DOUBLE RAMIFICATION CYCLES ON THE MODULI SPACES OF
ADMISSIBLE COVERS

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Abstract. We derive a formula for the virtual class of the moduli space of rubber maps
to \([\mathbb{P}^1/G]\) pushed forward to the moduli space of stable maps to \(BG\). As an application,
we show that the Gromov-Witten theory of \([\mathbb{P}^1/G]\) relative to 0 and \(\infty\) are determined by
known calculations.

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1. Introduction

1.1. \(\mathbb{P}^1\)-stacks. This paper is motivated by the study of Gromov-Witten theory of \(\mathbb{P}^1\)-stacks of the following form:

\[ [\mathbb{P}^1/G]. \]

Here, \(G\) is a finite group. The \(G\)-action on \(\mathbb{P}^1\) is given by the one-dimensional representation \(L = \mathbb{C}\),

\[ \varphi: G \rightarrow \mu_\alpha = \text{Im} \varphi \subset \mathbb{C}^* \cong GL(L), \]

together with the trivial one-dimensional representation \(\mathbb{C}\) via

\[ \mathbb{P}^1 = \mathbb{P}(L \oplus \mathbb{C}). \]

The \(\mathbb{C}^*\)-action on \(\mathbb{P}^1\) given by

\[ \lambda \cdot [z_0, z_1] := [z_0, \lambda z_1], \quad \lambda \in \mathbb{C}^*, [z_0, z_1] \in \mathbb{P}^1 \]

commutes with this \(G\)-action and induces a \(\mathbb{C}^*\)-action on \( [\mathbb{P}^1/G] \).

1.2. Stacky rubbers. The relative Gromov-Witten theory of the pairs

\[ (\mathbb{P}^1/G, [0/G]), (\mathbb{P}^1/G, [0/G] \cup [\infty/G]) \]

arise naturally in the pursuit of Leray-Hirsch type results in orbifold Gromov-Witten theory, see [11]. Indeed,

\[ [\mathbb{P}^1/G] = [\mathbb{P}(L \oplus \mathbb{C})/G] \rightarrow BG \]

can be viewed as the stacky \(\mathbb{P}^1\)-bundle associated to the line bundle

\[ L \rightarrow BG. \]

The relative Gromov-Witten theory of the pairs (1) may be studied using virtual localization with respect to the \(\mathbb{C}^*\)-action on \( [\mathbb{P}^1/G] \). Rubber invariants naturally arise in this approach. Let

\[ \overline{M}_{g,l}(\mathbb{P}^1/G, \mu_0, \mu_\infty)^- \]

be the moduli space of rubber maps, see Section 2 for precise definitions. Post-composition with \( [\mathbb{P}^1/G] \rightarrow BG \) defines a map

\[ \epsilon: \overline{M}_{g,l}(\mathbb{P}^1/G, \mu_0, \mu_\infty)^- \rightarrow \overline{M}_{g,l}(\mu_0) + l(\mu_\infty) + \#I(BG). \]

The cycle

\[ DR_g^G(\mu_0, \mu_\infty, I) := \epsilon_* \left[ \overline{M}_{g,l}(\mathbb{P}^1/G, \mu_0, \mu_\infty)^- \right]^{\text{vir}} \in A^g(\overline{M}_{g,l}(\mu_0) + l(\mu_\infty) + \#I(BG)) \]

is termed stacky double-ramification cycle. The main result of this paper is a formula for

\[ DR_g^G(\mu_0, \mu_\infty, I). \]

The formula, which involves complicated notations, is given in Theorem 3.9 below.

When \(G = \{1\}\), our formula reduces to Pixton’s formula for double ramification cycles, proven in [2]. Our proof, given in the bulk of this paper, closely follows that of [2].

The main application of the formula for \( DR_g^G(\mu_0, \mu_\infty, I) \) is the following

**Theorem 1.1.** The relative Gromov-Witten theory of

\[ ([\mathbb{P}^1/G], [0/G]) \text{ and } ([\mathbb{P}^1/G], [0/G] \cup [\infty/G]) \]

are completely determined.
Proof. Since evaluation maps on $\overline{M}_{g,t}(\mathbb{P}^1/G, \mu_0, \mu_\infty)$ factor through $\epsilon$, rubber invariants are all determined by the formula for $DR^G_{g}(\mu_0, \mu_\infty, I)$, together with the Gromov-Witten theory of $BG$ solved by [3].

Virtual localization reduces the calculation of relative Gromov-Witten invariants to calculating rubber invariants with target descendants. By rubber calculus in the fiber class case [5], rubber invariants with target descendants are determined by those without target descendants. The proof is complete. □

Theorem 1.1 is an evidence supporting [11, Conjecture 2.2], and we expect that Theorem 1.1 plays an important role in the general case.

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2. Stacky double ramification cycle

Let $G$ be a finite group and $L = \mathbb{C}$ a one dimensional $G$-representation given by the map $G \xrightarrow{\varphi} \mu_a = \text{Im} \varphi \subset GL(L) = \mathbb{C}^*$. Let $K := \ker \varphi$, we obtain the exact sequence

$$1 \to K \to G \xrightarrow{\varphi} \mu_a \to 1$$

Definition 2.1. For a conjugacy class $c \subset G$, define

$$r(c) \in \mathbb{N}$$

to be the order of any element of $c$. Define

$$a_c(L) \in \{0, \ldots, r(c) - 1\}$$

to be the unique integer such that each element of $c$ acts on $L$ by multiplication by $\exp \left( \frac{2\pi \sqrt{-1} a_c(L)}{r(c)} \right)$.

In other words, the representation $\varphi : G \to GL(L) = \mathbb{C}^*$ maps $c$ to $\exp \left( \frac{2\pi \sqrt{-1} a_c(L)}{r(c)} \right)$.

Consider the quotient stack $[\mathbb{P}^1/G]$, where the $G$-action on $\mathbb{P}^1$ is given by the 1-dimensional representation $\varphi$ together with the trivial one-dimensional representation $\mathbb{C}$ via $\mathbb{P}^1 = \mathbb{P}(L \oplus \mathbb{C})$.

Definition 2.2. Let $A$ denote the following data:

$$\mu_0 = \{(c_{0i}, f_{0i}, c_{0i})\}_i, \quad \mu_\infty = \{(c_{\infty i}, f_{\infty i}, c_{\infty i})\}_i, \quad I = \{c_1, \ldots, c_k\},$$

where $c_{0i}, c_{\infty i} \in \mathbb{Z}_{\geq 0}, f_{0i}, f_{\infty i} \in \mathbb{N}, c_{0i}, c_{\infty i}, c_1, \ldots, c_k \in \text{Conj}(G)$ such that

(i) $f_{0i}$ (resp. $f_{\infty i}$) is the order of any element in $c_{0i}$ (resp. $c_{\infty i}$).
(ii) $\sum_{i} c_{0i} f_{0i} = \sum_{j} c_{\infty j} f_{\infty j}.$
(iii) $\text{age}_{c_{0i}}(L) = \left( \frac{c_{0i}}{f_{0i}} \right), \quad \text{age}_{c_{\infty j}}(L) = \left( \frac{c_{\infty j}}{f_{\infty j}} \right).$

1Following [5], we treat disconnected invariants as products of connected ones.

2$a_c(L)$ is well-defined because $L$ is 1-dimensional.
(iv) \(\text{age} \xi_i(L) = 0, 1 \leq i \leq k\). So \(\xi_i \in \text{Conj}(K)\).

(v) Monodromy condition\(^3\) in genus \(g\) holds for \(\{\xi_0\} \cup \{\xi_{-1}\} \cup \{\xi_1, \ldots, \xi_k\}\).

Here the monodromy condition in genus \(g\) means the following.

**Definition 2.3** (Monodromy condition). Let \(H\) be a finite group. We say that the collection of conjugacy classes \(\xi_1, \ldots, \xi_n\) of \(H\) satisfy monodromy condition in genus \(g\) if there exist

\[
h_i \in \xi_i, 1 \leq i \leq n, \quad a_j, b_j \in H, 1 \leq j \leq g,
\]

such that

\[
\prod_{i=1}^{n} h_i = \prod_{j=1}^{g} [a_j, b_j].
\]

**Remark 2.4.** The data \(\mu_0, \mu_\infty\) are referred to as stacky partitions. The length of \(\mu_0\), denoted by \(l(\mu_0)\), is the number of triples in the partition \(\mu_0\).

The moduli space \(\overline{M}_{g,I}([\mathbb{P}^1/G], \mu_0, \mu_\infty)^-\) parametrizes stable relative maps of connected twisted curves of genus \(g\) to rubber with ramification profiles \(\mu_0, \mu_\infty\) over \([0/G]\) and \([\infty/G]\) respectively, and additional marked points whose stack structures are described by \(I\). As noted in [11, Appendix A. 2], rubber theory in the stack setting may be defined in the same way as e.g. [5, Section 1.5].

Set \(n = l(\mu_0) + l(\mu_\infty) + \#I\). A Riemann-Roch calculation\(^4\) shows that the virtual dimension of \(\overline{M}_{g,I}([\mathbb{P}^1/G], \mu_0, \mu_\infty)^-\) is

\[
\text{vdim} \overline{M}_{g,I}([\mathbb{P}^1/G], \mu_0, \mu_\infty)^- = 2g - 3 + n.
\]

The moduli space \(\overline{M}_{g,n}(BG)\) of \(n\)-pointed genus \(g\) stable maps to \(BG\) is smooth of dimension \(3g - 3 + n\). There is a morphism

\[
\epsilon : \overline{M}_{g,I}([\mathbb{P}^1/G], \mu_0, \mu_\infty)^- \rightarrow \overline{M}_{g,n}(BG)
\]

defined by post-composition with \([\mathbb{P}^1/G] \rightarrow BG\).

**Definition 2.5.** The stacky double ramification cycle is defined to be the push-forward

\[
DR^G_g(A) = \epsilon_*[\overline{M}_{g,I}([\mathbb{P}^1/G], \mu_0, \mu_\infty)^-]^{\text{vir}} \in A^g(\overline{M}_{g,n}(BG))
\]

**Remark 2.6.** The cycle \(DR^G_g(A)\) is supported on the component of \(\overline{M}_{g,n}(BG)\) parametrizing stable maps with orbifold structures at marked points given by \(\{\xi_0\} \cup \{\xi_{-1}\} \cup I\).

\(^3\)For a conjugacy class \((g)\), \((g)^{-1}\) stands for the conjugacy class \((g^{-1})\).

\(^4\)Note that the relative tangent bundle \(T_{[\mathbb{P}^1/G]}([-0/G] - [\infty/G])\) entering the Riemann-Roch formula is in fact trivial.
3. Total Chern class

3.1. **General case.** Let $H$ be a finite group and $V = \mathbb{C}$ a one-dimensional $H$-representation. The representation $H \to GL(V) = \mathbb{C}^*$ maps a conjugacy class $c$ to $exp \left( 2\pi \sqrt{-1} \frac{a_c(V)}{r(c)} \right)$, where $a_c(V) \in \{0, \ldots, r(c) - 1\}$.

We write $\text{Conj}(H)$ for the set of conjugacy classes of $H$. The inertia stack $\text{IB}H$ is decomposed as

$$\text{IB}H = \bigsqcup_{c = (h) \in \text{Conj}(H)} C_H(h)$$

where $C_H(h) \leq H$ is the centralizer of $h \in H$.

We write $\overline{M}_{g,n}(BH)$ for the moduli stack of stable $n$-pointed genus $g$ maps to $BH$. For $1 \leq i \leq n$, there is the $i$-th evaluation map

$$\text{ev}_i : \overline{M}_{g,n}(BH) \to \text{IB}H.$$ 

Pick $c_1, \ldots, c_n \in \text{Conj}(H)$, let

$$\overline{M}_{g,n}(BH; c_1, \ldots, c_n) := \prod_{i=1}^n \text{ev}_i^{-1}(BC_H(h_i)),$$

where $c_i = (h_i)$. Denote the universal family as follows:

$$\begin{array}{ccc}
C & \xrightarrow{f} & BH \\
\downarrow \pi & & \downarrow \pi \\
\overline{M}_{g,n}(BH; c_1, \ldots, c_n) & & \\
\end{array}$$

Consider the virtual bundle

$$V_{g,n} := \mathbb{R} \pi_* f^* V,$$

where $V$ is viewed as a line bundle on $BH$. The Chern character $ch(V_{g,n})$ was calculated in much greater generality in [10], by using Toen’s Grothendieck-Riemann-Roch formula for stacks [9]. Applied to the present situation, we find

\begin{equation}
ch(V_{g,n}) = \pi_* (ch(f^* V) Td' (\widetilde{L}_{n+1}))
= \sum_{i=1}^n \sum_{m \geq 1} \frac{ev_i^* A_m}{m!} \bar{\psi}_i^{m-1}
+ \frac{1}{2} (\pi \circ \iota)_* \sum_{m \geq 2} \frac{1}{m!} r_{\text{node}}^2 (ev_{\text{node}}^* A_m) \frac{\bar{\psi}_+^{m-1} + (-1)^m \bar{\psi}_-^{m-1}}{\psi_+ + \psi_-}
\end{equation}

The formula is explained and further processed as follows.

- $r_{\text{node}}$ is the order of the orbifold structure at the node.
- $ev_{\text{node}}$ is the evaluation map at the node defined in [10, Appendix B].
- $\bar{\psi}_+$ and $\bar{\psi}_-$ are the $\bar{\psi}$-classes associated to the branches of the node.
- Since $\dim BH = 0$, we have

$$ch(f^* V) = ch_0(f^* V) = \text{rank} V = 1.$$
• By definition, the Todd class is
\[ Td'(\bar{L}_{n+1}) = \frac{\bar{\psi}_{n+1}^r}{e^{\bar{\psi}_{n+1}} - 1} = \sum_{r \geq 0} \frac{B_r}{r!} \bar{\psi}_{n+1}^r, \]
where \( B_r \)'s are the Bernoulli numbers. Therefore,
\[ \pi_* (ch(f^*V)Td'(\bar{L}_{n+1})) = \sum_{r \geq 0} \frac{B_r}{r!} \pi_*(\bar{\psi}_{n+1}^r) \]

• \( A_m \) is defined in \([10]\) Definition 4.1.2. We have \( A_m \in H^*(IBH) \). For \( \mathfrak{c} = (h) \in \text{Conj}(H) \), the component of \( A_m \) in \( H^0(BC_H(h)) \subset H^*(IBH) \) is \( B_m \left( \frac{a_\mathfrak{c}(V)}{r(\mathfrak{c})} \right) \). Here \( B_m(x) \) are Bernoulli polynomials, defined by
\[ \frac{te^tx}{e^t - 1} = \sum_{m \geq 0} \frac{B_m(x)}{m!} t^m. \]

• The map \( i : \mathcal{Z}_{\text{node}} \to \mathcal{C} \) is the inclusion of the locus of the nodes. The last term of the right hand side of \([2]\) may be rewritten using the map
\[ B_{\text{node}} \overset{i}{\to} \overline{M}_{g,n}(BH; \mathfrak{c}_1, \ldots, \mathfrak{c}_n), \]
whose image is the locus of nodal curves. The map \( i \) exhibits \( B_{\text{node}} \) as the universal gerbe at the node, and hence degree of \( i \) is \( \frac{1}{r_{\text{node}}} \).

Given the above, we can write \( ch_m(V_{g,n}) \), the degree-\( 2m \) component of \( ch(V_{g,n}) \), as
\[ ch_m(V_{g,n}) = \frac{B_{m+1}}{(m+1)!} \pi_*(\bar{\psi}_{n+1}^{m+1}) \]
\[ + \sum_{i=1}^{n} \frac{1}{(m+1)!} B_{m+1} \left( \frac{a_\mathfrak{c}_i(V)}{r(\mathfrak{c}_i)} \right) \bar{\psi}_i^m \]
\[ + \frac{1}{2} \sum_{i \in \text{Conj}(H)} \frac{r(\mathfrak{c})}{(m+1)!} B_{m+1} \left( \frac{a_\mathfrak{c}(V)}{r(\mathfrak{c})} \right) \zeta_\mathfrak{c} \left( \frac{\bar{\psi}_+^m - (-\bar{\psi}_-)^m}{\bar{\psi}_+ + \bar{\psi}_-} \right) \]
where \( \zeta_\mathfrak{c} : B_{\text{node}, \mathfrak{c}} \to \overline{M}_{g,n}(BH; \mathfrak{c}_1, \ldots, \mathfrak{c}_n) \) is the universal gerbe at the node whose orbifold structure is given by \( \mathfrak{c} \).

Using the formula
\[ c(-E^\bullet) = \exp \left( \sum_{m \geq 1} (-1)^m (m-1)! ch_m(E^\bullet) \right), \quad E^\bullet \in D^b, \]
we can derive a formula for \( c(-V_{g,n}) \). To write this down we need more notations.

As in \([2]\), the strata of \( \overline{M}_{g,n} \) are indexed by stable graphs. The strata of \( \overline{M}_{g,n}(BH; \mathfrak{c}_1, \ldots, \mathfrak{c}_n) \) are indexed by stable graphs together with choices of conjugacy classes of \( H \) describing orbifold structures.

Let \( G_{g,n} \) be the set of stable graphs of genus \( g \) with \( n \) legs. Following \([2]\), a stable graph is denoted by
\[ \Gamma = (V, H, L, g : V \to \mathbb{Z}_{\geq 0}, v : H \to V, \iota : H \to H) \in G_{g,n}. \]
Properties in \([2]\) Section 0.3.2] are required for \( \Gamma \).
Remark 3.1. The set of legs $L(\Gamma)$ corresponds to the set of markings. The set of half edges $H(\Gamma)$ corresponds to the union of the set of a side of an edge and the set of legs. Each half edge is labelled with a vertex $v \in V(\Gamma)$. Each vertex $v \in V(\Gamma)$ is labelled with a nonnegative integer $g(v)$, called the genus.

Definition 3.2. We define $\chi_{\Gamma,H}$ to be the set of maps

$$\chi : H(\Gamma) \to \text{Conj}(H)$$

such that,

- $\chi$ maps the $i$-th leg $h_i$ to $c_i$, $1 \leq i \leq n$;
- for a vertex $v \in V(\Gamma)$, there exists $(\alpha_j), (\beta_j) \in \text{Conj}(H)$, for $1 \leq j \leq g(v)$, and $k_h \in \chi(h)$, for $h \in v$, such that
  $$\prod_{h \in v} k_h = \prod_{j=1}^{g(v)} [\alpha_j, \beta_j];$$
- for an edge $e = (h, h') \in E(\Gamma)$, there exists $k \in \chi(h)$, $k' \in \chi(h')$, such that
  $$kk' = \text{Id} \in H.$$ 

For each $\Gamma \in G_{g,n}$ and $\chi \in \chi_{\Gamma,H}$, there is a component $\overline{M}_{\Gamma,\chi} \subset B_{\text{node}}$ parametrizing maps with nodal domains of topological types given by $\Gamma$ and orbifold structures given by $\chi$. Let

$$\zeta_{\Gamma,\chi} : \overline{M}_{\Gamma,\chi} \longrightarrow \overline{M}_{g,n}(BH; c_1, \ldots, c_n)$$

be the restriction of $\chi$ to this component. Then $c(-V_{g,n})$ is

$$(3)$$

$$\sum_{\Gamma \in G_{g,n}} \sum_{\chi \in \chi_{\Gamma,H}} \frac{1}{|\text{Aut}(\Gamma)|} \zeta_{\Gamma,\chi}^* \left[ \prod_{v \in V(\Gamma)} \exp \left( - \sum_{m \geq 1} (-1)^{m-1} \frac{B_{m+1}}{m(m+1)} \kappa_m(v) \right) \times \prod_{i=1}^n \exp \left( \sum_{m \geq 1} (-1)^{m-1} \frac{1}{m(m+1)} B_{m+1} \left( \frac{a_{g,c}(V)}{r(c_i)} \right) \frac{\psi_m^m}{\psi_{h_i}} \right) \times \prod_{e \in E(\Gamma)} \frac{1}{\psi_{h_+} + \psi_{h_-}} \left( 1 - \exp \left( \sum_{m \geq 1} \frac{(-1)^{m-1}}{m(m+1)} B_{m+1} \left( \frac{a_{g,h_+}(V)}{r(\chi(h_+))} \right) \left( \frac{\psi_m^m}{\psi_{h_+}^m} + (-\psi_{h_-}^m) \right) \right) \right) \right].$$

Remark 3.3.

(i) For a half-edge $h$, $\bar{\psi}_h$ denotes the descendant at the marked point/node corresponding to $h$.
(ii) For a vertex $v$, let $\overline{M}_v(BH)$ be the moduli space of stable maps to $BH$ described by $v$ and let $\pi_v : C_v \to \overline{M}_v(BH)$ be the universal curve. Write $\bar{\psi}_v \in A^1(C_v)$ for the descendant corresponding to the additional marked point. Then define $\kappa_m(v) := \pi_v^* (\bar{\psi}_v^{m+1}).$
3.2. Cyclic extensions. Let \( r \in \mathbb{Z}_{>0} \), the \( r \)-th power map \( \mathbb{C}^* \rightarrow \mathbb{C}^*, \ z \mapsto z^r \) gives the map \( \mu_{ar} \rightarrow \mu_a \).

The kernel of the map is \( \mu_r \). Hence this gives the exact sequence

\[
1 \rightarrow \mu_r \xrightarrow{g} \mu_{ar} \xrightarrow{f} \mu_a \rightarrow 1,
\]

where

\[
g \left( \exp \left( \frac{2\pi \sqrt{-1} l}{r} \right) \right) = \exp \left( \frac{2\pi \sqrt{-1} l a}{r a} \right), \ 0 \leq l \leq r - 1,
\]

and

\[
f \left( \exp \left( \frac{2\pi \sqrt{-1} k}{ar} \right) \right) = \exp \left( \frac{2\pi \sqrt{-1} k a}{a} \right), \ 0 \leq k \leq r - 1.
\]

There is a unique finite group \( G(r) \) which fits into the following diagram with exact rows and columns:

\[
\begin{array}{c}
1 \\
\downarrow \\
\mu_r \\
\downarrow \\
1 \rightarrow K \xrightarrow{\alpha} G(r) \xrightarrow{\beta} \mu_{ar} \rightarrow 1 \\
\downarrow \\
1 \rightarrow K \xrightarrow{\phi} G \xrightarrow{\varphi} \mu_a \rightarrow 1 \\
\downarrow \\
1 \\
\end{array}
\]

Geometrically, the map \( \mu_{ar} \rightarrow \mu_a \) gives a \( \mu_r \)-gerbe over \( B\mu_a \),

\[
B\mu_{ar} \rightarrow B\mu_a.
\]

The map \( \varphi : G \rightarrow \mu_a \) gives a map

\[
BG \rightarrow B\mu_a.
\]

Pulling back the \( \mu_r \)-gerbe to \( BG \) using this map, we obtain the gerbe

\[
BG(r) \rightarrow BG.
\]

Moreover, when viewing the representation \( L \) as a line bundle on \( BG \), \( BG(r) \) is the gerbe of \( r \)-th roots of \( L \rightarrow BG \). The homomorphism

\[
G(r) \rightarrow \mu_{ar} \subset \mathbb{C}^*
\]

is a one-dimensional representation of \( G(r) \) which corresponds to the universal \( r \)-th root of \( L \) on \( BG(r) \). We denote this \( r \)-th root by

\[
L^{1/r} \rightarrow BG(r).
\]
Let $c \in \text{Conj}(G)$. Then $\varphi(c) \in \mu_a$ is a single number. The inverse image of $\varphi(c)$ under the $r$-th power map $\mu_a \rightarrow \mu_a$ has size $r$. The inverse image $\beta^{-1}(c) \subset G(r)$ can be partitioned into conjugacy classes of $G(r)$. Moreover, $\alpha$ maps these conjugacy classes to the set of inverse images of $\varphi(c)$, which has size $r$. So there are at least $r$ conjugacy classes in $\beta^{-1}(c)$. By the counting result [7, Example 3.4], there are at most $r$ conjugacy classes. Therefore, there are exactly $r$ conjugacy classes of $G(r)$ that map to $c$ and they are determined by their images under $G(r) \overset{\beta}{\rightarrow} \mu_a$.

A canonical splitting of

$$1 \rightarrow \mu_r \rightarrow \mu_{ar} \rightarrow \mu_a \rightarrow 1$$

is given by

$$\mu_a \rightarrow \mu_{ar}, \quad g \mapsto \exp \left( \frac{2\pi \sqrt{-1} \text{age}_g(L)}{r} \right).$$

(6)

Using this, for $g \in \mu_a$, we may identify the inverse image of $g$ under $\mu_{ar} \rightarrow \mu_a$ as

$$\left\{ \exp \left( \frac{2\pi \sqrt{-1} \text{age}_g(L) + e}{r} \right) \mid 0 \leq e \leq r - 1 \right\}$$

and hence with

$$\mu_r = \left\{ \exp \left( \frac{2\pi \sqrt{-1} e}{r} \right) \mid 0 \leq e \leq r - 1 \right\}.$$

In summary, given $c \in \text{Conj}(G)$, to specify the lifting $\tilde{c} \in \text{Conj}(G(r))$ such that $\beta(\tilde{c}) = c$ is equivalent to specifying $e \in \{0, \ldots, r - 1\}$.

Moreover, given $c_1, \ldots, c_n \in \text{Conj}(G)$ satisfying monodromy condition in genus $g$, selecting $\tilde{c}_1, \ldots, \tilde{c}_n \in \text{Conj}(G(r))$ with $\beta(\tilde{c}_i) = c_i$ satisfying monodromy condition in genus $g$ is equivalent to selecting $e_1, \ldots, e_n \in \{0, \ldots, r - 1\}$ such that

$$\sum_{i=1}^{n} e_i \equiv - \sum_{i=1}^{n} \text{age}_{c_i}(L) \mod r$$

(7)

This can be deduced from the lifting analysis in [8, Section 5]. We can also argue more directly as follows. Since $c_1, \ldots, c_n$ satisfy monodromy condition in genus $g$, there exists a stable map $f : \mathcal{C} \rightarrow BG$ with $\mathcal{C}$ smooth of genus $g$ and $\mathcal{C}$ has orbifold points described by $c_1, \ldots, c_n$. Calculating $\chi(\mathcal{C}, f^*L)$ by Riemann-Roch, we see that $\sum_{i=1}^{n} \text{age}_{c_i}(L) \in \mathbb{Z}$. Similarly, having the required $\tilde{c}_1, \ldots, \tilde{c}_n$ implies the existence of a stable map $\tilde{f} : \tilde{\mathcal{C}} \rightarrow BG(r)$ with $\tilde{\mathcal{C}}$ smooth of genus $g$ and $\tilde{\mathcal{C}}$ has orbifold points described by $\tilde{c}_1, \ldots, \tilde{c}_n$. Calculating $\chi(\tilde{\mathcal{C}}, \tilde{f}^*L^{1/r})$ by Riemann-Roch, we see that $\sum_{i=1}^{n} \text{age}_{\tilde{c}_i}(L^{1/r}) \in \mathbb{Z}$. Equation (7) follows because by construction $\text{age}_{\tilde{c}_i}(L^{1/r}) = (\text{age}_{\tilde{c}_i}(L) + e_i)/r$. This shows that equation (7) is necessary. That (7) is also sufficient can be seen by a direct calculation using the description of $G(r)$ as a set $G \times \mu_r$ endowed with the multiplication defined using the splitting (6), as in [6, Section 3]. We omit the details.

The above discussion allows us to split a sum over $\chi_{\Gamma,G(r)}$ as a double sum over $\chi_{\Gamma,G}$ and the set $W_{\Gamma,\chi,r}$ defined as follows.

**Definition 3.4.** A weighting mod $r$ associated to a stable graph $\Gamma$ and a map $\chi \in \chi_{\Gamma,G}$ is a function

$$w : \text{H}(\Gamma) \rightarrow \{0, \ldots, r - 1\}$$
such that

(i) For legs $h_1, \ldots, h_n$, $w(h_i) \equiv 0 \mod r$.
(ii) For $e = (h_+, h_-) \in E(\Gamma)$, if $\text{age}_{\chi(h_i)}(L) = 0$, then $w(h_+) + w(h_-) \equiv 0 \mod r$. If $\text{age}_{\chi(h_i)}(L) \neq 0$, then $w(h_+) + w(h_-) \equiv -1 \mod r$.
(iii) For $v \in V(\Gamma)$, $\sum_{h \in v} w(h) \equiv A(v, \chi) \mod r$, where $A(v, \chi) := -\sum_{h \in v} \text{age}_{\chi(h)}(L)$.

We write $W_{\Gamma, \chi, r}$ for the set of weightings mod $r$ associated to $\Gamma$ and $\chi$.

**Remark 3.5.**

(i) For $e = (h_+, h_-) \in E(\Gamma)$, the conditions on $w(h_i)$ ensure that

$$\left(\text{age}_{\chi(h_i)}(L) + w(h_-)\right)/r = 1 - \left(\text{age}_{\chi(h_i)}(L) + w(h_+)\right)/r.$$

(ii) For $v \in V(\Gamma)$, We have $A(v, \chi) \in \mathbb{Z}$ by applying Riemann-Roch to $\chi(f^*L)$, where $f : C \to BG$ is a stable map with $C$ smooth of genus $g(v)$ and orbifold marked points described by $\{\chi(h) | h \in v\}$.

### 3.3. Total Chern class on moduli spaces of stable maps to $BG(r)$.

We begin with the following notations.

**Definition 3.6 (Liftings).** For $\{c_{0i}\}_i, \{c_{\infty j}\}_j \subset \text{Conj}(G)$, we select liftings $\{\tilde{c}_{0i}\}_i, \{\tilde{c}_{\infty j}\}_j \subset \text{Conj}(G(r))$ by

$$\alpha(\tilde{c}_{0i}) = \exp\left(\frac{2\pi i \text{age}_{\chi_{0i}}(L)}{r}\right) \in \mu_{ar},$$

$$\alpha(\tilde{c}_{\infty j}) = \exp\left(\frac{2\pi i \text{age}_{\chi_{\infty j}}(L)}{r}\right) \in \mu_{ar}.$$ 

The lifts of $c_1, \ldots, c_k \in \text{Conj}(K) \subset \text{Conj}(G)$ are chosen to be themselves, viewed via $\text{Conj}(K) \subset \text{Conj}(G(r))$.

Let $\overline{M}_{g, \tilde{\mu}_0 + \tilde{\mu}_\infty + I}(BG(r))$ be the moduli space of stable maps to $BG(r)$ of genus $g$ whose marked points have orbifold structures given by $\{\tilde{c}_{0i}\} \cup \{\tilde{c}_{\infty j}^{-1}\} \cup \{c_1, \ldots, c_k\}$.

Let $\overline{M}_{g, \mu_0 + \mu_\infty + I}(BG)$ be similarly defined. There is a natural map

$$\epsilon : \overline{M}_{g, \tilde{\mu}_0 + \tilde{\mu}_\infty + I}(BG(r)) \to \overline{M}_{g, \mu_0 + \mu_\infty + I}(BG).$$

Strata of $\overline{M}_{g, \tilde{\mu}_0 + \tilde{\mu}_\infty + I}(BG)$ are indexed by pairs $\Gamma \in G_{g,n}$ and $\chi \in \chi_{\Gamma, G}$. Let $\zeta_{\Gamma, \chi}$ be the map from this stratum to $\overline{M}_{g, \mu_0 + \mu_\infty + I}(BG)$. Strata of $\overline{M}_{g, \tilde{\mu}_0 + \tilde{\mu}_\infty + I}(BG(r))$ are indexed by $\Gamma, \chi$, and $w \in W_{\Gamma, \chi, r}$. Let $\zeta_{\Gamma, \chi, w}$ be the natural map from the stratum to $\overline{M}_{g, \tilde{\mu}_0 + \tilde{\mu}_\infty + I}(BG(r))$. 

Applying the results of Section 5.1 we obtain the following formula for \( c(-L_{g,n}^{1/r}) \) on \( \overline{M}_{g,\bar{m}_0+\bar{m}_\infty+1}(BG(r)) \):

\[
\sum_{\Gamma \in G_{g,n}} \sum_{\chi \in \chi \Gamma} \sum_{w \in W_{\Gamma,X,r}} \frac{1}{|\text{Aut}(\Gamma)|} \left[ \prod_{v \in V(\Gamma)} \exp \left( - \sum_{m \geq 1} (-1)^{m-1} \frac{B_{m+1}}{m(m+1)} \kappa_m(v) \right) \right] \times \\
\times \prod_i \exp \left( \sum_{m \geq 1} (-1)^{m-1} \frac{1}{m(m+1)} B_{m+1} \left( \frac{\text{age}_{\chi_{\text{reg}}}(L)}{r} \psi_i^m \right) \right) \times \\
\times \prod_j \exp \left( \sum_{m \geq 1} (-1)^{m-1} \frac{1}{m(m+1)} B_{m+1} \left( 1 - \frac{\text{age}_{\text{reg}}(L)}{r} \psi_j^m \right) \right) \times \\
\times \prod_{l=1}^k \exp \left( \sum_{m \geq 1} (-1)^{m-1} \frac{1}{m(m+1)} B_{m+1} \psi_l^m \right) \\
\times \prod_{e \in E(\Gamma)} \frac{r(\chi(h_+))r}{\psi_{h_+} + \psi_{h_-}} \left( 1 - \exp \left( \sum_{m \geq 1} \frac{(-1)^{m-1}}{m(m+1)} B_{m+1} \left( \frac{\text{age}_{\chi_{\text{reg}}}(L)}{r} + \frac{w(h_+)}{r} \right) (\psi_{h_+}^m - (-\psi_{h_-})^m) \right) \right).
\]

By the calculation of Section 5], the degree of \( c \) on strata indexed by \( \Gamma \) is \( r \sum_{v \in V(\Gamma)} (2g(r) - 1) \). This yields the following formula for \( \epsilon_* c(-L_{g,n}^{1/r}) \):

\[
\sum_{\Gamma \in G_{g,n}} \sum_{\chi \in \chi \Gamma} \sum_{w \in W_{\Gamma,X,r}} \frac{2g-1-h^2(\Gamma)}{|\text{Aut}(\Gamma)|} \left[ \prod_{v \in V(\Gamma)} \exp \left( - \sum_{m \geq 1} (-1)^{m-1} \frac{B_{m+1}}{m(m+1)} \kappa_m(v) \right) \right] \times \\
\times \prod_i \exp \left( \sum_{m \geq 1} (-1)^{m-1} \frac{1}{m(m+1)} B_{m+1} \left( \frac{\text{age}_{\chi_{\text{reg}}}(L)}{r} \psi_i^m \right) \right) \times \\
\times \prod_j \exp \left( \sum_{m \geq 1} (-1)^{m-1} \frac{1}{m(m+1)} B_{m+1} \left( 1 - \frac{\text{age}_{\text{reg}}(L)}{r} \psi_j^m \right) \right) \times \\
\times \prod_{l=1}^k \exp \left( \sum_{m \geq 1} (-1)^{m-1} \frac{1}{m(m+1)} B_{m+1} \psi_l^m \right) \\
\times \prod_{e \in E(\Gamma)} \frac{r(\chi(h_+))r}{\psi_{h_+} + \psi_{h_-}} \left( 1 - \exp \left( \sum_{m \geq 1} \frac{(-1)^{m-1}}{m(m+1)} B_{m+1} \left( \frac{\text{age}_{\chi_{\text{reg}}}(L)}{r} + \frac{w(h_+)}{r} \right) (\psi_{h_+}^m - (-\psi_{h_-})^m) \right) \right).
\]

Note that we have

- \( \frac{\text{age}_{\chi_{\text{reg}}}(L)}{r} + \frac{w(h_+)}{r} = 1 - \frac{\text{age}_{\chi_{\text{reg}}}(L)}{r} - \frac{w(h_+)}{r} \) if \( \text{age}_{\chi_{\text{reg}}}(L) \neq 0 \).
- \( \frac{w(h_+)}{r} = 1 - \frac{w(h_+)}{r} \) if \( \text{age}_{\chi_{\text{reg}}}(L) = 0 \).

Bernoulli polynomials satisfy the following property

\[
B_m(x + y) = \sum_{k=0}^{m} \binom{m}{k} B_k(x) y^{m-k}.
\]

This implies that terms of \( \epsilon_* c(-L_{g,n}^{1/r}) \) depend polynomially on \( \{w(h) \mid h \in H(\Gamma)\} \). The proof of [2 Proposition 3"] may be modified to show that the polynomiality result remains valid for
Definition 3.6. Let $G$ be a finite group. When the finite group $G$ is trivial, the map $r$ is trivial. Therefore we may apply the arguments of [2, Proposition 5] to conclude the following.

**Proposition 3.7.** There exists a polynomial in $r$ which coincides with the cycle class $r^{2d-2g+1} \epsilon_s c_d(-L^{1/r}_{g,n})$ for $r \gg 1$ and prime.

**Remark 3.8.** The orbifold structure at $h_\ast$ has order $r(\chi(h_\ast))r$ when $r \gg 1$ are primes. For our purpose this suffices.

3.4. A formula for stacky double ramification cycle.

**Theorem 3.9.** Given a finite group $G$ and double ramification data $A = \{\mu_0, \mu_\infty, I\}$, the stacky double ramification cycle $DR^G_g(A)$ is the constant term in $r$ of the cycle class

$$a^{1-l(\mu_\infty)} r \cdot \epsilon_s c_g(-L^{1/r}_{g,n}) \in A^g(\overline{M}_{g,n}(BG)),$$

for $r$ sufficiently large. In other words,

$$DR^G_g(A) = a^{1-l(\mu_\infty)} \text{Coeff}_{r}[r \cdot \epsilon_s c_g(-L^{1/r}_{g,n})] \in A^g(\overline{M}_{g,n}(BG))$$

We denote by $P^G_{g,d,r}(A) \in A^d(\overline{M}_{g,n}(BG))$ the degree $d$ component of the class

$$\sum_{g \in G, n} \sum_{\chi \in \text{Ext}_G} \sum_{w \in \text{Aut}_G} \frac{1}{|\text{Aut}(T)|} \frac{1}{r^{h_j(T)}} \zeta_{g,s} \left[ \prod_i \exp(\text{age}_{\chi}(L)^2 \varphi_i) \prod_j \exp(\text{age}_{\chi}(L)^2 \psi_j) \right] \prod_{e \in E(T)} \frac{1 - \exp((\text{age}_{\chi}(L) + w(h_\ast))(\text{age}_{\chi}(L) + w(h_\ast)) (\varphi_{h_\ast} + \psi_{h_\ast}))}{\varphi_{h_\ast} + \psi_{h_\ast}}.$$ 

When the finite group $G$ is trivial, $P^G_{g,d,r}(A)$ reduces to Pixton’s polynomial in [2]. Arguing as in the proof of [2, Proposition 5], we see that $r^{2d-2g+1} \epsilon_s c_d(-L^{1/r}_{g,n})$ and $2^{-d} P^G_{g,d,r}(A)$ have the same constant term. Then the following corollary is a result of Theorem 3.9.

**Corollary 3.10.** The stacky double ramification cycle $DR^G_g(A)$ is the constant term in $r$ of

$$a^{1-l(\mu_\infty)} 2^{-d} P^G_{g,d,r}(A) \in A^g(\overline{M}_{g,n}(BG)),$$

for $r$ sufficiently large.

4. Localization analysis

In this section, we give a proof of Theorem 3.9 by virtual localization on the moduli space of stable relative maps to the target obtained from $[\mathbb{P}^1/G]$ by a root construction.

4.1. Set-up. Let $[\mathbb{P}^1/G]_{r,1}$ be the stack of $r$-th roots of $[\mathbb{P}^1/G]$ along the divisor $[0/G]$. By construction, there is a map $[\mathbb{P}^1/G]_{r,1} \to [\mathbb{P}^1/G]$. Over the divisor $[0/G] \simeq BG$, this map is the $\mu_r$-gerbe $BG(r) \to BG$ studied in Section 3.2.

Let $\mu_0, \mu_\infty, I$ be as in Definition 2.2. Let $\bar{\mu}_0 = \{(c_{0i}, f_{0i}, \bar{c}_{0i})\}_i$, where $\bar{c}_{0i}$ are given in Definition 3.6. Let

$$\overline{M}_{g,l,\mu_0}([\mathbb{P}^1/G]_{r,1}, \mu_\infty).$$
be the moduli space of stable relative maps to the pair \(([\mathbb{P}^1/G],_r,1, [\infty/G])\). The moduli space parametrizes connected, semistable, twisted curves \(C\) of genus \(g\) with non-relative marked points together with a map 

\[ f : C \to P \]

where \(P\) is an expansion of \([\mathbb{P}^1/G],_r\) over \([\infty/G]\) such that

(i) orbifold structures at the non-relative marked points are described by \(\bar{\mu}_0\) and \(\bar{I}\);
(ii) relative conditions over \([\infty/G]\) are described by \(\mu_\infty\).

(iii) The map \(f\) satisfies the ramification matching condition over the internal nodes of the destabilization \(P\).

By \([1]\), \(\overline{M}_{g, I, \mu_0}([\mathbb{P}^1/G],_r,1, \mu_\infty)\) has a perfect obstruction theory and its virtual fundamental class has complex dimension

\[ \text{vdim}_{\mathbb{C}}[\overline{M}_{g, I, \mu_0}([\mathbb{P}^1/G],_r,1, \mu_\infty)]^{\text{vir}} = 2g - 2 + n + \frac{|\mu_\infty|}{r} - \sum_{i=1}^{l(\mu_0)} \text{age}_{c_{0i}}(L^{1/r}) \]

where \(n = l(\mu_0) + l(\mu_\infty) + \#I\) and \(|\mu_\infty| := \sum_j c_{0j} \sum_{j \in I}^{c_{0j}}\).

For \(r >> 1\), we have \(\text{age}_{c_{0i}}(L^{1/r}) = c_{0i}/r f_{0i}\). In this case the virtual dimension is \(2g - 2 + n\).

In what follows, we assume that \(r\) is large and is a prime number.

4.2. Fixed loci. The standard \(\mathbb{C}^*\)-action on \(\mathbb{P}^1\) is given by

\[ \xi \cdot [z_0, z_1] := [z_0, \xi z_1], \quad \xi \in \mathbb{C}^*, \quad [z_0, z_1] \in \mathbb{P}^1. \]

This induces \(\mathbb{C}^*\)-actions on \([\mathbb{P}^1/G]\), \([\mathbb{P}^1/G],_r,1\), and \(\overline{M}_{g, I, \mu_0}([\mathbb{P}^1/G],_r,1, \mu_\infty)\). The \(\mathbb{C}^*\)-fixed loci of \(\overline{M}_{g, I, \mu_0}([\mathbb{P}^1/G],_r,1, \mu_\infty)\) are labeled by decorated graphs \(\Gamma\).

Notation 4.1. A decorated graph \(\Gamma\) is defined as follows:

(i) (Graph data)
- \(V(\Gamma)\): the set of vertices of \(\Gamma\);
- \(E(\Gamma)\): the set of edges of \(\Gamma\);
- \(F(\Gamma)\): the set of flags of \(\Gamma\), defined to be
  \[ F(\Gamma) = \{(e, v) \in E(\Gamma) \times V(\Gamma) | v \in e\}; \]
- \(L(\Gamma)\): the set of legs;

(ii) (Decoration data)
- each vertex \(v \in V(\Gamma)\) is assigned a genus \(g(v)\), a label of either \(\text{[0}/G(r)\) or \(\text{[\infty}/G\), and a group
  \[ G_v := \begin{cases} G(r) & \text{if } v \text{ is over } \text{[0}/G(r)\,,} \\ G & \text{if } v \text{ is over } \text{[\infty}/G\,.} \end{cases} \]
- each edge \(e \in E(\Gamma)\) is labelled with a conjugacy class \((k_e) \subset G_e := K\) and a positive integer \(d_e\), called the degree;
- each flag \((e, v)\) is labelled with a conjugacy class \((k_{(e,v)}) \subset G_v\);
- a map \(s : L(\Gamma) \to V(\Gamma)\) that assigns legs to vertices of \(\Gamma\);
legs are labelled with markings in \( \mu_0 \cup I \cup \mu_\infty \). Namely \( j \in L(\Gamma) \) is labelled with a conjugacy class \( (k_j) \in G_v \) where
\[
\begin{align*}
(k_j) & \in \{ \bar{c}_0 \}_i \cup I & \text{if } v \text{ is over } [0/G(r)] \\
(k_j) & \in \{ c_\infty \}_j & \text{if } v \text{ is over } [\infty/G].
\end{align*}
\]
The data above satisfy certain compatibility conditions. We omit them as they do not enter our analysis.

A vertex \( v \in V(\Gamma) \) over \([0/G(r)] \) corresponds to a contracted component mapping to \([0/G(r)] \) given by an element of the moduli space
\[
\mathcal{M}_v := \mathcal{M}_{g(v), I(v), \mu_0 (v)}(BG(r))
\]
of genus \( g(v) \) stable maps to \( BG(r) \) such that orbifold structures at marked points are given by corresponding entries of \( \bar{\mu}_0 \) and \( (k_{(e,v)})^{-1} \) for flags attached to \( v \). The dimension of \( \mathcal{M}_{g(v), I(v), \mu_0 (v)}(BG(r)) \) is \( 3g(v) - 3 + \#I(v) + l(\mu_0 (v)) + |E(v)| \).

The discussion on fixed stable maps over \([\infty/G] \in [\mathbb{P}^1/G]_{r,1} \) is similar to that in \([2, \text{Section 2.3}] \), we omit the details.

Let \( \mathcal{M}_\infty \) be the moduli space of stable maps to rubber. Its virtual class \( \mathcal{M}_\infty \)\(^{\text{vir}} \) has complex dimension \( 2g(\infty) - 3 + n(\infty) \), where \( g(\infty) \) is the domain genus and \( n(\infty) = \#I(\infty) + l(\mu_\infty) + |E(\Gamma)| \) is the total number of markings and incidence edges.

We write \( V_0^S(\Gamma) \) for the set of stable vertices of \( \Gamma \) over \([0/G(r)] \). If the target degenerates, define
\[
\mathcal{M}_\Gamma = \prod_{v \in V_0^S(\Gamma)} \mathcal{M}_{g(v), I(v), \mu_0 (v)}(BG(r)) \times \mathcal{M}_\infty,
\]
If the target does not degenerate, define
\[
\mathcal{M}_\Gamma = \prod_{v \in V_0^S(\Gamma)} \mathcal{M}_{g(v), I(v), \mu_0 (v)}(BG(r)).
\]
The fixed locus corresponding to \( \Gamma \) is isomorphic the quotient of \( \mathcal{M}_\Gamma \) quotiented by the automorphism group of \( \Gamma \) and the product of cyclic groups associated to the Galois covers of the edges. There is a natural map \( \iota : \mathcal{M}_\Gamma \to \mathcal{M}_{g, I, \mu_0}([\mathbb{P}^1/G]_{r,1}, \mu_\infty) \).

Assuming \( r >> 1 \), we may argue as in \([2, \text{Lemma 6}] \) to conclude that there are only two types of unstable vertices:
\[
\begin{itemize}
  \item \( v \) is mapped to \([0/G], g(v) = 0, v \) carries one marking and one incident edge;
  \item \( v \) is mapped to \([\infty/G], g(v) = 0, v \) carries one marking and one incident edge.
\end{itemize}
\]

**4.3. Contributions to localization formula.** By convention, the \( \mathbb{C}^* \)-equivariant Chow ring of a point is identified with \( \mathbb{Q}[t] \) where \( t \) is the first Chern class of the standard representation.

Let \( [f : C \to [\mathbb{P}^1/G]_{r,1}] \in \mathcal{M}_{\Gamma} \). The \( \mathbb{C}^* \)-equivariant Euler class of the virtual normal bundle in \( \mathcal{M}_{g, I, \mu_0}([\mathbb{P}^1/G]_{r,1}, \mu_\infty) \) to the \( \mathbb{C}^* \)-fixed locus indexed by \( \Gamma \) can be described as
\[
e(N^{\text{vir}})^{-1} = \frac{e(H^1(C, f^*T_{[\mathbb{P}^1/G]_{r,1}}(\cdot([-\infty/G]))) \cdot (\prod_i e(N_i))^{-1} e(N_\infty)^{-1} \cdot e(H^0(C, f^*T_{[\mathbb{P}^1/G]_{r,1}}(\cdot([-\infty/G]))) \cdot (\prod_i e(N_i))^{-1} e(N_\infty)^{-1}}{e(H^0(C, f^*T_{[\mathbb{P}^1/G]_{r,1}}(\cdot([-\infty/G]))) \cdot (\prod_i e(N_i))^{-1} e(N_\infty)^{-1}}.
\]
Let $V^S(\Gamma)$ be the set of stable vertices in $V(\Gamma)$. The set of stable flags is defined to be

$$F^S(\Gamma) = \{(e, v) \in F(\Gamma)| v \in V^S(\Gamma)\}.$$  

We have

$$\left[\overline{M}_{g,1,\mu_0}([\mathbb{P}^1/G]_{r,1}, \mu_\infty)\right]^{\text{vir}} = \sum_{\Gamma} \frac{1}{|\text{Aut}(\Gamma)|} \frac{1}{\prod_{e \in E(\Gamma) d_e|G_e|} \prod_{(e,v) \in F^S(\Gamma)} r_{(e,v)} \left|G_v\right|} \iota_* \left(\left[\overline{M}_1\right]^{\text{vir}}_{\text{vir}} \circ e(N_\text{vir})\right)$$

where $r_{(e,v)}$ is the order of $k_{(e,v)} \in G_v$.

The localization contributions are given as follows.

(i) Contributions to $e(H^1(C, f^*T[\mathbb{P}^1/G]_{r,1}(-[\infty/G])))$

$$c^*(\mathbb{C}^*) ((-L^{1/r}(v))) = \sum_{d \geq 0} c_d (-L^{1/r}_{g,n}) \left(\frac{t}{r(e,v)}\right)^{g(v) - 1 + |E(v)| - d}.$$

Here $r(e,v) = \frac{|G_v|}{|G_e|} = ar$, and the virtual rank $r$ is $g(v) - 1 + |E(v)|$. This follows from Riemann-Roch together with the observation that because $r >> 1$, the age terms in Riemann-Roch add up to $|E(v)|$.

(b) The two possible unstable vertices contribute to 1.

(c) The edge contribution is trivial because $r >> 1$.

(d) The contribution of a node $N$ over $[0/G(r)]$ is trivial.

(e) Nodes over $[\infty/G]$ contribute 1.

(ii) Contributions to $\prod_i e(N_i)$.

The product $\prod_i e(N_i)$ is formed by edges of $\Gamma$ attaching to vertices. If $N$ is such a node, then

$$e(N) = \frac{t}{r_{(e,v)} d_e} - \frac{\psi_e}{r_{(e,v)}}.$$

Hence, the contribution of this stable vertex $v$ is:

$$\prod_{e \in E(v)} \frac{1}{t} \frac{1}{r_{(e,v)} d_e} - \frac{\psi_e}{r_{(e,v)}} \sum_{d \geq 0} c_d (-L^{1/r}_{g,n}) \left(\frac{t}{r(e,v)}\right)^{g(v) - 1 + |E(v)| - d}.$$

(iii) Contributions to $e(N_\infty)$.

If the target degenerates, there is an additional factor

$$\frac{1}{e(N_\infty)} = \frac{\prod e(r_{(e,v)})}{t + \psi_\infty}.$$

4.4. **Extraction.** The virtual class of the moduli space of rubber maps has non-equivariant limit, and $\mathbb{C}^*$ acts trivially on $\overline{M}_{g,n}(BG)$. Therefore the $\mathbb{C}^*$-equivariant push-forward

$$\iota_* \left(\left[\overline{M}_{g,1,\mu_0}([\mathbb{P}^1/G]_{r,1}, \mu_\infty)\right]^{\text{vir}}\right)$$

via the natural map

$$\iota : \overline{M}_{g,1,\mu_0}([\mathbb{P}^1/G]_{r,1}, \mu_\infty) \to \overline{M}_{g,n}(BG)$$
is a polynomial in $t$. Hence its coefficient of $t^{-1}$ is equal to 0.

Set $s = tr$, we will extract the coefficient of $s^0 r^0$ in $\epsilon_* (\mathbb{M}_{g, I, \mu_0}([\mathbb{P}^1/G]_{r, 1}, \mu_\infty)]^{\vir})$. We denote the map

$$\epsilon : \mathbb{M}_{g(v), I(v), \mu(v)}(BG(r)) \to \mathbb{M}_{g(v), n(v)}(BG)$$

We write

$$\hat{c}_d = r(2d-2g(v)+1) \epsilon_* c_d \in A^d(\mathbb{M}_{g(v), n(v)}(BG)),$$

then by Proposition 3.7, $\hat{c}_d$ is a polynomial in $r$ for $r$ sufficiently large. So the operation of extracting the coefficient of $r^0$ is valid.

We have

$$\epsilon_* (t \mathbb{M}_{g, I, \mu_0}([\mathbb{P}^1/G]_{r, 1}, \mu_\infty)]^{\vir})$$

We have

$$\epsilon_* (t \mathbb{M}_{g, I, \mu_0}([\mathbb{P}^1/G]_{r, 1}, \mu_\infty)]^{\vir}) \approx s \cdot \sum_{\Gamma} \frac{1}{|Aut(\Gamma)|} \prod_{e \in E(\Gamma)} d_e |G_e| \prod_{(e,v) \in FS(\Gamma)} \frac{|G_v|}{r_{(e,v)}} \epsilon_* t^{\hat{R}_G} \left( \frac{[\mathbb{M}_r]}{e(N^{\vir})} \right),$$

where $\epsilon_* t^{\hat{R}_G} \left( \frac{[\mathbb{M}_r]}{e(N^{\vir})} \right)$ is the product of the following factors:

(i) For each stable vertex $v \in V(\Gamma)$ over 0, the factor is

$$\frac{r}{s} \prod_{e \in E(v)} \frac{d_e}{r_{(e,v)}} \sum_{d \geq 0} \hat{c}_d s^{g(v) - d} \cdot a^{-g(v) + 1 - E(v) + d}.$$  

Each edge contributes a factor $\frac{r_{(e,v)}}{r}$ which cancels with the factor $\frac{|G_v|}{r_{(e,v)}} = \frac{|G_v|}{r_{(e,v)}}$ in equation (8) which comes from the contribution of the automorphism group of the node labelled by $(k_{(e,v)})^{-1}$. Therefore, we have at least one positive power of $r$ for each stable vertex of the graph over 0.

(ii) When the target degenerates, there is a factor

$$\frac{-r}{s} \prod_{e \in E(\Gamma)} \frac{d_e}{1 + \frac{\psi_e}{\psi_\infty}}$$

we have at least one positive power of $r$ when the target degenerates.

There are only two graphs which have exactly one $r$ factor in the numerator:

- the graph with a stable vertex of genus $g$ over 0 and $l(\mu_\infty)$ unstable vertices over $\infty$;
- the graph with a stable vertex of genus $g$ over $\infty$ and $l(\mu_0)$ unstable vertices over 0.

Therefore, the $r^0$ coefficient is

$$\text{Coeff}_{r, 0}[\epsilon_* (t \mathbb{M}_{g, I, \mu_0}([\mathbb{P}^1/G]_{r, 1}, \mu_\infty)]^{\vir})] = \frac{|G_v|^{l(\mu_0)}}{|G_v|^{l(\mu_0)}} \cdot \text{Coeff}_{r, 0}[\sum_{d \geq 0} \hat{c}_d s^{g-d} \cdot a^{-g+1-l(\mu_0) + d}] - \frac{|G_v|^{l(\mu_\infty)}}{|G_v|^{l(\mu_\infty)}} DR^G_5 (A)$$

To extract the coefficient of $s^0$, we take $d = g$,

$$\text{Coeff}_{r, 0, 0}[\epsilon_* (t \mathbb{M}_{g, I, \mu_0}([\mathbb{P}^1/G]_{r, 1}, \mu_\infty)]^{\vir})] = \frac{|G_v|^{l(\mu_0)}}{|G_v|^{l(\mu_0)}} \cdot \text{Coeff}_{r, 0}[\hat{c}_g \cdot a^{-l(\mu_0)}] - \frac{|G_v|^{l(\mu_\infty)}}{|G_v|^{l(\mu_\infty)}} DR^G_g (A)$$
By the vanishing of $\text{Coeff}_{r,0,0}[\epsilon_s(t[\overline{M}_{g,1,\mu_0}(\mathbb{P}^1/G),r,1,\mu_\infty])^{\text{vir}})$, we have
\[ DR^G_g(A) = a^{1-l(\mu_\infty)} \text{Coeff}_{r,0}[r \cdot \epsilon_s c_g(-L^{1/r}_{g,n})] \in A^g(\overline{M}_{g,n}(BG)). \]

The proof is complete.

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