SHEAVES AND SYMPLECTIC GEOMETRY OF
COTANGENT BUNDLES

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Institut Fourier, CNRS and Université Grenoble Alpes. The author is also partially supported by the ANR project MICROLOCAL (ANR-15CE40-0007-01). Part of this paper was written during a stay at UMI 3457 in Montréal (CNRS - CRM - Université de Montréal).
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INTRODUCTION

As the title suggests this paper explains some applications of the microlocal theory of sheaves of Kashiwara and Schapira to the symplectic geometry of cotangent bundles. The main notions of the microlocal theory of sheaves are Sato’s microlocalization, introduced in the 70’s, and the notion of microsupport of a sheaf, introduced by Kashiwara and Schapira in the 80’s. These notions were motivated by the study of modules over the ring of (micro-)differential operators. The link with the symplectic geometry was noticed (a deep result of [30] says that the microsupport of any sheaf is coisotropic) but not used to study global aspects of symplectic geometry until the papers [42] of Nadler-Zaslow and [47] of Tamarkin. The paper [42], together with [40], show that the dg-category of constructible sheaves on a real analytic manifold $M$ is equivalent to the triangulated envelope of a version of the Fukaya category of $T^*M$. The paper [47] proves non-displaceability results in symplectic geometry using the properties of the microsupports of sheaves. Building on the ideas of this paper it is explained in [22] how to associate a sheaf with a Hamiltonian isotopy of a cotangent bundle. In this paper we go on in this direction and use sheaves to recover some classical results of symplectic geometry (the Gromov non-squeezing theorem and the Gromov-Eliashberg rigidity theorem) and a more recent result, which says that a compact exact Lagrangian submanifold of a cotangent bundle is homotopically equivalent to the base. We also prove a result about cusps of curves on the sphere (Arnol’d three cusps conjecture).

Before we give more details we recall some facts about the microsupport. Let $M$ be a manifold of class $C^\infty$ and let $k$ be a ring. We denote by $D(k_M)$ the derived category of sheaves of $k$-modules over $M$. The microsupport $SS(F)$ of an object $F$ of $D(k_M)$ is introduced
in [29]. It is a closed subset of the cotangent bundle $T^*M$, conic for the action of $\mathbb{R}^+$ on $T^*M$. It is defined as the closure of the set of singular directions with respect to $F$, where $(x; \xi) \in T^*M$ is said non singular if the restriction maps from a neighborhood $B$ of $x$ to $B \cap \{f < f(x)\}$ induce isomorphisms between $H^iF_x$ and $\lim_{B \ni x} H^i(B \cap \{f < f(x)\}; F)$, for all functions $f$ with $df(x) = \xi$ and all $i \in \mathbb{Z}$. The easiest example is $SS(k_N) = T^*_N M$, where $k_N$ is the constant sheaf on a submanifold $N$ of $M$. In general the microsupport can be a very singular set but it is coisotropic in some sense (see Theorem 1.3.6). If $M$ is real analytic and $F$ is constructible, then $SS(F)$ is Lagrangian. Any smooth conic Lagrangian submanifold of $T^*M$ is locally the microsupport of some sheaf on $M$.

The microsupport is well-behaved with respect to the standard sheaf operations. An important example is the composition. Let $M_i$, $i = 1, 2, 3$, be three manifolds and let $q_{ij}$ be the projection from $M_1 \times M_2 \times M_3$ to $M_i \times M_j$. For $K_1 \in D(k_{M_1 \times M_2})$ and $K_2 \in D(k_{M_2 \times M_3})$ we set $K_1 \circ K_2 = Rq_{13!}(q_{12}^{-1}K_1 \otimes q_{23}^{-1}K_2)$. We can define a set theoretic analog of the composition where direct and inverse images are replaced by the same set operations and the tensor product by the intersection. Then, under some geometric hypotheses, we have $SS(K_1 \circ K_2) \subset SS(K_1) \circ SS(K_2)$.

In [47] a sheaf version of the Chekanov-Sikorav theorem (see [10] and [45]) is given. A more functorial version is given in [22] as follows. Let $\Phi$ be an $\mathbb{R}_{>0}$-homogeneous Hamiltonian isotopy of $T^*M = T^*M \setminus T^*_0 M$. Then there exists a sheaf $K_{\Phi}$ on $M^2$ which is invertible for the composition (there exists $K'$ such that $K_{\Phi} \circ K' = k_{\Delta_M}$) and such that $\overline{SS}(K_{\Phi})$ is the graph of $\Phi$. Here we let $SS(F)$ be $SS(F)$ with the zero section removed. We then have $\overline{SS}(K_{\Phi} \circ F) = \Phi(\overline{SS}(F))$ for any $F \in D(k_M)$ and $F \mapsto K_{\Phi} \circ F$ is an auto-equivalence of $D(k_M)$. We recall this in Part 2.

Since the microsupport is conic it is rather related with the contact geometry of the sphere cotangent bundle than the symplectic geometry of the cotangent bundle. We can also consider a Legendrian submanifold of the 1-jet space $J^1(M)$ as a conic Lagrangian submanifold in $T^*(M \times \mathbb{R})$ contained in $\{\tau > 0\}$, where we use the coordinates $(t; \tau)$ on $T^*\mathbb{R}$. In [47] Tamarkin also remarks that a sheaf $F$ on $M \times \mathbb{R}$ with a microsupport in $\{\tau \geq 0\}$ comes with natural morphisms $\tau_c: F \to T_{\tau_c}(F)$, where $c \geq 0$ and $T_c$ is the vertical translation in $M \times \mathbb{R}$, $T_c(x, t) = (x, t + c)$. A useful invariant of $F$ is then $e(F) = \sup\{c \geq 0; \tau_c(F) \neq 0\}$. It is introduced in [47] and used in [7] to obtain displacement energy bounds. We use it in Part 6 to prove
classical nonsqueezing results. Our proof is a baby case of the proof of Chiu of a contact nonsqueezing theorem in [12].

In Part 6 we use some operations on sheaves introduced in [31] (cut-off lemmas) to reduce the size of a microsupport. These operations are compositions with the constant sheaf on a cone. We recall them in Part 3 where we also prove that a sheaf $F$ whose microsupport can be decomposed into two disjoint (and unknotted) subsets, say $\text{SS}(F) = S_1 \sqcup S_2$, can itself be locally decomposed, up to constant sheaves, as $F_1 \oplus F_2$ with $\text{SS}(F_i) = S_i$.

We also use the cut-off results in Part 5 to prove that a Legendrian submanifold of $J^1(M)$ has a graph selector as soon as it is the microsupport of a sheaf $F$ satisfying some conditions at infinity. The graph selector is given by the boundary of the support of a section of $F$.

In Part 7 we prove the Gromov-Eliashberg rigidity theorem as a consequence of the involutivity theorem of Kashiwara-Schapira. The starting point is very simple. Let $M$ be a manifold and let $\phi_n$ be a sequence of homogeneous Hamiltonian isotopies of $\hat{T}^*M$ which converges in $C^0$ norm to a diffeomorphism $\phi_\infty$ of $T^*M$. Let $K_n \in \mathcal{D}(k_M^2)$ be the sheaf associated with $\phi_n$ as recalled above. Then we can consider a kind of limit $K_\infty$ of $K_n$ and the microsupport of $K_\infty$ is contained in the graph of $\phi_\infty$. We deduce from the involutivity theorem that this graph is Lagrangian, hence that $\phi_\infty$ is a symplectic map. This idea does not work directly to prove a local statement but we can cut-off the microsupport (using the cut-off results recalled in Part 3) and make it work.

The main result of this paper is a sheaf theoretic proof that a compact exact Lagrangian submanifold $L$ of a cotangent bundle $T^*M$ is homotopically equivalent to $M$. This is done in Parts 9-13. This result was previously obtained with Floer homology methods (see the beginning of Part 13 for references). However we do not recover the more precise results of Abouzaid and Kragh, who proved in [4] that the map $L \to M$ is a simple homotopy equivalence and gave some conditions on the higher Maslov classes in [3] (we only prove the vanishing of the first two classes; for the other classes we should use sheaves of spectra – see [27, 26]).

An important tool for our proof is the Kashiwara-Schapira stack $\mu\text{Sh}(k_A)$ of a Lagrangian submanifold $\Lambda$ of a cotangent bundle $T^*M$. In [31] Kashiwara and Schapira consider the “microlocal” category $\mathcal{D}(k_M^2, \Omega)$ where $\Omega$ is a subset of $T^*M$. It is defined as the quotient of $\mathcal{D}(k_M^2)$ by the subcategory formed by the $F$ such that $\text{SS}(F) \cap \Omega = \emptyset$. When $\Omega$ runs over the open subsets of $T^*M$ this gives a prestack on
and we consider its associated stack, say $\mu\text{Sh}(k_{T^*M})$. In [31] it is proved that the $\mathcal{H}om$ sheaf in $\mu\text{Sh}(k_{T^*M})$ is $H^0\mu\text{hom}$ where $\mu\text{hom}$ is a variant of Sato’s microlocalization (our stack has a very poor structure because the triangulated structure does not survive in the stackification and we only obtain $H^0\mu\text{hom}$, not $\mu\text{hom}$). The stack $\mu\text{Sh}(k_{\Lambda})$ is the substack of $\mu\text{Sh}(k_{T^*M})$ formed by the objects with microsupport contained in $\Lambda$. One step in the proof of the homotopy equivalence $L \simeq M$ is the construction of a sheaf representing a given global object of $\mu\text{Sh}(k_{\Lambda})$, where $\Lambda$ is a conic Lagrangian submanifold of $T^*(M \times \mathbb{R})$ deduced from $L$ by adding one variable. This is done in Part 12. A similar result is obtained by Viterbo in [55] using Floer homology methods. More precisely, any given $F \in \mu\text{Sh}(k_{\Lambda})(\Lambda)$ is represented by $F \in D(k_{M \times \mathbb{R}})$ such that $F_- := F|_{M \times \{t\}, t \ll 0}$ vanishes and $F_+ := F|_{M \times \{t\}, t \gg 0}$, is locally constant. Then we prove that $F \mapsto F_+$ gives an equivalence between the category $\mathcal{D}_{[\Lambda],+}$ formed by the $F$ such that $S^*(F) \subset \Lambda$ and $F_- \simeq 0$ and the subcategory $\mathcal{D}_{lc}(k_{M})$ of $\mathcal{D}(k_{M})$ of locally constant sheaves. We prove another equivalence between $\mathcal{D}_{[\Lambda],+}$ and $\mathcal{D}_{lc}(k_{\Lambda})$. Hence $\mathcal{D}_{lc}(k_{M})$ is equivalent to $\mathcal{D}_{lc}(k_{\Lambda})$ and it follows that $L \to M$ is a homotopy equivalence. Actually the proof is not so straightforward. We first prove the fully faithfulness of $F \mapsto F_+$ in Part 12, then use this fully faithfulness in the beginning of Part 13 to prove some result on the fundamental groups and the vanishing of the first Maslov class. For this we need to know that $\mu\text{Sh}(k_{\Lambda})(\Lambda)$ has many objects. But the first Maslov class is an obstruction to the existence of a global object in $\mu\text{Sh}(k_{\Lambda})$. To bypass this problem we first work in the orbit category of sheaves (see Part 9) where there is no such obstruction. The orbit category contains less information than $\mathcal{D}(k_{M})$ but the above argument works well enough in this framework to obtain the vanishing of the Maslov class. Then we can go back to the usual category of sheaves and we can prove $\mathcal{D}_{lc}(k_{M}) \simeq \mathcal{D}_{lc}(k_{\Lambda})$, as claimed.

Acknowledgments. The starting point of this paper is a discussion with Claude Viterbo. He explained me his construction of a quantization in the sense of this paper using Floer homology and asked whether it was possible to obtain it with the methods of algebraic analysis. Masaki Kashiwara gave me the idea to glue locally defined simple sheaves under the assumption that the Maslov class vanishes. Claire Amiot explained me that we can use the orbit category to work without this vanishing assumption. The idea of applying the involutivity theorem to the $C^0$-rigidity emerged after several discussions with Claude Viterbo, Pierre Schapira and Vincent Humilière. The work on the three cusps conjecture was motivated by discussions with Emmanuel.
Giroux and Emmanuel Ferrand. I also thank Sylvain Courte, Pierre Schapira and Nicolas Vichery for several remarks and many stimulating discussions. I thank the anonymous referees for their careful reading of the paper, for pointing out several mistakes and for their constructive suggestions, making the exposition more clear.

Part 1. Microlocal theory of sheaves

We recall here some results of [31] that we will use very often. The notion of microsupport of a sheaf is in particular very important. We recall its definition and its behaviour under sheaf operations. We also recall quickly the definition of Sato’s microlocalization and the $\mu hom$ functor, later introduced by Kashiwara and Schapira; it will be used in particular in §10 to define a category of sheaves associated with a Lagrangian submanifold of a cotangent bundle.

When [31] was written, it was not well understood how to deal with unbounded derived categories. In particular the theory of microsupport is written for bounded derived categories of sheaves. Moreover one of the fundamental lemmas was proved using an induction on the cohomological degree and its extension to the unbounded case could be unclear. However this problem has been solved in [44] and we will state the results on the microsupport for unbounded categories (although the sheaves we consider later are always locally bounded).

1.1. Notations

We mainly follow the notations of [31].

Geometry. Unless otherwise specified the manifolds we consider here are not “manifolds with boundary”. (The reason is that the definition of the microsupport is not so meaningful at the boundary – if we want to deal with a sheaf $F$ on a manifold with boundary $M$, we consider an embedding $i: M \to M^+$, with $M^+$ a usual manifold, and look at the microsupport of the direct image $Ri_*(F)$.) When we say that a submanifold $N$ of $M$ is locally closed, closed or compact, we mean that it is locally closed (intersection of a closed subset and an open subset), closed or compact as a subset of $M$. We denote by $\pi_M: T^*M \to M$ the cotangent bundle of $M$. If $N \subset M$ is a submanifold, we denote by $T^*_N M$ its conormal bundle; it is the subbundle of the restriction of $T^*M$ over $N$ whose fiber over a point $x \in N$ is the orthogonal space of $T^*_x N$, that is, $(T^*_N M)_x = \{ \theta \in T^*_x M; \langle T^*_x N, \theta \rangle = 0 \}$. The zero-section of $T^*M$ will usually be denoted by $T^*_M M$, or $M$, if there is no ambiguity. We set $\hat{T}^*M = T^*M \setminus T^*_M M$ and we denote by $\hat{\pi}_M: \hat{T}^*M \to M$ the projection.
For any subset $A$ of $T^*M$ we define its antipodal $A^a = \{(x; \xi) \in T^*M; (x; -\xi) \in A\}$. We usually denote by $\Delta_M$ the diagonal of $M^2$.

We denote the normal bundle of $N$ by $T_N M$. It is defined as the quotient bundle $(N \times_M TM)/TN$, where $N \times_M TM$ is the restriction of the bundle $TM$ to $N$.

The cotangent bundle $T^*M$ carries an exact symplectic structure. We denote the Liouville 1-form by $\alpha$. It is given in local coordinates $(x; \xi)$ by $\alpha = \sum_i \xi_i dx_i$. The symplectic structure is then given by the non-degenerate 2-form $d\alpha$. Since it is non-degenerate, it induces an isomorphism $H: T^*T^*M \rightarrow TT^*M$, the Hamiltonian isomorphism. For the sections of $T^*T^*M$ it gives $H(dx_i) = -\partial/\partial \xi_i$ and $H(d\xi_i) = \partial/\partial x_i$. The Hamiltonian vector field of a function $h: T^*M \rightarrow \mathbb{R}$ is $X_h = H(dh)$; in local coordinates we have

$$X_h(x; \xi) = \sum_i \frac{\partial h}{\partial \xi_i} \frac{\partial}{\partial x_i} - \frac{\partial h}{\partial x_i} \frac{\partial}{\partial \xi_i}.$$ 

A submanifold $\Lambda$ of $T^*M$ is Lagrangian if, for each $p \in \Lambda$, the tangent space $T_p \Lambda$ is its own orthogonal with respect to the symplectic structure of $T_p T^*M$. If $N \subset M$ is a submanifold, then $T^*_N M$ is a Lagrangian submanifold of $T^*M$.

Let $f: M \rightarrow N$ be a morphism of manifolds. It induces morphisms on the cotangent bundles:

$$(1.1.1) \quad T^*M \overset{f^*}{\leftrightarrow} M \times_N T^*N \overset{f_*}{\rightarrow} T^*N.$$ 

Let $N \subset M$ be a submanifold and $A \subset M$ any subset. We denote by $C_N(A) \subset T_N M$ the cone of $A$ along $N$. We recall its definition in §1.3 using the normal deformation of $N$ in $M$; in the case where $M$ is a vector space, $x_0 \in N$ and $q: M \rightarrow T_{N,x_0}M$ denotes the natural quotient map, then

$$(1.1.2) \quad C_N(A) \cap T_{N,x_0}M = \bigcap_{x \in A \cap (U \setminus \{x_0\})} \bigcup q([x_0, x]),$$ 

where $U$ runs over the neighborhoods of $x_0$ and $[x_0, x)$ denotes the half line starting at $x_0$ and containing $x$.

If $A, B$ are two subsets of $M$, we set $C(A, B) = C_{\Delta_M}(A \times B)$. Identifying $T_{\Delta_M}(M \times M)$ with $TM$ through the first projection, we consider $C(A, B)$ as a subset of $TM$. If $M$ is a vector space and $x_0 \in M$, we
have
\[(1.1.3) \quad C(A, B) \cap T_{x_0}M = \bigcap_{U \in A \cap B} \bigcup_{y \in U, x \neq y} q([y, x]),\]

where \(U\) runs over the neighborhoods of \(x_0\).

**Sheaves.** We consider a commutative unital ring \(k\) of finite global dimension (we will use \(k = \mathbb{Z}\) or \(k = \mathbb{Z}/2\mathbb{Z}\)). We denote by \(\text{Mod}(k)\) the category of \(k\)-modules and by \(\text{Mod}(k_M)\) the category of sheaves of \(k\)-modules on \(M\). We denote by \(D(k_M)\) (resp. \(D^b(k_M), D^b_{lb}(k_M)\)) the derived category (resp. bounded derived category, locally bounded derived category) of \(\text{Mod}(k_M)\). (Hence \(D^b_{lb}(k_M)\) is the subcategory of \(D(k_M)\) formed by the \(F\) such that \(F|_C \in D^b(k_C)\), for any compact subset \(C \subset M\).) We recall that \(\text{Mod}(k_M)\) has a fully faithful embedding (that is, which preserves the Hom sets) into \(D(k_M)\) by sending a sheaf to a complex of sheaves concentrated in degree 0. We refer to [31] for results about derived categories and sheaves and also [32] for the unbounded case.

We recall some standard notations (following mainly [31]). For a morphism of manifolds \(f: M \to N\), we denote by \(Rf_*\), \(Rf!\): \(D(k_M) \to D(k_N)\) the direct image and proper direct image functors. We denote by \(f^{-1}, f^!\): \(D(k_N) \to D(k_M)\) their adjoint functors. We thus have adjunctions \((f^{-1}, Rf_*)\) and \((Rf!, f^!)\).

For \(L \in D(k)\) we denote by \(L_M = a^{-1}_M(L)\) the constant sheaf with stalk \(L\) on \(M\), where \(a_M: M \to \{\text{pt}\}\) is the map to a point. For the inclusion \(j: Z \to M\) of a subset of \(M\) and \(F \in D(k_M)\) we often write
\[F|_Z = j^{-1}F.\]

We say that \(F \in D(k_M)\) is locally constant if any point of \(M\) has a neighborhood \(U\) such that \(F|_U\) is constant. If \(F\) is concentrated in degree 0, we often say “\(F\) is a local system” instead of “\(F\) is a locally constant sheaf”. When \(Z\) is locally closed we define
\[F_Z = j_!j^{-1}F, \quad \Gamma_Z(F) = Rj_*j^{-1}F.\]

When \(F = L_M\) is the constant sheaf with stalk \(L \in D(k)\), we set for short \(L_Z = (L_M)_Z = j_!(L_Z)\). (In case of ambiguity we write \(L_{M,Z} = (L_M)_Z\), but in general the ambient manifold \(M\) is understood.) We recall that, if \(Z\) is closed and \(U\) is open, then \(\Gamma(U; k_Z) = \{f: Z \cap U \to k; f\text{ is locally constant}\}\). If \(Z'\) is locally closed and \(Z\) is a closed subset of \(Z'\), we have an exact sequence in \(\text{Mod}(k_M)\)
\[(1.1.4) \quad 0 \to k_{Z' \setminus Z} \to k_{Z'} \to k_Z \to 0.\]
We let \( \otimes \) and \( \mathcal{H}om \) denote the tensor product and internal Hom in \( \text{Mod}(k_M) \). We recall that, for \( F, G \in \text{Mod}(k_M) \), their internal Hom is the sheaf \( \mathcal{H}om(F, G) \) whose sections over an open subset \( U \) is given by \( \Gamma(U; \mathcal{H}om(F, G)) = \text{Hom}_{\text{Mod}(k_U)}(F|_U, G|_U) \). The derived functors of \( \otimes \) and \( \mathcal{H}om \) are denoted \( \overset{L}{\otimes} \) and \( \overset{L}{\text{R} \mathcal{H}om} \) (we sometimes write \( F \otimes G \) for \( F \overset{L}{\otimes} G \) when \( F \) or \( G \) has free stalks over \( k \), typically \( F \otimes k_Z \) for \( F \overset{L}{\otimes} k_Z \)).

We have an adjunction \( (\otimes, \overset{L}{\text{R} \mathcal{H}om}) \).

We have natural isomorphisms
\[
(1.1.5) \quad F_Z \simeq F \otimes k_Z, \quad \Gamma_Z(F) \simeq \overset{L}{\mathcal{H}om}(k_Z, F)
\]
and \( (1.1.4) \) gives the excision distinguished triangles, for \( F \in \text{D}(k_M) \),
\[
F_{Z' \setminus Z} \to F_{Z'} \to F_Z \overset{+1}{\to}, \quad \Gamma_F(Z) \to \Gamma_F(Z') \to \Gamma_F(Z' \setminus Z) \overset{+1}{\to}.
\]
If \( U \) is open, we let \( \Gamma(U; -) \) be the derived section functor. We set \( H^i(U; F) = H^i \Gamma(U; F) \). We let \( \Gamma_c(-) \) be the functor of sections with compact support and \( \Gamma_c(U) \) its derived functor. If \( a_M: M \to \{ \text{pt} \} \) is the map to a point, we thus have \( \Gamma(M; F) \simeq \text{Ra}_M(F) \) and for connected orientable open subset \( U \) of \( M \), \( \text{or}_M(U) \) is the rank 1 free module \( \langle o, o'; o = -o' \rangle \) where \( o, o' \) are the two possible orientations of \( U \). The duality functors are defined by
\[
(1.1.6) \quad \Gamma_Z(U; F) = \Gamma(U; \Gamma_Z(F)), \quad H^i_Z(U; F) = H^i \text{R} \Gamma_Z(U; F).
\]

We denote by \( \omega_M \) the dualizing complex on \( M \). Since \( M \) is a manifold, \( \omega_M \) is actually the orientation sheaf shifted by the dimension, that is, \( \omega_M \simeq o_M[d_M] \). (In general it is defined by \( \omega_M = a_M^\dagger(k_{\{\text{pt}\}}) \).) We recall that the sheaf \( o_M \) is locally constant with stalks \( k \) and, for a connected orientable open subset \( U \) of \( M \), \( o_M(U) \) is the rank 1 free module \( \langle o, o'; o = -o' \rangle \) where \( o, o' \) are the two possible orientations of \( U \). The duality functors are defined by
\[
(1.1.7) \quad D_M(\bullet) = \overset{L}{\text{R} \mathcal{H}om}(\bullet, \omega_M), \quad D'_M(\bullet) = \overset{L}{\text{R} \mathcal{H}om}(\bullet, k_M).
\]

For \( f: M \to N \) we also use the notation \( \omega_{M|N} = f^!(k_N) \). We have \( \omega_{M|N} \simeq f^!(\text{D}'_N(\omega_N)) \simeq or_M \otimes f^{-1}(or_N)[d_M - d_N] \) (we remark that \( \text{D}'_N(\text{or}_N) \simeq \text{or}_N \); indeed there exists a canonical isomorphism \( u: \text{or}_N \otimes \text{or}_N \simeq k_N \) such that, for any \( x \in N \) and any choice of orientation \( o \) around \( x \), we have \( u_x(o \otimes o) = 1 \).)

For two manifolds \( M, N \) and \( F \in \text{D}(k_M), G \in \text{D}(k_N) \) we define \( F \overset{L}{\otimes} G \in \text{D}(k_M \times k_N) \) by
\[
F \overset{L}{\otimes} G = q_1^{-1} F \otimes q_2^{-1} G,
\]
where \( q_i \ (i = 1, 2) \) is the \( i \)-th projection defined on \( M \times N \).
We recall some useful facts (see [31, §2, §3]).

**Proposition 1.1.1.** Let \( f : M \to N \) be a morphism of manifolds, \( F, G, H \in \mathcal{D}(k_M), \) \( F', G' \in \mathcal{D}(k_N) \). Then we have for any \( i \in \mathbb{Z} \)

(a) \( H^i \mathcal{R} \text{Hom}(F, G) \cong \text{Hom}(F, G[i]) \),

(b) \( \mathcal{R} \text{Hom}(k_U, F) \cong \mathcal{R} \Gamma(U; F), \) for \( U \subset M \) open,

(c) \( \mathcal{R} \Gamma(U; \mathcal{R} \text{Hom}(F, G)) \cong \mathcal{R} \text{Hom}(F|_U, G|_U), \) for \( U \subset M \) open,

(d) \( H^i F \) is the sheaf associated with \( V \mapsto H^i(V; F) \),

(e) \( H^i \mathcal{R} \text{Hom}(F, G) \) is the sheaf associated with \( V \mapsto \text{Hom}(F|_V, G|_V[i]) \),

(f) \( \mathcal{R} \text{Hom}(F \otimes G, H) \cong \mathcal{R} \text{Hom}(F, \mathcal{R} \text{Hom}(G, H)) \),

(g) \( \mathcal{R} f_!(F \otimes f^{-1} F') \cong (\mathcal{R} f_! F) \otimes F' \), (projection formula),

(h) \( f^! \mathcal{R} \text{Hom}(F', G') \cong \mathcal{R} \text{Hom}(f^{-1} F', f^! G') \),

(i) if \( f \) is an embedding, \( \mathcal{R} f_* \mathcal{R} \text{Hom}(F, G) \cong \mathcal{R} \text{Hom}(\mathcal{R} f_! F, \mathcal{R} f_* G) \),

(j) for a Cartesian diagram 

\[
\begin{array}{ccc}
M & \xrightarrow{f} & N \\
\downarrow g & & \downarrow g' \\
M' & \xrightarrow{f'} & N'
\end{array}
\]

formulas \( f'^{-1} \text{R} g'_!(F') \cong \text{R} g f^{-1}(F') \) and \( f'^* \text{R} g'_*(F') \cong \text{R} g f^!(F') \).

The adjunction between \( \otimes \) and \( \mathcal{R} \text{Hom} \) together with \( k_U \otimes k_U \cong k_U \) give \( \text{Hom}(k_U, D'(k_U)) \cong \text{Hom}(k_U, k_M) \cong H^0(U; k_M) \). The canonical section of this last group gives a morphism \( k_U \to D'(k_U) \). Similarly we have a natural morphism \( k_U \to D'(k_U) \). In the following case they are isomorphisms.

If the inclusion \( U \subset M \) is locally homeomorphic to the inclusion \([-\infty, 0] \times \mathbb{R}^{n-1} \subset \mathbb{R}^n \) (for example, if \( \partial U \) is smooth), then we have the first isomorphism in (1.1.8) below. Indeed, to prove that some morphism is an isomorphism we can work locally and thus assume that \( U = \{ x \in \mathbb{R}^n \mid x(n-1) < 0 \} \). For a point \( x \in \partial U \) and an open ball \( B \) around \( x \), we have

\[
\mathcal{R} \Gamma(B; D'(k_M)) \cong \mathcal{R} \text{Hom}(k_U|_B, k_M|_B) \cong \mathcal{R} \text{Hom}(k_U \cap B, k_B) \cong \mathcal{R} \Gamma(U \cap B; k_B) \cong k
\]

using (b) and (c). It follows that \( (D'(k_U))_x \cong (k_U)_x \). This also holds for \( x \in U \) or \( x \in M \setminus \overline{U} \) and we obtain the claim. The second isomorphism in (1.1.8) follows from the first one applied to \( M \setminus \overline{U} \) and the exact sequence \( 0 \to k_{M \setminus \overline{U}} \to k_M \to k_U \to 0 \).

(1.1.8) \[ k_U \cong D'(k_U), \quad k_U \cong D'(k_U). \]
1.2. Microsupport

1.2.1. Definition and first properties. We recall the definition of the microsupport (or singular support) $SS(F)$ of a sheaf $F$, introduced by M. Kashiwara and P. Schapira in [29] and [30].

Let $F \in D(k_M)$ and $p = (x_0; \xi_0) \in T^*M$ be given. We choose a real $C^1$-function $\phi$ on $M$ satisfying $d\phi(x_0) = \xi_0$ and we consider the restriction morphism “in the direction $p$” for a given degree $i \in \mathbb{Z}$:

$$H^iF_{x_0} \cong \lim_{U} H^i(U; F) \to \lim_{U} H^i(U \cap \{x; \phi(x) < \phi(x_0)\}; F),$$

where $U$ runs over the open neighborhoods of $x_0$. We are interested in the points $p$ where this morphism is not an isomorphism (for some $\phi$ and $i$). Taking the cone of the restriction morphism we obtain the following definition.

**Definition 1.2.1.** (see [31, Def. 5.1.2]) Let $F \in D(k_M)$. We define $SS(F) \subseteq T^*M$ as the closure of the set of points $(x_0; \xi_0) \in T^*M$ such that there exists a real $C^1$-function $\phi$ on $M$ satisfying $d\phi(x_0) = \xi_0$ and $(R\Gamma_{x_0; \phi(x) \geq \phi(x_0)}(F))_{x_0} \neq 0$.

We set $SS(F) = SS(F) \cap T^*M$.

The following properties are easy consequences of the definition:
(a) the microsupport is closed and $\mathbb{R}_{>0}$-conic, that is, invariant by the action of $(\mathbb{R}_{>0}, \times)$ on $T^*M$,
(b) $SS(F) \cap T^*_M = \pi_M(SS(F)) = supp(F)$,
(c) the microsupport satisfies the triangular inequality: if $F_1 \to F_2 \to F_3 \xrightarrow{\pm 1} \to$ is a distinguished triangle in $D(k_M)$, then $SS(F_i) \subseteq SS(F_j) \cup SS(F_k)$ for all $i, j, k \in \{1, 2, 3\}$ with $j \neq k$.

**Notation 1.2.2.** Following [31, §6.1], for an $\mathbb{R}_{>0}$-conic subset $S$ of $T^*M$, we denote by $D_S(k_M)$ the full subcategory of $D(k_M)$ formed by the $F$ with $SS(F) \subseteq S$. We also set $D_{[S]}(k_M) = D_{S \cup T^*_M}(k_M)$. We denote by $D_S(k_M)$ the full subcategory of $D(k_M)$ formed by the $F$ for which there exists a neighborhood $\Omega$ of $S$ in $T^*M$ such that $SS(F) \cap \Omega \subseteq S$. By the property (c) above, all these subcategories are triangulated.

**Example 1.2.3.** (i) If $F$ is a non-zero local system on a connected manifold $M$, then $SS(F) = T^*_M$, the zero-section. Conversely, if $SS(F) \subseteq T^*_M$, then the cohomology sheaves $H^i(F)$ are local systems, for all $i \in \mathbb{Z}$. (We say that $F$ is locally constant for short.) This follows for example from Proposition 1.2.9 below.

(ii) If $N$ is a closed submanifold (we only mean $N$ is a closed subset of $M$ — see §1.1) of $M$ and $F = k_N$, then $SS(F) = T^*_NM$. 

(iii) Let $U \subset M$ be an open subset with smooth boundary. Then
\[ SS(k_U) = (U \times_M T^*_M M) \cup T^{*, out}_{\partial U} M, \]
\[ SS(k_{\overline{\tau}}) = (U \times_M T^*_M M) \cup (T^{*, out}_{\partial U} M)^a, \]
where $T^{*, out}_{\partial U} M = \{(x; \lambda df(x)); f(x) = 0, \lambda \leq 0\}$, if $U = \{f > 0\}$ and $df \neq 0$ on $\partial U$ (see Fig. 1.2.1). To remember the directions of the microsupports we can take $\phi = f$ in Definition 1.2.1 and use $R\Gamma \{x; \phi(x) \geq 0\} (F) \simeq F$ for both $F = k_U$ and $F = k_{\overline{\tau}}$ (since $\text{supp}(F) \subset \{x; \phi(x) \geq 0\}$). This implies $(R\Gamma \{x; \phi(x) \geq 0\} (k_U))_{x_0} \simeq (k_U)_{x_0} \simeq 0$ and $(R\Gamma \{x; \phi(x) \geq 0\} (k_{\overline{\tau}}))_{x_0} \simeq (k_{\overline{\tau}})_{x_0} \simeq k$, for $x_0 \in \partial U$, and both formulas are consistent with the above description of the microsupport.

\[ SS(k_U) \quad SS(k_{\overline{\tau}}) \]

**Figure 1.2.1.** We identify covectors with vectors to draw the microsupport. The microsupport of the constant sheaf on an open (resp. closed) subset with smooth boundary points outward (resp. inward).

(iv) Let $\lambda$ be a closed convex cone with vertex at 0 in $E = \mathbb{R}^n$. By [31, Prop. 5.3.1] we have $SS(k_{\lambda}) \cap T_0^* E = \lambda^\circ$, where $\lambda^\circ$ is the polar cone of $\lambda$:
\[ \lambda^\circ = \{ \xi \in E^*; \langle v, \xi \rangle \geq 0 \text{ for all } v \in \lambda\}. \]

For a given $a > 0$, we define $h_a : E \rightarrow E, x \mapsto ax$. We clearly have $h_a^{-1}(k_{\lambda}) \simeq k_{\lambda}$. Hence $SS(k_{\lambda})$ is invariant by the map induced by $h_a$ on $T^* E$. Since the microsupport is closed, we deduce the rough bound $SS(k_{\lambda}) \subset E \times \lambda^\circ$ by letting $a \rightarrow 0$ (see Fig. 1.2.2).

\[ \begin{array}{c}
\vdots \\
\end{array} \]

**Figure 1.2.2.** At the vertex $SS(k_{\lambda})$ is the polar cone of $\lambda$.

(v) Let $F \in D(k_{\overline{\tau}})$ be such that $SS(F) = \{(0; \xi); \xi > 0\}$. On one hand, it follows from (i) that $F|_{\overline{\tau} \{0\}}$ is locally constant, hence $F|_{U_{\pm}}$ is constant, where $U_{\pm} = \{\pm x > 0\}$. In particular $(R\Gamma \{x; \phi(x) \geq \phi(x_0)\} (F))_{x_0} \simeq 0$
for any $x_0 \neq 0$ and any function $\phi$ with $d\phi(x_0) \neq 0$. On the other hand, by the definition of the microsupport there exist points $x_1$ arbitrarily close to 0 (maybe equal to 0) together with functions $\phi$ such that $d\phi(x_1) > 0$ and $(R\Gamma_{x; \phi(x) > \phi(x_1)}(F))_{x_1} \neq 0$. We have seen that the case $x_1 \neq 0$ is excluded. Hence $(R\Gamma_{x; \phi(x) > \phi(0)}(F))_0 \neq 0$ for some function $\phi$ with $d\phi(0) > 0$. This means $(R\Gamma_Z F)_0 \neq 0$, where $Z = \overline{U_+}$. Let us set $B_r = ]-r, r[$. Since $F|_{U_+}$ is constant, $R\Gamma(B_r; R\Gamma_Z F)$ is independent of $r > 0$. Hence we have in fact $(R\Gamma_Z F)_0 \simeq R\Gamma(\mathbb{R}; R\Gamma_Z F)$. Let us set $E = (R\Gamma_Z F)_0$. By the adjunction $(a^{-1}, R\ast)$ where $a: \mathbb{R} \to \{\text{pt}\}$ is the projection, the isomorphism $E \xrightarrow{\sim} R\Gamma(\mathbb{R}; R\Gamma_Z F) = Ra_*RGZF$ gives a morphism $u: E_\mathbb{R} \to R\Gamma_Z F$. By the adjunction $(\otimes, R\text{Hom})$ we obtain from $u$ another morphism $v: E_\mathbb{R} \to F$. We have $(R\Gamma_Z(E_\mathbb{R}))_0 \simeq E$ and the morphism $v$ induces an isomorphism $(R\Gamma_Z(E_\mathbb{R}))_0 \xrightarrow{\sim} (R\Gamma_Z F)_0$. In other words, defining $G$ by a distinguished triangle $G \to E_\mathbb{R} \xrightarrow{v} F \xrightarrow{1} E_\mathbb{R}$, we have $(R\Gamma_Z G)_0 \simeq 0$. By the triangle inequality we deduce that $SS(G) = \emptyset$, hence $G$ is a constant sheaf on $\mathbb{R}$ and we can write $G = E'_\mathbb{R}$ for some $E' \in D(k)$. In conclusion there exist $E, E' \in D(k)$ and a distinguished triangle

$$E'_\mathbb{R} \to E_\mathbb{R} \to F \xrightarrow{1} E_\mathbb{R}$$

Conversely a sheaf $F$ defined by such a distinguished triangle satisfies $SS(F) = \{(0; \xi); \xi > 0\}$ and $(R\Gamma_Z F)_0 \simeq E$.

1.2.2. Functorial operations.

**Proposition 1.2.4.** (See [31, Prop. 5.4.4].) Let $f: M \to N$ be a morphism of manifolds and let $F \in D(k_M)$. We assume that $f$ is proper on $supp(F)$. Then $SS(Rf_*F) \subset f_\pi f^{-1}_d SS(F)$, with equality when $f$ is a closed embedding (see the beginning of §1.14 for “closed”).

**Example 1.2.5.** With the notations of Proposition 1.2.4 we assume that $f$ is an embedding. Let $F \in D(k_N)$ be such that $SS(F) \subset T^*_M N$. Then there exists a locally constant $G \in D(k_M)$ such that $F \simeq Rf_*G$. Indeed, since $SS(F) \cap T^*(N \setminus M) = \emptyset$, we have $F|_N \setminus M \simeq 0$, by the property (b) after Definition 1.2.1. Hence $F \simeq Rf_*G$ where $G = f^{-1}F$. Then $SS(F) = f_\pi f^{-1}_d SS(G)$ by Proposition 1.2.4 and we deduce $SS(G) \subset T^*_N N$. Now the result follows from Example 1.2.3(i).

We recall some notations of [31, Def. 6.2.3].

Let $M$ and $N$ be two manifolds and $f: M \to N$ a morphism. Let $\Gamma_f \subset M \times N$ be the graph of $f$. We have $T^*_M (M \times N) \simeq M \times N T^* N$. If $X$ is a manifold and $\Lambda$ a Lagrangian submanifold of $T^* X$, the Hamiltonian isomorphism identifies $T_X T^* X$ with $T^* \Lambda$. Applying this to $X = M \times N$.
Indeed, we can identify $M_{N}(SS)$, where $f$ is an isomorphism, where (see also Example 1.2.7).

We also remark that $T^*M$ has a natural embedding in $T^*(M \times N T^*N)$. For a conic subset $A \subset T^*N$ we set

\[(1.2.3) \quad f^\sharp(A) = T^*M \cap C_{T^*_f(M \times N)}(T^*M \times A),\]

**Example 1.2.6.** Let $f$ be the embedding of $\mathbb{R}^m$ in $\mathbb{R}^{n+m}$. We take coordinates $(x', x''; \xi', \xi'')$ on $T^*\mathbb{R}^{n+m}$ such that $\mathbb{R}^m = \{x' = 0\}$. Then $(x''_{\infty}, \xi''_{\infty}) \in f^\sharp(A)$ if and only if there exists a sequence $\{(x'_n, x''_n; \xi'_n, \xi''_n)\}$

in $A$ such that $x'_n \to 0$, $x''_n \to x''_{\infty}$, $\xi'_n \to \xi''_{\infty}$ and $|x'_n|/|\xi'_n| \to 0$.

**Example 1.2.7.** With the notations (1.1.1) we say that $f$ and $A$ are non-characteristic if

\[f^{-1}(A) \cap f^{-1}(T^*_M M) \subset M \times N T^*_N N.\]

If $A \subset T^*N$ is closed conic and $f$ and $A$ are non-characteristic, then $f^\sharp(A) = f_d f^{-1}(A)$.

**Theorem 1.2.8.** (See [31, Cor. 6.4.4].) Let $f : M \to N$ be a morphism of manifolds and let $F \in D(k_N)$. Then, using the notation $f^\sharp$ of (1.2.3) (see also Example 1.2.7),

\[SS(f^{-1}F) \subset f^\sharp(SS(F)) \quad \text{and} \quad SS(f^\sharp F) \subset f^\sharp(SS(F)).\]

If $f$ is smooth, these inclusions are equalities. If $f$ is non-characteristic for $SS(F)$, then the natural morphism

\[f^{-1}F \otimes \omega_{M|N} \to f^\sharp(F)\]

is an isomorphism, where $\omega_{M|N} \simeq \text{or}_M \otimes f^{-1}(\text{or}_N)[d_M - d_N]$ is the relative dualizing complex.

**Proposition 1.2.9.** (See [31, Prop. 5.4.5].) Let $N, I$ be manifolds. We assume that $I$ is contractible. Let $f : N \times I \to N$ be the projection. Let $F \in D(k_{N \times I})$. Then $SS(F) \subset T^*N \times T^*_1 I$ if and only if $f^{-1}R_f{*}(F) \simeq F$.

**Example 1.2.10.** We set $M = \mathbb{R}^n$, $S = \{x_n = 0\}$, $Z = \{x_n \geq 0\}$ and $\Lambda = \{(x_1, \ldots, x_{n-1}, 0; 0, \xi_n); \xi_n > 0\}$. Let $F \in D(k_M)$ be such that $SS(F) = \Lambda$. Then there exist $E, E' \in D(k)$ and a distinguished triangle

\[E'M \to E_Z \to F \overset{\Delta}{\to} .\]

Indeed, we can identify $M$ with a product $I \times N$, where $I = \mathbb{R}^{n-1}$ and $N = \mathbb{R}$. Then $\Lambda = T^*_1 I \times \{(0; \xi_n); \xi_n > 0\}$. By Proposition 1.2.9 we can write $F \simeq f^{-1}G$ for some $G \in D(k_N)$, where $f : N \times I \to N$ is the
projection. By Theorem 1.2.8 we must have \( \hat{\text{SS}}(G) = \{(0; \xi_n); \xi_n > 0\} \)
and we conclude with Example 1.2.3 (v).

**Example 1.2.11 (Conic sheaves).**  (i) We set \( S^{n-1} = (\mathbb{R}^n \setminus \{0\})/\mathbb{R}_+ \)
and let \( q: \mathbb{R}^n \setminus \{0\} \to S^{n-1} \) be the quotient map. We say that \( F \in D(\mathbb{k}_{\mathbb{R}^n}) \) is conic if there exists \( G \in D(\mathbb{k}_{S^{n-1}}) \) such that \( F|_{\mathbb{R}^n \setminus \{0\}} \simeq q^{-1}G \). By Proposition 1.2.9 \( F \) is conic if and only if, for any \( x \neq 0 \in \mathbb{R}^n \), we have \( \hat{\text{SS}}(F) \cap T^*x\mathbb{R}^n \subset (T^*_x\mathbb{R}^n)_x \), where \( l = \mathbb{R} \cdot x \) is the line generated by \( x \).

(ii) In Example 1.2.3 (iv) we have given a bound for \( \text{SS}(\mathbb{k}_\lambda) \) when \( \lambda \) is a closed convex cone in \( \mathbb{R}^n \), namely \( \text{SS}(\mathbb{k}_\lambda) \subset \mathbb{R}^n \times \lambda^o \) and \( \text{SS}(\mathbb{k}_\lambda) \cap T^*_0\mathbb{R}^n = \lambda^o \), where \( \lambda^o \) is the polar cone of \( \lambda \). Since \( \mathbb{k}_\lambda \) is conic, the above discussion also implies \( \text{SS}(\mathbb{k}_\lambda) \cap T^*_x\mathbb{R}^n \subset (T^*_x\mathbb{R}^n)_x \), where \( l = \mathbb{R} \cdot x \), for any \( x \neq 0 \in \mathbb{R}^n \).

We remark that \( \text{Int}(\lambda^o) = \{\xi \in (\mathbb{R}^n)^*; \langle v, \xi \rangle > 0 \text{ for all } v \neq 0 \in \lambda\} \) (if \( \lambda \) contains a line, then \( \text{Int}(\lambda^o) = \emptyset \)). In particular, for \( x \neq 0 \in \lambda \), we have \( \text{SS}(\mathbb{k}_\lambda) \cap (T^*_x\mathbb{R}^n)_x = \emptyset \) and we deduce the more precise bound

\[
\text{SS}(\mathbb{k}_\lambda) \cap (\mathbb{R}^n \setminus \{0\}) \subset (\mathbb{R}^n \setminus \{0\}) \times \partial \lambda^o,
\]

where \( \partial \lambda^o = \lambda^o \setminus \text{Int}(\lambda^o) \).

Let \( \delta_M: M \to M^2 \) be the diagonal embedding. For conic subsets \( A, B \subset T^*M \) we set

\[
(1.2.5) \quad A \hat{\oplus} B = \delta_M^*(A \times B).
\]

In local coordinates \( A \hat{\oplus} B \) is the set of \((x; \xi)\) such that there exist two sequences \((x_n; \xi_n)\) in \( A \) and \((y_n; \eta_n)\) in \( B \) such that \( x_n, y_n \to x \), \( \xi_n + \eta_n \to \xi \) and \( |x_n - y_n|/|\xi_n| \to 0 \) when \( n \to \infty \).

**Example 1.2.12.** If \( A, B \subset T^*M \) are closed conic subsets such that \( A^a \cap B \subset T^*M \), then \( A \hat{\oplus} B = A + B \).

For the definition of **cohomologically constructible** we refer to [31, §3.4]. An example of a cohomologically constructible sheaf \( F \) is given by the case where \( F \) is constructible with respect to a Whitney stratification (that is, the restriction of \( F \) to each stratum is locally constant and of finite rank).

**Theorem 1.2.13.** (See [31, Cor. 6.4.5].) Let \( F, G \in D(\mathbb{k}_M) \). Then, using the notation \( \hat{\oplus} \) of (1.2.5) (see also Example 1.2.12),

\[
\text{SS}(F \hat{\otimes} G) \subset \text{SS}(F) \hat{\oplus} \text{SS}(G),
\]

\[
\text{SS}(R \text{Hom}(F, G)) \subset \text{SS}(F)^a \hat{\oplus} \text{SS}(G),
\]

\[
\text{SS}(D^f F) = \text{SS}(F)^a.
\]
If we assume that $\text{SS}(F) \cap \text{SS}(G) \subset T^*_M M$ and that $F$ is cohomologically constructible, then the natural morphism $D'F \otimes G \to R\text{Hom}(F,G)$ is an isomorphism.

**Remark 1.2.14.** Let $M, N$ be manifolds and let $q_1, q_2$ be the projections from $M \times N$ to $M, N$. For $F \in D(k_M)$ and $G \in D(k_N)$ Theorem 1.2.13 implies $\text{SS}(R\text{Hom}(q_1^{-1}F, q_2^{-1}G)) \subset \text{SS}(F)^a \times \text{SS}(G)$ and $\text{SS}(q_1^{-1}F \otimes q_2^{-1}G) \subset \text{SS}(F) \times \text{SS}(G)$.

Using the $\dot{-}$ operation we can give a version of Proposition 1.2.4 for an open embedding. We only state the case where the boundary of the open subset is smooth (see [31] for the general case).

**Theorem 1.2.15.** (See [31] Thm. 6.3.1.) Let $j: U \hookrightarrow M$ be the embedding of an open subset with a smooth boundary. Let $F \in D(k_U)$. Then $\text{SS}(Rj_* F) \subset \text{SS}(F) \dot{-} \text{SS}(k_U)^a$ and $\text{SS}(Rj_! F) \subset \text{SS}(F) \dot{-} \text{SS}(k_U)$.

The next result follows immediately from Proposition 1.2.4 and Example 1.2.3 (i). It is a particular case of a microlocal Morse result (see [31] Cor. 5.4.19), the classical theory corresponding to the constant sheaf $F = k_M$.

**Corollary 1.2.16.** Let $F \in D(k_M)$, let $\phi: M \to \mathbb{R}$ be a function of class $C^1$ and assume that $\phi$ is proper on $\text{supp}(F)$. Let $a < b$ in $\mathbb{R}$ and assume that $d\phi(x) \not\in \text{SS}(F)$ for $a \leq \phi(x) < b$. Then the natural morphisms $R\Gamma(\phi^{-1}([-\infty, b]); F) \to R\Gamma(\phi^{-1}([-\infty, a]); F)$ and $R\Gamma(\phi^{-1}([b, +\infty]); M; F) \to R\Gamma(\phi^{-1}([a, +\infty]); M; F)$ are isomorphisms.

Here is another useful consequence of the properties of the micro-support which appears in [11].

**Corollary 1.2.17.** Let $M$ be a manifold and $I$ an open interval of $\mathbb{R}$. Let $F, G \in D(k_{M \times I})$. We assume

1. the projection $\text{supp}(F) \cap \text{supp}(G) \to I$ is proper,
2. $F, G$ are non-characteristic for all maps $i_t: M \times \{t\} \to M \times I$, $t \in I$, that is, $\text{SS}(A) \cap (T^*_A M \times T^*_t I) = \emptyset$ for $A = F, G$,
3. setting $\Lambda_t = i_t^!(\text{SS}(F))$ and $\Lambda'_t = i_t^!(\text{SS}(G))$, we have $\Lambda_t \cap \Lambda'_t = \emptyset$ for all $t \in I$.

Then $R\text{Hom}(i_t!^{-1}F, i_t^{-1}G)$ is independent of $t$.

**Proof.** We recall that in the non-characteristic case we have $i_t! = i_t!d(i_t!)^{-1}$ (see Example 1.2.7). We set $H = R\text{Hom}(F,G)$. Since $\Lambda := \text{SS}(F)$ and $\Lambda' := \text{SS}(G)$ are non-characteristic for $i_t$ and $\Lambda_t \cap \Lambda'_t = \emptyset$, we can see that $\Lambda \cap \Lambda' \subset T^*_{M \times I}(M \times I)$ and then $\text{SS}(H) \subset \Lambda^a + \Lambda'$. We can see moreover that $\Lambda^a + \Lambda'$ is also non-characteristic for $i_t$. 

Hence Theorem 1.2.8 gives $i_t^{-1}G \simeq i_t^*G[1]$ and $i_t^{-1}H \simeq i_t^*H[1]$. Then it follows from Proposition 1.1.1(h) that $R\Hom(i_t^{-1}F, i_t^{-1}G) \simeq i_t^{-1}H$. By the base change formula we have $R\Gamma(M; i_t^{-1}H) \simeq (Rq_I)_t$, where $q: M \times I \to I$ is the projection. Hence it is enough to check that $Rq_IH$ is locally constant, that is, $SS(Rq_IH) \subset T^*_I I$. By Proposition 1.2.4 this follows from the fact that $SS(H)$ is non-characteristic for each $i_t$. □

Remark 1.2.18. The particular case $F = k_{M \times I}$ of Corollary 1.2.17 gives the following. Let $G \in D(k_{M \times I})$. We assume that $G$ is non-characteristic for all maps $i_t: M \times \{t\} \to M \times I$, $t \in I$, and that the map $\text{supp}(G) \to I$ is proper. Then the sections $R\Gamma(M; i_t^{-1}G)$ are independent of $t$.

1.2.3. Constructibility. We say a few words about the notion of constructibility for sheaves and its relation with the microsupport.

A stratification $\Sigma = \{\Sigma_i\}_{i \in I}$ of a manifold $M$ is a partition $M = \bigcup_{i \in I} \Sigma_i$ by locally closed subsets such that the closure of each stratum is a union of strata. We will always assume that our stratifications are locally finite (any compact subset meets finitely many strata). We say that $F \in D(k_M)$ is weakly constructible with respect to $\Sigma$ if $F|_{\Sigma_i}$ is locally constant for each $i \in I$. If $k$ is a Noetherian ring, we say that $F$ is constructible with respect to $\Sigma$ if $F \in D^b(k_M)$, $F$ is weakly constructible and, for each $x \in M$ and $j \in \mathbb{Z}$, the stalk $H^j(F)_x$ is finitely generated over $k$. We recall an important property of stratifications introduced by Kashiwara and Schapira because it helps to bound the microsupport of sheaves.

Definition 1.2.19 (Def. 8.3.19 of [31]). A stratification $\Sigma = \{\Sigma_i\}_{i \in I}$ of $M$ satisfies the $\mu$-condition if the strata are locally closed submanifolds and $\Lambda_\Sigma \overset{\sim}{=} \Lambda_\Sigma = \Lambda_\Sigma$, where $\Lambda_\Sigma = \bigcup_{i \in I} T^*_\Sigma M$.

In a subanalytic framework, the $\mu$-condition implies Whitney’s conditions (a) and (b) for any two strata (see Exercise VIII.12 of [31]). In Chapter 8 of [31] it is proved that, if $M$ is real analytic, any stratification by subanalytic subsets can be refined into a subanalytic stratification satisfying the $\mu$-condition. The following proposition appears in the same chapter but the proof does not require analyticity.

Proposition 1.2.20 (Prop. 8.4.1 in [31]). Let $\Sigma$ be a stratification of $M$ satisfying the $\mu$-condition. Then $F \in D(k_M)$ is weakly constructible with respect to $\Sigma$ if and only if $SS(F) \subset \Lambda_\Sigma$.

Proof. (i) We first assume that $SS(F) \subset \Lambda_\Sigma$. Let $\Sigma_i$ be a stratum and let $U$ be a neighborhood of $\Sigma_i$ such that $\Sigma_i$ is closed in $U$. By Theorem 1.2.13 we have $SS(F_{\Sigma_i}) \cap T^*U \subset (\Lambda_\Sigma \overset{\sim}{=} T^*_\Sigma M) \cap T^*U$. Since
\[ \Lambda_\Sigma \oplus T^*_\Sigma M \subset \Lambda_\Sigma \oplus \Lambda_\Sigma = \Lambda_\Sigma \] and \( F_{|\Sigma_i} \) vanishes outside \( \Sigma_i \) we deduce \( SS(F_{|\Sigma_i}) \cap T^*_U \subset T^*_\Sigma M \cap T^*U \). It then follows from Example 1.2.5 that \( F|_{\Sigma_i} (= F_{|\Sigma_i}) \) is locally constant.

(ii) Now we assume that \( F \) is weakly constructible with respect to \( \Sigma \). Since the problem is local on \( M \) we can assume that the stratification is finite, say \( \Sigma = \{ \Sigma_1, \ldots, \Sigma_N \} \). We also assume that \( \dim(\Sigma_i) \geq \dim(\Sigma_{i+1}) \). Hence \( \Sigma_1 = \bigsqcup_{i=1}^k \Sigma_i \) is an open subset of \( M \). We prove by induction on \( k \) that \( SS(F) \cap \pi^{-1}_M(\Sigma_k) \subset \Lambda_\Sigma \cap \pi^{-1}_M(\Sigma_k) \). The case \( k = 1 \) is clear because \( F|_{\Sigma_1} \) is locally constant. Let us assume that the claim is proved for \( k \). We have the excision distinguished triangle \( F_{\Sigma_k} \to F|_{\Sigma_{k+1}} \to F_{\Sigma_{k+1}} \to \). We recall that \( F_{\Sigma_k} = j_!j^*(F) \), where \( j \) is the inclusion of \( \Sigma_k \) in \( \Sigma_{k+1} \). By Theorem 1.2.15, the induction hypothesis and the \( \mu \)-condition, we have \( SS(F_{\Sigma_k}) \cap \pi^{-1}_M(\Sigma_{k+1}) \subset \Lambda_\Sigma \cap \pi^{-1}_M(\Sigma_{k+1}) \). Since \( F|_{\Sigma_{k+1}} \) is locally constant and \( \Sigma_{k+1} \) is smooth and closed in \( \Sigma_{k+1} \) we have \( SS(F_{\Sigma_{k+1}}) \cap \pi^{-1}_M(\Sigma_{k+1}) \subset T^*_\Sigma_{k+1}M \cap \pi^{-1}_M(\Sigma_{k+1}) \). Now the result follows from the triangular inequality for the microsupport.

\[ \square \]

**Corollary 1.2.21.** Let \( \Sigma \) be a stratification of \( M \) satisfying the \( \mu \)-condition and let \( F, G \in D^b(k_M) \) be given. We assume that \( F \) and \( G \) are weakly constructible with respect to \( \Sigma \). Then \( F \overset{L}{\otimes} G \) and \( R\mathcal{H}om(F, G) \) are also weakly constructible with respect to \( \Sigma \).

**Proof.** This follows from Proposition 1.2.20 and Theorem 1.2.13 (the case of \( F \overset{L}{\otimes} G \) also follows from the fact that \( \otimes \) commutes with the inverse image and does not need the \( \mu \)-condition). \[ \square \]

**Remark 1.2.22.** (a) If \( M \) is of dimension 1, a (locally finite) stratification \( \Sigma \) is the data of a discrete set of points, say \( \Sigma^0 \), and the connected components of \( M \setminus \Sigma^0 \). Then \( F \in D^b(k_M) \) is weakly constructible with respect to \( \Sigma \) if and only if \( F|_{M \setminus \Sigma^0} \) is locally constant (indeed the condition of being locally constant at the points of \( \Sigma^0 \) is empty). Since \( \otimes \) and \( R\mathcal{H}om \) both commute with the restriction to an open subset, Corollary 1.2.21 is obvious in the case of dimension 1.

(b) More generally, let us assume that the stratification \( \Sigma \) consists of a discrete set of disjoint closed (not only locally closed) submanifolds and the connected components of their complement. We see easily that \( \Sigma \) satisfies the \( \mu \)-condition. Moreover Proposition 1.2.20 follows directly from Proposition 1.2.9.

If \( M \) is a real analytic manifold and \( \Lambda \subset \hat{T}^*M \) is an \( \mathbb{R}_{>0} \)-conic real analytic Lagrangian submanifold, Corollary 8.3.22 of [31] says that...
there exists a stratification $\Sigma$ of $M$ satisfying the $\mu$-condition such that $\Lambda \subset \Lambda_{\Sigma}$. We can deform a $C^\infty$ Lagrangian submanifold into an analytic one and obtain the next proposition. We thank Sylvain Courte for his suggestions about this result.

**Proposition 1.2.23.** Let $M$ be a manifold and $\Lambda \subset \dot{T}^* M$ an $\mathbb{R}_{>0}$-conic closed Lagrangian submanifold (both $M$ and $\Lambda$ of class $C^\infty$). Then there exists an $\mathbb{R}_{>0}$-homogeneous Hamiltonian isotopy $\Psi: \dot{T}^* M \times [0, 1] \to \dot{T}^* M$ such that $\Psi_0 = \text{id}$ and $\Psi_1(\Lambda) \subset \Lambda_{\Sigma}$, for some stratification $\Sigma$ of $M$ satisfying the $\mu$-condition. Moreover $\Psi_1(\Lambda)$ can be chosen arbitrarily close to $\Lambda$.

**Proof.** We reduce to the real analytic case, following the ideas in [37], and apply results of [31].

(i) We set $\bar{\Lambda} = \Lambda/\mathbb{R}_{>0}$ and $S^* \bar{M} = \dot{T}^* \bar{M}/\mathbb{R}_{>0}$. Then $\bar{\Lambda}$ is a Legendrian submanifold of $S^* \bar{M}$ and the contact version of the Weinstein neighborhood theorem gives a contact embedding $j: U \to S^* \bar{M}$ of class $C^\infty$, where $U$ is an open neighborhood of $\bar{\Lambda}$ in $J^1(\bar{\Lambda}) = T^* \bar{\Lambda} \times \mathbb{R}$. We denote by $\xi_{\bar{\Lambda}}$ and $\xi_M$ the standard contact structures of $J^1(\bar{\Lambda})$ and $S^* \bar{M}$ and by $\alpha_{\bar{\Lambda}}$ and $\alpha_M$ the standard contact forms.

We know by [50] that there exist real analytic structures on $\bar{\Lambda}$ and $M$ compatible with their $C^\infty$ structures. They induce real analytic structures on $J^1(\bar{\Lambda})$, $S^* \bar{M}$ and $\alpha_{\bar{\Lambda}}$, $\alpha_M$ are analytic. We know also that we can find a $C^r$ map $\tilde{j}: U \times [0, 1] \to S^* \bar{M}$, for any integer $r$, such that, setting $j^s = \tilde{j}|_{U \times \{s\}}$, we have: $j^0 = j$, $j^1$ is an analytic open embedding and the $j^s$ are as close as required to $j$ in the compact open $C^r$-topology.

On $U$ we thus have a family of contact structures $\xi^s = j^{*s}(\xi_M)$, with $\xi^0 = j^*(\xi_M) = \xi_{\bar{\Lambda}|U}$ and $\xi^1$ analytic. For each given $s$ we consider a linear interpolation between $\xi^0$ and $\xi^s$:

$$\alpha^s_t = (1 - t)\alpha_{\bar{\Lambda}} + t j^{*s}(\alpha_M), \quad \xi^s_t = \ker(\alpha^s_t).$$

We remark that the $\alpha^s$ are analytic. Choosing $j^s$ $C^r$-close to $j$, we can assume that $j^{*s}(\alpha_M)$ is $C^{r-1}$-close to $\alpha_{\bar{\Lambda}}$. By Gray’s theorem, for each $s$ we can find an isotopy $\varphi^s_t$, $t \in [0, 1]$, such that $(\varphi^s_t)^*(\xi^s_t) = \xi^0$ near $\bar{\Lambda}$ as we recall in (ii) (see for example Theorem 2.2.2 of [18]).

(ii) For a given $s \in [0, 1]$, we define the vector field $X^s_t$, $t \in [0, 1]$, on $U$ uniquely determined by

$$X^s_t \in \xi^s_t \quad \text{and} \quad X^s_t \cdot d\alpha^s_t = -\dot{\alpha}^s_t|_{\xi^s_t},$$

(recall that $d\alpha^s_t|_{\xi^s_t}$ is non-degenerate). We remark that $X^s_t$ is analytic. We also remark that $d\alpha^s_t$ is close to $d\alpha_{\bar{\Lambda}}$ and $\dot{\alpha}^s_t = j^{*s}(\alpha_M) - \alpha_{\bar{\Lambda}}$ is close
to 0. Hence $X^s_t$ is as close to 0 as required and the flow $\varphi^s_t$ of $\{X^s_t\}_{t \in [0,1]}$ is defined on a set $V \times [0,1]$ where $V$ is some neighborhood of $\Lambda$ in $U$. Moreover $\varphi^1$ is analytic.

Now we check that $(\varphi^s_t)^*(\xi^s_t) = \xi^0$, where $\varphi^s_t = \varphi^s|_{V \times \{t\}}$. By (1.2.6) we have $X^s_t\alpha^s_t + \alpha^s_t = f^s_t\alpha^s_t$ for some function $f^s_t$ and

$$\frac{d}{dt}((\varphi^s_t)^*(\alpha^s_t) \wedge \alpha^s_t) = (\varphi^s_t)^*(L_{X^s_t}(\alpha^s_t) + \dot{\alpha}^s_t) \wedge \alpha^s_t$$

$$= (\varphi^s_t)^*(f^s_t\alpha^s_t) \wedge \alpha^s_t$$

$$= (f^s_t \circ \varphi^s_t)((\varphi^s_t)^*(\alpha^s_t) \wedge \alpha^s_t).$$

Since $(\varphi^s_0)^*(\alpha^s_0) \wedge \alpha^s_0 = 0$, we obtain $(\varphi^s_t)^*(\alpha^s_t) \wedge \alpha^s_t = 0$ for all $t \in [0,1]$, which implies $(\varphi^s_t)^*(\xi^s_t) = \xi^0$.

(iii) By construction the map $\psi^s = j^s \circ \varphi^s_1: V \to S^*M$ is contact. Moreover $\psi^1$ is analytic. Hence $\tilde{\Lambda}^s = \psi^s(\Lambda)$ defines a Legendrian deformation of $\Lambda$ with $\Lambda^1$ analytic. The existence of the required stratification for $\Lambda^1$ is given by Corollary 8.3.22 of [31]. Now a Legendrian deformation can be lifted to an ambient contact isotopy, which is the same as an $\mathbb{R}_{>0}$-homogeneous Hamiltonian isotopy of $T^*M$. \hfill $\Box$

### 1.3. Sato’s microlocalization

We quickly review the definition of the specialization and microlocalization functors as introduced in [31]. We first recall the notion of normal deformation. Let $M$ be a manifold and $N$ a closed submanifold of $N$. The normal deformation of $N$ in $M$ is a manifold $\tilde{M}_N$ together with three maps

$$s: T_NM \to \tilde{M}_N, \quad p: \tilde{M}_N \to M, \quad t: \tilde{M}_N \to \mathbb{R},$$

such that

$$\begin{align*}
&s \text{ is an embedding and } \text{im}(s) = t^{-1}(0), \\
p(\text{im}(s)) = N \text{ and } p \circ s \text{ is the projection } T_NM \to N, \\
p^{-1}(M \setminus N) \simeq (M \setminus N) \times (\mathbb{R} \setminus \{0\}), \\
t^{-1}(\mathbb{R} \setminus \{0\}) \simeq M \times (\mathbb{R} \setminus \{0\}), \\
p|_{t^{-1}(u)}: t^{-1}(u) \to M \text{ is a diffeomorphism, for all } u \neq 0.
\end{align*}$$

We can define $\tilde{M}_N$ as an open subset of some blow-up as follows. The blow-up $B_{N \times \{0\}}(M \times \mathbb{R})$ is set theoretically the union of $U = (M \times \mathbb{R}) \setminus (N \times \{0\})$ and $P(T_NM \times \mathbb{R})$, the projectivization of the vector bundle $T_NM \times \mathbb{R} \to N$. We consider $(M \setminus N) \times \{0\}$ as a subset of $U$ and $P(T_NM \times \{0\})$ as a subset of $P(T_NM \times \mathbb{R})$. We set

$$\tilde{M}_N = B_{N \times \{0\}}(M \times \mathbb{R}) \setminus (((M \setminus N) \times \{0\}) \cup P(T_NM \times \{0\})).$$
Now $B_{N \times \{0\}}(M \times \mathbb{R})$ comes with a map to $M \times \mathbb{R}$ and this map induces the maps $p, t$. The difference $P(T_N M \times \mathbb{R}) \setminus P(T_N M \times \{0\})$ is identified with $T_N M$ and this gives $s$.

We set $\Omega = t^{-1}([0, +\infty[)$. We let $j: \Omega \to \tilde{M}_N$ be the inclusion and set $p_+ = p \circ j$. We have the following formula for the cone of a subset $A \subset M$ along $N$: $C_N(A) = s^{-1}(p_+^{-1}(A))$. The sheaf counterpart of the cone construction is the following specialization functor.

**Definition 1.3.1.** (See [31, Def. 4.2.2].) With the above notations we define the functor $\nu_N: \mathcal{D}(k_M) \to \mathcal{D}(k_{T_N M})$ by $\nu_N(F) = s^{-1}Rj_+p_+^{-1}(F)$.

The sheaf $\nu_N(F)$ is conic, that is, invariant by the multiplicative action of $\mathbb{R}_{>0}$ on the fibers of $T_N M$. We can deduce a sheaf over $T_N^* M$ using the Fourier-Sato transform defined as follows.

**Definition 1.3.2.** (See [31, Def. 3.7.8].) Let $q_i$ be the $i$th projection from $T_N M \times_N T_N^* M$ and let $P \subset T_N M \times_N T_N^* M$ be the subset $P = \{ (\nu; \xi); \langle \nu, \xi \rangle \leq 0 \}$. For $F \in \mathcal{D}(k_{T_N M})$ we define $F^\wedge \in \mathcal{D}(k_{T_N^* M})$ by $F^\wedge = Rq_{2!}(q_1^{-1}F \otimes k_P)$.

In [31] the Fourier-Sato transform is actually defined for general vector bundles. It is proved that it gives an equivalence between conic sheaves on a vector bundle and conic sheaves on its dual.

**Definition 1.3.3.** (See [31, Def. 4.3.1].) The microlocalization functor $\mu_N: \mathcal{D}(k_M) \to \mathcal{D}(k_{T_N^* M})$ is defined by $\mu_N(F) = (\nu_N(F))^\wedge$.

If $V \subset T_N^* M$ is a convex open cone, we have, using notation (1.1.6),

$$H^i(V; \mu_N(F)) \simeq \lim_{U, Z} H^i_{Z \cap U}(U; F),$$

where $U$ runs over the open subsets of $M$ containing $\pi_M(V)$ and $Z$ over the closed subsets of $M$ such that $C_N(Z) \subset V^\circ$ (recall that $V^\circ$ is the polar cone of $V$).

In [31] we also find a generalization of Sato’s microlocalization which will be important when we consider the Kashiwara-Schapira stack. Let $\Delta_M$ be the diagonal of $M \times M$. Let $q_1, q_2: M \times M \to M$ be the projections. We identify $T^*_{\Delta_M}(M \times M)$ with $T^* M$ through the first projection.

**Definition 1.3.4.** (See [31, Def. 4.4.1].) For $F, G \in \mathcal{D}(k_M)$ we define $\mu_{\text{hom}}(F, G) \in \mathcal{D}(k_{T^* M})$ by

$$\mu_{\text{hom}}(F, G) = \mu_{\Delta_M}(R\text{Hom}(q_2^{-1}F, q_1^1G)).$$
For a submanifold $N$ of $M$ and $i: T^*_N M \to T^* M$ the inclusion, we have $i_* \mu_N(G) \simeq \mu hom(k_N, G)$, for any $G \in D(k_M)$. The functor $\mu hom$ is a refinement of the functor $R \mathcal{H} \text{om}$ in view of the following properties:

\begin{equation}
R \pi_{M*} \mu hom(F, G) \simeq R \mathcal{H} \text{om}(F, G),
\end{equation}

\begin{equation}
R \pi_{M*} \mu hom(F, G) \simeq \delta^{-1}_M R \mathcal{H} \text{om}(q_2^{-1} F, q_1^{-1} G),
\end{equation}

where $\delta_M: M \to M \times M$ is the diagonal embedding. For a conic sheaf $H$ on $T^* M$ (or on any vector bundle) we have a natural isomorphism $R \pi_{M!}(H) \simeq R \pi_{M*} R \Gamma_M(H)$ (where $M$ is here the zero section of $T^* M$) and the natural morphism $R \pi_{M!}(H) \to R \pi_{M*}(H)$ coincides with the morphism deduced from $R \Gamma_M(H) \to H$. The excision distinguished triangle associated with the inclusion $M \subset T^* M$ then gives a version of Sato’s distinguished triangle:

\begin{equation}
\delta^{-1}_M R \mathcal{H} \text{om}(q_2^{-1} F, q_1^{-1} G) \to R \mathcal{H} \text{om}(F, G)
\end{equation}

\begin{equation}
\to R \tilde{\pi}_{M*}(\mu hom(F, G)|_{T^* M}) \overset{+1}{\to} .
\end{equation}

If $F$ is cohomologically constructible, then the first term of (1.3.4) is isomorphic to $D'(F) \otimes G$ by Theorem 1.2.13 and we obtain

\begin{equation}
D'(F) \otimes G \to R \mathcal{H} \text{om}(F, G) \to R \tilde{\pi}_{M*}(\mu hom(F, G)|_{T^* M}) \overset{+1}{\to} .
\end{equation}

**Proposition 1.3.5.** (Cor. 6.4.3 of [31]) Let $F, G \in D(k_M)$. Then

\begin{equation}
\text{supp}(\mu hom(F, G)) \subset SS(F) \cap SS(G),
\end{equation}

\begin{equation}
SS(\mu hom(F, G)) \subset (H^{-1}(C(\text{SS}(G), \text{SS}(F))))^\theta,
\end{equation}

where $H$ is the Hamiltonian isomorphism.

When $F = G$, the inclusion (1.3.6) is an equality. More precisely, by (1.3.2), $id_F \in \text{Hom}(F, F)$ gives a global section of $\mu hom$, say

\begin{equation}
id^\mu_F \in H^0(T^* M; \mu hom(F, F))
\end{equation}

and [31] Cor. 6.1.3] says that

\begin{equation}
\text{supp}(id^\mu_F) = \text{supp} \mu hom(F, F) = SS(F).
\end{equation}

An important consequence of (1.3.7) and (1.3.9) is the following involutivity theorem.

**Theorem 1.3.6** (Thm. 6.5.4 of [31]). Let $M$ be a manifold and $F \in D(k_M)$. Then $S = SS(F)$ is a coisotropic subset of $T^* M$ in the sense that $C_p(S)$ contains the symplectic orthogonal of $C_p(S, S)$, for all $p \in S$.

When $\Lambda \subset T^* M$ is a Lagrangian submanifold we have $H^{-1}(T \Lambda) = T^* \Lambda T^* M$. Hence (1.3.6), (1.3.7) and Example 1.2.5 give the following result.
Corollary 1.3.7. Let $\Lambda$ be a conic Lagrangian submanifold of $\dot{T}^*M$. Let $F, G \in D(k_M)$. We assume that there exists a neighborhood $\Omega$ of $\Lambda$ such that $SS(F) \cap \Omega \subset \Lambda$ and $SS(G) \cap \Omega \subset \Lambda$. Then $\mu_{hom}(F,G)|_{\Omega}$ is supported on $\Lambda$ and is locally constant on $\Lambda$.

By the following result we can see $\mu_{hom}$ as a microlocal version of $R\mathcal{H}om$. Let $p \in T^*M$ be a given point. By the triangular inequality the full subcategory $N_p$ of $D(k_M)$ formed by the $F$ such that $p \notin SS(F)$ is triangulated and we can set $D(k_M;p) = D(k_M)/N_p$ (see the more general Definition 6.1.1 of [31]). The functor $\mu_{hom}(:,:,p)$ induces a bifunctor on $D(k_M;p)$ and we have

Theorem 1.3.8 (Theorem 6.1.2 of [31]). For all $F, G \in D(k_M)$, the morphism $\mathcal{H}om_{D(k_M;p)}(F,G) \to H^0(\mu_{hom}(F,G))_p$ is an isomorphism.

We also give the following useful consequence of Theorem 1.3.6.

Corollary 1.3.9. Let $\Lambda$ be a conic connected Lagrangian submanifold of $\dot{T}^*M$. Let $F \in D(k_M)$ be such that $\dot{SS}(F) \neq \emptyset$ and $\dot{SS}(F) \subset \Lambda$. Then $\dot{SS}(F) = \Lambda$.

Proof. Arguing by contradiction we assume that $U = \Lambda \setminus \dot{SS}(F)$ is non empty. The set $U$ is open in $\Lambda$ with a non empty boundary $\partial U$. We choose a chart $V$ in $\Lambda$ around a point of $\partial U$ and we choose a point $p_0 \in V \cap U$. Let $B$ be the open ball in $V$ with center $p_0$ and maximal radius such that $B \cap \dot{SS}(F) = \emptyset$. Then $\partial B \cap \dot{SS}(F)$ is non empty and we let $p$ be any of its points.

Since $\dot{SS}(F) \subset \Lambda$ and $\Lambda$ is smooth, we have $C_p(\dot{SS}(F),\dot{SS}(F)) \subset C_p(\Lambda,\Lambda) = T_p\Lambda$. Since $\Lambda$ is Lagrangian, it follows that the symplectic orthogonal of $C_p(\dot{SS}(F),\dot{SS}(F))$ contains $T_p\Lambda$. On the other hand $C_p(\dot{SS}(F))$ is contained in $C_p(V \setminus B)$ which is a half space of $T_p\Lambda$. Hence $C_p(\dot{SS}(F))$ does not contain $T_p\Lambda$ and this contradicts Theorem 1.3.6. \hfill $\square$

1.4. SIMPLE SHEAVES

Let $\Lambda$ be a closed conic Lagrangian submanifold of $\dot{T}^*M$. We recall the definition of simple and pure sheaves along $\Lambda$ and give some of their properties. We first recall some notations from [31]. For a function $\varphi: M \to \mathbb{R}$ of class $C^\infty$ we define

\begin{equation}
(1.4.1) \quad \Lambda_\varphi = \{ (x; d\varphi(x)); \ x \in M \}.
\end{equation}

We notice that $\Lambda_\varphi$ is a closed Lagrangian submanifold of $T^*M$. For a given point $p = (x; \xi) \in \Lambda \cap \Lambda_\varphi$ we have the following Lagrangian
subspaces of $T_p(T^* M)$

\[(1.4.2) \quad \lambda_0(p) = T_p(T^*_x M), \quad \lambda_\Lambda(p) = T_p\Lambda, \quad \lambda_\varphi(p) = T_p\Lambda_\varphi.\]

We recall the definition of the inertia index (see for example §A.3 in [31]). Let $(E, \sigma)$ be a symplectic vector space and let $\lambda_1, \lambda_2, \lambda_3$ be three Lagrangian subspaces of $E$. We define a quadratic form $q$ on $\lambda_1 \oplus \lambda_2 \oplus \lambda_3$ by $q(x_1, x_2, x_3) = \sigma(x_1, x_2) + \sigma(x_2, x_3) + \sigma(x_3, x_1)$ and

\[(1.4.3) \quad \tau_E(\lambda_1, \lambda_2, \lambda_3) = \text{sgn}(q)\]

where $\text{sgn}(q)$ is the signature of $q$, that is, $p_+ - p_-$, where $p_\pm$ is the number of $\pm 1$ in a diagonal form of $q$. We set

$$\tau_\varphi = \tau_{p,\varphi} = \tau_{T_pT^* M}(\lambda_0(p), \lambda_\Lambda(p), \lambda_\varphi(p)).$$

**Proposition 1.4.1** (Proposition 7.5.3 of [31]). Let $\varphi_0, \varphi_1: M \rightarrow \mathbb{R}$ be functions of class $C^\infty$, let $p = (x; \xi) \in \Lambda$ and let $F \in D(\Lambda(k_M))$ (see Notation 1.2.2. We assume that $\Lambda$ and $\Lambda_{\varphi_i}$ intersect transversely at $p$, for $i = 0, 1$. Then there exists an isomorphism

$$(R\Gamma_{\{\varphi_1 \geq \varphi_0(x)\}}(F))_x \simeq (R\Gamma_{\{\varphi_0 \geq \varphi_0(x)\}}(F))_x \left[\frac{1}{2}(\tau_{\varphi_0} - \tau_{\varphi_1})\right].$$

**Definition 1.4.2** (Definition 7.5.4 of [31]). In the situation of Proposition 1.4.1 we say that $F$ is pure at $p$ if $(R\Gamma_{\{\varphi_0 \geq \varphi_0(x)\}}(F))_x$ is concentrated in a single degree and free, that is, $(R\Gamma_{\{\varphi_0 \geq \varphi_0(x)\}}(F))_x \simeq L[d]$, for some free module $L \in \text{Mod}(k)$ and $d \in \mathbb{Z}$. The half integer

\[(1.4.4) \quad d + \frac{1}{2}d_M + \frac{1}{2}\tau_{\varphi_0}\]

is called the shift of $F$. If $L \simeq k$, we say that $F$ is simple at $p$.

If $F$ is pure (resp. simple) at all points of $\Lambda$ we say that it is pure (resp. simple) along $\Lambda$.

We denote by $D^p_{\Lambda}(k_M)$ the full subcategory of $D_{\Lambda}(k_M)$ formed by the $F$ such that $F$ is simple along $SS(F)$ and the stalks of $F$ at the points of $M \setminus \hat{\pi}_M(\Lambda)$ are finitely generated.

**Proposition 1.4.3** (Cor. 7.5.7 in [31]). We assume that $\Lambda$ is connected and $F \in D_{\Lambda}(k_M)$ is pure at some $p \in \Lambda$. Then $F$ is pure along $\Lambda$. Moreover the $L \in \text{Mod}(k)$ in the above definition is the same at every point.

**Remark 1.4.4.** In Proposition 10.5.3 below we will give a more precise result than Propositions 1.4.1 and 1.4.3.

If $k$ is a field, we know that $F$ is pure along $\Lambda$ if and only if $\mu \text{hom}(F, F)|_{T^* M}$ is concentrated in degree 0 and $F$ is simple along
Example 1.4.5. (1) Let $N$ be a submanifold of $M$ of codimension $d$. Then $k_N$ is simple with shift $\frac{1}{2}d$. To see that $k_N$ is simple we can assume $M = \mathbb{R}^n$, $N = \mathbb{R}^{n-d} \times \{0\}$ and $p = (x_0; \xi_0)$ with $x_0 = 0$, $\xi_0 = (0, \ldots, 0, 1)$. We choose the function $\varphi_0(x) = x_n + \sum_{i=1}^{n-1} x_i^2$. Then $R\Gamma_{\{\varphi_0 \geq 0\}}(k_N) \simeq k_N$ and $(R\Gamma_{\{\varphi_0 \geq 0\}}(k_N))_x \simeq k$. The computation of the shift is not difficult but lengthy. We refer to Example 7.5.5 of [31].

(2) In Example 1.2.3 (iii) the sheaves $k_U$ and $k_{\bar{U}}$ are simple; $k_U$ has shift $-1/2$ and $k_{\bar{U}}$ has shift $1/2$. In fact both compare to $k_{\partial U}$, which has shift $1/2$ by (1), through the distinguished triangle $k_U \to k_{\bar{U}} \to k_{\partial U} \xrightarrow{+1}$. Choosing $\varphi_0$ with $d\varphi_0 \in \mathring{SS}(k_U)$ we deduce from the triangle $R\Gamma_{\{\varphi_0 \geq \varphi_0(0)\}}(k_U) \simeq R\Gamma_{\{\varphi_0 \geq \varphi_0(0)\}}(k_{\partial U})[-1]$. The argument is similar for $k_{\bar{U}}$. We also refer to Example 7.5.5 of [31].

(3) For $i \in \mathbb{N}$ we let $\Lambda_i \subset \Lambda$ be the set of points such that the rank of $d\pi_M|_\Lambda$ is $(\dim M - 1 - i)$. For a generic closed conic Lagrangian connected submanifold $\Lambda$ in $T^*M$, $\Lambda_0$ is an open dense subset of $\Lambda$ and, for a given simple sheaf $F \in D(\Lambda)(k_M)$, the shift of $F$ at $p$ is locally constant on $\Lambda_0$ and changes by 1 when $p$ crosses $\Lambda_1$.

These properties correspond exactly to the definition of a Maslov potential for $\Lambda$. We recall that the Maslov class of $\Lambda$ (an element of $H^1(\Lambda; \mathbb{Z}_\Lambda)$) is the obstruction to the existence of a Maslov potential for $\Lambda$. Hence, if there exists a simple sheaf $F \in D(\Lambda)(k_M)$, then the Maslov class of $\Lambda$ vanishes and a Maslov potential is given by the shift of $F$.

We can also compute the stalks of $\mu hom$ for sheaves in $D(\Lambda)(k_M)$. Let $p = (x; \xi) \in \Lambda$ and $\varphi_0: M \to \mathbb{R}$ be as in Proposition 1.4.1. For $F, G \in D(\Lambda)(k_M)$, we have

$$\mu hom(F, G)_p \simeq R\text{Hom}((R\Gamma_{Z_0}(F))_x, (R\Gamma_{Z_0}(G))_x),$$

where $Z_0 = \{\varphi_0 \geq \varphi_0(x)\}$.

Now we prove that a simple sheaf belongs to $D(\Lambda)(k_M)$ as soon as its stalk at some given point is finitely generated. We set $Z_\Lambda = \{x \in \pi_M(\Lambda); \text{there exist a neighborhood } W \text{ of } x \text{ and a smooth hypersurface } S \subset W \text{ such that } \Lambda \cap T^*W \subset T^*_S W\}$. The transversality theorem implies the following result.
Lemma 1.4.6. Let \( x, y \in M \setminus \pi_M(\Lambda) \). Let \( I \) be an open interval containing 0 and 1. Then there exists a \( C^\infty \) embedding \( c: I \to M \) such that \( c(0) = x, c(1) = y \) and \( c([0, 1]) \) only meets \( \pi_M(\Lambda) \) at points of \( Z_\Lambda \), with a transverse intersection.

Lemma 1.4.7. Let \( F \in D_{|\Lambda|}(k_M) \) be a simple sheaf along \( \Lambda \). We set \( U = M \setminus \pi_M(\Lambda) \). We assume that \( M \) is connected and that there exists \( x_0 \in U \) such that the \( k \)-module \( \bigoplus_{i \in \mathbb{Z}} H^iF_{x_0} \) is finitely generated. Then, for any \( x \in U \), the \( k \)-module \( \bigoplus_{i \in \mathbb{Z}} H^iF_x \) is finitely generated. In other words \( F \) belongs to \( D^{\text{sf}}_{|\Lambda|}(k_M) \).

Proof. Let \( x \in U \) and let \( I \) be an open interval containing 0 and 1. By Lemma 1.4.6 we can choose a \( C^\infty \) path \( \gamma: I \to M \) such that \( \gamma(0) = x_0, \gamma(1) = x \) and \( \gamma([0, 1]) \) meets \( \pi_M(\Lambda) \) at finitely many points, all contained in \( Z_\Lambda \) and with a transverse intersection. We denote these points by \( \gamma(t_i) \), where \( 0 < t_1 < \cdots < t_k < 1 \).

Since \( F \) is locally constant on \( U \), the stalk \( F_{\gamma(t)} \) is constant for \( t \in [t_i, t_{i+1}] \). Near a point \( x_i = \gamma(t_i) \) we have a hypersurface \( S \) of \( M \) such that \( \Lambda \subset T^*_SM \). Let us first assume that \( \Lambda \) is one half of \( T^*_SM \). Using Example 1.2.10 and the fact that \( F \) is simple, there exist \( d \in \mathbb{Z} \), \( E' \in D(k) \) and a distinguished triangle, in some neighborhood of \( x_i \),

\[
E'_M \to k_Z[d] \to F \xrightarrow{\delta} +1,
\]

where \( Z \) is one of the closed half-spaces bounded by \( S \). It follows that the stalks \( F_{\gamma(t_i, -\varepsilon)} \) and \( F_{\gamma(t_i, +\varepsilon)} \) differ by \( k[d] \) or \( k[d + 1] \) for \( \varepsilon > 0 \) small enough. Now we assume that \( \Lambda = T^*_SM \). The argument in Example 1.2.10 works in the same way to reduce the situation to Example 1.2.3 (v), which also works in the same way and gives a distinguished triangle \( F' \to k_Z[d] \to F \xrightarrow{\delta} +1 \). Now \( F' \) is no longer constant, but \( SS(F') = SS(F) \setminus SS(k_Z[d]) \) is half of \( T^*_SM \). So we are back to the previous case. \( \square \)

1.5. Composition of sheaves

We will use several times a usual operation associated with sheaves called “composition” or “convolution”. We refer to [31 §3.6] for more details, or [22] §1.6, §1.10].

Let \( M_i, i = 1, 2, 3 \), be three manifolds. We denote by \( q_{ij} \) the projection from \( M_1 \times M_2 \times M_3 \) to \( M_i \times M_j \). For \( K_1 \in D(k_{M_1 \times M_2}) \) and \( K_2 \in D(k_{M_2 \times M_3}) \) we denote by \( K_1 \circ K_2 \in D(k_{M_1 \times M_3}) \) the composition of \( K_1 \) and \( K_2 \):

\[
K_1 \circ K_2 = Rq_{13!}(q_{12}^{-1}K_1 \otimes q_{23}^{-1}K_2).
\]
The Fourier-Sato transform of Definition 1.3.2 is an example of composition of sheaves.

Using the base change formula we see that the composition product is associative in the sense that, for another manifold $M_4$ and $K_3 \in \mathcal{D}(k_{M_3 \times M_4})$, we have a natural isomorphism $(K_1 \circ K_2) \circ K_3 \simeq K_1 \circ (K_2 \circ K_3)$. If $M_1 = M_2$, the constant sheaf on the diagonal $K_1 = k_{\Delta_{M_1}}$ is a left unit for this product: we have $k_{\Delta_{M_1}} \circ K_2 \simeq K_2$ for any $K_2 \in \mathcal{D}(k_{M_2 \times M_3})$. Similarly, if $M_2 = M_3$, the sheaf $k_{\Delta_{M_2}}$ is a right unit.

The base change formula gives a useful expression for the stalks of the composition. For $(x, z) \in M_1 \times M_3$ we have

\[(K_1 \circ K_2)_{(x, z)} \simeq \Gamma_\nu(K_2_{\{x\} \times M_2}) \otimes \Gamma_\nu(K_2_{M_2 \times \{z\}}).
\]

For $K_1 \in \mathcal{D}(k_{M_1 \times M_2})$ we have a natural candidate for an inverse, denoted $K_1^{-1}$ and defined as follows. Let $q_2: M_1 \times M_2 \to M_2$ be the projection and $\nu: M_1 \times M_2 \cong M_2 \times M_1$ the swap isomorphism. We recall that $q_2^! (k_{M_2}) \simeq \omega_{M_2} \otimes k_{M_2}$. We define

\[K_1^{-1} = \nu^! R\text{Hom}(K_1, q_2^! (k_{M_2})) \in \mathcal{D}(k_{M_2 \times M_1}).\]

Let $\delta_2: M_2 \to M_2 \times M_2$ and $\delta_2': M_1 \times M_2 \to M_2 \times M_1 \times M_2$ be the diagonal embeddings. The base change formula $\delta_2^{-1} R\text{q}_{23}! \simeq R\text{q}_{2}! \delta_2^{-1}$ implies

\[\delta_2^{-1} (K_1^{-1} \circ K_1) \cong R\text{q}_{2}! (K_1 \otimes R\text{Hom}(K_1, q_2^! (k_{M_2}))).\]

Using the contraction $K_1 \otimes R\text{Hom}(K_1, L) \to L$ and the adjunction morphisms for $(R\text{q}_{2}, q_2^!)$ and $(\delta_2^{-1}, R\text{δ}_{2})$ we deduce the first morphism in (1.5.4) below; the second morphism is obtained in the same way.

\[K_1^{-1} \circ K_1 \to k_{\Delta_{M_2}}, \quad K_1 \circ K_1^{-1} \to k_{\Delta_{M_1}}.\]

Using the bounds given by Proposition 1.2.4 and Theorem 1.2.8 for the behaviour of the microsupport under sheaves operations we obtain the following result. We denote by $p_{ij}$ the projections from $T^*(M_1 \times M_2 \times M_3)$ similar to the $q_{ij}$. We also define $a_2$ on $T^*(M_1 \times M_2)$ by $a_2(x, y; \xi, \eta) = (x, y; \xi, -\eta)$.

**Lemma 1.5.1.** Let $K_1 \in \mathcal{D}(k_{M_1 \times M_2})$ and $K_2 \in \mathcal{D}(k_{M_2 \times M_3})$ be given. We assume that $q_{13}$ is proper on $q_{12}^{-1} \text{supp}(K_1) \cap q_{23}^{-1} \text{supp}(K_2)$ and $p_{13}^{-1} a_2^{-1} \text{SS}(K_1) \cap p_{23}^{-1} \text{SS}(K_2) \cap (T^*_M M_1 \times T^*_M M_2 \times T^*_M M_3)$ is contained in the zero-section of $T^*(M_1 \times M_2 \times M_3)$. Then

\[\text{SS}(K_1 \circ K_2) \subset \text{SS}(K_1) \circ a \text{SS}(K_2),\]

where the operation $\circ a$ for $A_1 \subset T^*(M_1 \times M_2)$ and $A_2 \subset T^*(M_2 \times M_3)$ is defined by $A_1 \circ a A_2 = p_{13}(p_{12}^{-1} a_2^{-1}(A_1) \cap p_{23}^{-1}(A_2))$. 

We will also use a relative version of the composition. For a manifold \( I \) we denote by \( q_{ij} \) the projections from \( M_1 \times M_2 \times M_3 \times I \) to \( M_i \times M_j \times I \) similar to the \( q_{ij} \). For \( K_1 \in D(k_{M_1 \times M_2 \times I}) \) and \( K_2 \in D(k_{M_2 \times M_3 \times I}) \) we set
\[
(1.5.6) \quad K_1 \circ |_IK_2 = Rq_{13!}(q_{12}^{-1}K_1 \otimes q_{23}^{-1}K_2).
\]
The definition is chosen so that \((K_1 \circ |_IK_2)|_{M_1 \times M_3 \times \{t\}} \simeq K_{1,t} \circ K_{2,t} \) for all \( t \in I \), where \( K_{1,t} = K_1|_{M_1 \times M_2 \times \{t\}} \), \( K_{2,t} = K_2|_{M_2 \times M_3 \times \{t\}} \).

The previous results generalize to the relative setting (see \[22\]). In particular we can define \( K_1^{-1} = v^{-1}R\text{Hom}(K_1, q_2(k_{M_2 \times I})) \) (an object of \( D(k_{M_2 \times M_1 \times I}) \)) and we have natural morphisms
\[
(1.5.7) \quad K_1^{-1} \circ |_IK_1 \to k_{\Delta_{M_2 \times I}}, \quad K_1 \circ |_IK_1^{-1} \to k_{\Delta_{M_1 \times I}}.
\]
We also have \( K_1^{-1}|_{M_1 \times M_2 \times \{t\}} \simeq K_{1,t}^{-1} \).

**Remark 1.5.2.** For \( K_1 \in D(k_{M_1 \times M_2}) \) the composition \( \Phi_{K_1} : F \mapsto K \circ F \) is a functor \( D(k_{M_2}) \to D(k_{M_1}) \) and we have \( \Phi_{K_1 \circ K_2} \simeq \Phi_{K_1} \circ \Phi_{K_2} \). When \( M_1 = M_2 = M \), \( \Phi_{k_{\Delta_M}} \) is the identity functor. When \( M_1 = M_2 = M_3 = M \), if \( K_1 \circ K_2 \simeq k_{\Delta_M} \), then \( K_2 \circ K_1 \simeq k_{\Delta_M} \) and \( \Phi_{K_1}, \Phi_{K_2} \) are mutually inverse equivalences of categories.

## Part 2. Sheaves associated with Hamiltonian isotopies

In this part we recall the main result of \[22\], which, following ideas of Tamarkin in \[47\], gives a sheaf version of the Chekanov-Sikorav theorem about generating functions (see \[10\] and \[45\]). Let \( \Psi : J^1(N) \times I \to J^1(N) \) be a contact isotopy of the 1-jet bundle of some manifold \( N \). The Chekanov-Sikorav theorem says that, if a Legendrian submanifold \( L \) of \( J^1(N) \) has a generating function, so does \( \Psi_s(L) \), for any \( s \in I \). The sheaf version is more functorial. We can associate a sheaf \( K_\Psi \) on \( N^2 \times I \) with \( \Psi \). This sheaf acts by composition on \( D(k_N) \) and induces equivalences of categories, \( F \mapsto (K_\Psi|_{N^2 \times \{s\}}) \circ F \). We state this result with homogeneous Hamiltonian isotopies of \( T^*N \) (which is the same thing as contact isotopies of the sphere bundle of \( T^*N \)). We recall how it implies the non homogeneous case and give some complementary remarks.

### 2.1. Homogeneous case

Let \( N \) be a manifold and \( I \) an open ball of \( \mathbb{R}^d \) containing \( 0 \) (in general \( I \) will be an open interval of \( \mathbb{R} \) containing \( 0 \)). We consider a homogeneous Hamiltonian isotopy \( \Psi : T^*N \times I \to T^*N \) of class \( C^\infty \).

For \( s \in I, p \in T^*N \) we set \( \Psi_s(p) = \Psi(p, s) \). Hence \( \Psi_0 = \text{id}_{T^*N} \) and, for
each \( s \in I \), \( \Psi_s \) is a symplectic diffeomorphism such that \( \Psi_s(x; \lambda \xi) = \lambda \cdot \Psi_s(x; \xi) \), for all \( (x; \xi) \in \hat{T}^* N \) and \( \lambda > 0 \). We let \( \Lambda_{\Psi_s} \subset \hat{T}^* N^2 \) be the twisted graph of \( \Psi_s \), that is,
\[
\Lambda_{\Psi_s} = \{ (\Psi(x, \xi, s), (x; -\xi)) ; \ (x; \xi) \in \hat{T}^* N \}.
\]

We can see that there exists a unique conic Lagrangian submanifold \( \Lambda_{\Psi} \subset \hat{T}^* (N^2 \times I) \), described in (2.1.2) below, such that \( \Lambda_{\Psi_s} = i_s^*(\Lambda_{\Psi}) \), for all \( s \in I \), where \( i_s \) is the embedding \( N^2 \times \{ s \} \to N^2 \times I \). Our homogeneous Hamiltonian isotopy \( \Psi \) is the flow of some \( h : \hat{T}^* N \times I \to \mathbb{R} \) and there is a unique such \( h \) which is homogeneous of degree 1 in the variable \( \xi \). Then we have
\[
\Lambda_{\Psi} = \{ (\Psi(x, \xi, s), (x; -\xi), (s; -h(\Psi(x, \xi, s), s))) ; \ (x; \xi) \in \hat{T}^* N, \ s \in I \}.
\]

We also remark that \( \Lambda_{\Psi} \) is non-characteristic for the inclusion \( i_s \), for any \( s \in I \). We recall that \( D^{lb}(k_M) \) is the subcategory of \( D(k_M) \) of locally bounded complexes.

**Theorem 2.1.1** (Theorem. 3.7 and Remark 3.9 of [22] – see also Proposition 3.16 of [47]). There exists a unique \( K_{\Psi} \in D^{lb}(k_{N^2 \times I}) \) such that \( SS(K_{\Psi}) \subset \Lambda_{\Psi} \) and \( K_{\Psi}|_{N^2 \times \{ 0 \}} \simeq k_{\Delta N} \). Moreover we have \( SS(K_{\Psi}) = \Lambda_{\Psi} \), \( K_{\Psi} \) is simple along \( \Lambda_{\Psi} \), both projections \( \text{supp}(K_{\Psi}) \to N \times I \) are proper and the morphisms \( 1.5.7 \) are isomorphisms:
\[
K_{\Psi}^{-1} \circ |_IK_{\Psi} \sim \simeq k_{\Delta N \times I}, \quad K_{\Psi} \circ |_IK_{\Psi}^{-1} \sim \simeq k_{\Delta N \times I}.
\]

**Remarks 2.1.2.** (1) The equivalence between \( SS(K_{\Psi}) \subset \Lambda_{\Psi} \) and \( SS(K_{\Psi}) = \Lambda_{\Psi} \) follows from Corollary 1.3.9

(2-a) The fact that \( K_{\Psi} \) is simple along \( \Lambda_{\Psi} \) is not explicitly stated in [22] (although it is used in the proof of the Arnol’d conjecture about the intersection of the zero section of a cotangent bundle which its image under a Hamiltonian isotopy). However it is easily deduced from the construction of \( K_{\Psi} \) which is obtained as a composition of constant sheaves over open subsets with smooth boundaries. Such sheaves are simple and a composition of simple sheaves is simple by [31] Thm. 7.5.11.

(2-b) We can also check the simplicity without going back to the construction of \( K_{\Psi} \) and using only the properties \( SS(K_{\Psi}) = \Lambda_{\Psi} \) and \( K_{\Psi}|_{N^2 \times \{ 0 \}} \simeq k_{\Delta N} \). Indeed, by Proposition 1.4.3 it is enough to check the simplicity at one point of \( \Lambda_{\Psi} \). The map \( i_{s,\pi} : T^* N^2 \times T^*_s I \to T^* (N^2 \times I) \) is transverse to \( \Lambda_{\Psi} \), for any \( s \in I \), and the projection \( i_{s,d} : \Lambda_{\Psi} \cap (T^* N^2 \times T^*_s I) \to \Lambda_{\Psi_s} \) is a bijection. By [31] Cor. 7.5.13 it follows that the type of \( K_{\Psi}|_{N^2 \times \{ s \}} \) at a point \( p \in \Lambda_{\Psi_s} \) is the same as
the type of $K_\Psi$ at the point $p' \in \Lambda_\Psi$ such that $i_{s,d}(p') = p$. For $s = 0$ we deduce from $K_\Psi|_{N^2 \times \{0\}} \simeq k_{\Delta_N}$ that $K_\Psi$ is simple at any point of $\Lambda_\Psi \cap (T^*N^2 \times \{0\})$.

(3) We can define the isotopy $\Psi': \dot{T}^*N \times I \to \dot{T}^*N$ by $\Psi'_s = \Psi^{-1}_s$ for all $s \in I$. Using Lemma [1.5.4] we see that $\text{SS}(K_\Psi \circ |I K_\Psi') \subset T^*_{\Delta_N \times \{1\}}(N^2 \times I)$. Since $(K_\Psi \circ |I K_\Psi')|_{N^2 \times \{0\}} \simeq k_{\Delta_N}$, Proposition [1.2.3] gives $K_\Psi \circ |I K_\Psi' \simeq k_{\Delta_N \times \{1\}}$. The uniqueness of the inverse then implies that $K_\Psi^{-1} \simeq K_\Psi'$.

(4) By Remark [1.5.2] the composition with $K_{\Psi,s} = K_\Psi|_{N \times \{s\}}$ gives an equivalence of categories $\mathcal{D}(k_N) \to \mathcal{D}(k_\Psi)$, $F \mapsto K_{\Psi,s} \circ F$. Its inverse is given by $G \mapsto (K_{\Psi,s})^{-1} \circ G$; with the notations of (3) we have $(K_{\Psi,s})^{-1} \simeq K_{\Psi',s}$.

**Remark 2.1.3.** The link with the Chekanov-Sikorav theorem is as follows. Let $M$ be a manifold and let $\Lambda \subset J^1(M)$ be a Legendrian submanifold which admits a generating function $f: M \times \mathbb{R}^d \to \mathbb{R}$ quadratic at infinity. We define the epigraph of $f$, $\Gamma^+_f \subset M \times \mathbb{R}^d \times \mathbb{R}$, by $\Gamma^+_f = \{(x,v,t); t \geq f(x,v)\}$. We let $q: M \times \mathbb{R}^d \times \mathbb{R} \to M \times \mathbb{R}$ be the projection and set $F_f = Rq(k_{\Gamma^+_f})$. Then we can check that $\text{SS}(F_f) = \Lambda$, where we identify $\Lambda$ with a conic Lagrangian submanifold of $\dot{T}^*(M \times \mathbb{R})$ (this follows directly from Proposition [1.2.3] if $q$ is proper on $\Gamma^+_f$; in general we have to check that $q|_{\Gamma^+_f}$ has a good behaviour at infinity and we need some hypotheses on $f$ – quadratic at infinity for example).

Now we assume to be given a Legendrian isotopy $\Psi$ of $J^1(M)$, that we identify with a homogeneous Hamiltonian isotopy of $\dot{T}^*(M \times \mathbb{R})$. The Chekanov-Sikorav theorem says that, up to adding extra variables, we can modify $f$ into $f_s$, such that $f_s$ is a generating function of $\Psi_s(\Lambda)$. Then we have $F_{f_s} \simeq K_{\Psi,s} \circ F_f$.

**Example 2.1.4.** The easiest illustration of Theorem [2.1.1] is the sheaf associated with the (normalized) geodesic flow of $\dot{T}^*\mathbb{R}^n$. It is Example 3.10 of [22] and we recall it briefly. We set $I = \mathbb{R}$ and define $\Psi: \dot{T}^*\mathbb{R}^n \times I \to \dot{T}^*\mathbb{R}^n$, by $\Psi_s(x;\xi) = (x + s\frac{\xi}{||\xi||};\xi)$. It is the Hamiltonian flow of $h(x;\xi) = ||\xi||$ (it is $-||\xi||$ in [22], so our example is the same up to inverting time). For $s > 0$ we define $U_s = \{(x,y) \in \mathbb{R}^{2n}; ||x - y|| < s\}$. Using Example [1.2.3] (iii) we see directly that $\Lambda_{\Psi_s} = \text{SS}(k_{U_s})$. The formula [1.5.2] gives

$$(k_{U_s} \circ k_{U_t})(x,z) \simeq R\Gamma_c(\mathbb{R}^n; k_{B(x,s)} \otimes k_{B(z,t)}) \simeq R\Gamma_c(\mathbb{R}^n; k_{B(x,s) \cap B(z,t)})$$

where $B(x,s)$ is the open ball of radius $s$ centered at $x$. The intersection $B(x,s) \cap B(z,t)$ is empty or homeomorphic to an $n$-ball and we deduce
Lemma 4.4.4 below). The composition $K_s = K_{U_s}$ is locally constant on $U_{s+t}$, hence constant since $U_{s+t}$ is contractible. It follows $k_{U_s} \circ k_{U_t} \simeq k_{U_{s+t}}[-n]$. Defining $K_s = k_{U_s}[n]$ we thus have $K_s \circ K_t \simeq K_{s+t}$ and we can expect that the sheaf associated with $\Psi$ is given on $\{s > 0\}$ by $K = k_U[n]$, where $U = \{(x, y, s) \in \mathbb{R}^{2n+1}; s > 0, ||x - y|| < s\}$.

What about negative times? For $s > 0$ we have

$$K_s^{-1} \simeq v^{-1}R\mathcal{H}om(K_s, q_2^!(\mathbb{R}^n)) \simeq v^{-1}R\mathcal{H}om(K_s, k_{\mathbb{R}^{2n}}[n]) \simeq k_{\mathbb{R}^n}$$

(see (1.5.3) for the definition of $K_s^{-1}$ and use (1.1.8)). The composition $K_s^{-1} \circ K$ is a sheaf on $\mathbb{R}^{2n} \times \mathbb{R}_{>0}$. We shift it by $s$ to the left and get a sheaf, say $L(s)$, on $\mathbb{R}^{2n} \times ]-s, +\infty[\), whose restriction to $\mathbb{R}^{2n} \times \mathbb{R}_{>0}$ is $K$. We can compute its restriction to $\mathbb{R}^{2n} \times ]-s, 0]$ and find $L(s)|_{\mathbb{R}^{2n} \times ]-s, 0]} \simeq k_{\mathbb{R}^n}$ where $i(x, y, t) = (x, y, -t)$. The microsupport of $L(s)$ is again given by Lemma 1.5.1 and thus is the graph of $\Psi$. Since $L(s)|_{\mathbb{R}^{2n} \times \{0\}} \simeq k_{\Delta_{\mathbb{R}^n}}$ we deduce that $L(s) \simeq K_\Psi$. We can take $s$ arbitrarily big and obtain

$$\begin{cases} K_\Psi|_{\mathbb{R}^{2n} \times ]0, +\infty[} \simeq k_U[n], \\ K_\Psi|_{\mathbb{R}^{2n} \times ]-\infty, 0]} \simeq k_Z, \end{cases}$$

where $U = \{(x, y, s) \in \mathbb{R}^{2n+1}; s > 0, ||x - y|| < s\}$ and $Z = \{(x, y, s) \in \mathbb{R}^{2n+1}; s \leq 0, ||x - y|| \leq -s\}$. We remark that we have a distinguished triangle

$$k_U[n] \to K_\Psi \to k_Z \xrightarrow{u} k_U[n + 1]$$

which is not split, that is, $K_\Psi \not\simeq k_Z \oplus k_U[n]$. Indeed this would imply $SS(K_\Psi) = SS(k_U) \cup SS(k_Z)$. But Example 1.2.3 (iv) says that $SS(k_Z)$ is bigger than the graph of $\Psi$ at the points $T^*\mathbb{R}^{2n+1}$ above $\Delta_{\mathbb{R}^n} \times \{0\}$. In particular the morphism $u$ in (2.1.4) is non zero. The computation

$$R\mathcal{H}om(k_Z, k_U) \simeq R\mathcal{H}om(k_Z, D'(k_T)) \simeq R\mathcal{H}om(k_Z \otimes k_T, k_{\mathbb{R}^{2n+1}}) \simeq R\mathcal{H}om(k_{\Delta_{\mathbb{R}^n} \times \{0\}}, k_{\mathbb{R}^{2n+1}}) \simeq k_{\Delta_{\mathbb{R}^n} \times \{0\}}[-n - 1]$$

shows that Hom($k_Z, k_{U[n + 1]}$) $\simeq k$. Hence, if $k$ is a field, we have only one non trivial distinguished triangle like (2.1.4) up to isomorphism (see Lemma 4.4.4 below).

For a conic subset $A$ of $T^*N$ we let $D_{[A]}(k_N)$ be the full subcategory of $D(k_N)$ formed by the $F$ with $SS(F) \subset A$ (see Notation 1.2.2). Using
Remark 2.1.7.\] The notation $\circ^a$ of \[1.3.3\] we define $A' \subset \dot{T}^*(N \times I)$ by $A' = \Lambda_{\Psi} \circ^a A$. More explicitly we have

\[
A' = \{(\Psi(x, \xi, s), (s; -h(\Psi(x, \xi, s), s)); (x; \xi) \in A, s \in I\}.
\]

We see that $A'$ is non-characteristic for the inclusion $i_s$, for any $s \in I$. Moreover $i^2_s(A') = \Psi_s(A)$. We obtain an inverse image

\[
i^1_s : D|_{A'}(k_{N \times I}) \to D|_{\Psi_s(A)}(k_N).
\]

We can deduce from Theorem 2.1.1 that it is an equivalence (see [22, §3.4]), with an inverse induced by $K_{\Psi}$ (however the categories involved in (2.1.6) and the functor $i^{-1}_s$ only depend on the sets $\Psi_s(A), A'$, not on $\Psi$ itself):

**Corollary 2.1.5.** (i) For any $s \in I$ the composition $F \mapsto K_{\Psi,s} \circ F$ induces an equivalence of categories $D|_{A'}(k_N) \cong D|_{\Psi_s(A)}(k_N)$, where $K_{\Psi,s} = K_{\Psi}|_{N \times \{s\}}$.

(ii) For any $s \in I$ the inverse image functor (2.1.6) is an equivalence of categories, with an inverse given by $F \mapsto K_{\Psi} \circ K_{\Psi,s}^{-1} \circ F$. In particular, for any $F, G \in D(k_N)$, we have isomorphisms

\[
\text{Hom}(F, G) \cong \text{Hom}(K_{\Psi} \circ F, K_{\Psi} \circ G)
\]

\[
\cong \text{Hom}(K_{\Psi,s} \circ F, K_{\Psi,s} \circ G)
\]

and $R\Gamma(N; G) \cong R\Gamma(N \times I; K_{\Psi} \circ G) \cong R\Gamma(N; K_{\Psi,s} \circ G)$.

**Remark 2.1.6.** The functor (2.1.6) and Corollary 2.1.5 do not depend on the whole isotopy $\Psi$ but only on the deformation $\Psi_s(A), s \in I$, of $A$. It is only required that this deformation is given by some Hamiltonian isotopy.

**Remark 2.1.7.** Instead of composing with $K_{\Psi}$ on the left in Corollary 2.1.5 we can compose on the right. We can also add parameters, that is, consider $F, G$ on a product $N \times P$ or $P \times N$ for some other manifold $P$. We only quote the following formulas for later use: let $F, G \in D(k_{N \times P}), F', G' \in D(k_{P \times N})$ be given; then the restriction at time $0 \in I$ gives isomorphisms

\[
\text{Hom}_{D(k_{N \times P})}(K_{\Psi} \circ F, K_{\Psi} \circ G) \cong \text{Hom}_{D(k_{N \times P})}(F, G),
\]

\[
\text{Hom}_{D(k_{P \times N})}(F' \circ K_{\Psi}, G' \circ K_{\Psi}) \cong \text{Hom}_{D(k_{P \times N})}(F', G').
\]

### 2.2. Local behaviour

We check here that the restriction of $K_{\Psi,s} \circ F$ to an open subset $V$ of $N$ only depends on $F|_U$ for some bigger open subset $U$. 
Lemma 2.2.1. Let $U$, $V$ be two open subsets of $N$ and let $s \in I$ be given. We assume that $\Psi_t^{-1}(\tilde{T}^*V) \subset \tilde{T}^*U$ for all $t \in [0,s]$. Then, for any $F \in D(k_N)$, the morphism $F_U \rightarrow F$ induces an isomorphism

$$(K_{\Psi} \circ F_U)_{|V \times [0,s]} \simeq (K_{\Psi} \circ F)_{|V \times [0,s]}.$$  

In particular $(K_{\Psi,s} \circ F_U)|_V \simeq (K_{\Psi,s} \circ F)|_V$.

Proof. Let us set $G = K_{\Psi} \circ F_N\setminus U \in D(k_N \setminus I)$ and $G_t = G|_{N \times \{t\}}$. We have a distinguished triangle $K_{\Psi} \circ F_U \rightarrow K_{\Psi} \circ F \rightarrow G \rightarrow +1$ and the assertion of the lemma is equivalent to $G|_{V \times [0,s]} \simeq 0$. Since $G_0|_V \simeq 0$ it is enough to see that $G|_{V \times [0,s]}$ is locally constant.

We first remark that $G_t|_V$ is locally constant for each $t \in [0,s]$. Indeed $SS(G_t) = \Psi_t(SS(F_{N \setminus U})) \subset \Psi_t(\tilde{T}^*N \setminus \tilde{T}^*U)$ and the hypothesis $\tilde{T}^*V \subset \Psi_t(\tilde{T}^*U)$ implies $SS(G_t) \cap \tilde{T}^*V = \emptyset$. It follows that $G$ is locally on $V \times [0,s]$ of the form $G \simeq q^{-1}G'$ for some $G' \in D(k_I)$, where $q : N \times I \rightarrow I$ is the projection. Hence $SS(G|_{V \times [0,s]}) \subset T^*_N N \times \tilde{T}^*I$. On the other hand $G$ is non-characteristic for the inclusion of $N \times \{t\}$ in $N \times I$, for any $t \in I$. Hence $SS(G) = \emptyset$ as required.

Proposition 2.2.2. Let $F, G \in D(k_N)$. The restriction morphism $i_0^{-1}R\text{Hom}(K_{\Psi} \circ F, K_{\Psi} \circ G) \rightarrow R\text{Hom}(F,G)$

is an isomorphism.

Using the bound $A'$ of Corollary 1.2.17 with $A = \tilde{T}^*N$ we could argue as in the proof of Corollary 1.2.17. The main step is then to check that $(A')^a + A'$ is non-characteristic for $i_0$; here is another proof avoiding this computation.

Proof. We set $F' = K_{\Psi} \circ F$, $G' = K_{\Psi} \circ G$. It is enough to check that the restriction morphism induces an isomorphism in each degree of the stalks at any point $x \in N$. We recall that $H^0(U; R\text{Hom}(F,G)) \simeq \text{Hom}(F|_U, G|_U) \simeq \text{Hom}(F_U, G)$. Hence

$$(H^kR\text{Hom}(F,G))_x \simeq \lim_{x \in U} \text{Hom}(F|_U, G[k]|_U),$$

(2.2.1) 

$$(H^kR\text{Hom}(F', G'))_{(x,0)} \simeq \lim_{(x,0) \in U \times J} \text{Hom}(F'|_{U \times J}, G'[k]|_{U \times J}),$$

where $U$ runs over the open neighborhoods of $x \in N$ and $J$ over the neighborhoods of $0 \in I$. For such $U$ and $J$, with $J$ contractible, we have $\text{Hom}(K_{\Psi} \circ F_U|_{N \times J}, G'[k]|_{N \times J}) \simeq \text{Hom}(F_U, G[k])$ by (2.1.7). If $V \subset U$ satisfies $\Psi_t^{-1}(\tilde{T}^*V) \subset \tilde{T}^*U$ for all $t \in J$, then $K_{\Psi} \circ F_U|_{V \times J} \simeq K_{\Psi} \circ F|_{V \times J}$
by Lemma 2.2.1. We deduce a natural morphism \( \text{Hom}(F_U, G[k]) \rightarrow \text{Hom}(F'_V, G'[k]) \) which gives a commutative diagram

\[
\begin{array}{ccc}
\text{Hom}(F'_V, G'[k]) & \rightarrow & \text{Hom}(F_V, G[k]) \\
\downarrow & & \downarrow \\
\text{Hom}(F'_V, G'[k]) & \rightarrow & \text{Hom}(F_V, G[k]). \\
\end{array}
\]

It follows that the two limits in (2.2.1) are isomorphic, which proves the proposition.

Let \( A \) be a conic subset of \( \tilde{T}^*N \) and let \( A' \subset T^*(N \times I) \) be as in (2.1.5). Let \( U \) be an open subset of \( N \). We would like to find a neighborhood \( V \) of \( U \) in \( N \times I \) such that the inverse image by the inclusion of \( U \) in \( V \) is an equivalence of categories \( D_{[A' \cap T^*V]}(k_V) \simeq D_{[A \cap T^*V]}(k_V) \). We do not know if such a \( V \) always exists and give a weaker statement which will be sufficient for our purposes.

**Proposition 2.2.3.** Let \( A \subset \tilde{T}^*N \) and \( A' \subset T^*(N \times I) \) be as in (2.1.5). Let \( j: U \rightarrow N \) be the inclusion of an open subset.

(i) Let \( F \in D_{[A' \cap T^*V]}(k_U) \). Then there exist a neighborhood \( V \) of \( U \times \{0\} \) in \( N \times I \) and \( G \in D_{[A \cap T^*V]}(k_V) \) such that \( F \sim G|_{U \times \{0\}} \).

(ii) Let \( V \) be a neighborhood of \( U \times \{0\} \) in \( N \times I \) and \( G \in D_{[A \cap T^*V]}(k_V) \). Then there exists a smaller neighborhood \( V' \) of \( U \times \{0\} \) such that \( G|_{V'} \sim (K_{\Psi} \circ F)|_{V'} \), where \( F = R_j(G|_{U \times \{0\}}) \). In particular, for \( G, G' \in D_{[A' \cap T^*V]}(k_V) \) the inverse image by the inclusion gives an isomorphism

\[
\mathbf{RHom}(G, G')|_{U \times \{0\}} \xrightarrow{\sim} \mathbf{RHom}(G|_{U \times \{0\}}, G'|_{U \times \{0\}})
\]

and, if \( G|_{U \times \{0\}} \sim G'|_{U \times \{0\}} \), then there exists a smaller neighborhood \( V'' \) of \( U \times \{0\} \) such that \( G|_{V''} \sim G'|_{V''} \).

**Proof.** (i) We set \( F' = R_jF \) and \( G' = K_{\Psi} \circ F' \). We have the rough bounds \( \hat{S}(F') \subset A \cup \pi_N^{-1}(\partial U) \) and \( \hat{S}(G') \subset A' \cup \hat{\pi}_R^{-1}(Z) \) where \( Z = \bigsqcup_{s \in I} Z_s \times \{s\} \) and \( Z_s = \pi_N(\Psi_s(\hat{\pi}_R^{-1}(\partial U))) \). Then \( V = N \times R \setminus Z \) is a neighborhood of \( U \times \{0\} \) and \( G = G'|_{V} \) has the required property.

(ii) We let \( k: V \rightarrow N \times I \) be the inclusion and set \( G_1 = R_k G \) and \( H = K_{\Psi}^{-1} \circ |_I G_1 \). Then \( H \in D(k_{N \times I}) \) and \( H_s \simeq K_{\Psi,s}^{-1} \circ (G_1|_{N \times \{s\}}) \). Using the notation \( A_{\Psi} \circ \overset{\circ}{\partial} \) (see (2.1.5)) we have \( \hat{S}(H) \subset A_{\Psi} \circ \overset{\circ}{\partial} |_I \hat{S}(G_1) \). Then

\[
\hat{S}(H) \subset A_{\Psi} \circ \overset{\circ}{\partial} |_I \hat{S}(G_1)
\]

\[
\subset A_{\Psi} \circ \overset{\circ}{\partial} |_I (A' \cup \pi_N^{-1}(\partial V)) \subset (A \times T^*_I I) \cup B,
\]

\[
\overset{\circ}{\partial} \]
where $B = \Lambda_{\Phi^{-1} \circ |_\pi^{-1} \partial W}$. We remark that $W = (N \times I) \setminus \pi(B)$ is a neighborhood of $U \times \{0\}$. We let $V_1 \subset V \cap W \cap (U \times I)$ be a smaller neighborhood such that the fibers of the projection $p: V_1 \to U$ are intervals. Then $SS(H|_{V_1}) \subset A \times T^*_I I$ and Proposition 1.2.9 implies that $H|_{V_1} \simeq p^{-1}(F)$ for some $F \in D(k_U)$. Hence $K^{-1}_\psi \circ |_I Rk_i G$ is isomorphic to $Rr_j F \boxtimes k_f$ on some neighborhood of $U$ in $N \times I$. It follows easily that $Rk_i G$ is isomorphic to $K_\psi \circ |_I (Rr_j F \boxtimes k_f) \simeq K_\psi \circ Rr_j F$ on some smaller neighborhood $V'$. In the same way $G'$ can be written $G'|_{V'} \simeq (K_\psi \circ F')|_{V'}$, maybe up to shrinking $V'$. Now the last assertions of the proposition follow from Proposition 2.2.2 and Corollary 2.1.5. \qed

2.3. Non homogeneous case

Now we consider the case of a non homogeneous Hamiltonian isotopy on the cotangent bundle $T^* M$ of some manifold $M$. We reduce this case to the homogeneous framework by a common trick of adding one variable. Let $(x; \xi)$ and $(t; \tau)$ be the coordinates on $T^* M$ and $T^* \mathbb{R}$. We define $\rho_M: T^* M \times T^* \mathbb{R} \to T^* M$ by

\[(2.3.1) \quad \rho_M(x; t; \xi; \tau) = (x; \xi/\tau).\]

The fibers of $\rho_M$ have dimension 2. They are stable by the translation in the $t$ variable and are conic, that is, stable by the action of $\mathbb{R}_{>0}$ on the fibers of $T^*(M \times \mathbb{R})$. Let $\Phi: T^* M \times I \to T^* M$ be a Hamiltonian isotopy of class $C^\infty$. We assume that $\Phi$ has compact support, that is, there exists a compact subset $C \subset T^* M$ such that $\Phi(p, s) = p$ for all $p \in T^* M \setminus C$ and all $s \in I$. Then $\Phi$ is the Hamiltonian flow of a function $h: T^* M \times I \to \mathbb{R}$ such that $h$ is locally constant outside $C \times I$. In particular, if $M$ does not have a connected component diffeomorphic to the circle $S^1$, then any Hamiltonian isotopy on $T^* M$ with compact support can be defined by a Hamiltonian function $h$ with compact support.

**Proposition 2.3.1** (See Prop. A.6 of [22]). Let $\Phi: T^* M \times I \to T^* M$ be a Hamiltonian isotopy with compact support and let $h: T^* M \times I \to \mathbb{R}$ be a function with Hamiltonian flow $\Phi$.

(i) Let $h': T^*(M \times \mathbb{R}) \times I \to \mathbb{R}$ be the Hamiltonian function given by $h'(x; (t; \xi, \tau), s) = \tau h((x; \xi/\tau), s)$. Then the flow $\Phi'$ of $h'$ is a homogeneous Hamiltonian isotopy $\Phi': T^*(M \times \mathbb{R}) \times I \to T^*(M \times \mathbb{R})$ whose
restriction to $T^*M \times \hat{T}^*\mathbb{R} \times I$ gives the commutative diagram
\[
\begin{array}{ccc}
T^*M \times \hat{T}^*\mathbb{R} \times I & \xrightarrow{\Phi'} & T^*M \times \hat{T}^*\mathbb{R} \\
\downarrow \rho_M \times \text{id}_I & & \downarrow \rho_M \\
T^*M \times I & \xrightarrow{\Phi} & T^*M
\end{array}
\]
(2.3.2)

(ii) The isotopy $\Phi'$ preserves the subset $\{\tau > 0\}$ of $\hat{T}^*(M \times \mathbb{R})$ and commutes with the vertical translations $T_c$ of $\hat{T}^*(M \times \mathbb{R})$ given by $T_c(x, t; \xi, \tau) = (x, t + c; \xi, \tau)$, for all $c \in \mathbb{R}$. Defining $q: (M \times \mathbb{R})^2 \times I \to M^2 \times \mathbb{R} \times I$, $(x, t, x', t', s) \mapsto (x, x', t - t', s)$, the graph $\Lambda_{\Phi'}$ defined in (2.1.2) satisfies $\Lambda_{\Phi'} = q_d q_\pi^{-1}(\Lambda'_h)$, where $\Lambda'_h \subset \hat{T}^*(M^2 \times \mathbb{R} \times I)$ is given by $\Lambda'_h = q_\pi q_d^{-1}(\Lambda_{\Phi'})$.

Proof. (i) is Prop. A.6 of [22].

(ii) Since $h'$ does not depend on $t$, the isotopy $\Phi'$ commutes with the Hamiltonian flow of $\tau$ which is $T_c$. The flow $\Phi'$ also preserves the variable $\tau$, that is, $\Lambda_{\Phi'}$ is contained in $\Sigma := \{\tau + \tau' = 0\}$. Then $\Sigma = \text{im } q_d$ and the quotient map to the symplectic reduction of $\Sigma$ is $q_\pi$. Hence we can write $\Lambda_{\Phi'} = q_d q_\pi^{-1}(\Lambda'_h)$, where $\Lambda'_h = q_\pi q_d^{-1}(\Lambda_{\Phi'})$.

Corollary 2.3.2. Let $\Phi: T^*M \times I \to T^*M$ be a Hamiltonian isotopy with compact support and let $h: T^*M \times I \to \mathbb{R}$ be a function with Hamiltonian flow $\Phi$. Then there exists a unique $K \in \mathbf{D}^b(k_{M^2 \times \mathbb{R} \times I})$ such that $\text{SS}(K) = \Lambda'_h$ and $[K]_{M^2 \times \mathbb{R} \times \{0\}} \cong k_{\Delta M \times \{0\}}$, where $\Lambda'_h$ is defined in Proposition 2.3.1.

Proof. We use the notations of Proposition 2.3.1. By Proposition 2.2.9 the inverse image functor $q^{-1}$ gives an equivalence between $\{K' \in \mathbf{D}^b(k_{M^2 \times \mathbb{R} \times I}) ; \text{SS}(K') = \Lambda'_h\}$ and $\{K \in \mathbf{D}^b(k_{(M \times \mathbb{R})^2 \times I}) ; \text{SS}(K) = \Lambda_{\Phi'}\}$. Then the existence and uniqueness of $K$ follows from Theorem 2.1.1.

Part 3. Cut-off lemmas

In this part we recall several results of [31] which are called “(dual) cut-off lemmas”. We also give another version which removes some convexity hypotheses. We apply these results to decompose a sheaf with respect to a partition of its microsupport (see Proposition 3.3.2 below). The starting point is the following problem: for a given sheaf $F \in \mathbf{D}(k_V)$, with $V = \mathbb{R}^n$, and an open cone $C \subset T_0^*V = V^*$, find a sheaf $G$ such that $\text{SS}(G) = \text{SS}(F) \setminus (\text{SS}(F) \cap C)$ and $F \simeq G$ in the quotient category of $\mathbf{D}(k_V)$ by $\{H; (\text{SS}(H) \cap T_0^*V) \subset \overline{C}\}$. As we have seen in Corollary 1.3.9 the involutivity theorem 1.3.6 says this
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problem has no solution in general. However, for a given neighborhood \( \Omega \) of \( SS(F) \cap \partial C \) in \( T^*V \), it is possible to find \( G \) with \( SS(G) = (SS(F) \setminus (SS(F) \cap C)) \cup \Omega \) near \( T^*_0V \) (see Proposition 3.3.1 below). We first discuss a weaker version of this problem where we don’t ask that \( SS(G) = SS(F) \setminus (SS(F) \cap C) \) but we only ask for a “cut-off” functor \( F \mapsto P(F) \) so that \( SS(P(F)) \) is contained in \( V \times (V^* \setminus C) \). There are several ways to define such a \( P \) and we give four variations on the cut-off functors of [31] in (3.1.1). If we want \( P \) to be a projector (see Remark 3.1.9) these are in fact the only possible choices (see §3.5 for a quick discussion).

One version of the cut-off lemma says that the category of sheaves on \( V \) with microsupport in some convex cone \( \gamma \) is equivalent to the category of sheaves on \( V \) endowed with another topology. We use it to prove that \( SS(H^i(F)) \) is contained in the convex hull of \( SS(F) \) (see Corollary 3.4.3 below – this will be used to construct a graph selector in Part 5).

In the last paragraph we consider a relative version of the cut-off functor and recall some of its properties already considered by Tamarkin which we will use to recover non-squeezing results in §6.2.

3.1. Global cut-off

Let \( V \) be a vector space of dimension \( n \) and let \( \gamma \subset V \) be a closed convex cone (with vertex at 0). We denote by \( \gamma^a = -\gamma \) its opposite cone and by \( \gamma^c \subset V^* \) its polar cone (see (1.2.2)). We also define \( \tilde{\gamma} = \{(x, y) \in V^2; x - y \in \gamma \} \). Let \( q_i: V^2 \to V, i = 1, 2, \) be the projection to the \( i^{th} \) factor and let \( \Delta_V \) be the diagonal of \( V^2 \). The following functors are introduced in [31]:

\[
\begin{align*}
P_\gamma: D(k_V) & \to D(k_V), \quad F \mapsto Rq_{2*}(k_{\gamma^a} \otimes q_1^{-1}F), \\
Q_\gamma: D(k_V) & \to D(k_V), \quad F \mapsto Rq_{2!}(R\mathcal{H}om(k_{\gamma^a}, q_1^*F)), \\
P'_\gamma: D(k_V) & \to D(k_V), \quad F \mapsto Rq_{2!}(R\mathcal{H}om(k_{\gamma^a \setminus \Delta_V}[1], q_1^*F)), \\
Q'_\gamma: D(k_V) & \to D(k_V), \quad F \mapsto Rq_{2*}(k_{\gamma^a \setminus \Delta_V}[1] \otimes q_1^{-1}F).
\end{align*}
\]

We will mainly use the first three functors and introduce \( Q'_\gamma \) because these functors come in pairs (adjoint pairs or pairs of projectors – see §3.5). We will see the effect of these functors on the microsupport in (3.1.8) and Propositions 3.1.7, 3.1.8 and 3.1.10 below. For \( \gamma = \{0\} \), we have \( \tilde{\Delta_V} = \Delta_V \) and \( P_{\{0\}}(F) \simeq Q_{\{0\}}(F) \simeq F \). Using the distinguished triangle \( k_\gamma \to k_{\Delta_V} \to k_{\gamma^a \setminus \Delta_V}[1] \xrightarrow{+1} \) (and the same with
\( \gamma^a \) we obtain morphisms of functors
\[
\begin{align*}
  u_\gamma &: \gamma \to \id, & v_\gamma &: \id \to Q_\gamma, \\
  u'_\gamma &: P'_\gamma \to \id, & v'_\gamma &: \id \to Q'_\gamma
\end{align*}
\]
and, for any \( F \in D(k_V) \), the distinguished triangles
\[
\begin{align*}
P_\gamma(F) &\xrightarrow{u_\gamma(F)} F \xrightarrow{v'_\gamma(F)} Q'_\gamma(F) \xrightarrow{+1} , \\
P'_\gamma(F) &\xrightarrow{u'_\gamma(F)} F \xrightarrow{v_\gamma(F)} Q_\gamma(F) \xrightarrow{+1} .
\end{align*}
\]

The functors \( P_\gamma, Q_\gamma, P'_\gamma, Q'_\gamma \) have similarities with the composition operation \((1.5.1)\). Let us recall the composition: for three manifolds \( X, Y, Z \) and \( K \in D(k_{X \times Y}), L \in D(k_{Y \times Z}) \)

\[
K \circ L = Rq_{13!}(q_{12}^{-1}K \otimes q_{23}^{-1}L),
\]

where \( q_{ij} \) is the projection from \( X \times Y \times Z \) to the \( i \times j \)-factor. Taking \( X = \{pt\} \) and \( Y = Z = V \), we have, with the notations of \((3.1.1)\), \( F \circ k_\gamma = Rq_{21}(k_\gamma \otimes q^{-1}_1 F) \), which is almost \( P_\gamma(F) \) up to the change from \( Rq_{21} \) to \( Rq_{2*} \). The following lemma gives hypothesis on \( F \) so that both functors are the same.

**Lemma 3.1.1.** Let \( F \in D(k_V) \). We assume that supp(\( F \)) is compact or that there exists a linear function \( \xi : V \to \mathbb{R} \) such that \( \gamma \setminus \{0\} \subset \xi^{-1}([-\infty, 0]) \) and supp(\( F \)) \( \subset \xi^{-1}([a, \infty]) \), for some \( a \in \mathbb{R} \). Then \( P_\gamma(F) \simeq F \circ k_\gamma \) and supp(\( P_\gamma(F) \)) \( \subset \xi^{-1}([a, \infty]) \).

**Proof.** As was already remarked, the difference between \( P_\gamma(F) \) and \( F \circ k_\gamma \) is the change from \( Rq_{2*} \) to \( Rq_{2!} \) in \((3.1.1)\). Hence it is enough to prove that the projection \( q_2 \) is proper on \( S = \text{supp}(k_\gamma \otimes q^{-1}_1 F) \) to deduce the isomorphism. This is clear when \( \text{supp}(F) \) is compact. We assume the other hypothesis.

Let \( \| \cdot \| \) be some Euclidean norm on \( V \). Since \( \gamma \) is a closed cone, we actually have \( \gamma \subset \{ v \in V; \xi(v) \leq -c\|v\| \} \) for some \( c > 0 \). Hence \( S \subset \{ (v_1, v_2); \xi(v_1 - v_2) \leq -c\|v_1 - v_2\|, \xi(v_1) \geq a \} \). For a given \( v_2 \in V \) we see that \( S \cap q_2^{-1}\{v_2\} \subset \{ v_1; \|v_1 - v_2\| \leq c^{-1}(\xi(v_2) - a) \} \) is compact. Hence \( P_\gamma(F) \simeq F \circ k_\gamma \).

Moreover, if \( (P_\gamma(F))_{v_2} \neq 0 \), we must have \( S \cap q_2^{-1}\{v_2\} \neq \emptyset \), hence \( \xi(v_2) - a \geq 0 \). It follows that \( \text{supp}(P_\gamma(F)) \subset \xi^{-1}([a, \infty]) \). \( \Box \)

We can also reformulate \( Q_\gamma \) and \( P'_\gamma \) with the composition. Since \( \tilde{\gamma}^a = d^{-1}(\gamma^a) \) with \( d(x, y) = x - y \), we have \( k_{\tilde{\gamma}^a} \simeq d^{-1}(k_{\gamma^a}) \) and Theorem \((1.2.8)\) gives \( SS(k_{\tilde{\gamma}^a}) \subset \{ (x, y; \xi, -\xi) \} \). We also have \( SS(q^1_1 F) \subset \{ (x, y; \xi, 0) \} \). Hence the microsupports of \( k_{\tilde{\gamma}^a} \) and \( q^1_1 F \) do not intersect outside the
Examples 3.1.3. (i) The easiest example is the image of the skyscraper
sheaf \( V \) of \( \text{Int}(\gamma) \). We conclude with the remark that the automorphism \((x, y) \mapsto (y, x)\) of \( V^2 \) interchanges \( q_1 \), \( q_2 \) and \( \tilde{\gamma}^a \), \( \tilde{\gamma} \). □

We deduce the following adjunction properties:

**Lemma 3.1.2.** For any \( F, G \in D(k_V) \) we have \( \text{Hom}(Q_\gamma(F), G) \simeq \text{Hom}(F, P_\gamma(G)) \) and \( \text{Hom}(P_\gamma'(F), G) \simeq \text{Hom}(F, Q_\gamma'(G)) \).

**Proof.** We only prove the first isomorphism, the second one being similar. Using (3.1.5) we see \( Q_\gamma \) as a composition of left adjoint functors and we have

\[
\text{Hom}(Q_\gamma(F), G) \simeq \text{Hom}(Rq_2((D'(k_{\gamma^a}[n]) \otimes q_1^{-1}(F)), G) \\
\quad \simeq \text{Hom}(F, Rq_1*(R\text{Hom}((D'(k_{\gamma^a}[n]), q_2^1 G))).
\]

By a microsupport argument similar to the proof of (3.1.4) we obtain

\[
D'(D'(k_{\gamma^a}[n]) \otimes q_2^1 G \simeq R\text{Hom}((D'(k_{\gamma^a}[n]), q_2^1 G). Since \( k_{\gamma^a} \) is cohomologically constructible, \( D'(D'(k_{\gamma^a})) \simeq k_{\gamma^a} \) and we obtain

\[
\text{Hom}(Q_\gamma(F), G) \simeq \text{Hom}(F, Rq_1*(k_{\gamma^a} \otimes q_2^{-1} G)).
\]

We conclude with the remark that the automorphism \((x, y) \mapsto (y, x)\) of \( V^2 \) interchanges \( q_1, q_2 \) and \( \tilde{\gamma}^a, \tilde{\gamma} \). □

**Examples 3.1.3.** (i) The easiest example is the image of the skyscraper sheaf \( k_{\{x\}} \) for a given \( x \in V \). Using Lemma 3.1.1 and the formula (1.5.2) for the stalks of a composition, we find \( P_\gamma(k_{\{x\}}) \simeq k_{x-\gamma} \). If \( \text{Int}(\gamma) \neq \emptyset \), we have \( D'(k_{\gamma^a}) \simeq k_{\text{Int}(\gamma^a)} \) by (1.1.8). Hence (3.1.5) gives \( Q_\gamma(F) \simeq F \circ k_{\text{Int}(\gamma^a)}[n] \). By (1.5.2) again \( Q_\gamma(k_{\{x\}}) \simeq k_{\text{Int}(x+\gamma)}[n] \). We illustrate these results in Fig. 3.1.1 (for a sheaf of the type \( L_Z \), \( Z \) locally closed, we draw the set \( Z \) and label it with \( L \)).

The morphism \( v_\gamma(k_{\{x\}}) \) is given by the inverse image of 1 in the sequence of isomorphisms:

\[
\text{Hom}(k_{\{x\}}, k_{\text{Int}(x+\gamma)}[n]) \simeq \text{Hom}(k_{\{x\}}, D(k_{x+\gamma})) \\
\quad \simeq \text{Hom}(k_{\{x\}} \otimes k_{x+\gamma}, k_V[n]) \simeq H_{\{x\}}^0(k_V) \simeq k.
\]
(ii) Here are other easy examples in dimension 1. We set $\gamma = [−∞, 0]$.

| $F$ | $P_γ(F)$ | $Q_γ(F)$ | $P'_γ(F)$ | $Q'_γ(F)$ |
|-----|----------|----------|-----------|-----------|
| $k_{[0,1]}$ | $k_{[−∞,1]}$ | $(k_{−∞,0})[1]$ | $k_{−∞,1}$ | $(k_{1,∞})[1]$ |
| $k_{[0,1]}$ | $k_{[0,1]}$ | $k_{[0,1]}$ | $0$ | $0$ |
| $k_{[0,1]}$ | $0$ | $0$ | $k_{[0,1]}$ | $k_{[0,1]}$ |
| $k_{[0,1]}$ | $(k_{1,∞})[−1]$ | $k_{−∞,1}$ | $(k_{−∞,0})[−1]$ | $k_{[0,1]}$ |

It is easy to check here the properties stated in Proposition 3.1.10: if $(t; \tau)$ are the coordinates on $T^*\mathbb{R}$ we have $SS(P_γ(F)) = SS(F) \cap \{ \tau > 0 \}$, $SS(P'_γ(F)) = SS(F) \cap \{ \tau < 0 \}$ and $F$ is decomposed “up to a constant sheaf” as $P_γ(F) \oplus P'_γ(F)$ (triangle (3.1.14)); for example we have the triangle

$$k_\mathbb{R} \to k_{[0,1]} \oplus k_{[−∞,1]} \to k_{[0,1]} \xrightarrow{+1}.$$

We use the identifications $T_x^*V = V^*$ for any $x \in V$ and $T^*V = V \times V^*$.

**Lemma 3.1.4.** Let $F \in D(k_V)$. We assume that $F$ has a compact support. Then, for any $x \in V$,

\begin{align*}
(3.1.6) & \quad SS(P_γ(F)) \cap T^*_xV \subset p_2(SS(F) \cap SS(k_{x+γ}^a)), \\
(3.1.7) & \quad SS(Q_γ(F)) \cap T^*_xV \subset p_2(SS(F) \cap SS(k_{x+γ^a})),
\end{align*}

where $p_2: T^*V \to T^*_xV$ is the projection.

**Proof.** We only prove the first inclusion, the proof of the second one being similar. We set $G = k_\gamma \otimes q_1^{-1}F$, hence $P_γ(F) = Rq_2\gamma(G)$. Using $\tilde{γ} = d^{-1}(γ)$ with $d(x, y) = x − y$, we see by Theorems 1.2.8 and 1.2.13

$$SS(k_\gamma) = \{(x, y; ξ, −ξ); (x − y; ξ) \in SS(k_γ)\},$$

$$SS(G) \subset SS(k_\tilde{γ}) + (SS(F) \times T^*_xV).$$
Since $\text{supp}(F)$ is compact, Proposition 1.2.4 implies that $\text{SS}(P_\gamma(F))$ is bounded by $p_2^*(\text{SS}(G) \cap (T^*_V V \times T^*V))$ where $p_2^* : T^*V^2 \to T^*V$ is the second projection. A point $(x_1, x_2; 0, \eta)$ of $T^*_V V \times T^*V$ belongs to $\text{SS}(G)$ if and only if there exist $(x_1, x_2; \xi, -\xi)$ and $(x_1; \xi_1) \in \text{SS}(F)$ such that $(0, \eta) = (\xi, -\xi) + (\xi_1, 0)$. In other words $(x_1 - x_2; -\eta) \in \text{SS}(k_\gamma)$ and $(x_1; \eta) \in \text{SS}(F)$. Now, if $x_2 = x$ is fixed, we obtain $(x_1; \eta) \in \text{SS}(k_{x+\gamma}) \cap \text{SS}(F)$ and the result follows.

By Example 1.2.3 $\text{SS}(k_{x+\gamma^n}) \subset V \times \gamma^{o\alpha}$ and Lemma 3.1.4 gives $\text{SS}(P_\gamma(F)) \cup \text{SS}(Q_\gamma(F)) \subset V \times \gamma^{o\alpha}$ if $F$ has compact support; but this result holds without the compactness hypothesis:

Lemma 3.1.5. For any $F \in \mathcal{D}(k_V)$ we have

$$(3.1.8) \quad \text{SS}(P_\gamma(F)) \cup \text{SS}(Q_\gamma(F)) \subset V \times \gamma^{o\alpha}.$$ 

Proof. We prove the result for $P_\gamma(F)$, the case of $Q_\gamma(F)$ being similar. We set $G = k_\gamma \otimes q_1^{-1} F$ as in the proof of Lemma 3.1.4. We have already seen $\text{SS}(G) \subset \text{SS}(k_\gamma) + (\text{SS}(F) \times T^*_V V)$ (this didn’t require the compactness of supp$(F)$) and we deduce the rough bound $\text{SS}(G) \subset T^*V \times \gamma^{o\alpha}$. If $q_2$ were proper on supp$(G)$ we could conclude with Proposition 1.2.3. In our case we can use a compactification of $V$. We choose a diffeomorphism $\mathbb{R} \sim \to ]0,1[\text{ and deduce } V = \mathbb{R}^n \sim \to ]0,1[\times V$. We let $j: ]0,1[\times V \hookrightarrow \mathbb{R}^n \times V$ be the inclusion. Since $\varphi$ is proper, we can use Proposition 1.2.4 and we obtain $\text{SS}(R\varphi_*(G)) \subset T^*V \times \gamma^{o\alpha}$. Then Theorem 1.2.15 gives $\text{SS}(R_j^*R\varphi_*(G)) \subset T^*V \times \gamma^{o\alpha}$. Since $q_2$ is proper on $]0,1[\times V$, we can apply Proposition 1.2.4 again and, using $Rq_2^* R_j^* R\varphi_*(G) \simeq Rq_2^*(G)$, we obtain the result.

The first cut-off results say roughly that the bound (3.1.8) characterizes sheaves of the type $P_\gamma(F)$ or $Q_\gamma(F)$ (see Propositions 3.1.7 and 3.1.8).

For a given subset $\Omega$ of $T^*V$, a morphism $a : F \to G$ in $\mathcal{D}(k_V)$ is said to be an isomorphism on $\Omega$ if $\text{SS}(C(a)) \cap \Omega = \emptyset$, where $C(a)$ is given by the distinguished triangle $F \xrightarrow{a} G \xrightarrow{c_1} C(a) \xrightarrow{\sim}$. This implies that $\text{SS}(F) \cap \Omega = \text{SS}(G) \cap \Omega$. We will need the following composition result:

Lemma 3.1.6. If $a : F \to G$, $b : G \to H$ are isomorphisms on $\Omega$, then so is $b \circ a$.

Proof. The octahedron axiom implies that we have a distinguished triangle $C(a) \to C(b \circ a) \to C(b) \to C(a)[1]$, where $C(a)$, $C(b)$, $C(b \circ a)$ are the cones of $a$, $b$, $b \circ a$. Now the result follows from the triangular inequality for the microsupport.
The next two propositions are taken from [31] where they are stated using the $\gamma$-topology (see [3.4]). The link between the $\gamma$-topology and the functors $P_\gamma, Q_\gamma$ is also explained in Proposition 3.5.4 of [31].

**Proposition 3.1.7** (see [31] Prop. 5.2.3). For $F \in D(k_V)$ we let $u_\gamma(F) : P_\gamma(F) \to F$ be the morphism in (3.1.2). Then

(i) $u_\gamma(F)$ is an isomorphism on $V \times \text{Int}(\gamma^\alpha)$,

(ii) $u_\gamma(F)$ is an isomorphism if and only if $\text{SS}(F) \subset V \times \gamma^\alpha$.

**Proposition 3.1.8** (see [31] Lem. 6.1.5). We assume that $\gamma$ is proper (that is, $\gamma$ contains no line) and $\text{Int}(\gamma) \neq \emptyset$. Let $F \in D(k_V)$. We assume that $F$ has compact support. Then the morphism $v_\gamma(F) : F \to Q_\gamma(F)$ in (3.1.2) is an isomorphism on $V \times \text{Int}(\gamma^\alpha)$.

**Remark 3.1.9.** (1) The bound (3.1.8) and Proposition 3.1.7(ii) show that $u_\gamma(P_\gamma(F)) : P_\gamma(P_\gamma(F)) \to P_\gamma(F)$ is an isomorphism. In other words the morphism of functors $u_\gamma : P_\gamma \to \text{id}_{D(k_V)}$ induces an isomorphism $P_\gamma \circ P_\gamma \cong P_\gamma$. The functor $P_\gamma$ is then called a projector (see for example [32 §4.1]). More precisely it is the projector to the subcategory $D_{V \times \gamma^\alpha}(k_V)$ of $D(k_V)$ formed by the $F$ such that $\text{SS}(F) \subset V \times \gamma^\alpha$ (Notation 1.2.2). By the dual statements of [32 Prop. 4.1.3, 4.1.4] the functor $R : D(k_V) \to D_{V \times \gamma^\alpha}(k_V)$ induced by $P_\gamma$ is both right adjoint and left inverse to the inclusion $\iota$ of $D_{V \times \gamma^\alpha}(k_V)$ and we have $P_\gamma \cong \iota \circ R$.

We will see this again in a slightly different form in [3.4] where $D_{V \times \gamma^\alpha}(k_V)$ is identified with the category of sheaves on $V_\gamma$, which is the space $V$ with the “$\gamma$-topology” (see Proposition 3.4.11). Then $\iota, R$ correspond to $\phi_\gamma^{-1}, R\phi_\gamma$.

(2) Since $Q_\gamma$ is left adjoint to $P_\gamma$, it is also a projector. More precisely, $Q_\gamma \circ Q_\gamma$ is adjoint to $P_\gamma \circ P_\gamma \cong P_\gamma$ and by uniqueness of adjoints we have $Q_\gamma \circ Q_\gamma \cong Q_\gamma$. Instead of a morphism $P_\gamma \to \text{id}$ we have here a morphism $\text{id} \to Q_\gamma$. By [32 Prop. 4.1.3, 4.1.4] $Q_\gamma$ is then a projector to some full subcategory, say $D_\gamma$, of $D(k_V)$. Using the decomposition $P_\gamma \cong \iota \circ R$ in (1) and $R \circ \iota \cong \text{id}$ we can prove $\text{Hom}(P_\gamma(F), P_\gamma(G)) \cong \text{Hom}(P_\gamma(F), G)$, for any $F, G \in D(k_V)$. We deduce $\text{Hom}(Q_\gamma(P_\gamma(F)), G) \cong \text{Hom}(P_\gamma(F), G)$, proving that $Q_\gamma \circ P_\gamma \cong P_\gamma$. This implies that $D_{V \times \gamma^\alpha}(k_V)$ is stable by $Q_\gamma$. Hence we have $D_{V \times \gamma^\alpha}(k_V) \subset D$. In the same way we have $P_\gamma \circ Q_\gamma \cong Q_\gamma$ and we obtain finally $D_{V \times \gamma^\alpha}(k_V) = D$.

Hence $Q_\gamma$ is also a projector to $D_{V \times \gamma^\alpha}(k_V)$. The difference with $P_\gamma$ is that we could write $Q_\gamma \cong \iota \circ L$ where $L$ is left adjoint to the embedding $\iota$.

(3) Since, by (2), $Q_\gamma$ is a projector to $D_{V \times \gamma^\alpha}(k_V)$, we have as in Proposition 3.1.7(ii): $v_\gamma(F)$ is an isomorphism if and only if $\text{SS}(F) \subset V \times \gamma^\alpha$. 

for any $F \in \mathcal{D}(k_V)$ and without the additional hypothesis that $\text{Int}(\gamma) \neq \emptyset$. However we will only use the statement of Proposition 3.1.8.

A consequence of Proposition 3.1.7(i) is the bound
\begin{equation}
\text{SS}(P_\gamma(F)) \cap (V \times \text{Int}(\gamma^{oa})) = \text{SS}(F) \cap (V \times \text{Int}(\gamma^{oa})).
\end{equation}

However $\text{SS}(P_\gamma(F)) \cap (V \times \partial \gamma^{oa})$ could be bigger than $\text{SS}(F) \cap (V \times \partial \gamma^{oa})$. If we assume that $\text{SS}(F)$ does not meet $V \times \partial \gamma^{oa}$ and $\text{supp}(F)$ is compact, then $\text{SS}(P_\gamma(F))$ does not meet $V \times \partial \gamma^{oa}$ by (3.1.6). Hence (3.1.9) and (3.1.10) actually imply $\text{SS}(P_\gamma(F)) = \text{SS}(F) \cap (V \times \text{Int}(\gamma^{oa}))$. In Proposition 3.1.10 below we see the similar bound $\text{SS}(P_\gamma(F)) = \text{SS}(F) \cap (V \times (V \setminus \gamma^{oa}))$ and that $F$ is split as $P_\gamma(F) \oplus P_\gamma(F)$ up to a constant sheaf.

Actually, in Proposition 3.1.10 we give a weaker hypothesis than $\text{SS}(F) \cap (V \times \partial \gamma^{oa}) = \emptyset$. We want to split $F$ only on some open subset $W$ of $V$ and we can replace $V \times \partial \gamma^{oa}$ by some smaller set depending on $W$, namely $\bigcup_{x \in W} S^\gamma_x$ where $S^\gamma_x$ is defined below (remark that $\bigcup_{x \in V} S^\gamma_x = V \times \partial \gamma^{oa}$). This more precise result is used in the proof of Proposition 3.2.3.

The motivation to introduce $S^\gamma_x$ is the following. We first want to ensure that $\text{SS}(P_\gamma(F)) \cap T^*_x V$ coincides with $\tilde{\text{SS}}(F) \cap (V \times \text{Int}(\gamma^{oa})) \cap T^*_x V$. By (3.1.8) and Proposition 3.1.7 it is enough to consider $\text{SS}(P_\gamma(F)) \cap T^*_x V \cap (\{x\} \times (\partial \gamma^{oa})) = \emptyset$, where $\partial \gamma^{oa} = \gamma^{oa} \setminus \text{Int}(\gamma^{oa})$. Using (3.1.6) this holds if we assume that $\tilde{\text{SS}}(F)$ does not meet $(V \times \partial \gamma^{oa}) \cap \text{SS}(k_{x+\gamma})^a$, which is half of $S^\gamma_x$. The other half is introduced to bound $\text{SS}(Q_\gamma(F)) \cap T^*_x V$ in the same way, using (3.1.7).

Let $\gamma \subset V$ be a closed convex proper cone. For $x \in V$ we define $S^\gamma_x \subset T^*_x V$ by
\begin{equation}
S^\gamma_x = (\tilde{\text{SS}}(k_{x+\gamma})^a \cup \tilde{\text{SS}}(k_{x+\gamma})) \setminus (\{x\} \times \text{Int}(\gamma^{oa}))
= (\text{SS}(k_{x+\gamma})^a \cup \text{SS}(k_{x+\gamma})) \cap (V \times \partial \gamma^{oa}),
\end{equation}
where the second equality follows from (1.2.4).

In Fig. 3.1.2 we have pictured $\text{SS}(k_{x+\gamma})^a$, $\tilde{\text{SS}}(k_{x+\gamma})^a$ and $S^\gamma_x$ in dimension 2 (we identify $T^*\mathbb{R}^2$ with $T\mathbb{R}^2$; we note that $\tilde{\text{SS}}(k_{x+\gamma})^a$ and $\text{SS}(k_{x+\gamma})^a$ contain $\{x\} \times \text{Int}(\gamma^{oa})$). We can give an easy description of $S^\gamma_x$ when $\gamma$ has a smooth boundary away from 0, as in the following example (this is in fact the only case we need). For $n > 1$ we write the coordinates in $\mathbb{R}^n$ as $x = (x', x_n)$. We define $\gamma = \{(x', x_n) \in V; x_n \leq -||x'||\}$ and $C = \{(x', x_n) \in V; x_n = \pm ||x'||\}$. Then $(x_0 + \gamma) \cup (x_0 + \gamma^o)$ has boundary $C_{x_0} = x_0 + C$, which is a smooth hypersurface except at $x_0$. We set $C'_{x_0} = C_{x_0} \setminus \{x_0\}$ and define $\Lambda = \overline{T^*_{C_{x_0}} \mathbb{R}^n \cap T^*\mathbb{R}^n}$. Then
Figure 3.1.2.

\[ \Lambda \text{ is a smooth closed conic Lagrangian submanifold of } \dot{T}^*\mathbb{R}^n \text{ with two connected components and } S^\gamma_{x_0} \text{ is the component with } \xi_n > 0. \]

We deduce from (3.1.6), (3.1.7) and Example 1.2.3(iv) that, if \( F \in D(k_V) \) has compact support, then, for \( G = P_\gamma(F) \) or \( G = Q_\gamma(F) \),

\[ \dot{\text{SS}}(G) \cap (T^*_x V \setminus \text{Int}(\gamma^o)) \subset p_2(\dot{\text{SS}}(F) \cap S^\gamma_x). \]

Since \( u_\gamma(F) \) and \( v_\gamma(F) \) are isomorphisms on \( V \times \text{Int}(\gamma^o) \), we also have

\[ \dot{\text{SS}}(G) \cap (V \times \text{Int}(\gamma^o)) = \dot{\text{SS}}(F) \cap (V \times \text{Int}(\gamma^o)). \]

**Proposition 3.1.10.** Let \( F \in D(k_V) \) be such that \( \text{supp}(F) \) is compact and let \( W \subset V \) be an open subset such that

\[ S^\gamma_x \cap \dot{\text{SS}}(F) = \emptyset \quad \text{for any } x \in W. \]

Then \( v_\gamma(F) \circ u_\gamma(F)|_W : P_\gamma(F)|_W \to Q_\gamma(F)|_W \) is an isomorphism on \( T^*W \) and we have a distinguished triangle in \( D(k_W) \)

\[ P_\gamma(F)|_W \oplus P'_\gamma(F)|_W \xrightarrow{(u_\gamma(F), u'_\gamma(F))} F|_W \to L \to +1, \]

where \( L \in D(k_W) \) is locally constant and

\[ \dot{\text{SS}}(P_\gamma(F)|_W) = \dot{\text{SS}}(F) \cap (W \times \text{Int}(\gamma^o)), \]

\[ \dot{\text{SS}}(P'_\gamma(F)|_W) = \dot{\text{SS}}(F) \cap (W \times (V \setminus \gamma^o)). \]

**Proof.** (i) Let \( G = P_\gamma(F)|_W \) or \( Q_\gamma(F)|_W \). By (3.1.11) and (3.1.13) we have \( \dot{\text{SS}}(G) \subset W \times \text{Int}(\gamma^o) \). By (3.1.12) we deduce

\[ \dot{\text{SS}}(G) = \dot{\text{SS}}(F) \cap (W \times \text{Int}(\gamma^o)). \]

(ii) We define \( L \in D(k_W) \) by the distinguished triangle

\[ P_\gamma(F)|_W \xrightarrow{v_\gamma(F) \circ u_\gamma(F)} Q_\gamma(F)|_W \xrightarrow{\alpha} L \to +1. \]
By (i) we have $\hat{SS}(L) \subset W \times \text{Int}(\gamma^{\omega})$. On the other hand $v_\gamma(F) \circ u_\gamma(F)$ is an isomorphism on $W \times \text{Int}(\gamma^{\omega})$ by Propositions 3.1.7 and 3.1.8 and Lemma 3.1.6. Hence $SS(L) \cap (W \times \text{Int}(\gamma^{\omega})) = \emptyset$ and we find $SS(L) = \emptyset$. This proves the first assertion.

(iii) The distinguished triangle of the proposition follows from the triangles (3.1.3), (3.1.15) and Lemma 3.1.11 applied with $A = P_\gamma(F)|_W$, $B = F|_W$, $C = P'_\gamma(F)|_W$, $C' = Q_\gamma(F)|_W$, $D = L$.

Since $v_\gamma(F)$ is an isomorphism on $V \times \text{Int}(\gamma^{\omega})$, the second triangle in (3.1.3) gives $\hat{SS}(P'_\gamma(F)|_W) \cap (W \times \text{Int}(\gamma^{\omega})) = \emptyset$. It follows that $\hat{SS}(P'_\gamma(F)|_W)$ and $\hat{SS}(P_\gamma(F)|_W)$ are disjoint. Hence (3.1.14) implies

$$\hat{SS}(F|_W) = \hat{SS}(P'_\gamma(F)|_W) \sqcup \hat{SS}(P_\gamma(F)|_W).$$

Since (3.1.13) implies in particular $\hat{SS}(F) \cap (W \times \partial(\gamma^{\omega})) = \emptyset$, the triangular inequality for the microsupport implies the last equalities of the proposition. □

Lemma 3.1.11. We assume to be given a morphism $a: A \to B$ in a triangulated category and two distinguished triangles

$$C \xrightarrow{c} B \xrightarrow{c'} C' \xrightarrow{+1}, \quad A \xrightarrow{c \circ a} C' \xrightarrow{+1} D.$$

Then there exists a distinguished triangle

$$A \oplus C \xrightarrow{(a,c)} B \to D \xrightarrow{+1}.$$

Proof. We define $E$ by the distinguished triangle $A \oplus C \xrightarrow{(a,c)} B \to E \xrightarrow{+1}$. We also have the canonical triangle $C \xrightarrow{u} A \oplus C \xrightarrow{v} A \xrightarrow{+1}$ where $u = (0, \text{id}_C)$ and $v = (\text{id}_A, 0)$. Since $(a,c) \circ u = c: C \to B$, the octahedron axiom applied to this last two triangles and the first one in the statement gives a distinguished triangle

$$A \xrightarrow{w} C' \xrightarrow{+1} E \xrightarrow{+1}$$

such that $w \circ v = c' \circ (a,c)$. This implies $w = c' \circ a$. Hence $E \simeq D$ and the lemma follows. □

3.2. Local cut-off - special case

In this section we want to give local versions of Propositions 3.1.7 and 3.1.10. For example the conclusion $P_\gamma(F) \simeq F$ requires a bound for $SS(F)$ on $V$; in the same way the hypothesis (3.1.13) requires a knowledge of $SS(F)$ on some unbounded set. Here we only assume that we know $SS(F)$ over some open subset $U$ of $V$ and we will give results similar to the conclusions of the mentioned propositions over some smaller open subset $W \subset U$. 
In fact we will apply Propositions 3.1.7 and 3.1.10 to a sheaf $F_Z$ for some locally closed subset $Z \subset U$ such that $\overline{Z} \subset U$ (since $F_Z$ is considered as a sheaf on $V$ by extension by 0). Then $\text{SS}(F_Z)$ coincides with $\text{SS}(F)$ over $\text{Int}(Z)$ and is empty over $V \setminus \overline{Z}$. However, over $\overline{Z} \setminus \text{Int}(Z)$, we only have the bound $\text{SS}(F_Z) \subset \text{SS}(F) + \text{SS}(k_Z)$. We check in this section that we can choose $Z$ so that $F_Z$ satisfies the hypotheses of Propositions 3.1.7 and 3.1.10. This is done in [31, Prop. 6.1.4] for a convex cone $\gamma$. In Proposition 3.3.2 we will generalize this result to a more general cone, using Theorem 2.1.1 to reduce the situation to the case of a convex cone. Since we can as well reduce to a very special cone, we only explain a particular case of [31, Prop. 6.1.4] (the proof is not different from that of [31] but the exposition is easier).

We write $V = V' \times \mathbb{R}$, where $V' = \mathbb{R}^{n-1}$. We take coordinates $x = (x', x_n)$ on $V$ and we endow $V'$ and $V$ with the natural Euclidean structure. For $c > 0$ we let $\gamma_c \subset V$ be the cone
\[(3.2.1) \quad \gamma_c = \{(x', x_n) \in V; \ x_n \leq -c \|x'\|\}.
\]

Precised cut-off. We begin with an analog of Propositions 3.1.7, 3.1.8 we give a functor $R$ similar to $P_\gamma$ or $Q_\gamma$ which cuts the microsupport along $\{\xi_n \leq 0\}$, but locally defined (on $D(k_U)$ instead of $D(k_V)$, with $U$ open in $V$) and with a bound for $\text{SS}(R(F))$ up to the boundary $\{\xi_n = 0\}$. However $R(F)$ is only defined on a smaller subset $W \subset U$ and we don’t have the ideal bound $\text{SS}(R(F)) \subset \text{SS}(F) \cap \{\xi_n \leq 0\}$, but only $\text{SS}(R(F)) \subset (\text{SS}(F) \cap \{\xi_n \leq 0\}) \cup B'$, where $B'$ is an arbitrary neighborhood of $\text{SS}(F) \cap \{\xi_n = 0\}$ chosen in advance.

**Proposition 3.2.1.** Let $B \subset V^*$ be a closed conic subset and let $B' \subset V^*$ be a conic neighborhood of $(B \setminus \{0\}) \cap \{\xi_n = 0\}$. Let $U \subset V$ be a neighborhood of 0. Then there exist an open neighborhood $W$ of 0 in $U$ and a functor $R: D(k_U) \to D(k_W)$, together with a morphism of functors $(\cdot)|_W \to R$, such that, for any $F \in D(k_U)$,

(a) if $\text{SS}(F) \subset U \times B$, then $\text{SS}(R(F)) \subset W \times ((B \setminus \{\xi_n \geq 0\}) \cup B')$,

(b) if $\text{SS}(F) \subset U \times \{\xi_n \leq 0\}$, then $F|_W \tilde{\to} R(F)$.

Since the proof is a bit technical we explain the idea. We first cut $F$ by some open set $Y$ with $\overline{Y} \subset U$ (so that $F_Y$ can be consider as a sheaf on $V$ which is 0 outside $\overline{Y}$) and apply $Q_\gamma$ to $F_Y$. Using (3.1.6) we have $\text{SS}(Q_\gamma(F_Y)) \cap T^*_xV \subset p_2(\text{SS}(F_Y) \cap \text{SS}(k_{x+\gamma}))$ and we know that $\text{SS}(F_Y) \subset \text{SS}(F) \cap \text{SS}(k_Y)$. Of course this last subset could be much bigger than $\text{SS}(F)$, but it turns out that, if $\text{SS}(k_Y)$ is close enough to $\text{SS}(k_{x+\gamma})$ (at the points $x'$ where both $\text{SS}(k_Y) \cap T^*_xV$ and $\text{SS}(k_{x+\gamma}) \cap$
$T_x V$ are non trivial), then \(SS(Q'(F_Y))\) is not too much bigger than \(SS(F)\). This relies on Lemma 3.2.2 because, at the interesting points \(x'\), \(SS(k_Y) \cap T_x V\) and \(SS(k_{x+\gamma}) \cap T_x V\) are half-lines.

The examples we choose for \(Y\) and \(\gamma\) with the required property are for \(Y\) a vertical cylinder

\[
Y^h_r = \{ (x', x_n) \in V; \|x'\| < r, |x_n| < h \}
\]

and for \(\gamma\) a cone \(\gamma_c\) with big slope \(c\) such that \(\partial \gamma_c\) meets \(\partial Y^r\) along its vertical part, which means that \(h > rc\) (as in Fig. 3.2.1).

Figure 3.2.1.

**Lemma 3.2.2.** Let \(B, B'\) be as in Proposition 3.2.1. Let \(S\) be the unit sphere of \(V^*\). Then there exists \(\epsilon > 0\) such that, for any \((p_1, p_2) \in (S \cap \{\xi_n = 0\}) \times S\) with \(|p_1 - p_2| < \epsilon\), we have

\[
(B + \mathbb{R}_{\geq 0} \cdot (-p_1)) \cap (\mathbb{R}_{> 0} \cdot p_2) \subset B'.
\]

(In other words, if the intersection is non empty, then \(p_2 \in B'\).)

This lemma is an elaboration on the following remark: for \(p \in V^* \setminus B\) we have \(p \notin (B + \mathbb{R}_{\geq 0}(-p))\) and hence \((B + \mathbb{R}_{\geq 0}(-p)) \cap (\mathbb{R}_{> 0} \cdot p) = \emptyset\).

It follows that, for any \(p \in (S \cap \{\xi_n = 0\})\), we have \((B + \mathbb{R}_{\geq 0}(-p)) \cap (\mathbb{R}_{\geq 0} \cdot p) \subset (B \cap \{\xi_n = 0\})\). In the lemma \((p_1, p_2)\) is near the diagonal and \(B \cap \{\xi_n = 0\}\) is replaced by its neighborhood \(B'\).

**Proof.** For a subset \(C\) of the sphere \(S\), we set \(C_{S, \epsilon} = \{v \in S; \text{there exists } v' \in C, \|v - v'\| < \epsilon\}\). For a conic subset \(D\) of \(V^*\) we set \(D_\epsilon = \mathbb{R}_{> 0} \cdot (D \cap S)_{S, \epsilon}\). We remark that \(D_\epsilon = \{av_1 + bv_2; v_1 \in D \cap S, v_2 \in S, \|v_1 - v_2\| < \epsilon, a, b \geq 0\} \setminus \{0\}\).

A point \(p' \in (B + \mathbb{R}_{> 0}(-p_1)) \cap (\mathbb{R}_{> 0} \cdot p_2)\) is written \(p' = p - ap_1 = bp_2\), with \(p \in B, a, b \geq 0\). Then \(p = ap_1 + bp_2\) belongs to \(B \cap \{\xi_n = 0\}_{\epsilon}\).
The point \( q = \frac{1}{|p|}p \) belongs to the arc \( p_1p_2 \) on \( S \), hence \( ||q - p_2|| < \varepsilon \) and then \( p_2 \in (B \cap \{ \xi_n = 0 \})_\varepsilon \). Now, for \( \varepsilon \) small enough this last set is contained in \( B' \).

**Proof of Proposition 3.2.1** (i) We will use the functors \( P_\gamma, Q'_\gamma \) for a cone \( \gamma = \gamma_c \) to be chosen in (iii) and the distinguished triangle (3.1.2)

\[ P_\gamma(F) \to F \to Q'_\gamma(F) \xrightarrow{+1} \]

We have the bound, with the same proof as (3.1.6),

\[ SS(Q'_\gamma(F)) \cap T_x^*V \subset p_2(\text{SS}(F) \cap S_x^{\gamma_t}), \]

where \( S_x^{\gamma_t} = (\text{SS}(k_{(x+\gamma}_t\gamma \{x\}))^\ast \) and \( p_2: T^*V \to T_x^*V \simeq V^* \) is the projection.

(ii) We prove that \( S_x^{\gamma_t} \subset V \times (V^* \setminus \text{Int}(\gamma^{oa})) \). More precisely \( S_x^{\gamma_t} = (\text{SS}(k_{(x+\gamma}_t\gamma \{x\}))^\ast \) outside \( T_x^*V \) (which is obvious from the definition of \( S_x^{\gamma_t} \)) and \( S_x \cap T_x^*V = V^* \setminus \text{Int}(\gamma^{oa}) \). Indeed this can be computed directly from the definition of the microsupport, or we can use [31, Lem. 3.7.10, Thm. 5.5.5] as follows. The Fourier-Sato transform of a conic sheaf \( G \in \mathcal{D}(k_V) \) is \( G^\wedge \in \mathcal{D}(k_{V^*}) \) and, identifying \( T^*V = V \times V^* = T^*V^* \), we have \( \text{SS}(G^\wedge) = \text{SS}(G) \). In particular \( \text{SS}(G) \cap T_0^*V = \text{supp}(G^\wedge) \). We also have \( (k_{\lambda})^\wedge \simeq k_{\text{Int}(\lambda^a)} \) for any proper closed convex cone \( \lambda \), in particular \( \lambda = \gamma \) and \( \lambda = \{0\} \). Hence the exact sequence \( 0 \to k_{\gamma \setminus \{0\}} \to k_{\gamma} \to k_{\{0\}} \to 0 \) gives \( (k_{\gamma \setminus \{0\}})^\wedge \simeq k_{V^* \setminus \text{Int}(\gamma^a)} \) and the result follows.

(iii) We will choose \( c, r, h \) such that \( h > rc \) as explained after (3.2.2). Hence \( \partial(\gamma_c) \) meets \( \partial Y_r^h \) along its vertical part and the intersection, say \( I \), is the sphere \( \{(x', x_n) \in V; \|x'\| = r, x_n = -cr\} \). Let \( y \in I \), \( p_1, p_2 \in T_y^*V \) be the unit length vectors in \( T_y^*V \cap \text{SS}(k_{\gamma_c}), T_y^*V \cap S_y^{\gamma_c} \). Then \( \|p_1 - p_2\| \leq c^{-1} \). Let \( \varepsilon > 0 \) such that the conclusion of Lemma 3.2.2 holds. We choose \( c > 2\varepsilon^{-1} \) and \( h, r \) small enough so that \( Y_r^h \subset U \). We define \( R_0(F) = Q'_{\gamma_c}(F_{Y_r^h}) \). We have a natural morphism \( F \to R_0(F) \).

Let \( W \subset Y_r^h \) be the set of \( x \) such that \( \partial(x + \gamma_c) \) meets \( \partial Y_r^h \) along its vertical part and, for any \( y \in \partial(x + \gamma_c) \cap \partial Y_r^h \), the unit length vectors \( p_1 \in T_y^*V \cap \text{SS}(k_{\gamma_c}), p_2 \in T_y^*V \cap S_y^{\gamma_c} \) satisfy \( \|p_1 - p_2\| < \varepsilon \). The set \( W \) is open and contains \( 0 \) by construction.

By Theorem 1.2.13 we have \( \text{SS}(F_{Y_r^h}) \subset (V \times B) \hat{\oplus} \text{SS}(k_{Y_r^h}) \). We can use + instead of \( \hat{\oplus} \) in the last expression because the term \( V \times B \) is a product. By (i) and (ii) we have

\[
\text{SS}(R_0(F)) \cap T_x^*V \subset p_2(((V \times B) + \text{SS}(k_{Y_r^h})) \cap S_x^{\gamma_c})
\]

\[
\subset p_2((V \times B) \cap (V \times (V^* \setminus \text{Int}(\gamma_c^{oa}))))
\]

\[
\cup p_2(((V \times B) + \text{SS}(k_{Y_r^h})) \cap S_x^{\gamma_c}).
\]
By Lemma 3.2.2 and by our definition of $W$ we have $p_2((V \times B) + SS(k_{Y^h}) \cap S^{\gamma_c}_x) \subset B'$ if $x \in W$. Hence $SS(R_0(F)) \cap T^*_x V \subset (B \cap (V^* \setminus \text{Int}(\gamma^{oa}_c)) \cup B'$. Since $V^* \setminus \text{Int}(\gamma^{oa}_c) = \{\xi_n \geq c^{-1}\|\xi\|\}$, we have $B \cap (V^* \setminus \text{Int}(\gamma^{oa}_c)) \subset ((B \setminus \{\xi_n \geq 0\}) \cup B')$ for $c$ big enough. Now we set $R(F) = R_0(F)|_W$ and we obtain (a).

(iv) We prove that $P_{\gamma_c}(F_{Y^h})|_W \simeq 0$, which implies (b) by the triangle given in (i). By (1.5.2) we have $(P_{\gamma_c}(F_{Y^h}))_y \simeq R\Gamma(V; F_{E_y})$ where $E_y = Y^h \cap (y + \gamma_c)$. If we restrict to the open subset $\{x_n < h\}$, the micro-supports of $F$, $k_{Y^h}$ and $k_{y + \gamma_c}$, for $y \in V$, are all contained in $\{\xi_n \leq 0\}$. For $y \in W$ we have $\overline{E_y} \subset \{x_n < h\}$, hence $SS(F_{E_y}) \subset \{\xi_n \leq 0\}$ by Theorem 1.2.13. Since $\text{supp}(F_{E_y}) \subset \overline{Y^h}$ is compact, we deduce by Corollary 1.2.16 that $R\Gamma(V; F_{E_y}) \simeq 0$ (taking $\phi(x) = x_n$ in the corollary we obtain $R\Gamma(\phi^{-1}(-\infty, 2h)); F_{E_y}) \simeq R\Gamma(\phi^{-1}(-\infty, -2h)); F_{E_y}) \simeq 0$).

Hence $P_{\gamma_c}(F_{Y^h})|_W \simeq 0$ as claimed. \hfill \Box

Local splitting. Recall the cone $\gamma_c$ of (3.2.1) and the subset $S^{\gamma_c}_x \subset T^*V$ of (3.1.10). We have $\pi_V(S^{\gamma_c}_x) = \{(x', x_n; \xi', \xi_n); \ |x_n| = c \|x'\|\}$ and, for a non zero $y = (y', y_n) \in \pi_V(S^{\gamma_c}_0)$ we have

\begin{equation}
S^{\gamma_c}_0 \cap T^*_r V = \{(\lambda \tilde{y}', -c^{-1}\lambda \tilde{y}_n); \ \lambda y_n > 0\},
\end{equation}

where $(\tilde{y}', \tilde{y}_n) \in V^*$ corresponds to $y$ through the identification $V' \simeq (V')^*$ given by the Euclidean product. We will use a cylinder similar to $Y^h$

\begin{equation}
Z^h_r = \{(x', x_n) \in V; \ |x'| \leq r, -h \leq x_n < 0\}
\end{equation}

\begin{equation}
\sqcup \{(x', x_n) \in V; \ |x'| < r, 0 \leq x_n < h\}
\end{equation}

which is locally closed (and not open) and such that $SS(k_{Z^h})$ is close to $(S^{\gamma_c}_0)^a$ along $\partial Z^h \cap \pi_V(S^{\gamma_c}_0)$ for $c$ big. Indeed over the vertical boundary of $Z^h$, that is, for $y = (y', y_n)$ with $\|y'\| = r$ and $0 < |y_n| < h$ we have, by Example 1.2.3(iii),

\begin{equation}
SS(k_{Z^h}) \cap T^*_y V = \{(\lambda \tilde{y}', 0); \ \lambda y_n > 0\},
\end{equation}

where $\tilde{y}' \in (V')^*$ is as in the description of $S^{\gamma_c}_0$ (in Fig. 3.2.2 we have pictured $SS(k_{Z^h})$ and $S^{\gamma_c}_0$).

Proposition 3.2.3. Let $U \subset V$ be an open subset containing 0 and let $c > 0$ be given. Then there exist an open neighborhood $W$ of 0 in $U$ and two functors $P, P': D(k_U) \rightarrow D(k_W)$ together with morphisms of functors $u: P \rightarrow (\cdot)|_W$, $u': P' \rightarrow (\cdot)|_W$, such that, for any $F \in D(k_U)$ satisfying

\begin{equation}
\text{SS}(F) \subset U \times (\gamma^0_c \cup \gamma^{oa}_c),
\end{equation}
we have
\[
\dot{\text{SS}}(P(F)) = \dot{\text{SS}}(F|_W) \cap (W \times \gamma^0_c),
\]
\[
\dot{\text{SS}}(P'(F)) = \dot{\text{SS}}(F|_W) \cap (W \times \gamma^{oa}_c)
\]
and the object \(L \in \mathbb{D}(\mathbb{k}_W)\) given by the distinguished triangle
\[
P(F) \oplus P'(F) \xrightarrow{(a(F), a'(F))} F|_W \rightarrow L \rightarrow +1
\]
satisfies \(\dot{\text{SS}}(L) = \emptyset\).

**Proof.** (i) As in the proof of Proposition 3.2.1 we assume to be given \(d, r, h > 0\) such that \(h > rd\). Hence \(\pi_V(S^d_0)\) meets \(\partial Z^h_r\) along its vertical part, where \(Z^h_r\) is defined in (3.2.4), and the intersection is the union of the two spheres \(\{(x', x_n) \in V; \|x'\| = r, x_n = \pm dr\}\). Along these two spheres \(S^d_0\) and \(\text{SS}(\mathbb{k}_{Z^h_r})\) are the two half lines described in (3.2.3) and (3.2.5) (so they make an angle \(\arctan(d^{-1})\)). Using Lemma 3.2.2 with
\[
B = \gamma^0_c \cup \gamma^{oa}_c = \{(\xi', \xi_n); c \|\xi_n\| \leq ||\xi'||\}
\]
and \(B' = \emptyset\) we obtain
\[
S^d_0 \cap (\text{SS}(\mathbb{k}_{Z^h_r}) + V \times (\gamma^0_c \cup \gamma^{oa}_c)) = \emptyset
\]
for \(d\) big enough, whatever \(r, h\). We fix such a \(d\) and then choose \(r, h\) such that \(h > rd\) (as already assumed) and small enough so that \(\overline{Z^h_r} \subset U\).

(ii) We define \(W\) as the set of \(x \in \text{Int}(Z^h_r)\) such that \(\pi_V(S^d_x)\) meets \(\partial Z^h_r\) along its vertical part and (3.2.8) holds with \(S^d_0\) replaced by \(S^d_x\). Hence \(W\) is an open neighborhood of 0.
For $F \in \mathcal{D}(k_U)$ we can extend $F \otimes k_{Z^h_r}$ by 0 as an object of $\mathcal{D}(k_V)$. We define $P, P'$ by $P(F) = P_{\gamma}(F \otimes k_{Z^h_r})|_W$ and $P'(F) = P'_{\gamma}(F \otimes k_{Z^h_r})|_W$. The functors $u, u'$ are induced by $u_{\gamma}, u'_{\gamma}$ (we remark that $W \subset Z^h_r$, hence $(F \otimes k_{Z^h_r})|_W = F|_W$). If $F$ satisfies (3.2.6), then

\begin{equation}
SS(F \otimes k_{Z^h_r}) \subset (SS(k_{Z^h_r}) + V \times (\gamma_c^o \cup \gamma_c^\omega))
\end{equation}

by Theorem 1.2.13. Hence $F \otimes k_{Z^h_r}$ satisfies the hypothesis (3.1.13) of Proposition 3.3.2 by (3.2.8) (with $S_0^\omega$ replaced by $S_x^\omega$). Now the result follows from Proposition 3.3.10.

\section{3.3. Local cut-off - general case}

Here we extend the results of §3.2 replacing the cones $\{\xi_n \leq 0\}$ or $\gamma_c^o$ by more general ones. We deduce Propositions 3.3.1, 3.3.2 from Propositions 3.2.1, 3.2.3. The process of reduction to the results of §3.2 is the same for both propositions and we only prove the second one. (Moreover Proposition 3.3.1 is only a precised version of [31, Prop. 6.1.4] but Proposition 3.3.2 is new.)

\textbf{Proposition 3.3.1.} Let $U$ be an open subset of $V = \mathbb{R}^n$ and let $A \subset V^* \setminus \{0\}$ be an open cone. We assume that there exists a homotopy $\psi : (V^* \setminus \{0\}) \times [0, 1] \to V^* \setminus \{0\}$ of class $C^1$ and homogeneous of degree 1 such that $\psi_1(A) = \{\xi_n > 0\}$. Let $B \subset V^*$ be a closed conic subset and $B' \subset V^*$ be a conic neighborhood of $B \cap \partial A$. Let $x_0 \in U$ be given. Then there exist a neighborhood $W$ of $x_0$ and a functor $R : \mathcal{D}(k_U) \to \mathcal{D}(k_W)$, of the form $R(F) = K \circ F$ for some $K \in \mathcal{D}(k_{W \times U})$, together with a morphism of functors $(\cdot)|_W \to R$, such that, for any $F \in \mathcal{D}(k_U)$,

(i) if $SS(F) \subset U \times B$, then

$$SS(R(F)) \subset W \times ((B \setminus A) \cup B'),$$

(ii) if $\tilde{SS}(F) \cap (U \times A) = \emptyset$, then $F|_W \simeq R(F)$.

\textbf{Proposition 3.3.2.} Let $U$ be an open subset of $V = \mathbb{R}^n$ and let $A, A' \subset V^* \setminus \{0\}$ be two disjoint closed conic subsets. We assume that there exists a homotopy $\psi : (V^* \setminus \{0\}) \times [0, 1] \to V^* \setminus \{0\}$ of class $C^1$ and homogeneous of degree 1 such that $\psi_1(A)$ and $\psi_1(A')$ are separated by some hyperplane of $V^*$. Let $x_0 \in U$ be given. Then there exist a neighborhood $W$ of $x_0$ in $U$ and two functors $P, P' : \mathcal{D}(k_U) \to \mathcal{D}(k_W)$ together with morphisms of functors $u : P \to (\cdot)|_W$, $u' : P' \to (\cdot)|_W$ such that, for any $F \in \mathcal{D}(k_U)$ satisfying

\begin{equation}
\tilde{SS}(F) \subset U \times (A \cup A'),
\end{equation}
we have
\[ \hat{SS}(P(F)) = \hat{SS}(F|_W) \cap (W \times A) , \]
\[ \hat{SS}(P'(F)) = \hat{SS}(F|_W) \cap (W \times A') \]
and the object \( L \in D(k_W) \) given by the distinguished triangle
\[ P(F) \oplus P'(F) \xrightarrow{(u(F),u'(F))} F|_W \to L \xrightarrow{+1} \]
satisfies \( \hat{SS}(L) = \emptyset \).

Proof. (i) Let \( c > 0 \) be given. We recall the notation \( \gamma_c \) of \ref{3.2.1}. Up to a linear change of coordinates in \( \mathbb{R}^n \) we can assume that \( x_0 = 0 \), that the homotopy \( \psi \) in the hypothesis is defined on some open interval \( I \) containing \([0,1]\) and that \( \psi_1(A) \subset \gamma_c^\circ, \psi_1(A') \subset \gamma_c^{\circ a} \). We first extend \( \psi \) to a homogeneous Hamiltonian isotopy \( \Phi: \hat{T}^*U \times I \to \hat{T}^*U \) such that \( \Phi_1(T_0^*U) = T_0^*U \) for all \( t \in I \) and
\[ \Phi_1(\{0\} \times A) \subset \{0\} \times \gamma_c^\circ, \quad \Phi_1(\{0\} \times A') \subset \{0\} \times \gamma_c^{\circ a} . \]

To see that such a \( \Phi \) exists we choose local coordinates \((x;\xi)\) around 0 and write \( \partial h/\partial t(\xi,t) = \sum_{i=1}^n a_i(\xi,t)\partial_{\xi_i} \) (for \((\xi,t)\in V^* \times I\)). Then we choose a Hamiltonian function \( h: \hat{T}^*U \times I \to \mathbb{R} \) homogeneous of degree 1 in \( \xi \) and with support in \( C \times V^* \) for some compact subset \( C \) of \( U \) such that, near 0, we have \( h(x;\xi) = -\sum_{i=1}^n a_i(\xi,t)x_i \). Then the Hamiltonian flow \( \Phi \) of \( h \) satisfies the above relations.

We let \( R_\Phi: D(k_U) \to D(k_U), \quad F \mapsto K_{\Phi,1} \circ F, \) be the equivalence of categories given by Corollary \ref{2.1.5}. We have in particular \( SS(R_\Phi(F)) = \Phi_1(\hat{SS}(F)) \) for all \( F \in D(k_U) \).

(ii) We can find a neighborhood \( U_1 \) of 0 such that
\[ \Phi_1(U \times A) \cap \hat{T}^*U_1 \subset U_1 \times \gamma_{2c}^\circ, \quad \Phi_1(U \times A') \cap \hat{T}^*U_1 \subset U_1 \times \gamma_{2c}^{\circ a} . \]

Applying Proposition \ref{3.2.3} (with \( U_1 \) and \( 2c \) instead of \( U \) and \( c \)) we find a neighborhood of 0, say \( W_1 \), and functors \( P_1, P'_1: D(k_{W_1}) \to D(k_{W_1}) \) together with morphisms of functors \( u_1: P_1 \to (\cdot)|_{W_1}, \quad u'_1: P'_1 \to (\cdot)|_{W_1} \) satisfying the conclusion of Proposition \ref{3.2.3}.

(iii) We let \( W \) be an open neighborhood of 0 such that \( \Phi_t(\hat{T}^*W) \subset \hat{T}^*W_1 \) for all \( t \in [0,1] \). Let \( j \) denote the inclusion of \( W_1 \) in \( U \). We define the functor \( P \) by
\[ P(F) = (R_{\Phi_j}^{-1}j_*P_1(R_\Phi(F)|_{U_1}))|_W \]
and $P'$ from $P'_i$ by the same formula. We let $u$, $u'$ be the morphisms of functors induced by $u_1$, $u'_1$. By Proposition 3.2.3 we have a distinguished triangle on $W_1$

$$ P_1(R\phi(F)|_{U_1}) \oplus P'_1(R\phi(F)|_{U_1}) \to R\phi(F)|_{U_1} \to L^{+1}, $$

where $L$ is locally constant on $W_1$. Since $\Phi_1(T^*W) \subset \hat{T}^*W_1$ we see that $R\phi^{-1}((j)(L))|_{W}$ is locally constant. Applying $(R\phi^{-1}((j)(L)))|_{W}$ to the distinguished triangle, we find that $G = (R\phi^{-1}((R\phi(F))_{W_1}))|_{W}$ is isomorphic to $P(F) \oplus P'(F)$ up to a locally constant sheaf on $W$.

It only remains to check that $G$ is isomorphic to $F|_{W}$. Lemma 2.2.1 applied with $\Psi = \Phi^{-1}$, $U = W_1$ and $V = W$ gives $R\phi^{-1}(H_{W_1})|_{W} \simeq R\phi^{-1}(H)|_{W}$ for any $H$. Hence $G \simeq (R\phi^{-1}(R\phi(F)))|_{W} \simeq F|_{W}$, as required.

\[ \square \]

### 3.4. Cut-off and $\gamma$-topology

In [31] the Proposition 3.1.7 has another formulation in terms of the $\gamma$-topology. It gives directly a decomposition of the projector $P_\gamma$ as in Remark 3.1.9(1). As in the previous paragraphs, let $\gamma \subset V$ be a closed convex cone. The $\gamma$-topology is defined in [31] as follows. We say that an open subset $\Omega$ of $V$ is $\gamma$-stable if $x + y \in \Omega$ for all $(x, y) \in \Omega \times \gamma$. The $\gamma$-stable open subsets define a topology on $V$ and we denote by $V_\gamma$ this topological space. The identity map induces a continuous map $\phi_\gamma : V \to V_\gamma$. Then Propositions 3.5.3, 3.5.4 and 5.2.3 of [31] give

**Proposition 3.4.1.**

(i) For any $G \in D(k_{V_\gamma})$ the adjunction morphism $G \to R\phi_{\gamma*}\phi^{-1}_\gamma G$ is an isomorphism.

(ii) For $F \in D(k_V)$ the adjunction morphism $\phi^{-1}_\gamma R\phi_{\gamma*} F \to F$ is an isomorphism if and only if $SS(F) \subset V \times \gamma^{oa}$.

In particular for any $G \in D(k_{V_\gamma})$ we have $SS(\phi^{-1}_\gamma G) \subset V \times \gamma^{oa}$.

(iii) There exists an isomorphism of functors $P_\gamma \simeq \phi^{-1}_\gamma \circ R\phi_{\gamma*}$ and the morphism $u_\gamma$ of (3.1.2) corresponds to the adjunction morphism.

In other words, using the notation $D_{V \times \gamma^{oa}}(k_V)$ of Notation 1.2.2 the functors $\phi^{-1}_\gamma$ and $R\phi_{\gamma*}$ give mutually inverse equivalences of categories between $D(k_{V_\gamma})$ and $D_{V \times \gamma^{oa}}(k_V)$.

In Proposition 3.4.1 the conditions on the microsupport are global on $V$. However we can also consider local situations using the following lemma.

**Lemma 3.4.2.** Let $U$ be an open subset of $V$ and $F \in D(k_U)$. We assume that $SS(F) \subset U \times \gamma^{oa}$. Let $x_0 \in U$. Then there exist a neighborhood $U'$ of $x_0$ in $U$ and $G \in D(k_V)$ such that $G|_{U'} \simeq F|_{U'}$ and $SS(G) \subset V \times \gamma^{oa}$. 
Corollary 3.4.3. (ii) Let $x \in V$, $\xi \in \gamma^o$ and $t > 0$ we define the truncated cone $C = C_{x,\xi,t} = (x - \gamma) \cap \{y; \langle \xi, y - x \rangle < t\}$ with vertex at $x$. We have $SS(k_C) \subset V \times \gamma^o$ by Example 1.2.3 and Theorem 1.2.13.

We can find $x$, $\xi$, $t$ such that $x_0 \in \text{Int}(C)$ and $C \subset U$. By Theorem 1.2.13 the sheaf $G = F_C$ satisfies $SS(G) \subset U \times \gamma^o$. Since $G$ has a compact support we can extend it by 0 on $V$ and we still have $SS(G) \subset V \times \gamma^o$. Setting $U' = \text{Int}(C)$, we also have $G|_{U'} \simeq F|_{U'}$. □

We thank Pierre Schapira for the following useful result.

Corollary 3.4.3. (i) Let $U$ be an open subset of $V$ and $F \in D(k_U)$. We assume that $SS(F) \subset U \times \gamma^o$. Then $SS(H^iF) \subset U \times \gamma^o$ for all $i \in \mathbb{Z}$.

(ii) Let $M$ be a manifold and $F \in D(k_M)$. Let $C \subset T^*M$ be the convex hull of $SS(F)$ in the sense that $C$ is the intersection of all closed conic subsets $S$ of $T^*M$ which contain $SS(F)$ and are fiberwise convex ($S \cap T^*_xM$ is convex for any $x \in M$). We assume that $C$ does not contain any line. Then $SS(H^iF) \subset C$ for all $i \in \mathbb{Z}$.

Proof. (i) The statement is local on $U$. Let $x_0 \in U$ be given and let $U'$, $G$ be given by Lemma 3.4.2. By Proposition 3.4.1 there exists $G' \in D(k_{V_x})$ such that $G \simeq \phi_{\gamma}^{-1}G'$ ($G' = R\phi_{\gamma}G$). Since $\phi_{\gamma}^{-1}$ is exact we have $H^iG \simeq \phi_{\gamma}^{-1}H^iG'$. By Proposition 3.4.1 again we deduce $SS(H^iG) \subset V \times \gamma^o$. Since $G|_{U'} \simeq F|_{U'}$, this gives the required bound for $SS(H^iF)$ near $x_0$.

(ii) The statement is local on $M$ and we can assume that $M$ is open in a vector space $V$. Then $T^*M = M \times V^*$. For a given $x \in M$ we can find a closed convex cone $\delta$ in $V^*$, contained in an arbitrarily small neighborhood of $C \cap \{x\} \times V^*$, and a neighborhood $U$ of $x$ such that $SS(F|_U) \subset U \times \delta$. Then the result follows from (i). □

Remark 3.4.4. 1) Let $U$ be an open subset of $V$ and $F \in \text{Mod}(k_U)$. We assume that $SS(F) \subset U \times \gamma^o$. Then, for any $s \in F(U)$, $\text{supp}(s)$ is "locally $\gamma$-closed", that is, for any $x_0 \in U$, there exists a neighborhood $B$ of $x_0$ (for the usual topology) such that $B \cap \text{supp}(s) = B \cap Z$ for some $Z \subset V$ which is closed for the $\gamma$-topology.

Indeed, we consider $U'$, $G$ given by Lemma 3.4.2. Then $s|_{U'}$ is identified with a section of $G$. By Proposition 3.4.1 there exists $G' \in \text{Mod}(k_{V_x})$ such that $G \simeq \phi_{\gamma}^{-1}G'$. Since $G_{x_0} \simeq G'_{x_0}$, there exist a section $s'$ of $G'$ (over some $\gamma$-open set containing $x_0$) and a neighborhood $U''$ of $x_0$ in $V$ such that $s'|_{U''}$ is the section of $G$ induced by $s'$. In particular $s_x \simeq s'_x$ for all $x \in U''$ and we get $\text{supp}(s) \cap U'' = \text{supp}(s') \cap U''$. Since $\text{supp}(s')$ is closed in $V_\gamma$ we are done.
2) If a subset \( Z \) of \( V \) is locally \( \gamma \)-closed, then, for any \( x_0 \in Z \), there exists a neighborhood \( B \) of 0 in \( V \) such that \( x_0 - y \in Z \) for all \( y \in B \cap \gamma \).

3.5. Remarks on projectors - Tamarkin projector

We have seen in Remark 3.1.9 that the functors \( P_\gamma, Q_\gamma : D(k_V) \to D(k_V) \) are projectors to the subcategory \( D_{V \times \gamma \circ a}(k_V) \) of \( D(k_V) \) and induce adjoints to the embedding \( \iota : D_{V \times \gamma \circ a}(k_V) \to D(k_V) \); namely \( P_\gamma \)

induces the right adjoint to \( \iota \) and \( Q_\gamma \) the left adjoint. In this paragraph we give a similar interpretation to \( P'_\gamma, Q'_\gamma \) and give a relative version of \( P'_\gamma \).

It turns out that our projectors come in pairs. Using [32, Ex. 10.15] or [23, Prop. 4.21], we can deduce from the first triangle in (3.1.3) that \( Q'_\gamma \) is a projector to the right orthogonal of \( D_{V \times \gamma \circ a}(k_V) \):

\[
D_{V \times \gamma \circ a}^{\perp,r}(k_V) = \{ F \in D(k_V); \text{Hom}(G, F) \simeq 0 \text{ for all } G \in D_{V \times \gamma \circ a}(k_V) \}. \tag{3.5.1}
\]

Indeed, applying \( P_\gamma \) to (3.1.3) and using \( P_\gamma \circ P_\gamma \simeq P_\gamma \) we obtain \( P_\gamma \circ Q'_\gamma \simeq 0 \). Then, putting \( F = Q'_\gamma(G) \) in (3.1.3) we have \( Q'_\gamma \simeq Q'_\gamma \circ Q'_\gamma \). This proves that \( Q'_\gamma \) is a projector. Using \( \text{Hom}(P_\gamma(F), P_\gamma(G)) \simeq \text{Hom}(P_\gamma(F), Q'_\gamma(G)) \simeq 0 \) and we can deduce that the image of \( Q'_\gamma \) is indeed \( D_{V \times \gamma \circ a}^{\perp,r}(k_V) \). More precisely \( Q'_\gamma \) induces a left adjoint to the embedding of \( D_{V \times \gamma \circ a}(k_V) \).

In the same way, we can see that \( P'_\gamma \) is also a projector, with image the left orthogonal of \( D_{V \times \gamma \circ a}(k_V) \):

\[
D_{V \times \gamma \circ a}^{\perp,l}(k_V) = \{ F \in D(k_V); \text{Hom}(F, G) \simeq 0 \text{ for all } G \in D_{V \times \gamma \circ a}(k_V) \}. \tag{3.5.2}
\]

and that it induces a right adjoint to the embedding of \( D_{V \times \gamma \circ a}^{\perp,l}(k_V) \).

The relation between \( P'_\gamma, Q'_\gamma \) is not the same as the relation between \( P_\gamma, Q_\gamma \). In Remark 3.1.9 we deduced that \( Q_\gamma \) was a projector from the fact that its was adjoint to \( P_\gamma \). To deduce that they have the same image we used the more precise fact: \( P_\gamma \) induces the right adjoint to \( \iota \) and \( P_\gamma \) is the right adjoint to \( Q_\gamma \). Now \( P'_\gamma \) induces the right adjoint to the embedding of its image but \( P'_\gamma \) is left adjoint to \( Q'_\gamma \); so we cannot conclude that \( P'_\gamma \) and \( Q'_\gamma \) have the same image.

The fact that the embedding \( \iota \) has both a left and a right adjoint comes from the property that \( D_{V \times \gamma \circ a}(k_V) \) has arbitrary small sums and products and \( \iota \) commutes with sums and products (this is the Brown representability theorem – see for example [32, §14.2]). However \( D_{V \times \gamma \circ a}^{\perp,l}(k_V) \) is not stable by infinite products even for \( V = \mathbb{R}, \gamma = \)
[0, +\infty[ (see Example 3.5.1) and we cannot apply Brown theorem to find a new projector.

**Example 3.5.1.** In the case \( V = \mathbb{R} \), \( \gamma = [0, +\infty[ \), we have, for any \( x \in \mathbb{R} \), \( P'_\gamma(k_{[x, +\infty[}) \simeq k_{[x, +\infty[} \). Let us check that \( P'_\gamma(F) \not\cong F \), where \( F = \prod_{i \in \mathbb{N}} k_{[-i, +\infty[} \), thus proving that \( D^L_{V \times \gamma}(k_V) \) is not stable by infinite products.

We compute the stalks \( (P'_\gamma(F))_x \) using (3.1.5) and (1.5.2). We find \( (P'_\gamma(F))_x \simeq R\Gamma_c(\mathbb{R}; F_{I_\omega}) \), with \( I_\omega = ]-\infty, x[ \). Using the exact sequence \( 0 \to G \to k^N_{\mathbb{R}} \to F \to 0 \), with \( G = \prod_{i \in \mathbb{N}} k_{]-\infty, -i[} \), and \( R\Gamma_c(\mathbb{R}; L_{I_\omega}) \simeq 0 \) for any constant sheaf \( L \), we obtain \( (P'_\gamma(F))_x \simeq R\Gamma_c(\mathbb{R}; G_{I_\omega})[1] \). Now \( G \simeq \bigoplus_{i \in \mathbb{N}} k_{]-\infty, -i[} \) because this sum is locally finite. The functors \( R\Gamma_c(\mathbb{R}; \cdot) \) and \( (-)_{I_\omega} \) commute with direct sums. Hence \( R\Gamma_c(\mathbb{R}; G_{I_\omega}) \simeq k^{(\mathbb{N})} \) for \( x > 0 \). On the other hand we have \( F_x \simeq k^N \) for \( x > 0 \). Hence \( P'_\gamma(F) \not\cong F \).

It should be noted that the cut-off functor \( P_\gamma \) is defined in [31] in a relative situation where \( V \) is replaced by \( M \times V \) for some manifold \( M \). In this general setting, \( P_\gamma \) is defined on \( D(k_{M \times V}) \) and projects onto the subcategory \( D(k_{M \times V}) \) formed by the \( F \) such that \( SS(F) \subset T^*M \times (V \times \gamma^0) \) (Notation 1.2.2). We have \( P_\gamma(F) = Rq_{2*}(k_{M \times \gamma} \otimes q_1^{-1}(F)) \), where \( q_1, q_2 \) are now the projections \( M \times V^2 \to M \times V \) and \( \gamma \) is defined as in (3.1.1).

For later use we quote some properties of \( P_\gamma \) already considered by Tamarkin in [47] in the case \( V = \mathbb{R} \), \( \gamma = [0, +\infty[ \) or \( \gamma = ]-\infty, 0[ \). We introduce coordinates \( (t; \tau) \) on \( T^*\mathbb{R} \) and local coordinates \( (x, t; \xi, \tau) \) on \( T^*(M \times \mathbb{R}) \). We denote for short by \( \{\tau \geq 0\} \) or \( \{\tau > 0\} \) the subsets of \( T^*(M \times \mathbb{R}) \) defined by the corresponding conditions on \( \tau \). In this relative setting the projector \( P_{[-\infty, 0]} : D(k_{M \times \mathbb{R}}) \to D_{\{\tau \geq 0\}}(k_{M \times \mathbb{R}}) \) can be rewritten as

\[
P_{[-\infty, 0]}(F) = Rs_*(q_1^{-1}(F) \otimes q_2^{-1}(k_{[0, +\infty[})),
\]

where \( s, q_1 : M \times \mathbb{R}^2 \to M \times \mathbb{R} \) and \( q_2 : M \times \mathbb{R}^2 \to \mathbb{R} \) are defined by \( s(x, t_1, t_2) = (x, t_1 + t_2) \), \( q_1(x, t_1, t_2) = (x, t_1) \) and \( q_2(x, t_1, t_2) = t_2 \).

In [47] Tamarkin was rather interested in the projector \( P'_{[0, +\infty[} \), defined on the category \( D(k_{M \times \mathbb{R}}) \) with image

\[
D^L_{\{\tau \leq 0\}}(k_{M \times \mathbb{R}}) = \{ F \in D(k_{M \times \mathbb{R}}) ; \text{Hom}(F, G) \simeq 0 \text{ for all } G \in D_{\{\tau \leq 0\}}(k_{M \times \mathbb{R}}) \}
\]
as noted in (3.5.2). Using (3.1.5) and noticing that \( D'(k_{[0, +\infty[}) \simeq D'(k_{[0, +\infty[}) \simeq k_{[-\infty, 0]} \), we see that \( P_{[-\infty, 0]} \) and \( P'_{[0, +\infty[} \) have very similar
expressions. With the notations of (3.5.3) we can rewrite
\[ P'_{[0, +\infty]}(F) = \text{R} s_t(q_1^{-1}(F) \otimes q_2^{-1}(k_{[0, +\infty]})). \]
Using this formula we find the bound \( \text{SS}(P'_{[0, +\infty]}(F)) \subset \{ \tau \geq 0 \} \), like \( \text{SS}(P_{[-\infty, 0]}(F)) \) in (3.1.6). This proves that \( D_{t \geq 0}^{\perp} k_{M \times \mathbb{R}} \) is in fact contained in \( D_{\tau \geq 0} k_{M \times \mathbb{R}} \).

An important remark of Tamarkin is that the functors \( P_{[-\infty, 0]} \) or \( P'_{[0, +\infty]} \) have a natural morphism not only to the identity functor but also to the direct image functor \( T_{c*} \) for \( c \geq 0 \), where \( T_{c*} \) denotes the translation along \( \mathbb{R} \)
\[ T_{c*} : M \times \mathbb{R} \rightarrow M \times \mathbb{R}, \quad (x, t) \mapsto (x, t + c). \]
In other words the objects \( F \) of \( D_{\{ \tau \geq 0 \}} k_{M \times \mathbb{R}} \) or \( D_{\{ \tau \leq 0 \}}^{\perp} k_{M \times \mathbb{R}} \) come with a natural morphism \( \tau_c(F) : F \rightarrow T_{c*}(F) \) for \( c \geq 0 \).

To define \( \tau_c \) we first remark that we have an isomorphism of functors \( T_{c*} \circ P_{[-\infty, 0]} \simeq P_{[-\infty, 0]} \circ T_{c*} \) and that
\[ (3.5.5) \quad T_{c*} \circ P_{[-\infty, 0]}(F) \simeq \text{R} s_t(q_1^{-1}(F) \otimes q_2^{-1}(k_{[c, +\infty]})). \]
For \( c \geq 0 \) the natural morphism \( k_{[0, +\infty]} \rightarrow k_{[c, +\infty]} \) induces the morphism of functors (3.5.6) and hence the morphism (3.5.7) for sheaves in the image of \( P_{[-\infty, 0]} \):
\[ (3.5.6) \quad \tau_c : P_{[-\infty, 0]} \rightarrow T_{c*} \circ P_{[-\infty, 0]}, \]
\[ (3.5.7) \quad \tau_c(F) : F \rightarrow T_{c*}(F), \quad \text{for } F \in D_{\{ \tau \geq 0 \}} k_{M \times \mathbb{R}}. \]
Tamarkin then used this morphism \( \tau_c \) to give non-displaceability conditions on subsets of \( T^* M \). We will consider a variant of Tamarkin’s criterion by looking at the minimal \( c \) such that \( \tau_c(F) = 0 \) (see § 6.2).

Remark 3.5.2. Assuming \( \gamma \subset V \) is a closed convex cone which contains no line, we can also easily build projectors which cut the microsupport by the complement of \( \text{Int}(\gamma^o) \). Let us set \( Z_\gamma = V \times (V^* \setminus \text{Int}(\gamma^o)) \) and let \( D_{Z_\gamma}(k_V) \) be the subcategory of \( D(k_V) \) of sheaves with micro-support contained in \( Z_\gamma \). We can generalize Tamarkin’s construction in higher dimension and define projectors from \( D(k_V) \) to itself with image the subcategory \( D_{Z_\gamma}(k_V) \) or its left or right orthogonal. For example it is proved in [23 Prop. 4.21] that \( L_\gamma : F \mapsto F \circ k_\gamma^o \) is a projector on \( D(k_V) \) with image \( D_{Z_\gamma}^{\perp, o}(k_V) \). We also know that \( D_{Z_\gamma}^{\perp, o}(k_V) \) is contained in \( D_{V \times \gamma^o}(k_V) \), which is the image of \( P_\gamma^o \) (we can prove that \( \text{SS}(L_\gamma(F)) \subset V \times \gamma^o \) like (3.1.8), or we refer to loc. cit.). We recall that \( P_\gamma^o \) and \( L_\gamma \) often coincide (see Lemma 3.1.1). Of course, when \( V = \mathbb{R} \) we have \( Z_\gamma = V \times \gamma^o \), \( L_\gamma = P_\gamma^o \) and we recover Tamarkin’s projector (see around (3.5.4)).
Part 4. Constructible sheaves in dimension 1

In this part we apply Gabriel’s theorem to describe the constructible sheaves on the real line and the circle with coefficients in a field $k$.

4.1. Gabriel’s theorem

We give a quick reminder of a part of Gabriel’s theorem that we will use in this part. We follow the presentation of Brion’s lecture [9] on the subject and we refer the reader to [9] for further details. In this section $k$ is a field.

A quiver is a finite directed graph, that is, a quadruple $Q = (Q_0, Q_1, s, t)$ where $Q_0, Q_1$ are finite sets (the set of vertices, resp. arrows) and $s, t : Q_1 \rightarrow Q_0$ are maps assigning to each arrow its source, resp. target. A representation of a quiver $Q$ consists of a family of $k$-vector spaces $V_i$ indexed by the vertices $i \in Q_0$, together with a family of linear maps $f_{\alpha} : V_{s(\alpha)} \rightarrow V_{t(\alpha)}$ indexed by the arrows $\alpha \in Q_1$. For a representation $(\{V_i, f_{\alpha}\})$ the dimension vector is $(\dim V_i)_{i \in Q_0}$. The space $\mathbb{R}^{Q_0}$ of dimension vectors is endowed with the Tits form defined by $q_Q(d) = \sum_{i \in Q_0} d_i^2 - \sum_{\alpha \in Q_1} d_{s(\alpha)}d_{t(\alpha)}$.

A quiver is of finite orbit type if it has only finitely many isomorphism classes of representations of any prescribed dimension vector. Gabriel’s theorem describes the quivers of finite orbit type. It says that they are the quivers with a positive defined Tits form and also says that this is equivalent to be of type $A, D, E$.

Another part of Gabriel’s theorem gives the structure of the representations of the quivers of finite type. A representation is said indecomposable if it cannot be split as the sum of two non zero representations. A representation $V$ is Schur if $\text{Hom}(V, V) \simeq k \cdot \text{id}_V$.

**Theorem 4.1.1** (see Theorem 2.4.3 in [9]). Assume that the Tits form $q_Q$ is positive definite. Then:

(i) Every indecomposable representation is Schur and has no non-zero self-extensions.

(ii) The dimension vectors of the indecomposable representations are exactly those $d \in \mathbb{N}^{Q_0}$ such that $q_Q(d) = 1$.

(iii) Every indecomposable representation is uniquely determined by its dimension vector, up to isomorphism.

**Remark 4.1.2.** We are actually only interested in quivers of type $A_m$, that is, quivers whose underlying graph is the linear graph with $m$ vertices and $m - 1$ edges. In this case we have $Q_0 = \{0, \ldots, m - 1\}$
and we find
\[ q_Q(d) = \frac{1}{2}(d_0^2 + \sum_{i=1}^{m-1} (d_i - d_{i-1})^2 + d_{m-1}^2). \]

Hence a dimension vector \( d \) satisfies the condition \( q_Q(d) = 1 \) if and only if there exist \( i \leq j \in \{0, \ldots, m-1\} \) such that \( d_k = 1 \) if \( i \leq k \leq j \) and \( d_k = 0 \) else.

### 4.2. Constructible sheaves on the real line

We apply the results of Section 4.1 to sheaves on \( \mathbb{R} \) with coefficients in a field \( k \).

Let \( \mathcal{X} = \{x_1 < \cdots < x_n\} \) be a finite family of points in \( \mathbb{R} \). We denote by \( \text{Mod}_\mathcal{X}(k_\mathbb{R}) \) the category of constructible sheaves on \( \mathbb{R} \) with respect to the stratification induced by \( \mathcal{X} \). Setting \( x_0 = -\infty \) and \( x_{n+1} = +\infty \), a sheaf \( F \) belongs to \( \text{Mod}_\mathcal{X}(k_\mathbb{R}) \) if and only if the stalks \( F_y \) are finite dimensional for all \( y \in \mathbb{R} \) and the restrictions \( F|_{x_k, x_{k+1}} \) are constant for \( k = 0, \ldots, n \). If \( I \) is an interval of \( \mathbb{R} \) and \( x_1, \ldots, x_n \in I \) we define in the same way \( \text{Mod}_I(k_\mathbb{R}) \).

We say that a sheaf \( F \) on \( \mathbb{R} \) is constructible if, for any \( n > 0 \), \( F|_{-n,n} \in \text{Mod}_\mathcal{X}(k_\mathbb{R}) \) for some finite family \( \mathcal{X} = \{x_1 < \cdots < x_k\} \) of \( [-n, n] \). We denote by \( \text{Mod}_c(k_\mathbb{R}) \) the category of constructible sheaves on \( \mathbb{R} \).

A sheaf \( F \in \text{Mod}_\mathcal{X}(k_\mathbb{R}) \) is determined by the data of the spaces of sections
\[
\begin{align*}
V_{2i+1} &= F([x_i, x_{i+2}]) \isom F_{x_{i+1}}, \quad \text{for } i = 0, \ldots, n-1, \\
V_{2i} &= F([x_i, x_{i+1}]), \quad \text{for } i = 0, \ldots, n,
\end{align*}
\]
(together with the restriction maps \( V_{2i+1} \to V_{2i} \) and \( V_{2i+1} \to V_{2i+2} \) for \( i = 0, \ldots, n-1 \)). Conversely, any such family of vector spaces \( \{V_i\}_{i=0}^{2n} \) and linear maps defines a sheaf in \( \text{Mod}_\mathcal{X}(k_\mathbb{R}) \). Hence (4.2.1) gives an equivalence between \( \text{Mod}_\mathcal{X}(k_\mathbb{R}) \) and the category of representations of the quiver \( Q = (Q_0, Q_1, s, t) \) of type \( A_{2n+1} \) where \( Q_0 = \{0, \ldots, 2n\} \) and there is exactly one arrow in \( Q_1 \) from \( 2i-1 \) to \( 2i-2 \) and from \( 2i-1 \) to \( 2i \), for \( i = 1, \ldots, n \).

Since \( Q \) is of type \( A_{2n+1} \), we can apply Gabriel’s theorem and Remark 4.1.2. Hence the indecomposable representations of \( Q \) are in bijection with the dimension vectors \( d \) such that \( d_k = 1 \) if \( i \leq k \leq j \) and \( d_k = 0 \) else, for some \( i \leq j \in \{0, \ldots, 2n\} \). Through the equivalence (4.2.1) the corresponding sheaves in \( \text{Mod}_\mathcal{X}(k_\mathbb{R}) \) are the constant sheaves \( k_I \) on the intervals \( I \) with ends \( -\infty, x_1, \ldots, x_n \) or \( +\infty \) (the intervals can be open, closed or half-closed).
Gabriel’s theorem gives the following decomposition result for constructible sheaves with compact support. Extensions of Gabriel’s theorem gives the general case (see for example Theorem 1.1 in [13]). We can also deduce the general case from the case of compact support; this is done in [33, Thm. 1.17] and we reproduce the proof below.

**Corollary 4.2.1.** We recall that $k$ is a field. Let $F \in \text{Mod}_c(k_{\mathbb{R}})$. Then there exist a locally finite family of distinct intervals $\{I_a\}_{a \in A}$ and integers $\{n_a\}_{a \in A}$ such that

$$F \simeq \bigoplus_{a \in A} k_{n_a}^{I_a}.$$  \hspace{1cm} (4.2.2)

Moreover this decomposition is unique in the following sense. If we have another decomposition $F \simeq \bigoplus_{b \in B} k_{n_b}^{I_b}$ like (4.2.2), then there exists a bijection $\sigma: A \cong B$ such that $I_{\sigma(a)} = I_a$ and $m_{\sigma(a)} = n_a$ for all $a \in A$.

**Proof.** (i) For an integer $n \geq 1$ we set $U_n = [−n, n]$. Identifying $U_n$ with $\mathbb{R}$, the restriction $F|_{U_n}$ belongs to $\text{Mod}_c(k_{\mathbb{R}})$ for some finite set of points $x$. Through the equivalence (4.2.1) (and coming back to $U_n$) Gabriel’s theorem gives a decomposition $F|_{U_n} \simeq \bigoplus_{c \in C_n} k_{I_c}$, where $C_n$ is a finite set of intervals of $U_n$. Each factor $k_{I_c}$ has multiplicity 1, but we may have $I_c = I_d$ for $c, d \in C_n$. This decomposition is unique by (i) of Theorem 4.1.1 and by the Krull-Schmidt theorem.

The uniqueness implies that we can find an injective map $\gamma_n: C_n \rightarrow C_{n+1}$, for any $n \geq 1$, such that $I_c^n = U_n \cap I_{\gamma_n(c)}$ for any $c \in C_n$. We set $C = \limsup C_n$. For $c \in C$, represented by $\tilde{c} \in C_n$ for some $n$, we set $I_c = \bigcup_{m \geq n} I_{\gamma_n^{-1}(\tilde{c})}^m$, where $\gamma_n = \gamma_{n-1} \circ \cdots \circ \gamma_0$. We then have $I_c \cap U_m = I_{\gamma_n(c)}^m$. We remark that the obvious map $C_n \rightarrow C$ is injective, for any $n$.

(ii) Let $c \in C$ be given. We claim that we can find $i_c: k_{I_c} \rightarrow F$ and $p_c: F \rightarrow k_{I_c}$ such that $p_c \circ i_c = \text{id}_{k_{I_c}}$. For $n \geq 1$ we define

$$E_n = \text{Hom}(k_{I_c}|_{U_n}, F|_{U_n}) \times \text{Hom}(F|_{U_n}, k_{I_c}|_{U_n}),$$

$$E_n' = \{(i, p) \in E_n; \ p \circ i = \text{id}_{k_{I_c}|_{U_n}}\}.$$  \hspace{1cm} (4.2.3)

The restriction morphisms induce $e_n^m: E_m \rightarrow E_n$ for $m \geq n$. We clearly have $e_n^m(E_n) \subset E_n$. We remark that $E_n' \neq \emptyset$ because $k_{I_c \cap U_n}$ is a direct summand of $F|_{U_n}$. Let us prove that $e_n^m(E_n) = e_n^m(E_m) \cap E_n'$, for any $m \geq n$. This is clear when $I_c \cap U_n = \emptyset$ (in this case $E_n = E_n' = \{(0, 0)\}$). If $I_c \cap U_n \neq \emptyset$, then we have

$$k \simeq \text{Hom}(k_{I_c}|_{U_m}, k_{I_c}|_{U_n}) \xrightarrow{r} \text{Hom}(k_{I_c}|_{U_n}, k_{I_c}|_{U_n}) \simeq k,$$

and the restriction map $r$ is an isomorphism. In particular $r(u) = \text{id}_{k_{I_c}|_{U_m}}$ implies $u = \text{id}_{k_{I_c}|_{U_m}}$ and we get $e_n^m(E_m) = e_n^m(E_m) \cap E_n'$. Since
$F$ is constructible, the spaces $E_n$ are all finite dimensional. Hence, for a given $n$, the sequence $\{e^n_m(E'_m)\}_{m \geq n}$ stabilizes and it follows that $\{e^n_m(E'_m)\}_{m \geq n}$ also stabilizes. We set $E'^{\infty}_n = \bigcap_{m \geq n} e^n_m(E'_m)$. The maps $e^n_m$ induce surjective maps $E'^{\infty}_m \to E'^{\infty}_n$, for all $m \geq n$. Finally we have $E'^{\infty}_n \neq \emptyset$ for all $n$; indeed $E'^{\infty}_n = e^n_n(E'_n)$, for some $m$ big enough, and we have $E'_n \neq \emptyset$. Hence $\{E'^{\infty}_n\}_{n \geq 1}$ is a projective system of non empty sets, with surjective structural maps. It follows that $\lim_{\mathcal{N}} E'^{\infty}_n \neq \emptyset$ and any element in this limit is a sequence of compatible pairs of morphisms $(i_n, p_n) \in E'_n$ which glue into a pair $(i_c, p_c)$ as claimed.

(iii) Let $n \geq 1$ be an integer. We claim that we can write $F \simeq F_1 \oplus \bigoplus_{a \in A_1} k_{I_a}$, where $F_1|_{U_n} \simeq 0$ and $A_1$ is some finite family of intervals of $\mathbb{R}$.

If $F|_{U_n} \simeq 0$, the claim is trivial. If not, we pick $c \in C$ such that $I_c \cap U_n \neq \emptyset$. Using $(i_c, p_c)$ found in (ii) we write $F \simeq k_{I_c} \oplus F^1$. Then $F^1$ is constructible. If $F^1 \simeq 0$ we are done. If not, we apply the same argument to $F^1$ and write $F^1|_{U_n} \simeq k_{I_1} \oplus F^2$ for some interval $I_1$ meeting $U_n$ (the interval $I_1$ is in the family $C$ associated with $F$ but we do not need to know that). We go on with $F^2$ and write inductively $F \simeq k_{I_c} \oplus k_{I_1} \oplus \cdots \oplus k_{I_k} \oplus F^{k+1}$, where the $I_j$’s meet $U_n$, as long as $F^k|_{U_n} \simeq 0$. Since $F$ is constructible, the space $\text{Hom}(F|_{U_n}, F|_{U_n})$ is finite dimensional. Since the $I_j$’s meet $U_n$, the elements $\text{id}_{k_{I_j}}$ give a free family in $\text{Hom}(F|_{U_n}, F|_{U_n})$. Hence the process will stop after finitely many steps and the claim is proved.

(iv) Using (iii) we can write inductively for $k \geq 1$,

\[(D_k) \quad F \simeq F_k \oplus \bigoplus_{i=1}^{k} G_i,\]

where $F_k|_{U_k} \simeq 0$, $G_i = \bigoplus_{a \in A_i} k_{I_a}$, where $A_i$ is a finite family of intervals of $\mathbb{R}$ and $G_i|_{U_{i-1}} \simeq 0$ for $i \geq 2$. Indeed (iii) applied with $F$ and $n = 1$ gives the first step and (iii) applied with $F_k$ and $n = k + 1$ gives the $(k + 1)^{th}$ step. This inductive construction also makes the decompositions $(D_k)$ compatible in the sense that the projection $u_i: F \to G_i$ deduced from $(D_k)$ for $k \geq i$ is actually independent of $k$. Since $F_k|_{U_k} \simeq 0$, the sum $\sum_{i=1}^{k} u_i|_{U_k}: F|_{U_k} \to \bigoplus_{i=1}^{k} G_i|_{U_k}$ is an isomorphism. We set $G = \bigoplus_{i=1}^{\infty} G_i$ and define $u = \sum_{i=1}^{\infty} u_i: F \to G$. This last sum makes sense because, over each interval $U_k$, only the terms $u_i$ for $i = 1, \ldots, k$ are non zero. Since $u|_{U_k}$ is an isomorphism for each $k$, $u$ itself is an isomorphism. Putting together the intervals which appear several times (only finitely many times by constructibility) we obtain a decomposition of $F$ as stated in the corollary.
(v) To prove the uniqueness statement we first consider a bounded interval $I_a$, for $a \in A$. We choose an interval $U_n$ such that $\overline{I}_a \subset U_n$. Then $k_{I_a}$ appears in the decomposition of $F|_{U_n}$ with the same multiplicity as in $F$. By the uniqueness of the decomposition of $F|_{U_n}$ we deduce that $I_a = J_b$ for some $b \in B$ and $m_b = n_a$. Defining $A' = \{a \in A; I_a \text{ is bounded}\}$ and $B'$ similarly, we thus have a bijection between $A'$ and $B'$ with respects the multiplicities.

We are left with an isomorphism $\bigoplus_{a \in A \setminus A'} k_{I_a}^{n_a} \cong \bigoplus_{b \in B \setminus B'} k_{J_b}^{m_b}$. Now, an unbounded interval $I$ with one end $x$ is determined by $I \cap U_n$, as soon as $x \in U_n$. Defining $A'' = \{a \in A; I_a \text{ is unbounded and not equal to } \mathbb{R}\}$ and $B''$ similarly, we can then identify $A''$ with $B''$ as in the bounded case. Now we are left with the summand $k_{I_a}$, indexed by, say $a_0 \in A$, $b_0 \in B$, and we have $k_{I_a}^{n_{a_0}} \cong k_{I_{b_0}}^{m_{b_0}}$. Hence $n_{a_0} = m_{b_0}$ and this concludes the proof. □

We also remark that (i) of Theorem 4.1.1 is given in our case by the following easy result, which we quote for later use, and the fact that $\text{Ext}^1(k_I, k_I) \cong 0$.

**Lemma 4.2.2.** Let $I, J$ be two intervals of $\mathbb{R}$. Then

$$\text{Hom}(k_I, k_J) \cong \begin{cases} k & \text{if } I \cap J \neq \emptyset \text{ and } I \cap J \text{ is closed in } I \text{ and open in } J, \\ 0 & \text{else.} \end{cases}$$

In particular, if $I$ and $J$ are distinct, then we have $\text{Hom}(k_I, k_J) \cong 0$ or $\text{Hom}(k_J, k_I) \cong 0$.

### 4.3. Constructible sheaves on the circle

In this section we extend Corollary 4.2.1 to the circle. Like in the case of $\mathbb{R}$ the result is also a particular case of quiver representation theory and Auslander-Reiten theory (see for example [43] §3.6 p.153 and Theorem 5 p.158). However it is quicker to prove the facts we need than recall these general results.

We denote by $S^1$ the circle and we let $e : \mathbb{R} \to S^1 \cong \mathbb{R}/2\pi\mathbb{Z}$ be the quotient map. We use the coordinate $\theta$ on $S^1$ defined up to a multiple of $2\pi$. We also denote by $T : \mathbb{R} \to \mathbb{R}$ the translation $T(x) = x + 2\pi$. We recall that $k$ is a field. We say that $F \in \text{Mod}(k_{S^1})$ is constructible if $F|_I$ is constructible in the sense of §1.2 for any arc $I \subset S^1$. We denote by $\text{Mod}_c(k_{S^1})$ the category of such sheaves.

Since $e$ is a covering map, we have an isomorphism of functors $e^! \cong e^{-1}$, hence an adjunction $(e^!, e^{-1})$. For any $n \in \mathbb{Z}$ we have $e \circ T^n = e$, hence natural isomorphisms of functors $(T^n)^{-1} e^{-1} \cong e^{-1}$.
and $e_i T^n \simeq e_i$. For $G \in \text{Mod}(k_{\mathbb{R}})$ the isomorphism $e_i(T^n(G)) \cong e_i(G)$ gives by adjunction $i_n(G) : T^n(G) \to e^{-1}e_i(G)$. For $x \in \mathbb{R}$ we have $(e^{-1}e_i(G))_x \simeq (e_i(G))_e(x) \simeq \bigoplus_{n \in \mathbb{Z}} G_{T^n(x)}$. We deduce that the sum of the $i_n(G)$ gives an isomorphism

$$\bigoplus_{n \in \mathbb{Z}} T^n(G) \cong e^{-1}e_i(G).$$

Let $I$ be a bounded interval of $\mathbb{R}$. We have $e_i(k_I) \cong e_* (k_I)$. Let $A_I$ be the algebra $A_I = \text{Hom}(e_*(k_I), e_*(k_I))$. The adjunction $(e^{-1}, e_*)$ gives a morphism $e^{-1}e_*(k_I) \to k_I$. Using also the adjunction $(e_!, e^{-1})$ we obtain a natural morphism

$$\varepsilon_I : A_I \simeq \text{Hom}(k_I, e^{-1}e_*(k_I)) \to \text{Hom}(k_I, k_I) \simeq k.$$

**Lemma 4.3.1.** Let $I$ be a bounded interval of $\mathbb{R}$ and let $A_I$ be the algebra $A_I = \text{Hom}(e_*(k_I), e_*(k_I))$. Then the morphism $\varepsilon_I$ defined in (4.3.2) is an algebra morphism and $\ker(\varepsilon_I)$ is a nilpotent ideal of $A_I$. Moreover, a morphism $u \in A_I$ is an isomorphism if and only if $\varepsilon_I(u) \neq 0$.

More precisely, if $I$ is closed or open, then $\varepsilon_I$ is an isomorphism. If $I$ is half-closed, say $I = [a, x]$ or $I = [x, a]$ and we set $E_a = I \cap e^{-1}(e(a))$, then the identification $(e_*(k_I))_{e(a)} \simeq k^{E(a)}$ induces a morphism

$$A_I \to \text{Hom}(k^{E(a)}, k^{E(a)}), \quad \varphi \mapsto \varphi_{e(a)}$$

which identifies $A_I$ with the subalgebra of matrices generated by the standard nilpotent matrix of order $|E(a)|$.

**Proof.** Using $e^{-1}e_*(k_I) \simeq \bigoplus_{n \in \mathbb{Z}} T^n(k_I)$ and Lemma 4.2.2 the cases $I$ closed or open are obvious. If $I$ is half-closed of length $l$, we find $A_I \simeq k^{[E(a)]}$ as a vector space. Assuming $I = [a, x]$ (the case $[x, a]$ is similar), a basis of $A_I$ is given by the morphisms $e_*(u_n)$, for $n = 0, \ldots, |E(a)| - 1$, where $u_n : k_I \to k_{T^n(I)}$ is the natural morphism and we use the natural identification $\phi_n : e_*(k_{T^n(I)}) \simeq e_*(k_I)$. At the level of stalks $\phi_n$ identifies the summands $(k_I)_{a+2nk}$ of $(e_*(k_I))_{e(a)}$ and $(k_{T^n(I)})_{a+2nk}$ of $(e_*(k_{T^n(I)}))_{e(a)}$. We obtain that $(e_*(u_n))$ acts on $k^{E(a)}$ by $(s_1, s_2, \ldots) \mapsto (s_1 + n, s_2 + n, \ldots)$. We deduce that the image of $A_I$ in $\text{End}(k^{E(a)})$ is as claimed in the lemma.

The characterization of the isomorphisms then follows from the structure of $A_I$. \hfill $\square$

**Lemma 4.3.2.** Let $F \in \text{Mod}(k_{S_1})$. Let $I$ be a bounded interval of $\mathbb{R}$ such that $k_I$ is a direct summand of $e^{-1}(F)$. Then $e_*(k_I)$ is a direct summand of $F$.

**Proof.** We let $i_0 : k_I \to e^{-1}(F) \simeq e'(F)$ and $p_0 : e^{-1}(F) \to k_I$ be morphisms such that $p_0 \circ i_0 = \text{id}_{k_I}$ and we denote by $i'_0 : e_*(k_I) \simeq
Let \( p'_0 : F \to e_*(k_I) \) be the adjoint morphisms. Then we deduce that \( p'_0 \circ i'_0 = \text{id}_{e_*(k_I)} \) but it is enough to see that \( p'_0 \circ i'_0 \) is an isomorphism. For this we use Lemma 4.3.1. Let us compute \( \varepsilon (p'_0 \circ i'_0) \).

**Lemma 4.3.3.** Let \( F \in \text{Mod}_c(k_{S^1}) \). We assume that \( F \) is not locally constant. Then there exists a bounded interval \( I \) of \( \mathbb{R} \) such that \( k_I \) is a direct summand of \( e^{-1}(F) \).

**Proof.** By Corollary 4.2.1 there exist a locally finite family of intervals \( \{I_a\}_{a \in A} \) and integers \( \{n_a\}_{a \in A} \) such that \( e^{-1}(F) \simeq \bigoplus_{a \in A} k_{I_a}^{n_a} \). Since \( F \) is not locally constant, one of these intervals, say \( I \), is not \( \mathbb{R} \). Let \( T \) be the translation \( T(x) = x + 2\pi \). Since \( T^{-1}e^{-1}(F) \simeq e^{-1}(F) \), the intervals \( T^n(I) \) also appear in the decomposition, for all \( n \in \mathbb{Z} \). If \( I \) were not bounded, this would contradict the constructibility of \( F \).

**Proposition 4.3.4.** Let \( F \in \text{Mod}_c(k_{S^1}) \). Then there exist a finite family \( \{\{I_a, n_a\}\}_{a \in A} \), of bounded intervals and integers, and a locally constant sheaf of finite rank \( L \in \text{Mod}(k_{S^1}) \) such that

\[
F \simeq L \bigoplus \bigoplus_{a \in A} e_*(k_{I_a}^{n_a}).
\]

**Proof.** (i) We choose a finite stratification \( \{\Sigma_i\}_{i \in I} \) of \( S^1 \) such that \( F \) is constructible with respect to \( \{\Sigma_i\} \). We choose one point \( \theta_i \in \Sigma_i \) for each \( i \in I \) and set \( r(F) = \sum_{i \in I} \dim(F_{\theta_i}) \). We prove the proposition by induction on \( r(F) \). If \( r(F) = 0 \), then \( F \simeq 0 \) and the result is clear.

(ii) We assume \( r(F) \neq 0 \). If \( F \) is locally constant, the result is clear. Else, by Lemmas 4.3.3 and 4.3.2 there exists a bounded interval \( I \) of \( \mathbb{R} \) such that \( F \simeq e_*(k_I) \oplus F' \) for some \( F' \in \text{Mod}_c(k_{S^1}) \). Then \( r(F') < r(F) \) and the induction proceeds.

### 4.4. Cohomological dimension 1

The decomposition results for sheaves in dimension 1 extend to the derived category. For \( M = \mathbb{R} \) or \( M = S^1 \) we denote by \( D^b(k_M) \) the full subcategory of \( D^b(k_M) \) formed by the \( F \) such that \( H^i F \in \text{Mod}_c(k_M) \) for all \( i \in \mathbb{Z} \). We first recall a well-known decomposition result (see for example [32, Ex. 13.22]).
Lemma 4.4.1. Let $C$ be an abelian category and $X \in \mathbb{D}^b(C)$ a complex such that $\operatorname{Ext}^k(H^jX, H^jX) \simeq 0$ for all $i, j \in \mathbb{Z}$ and all $k \geq 2$. Then $X$ is split, that is, there exists an isomorphism $X \simeq \bigoplus_{i \in \mathbb{Z}} H^iX[-i]$ in $\mathbb{D}^b(C)$.

This applies in particular to constructible sheaves in dimension 1. Indeed, if $M = \mathbb{R}$ or $M = S^1$ and $k$ is a field, we have $\operatorname{Ext}^k(F, G) \simeq 0$ for all $F, G \in \operatorname{Mod}_c(k_M)$ and for all $k \geq 2$. We deduce:

Lemma 4.4.2. Let $M = \mathbb{R}$ or $M = S^1$ and let $k$ be a field. Then, for all $F \in \mathbb{D}^b(k_M)$ we have $F \simeq \bigoplus_{i \in \mathbb{Z}} H^iF[-i]$ in $\mathbb{D}^b(k_M)$.

Using Corollary 4.2.1 and Proposition 4.3.4 we obtain immediately:

Corollary 4.4.3. Let $k$ be a field. Let $M$ be $\mathbb{R}$ or $S^1$ and let $F \in \mathbb{D}^b(k_M)$ be a constructible object.

(i) If $M = \mathbb{R}$, then there exist a locally finite family of intervals $\{I_a\}_{a \in A}$ and integers $\{n_a\}_{a \in A}$, $\{d_a\}_{a \in A}$ such that

$$F \simeq \bigoplus_{a \in A} k_{n_a}^{d_a}[d_a].$$

(ii) If $M = S^1$, then there exist a finite family of bounded intervals $\{I_a\}_{a \in A}$, integers $\{n_a\}_{a \in A}$, $\{d_a\}_{a \in A}$ and $L \in \mathbb{D}^b(k_{S^1})$ with locally constant cohomology sheaves of finite rank such that

$$F \simeq L \oplus \bigoplus_{a \in A} e_* \left(k_{n_a}^{d_a}\right)[d_a],$$

where $e : \mathbb{R} \to S^1 \simeq \mathbb{R}/2\pi\mathbb{Z}$ is the quotient map.

The next lemma is related with the results of this section and was already used in Example 21.3. This is a classical result (see [32 Ex. 13.20, 13.21]). Let $C$ be an Abelian category. For $A \in \mathbb{D}(C)$ we let $\operatorname{Aut}(A) \subset \operatorname{Hom}(A, A)$ be the isomorphism group of $A$. For another $B \in \mathbb{D}(C)$ the product $\operatorname{Aut}(A) \times \operatorname{Aut}(B)$ acts on $\operatorname{Hom}(A, B)$ by composition. We recall that we have truncation functors $\tau_{\leq n}, \tau_{\geq n} : \mathbb{D}^b(C) \to \mathbb{D}^b(C)$ together with morphisms of functors $c_n$ which yield for any $X \in \mathbb{D}^b(C)$ and $n \in \mathbb{Z}$ a distinguished triangle

$$\tau_{\leq n}(X) \to X \to \tau_{\geq n+1}(X) \xrightarrow{c_n(X)} \tau_{\leq n}(X)[1].$$

For $A, B \in C$ and $n \geq 1$, we let $E^a_{A,B} \subset \mathbb{D}^b(C)$ be the full subcategory of objects $X$ such that $H^0(X) \simeq B$, $H^n(X) \simeq A$ and $H^i(X) \simeq 0$ for $i \neq 0, n$. For $X \in E^a_{A,B}$ and isomorphisms $a : A[-n] \cong \tau_{\geq 1}(X), b : B \cong \tau_{\leq 0}(X)$, we define $\phi_{a,b}(X) = (b[1])^{-1} \circ c_0(X) \circ a \in \operatorname{Hom}(A[-n], B[1]) \simeq \operatorname{Hom}(A, B[n + 1])$. 


Lemma 4.4.4. Let $\mathcal{C}$ be an abelian category and let $A, B \in \mathcal{C}$ and $n \geq 1$ be given. Let $E^n_{A,B}$ be the set of isomorphism classes in $E^n_{A,B}$. For a given $X \in E^n_{A,B}$ and isomorphisms $a: A[-n] \xrightarrow{\sim} \tau_{\geq 1}(X)$, $b: B \xrightarrow{\sim} \tau_{\leq 0}(X)$, the image of $\phi_{a,b}(X)$ in $\Hom(A, B[n+1])/\Aut(A) \times \Aut(B)$ is independent of the choice of $a$ and $b$ and yields a bijection between $\bar{E}^n_{A,B}$ and $\Hom(A, B[n+1])/\Aut(A) \times \Aut(B)$.

Proof. (i) Changing $a$ and $b$ modifies $(b[1])^{-1} \circ c_0(X) \circ a$ by the action of an element in $\Aut(A) \times \Aut(B)$. This proves that the image of $(b[1])^{-1} \circ c_0(X) \circ a$ in the quotient only depends on $X$.

(ii) If $u: X \rightarrow Y$ is an isomorphism in $E^n_{A,B}$, then $\tau_{\geq 1}(u)$ and $\tau_{\leq 0}(u)[1]$ are isomorphisms and make a commutative square with $c_0(X)$ and $c_0(Y)$. Hence $c_0(X)$ is conjugate to $c_0(Y)$ through $\Aut(A) \times \Aut(B)$. This defines a map $\bar{c}_0: \bar{E}^n_{A,B} \rightarrow \Hom(A, B[n+1])/\Aut(A) \times \Aut(B)$.

(iii) Let $X, Y \in E^n_{A,B}$ be given. If $\bar{c}_0(X) = \bar{c}_0(Y)$, then there exists $(\alpha, \beta)$ such that the square $(S)$ below commutes:

$$
\begin{array}{ccc}
\tau_{\leq 0}(X) & \xrightarrow{\alpha} & \tau_{\geq 1}(X) \\
\downarrow{i} & & \downarrow{\alpha} \\
\tau_{\leq 0}(Y) & \xrightarrow{\beta} & \tau_{\geq 1}(Y)
\end{array}
$$

By the axioms of triangulated categories, we deduce that $X \simeq Y$. Hence $\bar{c}_0$ is injective.

(iv) For $\phi \in \Hom(A, B[n+1])$ we define $X_\phi$ such that $X_\phi[1]$ is the cone of $\phi$. Then $\bar{c}_0(X_\phi) = [\phi]$ in $\Hom(A, B[n+1])/\Aut(A) \times \Aut(B)$. Hence $\bar{c}_0$ is surjective. \hfill $\square$

Part 5. Graph selectors

Let $M$ be a manifold and let $\Lambda \subset J^1(M)$ be a closed Legendrian submanifold. We let $\Gamma$ be the projection of $\Lambda$ on $M \times \mathbb{R}$. A graph selector for $\Lambda$ is a continuous function $\varphi: M \rightarrow \mathbb{R}$ whose graph is contained in $\Gamma$. If $\Lambda$ is generic, then $\Gamma$ is a union of transverse immersed hypersurfaces outside a set of codimension 1. In this case $\varphi$ is differentiable on a dense open set where the graph of $d\varphi$ is contained in the Lagrangian projection of $\Lambda$ in $T^*M$. We see $\Lambda$ as a closed conic Lagrangian submanifold of $T^*(M \times \mathbb{R})$ as follows. We choose coordinates $(t; \tau)$ on $T^*\mathbb{R}$ and denote by $T^*_{\tau>0}(M \times \mathbb{R})$ the open set $\{\tau > 0\}$ in $T^*(M \times \mathbb{R})$. Then the quotient by the multiplicative $\mathbb{R}_{>0}$-action in the fibers gives an identification $J^1(M) \simeq (T^*_{\tau>0}(M \times \mathbb{R}))/\mathbb{R}_{>0}$ and we abusively write $\Lambda$ for its inverse image by the quotient map; hence $\Lambda \subset T^*_{\tau>0}(M \times \mathbb{R})$. 


In this short section we prove that $\Lambda$ has a graph selector as soon as it is the microsupport of a sheaf $F$ satisfying some conditions at infinity; the graph of $\varphi$ is then the boundary of the support of a section of $F$. Using Theorem 1.3.5.1 below, we recover Theorem 1.2 of [5] which says that a compact exact Lagrangian submanifold of a cotangent bundle has a graph selector.

For a map $\varphi: M \to \mathbb{R}$ we set $\Gamma_{\varphi} = \{ t = \varphi(x) \}$ and $\Gamma_{\varphi}^+ = \{ t \geq \varphi(x) \}$. We assume in this section that $M$ is connected. For $F \in D(k_{M \times \mathbb{R}})$ we consider the following condition:

\begin{equation}
\text{(5.0.1)} \quad \text{there exists } A > 0 \text{ such that } \text{supp}(F) \subset M \times [-A, +\infty[\text{ and } \text{SS}(F) \subset T^*\tau_{t=0}(M \times [-A, A]).
\end{equation}

We recall that the category of sheaves $\text{Mod}(k_{M \times \mathbb{R}})$ is embedded in its derived category $D(k_{M \times \mathbb{R}})$ as the subcategory of complexes concentrated in degree 0. We remark the notion of support doesn’t make sense for a class $s \in H^0(M \times \mathbb{R}; F)$ for a general $F \in D(k_{M \times \mathbb{R}})$, but makes sense for $s \in H^0(M \times \mathbb{R}; F) = F(M \times \mathbb{R})$ when $F \in \text{Mod}(k_{M \times \mathbb{R}})$.

**Proposition 5.0.1.** Let $F \in \text{Mod}(k_{M \times \mathbb{R}})$ and let $s \in F(M \times \mathbb{R})$ be a non-zero section. We assume that $F$ satisfies (5.0.1). Then there exists a unique map $\varphi: M \to \mathbb{R}$ such that $\text{supp}(s) = \Gamma_{\varphi}^+$ and this map is continuous. More precisely, for a given chart $U$ in $M$, which is identified with a ball of $\mathbb{R}^n$ with coordinates $(x; \xi)$ on $T^*U$, if we have a bound $\text{SS}(F) \cap T^*(U \times \mathbb{R}) \subset \{ \tau \geq C||\xi|| \}$ for some $C > 0$, where $||\xi|| = (\sum_{i=1}^{n} \xi_i^2)^{1/2}$, then the map $\varphi$ is $C^{-1}$-Lipschitz on $U$, that is, $|\varphi(x) - \varphi(y)| \leq C^{-1}||x - y||$, for any $x, y \in U$, where $||\cdot||$ is again the norm induced by the coordinates.

The link between the Lipschitz condition and the assumption on the microsupport can also be found in [52] and [28].

**Proof.** (i) To prove that $\text{supp}(s)$ is of the form $\Gamma_{\varphi}^+$ it is enough to check that $\text{supp}(s) \cap \{ \{x\} \times \mathbb{R} \}$ is an interval of the form $[a, +\infty[$ for any $x \in M$. Let $z = (x_0, t_0) \in M \times \mathbb{R}$ such that $s_z \neq 0$. We set $i: \mathbb{R} \to M \times \mathbb{R}$, $t \mapsto (x_0, t)$ and $G = i^{-1}F$. Then $\text{SS}(G) \subset \{ \tau \geq 0 \}$ by the hypotheses on $F$ and by Theorem 1.2.8. The section $s$ induces a section $s'$ of $G$ over $\mathbb{R}$ and we have $s' = s_{i(t)}$. Hence $\text{supp}(s') = i^{-1}(\text{supp}(s))$ and this is non empty since $s_z \neq 0$.

By Corollary 1.2.16 the restriction map $H^0([a, c]; G) \to H^0([b, c]; G)$ is an isomorphism for any $a \leq b < c$. It follows that $s'_t = 0$ implies $s'_{t_1} = 0$ for all $t_1 \leq t$. Indeed, we can find $a, b, c$ such that $a < t_1$, $t_1 < b < t < c$ and $s'|_{b,c} = 0$. Then $s'|_{a,c} = 0$ and in particular
\[ s'_1 = 0. \] Since \( \text{supp}(s') \) is closed, non empty and contained in \([A, +\infty[\), this proves that it must be an interval of the form \([a, +\infty[\), as required.

It remains to check that, for any \( x \in M \), there exists \( t \) such that \( s(x,t) \neq 0 \). The hypothesis on \( \text{SS}(F) \) implies that \( F|_V \), where \( V = M \times ]A, +\infty[ \), is a locally constant sheaf. Since \( M \) is connected, the support of any global section of \( F|_V \) is either empty or \( V \). We have seen that \( \text{supp}(s) \) contains \( \{x_0\} \times [a, +\infty[ \) for some \( a \). Hence it must contain \( V \). Finally we obtain that \( \text{supp}(s) = \Gamma^+_\varphi \) for some function \( \varphi: M \to \mathbb{R} \).

(ii) Now we assume that we are given a chart \( U \) and \( C > 0 \) as in the second part of the statement; hence \( \text{SS}(F) \cap T^*(U \times \mathbb{R}) \subset (U \times \mathbb{R}) \times \{\tau \geq C||\xi||\} \). The chart \( U \) is identified with an open ball in \( \mathbb{R}^n \) and we let \( \gamma \subset \mathbb{R}^{n+1} \) be the closed convex cone \( \{t \leq -C^{-1}||x||\} \). We have \( \gamma^a = \{\tau \geq C||\xi||\} \). By Remark 3.4.4 \( \Gamma^+_\varphi = \text{supp}(s) \) is locally \( \gamma \)-closed, that is, for any \( z \in U \times \mathbb{R} \), there exists a neighborhood \( B \) of \( z \) (for the usual topology) such that \( B \cap \Gamma^+_\varphi = B \cap Z \) for some \( Z \subset \mathbb{R}^{n+1} \) which is closed for the \( \gamma \)-topology. By definition this means that \( \mathbb{R}^{n+1} \setminus Z = (\mathbb{R}^{n+1} \setminus Z) + \gamma \); it implies \( Z = Z + \gamma^a \).

Let \( x \in U \) be given and \( z = (x, \varphi(x)) \). Hence there exists a neighborhood \( B \) of \( z \) such that \( B \cap \Gamma^+_\varphi \) contains \( B_1 = B \cap (z + \gamma^a) \) and is contained in \( B \setminus B_2 \), where \( B_2 = B \cap (z + \gamma) \). Setting \( W = p(B_1) \cap p(B_2) \) we thus have \( ||\varphi(x) - \varphi(y)|| \leq C^{-1}||x - y|| \) for all \( y \in W \) and we can see that \( W \) is a neighborhood of \( x \) in \( U \). Since this holds for any \( x \in U \), we can deduce that \( \varphi \) is \( C^{-1} \)-Lipschitz on \( U \). Indeed for given \( x, x' \in U \), the segment \( [x, x'] \) is contained in \( U \) (\( U \) is a ball) and we can find finitely many points \( x_i \in [x, x'] \) with neighborhoods \( W_i \) as above, \( i = 1, \ldots, N \), such that \( [x, x'] \subset \bigcup_{i=1}^N W_i \). Then the result follows from the triangular inequality.

(iii) Since \( \text{SS}(F) \) is conic and closed in \( T^*(M \times \mathbb{R}) \) and contained in \( \{\tau > 0\} \), for any compact subset \( K \) of some coordinate chart of \( M \times \mathbb{R} \) we can find \( C > 0 \) such that \( \text{SS}(F) \cap \pi^{-1}_{M \times \mathbb{R}}(K) \subset K \times \{\tau \geq C||\xi||\} \). By (ii) it follows that \( \varphi \) is continuous everywhere.

In order to apply Proposition 5.0.1 in the situation of Corollary 5.0.3 we have to replace a complex of sheaves \( F \in \mathcal{D}(k_{M \times \mathbb{R}}) \) by a sheaf in \( \text{Mod}(k_{M \times \mathbb{R}}) \) — we will take \( H^0F \) — and ensure that our sheaf has a section. The next lemma implies that \( H^0F \) still satisfies the hypothesis of the proposition and Lemma 5.0.3 deals with the sections.

**Lemma 5.0.2.** Let \( F \in \mathcal{D}(k_{M \times \mathbb{R}}) \) which satisfies (5.0.1). Then, for any \( i \in \mathbb{Z} \), the sheaf \( H^iF \) also satisfies (5.0.1) and moreover we have \( \pi_{M \times \mathbb{R}}(\text{SS}(H^iF)) \subset \pi_{M \times \mathbb{R}}(\text{SS}(F)) \).
Proof. (i) Let us check (5.0.1) for $H^i F$. We have the general inclusion $\text{supp}(H^i F) \subset \text{supp}(F)$. Corollary 3.4.3 gives $\text{SS}(H^i F)) \subset T^*_{\tau>0}(M \times \mathbb{R})$. Since $\text{SS}(F)$ is contained in $T^*(M \times [-A, A])$, $F$ is locally constant outside $M \times [-A, A]$ and so is $H^i F$. Hence $\text{SS}(H^i F)$ is contained in $T^*(M \times [-A, A])$ and we have (5.0.1).

(ii) Now we prove the last assertion. We recall that, for any $G \in D(k_{M \times \mathbb{R}})$, a point $x$ is not in $\pi_{M \times \mathbb{R}}(\text{SS}(G))$ if and only if $G$ is constant in a neighborhood of $x$ (see Example 1.2.3(i)). Now, if $F$ is constant, so is $H^i F$. Hence, if $x \not\in \pi_{M \times \mathbb{R}}(\text{SS}(F))$, we have $x \not\in \pi_{M \times \mathbb{R}}(\text{SS}(H^i F))$, as required. □

Lemma 5.0.3. Let $F \in D(k_{M \times \mathbb{R}})$ which satisfies (5.0.1). Then the restriction morphism $R\Gamma(M \times \mathbb{R}; F) \to R\Gamma(M \times |B, +\infty[; F)$ is an isomorphism, for any $B \in \mathbb{R}$.

We remark that if $M$ were compact, this would follow directly from Corollary 1.2.16.

Proof. We set $Z = M \times [-\infty, B]$ and $U = M \times |B, +\infty[$. We have to prove that $r: R\Gamma(M \times \mathbb{R}; F) \to R\Gamma(M \times \mathbb{R}; R\Gamma_U(F))$ is an isomorphism. The cone of $r$ is $R\Gamma(M \times \mathbb{R}; R\Gamma_Z(F))$. Let $p: M \times \mathbb{R} \to M$ be the projection. It is enough to prove $R\Gamma_p(R\Gamma_Z(F)) \simeq 0$.

We remark that $p$ is proper on $\text{supp}(R\Gamma_Z(F))$. For a given $x \in M$ and $i_x: \mathbb{R} \to M \times \mathbb{R}$, $t \mapsto (x, t)$, we set $G = i_x^{-1}R\Gamma_Z(F)$. Then the base change formula gives $(R\Gamma_p(R\Gamma_Z(F)))_x \simeq R\Gamma(\mathbb{R}; G)$. By Theorems 1.2.13 and 1.2.8 we have $\text{SS}(G) \subset \{\tau \geq 0\}$. Since $G$ has compact support, we have $R\Gamma(\mathbb{R}; G) \simeq R\Gamma([a, +\infty[; G)$ for some $a \in \mathbb{R}$. By Corollary 1.2.16 we also have $R\Gamma([a, +\infty[; G) \simeq R\Gamma([b, +\infty[; G)$ for $a \leq b$ and this vanishes for $b \gg 0$ since $\text{supp}(G)$ is compact. □

Corollary 5.0.4. Let $\Lambda \subset T^*_{\tau>0}(M \times \mathbb{R})$ be a closed conic Lagrangian submanifold. We assume that $\Lambda \subset T^*_{\tau>0}(M \times [-A, A])$ for some $A$ and that there exists $F \in D(k_{M \times \mathbb{R}})$ such that $\text{SS}(F) = \Lambda$, $\text{supp}(F) \subset M \times [-A, +\infty[ \text{ and } F|_{M \times [-A, +\infty[} \simeq k_{M \times [-A, +\infty[}$. Then $\Lambda$ has a graph selector: there exists a continuous map $\varphi: M \to \mathbb{R}$ such that $\Gamma_\varphi \subset \pi_{M \times \mathbb{R}}(\Lambda)$.

Moreover, as in Proposition 5.0.7, for a given chart $U$ in $M$, which is identified with a ball of $\mathbb{R}^n$ with coordinates $(x; \xi)$ on $T^*U$, if we have a bound $\Lambda \cap T^*(U \times \mathbb{R}) \subset \{\tau > C||\xi||\}$ for some $C > 0$, then $\varphi$ is $C^{-1}$-Lipschitz on $U$.

Proof. (i) The sheaf $F|_{M \times [-A, +\infty[} \simeq k_{M \times [-A, +\infty[} \simeq H^0 F|_{M \times [-A, +\infty[}$ has a section over $M \times [-A, +\infty[$, say $s$, corresponding to $1 \in k$. By Lemma 5.0.2 $H^0 F$ satisfies (5.0.1), and by Lemma 5.0.3 the section
$s$ can be extended to $s \in H^0(M \times \mathbb{R}; H^0 F)$. Proposition 5.0.1 associates a continuous function $\varphi$ to $s$ and we have $\Gamma_\varphi \subset \pi_{M \times \mathbb{R}}(\Lambda)$ by Lemma 5.0.2 again.

(ii) Let $\Xi \subset T^*(M \times \mathbb{R})$ be the convex hull of $\Lambda$ in the sense that $\Xi$ is the intersection of all closed conic subsets $S$ of $T^*M$ which contain $\Lambda$ and are fiberwise convex. For a local chart as in the statement we also have $\Xi \cap T^*U \subset \{\tau > C||\xi||\}$. By Corollary 3.4.3 we have $\SS(H^0 F) \subset \Xi$. Hence the Lipschitz constant given in Proposition 5.0.1 for $H^0 F$ is the same as the one given in the current corollary.

An example of a sheaf satisfying the hypotheses of Corollary 5.0.4 is given by Corollary 2.1.5 as follows. We start with $\Lambda_0 = T^*_M(\{0\} \times \mathbb{R})$ which is the microsupport of $F_0 = k_{M \times [0, +\infty)}$. Let $\Phi_t$, $t \in I$, where $I$ is an open interval containing $[0, 1]$, be a homogeneous Hamiltonian isotopy of $\tilde{T}^*_M(\mathbb{R})$ which preserves $\tilde{T}^*_{\tau > 0}(\mathbb{R})$ (for example the homogeneous lift of a Hamiltonian isotopy of $T^*M$, whose support is proper over $M$, or the lift of a contact isotopy of $J^1 M$). Corollary 2.1.5 gives a sheaf $F$ on $M \times \mathbb{R} \times I$ with $F_0 = F^0$ and $\SS(F_t) = \Phi_t(\Lambda_0)$, where $F_t = F|_{M \times \mathbb{R} \times \{t\}}$. We remark that $\SS(F) \cap T^*N = \emptyset$ for $N = M \times [B, +\infty[ \times [0, 1]$ and $B \gg 0$. Hence $F|_N$ is locally constant. Since $F_0|_{M \times B, +\infty[} \simeq k_{M \times B, +\infty[}$, we deduce that $F_1|_{M \times B, +\infty[} \simeq k_{M \times B, +\infty[}$. So Corollary 5.0.4 implies that $\Phi_1(\Lambda_0)$ has a graph selector.

More generally, using Corollary 5.0.4 and Theorem 13.5.1 below, we recover Theorem 1.2 of [5]:

**Corollary 5.0.5.** Let $\Lambda \subset T^*_{\tau > 0}(M \times \mathbb{R})$ be a closed conic Lagrangian submanifold which is the conification of a compact exact Lagrangian submanifold of $T^*M$. Then $\Lambda$ has a graph selector.

**Part 6. The Gromov nonsqueezing theorem**

In this part we use the microlocal theory of sheaves to give a proof of the famous Gromov nonsqueezing theorem (see [20]), which says that there is no symplectomorphism of $\mathbb{R}^{2n}$ which sends the ball of radius $R$ into a cylinder $D_r \times \mathbb{R}^{2n-2}$, where $D_r$ is the disc of radius $r$, if $r < R$. There is already a proof with generating functions by Viterbo [53] and it is no wonder that we can also find a proof with sheaves. The proof we give is inspired by the papers of Chiu [12], to define a projector associated with a square in §6.1, and Tamarkin [47] to define a displacement energy (see Definition 6.2.1 – this displacement energy is also used in [7]). It is in fact a baby case of the main result of [12] which is the nonsqueezing in the contact setting (see also [57] for
a survey of Tamarkin’s and Chiu’s results and [15] for another proof. We also give a non squeezing result for a Lagrangian submanifold of the ball; this is related with a result of Théret in [48].

6.1. Cut-off in fiber and space directions

In this section we prove Corollary 6.1.6 which says that if a sheaf \( F \) on \( \mathbb{R}^n \) has its microsupport contained in some prescribed cone, then the natural morphism \( \tau_c(F) \) of (3.5.7) vanishes for \( c \) big enough. This will be used in the next section to recover classical non squeezing results. The idea is to construct explicitly a sheaf \( K_\infty \) on \( \mathbb{R}^{2n} \) such that \( F \circ K_\infty \sim -\rightarrow F \) for \( F \) as above and check that \( \tau_c(K_\infty) \) vanishes.

In Part 3 we have seen several functors on the category of sheaves whose effect is to change the microsupport and make it avoid some given set. For a vector space \( V \) and a closed convex cone \( \gamma \subset V \), we have seen \( P_\gamma : D(k_V) \to D(k_V) \) which is a projector, in the sense that \( P_\gamma \circ P_\gamma \simeq P_\gamma \), and satisfies \( SS(P_\gamma(F)) \subset V \times \gamma^{oa} \). More usual functors reduce the support of a sheaf, for example \( D(k_M) \to D(k_M) \), \( F \mapsto F_Z \), for a locally closed set \( Z \subset M \). This functor is also a projector and satisfies \( SS(F_Z) \subset Z \times V^* \). If \( Z \) is closed, a sheaf satisfying \( SS(F) \subset Z \times \gamma^{oa} \) is stable by both functors \( P_\gamma \) and \( (\_)_Z \), hence by the composition \( Q : F \mapsto P_\gamma(F_Z) \). This \( Q \) is not a projector but we will see in a special case that \( Q^{oi} \) converges when \( i \to \infty \) and gives a projector (in fact we will work with a variant of \( Q \)).

The construction we give in this section is in fact a baby case of a construction by Chiu in [12] who defines a projector corresponding to a subset \( C \) of \( T^* \mathbb{R}^{n+1} \) which is the cone over a ball in \( T^* \mathbb{R}^n \); here we do the case where \( C \) is the cone over a square \([−1, 1]^2 \) in \( T^* \mathbb{R} = \mathbb{R} \times \mathbb{R}^* \). We will use this projector to recover nonsqueezing results in the symplectic case.

It will be more convenient to use a composition functor \( F \mapsto F \circ k_\gamma \) rather than \( P_\gamma \) because the composition is associative. This is similar to the Tamarkin projector of §3.5 see Remark 3.5.2 where this functor is denoted \( L_\gamma \). We recall that, for a manifold \( M \) and a conic subset \( A \subset T^* M \), \( D_A(k_M) \) is the full subcategory of \( D(k_M) \) formed by the \( F \) with \( SS(F) \subset A \) (see Notation 1.2.2). The composition \( - \circ k_\gamma \) is still a projector and its image is \( D_{A_\gamma}^{\perp,l}(k_V) \), the left orthogonal of \( D_{A_\gamma}(k_V) \), where \( A_\gamma = V \times (V^* \setminus \text{Int}(\gamma^{oa})) \). We have \( D_{A_\gamma}^{\perp,l}(k_V) \subset D_{V \times \gamma^{oa}}(k_V) \). (See Remark 3.5.2 and [23] Prop. 4.10.) However we don’t use the results of [23]; the microsupport estimates will rely on the properties of \( P_\gamma \) and Lemma 3.1.1.
We introduce some notations. Let $n \geq 2$ be given. We consider the vector spaces $V' = \mathbb{R}^{n-2}$, $V = \mathbb{R}^2 \times V'$ and put coordinates $(x; \xi)$ on $T^*V$. We let $\gamma \subset V$ be the cone
\[
\gamma = \{(x) \in \mathbb{R}^n; \ x_2 \leq -|x_1|, \ x_3 = \cdots = x_n = 0 \}.
\]
Hence $\gamma^\circ \subset V^*$ is given by $\gamma^\circ = \{(\xi) \in \mathbb{R}^n; \ \xi_2 \leq -|\xi_1| \}$. We recall the notation $\tilde{\gamma} = \{(x, y) \in \mathbb{R}^2; \ x - y \in \gamma \}$ of \S \ref{subsec_nota}. We let $Z \subset V$ be the open strip $Z = [-1, 1[ \times \mathbb{R}^{n-1}$. Let $R_\gamma, R_Z: \mathcal{D}(k_V) \rightarrow \mathcal{D}(k_V)$ be the functors $R_\gamma(F) = F \circ k_\tilde{\gamma}$ and $R_Z(F) = F_Z$. They come with natural morphisms $R_\gamma(F) \rightarrow F$ and $R_Z(F) \rightarrow F$. We remark that $R_Z$ can also be written as a composition $R_Z(F) \simeq F \circ k_{\delta_V(Z)}$, where $\delta_V$ is the diagonal embedding. We define $R = R_Z \circ R_\gamma \circ R_Z$ and we find
\[
R(F) \simeq F \circ k_W,
\]
where
\[
W = \tilde{\gamma} \cap (Z \times Z)
\]
\[
= \{(x, y) \in \mathbb{R}^{2n}; \ y_2 - x_2 \geq |x_1 - y_1|, \ |x_1| < 1, \ |y_1| < 1, \ x_i = y_i, \ i = 3, \ldots, n \}.
\]
Since $W \cap \Delta_V$ is closed in $W$ and open in $\Delta_V$ we have a natural morphism $k_W \rightarrow k_{\Delta_V}$ which gives a morphism of functors $R \rightarrow \text{id}$. If a sheaf satisfies $R_\gamma(F) \xrightarrow[\sim]{\simeq} F$ and $F_Z \xrightarrow[\sim]{\simeq} F$, then $R(F) \xrightarrow[\sim]{\simeq} F$ and $R_\gamma(F) \xrightarrow[\sim]{\simeq} F$ for all $i \in \mathbb{N}$. We now compute $R^{\circ i}$; of course this is the composition with
\[
K_i = k_W \circ \cdots \circ k_W \quad (i \text{ factors } k_W).
\]
For $c \in \mathbb{R}$ we define $f, s, T_c: V \rightarrow V$ by
\[
f(y) = (-y_1, y_2 + 2, y_3, \ldots, y_n)
\]
\[
s(y) = (y_1, -y_2, y_3, \ldots, y_n)
\]
\[
T_c(y) = (y_1, y_2 + c, y_3, \ldots, y_n).
\]
Let $Z_\pm \subset \mathbb{R}^2$ be the closed half planes $Z_\pm = \{y_2 \geq \pm y_1\}$. We have natural morphisms $u_\pm: k_{\mathbb{R}^2} \rightarrow k_{Z_\pm}$ and we define $L \in \mathcal{D}(k_{\mathbb{R}^2})$ by
\[
(6.1.2) \quad L = 0 \rightarrow k_{\mathbb{R}^2} \rightarrow k_{Z_+} \oplus k_{Z_-} \rightarrow 0,
\]
where $k_{\mathbb{R}^2}$ is in degree 0. We have $H^0 L \simeq k_{\mathbb{R}^2 \setminus (Z_+ \cup Z_-)}$, $H^1 L \simeq k_{Z_+ \cap Z_-}$ and a distinguished triangle $k_{\mathbb{R}^2 \setminus (Z_+ \cup Z_-)} \rightarrow L \rightarrow k_{Z_+ \cap Z_-}[{-1}] \rightarrow k_{\mathbb{R}^2 \setminus (Z_+ \cup Z_-)}[1]$. (In fact we have already met $L$ in (2.1.4); we have $L \simeq K_{[0] \times \mathbb{R}^2}$ in the case $n = 1$.)
Lemma 6.1.1. We set $C_1 = W \cap (\text{id}_V \times f)((\text{id}_V \times s)(\text{Int}(W)))$ and $W_2 = (\text{id}_V \times f)(W)$. The sheaf $K_2 = k_W \circ k_W$ appears in a distinguished triangle

\[(6.1.3) \quad k_{C_1} \xrightarrow{\sim} K_2 \rightarrow k_{W_2}[-1] \rightarrow k_{C_1}[1] \]

and, in a small enough neighborhood of $\{y_1 = -x_1, y_2 = x_2 + 2\} \times \Delta V'$, we have $K_2 \simeq p^{-1}(L) \otimes k_{\Delta V'}$, with $p : \mathbb{R}^4 \rightarrow \mathbb{R}^2, (x_1, x_2, y_1, y_2) \mapsto (x_1 + y_1, y_2 - x_2 - 2)$. Moreover $k_{C_1} \circ k_W \simeq k_{C_1}$.

Proof. (i) Let us forget the variables $x_i, y_i$ for $i \geq 3$. Let $q : \mathbb{R}^6 \rightarrow \mathbb{R}^4$ be the projection $q(\underline{x}, y, z) = (\underline{x}, \underline{z})$. Then $K_2 = Rq(k_A)$ where $A = (W \times \mathbb{R}^2) \cap (\mathbb{R}^2 \times W)$. For a subset $E \subset \mathbb{R}^6$ and $(\underline{x}, \underline{z}) \in \mathbb{R}^4$ we set $E_{(\underline{x}, \underline{z})} = E \cap q^{-1}(\underline{x}, \underline{z})$. If $\underline{y} \notin Z$ or $\underline{z} \notin Z$ we have $A_{(\underline{x}, \underline{z})} = \emptyset$. This proves that $(K_2)_{Z \times \mathbb{R}^2} \simeq K_2$. Now we assume $\underline{(x}, \underline{z}) \in \mathbb{R}^2$ and we find $A_{(\underline{x}, \underline{z})} = \{y \in \mathbb{R}^2; y_x - y_z \geq |x_1 - y_1|, |y_1| < 1, z_2 - y_2 \geq |y_1 - z_1|\}$. Then $A_{(\underline{x}, \underline{z})}$ is a bounded convex polytope, but the cohomology of $k_{A_{(\underline{x}, \underline{z})}}$ depends on $(\underline{x}, \underline{z})$ because we have two types of boundary conditions (open or closed).

(ii) We define $A'$, $A^\pm \subset q^{-1}(Z^2) \subset \mathbb{R}^6$ by their fibers $A'_{(\underline{x}, \underline{z})} = \overline{A_{(\underline{x}, \underline{z})}}$ and $A^\pm_{(\underline{x}, \underline{z})} = \partial A_{(\underline{x}, \underline{z})} \cap \{y_1 = \pm 1\}$. We have an exact sequence $0 \rightarrow k_A \rightarrow k_{A'} \rightarrow k_{A^+} \oplus k_{A^-} \rightarrow 0$. Now the fibers of $A'$, $A^\pm$ are always compact convex polytopes. For any such polytope $B$ we have $Rq(k_B) \simeq k_{q(B)}$ and we deduce a resolution of $K_2$ as the complex $0 \rightarrow k_{q(A')} \xrightarrow{d_1} k_{q(A^+)} \oplus k_{q(A^-)} \rightarrow 0$. We have $q(A') = W$ and

\[q(A^+) = \{(x, z) \in \mathbb{R}^4; z_2 - x_2 \geq 2 \mp (x_1 + z_1), |x_1| < 1, |z_1| < 1\} \]

Hence $\ker(d_1) \simeq k_{C_1}$, $\coker(d_1) = k_{W_2}$ and, up to a change of coordinates, we have the complex (6.1.2) defining $L$.

(iii) To prove the last assertion we can compute $k_{C_1} \circ k_W$ directly or use the fact that $(k_{C_1})_Z \simeq k_{C_1}$, because $C_1 \subset Z^2$, and $Rq(k_{C_1}) \simeq k_{C_1}$, because $C_1$ is relatively compact and $\text{SS}(k_{C_1}) \subset T^*V \times (V \times \gamma_{\alpha_0})$. We deduce $k_{C_1} \circ k_W \simeq k_{C_1}$ and this proves the lemma.

We can see that $- \circ k_W$ commutes with the translation $T_c$. More precisely $T_{cs}(F) \circ k_W \simeq T_{cs}(F \circ k_W) \simeq F \circ k_{(\text{id}_V \times T_c)(W)}$. The same holds for the map $f$ and, since $W_2 = (\text{id}_V \times f)(W)$, we can deduce $k_{W_2} \circ k_W$ from the triangle (6.1.3). We find a similar triangle

\[(6.1.4) \quad k_{C_2} \rightarrow k_{W_2} \circ k_W \rightarrow k_{W_3}[-1] \rightarrow k_{C_2}[1], \]

where $C_2 = (\text{id}_V \times f)(C_1)$, $W_3 = (\text{id}_V \times f)(W_2)$. Since $k_{C_1} \circ k_W \simeq k_{C_1}$, applying $- \circ k_W$ to (6.1.3) gives the triangle

\[(6.1.5) \quad k_{C_1} \rightarrow K_3 \rightarrow k_{W_2} \circ k_W[-1] \rightarrow k_{C_1}[1]. \]
Since the morphism \( R \) gives \( H^0(K_3) \simeq k_{C_1} \), \( H^1(K_3) \simeq k_{C_2} \) and \( H^2(K_3) \simeq k_{W_3} \). Now we can compute \( k_{W_3} \circ k_W \) in the same way and an induction gives the following result.

**Proposition 6.1.2.** For \( i \geq 1 \) we define \( C_i = (\text{id}_Y \times f)^i(C_1) \) and \( W_i = (\text{id}_Y \times f)^{i-1}(W) \). Then, for any \( n \geq 1 \),

(i) we have a distinguished triangle

\[
\varinjlim C_i \rightarrow (\text{id}_Y \times f)_*(K_n)[1] \rightarrow k_{C_i}[1],
\]

(ii) \( K_n \) is concentrated in degrees 0, \ldots, \( n - 1 \) and \( H^i K_n \simeq k_{C_i+1} \), for \( i = 0, \ldots, n - 2 \), \( H^{n-1} K_n \simeq k_{W_n} \),

(iii) setting \( \Delta_i^t := \{ y_1 = (-1)^i x_1, y_2 = x_2 + 2i \} \times \Delta Y \) and \( p_i : \mathbb{R}^2 \rightarrow \mathbb{R}^2, (x_1, x_2, y_1, y_2) \mapsto (x_1 - (-1)^i y_1, y_2 - x_2 - 2i) \), we have, in a small enough neighborhood of \( \Delta_i^t \) and for \( i = 1, \ldots, n - 1 \):

\[
K_n \simeq (p_i^{-1}(L) \otimes k_{\Delta Y_i})[1 - i],
\]

(iv) setting \( Y_i = \{ y_2 - x_2 \geq 2i \} \), we have, for \( i = 1, \ldots, n - 1 \), \( (K_n)_{Y_i} \simeq (\text{id}_Y \times f)_*(K_{n-i})[-i]. \)

**Lemma 6.1.3.** We set \( S_n = \text{supp}(K_n) = \overline{W_n \cup \bigcup_{i=1}^{n-1} C_i} \). Then we have an isomorphism \( k_{S_n} \simeq \text{RHom}(K_n, K_n) \) mapping 1 to the identity morphism.

**Proof.** (i) The statement is local on \( V^2 \). In a neighborhood of a point \( x_0 \) which is away from the sets \( \Delta_i^t \) defined in Proposition 6.1.2 our sheaf \( K_n \) is up to shift a constant sheaf on some subset \( Y \) of \( V^2 \) with \( Y = C_i \) or \( Y = W_n \). Since the sets \( C_i \) and \( W_n \) are locally closed convex, we get \( \text{RHom}(K_n, K_n) \simeq k_{S_n} \) near \( x_0 \).

(ii) In a neighborhood of some \( \Delta_i^t \) the lemma is reduced to the proof of \( \text{RHom}(L, L) \simeq k_D \), where \( D = \text{supp}(L) \). Let us set \( H = \text{RHom}(L, L) \). Since the morphism \( k_D \rightarrow H \) is given, it only remains to check \( H_x \simeq k \) at any point \( x \in D \). This is clear for \( x \neq 0 \). We can compute \( H_0 \) directly from the definition (6.1.2). Alternatively we can remark that \( \text{SS}(L) = \Lambda \Psi \circ a \cdot T^*_0 \mathbb{R} \) (see the notation (2.1.5)), where \( \Psi \) is the Hamiltonian isotopy of \( T^* \mathbb{R} \) defined by \( \Psi_s(x; \xi) = (x - s \xi / |\xi|; \xi) \). It follows from Corollary 2.1.5 that the restriction morphisms \( \text{RHom}(L, L) \rightarrow \text{RHom}(L_s, L_s) \), where \( L_s = L|_{\mathbb{R} \times \{s\}} \) are all isomorphisms. We deduce that \( \text{RG}(\mathbb{R}^2; H) \simeq k \). Since \( H \) is a conic sheaf, we have \( H_0 \simeq \text{RG}(\mathbb{R}^2; H) \), hence \( H_0 \simeq k \).

We define an increasing sequence of open subsets \( U_i = \{ y_2 - x_2 < 2i \} \subset \mathbb{R}^{2n}, i \in \mathbb{Z} \). By Lemma 6.1.3 the natural morphism \( u_j : K_{j+1} \rightarrow \)
$K_j$, induced by $k_W \to k_{\Delta V}$, gives an isomorphism $K_{j+1}|_{U_i} \simeq K_j|_{U_i}$ if $j > i$. Hence we can define a sheaf $K_\infty \in D(k_{\mathbb{R}^n})$ such that $K_\infty|_{U_i} \simeq K_j|_{U_i}$ if $j > i$, for example as follows. We set $U'_i = U_i \setminus \overline{U}_{i-1}$ and $U''_i = U_i \setminus \overline{U}_{i-2}$. The natural restriction morphism and $u_j$ induce $v_j: (K_{j+1})_{U'_{i-1}} \to (K_{j+1})_{U''_i}$ and $v'_j: (K_{j+1})_{U'_{i-1}} \to \oplus (K_i)_{U''_i-1}$, whose restrictions to $U''_i$ are isomorphisms. Now we define $K_\infty$ by the distinguished triangle

\[(6.1.6) \bigoplus_{i \in \mathbb{Z}} (K_i)_{U'_{i-2}} \xrightarrow{v} \bigoplus_{i \in \mathbb{Z}} (K_i)_{U''_i-1} \to K_\infty \xrightarrow{+1}, \quad v = \bigoplus_i \left( v_{i-1} \right)_{U''_i-1}.\]

**Proposition 6.1.4.** We set $S_\infty = \text{supp}(K_\infty) = \bigcup_{i=1}^{\infty} C_i$ and $Y_i = \{y_2 - x_2 \geq 2i\}$. Then we have $R\text{Hom}(K_\infty, K_\infty) \simeq k_{S_\infty}$ and $(K_\infty)_{Y_i} \simeq R\Gamma_{\text{Int}(Y_i)}(K_\infty) \simeq (id_V \times f)^*(K_\infty)[-i]$, for $i \geq 1$. In particular

\[(6.1.7) R\text{Hom}(K_\infty, T_{4s}(K_\infty)) \simeq k[2].\]

**Proof.** The first assertion follows from Lemma 6.1.3, Proposition 6.1.2 (iv) gives $(K_\infty)_{Y_i} \simeq (id_V \times f)^*(K_\infty)[-i]$. The inductive description of $K_n$ in (i) of the same proposition shows that $SS(K_\infty) \subset \{\xi_2 + \eta_2 \geq 0\}$. Hence $SS(K_\infty)$ does not meet $SS(k_{\text{Int}(Y_i)})$ and Theorem 1.2.13 gives $(K_\infty)_{Y_i} \simeq R\Gamma_{\text{Int}(Y_i)}(K_\infty)$. For the last assertion we remark that $T_4 = (id_V \times f)^2$. Hence

\[
\begin{align*}
R\text{Hom}(K_\infty, T_{4s}(K_\infty)) &\simeq R\text{Hom}(K_\infty, R\Gamma_{\text{Int}(Y_2)}(K_\infty))[2] \\
&\simeq R\Gamma_{\text{Int}(Y_2)}R\text{Hom}(K_\infty, K_\infty)[2] \\
&\simeq R\Gamma_{\text{Int}(Y_2)}(k_{S_\infty})[2] \\
&\simeq (k_{S_\infty \cap Y_2})[2]
\end{align*}
\]

and the result follows. \hfill \Box

The morphism $k_W \to k_{\Delta V}$ gives by iteration $K_i \to k_{\Delta V}$ and then $K_\infty \to k_{\Delta V}$. In particular for any $F \in D(k_V)$ we have a natural morphism $F \circ K_\infty \to F$.

We recall that we used the composition functor $F \mapsto F \circ k_\xi$ rather than the cut-off functor $P_\xi$ since the composition is associative. The difference between these two functors is a switch between proper and usual direct image in the definition (1.5.1) of the composition. Here we will also use the usual direct image. For later use we introduce a space of parameters which is a manifold $N$. Let $q_1, q_2: N \times V^2 \to N \times V$ and $q_3: N \times V^2 \to V^2$ be the projections. For $F \in D(k_{N \times V})$ we define
\( F \circ K_\infty, F \circ_{np} K_\infty \in D(k_{N \times V}) \) by
\[
(6.1.8) \quad F \circ K_\infty = Rq_2(q_1^{-1}F \otimes q_3^{-1}K_\infty),
\]
\[
(6.1.9) \quad F \circ_{np} K_\infty = Rq_{2*}(q_1^{-1}F \otimes q_3^{-1}K_\infty).
\]

**Proposition 6.1.5.** Let \( F \in D(k_{N \times V}) \). We assume \( \text{SS}(F) \subset T^* N \times (V \times \gamma^{2n}) \) and \( \text{supp}(F) \subset N \times \overline{Z} \) (recall \( Z = [-1, 1] \times \mathbb{R}^{n-1} \). Then \( F \circ_{np} K_\infty \simeq F \).

If, moreover, there exists \( x_2^0 \in \mathbb{R} \) such that \( \text{supp}(F) \subset \{ x_2 \geq x_2^0 \} \), then \( F \circ K_\infty \simeq F \).

**Proof.** The second assertion is a particular case of the first one, since the hypothesis gives that the map \( q_2 \) in \((6.1.9)\) is proper on the support of \( q_1^{-1}F \otimes q_3^{-1}K_\infty \). However we first prove the second assertion in (i) and (ii) below and we deduce the first one in (iii).

(i) To prove that the morphism \( F \circ K_\infty \rightarrow F \) is an isomorphism, it is enough to see that the restrictions to \( \{ p \} \times V \), for any \( p \in N \), are isomorphisms. Using the base change formula and the bound of Theorem 1.2.8 for \( \text{SS}(F|_{\{ p \} \times V}) \), this means that we can assume that \( N \) is a point.

By Proposition 3.1.7 we have \( P_\gamma(F) \simeq F \), where \( P_\gamma \) is defined in \((3.1.1)\) by \( P_\gamma(F) = Rq_{2*}(k_\gamma \otimes q_1^{-1}F) \). Since \( \text{supp}(F) \subset \{ x_2 \geq x_2^0 \} \) we can replace \( Rq_{2*} \) by \( Rq_2 \) in this expression and we find \( F \circ K_\infty \simeq F \).

We also have \( F|_Z \simeq F \). Indeed, the support condition implies \( F \simeq F|_Z \) and \( \Gamma_{\{ \varphi \geq 0 \}}F \simeq F \) where \( \varphi(x) = 1 + x_1 \). Now the micro-support condition implies \( (\Gamma_{\{ \varphi \geq 0 \}}F)_x \simeq 0 \) if \( x \in \{ x_1 = -1 \} \). Hence \( F|_{\{ x_1 = -1 \}} \simeq 0 \). The same result holds for \( x_1 = 1 \) and we get \( F|_{\partial Z} \simeq 0 \). Finally \( F|_Z \simeq F \) as claimed.

We deduce \( F \circ K_i \simeq F \) for all \( i \in \mathbb{N} \).

(ii) We recall that \( K_\infty|_{U_i} \simeq K_j|_{U_j} \) if \( j > i \), where \( U_i = \{ y_2 - x_2 < 2i \} \subset \mathbb{R}^{2n}, i \in \mathbb{Z} \). For any sheaf \( K \) on \( V^2 \) supported in \( V^2 \setminus U_i \) we have \( \text{supp}(F \circ K) \subset \{ x_2 \geq x_2^0 + 2i \} \). Using the triangle \( (K_\infty|_{U_i} \rightarrow K_\infty \rightarrow (K_\infty|_{V^2 \setminus U_i} \rightarrow 1) \) and the same one with \( K_\infty \) replaced by \( K_i \), we deduce that, for any given \( x' \in V \), if \( j > i > x_2' - x_2^0 \), we have \( (F \circ K_\infty)|_{x'} \simeq (F \circ (K_\infty|_{U_i})|_{x'} \simeq (F \circ K_j)|_{x'} \). Hence \( (F \circ K_\infty)|_{x'} \simeq F_{x'} \) which proves that the morphism \( F \circ K_\infty \rightarrow F \) is an isomorphism.

(iii) Now we deduce the first assertion. The idea is to write \( F \) as the limit of \( F_{Z_k}, k \in \mathbb{N} \), where \( Z_k = \{ x_2 \geq -k \} \), and check that \( -\circ_{np} q_3^{-1}K_\infty \) commutes with limits. Since \( F_{Z_k} \circ_{np} q_3^{-1}K_\infty \simeq F_{Z_k} \circ q_3^{-1}K_\infty \), the result follows.
Since we work in the derived category (and not the dg-derived category), we cannot use limits. Instead we use the distinguished triangle $F \to \prod_{k \in \mathbb{N}} F_{Z_k} \xrightarrow{u} \prod_{k \in \mathbb{N}} F_{Z_k} \xrightarrow{1}$, where $u = \text{id} - s$ and $s$ is the product of the natural morphisms $F_{Z_k} \to F_{Z_k-1}$. It is then enough to check that $- \circ np \; q_3^{-1} \mathcal{K}_\infty$ commutes with products. The definition (6.1.9) is the composition of the three functors $q_1^{-1}, \otimes$ and $Rq_{2*}$. Since $q_1$ is a submersion we have $q_1^{-1} \simeq q_1^{-1} [-n]$ and this is a right adjoint. Hence $q_1^{-1}$ commutes with products; the same holds for $Rq_{2*}$. The tensor product is not a right adjoint, but we can write $q_1^{-1}F \otimes q_3^{-1} \mathcal{K}_\infty \simeq R\text{Hom}(D'(q_3^{-1} \mathcal{K}_\infty), q_1^{-1}F)$ and $q_1^{-1}F_{Z_k} \otimes q_3^{-1} \mathcal{K}_\infty \simeq R\text{Hom}(D'(q_3^{-1} \mathcal{K}_\infty), q_1^{-1}F_{Z_k})$. Indeed the microsupports of $F$, $k_{Z_k}$ and $\mathcal{K}_\infty$ are all contained in $\{ \xi_2 > 0 \}$ (away from the zero section) and $\mathcal{K}_\infty$ is constructible. Hence the claimed isomorphisms follow from several applications of Theorem 1.2.13.

Now we obtain $F \circ np \; q_3^{-1} \mathcal{K}_\infty \simeq Rq_{2*}(R\text{Hom}(D'(q_3^{-1} \mathcal{K}_\infty), q_1^{-1}F[-n]))$ (and the same with $F_{Z_k}$ instead of $F$) and this is a composition of right adjoint functors. Hence it commutes with products and the result follows.

For a real number $c$ we let $T_c : N \times V \to N \times V$ be the translation along the direction $y_2$ of $V$, $T_c(p, y_1, \ldots, y_n) = (p, y_1, y_2 + c, y_3, \ldots, y_n)$ ($p \in N$). For $F \in D(k_{N \times V})$ such that $SS(F) \subset \{ \eta_2 \geq 0 \}$ we let $\gamma_c(F) : F \to T_{c*}(F)$ be the morphism (3.5.7).

**Corollary 6.1.6.** We have $\gamma_c(\mathcal{K}_\infty) = 0$ for all $c \geq 4$. In particular, for any manifold $N$ and $F \in D(k_{N \times V})$ such that $SS(F) \subset T^*N \times (V \times \gamma_0)$ and supp$(F) \subset N \times \overline{Z}$, we have $\gamma_c(F) = 0$ for all $c \geq 4$.

**Proof.** By (6.1.7) we have $\text{Hom}(\mathcal{K}_\infty, T_{c*} \mathcal{K}_\infty) \simeq 0$ if $c \geq 4$ and a fortiori $\gamma_c(\mathcal{K}_\infty) = 0$. The second assertion then follows from the isomorphism $F \circ np \; K_\infty \cong F$ of Proposition 6.1.5 and the fact that $\gamma_c(F)$ coincides with $\text{id}_F \circ np \; \gamma_c(\mathcal{K}_\infty)$.

**6.2. Nonsqueezing results**

Here we use Corollary 6.1.6 to prove classical nonsqueezing results in the symplectic case. The morphism $\gamma_c$ of (3.5.6), introduced by Tamarkin in [47], gives the following invariant, that we can call a displacement energy. We refer to [7] where a similar (more refined) invariant is used to obtain bounds on the displacement energy of some subsets of a cotangent bundle and to [57] for a survey of Tamarkin’s and Chiu’s results. We put coordinates $(t; \tau)$ on $T^*\mathbb{R}$. For a manifold $M$ we thus consider $\{ \tau \geq 0 \}$ as a subset of $T^*(M \times \mathbb{R})$ and we denote by $D_{\tau \geq 0}(k_{M \times \mathbb{R}})$ the category of sheaves $F$ on $M \times \mathbb{R}$ with $SS(F) \subset \{ \tau \geq 0 \}$.
Definition 6.2.1. Let $M$ be a manifold and $F \in D_{\tau \geq 0}(k_{M \times \mathbb{R}})$. We set $e(F) = \sup\{c \geq 0; \tau_c(F) \neq 0\}$.

We check in Proposition 6.2.3 below that $e(F)$ is invariant by Hamiltonian isotopies of $T^*M$ with compact support. We can reformulate Corollary 6.1.6 as follows: for any $F \in D(k_{N \times V})$ such that $SS(F) \subset (N \times \mathbb{Z}) \times (V \times \gamma^a)$ we have $e(F) \leq 4$.

Remark 6.2.2. The morphism $\tau_c(F)$ for $F \in D_{\tau \geq 0}(k_{M \times \mathbb{R}})$ is functorial in $M$: if $N \subset M$ is a submanifold, then $\tau_c(F|_{N \times \mathbb{R}}) = (\tau_c(F))|_{N \times \mathbb{R}}$. This implies $e(F|_{N \times \mathbb{R}}) \leq e(F)$.

6.2.1. Invariance of the displacement energy. Let $M$ be a manifold and let $h: T^*M \times I \rightarrow \mathbb{R}$ be a function of class $C^\infty$. We assume that its Hamiltonian flow $\Phi: T^*M \times I \rightarrow T^*M$ is defined and has compact support. As in Proposition 2.3.1 we associate with $h$ a homogeneous Hamiltonian isotopy $\Phi': \hat{T}^*(M \times \mathbb{R}) \times I \rightarrow \hat{T}^*(M \times \mathbb{R})$ lifting $\Phi$ and we let $K_{\Phi'} \in D(k_{(M \times \mathbb{R})^2 \times I})$ be the sheaf given by Theorem 2.1.1. The isotopy $\Phi'$ preserves the subset $\{\tau \geq 0\}$ of $\hat{T}^*(M \times \mathbb{R})$ and commutes with the transpose derivative of $T_c$ acting on $\hat{T}^*(M \times \mathbb{R})$ (also denoted $T_c$ abusively, so $T_c(x, t; \xi, \tau) = (x, t + c; \xi, \tau)$) for all $c \in \mathbb{R}$.

In other words $\Phi' = T_{-c} \circ \Phi' \circ T_c$ and the functor $K_{\Phi'} \circ -$ coincides with $T_{-c*} \circ (K_{\Phi'} \circ -) \circ T_{c*}$. We obtain: for any $F \in D_{\tau \geq 0}(k_{M \times \mathbb{R}})$ we have $K_{\Phi'} \circ F \in D_{\tau \geq 0}(k_{M \times \mathbb{R} \times I})$ and an isomorphism $\alpha: K_{\Phi'} \circ T_{c*}F \simeq T_{c*}(K_{\Phi'} \circ F)$. We can then ask if the image of $\tau_c(F)$ by the composition functor $K_{\Phi'} \circ -$ coincides with $\tau_c(K_{\Phi'} \circ F)$ through $\alpha$. To see this we recall the equivalence (2.1.6) of Corollary 2.1.5.

We use (2.1.6) with $N = M \times \mathbb{R}$, $A = \{\tau \geq 0\} \setminus T^*(M \times \mathbb{R}) \subset T^*(M \times \mathbb{R})$ and $A' \subset T^*(M \times \mathbb{R})$ associated with $A$ by $\Phi'$ in the sense of (2.1.3). We let $A''$ be the union of $A'$ and the zero section. Then $A''$ is half of the hypersurface of $T^*(M \times \mathbb{R})$ defined by the graph of the lift of $h$: $A'' = \{(x, t; \xi, \tau), (s; -\tau h(x, \xi/\tau, s)); \tau \geq 0\}$. However we do not need a precise expression of $A'$, we only need to know that $i_s^*(A') = \Phi'_s(A)$ coincides with $A$, for all $s \in I$, where $i_s$ is the inclusion of $M \times \mathbb{R} \times \{s\}$ in $M \times \mathbb{R} \times I$. Corollary 2.1.5 says that, for any $s \in I$, the inverse image functor

$$ \hat{i}_s^{-1}: D_{A''}(k_{M \times \mathbb{R} \times I}) \rightarrow D_{\tau \geq 0}(k_{M \times \mathbb{R}}) $$

is an equivalence of categories. Now we have $A'' \subset \{\tau \geq 0\}$ and any $G \in D_{A''}(k_{M \times \mathbb{R} \times I})$ also comes with the morphism $\tau_c(G): G \rightarrow T_{c*}G$. By the functoriality of $\tau_c$ (Remark 6.2.2), we have $\tau_c(i_s^{-1}(G)) = i_s^{-1}(\tau_c(G))$.

Since $i_s^{-1}$ is an equivalence, we deduce that, for any $s \in I$, $\tau_c(i_s^{-1}(G))$ vanishes if and only if $\tau_c(G)$ vanishes. We thus get the invariance of the displacement energy:
Proposition 6.2.3. Let \( \Phi' : \tilde{T}^*(M \times \mathbb{R}) \times I \to \tilde{T}^*(M \times \mathbb{R}) \) be a homogeneous Hamiltonian isotopy lifting some Hamiltonian isotopy of \( T^*M \) in the sense of Proposition 2.3.1. Let \( F \in D_{r \geq 0}(k_{M \times \mathbb{R}}) \). Then \( e(F) = e(K_{\Phi'} \circ F) = e(K_{\Phi'},s \circ F) \), for any \( s \in I \).

6.2.2. Nonsqueezing for a flying saucer. We put the natural Euclidean structure on \( \mathbb{R}^n \) and \( T^* \mathbb{R}^n \simeq \mathbb{R}^{2n} \) and we denote by \( B_1(E) \) and \( S_1(E) \) the closed unit ball and unit sphere of an Euclidean space \( E \). We first define a subset \( \Lambda_0 \) of \( S_1(T^* \mathbb{R}^n) \) which is the image of a Legendrian of \( J^1(\mathbb{R}^n) \) whose front projection in \( \mathbb{R}^{n+1} \) is a flying saucer with conic points.

We define \( \Lambda_0 \) as the union of the graphs of the differential of two functions \( f_1, f_2 : B_1(\mathbb{R}^n) \to \mathbb{R} \). We choose these functions to be rotation invariant with a differential belonging to the unit sphere of \( T^* \mathbb{R}^n \). In other words we write \( f_i(x) = g_i(||x||) \) for a function \( g_i \) such that \( r^2 + (g_i(r))^2 = 1 \). This determines \( f_i \) up to a constant and we find \( f_1(x) = \int_0^{||x||} \sqrt{1 - u^2} du \) and \( f_2(x) = \pi/2 - f_1(x) \). We let \( W_0 = \{ f_1(||x||) \leq t < f_2(||x||) \} \) be the region in \( \mathbb{R}^{n+1} \) bounded by the graphs of \( f_1, f_2 \) and define

\[
(6.2.2) \quad \Lambda_0 = \rho_{\mathbb{R}^n}(\tilde{S}\mathcal{S}(k_{W_0})) \subset T^* \mathbb{R}^n.
\]

The functions \( f_i \) are not differentiable at 0 and \( W_0 \) has a conic point at \((0,0)\) and \((0,\pi/2)\):

\[
\mathcal{S}\mathcal{S}(k_{W_0}) \cap T^*_0 \mathbb{R}^{n+1} = \mathcal{S}\mathcal{S}(k_{W_0}) \cap T^*_0 T^*_0 \mathbb{R}^{n+1} = \{ (\xi, \tau) : \tau \geq ||\xi|| \}
\]

and we have \( df_{i,x} = g'_i(||x||) \cdot \frac{\xi}{||\xi||} \) for a non zero \( x \). We deduce

\[
(6.2.3) \quad \begin{cases}
\Lambda_0 = \Lambda_0 \cup B_1(T^* \mathbb{R}^n), \\
\Lambda_0 := \{ (x,\xi) \in S_1(T^* \mathbb{R}^n) : \xi \text{ is a scalar multiple of } x \}.
\end{cases}
\]

We remark that \( \Lambda_0 \) is the image of \( S_1(T^* \mathbb{R}^n) \) by the characteristic flow of \( S_1(T^* \mathbb{R}^n) \). More precisely we have \( S_1(T^* \mathbb{R}^n) = \{ h = 1 \} \) where \( h(x;\xi) = ||x||^2 + ||\xi||^2 \). The flow of \( X_h = 2 \sum_i \xi_i \frac{\partial}{\partial x_i} - x_i \frac{\partial}{\partial \xi_i} \) has orbits of period \( \pi \). Identifying \( T^* \mathbb{R}^n \) with \( \mathbb{C}^n \) by \( (x,\xi) \mapsto x + i\xi \) the flow is given by \( \Phi_{h,s}(x + i\xi) = \exp(2si) \cdot (x + i\xi) \). For an orbit \( \gamma \) of the flow the action is \( A = \int_{\gamma} \alpha \) where \( \alpha = \sum \xi_i dx_i \) is the Liouville form. Here we find \( A = \pi \) for all orbits in \( S_1(T^* \mathbb{R}^n) \). We thus have

\[
(6.2.4) \quad \Lambda_0 = \{ \exp(2si) \cdot (0;\xi) \in T^* \mathbb{R}^n \simeq \mathbb{C}^n : s \in [0,\pi], ||\xi|| = 1 \}.
\]

If we smooth our functions \( f_i \) near the origin (in which case the front looks like a flying saucer), we obtain an approximation of \( \Lambda_0 \) by an immersed Lagrangian sphere with one double point.
Proposition 6.2.4. Let \( r < 1/\sqrt{2} \) be given and let \( D_r \subset \mathbb{R}^2 \) be the closed disc of radius \( r \). There is no Hamiltonian isotopy \( \Phi : \mathbb{R}^{2n} \times I \to \mathbb{R}^{2n} \) such that \( \Phi_1(\Lambda_0) \subset D_r \times \mathbb{R}^{2n-2} \), where \( \Lambda_0 \) is defined in (6.2.2).

Proof. (i) Let us assume that such an isotopy \( \Phi \) exists. We can find an isotopy \( \Psi \) of \( \mathbb{R}^{2n} \to T^*\mathbb{R}^n \) such that \( \Psi_1(D_r \times \mathbb{R}^{2n-2}) \subset ]-a, a[^{2n} \times T^*\mathbb{R}^{n-1} \), for some \( a \) with \((2a)^2 < \pi/2\). Hence we can as well assume that \( \Phi_1(\Lambda_0) \subset ]-a, a[^{2n} \times T^*\mathbb{R}^{n-1} \). We can also assume that \( \Phi \) has compact support. We let \( \Phi' \) be a homogeneous isotopy of \( T^*\mathbb{R}^{n+1} \) lifting \( \Phi \) as in Proposition 2.3.1 and we let \( K_{\Phi'} \in \text{D}(k_{\mathbb{R}^{2n+2} I}) \) be the sheaf associated with \( \Phi' \) by Theorem 2.1.1.

(ii) We set \( F_0 = k_{\mathbb{W}_0} \) and define \( F_1 = K_{\Phi',1} \circ F_0 \). Hence \( SS(F_1) = \Phi'(SS(F_0)) \) and \( F_1 \) has compact support. In particular \( \rho_{\mathbb{R}^n}(SS(F_1)) = \Phi_1(\Lambda_0) \). By Proposition 6.2.3 we have \( e(F_0) = e(F_1) \).

(iii) Let \( \Gamma_1 = \pi_{\mathbb{R}^{2n+1}}(SS(F_1)) \) be the projection of \( SS(F_1) \) to the base. Since \( \rho_{\mathbb{R}^n}(SS(F_1)) \subset ]-a, a[^{2n} \times T^*\mathbb{R}^{n-1} \), we have \( \Gamma_1 \subset Z_a = ]-a, a[ \times \mathbb{R}^n \). Hence \( F_1 \) is locally constant outside \( Z_a \). Since it has compact support, it has to vanish outside \( Z_a \). Hence \( F_1 \) satisfies the hypotheses of Corollary 6.1.6 up to a rescaling of \( Z \) and \( \gamma \) by \( a \) and we obtain
\[
e(F_1) \leq 4a^2 < \pi/2.
\]

On the other hand \( R\text{Hom}(F_0, T_{c*}(F_0)) \cong k_{\mathbb{W}_0[T_{c*}(W_0)]} \) for \( c \in [0, \pi/2[ \). Since the topology of \( \overline{W_0 \cap T_c(W_0)} \) is unchanged when \( c \) runs over \([0, \pi/2[ \) we deduce that \( e(F_0) = \pi/2 \). We thus have a contradiction. \( \square \)

6.2.3. Nonsqueezing for \( L_0 \). By (6.2.3) the Lagrangian subset \( \Lambda_0 \) of \( B_1(T^*\mathbb{R}^n) \) consists of two parts, \( B_1(T_0^*\mathbb{R}^n) \) and \( L_0 \) which can be identified with the image of \( S_1(T_0^*\mathbb{R}^n) \) by the geodesic flow. The part \( B_1(T_0^*\mathbb{R}^n) \) corresponds to the microsupport of \( k_{\mathbb{W}_0} \) at the conic points \((0, 0)\) and \((0, \pi/2)\). We can define another sheaf \( G_0 \) on \( \mathbb{R}^{n+1} \) such that \( \rho_{\mathbb{R}^n}(SS(G_0)) = L_0 \). Let us recall the sheaf \( K_{\Psi} \in \text{D}(k_{\mathbb{R}^{2n+1}}) \) of Example 2.1.4 associated with the normalized geodesic flow of \( T^*\mathbb{R}^n \).

It fits in the distinguished triangle (2.1.4) which is an extension between constant sheaves on two opposite cones (one closed, the other open). If we restrict \( K_{\Psi} \) to a slice \( V_0 = \mathbb{R}^n \times \{0\} \times \mathbb{R} \), we obtain a picture similar to the following. We define \( W_1 = T_{\pi/2}(W_0) \). Then there exists a diffeomorphism \( f \) from a neighborhood \( \Omega \) of \((0, \pi/2)\) in \( \mathbb{R}^{n+1} \) to a neighborhood of 0 in \( \mathbb{R}^{n+1} \) such that \( f(W_0) = U \cap V_0 \) and \( f(W_1) = Z \cap V_0 \) (with the notations \( U, Z \) of (2.1.4)). Then \( f^{-1}(K_{\Psi}|_{V_0})[-n] \) extends as a sheaf \( F \) which fits in a distinguished triangle
\[
k_{\mathbb{W}_0} \to F \to k_{\mathbb{W}_1}[-n] \to k_{\mathbb{W}_0}[1].
\]
The microsupport of $F$ is the image of $\mathrm{SS}(K_\varphi|_{V_0})$ by $df$ and we find $\rho_{\mathbb{R}^n}(\hat{\mathrm{SS}}(F|\Omega)) \subset L_0$, as required.

However $F$ still has a too big microsupport at the conic points $(0, 0)$ and $(0, \pi)$. We can repeat the above gluing process at these points and iterate. We obtain a sheaf $G_0 \in \mathcal{D}_{\mathrm{lb}}^!(k_{\mathbb{R}^{n+1}})$, in the same way we defined $K_{\infty}$ (see (6.1.9)), with the following property

\begin{equation}
G_0|_{\Omega_k} \simeq (T_{n,\pi})_{|*(F)}[\pi k|\Omega_k], \text{ for any } k \in \mathbb{Z}, \text{ where } F \text{ is defined in (6.2.5)} \text{ and } \Omega_k = \mathbb{R}^n \times ]k\pi/2, (k+2)\pi/2[\]
\end{equation}

and we have $\rho_{\mathbb{R}^n}(\hat{\mathrm{SS}}(G_0)) = L_0$.

**Proposition 6.2.5.** Proposition 6.2.4 holds with $\Lambda_0$ replaced by $L_0$.

**Proof.** It is enough to prove that $e(G_0) \geq \pi/2$; the rest of the proof of Proposition 6.2.4 works the same. By Remark 6.2.2 we have $e(G_0) \geq e(G_0|_{(x_0) \times \mathbb{R}})$ for any $x_0 \in \mathbb{R}^n$. The above description of $G_0$ implies $G_0|_{(0) \times \mathbb{R}} \simeq \bigoplus_{k \in \mathbb{Z}} k\pi/2, (k+1)\pi/2 ][\pi k]$. Since $e(k_{[a,b]}) = b - a$ we obtain $e(G_0) \geq \pi/2$, as claimed. \qed

### 6.2.4. Nonsqueezing for the ball.

We set $B = \mathrm{Int}(B_1(\mathbb{R}^n))$ and $S = S_1(T^*\mathbb{R}^n)$.

For $y \in B$, let $L_y \subset S$ be the image of $T^*\mathbb{R}^n \cap S$ by the characteristic flow of $S$. Then $L_y$ is a Lagrangian submanifold of $T^*\mathbb{R}^n$. To prove the nonsqueezing for the ball we use the Lagrangian submanifolds $L_y$, $y \in B$, in family. We first prove that there exists a sheaf $G$ on $B \times \mathbb{R}^{n+1}$ such that $\rho_{\mathbb{R}^n}(\hat{\mathrm{SS}}(G_y)) = L_y$, where $G_y = G|_{(y) \times \mathbb{R}^{n+1}}$.

Let us describe the family $L_y$. For a given $y \in B$ we can see $L_y$ as in (6.2.4) as the image of a map

\[f_y: (\mathbb{R}/\pi\mathbb{Z}) \times S_r(T^*_y\mathbb{R}^n) \to T^*\mathbb{R}^n, \ (s, \xi) \mapsto \exp(2si) \cdot (y; \xi),\]

where $r = \sqrt{1 - ||y||^2}$. We set $L^0 = \bigsqcup_{y \in B} L_y \subset \mathbb{R}^n \times T^*\mathbb{R}^n$. We can describe a Lagrangian submanifold $L \subset (T^*\mathbb{R}^n)^2$ above $L^0$ as follows. In $T^*\mathbb{R}^{2n}$ we consider the hypersurface $Z = T^*\mathbb{R}^n \times S$ and the Lagrangian submanifold $T^*_x\mathbb{R}^{2n}$. Then $L^1 = Z \cap T^*_x\mathbb{R}^{2n}$ is isotropic and its image $L$ by the characteristic flow of $Z$ is still isotropic, hence Lagrangian since it is of dimension $2n$. We can see that $L \cap (T^*B \times T^*\mathbb{R}^n)$ maps bijectively onto $L^0$ through $T^*B \times T^*\mathbb{R}^n \to B \times T^*\mathbb{R}^n$. Like $L_y$ the Lagrangian $L$ is the image of a map

\begin{equation}
f: (\mathbb{R}/\pi\mathbb{Z}) \times S \to (T^*\mathbb{R}^n)^2,
(s, (y; \xi)) \mapsto ((y; -\xi), \exp(2si) \cdot (y; \xi)).\]
\end{equation}

The important point is that $f$ is injective whereas $f_y$ is not. Let $L' \subset J^1(\mathbb{R}^{2n}) = (T^*\mathbb{R}^{2n}) \times \mathbb{R}$ be a Legendrian lift of $L$. Then $L' \to L$ is an
infinite cyclic cover. A loop on $L$ given by a characteristic flow line lifts to a path $l$ on $L'$ with $l(1) = T_\pi(l(0))$, where $T_\pi$ is as before the translation in the last variable by $\pi$. Hence $T_\pi(L') = L'$. Since $f$ is injective we have

$$T_c(L') \cap L' = \emptyset \quad \text{for } 0 < c < \pi.$$  

Moreover, we remark in (6.2.7) that, if $y \in \partial B$, then $\xi = 0$ (because $(y; \xi) \in S$). Hence above $\partial B \times \mathbb{R}^n$ we have

$$L' \cap ((\partial B \times \mathbb{R}^n) \times \mathbb{R}^{2n}) J^1(\mathbb{R}^{2n})) \subset T^*_B B \times T^* \mathbb{R}^n \times \mathbb{R}.$$  

For $y \in B$ we let $L'_y \subset J^1(\{y\} \times \mathbb{R}^n)$ be the projection of $L' \cap (\{y\} \times \mathbb{R}^n)$ to $J^1(\{y\} \times \mathbb{R}^n)$. Then $L'_y$ is a Legendrian lift of $L_y$. For $y \in B$ these manifolds $L'_y$ are all diffeomorphic to $L'_0$ and there exists a $C^\infty$ map $u : B \times L'_0 \to J^1(\mathbb{R}^n)$ such that $u(\{y\} \times L'_0) = L'_y$. We can lift this map $u$ to a contact isotopy (see for example Theorem 2.6.2 of [18]). Indeed this is possible for a family of compact Legendrian manifolds. However, since $T_\pi(L'_0) = L'_y$ for all $y \in B$, we can find a family $L''_y$, $y \in B$, of compact Legendrian submanifolds of $(T^* \mathbb{R}^n) \times (\mathbb{R}/\pi \mathbb{Z})$ such that $L'_y$ is a covering of $L''_y$. Then we find a contact isotopy of $(T^* \mathbb{R}^n) \times (\mathbb{R}/\pi \mathbb{Z})$ and lift it to $J^1(\mathbb{R}^n)$. We thus have a contact isotopy $\Phi : B \times J^1(\mathbb{R}^n) \to J^1(\mathbb{R}^n)$ such that $\Phi_y(L'_0) = L''_y$ for any $y \in B$.

A Legendrian submanifold of $J^1(M)$ gives a conic Lagrangian submanifold of $\{\tau > 0\} \subset T^*(M \times \mathbb{R})$. Hence $L'$, $L'_y$ give conic Lagrangian submanifolds of $\hat{T}^* \mathbb{R}^{2n+1}$ or $\hat{T}^* \mathbb{R}^{n+1}$ that we also denote $L'$, $L'_y$. The equivalence of categories (2.1.6) of Corollary 2.1.5 applied with $I = B$, $A_0 = L$, $A = L'$ gives a unique sheaf $G \in D_{\{L\}}(k_{B \times \mathbb{R}^{n+1}})$ such that $G|_{\{0\} \times \mathbb{R}^{n+1}} \simeq G_0$, where $G_0$ is defined in (6.2.6).

**Remark 6.2.6.** A sheaf similar to $G$ is constructed in another way in [12] and is shown to be a projector from sheaves on $\mathbb{R}^{n+1}$ to sheaves with a microsupport contained in $\rho_{\mathbb{R}^n}^{-1}(B_1(T^* \mathbb{R}^n))$.

**Proposition 6.2.7.** Let $r < 1$ be given and let $D_r \subset \mathbb{R}^2$ be the closed disc of radius $r$. There is no Hamiltonian isotopy $\Phi : \mathbb{R}^{2n} \times I \to \mathbb{R}^{2n}$ such that $\Phi_1(B_1(\mathbb{R}^{2n})) \subset D_r \times \mathbb{R}^{2n-2}$.

**Proof.** It is enough to prove that $e(G) \geq \pi$; the rest of the proof of Proposition 6.2.4 works the same (recall that Corollary 6.1.6 works with a parameter space $N$ which is $B$ in our case). We set $\Omega = B \times \mathbb{R}^{n+1}$ and let $j : \Omega \to \mathbb{R}^{2n+1}$ be the inclusion. We let $u \in \text{Hom}(R_jG, R_juG)$ be the natural morphism. It is enough to prove that $u \subset := R_ju(\tau_c(G)) \circ u$ is non zero for $0 \leq c < \pi$. 

As in the proof of Proposition 6.2.5 we have $u_\epsilon \neq 0$ for small $c$ (in fact for $c < \pi/2$). We have $u_\epsilon \in H_\epsilon := \text{Hom}(R_j\pi, T_{c\epsilon}R_j\pi)$. It is thus enough to see that the morphism $H_\epsilon \to H_\epsilon$ is an isomorphism for $0 < \epsilon \leq c < \pi$.

We set $L_\Omega = \text{SS}(k_\Omega)$. We recall that $L_\Omega$ is the union of the zero section over $\Omega$ and one half of $T_{\partial\Omega}^\ast \mathbb{R}^{2n+1}$. By Theorem 1.2.15 we have $\text{SS}(R_j\pi) \subset L' \setminus L_\Omega$ and $\text{SS}(R_j\pi) \subset L' \setminus L_\Omega$. Hence $\text{SS}(R_j\pi) \cap \text{SS}(T_{c\epsilon}R_j\pi) = \emptyset$ for $0 < c < \pi$: this follows from (6.2.8) above $\Omega$ and from (6.2.8) and (6.2.9) above $\partial\Omega$. The hypotheses of Corollary 1.2.17 are then almost satisfied except for the compactness of $\text{supp}(R_j\pi)$. However $G$ is $\pi$-periodic up to shift in the sense that $T_{x_\pi}(G) \simeq G[-2n]$ and we deduce that $\text{RHom}(R_j\pi, T_{c\epsilon}R_j\pi)$ is $\pi$-periodic. Hence we can apply Lemma 6.2.8 below which a variant of Corollary 1.2.17.

**Lemma 6.2.8.** Let $M$ be a manifold and $J$ an open interval of $\mathbb{R}$. Let $F, G \in D(k_{M \times \mathbb{R} \times J})$. We assume

1. the projection $\text{supp}(F) \cap \text{supp}(G) \to \mathbb{R} \times J$ is proper,
2. $F, G$ are non-characteristic for all maps $i_u: M \times \mathbb{R} \times \{u\} \to M \times \mathbb{R} \times J$, $u \in J$, that is, $\text{SS}(A) \cap (T^\ast_{M \times \mathbb{R}}(M \times \mathbb{R}) \times T^\ast_t J) = \emptyset$ for $A = F, G$,
3. setting $\Lambda_u = i^\ast_u(\text{SS}(F))$ and $\Lambda'_u = i^\ast_u(\text{SS}(G))$, we have $\Lambda_u \cap \Lambda'_u = \emptyset$ for all $u \in J$,
4. $H := \text{RHom}(F, G)$ is $a$-periodic for some $a \in \mathbb{R}$ in the sense that $T_{-1}aH \simeq H$, where $T_a$ is the translation $T_a(x, t, u) = (x, t + a, u)$.

Then $\text{RHom}(i^{-1}_u F, i^{-1}_u G)$ is independent of $u \in J$.

**Proof.** As in the proof of Corollary 1.2.17 we set $H = \text{RHom}(F, G)$ and the microsupport estimates imply $\text{RHom}(i^{-1}_u F, i^{-1}_u G) \simeq i^{-1}_u H$ for all $u \in J$. Hence $\text{RHom}(i^{-1}_u F, i^{-1}_u G) \simeq \text{R}\Gamma(M \times \mathbb{R}; i^{-1}_u H)$.

Let $e: \mathbb{R} \to S^1 = \mathbb{R}/a\mathbb{Z}$ be the projection. We also write $e$ for $\text{id}_M \times e$ or $\text{id}_M \times e \times \text{id}_J$. Since $H$ is periodic there exists $H' \in D(k_{M \times S^1 \times J})$ such that $H \simeq e^{-1}H'$. Since $e$ is a covering map, we have $e_\ast e^{-1}H' \simeq H' \otimes L_{M \times S^1 \times J}$, where $L_{M \times S^1 \times J}$ is the local system $L_{M \times S^1 \times J} = e_\ast(k_{M \times \mathbb{R} \times J})$. In the same way $e_\ast e^{-1}j^{-1}_u H' \simeq j^{-1}_u H' \otimes L_{M \times S^1}$, where $j_u$ is the inclusion of $M \times S^1 \times \{u\}$. Finally

$$\text{RHom}(i^{-1}_u F, i^{-1}_u G) \simeq \text{R}\Gamma(M \times \mathbb{R}; e_\ast e^{-1}j^{-1}_u H')$$

$$\simeq \text{R}\Gamma(M \times S^1; j^{-1}_u(H' \otimes L_{M \times S^1 \times J})).$$

Let $q: M \times S^1 \times J \to J$ be the projection. Then $q$ is proper on $\text{supp}(H')$ and the base change formula gives $\text{RHom}(i^{-1}_u F, i^{-1}_u G) \simeq (\text{R}q_\ast(H' \otimes L_{M \times S^1 \times J}))_u$. Hence it is enough to see that $\text{R}q_\ast(H' \otimes L_{M \times S^1 \times J})$ is
locally constant, which is proved by microsupport estimates as in the proof of Corollary [2.17] (we remark that $H'$ and $H' \otimes L_{M \times S^1 \times J}$ have the same microsupport since $L_{M \times S^1 \times J}$ is locally constant).

□

Part 7. The Gromov-Eliashberg theorem

The Gromov-Eliashberg theorem (see [14, 21]) says that the group of symplectomorphisms of a symplectic manifold is $C^0$-closed in the group of diffeomorphisms. This can be translated into a statement about the Lagrangian submanifolds which are graphs of symplectomorphisms. It can be deduced from the Gromov nonsqueezing theorem but we want to stress the relation with the involutivity theorem of Kashiwara-Schapira (stated here as Theorem 1.3.6).

Let us explain the idea of the proof. We assume for a while a stronger assumption than the Gromov-Eliashberg theorem: let $M$ be a manifold and let $\phi_n$ be a sequence of homogeneous Hamiltonian isotopies of $\tilde{T}^* M$ which converges in $C^0$ norm to a diffeomorphism $\phi_\infty$ of $\tilde{T}^* M$. Let $K_n \in \mathcal{D}(\mathbf{k}_M)$ be the sheaf associated with $\phi_n$ by Theorem 2.1.1. Hence $\mathcal{S}\mathcal{S}(K_n)$ is $\Gamma_{\phi_n}$, the graph of $\phi_n$, and $H^0(M^2; K_n) \simeq \mathbf{k}$. We define a kind of limit $K_\infty$ by the distinguished triangle $\bigoplus_{n \in \mathbb{N}} K_n \to \prod_{n \in \mathbb{N}} K_n \to K_\infty \to$. Then we can check that $\mathcal{S}\mathcal{S}(K_\infty) \subset \Gamma_{\phi_\infty}$ and $H^0(M^2; K_\infty) \simeq \prod_{k \in \mathbb{N}} \mathbf{k}/\bigoplus_{k \in \mathbb{N}} \mathbf{k}$. Hence $K_\infty$ is not the zero sheaf. If we could prove moreover that $K_\infty$ is not locally constant, we would deduce from the involutivity theorem that $\Gamma_{\phi_\infty}$ is coisotropic, then that $\phi_\infty$ is a symplectic map.

The Gromov-Eliashberg theorem is a local non homogeneous version of the previous (unproved) statement and the $\phi_n$ are only symplectic diffeomorphisms. It is not difficult to modify the $\phi_n$ away from a given point to turn them into Hamiltonian isotopies and make them homogeneous by adding a variable. Our sheaves $K_n$ live now on $M^2 \times \mathbb{R}$. The problem is that the convergence of $\Gamma_{\phi_n}$ to $\Gamma_{\phi_\infty}$ is now true only in a neighborhood of a point and we have no control on $\Gamma_{\phi_n}$ away from this point; more precisely we have a subset $\Gamma'_n$ of $\Gamma_{\phi_n}$ such that $\Gamma'_n$ converges to a subset of $\Gamma_{\phi_\infty}$. We use a cut-off lemma of Part 3 to split $K_n$ in a small ball $B$ as $K_n = K'_n \oplus K''_n$, with $\mathcal{S}\mathcal{S}(K'_n) \subset \Gamma'_n$. The main difficulty is to prove that the above “limit” of $K'_n$ is not locally constant. For this we restrict $K_n$ to a line $D = \{x_0\} \times \mathbb{R}$ and decompose $K_n$ as a sum of constant sheaves on intervals using Corollary 4.4.3 say $K_n|_D \simeq \bigoplus_{a \in A_n} \mathbf{k}[I_a][d^n_a]$. To ensure that $K'_n$ does not vanish when $n \to \infty$, we prove that there are intervals $I^n_a$ bigger than $B \cap D$ as follows. Let $\pi$ be the projection from $T^*(M^2 \times \mathbb{R})$ to the base. If an interval $I^n_a$ is contained in $B$, the splitting $K_n = K'_n \oplus K''_n$ prevents it
from having one end in \( \pi(\Gamma'_n) \) and the other in \( \pi(\Gamma'_n \setminus \Gamma'_n) \). In other word, the intervals with exactly one end in \( \pi(\Gamma'_n) \) are big. Hence it is enough to see that the projection of \( \Gamma'_n \) to \( M^2 \) is of degree one to find a big interval.

7.1. The involutivity theorem

The main tool in our proof of the Gromov-Eliashberg theorem is the involutivity theorem of [31]. We recall its statement (see Theorem 6.5.4 of [31] restated here as Theorem 1.3.6). For a manifold \( N \), a subset \( S \) of \( N \) and \( p \in N \), we use the notations \( C_p(S), C_p(S, S) = C(S, S) \cap T_p N \) of (1.1.2) and (1.1.3) for the tangent cones of \( S \) at \( p \). Let \( M \) be a manifold, \( k \) any coefficient ring and \( F \in D(k_M) \). The involutivity theorem says that the microsupport \( S = SS(F) \) of \( F \) is a coisotropic subset of \( T^* M \) in the sense that \( (C_p(S, S))^\perp \subset C_p(S) \), for all \( p \in S \). We quote the following lemmas.

**Lemma 7.1.1.** Let \( X \) be a symplectic manifold and let \( S \subset S' \) be locally closed subsets of \( X \). Let \( p \in S \). We assume that \( S \) is coisotropic at \( p \). Then \( S' \) is also coisotropic at \( p \).

**Proof.** This is obvious since we have the inclusions
\[
(C_p(S', S'))^{\perp_{\omega_p}} \subset (C_p(S, S))^{\perp_{\omega_p}} \subset C_p(S) \subset C_p(S').
\]
\( \square \)

We recall the map \( \rho_M : T^* M \times T^* \mathbb{R} \to T^* M, (x, t; \xi, \tau) \mapsto (x; \xi/\tau) \), defined in (2.3.1).

**Lemma 7.1.2.** Let \( M \) be a manifold and \( S \subset T^* M \) a locally closed subset. Let \( p \in S \) and \( q \in \rho_M^{-1}(p) \). Then \( S \) is coisotropic at \( p \) if and only if \( \rho_M^{-1}(S) \) is coisotropic at \( q \).

**Proof.** We use coordinates \( (x, t; \xi, \tau) \) on \( T^*(M \times \mathbb{R}) \) and corresponding coordinates \( (X, T; \Xi, \Sigma) \) on \( T_q T^*(M \times \mathbb{R}) \). We set \( S' = \rho_M^{-1}(S) \) and we write \( q = (x_0, t_0; \xi_0, \tau_0) \). We have \( d\rho_M(X, T; \Xi, \Sigma) = (X; \Xi_0 - \frac{\xi_0}{\tau_0} \Sigma) \). Hence \( S' \) is conic, we may assume \( \tau_0 = 1 \). Using the symplectic transformations \( (x; \xi) \mapsto (x; \xi - \xi_0) \) on \( T^* M \) and \( (x, t; \xi, \tau) \mapsto (x, t + \langle \xi_0, x \rangle; \xi - \tau \xi_0, \tau) \) on \( T^*(M \times \mathbb{R}) \), which commute with \( \rho_M \), we may also assume \( \xi_0 = 0 \). Then we have \( d\rho_M(X, T; \Xi, \Sigma) = (X; \Xi) \) and we deduce \( C_q(S') = C_p(S) \times T_{(t_0;1)} T^* \mathbb{R} \) and \( C_q(S', S'') = C_p(S, S) \times T_{(t_0;1)} T^* \mathbb{R} \). Now the result follows easily. \( \square \)

7.2. Approximation of symplectic maps

Let \( (E, \omega) \) be a symplectic vector space which we identify with \( \mathbb{R}^{2n} \). We recall a standard application of the Alexander trick which says that
a symplectic map \( \varphi : B^E_R \to E \) defined on some ball of \( E \) coincides with a Hamiltonian isotopy of \( E \) on some smaller ball \( B^E_r \).

We endow \( E \) with the Euclidean norm of \( \mathbb{R}^{2n} \). For an open subset \( U \subset E \) and a map \( \psi : U \to E \) we set

\[
(7.2.1) \quad \| \psi \|_U = \sup\{ \| \psi(x) \| ; \quad x \in U \},
\]

\[
(7.2.2) \quad \| \psi \|^1_U = \sup\{ \| \psi(x) \|, \| d\psi_x(v) \| ; \quad x \in U, \| v \| = 1 \}, \text{ if } \psi \text{ is } C^1.
\]

**Lemma 7.2.1.** Let \( R > r \) and \( \varepsilon \) be positive numbers. Let \( \varphi : B^E_R \to E \) be a symplectic map of class \( C^1 \). Then there exists \( R' > r \) and a symplectic map \( \psi : B^E_{R'} \to E \) which is of class \( C^\infty \) such that \( \| \varphi - \psi \|_{B^E_1} \leq \varepsilon \).

**Proof.** We set \( r_1 = (R + r)/2 \) and we choose a (non symplectic) map \( \varphi' : B^E_R \to E \) of class \( C^\infty \) such that \( \| \varphi - \varphi' \|_{B^E_1} \leq \varepsilon \). We set \( \omega' = \varphi'^*(\omega) \).

We have \( \omega - \omega' = (\varphi - \varphi')^*\omega \). Hence, if we consider \( \omega \) and \( \omega' \) as maps from \( E \) to \( \wedge^2 E \) and we endow \( \wedge^2 E \) with the Euclidean structure induced by \( E \), we have \( \| \omega - \omega' \|_{B^E_1} \leq C\varepsilon \), where the constant \( C \) only depends on \( n \).

We set \( r_2 = (r_1 + r)/2 \). By Moser’s lemma we can find a flow \( \Phi : B^E_{r_2} \times [0, 1] \to E \) such that \( \Phi_t(B^E_{r_2}) \subset B^E_{r_1} \) for all \( t \in [0, 1] \) and \( \varphi|_{B^E_{r_2}} = \Phi_1(\omega')|_{B^E_{r_2}} \). The flow \( \Phi \) is the flow of a vector field \( X_t \) which satisfies

\[ \iota_{X_t}(\omega_t) = -\alpha \text{ over } B^E_{r_1}, \]

where \( \omega_t = t\omega' - (1-t)\omega \) and \( d\alpha = \omega' - \omega \).

We can assume that \( \alpha \) satisfies the bound \( \| \alpha \|_{B^E_{r_1}} \leq C' \| \omega' - \omega \|_{B^E_{r_1}} \) for some \( C' > 0 \) only depending on \( r_1 \). Hence \( X_t \) satisfies \( \| X_t \|_{B^E_1} \leq C''\varepsilon \), for some constant \( C'' > 0 \) and all \( t \in [0, 1] \).

We may assume from the beginning that \( C''\varepsilon < r_1 - r_2 \). Hence \( \Phi_1(B^E_{r_2}) \subset B^E_{r_1} \) and we have \( \| \Phi_1 - \text{id} \|_{B^E_1} \leq C''\varepsilon \). The map \( \psi = \varphi' \circ \Phi_1 : B^E_{r_2} \to E \) is a symplectic map such that \( \| \varphi - \psi \|_{B^E} \leq (1 + C'')\varepsilon \), which gives the lemma (up to replacing \( \varepsilon \) by \( \varepsilon/(1 + C'') \)).

**Proposition 7.2.2.** Let \( R > r > 0 \) be given. Let \( \varphi : B^E_R \to E \) be a symplectic map of class \( C^\infty \). Then there exists a Hamiltonian isotopy \( \Phi : E \times \mathbb{R} \to E \) of class \( C^\infty \) and with compact support such that \( \Phi_1|_{B^E} = \varphi|_{B^E} \).

**Proof.** (i) Up to composing with a translation or a symplectic linear map, we may assume that \( \varphi(0) = 0 \) and \( d\varphi_0 = \text{id}_E \).

We first show that there exists a Hamiltonian isotopy \( \Phi : E \times \mathbb{R} \to E \) with compact support such that \( \Phi^{-1}_1 \circ \varphi = \text{id}_E \) near 0. We choose a symplectic isomorphism \( E^2 \simeq T^*\Delta \) where \( \Delta \) is the diagonal. Through this isomorphism the graph of \( \varphi \) is a Lagrangian subset, say \( \Lambda \), of \( T^*\Delta \). Since \( \varphi(0) = 0 \) and \( d\varphi_0 = \text{id}_E \), the set \( \Lambda \) is tangent to the zero section...
Moreover we can find $\|\psi\|$. Then $(\varphi - (ii)$ By (i) we can assume that $\varphi_t \Phi$ is the graph of $\psi$ and $\Phi_1^{-1} \circ \varphi = \text{id}_E$ near 0.

(ii) By (i) we can assume that $\varphi|_{B^E_\varepsilon} = \text{id}_E$ for some small ball $B^E_\varepsilon$. We define $U \subset E \times ]0, +\infty[ \text{ and } \psi : U \to E$ by

$$U = \{(x, t) ; \|x\| < R/t\}, \quad \psi(x, t) = t^{-1} \varphi(tx).$$

Then $\psi(\cdot, t)$ is a symplectic map for all $t > 0$. We let $V = \{((\varphi(x, t), t); (x, t) \in U\}$ be the image of $U$ by $\psi \times \text{id}_{\mathbb{R}}$. Then $V$ is contractible and we can find $h : V \to \mathbb{R}$ such that $\psi$ is the Hamiltonian flow of $h$.

We define $U_0 \subset U$ by $U_0 = \{(x, t) ; t > 0, \|x\| < \varepsilon/t\}$. Since $\varphi|_{B^E_\varepsilon} = \text{id}_E$, we have $\psi(x, t) = x$ for all $(x, t) \in U_0$. Hence $U_0 \subset V$. Moreover $h$ is constant on $U_0$. We can assume $h|_{U_0} = 0$ and extend $h$ by 0 to a $C^\infty$ function defined on $V' = ]-\infty, 0] \cup V$.

We set $Z = \{((\psi(x, t), t); t \in ]0, 1[ \text{ and } \|x\| \leq r\}$. For $t \leq \varepsilon/r$ and $\|x\| \leq r$ we have $\psi(x, t) = x$. Hence

$$Z = (B^E_\varepsilon \times [0, \varepsilon/r]) \cup (\psi \times \text{id}_{\mathbb{R}})(B^E_\varepsilon \times [\varepsilon/r, 1])$$

and it follows that $\overline{Z}$ is compact. We choose a $C^\infty$ function $g : E \times \mathbb{R} \to \mathbb{R}$ with compact support such that $g = h$ on $Z$. Then the Hamiltonian isotopy $\Phi$ defined by $g$ has compact support contained in $C$ and satisfies $\Phi_1 = \varphi$ on $B^E_\varepsilon$. This proves the proposition. \hfill $\Box$

### 7.3. Degree of a continuous map

We recall the definition of the degree of a continuous map. Let $M, N$ be two oriented manifolds of the same dimension, say $d$. We assume that $N$ is connected. We have a morphism $H^d_c(M; \mathbb{Z}_M) \to \mathbb{Z}$ and an isomorphism $H^d_c(N; \mathbb{Z}_N) \cong \mathbb{Z}$. Let $f : M \to N$ be a proper continuous map. Applying $H^d_c(N; \cdot)$ to the morphism $\mathbb{Z}_N \to Rf_*f^{-1}\mathbb{Z}_N \cong Rf_*\mathbb{Z}_M$ we find

$$\mathbb{Z} \cong H^d_c(N; \mathbb{Z}_N) \to H^d_c(M; \mathbb{Z}_M) \to \mathbb{Z}.$$

The degree of $f$, denoted $\text{deg } f$, is the image of 1 by this morphism.

**Lemma 7.3.1.** Let $M, N$ be two oriented manifolds of dimension $d$. We assume that $N$ is connected.

(i) Let $f : M \to N$ be a proper continuous map and let $V \subset N$ be a connected open subset. Then $\text{deg } f = \text{deg } (f|_{f^{-1}(V)} : f^{-1}(V) \to V)$.
(ii) Let $I \subset \mathbb{R}$ be an interval. Let $U \subset M \times I$, $V \subset N \times I$ be open subsets and let $f: U \to V$ be a continuous map which commutes with the projections $U \to I$ and $V \to I$. We set $U_t = U \cap (M \times \{t\})$, $V_t = V \cap (N \times \{t\})$ and $f_t = f|_{U_t}: U_t \to V_t$, for all $t \in I$. We assume that $f$ is proper and that $V$ and all $V_t$, $t \in I$, are non empty and connected. Then $\deg f = \deg f_t$, for all $t \in I$.

Proof. (i) and (ii) follow respectively from the commutative diagrams

\[
\begin{array}{cccc}
\mathbb{Z} & \xleftarrow{\sim} & H^d_c(N; \mathbb{Z}_V) & \longrightarrow & H^d_c(M; \mathbb{Z}_{f^{-1}(V)}) & \longrightarrow & \mathbb{Z} \\
\downarrow & & \downarrow & & \downarrow & & \\
\mathbb{Z} & \xleftarrow{\sim} & H^d_c(N; \mathbb{Z}_N) & \longrightarrow & H^d_c(M; \mathbb{Z}_M) & \longrightarrow & \mathbb{Z}, \\
\%
\downarrow & & \downarrow & & \downarrow & & \\
\mathbb{Z} & \xleftarrow{\sim} & H^d_c(V_t; \mathbb{Z}_{V_t}) & \longrightarrow & H^d_c(U_t; \mathbb{Z}_{U_t}) & \longrightarrow & \mathbb{Z} \\
\%
\downarrow & & \downarrow & & \downarrow & & \\
\mathbb{Z} & \xleftarrow{\sim} & H^{d+1}_c(V; \mathbb{Z}_V) & \longrightarrow & H^{d+1}_c(U; \mathbb{Z}_U) & \longrightarrow & \mathbb{Z}.
\end{array}
\]

\[\square\]

**Proposition 7.3.2.** Let $B_R$ be the open ball of radius $R$ in $\mathbb{R}^d$. Let $U, V \subset \mathbb{R}^d$ be open subsets and let $f: U \to B_R$, $g: V \to B_R$ be proper continuous maps. We assume that there exists $r < R$ such that $f^{-1}(B_r) \subset U \cap V$. Then $\deg f = \deg g$.

Proof. (i) We define $h: (U \cap V) \times [0, 1] \to \mathbb{R}^{d+1}$ by $h(x, t) = (tf(x) + (1-t)g(x), t)$. Let us prove that $h^{-1}(B_{r/2} \times [0, 1])$ is compact. Since $h^{-1}(B_{r/2} \times [0, 1])$ is compact and contained in $U \cap V$, it enough to prove that $h^{-1}(B_{r/2} \times [0, 1]) \subset h^{-1}(B_r) \times [0, 1]$. Let $(x, t) \in (U \cap V) \times [0, 1]$ be such that $\|h(x, t)\| \leq r/2$. Since $h(x, t)$ belongs to the line segment $[f(x), g(x)]$ which is of length $< r/2$, we deduce $f(x) \in B_r$, as required.

(ii) We define $W = h^{-1}(B_{r/2} \times [0, 1])$, $W_t = W \cap (\mathbb{R}^d \times \{t\})$ for $t \in [0, 1]$ and $h'_t = h|_{W_t}: W_t \to B_{r/2}$. By (i) $h|_W: W \to B_{r/2} \times [0, 1]$ is proper. Hence Lemma 7.3.1 (ii) implies that $\deg h'_0 = \deg h'_1$. We conclude with Lemma 7.3.1 (i) which implies $\deg h'_0 = \deg g$ and $\deg h'_1 = \deg f$. \[\square\]

**7.4. The Gromov-Eliashberg theorem**

Let $(E, \omega)$ be a symplectic vector space which we identify with $\mathbb{R}^{2n}$. We endow $E$ with the Euclidean norm of $\mathbb{R}^{2n}$. For $R > 0$ we let $B^E_R$ be the open ball of radius $R$ and center $0$. For a map $\psi: B^E_R \to E$ we set $\|\psi\|_{B^E_R} = \sup \{\|\psi(x)\|; x \in B^E_R\}$. 
Lemma 7.4.1. Let $V = \mathbb{R}^n$ and $E = T^*V$. Let $f: E \to E$ be a map of class $C^1$ and $0 < R$ be given. Then there exist a Hamiltonian isotopy $\Theta: T^*V^2 \times \mathbb{R} \to T^*V^2$ with compact support, $\varepsilon > 0$ and three balls centered at $0$: $B_V \subset V^2$ and $B^*_0 \subset B_1^* \subset (V^*)^2$ such that for any other map $g: E \to E$ with $\|f-g\|_{B^R_2} < \varepsilon$ we have

(i) $\Theta \circ i_f(0) = (0; 0)$,

(ii) $\Theta_1(\Gamma_g) \cap (B_V \times (B_1^* \setminus B_0^*)) = \emptyset$,

(iii) $\Gamma'_g := \Theta_1(\Gamma_g) \cap (B_V \times B_1^*)$ is contained in $\Theta_1(i_g(B^R_2))$,

(iv) the restriction of $\pi_{V^2}$ to $\Gamma'_g$ gives a proper map $\Gamma'_g \to B_V$ of degree $1$.

Proof. (i) We set $p = (0, f(0)) \in E^2$, $q = (0; 0) \in T^*V^2$ and $F = T_q \Gamma_f$. We can find a symplectic map $\psi: T_pE^2 \to T_qE^2$ such that $d(\pi_{V^2})_q: T_qE^2 \to T_0V^2$ induces an isomorphism $\psi(F) \xrightarrow{\sim} T_0V^2$. We choose a Hamiltonian isotopy $\Theta$ such that $\Theta_1(p) = q$ and $d\Theta_1 = \psi$.

(ii) We can find balls $B_V$, $B_0^*$, $B_1^*$ such that (b-d) hold for $g = f$. Indeed we first choose a neighborhood $W \subset B^R_2$ of $0$ in $E$ such that $\pi_{V^2} \circ \Theta_1 \circ i_f$ induces a diffeomorphism from $W$ to $W' = \pi_{V^2}(\Theta_1(i_f(W)))$. Then $\pi_{V^2}$ is a diffeomorphism from $\Gamma' = \Theta_1(i_f(W))$ to $W'$ and it is easy to find $B_V$, $B_0^*$ such that (c) and (d) hold. Up to shrinking $B_V$, $B_0^*$, we can also find $B_1^*$ such that (b-d) hold.

(iii) For $g$ close enough to $f$ the property (d) holds by Proposition 7.3.2 (apply the proposition with $f'$, $g'$, $U'$, $V'$ where $f' := \pi_{V^2} \circ \Theta_1 \circ i_f$, $g' := \pi_{V^2} \circ \Theta_1 \circ i_g$, $U' = f'^{-1}(B_V)$, $V' = g'^{-1}(B_V)$).

To check (c), we ask the additional condition $\Theta_1^{-1}(B_V \times B_1^*) \subset B^R_2 \times E$, which is satisfied up to shrinking $B_V$, $B_1^*$.

For (b) we choose balls $B'_V$, $B'_1$ slightly smaller than $B_V$, $B_1^*$ and $B'_0$ slightly bigger than $B_0^*$. Let us assume that there exists $x \in E$ such that $\Theta_1(x, g(x)) \in B'_V \times (B'_1 \setminus B'_0)$. Then $(x, g(x)) \in B^R_2 \times E$ and $\|f(x) - g(x)\| < \varepsilon$. For $\varepsilon$ small enough this implies $\Theta_1(x, f(x)) \in B_V \times (B_1^* \setminus B_0^*)$ which is empty. Hence, up to replacing $B_V$, $B_0^*$, $B_1^*$ by $B'_V$, $B'_0$, $B'_1$, we also have (b) for $g$. \qed

Now we can give a proof of the Gromov-Eliashberg rigidity theorem (see [14, 21]).

Theorem 7.4.2. Let $R > 0$. Let $\varphi_k: B^E_R \to E$, $k \in \mathbb{N}$, and $\varphi_\infty: B^E_R \to E$ be $C^1$ maps. We assume

(i) $\varphi_k$ is a symplectic map, that is, $\varphi_k^*(\omega) = \omega$, for all $k \in \mathbb{N}$,
homogeneous lift of $\Phi_k = \gamma_k$. Then

(i) We will prove $d\varphi_\infty (x) E \to T\varphi_\infty (x) E$ is an isomorphism, for all $x \in B^E_R$. Then $\varphi_\infty|B^E_R$ is a symplectic map.

Proof. (i) We will prove $d\varphi_\infty|B^E_R$ is a symplectic linear map for any given $x \in B^E_R$. Up to composition with a translation we can as well assume $x = 0$. By Lemma 7.2.1 and Proposition 7.2.2 we can also assume, up to shrinking $R$, that $\varphi_k = \Phi_k|B^E_R$ is the restriction of (the time 1 of) a globally defined Hamiltonian isotopy $\Phi^k: E \times \mathbb{R} \to E$ with compact support, for each $k \in \mathbb{N}$.

Let us choose an isomorphism $E \simeq T^*V$ where $V = \mathbb{R}^n$. We let $\Gamma_k \subset T^*V^2$ be the graph of $\Phi^k_1$, twisted by $(x, x'; \xi, \xi') \mapsto (x, x'; \xi, -\xi')$. We let $\Gamma_\infty$ be the graph of $\varphi_\infty$ with the same twist. It is enough to prove that $\Gamma_\infty$ is coisotropic at $(0; \varphi_\infty(0))$.

We assume $n \geq 2$ (the case $n = 1$ is about volume preserving maps and is easy). Then $\Phi^k$ can be defined by a compactly supported Hamiltonian function.

(ii) By Corollary 2.3.2 there exists $F_k \in D^b(k_{V^2 \times \mathbb{R}})$ such that $\mathbb{S}\mathbb{S}(F_k)$ is a conic Lagrangian submanifold of $\hat{T}^*(V^2 \times \mathbb{R})$ which is the graph of a homogeneous lift of $\Phi^k_1$. Applying the cut-off functor $P_\gamma$ of (3.1.1), with $\gamma = \{(0, t) \in V^2 \times \mathbb{R}; t \leq 0\}$, we can as well assume $\mathbb{S}\mathbb{S}(F_k) \subset \{\tau > 0\}$. We recall the map $\rho: T^*V^2 \times T^*\mathbb{R} \to T^*V^2$, $(x, t; \xi, \tau) = (x; \xi/\tau)$, defined in (2.3.1). Then $\rho(\mathbb{S}\mathbb{S}(F_k)) = \Gamma_k$.

(iii) We apply Lemma 7.4.1 with the function $f: E \to E$ given by $f(x; \xi) = \varphi_\infty(x; \xi)^a$. It yields a Hamiltonian isotopy $\Theta$ with compact support and three balls centered at 0: $B_V \subset V^2$ and $B^*_0 \subset B^*_1 \subset (V^*)^2$ such that the conditions (a-d) of the lemma hold for $g(x; \xi) = \varphi_k(x; \xi)^a$ if $k$ is big enough.

By Proposition 2.3.1 we can lift $\Theta$ to a homogeneous Hamiltonian isotopy $\tilde{\Theta}$ of $\hat{T}^*(V^2 \times \mathbb{R})$ such that the diagram (2.3.2) commutes. Then Theorem 2.1.1 gives $K \in D^b(k_{(V^2 \times \mathbb{R})^2})$ such that $\mathbb{S}\mathbb{S}(K)$ is the graph of $\tilde{\Theta}_1$. We set $G_k = K \circ F_k$ and $\Lambda_k = \mathbb{S}\mathbb{S}(G_k)$ for each $k \in \mathbb{N}$. Then $G_k \in D^b(k_{(V^2 \times \mathbb{R})^2})$ and $\Lambda_k = \tilde{\Theta}_1(\mathbb{S}\mathbb{S}(F_k))$. We still have $\Lambda_k \subset \{\tau > 0\}$, $\rho(\Lambda_k) = \Theta_1(\Gamma_k)$.

(iv) We can find a point $x_0 \in B_V \subset V^2$, as closed to 0 as we want, such that, for any $k$, the Lagrangian submanifold $\Lambda_k \subset T^*(V^2 \times \mathbb{R})$ is in generic position with respect to the line $D = \{x_0\} \times \mathbb{R}$ in the following sense: there exists a neighborhood $W_k$ of $D$ and a hypersurface $S_k$ of $W_k$ meeting $D$ transversely such that $\Lambda_k \cap \hat{T}^*W_k = \hat{T}^*_kW_k \cap \{\tau > 0\}$. 
in bijection with

Then \( G_k|_D \) is a constructible sheaf and we can decompose it as a finite sum of constant sheaves on intervals \( G_k|_D \approx \bigoplus_{\alpha \in A_k} k_{I^k_\alpha}[d^k_\alpha] \) by Corollary 4.4.3. Since \( \dot{\mathrm{SS}}(G_k) \subset \{ \tau \geq 0 \} \), the \( I^k_\alpha \) are of the form \([a,b]\) or \([0,b]\] with \( b \in \mathbb{R} \). One of these possibilities occurs infinitely many times and, up to taking a subsequence, we assume

\[
I^k_\alpha = [0,b_k] \text{ or } [b_k,0].
\]

By Lemma 7.4.1, the map \( \Gamma^0_k \rightarrow B_V \) is of degree 1 and it follows that \( E^0_k \) is of odd cardinality. Hence there exists one interval \( I^k_\alpha \) with one end, say \( a_k \), in \( E^0_k \) and the other, say \( b_k \), in \( (E_k \setminus E^0_k) \cup \{ \pm \infty \} \). We translate \( G_k \) vertically so that \( a_k = 0 \) and we shift its degree so that \( d^k_\alpha = 0 \) (we still have \( \rho(I^k_\alpha) = \Theta_1(\Gamma^0_k) \)).

Now \( G_k|_D \) has one direct summand which is \( k_{I^k_\alpha} \) with \( I^k_\alpha = [0,b_k] \) or \([b_k,0] \), \( b_k \) finite or infinite. One of these possibilities occurs infinitely many times and, up to taking a subsequence, we assume \( I^k_\alpha = [0,b_k] \) with \( b_k \in \mathbb{R} \), for all \( k \), the other cases being similar.

By Lemma 7.4.1, again \( \rho(\dot{\mathrm{SS}}(G_k)) \cap (B_V \times (B^*_V \setminus B^*_0)) = \emptyset \). Hence, by Proposition 3.3.2, there exists an open ball \( W \) with center \( (x_0,0) \) and radius \( r \) such that, for any \( k \), we have a distinguished triangle on \( W \)

\[
G'_k \oplus G''_k \rightarrow G_k|_W \rightarrow L_k \xrightarrow{+1}.
\]

where \( L_k \) is a constant sheaf, \( \dot{\mathrm{SS}}(G_k') = \Lambda'_k \) with

\[
\Lambda'_k = \Lambda_k \cap \rho^{-1}(B_V \times B^*_0) \cap T^*W
\]

and \( \dot{\mathrm{SS}}(G''_k) = (\Lambda_k \cap T^*W) \setminus \Lambda'_k \). In particular \( \dot{\mathrm{SS}}(G''_k) \) does not meet \( T^{x_0,0})W \) and \( G''_k \) is constant near \((x_0,0)\). In the same way, if \((x_0,b_k)\) belongs to \( W \), then \( G'_k \) is constant near \((x_0,b_k)\).

On the other hand, if \((x_0,b_k)\) belongs to \( W \), Lemma 7.4.3 below implies that the direct summand \( k_{[0,b_k]})|_{D\cap W} \) of \( G_k|_{D\cap W} \) is also a direct summand of \( H \) with \( H = G'_k|_{D\cap W} \) or \( H = G''_k|_{D\cap W} \). Then \( H \) would be non constant at \((x_0,0)\) and \((x_0,b_k)\), which gives a contradiction.

It follows that \((x_0,b_k) \notin W \) and \( G'_k|_{D\cap W} \) has a direct summand which is \( k_{[0,r]} \) (recall \( r \) is the radius of \( W \)).

We define \( G \in D(k_W) \) by the distinguished triangle

\[
(7.4.1) \quad \bigoplus_{k \in \mathbb{N}} G'_k \rightarrow \prod_{k \in \mathbb{N}} G'_k \rightarrow G \xrightarrow{+1}.
\]
For any \( N \in \mathbb{N} \) we also have the triangle \( \bigoplus_{k \geq N} G'_k \to \prod_{k \geq N} G'_k \to G \to \).

We have \( \hat{\SS}(\bigoplus_{k \geq N} G'_k) \subset \bigcup_{k \geq N} N'_k \) and the same bound holds for \( \hat{\SS}(\prod_{k \geq N} G'_k) \). Hence the same bound also holds for \( \hat{\SS}(G) \) and, when \( N \to \infty \), we obtain

\[
\hat{\SS}(G) \subset \rho^{-1}(\Theta_1(\Gamma_\infty) \cap (W \times B^*_0)).
\]

We let \( i: D \cap W \to W \) be the inclusion. We remark that \( i \) is non-characteristic for \( \rho^{-1}(W \times B^*_0) \). Hence \( i \) is non-characteristic for \( F = G'_k \) or \( F = \prod_{k \in \mathbb{N}} G'_k \) and Theorem 7.2.8 gives \( i^{-1} F \simeq i' F[1] \). Since \( i^{-1} \) commutes with \( \bigoplus \) and \( i' \) with \( \prod \), we deduce the following distinguished triangle by applying \( i^{-1} \) to \( (7.4.1) \)

\[
\bigoplus_{k \in \mathbb{N}} (G'_k|_D) \to \prod_{k \in \mathbb{N}} (G'_k|_D) \to G|_D \overset{+1}{\to}.
\]

We set \( l = \prod_{k \in \mathbb{N}} k / \bigoplus_{k \in \mathbb{N}} k \). Since \( k_{[0,r]} \) is a direct summand of \( G'_k|_{D\cap W} \), the sheaf \( l_{[0,r]} \) is a direct summand of \( G|_{D\cap W} \). In particular \( G \) is not constant around \((x_0, 0)\) and \( \hat{\SS}(G) \cap T^*_{(x_0,0)}(V^2 \times \mathbb{R}) \neq \emptyset \).

We choose \( p \in \hat{\SS}(G) \cap T^*_{(x_0,0)}(V^2 \times \mathbb{R}) \). By the involutivity Theorem and Lemma 7.1.2 we obtain that \( \Theta_1(\Gamma_\infty) \) is coisotropic at \( \rho(p) \). Since \( \rho(p) \in T^*_{x_0} V^2 \) and \( \Theta_1(\Gamma_\infty) \cap T^*_{x_0} V^2 \) consists of a single point, we have \( \rho(p) = (x_0; \xi(x_0)) \). The point \( x_0 \) can be chosen arbitrarily close to 0. Hence \( \Theta_1(\Gamma_\infty) \) is coisotropic at \((0; \xi(0))\) and it follows that \( \Gamma_\infty \) is coisotropic at \((0; \varphi_\infty(0))\), as required.

\[\square\]

**Lemma 7.4.3.** Let \( k \) be a field. Let \( I \) be an interval in \( \mathbb{R} \) and let \( a, b \in I \) with \( a < b \). Let \( F_1, F_2, F, L \in \mathcal{D}^0(k_I) \). We assume that \( \hat{\SS}(L) = \emptyset \), that we have a distinguished triangle

\[
F_1 \oplus F_2 \overset{u}{\to} F \overset{v}{\to} L \overset{-1}{\to},
\]

and that \( k_{[a,b]} \) is a direct summand of \( F \). Then \( k_{[a,b]} \) is a direct summand of \( F_1 \) or \( F_2 \).

**Proof.** By hypotheses there exist \( i: k_{[a,b]} \to F \) and \( p: F \to k_{[a,b]} \) such that \( p \circ i = \text{id}_{k_{[a,b]}} \). Since \( L \) has constant cohomology sheaves, we have \( \text{Hom}(k_{[a,b]}, L) \simeq 0 \). Hence \( v \circ i = 0 \) and there exists \( i' = (i_1^{[a,b]}) : k_{[a,b]} \to F_1 \oplus F_2 \) such that \( i = u \circ i' \). Let \( p_1: F_1 \to k_{[a,b]} \), \( p_2: F_2 \to k_{[a,b]} \) be the components of \( p \circ u \). Then \( \text{id}_{k_{[a,b]}} = p_1 \circ i_1 + p_2 \circ i_2 \) and we deduce \( p_1 \circ i_1 \neq 0 \) or \( p_2 \circ i_2 \neq 0 \). Since \( \text{Hom}(k_{[a,b]}, k_{[a,b]}) = k \), we can multiply our morphisms by a scalar to have \( p_1 \circ i_1 \) or \( p_2 \circ i_2 \) equals id, proving the lemma.

\[\square\]
Part 8. The three cusps conjecture

In [6] Arnol’d states a theorem of Möbius “a closed smooth curve sufficiently close to the projective line (in the projective plane) has at least three points of inflection” and conjectures that “the three points of flattening of an immersed curve are preserved so long as under the deformation there does not arise a tangency of similarly oriented branches”. This is a statement about oriented curves in \( \mathbb{R}P^2 \). Under the projective duality it is turned into a statement about Legendrian curves in the projectivized cotangent bundle of \( \mathbb{R}P^2 \): if \( \{ \Lambda_s \}_{s \in [0,1]} \) is a generic path in the space of Legendrian knots in \( PT^*(\mathbb{R}P^2) = (T^*\mathbb{R}P^2 \setminus \mathbb{R}P^2)/\mathbb{R}^\times \) such that \( \Lambda_0 \) is a fiber of the projection \( \pi: PT^*(\mathbb{R}P^2) \to \mathbb{R}P^2 \), then the front \( \pi(\Lambda_s) \) has at least three cusps. This statement is given by Chekanov and Pushkar in [11], where the authors prove a local version, replacing \( \mathbb{R}P^2 \) by the plane \( \mathbb{R}^2 \), and also another similar conjecture “the four cusps conjecture”. Here we prove the conjecture when the base is the sphere \( S^2 \) (which implies the case of \( \mathbb{R}P^2 \) since we can lift a deformation in \( PT^*(\mathbb{R}P^2) \) to a deformation in \( PT^*S^2 = \tilde{T}^*S^2/\mathbb{R}^\times \)).

Let us give the precise statement. Let \( \{ \overline{\Lambda}_s \}_{s \in [0,1]} \) be a path of Legendrians in \( PT^*S^2 \) starting at \( \overline{\Lambda}_0 = \tilde{T}_{x_0}^*S^2/\mathbb{R}^\times \) for some point \( x_0 \in S^2 \). We let \( \Sigma_s \) be the projection of \( \overline{\Lambda}_s \) to \( S^2 \).

**Theorem 8.0.1.** Let \( s \in [0,1] \) be such that \( \Sigma_s \) is a curve with only cusps and double points as singularities. Then \( \Sigma_s \) has at least three cusps.

Here is a sketch of the proof. A first remark is that, for a connected curve \( \Sigma \) in \( S^2 \) with only cusps and double points as singularities and smooth part \( \Sigma_{reg} \), the closure of \( \tilde{T}_{\Sigma_{reg}}^*S^2 \) in \( \tilde{T}^*S^2 \) is connected if and only if the number of cusps is odd. Hence we only have to show that \( \Sigma_s \) does not have only one cusp. For any path of Legendrians \( \{ \overline{\Lambda}_s \}_{s \in [0,1]} \) there exists a contact isotopy \( \{ \overline{\Phi}_s \}_{s \in [0,1]} \) of the ambient contact manifold such that \( \overline{\Lambda}_s = \overline{\Phi}_s(\overline{\Lambda}_0) \) (see for example Theorem 2.6.2 of [18]). We will apply Theorem 2.7.1 to this contact isotopy. To fit with the framework of this theorem we lift the isotopy \( \overline{\Phi}: PT^*S^2 \times [0,1] \to PT^*S^2 \) to a homogeneous Hamiltonian isotopy \( \Phi: \tilde{T}^*S^2 \times [0,1] \to \tilde{T}^*S^2 \) which satisfies \( \Phi_0 = id \) and \( \Phi_s(x; \lambda \xi) = \lambda \cdot \Phi_s(x; \xi) \) for all \( \lambda \in \mathbb{R}^\times \) (not only for \( \lambda \in \mathbb{R}_{>0} \)) and all \( (x; \xi) \in \tilde{T}^*S^2 \). By Theorem 2.7.1 for each \( s \in [0,1] \) we obtain an auto-equivalence, say \( R_{\Phi,s} = K_{\Phi,s} \circ -- \), of the category \( D(k_{S^2}) \), with the property that \( SS(R_{\Phi,s}(F)) = \Phi_s(SS(F)) \) for all \( F \in D(k_{S^2}) \). We set \( F_0 = k_{(x_0)} \) and \( F_s = R_{\Phi,s}(F_0) \). Hence \( SS(F_0) = T_{x_0}S^2 \) and it follows that \( SS(F_s) = \Phi_s(T_{x_0}S^2) \).
The fact that \( \Phi \) is homogeneous for the action of \( \mathbb{R}^X \), and not only \( \mathbb{R}_{>0} \), implies that \( R\Phi_s \) commutes with the duality functor \( D_{S^2} \). The sheaf \( F_0 \) is self-dual and simple. Hence so is \( F_s \). Now we prove the following result (see Theorem 8.6.1 and the beginning of the proof of Theorem 8.7.3): if \( F \in D(k_{S^2}) \) satisfies

(a) \( F \) is self-dual,
(b) \( F \) is simple,
(c) \( R\Gamma(S^2;F) \cong k \),
(d) \( \Lambda = SS(F)/\mathbb{R}_{>0} \) is a smooth curve whose projection to \( S^2 \) is a generic curve \( \Sigma \) with only one cusp,

then \( \text{Hom}(F,F[1]) \neq 0 \). Since \( \text{Hom}(F_0,F_0[1]) = 0 \) there cannot exist an auto-equivalence \( R \) of \( D(k_{S^2}) \) such that \( R(F_0) = F \).

We describe the hypotheses of Theorem 8.6.1 and see at the same time how (a-d) imply them. We choose a Morse function \( q: S^2 \to \mathbb{R} \) with only two critical points and sufficiently generic with respect to \( \Sigma \). We decompose \( Rq_*(F) \) using Corollary 4.4.3. The hypothesis (c) implies that all intervals, but one, appearing in the corollary are half-closed. Then (a) implies that the non half-closed interval is reduced to a point. We thus have \( Rq_*(F) \cong k_{\{0\}} \oplus \bigoplus_{a \in A} k_{I_a}^{d_a} \) where the \( I_a \) are half-closed intervals, \( d_a \in \mathbb{Z} \) and \( t_0 \) is some point in \( \mathbb{R} \). We can assume \( t_0 = 0 \) and, choosing \( q \) sufficiently generic with respect to \( \Sigma \), we can assume that \( 0 \notin T_a \), for all \( a \). Then the decomposition of \( Rq_*(F) \), restricted to a neighborhood of 0, gives the hypothesis (8.2.8) of Theorem 8.6.1 (up to rescalling), that is, \( \text{Rq}_*(F)|_{-1,1} \cong k_{\{0\}} \oplus B_{-1,1} \) for some \( B \in D^b(k) \).

We write \( C_t = q^{-1}(t) \). The decomposition of \( Rq_*(F) \) and Proposition 1.2.4 imply that \( \Sigma \) is tangent to \( C_0 \). For \( q \) generic, we can choose a diffeomorphism \( q^{-1}([-1,1]) \cong S^1 \times [-1,1] \) such that \( \Sigma \cap q^{-1}([-1,1]) \) is the union of one branch, say \( \Gamma_0 \), tangent to \( C_0 \) and contained in \( q^{-1}([0,1]) \), and other branches of the form \( \{\theta\} \times [-1,1] \) (see 8.2.5, 8.2.6, 8.2.7 and Fig. 8.2.1). We are now in the settings of Theorem 8.6.1 which says that, if \( \Sigma \) only has one cusp (hypothesis (d) above), then \( \text{Hom}(F,F[1]) \neq 0 \).

The proof of Theorem 8.6.1 distinguishes two cases and uses two criteria to ensure the non-vanishing of \( \text{Hom}(F,F[1]) \). We decompose \( F|_{C_{1/2}} \) using Corollary 1.4.3 as \( F|_{C_{1/2}} \cong L \oplus \bigoplus_{a \in A'} e_*(k_{I_a}^{d_a}) \), where \( L \) is locally constant, the \( I_a \) are intervals of \( \mathbb{R} \), \( d_a \in \mathbb{Z} \) and \( e: \mathbb{R} \to C_{1/2} \cong \mathbb{R}/2\pi \mathbb{Z} \) is the quotient map. The branch \( \Gamma_0 \) meets \( C_{1/2} \) in two points, say \( \theta_1, \theta_2 \), and some intervals \( I_a \) have one end in \( e^{-1}(\{\theta_1,\theta_2\}) \) (a priori there are four such intervals but some could coincide). Now the two cases depend on these intervals.
If one of the intervals with one end in $e^{-1}(\{\theta_1, \theta_2\})$ is closed or open, then the non vanishing of $\text{Hom}(F, F[1])$ is given by Propositions 8.2.3 and 8.2.5. This result is in fact local around $C_0$ and we do not use the fact that $\Sigma$ only has one cusp.

In the other case all intervals with one end in $e^{-1}(\{\theta_1, \theta_2\})$ are half-closed. Here we apply a more global criterion to ensure $\text{Hom}(F, F[1]) \neq 0$. We define notions of $F$-linked and $F$-conjugate points of $\Lambda$ in Section 8.3. Our criterion is roughly that, if there exist a pair $(p_0, p_1)$ of $F$-linked points and a pair $(q, q')$ of $F$-conjugate points such that $(p_0, p_1)$ and $(q, q')$ are intertwined on the circle $\Lambda/\mathbb{R}_{>0}$, then $\text{Hom}(F, F[1]) \neq 0$ (see Proposition 8.3.5). Examples of conjugate points are the points of $\Lambda$ corresponding to the ends of one interval $I_a$ in the decomposition of $F|_{C_{1/2}}$ recalled above. In Proposition 8.3.1 we see that in our situation we can situate some $F$-linked points. The hypothesis that $\Sigma$ has only one cusp is used as follows. The sheaf $F$ has a shift (or Maslov potential – see Example 1.4.5) at each point of $\Lambda$, which is a half integer and changes by 1 when we cross a cusp. If $\Sigma$ has only one cusp, then $\Lambda/\mathbb{R}_{>0}$ is decomposed in two arcs, say $\Lambda_+$ and $\Lambda_-$, according to the value of the shift. In the above decomposition of $F|_{C_{1/2}}$, the points above the ends of an interval $I_a'$ which is half-closed are not in the same component $\Lambda_\pm$. This gives several pairs of conjugate points $(q_+, q_-)$ with $q_\pm \in \Lambda_\pm$. Using the linked points of Proposition 8.3.1 it is then possible to find two intertwined pairs of linked/conjugate points and apply Proposition 8.3.5.

This part is organized as follows. In Section 8.1 we describe the sheaf associated with the standard curve with three cusps and give an example of a curve with one cusp whose conormal bundle is the microsupport of a sheaf. In Section 8.2 we give the first criterion (local around $C_0$) for the non vanishing of $\text{Hom}(F, F[1])$. In Section 8.3 we introduce the notions of linked and conjugate points and prove that the existence of two intertwined pairs of linked/conjugate points also implies the non vanishing of $\text{Hom}(F, F[1])$. Examples of conjugate points are given in the next section. In Section 8.5 we prove the existence of three linked points (under the same hypotheses as in Section 8.2). In the next two sections we apply these criteria to prove the three cusps conjecture. In the last section we give a sketch of proof of the four cusp conjecture with the same method.

In this part we assume that $k$ is an infinite field (we use it in Proposition 8.5.1 and we use Gabriel’s theorem).
8.1. Examples

A first idea to prove Theorem 8.0.1 could be the more ambitious statement: if \( \Lambda \subset \dot{T}^*S^2 \) is a smooth conic Lagrangian submanifold and \( \Sigma = \pi_{S^2}(\Lambda) \) is a curve with only one cusp and otherwise only double points as singularities, then there is no \( F \in D(k_{S^2}) \) such that \( SS(F) = \Lambda \). Indeed Theorem 2.1.1 would imply that there is no homogeneous isotopy \( \Phi \) of \( \dot{T}^*S^2 \) such that \( \Lambda = \Phi_1(\dot{T}^*x_0S^2) \). But this statement is false; we give a counterexample in this section. The examples given here are in fact in \( \mathbb{R}^2 \). We put coordinates \((x,y)\) on \( \mathbb{R}^2 \) and \((x,y;\xi,\eta)\) on \( T^*\mathbb{R}^2 \).

**Sheaf associated with a cusp.** Let \( \Sigma_{\text{cusp}} = \{(x,y); x^2 = y^3\} \) be the ordinary cusp in \( \mathbb{R}^2 \). Let \( \Lambda_{\text{cusp}} \) be the closure of \( \dot{T}^*\Sigma_{\text{cusp}} \setminus \{0\} \) in \( \dot{T}^*\mathbb{R}^2 \). Then \( \Lambda_{\text{cusp}} \) is a smooth conic Lagrangian submanifold of \( \dot{T}^*\mathbb{R}^2 \) consisting of two connected components, say \( \Lambda_1 = \{(t^3, t^2; -2u, 3tu); t \in \mathbb{R}, u \in \mathbb{R}_{\geq 0}\} \) and \( \Lambda_2 = \Lambda_1^a \). We set \( W_1 = \{(x,y); -y^{3/2} < x \leq y^{3/2}, y > 0\} \) and \( W_2 = \{(x,y); -y^{3/2} \leq x < y^{3/2}, y > 0\} \) (see Fig. 8.1.1). By Example 5.3.4 of [31] we know that \( SS(k_{W_i}) = \Lambda_i \), for \( i = 1, 2 \) (outside \( T_0^*\mathbb{R}^2 \) this follows from Example 1.2.3-(iii) and a direct computation gives the microsupport over 0).

![Figure 8.1.1](image.png)

**Lemma 8.1.1.** Let \( F \in D(k_{\mathbb{R}^2}) \) be such that \( SS(F) = \Lambda_{\text{cusp}} \) and \( F \) is simple along \( \Lambda_{\text{cusp}} \). Then there exist \( E \in D(k) \) and \( d_1, d_2 \in \mathbb{Z} \) such that \( F \simeq k_{W_1}[d_1] \oplus k_{W_2}[d_2] \).

With the notations of Lemma 8.1.1 if \( \text{supp}(F) \subset W_1 \) and \( L \) is a horizontal line cutting \( \Sigma_{\text{cusp}} \) in two points \( z, z' \), then \( F|_L \simeq k_{z,z'}[d_1] \oplus k_{z,z'}[d_2] \).

**Proof.** This follows from the description of sheaves on the plane in [46] (see for example (3.3) in the proof of Thm. 3.12 or Prop. 5.8) but we give a sketch of proof for the reader convenience.

(i) We first consider the case where \( SS(F) = \Lambda_1 \) and \( \text{supp}(F) \) is contained in \( W_1 \). The set \( U = W_2 \cup \{x < 0\} \) is open with a boundary of class \( C^1 \). We set \( Z = \mathbb{R}^2 \setminus U \). The microsupport condition
gives \((\mathsf{R} \mathcal{Z}(F))_p \simeq 0\) for all \(p \in \partial U\), and then for all \(p \in Z\) because \(\text{supp}(F) \subset \overline{U}\). Hence \(\mathsf{R} \mathcal{Z}(F) \simeq 0\) and we obtain, by excision, \(F \simeq \mathsf{R} \mathcal{Z}(U(F)) \simeq R_j \mathcal{Z}(F|_U)\), where \(j\) is the inclusion of \(U\) in \(\mathbb{R}^2\). Using Example 1.2.10 and the fact that \(F\) is simple, we see that \(F|_U \simeq k_{U,W}[d_1]\) for some \(d_1 \in \mathbb{Z}\). We deduce \(F \simeq k_{\mathbb{R}^2,W_1}[d_1]\).

(ii) Now we assume \(\mathsf{SS}(F) = \Lambda_1\) but we assume nothing on \(\text{supp}(F)\). We set \(q = (0,-1)\) and let \(B_q\) be the open disc with center \(q\) and radius \(r\). By Corollary 1.2.16 we have isomorphisms \(\mathsf{R} \mathcal{Z}(B_q; F) \simeq \mathsf{R} \mathcal{Z}(\mathbb{R}^2; F)\) for all \(r \geq s > 0\). We deduce \(\mathsf{R} \mathcal{Z}(\mathbb{R}^2; F) \simeq F_q\). We set \(E' = \mathsf{R} \mathcal{Z}(\mathbb{R}^2; F)\). We have a natural morphism \(u : E'_{\mathbb{R}^2} \rightarrow F\) and \(u_q\) is an isomorphism. Hence the cone of \(u\), say \(F'\), satisfies \(\mathsf{SS}(F') = \Lambda_1\) and \(F'_q \simeq 0\). By (i) we have \(F' \simeq k_{W_1}[d_1]\) for some \(d_1 \in \mathbb{Z}\). We can check that \(\text{Hom}(F', E'_{\mathbb{R}^2}[1]) \simeq 0\) and it follows that \(F \simeq F' \oplus E'_{\mathbb{R}^2}\).

(iii) Now we assume \(\mathsf{SS}(F) = \Lambda_{\text{cusp}}\). By Proposition 3.3.2 there exists a neighborhood \(V\) of \(0\) and a distinguished triangle in \(V\), \(F_1 \oplus F_2 \rightarrow F|_V \rightarrow E''_{\mathbb{R}^2} \rightarrow \cdots\), where \(\mathsf{SS}(F_i) = \Lambda_1\) and \(E''_{\mathbb{R}^2}\) is a constant sheaf. We can find an isotopy from \(\mathbb{R}^2\) to a small neighborhood of \(0\) which preserves \(\Lambda_{\text{cusp}}\). Hence Proposition 1.2.9 implies that the distinguished triangle can be extended to \(\mathbb{R}^2\). By (ii) we know \(F_1\) and \(F_2\) and, using \(\text{Hom}(k_{\mathbb{R}^2}, F_1[1]) \simeq 0\), we deduce the result.

**Sheaf associated with a deltoid.** We deform \(\hat{T}_0 \mathbb{R}^2\) by the Hamiltonian flow, say \(\Phi\), of the function \(h(x,y; \xi,\eta) = \eta^2/(\xi^2 + \eta^2)\) (which is one of the simplest example of function homogeneous of degree 1). We set \(\Lambda_{\text{delt}} = \Phi_1(\hat{T}_0 \mathbb{R}^2)\) and \(\Sigma_{\text{delt}} = \pi_{\mathbb{R}^2}(\Lambda_{\text{delt}})\). Then \(\Sigma_{\text{delt}}\) is a curve which bounds a star shaped domain, say \(D\), and which has three cusps, all pointing to the outward direction (\(\Sigma_{\text{delt}}\) is a kind of deltoid):

Let \(K_{\Phi}\) be the sheaf associated with \(\Phi\) by Theorem 2.1.1. The composition with \(K_{\Phi,1}\) gives an equivalence between simple sheaves with microsupport \(\hat{T}_0 \mathbb{R}^2\) and simple sheaves with microsupport \(\Lambda_{\text{delt}}\). We set \(F = K_{\Phi,1} \circ k_{(0)}\). Then \(\mathsf{SS}(F) = \Phi_1(\mathsf{SS}(k_{(0)})) = \Lambda_{\text{delt}}\) and \(F\) is simple along \(\Lambda_{\text{delt}}\). Since \(k_{(0)}\) has compact support, the same holds for \(F\). Since \(\mathsf{SS}(F) = \Lambda_{\text{delt}}\), \(F\) is locally constant outside \(\Sigma_{\text{delt}}\). Hence \(F\) must be zero outside \(D\). By Lemma 8.1.1 we have \(\text{supp}(F) = \overline{D}\) and, if \(L\) is a line cutting \(\Sigma_{\text{delt}}\) transversely in two points \(z, z'\), then \(F|_L \simeq k_{x,z'}[d_1] \oplus k_{x,z}[d_2]\) for some integers \(d_1, d_2\).
**Lemma 8.1.2.** Let $F = K_{i,1} \circ k_{\{0\}} \in D(\mathbb{K}_{\mathbb{R}^2})$ be the simple sheaf along $\Lambda_{delt}$ corresponding to $k_{\{0\}}$. Then $F$ is concentrated in degree $-1$ and its restriction to a line cutting transversely $\Sigma_{delt}$ at $z, z'$ is $(k_{[z, z']^\ast} \oplus k_{[z, z']})[1]$.

We summarize these results in the following picture (where the deltoid is slightly deformed)

![Diagram](image)

(F[−1])|_{L_i} \simeq k_{\ldots} \oplus k_{\ldots}

**Figure 8.1.2.**

**Proof.** We have already seen that the restriction of $F$ to a transverse line is of the form $k_{[z, z'][d_1]} \oplus k_{[z, z'][d_2]}$ for some integers $d_1, d_2$.

Let us first prove that $d_1 = d_2$. Let $E_1, E_2, E_3$ be the edges of the domain $D$. We set $U_i = \mathbb{R}^2 \setminus (E_{i-1} \cup E_{i+1})$ (with $E_4 = E_1$). Then $F|_{U_i}$ is of the form $F|_{U_i}^{d_1,d_2} := k_{U_i \cap D}[d_1] \oplus k_{U_i \cap D}[d_2]$ or $F|_{U_i}^{d_2,d_1}$. Moreover, if $F|_{U_i} \simeq F|_{U_i}^{d_1,d_2}$, then $F|_{U_{i+1}} \simeq F|_{U_{i+1}}^{d_2,d_1}$. Turning once around $D$ we obtain $F|_{U_i}^{d_1,d_2} \simeq F|_{U_{i+1}}^{d_2,d_1}$, which gives $d_1 = d_2$.

Now we prove that $d_1 = 1$. Let $q: \mathbb{R}^2 \to \mathbb{R}$ be the projection $q(x, y) = y$. By Corollary 2.1.5 we know that $R\Gamma(\mathbb{R}^2; F) \simeq R\Gamma(\mathbb{R}^2; k_{\{0\}}) \simeq k$. We compute $R\Gamma(\mathbb{R}^2; F)$ using $R\Gamma(\mathbb{R}^2; F) \simeq R\Gamma(\mathbb{R}; Rq_\ast(F))$ and we will deduce $d_1$.

The line $q^{-1}(y)$ is transverse to $\Sigma_{delt}$ except for one value $y = y_0$. For $y \neq y_0$, $F|_{q^{-1}(y)}$ is a sheaf on $\mathbb{R}$ which is the sum of two or four sheaves of the type $k_I[d_1]$, where $I$ is a half closed interval. Hence $(Rq_\ast(F))_y \simeq R\Gamma(q^{-1}(y); F|_{q^{-1}(y)}) \simeq 0$ for $y \neq y_0$. It follows that $R\Gamma(\mathbb{R}; Rq_\ast(F)) \simeq (Rq_\ast(F))_{y_0}$. Let $z, z'$ be the transverse intersections of $\Sigma_{delt}$ and $q^{-1}(y_0)$ and let $z''$ be their tangent intersection. Then $F|_{q^{-1}(y_0)}$ is of the type

\[
(k_{[z,-]} \oplus k_{[z,-]}')[d_1] \text{ near } z, \quad (k_{[z', -]} \oplus k_{[z', -']}')[d_1] \text{ near } z',
\]

\[
(k_{[z', -]} \oplus k_{[z', -]} )[d_1] \text{ near } z''.
\]

By Corollary 4.1.3 we have $F|_{q^{-1}(y_0)} \simeq (k_{I_1} \oplus k_{I_2} \oplus k_{I_3})[d_1]$, where the intervals $I_1, I_2, I_3$ have two closed ends and four open ends. This leaves two possibilities: (1) two of these intervals are half-closed and one is open, or (2) two are open and one is closed. In the first case we find $(Rq_\ast(F))_{y_0} \simeq k[d_1 - 1]$ and in the second case $(Rq_\ast(F))_{y_0} \simeq k[d_1 - 1]$.
Since the result is $k$, this excludes the second case and we obtain $d_1 = 1$. □

Sheaf associated with a curve with one cusp. We define a new curve $\Sigma$ from $\Sigma_{\text{delt}}$ as follows. We cut $\Sigma_{\text{delt}}$ by the rectangle $B$ of Fig. 8.1.2; the bottom edge $\partial^- B$ of $B$ cuts $\Sigma_{\text{delt}}$ in one pair of points near the bottom cusp and the top edge $\partial^+ B$ of $B$ cuts $\Sigma_{\text{delt}}$ in two pairs of points near the top cusps. We put two small copies of $B$, say $B_2$ and $B_3$, above another copy $B_1$, so that the corresponding copies of $\Sigma_{\text{delt}} \cap B$ glue together, as in Fig. 8.1.3. We attach a cusp to the pair of points of $\Sigma_{\text{delt}} \cap \partial^- B_1$ and we attach four arcs of circles, say $C_1, \ldots, C_4$, to the four pairs of points of $\Sigma_{\text{delt}} \cap (\partial^+ B_2 \cup \partial^+ B_3)$ so that the resulting curve, say $\Sigma$, is connected. The closure of the conormal of the regular part of $\Sigma$ is a smooth conic Lagrangian submanifold of $\mathcal{T}^* \mathbb{R}^2$, say $\Lambda$.

Let $F$ be the sheaf described in Lemma 8.1.2. Let $F_i$ be a copy of $F|_B$ in the rectangle $B_i$. By the description of $F$, the sheaves $F_1, F_2, F_3$ glue into a sheaf $G$, on $B_1 \cup B_2 \cup B_3$. We can then glue $G$ with a sheaf associated with the cusp to obtain a sheaf $G'$. We let $D \simeq C_1 \times [0,1]$ be the domain bounded by $C_1, C_2$ and we set $D_1 = D \cup C_1, D_2 = D \cup C_2$. We define similarly the strips $D', D'_3, D'_4$ associated with the two other arcs $C_3, C_4$. Now we glue $G'$ with the sheaf $(k_{D_1} \oplus k_{D_2} \oplus k_{D'_3} \oplus k_{D'_4})[1]$ and we obtain a sheaf $F'$ such that $\mathrm{SS}(F') = \Lambda$.

8.2. Simple sheaf at a generic tangent point

In this section we consider a sheaf $F$ on a surface $M$ whose micro-support $\mathrm{SS}(F) = \Lambda \subset \mathcal{T}^* M$ is a smooth Lagrangian submanifold. We
give a criterion for the non vanishing of \( \text{Hom}(F, F[1]) \) (see Propositions 8.2.3 and 8.2.5). The geometric situation and the hypotheses on \( F \) are described in Section 8.2.2. Our criterion is in fact local around an embedded circle of \( M \). We begin with a general statement to go from a local non vanishing to a global one (Lemma 8.2.1).

8.2.1. **Local cohomology.** We make general remarks about the (local) extensions of sheaves. Let \( M \) be a manifold and \( Z \) a closed subset of \( M \). Let \( F, G \in \mathcal{D}(k_M) \) be given. We will often use the isomorphisms (see Proposition 1.1.1-(a-c))

\[
\text{Hom}(F, G)[i] \simeq H^i \text{RHom}(F, G) \simeq H^i(M; \text{RHom}(F, G)),
\]

where \( \text{RHom} \) denotes the derived Hom functor, with values in \( \mathcal{D}(k) \), and \( \text{Hom} \) denotes the Hom functor in \( \mathcal{D}(k_M) \), with values in \( \text{Mod}(k) \). We also have the following isomorphisms, deduced from (1.1.5) and the adjunction \((\otimes, \text{RHom})\)

\[
\text{R}\Gamma_Z \text{RHom}(F, G) \simeq \text{RHom}(F, \text{R}\Gamma_Z(G)) \simeq \text{RHom}(F_Z, G).
\]

Since \( \text{R}\Gamma_M(F) \simeq F \simeq k_M \), the morphism \( k_M \to k_Z \) induces the natural morphisms (8.2.3) and then (8.2.4) by composition

\[
\begin{align*}
i_Z(F) &: \text{R}\Gamma_Z(F) \to F; \\
j_Z(F) &: F \to F_Z,
\end{align*}
\]

\[
\begin{align*}
H^i \text{RHom}(F_Z, \text{R}\Gamma_Z(G)) &\to H^i \text{RHom}(F, G).
\end{align*}
\]

The next result gives conditions about some morphisms of sheaves supported on \( Z \) to ensure the non triviality of \( H^i \text{RHom}(F, G) \).

**Lemma 8.2.1.** Let \( i \in \mathbb{Z} \) and \( u: F_Z \to \text{R}\Gamma_Z(G)[i] \) be given. We assume that \( j_Z(G) \circ i_Z(G) \circ (u[-i]) \neq 0 \). Then the image of \( u \) in \( H^i \text{RHom}(F, G) \) by (8.2.4) is non zero.

**Proof.** Composing with \( j_Z(F), i_Z(G) \) and \( j_Z(G) \) gives the commutative diagram

\[
\begin{array}{ccc}
\text{RHom}(F_Z, \text{R}\Gamma_Z(G)) & \xrightarrow{i_Z(G) \circ o j_Z(F)} & \text{RHom}(F, G) \\
\downarrow j_Z(G) \circ o i_Z(G) & & \downarrow j_Z(G) \circ o - \\
\text{RHom}(F_Z, G_Z) & \xrightarrow{o j_Z(F)} & \text{RHom}(F, G_Z),
\end{array}
\]

where the top arrow is (8.2.4) and the left arrow is composing with \( j_Z(G) \circ i_Z(G) \). Since \( G_Z \) is supported in \( Z \), we have \( \text{R}\Gamma_Z(G_Z) \simeq G_Z \) and the isomorphisms (8.2.2) show that the bottom arrow is an isomorphism. Taking \( H^i(-) \) gives the result. \( \square \)
8.2.2. Generic tangent point - notations and hypotheses. We consider a surface \( M \), a smooth closed conic Lagrangian submanifold \( \Lambda \subset T^*M \) and \( F \in D^b(k_M) \) such that \( SS(F) = \Lambda \). We introduce some hypotheses on \( \Lambda \cap T^*U \) and \( F|_U \), where \( U \) is an open subset of \( M \) diffeomorphic to a cylinder.

Since these hypotheses are not completely obvious, we give a quick justification why we introduce them (see also the introduction and Lemma [8.7.2]). Let us assume for a while (as we will do when we apply the results of this section) that \( \Sigma = \dot{\pi}_M(\Lambda) \) is an immersed curve, with only cusps and transverse double points as singularities. We choose a Morse function \( q: M \to \mathbb{R} \) generic enough so that the fibers \( q^{-1}(t) \) are tangent to \( \Sigma \) only for finitely many values \( t_1, \ldots, t_N \) and that \( q^{-1}(t_i) \) only contains one tangent point and no cusp nor double points. By Corollary [1.4.3] there exists a decomposition of \( R_q(F) \) as a finite sum \( R_q(F) \simeq \bigoplus_{a \in A} k^{n_a}_a[d_a] \), where the \( I_a \)'s are intervals.

By Proposition [1.2.4] the ends of these intervals can only be the points \( t_i \) and the critical values of \( q \). In our application the sheaf \( F \in D^b(k_M) \) will satisfy \( D_M(F) \simeq F \) (for this, it is necessary that \( \Lambda^a = \Lambda \), by Theorem [1.2.13]) and \( R\Gamma(M;F) \simeq k \). By Lemma [8.7.2] this implies that all intervals \( I_a \) are half-closed except one which is reduced to a point. We can check that this point cannot be a critical value, hence it is \( t_{i_0} \) for some \( i_0 \). Thus there is one special tangent fiber, with respect to \( F \), and in this section we try to understand \( F \) around this fiber. Up to adding \( -t_{i_0} \) to \( q \) we assume \( t_{i_0} = 0 \). Then \( q^{-1}(0) \) is a union of circles, one of which is tangent to \( \Sigma \). Restricting to a neighborhood \( U \) of this circle, we obtain the situation described in (8.2.5) and (8.2.6) below, with \( \Gamma = \Sigma \cap U \), and \( F|_U \) satisfies (8.2.7) and (8.2.8) (we restrict the isomorphism \( R_q(F) \simeq \bigoplus_{a \in A} k^{n_a}_a[d_a] \) to a neighborhood \( J \) of 0 which does not contain the ends of the \( I_a \)'s, for \( I_a \neq \{0\} \), and rescale to have \( J = ]-1,1[ \)). In Fig. [8.2.1] we give a typical curve \( \dot{\pi}_M(\Lambda) \) in \( M \) and more precisely the situation in the open subset \( U \).

We set \( J = ]-1,1[ \) and choose a diffeomorphism \( U \simeq S^1 \times J \). We set

\[
q: U \to J
\]

be the projection. We take the coordinates \( \theta \in ]-\pi,\pi] \) on \( S^1 \), \( t \) on \( J \) and \( (\theta;\xi) \) on \( T^*S^1 \), \( (t;\tau) \) on \( T^*J \). We set

\[
C_t = q^{-1}(t), \quad t \in J, \quad U_0 = ]-1,1[ \times J, \quad \Omega = \{ (\theta,t) \in U_0; \; t > \theta^2 \}, \quad p^\pm = (0,0;0,\pm 1) \in \Lambda_0.
\]

\[
\Gamma_0 = \{(\theta,t) \in U_0; \; t = \theta^2 \}, \quad \Omega = \{ (\theta,t) \in U_0; \; t > \theta^2 \}, \quad p^\pm = (0,0;0,\pm 1) \in \Lambda_0.
\]
We assume that $\Lambda$ satisfies

\begin{equation}
\begin{cases}
- \Lambda^a = \Lambda \text{ (that is, $\Lambda$ is stable by the antipodal map $(x; \xi) \mapsto (x; -\xi)$)}, \\
- \Lambda \cap T^*U = T^*\Gamma U, \text{ where } \Gamma = (\{\theta_1, \ldots, \theta_N\} \times \mathbb{J}) \cup \Gamma_0 \\
\text{and } \theta_1, \ldots, \theta_N \in S^1 \setminus [-1, 1].
\end{cases}
\end{equation}

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure.png}
\caption{Figure 8.2.1.}
\end{figure}

In fact, if $\Lambda \subset \dot{T}^*M$ is any smooth closed conic Lagrangian submanifold such that $\Lambda^a = \Lambda$ and $\pi_M(\Lambda)$ is a smooth curve in a neighborhood of $C_0$ which has one tangent point with $C_0$ with a tangency of order 1, then we can find an isotopy $\{\varphi_t\}_{t \in [0, 1]}$ of $M$ such that $d\varphi_1(\Lambda)$ satisfies [8.2.6], up to maybe changing $q$ to $-q$.

We will often make the following hypotheses on the restriction of $F$ to $\mathcal{U}$

\begin{equation}
D_\mathcal{U}(F|_\mathcal{U}) \simeq F|_\mathcal{U},
\end{equation}

\begin{equation}
Rq_*(F|_\mathcal{U}) \simeq k_{\{0\}} \oplus B_J, \quad \text{for some } B \in D^b(k),
\end{equation}

where $B_J$ denotes the constant sheaf on $\mathbb{J}$ with stalk $B$ (that is, $B_J = a_{-1}(B)$ where $a_J: \mathbb{J} \to \{\text{pt}\}$ is the map to a point).

8.2.3. **Local study around $C_0$.** We prove that $H^1R\text{Hom}(F, F)$ is non zero for a sheaf $F$ satisfying (8.2.7), (8.2.8) and some additional hypothesis (see Proposition [8.2.5]). For this we use Lemma 8.2.1 with
$G = F$ and $Z = C_0$. To construct the morphism $u$ of the lemma we use the decomposition of $F_{C_0}$ and $R\Gamma_{C_0}(F)$ (which are sheaves on the circle) given by Corollary 4.4.3 and, more specifically, the fact that a non zero locally constant sheaf with unipotent monodromy appears in the decomposition of $F_{C_0}$. This fact is proved in Proposition 8.2.3. We begin with a lemma which describes the restriction of $F$ to the open set $U_0$ defined in (8.2.5). For this lemma and for Proposition 8.2.3 we consider a sheaf satisfying (8.2.3) but maybe not (8.2.7) (the argument of the proof constructs an intermediate sheaf which inherits (8.2.8) but a priori not (8.2.7)).

In this section we use the notations (8.2.5) and (8.2.6), in particular $U, U_0, \Gamma, \Omega, \Lambda, \Lambda_0$.

Lemma 8.2.2. Let $\Lambda \subset \hat{T}^*M$ be a smooth closed conic Lagrangian submanifold satisfying (8.2.6). Let $F \in D^b_{\Lambda \cap \hat{T}^*U}(k_U)$ be given. Then $F$ is decomposed as $F \simeq \bigoplus_{i \in \mathbb{Z}} (H^i F)[-i]$ and $\hat{SS}(H^i F) \subset \Lambda \cap T^*U$, for each $i \in \mathbb{Z}$. Moreover, if $F$ satisfies (8.2.8), then

1. $(H^{-1} F)[1]$ satisfies (8.2.8) (for some other $B \in D(k)$),
2. $Rq_* (H^i F)$ is a constant sheaf on $\mathbb{J}$, for all $i \neq -1$,
3. $H^{-1} F|_{U_0} \simeq k_{t_0}^p \oplus k_{t_0}^q \oplus k_{t_0}^r$ with $p, q, r \in \mathbb{N}$, $q, r \geq 1$.

Proof. (i) Since $\Lambda \cap T^*U \subset T^*U$ where $\Gamma$ is a smooth curve, $F$ is weakly constructible for the stratification $U = \Gamma \cup (U \setminus \Gamma)$ by Proposition 1.2.20 and Remark 1.2.22. It follows that the same holds for all $H^i F$. By Corollary 1.2.21 for two such weakly constructible sheaves $G, G'$ the complex $H = R\mathcal{H}om(G, G')$ is also weakly constructible for the same stratification. If $G, G'$ are concentrated in degree 0, then $H_{U \cap \Gamma}$ is concentrated in degree 0 and $H_{\Gamma}$ in degrees 0 and 1. Moreover $H_{\Gamma}$ has stalks 0 outside $\Gamma$ and $H_{\Gamma}|_{\Gamma}$ is locally constant; similarly $H_{U \cap \Gamma}$ has stalks 0 outside $U \setminus \Gamma$ and $H_{(U \cap \Gamma)|_{\Gamma}}$ is locally constant. Since $\Gamma$ consists only of segments, we deduce $H^2(U; H_{\Gamma}) \simeq 0$. We can check that the same vanishing holds for $H^2(U; H_{U \cap \Gamma})$ and we deduce by excision that $\text{Ext}^2(G, G') \simeq 0$. Lemma 1.4.1 gives $F \simeq \bigoplus_{i \in \mathbb{Z}} (H^i F)[-i]$ and this implies the bound for $\hat{SS}(H^i (F))$.

(ii) Now we assume that $F$ satisfies (8.2.8). By (i) we have $Rq_* F \simeq \bigoplus_{i \in \mathbb{Z}} Rq_*(H^i F)[-i]$. Each $Rq_*(H^i F)$ is weakly constructible for the stratification $\mathbb{J} = \{0\} \cup (\mathbb{J} \setminus \{0\})$ and the same argument as in (i) shows that $Rq_*(H^i F)$ is decomposed by the cohomological degree. Hence there exists $i_0$ such that $Rq_*(H^{i_0} F)[-i_0]$ has $k_{\{0\}}$ as a direct summand and $Rq_*(H^i F)$ are constant sheaves on $\mathbb{J}$ for $i \neq i_0$. Let us check that $i_0 = -1$. We see $H^{i_0} F$ as an object of $D^b(k_U)$ concentrated in degree 0. Since the fibers of $q$ have dimension 1, the direct image $Rq_*(H^{i_0} F)$ is
concentrated in degrees 0 and 1 and so is its summand $k_{(0)}[i_0]$. Hence $i_0$
can only be 0 or $-1$. If $i_0 = 0$, then taking $H^0$ of $Rq_*(H^0 F)$ we find that
$k_{(0)}$ is a direct summand of $H^0 Rq_*(H^0 F)$ (in the category $\text{Mod}(k_2)$ of
non derived sheaves). We recall that $H^0 Rq_*(H^0 F) \simeq q_*(H^0 F)$ (the non
derived direct image). It follows that there exists $s \in \Gamma(\mathcal{J}; q_*(H^0 F))$
with $\text{supp}(s) = \{0\}$. Since $\Gamma(\mathcal{J}; q_*(H^0 F)) \simeq \Gamma(\mathcal{U}; H^0 F)$
this section $s$ gives a section $s'$ of $H^0 F$ with $\text{supp}(s') \subset q^{-1}(0)$. But we have seen that
$H^0 F$ is constructible for the stratification $\mathcal{U} = \Gamma \cup (\mathcal{U} \setminus \Gamma)$. The support
of $s'$ should be a union of strata and there is no stratum contained in $q^{-1}(0)$. This excludes the case $i_0 = 0$ and we have $i_0 = -1$.

(iii) We recall that $\mathcal{U}_0$ is an open subset of $\mathcal{U}$ which contains the curve
$\Gamma_0$ and no other component of $\Gamma$ (see (8.2.5)). We can find a submersion
$f : \mathcal{U}_0 \to \mathbb{R}$ whose fibers are intervals and such that $\Omega = f^{-1}(]0, +\infty[)$. Since $\mathcal{S}\mathcal{S}(H^{-1}(F)) \subset \Lambda$, Proposition 1.2.9 implies that $H^{-1}(F)|_{\mathcal{U}_0} \simeq f^{-1}G$ for some sheaf $G$ on $\mathbb{R}$. Then $G$ must be concentrated in degree 0 and we have $\mathcal{S}\mathcal{S}(G) \subset \mathcal{T}_0^* \mathbb{R}$. By Corollary 4.4.3 $G$ is decomposed as
a sum of the sheaves $k_\mathbb{R}$, $k_{[0, +\infty[}$, $k_{]0, +\infty[}$, $k_{(0)}$, $k_{]0, -\infty[}$ and $k_{]0, -\infty[}$
with some multiplicity. Hence $H^{-1}(F)|_{\mathcal{U}_0}$ is decomposed as a sum of the sheaves
$k_{\mathcal{U}_0}$, $k_{\mathcal{U}_0}$, $k_{\mathcal{U}_0}$, $k_{\mathcal{U}_0 \setminus \mathcal{U}}$ and $k_{\mathcal{U}_0 \setminus \mathcal{U}}$. The sheaves $k_{\mathcal{U}_0}$, $k_{\mathcal{U}_0}$, $k_{\mathcal{U}_0}$ have
a support which is closed in $\mathcal{U}$ (not only in $\mathcal{U}_0$). It follows that, if one of them,
say $F'$, is a summand of $H^{-1}(F)|_{\mathcal{U}_0}$, then it is also a summand of $H^{-1}(F)$ and $Rq_*(F')$
is a summand of $Rq_*(H^{-1}(F))$. Since $H^{-1}(F)[1]$ satisfies (8.2.8), the summands of $Rq_*(H^{-1}(F))$
can only have $\{0\}$ or $\mathbb{J}$ as possible support. On the other hand $Rq_* k_{\mathcal{U}_0}$, $Rq_* k_{\mathcal{U}_0}$, $Rq_* k_{\mathcal{U}_0}$
all have support $[0, 1]$. Hence $k_{\mathcal{U}_0}$, $k_{\mathcal{U}_0}$, $k_{\mathcal{U}_0}$ cannot appear in the decomposition
of $H^{-1}(F)|_{\mathcal{U}_0}$.

The only possible summands of $H^{-1}(F)|_{\mathcal{U}_0}$ are then $k_{\mathcal{U}_0}$, $k_{\mathcal{U}_0}$, $k_{\mathcal{U}_0}$
and $k_{\mathcal{U}_0 \setminus \mathcal{U}}$. If the last two do not appear both, then $\mathcal{S}\mathcal{S}(H^{-1}(F))$
does not meet $\Lambda_0$ or only contains one of the two components of $\Lambda_0$. By
Proposition 1.2.3 we obtain that $\mathcal{S}\mathcal{S}(Rq_*(H^{-1}(F)))$ does not meet $\mathcal{T}_0^* \mathbb{R}$
or only contains one half of $\mathcal{T}_0^* \mathbb{R}$, which contradicts the fact that $k_{(0)}[-1]$
is a summand of $Rq_*(H^{-1}(F))$. This concludes the proof. \qed

Let $F \in \text{D}^b(k_2)$ be given such that $\mathcal{S}\mathcal{S}(F) = \Lambda \cap \mathcal{T}^* \mathcal{U}$ and $F$ is simple.
We assume that $F$ is constructible (for the stratification $\mathcal{U} = \Gamma \cup (\mathcal{U} \setminus \Gamma)$
– see Section 1.2.3). Then $F|_{C_{1/2}}$ is constructible and we can decompose
$F|_{C_{1/2}}$ according to Corollary 4.4.3 there exist $L \in \text{D}^b(k_{C_{1/2}})$ and
$\{(I_a, d_a)\}_{a \in A}$, where $L$ has locally constant cohomology sheaves of finite
rank and $A$ is a finite family of bounded intervals and integers, such
that

\[(8.2.9)\quad F|_{C_{1/2}} \simeq L \oplus \bigoplus_{a \in A} e_*(k_{I_a})[d_a],\]

where \(e: \mathbb{R} \to C_{1/2} \simeq \mathbb{R}/2\pi\mathbb{Z}\) is the quotient map. Since \(F\) is simple, so is \(F|_{C_{1/2}}\) by Corollary 7.5.13 of [31] (see also Lemma 8.4.2 below). In particular, the intervals \(I_a\) appear with multiplicity 1 and, for any two distinct intervals \(I_a, I_b\), we have \(\mathcal{S}\mathcal{S}(e_*(k_{I_a})) \cap \mathcal{S}\mathcal{S}(e_*(k_{I_b})) = \emptyset\). This implies:

\[(8.2.10)\]

if \(x\) is an end of \(I_a\), \(y\) an end of \(I_b\) and there exists \(\pi > \varepsilon > 0\) such that \(e(I_a \cap ]x - \varepsilon, x + \varepsilon[) = e(I_b \cap ]y - \varepsilon, y + \varepsilon[)\), then \(I_a = I_b\).

In the next proposition we consider a sheaf \(F \in \mathcal{D}^*_b(k_U)\) satisfying \((8.2.8)\). By Lemma 8.2.2 we know that \(F\) is decomposed according to the cohomological degree and that \(H^{-1}F|_{U_0}\) has \(k_{U_0}\setminus \Omega\) and \(k_{U_0}\setminus \overline{\Omega}\) as direct summands. We remark that \(C_{1/2} \cap (U_0 \setminus \Omega)\) is the union of the two intervals \([-1, -1/\sqrt{2}]\) and \([1/\sqrt{2}, 1]\) of length \(l = 1 - 1/\sqrt{2}\). By \((8.2.10)\), in the decomposition \((8.2.9)\) of \(F|_{C_{1/2}}\), there are two intervals (uniquely defined but may be equal) in the family \(\{I_a\}_{a \in A}\), say \(I_b, I_c\), with right end \(x\) and \(I_c\), with left end \(y\), such that

\[(8.2.11)\]

\[e(I_b \cap ]x - l, +\infty[) = [-1, -1/\sqrt{2}],\]
\[e(I_c \cap ]-\infty, y + l[) = [1/\sqrt{2}, 1[.\]

We recall that a locally constant sheaf \(G\) on the circle is described up to isomorphism by its stalk \(G_\theta\) at a given point and its monodromy \(m: G_\theta \to G_\theta\). We then have \(H^0(C_0; G) \simeq \{v \in G_\theta; m(v) = v\}\). Hence a locally constant sheaf has a section if and only if its monodromy has a unipotent factor.

**Proposition 8.2.3.** Let \(\Lambda \subset \hat{T}^*M\) be a smooth closed conic Lagrangian submanifold satisfying \((8.2.6)\). We use the notations \((8.2.5)\). Let \(F \in \mathcal{D}^b_\Lambda(k_U)\) be a simple sheaf satisfying \((8.2.8)\). We assume that \(F\) is constructible (see Section 1.2.3). We consider the intervals \(I_b, I_c\) appearing in the decomposition \((8.2.9)\) of \(F|_{C_{1/2}}\) which are defined in \((8.2.10)\). We assume that \(I_b\) or \(I_c\) is closed. Then there exists a decomposition \(F|_{C_0} \simeq F' \oplus L[1]\) such that \(L \in \text{Mod}(k_{C_0})\) is a locally constant sheaf with unipotent monodromy and \(L \neq 0\).

**Proof.** (i) We assume that \(I_b\) is closed, the other case being similar. We argue by contradiction and assume that there exists \(F\) satisfying the hypotheses of the proposition but not the conclusion. We choose
$F$ (among those $F$ contradicting the proposition) so that the integer \([l(I_b)/2\pi]\) is minimal, where \(l(I_b)\) is the length of \(I_b\). We write \(I_b = [\alpha, \beta]\). We set \(C = C_{1/2}\) for short. By Lemma 8.2.2 we know that $F$ is decomposed according to the cohomological degree, that $(H^{-1}F)[1]$ satisfies (8.2.8) and that $H^{-1}F|_{U_0}$ has $k_{t_0} \cap \Omega$ and $k_{t_0} \setminus \Omega$ as direct summands. Since $F$ is simple, only this degree $-1$ term involves intervals with one end in $e^{-1}(\Gamma_0 \cap C)$. We can then assume from the beginning that $F$ is concentrated in degree $-1$ and we write $F = F_0[1]$, where $F_0 \in \text{Mod}(k_{t_0})$ is seen as usual as an object of $\mathcal{D}^b(k_{t_0})$ that is concentrated in degree $0$. The hypotheses say that we can write $F_0|_C \simeq e_* (k_{[\alpha, \beta]}) \oplus F_1$. We define $s = (1, 0) \in H^0(C; F_0|_C)$ according to this decomposition, where $1$ is the natural section of $e_*(k_{[\alpha, \beta]})$.

By the base change formula we have $H^0(C; F_0|_C) \simeq (q_*(F_0))(1/2)$ (the stalk of the non derived direct image $q_*(F_0)$ at the point $1/2 \in \mathbb{J}$). The hypothesis (8.2.8) implies that $q_*(F_0) \simeq H^0(Rq_*(F_0))$ is a constant sheaf on $\mathbb{J}$, hence the restriction morphism $H^0(\mathbb{J}; q_*(F_0)) \to (q_*(F_0))(1/2)$ is an isomorphism. Since $H^0(\mathbb{J}; q_*(F_0)) = H^0(\mathcal{U}; F_0)$, there exists $s' \in H^0(\mathcal{U}; F_0)$ such that $s'|_C = s$. We interpret the sections $s$ and $s'$ as morphisms

$$u: k_C \to F_0|_C, \quad u': k_{t_0} \to F_0.$$

(ii) If $[(\beta - \alpha)/2\pi] = 0$, then $I' = e([\alpha, \beta])$ is an arc of $C$ (smaller than $C$). Then $S = \text{supp}(s')$ is a closed subset of $\mathcal{U}$ which satisfies $S \cap C = I'$. By Lemma 8.2.2 and the fact that $F$ is simple we have $F_0|_{U_0} \simeq k_{t_0} + k_{t_0} \cap \Omega + k_{t_0} \setminus \Omega$. Hence $S \cap U_0$ can only be $\emptyset$, $U_0$ or $U_0 \setminus \Omega$. The first two cases are excluded because one end of the arc $I'$ is in $\partial\Omega$ and $I' \neq C$. Hence $S \cap U_0 = U_0 \setminus \Omega$. It follows that $I' = C \setminus (C \cap \Omega)$.

Outside $U_0$ the sheaf $F_0$ is constant on the vertical segments $\{\theta\} \times \mathbb{J}$, by Theorem 1.2.8 and the hypotheses (8.2.6) on $\Lambda$, and $S \setminus (S \cap U_0)$ must be a union of such segments. Hence we have $S = U \setminus \Omega$.

The morphism $u'$ factorizes through $k_{t_0} \to k_S$ and gives $v: k_S \to F_0$. By construction the restriction of $v$ to $U_0$, $v|_{U_0}: k_{t_0} \cap \Omega \to F_0|_{U_0}$, is the morphism induced by the above decomposition of $F_0|_{U_0}$. Let us define $G \in \mathcal{D}^b(k_{t_0})$ by the distinguished triangle

$$(8.2.12) \quad k_S \xrightarrow{u} F_0 \to G \xrightarrow{+1}.$$

Then $G|_{U_0} \simeq k_{t_0} + k_{t_0} \setminus \Omega$. The image of the triangle (8.2.12) by $Rq_*$ gives

$$(8.2.13) \quad Rq_*(k_S) \xrightarrow{w} k_{[0]}[-1] \oplus B_2[-1] \to Rq_*(G) \xrightarrow{+1},$$

with $B \in \mathcal{D}^b(k)$ as in (8.2.8) and $w = Rq_*(v)$. Let us write $w = (w_0/w_1)$. The morphism $w_0$ has target $k_{[0]}[-1]$ and thus is determined.
by $H^1(w_{|0})$. By the base change formula $H^1(w_{|0}) = H^1(C_0; v_{|C_0})$. We have $k_s|_{C_0} \simeq k_{C_0}$ and we have assumed that $F_0|_{C_0}$ does not have a direct summand which is locally constant with unipotent monodromy. This implies that $H^1(C_0; v_{|C_0})$ vanishes. Indeed, decomposing $F_0|_{C_0} \simeq L_0 \oplus \bigoplus_{a \in A'} e_*(k_{j_a}^{u_a})$ as in (4.3.3), we write $v|_{C_0} = v_0 + \sum a \in A' v_a$. Our hypothesis says that $L_0$ has no global section, hence $v_0 = 0$. If $v_a \neq 0$, then the corresponding interval $J_a$ is closed and $H^1(C_0; e_*(k_{j_a}^{u_a})) \simeq 0$, hence $H^1(C_0; v_a) = 0$. Finally $H^1(C_0; v_{|C_0}) = 0$, as claimed.

We thus have $w_0 = 0$ and we deduce from (8.2.13) that $k_{|0}[-1]$ is a direct summand of $Rq_*(G)$. On the other hand, the triangular inequality for the microsupport together with (8.2.12) and $k|_{\bar{U}} \simeq k|_{\bar{U}} \oplus k|_{\bar{U}}$ give the bound $\bar{S}(G) \subset (\Lambda \cap T^*U)|_{\Lambda_0}$, where $\Lambda_0$ is the connected component of $\Lambda$ which contains $p^-$. This implies $\bar{S}(Rq_*(G)) \subset \{(0; \tau); \tau > 0\}$ and contradicts the fact that $k_{|0}[-1]$ is a direct summand of $Rq_*(G)$.

(iii) Now we assume $|((\beta - \alpha)/2\pi)| > 0$. We define $G \in \mathcal{D}^b(k_{\mathcal{U}})$ by the distinguished triangle $k_{\mathcal{U}} \to F_0 \to G$. Applying $Rq_*$ gives the distinguished triangle

$$
(8.2.14) \quad Rq_*(k_{\mathcal{U}}) \to k_{|0}[-1] \oplus B_J[-1] \to Rq_*(G) \xrightarrow{\pm 1}.
$$

The same argument as in (ii) works and we find that $k_{|0}[-1]$ is a direct summand of $Rq_*(G)$. Since $Rq_*(k_{\mathcal{U}}) \simeq k_J \oplus k_{|0}[-1]$ the other summands of $Rq_*(G)$ are constant sheaves on $\mathcal{J}$. Hence $G|_C$ satisfies (8.2.8). Since $\bar{S}(k_{\mathcal{U}}) = \emptyset$, we also have $\bar{S}(G) = \bar{S}(F_0)$ and $G$ is simple. By (iv) below we have $G|_C \simeq e_*(k_{|0} \oplus k_{|1}) \oplus F_1$. Hence $G|_C$ satisfies the same hypotheses as $F$ with $(\beta - \alpha)/2\pi$ replaced by $(\beta - \alpha)/2\pi - 1$. Hence $G|_{C_0}$ has a direct summand, say $L_u$, which is a locally constant sheaf with unipotent monodromy (recall that $F$ was chosen to contradict the result with $\beta - \alpha$ minimal). As in (ii) we write $F_0|_{C_0} \simeq L_0 \oplus \bigoplus_{a \in A'} e_*(k_{j_a}^{u_a})$ and we write $u'|_{C_0} = \sum a \in A u'_a$ where $A'' \subset A'$ is the set of closed intervals. Setting $F_0' = L_0 \oplus \bigoplus_{a \in A''} e_*(k_{j_a}^{u_a})$, $F_0'' = \bigoplus_{a \in A'} e_*(k_{j_a}^{u_a})$ we have $F_0|_{C_0} \simeq F_0' \oplus F_0''$ and $u'|_{C_0} = (0, u'')$. Hence $G|_{C_0} \simeq F_0' \oplus \text{coker}(u'')$. The summand $L_u$ of $G|_{C_0}$ cannot appear in $F_0'$ (by the assumption on $F_0|_{C_0}$) hence it appears in $\text{coker}(u'')$. Hence $L_u$ is a quotient of $F_0''$. On the other hand, if $J_a$ is closed, then $\text{Hom}(e_*(k_{j_a}^{u_a}), L_u) = 0$ and we have a contradiction. This proves the proposition.

(iv) It remains to explain the decomposition of $G|_C$. We have $\beta - \alpha \geq 2\pi$ and we set $I = [\alpha, \beta]$, $J = [\alpha, \beta - 2\pi]$, $J' = [\alpha + 2\pi, \beta]$. We have the canonical isomorphisms $\text{Hom}(k_{\mathcal{C}}, e_*(k_{\mathcal{I}})) \simeq \text{Hom}(k_{\mathcal{C}^*}, k_{\mathcal{I}}) \simeq \Gamma(\mathbb{R}; k_{\mathcal{I}}) \simeq k$ where the first one is given by the adjunction $(e^{-1}, e_*)$. 


Let $i: k_C \to e_*(k_I)$ be the morphism corresponding to $1 \in k$. We have a natural isomorphism $\varphi: e_*(k_J) \simeq e_*(k_{J'})$ and two restriction morphisms $r: k_I \to k_J$, $r': k_I \to k_{J'}$. For $\theta \in C$ the vector space $(e_*(k_I))_\theta$ has a basis $\{p_1, \ldots, p_n\}$ identified with $I \cap e^{-1}(\theta)$ ($n$ depends on $\theta$). We order this basis by the order induced by $\mathbb{R}$, that is, $I \cap e^{-1}(\theta) = \{p_1 < \cdots < p_n\}$. In the same way, a basis of $(e_*(k_J))_\theta$ is $\{p_1, \ldots, p_{n-1}\}$. The morphisms $e_*(r)$ and $\varphi^{-1} \circ e_*(r')$ induce in the stalks the morphisms $(x_1, \ldots, x_n) \mapsto (x_1, \ldots, x_{n-1})$ and $(x_1, \ldots, x_n) \mapsto (x_2, \ldots, x_n)$. We also have $(k_C)_\theta \simeq k$ and the morphism $i$ induces the diagonal morphism in the stalks, $x \mapsto (x, \ldots, x)$. It follows that the sequence

$$0 \to k_C \xrightarrow{i} e_*(k_I) \xrightarrow{e_*(r) - \varphi^{-1} \circ e_*(r')} e_*(k_J) \to 0$$

is exact in the stalks, hence exact. \hfill \Box

**Example 8.2.4.** Here is an example of a sheaf satisfying the hypotheses of Proposition 8.2.3. We consider the closed subset $S = U \setminus \Omega$ of $U$ introduced in part (ii) of the proof and define its interior $S' = U \setminus \Omega$. Then $\text{Hom}(k_S, k_S[1]) \simeq H^1(S'; k_S) \simeq k$ and we define $F \in \text{D}^b(k_U)$ by the distinguished triangle $k_S' \xrightarrow{u} k_S[1] \to F \xrightarrow{+1}$. where $u$ is the image of 1 by this isomorphism. The image of the triangle by $Rq_*F$ gives the distinguished triangle

$$k_I[-1] \oplus k_{-1,0} \xrightarrow{v} k_J[1] \oplus k_{-1,0} \to Rq_*F \xrightarrow{+1},$$

where the morphism $v$ is non zero because its stalk at a point $t \in [-1,0]$ is the image of the morphism $k_S' \to k_S'[1]$ by the functor of global sections and this is non zero. Writing $v$ as a $2 \times 2$ matrix of morphisms, the only possibly non zero entry is $v_{22}: k_{-1,0} \to k_{-1,0}$ and $v_{22}$ must be a multiple of the canonical morphism $k_{-1,0} \to k_{-1,0}$ whose cone is $k_{(0)}$. Since the other entries of $v$ are zero, we find $Rq_*F \simeq k_{(0)} \oplus k_I \oplus k_{J}[1]$, which is the hypothesis (8.2.8).

**Proposition 8.2.5.** Let $\Lambda \subset T^*M$ be a Lagrangian submanifold satisfying (8.2.6) and let $F \in \text{D}^b_{[\Lambda]}(k_U)$ be a simple sheaf. We assume that $F$ is constructible (see Section 7.2.3) and satisfies (8.2.7) and (8.2.8). We also assume that $H^{-1}F|_{C_0}$ has a direct summand which is a (non zero) locally constant sheaf with unipotent monodromy. Then $H^1R\text{Hom}(F,F)$ is non zero.

**Proof.** (i) We will use Lemma 8.2.1 and look for $u: F_{C_0}[-1] \to R\Gamma_{C_0}(F)$ such that $i \circ u \neq 0$, where $i = j_{C_0}(F) \circ i_{C_0}(F)$ is the composition of the natural morphisms $i_{C_0}(F): R\Gamma_{C_0}(F) \to F$, $j_{C_0}(F): F \to F_{C_0}$. In fact, to ensure that $i \circ u \neq 0$ we first find $w: k_{C_0} \to R\Gamma_{C_0}(F)$ satisfying
\(i \circ w \neq 0\) and prove the existence of a factorization \(w = u \circ v\) as in the diagram

\[
\begin{array}{ccc}
\mathbf{k}_{\mathcal{C}_0} & \xrightarrow{w} & \Gamma\mathcal{C}_0(F) \xrightarrow{i} F_{\mathcal{C}_0} \\
\downarrow v & & \downarrow i \\
F_{\mathcal{C}_0}[-1] & \xrightarrow{u} & \\
\end{array}
\]

It is then clear that \(i \circ w \neq 0\). We first define \(w\) such that \(i \circ w \neq 0\). Then in parts (ii-iv) of the proof we actually show that any \(w: \mathbf{k}_{\mathcal{C}_0} \to \Gamma\mathcal{C}_0(F)\) can be written \(w = u \circ v\) as above.

The decomposition (8.2.8) gives a morphism \(\mathbf{k}_{\{0\}} \to Rq_*(F)\), hence \(w': \mathbf{k}_{\{0\}} \to \Gamma\mathcal{C}_0(Rq_*(F))\). Using \(\Gamma\mathcal{C}_0(Rq_*(F)) \cong Rq_*\Gamma\mathcal{C}_0(F)\) and the adjunction \((q^{-1}, Rq_*)\), we see that \(w'\) induces a morphism \(w: \mathbf{k}_{\mathcal{C}_0} \to \Gamma\mathcal{C}_0(F)\). Then \(i \circ w: q^{-1}(\mathbf{k}_{\{0\}}) \cong \mathbf{k}_{\mathcal{C}_0} \to F_{\mathcal{C}_0}\) corresponds to \(Rq_*(i) \circ w': \mathbf{k}_{\{0\}} \to Rq_*(F_{\mathcal{C}_0})\) via the adjunction. Through the isomorphisms \(\Gamma\mathcal{C}_0(Rq_*(F)) \cong Rq_*\Gamma\mathcal{C}_0(F)\) and \((Rq_*(F))_{\{0\}} \cong Rq_*(F_{\mathcal{C}_0})\) the direct image \(Rq_*(i): \Gamma\mathcal{C}_0(Rq_*(F)) \to (Rq_*(F))_{\{0\}}\) coincides with the natural morphism \(j_{\{0\}}(Rq_*(F)) \circ i_{\{0\}}(Rq_*(F))\). It follows that \(Rq_*(i) \circ w'\) induces the identity morphism on the summand \(\mathbf{k}_{\{0\}}\) of \(Rq_*(F)\). Hence \(Rq_*(i) \circ w'\) is non zero and its image by the adjunction, \(i \circ w\), is also non zero.

(ii) The existence of \(v, u\) only depend on the restriction of \(F\) to a neighborhood of \(\mathcal{C}_0\). So from now on we restrict over \(\mathcal{U}\), but still write \(F\) instead of \(F_{\mid \mathcal{U}}\). We recall that \(DF = D_{\mid \mathcal{U}} F = R\mathcal{H}om(F, \omega_\mathcal{U})\) and, since \(\mathcal{U}\) is oriented of dimension 2, \(D_{\mid \mathcal{U}} F \cong (D_{\mid \mathcal{U}} F)_2\), where \(D_{\mid \mathcal{U}} F = R\mathcal{H}om(F, \mathbf{k}_\mathcal{U})\).

Using Lemma 8.2.2 we have \(F \cong \bigoplus_{i \in \mathbb{Z}} H^i F[-i]\) and \(H^{-1} F\) satisfies (8.2.8). We deduce \(DF \cong \bigoplus_{i \in \mathbb{Z}} D(H^i F[-i])\), where \(D(H^i F[-i])\) is concentrated in degree 2 \(\minus i\). Hence \(H^{-1} F\) satisfies (8.2.14). Since \(R\mathcal{H}om(H^{-1} F[-1], H^{-1} F[-1])\) is a direct summand of \(R\mathcal{H}om(F, F)\), we may assume from the beginning that \(F\) is concentrated in degree \(-1\).

(iii) We recall that a locally constant sheaf \(G\) on the circle is described up to isomorphism by its stalk \(G_\theta\) at a given point and its monodromy \(m: G_\theta \to G_\theta\). The stalk of \(D' G\) is \(\text{Hom}(G_\theta, \mathbf{k})\) and its monodromy is \(t^i m^{-1}\). Two locally constant sheaves with isomorphic stalks and conjugate monodromies are isomorphic. In particular, if \(G\) has unipotent monodromy, then \(D' G \cong G\). Using Corollary 4.1.3 we decompose \(F_{\mid \mathcal{C}_0}\) as

\[
(8.2.15) \quad F\mid_{\mathcal{C}_0} \cong L_{\text{uni}}[1] \oplus L_{\text{nu}}[1] \oplus \bigoplus_{a \in A'} e_s(\mathbf{k}_s)[1],
\]
where $L_{\text{uni}}$, $L_{\text{nu}}$ are locally constant sheaves, $L_{\text{uni}}$ has unipotent monodromy, no summand of $L_{\text{nu}}$ has unipotent monodromy, and $A'$ is a finite family of bounded intervals or $\mathbb{R}$. Since $D_u F \simeq F$ we have $R\Gamma_c_0(F)|_{C_0} \simeq R\Gamma_c_0(D_u F)|_{C_0} \simeq D_{C_0}(F|_{C_0})$. Since $D'L_{\text{uni}} \simeq L_{\text{uni}}$, we obtain

$$R\Gamma_c_0(F)|_{C_0} \simeq L_{\text{uni}} \oplus D'L_{\text{nu}} \oplus \bigoplus_{a \in A'} e_*(k_{I_a^*}),$$

where $I_a^*$ is the interval such that $D'k_{I_a} \simeq k_{I_a^*}$.

(iv) We see $w$ as a section of $R\Gamma_c_0(F)$ using $\text{Hom}(k_{C_0}, R\Gamma_c_0(F)) \simeq H^0(C_0; R\Gamma_c_0(F))$. For a locally constant sheaf $G$ as in (iii) with monodromy $m: G_0 \to G_0$, we have $H^0(C_0; G) \simeq \{v \in G_0; m(v) = v\}$. Hence a locally constant sheaf has a section if and only if its monodromy has a unipotent factor. Using the decomposition (8.2.16) we can then write $w = (w_{\text{uni}}, 0, \sum a w_a)$ where $w_{\text{uni}}$ is a section of $L_{\text{uni}}$ and $w_a$ a section of $e_*(k_{I_a^*})$.

We choose a non zero section $v_{\text{uni}}^1$ of $L_{\text{uni}}$ (recall that $L_{\text{uni}} \neq 0$ by assumption) and define $v_{\text{uni}}$ by

$$v_{\text{uni}} = w_{\text{uni}} \text{ if } w_{\text{uni}} \neq 0, \quad v_{\text{uni}} = v_{\text{uni}}^1 \text{ if } w_{\text{uni}} = 0.$$

Using the decomposition (8.2.13) we set $v = (v_{\text{uni}}, 0, 0): k_{C_0} \to F_{C_0}[-1]$ (viewing $v_{\text{uni}}$ as a morphism $k_{C_0} \to L_{\text{uni}}$).

Now, for each $a \in A'$, we look for $u_a: L_{\text{uni}} \to e_*(k_{I_a^*})$ such that $w_a = u_a \circ v_{\text{uni}}$. The morphism $v_{\text{uni}}$ is injective (at each stalk, it is a non zero morphism $(k_{C_0})_0 = k \to (L_{\text{uni}})_0$, hence injective). Let $L'$ be its cokernel. For each $a \in A'$, the exact sequence $0 \to k_{C_0} \to L_{\text{uni}} \to L' \to 0$ yields the following part of a long exact sequence

$$\text{Hom}(L_{\text{uni}}, e_*(k_{I_a^*})) \xrightarrow{\varphi} \text{Hom}(k_{C_0}, e_*(k_{I_a^*})) \to \text{Ext}^1(L', e_*(k_{I_a^*})),
$$

where $\varphi$ is the morphism $f \mapsto f \circ v_{\text{uni}}$. By the adjunction $(e^{-1}, e_*)$ we have

$$\text{Ext}^1(L', e_*(k_{I_a^*})) \simeq \text{Hom}(L', e_*(k_{I_a^*})[1]) \simeq \text{Hom}(e^{-1}(L'), k_{I_a^*}[1]).$$

Now $L'$ is locally constant, hence $e^{-1}(L')$ is constant, say $e^{-1}(L') \simeq k_N^\mathbb{R}$, and we have $\text{Hom}(e^{-1}(L'), k_{I_a^*}[1]) \simeq (H^1(\mathbb{R}; k_{I_a^*}^\mathbb{R}))^N$. We remark that a sheaf $k_{I_a^*}^\mathbb{R}$ has a non zero section if and only if $I_a^*$ is closed, in which case we have $H^1(\mathbb{R}; k_{I_a^*}) \simeq 0$ and the morphism $\varphi$ is surjective. Hence for any $a \in A'$ there exists $u_a: L_{\text{uni}} \to e_*(k_{I_a^*})$ such that $w_a = u_a \circ v_{\text{uni}}$.

Using the decompositions (8.2.13), (8.2.16) we define $u: F_{C_0}[-1] \to R\Gamma_c_0(F)$ by

$$u = \begin{pmatrix} \text{id}_{L_{\text{uni}}} & 0 & 0 \\ 0 & 0 & 0 \\ \sum u_a & 0 & 0 \end{pmatrix} \text{ if } w_{\text{uni}} \neq 0, \quad u = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \sum u_a & 0 & 0 \end{pmatrix} \text{ if } w_{\text{uni}} = 0.$$
The equality \( u \circ v = w \) follows; indeed it reads respectively in the cases \( w_{uni} \neq 0 \) and \( w_{uni} = 0 \):

\[
\begin{pmatrix}
\text{id}_{\sum_{u_a} w_{uni}} & 0 & 0 \\
0 & \sum_{u_a} w_{uni} & 0 \\
0 & 0 & \sum_{u_a} w_{uni}
\end{pmatrix}
\begin{pmatrix}
w_{uni} \\
0 \\
0
\end{pmatrix}
= 
\begin{pmatrix}
0 \\
0 \\
0
\end{pmatrix}
\].

\[\square\]

### 8.3. Microlocal linked points

Let \( M \) be a manifold (in Proposition 8.3.5 \( M \) will be a surface) and let \( \Lambda \) be a smooth conic Lagrangian submanifold of \( \dot{T}^* M \). We consider \( F \in D^b(k_M) \) such that \( SS(F) = \Lambda \) and \( F \) is simple along \( \Lambda \). In this section we give a criterion which implies that \( H^1_{\text{RHom}}(F,F) \) is non-zero (see Proposition 8.3.5). For \( U, V \) two open subsets of \( M \) such that \( M = U \cup V \) and for \( G \in D^b(k_M) \) we denote by

\[
H^1({\{U, V\}}; G) = H^0(U \cap V; G)/(H^0(U; G) \times H^0(V; G))
\]

the first \( \check{\text{Cech}} \) group of \( G \) associated with the covering \( \{U, V\} \) of \( M \) (we do not assume that \( U \) and \( V \) are connected). By the Mayer-Vietoris long exact sequence we have an injective map

\[
H^1({\{U, V\}}; G) \hookrightarrow H^1(M; G).
\]

In particular it is enough for our purpose to find a covering \( \{U, V\} \) with \( H^1({\{U, V\}; \text{RHom}(F, F)}) \neq 0 \). To better understand this latter group, we use the natural morphism from \( \text{RHom}(F, F) \) to \( R\pi_M^* \muhom(F, F) \).

Since \( F \) is simple we have a canonical isomorphism

\[
k_\Lambda \simeq \muhom(F, F)|_{\dot{T}^* M}
\]

sending 1 to \( \text{id}_F \). We deduce morphisms \( \text{RHom}(F, F) \to R\pi_M^* (k_\Lambda) \) and

\[
H^1({\{U, V\}; \text{RHom}(F, F)}) \to H^1({\{U', V'\}; k_\Lambda}),
\]

where \( U' = T^* U \cap \Lambda, V' = T^* V \cap \Lambda \). However we lose too much information in this way and we consider a map to another \( \check{\text{Cech}} \) group (see (8.3.9)). The construction of this map relies on the notion of linked points in \( \Lambda \) introduced in Definition 8.3.1 below.

We first introduce a general notation. For \( G, G' \in D^b(k_M) \) we recall the canonical isomorphism

\[
\text{RHom}(G, G') \simeq R\pi_M^* \muhom(G, G').
\]

For an open subset \( W \) of \( M \) and \( p \in T^* W \), we deduce the morphisms

\[
\text{Hom}(G|_W, G'|_W) \to H^0(T^* W; \muhom(G, G')), \quad u \mapsto u^{u^+},
\]

\[
\text{Hom}(G|_W, G'|_W) \to H^0 \muhom(G, G')_p, \quad u \mapsto u^{u^+}.
\]
For $G = G' = F$ with $F$ simple and for $p \in \hat{\SS}(F)$ we obtain

\begin{align}
(8.3.6) & \quad \Hom(F|_W, F|_W) \to H^0(T^*W; k_\Lambda), \quad u \mapsto u^\mu, \\
(8.3.7) & \quad \Hom(F|_W, F|_W) \to k, \quad u \mapsto u^\mu_p
\end{align}

and we have $u^{\mu+}_p = u^\mu_p \cdot (\id_F)^{\mu+}$, where $\cdot$ is the scalar multiplication (indeed the identification $H^0 \muhom(F, F)_p \simeq k$ sends $(\id_F)^{\mu+}$ to 1).

**Definition 8.3.1.** Let $W \subset M$ be an open subset and let $p, q \in \Lambda \cap T^*W$ be given points. We say that $p$ and $q$ are $F$-linked over $W$ if $u^\mu_p = u^\mu_q$ for all $u \in \Hom(F|_W, F|_W)$.

Because of the isomorphism $k_\Lambda \cong \muhom(F, F)|_{\hat{\J}^*_M}$ the scalar $u^\mu_p$ only depends on the component of $\Lambda \cap T^*W$ containing $p$ and we could also speak of $F$-linked connected components of $\Lambda \cap T^*W$. In particular, if $\Lambda$ is connected and $W = M$ all points of $\Lambda$ are $F$-linked over $M$. The notion of $F$-linked points will be used when $\Lambda \cap T^*W$ is a priori non-connected.

**Remark 8.3.2.** (a) Let $p = (x; \xi) \in \Lambda$. Let $\varphi: M \to \mathbb{R}$ be a function of class $C^\infty$ such that $\Lambda$ and $\Lambda_\varphi$ intersect transversely at $p$. We set $Z = \{ \varphi \geq \varphi(x) \}$. For $G, G' \in D^b_\Lambda(k_M)$, we have by (1.4.6)

$$H^0 \muhom(G, G')_p \simeq \Hom((\Gamma_Z(G))_x, (\Gamma_Z(G'))_x).$$

If $G = G' = F$ with $F$ simple at $p$, then $(\Gamma_Z(F))_x$ is $k$ in some degree. For $u \in \Hom(F, F)$ the morphism $(\Gamma_Z(u))_x$ is the multiplication by the scalar $u^\mu_p$.

(b) We keep the notations in (a). We assume that $u^\mu_p \neq 0$. Defining $H \in D^b_\Lambda(k_M)$ by the distinguished triangle $F \xrightarrow{u} F \to H \xrightarrow{+1}$, we thus have $(\Gamma_Z(H))_x \simeq 0$. On the other hand $\hat{\SS}(H) \subset \Lambda$ by the triangular inequality for the microsupport and the vanishing of $(\Gamma_Z(H))_x$ implies that $\hat{\SS}(H)$ does not meet the connected component of $\Lambda$ containing $p$.

(c) We keep the notations in (a). We assume that $u$ factorizes through $H \in D^b(k_M)$ and $p \notin \SS(H)$. Then $(\Gamma_Z(u))_x$ factorizes through $(\Gamma_Z(H))_x \simeq 0$ and we obtain $u^\mu_p = 0$.

By Theorem 1.3.8 the functors $u \mapsto u^{\mu+}_p$ or $u \mapsto u^\mu_p$ are well-defined in the quotient category $D^b(k_M; p)$. We can also express this result as follows.

**Lemma 8.3.3.** Let $G, G' \in D^b(k_M)$ be such that $\SS(G), \hat{\SS}(G') \subset \Lambda$ and let $p \in \Lambda$ be given. We assume that there exists a distinguished triangle $G \xrightarrow{q} G' \to H \xrightarrow{+1}$ with $p \notin \SS(H)$. Then the composition with
$g$ induces isomorphisms
\[ \mu_{\text{hom}}(G,G)_p \xrightarrow{\sim} \mu_{\text{hom}}(G,G')_p \xleftarrow{\sim} \mu_{\text{hom}}(G',G')_p \]
and we have $a((\text{id}_G)_p^+) = g_p^+ = b((\text{id}_{G'})_p^+)$. In particular, if $G$ and $G'$ are simple and $u: G \to G$ and $v: G' \to G'$ satisfy $v \circ g = g \circ u$ then $u_p^+ = v_p^+$.

**Proof.** We apply the functor $\mu_{\text{hom}}(G, \cdot)$ to the given distinguished triangle and we take the stalks at $p$. By the bound (1.3.6) we have $\mu_{\text{hom}}(G,H)_p \simeq 0$ and we deduce the isomorphism $a$. The isomorphism $b$ is obtained in the same way. Then the last assertion follows from the relations $u_p^+ = u_p^+ \cdot (\text{id}_G)_p^+$ and $v_p^+ = v_p^+ \cdot (\text{id}_{G'})_p^+$. \hfill $\square$

We recall that $U, V$ are two open subsets of $M$ such that $M = U \cup V$. Let $\gamma: [0,1] \to \Lambda$ be a path such that
\[ p_0 = \gamma(0) \text{ and } p_1 = \gamma(1) \text{ belong to } (T^*U \setminus T^*V) \cap \Lambda \]
and are $F$-linked over $U$; moreover $\text{im}(\gamma)$ is not entirely contained in $T^*U \cap \Lambda$.

We define a circle $C$ by identifying 0 and 1 in $[0,1]$ and let $r: [0,1] \to C$ be the quotient map. The natural orientation of $[0,1]$ induces an orientation on $C$. We set $U' = r(\gamma^{-1}(T^*U \cap \Lambda))$, $V' = r(\gamma^{-1}(T^*V \cap \Lambda))$. We have $U' \cup V' = C$ and, by (8.3.8), $U'$ and $V'$ are neither empty nor equal to $C$. Hence they are unions of non trivial arcs of $C$ and we have a canonical isomorphism $H^1(\{U', V'\}; k_C) \simeq H^1(C; k_C) \simeq k$.

For $u \in H^0(U; \text{RHom}(F,F))$ the inverse image of $u^+$ by $\gamma$ gives a well-defined section of $H^0(U'; k_C)$ because $p_0$ and $p_1$ are $F$-linked over $U$. An element of $H^0(V; \text{RHom}(F,F))$ also induces a section of $H^0(V'; k_C)$ because $p_0, p_1 \notin T^*V \cap \Lambda$. We deduce a well-defined map
\[ m_\gamma: H^1(\{U, V\}; \text{RHom}(F,F)) \to H^1(\{U', V'\}; k_C) \simeq k. \]

Now we describe a situation where the map $m_\gamma$ is surjective. We first introduce a notion of conjugate pair of points.

**Definition 8.3.4.** Let $M$ be a manifold, $k$ a field and $F \in \mathbb{D}^b(k_M)$. Let $q_0 = (x_0; \xi_0)$ and $q_1 = (x_1; \xi_1)$ be given points of $SS(F)$, generating two distinct half-lines $\mathbb{R}_{>0} \cdot q_0 \neq \mathbb{R}_{>0} \cdot q_1$. Let $I$ be either an open interval of $\mathbb{R}$ or the circle $S^1$ and let $i: I \to M$ be an embedding. We say that $q_0$ and $q_1$ are $F$-conjugate with respect to $i$ if
(i) $x_0, x_1 \in \text{im}(i)$; we write $t_0 = i^{-1}(x_0)$, $t_1 = i^{-1}(x_1)$,
(ii) for $k = 0, 1$, there exist a neighborhood $U_k \subset M$ of $x_k$ and a hypersurface $N_k \subset U_k$ such that $SS(F) \cap T^*U_k \subset T^*N_k U_k$ and $i$ is transverse to $N_k$ at $t_k$,
(iii) $F$ is simple along $SS(F)$ at $q_k$ for $k = 0, 1$. 

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(iv) $i^{-1}F$ has a direct summand $F'$ such that $(t_k; i_d(\xi_k)) \in \text{SS}(F')$ for $k = 0, 1$ and $F'$ is isomorphic to

\[
\begin{cases}
k_J[d] & \text{for some interval } J \text{ of } I \quad \text{if } I \text{ is an interval},
\\e_*(k_I)[d] & \text{for some interval } J \text{ of } \mathbb{R} \quad \text{if } I = S^1,
\end{cases}
\]

where $d \in \mathbb{Z}$ and $e: \mathbb{R} \to S^1$ is the covering map.

We will see in Proposition 8.4.4 that the notion of conjugate and linked points are related.

**Proposition 8.3.5.** Let $M$ be a surface and let $\Lambda$ be a smooth conic Lagrangian submanifold of $T^*M$ such that $\Lambda/\mathbb{R}_{>0}$ is a circle. Let $F \in D^b(\Lambda)(k_M)$ such that $F$ is simple along $\Lambda$. We assume that there exists an embedding of the circle $i: S^1 \to M$ and $p_0, p_1, q_0, q_1 \in \Lambda$ such that

(a) $\text{im}(i)$ meets $\pi_M(\Lambda)$ at smooth points and transversely,
(b) $M \setminus \text{im}(i)$ has two connected components $U, V$,
(c) $q_0$ and $q_1$ are $F$-conjugate with respect to $i$,
(d) $p_0$ and $p_1$ are $F$-linked over $U$,
(e) the pairs $\{[p_0], [p_1]\}$ and $\{[q_0], [q_1]\}$ in $\Lambda/\mathbb{R}_{>0}$ are intertwined, i.e. each of the two arcs from $[p_0]$ to $[p_1]$ contains exactly one point of $\{[q_0], [q_1]\}$ (where $[p]$ denotes the image of $p \in \Lambda$ in $\Lambda/\mathbb{R}_{>0}$).

Let $\gamma$ be a path from $p_0$ to $p_1$ in $\Lambda$ lifting an arc in $\Lambda/\mathbb{R}_{>0}$. Then we can increase $U, V$ so that the map $m_\gamma$ of (8.3.9) is defined and surjective. In particular $H^1_R\text{Hom}(F, F) \neq 0$.

The hypotheses are illustrated in Fig. 8.6.1 and 8.6.2 for the proof of Theorem 8.6.1, the embedding $i$ is the inclusion of the circle $C_{1/2}$ and the points $p^+_i, q_i$ are $F$-conjugate with respect to $i$; the points $p^+, p_0$ are $F$-linked over the lower open set bounded by $C_{1/2}$; the dotted path in Fig. 8.6.1 is $\text{im}(\pi_M \circ \gamma)$.

**Proof.** (i) By the hypothesis (a) we can assume that $\text{im}(i)$ has a neighborhood $\mathcal{N}$ of the type $\mathcal{N} = S^1 \times K$, where $K = ]-1, 1[$ and $i$ corresponds to the embedding of $S^1 \times \{0\}$, such that $\Lambda \cap T^*\mathcal{N} = \Lambda_0 \times T^*_K K$ for some subset $\Lambda_0 \subset \mathring{T}^*S^1$. Since $p_0, p_1 \in T^*(M \setminus S^1)$ we can also assume that $p_0, p_1 \not\in \mathring{T}^*\mathcal{N}$. By Theorem 1.2.8 we can write $F|_\mathcal{N} \simeq p^{-1}F_0$ where $p: \mathcal{N} \to S^1$ is the projection and $F_0 \in D(k_M)$. We must have $F_0 = i^{-1}F$. We write $q_0 = (a, 0; \alpha, 0), q_1 = (b, 0; \beta, 0)$ with $a, b \in S^1$ and $\alpha, \beta \neq 0$. By the definition of conjugate points, $F_0$ has a direct summand isomorphic to $e_*(k_J)[d]$, for some interval $J$ of $\mathbb{R}$ whose endpoints are mapped to $a$ and $b$ by $e: \mathbb{R} \to S^1$ and such that
\[ \tilde{S}(e_\ast(k_j)) = \mathbb{R}_{>0} \cdot (a; \alpha) \sqcup \mathbb{R}_{>0} \cdot (b; \beta). \] Hence we can write
\[ (8.3.10) \quad F|_\mathcal{N} \simeq F' \oplus F'', \]
where \( F' = p^{-1}(e_\ast(k_j)[d]) \) and \( \tilde{S}(F') = \tilde{S}(e_\ast(k_j)) \times T^*_K K \). In other words \( \tilde{S}(F') \) consists of the two components of \( \Lambda \cap T^*_N \), containing \( q_0 \) and \( q_1 \). We remark that the components of \( \Lambda \cap T^*_N \) are of the form \( \Lambda_i = \mathbb{R}_{>0} \cdot (a_i; \alpha_i) \times T^*_K K, \ i = 1, \ldots, n, \) with \((a_i; \alpha_i) \in T^*S^1 \). Since \( \gamma: [0, 1] \to \Lambda \) lifts an arc of \( \Lambda/\mathbb{R}_{>0} \) whose two ends, \( p_0 \) and \( p_1 \), do not belong to \( T^*_N \), it follows that \( I_i = \gamma^{-1}(\Lambda_i) \) is either empty or an interval of \([0, 1] \). Moreover \( T_i \) does not contain 0, 1. The hypothesis (e) says that \( \text{im}(\gamma) \) meets exactly one of the two components of \( \tilde{S}(F') \).

Hence \( \gamma^{-1}(\tilde{S}(F')) \) consists of a single interval. We choose the indices so that \( \gamma^{-1}(\tilde{S}(F')) = I_1, \) \( \text{im}(\gamma) \) meets \( \Lambda_1, \ldots, \Lambda_{n'} \) and does not meet \( \Lambda_{n'+1}, \ldots, \Lambda_n \), for some \( n' \leq n \).

(ii) We set \( U^+ = U \cup \mathcal{N} \) and \( V^+ = V \cup \mathcal{N} \). Hence \( U^+ \cap V^+ = \mathcal{N} \). Let \( x \in k \) be given. Using the decomposition (8.3.10) we define a morphism \( \alpha(x): F|_\mathcal{N} \to F|_\mathcal{N} \) by
\[ (8.3.11) \quad \alpha(x) = \begin{pmatrix} x \cdot \text{id}_{F'} \\ 0_{F''} \end{pmatrix}. \]

As described after (8.3.8) we set \( C = [0, 1]/(0 \sim 1) \) and let \( r: [0, 1] \to C \) be the quotient map. We set \( U' = r(\gamma^{-1}(T^*U^+ \cap \Lambda)) \) and \( V' = r(\gamma^{-1}(T^*V^+ \cap \Lambda)) \). Then \( U' \) and \( V' \) are unions of possibly several arcs of \( C \), say \( U' = \bigsqcup_{i=1}^l U'_i, \ V' = \bigsqcup_{i=1}^m V'_i \). We have \( U' \cap V' = r(\gamma^{-1}(\Lambda \cap T^*_N \Lambda)) = \bigsqcup_{i=1}^{n'} r(I_i) \) and in fact \( m = l = n'/2 \). By definition \( H^1(\{U', V'\}; k_C) \) is the cokernel of the morphism
\[ d: \bigoplus_{i=1}^l H^0(U'_i; k_C) \oplus \bigoplus_{i=1}^m H^0(V'_i; k_C) \to \bigoplus_{i=1}^{n'} H^0(r(I_i); k_C), \]

where each \( H^0(\cdot; k_C) \) is isomorphic to \( k \) and \( d \) is induced by the obvious restriction maps (we remark that \( H^0(U'_i; k_C) \to H^0(r(I_j); k_C) \) is the identity map if \( r(I_j) \subset U'_i \), the zero map else). We know that \( H^1(\{U', V'\}; k_C) \simeq H^1(C, k_C) \simeq k \) and the quotient map induces \( H^0(r(I_i); k_C) \xrightarrow{\sim} k \), for each \( i = 1, \ldots, n' \).

By the definition of \( \alpha(x) \), the section \( \alpha(x)^{\mu} \) of \( \mu_{\text{hom}}(F, F) \simeq k_\Lambda \) is the scalar \( x \) over the two components of \( \Lambda \cap T^*_N \) given by \( \tilde{S}(F') \) and 0 over all other components of \( \Lambda \cap T^*_N \). Taking the pull-back by \( \gamma \) and the image by \( r \), and remembering that \( \gamma^{-1}(\tilde{S}(F')) = I_1 \) is a single interval, we obtain that \( m_\gamma([\alpha(x)]) \) is represented by \( (x_i)_{i=1}^{n'} \).
\[ \bigoplus_{i=1}^{n'} H^0(r(I_i); \mathcal{K}_C) \] with \( x_1 = x \) and \( x_i = 0 \) for \( i \neq 1 \). It follows that \( m_\gamma(\alpha(x)) \) is non zero if \( x \neq 0 \), hence \( m_\gamma \) is surjective.

In particular \( H^1(\{U, V\}; R\text{Hom}(F, F)) \neq 0 \) and, by \([8.3.2]\), we have \( H^1 R\text{Hom}(F, F) \simeq H^1(M; R\text{Hom}(F, F)) \neq 0 \). \( \square \)

8.4. Examples of microlocal linked points

We begin with the following easy example over \( \mathbb{R} \). In higher dimension we give Propositions 8.4.4 and 8.4.5 below which reduce to this case by inverse or direct image.

**Proposition 8.4.1.** Let \( t_0 \leq t_1 \in \mathbb{R} \) and \( p_0 = (t_0; \tau_0), p_1 = (t_1; \tau_1) \in \mathcal{T}^* \mathbb{R} \) be given. Let \( F \in \mathcal{D}^b(\mathcal{K}_F) \) be a constructible sheaf with \( p_0, p_1 \in \text{SS}(F) \) such that \( F \) is simple at \( p_0, p_1 \). We assume that there exists a decomposition \( F \simeq G \oplus \mathcal{K}_I[d] \) where \( G \in \mathcal{D}^b(\mathcal{K}_F) \), \( d \in \mathbb{Z} \) and \( I \) is an interval with ends \( t_0, t_1 \) such that \( p_0, p_1 \in \text{SS}(\mathcal{K}_I) \). Then \( p_0 \) and \( p_1 \) are \( F \)-linked over any open interval containing \( \mathcal{T} \).

**Proof.** Let \( i: \mathcal{K}_I[d] \to F \) and \( q: F \to \mathcal{K}_I[d] \) be the morphism associated with the decomposition of \( F \). Then \( i \) and \( q \) give a morphism \( \text{Hom}(F, F) \to \text{Hom}(\mathcal{K}_I, \mathcal{K}_I) \simeq \mathcal{K} \). Since \( F \) is simple at \( p_k \), for \( k = 0, 1 \), and \( p_k \in \text{SS}(\mathcal{K}_I) \), we have \( p_k \not\in \text{SS}(G) \). Let \( u: F \to F \) be given and let \( i u: \mathcal{K}_I[d] \to \mathcal{K}_I[d] \) be the morphism induced by \( u \). Then \( u^\mu_{p_k} = (i u)^\mu_{p_k} \) by Lemma \([8.3.3] \). Since \( \text{Hom}(\mathcal{K}_I, \mathcal{K}_I) \simeq \mathcal{K} \) we have \( (i u)^\mu_{p_0} = (i u)^\mu_{p_1} \) for any \( u \) defined in a neighborhood of \( \mathcal{T} \). The result follows. \( \square \)

Let \( f: M \to N \) be a morphism of manifolds. We recall the notations \( f_d: M \times_N T^* N \to T^* M \) and \( f_\tau: M \times_N T^* N \to T^* N \) for the natural maps induced on the cotangent bundles.

We study easy cases of inverse or direct images of simple sheaves. We let \( F \in \mathcal{D}^b(\mathcal{K}_F) \) be such that \( \Lambda = \text{SS}(F) \) is a smooth Lagrangian submanifold and \( F \) is simple along \( \Lambda \). More general, but local, statements are given in \([31] \) (see Corollaries 7.5.12 and 7.5.13).

**Lemma 8.4.2.** Let \( i: L \to M \) be an embedding and let \( x_0 \) be a point of \( L \). We assume that there exist a neighborhood \( U \subset M \) of \( i(x_0) \) and a submanifold \( N \subset U \) such that \( \Lambda \cap \mathcal{T}^* U \subset T^*_N U \) and \( N \) is transverse to \( L \) at \( x_0 \). Then, up to shrinking \( U \) around \( x_0 \), the set \( \Lambda' = \text{SS}((i^{-1} F)|_{U \cap L}) \) is contained in \( \mathcal{T}^*_N L \) and \( (i^{-1} F)|_{U \cap L} \) is simple along \( \Lambda' \).

Moreover, if \( u: F \to F \) is defined in a neighborhood of \( x_0 \) and \( v: i^* F \to i^{-1} F \) is the induced morphism, then for any \( p = (i(x_0); \xi_0) \in \Lambda \) and \( q = (x_0; id(\xi_0)) \) we have \( v_q^\mu = u_p^\mu \).
We note that the inclusion $\Lambda \cap T^* U \subset \hat{T}_N^* U$ implies that $\Lambda$ is a union of components of $\hat{T}_N^* U$ (hence this is an equality if $N$ is connected of codimension $\geq 2$).

**Proof.** Up to shrinking $U$ we can find a submersion $f: U \to L \cap U$ such that $N = f^{-1}(L \cap N)$ and $f \circ i = \text{id}_{L \cap U}$. By Proposition 1.2.9 we can write $F|_U \simeq f^{-1}G$ for some $G \in D^b(\mathbb{k}_{L \cap U})$. We must have $G = (i^{-1}F)|(L \cap U)$. Then $\mu_{\text{hom}}(F, F) \simeq f_\ast, f^{-1}_\ast(\mu_{\text{hom}}(G, G))$ over $U$ and the lemma follows. □

**Lemma 8.4.3.** Let $q: M \to \mathbb{R}$ be a function of class $C^2$ and let $t_0$ be a regular value of $q$. We assume that there exist an open interval $J$ around $t_0$, a connected hypersurface $L$ of $U := q^{-1}(J)$ and a connected component $\Lambda_0$ of $\hat{T}_L^* U$ such that

(a) $q|_U: U \to J$ is proper on $\text{supp} F$,
(b) $q|_L$ is Morse with a single critical point $x_0$ and $q(x_0) = t_0$,
(c) $\Lambda_0 \subset \Lambda \cap T^* U$ and $\hat{T}_{\Lambda_0}^* M \cap \Lambda = \hat{T}_{\Lambda_0}^* M \cap \Lambda_0$,
(d) $((\Lambda \cap T^* U) \setminus \Lambda_0) \cap \text{supp}(M \times \mathbb{R}^\ast T^* \mathbb{R}) = \emptyset$.

Let $p = (x_0; \xi_0)$ be the point of $\Lambda_0$ ($\xi_0$ is unique up to a positive scalar) above $x_0$. We have $p \in \text{im}(q_d)$ and we set $p' = (t_0; \tau_0) = q_\ast(q^{-1}_d(p))$. Then $p' \in \text{SS}(R_{q_\ast}F)$ and $R_{q_\ast}(F)$ is simple at $p'$. Moreover, for any morphism $u: F|_U \to F|_L$, denoting by $v = R_{q_\ast}(u): R_{q_\ast}F|_L \to R_{q_\ast}F|_L$ the induced morphism, we have $v_{p'}^u = v_{p'}^u$.

**Proof.** (i) We set $N = q^{-1}(t_0)$. Then $N$ is a smooth hypersurface of $M$ and, up to shrinking $J$ and restricting to a neighborhood of $\text{supp}(F) \cap N$, we can assume that $U = N \times J$. The hypersurfaces $N$ and $L$ of $U$ are tangent at the point $x_0$. We choose a distance function $d_N$ on $N$ and define $f: U \to \mathbb{R}$, $(x, t) \mapsto d_N(x_0, x)$. For $r > 0$ we set $V_r = f^{-1}([0, r])$ and $Z_r = U \setminus V_r$. We have a distinguished triangle $F_{V_r} \to F \to F_{Z_r} \to$. By (c) we can find a ball $B$ around $x_0$ in $U$ such that $T^* B \cap \Lambda = T^* B \cap \Lambda_0$. By Theorem 1.2.13 we obtain $\text{SS}(F_{Z_r}) \cap T^* B \subset \Lambda(r)$ where

$$\Lambda(r) = ((\Lambda_0 \cap \pi_{U_1}^{-1}(Z_r)) \cup T_{\partial V_r}^* U \cup (\Lambda_0 + T_{\partial V_r}^* U)) \cap T^* B,$$

and $T_{\partial V_r}^* U$ is the outer conormal bundle of $\partial V_r$ in $U$. We claim that $\Lambda(r) \cap T_N^* U = \emptyset$ for a dense set of $r$. Indeed it is enough to see that $(T_{L \cap U}^* U + T_{f^{-1}(r)}^* U) \cap T_N^* U \cap T^* B = \emptyset$. Since $x_0$ is the only critical point of $q|_L$, $L \cap N$ is smooth outside $x_0$ and $(T_{L \cap U}^* U + T_{f^{-1}(r)}^* U) \cap T_N^* U$ is empty if and only if $T_{L \cap U}^* U \cap T_{f^{-1}(r)}^* U$ is empty; this means that $r$ is a regular value of $f|_{L \cap N}$ and happens for a dense set of $r$. 


(ii) We choose \( q \in F \) and we can assume from the beginning that \( r \Lambda(N \setminus [-\varepsilon, \varepsilon]) \times \mathbb{R} \mathbb{T}^* \mathbb{R} = \emptyset \). It follows from (d) that \( \dot{\mathcal{S}} \mathcal{S}(F_{Z_{r}}) \cap q_{d}((N \times [-\varepsilon, \varepsilon]) \times \mathbb{R} \mathbb{T}^* \mathbb{R}) = \emptyset \). Hence \( Rq_{r}(F_{Z_{r}})|_{[-\varepsilon, \varepsilon]} \) is a constant sheaf and we can assume from the beginning that \( F = F_{Y} \).

(iii) We set \( W = V_{r} \cap (N \times [-\varepsilon, \varepsilon]) \). Taking \( r \) and \( \varepsilon \) smaller if necessary we are in the situation of Example 1.2.10 and there exist \( E, E' \) and a distinguished triangle \( E' \) we must have \( r = \varepsilon, \varepsilon \) is constant, we deduce \( (R\Gamma_{V_{r}}(Rq_{r}(F_{V_{r}})))_{t_{0}} \simeq (R\Gamma_{V_{r}}(Rq_{r}(k_{W'}[d])))_{t_{0}} \), where \( Y = ]-\infty, 0[ \) or \( Y = [0, +\infty[ \) according to the sign of \( \tau_{0} \). Now the problem is reduced to the computation of \( Rq_{*}(k_{W'}) \) in a neighborhood of \( t_{0} \), which is a classical computation in Morse theory (in our case this is done in the proof of Proposition 7.4.2 of [31]).

Proposition 8.4.1 and Lemmas 8.4.2 and 8.4.3 imply the following results.

**Proposition 8.4.4.** Let \( M \) be a manifold, \( k \) a field and \( F \in D^{b}(k_{M}) \). Let \( p_{0}, p_{1} \in \mathcal{S}(F) \) be given. We assume that there exists an embedding \( i: I \rightarrow M \) of the circle or an open interval of \( \mathbb{R} \) such that \( p_{0} \) and \( p_{1} \) are \( F \)-conjugate with respect to \( i \). Then \( p_{0} \) and \( p_{1} \) are \( F \)-linked over any open subset containing \( i(I) \).

**Proposition 8.4.5.** Let \( q: M \rightarrow \mathbb{R} \) be a map of class \( C^{2} \) and let \( t_{0} \leq t_{1} \) be regular values of \( q \). For \( k = 0, 1 \) we assume that there exist an open interval \( J_{k} \) around \( t_{k} \), a connected hypersurface \( L_{k} \) of \( U_{k} := q^{-1}(J_{k}) \) and a connected component \( \Lambda_{k} \) of \( T_{L_{k}}^{*}U_{k} \) such that the hypotheses (a)-(d) of Lemma 8.4.3 are satisfied. We assume moreover that \( Rq_{*}F \) has a decomposition \( Rq_{*}F \simeq G \oplus k_{J}[d] \) where \( G \in D^{b}(k_{\mathbb{R}}) \), \( d \in \mathbb{Z} \) and \( J \) is an interval with ends \( t_{0}, t_{1} \). Let \( p_{k} = (x_{k}; \xi_{k}) \in \Lambda \) be such that \( q(x_{k}) = t_{k} \) and \( q_{p}(q_{d}^{-1}(p_{k})) \in \mathcal{S}(\mathcal{S}(F)) \). Then \( p_{0} \) and \( p_{1} \) are \( F \)-linked over any open subset containing \( q^{-1}([t_{0}, t_{1}]) \).

Now we check that in a generic situation a given point has a unique conjugate.

**Proposition 8.4.6.** Let \( M \) be a manifold, let \( \Lambda \subset \dot{T}^{*}M \) be a smooth conic Lagrangian submanifold and let \( i: S^{1} \rightarrow M \) be an embedding of the circle. We assume that, in some neighborhood \( U \) of \( i(S^{1}) \), there exists a hypersurface \( N \) of \( U \) which is transverse to \( i \) and such that \( \Lambda \subset T_{N}^{*}U \). Let \( F \in D^{b}(k_{M}) \) be such that \( \mathcal{S}(F) = \Lambda \), \( F \) is simple along
Λ and \( i^{-1}F \) is constructible. Then, for any \( q_0 \in \Lambda \cap (i(S^1) \times_MT^*M) \) there exists a unique \( q_1 \in \Lambda \cap (i(S^1) \times_MT^*M) \) (unique up to the action of \( \mathbb{R}_{>0} \)) such that \( q_0 \) and \( q_1 \) are \( F \)-conjugate with respect to \( i \).

**Proof.** We set \( \Lambda' = i_{\partial i}^{-1}(\Lambda) \). The hypothesis on \( \Lambda \) implies that \( \Lambda' \) is in bijection with \( \Lambda \cap (i(S^1) \times_MT^*M) \). Up to shrinking \( U \) we can find a diffeomorphism \( f: U \xrightarrow{\sim} S^1 \times \mathbb{R}^d \), \( d = \dim M - 1 \), and a finite subset \( Z \) of \( S^1 \) such that \( f(N) = Z \times \mathbb{R}^d \) and \( f \circ i \) is the inclusion of \( S^1 \times \{0\} \). By Proposition \( 1.2.9 \) we can write \( f_* (F|_U) \simeq p^{-1}i^{-1}(F) \) where \( p: S^1 \times \mathbb{R}^d \to S^1 \) is the projection. It follows that \( SS(i^{-1}(F)) = \Lambda' \). Moreover \( i^{-1}(F) \) is simple along \( \Lambda' \) by Lemma \( 8.4.2 \).

Using Corollary \( 1.4.3 \), we write \( i^{-1}(F) \simeq L \oplus \bigoplus_{a \in A} e_*(k^{n_a}_{\lambda_a})[d_a] \), where \( L \) is locally constant and \( A \) is a finite family of bounded intervals of \( \mathbb{R} \). Since the \( I_a \)'s are bounded we can identify the set of ends of these intervals with \( B = A \times \{\text{left}, \text{right}\} \). Taking the microsupport gives a map \( \mu \) from \( B \) to the set of half-lines of \( \hat{T}^*S^1 \). More precisely, for an interval \( I_a \) with \( T_a = [x,y] \) we have \( \mu(I_a, \text{left}) = \{(e(x); \xi); \varepsilon \xi > 0\} \) with \( \varepsilon = 1 \) if \( I_a \) is closed near \( x \), \( \varepsilon = -1 \) if \( I_a \) is open near \( x \), and \( \mu(I_a, \text{right}) = \{(e(y); \xi); \varepsilon \xi > 0\} \) with \( \varepsilon = -1 \) if \( I_a \) is closed near \( y \), \( \varepsilon = 1 \) if \( I_a \) is open near \( y \).

Since \( i^{-1}(F) \) is simple, the integers \( n_a \) are all equal to 1 and the map \( \mu \) is injective, inducing a bijection between \( B \) and \( \Lambda' / \mathbb{R}_{>0} \). Now, two points \( q_0, q_1 \in \Lambda \cap (i(S^1) \times_MT^*M) \) are \( F \)-conjugate with respect to \( i \) if and only if their images in \( \Lambda' \) correspond, via \( \mu \), to the right and left ends of the same interval. The result follows. \( \square \)

**8.5. Generic tangent point - global study**

In this section we find some \( F \)-linked points in order to be able to apply Proposition \( 8.3.5 \) in the proof of Theorem \( 8.6.1 \). We first describe the geometric situation. We assume now that \( M = S^2 \) is the sphere. We keep the notations and assumptions of \( 8.2.2 \). \( U \subset M \) is an open subset diffeomorphic to \( S^1 \times \mathbb{J} \), with \( \mathbb{J} = ]-1,1[ \), and \( \Lambda \subset \hat{T}^*M \) is a smooth closed conic Lagrangian submanifold satisfying \( (8.2.6) \). We will also use the notations \( \Gamma, \Omega, \Lambda_0 \) and \( p^\pm \in \Lambda_0 \) in \( 8.2.3 \) and \( 8.2.6 \).

Since \( M \) is the sphere, the boundary \( S^1 \times \{-1\} \) of \( U \) bounds a disc \( D \). For \( t \in [-1,1] \) we set

\[
(8.5.1) \quad M_t = D \cup (S^1 \times [-1, t[).
\]

**Proposition 8.5.1.** Let \( F \in D^b_M(k_M) \) be a simple sheaf. We assume that \( F|_U \) is constructible (see Section \( 1.2.3 \)) and satisfies \( (8.2.8) \). Then there exists \( p \in \Lambda \cap T^*M_0 \) such that \( p, p^- \) and \( p^+ \) are \( F \)-linked over \( M_t \), for any \( t \in [0,1] \).
Proof. (i) By (8.2.5) and (8.2.6) the group \( \text{Hom}(F|_{M_t}, F|_{M_t}) \) is independent of \( t \in [0, 1] \) and we can assume \( t = 1 \). By Proposition 8.4.5 (applied with \( t_0 = t_1 = 0 \) and \( J = \{0\} \)) we know that \( p^- \) and \( p^+ \) are \( F \)-linked over \( M_1 \). Hence it is enough to find \( p \in \Lambda \cap T^*M_0 \) which is \( F \)-linked with \( p^+ \) over \( M_1 \). We argue by contradiction and assume that there exists no such \( p \). Since \( \Lambda/\mathbb{R}_{>0} \) is compact and \( \Lambda \cap T^*\mathcal{U} \) has the form given in (8.2.6), we see that \( \Lambda \cap T^*M_1 \) is a finite union of connected components, say

\[
\Lambda \cap T^*M_1 = \bigcup_{i=0}^{\mathcal{N}} \Lambda_i,
\]

where \( \Lambda_0 \) is already defined in (8.2.5). The components of \( \Lambda \cap T^*M_0 \) are then \( \Lambda_i \cap T^*M_0 \), \( i = 1, \ldots, \mathcal{N} \).

Before we go on we discuss the idea of the proof since it is rather long. The statement implies in particular that \( \mathcal{N} \geq 1 \). We could indeed see now that \( \mathcal{N} \geq 1 \) as follows: if \( \mathcal{N} = 0 \), then \( F|_{M_1} \) is a simple sheaf on the disc \( M_1 \) with \( \mathcal{S}(F|_{M_1}) = \mathcal{L} \). Using Example 1.2.10 we could describe \( F \) and see that it cannot satisfy the decomposition hypothesis (8.2.8). The idea is to try to reduce to this situation and obtain a contradiction. We define \( u : F|_{M_1} \to F|_{M_1} \) satisfying (8.5.2) below. The cone of \( u \), say \( C(u) \), is such that \( \mathcal{S}(C(u)) = \mathcal{L} \) but a priori \( C(u) \) does not satisfy (8.2.8). We modify \( u \) in (ii) and (iii) below to obtain another morphism, say \( u^1 \), \((u^1 = u + v \text{ in } (8.5.6))\) such that \( u^1 \) is an isomorphism on \( M_0 \), and not only a “microlocal isomorphism” along \( \Lambda \cap T^*M_0 \) as is the case for \( u \). (For this we modify \( u \) by a morphism \( v \) which factorizes through a constant sheaf.) Now the cone of \( u^1 \) still does not satisfy (8.2.8) but it is zero on \( M_0 \) and we easily obtain a contradiction in (iv), using (8.2.8) for \( F \) (in fact the case \( \mathcal{N} = 0 \) was the motivation for introducing \( u \), \( u^1 \) and their cones, but the proof is finally not a reduction to \( \mathcal{N} = 0 \)).

Now we resume the course of the proof and we build \( u \). We have \( p^+ \in \Lambda_0 \) and we choose \( p_i \in \Lambda_i \cap T^*M_0 \) for each \( i = 1, \ldots, \mathcal{N} \). By Definition 8.3.1 there exists \( u_i : F|_{M_i} \to F|_{M_i} \) such that \((u_i)^{\mu}_{p_i} \neq (u_i)^{\mu}_{p^+} \), for each \( i = 1, \ldots, \mathcal{N} \). Adding a multiple of \( \text{id}_F \) and rescaling we can even assume \((u_i)^{\mu}_{p_i} = 1 \) and \((u_i)^{\mu}_{p^+} = 0 \).

Since \( k \) is infinite, we can find \( \overline{\mathfrak{a}} = (a_i)_{i=1, \ldots, \mathcal{N}} \in k^\mathcal{N} \) outside the union of hyperplanes \( \bigcup_{i=1, \ldots, \mathcal{N}} P_i \), where \( P_i = \{ \overline{\mathfrak{a}} : \sum_{j=1, \ldots, \mathcal{N}} a_j (u_j)^{\mu}_{p_i} = 0 \} \) (we remark that \( P_i \neq k^\mathcal{N} \) because the coefficient of \( a_i \) is 1). Then \( u = \sum_{j=1, \ldots, \mathcal{N}} a_j u_j : F|_{M_1} \to F|_{M_1} \) satisfies

\[
(8.5.2) \quad u^{\mu}_{p^+} = 0 \text{ and } u^{\mu}_{p_i} \neq 0 \text{ for all } i = 1, \ldots, \mathcal{N}.
\]
(ii) Since $F|_{\mathcal{U}}$ is constructible the algebra $\text{Hom}(F|_{\mathcal{U}}, F|_{\mathcal{U}})$ is finite dimensional and there exists a non zero polynomial $P \in k[X]$ such that $P(u|_{\mathcal{U}}) = 0$. Let us write $P(X) = Q(X) \cdot X^k$, where $Q(0) \neq 0$. Then $R(X) = Q(X) + X$ is prime with $P(X)$ and it follows that $R(u|_{\mathcal{U}})$ is an isomorphism (we can find $A, B \in k[X]$ with $AR + BP = 1$ and we obtain $A(u)R(u) = \text{id}_F$). It follows that $R(u)$ is an isomorphism (on $M_1$). Indeed, let $F'$ be the cone of $R(u)$. By the triangular inequality for the microsupport we have $\text{SS}(F') \subseteq \Lambda \cap T^*M_1$. Since $R(u|_{\mathcal{U}})$ is an isomorphism we also have $\text{SS}(F') \cap T^*\mathcal{U} = \emptyset$. But a microsupport cannot be a proper subset of a smooth connected Lagrangian submanifold by Corollary 1.3.9. Since all components of $\Lambda \cap T^*M_1$ meet $T^*\mathcal{U}$ we deduce $\text{SS}(F') = \emptyset$. Hence $F'$ is locally constant on $M_1$. Since it vanishes on $\mathcal{U}$, it vanishes everywhere on $M_1$. This means that $R(u)$ is an isomorphism on $M_1$.

(iii) We claim that there exists $v: F|_{M_1} \to F|_{M_1}$ such that

$$v|_{M_0} = Q(u)|_{M_0} \text{ and } v^\mu_p = 0 \text{ for any } p \in \Lambda \cap T^*M_1.$$  

To construct $v$ we first define $G \in D^b(k_{M_1})$ and $u'$ by the distinguished triangle

$$G \xrightarrow{w'} F|_{M_1} \xrightarrow{u}$$(8.5.3)\xrightarrow{1} F|_{M_1} \to G[1].$$

By the triangular inequality for the microsupport we have $\text{SS}(G) \subseteq \Lambda \cap T^*M_1$. By (8.5.2) and Remark 8.3.2 (b) $\text{SS}(G)$ avoids the components $\Lambda_i$ for $i = 1, \ldots, N$. Hence $\text{SS}(G) \subseteq \Lambda_0$. In particular $\text{SS}(G|_{M_0})$ is empty and $G|_{M_0}$ is locally constant, hence constant since $M_0$ is a disc. Let us write $G|_{M_0} = K_{M_0} = a_{M_0}^{-1}(K)$ for some $K \in D^b(k)$ where $a_{M_0}: M_0 \to \{\text{pt}\}$ is the projection.

Since $u^k \circ Q(u) = 0$ there exists, $w': F \to G$ such that $Q(u) = u' \circ w'$. Now both morphisms $u'|_{M_0}: K_{M_0} \to F|_{M_0}$ and $w'|_{M_0}: F|_{M_0} \to K_{M_0}$ extend to $M_1$ as $u'': K_{M_1} \to F|_{M_1}$ and $w'': F|_{M_1} \to K_{M_1}$. To see this we use $K_{M_0} = a_{M_0}^{-1}(K)$, $K_{M_0} \simeq a_{M_0}^{-1}(K)[-2]$ and the adjunctions $(a_{M_0}^{-1}, \Gamma(M_0; -)$ and $(\Gamma_c(M_0; -), a_{M_0}^{-1})$, which yield

$$\text{Hom}(K_{M_0}, F|_{M_0}) \simeq \text{Hom}(K, \Gamma(M_0; F)),$$

(8.5.4)

(8.5.5)\text{Hom}(F|_{M_0}, K_{M_0}) \simeq \text{Hom}(\Gamma_c(M_0; F), K[-2]).$$

We have similar isomorphisms with $M_0$ replaced by $M_1$. We extend the projection $q: \mathcal{U} \to \mathcal{Y}$ as $q: M_1 \to \mathbb{R}$ by setting $q(D) = -1$. Using the hypothesis (8.2.2) and $\Gamma(M_1; F) \simeq \Gamma(\mathbb{R}; q_*F)$, we have $\Gamma(M_1; F) \simeq \Gamma(M_0; F) \oplus k$. Hence the image of $u'$ by (8.5.4) can be extended to $M_1$ and thus $u'$ also can be extended to $M_1$. In the same way $\Gamma_c(M_1; F) \simeq \Gamma_c(M_0; F) \oplus k$ and $u'$ can be extended to $M_1$. 

We deduce that using Proposition 1.2.9 as in the proof of Lemma 8.2.2.

Then \( \dot{S} S(\cdot) \) \( k \) \( v \)

(iv) We define \( G' \in \mathbb{D}^b(k_{M_1}) \) by the distinguished triangle

\[
G' \to F|_{M_1} \xrightarrow{u+v} F|_{M_1} \to G'[1].
\]

Then \( SS(G') \cap T^*M_1 \subset \Lambda \cap T^*M_1 \). Since \( (u+v)|_{M_0} = (u+Q(u))|_{M_0} = (R(u))|_{M_0} \) is an isomorphism by (ii), we have \( G' \vert_{M_0} \simeq 0 \). In particular \( SS(G') \cap T^*M_0 = \emptyset \) and we obtain that \( SS(G') \cap T^*M_1 \subset \Lambda_0 \). Then \( G' \) is locally constant on \( M_1 \setminus \overline{\Omega} \), hence zero since it is zero on \( M_0 \). Using Proposition 8.2.9 as in the part (iii) of the proof of Lemma 8.2.2 we deduce that \( G' \) is a direct sum of sheaves of the type \( k_{\Omega_2}, k_{\Omega} \) and \( k_{\Omega_0} \) with some shifts in degree. It follows easily that either \( G' \simeq 0 \) or \( supp(Rq_\ast(G')) = [0,1] \).

On the other hand, applying \( Rq_\ast \) to the triangle \((8.5.6)\), we obtain the distinguished triangle \( Rq_\ast(G')|_J \to (k_{\{0\}} \oplus B_1) \xrightarrow{w} (k_{\{1\}} \oplus B_1) \xrightarrow{+1} \). Whatever the morphism \( w \) this implies \( supp(Rq_\ast(G')) \cap J = J, \{0\} \) or \( \emptyset \). This prevents \( supp(Rq_\ast(G')) = [0,1] \) and proves that \( G' \simeq 0 \). Hence \( u+v \) is an isomorphism. But we also have \( (u+v)_p^\mu = u_p^\mu + v_p^\mu = 0 \) for any \( p \in \Lambda_0 \). This gives a contradiction and concludes the proof.

8.6. FRONT WITH ONE CUSP

In this section we keep the hypotheses of Section 8.3 and add genericity hypotheses on \( \Sigma = \pi_M(\Lambda) \). So \( M = \mathbb{S}^2 \) is the sphere, \( \Lambda \subset \widetilde{T}^*M \) is a smooth closed conic Lagrangian submanifold and there exists a cylinder \( U = S^1 \times \mathbb{J} \) in \( M \) such that \( \Lambda \) satisfies \((8.2.6)\). We assume moreover that \( \Sigma \) has exactly one simple cusp \( c \) and that \( \Sigma \cap (M \setminus \{c\}) \) is an immersed curve, with transverse double points. Since \( \Lambda \) is stable by the antipodal map, \( \Lambda \cap T^*_c M \) consists of two opposite half lines and \( \Lambda \setminus (\Lambda \cap T^*_c M) \) has two connected components. The antipodal map has no fixed point and it follows that it exchanges the two connected components of \( \Lambda \setminus (\Lambda \cap T^*_c M) \).

We consider \( F \in \mathbb{D}^b_{[\Lambda]}(k_M) \) satisfying \((8.2.7)\) and \((8.2.8)\). Since \( F \) is simple along \( \Lambda \), it has a shift \( s(p) \in \frac{1}{2} \mathbb{Z} \) at any \( p \in \Lambda \) (see \((1.4.4)\)). As recalled in Example 1.4.5, the shift is locally constant outside the cusps and changes by 1 when \( p \) crosses a cusp. Hence \( s(p) \) takes two distinct values over \( \Lambda \setminus (\Lambda \cap T_c^* M) \), one for each connected component. We recall that we have distinguished two points \( p^\pm \) of \( \Lambda \) (see notations \((8.2.5)\)) and we have \( p^{+a} = p^- \). Hence \( F \) has different shifts at the points \( p^+ \)
and \( p^- \). We denote by \( \Lambda^\pm \) the connected component of \( \Lambda \setminus (\Lambda \cap T^*_c M) \) containing \( p^\pm \).

By Example 1.4.5 the sheaf \( k|_{-\infty,0] \) on \( \mathbb{R} \) has shift \(-\frac{1}{2}\) at \((0;1)\) and the sheaf \( k|_{-\infty,0] \) has shift \( \frac{1}{2} \) at \((0;-1)\). In particular the constant sheaf on a closed or open interval has the same shifts at both ends. The constant sheaf on a half-closed interval has different shifts at both ends.

\[
\text{Figure 8.6.1.}
\]

**Theorem 8.6.1.** Let \( M \) be the sphere and let \( \Lambda \subset \dot{T}^* M \) be a closed conic Lagrangian submanifold such that \( \Lambda \) satisfies (8.2.6). We assume that \( \Sigma = \dot{\pi}_M(\Lambda) \) is a curve with only cusps and ordinary double points as singularities. We assume that \( \Sigma \) has exactly one cusp. Let \( F \in D^b_{(\Lambda)}(k_M) \) be a simple sheaf. We assume that \( F|_{U} \) is constructible and satisfies (8.2.7) and (8.2.8). Then \( H^1(R\text{Hom}(F,F)) \neq 0 \).

**Proof.** The notations introduced in (i) are illustrated in Fig. 8.6.1, where the curve is \( \Sigma \), the dotted path is \( \text{im}(\pi_M \circ \gamma) \) and \( \varepsilon = + \).

(i) We let \( i \) be the inclusion of \( S^1 = C_{1/2} \) in \( M \). We use the notation \( M_t \) of (8.5.1). By Proposition 8.5.1 there exists \( p_0 = (z_0; \xi_0) \in \Lambda \cap T^*_c M_0 \) such that \( p^+, p^- \) and \( p_0 \) are \( F \)-linked over \( M_{1/2} \). The point \( p_0 \) is in \( \Lambda^+ \) or \( \Lambda^- \). We define \( \varepsilon = + \) or \( \varepsilon = - \) so that \( p_0 \in \Lambda^\varepsilon \).

We denote by \( z_l = (-1/\sqrt{2}, 1/2) \), \( z_r = (1/\sqrt{2}, 1/2) \) the intersections of \( \Gamma_0 \) and \( C_{1/2} \). We also denote by \( p^l_0, p^r_0 \) the points of \( \Lambda_0 \) (well-defined up to a positive multiple) above \( z_l, z_r \) in the same connected component as \( p^\varepsilon_0 \).

Let \( \gamma: [0,1] \to \Lambda \) be an embedding such that \( \gamma(0) = p^\varepsilon_0 \), \( \gamma(1) = p_0 \), \( \gamma([0,1]) \subset \Lambda^\varepsilon \) and \( \pi_M \circ \gamma \) is an immersion (hence \( \pi_M \circ \gamma \) describes the
portion of the curve $\Sigma$ between $(0, 0)$ and $z_0$ which does not contain the cusp. Then the image of $\gamma$ meets either the half-line $\mathbb{R}^+ \cdot p_l^e$ or the half-line $\mathbb{R}^+ \cdot p_r^e$. We assume it meets $\mathbb{R}^+ \cdot p_l^e$, the other case being similar. (In Fig. 8.6.2 the circle represents $\Lambda/\mathbb{R}_{>0}$ and the positions of the points correspond to Fig. 8.6.1; the points $c', c''$ are the preimages of the cusp $c$; we have $\varepsilon = +$, the image of the path $\gamma$ is the arc $(p^+ p_0)$, which contains $p_l^e$.)

By Proposition 8.4.6 there exists a unique $q_l \in \Lambda$ (up to a positive scalar) which is $F$-conjugate to $p_l^e$ with respect to the embedding $i$ of $C_{1/2}$ in the sense of Definition 8.3.4. This means that there exists an interval $I_a$ in the decomposition (8.2.9) of $F|_{C_{1/2}}$ such that $SS(e_*(k_{I_a})) = i d^{-1}_\pi(R^+ \cdot p_l^e \sqcup R^+ \cdot q_l)$. Let $x$ be the right end of $I_a$. Then $e(x) = z_l$. If $I_a$ is closed near $x$, then $I_a$ is the same as $I_b$ introduced in (8.2.11).

(ii) If the interval $I_a$ is closed, then the non vanishing of $H^1 \text{RHom}(F, F)$ follows from Propositions 8.2.3 and 8.2.5.

(iii) We assume that $I_a$ is open. Then $D_{C_{1/2}}(e_*(k_{I_a})) \simeq e_*(k_{\overline{T_a}})$. Since $D_M(F) \simeq F$, the interval $\overline{T_a}$, maybe translated by a multiple of $2\pi$, also appears in the decomposition of $F|_{C_{1/2}}$. Hence we can still apply Propositions 8.2.3 and 8.2.5.

(iv) If the interval $I_a$ is half-closed, then $F$ has different shifts at the points $p_l^e$ and $q_l$. Hence $q_l$ belongs to $\Lambda^{-\varepsilon}$ and the path $\gamma$ does not meet $\mathbb{R}^+ \cdot q_l$. This means that the pairs $\{[p_l^e], [q_l]\}$ and $\{[p_l^e], [p_0]\}$ are intertwined in $\Lambda/\mathbb{R}_{>0}$ (see Fig. 8.6.2). The result follows from Proposition 8.3.5 (applied with the embedding $i$ of $C_{1/2}$ in $M$ and the path $\gamma$).

\[\text{Figure 8.6.2.}\]
8.7. Proof of the three cusps conjecture

We first make a general remark about the sheaf $K_\Phi$ associated with a homogeneous Hamiltonian isotopy. Lemma \[8.7.1\] is a general statement and $M$ is any manifold (we come back to $M = S^2$ for the three cusps conjecture), $I$ is an open interval containing 0 and $\Phi : T^*M \times I \to T^*M$ is a Hamiltonian isotopy such that $\Phi_0 = \text{id}$. We assume that $\Phi_s(x; \lambda \xi) = \lambda \cdot \Phi_s(x; \xi)$ for all $\lambda \in \mathbb{R}^\times$ (not only $\lambda \in \mathbb{R}_{>0}$) and all $(x; \xi) \in T^*M$, where we set as usual $\Phi_s(\cdot) = \Phi(\cdot, s)$. Theorem \[2.1.1\] associates with $\Phi$ a sheaf $K_\Phi \in D^b(k_{M^2 \times \mathbb{R}})$ and the next lemma says that it is self-dual in the sense $K_\Phi \simeq R\mathcal{H}om(K_\Phi, \omega_M \boxtimes k_{M \times \mathbb{R}})$. Let us first recall some fact about the dualizing sheaf. For a manifold $M$ we set $\omega_M = a^!_M(k)$, where $a_M : M \to \{\text{pt}\}$ is the projection. We have $\omega_M \simeq or_M(d_M)$, where $or_M$ is the orientation sheaf and $d_M$ the dimension of $M$. In our relative case $\omega_M \boxtimes k_{M \times \mathbb{R}} \simeq p^!(k_{M \times \mathbb{R}})$, where $p : M^2 \times I \to M \times I$ is $(x, y) \mapsto (y, s)$.

Lemma \[8.7.1\]. Let $K_\Phi \in D^b(k_{M^2 \times \mathbb{R}})$ be the sheaf associated with $\Phi$ by Theorem \[2.1.1\]. Then $K_\Phi \simeq R\mathcal{H}om(K_\Phi, \omega_M \boxtimes k_{M \times \mathbb{R}})$. Moreover, for any $F \in D(k_M)$ and any $s \in I$, we have $D_M(K_{\Phi, s} \circ F) \simeq K_{\Phi, s} \circ D_M(F)$.

Proof. (i) We use the uniqueness part of Theorem \[2.1.1\]. We set $K' = R\mathcal{H}om(K_\Phi, \omega_M \boxtimes k_{M \times \mathbb{R}})$. Since $\omega_M$ is locally constant, Theorem \[1.2.13\] gives $SS(K') \simeq (SS(K_\Phi))^a = \Lambda^\circ_\Phi$, where $\Lambda_\Phi$ is the graph of $\Phi$, as defined in \[2.1.2\]. The hypothesis on $\Phi$ implies $\Lambda^\circ_\Phi = \Lambda_\Phi$, hence $SS(K') = \Lambda_\Phi$.

In particular the inclusion $i : M^2 \times \{0\} \to M^2 \times I$ is non-characteristic for $SS(K')$ and Theorem \[1.2.8\] gives $i^!(K') \simeq i^{-1}(K') \otimes \omega_i$, where $\omega_i = i^!(k_{M^2 \times \mathbb{R}}) \simeq k_{M^2 [-1]}$. By adjunction we have

\[
i^!(K') \simeq R\mathcal{H}om(i^{-1}K_\Phi, i^!(\omega_M \boxtimes k_{M \times \mathbb{R}})) \\
\simeq R\mathcal{H}om(k_{\Delta M}, \omega_M \boxtimes k_{M [-1]}).
\]

Now $R\mathcal{H}om(k_{\Delta M}, -) \simeq R\delta_\delta^!(-)$, where $\delta$ is the inclusion of $\Delta_M$. We also have $\omega_M \boxtimes k_M \simeq p^!_2(k_M)$, where $p_2 : M^2 \to M$ is the second projection. Since $p_2 \circ \delta = \text{id}_M$ we obtain $R\mathcal{H}om(k_{\Delta M}, \omega_M \boxtimes k_M) \simeq k_{\Delta M}$. Hence $i^{-1}(K') \simeq k_{\Delta M}$. Summing up we have $SS(K') = \Lambda_\Phi$ and $K'_{\mid M^2 \times \{0\}} \simeq k_{\Delta M}$. The uniqueness property of $K_\Phi$ gives $K' \simeq K_\Phi$.

(ii) The proof of the second statement is the same as (i). We define two sheaves $F', F''$ on $M \times I$ by $F' = K_\Phi \circ D_M(F)$ and $F'' = R\mathcal{H}om(K_\Phi \circ F, \omega_M \boxtimes k_I)$. Then $F'_{\mid M \times \{0\}} \simeq F''_{\mid M \times \{0\}}$ and $F', F''$ have the same microsupport, which is $\Lambda_\Phi \circ SS(F)$ in the notation \[2.1.5\]. Then Corollary \[2.1.3\] gives $F' \simeq F''$. In particular $F'_{\mid M \times \{s\}} \simeq F''_{\mid M \times \{s\}}$, which is the required isomorphism. \qed
Lemma 8.7.2. Let $G \in D^b(k_\mathbb{A})$ be a constructible sheaf. We assume that $G$ has compact support, that there exists an isomorphism $G \simeq D_\mathbb{R}(G)$ and that $R\Gamma(\mathbb{R}; G) \simeq k$. Then there exist $t_0 \in \mathbb{R}$ and a decomposition $G \simeq k_{(t_0)} \oplus \bigoplus_{a \in A} k_{t_a}^{na}[d_a]$ where the $I_a$ are half-closed intervals.

Proof. By Corollary 4.4.3 there exists a decomposition of $G$ as a finite sum $G \cong \bigoplus_{a \in A} k_{I_a}^{na}[d_a]$. Then $R\Gamma(\mathbb{R}; G) \simeq \bigoplus_{a \in A} R\Gamma(\mathbb{R}; k_{t_a})^{na}[d_a]$. Since $R\Gamma(\mathbb{R}; G) \cong k$ all the intervals $I_a$ but one, say $I_b$, have cohomology zero, which means that they are half-closed. For $\alpha < \beta$ and $I = [\alpha, \beta]$, $\lambda = [\alpha, \beta]$, we set $I^* = J, J^* = I$. We have $D_\mathbb{R}(k_{t_b}[d]) \simeq k_{I^*}[1 - d]$. If the remaining interval $I_b$ is open or is closed with non empty interior, then $I_b$ also appears in the family $I_a, a \in A$, because $D_\mathbb{R}(G) \simeq G$. Then $k_{I_b}$ also contributes to $R\Gamma(\mathbb{R}; G)$ and it would imply that $\bigoplus_{k \in \mathbb{Z}} H^k(\mathbb{R}; G)$ has dimension at least 2. Hence $I_b$ is reduced to one point, say $t_0$. Using $D_\mathbb{R}(k_{(t_0)}[d]) \simeq k_{(t_0)}[-d]$ we deduce the lemma. \qed

Now we can prove the three cusps conjecture. We let $PT^*\mathbb{S}^2 = T^*\mathbb{S}^2/\mathbb{R}_\times$ be the projectivized cotangent bundle of $\mathbb{S}^2$. Let $\Pi$ be an open interval containing 0 and let $\tilde{\Phi}: PT^*\mathbb{S}^2 \times \Pi \to PT^*\mathbb{S}^2$ be a contact isotopy of $PT^*\mathbb{S}^2$. It lifts to a homogeneous Hamiltonian isotopy $\Phi: \tilde{T}^*\mathbb{S}^2 \times \Pi \to T^*\mathbb{S}^2$ which satisfies $\Phi_0 = \text{id}$ and $\Phi_\lambda(x; \lambda \xi) = \lambda \cdot \Phi_\lambda(x; \xi)$ for all $\lambda \in \mathbb{R}_\times$ (not only for $\lambda \in \mathbb{R}_{>0}$) and all $(x; \xi) \in \tilde{T}^*\mathbb{S}^2$. Let $x_0 \in \mathbb{S}^2$ be a given point. We set $\Lambda_0 = \tilde{T}^*_{x_0} \mathbb{S}^2, \Lambda_s = \Phi_\lambda(x_0)$ and $\Sigma_s = \pi_{\mathbb{S}^2}(\Lambda_s)$.

Theorem 8.7.3. Let $s \in \Pi$ be such that $\Sigma_s$ is a curve with only cusps and double points as singularities. Then $\Sigma_s$ has at least three cusps.

Proof. (i) We remark that the number of cusps of $\Sigma_s$ is odd because $\Lambda_s$ is connected. Hence we have to prove that $\Sigma_s$ does not have one cusp.

(ii) Let $K = K_\Phi \in D^b(k_{(2)^2 \times 1})$ be the sheaf associated with $\Phi$ by Theorem 2.1.1. For $s \in \Pi$ we set $K_s = K|_{(2)^2 \times \{s\}} \in D^b(k_{(2)^2})$. We also set $F_0 = k_{\{x_0\}}$ and $F_s = K_s \circ F_0$. Then $SS(F_s) = \Lambda_s$. We have $D_\mathbb{S}(F_0) \simeq F_0$ and by Lemma 8.7.1 we deduce that $D_\mathbb{S}(F_s) \simeq F_s$. By Corollary 2.1.5 we also have $R\Gamma(\mathbb{S}^2; F_s) \simeq R\Gamma(\mathbb{S}^2; F_0) \simeq k$ for all $s \in \Pi$.

(iii) Now we consider a given $s \in \Pi$ so that $\Sigma_s$ satisfies the hypotheses of the theorem. We choose a Morse function $q: \mathbb{S}^2 \to \mathbb{R}$ with only one minimum $x_-$ and one maximum $x_+$ such that $x_-, x_+ \not\in \Sigma_s$. We can choose $q$ such that any fiber $q^{-1}(t), t \in \mathbb{R}$, contains at most one of the following special points: tangent point between $q^{-1}(t)$ and $\Sigma_s$, double point or cusp (in particular a cusp is not tangent to $q^{-1}(t)$).
We set $G = Rq_ s(F s)$. We have isomorphisms $D_R(G) \simeq Rq_ s(DG) \simeq G$ and $R\Gamma(S^2; F s) \simeq k$. By Lemma 8.7.2 there exist $t_0 \in \mathbb{R}$ and a decomposition $G \simeq k_{(t_0)} \oplus \bigoplus_{a \in A} k_{t_0}^{\alpha_a} [d_a]$ where the $I_a$ are half-closed intervals. We set $t_\pm = q(x_\pm)$. Since $F_s$ is constant near $x_\pm$, we have $G \simeq L \otimes (k_{[t_-] \cup \infty} \oplus k_{[t_-, \infty]})$ near $t_\pm$, where $L = (F_s)_{x_\pm}$. The same holds near $t_\pm$. Hence $t_0 \neq t_\pm$ and $t_0 \neq t_+$.

Since $k_{(t_0)}$ is a direct summand of $G$, we have $T^*_t \mathbb{R} \subset SS(G)$. By Proposition 1.2.20 this implies $\Lambda_1 \cap T^*_q \mathbb{R} \neq \emptyset$. By the assumption on $q$ it follows that $q^{-1}(t_0)$ is tangent to $\Sigma_1$ at a single point and that $q^{-1}(t_0)$ contains no cusp or double point. By Lemma 8.4.3 the intervals $I_0$ cannot have an end at $t_0$.

Up to a change of coordinates we can assume that $t_0 = 0$ and that $\Lambda \cap T^* U$, with $U = q^{-1}(\{0\}] \cup \{1\})$, satisfies (8.2.6) and $F_s$ satisfies (8.2.7) and (8.2.8). In particular $F_s|_U$ is weakly constructible (see Proposition 1.2.20 and Remark 1.2.22). Moreover $F = K \circ F_0 \in \mathcal{D}^b(k_{S^2 \times 1})$ has finite dimensional stalks at any point of $(S^2 \times \mathbb{I}) \setminus \pi_{S^2 \times 1}(SS(F))$ by Lemma 1.4.7. Hence $F_s|_U$ is constructible. By Theorem 8.6.1 we deduce $\text{Hom}(F_s, F_s[1]) \neq 0$. By Corollary 2.1.3 we have

$$\text{Hom}(F_s, F_s[1]) \simeq \text{Hom}(F_0, F_0[1]) \simeq 0$$

and this gives a contradiction. $\square$

### 8.8. The Four Cusps Conjecture

In this short section we sketch a proof of Arnol’d’s four cusps conjecture along the same lines as the proof of Theorem 8.7.3. It is proved by Chekanov and Pushkar in [11] and says the following. Let $M = \mathbb{R}^2$ and $f: M \to \mathbb{R}$, $x \mapsto ||x||$. Let $S^1 = f^{-1}(1)$ be the unit circle and let $\Lambda^\pm = \{(x; \lambda df); x \in S^1, \pm \lambda > 0\}$ be the outer and inner conormal bundles of $S^1$. Let $\Phi: \dot{T}^* M \times \mathbb{I} \to \dot{T}^* M$ be a homogeneous Hamiltonian isotopy with $0, 1 \in \mathbb{I}$. We set $\Lambda_s = \Phi_s(\Lambda^+)$ for $s \in \mathbb{I}$ and we assume that $\Lambda_1 = \Lambda^+$. We make the same genericity hypotheses as in [11] (C):

- For $s \in \mathbb{I}$ outside a finite set $\{s_1, \ldots, s_N\}$, $\Sigma_s = \pi_M(\Lambda_s)$ is a curve with only cusps and ordinary double points as singularities,
- For $s \in \{s_1, \ldots, s_N\}$, $\Sigma_s$ may have a birth/death of swallowtail, a triple point, a self-tangency or a double point where one of the points is a cusp.

Then there exists $s \in \mathbb{I}$ such that $\Sigma_s$ has at least four cusps.

Here is a sketch of proof.

(i) We let $F_0 = k_{\{f \leq 1\}}$ be the constant sheaf on the closed unit ball. Hence $SS(F_0) = \Lambda_0$. We let $\Lambda' \subset \dot{T}^* (M \times \mathbb{I})$ be the image of $\Lambda_0$ by the...
whole isotopy (see (2.1.5)). It is a conic Lagrangian submanifold which is non-characteristic for all inclusions \( i_s : M \times \{s\} \to M \times \mathbb{I} \) (that is, \( \Lambda' \cap (T^*_M M \times T^*_\mathbb{I}) = \emptyset \)) and satisfies \( i^*_s(\Lambda') = \Lambda_s \). By Corollary 2.1.5 there exists a unique \( F \in \mathcal{D}(k_{M \times \mathbb{I}}) \) such that \( \text{SS}(F) = \Lambda' \) and \( i^*_0 F \simeq F_0 \). We set \( F_s = i^*_s F \). We know that \( F \) is simple along \( \Lambda' \), \( F_s \) has compact support, \( \text{SS}(F_s) = \Lambda_s \) and \( \Gamma(M; F_s) \) is independent of \( s \).

(ii) We compute \( F_1 \). We have \( \hat{\text{SS}}(F_1) = \Lambda^+ \), hence \( F_1 \) is locally constant on \( M \setminus S^1 \). Since \( F_1 \) has compact support, it must be supported on the unit ball. The microsupport condition then implies that it is of the form \( F_1 \simeq E_{\{f < 1\}} \) for some \( E \in \mathcal{D}(k) \). Since \( F_1 \) is simple, we have \( E = k[d] \) for some \( d \in \mathbb{Z} \). Finally the condition \( \Gamma(M; F_1) \simeq \Gamma(M; F_0) \simeq k \) gives \( d = 2 \) and we have \( F_1 \simeq k_{\{f < 1\}}[2] \).

(iii) We choose a projection \( q : M \to \mathbb{R} \) which is generic with respect to the family of curves \( \Sigma_s \) (in particular, for all but finitely many \( s \), \( \Sigma_s \) has no more than one tangent point to a fiber of \( q \) and, for the remaining values of \( s \), \( \Sigma_s \) has no more than two tangent points to a fiber of \( q \)). We set \( G = R(q \times \text{id})_*(F) \in \mathcal{D}(k_{\mathbb{R} \times \mathbb{I}}) \), \( G_s = G|_{\mathbb{R} \times \{s\}} \simeq Rq_s(F_s) \in \mathcal{D}(k_{\mathbb{R}}) \). We have \( \Gamma_{\mathbb{R}}(\mathbb{R}; G_s) \simeq \Gamma(M; F_s) \simeq k \) for all \( s \in \mathbb{I} \). We can decompose \( G_s \) as a sum of constant sheaves on intervals, with degree shifts, using Corollary 4.4.3. Since \( \Gamma_{\mathbb{R}}(\mathbb{R}; G_s) \simeq k \) all these intervals are half-closed but one, say \( I(s) \). Then either \( I(s) \) is closed and \( G_s \) has \( k_{I(s)} \) as a direct summand (this is the case for \( s \) close to 0), or \( I(s) \) is open and \( G_s \) has \( k_{I(s)}[1] \) as a direct summand (this is the case for \( s \) close to 1).

(iv) We prove that there exists \( s_0 \in \mathbb{I} \) such that \( I(s_0) \) is reduced to a point. For this we argue as in Part 5. We set \( G'_s = H^0 G_s \). We have a natural morphism \( H^0(\mathbb{R} \times \mathbb{I}; G) \to H^0(\mathbb{R} \times \mathbb{I}; \tau_{\geq 0} G) \simeq H^0(\mathbb{R} \times \mathbb{I}; G') \). Hence the element \( \sigma \in H^0(\mathbb{R} \times \mathbb{I}; G) \simeq k \) corresponding to 1 induces a section \( \sigma' \) of \( G' \). We consider its support, say \( Z = \text{supp}(\sigma') \). Then \( Z \) is a closed set and coincides with the set of \((z, s) \in \mathbb{R} \times \mathbb{I} \) such that \( \sigma'_{(z, s)} \neq 0 \). We set \( G'_s = G'|_{\mathbb{R} \times \{s\}} \simeq H^0 G_s \) and we let \( \sigma'(s) \in H^0(\mathbb{R}; G'_s) \) be the section of \( G'_s \) induced by \( \sigma' \). We have \( \text{supp}(\sigma'(s)) = Z \cap (\mathbb{R} \times \{s\}) \). By Corollary 4.4.3 \( G'_s \) is the direct sum of the summands of \( G_s \) which are in degree 0. A sheaf \( k_J J \) some interval, has a global section if and only if \( J \) is closed; in this case the support of the section is \( J \). Hence we deduce from (iii) that, if \( \sigma'(s) \neq 0 \), then \( I(s) \) is closed and \( \text{supp}(\sigma'(s)) = I(s) \). We define \( \mathbb{I}' \subset \mathbb{I} \) by \( \mathbb{I}' = \{s; \sigma'(s) \neq 0\} \). Then \( \mathbb{I}' \) is closed and \( Z = \bigsqcup_{s \in \mathbb{I}'} (I(s) \times \{s\}) \). We define \( s_0 \) as the upper bound of \( \mathbb{I}' \cap [0, 1] \). We remark that \( s_0 < 1 \) because \( I(s) \) is open for \( s \) close to 1.

On the other hand \( \text{SS}(G) \) is bounded by \( \Xi = (q \times \text{id})_*(q \times \text{id})^{-1}_1(\Lambda') \). We set \( C = \hat{n}_{\mathbb{R} \times 1}(\Xi) \) (see Fig. 8.8.1). If \( q \) is generic with respect to \( \Lambda' \),
then $C$ is a (singular) curve of $\mathbb{R} \times I$, with a finite set, say $C_{\text{sing}}$, of singular points (cusps or multiple points). We set $C_{\text{reg}} = C \setminus C_{\text{sing}}$, which is a finite union of disjoint smooth arcs. If $C'$ is a connected component of $C_{\text{reg}}$, then $\Xi$ must contain one of the two connected components of $T^*_C(\mathbb{R} \times I)$. Using the fact that $\Lambda'$ is non-characteristic for all inclusions $i_s$, we can see that $\Xi \cap (T^*_\mathbb{R} \times T^*I) = \emptyset$. It follows that $C'$ is not tangent to any line $\mathbb{R} \times \{s\}$, $s \in I$.

Now $G$, hence $H^0G$, is weakly constructible with respect to the stratification induced by $C$ (see Section 1.2.3). The strata are the components of $(\mathbb{R} \times I) \setminus C$, the components of $C_{\text{reg}}$ and the points of $C_{\text{sing}}$. The set $Z$, which is the support of a section of $H^0G$, is a union of strata. Hence, if $I(s_0) \times \{s_0\}$ meets an open stratum, then $Z \cap (\mathbb{R} \times \{s_0 + \varepsilon\})$ is not empty for small $\varepsilon > 0$, which contradicts the definition of $s_0$. Hence $I(s_0) \times \{s_0\}$ is contained in $C$. If $I(s_0)$ is not reduced to a point, then $C_{\text{reg}} \cap (I(s_0) \times \{s_0\})$ is non trivial and $C_{\text{reg}}$ is tangent to $\mathbb{R} \times \{s_0\}$, which is impossible as already remarked. Hence $I(s_0)$ is a point.

![Figure 8.8.1. The curve C and the support Z of σ'](image)

(v) The situation at time $s_0$ is similar to the setting of §8.2.2. In particular, assuming $I(s_0) = \{0\}$, we have $(Rq_*(F_{s_0}))|_J \simeq k_{(0)} \oplus B_3$, for some interval $J$ around 0 and some $B \in D^b(k)$ (like (8.2.8)). We can see also that $\Lambda_{s_0}$ meets $T^*_{q^{-1}(0)}M$ along two half lines, generated by, say, $p^\pm$.

These two points belong to the conormal bundles of two branches of the intersection of $\Sigma_{s_0}$ with some neighborhood of $q^{-1}(0)$, say $\Gamma^\pm$, tangent to $q^{-1}(0)$ (unlike the case of the three cusps conjecture $\Lambda_{s_0}$ is not equal to $\Lambda^a_{s_0}$ and the map $\Lambda^a_{s_0}/\mathbb{R}_{>0} \to M$ is generically one to one). The fact that $M$ is $\mathbb{R}^2$ and not $\mathbb{S}^2$ makes the situation easier. In particular an argument as in the step (ii) in the proof of Proposition 8.2.3 gives the following: let $p^+_0$, $p^+_1$ be the two points in $\Lambda_{s_0} \cap T^*_\Gamma M \cap \pi^{-1}q^{-1}(\pm \varepsilon)$,
for $\varepsilon > 0$ small enough (we write $\pm \varepsilon$ because we do not know which side of $q^{-1}(0)$ the branch $\Gamma^+$ is situated). Let $q_0^+$, $q_1^+$ be the conjugate points of $p_0^+$, $p_1^+$ with respect to $q^{-1}(\pm \varepsilon)$. Then $F_{\Sigma}$ has different shifts at $p_i^+$ and $q_i^+$. (See Fig. 8.8.2 where we give two examples of possible $\Sigma_{\Sigma_{\Sigma}}$ and some notations – we do not claim that there actually exist sheaves corresponding to these pictures.)

We have similar pairs $\{p_i^-, q_i^\}$. Using these four pairs of conjugate points we can now prove directly an analog of Theorem 8.6.1 (there is no need for a third linked point as in Proposition 8.5.1) as follows. Let us assume that $\Sigma_{\Sigma}$ has only two cusps, say $c_1$, $c_2$. Then $\Lambda_{\Sigma} \setminus (\Lambda_{\Sigma} \cap (T_{c_1}^* M \cup T_{c_2}^* M))$ consists of two connected components, corresponding to two different shifts, say $\Lambda^0$, $\Lambda^1$. We assume that $p^+ \in \Lambda^0$. By Proposition 8.4.5 (with $t_0 = t_1 = 0$ and $J = \{0\}$) we know that $p^-$ and $p^+$ are $F$-linked over any open subset containing $q^{-1}(0)$. We distinguish
three cases, (a), (b-i), (b-ii) (see Fig. 8.8.3 – the first picture in Fig. 8.8.2 is compatible with (b-i) and the second one with (b-ii)):

(a) If $p^- \in \Lambda^0$, the arc of $\Lambda^0$ from $p^+$ to $p^-$ contains $p_0^+$ or $p_1^+$. We choose the notations so that it contains $p_0^+$ (see Fig. 8.8.3 (a)). Then $q_0^+ \in \Lambda^1$ and we see that the pairs $\{p^+, [p^-]\}$ and $\{[p_0^+], [q_0^+]\}$ in $\Lambda_{s_0}/\mathbb{R}_{>0}$ are intertwined. By Proposition 8.3.5, we deduce $H^1\text{RHom}(F_{s_0}, F_{s_0}) \neq 0$, contradicting $H^1\text{RHom}(F_0, F_0) = 0$.

(b) If $p^- \in \Lambda^1$, we choose the notations so that $p_0^+$, $p_0^-$ and $c_1$ belong to the same arc of $\Lambda_{s_0}/\mathbb{R}_{>0}$ joining $p^+$ and $p^-$. We remark that the open arc $\{p_0^+, p_1^-\}$ is contained in $q^{-1}(]-\varepsilon, \varepsilon[)$. Since $q(q_0^+) = \pm \varepsilon$, $q_0^+$ belongs to the arc $\{c_1, p_0^-\}$ or to the arc $\{c_2, p_1^-\}$. It could happen that $q_0^+ = p_0^-$, but this implies that $\Gamma^+$ and $\Gamma^-$ are on the same side of $q^{-1}(0)$ and that $\pi_M(p_1^+) < \pi_M(p_0^+) < \pi_M(p_0^-) < \pi_M(p_1^-)$ for the order on the line $q^{-1}(\varepsilon)$; hence we cannot have both $q_0^+ = p_0^-$ and $q_0^- = p_1^-$. Up to switching 0 and 1, we can assume $q_0^+ \neq p_0^-$. In other words $p_0^+$ and $p_0^-$ are not conjugate, and we also have $q_0^- \neq q_0^+$. Similarly as $q_0^+$, $q_0^-$ belongs to the arc $\{c_1, p_0^-\}$ or to the arc $\{c_2, p_1^+\}$.

(b-i) If $q_0^+$ belongs to $\{c_2, p_1^-\}$ (see Fig. 8.8.3 (b-i)), then the pairs $\{[p^+], [p^-]\}$ and $\{[p_0^+], [q_0^+]\}$ are intertwined. If $q_0^-$ belongs to $\{c_2, p_1^+\}$, then the pairs $\{[p^+], [p^-]\}$ and $\{[p_0^-], [q_0^-]\}$ are intertwined. In both cases we conclude as in case (a).

(b-ii) If none of these two cases occurs, then $q_0^+$ belongs to $\{c_1, p_0^-\}$, $q_0^-$ belongs to $\{c_1, p_0^+\}$ and the pairs $\{[p_0^+], [q_0^+]\}$ and $\{[p_0^-], [q_0^-]\}$ are intertwined (see Fig. 8.8.3 (b-ii)). Since $F$-conjugate points are $F$-linked (see Proposition 8.4.4), we can again apply Proposition 8.3.5 after moving the pairs so that they are not on the same line (choose $p_0^-$ on a line $q^{-1}(\varepsilon')$ for $\varepsilon'$ slightly different from $\varepsilon$).

Figure 8.8.3.
Part 9. Triangulated orbit categories for sheaves

In the study of Lagrangian exact submanifolds of cotangent bundles in Part 13 we will use triangulated orbit categories. For a triangulated category $\mathcal{T}$, the triangulated envelope of the orbit category of $\mathcal{T}$ is a triangulated category $\mathcal{T}'$ with a functor $\iota: \mathcal{T} \to \mathcal{T}'$ such that $\iota(F) \simeq \iota(F)[1]$ for any $F \in \mathcal{T}$ and $\text{Hom}_{\mathcal{T}'}(\iota(F), \iota(G)) \simeq \bigoplus_{i \in \mathbb{Z}} \text{Hom}_{\mathcal{T}}(F, G[i])$. Such categories are constructed by Keller in \cite{34}. We specify his construction in the case of categories of sheaves and check that we can define a microsupport in this framework.

9.1. Definition of triangulated orbit categories

We will use a very special case of the triangulated hull of an orbit category as described by Keller in \cite{34}. More precisely Definition 9.1.1 below is inspired by §7 of \cite{34} that we apply to the simple case where we quotient $\mathcal{D}^b(kM)$ by the autoequivalence $F \mapsto F[1]$ (in \cite{34} much more general equivalences are considered). However we apply this construction for sheaves instead of modules over an algebra. We use the name triangulated orbit category for the category $\mathcal{D}^b/_{[1]}(kM)$ introduced in Definition 9.1.1 by analogy with the categories introduced by Keller. However we only show that we have a functor $\iota_M: \mathcal{D}^b(kM) \to \mathcal{D}^b/_{[1]}(kM)$ such that $\iota_M(F) \simeq \iota_M(F)[1]$ and which satisfies Corollary 9.1.9 below (the proof of this result is also inspired by \cite{34}). In this section we assume $k = \mathbb{Z}/2\mathbb{Z}$.

Quick reminder on localization. We recall quickly some facts about quotients of triangulated categories. We follow the exposition of \cite{31, §1.6} or \cite{32, §10.2}. Let $\mathcal{D}$ be a triangulated category. A multiplicative system $\mathcal{S}$ in $\mathcal{D}$ is a family of morphisms such that

(S1) any isomorphism is in $\mathcal{S}$,
(S2) $\mathcal{S}$ is stable by composition of morphisms,
(S3) for given $f: X \to Y$ and $s: X \to X'$, with $s \in \mathcal{S}$, there exist $t: Y \to Y'$ and $g: X' \to Y'$, with $t \in \mathcal{S}$, such that $g \circ s = t \circ f$, and the same holds with all arrows reversed (visualized by $X \xrightarrow{f} Y$ and $X \xleftarrow{f} Y$)
(S4) For $f,g: X \Rightarrow Y$, there exists $s: W \to X$ in $\mathcal{S}$ such that $f \circ s = g \circ s$ if and only if there exists $t: Y \to Z$ in $\mathcal{S}$ such that $t \circ f = t \circ g$. 


The localization of \( D \) by \( S \) is then the category \( D_S \) with the same objects as \( D \) and with morphisms \( \text{Hom}_{D_S}(X,Y) = \{(X',s,f); s: X' \to X, f: X' \to Y \text{ with } s \in S, f \text{ any morphism}\}/\sim \), where the equivalence relation \( \sim \) is generated by: \((X',s,f) \sim (X'',t,g)\) if there exists a commutative diagram

\[
\begin{array}{ccc}
X & \xleftarrow{s} & X' \\
\downarrow{t} & & \downarrow{f} \\
X'' & \xrightarrow{g} & Y
\end{array}
\]

defined using the property (S3). The localization comes with a functor \( Q_S: D \to D_S \) such that, for any \( s \in S \), \( Q_S(s) \) is an isomorphism. Moreover the pair \((D_S, Q_S)\) is universal with respect to this property. (Up to now we did not use the triangulated structure.) We now define a distinguished triangle in \( D_S \) as a triangle \( A \to B \to C \to A[1] \) which is isomorphic in \( D_S \) to the image by \( Q_S \) of a distinguished triangle of \( D \). This turns \( D_S \) into a triangulated category.

In the framework of triangulated categories, a convenient way to obtain a multiplicative system is to start with a full triangulated subcategory \( \mathcal{N} \) of \( D \) which is saturated in the following sense: for any isomorphism \( X \cong Y \) in \( D \), if \( X \in \mathcal{N} \), then \( Y \in \mathcal{N} \). We then define \( S_{\mathcal{N}} \) as the family of morphisms \( s \) in \( D \) such that the cone of \( s \) belongs to \( \mathcal{N} \). We can check that \( S_{\mathcal{N}} \) is a multiplicative system and consider the localization \( D_{S_{\mathcal{N}}} \). Then, for any \( N \in \mathcal{N} \), \( Q_{S_{\mathcal{N}}}(N) \cong 0 \); moreover \((D_{S_{\mathcal{N}}}, Q_{S_{\mathcal{N}}})\) is universal with respect to this property. We call \( D_{S_{\mathcal{N}}} \) the quotient of \( D \) by \( \mathcal{N} \) and denote it by \( D/\mathcal{N} \).

The localization is used to define the derived category. If \( \mathcal{C} \) is an abelian category, we let \( \mathcal{C}(\mathcal{C}) \) be the category of complexes of objects of \( \mathcal{C} \) and \( \mathcal{C}^b(\mathcal{C}) \) its full subcategory of bounded complexes. We define \( \mathcal{K}(\mathcal{C}) \) as the category with the same objects as \( \mathcal{C}(\mathcal{C}) \) and morphisms \( \text{Hom}_{\mathcal{K}(\mathcal{C})}(X,Y) = \text{Hom}_{\mathcal{C}(\mathcal{C})}(X,Y)/\sim \) where \( \sim \) is the homotopy equivalence of morphisms. We define \( \mathcal{K}^b(\mathcal{C}) \) as the full subcategory of \( \mathcal{K}(\mathcal{C}) \) of bounded complexes. These categories \( \mathcal{K}(\mathcal{C}), \mathcal{K}^b(\mathcal{C}) \) are triangulated. We recall that \( u: X \to Y \) is a quasi-isomorphism if \( H^n(u): H^n(X) \to H^n(Y) \) is an isomorphism for all \( n \in \mathbb{Z} \). Then \( \mathcal{D}(\mathcal{C}) = (\mathcal{K}(\mathcal{C}))_{\text{qis}} \) and \( \mathcal{D}^b(\mathcal{C}) = (\mathcal{K}^b(\mathcal{C}))_{\text{qis}}, \) where \( \text{qis} \) denotes the family of quasi-isomorphisms. We have a natural functor \( \mathcal{D}^b(\mathcal{C}) \to \mathcal{D}(\mathcal{C}) \) which identifies \( \mathcal{D}^b(\mathcal{C}) \) as a full subcategory of \( \mathcal{D}(\mathcal{C}) \). It is classical that \( \mathcal{D}^b(\mathcal{C}) \) is also equivalent to the full subcategory of \( \mathcal{D}(\mathcal{C}) \):

\[
(9.1.1) \quad \{X \in \mathcal{D}(\mathcal{C}); H^n(X) \cong 0 \text{ for } n \ll 0 \text{ and } n \gg 0\}.
\]

We use similar constructions to define the derived categories of complexes bounded from below, \( \mathcal{D}^+(\mathcal{C}) \), or above, \( \mathcal{D}^-(\mathcal{C}) \).
If $C'$ is another abelian category, any additive functor $F: C \to C'$ induces a functor, $F_1: K(C) \to K(C')$, compatible with the triangulated structures.

If $F$ is exact, then $F_1$ sends quasi-isomorphisms to quasi-isomorphisms and $Q \circ F_1$ sends quasi-isomorphisms to isomorphisms, where $Q: K(C') \to D(C')$ is the localization functor. By the universal property, $Q \circ F_1$ factorizes through a functor

$$F: D(C) \to D(C')$$

(compatible with the triangulated structures). In the same way $F$ induces functors, all denoted $F$, $D^+(C) \to D^+(C')$, where $*$ stands for $b$, $+$ or $-$. In general, if $F$ is only additive, $Q \circ F_1$ does not factorizes through $D(C)$ but, under some assumptions (for example, $C$ has enough injectives), we can define a “best possible approximation” to a factorization, which is called the right derived functor of $F$,

$$R_F: D^+(C) \to D^+(C').$$

If $F$ is left exact, we have $F \simeq H^0 \circ RF \circ \iota$, where $\iota$ is the embedding of $C$ as the full subcategory of $D^+(C)$ of complexes concentrated in degree 0 (if $F$ is not left exact, $F$ and $RF$ are not related in a useful way). If $F$ is exact, then $F \simeq RF$.

If $C$ has enough projectives, we can also derive functors on the left. However we are interested in categories of sheaves, which do not have enough projectives. For a specific functor $F$, we can still derive $F$ on the left, if $C$ has enough $F$-projective objects. We are interested in the case of the tensor product of sheaves; the $\otimes$-projective sheaves are the flat sheaves and there are enough flat sheaves. So we can define $\otimes$ on the bounded from above derived categories of sheaves:

$$\otimes^L_{k_M}: D^-(k_M) \times D^-(k_M) \to D^-(k_M),$$

where $M$ is any manifold and $k$ any ring.

**Definition of the orbit category.** We recall that $k = \mathbb{Z}/2\mathbb{Z}$ and we set $\mathbb{K} = k[X]/(X^2)$. We let $\varepsilon$ be the image of $X$ in $\mathbb{K}$. Hence $\mathbb{K} = k[\varepsilon]$ with $\varepsilon^2 = 0$. Let $M$ be a manifold. We denote as usual by $\text{Mod}(k_M)$ (resp. $\text{Mod}(\mathbb{K}_M)$) the category of sheaves of $k$-modules (resp. $\mathbb{K}$-modules) on $M$. We set $D^b(k_M) = D^b(\text{Mod}(k_M))$ and $D^b(\mathbb{K}_M) = D^b(\text{Mod}(\mathbb{K}_M))$, in the sense of (9.1.1).

The natural ring morphisms $k \to \mathbb{K}$ and $\mathbb{K} \to k$ induce two pairs of adjoint functors $(e_M, r_M)$ and $(E_M, R_M)$, where $e_M, E_M$ are scalar
extensions and \( r_M, R_M \) restrictions of scalars (or rather, their derived versions, see the discussion around (9.1.2)-(9.1.4) for a quick reminder):

\[
\begin{align*}
\mathcal{D}^b(k_M) & \xrightarrow{e_M} \mathcal{D}^b(k), \quad e_M(F) = k_M \otimes_{k_M} F, \quad r_M(G) = G, \\
\mathcal{D}^b(k_M) & \xleftarrow{R_M} \mathcal{D}^b(k_M), \quad R_M(F) = F, \\
\mathcal{D}^+(k_M) & \xrightarrow{E_M} \mathcal{D}^+(k_M), \quad E_M(G) = k_M \otimes_{k_M} G, \quad R_M(F) = F.
\end{align*}
\]

We will sometimes use the fact that \( r_M \) is conservative, that is, for a morphism \( u : F \to G \) in \( \mathcal{D}^b(k_M) \), if \( r_M(u) \) is an isomorphism, so is \( u \). Indeed \( r_M \) is the derived functor of an exact functor and the assertion follows from the case of the module categories \( \text{Mod}(k_M), \text{Mod}(k) \), where it is clear.

The ring morphism \( \mathbb{K} \to k \) gives a basis of \( \text{Hom}_k(\mathbb{K}, k) \) (which is a free \( \mathbb{K} \)-module). It induces \( \mathbb{K}_M \xrightarrow{\sim} \mathcal{RHom}_{k_M}(\mathbb{K}_M, k_M) \) and then a canonical isomorphism, for all \( F \in \mathcal{D}^b(k_M) \),

\[
(9.1.5) \quad k_M \otimes_{k_M} F \simeq \mathcal{RHom}_{k_M}(\mathbb{K}_M, F)
\]

and we deduce the isomorphisms:

\[
\begin{align*}
\text{Hom}_{\mathcal{D}^b(k_M)}(G, e_M(F)) & \simeq \text{Hom}_{\mathcal{D}^b(k_M)}(G, \mathcal{RHom}_{k_M}(\mathbb{K}_M, F)) \\
& \simeq \text{Hom}_{\mathcal{D}^b(k_M)}(G \otimes_{k_M} \mathbb{K}_M, F) \\
& \simeq \text{Hom}_{\mathcal{D}^b(k)}(r_M(G), F).
\end{align*}
\]

**Definition 9.1.1.** We let \( \mathcal{E}(\mathbb{K}_M) \) be the full triangulated subcategory of \( \mathcal{D}^b(\mathbb{K}_M) \) generated by the image of \( e_M \), that is, by the objects of the form \( \mathbb{K}_M \otimes_{k_M} F \) with \( F \in \mathcal{D}^b(k_M) \).

We denote by \( \mathcal{D}/[1](k_M) \) the quotient category \( \mathcal{D}^b(k_M)/\mathcal{E}(\mathbb{K}_M) \). We let \( Q_M : \mathcal{D}^b(k_M) \to \mathcal{D}/[1](k_M) \) be the quotient functor and we set \( r_M = Q_M \circ R_M : \mathcal{D}^b(k_M) \to \mathcal{D}/[1](k_M) \).

An object of \( \mathcal{E}(\mathbb{K}_M) \) is obtained by taking iterated cones of objects in \( e_M(\mathcal{D}^b(k_M)) \). There are objects in \( \mathcal{E}(\mathbb{K}_M) \) which are not of the form \( e_M(F) \). For example, when \( M \) is a point, the objects of \( \mathcal{D}^b(k) \) are split (sum of their cohomology) and so are the objects of \( e_{[\pi]}(\mathcal{D}^b(k)) \), but the object \( L^{p,q} \in \mathcal{E}(\mathbb{K}) \) defined in (9.1.17) below is not split.

Let \( \mathcal{E}'(\mathbb{K}_M) \) be the subcategory of \( \mathcal{D}^b(\mathbb{K}_M) \) formed by the \( P \) such that \( Q_M(P) \simeq 0 \). Then \( \mathcal{E}(\mathbb{K}_M) \subset \mathcal{E}'(\mathbb{K}_M) \) and we also have \( \mathcal{D}/[1](k_M) \simeq \mathcal{D}^b(\mathbb{K}_M)/\mathcal{E}'(\mathbb{K}_M) \). We do not know whether \( \mathcal{E}'(\mathbb{K}_M) = \mathcal{E}(\mathbb{K}_M) \). A general result (see [32] Ex. 10.11) says that \( P \in \mathcal{E}'(\mathbb{K}_M) \) if and only if \( P \oplus P[1] \in \mathcal{E}(\mathbb{K}_M) \).
Notation 9.1.2. If the context is clear, we will not write the functor \(Q_M\) or \(R_M\), that is, for \(F \in \mathbb{D}^b(k_M)\) we often write \(F\) instead of \(R_M(F)\), and, for \(G \in \mathbb{D}^b(\mathbb{K}_M)\), we often write \(G\) instead of \(Q_M(G)\). In particular for a locally closed subset \(Z \subset M\), we consider \(k_Z = R_M(k_Z) \in \mathbb{D}^b(\mathbb{K}_M)\) and \(\mathbb{K}_Z = Q_M(\mathbb{K}_Z) \in \mathbb{D}^b(\mathbb{K}_M)\).

The exact sequence of \(\mathbb{K}\)-modules \(0 \to k \to \mathbb{K} \to k \to 0\) induces a morphism
\[
(9.1.7) \quad s_M: k_M \to k_M[1] \quad \text{in } \mathbb{D}^b(\mathbb{K}_M)
\]
and a distinguished triangle, for any \(F \in \mathbb{D}^b(k_M)\),

\[
(9.1.8) \quad R_M(F) \to e_M(F) \to R_M(F) \xrightarrow{s_M \otimes \text{id}_F} R_M(F)[1].
\]

We thus obtain an isomorphism \(s_M \otimes \text{id}_F: R_M(F) \xrightarrow{\sim} R_M(F)[1]\) in \(\mathbb{D}^b(\mathbb{K}_M)\), for any \(F \in \mathbb{D}^b(k_M)\). This would work for any field \(k\). In characteristic 2 we can generalize this isomorphism to any \(F \in \mathbb{D}^b(\mathbb{K}_M)\) (see Remark 9.1.4).

Internal tensor product and homomorphism. For two \(\mathbb{K}\)-modules \(E_1, E_2\) we define \(E_1 \otimes_k^\varepsilon E_2 \in \text{Mod}(\mathbb{K})\) as follows. The underlying \(k\)-vector space is \(E_1 \otimes_k E_2\) and \(\varepsilon\) acts by

\[
(9.1.9) \quad \varepsilon \cdot (x \otimes y) = (\varepsilon x) \otimes y + x \otimes (\varepsilon y).
\]

Since the characteristic is 2, we can check that \(\varepsilon^2\) acts by 0 and this defines an object of \(\text{Mod}(\mathbb{K})\) that we denote \(E_1 \otimes_k^\varepsilon E_2\). We obtain in this way a bifunctor \(\otimes_k^\varepsilon: \text{Mod}(\mathbb{K}) \times \text{Mod}(\mathbb{K}) \to \text{Mod}(\mathbb{K})\). For \(F_1, F_2 \in \text{Mod}(\mathbb{K}_M)\), we define \(F_1 \otimes_k^\varepsilon F_2 \in \text{Mod}(\mathbb{K}_M)\) as the sheaf associated with the presheaf \(U \mapsto F_1(U) \otimes_k^\varepsilon F_2(U)\). We remark that \(r_M(F_1 \otimes_k^\varepsilon F_2) \simeq r_M(F_1) \otimes_k r_M(F_2)\), where \(r_M\) is seen here as a functor \(\text{Mod}(\mathbb{K}_M) \to \text{Mod}(k_M)\), and it follows easily that \(\otimes_k^\varepsilon\) is an exact functor. Hence it induces a functor on the derived category (see the discussion around (9.1.2)), denoted in the same way:

\[
(9.1.10) \quad \otimes_k^\varepsilon: \mathbb{D}^b(\mathbb{K}_M) \times \mathbb{D}^b(\mathbb{K}_M) \to \mathbb{D}^b(\mathbb{K}_M).
\]

For any \(F, G \in \mathbb{D}^b(\mathbb{K}_M)\) we have canonical isomorphisms
\[
(9.1.11) \quad k_M \otimes_k^\varepsilon k_M \simeq F \otimes_k^\varepsilon k_M \simeq F \quad \text{in } \mathbb{D}^b(\mathbb{K}_M),
\]
\[
(9.1.12) \quad r_M(F \otimes_k^\varepsilon k_M G) \simeq r_M(F) \otimes_k r_M(G) \quad \text{in } \mathbb{D}^b(k_M).
\]

Using (9.1.10) and the exact sequence \(0 \to k \to \mathbb{K} \to k \to 0\), we obtain as in (9.1.7)-(9.1.8) a morphism \(s_M(F): F \to F[1]\), for any \(F \in \mathbb{D}^b(\mathbb{K}_M)\), and a distinguished triangle

\[
(9.1.13) \quad F \to \mathbb{K}_M \otimes_k^\varepsilon F \to F \xrightarrow{s_M(F)} F[1].
\]
Using the adjunction \((e_M, r_M)\) and \((9.1.11)\) we have the isomorphism, for any \(F \in \mathcal{D}^b(k_M)\) and \(G \in \mathcal{D}^b(\mathbb{K}_M)\),
\[
\text{Hom}_{\mathcal{D}^b(\mathbb{K}_M)}(e_M(F \otimes_{k_M} r_M(G)), e_M(F) \otimes_{k_M} G) \\
\cong \text{Hom}_{\mathcal{D}^b(k_M)}(F \otimes_{k_M} r_M(G), (r_M e_M(F)) \otimes_{k_M} r_M(G)).
\] (9.1.13)

By adjunction we have a morphism \(a_F : F \to r_M e_M(F)\). The inverse image of \(a_F \otimes \text{id}_{r_M(G)}\) by \((9.1.13)\) gives a canonical morphism, for any \(F \in \mathcal{D}^b(k_M)\) and \(G \in \mathcal{D}^b(\mathbb{K}_M)\),
\[
e_M(F \otimes_{k_M} r_M(G)) \to e_M(F) \otimes_{k_M} G.
\] (9.1.14)

Lemma 9.1.3. Let \(F \in \mathcal{D}^b(k_M)\) and \(G \in \mathcal{D}^b(\mathbb{K}_M)\). Then the morphism \((9.1.14)\) is an isomorphism. In the same way \(\mathbb{K} \otimes_{k_M} e_M(F) \cong e_M(r_M(G) \otimes_{k_M} F)\). In particular, for \(F, G \in \mathcal{D}^b(\mathbb{K}_M)\) such that \(F\) or \(G\) belongs to \(E(\mathbb{K}_M)\), we have \(F \otimes_{k_M} G \in \mathcal{E}(\mathbb{K}_M)\) and \(\otimes_{k_M} \) induces a functor

\[
\otimes_{k_M} : \mathcal{D}_{/[1]}(k_M) \times \mathcal{D}_{/[1]}(k_M) \to \mathcal{D}_{/[1]}(k_M).
\]

Proof. (i) Let us denote by \(u_{F,G}\) the morphism \((9.1.14)\). Using the distinguished triangle \(\tau_{\leq i} F \to F \to \tau_{> i} F \to +1\) and the similar one for \(G\) we can argue by induction on the length of \(F\) and \(G\) to prove that \(u_{F,G}\) is an isomorphism. Then we are reduced to the case where \(F\) and \(G\) are concentrated in degree 0. Writing \(\mathbb{K} = k \oplus e k\) we have \(e_M(F \otimes_{k_M} r_M(G)) = (F \otimes G) \oplus \varepsilon(F \otimes G)\) and \(e_M(F) \otimes_{k_M} G = (F \oplus \varepsilon F) \otimes G\). For sections \(f\) and \(g\) of \(F\) and \(G\) we have
\[
u_{F,G}(f \otimes g) = f \otimes g, \\
u_{F,G}(\varepsilon(f \otimes g)) = \varepsilon \cdot (f \otimes g) = (\varepsilon f) \otimes g + f \otimes (\varepsilon g)
\]
and we can check directly that \(u_{F,G}\) is an isomorphism with inverse
\[
u_{F,G}^{-1}((f_0 + \varepsilon f_1) \otimes g) = (f_0 \otimes g + f_1 \otimes (\varepsilon g)) + \varepsilon(f_1 \otimes g).
\]

(ii) For any \(F \in \mathcal{E}(\mathbb{K}_M)\), there exist a sequence \(F_1, F_2, \ldots, F_n = F \in \mathcal{D}^b(\mathbb{K}_M)\) and distinguished triangles \(F_i \to F_i' \to F_i'' \to +1\) in \(\mathcal{D}^b(\mathbb{K}_M)\), \(i = 1, \ldots, n\), such that \(F_i', F_i''\) belong to \(e_M(\mathcal{D}^b(k_M)) \cup \{F_1, F_2, \ldots, F_{i-1}\}\). Then (i) and an induction on \(n\) give \(F \otimes_{k_M} G \in \mathcal{E}(\mathbb{K}_M)\). \(\square\)

Remark 9.1.4. An easy case of Lemma 9.1.3 is \(F = k_M\) in \((9.1.14)\).

We obtain \(e_M r_M(G) \cong k_M \otimes_{k_M} G\), for any \(G \in \mathcal{D}^b(\mathbb{K}_M)\). Hence the distinguished triangle \((9.1.12)\) becomes
\[
F \to e_M r_M(F) \to F \xrightarrow{s_M(F)} F[1] \quad \text{for any } F \in \mathcal{D}^b(\mathbb{K}_M).
\] (9.1.15)

Applying \(Q_M\) to this triangle gives an isomorphism \(s_M(F) : F \cong F[1]\) in \(\mathcal{D}_{/[1]}(k_M)\).
We can define an adjoint $\mathcal{H}om^\varepsilon$ to $\otimes^\varepsilon$ by a similar construction. For $F_1, F_2 \in \text{Mod}(\mathbb{K}_M)$, we define $\mathcal{H}om^\varepsilon(F_1, F_2) \in \text{Mod}(\mathbb{K}_M)$ as the sheaf of $k$-vector spaces $\mathcal{H}om_k(F_1, F_2)$ with the action of $\varepsilon$ given by
$$
(\varepsilon \cdot \varphi)(x) = \varepsilon\varphi(x) + \varphi(\varepsilon x),
$$
where $\varphi$ is a section of $\mathcal{H}om_k(F_1, F_2)$ over an open set $U$ and $x$ a section of $F_1$ over a subset $V$ of $U$. Then we see that $\mathcal{H}om^\varepsilon$ is right adjoint to $\otimes^\varepsilon$, hence left exact. We check also that its derived functor $R\mathcal{H}om^\varepsilon$ is right adjoint to $\otimes^\varepsilon$ in $D^{b}(\mathbb{K}_M)$ and that, for any $F, G \in D^{b}(\mathbb{K}_M),$ $\mathsf{r}_M(R\mathcal{H}om^\varepsilon(F, G)) \simeq R\mathcal{H}om(r_M(F), r_M(G)).$

We have the similar result as Lemma 9.1.3.

**Lemma 9.1.5.** Let $F, G \in D^{b}(\mathbb{K}_M)$. We assume that $F$ or $G$ belongs to $E(\mathbb{K}_M)$. Then $R\mathcal{H}om^\varepsilon(F, G) \in E(\mathbb{K}_M)$. The induced functor
$$
R\mathcal{H}om^\varepsilon : \text{D}_{/|1|}(\mathbb{K}_M)^{\text{op}} \times \text{D}_{/|1|}(\mathbb{K}_M) \to \text{D}_{/|1|}(\mathbb{K}_M).
$$

is right adjoint to $\otimes^\varepsilon_{k_M}$.

**Morphisms in the triangulated orbit category.** We prove the formula (9.1.20) which describes the morphisms in $D_{/|1|}(k_M)$.

**Lemma 9.1.6.** Let $F, P \in D^{b}(\mathbb{K}_M)$. We assume that $P \in E(\mathbb{K}_M)$. Then $R\mathcal{H}om(P, F)$ and $R\mathcal{H}om(F, P)$ belong to $D^{b}(\mathbb{K})$.

**Proof.** Since $E(\mathbb{K}_M)$ is generated by $e_M(D^{b}(\mathbb{K}_M))$, the same argument as in (ii) of the proof of Lemma 9.1.3 implies that we can assume $P = e_M(Q)$, for some $Q \in D^{b}(\mathbb{K}_M)$. Then $R\mathcal{H}om_{k}(P, F)$ is isomorphic to $R\mathcal{H}om_{k}(Q, r_M(F))$ and is bounded. Using (9.1.6), the same proof gives that $R\mathcal{H}om(F, P)$ is bounded.

We define the following objects $L^{p,q}$ of $D^{b}(\mathbb{K})$, for any two integers $p \leq q$, by
$$
L^{p,q} = 0 \to \mathbb{K} \xrightarrow{\varepsilon} \mathbb{K} \xrightarrow{\varepsilon} \ldots \xrightarrow{\varepsilon} \mathbb{K} \to 0,
$$
where the first $\mathbb{K}$ is in degree $p$ and the last one in degree $q$. Then $L^{p,q} \in E(\mathbb{K})$ and there is a distinguished triangle in $D^{b}(\mathbb{K}),$
$$
(9.1.17) \quad L^{p,q} \to L^{p,q} \to \mathbb{K}[-p] \xrightarrow{s^{p,q}} \mathbb{K}[-p+1],
$$
with $s^{p,q} = s^{p,q}_{\{pt\}}[-q]$, where $s_{\{pt\}}$ is (9.1.7). For $F \in D^{b}(\mathbb{K}_M)$ we define $s^{p,q}_{M}(F) := s^{p,q}_{\{pt\}} \otimes \text{id}_F : F[-q] \to F[-p+1]$. We deduce the triangle, for any $F \in D^{b}(\mathbb{K}_M)$ and any $n \geq 1,$
$$
L^{1,n}_{M} \otimes_{k_M}^\varepsilon F \to F[-n] \xrightarrow{s^{1,n}_{M}(F)} F \xrightarrow{+1}. 
$$
Lemma 9.1.7. We consider a distinguished triangle \( P \to F' \to F \xrightarrow{+1} \) in \( D^b(K_M) \) and we assume that \( P \in E(K_M) \). Then there exist \( n \in \mathbb{N} \) and a morphism of triangles

\[
\begin{array}{cccccc}
L^n_{M} \otimes_{K_M} F & \longrightarrow & F[-n] & \longrightarrow & s^n_{M}(F) & \longrightarrow & F \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\longrightarrow & & P & \longrightarrow & F' & \longrightarrow & F \xrightarrow{+1} \\
\end{array}
\]

Proof. We set for short \( s_n = s^n_{M}(F) \). We consider the diagram

\[
\begin{array}{cccccc}
F[-n] & \longrightarrow & F \\
\downarrow & & \downarrow \\
P & \longrightarrow & F' & \longrightarrow & F & w \longrightarrow & P[1].
\end{array}
\]

We have \( w \circ s_n \in \text{Hom}(F[-n], P[1]) \). Since \( P \in E(K_M) \) this group vanishes for \( n \) big enough, by Lemma 9.1.6. In this case we have \( w \circ s_n = 0 \) and there exists a morphism \( a \) as in the diagram making the square commute. Then we can extend the square to a commutative diagram as in the lemma.

For \( F \in D^b(K_M) \) we have \( s^n_{M}(F) : F[-n] \to F[-n+1] \). Then \( \{F[-n], s^n_{M}(F)\}_{n \in \mathbb{N}} \) gives a projective system. For \( G \in D^b(K_M) \) we define

\[
(9.1.19) \quad \lim_{n \in \mathbb{N}} \text{Hom}_{D^b(K_M)}(F[-n], G) \to \text{Hom}_{D/|1|}(k_M)(F, G),
\]

by sending \( \varphi_n : F[-n] \to G \) to \( \varphi_n \circ (s^n_{M}(F))^{-1} \). This is well defined since \( s^n_{M}(F) \) becomes invertible in \( D/|1|_M(k_M) \) and \( s^n_{M}(F) = s_{M}^{1,n-1}(F) \circ s_{M}^{n,n}(F) \).

Proposition 9.1.8. Let \( F, G \in D^b(K_M) \). Then the inductive limit in the left hand side of \( (9.1.19) \) stabilizes and the morphism \( (9.1.19) \) is an isomorphism. More precisely, if \( H^i(F) = H^i(G) = 0 \) for all \( i \) outside an interval \([a,b]\), then

\[
(9.1.20) \quad \text{Hom}_{D^b(K_M)}(F[-n], G) \xrightarrow{\sim} \text{Hom}_{D/|1|}(k_M)(F, G),
\]

for all \( n > b - a + \dim M + 1 \).

Proof. (i) We prove that the limit stabilizes. We chose \( a \leq b \) such that \( H^i(F) = H^i(G) = 0 \) for all \( i \) outside \([a,b]\). By \( (9.1.15) \) we have the
for all $n \in \mathbb{Z}$. By adjunction we have
\[
\text{Hom}_{\mathbb{D}^b(\mathbb{K}_M)}(e_M r_M(F)[-n], G) \cong \text{Hom}_{\mathbb{D}^b(\mathbb{K}_M)}(F[-n], G)
\]
(9.1.21)
\[
\xrightarrow{s'_n} \text{Hom}_{\mathbb{D}^b(\mathbb{K}_M)}(F[-n-1], G) \xrightarrow{+1},
\]
for all $n \in \mathbb{Z}$. By adjunction we have
\[
\text{Hom}_{\mathbb{D}^b(\mathbb{K}_M)}(e_M r_M(F)[-n], G) \cong \text{Hom}_{\mathbb{D}^b(\mathbb{K}_M)}(r_M(F)[-n], r_M(G))
\]
and this is zero for $n > b - a + \dim M + 1$ (recall that the flabby dimension of a manifold $M$ is $\dim M + 1$ and that injective over a field is the same as flabby). It follows that the morphism $s'_n$ in (9.1.21) is an isomorphism for $n > b - a + \dim M + 1$.

(ii) We prove that (9.1.19) is an isomorphism. We recall that
\[
\text{Hom}_{\mathbb{D}^b(\mathbb{K}_M)}(F, G) \simeq \lim_{i: F' \to F} \text{Hom}_{\mathbb{D}^b(\mathbb{K}_M)}(F', G),
\]
where the limit runs over the morphisms $i: F' \to F$ whose cone belongs to $\mathbb{E}(\mathbb{K}_M)$ (and a morphism $u: F' \to G$ is send to $u \circ i^{-1}$ in $\mathbb{D}_{/[1]}(\mathbb{K}_M)$). By Lemma 9.1.7 we can restrict to the family of morphisms $s_{M,n}^n(F): F[-n] \to F$ for $n \in \mathbb{N}$. This gives the result.

\begin{corollary}
Let $F, G \in \mathbb{D}^b(\mathbb{K}_M)$. Let $\iota_M = Q_M \circ R_M$. We have
\[
\text{Hom}_{\mathbb{D}_{/[1]}(\mathbb{K}_M)}(\iota_M(F), \iota_M(G)) \simeq \bigoplus_{n \in \mathbb{Z}} \text{Hom}_{\mathbb{D}^b(\mathbb{K}_M)}(F[-n], G).
\]
\end{corollary}

\begin{proof}
By Proposition 9.1.8 the left hand side of the formula is isomorphic to
\[
\text{Hom}_{\mathbb{D}^b(\mathbb{K}_M)}(R_M(F)[-n_0], R_M(G)) \simeq \text{Hom}_{\mathbb{D}^b(\mathbb{K}_M)}(E_M R_M(F)[-n_0], G),
\]
for any big enough $n_0 \in \mathbb{N}$. Using the resolution of $\mathbb{k}$ as a $\mathbb{K}$-module given by $\cdots \to \mathbb{K} \xrightarrow{\xi} \mathbb{K} \xrightarrow{\xi} \mathbb{K} \to \mathbb{k} \to 0$, we see that $E_M R_M(F) \simeq \bigoplus_{i \in \mathbb{N}} F[i]$. The result follows easily.
\end{proof}

This last result says in particular that $\iota_M$ is faithful. We define $\iota_M^0: \text{Mod}(\mathbb{K}_M) \to \mathbb{D}_{/[1]}(\mathbb{K}_M)$ as the composition of $\iota_M$ and the embedding $\text{Mod}(\mathbb{K}_M) \to \mathbb{D}^b(\mathbb{K}_M)$ which sends a sheaf to a complex concentrated in degree 0. For $F \in \mathbb{D}_{/[1]}(\mathbb{K}_M)$ we let $h_M^0(F)$ be the sheaf associated with the presheaf $U \mapsto \text{Hom}_{\mathbb{D}_{/[1]}(\mathbb{K}_U)}(\mathbb{k}_U, F|_U)$. This defines a functor $h_M^0: \mathbb{D}_{/[1]}(\mathbb{K}_M) \to \text{Mod}(\mathbb{K}_M)$.

\begin{corollary}
For all $F \in \text{Mod}(\mathbb{K}_M)$ we have $h_M^0(\iota_M^0(F)) \simeq F$. When $M$ is a point, the functors $\iota^0$ and $h^0$ are mutually inverse equivalences of categories between $\text{Mod}(\mathbb{k})$ and $\mathbb{D}_{/[1]}(\mathbb{k})$.
\end{corollary}
Proof. (i) We have \( \text{Hom}_{\mathcal{D}/|U|}(k_U, \iota_M(F)|_U) \simeq \bigoplus_{n \in \mathbb{Z}} H^n(U; F) \) by Corollary 9.1.9. The sheaf associated with \( U \mapsto H^n(U; F) \) is \( H^n F \) and we obtain \( h^0_M(\iota_M(F)) \simeq F \) when \( F \) is in degree 0.

(ii) When \( M \) is a point, Corollary 9.1.9 says that \( \iota^0 \) is fully faithful. Let us prove that it is essentially surjective. Any \( G \in \text{Mod}(\mathbb{K}) \) can be written as an extension \( 0 \to R(F) \to G \to R(F') \to 0 \) with \( F, F' \in \text{Mod}(k) \). It follows that any object in \( \mathcal{D}/|I|((\mathbb{K})) \) is obtained from objects in \( \iota^0(\text{Mod}(k)) \) by taking iterated cones. Hence to prove that \( \iota^0 \) is essentially surjective, it is enough to see that if we have a distinguished triangle \( \iota^0(F) \xrightarrow{a} \iota^0(F') \to G \xrightarrow{+1} \) in \( \mathcal{D}/|I|((\mathbb{K})) \), with \( F, F' \in \text{Mod}(k) \), then \( G \simeq \iota^0(F'') \) for some \( F'' \in \text{Mod}(k) \). Since \( \iota^0 \) is fully faithful, we have \( u \in \text{Hom}_{\text{Mod}(k)}(F, F') \). Then \( G \simeq \text{coker}(u) \oplus \text{ker}(u)[1] \simeq \text{coker}(u) \oplus \text{ker}(u) \) and the result follows.

**Direct sums.** We recall that we denote by \( Q_M \) the quotient functor \( \mathcal{D}^b(\mathbb{K}_M) \to \mathcal{D}/|I|((\mathbb{K}_M)). \)

**Lemma 9.1.11.** Let \( I \) be a small set and \( \{F_i\}_{i \in I} \) a family in \( \mathcal{D}^b(\mathbb{K}_M) \). We assume that there exist two integers \( a \leq b \) such that \( H^k(F_i) = 0 \) for all \( k \) outside \( [a, b] \) and all \( i \in I \). Then \( \bigoplus_{i \in I} Q_M(F_i) \) exists in \( \mathcal{D}/|I|((\mathbb{K}_M)) \) and \( \bigoplus_{i \in I} Q_M(F_i) \simeq Q_M(\bigoplus_{i \in I} F_i) \).

**Proof.** By the hypothesis on the degrees the sum \( \bigoplus_{i \in I} F_i \) exists in \( \mathcal{D}^b(\mathbb{K}_M) \) and we have \( H^k(\bigoplus_{i \in I} F_i) = 0 \) for \( k \) outside \( [a, b] \). Let \( G \in \mathcal{D}^b(\mathbb{K}_M) \) and let \( a' \leq a, b' \geq b \) be such that \( H^k(G) = 0 \) for \( k \) outside \( [a', b'] \). We set \( n = b' - a' + \dim M + 2 \). If \( F = F_i \) for some \( i \in I \), or \( F = \bigoplus_{i \in I} F_i \), we have

\[
\text{Hom}_{\mathcal{D}^b(\mathbb{K}_M)}(F[-n], G) \xrightarrow{\sim} \text{Hom}_{\mathcal{D}/|I|((\mathbb{K}_M))}(Q_M(F), Q_M(G))
\]

by Proposition 9.1.8. Now the lemma follows from the universal property of the sum.

**Direct and inverse images.** Let \( f: M \to N \) be a morphism of manifolds. We have functors \( Rf_* \), \( Rf_! \), \( f^{-1} \), and \( f^! \), between \( \mathcal{D}^b(\mathbb{K}_M) \) and \( \mathcal{D}^b(\mathbb{K}_N) \). Indeed, the functors \( Rf_* \), \( Rf_! \): \( \mathcal{D}(\mathbb{K}_M) \to \mathcal{D}(\mathbb{K}_N) \) commute with the functors \( r_N: \mathcal{D}(\mathbb{K}_N) \to \mathcal{D}(\mathbb{K}_N) \) and \( r_M \). We remark that, for \( G \in \mathcal{D}(\mathbb{K}_N) \), if \( r_N(G) \in \mathcal{D}^b(\mathbb{K}_N) \), then \( G \in \mathcal{D}^b(\mathbb{K}_N) \). Hence \( Rf_* \) and \( Rf_! \) induce functors \( \mathcal{D}^b(\mathbb{K}_M) \to \mathcal{D}^b(\mathbb{K}_N) \) between the bounded categories.

The case of \( f^{-1} \) is clear since it is an exact functor. To prove the existence of \( f^! \), right adjoint to \( Rf_! \), it is enough, by factorizing through the graph embedding, to consider the cases where \( f \) is an embedding or \( f \) is a submersion. The usual formulas work in our case. If \( f \) is an embedding, then \( f^!(\cdot) = f^{-1}R\text{Hom}_e(\mathbb{K}_M, \cdot) \) is adjoint to \( Rf_! \). If \( f \) is a
submersion, then \( f^! (\cdot) = f^{-1} (\cdot) \otimes_{\kappa} \omega_{M|N} \). We also remark that \( f^{-1} \) and \( f^! \) commute with \( r_N \) and \( r_M \).

**Lemma 9.1.12.** Let \( f: M \rightarrow N \) be a morphism of manifolds. Then the functors \( Rf_* \), \( Rf_! \), \( f^{-1} \), \( f^! \), and \( f^* \), between \( \mathcal{D}^{b}(\kappa_M) \) and \( \mathcal{D}^{b}(\kappa_N) \), preserves the categories \( \mathcal{E}(\kappa_M) \) and \( \mathcal{E}(\kappa_N) \). They induce pairs of adjoint functors (that we denote in the same way) between \( \mathcal{D}_{/[I]}(\kappa_M) \) and \( \mathcal{D}_{/[I]}(\kappa_N) \).

**Proof.** We only consider the case of \( Rf_* \), the other cases being similar. For \( F \in \mathcal{D}^{b}(\kappa_M) \), we have a natural morphism \( u: \kappa_N \otimes_{\kappa_N} Rf_* F \rightarrow Rf_*(\kappa_M \otimes_{\kappa_M} F) \). Since \( r_N(\kappa_N) \approx \kappa^2_N \), we see easily that \( r_N(u) \) is an isomorphism. Since \( r_N \) is conservative, \( u \) is an isomorphism and we obtain \( Rf_*(\kappa_M \otimes_{\kappa_M} F) \in \mathcal{E}(\kappa_N) \). It follows as in (ii) of the proof of Lemma 9.1.13 that \( Rf_*(\mathcal{E}(\kappa_M)) \subset \mathcal{E}(\kappa_N) \), as required.

For \( F \in \mathcal{D}_{/[I]}(\kappa_M) \) and \( j: U \rightarrow M \) the inclusion of a locally closed subset, we use the standard notations \( F|_U = j^{-1}F \), \( F_U = R j_! j^{-1}F \), \( R\Gamma_U(F) = R j_* j^{-1}F \), \( R\Gamma(U; F) = Ra_U(F) \in \mathcal{D}_{/[I]}(\kappa) \), where \( a_U \) is \( U \rightarrow \{ pt \} \), and \( R\Gamma_{c}(U; F) = Ra_U(F) \in \mathcal{D}_{/[I]}(\kappa) \). We have the same formulas as in \( \mathcal{D}^{b}(\kappa_M) \):

\[
F_U \simeq F \otimes_{\kappa_M} k_U, \quad R\Gamma_U(F) \simeq R\text{Hom}^\varepsilon(k_U, F).
\]

We also define \( R\text{Hom}^\varepsilon(F, G) = R\Gamma(M; R\text{Hom}^\varepsilon(F, G)) \in \mathcal{D}_{/[I]}(\kappa) \). The adjunctions \( (a^{-1}_M, Ra_M) \) and \( (\otimes_{\kappa_M}, R\text{Hom}^\varepsilon) \) give

\[
\text{Hom}_{\mathcal{D}_{/[I]}(\kappa_M)}(F, G) \simeq \text{Hom}_{\mathcal{D}_{/[I]}(\kappa)}(k, R\text{Hom}^\varepsilon(F, G)) \simeq \text{Hom}_{\mathcal{D}_{/[I]}(\kappa_M)}(k_M, R\text{Hom}^\varepsilon(F, G)).
\]

Let \( N \) be a submanifold of \( M \). We recall that Sato’s microlocalization is a functor \( \mu_N : \mathcal{D}^{b}(\kappa_M) \rightarrow \mathcal{D}^{b}(\kappa_{T^*_NM}) \). It is defined by composing direct and inverse images functors and Lemma 9.1.12 implies that it induces functors, denoted in the same way:

\[
(9.1.23) \quad \mu_N : \mathcal{D}^{b}(\kappa_M) \rightarrow \mathcal{D}^{b}(\kappa_{T^*_NM}),
\]

\[
(9.1.24) \quad \mu_N : \mathcal{D}_{/[I]}(\kappa_M) \rightarrow \mathcal{D}_{/[I]}(\kappa_{T^*_NM}).
\]

**Definition 9.1.13.** Let \( q_1, q_2 : M \times M \rightarrow M \) be the projections. We identify \( T^*_{\Delta M}(M \times M) \) with \( T^*M \) through the first projection. For \( F, G \in \mathcal{D}_{/[I]}(\kappa_M) \) we define as in (1.3.1)

\[
\mu_{\text{hom}^\varepsilon}(F, G) = \mu_{\Delta M}(R\text{Hom}^\varepsilon(q_2^{-1}F, q_1^!G) \in \mathcal{D}_{/[I]}(\kappa_{T^*M}).
\]

The following result follows from the analogous one in \( \mathcal{D}^{b}(\kappa_M) \).

**Lemma 9.1.14.** Let \( \{ F_i \}_{i \in I} \) a small family in \( \mathcal{D}^{b}(\kappa_M) \) satisfying the hypotheses of Lemma 9.1.11. Let \( f : M' \rightarrow M \) and \( g: M \rightarrow M'' \) be
morphismis of manifolds and let $G \in D_{/I[1]}(k_M)$. Then we have canonical isomorphisms

$$
\begin{align*}
&Q \simeq Q \circ f^{-1} \bigoplus_{i \in I} Q_M(F_i), \\
&Rg(\bigoplus_{i \in I} Q_M(F_i)) \simeq \bigoplus_{i \in I} Rg(F_i), \\
&\bigoplus_{i \in I} Q_M(F_i) \otimes G \simeq \bigoplus_{i \in I} (Q_M(F_i) \otimes G).
\end{align*}
$$

**Lemma 9.1.15.** Let $U$ be an open subset of $M$. Let $F \in D^b(K_M)$ and $F' \in D^b(K_U)$. We assume that there exists an isomorphism $F|_U \simeq F'$ in $D_{/I[1]}(k_U)$. Then there exists $F_1 \in D^b(K_M)$ such that $F_1|_U \simeq F'$ in $D^b(K_U)$ and $F_1 \simeq F$ in $D_{/I[1]}(k_M)$.

**Proof.** We let $j: U \to M$ be the inclusion and we set $Z = M \setminus U$. Let $u: F|_U \to F'$ be an isomorphism in $D_{/I[1]}(k_U)$. By Proposition 9.1.8 there exist $n \in Z$ and a morphism $F[-n] \to F'$ in $D^b(K_M)$ which represents $u$. Defining $P$ by the distinguished triangle $F|_U[-n] \to F' \to P \map{+1}$ we have $Q_U(P) \simeq 0$. We apply $j_!$ to this triangle and get (9.1.26) below; we also consider the excision triangle (9.1.25) and the triangle (9.1.27) built on the composition $F_Z[-n-1] \to F_U[-n] \to j_!F'$:

1. (9.1.25) $F_Z[-n-1] \map{a} F_U[-n] \to F[-n] \map{+1},$
2. (9.1.26) $F_U[-n] \map{b} j_!F' \to j_!P \map{+1},$
3. (9.1.27) $F_Z[-n-1] \map{\text{seq}} j_!F' \to F_1 \map{+1}.$

Then the octahedron axiom gives the triangle $F[-n] \to F_1 \to j_!P \map{+1}$. We have $Q_M(j_!P) \simeq j_!Q_M(P) \simeq 0$, hence $F_1 \simeq F[-n] \simeq F$ in $D_{/I[1]}(k_M)$. Applying $j^{-1}$ to the triangle (9.1.27) gives $F' \simeq F|_U$, as required.

**Definition 9.1.16.** For $F \in D_{/I[1]}(k_M)$ we define supp$^{\text{orb}}(F)$ as the complement of the union of the open subsets $U \subset M$ such that $F|_U \simeq 0$.

For an open subset $U \subset M$ we have $F|_U \simeq 0$ in $D_{/I[1]}(k_U)$ if and only if $F_U \simeq 0$ in $D_{/I[1]}(k_M)$. By the Mayer-Vietoris triangle we deduce that, for a finite covering $U = \bigcup_{i=1}^n U_i$, we have $F|_U \simeq 0$ if and only if $F|_{U_i} \simeq 0$ for all $i$. For an increasing countable union $U = \bigcup_{i=1}^\infty U_i$ we have an exact sequence

$$
0 \to \bigoplus_{i=1}^\infty k_{U_i} \map{id-l} \bigoplus_{i=1}^\infty k_{U_i} \map{b} k_U \to 0
$$
in \( \text{Mod}(k_M) \), where \( t \) is the sum of the morphisms \( t_i : k_{U_i} \to k_{U_{i+1}} \)
induced by the inclusions \( U_i \subset U_{i+1} \) and \( s \) the sum of the morphisms \( s_i : k_{U_i} \to k_U \) (the exactness is easily checked in the stalks). Turning this sequence into a distinguished triangle in \( D(k_M) \), then in \( D_{/[1]}(k_M) \), and applying \( F \otimes^\xi_{k_M} \) — we obtain a similar triangle

\[
\bigoplus_{i=1}^{\infty} F_{U_i} \to \bigoplus_{i=1}^{\infty} F_{U_i} \to F_U \xrightarrow{+1} \]

and we deduce as in the finite case that \( F|_U \simeq 0 \) if \( F|_{U_i} \simeq 0 \) for all \( i \).

We obtain finally, for any \( F \in D_{/[1]}(k_M) \),

\[
F|_{M \setminus \text{supp}^\text{orb}(F)} \simeq 0 \quad \text{and} \quad F \xrightarrow{\sim} F_{\text{supp}^\text{orb}(F)}. \tag{9.1.29}
\]

### 9.2. Microsupport in the triangulated orbit categories

We define the microsupport of objects of \( D_{/[1]}(k_M) \) and check that it satisfies the same properties as the usual microsupport.

We recall that the microsupport is invariant by restriction of scalars, that is, for \( F \in D(K_M) \), we have \( \text{SS}(r_M(F)) = \text{SS}(F) \). We deduce that Theorem 1.2.13 about the microsupports of \( F \otimes G \) and \( R\text{Hom}(F,G) \), for \( F, G \in D(K_M) \), is still true if we replace \( \otimes \) and \( R\text{Hom} \) by \( \otimes^\xi_{k_M} \) and \( R\text{Hom}^\xi \), because of (9.1.11) and (9.1.16).

#### 9.2.1. Definition and first properties

We define the microsupport \( \text{SS}^\text{orb}(F) \) of an object of \( D_{/[1]}(k_M) \) from the microsupports of its representatives in \( D^b(K_M) \). We prove in Proposition 9.2.3 that, for a given \( x_0 \in M \) and \( F \in D_{/[1]}(k_M) \), we can find a representative \( F' \in D^b(K_M) \) with \( T_{x_0}^* M \cap \text{SS}(F') \) contained in an arbitrary neighborhood of \( T_{x_0}^* M \cap \text{SS}^\text{orb}(F) \).

**Definition 9.2.1.** Let \( F \in D_{/[1]}(k_M) \). We define \( \text{SS}^\text{orb}(F) \subset T^* M \) by \( \text{SS}^\text{orb}(F) = \bigcap_{F'} \text{SS}(F') \) where \( F' \) runs over the objects of \( D^b(K_M) \) such that \( F' \simeq F \) in \( D_{/[1]}(k_M) \). We set \( \text{SS}^\text{orb}(F) = \text{SS}^\text{orb}(F) \cap T^* M \).

We remark that \( \text{SS}^\text{orb}(F) \) is a closed conic subset of \( T^* M \). We deduce from Lemma 9.1.15 that \( \text{SS}^\text{orb}(F) \) is a local notion, that is, for \( U \subset M \) open, we have

\[
\text{SS}^\text{orb}(F|_U) = \text{SS}^\text{orb}(F) \cap T^* U. \tag{9.2.1}
\]

In other words \( p = (x; \xi) \notin \text{SS}^\text{orb}(F) \) if and only if there exist a neighborhood \( U \) of \( x \) and \( F' \in D^b(K_U) \) such that \( F|_U \simeq F' \) in \( D_{/[1]}(k_U) \) and \( p \notin \text{SS}(F') \). We also have \( \text{supp}^\text{orb}(F) = T^*_M M \cap \text{SS}^\text{orb}(F) \). Indeed we have \( T^*_M M \cap \text{SS}^\text{orb}(F) \subset \text{supp}^\text{orb}(F) \) by (9.2.1). Conversely, if
(x; 0) \notin \text{SS}^{\text{orb}}(F)$, then $F$ has a representative $F' \in \text{D}^b(\mathbb{K}_M)$ such that $(x; 0) \notin \text{SS}(F')$. Hence $F'$, and thus $F$, vanishes in some neighborhood of $x$.

**Lemma 9.2.2.** Let $F \in \text{D}/[[1]](k_M)$ and $F', F'' \in \text{D}^b(k_M)$ such that $Q_M(F') \simeq Q_M(F'') \simeq F$. Let $x_0 \in M$ and let $A \subset T_{x_0}^* M$ be an open contractible cone with a smooth boundary such that $(\overline{A} \setminus \{x_0\}) \cap \text{SS}(F'') = \emptyset$. Let $C \subset T_{x_0}^* M$ be a conic neighborhood of $\text{SS}(F') \cap \partial A$. Then there exists $G \in \text{D}^b(k_M)$ such that $Q_M(G) \simeq F$ and $\text{SS}(G) \cap T_{x_0}^* M \subset (\text{SS}(F') \setminus A) \cup C$.

**Proof.** Let $U$ be a chart around $x_0$ such that $T^* U \simeq U \times T^* x_0 M$ and $(U \times A) \cap \text{SS}(F'') = \emptyset$. We apply Proposition 3.3.1 with $B = \text{SS}(F') \cap T^* x_0 M$ and $B' = C$. We obtain a neighborhood $W$ of $x_0$ and a functor $R: \text{D}(k_U) \to \text{D}(k_W)$ together with a morphism of functors $(-)|_W \to R$. This functor $R$ is a composition of usual sheaf operations and, by the results of the previous section, it induces a functor that we denote by the same letter $R: \text{D}/[[1]](k_U) \to \text{D}/[[1]](k_W)$. By (ii) of Proposition 3.3.1 we have $F'|_W \sim R(F'|_U)$. Hence $F'|_W \sim R(F'|_U)$ and it follows that $Q_W(R(F'|_U)) \simeq R(Q_U(F'|_U)) \simeq F'|_W$. By (i) of Proposition 3.3.1 the sheaf $R(F'|_U)$, defined on $W$, satisfies the conclusion of the lemma. By Lemma 9.1.15 we can extend $R(F'|_U)$ into $G$, defined on $M$. \hfill \box

**Proposition 9.2.3.** Let $F \in \text{D}/[[1]](k_M)$. Let $x_0 \in M$ be given and let $B \subset T_{x_0}^* M$ be a closed conic subset such that $\text{SS}^{\text{orb}}(F) \cap B = \emptyset$. Then there exists $F' \in \text{D}^b(k_M)$ such that $Q_M(F') \simeq F$ and $\text{SS}(F') \cap B = \emptyset$.

**Proof.** (i) We first prove the following claim to reduce the problem to Lemma 9.2.2.

Let $A_1, \ldots, A_n$ be open cones and $S$ a closed cone in $T_{x_0}^* M$. Let $C$ be a conic neighborhood of $S \cap \partial \left( \bigcup_{i=1}^n A_i \right)$. Then there exist conic subsets $C_1, \ldots, C_n$ of $T_{x_0}^* M$ such that, defining inductively $S_0 = S$ and $S_i = (S_{i-1} \setminus A_i) \cup C_i$, we have: $C_i$ is a neighborhood of $S_{i-1} \cap \partial A_i$ and $S_n \subset (S \setminus \bigcup_{i=1}^n A_i) \cup C$.

We prove the claim by induction on $n$. For $n = 1$ we take $C_1 = C$ and the claim is clear. Let us assume we have proved it for $n - 1$. Since $(S \cap \partial A_1 \cap \partial \left( \bigcup_{i=2}^n A_i \right)) \subset C$ and $(S \cap \partial A_1) \subset C \cup \bigcup_{i=2}^n A_i$, we can choose a neighborhood $C_1$ of $S \cap \partial A_1$ such that $(C_1 \setminus \partial \left( \bigcup_{i=2}^n A_i \right)) \subset C$ and $C_1 \subset C \cup \bigcup_{i=2}^n A_i$. We set $S_1 = (S \setminus A_1) \cup C_1$. Then $C$ is a neighborhood of $S_1 \cap \partial \left( \bigcup_{i=2}^n A_i \right)$ and we can apply the induction hypothesis with $A_2, \ldots, A_n$, $S = S_1$ and $C$. The claim follows (using $(S_1 \setminus \bigcup_{i=2}^n A_i) \cup C \subset (S \setminus \bigcup_{i=1}^n A_i) \cup C$).
(ii) For any \( p \in B \) we can find \( F^p \in \text{D}^b(\mathbb{K}_M) \) representing \( F \) such that \( p \not\in \text{SS}(F^p) \). We can find an open convex cone \( A^p \) around \( p \) such that \((A^p \setminus \{x_0\}) \cap \text{SS}(F^p) = \emptyset\). We can choose finitely many such cones, say \( A_1, \ldots, A_n \), such that \( B \subset \bigcup_{i=1}^n A_i \) and we let \( F_1, \ldots, F_n \in \text{D}^b(\mathbb{K}_M) \) be the corresponding sheaves. We also choose an arbitrary representative \( F_0 \) of \( F \) and set \( S = \text{SS}(F_0) \). Finally we choose a neighborhood \( C \) of \( S \cap \partial(\bigcup_{i=1}^n A_i) \) such that \( C \cap B = \emptyset \).

We let \( C_1, \ldots, C_n \) be the subsets of \( T^*_x M \) given by the claim in (i).

Now we define \( G_i \in \text{D}^b(\mathbb{K}_M) \), \( i = 0, \ldots, n \) inductively: we start with \( G_0 = F_0 \), and using Lemma 9.2.2 with \( F' = G_i-1 \), \( F'' = F_i \), \( A = A_i \), \( C = C_i \), we set \( G_i = G \) (\( G \) given by the lemma). Then \( G_i \) satisfies \( \text{SS}(G_i) \subset S_i \), for each \( i \). In particular the sheaf \( G_n \) satisfies the conclusion of the proposition. \( \square \)

**Proposition 9.2.4.** Let \( F \to F' \to F'' \xrightarrow{id} \) be a distinguished triangle in \( \text{D}_{/[1]}(\mathbb{K}_M) \). Then \( \text{SS}^{\text{orb}}(F'') \subset \text{SS}^{\text{orb}}(F) \cup \text{SS}^{\text{orb}}(F') \).

*Proof.* Let \( p = (x_0; \xi_0) \in T^* M \) be given. We assume that \( p \not\in \text{SS}^{\text{orb}}(F) \cup \text{SS}^{\text{orb}}(F') \). Let \( u: F \to F' \) be the morphism of the triangle. By definition we can find \( F_0, F'_0 \in \text{D}^b(\mathbb{K}_M) \) such that \( Q_M(F_0) \simeq F \), \( Q_M(F'_0) \simeq F' \) and \( p \not\in \text{SS}(F_0), p \not\in \text{SS}(F'_0) \). By Proposition 9.1.8 there exist \( n \in \mathbb{Z} \) and a morphism \( u_0 : F_0[n] \to F'_0 \) such that \( Q_M(u_0) = u \). Then the cone of \( u_0 \), say \( F''_0 \), represents \( F'' \) and \( p \not\in \text{SS}(F''_0) \) by the triangular inequality for the usual microsupport. The result follows. \( \square \)

**9.2.2. Functorial behavior.** We prove that \( \text{SS}^{\text{orb}}(\cdot) \) satisfies the same properties as \( \text{SS}((\cdot) \) with respect to the usual sheaf operations.

**Proposition 9.2.5.** Let \( f: M \to N \) be a morphism of manifolds. Let \( G \in \text{D}_{/[1]}(\mathbb{K}_N) \). We assume that \( f \) is non-characteristic for \( \text{SS}^{\text{orb}}(G) \). Then \( \text{SS}^{\text{orb}}(f^{-1}G) \cup \text{SS}^{\text{orb}}(f'G) \subset f_d f^{-1} \text{SS}^{\text{orb}}(G) \).

*Proof.* (i) The cases of \( f^{-1} \) and \( f' \) are similar and we only consider \( f^{-1} \). We can write \( f = p \circ i \) where \( i: M \to M \times N \) is the graph embedding and \( p: M \times N \to N \) is the projection. Since the result is compatible with the composition it is enough to consider the case of an embedding and a submersion separately.

(ii) We assume that \( f \) is a submersion. Let \( x \in M \) and set \( y = f(x) \).

Then \( \text{SS}^{\text{orb}}(f^{-1}G) \cap T^*_x M = \bigcap_{F'} \text{SS}(F') \cap T^*_x M \subset \bigcap_{G'} \text{SS}(f^{-1}G') \cap T^*_x M \), where \( F' \) runs over the objects of \( \text{D}^b(\mathbb{K}_M) \) such that \( F' \simeq f^{-1}G \) in \( \text{D}_{/[1]}(\mathbb{K}_M) \) and \( G' \) over the objects of \( \text{D}^b(\mathbb{K}_N) \) such that \( G' \simeq G \) in
\( D_{/|1|}(k_N) \). Now the result follows from Theorem 1.2.8 and the fact that 
\( t' f'_x: T^*_y N \to T^*_x M \) is injective.

(iii) We assume that \( f \) is an embedding. Let \((x_0; \xi_0) \in T^*_x M\) be such that \((x_0; \xi_0) \notin f_d(\text{SS}^{\text{orb}}(G) \cap T^*_x M)\). Let \( l\) be the half line \( \mathbb{R}_{\geq 0} \cdot (x_0; \xi_0) \).

Since \( f \) is non-characteristic for \( \text{SS}^{\text{orb}}(G) \), we have \( f_d^{-1}(l) \cap \text{SS}^{\text{orb}}(G) \subset \{x_0\} \). By Proposition 9.2.3 there exist a neighborhood \( V \) of \( x_0 \) and \( G' \subset D^b(\mathbb{K}_V) \) such that \( G' \simeq G \) in \( D_{/|1|}(k_V) \) and \( f_d^{-1}(l) \cap \text{SS}(G') \subset \{x_0\} \). Then \( f^{-1}G' \simeq f^{-1}G \) in \( D_{/|1|}(k_{M|V}) \) and \((x_0; \xi_0) \notin f_d(\text{SS}(G') \cap T^*_x N)\). This proves \((x_0; \xi_0) \notin \text{SS}^{\text{orb}}(f^{-1}G)\), hence the inclusion of the proposition.

\[ \square \]

**Proposition 9.2.6.** Let \( f: M \to N \) be a morphism of manifolds. Let \( F \in D_{/|1|}(k_M) \). We assume that \( f \) is proper on \( \text{supp}^{\text{orb}}(F) \). Then \( \text{SS}^{\text{orb}}(Rf_!F) \subset f_*f_d^{-1}\text{SS}^{\text{orb}}(F) \).

**Proof.** (i) As in the proof of Proposition 9.2.5 we can reduce the problem to the cases where \( f \) is an embedding or a projection. The case of an embedding is similar to the part (ii) of the proof of Proposition 9.2.5.

(ii) We assume now that \( M = M' \times N \) and \( f \) is the projection. Let \( q = (y; \eta) \in T^*N \) be such that \( q \notin f_*f_d^{-1}\text{SS}^{\text{orb}}(F) \). Let us prove that \( q \notin \text{SS}^{\text{orb}}(Rf_!F) \). Since \( f \) is proper on \( \text{supp}^{\text{orb}}(F) \), we can assume, up to restricting to a neighborhood of \( y \), that \( \text{supp}^{\text{orb}}(F) \subset C \times N \), for some compact set \( C \subset M' \). We can then find an open convex cone \( A \) in \( T^*_y N \) containing \( \eta \) and a neighborhood \( \Omega \) of \( C \) such that, for any \( x \in \Omega \) there exists \( G \in \text{D}^b(k_M) \) representing \( F \) with \( \text{SS}(G) \cap (T^*_x M' \times A) = \emptyset \).

As in the proof of Lemma 9.2.2 we apply Proposition 3.3.1 to obtain two neighborhoods \( W \subset U \) of \( y \) in \( N \) and a functor \( R^+: \text{D}^b(\mathbb{K}_{M' \times U}) \to \text{D}^b(\mathbb{K}_{M' \times W}) \), of the form \( R^+(H) = K \circ H \), where \( K \in \text{D}^b(k_{W \times U}) \) is given by the proposition, which satisfies

(a) \( R^+ \) induces a functor on \( D_{/|1|}(k_{M' \times U}) \),

(b) for \( G \in \text{D}^b(\mathbb{K}_{M' \times U}) \) and \( x \in M' \) such that \( \text{SS}(G) \cap (T^*_x M' \times A) = \emptyset \) we have \( R^+(G) \simeq G|_W \) around \( \{x\} \times W \),

(c) for any \( G \in \text{D}^b(\mathbb{K}_{M' \times U}) \) we have \( \text{SS}(R^+(G)) \cap (T^*_x M' \times \{p\}) = \emptyset \).

Then (a) and (b) imply \( R^+(F) \simeq F|_W \). Now we choose a representative \( F' \in \text{D}^b(k_M) \) of \( F \). Then \( R^+(F') \) is another representative of \( F \) and we deduce the result from the condition (c) and Proposition 1.2.4. \[ \square \]

**Proposition 9.2.7.** Let \( F, G \in D_{/|1|}(k_M) \).

(i) We assume that \( \text{SS}^{\text{orb}}(F) \cap \text{SS}^{\text{orb}}(G) \forceq T^*_M M \). Then \( \text{SS}^{\text{orb}}(F \otimes_k G) \subset \text{SS}^{\text{orb}}(F) + \text{SS}^{\text{orb}}(G) \).
(ii) We assume that $$\text{SS}^{\text{orb}}(F) \cap \text{SS}^{\text{orb}}(G) \subset T^*_x M$$. Then
$$\text{SS}^{\text{orb}}(R\text{Hom}^\varepsilon(F, G)) \subset \text{SS}^{\text{orb}}(F)^{a} + \text{SS}^{\text{orb}}(G).$$

Proof. Let us prove (i). Let $$x_0 \in M$$ and let $$A, B \subset T^*_x M$$ be conic neighborhoods of $$\text{SS}^{\text{orb}}(F) \cap T^*_x M$$, $$\text{SS}^{\text{orb}}(G) \cap T^*_x M$$ such that $$A \cap B^a \subset \{x_0\}$$. By Proposition 9.2.3 we can find representatives $$F', G' \in D^b(K_M)$$ of $$F, G$$ such that $$\text{SS}(F') \cap T^*_x M \subset A$$, $$\text{SS}(G') \cap T^*_x M \subset B$$. Since microsupports are closed we have $$\text{SS}(F') \cap \text{SS}(G')^a \subset T^*_0 U$$ for some neighborhood $$U$$ of $$x_0$$. Then Theorem 1.2.13 gives $$\text{SS}(F' \otimes_{K_M} G') \cap T^*_x M \subset A + B$$. Since $$A$$ and $$B$$ are arbitrarily close to our microsupports we deduce (i). The proof of (ii) is the same. □

9.2.3. Microsupport in the zero section. In Proposition 9.2.10 we give a special case of Proposition 1.2.9 for $$\text{SS}^{\text{orb}}$$.

Lemma 9.2.8. Let $$C = [a, b]^d$$ be a compact cube in $$\mathbb{R}^d$$ and let $$\{U_i\}_{i \in I}$$ be a family of open subsets of $$\mathbb{R}^d$$ such that $$C \subset \bigcup_{i \in I} U_i$$. Then there exists a finite family of open subsets $$\{V_n\}_n$$, $$n = 1, \ldots, N$$, such that
(i) for each $$n = 1, \ldots, N$$ there exists $$i \in I$$ such that $$V_n \subset U_i$$,
(ii) $$C \subset \bigcup_{n=1}^N V_n$$,
(iii) $$\left(\bigcup_{k=1}^n V_k\right) \cap V_{n+1}$$ is contractible, for each $$n = 1, \ldots, N - 1$$.

Proof. For $$x \in \mathbb{R}^d$$ and $$\varepsilon > 0$$ we set $$C^\varepsilon_x = x + [-\varepsilon, \varepsilon]^d$$. We can choose $$\varepsilon > 0$$ such that, for any $$x \in C$$, there exists $$i \in I$$ satisfying $$C^\varepsilon_x \subset U_i$$ ($$\varepsilon$$ is a Lebesgue number of the covering). We let $$x_n, n = 1, \ldots, N$$, be the points of the lattice $$C \cap (\varepsilon\mathbb{Z})^d$$ ordered by the lexicographic order of their coordinates. Then the family $$V_n = C^\varepsilon_{x_n}, n = 1, \ldots, N$$, satisfies the required properties. □

Lemma 9.2.9. Let $$M$$ be a manifold and $$U, V \subset M$$ two open subsets. Let $$F \in D^b_{/U}(K_M)$$. We assume that $$U \cap V$$ is contractible and that there exist $$A, A' \in \text{Mod}(k)$$ such that $$F|_U \simeq A_U$$ and $$F|_V \simeq A'_V$$. Then $$A \simeq A'$$ and $$F|_{U \cup V} \simeq A|_{U \cup V}$$.

Proof. We set $$W = U \cap V$$ and $$X = U \cup V$$. Taking the stalks at some $$x \in W$$ gives $$A \simeq A'$$. The Mayer-Vietoris triangle yields in our case $$A_W \xrightarrow{u} A_U \oplus A_V \rightarrow F_X \xrightarrow{\partial}$$, where $$u$$ is of the form $$(1, v)$$ for some isomorphism $$v \in \text{Hom}_{D^b(K_M)}(A_W, A_W)$$. By Corollary 9.1.9 we have
$$\text{Hom}_{D^b_{/U}(K_M)}(A_V, A_V) \simeq \bigoplus_{i \in \mathbb{Z}} \text{Hom}_{D^b(K_M)}(A_V, A_V[i]) \simeq \text{Hom}(A, A) \otimes \bigoplus_{i \in \mathbb{N}} H^i(V; k).$$
In the same way Hom$_{D/|1|}(A_W, A_W) \simeq \text{Hom}(A, A)$ since $W$ is contractible. Hence $v$ can be extended as an isomorphism to $V$ and we can define the commutative square (C) below:

$$
\begin{array}{cccc}
A_W & \xrightarrow{(1,v)} & A_U \oplus A_V & \xrightarrow{+1} F_X \\
\| & & \| & \\
A_W & \xrightarrow{(1,1)} & A_U \oplus A_V & \xrightarrow{+1} A_X \\
\end{array}
$$

We extend this square to an isomorphism of triangles and obtain $F_X \simeq A_X$, as required.

**Proposition 9.2.10.** Let $E = \mathbb{R}^d$ and $F \in D_{/|1|}(k_E)$. We assume that SS$^{\text{orb}}(F) \subset T_{0}^*E$. Then there exists $A \in \text{Mod}(k)$ such that $F \simeq A_E$.

**Proof.** (i) By Proposition 9.2.3 for any $x \in E$ there exists a representative of $F$, say $F^x$, in $D^b(k_E)$ such that $SS(F^x) \cap T^*_x E \subset \{x\}$. Since microsupports are closed there exists an open neighborhood of $x$, say $U_x$, such that $SS(F^x) \cap \overline{T^*U_x} = \emptyset$, that is, $F^x|_{U_x}$ is constant. In other words, there exists $B^x \in D^b(k)$ such that $F^x|_{U_x} \simeq B^x_{U_x}$. By Corollary 9.1.10 there exists $A_x \in \text{Mod}(k)$ such that $B^x \simeq A^x$ in $D_{/|1|}(k)$ and we have $F|_{U_x} \simeq A^x_{U_x}$ in $D_{/|1|}(k_{U_x})$.

(ii) We set $I_n = ]-n, n[^d$ for $n \in \mathbb{N} \setminus \{0\}$. The family $\{U_x\}_{x \in \overline{T_n}}$ covers $\overline{T_n}$. Hence, by Lemmas 9.2.8 and 9.2.9 there exists $A \in \text{Mod}(k)$ such that $F_{I_n} \simeq A_{I_n}$ for all $n \in \mathbb{N} \setminus \{0\}$. We can assume that these isomorphisms are compatible with the morphisms $i_n: \cdot|_{I_n} \to \cdot|_{I_{n+1}}$. We obtain the commutative square (D) below between triangles deduced as in (9.1.28)

$$
\begin{array}{cccc}
\oplus_n F_{I_n} & \xrightarrow{u} & \oplus_n F_{I_n} & \xrightarrow{+1} F \\
\downarrow & & \downarrow & \\
\oplus_n A_{I_n} & \xrightarrow{u} & \oplus_n A_{I_n} & \xrightarrow{+1} A_E \\
\end{array}
$$

where the $n^{th}$-component of $u$ is $\text{id} - i_n$. We extend this square to an isomorphism of triangles and we see that $F \simeq A_E$. $\square$

We deduce a version of the Corollary 1.2.16 for the orbit category. We state a particular case on the real line (the only case we will use), but the statement of Corollary 1.2.16 follows by applying this case to $R\phi_*F$ and using the triangle $k_{[-\infty, a[} \to k_{-\infty, b[} \to k_{[a, b]} \xrightarrow{+1}$.

**Corollary 9.2.11.** Let $F \in D_{/|1|}(k_{\mathbb{R}})$ and let $a < b$ be given. We assume that SS$^{\text{orb}}(F) \cap \pi_{-1}^*(-([a, b]))$ is contained in $T^*_\tau_{\leq 0} \mathbb{R} = \{(t; \tau) \in T^*_\tau \mathbb{R}; \tau \leq 0\}$. Then $R\text{Hom}^x(k_{[a, b]}, F) \simeq 0$.
Proof. We set \( G = R_s \mathbb{R}Hom^\varepsilon( q_2^{-1} k_{[0,\infty]}, q_1^! \mathbb{R}Hom^\varepsilon(k_{[a,b]}, F)) \), where \( q_1, q_2, s : \mathbb{R}^2 \to \mathbb{R} \) are the two projections and the sum. Using the bounds in Propositions 9.2.5, 9.2.7 and 9.2.6 we obtain

\[
\text{SS}^{\text{orb}}(\mathbb{R}Hom^\varepsilon( k_{[a,b]}, F)) \subset \{(t; \tau); \ a \leq t < b, \ \tau \leq 0\} \cup \mathbb{T}_b^* \mathbb{R},
\]

\[
\text{SS}^{\text{orb}}(G) \subset \{(t; 0); \ t \neq b\} \cup \{(b; \tau); \ \tau \geq 0\}.
\]

We deduce that \( G|_{-\infty,b} \) is constant by Proposition 9.2.10. Let \( i_t : \mathbb{R} \to \mathbb{R}^2, \ t' \mapsto (t', t - t') \) be the inclusion of \( s^{-1}(t) \). Then \( q_1 \circ i_t = \text{id} \) and \( q_2 \circ i_t \) is the reflexion along \( \frac{t}{2} \). Hence Proposition 1.1.1-(h-j) gives

\[
\mathbb{R}\Gamma_{(t)} G \simeq \mathbb{R}Hom^\varepsilon( k_{-\infty,t}, \mathbb{R}Hom^\varepsilon(k_{[a,b]}, F))
\]

\[
\simeq \mathbb{R}Hom^\varepsilon( k_{-\infty,t} \otimes^\varepsilon k_{[a,b]}, F)
\]

\[
\simeq \mathbb{R}Hom^\varepsilon(k_{[a,c]}, F),
\]

where \( c = \min\{b, t\} \). In particular \( \mathbb{R}\Gamma_{(t)} G \simeq 0 \) for \( t < a \) and hence \( G|_{-\infty,b} \) vanishes. We also obtain \( \mathbb{R}\Gamma_{(b)} G \simeq \mathbb{R}Hom^\varepsilon(k_{[a,b]}, F) \).

We can represent \( G \) by \( \hat{G} \in \mathcal{D}(\mathbb{K}_\mathbb{R}) \) with \( (b; -1) \notin \text{SS}(\hat{G}) \). The definition of the microsupport implies \( (\mathbb{R}\Gamma_{-\infty,b}\hat{G})_b[-1] \simeq (\mathbb{R}\Gamma_{(b)} \hat{G})_b \). The same isomorphism holds in \( D/|1|(\mathbb{K}_\mathbb{R}) \) and we deduce \( \mathbb{R}\Gamma_{(b)} G \simeq 0 \) because \( G|_{-\infty,b} \) vanishes.

\[
\square
\]

Part 10. The Kashiwara-Schapira stack

Let \( M \) be a manifold and \( \Lambda \) a locally closed conic Lagrangian submanifold of \( \overline{T}^* M \). In this part we define the Kashiwara-Schapira stack \( \mu \text{Sh}(k_{\Lambda}) \) and its orbit category version. It is obtained by quotienting the category of sheaves on \( M \) by subcategories defined by microsupport conditions. These categories are introduced by Kashiwara-Schapira in [31]. For a conic subset \( S \) of \( T^* M \) we denote by \( D^b(k_S; S) \) the quotient of \( D^b(k_M) \) by the subcategory of sheaves \( F \) with \( \text{SS}(F) \cap S = \emptyset \). Then \( \mu \text{Sh}(k_{\Lambda}) \) is the stack on \( \Lambda \) associated with the prestack whose value over \( \Lambda_0 \subset \Lambda \) is the subcategory of \( D^b(k_M; \Lambda_0) \) generated by the \( F \) such that there exists a neighborhood \( \Omega \) of \( \Lambda_0 \) in \( T^* M \) with \( \text{SS}(F) \cap \Omega \subset \Lambda_0 \). We remark that we lose the triangulated structure in the stackification process.

An important result of [31] says that the \( \mathcal{H}om \) sheaf in \( \mu \text{Sh}(k_{\Lambda}) \) is given by \( H^b \mu \text{hom} \) (see Corollary 10.1.6). It is then possible to describe an object of \( \mu \text{Sh}(k_{\Lambda}) \) by local data (see Remark 10.1.7). We check in Lemma 10.2.26 that, locally on \( M \), there exist simple sheaves with microsupport \( \Lambda \): if \( B \) is a small enough ball in \( M \) and \( \Lambda \) is in generic position, there exist simple sheaves on \( B \) with microsupport \( \Lambda \cap T^* B \). We define two classes \( \mu^s_1(\Lambda) \in H^1(\Lambda; \mathbb{Z}) \) and \( \mu^s_2(\Lambda) \in H^2(\Lambda; k^x) \)
which are obstructions for the existence of a global object of $\mu \mathbf{Sh}(k_\Lambda)$ (that is, an object of $\mu \mathbf{Sh}(k_\Lambda)(\Lambda)$). In the last two sections we make the link between these classes and the usual Maslov class $\mu_1(\Lambda)$ and another obstruction class $\mu_2^{gf}(\Lambda)$ (an obstruction class to trivialize the Gauss map of $\Lambda$). In fact it would not be difficult to deduce $\mu_1^{sh}(\Lambda) = \mu_1(\Lambda)$ directly from Proposition 1.4.1 and the description of the Maslov class by Chech cohomology (for example in [24]). However the class $\mu_2^{sh}(\Lambda)$ requires more work; we only prove the useful implication that the vanishing of $\mu_2^{sh}(\Lambda)$ implies the vanishing of $\mu_2^{gf}(\Lambda)$.

When we work with sheaves of $k$-modules we only have the obstruction classes $\mu_1^{sh}(\Lambda)$ and $\mu_2^{sh}(\Lambda)$ for the existence of a global section of $\mu \mathbf{Sh}(k_\Lambda)$. If we work with sheaves of spectra, there are infinitely many of them. Of course this requires to work with dg-categories or infinity categories (see [49, 50] or [39]). In this framework the stack $\mu \mathbf{Sh}(k_\Lambda)$ has the structure of a stable category, like a category of sheaves (whereas the triangulated structures are not suited for stackification). This is explained in [26] where the higher classes $\mu_i^{sh}(\Lambda)$ are described (they do not coincide with the obstruction classes $\mu_i^{gf}(\Lambda)$ – see also [27]).

In [3] it is proved that these classes vanish when $\Lambda$ is a Lagrangian embedding in $T^*M$ in the homotopy class of the base.

**Notation 10.0.1.** For a conic subset $S$ of $T^*M$ we recall the categories $D^*_S(k_M)$, $D^*_{[S]}(k_M)$ and $D^*_S(k_M)$ of Notation 1.2.2. We also denote by $D(k_M; S)$ the quotient of $D(k_M)$ by $D_{T^*M \backslash S}(k_M)$ (see the reminder on localization in section 9.1).

We use similar notations for the (locally) bounded derived categories: $D^*_S(k_M)$, $D^*_{[S]}(k_M)$, $D^*_S(k_M)$ and $D^*_S(k_M)$, where $* = b$ or $* = lb$. We also define $D_{/[1], S}(k_M)$, $D_{/[1], [S]}(k_M)$, $D_{/[1], (S)}(k_M)$, $D_{/[1]}(k_M)$ and $D_{/[1]}(k_M; S)$ in the same way, replacing $D$ by $D_{/[1]}$ and $SS$ by $SS^{orb}$.

As recalled in section 9.1 the objects of $D^b(k_M; S)$ are those of $D^b(k_M)$. A morphism $u : F \to G$ in $D^b(k_M)$ is represented by a triple $(F', s, u')$ where $F' \in D^b(k_M)$ and $s, u'$ are morphisms

\[(10.0.1)\]

such that the $L$ defined (up to isomorphism) by the distinguished triangle $F' \xrightarrow{\Delta} F \to L \xrightarrow{+1}$ satisfies $S \cap SS(L) = \emptyset$. Two such triples $(F'_i, s_i, u'_i)$, $i = 1, 2$, represent the same morphism if there exists a third triple $(F'_j, s, u')$ and two morphisms $v_1 : F' \to F'_i$ such that $s = s_i \circ v_i$, $i = 1, 2$, and $u' \circ v_1 = u'_j \circ v_2$.

The notion of stack used here is that of “sheaf of categories”. We refer for example to [32, §19]. A prestack $\mathcal{C}$ on a topological space $X$
consists of the data of a category $\mathcal{C}(U)$, for each open subset $U$ of $X$, restriction functors $r_{V,U}: \mathcal{C}(U) \to \mathcal{C}(V)$, for $V \subseteq U$, and isomorphisms of functors $r_{W,V} \circ r_{V,U} \simeq r_{W,U}$, for $W \subseteq V \subseteq U$, satisfying compatibility conditions.

A stack is a prestack satisfying some gluing conditions. In particular, if $\mathcal{C}$ is a stack and $A, B \in \mathcal{C}(U)$, then the presheaf $V \mapsto \text{Hom}_{\mathcal{C}(V)}(A|_V, B|_V)$ is a sheaf on $U$. Moreover, if $U = \bigcup_{i \in I} U_i$ and $A_i \in \mathcal{C}(U_i)$ are given objects with compatible isomorphisms between their restrictions on the intersections $U_i \cap U_j$, then these objects glue into an object of $\mathcal{C}(U)$.

For any given prestack we can construct its associated stack, similar to the associated sheaf of a presheaf.

10.1. Definition of the Kashiwara-Schapira stack

We use the categories associated with a subset of $T^*M$ introduced in Notation 10.0.1.

Definition 10.1.1. Let $\Lambda \subset T^*M$ be a locally closed conic subset. We define a prestack $\mu\text{Sh}_0^0(\Lambda)$ on $\Lambda$ as follows. Over an open subset $\Lambda_0$ of $\Lambda$ the objects of $\mu\text{Sh}_0^0(\Lambda_0)$ are those of $\text{D}^b(\Lambda_0)(k_M)$. For $F, G \in \mu\text{Sh}_0^0(\Lambda_0)$ we set

$$\text{Hom}_{\mu\text{Sh}_0^0(\Lambda_0)}(F, G) := \text{Hom}_{\text{D}^b(k_M;\Lambda_0)}(F, G).$$

We define the Kashiwara-Schapira stack of $\Lambda$ as the stack associated with $\mu\text{Sh}_0^0$. We denote it by $\mu\text{Sh}(k_\Lambda)$. For $\Lambda_0 \subset \Lambda$ we usually write abusively $\mu\text{Sh}(k_{\Lambda_0})$ instead of $\mu\text{Sh}(k_\Lambda)(\Lambda_0)$.

We denote by $m_\Lambda: \text{D}^b(\Lambda)(k_M) \to \mu\text{Sh}(k_\Lambda)$ the obvious functor. However, for $F \in \text{D}^b(\Lambda)(k_M)$, we often write $F$ instead of $m_\Lambda(F)$ if there is no risk of ambiguity.

Several results in the next sections give links between $\mu\text{Sh}(k_\Lambda)$ and stacks of the following type.

Definition 10.1.2. Let $X$ be a topological space. We let $\text{DL}^0(k_X)$ be the subprestack of $U \mapsto \text{D}^b(k_U)$, $U$ open in $X$, formed by the $F \in \text{D}^b(k_U)$ with locally constant cohomologically sheaves. We let $\text{DL}(k_X)$ be the stack associated with $\text{DL}^0(k_X)$. We denote by $\text{Loc}(k_X)$ the substack of $\text{Mod}(k_X)$ formed by the locally constant sheaves.

Remark 10.1.3. We remark that $\text{DL}(k_X)$ is only a stack of additive categories (the triangulated structure is of course lost in the “stackification”). However the cohomological functors $H^i: \text{D}^b(k_U) \to \text{Mod}(k_U)$ induce functors of stacks $H^i: \text{DL}(k_X) \to \text{Loc}(k_X)$ and the natural embedding $\text{Mod}(k_U) \hookrightarrow \text{D}^b(k_U)$ induces $i: \text{Loc}(k_X) \to \text{DL}(k_X)$. We have
$H^0 \circ i \simeq \text{id}_{\text{Loc}(k_X)}$. Hence $i$ is faithful and $\text{Loc}(k_X)$ is a subcategory of $\text{DL}(k_X)$.

**Link with microlocalization.** We recall now some results of [31] which explain the link between the localized categories $\text{D}^b(k_M; S)$, for $S \subset T^*M$, and the microlocalization, making these categories easier to describe.

First we recall some properties of the functor $\mu_{\text{hom}}$. For $U \subset M$ and $F, G \in \text{D}(k_U)$, the isomorphism $R\text{Hom}(F, G) \simeq R\pi_{M*}\mu_{\text{hom}}(F, G)$ of (1.3.2) implies $\text{Hom}(F, G) \xrightarrow{\sim} H^0(T^*U; \mu_{\text{hom}}(F, G))$; for $u: F \to G$ we denote by

$$u^\mu \in H^0(T^*U; \mu_{\text{hom}}(F, G))$$

the image of $u$ by this isomorphism (this notation has been introduced in (8.3.6)). For $F, G, H \in \text{D}^b(k_M)$ we have a composition morphism

$$\mu_{\text{hom}}(F, G) \otimes \mu_{\text{hom}}(G, H) \to \mu_{\text{hom}}(F, H).$$

(With the notations of Definition 1.3.4, it is induced by a natural morphism $\mu_{\Delta_M}(A) \otimes \mu_{\Delta_M}(B) \to \mu_{\Delta_M}(A \otimes B)$ and the composition morphism for $R\text{Hom}(q_{2\dagger}F, q_{1\dagger}G)$.)

**Notation 10.1.4.** For $F, G, H \in \text{D}^b(k_M)$, $S \subset T^*M$ and sections of $\mu_{\text{hom}}(-, -)$, say $a \in H^i(S; \mu_{\text{hom}}(F, G)|_S)$, $b \in H^j(S; \mu_{\text{hom}}(G, H)|_S)$, we denote by

$$b \circ a \in H^{i+j}(S; \mu_{\text{hom}}(F, H)|_S)$$

the image of $a \otimes b$ by the morphism induced by (10.1.2) on sections.

By construction $\circ$ is compatible with the usual composition morphism for $R\text{Hom}$ through the isomorphism (1.3.2): for $u: F \to G$, $v: G \to H$ we have $(v \circ u)^\mu = v^\mu \circ u^\mu$.

In the same way $\circ$ is also compatible with the functoriality of $\mu_{\text{hom}}$ as follows. A morphism $v: G \to H$ induces

$$\mu_{\text{hom}}(F, v): \mu_{\text{hom}}(F, G) \to \mu_{\text{hom}}(F, H)$$

and we have, for a section $a$ of $\mu_{\text{hom}}(F, G)$,

$$\mu_{\text{hom}}(F, v)(a) = v^\mu \circ a.$$
$F' \xrightarrow{\sigma} F \rightarrow L \xrightarrow{\phi}$. Then $L$ satisfies $S \cap SS(L) = \emptyset$. By (1.3.6) we have $\mu_{hom}(L, G)|_S \simeq 0$ and $\mu_{hom}(s, G)$ as in (10.1.3) gives an isomorphism

$$\mu_{hom}(s, G) : \mu_{hom}(F, G)|_S \simeq \mu_{hom}(F', G)|_S.$$  

Hence the triple $(F', s, u')$ yields a section $(\mu_{hom}(s, G))^{-1}(u'^\mu)$ of $H^0(S; \mu_{hom}(F, G)|_S)$. We can check that this section only depends on $u$ and not its representative $(F', s, u')$. Thus we obtain a well-defined morphism

$$(10.1.5) \quad \text{Hom}_{\mathbb{D}^b(k_M; S)}(F, G) \to H^0(S; \mu_{hom}(F, G)|_S).$$

**Theorem 10.1.5** (Thm. 6.1.2 of [31]). If $S = \mathbb{R}_{>0} \cdot p$ for some $p \in T^* M$, then (10.1.5) is an isomorphism.

When $p \in M \simeq T^*_M$ we have $S = \{p\}$ and the statement has the following meaning: we remark that $\mathbb{D}^b(k_M; \{p\})$ is the localization of $\mathbb{D}^b(k_M)$ by the subcategory of sheaves which are zero in some neighborhood of $p$. We can then see that $\text{Hom}_{\mathbb{D}^b(k_M; \{p\})}(F, G)$ identifies with $H^0(R\text{Hom}(F, G))_p$. We also have $\mu_{hom}(F, G)|_{T^*_M} \simeq R\text{Hom}(F, G)$ (use (1.3.2) and the fact that $\mu_{hom}$ is conic), hence $(\mu_{hom}(F, G))_p \simeq (R\text{Hom}(F, G))_p$.

Let $F, G \in \mathbb{D}^b(k_M)$ be given. It follows from Theorem (10.1.5) that the sheaf associated with the presheaf $\Omega \mapsto \text{Hom}_{\mathbb{D}^b(k_M; \mathbb{R}_{>0} \cdot \Omega)}(F, G)$, where $\Omega$ runs over the open subsets of $T^*_M$, is $H^0\mu_{hom}(F, G)$ (here $\mathbb{R}_{>0} \cdot \Omega$ is the image of $\mathbb{R}_{>0} \times \Omega \mapsto T^*_M$, $(a, p) \mapsto a \cdot p$). We obtain an alternative definition of $\mu_{Sh}(k_\Lambda)$:

**Corollary 10.1.6.** Let $\Lambda \subset T^*_M$ be as in Definition (10.1.4). We define a prestack $\mu_{Sh}^1(\Lambda)$ on $\Lambda$ as follows. Over an open subset $\Lambda_0$ of $\Lambda$ the objects of $\mu_{Sh}^1(\Lambda_0)$ are those of $\mathbb{D}^b(\Lambda_0)(k_M)$. For $F, G \in \mu_{Sh}^1(\Lambda_0)$ we set $\text{Hom}_{\mu_{Sh}^1(\Lambda_0)}(F, G) := H^0(\Lambda_0; \mu_{hom}(F, G)|_{\Lambda_0})$. The composition is induced by (10.1.2). Then, the natural functor of prestacks $\mu_{Sh}^1(\Lambda) \rightarrow \mu_{Sh}^1(\Lambda)$ induces an isomorphism on the associated stacks.

**Remark 10.1.7.** By Corollary (10.1.6) an object of $\mu_{Sh}(k_\Lambda)$ is determined by the data of an open covering $\{\Lambda_i\}_{i \in I}$ of $\Lambda$, objects $F_i \in \mathbb{D}^b(\Lambda_i)(k_M)$, for any $i \in I$, and sections $u_{ji} \in H^0(\Lambda_{ij}; \mu_{hom}(F_i, F_j)|_{\Lambda_{ij}})$, for any $i, j \in I$, such that

(i) $u_{ii}$ is induced by $\text{id}_{F_i}$, for any $i \in I$,

(ii) $u_{kj} \circ u_{ji} = u_{ki}$, for any $i, j, k \in I$.

For a complex of sheaves $A$, the sheaf associated with the presheaf $U \mapsto H^0(U; A)$ is $H^0 A$. Hence, for $F, G \in \mathbb{D}^b(\Lambda)(k_M)$, we find that the homomorphism sheaf $\mathcal{H}om_{\mu_{Sh}(k_\Lambda)}(m_\Lambda(F), m_\Lambda(G))$ is $H^0(\mu_{hom}(F, G))|_\Lambda$. In
particular, for an open subset $\Lambda_0 \subset \Lambda$,
\begin{equation}
(10.1.6) \quad \text{Hom}(m_{\Lambda}(F)|_{\Lambda_0}, m_{\Lambda}(G)|_{\Lambda_0}) \simeq H^0(\Lambda_0; H^0(\mu_{\text{hom}}(F, G))).
\end{equation}

**Remark 10.1.8.** We will only consider $\mu\text{Sh}(k_{\Lambda})$ when $\Lambda$ is conic Lagrangian submanifold of $T^*M$. In this case, for $F, G \in D^b_{(\Lambda)}(k_M)$ we know by Corollary 1.3.7 that $\mu_{\text{hom}}(F, G)$ has locally constant cohomology sheaves on $\Lambda$. Moreover, for a given $p = (x; \xi) \in \Lambda$ we have

$$
\mu_{\text{hom}}(F, G)_p \simeq \text{RHom}((\text{R} \Gamma_{\{\varphi_0 \geq 0\}}(F))_x, (\text{R} \Gamma_{\{\varphi_0 \geq 0\}}(G))_x),
$$

where $\varphi_0$ is such that $\Lambda$ and $\Lambda_{\varphi_0}$ intersect transversely at $p$ (see (1.4.6)). Hence, if $F$ and $G$ are pure along $\Lambda$, $\mu_{\text{hom}}(F, G)$ is concentrated in one degree. If this degree is 0, the right hand side of (10.1.6) coincides with $H^0(\Lambda_0; \mu_{\text{hom}}(F, G))$. If $F$ and $G$ are simple, then $\mu_{\text{hom}}(F, G)$ is moreover of rank one.

**Remark 10.1.9.** For a conic subset $\Omega$ of $T^*M$, we recall that a morphism $a: F \to G$ in $D(k_M)$ is an isomorphism on $\Omega$ if $\text{SS}(C(a)) \cap \Omega = \emptyset$, where $C(a)$ is given by the distinguished triangle $F \xrightarrow{a} G \to C(a) \xrightarrow{+1}$. In particular $a$ induces an isomorphism in $D^b(\kappa_M; \Omega)$. We assume that $\Lambda$ is a conic Lagrangian submanifold and $F, G \in D^b_{(\Lambda)}(k_M)$. Let $\Lambda_0$ be an open subset of $\Lambda$ and let $a: F \to G$ be an isomorphism on $\Lambda_0$. Then the morphism $\mathfrak{m}_{\Lambda}(a)|_{\Lambda_0}: \mathfrak{m}_{\Lambda}(F)|_{\Lambda_0} \to \mathfrak{m}_{\Lambda}(G)|_{\Lambda_0}$ is an isomorphism. It follows from Remark 10.1.8 that there exists $b \in H^0(\Lambda_0; H^0(\mu_{\text{hom}}(F, G)))$ such that

$$
a'' \circ b = \text{id}^\mu_G, \quad b \circ a'' = \text{id}^\mu_F.
$$

The functor $\mu\text{Sh}_{\Lambda}^{0, \text{op}} \times \mu\text{Sh}_{\Lambda}^{0} \to D^b(\kappa_{\Lambda}), (F, G) \mapsto \mu_{\text{hom}}(F, G)$ induces a functor of stacks
\begin{equation}
(10.1.7) \quad \overline{\mu_{\text{hom}}}: \mu\text{Sh}(\kappa_{\Lambda})^{\text{op}} \times \mu\text{Sh}(\kappa_{\Lambda}) \to \text{DL}(\kappa_{\Lambda}).
\end{equation}

### 10.2. Simple sheaves

In this section we assume that $\Lambda$ is a locally closed conic Lagrangian submanifold of $T^*M$. We have seen the notion of pure and simple sheaves along $\Lambda$ in Section 1.4. We give here some additional properties.

It is easy to describe the simple sheaves along a Lagrangian submanifold at a generic point. They are given in the following example.

**Example 10.2.1.** We consider the hypersurface $S = \mathbb{R}^{n-1} \times \{0\}$ in $M = \mathbb{R}^n$. We let $\Lambda = \{(x, 0; 0, \xi_n); \xi_n > 0\}$ be the “positive” half part of $T^*_S M$. We set $Z = \mathbb{R}^{n-1} \times \mathbb{R}_{\geq 0}$. The sheaf $k_Z$ is simple along $\Lambda$. 

More generally, by Example 1.2.10 the simple sheaves $F$ along $\Lambda$ fit in a distinguished triangle

$$E'_M \to k_Z[i] \to F \xrightarrow{+1}$$

for some integer $i$ and some $E' \in D(k)$. Let $F \in D_b(\Lambda)$. Then, there exists $L \in D^b(k)$ such that the image of $F$ in the quotient category $D^b(k_M; \hat{T}^*M)$ is isomorphic to $L_Z = L_M \otimes k_Z$. The pure sheaves correspond to the case where $L$ is concentrated in one degree and free. The simple sheaves correspond to the case where $L \cong k[i]$ for some degree $i \in \mathbb{Z}$.

For any $p \in \Lambda$ we can find a homogeneous Hamiltonian isotopy that sends a neighborhood of $p$ in $\Lambda$ to the conormal bundle of a smooth hypersurface. Then Theorem 2.1.1 reduces the general case to Example 10.2.1 (since this is a local statement in $T^*M$ we can also use Theorem 7.2.1 of [31]). We deduce:

**Lemma 10.2.2.** Let $p = (x; \xi)$ be a given point of $\Lambda$. Then there exists a neighborhood $\Lambda_0$ of $p$ in $\Lambda$ such that

(i) there exists $F \in D^b_{\Lambda_0}(k_M)$ which is simple along $\Lambda_0$,

(ii) for any $G \in D^b_{\Lambda_0}(k_M)$ there exist a neighborhood $\Omega$ of $\Lambda_0$ in $T^*M$ and an isomorphism $F \otimes L_M \cong G$ in $D^b(k_M; \Omega)$, where $L \in D^b(k)$ is given by $L = \mu_{\text{hom}}(F, G)_p$.

The subset $\Lambda_0$ of the lemma is contractible by construction if it is obtained from the $\Lambda$ of Example 10.2.1 by a homogeneous Hamiltonian isotopy. Conversely if the condition (ii) of the lemma is satisfied, we must have $\pi_1(\Lambda_0) = 0$ and $H^i(\Lambda_0; k) \cong 0$ for all $i > 0$.

**Definition 10.2.3.** Let $\Lambda \subset \hat{T}^*M$ be a locally closed conic Lagrangian submanifold. We let $\mu_{\text{Sh}}(k_\Lambda)$ (resp. $\mu_{\text{Sh}}^s(k_\Lambda)$) be the substack of $\mu_{\text{Sh}}(k_\Lambda)$ formed by the pure (resp. simple) sheaves along $\Lambda$.

Lemma 10.2.2 implies the following result.

**Proposition 10.2.4.** Let $\Lambda \subset \hat{T}^*M$ be a locally closed conic Lagrangian submanifold. We assume that there exists a simple sheaf $F \in \mu_{\text{Sh}}(k_\Lambda)$. Then the functor $\mu_{\text{hom}}$ defined in 10.1.7 induces an equivalence of stacks

$\overline{\mu_{\text{hom}}(F, \cdot)}: \mu_{\text{Sh}}(k_\Lambda) \cong \text{DL}(k_\Lambda), \quad G \mapsto \overline{\mu_{\text{hom}}(F, G)}$.

By Lemma 10.2.2 we can find a simple sheaf with microsupport $\Lambda$ locally around a given point $p \in \Lambda$. When $\Lambda$ is in a good position we can improve this result as follows.
Lemma 10.2.5. Let $M$ be a manifold and let $\Lambda \subset T^*M$ be a locally closed conic Lagrangian submanifold such that the projection $\Lambda/\mathbb{R}_{>0} \to M$ has finite fibers. Let $p = (x; \xi) \in \Lambda$. Then there exist a neighborhood $U$ of $x$ and $F \in \mathcal{D}^b(k_U)$ such that $SS(F) = \Lambda \cap T^*U$ and $F$ is simple along $\Lambda \cap T^*U$.

Proof. (i) By hypotheses $\Lambda \cap T^*_x M$ consists of finitely many half-lines, say $\mathbb{R}_{>0} \cdot p_i$, with $p_i = (x; \xi_i)$, $i = 1, \ldots, n$. Up to a restriction to a neighborhood of $x$ we can assume that the $p_i$ belong to distinct connected components of $\Lambda$, say $\Lambda_i$, $i = 1, \ldots, n$. If $F_i$ is simple along $\Lambda_i$, then the direct sum $\oplus_i F_i$ is simple along $\Lambda$. Hence we can assume that $\Lambda \cap T^*_x M = \mathbb{R}_{>0} \cdot p$ for some $p = (x; \xi)$.

(ii) By Lemma 10.2.2 there exists a neighborhood $\Omega$ of $p$ in $T^*M$ and $F_0 \in \mathcal{D}^b(k_M)$ such that $SS(F_0) \cap \Omega = \Lambda$ and $F_0$ is simple along $\Lambda$ at $p$. For a neighborhood $V$ of $x$ we choose a trivialization $T^*V \simeq V \times T^*_x M$. Up to shrinking $V$ we can find two disjoint closed conic contractible subsets $A, A' \subset T^*_x M$ such that $SS(F_0) \subset V \times (A \cup A')$ and $SS(F_0) \cap (V \times A) = \Lambda \cap T^*V$. By Proposition 3.3.2 there exists a distinguished triangle $F \oplus F_1 \to F_0|_U \to L \xrightarrow{\quad \pm 1 \quad} \mathcal{D}(k_U)$ on some smaller neighborhood $U$ of $x$ such that $SS(F) = SS(F_0) \cap (U \times A)$, $SS(F_1) = SS(F_0) \cap (U \times A')$ and $L$ is constant. Then $SS(F) = \Lambda \cap T^*U$ and $F$ is simple.

10.3. Obstruction classes

In this section we see that there are two obstructions to the existence of a global simple object in $\mu\text{Sh}(k\Lambda)$ (by this we mean an object of $\mu\text{Sh}^*(k\Lambda)\Lambda$). They are classes $\mu_1^{sh}(\Lambda) \in H^1(\Lambda; \mathbb{Z})$ and $\mu_2^{sh}(\Lambda) \in H^2(\Lambda; k^\times)$, where $k^\times$ is the group of units in $k$ (when $k = \mathbb{Z}/2\mathbb{Z}$ this last class is automatically zero).

On the other hand, it is proved in [19] that $\Lambda$ has a local generating function if and only if the stable Gauss map $g: \Lambda \to U/O$ is homotopic to the constant map $v: \Lambda \to U/O$ which sends a point to the vertical fiber of $T^*M$. Here $U/O = \lim_{\to n} U(n)/O(n)$ is the (stable) Lagrangian Grassmannian (some choices are needed to define the Gauss map, but it is well-defined up to homotopy). The obstruction classes to find such a homotopy are classes $\mu^{gf}_i(L) \in H^i(\Lambda; \pi_i(U/O))$ for $i = 1, \ldots, \dim M$ ($gf$ stands for generating function). The first class is the Maslov class $\mu_1^{gf}(\Lambda) = \mu_1(\Lambda)$. We will see in §10.6 that, for $i = 1, 2$, the vanishing of $\mu_1^{sh}(\Lambda)$ implies the vanishing of $\mu_1^{gf}(\Lambda)$.

To define the classes $\mu_i^{sh}(\Lambda)$ we recall how we can describe an object of $\mu\text{Sh}^*(k\Lambda)\Lambda$. By Remark 10.1.7 a global simple object of $\mu\text{Sh}(k\Lambda)$
is determined by the data of an open covering \( \{ \Lambda_i \}_{i \in I} \) of \( \Lambda \), objects \( F_i \in D^b(\Lambda_i)(k_M) \), for all \( i \in I \), which are simple along \( \Lambda_i \), and sections \( u_{ij} \in H^0(\Lambda_{ij}; \mu_{\text{hom}}(F_i, F_j)|_{\Lambda_{ij}}) \), for any \( i, j \in I \), such that

1. \( u_{ii} \) is induced by \( \text{id}_{F_i} \), for any \( i \in I \),
2. \( u_{kj} \circ u_{ji} = u_{ki} \), for any \( i, j, k \in I \).

We deduce that the \( \check{\text{C}} \)ech cochain \( d \) for some family of integers \( \mu \) is another simple sheaf along \( \Lambda \) with coefficient ring \( k \). Moreover, for each \( i \in I \), \( F_i' \in D^b(\Lambda_i)(k_M) \) is another simple sheaf along \( \Lambda \) we have \( m_{\Lambda_i}(F'_i) \simeq m_{\Lambda_i}(F_i)[d] \) for some shift \( d \).

The sheaf \( \mu_{\text{hom}}(F_i, F_j)|_{\Lambda_{ij}} \) is constant on \( \Lambda_{ij} \), free of rank one. Hence there exist isomorphisms \( \varphi_{ji}: \mu_{\text{hom}}(F_i, F_j)|_{\Lambda_{ij}} \simeq k_{\Lambda_{ij}}[-d_{ij}] \), for some integers \( d_{ij} \). They induce \( \varphi_{ji} \in H^0(\Lambda_{ij}; \mu_{\text{hom}}(F_i, F_j)[d_{ij}]) \). In view of (10.1.6) the \( \varphi_{ji} \)'s give a \( \check{\text{C}} \)ech cochain \( \{ d_{ij} \}_{i,j \in I} \) is a cocycle and defines

\[
\mu^s_1(\Lambda) = \{ [d_{ij}] \} \in H^1(\Lambda; \mathbb{Z}).
\]

By the remark that \( m_{\Lambda_i}(F_i) \) is well defined up to shift, this class only depends on \( \Lambda \). If there exists a global simple object \( \mathcal{F} \) in \( \mu\text{Sh}(k_\Lambda) \), for some coefficient ring \( k \), then we can choose \( F_i \)'s which represent \( \mathcal{F}|_{\Lambda_i} \) and this implies \( d_{ij} = 0 \) for all \( i, j \) and thus \( \mu^s_1(\Lambda) = 0 \).

Let us assume that \( \mu^s_1(\Lambda) = 0 \). Then we can write \( d_{ij} = d_j - d_i \) for some family of integers \( d_i, i \in I \). We set \( F'_i = F_i[d_i] \) and obtain isomorphisms

\[
w_{ji}: m_{\Lambda_i}(F'_i)|_{\Lambda_{ij}} \xrightarrow{\sim} m_{\Lambda_j}(F'_j)|_{\Lambda_{ij}}.
\]

For \( i, j, k \in I \) we define an automorphism \( c_{ijk} \) of \( m_{\Lambda_i}(F'_i)|_{\Lambda_{ijk}} \) by \( c_{ijk} = w_{ik} \circ w_{kj} \circ w_{ij} \). Since

\[
\mathcal{H}om(m_{\Lambda_i}(F'_i), m_{\Lambda_i}(F'_i)) \simeq H^0_{\text{hom}}(F'_i, F'_i) \simeq k_{\Lambda_i},
\]

we can canonically identify the automorphisms group of \( m_{\Lambda_i}(F'_i)|_{\Lambda_{ijk}} \) with \( k^\times \subset H^0(\Lambda_{ijk}; k_{\Lambda_{ijk}}) \simeq k \). Hence the \( c_{ijk} \)'s give a \( \check{\text{C}} \)ech cochain with coefficient in \( k^\times \). It is easy to see that it is a cocycle and defines

\[
\mu^s_2(\Lambda) = \{ [c_{ijk}] \} \in H^2(\Lambda; k^\times).
\]

The isomorphism \( H^0_{\text{hom}}(F'_i, F'_i) \simeq k_{\Lambda_i} \) also implies that \( w_{ji} \) is well defined up to multiplication by a unit. It follows that \( \mu^s_2(\Lambda) \) only
depends on $\Lambda$. If $\mu^b_2(\Lambda) = 0$, then we can write $\{c_{ijk}\}$ as the boundary of a 2-cochain, say $\{b_{ij}\}$, with $b_{ij} \in k^\times$. Defining 

$$w'_{ji} = b^{-1}_{ji}w_{ji}: m_{\Lambda}(F'_j)|_{\Lambda_{ij}} \rightarrow m_{\Lambda}(F'_j)|_{\Lambda_{ij}}$$

we have $w'_{ik} \circ w'_{kj} = w'_{ij}$ and the $m_{\Lambda}(F'_j)$ glue into a global simple object of $\mu Sh(k_\Lambda)$.

### 10.4. The Kashiwara-Schapira stack for orbit categories

In this section we set $k = \mathbb{Z}/2\mathbb{Z}$. We have defined the usual sheaf operations for the triangulated orbit categories $D_{/\{1\}}(k_M)$ and we can define a Kashiwara-Schapira stack in this situation. We give quickly the analogs of the results obtained in the previous sections. For a conic subset $S$ of $T^* M$ we recall the categories $D_{/\{1\}}(k_M)$, $D_{/\{1\},[S]}(k_M)$, $D_{/\{1\},(S)}(k_M)$ and $D_{/\{1\}}(k_M;S)$ (see Notation 10.0.1).

Let $\Lambda \subset T^* M$ be a locally closed conic Lagrangian submanifold. We define a stack $\mu Sh_{/\{1\}}(k_\Lambda)$ on $\Lambda$ as in Definition 10.1.1 replacing $D^b$ by $D_{/\{1\}}$. It comes with a functor $m_{/\{1\},\Lambda}: D_{/\{1\},(\Lambda)}(k_M) \rightarrow \mu Sh_{/\{1\}}(k_\Lambda)$.

We say that $F \in D_{/\{1\}}(k_M)$ is simple along $\Lambda$ if $SS^{\text{orb}}(F) \cap \hat{T}^* M \subset \Lambda$ and, for any $p \in \Lambda$, there exists $F' \in D^b(k_M)$ such that $\iota_M(F') \simeq F$, $SS(F') = \Lambda$ in a neighborhood of $p$ and $F'$ is simple along $\Lambda$ at $p$. As in section 10.1 we can define the substack $\mu Sh^*_{/\{1\}}(k_\Lambda)$ of $\mu Sh_{/\{1\}}(k_\Lambda)$ associated with the simple sheaves. For $\Omega \subset T^* M$, we have a morphism similar to (10.1.5)

$$\text{Hom}_{D_{/\{1\}}(k_M;\Omega)}(F,G) \rightarrow \text{Hom}_{D_{/\{1\}}(k_\Omega)}(k_\Omega, \mu \text{hom}^c(F,G)|_{\Omega})$$

and, as in Theorem 10.1.5 it is an isomorphism if $\Omega = \{p\}$ for some $p \in T^* M$.

We remark that Proposition 9.2.10 implies in particular that, if $B$ is homeomorphic to a ball and $L \in D_{/\{1\}}(k_B)$ is locally of the form $k_U$, then there exists an isomorphism $u: L \simeq k_B$. Moreover $\text{Hom}_{D_{/\{1\}}(k_B)}(k_B,k_B) \simeq k$ (and $k = \mathbb{Z}/2\mathbb{Z}$), hence $u$ is unique. For simple sheaves this gives:

**Lemma 10.4.1.** Let $\Lambda \subset \hat{T}^* M$ be a locally closed conic Lagrangian submanifold. We assume that $\Lambda$ is contractible. Let $F, F' \in D_{/\{1\}}(k_M)$ be two simple sheaves along $\Lambda$ and let $\Omega$ be a neighborhood of $\Lambda$ such that $SS^{\text{orb}}(F) \cap \Omega = SS^{\text{orb}}(F') \cap \Omega = \Lambda$. Then we have a unique isomorphism $\mu \text{hom}^c(F,F')|_{\Omega} \simeq k_{\Lambda}$ in $D_{/\{1\}}(k_\Omega)$.

By Lemma 10.4.1 there exists a unique simple sheaf in $\mu Sh_{/\{1\}}(k_{\Lambda_0})$ for any contractible open subset $\Lambda_0 \subset \Lambda$, up to a unique isomorphism.
In other words \( \mu_sh_{/[1]}(k_A) \) has locally a unique object with the identity as unique isomorphism. Hence gluing is trivial. Since \( \mu_sh_{/[1]}(k_A) \) is a stack it follows that it has a unique global object.

We recall that \( \text{Loc}(k_X) \) is the substack of \( \text{Mod}(k_X) \) formed by the locally constant sheaves.

**Definition 10.4.2.** Let \( X \) be a manifold. We let \( OL^0(k_X) \) be the subprestack of \( U \mapsto D_{/[1]}(k_U) \), \( U \) open in \( X \), formed by the \( F \in D_{/[1]}(k_U) \) such that \( SS^{\text{orb}}(F) \subset T^*_U U \). We let \( OL(k_X) \) be the stack associated with \( OL^0(k_X) \).

By Proposition 9.2.10 the condition \( SS^{\text{orb}}(F) \subset T^*_U U \) is equivalent to: \( F \) is locally isomorphic to \( A_U \) for some \( A \in \text{Mod}(k) \). We recall the functors \( \iota^0 : \text{Mod}(k_X) \to D_{/[1]}(k_X) \) and \( h_0 : D_{/[1]}(k_X) \to \text{Mod}(k_X) \) defined before Corollary 9.1.10. They induce functors of stacks \( i_X : \text{Loc}(k_X) \to OL(k_X) \) and \( h_X : OL(k_X) \to \text{Loc}(k_X) \).

**Lemma 10.4.3.** The functors \( i_X \) and \( h_X \) are mutually inverse equivalences of stacks.

**Proof.** We have seen in Corollary 9.1.10 that \( h_X \circ i_X \simeq \text{id}_{\text{Loc}(k_X)} \). Hence it is enough to see that \( i_X \) is locally an equivalence, that is, essentially surjective and fully faithful. Let \( U \subset X \) be an open subset homeomorphic to a ball. By Proposition 9.2.10 the functor \( i_U \) is essentially surjective and, for \( F,G \in \text{Loc}(k_U) \), Corollary 9.1.9 gives

\[
\text{Hom}_{D_{/[1]}(k_U)}(i_U(F), i_U(G)) \simeq \bigoplus_{n \in \mathbb{Z}} \text{Hom}_{D^b(k_U)}(F[-n], G)
\]

\[
\simeq \text{Hom}_{\text{Loc}(k_U)}(F, G),
\]

which proves that \( i_U \) is fully faithful.

As we remarked after Lemma 10.4.1 \( \mu_sh_{/[1]}(k_A) \) has a unique global object. As in 10.1.7 the functor \( \mu_{\text{hom}}^\varepsilon \) induces a functors of stacks

\[
\mu_{\text{hom}}^\varepsilon : (\mu_{sh_{/[1]}(k_A)})^{\text{op}} \times \mu_{sh_{/[1]}(k_A)} \to \text{OL}(k_A) \simeq \text{Loc}(k_A).
\]

We have an analog of Proposition 10.2.4 in the orbit category case.

**Proposition 10.4.4.** The stack \( \mu_{sh_{/[1]}(k_A)} \) has a unique object, say \( F_0 \), defined over \( A \). Moreover the functor \( \mu_{\text{hom}}^\varepsilon(F_0, -) \) induces an equivalence of stacks \( \mu_{sh_{/[1]}(k_A)} \simeq \text{Loc}(k_A) \).

10.5. Microlocal germs

In this section and the next one we see the link between the classes introduced in §10.3, \( \mu^h_1(A) \in H^1(A; \mathbb{Z}) \), \( \mu^h_2(A) \in H^2(A; \mathbb{K}) \) and
\( \mu_1^{gf}(\Lambda) \in H^1(\Lambda; \mathbb{Z}) \), \( \mu_2^{gf}(\Lambda) \in H^2(\Lambda; \mathbb{Z}/2\mathbb{Z}) \). We only prove the useful implication that the vanishing of \( \mu_i^{sh}(\Lambda) \) (for a ring \( \mathbb{k} \) with \( 2 \neq 0 \)) implies the vanishing of \( \mu_i^{gf}(\Lambda) \) but a little more work would show that they coincide.

In the definition of the microsupport of a sheaf \( F \) we consider whether some local cohomology group vanishes, namely \( (R\Gamma_{x; \phi(x) \geq \phi(x_0)}(F))_{x_0} \) for some function \( \phi \). A natural question is then how this group depends on \( \phi \) and not only on \( \xi_0 = d\phi(x_0) \). In general it really depends on \( \phi \), but it is proved in Proposition 7.5.3 of [31] that it is independent of \( \phi \) (up to a shift \( d\phi \)) if we assume that \( \Lambda = \text{SS}(F) \) is a Lagrangian submanifold near \( (x_0; \xi_0) \) and that \( \Gamma_{d\phi} \) is transverse to \( \Lambda \) at \( (x_0; \xi_0) \) (see Proposition 1.4.1). Moreover the shift \( d\phi \) is related with the Maslov index of the three Lagrangian subspaces of \( T_{x_0; \xi_0}T^*M \) given by \( \lambda = T_{x_0; \xi_0} \Lambda, \lambda_{\phi} = T_{x_0; \xi_0} \Gamma_{d\phi} \) and \( \lambda_0 = T_{x_0; \xi_0}(\pi^{-1}(x_0)) \). In particular \( (R\Gamma_{x; \phi(x) \geq \phi(x_0)}(F))_{x_0} \) only depends on \( \lambda_{\phi} \) which is a Lagrangian subspace of \( T_{x_0; \xi_0}T^*M \) transverse both to \( \lambda \) and \( \lambda_0 \). In this section we precise a little bit this result and prove that there exists a locally constant sheaf on some open subset of the Lagrangian Grassmannian of \( \Lambda \times T^* M \) whose stalks at \( \lambda_{\phi} \) is \( (R\Gamma_{x; \phi(x) \geq \phi(x_0)}(F))_{x_0} \). We will also see in §10.6 that it has a non trivial monodromy.

We first introduce some notations. In this section \( M \) is a manifold of dimension \( n \) and \( \Lambda \) is a locally closed conic Lagrangian submanifold of \( T^* M \). We recall the notations (1.4.2): for a given point \( p = (x; \xi) \in \Lambda \) we have the following Lagrangian subspaces of \( T_p(T^*M) \)

\[
\lambda_0(p) = T_p(T_p^* M), \quad \lambda_\lambda(p) = T_p \Lambda
\]

and, for a function \( \varphi : M \to \mathbb{R} \), we set \( \Lambda_\varphi = \Gamma_{d\varphi} = \{(x; d\varphi(x)); \; x \in M\} \) and

\[
\lambda_\varphi(p) = T_p \Lambda_\varphi.
\]

We let

\[
(10.5.1) \quad \sigma_{T^*M} : \mathcal{L}_M \to T^* M, \quad \sigma^0_{T^*M} : \mathcal{L}^0_M \to T^* M
\]

be respectively the fiber bundle of Lagrangian Grassmannian of \( T^* M \) and the subbundle whose fiber over \( p \in T^* M \) is the set of Lagrangian subspaces of \( T_p T^* M \) which are transverse to \( \lambda_0(p) \). Then \( \mathcal{L}^0_M \) is an open subset of \( \mathcal{L}_M \). For a given \( p \in T^* M \) we set \( V = T_{\pi_M(p)} M \) and we identify \( T_p T^* M \) with \( V \times V^* \). We use coordinates \( (\nu; \eta) \) on \( T_p T^* M \). Then we can see that any \( l \in \mathcal{L}^0_M(p) \) is of the form

\[
(10.5.2) \quad l = \{(\nu; \eta) \in T_p T^* M; \; \eta = A \cdot \nu\},
\]
where $A: V \to V^*$ is a symmetric matrix. This identifies the fiber $(\mathcal{L}_M^0)_p$ with the space of $n \times n$-symmetric matrices.

For a function $\varphi$ defined on a product $X \times Y$ and for a given $x \in X$ we use the general notation $\varphi_x = \varphi|_{\{x\} \times Y}$.

**Lemma 10.5.1.** There exists a function $\psi: \mathcal{L}_M^0 \times M \to \mathbb{R}$ of class $C^\infty$ such that, for any $l \in \mathcal{L}_M^0$ with $\sigma_{T^* M}(l) = (x; \xi)$,

$$\psi_l(x) = 0, \quad d\psi_l(x) = \xi, \quad \lambda_l(\sigma_{T^* M}(l)) = l.$$ 

**Proof.** (i) We first assume that $M$ is the vector space $V = \mathbb{R}^n$. We identify $T^* M$ and $M \times V^*$. For $p = (x; \xi) \in M \times V^*$ the fiber $(\mathcal{L}_M^0)_p$ is identified with the space of quadratic forms on $V$ through $(10.5.2)$. For $l \in (\mathcal{L}_M^0)_p$ we let $q_l$ be the corresponding quadratic form. Now we define $\psi_0$ by

$$\psi_0(l, y) = \langle y - x, \xi \rangle + \frac{1}{2} q_l(y - x), \quad \text{where } (x; \xi) = \sigma^0_{T^* M}(l).$$

We can check that $\psi_0$ satisfies the conclusion of the lemma.

(ii) In general we choose an embedding $i: M \hookrightarrow \mathbb{R}^N$. For a given $p' = (x; \xi') \in M \times \mathbb{R}^N T^* \mathbb{R}^N$ the subspace $T_{p'}(M \times \mathbb{R}^N T^* \mathbb{R}^N)$ of $T_{p'} T^* \mathbb{R}^N$ is coisotropic. The symplectic reduction of $T_{p'} T^* \mathbb{R}^N$ by $T_{p'}(M \times \mathbb{R}^N T^* \mathbb{R}^N)$ is canonically identified with $T_{p'} T^* M$, where $p = i_d(p')$. The symplectic reduction sends Lagrangian subspaces to Lagrangian subspaces and we deduce a map, say $r_{p'}: \mathcal{L}_{\mathbb{R}^N, p'} \to \mathcal{L}_{M, p'}$. The restriction of $r_{p'}$ to the set of Lagrangian subspaces which are transverse to $T_{p'}(M \times \mathbb{R}^N T^* \mathbb{R}^N)$ is an actual morphism of manifolds. In particular it induces a morphism $r^0_{p'}: \mathcal{L}_{\mathbb{R}^N, p'}^0 \to \mathcal{L}_{M, p'}^0$. We can see that $r^0_{p'}$ is onto and is a submersion. When $p'$ runs over $M \times \mathbb{R}^N T^* \mathbb{R}^N$ we obtain a surjective morphism of bundles, say $r$:

$$\mathcal{L}_{\mathbb{R}^N}^0|_{M \times \mathbb{R}^N T^* \mathbb{R}^N} \xrightarrow{r} \mathcal{L}_M^0 \quad \text{and} \quad M \times \mathbb{R}^N T^* \mathbb{R}^N \xrightarrow{id} T^* M.$$

We can see that $r$ is a fiber bundle, with fiber an affine space. Hence we can find a section, say $j: \mathcal{L}_M^0 \to \mathcal{L}_{\mathbb{R}^N}^0$. For $(l, x) \in \mathcal{L}_M^0 \times M$ we set $\psi(l, x) = \psi_0(j(l), i(x))$, where $\psi_0$ is defined in (i). Then $\psi$ satisfies the conclusion of the lemma. 

We come back to the Lagrangian submanifold $\Lambda$ of $T^* M$. We let

$$U_\Lambda \subset \mathcal{L}_M^0|_\Lambda$$

be the subset of $\mathcal{L}_M^0|_\Lambda$ consisting of Lagrangian subspaces of $T_p T^* M$ which are transverse to $\lambda_\Lambda(p)$. We define $\sigma_\Lambda = \sigma_{T^* M}|_{U_\Lambda}$, $\tau_M = \pi_M|_\Lambda \circ \sigma_\Lambda$ where $\pi_M$ is the canonical projection $M \to \Lambda$. 

(10.5.3)
We note that $U \lra U \times M$, $l \mapsto (l, \tau_M(l))$:

\[ U \xrightarrow{\tau_M} \Lambda \]
\[ U \times M \xrightarrow{q_2} M. \]

We define $I \subseteq \mathbb{R}$. Let $\psi \colon \mathcal{L}_M^0 \times M \to \mathbb{R}$ be a function satisfying the conclusions of Lemma 10.5.1 and let $\varphi : U \times M \rightarrow \mathbb{R}$ be its restriction to $U \times M$. For $F \in \mathcal{D}_M^b(k_M)$ we define $m_\varphi(F) \in \mathcal{D}_U^b(k_U)$ by

\[ m_\varphi(F) = i_\Lambda^{-1}(R\Gamma_{\varphi^{-1} ([0, +\infty])}(q_2^{-1}F)), \]

where $q_2 : U \times M \rightarrow M$ is the projection.

**Proposition 10.5.3.** Let $F \in \mathcal{D}_M^b(k_M)$. Let $\varphi : U \times M \rightarrow \mathbb{R}$ be as in Definition 10.5.2. Then the object $m_\varphi(F) \in \mathcal{D}_U^b(k_U)$ has locally constant cohomology sheaves and its stalks are

\[ (m_\varphi(F))_l \simeq (R\Gamma_{\varphi^{-1} ([0, +\infty])}(F))_x, \]

for any $l \in U$ and $x = \tau_M(l)$.

**Proof.** (i) We first prove that $m_\varphi(F)$ is locally constant. For this we give another expression of $m_\varphi(F)$. We define $G \in \mathcal{D}_U^b(k_{\hat{T}^*U \times M})$ by $G = \mu \text{hom}(k_{\varphi^{-1} ([0, +\infty])}, q_2^{-1}F)$. We use the notations in (10.5.4) and we define $I_\Lambda = \text{im}(i_\Lambda) \subseteq U \times M$ and $J_\Lambda \subseteq \hat{T}^*(U \times M)$, $J_\Lambda = \{(l, x; 0, \lambda \xi); (x; \xi) = \sigma_\Lambda(l), \lambda > 0\}$. We remark that $J_\Lambda$ is a fiber bundle over $I_\Lambda$ with fiber $\mathbb{R}_{>0}$. We prove in (ii) and (iii) below that there exists a neighborhood $V$ of $I_\Lambda$ in $U \times M$ such that

(a) $\text{supp}(G) \cap \hat{T}^*V \subseteq J_\Lambda$,
(b) $SS(G|_{\hat{T}^*V}) \subseteq T^*_\Lambda \hat{T}^*(U \times M)$,
(c) $R\Gamma_{\varphi^{-1} ([0, +\infty])}(q_2^{-1}F))_I_\Lambda \simeq \pi_\Lambda^* (\varphi_{\hat{T}^*V}^*(G|_{\hat{T}^*V}))$.

By Proposition 1.2.9 the properties (a-b) imply that $G|_{\hat{T}^*V}$ has support in $J_\Lambda$ and is locally constant along $J_\Lambda$. Since $J_\Lambda$ is a fiber bundle over $I_\Lambda$, we deduce by (c) that $R\Gamma_{\varphi^{-1} ([0, +\infty])}(q_2^{-1}F))_I_\Lambda$ is locally constant on $I_\Lambda$, hence $m_\varphi(F)$ is locally constant on $U_\Lambda$.

(ii) We prove (i-a) and (i-b). By Proposition 1.3.5 and Lemma 10.5.5 below we have: for $\Lambda_1, \Lambda_2$ two conic Lagrangian submanifolds of a cotangent bundle $\hat{T}^*X$ with a clean intersection $\Xi = \Lambda_1 \cap \Lambda_2$ and for
\[ F_i \in D_{[\Lambda]}^b(k_X), \ i = 1, 2, \] we have \( \text{supp}(\mu \text{hom}(F_1, F_2)|_{T^*X}) \subset \Xi \) and \( \text{SS}(\mu \text{hom}(F_1, F_2)|_{T^*X}) \subset T^*_\Xi T^*X. \)

By Example 1.3.2 (iii) we have \( \text{SS}(k_{\varphi^{-1}(0, +\infty)}) = \Lambda'_\varphi, \) where

\[ \Lambda'_\varphi = \{(l, x; \lambda \cdot \varphi(l, x)); \ (l, x) \in U_\Lambda \times M, \ \lambda > 0, \ \varphi(l, x) = 0\}. \]

We also have \( \text{SS}(q_2^{-1}F) = T^*_U U_\Lambda \times \Lambda. \) It is then enough to find a neighborhood \( V \) of \( I_\Lambda \) in \( U_\Lambda \times M \) such that \( T^*V \cap \Lambda'_\varphi \) and \( T^*V \cap (T^*U_\Lambda U_\Lambda \times \Lambda) \) have a clean intersection, which is \( J_\Lambda. \)

Let us first prove that \( \Lambda'_\varphi = (\Gamma_{\text{d}\varphi}) \) is transverse to \( T^*U_\Lambda \times \Lambda. \) For \( l_0 \in U_\Lambda, \) with \( \sigma_\Lambda(l_0) = (x_0; \xi_0), \) we know that \( \Lambda_{\varphi l_0} \) is transverse to \( \Lambda \) at the point \( (x_0; \xi_0) \) and \( \xi_0 = d\varphi l_0(x_0). \) Hence we can find a neighborhood \( V_{l_0} \) of \( x_0 \) in \( M \) such that \( \Lambda_{\varphi l_0} \cap \Lambda \cap T^*V_{l_0} = \{(x_0; \xi_0)\}. \) Since \( \Lambda_{\varphi l_0} \) is the projection of \( (T^*_0 U_\Lambda \times T^*M) \cap \Lambda_{\varphi} \) to \( T^*M, \) it follows that, in \( T^*_0 U_\Lambda \times T^*V_{l_0}, \) the submanifolds \( (T^*_0 U_\Lambda \times T^*V_{l_0}) \cap \Lambda_{\varphi} \) and \( T^*_0 U_\Lambda \times (T^*V_{l_0} \cap \Lambda) \) are transverse at the point \( (l_0, x_0; \frac{\partial \varphi}{\partial t}, \frac{\partial \varphi}{\partial x}) \) (and this is the only intersection point). We can make \( V_{l_0} \) move nicely enough with \( l_0 \) so that \( V = \bigsqcup_{l \in U_\Lambda} \{l\} \times V_l \) is a neighborhood of \( I_\Lambda \) in \( U_\Lambda \times M. \) Then \( T^*V \cap \Lambda_{\varphi} \) is transverse to \( T^*V \cap (T^*U_\Lambda U_\Lambda \times \Lambda), \) with intersection

\[ J^1_\Lambda = \{(l, x; \frac{\partial \varphi}{\partial l}, \frac{\partial \varphi}{\partial x}); \ l \in U_\Lambda, \ x = \tau_M(l)\}. \]

Let us prove that \( J_\Lambda = \mathbb{R}_{>0} \cdot J^1_\Lambda. \) For \( l_0 \in U_\Lambda \) and \( x_0 = \tau_M(l_0) \) we have \( (x_0; \frac{\partial \varphi}{\partial x}(l_0, x_0)) = \sigma_\Lambda(l_0) \) by the definition of \( \varphi. \) It remains to see that \( \frac{\partial \varphi}{\partial t}(l_0, x_0) = 0. \) We recall that \( \varphi(l, \tau_M(l)) = 0 \) for all \( l \in U_\Lambda. \)

Differentiating this relation we obtain \( \frac{\partial \varphi}{\partial t}(l_0, x_0) + \xi_0 \circ d\tau_M(l_0) = 0, \) for all \( l_0 \in U_\Lambda \) and \( (x_0; \xi_0) = \sigma_\Lambda(l_0) \) (here we view \( \xi_0 = \frac{\partial \varphi}{\partial x}(l_0, x_0) \) as a map from \( T_{x_0} \mathbb{R} \)) Since the map \( \tau_M: U_\Lambda \to \mathbb{R} \) factorizes through \( \sigma_\Lambda: U_\Lambda \to \Lambda \subset T^*M, \) we have

\[ \xi_0 \circ d\tau_M(l_0) = \xi_0 \circ d\pi_M(x_0; \xi_0) \circ d\sigma_M(l_0) = \alpha_M \circ d\sigma_M(l_0), \]

where \( \alpha_M \) is the Liouville 1-form on \( T^*M. \) Now \( \Lambda \) is conic Lagrangian, hence the pull-back of \( \alpha_M \) to \( \Lambda \) vanishes and we obtain \( \xi_0 \circ d\tau_M(l_0) = 0, \) hence \( \frac{\partial \varphi}{\partial t}(l_0, x_0) = 0, \) as required.

It follows from this discussion that \( A = T^*V \cap \mathbb{R}_{>0} \cdot \Lambda_{\varphi} \) is transverse to \( B = T^*V \cap (T^*U_\Lambda U_\Lambda \times \Lambda) \) with intersection \( J_\Lambda. \) Now \( J_\Lambda \) is contained in \( A_1 = \{\varphi = 0\} \) and in \( B_1 = T^*_U U_\Lambda \times \Lambda. \) We deduce that \( A \cap A_1 \) and \( B \cap B_1 \) have a clean intersection which is still \( J_\Lambda. \) Since \( A \cap A_1 = T^*V \cap \Lambda'_{\varphi}, \) and \( B \cap B_1 = T^*V \cap (T^*U_\Lambda U_\Lambda \times \Lambda), \) this concludes the proof of (i-a) and (i-b).
(iii) Now we prove the claim (c) of (i). Sato’s triangle \([1.3.5]\) gives
\[
(D'(k_{\varphi^{-1}(0, +\infty)})) \otimes q_2^{-1}F)_{|I_A} \to (R\Gamma_{\varphi^{-1}(0, +\infty)}(q_2^{-1}F))_{|I_A}
\]
\[
\to R\hat{\pi}_{U \times M_*(G)}(G)_{|I_A} \xrightarrow{\sim}.
\]

By definition \(d\varphi \) does not vanish in a neighborhood of \(I_A \). Hence \(\varphi^{-1}(0) \) is a smooth hypersurface near \(I_A \) and \(D'(k_{\varphi^{-1}(0, +\infty)}) \approx k_{\varphi^{-1}(0, +\infty)} \).
Since \(I_A \subset \varphi^{-1}(0) \), the first term of the above triangle is zero. By (i-a)
the support of \(R\hat{\pi}_{V_*}(G|_{T^*_V}) \) is already contained in \(I_A \). So we can
forget the subscript \(I_A \) in the third term and we obtain (i-c).

(iv) We prove the last assertion of the proposition. Let \(l_0 \in U_\Lambda \) be
given and \((x_0, \xi_0) = \sigma(l_0) \). Since \(\Lambda'_\varphi \) is transverse to \(\Lambda \) at \((x, \xi) = \sigma(l) \),
by Lemma \([10.5.6]\) below we can find neighborhoods \(U \) of \(l_0 \) and \(W \) of \(x_0 \) and a homogeneous
Hamiltonian isotopy of \(T^*W \) parameterized by \(U \), say \(\Psi : U \times T^*W \to T^*W \), such that \(\Psi_l(\Lambda) \cap T^*W = \Lambda \cap T^*W \) and
\(\Psi_l(\Lambda'_\varphi \cap T^*W = \Lambda'_\varphi \cap T^*W \), for all \(l \in U \). We set \(\Lambda^+ = \Lambda'_\varphi \cup (T^*_l U \cap \Lambda) \)
and \(\Lambda^+_l = \Lambda'_\varphi \cup \Lambda \). Then \(k_{\varphi^{-1}(0, +\infty)} \) and \(q_2^{-1}F \) belong to \(D_{|\Lambda^+}(k_{U \times W}) \)
and \(\Psi_l(\Lambda^+_l) = \Lambda^+_l \). By Proposition \([2.2.3 (ii) \] we deduce

\[
R\text{Hom}(k_{\varphi^{-1}(0, +\infty)}(q_2^{-1}F))_{|(l_0) \times W}
\]
\[
\xrightarrow{\sim} R\text{Hom}(k_{\varphi^{-1}(0, +\infty)}(q_2^{-1}F))_{|(l_0) \times W}
\]
\[
\times R\text{Hom}(k_{\varphi^{-1}(0, +\infty)}, F)_{|W}.
\]

Taking the germs at \(x_0 \in W \) we obtain the required isomorphism.  

Let \(X \) be a manifold and \(Y, Z \) two submanifolds of \(X \). We recall that
\(Y \) and \(Z \) have a clean intersection if \(W = Y \cap Z \) is a submanifold of \(X \)
and \(TW = TY \cap TZ \). This means that we can find local coordinates
\((x, y, \bar{x}, \bar{y}, w) \) such that \(Y = \{x = \bar{x} = 0 \} \) and \(Z = \{x = y = 0 \}. \) Using
these coordinates the following lemma is easy.

**Lemma 10.5.4.** Let \(X \) be a manifold and \(Y, Z \) two submanifolds of \(X \)
which have a clean intersection. We set \(W = Y \cap Z \). Then \(C(Y, Z) =\)
\(W \times_X TY + W \times_X TZ \).

**Lemma 10.5.5.** Let \(X \) be a manifold and \(\Lambda_1, \Lambda_2 \) be two Lagrangian
submanifolds of \(T^*X \). Let \(F_1 \in D_{(A_1)}^b(k_X) \) and \(F_2 \in D_{(A_2)}^b(k_X) \). We
assume that \(\Lambda_1 \) and \(\Lambda_2 \) have a clean intersection and we set \(\Xi =\)
\(\Lambda_1 \cap \Lambda_2 \). Then there exists a neighborhood \(U \) of \(\Xi \) in \(T^*X \) such that
\(SS(\mu_{\text{hom}}(F_1, F_2)|_U) \subset T^*_\Xi T^*X \), that is, \(\mu_{\text{hom}}(F_1, F_2)|_U \) is supported on
\(\Xi \) and has locally constant cohomology sheaves on \(\Xi \).

**Proof.** We have \(SS(\mu_{\text{hom}}(F_1, F_2)) \subset (H^{-1}(C(SS(F_2), SS(F_1))))^a \) by the
bound \([1.3.7]\). Let \(U_i \) be a neighborhood of \(\Lambda_i \) such that \(SS(F_i) \cap U_i \subset\)

\(\Xi \), then \(\mu_{\text{hom}}(F_1, F_2)|_U \) is supported on \(\Xi \) and has locally constant cohomology sheaves on \(\Xi \).
Let $B$ be a neighborhood of 0 in $\mathbb{R}^n$. Let $\varphi: B \times \mathbb{R}^n \to \mathbb{R}$ be a family of functions.

(i) We assume that $\Gamma_{d\varphi_b}$ is transverse to the zero-section $T^*\mathbb{R}^n$ of $T^*\mathbb{R}^n$ and $\Gamma_{d\varphi_0} \cap T^*_0 \mathbb{R}^n = \{0\}$. Then there exist neighborhoods $B'$ of 0 in $\mathbb{R}^n$ and $V$ of 0 in $\mathbb{R}^n$ and a family of Hamiltonian isotopies of $T^*\mathbb{R}^n$ parameterized by $B'$, say $\Psi_b: B' \times T^*\mathbb{R}^n \to T^*\mathbb{R}^n$, such that $\Psi_b(T^*\mathbb{R}^n, r) = T^*\mathbb{R}^n$ and $\Psi_b(\Gamma_{d\varphi_0}) \cap T^*V = \Gamma_{d\varphi_0} \cap T^*V$, for all $b \in B'$.

(ii) Let $\Lambda \subset T^*\mathbb{R}^n$ be a closed conic Lagrangian submanifold. We assume that $\Gamma_{d\varphi_0}$ is transverse to $\Lambda$ with $\Gamma_{d\varphi_0} \cap \Lambda = \{(0; \xi_0)\}$, $\Gamma_{d\varphi_b} \cap \Lambda = \{(x_b; \xi_b)\}$ and $\varphi_b(x_b) = 0$ for all $b \in B$. Then there exist neighborhoods $B'$ of 0 in $\mathbb{R}^n$ and $V$ of 0 in $\mathbb{R}^n$ and a family of homogeneous Hamiltonian isotopies of $\tilde{T}^*\mathbb{R}^n$ parameterized by $B'$, say $\Psi_b: B' \times \tilde{T}^*\mathbb{R}^n \to \tilde{T}^*\mathbb{R}^n$, such that $\Psi_b(\Lambda) \cap \tilde{T}^*V = \Lambda \cap \tilde{T}^*V$ and $\Psi_b(\Lambda'_0) \cap \tilde{T}^*V = \Lambda'_0 \cap \tilde{T}^*V$, for all $b \in B'$, where $\Lambda'_0 = \{(x; \lambda \cdot d\varphi_b(x)); \; \lambda > 0, \; \varphi_b(x) = 0\}$.

Proof. (i) The transversality hypothesis implies that $d\varphi_b$ viewed as a function from $\mathbb{R}^n$ to $(\mathbb{R}^n)^*$ is invertible near 0, for $b$ small enough. We set $\theta_b = (d\varphi_b)^{-1}$ and view $\theta_b$ as a 1-form on $(\mathbb{R}^n)^*$ defined in some neighborhood of 0. Since the graph of $\theta_b$ is Lagrangian, it is a closed 1-form and we can write $\theta_b = dh_b$ near 0. We consider $h_b(\xi)$ as a Hamiltonian function on $T^*\mathbb{R}^n$. By construction its Hamiltonian vector field is $X_{h_b}(x; \xi) = \sum_i (\theta_b)_i(\xi) \partial x_i$ and the time 1 of its flow satisfies $\phi^1_{h_b}(\{0\} \times (\mathbb{R}^n)^*) = \Gamma_{d\varphi_0} \cap T^*V$ near 0. Moreover $\phi^1_{h_b}(\mathbb{R}^n \times \{0\}) \subset \mathbb{R}^n \times \{0\}$. Now we set $\Psi_b = \phi^1_{h_b} \circ (\phi^1_{h_0})^{-1}$.

(ii) We can find a homogeneous Hamiltonian isotopy $\Phi$ arbitrarily close to id and a neighborhood $W$ of $\Phi(0; \xi_0)$ such that $T^*W \cap \Phi(\Lambda)$ is half of the conormal bundle of a smooth hypersurface $X$ and $T^*W \cap \Phi(\Gamma_{d\varphi_0})$ is still the graph of a function for $b$ close enough to 0. We take coordinates on $W$ such that $X = \mathbb{R}^{n-1} \times \{0\}$, $\Phi(0; \xi_0) = (0; 0; 0; 1)$ and we write $\Phi(x_b; \xi_0) = (y_b; 0; 0; \eta_b)$. Then $\Phi(\Lambda'_0)$ is the conormal bundle of a hypersurface which is the graph of a function $\varphi'_b: \mathbb{R}^{n-1} \to \mathbb{R}$. Moreover $\Gamma_{d\varphi'_b}$ is transverse to $T^*\mathbb{R}^{n-1}$. Part (i) of the proof gives a family $\Psi'$ of Hamiltonian isotopies of $T^*\mathbb{R}^{n-1}$ such that $\Psi'_b(T^*\mathbb{R}^{n-1}) = T^*\mathbb{R}^{n-1}$ and $\Psi'_b(\Gamma_{d\varphi'_0}) \cap T^*V = \Gamma_{d\varphi'_0} \cap T^*V$. We lift $\Psi'$ into a family $\Psi''$ of homogeneous Hamiltonian isotopies of $\tilde{T}^*\mathbb{R}^n$ and we set $\Psi_b = \Phi^{-1} \circ \Psi'' \circ \Phi$.

\[\square\]
Remark 10.5.7. The assignment \( F \mapsto m_{\varphi}(F) \) of Definition 10.5.2 is a functor \( m_{\varphi}: D_{b\Lambda}(k_M) \to D^b(k_{U_\Lambda}). \) By Proposition 10.5.3 it factorizes through \( DL^0(k_{U_\Lambda}) \) (see Definition 10.1.2). Let us check that \( m_{\varphi} \) induces a functor of stacks \( m_{\varphi}: \mu Sh(k_{\Lambda}) \to DL(k_{U_\Lambda}). \)

We recall that \( \mu Sh(k_{\Lambda}) \) is associated with some prestack \( \mu Sh^0_{\Lambda}; \) the objects of \( \mu Sh^0_{\Lambda}(\Lambda_0) \) are those of \( D_{b\lambda}(k_M) \) and the Hom set between \( F \) and \( G \) is \( \text{Hom}_{D_{b\lambda}(k_M;\Lambda)}(F,G) \) (see Definition 10.1.1). Now we remark that the definition of \( m_{\varphi}(F) \) applies to any \( F \in D^b(k_M) \) and gives in fact \( m_{\varphi}: D^b(k_M) \to D^b(k_{U_\Lambda}). \) By the definition of the microsupport we see that \( m_{\varphi}(F) \simeq 0 \) if \( SS(F) \cap \Lambda = \emptyset \), that is, \( m_{\varphi}(F) \simeq 0 \) if \( F \in D_{T^*M,\Lambda}(k_M). \) By the definition of \( D^b(k_M;\Lambda) \) this means that \( m_{\varphi} \) factorizes through \( D^b(k_M;\Lambda) \) (see the reminder on localization in section 9.1). Hence \( m_{\varphi} \) induces a functor from \( \mu Sh^0_{\Lambda}(\Lambda) \) to \( D^b(k_{U_\Lambda}), \) still with image in \( DL^0(k_{U_\Lambda}) \). We can replace \( \Lambda \) by any of its open subset \( \Lambda_0 \) and obtain in this way a functor of prestacks from \( \mu Sh^0_{\Lambda} \) to \( DL^0(k_{U_\Lambda}) \). Passing to the associated stacks it induces a functor of stacks \( m_{\varphi}: \mu Sh(k_{\Lambda}) \to DL(k_{U_\Lambda}). \)

10.6. Monodromy morphism

We keep the notations of Definition 10.5.2 and Remark 10.5.7. In particular we have a choice of function \( \varphi: U_\Lambda \times M \to \mathbb{R} \) which gives a functor \( m_{\varphi}: D_{b\Lambda}(k_M) \to D^b(k_{U_\Lambda}). \) For \( F \in D_{b\lambda}(k_M) \) we know that \( m_{\varphi}(F) \) is a locally constant object on \( U_\Lambda. \) Here we describe the monodromy of its restriction to a fiber \( U_{\Lambda,p} \) of \( \sigma_\lambda: U_\Lambda \to \Lambda. \)

We first recall well-known results on locally constant sheaves and introduce some notations. Let \( X \) be a manifold and \( L \in D^b(k_X) \) such that \( SS(L) \subset T^*_X X \), that is, \( L \) has locally constant cohomology sheaves. Then any path \( \gamma: [0,1] \to X \) induces an isomorphism

\[
M_\gamma(L): L_{\gamma(0)} \xhookrightarrow{\sim} R\Gamma([0,1]; \gamma^{-1}L) \xrightarrow{\sim} L_{\gamma(1)}.
\]

Moreover, \( M_\gamma(L) \) only depends on the homotopy class of \( \gamma \) with fixed ends and \( M_\gamma(L) \circ M_{\gamma'}(L) = M_{\gamma \circ \gamma'}(L) \) if \( \gamma, \gamma' \) are composable. In particular, if we fix a base point \( x_0 \in X \), we obtain the monodromy morphism

\[
M(L): \pi_1(X; x_0) \to \text{Iso}(L_{x_0})
\]

\[
\gamma \mapsto M_\gamma(L),
\]

where \( \text{Iso}(L_{x_0}) \) is the group of isomorphisms of \( L_{x_0} \) in \( D^b(k) \).
Now we go back to the situation of section 10.3. For a given $p \in \Lambda$ we set $U_{\Lambda,p} = \sigma_\Lambda^{-1}(p)$. This is the open subset of Lagrangian Grassmannian manifold $\mathcal{L}(T_p T^* M)$ formed by the Lagrangian subspaces of $T_p \Lambda$ which are transverse to $\lambda_0(p)$ and $\lambda_\Lambda(p)$. Let us describe its connected components and their fundamental groups.

Let $(V, \omega)$ be a symplectic vector space of dimension $2n$. Let $l_1, l_2$ be two Lagrangian subspaces. We can assume that $V = \mathbb{R}^{2n}$ with $\omega = \sum e_i \wedge f_i$ in the canonical base $(e_1, \ldots, e_n, f_1, \ldots, f_n)$ and $l_1 = \langle e_1, \ldots, e_n \rangle$, $l_2 = \langle e_1, \ldots, e_k, f_{k+1}, \ldots, f_n \rangle$. Let $U(l_1) \subset \mathcal{L}(V)$ be the open subset of Lagrangian subspaces which are transverse to $l_1$. Then $U(l_1)$ is diffeomorphic to $\text{Sym}_n$, the space of symmetric matrices of size $n \times n$, through $M \mapsto l_M := \{(My, y); \ y = (y_1, \ldots, y_n)\}$. Writing $M = \begin{pmatrix} A & B \\ tB & C \end{pmatrix}$, with $A$ of size $k \times k$, we see that $l_M$ is also transverse to $l_2$ if and only if $C$ is invertible. Hence $U(l_1) \cap U(l_2)$ is diffeomorphic to $\mathbb{R}^d \times \text{Sym}_{n-k}^0$ where $d$ is some integer and $\text{Sym}_{n-k}^0$ is the subset of invertible matrices in $\text{Sym}_{n-k}$.

Let us recall the topology of $\text{Sym}_n^{p,q}$ the subset of $\text{Sym}_n$ of matrices with $p$ positive eigenvalues and $q$ negative eigenvalues, $p + q = n$. The action of $\text{GL}_n^+$ on $\text{Sym}_n^{p,q}$, $A \cdot M = ^tAMA$, gives $\text{Sym}_n^{p,q} \simeq \text{GL}_n^+ / \text{SO}(p, q)$. In particular $\text{Sym}_n^{p,q}$ is connected. Now a maximal compact subgroup of $\text{SO}(p, q)$ is $K = S(\text{O}(p) \times \text{O}(q))$ and $\text{SO}(p, q)$ is diffeomorphic to $K \times \mathbb{R}^d$ for some $d$. We deduce an exact sequence of fundamental groups:

$$\pi_1(K) \to \pi_1(\text{GL}_n^+) \to \pi_1(\text{Sym}_n^{p,q}) \to \pi_0(K) \to \pi_0(\text{GL}_n^+) = \{1\}.$$  

We recall that $\pi_1(\text{SO}(n)) = \pi_1(\text{GL}_n^+)$ is $\mathbb{Z}$ for $n = 2$ and $\mathbb{Z}/2\mathbb{Z}$ for $n \geq 3$. We will only need the case where $n$ is big. In particular we can assume $n \geq 3$ and $p$ or $q \geq 2$. Since $K$ contains $\text{SO}(p) \times \text{SO}(q)$ it follows that the map $\pi_1(K) \to \pi_1(\text{GL}_n^+)$ is surjective. Hence $\pi_1(\text{Sym}_n^{p,q}) \simeq \pi_0(K)$. If $p$ and $q$ are both $\geq 1$, then $\text{O}(p) \times \text{O}(q)$ has four components and $\pi_0(\text{SO}(p) \times \text{O}(q))) = \mathbb{Z}/2\mathbb{Z}$. If $p$ or $q$ vanishes, then $\text{SO}(p, q) = \text{SO}(n)$ and $\text{Sym}_n^{p,q}$ is contractible. In conclusion, for $n \geq 3$ we have $\pi_1(\text{Sym}_n^{p,q}) \simeq \mathbb{Z}/2\mathbb{Z}$ if $p \neq 0$ and $q \neq 0$ and $\pi_1(\text{Sym}_n^{p,q}) \simeq 0$ if $p = 0$ or $q = 0$.

Let us write down the above results for $U_{\Lambda,p} = \sigma_\Lambda^{-1}(p)$. We set $n = \dim M$ and $k = \dim (\lambda_0(p) \cap \lambda_\Lambda(p))$. We assume $n - k \geq 3$. Then $U_{\Lambda,p}$ is topologically equivalent to $\text{Sym}_{n-k}^0 \cup_{p=0}^{n-k} \text{Sym}_{n-k}^{p,n-k-p}$, which has $n-k+1$ components, two of them being contractible and the other ones having $\pi_1 = \mathbb{Z}/2\mathbb{Z}$.
The inertia index $\tau_{T_p T^* M}$ introduced in (1.4.3) gives a function on 
$$\tau_p: U_{\Lambda,p} \rightarrow \mathbb{Z}, \quad l \mapsto \tau_{T_p T^* M}(\lambda_0(p), \lambda_\Lambda(p), l)$$
which is constant on each component of $U_{\Lambda,p}$. In the coordinates chosen above for $l_1, l_2$ and with $l = l_M$, $M = \begin{pmatrix} A & B \\ tB & C \end{pmatrix}$ we can see that $\tau_p(l_M) = \text{sgn}(C)$, using Lemma A.3.3 of [31] (here $\text{sgn}(C)$ is $p_+ - p_-$, where $p_+$ and $p_-$ are the numbers of positive and negative eigenvalues of $C$). Since $C$ is an invertible symmetric matrix of size $(n - k)$, we obtain that the values of $\tau_p$ are $\{-n+k,-n+k+2,\ldots,n-k\}$. Hence $\tau_p$ distinguishes the components of $U_{\Lambda,p}$ and we can index them as follows:

\begin{equation}
(10.6.3) \quad U^i_{\Lambda,p} \text{ is the connected component of } U_{\Lambda,p} \text{ where } \tau_p = i.
\end{equation}

We have to be careful that the components of $U_\Lambda$ cannot be indexed in this way: the function $\tau_p$ is locally constant on $U_{\Lambda,p}$ but the function $l \mapsto \tau_{T_{\delta(l)}}(\lambda_0(\sigma(l)), \lambda_\Lambda(\sigma(l)), l)$ is not locally constant on $U_\Lambda$. For example, when $\dim(\lambda_0(p) \cap \lambda_\Lambda(p))$ changes by 1 (which happens when $p$ moves from a generic point to a cusp), the parity of the possible values of $\tau_p$ also changes. However we have the following result.

**Lemma 10.6.1.** The function $\delta: U_\Lambda \times_\Lambda U_\Lambda \rightarrow \mathbb{Z}$, $(l, l') \mapsto \tau_p(l) - \tau_p(l')$, where $p = \sigma(l) = \sigma(l')$, is locally constant on $U_\Lambda \times_\Lambda U_\Lambda$.

**Proof.** We recall that $\tau$ satisfies a cocycle relation (see for example [31], Thm A.3.2) $\tau(l_1, l_2, l_3) - \tau(l_2, l_3, l_4) + \tau(l_3, l_4, l_1) - \tau(l_4, l_1, l_2) = 0$ and that $\tau(l_1, l_2, l_3)$ is constant when $(l_1, l_2, l_3)$ moves but the dimensions of $l_i \cap l_j$, $i, j = 1, 2, 3$, and $l_1 \cap l_2 \cap l_3$ do not change.

The function $\delta$ is locally constant on $U_{\Lambda,p} \times U_{\Lambda,p}$ for a given $p$ because so are $\tau_p(l)$ and $\tau_p(l')$. When $p$ moves we choose a local trivialization of $T^* M$ and we consider $l, l'$, $\lambda_0(p)$, $\lambda_\Lambda(p)$ as subspaces of a fixed symplectic space. Then $\tau_p(l) - \tau_p(l') = \tau(l, l', \lambda_0(p)) - \tau(l, l', \lambda_\Lambda(p))$ is constant for $l, l'$ fixed and $\lambda_0(p), \lambda_\Lambda(p)$ remaining transverse to $l, l'$.

**Proposition 10.6.2.** Let $F \in \mathcal{D}_{(\Lambda)}^b(k_M)$. For $p \in \Lambda$ let $U^i_{\Lambda,p}$ and $U^j_{\Lambda,p}$ be two components of $U_{\Lambda,p}$ (see (10.6.3)). Then

(a) for $l \in U^i_{\Lambda,p}$ and $l' \in U^j_{\Lambda,p}$ we have $m_\varphi(F)_l \simeq m_\varphi(F)_{l'}[(i - j)/2],$

(b) if $\pi_1(U^i_{\Lambda,p}) = \mathbb{Z}/2\mathbb{Z}$, the monodromy of $m_\varphi(F)|_{U^i_{\Lambda,p}}$ along the non trivial loop is the multiplication by $-1$.

By the discussion before (10.6.3) we have $\pi_1(U^i_{\Lambda,p}) = \mathbb{Z}/2\mathbb{Z}$ if $|i|$ is not maximal, that is, $|i| \neq \dim M - \dim(\lambda_0(p) \cap \lambda_\Lambda(p))$.

**Proof.** The point (a) is already proved in [31] and stated here as Proposition 1.4.1. It will be recovered in the course of the proof of (b).
(i) Since \( m_\varphi(F) \) is locally constant on \( U_\Lambda \) and \( j-i \) is well-defined in a neighborhood of \( U_{\Lambda,p}^i \times U_{\Lambda,p}^j \) by Lemma [10.6.1], we can assume for the proof of (a) that \( p \) is a generic point of \( \Lambda \). This also works for the proof of (b) since any loop in \( U_{\Lambda,p}^i \) can be deformed into a loop in a nearby fiber \( U_{\Lambda,q}^i \).

Hence we assume that \( \Lambda = T_\ast M \) in a neighborhood of \( p \), for some submanifold \( N \subset M \). By Example [10.4.5] we know that \( k_N \) is simple. By Lemma [10.2.2] there exists a neighborhood \( \Omega \) of \( p \) in \( T_\ast M \) such that \( F \) is isomorphic to \( k_N \otimes L_M \simeq \mathbb{Z}_N \otimes L_M \) in \( D^b(k_M; \Omega) \), for some \( L \in D^b(k) \). Hence \( m_\varphi(F) \simeq m_\varphi(\mathbb{Z}_N) \otimes L_{U_\Lambda} \) and we can assume that \( k = \mathbb{Z} \) and \( F = \mathbb{Z}_N \).

(ii) We take coordinates \((x_1, \ldots, x_n)\) so that \( N = \{x_1 = \cdots = x_k = 0\} \) and \( p = (0; 1, 0) \). We identify \((\mathcal{L}^0_M)_p\) with a space of matrices as in (10.5.2). Then \( U_{\Lambda,p}^i \) is the space of symmetric matrices \( A \) such that \( \text{det}(A_k) \neq 0 \), where \( A_k \) is the matrix obtained from \( A \) by deleting the \( k \) first lines and columns. The component \( U_{\Lambda,p}^i \) is defined by \( \text{sgn}(A_k) = i \). We can choose a base point \( B \in U_{\Lambda,p}^i \) represented by a diagonal matrix \( B \) with entries 0, 1 or \(-1\). More precisely the diagonal consists of \( k \) 0’s, \( \alpha \) 1’s and \( \beta \) \(-1\)’s. We have \( \alpha - \beta = i \), hence \( 2\alpha = i + n - k \). Since \( \pi_1(U_{\Lambda,q}^i) \neq 0 \) we also have \( \alpha, \beta \geq 1 \).

We choose \( a, b \) such that \( B_{aa} = 1 \) and \( B_{bb} = -1 \). For \( \theta \in [0, 2\pi] \), we define the matrix \( B(\theta) \) which is equal to \( B \) except

\[
\begin{pmatrix}
B_{aa}(\theta) & B_{ab}(\theta) \\
B_{ba}(\theta) & B_{bb}(\theta)
\end{pmatrix} = \begin{pmatrix}
\cos(\theta) & \sin(\theta) \\
\sin(\theta) & -\cos(\theta)
\end{pmatrix}.
\]

Then \( \gamma: \theta \mapsto B(\theta) \) defines a non trivial loop in \( U_{\Lambda,p}^i \) and we want to prove that the monodromy of \( m_\varphi(\mathbb{Z}_N) \) around \( \gamma \) is the multiplication by \(-1\). Since \( m_\varphi(\mathbb{Z}_N) \) has stalk \( \mathbb{Z} \) up to some shift, we only have to check that this monodromy is not trivial.

(iii) We define \( \varphi: [0, 2\pi] \times M \to \mathbb{R} \) by \( \varphi(\theta, x) = x_1 + x B(\theta) \cdot x \). Then \( C_\theta = \{ \varphi_\theta \geq 0 \} \cap N \) is a quadratic cone and the pair \((N, C_\theta)\) is homotopically equivalent to the pair \((N, V_\theta)\), where \( V_\theta \) is the subspace

\[
V_\theta = \langle e_\theta, e_p; p = k + 1, \ldots, k + \alpha, p \neq a \rangle,
\]

of dimension \( \alpha \) with \( e_\theta = (0, \cos(\theta/a), 0, \sin(\theta/b), 0) \) and \( e_p = (0, 1, 0) \). The stalk of \( m_\varphi(\mathbb{Z}_N) \) at \( B(\theta) \in U_{\Lambda,p}^i \) is

\[
m_\varphi(\mathbb{Z}_N)_{B(\theta)} \simeq R\Gamma_{\{\varphi_\theta \geq 0\}}(\mathbb{Z}_N) \simeq (H^{n-k-\alpha}_{V_\theta}(\mathbb{Z}_N))_0[k + \alpha - n]
\]

and a non zero germ \( s_\theta \in m_\varphi(\mathbb{Z}_N)_{B(\theta)} \) gives a choice of relative orientation of \( V_\theta \). In particular a non zero section of \( m_\varphi(\mathbb{Z}_N) \) defined on some
neighborhood of $\gamma$ would induce relative orientations of all $V_\theta$ together for $\theta \in [0, 2\pi]$, varying continuously with $\theta$ and coinciding for $\theta = 0$ and $\theta = 2\pi$, which is impossible. Hence the monodromy of $m_\varphi(Z_N)$ is not 1, which proves (b).

The part (a) follows from the fact that $m_\varphi(Z_N)_{B(\theta)}$ is concentrated in cohomological degree $n - k - \alpha = (n - k)/2 - i/2$. \hfill $\square$

Let $M$ be a manifold and $L \subset T^*M$ a closed Lagrangian submanifold. We set $L = L_M|_L$. We have already defined the Gauss map $g: L \to L$ as $g(p) = \lambda_\Lambda(p)$. We have also considered the tangent to the vertical fiber and we see it now as a map $v: L \to L$, $p \mapsto \lambda_0(p)$. We can define the same maps after stabilisation. For an integer $N$ we let $V_N$ be the symplectic vector space $V_N = \mathbb{C}^N$ and we let $l_0^N = \mathbb{R}^N$, $l_1^N = i\mathbb{R}^N$ be two Lagrangian subspaces. We define $L_N \to L$ the fiber bundle whose fiber at $p \in L$ is $L(T_pT^*M \oplus V_N)$. We extend $g$ and $v$ into two sections $g_N, v_N: L \to L_N$ defined by $g_N(p) = \lambda_\Lambda(p) \oplus l_1^N$ and $v_N(p) = \lambda_0(p) \oplus l_0^N$.

Taking the limit for $N \to \infty$, we set $L_\infty = \lim_{N \to \infty} L_N$ and obtain the sections $g_\infty$ and $v_\infty$ of $L_\infty$. As recalled in (10.3) it is proved in [19] that $\Lambda$ has a local generating function if and only if the sections $g_\infty$ and $v_\infty$ are homotopic.

In this situation it is classical to consider the obstructions classes to find a homotopy of sections between $g_\infty$ and $v_\infty$. The $i^{th}$ class belongs to $H^i(L; \pi_i(U/O))$; we denote it by $\mu^{g_i}_i(L)$ (more precisely, these classes are defined inductively and we need the vanishing of the first $i - 1$ classes to define the $i^{th}$ class). Let us assume that $L$ has a triangulation and denote by $S_k(L)$ the $k$-skeleton of $L$. Then $\mu^{g_i}_i = 0$ for $i = 1, \ldots, k$, if and only if there exists a homotopy between $g_\infty|_{S_k}$ and $v_\infty|_{S_k}$.

The question of finding an isotopy between $g$ and $v$ is related to finding a section of the map $U_L \to L$, where $U_L$ is the open subset of $L_M|_L$ introduced in (10.3.3). Indeed, for $p \in L$ and $l \in U_{L,P}$ the open subset $U(l)$ of $L_{M,P}$ consisting of Lagrangian subspaces transverse to $l$ is an affine chart of $L_{M,P}$ isomorphic to a space of symmetric matrices. It has a natural structure of affine space. Since $g(p)$ and $v(p)$ belong to $U(l)$, we obtain an isotopy $h_t: t \mapsto tg(p) + (1 - t)v(p)$, $t \in [0, 1]$. Hence any section of the map $\sigma: U_L \to L$ over a subset $S$ of $L$ gives an isotopy between $g|_S$ and $v|_S$.

For an integer $N$ we set $\Xi_N = \tilde{T}^*_{\mathbb{R}^N}\mathbb{R}^{N+1}$ and $\xi_N = (0, 0; 0, 1) \in \Xi_N$. For $F \in \mathcal{D}(\mathbb{k}_M)$ we consider $F \boxtimes \mathbb{k}_{\mathbb{R}^N} \in \mathcal{D}(\mathbb{k}_{M \times \mathbb{R}^{N+1}})$. Since $F \boxtimes \mathbb{k}_{\mathbb{R}^N} \simeq i_*p^{-1}F$, where $p: M \times \mathbb{R}^N \to M$ is the projection and $i: M \times \mathbb{R}^N \to M \times \mathbb{R}^{N+1}$ the inclusion, we have $SS(F \boxtimes \mathbb{k}_{\mathbb{R}^N}) = SS(F) \times \Xi_N$. If $SS(F) = \Lambda$, then $SS(F \boxtimes \mathbb{k}_{\mathbb{R}^N})$ contains $\Lambda \times \Xi_N$ as an open set and $\Lambda \times \Xi_N$ is a neighborhood of $\Lambda \simeq \Lambda \times \{\xi_N\}$ which retracts to $\Lambda$. We
can identify $\mathcal{L}(M \times \mathbb{R}^n)|_{\Lambda \times \{\xi\}}$ with the Lagrangian Grassmannian $L_\Lambda$ of the stabilisation of $T^*M$ introduced above.

Summing up the discussion we obtain the following result. We assume that $\Lambda$ is triangulated. If the map $\sigma: U_{\Lambda \times \Xi_N} \to \Lambda \times \Xi_N$ has a section over the $k$-skeleton of $\Lambda \times \{\xi\}$, then $\mu_i^\sigma(\Lambda) = 0$ for $i \leq k$.

**Corollary 10.6.3.** Let $\Lambda \subset T^*M$ be a closed conic Lagrangian submanifold. We assume that there exists $F \in \mu\text{Sh}(k_\Lambda)(\Lambda)$ which is simple along $\Lambda$. Then $\mu_i^\sigma(\Lambda) = 0$. Moreover, if $k = \mathbb{Z}$, then $\mu_2^\sigma(\Lambda) = 0$.

**Proof.** (i) We choose a triangulation of $\Lambda$. By the discussion before the corollary it is enough to prove that, for $N$ big enough, the map $\sigma: U_{\Lambda \times \Xi_N} \to \Lambda \times \Xi_N$ has a section over the $k$-skeleton of $\Lambda \times \{\xi\}$, for $k = 1, 2$.

We consider the connected components of $U_{\Lambda \times \Xi_N}|_{\Lambda \times \{\xi\}}$, that we denote by $U_{a, N}$, $a \in A_N$. We let $U_{a, N}$ be the connected components of $U_{\Lambda}$. To prove the vanishing of $\mu_1$ it is enough to see that there exists $N$ and $U_{a, N}'$ such that $\sigma|_{U_{a, N}'}: U_{a, N}' \to \Lambda$ is surjective with connected fibers (then it is possible to find a section on the $1$-skeleton, since $U_{a, N}'$ is an open subset of a fiber bundle over $\Lambda$).

(ii) By Remark 10.5.7 we can define $m_\phi(F) \in D(L(k_\Lambda))$. Since $F$ is simple, $m_\phi(F)$ is (locally) concentrated in one degree with germs $k$ in this degree. Hence, for any $a \in A$ there exists $d_a = d_a(F) \in \mathbb{Z}$ such that $m_\phi(F)|_{U_{a, N}}[d_a]$ is locally constant with germs $k$.

Let $a, a' \in A$. We recall that the connected components of $U_{a, p}$ are distinguished by the inertia index and that we denote by $U_{a, p}^{(i)}$ the component with index $i$. We assume that $U_{a, p}^{(i)}$ is a component of $U_a \cap U_{a, p}$ and $U_{a', p}^{(i')}$ is a component of $U_{a'} \cap U_{a, p}$. By Proposition 10.6.2 (a) this implies $i - i' = 2(d_a - d_{a'})$. We thus obtain

(ii-a) $U_a$ cannot intersect $U_{a, p}$ in more than one connected component,

(ii-b) if $d_a(F) = d_{a'}(F)$ and $\sigma(U_a) \cap \sigma(U_{a'}) \neq \emptyset$, then $U_a = U_{a'}$.

Applying (ii-a) to $F \boxtimes k_{\mathbb{R}^N}$ (writing abusively $k_{\mathbb{R}^N}$ for $m_{\Xi_N}(k_{\mathbb{R}^N})$) we obtain that $\sigma|_{U_{a, N}}$ has connected fibers, for any component $U_{a, N}$ introduced in (i).

(iii) We denote by $V_{b, N}$, $b \in B_N$, the components of $U_{\Xi_N}$ and by $d_b(k_{\mathbb{R}^N})$ (as in (ii)) the cohomological degree such that $m_{\varphi}(k_{\mathbb{R}^N})$ has germs $k$ in degree $d_b(k_{\mathbb{R}^N})$, for a function $\varphi: U_{\Xi_N} \times \mathbb{R}^{N+1} \to \mathbb{R}$ similar to $\varphi$. We can see directly on the formula of Proposition 10.5.3 that $d_b(k_{\mathbb{R}^N})$ takes the values $0$, $1$, $\ldots$, $N$ (more precisely $H^*_i(U_{\Xi_N})k_{\mathbb{R}^N}$ is concentrated in degree $N - i$ when $f(x) = x_{N+1} + q(x_1, \ldots, x_N)$ and $q$ is a non
can be extended to the 2-skeleton if and only if the chain \( Z \) is locally constant with germs \( S \)−monodromy vanishes. By Proposition 10.6.2–(b) the sheaf \( U_\sigma \) of \( b \) is defined as follows. The boundary of each triangle and \( p \) identify \( i \sigma \). Hence \( (iv) \) Now we assume \( k = \mathbb{Z} \) and prove \( \mu_2^{gf} = 0 \). Let us set \( U_0^N = U_{\sigma(a)}^N \cap \sigma_{\Lambda_X} \Lambda \setminus \{ \xi \} \) (\( U_{\sigma(a)}^N \) is the component of \( U_{\Lambda_X} \) found in (iii)). Hence \( \sigma := \sigma_{\Lambda_X} \Lambda \setminus \{ \xi \} : U_0^N \to \Lambda \) is surjective with connected fibers. We recall that the components of \( U_{\Lambda,p} \) have fundamental group \( \mathbb{Z} / 2\mathbb{Z} \) except the two with extremal inertia index. Hence, up to taking the product with \( V \) (the component of \( U_{\Lambda,2} \) with index 1 − see (iii)), we can assume that the fibers of \( \sigma|_{U_0^N} \) have fundamental group \( \mathbb{Z} / 2\mathbb{Z} \).

Let us recall how \( \mu_2^{gf} \) is defined. We assume that our triangulation is fine enough so that \( \sigma_1(\sigma_1(T)) = \sigma_1(\sigma_1(p)) = \mathbb{Z} / 2\mathbb{Z} \) for any triangle \( T \) and \( p \in T \). We choose a section of \( \sigma \) on the 1-skeleton, say \( i : S_1(\Lambda) \to U_0^N \). Then \( \mu_2^{gf} \) is the obstruction to extend it to the 2-skeleton and is defined as follows. The boundary of each triangle \( T \) gives a loop \( i \) in \( \sigma_1(T) \), hence an element \( c(T) \in \pi_1(\sigma_1(T)) = \mathbb{Z} / 2\mathbb{Z} \). Then \( i \) can be extended to the 2-skeleton if and only if the chain \( c : T \to c(T) \) vanishes. It is easy to see that \( c \) is a cocycle and, by definition \( \mu_2^{gf} = [c] \).

If \( [c] = 0 \), we write \( c = \partial b \) where \( b \) is a 1-chain and we can modify \( i \) by \( b \) and obtain a new section \( i' : S_1(\Lambda) \to U_0^N \) which can be extended to \( S_2(\Lambda) \).

Now we see how we can use our sheaf to define \( i \) such that the chain \( c \) vanishes. By Proposition 10.6.2–(b) the sheaf \( G = m_{\varphi}(\mathcal{F} \boxtimes \mathbb{Z}_{\pi \Lambda})|_{U_0^N} \) is locally constant with germs \( \mathbb{Z} \) and its restriction to the fibers has monodromy \( -1 \). We first define \( i \) on \( S_0(\Lambda) \) arbitrarily. For each \( p \in S_0(\Lambda) \) we also choose a generator \( u_p \) of \( G_{\varphi(p)} \simeq \mathbb{Z} \). We have two such generators \( u_p \) and \( -u_p \).

Let \( E \subset S_1(\Lambda) \) be an edge with boundaries \( p, p' \). The fundamental group of \( \varphi^{-1}(E) \) is \( \pi_1(\varphi^{-1}(E)) = \mathbb{Z} / 2\mathbb{Z} \). Hence we have two sections \( j, j' \) of \( \varphi \) over \( E \) up to homotopy such that \( j(p) = j'(p) = i(p) \) and
Using again $π_i$ we can extend $j$ to $G_{i(p)}$. Hence we can extend $j$ to $G_{i(p')}$. Since $G$ has monodromy $-1$, we have $a_j = -a_j'$. Hence we can choose one section $j$ or $j'$, that we call $i$, such that $a_i(u_p) = u_{p'}$.

We do this for all edges and we obtain $i: S_1(Λ) → U_0^N$. With this definition the monodromy of $G$ along $i(∂T)$ is $1$, for any triangle $T$. Using again $π_1(σ^{-1}(T)) = Z/2Z$ and the fact that $G$ has monodromy $-1$ along the non trivial loop, we deduce that $i(∂T)$ is a trivial loop. Hence we can extend $i$ to $S_2(Λ)$. This proves $ψ_{ij}^2 = 0$. □

**Part 11. Convolution and microlocalization**

Let $N$ be a manifold and $Λ ⊂ T^*N$ a locally closed conic Lagrangian submanifold. As explained in Remark 11.1.7 the objects of $µSh^*(k_A)$ are described by simple sheaves along a covering $Λ_i$ of $Λ$, say $F_i ∈ D^b_{(Λ_i)}(k_N)$, and gluing “isomorphisms” $u_{ji} ∈ H^0(Λ_{ij}; µhom(F_i, F_j)|_{Λ_{ij}})$. For a given object of $µSh^*(k_A)$ we want to find a representative in $D^b(Λ_N)$, or better, $D^b_{Λ}(k_N)$. For this we would like to glue the $F_i$’s in the category $D^b(k_N)$ instead of $µSh^*(k_A)$. A first step for this is to find other representatives of the $m_{Λ_i}(F_i)$’s for which the $u_{ji}$ arise from morphisms in $D^b(k_N)$. In this part we introduce a functor, $Ψ$, which gives an answer to this question (see Proposition 11.3.7 and Corollary 11.3.8 below). This functor is a variation on Tamarkin’s projector of §3.5. To define $Ψ$ we need to choose a direction on $N$ and we assume that $N$ is a product $N = M × R$. For an open subset $U$ of $N$ we define $Ψ_U: D(k_U) → D(k_{U × ]0, +∞[})$ with the following properties: setting $Ψ_U^ε(F) = Ψ_U(F)|_{U × [ε]}$ for $ε > 0$, we have, for $F, G ∈ D(k_U)$, with a microsupport contained in $T^∗M × \{τ \geq 0\}$ ($τ$ the fiber coordinate of $T^*R$ – see §11.0.1 below),

(i) $\hat{SS}(Ψ_U^ε(F)) = SS(F) ∧ T_ε(\hat{SS}(F))$, where $T_ε$ is the translation by $ε$ along $R$,

(ii) $Ψ_U^ε(F)$ is isomorphic to $F$ along $\hat{SS}(F)$ in the sense that we have a triangle $Ψ_U^ε(F) → F → H \overset{+1}{→}$ with $SS(H) ∩ \hat{SS}(F) = ∅$,

(iii) if $F ∼ G$ in $D(k_U; T^∗U)$, then $Ψ_U(F) ∼ Ψ_U(G)$,

(iv) $H^0(T^∗V; µhom(F, G)) ∼ \lim_{ε → 0} Hom(Ψ_U^ε(F)|_V, Ψ_U^ε(G)|_V)$, if $V$ is a relatively compact open subset of $U$.

The property (iv) will be used in the next part to glue representatives of objects of $µSh^*(k_A)$ as follows. For $W ⊂ M × R$ we set $Λ_W =...
\( \Lambda \cap T^*W \). We are given \( \mathcal{F} \in \mu \text{Sh}^s(k_\Lambda)(\Lambda_0) \), a covering \( W = W_1 \cup W_2 \) and \( F_i \in \mathcal{D}(k_{W_i}) \) representing \( \mathcal{F}|_{\Lambda W_i} \). We set \( U = W_1 \cap W_2 \). We then have isomorphisms \( m_{\Lambda U}(F_1|_U) \simeq \mathcal{F}|_{\Lambda U} \simeq m_{\Lambda U}(F_2|_U) \), hence a section of \( H^0(\mu \text{hom}(F_1, F_2)) \) over \( \dot{T}^*U \). Using the property (iv) we deduce an isomorphism \( \Psi_{\dot{U}}(F_1) \simeq \Psi_{\dot{U}}(F_2) \) and we can glue \( \Psi_{\dot{U}}(F_1) \) and \( \Psi_{\dot{U}}(F_2) \) into a representative of \( \mathcal{F} \) over \( W \).

With this procedure we can construct sheaves representing objects of \( \mu \text{Sh}^s(k_\Lambda)(\Lambda_0) \) when \( \Lambda_0 \subset \Lambda \) is of the type \( \Lambda \cap T^*U_0 \), \( U_0 \) open in \( M \times \mathbb{R} \). Unfortunately we will need to consider more general open subsets \( \Lambda_0 \) in \( W \).

Our \( \Lambda_0 \) will not be a union of subsets \( \Lambda \cap T^*W_i \), but only a union of connected components of \( \Lambda \cap T^*W_i \). This means that \( \pi_{M \times \mathbb{R}}(\Lambda_0) \cap \pi_{M \times \mathbb{R}}(\partial \Lambda_0) \) is a priori non empty. Let \( p = (x; \xi) \in \partial \Lambda_0 \), \( \Xi \) a neighborhood of \( p \) in \( \Lambda \) and \( U = \pi_{M \times \mathbb{R}}(\Xi) \), \( U_0 = \pi_{M \times \mathbb{R}}(\Xi \cap \Lambda_0) \). Near \( x \) we will have to consider sheaves of the form \( F = R\Gamma_{U_0}(F') \) with \( F' \in \mathcal{D}(k_U) \), \( SS(F') = \Xi \). For sheaves of this kind the above formula (iv) may still hold but the microsupport of \( F \) is bigger than \( \Xi \cap T^*U_0 \).

If \( G = R\Gamma_{U_1}(G') \) is another sheaf of the same type, what we really need in (iv) is not \( H^0(\dot{T}^*V; \mu \text{hom}(F, G)) \) (for some \( V \subset U \)) but \( H^0(\dot{T}^*V \cap \Lambda_0; \mu \text{hom}(F, G)|_{\Lambda_0}) \). In fact, for generic open subsets \( U_0, U_1 \) (in the sense that the conormal bundles of their boundaries are well positioned with respect to \( \Lambda \)), these two groups are isomorphic. We check this in \( \S \). We introduce a category of sheaves which are locally of the form \( \Psi_{U}(R\Gamma_{U_0}(F')) \).

We introduce some notations. We set for short \( \mathbb{R}_{>0} = [0, +\infty[ \) and \( \mathbb{R}_{\geq 0} = [0, +\infty] \). We usually endow the factor \( \mathbb{R} \) in \( M \times \mathbb{R} \) with the coordinate \( t \). We will need an extra parameter, usually denoted \( u \), running over \( \mathbb{R}_{>0} \) or \( \mathbb{R}_{\geq 0} \). The associated coordinates in the cotangent bundles are \( (t; \tau) \) for \( T^*\mathbb{R} \) and \( (u; v) \) for \( T^*\mathbb{R}_{\geq 0} \). We set \( T^*_{\tau \geq 0} = \{(t; \tau) \in T^*\mathbb{R}; \tau \geq 0\} \) and we define \( T^*_{\tau > 0} \) similarly. For a manifold \( M \) and an open subset \( U \subset M \times \mathbb{R} \) we define

\[
T^*_{\tau \geq 0}U = (T^*M \times T^*_{\tau \geq 0}\mathbb{R}) \cap T^*U, \quad T^*_{\tau < 0}U = (T^*_{\tau < 0}\mathbb{R}) \cap T^*U, \quad T^*_{\tau = 0}U = (T^*_{\tau = 0}\mathbb{R}) \cap T^*U.
\]

\textbf{Definition 11.0.1.} Let \( U \) be an open subset of \( M \times \mathbb{R} \). For \( * = 0, b \) or \( lb \), we let \( D^*_r(U \mathcal{D}(k_U)) \) (resp. \( D^*_r(U \mathcal{D}(k_U)^0) \)) be the full subcategory of \( \mathcal{D}(k_U) \) of sheaves \( F \) satisfying \( SS(F) \subset T^*_{\tau > 0}U \) (resp. \( SS(F) \subset T^*_{\tau > 0}U \)).

\textbf{11.1. The functor } \( \Psi \)

The convolution product is a variant of the “composition of kernels” considered in \( [31] \) (denoted by \( \circ \) – see \( (1.5.1) \)). It is used in \( [47] \) to
study the localization of \( \mathcal{D}(\mathbf{k}_{M \times \mathbb{R}}) \) by the objects with microsupport in \( T^*_\tau(M \times \mathbb{R}) \), in a framework similar to the present one. Namely, Tamarkin proves that the functor \( F \mapsto \mathbf{k}_{[0, +\infty[} \star F \) is a projector from \( \mathcal{D}(\mathbf{k}_{M \times \mathbb{R}}) \) to the left orthogonal of the subcategory \( \mathcal{D}_{T^*_\tau(M \times \mathbb{R})}(\mathbf{k}_{M \times \mathbb{R}}) \) of objects with microsupport in \( T^*_\tau(M \times \mathbb{R}) \) (see [23] for a survey). We will use a variant of Tamarkin’s definition.

We will use the product \( \star \) in the following special situation. We define the subsets of \( \mathbb{R} \times \mathbb{R}_{>0} \):

\[
\gamma = \{(t, u); \ 0 \leq t < u\}, \\
\lambda_0 = \{(0) \times \mathbb{R}_{>0}\}, \quad \lambda_1 = \{(t, u) \in \mathbb{R} \times \mathbb{R}_{>0}; \ t = u\}.
\]

**Definition 11.1.1.** Let \( M \) be a manifold and let \( U \subset M \times \mathbb{R} \) be an open subset. We define \( U_\gamma \subset M \times \mathbb{R} \times \mathbb{R}_{>0} \) by

\[
U_\gamma = \{(x, t, u) \in M \times \mathbb{R} \times \mathbb{R}_{>0}; \ \{x\} \times \{t - u, t\} \subset U\}.
\]

For \( F \in \mathcal{D}(\mathbf{k}_U) \) and \( G \in \mathcal{D}(\mathbf{k}_{\mathbb{R} \times \mathbb{R}_{>0}}) \) with \( \text{supp}(G) \subset \overline{\gamma} \), we define \( G \star F \in \mathcal{D}(\mathbf{k}_{U_\gamma}) \) by

\[
G \star F = (\mathcal{R}s_! (F \boxtimes G))[_{U_\gamma}],
\]

where \( s = s_U: U \times \mathbb{R} \times \mathbb{R}_{>0} \to M \times \mathbb{R} \times \mathbb{R}_{>0} \) is the sum \( s(x, t_1, t_2, u) = (x, t_1 + t_2, u) \) (in general \( U \) is understood and we write \( s \) for \( s_U \)). We define the functor \( \Psi_U: \mathcal{D}(\mathbf{k}_U) \to \mathcal{D}(\mathbf{k}_{U_\gamma}) \) by

\[
\Psi_U(F) = k_\gamma \star F = (\mathcal{R}s_! (F \boxtimes k_{(t, u); 0 \leq t < u}))[_{U_\gamma}],
\]

**Remark 11.1.2.**

(i) We see easily from the definition of \( U_\gamma \) that, for any submanifold \( M' \) of \( M \), we have

\[
U_\gamma \cap (M' \times \mathbb{R} \times \mathbb{R}_{>0}) = (U \cap (M' \times \mathbb{R}))_\gamma.
\]

In particular \( U_\gamma \cap \{(x) \times \mathbb{R} \times \mathbb{R}_{>0}\} = (U \cap \{(x) \times \mathbb{R}\})_\gamma \), for any point \( x \in M \), and we can write \( U_\gamma = \bigsqcup_{x \in M} \{(x) \times (U \cap \{(x) \times \mathbb{R}\})\}_\gamma \).

For a disjoint union \( U = \bigsqcup_{i \in I} U_i \) we have \( U_\gamma = \bigsqcup_{i \in I} U_{i, \gamma} \). Hence we can reduce the description of \( U_\gamma \) to the case where \( M \) is a point and \( U = [a, b[ \) is an interval of \( \mathbb{R} \). Then we have

\[
[a, b[)_\gamma = \{(t, u) \in \mathbb{R} \times \mathbb{R}_{>0}; \ a + u < t < b\}.
\]

(ii) We have \( s^{-1}(U_\gamma) \cap (M \times \mathbb{R} \times \mathbb{R}) \subset U \times \overline{\gamma} \). Since \( \text{supp}(G) \subset \overline{\gamma} \), it follows that we also have \( G \star F = (\mathcal{R}s_{M \times \mathbb{R}_!} (F' \boxtimes G))|[_{U_\gamma} \) where \( F' \in \mathcal{D}(\mathbf{k}_{M \times \mathbb{R}}) \) is any object such that \( F'|_U = F \).

(iii) For the same reason the restriction of \( s \) to \( s^{-1}(U_\gamma) \cap (M \times \mathbb{R} \times \overline{\gamma}) \) is a proper map. Hence we can replace \( \mathcal{R}s_! \) by \( \mathcal{R}s_* \) in (11.1.2).
Example 11.1.3. In Fig. 11.1.1 we give the easiest examples of $\Psi_U(F)$ when $M$ is a point and $U = \mathbb{R}$. The three pictures describe $\Psi_U(F)$ respectively for $F_1 = \mathbb{k}_{[0, +\infty[}, F_2 = (\mathbb{k}_{]-\infty,0[})[1], F_3 = \mathbb{k}_{[0,1]}$. We have $\Psi_U(F_1) \simeq \Psi_U(F_2) \simeq \mathbb{k}_W$, where $W = \{(t, u); 0 \leq t < u\} \subset \mathbb{R} \times \mathbb{R}_{>0}$, and $\Psi_U(F_3)$ is concentrated in degrees $0, 1$, with $H^0\Psi_U(F_3) = \mathbb{k}_{W_0}$, $H^1\Psi_U(F_3) = \mathbb{k}_{W_1}$, for some locally closed subsets $W_0, W_1$ of $\mathbb{R} \times \mathbb{R}_{>0}$ (however $\Psi_U(F_3) \not\simeq \mathbb{k}_{W_0} \oplus \mathbb{k}_{W_1}[-1]$). We have sketched $W$, $W_0$, $W_1$ in the pictures.

We remark that we have a distinguished triangle $\mathbb{k}_\mathbb{R} \to F_1 \to F_2 \xrightarrow{+1}$ and Lemma 11.1.5 below implies $\Psi_U(\mathbb{k}_\mathbb{R}) \simeq 0$, hence $\Psi_U(F_1) \simeq \Psi_U(F_2)$ as we have said. Setting $\Lambda_{x_0} = \{(x_0; \tau), \tau > 0\} \subset \mathbb{T}^{*}\mathbb{R}$, for $x_0 \in \mathbb{R}$, we also have $\mathbb{SS}(F_1) = \mathbb{SS}(F_2) = \Lambda_0$ and $\mathbb{m}_{\Lambda_0}(F_1) \simeq \mathbb{m}_{\Lambda_0}(F_2)$.

Similar things happen for $F_3$ and $F_4 := F_1 \oplus \mathbb{k}_{[1, +\infty[}[-1]$. We have $\mathbb{SS}(F_3) = \mathbb{SS}(F_4) = \Lambda_0 \sqcup \Lambda_1$ and $\mathbb{m}_{\Lambda_0\sqcup\Lambda_1}(F_3) \simeq \mathbb{m}_{\Lambda_0\sqcup\Lambda_1}(F_4)$. However $\Psi_U(F_3) \not\simeq \Psi_U(F_4)$; indeed $\Psi_U$ is additive and the result for $F_1$ gives here $\Psi_U(F_4) \simeq \mathbb{k}_W \oplus \mathbb{k}_{W'}[-1]$, where $W' = \{(t, u); 1 \leq t < 1 + u\}$. What remains true is that, for $\varepsilon > 0$ small enough (here $\varepsilon \leq 1$), $\Psi_U(F_3)|_{\mathbb{R} \times \mathbb{R}_{>0}[\varepsilon]} \simeq \Psi_U(F_4)|_{\mathbb{R} \times \mathbb{R}_{>0}[\varepsilon]}$.

In Corollary 11.3.8 we will see: for a closed conic Lagrangian submanifold $\Lambda \subset T_{>0}\gamma U$ and $F, G \in D_{[\Lambda]}(\mathbb{k}_U)$, if $\mathbb{m}_{\Lambda}(F) \simeq \mathbb{m}_{\Lambda}(G)$, then $\Psi_U(F)|_V \simeq \Psi_U(G)|_V$, for some open subset $V$ of $U_\gamma$ such that $(U \times \{0\}) \cap V$ is open in $U \times \mathbb{R}_{>0}$. So $\Psi_U$ gives canonical representatives of objects of $\mu\mathbb{Sh}(\mathbb{k}_\Lambda)$, but with microsupport “doubled” (see Remark 11.1.8 – the problem is addressed in §12.3).

We define the projections
\begin{equation}
q: M \times \mathbb{R} \times \mathbb{R}_{>0} \to M \times \mathbb{R}, \quad (x, t, u) \mapsto (x, t),
\end{equation}
\begin{equation}
r: M \times \mathbb{R} \times \mathbb{R}_{>0} \to M \times \mathbb{R}, \quad (x, t, u) \mapsto (x, t - u)
\end{equation}
and, for an open subset $U$ of $M \times \mathbb{R}$, we denote by
\begin{equation}
q_U, r_U: U_{\gamma} \to U
\end{equation}
the restrictions of $q, r$ to $U_{\gamma}$. Using the notations (11.1.1) we have $\mathbb{k}_{\Lambda_0} \star F \simeq q_U^{-1}(F)$ and $\mathbb{k}_{\Lambda_1} \star F \simeq r_U^{-1}(F)$, for any $F \in D(\mathbb{k}_U)$. The closed inclusions $\lambda_0 \subset \gamma$, $\lambda_1 \subset \gamma$ and $\lambda_1 \subset \gamma \setminus \lambda_0$ give excision distinguished triangles
\[
\mathbb{k}_{\text{Int}(\gamma)} \xrightarrow{\nu'} \mathbb{k}_\gamma \xrightarrow{a} \mathbb{k}_{\lambda_0} \xrightarrow{+1} \gamma, \quad \mathbb{k}_{\lambda_1}[-1] \xrightarrow{b} \mathbb{k}_\gamma \xrightarrow{\nu} \mathbb{k}_{\gamma \setminus \lambda_0} \xrightarrow{+1} \gamma
\]
and we have \( b = b'' \circ b' \). The convolution \(- \ast F\) turns the morphisms \( a, b, b', b''\) into morphisms of functors:

\[
\begin{align*}
\alpha(F) &: \Psi_U(F) \to q_U^{-1}(F), \\
\beta(F) &: r_U^{-1}(F)[-1] \to \Psi_U(F), \\
\end{align*}
\]

and \( \beta'(F) : r_U^{-1}(F)[-1] \to k_{\text{Int}(\gamma)} \ast F, \beta''(F) : k_{\text{Int}(\gamma)} \ast F \to \Psi_U(F) \). We have \( \beta(F) = \beta''(F) \circ \beta'(F) \) and two distinguished triangles

\[
\begin{align*}
(11.1.8) & \quad k_{\text{Int}(\gamma)} \ast F \xrightarrow{\beta''(F)} \Psi_U(F) \xrightarrow{\alpha(F)} q_U^{-1}(F) \xrightarrow{\pm 1}, \\
(11.1.9) & \quad r_U^{-1}(F)[-1] \xrightarrow{\beta'(F)} k_{\text{Int}(\gamma)} \ast F \to k_{\gamma \setminus \lambda_0} \ast F \xrightarrow{+1}.
\end{align*}
\]

**Lemma 11.1.4.** For \( F \in D_{r \geq 0}(k_U) \) the morphism \( \beta'(F) \) is an isomorphism and \( (11.1.8) \) gives the distinguished triangle

\[
\begin{align*}
(11.1.10) & \quad r_U^{-1}(F)[-1] \xrightarrow{\beta(F)} \Psi_U(F) \xrightarrow{\alpha(F)} q_U^{-1}(F) \xrightarrow{\pm 1}.
\end{align*}
\]

**Proof.** Since \( \beta(F) = \beta''(F) \circ \beta'(F) \) the second part of the lemma follows from the claim that \( \beta'(F) \) is an isomorphism. In view of \((11.1.9)\) we only have to prove that \( k_{\gamma \setminus \lambda_0} \ast F \simeq 0 \). For \((x, t, u) \in U_\gamma\) we have the Cartesian square

\[
\begin{array}{ccc}
\mathbb{R} & \xrightarrow{i(x,t,u)} & M \times \mathbb{R}^2 \times \mathbb{R}_{>0} \\
\downarrow & & \downarrow s \\
\{(x,t,u)\} & \longrightarrow & M \times \mathbb{R} \times \mathbb{R}_{>0},
\end{array}
\]
where $i_{(x,t,u)}(t') = (x, t', t - t', u)$. Writing $i_{(x,t,u)} = (i_x \times i_{(t,u)}) \circ \delta$, with $\delta(t') = (t', t')$, $i_x(t') = (x, t')$ and $i_{(t,u)}(t') = (t - t', u)$, we deduce by the base change formula that

$$(k\gamma_{\lambda_0} \ast F)_{(x,t,u)} \simeq R\Gamma_c(\mathbb{R}; i_x^{-1}F \otimes i_{(t,u)}^{-1}k\gamma_{\lambda_0})$$

\[ \simeq R\Gamma_c(\mathbb{R}; i_x^{-1}F \otimes k_{[t-u,t]}) \, . \]

Then $G = i_x^{-1}F \otimes k_{[t-u,t]}$ has compact support and satisfies $SS(G) \subset T_{\tau \geq 0} \mathbb{R}$ by Theorems 12.8 and 12.13. Using Corollary 12.10 we deduce that $R\Gamma_c(\mathbb{R}; G) \simeq 0$. It follows that $k\gamma_{\lambda_0} \ast F \simeq 0$. □

Let $U$ be an open subset of $M \times \mathbb{R}$ and $V$ an open subset of $U$. Let $X$ be a submanifold of $M$ and $U' = U \cap (X \times \mathbb{R})$. We have

(11.1.11) $\Psi_V(F|_V) \simeq (\Psi_U(F)|_V)_\gamma$,  
(11.1.12) $\Psi_{U'}(F|_{U'}) \simeq (\Psi_U(F)|_{U'})_{U'}$,  

where the first isomorphism follows from supports estimates as in Remark 11.1.2 (ii) and the second one follows from the base change formula.

In the next lemma we use an analog of the convolution for sets. For $A \subset M \times \mathbb{R}$ and $B \subset \mathbb{R} \times \mathbb{R}_{>0}$ we define $B \ast A \subset M \times \mathbb{R} \times \mathbb{R}_{>0}$ by

(11.1.13) $B \ast A = s_{M \times \mathbb{R}}(A \times B)$,

where $s_{M \times \mathbb{R}}: M \times \mathbb{R}^2 \times \mathbb{R}_{>0} \to M \times \mathbb{R} \times \mathbb{R}_{>0}$ is the sum as in Definition 11.3.

**Lemma 11.1.5.** Let $F \in \mathcal{D}(k_U)$ and let $V \subset U$ be an open subset. We assume that

(11.1.14) $F|_{V \cap \{(x) \times \mathbb{R}\}}$ is locally constant, for any $x \in M$.

Then $\Psi_U(F)|_{V_\gamma} \simeq 0$. As a special case, if $SS(F|_V) \subset T_{\tau}^* V$, then $\Psi_U(F)|_{V_\gamma} \simeq 0$. In particular $\text{supp}(\Psi_U(F)) \subset (\gamma \ast \check{\pi}_U(\text{SS}(F))) \cap U_\gamma$.

**Proof.** We set $V_x = V \cap \{(x) \times \mathbb{R}\}$. Then $V_\gamma = \bigsqcup_{x \in M} \{(x) \times (V_x)_\gamma\}$ and we have to prove $\Psi_U(F)|_{\{(x) \times (V_x)_\gamma\}} \simeq 0$, for all $x \in M$. By (11.1.12) we have $\Psi_U(F)|_{\{(x) \times (V_x)_\gamma\}} \simeq \Psi_{V_x}(F|_{V_x})$. The set $V_x$ is a disjoint union of open intervals of $\mathbb{R}$ and $F|_{V_x}$ is constant on each of these intervals. A direct computation gives $\Psi_{V_x}(F|_{V_x}) \simeq 0$ and we obtain the result. □

**Lemma 11.1.6.** Let $F \in \mathcal{D}(k_U)$.

(i) We have $Rq_Uq_U^!(F) \simeq F$ and $Rr_U^!(\Psi_U(F)) \simeq 0$.

(ii) If $F \in \mathcal{D}_{T_{\tau \geq 0}}(k_U)$, then $Rq_Ur_U^{-1}(F)$ satisfies (11.1.14) (with $V = U$). In particular $\Psi_U(Rq_Ur_U^{-1}(F)) \simeq 0$. 

(iii) We assume that $U = M \times \mathbb{R}$, that $F \in D_{r \geq 0}(k_u)$ and that $\operatorname{supp}(F) \subset M \times [a, +\infty[$ for some $a \in \mathbb{R}$. Then $R_{qU}! r_U^1 (F) \simeq 0$.

**Proof.** (i) The first morphism is the adjunction for $(R_{qU}, q_U^1)$. It is an isomorphism because the fibers of $q_U$ are intervals. Let us prove the second isomorphism. By the base change formula (see (11.1.12)) we may as well assume that $M$ is a point. Then $U$ is an open subset of $\mathbb{R}$ and we can restrict to one component of $U$. Hence we assume $U$ is an interval. We define $r' : \mathbb{R}^2 \times \mathbb{R}_{>0} \to \mathbb{R}^2$, $(t_1, t_2, u) \mapsto (t_1, t_2 - u)$ and $s' : \mathbb{R}^2 \to \mathbb{R}$, $(t_1, t_2) \mapsto (t_1 + t_2)$. We have the commutative diagram

$$
\begin{array}{ccc}
\mathbb{R}^2 \times \mathbb{R}_{>0} & \xrightarrow{r'} & \mathbb{R}^2 \\
\downarrow s & & \downarrow s' \\
\mathbb{R} \times \mathbb{R}_{>0} & \xrightarrow{r} & \mathbb{R}.
\end{array}
$$

We let $j : U \to \mathbb{R}$ be the inclusion and we set $F' = j_! F$. Then $\Psi_U(F) \simeq (k_{\gamma} \star F')|_{U_{\gamma}}$. Let $q_2 : \mathbb{R} \times \mathbb{R} \times \mathbb{R}_{>0} \to \mathbb{R} \times \mathbb{R}_{>0}$ be the projection on the last two factors. We have

$$
Rr_{U!}(\Psi_U(F)) \simeq Rr_{U!}(Rs_{!}((F' \boxtimes k_{\gamma}) \otimes k_{U_{\gamma}})) \\
\simeq R(r \circ s)_{!}((F' \boxtimes k_{\mathbb{R} \times \mathbb{R}_{>0}}) \otimes k_{q_2^{-1} \gamma \cap s^{-1} U_{\gamma}}) \\
\simeq R(s' \circ r')_{!}(r'^{-1}(F' \boxtimes k_{\mathbb{R}}) \otimes k_{q_2^{-1} \gamma \cap s^{-1} U_{\gamma}}) \\
\simeq Rs'_{!}((F' \boxtimes k_{\mathbb{R}}) \otimes Rr'_{!}(k_{q_2^{-1} \gamma \cap s^{-1} U_{\gamma}})).
$$

Hence it is enough to prove that $Rr'_{!}(k_{q_2^{-1} \gamma \cap s^{-1} U_{\gamma}}) \simeq 0$. We write $U = [a, b[$. Then we have $U_{\gamma} = \{(t, u) \in \mathbb{R} \times \mathbb{R}_{>0}; a + u < t < b\}$. For any $(t_1, t_2) \in \mathbb{R}^2$ the fiber $r'^{-1}(t_1, t_2) \cap (q_2^{-1} \gamma \cap s^{-1} U_{\gamma})$ is identified with

$$
\begin{align*}
\{u > 0; (t_1, t_2 + u, u) \in q_2^{-1} \gamma \cap s^{-1} U_{\gamma}\} \\
&= \{u > 0; 0 \leq t_2 + u < u \text{ and } (t_1 + t_2 + u, u) \in U_{\gamma}\} \\
&= \{u > 0; -t_2 \leq u \text{ and } u < b - t_1 - t_2\},
\end{align*}
$$

where we assume $t_2 < 0$ and $a < t_1 + t_2$ (otherwise the fiber is empty).

Since $-t_2 > 0$ we see that the fiber is either empty or a half closed interval. This implies $Rr'_{!}(k_{q_2^{-1} \gamma \cap s^{-1} U_{\gamma}}) \simeq 0$, as required.

(ii) We choose $x \in M$ and we set $U_x = U \cap \{\{x\} \times \mathbb{R}\}$. By the base change formula we have $(R_{qU}! r_U^{-1}(F)|_{U_x}) \simeq R_{qU_x}! r_U^{-1}(F|_{U_x})$. Hence we can assume that $M$ is a point and that $U$ is an interval. It is enough to prove that $(R_{qU}! r_U^{-1}(F))|_{[a, b]}$ is constant for any $a, b \in U$. We set $V = U \cap ]-\infty, a[$, $Z = U \setminus V$. In view of the distinguished triangle $F_V \to F \to F_Z \xrightarrow{+1}$ it is enough to see that $G_1 = (R_{qU}! r_U^{-1}(F_V))|_{[a, b]}$ and $G_2 = (R_{qU}! r_U^{-1}(F_Z))|_{[a, b]}$ are constant.
The maps $q_U$ and $r_U$ restricted to $W = q_U^{-1}([a, b]) \cap r_U^{-1}(V)$ identify $W$ with $[a, b] \times V$ and we deduce, by the base change formula, that $G_1$ is constant with stalks $R\Gamma_c(V; F|_V)$.

Let $j: U \to \mathbb{R}$ be the inclusion and let $q', r': \mathbb{R}^2 \to \mathbb{R}$ be the maps $(t, u) \mapsto t$, $(t, u) \mapsto t - u$. We set $F' = Rj(F_Z)$. Then we have $G_2 \simeq (Rq'_1(r'^{-1}(F') \otimes k_{\mathbb{R}\times\mathbb{R}^2 }))|_{[a, b]}$ and $SS(F') \subset T^*_{\mathbb{R}^2 \mathbb{R}^2}$. Using Theorems 1.2.13 and 1.2.8 we obtain, with coordinates $(t, u; \tau, v)$ on $T^*\mathbb{R}^2$:

$$SS(r'^{-1}(F')|_{U \times \mathbb{R}}) \subset \{(t, u; \tau, -\tau); \tau \geq 0\},$$

$$SS((r'^{-1}(F') \otimes k_{\mathbb{R}\times\mathbb{R}^2})|_{U \times \mathbb{R}}) \subset \{(t, u; \tau, -\tau + v); \tau \geq 0, 0 \leq v \leq 0\}.$$ 

This last set intersects $T^*\mathbb{R} \times T^*\mathbb{R}$ along $T^*_{\mathbb{R}^2 \mathbb{R}^2}$. Since $q'$ is proper on $r'^{-1}(Z) \cap (\mathbb{R} \times \mathbb{R}^2)$ it follows from Proposition 1.2.4 that $SS(G_2)$ is contained in the zero section. Hence $G_2$ is constant.

(iii) By (ii) $Rq_U r_U^{-1}(F)$ is constant on the fibers $\{x\} \times \mathbb{R}$ for all $x \in M$. By the hypothesis its restriction to $M \times \{a - 1\}$ vanishes. Hence it is zero.

**Lemma 11.1.7.** We let $j: U \times \mathbb{R} > 0 \to U \times \mathbb{R}$ be the inclusion. We set $A = (U \times \{0\}) \times_{U \times \mathbb{R}} T^*(U \times \mathbb{R})$. We use the maps $q_U, r_U$ of (11.1.6) and the notations (11.1.1). Let $F \in D_{\mathbb{R}^2}(k_U)$. Then

$$SS(\Psi_U(F)) \subset q_U, d q_U^{-1}(SS(F)) \cup r_U, d r_U^{-1}(SS(F)),$$

$$SS(\Psi_U(F)) \cap A \subset \{(x, t, 0; \xi, \tau, v); (x, t; \xi, \tau) \in SS(F), -\tau \leq v \leq 0\}.$$ 

In particular $Rj_! \Psi_U(F) \simeq Rj_! \Psi_U(F)$.

**Remark 11.1.8.** If we restrict to a “slice” $U_\gamma^\xi = U_\gamma \cap (U \times \{\xi\})$, we obtain $SS(\Psi_U(F)|_{U_\gamma^\xi}) \subset (SS(F) \cup T_\xi(\bar{SS}(F))) \cap T^*U_\gamma^\xi$, where $T_\xi$ is the translation $T_\xi(x, t; \xi, \tau) = (x, t + c; \xi, \tau)$.

**Proof.** (i) The first inclusion follows from the triangle (11.1.10) and the triangular inequality for the microsupport. To prove the second inclusion we consider $\gamma$ as a subset of $\mathbb{R}^2$ rather than $\mathbb{R} \times \mathbb{R} > 0$. We also consider the sum $s, (x, t_1, t_2, u) \mapsto (x, t_1 + t_2, u)$, as a map from $U \times \mathbb{R}^3$ to $M \times \mathbb{R}^2$. Then by the base change formula we have $Rj_! \Psi_U(F) \simeq Rs_!(F \boxtimes k_\gamma)$.

Setting $V = (U \times [-\infty, 0]) \cup U_\gamma$ we see that $s$ is proper as a map from $s^{-1}(V) \cap (U \times \gamma)$ to $V$. Hence we can use Proposition 1.2.4 to bound $SS(Rs_!(F \boxtimes k_\gamma)|_V)$. We see that we only need to know $SS(F \boxtimes k_\gamma)$ above $U \times \mathbb{R} \times \{0\}$. Now $SS(k_\gamma) \cap T^*_{(t, 0)}\mathbb{R}^2$ is empty for $t \neq 0$ and is the cone $\{-\tau \leq v \leq 0\}$ for $t = 0$. Hence $SS(F \boxtimes k_\gamma) \cap \{u =
by the excision distinguished triangle, we are reduced to prove that
\[ W(x, t_1; \xi, \tau_1) \in SS(F), -\tau_2 \leq v \leq 0. \]
We conclude with Proposition 1.2.4.

(ii) We deduce \( Rj_! \Psi_U(F) \cong Rj_! \Psi_U(F) \). Let us set \( Z = U \times ]-\infty, 0]\) and \( W = U \times ]0, +\infty[. \) Then \( Rj_! \Psi_U(F) \cong R\Gamma W(Rj_! \Psi_U(F)) \) and, by the excision distinguished triangle, we are reduced to prove that \( R\Gamma Z(Rj_! \Psi_U(F)) \cong 0. \) Since the support of \( Rj_! \Psi_U(F) \) is contained in \( \overline{W} \), it is enough to prove \( (R\Gamma Z(Rj_! \Psi_U(F)))_z \cong 0 \) for any \( z \in U \times \{0\}. \) Going back to the definition of the microsupport, this follows from \( (z, 0; 0, -1) \not\in SS(Rj_! \Psi_U(F)). \)

**Proposition 11.1.9.** Let \( F, G \in D_{\tau \geq 0}(k_U) \). We set \( \Omega = T^*_\tau \gamma U^\gamma \) for short. Then we have a natural decomposition
\[
\mu_{\text{hom}}(\Psi_U(F), \Psi_U(G))|_{\Omega} \cong \mu_{\text{hom}}(q_U^{-1}(F), q_U^{-1}(G))|_{\Omega} \\
\oplus \mu_{\text{hom}}(r_U^{-1}(F), r_U^{-1}(G))|_{\Omega}
\]
such that the corresponding projection from \( \mu_{\text{hom}}(\Psi_U(F), \Psi_U(G))|_{\Omega} \) to \( \mu_{\text{hom}}(q_U^{-1}(F), q_U^{-1}(G))|_{\Omega} \) coincides with \( \alpha_2 \circ \alpha_1 \), where \( \alpha_1, \alpha_2 \) are respectively induced by the morphisms \( \alpha(G) \) and \( \alpha(F) \) of (11.1.7) as follows:
\[
\mu_{\text{hom}}(\Psi_U(F), \Psi_U(G))|_{\Omega} \xrightarrow{\alpha_1} \mu_{\text{hom}}(\Psi_U(F), q_U^{-1}(G))|_{\Omega} \\
\xrightarrow{\alpha_2} \mu_{\text{hom}}(q_U^{-1}(F), q_U^{-1}(G))|_{\Omega}.
\]

**Proof.** (i) We set for short \( h(A) = \mu_{\text{hom}}(\Psi_U(F), A)|_{\Omega} \) for a sheaf \( A \) on \( U^\gamma \). The triangle (11.1.10) induces a distinguished triangle
\[
h(r_U^{-1}(G))|_{[-1]} \rightarrow h(\Psi_U(G)) \xrightarrow{\alpha_1} h(q_U^{-1}(G)) \rightarrow h(r_U^{-1}(G))
\]
on \( \Omega \). We recall the bound \( \text{supp} \mu_{\text{hom}}(F_1, F_2) \subset SS(F_1) \cap SS(F_2) \). By Theorem 1.2.8 the microsupports of \( q_U^{-1}(G) \) and \( r_U^{-1}(F) \) are respectively contained in \( \{(x, t, u; \xi, \tau, 0)\} \) and \( \{(x, t, u; \xi, \tau, -\tau)\} \). Hence their intersection is contained in \( \{
\tau = 0\} \). Since we work on \( \Omega = T^*_\tau \gamma U^\gamma \) it follows that the supports of \( h(q_U^{-1}(G)) \) and \( h(r_U^{-1}(G)) \) are disjoint, hence that \( \gamma_1 = 0 \). This proves that \( h(\Psi_U(G)) \cong h(q_U^{-1}(G)) \oplus h(r_U^{-1}(G)) \).

(ii) Now we check that \( \alpha_2: \mu_{\text{hom}}(q_U^{-1}(F), q_U^{-1}(G))|_{\Omega} \rightarrow h(q_U^{-1}(G)) \) is an isomorphism. Using the triangle (11.1.10) again, we see that the cone of \( \alpha_2 \) is \( \mu_{\text{hom}}(r_U^{-1}(F), q_U^{-1}(G))|_{\Omega} \) whose support is contained in \( SS(r_U^{-1}(F)) \cap SS(q_U^{-1}(G)) \). The same bound as in (i) shows that this set is contained in \( \{\tau = 0\} \) and does not meet \( \Omega \). Hence the cone of \( \alpha_2 \) vanishes and \( \alpha_2 \) is an isomorphism.

We obtain \( \mu_{\text{hom}}(r_U^{-1}(F), r_U^{-1}(G))|_{\Omega} \cong h(r_U^{-1}(G)) \) by swapping \( q_U \) and \( r_U \) in the previous argument. This concludes the proof of the proposition. \( \square \)
11.2. Adjoint properties

Let $U \subset M \times \mathbb{R}$ be an open subset. We let $D^a_{\geq 0}(k_U)$ be the full subcategory of $D_{\geq 0}(k_U)$ consisting of $Rr_U$-acyclic objects, that is, the objects $G$ such that $Rr_U G \simeq 0$. This is a triangulated category. By Lemma 11.1.6 (i) the functor $\Psi U$ takes values in $D^a_{\geq 0}(k_U)$. Using Theorem 1.2.15 for the embedding $q_U$ we see that the functor $R\alpha$ is adjoint to $\Psi U$.

By Lemma 11.2.1, the functor $\Psi U$ is adjoint to $R\Psi U$.\[\begin{equation}
(11.2.1)\end{equation}\]

Moreover the morphism of functors $\beta : \Psi U \rightarrow q_U^{-1}$ in $\Psi U$ and the adjunction morphism $Rq_U q_U^{-1} \simeq Rq_U q_U^{-1}[1] \rightarrow \text{id}$ induce:

(11.2.1) $b_U(F) : Rq_U \Psi U(F)[1] \rightarrow F$, for all $F \in D_{\geq 0}(k_U)$.

**Lemma 11.2.1.** The functor $Rq_U[1] : D^a_{\geq 0}(k_U) \rightarrow D_{\geq 0}(k_U)$ is left adjoint to $\Psi U : D_{\geq 0}(k_U) \rightarrow D^a_{\geq 0}(k_U)$. In particular we have an adjunction morphism

(11.2.2) $b'_U(G) : G \rightarrow \Psi U Rq_U(G)[1]$, for all $G \in D^a_{\geq 0}(k_U)$.

**Proof.** Since $D_{\geq 0}(k_U)$ and $D^a_{\geq 0}(k_U)$ are full subcategories of $D(k_U)$ and $D(k_U)$, it is enough to prove

(11.2.3) $\text{Hom}_D(k_U)(G, \Psi U(F)) \simeq \text{Hom}_D(k_U)(Rq_U G[1], F)$

for any $F \in D_{\geq 0}(k_U)$ and $G \in D^a_{\geq 0}(k_U)$. Since $r_U$ is a smooth map with fibers homeomorphic to $\mathbb{R}$ we have a canonical isomorphism of functors $r'_U \simeq r_U^{-1}[1]$; hence an adjunction $(Rr_U, r_U^{-1}[1])$. The same holds for $q_U$. Applying $R\text{Hom}(G, \cdot)$ to $(11.1.10)$ we obtain the distinguished triangle

$$\text{RHom}(G, r_U^{-1} F[-1]) \rightarrow \text{RHom}(G, \Psi U(F)) \rightarrow \text{RHom}(G, q_U^{-1} F) \rightarrow .$$

The adjunction $(Rr_U, r_U^{-1}[1])$ and the hypothesis $G \in D^a_{\geq 0}(k_U)$ give $\text{RHom}(G, r_U^{-1} F[-1]) \simeq 0$. We deduce

$$\text{RHom}(G, \Psi U(F)) \simeq \text{RHom}(G, q_U^{-1} F) \simeq \text{RHom}(Rq_U G[1], F),$$

which implies $(11.2.3)$. $\square$

**Lemma 11.2.2.** Let $F \in D_{\geq 0}(k_U)$. Then the morphisms $\Psi U(b_U(F))$ and $b'_U(\Psi U(F))$ are mutually inverse isomorphisms:

$$\Psi U(b_U(F)) : \Psi U Rq_U \Psi U(F)[1] \rightarrow \Psi U(F),$$

$$b'_U(\Psi U(F)) : \Psi U(F) \rightarrow \Psi U Rq_U \Psi U(F)[1].$$

**Proof.** (i) We prove the first isomorphism. We apply $Rq_U[1]$ to the distinguished triangle $(11.1.10)$. Since $q_U$ has fibers isomorphic to $\mathbb{R}$
the adjunction morphism $Rq_U q_U^{-1}(F)[1] \simeq Rq_U q_U^{-1}(F) \to F$ is an isomorphism and we obtain the distinguished triangle:

(11.2.4) \[ L \to Rq_U \Psi_U(F)[1] \xrightarrow{b_U(F)} F \to 1, \]

where $L = Rq_U q_U^{-1}(F)$. By Lemma 11.1.6 (i) and (iii) we have $\Psi_U(L) \simeq 0$. Hence applying $\Psi_U$ to (11.2.4) gives the lemma.

(ii) The composition $\Psi_U(b_U(F)) \circ b_U(\Psi_U(F))$ is the identity morphism of $\Psi_U(F)$, by general properties of adjunctions. Hence the lemma follows from (i).

**Proposition 11.2.3.** We assume that $U = M \times \mathbb{R}$, that $F \in D_{\geq 0}(k_U)$ and that $\text{supp}(F) \subset M \times [a, +\infty[\text{ for some } a \in \mathbb{R}$. Then the adjunction morphism $b_U(F) : Rq_U \Psi_U(F)[1] \to F$ of (11.2.1) is an isomorphism and for any $G \in D_{\geq 0}(k_U)$ we have

\[ \text{Hom}(F, G) \simeq \text{Hom}(\Psi_U(F), \Psi_U(G)). \]

**Proof.** By Lemma 11.1.6 (i) and (iii) we have $Rq_U q_U^{-1}(F) \simeq F$ and $Rq_U q_U^{-1}(F) \simeq 0$. Hence the first part follows from the distinguished triangle (11.1.10). Then the second part is given by the adjunction $(Rq_U, \Psi_U)$ of Lemma 11.2.1. \qed

### 11.3. Link with microlocalization

In this section we prove Proposition 11.3.7 below which says that the group of homomorphisms from $\Psi_U(F)$ to $\Psi_U(G)$ is isomorphic to the group of sections of $\muhom(F, G)$ outside the zero section. We first deduce from Proposition 11.1.9 a morphism from $\muhom(\Psi_U(F), \Psi_U(G))$ to $\muhom(F, G)$. Then we use “boundary values” of sheaves on $U_\gamma$ in the following sense. Let $U \subset M \times \mathbb{R}$ and $V \subset M \times \mathbb{R} \times \mathbb{R}_{>0}$ be open subsets satisfying

(11.3.1) \[ i_U(U) \sqcup j_V(V) \text{ is open in } M \times \mathbb{R} \times \mathbb{R}_{\geq 0}, \]

where $i_U, j_V$ are the natural inclusions

(11.3.2) \[ i_U : U \to M \times \mathbb{R} \times \mathbb{R}, \quad (x, t) \mapsto (x, t, 0), \]
\[ j_V : V \to M \times \mathbb{R} \times \mathbb{R}. \]

Then, for $G \in D(k_V)$, its boundary value is $i_U^{-1} R j_{V*} (G) \in D(k_U)$. We remark that we can shrink $V$ as long as (11.3.1) is satisfied:

**Lemma 11.3.1.** Let $V' \subset V$ be an open subset such that $i_U(U) \sqcup j_{V'}(V')$ is open in $M \times \mathbb{R} \times \mathbb{R}_{\geq 0}$. Then the morphism $R j_{V*}(G) \to R j_{V'*} (G|_{V'})$, obtained from $j_{V'}^{-1} R j_{V*}(G) \simeq G|_{V'}$ by the adjunction $(j_{V'}^{-1}, R j_{V*})$, induces an isomorphism $i_U^{-1} R j_{V*}(G) \simeq i_U^{-1} R j_{V'*}(G|_{V'})$. 

Proof. To check that a given morphism is an isomorphism it is enough to see that it gives an isomorphism on the cohomology of the stalks at any point. For \((x, t) \in U\) and \(k \in \mathbb{Z}\) we have \(H^k(i_U^{-1}Rj_{*V*}(G))(x,t) \simeq \lim_{\rightarrow W}H^k(W \cap V; G)\), where \(W\) runs over the neighborhoods of \((x, t, 0)\) in \(M \times \mathbb{R} \times \mathbb{R}_{>0}\). For a given \((x, t) \in U\) we have \(W \cap V = W \cap V'\) when \(W\) is small enough and the result follows. 

\[\square\]

**Notation 11.3.2.** Because of Lemma [11.3.1] we can forget the subscript \(V\) and write the boundary value as follows.

Let \(\pi_U^{>0}: T_{\tau>0}U \to U\), \(\pi_U^{>0}: T_{\tau>0}U_\gamma \to U_\gamma\) be the projections. Proposition [11.1.9] yields a morphism, for \(F, G \in \mathcal{D}_{\tau\geq 0}(k_U)\),

\[\text{RHom}(\Psi_U(F), \Psi_U(G))\]
\[\to \text{R}\pi_U^{>0}(\muhom(q_U^{-1}(F), q_U^{-1}(G))|_{T_{\tau>0}U_\gamma})\]

which can be written as the composition

\[\text{RHom}(\Psi_U(F), \Psi_U(G))\]
\[\to \text{RHom}(\Psi_U(F), q_U^{-1}(G))\]
\[\simeq \text{R}\pi_U(\muhom(F, q_U^{-1}(G)))\]
\[\to \text{R}\pi_U^{>0}(\muhom(F, q_U^{-1}(G))|_{T_{\tau>0}U_\gamma})\]
\[\simeq \text{R}\pi_U^{>0}(\muhom(F, q_U^{-1}(G))|_{T_{\tau>0}U_\gamma})\]

where the first line is induced by \(\alpha(G): \Psi_U(G) \to q_U^{-1}(G)\), the second line is [11.3.2], the third line is the restriction to \(T_{\tau>0}U_\gamma\) and the fourth line is given by Proposition [11.1.9]. Now it is easy to describe the boundary value of the right hand side of [11.3.3] as follows.

**Lemma 11.3.3.** For \(F, G, H \in \mathcal{D}(k_U)\) and \(\mathcal{F} \in \mathcal{D}(k_{T^*U})\) we have natural isomorphisms

\[\muhom(q_U^{-1}(F), q_U^{-1}(G)) \simeq q_{U,d| q_{U,\pi} \muhom(F, G)},\]
\[\text{R}\pi_U^{>0}((q_{U,d| q_{U,\pi} \muhom(F)}|_{T_{\tau>0}U_\gamma}) \simeq q_U^{-1}\text{R}\pi_U^{>0}(\mathcal{F}|_{T_{\tau>0}U}),\]
\[i_U^{-1}Rj_{*q_U^{-1}H} \simeq H\]

which induce

\[i_U^{-1}Rj_{*}\pi_U^{>0}(\muhom(q_U^{-1}(F), q_U^{-1}(G))|_{T_{\tau>0}U_\gamma})\]
\[\simeq \text{R}\pi_U^{>0}(\muhom(F, G)|_{T_{\tau>0}U}).\]
Proof. (i) The behaviour of $\mu\hom$ under an inverse image by a submersion is described in [31, Prop. 4.4.7] and gives in our case the isomorphism $\text{(11.3.6)}$.

(ii) We remark that $q_U, d_1 = q_{U, d_1}$, since $q_{U, d}$ is an embedding, and $q^{-1}_U \simeq q^{-1}_U[-1]$, $q_{U, \pi} \simeq q^{-1}_{U, \pi}[1]$, since $q_U$ is a projection with fibers $\mathbb{R}$. Now $\text{(11.3.7)}$ follows from the base change formula $R\alpha q^{-1}_U \simeq q^{-1}_U R\pi_{U, *}$ with $a = \pi_{U, *} \circ q_{U, d}$.

(iii) We have $i^{-1}_U R j_* q^{-1}_U H \simeq i^{-1}_U R \Gamma_{U \times \mathbb{R}_>0}(q^{-1}_1 H)$, where $q_1 : U \times \mathbb{R} \to U$ is the projection. Since $SS(k_{U \times \mathbb{R}_>0}) \subset T_1 U \times T^* \mathbb{R}$, Theorem $\text{(1.2.13)}$ gives

$$R\Gamma_{U \times \mathbb{R}_>0}(q^{-1}_1 H) \simeq R\Gamma_{U \times \mathbb{R}_>0}(q^{-1}_1 H) \simeq k_{U \times \mathbb{R}_>0} \otimes q^{-1}_1 H$$

and we obtain $i^{-1}_U R j_* q^{-1}_U H \simeq i^{-1}_U (k_{U \times \mathbb{R}_>0} \otimes q^{-1}_1 H) \simeq H$. \hfill $\square$

Definition 11.3.4. For $F, G \in D_{\tau \geq 0}(k_U)$ we define the morphism (functorial in $F$ and $G$)

$$b(F, G) : i^{-1}_U R j_* R\Gamma_{U \times \mathbb{R}_>0}(q^{-1}_1 H) \to R\pi_{U, *}(\mu\hom(F, G)|_{T^*_\tau U})$$

as the composition of $\text{(11.3.3)}$ and $\text{(11.3.9)}$.

We prove below that $b(F, G)$ is an isomorphism if $G \in D_{\tau > 0}(k_U)$. We need some remarks on the $\mu\hom$ functor. We will use Sato’s distinguished triangle $\text{(1.3.3)}$ and introduce the following notation.

Notation 11.3.5. Let $X$ be a manifold. Let $q_{X, 1}, q_{X, 2} : X \times X \to X$ be the projections and $\delta_X : X \to X \times X$ the diagonal embedding. Let $F, F' \in D(k_X)$. We set

$$\text{(11.3.10)}$$

$$\mathcal{Hom}'(F, F') := \delta_X^{-1} R\Gamma_{U \times \mathbb{R}_>0}(q^{-1}_{X, 2} F, q^{-1}_{X, 1} F').$$

Then Sato’s distinguished triangle becomes, for $F, F' \in D(k_X),$

$$\mathcal{Hom}'(F, F') \to R\Gamma_{U \times \mathbb{R}_>0}(q^{-1}_{X, 1} F, q^{-1}_{X, 2} F')$$

$$\to R\pi_{X, *}(\mu\hom(F, F')|_{T^*_\tau X}) \to.$$

Lemma 11.3.6. (i) Let $f : X \to Y$ be a morphism of manifolds. Let $F, F' \in D(k_Y)$ such that $f$ is non-characteristic for $SS(F)$ and $SS(F')$. Then

$$f^{-1}\mathcal{Hom}'(F, F') \simeq \mathcal{Hom}'(f^{-1} F, f^{-1} F').$$

(ii) For $F, F' \in D(k_X)$ we have $SS(\mathcal{Hom}'(F, F')) \subset SS(F)^a \cap SS(F')$. 


Proof. (i) We set $G = R\mathcal{H}om(q_{12}^{-1}F, q_{11}^{-1}F')$ (using similar notations as in (11.3.10)). Then $\text{SS}(q_{11}^{-1}F')$ and $\text{SS}(G) \subset \text{SS}(F)^{\circ} \times \text{SS}(F')$ are non-characteristic for $f \times f$. By Theorem 1.2.8 we deduce
\[
(f \times f)q_{11}^{-1}F' \simeq (f \times f)^{-1}q_{11}^{-1}F' \otimes \omega_{X \times X|Y \times Y},
\]
\[
(f \times f)^{-1}G \simeq (f \times f)^{-1}G \otimes \omega_{X \times X|Y \times Y}^{-1}.
\]

By Proposition 1.1.1(h) this gives the third isomorphism in the following sequence:
\[
f^{-1}\mathcal{H}om'(F, F') \simeq f^{-1}\delta^{-1}R\mathcal{H}om(q_{12}^{-1}F, q_{11}^{-1}F')
\]
\[
\simeq \delta_X^{-1}(f \times f)^{-1}R\mathcal{H}om(q_{12}^{-1}F, q_{11}^{-1}F')
\]
\[
\simeq \delta_X^{-1}R\mathcal{H}om((f \times f)^{-1}q_{12}^{-1}F, (f \times f)^{-1}q_{11}^{-1}F')
\]
\[
\simeq \delta_X^{-1}R\mathcal{H}om(q_{12}^{-1}f^{-1}F, q_{11}^{-1}f^{-1}F').
\]

(ii) follows from Remark 1.2.14 and Theorem 1.2.8. □

Proposition 11.3.7. Let $F, G \in D_{r \geq 0}(k_U)$. If $\check{\text{SS}}(F) \cap \text{SS}(G)$ is contained in $\{\tau > 0\}$, then we have
\[
R\pi_U^*(\mu\text{hom}(F, G)|_{\check{T}_{r \cdot U}}) \cong R\pi_U^*(\mu\text{hom}(F, G)|_{T_{r \cdot U}})
\]
and the morphism $b(F, G)$ of Definition 11.3.4 is an isomorphism.

Proof. (i) We recall that $\text{supp} \mu\text{hom}(F_1, F_2) \subset \text{SS}(F_1) \cap \text{SS}(F_2)$. Hence the hypothesis implies that $\mu\text{hom}(F, G)|_{\check{T}_{r \cdot U}}$ is supported in $T_{r \cdot U}^{r \cdot 0}$ and this gives the first isomorphism. Let us call $u$ and $v$ the morphisms (11.3.4) and (11.3.5). To see that $b(F, G)$ is an isomorphism, we prove that the induced morphisms $i_U^{-1}Rj_u(u)$ and $i_U^{-1}Rj_v(v)$ are isomorphisms, respectively in (ii) and (iii) below.

(ii) Let us prove that $i_U^{-1}Rj_u(u)$ is an isomorphism. By the distinguished triangle (11.1.10) its cone is $A = i_U^{-1}Rj_uR\mathcal{H}om(\Psi_U(F), r_U^{-1}(G))$ and we have to prove the vanishing of $A$. For a given $(x, t) \in U$ and $k \in \mathbb{Z}$ we have
\[
H^k(A)_{(x,t)} \simeq \lim_{\substack{\longrightarrow \atop W}} \text{Hom}(\Psi_U(F)|_{W}, r_U^{-1}(G)|_{W[k]}),
\]
where $W$ runs over the open subsets of $M \times \mathbb{R} \times \mathbb{R}_{\geq 0}$ such that $\overline{W}$ is a neighborhood of $(x, t, 0)$ in $U \times [0, +\infty]$. We may take $W = V_\gamma$, where $V$ runs over the open neighborhoods of $(x, t)$ in $U$. By (11.1.11) we have $\Psi_U(F)|_{V_\gamma} \simeq \Psi_U(F|_{V})$. We also have $r_U^{-1}(G)|_{V_\gamma} \simeq r_V^{-1}(G|_{V})$. Since $r_V^1 \simeq r_V^{-1}[1]$, the adjunction $(Rr_V^1, r_V^{-1})$ gives
\[
\text{Hom}(\Psi_U(F|_{V}), r_V^{-1}(G|_{V})[k]) \simeq \text{Hom}(Rr_V^1\Psi_U(F|_{V}), G|_{V}[k-1]).
\]
By Lemma [11.1.6] we have \( Rr_{V!}\Psi_V(F|_V) \simeq 0 \) and we deduce the vanishing of \([11.3.12]\) for all \((x,t)\) in \(U\). Hence \(A \simeq 0\), as required.

(iii) Now we prove that \(i_U^{-1}Rj_* (v)\) is an isomorphism. Using the hypothesis on the microsupport, we remark as in (i) that the right hand side of \([11.3.5]\) is isomorphic to \(R\hat{\pi}_{U \gamma*}(\mu\text{hom}(\Psi_U(F), q_U^{-1}(G)))|_{T^*U_{\gamma}}\). Hence, by the triangle \([11.3.11]\), the cone of \(i_U^{-1}Rj_* (v)\) is (up to a shift by 1) \(B := i_U^{-1}Rj_*\text{Hom}'(\Psi_U(F), q_U^{-1}(G))\). Let us prove that \(B\) vanishes.

Let \(p_U: U \times \mathbb{R} \to U\) be the projection. We set \(U_+ = U \times \mathbb{R}_{>0}\) and \(C = \text{Hom}'(Rj_!\Psi_U(F), p_U^{-1}(G))\). Then

\[
Rj_*\text{Hom}'(\Psi_U(F), q_U^{-1}(G)) \simeq R\Gamma_{U_+}C.
\]

By Lemma [11.1.7] we have \(\text{SS}(Rj_!\Psi_U(F)) \subset \{(\xi, \tau, v); -\tau \leq v \leq 0\}\). We also have \(\text{SS}(p_U^{-1}(G)) \subset \{(\xi, \tau, 0); \tau \geq 0\}\) and we obtain \(\text{SS}(C) \subset \{v \geq 0\}\) by Lemma [11.3.6]. Since \(\text{SS}(k_{U_+}) = T^*_U U \times \{v < 0\}\), we deduce \(R\Gamma_{U_+} C \simeq D'(k_{U_+}) \otimes C \simeq C|_{U_+}\) by Theorem [11.2.13]. In particular \(B \simeq i_U^{-1}C\).

We also obtain that \(i_U\) is non-characteristic for \(\text{SS}(Rj_!\Psi_U(F))\) and \(\text{SS}(p_U^{-1}(G))\). Hence \(i_U^{-1}C \simeq \text{Hom}'(i_U^{-1}Rj_!\Psi_U(F), G)\) by Lemma [11.3.6]. Since \(i_U^{-1}Rj_!\Psi_U(F) \simeq 0\), we obtain \(i_U^{-1}C \simeq 0\), as required. \(\square\)

For the next result we use the notion of pure sheaves (see Definition [1.4.2]) along a Lagrangian submanifold \(\Lambda \subset T^*U\) and the stack \(\mu\text{Sh}(k_{\Lambda})\) together with the functor \(m_{\Lambda}: D^b_{(\Lambda)}(k_{\Lambda}) \to \mu\text{Sh}(k_{\Lambda})\) (see Definition [10.1.1]).

**Corollary 11.3.8.** Let \(\Lambda \subset T_{>0}^*U\) be a closed conic Lagrangian submanifold. Let \(F, G \in D_{(\Lambda)}(k_{\Lambda})\) be pure sheaves with the same shift. Then

\[
\lim_{V \to V'} \text{Hom}(\Psi_U(F)|_V, \Psi_U(G)|_V) \simeq \text{Hom}(m_{\Lambda}(F), m_{\Lambda}(G)),
\]

where \(V\) runs over the open subsets of \(U_\gamma\) such that \((U \times \{0\}) \cup V\) is open in \(M \times \mathbb{R} \times \mathbb{R}_{>0}\). In particular, if \(m_{\Lambda}(F) \simeq m_{\Lambda}(G)\), then there exists such an open subset \(V\) such that \(\Psi_U(F)|_V \simeq \Psi_U(G)|_V\).

**Proof.** (i) We recall that the \(\text{Hom}\) sheaf in \(\mu\text{Sh}(k_{\Lambda})\) is induced by \(H^0\mu\text{hom}\). By the purity hypothesis \(\mu\text{hom}(F, G)\) is concentrated in degree 0 and \([10.1.6]\) gives \(\text{Hom}(m_{\Lambda}(F), m_{\Lambda}(G)) \simeq H^0(\Lambda; \mu\text{hom}(F, G))\). Hence the isomorphism of the corollary follows from Proposition [11.3.7] and the remark that, for any \(F' \in D(k_{U_\gamma})\),

\[
H^0(U; i_U^{-1}Rj_*(F')) \simeq \lim_{W \to W'} H^k(W \cap U_\gamma; F'),
\]

where \(W\) runs over the neighborhoods of \(U\) in \(M \times \mathbb{R} \times \mathbb{R}_{>0}\).
Let \( u : \mathfrak{m}_A(F) \cong \mathfrak{m}_A(G) \) be an isomorphism. By (i) there exist \( V \) as in the statement of the corollary and \( a : \Psi_U(F)|_V \to \Psi_U(G)|_V, \ b : \Psi_U(G)|_V \to \Psi_U(U)|_V \) representing \( u, u^{-1} \). Using (i) again, and maybe shrinking \( V \), the relations \( u \circ u^{-1} = id, \ u^{-1} \circ u = id \) give \( a \circ b = id, \ b \circ a = id \). Hence \( a \) is an isomorphism.

\[ \square \]

### 11.4. Doubled sheaves

The isomorphism of Proposition [11.3.7] will be used mainly when \( SS(F) = SS(G) = \Lambda \) is a smooth Lagrangian submanifold of \( T^*_\tau(M \times \mathbb{R}) \). In this case \( \mu \text{hom}(F,G)|_{T^*_\tau(M \times \mathbb{R})} \) is a locally constant sheaf on \( \Lambda \).

Unfortunately we will also need to consider the case where \( F \) and \( G \) have microsupport \( \Lambda \) only in a neighborhood of some open subset \( \Lambda_0 \) of \( \Lambda \). To easily handle this case we actually assume that \( \Lambda \) have microsupport only in a neighborhood of some open subset \( \Lambda_0 \).

In this section we use adapted families of open subsets to cut sheaves defined on an open subset \( W \) of \( M \times \mathbb{R} \). To avoid

**Notation 11.4.1.** Let \( \Lambda \subset T^*_\tau(M \times \mathbb{R}) \) be a conic Lagrangian submanifold such that \( \Lambda/\mathbb{R}_{>0} \) is compact, the map \( \Lambda/\mathbb{R}_{>0} \to M \) has finite fibers. A finite family \( U = \{ U_a ; a \in A \} \) of open subsets of \( M \times \mathbb{R} \) is said to be adapted to \( \Lambda \) if the following conditions hold:

(i) for each \( a \in A \) we have \( V_a = U_a \times I_a \) and \( \Lambda \cap T^*V_a \) is contained in \( \pi^{-1}_{M \times \mathbb{R}}(U_a \times K) \) for some compact interval \( K \) of \( I_a \),

(ii) for all \( B, B' \subset A \) we have \( D'(k_{V_B}) \cong k_{V_B} \) and, setting \( \Lambda_+ = \Lambda \cup T^*_\tau(M \times \mathbb{R}) \),

\[ (SS(k_{V_B}) \oplus SS(k_{V_B}^n)) \cap (\Lambda_+^a \oplus \Lambda_+) \subset T^*_\tau(M \times \mathbb{R}). \]

**Notation 11.4.2.** In this section we use adapted families of open subsets to cut sheaves defined on an open subset \( W \) of \( M \times \mathbb{R} \). To avoid
too heavy notations we set abusively, for $F \in \mathcal{D}(k_W)$ and $B \subset A$:
\[ \Gamma_{V,B}(F) := \Gamma_{W \cap V,B}(F) \in \mathcal{D}(k_W) \]
and similarly $\Gamma_{V,B \times \mathbb{R}_{>0}}(-) := \Gamma_{W \cap (V,B \times \mathbb{R}_{>0})}(-)$ for sheaves defined over $W$, or $\Gamma_{T^*V,B}(-) := \Gamma_{T^*W \cap T^*V}(-)$ over $T^*W$.

We first check that we have enough such adapted families.

**Lemma 11.4.3.** Let $\Lambda \subset T^*_M(M \times \mathbb{R})$ be a conic Lagrangian submanifold such that $\Lambda/\mathbb{R}_{>0}$ is compact and let $\Lambda = \bigcup_{i \in I} \Lambda_i$ be a finite covering by conic open subsets. Then there exists a Hamiltonian isotopy $\Phi$, as closed to $\text{id}$ as desired, and a finite family $\{V_a\}_{a \in A}$ of open subsets of $M \times \mathbb{R}$ which is adapted to $\Lambda' := \Phi_1(\Lambda)$ such that each connected component of $\Lambda' \cap T^*V_a$, for $a \in A$, is contained in $\Phi_1(\Lambda_i)$, for some $i \in I$. Moreover we can assume that the family $\{V_a\}_{a \in A}$ is stable by intersection and that $\Lambda' \cap T^*V_a$ has finitely many connected components, for each $a \in A$.

**Proof.** (i) We recall Definition 1.2.19 a stratification $\Sigma = \{\Sigma_j, j \in J\}$ of $M \times \mathbb{R}$ satisfies the $\mu$-condition if $\Lambda_0 \subset \Sigma \subset \Lambda$, where $\Lambda_0 = \bigcup_{j \in J} T_{\Sigma_j}(M \times \mathbb{R})$. By Proposition 1.2.20 any sheaf $F \in \mathcal{D}(k_{M \times \mathbb{R}})$ which is constructible with respect to $\Sigma$ satisfies $SS(F) \subset \Lambda$. By Proposition 1.2.23 there exists an $\mathbb{R}_{>0}$-homogeneous Hamiltonian isotopy $\Phi$ and a stratification $\Sigma$ of $M \times \mathbb{R}$ satisfying the $\mu$-condition such that $\Lambda' := \Phi_1(\Lambda) \subset \Lambda$. Now to have (11.4.1) it is enough that the family $\{V_a\}_{a \in A}$ satisfies
\[ \left( SS(k_{V,B}) \supset SS(k_{V,B'}) \right) \cap \Lambda_0 \cap \pi_{M \times \mathbb{R}}^{-1}(\pi_{M \times \mathbb{R}}(\Lambda')) \subset T^*_{M \times \mathbb{R}}(M \times \mathbb{R}). \]

(ii) We can assume that $\Lambda'$ is also contained in $T^*_M(M \times \mathbb{R})$. Hence, for a given point $y_0 = (x_0, t_0)$ in $\hat{\pi}_{M \times \mathbb{R}}(\Lambda')$ the strata $\Sigma_j$ such that $y_0 \in \Sigma_j \subset \hat{\pi}_{M \times \mathbb{R}}(\Lambda')$ do not meet the truncated cone given in local coordinates by $\{ \eta > |t - t_0| > C||x - x_0|| \}$, for $\eta > 0$ small enough and $C$ big enough. Then, for $U = \{ ||x - x_0|| < \eta/2C \} \subset M$ and $I = \{ |t - t_0| < \eta \} \subset \mathbb{R}$, we have $y_0 \in U \times I$ and $\hat{\pi}_{M \times \mathbb{R}}(\Lambda') \cap (U \times \partial I) = \emptyset$. We can then cover $\pi_{M \times \mathbb{R}}(\Lambda')$ by open subsets of this kind and, taking a finite subcover, we obtain a finite family of open subsets of $M \times \mathbb{R}$, $W_a$, $a \in A$, such that $\pi_{M \times \mathbb{R}}(\Lambda') \subset \bigcup_{a \in A} W_a$, $W_a = U_a \times I_a$ as in (i) of Definition 11.4.1 and $U_a = \{ f_a < 0 \}$ for a $C^\infty$ function $f_a: M \to \mathbb{R}$. Moreover, for each $a \in A$ there exists a compact interval $K_a \subset I_a$ such that $\Lambda' \cap T^*W_a$ is contained in $\pi_{M \times \mathbb{R}}^{-1}(U_a \times K_a)$. We can also assume that each $W_a$ is small enough so that each component of $\Lambda' \cap T^*W_a$ is contained in $\Phi_1(\Lambda_i)$, for some $i \in I$. 

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Let \( \varepsilon = \{\varepsilon_a; a \in A\} \) be a family of negative numbers. We define \( W_{a,\varepsilon_a} = \{f_a < \varepsilon_a\} \times I_a \). We choose \( \delta > 0 \) such that \( \pi_{M \times \mathbb{R}}(\Lambda') \subset \bigcup_{a \in A} W_{a,\varepsilon_a} \). Let \( E \subset [-\delta,0[^A \) be the subset formed by the \( \varepsilon \) such that:

(a) the hypersurfaces \( X_{a,\varepsilon_a} = \{f_a = \varepsilon_a\}, a \in A, \) are smooth and intersect transversely, in the sense that their union is locally diffeomorphic to the embedding of coordinates hyperplanes in \( \mathbb{R}^n \),

(b) for any \( B \subset A \), the manifold \( X_{B,\varepsilon} = \bigcap_{a \in B}(X_{a,\varepsilon_a} \times I_a) \) intersects each stratum \( \Sigma_j \) of \( \Sigma \) transversely.

By the transversality theorem \( E \) is dense in \([-\delta,0[^A \). Hence we only have to prove that, for a given \( \varepsilon \in E \), the family \( V_a = W_{a,\varepsilon_a}, a \in A, \) satisfies the conclusion of the lemma.

The condition \( D'(k_{V'B}) \simeq k_{V'B} \) is local on \( M \times \mathbb{R} \) and follows from the above condition (a), up to modifying slightly the intervals \( I_a \)'s so that they have distinct ends.

We set \( Y = \bigcup_{a \in A}(X_{a,\varepsilon_a} \times \partial I_a) \) and \( \Omega = (M \times \mathbb{R}) \setminus Y \). We remark that \( \pi_{M \times \mathbb{R}}(\Lambda') \cap Y = \emptyset \). Hence, to prove \((11.4.2)\) it is enough to see \((SS(k_{V'B})) \cap \Lambda \cap T^\ast \Omega \subset \mathbb{T}^\ast \Omega \).

In \( \Omega \) we have the family of hypersurfaces \( (X_{a,\varepsilon_a} \times I_a) \cap \Omega, a \in A, \) which are (not only locally closed), smooth and intersect transversely. This family generates a stratification \( \Sigma'(\varepsilon) \) of \( \Omega \) such that the closures of the strata are the \( X_{B,\varepsilon} \cap \Omega, B \subset A \). By the transversality assumptions \( \Lambda \subset \Sigma'(\varepsilon) \) and \( \Lambda \cap T^\ast \Omega \subset \mathbb{T}^\ast \Omega \). For \( B \subset A \) the open set \( V_B \cap \Omega \) is constructible with respect to \( \Sigma'(\varepsilon) \) and we deduce \((SS(k_{V'B}) \cap T^\ast \Omega \subset \Lambda \cap T^\ast \Omega) \). This gives \((11.4.2)\) and the family \( \{V_a\}_{a \in A} \) is adapted to \( \Lambda' \).

(iii) Let \( A' \) be the set of subsets of \( A \) and, for \( a' \in A' \), set \( \tilde{V}_{a'} = \bigcap_{a \in A'} V_a \). An open subset \( \tilde{V}'_{a'} \), for \( B' \subset A' \), is still bounded by the hypersurfaces \( X_{a,\varepsilon_a} \) introduced in the condition (a) of (ii). So we also have \( D'(k_{\tilde{V}'_{a'}}) \simeq k_{\tilde{V}'_{a'}} \) and \( SS(k_{\tilde{V}'_{a'}}) \subset \Lambda \cap T^\ast \Omega \) as in (ii). Replacing \( A \) by \( A' \) we have an adapted family which is stable by intersection. To ensure that \( \Lambda \cap T^\ast \Omega \) has finitely many connected components, for each \( a' \in A' \), we choose \( \varepsilon_a \) so that \( \varepsilon_a \) is a regular value of \( f_a \circ \pi: \Lambda' \cap T^\ast W_a \rightarrow \mathbb{R}, \) for each \( a \in A, \) where \( \pi \) is the projection \( A' \rightarrow M. \)

\[\text{Definition 11.4.4.} \hspace{1em} \text{Let } \Lambda \text{ and } \mathcal{V} = \{V_a; a \in A\}, \text{ be as in Definition 11.4.1.} \]

Let \( V \subset M \times \mathbb{R} \) be an open subset. We denote by \( D_{\Lambda,\mathcal{V}}^b(k_{\mathcal{V}}) \) the subcategory of \( D^b(k_{\mathcal{V}}) \) formed by the \( G \) such that any point of \( V \) has an open neighborhood \( W \subset V \) such that \( \Lambda \cap T^\ast W \) has finitely many connected components, say \( \{\Lambda_i\}_{i \in I} \), and for each \( i \in I \) there
exist $A_i \subset A$ and $F_i \in D^b_{[A_i]}(k_W)$ satisfying
\begin{equation}
G|_{W_\gamma} \simeq \bigoplus_{i \in I} R\Gamma_{V^A_i \times \mathbb{R}_{>0}}(\Psi_W(F_i))
\end{equation}
where $V^A_i = \bigcup_{a \in A_i} V_a$ and we use the abusive Notation 11.4.2.

For $G$ and $W$ as in (11.4.3) we have $\text{supp}(G) \subset W_\gamma \cap (\mathfrak{T} \star \hat{\pi}_{M \times \mathbb{R}}(\Lambda))$ by Lemma 11.1.5 (see (11.1.13) for $\star$). We can cover $V$ by open subsets $W_j$, $j \in J$, for which a decomposition (11.4.3) holds; setting $V' = \bigcup_{j \in J} W_{j,\gamma}$ we then have
\begin{equation}
\text{supp}(G) \cap V' \subset (\mathfrak{T} \star \hat{\pi}_{M \times \mathbb{R}}(\Lambda)) \cap V'
\end{equation}
and $(V \times \{0\}) \sqcup V'$ is open in $M \times \mathbb{R} \times \mathbb{R}_{\geq 0}$ (as in (11.3.1)).

For an open subset $W \subset M \times \mathbb{R}$ we recall the maps $q_W, r_W: W_\gamma \to M \times \mathbb{R}$ of (11.1.6). For subsets $U$ of $M \times \mathbb{R}$ or $\Xi$ of $T^*(M \times \mathbb{R})$ we introduce the notations
\begin{equation}
\begin{split}
&U^q = U^{q_W} = q_W^{-1}(U \cap W), \\
&\Xi^q = \Xi^{q_W} = q_{W,d}q_{W,\pi}^{-1}(\Xi \cap T^*W) = (\Xi \times T^*_{\mathbb{R}_{>0}}) \cap T^*W, \\
&\Xi^r = \Xi^{r_W} = r_{W,d}r_{W,\pi}^{-1}(\Xi \cap T^*W).
\end{split}
\end{equation}

($W$ is in general suppressed from notation if there is no ambiguity.) The next lemma gives results about the sheaves appearing in (11.4.3).

**Lemma 11.4.5.** Let $\Lambda$ and $\mathcal{V} = \{V_a; a \in A\}$ be as in Definition 11.4.1. Let $W$ be an open subset of $M \times \mathbb{R}$ and let $\Lambda_1$ be a connected component of $\Lambda \cap T^*W$. Let $F \in D^b_{[\Lambda_1]}(k_W)$ and $B \subset A$ be given. Then, using Notation 11.3.2 and (11.4.5), we have
\begin{enumerate}
  
  
  
  
  \begin{equation}
  \begin{split}
  (\text{i})\ R\Gamma_{V^B,q}(\Psi_W(F)) &\simeq (\Psi_W(F))_{V^B,q}, \\
  (\text{ii})\ SS(R\Gamma_{V^B,q}(\Psi_W(F))) &\cap \Lambda^q \subset (\Lambda_1 \cap T^*V^B)^q, \\
  (\text{iii})\ \text{for any } G \in D^b_{[\Lambda_1,V]}(k_W) \text{ we have}
  \end{split}
  \end{equation}
\end{enumerate}
\begin{equation}
\begin{split}
\sum_{i=1}^{\infty} Rj_*(R\text{Hom}(G, R\Gamma_{V^B,q}(\Psi_W(F)))) \\
\simeq R\Gamma_{V,\Lambda} (i_W^* Rj_*(R\text{Hom}(G, \Psi_W(F))))
\end{split}
\end{equation}

\begin{enumerate}
  \begin{itemize}
  \item For any $x \in W \setminus \hat{\pi}_{M \times \mathbb{R}}(\Lambda_1)$ there exists a neighborhood $W'$ of $x$ such that $(R\Gamma_{V^B,q}(\Psi_W(F)))|_{W'_x} \simeq 0,$
  \item For any $x \in W \cap \hat{\pi}_{M \times \mathbb{R}}(\Lambda_1)$ there exists a neighborhood $W'$ of $x$ such that
  \begin{equation}
  (R\Gamma_{V^B,q}(\Psi_W(F)))|_{W'_x} \simeq \Psi_{W'}(R\Gamma_{W' \cap V^B}(F|_{W'}))
  \end{equation}
and $SS(R\Gamma_{W' \cap V^B}(F|_{W'})) \subset T^*_{\mathbb{R}_{\geq 0}}W'.$
  \end{itemize}
\end{enumerate}
Proof. (i)-(ii) We will show that the first two claims follow from Theorem 12.13 the bound \( \dot{SS}(\Psi_W(F)) \subset \Lambda_1^q \cup \Lambda_1^q \) of Lemma 11.1.7 and the hypotheses of Definition 11.4.1.

More precisely, to deduce (i) from Theorem 12.13 we need to check that \( \dot{SS}(\Psi_W(F)) \) and \( \dot{SS}(k_{V_B,a}) = (\dot{SS}(k_{V_B}))^q \) do not intersect (here \( F \) is a sheaf on \( W \) and \( k_{V_B} \) means \( k_{V_B}|_W = k_{V_B \cap W} \) with the same abuse as in Notation 11.3.2). We take coordinates \( (\xi, \tau, \upsilon) \) in a fiber of \( T^* (M \times \mathbb{R} \times \mathbb{R}_{>0}) \). The points of \( \Lambda_1^q, \Lambda_1^q \) and \( \dot{SS}(k_{V_B,a}) \) are respectively of the form \( (\xi_1, \tau_1, 0), (\xi_2, \tau_2, -\tau_2), (\xi_3, \tau_3, 0) \) with \( \tau_1, \tau_2 > 0 \) (since \( \Lambda_1 \subset \Lambda \)). Hence the subsets \( \Lambda_1^q \) and \( \dot{SS}(k_{V_B,a}) \) cannot intersect. So it remains to check that \( SS(k_{V_B}) \) and \( SS(F) \) do not intersect. We recall that \( SS(k_{V_B}) \) is contained in the fibers over \( \partial V^B \) and that \( V^B \) is a finite union of products \( U_a \times I_a \). We remark that \( \partial V^B \subset \bigcup_{a \in B} \partial (U_a \times I_a) \). Let us call the “vertical part” of \( \partial V^B \) its subset \( \partial V^B \setminus (\partial V^B \cap \bigcup_{a \in B} (U_a \times \partial I_a)) \).

Near a point in the vertical part \( \partial V^B \) is then of the form \( \partial V \times I \) for some open subsets \( V \subset M, I \subset \mathbb{R} \) and a point of \( SS(k_{V_B,a}) \) is of the form \( (\xi_0, 0, 0) \). The hypothesis (i) in Definition 11.4.1 implies that \( \pi_{M \times \mathbb{R}}(\Lambda) \) and \( \partial V^B \) can only meet at points in the vertical part of \( \partial V^B \).

Near such a point \( SS(k_{V_B}) \) is contained in \( \{ \tau = 0 \} \) and cannot meet \( \Lambda \). This proves (i).

Since \( \dot{SS}(\Psi_W(F)) \) and \( \dot{SS}(k_{V_B,a}) \) do not intersect, we know by Theorem 12.13 and Example 12.12 that \( SS(R\Gamma_{V_B,a}(\Psi_W(F))) \) is bounded by the pointwise sum \( \Xi = \Lambda_1^q + \dot{SS}(k_{V_B,a}) \), where \( \Lambda_1^q = \Lambda_1^q \cup \Lambda_1^q \cup T_{W,\gamma}^* W_\gamma \).

We need to understand the intersection of \( \Xi \) and \( \Lambda^q \). It is clear that \( \Lambda_1^q + (T_{V_B}^* V^B)^q \) is equal to \( \Lambda_1^q \cap T^* (q^{-1}(V^B)) \) and this satisfies the required bound. Hence we can concentrate on the part of \( (SS(k_{V_B}))^q \) outside the zero section, hence above \( q^{-1}(\partial V^B) \). Since we are interested in \( \Lambda^q \cap \Xi \), by the preceding discussion we only need to consider points above the vertical part of \( q^{-1}(\partial V^B) \) and a point of \( (SS(k_{V_B}))^q \) is then of the form \( (\xi_0, 0, 0) \). It follows that a point of \( \Lambda_1^q + (SS(k_{V_B}))^q \) is of the form \( (\xi, \tau, -\tau), \tau > 0 \), and cannot belong to \( \Lambda^q \). We also see that \( T_{V,\gamma}^* W_\gamma + (SS(k_{V_B}))^q \) does not meet \( \Lambda^q \). So it only remains to understand \( (\Lambda_1^q + (SS(k_{V_B}))^q) \cap \Lambda^q \) and the statement follows from the hypothesis (ii) of Definition 11.4.1.

(iii-a) The isomorphism (iii) is local on \( W \) and we can shrink \( W \) if necessary. Since \( G \in D^{\Sigma}_{\Lambda_0,\gamma}(k_W) \) we can thus assume that \( G \) satisfies (11.4.3). Taking one summand we assume in fact \( G = R\Gamma_{V_B,a}(\Psi_W(F')) \), for
some $B' \subset A$ and $F' \in D^b_{\mathcal{A}}(k_W)$. We will prove

\begin{equation}
\begin{aligned}
& i_{W}^{-1}R_{j_{*}}R\mathcal{H}om(\mathcal{R}_{V_{B'\cap V_{B}}}P_{W}(F'), \mathcal{R}_{V_{B'\cap V_{B}}}P_{W}(F))) \\
& \simeq \mathcal{R}\mathcal{H}_{V_{B'\cap V_{B}}}P_{W}(i_{W}^{-1}R_{j_{*}}R\mathcal{H}om(P_{W}(F'), P_{W}(F)))
\end{aligned}
\end{equation}

(11.4.8)

This implies (11.4.6): use (11.4.8) as it is stated and also in the case where $V_{B} \supset W$, together with $\mathcal{R}\mathcal{H}_{V_{B'\cap V_{B}}}P_{W}(-) \simeq \mathcal{R}\mathcal{H}_{V_{B'}}P_{W}(-)$.

We set $H = R\mathcal{H}om(P_{W}(F'), P_{W}(F))$. Using part (i) of the lemma and the isomorphism $R\mathcal{H}om((-)_{Z'}, \mathcal{R}\mathcal{H}_{Z_{\cap Z'}}P_{W}(-, -)$, we deduce that the left hand side of (11.4.8) is

\begin{equation}
\begin{aligned}
i_{W}^{-1}R_{j_{*}}R\mathcal{H}_{(V_{B'\cap V_{B}})_{\wedge}}P_{W}(H) & \simeq i_{W}^{-1}\mathcal{R}\mathcal{H}_{(V_{B'\cap V_{B}})_{\times}}P_{W}(R_{j_{*}}H)
\end{aligned}
\end{equation}

and we are reduced to prove

\begin{equation}
\begin{aligned}
i_{W}^{-1}\mathcal{R}\mathcal{H}_{(V_{B'\cap V_{B}})_{\times}}P_{W}(R_{j_{*}}H) & \simeq \mathcal{R}\mathcal{H}_{V_{B'\cap V_{B}}}(i_{W}^{-1}R_{j_{*}}H).
\end{aligned}
\end{equation}

(11.4.9)

For this it is enough to check, for some $\varepsilon > 0$, setting $J_{\varepsilon} = [-\infty, \varepsilon]$, 

\begin{equation}
\begin{aligned}
& SS(k_{(V_{B'\cap V_{B}})_{\times}}P_{W}(T_{J_{\varepsilon}})) \cap SS(R_{j_{*}}P_{W}(H)) = \emptyset,
& SS(k_{(V_{B'\cap V_{B}})_{\times}}P_{W}(i_{W}^{-1}R_{j_{*}}H)) = \emptyset.
\end{aligned}
\end{equation}

Indeed, (11.4.9), (11.4.11) and Theorem 12.13 imply the isomorphisms

\begin{equation}
\begin{aligned}
& \mathcal{R}\mathcal{H}_{(V_{B'\cap V_{B}})_{\times}}P_{W}(R_{j_{*}}H) \simeq D'(k_{(V_{B'\cap V_{B}})_{\times}}P_{W}) \otimes R_{j_{*}}H \\
& \simeq (D'(k_{(V_{B'\cap V_{B}})_{\times}}P_{W}) \boxtimes k_{T_{\varepsilon}}) \otimes R_{j_{*}}H,
\end{aligned}
\end{equation}

\begin{equation}
\begin{aligned}
& \mathcal{R}\mathcal{H}_{(V_{B'\cap V_{B}})_{\times}}P_{W}(i_{W}^{-1}R_{j_{*}}H) \simeq D'(k_{(V_{B'\cap V_{B}})_{\times}}P_{W}) \otimes i_{W}^{-1}R_{j_{*}}H.
\end{aligned}
\end{equation}

(11.4.10)

We then use the commutativity of $i_{W}^{-1}$ with the tensor product.

(iii-b) Let us prove (11.4.9). Proposition 11.1(i) gives

\begin{equation}
R_{j_{*}}H \simeq R\mathcal{H}om(R_{j_{*}}P_{W}(F''), R_{j_{*}}P_{W}(F)).
\end{equation}

For $F'' \in D_{r > 0}(k_W)$ Lemma 11.1.7 says $R_{j_{*}}P_{W}(F'') \simeq R_{j_{*}}P_{W}(F''')$ and gives a bound for $SS(R_{j_{*}}P_{W}(F'''))$. We deduce the following less precise bound which is easier to handle. For $S \subset T^{*}W$ and $\varepsilon > 0$ we define $N_{\varepsilon}(S) = \bigcup_{c \in [0, \varepsilon]} T_{c}(S)$ where $T_{c}$ is the vertical translation $T_{c}(x, t; \xi, \tau) = (x, t + c; \xi, \tau)$. Lemma 11.1.7 implies

\begin{equation}
SS((R_{j_{*}}P_{W}(F''))_{W \times J_{\varepsilon}}) \subset N_{\varepsilon}(SS(F'')) \times T_{J_{\varepsilon}}.
\end{equation}

By Theorem 12.13 we deduce

\begin{equation}
SS(R_{j_{*}}(H)) \subset N_{\varepsilon}(\Lambda_{+} \hat{\Lambda}_{+}) \times T_{J_{\varepsilon}}.
\end{equation}
When we take the inverse image by $i_W$ in (11.4.10) we can assume $\varepsilon$ as small as we want and Theorem 1.2.8 gives

$$SS(i_W^{-1}Rf_* (H)) \subset \Lambda^+_\tau \cap \Lambda^+. \tag{11.4.11}$$

Finally we remark $SS(k_{V_B \cap V'}) \subset SS(k_{V'B}) \cap SS(k_{V'B'})^a$. Now we deduce the relations (11.4.9) from (11.4.10) (taking $\varepsilon$ as small as required), (11.4.11) and (11.4.1).

(iv) follows from the bound (11.4.3) and the remark that, for any subset $S$ of $M \times \mathbb{R}$ and any $\varepsilon > 0$,

$$(\tau * S) \cap (M \times \mathbb{R} \times [0, \varepsilon]) \subset \left( \bigcup_{c \in [0, \varepsilon]} t_c(S) \right) \times [0, \varepsilon],$$

where $t_c$ is the vertical translation $t_c(x, t) = (x, t + c)$.

(v) If $x \in V_B$, then we choose $W' = W \cap V_B$ and the result is obvious. If $x \not\in V_B$, we choose $W'$ such that $W' \cap V_B = \emptyset$ and both sides are zero. So we can assume $x \in \partial V_B$. Since $x \in \pi_M \times \mathbb{R} (\Lambda_1)$ we have seen in the proof of (i) that $x$ belongs to the “vertical part” of $\partial V_B$. Hence we can find a neighborhood $W'$ of $x$ such that $W' \cap V_B = W' \cap (U \times \mathbb{R})$, for some open subset $U$ of $M$: recalling that $V_B = \bigcup_{a \in B} (U_a \times \mathbb{R})$, we have $U = \bigcup_{a \in B, x \in U_a \times I_a} U_a$. It follows from the projection formula (see Proposition 11.4(a)) that $\Psi_W((H)_{U \cap (Z \times \mathbb{R})}) \cong (\Psi_W(H))_{U \cap (Z \times \mathbb{R} \times \mathbb{R} > 0)}$, for any sheaf $H$ on $W'$ and any locally closed subset $Z$ of $M$. Using part (i) of the lemma (and the similar isomorphism $R\Gamma_{V_B}(F) \cong (F)_{\tau \tau}$) we deduce the result.

We will state the analog of Proposition 11.3.7 for $D_{\Lambda, \tau}^{db}(k_V)$. We recall that the restriction of $\mu_{\text{hom}}(\Psi_U(F), \Psi_U(G))$ to $\{\tau > 0\}$ is decomposed, by Proposition 11.1.9, as the sum of $\mu_{\text{hom}}(q_{U}^{-1}(F), q_{U}^{-1}(G))$ and $\mu_{\text{hom}}(r_{U}^{-1}(F), r_{U}^{-1}(G))$. Hence in an analog of Proposition 11.3.7 we can expect that $\mu_{\text{hom}}(F, G)$ (for $F, G \in D_{\Lambda, \tau}^{db}(k_V)$ should be replaced by $\mu_{\text{hom}}(F', G')|_{\Lambda^\tau}$ (for $F', G' \in D_{\Lambda, \tau}^{db}(k_V)$)). We will make this more precise soon and we will use the following lemma.

**Lemma 11.4.6.** Let $G, G' \in D_{\Lambda, \tau}^{db}(k_V)$. Then there exist a uniquely defined sheaf $\mu_{\text{hom}}^{db}(G, G')$ in $D^b(k_{\Lambda \cap T' \cdot V})$ and an open subset $V'$ of $V$ such that $(V \times \{0\}) \cup V'$ is open in $M \times \mathbb{R} \times \mathbb{R}_{>0}$ and

$$\mu_{\text{hom}}(G, G')|_{\Lambda \cap T' \cdot V'} \cong p_\Lambda^{-1}(\mu_{\text{hom}}^{db}(G, G'))|_{\Lambda \cap T' \cdot V'},$$

where $p_\Lambda: \Lambda^\tau \cap T' \cdot V' \to \Lambda$ is the projection. Moreover, for $W \subset V$ open, if $G|_{W_\tau} \cong \Psi_W(F)$, $G'|_{W_\tau} \cong \Psi_W(F')$ for some $F, F' \in D_{r_{>0}}(k_W)$, then $\mu_{\text{hom}}^{db}(G, G')|_{\Lambda \cap T' \cdot W} \cong \mu_{\text{hom}}(F, F')|_{\Lambda \cap T' \cdot W}$.
Proof. By Lemma 11.4.3(iv-v), any point \(x\in V\) has a neighborhood \(W\) such that \(G\mid_{W}\times R\) are of the form \(\Psi_{W}(F), \Psi_{W}(F')\) for some \(F, F'\in D_{r\geq 0}(k_{W})\). Then Proposition 11.1.9 implies that the restriction of \(\mu_{hom}(G, G')\) to \(\Lambda^{q}\cap T^{*}W_{\gamma}\) is of the form \(\mu_{hom}^{1}(q_{W}^{-1}(F), q_{W}^{-1}(F'))\cong q_{W,\pi}^{-1}\mu_{hom}(F, F')\). In particular
\[
\mu_{hom}(G, G')|_{\Lambda^{q}\cap T^{*}W_{\gamma}}\cong (p_{A}|_{\Lambda^{q}\cap T^{*}W_{\gamma}})^{-1}\mu_{hom}(F, F'),
\]
which proves the last assertion and shows that \(\mu_{hom}(G, G')|_{\Lambda^{q}\cap T^{*}W_{\gamma}}\) is constant on the fibers of the projection \(\Lambda^{q}\cap T^{*}W_{\gamma}\to\Lambda\).

Now we choose a family \(W_{i}, i\in I\), of such open subsets \(W\) which covers \(V\) and is locally finite (each compact subset of \(V\) meets finitely many \(W_{i}\)'s). We set \(V'=\bigcup_{i\in I}W_{i,\gamma}\). Then \((V\times\{0\})\cup V'\) is open in \(M\times R\times R_{\geq 0}\) and, for any \(x\in V\), there exists \(i\in I\) such that \((\{x\}\times R_{\geq 0})\cap W_{i,\gamma}=\{x\}\times R_{\geq 0}\cap W_{i,\gamma}\). It follows that \(\mu_{hom}(G, G')|_{\Lambda^{q}\cap T^{*}V'}\) is constant on the fibers of the projection \(p_{A}: \Lambda^{q}\cap T^{*}V'\to\Lambda\), which are open intervals. Hence it is the inverse image of some uniquely defined sheaf \(\mu_{hom}^{db}(G, G')\) by \(p_{A}\) (and we have \(\mu_{hom}^{db}(G, G')=Rp_{A,*}(\mu_{hom}(G, G')|_{\Lambda^{q}\cap T^{*}V'})\)).

Equality (11.4.4) holds for any \(x\in V\) which has a neighborhood \(\Omega\) of \(\Lambda_{G}\) in \(T^{*}V\) such that \((V\times\{0\})\cup V'\) is open in \(M\times R\times R_{\geq 0}\) and \(SS(G)\cap \Lambda^{q}\cap T^{*}V'=(\Lambda^{q}\cap T^{*}V')\cap T^{*}V'\). Namely, the microsupport of \(G\in D^{db}_{A, V}(k_{V})\) is in general bigger than \((SS^{db}(G))^{q}\cup(\mu_{hom}^{db}(G))^{q}\); the last formula in Lemma 11.4.7 says \(SS(G)^{q}\) coincides with \((SS^{db}(G))^{q}\) in some neighborhood of \((SS^{db}(G))^{q}\), but \(SS(G)^{q}\) is a priori not contained in \(\Lambda^{q}\cap T^{*}V'\). Hence, the microsuppport of \(G\) could be as big as the bound given in the proof of Lemma 11.4.3(ii).

Remark 11.4.8. We can define the notions of pure or simple doubled sheaf and also \(m_{\Lambda}(G)\) for a doubled sheaf \(G\); this will be used in Part 12. Let \(G\in D^{db}_{A, V}(k_{V})\). Lemma 11.4.7 defines an open subset \(SS^{db}(G)^{q}\) of \(\Lambda\). Let \(\Lambda_{0}\)
be an open subset of $SS^{dbl}(G)$. With $V'$ as in the lemma we have $G|_{V'} \in D^b(\Lambda^{\emptyset}_0 \cap T^*V')(k_{V'})$ (we recall that this means that $SS(G)$ coincides with $\Lambda^{\emptyset}_0$ on some neighborhood of $\Lambda^{\emptyset}_0$). We say that $G$ is pure (or simple) along $\Lambda_0$ if $G|_{V'}$ is pure (or simple) in the usual sense along $\Lambda^{\emptyset}_0 \cap T^*V'$. Since $G|_{V'} \in D^b(\Lambda^{\emptyset}_0 \cap T^*V')(k_{V'})$, we can also consider $m_{\Lambda^{\emptyset}_0 \cap T^*V'}(G|_{V'})$ which is an object of $\mu\text{Sh}(k_{\Lambda^{\emptyset}_0 \cap T^*V'})$. Up to shrinking $V'$ we can assume that the fibers of the projection $p_{\Lambda_0}: \Lambda^{\emptyset}_0 \cap T^*V' \to \Lambda_0$ are open intervals. Then the inverse image by $p_{\Lambda_0}$ induces an equivalence

$$\text{(11.4.12)} \quad \mu\text{Sh}(k_{\Lambda_0}) \sim \mu\text{Sh}(k_{\Lambda^{\emptyset}_0 \cap T^*V'}).$$

In this way we can identify $m_{\Lambda^{\emptyset}_0 \cap T^*V'}(G|_{V'})$ with an object of $\mu\text{Sh}(k_{\Lambda_0})$ that we denote by $m_{\Lambda_0}^{dbl}(G)$. We obtain a functor, for any open subset $\Lambda_0$ of $\Lambda \cap T^*V$,

$$\text{(11.4.13)} \quad m_{\Lambda_0}^{dbl}: \{G \in D^b_{\Lambda,V}(k_{V}); \Lambda_0 \subset SS^{dbl}(G)\} \to \mu\text{Sh}(k_{\Lambda_0})$$

and $(10.1.6)$ together with $(11.4.12)$ give, for $G, G' \in D^b_{\Lambda,V}(k_{V})$ such that $\Lambda_0 \subset SS^{dbl}(G), \Lambda_0 \subset SS^{dbl}(G')$,

$$\text{(11.4.14)} \quad \text{Hom}(m_{\Lambda_0}^{dbl}(G), m_{\Lambda_0}^{dbl}(G')) \simeq H^0(\Lambda_0; H^0(\mu\text{hom}^{dbl}(G, G'))) \text{.}$$

(Here the condition $\Lambda_0 \subset SS^{dbl}(G)$ ensures that $G \in D^b_{\Lambda_0}(k_{V'}).$) Let us set $V_u = V \cap (M \times \mathbb{R} \times \{u\})$ for $u > 0$. We have a natural inclusion $T^*V_u \subset T^*V_\gamma$. We remark that, if $V = V^B$ for some $B \subset A$ and $u$ is small enough, we have $\Lambda^\emptyset_0 \cap T^*V_u = \Lambda_0$. Then $G|_{V_u}$ belongs to $D^b_{(\Lambda_0)}(k_{V_u})$ and we have, by the construction of $m_{\Lambda_0}^{dbl}$,

$$\text{(11.4.15)} \quad m_{\Lambda_0}(G|_{V_u}) \simeq m_{\Lambda_0}^{dbl}(G) \text{.}$$

Now we define a version of the morphism $b(F, G)$ of Definition 11.3.4. Let $G, G' \in D^b_{\Lambda,V}(k_{V'})$ and set $\Lambda_0 = SS^{dbl}(G), \Lambda'_0 = SS^{dbl}(G')$ (they are open subsets of $\Lambda$). We choose an open subset $V'$ of $W_\gamma$ satisfying the conclusions of Lemmas 11.4.6 and 11.4.7 for $G, G'$. We denote by $\pi_{A^\#}$, $\pi_\Lambda$ the projections from $\Lambda^\emptyset \cap T^*V'$, $\Lambda \cap T^*V$ respectively to $V', V$. The support of $\mu\text{hom}(G, G')(\Lambda^{\emptyset}_0 \cap T^*V')$ is contained in $\Lambda^\emptyset \cap T^*V' \cap \Lambda^{\emptyset}_0 \cap \Lambda_0$, hence the support of $\mu\text{hom}^{dbl}(G, G')$ is contained in $\Lambda^{\emptyset}_0 \cap \Lambda_0$. Since $\Lambda_0$ is open in $\Lambda$ we have a natural morphism $(-) \to R\Gamma_{\Lambda_0}(-)$ and we deduce the following sequence of morphisms

$$\text{RHom}(G, G')|_{V'} \simeq R\pi_{V'}(\mu\text{hom}(G, G')|_{T^*V'})$$

$$\to R\pi_{A^\#}(\mu\text{hom}(G, G')|_{\Lambda^\emptyset \cap T^*V'})$$

$$\simeq (q_V^{-1}R\pi_{A^\#} \mu\text{hom}^{dbl}(G, G'))|_{V'}$$

$$\to (q_V^{-1}R\pi_{\Lambda} R\Gamma_{\Lambda_0} \mu\text{hom}^{dbl}(G, G'))|_{V'} \text{.}$$
Let $i_V : V \times \{0\} \to M \times \mathbb{R}^2$ and $j : V' \to M \times \mathbb{R}^2$ be the inclusions. We apply the functor $i_V^{-1}Rj_*$ to the above sequence and we obtain a version of the morphism $b(F,G)$ of Definition [11.3.4] for $D^b_{\Lambda_V}(k_V)$:

$$b'(G,G') : i_V^{-1}Rj_*RHom(G,G')$$

$$\to R\pi_{\Lambda V}^*R\Gamma^*_{\Lambda_V} \mu_{hom}^{dbl}(G,G'),$$

where $\Lambda' = SS^{dbl}(G')$. The Proposition [11.3.7] generalizes to this setting as follows.

**Proposition 11.4.9.** Let $G, G' \in D^b_{\Lambda V}(k_V)$. Then the morphism $b'(G,G')$ in (11.4.16) is an isomorphism. In particular, taking global sections gives

$$\lim_{V'} \text{Hom}(G|_{V'}, G'|_{V'}) \to H^0(\Lambda'_0; \mu_{hom}^{dbl}(G,G')),$$

where $V'$ runs over the open subsets of $M \times \mathbb{R} \times \mathbb{R}_{>0}$ such that $(V \times \{0\}) \cup V'$ is open in $M \times \mathbb{R} \times \mathbb{R}_{>0}$ and $\Lambda'_0 = SS^{dbl}(G')$ is the open subset of $\Lambda \cap T^*V$ defined in Lemma [11.4.7].

**Proof.** Since the statement is local on $V$ we may as well assume that $G$ and $G'$ are decomposed as in (11.4.3) and it is enough to consider one summand in their decompositions. By Lemma [11.4.5](iv-v) we can assume that $G = \Psi_W(R\Gamma^*_{V'}(F))$ and $G' = \Psi_W(R\Gamma^*_{V'}(F'))$ for some open subset $W \subset V$, $F, F' \in D^b_{[\Lambda]}(k_W)$ and $B, B' \subset A$. By Lemma [11.4.5](iii) the restriction to $W$ of the left hand side of (11.4.16) becomes

$$\lim_{V'} i_W^{-1}Rj_*R\Hom(G,G') \to R\Gamma^*_{V'} \mu_{hom}^{dbl}(i_W^{-1}Rj_*R\Hom(G, \Psi_W(F'))).$$

We have $\mu_{hom}^{dbl}(G,G') \simeq \mu_{hom}(R\Gamma^*_{V'}(F), R\Gamma^*_{V'}(F'))|_{\Lambda}$ by the construction of $\mu_{hom}^{dbl}$ in Lemma [11.4.6]. Let us set $\Lambda' = SS(F')$. We have $SS^{dbl}(G') = \Lambda' \cap T^*V^{B'}$ and the restriction to $W$ of the right hand side of (11.4.16) becomes

$$R\pi_{\Lambda V}^*R\Gamma^*_{\Lambda \cap T^*V^{B'}} (\mu_{hom}(R\Gamma^*_{V'}(F), R\Gamma^*_{V'}(F'))|_{\Lambda})$$

$$\simeq R\pi_{\Lambda V}^*R\Gamma^*_{\Lambda \cap T^*V^{B'}} (\mu_{hom}(R\Gamma^*_{V'}(F), F')|_{\Lambda})$$

$$\simeq R\pi_{\Lambda V}^*R\Gamma^*_{T^*V^{B'}} (\mu_{hom}(R\Gamma^*_{V'}(F), F')|_{\Lambda})$$

$$\simeq R\Gamma^*_{V'} R\pi_{\Lambda V}^* \mu_{hom}(R\Gamma^*_{V'}(F), F')|_{\Lambda},$$

where the first isomorphism follows from $R\Gamma^*_{\Lambda \cap T^*V^{B'}} \simeq R\Gamma_{\Lambda V} R\Gamma^*_{T^*V^{B'}}$ and the fact that $R\Gamma_U(H)$ only depends on $H|_U$ when $U$ is open (here $U = T^*V^{B'}$) and the second isomorphism follows from the inclusion $\text{supp}(\mu_{hom}(H,F')|_{\Lambda}) \subset SS(F') = \Lambda'$, whatever $H$. Since $SS(F') \subset
\{\tau > 0\} we can apply Proposition 11.3.7 and we obtain that the right hand side of (11.4.16) is isomorphic to
\[ \text{R}\Gamma_{V'}(i_{W}^{-1}R_{j_{*}}R\text{Hom}(\Psi_{W}(\text{R}\Gamma_{V}(\mathcal{F})), \Psi_{W}(\mathcal{F}'))). \]
Comparing with (11.4.17) we obtain the result.

The last assertion follows by applying the functor \( \text{H}^{0}(V; -) \) to both sides of (11.4.16) and using \( \text{H}^{0}(V'; R\text{Hom}(G, G')) \simeq \text{H}\text{om}(G|_{V'}, G'|_{V'}). \)

\[ \square \]

We have seen a notion of pure doubled sheaf and we have defined \( m_{\Lambda_{0}}^{dbl} \) in the paragraph before (11.4.13). In the same way that we deduced Corollary 11.3.8 from Proposition 11.3.7, we obtain the following result, using Proposition 11.4.9 and (11.4.14):

**Corollary 11.4.10.** Let \( \Lambda_{0} \subset \Lambda \cap T^{*}V \) be an open subset. Let \( G, G' \in \mathcal{D}_{\Lambda_{0}}^{\text{pp}}(\mathcal{A}_{V}) \) be such that \( \Lambda_{0} \subset \text{SS}^{\text{dbl}}(G) \) and \( \text{SS}^{\text{dbl}}(G') = \Lambda_{0} \). We assume that \( G \) and \( G' \) are pure with the same shift. Then
\[ (11.4.18) \lim_{V} \text{Hom}(G|_{V}, G'|_{V}) \simeq \text{Hom}(m_{\Lambda_{0}}^{\text{dbl}}(G), m_{\Lambda_{0}}^{\text{dbl}}(G')), \]
where \( V \) runs over the open subsets of \( U_{\gamma} \) such that \((U \times \{0\}) \cup V \) is open in \( M \times \mathbb{R} \times \mathbb{R}_{\geq 0} \). In particular, if \( \text{SS}^{\text{dbl}}(G) = \text{SS}^{\text{dbl}}(G') = \Lambda_{0} \) and \( m_{\Lambda_{0}}^{\text{dbl}}(G) \simeq m_{\Lambda_{0}}^{\text{dbl}}(G') \), then there exists such an open subset \( V \) such that \( G|_{V} \simeq G'|_{V} \).

We remark that the hypothesis \( \text{SS}^{\text{dbl}}(G') = \Lambda_{0} \) implies that \( \Lambda_{0} \) is not any open subset of \( \Lambda \) but of the form described in (i) of Lemma 11.4.7. In Lemma 11.4.13 below we describe the isomorphism (11.4.18) locally when we have a decomposition of \( G \) and \( G' \) as in (11.4.3). Before that we show that the subsets \( \Lambda \cap T^{*}V^{B}, B \subset A \), have a local connectedness property near the boundary. For example the following situation is excluded. Assume \( M = \mathbb{R} \) and \( \Lambda \) is half of the conormal bundle of the cusp \( \{(x, t); x^{3} = t^{2}\} \), say \( \Lambda = \{(z^{2}, z^{3}; -\frac{3}{2}z\tau, \tau); z \in \mathbb{R}, \tau > 0\} \) (this example is explained in Lemma 8.1.1) and set \( V = ]0, 1[ \times ]-1, 1[. \) Then \( \Lambda \cap T^{*}V \) has two connected components but \( \Lambda \cap \pi^{-1}_{M \times \mathbb{R}}(V) \) has only one component. In fact \( V \) cannot belong to an adapted family for \( \Lambda \). The next lemma says a bit more.

**Lemma 11.4.11.** Let \( \Lambda \) and \( V = \{V_{a}; a \in A\} \) be as in Definition 11.4.7. Then, for any \( B \subset A \), \( \Lambda \cap T^{*}V^{B} \) has the following local connectedness property. For any \((x, t) \in M \times \mathbb{R}\) and any small enough neighborhood \( W \) of \((x, t)\), denoting by \( \Lambda \cap T^{*}W = \bigcup_{j \in J} \Lambda_{j} \) and \( \Lambda_{j} \cap T^{*}V^{B} = \bigcup_{k \in K_{j}} \Lambda_{k} \) the decompositions into connected components we have: for any \( j \in J \) there exists at most one \( k \in K_{j} \) such that
\( \Lambda_j^k \cap T_{(x,t)}^*(M \times \mathbb{R}) \) is non empty. In other words, there exists a smaller neighborhood \( W' \) of \( (x,t) \) such that the inclusion of \( \Lambda_j \cap T^*V^B \cap T^*W' \) in \( \Lambda_j^k \cap T^*V^B \) factorizes through a connected set.

**Remark 11.4.12.** A stronger statement would be that the subsets \( \Lambda_j \cap T^*V^B \) in the lemma are connected (that is, \( K_j \) is a singleton), but this would require a good choice of \( W \). The lemma only says that, when restricting to a smaller neighborhood of \( T_{(x,t)}(M \times \mathbb{R}) \), at most one component of \( \Lambda_j \cap T^*V^B \) survives.

**Proof.** (i) We set \( F_\Lambda = R(\pi_{M \times \mathbb{R}})_*(k_\Lambda) \). Let us prove that the natural morphism \( u: (F_\Lambda)(U) \to RT^*V^B(F_\Lambda) \) is an isomorphism, for any \( B \subset A \). By Theorem 1.2.13 and the condition \( D'(k_{V^B}) \simeq k_{V^B}^{-1} \), it is enough to see that \( SS(k_{V^B}) \) and \( SS(F_\Lambda) \) do not meet outside the zero section. This follows from (11.4.11) and the inclusion \( SS(F_\Lambda) \subset \Lambda_+^1 \Lambda_+^0 \), where \( \Lambda_+^1 \Lambda_+^0 \) is connected, that is, \( \Lambda_1^0 \Lambda_+^0 \) is a singleton), and \( \muhom(F,F)|_{\Lambda_1^0 \Lambda_+^0} \simeq k_\Lambda \) (see (1.4.9)). Then the inclusion follows from the triangle (11.3.11), the triangular inequality for the microsupport and the bounds in Lemma 10.2.3 and Theorem 1.2.13.

(ii) Let \( J \) be as in the lemma and let \( J' \) be the set of half lines in \( \Lambda \cap T_{(x,t)}(M \times \mathbb{R}) \). We choose \( W \) small enough so that the obvious map \( J' \to J \) is injective. Applying \( H^0(W; -) \) to the isomorphism \( u \) of (i) we obtain

\[
H^0(T^*W; k_{\Lambda \cap \pi_{M \times \mathbb{R}}^{-1}(V^B)}) \simeq H^0(T^*W \cap \pi_{M \times \mathbb{R}}^{-1}(V^B); k_\Lambda).
\]

This says that the connected components of \( \Lambda \cap \pi_{M \times \mathbb{R}}^{-1}(V^B) \cap T^*W \) and \( \Lambda \cap \pi_{M \times \mathbb{R}}^{-1}(V^B) \cap T^*W \) are in bijection. Now we assume that there exist \( j \in J \) and \( k, k' \in K_j \) such that \( \Lambda_j^k \) and \( \Lambda_j^k \) both meet \( T_{(x,t)}(M \times \mathbb{R}) \). Then \( \Lambda_j^k \) and \( \Lambda_j^k \) both contain the same half line of \( J' \). In particular they have a non empty intersection and must be the same connected component of \( \Lambda \cap \pi_{M \times \mathbb{R}}^{-1}(V^B) \cap T^*W \), that is, \( k = k' \), as required. \( \square \)

The next result says that the local decomposition (11.4.3) can be written in a canonical way and describe the morphism (11.4.13) locally when we have such a decomposition.

**Lemma 11.4.13.** Let \( V \) be an open subset of \( M \times \mathbb{R} \) and let \( (x,t) \in V \) be given. Let \( \{ \lambda_i \}_{i \in I} \) be the set of half lines in \( \Lambda \cap T_{(x,t)}(M \times \mathbb{R}) \). Let \( W_0 \) be a neighborhood of \( (x,t) \) small enough so that the map \( I \to \pi_0(\Lambda \cap T^*W_0) \) is injective; let \( \Lambda_i \) be the connected component of \( \Lambda \cap T^*W_0 \) containing \( \lambda_i \). We assume that, for each \( i \in I \), there exists a simple
sheaf $F_i \in \mathcal{D}_{\{|\Lambda|\}}(k_{W_0})$. Now let $G, G' \in \mathcal{D}_{\lambda_0}(k_{V'})$ be pure objects with the same shift. We set $\Lambda_0 = SS^{dbl}(G), \Lambda'_0 = SS^{dbl}(G')$. Then we have:

(i) There exists an isomorphism, for some smaller neighborhood $W$ of $(x, t)$,

$$G|_{W_i} \simeq \bigoplus_{i \in I} R\Gamma_{W_i \times \mathbb{R}_{>0}}(\Psi_W(F_i \otimes (E_i)_W)),$$

where $W_i = W \cap \pi_{M \times \mathbb{R}}(\Lambda_i \cap \Lambda_0)$ and $E_i \in \mathcal{D}(k)$ is given by $E_i = (\muhom^{dbl}(\Psi_W(F_i), G))_{p_i}$ for any $p_i \in \Lambda_i \cap \Lambda_0$ (and $E_i = 0$ if $\Lambda_i \cap \Lambda_0$ is empty).

(ii) We assume that $\Lambda'_0 \subset \Lambda_0$ and we define $W'_i, E'_i$ like $W_i, E_i$ in (i), choosing the same $p_i$ for $G$ and $G'$ when $\Lambda_i \cap \Lambda'_0 \neq \emptyset$. For a given $u$ in

$$\text{Hom}(\mathfrak{m}^{dbl}_{\Lambda_0}(G), \mathfrak{m}^{dbl}_{\Lambda'_0}(G')) \simeq H^0(\Lambda'_0; \muhom^{dbl}(G, G'))$$

we let $u_i: E_i \to E'_i$, $e \mapsto u_{p_i} \circ e$, be the morphism induced by the composition (10.1.2) (we use the notation 10.1.4). Then, up to shrinking $W$, the inverse image of $u|_{\pi_{W'}} \in \text{Hom}(\mathfrak{m}^{dbl}_{\Lambda_i \cap \Lambda'_0}(G), \mathfrak{m}^{dbl}_{\Lambda_i \cap \Lambda'_0}(G'))$ through (11.4.18) (replacing $V$ in the corollary by $W$) is represented by $\bigoplus_{i \in I} v_i$, where

$$v_i: R\Gamma_{W_i \times \mathbb{R}_{>0}}(\Psi_W(F_i \otimes (E_i)_W)) \to R\Gamma_{W'_i \times \mathbb{R}_{>0}}(\Psi_W(F'_i \otimes (E'_i)_W))$$

is the composition of the morphism $R\Gamma_{W_i \times \mathbb{R}_{>0}}(-) \to R\Gamma_{W'_i \times \mathbb{R}_{>0}}(-)$ induced by the open inclusion $W'_i \subset W_i$ and the morphism $\Psi_W(F_i \otimes (E_i)_W) \to \Psi_W(F'_i \otimes (E'_i)_W)$ induced by $u_i$.

Proof. (i) We first take $W$ so that we have a decomposition (11.4.3)

$G|_{W_i} \simeq \bigoplus_{i \in I'} R\Gamma_{\pi^{-1}(\Lambda_i \cap T^*W)}(\Psi_W(F'_i))$ where $I' = \pi_0(\Lambda \cap T^*W)$ and $F'_i \in \mathcal{D}_{\{|\Lambda|\}}(k_{W_0})$ ($I'$ contains $I$ but could be bigger). Shrinking $W$ we can forget the components in $I' \setminus I$ (maybe $\Lambda_i \cap T^*W$ will be no longer connected but we don’t care). Hence we can assume $I' = I$.

Up to shrinking $W$ several times, Lemma 10.2.2 implies that, for each $i \in I$, $m_{\Lambda_i}(F'_i)$ is isomorphic to $m_{\Lambda_i}(F_i \otimes (E'_0)_W)$, where $E'_0 = (\muhom(F_i, F'_i))_{p_i}$, and Corollary 11.3.8 implies $\Psi_W(F'_i) \simeq \Psi_W(F_i \otimes (E'_0)_W)$. We remark that

$$\muhom(F_i, F'_i)|_{\Xi_i} \simeq \muhom^{dbl}(\Psi_W(F_i), \Psi_W(F'_i))|_{\Xi_i} \simeq \muhom^{dbl}(\Psi_W(F_i), G)|_{\Xi_i},$$

where $\Xi_i = \Lambda_i \cap SS^{dbl}(G) \cap T^*W$. Hence $E'_0 \simeq E_i$. Finally $W \cap V^{A_1} = W \cap \pi_{M \times \mathbb{R}}(\Lambda_i \cap \Lambda_0)$ by Lemma 11.4.7 (i), proving the formula in (i).
(ii) Using the decomposition in (i) for $G$ and $G'$, we can assume that $G = \mathrm{R}\Gamma_{\mathcal{W}_i \times \mathbb{R}_{>0}}(\Psi \mathcal{W}(F_i \otimes (E_i \mathcal{W})))$, $G' = \mathrm{R}\Gamma_{\mathcal{W}_i' \times \mathbb{R}_{>0}}(\Psi \mathcal{W}(F_i \otimes (E_i' \mathcal{W})))$. The hypothesis that $G, G'$ are pure with the same shifts says that the complexes $E_i, E_i'$ are concentrated in the same degree. Hence $\mu\hom^{\text{dbl}}(G, G')|_{\Lambda_0'^{\text{dbl}} \cap \Lambda_i}$ is a constant sheaf concentrated in degree 0 with stalks $\hom(E_i, E_i')$. By Lemma [11.4.11] we may consider that $\Lambda_0\cap \Lambda_i$ is connected, up to shrinking $W$. Then $u$ is determined by its germ at $p_i$. By the construction of $v_i$ in the lemma, the morphism $v_i^\mu$ has the same germ as $u$ at $p_i$. Hence $v_i$ represents $u$. 

**Corollary 11.4.14.** Let $G \in D_{\Lambda,Y}^\text{dbl}(k_Y)$ and $\Lambda_0 = SS^\text{dbl}(G) \subset T^*V$ (see Lemma [11.4.7]). Let $\Lambda_0 = \Lambda_0^1 \cup \Lambda_0^2$ be a decomposition of $\Lambda_0$ into two open and closed subsets. Then there exists an open subset $V'$ of $V$, such that $(V \times \{0\}) \cup V'$ is open in $M \times \mathbb{R} \times \mathbb{R}_{\geq 0}$ and there exist $G_1, G_2 \in D_{\Lambda,Y}^\text{dbl}(k_Y)$ such that $SS^\text{dbl}(G_i) = \Lambda_0^i \cap T^*V'$, $i = 1, 2$, and $G|_{V'} \simeq G_1|_{V'} \oplus G_2|_{V'}$.

**Proof.** (i) By Proposition [11.4.9] we have

$$\lim_{V'} \hom(G|_{V'}, G|_{V'}) \Rightarrow H^0(\Lambda_0; \mu\hom^{\text{dbl}}(G, G)),$$

where $V'$ runs over the open subsets of $M \times \mathbb{R} \times \mathbb{R}_{\geq 0}$ such that $(V \times \{0\}) \cup V'$ is open in $M \times \mathbb{R} \times \mathbb{R}_{\geq 0}$. The identity morphism of $G$ induces an element $1_G$ of the left hand side of (11.4.19). We let $1_G^\mu$ be the corresponding section of $\mu\hom^{\text{dbl}}(G, G)$. Since $\Lambda_0$ is split into two open and closed subsets we have

$$H^0(\Lambda_0; \mu\hom^{\text{dbl}}(G, G)) \simeq \bigoplus_{i=1,2} H^0(\Lambda_0^i; \mu\hom^{\text{dbl}}(G, G))$$

and we write $1_G^\mu = e_1 + e_2$ according to this decomposition. Since $\Lambda_0^1$ and $\Lambda_0^2$ are disjoint we have $e_1e_2 = e_2e_1 = 0$. Hence $e_1$ and $e_2$ are orthogonal idempotents. By (11.4.19) again we deduce a decomposition $1_G = f_1 + f_2$ where $f_1$ and $f_2$ are also orthogonal idempotents.

(ii) We can find $V'$ as in (11.4.19) and $f_1', f_2'$ in $\hom(G|_{V'}, G|_{V'})$ which represent $f_1, f_2$. Up to shrinking $V'$ we can also assume that $f_1', f_2'$ are orthogonal idempotents (using (11.4.19) again). By [8, Prop. 3.2] we deduce a corresponding decomposition $G|_{V'} \simeq G_1 \oplus G_2$ in $D^b(k_Y)$.

(iii) We extend $G_1, G_2$ arbitrarily to $V$. It remains to check that $G_i \in D_{\Lambda,Y}^\text{dbl}(k_Y)$ and $SS^\text{dbl}(G_i) = \Lambda_0^i \cap T^*V'$. We will prove that we have a local decomposition (11.4.3) of $G_i$ in a neighborhood $W$ of any given point $(x, t) \in V$. We first choose $W$ so that $W_{x'} \subset V'$ and a decomposition (11.4.3) holds for $G|_{W_{x'}}$ (and we use the corresponding
notations $I, A_i$). For $i \in I$ the set $\Lambda_i \cap T^* V^{A_i}$ may be non connected, but, by Lemma 11.4.11 if we start with $W$ small enough, at most one component of $\Lambda_i \cap T^* V^{A_i}$ can meet $T^* W'$ for some smaller neighborhood $W'$. Hence $\Lambda_i \cap T^* V^{A_i} \cap T^* W'$ is contained in either $\Lambda_0^1$ or $\Lambda_0^2$. We choose $W'$ so that this holds for all $i \in I$.

We define $G'_1$ (resp. $G'_2$) to be the sum of the summands of $G|_{W'_i}$ in (11.4.3) indexed by the $j \in I$ so that $\Lambda_j \cap T^* V^{A_j} \cap T^* W'$ is contained in $\Lambda_0^1$ (resp. $\Lambda_0^2$). Then $G' \in D^{bd}_{\Lambda, \mathcal{V}}(k)$. This gives another decomposition of $G|_{W'_i}$ and corresponding idempotents $f''_i, f'_i$. Since $\mathrm{SS}(G'_i) \cap \Lambda = T^* W'_i \cap \Lambda_0^0$, the section of $\mu \hom_{\mathcal{D}^{bd}}(G, G)|_{T^* W'_i \cap \Lambda}$ associated with $f''$ by (11.4.19) must be $e_i|_{T^* W'_i \cap \Lambda}$. Up to restricting $W'$ once more, we deduce $f_i'' = f_i'|_{W'_i}$ and then $G_i' \simeq G_i|_{W \times 0, \varepsilon}$, which proves the result.

\[ \square \]

Part 12. Quantization

The main result of this part is that, for any global object $\mathcal{F}$ of the Kashiwara-Schapira stack $\mu \mathcal{S}(k)$, there exists $F \in D^b(k_{M \times \mathbb{R}})$ with $\mathrm{SS}(F) = \Lambda$ which represents $\mathcal{F}$, when $\Lambda$ is the conification of a compact exact Lagrangian submanifold of $T^* M$ (see (12.0.1) below). This recovers a result of Viterbo in [55] who proves the existence of such a sheaf using Floer theory (the proof of [55] was sketched in [54] in 2011).

We first consider a compact Legendrian submanifold of $J^1 M$, or equivalently, a closed conic Lagrangian submanifold $\Lambda$ of $T^*_{\tau > 0}(M \times \mathbb{R})$ such that $\Lambda/\mathbb{R}_{> 0}$ is compact. We apply the procedure sketched in the introduction of Part 11 and we prove that any object $\mathcal{F} \in \mu \mathcal{S}(k)$, or $\mathcal{F} \in \mu \mathcal{S}(\mathcal{T}^{l}[1])$, is represented by some $F \in D^b(k_{M \times \mathbb{R}})$, or $F \in D^b(\mathcal{T}^{l}[1])$, such that $\mathrm{SS}(F) = \Lambda \sqcup T_{\varepsilon}(\Lambda)$ for $\varepsilon > 0$ small, where $T_{\varepsilon}$ is the translation along the factor $\mathbb{R}$. Then, assuming that $\Lambda/\mathbb{R}_{> 0}$ has no Reeb chord we prove that there exists another representative $F'$ such that $\mathrm{SS}(F') = \Lambda$. We recall that “having no Reeb chord” means that $\rho_M : T^*_{\tau > 0}(M \times \mathbb{R}) \to T^* M, (x, t; \xi, \tau) \mapsto (x; \xi/\tau)$, induces an injection $\Lambda/\mathbb{R}_{> 0} \hookrightarrow T^* M$. The image $\tilde{\Lambda} = \rho_M(\Lambda)$ is then a compact exact Lagrangian submanifold of $T^* M$ in the sense that $\alpha_M|_{\tilde{\Lambda}}$ is exact. The link between $\Lambda$ and $\tilde{\Lambda}$ is given by

\[ (12.0.1) \quad \Lambda = \{(x, t; \xi, \tau); \tau > 0, (x; \xi/\tau) \in \tilde{\Lambda}, t = -f(x; \xi/\tau)\}, \]

where $f$ is a primitive of $\alpha_M|_{\tilde{\Lambda}}$.

Now we introduce some notations. We first remark that, by Theorem 2.1.1 we can as well move $\Lambda$ by any contact isotopy of $J^1 M$ and
assume from the beginning that it satisfies some genericity hypotheses. Hence, we can assume that the map $\Lambda/\mathbb{R}_{>0} \to M$ has finite fibers and, by Lemma 11.4.3, we can assume that there exists an adapted family $\mathcal{V} = \{V_a; a \in A\}$ in the sense of Definition 11.4.1. We let $\Lambda_0, b \in B_a$, be the family of components of $\Lambda \cap T^*V_a$. We set $B = \bigsqcup_{a \in A} B_a$.

We have introduced the subcategory $\mathcal{D}^{dbl}_{\Lambda, \mathcal{V}}(k_{M \times \mathbb{R}})$ of $\mathcal{D}(k_{M \times \mathbb{R} \times \mathbb{R}_{>0}})$ in Definition 11.4.4. For $G \in \mathcal{D}^{dbl}_{\Lambda, \mathcal{V}}(k_{M \times \mathbb{R}})$ we have defined $SS^{dbl}(G)$ in Lemma 11.4.7 it is an open subset of $\Lambda$ of the form $\bigcup_{b \in B} \Lambda_b$ for some $B \subset B$. For an open subset $\Lambda_0$ of $\Lambda$ we also have a functor (see 11.4.13)

$$m^{dbl}_{\Lambda_0}: \{G \in \mathcal{D}^{dbl}_{\Lambda, \mathcal{V}}(k_V); \Lambda_0 \subset SS^{dbl}(G)\} \to \mu \text{Sh}(k_{\Lambda_0}).$$

The same definitions make sense for the orbit category. We define the subcategory $\mathcal{D}^{dbl}_{/\Lambda, \mathcal{V}}(k_{M \times \mathbb{R}})$ of $\mathcal{D}(\Lambda, (k_{M \times \mathbb{R} \times \mathbb{R}_{>0}})$ as in Definition 11.4.4 replacing everywhere $\mathcal{D}(k_X)$ by $\mathcal{D}(\Lambda, (k_X)$. We can also define $SS^{dbl}(G)$ for $G \in \mathcal{D}^{dbl}_{/\Lambda, \mathcal{V}}(k_{M \times \mathbb{R}})$ and a functor

$$m^{dbl}_{/\Lambda, \mathcal{V}}: \{G \in \mathcal{D}^{dbl}_{/\Lambda, \mathcal{V}}(k_V); \Lambda_0 \subset SS^{dbl}(G)\} \to \mu \text{Sh}(k_{\Lambda_0}).$$

In Theorems 12.1.1 and 12.2.2 we see that the functors $m^{dbl}_{\Lambda_0}$ and $m^{dbl}_{/\Lambda, \mathcal{V}}$ are essentially surjective. For a given $G \in \mathcal{D}^{dbl}_{\Lambda, \mathcal{V}}(k_{M \times \mathbb{R}})$ the microsupport of $G|_{M \times \mathbb{R} \times \{\varepsilon\}}$, for $\varepsilon > 0$ small enough, is made of two copies of $\Lambda$. In Corollary 12.3.2 we see that, if we have no Reeb chords, we can translate one copy of $\Lambda$ vertically using a Hamiltonian isotopy and obtain an object of $\mathcal{D}^b_{\Lambda, \mathcal{V}}(k_{M \times \mathbb{R}})$ from a given one in $\mathcal{D}^{dbl}_{\Lambda, \mathcal{V}}(k_{M \times \mathbb{R}})$. In 12.4 we see a relation between objects of $\mathcal{D}^b_{\Lambda, \mathcal{V}}(k_{M \times \mathbb{R}})$, their restrictions to $M \times \{t_0\}, t_0 \gg 0$, and their microlocalizations; in particular we see that $F \mapsto F|_{M \times \{t_0\}}$ induces a fully faithful functor from sheaves on $M \times \mathbb{R}$ with microsupport $\Lambda$ and vanishing on $M \times \{t\}, t \ll 0$, and locally constant sheaves on $M$.

12.1. Quantization for the doubled Legendrian

Let $\Lambda$ be a closed conic Lagrangian submanifold of $T^*_{>0}(M \times \mathbb{R})$ such that $\Lambda/\mathbb{R}_{>0}$ is compact. By Lemma 11.4.3 we can assume that there exists an adapted family $\mathcal{V} = \{V_a; a \in A\}$ for $\Lambda$ which is stable by intersection. We can also assume that $\Lambda \cap T^*V_a$ has finitely many connected components, for each $a \in A$. We let $\Lambda_b, b \in B_a$, be the family of components of $\Lambda \cap T^*V_a$. We set $B = \bigsqcup_{a \in A} B_a$ and we let $\sigma: B \to A$ be the obvious map. Hence $\Lambda_b$ is a component of $\Lambda \cap T^*V_{\sigma(b)}$. When we use Lemma 11.4.3 we can also assume that the family $\{\Lambda_b\}_{b \in B}$ refines any given family. Since we know that simple sheaves along $\Lambda$ locally
exist (Lemma 10.2.5), we can assume that, for each \( b \in B \), there exist a neighborhood \( V'_b \) of \( V_{\sigma(b)} \), a contractible component \( \Lambda'_b \) of \( \Lambda \cap T^* V'_b \) and \( F_b \in D^b(k_{V'_b}) \) such that \( \Lambda_b \) is a component of \( \Lambda'_b \cap T^* V_{\sigma(b)} \), \( \mathcal{S}(F_b) = \Lambda'_b \) and \( F_b \) is simple.

**Theorem 12.1.1.** In the above setting let \( B' \) be a subset of \( B \) and set \( \Lambda_0 = \bigcup_{b \in B'} \Lambda_b \), \( \Lambda_0' = \bigcup_{b \in B'} \Lambda'_b \). Then, for any pure object \( \mathcal{F} \in \mu \mathcal{S}(k_{\Lambda_0'}) \) there exists \( F \in D_{\Lambda,V}^{db}(k_{M \times R}) \) such that \( \mathcal{S}^{db}(F) = \Lambda_0 \) and \( \mathfrak{m}^{db}_{\Lambda_0'}(F) \cong \mathcal{F}|_{\Lambda_0} \), where \( \mathfrak{m}^{db}_{\Lambda_0} \) is defined in (11.4.13).

**Proof.** (i) We proceed by induction on \( |B'| \). Let \( b \in B \). Since \( \Lambda'_b \) is contractible, the objects of \( \mu \mathcal{S}(k_{\Lambda'_b}) \) are of the form \( \mathfrak{m}_{\Lambda'_b}(E_{V'_b} \otimes F_b) \) for some \( E \in D^b(k) \). So we write \( \mathcal{F}|_{\Lambda'_b} = \mathfrak{m}_{\Lambda'_b}(E_{V'_b} \otimes F_b) \) and we set

\[
G = \mathcal{R}_{(V_{\sigma(b)} \times R_{\geq 0}) \cap (V'_b)_0}(V_{\sigma(b)} \times R_{\geq 0}).
\]

extended by zero outside \( (V'_b)_0 \) (the formula defines \( G \) on \( (V'_b)_0 \) with a support contained in \( (V_{\sigma(b)} \times R_{\geq 0}) \)). Let us prove that \( G \) belongs to \( D_{\Lambda,V}^{db}(k_{M \times R}) \). We check Definition 11.4.4 around a point \((x, t) \in M \times R\):

If \((x, t) \not\in V_{\sigma(b)}\), we choose \( W \) such that \( W \cap V_{\sigma(b)} = \emptyset \) and we have \( G|_{W_0} = 0 \) so (11.4.3) is trivial.

If \((x, t) \in V'_b\), we choose \( W = V'_b \) and the defining formula for \( G \) satisfies (11.4.3). We remark that the family \( I \) in (11.4.3) consists of one element, say \( I = \{i_0\} \), and \( A_{i_0} = \{\sigma(b)\} \).

We have \( \mathcal{S}^{db}(G) = \Lambda'_b \cap T^* V_{\sigma(b)} \) and \( \Lambda_b \) is a connected component of \( \mathcal{S}^{db}(G) \). By Corollary 11.4.14 there exists \( G_b, G' \in D_{\Lambda,V}^{db}(k_{M \times R}) \) such that \( \mathcal{S}^{db}(G_b) = \Lambda_b \), \( \mathcal{S}^{db}(G') = \mathcal{S}^{db}(G) \setminus \Lambda_b \) and \( G|_O \cong G_b|_O \oplus G'|_O \) for some open subset \( O \) of \( M \times R \) satisfying

\[
(M \times R \times \{0\}) \cup O \text{ is open in } M \times R \times R_{\geq 0}.
\]

By construction we have \( \mathfrak{m}_{\Lambda_b}(G_b) \cong \mathcal{F}|_{\Lambda_{\sigma(b)}} \), which proves the case \( B' = \{b\} \).

(ii) Now we write \( B' = B'' \cup \{b\} \) and assume that the result holds for \( B'' \). We set \( \Lambda_1 = \bigcup_{c \in B''} \Lambda_c \) and \( \Lambda_{1b} = \Lambda_1 \cap \Lambda_b \). By the induction hypothesis and by (i) there exist \( G_1, G_b \in D_{\Lambda,V}^{db}(k_{M \times R}) \) with \( \mathcal{S}^{db}(G_1) = \Lambda_1 \), \( \mathcal{S}^{db}(G_b) = \Lambda_b \) and isomorphisms \( \varphi_1: \mathfrak{m}_{\Lambda_1}(G_1) \cong \mathcal{F}|_{\Lambda_1}, \varphi_b: \mathfrak{m}_{\Lambda_b}(G_b) \cong \mathcal{F}|_{\Lambda_b} \). We have assumed that the adapted family \( \{V_a\}_{a \in A} \) is stable by intersection. Hence

\[
G_1' = \mathcal{R}_{(V_{\sigma(b)} \times R_{\geq 0})}(G_1)
\]
belongs to $\mathbf{D}_{\Lambda,\mathbb{V}}^{\text{dib}}(\mathbb{k}_{M \times \mathbb{R}})$ (this can be checked directly on the local decomposition (11.4.3)). We have $SS^{\text{dib}}(G_1) = \Lambda_1 \cap T^*V_{\sigma(b)}$ and $\Lambda_{1b}$ is open and closed in $SS^{\text{dib}}(G_1)$. As in (i) we use Corollary 11.4.14 to find $G_{1b} \in \mathbf{D}_{\Lambda,\mathbb{V}}^{\text{dib}}(\mathbb{k}_{M \times \mathbb{R}})$ such that $SS^{\text{dib}}(G_{1b}) = \Lambda_{1b}$ and $G_{1b}|_{O'}$ is the summand of $G_1'|_{O'}$ corresponding to $\Lambda_{1b}$, for some open subset $O'$ satisfying (12.1.1).

We have a natural morphism $g_1: G_1|_{O'} \to G_{1b}|_{O'}$ given by the composition of the natural morphism $G_1 \to G_1'$ and the projection to the summand $G_1'|_{O'} \to G_{1b}|_{O'}$. By construction we have an isomorphism $\varphi_{1b}: m_{\Lambda_{1b}}^{\text{dib}}(G_{1b}) \cong \mathcal{F}|_{\Lambda_{1b}}$ compatible with $\varphi_1$ in the sense that $\varphi_1|_{\Lambda_{1b}} = \varphi_{1b} \circ m_{\Lambda_{1b}}(g_1)$ (we use $m_{\Lambda_{1b}}^{\text{dib}}(G_1) = m_{\Lambda_1}^{\text{dib}}(G_1)|_{\Lambda_{1b}}$).

We define $g_b = \varphi_1^{-1} \circ (\varphi_1|_{\Lambda_{1b}}): m_{\Lambda_{1b}}^{\text{dib}}(G_b) \cong m_{\Lambda_{1b}}^{\text{dib}}(G_{1b})$. Since $\mathcal{F}$ is pure, we can apply Corollary 11.4.10 and, up to shrinking $O'$, we obtain $g_b: G_b|_{O'} \to G_{1b}|_{O'}$ which represents $\tilde{g}_b$. Finally we define $G \in \mathbf{D}^b(\mathbb{k}_{M \times \mathbb{R} \times \mathbb{R}^\geq_0})$ by the distinguished triangle on $O'$

\[(12.1.2) \quad G \to G_1|_{O'} \oplus G_{1b}|_{O'} \xrightarrow{(g_1,-g_b)} G_{1b}|_{O'} \xrightarrow{+1} \]

and by extending $G$ by zero outside $O'$. We prove in (iii) that $G$ represents $\mathcal{F}$ over $\Lambda_0 = \Lambda_1 \cup \Lambda_b$.

(iii) We first have to check that $G$ belongs to $\mathbf{D}_{\Lambda,\mathbb{V}}^{\text{dib}}(\mathbb{k}_{M \times \mathbb{R}})$, which means that it can be decomposed as in (11.4.3), around any given point $(x, t) \in M \times \mathbb{R}$. We use Lemma 11.4.13 to describe $G_*$, for $* = 1, *, b$ or $* = 1b$, over a small enough neighborhood $W$ of $(x, t)$. We have a partition $\Lambda \cap T^*W = \bigsqcup_{i \in I} \Xi_i$ into connected components and a simple sheaf $F_i \in \mathbf{D}_{\Xi_i}(\mathbb{k}_W)$ for each $i \in I$. We choose $p_i \in \Xi_i$ and set $E_i = (\mathcal{H}om(m_{\Xi_i}(F_i), F))_{p_i}$ (this does not depend on $p_i$). Over $SS^{\text{dib}}(G_*)$ we have

\[
\mu^{\text{dib}}(\Psi_W(F_i), G_*) \simeq \mathcal{H}om(m_{\Xi_i}^{\text{dib}}(\Psi_W(F_i)), m_{\Xi_i}^{\text{dib}}(G_*))
\simeq \mathcal{H}om(m_{\Xi_i}^{\text{dib}}(\Psi_W(F_i)), \mathcal{F})
\simeq \mathcal{H}om(m_{\Xi_i}(F_i), \mathcal{F}).
\]

Hence, setting for short $H_i = \Psi_W(F_i)^{\mathbb{L}} \otimes (E_i)_W$ and maybe shrinking $W$, we have by Lemma 11.4.13 (i)

\[
G_*|_{W_i} \simeq \bigoplus_{i \in I} \Gamma W^*_i \times \mathbb{R}^\geq (H_i, \quad * = 1, b, 1b),
\]

where $W_i^* = W \cap \hat{\pi}_M \times \mathbb{R}(\Xi_i \cap SS^{\text{dib}}(G_*)) = W \cap \hat{\pi}_M \times \mathbb{R}(\Xi_i \cap \Lambda_*)$.

By Lemma 11.4.13 (ii) the morphisms $g_1, g_b$ in (12.1.2) are obtained by composing morphisms $\alpha_i^*: \Gamma W^*_i \times \mathbb{R}^\geq(-) \to \Gamma W^*_i \times \mathbb{R}^\geq(-),$
there exists a map,
Lemma 12.2.1. open subsets with vanishing Maslov classes.

We prove the result for the triangulated orbit category by gluing two

We checked in Sections 9.1 and 9.2 that the results we used in the
category $D^b(k_M)$ have analogs in the category $D_{[1]}(k_M)$. In particular
we have already defined a Kashiwara-Schapira stack $\mu\text{Sh}_{[1]}(k_\Lambda)$ in this
situation. However the proof of Theorem 12.1.1 does not work because
we used the fact that some $\mu\text{hom}$ sheaf was concentrated in degree 0
(we used Corollary 11.4.10), which makes no sense in the orbit category.
We prove the result for the triangulated orbit category by gluing two
sheaves obtained on two open subsets by Theorem 12.1.1. For this we
first remark in Lemma 12.2.1 below that we can decompose $\Lambda$ in two
open subsets with vanishing Maslov classes.

**Lemma 12.2.1.** Let $X$ be a manifold and let $c \in H^1(X; \mathbb{Z}_X)$. Then
there exists a map $f : X \to S^1$ of class $C^\infty$ such that $c = f^*(\delta)$, where
$\delta \in H^1(S^1; \mathbb{Z}_{S^1})$ is the canonical class. In particular, for any
covering $S_1 = I_1 \cup I_2$ by two open intervals, the restrictions $c|_{f^{-1}(I_i)} \in
H^1(f^{-1}(I_i); \mathbb{Z}_{f^{-1}(I_i)})$ vanish, for $i = 1, 2$. 
Proof. Since $H^1(X; \mathbb{Z}) \to H^1(X; \mathbb{R})$ is injective, it is enough to prove the result for the image of $c$ in $H^1(X; \mathbb{R})$, which we represent by a 1-form $\alpha$. Let $r: X' \to X$ be the universal covering of $X$ and let $g: X' \to \mathbb{R}$ be a primitive of $r^*(\alpha)$. Then, for any $x_1, x_2 \in X'$ such that $r(x_1) = r(x_2)$ we have $g(x_1) - g(x_2) = \langle c, \gamma \rangle$, where $\gamma$ is the loop at $g(x_1)$ determined by $x_1, x_2$. Hence $g(x_1) - g(x_2)$ is an integer and $g$ descends to a map $f: X \to S^1$ which satisfies the conclusion of the lemma. \[ \square \]

For the next result the ring is $k = \mathbb{Z}/2\mathbb{Z}$.

**Theorem 12.2.2.** For any global object $F$ of $\mu \text{Sh}_{/\mathbb{R}}(k_\Lambda)$ there exists an object $F \in D_{\mathbf{bd}[[1]], \mathcal{V}}(k_{M \times \mathbb{R}})$, for some finite family $\mathcal{V}$ of open subsets of $M \times \mathbb{R}$ which is adapted to $\Lambda$, such that $m_{\mathbf{bd}[[1]]}(F) \simeq F$.

**Proof.** (i) By Proposition 10.4.4 $\mu \text{Sh}_{/\mathbb{R}}(k_\Lambda)$ has a unique simple object, $F_0$, and $F$ is of the type $F \simeq F_0 \otimes L$ for some $L \in \text{Loc}(k_\Lambda)$.

We let $\mu_1(\Lambda) \in H^1(\Lambda; \mathbb{Z}_\Lambda)$ be the sheaf obstruction class of $\Lambda$ (which coincides with the Maslov class). We apply Lemma 12.2.1 to obtain $f: \Lambda \to S^1$ such that $\mu_1(\Lambda)$ is the pull-back of the fundamental class of $S^1$. We choose a covering $S_1 = I_+ \cup I_-$ by two open intervals and set $\Lambda_\pm = f^{-1}(I_\pm)$. Then $\mu_1(\Lambda)|_{\Lambda_\pm} = 0$ and the categories $\mu \text{Sh}(k_{\Lambda_\pm})$ have simple objects, say $F_{0,\pm}$. Their images in $\mu \text{Sh}_{/\mathbb{R}}(k_{\Lambda_\pm})$ are $F_{0}|_{\Lambda_\pm}$ because $\mu \text{Sh}_{/\mathbb{R}}(k_{\Lambda_\pm})$ has a unique simple object. The intersection $I_+ \cap I_-$ is the union of two intervals, say $I_a$, $I_b$. We set $\Lambda_a = f^{-1}(I_a)$, $\Lambda_b = f^{-1}(I_b)$ and $F_{0,a} = F_{0,+}|_{\Lambda_a}$, $F_{0,b} = F_{0,+}|_{\Lambda_b}$. We remark that simple objects in $\mu \text{Sh}(k_{\Xi})$, for $\Xi$ open in $\Lambda$, with coefficients $k = \mathbb{Z}/2\mathbb{Z}$, are unique up to shift and up to a unique isomorphism. Hence we have the following canonical isomorphisms

$$\varphi_+^a = \text{id}: F_{0,a} \cong F_{0,+}|_{\Lambda_a}, \quad \varphi_+^b = \text{id}: F_{0,b} \cong F_{0,+}|_{\Lambda_b},$$

$$\varphi_-^a: F_{0,a} \cong F_{0,-}|_{\Lambda_a}[d_a], \quad \varphi_-^b: F_{0,b} \cong F_{0,-}|_{\Lambda_b}[d_b],$$

where $d_a, d_b \in \mathbb{Z}$ are locally constant functions on $\Lambda_a$, $\Lambda_b$. We set $F_\pm = F_{0,\pm} \otimes L$, $F_a = F_{0,a} \otimes L$, $F_b = F_{0,b} \otimes L$ and $\Phi_\pm = \varphi_\pm^* \otimes \text{id}_L$, for $* = a, b$.

(ii) Up to shrinking $\Lambda_0$ and $\Lambda_1$ we can find a finite family $\mathcal{V} = \{V_a; a \in A\}$ of open subsets of $M \times \mathbb{R}$ which is adapted to $\Lambda$ such that $\Lambda_+$ and $\Lambda_-$ are of the form $\Lambda^B$ for some $B \subseteq A$ (see Lemma 11.4.3).

We can also assume that the adapted family is stable by intersection. Hence $\Lambda_a$ and $\Lambda_b$ are also of the form $\Lambda^B$.\]
By Theorem 12.1.1 there exist $F_\bullet \in D^{dbl}_{\Lambda, V}(k_{M \times \mathbb{R}})$, for $\bullet = +, -, a, b$, such that $SS^{dbl}(F_\bullet) = \Lambda_\bullet$ and $m^{dbl}_{/[1], \Lambda}(F_\bullet) \simeq F_\bullet$. Moreover, by Proposition 11.4.9 we have morphisms, for $* = a, b$, $\Psi^+_*: F_+|_V \to F_+|_V$ and $\Psi^-*: F_-|_V \to F_-|_V|[d_\ast]$ representing $\Phi^*_\pm$, where $V$ is some open subset of $M \times \mathbb{R} \times \mathbb{R}_{>0}$ satisfying (12.1.1). In $D_{/[1]}(k_V)$ we have $F_- \simeq F_-|_V[d_\ast]$ and we can define $F \in D_{/[1]}(k_V)$ by the distinguished triangle

$$F_a \oplus F_b \xrightarrow{(\Psi^+_a, \Psi^+_b)} F_+ \oplus F_- \to F \oplus_1 \to .$$

We extend $F$ by zero outside $V$. We can then check as in part (iii) of the proof of Theorem 12.1.1 that $F \in D^{dbl}_{/[1], \Lambda, V}(k_{M \times \mathbb{R}})$ and that $m^{dbl}_{/[1], \Lambda}(F) \simeq F$.

**Remark 12.2.3.** By Proposition 10.4.4 $\mu \text{Sh}_{/[1]}(k_\Lambda)$ has a simple object. Hence Theorem 12.2.2 gives the existence of a simple object in $D^{dbl}_{/[1], \Lambda, V}(k_{M \times \mathbb{R}})$, say $F_0$. In the case where $\Lambda$ has no Reeb chord, we will see that $\text{Loc}(k_\Lambda)$ is equivalent to $\text{Loc}(k_M)$ and all objects of $D^{dbl}_{/[1], \Lambda, V}(k_{M \times \mathbb{R}})$ are of the form $F_0 \otimes_{k_{M \times \mathbb{R}}} L$ for some $L \in \text{Loc}(k_{M \times \mathbb{R}})$. (However in the course of the proof of the previous theorem this is unknown, so we cannot deduce a representative for $F$ from one for $F_0$.)

### 12.3. Translation of the microsupport

Now we assume that the Legendrian submanifold $\Lambda/\mathbb{R}_{>0}$ of $J^1 M$ has no Reeb chord, that is, the map $T^*_r(M \times \mathbb{R}) \to T^*_r M$, $(x, t; \xi, \tau) \mapsto (x, \xi/\tau)$, induces an embedding $\Lambda/\mathbb{R}_{>0} \hookrightarrow T^*_r M$ (see (12.0.1)).

For $u \in \mathbb{R}$ we define the translation $T_u: M \times \mathbb{R} \to M \times \mathbb{R}$, $(x, t) \mapsto (x, t + u)$. We denote by $T^*_u: T^*(M \times \mathbb{R}) \to T^*(M \times \mathbb{R})$, $(x, t; \xi, \tau) \mapsto (x, t + u; \xi, \tau)$, the induced map on the cotangent bundle. We also introduce some notations, for $\Lambda \subset T^*_r M$:

$$\Lambda_u = \Lambda \cup T^*_u(\Lambda) \subset T^*_r(M \times \mathbb{R}), \quad \text{for } u > 0,$$

$$\Lambda^+ = q_d q^{-1}_r(\Lambda) \cup r_d r^{-1}_r(\Lambda) \subset T^*_r(M \times \mathbb{R} \times \mathbb{R}_{>0}),$$

where $q, r: M \times \mathbb{R} \times \mathbb{R}_{>0} \to M \times \mathbb{R}$ are given by $q(x, t, u) = (x, t)$, $r(x, t, u) = (x, t - u)$. We remark that $\Lambda^+$ is non-characteristic for the inclusions $i_u: M \times \mathbb{R} \times \{u\} \to M \times \mathbb{R} \times \mathbb{R}_{>0}$, $u > 0$, and that $\Lambda_u = (i_u)_d((i_u)_r^{-1}(\Lambda^+))$.

**Lemma 12.3.1.** There exists $\phi: \tilde{T}^*(M \times \mathbb{R}) \times \mathbb{R}_{>0} \to \tilde{T}^*(M \times \mathbb{R})$, a homogeneous Hamiltonian isotopy, such that $\phi_1 = \text{id}$ and, using the
notations \((1.5.5)\) and \((12.3.1)\), we have \(\Gamma_\phi \circ^a \Lambda_1 = \Lambda^+\). In particular \(\phi_\alpha(\Lambda_1) = \Lambda_u\), for all \(u > 0\).

**Proof.** (i) We set \(I = \mathbb{R}_{>0}\). Since the map \((x, t; \xi, \tau) \mapsto (x; \xi/\tau)\) induces an injection \(\Lambda/\mathbb{R}_{>0} \hookrightarrow T^*M\), the sets \(\Lambda\) and \(T_u'(\Lambda)\) are disjoint for all \(u > 0\). Considering all \(u > 0\) at once we define the following closed subsets of \(T^*(M \times \mathbb{R}) \times I\):

\[
\Lambda^0 = \Lambda \times \mathbb{R}_{>0}, \quad \Lambda^1 = \bigcup_{u>0} (T_u'(\Lambda) \times \{u\}).
\]

Then \(\Lambda^0\) and \(\Lambda^1\) are disjoint and the projections \(\Lambda^i/\mathbb{R}_{>0} \to I\) are proper for \(i = 0, 1\). Hence we can find a conic neighborhood \(\Omega\) of \(\Lambda^1\) in \(\tilde{T}^*(M \times \mathbb{R}) \times I\) such that \(\Omega \cap \Lambda^0 = \emptyset\) and the projection \(\Omega/\mathbb{R}_{>0} \to I\) is proper, that is, \(\Omega \cap (\tilde{T}^*(M \times \mathbb{R}) \times \{u\})\) is compact for all \(u > 0\).

(ii) We choose a \(C^\infty\)-function \(h: \tilde{T}^*(M \times \mathbb{R}) \times I \to \mathbb{R}\) such that

(a) \(h_u := h|_{\tilde{T}^*(M \times \mathbb{R}) \times \{u\}}\) is homogeneous of degree 1, for all \(u \in I\),

(b) \(h\) vanishes outside \(\Omega\),

(c) there exists a neighborhood \(\Omega'\) of \(\Lambda^1\) such that \(h(x, t; \xi, \tau) = -\tau\), for all \((x, t; \xi, \tau) \in \Omega'\).

By (a), (b) and the compactness of \((\Omega \cap (\tilde{T}^*(M \times \mathbb{R}) \times \{u\}))/\mathbb{R}_{>0}\), the Hamiltonian flow of \(h\), say \(\phi\), is defined on \(I\) (the initial time here is \(t_0 = 1\)). Then \(\phi_u\) is the identity map outside \(\Omega\) for all \(u \in I\) and \(\phi_u(x, t; \xi, \tau) = (x, t + u - 1; \xi, \tau)\), for all \(((x, t; \xi, \tau), u) \in \Omega'\). Since \(\Lambda^0 \subset (\tilde{T}^*(M \times \mathbb{R}) \setminus \Omega)\) and \(\Lambda^1 \subset \Omega'\), the lemma follows. \(\square\)

**Corollary 12.3.2.** Let \(\Lambda\) be a closed conic Lagrangian submanifold of \(T^*_\Lambda(M \times \mathbb{R})\) coming from a compact exact submanifold of \(T^*M\) as in \((12.0.1)\). Then, for any pure object \(\mathcal{F} \in \mu\text{Sh}(k_\Lambda)\) there exists \(F \in D_{/|\Lambda|}(k_{M \times \mathbb{R}})\) such that \(m_\Lambda(F) \simeq \mathcal{F}\) and \(F|_{M \times \{t\}} \simeq 0\) for \(t \ll 0\). In the same way, for any \(\mathcal{F} \in \mu\text{Sh}_/|\Lambda|(k_\Lambda)\) there exists \(F \in D_{/|\Lambda|,\Lambda}(k_{M \times \mathbb{R}})\) such that \(m_\Lambda(F) \simeq \mathcal{F}\) and \(F|_{M \times \{t\}} \simeq 0\) for \(t \ll 0\).

**Proof.** (i) The proofs are the same for \(\mathcal{F} \in \mu\text{Sh}(k_\Lambda)\) or \(\mathcal{F} \in \mu\text{Sh}_/|\Lambda|(k_\Lambda)\). We first consider \(\mathcal{F} \in \mu\text{Sh}(k_\Lambda)\) and then emphasize some points for the other case.

By Theorem \((12.1.1)\) there exists \(F_0 \in D_{/\Lambda}(k_{M \times \mathbb{R}})\), for some finite family \(\mathcal{V}\) of open subsets of \(M \times \mathbb{R}\) which is adapted to \(\Lambda\), such that \(\text{SS}^{\text{bd}}(F_0) = \Lambda\) and \(m_\Lambda^{\text{bd}}(F_0) \simeq \mathcal{F}\). By the definition of \(D_{/\Lambda}(k_{M \times \mathbb{R}})\) any point \(x \in M \times \mathbb{R}\) has a neighborhood \(W_x\) such that a decomposition \((11.4.3)\) holds for \(F_0|_{W_x,\gamma}\). Since \(\text{SS}^{\text{bd}}(F_0) = \Lambda\), we have in fact \(F_0|_{W_x,\gamma} \simeq \Psi_{W_x}(F_x)\) for some \(F_x \in D_{/\Lambda}(k_{W_x})\) (in Lemma \((11.4.13)\) (i) all \(W_i\) coincide with \(W_x\)). Setting \(V = \bigcup_{x \in M \times \mathbb{R}} W_{x,\gamma}\) we then have
supp($F_0$) ∩ $V \subseteq (\tilde{\Lambda} \ast \tilde{T}_{M \times \mathbb{R}}(\Lambda)) \cap V$ and $\tilde{SS}(F_0) \cap T^*V \subseteq \Lambda^+ \cap T^*V$ by Lemmas 11.4.15 and 11.4.17. In particular we can find a compact neighborhood $C$ of $\tilde{T}_{M \times \mathbb{R}}(\Lambda)$ such that, for $u \leq 1$, the support of $F_0[V \cap (M \times \mathbb{R} \times [0,u])$ is contained in $C \times [0, u]$. For $u > 0$ small enough we have $(C \times [0, u]) \subseteq V$ and we obtain $F_1 \in D^b_{\Lambda^+}|(M \times \mathbb{R} \times [0,u])$ by extending $F_0[V \cap (M \times \mathbb{R} \times [0,u])$ by zero. For any $v \in [0,u]$, we have $\tilde{SS}(F_1|_{M \times \mathbb{R} \times \{v\}}) = \Lambda_v$ and it follows from (11.4.15) that $m_{\Lambda}(F_1|_{M \times \mathbb{R} \times \{v\}}) \simeq F$.

(ii) We set $F_2 = F_1|_{M \times \mathbb{R} \times \{\frac{1}{2}u\}}$. Then $\tilde{SS}(F_2) = \Lambda_{\frac{1}{2}u}$. We consider the isotopy $\phi$ of Lemma 12.3.1 and let $K_\phi \in D^b(k(M \times \mathbb{R})^2 \times \mathbb{R}_{>0})$ be the sheaf associated with $\phi$ by Theorem 2.1.1. For given $0 < u_1 < u_2$ the restriction of $\phi$ to $T^*(M \times \mathbb{R} \times [u_1, u_2])$ has compact support (after quotienting by the $\mathbb{R}_{>0}$ action in the fibers) and $K_\phi|_{(M \times \mathbb{R})^2 \times [u_1, u_2]}$ coincides with $k_{\Delta_{M \times \mathbb{R}} \times [u_1, u_2]}$ outside a compact set, hence is bounded. In particular, for any $v > 0$, the sheaf $F_{2,v} = K_{\phi,v} \circ K_{\phi,\frac{1}{2}u}^{-1} \circ F_2$ belongs to $D^b(k(M \times \mathbb{R}))$ and satisfies $\tilde{SS}(F_{2,v}) = (\phi_v \circ \phi_{\frac{1}{2}u}^{-1})(\Lambda_{\frac{1}{2}u}) = \Lambda_v$. Moreover, for given $0 < u_1 < u_2$, $\phi_0$ is the identity map on some neighborhood $\Omega$ of $\Lambda$, for any $u \in [u_1, u_2]$, and it follows that $K_{\phi,u} \circ -$ induces the identity functor on $D(k_{M \times \mathbb{R}}; \Omega)$. Hence $m_{\Lambda}(F_{2,v}) \simeq m_{\Lambda}(F_2) \simeq \mathcal{F}$.

(iii) Since $\Lambda \times \mathbb{R}_{>0}$ is compact we can choose $A > 0$ such that $\Lambda \subseteq T^*_\mathbb{R}_{>0}(M \times ]-A,A[)$. For $v \geq 2A + 1$ we have $T'_v(\Lambda) \subseteq T^*_\mathbb{R}_{>0}(M \times ]A + 1, +\infty[)$. We set $F_3 = F_2|_{M \times ]-\infty,A]}_A$. Then $\tilde{SS}(F_3) = \Lambda$. We choose a diffeomorphism $f : ]-\infty, A + 1[ \simeq \mathbb{R}$ such that $f$ is the identity map on $]A, A[$. Then $F = R(\text{id}_M \times f)_*(F_3)$ satisfies $\tilde{SS}(F) = \Lambda$ and $m_{\Lambda}(F) \simeq \mathcal{F}$.

We set $U_A = M \times ]-\infty, -A[$. We remark that $\tilde{SS}(F_{2,v}|_{U_A})$ is empty for all $v > 0$, hence $F_{2,v}$ is locally constant on $U_A$. If $F_{2,v}|_{U_A}$ does not vanish, then supp($F_{2,v}$) must contain $U_A$. We have seen that supp($F_1$) is contained in $C \times [0, u]$ for some compact set $C$. Hence $F_2$ and all $F_{2,v}$ have compact support. It follows that $F_{2,v}|_{U_A}$ vanishes and so does $F|_{U_A}$. This concludes the proof for the case $\mathcal{F} \in \mu \text{Sh}_{/\Lambda}(k_{\Lambda})$.

(iv) Now we assume $\mathcal{F} \in \mu \text{Sh}_{/\Lambda}(k_{\Lambda})$. We use Theorem 12.2.2 instead of 12.2.1 in (i). We didn’t state Theorem 2.1.1 for the orbit category but there is nothing really new to say: as soon as the sheaf $K_\phi$ belongs to $D^b(k(M \times \mathbb{R})^2 \times [u_1, u_2])$ (rather than the locally bounded category) we can take its image in $D_{/1}\left(k(M \times \mathbb{R})^2 \times [u_1, u_2]\right)$.

The relation $K_{\phi,u} \circ K_{\phi,u}^{-1} \simeq k_{\Delta_{M \times \mathbb{R}}}$ still holds in the orbit category and we deduce the same equivalence $K_{\phi,u} \circ -$ : $D_{/1},[A]\left(k_{M \times \mathbb{R}}\right) \simeq D_{/1},[\phi_0(A)]\left(k_{M \times \mathbb{R}}\right)$, for any $A \subset T^*(M \times \mathbb{R})$. Now the step (ii) works the same way. The
In the proof of Theorem 12.1.1 the compactness of Remark 12.3.3.

Remark 12.3.3. In the proof of Theorem 12.1.1 the compactness of \( \Lambda / \mathbb{R}_{>0} \) was used since we glued doubled sheaves defined on open subsets of a covering of \( \pi_M \times \mathbb{R}(\Lambda) \) by a finite induction. We could probably use a countable covering (with some care to take the limit) and the compactness of \( \Lambda / \mathbb{R}_{>0} \) was not essential. However in Corollary 12.3.2 this compactness is necessary to ensure that the sheaf given by Theorem 12.1.1 can be modified to have the required microsupport over \( M \times \mathbb{R} \times [0, u] \) for some \( u > 0 \).

Remark 12.3.4. Using Remark 12.3.3 and Corollary 12.3.2 we see that, if \( \Lambda \) comes from a compact exact submanifold of \( T^*M \), then there exists a simple object \( F \in D_{/\{1, \Lambda\}|}(kM \times \mathbb{R}) \) such that \( F_{/M \times \{t\}} \simeq 0 \) for \( t \ll 0 \).

12.4. Restriction at infinity

As in \([12.3]\) we assume that \( \Lambda / \mathbb{R}_{>0} \) has no Reeb chord. Since \( \Lambda / \mathbb{R}_{>0} \) is compact, we can choose \( A > 0 \) such that \( \Lambda \subset T^*_{>0}(M \times [-A, A]) \). Then, for any \( F \in D_{/\{1\}, \Lambda}(kM \times \mathbb{R}) \), the restrictions \( F_{/M \times [-\infty, -A]} \) and \( F_{/M \times A, +\infty} \) have locally constant cohomology sheaves.

Definition 12.4.1. For \( F \in D_{/\{1\}, \Lambda}(kM \times \mathbb{R}) \) we let \( F_-, F_+ \in D_{/\{1\}, \Lambda}(kM) \) be the restrictions at infinity \( F_- = F_{/M \times (-t)} \), \( F_+ = F_{/M \times \{t\}} \), for any \( t \in [A, +\infty) \). Then \( F_-, F_+ \) are indeed independent of \( t \in [A, +\infty) \) and have locally constant cohomology sheaves. We let \( D_{/\{1\}, \Lambda}^{lb}(kM \times \mathbb{R}) \) be the full subcategory of \( D_{/\{1\}, \Lambda}(kM \times \mathbb{R}) \) consisting of the \( F \) such that \( F_- \simeq 0 \).

Replacing \( D_{/\{1\}, \Lambda}^{lb}(\_\_\_\_\_\_\_\_\_) \) by \( D_{/\{1\}, \Lambda}(\_\_\_\_\_\_\_\_) \) we also define \( F_-\), \( F_+ \in D_{/\{1\}, \Lambda}(kM) \) for \( F \in D_{/\{1\}, \Lambda}(kM \times \mathbb{R}) \) and a similar category \( D_{/\{1\}, \Lambda}^+ \)(\( kM \times \mathbb{R} \)).

For \( F \in D_{/\{1\}, \Lambda}^{lb}(kM \times \mathbb{R}) \) we have by definition

\[
(12.4.1) \quad F_{/M \times [A, +\infty] \simeq F_+ \boxtimes k_{[A, +\infty]}}, \quad F_{/M \times [-\infty, -A]} \simeq 0 \quad \text{for} \quad A \gg 0.
\]

Lemma 12.4.2. Let \( F \in D_{/\{1\}}^{lb}(kM \times \mathbb{R}) \) (or \( F \in D_{/\{1\}}^{lb}(kM \times \mathbb{R}) \)). We assume that there exists \( A > 0 \) such that \( \text{supp}(F) \subset M \times [-A, A] \). We also assume either \( \text{SS}(F) \subset T^*_{>0}(M \times \mathbb{R}) \) or \( \text{SS}(F) \subset T^*_{<0}(M \times \mathbb{R}) \). Let \( p_M : M \times \mathbb{R} \to M \) be the projection. Then \( Rp_M F \simeq Rp_M^* F \simeq 0 \).

Proof. By Proposition 1.2.4 (or 9.2.6), we have \( \text{SS}(Rp_M F) \subset T^*_M M \). Hence \( Rp_M F \) is locally constant (for \( F \in D_{/\{1\}}(kM \times \mathbb{R}) \), use Proposition 9.2.10 and it is enough to prove that \( (Rp_M F)_x \simeq 0 \) for one \( x \in M \).
The base change formula gives \((R_p M F)_x \simeq R\Gamma(R; F|^x_{x \times \mathbb{R}})\). Now \(F^{|x \times \mathbb{R}}\) has a compact support and a microsupport in \(T_{\tau \geq 0}^*\) (or \(T_{\tau \leq 0}^*\)) and the result follows from Corollary 1.2.16 (or Corollary 9.2.11 for the case of \(D_{/[1]}(k_{\mathbb{R}})\): take \(a, b\) in the corollary so that \([a, b]\) contains the support of \(F\)).

**Lemma 12.4.3.** Let \(F \in D^b_{/[1]}(k_{\mathbb{R}})\) (or \(F \in D_{/[1],[[A],+(k_{\mathbb{R}})]})\). We have \(R_p M F \simeq F_+\) and \(R_p M F \simeq 0\).

**Proof.** Let us choose \(A > 0\) so that (12.4.1) holds. We set \(G = F \otimes k_{\mathbb{R} \times [-\infty, +\infty]}\). Then supp\((G) \subset [-A, A + 1]\) and SS\((G) \subset T_{\tau > 0}^*(M \times \mathbb{R})\) by Theorem 1.2.13. By Lemma 12.4.2 we obtain \(R_p M F \simeq 0\).

By (12.4.1) we have the distinguished triangle \(G \to F \to F \otimes k_{[A+1, +\infty]} \to \cdot\) and we deduce \(R_p M F \simeq R_p M_+ (F_+ \otimes k_{[A+1, +\infty]}) \simeq F_+\) and \(R_p M F \simeq R_p M_! (F_+ \otimes k_{[A+1, +\infty]}) \simeq 0\).

We recall the maps \(q, r, T_u\) (see around (12.3.1)). By Lemma 11.1.4 we have a morphism \(q^{-1} F \to r^{-1} F\), for any \(F \in D_{r \geq 0}(k_{\mathbb{R}})\). Restricting to \(M \times \mathbb{R} \times \{u\}\) we obtain a morphism \(F \to T_{u*} F\).

**Lemma 12.4.4.** For all \(F, F' \in D^b_{/[1]}(k_{\mathbb{R}})\) (or in \(D_{/[1],[[A],+(k_{\mathbb{R}})]})\) and any \(u \geq 0\), the morphism \(F \to T_{u*} F'\) induces the isomorphism \(R\text{Hom}(F, F') \simeq R\text{Hom}(F, T_{u*} F')\).

Moreover, for any \(u > 0\), we have \(R\text{Hom}(T_{u*} F, F') \simeq 0\).

**Proof.** (i) We extend \(q, r\) to \(M \times \mathbb{R} \times \mathbb{R}\) (with the same formulas) and we define \(p_2: M \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}, (x, t, u) \mapsto u\). We introduce \(G = R\text{Hom}(q^{-1} F, r^j F')\). For \(u \in \mathbb{R}\), letting \(i_u: M \times \mathbb{R} \times \{u\} \to M \times \mathbb{R} \times \mathbb{R}\) be the inclusion we have \(q \circ i_u = \text{id}, r \circ i_u = T_{-u}\), and hence, by Proposition 11.1.1, \(R\Gamma_{\{u\}}(G) \simeq R\Gamma(M \times \mathbb{R}; i_u^! R\text{Hom}(q^{-1} F, r^j F'))\).

Using the microsupport bounds of 9.2 or 9.2 we obtain

\[
\text{SS}(F) \subset \{(u; v); \exists (x, t; \xi, \tau) \in \text{SS}(F), \exists (x', t'; \xi', \tau') \in \text{SS}(F'), x = x', t - u = t', (-\xi, -\tau, 0) + (\xi', \tau', -\tau') = (0, 0, v)\}.
\]

Using \(\text{SS}(F) = \text{SS}(F') = \Lambda\) and the fact that \(\Lambda\) has no Reeb chord (hence \(\Lambda \cap T_u(\Lambda) = \emptyset\) for \(u \neq 0\)), we deduce \(\text{SS}(G) \subset \{(0; v); v < 0\}\).

(ii) Using Corollary 12.16 (or Corollary 9.2.11 for \(D_{/[1]}(k_{\mathbb{R}})\)) we deduce \(R\Gamma_{[a, b]}(\mathbb{R}; G) \simeq 0\) for any \(a < b\) and \(R\Gamma_{[a', b']}(\mathbb{R}; G) \simeq 0\) for any...
0 ≤ a' < b'. In particular \( \text{R} \Gamma_{\{u\}}(G) \cong \text{R} \Gamma_{[u,u']}(\mathbb{R}; G) \cong \text{R} \Gamma_{\{u\}}(G) \) for \( 0 ≤ u ≤ u' \) and the first isomorphism follows.

By the same argument \( \text{RHom}(T_uF,F') \cong \text{RHom}(F,T_{-u}F') \) is independent of \( u > 0 \). If we choose \( u > 2A \), where \( A \) satisfies \( \langle 12.4.1 \rangle \) both for \( F \) and \( F' \), then \( T_{-u}F' \) coincides with \( p_M^{-1}F'_+ \) over \( \text{supp}(F) \), where \( p_M \) is the projection \( M \times \mathbb{R} \rightarrow M \). Hence

\[
\text{RHom}(F,T_{-u}F') \cong \text{RHom}(F,p_M^{-1}F'_+) \cong \text{RHom}(R\text{p}_M!F,F'_+)[-1]
\]

vanishes by Lemma \( 12.4.3 \).

**Theorem 12.4.5.** Let \( F,F' \in \mathbb{D}^{[A]_+,+}_M(k_M) \). We let \( F_+,F'_+ \in \mathbb{D}^b(k_M) \) be their restrictions to \( M \times \{t\}, t \gg 0 \), as in Definition \( 12.4.1 \). Then (12.4.2) \( \text{RHom}(F,F') \cong \text{RHom}(F_+,F'_+) \).

In particular the functor \( \mathbb{D}^{[A]_+,+}_M(k_M) \rightarrow \mathbb{D}^b(k_M) \) given by \( F \mapsto F_+ \) is fully faithful and we have: \( F \cong F' \) if and only if \( F_+ \cong F'_+ \).

The same statement holds with \( \mathbb{D}^{[A]_+,+}_M(k_M) \), \( \mathbb{D}^b(k_M) \) replaced by \( \mathbb{D}^{[1][1],[A]_+,+}(k_M), D^{[1][1]}(k_M) \) (using \( \text{RHom}^\nu \) defined before \( 9.1.23 \)).

**Proof.** Let \( p_M: M \times \mathbb{R} \rightarrow M \) be the projection. Let us choose \( A > 0 \) so that \( \langle 12.4.1 \rangle \) holds for \( F \) and \( F' \) and let \( u > 2A \). Then

\[
\text{RHom}(F,F') \cong \text{RHom}(F,T_uF')
\]

\[
\cong \text{RHom}(p_M^{-1}F_+,T_uF')
\]

\[
\cong \text{RHom}(F_+,R\text{p}_M!T_uF')
\]

\[
\cong \text{RHom}(F_+,F'_+)
\]

where the first isomorphism follows from Lemma \( 12.4.4 \) the second one from \( \text{supp}(T_uF') \subset M \times ]A, +\infty[ \) and the last one from Lemma \( 12.4.3 \). □

With Theorem \( 12.4.5 \) we can recover a classical result \( [38] \) of Lalonde and Sikorav:

**Corollary 12.4.6.** Let \( \tilde{\Lambda} \subset T^*M \) be a compact exact Lagrangian submanifold. Then the map \( \tilde{\Lambda} \rightarrow M \) is onto. In particular \( M \) is compact.

**Proof.** We let \( \Lambda \subset \tilde{T}^*(M \times \mathbb{R}) \) be the conification of \( \tilde{\Lambda} \) as in \( \langle 12.0.1 \rangle \). By Remark \( 12.3.4 \) there exists a simple object \( F \in \mathbb{D}^{[1][1],[A]_+,+}(k_M) \). In particular \( F \neq 0 \). By Theorem \( 12.4.5 \) (for the case of the orbit category) we have \( \text{RHom}(F_+,F_+) \neq 0 \), hence \( F_+ \neq 0 \). Let us assume that \( \tilde{\Lambda} \rightarrow M \) is not onto. Then there exists an open subset \( U \) of \( M \) such that \( \Lambda \cap \tilde{T}^*(U \times \mathbb{R}) = \emptyset \). Hence \( F|_{U \times \mathbb{R}} \) is locally constant, \( F|_{U \times \{t\}} \cong 0 \) for \( t ≪ 0 \) and \( F|_{U \times \{t\}} \neq 0 \) for \( t \gg 0 \), which is a contradiction. □
Theorem 12.4.7. Let $F, F' \in \mathcal{D}^b_{[\Lambda],+}(\mathbf{k}_M \times \mathbb{R})$. Then we have an isomorphism
\[(12.4.3) \quad R\text{Hom}(F, F') \cong R\Gamma(\Lambda; \mu\text{hom}(F, F')).\]
Its composition with \[(12.4.2) \quad R\text{Hom}(\Lambda; \mu\text{hom}(F, F')).\]
gives a canonical isomorphism \[(12.4.4) \quad R\text{Hom}(F, F') \cong R\Gamma(\Lambda; \mu\text{hom}(F, F')).\]

As in Theorem 12.4.5 the same statement holds with $\mathcal{D}^b_{[\Lambda],+}(\mathbf{k}_M \times \mathbb{R})$ replaced by $\mathcal{D}/[1],[\Lambda],+((\mathbf{k}_M \times \mathbb{R})$.

Proof. (i) For an interval $I$ of $\mathbb{R}_{>0}$ we set $N_I = M \times \mathbb{R} \times I$. Proposition 11.3.7 and the fact that $N_{0,\epsilon}$ is open give the isomorphisms, for any $i \in \mathbb{Z}$,
\[
H^iR\Gamma(\Lambda; \mu\text{hom}(F, F'))
\cong \lim_{\epsilon > 0} H^iR\text{Hom}(\Psi_{M \times \mathbb{R}}(F)|_{N_{0,\epsilon}}, \Psi_{M \times \mathbb{R}}(F')|_{N_{0,\epsilon}}) \cong \lim_{\epsilon > 0} H^iA_{\epsilon},
\]
where we set $A_{\epsilon} = R\text{Hom}((\Psi_{M \times \mathbb{R}}(F))_{N_{0,\epsilon}}, \Psi_{M \times \mathbb{R}}(F'))$. It is enough to prove that $A_{\epsilon} \cong R\text{Hom}(F, F')$ for all $\epsilon > 0$. The triangle \[(11.1.10), \quad \Psi_{M \times \mathbb{R}}(F') \to q^{-1}(F') \to r^{-1}(F') \overset{+1}{\to}, \]
yields another distinguished triangle $A_{\epsilon} \to B_{\epsilon} \to C_{\epsilon} \overset{+1}{\to}$ with
\[
B_{\epsilon} = R\text{Hom}((\Psi_{M \times \mathbb{R}}(F))_{N_{0,\epsilon}}, q^{-1}(F'))
\cong R\text{Hom}(R\text{q}((\Psi_{M \times \mathbb{R}}(F))_{N_{0,\epsilon}}), F')[-1],
\]
\[
C_{\epsilon} = R\text{Hom}((\Psi_{M \times \mathbb{R}}(F))_{N_{0,\epsilon}}, r^{-1}(F'))
\cong R\text{Hom}(R\text{r}((\Psi_{M \times \mathbb{R}}(F))_{N_{0,\epsilon}}), F')[-1].
\]

We check in (ii) below that $B_{\epsilon} \cong R\text{Hom}(F, F')$ and in (iii) that $C_{\epsilon} \cong 0$, which proves the theorem.

(ii) We compute $B_{\epsilon}$. The triangle \[(11.1.10) \quad \text{again gives the distinguished triangle}
\]
\[
R\text{q}((\Psi_{M \times \mathbb{R}}(F))_{N_{0,\epsilon}}) \to R\text{q}((q^{-1}F)_{N_{0,\epsilon}})
\overset{+1}{\to} R\text{q}((r^{-1}F)_{N_{0,\epsilon}})
\]
\[(12.4.5) \quad \to R\text{q}((r^{-1}F)_{N_{0,\epsilon}}) \overset{+1}{\to} .
\]
In the following computations we consider $q, r$ as maps defined on $M \times \mathbb{R}^2$ (with the same formulas – see after \[(12.3.1) \text{). The projection formula}\]
(Proposition 1.1.1-(g)) gives $R\text{q}((q^{-1}F)_{N_{0,\epsilon}}) \cong F[-1]$. We have $SS(r^{-1}F) \subset \{\nu = \tau > 0\}$ and $SS(k_{N_{0,\epsilon}}) \subset \{\nu \leq 0\}$. Hence $SS((r^{-1}F)_{N_{0,\epsilon}}) \subset \{\nu < 0\}$ and Lemma 12.4.2 (used with $q: M \times \mathbb{R}^2 \to$
\(M \times \mathbb{R}\) instead of \(p_M : M \times \mathbb{R} \to M\) gives \(Rq!((r^{-1}F)_{N[0,\varepsilon]}) \simeq 0\). Using the triangle \(k_{N[\varepsilon]}[-1] \to k_{N[0,\varepsilon]} \to k_{N[0,\varepsilon]} \oplus 1\) we deduce
\[Rq!((r^{-1}F)_{N[0,\varepsilon]}) \simeq Rq!((r^{-1}F)_{N[\varepsilon]}[-1]) \simeq T_{\varepsilon}^*F[-1].\]

Lemma \[12.4.3\] gives \(R\text{Hom}(T_{\varepsilon}^*F, F') \simeq 0\) for any \(\varepsilon > 0\). Applying \(R\text{Hom}(\cdot, F')\) to the triangle \[12.4.5\] we deduce \(B_{\varepsilon} \simeq R\text{Hom}(F, F')\), as claimed.

(iii) Now we prove \(Rr!(((\Psi_{M \times \mathbb{R}}(F))_{N[0,\varepsilon]}) \simeq 0\), which implies \(C_{\varepsilon} \simeq 0\). As in (ii) we have the triangle
\[Rr!((\Psi_{M \times \mathbb{R}}(F))_{N[0,\varepsilon]}) \to Rr!((q^{-1}F)_{N[0,\varepsilon]}) \Rightarrow Rr!((r^{-1}F)_{N[0,\varepsilon]}) \oplus 1\]
and an isomorphism \(Rr!((r^{-1}F)_{N[0,\varepsilon]}) \simeq F[-1]\). The microsupport bound \(\mathbb{S}((q^{-1}F)_{N[0,\varepsilon]}) \subset \{\tau > 0, \upsilon \geq 0\}\) and Lemma \[12.4.2\] again give \(Rr!((q^{-1}F)_{N[0,\varepsilon]}) \simeq 0\). We deduce
\[Rr!((q^{-1}F)_{N[0,\varepsilon]}) \simeq Rq!((r^{-1}F)_{N[0,\varepsilon]}[-1]) \simeq F[-1].\]
Moreover the morphism \(\alpha\) in \[12.4.6\] corresponds to \(\text{id}_F\) through these isomorphisms and we deduce \(Rr!(((\Psi_{M \times \mathbb{R}}(F))_{N[0,\varepsilon]}) \simeq 0\).

\[\square\]

Remark 12.4.8. We have recalled in \[10.1.2\] that \(\mu_{\text{hom}}\) admits a composition morphism (denoted by \(\circ\) in Notation \[10.1.4\]) compatible with the composition morphism for \(R\text{Hom}\). In particular the isomorphism \[12.4.3\] is compatible with the composition morphisms \(\circ\) and \(\circ\). Since \[12.4.2\] is clearly compatible with \(\circ\), we deduce that \[12.4.4\] also is compatible with \(\circ\) and \(\circ\).

Part 13. Exact Lagrangian submanifolds in cotangent bundles

In this part \(M\) is a connected manifold and \(\Lambda_0 \subset T^*M\) is a compact exact Lagrangian submanifold. We use the sheaves constructed in Corollary \[12.3.2\] and Theorems \[12.4.5\] \[12.4.7\] to recover some results on the topology of \(\Lambda_0\), namely that the projection \(\Lambda_0 \to M\) is a homotopy equivalence and that the first and second Maslov classes of \(\Lambda_0\) vanish.

The fact that \(\Lambda\) and \(M\) have the same homology was proved by Fukaya, Seidel, Smith in \[16\] and also by Nadler in \[40\] using the Fukaya category of the cotangent bundle and assuming that the first Maslov class vanishes. The fact that the projection \(\Lambda_0 \to M\) is a homotopy equivalence was proved by Abouzaid in \[2\], also assuming the vanishing
of the Maslov class. Then Kragh in \[35\] proved that this vanishing holds.

Abouzaid and Kragh gave more precise results on the topology of \(\Lambda\). In \[4\] they proved that the map \(\Lambda_0 \to M\) is a simple homotopy equivalence and in \[3\] they proved a vanishing result for the higher Maslov classes (very roughly the images of the Maslov classes by the map \(BO \to BH\) vanish, where \(H\) is the group of homotopy equivalences of the sphere – this gives the vanishing of obstructions for the existence of a sheaf in spectra – see \[27, 26\]).

In the following we choose \(f: \Lambda_0 \to \mathbb{R}\) such that \(df = \alpha_M|_{\Lambda_0}\) and define as in \((12.0.1)\)

\[
\Lambda = \{(x, t; \xi, \tau); \tau > 0, (x; \xi/\tau) \in \Lambda_0, t = -f(x; \xi/\tau)\}.
\]

We have \(\Lambda/\mathbb{R}_{>0} = \Lambda_0\) and we work with \(\Lambda\) instead of \(\Lambda_0\).

### 13.1. Fundamental groups

We let \(\pi_\Lambda: \Lambda \to M\) be the projection to the base and we denote by \(\pi_1(\pi_\Lambda): \pi_1(\Lambda) \to \pi_1(M)\) the induced morphism of fundamental groups.

**Proposition 13.1.1.** The morphism \(\pi_1(\pi_\Lambda): \pi_1(\Lambda) \to \pi_1(M)\) is injective.

**Proof.** (i) We set \(k = \mathbb{Z}/2\mathbb{Z}\) and \(G = \pi_1(\Lambda)\). We let \(\rho: G \to GL(k[G])\) be the regular representation of \(G\). This means that \(k[G]\) is the vector space with basis \(\{e_g\}_{g \in G}\) and the action of \(G\) is given by \(g \cdot e_h = e_{gh}\), for all \(g, h \in G\). We let \(\mathcal{L}_\rho\) be the local system on \(\Lambda\) with stalks \(k[G]\) corresponding to this representation \(\rho\).

(ii) We recall some results of \[10.4\]. The \(\mathcal{H}om\) sheaf in \(\muSh_{/[1]}(k_{\Lambda})\) is induced by \(\mu hom\) through the equivalence \(h_\Lambda: \mathcal{O}L(k_{\Lambda}) \simeq \text{Loc}(k_{\Lambda})\) of Lemma \[10.4.3\]. By Proposition \[10.4.4\] there exists a simple object \(\mathcal{F}_0 \in \muSh_{/[1]}(k_{\Lambda})\) and \(\mathcal{H}om(\mathcal{F}_0, \cdot)\) gives an equivalence \(\muSh_{/[1]}(k_{\Lambda}) \xrightarrow{\sim} \text{Loc}(k_{\Lambda})\).

We let \(\mathcal{F}_\rho \in \muSh_{/[1]}(k_{\Lambda})\) be the object associated with \(\mathcal{L}_\rho\) by this equivalence. By Corollary \[12.3.2\] there exist \(F_0, F_\rho \in D_{/[1],[\Lambda],+(k_{M \times \mathbb{R}})}\) such that \(m_{/[1],\Lambda}(F_0) \simeq \mathcal{F}_0\) and \(m_{/[1],\Lambda}(F_\rho) \simeq \mathcal{F}_\rho\). Moreover

\[
\mathcal{L}_\rho \simeq \mathcal{H}om(\mathcal{F}_0, \mathcal{F}_\rho) \simeq h_\Lambda(\mu hom^c(F_0, F_\rho)|_{\Lambda}).
\]

We define \(L_0, L_1 \in D_{/[1]}(k_M)\) by \(L_0 = F_0|_{M \times \{t\}}\) and \(L_1 = F_\rho|_{M \times \{t\}}\) for \(t \gg 0\). We let \(p: M \times \mathbb{R} \to M\) be the projection and we set \(F = F_0 \otimes_{k_{M \times \mathbb{R}}} p^{-1}L_1\) and \(F' = F_\rho \otimes_{k_{M \times \mathbb{R}}} p^{-1}L_0\). Taking the tensor product with a locally constant sheaf (like \(L_0, L_1\) does not increase the microsupport and we still have \(F, F' \in D_{/[1],\Lambda},+(k_{M \times \mathbb{R}})\). Then
F|_{M \times \{t\}} \simeq L_0 \otimes_{k_M} L_1 \simeq F'|_{M \times \{t\}} \quad \text{for } t \gg 0 \quad \text{and Theorem 12.4.3 (for the orbit category) implies}

(13.1.1) \quad F_0 \otimes_{k_{M \times \mathbb{R}}} p^{-1} L_1 \simeq F_\rho \otimes_{k_{M \times \mathbb{R}}} p^{-1} L_0.

(iii) Applying $m_{/[1], \Lambda}$ to (13.1.1) we find $F_0 \otimes_{k_\Lambda} \pi^{-1}_\Lambda L_1 \simeq F_\rho \otimes_{k_\Lambda} \pi^{-1}_\Lambda L_0$ in $\mu \text{Sh}_{/[1]}(k_\Lambda)$. Using the equivalences $\text{Hom}(F_0, \cdot)$ and $h_\Lambda$ recalled in (ii) we obtain $\pi^{-1}_\Lambda L_1 \simeq \mathcal{L}_\rho \otimes \pi^{-1}_\Lambda L_0$, where $L'_i = h_{M, i}$, for $i = 0, 1$ (see Lemma 10.4.3). $L'_i$ is the locally constant sheaf on $M$ associated with the presheaf $U \mapsto \text{Hom}_{/[1]}(k_U)(k_U, L_i)$.

(iv) We let $\rho'_0$ and $\rho'_1$ be the representations of $\pi_1(M)$ corresponding to the local systems $L'_0$ and $L'_1$. They induce representations of $G = \pi_1(\Lambda)$, say $\rho''_0$ and $\rho''_1$, through the morphism $\pi_1(\pi_\Lambda)$. Then the result of (iii) gives the isomorphism of representations of $G$; $\rho'_i \simeq \rho \otimes \rho''_0$. We restrict these representations to the subgroup $K = \ker(\pi_1(\pi_\Lambda))$ of $G$. Then $\rho''_0|_K$ and $\rho'_1|_K$ are trivial representations and we deduce that $\rho|_K$ also is trivial. Since $\rho$ is a faithful representation of $G$, this gives $K = \{1\}$, as required.

We will see later (see Theorem 13.5.1 and Proposition 13.5.2) that the sheaf $F_0$ introduced in part (ii) of the proof of Proposition 13.1.1 satisfies $F_0|_{M \times \{t\}} \simeq k_M$ for $t \gg 0$ (in general it is of rank one but here $k = \mathbb{Z}/2\mathbb{Z}$ so it is constant). If we already knew this, (13.1.1) would give $F_0 \otimes_{k_{M \times \mathbb{R}}} p^{-1} L_1 \simeq F_\rho$ and we would have directly $\pi^{-1}_\Lambda L'_1 \simeq \mathcal{L}_\rho$, simplifying the end of the proof.

Let $r: M' \to M$ be a covering. The derivative of $r$ induces a covering $r': T^* M' \to T^* M$. We let $\Lambda'_0$ be a connected component of $r'^{-1}(\Lambda)$. Then $\Lambda'_0 \to \Lambda$ is a covering and $\pi_1(\Lambda'_0)$ is a subgroup of $\pi_1(\Lambda)$. We have the commutative diagram

\[
\begin{array}{ccc}
\pi_1(\Lambda'_0) & \longrightarrow & \pi_1(\Lambda) \\
\downarrow & & \downarrow_{\pi_1(\pi_\Lambda)} \\
\pi_1(M') & \longrightarrow & \pi_1(M),
\end{array}
\]

where $\pi_1(\pi_\Lambda)$ is injective by Proposition 13.1.1. This implies that the morphism $\pi_1(\Lambda'_0) \to \pi_1(M')$ is injective. In particular, if $M'$ is the universal cover of $M$, then $\pi_1(\Lambda'_0)$ vanishes, that is, $\Lambda'_0$ is the universal cover of $\Lambda$.

We let $m_\Lambda: \pi_1(\Lambda) \to \mathbb{Z}$ be the group morphism induced by the Maslov class $\mu^sh_1(\Lambda) \in H^1(\Lambda; \mathbb{Z}_\Lambda)$ introduced in §10.3. We remark that $m_\Lambda$ determines $\mu^sh_1(\Lambda)$. 
Corollary 13.1.2. There exists a covering map $r: M' \to M$ and a closed conic connected Lagrangian submanifold $\mathcal{N} \subset \tilde{T}^*(M' \times \mathbb{R})$ such that the derivative of $r$ and the projection $\Lambda' \to M'$ induce isomorphisms $\mathcal{N} \xrightarrow{\sim} \Lambda$ and $\pi_1(\Lambda) \xleftarrow{\sim} \pi_1(\mathcal{N}) \xrightarrow{\sim} \pi_1(M')$:

$$
\begin{array}{ccc}
\mathcal{N} & \xrightarrow{\sim} & \Lambda \\
\downarrow & & \downarrow \\
M' & \xrightarrow{r} & M.
\end{array}
$$

Moreover the isomorphism $\mathcal{N} \xrightarrow{\sim} \Lambda$ identifies $\mu_{1h}(\mathcal{N})$ and $\mu_{1h}(\Lambda)$.

By Corollary 12.4.6 $M$ and its covering $M'$ are compact.

Proof. As noticed around the diagram (13.1.2), if $\tilde{r}: \tilde{M} \to M$ is the universal cover of $M$ and $\Lambda'_0$ a connected component of the pull-back of $\Lambda$ by $\tilde{r}$, then $\Lambda'_0$ is the universal cover of $\Lambda$. We can see that the action of $\pi_1(\Lambda)$ on $\tilde{M}$ (via the inclusion $\pi_1(\Lambda) \subset \pi_1(M)$) induces an action on $T^*\tilde{M}$ which preserves $\Lambda'_0$. The result follows by setting $M' = \tilde{M}/\pi_1(\Lambda)$ and $\Lambda' = \Lambda'_0/\pi_1(\Lambda)$.

13.2. Vanishing of the Maslov class

By Corollary 10.6.3 the vanishing of $\mu_{1h}(\Lambda)$ implies the vanishing of the usual Maslov class $\mu_1(\Lambda)$. We recall that $m_\Lambda: \pi_1(\Lambda) \to \mathbb{Z}$ is the group morphism induced by $\mu_{1h}(\Lambda)$ and that $m_\Lambda$ determines $\mu_{1h}(\Lambda)$. In this section we prove the vanishing of $\mu_{1h}(\Lambda)$ as follows. Assuming $\mu_{1h}(\Lambda) \neq 0$ we consider the cyclic cover $M'$ of $M$ associated with $\mu_{1h}(\Lambda)$. We let $\psi_n$ be the action of $\mathbb{Z}$ on $M'$ and we let $\Lambda'$ be the pull-back of $\Lambda$. We construct $G \in \mathbb{D}_{[\mathcal{N}]}(k_{M' \times \mathbb{R}})$ which is quasi-periodic in the sense $\psi_n^{-1}(G) \simeq G[-n]$, for each $n \in \mathbb{Z}$. We then check that $G|_{M' \times \{t\}}$ is bounded for $t \gg 0$, obtaining a contradiction. Since $\Lambda'$ is not compact, we cannot apply Corollary 12.3.2 immediately and we will use the $\mathbb{Z}$-action to construct $G$.

We first give the statement at the level of stacks. In this section we take $k = \mathbb{Z}/2\mathbb{Z}$. By Corollary 13.1.2 up to replacing $M$ by some covering (and without changing $\Lambda$), we can assume that $\pi_1(\Lambda) \xrightarrow{\sim} \pi_1(M)$. Hence we have $\pi_1^*: H^1(M; \mathbb{Z}_M) \xrightarrow{\sim} H^1(\Lambda; \mathbb{Z}_\Lambda)$. By Lemma 12.2.1 there exists a map $f: M \to S^1$ such that $\mu_{1h}(\Lambda) = \pi_1^*f^*(\delta)$, where $\delta \in H^1(S^1; \mathbb{Z}_{S^1})$ is the canonical class. We set $M' = M \times_{S^1} \mathbb{R}$ and
\( \Lambda' = \Lambda \times_{\mathbb{R}} \mathbb{R} \)

\[
\begin{array}{ccc}
\Lambda' & \longrightarrow & M' \\
& \downarrow{f'} & \downarrow{r} \\
\Lambda & \longrightarrow & M \\
& \downarrow{\pi_{\Lambda}} & \downarrow{f} \\
& S^1 & \end{array}
\]

and we have \( \mu^h_{\Lambda'} = r^* \mu^h_\Lambda = 0 \). By construction \( M' \) comes with a \( \mathbb{Z} \)-action, denoted \( \psi_n : M' \to M' \) for \( n \in \mathbb{Z} \), which translates by \( n \) in the fibers of \( r \). We abusively denote by \( \psi_n \) the diffeomorphisms induced by \( \psi_n \) on \( M' \times \mathbb{R} \), \( M' \times \mathbb{R} \times \mathbb{R}_{>0} \) or \( \Lambda' \).

**Lemma 13.2.1.** There exists \( \mathcal{F} \in \mu\text{Sh}(\mathcal{K}_\Lambda) \) which is simple and satisfies \( \psi_n^{-1}(\mathcal{F}) \simeq \mathcal{F}[-n] \), for each \( n \in \mathbb{Z} \).

If \( \Lambda' \) is connected (that is, if \( \mu^h_\Lambda(\Lambda) \neq 0 \)), Proposition 10.2.4 then implies that all objects \( \mathcal{F} \in \mu\text{Sh}(\mathcal{K}_\Lambda) \) satisfy \( \psi_n^{-1}(\mathcal{F}') \simeq \mathcal{F}'[-n] \).

**Proof.** In §10.3 we have defined \( \mu^h_\Lambda(\Lambda) \) by a Čech cocycle as follows. First we choose a covering \( \{\Lambda_i\}_{i \in I} \) of \( \Lambda \) by suitable small open subsets. We choose \( F_i \in D^b_{(\Lambda)}(k_{M \times \mathbb{R}}) \), for each \( i \in I \), which is simple along \( \Lambda_i \). We have seen that there exist isomorphisms \( \mathcal{m}_{\Lambda_i}(F_i)|_{\Lambda_{ij}} \simeq \mathcal{m}_{\Lambda_j}(F_j)|_{\Lambda_{ij}}[d_{ij}] \), for some integers \( d_{ij} \), and that the cocycle \( \{d_{ij}\}_{i,j \in I} \) is a cocycle whose cohomology class, \( \mu^h(\Lambda) \), is independent of choices.

The relation \( \mu^h(\Lambda) = \pi_{\Lambda}^* f^* (\delta) \) gives a representative of \( \mu^h(\Lambda) \), as follows. We cover \( S^1 \) by three arcs, say \( A_0, A_1, A_2 \), and represent \( \delta \) by the cocycle \( \{\delta_{ij}\}_{i,j=0,1,2}, \delta_{00} = \delta_{12} = 0, \delta_{20} = 1 \) (and \( \delta_{ij} = -\delta_{ij} \)). Taking the \( \Lambda_i \)'s small enough, we can assume that there exists \( \sigma : I \to \{0, 1, 2\} \) such that \( f(\pi_{\Lambda}(\Lambda_i)) \subset A_{\sigma(i)} \) for each \( i \in I \). Then \( d'_{ij} := \delta_{\sigma(i), \sigma(j)} \) defines a cocycle representing \( \mu^h(\Lambda) \). Hence \( \{d'_{ij}\}_{i,j \in I} \) and \( \{d''_{ij}\}_{i,j \in I} \) differ by a coboundary and, shifting the \( F_i \)'s by this coboundary, we obtain

\[
\mathcal{m}_{\Lambda_i}(F_i)|_{\Lambda_{ij}} \simeq \mathcal{m}_{\Lambda_j}(F_j)|_{\Lambda_{ij}}[d'_{ij}].
\]

We write the pull-back of \( A_i \) to \( \mathbb{R} \) as \( \bigsqcup_{n \in \mathbb{Z}} A^n_i \) in such a way that \( A_n^{i+1} = 1 + A^n_i \) and \( A^n_0 \) meets \( A_0^0 \) and \( A_0^2 \). We numerate the pull-backs of the \( \Lambda_i \)'s to \( T^*(M' \times \mathbb{R}) \) accordingly and obtain a covering \( \{\Lambda^n_i\}_{i \in I, n \in \mathbb{Z}} \) of \( \Lambda' \). The pull-back of \( F_i \) yields \( F^n_i \in D^b_{(\Lambda^n_i)}(k_{M' \times \mathbb{R}}) \), \( n \in \mathbb{Z} \) (actually \( F^n_i = F^n_i \) but we see \( F^n_i \) and \( F^m_i \) in different categories), and we set \( \mathcal{F}^n_i = \mathcal{m}_{\Lambda^n_i}(F^n_i)[n] \). Then the relation (13.2.2) gives \( \mathcal{F}^n_i|_{\Lambda^n_i \cap \Lambda^n_j} \simeq \mathcal{F}^m_j|_{\Lambda^n_i \cap \Lambda^n_j} \). Since \( k = \mathbb{Z}/2\mathbb{Z} \) and the \( \mathcal{F}^n_i \)'s are simple, the compatibility conditions on the triple intersections are trivial. Hence the \( \mathcal{F}_i \)'s glue together in an object \( \mathcal{F} \) which satisfies \( \psi_n^{-1}(\mathcal{F}) \simeq \mathcal{F}[-n] \), for each \( n \in \mathbb{Z} \). \( \Box \)
Theorem 13.2.2. We have $\mu_1(\Lambda) = 0$.

Proof. Let $\mathcal{F} \in \mu\text{Sh}(k_\Lambda)$ be given by Lemma 13.2.1. We first build in (i) a doubled sheaf $F$ on $M' \times \mathbb{R}$ which represents $\mathcal{F}$. Then in (ii) we deduce from $F$ a usual sheaf, as in the proof of Corollary 12.3.2. and in (iii) we prove that $\mu_1^{ab}(\Lambda)$ vanishes.

(i) As in 12.1 we choose an adapted family $\mathcal{V} = \{V_a; a \in A\}$ for $\Lambda$ which is stable by intersection. We can assume $f(V_a) \neq S^1$, for each $a \in A$, and hence $(r \times \text{id}_\mathbb{R})^{-1}(V_a)$ decomposes as a disjoint union $\bigsqcup_{n \in \mathbb{Z}} V_{a,n}$ with $V_{a,n} \sim V_a$, for each $n$. In this way the pull-back of $\mathcal{F}$ to $M' \times \mathbb{R}$ gives an adapted family $\mathcal{V}' = \{V'_b; b \in B\}$ for $\Lambda'$ which is stable by intersection and by the $\mathbb{Z}$-action.

We set $B_0 = \{b \in B; V'_b \cap f^{-1}([0, 1]) \neq \emptyset\}$ and $U_0 = \bigcup_{b \in B_0} V'_b$, $\Lambda_0 = \Lambda' \cap T^*U_0$, $U_n = \psi_n^{-1}(U_0)$, $\Lambda'_n = \psi_n^{-1}(\Lambda'_0)$. By Theorem 12.1 there exists $F_0 \in D^{dbb}_{A',V'}(k_{M' \times \mathbb{R}})$ such that $SS^{dbb}(F_0) = \Lambda_0$ and $\mathfrak{m}_0^{dbb}(F_0) \simeq \mathcal{F}|_{\Lambda_0}$. By $\mathbb{Z}$-invariance $F_1 := \psi_1^{-1}F_0$ belongs to $D^{dbb}_{A',V'}(k_{M' \times \mathbb{R}})$, $SS^{dbb}(F_1) = \Lambda'_1$, and $\mathfrak{m}_0^{dbb}(F_1) \simeq (\psi_1^{-1}\mathcal{F})|_{\Lambda'_1} \simeq \mathcal{F}[-1]|_{\Lambda'_1}$.

We set $U^+_n = U_n \times \mathbb{R}_{>0}$, $U^+_0 = U^+_n \cap U^+_1$ and $\Lambda'_0 = \Lambda'_n \cap \Lambda'_1$. Since $\mathcal{V}'$ is stable by intersection, $R\Gamma_{U^+_0}F_0$ and $R\Gamma_{U^+_1}F_1$ belong to $D^{dbb}_{A',V'}(k_{M' \times \mathbb{R}})$ and we have $SS^{dbb}(R\Gamma_{U^+_0}F_0) = SS^{dbb}(R\Gamma_{U^+_1}F_1) = \Lambda'_0$ and $\mathfrak{m}_0^{dbb}(R\Gamma_{U^+_0}F_0) \simeq \mathfrak{m}_0^{dbb}(R\Gamma_{U^+_1}F_1)[1] \simeq \mathcal{F}|_{\Lambda'_0}$. By Corollary 11.4.10 there exists an open subset $V$ of $M' \times \mathbb{R} \times \mathbb{R}_{>0}$ such that $(M' \times \mathbb{R} \times \{0\}) \cup V$ is open in $M' \times \mathbb{R} \times \mathbb{R}_{>0}$ and an isomorphism $u: (R\Gamma_{U^+_0}F_0)[1]|_V \sim R\Gamma_{(U^+_0 \cup F_0)}(R\Gamma_{U^+_0}F_0)|_V$.

Now we can glue $F_0$ and $F_1$ and define $F^{01} \in D(k_{M' \times \mathbb{R} \times \mathbb{R}_{>0}})$ by the triangle $F^{01} \to \Gamma_V R\Gamma_{U^+_0}F_0 \oplus \Gamma_V R\Gamma_{U^+_1}F_1[1] \xrightarrow{u'} \Gamma_V R\Gamma_{U^+_0}F_0 \xrightarrow{\psi_1^{-1}}$, where $u' = (\text{id}, -u)$. Then $F^{01}|_{(U^+_0 \times \mathbb{R}_{>0}) \cap V} \simeq F_1|_{(U^+_0 \times \mathbb{R}_{>0}) \cap V}$ for $i = 0, 1$ and hence $F^{01}$ belongs to $D^{dbb}_{A',V'}(k_{M' \times \mathbb{R}})$ and represents $\mathcal{F}|_{\Lambda'_0 \cup \Lambda'_1}$. More generally, assuming that the $V'_b$’s are small enough so that $U_0 \cap U_2 = \emptyset$, we can glue all translates of $F_0$ at once and we define $F$ by the triangle

$$F \to \bigoplus_{n \in \mathbb{Z}} \psi_n^{-1}(R\Gamma_{U^+_0}F_0)[n] \xrightarrow{v} \bigoplus_{n \in \mathbb{Z}} \psi_n^{-1}(R\Gamma_{U^+_1}F_0)[n] \xrightarrow{\psi_n^{-1}}$$

where the restriction of $v$ to the summand $\psi_n^{-1}(R\Gamma_{U^+_0}F_0)$ is $\psi_n^{-1}(\text{id}) \oplus \psi_n^{-1}(-u)$. Then $F$ belongs to $D^{dbb}_{A',V'}(k_{M' \times \mathbb{R}})$ (but is only locally bounded in cohomological degrees) and represents $\mathcal{F}$. Moreover, we have by construction $\psi_n^{-1}F \simeq F[-n]$, for all $n \in \mathbb{Z}$.

(ii) As in (i) of the proof of Corollary 12.3.2, we can see that there exists $u > 0$ such that, truncating $F$ near $(\mathcal{F} \star \pi_{M' \times \mathbb{R}}(\Lambda'))$, we obtain $G \in$
Proposition 13.3.1. We assume that $\mathbf{k} = \mathbb{Z}$ or $\mathbf{k}$ is a finite field. Let $F \in \mathcal{D}_{[\Lambda],+}^{b}(\mathbf{k}_{M\times \mathbb{R}})$. We assume that $F$ is simple along $\Lambda$. Then $F_+$ is concentrated in one degree, say $i$, and $H^i F_+$ is a local system with stalks isomorphic to $\mathbf{k}$.

Proof. (i) We first assume that $\mathbf{k}$ is a finite field. Let us prove that $F_+$ is concentrated in one degree. Let $a \leq b$ be respectively the minimal and maximal integers $i$ such that $H^i F_+ \neq 0$. By Lemma [12.4.7] the local systems $H^i F_+$ are of finite rank. Since $\mathbf{k}$ is finite we can find a finite cover $r: M' \rightarrow M$ such that $r^{-1}(H^i F_+)$ is a constant sheaf, for $i = a, b$. We set $F' = (r \times \text{id}_{\mathbb{R}})^{-1} F$ and $\Lambda' = d(r \times \text{id}_{\mathbb{R}})^{-1}(\Lambda)$. Then $r^{-1}(H^i F_+) \simeq H^i F'_+$, $F'$ is simple along $\Lambda'$ and we have $\mu_{\text{hom}}(F', F') \simeq \mathbf{k}_{\Lambda'}$. Since $\Lambda'/\mathbb{R}_{>0}$ is compact, Theorem [12.4.7] gives 

\begin{equation}
\text{RHom}(F'_+, F'_+) \simeq \Gamma(\Lambda'; \mathbf{k}_{\Lambda'}).
\end{equation}

On the other hand the complex $G = \mathcal{RHom}(F'_+, F'_+)$ is concentrated in degrees greater than $a - b$ and $H^{a-b} G \simeq \mathcal{Hom}(H^a F'_+, H^b F'_+)$ is a non zero constant sheaf. Hence $H^{a-b} \mathcal{RHom}(F'_+, F'_+)$ is non zero. By (13.3.1) we deduce that $H^{a-b} \Gamma(\Lambda'; \mathbf{k}_{\Lambda'})$ also is non zero, which implies $a - b \geq 0$. Hence $a = b$ and $F_+$ is concentrated in a single degree.
(ii) Now we prove that $H^aF_+$ is of rank one, that is, $H^aF_+ \simeq k_{M'}$. There exists $d \geq 1$ such that $H^aF_+ \simeq k^d_{M'}$. The isomorphism (13.3.1) gives in degree 0:

$$(13.3.2) \quad \text{Hom}(k^d, k^d) \simeq H^0(\Lambda'; k_{\Lambda'}).$$

By Remark 12.4.8 this isomorphism is compatible with the algebra structures of both terms. Let $I$ be the set of connected components of $\Lambda'$. We obtain $|I| = d^2$. The natural decomposition $H^0(\Lambda'; k_{\Lambda'}) \simeq \bigoplus_{i \in I} H^0(\Lambda'_i; k_{\Lambda'_i})$ gives an expression of the unit as a sum of orthogonal idempotents, $1 = \sum_{i \in I} e_i$, where $e_i$ is the projection $e_i: H^0(\Lambda'; k_{\Lambda'}) \to H^0(\Lambda'_i; k_{\Lambda'_i}), \quad i \in I$.

We let $m_i \in \text{Hom}(k^d, k^d) = \text{Mat}_{d \times d}(k)$ be the image of $e_i$ by (13.3.2). The relation $1 = \sum_{i \in I} e_i$ gives a decomposition of the identity matrix $I_d = \sum_{i \in I} m_i$ as a sum of $|I|$ non-zero orthogonal projections, that is, $m^2_i = m_i$ and $m_im_j = 0$, for $i \neq j$. We deduce that $|I| \leq d$, that is, $d^2 \leq d$. Hence $d = 1$, as claimed.

(iii) Now we assume that $k = \mathbb{Z}$. By Lemma 13.3.7 the stalks of $F_+$ are of finite rank over $\mathbb{Z}$. We recall that any object of $\text{D}^b(\mathbb{Z})$ is the sum of its cohomology. Hence, for $z = (x, t)$, $t \gg 0$, we can write $(F_+)_z = \bigoplus_{i=a}^{b} M_i[-i]$, where the $M_i$'s are abelian groups of finite rank.

We first prove that the $M_i$'s are free. If this is not the case, there exist $i \in \mathbb{Z}$ and a prime $p$ such that $M_i$ has $p$-torsion. We set $G = F \otimes_{\mathbb{Z}} \mathbb{Z}/p\mathbb{Z}$. Then $G$ is simple along $\Lambda$ and $(G_+)_z \simeq (F_+)_z \otimes_{\mathbb{Z}} \mathbb{Z}/p\mathbb{Z}$. We have seen that $(G_+)_z$ is isomorphic to $\mathbb{Z}/p\mathbb{Z}[j]$ for some shift $j$. On the other hand $H^{-1}(M_i \otimes_{\mathbb{Z}} \mathbb{Z}/p\mathbb{Z})$ and $H^0(M_i \otimes_{\mathbb{Z}} \mathbb{Z}/p\mathbb{Z})$ are both non zero, since $M_i$ has $p$-torsion. This gives a contradiction and proves that, for each $i$, we have $M_i \simeq \mathbb{Z}^{d_i}$ for some $d_i$.

We set $G = F \otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z}$. We have again $(G_+)_z \simeq (F_+)_z \otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z} \simeq \mathbb{Z}/2\mathbb{Z}[j]$. Hence $d_i = 0$ for all $i \neq j$ and $d_j = 1$, as claimed.

**Corollary 13.3.2.** We assume that $k = \mathbb{Z}$ or $k$ is a finite field and that $\mu\text{Sh}(k\Lambda)$ has at least one global simple object. Then the projection $\Lambda \to M$ induces an isomorphism $R\Gamma(M; k_M) \simeq R\Gamma(\Lambda; k\Lambda)$.

**Proof.** We choose a simple object $F \in \mu\text{Sh}(k\Lambda)$. By Corollary 12.3.2 there exists $F \in D^b_{[\Lambda],+}(k_{M \times \mathbb{R}})$ such that $m\Lambda(F) \simeq F$. By Proposition 13.3.1 $F_+$ is concentrated in one degree, say $i$, and $H^iF_+$ is a local system with stalks isomorphic to $k$. Hence $R\text{Hom}(F_+, F_+) \simeq k_M$ and $R\text{Hom}(F_+, F_+) \simeq R\Gamma(M; k_M)$. Since $F$ is simple we also have
\( \mu_{\text{hom}}(F,F)|_{\Lambda} \simeq k_{\Lambda} \). By Theorem 12.4.7 we deduce an isomorphism
\[
R\Gamma(M; k_{M}) \simeq R\Gamma(\Lambda; k_{\Lambda}).
\]
By construction (13.3.3) is given by taking the global sections in the bottom morphism of the commutative diagram:
\[
\begin{array}{ccc}
k_{M \times R} & \xrightarrow{a} & R\hat{\pi}_{*}(k_{\Lambda}) \\
\downarrow b & & \downarrow c \\
R\mathcal{H}om(F, F) & \xrightarrow{\sim} & R\hat{\pi}_{*}\mu_{\text{hom}}(F, F) & \xrightarrow{\sim} & R\hat{\pi}_{*}(\mu_{\text{hom}}(F, F)|_{\Lambda}),
\end{array}
\]
where \( \pi = \pi_{M \times R} \), \( b \) and \( c \) map the sections 1 to the identity morphisms. When taking global sections, \( b \) and \( c \) induce isomorphisms and \( a \) induces the natural morphism \( R\Gamma(M; k_{M}) \rightarrow R\Gamma(\Lambda; k_{\Lambda}) \) given by the projection of \( \Lambda \) to the base \( M \). The bottom horizontal arrow induces (13.3.3). This shows that (13.3.3) is indeed induced by the projection to the base. \( \square \)

**Remark 13.3.3.** By Theorem 13.2.2 the first Maslov class of \( \Lambda \) vanishes. Hence, when \( k = \mathbb{Z}/2\mathbb{Z} \) the stack \( \mu_{\text{Sh}}(k_{\Lambda}) \) has a global simple object and Corollary 13.3.2 applies: the projection \( \Lambda \rightarrow M \) induces an isomorphism
\[
R\Gamma(M; \mathbb{Z}/2\mathbb{Z}_{M}) \xrightarrow{\sim} R\Gamma(\Lambda; \mathbb{Z}/2\mathbb{Z}_{\Lambda}).
\]

### 13.4. Vanishing of the second obstruction class

We have seen the class \( \mu_{2}^{\text{sh}}(\Lambda) \in H^{2}(\Lambda; k^{\times}) \) in §10.3. By Corollary 10.6.3 if \( k = \mathbb{Z} \), the vanishing of \( \mu_{2}^{\text{sh}}(\Lambda) \) implies the vanishing of the usual obstruction class \( \mu_{2}^{\text{UH}}(\Lambda) \). Here we prove that \( \mu_{2}^{\text{sh}}(\Lambda) \in H^{2}(\Lambda; \mathbb{Z}/2\mathbb{Z}_{\Lambda}) \) vanishes. For this we will use Corollary 12.3.2 in the framework of twisted sheaves. Let \( c \in H^{2}(M; \mathbb{Z}/2\mathbb{Z}) \) be given and let \( \hat{c} = \{c_{ijk}\} \), \( i,j,k \in \mathbb{Z} \), be a Čech cocycle representing \( c \) with respect to a finite covering \( \{U_{i}\}_{i \in I} \) of \( M \). We view \( \mathbb{Z}/2\mathbb{Z} \) as the multiplicative group \( \{\pm 1\} \) and \( c_{ijk} = \pm 1 \), for all \( i,j,k \).

**Definition 13.4.1.** A \( \hat{c} \)-twisted sheaf \( F \) on \( M \) is the data of sheaves \( F_{i} \in \text{Mod}(k_{U_{i}}) \) and isomorphisms \( \varphi_{ij} : F_{j}|_{U_{ij}} \xrightarrow{\sim} F_{i}|_{U_{ij}} \) satisfying the condition
\[
\varphi_{ij} \circ \varphi_{jk} = c_{ijk} \varphi_{ik}.
\]
The \( \hat{c} \)-twisted sheaves form an abelian category that we denote by \( \text{Mod}(k_{M}^{\hat{c}}) \). We denote by \( \text{D}^{b}(k_{M}^{\hat{c}}) \) its derived category.

The prestack \( U \mapsto \text{Mod}(k_{M}^{\hat{c}}) \) is a stack which is locally equivalent to the stack of sheaves. The usual operations on sheaves extend to twisted sheaves. In particular if \( \hat{c}, \hat{d} \) are Čech cocycles on \( M \) and
\[ F \in \mathcal{D}^b(k_M^c), \ F' \in \mathcal{D}^b(k_M^d), \ \text{we have a tensor product} \ F \otimes F' \in \mathcal{D}(k_M^{d+c}) \]

and a homomorphism sheaf \( R\mathcal{H}om(F;F') \in \mathcal{D}(k_M^{d-c}) \). If \( f: M \to N \) is a morphism of manifolds and \( \mathcal{D} \) is a Čech cocycle on \( N \) with values in \( \{ \pm 1 \} \), we have inverse images \( f^{-1}, f^! : \mathcal{D}(k_N^{d-c}) \to \mathcal{D}(k_M^{d-c}) \) and direct images \( Rf_*, Rf! : \mathcal{D}(k_M^{d-c}) \to \mathcal{D}(k_N^{d-c}) \) with the usual adjunction properties. The notion of microsupport also generalizes to the twisted operations.

We can define a Kashiwara-Schapira stack \( \mu \mathcal{S}(k_M^c) \) and formulate a version of Corollary 13.3.2 in this framework: for \( \mathcal{F} \in \mu \mathcal{S}(k_M^c) \) there exists \( F \in \mathcal{D}(k_{M \times \mathbb{R}}^{c}) \) such that \( SS(F) = \Lambda, \ F|_{M \times \{ t \}} \simeq 0 \) for \( t \ll 0 \) and \( m^t_\Lambda(F) \simeq \mathcal{F} \).

**Proposition 13.4.2.** The class \( \mu^t_\Lambda(\Lambda) \in H^2(\Lambda; \mathbb{Z}/2\mathbb{Z}_A) \) is zero.

**Proof.** (i) By Corollary 13.3.2 and Remark 13.3.3 we have an isomorphism \( H^2(M; \mathbb{Z}/2\mathbb{Z}_M) \simeq H^2(\Lambda; \mathbb{Z}/2\mathbb{Z}_A) \). We let \( c \in H^2(M; \mathbb{Z}/2\mathbb{Z}_M) \) be the inverse image of \( \mu^t_\Lambda(\Lambda) \) by this isomorphism and we choose a Čech cocycle \( \tilde{c} \) representing \( c \). Then the twisted Kashiwara-Schapira stack \( \mu \mathcal{S}(\mathbb{Z}_M^c) \) has a simple global object and the twisted version of Corollary 13.3.2 gives \( F \in \mathcal{D}_{[\Lambda]}^b(\mathbb{Z}_M^{c}) \) which is simple along \( \Lambda \). By Proposition 13.3.1 we have \( F_+ \simeq L[d] \) where \( L \in \text{Mod}(\mathbb{Z}_M^c) \) is a twisted locally constant sheaf with stalks isomorphic to \( \mathbb{Z} \) and \( d \) is some integer.

(ii) Now we prove that the existence of \( L \in \text{Mod}(\mathbb{Z}_M^c) \) as in (i) implies that \( \tilde{c} \) is a boundary, that is, \( \mu^{t-bh}_\Lambda(\Lambda) = 0 \). The cocycle \( \tilde{c} \) is associated with a covering \( \{ U_i \} \subseteq I \) of \( M \). The object \( L \in \text{Mod}(\mathbb{Z}_M^c) \) is given by sheaves \( L_i \in \text{Mod}(\mathbb{Z}_{U_i}) \) and isomorphisms \( \varphi_{ij} : L_j|_{U_{ij}} \simeq L_i|_{U_{ij}} \) for any \( i, j \in I \), such that \( \varphi_{ij} \circ \varphi_{jk} = c_{ijk} \varphi_{ik} \) for all \( i, j, k \in I \). We can assume that \( U_i \) is contractible and that \( U_{ij} \) is connected for any \( i, j \in I \). Since \( L \) is locally constant, we can choose an isomorphism \( \varphi_i : L|_{U_i} \simeq \mathbb{Z}_{U_i} \) for each \( i \in I \). Then the composition \( b_{ij} = \varphi_i \varphi_{ij}^{-1} \) is an isomorphism \( \mathbb{Z} \simeq \mathbb{Z} \), that is, \( b_{ij} = \pm 1 \). We let \( \bar{b} \) be the 1-cochain defined by \( \{ b_{ij} \}_{i, j \in I} \). Then the equality \( \varphi_{ij} \circ \varphi_{jk} = c_{ijk} \varphi_{ik} \) says that \( \tilde{c} \) is the boundary of \( \bar{b} \), as required. \( \square \)

### 13.5. Homotopy equivalence

Here we recover a result of [2] that the projection \( \pi_\Lambda : \Lambda \to M \) is a homotopy equivalence, that is, it induces an isomorphism of all fundamental groups. It is well know that this is equivalent to the following:
π₁(Λ) ∼→ π₁(M) and, for any local system L on M, RΓ(M; L) ∼→ RΓ(Λ; π⁻¹ L).

We recall the subcategory D_{[A]}^b(k_M × R) of D_{[A]}^b(k_M × R) introduced in Definition 12.4.1. It consists of the F such that F_− ≃ 0, where we have set F_± = F|_{M×{-t}} for t ≫ 0.

**Theorem 13.5.1.** Let k be a ring with finite global dimension.

(i) There exists F ∈ D_{[A],+}^b(k_M × R) such that F_+ ≃ k_M; it is a simple sheaf.

(ii) The functor G ↦ G_+, D_{[A],+}^b(k_M × R) → D^b(k_M), induces an equivalence between D_{[A],+}^b(k_M × R) and the full subcategory of D^b(k_M) formed by the locally constant sheaves. In particular the object F in (i) is unique up to a unique isomorphism.

(iii) The projection π_Λ: Λ → M induces an isomorphism in cohomology RΓ(M; k_M) ∼→ RΓ(Λ; k_Λ).

(iv) More generally we have RΓ(M; L) ∼→ RΓ(Λ; π⁻¹ L) for any local system L on M.

**Proof.** (i) We first assume that k = Z. By Theorem 13.2.2 and Proposition 13.4.2 we know that μ₁(Λ) = 0 and μ_{dcoh}(Λ) = 0. By Corollary 12.3.2 there exists F_0 ∈ D_{[A],+}^b(k_M × R) which is simple along Λ. By Proposition 13.3.1 we have F_0 ≃ L[d] where L ∈ Mod(k_M) is locally constant with stalks isomorphic to Z and d is some integer. Let p: M × R → M be the projection. Then F_1 = F_0 ⊗ p⁻¹ L⊗⁻¹[-d] satisfies the required properties. For a general ring k we set F = F_1 ⊗_{Z, M × R} k_M × R.

(ii) By Theorem 12.4.5 the functor G ↦ G_+ is fully faithful. Let L ∈ D^b(k_M) be a locally constant sheaf. Then F ⊗ p⁻¹ L belongs to D_{[A],+}^b(k_M × R) and we have (F ⊗ p⁻¹ L)_+ ≃ F_+ ⊗ L ≃ L, which proves that G → G_+ is essentially surjective.

(iii) follows from Corollary 13.3.2.

(iv) We apply Theorem 12.4.7 with F given by (i) and F' = F ⊗ (L ⊗ k_R). We have in this case F_+ ≃ F_+ ⊗ L ≃ L, hence RHom(F_+, F_+) ≃ L, and also μ_{coh}(F, F') ≃ μ_{coh}(F, F) ⊗ π⁻¹ L ≃ π⁻¹ L. The theorem then gives RΓ(M; L) ≃ RΓ(Λ; π⁻¹ L).

Now we prove that π₁(Λ) → π₁(M) is an isomorphism. It is equivalent to show that the inverse image by π_Λ induces an equivalence of categories Loc(k_M) ∼→ Loc(k_Λ), for some field k.
**Proposition 13.5.2.** Let \( k \) be a field. Let \( \pi_\Lambda : \Lambda \to M \) be the projection. Then the inverse image functor \( \pi_\Lambda^{-1} : \text{Loc}(k_M) \to \text{Loc}(k_\Lambda) \) is an equivalence of categories.

**Proof.** (i) We first prove that \( \pi_\Lambda^{-1} \) is fully faithful. Let \( F \in D^b_{[\Lambda],+}(k_M \times \mathbb{R}) \) be the simple object given by Theorem 13.5.1. Since \( F \) is simple we have \( \muhom(F, F)|_\Lambda \simeq k_\Lambda \) and we deduce, for \( L, L' \in \text{Loc}(k_M) \),

\[
\muhom(F \otimes p^{-1}L, F \otimes p^{-1}L') \simeq \mathcal{H}om(\pi_\Lambda^{-1}L, \pi_\Lambda^{-1}L'),
\]

where \( p : M \times \mathbb{R} \to M \) is the projection. Using \( (F \otimes p^{-1}L)_+ \simeq L \) and the isomorphisms (12.4.4) and (13.5.1), we obtain

\[
\text{Hom}(L, L') \simeq H^0(\Lambda; \muhom(F \otimes p^{-1}L, F \otimes p^{-1}L'))
\]

\[
\simeq \text{Hom}(\pi_\Lambda^{-1}L, \pi_\Lambda^{-1}L'),
\]

which means that \( \pi_\Lambda^{-1} \) is fully faithful.

(ii) We prove that \( \pi_\Lambda^{-1} \) is essentially surjective. Let \( L_1 \in \text{Loc}(k_\Lambda) \) be given. We view \( \text{Loc}(k_\Lambda) \) as a subcategory of \( D^L(k_\Lambda) \) of objects concentrated in degree 0, as justified by Remark 10.1.3. We recall that the functor \( \muhom(F, \cdot) \) induces an equivalence \( \muSh(k_\Lambda) \simeq D^L(k_\Lambda) \) (see Proposition 10.2.4 where the induced functor is denoted \( \muhom(F, \cdot) \)). Hence there exists \( L_1 \in \muSh(k_\Lambda) \) such that \( \muhom(F, L_1) \simeq L_1 \). By Corollary 12.3.2 there exists \( F_1 \in D^b(k_M \times \mathbb{R}) \) such that \( m_\Lambda(F_1) \simeq L_1 \). Then we have an isomorphism in \( D^b(k_\Lambda) \)

\[
\muhom(F, F_1)|_\Lambda \simeq L_1.
\]

Indeed it holds first in \( D^L(k_\Lambda) \). Then, applying the functors \( H^i \) of Remark 10.1.3 to (13.5.2), we see that \( \muhom(F, F_1)|_\Lambda \) is concentrated in degree 0. Hence (13.5.2) is an isomorphism in \( \text{Loc}(k_\Lambda) \) and then in \( D^b(k_\Lambda) \) because \( \text{Loc}(k_\Lambda) \) is also a subcategory of \( D^b(k_\Lambda) \).

We set \( L = (F_1)_+ \in D^b(k_M) \). Then \( \mathcal{S}(L) = \emptyset \) and, since \( F_+ \simeq k_M \), we also have \( L \simeq (F \otimes p^{-1}L)_+ \). Hence \( (F_1)_+ \simeq (F \otimes p^{-1}L)_+ \) and Theorem 12.4.5 gives \( F_1 \simeq F \otimes p^{-1}L \). We deduce

\[
\muhom(F, F_1)|_\Lambda \simeq \muhom(F, F \otimes p^{-1}L)|_\Lambda \simeq \pi_\Lambda^{-1}L.
\]

Hence \( L_1 \simeq \pi_\Lambda^{-1}L \). Taking \( H^0 \) of both sides we have \( L_1 \simeq \pi_\Lambda^{-1}H^0L \). Since \( H^0L \in \text{Loc}(k_M) \), we have \( L_1 \in \pi_\Lambda^{-1}(\text{Loc}(k_M)) \), as required. (We could see in fact that \( L \) is concentrated in degree 0.)

**Corollary 13.5.3.** The projection \( \Lambda \to M \) is a homotopy equivalence.

**Proof.** As already recalled this follows from Theorem 13.5.1(iv) and Proposition 13.5.2. ■
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