Differential stability of convex discrete optimal control problems with possibly empty solution sets

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ABSTRACT

As a complement to two recent papers by Toan and Yao (Mordukhovich subgradients of the value function to a parametric discrete optimal control problem. J Global Optim. 2014;58:595–612), and by An and Toan (Differential stability of convex discrete optimal control problems. Acta Math Vietnam. 2018;43:201–217) on subdifferentials of the optimal value function of discrete optimal control problems, this paper studies the differential stability of convex discrete optimal control problems under control constraints, where the solution set may be empty. By using a suitable sum rule for \( \varepsilon \)-subdifferentials and a suitable product rule for \( \varepsilon \)-normal directions, we obtain formulas for computing the \( \varepsilon \)-subdifferential of the optimal value function. Several illustrative examples are also given.

1. Introduction

Differential stability of parametric optimization problems is an important topic in variational analysis and optimization. Studying differential stability of optimization problems usually means to study differentiability properties of the optimal value function in parametric mathematical programming.

According to Penot [1, Chapter 3], the class of convex functions is an important class that enjoys striking and useful properties. The consideration of directional derivative makes it possible to reduce this class to the subclass of sublinear functions. This subclass is next to the family of linear functions in terms of simplicity: The epigraph of a sublinear function is the convex cone, a notion almost as simple and useful as the notion of linear subspace.

The concept of the \( \varepsilon \)-subdifferential or approximate subdifferential was first introduced by Brøndsted and Rockafellar [2]. It has become an essential tool in convex analysis. For example, approximate minima and approximate subdifferentials are linked together by Legendre–Fenchel transforms (see, e.g. [3]). Like for the subdifferential, calculus rules on the \( \varepsilon \)-subdifferential are of importance.
and attract the attention of many researchers (see, e.g. [3–11] and the references therein).

In [12], Mordukhovich et al. gave formulas for computing and estimating the Fréchet subdifferential, the Mordukhovich subdifferential, and the singular subdifferential of the optimal value function in parametric mathematical programming problems under inclusion constraints. When the problem is convex, by using the Moreau–Rockafellar theorem and appropriate regularity conditions, An and Yao [13], An and Yen [14] have obtained formulas for computing subdifferentials of the optimal value function. In some sense, the results of [13,14] show that the preceding results of [12] admit a simpler form where several assumptions used in the general nonconvex case can be dropped. In both papers, the authors assumed that the original convex program has a nonempty solution set.

A natural question arises: Is there any analogous version of the formulas given in [13,14] for the case where the solution set can be empty? An and Yao [15] have provided formulas for the \( \varepsilon \)-subdifferential of the optimal value function of convex optimization problems under inclusion constraints in the case, where the solution set may be empty. However, we did not see formulas for the \( \varepsilon \)-subdifferential of the optimal value function of convex optimization problems under geometrical and inclusion constraints, where the solution set may be empty.

Besides the study on differential stability of parametric optimization problems, the study on differential stability of discrete optimal control problems is also an issue of importance (see, e.g. [9,11,16–20] and the references therein). Recently, Chieu and Yao [17], Toan and Yao [20] have derived formulas for computing the Fréchet subdifferential and the Mordukhovich subdifferential of the value function for the case where the solution map admits locally upper Lipschitzian selection. In [16], An and Toan have studied the first-order behaviour of the value function to a parametric convex discrete optimal control problem with convex cost functions and linear state equations by giving a shaper formula for computing the subdifferential of the value function. By virtue of the convexity, the assumption used in [17,20], like the existence of a local upper Lipschitzian selection of the solution map, is no longer needed. In all three papers, the authors assumed that the original optimal control problem has a nonempty solution set.

In this paper, we provide formulas for computing the \( \varepsilon \)-subdifferential of the optimal value function of convex discrete optimal control problems under linear state equations and control constraints, where the solution set may be empty. In order to prove the main result, we first reduce the problem to a convex optimization problem and using a suitable sum rule for \( \varepsilon \)-subdifferentials, a suitable product rule for \( \varepsilon \)-normal directions and appropriate regularity conditions, we obtain formulas for computing \( \varepsilon \)-subdifferentials of the optimal value function of convex optimization problems under geometrical and functional constraints, where the solution set may be empty. We then apply the
obtained result to our problem together with using some techniques of functional analysis.

2. Problem formulation and statement of the main result

A wide variety of the problems in discrete optimal control can be posed in the following form.

Determine a pair \((x, u)\) of a path \(x = (x_0, x_1, \ldots, x_N) \in \mathbb{R}^m \times \mathbb{R}^m \times \cdots \times \mathbb{R}^m\) and a control vector \(u = (u_0, u_1, \ldots, u_{N-1}) \in \mathbb{R}^n \times \mathbb{R}^n \times \cdots \times \mathbb{R}^n\) which minimize the cost

\[
f(x, u, w) = \sum_{k=0}^{N-1} h_k(x_k, u_k, w_k) + h_N(x_N) \quad (1)
\]

and which satisfy the state equation

\[
x_{k+1} = A_kx_k + B_ku_k + w_k, \quad k = 0, 1, \ldots, N - 1,
\]

the initial condition

\[
x_0 \in C,
\]

and the constraints

\[
u_k \in \Omega_k, \quad k = 0, 1, \ldots, N - 1.
\]

The notations in (1)–(4) have the following meanings:

- \(k\) indexes the discrete time, \(N\) is the horizon or number times control applied,
- \(x_k\) is the state of the system which summarizes past information that is relevant to future optimization,
- \(u_k\) is the control variable to be selected at time \(k\) with the knowledge of the state \(x_k\),
- \(w = (w_0, w_1, \ldots, w_{N-1}) \in \mathbb{R}^m \times \mathbb{R}^m \times \cdots \times \mathbb{R}^m\) is a random parameter (also called disturbance or noise),
- \(A_k \in M(m, m)\) and \(B_k \in M(m, n)\) are given matrices, \(M(m, n)\) denotes the set of \(m \times n\) matrices,
- \(\Omega_k\) is a nonempty convex set in \(\mathbb{R}^n\), \(C\) is a nonempty convex set in \(\mathbb{R}^m\),
- \(h_N : \mathbb{R}^m \rightarrow \mathbb{R}\) is a proper convex function, \(h_k : \mathbb{R}^m \times \mathbb{R}^n \times \mathbb{R}^m \rightarrow \mathbb{R}\) is a proper convex function.

Problems of this type were considered in many papers (see [9,11,16–21] and the references therein). A classical example for the problem (1)–(4) is the economic stabilization problem (see [22,23]).

Put \(X = \mathbb{R}^{(N+1)m}\), \(U = \mathbb{R}^{Nn}\), \(Z = X \times U\), and \(W = \mathbb{R}^{Nm}\). For each \(w \in W\), we denote by \(V(w)\) the optimal value of the problem (1)–(4) corresponding...
to parameters $w = (w_0, w_1, \ldots, w_{N-1}) \in W$, and $S(w)$ the solution set of the problem (1)–(4) corresponding to the parameter $w \in W$. Thus,

$$V : W \rightarrow \bar{\mathbb{R}}$$

is an extended real-valued function which is called the value function of the problem (1)–(4) defined by

$$V(w) := \inf \{ f(x, u, w) : \text{the conditions (2)–(4) are satisfied are satisfied} \}, \quad (5)$$

and $S : W \Rightarrow X \times U$ defined by $S(\bar{w}) := \{(x, u) \in X \times U : V(\bar{w}) = f(\bar{w}, x, u)\}$. For $\mu > 0$, one calls $S_\mu(\bar{w}) := \{(x, u) \in X \times U : V(\bar{w}) \leq f(\bar{w}, x, u) + \mu\}$ the approximate solution set of the problem (1)–(4).

Let $Z = X \times U$, $\Omega = \Omega_0 \times \Omega_1 \times \cdots \times \Omega_{N-1}$, $\bar{X} = \mathbb{R}^{Nm}$, and $K = C \times \bar{X} \times \Omega$. Then, the problem (1)–(4) can be written as the following form:

$$V(w) = \inf_{z \in G(w) \cap K} f(z, w),$$

where

$$G(w) = \{z = (x, u) \in Z : Mz = w\}, \quad (6)$$

$M : Z \rightarrow W$ is defined by

$$Mz = \begin{bmatrix}
-A_0 & I & 0 & 0 & \cdots & 0 & 0 & -B_0 & 0 & 0 & \cdots & 0 \\
0 & -A_1 & I & 0 & \cdots & 0 & 0 & 0 & -B_1 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & -A_{N-1} & I & 0 & 0 & 0 & \cdots & -B_{N-1} \\
\end{bmatrix} \times \begin{bmatrix}
x_0 \\
x_1 \\
\vdots \\
x_N \\
u_0 \\
u_1 \\
\vdots \\
u_{N-1}
\end{bmatrix}.$$
From the formula of $M$, we have

$$M^* y^* = \begin{bmatrix} -A_0^* & 0 & 0 & \ldots & 0 \\ I & -A_1^* & 0 & \ldots & 0 \\ 0 & I & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & -A_{N-1}^* \end{bmatrix} \begin{bmatrix} y_0^* \\ y_1^* \\ \vdots \\ y_{N-1}^* \end{bmatrix},$$

where $M^*$ is the adjoint operator of $M$.

For any $\varepsilon \geq 0$ and $\eta \geq 0$, define by $\Gamma(\mu + \varepsilon)$ the set

$$\Gamma(\mu + \varepsilon) = \{ (\gamma_1, \gamma_2) : \gamma_1 \geq 0, \gamma_2 \geq 0, \gamma_1 + \gamma_2 \leq \mu + \varepsilon \}.$$

We are now ready to state our main result.

**Theorem 2.1:** Let the optimal value function $V(\cdot)$ in (5) be finite at $\tilde{w} = (\tilde{w}_0, \tilde{w}_1, \ldots, \tilde{w}_{N-1}) \in W$. Suppose that $h_k$ are continuous at $(x_0^0, u_0^0, \tilde{w}_0^0)(k = 0, 1, \ldots, N - 1)$, $h_N$ is continuous at $x_N^0$ with $(w^0, x^0, u^0) = (w_0^0, \ldots, w_{N-1}^0, x_0^0, \ldots, x_N^0, u_0^0, \ldots, u_{N-1}^0) \in (W \times K) \cap gph G$ and $\text{int}(W \times K) \cap gph G \neq \emptyset$. Then for each $\varepsilon \geq 0$, $w^* = (w_0^*, w_1^*, \ldots, w_{N-1}^*) \in W$ to be an $\varepsilon$-subdifferential of $V$ at $\tilde{w}$, it is necessary that for any $\mu > 0$ there exist $\tilde{z} = (\tilde{x}, \tilde{u}) \in Z, (\gamma_1, \gamma_2) \in \Gamma(\mu + \varepsilon), x_0^* \in N_{\gamma_2}(\tilde{x}_0; C), u_k^* \in N_{\gamma_2}(\tilde{u}_k; \Omega_k) (k = 0, 1, \ldots, N - 1)$ and $\tilde{w}^* = (\tilde{w}_0^*, \tilde{w}_0^*, \ldots, \tilde{w}_{N-1}^*) \in W$ such that

$$w_k^* + \tilde{w}_k^* \in \left( \frac{\partial \gamma_1 h_k}{\partial w_k} \right)(\tilde{x}_k, \tilde{u}_k, \tilde{w}_k) \quad \text{for} \quad k = 0, 1, \ldots, N - 1,$$

$$-x_0^* + A_0^* \tilde{w}_0^* \in \left( \frac{\partial \gamma_1 h_0}{\partial x_0} \right)(\tilde{x}_0, \tilde{u}_0, \tilde{w}_0), \quad -\tilde{w}_{N-1}^* \in \left( \frac{\partial \gamma_1 h_N}{\partial x_N} \right)(\tilde{x}_N),$$

$$-\tilde{w}_{k-1}^* + A_k^* \tilde{w}_k^* \in \left( \frac{\partial \gamma_1 h_k}{\partial x_k} \right)(\tilde{x}_k, \tilde{u}_k, \tilde{w}_k) \quad \text{for} \quad k = 1, 2, \ldots, N - 1, \tag{7}$$

$$-x_k^* + B_k^* \tilde{w}_k^* \in \left( \frac{\partial \gamma_1 h_k}{\partial u_k} \right)(\tilde{x}_k, \tilde{u}_k, \tilde{w}_k) \quad \text{for} \quad k = 0, 1, \ldots, N - 1.$$

Moreover, if $S(\tilde{w}) \neq \emptyset$, then for each $\varepsilon \geq 0$, $w^* = (w_0^*, w_1^*, \ldots, w_{N-1}^*) \in W$ to be an $\varepsilon$-subdifferential of $V$ at $\tilde{w}$, it is necessary that there exist $(\gamma_1, \gamma_2) \in \Gamma(\varepsilon), x_0^* \in N_{\gamma_2}(\tilde{x}_0; C), u_k^* \in N_{\gamma_2}(\tilde{u}_k; \Omega_k) (k = 0, 1, \ldots, N - 1)$ and $\tilde{w}^* = (\tilde{w}_0^*, \tilde{w}_0^*, \ldots, \tilde{w}_{N-1}^*) \in W$ such that the system Equations (7) are satisfied for all $\tilde{z} = (\tilde{x}, \tilde{u}) \in S(\tilde{w})$. 
3. Preliminaries

In this section, we recall some notions and facts from variational analysis and generalized differentiation, which will be used in the sequel. These notations and facts can be found in [4,5,24–26].

Let $X$ and $Y$ be finite-dimensional Euclidean spaces. Let $f : X \to \overline{\mathbb{R}}$, where $\overline{\mathbb{R}} := [-\infty, +\infty] = \mathbb{R} \cup \{+\infty\} \cup \{-\infty\}$ be an extended real-valued function. One says that $f$ is proper if the domain

$$\text{dom} f := \{x \in X : f(x) < +\infty\}$$

is nonempty, and if $f(x) > -\infty$ for all $x \in X$. It is well known that if $\text{epi} f$ of $f$ is convex, then $f$ is said to be a convex function, where

$$\text{epi} f := \{(x, \alpha) \in X \times \mathbb{R} : \alpha \geq f(x)\}.$$ 

If $\text{epi} f$ is a closed subset of $X \times \mathbb{R}$, $f$ is said to be a closed function. Denoting the set of all the neighbourhoods of $x$ by $\mathcal{N}(x)$, one says that $f$ is lower semicontinuous (l.s.c.) at $x \in X$ if for every $\varepsilon > 0$ there exists $U \in \mathcal{N}(x)$ such that $f(x') \geq f(x) - \varepsilon$ for any $x' \in U$. If $f$ is l.s.c. at every $x \in X$, $f$ is said to be l.s.c. on $X$. It is easy to show that $f$ is l.s.c. on $X$ if and only if $f$ is closed and $\text{dom} f$ is closed too.

It is convenient to denote the set of all proper lower semicontinuous convex functions on $X$ by $\Gamma_0(X)$.

**Definition 3.1:** Let $f$ be a convex function defined on $X$, $\bar{x} \in \text{dom} f$, and $\varepsilon \geq 0$. The $\varepsilon$-subdifferential of $f$ at $\bar{x}$ is the set

$$\partial_{\varepsilon} f(\bar{x}) = \{x^* \in X : \langle x^*, x - \bar{x} \rangle \leq f(x) - f(\bar{x}) + \varepsilon, \forall x \in X\}.$$ 

The set $\partial_{\varepsilon} f(\bar{x})$ reduces to the subdifferential $\partial f(\bar{x})$ when $\varepsilon = 0$. From the definition it follows that $\partial_{\varepsilon} f(\bar{x})$ is a closed, convex set. In addition, for any nonnegative values $\varepsilon_1, \varepsilon_2$ with $\varepsilon_1 \leq \varepsilon_2$, one has $\partial_{\varepsilon_1} f(\bar{x}) \subseteq \partial_{\varepsilon_2} f(\bar{x})$. Moreover,

$$\partial f(\bar{x}) = \partial_{0} f(\bar{x}) = \bigcap_{\varepsilon > 0} \partial_{\varepsilon} f(\bar{x}).$$ 

If $f \in \Gamma_0(X)$, then $\partial_{\varepsilon} f(\bar{x})$ is nonempty for every $\bar{x} \in \text{dom} f$ and $\varepsilon > 0$ (see [4]). The following example shows that the traditional subdifferential $\partial f(\bar{x})$ may be empty, while $\partial_{\varepsilon} f(\bar{x}) \neq \emptyset$ for all $\varepsilon > 0$. 
Example 3.2: Let $X = \mathbb{R}$ and $\bar{x} = 0$. Clearly, the functions $f, g : X \to \mathbb{R}$ given by

$$f(x) = \begin{cases} -\sqrt{x} & \text{if } x \geq 0, \\ +\infty & \text{otherwise} \end{cases}$$

and $g(x) = x^2$ belong to $\Gamma_0(X)$ and $\bar{x} \in \text{dom } f \cap \text{dom } g$. For every $\varepsilon > 0$, one has

$$\partial_\varepsilon f(\bar{x}) = \{ x^* \in X^* : \langle x^*, x - \bar{x} \rangle \leq f(x) - f(\bar{x}) + \varepsilon, \forall x \in X \}$$

$$= \{ x^* \in \mathbb{R} : x^* x \leq -4\sqrt{x} + \varepsilon, \forall x \geq 0 \}$$

$$= \left(-\infty, -\frac{27}{256\varepsilon^3}\right]$$

and

$$\partial_\varepsilon g(\bar{x}) = \{ x^* \in X^* : \langle x^*, x - \bar{x} \rangle \leq g(x) - g(\bar{x}) + \varepsilon, \forall x \in X \}$$

$$= \{ x^* \in \mathbb{R} : x^* x \leq x^2 + \varepsilon, \forall x \in \mathbb{R} \}$$

$$= \left[-2\sqrt{\varepsilon}, 2\sqrt{\varepsilon}\right].$$

Meanwhile, it is easy to verify that $\partial f(\bar{x}) = \emptyset$, $\partial g(\bar{x}) = \{0\}$.

If $\bar{x}_1 = -1, \bar{x}_2 = 1$, then

$$\partial_\varepsilon g(\bar{x}_1) = \left[2(-1 - \sqrt{\varepsilon}), 2(-1 + \sqrt{\varepsilon})\right]$$

and

$$\partial_\varepsilon g(\bar{x}_2) = \left[2(1 - \sqrt{\varepsilon}), 2(1 + \sqrt{\varepsilon})\right].$$

The following example is taken from [7, pp. 93–94].

Example 3.3: Let $f(x) = |x|$ for all $x \in \mathbb{R}$ and $\varepsilon \geq 0$. We have

$$\partial_\varepsilon f(x) = \begin{cases} \left[-1, -1 - \frac{\varepsilon}{x}\right] & \text{if } x < -\frac{\varepsilon}{2}, \\ [-1, 1] & \text{if } -\frac{\varepsilon}{2} \leq x \leq \frac{\varepsilon}{2}, \\ \left[1 - \frac{\varepsilon}{x}, 1\right] & \text{if } x > \frac{\varepsilon}{2}. \end{cases}$$

In the sequel, we will also need the notion of a conjugate function. By definition, the function $f^* : X \to \overline{\mathbb{R}}$ given by

$$f^*(x^*) = \sup_{x \in X} \left[\langle x^*, x \rangle - f(x) \right], \quad x^* \in X,$$

is said to be the conjugate function (also called the Young–Fenchel transform, the Legendre–Fenchel conjugate) of $f : X \to \overline{\mathbb{R}}$. The conjugate function of $f^*$, denoted
by $f^{**}$, is a function defined on $X$ and has values in $\mathbb{R}$:

$$f^{**}(x) = \sup_{x^* \in X} [(x^*, x) - f^*(x^*)], \quad x \in X.$$ 

Clearly, the function $f^{**}$ is convex and closed (in the sense that $\text{epi} f^{**}$ is closed in $X \times \mathbb{R}$ or, in other words, $f^{**}$ is lower semicontinuous). According to the Fenchel–Moreau theorem (see [24, Theorem 1, p. 175]), if $f$ is a function on $X$ everywhere greater than $-\infty$, then $f = f^{**}$ if and only if $f$ is closed and convex.

According to [4], there are two basic ways to describe $\partial_{\epsilon} f(\bar{x})$:

(a) Via the conjugate function $f^*$ of $f$;
(b) Via the support function $\delta^*(v; \partial_{\epsilon} f(\bar{x})) := \sup \{\langle x^*, x \rangle : x^* \in \partial_{\epsilon} f(\bar{x})\}$ of $\partial_{\epsilon} f(\bar{x})$.

**Proposition 3.4** (See [4, Propositions 1.1 and 1.2]): The following holds:

(i) If $\bar{x} \in \text{dom} f$ and $\epsilon \geq 0$, then

$$x^* \in \partial_{\epsilon} f(\bar{x}) \iff f^*(x^*) + f(\bar{x}) \leq \langle x^*, \bar{x} \rangle + \epsilon.$$ 

(ii) If $f \in \Gamma(X), \bar{x} \in \text{dom} f$ and $\epsilon \geq 0$, then

$$\delta^*(v; \partial_{\epsilon} f(\bar{x})) = \inf_{t > 0} \frac{f(\bar{x} + tv) - f(\bar{x}) + \epsilon}{t}, \quad v \in X.$$ 

To deal with constrained optimization problems, we will need some results on $\epsilon$-normal directions from [5]. Let $C$ be a nonempty convex set in a finite-dimensional Euclidean space $X$.

**Definition 3.5**: The set $N_{\epsilon}(\bar{x}; C)$ of $\epsilon$-normal directions to $C$ at $\bar{x} \in C$ is defined by

$$N_{\epsilon}(\bar{x}; C) = \{x^* \in X : \langle x^*, x - \bar{x} \rangle \leq \epsilon, \forall x \in C\}.$$ 

As usual, the indicator function $\delta(\cdot; C)$ of $C$ is defined by setting $\delta(x; C) = 0$ if $x \in C$ and $\delta(x; C) = +\infty$ if $x \notin C$. It is easy to see that $N_{\epsilon}(\bar{x}; C) = \partial_{\epsilon} \delta(\bar{x}; C)$ for every $\epsilon \geq 0$. Moreover, when $\epsilon = 0$, $N_{\epsilon}(\bar{x}; C)$ reduces to the normal cone of $C$ at $\bar{x}$, which is denoted by $N(\bar{x}; C)$. However, as a general rule, $N_{\epsilon}(\bar{x}; C)$ is not a cone when $\epsilon > 0$.

The polar set of $A \subset X$ is defined by

$$A^0 = \{x^* \in X : \langle x^*, x \rangle \leq 1, \forall x \in A\}.$$ 

**Proposition 3.6** (See [5, p. 222]): The following properties of $\epsilon$-normal directions are valid:
The first assertion of Proposition 3.6 shows that the set of the $\varepsilon$-normal directions $N_\varepsilon(x; C)$ can be computed via the polar set of a set containing 0. Provided that the set $N_\varepsilon(x; C)$ has been found, by using the second assertion of Proposition 3.6, one can compute the normal cone $N(x; C)$. Due to the importance of the polar sets of sets containing the origin, it is reasonable to consider an illustrative example. Let $X = \mathbb{R}^2$ and $\overline{B}_2$ be the unit closed ball in $\mathbb{R}^2$.

**Example 3.7:** Consider the set $A = \overline{B}((2,0); 1) = \{(x_1, x_2) \in \mathbb{R}^2 : (x_1 - 2)^2 + x_2^2 \leq 1\}$, we have $A^0 = \{x^* = (x_1^*, x_2^*) \in \mathbb{R}^2 : 2x_1^* + ||x^*||^2 \leq 1\}$, where $||x^*|| = \sqrt{x_1^2 + x_2^2}$. Indeed, since $A = (2, 0) + \overline{B}_2$, we have

$$A^0 = \{x^* = (x_1^*, x_2^*) \in \mathbb{R}^2 : (x_1^*, x_2^*), (2, 0) + v \leq 1, \forall v \in \overline{B}_2\}$$

$$= \{x^* \in \mathbb{R}^2 : 2x_1^* + ||x^*|| \leq 1\}.$$

Now, consider a proper convex function $f : X \to \overline{\mathbb{R}}$ and suppose that $\bar{x} \in \text{dom } f$. The relationship between $\partial_\varepsilon f(\bar{x})$ and $N_\varepsilon((\bar{x}, f(\bar{x})); \text{epi } f)$ is described [5, p. 224] as follows:

$$\partial_\varepsilon f(\bar{x}) = \{x^* \in X^* : (x^*, -1) \in N_\varepsilon((\bar{x}, f(\bar{x})); \text{epi } f)\} \quad (\varepsilon \geq 0). \quad (8)$$

Taking $\varepsilon = 0$, from (8) we recover the following fundamental formula in convex analysis, which relates subdifferentials of a given convex function to the normal cones of its epigraph:

$$\partial f(x) = \{x^* \in X^* : (x^*, -1) \in N((x, f(x)); \text{epi } f)\} \quad (\forall x \in \text{ dom } f).$$

In convex analysis and optimization, summing two functions is a key operation. The Moreau–Rockafellar theorem can be viewed as a well-known result, which describes the subdifferential of the sum of two subdifferentiable functions. Invoking a result on the infimal convolution of two functions, one gets a sum rule for $\varepsilon$-subdifferentials. In the sequel, we will need next fundamental sum rule for $\varepsilon$-subdifferentials.

**Theorem 3.8 (See [4, Theorem 2.1]):** Suppose that $f_1, f_2 : X \to \overline{\mathbb{R}}$ are two proper convex functions on a finite-dimensional Euclidean space $X$ and the qualification condition

$$(f_1 + f_2)^*(x^*) = \min\{f_1^*(x_1^*) + f_2^*(x_2^*) : x_1^*, x_2^* \in X, x_1^* + x_2^* = x^*\} \quad (\forall x^* \in X) \quad (9)$$

holds. Then, for every $\bar{x} \in \text{dom } f_1 \cap \text{dom } f_2$ and $\varepsilon > 0$, one has

$$\partial_\varepsilon (f_1 + f_2)(\bar{x}) = \bigcup_{\varepsilon_1 \geq 0, \varepsilon_2 \geq 0, \varepsilon_1 + \varepsilon_2 = \varepsilon} \{\partial_{\varepsilon_1} f_1(\bar{x}) + \partial_{\varepsilon_2} f_2(\bar{x})\}. \quad (10)$$
Condition (9) means that, for every $x^* \in X$, one has

$$(f_1 + f_2)^*(x^*) = \inf \{ f_1^*(x_1^*) + f_2^*(x_2^*) : x_1^*, x_2^* \in X^*, x_1^* + x_2^* = x^* \},$$

and the infimum is attained, i.e. there exist $\tilde{x}_1^*$, $\tilde{x}_2^*$ from $X$ with $\tilde{x}_1^* + \tilde{x}_2^* = x^*$ such that

$$f_1^*(\tilde{x}_1^*) + f_2^*(\tilde{x}_2^*) = \inf \{ f_1^*(x_1^*) + f_2^*(x_2^*) : x_1^* + x_2^* = x^* \}.$$  \(12\)

A deeper understanding of condition (9) is achieved via the notion of infimal convolution [24, p. 168] of convex functions.

The infimal convolution $f_1 \oplus f_2$ of proper convex functions $f_1 : X \to \overline{\mathbb{R}}$ and $f_2 : X \to \overline{\mathbb{R}}$ is defined by

$$(f_1 \oplus f_2)(x) := \inf \{ f_1(x_1) + f_2(x_2) : x_1 + x_2 = x \} \quad (x \in X).$$

Applying this construction to the functions $f_1^* : X \to \overline{\mathbb{R}}$ and $f_2^* : X \to \overline{\mathbb{R}}$, we have

$$(f_1^* \oplus f_2^*)(x^*) = \inf \{ f_1^*(x_1^*) + f_2^*(x_2^*) : x_1^* + x_2^* = x^* \}.$$  \(13\)

The attainment of the infimum on the right-hand side of (13) at a point $x^*$ is a kind of qualification on the functions $f_1$, $f_2$ in a dual space setting. The writing $(f_1^* \oplus f_2^*)(x^*) = \min\{f_1^*(x_1^*) + f_2^*(x_2^*) : x_1^* + x_2^* = x^* \}$ means that there exist $\tilde{x}_1^*$, $\tilde{x}_2^*$ from $X$ with $x^* = \tilde{x}_1^* + \tilde{x}_2^*$ and $(f_1^* \oplus f_2^*)(x^*) = f_1^*(\tilde{x}_1^*) + f_2^*(\tilde{x}_2^*)$.

According to [24, p. 168], the infimal convolution of proper convex functions is a convex function. However, the latter can fail to be proper. For example, if $f_1$ and $f_2$ are linear functions not equal to one another, then their infimal convolution is identically $-\infty$.

By the definition of conjugate function, we have

$$(f_1 + f_2)^*(x^*) = \sup_{x \in X} \{ \langle x^*, x \rangle - (f_1 + f_2)(x) \}.$$  \(11\)

So, substituting $x^* = x_1^* + x_2^*$ with $x_1^* \in X$ and $x_2^* \in X$ yields

$$(f_1 + f_2)^*(x^*) = \sup_{x \in X} \{ \langle x_1^* + x_2^*, x \rangle - f_1(x) - f_2(x) \}
= \sup_{x \in X} \{ \langle x_1^*, x \rangle - f_1(x) + \langle x_2^*, x \rangle - f_2(x) \}
\leq \sup_{x \in X} \{ \langle x_1^*, x \rangle - f_1(x) \} + \sup_{x \in X} \{ \langle x_2^*, x \rangle - f_2(x) \}.$$

Thus, the inequality

$$(f_1 + f_2)^*(x^*) \leq f_1^*(x_1^*) + f_2^*(x_2^*)$$  \(14\)

holds for all $x^*, x_1^*, x_2^* \in X$ satisfying $x^* = x_1^* + x_2^*$. For any $x^* \in X$, taking infimum of both sides of (14) on the set of all $(x_1^*, x_2^*)$ with $x_1^* + x_2^* = x^*$, we
get

\[(f_1 + f_2)^*(x^*) \leq (f_1^* + f_2^*)(x^*); \tag{15}\]

see [24, p. 181]. Since (11) can be rewritten as

\[(f_1 + f_2)^*(x^*) = (f_1^* + f_2^*)(x^*), \tag{16}\]

condition (9) requires that, for the functions \(f_1\) and \(f_2\) in question, the inequality in (15) holds as equality for all \(x^* \in X\). Luckily, this requirement is satisfied under some verifiable regularity conditions. The following theorem describes a condition of this type.

**Theorem 3.9 (See [24, Theorem 1, p. 178]):** Suppose that \(f_1, f_2\) are proper convex functions. If one of the functions \(f_1, f_2\) is continuous at a point belonging to the effective domain of the other, then the equality \((f_1 + f_2)^*(x^*) = (f_1^* + f_2^*)(x^*)\) holds for every \(x^* \in X\). Moreover, for every \(x^* \in \text{dom}(f_1 + f_2)^*\), there exist points \(\bar{x}_i^* \in \text{dom} f_i^*, \ i = 1, 2\), such that \(\bar{x}_1^* + \bar{x}_2^* = x^*\) and

\[f_1^*(\bar{x}_1^*) + f_2^*(\bar{x}_2^*) = (f_1 + f_2)^*(x^*).\]

**Remark 3.10:** Under the assumptions of Theorem 3.9, condition (9) is satisfied. Indeed, suppose that one of the proper convex functions \(f_1, f_2\) is continuous at a point \(x^0\) belonging to the effective domain of the other. Then, one has \(x^0 \in \text{dom}(f_1 + f_2)\). It follows that \((f_1 + f_2)^*(x^*)\) is everywhere greater than \(-\infty\) for all \(x^* \in X\). If \(x^* \notin \text{dom}(f_1 + f_2)^*\), then \((f_1 + f_2)^*(x^*) = +\infty\). Choose \(\bar{x}_1^*, \bar{x}_2^* \in X\) such that \(x^* = \bar{x}_1^* + \bar{x}_2^*\). By (14), \(+\infty = (f_1 + f_2)^*(x^*) \leq f_1^*(\bar{x}_1^*) + f_2^*(\bar{x}_2^*)\). Noting that \(f_1^*(\bar{x}_1^*) > -\infty\) and \(f_2^*(\bar{x}_2^*) > -\infty\) because \(f_1, f_2\) are proper functions, from this we infer that at least one of the values \(f_1^*(\bar{x}_1^*)\) and \(f_2^*(\bar{x}_2^*)\) must be \(+\infty\). Combining this with (14) yields (12). Since (11) is equivalent to (16), and the latter is fulfilled. Thanks to Theorem 3.9, we have thus proved that the equality in (9) is satisfied for every \(x^* \notin \text{dom}(f_1 + f_2)^*\). If \(x^* \in \text{dom}(f_1 + f_2)^*\), then the equality in (9) follows immediately from Theorem 3.9.

**4. The optimal control problem as a programming problem**

In this section, we suppose that \(X, W\) and \(Z\) are finite-dimensional Euclidean spaces. Assume that \(M : Z \to W\) is a continuous linear mapping and \(M^* : W \to Z\) is an adjoint mapping of \(M\). Let function \(f : W \times Z \to \mathbb{R}\) be a proper convex and \(\Omega\) be a convex subset of \(Z\), with \(\text{int} \Omega \neq \emptyset\). For each \(w \in W\), we put

\[H(w) := \{z \in Z : Mz = w\}.\]

Consider the parametric optimization problem under inclusion and geometrical constraints

\[\min \{f(w, z) : z \in H(w) \cap \Omega\} \tag{17}\]
depending on the parameter $w$. The function $f$ (resp., the multifunction $H$, the set $\Omega$) is called the objective function (resp., the constraint multifunction, the constraint set) of (17). The optimal value function $h : W \to \mathbb{R}$ of (17) is
\[
h(w) := \inf_{z \in H(w) \cap \Omega} f(w, z). \tag{18}
\]
The solution set of (17) is defined by $\hat{\mathcal{S}}(\bar{w}) := \{z \in Z : h(\bar{w}) = f(\bar{w}, z)\}$. For $\mu > 0$, one calls
\[
\hat{\mathcal{S}}_\mu(\bar{w}) := \{z \in Z : f(\bar{w}, z) \leq h(\bar{w}) + \mu\} \tag{19}
\]
the approximate solution set of (17).

For any $\varepsilon \geq 0$ and $\eta \geq 0$, define by $\hat{\mathcal{F}}(\mu + \varepsilon)$ the set
\[
\hat{\mathcal{F}}(\mu + \varepsilon) = \{(\gamma_1, \gamma_2) : \gamma_1 \geq 0, \gamma_2 \geq 0, \gamma_1 + \gamma_2 \leq \mu + \varepsilon\}.
\]
We now obtain formulas for computing the $\varepsilon$-subdifferential of $h(\cdot)$.

**Theorem 4.1:** Suppose that the optimal value function $h(\cdot)$ in (18) is finite at $\bar{w} \in W$. If at least one of the following regularity conditions is satisfied:

(a) $\text{int}(W \times \Omega) \cap \text{int}(\text{gph } H) \cap \text{dom } f \neq \emptyset$,
(b) $f$ is continuous at a point $(x^0, y^0) \in (W \times \Omega) \cap \text{gph } H$ and $\text{int}(W \times \Omega) \cap \text{gph } H \neq \emptyset$, then, for every $\varepsilon \geq 0$, we have
\[
\partial_\varepsilon h(\bar{x}) = \bigcap_{\mu > 0} \bigcap_{z \in \hat{\mathcal{S}}_\mu(\bar{w})} \bigcup_{w^*_1 \in W} \{w^* \in W : (w^* + w^*_1, -M^* w^*_1) \in \partial_{\gamma_1} f(\bar{w}, z) + N_{\gamma_2}(\bar{w}, z; W \times \Omega)\} \tag{20}
\]
where $\hat{\mathcal{S}}_\mu(\bar{w})$ is given in (19).

In particular,
\[
\partial h(\bar{x}) = \bigcap_{\mu > 0} \bigcap_{z \in \hat{\mathcal{S}}_\mu(\bar{w})} \bigcup_{w^*_1 \in W} \{w^* \in W : (w^* + w^*_1, -M^* w^*_1) \in \partial_{\gamma_1} f(\bar{w}, z) + N_{\gamma_2}(\bar{w}, z; W \times \Omega)\}
\]
\[
eq \bigcap_{\mu > 0} \bigcap_{z \in \hat{\mathcal{S}}_\mu(\bar{w})} \bigcup_{w^*_1 \in W} \{w^* \in W : (w^* + w^*_1, -M^* w^*_1) \in \partial_{\gamma_1} f(\bar{w}, z) + N_{\gamma_2}(\bar{w}, z; W \times \Omega)\}.
\]
Moreover, if \( \hat{S}(\tilde{w}) \neq \emptyset \), then for every \( \varepsilon \geq 0 \), one has

\[
\partial_{\varepsilon} h(\bar{x}) = \bigcup_{w^*_1 \in W} \bigcup_{(\gamma_1, \gamma_2) \in \Gamma(\varepsilon)} \{ w^* \in W : (w^* + w^*_1, -M^*w^*_1) \in \partial_{\gamma_1} f(\tilde{w}, z) + N_{\gamma_2}((\tilde{w}, z); W \times \Omega) \},
\]

for all \( z \in \hat{S}(\tilde{w}) \).

For the proof of this theorem, we need the following lemmas.

**Lemma 4.2:** Suppose that assumptions of Theorem 4.1 are satisfied. Then for each \((\tilde{w}, \tilde{z}) \in gph H \) one has

\[
N_\varepsilon((\tilde{w}, \tilde{z}); gph H) = N((\tilde{w}, \tilde{z}); gph H) = \{(-w^*, M^*w^*) : w^* \in W\}, \quad \forall \varepsilon \geq 0.
\]

**Proof:** For any \( \varepsilon \geq 0 \). By the definition of \( \varepsilon \)-normal directions, we have

\[
N_\varepsilon((\tilde{w}, \tilde{z}); gph H)
\]

\[
= \{(w^*, z^*) \in W \times Z : ((w^*, z^*), (w, z) - (\tilde{w}, \tilde{z})) \leq \varepsilon, \forall (w, z) \in gph H \}
\]

\[
= \{(w^*, z^*) \in W \times Z : ((w^*, z^*), (Mz, z) - (M(\tilde{z}), \tilde{z})) \leq \varepsilon, \forall z \in Z \}
\]

\[
= \{(w^*, z^*) \in W \times Z : (w^*, Mz - M(\tilde{z})) + (z^*, z - \tilde{z}) \leq \varepsilon, \forall z \in Z \}
\]

\[
= \{(w^*, z^*) \in W \times Z : (M^*w^*, z - \tilde{z}) + (z^*, z - \tilde{z}) \leq \varepsilon, \forall z \in Z \}
\]

\[
= \{(w^*, z^*) \in W \times Z : (M^*w^* + z^*, z - \tilde{z}) \leq \varepsilon, \forall z \in Z \},
\]

this is equivalent to \( z^* = -M^*(w^*) \) and the proof of the lemma is complete. \( \blacksquare \)

**Lemma 4.3:** Let \( Z \) be a finite-dimensional Euclidean space and \( A, B \) be convex subsets of \( Z \). Then, for any \( \varepsilon \geq 0 \) and \( \bar{z} = (\tilde{z}_1, \tilde{z}_2) \in A \times B \), one has

\[
N_\varepsilon(\bar{z}; A \times B) \subset N_\varepsilon(\tilde{z}_1; A) \times N_\varepsilon(\tilde{z}_2; B) \subset N_{2\varepsilon}(\tilde{z}; A \times B). \quad (21)
\]

Moreover, if \( \tilde{z}_1 \in \text{int} A \), then \( N_\varepsilon(\tilde{z}_1; A) = \{0\} \).

**Proof:** Suppose that \( z^* = (z^*_1, z^*_2) \in N_\varepsilon(\tilde{z}; A \times B) \) for some \( \varepsilon \geq 0 \). Then, we have

\[
((z^*_1, z^*_2), (z_1, z_2) - (\tilde{z}_1, \tilde{z}_2)) \leq \varepsilon, \forall (z_1, z_2) \in A \times B. \quad (22)
\]

By our assumption, \( (22) \) is equivalent to

\[
(z^*_1, z_1 - \tilde{z}_1) + (z^*_2, z_2 - \tilde{z}_2) \leq \varepsilon, \quad \forall (z_1, z_2) \in A \times B. \quad (23)
\]

On the one hand, substituting \( z_2 = \tilde{z}_2 \) into \( (23) \), we get \( z^*_1 \in N_\varepsilon(\tilde{z}_1; A) \). On the other hand, taking \( z_1 = \tilde{z}_1 \) into \( (23) \), we have \( z^*_2 \in N_\varepsilon(\tilde{z}_2; B) \). Therefore, for any
\( \varepsilon \geq 0, \)
\[
N_{\varepsilon} (\tilde{z}; A \times B) \subset N_{\varepsilon} (\tilde{z}_1; A) \times N_{\varepsilon} (\tilde{z}_2; B).
\]
The second inclusion in (21) can be obtained easily by the definition of \( \varepsilon \)-normal directions. It is easy to show that if \( \tilde{z}_1 \in \text{int } A \), then \( N_{\varepsilon}(\tilde{z}_1; A) = \{0\} \). Thus proof of lemma is complete.

**Proof of Theorem 4.1:** We put
\[
\mathcal{M}_\mu (\tilde{w}) = \bigcap_{z \in \hat{S}_\mu (\tilde{w})} \bigcup_{w^*_1 \in W} \bigcup_{(\gamma_1, \gamma_2) \in \hat{\Gamma}(\mu + \varepsilon)} \{w^* \in W : (w^* + w^*_1, -M^*w^*_1) \in \partial_{\gamma_1} f(\tilde{w}, z) + N_{\gamma_2} ((\tilde{w}, z); W \times \Omega) \}
\]
and
\[
\mathcal{N}_\mu (\tilde{w}) = \bigcup_{z \in \tilde{z}} \bigcup_{w^*_1 \in W} \bigcup_{(\gamma_1, \gamma_2) \in \hat{\Gamma}(\mu + \varepsilon)} \{w^* \in W : (w^* + w^*_1, -M^*w^*_1) \in \partial_{\gamma_1} f(\tilde{w}, z) + N_{\gamma_2} ((\tilde{w}, z); W \times \Omega) \}.
\]
Since \( h(\tilde{w}) = \inf_{z \in Z} \{ f(\tilde{w}, z) + \delta((\tilde{w}, z); \text{gph } H \cap (W \times \Omega)) \} \) by (18), the set \( \hat{S}_\mu (\tilde{w}) \) is nonempty for every \( \mu > 0 \). Thus, one has \( \mathcal{M}_\mu (\tilde{w}) \subset \mathcal{N}_\mu (\tilde{w}) \) for all \( \mu > 0 \). Hence \( \bigcap_{\mu > 0} \mathcal{M}_\mu (\tilde{w}) \subset \bigcap_{\mu > 0} \mathcal{N}_\mu (\tilde{w}) \). So, the equalities in (20) will be proved, if we can show that
\[
\partial_{\mu} h(\tilde{w}) \subset \bigcap_{\mu > 0} \mathcal{M}_\mu (\tilde{w}) \tag{24}
\]
and
\[
\bigcap_{\mu > 0} \mathcal{N}_\mu (\tilde{w}) \subset \partial_{\mu} h(\tilde{w}) \tag{25}
\]
We now prove (24). Take any \( w^* \in \partial_{\mu} h(\tilde{w}), \mu > 0 \) and \( z \in \hat{S}_\mu (\tilde{w}) \). By Proposition 3.4,
\[
h(\tilde{w}) + h^*(w^*) \leq \langle w^*, \tilde{w} \rangle + \varepsilon. \tag{26}
\]
Adding \( \mu > 0 \) to both sides of (26) yields
\[
h(\tilde{w}) + h^*(w^*) + \mu \leq \langle w^*, \tilde{w} \rangle + \varepsilon + \mu. \tag{27}
\]
Since \( z \in \hat{S}_\mu (\tilde{w}) \), one has \( f(\tilde{w}, z) + \delta((\tilde{w}, z); \text{gph } H \cap (W \times \Omega)) \leq h(\tilde{w}) + \mu \). So, (27) gives
\[
f(\tilde{w}, z) + \delta((\tilde{w}, z); \text{gph } H \cap (W \times \Omega)) + h^*(w^*) \leq \langle w^*, \tilde{w} \rangle + \varepsilon + \mu. \tag{28}
\]
We will prove that if \( v^* \in Z \), then

\[
h^*(v^*) = (f + \delta(\cdot; \text{gph } H \cap (W \times \Omega)))^*(v^*, 0).
\]

Indeed, by the definition of conjugate function,

\[
h^*(v^*) = \sup_{w \in W} \{ \langle v^*, w \rangle - h(w) \}
= \sup_{w \in W} \{ \langle v^*, w \rangle - \inf_{(w, z) \in \text{gph } H \cap (W \times \Omega)} f(w, z) \}
= \sup_{w \in W} \{ \langle v^*, w \rangle - \inf_{z \in Z} f(w, z) + \delta((w, z); \text{gph } H \cap (W \times \Omega)) \}
= \sup_{(w, z) \in W \times Z} \{ \langle v^*, w \rangle - (f(w, z) + \delta((w, z); \text{gph } H \cap (W \times \Omega))) \}
= (f + \delta(\cdot; \text{gph } H \cap (W \times \Omega)))^*(v^*, 0).
\]

Substituting \( h^*(w^*) = (f + \delta(\cdot; \text{gph } H \cap (W \times \Omega)))^*(w^*, 0) \) into (28), we have

\[
f(\tilde{w}, z) + \delta((\tilde{w}, z); \text{gph } H \cap (W \times \Omega)) + (f + \delta(\cdot; \text{gph } H \cap (W \times \Omega)))^*(w^*, 0) \\
\leq \langle w^*, \tilde{w} \rangle + \varepsilon + \mu.
\]

According to Proposition 3.4, we have

\[
(w^*, 0) \in \partial_{\varepsilon + \mu} (f + \delta(\cdot; \text{gph } H \cap (W \times \Omega)))(\tilde{w}, z),
\]

for all \( \mu > 0 \) and \( z \in \hat{S}_\mu(\tilde{w}) \). We will prove that

\[
\partial_{\varepsilon + \mu} (f + \delta(\cdot; \text{gph } H \cap (W \times \Omega)))(\tilde{w}, z)
= \bigcup_{w^*_1 \in W} \bigcup_{(\gamma_1, \gamma_2) \in \hat{\Gamma}(\mu + \varepsilon)} \{( -w^*_1, M^*w^*_1^* ) + \partial_{\gamma_1} f(\tilde{w}, z) + N_{\gamma_2}((\tilde{w}, z); W \times \Omega) \},
\]

where

\[
\hat{\Gamma}(\mu + \varepsilon) = \{ (\gamma_1, \gamma_2) : \gamma_1 \geq 0, \gamma_2 \geq 0, \gamma_1 + \gamma_2 \leq \mu + \varepsilon \}.
\]

Indeed, suppose that at least one of the regularity conditions (a) or (b) is fulfilled. From \( H \) is linear mapping, \( \text{gph } H \cap (W \times \Omega) \) is convex. So, \( \delta(\cdot; \text{gph } H \cap (W \times \Omega) : W \times Z \to \mathbb{R} \) is convex. Obviously, \( \delta(\cdot; \text{gph } H \cap (W \times \Omega) \) is continuous at every point belonging to \( \text{int}(\text{gph } H \cap (W \times \Omega)) \). Hence, if the regularity condition (a) is satisfied, then \( \delta(\cdot; \text{gph } H \cap (W \times \Omega) \) is continuous at a point in \( \text{dom} f \). Consider the case where the regularity condition (b) is fulfilled. Since
dom \( \delta(\cdot; \text{gph } H \cap (W \times \Omega)) = \text{gph } H \cap (W \times \Omega) \). From (b), it follows that \( f \) is continuous at a point in dom \( \delta(\cdot; \text{gph } H \cap (W \times \Omega)) \). So, in both cases, we have

\[
(f + \delta(\cdot; \text{gph } H \cap (W \times \Omega)))^*(w^*, z^*) = \min \{f^*(w_1^*, z_1^*) + \delta^*((w_2^*, z_2^*); \text{gph } H \cap (W \times \Omega)) : (w^*, z^*) = (w_1^*, z_1^*) + (w_2^*, z_2^*) \},
\]

for all \( (w^*, z^*) \in W \times Z \) by Theorem 3.9 and Remark 3.10. So, all assumptions of Theorem 3.8 are satisfied. Therefore,

\[
\partial_{\varepsilon + \mu} (f + \delta(\cdot; \text{gph } H \cap (W \times \Omega)))(\bar{w}, z)
= \bigcup_{\gamma_1, \gamma_2 \geq 0, \gamma_1 + \gamma_2 = \varepsilon + \mu} \{\partial_{\gamma_1} f(\bar{w}, z) + N_{\gamma_2} ((\bar{w}, z); \text{gph } H \cap (W \times \Omega))\}, \tag{31}
\]

for any \( (\bar{w}, z) \in \text{dom } f \cap \text{gph } H \cap (W \times \Omega) \). Moreover, by the regularity condition (a) or the regularity condition (b), \( \delta(\cdot; \text{gph } H) \) is continuous at a point in dom \( \delta(\cdot; W \times \Omega) = W \times \Omega \) and \( \delta(\cdot; W \times \Omega) \) is continuous at a point in dom \( \delta(\cdot; \text{gph } H) = \text{gph } H \). Also by Theorem 3.9 and Remark 3.10, we have

\[
(\delta(\cdot; \text{gph } H) + \delta(\cdot; W \times \Omega))^*(w^*, z^*) = \min \{\delta^*((w_1^*, z_1^*); \text{gph } H) + \delta^*((w_2^*, z_2^*); W \times \Omega) : (w^*, z^*) = (w_1^*, z_1^*) + (w_2^*, z_2^*) \},
\]

for all \( (w^*, z^*) \in W \times Z \). So, all assumptions of Theorem 3.8 are also satisfied. Hence,

\[
\partial_{\mu} \delta((\bar{w}, z); \text{gph } H \cap (W \times \Omega)) = \partial_{\mu} \delta((\bar{w}, z); \text{gph } H) + \delta((\bar{w}, z); W \times \Omega) = \bigcup_{\gamma_2, \gamma_3 \geq 0, \gamma_2 + \gamma_3 = \mu} \{\partial_{\gamma_2} \delta((\bar{w}, z); \text{gph } H) + \partial_{\gamma_3} \delta((\bar{w}, z); W \times \Omega)\}. \tag{32}
\]

Moreover, \( \partial_{\gamma_2} \delta((\bar{w}, z); W \times \Omega) = N_{\gamma_2} ((\bar{w}, z); W \times \Omega) \) and

\[
\partial_{\gamma_3} \delta((\bar{w}, z); \text{gph } H) = \{(-w_1^*, M^* w_1^*) : w_1^* \in W\}, \tag{33}
\]

for all \( \gamma_3 \geq 0 \) by Lemma 4.2. Substituting (32) and (33) into (31), we obtain (30). By (29) and (30), there exists \( w_1^* \in W \) such that

\[
(w^* + w_1^*, -M^* w_1^*) \in \bigcup_{(\gamma_2, \gamma_2) \in \Gamma(\mu + \varepsilon)} \{\partial_{\gamma_2} f(\bar{w}, z) + \partial_{\gamma_2} \delta((\bar{w}, z); W \times \Omega)\},
\]

for all \( \mu \geq 0 \) and \( z \in \hat{S}_\mu(\bar{w}) \). This means that \( w^* \in \bigcap_{\mu > 0} M_\mu(\bar{w}) \), so (24) is valid.
Next, to prove (25), take any \( w^* \in \bigcap_{\mu > 0} N_{\mu}(\bar{w}). \) Then, for every \( \mu > 0 \), there exist \( z \in Z, w_1^* \in W, \) and \( (\gamma_1, \gamma_2) \in \tilde{F}(\mu + \epsilon) \) such that

\[
\langle w^* + w_1^*, -M^* w_1^* \rangle \in \partial_{\gamma_1} f(\bar{w}, z) + \partial_{\gamma_2} \delta((\bar{w}, z); W \times \Omega).
\]

So, \( (w^*, 0) \in \{(-w_1^*, M^* w_1^*)\} + \partial_{\gamma_1} f(\bar{w}, z) + \partial_{\gamma_2} \delta((\bar{w}, z); W \times \Omega). \) By the above argument, we have \( (w^*, 0) \in \partial_{\epsilon + \mu}(f + \delta(\cdot; \text{gph } H \cap (W \times \Omega))(\bar{w}, z)). \) By Proposition 3.4, we get

\[
(f + \delta(\cdot; \text{gph } H \cap (W \times \Omega)))^*(w^*, 0) + f(\bar{w}, z) + \delta((\bar{w}, z); \text{gph } H \cap (W \times \Omega))
- \langle (w^*, 0), (\bar{w}, z) \rangle \leq \epsilon + \mu.
\]

So,

\[
(f + \delta(\cdot; \text{gph } H \cap (W \times \Omega)))^*(w^*, 0) + f(\bar{w}, z) + \delta((\bar{w}, z); \text{gph } H \cap (W \times \Omega))
- \langle w^*, \bar{w} \rangle \leq \epsilon + \mu. \tag{34}
\]

Since \( (f + \delta(\cdot; \text{gph } H \cap (W \times \Omega)))^*(w^*, 0) = h^*(w^*) \) and \( h(\bar{w}) \leq f(\bar{w}, z) + \delta((\bar{w}, z); \text{gph } H \cap (W \times \Omega)), \) (34) implies that

\[
h^*(w^*) + h(\bar{w}) - \langle w^*, \bar{w} \rangle \leq \epsilon + \mu. \tag{35}
\]

As (35) holds for every \( \mu > 0 \), letting \( \mu \to 0 \) yields

\[
h^*(w^*) + h(\bar{w}) - \langle w^*, \bar{w} \rangle \leq \epsilon.
\]

This is equivalent to \( w^* \in \partial_{\epsilon} h(\bar{w}). \) Therefore, (25) is fulfilled. The proof of theorem is complete. \[\blacksquare\]

To give the illustrative example for Theorem 4.1, we first consider the following example.

**Example 4.4:** Let \( Z = \mathbb{R}, f : Z \to \overline{\mathbb{R}} \) given by

\[
f(z) = \begin{cases} 
-\sqrt{z} & \text{if } z \geq 0, \\
+\infty & \text{otherwise}.
\end{cases}
\]

Then, for every \( \epsilon > 0 \) and \( \bar{z} \geq 0, \) one has

\[
\partial_{\epsilon} f(\bar{z}) = \{z^* \in Z^* : \langle z^*, z - \bar{z} \rangle \leq f(z) - f(\bar{z}) + \epsilon, \forall z \in Z\}
= \left\{ z^* \in \mathbb{R} : z^*(z - \bar{z}) \leq -\sqrt{z} + \sqrt{\bar{z}} + \epsilon, \forall z \geq 0 \right\}.
\]
By computing directly, we see that

$$\partial \varepsilon f(\bar{z}) = \begin{cases} 
\emptyset & \text{if } \bar{z} < 0, \\
(-\infty, -\frac{1}{4\varepsilon}] & \text{if } \bar{z} = 0, \\
\frac{1}{2\left[\sqrt{\varepsilon^2 + 2\sqrt{\bar{z}\varepsilon}} - \left(\sqrt{\bar{z}} + \varepsilon\right)\right]} & \text{if } \bar{z} > 0.
\end{cases}$$

We now give an illustrative example for Theorem 4.1.

**Example 4.5:** Let $Z = \mathbb{R}^2$, $W = \mathbb{R}$, $\Omega = [1, a) \times \mathbb{R}$, $a > 1$,

$$f(w, z) = -\sqrt{z_1} + |w|$$

and $H(w) = \{(z_1, z_2) \in \mathbb{R}^2 : z_1 + z_2 = w\}$. Assume that $\bar{w} = 1$. Then the optimal value function (18) of the parametric problem (17) is $h(w) = |w| - \sqrt{a}$. By Example 3.3, for any $\varepsilon \geq 0$, one has

$$\partial \varepsilon h(\bar{w}) = [1 - \varepsilon, 1].$$

In this case, $\bar{z} = (a, 1 - a) \notin \hat{S}(\bar{w}) = \emptyset$, so we will clarify equality (20).

It is easy to see that $f(w, z)$ is continuous and $\text{int}(W \times \Omega) \cap \text{gph} H \neq \emptyset$. For any $\varepsilon \geq 0$, by Theorem 4.1, we get

$$\partial \varepsilon h(\bar{x}) = \bigcap_{\mu > 0} \bigcup_{z \in Z} \bigcup_{\bar{w}^* \in W} \bigcup_{(\gamma_1, \gamma_2) \in \hat{\Gamma}(\varepsilon + \mu)} \{w^* \in W : (w^* + \bar{w}^*, -M^* \bar{w}^*) \\
\in \partial \gamma_1 f(\bar{w}, z) + N_{\gamma_2}((\bar{w}, z); W \times \Omega)\},$$

with $M : \mathbb{R}^2 \to \mathbb{R}$ be the mappings defined by

$$Mz = \begin{bmatrix} 1 & 1 \\
\end{bmatrix} \begin{bmatrix} z_1 \\
z_2 \end{bmatrix}.$$ 

So, $M^* : \mathbb{R} \to \mathbb{R}^2$ be the mappings defined by

$$M^* \bar{w}^* = \begin{bmatrix} 1 \\
1 \end{bmatrix} \begin{bmatrix} \bar{w}^* \\
\end{bmatrix} = (\bar{w}^*, \bar{w}^*).$$

By [15, Proposition 4.1], one has

$$\partial \gamma f(\bar{w}, z) \subset \partial \gamma_1 f(\bar{w}) \times \partial \gamma f(z_1) \times \partial \gamma f(z_2),$$
where \( f(w) = |w|, f(z_1) = -\sqrt{z_1}, f(z_2) = 0 \). By Example 3.3 and Example 4.4, 
\( \partial_{\gamma_2}f(\tilde{w}_1) = [1 - \gamma_1, 1], \partial_{\gamma_1}f(\tilde{z}_2) = \{0\} \) and

\[
\partial_{\gamma_1}f(z_1) = \begin{cases} 
\emptyset & \text{if } \tilde{z} < 0, \\
\left(-\infty, -\frac{1}{4\gamma_1}\right] & \text{if } \tilde{z} = 0, \\
\frac{1}{2\left[\sqrt{\gamma_1^2 + 2\sqrt{z}\gamma_1} - \left(\sqrt{z} + \gamma_1\right)\right]} & \text{if } \tilde{z} > 0, \\
\frac{-1}{2\left[\sqrt{\gamma_1^2 + 2\sqrt{z}\gamma_1} + \left(\sqrt{z} + \gamma_1\right)\right]} & \text{if } \tilde{z} = 0.
\end{cases}
\]

By Lemma 4.3,

\[
N_{\gamma_2}(\tilde{w}, \tilde{z}; W \times \Omega) \subset \{0\} \times N_{\gamma_2}(z_1; \Omega_1) \times \{0\},
\]

where

\[
N_{\gamma_2}(z_1; \Omega_1) = \begin{cases} 
\{0\} & \text{if } 1 < z_1 < a, \\
\left(-\infty, \frac{\gamma_1}{a-1}\right) & \text{if } z_1 = 1, \\
\left[\frac{\gamma_1}{1-a}, +\infty\right) & \text{if } z_1 = a, \\
\emptyset & \text{if } z_1 < 1 \text{ or } z_1 > a.
\end{cases}
\]

Hence,

\[
\partial_{\varepsilon} h(\tilde{x}) \subset \bigcap_{\mu > 0} \bigcup_{z_1 \in \Omega_1} \bigcup_{\tilde{w}^* \in \tilde{W}(\gamma_1, \gamma_2) \in \hat{F}(\varepsilon + \mu)} \{w^* \in W : (w^* + \tilde{w}^*, -\tilde{w}^*, -\tilde{w}^*) \in [1 - \gamma_1, 1] \times [\partial_{\gamma_1}f(z_1) + N_{\gamma_2}(z_1; \Omega_1)] \times \{0\}\}.
\]

This implies that \( \tilde{w}^* = 0 \) and

\[
\partial_{\varepsilon} h(\tilde{x}) \subset \bigcap_{\mu > 0} \bigcup_{(\gamma_1, \gamma_2) \in \hat{F}(\varepsilon + \mu)} [1 - \gamma_1, 1] \subset [1 - \varepsilon, 1].
\]

Therefore, the conclusion of Theorem 4.1 is justified.

5. **Proof of the main result**

To prove Theorem 2.1, we need the following lemma.
Lemma 5.1: Let $X, Y$ be finite-dimensional Euclidean spaces, $\varphi : X \times Y \to \mathbb{R}$ be a convex function. Then, for any $\varepsilon \geq 0$ and $(\bar{x}, \bar{y}) \in X \times Y$, one has

$$\partial_\varepsilon \varphi(\bar{x}, \bar{y}) \subset \frac{\partial_\varepsilon \varphi}{\partial x}(\bar{x}, \bar{y}) \times \frac{\partial_\varepsilon \varphi}{\partial y}(\bar{x}, \bar{y}),$$

where

$$\frac{\partial_\varepsilon \varphi}{\partial x}(\bar{x}, \bar{y}) = \{x^* \in X : (x^*, x - \bar{x}) \leq \varphi(x, \bar{y}) - \varphi(\bar{x}, \bar{y}) + \varepsilon\}.$$

Proof: Suppose that $(x^*, y^*) \in \partial_\varepsilon \varphi(\bar{x}, \bar{y})$ for some $\varepsilon \geq 0$. Then, we have

$$\langle (x^*, y^*), (x, y) - (\bar{x}, \bar{y}) \rangle \leq \varphi(x, y) - \varphi(\bar{x}, \bar{y}) + \varepsilon, \quad \forall (x, y) \in X \times Y.$$  

(37)

By our assumption, (37) is equivalent to

$$\langle x^*, x - \bar{x} \rangle + \langle y^*, y - \bar{y} \rangle \leq \varphi(x, y) - \varphi(\bar{x}, \bar{y}) + \varepsilon, \quad \forall (x, y) \in X \times Y.$$  

(38)

On one hand, substituting $y = \bar{y}$ into (38), we get $x^* \in \frac{\partial \varphi}{\partial x}(\bar{x}, \bar{y})$. On the other hand, taking $x = \bar{x}$, from (38) we have $y^* \in \frac{\partial \varphi}{\partial y}(\bar{x}, \bar{y})$. Therefore, for any $\varepsilon \geq 0$,

$$\partial_\varepsilon \varphi(\bar{x}, \bar{y}) \subset \frac{\partial_\varepsilon \varphi}{\partial x}(\bar{x}, \bar{y}) \times \frac{\partial_\varepsilon \varphi}{\partial y}(\bar{x}, \bar{y}).$$

The proof of the lemma is complete. 

We now return to the proof of Theorem 2.1, our main result.

From the assumptions of Theorem 2.1, we have that the regularity condition (b) of Theorem 4.1 is satisfied. According to Theorem 4.1 and Lemma 4.3, it follows that if $w^* \in \partial_\varepsilon V(\bar{w})$, then for any $\mu > 0$ there exist $(\bar{x}, \bar{u}) \in Z$, $(\gamma_1, \gamma_2) \in \Gamma(\mu + \varepsilon)$,

$$(0, x_0^*, 0, \bar{x}, u_0^*, \ldots, u_{N-1}^*) \in N_{\gamma_2}((\bar{w}, \bar{z}); W \times K) \subset \{0\} \times N_{\gamma_2}(\bar{x}_0; C) \times \{0\} \times N_{\gamma_2}(\bar{u}_0; \Omega_0) \times \ldots \times N_{\gamma_2}(\bar{u}_{N-1}; \Omega_{N-1}) \subset N_{(3N+1)\gamma_2}((\bar{w}, \bar{z}); W \times K)$$

and $\bar{w}^* = (\bar{w}_0^*, \bar{w}_1^*, \ldots, \bar{w}_{N-1}^*) \in W$ such that

$$(w^* + \bar{w}^*, -M^* \bar{w}^*) - (0, x_0^*, 0, \bar{x}, u_0^*, \ldots, u_{N-1}^*) \in \partial_{\gamma_2} f(\bar{w}, \bar{z})$$

(39)

Since $f(x, u, w) = \sum_{k=0}^{N-1} h_k(x_k, u_k, w_k) + h_N(x_N)$, Lemma 5.1 and [15, Proposition 4.1], we have

$$\partial_{\gamma_2} f(\bar{w}, \bar{z}) \subset \prod_{k=0}^{N-1} \left(\frac{\partial_{\gamma_2} h_k}{\partial w_k}\right)(\bar{x}_k, \bar{u}_k, \bar{w}_k) \times \prod_{k=0}^{N-1} \left(\frac{\partial_{\gamma_2} h_k}{\partial x_k}\right)(\bar{x}_k, \bar{u}_k, \bar{w}_k) \times \prod_{k=0}^{N-1} \left(\frac{\partial_{\gamma_2} h_N}{\partial w_k}\right)(\bar{x}_N, \bar{u}_k, \bar{w}_k).$$

(40)
We note that
\[
-M^* \bar{w}^*(A_0^* \bar{w}_0^* - \bar{w}_0^* + A_1^* \bar{w}_1^* + \ldots, -\bar{w}_N^*) \\
+ A_{N-1}^* \bar{w}_{N-1}^* - \bar{w}_{N-1}^*, B_0^* \bar{w}_0^* + \ldots, B_{N-1}^* \bar{w}_{N-1}^*).
\]

Since (39) and (40), we obtain
\[
w_k^* + \bar{w}_k^* \in \left(\frac{\partial \gamma_i h_k}{\partial w_k}\right)(\tilde{x}_k, \bar{u}_k, \bar{w}_k) \quad \text{for } k = 0, 1, \ldots, N - 1,
\]
\[-x_0^* + A_0^* \bar{w}_0^* \in \left(\frac{\partial \gamma_i h_0}{\partial x_0}\right)(\tilde{x}_0, \bar{u}_0, \bar{w}_0), \\
-\bar{w}_{N-1}^* \in \left(\frac{\partial \gamma_i h_{N}}{\partial x_N}\right)(\tilde{x}_N),
\]
\[-\bar{w}_{k-1}^* + A_k^* \bar{w}_k^* \in \left(\frac{\partial \gamma_i h_k}{\partial x_k}\right)(\tilde{x}_k, \bar{u}_k, \bar{w}_k) \quad \text{for } k = 1, 2, \ldots, N - 1,
\]
\[-u_k^* + B_k^* \bar{w}_k^* \in \left(\frac{\partial \gamma_i h_k}{\partial u_k}\right)(\tilde{x}_k, \bar{u}_k, \bar{w}_k) \quad \text{for } k = 0, 1, \ldots, N - 1.
\]

The proof of the theorem is complete. \(\blacksquare\)

Note that the inverse inclusion of (36) is not true in general. So, we have not got sufficient conditions for \(w^* \in \partial \varepsilon V(\bar{w})\).

To give the illustrative example for Theorem 2.1. We first consider the following example.

**Example 5.2:** Let \(Z = \mathbb{R}, f : Z \rightarrow \mathbb{R}\) given by \(f(z) = |z - 1|, \forall z \in \mathbb{R}\). Then, for every \(\varepsilon > 0\), one has
\[
\partial \varepsilon f(\bar{z}) = \{z^* \in Z^* : \langle z^*, z - \bar{z} \rangle \leq f(z) - f(\bar{z}) + \varepsilon, \forall z \in Z\}
\]
\[
= \{z^* \in \mathbb{R} : z^*(z - \bar{z}) \leq |z - 1| - |\bar{z} - 1| + \varepsilon, \forall z \in \mathbb{R}\}.
\]

By computing directly, we see that
\[
\partial \varepsilon f(\bar{z}) = \begin{cases}
\left[-1, -1 + \frac{\varepsilon}{1 - \bar{z}}\right] & \text{if } \bar{z} < 1 - \frac{\varepsilon}{2}, \\
[-1, 1] & \text{if } 1 - \frac{\varepsilon}{2} \leq \bar{z} \leq 1 + \frac{\varepsilon}{2}, \\
\left[1 + \frac{\varepsilon}{1 - \bar{z}}, 1\right] & \text{if } \bar{z} > 1 + \frac{\varepsilon}{2}.
\end{cases}
\]

**Example 5.3:** We will illustrate the main result-Theorem 2.1 by a concrete problem
\[
f(x, u, w) = (x_0 + u_0)^2 + w_0^2 + |x_1 - 1| + |w_1| + |x_2| \rightarrow \inf,
\]
\[
x_1 = -x_0 + w_0, \quad x_2 = x_1 - u_1 + w_1, \quad x(0) \in (-\infty, 1].
\]

Let \(\bar{w} = (\bar{w}_0, \bar{w}_1) = (0, 0)\). Then, for any \(\varepsilon > 0\), one has
\[
V_\varepsilon(\bar{w}) \subset [-4\sqrt{\varepsilon}, 4\sqrt{\varepsilon}] \times [-1, 1].
\]
Indeed, for $\tilde{w} = (0, 0)$, the problem becomes
\[
\begin{align*}
    f(x, u, w) &= (x_0 + u_0)^2 + |x_1 - 1| + |x_2| \to \inf, \\
    x_1 &= -x_0, \quad x_2 = x_1 - u_1, \quad x(0) \in (-\infty, 1].
\end{align*}
\]

Using [16, Proposition 4], it is not difficult to see that $S(\tilde{w}) = \{\tilde{z}\}$, where $\tilde{z} = (\tilde{x}_0, \tilde{x}_1, \tilde{x}_2, \tilde{u}_0, \tilde{u}_1) = (-1, 1, 0, 0, 1)$. So, $V(\tilde{w}) = 0$. It is easy to see that $h_k (k = 0, 1, 2)$ are continuous and $\text{int}(W \times K) \cap \text{gph} \ G \neq \emptyset$, with $W = \mathbb{R}^2, K = (-\infty, 1] \times \mathbb{R}^4$ and $G(w) = \{z = (x, u) : Mz = w\}$,

\[
Mz = \begin{bmatrix}
1 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
\end{bmatrix}
\begin{bmatrix}
x_0 \\
x_1 \\
x_2 \\
u_0 \\
u_1 \\
\end{bmatrix}
= \begin{bmatrix}
x_0 + x_1 \\
x_0 + x_2 + u_1 \\
\end{bmatrix}
\]

We note that

\[
\begin{align*}
    h_0(x_0, u_0, w_0) &= (x_0 + u_0)^2 + w_0^2, \\
    h_1(x_1, u_1, w_1) &= |x_1 - 1| + |w_1| \quad \text{and} \quad h_2(x_2) = |x_2|.
\end{align*}
\]

So,

\[
\begin{align*}
    \left(\frac{\partial \gamma_1 h_0}{\partial w_0}\right) (\tilde{x}_0, \tilde{u}_0, \tilde{w}_0) &= [-2\sqrt{\gamma_1}, 2\sqrt{\gamma_1}], \quad \left(\frac{\partial \gamma_1 h_1}{\partial w_1}\right) (\tilde{x}_1, \tilde{u}_1, \tilde{w}_1) = [-1, 1], \\
    \left(\frac{\partial \gamma_1 h_0}{\partial x_0}\right) (\tilde{x}_0, \tilde{u}_0, \tilde{w}_0) &= [-2\sqrt{\gamma_1}, 2\sqrt{\gamma_1}], \quad \left(\frac{\partial \gamma_1 h_2}{\partial x_2}\right) (\tilde{x}_2) = [-1, 1], \\
    \left(\frac{\partial \gamma_1 h_1}{\partial x_1}\right) (\tilde{x}_1, \tilde{u}_1, \tilde{w}_1) &= [-1, 1], \quad \left(\frac{\partial \gamma_1 h_1}{\partial u_1}\right) (\tilde{x}_1, \tilde{u}_1, \tilde{w}_1) = [0].
\end{align*}
\]

Assume that $\varepsilon \geq 0$ and $w^* = (w_0^*, w_2^*) \in \partial \varepsilon V(\tilde{w})$. By Theorem 2.1, there exist $(\gamma_1, \gamma_2) \in \Gamma(\varepsilon), x_0^* \in N_{\gamma_2}(\tilde{x}_0; C) = \{0\}$ and $\tilde{w}^* = (\tilde{w}_0^*, \tilde{w}_2^*) \in \mathbb{R}^2$ such that

\[
\begin{align*}
    w_0^* + \tilde{w}_0^* \in \left(\frac{\partial \gamma_1 h_0}{\partial w_0}\right) (\tilde{x}_0, \tilde{u}_0, \tilde{w}_0), \quad w_1^* + \tilde{w}_1^* \in \left(\frac{\partial \gamma_1 h_1}{\partial w_1}\right) (\tilde{x}_1, \tilde{u}_1, \tilde{w}_1), \\
    -x_0^* - \tilde{w}_0^* \in \left(\frac{\partial \gamma_1 h_0}{\partial x_0}\right) (\tilde{x}_0, \tilde{u}_0, \tilde{w}_0), \quad -\tilde{w}_1^* \in \left(\frac{\partial \gamma_1 h_2}{\partial x_2}\right) (\tilde{x}_2), \\
    -\tilde{w}_0^* + \tilde{w}_1^* \in \left(\frac{\partial \gamma_1 h_1}{\partial x_1}\right) (\tilde{x}_1, \tilde{u}_1, \tilde{w}_1), \quad -\tilde{w}_1^* \in \left(\frac{\partial \gamma_1 h_1}{\partial u_1}\right) (\tilde{x}_1, \tilde{u}_1, \tilde{w}_1).
\end{align*}
\]

So, (41) implies that $\tilde{w}_0^* \in [-2\sqrt{\gamma_1}, 2\sqrt{\gamma_1}], \quad \tilde{w}_1^* = 0$ and $w_0^* \in [-4\sqrt{\gamma_1}, 4\sqrt{\gamma_1}], \quad w_1^* \in [-1, 1]$. Hence,

\[
\partial \varepsilon V(\tilde{w}) \subset [-4\sqrt{\varepsilon}, 4\sqrt{\varepsilon}] \times [-1, 1].
\]

**Disclosure statement**

No potential conflict of interest was reported by the author(s).
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