Quantum Gravity and background field formalism

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Abstract

We analyze the problem of general covariance for quantum gravity theories in the background field formalism with respect to gauge fixing procedure. We prove that the background effective action is not invariant under general coordinate transformations of background metric tensor in non-linear gauges.

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1 Introduction

The background field formalism [1, 2, 3] is one of the most popular methods for quantum studies and calculations in gauge theories because it allows to work with the effective action invariant under the gauge transformations of background fields and to reproduce all usual physical results by choosing a special background gauge fixing condition. Various aspects of quantum properties of Yang-Mills theories have been successfully studied in this technique [4, 5, 6, 7, 8, 9, 10, 11, 12].

A classical action of all quantum gravity theories obeys the property of invariance under general coordinate transformations and can be considered as an example of gauge theory with closed gauge algebra and with structure coefficients independent on fields (the metric tensor). For such kind of theories the quantization can be performed in the form of Faddeev-Popov procedure [13]. Because similarity between Yang-Mills theories and gravity theories as gauge theories it seems naturally to apply the background field formalism being very successfully in the case of Yang-Mills fields to study their quantum properties. We are going to consider more detailed the problem of general covariance of the background effective action for quantum gravity theories with respect to gauge fixing procedure.

In the present paper we analyze the general covariance of the effective action for any initial classical gravity action in the background field formalism. Application of this method to Yang-Mills theories gives rise two important advantages of the effective action: gauge invariance and gauge independence on its extremals. In the case of Quantum Gravity formulated within the background field formalism we confirm the property of gauge independence of the effective action on its extremals for all admissible gauges but we point out that the gauge invariance of the background effective action is supported by linear vector gauges only.

The paper is organized as follows. In Section 2 we fix notations and represent arbitrary gravity theories in the background field formalism in any admissible gauges to confirm gauge independence of vacuum functional on gauge conditions and as a consequence the same property of effective action on its extremals. In Section 3 the general covariance of vacuum functional is analyzed. It is found that the general covariance can be arrived at the propositions concerning validity of tensor transformations of gauge fixing functions and its linear dependence on quantum gravitational fields. In Section 4 the simplest case of gauge fixing condition is considered to check previous assumptions. It is shown that the standard choice of gauge fixing functions satisfies the required propositions. In Section 5 concluding discussions are given.

In the paper the DeWitt’s condensed notations are used [14]. We employ the notation $\varepsilon(A)$ for the Grassmann parity and the $\text{gh}(A)$ for the ghost number of any quantity $A$. All functional derivatives are taken from the left. The functional right derivatives with respect to fields are marked by special symbol " $\leftrightarrow$ ".

2
2 Background field formalism for Quantum Gravity: gauge independence

Our starting point is an arbitrary action of a Riemann’s metric, \( S_0 = S_0(g), g = \{ g_{\mu\nu} \} \) invariant under the general coordinate transformations,\(^2\)

\[
x'\mu = f^{\mu}(x) \rightarrow x^\mu = x^\mu(x'), \quad g_{\mu\nu} \rightarrow g'_{\mu\nu}(x') = g_{\alpha\beta}(x) \frac{\partial x^\alpha}{\partial x'^\mu} \frac{\partial x^\beta}{\partial x'^\nu}.
\]

(2.1)

In the infinitesimal form the transformations (2.1) read

\[
x'^\mu = x^\mu + \omega^\mu(x) \rightarrow x^\mu = x'^\mu - \omega^\mu(x'), \quad g_{\mu\nu} \rightarrow g'_{\mu\nu}(x) = g_{\mu\nu}(x) + \delta_\omega g_{\mu\nu}(x),
\]

(2.2)

where

\[
\delta_\omega g_{\mu\nu}(x) = -\omega^\sigma(x) \partial_\sigma g_{\mu\nu}(x) - g_{\mu\sigma}(x) \partial_\nu \omega^\sigma(x) - g_{\sigma\nu}(x) \partial_\mu \omega^\sigma(x).
\]

(2.3)

The action \( S_0(g) \) is invariant under the transformations (2.3)

\[
\int dx \frac{\delta S_0(g)}{\delta g_{\mu\nu}(x)} \delta_\omega g_{\mu\nu}(x) = 0.
\]

(2.4)

For any tensor fields \( A_\mu(x), A^{\mu}(x), A_{\mu\nu\lambda}(x) \) of types \((0, 1), (1, 0), (0, 3), (1, 2)\), respectively, on a given manifold the infinitesimal form of general coordinate transformations is

\[
\delta_\omega A_\mu(x) = -\omega^\sigma(x) \partial_\sigma A_\mu(x) - A_\sigma(x) \partial_\mu \omega^\sigma(x),
\]

(2.5)

\[
\delta_\omega A^{\mu}(x) = -\omega^\sigma(x) \partial_\sigma A^{\mu}(x) + A^\sigma(x) \partial_\mu \omega^\sigma(x),
\]

(2.6)

\[
\delta_\omega A_{\mu\nu\lambda}(x) = -\omega^\sigma(x) \partial_\sigma A_{\mu\nu\lambda}(x) - A_{\mu\sigma\lambda}(x) \partial_\nu \omega^\sigma(x) - A_{\nu\mu\lambda}(x) \partial_\sigma \omega^\sigma(x) - A_{\sigma\nu\lambda}(x) \partial_\mu \omega^\sigma(x),
\]

(2.7)

\[
\delta_\omega A^{\lambda}_{\mu\nu}(x) = -\omega^\sigma(x) \partial_\sigma A^{\lambda}_{\mu\nu}(x) + A^{\sigma}_{\mu\nu}(x) \partial_\lambda \omega^\sigma(x) - A^{\lambda}_{\mu\sigma}(x) \partial_\nu \omega^\sigma(x) - A^{\lambda}_{\sigma\nu}(x) \partial_\mu \omega^\sigma(x).
\]

(2.8)

Let us represent the transformations (2.3) in the form

\[
\delta_\omega g_{\mu\nu}(x) = \int dy R_{\mu\nu\sigma}(x, y; g) \omega^\sigma(y),
\]

(2.9)

where

\[
R_{\mu\nu\sigma}(x, y; g) = -\delta(x - y) \partial_\sigma g_{\mu\nu}(x) - g_{\mu\sigma}(x) \partial_\nu \delta(x - y) - g_{\sigma\nu}(x) \partial_\mu \delta(x - y)
\]

(2.10)

\(^2\)Standard examples are Einstein gravity, \( S_0(g) = \kappa^{-2} \int dx \sqrt{-g} R \), and \( R^2 \) gravity, \( S_0(g) = \int dx \sqrt{-g} (\lambda_1 R^2 + \lambda_2 R^{\mu\nu} R_{\mu\nu} + \kappa^{-2} R) \).
can be considered as the generators of gauge transformations of the metric tensor $g_{\mu\nu}$ with gauge parameters $\omega^\sigma(x)$. The algebra of gauge transformations has the following form

$$
\int du \left( \frac{\delta R_{\mu\nu\sigma}(x, y; g)}{\delta g_{\alpha\beta}(u)} R_{\alpha\beta\gamma}(u, z; g) - \frac{\delta R_{\mu\nu\gamma}(x, z; g)}{\delta g_{\alpha\beta}(u)} R_{\alpha\beta\sigma}(u, y; g) \right) =
$$

$$= -\int du R_{\mu\nu\lambda}(x, u; g) F^\lambda_{\sigma\gamma}(u, y, z),$$

where

$$F^\lambda_{\alpha\beta}(x, y, z) = \delta(x - y) \delta^\lambda_\beta \partial_\alpha \delta(x - z) - \delta(x - z) \delta^\lambda_\alpha \partial_\beta \delta(x - y)$$

are structure functions of the gauge algebra which do not depend on the metric tensor $g_{\mu\nu}$. Therefore, any theory of gravity looks like a gauge theory with closed gauge algebra and structure functions independent on fields (metric tensor), i.e. as an Yang-Mills type theory. In what follows we will omit the space - time argument $x$ of fields and gauge parameters when this does not lead to misunderstandings in the formulas and relations employing the DeWitt’s condensed notations [1]. Then the relations (2.9), (2.11) are presented in the form

$$\delta_\omega g_{\mu\nu} = R_{\mu\nu\sigma}(g) \omega^\sigma,$$

$$\frac{\delta R_{\mu\nu\sigma}(g)}{\delta g_{\alpha\beta}} R_{\alpha\beta\gamma}(g) - \frac{\delta R_{\mu\nu\gamma}(g)}{\delta g_{\alpha\beta}} R_{\alpha\beta\sigma}(g) = -R_{\mu\nu\lambda}(g) F^\lambda_{\sigma\gamma}. \quad (2.14)$$

In the background field formalism [1, 2] the metric tensor $g_{\mu\nu}$ appearing in classical action $S_0(g)$, is replaced by $\bar{g}_{\mu\nu} + h_{\mu\nu}$,

$$S_0(g) \to S_0(\bar{g} + h),$$

where $\bar{g}_{\mu\nu}$ is considered as a background metric tensor while $h_{\mu\nu}$ present the quantum fields as integration variables in functional integrals for generating functionals of Green functions.

The action $S_0(\bar{g} + h)$ obeys obviously the gauge invariance,

$$\delta_\omega S_0(\bar{g} + h) = 0, \quad \delta_\omega h_{\mu\nu} = R_{\mu\nu\sigma}(h) \omega^\sigma, \quad \delta_\omega \bar{g}_{\mu\nu} = R_{\mu\nu\sigma}(\bar{g}) \omega^\sigma. \quad (2.16)$$

The corresponding Faddeev-Popov action $S_{FP} = S_{FP}(\phi, \bar{g})$ is written as [13]

$$S_{FP} = S_0(\bar{g} + h) + S_{gh}(\phi, \bar{g}) + S_{gf}(\phi, \bar{g}), \quad (2.17)$$

where $S_{gh}(\phi, \bar{g})$ is the ghost action

$$S_{gh}(\phi, \bar{g}) = \int dx \sqrt{-\bar{g}} \bar{C}^\alpha G^{\beta\gamma}_\alpha(\bar{g}, h) R_{\beta\gamma\sigma}(\bar{g} + h) C^\sigma, \quad (2.18)$$

with the notation

$$G^{\beta\gamma}_\alpha(\bar{g}, h) = \frac{\delta X_\alpha(\bar{g}, h)}{\delta h_{\beta\gamma}}. \quad (2.19)$$
The $S_{gf}(\bar{g}, h)$ is the gauge fixing action

$$S_{gf}(\phi, \bar{g}) = \int dx \sqrt{-\bar{g}} B^\alpha \chi_\alpha(\bar{g}, h).$$

(2.20)

Here $\chi_\alpha(\bar{g}, h)$ are functions lifting the degeneracy of the action $S_0$, $\phi = \{\phi^i\}$ is the set of all fields $\phi^i = (h_{\mu\nu}, B^\alpha, C^\alpha, \bar{C}^\alpha) (\varepsilon(\phi^i) = \varepsilon_i)$ with the Faddeev-Popov ghost and anti-ghost fields $C^\alpha, \bar{C}^\alpha (\varepsilon(C^\alpha) = \varepsilon(\bar{C}^\alpha) = 1, \ gh(C^\alpha) = -gh(\bar{C}^\alpha) = 1)$, respectively, and the Nakanishi-Lautrup auxiliary fields $B^\alpha (\varepsilon(B^\alpha) = 0, gh(B^\alpha) = 0)$.

For any admissible choice of gauge fixing functions $\chi_\alpha(\bar{g}, h)$ the action (2.12) is invariant under global supersymmetry (BRST symmetry) \[15, 16, \]

$$\delta_B h_{\mu\nu} = R_{\mu\nu\alpha}(\bar{g} + h) C^\alpha \Lambda, \quad \delta_B B^\alpha = 0, \quad \delta_B C^\alpha = -C^\sigma \partial_\sigma C^\alpha \Lambda, \quad \delta_B \bar{C}^\alpha = B^\alpha \Lambda,$$

(2.21)

where $\Lambda$ is a constant Grassmann parameter. Let us present the BRST transformations (2.21) in the form

$$\delta_B \phi^i = R^i(\phi, \bar{g}) \Lambda, \quad \varepsilon(R^i(\phi, \bar{g})) = \varepsilon_i + 1,$$

(2.22)

where

$$R^i(\phi, \bar{g}) = \left( R_{\mu\nu\alpha}(\bar{g} + h) C^\sigma, 0, -C^\sigma \partial_\sigma C^\alpha, B^\alpha \right).$$

(2.23)

Introducing the gauge fixing functional $\Psi = \Psi(\phi, \bar{g})$,

$$\Psi = \int dx \sqrt{-\bar{g}} \bar{C}^\alpha \chi_\alpha(\bar{g}, h),$$

(2.24)

the action (2.16) rewrites as

$$S_{FP}(\phi, \bar{g}) = S_0(\bar{g} + h) + \Psi(\phi, \bar{g}) \hat{R}(\phi, \bar{g}), \quad S_0(\bar{g} + h) \hat{R}(\phi, \bar{g}) = 0,$$

(2.25)

where

$$\hat{R}(\phi, \bar{g}) = \int dx \frac{\varepsilon}{\delta \phi^i} R^i(\phi, \bar{g})$$

(2.26)

is the generator of BRST transformations. Due to the nilpotency property of $\hat{R}$, $\hat{R}^2 = 0$, the BRST symmetry of $S_{FP}$ follows from the presentation (2.25) immediately,

$$S_{FP}(\phi, \bar{g}) \hat{R}(\phi, \bar{g}) = 0.$$

(2.27)

The generating functional of Green functions in the background field method is defined in the form of functional integral

$$Z(J, \bar{g}) = \int d\phi \ \exp \left\{ \frac{i}{\hbar} \left[ S_{FP}(\phi, \bar{g}) + J\phi \right] \right\} = \exp \left\{ \frac{i}{\hbar} W(J, \bar{g}) \right\},$$

(2.28)

\[3\]The gravitational BRST transformations were introduced in \cite{17, 18, 19}.
where $W(J, \bar{g})$ is the generating functional of connected Green functions. In (2.28) the notations

$$J \phi = \int dx \sqrt{-g} J_i(x) \phi^i(x), \quad J_i(x) = (J^{\mu\nu}(x), J^{(B)}_\alpha(x), \bar{J}_\alpha(x), J_\alpha(x))$$

are used and $J_i(x)$ ($\varepsilon(J_i(x)) = \varepsilon_i, \ gh(J_i(x)) = gh(\phi^i(x))$) are external sources to fields $\phi^i(x)$.

Let $Z_\Psi(\bar{g})$ be the vacuum functional which corresponds to the choice of gauge fixing functional (2.24) in the presence of external fields $\bar{g}$,

$$Z_\Psi(\bar{g}) = \int d\phi \ exp \left\{ \frac{i}{\hbar} \left[ S_0(\bar{g} + h) + \Psi(\phi, \bar{g}) \hat{R}(\phi, \bar{g}) \right] \right\} = \int d\phi \ exp \left\{ \frac{i}{\hbar} S_{FP}(\phi, \bar{g}) \right\} = \exp \left\{ \frac{i}{\hbar} W_\Psi(\bar{g}) \right\}. \quad (2.30)$$

In turn, let $Z_{\Psi+\delta \Psi}$ be the vacuum functional corresponding to a gauge fixing functional $\Psi(\phi, \bar{g}) + \delta \Psi(\phi, \bar{g})$,

$$Z_{\Psi+\delta \Psi}(\bar{g}) = \int d\phi \ exp \left\{ \frac{i}{\hbar} \left[ S_{FP}(\phi, \bar{g}) + \delta \Psi(\phi, \bar{g}) \hat{R}(\phi, \bar{g}) \right] \right\}. \quad (2.31)$$

Here, $\delta \Psi(\phi, \bar{g})$ is an arbitrary infinitesimal odd functional which may in general has a form differing on (2.24). Making use of the change of variables $\phi^i$ in the form of BRST transformations (2.21) but with replacement of the constant parameter $\Lambda$ by the following functional

$$\Lambda = \Lambda(\phi, \bar{g}) = \frac{i}{\hbar} \delta \Psi(\phi, \bar{g}), \quad (2.32)$$

and taking into account that the Jacobian of transformations is equal to

$$J = \exp\{-\Lambda(\phi, \bar{g}) \hat{R}(\phi, \bar{g})\}, \quad (2.33)$$

we find the gauge independence of the vacuum functional

$$Z_\Psi(\bar{g}) = Z_{\Psi+\delta \Psi}(\bar{g}), \quad (2.34)$$

so that

$$\delta_\Psi Z(\bar{g}) = 0 \rightarrow \delta_\Psi W(\bar{g}) = 0. \quad (2.35)$$

The property (2.34) was a reason to omit the label $\Psi$ in the definition of generating functionals (2.28). In deriving (2.29) the relation

$$(-1)^{\varepsilon_i} \frac{\partial}{\partial \phi^i} R^i(\phi, \bar{g}) = 0, \quad (2.36)$$

Using the finite BRST transformations one can connect the description of any gauge theory in two arbitrary admissible gauges [20, 21, 22].
was used. The property (2.29) means that due to the equivalence theorem [23] the physical S-matrix does not depend on the gauge fixing. In terms of the effective action $\Gamma(\Phi, \bar{g})$ which is defined with the help of Legendre transformation

$$
\Gamma(\Phi, \bar{g}) = W(J, \bar{g}) - J\Phi, \quad \frac{\delta W(J, \bar{g})}{\delta J_i} = \sqrt{-\bar{g}} \Phi^i, \quad J\Phi = \int dx \sqrt{-\bar{g}} J_i(x) \Phi^i(x),
$$

(2.37)

the property (2.35) reads

$$
\delta_{\Phi} \Gamma(\Phi, \bar{g}) \bigg|_{\frac{\delta \Gamma(\Phi, \bar{g})}{\delta \Phi} = 0} = 0,
$$

(2.38)

i.e. the effective action evaluated on its extremal does not depend on gauge.

3 Gauge invariance

The gauge independence of the vacuum functional for Quantum Gravity in the background field formalism repeats the corresponding property for the vacuum functional for Yang-Mills theories. Moreover that vacuum functional for Yang-Mills theories obeys additional important invariance property under the gauge transformations of external vector fields [3]. All quantum gravity theories look like as special type of gauge theories (similar to Yang-Mills theories) with closed gauge algebra and with structure coefficients independent on fields. Therefore it is natural to expect invariance of vacuum functional for Quantum Gravity in the background field formalism under general coordinate transformations on manifolds with an external metric tensor $\bar{g}_{\mu\nu}$ [9].

Consider a variation of $Z(\bar{g})$ under general coordinates transformations of external metric tensor $\bar{g}_{\mu\nu}$,

$$
\delta^{(c)}_{\omega} \bar{g}_{\mu\nu} = R_{\mu\nu\sigma}(\bar{g}) \omega^\sigma.
$$

(3.1)

Then we have

$$
\delta^{(c)}_{\omega} Z(\bar{g}) = \frac{i}{\hbar} \int d\phi \left[ \delta^{(c)}_{\omega} S_0(\bar{g} + h) + \delta^{(c)}_{\omega} S_{gh}(\phi, \bar{g}) + \delta^{(c)}_{\omega} S_{gf}(\phi, \bar{g}) \right] \exp \left\{ \frac{i}{\hbar} S_{FP}(\phi, \bar{g}) \right\}. \tag{3.2}
$$

Now, using a change of variables in the functional integral (3.2) one should try to arrive at the relation $\delta^{(c)}_{\omega} Z(\bar{g}) = 0$ to prove invariance of $Z(\bar{g})$ under the transformations (3.1). In the sector of fields $h_{\mu\nu}$ the form of this transformations is dictated by the invariance property of $S_0(\bar{g} + h)$ and reads

$$
\delta^{(q)}_{\omega} h_{\mu\nu} = \bar{R}_{\mu\nu\sigma}(h) \omega^\sigma = -\omega^\sigma \partial_\sigma h_{\mu\nu} - h_{\mu\sigma} \partial_\nu \omega^\sigma - h_{\nu\sigma} \partial_\mu \omega^\sigma, \tag{3.3}
$$

so that

$$
\delta_{\omega} S_0(\bar{g} + h) = 0, \quad \delta_{\omega} = (\delta^{(c)}_{\omega} + \delta^{(q)}_{\omega}). \tag{3.4}
$$
Notice that on this stage there exists a difference between the Yang-Mills theories formulated in the background field formalism and Quantum Gravity theories under consideration. The change (3.3) is just a gauge transformation of quantum fields \( h_{\mu \nu} \) while in the case of Yang-Mills theories the corresponding change has the form of tensor transformations of quantum vector fields [3]. It is the reason for us to consider the invariance property of the action (2.18) in detail.

Next step is related with analysis of the gauge fixing action \( S_{gf}(\phi, \bar{g}) \) because it depends only on three variables \( h_{\mu \nu}, B^\alpha, \bar{g}_{\mu \nu} \) and for two of them, \( h_{\mu \nu}, \bar{g}_{\mu \nu} \), the transformation law is already defined (3.1), (3.3). Let \( \delta_\omega B^\alpha \) be at the moment unknown transformation of fields \( B^\alpha \). The explicit form of \( \delta_\omega B^\alpha \) should be chosen in a such of way to compensate the variation of \( S_{gf}(\phi, \bar{g}) \) caused by transformations \( \bar{g}_{\mu \nu} \) and \( h_{\mu \nu} \). In the case of Yang-Mills theories it can be done with success in the form of tensor transformations of \( B^\alpha \) [3]. In the case under consideration we have

\[
\delta_\omega S_{gf} = \int dx \sqrt{-\bar{g}} \left[ (\delta_\omega B^\alpha + \omega^\sigma \partial_\sigma B^\alpha) \chi_\alpha(\bar{g}, h) + B^\alpha \omega^\sigma \partial_\sigma \chi_\alpha(\bar{g}, h) + B^\alpha \delta_\omega \chi_\alpha(\bar{g}, h) \right]. \tag{3.5}
\]

Suppose that the variation of gauge fixing functions \( \chi_\alpha \) under gauge transformations (3.1), (3.3) has the form

\[
\delta_\omega \chi_\alpha = -\omega^\sigma \partial_\sigma \chi_\alpha - \partial_\alpha \omega^\sigma \chi_\sigma, \tag{3.6}
\]

which corresponds to the transformation of vector fields of type \((0, 1)\) (2.5). Then choosing the transformation law for \( B^\alpha \) in the form

\[
\delta_\omega B^\alpha = -\omega^\sigma \partial_\sigma B^\alpha + B^\sigma \partial_\sigma \omega^\alpha, \tag{3.7}
\]

we arrive at the desired relation

\[
\delta_\omega S_{gf} = 0. \tag{3.8}
\]

Notice that the transformation (3.7) coincides with the corresponding rule for tensor fields of type \((1, 0)\), (2.6).

Due to the non-locality representation of the ghost action the its variation should be presented in detail

\[
\delta_\omega S_{gh} = \int dx dy dz \sqrt{-\bar{g}(x)} \left[ (\delta_\omega \bar{C}_\alpha(x) + \omega^\rho(x) \partial_\rho \bar{C}_\alpha(x)) \bar{C}_\beta^\gamma(x, y) R_{\beta \gamma \rho}(y, z) C^\rho(z) + \bar{C}_\alpha(x) \omega^\rho(x) \partial_\rho G^\beta_\alpha^\gamma(x, y) \delta_\omega C^\rho(z) + \bar{C}_\alpha(x) G^\beta_\alpha^\gamma(x, y) \delta_\omega R_{\beta \gamma \rho}(y, z) C^\rho(z) + \bar{C}_\alpha(x) \delta_\omega G^\beta_\alpha^\gamma(x, y) R_{\beta \gamma \sigma}(y, z) C^\sigma(z) + \bar{C}_\alpha(x) \delta_\omega (R_{\beta \gamma \sigma}(y, z)) C^\sigma(z) \right]. \tag{3.9}
\]
For variation of the gauge generators we find

$$\delta \omega R_{\beta \gamma \sigma}(y, z) = \int du dv \left[ \frac{\delta R_{\beta \gamma \rho}(y, v)}{\delta g_{\mu \nu}(u)} \omega^\rho(v) R_{\mu \nu \sigma}(u, z) - R_{\beta \gamma \lambda}(y, u) F^\lambda_{\sigma \rho}(u, z, v) \omega^\rho(v) \right] =$$

$$= -\omega^\rho(y) \partial^\rho_{\mu} R_{\beta \gamma \sigma}(y, z) - \partial^\beta \omega^\rho(y) R_{\gamma \rho \sigma}(y, z) - \partial^\gamma \omega^\rho(y) R_{\beta \rho \sigma}(y, z) -$$

$$- R_{\beta \gamma \lambda}(y, z) \partial_{\sigma} \omega^\lambda(z) - \partial^\rho_{\mu} \left( R_{\beta \gamma \sigma}(y, z) \omega^\rho(z) \right). \quad (3.10)$$

The gauge transformation of $R_{\beta \gamma \sigma}(y, z)$ by itself differs of the transformation law for the tensor field of type (0, 3). It is no wonder because of its non-locality nature but it differs as well of the tensor transformations of product of two tensors like $A_{\beta \gamma}(x) B_{\sigma}(y)$.

Having in mind the conditions (3.6) we can study a variation of the operator (2.19) under the gauge transformations (3.1) and (3.3). The result looks like more complicated than (3.10) and takes the form

$$\delta \omega G_{\alpha \beta \gamma}(x, y) = \int dz G_{\alpha \mu \nu}(x, z) \frac{\delta \omega h_{\mu \nu}(z)}{\delta h_{\beta \gamma}(y)} - \int dz \frac{\delta G_{\alpha \beta \gamma}(x, y)}{\delta h_{\mu \nu}(z)} \delta \omega h_{\mu \nu}(z),$$

or

$$\delta \omega G_{\alpha \beta \gamma}(x, y) = -\omega^\sigma(x) \partial^\sigma_{\alpha} G_{\beta \gamma}(x, y) - \partial^\beta \omega^\sigma(x) G_{\sigma \gamma}(x, y) +$$

$$+ G_{\alpha \sigma}(x, y) \partial_{\sigma} \omega^\gamma(y) + G_{\alpha \gamma}(x, y) \partial_{\sigma} \omega^\beta(y) - \partial^\gamma \left( G_{\alpha \beta}(x, y) \omega^\sigma(y) \right) -$$

$$- \int dz \frac{\delta G_{\alpha \beta \gamma}(x, y)}{\delta h_{\mu \nu}(z)} \delta \omega h_{\mu \nu}(z). \quad (3.11)$$

This transformations are again a far from the tensor transformation of type (1, 2) and of the product of tensors like $A_{\alpha}(x) B^{\beta \gamma}(y)$.

In the case of linear gauge fixing functions $\chi_{\alpha}$,

$$\frac{\delta G_{\alpha \beta \gamma}(x, y)}{\delta h_{\mu \nu}(z)} = 0, \quad (3.12)$$

the transformations (3.11) is simplified and we have
\[ \delta_\omega S_{gh} = \int dxdydz \sqrt{-\bar{g}(x)} \left[ (\delta_\omega \bar{C}^\alpha(x) + \omega^\sigma(x) \partial_\sigma \bar{C}^\alpha(x)) G^\beta_\alpha(x,y) R^\gamma_\beta_\gamma_\rho(y,z) C^\rho(z) + 
\right. \\
\left. + \bar{C}^\alpha(x) \omega^\rho(x) \partial^\rho_\beta G^\beta_\gamma_\alpha(x,y) R^\gamma_\beta_\gamma_\rho(y,z) C^\rho(z) + 
\right. \\
\left. + \bar{C}^\alpha(x) G^\beta_\gamma_\alpha(x,y) R^\gamma_\beta_\gamma_\sigma(y,z) \delta_\omega C^\sigma(z) - 
\right. \\
\left. - \bar{C}^\alpha(x) \omega^\rho(x) \partial^\rho_\beta G^\beta_\gamma_\alpha(x,y) R^\gamma_\beta_\gamma_\rho(y,z) C^\rho(z) - 
\right. \\
\left. - \bar{C}^\alpha(x) \partial_\alpha \omega^\rho(x) G^\beta_\rho_\beta_\gamma_\alpha(x,y) R^\gamma_\beta_\gamma_\sigma(y,z) C^\sigma(z) + 
\right. \\
\left. + \bar{C}^\alpha(x) G^\beta_\rho_\beta_\gamma_\alpha(x,y) \partial_\rho \omega^\gamma(y) R^\gamma_\beta_\gamma_\sigma(y,z) C^\sigma(z) + 
\right. \\
\left. + \bar{C}^\alpha(x) G^\beta_\gamma_\alpha(y,x) \partial_\gamma \omega^\rho(y) \partial^\rho_\beta R^\gamma_\beta_\gamma_\sigma(y,z) C^\sigma(z) - 
\right. \\
\left. - \bar{C}^\alpha(x) G^\beta_\gamma_\alpha(x,y) \partial_\beta \omega^\rho(y) \partial^\rho_\gamma R^\gamma_\beta_\gamma_\sigma(y,z) C^\sigma(z) - 
\right. \\
\left. - \bar{C}^\alpha(x) G^\beta_\gamma_\alpha(x,y) \partial_\gamma \omega^\rho(y) R^\gamma_\beta_\gamma_\lambda(y,z) \partial_\lambda \omega^\sigma(z) C^\sigma(z) + 
\right. \\
\left. + \bar{C}^\alpha(x) G^\beta_\gamma_\alpha(x,y) R^\gamma_\beta_\gamma_\sigma(y,z) \omega^\rho(z) \partial_\rho C^\sigma(z) \right]. 
\] (3.13)

Finally
\[ \delta_\omega S_{gh} = \int dxdydz \sqrt{-\bar{g}(x)} \left[ (\delta_\omega \bar{C}^\alpha(x) + \omega^\sigma(x) \partial_\sigma \bar{C}^\alpha(x)) - 
\right. \\
\left. - \bar{C}^\alpha \partial_\rho \omega^\rho(x)) G^\beta_\gamma_\alpha(x,y) R^\gamma_\beta_\gamma_\rho(y,z) C^\rho(z) + 
\right. \\
\left. + \bar{C}^\alpha(x) G^\beta_\gamma_\alpha(x,y) R^\gamma_\beta_\gamma_\sigma(y,z) (\delta_\omega C^\sigma(z) + \omega^\sigma(z) \partial_\rho C^\sigma(z) - \partial_\rho \omega^\sigma(z) C^\rho(z)) \right]. 
\] (3.14)

Choosing the tensor transformation law for the ghost fields \( \bar{C}^\alpha, C^\alpha \)
\[ \delta_\omega \bar{C}^\alpha(x) = -\omega^\sigma(x) \partial_\sigma \bar{C}^\alpha(x) + \bar{C}^\rho \partial_\rho \omega^\alpha(x), \] (3.15)
\[ \delta_\omega C^\alpha(x) = -\omega^\sigma(x) \partial_\sigma C^\alpha(x) + C^\rho \partial_\rho \omega^\alpha(x), \] (3.16)
we arrive at the invariance of the ghost action
\[ \delta_\omega S_{gh} = 0. \] (3.17)

Finally we conclude that the Faddeev-Popov action \( S_{FP} \),
\[ \delta_\omega S_{FP} = 0, \] (3.18)

is invariant under the background transformations of all fields \( \phi, \bar{g}, \)
\[ \delta_\omega (\bar{g})_{\mu\nu} = R_{\mu\nu\sigma}(\bar{g}) \omega^\sigma, \quad \delta_\omega h_{\mu\nu} = R_{\mu\nu\sigma}(h) \omega^\sigma, \] (3.19)
\[ \delta_\omega B^\alpha = -\omega^\sigma \partial_\sigma B^\alpha + B^\alpha \partial_\rho \omega^\alpha, \quad \delta_\omega \bar{C}^\alpha = -\omega^\sigma \partial_\sigma \bar{C}^\alpha + \bar{C}^\sigma \partial_\sigma \omega^\alpha, \] (3.20)
\[ \delta_\omega C^\rho = -\omega^\sigma \partial_\sigma C^\rho + \partial_\rho \omega^\sigma C^\alpha. \] (3.21)
As the consequence of (3.18) the gauge invariance of the vacuum functional follows

\[ \delta_\omega Z(\bar{g}) = 0. \tag{3.22} \]

The same statement is valid for background effective action \( \Gamma(\bar{g}) = \Gamma(\Phi = 0, \bar{g}) \)

\[ \delta_\omega \Gamma(\bar{g}) = 0. \tag{3.23} \]

We see that the gauge invariance for quantum gravity theories in the background field formalism can be achieved if the two essential propositions related to the transformation law for gauge fixing functions (3.6) and to the linearity of these functions. If the gauge fixing functions are not linear in quantum fields \( h_{\mu\nu} \),

\[ \frac{\delta^2 \chi_\alpha(x)}{\delta h_{\beta\gamma}(y) \delta h_{\mu\nu}(z)} \neq 0, \tag{3.24} \]

then the tensor transformations (3.19)-(3.21) cannot cancel the additional contribution (3.11) appearing in the variation of the ghost action \( S_{gh} \). Fulfilment or not fulfilment of these requirements is closely related to a choice of gauge fixing functions \( \chi_\alpha = \chi_\alpha(\bar{g}, h) \).

4 Special choice of gauge fixing condition

A standard choice of \( \chi_\alpha(\phi, \bar{g}) \) corresponding to the background field gauge condition [9] reads

\[ \chi_\alpha(\bar{g}, h) = -\bar{g}^{\mu\lambda}(a\nabla_\lambda h_{\mu\alpha} + b\nabla_\alpha h_{\mu\lambda}), \tag{4.1} \]

where \( \nabla_\sigma \) is the covariant derivative corresponding the external metric tensor \( \bar{g}_{\mu\nu} \) and \( a, b \) are constants. The popular de Donder gauge condition corresponds to the case when \( a = 1, b = -1/2 \).

The choice (4.1) corresponds to linear dependence on quantum fields \( h_{\mu\nu} \) so that we need to check the transformation law (3.6) only. The \( \chi_\alpha = \chi_\alpha(\bar{g}, h) \) are point functions of space-time coordinates \( x \), \( \chi_\alpha = \chi_\alpha(x) \), constructed with the help of second-rank tensor fields \( \bar{g}^{\mu\lambda} \) of type (2, 0) and third-rank tensor fields \( \nabla_\lambda h_{\mu\alpha} \) of type (0, 3) by contracting indices \( \mu, \lambda \). Therefore \( \chi_\alpha(x) \) (4.1) is the tensor field of type (0, 1) with transformation law (2.5) that confirms the transformation proposed (3.6). The same result can be obtained by explicit calculations of gauge variation of functions (4.1). We demonstrate this fact in the simplest case of a choice of \( \chi_\alpha \) when \( a = 0, b = -1 \) so that

\[ \chi_\alpha(\bar{g}, h) = \bar{g}^{\mu\lambda}\nabla_\alpha h_{\mu\lambda} = \nabla_\alpha(\bar{g}^{\mu\lambda} h_{\mu\lambda}) = \partial_\alpha(\bar{g}^{\mu\lambda} h_{\mu\lambda}). \tag{4.2} \]

Consider the gauge variation of (4.2)

\[ \delta_\omega \chi_\alpha(\bar{g}, h) = \partial_\alpha(\delta_\omega(\bar{g}^{\mu\lambda}) h_{\mu\lambda}) + \delta_\omega(\bar{g}^{\mu\lambda}) \partial_\alpha h_{\mu\lambda} + + \partial_\alpha(\bar{g}^{\mu\lambda} \delta_\omega h_{\mu\lambda} + \bar{g}^{\mu\lambda} \partial_\alpha(\delta_\omega h_{\mu\lambda}), \tag{4.3} \]
Finally we have the result
\[ \delta_{\omega} g^{\mu \lambda} = -\bar{g}^{\mu \alpha} (\delta_{\omega} \bar{g}_{\alpha \beta}) \bar{g}^{\beta \lambda} = -\omega^\sigma \partial_\sigma g^{\mu \lambda} + \bar{g}^{\sigma \lambda} \partial_\sigma \omega^\mu + \bar{g}^{\mu \sigma} \partial_\sigma \omega^\lambda, \] (4.4)
and \( \delta_{\omega} h_{\mu \lambda} \) is given in (3.3). The set of terms in (4.3) without derivatives of functions \( \omega^\sigma \) is
\[ -\omega^\sigma (h_{\mu \lambda} \partial_\alpha \partial_\sigma \bar{g}^{\mu \lambda} + \partial_\sigma \bar{g}^{\mu \lambda} \partial_\alpha h_{\mu \lambda} + \partial_\alpha \bar{g}^{\mu \lambda} \partial_\sigma h_{\mu \lambda} + \bar{g}^{\mu \lambda} \partial_\alpha \partial_\sigma h_{\mu \lambda}) = -\omega^\sigma \partial_\sigma \partial_\alpha (\bar{g}^{\mu \lambda} h_{\mu \lambda}) = -\omega^\sigma \partial_\sigma \chi_\alpha. \] (4.5)
As to terms containing the second derivatives of \( \omega \) in (4.3) we have
\[ \bar{g}^{\sigma \lambda} h_{\mu \lambda} \partial_\alpha \partial_\sigma \omega^\mu + \bar{g}^{\mu \sigma} h_{\mu \lambda} \partial_\alpha \partial_\sigma \omega^\lambda - \bar{g}^{\mu \lambda} h_{\mu \sigma} \partial_\alpha \partial_\lambda \omega^\sigma - \bar{g}^{\mu \lambda} h_{\sigma \lambda} \partial_\alpha \partial_\mu \omega^\sigma = 0. \] (4.6)
Collection of terms of the structure \( \partial \bar{g} \partial \omega \partial h \) in (4.3) reads
\[ -\partial_\sigma \bar{g}^{\mu \lambda} \partial_\alpha \omega^\mu h_{\mu \lambda} + \partial_\alpha \bar{g}^{\mu \sigma} \partial_\sigma \omega^\mu h_{\mu \lambda} + \partial_\alpha \bar{g}^{\sigma \lambda} \partial_\sigma \omega^\lambda h_{\mu \lambda} - \partial_\alpha \bar{g}^{\mu \lambda} \partial_\sigma \omega^\mu h_{\mu \sigma} - \partial_\alpha \bar{g}^{\mu \lambda} \partial_\sigma \omega^\sigma h_{\mu \lambda} = -\partial_\sigma \bar{g}^{\mu \lambda} \partial_\alpha \omega^\sigma h_{\mu \lambda}. \] (4.7)
In its turn the terms of the structure \( \bar{g} \partial \omega \partial h \) enter in (4.3) in the form
\[ \bar{g}^{\mu \sigma} \partial_\sigma \omega^\lambda \partial_\alpha h_{\mu \lambda} - \bar{g}^{\mu \lambda} \partial_\alpha \omega^\sigma h_{\mu \lambda} - \bar{g}^{\mu \lambda} \partial_\lambda \omega^\sigma \partial_\alpha h_{\mu \sigma} - \bar{g}^{\mu \lambda} \partial_\lambda \omega^\mu \partial_\alpha h_{\mu \sigma} = -\bar{g}^{\sigma \lambda} \partial_\alpha \omega^\sigma h_{\mu \lambda}. \] (4.8)
Finally we have the result
\[ \delta_{\omega} \chi_\alpha = -\omega^\sigma \partial_\sigma \chi_\alpha - \partial_\sigma \bar{g}^{\mu \lambda} \partial_\alpha \omega^\mu h_{\mu \lambda} - \bar{g}^{\sigma \lambda} \partial_\sigma \omega^\sigma h_{\mu \lambda} = -\omega^\sigma \partial_\sigma \chi_\alpha - \partial_\sigma \omega^\sigma \partial_\alpha (\bar{g}^{\mu \lambda} h_{\mu \lambda}) = -\omega^\sigma \partial_\sigma \chi_\alpha - \partial_\sigma \omega^\sigma \chi_\alpha, \] (4.9)
which confirms the transformations (3.6). In a similar way one can check the rightness of (3.6) for (4.1) when \( a \neq 0 \) but corresponding calculations look more complicated due to the covariant derivative \( \tilde{\nabla}_\lambda \) and we omit them.

5 Discussion

In the present paper we have considered the background field formalism for Quantum Gravity from point of view of choice of the gauge fixing condition. Application of this formalism to the Yang-Mills theories is very effective means in quantum region (among recent investigations see, for example, [9, 11]) because it allows to support gauge invariance on all stages of calculations. The quantum gravity theories look like as special type of gauge theories of Yang-Mills fields with closed gauge algebra and with structure coefficients independent on fields and therefore they can be quantized in the form of the Faddeev-Popov procedure. Then for all admissible choice of gauge condition both the vacuum functional and the background effective action on its extremals are gauge independent. The property of gauge invariance of the vacuum functional and the background effective action is more sensitive to the choice of gauges. It has been verified explicitly that the gauge invariance can be arrived at the fulfillment of two conditions: a) the linearity of gauge fixing functions with respect to quantum gravitational fields and b) the tensor transformations for gauge fixing functions under the background gauge transformations.
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