Invariant distributions and collisionless equilibria

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Abstract

This paper discusses the possibility of constructing time-independent solutions to the collisionless Boltzmann equation which depend on quantities other than global isolating integrals such as energy and angular momentum. The key point is that, at least in principle, a self-consistent equilibrium can be constructed from any set of time-independent phase space building blocks which, when combined, generate the mass distribution associated with an assumed time-independent potential. This approach provides a way to justify Schwarzschild’s (1979) method for the numerical construction of self-consistent equilibria with arbitrary time-independent potentials, generalising thereby an approach developed by Vandervoort (1984) for integrable potentials. As a simple illustration, Schwarzschild’s method is reformulated to allow for a straightforward computation of equilibria which depend only on one or two global integrals and no other quantities, as is reasonable, e.g., for modeling axisymmetric configurations characterised by a nonintegrable potential.

1 MOTIVATION

Conventional wisdom holds that galaxies in or near equilibrium can be modeled as time-independent solutions to the collisionless Boltzmann equation. In this context, the modeling of galaxies would seem to break logically into two reasonably distinct components, namely (i) constructing time-independent solutions to the collisionless Boltzmann equation and then (ii) determining whether said solutions are linearly stable and otherwise viable as reasonable models of what one actually sees. The principal focus of this paper is primarily on the former component, the construction of time-independent solutions, although the concluding section will comment on issues related to viability.

Given an assumed equilibrium mass distribution $\rho_0$, and an associated potential $\Phi_0$ generated by the gravitational Poisson equation,

$$\nabla^2 \Phi_0 = 4\pi G \rho_0, \quad (1)$$

there is a globally conserved quantity (isolating integral) $E$ reflecting time translational invariance. If the configuration is time-independent in an inertial frame, this quantity is the ordinary energy $E = \frac{1}{2}v^2 + \Phi_0$ (here, and henceforth, units have been chosen so that the stellar mass $m = 1$). If instead the configuration is time-independent in a suitably chosen rotating frame, $E$ is the Jacobi integral. If $\Phi_0$ manifests other continuous symmetries as

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well (e.g., spherical or axial symmetry), Noether’s Theorem (cf. Arnold 1989) implies that there will be one or more additional globally conserved quantities, say \( \{I_i\} \). If there exist three independent global integrals, motion in the potential \( \Phi_0 \) is integrable. If fewer than three independent global integrals exist, the motion is nonintegrable.

Jeans’ Theorem implies that any function \( f_0(E, \{I_i\}) \) that reproduces the assumed \( \rho_0 \), i.e., for which

\[
\rho_0 = \int d^3v f_0(E, \{I_i\}),
\]

yields a self-consistent equilibrium (cf. Binney and Tremaine 1987). Indeed, many workers have gone further and assumed that all time-independent equilibria must depend on such global isolating integrals. If, however, one demands that every \( f_0 \) be a function only of some set of global isolating integrals, there is an obvious critique which can be leveled towards numerical model-building based on Schwarzschild’s (1979) method. This method, which involves selecting ensembles of orbit segments that self-consistently reproduce the mass density associated with an imposed potential, says nothing \textit{a priori} about any global integrals; and, as such, for generic potentials nothing intrinsic to the method imposes explicitly the demand that the equilibrium \( f_0 \) be a function of one or more global isolating integrals.

For the special case of integrable potentials, e.g., spherical configurations or triaxial equilibria characterised by Staeckel (1890) potentials, there \textit{is} a direct, albeit not completely trivial, connection between Schwarzschild’s method and global integrals. As discussed by Vandervoort (1984) in a slightly different language, if orbits are constrained by three independent global integrals, say \( I_1(r, v) \), \( I_2(r, v) \), and \( I_3(r, v) \), fixing the values of these integrals as \( I_{1,0}, I_{2,0}, \) and \( I_{3,0} \) determines completely a collection of one or more multiply periodic orbits in the three-dimensional configuration space, each of which must in principle be included in an equilibrium model with the proper relative weight.\footnote{Consider, for example, a spherical system with global integrals \( E, J^2, \) and \( J_z \), and focus on motion in the equatorial plane, i.e., \( J_x = J_y = 0 \). Here there are an infinite number of possible orbits, characterised by initial conditions corresponding to all possible points \((r, \varphi)\) located in an annulus with inner and outer radii fixed by \( E \) and \( J_z^2 \). The weighting implicit in eq. (3) means that all values of \( \varphi \) should be treated equally, but that the relative weighting of different \( r \)’s must reflect the fact that, because of conservation of angular momentum, orbits spend different amounts of time at different radii. As a practical matter, however, this subtlety is arguably unimportant, and it may suffice computationally to consider a single orbit: If the radial and azimuthal periods are incommensurate, any \((x, y)\) yields an orbit that densely fills the annulus with the proper weight; and, even if the periods are not incommensurate, unless they are in a relatively low order resonance the orbit will generally fill a region which, to the level of accuracy associated with one’s configuration space discretisation, is essentially dense in the annulus.}

More precisely, specifying a triplet \( \{I_{1,0}, I_{2,0}, I_{3,0}\} \) defines a phase space density

\[
g_{I_{1,0},I_{2,0},I_{3,0}}(r, v) \propto \delta_D[I_1(r, v) - I_{1,0}] \delta_D[I_2(r, v) - I_{2,0}] \delta_D[I_3(r, v) - I_{3,0}]
\]

and a corresponding configuration space density

\[
n_{I_{1,0},I_{2,0},I_{3,0}}(r) = \int d^3v g_{I_{1,0},I_{2,0},I_{3,0}}(r, v)
\]

\[
= \int \int \int dI_1 dI_2 dI_3 \frac{\partial(v_1, v_2, v_3)}{\partial(I_1, I_2, I_3)} g_{I_{1,0},I_{2,0},I_{3,0}}(r, v),
\]

which, when evolved into the future using the Hamilton equations associated with \( \Phi_0 \), remains unchanged. The assumption that the desired \( \rho_0 \) is generated from some \( f_0(I_1, I_2, I_3) \)
depending only on global integrals means that the equilibrium distribution must be constructed as a superposition of solutions of the form given by eq. (3). In this context, the proper construction of an equilibrium model using Schwarzschild’s method thus entails three stages, namely (i) selecting all the orbits entering into each \( g_{I_1,0,I_2,0,I_3,0}(r,v) \) with the weights implicit in eq. (3), (ii) performing the integration of eq. (4) to extract \( n_{I_1,0,I_2,0,I_3,0}(r) \), which incorporates the fact that each point in configuration space is weighted in proportion to the amount of time orbits spend there, and then (iii) choosing a superposition of \( n_{I_1,0,I_2,0,I_3,0}(r) \)’s which yields the imposed \( \rho_0 \).

At least in the specific setting described by Vandervoort (1984), this interpretation breaks down for the case of nonintegrable potentials, where one cannot identify three global integrals, or, more generally, whenever one relaxes the demand that \( f_0 \) be realisable as a function of global isolating integrals. However, as described more carefully later on in this paper, one can still capture the essential aspect of Vanderv oort’s analysis, namely that appropriately identified orbit segments yield the natural time-independent building blocks in terms of which to construct a self-consistent model.

Nonintegrable potentials generically admit two different classes of orbits, namely regular and chaotic. Regular orbits in a nonintegrable potential behave qualitatively like orbits in an integrable potential in that they are multiply periodic and, even more importantly, are restricted to a three- (or lower-)dimensional hypersurface in the six-dimensional phase space. It follows that, even though they do not admit three global integrals, they must be constrained (cf. Lichtenberg and Lieberman 1992) by what are sometimes termed “local integrals.” For this reason, regular orbits in nonintegrable potentials define time-independent building blocks in the same sense as do orbits in integrable potentials. Chaotic orbits are very different in that they are intrinsically aperiodic and densely fill phase space regions that are necessarily higher than three-dimensional. However, chaotic phase space regions can still be characterised by invariant distributions which, when evolved into the future, remain unchanged; and one can use these invariant distributions as an additional set of building blocks when constructing self-consistent equilibria.

This alternative “orbital” interpretation of the building blocks for self-consistent equilibria is important for at least two reasons. (1) It is possible to generate analytically exact two-integral models for axisymmetric configurations which are characterised by nonintegrable potentials (cf. Hunter and Qian 1993), including potentials where motion in the meridional plane is chaotic. However, reproducing these models numerically using Schwarzschild’s method must entail sampling a collection of time-independent building blocks more complex than those associated with integrable potentials. (2) Because global integrals are associated with continuous symmetries, one might expect generically that, for genuinely three-dimensional potentials, there will not exist any global integral aside from the energy (or the Jacobi integral). If, however, one demands that the equilibrium distribution be a function only of the energy \( E \), so that \( f_0 = f_0(E) \), one concludes (cf. Binney and Tremaine 1987) that the mass distribution \( \rho_0 \) must be spherical. (This is the analogue of the well known theorem from stellar structure that all static perfect fluid stars are spherical.)

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\(^3\) A good example of a local integral is the so-called third integral associated in some cases with a nonintegrable, time-independent, axisymmetric potential, where the only global integrals are energy \( E \) and rotational angular momentum \( J_z \). How regular orbits in a nonintegrable potential differ from orbits in an integrable potential is clearly stated on p. 49 of Lichtenberg and Lieberman (1992): “Since the regular trajectories depend discontinuously on initial conditions, their presence does not imply the existence of an isolating integral (global invariant) or symmetry of the system. However, regular trajectories, when they exist, represent exact invariants of the motion.”
other words, every nonrotating triaxial equilibrium must depend on something other than the energy, either additional global integrals, as for Staeckel potentials, or “local” integrals, as is implicit in the construction of cuspy triaxial models by Merritt and Fridman (1996) or Siopis (1997).

Section 2 discusses more carefully the basic building blocks of a self-consistent equilibrium, allowing explicitly for the possibility of nonintegrable potentials that admit chaotic orbits. Section 3 then illustrates how, for the simple case of nonintegrable equilibria depending only on one or two global isolating integrals (and nothing else), Schwarzschild’s method can be reformulated in terms of an appropriate set of time-independent building blocks. Section 4 concludes by describing straightforward generalisations to construct equilibria that do not depend simply on global integrals and then discussing the issue of whether such “more complex” equilibria are physically viable.

2 INVARIANT DISTRIBUTIONS AND SELF-CONSISTENT EQUILIBRIA

It is often asserted glibly that “any equilibrium solution $f_0$ to the collisionless Boltzmann equation must be given as a function of the time-independent integrals of the motion associated with the potential $\Phi_0$ generated self-consistently from $f_0$.” In point of fact, however, this statement is an oversimplification and requires some careful thought. Should one demand, as is often assumed, at least tacitly, that $f_0$ depends only on the global isolating integrals, such as energy $E$ (or the Jacobi integral for a rotating configuration) and angular momentum $J_z$, or can one instead allow for equilibria $f_0$ that depend on the “local” isolating integrals which, in a generic nonintegrable potential, make regular orbits regular, i.e., restrict them to a lower-dimensional phase space hypersurface (cf. Lichtenberg and Lieberman 1992)? Could one, for example, try to construct models which assign regular and chaotic orbits on the same $E$-$J_z$ hypersurface different weights, or must one sample each constant $E$-$J_z$ hypersurface uniformly?

Arguably, the only crucial point is that a time-independent solution to the collisionless Boltzmann equation must be constructed from time-independent building blocks, so as to ensure that, if initial data be evolved into the future along the characteristics associated with the self-consistent potential, the form of $f_0$ will remain unchanged. The easiest way to do this, both conceptually and practically, is to demand that $f_0$ be given as a function of one or more global isolating integrals, say $E$ and $I$. The obvious point is that any $f_0(E, I)$ which implies the proper mass density $\rho_0$, and hence the proper potential $\Phi_0$, will yield a time-independent solution since, by assumption, both $E$ and $I$ are time-independent constants of the motion. In other words, $dE/dt = dI/dt = 0$ implies $\partial f_0/\partial t = 0$. This is of course the standard way of showing that time-independent conserved quantities can be used to construct a self-consistent equilibrium.

However, there is another, more “microscopic,” viewpoint. Specifically, viewed in terms of the orbits associated with the equilibrium (i.e., characteristics associated with the Boltzmann equation), this construction works because such an $f_0$ implies that the phase space number density is constant on hypersurfaces of constant $E$ and $I$. This constancy means that the orbit ensemble that generates $f_0$ must involve a uniform, i.e., microcanonical, sampling of each constant $E-I$ phase space hypersurface, but Hamilton’s equations for motion in a fixed $\Phi_0$ imply that such a population is invariant under time translation.

It would seem clear from this latter viewpoint that choosing $f_0$ to be a function only of global isolating integrals is not necessary, at least in principle. A priori, a self-consistent equilibrium $f_0$ can be constructed from any collection of time-independent building blocks.
which successfully reproduces the assumed potential $\Phi_0$. The key point, then, as stressed, e.g., by Ott (1993), is that, assuming the validity of the Ergodic Theorem, for flows in a fixed time-independent potential any orbit is ergodic in an appropriately interpreted subspace, so that a microcanonical population of the appropriate subspace yields a time-independent building block.

Regular orbits are multiply periodic and, as such, are characterised (in an orbit-averaged sense) by a density that is time-independent, so that they can be treated individually as time-independent building blocks, with a density $\rho$ proportional to the time that the orbit spends in the neighborhood of each point. In this sense, regular orbits in a nonintegrable potential can be exploited in the same way as the orbits in an integrable potential, even though it is seemingly impossible to identify explicitly the forms of the “local integrals,” and even though regularity is not attributable directly to a continuous symmetry.

Chaotic orbits are not periodic, so that this naive argument does not hold. However, it would still seem possible to identify an appropriate set of time-independent chaotic building blocks. For a fixed value of $E$ (and any other global integral), the constant $E$ (or $E-I$) phase space hypersurface divides naturally into regular and chaotic regions. The chaotic region divides in turn into one or more subregions which are connected in the sense that an orbit starting in any part of a subregion will eventually pass arbitrarily close to every other part of that subregion. The important point then is that a uniform, i.e., microcanonical, population of any connected chaotic domain defines a time-independent building block. Why this should be so is easy to understand: Time translation using Hamilton’s equations moves each phase space point in the chaotic domain to another point in the same domain, but the only initial distribution invariant under time translation using Hamilton’s equations is a constant density distribution. Integrating this phase space building block over the allowed range of velocities yields a configuration space density which, as for the regular orbits, is proportional to the amount of time that a representative chaotic orbit in the domain spends in the neighborhood of each point $r$.

Assuming the validity of the Ergodic Theorem for individual connected chaotic domains, it is relatively simple to generate such invariant distributions numerically. Specifically, one knows that, when evolved into the future, a generic ensemble of initial conditions located anywhere in the domain will evidence a coarse-grained approach towards the invariant microcanonical distribution. In fact, this has been confirmed by numerical experiments (cf. Kandrup and Mahon 1994, Mahon, Abernathy, Bradley, and Kandrup 1995, Merritt and Valluri 1996) which have shown that, for chaotic flows in a variety of different potentials, reduced distribution functions (like $f(r)$ or $f(v)$) exhibit an apparent exponential in time approach towards an invariant reduced distribution on a time scale $\tau$ that correlates with the value of the largest Lyapunov exponent.

Viewed in this fashion, regular and chaotic orbits can be used to define time-independent building blocks in exactly the same way, the only difference being that chaotic building blocks are intrinsically higher-dimensional. When evolved into the future, initial conditions corresponding to regular and chaotic orbits both yield trajectories which, in an asymptotic, late time limit, converge towards time-independent invariant distributions.

The crucial point in all of this is that, in principle, a library comprised of all possible invariant distributions, both those corresponding to individual regular orbits and those corresponding to individual connected chaotic phase space regions, should constitute a complete set of building blocks in terms of which to construct self-consistent models of a galaxy with the specified potential $\Phi_0$. In the real world, one cannot construct such
a library, which would contain an infinite number of building blocks. However, one can construct a large, but finite, library and then sample that library in an attempt to select appropriate combinations that reproduce a suitably discretised version of the assumed $\Phi_0$. This is the essence of what Schwarzschild’s (1979) method can, and should, do when applied to a generic nonintegrable potential that admits both regular and chaotic orbits. It is also evident that, in principle, nothing stops one from trying to construct equilibria that contain only regular (or perhaps only chaotic) orbits, although one might imagine that it would be very hard to reproduce a smooth potential $\Phi_0$ with a collection of orbits that systematically avoids significant phase space regions or, especially, probes the phase space in an exceedingly irregular fashion.

For the case of a generic rotating, axisymmetric equilibrium, there are two global isolating integrals, namely $E$ and $J_z$, associated respectively via Noether’s Theorem with symmetries with respect to time translations and rotations about the $z$-axis. The potential may in fact be integrable, so that there are three global isolating integrals, but this is not necessarily the case. In general, a rotating, axisymmetric, time-independent potential will be nonintegrable and admit both chaotic and regular orbits, each of which defines time-independent building blocks. It follows that, if one allows for local integrals, one can, at least in principle, try to construct equilibria that do not sample constant $E$-$J_z$ surfaces uniformly. One could, e.g., try to construct models which exclude all chaotic orbits. Analytic approaches to constructing self-consistent axisymmetric equilibria, as developed, e.g., by Hunter and Qian (1994), neglect this possibility altogether and focus exclusively on solutions for which $f_0 = f_0(E, J_z)$, so that the phase space density is constant on hypersurfaces of constant $E$ and $J_z$. Whether this is well motivated physically, or whether this is only a useful analytic simplification, is not completely clear at the present time.

It should be stressed that great care must be taken in identifying the invariant distributions associated with (ensembles of) chaotic orbits. As has long been known from numerical investigations of simple maps (cf. Lieberman and Lichtenberg 1972), the presence of cantori (cf. Aubry and Andre 1978, Mather 1982) or an Arnold (1964) web allows for the possibility of chaotic near-invariant distributions which, albeit not strictly time-independent, can, at least in the absence of any perturbations, behave for very long times as if they were essentially time-independent distributions. As discussed, e.g., in Mahon, Abernathy, Bradley, and Kandrup (1995), the crucial point here that cantori and Arnold webs serve as partial obstructions which, although they cannot completely block motion between different phase space regions, can significantly impede phase space transport. It follows that, even if a single chaotic region is connected, it may appear partitioned into disjoint regions even over relatively long time scales.

This phenomenon is problematic. A putative self-consistent equilibrium generated with a near-, rather than true, invariant distribution cannot be a true self-consistent equilibrium. On sufficiently long time scales, the orbital population associated with the distribution will change, occasioning changes in the mass distribution, the potential, and so forth. One might nevertheless want to argue that, if the time scale associated with this phenomenon is sufficiently long, this very slow effect will be irrelevant astronomically, so that one can speak of nearly self-consistent equilibria that can exist for times much longer than the age of the Universe, $t_H$. However, this argument is probably wrong. Real astronomical systems involve $N$-body realisations of self-consistent equilibria which, heuristically, are presumed to behave like smooth three-dimensional Hamiltonian systems perturbed by friction and noise. However, numerical experiments involving perturbations of motion in a fixed poten-
tial indicate (Habib, Kandrup, and Mahon 1996, 1997) that even very weak friction and noise can dramatically accelerate phase space transport through cantori or along an Arnold web, occasioning significant changes in an initial near-invariant distribution on comparatively short time scales. Trying to estimate the longevity of a near-invariant distribution without allowing for the effects of perturbations is unquestionably a very bad idea.

3 A VARIANT OF SCHWARZSCHILD’S METHOD FOR ONE- AND TWO-INTEGRAL DISTRIBUTIONS

The objective of this Section is to reformulate Schwarzschild’s method in terms of the natural set of building blocks so as to permit the construction of equilibrium models $f_0$ which are assumed to depend on one or two global integrals and to exhibit no additional dependence on any nonclassical local integrals. This is a straightforward generalisation of an approach proposed by Vandervoort (1984) for the construction of three-integral equilibria.

Start by specifying a time-independent potential $\Phi_0(r)$, and hence a configuration space density $\rho_0(r)$ which admits (say) two constants of the motion, $E$ and $I$, where $E$ is the particle energy (or, perhaps, the Jacobi integral) and $I$ is some other isolating integral, the form of which is assumed to be known explicitly. By assumption, the desired equilibrium (or equilibria) $f_0$ must be given exactly as a function $f_0 = f_0(E, I)$. The object, therefore, is to construct a discretised approximation to a smooth $f_0$ of this form which reproduces the assumed $\rho(r)$ self-consistently. This can be done in two stages, viz:

1. First grid $E$-$I$ space into a collection of cells and, for each pair $\{E_i, I_j\}$, write down the invariant distribution $g_{ij}(r, v)$ on the constant $E_i$-$I_j$ hypersurface. Use these $g_{ij}$’s to derive reduced configuration space densities $n_{ij}(r)$.

2. Then construct the desired numerical approximation to $f_0$ as a sum of contributions from the different invariant distributions $g_{ij}$, with the relative weights of the different $g_{ij}$’s fixed by the requirement of self-consistency for the configuration space density.

This construction proceeds without explicit reference to individual orbits and, as such, provides no insight into the orbital building blocks entering into the equilibrium. If this be perceived as a serious lacuna, the natural tack numerically is to consider separately the different constant $E$-$I$ hypersurfaces and, on each hypersurface, to construct ensembles of orbit segments that reproduce self-consistently the invariant $g_{ij}$’s.

3.1 Construction of the invariant distribution for fixed $E$ and $I$

A uniform population of the phase space hypersurface of fixed $E_i$ and $I_j$ corresponds to an invariant distribution of the form

$$g(E_i, I_j) \equiv g_{ij}(r, v) = K \delta_D[E_i - E(r, v)] \delta_D[I_j - I(r, v)],$$

where $\delta_D$ denotes a Dirac delta, and the quantities $E$ and $I$ are viewed explicitly as functions of the phase space coordinates. The quantity $K$ is a constant, whose value is fixed by the

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4 Although it is the density, rather than the potential, that astronomers are wont to specify, it is more natural conceptually to view $\Phi_0$ as the fundamental object, since it is the Hamiltonian associated with $\Phi_0$ that defines the time-invariant building blocks.

5 Reformulating the following for equilibria admitting only one isolating integral is completely straightforward. If the equilibrium admits three independent integrals, it is integrable and can be addressed using the approach described by Vandervoort (1984).
normalisation
\[
\int d^3x \int d^3v \, g_{ij}(\mathbf{r}, \mathbf{v}) = 1,
\]
where the integral extends over the allowed phase space regions. In other words, the invariant distribution corresponds to a normalised microcanonical population of the constant \(E_i-I_j\) hypersurface.

Given such a \(g_{ij}(\mathbf{r}, \mathbf{v})\), it is straightforward to integrate over the velocity dependence to extract a reduced configuration space density \(n_{ij}(x, y, z)\). Because \(E_i\) and \(I_j\) are known functions of \(r\) and \(v\), one can choose to view any two of the phase space coordinates, say \(v_y\) and \(v_z\), as functions of \(E\), \(I\), and the remaining four phase space coordinates. However, the Dirac deltas in eq. (5) make \(dv_y\) and \(dv_z\) integrations trivial, so that one can immediately write down analytically a reduced \(\tilde{g}_{ij}(x, y, z, v_x)\)
\[
\tilde{g}_{ij}(x, y, z, v_x) \equiv K \int dv_y dv_z \delta_D[E_i - E(\mathbf{r}, \mathbf{v})] \delta_D[I_j - I(\mathbf{r}, \mathbf{v})].
\]

It follows that the configuration space density,
\[
n_{ij}(x, y, z) = \int dv_x \, \tilde{g}_{ij}(x, y, z, v_x),
\]
associated with each constant \(\{E_i, I_j\}\) pair is given as a simple quadrature.

In general, it may be impossible to perform the integral in eq. (8) analytically. This, however, is not a serious difficulty. Even if known analytically, the \(n_{ij}\)'s must eventually be approximated by a set of values on a configuration space grid so as to facilitate a comparison between the imposed density \(\rho_0(\mathbf{r})\) and the inferred density \(n(\mathbf{r})\) constructed from the invariant \(n_{ij}\)'s.

### 3.2 Construction of \(f_0(E, I)\) from the invariant distributions

In the continuum limit, one knows that the true equilibrium distribution
\[
f_0(\mathbf{r}, \mathbf{v}) = \int \int dE dI \, A(E, I) \, g_{E,I}(\mathbf{r}, \mathbf{v}),
\]
where \(g_{E,I}\) is the invariant distribution for fixed \(E\) and \(I\), viewed as a function of \(\mathbf{r}\) and \(\mathbf{v}\), and \(A(E, I)\) is an expansion coefficient, which gives the relative weights of the different values of \(E\) and \(I\) entering into \(f_0\). The discretised construction thus involves
\[
f_0(\mathbf{r}, \mathbf{v}) = \sum_i \sum_j A_{ij} g_{ij}(\mathbf{r}, \mathbf{v}).
\]

The proper choice of weights \(A_{ij}\) derives from the demand of self-consistency: Given \(f_0\), one can define a density
\[
n(\mathbf{r}) = \int \, d^3v \, f_0(\mathbf{r}, \mathbf{v})
\]
which, when discretised, becomes
\[
n(x, y, z) = \sum_i \sum_j A_{ij} n_{ij}(x, y, z)
\]

\(^6\) The choice of Cartesian coordinates, implicit in the following, is only for specificity: as far as this algorithm is concerned, the coordinate system is completely irrelevant.
in terms of the unknown expansion coefficients $A_{ij}$. However, demanding that this $n(x, y, z)$
correspond as closely as possible to the density

$$\rho_0(x, y, z) = \frac{1}{4\pi G} \nabla^2 \Phi_0$$

(13)

associated with the assumed potential $\Phi_0$ then enables one to determine the “best” values for
the $A_{ij}$’s. This construction is very much analogous to the ordinary Schwarzschild method,
save only that the basic building blocks are now the reduced invariant distributions $n_{ij}$,
rather than individual orbits.

### 3.3 Orbital building blocks for the invariant distributions

One way in which to obtain insights into the orbital building blocks of a self-consistent
model constructed using this algorithm is by proceeding numerically to construct ensem-
bles of orbit segments which reproduce self-consistently the invariant distributions $g_{ij}$. In
general, $g_{ij}$ will contain contributions from both regular and chaotic orbits, each of which
is characterised separately by its own invariant distribution. The easiest way to construct
$g_{ij}$ is probably to (1) obtain an invariant distribution for the chaotic orbits and then (2)
augment this by another (sub)distribution comprised of segments of regular orbits, the latter so chosen that the combination of regular and chaotic orbits yields a satisfactory
approximation to the true invariant distribution.

The invariant distribution is approximated numerically by binning the six-dimen-sional
phase space into a collection of six-dimensional hypercubes, and then assigning occupation
numbers to the different hypercubes which are proportional to the time that orbits sampling
the true invariant distribution reside in each cell. (This is justified by the Ergodic Theorem
[cf. Lichtenberg and Lieberman 1992].) In the continuum limit, the invariant distribution
corresponds to a uniform population on a four-dimensional phase space hypersurface.
Given a discretisation of the phase space coordinates, the invariant distribution corresponds
instead to a four-dimensional shell in the six-dimensional phase space.

The invariant (sub-)distribution of chaotic orbits is especially easy to compute if, as
is often the case, for fixed $E_i$ and $I_j$ the entire chaotic region is connected in the sense
that, eventually, every chaotic orbit will pass arbitrarily close to every point in the chaotic
region. (For simplicity, ignore the tiny measure of chaotic orbits trapped permanently inside
invariant KAM tori.) All that one need do is specify a (more or less arbitrary) ensemble of
initial conditions, each corresponding to a chaotic orbit, evolve each initial condition into
the future, and wait until the evolved ensemble approaches an invariant distribution, i.e.,
a uniform sampling of the chaotic portions of the $E_i$-$I_j$ hypersurface (cf. Kandrup and
Mahon 1994, Mahon, Abernathy, Bradley, and Kandrup 1995).

To expedite the calculation, it is useful to evolve the initial conditions in the presence
of very weak amplitude friction and noise, sufficiently weak that the values of $E$ and $I$ are
almost constant (cf. Habib, Kandrup, and Mahon 1996, 1997). The advantage of intro-
ducing weak friction and noise is that such small perturbations can dramatically accelerate
the overall approach towards a true invariant distribution by facilitating extrinsic diffusion
through cantori and/or along an Arnold web (cf. Lichtenberg and Lieberman 1992). If one
does not either (a) integrate for a very long time and/or (b) allow for such perturbing influ-
ences, one faces the problem that the initial ensemble may evolve towards a near-invariant
distribution which, albeit not strictly time-independent, only changes significantly on a very
long time scale.

The contribution of different regular orbits to the invariant distribution can be generated
using an analogue of the original Schwarzschild method. Specify a large number of regular initial conditions and integrate each into the future to generate a library of regular orbits. Then use a linear programming algorithm, or some variant thereof, to select a weighted ensemble of regular orbits which, when combined with the chaotic (sub)distribution, yields a satisfactory approximation to the true invariant distribution on the constant $E_i - I_j$ hypersurface.

This construction of invariant distributions $g_{ij}$, and the corresponding densities $n_{ij}$, is admittedly tedious numerically (albeit presumably straightforward), since it involves repeating Schwarzschild's method for each pair $\{E_i, I_j\}$. However, it is arguably a crucial step in obtaining a proper understanding of the orbital structure associated with the self-consistent model since, as discussed already, one knows that the $g_{ij}$'s are the proper building blocks in terms of which to construct an equilibrium $f_0(E, I)$.

### 3.4 A simple two-dimensional example

Consider as a pedagogical example the case of two-dimensional gravity, this corresponding physically to a collection of infinite rods aligned along the $z$-axis, and suppose that the configuration is rotating uniformly about the $z$-axis with angular velocity $\Omega$. It then follows that, in the rotating frame, the configuration is characterised by a potential $\Psi(x, y)$ and a surface density $\sigma(x, y)$ related by

$$\nabla^2 \Psi(x, y) = 4\pi G \sigma(x, y).$$ (15)

Suppose then that there is only one global isolating integral, namely the Jacobi integral $E$, which, in terms of phase space coordinates defined in the rotating frame, takes the form

$$E = \frac{1}{2} v_x^2 + \frac{1}{2} v_y^2 + \Psi(x, y) - \frac{1}{2} \Omega^2 (x^2 + y^2) = \frac{1}{2} v_x^2 + \frac{1}{2} v_y^2 + \Psi_{eff}(x, y).$$ (16)

To the extent that one demands that any equilibrium $f_0$ associated with this mass distribution be a function only of the isolating integral $E$, the fundamental building block is the microcanonical phase space density on a constant Jacobi integral hypersurface, which, for any $E_i$, takes the form

$$g(E_i) = g_i(r, v) = K \delta_D[E_i - E(r, v)].$$ (17)

As will be evident from below, the normalisation constant $K$ can be written as

$$K = \frac{1}{2\pi V(E_i)}$$ (18)

in terms of $V(E_i)$, the area of the configuration space region with $\Psi_{eff} \leq E_i$. The reduced configuration space density $n_i$ associated with this $g_i$ satisfies

$$n_i(x, y) = K \int \int dv_x dv_y \delta_D[E_i - E(r, v)],$$ (19)

where the integrals extend over the values of $v_x$ and $v_y$ that are allowed energetically, i.e., for which $E_i \geq \Psi_{eff}$. The $dv_y$ integration can be performed trivially by implementing the Dirac delta, allowing for nonzero contributions at two values, namely $v_y = \pm \sqrt{2(E_i - \Psi_{eff}) - v_x^2} \equiv \alpha$. It follows that, for those regions in configuration space for which $\Psi_{eff}(x, y) \leq E_i$,

$$n(x, y) = 2K \int_{-\alpha}^{\alpha} \frac{dv_x}{\sqrt{\alpha^2 - v_x^2}}.$$ (20)
The remaining integral can then be performed trivially, leading to a reduced configuration space density on the constant $E_i$ hypersurface of the form

$$n_i(x, y) = 2\pi K \Theta[E_i - \Psi_{\text{eff}}(x, y)] = \frac{1}{V(E_i)} \Theta[E_i - \Psi_{\text{eff}}(x, y)],$$

(21)

where $\Theta(z) = 1$ for $z \geq 0$ and $\Theta = 0$ otherwise.

It follows from eq. (21) that, independent of the specific form of the potential $\Psi(x, y)$, the total configuration space surface density, given as a sum of contributions on different constant Jacobi integral hypersurfaces, must be of the form

$$n(x, y) = \sum_i A_i \frac{1}{V(E_i)} \Theta[E_i - \Psi_{\text{eff}}(x, y)].$$

(22)

where the $A_i$'s give the relative weights of the different $E_i$ hypersurfaces. The demand that the $n(x, y)$ of eq. (22) agree as closely as possible with the $\sigma(x, y)$ associated with $\Psi$ may then be used to identify the “best” values of the $A_i$’s.

4 DISCUSSION

There are a number of different ways in which the algorithm described in the preceding Section can be generalised to permit the construction of more complex equilibria, which do not depend simply on the global integrals $E$ and $I$. For fixed values of $E$ and $I$, it is straightforward to locate the general locations of (at least the large) chaotic regions and, by evolving arbitrary ensembles of initial conditions located in these regions into the future, it is easy to derive a numerical approximation to the invariant distribution associated with each of these chaotic regions. Given these invariant distributions, one can then integrate over velocities to extract the chaotic contribution $n_{ij}^c(x, y, z)$ to total density $n_{ij}(x, y, z)$ associated with any pair $E_i$ and $I_j$. Subtracting $n_{ij}^c$ from the full $n_{ij}$ then yields the regular contribution $n_{ij}^r(x, y, z)$ to the density. However, given a knowledge of $n_{ij}^c$ and $n_{ij}^r$ separately, one can then attempt to construct models which assign different relative weights to the regular and chaotic portions of the $E_i-I_j$ hypersurface, thus allowing one to test the prejudice of some workers that self-consistent models should contain few, if any, chaotic orbits.

Similarly, one can identify those portions of the $E_i-I_j$ hypersurface that correspond to different types of regular orbits, e.g., boxes and tubes, and compute their relative densities, say $n_{ij}^b$ and $n_{ij}^t$, which can in turn be used as separate building blocks. In particular, given such a collection one can try to construct models which associate different relative weights to boxy and/or tuby and/or chaotic orbits, and, to the extent that such models can be constructed, one can investigate whether the different phase space densities $f_0$ have obvious observational signatures which could be compared with real astronomical data. Is there, e.g., some natural signature which, when observed in real galaxies, can be interpreted as evidence that $f_0$ contains a significant measure of chaotic orbits?

In principle, one can continue this process of refinement more or less ad infinitum, identifying increasing numbers of time-independent building blocks associated with different regular orbits, although one’s freedom to deal with chaotic orbits is limited by the fact that there is only one natural notion of a time-independent invariant distribution. However, it is not clear that such a procedure is well motivated physically. At least heuristically, it would seem that building an equilibrium by “picking and choosing” amongst individual orbits in a strongly nonintegrable potential with different values of local isolating integrals is akin to
selecting orbits in an integrable potential which yield a distribution function that is a highly irregular function of the \( I_i \)'s. This latter procedure might strike one as contrived and, in any event, one knows that, in many cases, such irregular \( f_0 \)'s are linearly unstable. Thus, e.g., it is well known that, for a spherical equilibrium with \( f_0 = f_0(E, J^2) \), stability or lack thereof often correlates with the sign of the partial derivatives \( \partial f_0 / \partial E \) and/or \( \partial f_0 / \partial J^2 \). In particular, population inversions can trigger instabilities.

If any discrete construction based on Schwarzschild's method is to be reasonable, there must be a sense in which, as the discretisation of the density becomes more refined and as the number of building blocks becomes larger, the solution constructed numerically converges towards a continuous self-consistent equilibrium. However, identifying the precise sense in which this is so would most likely be very difficult. Mathematically, establishing such a convergence would probably involve a study of sequences of discrete Banach spaces, along the lines that have been used to study the convergence of finite difference schemes for solving partial differential equations. In that setting, a good deal is known about linear differential equations but, if one incorporates nonlinearities and/or allows for an integro-differential equation – recall that the collisionless Boltzmann is a quadratically nonlinear integro-differential equation – things become much harder!

It is evident, both intuitively and from painful experience (cf. Siopis, Athanassoula, and Kandrup 1997), that it is easier to approximate comparatively smooth quantities on a finite lattice than quantities that manifest intricate structures on a variety of different scales. For this reason, one might expect that it is much easier to construct a satisfactory numerical representation of an \( f_0 \) that is a function only of smoothly varying global isolating integrals than an \( f_0 \) that depends sensitively on “local” integrals that manifest the details of the complex phase space structure associated with a generic nonintegrable potential. Moreover, even if one allows for local integrals, the numerical construction should be more straightforward if, for example, on a constant energy hypersurface, the phase space population is reasonably smooth, e.g., perhaps avoiding chaotic regions but weighting different regular regions in a fashion that varies smoothly with their phase space location.

Suppose that there is in fact a “true” \( f_0 \) involving local integrals, generated (in principle) as an exact time-independent solution to the collisionless Boltzmann equation, to which one has constructed a latticized \( f_0 \) via some analogue of Schwarzschild’s method, and that this latticized \( f_0 \) has been used to generate an ensemble of initial conditions to populate an \( N \)-body realisation of the model. There are then two potentially serious sources of error: (1) The latticized approximation to \( f_0 \) could miss important microscopic structures associated with local integrals. If the “true” \( f_0 \) is a function only of smoothly varying integrals like the energy \( E \), allowing for as few as 20 different energies, as did Schwarzschild (1979), Merritt and Fridman (1996), and Siopis (1997), may be adequate to capture the essence of the analytic model. If, however, \( f_0 \) involves a complex combination of local integrals as well as the energy, allowing for 20 values may not be enough. (2) Even if most/all the important microscopic structures are adequately represented in the discretised model, the \( N \)-body realisation could fail to sample them adequately. Even for very large particle number, \( N \sim 10^6 \) or more, there is no guarantee that a complex phase space will be adequately sampled.

If, however, it is difficult for galactic astronomers to construct \( N \)-body realisations of “complex” equilibria \( f_0 \) that involve local integrals in a highly irregular way, nature too may find it hard. If \( N \sim 10^6 \) is not enough to generate a “fair” sampling, \( N \sim 10^{11} \) may also be inadequate to probe the complex phase space associated with a smooth potential.
generated as a Boltzmann equilibrium. Both statistical fluctuations, which will obviously
be present, and small non-Hamiltonian irregularities, which can be important in complex
Hamiltonian systems (cf. Lichtenberg and Lieberman 1992) may tend to “smooth out” a
complex would-be equilibrium into something substantially simpler.

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