ON THE M-EIGENVALUE ESTIMATION OF FOURTH-ORDER PARTIALLY SYMMETRIC TENSORS

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ABSTRACT. In this article, the M-eigenvalue of fourth-order partially symmetric tensors is estimated by choosing different components of M-eigenvector. As an application, some upper bounds for the M-spectral radius of nonnegative fourth-order partially symmetric tensors are discussed, which are sharper than existing upper bounds. Finally, numerical examples are reported to verify the obtained results.

1. Introduction. Consider the following bi-quadratic optimization problem

\[
\begin{align*}
\max f(x, y) = C_{xyxy} &= \sum_{i,k \in [m]} \sum_{j,l \in [n]} c_{ijkl}x_iy_jx_ky_l, \\
\text{s.t.} & \quad x^T x = 1, \quad y^T y = 1, \\
& \quad x \in \mathbb{R}^m, \quad y \in \mathbb{R}^n,
\end{align*}
\]

(1.1)

where \( m \) and \( n \) are positive natural numbers and write \([n] = \{1, 2, \cdots, n\}\), the coefficients \( c_{ijkl} \) satisfy the following symmetric property

\( c_{ijkl} = c_{kjil} = c_{ilkj} = c_{klij}, \quad i, k \in [m], \quad j, l \in [n] \).

In this sense, tensor \( C = (c_{ijkl}) \) is called partially symmetric.

Problem (1.1) finds applications in such as nonlinear elastic materials analysis, the ordinary ellipticity and strong ellipticity [13, 14, 20]. It is known that an elastic material has wave propagation and instabilities [7, 16]. Hence, characterizing the ordinary and strong ellipticity is an important issue in practice, and some necessary and/or sufficient conditions are established in the literature [8, 10, 17].

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To establish the criteria in identifying the strong ellipticity in elastic mechanics, Qi et al. [17] introduced the following definition [15]. For \( \lambda \in \mathbb{R}, \ x \in \mathbb{R}^m, \ y \in \mathbb{R}^n, \) if

\[
\begin{aligned}
C \cdot yxy &= \lambda x, \\
C xyx &= \lambda y, \\
x^T x &= 1, \\
y^T y &= 1,
\end{aligned}
\]  

(1.2)

where \((C \cdot yxy)_i = \sum_{k \in [m], j \in [n]} c_{ijkl} y_j x_k y_i, \) and \((C xyx)_i = \sum_{i, k \in [m], j \in [n]} c_{ijkl} x_i y_j x_k, \) then the scalar \( \lambda \) is called an M-eigenvalue of the tensor \( C, \) and \( x \) and \( y \) are called left and right M-eigenvectors of \( C, \) respectively, which associated with the M-eigenvalue \( \lambda. \)

Based on the M-eigenvalue with the strong ellipticity [13, 14, 20], Han et al. [10] proposed the strong ellipticity condition to the rank-one positive definiteness of three second-order tensors, three fourth-order tensors, and a sixth-order tensor. Wang et al. [23] presented a practical method to compute the largest M-eigenvalue of a fourth-order partially symmetric tensor. The research in [17] exhibits that the strong ellipticity holds if and only if all M-eigenvalues of the ellipticity tensor is positive.

Another important similar concept in tensor analysis is Z-eigenvalue problem [18, 17] which collapses to M-eigenvalue when the underlying tensor is partially symmetric [17]. The Z-eigenvalue plays an important role in best rank-one approximation, which has a wide range of practical applications in statistical data analysis and engineering [15, 26]. For this, Chang et al. [1] proposed upper bounds for Z-spectral radius of nonnegative tensors. Moreover, Song et al. [19] improved the upper bounds for Z-spectral radius based on the relationship between the Gelfand formula and the spectral radius. For weakly symmetric and positive tensors, He et al. [11] presented the Ledermann-like upper bound for the largest Z-eigenvalue. For general tensors, Wang et al. [25] established Z-eigenvalue inclusion theorems, and the upper bounds for the largest Z-eigenvalue of a weakly symmetric nonnegative tensor was obtained.

Generally speaking, the study on high order tensors have attracted much attention of researchers, which made tensor analysis an important tool in theoretical physics, continuum mechanics and many other areas of science and engineering [2, 3, 4, 5, 6, 21, 24, 28, 27, 22, 12]. Particulary, Wang et al. [25] established Z-eigenvalue inclusion theorems, and the upper bounds for the largest Z-eigenvalue of a weakly symmetric nonnegative tensor was obtained. Since Z-eigenvalue is a special kind of M-eigenvalue, the research on Z-eigenvalue motivates us to consider the estimation on M-eigenvalues. This constitutes the main issue considered in this paper. More precisely, in this paper, several M-eigenvalue localization sets for tensors are obtained by choosing different components of M-eigenvector. As an application, some upper bounds for the M-spectral radius of nonnegative tensors are discussed. Finally, numerical examples are proposed to verify the theoretical results.

The remainder of this paper is organized as follows. In Section 2, we establish some M-eigenvalue inclusion theorems and give comparisons among these eigenvalue inclusion sets. In Section 3, we apply these inclusion theorems to estimate upper bounds of the largest M-eigenvalue for nonnegative tensors.
2. M-eigenvalue inclusion theorems. In this section, we discuss several M-eigenvalue inclusion theorems of fourth-order partially symmetric tensors. Furthermore, we establish comparisons among different M-eigenvalue inclusion sets. The M-spectral radius $\rho(C)$ of $C$ is defined as

$$\rho(C) = \max \{|\lambda| : \lambda \in \sigma(C)\}$$

where $\sigma(C)$ is the M-spectrum of $C$, which contains all M-eigenvalues of $C$.

Inspired by the ideas of H-eigenvalue inclusion theorem [18] and Z-eigenvalue inclusion theorems [25], we can establish the following M-eigenvalue inclusion theorems.

**Theorem 2.1.** Suppose $C = (c_{ijkl})$ is a partially symmetric tensor with $i, k \in [m], j, l \in [n]$. Then

$$\sigma(C) \subseteq \Gamma(C) = \bigcup_{i \in [m]} \Gamma_i(C),$$

where $\Gamma_i(C) = \{z \in \mathbb{C} : |z| \leq R_i(C)\}$, and $R_i(C) = \sum_{k \in [m], j, l \in [n]} |c_{ijkl}|$.

**Proof.** Let $\lambda$ be an M-eigenvalue of the tensor $C$ with the associated left M-eigenvector $x \in \mathbb{R}^m$ and right M-eigenvector $y \in \mathbb{R}^n$. Then there exists index $1 \leq t \leq n$ such that $|x_t| = \max_{p \in [m]} |x_p| > 0$.

It follows from (1.2) that

$$\lambda x_t = (C : yxy)_t = \sum_{k \in [m], j, l \in [n]} c_{ijkl} y_j x_k y_l.$$ 

Furthermore, since $y^T y = 1$, one has $|y_j| \leq 1$ for any $j \in [n]$. Thus,

$$|\lambda| \leq \sum_{k \in [m], j, l \in [n]} |c_{ijkl}| |y_j| \frac{|x_k|}{|x_t|} y_l \leq \sum_{k \in [m], j, l \in [n]} |c_{ijkl}|,$$  

which implies $\lambda \in \Gamma(C)$ and the desired result holds.

**Theorem 2.2.** Let $C = (c_{ijkl})$ be a partially symmetric tensor with $i, k \in [m], j, l \in [n]$. Then

$$\sigma(C) \subseteq \mathcal{L}(C) = \bigcup_{i \in [m]} \left( \bigcap_{k \in [m], k \neq i} \mathcal{L}_{i,k}(C) \right),$$

where $\mathcal{L}_{i,k}(C) = \{z \in \mathbb{C} : (|z| - (R_i(C) - R^k_i(C)))|z| \leq R^k_i(C)R_k(C)\}$, and $R^k_i(C) = \sum_{j, l \in [n]} |c_{ijkl}|$.

**Proof.** From (1.2), we can assume that left M-eigenvector $x$ has at least one nonzero component, without loss of generality, let $|x_t| = \max_{p \in [m]} |x_p| > 0$. 

Therefore,
\[ r \| \lambda x_t = (C \cdot yxy)_t, \]
\[ = \sum_{k \in [m], j, t \in [n]} c_{tjk} y_j x_k y_t \]
\[ = \sum_{k \in [m], k \neq s, j, t \in [n]} c_{tjk} y_j x_k y_t + \sum_{j, t \in [n]} c_{tjs} y_j x_s y_t, \quad s \in [m], \ s \neq t, \]
which is equivalent with
\[ |\lambda| \leq \sum_{k \in [m], k \neq s, j, t \in [n]} |c_{tjk}| + \sum_{j, t \in [n]} |c_{tjs}| \left| \frac{x_t}{x_s} \right|. \quad (2.4) \]

If \(|x_s| = 0\), then \(|\lambda| - \sum_{k \in [m], k \neq s, j, t \in [n]} |c_{tjk}| \leq 0\), which means \(\lambda \in \mathcal{L}_{t,s}(C) \subseteq \mathcal{L}(C)\).

Otherwise, for \(|x_s| > 0\), we have
\[ \lambda x_a = (C \cdot yxy)_a = \sum_{k \in [m], j, t \in [n]} c_{sjk} y_j x_k y_t. \]

Further,
\[ |\lambda| \leq \sum_{k \in [m], j, t \in [n]} |c_{sjk}| \left| \frac{x_k}{x_s} \right| \leq \sum_{k \in [m], j, t \in [n]} |c_{sjk}| \left| \frac{x_t}{x_s} \right|. \quad (2.5) \]

Multiplying (2.4) with (2.5) yields
\[ \left( |\lambda| - \sum_{k \in [m], k \neq s, j, t \in [n]} |c_{tjk}| \right) |\lambda| \leq \sum_{j, t \in [n]} |c_{tjs}| \sum_{k \in [m], j, t \in [n]} |c_{sjk}|. \]

Consequently,
\[ (|\lambda| - (R_t(C) - R^*_t(C)))|\lambda| \leq R^*_t(C)R_s(C), \]
which implies that \(\lambda \in \mathcal{L}_{t,s}(C)\). From the arbitrariness of \(s\), it follows that \(\lambda \in \bigcap_{k \in [m], k \neq t} \mathcal{L}_{t,k}(C)\), and the desired result follows.

The following conclusion exhibits the relationship between \(\sigma(C), \mathcal{L}(C)\) and \(\Gamma(C)\).

**Theorem 2.3.** Let \(C\) be defined as in Theorem 2.2. Then
\[ \sigma(C) \subseteq \mathcal{L}(C) \subseteq \Gamma(C). \]

**Proof.** By Theorem 2.1 and Theorem 2.2, it is sufficient to prove \(\mathcal{L}(C) \subseteq \Gamma(C)\). Without loss of generality, for any \(\lambda \in \mathcal{L}(C)\), there exists an index \(t \in [m]\) such that \(\lambda \in \mathcal{L}_{t,s}(C)\), for all \(s \neq t\). Thus
\[ (|\lambda| - (R_t(C) - R^*_t(C)))|\lambda| \leq R^*_t(C)R_s(C). \]

We now break up the argument into two cases.

**Case 1.** If \(R^*_t(C)R_s(C) = 0\), then
\[ |\lambda| - (R_t(C) - R^*_t(C)) \leq 0, \quad \text{or} \quad \lambda = 0. \]

Hence
\[ |\lambda| \leq R_t(C) - R^*_t(C) \leq R_t(C), \quad \text{or} \quad \lambda = 0. \]

Therefore, \(\lambda \in \Gamma_t(C)\).
Case 2. If \( R_s^-(C)R_s^+(C) > 0 \), then

\[
\frac{|\lambda| - (R_t(C) - R_t^+(C))}{R_t^+(C)} \leq 1,
\]

which means that

\[
\frac{|\lambda| - (R_t(C) - R_t^+(C))}{R_t^+(C)} \leq 1
\]
or

\[
\frac{|\lambda|}{R_s^+(C)} \leq 1.
\]

Thus, \( \lambda \in \Gamma_t(C) \cup \Gamma_s(C) \).

In summary, \( \sigma(C) \subseteq \mathcal{L}(C) \subseteq \Gamma(C) \) and the desired result follows.

In what follows, let \( x_s \) denote the component of the left M-eigenvector \( x \) with the second largest modulus. Then we can obtain the following technical results for \( \sigma(C) \).

**Theorem 2.4.** Suppose \( C = (c_{ijkl}) \) is a partially symmetric tensor with \( i, k \in [m], j, l \in [n] \). Then

\[
\sigma(C) \subseteq \mathcal{M}(C) = \bigcup_{i,k \in [m], \ k \neq i} \left( \mathcal{M}_{i,k}(C) \bigcup \mathcal{H}_{i,k}(C) \right),
\]

where \( \mathcal{M}_{i,k}(C) = \{z \in C : (|z| - (R_i(C) - R_k^+(C)))(|z| - R_k^+(C)) \leq R_t^+(C)(R_k(C) - R_k^+(C)) \} \), and \( \mathcal{H}_{i,k}(C) = \{z \in C : |z| - (R_i(C) - R_k^+(C)) \leq 0, |z| - R_k^+(C) < 0 \} \).

**Proof.** From (1.2), we know that the left M-eigenvector \( x \) has at least one nonzero component, thus we can assume that \( x_t \) is a component of \( x \) with the largest absolute value and \( x_s \) is a component of \( x \) with the second largest absolute value. It is easy to check that \( |x_t| > 0 \).

Following the argument of (2.4), we have

\[
|\lambda| \leq \sum_{k \in [m], \ k \neq s, j,l \in [n]} |c_{ijkl}| + \sum_{j,l \in [n]} |c_{j,kl}| \frac{|x_s|}{|x_t|}. \tag{2.6}
\]

Thus, if \( |x_s| = 0 \), then \( |\lambda| - \sum_{k \in [m], \ k \neq s, j,l \in [n]} |c_{ijkl}| = |\lambda| - (R_t(C) - R_t^+(C)) \leq 0 \). If \( |\lambda| - R_s^+(C) \geq 0 \), then \( \lambda \in \mathcal{M}_{s,t}(C) \), and if \( |\lambda| - R_s^+(C) < 0 \), then \( \lambda \in \mathcal{H}_{t,s}(C) \).

Conversely, for \( |x_s| > 0 \), one has

\[
\lambda x_s = (C \cdot xy) x_s = \sum_{k \in [m], \ j,l \in [n]} c_{j,kl} y_j x_k y_l
\]

\[
= \sum_{j,l \in [n]} c_{j,kl} y_j x_s y_l + \sum_{k \in [m], \ k \neq s, j,l \in [n]} c_{j,kl} y_j x_k y_l.
\]

Moreover,

\[
|\lambda| \leq \sum_{j,l \in [n]} |c_{j,kl}| |y_j| |y_l| + \sum_{k \in [m], \ k \neq s, j,l \in [n]} |c_{j,kl}| |y_j| \frac{|x_k|}{|x_s|} |y_l| \tag{2.7}
\]

\[
\leq \sum_{j,l \in [n]} |c_{j,kl}| + \sum_{k \in [m], \ k \neq s, j,l \in [n]} |c_{j,kl}| \frac{|x_k|}{|x_s|}.
\]
If $|\lambda| - (R_t(C) - R_t^*(C)) \geq 0$ or $|\lambda| - R_s^*(C) \geq 0$, then multiplying (2.6) with (2.7) yields
\[
(|\lambda| - \sum_{k \in [m], k \neq s, j, l \in [n]} |c_{t,kl}|)(|\lambda| - \sum_{j, l \in [n]} |c_{sj,tl}|) \leq \sum_{j, l \in [n]} |c_{t,kl}| \sum_{k \in [m], k \neq s, j, l \in [n]} |c_{sj,tl}|,
\]
which is equivalent to
\[
(|\lambda| - (R_t(C) - R_t^*(C)))(|\lambda| - R_s^*(C)) \leq R_t^*(C)(R_s(C) - R_s^*(C)).
\]
This implies that $\lambda \in M_{s,t}(C) \subseteq M(C)$.

If $|\lambda| - (R_t(C) - R_t^*(C)) < 0$ and $|\lambda| - R_s^*(C) < 0$, then $\lambda \in H_{s,t}(C) \subseteq M(C)$. Thus, the desired results follow.

From Theorems 2.1 and 2.4, one has the following conclusion on the relationship between $\sigma(C), M(C)$ and $\Gamma(C)$.

**Theorem 2.5.** Suppose $C$ is a partially symmetric tensor as in Theorem 2.4. Then $\sigma(C) \subseteq M(C) \subseteq \Gamma(C)$.

**Proof.** For any $\lambda \in M(C)$, we break the proof into two cases.

**Case 1.** $\lambda \in M_{s,t}(C)$. In this case, there exist $s, t \in [m]$, $s \neq t$ such that
\[
(|\lambda| - (R_t(C) - R_t^*(C)))(|\lambda| - R_s^*(C)) \leq R_t^*(C)(R_s(C) - R_s^*(C)).
\]
If $R_t^*(C)(R_s(C) - R_s^*(C)) = 0$, then
\[
|\lambda| - (R_t(C) - R_t^*(C)) \leq 0
\]
or
\[
|\lambda| - R_s^*(C) \leq 0.
\]
Thus, $\lambda \in \Gamma_t(C) \cup \Gamma_s(C) \subseteq \Gamma(C)$.

If $R_t^*(C)(R_s(C) - R_s^*(C)) > 0$, then it follows from (2.8) that
\[
\frac{|\lambda| - (R_t(C) - R_t^*(C))}{R_t^*(C)} \frac{|\lambda| - R_s^*(C)}{R_s(C) - R_s^*(C)} \leq 1.
\]
Further,
\[
\frac{|\lambda| - (R_t(C) - R_t^*(C))}{R_t^*(C)} \leq 1, \text{ or } \frac{|\lambda| - R_s^*(C)}{R_s(C) - R_s^*(C)} \leq 1.
\]
Thus, $\lambda \in \Gamma_t(C) \cup \Gamma_s(C) \subseteq \Gamma(C)$.

**Case 2.** $\lambda \in H_{s,t}(C)$. Then there exist $s, t \in [m]$, $s \neq t$ such that
\[
|\lambda| - (R_t(C) - R_t^*(C)) \leq 0,
\]
and
\[
|\lambda| - R_s^*(C) < 0.
\]
Consequently, $\lambda \in \Gamma_t(C) \cup \Gamma_s(C) \subseteq \Gamma(C)$.

Combining cases 1 and 2 yields that $\sigma(C) \subseteq M(C) \subseteq \Gamma(C)$ and the desired result follows.

Following the argument of Theorem 2.4, we can obtain the following M-eigenvalue inclusion theorem.
Theorem 2.6. Let $\mathcal{C} = (c_{ijkl})$ be a partially symmetric tensor with $i, k \in [m], j, l \in [n]$. Then

$$\sigma(\mathcal{C}) \subseteq \mathcal{N}(\mathcal{C}) = \bigcup_{i,k\in[m],\,k\neq i} \mathcal{N}_{i,k}(\mathcal{C}),$$

where $\mathcal{N}_{i,k}(\mathcal{C}) = \{ z \in \mathbb{C} : (|\lambda| - R_i(\mathcal{C}))|\lambda| \leq (R_i(\mathcal{C}) - R_k(\mathcal{C}))R_k(\mathcal{C}) \}$.

Proof. From (1.2), for any left $M$-eigenvector $x$, without loss of generality, we assume that $x_t$ is the component with largest absolute value and $x_s$ is the component such that $x_t \geq x_s = \max_{p \in [m], p \neq t} |x_p|$. Surely, $|x_t| > 0$. From (1.2), it yields that

$$\lambda x_t = (\mathcal{C} \cdot x y) t = \sum_{k \in [m], j, l \in [n]} c_{tjkl} y_j x_k y_l = \sum_{j, l \in [n]} c_{tjkl} y_j x_k y_l + \sum_{k \in [m], k \neq t, j, l \in [n]} c_{tjkl} y_j x_k y_l.$$

Therefore,

$$|\lambda x_t| = \left| (\mathcal{C} \cdot x y) t \right| = \left| \sum_{k \in [m], j, l \in [n]} c_{tjkl} y_j x_k y_l \right| \leq \sum_{j, l \in [n]} |c_{tjkl}| |y_j x_k y_l| + \sum_{k \in [m], k \neq t, j, l \in [n]} |c_{tjkl}| |y_j x_k y_l| \leq \sum_{j, l \in [n]} |c_{tjkl}| |y_j x_k y_l| + \sum_{k \in [m], k \neq t, j, l \in [n]} |c_{tjkl}| |y_j x_s y_l|.$$

Furthermore,

$$|\lambda| \leq \sum_{j, l \in [n]} |c_{tjkl}| |y_j y_l| + \sum_{k \in [m], k \neq t, j, l \in [n]} |c_{tjkl}| \frac{|x_s|}{|x_t|} |y_l| \leq \sum_{j, l \in [n]} |c_{tjkl}| + \sum_{k \in [m], k \neq t, j, l \in [n]} |c_{tjkl}| \frac{|x_s|}{|x_t|} \tag{2.9}$$

If $|x_s| = 0$, then $|\lambda| - \sum_{j, l \in [n]} |c_{tjkl}| \leq 0$. It holds that $\lambda \in \mathcal{N}_{t,s}(\mathcal{C}) \subseteq \mathcal{N}(\mathcal{C})$.

Otherwise, for $|x_s| > 0$, multiplying (2.5) with (2.9) yields that

$$\left( |\lambda| - \sum_{j, l \in [n]} |c_{tjkl}| \right) |\lambda| \leq \sum_{k \neq t, j, l \in [n]} |c_{tjkl}| \sum_{k \in [m], j, l \in [n]} |c_{sijkl}|,$$

which is equivalent to

$$\left( |\lambda| - R_i(\mathcal{C}) \right) |\lambda| \leq (R_i(\mathcal{C}) - R_k(\mathcal{C})) R_k(\mathcal{C}).$$

Thus, $\lambda \in \mathcal{N}_{t,s}(\mathcal{C}) \subseteq \mathcal{N}(\mathcal{C})$ and the desired result follows.

Similar to the argument in Theorems 2.3 and 2.5, we can obtain the following conclusion.

Theorem 2.7. Suppose $\mathcal{C}$ is a fourth-order partially symmetric tensor. Then

$$\sigma(\mathcal{C}) \subseteq \mathcal{N}(\mathcal{C}) \subseteq \Gamma(\mathcal{C}).$$

Now, we present the following examples to illustrate the $M$-eigenvalue inclusion sets $\Gamma(\mathcal{C}), \mathcal{E}(\mathcal{C}), \mathcal{M}(\mathcal{C})$ and $\mathcal{N}(\mathcal{C})$. 


Example 2.1. Consider the following fourth-order partially symmetric tensor
\[ c_{ijkl} = \begin{cases} 
  c_{1111} = -1, & c_{1112} = 2, & c_{1131} = 3, & c_{1121} = -1, & c_{1211} = 2, & c_{1221} = 1, & c_{1122} = 1, \\
  c_{2111} = -1, & c_{2211} = 1, & c_{2112} = 1, & c_{2131} = -2, & c_{2222} = 2, \\
  c_{3111} = 3, & c_{3232} = -1, & c_{3131} = -2, \\
  c_{ijkl} = 0, & \text{otherwise}.
\end{cases} \]

By computation, we obtain the corresponding M-eigenvalues is -0.8805. From Theorem 2.1, it holds that
\[ \Gamma(C) = \bigcup_{i \in [m]} \Gamma_i(C) = \{ \lambda \in C : |\lambda| \leq 11 \}. \]
From Theorem 2.2, we obtain
\[ L(C) = \bigcup_{i \in [m]} \left( \bigcap_{k \in [m], k \neq i} L_{i,k}(C) \right) = \{ \lambda \in C : |\lambda| \leq 4 + \sqrt{34} \}, \]
where
\[ L_{1,2}(C) = \left\{ \lambda \in C : |\lambda| \leq 4 + \sqrt{37} \right\}, \quad L_{1,3}(C) = \left\{ \lambda \in C : |\lambda| \leq 4 + \sqrt{34} \right\}, \]
\[ L_{2,1}(C) = \left\{ \lambda \in C : |\lambda| \leq \frac{7 + \sqrt{181}}{2} \right\}, \quad L_{2,3}(C) = \left\{ \lambda \in C : |\lambda| \leq \frac{5 + \sqrt{73}}{2} \right\}, \]
\[ L_{3,1}(C) = \left\{ \lambda \in C : |\lambda| \leq \frac{3 + \sqrt{141}}{2} \right\}, \quad L_{3,2}(C) = \left\{ \lambda \in C : |\lambda| \leq 2 + 3\sqrt{2} \right\}. \]
From Theorem 2.4, we obtain
\[ M(C) = \bigcup_{i,k \in [m], k \neq i} \left( M_{i,k}(C) \bigcup \mathcal{H}_{i,k}(C) \right) = \{ \lambda \in C : |\lambda| \leq 5 + 2\sqrt{6} \}, \]
where
\[ M_{1,2}(C) = \{ \lambda \in C : 5 - 2\sqrt{6} \leq |\lambda| \leq 5 + 2\sqrt{6} \}, \]
\[ M_{1,3}(C) = \left\{ \lambda \in C : \frac{11 - \sqrt{61}}{2} \leq |\lambda| \leq \frac{11 + \sqrt{61}}{2} \right\}, \]
\[ M_{2,1}(C) = \left\{ \lambda \in C : \frac{9 - \sqrt{73}}{2} \leq |\lambda| \leq \frac{9 + \sqrt{73}}{2} \right\}, \]
\[ M_{2,3}(C) = \left\{ \lambda \in C : 4 - \sqrt{7} \leq |\lambda| \leq 4 + \sqrt{7} \right\}, \]
\[ M_{3,1}(C) = \left\{ \lambda \in C : 4 - \sqrt{19} \leq |\lambda| \leq 4 + \sqrt{19} \right\}, \quad M_{3,2}(C) = \{ \lambda \in C : 2 \leq |\lambda| \leq 6 \}, \]
\[ \mathcal{H}_{1,2}(C) = \{ \lambda \in C : |\lambda| < 2 \}, \quad \mathcal{H}_{1,3}(C) = \{ \lambda \in C : |\lambda| < 3 \}, \]
\[ \mathcal{H}_{2,1}(C) = \{ \lambda \in C : |\lambda| \leq 4 \}, \quad \mathcal{H}_{2,3}(C) = \{ \lambda \in C : |\lambda| < 3 \}, \]
\[ \mathcal{H}_{3,1}(C) = \{ \lambda \in C : |\lambda| \leq 3 \}, \quad \mathcal{H}_{3,2}(C) = \{ \lambda \in C : |\lambda| < 2 \}. \]
From Theorem 2.6, we obtain
\[ \mathcal{N}(C) = \bigcup_{i,k \in [m], k \neq i} \mathcal{N}_{i,k}(C) = \left\{ \lambda \in C : |\lambda| \leq \frac{5 + \sqrt{193}}{2} \right\}, \]
where
\[ N_{1,2}(C) = \left\{ \lambda \in C : |\lambda| \leq \frac{5 + \sqrt{193}}{2} \right\}, \quad N_{1,3}(C) = \left\{ \lambda \in C : |\lambda| \leq 9 \right\}, \]
\[ N_{2,1}(C) = \left\{ \lambda \in C : |\lambda| \leq 1 + 2\sqrt{14} \right\}, \quad N_{2,3}(C) = \left\{ \lambda \in C : |\lambda| \leq 1 + \sqrt{31} \right\}, \]
\[ N_{3,1}(C) = \left\{ \lambda \in C : |\lambda| \leq \frac{1 + \sqrt{221}}{2} \right\}, \quad N_{3,2}(C) = \left\{ \lambda \in C : |\lambda| \leq \frac{1 + \sqrt{141}}{2} \right\}. \]

**Figure 1.** The comparisons of \( \Gamma(C), L(C), M(C) \) and \( N(C) \)

**Example 2.2.** Consider 4th order 2 dimensional tensor \( C = (c_{ijkl}) \) defined by

\[
c_{ijkl} = \begin{cases} 
  c_{1111} = 1, & c_{1112} = 2, \\
  c_{1222} = 5, & c_{2221} = 5, \\
  c_{2222} = 6. & 
\end{cases}
\]

It is easy to compute the corresponding M-eigenvalues are, respectively, 0.0710, 15.2091, 0.3437, 0.1242, -1.2765, -1.2765, 0.2765, 0.2765.

From Theorem 2.1, it holds that
\[ \Gamma(C) = \bigcup_{i \in [m]} \Gamma_i(C) = \left\{ \lambda \in C : |\lambda| \leq 34 \right\}. \]

From Theorem 2.2, one has
\[ L(C) = \bigcup_{i \in [m]} \left( \bigcap_{k \in [m], k \neq i} L_{i,k}(C) \right) = \left\{ \lambda \in C : |\lambda| \leq \frac{19 + \sqrt{1741}}{2} \right\}, \]
where
\[ L_{1,2}(C) = \left\{ \lambda \in C : |\lambda| \leq 4 + \sqrt{233} \right\}, \quad L_{2,1}(C) = \left\{ \lambda \in C : |\lambda| \leq \frac{19 + \sqrt{1741}}{2} \right\}. \]

From Theorem 2.4, we have
\[ M(C) = \bigcup_{i,k \in [m], k \neq i} \left( M_{i,k}(C) \bigcup H_{i,k}(C) \right) = \left\{ \lambda \in C : |\lambda| \leq \frac{27 + \sqrt{1021}}{2} \right\}. \]
where
\[ M_{1,2}(C) = \{ \lambda \in C : 0 \leq |\lambda| \leq \frac{27 + \sqrt{1021}}{2} \}, \]
\[ M_{2,1}(C) = \left\{ \lambda \in C : 0 \leq |\lambda| \leq \frac{27 + \sqrt{1021}}{2} \right\}, \]
\[ H_{1,2}(C) = \{ \lambda \in C : |\lambda| \leq 8 \}, \]
\[ H_{2,1}(C) = \{ \lambda \in C : |\lambda| < 8 \}. \]

It follows from Theorem 2.6 that
\[ N(C) = \bigcup_{i,k \in [m], k \neq i} N_{i,k}(C) = \left\{ \lambda \in C : |\lambda| \leq \frac{19 + \sqrt{1741}}{2} \right\}, \]
where
\[ N_{1,2}(C) = \{ \lambda \in C : |\lambda| \leq 4 + \sqrt{233} \}, \]
\[ N_{2,1}(C) = \left\{ \lambda \in C : |\lambda| \leq \frac{19 + \sqrt{1741}}{2} \right\}. \]

**Figure 2.** The comparisons of \( \Gamma(C) \), \( \mathcal{L}(C) \) and \( \mathcal{M}(C) \)

**Example 2.3.** Consider the following fourth-order partially symmetric tensor
\[ c_{ijkl} = \begin{cases} c_{1111} = 1, c_{1112} = 2, c_{1122} = 4, c_{1212} = 12, \\ c_{1222} = 1, c_{1121} = -1, c_{1211} = 2, c_{1221} = 4. \end{cases} \]

By computation, we can obtain that the corresponding M-eigenvalues are, respectively,
0.5837, 7.8222, 0.3437, 2.7964, 13.7558, 2.7964.

From Theorem 2.1, it holds that
\[ \Gamma(C) = \bigcup_{i \in [m]} \Gamma_i(C) = \{ \lambda \in C : |\lambda| \leq 28 \}. \]

From Theorem 2.2, one has
\[ \mathcal{L}(C) = \bigcup_{i \in [m]} \left( \bigcap_{k \in [m], k \neq i} \mathcal{L}_{i,k}(C) \right) = \{ \lambda \in C : |\lambda| \leq \frac{17 + \sqrt{1433}}{2} \}, \]
where
\[ \mathcal{L}_{1,2}(C) = \left\{ \lambda \in C : |\lambda| \leq \frac{17 + \sqrt{1433}}{2} \right\}, \]
\[ \mathcal{L}_{2,1}(C) = \left\{ \lambda \in C : |\lambda| \leq \frac{15 + \sqrt{1457}}{2} \right\}, \]
\[ \mathcal{L}_{3,1}(C) = \left\{ \lambda \in C : |\lambda| \leq \frac{3 + \sqrt{141}}{2} \right\}, \quad \mathcal{L}_{3,2}(C) = \left\{ \lambda \in C : |\lambda| \leq 2 + 3\sqrt{2} \right\}, \]

From Theorem 2.4, we have
\[
\mathcal{M}(C) = \bigcup_{i,k \in [m], \ k \neq i} \left( \mathcal{M}_{i,k}(C) \cup \mathcal{H}_{i,k}(C) \right) = \left\{ \lambda \in C : |\lambda| \leq 16 + 2\sqrt{61} \right\},
\]
where
\[
\mathcal{M}_{1,2}(C) = \left\{ \lambda \in C : 16 - 2\sqrt{61} \leq |\lambda| \leq 16 + 2\sqrt{61} \right\},
\]
\[
\mathcal{M}_{2,1}(C) = \left\{ \lambda \in C : 16 - 2\sqrt{61} \leq |\lambda| \leq 16 + 2\sqrt{61} \right\},
\]
\[
\mathcal{H}_{1,2}(C) = \left\{ \lambda \in C : |\lambda| < 15 \right\}, \quad \mathcal{H}_{2,1}(C) = \left\{ \lambda \in C : |\lambda| \leq 15 \right\},
\]
It follows from Theorem 2.6 that
\[
\mathcal{N}(C) = \bigcup_{i,k \in [m], \ k \neq i} \mathcal{N}_{i,k}(C) = \left\{ \lambda \in C : |\lambda| \leq \frac{17 + \sqrt{1433}}{2} \right\}, \]
where
\[
\mathcal{N}_{1,2}(C) = \left\{ \lambda \in C : |\lambda| \leq \frac{17 + \sqrt{1433}}{2} \right\}, \quad \mathcal{N}_{2,1}(C) = \left\{ \lambda \in C : |\lambda| \leq \frac{15 + \sqrt{1457}}{2} \right\}.
\]

**Figure 3.** The comparisons of \( \Gamma(C) \), \( \mathcal{L}(C) \) and \( \mathcal{M}(C) \)

**Remark 2.1.** The M-eigenvalue inclusion sets \( \Gamma(C) \), \( \mathcal{L}(C) \), \( \mathcal{M}(C) \) and \( \mathcal{N}(C) \) of example 2.1 are drawn in Figure 1, where \( \Gamma(C) \), \( \mathcal{L}(C) \), \( \mathcal{M}(C) \) and \( \mathcal{N}(C) \) are represented by red, blue, green and black boundary, respectively, and the exact eigenvalues is plotted by *. From Figure 1, the example shows the M-eigenvalue inclusion sets \( \Gamma(C) \), \( \mathcal{L}(C) \), \( \mathcal{M}(C) \) and \( \mathcal{N}(C) \) are different. From Theorem 2.2 and 2.6, we observe \( \mathcal{L}(C) = \mathcal{N}(C) \) when \( n = 2 \). The M-eigenvalue inclusion sets \( \Gamma(C) \), \( \mathcal{L}(C) \), and \( \mathcal{M}(C) \) of example 2.2 and 2.3 are drawn in Figure 2 and Figure 3, respectively. The M-eigenvalue inclusion sets \( \Gamma(C) \), \( \mathcal{L}(C) \) and \( \mathcal{M}(C) \) are represented by red, blue and green boundary, respectively. From Figure 2 and 3, the examples show that the M-eigenvalue inclusion sets \( \Gamma(C) \), \( \mathcal{L}(C) \) and \( \mathcal{M}(C) \) are different.
3. Bounds on the largest M-eigenvalue of nonnegative fourth-order partially symmetric tensors. Based on the obtained results in last section, we present some sharp upper bounds estimation on M-spectral radius of nonnegative fourth-order partially symmetric tensors, which improves the corresponding results in [1, 19]. To proceed, we first recall some fundamental results on nonnegative tensors. Here, \( R_t(C) \) and \( R^k_t(C) \) are defined as in Theorems 2.1 and 2.2, respectively.

Lemma 3.1. [1] Let \( A \) be an \( m \)-th order \( n \)-dimensional nonnegative tensor. Then
\[
\rho(A) \leq \max_{i \in N} \sqrt[n]{R_t(A)}.
\]

Lemma 3.2. [19] Let \( A \) be an \( m \)-th order \( n \)-dimensional nonnegative tensor. Then
\[
\rho(A) \leq \max_{i \in N} R_t(A).
\]

Lemma 3.3. [9] The M-spectral radius of any nonnegative partially symmetric tensor is exactly its greatest M-eigenvalue. Furthermore, there is a pair of nonnegative M-eigenvectors corresponding to the M-spectral radius.

With the help of Theorem 2.2, we can present a sharp bound estimation on the largest M-eigenvalue for nonnegative fourth-order partially symmetric tensors.

Theorem 3.1. Suppose the tensor \( C \) is a nonnegative fourth-order partially symmetric tensor. Then
\[
\rho(C) \leq \max_{i \in [m]} \min_{k \in [m], k \neq i} \frac{1}{2} \left\{ R_i(C) - R^k_i(C) + \sqrt{(R_i(C) - R^k_i(C))^2 + 4R^k_i(C)R_k(C)} \right\}.
\]

Proof. By Lemma 3.3, we can assume that \( \rho(C) \) is the largest M-eigenvalue of \( \mathcal{C} \). It follows from Theorem 2.2 that there exists an index \( t \in [m] \) such that
\[
(\rho(C) - (R_t(C) - R^t_s(C)))\rho(C) \leq R^s_t(C)R_s(C), \forall s \in [m], s \neq t.
\]

Then
\[
\rho(C) \leq \frac{1}{2} \left\{ (R_t(C) - R^t_s(C)) + \sqrt{(R_t(C) - R^t_s(C))^2 + 4R^t_s(C)R_s(C)} \right\}.
\]

Since \( s \in [m] \) is arbitrary, we have
\[
\rho(C) \leq \min_{k \in [m], k \neq t} \left\{ \frac{1}{2} (R_k(C) - R^k_t(C)) + \sqrt{(R_k(C) - R^k_t(C))^2 + 4R^k_t(C)R_k(C)} \right\}.
\]

Moreover,
\[
\rho(C) \leq \max_{i \in [m]} \min_{k \in [m], k \neq i} \frac{1}{2} \left\{ (R_i(C) - R^k_i(C)) + \sqrt{(R_i(C) - R^k_i(C))^2 + 4R^k_i(C)R_k(C)} \right\},
\]
and the desired result follows.

From Theorem 3.1, the following comparison conclusion can be readily obtained.

Theorem 3.2. Suppose \( C \) is a nonnegative fourth-order partially symmetric tensor. Then
\[
\rho(C) \leq \max_{i \in [m]} \min_{k \in [m], k \neq i} \frac{1}{2} \left\{ R_i(C) - R^k_i(C) + \sqrt{(R_i(C) - R^k_i(C))^2 + 4R^k_i(C)R_k(C)} \right\} \\
\leq \max_{i \in [m]} R_t(C).
\]
Proof. We break the proof into two cases.

**Case 1.** For \( i, k \in [m], i \neq k, R_i(C) \geq R_k(C) \). In this case, one has

\[
4R_i^k(C)R_k(C) \leq 4R_i^k(C)R_i(C),
\]

which leads to

\[
\frac{1}{2} \left\{ R_i(C) - R_i^k(C) + \sqrt{(R_i(C) - R_i^k(C))^2 + 4R_i^k(C)R_k(C)} \right\} \\
\leq \frac{1}{2} \left\{ R_i(C) - R_i^k(C) + \sqrt{(R_i(C) - R_i^k(C))^2 + 4R_i^k(C)R_i(C)} \right\} \\
= R_i(C).
\]

Furthermore,

\[
\min_{k \in [m], \ k \neq i} \frac{1}{2} \left\{ R_i(C) - R_i^k(C) + \sqrt{(R_i(C) - R_i^k(C))^2 + 4R_i^k(C)R_k(C)} \right\} \leq R_i(C).
\]

Thus,

\[
\max_{i \in [m]} \min_{k \in [m], \ k \neq i} \frac{1}{2} \left\{ R_i(C) - R_i^k(C) + \sqrt{(R_i(C) - R_i^k(C))^2 + 4R_i^k(C)R_k(C)} \right\} \\
\leq \max_{i \in [m]} R_i(C).
\]

**Case 2.** For \( i, k \in [m], i \neq k, R_i(C) \leq R_k(C) \). In this case, it holds that

\[
0 \leq R_i(C) - R_i^k(C) \leq R_k(C) - R_i^k(C).
\]

Hence

\[
\frac{1}{2} \left\{ R_i(C) - R_i^k(C) + \sqrt{(R_i(C) - R_i^k(C))^2 + 4R_i^k(C)R_k(C)} \right\} \\
\leq \frac{1}{2} \left\{ R_k(C) - R_i^k(C) + \sqrt{(R_k(C) - R_i^k(C))^2 + 4R_i^k(C)R_k(C)} \right\} \\
\leq R_k(C).
\]

Then,

\[
\max_{i \in [m]} \min_{k \in [m], \ k \neq i} \frac{1}{2} \left\{ R_i(C) - R_i^k(C) + \sqrt{(R_i(C) - R_i^k(C))^2 + 4R_i^k(C)R_k(C)} \right\} \\
\leq \max_{i \in [m]} R_k(C),
\]

which implies the desired result holds.

By Theorem 2.4, we can establish a sharp bound estimation of the largest M-eigenvalue for nonnegative fourth-order partially symmetric tensors.

**Theorem 3.3.** Suppose \( C \) is a nonnegative fourth-order partially symmetric tensor. Then

\[
\rho(C) \leq \max_{i, k \in [m], \ k \neq i} \left\{ \frac{1}{2}(R_i(C) - R_i^k(C) + R_k^i(C) + \delta_i^k), \ R_i(C) - R_i^k(C), \ R_k^i(C) \right\}
\]

where \( \delta_i^k = \sqrt{(R_i(C) - R_i^k(C) + R_k^i(C))^2 - 4(R_i^k(C)R_i(C) - R_i^k(C)R_k(C))} \).

**Proof.** Suppose \( \rho(C) \) is the largest M-eigenvalue of \( C \). We break the proof into two cases.

**Case 1.** There exist \( t, s \in [m], s \neq t \) such that \( \rho(C) \in M_{t,s}(C) \). In this case, one has

\[
(\rho(C) - (R_t(C) - R_t^s(C)))(\rho(C) - R_t^s(C)) \leq R_t^s(C)(R_s(C) - R_t^s(C)),
\]

\[
\frac{1}{2} \left\{ R_t(C) - R_t^s(C) + \sqrt{(R_t(C) - R_t^s(C))^2 + 4R_t^s(C)R_s(C)} \right\} \\
\leq \frac{1}{2} \left\{ R_s(C) - R_t^s(C) + \sqrt{(R_s(C) - R_t^s(C))^2 + 4R_t^s(C)R_s(C)} \right\} \\
= R_t^s(C).
\]

Furthermore, \( \min_{s \in [m], \ s \neq t} \frac{1}{2} \left\{ R_t(C) - R_t^s(C) + \sqrt{(R_t(C) - R_t^s(C))^2 + 4R_t^s(C)R_s(C)} \right\} \leq R_t^s(C) \).

Thus,

\[
\max_{t \in [m]} \min_{s \in [m], \ s \neq t} \frac{1}{2} \left\{ R_t(C) - R_t^s(C) + \sqrt{(R_t(C) - R_t^s(C))^2 + 4R_t^s(C)R_s(C)} \right\} \\
\leq \max_{t \in [m]} R_t^s(C),
\]

which implies the desired result holds.
which yields that
\[ \rho(C) \leq \frac{1}{2} (R_i(C) - R_t^*(C) + R_t^*(C) + \delta_t^*) \]
\[ \leq \max_{i,k \in [m], k \neq t} \frac{1}{2} (R_i(C) - R_t^*(C) + R_t^*(C) + \delta_t^*). \]

**Case 2.** There exist \( t, s \in [m], s \neq t \) such that \( \rho(C) \in H_{t,s}(C) \). In this case, one has
\[ \rho(C) \leq R_t(C) - R_s^*(C), \]
and
\[ \rho(C) < R_s^*(C). \]

Thus, the desired result holds. \( \square \)

Similar to the proof of Theorem 3.2, one has the following conclusion.

**Theorem 3.4.** Suppose the tensor \( C \) is a nonnegative fourth-order partially symmetric tensor. Then
\[
\rho(C) \leq \max_{i,k \in [m], k \neq t} \left\{ \frac{1}{2} (R_i(C) - R_t^*(C) + R_t^*(C) + \delta_t^*), R_t(C) - R_t^*(C), R_t^*(C) \right\}
\]
\[
\leq \max_{i \in [m]} R_t(C),
\]
where \( \delta_t^* = \sqrt{(R_i(C) - R_t^*(C) + R_t^*(C))^2 - 4(R_t^*(C))R_t(C) - R_t^*(C)R_t(C))}. \)

From Theorem 2.6, we have the following conclusion, and the proof is omitted.

**Theorem 3.5.** Suppose the tensor \( C \) is a nonnegative fourth-order partially symmetric tensor. Then
\[
\rho(C) \leq \max_{i,k \in [m], k \neq t} \left\{ \frac{1}{2} (R_i(C) + \sqrt{(R_i(C))^2 + 4(R_i(C) - R_t^*(C))R_t(C))} \right\}
\]
\[
\leq \max_{i \in [m]} R_t(C),
\]

Now, we present some running examples [1, 17] to illustrate the improvement of the obtained results.

**Example 3.1.** Consider 4th order 2 dimensional tensor \( C = (c_{ijkl}) \) defined by
\[
c_{ijkl} = \begin{cases} 
  c_{1111} = \frac{1}{2}, c_{2222} = 3, \\
  c_{ijkl} = \frac{1}{3}, \text{otherwise.} 
\end{cases}
\]

It is easy to compute that \( \rho(C) = 3.1122 \), and by Lemma 3.1, we have \( \rho(C) \leq 7.5432 \). By Lemma 3.2, we have \( \rho(C) \leq 5.3333 \). Since Theorems 4.5, 4.6 and 4.7 in [25] are equivalent when \( n = 2 \), we have \( \rho(C) \leq 5.1822 \).

On the other hand, by Theorem 3.1, we have \( \rho(C) \leq 4.7889 \), and by Theorem 3.3, we have \( \rho(C) \leq 4.5776 \). By Theorem 3.5, we have \( \rho(C) \leq 4.7889 \).

**Example 3.2.** Consider 4th order 2 dimensional tensor \( C = (c_{ijkl}) \) defined by
\[
c_{ijkl} = \begin{cases} 
  c_{1111} = 1, c_{1112} = 2, c_{1121} = 2, c_{1212} = 3, \\
  c_{2222} = 5, c_{1211} = 2, c_{1122} = 4, c_{1221} = 4, \\
  c_{2111} = 2, c_{2112} = 4, c_{2121} = 3, c_{2122} = 5, \\
  c_{2211} = 4, c_{2221} = 5, c_{2221} = 5, c_{2222} = 6. 
\end{cases}
\]
It is easy to compute the corresponding M-eigenvalues are, respectively,
\[0.0710, 15.2091, 0.3437, 0.1242, -1.2765, -1.2765, 0.2765, 0.2765.\]

Then, \(\rho(C) = 15.2091\). On the other way, by Lemma 3.1, we have \(\rho(C) \leq 48.0833\).
By Lemma 3.2, we have \(\rho(C) \leq 34\). By Theorem 3.1, we have \(\rho(C) \leq 30.3626\). By Theorem 3.3, we have \(\rho(C) \leq 29.4765\). By Theorem 3.5, we have \(\rho(C) \leq 30.3626\).

4. Conclusions. In this article, several M-eigenvalue localization sets for fourth-order partially symmetric tensors were obtained. As an application, several upper bounds for the M-spectral radius of nonnegative fourth-order partially symmetric tensors were discussed, which improve the existing corresponding results.

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