Extracting the low-energy constant $L'_0$ at three flavors from pion-kaon scattering

Chaitra Kalmahalli Guruswamy$^1$, Ulf-G Meißner$^{1,2,3}$ and Chien-Yeaw Seng$^1$

$^1$Helmholtz-Institut für Strahlen- und Kernphysik and Bethe Center for Theoretical Physics, Universität Bonn, D-53115 Bonn, Germany
$^2$Institute for Advanced Simulation, Institut für Kernphysik and Jülich Center for Hadron Physics, Forschungszentrum Jülich, D-52425 Jülich, Germany
$^3$Tbilisi State University, 0186 Tbilisi, Georgia

E-mail: meissner@hiskp.uni-bonn.de

Received 30 March 2022, revised 25 April 2022
Accepted for publication 6 May 2022
Published 13 June 2022

Abstract

Based on our analysis of the contributions from the connected and disconnected contraction diagrams to the pion-kaon scattering amplitude, we provide the first determination of the only free low-energy constant at $\mathcal{O}(p^4)$, known as $L'_0$, in SU(4) Partially-Quenched Chiral Perturbation theory using the data from the Extended Twisted Mass collaboration, $L'_0(\mu = M_\pi) = 0.77(20)(25)(7)(7)(2) \cdot 10^{-3}$. The theory uncertainties originate from the unphysical scattering length, the physical low-energy constants, the higher-order chiral corrections, the (lattice) meson masses and the pion decay constant, respectively.

Keywords: lattice QCD, partially quenched chiral perturbation theory, pion-kaon scattering

1. Introduction

Partially-Quenched Quantum Chromodynamics and its low-energy effective field theory (EFT), Partially-Quenched Chiral Perturbation theory (PQChPT) [1–7], are powerful tools to assist the first-principles calculations of hadronic observables using lattice QCD. They were originally created to handle the so-called partially-quenched approximation in early lattice studies, where the ‘valence’ and ‘sea’ quark masses were made distinct in order to simplify the calculation of the fermion determinant. Nowadays such an approximation has been largely abandoned, but the devised theory frameworks have found their own ways to continue being useful. In particular, it was recently realized that since additional quark flavors are introduced in PQChPT, it allows an EFT description of each individual quark contraction diagram in the lattice simulation of a given physical observable, which is very useful for getting a better handle on the noiser and computationally-expensive contractions (the so-called ‘disconnected diagrams’). This idea was applied initially to the study of the hadronic vacuum polarization [8] and the pion scalar form factor [9], and was later extended to pion-pion scattering [10, 11] and the parity-odd pion-nucleon coupling constant [12].

As in any EFT, the full predictive power of PQChPT to a given order is guaranteed only when all the low-energy constants (LECs) at that order are fixed. This is a non-trivial task since some of them are not constrained by any physical experiment and can only be determined through lattice simulations. In particular, in an extended flavor sector with $N_f \geq 4$, the Cayley–Hamilton relation for $3 \times 3$ matrices used in SU(3) ChPT cannot be applied anymore, which leads to the following extra term in the PQChPT Lagrangian at $\mathcal{O}(p^4)$:

$$
\mathcal{L}^{(4)} = L_0 \text{Str}(\partial_\nu U^\dagger \partial_\mu U \partial_\nu U^\dagger \partial_\mu U),
$$

where ‘Str’ denotes the supertrace over the extended flavor space, $U$ is the standard exponential representation of the pseudo-Nambu-Goldstone bosons, and $L_0$ is a new LEC that does not appear in ordinary two-flavor and three-flavor ChPT. Despite appearing at the next-to-leading order (NLO), this LEC contributes to static quantities such as the pion mass and decay constant only at the next-to-next-to-leading order (NNLO), when the valence and sea quark masses are kept distinct. Based on this, [13] has determined the renormalized LEC $L'_0$ in the SU(4) PQChPT, which is equivalent to a two-flavor ChPT in
computations of physical processes. The quoted result is \( L_0^q (\mu = 1 \text{ GeV}) = 1.0(1.1) \cdot 10^{-3} \), using dimensional regularization.

[10, 11] provided an alternative determination of \( L_0^q \) in SU(4(2)) using the fact that it appears in separate contraction diagrams in the \( \pi \pi \) scattering amplitude at NLO, and therefore can be obtained from the unphysical scattering length defined through an appropriate linear combination of contraction diagrams. The advantage of this new method is that, since only light quarks are involved in the procedure, the higher-order chiral corrections that scale generically as \( M_F^2/(4\pi F)^2 \) are expected to be small so the NLO fitting is more stable than NNLO, and of course the number of unknown LECs in the former is also smaller. Using the lattice data of connected \( \pi \pi \) correlation functions by the Extended Twisted Mass (ETM) collaboration [11, 14] reported \( L_0^q (\mu = 1 \text{ GeV}) = 5.7(1.9) \cdot 10^{-3} \). The significant difference from the result of the NNLO fit in [13] is yet to be understood. Resolving such a discrepancy may improve our knowledge of the chiral dynamics at low energies, or even reveal some unexpected lattice systematics that could play important roles in precision physics.

The analysis in [10, 11] could be straightforwardly generalized to the three-flavor case. In particular, the so-called SU(4)(1) PQChPT is of special interest because it is the simplest extension of the original three-flavor ChPT. Moreover, among all its renormalized LECs at \( \mathcal{O}(p^4) \), only \( L_0^q \) is undetermined, while all the others are identical to those in SU(3). This implies that the theory would be fully predictive at \( \mathcal{O}(p^4) \) once \( L_0^q \) is fixed, and could then be used to aid the lattice studies of interesting hadronic processes involving strange quarks, such as \( K\pi \to K\pi, \pi\pi \to KK \) and \( K\eta \to K\eta \) scatterings, in particular channels where disconnected diagrams appear (e.g. the \( I = 1/2 \) channel of \( K\pi \) scattering).

In this work, we present the first-ever numerical determination of \( L_0^q \) in SU(4)(1) based on the method outlined in our previous paper, [15]. The main idea is that, by switching the relative sign between the two types of connected contraction diagrams that occur in the \( I = 3/2 \) \( K\pi \) scattering, one obtains effectively an unphysical single-channel scattering amplitude \( T_{gh} \) of which the scattering length depends on \( L_0^q \) at \( \mathcal{O}(p^4) \). Invoking the recent lattice data by the ETM collaboration [16], the unphysical scattering length is obtained through the usual Lüscher analysis of the discrete energy states [17], which consequently fixes \( L_0^q \). This completes the SU(4)(1) PQChPT Lagrangian at \( \mathcal{O}(p^4) \) and is potentially useful for all the purposes mentioned above. Besides, it serves as a prototype for future analyses of more complicated PQ-extensions of three-flavor ChPT. Such theories, after integrating out the strange quark, reduce to PQChPT with only two light flavors. This may give an independent check of value of \( L_0^q \) in SU(4)(2), and provide some hints towards the solution to the aforementioned discrepancy.

2. Formalism

As detailed in [15], the pertinent two contractions in \( K\pi \) scattering with total isospin \( I = 3/2 \), as depicted in figure 1, are related to the amplitudes in the SU(4(1)) PQChPT:

\[
T_a(s, t, u) = T_{a(ad)(df)}(s, t, u), \quad T_b(s, t, u) = T_{a(df)(ad)}(s, t, u),
\]

with \( s, t, u \) the conventional Mandelstam variables subject to the constraint \( s + t + u = 2(M_K^2 + M_{\pi}^2) \), \( u, d, s \) the physical quarks and \( j \) denotes the additional valence quark (which comes together with a ghost quark \( \bar{j} \)) in PQChPT. The assigned quark masses are \( m_u = m_d = m_l = m_s < m_c \). For total isospin \( I = 1/2 \), there is one additional scattering amplitude \( T_c \) that is related to \( T_b \) through a simple crossing, \( T_b(s, t, u) \equiv T_b(u, t, s) \). Although such a crossing is analytically straightforward, from a lattice point of view \( T_a \) and \( T_b \) are relatively easy to evaluate, whereas \( T_c \) involves a pair of quark propagators that start and end on the same temporal slice, and is exactly what we call a ‘disconnected diagram’. Such a diagram suffers from a low signal-to-noise ratio and represents a fundamental challenge in the first-principles study of the \( K\pi \) scattering in the \( I = 1/2 \) channel.

From the above one may define three effective single-channel scattering amplitudes:

\[
T_a(s, t, u) = T_a(s, t, u) + T_b(s, t, u)
\]

\[
T_b(s, t, u) = T_b(s, t, u) - T_b(s, t, u)
\]

\[
T_c(s, t, u) = T_a(s, t, u) - \frac{1}{2} T_b(s, t, u) + \frac{3}{2} T_c(s, t, u),
\]

where \( T_a \) and \( T_c \) correspond to the physical \( I = 3/2 \) and \( I = 1/2 \) scattering amplitudes, respectively, while \( T_b \) is an unphysical amplitude. The S-wave scattering lengths are defined through the threshold values of the amplitudes:

\[
a_\alpha^a = -\frac{1}{8\pi s_0} T_a(s_0, t_0, u_0) \quad (\alpha = \alpha, \beta, \gamma), \]

with \( s_0 = (M_K + M_{\pi})^2 \), \( t_0 = 0 \) and \( u_0 = (M_K - M_{\pi})^2 \). Obviously, only the unphysical scattering length \( a_\beta^b \) can depend on the unphysical LEC \( L_0^q \). Its explicit expression at
\( \mathcal{O}(p^4) \) reads:

\[
\begin{align*}
\alpha_0^\beta &= -\frac{1}{8\pi\sqrt{30}} \left[ \frac{M_2^2 M_K^2}{F_\pi^4} (-96L_0^\prime) \\
+ 32L_1^\prime + 32L_2^\prime - 16L_3^\prime - 32L_4^\prime \\
+ 8L_5^\prime + 32L_6^\prime - 16L_7^\prime - \frac{5}{576\pi^2} \right] \\
+ \frac{8M_2^2 M_K^2}{F_\pi^4} + \frac{\mu_p}{F_\pi^4 (M_K^2 - M_\pi^2)} \\
\times \left( -\frac{M_2^2}{2} - \frac{1}{2} M_\pi^3 M_K^2 + \frac{13}{4} M_\pi^2 M_K^2 + \frac{5}{2} M_\pi^3 M_K \right) \\
+ \frac{\mu_K}{F_\pi^4} \left( -\frac{M_2^2}{2 M_K - M_\pi} - \frac{M_\pi^3}{2 M_K - M_\pi} - \frac{4 M_\pi^2 M_K^2}{M_K^2 - M_\pi^2} \right) \\
+ \frac{M_2^2 M_K^2}{2(M_K - M_\pi) + 2(M_K - M_\pi)} \\
+ \frac{\mu_p}{F_\pi^4 (M_K^2 - M_\pi^2)} \left( -\frac{4 M_2^2 M_K^2}{9 M_\pi^2} - \frac{M_3^2 M_K^2}{18 M_\pi^2} \right) \\
+ \frac{3 M_2^2 M_K^2}{4} + \frac{3}{2} M_\pi M_K^2 + \frac{11}{4} M_\pi^2 M_K^2 - \frac{3}{2} M_\pi^3 M_K \\
+ \frac{M_2^2 M_K^2 J_{K(s_0)}(u_0)}{F_\pi^4} + \frac{J_{K(u_0)}(M_\pi^2)}{F_\pi^4} \\
\times \left( \frac{M_2^2}{8} + \frac{1}{2} M_\pi M_K^2 - \frac{7}{12} M_2^2 M_K^2 \right) \\
+ \frac{1}{18} M_2^2 M_K + \frac{M_\pi M_K^2}{72} \\
+ \frac{1}{2} M_2^2 M_K + \frac{M_\pi M_K^2}{72} \\
+ \frac{J_{K(u_0)}(M_\pi^2)}{F_\pi^4} \left( -\frac{M_2^2}{8} - \frac{1}{2} M_\pi M_K^2 \right) \\
+ \frac{3}{2} M_2^2 M_K - \frac{1}{2} M_\pi M_K^2 - \frac{M_4^2}{8} \\
+ \frac{J_{K(u_0)}(M_\pi^2)}{F_\pi^4} \left( \frac{M_2^2}{72} - \frac{1}{2} M_\pi M_K^2 \right) - \frac{1}{12} M_2^2 M_K^2 \\
- \frac{1}{18} M_2^2 M_K - \frac{M_\pi M_K^2}{72} - \frac{5 M_2^4}{384\pi^2 F_\pi^2} - \frac{5 M_\pi M_K^2}{192\pi^2 F_\pi^2} \\
- \frac{M_3^2 M_K}{192\pi^2 F_\pi^2} + \frac{M_2 M_K}{F_\pi^2} - \frac{M_4^2}{384\pi^2 F_\pi^2} \right], \\
\end{align*}
\]

where \( \mu_p = (M_2^2/32\pi^2 F_\pi^2) \ln(M_2^2/\mu^2) \), with \( \mu \) the renormalization scale, and the functions \( J_{Q(s)}(u) \), \( \tilde{J}_{Q(s)}(u) \) are given in [15]. The \( \{L_i^\prime\} \) are the \( \mathcal{O}(p^4) \) LECs in SU(4(1)) PQChPT, among which \( L_{-8}^\prime \) are numerically identical with those in the ordinary three-flavor ChPT. Only \( L_0^\prime \) is new, and can be readily solved from the equation above.

| Ensemble | \( aM_\pi \) | \( aM_K \) | \( aM_\eta \) | \( aF_\pi \) |
|----------|-------------|-------------|-------------|-------------|
| A30.32   | 0.1239(3)   | 0.236(7)    | 0.314(17)   | 0.064(52)   |
| A40.24   | 0.1452(5)   | 0.241(7)    | 0.317(7)    | 0.065(77)   |

| Ensemble | \( \mu_{\pi K} a_0^\beta \) |
|----------|-------------------------------|
| A30.32   | -0.0956(91)                  |
| A40.24   | -0.1152(144)                 |

3. Extraction of \( L_0^\prime \)

The unphysical scattering length \( a_0^\beta \) is an essential input in our study. It is obtained from the analysis of the lattice-volume-dependence of the discrete energy levels corresponding to the combination \( T_\pi - T_0 \) through the standard Lüschere formula [17]. In this work, we utilize the results in [16] for the \( \kappa_\pi \) system in the \( I = 3/2 \) channel where the correlation functions \( C_\alpha(\tau) \) and \( C_k(\tau) \) corresponding to the two contractions in figure 1 were separately calculated. The calculation was based on the \( N_f = 2 + 1 + 1 \) twisted mass lattice QCD. For our analysis, we have considered the ensembles A30.32 and A40.24 for the determination of the discrete ground-state energies of the \( \kappa_\pi \) system. The basic parameters of each ensemble are summarized in Table 1.

Only results that correspond to the physical combination \( C_\alpha(\tau) + C_k(\tau) \) were displayed in [16] for obvious reasons. In this project, we acquired the unphysical S-wave scattering length \( a_0^\beta \) directly from the authors of that paper, who obtained its value through an unpublished analysis of the volume-dependence of the discrete energy levels extracted from the unphysical combination \( C_\alpha(\tau) = C_\alpha(\tau) - C_k(\tau) \) [18]; first, the energy shift \( \delta E_\beta = E_\beta^S - M_\pi - M_K \) was obtained as a function of the lattice size \( L \) from the exponential behavior of \( C_\alpha(\tau) \) at large Euclidean time \( \tau \). The scattering length \( a_0^\beta \) at infinite volume was then computed using the single-channel, Taylor-expanded Lüscher formula

\[
\delta E_\beta = -\frac{2\pi a_0^\beta}{\mu_{\pi K} L^2} \left( 1 + c_1 a_0^\beta \frac{a_0^\beta}{L^2} + c_2 \frac{(a_0^\beta)^2}{L^2} \right) + \mathcal{O}(L^{-6}),
\]

as in equation (14) of [16]. Here, \( \mu_{\pi K} \) is the reduced mass of the \( \kappa_\pi \) system and \( c_{1,2} \) are known coefficients. The final outcomes are summarized in Table 2. The main uncertainty of \( a_0^\beta \) is systematic, which comes from the unwanted time-dependent contributions at finite \( \tau \) (i.e. the ‘thermal pollutions’). Their effects were studied using two different
methods labeled as E1 (weighting and shifting) and E2 (dividing out the pollution), respectively [19].

To solve for $L_3^\rho$ using equation (5), we further require the values of all the physical LECs. We took their values at $\mu = M_\rho = 770$ MeV from [20]:

$$
\begin{align*}
10^2 L_1^\rho &= 1.11(10), & 10^3 L_4^\rho &= 1.05(17), \\
10^2 L_2^\rho &= -3.82(30), & 10^2 L_3^\rho &= 1.87(53), \\
10^2 L_4^\rho &= 1.22(06), & 10^3 L_6^\rho &= 1.46(46), \\
10^3 L_6^\rho &= 0.65(07). \\
\end{align*}
$$

(7)

With all the above, we may now compute $L_3^\rho$ straightforwardly. The outcome at $\mu = M_\rho$ reads:

A30.32 , Method E1: $10^2 L_3^\rho = 0.78(21)(25)(7)(7)(2)$
A30.32 , Method E2: $10^2 L_3^\rho = 0.77(20)(25)(7)(7)(2)$
A40.24 , Method E1: $10^2 L_3^\rho = 0.86(25)(25)(7)(4)(2)$
A40.24 , Method E2: $10^2 L_3^\rho = 0.87(22)(25)(7)(4)(2)$,

(8)

where the uncertainties come from $a_i^{\rho4}$, the physical LECs, the higher-order ChPT corrections, the (lattice) meson masses and $F_\pi$, respectively. In particular, the higher-order ChPT corrections are estimated by multiplying the central value with the usual chiral suppression factor $M_\rho^2/(4\pi F_\pi)^2$. We see that the values of $L_3^\rho$ from all four determinations are consistent with each other within the error bars, so we may simply quote the number with the smallest theory uncertainty, namely the one from the ensemble A30.32 with method E2:

$$
L_3^\rho(M_\rho) = 0.77(33) \times 10^{-3}.
$$

(9)

Finally, we comment on the relation between this result and the corresponding LEC in SU(4|2). In principle, relations between LECs in the two-flavor and three-flavor ChPT can be obtained by integrating out the strange quark in the latter. Possible PQChPT extensions of an ordinary two-flavor ChPT are SU(3|1), SU(4|2), SU(5|3)... etc, but only SU(4|2) onwards possess an $L_3^\rho$-dependence at tree-level as it requires at least four fermionic quarks. Similarly, possible PQChPT extensions of a three-flavor ChPT are SU(4|1), SU(5|2), SU(6|3)... . In [15] we chose the simplest version which is SU(4|1). After integrating out the strange quark, it reduces to SU(3|1) that does not depend on $L_3$ at tree-level. Therefore, it is not possible to discuss the matching between $L_3^\rho$ in two- and three-flavors based on the theory setup in [15]. For that, one would have to repeat the calculations using a larger graded algebra, such as SU(5|2). This is of great interest because it may provide new insights into the apparent disagreement between the determination of $L_3^\rho$ at SU(4|2) from NLO and NNLO. However, it goes beyond the scope of this work and will be carried out in follow-up studies.

4. Summary

In this work, we have for the first time determined the unphysical LEC $L_3^\rho(M_\rho)$ in the simplest PQ-extension of the three-flavor ChPT through an NLO analysis of contraction diagrams in $K\pi$ scattering, $L_3^\rho = 0.77(33) \times 10^{-3}$. Utilizing the precise data from the ETM collaboration for $K\pi$ scattering in the $f = 3/2$ channel, we control the absolute uncertainty in this LEC to $3.3 \times 10^{-4}$, which is better than the previous determinations of the corresponding LEC for two flavors in [13] that made use of an NNLO fitting, and in the NLO fitting to the $\pi\pi$ scattering amplitudes in [10, 11] that depends on more unknown LECs. The major sources of uncertainty in this study are the systematic errors in the lattice extraction of the unphysical scattering length $a_i^{\rho4}$, as well as the physical LECs $L_i^{\rho4}$. Our work completes the PQChPT Lagrangian at $O(p^6)$ and prepares it for future applications in studies of interesting hadronic observables involving strange quarks, in synergy with lattice QCD.

Acknowledgments

We are very grateful to Ferenc Piller for making the ETM collaboration data available to us and for his detailed explanations concerning these. We thank Hans Bijnens for a useful communication. This work is supported in part by the DFG (Projektnummer 196 253 076—TRR 110) and the NSFC (Grant No. 11 621 131 001) through the funds provided to the Sino-German CRC 110 ‘Symmetries and the Emergence of Structure in QCD’, by the Alexander von Humboldt Foundation through the Humboldt Research Fellowship, by the Chinese Academy of Sciences (CAS) through a President’s International Fellowship Initiative (PIFI) (Grant No. 2018DM00034), by the VolkswagenStiftung (Grant No. 93 562), and by the EU Horizon 2020 research and innovation programme, STRONG-2020 project under grant agreement No. 824 093.

References

[1] Bernard C W and Golterman M F L 1994 Partially quenched gauge theories and an application to staggered fermions Phys. Rev. D 49 486–94
[2] Sharpe S R and Shores N 2000 Partially quenched QCD with nondegenerate dynamical quarks Nucl. Phys. B 83 968–70
[3] Sharpe S R and Shores N 2000 Physical results from unphysical simulations Phys. Rev. D 62 094503
[4] Sharpe S R and Shores N 2001 Partially quenched chiral perturbation theory without PhD Phys. Rev. D 64 114510
[5] Bernard C and Golterman M 2013 On the foundations of partially quenched chiral perturbation theory Phys. Rev. D 88 014004
[6] Sharpe S R arXiv:hep-lat/0607016 [hep-lat]
[7] Golterman M arXiv:0912.4042 [hep-lat]
[8] Della Morte B and Jüttner A 2010 Quark disconnected diagrams in chiral perturbation theory J. High Energy Phys. 11 154
[9] Jüttner A 2012 Revisiting the pion’s scalar form factor in chiral perturbation theory J. High Energy Phys. 01 JHEP01 (2012)007
[10] Acharya N R, Guo F K, Meißner U-G and Seng C Y 2017 Connected and disconnected contractions in pion–pion scattering Nucl. Phys. B 922 480–98

[11] Acharya N R, Guo F K, Meißner U-G and Seng C Y 2019 Constraints on disconnected contributions in ππ scattering J. High Energy Phys. 04 JHEP04(2019)165

[12] Guo F K and Seng C Y 2019 Effective field theory in the study of long range nuclear parity violation on lattice Eur. Phys. J. C 79 22

[13] Boyle P A et al 2016 Low energy constants of SU(2) partially quenched chiral perturbation theory from \( N_f = 2 + 1 \) domain wall QCD Phys. Rev. D 93 054502

[14] Helmes C(ETM) et al 2015 Hadron-hadron interactions from \( N_f = 2 + 1 + 1 \) lattice QCD: isospin-2 \( \pi \pi \) scattering length J. High Energy Phys. 09 109

[15] Kalmaahalli Guruswamy C, Meißner U-G and Seng C Y 2020 Contraction diagram analysis in pion-kaon scattering Nucl. Phys. B 957 115091

[16] Helmes C et al (ETM) 2018 Hadron–Hadron interactions from \( N_f = 2 + 1 + 1 \) lattice QCD: \( I = 3/2 \) \( \pi K \) scattering length Phys. Rev. D 98 114511

[17] Lüscher M 1986 Volume dependence of the energy spectrum in massive quantum field theories. 2. scattering states Commun. Math. Phys. 105 153–88

[18] Pitler F 2020 private communication

[19] Dudek J J, Edwards R G and Thomas C E 2012 S and D-wave phase shifts in isospin-2 \( \pi \pi \) scattering from lattice QCD Phys. Rev. D 86 034031

[20] Bijnens J and Ecker G 2014 Mesonic low-energy constants Ann. Rev. Nucl. Part. Sci. 64 149–74