Time-resolved spectrum output from a grating spectrometer

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Time-resolved spectra are often recorded in modern Thomson scattering experiments of laser-produced plasmas. In this essay, we discuss the exact meaning of time-resolved spectra output from a grating spectrometer. Our calculations show that the streak image of a Thomson scattering spectrum does not directly give the time-resolved plasma parameters.

In a modern experiment of optical Thomson scattering\textsuperscript{1} off laser-produced plasmas, scattered light waves are usually collected and relayed into a grating spectrometer coupled with a streak camera\textsuperscript{2–9}. With this kind of experimental setup, time-resolved scattering spectra are obtained. Experimental data then are fitted with theoretical ones in the default assumption that the recorded time-resolved spectra are simply proportional to the instant dynamic form factor, from which the temporal evolution of plasma parameters are inferred. Only in a very recent article, Davies et al\textsuperscript{10} applied new fitting scheme in which the authors take the issue into account. However, to the best knowledge of the authors, the exact meaning of time-resolved spectrum obtained with above mentioned method is never clarified. In this short note, we address this topic and discuss its effect on data analysis.

We assume that a point light source after aligned with an ideal lens impacts onto a grating. The signal then diffracts via a grating, and then is focused with another lens onto the slit of a streak camera. As seen in Fig. 1, the incident angle is $\phi$ and the diffraction angle is $\theta$.

The signal from the point source is described with a function $E(t)$. The output signal at the spectral plane of the spectrometer is given with

$$I(t, \theta) = \left| \frac{1}{N} \sum_{k=-N/2}^{N/2-1} E(t - k\tau_\theta) \right|^2,$$  \hspace{1cm} (1)

where $\tau_\theta$ is time delay between two adjacent groove lines of the grating,

$$\tau(\theta) = \frac{d}{c} (\sin \phi + \sin \theta).$$  \hspace{1cm} (2)

Here $d$ is the grating constant, $c$ is the light speed, and $N$ is the effective groove line number of the grating. We represent the input signal $E(t)$ with its Fourier component $\hat{E}(\omega)$,

$$E(t) = \int_{-\infty}^{\infty} \hat{E}(\omega)e^{i\omega t} \frac{d\omega}{2\pi}. \hspace{1cm} (3)$$

The output signal at the spectral plane of the grating spectrometer is

$$I(t, \theta) = \left| \int_{-\infty}^{\infty} e^{-i\omega t} \hat{E}(\omega) G(\omega\tau_\theta) \frac{d\omega}{2\pi} \right|^2. \hspace{1cm} (4)$$

where $G(\omega\tau_\theta)$ is the grating function defined as\textsuperscript{11}

$$G(\omega\tau_\theta) = \frac{\sin(N\omega\tau_\theta/2)}{N\sin(\omega\tau_\theta/2)). \hspace{1cm} (5)$$

Eqs. (1) and (4) describes the exact meaning of time-resolved spectrum output from a grating spectrometer.

If the grating spectrometer is coupled with a time-integrated detector, the recorded signal is given by

$$I(\theta) = \int_{-\infty}^{\infty} I(t, \theta) dt = \int_{-\infty}^{\infty} |\hat{E}(\omega)|^2 G^2(\omega\tau_\theta) \frac{d\omega}{2\pi} \hspace{1cm} (6)$$

We can see that the detected time-integrated signal is just the convolution of the spectral power density with the square of grating function. When $N \to \infty$, the grating function (5) can be approximated as the Dirac $\delta$-function,

$$G^2(\omega\tau) \rightarrow \delta(\omega\tau_\theta - 2\pi).$$

Here we just keep the first order diffraction term. Thus we obtain

$$I(\theta) = |\hat{E}(2\pi/\tau_\theta)|^2. \hspace{1cm} (7)$$

The time-integrated signal is proportional to the spectral density of the source.

We are interested in the case that the frequency of the input signal is dependent of time. As an example, we first consider the experiment that a laser beam is scattered from a beam of accelerating electrons. We assume that the laser pulse is a gaussian and that the velocity of the electron beam is given by

$$v = at \hspace{1cm} and \hspace{1cm} v \ll c. \hspace{1cm} (8)$$

Here $c$ is the light speed in vacuum. In this case, the scattering process can be approximated with the dipole...
radiation, and the signal can be described with the following equation,
\[ E(t) = e^{-\tau^2/2\tau^2} \cos \left[ (\omega_0 + k \cdot v)t \right], \]  
where \( \omega_0 \) is the frequency of the laser probe, \( \tau_1 \) is its pulse duration, and \( k \) is the differential wave number of the scattering. We define the nominal frequency \( \omega_n \) of Eq. (9) as
\[ \omega_n = \omega_0 + k \cdot v. \]  
The instant frequency consists of the Doppler effect of the moving electrons, and is the consequence of energy and momentum conservation laws in the scattering process. Due to the acceleration of the electrons, however, the instant frequency \( \omega_s \) of the scattering light is given by,
\[ \omega_s = \frac{\partial}{\partial t}(\omega_n t) = \omega_0 + \kappa t, \]  
where \( \kappa = 2k \cdot a. \) Comparing Eq. (11) with Eq. (10), we can see that the two quantities are different from each other except that the electron velocity is a constant. What we want to know is the output signal of a grating spectrometer with input signal of Eq. (9).

The spectral density of the signal (9) is
\[ \hat{E}(\omega) = \frac{\sqrt{2\pi} \tau_1}{2} \left\{ \frac{1}{\sqrt{1 + i\kappa \tau_1^2}} e^{-\frac{1}{2} \left\{ \frac{(\omega - \omega_0)^2}{1 + i\kappa \tau_1^2} + \frac{1}{\sqrt{1 - i\kappa \tau_1^2}} e^{-\frac{1}{2} \left\{ \frac{(\omega + \omega_0)^2}{1 - i\kappa \tau_1^2} \right\}} \right\}} \right\}. \]  

In order to give an analytical calculation, we further assume that the grating function can be approximated with a sum of gaussians,
\[ G(\omega \tau) \approx e^{-\omega^2/\tau_s^2} + e^{-\omega^2/\tau_s^2}. \]  

Here we already omit the oscillation terms dependent of time. In real experiment, we can omit the second term in Eq. (14) via suitable experimental setup. At given time \( t \), the output signal has a maximum locating at
\[ \omega_0^* = \omega_0 + \kappa t \left( 1 + \frac{\tau_1}{\tau_g} \right)^2. \]  
It is \( \omega_0^* \) that is regarded as the measured instant frequency of the signal. Comparing Eq. (15) with the real instant frequency (11) of the input signal, we can make the conclusion that \( \omega_0^* \) is indeed a good approximation of the instant frequency \( \omega_s \), i.e.,
\[ \omega_0^* \approx \omega_s \text{ if } \tau_g/\tau_1 \ll 1. \]  
As seen in Eq. (10), the motion of the electrons is in fact described with the nominal frequency \( \omega_n \) rather than the instant frequency \( \omega_s(t) \). Therefore, after we obtain \( \omega_0^* \), we need to integrate the following equation to obtain the nominal frequency,
\[ \frac{\partial}{\partial t}(\omega_n t) = \omega_0^*. \]  

The conclusion we make above does not depend on the approximation of Eq. (13). Here we give a numerical result. We assume the input signal is given by
\[ E(t) = \frac{1}{1 + (t/\tau_1)^2} \exp \left\{ t \left[ \omega_0 + \frac{\Delta \omega}{2} \tanh(t/\tau_2) \right] \right\}. \]  

| \( N \) | \( \sin \phi \) | \( \sin \theta_0 \) | \( \tau_1 \) \text{ (ns)} | \( \tau_g \) \text{ (ns)} | \( \omega_0 \) \text{ (Hz)} | \( \Delta \omega/\omega_0 \) |
|---|---|---|---|---|---|---|
| 2 | 1/2 | 1/2 | 7.154 \times 10^{-15} | 1/2 |

TABLE I. Calculation parameters in Fig. 2.
FIG. 2. Streak image of the output signal from a grating spectrometer in the case that the input signal is given by Eq. (17).

The instant frequency of this signal is

\[ \omega_s = \omega_0 + \Delta \omega \left[ \tanh(t/\tau_2) + (t/\tau_2) \cosh^{-2}(t/\tau_2) \right]. \]  

(18)

Based on Eq. (1), we can compute the streak image of the output signal in the case of input signal of Eq. (17). The result is presented in Fig. 2, and the parameters of the computation is list in Table I.

As seen in Fig. 2, at any time \( t \), the streak image has a maximum around the instant frequency \( \omega_s \). We pick out the instant frequency \( \omega_s \) from Fig. 2, and plot it versus time in Fig. 3. For the sake of comparison, we also plot in Fig. 3 the instant frequency of Eq. (18). We can see that the former agrees with the latter very well.

In an experiment of Thomson scattering off laser-produced plasmas, the observed scattering spectra usually vary with time due to the temporal evolutions of plasma parameters such as electron temperature and plasma flow velocity. It is the nominal frequency that directly depends on plasma parameters. For example, the scattering spectrum could have two peaks at the nominal frequencies

\[ \omega_n = \omega_0 \pm \omega_{ia} + \mathbf{k} \cdot \mathbf{u} \]

where \( \omega_{ia} \) is the ion-acoustic wave frequency of the plasmas, and \( \mathbf{u} \) is the plasma flow velocity. For a laser-produced plasma, both \( \omega_{ia} \) and \( \mathbf{u} \) depend on time, due to the rapid evolution of the plasma. However, the streak image of scattering spectrum gives the instant frequency \( \omega_s \). When plasma parameters vary with time, the two frequencies are different from each other. One should integrate Eq. (16) to obtain the nominal frequency and then infer plasma parameters. In this procedure, we have to know the initial condition, which is usually unavailable. However, in the case that the variation of instant frequency is very slow, we may make the approximation,

\[ \omega_n(t) \approx \omega_s^0(t) - t \frac{\partial}{\partial t} \omega_s^0(t). \]

By this way, we could refine the data analysis.

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