Nonlocality Distillation for High-Dimensional System

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The intriguing and powerful capability of nonlocality in communication field ignites the research of the nonlocality distillation. The first protocol presented in Ref[Phys. Rev. Lett. 102, 120401] shows that the nonlocality of bipartite binary-input and binary-output nonsignaling correlated boxes could be amplified by ‘wiring’ two copies of weaker-nonlocality boxes. Several optimized distillation protocols were presented later for bipartite binary-input and binary-output nonsignaling correlated boxes. In this paper, we focus on the bipartite binary-input and multi-nary-output nonsignaling correlated boxes—high-dimensional boxes, and design comparators-based protocols to achieve the distillation of high-dimensional nonlocality. The results show that the high-dimensional nonlocality can be distilled in different ways, and we find that the efficiencies of the protocols are influenced not only by the wirings but also by the classes the initial nonlocality boxes belong to. Here, the initial nonlocalities may have the same violation of the high-dimensional Bell-type inequality, but they can fall into different classes, which shows that the value of the violation of the high-dimensional Bell-type inequality is not the only representation of nonlocality, and the combination manner(classes) in the expression of the correlated boxes is another important nonlocal representation too. The current protocols are compatible with the previous two-dimensional nonlocality distillation protocols, but our protocols are more powerful and universal than previous ones in the sense that the current protocols can be applied to the system with any dimension rather than the only two-dimension system in the previous protocols.

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I. INTRODUCTION

The result of the measurements on spatially separated maximal entangled state runs counter to local realism, which was named as quantum nonlocality. Quantum nonlocality was the focus of the query of quantum mechanics (QM) in the original time when QM was born[1]. After Bell[2] gave the first inequality in his theory which classical correlation must obey but QM doesn’t, quantum nonlocality could be directly perceived through the violation of the corresponding Bell-type inequality[3].

Tsirelson[4]’s study shows that quantum correlations are limited by $BQ = 2\sqrt{2}$ under the Clauser-Horne-Shimony-Holt[5] (CHSH) expression of Bell’s theory. Popescu and Rohrlich[6] (PR) presented an unnatural correlation which violates CHSH inequality by its algebraic maximum 4. All these results illustrate that the set of quantum correlations is bigger than the set of local correlations bounded by Bell-type inequalities, but it is still included in a bigger set of the general nonsignaling correlations, which can reach the algebraic maximal violation of Bell-type inequalities[7].

As an information-theoretic resource, the general nonsignaling models in the form of nonsignaling boxes were presented for convenient study of the set of different correlations[2]. The nonsignaling boxes not only can give simulations of quantum nonlocal correlations (QNC) but also can indicate the correlations beyond QNC—postquantum correlations (PQC), as PR correlations for example. Some extraordinary information processing abilities are revealed by the study of postquantum theories. The PQC can make communication complexity trivial[8–10], simulate quantum entanglement without communication[12, 13], and make dynamics rich in the process of nonlocality swapping[11]. ‘Nonlocal computation’ is allowed by the PQC[14], and ‘information causality’ is also violated[15].

Generally, The stronger nonlocality the resources have, the more useful they are. The question on how to obtain more nonlocality from weak ones is naturally raised to us. Foster et.al[16] gave an affirmative answer to this question by presenting a deterministic nonlocality distillation protocol (FWW) for bipartite binary-input and binary-output correlated nonlocal boxes. Then Brunner et.al[10] made an improved protocol (BS) where one box’s outputs are added into another box’s inputs. If these two boxes are identical, this kind of wiring can be regarded as an kind feedback in this sense. Later Allcock et.al[17] gave a more efficient distillation protocol (ABLPsv) by changing the nonlocal elements, and Hoyer et.al[18] made a ‘depth-3’ method (HR) by using three boxes in each distillation round. All these protocols are working fine for the bipartite correlated nonlocal boxes in certain scope. Recently, the distillation protocol for multipartite nonlocality was discussed, where Li-Yi Hsu and Keng-ShuoWu[19] generalized the bipartite FWW protocol into the n-partite case. Brunner et.al[20] discussed the bound nonlocality and activation of distillable nonlocality under the Elitzur-Popescu-Rohrl[21] decomposition. Forster[22] gave a way to bound...
the distillable nonlocality of a resource by solving a related optimization problem.

Binary-outputs of boxes means that the boxes are two dimensional ones, and so far, all the above-mentioned distillation protocols only work for the two dimensional boxes. Here we will present a novel nonlocality distillation protocol for higher dimensional boxes. In our protocol, the boxes to be distilled is replaced by arbitrary high-dimensional boxes and new wiring method is applied on it. The results show that our protocol distills different boxes with different efficiencies, and the protocol is applied on it. The results show that our protocol distills different boxes with different efficiencies, and the protocol also works for the more general ‘noisy’ correlated nonlocal boxes.

II. BIPARTITE HIGH-DIMENSIONAL NONSIGNALLING CORRELATED BOXES

QM is nonlocal and the nonsignaling models can give simulations of the nonlocal correlations from the respective measurements on spatially separated quantum states. As shown in Fig. 1 a nonsignaling model is a black box which has two input terminals and two output terminals on the two sides belonging to two spatially separated users-Alice and Bob. The inputs $x$ and $y$ simulate Alice and Bob’s possible measurements in quantum measuring process, and the outputs $a$ and $b$ simulate the possible outcomes after Alice and Bob’s measurements. Each user can carry out two possible measurements when the inputs $x$ and $y$ are both binary. The fact that the outputs $a$ and $b$ can be 0, 1, ..., $d$ - 1 means that each measurement of their two choices may have $d$ different outcomes, i.e., the system is $d$-dimensional. In the generalized nonsignaling theories, when $d = 2$, the box is a two-dimensional one, i.e., the one being widely discussed before. $d > 2$ means the box is a $d$-dimensional one, which is in accordance with $d$-dimensional quantum system.

The joint probability distribution $P(ab|xy)$ characterizes the nonlocality of the box and the corresponding quantum system the box simulates. The nonlocality of the box is directly reflected by the violation of the high-dimensional Bell inequality—the $d$-dimensional generalization of CHSH inequality (CGLMP), which was developed in Ref.[23]. We adopt the scalar product-type expression in Ref.[11] and denote the box’s CGLMP value as $CGLMP \cdot \bar{P}(ab|xy)$. For a $d$-dimensional system, the inequality has the form:

$$CGLMP \cdot \bar{P}(ab|xy) = E_{xy} + E_{xy} + E_{xy} - E_{xy} \leq 2,$$

where $\bar{x}$ and $\bar{y}$ indicate bit flips, and $E_{xy}$ is defined as

$$E_{xy} = \frac{d}{2} - \sum_{k=0}^{d/2-1} \left(1 - \frac{2k}{d - 1}\right) |P((b - a) \mod d = -k|xy) - P((b - a) \mod d = k + 1|xy)|.$$ (2)

The local bound is always 2 and the algebraic maximal violation of CGLMP inequality is also 4 for arbitrary high-dimensional system.

If the inputs $x, y \in \{0,1\}$ and outputs $a, b \in \{0,1,...,d - 1\}$ as depicted in Fig. 1 the nonlocal vertex is given by

$$P_{NL}^d(ab|xy) = \begin{cases} 1/d, & \text{if } (b-a) \mod d = x, \\ 0, & \text{otherwise}. \end{cases}$$ (3)

It has an extremal violation 4 of CGLMP inequality. When $d = 2$, it is the PR correlation.

The $d$-dimensional local correlated box is defined as:

$$P_{Lc}^d(ab|xy) = \begin{cases} 1/d, & \text{if } (b-a) \mod d = 0, \\ 0, & \text{otherwise}. \end{cases}$$ (4)

Its outputs are independent of the inputs and it violates the CGLMP inequality by 2. When $d = 2$, it is the box ‘$P_{Lc}^2$’ in the earliest two distillation protocols[10,16] of 2-dimensional boxes.

The $d$-dimensional local deterministic box is described as:

$$P_{Ld}^d(ab|xy) = \begin{cases} 1, & \text{if } a = d - 1, b = d - 1, \\ 0, & \text{otherwise}. \end{cases}$$ (5)

It also violates CGLMP inequality by 2 as $P_{Ld}^d(ab|xy)$, and it is the box ‘$P_{Ld}^2$’ appearing in the third protocol[17] when $d = 2$.

The $d$-dimensional fully mixed box $I_d(ab|xy) = 1/d^2$, $\forall a, b, x, y$, and, for simplicity, it will be written as $I$ hereafter.

III. NONLOCALITY DISTILLATION OF $d$-DIMENSIONAL BOXES

A. A brief review of nonlocality distillation in bipartite two-dimensional boxes

One nonsignaling correlated box with more nonlocality could be generated by certain local operations (local classical operations such as connecting users’ respective inputs and outputs etc.) on many ones without classical communication, which is the so-called noncoality distillation. Refs.[7,10,17] pointed out that the distillation protocol can be viewed as classical circuitry, i.e., the operations on the inputs and outputs are considered as wirings, and the nonsignaling correlated boxes correspond to components.

In the earliest protocol, FWW protocol, two users Alice and Bob made the wiring between two copies of box $I_{c,c}^2$. 

![FIG. 1: The $d$-dimensional bipartite nonsignalling correlated box. The two inputs of the box are binary and the two outputs are $d$-ary.](image-url)
Consider the class of nonsignaling correlated boxes defined as follows:

$$P_{e,c}^{d} = \epsilon P_{NL}^{d} + (1 - \epsilon) P_{LC}^{d},$$

where $0 \leq \epsilon \leq 1$ and $d > 2$. The box $P_{e,c}^{d}$ has a CGLMP value $\text{CGLMP} \cdot \tilde{P}_{e,c}^{d} = 2 + 2\epsilon$. As shown in Fig. 2 (ignore the illustration by red color), Alice and Bob have two copies of $P_{e,c}^{d}$, and the state before distillation is

$$P_{e,c}^{d} P_{e,c}^{d} = \epsilon^2 P_{NL}^{d} P_{NL}^{d} + (1 - \epsilon)(P_{NL}^{d} P_{LC}^{d} + P_{LC}^{d} P_{NL}^{d}) + (1 - \epsilon)^2 P_{LC}^{d} P_{LC}^{d}.$$

The users proceed in each side as follows: $x_1 = x, x_2 = \frac{a_1}{d-1}, a = a_1 + a_2, y_1 = y, y_2 = \frac{b_1}{d-1}, y = b_1 + b_2$, where $+$ in the wirings denotes the addition modulo $d$, and $x_1, y_1, a_i, b_i$ denote the inputs and outputs bits of the $i$th box. As the key of the protocol, \[
\frac{k}{d-1} = \begin{cases} 1, & \text{if } k = d-1, \\ 0, & \text{if } k < d-1. \end{cases}
\]

which is similar to a comparator in classical circuitry.

The wirings defined above take two initial boxes to one final box, so there will be four cases of $P, P' \rightarrow P_{f}$:

a) $p_{NL}^{d} P_{NL}^{d} \rightarrow P_{NL}^{d}$. We can get $(b_1 - a_1) \mod d = xy$ from the first box, i.e. $b_1 = (xy + a_1) \mod d$. For the second box we have $(b_2 - a_2) \mod d = \left(\frac{a_1}{d-1}\right) \mod d$ (mod $d = d$).

b) $P_{NL}^{d} P_{LC}^{d} \rightarrow P_{NL}^{d}$. For the first box we have $(b_1 - a_1) \mod d = xy$, and for the second box we have $(b_2 - a_2) \mod d = 0$, so the final relation $(b - a) \mod d = xy$ too.

c) $P_{LC}^{d} P_{NL}^{d} \rightarrow \frac{1}{2} P_{NL}^{d} + (1 - \frac{1}{2}) P_{LC}^{d}$. The first box given by $(b_1 - a_1) \mod d = 0$, implies $b_1 = a_1$. For the second box, $(b_2 - a_2) \mod d = \left(\frac{a_1}{d-1}\right) \mod d = xy$. Finally, we get $(b - a) \mod d = xy$ where $a_1$ is random. The final relation will be $(b - a) \mod d = xy$ when $a_1 = d-1$ with probability $1/d$, and $(b - a) \mod d = 0$ otherwise.

d) $P_{LC}^{d} P_{LC}^{d} \rightarrow P_{LC}^{d}$. Here $(b_1 - a_1) \mod d = 0$ and $(b_2 - a_2) \mod d = 0$, so we have $(b - a) \mod d = 0$.

After the above four logical calculations, the final state of the box is given by

$$P_{e,c}^{d} = \left((1 + \frac{1}{d})\epsilon - \frac{1}{d}\epsilon^2\right) P_{NL}^{d} + \left[1 - (1 + \frac{1}{d})\epsilon + \frac{1}{d}\epsilon^2\right] P_{LC}^{d},$$

where $\epsilon' = (1 + \frac{1}{d})\epsilon - \frac{1}{d}\epsilon^2$. CGLMP $\cdot \tilde{P}_{e,c}^{d} = \tilde{P}_{e,c}^{d} = 2 + 2\epsilon' - (2 + 2\epsilon') > 0$ is always tailable with $0 < \epsilon < 1$ and a finite dimension, which implies the fraction of the nonlocal component is increased and the distillation protocol succeeds.

C. Protocol B: distillation of $P_{e,d}^{d}$

Suppose we use another superposition of different components as the initial boxes, i.e.

$$P_{e,d}^{d} = \epsilon P_{NL}^{d} + (1 - \epsilon) P_{LD}^{d},$$

where $0 \leq \epsilon \leq 1$ and $d > 2$. Notice that this kind of superposition has the same CGLMP value as that of the superpositions.
in Eq. (10) in protocol A. The initial state is

\[
P_{e,d}^{d}P_{e,d}^{d} = \epsilon^2 P_{NL,d}^{d} + (1 - \epsilon)(P_{NL,d}^{d} P_{d}^{d} + P_{d}^{d} P_{d}^{d}) + (1 - \epsilon)^2 P_{d}^{d} P_{d}^{d}.
\]

(11)

Through the wirings shown in Fig 2 (ignore the blue part): 

\[
x_1 = x, x_2 = \left[ \frac{a_1}{d-1} \right], a = a_1 + a_2 + 1, y_1 = y, y_2 = \left[ \frac{b_1}{d-1} \right], b = b_1 + b_2 + 1,
\]

we could get the following transformations: \(P_{d}^{d} P_{NL,d}^{d} \rightarrow P_{d}^{d} P_{d}^{d}, P_{d}^{d} P_{NL,d}^{d} \rightarrow P_{d}^{d} P_{d}^{d}, P_{d}^{d} P_{d}^{d} \rightarrow P_{d}^{d} P_{d}^{d}, \) The final box is

\[
P_{e,d}^{d} = (2\epsilon - \epsilon^2)P_{d}^{d} + (1 - 2\epsilon + \epsilon^2)P_{d}^{d}.
\]

(12)

Obviously for \(0 < \epsilon < 1, \epsilon' = 2\epsilon - \epsilon^2 > \epsilon, \) and the nonlocality of the final box is bigger than the initial ones.

The initial boxes in the two protocols have the same violation of CGLMP inequality, but they induce different efficiency of distillation in the similar wirings. The fact that protocol B has a better efficiency than protocol A is revealed immediately from the comparison shown in Fig 2.

It is easy to see that, our protocol A is compatible with the BS protocol, and protocol B has an equal efficiency as the ABLPSV protocol when \(d = 2, \) so our protocol is effective for arbitrary dimensional boxes. In the protocol A, the dimension of the boxes appears in the coefficient of the final nonlocal fraction, that is to say, the distillation efficiency is a function of the dimension of the boxes (the protocol becomes trivial when \(d \rightarrow \infty \)). But in the protocol B, the final box’s nonlocal fraction is not varying with the dimension of the boxes, and is always bigger than protocol A.

IV. NONLOCALITY DISTILLATION IN MORE GENERAL CASES

Our protocol can also work for the more general \(d\)-dimensional nonlocal boxes, i.e. the noisy correlated boxes as such a proxytope:

\[
P_{\xi,\gamma}^{d} = \xi P_{NL,d}^{d} + \gamma P_{d}^{d} + (1 - \xi - \gamma) \mathbf{1},
\]

(13)

where \(\xi, \gamma \geq 0\) and \(\xi + \gamma \leq 1, \mathbf{1}\) is the \(d\)-dimensional fully mixed box. After applying the wirings presented in protocol B on two copies of the noisy box, and noticing the extra transformations \(P_{NL,d}^{d} \mathbf{1} \rightarrow \mathbf{1}, \mathbf{1} P_{d}^{d} \mathbf{1} \rightarrow \frac{1}{2}P_{d}^{d} + \mathbf{1} - \frac{1}{2}P_{d}^{d} \mathbf{1}, P_{d}^{d} \mathbf{1} \rightarrow \mathbf{1}, \) \(\mathbf{1} P_{d}^{d} \mathbf{1} \rightarrow \mathbf{1}, \mathbf{1} \mathbf{1} \rightarrow \mathbf{1}, \mathbf{1}, \) we can get the final box after distillation:

\[
P_{d}^{d} = \frac{1}{2} + (1 + \xi + \gamma)(1 - \xi - \gamma) \mathbf{1} - \frac{1}{2d^2}(1 - \xi - \gamma) P_{NL,d}^{d}.
\]

(14)

The CGLMP value of the initial box is \(4\xi + 2\gamma. \) After distillation, we get a box with CGLMP value \((4 + \frac{2}{\xi+\gamma})\xi^2 + (8 - \frac{2}{\xi+\gamma})\xi + \frac{2}{\xi+\gamma} + 2\gamma^2.\) For a fixed \(d, \) it is easy to compare the strength of nonlocality of the initial box and the final box. Consider the case \(d \rightarrow \infty, \) the corresponding final box has the CGLMP value of \(4\xi^2 + 2\gamma + 2\gamma^2. \) When the weight of the non-local part \(\xi\) and local part \(\gamma\) range in the shaded areas in Fig 3(b), the protocol works well for the \(d \rightarrow \infty\) case.

V. CONCLUSIONS

By adding the comparators into the wirings of the nonlocality distillation, we realized the nonlocality distillation of high-dimensional nonsignaling correlated nonlocal boxes. It shows that high-dimensional nonlocality in general nonsignaling theory measured by the CGLMP inequality also can be distilled by local classical operation. An arbitrarily small violation of the CGLMP inequality can be amplified to the asymptotic extremal violation through a finite number of our distillation protocols. We used different local compositions in the initial boxes, and found that the initial boxes superposed by local deterministic boxes and nonlocal extremal boxes can be distilled with a higher efficiency than that of the case with the initial boxes being superposition of local correlated boxes and non-local extremal boxes. The protocol distilling different boxes with different efficiencies showed that the nonlocality distillation is both wirings- and components-selective.

Furthermore, in sec. IV we showed that our protocol also works for the more general high-dimensional boxes. We also studied the distillation of another general superposition: \(P_{\xi,\gamma}^{d} = \xi P_{NL,d}^{d} + \gamma P_{d}^{d} + (1 - \xi - \gamma) \mathbf{1}, \) and the nonlocality of the final one will be \((4 + \frac{2}{\xi+\gamma})\xi^2 + (6 + \frac{2}{\xi+\gamma})\xi + \frac{2}{\xi+\gamma} + 2\gamma^2\) after the wirings in protocol A, which is similar to the protocol we discussed in sec IV.

It is worth while to discuss some further problems. Are there protocols distilling better for more high-dimensional
FIG. 4: (a) Comparison of nonlocalities before and after distilling 'noisy correlated boxes' for the case $d \to \infty$. In the figure, the gray surface represents the set of initial boxes' CGLMP value and the colorful surface represents the set of final boxes' CGLMP value. (b) The area our protocol works for the case of $d \to \infty$. In the figure, the blue line corresponds to the equation: $\xi + \gamma = 1$, and the red line corresponds to the equation: $4\xi^2 + 2\gamma^2 + 8\xi\gamma = 4\xi + 2\gamma$. In the shaded area our protocol works well.

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