OSCILLATORY BREUER-MAJOR THEOREM WITH APPLICATION TO THE RANDOM CORRECTOR PROBLEM

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Abstract. In this paper, we present an oscillatory version of the celebrated Breuer-Major theorem that is motivated by the random corrector problem. As an application, we are able to prove new results concerning the Gaussian fluctuation of the random corrector. We also provide a variant of this theorem involving homogeneous measures.

1. Introduction and main results

Our work is motivated by the following random homogenization problem. Consider a one-dimensional equation with highly oscillatory coefficients of the form

\[
\begin{cases}
-\frac{d}{dx} \left( a(x/\varepsilon, \omega) \frac{d}{dx} u_\varepsilon(x, \omega) \right) = f \in L^1([0,1], dx) \\
u_\varepsilon(0, \omega) = 0, \quad u_\varepsilon(1, \omega) = b \in \mathbb{R},
\end{cases}
\]

(1.1)

where \( \varepsilon \in (0,1] \). In the literature (see e.g. [1, 2, 8, 10]), the random potential \( a \) is often assumed to be ergodic, uniformly elliptic (i.e. positive and bounded with bounded inverse). Notice that, under the following hypothesis:

For all \( \varepsilon \in (0,1] \), \( \int_0^{1/\varepsilon} \frac{1}{|a(x)|} dx < \infty \) and \( \int_0^{1/\varepsilon} \frac{1}{a(x)} dx \neq 0 \) almost surely, \textbf{(H)}

we can solve (1.1) explicitly:

\[
u_\varepsilon(x, \omega) = c_\varepsilon(\omega) \int_0^x \frac{1}{a(y/\varepsilon, \omega)} dy - \int_0^x \frac{F(y)}{a(y/\varepsilon, \omega)} dy,
\]

(1.2)

where \( F(x) := \int_0^x f(y) dy \) is the antiderivative of \( f \) vanishing at zero and

\[
c_\varepsilon(\omega) := \left( b + \int_0^1 \frac{F(y)}{a(y/\varepsilon, \omega)} dy \right) \left( \int_0^1 \frac{1}{a(y/\varepsilon, \omega)} dy \right)^{-1}.
\]

Throughout this note, we assume that \( a \) satisfies \( \textbf{(H)} \) and it has the following form

\[ q(x) \equiv \frac{1}{a(x)} - \frac{1}{a^*} = \Phi(W_x), \]

where

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(i) \{W_x, x \in \mathbb{R}\} is a centered stationary Gaussian process with a correlation given by \(\rho(x - y) = \mathbb{E}[W_xW_y]\), and we assume that \(\rho\) is continuous with \(\rho(0) = 1\);  
(ii) \(\Phi \in L^2(\mathbb{R}, e^{-x^2/2}dx)\) has the following orthogonal expansion  
\[
\Phi(x) = \sum_{q \geq m} c_q H_q, 
\]  
with \(H_q(x) = (-1)^q e^{x^2/2} \frac{d^q}{dx^q} e^{-x^2/2}\) denoting the \(q\)th Hermite polynomial. Here \(c_m \neq 0\) and \(m \geq 1\) is called the Hermite rank of \(\Phi\).

The quantity \(a^* := 1/\mathbb{E}[1/a(0)]\) is known as the harmonic mean or effective diffusion coefficient of the random potential, see [10, 15].

Remark 1. Assuming the structure \(a(x)^{-1} = \Phi(W_x) + (a^*)^{-1}\), our hypothesis (H) holds provided \(\int_0^{1/\varepsilon} ((a^*)^{-1} + \Phi(W_x)) \, dx \neq 0\) almost surely, for all \(\varepsilon \in (0, 1]\). Note that the local integrability of \(a(x)^{-1}\) follows immediately from its structure: Indeed, for any \(\varepsilon > 0\),  
\[
\mathbb{E} \int_0^{1/\varepsilon} \frac{1}{|a(x)|} \, dx \leq \frac{1}{|a^*|\varepsilon} + \int_0^{1/\varepsilon} \mathbb{E}[|\Phi(W_x)|] \, dx = \frac{1}{|a^*|\varepsilon} + \frac{1}{\varepsilon} \mathbb{E} [\Phi(W_1)] < +\infty,
\]  
which implies that \(\int_0^{1/\varepsilon} |a(x)|^{-1} \, dx\) is almost surely finite. It is clear that our hypothesis (H) holds in presence of uniform ellipticity of \(a\) and the latter is equivalent to the boundedness of \(\Phi\); as one can see from page 276-277 in [11], one can easily construct bounded measurable function \(\Phi\) with given Hermite rank.

Under some mild assumptions on \(\rho\) and \(\Phi\), we can derive the following result concerning the asymptotic behavior of \(u_\varepsilon\) as well as the associated fluctuation.

**Theorem 1.1.** Let previous notation and assumptions hold and we assume that \(\rho \in L^m(\mathbb{R}, dx)\). Then the following statements hold true:

1. For every \(x \in [0, 1]\), \(u_\varepsilon(x)\) converges in probability to \(\bar{u}(x)\), as \(\varepsilon \downarrow 0\), where \(\bar{u}(x)\) solves the following (deterministic) homogenized equation  
\[
\begin{cases}
-\frac{d}{dx} \left( a^* \frac{d}{dx} \bar{u}(x) \right) = f \\
\bar{u}(0) = 0, \quad \bar{u}(1) = b.
\end{cases}
\]  

2. For every \(x \in (0, 1)\), with \(\mu^2 := \sum_{q \geq m} c_q q ! \int_\mathbb{R} \rho(t)^q dt \in [0, \infty)\),  
\[
\frac{u_\varepsilon(x) - \bar{u}(x)}{\sqrt{\varepsilon}} \overset{\mathbb{P}}{\rightarrow} N(0, \mu^2 \int_0^1 F(x, y)^2 \, dy).
\]  

Moreover, if in addition \(\Phi \in L^p(\mathbb{R}, e^{-x^2/2}dx)\) for some \(p > 2\), then  
\[
\left\{ \frac{u_\varepsilon(x) - \bar{u}(x)}{\sqrt{\varepsilon}}, \; x \in [0, 1] \right\} \overset{\mathbb{P}}{\rightarrow} \left\{ \mu \int_0^1 F(x, y) dA_y, \; x \in [0, 1] \right\},
\]
where the above weak convergence takes place in $C([0,1])$,

$$F(x,y) := (c^* - F(y))1_{[0,x]}(y) + x(F(y) - c^*)$$

for $x,y \in [0,1]$ and $\{A_y, y \in [0,1]\}$ is a standard Brownian motion. Here $c^* := ba^* + \int_0^1 F(z)dz$.

The difference $u_\varepsilon - \bar{u}$ is known as the random corrector in the homogenization theory, see [1] and references therein. Our Theorem 1.1 complements findings in the literature, see the following Remark 2: Points (i)-(iii) sketch some relevant history and point (iv) summarizes the novelty of our results.

Remark 2. (i) The authors of [4] considered the case where the random potential had integrable correlation and satisfies certain (strong) mixing conditions: They showed that the random corrector $u_\varepsilon - \bar{u}$ is of order $\sqrt{\varepsilon}$ and converges, after proper scaling, to a Wiener integral with respect to Brownian motion; see also Theorem 2.6 in [1].

(ii) In [1], the result has been extended to a large family of random potential with long-range correlation (i.e. $\rho(\tau) \sim \text{constant} \cdot \tau^{-\alpha}$ for some $\alpha \in (0,1)$): It was shown that when the Hermite rank of $\Phi$ is one, the corrector’s amplitude is of order $\varepsilon^{\alpha/2}$ and after properly scaled, the random corrector converges in law to a stochastic integral with respect to the fractional Brownian motion with Hurst parameter $(2 - \alpha)/2$; see also Theorem 2.3 in [8].

(iii) Following [1], the authors of [8] studied the random corrector problem for the case where the Hermite rank of $\Phi$ is two and $\rho(\tau) \sim \text{constant} \cdot |\tau|^{-\alpha}$ as $\tau \to \infty$, with $\alpha \in (0,1/2)$. They established that the corrector’s amplitude is of order $\varepsilon^\alpha$ and the random corrector, after proper rescaling, converges in law to a stochastic integral with respect to the fractional Brownian motion with Hurst parameter $(2 - \alpha)/2$; see [8, Theorem 2.2]. In the end of the paper [8], the authors conjectured that when the Hermite rank of $\Phi$ is three or higher, the properly rescaled corrector is expected to converge in law to some stochastic integral with respect to the so-called Hermite process and this is confirmed in the work [11].

(iv) Note that all the references mentioned in (i)-(iii) assume that $a$ is stationary ergodic such that $0 < c_1 \leq a(x) \leq c_2$ almost surely for some numerical constants $c_1, c_2$ (so $\Phi$ is bounded), while we do not assume the uniform boundedness of $\Phi$. Instead, we only assume hypothesis (H) and $\Phi \in L^p(\mathbb{R}, e^{-x^2/2}dx)$ for some $p > 2$. Moreover, we consider the case where the Hermite rank of $\Phi$ can be any integer $m \geq 1$, and in this case we establish that the corrector’s amplitude is of order $\sqrt{\varepsilon}$ and properly rescaled corrector converges in law to a Gaussian process, provided the correlation function $\rho \in L^m(\mathbb{R}, dx)$. This is different from the settings in [1,11].

Our Theorem 1.1 is a special case of the following more general result. We denote by $B_b$ the collection of bounded closed sets in $\mathbb{R}^d$. For any $R \geq 0$ we put $B_R := \{x \in \mathbb{R}^d : \|x\| \leq R\}$. Also, f.d.d. means convergence of the finite-dimensional distributions of a given family of random variables depending on a parameter $R$, which tends to $+\infty$. 
Theorem 1.2. Let \( \{W_x, x \in \mathbb{R}^d\} \) be a centered Gaussian stationary process with continuous covariance \( \rho(x - y) := \mathbb{E}[W_x W_y] \) such that \( \rho(0) = 1 \) and \( \rho \in L^m(\mathbb{R}^d, dx) \). Let \( \Phi \) be given as in (1.3) with Hermite rank \( m \geq 1 \). Then, with \( h \in C(\mathbb{R}^d) \), we have

\[
\left\{ R^{d/2} \int_B \Phi(W_x) h(x) \, dx \right\}_{B \in B_b} \xrightarrow{\text{law}} \left\{ \sigma \int_B h(x) dZ_x \right\}_{B \in B_b} ,
\]

where \( Z \) denotes the standard Gaussian white noise on \( \mathbb{R}^d \) and

\[
\sigma^2 = \sum_{q=m}^{\infty} q! c_2^q \int_{\mathbb{R}^d} \rho(z)^q \, dz \in [0, +\infty) .
\]

If in addition \( \Phi \in L^p(\mathbb{R}, e^{-x^2/2} dx) \) for some \( p > 2 \). Then, the following functional central limit theorems hold true:

1. \[
\left\{ R^{d/2} \int_{[0,\zeta]} g(W_x) h(x) \, dx \right\}_{\zeta \in \mathbb{R}^d_+} \xrightarrow{\text{law}} \left\{ \sigma \int_{[0,\zeta]} h(x) dZ_x \right\}_{\zeta \in \mathbb{R}^d_+} ,
\]
   where the above weak convergence holds on the space \( C(\mathbb{R}^d_+) \) and \( [0,\zeta] = \prod_{j=1}^{\ell} [0, z_j] \) given \( \zeta = (z_1, \ldots, z_d) \in \mathbb{R}^d_+ \);

2. \[
\left\{ R^{d/2} \int_{B_t} g(W_x) h(x) \, dx \right\}_{t \geq 0} \xrightarrow{\text{law}} \left\{ \sigma \int_{B_t} h(x) dZ_x \right\}_{t \geq 0} ,
\]
   where the above weak convergence takes place on \( C(\mathbb{R}_+) \).

Roughly speaking, the random corrector \( u_\varepsilon(x) - \bar{u}(x) \) from Theorem 1.1 can be written as a sum of an oscillatory integral and a negligible term so that an easy application of Theorem 1.2 gives us Theorem 1.1, see Section 3 for more details. We will proceed the proof of (1.7) by following the usual arguments for the chaotic central limit theorem (see e.g. [9, 13]), while the functional central limit theorem in (1.9) is established with the help of Malliavin calculus techniques, notably Meyer’s inequality (see [6, 12]).

Remark 3. (i) Theorem 1.2 is a generalization of the celebrated Breuer-Major theorem [5] that corresponds to the case where \( h = 1 \), see also [6, 12]. So we call our result an oscillatory Breuer-Major theorem and this explains our title.

(ii) The functional limit theorem described in (1.8) is new and the limit is a \( d \)-parameter Gaussian process with covariance given by

\[
\sigma^2 \int_{[0,\zeta] \cap [0,\eta]} h^2(x) \, dx ,
\]

while the limit in (1.9) is a Gaussian martingale with quadratic variation given by

\[
\sigma^2 \int_{B_t} h^2(x) \, dx .
\]
Our approach is quite flexible and we can provide another variant of Breuer-Major’s theorem that involves an *homogeneous measure*. Let us first recall the definition of homogeneous measure (see e.g. [7]).

**Definition 1.3.** Given \( \alpha \in \mathbb{R} \setminus \{0\} \), a measure \( \nu \) on \( \mathbb{R}^d \) is said to be \( \alpha \)-**homogeneous** if
\[
\nu(sA) = s^\alpha \nu(A), \quad \text{for any} \quad s > 0 \quad \text{and} \quad A \subset \mathbb{R}^d \quad \text{Borel measurable},
\]
where \( sA := \{ x \in \mathbb{R}^d : s^{-1}x \in A \} \). For example, \( \mu(dx) = |x|^{-\beta}dx \) defines a \((d - \beta)\)-homogeneous measure on \( \mathbb{R}^d \) for any \( \beta \neq d \). Note that for general \( h \in C(\mathbb{R}^d) \), the measure \( \gamma(dx) = h(x)dx \) is not necessarily homogeneous.

**Theorem 1.4.** Fix \( \alpha \in (0, \infty) \) and consider an \( \alpha \)-homogeneous measure \( \nu \) on \( \mathbb{R}^d \) such that \( 0 < \nu(B_1) < \infty \). Let \( \Phi \) be given as in (1.3) with Hermite rank \( m \geq 1 \) and let \( \{W_x, x \in \mathbb{R}^d\} \) be a centered Gaussian stationary process with continuous covariance \( \rho(x-y) := \mathbb{E}[W_xW_y] \) such that \( \rho(0) = 1 \) and \( \rho \in L^m(\mathbb{R}^d, d\nu) \). Then
\[
\left\{ \frac{R^{\alpha/2}}{B} \int_B \Phi(W_{xR})\nu(dx) \right\} \xrightarrow{\text{law} \quad R \to +\infty} \left\{ \sigma_\nu Z(B) \right\}_{B \in \mathcal{B}}
\]
where \( Z \) stands for the Gaussian random measure with intensity \( \nu \) on \( \mathbb{R}^d \) and
\[
\sigma^2_\nu := \sum_{q \geq m} c_q^2 q! \int_{\mathbb{R}^d} \rho(z)^q \nu(dz) \in [0, +\infty).
\]

Moreover, we have the following functional central limit theorems:

1. If in addition, \( \Phi \in L^p(\mathbb{R}, e^{-x^2/2}dx) \) for some \( p > 2 \) and \( \alpha p > 2d \), then
\[
\left\{ \frac{R^{\alpha/2}}{[0,z]} \int_{[0,z]} \Phi(W_{xR})\nu(dx) \right\} \xrightarrow{\text{law} \quad R \to +\infty} \left\{ \sigma_\nu Z([0,z]) \right\}_{z \in \mathbb{R}^d_+}.
\]

2. If in addition, \( \Phi \in L^p(\mathbb{R}, e^{-x^2/2}dx) \) for some \( p > 2 \) and \( \alpha p > 2 \), then
\[
\left\{ \frac{R^{\alpha/2}}{B_t} \int_{B_t} g(W_{xR})\nu(dx) \right\} \xrightarrow{\text{law} \quad R \to +\infty} \left\{ \sigma_\nu Z(B_t) \right\}_{t \geq 0}.
\]

One can refer to the book [14] for any unexplained notation and definition. The rest of this article consists of three more sections: Section 2 is devoted to some preliminary material. In Section 3, we present the proof of Theorem 1.2 and then as anticipated, we demonstrate how Theorem 1.2 implies Theorem 1.1. We will sketch the proof of Theorem 1.4 in Section 4.

Note that all random objects in this note are assumed to be defined on a common probability space \( (\Omega, \mathcal{F}, \mathbb{P}) \) and we will use \( C \) to denote a generic constant that is immaterial to our estimates and it may vary from line to line.
2. Preliminaries

Recall that \( \{W_x, x \in \mathbb{R}^d\} \) is a centered stationary Gaussian process with a continuous covariance function \( \rho \). Without losing any generality, we assume that \( W_x = X(e_x) \), where \( X = \{X(h), h \in \mathcal{H}\} \) is an isonormal Gaussian process over a real Hilbert space \( \mathcal{H} \) and \( e_x \in \mathcal{H} \) are elements such that \( \langle e_x, e_y \rangle_{\mathcal{H}} = \rho(x - y) \).

In what follows, we introduce some standard notation from Malliavin calculus; see also the monograph [1] for more details. For a smooth and cylindrical random variable \( F = f(X(h_1), \ldots, X(h_n)) \) with \( h_i \in \mathcal{H} \) and \( f \in C_0^\infty(\mathbb{R}^n) \), we define its Malliavin derivative as the \( \mathcal{H} \)-valued random variable given by

\[
DF = \sum_{i=1}^n \frac{\partial}{\partial x_i} f(X(h_1), \ldots, X(h_n)) h_i.
\]

By iteration, we can define the \( k \)th Malliavin derivative of \( F \) as an element in \( L^2(\Omega; \mathcal{H}^\otimes k) \). Here \( \mathcal{H}^\otimes k \) denotes the \( k \)th tensor product of \( \mathcal{H} \) and we denote by \( \mathcal{H}^\otimes \) the space of symmetric tensors in \( \mathcal{H}^\otimes k \). For any \( k \in \mathbb{N} \) and \( p \in [1, \infty) \), we define the Sobolev space \( \mathbb{D}^{k,p} \) as the closure of the space of smooth and cylindrical random variables with respect to the norm \( \| \cdot \|_{k,p} \) defined by

\[
\|F\|_{k,p}^p = \mathbb{E}(|F|^p) + \sum_{i=1}^k \mathbb{E}(\|D^i F\|^p_{\mathcal{H}^\otimes i}).
\]

The divergence operator \( \delta \) is defined as the adjoint of the derivative operator \( D \). An element \( u \in L^2(\Omega; \mathcal{H}) \) belongs to the domain of \( \delta \), denoted by \( \text{dom}(\delta) \) if there is a constant \( c_u \) that only depends on \( u \) such that

\[
|\mathbb{E}[\langle DF, u \rangle_{\mathcal{H}}]| \leq c_u \sqrt{\mathbb{E}[F^2]} \quad \text{for any } F \in \mathbb{D}^{1,2}.
\]

For \( u \in \text{dom}(\delta) \), the existence of \( \delta(u) \) is guaranteed by the Riesz representation theorem and it satisfies the following duality relation

\[
\mathbb{E}[\langle DF, u \rangle_{\mathcal{H}}] = \mathbb{E}[F \delta(u)] \quad \text{for any } F \in \mathbb{D}^{1,2}.
\]

Similarly, we can define the iterated divergence \( \delta^k \): For \( u \in \text{dom}(\delta^k) \subset L^2(\Omega; \mathcal{H}^\otimes k) \), \( \delta^k(u) \) is characterized by the following duality relation

\[
\mathbb{E}[\langle D^k F, u \rangle_{\mathcal{H}}] = \mathbb{E}[F \delta^k(u)] \quad \text{for any } F \in \mathbb{D}^{k,2}.
\]

The well-known Wiener-Itô chaos decomposition states that any \( F \in L^2(\Omega, \sigma\{W\}, \mathbb{P}) \) admits the following expression

\[
F = \mathbb{E}[F] + \sum_{p \geq 1} \mathbb{E}^p(f_p), \quad \text{(2.1)}
\]

with \( f_p \in \mathcal{H}^\otimes p \) uniquely determined by \( F \). Note that given any unit vector \( e \in \mathcal{H} \), we have \( H_p(X(e)) = \delta^p(e^\otimes p) \). We call \( \mathcal{C}_p = \{H_p(X(e)) : e \in \mathcal{H} \text{ and } \|e\|_{\mathcal{H}} = 1\} \) the \( p \)th Wiener chaos associated with the isonormal Gaussian process \( X \) and we write \( J_p \).
for the projection operator onto \( C_p \). Then we define Ornstein-Uhlenbeck semigroup 
\( (P_t, t \in \mathbb{R}_+) \) and its generator \( L \) by putting
\[
P_t = \sum_{p \geq 0} e^{-pt} J_p \quad \text{and} \quad L = \sum_{p \geq 1} -p J_p ,
\]
and we write \( L^{-1} \) for the pseudo-inverse of \( L \), that is,
\[
L^{-1} F = -\sum_{p \geq 1} \frac{1}{p} J_p F \quad \text{for any centered} \quad F \in L^2(\Omega, \sigma\{W\}, \mathbb{P}).
\]
Note that these operators enjoy the following nice relation: \( F = -\delta DL^{-1} F \) for any centered \( F \in L^2(\Omega, \sigma\{W\}, \mathbb{P}) \). Now let us record an important consequence of this relation. Let \( \Phi \) be given as in [13] and have Hermite rank \( m \geq 1 \). We define the shifted function
\[
\Phi_m(x) = \sum_{q \geq m} c_q H_{q-m}(x) ,
\]
which satisfies the following properties:
(A) \( \Phi_m(W_x) = \Phi_m(X(e_x)) \in \mathbb{D}^{m,2} \) and \( \Phi(W_x) = \delta^m \left( \Phi_m(W_x)e_x^{\otimes m} \right) \) for any \( x \in \mathbb{R}^d \);
(B) \( \Phi_m(W_x)e_x^{\otimes m} = (-DL^{-1})^m \Phi(W_x) \) and as a consequence of Meyer’s inequality, we have for every \( k \in \{0,1,\ldots,m\} \), \( x \in \mathbb{R}^d \) and \( p > 1 \),
\[
\| D^k(\Phi_m(W_x)) \|_{L^p(\Omega,\delta^{\otimes k})} \leq C \| \Phi(W_x) \|_{L^p(\Omega)} ; \tag{2.2}
\]
see also Lemma 2.1 and Lemma 2.2 in [12].

Let \( \{ \varepsilon_i, i \in \mathbb{N} \} \) be an orthonormal basis of \( \mathfrak{H} \). For \( f \in \mathfrak{H}^{\circ p} \) and \( g \in \mathfrak{H}^{\circ q} \) \((p,q \in \mathbb{N})\), we define the \( r \)-contraction as the element in \( \mathfrak{H}^{\circ p+q-2r} \) \((r \in \{0,\ldots,p \land q\})\) given by
\[
f \otimes_r g = \sum_{i_1,\ldots,i_r \in \mathbb{N}} \langle f, \varepsilon_{i_1} \otimes \varepsilon_{i_2} \otimes \cdots \otimes \varepsilon_{i_r} \rangle_{\mathfrak{H}^{\circ r}} \langle g, \varepsilon_{i_1} \otimes \varepsilon_{i_2} \otimes \cdots \otimes \varepsilon_{i_r} \rangle_{\mathfrak{H}^{\circ r}} .
\]
In particular, \( f \otimes_0 g = f \otimes g \) and if \( p = q \), \( f \otimes_p g = \langle f, g \rangle_{\mathfrak{H}^{\circ p}} \).

In the end of this section, we present a multivariate version of the chaotic central limit theorem [9] that we borrow from [6] Theorem 2.1.

**Proposition 2.1.** Fix an integer \( n \geq 1 \) and consider a family \( \{G_R, R > 0\} \) of random vectors in \( \mathbb{R}^n \) such that each component of \( G_R = (G_{R,1}, \ldots, G_{R,n}) \) belongs to \( L^2(\Omega, \sigma\{W\}, \mathbb{P}) \) and has the following chaos expansion
\[
G_{R,j} = \sum_{q \geq 1} \delta^q(g_{q,j,R}) \quad \text{with} \quad g_{q,j,R} \in \mathfrak{H}^{\circ q} \text{ deterministic}.
\]
Suppose the following conditions (a)-(d) hold:
(a) For each \( i,j \in \{1,\ldots,n\} \) and for every \( q \geq 1 \), \( q! \| g_{q,i,R} g_{q,j,R} \|_{\mathfrak{H}^{\circ q}} \) converges to some \( \sigma_{i,j,q} \in \mathbb{R} \), as \( R \to +\infty \).
(b) For each \( i \in \{1,\ldots,n\} \), \( \sum_{q \geq 1} \sigma_{i,i,q} < +\infty \).
(c) For each \( i \in \{1,\ldots,n\} \), \( q \geq 2 \) and \( r \in \{1,\ldots,q-1\} \), we have that, as \( R \to +\infty \), \( \| g_{q,i,R} \otimes_r g_{q+1,i,R} \|_{\mathfrak{H}^{\circ q+2r-2r}} \) converges to zero.
(d) For each $i \in \{1, \ldots, n\}$, \(\lim_{N \to +\infty} \sup_{R > 0} \sum_{q \geq N+1} q! \|g_{q,i,R}\|^2_{\mathcal{G}^{\otimes q}} = 0\).

Then \(G_R\) converges in law to \(N(0, \Sigma)\) as \(R \to +\infty\), where \(\Sigma = (\sigma_{i,j})_{i,j=1}^n\) is given by \(\sigma_{i,j} = \sum_{q \geq 1} \sigma_{i,j,q}\).

3. Proof of Theorem 1.2 and Theorem 1.1

In this section, we first prove the convergence of finite-dimensional distributions in the framework of Theorem 1.2. Next, we will establish the tightness property under the additional assumption that \(\Phi \in L^p(\mathbb{R}, e^{-x^2/2}dx)\) for some \(p > 2\), which is needed to establish (1.8) and (1.9). These two steps will conclude the proof of Theorem 1.2, and in the end of this section, we demonstrate how one can derive Theorem 1.1 from Theorem 1.2.

3.1. Convergence of finite-dimensional distributions. For each \(R > 0\) and \(B \in \mathcal{B}_b\), we put

\[
G_R(B) = R^{d/2} \int_B \Phi(W_xR)h(x) \, dx.
\]

Then, it is enough to consider bounded Borel sets \(B_i \in \mathcal{B}_b, i = 1, \ldots, n\), and establish the following limit result

\[
(G_R(B_1), \ldots, G_R(B_n)) \xrightarrow{\text{law}} N(0, \Sigma),
\]

where \(\Sigma = (\sigma_{i,j})_{i,j=1}^n\) is defined by

\[
\sigma_{i,j} = \sigma^2 \int_{B_i \cap B_j} h(x)^2 \, dx.
\]

For \(j \in \{1, \ldots, n\}\), we can rewrite \(G_R(B_j)\) using the Hermite expansion (1.3) as follows:

\[
G_R(B_j) = R^{d/2} \int_{B_j} \sum_{q \geq m} c_q H_q(W_xR)h(x) \, dx = R^{d/2} \int_{B_j} \sum_{q \geq m} \delta^q(c_q e_{xR}^{\otimes q})h(x) \, dx
\]

\[
= \sum_{q \geq m} \delta^q \left( c_q R^{d/2} \int_{B_j} e_{xR}^{\otimes q}h(x) \, dx \right) =: \sum_{q \geq m} \delta^q (g_{q,j,R}).
\]

(a) For any \(i, j \in \{1, \ldots, n\}\), we have

\[
q! \langle g_{q,i,R}, g_{q,j,R} \rangle_{\mathcal{G}^{\otimes q}} = q! c_q^2 R^d \int_{B_i \cap B_j} \rho(xR - yR)^q h(x)h(y) \, dxdy
\]

\[
= q! c_q^2 R^{-d} \int_{Rj \cap RB_i} \rho(x - y)^q h((x/R)h(y/R) \, dxdy
\]

\[
= q! c_q^2 R^{-d} \int_{\{x \in RB_i, x - z \in RB_j\}} \rho(z)^q h(x/R)h((x - z)/R) \, dxdz.
\]
Making the change of variables $x/R = y$ yields
\[
q!\langle g_{q,i,R}, g_{q,j,R} \rangle_{\mathcal{S}_q^0} = C_q^2 q! \int_{\{y \in B_i, y - zR^{-1} \in B_j\}} \rho(z)^q \left( h(y - zR^{-1})h(y) \right) dy dz.
\]

Taking into account that $h$ is continuous and $B_j$ is closed, we deduce from the dominated convergence theorem that
\[
q!\langle g_{q,i,R}, g_{q,j,R} \rangle_{\mathcal{S}^0_q} \xrightarrow{R \to +\infty} C_q^2 q! \left( \int_{\mathbb{R}^d} \rho(z)^q dz \right) \int_{B_i \cap B_j} h(y)^2 dy =: \sigma_{i,j,q}.
\]

(b) For each $i \in \{1, \ldots, n\}$,
\[
\sum_{q \geq m} \sigma_{i,i,q} = \left( \int_{B_i} h(y)^2 dy \right) \sum_{q \geq m} C_q^2 q! \left( \int_{\mathbb{R}^d} \rho(z)^q dz \right) = \sigma^2 \int_{B_i} h(y)^2 dy.
\]

Note that the quantity $\sigma^2$ as defined in the statement of Theorem 1.2 is finite, because $\int_{\mathbb{R}^d} \rho(z)^q dz$ is bounded by $\int_{\mathbb{R}^d} |\rho(z)|^m dz$ and $\sum_{q \geq m} C_q^2 q! < +\infty$. So we just verified the condition (b).

(c) For each $i \in \{1, \ldots, n\}$, $q \geq 2$ and $r \in \{1, \ldots, q - 1\}$, we have,
\[
\|g_{q,i,R} \otimes_r g_{q,i,R}\|_{\mathcal{S}^0_{2q - 2r}}^2 = C_q^4 R^{2d} \int_{B_i^4} \rho(Rx_1 - Rx_2)^r \rho(Rx_3 - Rx_4)^r \rho(Rx_1 - Rx_3)^{q - r} \times \rho(Rx_2 - Rx_4)^{q - r} \prod_{i=1}^4 h(x_i) \, d\mathbf{x}
\]
\[
= \frac{C_q^4}{R^{2d}} \int_{(RB_i)^4} \rho(x_1 - x_2)^r \rho(x_3 - x_4)^r \rho(x_1 - x_3)^{q - r} \rho(x_2 - x_4)^{q - r} \prod_{i=1}^4 h(x_i/R) \, d\mathbf{x},
\]

where $\mathbf{x} = (x_1, x_2, x_3, x_4)$. In view of the elementary inequality $a^rb^{q-r} \leq a^q + b^q$ for any $a, b \in \mathbb{R}_+$, we can write
\[
\|g_{q,i,R} \otimes_r g_{q,i,R}\|_{\mathcal{S}^0_{2q - 2r}}^2 \leq \frac{C_q^4}{R^{2d}} \int_{(RB_i)^4} \left( |\rho(x_1 - x_2)|^q + |\rho(x_1 - x_3)|^q \right) \times |\rho(x_3 - x_4)|^r |\rho(x_2 - x_4)|^{q - r} \prod_{i=1}^4 |h(x_i/R)| \, d\mathbf{x}
\]
\[
= \frac{2C_q^4}{R^{2d}} \int_{(RB_i)^4} |\rho(x_1 - x_2)|^q |\rho(x_3 - x_4)|^r |\rho(x_2 - x_4)|^{q - r} \prod_{i=1}^4 |h(x_i/R)| \, d\mathbf{x}.
\]
Taking into account that $h$ is continuous and $B_i$ is bounded yields
\[
\|g_{q,i,R} \otimes_R g_{q,i,R}\|_{\mathcal{B}^{2q-2r}}^2 \leq C \frac{1}{R^{2d}} \left( \int_{(RB_i)^3} \rho(x_1 - x_2)^q \rho(x_3 - x_4)^r \rho(x_2 - x_4)^{q-r} \, dx \right)
\]
\[
\leq C \frac{1}{R^{2d}} \left( \int_{\mathbb{R}^d} |\rho(z)|^q \, dz \right) \int_{(RB_i)^3} \rho(x_3 - x_4)^r \rho(x_2 - x_4)^{q-r} \, dx_2 \, dx_3 \, dx_4
\]
\[
\leq C \frac{1}{R^{2d}} \left( \int_{\mathbb{R}^d} |\rho(z)|^q \, dz \right) \left( \int_{RE_i} |\rho(x)|^r \, dx \right) \left( \int_{RE_i} |\rho(y)|^{q-r} \, dy \right),
\]
where $E_i := B_i - B_i = \{x - y : x, y \in B_i\}$. It is clear that $E_i$ is also bounded.

It suffices to show that for each $r = 1, \ldots, q - 1$,
\[
\frac{1}{R^{d(1-rq^{-1})}} \int_{RE_i} |\rho(x)|^r \, dx \xrightarrow{R \to +\infty} 0. \tag{3.3}
\]
One can establish the above limit as follows. Fix $\delta \in (0,1)$, we deduce from the Hölder’s inequality that
\[
\frac{1}{R^{d(1-rq^{-1})}} \int_{RE_i} |\rho(x)|^r \, dx = \frac{\int_{\delta RE_i} |\rho(x)|^r \, dx}{R^{d(1-rq^{-1})}} + \frac{\int_{(RE_i) \setminus (\delta RE_i)} |\rho(x)|^r \, dx}{R^{d(1-rq^{-1})}}
\]
\[
\leq C \delta^{d(1-rq^{-1})} \left( \int_{\mathbb{R}^d} |\rho(x)|^q \, dx \right)^{r/q} + C(1 - \delta^{d(1-rq^{-1})}) \left( \int_{RB_i \setminus \delta RB_i} |\rho(x)|^q \, dx \right)^{r/q}.
\]
Note that for any fixed $\delta \in (0,1)$, the second term goes to zero, as $R \to +\infty$ and the first term can be made arbitrarily small by choosing sufficiently small $\delta$. This completes our verification of condition (c) from Proposition 2.1.

(d) For each $i \in \{1, \ldots, n\}$, we can see from the computations from step (a) that
\[
\sum_{q \geq N+1} q! \|g_{q,i,R}\|_{\mathcal{B}^{2q-2r}}^2 = \sum_{q \geq N+1} c_q^2 q! \int_{\{y \in B_i \mid y - zR^{-1} \in B_i\}} \rho(z)^q h(y - zR^{-1}) h(y) \, dy \, dz
\]
\[
\leq \left( \int_{\mathbb{R}^d} 1_{B_i}(x) \, dx \right) \left( \sup_{z \in B_i} |h(z)|^2 \right) \sum_{q \geq N+1} c_q^2 q! \int_{\mathbb{R}^d} |\rho(z)|^m \, dz,
\]
which converges to zero (uniformly in $R$), as $N$ goes to infinity.

Therefore, the limit in (3.2) is proved. In particular, (1.7) is established. \qed

Remark 4. If we only assume that $h : \mathbb{R}^d \to \mathbb{R}$ is continuous except at finitely many points, we can still obtain (1.7). This observation will be helpful in the proof of Theorem 1.1.
3.2. Tightness. This part is split into two portions, dealing with proofs of (1.9) and (1.8) respectively.

Proof of (1.9). For each $t \geq 0$, we put

$$X_R(t) = R^{d/2} \int_{B_t} \Phi(W_{xR})h(x)dx.$$ 

Clearly $X_R$ is a random variable with values in $C(\mathbb{R}_+)$. We know from Billingsley’s book [3] that in order to have the tightness of $\{X_R, R > 0\}$, it is sufficient to prove the following moment estimate: There exists some constant $C_T > 0$ such that for any $0 < s < t \leq T$,

$$\|X_R(t) - X_R(s)\|_{L^p(\Omega)} \leq C_T \sqrt{t - s}, \quad (3.4)$$

where $p > 2$ is the fixed index in the statement of Theorem 1.2. To simplify the proof, we assume that $T = 1$.

Using the notation from Section 2 we first write $\Phi(W_{xR}) = \delta^m(\Phi_m(W_{xR})e_{xR}^m)$. Then for any $0 < s < t \leq 1$,

$$\|X_R(t) - X_R(s)\|_{L^p(\Omega)}$$

$$= R^{d/2} \left\| \int_{B_t \setminus B_s} \Phi(W_{xR})h(x)dx \right\|_{L^p(\Omega)}$$

$$= R^{d/2} \left\| \int_{B_t \setminus B_s} \delta^m(\Phi_m(W_{xR})e_{xR}^m)h(x)dx \right\|_{L^p(\Omega)}$$

$$= R^{d/2} \left\| \delta^m \left( \int_{B_t \setminus B_s} \Phi_m(W_{xR})e_{xR}^m h(x)dx \right) \right\|_{L^p(\Omega)}$$

$$=: \|\delta^m(v_R)\|_{L^p(\Omega)},$$

with $v_R = R^{d/2} \int_{B_t \setminus B_s} \Phi_m(W_{xR})e_{xR}^m h(x)dx$. Now we apply the Meyer’s inequality (see [14] Proposition 1.5.4), to get

$$\|\delta^m(v_R)\|_{L^p(\Omega)} \leq C \sum_{k=0}^{m} \|D^k v_R\|_{L^p(\Omega; \delta^{k+m})} \quad \text{see also [12] (2.8)}$$

$$\leq C \sum_{k=0}^{m} \left\| R^{d/2} \int_{B_t \setminus B_s} D^k(\Phi_m(W_{xR})e_{xR}^m) h(x)dx \right\|_{L^p(\Omega; \delta^{k+m})}. $$
Following the same arguments as in the proof of (1.9), we have
\begin{equation}
R^{d/2} \int_{B_t \setminus B_s} D^k \left( \Phi_m (W_x e^{\xi_R}) \right) h(x) dx \\|_{L^p(\Omega;\mathbb{R}^{m+k})}^2
\end{equation}
where we also applied Minkowski's inequality in the last inequality. Therefore, Cauchy-Schwarz inequality and property (B) from Section 2 imply that the quantity in (3.5) is bounded by
\begin{equation}
CR^d \int_{(B_t \setminus B_s)^2} \left\| D^k (\Phi_m (W_x)), D^k (\Phi_m (W_y)) \right\|_{L^2(\Omega)} \| \rho (xR - yR)^m h(x) h(y) dx dy \leq C(t^d - s^d) \int_{\mathbb{R}^d} \| \rho(z) \|^m dz .
\end{equation}
It follows that
\begin{equation}
\| X_R(t) - X_R(s) \|_{L^p(\Omega)} \leq C\sqrt{t^d - s^d} \leq C\sqrt{t - s} .
\end{equation}

Now we show the weak convergence described in (1.8).

**Proof of (1.8).** For $z \in \mathbb{R}_+$, we put
\begin{equation}
Y_R(z) = R^{d/2} \int_{[0,z]} \Phi(W_x) h(x) dx
\end{equation}
and in what follows, we will focus on establishing the tightness of $\{Y_R, R > 0\}$ by proving the following estimate
\begin{equation}
\| Y_R(z) - Y_R(y) \|_{L^p(\Omega)} \leq C \| z - y \|^{d/2} ,
\end{equation}
here $\| \cdot \|$ denotes the Euclidean norm. We write
\begin{equation}
Y_R(z) - Y_R(y) = R^{d/2} \int_{[0,z] \setminus [0,y]} g(W_x) h(x) dx - R^{d/2} \int_{[0,y] \setminus [0,z]} g(W_x) h(x) dx =: A_1 - A_2 .
\end{equation}
Following the same arguments as in the proof of (1.9), we have
\begin{equation}
\| A_1 \|_{L^p(\Omega)} \leq C R^d \int_{[0,z] \setminus [0,y]} \int_{[0,z] \setminus [0,y]} \| \rho (Rx - Ry) \|^m dx dy \leq C \| \rho \|_{L^m(\mathbb{R}^d, dx)}^m \prod_{j=1}^d |y_j - z_j| .
\end{equation}
It is clear that $\prod_{j=1}^d |y_j - z_j| \leq \| y - z \|^{d}$. It follows that $\| A_1 \|_{L^p(\Omega)} \leq C \| y - z \|^{d/2}$. In the same way, we can obtain the same estimate for $\| A_2 \|_{L^p(\Omega)}$, so that (3.6) holds true. \hfill \Box
3.3. Proof of Theorem 1.1. Let us recall that \( q(x) = a(x)^{-1} - (1/a^*) = \Phi(W_x) \) and the solution to (1.1) is given by

\[
\bar{u}_\varepsilon(x) = c_\varepsilon(\omega) \int_0^x \frac{1}{a(y/\varepsilon)} dy - \int_0^x \frac{F(y)}{a(y/\varepsilon)} dy,
\]

where \( F(x) := \int_0^x f(y) dy \) and

\[
c_\varepsilon(\omega) := \left( b + \int_0^1 \frac{F(y)}{a(y/\varepsilon)} dy \right) \left( \int_0^1 \frac{1}{a(y/\varepsilon)} dy \right)^{-1}.
\]

Note that for any \( h \in C([0,1]) \) and each \( v \in (0,1] \), we obtain, by using the Hermite expansion, that

\[
\left\| \int_0^v \frac{q(y/\varepsilon)}{a(y/\varepsilon)} h(y) dy \right\|^2_{L^2(\Omega)} = \sum_{q \geq m} c_q^2 q! \int_0^v \int_0^v \left| \rho \left( \frac{y-x}{\varepsilon} \right) \right|^m dy dx dy \\
\leq \sum_{q \geq m} c_q^2 q! \int_0^v \int_0^v \left| \rho \left( \frac{y-x}{\varepsilon} \right) \right|^m dy dx dy \\
\leq \sum_{q \geq m} c_q^2 q! \int_0^v \int_0^v \left| \rho \left( \frac{y-x}{\varepsilon} \right) \right|^m dy dx dy \\
\leq \left( \sum_{q \geq m} c_q^2 q! \int_\mathbb{R} |\rho(z)|^m dz \right) \varepsilon \leq C \varepsilon.
\]

It follows that

\[
\int_0^v \frac{1}{a(y/\varepsilon)} h(y) dy \text{ converges in } L^2(\Omega) \text{ to } \frac{1}{a^*} \int_0^1 h(y) dy, \text{ as } \varepsilon \downarrow 0.
\]

In particular, the random vector

\[
\bar{J}_\varepsilon(x) := \left( \int_0^x \frac{1}{a(y/\varepsilon)} dy, \int_0^x \frac{F(y)}{a(y/\varepsilon)} dy, \int_0^1 \frac{F(y)}{a(y/\varepsilon)} dy, \int_0^1 \frac{1}{a(y/\varepsilon)} dy \right)
\]

converges in \( L^2(\Omega; \mathbb{R}^4) \) to

\[
J(x) := \left( \frac{x}{a^*}, \int_0^x \frac{F(y)}{a^*} dy, \int_0^1 \frac{F(y)}{a^*} dy, \frac{1}{a^*} \right).
\]

Put \( M(z_1, z_2, z_3, z_4) = (b + z_4)z_1z_4^{-1} - z_2 \), then it follows from continuous mapping theorem that \( u_\varepsilon(x) = M(\bar{J}_\varepsilon(x)) \rightarrow M(J(x)) = \bar{u}(x) \) in probability, as \( \varepsilon \downarrow 0 \), where

\[
\bar{u}(x) = c^* \frac{x}{a^*} - \int_0^x \frac{F(y)}{a^*} dy \quad \text{with} \quad c^* = ba^* + \int_0^1 F(y) dy.
\]

It is easy to see that \( \bar{u} \) solve equation (1.1), so part (1) of Theorem 1.1 is established.

Following the decomposition given in [8] pages 1082-1085], we rewrite the rescaled corrector as follows:

\[
\frac{u_\varepsilon(x) - \bar{u}(x)}{\sqrt{\varepsilon}} = U_\varepsilon(x) + r_\varepsilon(x) + \frac{\rho_\varepsilon(x)}{\sqrt{\varepsilon}}, \quad (3.7)
\]
Estimation of immediately that weakly converge to some processes in $C\{\text{random variable and } F \text{ with convergence to zero in probability, while both }\phi\text{ and conclude that, as } \epsilon \downarrow 0,\text{ we can apply \cite{18} with } d = 1\text{ and } R = 1/\epsilon\text{ and conclude that, as } \epsilon \downarrow 0,\
\left\{\frac{1}{\sqrt{\epsilon}} \int_0^1 q(y/\epsilon) \, dy, x \in [0, 1]\right\}\text{ converges in law to a Gaussian process.}\nThus, the process \{r_\epsilon(x), x \in [0, 1]\} converges in law, hence also in probability, to the zero process.\n(ii) Estimation of \rho_\epsilon(x): Similarly,
\begin{align*}
\frac{\rho_\epsilon(x)}{\sqrt{\epsilon}} &= c^* \sqrt{\epsilon} \left(\int_0^1 \frac{1}{a(y/\epsilon)} \, dy\right)^{-1} \left(\frac{1}{\sqrt{\epsilon}} \int_0^1 q(y/\epsilon) \, dy\right)^2 x \\
&\quad - \left(\int_0^1 q(y/\epsilon) \, dy\right) \left(\int_0^1 \frac{1}{a(y/\epsilon)} \, dy\right)^{-1} \left(\frac{1}{\sqrt{\epsilon}} \int_0^1 F(y) q(y/\epsilon) \, dy\right) x.
\end{align*}
It is clear that both
\begin{align*}
c^* \sqrt{\epsilon} \left(\int_0^1 \frac{1}{a(y/\epsilon)} \, dy\right)^{-1} \quad \text{and} \quad \left(\int_0^1 q(y/\epsilon) \, dy\right) \left(\int_0^1 \frac{1}{a(y/\epsilon)} \, dy\right)^{-1}
\end{align*}
converge to zero in probability, while both
\begin{align*}
\left\{\frac{1}{\sqrt{\epsilon}} \int_0^1 q(y/\epsilon) \, dy \cdot x\right\}_{x \in [0, 1]} \quad \text{and} \quad \left\{\frac{1}{\sqrt{\epsilon}} \int_0^1 F(y) q(y/\epsilon) \, dy \cdot x\right\}_{x \in [0, 1]}
\end{align*}
weakly converge to some processes in $C([0, 1])$, as $\epsilon \downarrow 0$. This implies the process \{\epsilon^{-1/2} \rho_\epsilon(x) : x \in [0, 1]\} converges in probability to the zero process. Then it follows immediately that
\[ r_\epsilon(x) + \frac{\rho_\epsilon(x)}{\sqrt{\epsilon}} \text{ converges in probability to the zero, for every } x \in [0, 1]; \]
and under the additional assumption $\Phi \in L^p(\mathbb{R}, e^{-x^2/2} \, dx),\]
\[ \left\{ r_\varepsilon(x) + \frac{\rho_\varepsilon(x)}{\sqrt{\varepsilon}}, x \in [0,1] \right\} \text{ converges in probability to the zero process.} \]

Endgame: In view of Slutsky’s theorem, we have just established (1.5) and to reach (1.6), it suffices to prove as \( \varepsilon \downarrow 0 \),

\[ U_\varepsilon \xrightarrow{\text{law}} \left\{ \mu \int_0^1 F(x, y) dA_y : x \in [0,1] \right\}, \]

where \( A \) is a standard Brownian motion on \([0,1]\). Now we write for every \( x \in [0,1] \),

\[ U_\varepsilon(x) = \frac{1}{\sqrt{\varepsilon}} \int_0^x \left( c^* - F(y) \right) q(y/\varepsilon) dy + \frac{x}{\sqrt{\varepsilon}} \int_0^1 \left( F(y) - c^* \right) q(y/\varepsilon) dy \]

\[ =: \mathcal{V}_{1,\varepsilon}(x) + \mathcal{V}_{2,\varepsilon}(x). \]

Then applying (1.8) again yields

\[ \mathcal{V}_{1,\varepsilon} \xrightarrow{\text{law}} \mathcal{V}_1 := \left\{ \mu \int_0^x \left( c^* - F(y) \right) dA_y : x \in [0,1] \right\}. \]

It is also clear that as \( \varepsilon \downarrow 0 \),

\[ \mathcal{V}_{2,\varepsilon} \xrightarrow{\text{law}} \mathcal{V}_2 := \left\{ \mu x \int_0^1 \left( F(y) - c^* \right) dA_y : x \in [0,1] \right\}. \]

It follows that the sequence \((\mathcal{V}_{1,\varepsilon}, \mathcal{V}_{2,\varepsilon})\) is tight, and so is \( \mathcal{V}_{1,\varepsilon} + \mathcal{V}_{2,\varepsilon} \). That is, \( U_\varepsilon \) is tight. Now consider \( \lambda_k \in \mathbb{R} \) and \( x_k \in [0,1] \) for \( k \in \{1, \ldots, \ell\} \) and any \( \ell \geq 1 \). We have

\[ \sum_{k=1}^\ell \lambda_k U_\varepsilon(x_k) = \frac{1}{\sqrt{\varepsilon}} \int_0^1 \sum_{k=1}^\ell \lambda_k F(x_k, y) q(y/\varepsilon) dy \xrightarrow{\text{law}} \sum_{k=1}^\ell \lambda_k \int_0^1 F(x_k, y) dA_y. \]

This proves the convergence of the finite-dimensional distributions for \( U_\varepsilon \) and conclude our proof with the above tightness of \( \{U_\varepsilon, \varepsilon > 0\} \).

\[ \blacksquare \]

4. Proof of Theorem 1.4

The proof follows similar arguments as in the proof of Theorem 1.2. Here we first sketch the proof of (1.10). For any \( B \in \mathcal{B}_b \), we first rewrite using Hermite expansions

\[ \hat{G}_R(B) := R^{\alpha/2} \int_B \Phi(W_{xR}) \nu(dx) = R^{\alpha/2} \int_B \sum_{q \geq m} c_q H_q(W_{xR}) \nu(dx) \]

\[ = \sum_{q \geq m} \delta^q \left( c_q R^{\alpha/2} \int_B e^{2q} \nu(dx) \right). \]
By the orthogonality of Hermite polynomials, we have
\[
\mathbb{E}[\hat{G}_R(B)^2] = R^\alpha \sum_{q \geq m} c_q^2 q! \int_{B^2} \rho(xR - yR)^q \nu(dx) \nu(dy)
\]
\[
= R^{-\alpha} \sum_{q \geq m} c_q^2 q! \int_{(RB)^2} \rho(x - y)^q \nu(dx) \nu(dy),
\]
where we used the \(\alpha\)-homogeneity and made a change of variable in the last equality: \((xR, yR) \rightarrow (x, y)\). Making another change of variable \((x = x, z = x - y)\) yields
\[
\int_{(RB)^2} \rho(x - y)^q \nu(dx) \nu(dy) = \int_{\mathbb{R}^d} \rho(z)^q \nu[(RB) \cap (z + RB)] \nu(dz)
\]
\[
= R^\alpha \nu(B) \int_{\mathbb{R}^d} \rho(z)^q \nu[(RB) \cap (z + RB)] \nu(dz).
\]
In view of the \(\alpha\)-homogeneity, the quantity \(\nu[(RB) \cap (z + RB)] / \nu(RB)\) converges to 1 as \(R \rightarrow +\infty\), for each \(z \in \mathbb{R}^d\). Indeed, given \(z \in \mathbb{R}\), we can write
\[
\frac{\nu[(RB) \cap (z + RB)]}{\nu(RB)} = \frac{\nu[B \cap (R^{-1}z + B)]}{\nu(B)}.
\]
By the dominated convergence theorem, our assumptions ensure that
\[
R^\alpha \mathbb{E}[\hat{G}_R(B)^2] = \nu(B) \sum_{q \geq m} c_q^2 q! \int_{\mathbb{R}^d} \rho(z)^q \nu[(RB) \cap (z + RB)] \nu(dz)
\]
\[
\xrightarrow{R \rightarrow +\infty} \nu(B) \sum_{q \geq m} c_q^2 q! \int_{\mathbb{R}^d} \rho(z)^q \nu(dz).
\]
This gives us the limiting variance. To show the central convergence, it is routine to verify the contraction conditions, which can be done in the same way as before. We omit the details here and point out that we need to use the following limiting result instead of (3.3): for each \(r \in \{1, \ldots, q - 1\},\)
\[
\frac{1}{R^{\alpha(1 - rq^{-1})}} \int_{RB} |\rho(x)|^r \nu(dx) \xrightarrow{R \rightarrow +\infty} 0.
\]
The above limit can be verified in the same way, by using Hölder’s inequality and the fact that \(\rho \in L^m(\mathbb{R}, d\nu)\). In this way, we can obtain the f.d.d. convergence described in (1.10), and we leave this as an easy exercise for the interested readers.

In the following, we sketch the arguments for tightness. For every \(z \in \mathbb{R}^d_+\) and \(t \geq 0\), we put
\[
\hat{X}_R(z) = R^{\alpha/2} \int_{B_t} \Phi(W_{xR}) \nu(dx)
\]
and
\[
\hat{Y}_R(z) = R^{\alpha/2} \int_{[0, z]} \Phi(W_{xR}) \nu(dx).
\]
In the sequel, we show the tightness for both \( \{\hat{X}_R, R > 0\} \) and \( \{\hat{Y}_R, R > 0\} \).

For fixed \( 0 < s < t \leq 1 \), we can obtain, by similar arguments as before, that

\[
\|\hat{X}_R(t) - \hat{X}_R(s)\|_{L^p(\Omega)} = R^{\alpha/2} \left\| \delta^m \left( \int_{B_1 \setminus B_s} \Phi_m(W_{xR})e^{\omega_m(xR)} \nu(dx) \right) \right\|_{L^p(\Omega)}
\]

\[
\leq C \left( R^{\alpha} \int_{(B_1 \setminus B_s)^2} |\rho(xR - yR)|^m \nu(dx) \nu(dy) \right)^{1/2}
\]

\[
\leq C \left( R^{-\alpha} \nu(B_{1R} \setminus B_{sR}) \right)^{1/2} \leq C \left( t^\alpha - s^\alpha \right)^{1/2};
\]

see proof of \([1.9]\). Note that for any \( a, b \in \mathbb{R}_+ \) and any \( \beta \in (0,1) \), it holds that \( (a + b)\beta \leq a^\beta + b^\beta \); for any \( a, b \in [0,1] \) and \( \beta \in (1, +\infty) \), there exists a constant \( C_\beta \) that only depends on \( \beta \) such that \( |a^\beta - b^\beta| \leq C_\beta |a - b| \). This gives us

\[
\|\hat{X}_R(t) - \hat{X}_R(s)\|_{L^p(\Omega)} \leq C(t - s)^{(\alpha \wedge 1)/2}.
\]

Since \( \alpha p > 2 \), we can deduce the tightness of \( \{\hat{X}_R, R > 0\} \).

Similarly for \( \{\hat{Y}_R, R > 0\} \), we can write

\[
\|\hat{Y}_R(z) - \hat{Y}_R(y)\|_{L^p(\Omega)} \leq C \sqrt{\nu([0,z] \setminus [0,y] + \nu([0,y] \setminus [0,z])}
\]

\[
\leq C \max_{j=1,\ldots,d} |y_j - z_j|^{\alpha/2} \leq C\|y - z\|^{\alpha/2};
\]

see the proof of \([1.3]\). Since \( \alpha p > 2d \), we obtain the tightness of \( \{\hat{Y}_R, R > 0\} \). \( \square \)

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