The common handsaw can be converted into a bowed musical instrument capable of producing exquisitely susurral notes when its blade is appropriately bent. Acoustic modes located on an inflection point are known to underlie the saw’s sonorous quality, yet the origin of localization remains mysterious. Here we uncover a topological basis for the existence of localized modes that rely on and are protected by spatial asymmetry. By combining experimental demonstrations, theory, and computation, we show that spatial variations in blade curvature control the localization of these trapped states, allowing the saw to function as a geometrically tunable high-quality oscillator. Our work establishes an unexpected connection between the dynamics of thin elastic shells and topological insulators and offers a robust principle for designing high-quality resonators across scales, from continuum macroscopic systems to nanoscale devices, simply through geometry.

**Significance**

The ability to sustain notes or vibrations underlies the design of most acoustic devices, ranging from musical instruments to nanomechanical resonators. Inspired by the singing saw that acquires its musical quality from its blade being unusually bent, we ask how geometry can be used to trap and insulate acoustic modes from dissipative decay in a continuum elastic medium. By using experiments and theoretical and numerical analysis, we demonstrate that spatially varying curvature in a thinnish shell can localize topologically protected states at inflection points, akin to exotic edge states in topological insulators. A key feature is the ability to geometrically control both spatial localization and the dynamics of oscillations in thin shells. Our work uncovers an unusual mechanism for designing robust, yet reconfigurable, high-quality resonators across scales.

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**Musical Instruments, even those made from everyday objects such as sticks, saws, pans, and bowls ([1]), must have the ability to create sustained notes for them to be effective. While this ability is often built into the design of the instruments, the musical saw, used to make music across the world for over a century and a half ([2]), is unusual in that it is just a carpenter’s saw but held in an unconventional manner to allow it to sing. When a saw (Fig. 1A) is either bowed or struck by a mallet, it produces a sustained sound that mimics a soprano’s lyric trill ([3]). Importantly, for such a note to be produced, the blade cannot be flat or bent into a J-shape (Fig. 1B) but must be bent into an S-shape (Fig. 1C).

This geometric transformation allows the saw to sing and is well known to musicians who describe the presence of “sweet spot,” i.e., the inflection curve in the S-shaped blade; bowing near it produces the clearest notes, while bowing far from it causes the saw to fall silent ([3]). Early works ([4, 5]), including notably Scott and Woodhouse ([6]), attempted to understand this peculiar feature by analyzing the linearized vibrational modes of a thin elastic shell ([7, 8]). Through a simplified asymptotic analysis, they showed that a localized vibrational eigenmode emerges at an inflection point in a shell with spatially varying curvature and is responsible for the musicality of the saw. Recent works have reproduced this result using numerical simulations ([9, 10]), but a deeper understanding of the origin of localization has remained elusive.

A simple demonstration of playing the saw quickly reveals the robustness of its musical quality to imperfections in the saw, irregularities in its shape, and the precise details of how the blade is flexed. Fig. 1D shows a time trace and spectrogram of the saw clamped in either a J-shape or an S-shape (Fig. 1B and C) each of which produces the fullest note ([11] and [12], respectively) and [13]).

The lack of sensitivity to these details suggests a topological origin for the localized mode responsible for the saw’s striking sonority. That topology can have implications for band structures and the presence of edge conducting states even when the bulk is insulating. The dull and short-lived sound (Audio 1) associated with the J-shape might be contrasted to the S-shape, produced in the absence of driven or active elements, spatial symmetries of a unit cell can also be used to achieve topological modes via acoustic analogs of the quantum spin or valley Hall effect ([14, 15]), although these examples rely on carefully engineered periodic lattices. Here we expand the use of topological ideas to continuum shells and show that
underlying the time-reversible Newtonian dynamics of theseinging
saw is a topological invariant that characterizes the propagation of waves in thin shells, arising from the breaking of up-down
inversion symmetry by curvature.

Results

Continuum Model of ThinShell Dynamics. This saw is modeled as a very thin rectangular elastic shell (thickness h/4; W L, where W, L are the width and length of the strip) made of a material with Young’s modulus Y, Poisson ratio v, and density p (Fig. IE). Its geometry is characterized by a spatially varying curvature tensor (second fundamental form) b(x), where x = (x, y) is the spatial coordinate in the plane. As the saw is bent only along the (long) x axis, b(x) is the sole nonvanishing curvature. To describe its dynamical response, we take advantage of its slenderness and treat the saw as a thin elastic shell that can be bent, stretched, sheared, and twisted. Before moving to a computational model that accounts for these modes of deformation as well as real boundary conditions, to gain some insight into the problem and expose the topological nature of elastic waves, it is instructive to instead consider a simplified description valid for shallow shells with slowly varying curvature.

In a thin shallow shell (h b, l/4), as bending is energetically cheaper than stretching (30), beam becomes negligible (Q100; Fig. IE) and in-plane deformations propagate much more rapidly (at the speed of sound c = JcM/4) so that the depth-averaged stresses can be assumed to equilibrated, i.e., b, b, = 0 (6, 31). In this limit, we solve the equations in terms of the long stress function x (u, = P f x, where P f 1 = 6 - b, 0 f V is a projection operator; SF Appendix, section 3). The in-plane compatibility relation and the linearized dynamical equations for transverse motions can be written as (7, 32)

\[ \frac{1}{\rho h^2} \frac{\partial^2 x}{\partial t^2} = -\frac{1}{c^2} \nabla \cdot (b(x) \nabla x), \] (1)

\[ \rho b f^2 = -\frac{1}{c^2} \nabla \cdot (b(x) \nabla x). \] (2)

Here f is the out-of-plane deflection of the shell (Fig. IE) and the bending rigidity \( K = \frac{Y h^2}{12(1 - v^2)} \). Crucially, in-plane and flexural (out-of-plane) modes remain geometrically coupled in the presence of curvature even in the linearized setting (Eqs. 1 and 2). For a shell bent with constant curvature along the x axis, i.e., a section of a uniform cylinder, b(x) = b(x) is a constant. In the bulk of the system, disregarding boundaries, we can Fourier transform Eqs. 1 and 2 using the solution ansatz \( x = \hat{x}(\omega, q, \xi) = \sum_{\omega, q, \xi} \hat{x}(\omega, q, \xi) e^{i(\omega t, q \cdot x)} \). When \( \hat{x} = 0 \), i.e., the sheet is undeformed in the transverse direction, it remains developable (with generators that run parallel to the y direction), and the bending waves are gapless, i.e., w = 0 as \( q = 0 \). However, when \( q, f > 0 \), a finite frequency gap \( \Delta \) (in addition to the finite \( \omega \) corrections) controlled by the speed of sound and the curvature of the shell emerges as \( q = 0 \) (Fig. 2A).

Intuitively, this arises due to the geometric coupling between bending and stretching deformations in acurved shell which leads to an effective stiffening that forbids wave propagation below a frequency threshold. Similar spectral gaps appear in curved filaments and doubly curved shells as well (31, 33).

For the -shaped saw, curvature scales of \( b b0.4 \) to 0.8 m\(^{-1}\) are easily achievable (as in Fig. 1 B and C), while the typical sound speed in steel is \( c \approx 5 \times 10^3 \text{ms}^{-1} \) so that the frequency gap is of order 2 to 5 kHz. Comparing these estimates to the spectrogram in Fig. ID (further quantified in Fig. 3) suggests that the localized mode excited upon bowing the -shaped saw (Fig. 1C) lies within the frequency gap. The |shaped saw (Fig. 1B) also exhibits low-fqencies (compared to the gap) whenever, presumably through the \( \nu = 0 \) branch of delocalized flexural modes, although higher frequencies above the band gap can be excited by careful bowing (SF Appendix, Fig. 3A and B).

Curvature-Induced Z_2 Topological Invariant. To unveil the topological structure of the vibration spectrum of the saw, we cast the second-order dynamical equations (Eqs. 1 and 2) in terms of first-order equations by taking the square root of the dynamical matrix (14, 34). Focusing on the flexural modes alone, we obtain a Schrödinger-like equation for the transverse deflection of a shallow shell (SF Appendix, section 3)

\[ i \frac{\partial \psi}{\partial t} = \mathcal{H} \psi, \quad \mathcal{H} = V \nabla^2 + \psi, \] (3)

where \( \psi = (c^2 f, i o,)/V \) and \( V = i b(x) \). The eigenvalues of the effective Hamiltonian \( \mathcal{H} \) are given by the previously derived \( \omega = \pm q \), and its complex eigenvectors \( \psi(\omega) \) encode the topology of the band structure. The singularities in the arbitrary phase of the eigenvectors signals nontrivial band topology. To understand the phase of eigenvectors along the saw’s long direction, we can consider fixing the transverse wave vector \( \nu, f 0 \), leading to an effective one-dimensional (ID) system along the x axis. Then the obstruction to continuously define the phase of the eigenvectors at every \( x, f 0 \) in Fourier space while respecting all the symmetries of the problem is quantified by the ID Berry connection \( A(q) = i \mathcal{N}(q) \) (Fig. 2A). The curvature domain wall, the jump in the topological invariant is well defined independent of microscopic details. Across an interface at which curvature changes sign, i.e., a curvature domain wall, the jump in the topological invariant is given by

\[ \Delta \omega = \omega - \omega = \pm \frac{\pi}{2}. \] (4)

similiar to topological insulators with crystalline symmetries (37-39). Pf(W) denotes the Pfaffian of the antisymmetric overlap matrix \( W(q) = \psi(\omega, q) / \mathcal{C} \langle \mathcal{T} \psi(\omega, q) (\nu, \nu) = \omega \) (Fig. 2A). We note that unlike the mechanical Su-Schrieffer-Heeger chain (14) that exhibits a topological polarization in ID, the emergent tangent-plane spatial reflection symmetry in our problem forces this polarization to vanish (SF Appendix, section 3).

\[ \text{As we work in the continuum, only differences in the topological invariant are well defined independent of microscopic details. Across an interface at which curvature changes sign, i.e., a curvature domain wall, the jump in the topological invariant is given by} \] (5)

\[ \Delta \omega = \Delta \omega = \pm \frac{\pi}{2}. \] (4)
ill-conditioning commonly seen in high-order continuum theories for slender plates and shells while allowing for numerical methods that require less smoothness and are easier to implement (SJ Appendix: section 3). Together, these allow for better computational accuracy. This framework forms the basis for the Naghdishell model (40) (see SJ Appendix, section 2, for details) and accounts for an in-plane displacement vector along the midplane $u(x,t)$, an out-of-plane deflection $w(x,t)$ normal to the shell, and an additional rotation $\Omega (x, t)$ of the local normal itself (Fig. IE). These modes of deformations lead to depth-averaged stress resultants associated with stretching ($\langle \rangle$), bending ($\langle \rangle$), and shear ($\langle \rangle$) as shown in Fig. IE. The resulting covariant nonlinear shell theory along with inertial Newtonian dynamics provides an accurate and computationally tractable description of the elastodynamics of thin shells (Fig. IE and SJ Appendix, section 2). To highlight the topological robustness of our results, in our calculations we vary both the boundary conditions and curvature profiles imposed on the saw.
within the spectral gap for increasing curvature (solid red line, Fig. 2B). This midgap state (shown here for \( m = 2 \)) is a localized mode that is trapped in the neighborhood of the inflection line (Fig. 2C, Bom,m). For increasing mode number \( m \geq 2 \), similar topological modes appear within the bulk bandgap, with growing localization lengths (Fig. 2D, blue). As predicted analytically (SI Appendix, section 3), qualitatively, the presence of an inflection line in the S-shaped saw makes it geometrically soft; the generators of the cylindrical modes are now along the length of the saw, and the curved regions on either side are geometrically stiff serve to insulate the soft internal edge from the real damped edges.

Of panicular note is that the localized modes, unlike the extended states, are visually unaffected by the boundaries and the conditions there (see SI Appendix, Fig. S2A, for eigenmodes in a strip with asymmetric boundary conditions where the left edge is clamped and the right edge is free). Spatial gradients in curvature, however, do impact the extent of localization. We demonstrate this using a piecewise continuous curvilinear profile that has a constant linear gradient \( b' \) over a Length \( t \), zero the origin and adopts a constant curvature outside this region. By varying both the curvature gradient \( b' \) and the length scale \( t \), we can tune the localization of the lowest topological mode (same as Fig. 2C, Bom,m), quantified by the inverse participation ratio \( \text{IPR} = \frac{\langle J(x) \rangle^2}{\langle J(x)^2 \rangle} \) (Fig. 2E). Strong localization (high

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In Fig. 2B, the distribution of eigenmodes as a function of frequency is shown in the integrated density of states for a constant curvature shell with a smooth curvature profile \( \alpha(x) = \beta x \) and a S-shaped shell with a smooth curvature profile \( \alpha(x) = \beta x \) for increasing mode number \( m \). In both cases, the ends of the strip are kept clamped, and the spectra are calculated using an open-source code based on the finite element method (41, 42). As the curvature of the S shape approaches a constant value \( \pm \alpha \) from the origin, the bulk spectral gap and delocalized modes match that of the constant curvature case. Figure 2C shows modes that vary at most linearly in direction \( y \) (labeled by discrete mode numbers \( n = 0, 1 \)) and are delocalized over the entire ribbon (Fig. 2C, Top and Middle) and populate states all the way to zero frequency, i.e., with a gapless spectrum. This is true for both constant curvature (dashed blue line, Fig. 2B) and the S-shaped shell (solid blue line, Fig. 2B) as these bulk modes are unaffected by curvature. In contrast, all other modes that bend in both directions \( (m = 2) \) are generically gapped for constant curvature profile (dashed red line, Fig. 2C) as expected. However, for the S shape, in addition to the gapped bulk modes, a new mode appears within the spectral gap (solid red line, Fig. 2B). This midgap state (shown here for \( m = 2 \)) is a localized mode that is trapped in the neighborhood of the inflection line (Fig. 2C, Bom,m). For increasing mode number \( m \geq 2 \), similar topological modes appear within the bulk bandgap, with growing localization lengths (Fig. 2D, blue). As predicted analytically (SI Appendix, section 3), qualitatively, the presence of an inflection line in the S-shaped saw makes it geometrically soft; the generators of the cylindrical modes are now along the length of the saw, and the curved regions on either side are geometrically stiff serve to insulate the soft internal edge from the real damped edges.

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Fig. 3. Dissipative dynamics and high-quality oscillators. (A) Resonance waves for a shell with a linear curvature profile (Ins.et) periodically driven at the inflection point ($x=0$; red) and away, from it ($x=0.4L$; black) for varying frequency ($w$, 740 Hz corresponds to the first localized mode). (B) Numerically computed Q factor shows dramatic enhancement at localized mode frequencies (red) over delocalized modes (blue). (C and D) Experimental measurement of Q factor (see SI Appendix, section 1, for details) for the musical saw (QJ shape (Fig. 1B) and OS shape (Fig. 1C) (Top) Note the normalized Fourier spectrum amplitude in log scale below 0.1 and linear above, with the peak frequency marked as $f$. (Bottom) The average signal decay (blue curve) is fit to a single exponential (black curve). The shaded region is the SE in both C and D.

IPR (interference probability ratio) is quickly achieved for sharp gradients in curvature ($IPR \approx \frac{J}{L}/\hbar$; SI Appendix, section 3) as long as the length scale of curvature variation is not too small ($L/\hbar < 0.1$, Fig. 2E), corresponding to a diffuse domain wall. In the opposite limit $L/\hbar \to 0$ for $\hbar' = \hbar$ fixed, i.e., a sharp domain wall with a discontinuous curvature profile $b(x) = \text{sgn}(x)$, strong localization persists (SI Appendix, Fig. S3), consistent with our topological prediction and demonstrating the ease of geometric control of localization.

Geometrically Tunable High-Quality Oscillators. The boundary insensitivity of topologically localized modes has important dynamic consequences that can be harnessed to produce high-quality resonators. The primary mode of dissipation in the saw, as in nanoelectromechanical devices (43), is through substrate or anchoring losses at the boundary. Internal dissipation mechanisms (from, e.g., plasticity, thermoelastic effects, and radiation losses), although present, are considerably weaker and neglected here. To model dissipative dynamics, we retain damped boundary conditions on the left end and augment the right boundary to include a restoring spring ($k$) and dissipative friction ($-\gamma$) for both the in-plane forces and bending moments (Fig. 3A, Inset, and SI Appendix, section 2). Informed by Fig. 2E, we choose a linear curvature profile spanning the entire length of the shell to obtain a strongly localized mode. Upon driving the shell into steady oscillations, with a periodic point force applied at the inflection point ($x = 0$; Fig. 3A, red curve), we see an extremely sharp resonance peak right at the frequency of the first localized mode (Fig. 3A). In contrast, when the shell is driven closer to the boundary ($x = 0.4L$; Fig. 3A, black curve), the response is at least six-orders of magnitude weaker as the localized mode is not excited and only the delocalized modes contribute. Localization hence protects the mode from dissipative decay, unlike extended states that dampen rapidly through the boundaries. We further quantify this using a Q factor computed from unclamped relaxation of the shell initialized in a given eigenmode (SI Appendix, section 2). Ultrahigh values of $Q'' = 10^5$ to 106 are easily attained when a localized mode is excited (Fig. 3B, red), well over the Q factor of all other modes (Fig. 3B, blue). Similar results are obtained for other curvature profiles as well, such as a sigmoid curve (SI Appendix, Fig. S2B).
To compare these computational results with experiments, we perform ringdown measurements on a musical saw (see SI Appendix, section I, for details) damped in both the J shape (Fig. 1B) and the S shape (Fig. 1C). As indicated by Eq. 5, the key distinguishing feature of the S-shaped saw (compared to the J shape) is the presence of an inflection line (curvature domain wall) that engenders a well-localized domain wall mode capable of sustaining long-lived oscillations. The normalized 2D Fourier spectra and exponential decay [-(t) of the signal envelope are shown in Fig. 3C (U shape) and Fig. 3D (S shape) with the dominant frequency (w0) marked. We find a factor ~15 enhancement in the Q factor Q = w0/2π for the S-shaped saw (Q "= 150; Fig. 3D, Left) over the J shape (Q "= 10; Fig. 3C, Left). We emphasize that this is a significant factor improvement, although not as dramatic as the numerically computed Q factors (Fig. 3B). It is still striking given the initial impulse (mallet strike) for J shape and bow for S shape; see SI Appendix, Fig. SI, for other cases) exists an uncontrollable range of frequencies and sources of energy loss including internal damping are presumably also present.

Discussion and Conclusion

Our combination of analysis, finite element simulations and experiments has demonstrated that a saw's ability to control sound waves is due to the structural properties of the saw blades themselves, not just to the technique of playing the saw. We have shown that the S shape, bow for S shape, and an S shape with a metal contact, can be used to achieve high-Q factors in the musical saw, and that these properties are achieved through the clamped-ended condition. The saw blade is a useful model for understanding the behavior of thin elastic sheets.

Materials and Methods

Extended data on the experiments and the details of the numerical modeling and theoretical calculations are provided in SI Appendix.

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