On (local) metric dimension of graphs with $m$-pendant points

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ABSTRACT. An ordered set of vertices $S$ is called as a (local) resolving set of a connected graph $G = (V_G, E_G)$ if for any two adjacent vertices $s \neq t \in V_G$ have distinct representation with respect to $S$, that is $r(s \mid S) \neq r(t \mid S)$. A representation of a vertex in $G$ is a vector of distances to vertices in $S$. The minimum (local) resolving set for $G$ is called as a (local) basis of $G$. A (local) metric dimension for $G$ denoted by $\text{dim}(G)$, is the cardinality of vertices in a basis for $G$, and its local variant by $\text{dim}_l(G)$.

Given two graphs, $G$ with vertices $s_1, s_2, ..., s_p$ and edges $e_1, e_2, ..., e_q$, and $H$. An edge-corona of $G$ and $H$, $G \circ H$ is defined as a graph obtained by taking a copy of $G$ and $q$ copies of $H$ and for each edge $e_j = s_i s_j$ of $G$ joining edges between the two end-vertices $s_i, s_j$ of $e_j$ and each vertex of $j$-copy of $H$.

In this paper, we determine and compare between the metric dimension of graphs with $m$-pendant points, $G \circ mK_1$, and its local variant for any connected graph $G$. We give an upper bound of the dimensions.

1. Introduction

Metric dimension is an attractive parameter in Graph Theory. This is shown by many applications and research results about it, in the last two decade recently. One of them is related to the structure of chemistry compound, such that there are some researchs by using notion in Chemistry such as: metric dimension of corona graph [5, 10] and its local variant [1, 6, 7, 8].

The development of the last result is local metric dimension of edge-corona graphs [9], that is for operation of graphs, $G$ and $H$, with $H$ of order greater than equal to two. It is important to motivate researcher in Graph Theory to produce the metric dimension of graphs.

Although edge-corona graph has appeared in [4] for case spectrum of graphs, at that time there is no result about metric dimension of this graph. Moreover, specially for the other variant.

In this paper, we go on our research about (local) metric dimension edge-corona graph $G$ and $H$, with $H$ of order one. We call this graph as graph with pendant points. We determine and compare between the metric dimension of graphs with $m$-pendant points and its local variant for any connected graph $G$. We give an upper bound of the dimensions.

2. Material and method

In this paper, we use a simple, finite and connected graph $G = (V_G, E_G)$, with $V_G$ is a set of vertices in $G$ and $V_G$ is a set of edges in $G$. An ordered set of vertices $S = \{s_1, s_2, ..., s_k\} \subseteq V_G$ is called a
resolving set of $G$ if for every two distinct vertices $u,v \in V_G$ hold \( r(u \mid S) \neq r(v \mid S) \). The representation of a vertex $x \in V_G$ with respect to $S$, denoted by $r(x \mid S)$, is defined as \( r(x \mid S) = (d(x,s_1),d(x,s_2),...,d(x,s_k)) \), where $d(x,s_i)$ is the distance between vertices $x,s_i \in V_G$. If $u,v \in V_G$ are adjacent then $S$ is called a local resolving set of $G$. A minimal (local) resolving set of $G$ is a (local) basis of $G$ and its cardinality, $\dim_l(G)$, called by a (local) metric dimension of $G$.

We use the following results that have been obtained from some researchers before.

**Theorem 2.1.** [3] Let $G$ be a connected graph of order $n \geq 2$.

(i). $\dim(G) = 1$ if only if $G = P_n$.

(ii). $\dim(G) = n - 1$ if only if $G = K_n$.

(iii). For $n \geq 4$, $\dim(G) = n - 2$ if only if $G = K_n$, \( r, s \geq 1 \), $G = K_r + K_s$ (where $r \geq 1$, $s \geq 2$) or $G = K_r + (K_1 \cup K_t)$, \( r,s \geq 1 \).

(iv). For $n \geq 3$, $\dim(C_n) = 2$.

**Theorem 2.2.**[6] Let $G$ be a connected graph of order $n \geq 2$ and diameter $k$, then

(i). $\dim(G) = 1$ if only if $G$ is a bipartite graph.

(ii). $\dim(G) = n - 1$ if only if $G = K_n$.

**Theorem 2.3.**[9] For any connected graph which is not an incomplete-bipartite graph $G$ of order $n_1 \geq 3$ with $|E(G)| = m$ and any graph complement of complete graph $H = \overline{K}_{n_2}$ hold $\dim(G \cup H) = \dim(G)$.

In this study, we determine and compare the metric dimension of graphs with pendant points and their local variant. We use a simple, finite and connected graph $G = (V_G,E_G)$, with $V_G$ is a set of vertices in $G$ and $V_G$ is a set of edges in $G$. We start with basic concepts and notations. Notation $u \sim v$, we mean that $u$ adjacent to $v$.

Given two graphs, $G$ with vertices $s_1, s_2, ..., s_p$ and edges $e_1, e_2, ..., e_p$, and $H$. An edge-corona $G \cup H$ graph of $G$ and $H$ graphs, is defined as a graph obtained by taking a copy of $G$ and $q$ copies of $H$ and for each edge $e_j = s_i s_{i+1}$ of $G$ joining edges between the two end-vertices $s_i, s_{i+1}$ of $e_j$ and each vertex of $j$-copy of $H$[4]. If we take $H \cong mK_1, m \geq 1$, we call a $G \cup H$ graph as graph with pendant points. The set of vertices of $G$ will be denoted by $\{x_1, x_2, ..., x_p\}$ and the vertex of a trivial graph $K_1$ (that is a pendant point) will be denoted by $y_j$ such that $y_1 \sim x_1$ and $y_2 \sim x_2$, too, with $e_j = x_1 x_2, i \neq h \in \{1,2, ..., p\}, j \in \{1,2, ..., q\}$.

3. Main Results

In this part, we want to determine the metric dimension of $G \cup H$ for $H \cong K_1$. We determine the exact values of $\dim(G \cup K_1)$ for some simple connected graphs $G$. We obtain general bound of $\dim(G \cup mK_1)$.

3.1. The Value of $\dim(G \cup mK_1)$

Here, as $G$, we use $P_n, C_n, K_n, S_n$, and $K_r, r + s = n, r \geq 2, s \geq 1$.

3.1.1. The exact value of $\dim(G \cup mK_1)$ when $G$ is a path graph

Let $P_p$ is a path of order $p$ and measure $q = p - 1$, and $K_1$ is a trivial graph. A $G$ graph, is an edge-corona graph of $P_p$ and $K_1$ denoted by $G \cong P_p \cup K_1$, has $q$ copies of a trivial graph $K_1$. For a simple way, a
copy of a \(^p\)th trivial graph \(K_1\) denoted by \(H_j = y_j\) for some \(j \in \{1, 2, \ldots, q\}\). An edge-corona graph \(G \cong P_p \bowtie K_1\) has a set of vertices \(V_G = \{x_1, x_2, \ldots, x_p\} \cup \{y_1, y_2, \ldots, y_q\}\), and a set of edges \(E_G = \{x_i x_{i+1} | 1 \leq i \leq p-1\} \cup \{x_i y_j, x_i y_{j+1} | 1 \leq i = j \leq p-1\}\). We consider \(\{e_j\} \circ H_j \subset P_p \bowtie K_1\). The metric dimension of \(P_p \bowtie K_1\), with \(p \geq 3\) given by the following Theorem 3.1.1.

**Theorem 3.1.1.** Given a path of order \(p \mid V_{p}\) with measure \(q \mid E_{p}\), and a trivial graph \(K_1\). For \(p \geq 3\), the metric dimension of edge-corona of \(P_p\) and \(K_1\), \(\dim(P_p \bowtie K_1)\), is two.

**Proof.** 1). According to Theorem 2.1., because \(G \cong P_p \bowtie K_1\) is not a path then \(\dim(G) \geq 2\).

  2). We will show \(\dim(G) \leq 2\).

  Choose a set \(W = \{y_1, y_2\} \subset V_G\) and a vertex \(v \in V_G\). The representation of all vertices in \(V_G \setminus \{y_1, y_2\}\) are:

  \[
  \begin{align*}
  r(x_i|W) &= (1, p-1) \\
  r(x_i|W) &= (p-1, 1) \\
  r(x_i|W) &= (i-1, p-i); \; 2 \leq i \leq p-1 \\
  r(y_j|W) &= (1, q-h+1); \; 2 \leq h \leq q-1
  \end{align*}
  \]

  We can show that all of the representation above are distinct. For example, \((s-1, p-s) = (t, q-t+1)\) for some \(s\) and \(t\) fixed. Then \(s = t+1\) and \(s = n-q+t-1\), a contradiction. So \(W\) is a resolving set of \(G\) with minimal cardinality. Hence \(\dim(P_p \bowtie K_1) = 2\). \(\square\)

**3.1.2. The exact value of \(\dim(G \bowtie mK_1)\) when \(G\) is a cycle graph**

Given a cycle \(C_p\) of order \(p\) and measure \(q=p\), and a trivial graph \(K_1\). A graph \(G\), is an edge-corona graph of \(C_p\) and \(K_1\) denoted by \(G \cong C_p \bowtie K_1\), has \(q\) copies of a trivial graph \(K_1\). For a simplification, a copy of a \(^p\)th trivial graph denoted by \(H_j = y_j\) for some \(j \in \{1, 2, \ldots, q\}\). An edge-corona graph \(G \cong C_p \bowtie K_1\) has a set of vertices \(V_G = \{x_1, x_2, \ldots, x_p\} \cup \{y_1, y_2, \ldots, y_q\}\) and a set of edges \(E_G = \{x_i x_{i+1} | 1 \leq i \leq p-1, x_p x_1\}\) \(\cup \{x_i y_j, x_{i+1} y_j | 1 \leq i \leq n-1, 1 \leq h \leq q\}\). We consider \(\{e_j\} \circ H_j \subset C_p \bowtie K_1\). The metric dimension of \(C_p \bowtie K_1\), with \(p \geq 4\) given by the following Theorem 3.1.2.

**Theorem 3.1.2.** Given a cycle \(C_p\) of order \(p\) and measure \(q\) and a trivial graph \(K_1\), For \(p \geq 4\), the metric dimension of an edge-corona graph of \(C_p\) and \(K_1\),

\[
\dim(C_p \bowtie K_1) = \begin{cases} 
 2, & \text{for even } p \\
 3, & \text{for odd } p.
\end{cases}
\]

**Proof.** a). When \(p = 2k\) for \(k \in \mathbb{N}\).

  1). According to Theorem 2.1., because \(G \cong C_p \bowtie K_1\) is not a path then \(\dim(G) \geq 2\).

  2). We will show \(\dim(G) \leq 2\).

  Choose a set \(W = \{y_1, y_2\} \subset V_G\) and a vertex \(v \in V_G\). Representations of all vertices in \(V_G \setminus \{y_1, y_2\}\) are:
\[
 r(x_i | W) = \begin{cases} 
 (1, k) & , i = 1 \\
 (i - 1, k - i + 1) & , 2 \leq i \leq k \\
 (k, 1) & , i = k + 1 \\
 (p - i + 2, i - k) & , k + 2 \leq i \leq p. 
\end{cases}
\]

All of the above representations are distinct. We can check for some fixed \( s \) and \( t \) as follows (\( s - 1, k - s + 1, k, s + 1 \)) = (\( p - t + 2, t - k \)). Then \( s - 1 = p - t + 2 \) and \( k - s + 1 = t - k \), a contradiction. Thus, \( W \) is a resolving set for \( G \) and obviously from minimal cardinality. Hence, \( \dim(C_p \circ K_1) = 2. \)

Since \( K^s \circ K_1 \cong K_{n+1} \) then \( \dim(K^s \circ K_1) = (n + 1) - 1 = n \) (Theorem 2.1.). Since \( S^s \circ K^1 \cong f^s \) and \( \dim(f^s) = n \) then \( \dim(S^s \circ K_1) = n. \) For \( r \geq 2, s = 1, K_{2,2} \cong P_3 \) so that \( \dim(K_{2,2} \circ K_1) = 2 \) (Theorem 3.1.1.). For \( r = 2, s = 2, K_{2,2} \cong C_4 \) so that \( \dim(K_{2,2} \circ K_1) = 2 \) (Theorem 3.1.2.). Moreover, we just need to show \( \dim(K_{r,s} \circ K_1) \) for \( r \geq 2, s \geq 3 \), similar to the proof in the following theorem.

### 3.1.3. The exact value of \( \dim(G \circ mK_1) \) when \( G \) is a bipartite graph

Given a bipartite graph \( K_{r,s}, r \geq 2, s \geq 3 \), of order \( p = r + s \) and measure \( q \), and a trivial graph \( K_1 \). The set of vertices of \( K_{r,s} \) is \( V_{K_{r,s}} = V_1 \cup V_2 = \{ x_i \mid i = 1,2,\ldots,r \} \cup \{ x_{2h} \mid h = 1,2,\ldots,s \} \) and the set of edges is \( E_{K_{r,s}} = \{ x_i x_{2h} \mid i = 1,2,\ldots,r; h = 1,2,\ldots,s \} \). A graph \( G \), is an edge-corona graph of \( K_{r,s} \) and \( K_1 \) denoted by \( G \cong K_{r,s} \circ K_1 \), has \( q \) copies of a trivial graph \( K_1 \). As above, a copy of a \( j \)th trivial graph denoted by \( H_j = y_j = y_{ih}, \quad i = 1,2,\ldots,r; h = 1,2,\ldots,s \). An Edge-corona graph \( G \cong K_{r,s} \circ K_1 \) has a set of vertices \( V_G = V_{K_{r,s}} \cup \bigcup_{h=1}^{s} \bigcup_{i=1}^{r} \{ y_{ih} \} \) and a set of edges \( E_G = E_{K_{r,s}} \cup \{ x_{2h} y_{ih}, x_i y_{ih} \mid 1 \leq i \leq p-1, 1 \leq h \leq p \} \).

We consider \( \{ e_j \} \circ H_j \subset K_{r,s} \circ K_1 \). The metric dimension of \( K_{r,s} \circ K_1 \), with \( p \geq 4 \) given by the following Theorem 3.1.3.

**Theorem 3.1.3.** Given a bipartite graph \( K_{r,s} \) of order \( p = r + s \) and measure \( q \) and a trivial graph \( K_1 \). For any integer \( r, s \) with \( r \geq 2, s \geq 3 \), the metric dimension of an edge-corona graph of \( K_{r,s} \) and \( K_1 \), \( \dim(K_{r,s} \circ K_1) = p - 2. \)

The following results is the metric dimension of graphs with \( m \)-pendant points.

**Theorem 3.1.4.** Given a connected graph \( G \) of order \( p \geq 4 \) and measure \( q \geq 2 \). Then,

\[
 \dim(G \circ mK_1) = \begin{cases} 
 mq - 1 & , \text{ for } G \cong S_n \\
 q(m - 1) & , \text{otherwise.} 
\end{cases}
\]

It is mentioned in [2] that if we add a point at a graph \( G \) such that there are greater than one edge incident to this point then metric dimension of this graph maybe fixed, decrease or increase from metric dimension of \( G \). From this statement and the results in Theorem 3.1.1. until Theorem 3.1.3., we give the following lemma.

**Lemma 3.1.5.** If \( G' \) is a graph obtained by adding a pendant point to every edge of a connected graph \( G \) of order \( p \geq 2 \), then \( \dim(G) \leq \dim(G') \leq \dim(G) + 1. \)
Lemma 3.1.5. useful to obtain the following theorem.

**Theorem 3.1.6.** For every connected graph $G$ of order $p \geq 3$ with $|E(G)| = q \geq 2$, $\dim(G \hat{\circ} mK_1) \leq q(m - 1)$.

The above results is an upper bound of $\dim(G \hat{\circ} mK_1)$. Now, we determine the local metric dimension of $G \hat{\circ} H$ for $H \cong mK_1$.

### 3.2. The Exact Value of $\dim_e(G \hat{\circ} mK_1)$

As mentioned above, a graph $G$ that we use in this paper are $P_n$, $C_n$, $K_n$, $S_n$ and $K_{r,s}$, $r \geq 2, s \geq 1$. In this part, we determine the exact values of $\dim_e(G \hat{\circ} mK_1)$ for some simple connected graphs $G$. By Theorem 2.3., we only show $\dim_e(G \hat{\circ} mK_1)$ with $G$ is a bipartite graph that is $P_n$, $S_n$ and $K_{r,s}$, $r \geq 2, s \geq 1$.

#### 3.2.1. The exact value of $\dim_e(G \hat{\circ} mK_1)$ when $G$ is a path graph

Let $P_p$ be a path with vertex set $V_{P_p} = \{x_i | 1 \leq i \leq p, i \in N\}$ and edge set $E_{P_p} = \{x_ix_h | 1 \leq i \leq p - 1, h = i + 1; i \in N\}$ and $K_1$ is a trivial graph. A graph $G \cong P_p \hat{\circ} mK_1$, has $mq$ copies of a trivial graph $K_1$. We denote by $y_{jk}$, the pendant point of edge $e_j = x_{ik}$, $1 \leq i = j \leq p - 1; h = i + 1$. Here, a copy of $i^h \cdot mK_1$ denoted by $H_i = (mK_1)_i = \bigcup_{k=1}^{m} \{y_{jk}\}$ for some $j \in \{1,2,...,q\}$. A graph $G \cong P_p \hat{\circ} mK_1$ has a set of vertices $V_G = V_{P_p} \cup \bigcup_{k=1}^{m} y_{jk}$ and a set of edges $E_G = E_{P_p} \cup \{x_{ik}, x_{i+1}y_{jk} | 1 \leq i = j \leq p - 1, 1 \leq k \leq m\}$. We prove the following theorem.

**Theorem 3.2.1.** Let $P_p$ be a path graph of order $p \geq 4$, and $K_1$ is a trivial graph. For all integer $m \geq 1$, $\dim_e(P_p \hat{\circ} mK_1) = 2$.

**Proof.** Let $G \cong P_p \hat{\circ} mK_1$. Since $G$ is not a bipartite graph, then by Theorem 2.3., we obtain $\dim_e(G) \geq 2$. Moreover, we just need to show $\dim_e(G) \leq 2$. We will show that $W = \{x_1, x_m\}$ is a resolving set of $G$. The representation of all vertices in $V_G - W$ are

- $r(x_i | W) = (i - 1, p - i), \ 2 \leq i \leq p - 2$
- $r(y_{jk} | W) = (j, p - j), \ 1 \leq j \leq p - 1; \ 1 \leq k \leq m$.

All the above representations are distinct. Suppose that $(s - 1, p - s) = (t, p - t)$ for some fixed $s$ and $t$. Then $s = t + 1$ and $s = t$, a contradiction. Thus, representation of any two adjacent vertices are distinct, too. Therefore, $W$ is a local resolving set of $G$. Obviously cardinality of $W$ is minimal. So, $\dim_e(G) \leq 2$. \[\square\]

#### 3.2.2. The exact value of $\dim_e(G \hat{\circ} mK_1)$ when $G$ is a star graph

Let $S_p$ be a star graph with vertex set $V_{S_p} = \{c, x_i | 1 \leq i \leq p, i \in N\}$ and edge set $E_{S_p} = \{cx_i | 1 \leq i \leq p, i \in N\}$ and $K_1$ is a trivial graph. A graph $G \cong S_p \hat{\circ} pK_1$, has $mp$ copies of a trivial graph $K_1$. We denote by $y_{jk}$, the pendant point of edge $e_i = cx_i, 1 \leq i \leq p$. We denote a copy of $i^h \cdot mK_1$ by $H_i = (mK_1)_i = \bigcup_{k=1}^{m} \{y_{jk}\}$ for some $i \in \{1,2,...,p\}$. A graph $G \cong S_p \hat{\circ} mK_1$ has a set of vertices $V_G = V_{S_p} \cup$
\[ \bigcup_{k=1}^{m} y_{ik}, \ 1 \leq i \leq p \] and a set of edges \( E_G = E_{S_p} \cup \{e_{y_{ik}, x_{ij}y_{ik}} \mid 1 \leq i \leq p, \ 1 \leq k \leq m \}. \) The following is Theorem 3.2.2. and its proof.

**Theorem 3.2.2.** Let \( S_p \) be a star graph of order \( n \geq 4 \), and \( K_1 \) is a trivial graph. For all integer \( m \geq 1 \), \( \dim_{r}(S_p \otimes mK_1) = p - 1 \).

**Proof.** Let \( G \cong S_p \otimes mK_1 \). By Theorem 2.2., since \( G \) is not a complete graph, then \( \dim_{r}(G) \geq p - 1 \).

We will show that \( W = \{x_i \mid 1 \leq i \leq p, p \in N\} \) is a local resolving set of \( G \). The representation of all vertices in \( V_G - W \) are

\[
\begin{align*}
    r(c_{W}) &= (1, 1, ..., 1) \\
    r(y_{ik}|W) &= (2, 2, ..., \frac{1}{1}, 2, 2, ..., 2), \ 1 \leq i \leq p; \ 1 \leq k \leq m.
\end{align*}
\]

From direct observation, we can see that all the above representations are distinct. Such that for any two adjacent vertices in \( G \) also have distinct representations. So, \( W \) is a local resolving set of \( G \) and \( |W| < p - 1 \).

Suppose that \( B \) is a local basis of \( G \) with \( |B| < |W| \). Because \( H \) is a null graph, if there is \( z \in B \cap V_H \), for some \( i \), then the pairs of vertices of \( G \) which are distinguished by \( z \) can be distinguished also by \( x_i \). Let the set \( B_1 \) obtained from \( B \) by replacing \( x_i \) by \( z \in B \cap V_H \), with \( i \in \{1, 2, ..., p\} \). So, \( B_1 \) is a local resolving set of \( G \) and \( |B_1| \leq |B| < |W| = \dim_{r}(G) \), a contradiction. Thus, \( W \) is a local basis of \( G \). \( \square \)

**3.2.3. The exact value of \( \dim_{r}(G \otimes mK_1) \) when \( G \) is a bipartite graph**

Let \( K_{r,s}, r \geq 2, s \geq 1 \) be a bipartite graph of order \( p = r + s \) and measure \( q = r.s \) with vertex set \( V_{K_{r,s}} = V_1 \cup V_2 \), where \( V_1 = \{x_i \mid 1 \leq i \leq r, r \in N\} \), \( V_2 = \{y_h \mid 1 \leq h \leq s; s \in N\} \), and edge set \( E_{K_{r,s}} = \{x_iy_h \mid 1 \leq i \leq r, 1 \leq h \leq s; i, h \in N\} \). \( K_1 \) is a trivial graph. A graph \( G \cong K_{r,s} \otimes mK_1 \) has \( mq \) copies of a trivial graph \( K_1 \). We denote by \( z_{i(h(k))} = 1 \leq k \leq m \); the pendant point \( K_1 \) of edge \( e_{i(h)} = x_iy_h \).

For some \( 1 \leq h \leq s; \ 1 \leq i \leq r \). We denote a copy of \( j^{th}-mK_1 \) by \( H_j = ((mK_1)_j) = \bigcup_{k=1}^{m} \{z_{i(h(k))} = z_{jk} \} \) for some \( j \in \{1, 2, ..., q\} \), with \( j = \left\{ \begin{array}{ll}
        h & \text{if } i = 1 \\
        (i-1)r + h & \text{if } i \geq 2.
        \end{array} \right. \)

A graph \( G \cong K_{r,s} \otimes mK_1 \) has a set of vertices \( V_G = V_{K_{r,s}} \cup \bigcup_{j=1}^{m} z_{jk}, \ 1 \leq j \leq q \) and a set of edges \( E_G = E_{K_{r,s}} \cup \{x_{ij}z_{jk}, x_{2i}z_{jk} \mid 1 \leq i \leq r, 1 \leq j \leq q, 1 \leq k \leq m\} \). The following is Theorem 3.2.3. and its proof.

**Theorem 3.2.3.** Let \( K_{r,s} \) with \( r \geq 2, s \geq 1 \), is a bipartite graph of order \( n \geq 4 \), and \( K_1 \) is a trivial graph. For any integer \( m \geq 1 \), \( \dim_{r}(K_{r,s} \otimes mK_1) = \max\{r, s\} \).

**Proof.** Let \( G \cong K_{r,s} \otimes mK_1 \). We suppose that \( s \geq t \). We will prove that \( W = \{z_{i(h(k))} \mid 1 \leq h \leq s\} \) is a local resolving set of \( G \). The representation of all vertices in \( V_G - W \) are

\[
\begin{align*}
    r(x_i|W) &= \left\{ \begin{array}{ll}
        (1, 1, ..., 1) & \text{if } i = 1 \\
        (2, 2, ..., 2) & \text{if } 2 \leq i \leq r.
        \end{array} \right.
\end{align*}
\]
\begin{equation}
\begin{aligned}
r(y_h | W) &= (2, 2, \ldots, 1, 2, \ldots, 2), \quad 1 \leq h \leq s. \\
r(z_{i(h(k))} | W) &= (2, 2, \ldots, 2), \quad 1 \leq h \leq s; \quad 2 \leq k \leq m. \\
r(z_{i(h(k))} | W) &= (3, 3, \ldots, 2, 3, 3, \ldots, 3), \quad 2 \leq i \leq r; \quad 1 \leq h \leq s; \quad 1 \leq k \leq m.
\end{aligned}
\end{equation}

Among pendant points \( z_{i(h(k))} \) are not adjacent. For every \( 2 \leq j \neq l \leq r \), \( x_j \neq x_l \), and for every \( 1 \leq j \neq l \leq s \), \( y_j \neq y_l \). So, for every pairs of adjacent vertices in \( G \) have distinct representations. Therefore, \( W \) is a local resolving set of \( G \).

Suppose that \( A \) is a local basis of \( G \) with \( |A| < |W| \). Because \( H \) is a null graph, if there is \( v \in A \cap V_{H_j} \), for some \( j \), then the pairs of vertices of \( G \) which are distinguished by \( v \) can be distinguished also by \( x_j \) or \( y_j \). Let the set \( A_1 \) obtained from \( A \) by replacing \( x_j \) or \( y_j \) by \( v \in A \cap V_{H_j} \), with \( j \in \{1, 2, \ldots, q\} \). So, \( A_1 \) is a local resolving set of \( G \) and \( |A_1| \leq |A| < |W| = \dim_t(G) \), a contradiction. Thus, \( W \) is a local basis of \( G \).

From the results that have discussed in part 3.1 and 3.2, we can conclude as follows.

**Theorem 3.2.4.** Let \( G \) be a connected graph of order \( p \geq 4 \). For any integer \( m \geq 1 \),
\[
\dim_t(G \odot mK_j) \leq \dim(G \odot mK_j).
\]

4. Conclusions.
We have discussed the (local) metric dimension of edge-corona of some connected graphs \( G \) and null graph \( \overline{K}_m \) or \( mK_1 \) that is a graph with pendant points. As metric dimension, if we add a point at a graph \( G \) such that there are greater than one edge incident to this point, then the local metric dimension of this graph can be fixed, decrease or increase from local metric dimension of \( G \) and \( \dim_t(G \odot mK_j) \leq \dim(G \odot mK_j) \).

**Acknowledgement**
Authors would like to thank the anonymous referee for all of their comments which is very useful to improve this paper.

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