Hitting time statistics for observations of dynamical systems

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Abstract
In this paper we study the distributions of hitting and return times for observations of dynamical systems. We apply the results to get an exponential law for the distributions of hitting and return times for rapidly mixing random dynamical systems. In particular, this allows us to obtain an exponential law for random expanding maps, random circle maps expanding on average and randomly perturbed dynamical systems.

Keywords: Poincaré recurrence, hitting times, exponential law, random dynamical systems, decay of correlations
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1. Introduction

When trying to study reality, experimentalists often have to approximate their system or use a simplified model to get a lower dimensional system that is more suitable for analysing. With the same philosophy, when working on a high dimensional system, experimentalists are generally just interested in gauging different quantities (temperature, pressure, wind speed, wave height, and so on) or in trying to use a measurement or observation of the system to get information on the whole system. Following these ideas, a Platonic formalism for dynamical systems is given in [23]. More precisely, one may wish to determine what information can be learned about an attractor of a high dimensional dynamical system from just the knowledge of its image under a function taking values in a lower dimensional space, called an observation. In investigating the statistical properties of dynamical systems, the quantitative description of Poincaré recurrence behaviour plays an essential part, and it has been widely studied (see...
e.g. the review of Saussol [29]). Our aim in this article is to examine Poincaré recurrence for observations of dynamical systems.

The study of quantitative Poincaré recurrence for observations began with the work of Boshernitzan [8], where it was shown that for a dynamical system \((X, T, \mu)\) and an observation \(f\) from \(X\) to a metric space \((Y, d)\), if the \(\alpha\)-dimensional Hausdorff measure is \(\sigma\)-finite on \(Y\), then

\[
\liminf_{n \to \infty} n^{1/\alpha} d \left( f(x), f(T^n x) \right) < \infty \quad \text{for } \mu\text{-almost every } x. \tag{1}
\]

Following this work, a definition of the return time for the observation was given in [26] and the asymptotic behaviour of the return time was analysed. More precisely, for a measurable function \(f : X \to Y\), a point \(x \in X\) and \(r > 0\), the return time for the observation is defined by

\[
\tau_f^r (x) := \inf \left\{ k \in \mathbb{N}^+ : f(T^k x) \in B(f(x), r) \right\},
\]

and it was proved, under some aperiodicity condition, that if the system is rapidly mixing and if the observation \(f\) is Lipschitz, then \(\tau_f^r (x) \sim r^{-d}\), where \(d\) is the pointwise dimension of the pushforward measure \(f_* \mu\). In [25], these results were extended to continuous time and applied to the geodesic flow.

In [26], a short example was given to show that one can obtain results in the quantitative study of Poincaré recurrence for random dynamical systems using the study of recurrence for observations of dynamical systems. This idea was developed in [22] where, to the best of our knowledge for the first time, annealed and quenched return times for random dynamical systems were defined and studied. It was proved that for super-polynomially mixing random dynamical systems presenting some kind of aperiodicity, the random recurrence rates are equal to the pointwise dimensions of the stationary measure (recently the same kinds of results were obtained in [4] for random hitting time).

The distributions of the return time and hitting time statistics are other aspects of Poincaré recurrence, and have been extensively considered for deterministic dynamical systems (one could see the reviews [2, 10, 16, 29]) and an exponential law has been proved for numerous systems with chaotic behaviour (see e.g. [1, 3, 9, 11, 12, 17, 18, 19, 24]). Recently, a relation between hitting time statistics and extreme value theory has been obtained by Freitas, Freitas and Todd [13, 14]. One could also mention the works of Keller and Liverani, in which spectral perturbation is used to obtain exponential hitting time distributions [21, 20].

In the past few months, a couple of papers have appeared on the laws of rare events for random dynamical systems. Indeed, in [5] they link the distribution of hitting times and extreme value laws for randomly perturbed dynamical systems, study the convergence of rare event point processes, and prove an exponential law for some randomly perturbed dynamical systems. An exponential distribution for hitting times is also proved in [27] for rapidly mixing random subshifts of finite type and random expanding maps. We emphasize that the exponential law is not proved with respect to the same measure in the two cases: the product measure of the skew-product is considered in [5] while [27] works with the sample measures.

Our aim in this paper is to continue the quantitative study of Poincaré recurrence for observations, and thus the quantitative study of recurrence of random dynamical systems, investigating the distributions of hitting and return times for the observations. An exponential law for the distributions of the hitting and return times for the observation is given in section 2 and proved in section 4, for rapidly mixing dynamical systems, presenting some kind of aperiodicity. This allows us to obtain in section 3—while the proofs are given in section 5—an exponential law for super-polynomially mixing random dynamical systems, and apply this result to random expanding maps, random circle maps expanding on average and randomly perturbed dynamical systems.
2. Exponential law for observations of dynamical systems

Let \((X, A, \mu, T)\) be a measure-preserving system (m.p.s.), i.e. \(A\) is a \(\sigma\)-algebra, \(\mu\) is a probability measure on \((X, A)\) and \(\mu\) is invariant under \(T\) (i.e. \(\mu(T^{-1}A) = \mu(A)\) for all \(A \in A\)) where \(T : X \to X\). We assume that \(X\) is a metric space and \(A\) is its Borel \(\sigma\)-algebra.

We introduce the hitting and return times for the observation and the associated recurrence rates.

**Definition 1.** Let \(f : X \to \mathbb{R}^N\) be a measurable function, called the observation, and \(A \subset \mathbb{R}^N\); we define for \(x \in X\) the hitting time for the observation of \(x\) in \(A\):

\[
\tau_f^A(x) := \inf \{ k \in \mathbb{N}^* : f(T^k x) \in A \}.
\]

When we study the hitting time in a ball, we define for \(x_0 \in X\) and \(x \in X\) the hitting time for the observation:

\[
\tau_f^r(x, x_0) := \inf \{ k \in \mathbb{N}^* : f(T^k x) \in B(f(x_0), r) \},
\]

where \(B(y, r)\) denotes the ball centred at \(y\) with radius \(r\). Also, we define for \(x \in X\) the return time for the observation:

\[
\tau_f^r(x) := \inf \{ k \in \mathbb{N}^* : f(T^k x) \in B(f(x), r) \}.
\]

We then define the lower and upper recurrence rates for the observation:

\[
R_f^l(x) := \liminf_{r \to 0} \frac{\log \tau_f^r(x)}{-\log r} \quad \text{and} \quad R_f^u(x) := \limsup_{r \to 0} \frac{\log \tau_f^r(x)}{-\log r}.
\]

To obtain optimal results for the return and hitting times for the observation, we need to assume that the system presents some kind of aperiodicity:

**Definition 2.** An m.p.s. \((X, A, \mu, T)\) is called \(\mu\)-almost aperiodic for the observation \(f\) if

\[
\mu(\{ x \in X : \exists n \in \mathbb{N}^* \text{ such that } f(T^n x) = f(x) \}) = 0.
\]

We emphasize that this condition can be removed by introducing non-instantaneous return times [26].

Since we wish to link the behaviour of the return time to the behaviour of the measure of shrinking sets, we recall the definition of the lower and upper pointwise or local dimensions of a Borel probability measure \(\nu\) on \(Y\) at a point \(y \in Y\):

\[
d_\nu(y) = \lim_{r \to 0} \frac{\log \nu(B(y, r))}{\log r} \quad \text{and} \quad \overline{d}_\nu(y) = \lim_{r \to 0} \frac{\log \nu(B(y, r))}{\log r}.
\]

We also recall that the conditional measure is

\[
\nu_A(B) = \frac{\nu(A \cap B)}{\nu(A)}
\]

and the pushforward measure is \(f_* \nu(\cdot) := \nu(f^{-1}(\cdot))\).

In order to study the behaviour of the return time, we need a rapid mixing condition:

**Definition 3.** \((X, T, \mu)\) has a super-polynomial decay of correlations if, for all \(\psi\) Lipschitz functions from \(X\) to \(\mathbb{R}\), for all \(\phi\) measurable bounded functions from \(X\) to \(\mathbb{R}\) and for all \(n \in \mathbb{N}^*\), we have

\[
\left| \int_X \psi \circ T^n d\mu - \int_X \psi \, d\mu \int_X \phi \, d\mu \right| \leq \|\psi\|_{\text{Lip}} \|\phi\|_{\infty} \theta_n^p,
\]

with \(\lim_{n \to \infty} \theta_n n^p = 0\) for all \(p > 0\).
In [26], the authors proved that the recurrence rates for the observation are linked to the local dimension of the pushforward measure:

**Theorem 4 ([26]).** Let \((X, A, \mu, T)\) be an m.p.s. and \(f : X \to \mathbb{R}^N\) a measurable observation such that the system is \(\mu\)-almost aperiodic for \(f\). Then, for \(\mu\)-almost every \(x \in X\),

\[
\mathcal{R}_f(x) \leq d_{f,\mu}(f(x)) \quad \text{and} \quad \overline{\mathcal{R}}_f(x) \leq \overline{d}_{f,\mu}(f(x)).
\]

Moreover, if the system has a super-polynomial decay of correlations and \(f\) is Lipschitz, then

\[
\mathcal{R}_f(x) = d_{f,\mu}(f(x)) \quad \text{and} \quad \overline{\mathcal{R}}_f(x) = \overline{d}_{f,\mu}(f(x))
\]

for \(\mu\)-almost every \(x \in X\) such that \(d_{f,\mu}(f(x)) > 0\).

One can observe that in [26], they assumed that the observables \(\phi\), in definition 3, are Lipschitz functions. A simple modification of their proof allows us to state their theorem with measurable bounded functions, a necessary assumption in the proof of our main theorem.

To obtain information on the fluctuation of the return time, we will need the system to fulfil the assumptions of the previous theorem, and we will also need a hypothesis on the measure:

**Hypothesis (I).** For \(f_{*}\mu\)-almost every \(y \in \mathbb{R}^n\), there exist \(a > 0\) and \(b \geq 0\) such that

\[
f_{*}\mu(B(y, r) \setminus B(y, r - \rho)) \leq r^{-b} \rho^a
\]

for any \(r > 0\) sufficiently small and any \(0 < \rho < r\).

In section 3 we will apply our results to random dynamical systems and give some examples where all the assumptions are fulfilled. Some examples of measures which fulfil the hypothesis (I) are given in lemma 44 of [29].

Our theorem on the fluctuations of the return time for the observations is the following:

**Theorem 5.** Let \((X, A, \mu, T)\) be an m.p.s. with a super-polynomial decay of correlations and \(f : X \to \mathbb{R}^N\) a Lipschitz observation such that the system is \(\mu\)-almost aperiodic for the observation \(f\). If hypothesis (I) is satisfied, then for every \(t \geq 0\) and \(\mu\)-almost every \(x_0 \in X\) such that \(d_{\mu}(x_0) > 0\),

\[
\lim_{r \to 0} \mu_x \left( x \in X, \tau_r(x, x_0) > \frac{t}{f_{*}\mu (B(f(x_0), r))} \right) = e^{-t}
\]

and

\[
\lim_{r \to 0} \mu_{f^{-1}B(f(x_0), r)} \left( x \in X, \tau_r(x, x_0) > \frac{t}{f_{*}\mu (B(f(x_0), r))} \right) = e^{-t}.
\]

One can observe that, for a rapidly mixing dynamical system, we can apply this theorem to the observation \(f = \text{id}\) and obtain some of the results cited in the introduction and, in particular, a generalization of theorem 40 of [29]:

**Corollary 6.** Let \(X \subset \mathbb{R}^N\) and let \((X, A, \mu, T)\) be an m.p.s. with a super-polynomial decay of correlations. Let us suppose that for \(\mu\)-almost every \(y \in X\), there exist \(a > 0\) and \(b \geq 0\) such that \(\mu(B(y, r) \setminus B(y, r - \rho)) \leq r^{-b} \rho^a\) for any \(r > 0\) sufficiently small and any \(0 < \rho < r\).

Then for every \(t \geq 0\) and for \(\mu\)-almost every \(x_0 \in X\) such that \(d_{\mu}(x_0) > 0\),

\[
\lim_{r \to 0} \mu_x \left( x \in X, \tau_r(x, x_0) > \frac{t}{\mu (B(x_0, r))} \right) = e^{-t}
\]

and

\[
\lim_{r \to 0} \mu_{B(x_0, r)} \left( x \in X, \tau_r(x, x_0) > \frac{t}{\mu (B(x_0, r))} \right) = e^{-t}
\]

where \(\tau_r(x, x_0) := \inf \{k \in \mathbb{N}^* : T^k x \in B(x_0, r)\}\).

One can remark that, in this corollary, we do not need the assumption of the \(\mu\)-almost aperiodicity, since the system is mixing.
3. Hitting time statistics for random dynamical systems

In [26], it was observed that the study of recurrence for observations of dynamical systems allows one to study recurrence for random dynamical systems. Following this remark, random return times for dynamical systems were defined in [22] and the random recurrence rates were linked to the local dimension of the stationary measure. Recently, a few works on this theme have been emerging; indeed, random hitting time indicators are studied in [4], while [5] linked the distribution of hitting time for randomly perturbed dynamical systems and extreme value laws, and an exponential distribution for hitting times is proved in [27] for random subshifts of finite type.

In this section, we will use our results for observations of dynamical systems to prove an exponential law for random dynamical systems presenting some rapidly mixing conditions, and apply this result to random expanding maps, random circle maps expanding on average and randomly perturbed dynamical systems.

3.1. Exponential law for random dynamical systems

Let $\Omega$ be a metric space and $B(\Omega)$ its Borelian $\sigma$-algebra. Let $\vartheta : \Omega \to \Omega$ be a measurable transformation on $\Omega$ preserving some probability measure $P$. Given a compact metric space $X$ and a family $T = (T_\omega)_{\omega \in \Omega}$ of transformations $T_\omega : X \to X$, we say that it defines a random dynamical system over $(\Omega, B(\Omega), P, \vartheta)$ via $T^n_\omega = T^{\vartheta^{n-1}(\omega)} \circ \ldots \circ T^{\vartheta(\omega)} \circ T_\omega$ for every $n \geq 1$ and $T^0_\omega = \text{Id}$.

The dynamics of the random dynamical systems generated by $T$ over $(\Omega, B(\Omega), P, \vartheta)$ is given by the skew-product:

$$S : \Omega \times X \to \Omega \times X \quad (\omega, x) \mapsto (\vartheta(\omega), T_\omega(x)).$$

A probability measure $\mu$ is invariant under the random dynamical system if it is $S$-invariant and $\pi_* \mu = P$, where $\pi : \Omega \times X \to \Omega$ is the canonical projection. Henceforth, we denote by $\nu$ the marginal of $\mu$ on $X$, i.e. $\nu = \int \mu_\omega \, dP$ where $(\mu_\omega)_{\omega}$ denotes the decomposition of $\mu$ on $X$, that is, $d\mu(\omega, x) = d\mu_\omega(x) \, dP(\omega)$.

When the skew-product invariant measure is a product measure $\mu = P \otimes \nu$, we will say that $\nu$ is a stationary measure for the random dynamical system. One can observe that it includes the case where the maps $T_\omega$ are chosen independently and with the same distribution.

Now we shall define return and hitting times for random dynamical systems. For a fixed $\omega \in \Omega$, the quenched random hitting time in a measurable subset $A \subset X$ of the random orbit starting from a point $x \in X$ is

$$\tau^\omega_A(x) = \inf \{ n > 0 : T^n_\omega x \in A \}.$$  

From now on we assume that $X \subset \mathbb{R}^N$ for some $N \in \mathbb{N}$. For $\omega \in \Omega$ and $x_0 \in X$, we are interested in the behaviour as $r \to 0$ of the quenched random hitting time of a point $x \in X$ in the open ball $B(x_0, r)$, defined by

$$\tau^\omega(x, x_0) := \inf \{ n > 0 : T^n_\omega x \in B(x_0, r) \}$$

and the quenched random return time of a point $x \in X$ in the open ball $B(x, r)$, defined by

$$\tau^\omega(x) := \inf \{ n > 0 : T^n_\omega x \in B(x, r) \}.$$  

As previously, we will deal with systems presenting some kind of aperiodicity:
Definition 7. The random dynamical system $T$ on $X$ over $(\Omega, B(\Omega), \mathbb{P}, \vartheta)$ with an invariant measure $\mu$ is called random aperiodic if
\[ \mu \left( (\omega, x) \in \Omega \times X : \exists n \in \mathbb{N}^*, T^n\omega x = x \right) = 0. \]

More details on such systems can be found in section 2.3 of [26] and the sections 4 and 5 of [22]. We also need an assumption on the measure and an assumption on the decay of correlations, which will be weaker than assuming super-polynomial decay of correlations for the skew-product:

(a) For $\nu$-almost every $x \in X$, there exist $a > 0$ and $b \geq 0$ such that
\[ v \left( B(\omega, x) \setminus B(x, r - \rho) \right) \leq r^{-b} \rho^a \]
for any $r > 0$ sufficiently small and any $0 < \rho < r$.

(b) For all $n \in \mathbb{N}^*$, $\psi$ Lipschitz observables from $X$ to $\mathbb{R}$ and $\varphi$ measurable bounded functions from $\Omega \times X$ to $\mathbb{R}$,
\[ \left| \int_{\Omega \times X} \psi(x)\varphi(S^n(\omega, x)) \, d\mu - \int_X \psi \, d\nu \int_{\Omega \times X} \varphi \, d\mu \right| \leq \|\psi\|_{Lip} \|\varphi\|_{\infty} \theta_n \]
with $\lim_{n \to \infty} \theta_n n^p = 0$ for any $p > 0$.

Theorem 8. Let $T$ be a random dynamical system on $X$ over $(\Omega, B(\Omega), \mathbb{P}, \vartheta)$ with an invariant measure $\mu$. If the random dynamical system is random aperiodic and satisfies hypothesis (a) and (b), then for every $t \geq 0$ and for $\nu$-almost every $x_0 \in X$ such that $d_\nu(x_0) > 0$,
\[ \lim_{r \to 0} \mu \left( (\omega, x) \in \Omega \times X, \tau^{(\omega, x_0)}_r > \frac{t}{v(B(x_0, r))} \right) = e^{-t} \]
and
\[ \lim_{r \to 0} \mu_{\Omega \times B(x_0, r)} \left( (\omega, x) \in \Omega \times X, \tau^{(\omega, x_0)}_r > \frac{t}{v(B(x_0, r))} \right) = e^{-t}. \]

The basic idea for proving this theorem, which was already used in [22, 26], is to apply theorem 5 to the dynamical system $(\Omega \times X, B(\Omega \times X), \mu, S)$ with the specific observation $f$ defined by
\[ f : \Omega \times X \to X \]
\[ (\omega, x) \mapsto x. \]

Indeed, with this observation, the hitting time for the observation and the hitting time for the random dynamical system are equal:
\[ \tau^{(\omega, x_0)}_r = \tau^{(\omega, x_0)}_r (\omega, x) = \tau^{(\omega, x_0)}_r (x). \]

The complete proof of the theorem will be given in section 5.

Remark 9. One can observe that this result is complementary to the ones proved in [27]. Indeed, in our result only the point $x_0$ is fixed and we have proved an exponential law with respect to the invariant measure $\mu$.

In [27], both the target $x_0$ and the realization $\omega$ are fixed, and the exponential law is proved with respect to the sample measures $\mu_{\omega,x_0}$ and also with respect to the marginal $v$.

We emphasize that in [5], an exponential law with respect to the invariant measure $\mu$ is also obtained for randomly perturbed dynamical systems.
For i.i.d. random dynamical systems, to obtain an exponential law, we just need to assume a super-polynomial decay of correlations for the random dynamical system, i.e. our observables are from $X$ to $\mathbb{R}$, which is a more natural assumption than hypothesis (b).

More precisely, let $\{T_\lambda\}_{\lambda \in \Lambda}$ be a family of transformations defined on a compact Riemannian manifold $X$ and let $P$ be a probability measure on a metric space $\Lambda$. We will consider $T$, a random dynamical system on $X$ over $(\Lambda^N, P^N, \sigma)$ with a stationary measure $\nu$, where $\sigma$ is the shift. That is, for an i.i.d. stochastic process $\lambda = (\lambda_n)_{n \geq 1} \in \Lambda^N$ with common distribution $P$, a random evolution of an initial state $x \in X$ will be $T^\lambda_n x = T_{\lambda_n} \circ \ldots \circ T_{\lambda_1} x$ for every $n \geq 0$.

**Definition 10.** The i.i.d. random dynamical system has a super-polynomial decay of correlations if, for all $n \in \mathbb{N}^*$, $\psi$ Lipschitz observables from $X$ to $\mathbb{R}$ and $\phi$ measurable bounded functions from $X$ to $\mathbb{R}$,

$$\left| \int_{X^N \times X} \psi(x) \phi(T_n^\lambda x) \, dP^\nu - \int_X \psi \, d\nu \right| \leq \|\psi\|_{Lip} \|\phi\|_{\infty} \theta_n,$$

with $\lim_{n \to \infty} \theta_n n^p = 0$ for any $p > 0$.

**Theorem 11.** Let $T$ be an i.i.d. random dynamical system on $X$ over $(\Lambda^N, P^N, \sigma)$ with a stationary measure $\nu$. If the random dynamical system is random aperiodic, satisfies hypothesis (a) and has a super-polynomial decay of correlations, then for every $t \geq 0$ and for $\nu$-almost every $x_0 \in X$ such that $d_\nu(x_0) > 0$,

$$\lim_{r \to 0} P^N \otimes \nu \left( (\lambda, x) \in \Lambda^N \times X, \tau^\lambda_n(x, x_0) > \frac{t}{\nu(B(x_0, r))} \right) = e^{-t}$$

and

$$\lim_{r \to 0} P^N \otimes \nu_{\Lambda^N \times B(x_0, r)} \left( (\lambda, x) \in \Lambda^N \times X, \tau^\lambda_n(x, x_0) > \frac{t}{\nu(B(x_0, r))} \right) = e^{-t}.$$

**Remark 12.** We emphasize that this result extends the result of [5] for randomly perturbed dynamical systems. The principal generalization lies in the decay of correlations. First of all, they need polynomial decay of correlations against $L^1$ observables where here we just need super-polynomial decay of correlations against $L^\infty$ observables. Besides this, for the observables $\psi$, we do not assume that indicator functions of balls are bounded in the Banach space.

Moreover, they study randomly perturbed dynamical systems, or more precisely, they perturbed an original map with random additive noise, where in our setting we can study more general random dynamical systems, as shown in the following examples.

In the following subsections, we will give examples of random dynamical systems where we can apply our results. The first example was given in [22] as an example of non-i.i.d. random dynamical systems for which the recurrence rates can be computed.

### 3.2. Non-i.i.d. random expanding maps

Let $T_1$ and $T_2$ be the following two maps defined on the one-dimensional torus $X = \mathbb{T}^1$:

$T_1 : X \mapsto X$ and $T_2 : X \mapsto X$

$x \mapsto 2x$ and $x \mapsto 3x$. 2383
The dynamics of the random dynamical system is given by the following skew-product:

\[ S : \Omega \times X \longrightarrow \Omega \times X \]

\[ (\omega, x) \mapsto (\vartheta(\omega), T_\omega x) \]

with \( \Omega = [0, 1] \), \( T_\omega = T_1 \) if \( \omega \in [0, 2/5) \), and \( T_\omega = T_2 \) if \( \omega \in [2/5, 1] \), and where \( \vartheta \) is the following piecewise linear map:

\[ \vartheta(\omega) = \begin{cases} 
2\omega & \text{if } \omega \in [0, 1/5) \\
3\omega - 1/5 & \text{if } \omega \in [1/5, 2/5) \\
2\omega - 4/5 & \text{if } \omega \in [2/5, 3/5) \\
3\omega/2 - 1/2 & \text{if } \omega \in [3/5, 1].
\end{cases} \]

One can observe that the random orbit is constructed by choosing the maps \( T_1 \) and \( T_2 \) following a Markov process with the stochastic matrix

\[ A = \begin{pmatrix} 1/2 & 1/2 \\
1/3 & 2/3 \end{pmatrix}. \]

It was proved in [22] that the associated skew-product is \( \text{Leb} \otimes \text{Leb} \)-invariant, is random aperiodic and has an exponential decay of correlations.

We can verify that the Lebesgue measure satisfies hypothesis (a) and thus theorem 8 applies, i.e. for every \( t \geq 0 \) and for \( \text{Leb} \)-almost every \( x_0 \in T^1 \),

\[ \lim_{r \to 0} \text{Leb} \otimes \text{Leb} \left( \tau_\omega^r(x, x_0) > \frac{t}{r} \right) = e^{-t} \]

and

\[ \lim_{r \to 0} \text{Leb} \otimes \text{Leb}_{[0,1] \times B(x_0, r)} \left( \tau_\omega^r(x, x_0) > \frac{t}{r} \right) = e^{-t}. \]

### 3.3. Random circle maps expanding on average

Let \( \Lambda \) be a metric space with a probability measure \( \mathcal{P} \). For every \( \lambda \in \Lambda \), let \( T_\lambda : T^1 \to T^1 \) be an application \( C^2 \) without a critical point. We will assume that

\[ \int_\Omega \frac{1}{\inf |T_\lambda|} \, d\mathcal{P}(\lambda) < 1 \]

and that

\[ \int_\Omega \left\| \frac{T_\lambda^n}{(T_\lambda)^2} \right\|_\infty \, d\mathcal{P}(\lambda) < +\infty. \]

It has been proved (see e.g. [30] and references therein) that for the i.i.d. random dynamical system on \( T^1 \) over \( (\Lambda^N, \mathcal{P}^N, \sigma) \), there exists an absolutely continuous stationary measure, and that the random dynamical system has an exponential decay of correlations for Hölder observables (nevertheless one can use for example [15] to go from Hölder to Lipschitz observables). Since absolutely continuous invariant measures satisfy hypothesis (a), we obtain by theorem 11 that if the system is random aperiodic, we have an exponential law for the return time and for the hitting time.

For example, one can apply these results for random \( \beta \)-transformations. More precisely, for every \( \lambda \in \Lambda \), let \( T_\lambda : T^1 \to T^1 \) be such that \( T_\lambda x = \beta_\lambda x \) where \( \beta_\lambda > 1 \). To prove that this system has an exponential law, we just need to prove that it is random aperiodic. For any \( \lambda \in \Lambda^N \) and any \( n \in \mathbb{N} \), \( T^n_\lambda \) is a \( \beta \)-transformation; thus

\[ \text{Card} \{ x \in T^1 : T^n_\lambda x = x \} < +\infty \]
and so for every \( \lambda \in \Lambda^N \),

\[
\text{Leb}(\{x \in T^d : \exists n \in \mathbb{N}^*, T^n_\lambda x = x\}) = 0.
\]

Then, since the stationary measure is absolutely continuous, we obtain that

\[
P^N \otimes \nu(\{(\lambda, x) \in \Lambda^N \times T^d : \exists n \in \mathbb{N}^*, T^n_\lambda x = x\}) = 0
\]

and therefore, one can apply theorem 11.

3.4. Randomly perturbed dynamical systems

In this section, we just give a short overview of some randomly perturbed dynamical systems (see e.g. [6, 7]) for which our results apply, obtaining a generalization of [5].

Let \( X \) be a compact Riemannian manifold and let \((X, T, \mu)\) be a deterministic dynamical system. We build our random dynamical systems by perturbing our transformation \( T \) with random additive noise. More precisely, for \( \varepsilon > 0 \), let \( \Lambda_\varepsilon = B(0, \varepsilon) \) and \( P_\varepsilon \) a probability measure on \( \Lambda_\varepsilon \). The family of transformations \( \{T_\lambda\}_{\lambda \in \Lambda_\varepsilon} \), where \( T_\lambda : X \to X \), are defined by

\[
T_\lambda(x) = T(x) + \lambda.
\]

We consider \( T \) the i.i.d. random dynamical system on \( X \) over \((\Lambda_\varepsilon^N, P_\varepsilon^N, \sigma)\).

For \( X = T^d \), it has been proved (see e.g. [5, 6, 31]) that for some expanding and piecewise expanding maps, if \( \varepsilon \) is small enough, the random dynamical system admits a stationary measure \( \nu_\varepsilon \) absolutely continuous with respect to the Lebesgue measure (thus satisfying hypothesis (a)) and that the system has a super-polynomial decay of correlations.

Moreover, since these systems are random aperiodic [22], one can apply theorem 11 and obtain an exponential law.

This gives, for example, an exponential law for perturbation of expanding and piecewise expanding maps of the circle with a finite number of discontinuities (see [31] or [5] for precise definitions) and also an exponential law for perturbations of expanding and piecewise expanding maps of \( T^d \) (see e.g. [5, 28]).

4. Proof of the exponential law for observations of dynamical systems

This section is dedicated to the proof of theorem 5.

The proof of theorem 5 follows the ideas of [19, 29]. For a measurable subset \( A \subset \mathbb{R}^N \) such that \( f_*\mu(A) > 0 \), we define

\[
\delta(A) = \sup_{k \in \mathbb{N}} \left| \mu\left( \tau^f_A > k \right) - \mu f^{-1}\mu\left( \tau^f_A > k \right) \right|.
\]

Lemma 13. Let \( A \subset \mathbb{R}^N \) be such that \( f_*\mu(A) > 0 \). For any \( n \in \mathbb{N} \), we have

\[
\left| \mu\left( \tau^f_A > n \right) - (1 - f_*\mu(A))^n \right| \leq \delta(A).
\]

Proof. Using the fact that \( \tau^f_A(x) = \tau_{f^{-1}A}(x) := \inf\{k > 0, T^k x \in f^{-1}A\} \), one can apply lemma 41 of [29] to the set \( f^{-1}A \).

To prove the exponential law for the hitting and return times, we will need to estimate the decay of \( \delta \):

\[
\text{Leb}(\{x \in T^d : \exists n \in \mathbb{N}^*, T^n_\lambda x = x\}) = 0.
\]
Lemma 14. Under the assumption of theorem 5, we have
\[ \lim_{r \to 0} \delta (B(f(x_0), r)) = 0 \]
for \( \mu \)-almost every \( x_0 \) such that \( d_{f, \mu}(f(x_0)) > 0 \).

**Proof.** Let \( x_0 \in X, r > 0, 0 < \rho < r \) and \( n \in \mathbb{N} \). Let \( A = B(f(x_0), r) \), \( B_n = \{ \tau'_A > n \} \) and \( g \geq n \). Let \( \phi : X \to \mathbb{R} \) be a function to be defined later. We have
\[
|\mu(f^{-1}A \cap B_n) - \mu(f^{-1}A)\mu(B_n)| \leq |\mu(f^{-1}A \cap T^{-\varepsilon}B_{n-\varepsilon})| = \mu(f^{-1}A \cap T^{-\varepsilon}B_{n-\varepsilon} \cap \tau_A \leq g)|
\]
\[
\leq \mu(f^{-1}A \cap \tau_A \leq g).
\]
Let us estimate each term of this inequality. First of all, we use the definition of \( B_n \) to estimate the first term:
\[
|\mu(f^{-1}A \cap B_n) - \mu(f^{-1}A)\mu(B_n)| = |\mu(f^{-1}A \cap T^{-\varepsilon}B_{n-\varepsilon} \cap \tau_A \leq g)|
\]
\[
\leq \mu(f^{-1}A \cap \tau_A \leq g).
\]
Let \( C = B(f(x_0), r - \rho) \) and define \( \phi(x) := \max(0, 1 - \frac{1}{\rho}d(f(x), C)) \). One can observe that \( 1_{f^{-1}C} \leq \phi \leq 1_{f^{-1}A} \) and that \( \phi \) is \( \frac{1}{\rho} \)-Lipschitz where \( K = \max(1, |f|_{Lip}) \); thus
\[
|\mu(f^{-1}A \cap T^{-\varepsilon}B_{n-\varepsilon}) - \int \phi \cdot 1_{B_{n-\varepsilon}} \circ T^\varepsilon d\mu| = \left| \int 1_{f^{-1}A} \cdot 1_{B_{n-\varepsilon}} \circ T^\varepsilon d\mu - \int \phi \cdot 1_{B_{n-\varepsilon}} \circ T^\varepsilon d\mu \right|
\]
\[
\leq \int |1_{f^{-1}A} - \phi| d\mu
\]
\[
\leq \int 1_{f^{-1}A} - 1_{f^{-1}C} d\mu = \mu(f^{-1}A \setminus f^{-1}C)
\]
\[
\leq f_\star \mu(A \setminus C).
\]
The information on the decay of correlation gives us the estimate of the third term:
\[
|\int \phi \cdot 1_{B_{n-\varepsilon}} \circ T^\varepsilon d\mu - \mu(B_{n-\varepsilon})\int \phi d\mu| \leq \|\phi\|_{Lip}\|1_{B_{n-\varepsilon}}\|_\infty \theta_\varepsilon
\]
\[
\leq K \frac{1}{\rho} \theta_\varepsilon. \tag{3}
\]
The estimate of the fourth term is given using the same idea as for the second term:
\[
|\mu(B_{n-\varepsilon})\int \phi d\mu - \mu(B_{n-\varepsilon})\mu(f^{-1}A)| \leq |\int \phi - 1_{f^{-1}A} d\mu|
\]
\[
\leq f_\star \mu(A \setminus C).
\]
Finally, the last term can be estimated using the invariance of the measure and the definition of $B_n$:
\[ |\mu(B_{n-2}^c) \mu(f^{-1} A) - \mu(f^{-1} A) \mu(B_n)| = \mu(f^{-1} A) \left( \mu(T^{-x} B_{n-2}) - \mu(B_n) \right) \leq \mu(f^{-1} A) \mu(\tau^f_A) \leq g \].

These estimates give us
\[ |\mu_f^{-1} (B_n) - \mu(B_n)| \leq \mu_f^{-1} \left( \tau^f_A \leq g \right) + 2 \frac{f_A \mu(A \cap C)}{f_A \mu(A)} + \frac{K}{\rho} \frac{\theta_g}{f_A \mu(A)} + \mu \left( \tau^f_A \leq g \right). \tag{4} \]

One can observe that this inequality is still satisfied if $n \leq g$. Indeed, when $n \leq g$, we have
\[ |\mu_f^{-1} (B_n) - \mu(B_n)| \leq |1 - \mu_f^{-1} (B_n)| + |1 - \mu(B_n)| \leq \mu_f^{-1} \left( \tau^f_A < n \right) \mu \left( \tau^f_A < n \right) \leq \mu_f^{-1} \left( \tau^f_A \leq g \right) \mu \left( \tau^f_A \leq g \right). \]

Then, (4) holds for every $n \in \mathbb{N}$ and gives us an upper bound for $\delta(B(f(x_0), r))$: \[
\delta(B(f(x_0), r)) \leq \mu_f^{-1} \left( \tau^f_A \leq g \right) + 2 \frac{f_A \mu(\text{B}(\text{f}(x_0), r))}{f_A \mu(\text{B}(\text{f}(x_0), r))} + \frac{K}{\rho} \frac{\theta_g}{f_A \mu(\text{B}(\text{f}(x_0), r))} + \mu \left( \tau^f_A \leq g \right). \tag{5} \]

To prove that $\delta(B(f(x_0), r)) \to 0$ for $\mu$-almost every $x_0 \in X$ such that $d_{\mu, \mu}(f(x_0)) > 0$, we need the two following lemmas:

**Lemma 15.** For every $x_0 \in X$ such that $d_{\mu, \mu}(f(x_0)) > 0$, for any $d \in (0, d_{\mu, \mu}(f(x_0)))$, we have
\[ \mu \left( \tau^f_A(x, x_0) \leq r^{-d} \right) \to 0 \quad \text{as } r \to 0. \]

**Proof.** One can observe that for any measurable subset $A \subset X$ and for any $n \in \mathbb{N}$, we have
\[ \mu(\tau_A \leq n) = \mu(T^{-1} A \cup T^{-2} A \cup \ldots \cup T^{-n} A) \leq \mu(T^{-1} A) + \mu(T^{-2} A) + \ldots + \mu(T^{-n} A) \leq n \mu(A). \]

This implies that for every $x_0 \in X$ such that $d_{\mu, \mu}(f(x_0)) > 0$ and for any $d \in (0, d_{\mu, \mu}(f(x_0)))$, we have
\[ \mu \left( \tau^f_A(x, x_0) \leq r^{-d} \right) = \mu \left( \tau^f_{A} \mu \left( f(x_0), r \right) \right) \leq r^{-d} \mu \left( f(x_0), r \right) \to 0 \quad \text{as } r \to 0. \]

Since $0 < d < d_{\mu, \mu}(f(x_0))$, we have $r^{-d} f_A \mu(B(f(x_0), r)) \to 0$ as $r \to 0$ and the lemma is proved. \hfill \Box

**Lemma 16.** Under the assumptions of theorem 5, for $\mu$-almost every $x_0 \in X$ such that $d_{\mu, \mu}(f(x_0)) > 0$, for any $d \in (0, d_{\mu, \mu}(f(x_0)))$, we have
\[ \mu_f^{-1} \left( \tau^f_A(x, x_0) \leq r^{-d} \right) \to 0 \quad \text{as } r \to 0. \tag{6} \]
Proof. For $a > 0$, let us define $Y_a = \{ y \in \mathbb{R}^N, d_{f,\mu}(y) > a \}$. One can observe that theorem 4 gives us that
\[
\liminf_{r \to 0} \frac{\log \tau_f^r(x)}{-\log r} \geq d_{f,\mu}(f(x)) > a
\]
for $\mu$-a.e. $x \in f^{-1}(Y_a)$. Let $r_0 > 0$ and define for $y \in Y_a$
\[
A(r_0, y) = \{ x \in X : f(x) = y \text{ and } \exists r < r_0, \tau_f^r(x) < r^{-a/2} \}.
\]
Let $\varepsilon > 0$ and set
\[
D_\varepsilon(r_0) = \{ y \in Y_a : \mu(A(r_0, y)) \leq \varepsilon \}.
\]
Let $x_0 \in X$ be such that $f(x_0)$ is a Lebesgue density point of the set $D_\varepsilon(r_0)$ for the measure $f_*\mu$, i.e.,
\[
\frac{f_*\mu(B(f(x_0), r) \cap D_\varepsilon(r_0))}{f_*\mu(B(f(x_0), r))} \to 1
\]
as $r \to 0$. Hence there exists $r_1 < r_0$ such that for any $r < r_1$,
\[
f_*\mu(B(f(x_0), r) \cap D_\varepsilon(r_0)) \leq \varepsilon f_*\mu(B(f(x_0), r)).
\]
Let $r < r_1$ and $d > a$. We get
\[
\mu_{f^{-1}B(f(x_0), r)}(\tau_f^r(x, x_0) \leq r^{-d})
\]
\[
= \frac{1}{f_*\mu(B(f(x_0), r))} \int_X 1_{f^{-1}B(f(x_0), r)}(x) 1_{[\tau_f^r(x, x_0) \leq r^{-d}]}(x) \, d\mu(x)
\]
\[
\leq \frac{1}{f_*\mu(B(f(x_0), r))} \int_X 1_{B(f(x_0), r)}(f(x)) 1_{[\tau_f^r(x, x_0) \leq r^{-d}]}(x) \, d\mu(x)
\]
\[
= \frac{1}{f_*\mu(B(f(x_0), r))} \int_X 1_{B(f(x_0), r)}(f(x)) E_{\mu}(1_{[\tau_f^r(x, x_0) \leq r^{-d}]} | f) \, d\mu(x)
\]
\[
= \frac{1}{f_*\mu(B(f(x_0), r))} \int_{\mathbb{R}^N} 1_{B(f(x_0), r)}(y) E_{\mu}(1_{[\tau_f^r(x, x_0) \leq r^{-d}]} | f = y) \, df_*\mu(y)
\]
\[
\leq \frac{1}{f_*\mu(B(f(x_0), r))} \int_{\mathbb{R}^N} 1_{B(f(x_0), r)}(y) \mu(A(r_0, y)) \, df_*\mu(y)
\]
\[
\leq \frac{1}{f_*\mu(B(f(x_0), r))} 2\varepsilon f_*\mu(B(f(x_0), r)) = 2\varepsilon.
\]
Since $\varepsilon$ is arbitrary and the measure of $D_\varepsilon(r_0)$ can be made arbitrarily close to the measure of $Y_a$, this shows that $\mu_{f^{-1}B(f(x_0), r)}(\tau_f^r(x, x_0) \leq r^{-d}) \to 0$ for $\mu$-a.e. $x_0 \in f^{-1}(Y_a)$ and for any $d > a$. The lemma is proved since $a$ can be chosen arbitrarily small. \qed

We now have all the ingredients to finish the proof of the lemma. Let $x_0 \in X$ be such that $d_{f,\mu}(f(x_0)) > 0$, such that $d_{f,\mu}(f(x_0)) \leq N$ and such that (2) and (6) are satisfied. Since the upper local dimension of a measure is almost everywhere smaller than the dimension of the ambient space, we obtain that $\mu(x \in X : d_{f,\mu}(f(x)) \leq N) = 1$.

Let $0 < d < d_{f,\mu}(f(x_0))$. Let us choose $g = \lfloor r^{-d} \rfloor$ and $\rho = \theta_{g/2}$.

The choices of $x_0$ and $g$ and lemma 16 give us that
\[
\mu_{f^{-1}B(f(x_0), r)}(\tau_f^r(\cdot, x_0) \leq g) \to 0 \quad \text{as } r \to 0
\]
and by lemma 15, we have
\[ \mu \left( \tau^f_{B(f(x_0), r)} \right) \leq g \rightarrow 0 \quad \text{as } r \to 0. \] (8)

Using (2), we have that for \( r \) sufficiently small,
\[ f_* \mu \left( B \left( f(x_0), r \right) \setminus B \left( f(x_0), r - \rho \right) \right) \leq r^{-b} \rho^a \]
and
\[ f_* \mu \left( B(f(x_0), r) \right) \leq r^{N+1}, \] (9)

since \( x_0 \) satisfies \( \text{d} f_* \mu(f(x_0)) > 0 \), which implies that
\[ f_* \mu \left( B(f(x_0), r) \setminus B(f(x_0), r - \rho) \right) \to 0 \quad \text{as } r \to 0. \] (10)

The choices of \( g \) and \( \rho \) together with (9) give us
\[ \theta_g \to 0 \quad \text{as } r \to 0. \] (11)

Finally, using hypothesis (i) and (5) together with (7), (8), (10) and (11), we obtain that
\[ \delta(B(f(x_0), r)) \to 0 \quad \text{as } r \to 0 \] for \( \mu \)-almost every \( x_0 \in X \) such that \( d f_* \mu(f(x_0)) > 0 \), which concludes the proof of the lemma. \[ \square \]

**Proof of theorem 5.** Let \( t > 0 \). Let us define \( n = \left\lfloor \frac{t}{f_* \mu(B(f(x_0), r))} \right\rfloor \) and \( A = B(f(x_0), r) \). We observe that
\[ \left| \mu \left( \tau^f_{B(f(x_0), r)} > \frac{t}{f_* \mu(B(f(x_0), r))} \right) - e^{-t} \right| = \left| \mu \left( \tau^f_A > n \right) - (1 - f_* \mu(A))^n + (1 - f_* \mu(A))^n - e^{-t} \right| \leq \delta(A) + |(1 - f_* \mu(A))^n - e^{-t}|. \]

Moreover, we have
\[ |(1 - f_* \mu(A))^n - e^{-t}| \leq \left| (1 - f_* \mu(A))^n - \left( 1 - \frac{t}{n} \right)^n \right| + \left| \left( 1 - \frac{t}{n} \right)^n - e^{-t} \right|. \] (12)

Using the mean value theorem and the definition of \( n \), we obtain that
\[ \left| (1 - f_* \mu(A))^n - \left( 1 - \frac{t}{n} \right)^n \right| \leq n \left| f_* \mu(A) - \frac{t}{n} \right| \leq \frac{t}{n}. \] (13)

Since it is well-known that \( |(1 - \frac{t}{n})^n - e^{-t}| \to 0 \) as \( n \to \infty \), (12) together with (13) implies that
\[ |(1 - f_* \mu(A))^n - e^{-t}| \to 0 \quad \text{as } n \to \infty. \] (14)

Finally the first part of the theorem is proved using (14) and the fact that by lemma 14, \( \delta(A) \to 0 \) as \( r \to 0 \) for \( \mu \)-almost every \( x_0 \in X \) such that \( d_{f_* \mu}(f(x_0)) > 0 \).

To prove the second part of the theorem, we just need to observe that
\[ \left| \mu_{f_* \mu}(\tau^f_A > n) - e^{-t} \right| \leq \left| \mu_{f_* \mu}(\tau^f_A > n) - \mu(\tau^f_A > n) \right| + \left| \mu(\tau^f_A > n) - e^{-t} \right| \leq \delta(A) + \left| \mu(\tau^f_A > n) - e^{-t} \right| \]
and lemma 14 and the first part of the theorem give us that the right-hand side of the inequality goes to zero as \( r \) goes to zero. \[ \square \]
5. Proof of the exponential law for random dynamical systems

In this section, we will prove theorem 8 and theorem 11.

**Proof of theorem 8.** This theorem is proved using theorem 5 applied to the dynamical system 
\((\Omega \times X, B(\Omega \times X), \mu, S)\) with the observation \(f\) defined by

\[
\begin{align*}
  f & : \Omega \times X \longrightarrow X \\
  (\omega, x) & \longmapsto x.
\end{align*}
\]

With this observation, for all \((\omega, x) \in \Omega \times X\) and for all \(r > 0\), we can link the hitting time for the observation and the hitting time for the random dynamical system:

\[
\tau_{B(f(x_0, \omega_0), r)}^f(\omega, x) = \tau_{B(x_0, r)}^\omega(x),
\]

we can identify the pushforward measure:

\[
f_\ast \mu = \nu,
\]

and for the pointwise dimensions we can observe that

\[
d_{f_\ast \mu}(f(x_0, \omega_0)) = d_\mu(x_0) \quad \text{and} \quad d_{f_\ast \mu}(f(x_0, \omega_0)) = d_{\nu}(x_0).
\]

Moreover, the random dynamical system is random aperiodic if and only if the system is \(\mu\)-almost aperiodic for \(f\).

Finally, in the proof of theorem 5, one can observe that hypothesis (b) is sufficient for proving (3) and thus the theorem is proved. \(\square\)

**Proof of theorem 11.** One can see that the difference between theorem 11 and theorem 8 lies in the decay of correlations and that the decay of correlations is only used to obtain equation (3). Thus, we will prove that equation (3) is still satisfied under the condition of theorem 11.

As observed in the proof of theorem 8, the observable \(\phi(\omega, x)\) does not depend on \(\omega\) and can be replaced by an observable \(\phi(x)\). Then, using the setting of theorem 11, we have

\[
\int X \int X 1_{\tau f_{\omega_0}^\omega, f(x_0, \omega_0) > n - \varepsilon} d\nu d\mu = \int X \int X 1_{\tau f_{\omega_0}^\omega, f(x_0, \omega_0) > n - \varepsilon} d\nu d\mu.
\]

Indeed, the fact that the \((\lambda_i)\) are chosen i.i.d. gives us

\[
\int X \int X 1_{\tau f_{\omega_0}^\omega, f(x_0, \omega_0) > n - \varepsilon} d\nu d\mu = \int X \int X 1_{\tau f_{\omega_0}^\omega, f(x_0, \omega_0) > n - \varepsilon} d\nu d\mu.
\]
where $B$ stands for $B(x_0, r)$. Thus, we obtain that

$$
\int \phi \cdot 1_{B_{n-g}} \circ T^g \, d\mu = \int_X \psi(x) \int_{\Lambda^n} \psi(T^g_{\lambda} x) \, d\mathcal{P}_N(\lambda) \, dv
$$

where

$$
\psi(x) = \int_{\Lambda^n} 1_{\left\{ \tau_{\tilde{\lambda}}^g(x, x_0) > n - g \right\}} \, d\mathcal{P}_N(\lambda).
$$

Since one can easily observe that

$$
\mu(B_{n-g}) = \int_X \psi(x) \, dv,
$$

we can use the hypothesis on the decay of correlations for the random dynamical systems to obtain the equivalent of equation (3) in this setting:

$$
\left| \int \phi \cdot 1_{B_{n-g}} \circ T^g \, d\mu - \mu(B_{n-g}) \int \phi \, d\mu \right| = \int_X \psi(x) \int_{\Lambda^n} \psi(T^g_{\lambda} x) \, d\mathcal{P}_N \, dv - \int_X \psi \, dv \int_X \psi \, dv \\
\leq \|\psi\|_{L^p} \|\psi\|_{\infty} \partial_g \\
\leq K \rho \theta_g,
$$

and the theorem is proved as a corollary of theorem 8.

\[\square\]

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**References**

[1] Abadi M 2006 Hitting, returning and the short correlation function Bull. Braz. Math. Soc. (N.S.) 37 593–609
[2] Abadi M and Galves A 2001 Inequalities for the occurrence times of rare events in mixing processes. The state of the art Markov Processes Related Fields vol 7 pp 97–112 Inhomogeneous Random Systems (Cergy-Pontoise, France, 2000)
[3] Abadi M and Saussol B 2011 Hitting, returning to rare events for all alpha-mixing processes Stochastic Process. Appl. 121 314–23
[4] Arbieto A, Junqueira A and Soares R 2013 Hitting times for random dynamical systems Dyn. Syst. 28 484–500
[5] Aytac H, Freitas J and Vaienti S 2012 Laws of rare events for deterministic, random dynamical systems (arXiv:1207.5188)
[6] Baladi V and Young L-S 1993 On the spectra of randomly perturbed expanding maps Commun. Math. Phys. 156 355–85
[7] Baladi V, Benedicks M and Maume-Deschamps V 2003 Almost sure rates of mixing for i.i.d. unimodal maps Ann. Sci. Éc. Norm. Supér. (4) 35 77–126
[8] Boshernitzan M D 1993 Quantitative recurrence results Invent. Math. 113 617–31
[9] Bruin H and Todd M 2009 Return time statistics of invariant measures for interval maps with positive Lyapunov exponent Stochastics Dyn. 9 81–100
[10] Coelho Z 2000 Asymptotic laws for symbolic dynamical systems Topics in symbolic dynamics and applications (Temuco, 1997) (London Mathematical Society Lecture Note Series vol 279) (Cambridge: Cambridge University Press) pp 123–65
[11] Collet P 1996 Some ergodic properties of maps of the interval Dynamical systems (Temuco, 1991/1992) vol 52 Travaux en Cours, Hermann, Paris pp 55–91
[12] Collet P, Galves A and Schmitt B 1992 Unpredictability of the occurrence time of a long laminar period in a model of temporal intermittency Ann. Inst. H. Poincaré Phys. Théor. 57 319–31
[13] Freitas A C M, Freitas J M and Todd M 2010 Hitting time statistics, extreme value theory *Probab. Theory Relat. Fields* 147 675–710

[14] Freitas A C M, Freitas J M and Todd M 2011 Extreme value laws in dynamical systems for non-smooth observations *J. Stat. Phys.* 142 108–26

[15] Galatolo S, Rousseau J and Saussol B 2014 Skew products, quantitative recurrence, shrinking targets, decay of correlations *Ergod. Theory Dyn. Syst.* at press doi:10.1017/etds.2014.10

[16] Haydn N 2013 Entry, return times distribution *Dyn. Syst.* 28 333–53

[17] Haydn N 2000 Statistical properties of equilibrium states for rational maps *Ergod. Theory Dyn. Syst.* 20 1371–90

[18] Hirata M 1993 Poisson law for axiom A diffeomorphisms *Ergod. Theory Dyn. Syst.* 13 533–56

[19] Hirata M, Saussol B and Vaienti S 1999 Statistics of return times: a general framework, new applications *Commun. Math. Phys.* 206 33–55

[20] Keller G 2012 Rare events, exponential hitting times, extremal indices via spectral perturbation *Dyn. Syst.* 27 11–27

[21] Keller G and Liverani C 2009 Rare events, escape rates, quasistationarity: some exact formulae *J. Stat. Phys.* 135 519–34

[22] Marie P and Rousseau J 2011 Recurrence for random dynamical systems *Discrete Contin. Dyn. Syst.* 30 1–16

[23] Ott W and Yorke J A 2003 Learning about reality from observation *SIAM J. Appl. Dyn. Syst.* 2 297–322 (electronic)

[24] Pitskel B 1991 Poisson limit law for Markov chains *Ergod. Theory Dyn. Syst.* 11 501–13

[25] Rousseau J 2012 Recurrence rates for observations of flows *Ergod. Theory Dyn. Syst.* 32 1727–51

[26] Rousseau J and Saussol B 2010 Poincaré recurrence for observations *Trans. Am. Math. Soc.* 362 5845–59

[27] Rousseau J, Saussol B and Varandas P 2014 Exponential law for random subshifts of finite type *Stochastic Process. Appl.* 124 3260–76

[28] Saussol B 2000 Absolutely continuous invariant measures for multidimensional expanding maps *Isr. J. Math.* 116 225–48

[29] Saussol B 2009 An introduction to quantitative Poincaré recurrence in dynamical systems *Rev. Math. Phys.* 21 949–79

[30] Stenlund M and Sulku H 2014 A coupling approach to random circle maps expanding on the average *Stoch. Dyn.* 14

[31] Viana M 1997 Stochastic dynamics of deterministic systems *Brazilian Math. Colloquium, IMPA*