On the Riesz Basisness of Systems Composed of Root Functions of Periodic Boundary Value Problems

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Abstract In this paper, we consider the nonself-adjoint Sturm-Liouville operator with \( q \in L_1[0, 1] \) and either periodic, or anti-periodic boundary conditions. We obtain necessary and sufficient conditions for systems of root functions of these operators to be a Riesz basis in \( L_2[0, 1] \) in terms of the Fourier coefficients of \( q \).

Keywords periodic Sturm-Liouville problem · Riesz basis · Jordan chain · simple eigenvalues

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1 Introduction

Let \( L \) be Sturm-Liouville operator generated in \( L_2[0, 1] \) by the expression

\[
y'' + (\lambda - q)y = 0, \quad (1)
\]
either with the periodic boundary conditions

\[
y(1) = y(0), \quad y'(1) = y'(0), \quad (2)
\]
or with the anti-periodic boundary conditions

\[
y(1) = -y(0), \quad y'(1) = -y'(0). \quad (3)
\]

where \( q \) is a complex-valued summable function on \([0, 1]\). We will consider only the periodic problem. The anti-periodic problem is completely similar. The operator \( L \) is regular, but not strongly regular. It is well known \([6,14]\) that
the system of root functions of an ordinary differential operator with strongly
regular boundary conditions forms a Riesz basis in $L_2[0, 1]$. Generally, the
normalized eigenfunctions and associated functions, that is, the root functions
of the operator with only regular boundary conditions do not form a Riesz
basis. Nevertheless, Shkalikov [16,17] showed that the system of root functions
of an ordinary differential operator with regular boundary conditions forms a
basis with parentheses. In [9], they proved that under the conditions

$$q(1) \neq q(0), \quad q \in C^{(4)}[0, 1]$$

(4)

the system of root functions of $L$ forms a Riesz basis of $L_2[0, 1]$. A new approach
in terms of the Fourier coefficients of $q$ is due to Dernek and Veliev [1]. They
proved that if the following conditions

$$q_{2m} \sim q_{-2m}, \quad \lim_{m \to \infty} \frac{\ln|m|}{mq_{2m}} = 0,$$

(5)

hold, then the root functions of $L$ form a Riesz basis in $L_2[0, 1]$, where

$$q_m =: (q, e^{i2m\pi x}) =: \int_0^1 q(x) e^{-i2m\pi x} dx$$

is the Fourier coefficient of $q$ and without loss of generality we always suppose
that $q_0 = 0$ and the notation $a_m \sim b_m$ means that there exist constants $c_1,
c_2$ such that $0 < c_1 < c_2$ and $c_1 < |a_m/b_m| < c_2$ for all large $m$. Makin [12]
extended this result as follows:

Let the first condition (5) hold. But the second condition (5) is replaced by
a less restrictive one: $q \in W_1^1[0, 1]$. They proved that if the following conditions

$$|q_{2m}| > \varepsilon m^{-s-1} \quad \text{or} \quad |q_{-2m}| > \varepsilon m^{-s-1} \quad \text{for all large } m$$

(7)

with some $\varepsilon > 0$. Then a normal system of root functions of the operator $L$
forms a Riesz basis if and only if $q_{2m} \sim q_{-2m}$. 

Let $p \geq 0$ be an arbitrary integer, $q \in W_p^1[0, 1]$ and (6) holds with some
$s \leq p$, and let one of the following conditions hold:

$$|q_{2m}| > \varepsilon m^{-s-1} \quad \text{or} \quad |q_{-2m}| > \varepsilon m^{-s-1} \quad \text{for all large } m$$

(7)

with some $\varepsilon > 0$. Then a normal system of root functions of the operator $L$
forms a Riesz basis if and only if $q_{2m} \sim q_{-2m}$. 

In addition, some conditions which imply that the system of root functions
does not form a Riesz basis of $L_2[0, 1]$ were established in [12] (see also [2,3,4]).
In [11], we proved that the Riesz basis property is valid if the first condition
(4) holds, but the second is replaced by $q \in W_1^1[0, 1]$. The results of Shkalilov
and Veliev [18] are more general and inclusive. The assertions in various forms
concerning the Riesz basis property were proved. One of the basic results in
the paper [18] is the following statement:

Let $p \geq 0$ be an arbitrary integer, $q \in W_p^1[0, 1]$ and (6) holds with some
$s \leq p$, and let one of the following conditions hold:

$$|q_{2m}| > \varepsilon m^{-s-1} \quad \text{or} \quad |q_{-2m}| > \varepsilon m^{-s-1} \quad \text{for all large } m$$

(7)

with some $\varepsilon > 0$. Then a normal system of root functions of the operator $L$
forms a Riesz basis if and only if $q_{2m} \sim q_{-2m}$. 

Here, for large $m$, denote by $\Psi_{m,j}(x)$ for $j = 1, 2$ the normalized eigenfunctions corresponding to the simple eigenvalues $\lambda_{m,j}$. If the multiplicities of these eigenvalues equal to 2, then the root subspace consists either of two eigenfunctions, or of Jordan chains comprising one eigenfunction and one associated function. First, if the multiple eigenvalue $\lambda_{m,1} = \lambda_{m,2}$ has geometric multiplicity 2, we take the normalized eigenfunctions $\Psi_{m,1}(x), \Psi_{m,2}(x)$. Secondly, if there is one eigenfunction $\Psi_{m,1}(x)$ corresponding to the multiple eigenvalue $\lambda_{m,1} = \lambda_{m,2}$, then we take the Jordan chain consisting of a normalized eigenfunction $\Psi_{m,1}(x)$ and corresponding associated function denoted again by $\Psi_{m,2}(x)$ and orthogonal to $\Psi_{m,1}(x)$. Thus the system of root functions obtained in this way will be called a normal system.

Moreover, for the other interesting results about the Riesz basis property of root functions of the periodic and anti-periodic problems, we refer in particular to [5,7,10,13] and [19,20].

In this paper, we prove the following main result:

**Theorem 1** Let $q \in L_1[0, 1]$ be arbitrary complex-valued function and suppose that at least one of the conditions

$$
\lim_{m \to \infty} \rho(m) = 0, \quad \lim_{m \to \infty} \rho(m) = 0
$$

is satisfied, where $\rho(m)$, defined in (30), is a common order of the Fourier coefficients $q_{2m}$ and $q_{-2m}$ of $q$.

Then a normal system of root functions of the operator $L$ forms a Riesz basis if and only if

$$
q_{2m} \sim q_{-2m}.
$$

This form of Theorem 1 is not novel (see, for example, [15]). The novelty is in the term $\rho(m)$ defined in (30) (see also Lemma 2). Indeed, if we take $p = 0$ in the Sobolev space $W^p[0, 1]$ given above in [15], that is, if $q \in L_1[0, 1]$ then the nonnegative integer $s$ in the conditions (7) must be zero and the assertion on the Riesz basis property remains valid with a less restrictive condition (5) instead of (7). For example, let $\rho(m) = o(m^{-1/2})$. If instead of (5) we suppose that at least one of the following conditions holds

$$
|q_{2m}| > \varepsilon m^{-3/2} \text{ or } |q_{-2m}| > \varepsilon m^{-3/2}
$$

for all large $m$ with some $\varepsilon$, then the assertion of Theorem 1 is obvious.

It is well known (see, e.g., [15], Theorem 2 in page 64) that the periodic eigenvalues $\lambda_{m,1}, \lambda_{m,2}$ are located in pairs, satisfying the following asymptotic formula

$$
\lambda_{m,1} = \lambda_{m,2} + O(m^{1/2}) = (2m\pi)^2 + O(m^{1/2}),
$$

for $m \geq N$. Here, by $N \gg 1$, we denote large enough positive integer. From this formula, the pair of the eigenvalues $\{\lambda_{m,1}, \lambda_{m,2}\}$ is close to the number $(2m\pi)^2$ and isolated from the remaining eigenvalues of $L$ by a distance $m$. That is, we have, for $j = 1, 2$,

$$
|\lambda_{m,j} - (2(m-k)\pi)^2| > |k||2m - k| > Cm,
$$

(10)
for all $k \neq 0, 2m$ and $k \in \mathbb{Z}$, where $m \geq N$ and, here and in subsequent relations, $C$ is some positive constant whose exact value is not essential. For the potential $q = 0$ and $m \geq 1$, clearly, the system $\{e^{-i2m\pi x}, e^{i2m\pi x}\}$ is a basis of the eigenspace corresponding to the eigenvalue $(2m\pi)^2$ of the periodic boundary value problems.

Finally, let us state the following relevant theorem which will be used in the proof of Theorem 1.

**Theorem 2** (see [18]) The following assertions are equivalent:

i) a normal system of root functions of the operator $L$ forms a Riesz basis in the space $L_2[0, 1]$;

ii) the number of Jordan chains is finite and the relation

\[ u_{m,j} \sim v_{m,j} \]  

holds for all indices $m$ and $j$ corresponding only to the simple eigenvalues $\lambda_{m,j}$ for $j = 1, 2$, where $u_{m,j}, v_{m,j}$ are the Fourier coefficients defined in (18);

iii) the number of Jordan chains is finite and the relation (11) for either $j = 1$, or $j = 2$ holds.

2 Preliminaries

The following well-known relation will be used to obtain, for large $m$, the asymptotic formulas for periodic eigenvalues $\lambda_{m,j}$ corresponding to the normalized eigenfunctions $\Psi_{m,j}(x)$:

\[ A_{m-k,j}(\Psi_{m,j}, e^{i2(m-k)\pi x}) = (q \Psi_{m,j}, e^{i2(m-k)\pi x}), \]  

where $A_{m-k,j} = \lambda_{m,j} - (2(m-k)\pi)^2$, $j = 1, 2$. From Lemma 1 in [21], we iterate (12) by using the following relations

\[ (q \Psi_{m,j}, e^{i2m\pi x}) = \sum_{m_1 = -\infty}^{\infty} q_{m_1}(\Psi_{m,j}, e^{i2(m-m_1)\pi x}), \]  

\[ |(q \Psi_{m,j}, e^{i2(m-m_1)\pi x})| < 3M, \]  

where for all $m \geq N$, $m_1 \in \mathbb{Z}$ and $j = 1, 2$, where $M = \sup_{m \in \mathbb{Z}} |q_m|$.

Hence, substituting (13) in (12) for $k = 0$ and then isolating the terms with indices $m_1 = 0, 2m$, we deduce, in view of $q_0 = 0$, that

\[ A_{m,j}(\Psi_{m,j}, e^{i2m\pi x}) = q_{2m}(\Psi_{m,j}, e^{-i2m\pi x}) + \sum_{m_1 \neq 0, 2m} q_{m_1}(\Psi_{m,j}, e^{i2(m-m_1)\pi x}). \]  

(15)

First, we use (12) for $k = m_1$ in the right-hand side of (15). Then, considering (13) with the indices $m_2$ and isolating the terms with indices $m_1 + m_2 = 0, 2m$, we get

\[ [A_{m,j} - a_1(\lambda_{m,j})]u_{m,j} = [q_{2m} + b_1(\lambda_{m,j})]v_{m,j} + R_1(m), \]  

(16)
by repeating this procedure once again, and
\[ [A_{m,j} - a_1(\lambda_{m,j}) - a_2(\lambda_{m,j})]u_{m,j} = [q_{2m} + b_1(\lambda_{m,j}) + b_2(\lambda_{m,j})]v_{m,j} + R_2(m), \]
where \( j = 1, 2, \)
\[ u_{m,j} = (\psi_{m,j}, e^{i2m\pi x}), \quad v_{m,j} = (\psi_{m,j}, e^{-i2m\pi x}), \]
\[ a_1(\lambda_{m,j}) = \sum_{m_1} q_{m_1} q_{m_1 - m_1 - 1}, \quad a_2(\lambda_{m,j}) = \sum_{m_1, m_2} q_{m_1} q_{m_2} q_{m_1 - m_2}, \]
\[ b_1(\lambda_{m,j}) = \sum_{m_1} q_{m_1} q_{2m_1 - m_1}, \quad b_2(\lambda_{m,j}) = \sum_{m_1, m_2} q_{m_1} q_{m_2} q_{2m_1 - m_2}. \]
\[ R_1(m) = \sum_{m_1, m_2} q_{m_1} q_{m_2} (\psi_{m,j}, e^{i2(m_1 - m_2)\pi x}), \]
\[ R_2(m) = \sum_{m_1, m_2, m_3} q_{m_1} q_{m_2} q_{m_3} (\psi_{m,j}, e^{i2(m_1 - m_2 - m_3)\pi x}), \]
\( m_j \neq 0, \forall i; \sum_{i=1}^{k} m_i \neq 0, 2m, \quad \forall k = 1, 2, 3. \]

Using (10), (14) and the relation
\[ \sum_{m_1 \neq 0, 2m} \frac{1}{|m_1||2m - m_1|} = O\left( \frac{\ln|m|}{m} \right) \]
one can prove the estimates
\[ R_i(m) = O\left( \frac{\ln|m|}{m} \right)^{i+1}, \quad i = 1, 2. \]

In the same way, by using the eigenfunction \( e^{-i2m\pi x} \) of the operator \( L \) for \( q = 0 \), we can obtain the relations
\[ [A_{m,j} - a_1'(\lambda_{m,j}) - a_2'(\lambda_{m,j})]v_{m,j} = [q_{2m} + b_1'(\lambda_{m,j}) + b_2'(\lambda_{m,j})]u_{m,j} + R_2'(m), \]
where
\[ a_1'(\lambda_{m,j}) = \sum_{m_1} q_{m_1} q_{m_1 - m_1}, \quad a_2'(\lambda_{m,j}) = \sum_{m_1, m_2} q_{m_1} q_{m_2} q_{m_1 - m_2}, \]
\[ b_1'(\lambda_{m,j}) = \sum_{m_1} q_{m_1} q_{2m_1 - m_1}, \quad b_2'(\lambda_{m,j}) = \sum_{m_1, m_2} q_{m_1} q_{m_2} q_{2m_1 - m_2}, \]
\[ R_1'(m) = \sum_{m_1, m_2} q_{m_1} q_{m_2} (\psi_{m,j}, e^{i2(m_1 + m_2)\pi x}). \]
\[ R'_2(m) = \sum_{m_1, m_2, m_3} \frac{q_{m_1} q_{m_2} q_{m_3} (q_{\Psi_{m,j}} e^{i2(m+m_1+m_2+m_3)\pi x})}{A_{m+m_1,j} A_{m+m_2,m_2,j} A_{m+m_1+m_2+m_3,j}}, \quad (27) \]

\[ m_i \neq 0, \quad \sum_{i=1}^{k} m_i \neq 0, -2m, \quad \forall k = 1, 2, 3. \]

Here the similar estimates as in \((23)\) are valid for \(R'_i(m), i = 1, 2, \).

In addition, by using \((10), (12)\) and \((14)\), we get

\[ \sum_{k \in \mathbb{Z}, k \neq \pm m} \left| (\Psi_{m,j}, e^{i2k\pi x}) \right|^2 = O\left(\frac{1}{m^2}\right). \]

Thus, we obtain that the normalized eigenfunctions \(\Psi_{m,j}(x)\) by the basis \(\{e^{i2k\pi x} : k \in \mathbb{Z}\}\) on \([0, 1]\) has the following expansion

\[ \Psi_{m,j}(x) = u_{m,j} e^{i2m\pi x} + v_{m,j} e^{-i2m\pi x} + h_m(x), \quad (28) \]

where

\[ (h_m, e^{i2m\pi x}) = 0, \quad \|h_m(x)\| = O(m^{-1}), \]

\[ |u_{m,j}|^2 + |v_{m,j}|^2 = 1 + O\left(m^{-2}\right). \quad (29) \]

Now, let us consider the following form of the Riemann-Lebesgue lemma. By this we set

\[ \rho(m) := \max \left\{ \sup_{0 \leq x \leq 1} \left| \int_0^x q(t) e^{-i2(2m)\pi t} dt \right|, \sup_{0 \leq x \leq 1} \left| \int_0^x q(t) e^{i2(2m)\pi t} dt \right| \right\}, \quad (30) \]

and clearly \(\rho(m) \to 0\) as \(m \to \infty\). As the proof of lemma is similar to that of Lemma 6 in \([8]\), we pass to the proof.

**Lemma 1** If \(q \in L^1[0, 1]\) then \(\int_0^x q(t) e^{i2m\pi t} dt \to 0\) as \(|m| \to \infty\) uniformly in \(x\).

**3 Main results**

To prove the main results of the paper we need the following lemmas.

**Lemma 2** The eigenvalues \(\lambda_{m,j}\) of the operator \(L\) for \(m \geq N\) and \(j = 1, 2,\)

\[ \lambda_{m,j} = (2m\pi)^2 + O(\rho(m)), \quad (31) \]

where \(\rho(m)\) is defined in \((30)\).
Proof For the proof we have to estimate the terms of (16) and (24). It is easily seen that
\[
\sum_{m_1 \neq 0, \pm 2m} \left| \frac{1}{A_{m_1}} - \frac{1}{A_0} \right| = O\left( \frac{A_{m,j}}{m^2} \right),
\]
where \(A_0 = (2m\pi)^2 - (2(m \mp m_1)\pi)^2\). Thus, we get
\[
a_1(\lambda_{m,j}) = \frac{1}{4\pi^2} \sum_{m_1 \neq 0, 2m} \frac{q_m q_{-m_1}}{m_1 (2m - m_1)} + O\left( \frac{A_{m,j}}{m^2} \right).
\]
From the argument in Lemma 2(a) of [20] we deduce, with our notations,
\[
a_1(\lambda_{m,j}) = \frac{1}{2\pi^2} \sum_{m_1 > 0, m_1 \neq 2m} \frac{q_m q_{-m_1}}{(2m + m_1)(2m - m_1)} + O\left( \frac{A_{m,j}}{m^2} \right)
\]
where
\[
G(x, m) = \int_0^x \frac{q(t)}{2\pi m_1} e^{i2(2m \pi t)} dt - q_2m^2,
\]
\[
G_{m_1}(m) = (G(x, m), e^{i2m_1 \pi x}) = \frac{q_{2m + m_1}}{2\pi m_1}
\]
for \(m_1 \neq 0\) and
\[
G(x, m) - G_0(m) = \sum_{m_1 \neq 2m} \frac{q_{m_1}}{2\pi (m_1 - 2m)} e^{i2(m_1 - 2m)\pi x}.
\]
Thus, from the equalities
\[
G(x, m) - G_0(m) = O(\rho(m)), \quad G(1, m) = G(0, m) = 0
\]
(see (30) and (34)) and since \(q \in L^1[0, a]\), integration by parts gives for the integral in (33) the estimate
\[
a_1(\lambda_{m,j}) = O\left( \frac{\rho(m)}{m} \right) + O\left( \frac{A_{m,j}}{m^2} \right)
\]
for large \(m\). It is easily seen by substituting \(m_1 = -k\) into the relation for \(a_1'(\lambda_{m,j})\) (see (21)) that
\[
a_1(\lambda_{m,j}) = a_1'(\lambda_{m,j}).
\]
In a similar way, by \(32\), etc., we get
\[
b_1(\lambda_{m,j}) = \frac{1}{4\pi^2} \sum_{m_1 \neq 0, 2m} \frac{q_{m_1} q_{2m - m_1}}{m_1 (2m - m_1)} + O\left( \frac{A_{m,j}}{m^2} \right)
\]
\[
= - \int_0^1 (Q(x) - Q_0)^2 e^{-i2(2m\pi x)} dx + O\left( \frac{A_{m,j}}{m^2} \right)
\]
where
\[
G(x, m) = \int_0^x q(t) e^{i2(2m \pi t)} dt - q_2m^2,
\]
\[
G_{m_1}(m) = (G(x, m), e^{i2m_1 \pi x}) = \frac{q_{2m + m_1}}{2\pi m_1}
\]
for \(m_1 \neq 0\) and
\[
G(x, m) - G_0(m) = \sum_{m_1 \neq 2m} \frac{q_{m_1}}{2\pi (m_1 - 2m)} e^{i2(m_1 - 2m)\pi x}.
\]
\[ \frac{-1}{i2\pi(2m)} \int_0^1 2(Q(x) - Q_0) q(x) e^{-i2(2m)\pi x} dx + O\left(\frac{A_{m,j}}{m^2}\right), \quad (39) \]

where \( Q(x) = \int_0^x q(t) dt, \quad Q_{m_1} = (Q(x), e^{i2m_1\pi x}) = \frac{q_{m_1}}{i2\pi m_1} \) if \( m_1 \neq 0, \)

\[ Q(x) - Q_0 = \sum_{m_1 \neq 0} Q_{m_1} e^{i2m_1\pi x}. \quad (40) \]

Thus, by using \( Q(1) = q_0 = 0 \) and \( (30), \) integration by parts again gives for the integral in \( (39) \) the following estimate

\[ b_1(\lambda_{m,j}) = O\left(\frac{\rho(m)}{m}\right) + O\left(\frac{A_{m,j}}{m^2}\right). \quad (41) \]

Similarly

\[ b_1'(\lambda_{m,j}) = O\left(\frac{\rho(m)}{m}\right) + O\left(\frac{A_{m,j}}{m^2}\right). \quad (42) \]

To estimate \( R_1(m) = o(\rho(m)) \) (see \( (20) \)), let us show that

\[ \rho(m) > C m^{-1} \quad (43) \]

for \( m \geq N \) and some \( C > 0. \) Since \( q(x) \neq 0 \) is summable function on \([0, 1],\) there exists \( x \in [0, 1] \) such that

\[ \int_0^x q(t) dt \neq 0 \quad (44) \]

and the integral \( (44) \) is bounded for all \( x \in [0, 1]. \) Hence, multiplying the integrand of \( (44) \) by \( e^{-i2(2m)\pi x} e^{i2(2m)\pi x}, \) and then using integration by parts, we get

\[ \sup_{0 \leq x \leq 1} \left| \int_0^x q(t) dt \right| \leq C (\rho(m) + mp(m)) \leq C mp(m) \]

which implies \( (43). \)

Thus by \( (10), (14) \) and relation \( (22), \) we deduce that

\[ |R_1(m)| \leq C \left(\frac{\ln|m|}{m^2}\right)^2 = o(\rho(m)). \]

Also \( R_1'(m) = o(\rho(m)). \)

From the relation \( (29), \) for large \( m, \) it follows that either \( |u_{m,j}| > 1/2 \) or \( |v_{m,j}| > 1/2. \) We first consider the case when \( |u_{m,j}| > 1/2. \) Hence, by using \( (10), (37) \) and \( (11) \) with \( R_1(m) = o(\rho(m)) \) we obtain

\[ A_{m,j}(1 + O(m^{-2})) = q_{2m} \frac{v_{m,j}}{u_{m,j}} + o(\rho(m)). \]

This with the definition \( (30) \) gives \( A_{m,j} = O(\rho(m)). \) Similarly, for the other case \( |v_{m,j}| > 1/2, \) by using \( (24), (37), (12) \) and \( R_1'(m) = o(\rho(m)), \) we get \( (31). \)

The lemma is proved. \( \Box \)
Lemma 3 For all large \( m \), we have the following estimates (see, respectively, (19), (20) and (21), (27))

\[ b_2(\lambda_{m,j}) \leq R_2(m) = O \left( \rho(m)m^{-2} \right) \]

Proof Let us estimate the sum \( R_2(m) \). By using the estimate (23) and the inequality (13) for large \( m \), we deduce that

\[ |R_2(m)| \leq C \left( \frac{\ln|m|}{m^3} \right) = O \left( \rho(m)m^{-1} \right). \]

In the same way \( R'_2(m) = O \left( \rho(m)m^{-1} \right) \).

Arguing as in (18) (see the proof of Lemma 6), let us now estimate the sum \( b_2(\lambda_{m,j}) \). Taking into account (42) and Lemma 2 we have

\[ b_2(\lambda_{m,j}) = \frac{1}{(2\pi)^4} I(m) + O \left( \frac{\rho(m)}{m^3} \right), \]

where

\[ I(m) = \sum_{m_1, m_2} \frac{q_{m_1} q_{m_2} q_{2m-m_1-m_2}}{m_1(2m-m_1)(m_1+m_2)(2m-m_1-m_2)}. \]

By using the identity

\[ \frac{1}{k(2m-k)} = \frac{1}{2m} \left( \frac{1}{k} + \frac{1}{2m-k} \right), \]

and the substitutions \( k_1 = m_1, k_2 = 2m-m_1-m_2 \) in the formula \( I(m) \), we obtain \( I(m) \) with the indices \( m_1, m_2 \) in the following form

\[ I(m) = \frac{1}{(2m)^2} (I_1 + 2I_2 + I_3), \]

where

\[ I_1 = \sum_{m_1, m_2} \frac{q_{m_1} q_{m_2} q_{2m-m_1-m_2}}{m_1 m_2}, \quad I_2 = \sum_{m_1, m_2} \frac{q_{m_1} q_{m_2} q_{2m-m_1-m_2}}{m_2(2m-m_1)}, \quad I_3 = \sum_{m_1, m_2} \frac{q_{m_1} q_{m_2} q_{2m-m_1-m_2}}{(2m-m_1)(2m-m_2)}. \]

From (43) - (46), (49), \( 2I_2(m) = I_1(m) \) and using integration by parts only in \( I_1 \), we obtain the following estimates

\[ \begin{aligned}
   I_1 &= -4\pi^2 \int_0^1 (Q(x) - Q_0)^2 q(x) e^{-i2(2m)\pi x} dx = O \left( \rho(m) \right), \\
   I_3 &= -4\pi^2 \int_0^1 (G(x,m) - G_0(m))^2 q(x) e^{i2(2m)\pi x} dx = O \left( \rho(m) \right).
\end{aligned} \]

Then, in view of (47) and (48), \( I(m) = O \left( \rho(m)m^{-2} \right) \). This with the equality (45) implies that \( b_2(\lambda_{m,j}) = O \left( \rho(m)m^{-2} \right). \) In the same way \( b'_2(\lambda_{m,j}) \) satisfies the same estimate. The lemma is proved. \( \square \)
Thus by using Lemma \textsuperscript{2} and an argument similar to that of Theorem 2 in \cite{18} under the conditions \textsuperscript{8}, let us prove the following main result.

**Proof of Theorem \textsuperscript{1}**

In view of Lemma \textsuperscript{2}, substituting the values of

\[ b_1(\lambda_{m,j}), b'_1(\lambda_{m,j}) = O(\rho(m)m^{-1}) , \]
\[ b_2(\lambda_{m,j}), b'_2(\lambda_{m,j})R_2(m), R'_2(m) = O(\rho(m)m^{-2}) \]
given by \textsuperscript{11}, \textsuperscript{12}, \textsuperscript{15} in the relations \textsuperscript{17} and \textsuperscript{25}, we get the following reversion of the relations

\[ [A_{m,j} - a_1(\lambda_{m,j}) - a_2(\lambda_{m,j})] u_{m,j} = [q_{2m} + O(\rho(m)m^{-1})] v_{m,j} + O(\rho(m)m^{-2}) , \]
\[ [A_{m,j} - a'_1(\lambda_{m,j}) - a'_2(\lambda_{m,j})] v_{m,j} = [q_{-2m} + O(\rho(m)m^{-1})] u_{m,j} + O(\rho(m)m^{-2}) \]

for \( j = 1, 2 \).

It is easily seen again by substituting \( m_1 + m_2 = -k_1, m_2 = k_2 \) in the sum \( a'_2(\lambda_{m,j}) \) (see \textsuperscript{24}) and using \textsuperscript{35} that \( a_i(\lambda_{m,j}) = a'_i(\lambda_{m,j}) \) for \( i = 1, 2 \).

Hence, multiplying \textsuperscript{49} by \( v_{m,j} \) and \textsuperscript{50} by \( u_{m,j} \) and subtracting we obtain the following equality

\[ q_{2m} v_{m,j}^2 - q_{-2m} u_{m,j}^2 = O(\rho(m)m^{-1}) . \]

Suppose, for example, that \( q_{2m} \ satisfies the condition in \textsuperscript{8}. Then using this equality we get

\[ v_{m,j}^2 - \kappa_m u_{m,j}^2 = o(1), \quad \kappa_m = \frac{q_{-2m}}{q_{2m}} \]

for \( j = 1, 2 \). In addition, for large \( m \), the condition \textsuperscript{8} for \( q_{2m} \ implies that the geometric multiplicity of the eigenvalue \( \lambda_{m,j} \) is 1. Arguing as in Lemma 4 of \cite{18}, if there exist mutually orthogonal two eigenfunctions \( \Psi_{m,j}(x) \) corresponding to \( \lambda_{m,1} = \lambda_{m,2} \), then one can choose an eigenfunction \( \Psi_{m,j}(x) \) such that \( u_{m,j} = 0 \). Thus combining this with \textsuperscript{29} and \textsuperscript{51}, we get \( q_{2m} = O(\rho(m)m^{-1}) \) which contradicts \textsuperscript{8}.

Let the normal system of root functions form a Riesz basis. To prove \( \kappa_m \sim 1, \) from \textsuperscript{52} it is enough to show that all the large periodic eigenvalues \( \lambda_{m,j} \) are simple, since in this case we have, by Theorem \textsuperscript{2}

\[ u_{m,j} \sim v_{m,j} \sim 1 \]

for \( j = 1, 2 \). For large \( m \), again by Theorem \textsuperscript{2} and the condition \textsuperscript{8} for \( q_{2m} \), respectively, the number of Jordan chains and the eigenvalues of geometric multiplicity 2 is finite, that is, all large eigenvalues are simple.

Now let \( q_{2m} \sim q_{-2m} \). From the second formula of \textsuperscript{52}, we obtain that \( \kappa_m \sim 1 \) and then, from the first, that \textsuperscript{53} for the eigenfunction \( \Psi_{m,1}(x) \), that is, \( j = 1 \) which implies that the number of Jordan chains is finite. In
fact, if there exists a Jordan chain consists of an eigenfunction $\Psi_{m,1}(x)$ and an associated function $\Psi_{m,2}(x)$ corresponding to the eigenvalue $\lambda_{m,1} = \lambda_{m,2}$, then, for example for $\lambda_{m,1}$, using the eigenfunction $\Psi_{m,1}(x)$ of the adjoint operator $L^*$ and the relation

$$(L - \lambda_{m,1})\Psi_{m,2}(x) = \Psi_{m,1}(x),$$

we obtain that $(\Psi_{m,1}, \Psi_{m,1}) = 0$. Thus, from the expansion (28) for $j = 1$, we get $u_{m,1}v_{m,1} = O(m^{-2})$ which contradicts (53) for $j = 1$. Thus, using Theorem 2 we prove that a normal system of root functions of the operator $L$ forms a Riesz basis. 

Arguing as in the proof of Theorem 1, we obtain a similar result established below for the anti-periodic problems.

**Theorem 3** Let $q \in L_1[0,1]$ be arbitrary complex-valued function and suppose that at least one of the conditions

$$\lim_{m \to \infty} m q_{2m+1}^{\rho(m)} = 0, \quad \lim_{m \to \infty} m q_{-2m-1}^{\rho(m)} = 0$$

is satisfied, where $\rho(m)$ is obtained from (30) by replacing $2m$ with $2m+1$ and a common order of both Fourier coefficients $q_{2m+1}$ and $q_{-2m-1}$ of $q$.

Then a normal system of root functions of the operator $L$ with anti-periodic boundary conditions forms a Riesz basis if and only if $q_{2m+1} \sim q_{-2m-1}$.

**Remark 1** Clearly if instead of (3) we assume that at least one of the conditions

$$\rho(m) \sim q_{2m}, \quad \rho(m) \sim q_{-2m}$$

holds, then the assertion of Theorem 1 is satisfied. In this way one can easily write a similar result for the anti-periodic problem.

In addition to all the above results, we note that if either the first condition of (3) and (4), or the second condition of (3) and (4) hold then all the periodic eigenvalues are asymptotically simple. We can write a similar result for the anti-periodic problem.

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