}{\epsilon_0 \rho(\mu)},$$

in Thomas-Fermi approximation, where $\rho(\mu)$ is total density of states of electrons per unit volume at chemical potential $\mu$. In layer systems, the total density of states per unit volume $\rho(\mu)$ is written in terms of the total density of states per unit area $\rho_{\parallel}^{2D}(\mu)$ in each layers as

$$\rho(\mu) = \rho_{\parallel}^{2D}(\mu)/b.$$

Here it is found that the screening becomes weaker when the layer spacing increases, because the volume density of electrons which contribute to screening decreases when the layer spacing increases. However, it should be noted that the screening length within a layer changes by the change of the inter-layer spacing $b$, even when the lattice constant $a$ in the layers is unchanged. Therefore the behavior of the screening length examined is not derived by a simple scale transformation in terms of $a$ and $b$ as the length scales.

We define a length scale $b_0$ as

$$\alpha = 1 + \frac{q_s^2}{2k_F^2} \equiv 1 + \frac{b_0}{b},$$

from eqs. (20) and (21). In a simple case, the length scale $b_0$ is estimated as follows. Assuming non-interacting two dimensional electron gas in $\rho_{\parallel}^{2D}(\mu)$, we obtain

$$b_0 = \frac{a^2}{\pi n_a H},$$

where $n_a$ is the density of states per unit area at $\mu$. Therefore the behavior of the screening length examined is not derived by a simple scale transformation in terms of $a$ and $b$ as the length scales.
since $\rho_0^{(2D)}(\mu) = m/\pi \hbar^2$ and $k_F a = \sqrt{2\pi n}$, where $n$ is the electron number per site. Here $a_H$ denotes Bohr radius $a_H = 4\pi\epsilon_0 \hbar^2/me^2 = 0.5292\text{Å}$. As an example, if $a \sim 4\text{Å}$ and $n \sim 1$ we have $b_0 \sim 9.6\text{Å}$ as a crude estimation.

Since the basic length scale $a_H$ which is independent of the lattice constants $a$ and $b$ comes in eq. (23), changes not only of the ratio $b/a$ but also of the absolute values of $a$ and $b$ give rise to changes in the qualitative results.

FIG. 1. The dimensionless coupling constants $\lambda_m$ as a function of the layer spacing $b$. The solid and dashed lines show the results for $p$-wave ($m = 1$) and $d$-wave ($m = 2$), respectively. In the inset, the short dashed line shows the result for $s$-wave ($m = 0$).

FIG. 2. The phase diagram at $T = 0$ in $b\cdot U$ plane. SC stands for superconductivity.

Figure 2 shows the result of $\lambda_m$ as a function of the layer spacing $b$. It is seen that both $p$-wave and $d$-wave components of the pairing interactions increase with the layer spacing $b$. In particular, it is found that the $p$-wave components increase rapidly in the region $0 < b \lesssim b_0$. As the inset shows, the $s$-wave component $\lambda_0/gN(0)$ is equal to 1 in the limit of $b = 0$ and decreases with $b$. It remains larger than the other anisotropic components, but if the additional short-range Coulomb energy $U$ is sufficiently large so that $\lambda_0 - \tilde{u} < \lambda_1$, $p$-wave pairing occurs instead of $s$-wave pairing.

Figure 3 is the phase diagram at $T = 0$ in $b\cdot U$ plane. SC denotes Bohr radius.

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IV. INTER-LAYER PAIRING

In this section we consider a situation in which the inter-layer coupling constant $g'$ is of the same order as the intra-layer coupling constant $g$. The coupling constants would depend on the layer spacing $b$, but here we regard them as independent parameters. The condition $g' \sim g$ would actually be satisfied when $b$ is not too large. Then, we must consider the gap function of the form $\Delta(k) = \Delta(\parallel)\eta(k_z)\eta(k_z)$ with $\eta(k_z) = \sqrt{2}\cos k_z b$ or $\sqrt{2}\sin k_z b$. The expansion of $\Delta(\parallel)$ by eq. (12) holds also in this case.

Figure 4 shows the dimensionless coupling constants $\lambda'_m$. A set of parameters, $g' = 0.8g$, $U = 0.4g$, and $q'_s = q_s$ are taken as an example. For $b/b_0 \lesssim 0.6$ and $b/b_0 \gtrsim 2.2$, intra-layer pairing (of $s$-wave and $p$-wave in each region, respectively) is favored. On the other hand, for $0.6 \lesssim b/b_0 \lesssim 2.2$, inter-layer pairing with $m = 0$ is favored. The gap function has a form such as

$$
\begin{align*}
\Delta(k) &= \Delta_0 \sin k_z b \\
\Delta(k) &= \Delta_0 \cos k_z b.
\end{align*}
$$

The former is an order parameter of triplet pairing, while the latter is that of singlet pairing. These gap functions have horizontal line nodes at $k_z = 0, \pm \pi/b$ and at $k_z = \pm \pi/2b$, respectively, but they are isotropic in the layers.
inter-layer pairing with best neighbor layer pairing as a function of the layer spacing. The thin dashed and short dashed lines show the results of the momentum. We will show that the electron dispersion depends on the direction \(\phi\) where \(\lambda\) is the angle between a momentum \(p\) and \(p_x\)-axis. In addition, we regard \(k_F\) as being constant, for simplicity.

Figure 3 shows a verification of this simplified model in the square lattice tight binding model with a nearest neighbor hopping energy \(t\) at \(\mu = -t\). Although the Fermi-surface is nearly isotropic, the density of states \(\rho(\epsilon, \phi)\) varies with the direction \(\phi\). For example, when \(\mu = -t\), \(\rho_0 \approx 0.142\) and \(\rho_1 \approx 0.040\) are estimated.

Regarding eq. (25) as an expansion of \(\rho(\phi, 0)\), we could extend it into more general forms by adding terms \(\rho_{4n} \cos(4n\phi)\) with \(n \geq 2\). Then the terms of \(\rho_{4n}\) mix \(\Delta_m\) of a small \(m\) with \(\Delta_{m'}\) of a large \(m' = |m \pm 4n|\). However, since \(V_m\) decreases rapidly with \(m\) as seen by eq. (16), \(\Delta_{m'}\) of such large \(m'\) are small. Therefore the higher order terms in the expansion of \(\rho(\phi, 0)\) can be omitted in practice.

In the gap equation, the anisotropic term proportional to \(\rho_4 \cos 4\phi\) of the angle-dependent density of states mixes \(\Delta_m \cos m\phi\) with \(\Delta_{m-4|} \cos((m-4)\phi)\) and \(\Delta_{m+4} \cos((m+4)\phi)\), but it does not affect equations for \(\Delta_m \sin m\phi\). Therefore we only consider equations for \(\Delta_m\). For general \(m\), we can write the gap equation at \(T = T_c\) as

\[
\Delta_m = -\lambda_m^{(0)} \log \frac{2e^\gamma \omega_D}{\pi T_c} \times \left\{ \Delta_m + \frac{\rho_4}{2\rho_0} \left\{ \frac{n_m}{n_{m+4}} \Delta_{m+4} + \frac{n_{m-4|}}{n_{m-4|}} \Delta_{m-4|} \right\} \right\},
\]

(26)

where we define \(\lambda_m^{(0)} \equiv V_m \rho_0 = V_m N(0)\) is the dimensionless coupling constant for the isotropic case.

In this section, we consider the square lattice systems, in which the electron dispersion depends on the direction of the momentum. We will show that the \(d\)-wave coupling constant \(\lambda_2\) is enhanced for \(d_{x^2-y^2}\) symmetry, but not for \(d_{xy}\) symmetry, due to the anisotropy of the density of single-particle states. We define an angle-dependent density of states \(\rho(\epsilon, \phi)\) as a density of single-particle states per unit energy and unit angle.

In the square lattice system, the angle-dependent density of states at the Fermi-energy \(\rho(0, \phi)\) can be approximated by

\[
\rho(0, \phi) \approx \rho_0 + \rho_4 \cos(4\phi),
\]

(25)

where \(\phi\) is the angle between a momentum \(p\) and \(p_x\)-axis. In addition, we regard \(k_F\) as being constant, for simplicity.

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Regarding eq. (25) as an expansion of \(\rho(\phi, 0)\), we could extend it into more general forms by adding terms \(\rho_{4n} \cos(4n\phi)\) with \(n \geq 2\). Then the terms of \(\rho_{4n}\) mix \(\Delta_m\) of a small \(m\) with \(\Delta_{m'}\) of a large \(m' = |m \pm 4n|\). However, since \(V_m\) decreases rapidly with \(m\) as seen by eq. (16), \(\Delta_{m'}\) of such large \(m'\) are small. Therefore the higher order terms in the expansion of \(\rho(\phi, 0)\) can be omitted in practice.

Since \(V_3, V_4, V_5 \cdots\) are much smaller than \(V_0\) and \(V_1\), the terms proportional to \(\rho_4\) can be neglected in eq. (26) for \(m = 0\) and 1. Hence, \(\lambda_0\) and \(\lambda_1\) are not modified by \(\rho_4\). On the other hand, for \(m = 2\), we cannot omit the term of \(\Delta_{m=4}\) in eq. (26) since \(|m-4| = 2\). Neglecting the term of \(\Delta_{m+4} = \Delta_6\) because \(V_6 \ll V_2\), we obtain

\[
\Delta_2 = \lambda_2^{(0)} \left[ 1 + \frac{\rho_4}{2\rho_0} \right] \log \frac{2e^\gamma \omega_D}{\pi T_c} \Delta_2
\]

\[
\approx \lambda_2 \log \frac{2e^\gamma \omega_D}{\pi T_c} \Delta_2,
\]

(27)

where we define an effective coupling constant \(\lambda_2 \equiv \lambda_2^{(0)}(1 + \rho_4/2\rho_0)\), which gives \(T_c\) by eq. (17).

Therefore, it is found that \(d_{x^2-y^2}\)-wave pairing is favored more than \(d_{xy}\)-wave pairing by the enhancement factor \((1 + \rho_4/2\rho_0)\), since \(\rho_4\) changes the equation for \(\Delta_m\)
but does not for $\Delta_m$ as mentioned above. The enhancement factor $1 + \rho_4/2\rho_0$ is estimated to be 1.14 for $\mu = -t$, and 1.22 for $\mu = -0.5t$. On the competition with $p$-wave pairing, those values are not large enough to change the sign of $\lambda_2 - \lambda_1$. Therefore, another non-phonon contribution seems to be needed for $d$-wave pairing to occur.

Figure 4 are phase diagram in the absence of $p$-wave superconductivity. It is found that $d$-wave superconductivity is favored when the layer spacing $b$ is larger and the short-range Coulomb repulsion $\tilde{U}$ is stronger. If there are additional contributions to $d$-wave superconductivity mentioned above, the phase boundary is shifted downward.

![Phase Diagram](image)

**Fig. 5.** The phase diagram of $s$-wave superconductivity and $d_{x^2-y^2}$-wave superconductivity at $T = 0$ in $b-\tilde{U}$ plane in the absence of $p$-wave superconductivity. SC stands for superconductivity.

VI. DISCUSSION AND SUMMARY

We have examined pairing interactions mediated by phonons in the layer systems. The screening of Coulomb interactions becomes weaker when the layer spacing $b$ increases. Then anisotropic components of the pairing interactions increase with the layer spacing $b$ since the momentum dependence of the interactions changes. In particular, $p$-wave superconductivity occurs for large $b$ and strong short-range Coulomb repulsion $\tilde{U}$, even in the absence of any additional non-phonon interactions.

It was found that the $p$-wave coupling constant $\lambda_1$ increases rapidly with the layer spacing $b$ in the region $b \lesssim b_0$, where $b_0$ is a length scale defined by eq. (22). For the rapid increase of $\lambda_1$, the condition $\lambda_0 - \tilde{U} < \lambda_1$ is realized more easily in layer systems than in usual three dimensional systems, where $\lambda_0$ denotes the $s$-wave coupling constant and $-\tilde{U}$ is a negative contribution to $s$-wave pairing due to the short-range Coulomb repulsion discussed near eq. (19). Hence triplet pairing superconductivity is favored in layered compounds.

We have also examined inter-layer pairing. In some region of the parameter space, for example $0.6 \lesssim b/b_0 \lesssim 2.2$ for the parameters indicated in Fig. 3, the gap function has horizontal line nodes parallel to the layers. In this case, the solutions of singlet pairing and triplet pairing of eq. (24) degenerate. If some effect due to spin fluctuations, ferromagnetic correlations, or spin-orbit coupling removes this degeneracy, inter-layer triplet pairing may occur.

In Sr$_2$RuO$_4$ compounds, existence of the line nodes was supported by some experiments such as temperature dependences of specific heat and NMR relaxation rate $\gamma$. However, direction of the line nodes does not seem clear at the present. Line nodes vertical to the layers were indicated by ultrasound attenuation [28], whereas almost isotropic state was indicated by thermal conductivity [29]. The isotropic state can be consistent with the specific heat and NMR experiments, if the horizontal line nodes are assumed [30].

In the present theory, inter-layer triplet pairing reproduces the horizontal line nodes, while the intra-layer triplet pairing is a candidate for the vertical line nodes. For the latter pairing, we need some additional mechanism for the vertical line nodes to occur, because isotropic states such as $p_x + ip_y$ have the lowest free energy in the present isotropic system. Consistent explanation of the experimental results within the present theory remains for a future study.

In order to discuss the reality of the phonon-mediated anisotropic superconductivity, we crudely estimate the parameters for the Sr$_2$RuO$_4$ compound and quasi-one-dimensional organic superconductors from the observed transition temperature $T_c \sim 1.5K$. We assume triplet pairing here, although for the organics it might be rather controversial. The results of the parameter values do not strongly depend on the direction of the line nodes.

Roughly speaking, $b \gtrsim b_0$ is satisfied in both kinds of compounds. Thus we have $\lambda_1 \lesssim 0.15 \times gN(0)$ for intra-layer triplet pairing from Fig. 4, while $\lambda_0 \lesssim 0.22 \times gN(0)$ for inter-layer pairing from Fig. 3. On the other hand, if we assume $\omega_D \sim 1000K$ and $T_c \sim 1.5K$, we have $\lambda_1 \sim 0.15 \times gN(0)$ (or $\lambda_0 \sim 0.15 \times gN(0)$). Therefore, we obtain $gN(0) \sim 1.0$ and 0.69, respectively. For such choices of parameter values, it is likely that $s$-wave pairing is suppressed. For example, for intra-layer pairing, since we obtained $gN(0) \sim 1$ above, the $s$-wave coupling constant is estimated as $\lambda_0 \approx 0.58 \times gN(0)$ at $b = b_0$, which corresponds to $V_0 = V_1 \approx 0.42/N(0) \sim 0.42W$, where $W$ is the band width. Therefore $s$-wave pairing is suppressed when the magnitude of the negative contribution $\tilde{U}$ due to the short-range Coulomb repulsion is larger than $0.42W \sim W/2$. Although this estimation is crude, the value $W/2$ seems realistic as the order of the magnitude.

On the other hand, for $d$-wave superconductivity to occur in the present model, it is needed that $p$-wave pairing is suppressed for some extra reason as well as $s$-wave
pairing, or that there are some additional contributions to d-wave pairing. For this problem, we examined effect of the anisotropy of the electron dispersion. It was found that the d-wave coupling constant $\lambda_2$ is enhanced by the anisotropy for $d_{x^2-y^2}$ symmetry, while not for $d_{xy}$ symmetry and $p_x$, $p_y$ symmetries. However, the enhancement does not seem to be large enough to realize the d-wave superconductivity.

This might suggest an existence of a non-phonon contribution to the d-wave pairing interaction in the cuprates. For example, many authors discussed that antiferromagnetic spin fluctuations may contribute to d-wave pairing interactions \([31,32,8,9,33,20,21]\). In particular, such interactions enhance d-wave pairing efficiently in the presence of the van Hove singularities \([9]\) in the square lattice systems.

However, even if we assume that a non-phonon contribution is indispensable for high-\(T_c\) the present theory suggests that there is a large phonon contribution to the d-wave pairing interactions especially in layer systems for the weak screening. This result is consistent with the observed large shifts of \(T_c\) as absolute values due to the isotope effect \([10,11]\).

It was also found that the coupling constant $\lambda_2$ increases with the layer spacing \(b\). This behavior might be a reason why the transition temperature of \(\text{Bi}_2\text{Sr}_2\text{CaCu}_2\text{O}_{8+\delta}\) is much higher than that of \(\text{La}_{2-x}\text{Sr}_x\text{CuO}_4\). Since \(T_c\) is a sensitive function of $\lambda_m$, such a slight enhancement of $\lambda_2$ may increase $T_c$ considerably.

In conclusion, phonon-mediated pairing interactions include anisotropic components of various symmetries. In particular, they are large when the screening effect is weak due to the layered structure of the system. As a result, the interactions induce various types of anisotropic superconductivity depending on values of the energy parameters and lattice constants. In particular, triplet superconductivity is favored for large layer spacing and strong short-range repulsion. For anisotropic superconductivity, $T_c$ increases with the layer spacing \(b\). The phonon-mediated pairing interactions may play some essential roles in the anisotropic superconductivity of layered compounds, such as \(\text{Sr}_2\text{RuO}_4\), the organic superconductors, and the high-$T_c$ cuprates.

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