Renormalisation group calculation of correlation functions for the 2D random bond Ising and Potts models

Vladimir Dotsenko, Marco Picco and Pierre Pujol

LPTHE

Université Pierre et Marie Curie, PARIS VI
Université Denis Diderot, PARIS VII
Boîte 126, Tour 16, 1er étage
4 place Jussieu
F-75252 Paris CEDEX 05, FRANCE
dotsenko,picco,pujol@lpthe.jussieu.fr

ABSTRACT

We find the cross-over behavior for the spin-spin correlation function for the 2D Ising and 3-states Potts model with random bonds at the critical point. The procedure employed is the renormalisation approach of the perturbation series around the conformal field theories representing the pure models. We obtain a crossover in the amplitude for the correlation function for the Ising model which doesn’t change the critical exponent, and a shift in the critical exponent produced by randomness in the case of the Potts model. A comparison with numerical data is discussed briefly.

*Also at the Landau Institute for Theoretical Physics, Moscow
†Laboratoire associé No. 280 au CNRS
1 Introduction

In the studies of critical phenomena in real physical systems, impurities and inhomogeneities are always present. Many theoretical models have been proposed for the study of random models. In most of these cases the replica trick for quenched systems is employed. This corresponds to computing the averaged free energy by taking $n$ copies of the system and going to the limit $n \to 0$. For models with random bonds, the main problem is to determine if the randomness leaves unchanged the critical properties of the pure system or if the singularities of the thermodynamical functions are eliminated. First results, which suggest an intermediate situation, have been obtained by Harris and Lubensky [1], Grinstein and Luther [2] and Khmelnitskii [3] using the standard $\phi^4$ theory. Other cases, like long-range correlated quenched defects have also been considered, (see for example [4, 5].) A first step in the understanding of the relevance of randomness was given by the Harris criterion [6]: the randomness is relevant (irrelevant) if the specific heat exponent of the pure model is positive (negative). Two dimensional systems are particularly interesting because of the rich structure of conformal invariance in this dimension. Assuming that a random model has a critical point with second order phase transition, the main interesting problem is to determine which conformal field theory represents this model at the infrared fixed point. Let us also mention that an exact result has been obtained by McCoy and Wu [7] who considered a two-dimensional Ising model where only vertical bonds on a square lattice were allowed to acquire the same random value. They found that the logarithmic singularity of the specific heat disappeared completely.

The models that we will study in this paper are the two dimensional Ising and Potts models with random bonds. For the case of Ising model, Harris criterion doesn’t provide a qualitative answer of the relevance of the randomness (because the specific heat exponent for the $2 - D$ Ising model is 0). First results were obtained by Dotsenko and Dotsenko [8] who showed that near the critical point, this model can be represented by an $n = 0$ Gross-Neveu model [9]. With this technique, they found that the specific heat singularity get smoothed as $ln(ln(\frac{1}{|t|}))$ where $|t|$ is the reduced temperature. Calculation of spin-spin correlation function by this technique which involves non-local fermionic representation of
\[ <\sigma\sigma>, \] was later questioned by Shalaev and Shankar \cite{10, 11} who gave arguments that the asymptotic behavior of this correlation function is unchanged by the randomness (see also \cite{12}). Some arguments on why computations using non-local fermionic representation fail to give the correct result for the spin-spin correlation function in the random case will be given at the end of the paper. More recently, this situation was questioned by Ziegler \cite{13}. This author claims that non-perturbative effects introduce an intermediate phase around the critical point of the pure model. On the other hand, numerical simulations of the Ising model \cite{15, 14} seem to confirm the theoretical predictions of the specific heat and spin-spin correlation function asymptotic behavior. For the Potts model, Harris criterion predicts that randomness is relevant and changes the critical behavior. Using conformal field theory techniques, Ludwig \cite{16} also perturbatively computed a shift in the critical exponent of the energy operator in the case of the random Potts model (this critical exponent for the Ising model is also unchanged).

The paper, which complete the work presented in \cite{17}, is organized as follows. In section 2 we explain how the replica trick method can be used to deal with the partition function of the 2D Ising and Potts models with random bonds. Near the critical point, these models are represented by perturbed conformal field theories. We give a brief summary of the Coulomb Gas representation of conformal field theories needed to pertubatively compute correlation functions. We recall also a kind of “\(\epsilon\)” regularization which consists of shifting the central charge of the pure model. Then, in section 3, we sketch the renormalisation group equations for the coupling constant and correlation function. Specializing then to the particular case of the Ising model, we find a cross-over in the amplitude at intermediate distances produced by randomness. In the case of the Potts model, we show that the spin exponent is modified in a power series of \(\epsilon\) due to the randomness. Finally, section 4 contains a summary of our results and a discussion. Technical details of the renormalisation group calculation and the calculation of integrals are put into Appendices.
2 The model

Our starting point is the 2-D Ising-Potts Hamiltonian at the critical temperature:

\[ S_0 = \sum_{<i,j>} (\beta J_0) \delta_{s_i,s_j} \]  

(2.1)

where \( J_0 \) is the coupling between spins, \( s_i \) are the spin variables and \( < i, j > \) means that the sum is restricted to nearest neighbor spins. The unperturbed partition function is:

\[ Z_0 = Tr_{s_i} e^{-S_0} \]

Here, \( Tr_{s_i} \) means a summation over all spin configurations. The addition of a position dependent random coupling constant produces the following change in the partition function: \( J_0 \rightarrow J_0 + \Delta J_{i,j} \). It is well known that at the critical point the unperturbed models are represented in the continuum by unitary minimal conformal field theories in which the relevant operators are the spin operator \( \sigma \) corresponding to \( s_i \) and the energy operator \( \varepsilon \) corresponding to \( \delta_{s_i,s_j} \). In the continuum the term \( \Delta J_{i,j} \) can be written as a position dependent mass term \( m(z) \), which gives the total partition function in the form:

\[ Z = Tr_{s_i} e^{-S_0 - \int m(z) \varepsilon(z) d^2 z} \]  

(2.2)

In order to compute the free energy, we use the replica method. Taking the partition function of \( n \) identical copies of the system and analytically continuing to the limit \( n \rightarrow 0 \) gives the quenched free energy

\[ -\beta F = \ln(Z) = \lim_{n \rightarrow 0} \frac{Z^n - 1}{n} \]

where:

\[ Z^n = \prod_{a=1}^{n} Tr_{a,s_i} e^{-S_{0,a} - \int m(z) \sum_{a=1}^{n} \varepsilon_a(z) d^2 z} \]  

(2.3)

the average of \( Z^n \) is made with a Gaussian distribution for \( m(z) \):

\[ \overline{Z^n} = \int dm(z) Z^n e^{-\frac{1}{2g_0^2} (m(z) - m_0)^2} \]

which gives:

\[ \overline{Z^n} = \prod_{a=1}^{n} Tr_{a,s_i} e^{-S_{0,a} + g_0 \int \sum_{a,b=1}^{n} \varepsilon_a(z) \varepsilon_b(z) d^2 z - m_0 \int \sum_{a=1}^{n} \varepsilon_a(z) d^2 z} \]  

(2.4)
The average with a more complicated distribution gives higher order cumulants, producing in (2.4) terms which can be shown to be irrelevant by power counting [16]. Also, the terms in \( \int \sum_{a,b=1}^{n} \varepsilon_a(z)\varepsilon_b(z) d^2z \) containing the same replica label produce irrelevant operators and can be omitted. We obtain then the following partition function:

\[
\prod_{a=1}^{n} Tr_{a,s} e^{-\sum_{a=1}^{n} S_{0,a} + g_0 \int \sum_{a \neq b} \varepsilon_a(z)\varepsilon_b(z) d^2z - m_0 \int \sum_{a=1}^{n} \varepsilon_a(z) d^2z}
\]  

(2.5)

In the limit \( m_0 \to 0 \), this model corresponds to a conformal field theory perturbed by the term that is quadratic in the \( \varepsilon \) operator. Then the “evolution” of the coupling constants \( g_0 \) and \( m_0 \) under a renormalisation group (R.G.) transformation can be analyzed as well as the behavior of the correlation functions. In the calculation of correlation functions \( < O(0)O(R) > \), where \( O \) is some local operator, we will proceed perturbatively:

\[
< O(0)O(R) > = < O(0)O(R) >_0 + < S_I O(0)O(R) >_0 + \frac{1}{2} < S_I^2 O(0)O(R) >_0 + ... 
\]

where \( <>_0 \) means the expectation value taken with respect to \( S_0 \) and

\[
S_I = \int H_I(z) d^2z = g_0 \int \sum_{a \neq b} \varepsilon_a(z)\varepsilon_b(z) d^2z
\]  

(2.6)

The operator \( O \) is then renormalised as

\[
O \to O(1 + A_1 g_0 + A_2 g_0^2 + A_3 g_0^3 + \cdots) \equiv Z_0 O
\]

The integrals of correlation functions involved in the calculation can be performed by analytic continuation with the Coulomb-gas representation of a conformal field theory [18] where the central charge is \( c = \frac{1}{2} + \epsilon' \). The \( \epsilon' \) term corresponds to a short distance regulator for the integrals. In addition, we also used an infrared (I.R.) cut-off \( r \). The result is then expressed as an \( \epsilon' \) series with coefficients depending on \( r \). The limit \( \epsilon' \to 0 \) corresponds to the pure Ising model at the critical point while the Potts model is obtained for some finite value of \( \epsilon' \). We recall here some notations of the Coulomb-gas representation for the vertex operators [18]. The central charge \( c \) will be characterized in the following by the parameter \( \alpha_+^2 = \frac{2p}{2p-1} = \frac{4}{3} + \epsilon \) with

\[
c = 1 - 24\alpha_0^2; \quad \alpha_\pm = \alpha_0 \pm \sqrt{\alpha_0^2 + 1} \tag{2.7}
\]

\[
\alpha_+\alpha_- = -1
\]
Note that for the pure 2D Ising model $\alpha_+^2 = \frac{4}{3}$ and $c = \frac{1}{2}$ while for the 3-state Potts model $\alpha_+^2 = \frac{6}{5}$, $c = \frac{4}{5}$ and $\epsilon = -\frac{2}{15}$. The vertex operators are defined by

$$V_{nm}(x) = e^{i\alpha_{nm}\phi(x)}$$

where $\phi(x)$ is a free scalar field and where the $\alpha_{nm}$ are given by

$$\alpha_{nm} = \frac{1}{2} (1 - n)\alpha_- + \frac{1}{2} (1 - m)\alpha_+$$

The conformal dimension of an operator $V_{nm}(x)$ is $\Delta_{nm} = -\alpha_{mm}\alpha_{nm}$ with

$$\alpha_{mm} = 2\alpha_0 - \alpha_{nm} = \frac{1}{2} (1 + n)\alpha_- + \frac{1}{2} (1 + m)\alpha_+$$

The spin field $\sigma$ can be represented by the vertex operator $V_{p,p-1}$ whereas $V_{1,2}$ corresponds to the energy operator $\varepsilon$. In the same way, we associate $e^{i\alpha_+\phi(x)}$ to the screening charge operator $V_+(x)$. Note that in the Ising case the $\sigma$ operator can also be represented by the $V_{21}$ operator (since both operators coincide in the limit $\epsilon \to 0$). So, we can represent our spin operator by $V_{k,k-1}$ where $k = \frac{2 + 3\lambda\epsilon}{1 + 3\epsilon}$. We have $\lambda = 2$ for $V_{21}$ and $\lambda = \frac{1}{2}$ for $V_{p,p-1}$.

### 3 Renormalisation Group Equations

In this section, we will deal with the computation of correlation functions of operators $\varepsilon$ and $\sigma$. To compute them, one needs to determine the effect of the random coupling on the operators $\varepsilon$ and $\sigma$ and compute the renormalised operators $\varepsilon'$ and $\sigma'$. This means that we want to compute the functions $Z_\varepsilon$ and $Z_\sigma$ such that

$$\varepsilon' = Z_\varepsilon \varepsilon \quad \text{and} \quad \sigma' = Z_\sigma \sigma$$

A convenient way to define $Z_\varepsilon$ and $Z_\sigma$ is to consider the more general action

$$\sum_{a=1}^n S_{0,a} - g_0 \int \sum_{a,b=1}^n \varepsilon_a(z)\varepsilon_b(z)d^2z + m_0 \int \sum_{a=1}^n \varepsilon_a(z)d^2z - h_0 \int \sum_{a=1}^n \sigma_a(z)d^2z$$

This merely corresponds to the action used in (2.4) with an additional coupling of the $\sigma$ field. Then, with the help of the operator algebra (O.A.) coming from contractions
between $\varepsilon$ and $\sigma$ operators, we will compute the effect of the $g_0 \int \sum_{a,b=1}^{n} \varepsilon_a(z)\varepsilon_b(z) d^2 z$ term (i.e. the random part of the model) on the coupling terms $m_0 \int \sum_{a=1}^{n} \varepsilon_a(z) d^2 z$ and $h_0 \int \sum_{a=1}^{n} \sigma_a(z) d^2 z$. More precisely, we will compute

$$\sum_i \left( \left[ g_0 \int \sum_{a,b=1}^{n} \varepsilon_a(z)\varepsilon_b(z) d^2 z \right]^i \right) m_0 \int \sum_{a=1}^{n} \varepsilon_a(z) d^2 z \simeq m \int \sum_{a=1}^{n} \varepsilon_a(z) d^2 z$$

and

$$\sum_i \left( \left[ g_0 \int \sum_{a,b=1}^{n} \varepsilon_a(z)\varepsilon_b(z) d^2 z \right]^i \right) h_0 \int \sum_{a=1}^{n} \sigma_a(z) d^2 z \simeq h \int \sum_{a=1}^{n} \sigma_a(z) d^2 z$$

$m$ and $h$ being the renormalised coupling constants. Obviously, this computation will be perturbatively made only up to some finite power in $g_0$. In fact, the first step of the computation will be to determine the renormalised $g$ constant on which $Z_{\varepsilon}$ and $Z_{\sigma}$ depend.

### 3.1 Renormalisation of the coupling constant $g$.

The renormalisation of the coupling constant $g$ will be determined directly by a perturbative computation. $g$ is also given by the O.A. producing

$$g_0 \int \sum_{a,b=1}^{n} \varepsilon_a(z)\varepsilon_b(z) d^2 z + \frac{1}{2} \left( g_0 \int \sum_{a,b=1}^{n} \varepsilon_a(z)\varepsilon_b(z) d^2 z \right)^2 + \frac{1}{6} \left( g_0 \int \sum_{a,b=1}^{n} \varepsilon_a(z)\varepsilon_b(z) d^2 z \right)^3 + \cdots = g \int \sum_{a,b=1}^{n} \varepsilon_a(z)\varepsilon_b(z) d^2 z$$

with $g = g_0 + A_2 g_0^2 + A_3 g_0^3 + \cdots$ where $A_2$ comes from

$$\frac{1}{2} \int \sum_{a,b=1}^{n} \varepsilon_a(z)\varepsilon_b(z) d^2 z \int \sum_{a,b=1}^{n} \varepsilon_a(z)\varepsilon_b(z) d^2 z = A_2 \int \sum_{a,b=1}^{n} \varepsilon_a(z)\varepsilon_b(z) d^2 z + \cdots$$

and $A_3$ from

$$\frac{1}{6} \int \sum_{a,b=1}^{n} \varepsilon_a(z)\varepsilon_b(z) d^2 z \int \sum_{a,b=1}^{n} \varepsilon_a(z)\varepsilon_b(z) d^2 z \int \sum_{a,b=1}^{n} \varepsilon_a(z)\varepsilon_b(z) d^2 z$$

$$= A_3 \int \sum_{a,b=1}^{n} \varepsilon_a(z)\varepsilon_b(z) d^2 z + \cdots$$
Here, we will only perform the computations up to the third order in $g_0$. The technical
details of the computations of $A_2$ and $A_3$ are given in the appendix A with the following
result for $g(r)$:

$$g(r) = r^{-3\epsilon} \left( g_0 - g_0^2 4\pi (n-2) \frac{r^{-3\epsilon}}{3\epsilon} + g_0^3 8\pi^2 (n-2) \frac{r^{-6\epsilon}}{3\epsilon} \left(1 + \frac{2(n-2)}{3\epsilon}\right) \right)$$  \hspace{1cm} (3.8)

Note that we multiply the result by $r^{-3\epsilon}$ in order to obtain a dimensionless coupling
constant $g(r)$.

In the computation of correlation functions, we will need the $\beta$-function associated to
the renormalisation group equations of $g(r)$. It can be derived directly from the previous
expression of $g(r)$ with the result :

$$\beta(g) = \frac{dg}{d\ln(r)} = -3\epsilon g(r) + 4\pi(n-2)g^2(r) - 16\pi^2(n-2)g^3(r) + O(g^4(r))$$  \hspace{1cm} (3.9)

Finally, taking the limit $n \to 0$, we obtain for the $\beta$-function up to the third order :

$$\beta(g) = -3\epsilon g - 8\pi g^2 + 32\pi^2 g^3$$  \hspace{1cm} (3.10)

We can then immediately note that, in the limit $\epsilon \to 0$ (i.e. the random Ising model),
$\beta(g)$ has an infrared fixed point at $g = 0$. In the case where $\epsilon < 0$ (i.e. the random Potts
model) the infrared fixed point is located at $g_c = -\frac{3\epsilon}{8\pi} + \frac{9\epsilon^2}{16\pi} + O(\epsilon^3)$.

### 3.2 Renormalisation of $\sigma$ and $\epsilon$

In order to be able to compute the correlation functions of $\sigma$ and $\epsilon$, the second step is
to determine the effect of the renormalisation on these operators. One needs to compute
the multiplicative functions $Z_\sigma$ and $Z_\epsilon$. This will be made by computing the renor-
malised coupling terms $m \int \sum_{a=1}^{n} \epsilon_a(z)d^2z = (m_0 Z_\epsilon) \int \sum_{a=1}^{n} \epsilon_a(z)d^2z$ and $h \int \sum_{a=1}^{n} \sigma_a(z)d^2z = (h_0 Z_\sigma) \int \sum_{a=1}^{n} \sigma_a(z)d^2z$. A direct computation of both $m$ and $h$ will provide us with the functions $Z_\sigma$ and $Z_\epsilon$. As for the computation of $g = Z_g g_0$, we will compute in perturba-
tion :

$$m_0 \int \sum_{a=1}^{n} \epsilon_a(z)d^2z + g_0 m_0 \int \sum_{a,b=1}^{n} \epsilon_a(z)\epsilon_b(z)d^2z \int \sum_{a=1}^{n} \epsilon_a(z)d^2z +$$  \hspace{1cm} (3.11)
\[ \frac{g_0^2}{2} m_0 \left( \int \sum_{a, b=1}^{n} \varepsilon_a(z) \varepsilon_b(z) \right)^2 \int \sum_{a=1}^{n} \varepsilon_a(z) d^2 z + \cdots = m \int \sum_{a=1}^{n} \varepsilon_a(z) d^2 z \]

and \( m = m_0(1 + B_1 g_0 + B_2 g_0^2 + \cdots) \) with \( B_1 \) defined by

\[ \int \sum_{a, b=1}^{n} \varepsilon_a(z) \varepsilon_b(z) d^2 z \int \sum_{a=1}^{n} \varepsilon_a(z) d^2 z = B_1 \int \sum_{a=1}^{n} \varepsilon_a(z) d^2 z \]

and \( B_2 \) by

\[ \left( \int \sum_{a, b=1}^{n} \varepsilon_a(z) \varepsilon_b(z) \right)^2 \int \sum_{a=1}^{n} \varepsilon_a(z) d^2 z = B_2 \int \sum_{a=1}^{n} \varepsilon_a(z) d^2 z \]

The details of the computation are presented in the appendix B, with the result

\[ r^{-1 + \frac{8}{3} \epsilon} m(r) = m_0 \left( 1 - 4 \pi (n-1) g_0 \frac{r^{-3 \epsilon}}{3 \epsilon} + 4 \pi^2 (n-1) g_0^2 \frac{r^{-6 \epsilon}}{3 \epsilon} \left( 1 + \frac{4n-6}{3 \epsilon} \right) \right) \]

Here again, we multiply \( m(r) \) by \( r^{-1 + \frac{8}{3} \epsilon} \) in order to obtain a dimensionless coupling constant. We also give the R.G. equation for \( Z_\epsilon \), which we will need later.

\[ \frac{d \ln(Z_\epsilon(r))}{d \ln(n)} = 4 \pi (n-1) g - 8 \pi^2 (n-1) g^2 \] (3.12)

Similarly, for the coupling constant \( h_0 \), we compute up to the third order :

\[ \left( \int \sum_{a, b=1}^{n} \varepsilon_a(z) \varepsilon_b(z) \right)^i \int \sum_{a=1}^{n} \sigma_a(z) d^2 z = C_i \int \sum_{a=1}^{n} \sigma_a(z) d^2 z \]

We give here directly the result (see eq. (C.7) in appendix C):

\[ r^{-\frac{16}{9} - a(\epsilon)} h(r) = h_0 \left( 1 + (n-1) g_0^2 \frac{r^{-6 \epsilon}}{2} \right) \left[ 1 + \frac{4}{3} (2 - \lambda) \frac{\Gamma^2(-\frac{2}{3}) \Gamma^2(\frac{1}{3})}{\Gamma^2(-\frac{1}{3}) \Gamma^2(-\frac{1}{6})} \right] \]

\[ -12(n-1)(n-2) g_0^3 \frac{r^{-9 \epsilon}}{9 \epsilon} \left[ 1 + \frac{8}{9} (2 - \lambda) \frac{\Gamma^2(-\frac{2}{3}) \Gamma^2(\frac{1}{3})}{\Gamma^2(-\frac{1}{3}) \Gamma^2(-\frac{1}{6})} \right] \] (3.13)

The multiplicative term \( r^{-\frac{16}{9} - a(\epsilon)} \) in front of \( h(r) \) is introduced in order to make this parameter dimensionless. Here, \( a(\epsilon) \) is a function of \( \epsilon \) depending on which representation of the spin field we are taking in the Coulomb gas picture (see section 2). Its explicit form will be irrelevant in the following. The corresponding R.G. equation for \( Z_\sigma \) will be given by

\[ \frac{d \ln(Z_\sigma(r))}{d \ln(r)} = -3(n-1) g^2(r) \pi^2 \epsilon \left[ 1 + \frac{4}{3} (2 - \lambda) \frac{\Gamma^2(-\frac{2}{3}) \Gamma^2(\frac{1}{3})}{\Gamma^2(-\frac{1}{3}) \Gamma^2(-\frac{1}{6})} \right] \]

\[ + 4(n-1)(n-2) \pi^3 g^3(r) \] (3.14)
4 Correlation Functions

We now have all the ingredients needed in order to compute the correlation functions. They will be calculated with the help of the R.G. equations, for the theory with \( m_0, h_0 \to 0 \). From the R.G. equations, we have:

\[
< \varepsilon(0) \varepsilon(sR) >_{r,g(r)} = \frac{Z_{\varepsilon}^2(sR, g(s))}{Z_{\varepsilon}^2(r, g(r))} s^{-2\Delta\varepsilon} < \varepsilon(0) \varepsilon(R) >_{r,g(sr)}
\]

This can be written as:

\[
< \varepsilon(0) \varepsilon(sR) >_{r,g(r)} = e^{g(s) \frac{\gamma_\varepsilon(g)}{\beta(g)}} s^{-2\Delta\varepsilon} < \varepsilon(0) \varepsilon(R) >_{r,g(sr)}
\] (4.1)

where we used the notation:

\[
\frac{d\ln Z_{\varepsilon}}{d\ln r} = \gamma_\varepsilon(g)
\] (4.2)

and \( g(s) = g(sr); g_0 = g(r) \). We assume now \( r \) to be a lattice cut-off scale. In a similar way for \( < \sigma(0) \sigma(R) > \) the R.G. equation is:

\[
< \sigma(0) \sigma(sR) >_{r,g(r)} = e^{g(s) \frac{\gamma_\sigma(g)}{\beta(g)}} s^{-2\Delta\sigma} < \sigma(0) \sigma(R) >_{r,g(sr)}
\] (4.3)

with

\[
\frac{d\ln Z_{\sigma}}{d\ln r} = \gamma_\sigma(g)
\] (4.4)

In equations (4.1)-(4.3), \( R \) is an arbitrary scale which can be fixed to one lattice spacing \( r \) of a true statistical model. The dependence of \( < \sigma(0) \sigma(r) >_{r,g(s)} \) on \( s \) will then be negligible, assuming that there are not interactions on distances smaller than \( r \). Therefore, it reduces to a constant. Then, \( s \) will measure the number of lattice spacings between two spins in \( < \sigma(0) \sigma(sR) > \). In the following, when we treat separately the Ising model and the 3-states Potts model, we adopt the choice \( r = 1 \).

4.1 The Ising model

The Ising model corresponds to the case \( \varepsilon \to 0 \) and so the \( \beta \) function is:

\[
\beta(g) = -8\pi g^2 + 32\pi^2 g^3
\] (4.5)
Therefore, we can see that the I.R. fixed point is located at $g = 0$. Also we have, by eqs. (3.12), (3.14) for $n = 0, \epsilon = 0$ and definitions (4.2), (4.4),

$$\gamma_\epsilon(g) = -4\pi g + 8\pi^2 g^2 \quad (4.6)$$

$$\gamma_\sigma(g) = 8\pi g^3 \quad (4.7)$$

The integral for the $\epsilon$ correlation function, eq.(4.1), gives:

$$2 \int_{g_0}^{g(s)} \frac{\gamma_\epsilon(g)}{\beta(g)} dg = \int_{g_0}^{g(s)} \frac{1 - 2\pi g}{1 - 4\pi g} \approx \int_{g_0}^{g(s)} \frac{1 + 2\pi g}{g} \frac{dg}{g}$$

$$= 2\pi (g(s) - g_0) + \ln \left( \frac{g(s)}{g_0} \right) \quad (4.8)$$

Now, we need to compute $g(s)$. The integration of equation $\beta(g) = \frac{dg}{d \ln(r)}$ gives:

$$\int_{g_0}^{g(s)} \frac{dg}{-8\pi g^2 + 32\pi^2 g^3} = \int_{r}^{s^r} d\ln(r)$$

with the following solution up to the second order:

$$g(s) = \frac{g_0}{1 + 8\pi g_0 ln(1 + 8\pi g_0 ln(s))} + O(g_0^3) \quad (4.9)$$

So, $< \epsilon \epsilon >$ correlation function is given by:

$$< \epsilon(0) \epsilon(s) > \sim \frac{g(s)}{g_0} \frac{1}{1 - 2\pi \left(g_0 - g(s)\right)} s^{-2\Delta_\epsilon} \quad (4.10)$$

$$\sim \frac{1}{1 + 8\pi g_0 ln(s)} \left( 1 + \frac{4\pi g_0}{1 + 8\pi g_0 ln(s)} \left( \ln(1 + 8\pi g_0 ln(s)) - 4\pi g_0 ln(s) \right) \right) s^{-2\Delta_\epsilon} + O(g_0^2)$$

For the $\sigma$ correlation function, the computation is similar. In fact, in that case, we have $\gamma_\sigma(g) = 8\pi^3 g^3$. Thus, keeping only the first order of the $\beta$-function (i.e. $\beta(g) = -8\pi g^2$), we obtain:

$$2 \int_{g_0}^{g(s)} \frac{\gamma_\sigma(g)}{\beta(g)} dg = -2\pi^2 \int_{g_0}^{g(s)} g dg = -\pi^2 \left( g(s)^2 - g_0^2 \right) + O(g_0^3) \quad (4.11)$$

The $< \sigma \sigma >$ correlation function is then found to be given by:

$$< \sigma(0) \sigma(s) > \sim \left( 1 + \pi^2 \left( g_0^2 - g(s)^2 \right) \right) s^{-2\Delta_\sigma} \quad (4.12)$$

$$\sim \left( 1 + \pi^2 g_0^2 \left( 1 - \frac{1}{(1 + 8\pi g_0 ln(s))^2} \right) \right) s^{-2\Delta_\sigma} + O(g_0^3)$$
The calculation of the $g^3$ term in the $\beta$ function and the $g^2$ term in the renormalisation of $\varepsilon$ was already done in [16], extending the one loop result of [8]. We recovered these higher order corrections using a different technique, which allowed us to calculate also the modified correlation function of the spin operators.

### 4.2 The Potts model

We now consider the 3-state Potts model. With the convention that we use this case corresponds to $\varepsilon = -\frac{2}{15}$. $\beta(g)$ is given in eq.(3.10), and

$$\gamma_\varepsilon(g) = -4\pi g + 8\pi^2 g^2$$  \hspace{1cm} (4.13)

$$\gamma_\sigma(g) = 3\pi^2 \varepsilon \left(1 + \frac{4}{3}(2 - \lambda) \frac{\Gamma^2(-\frac{2}{3})\Gamma^2(\frac{1}{6})}{\Gamma^2(-\frac{4}{3})\Gamma^2(-\frac{1}{6})}\right) g^2 + 8\pi^3 g^3$$  \hspace{1cm} (4.14)

At long distances, the integrals in eqs.(4.1), (4.3) will be dominated by the region $g \sim g_c$, with $g_c = -\frac{3\varepsilon}{8\pi} + \frac{g^2_\varepsilon}{16\pi} + O(\varepsilon^3)$. This is different from the Ising model, because here $\gamma_\varepsilon(g_c)$ and $\gamma_\sigma(g_c)$ have finite values for $g = g_c$. Thus,

$$\int_{g_0}^{g(s)} \frac{\gamma_\varepsilon(g)}{\beta(g)} dg \approx \gamma_\varepsilon(g_c) \ln(s)$$  \hspace{1cm} and  \hspace{1cm} $$\int_{g_0}^{g(s)} \frac{\gamma_\sigma(g)}{\beta(g)} dg \approx \gamma_\varepsilon(g_c) \ln(s)$$  \hspace{1cm} (4.15)

The correlation functions can then be deduced directly:

$$\langle \varepsilon(0)\varepsilon(s) \rangle_{g_0} \sim s^{-(2\Delta_\varepsilon - 2\gamma_\varepsilon(g_c))} \quad \text{and} \quad \langle \sigma(0)\sigma(s) \rangle_{g_0} \sim s^{-(2\Delta_\sigma - 2\gamma_\sigma(g_c))}$$  \hspace{1cm} (4.16)

So, we can see that a direct consequence of the new IR fixed point is a modification of the critical exponents $\Delta_\varepsilon$ and $\Delta_\sigma$. A straightforward computation will give these new exponents:

$$2\Delta'_\varepsilon = 2\Delta_\varepsilon - 2\gamma_\varepsilon(g_c) = 2\Delta_\varepsilon + 8\pi g_c - 16\pi^2 g_c^2$$

$$= 2\Delta_\varepsilon - 3\varepsilon + \frac{9}{4} \varepsilon^2 + O(\varepsilon^3)$$  \hspace{1cm} (4.17)

and

$$2\Delta'_\sigma = 2\Delta_\sigma - 2\gamma_\sigma(g_c) = 2\Delta_\sigma - 6\pi^2 g_c^2 \left[1 + \frac{4}{3}(2 - \lambda) \frac{\Gamma^2(-\frac{2}{3})\Gamma^2(\frac{1}{6})}{\Gamma^2(-\frac{4}{3})\Gamma^2(-\frac{1}{6})}\right] - 16\pi^3 g_c^3$$

$$= 2\Delta_\sigma - \frac{9}{8}(2 - \lambda) \frac{\Gamma^2(-\frac{2}{3})\Gamma^2(\frac{1}{6})}{\Gamma^2(-\frac{4}{3})\Gamma^2(-\frac{1}{6})} \varepsilon^3 + O(\varepsilon^4)$$  \hspace{1cm} (4.18)
\[ \lambda = \frac{1}{2} \] and the final result for the new critical exponent is:

\[
2\Delta'_\sigma = 2\Delta_\sigma - \frac{27}{16} \frac{\Gamma^2\left(-\frac{2}{3}\right)\Gamma^2\left(-\frac{1}{6}\right)}{\Gamma^2\left(-\frac{1}{2}\right)\Gamma^2\left(-\frac{1}{6}\right)} \epsilon^3 + O(\epsilon^4)
\]

(4.19)

The value of the critical exponent for the energy operator was already computed, up to the second order, by Ludwig [16]. For the spin operator, deviation of the critical exponent from the pure case appears only at the third order, which we compute. This deviation is in fact very small. While in case of 3-state Potts model without disorder \(2\Delta_\sigma = \frac{4}{15}\), we obtain the new critical exponent \(2\Delta'_\sigma = \frac{4}{15} + 0.00264 = 0.26931\). Thus, the deviation corresponds to a modification of 1%.

5 Conclusions

In case of Ising model spin-spin function the calculation of up to third order of the renormalisation group was needed to find the deviation from the perfect model case, in the form of the cross-over in the amplitude. This completes the observations of [10, 11, 12], based on absence of renormalisation of this function in the first order, that asymptotically the spin-spin function has the same exponent as in case of the perfect lattice model.

Earlier calculation of the spin-spin function in the second reference of [8], in which a non-local fermionic representation had been used, gave a different result, which was not confirmed neither by further theoretical calculations, based on local renormalisation directly of spin operators, nor by the numerical simulations in [14, 15]. In calculation of [8] a double summation series had been involved, both sums being log divergent. One sum was due to non-local representation of the spin-spin function, another was due to randomness-caused interactions. In the calculations, a particular ordering of the two summations had been employed: renormalizing the terms of the first series and summing up next. We may guess, lacking a better established reason, that such a treatment might not be justified.

Recently the numerical simulations of the random Ising model has been performed which measure directly the deviation of \(<\sigma\sigma>\) from the pure Ising model at the critical
point [15]. These measurements were made for disorder such that $8\pi g_0 \approx 0.3$. Deviations predicted by our computations are very small. They correspond to 0.1%. The deviations obtained in numerical simulations are around ten times larger, and they are of opposite sign, \textit{i.e.} correspond to an extra decrease of the spin-spin function with distance $r$.

In [15], it has been checked that this decrease corresponds, within the accuracy of the measurements, to a factor function of the ratio $r/L$, $F(r/L)$, $r$ being the distance between the spins and $L$ is the lattice size. So they correspond to finite size effects, being different for perfect and random models. We would suggest, on the bases of our calculation of the $r$ dependence of the spin-spin function on an infinite lattice, that numerical deviations will continue to be plotted by the same curve $F(r/L)$, if one measures $<\sigma\sigma>$ for different lattice sizes as it has been done in [15], until the accuracy reaches the value of the $r$-deviation which we calculated here. Only then the curves for different $L$ will split.

In case of Potts model, the deviation of $<\sigma\sigma>$ appear also only in the third order like for the Ising model, this time in the exponent of the asymptotic function, eq.(4.19). This might be easier to check with numerical simulations [20]. We refered to the 3-state Potts model in Sec.4.2 because this particular case might be easier to realize numerically, and also because we expect that $\epsilon$-expansion should be well defined in this case. Extending our result to the 4-state Potts model might be questionable, as this is actually the limiting case in the range of minimal conformal theories which are parametrized by $\epsilon$.

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A Renormalisation of $g$

In this appendix, we shall describe the calculations for the renormalisation of $g$ up to the second order.

A.1 First order

The first order correction to $g$ comes from the contraction of two ($\varepsilon\varepsilon$) terms (plus some combinatorial factor corresponding to all of the possible contractions):

$$
\frac{g_0^2}{2} \int_{|x-y|<r} \sum_{a\neq b} \varepsilon_a(x)\varepsilon_b(x) \sum_{c\neq d} \varepsilon_c(y)\varepsilon_d(y) d^2 x d^2 y
$$

When $b = c \neq a, d$ this contraction gives:

$$
A_2(r, \varepsilon) g_0^2 \int \sum_{a \neq d} \varepsilon_a(x)\varepsilon_d(x) d^2 x
$$

Here, $A_2(r, \varepsilon)$ is the result of the integral produced by the contraction of two $\varepsilon$ operators:

$$
A_2(r, \varepsilon) = 2(n - 2) \int_{|x-y|<r} <\varepsilon(x)\varepsilon(y)>_0 d^2 y = 4\pi(n - 2) \int_{y<r} \left( \frac{dy}{y^{1+3\epsilon}} \right)
$$

$$
= -4\pi(n - 2) \left( \frac{r^{-3\epsilon}}{3\epsilon} \right)
$$

$<\varepsilon(x)\varepsilon(y)>_0$ being the standard correlation function of the unperturbed conformal field theory.

A.2 Second order

We will now compute the second order corrections to $g$, produced by the contractions of three ($\varepsilon\varepsilon$) terms. Here, there will be two contributions coming from:

$$
\frac{g_0^3}{3!} \int_{|x-y|,|x-z|<r} \sum_{a\neq b} \varepsilon_a(x)\varepsilon_b(x) \sum_{c\neq d} \varepsilon_c(y)\varepsilon_d(y) \sum_{e\neq f} \varepsilon_e(z)\varepsilon_f(z) d^2 x d^2 y d^2 z
$$

The first one corresponds to the case when $b = c; d = e; b, d \neq a, f$. We will denote this term by:

$$
A_{3,1}(r, \varepsilon) g_0^3 \int \sum_{a \neq f} \varepsilon_a(x)\varepsilon_f(x) d^2 x
$$
where \( A_{3,1}(r, \epsilon) \) corresponds to the following:

\[
A_{3,1}(r, \epsilon) = 4(n - 2)(n - 3) \int_{|y - x|, |z - x| < r} < \epsilon(x)\epsilon(y) >_0 < \epsilon(y)\epsilon(z) >_0 d^2zd^2y \quad (A.3)
\]

Now, replacing \(< \epsilon(x)\epsilon(y) >_0 \) by \(|x - y|^{-2-3\epsilon} \) and performing a trivial change of variable, this gives:

\[
A_{3,1}(r, \epsilon) = 8\pi(n - 2)(n - 3) \int_{y < r} \left(\frac{dy}{y^{1+6\epsilon}}\right) \int |z|^{-2-3\epsilon}|z - 1|^{-2-3\epsilon}dz
\]

\[
= 16\pi^2(n - 2)(n - 3) \left(\frac{r^{-6\epsilon}}{9\epsilon^2}\right) \quad (A.4)
\]

The second term is produced when \( a = c = e \neq f; b = d \neq f \). This will be denoted by:

\[
A_{3,2}(r, \epsilon)g_0^3 \int \sum_{a \neq f} \epsilon_a(x)\epsilon_f(x)d^2x
\]

with:

\[
A_{3,2}(r, \epsilon) = 4(n - 2) \int_{|y - x|, |z - x| < r} < \epsilon(x)\epsilon(y)\epsilon(z)(\infty) >_0 < \epsilon(y)\epsilon(z) >_0 d^2yd^2z \quad (A.5)
\]

Here, the \(< \epsilon(x)\epsilon(y)\epsilon(z)(\infty) >_0 \) term corresponds to the result of \( \varepsilon\varepsilon\varepsilon \rightarrow \varepsilon \) obtained by projecting \( \varepsilon\varepsilon\varepsilon \) over \( \varepsilon(\infty) \). In order to compute \( A_{3,2}(r, \epsilon) \), we need to use the Coulomb gas representation for this 4-points correlation function:

\[
A_{3,2}(r, \epsilon) = 4(n - 2)N \times \int_{|y - x|, |z - x| < r} < V_{12}(x)V_{12}(y)V_{12}(z)V_{12}(\infty) > |y - z|^{-2-3\epsilon}d^2yd^2zd^2u \quad (A.6)
\]

\( N \) is a normalization constant which can be fixed by the OA relation in the following way. We have:

\[
< \epsilon(0)\epsilon(R)\epsilon(x)\epsilon(y) >= N \int < V_{12}(0)V_{12}(R)V_{12}(x)V_{12}(y) > d^2u
\]

In the limit \( R \rightarrow 0 \):

\[
\epsilon(0)\epsilon(R) = \frac{1}{R^{4\Delta_{12}}} I + ...
\]

( \( I \) is the identity operator) and

\[
< \epsilon(0)\epsilon(R)\epsilon(x)\epsilon(y) >= \frac{1}{R^{4\Delta_{12}}|x - y|^{4\Delta_{12}}} < \epsilon(x)\epsilon(y) >= \frac{1}{R^{4\Delta_{12}}|x - y|^{4\Delta_{12}}}
\]
For the r.h.s. we obtain

\[
N \frac{1}{R^{4\Delta_{12}}} \int < V_{20\alpha}(0)V_{12}(x)V_{12}(y)V^{+}(u) > d^{2}u
\]

which produces

\[
N \frac{1}{R^{4\Delta_{12}}|x - y|^{4\Delta_{12}}} \int |u|^{4\alpha_{12}\alpha_{+}}|u - 1|^{8\alpha_{0}\alpha_{+}}d^{2}u
\]

Comparing both results we obtain:

\[
N = \left( \int |u|^{4\alpha_{12}\alpha_{+}}|u - 1|^{8\alpha_{0}\alpha_{+}}d^{2}u \right)^{-1} = \frac{2}{\sqrt{3}} \frac{(\Gamma(-\frac{2}{3}))^{2}}{(\Gamma(-\frac{5}{3}))^{4}} \quad (A.7)
\]

We now return to the computation of \(A_{3,2}(r, \epsilon)\). By redefining \(y \rightarrow y - x; z \rightarrow z - x\), \(A_{3,2}(r, \epsilon)\) is transformed into the following integral

\[
A_{3,2}(r, \epsilon) = 8\pi(n - 2)N \times \int \left( \frac{dy}{y^{1+6\epsilon}} \right) \int |z|^{-4\Delta_{12}}|z - 1|^{-4\Delta_{12} + 4\alpha_{+}^{2}}|u|^{4\alpha_{+}\alpha_{12}}|u - 1|^{4\alpha_{+}\alpha_{12}}|u - z|^{4\alpha_{+}\alpha_{12}}d^{2}z d^{2}u
\]

The first integral equals \(-\frac{r^{-6\epsilon}}{6\epsilon}\). The second integral is more complicated. We will show in appendix D how to calculate it. In fact, it equals (cf. (D.10)) \(3\pi \frac{\Gamma^{4}(-\frac{1}{3})}{\Gamma^{4}(-\frac{5}{3})}\). We obtain:

\[
A_{3,2}(r, \epsilon) = 8\pi^{2}(n - 2) \left( \frac{r^{-6\epsilon}}{3\epsilon} \right) \quad (A.9)
\]

In fact, in appendix D, the \(z\) integration is performed over the whole complex plane, while in the integral (A.8), the domain of integration is restricted to the disk \(|z| < \frac{r}{|y|}\). This introduces a new singularity at infinity, which we need to subtract. However, we will show that this singularity is cancelled by another singularity at the origin, in the computation that we perform in appendix D. Thus, we need only to compute the singularity at the origin, and this singularity must be added to the result. That the singularity at the origin cancels the one at infinity is obvious if we isolate these two parts in

\[
\int < \epsilon(0)\epsilon(1)\epsilon(z)\epsilon(\infty) > < \epsilon(1)\epsilon(z) > d^{2}z
\]

The singularity in \(z \rightarrow 0\) is \(\int_{|z|<r/|y|} d^{2}z|z|^{-2-3\epsilon} = 2\pi \frac{(r/|y|)^{-3\epsilon}}{-3\epsilon}\) and the one in \(z \rightarrow \infty\) is \(\int_{|z|>r/|y|} d^{2}z|z|^{-2-3\epsilon} = 2\pi \frac{(r/|y|)^{-3\epsilon}}{3\epsilon}\), confirming that these contributions cancel each other.
We still need to compute the extra contribution from \( z \to 0 \). Some easy manipulations show that this contribution is the same as the one from the integral in equation (A.3). So the third second order term is:

\[
A_{3,3}(r, \epsilon) = 16\pi^2(n - 2) \left( \frac{r^{-6\epsilon}}{9\epsilon^2} \right) \quad (A.10)
\]

Finally, in the case \( a = c = e \) and \( b = d = f \), a straightforward calculation of the integral (A.2) gives us a result of the form \( \int_{|z|<r} d^2z z^{-4-6\epsilon} \). This does not contain any singularities. Collecting all of our results together, we obtain the re-normalized coupling constant:

\[
g(r) = g_0 + g_0^2 A_2(r, \epsilon) + g_0^3 (A_{3,1}(r, \epsilon) + A_{3,2}(r, \epsilon) + A_{3,3}(r, \epsilon))
\]

\[
= g_0 \left( 1 - g_0 4\pi(n - 2) \frac{r^{-3\epsilon}}{3\epsilon} + g_0^2 8\pi^2(n - 2) \frac{r^{-6\epsilon}}{3\epsilon} \left( 1 + 2 \frac{(n - 2)}{3\epsilon} \right) \right) \quad (A.11)
\]

## B  Renormalisation of \( m \)

This appendix is devoted to the computation of the renormalisation of the coupling constant \( m \) associated to the energy operator. As in appendix A, we will compute perturbatively by contracting \( \epsilon \) operators.

### B.1 First order

The first order correction to \( m \) is produced by the product of an \( g_0(\epsilon \epsilon) \) operator with an \( m_0(\epsilon) \) operator:

\[
g_0 m_0 \int_{|x-y|<r} \sum_{a \neq b} \epsilon_a(x) \epsilon_b(x) \sum_c \epsilon_c(y) d^2x d^2y
\]

When \( b = c \), we can contract two \( \epsilon \) operators:

\[
g_0 m_0 B_1(r, \epsilon) \int \sum_a \epsilon_a(y) d^2y
\]

with

\[
B_1(r, \epsilon) = 2(n - 1) \int_{|y-x|<r} <\epsilon(x)\epsilon(y)>_0 d^2y
\]
\[ = 4\pi(n - 1) \int_{y < r} \left( \frac{dy}{y^{1+3\epsilon}} \right) = -4\pi(n - 1) \left( \frac{r^{-3\epsilon}}{3\epsilon} \right) \]  

(B.1)

### B.2 Second order

At the second order, corresponding to the product of two \(g_0(\varepsilon\varepsilon)\) operators with an \(m_0(\varepsilon)\) one, we have to compute:

\[
g_0^2 m_0 \int_{|x-y|,|x-z|<r} \sum_{a\neq b} \varepsilon_a(x)\varepsilon_b(x) \sum_{c \neq d} \varepsilon_c(y)\varepsilon_d(y) \sum_{e} \varepsilon_e(z) d^2xd^2ydz \tag{B.2}
\]

This expression is, in fact, very similar to the one in eq.(A.2). Again there will be two contributions. The first one corresponds to \(b = c \neq d = e\). We will denote this case by:

\[
g_0^2 m_0 B_{2,1}(r, \epsilon) \int \sum_{a} \varepsilon_a(y) d^2y
\]

\(B_{2,1}(r, \epsilon)\) corresponds to \(A_{3,1}(r, \epsilon)\) of appendix A, where the combinatorial term \((n-2)(n-3)\) has been replaced by \((n-1)(n-2)\). From eq.(A.4) we can read the result:

\[
B_{2,1}(r, \epsilon) = 16\pi^2(n-1)(n-2) \left( \frac{r^{-6\epsilon}}{9\epsilon^2} \right) \tag{B.3}
\]

The second case is produced when \(b = c = e = a = d\):

\[
g_0^2 m_0 B_{2,2}(r, \epsilon) \int \sum_{a} \varepsilon_a(y) d^2y
\]

Again, \(B_{2,2}(r, \epsilon)\) corresponds to \(A_{3,2}(r, \epsilon)\) computed in appendix A where \((n-2)\) has been replaced by \(\frac{(n-1)}{2}\). From eq.(A.9), we have

\[
B_{2,2}(r, \epsilon) = 4\pi^2(n-1) \left( \frac{r^{-6\epsilon}}{3\epsilon} \right) \tag{B.4}
\]

Again, the \(B_{2,2}(r, \epsilon)\) term must be completed by a \(B_{2,3}(r, \epsilon)\) term (the equivalent of \(A_{2,3}(r, \epsilon)\)) which is:

\[
B_{2,3}(r, \epsilon) = 8\pi^2(n-1) \left( \frac{r^{-6\epsilon}}{9\epsilon^2} \right) \tag{B.5}
\]

Collecting all of these results together, we then obtain for \(m(r) = m_0(1 + B_1(r, \epsilon)g_0 + g_0^2(B_{2,1}(r, \epsilon) + B_{2,2}(r, \epsilon) + B_{2,3}(r, \epsilon))\) at the second order in \(g(r)\):

\[
m(r) = m_0 \left( 1 - 4\pi(n-1)g_0 \frac{r^{-3\epsilon}}{3\epsilon} + 4\pi^2(n-1)g_0^2 \left( \frac{r^{-6\epsilon}}{3\epsilon} \right) (1 + \frac{4n-6}{3\epsilon}) \right) \tag{B.6}
\]
C Renormalisation of $h$

We now turn to the computation of the renormalisation of the coupling constant $h(r)$ associated to the $\sigma$ operator. Here the situation is a little bit different, because the $\varepsilon$ operator and the $\sigma$ operator are involved. At the first order, we must compute the product of $g_0(\varepsilon\varepsilon)$ on $h_0(\sigma)$. This product will give in fact no contribution (the OPA of $\sigma$ and $\varepsilon$ does not contain $\sigma$ operators at the first order) and so we go directly to the second order.

C.1 Second order

Here, we will compute the product of two $g_0(\varepsilon\varepsilon)$ with an $h_0(\sigma)$ operator:

$$\frac{g_0^2}{2}h_0\int \sum_{a\neq b}\varepsilon_a(x)\varepsilon_b(x) \sum_{c\neq d}\varepsilon_c(y)\varepsilon_d(y) \sum_e \sigma_e(z) d^2xd^2yd^2z$$

When $a = c = e \neq b = d$ the product $\sigma\varepsilon\varepsilon$ will contain a $\sigma$ operator. By projecting $\sigma\varepsilon\varepsilon$ on a $\sigma(\infty)$ operator, we obtain:

$$h_0g_0^2C_2(r, \varepsilon) \int \sum_{a=1}^n \sigma_a(z) d^2z$$

where

$$C_2(r, \varepsilon) = 2(n - 1) \int_{|y-x|,|z-x|<r} <\sigma(x)\varepsilon(y)\varepsilon(z)\sigma(\infty)>_0 <\varepsilon(y)\varepsilon(z)>_0 d^2yd^2z \quad \text{(C.1)}$$

Using the Coulomb gas representation, this integral becomes:

$$4(n - 1)N\pi \int_{y<r} \left( \frac{dy}{y^{1+6\varepsilon}} \right) \times$$

$$\int |z|^{4\alpha_{12}\alpha_{k,k'}}|z - 1|^{-4\Delta_{12}+4\alpha_{12}}|u|^{4\alpha_{12}\alpha_{k,k'}}|u-1|^{4\alpha_{12}\alpha_{12}}|u-z|^{4\alpha_{12}d^2}z^d u \quad \text{(C.2)}$$

where the normalisation $N$ is the same as the one computed in the appendix A, see eq.(A.7). The first part of this expression (the $y$ integral) gives $(-\frac{r^{-6\varepsilon}}{6\varepsilon})$, while the second one is computed in appendix D, see (D.13). We obtain:

$$C_2(r, \varepsilon) = (n - 1)\pi^2 \frac{r^{-6\varepsilon}}{2} \left[ 1 + \frac{4}{3}(2 - \lambda) \frac{\Gamma^2(-\frac{2}{3})\Gamma^2(\frac{1}{6})}{\Gamma^2(-\frac{1}{3})\Gamma^2(-\frac{1}{6})} \right] \quad \text{(C.3)}$$

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C.2 Third order

At the third order, corresponding to the product of three $g_0(\varepsilon \varepsilon)$ operators with a $h_0(\sigma)$ one, we have to compute:

$$\frac{g_0^3}{3!} h_0 \int_{|x-y|,|x-z|,|x-u|<r} \sum_{a \neq b} \varepsilon_a(x) \varepsilon_b(x) \sum_{c \neq d} \varepsilon_c(y) \varepsilon_d(y) \sum_{e \neq f} \varepsilon_e(x) \varepsilon_f(x) \sum_{g} \sigma_g(z) d^2 x d^2 y d^2 z d^2 u$$

Here contributions will come from the following contractions $a = c = g; b = e; d = f$ and produce:

$$h_0 g_0^2 C_3(r, \varepsilon) \int \sum_{a=1}^{n} \sigma_a(z) d^2 z$$

where

$$C_3(r, \varepsilon) = 4(n-1)(n-2) \times (C.4)$$

$$\int_{|y-x|,|z-x|,|w-x|<r} <\sigma(x) \varepsilon(y) \varepsilon(z) \sigma(\infty) >_0 <\varepsilon(y) \varepsilon(w) >_0 <\varepsilon(w) \varepsilon(z) >_0 d^2 y d^2 z d^2 w$$

Again, this expression can be computed in the Coulomb gas representation

$$8(n-1)(n-2) N \pi \int |w|^{-2-3\kappa} |w-1|^{-2-3\kappa} d^2 w \int_{y<0} \left( \frac{dy}{y^{1+9\kappa}} \right)$$

$$\int |z|^{4\alpha_{12} \alpha_{k,k-1}} |z-1|^{-8\Delta_{12}+4\alpha_{12}^2+2} |u|^{4\alpha_{12} \alpha_{k,k-1}} |u-1|^{4\alpha_{12}} |u-z|^{4\alpha_{12} \alpha_{12}} d^2 z d^2 u$$

(C.5)

where we also factorize the $y$ and $w$ integrations. The first two integrations will then produce $\frac{24 \pi r^{-9\kappa}}{81 \kappa^2}$ while the results of the $z$ and $u$ integrations are given by eq.(D.13) of appendix D. So :

$$C_3(r, \varepsilon) = -12(n-1)(n-2) \pi^3 \left( \frac{r^{-9\kappa}}{9 \kappa} \right) \left[ 1 + \frac{8}{9} (2 - \lambda) \frac{\Gamma^2(-\frac{2}{3}) \Gamma^2(\frac{1}{3})}{\Gamma^2(-\frac{1}{3}) \Gamma^2(-\frac{1}{6})} \right]$$

(C.6)

Putting together the $C_i$, we obtain for $h(r)$, up to the third order :

$$h(r) = h_0 (1 + (n-1) g_0^2 \pi^3 \frac{r^{-6\kappa}}{2} \left[ 1 + \frac{4}{3} (2 - \lambda) \frac{\Gamma^2(-\frac{2}{3}) \Gamma^2(\frac{1}{3})}{\Gamma^2(-\frac{1}{3}) \Gamma^2(-\frac{1}{6})} \right]$$

$$-12(n-1)(n-2) g_0^3 \pi^3 \left( \frac{r^{-9\kappa}}{9 \kappa} \right) \left[ 1 + \frac{8}{9} (2 - \lambda) \frac{\Gamma^2(-\frac{2}{3}) \Gamma^2(\frac{1}{3})}{\Gamma^2(-\frac{1}{3}) \Gamma^2(-\frac{1}{6})} \right]$$

(C.7)
D Computation of integrals

In this appendix, we briefly present the general method to compute integrals of the following form

\[ I = \int |x|^{2a} |y|^{2a'} (y - 1)^{2b'} d^2xd^2y \]  \hspace{1cm} (D.1)

where the integration is performed over the whole complex plane. Using the techniques of ref. [18] (see also [19]), this integral can be decomposed into holomorphic and antiholomorphic parts:

\[ I = s(b)s(b') \left[ J_1^+ J_1^- + J_2^+ J_2^- \right] + s(b)s(2g + a' + b') J_1^+ J_2^- + s(b + 2g) s(b') J_1^+ J_1^- \]  \hspace{1cm} (D.2)

where \( s(x) \) corresponds to \( \sin(\pi x) \) and

\[ J_1^+ = J(a, b, a', b', g) \hspace{1cm} J_2^+ = J(b, a, b', a', g) \]

\[ J_1^- = J(b, -2 - a - b - 2g, b', -2 - a' - b' - 2g, g) \hspace{1cm} (D.3) \]

\[ J_2^- = J(-2 - a - b - 2g, b, -2 - a' - b' - 2g, b', g) \]

Here, we used the notation

\[ J(a, b, a', b', g) = \int_0^1 du \int_0^1 dv \ \frac{u^{a + a' + 2g + 1} (1 - u)^b v^{a'} (1 - v)^{2g} (1 - uv)^{b'}}{(1 - u) (1 - v)} \]  \hspace{1cm} (D.4)

\[ = \frac{\Gamma(2 + a + a' + 2g) \Gamma(1 + b) \Gamma(1 + a') \Gamma(1 + 2g)}{\Gamma(3 + a + a' + b + 2g) \Gamma(2 + a' + 2g)} \sum_{k=0}^{\infty} \frac{(-b')_k (2 + a + a' + 2g)_k (1 + a')_k}{k! (3 + a + a' + b + 2g)_k (2 + a' + 2g)_k} \]

and

\[ (a)_k = a(a + 1) \ldots (a + k - 1) \]

The \( J \) integrals appearing in (D.2) are not all independent. Using contour deformation of integrals it can be shown that we have the following relations:

\[ s(2g + a + b) J_1^- + s(a + b) J_2^- = \frac{s(a)}{s(2g + a' + b')} \left( s(a') J_1^+ + s(2g + a') J_2^+ \right) \]  \hspace{1cm} (D.5)

\[ s(2g + a' + b') J_2^- + s(a' + b') J_1^- = \frac{s(a')}{s(2g + a + b)} \left( s(a) J_2^+ + s(2g + a) J_1^+ \right) \]  \hspace{1cm} (D.6)

With these formulas, the integrals appearing in (A.8), (C.2), (C.3) can be calculated as a power series of \( \epsilon \). We will now compute explicitly these three cases:
We first compute the integral (A.8). The coefficients corresponding to that case are
\[ a = -2a' = 2 + 3\epsilon \quad b' = -\frac{1}{3} - \epsilon \quad b = -2g = -\frac{2}{3} - \frac{\epsilon}{2} \]  \hspace{1cm} (D.7)

Substituting these values in (D.2), we obtain at the first order in \( \epsilon \)
\[ I = \frac{3}{4}(J_2^+ - J_1^+)(J_1^- - J_2^-) - \frac{\sqrt{3}\pi \epsilon}{4} \left( J_1^+(2J_1^- + J_2^-) + J_2^+(J_1^- + 2J_2^-) \right) \]  \hspace{1cm} (D.8)

Now writing the \( J_i^- \) as functions of the \( J_i^+ \), with the help of relations (D.5, D.6), this simplifies to
\[ I = 3\sqrt{3}\pi \epsilon J_1^+ J_2^+ - \frac{9}{2}\pi^2 \epsilon^2 (J_2^+)^2 \]  \hspace{1cm} (D.9)

The \( J_1^+ \) and \( J_2^+ \) can now be computed explicitly using formulas (D.3, D.4). We obtain:
\[ J_1^+ = \frac{2}{9}\left(\frac{3}{2} + \pi \sqrt{3}\right) \frac{\Gamma^2(-\frac{1}{3})}{\Gamma(\frac{1}{3})} + O(\epsilon) \]
\[ J_2^+ = \frac{4}{9\epsilon} \frac{\Gamma^2(-\frac{1}{3})}{\Gamma(\frac{1}{3})} + O(cst) \]

Collecting all of these pieces together, we obtain, at the lowest order in \( \epsilon \):
\[ I = \sqrt{3}\pi \frac{\Gamma^4(-\frac{1}{3})}{\Gamma^2(-\frac{2}{3})} \]  \hspace{1cm} (D.10)

We now compute the integral (C.2). Here, we have:
\[ a = -\frac{1}{3} + \lambda \epsilon = -\frac{1}{2}a' \quad b = 2g = b' - 1 = -\frac{4}{3} - \epsilon \]  \hspace{1cm} (D.11)

Then, relating the \( J_i^- \) to the \( J_i^+ \), we obtain for \( I \):
\[ I = \frac{3^2\pi \epsilon}{2} \left( \frac{J_2^-}{2} \right)^2 + \frac{3^2(2-\lambda)\pi \epsilon}{2} \left( J_2^- - J_1^- \right)^2 + O(\epsilon^2) \]  \hspace{1cm} (D.12)

These \( J_i^- \) can again be computed with the help of (D.3, D.4):
\[ J_1^- = J_2^- + \frac{\Gamma(-\frac{1}{3})\Gamma(\frac{1}{6})}{\Gamma(-\frac{1}{6})} = \frac{1}{2} \frac{\Gamma^2(-\frac{1}{3})}{\Gamma(-\frac{1}{3})} + \frac{\Gamma(-\frac{1}{3})\Gamma(\frac{1}{6})}{\Gamma(-\frac{1}{6})} + O(\epsilon) \]

Thus:
\[ I = \frac{3^2\pi \epsilon}{8} \frac{\Gamma^4(-\frac{1}{3})}{\Gamma^2(-\frac{2}{3})} + \frac{3^2(2-\lambda)\pi \epsilon \Gamma^2(-\frac{1}{3})\Gamma^2(\frac{1}{6})}{2 \Gamma^2(-\frac{1}{6})} + O(\epsilon^2) \]  \hspace{1cm} (D.13)
The last case, corresponding to (C.3), use the following parameters

\[ a = -\frac{1}{3} + \lambda \epsilon = -\frac{1}{2} a' \; ; \; \; b = 2g = -\frac{4}{3} - \epsilon \; ; \; \; b' = -\frac{1}{3} - \frac{5}{2} \epsilon \]  \hspace{1cm} (D.14)

Performing the same manipulations, we immediately obtain

\[
I = \frac{9\sqrt{3}\pi \epsilon}{16} \frac{\Gamma^4(-\frac{1}{3})}{\Gamma^2(-\frac{2}{3})} + \frac{3(2 - \lambda)\pi \epsilon}{2} \frac{\Gamma^2(-\frac{1}{3})\Gamma^2(-\frac{1}{2})}{\Gamma^2(-\frac{1}{6})} + O(\epsilon^2) \]  \hspace{1cm} (D.15)
References

[1] A. B. Harris and T. C. Lubensky, Phys. Rev. Lett. 33, 1540 (1974).
[2] G. Grinstein and A. Luther, Phys. Rev. B13, 1329 (1976).
[3] D. E. Khmelnitskii, Zh. Eksp. Teor. Fiz. 68, 1960 (1975) [Sov. Phys. JETP 41, 981 (1975)].
[4] A. Weinrib and B. I. Halperin, Phys. Rev. B27, 413 (1983).
[5] A. L. Korzhenevskii, A. A. Lushtkov and W. Schirmacher, Phys. Rev. B50, 3661 (1994).
[6] A. B. Harris, J. Phys. C7, 1671 (1974).
[7] B. M. McCoy and T. T. Wu, Phys. Rev. 176, 631 (1968).
[8] Vik. S. Dotsenko and Vl. S. Dotsenko, Sov. Phys. JETP Lett. 33, 37 (1981); Adv. Phys. 32, 129 (1983).
[9] D. J. Gross and A. Neveu, Phys. Rev. D 10, 3235 (1974).
[10] B. N. Shalaev, Sov. Phys. Solid State 26, 1811 (1984).
[11] R. Shankar, Phys. Rev. Lett. 58, 2466 (1987).
[12] A. W. W. Ludwig, Nucl. Phys. B330, 639 (1990).
[13] K. Ziegler, cond-mat@xxx.lanl.gov No. 9312017
[14] V. B. Andreichenko, Vl. S. Dotsenko, W. Selke and J. -S. Wang, Nucl. Phys. B344, 531 (1990); J. -S. Wang, W. Selke, Vl. S. Dotsenko and V. B. Andreichenko, Europhys. Lett. 11, 301 (1990); J. -S. Wang, W. Selke, Vl. S. Dotsenko and V. B. Andreichenko, Physica A 164, 221 (1990).
[15] A. L. Talapov and L. N. Shchur, hep-lat@xxx.lanl.gov No. 9404002
[16] A. W. W. Ludwig, Nucl. Phys. B285, 97 (1987).
[17] Vl. S. Dotsenko, M. Picco and P. Pujol, hep-th@xxx.lanl.gov No. 9405003
[18] Vl. S. Dotsenko and V. A. Fateev, Nucl. Phys. B240, 312 (1984), B251, 691 (1985).
[19] Vl. S. Dotsenko, Advanced Studies in Pure Mathematics 16, 123 (1988).
[20] Vl. S. Dotsenko, M. Picco and P. Pujol, work in progress.