Blowing up paraxial beams, vortex beams, and cylindrical beams of light with finite power in linear media, and their enhanced longitudinal field

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We present paraxial light beams with cylindrical symmetry, smooth and localized transversal profile carrying finite power, that develop intensity singularities when they are focused in a linear medium, such as vacuum. They include beams with orbital angular momentum and with radial polarization, in which case they develop punctual phase and polarization singularities surrounded by infinitely bright rings, and singular longitudinal component. In practice, these effects are manifested in focal intensities and spot sizes, vortex bright ring intensities and radii, and strengths of the longitudinal component of the electric field, that strongly increase with, and hence can be controlled by, the lens aperture radius, without changing the illuminating beam and collected power, and while keeping paraxial focusing conditions.

I. INTRODUCTION

Inspired by what happens to some wave functions in quantum mechanics [1], Aiello has recently introduced a class of paraxial light beam with a localized transversal profile and finite power that develops a singularity when its is ideally focused [2]. Real, apertured versions of these beams have subsequently been demonstrated in experiments [2]. These beams, even if apertured, outperform standard Gaussian beams of similar intensity and power in terms of focal intensity and resolution. In a sense, these beams are able to reproduce with a finite amount of power the focusing behavior of plane waves, though their finite-aperture realizations do not exceed the diffraction limit [2]. The electric field of these beams is factorized in the coordinates x and y in the transversal plane, i.e., it has a rectangular geometry, which greatly facilitates its analytical treatment.

In many experimental settings, however, cylindrical symmetry is advisable, if not mandatory; in particular, lenses, whose aperture plays a crucial role in the behavior of these beams, are most often circular. In this paper we describe cylindrical symmetric beams with finite power and similar blowing up behavior in their intensity when they are focused. Considering beams with cylindrical geometry, we can also examine other types of beams that are of great interest today. We describe blowing up beams with orbital angular momentum, and blowing up radially polarized beams, both with finite power. They form a punctual dark vortex surrounded by an infinitely bright ring, accompanied by an infinitely strong longitudinal component.

Their apertured versions approach the above ideal behavior as the lens aperture is increased, which offers a practical way to control these properties without changing the illuminating beam, including the collected power, e.g., to enhance the amplitude of the axial component up to 66 percent of that of the transverse component under paraxial conditions. These blowing up beams, vortex beams and radially polarized beams with large, but paraxial apertures, greatly outclass the performance of standard illuminating fields such as Gaussian or Laguerre-Gaussian beams of similar power and intensity. As their Cartesian counterparts, blowing up beams of finite aperture do not beat the diffraction limit for plane wave illumination, but approach it closely, whose practical advantages versus uniform illumination is discussed at the end of this paper.

Of course, the blowing up behavior of these light beams has nothing to do with blowing up light beams in self-focusing nonlinear media [4,5]. Yet, they are somewhat similar in that they both appear in paraxial fields, are dissolved when more precise models are considered, and their mathematical existence has physical repercussions. The results presented here can straightforwardly extended to waves in other areas of physics such as probability waves in quantum mechanics, or mechanical waves in acoustics, and to matter waves and electron beams.

II. IDEAL BLOWING UP BEAMS AND VORTEX BEAMS

In the Fresnel regime, the focused electric field $\psi(x, y, z) e^{i(kz - \omega t)}$ of frequency $\omega$ and propagation constant $k$ (e.g., $k = \omega/c$ in vacuum), and the field $\psi(x, y, 0) e^{-i\omega t}$ at the output plane of a thin lens of focal length $f$, are related by

$$
\psi(x, y, z) = \frac{k}{2\pi i z} \int dx'y' \psi(x', y', 0) e^{\frac{ik}{2f}((x-x')^2 + (y-y')^2)},
$$

(1)

where $\psi(x, y, 0) = \psi(x, y)e^{-ik(x^2 + y^2)/2f}$, $\psi(x, y)$ is the illuminating field at the input plane of the lens, and the integral extends over the aperture of the lens. We are interested in an illuminating field of the form $\psi(x, y) = \psi(r)e^{i\varphi}$, where $(r, \varphi)$ are polar coordinates in the transversal plane, and $s = 0, \pm 1, \pm 2, \ldots$ i.e., in light beams with revolution symmetry about the propagation axis without and with orbital angular momentum associated with vortex of topological charge $s$. For these fields, Fresnel integral in [1] can more conveniently be written
as
\[
\psi(r, z) e^{i\varphi} = \frac{k e^{i\varphi}}{i |s| + 1} e^{i\frac{\sigma r^2}{2}} \int_0^R dr' r' \psi(r')
\times e^{-i \frac{\sigma r'^2}{2} + \frac{k}{z} r'} J_n\left(\frac{k}{z} r'\right),
\]
(2)
where \(J_n(\cdot)\) is the Bessel function of the first kind and order \(n\), and \(R\) is the radius of the lens aperture.

We consider the illuminating, collimated field with transversal distribution of amplitude
\[
\psi(r) = \sqrt{\frac{P}{A}} \frac{(r/\sigma)|s|}{(1 + r^2/\sigma^2)^{\mu+1}},
\]
(3)
where the length \(\sigma\) scales the field transversally, and \(P\) and \(A\) are constants to be conveniently fixed. With the real number \(\mu\) satisfying \(\mu > (|s| - 1)/2\), the power of the illuminating field in the entire transversal plane, \(2\pi \int_0^\infty dr |\psi(r)|^2\), is finite, and therefore this field is in principle physically realizable. With the choice \(A = \pi \sigma \Gamma(|s| + 1) \Gamma(2\mu + 1 - |s|)/\Gamma(2\mu + 2)\), where \(\Gamma(\cdot)\) is the Gamma function, the beam power is just the constant \(P\) appearing in (3). Examples of (3) with \(s = 0\) and \(|s| = 1\) are depicted as solid curves in Figs. 1(a) and 2(a).

At the focal plane, \(z = f\), the integral in (2) with (3) can be carried out analytically (see 6.565.4 in Ref. [6]) when neglecting the finiteness of the lens aperture, i.e., with \(R = \infty\):
\[
\psi(r, f) e^{i\varphi} = \sqrt{\frac{P}{A}} \frac{k \sigma^2}{i |s| + 1} f 2^{\mu} \Gamma(\mu + 1) \times e^{i\frac{\sigma r^2}{2}} (\frac{k}{\sigma} r)^\mu K_{|s| - \nu} (\frac{k}{\sigma} r) e^{i\varphi},
\]
(4)
for \(r > 0\), where \(K_{\nu}(\cdot)\) is the modified Bessel function of the second kind and order \(\nu\), \(f\). Using that \(K_{-\nu}(s) = K_{\nu}(s)\), and the asymptotic behavior \(K_{\nu}(\alpha) \simeq (1/2\nu) \Gamma(\nu) (\alpha/2)^{-\nu}\) for \(\nu > 0\) for small argument \(\alpha\), (4) is seen to approach infinity for \(r \to 0\) when \(\mu < |s|/2\). For these values of \(\mu\), the vortex-less beam (s = 0) reaches infinitely large amplitude at the beam center \(r = 0\), and the vortex beam \((s \neq 0)\) vanishes at \(r = 0\), since \(J_{\nu}(0) = 0\) for \(s \neq 0\), i.e., has an punctual vortex surrounded by an infinitely bright ring. In short, the unapertured focused field of illumination in (3) carries finite power and produces infinitely intense field approaching the focal point if
\[
\frac{|s| - 1}{2} < \mu < \frac{|s|}{2},
\]
(5)
and this unbounded field surrounds an infinitely narrow, dark vortex when \(s \neq 0\). The singularity originates from the small but persistent tails of (3) at large radius that are superposed with uniform phases at the focal point, whose contribution to the beam power is nevertheless finite and small.

Figures 1(a) and (b) compare the blowing up illuminating field in (3) at \(z = 0\) and blown up field in (4) at \(z = f\) for \(s = 0\) and \(\mu = -0.25\) (solid curves) with illuminating Gaussian beam of the same power and peak intensity and its focused field (dashed curves). Figures 1(c), (d) and (e) show peak intensities at \(r = 0\) versus \(z\) and radial profiles at values of \(z\) close to \(f\) illustrating how the beam profile approaches the blown up profile in (4), compared to the same quantities for the Gaussian illumination (dashed curves). The singularity is only formed at the focal plane because the fast oscillations \(e^{-i \frac{\sigma r'^2}{2} + \frac{k}{z} r'}\) in (2) out of focus makes the integral to converge to finite values. Compared to the Gaussian illumination, the singularity develops explosively in a tiny axial region of the standard depth of focus [vertical lines in (c)].

Thus, as already pointed out in 2 and 3, the blow-
Intensities at the unapertured blown up field at the focal plane numerically evaluated from (2) with (3), versus $z$ for Laguerre-Gauss illumination with $s = 1$ and zero radial order, having the same power and peak intensity, and its focused field (dashed curves). The peak intensity at the radius of the bright ring surrounding the vortex grows up explosively with $z$ to infinity at $f$, as seen in Fig. 2(c) (solid black curve), and the same time that the bright ring shrinks acceleratedly down to zero to form a punctual vortex at $f$, as also seen in Fig. 2(c) (solid black curve). For comparison, Fig. 2(c) also shows the peak intensity and radius of the bright ring for the Laguerre-Gauss illumination (dashed curves). The ripples in the explosively growing and shrinking radial profiles at planes close to the focal plane, observed in Fig. 2(d) and (e), disappear in the smoother, blown up profile at the focal plane.

III. OBSERVABLE EFFECTS OF BLOWING UP BEHAVIOR

The above description is an ideal behavior that originates from ignoring the finite aperture of focusing systems, and that disappears when finite transversal extents are considered. Still, the existence of these blowing up beams in a linear medium has observable effects in experiments that are absent with standard illuminating beams. If a Gaussian or any other standard light beam carrying finite power is focused, the width of the focal spot and its intensity do not change once the lens aperture is appreciably larger than the beam spot size; by contrast, focusing the blowing up field in (3) produces, in principle, a brighter and narrower focal spot as the aperture radius $R$ of the focusing system is opened. This behavior also mimics what happens when a plane wave illuminates the lens of increasing radius $R$, with the important difference that the focused power does not change significantly when opening the aperture (once $R \gg \sigma$), since the power in the beam tails is increasingly negligible. In addition, focusing the blowing up vortex fields in (3) produces increasingly brighter and narrower ring surrounding the vortex at the focal plane when opening the aperture without significantly changing the beam power. These effects are described below under conditions of paraxial focusing, conditions under which they are supposed to be physically valid.

For beams without orbital angular momentum ($s = 0$, hence $-1/2 < \mu < 0$), the intensity at the focal point $(r, z) = (0, f)$ with a lens of aperture radius $R$ can easily be evaluated from (2) with (3) to be

$$|\psi(0, f)|^2 = \frac{P}{A} \left( \frac{k R^2}{2 f |\mu|} \right)^2 \left[ 1 - (1 + \rho^2 )^{1/2} \right]^2 ,$$

where $\rho = R/\sigma$, which is seen in Fig. 3(a) to grow without bound with $R$ for any allowed value of $\mu$ even when the aperture radius $R$ is much larger than the beam spot size of (3) on the lens, measured by $\sigma$. The curves end...
when the angle of a marginal ray as seen from the focus is 30°, taken, according to Siegman [7], as a limit of validity of the paraxial approximation, i. e., for a numerical aperture of NA = 0.5. Evaluation of the derivative of (6) with respect to μ reveals more pronounced intensity enhancements (at large values R/σ) when using values of μ about −0.42, as can also be appreciated in Fig. 3(a). By contrast, the intensity at the focal point of Gaussian illumination approaches a constant value as soon as the aperture radius R is slightly larger than its spot size, as seen in Fig. 3(b). Note the logarithmic vertical scale to visualize the tiny focal intensity of the focused Gaussian illumination of the same power and peak intensity. Intensities two orders of magnitude higher than those attainable with the Gaussian illumination are obtained while keeping paraxial focusing conditions. Also, the diameter of the focal spot decreases as the aperture radius is increased up to the limit imposed by the paraxial approximation, in contrast with the constant and much larger focal diameter with the Gaussian beam of the same power and peak intensity, as seen in Fig. 3(c). Bright spots of area two orders of magnitude smaller than with the Gaussian illumination are obtained. Two examples of focused transversal profiles with numerical apertures 0.5 and 0.25 are compared to the low-intensity and much broader focused Gaussian beam in Fig. 3(d).

Figures 3(c) and (d) include also diffraction-limited focal diameters and transversal profiles (dotted curves) corresponding to plane wave illumination on the lens of radius R. It is clear that focusing the blowing up profile with finite power imitates focusing of a plane wave, and in the sense that increasing the numerical aperture results in tighter focal spots. The peak intensities of the focused transversal profiles with illumination (3) and a plane wave are equated in Fig. 3(d) to visualize that the width of the focal spot is, for any aperture radius, only slightly above the diffraction limit, in contrast to what happens with the Gaussian illumination.

For blowing up beams with orbital angular momentum |s| ≠ 0, hence (|s| − 1)/2 < μ < |s|/2) the intensity of the bright ring at the focal plane grows similarly when opening the aperture, in contrast to what happens to Laguerre-Gauss beams, as seen in Fig. 4(a). As with s = 0, this enhancement is further magnified by using the value μ ≃ (|s| − 1)/2 + 0.08, e. g., the value μ = 0.08 for |s| = 1 chosen in Fig. 4. Again focal intensities about two orders of magnitude higher than with Laguerre-Gaussian illumination of the same power, peak intensity and vorticity on the lens are easily attained. Figure 4(b) shows the shrinking radius of the bright ring at the focal plane when increasing the aperture radius R, in contrast to the constant radius with the Laguerre-Gauss illumination, and mimicking the shrinking diffraction-limited radius for plane wave illumination with a punctual vortex (solid, dashed and dotted curves, respectively). The gray curves of the same type locate the inner and outer radius of the corresponding bright rings, in order to visualize the constant thickness of the bright ring at the focal plane for the Laguerre-Gauss illumination, and the shrinking thicknesses for blowing up and plane wave illuminations. Two examples of transversal profiles at the focal plane with numerical apertures 0.5 and 0.25 are seen in Fig. 4(c), where they are compared to the much broader and less intense focused Laguerre-Gauss, and are also seen to be slightly broader than the diffraction-limited profiles of the same numerical apertures. Interestingly, the blowing up beam focused by a lens of finite radius behaves very approximately as the ideal blowing up beam from an infinite aperture out of the focal plane, as illustrated in Fig. 4(d) plotting the intensity of the maximum of the bright ring and its radius about the focus. Only in a tiny fraction (about one wavelength) of the focal region (of about 0.5 mm) the bright ring intensity and its radius depart from the ideal behavior, reaching at the focus finite and non-zero values, respectively.
FIG. 4. Physical properties of finite-aperture blowing up vortex beams. (a) Focal intensity when blowing up profiles [1] with $|s| = 1$, $\sigma = 1$ mm, power $P = 10$ W, frequency $\omega = 2.5$ fs$^{-1}$ and $\mu = 0.08$ illuminate a lens of focal length $f = 200$ mm and increasing radius $R$ (solid curve) compared to the same quantity for Laguerre-Gauss (LG) illumination of the same power and peak intensity on the lens. (b) Radius of the maximum of the bright ring at the focal plane produced by the blowing profile [1] (solid), by the Laguerre-Gauss profile (dashed), and by an uniform plane wave with a punctual vortex (dotted), as functions of the radius of the lens. The gray curves of the same type represent the inner radius and outer radius (half width at half maximum of the maximum intensity) for the corresponding illuminations. (c) Radial profiles of intensity at the focal plane due to blowing up (solid), Laguerre-Gauss of the same peak intensity and power (dashed), and vortex plane wave (dotted) illuminations for lens radii $R = 51.65$ mm (0.25 NA) and $R = 115.47$ mm (0.5 NA). For the Gaussian illumination there is no change with $R$. (d) Intensity of the bright ring and its radius as a function of propagation distance about the focus for the blowing up beam with radius $R = 115.47$ mm (solid curves) compared to the ideal blowing up beam. The axial region is so short that the same quantities for Laguerre-Gauss beams do not appreciably change (dashed lines).

IV. ENHANCED LONGITUDINAL COMPONENT

The ideally infinite gradients of blowing up beams and vortex beams at the focal plane suggests the presence of a strong longitudinal or axial component of the electric field, even under paraxial conditions, since the axial component is directly related to the gradient of the transversal part. Enhancement of the axial component commonly relies on nonparaxial focusing and the use of specific fields such as radially polarized beams [8, 9]. While paraxial, standard, radially polarized fields present quite small longitudinal components, blowing up radially polarized fields have longitudinal components comparable in amplitude to that of the transversal component.

For a paraxial beam of transversal components $\psi_\perp = \psi_x u_x + \psi_y u_y$, the axial component can be evaluated from

$$\psi_z = \frac{i}{k} \nabla_\perp \cdot \psi_\perp = \frac{i}{k} \left( \frac{\partial \psi_x}{\partial x} + \frac{\partial \psi_y}{\partial y} \right).$$

(7)

In order to maintain the cylindrical symmetry of the intensity of the vectorial field, we consider left or right handed circular polarizations, $\psi_\perp = \psi(r, z)e^{i\sigma \varphi}u_{l,r}$, where $u_{l,r} = (u_x \pm iu_y)/\sqrt{2}$, and the upper and lower signs stand for left and right handed polarizations, respectively, and where $\psi(r, z)$ is given by Fresnel integral in [1] with [10]. Some calculation of derivatives leads to

$$\psi_z(r, z)e^{i(s+1)\varphi} = \frac{1}{\sqrt{2}k} \left[ \frac{\partial \psi(r, z)}{\partial r} + \frac{s}{r} \psi(r, z) \right] e^{i(s+1)\varphi},$$

(8)

With $s = -1$ (hence $0 < \mu < 1/2$) and left handed circular polarization, and with $s = +1$ (also $0 < \mu < 1/2$) and right handed polarization, the axial components do not carry any vorticity, can then have maxima at the beam center (where the transversal components vanish), which are equal in both cases and given by $\psi_z(r, z) = (i/\sqrt{2k})[\partial \psi(r, z)/\partial r + \psi(r, z)/r]$.

We can now easily consider the radially or azimuthally polarized beams

$$\psi_\perp = \frac{\psi(r, z)}{\sqrt{2}} \left[ e^{-i\sigma \varphi}u_l \pm e^{i\sigma \varphi}u_r \right],$$

(9)

having the same amplitude associated with the transversal components, $|\psi_\perp| = |\psi(r, z)|$, as that of the individual vortex beams with circular polarizations. The axial component vanishes for azimuthal polarization [for the minus sign in (4)], and is given by

$$\psi_z = \frac{i}{k} \left[ \frac{\partial \psi(r, z)}{\partial r} + \frac{1}{r} \psi(r, z) \right] = \frac{i}{k} \frac{\partial \psi(r, z)}{\partial r},$$

(10)

for radial polarization [for the plus sign in (4)]. Writing the ideal focal profile in (4) for $|s| = 1$ in the compact form $\psi(r, f) = C\alpha^\mu K_{1-\mu}(\alpha)$, where $\alpha = kr/\ell$ and $C$ is the constant in the first row in (4), and where we have neglected the negligible curvature factor $e^{ikr^2/2f}$ in the Debye approximation (symmetric focused field about the focal plane), one can readily evaluate the axial component in (10) as

$$\psi_z = \frac{i\sigma}{f} \int C\alpha^\mu \left[ K_{2-\mu}(\alpha) - \frac{K_{1-\mu}(\alpha)}{\alpha} \right],$$

(11)

which is singular at $\alpha = 0$, i.e., at $r = 0$. Indeed the ratio of amplitudes of the axial and transversal components close to the origin behaves as $|\psi_z|/|\psi| = 2\sigma\mu/f\alpha =$...


2μ/kr, implying that the singularity in the axial field is stronger than that of the transversal field. Of course, these are mathematical equations for ideal blowing up radially polarized beams, but they have physical manifestations in their apertured versions. From (10), with finite radius \( R \) and illumination \( |s| = 1 \), the amplitude of the axial component of the radially polarized beam at the focal plane can be evaluated to be

\[
|\psi_z(0, f)| = \frac{k^2}{f^2} \sqrt{\frac{P \sigma^3}{A \cdot 2\mu}} \left[ \frac{\rho^2}{(1 + \rho^2)^\mu} + \frac{1 - (1 + \rho^2)^{-\mu}}{1 - \mu} \right],
\]

where again \( \rho = R/\sigma \). This amplitude grows as \((R/\sigma)^{2-2\mu}\) at large \( R/\sigma \), while the amplitude of the transversal component at the bright ring surrounding the polarization singularity grows with the potential law \((R/\sigma)^{1-2\mu}\) (also at large \( R/\sigma \)) of lower power, as can be seen in Fig. 5(a). This difference implies a linear increase with \( R/\sigma \) of the ratio of the axial and transversal amplitudes, as can be appreciated in Fig. 5(b). For illustration Figs. 5(c) and (d) depict transversal profiles of amplitude of the axial and transversal components at the focal plane for NA = 0.25 and 0.5. The above analysis indicates that the individual, circularly polarized blowing up vortex beams have also enhanced axial components, but we have ignored them because they simply are \( \sqrt{2} \) smaller than that of the radially polarized beam.

V. BLOWING UP VERSUS UNIFORM ILLUMINATION

From what is exposed so far, it arises the practical question of the possible advantages of using blowing up illumination over uniform illumination (of finite transversal extent). Of course the response depends on the particular application. Smooth profiles are more convenient in nonlinear optics applications to avoid instabilities and collapse. Reshaping Gaussian-like beam profiles into uniform flat-top profiles involves large power losses if based on truncation; otherwise it requires using sophisticated diffractive and refractive optics [11], or more recently developed techniques [12], but uniform illumination provides better resolution [13] in applications such as microscopy.

Suppose here we dispose of the fixed amount of power \( P \) supplied by a laser system which can be shaped as the blowing up profile or as a flat-top profile, and wish to increase the intensity and/or diminish the size of the focal spot by opening the aperture, or simply to control these properties. As seen in Fig. 3 (a) and (b) the intensity is considerably higher and the spot size slightly smaller (the diffraction limit) with a flat-top profile that always fills the aperture (dotted curves) than with blowing up illumination (solid curves), but this arrangement requires reshaping the flat-top profile each time the aperture radius is increased. The situation is different in a setting where reshaping of the illumination is difficult or not possible (dashed curves). The peak intensity of the flat-top illumination of the power \( P \) and fixed radius \( R_0 \) is at first lower than with the blowing up illumination because of truncation losses, then higher when more power is collected, and finally constant once \( R_0 \) is smaller than the aperture radius \( R \), becoming eventually smaller than the focal intensity with the blowing up illumination.

Similarly, the focal spot is tighter but then constant and wider than with the blowing up illumination of increasing aperture radius. Thus, the blowing up profile offers the possibility of a smoother control of the peak intensity.
and width of the focusing spot over wider ranges without the necessity of reshaping the input beam. Similar conclusion holds for the intensity and thickness of blowing up vortex beams, and for the strength of the axial component of radially polarized blowing up beams.

VI. CONCLUSIONS

In conclusion, we have reported the existence of cylindrically symmetric, paraxial beams, vortex beams, and radially polarized beams of light that produce singular (infinite) intensities when they are ideally focused. They mimic the behavior of ideally focused plane waves but with a localized transversal profile that carries a finite amount of power.

With real focusing systems of finite lateral extent, this peculiar behavior manifests as focused beams whose intensity and spot size, vortex intensity and radius, and strength of the longitudinal component strongly increase with the aperture size, even if the focused power is not significantly altered. With the same blowing up illumination, focal intensities exceeding by two orders of magnitude, focal spots and vortex radii smaller by two orders of magnitude, and longitudinal component one order of magnitude higher than the same properties for standard illuminating beams of similar power and intensity can be achieved by increasing the aperture radius while keeping paraxial focusing.

These effects are absent with standard beams, and can find application in linear or nonlinear optics experiments where a precise control of the width and intensity of the focal spot, of the radius and intensity of the vortex bright ring, or the strength of the axial component of the electric field, are crucial, e.g., in second harmonic generation [14], also in surfaces [15], acceleration of electrons [16, 17], particle trapping [18], laser material processing [19], etc.

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