SPECTRAL DECOMPOSITION FORMULA AND
MOMENTS OF SYMMETRIC SQUARE L-FUNCTIONS

OLGA BALKANOVA

Abstract. We prove a spectral decomposition formula for averages of Zagier L-series in terms of moments of symmetric square L-functions associated to Maass and holomorphic cusp forms of levels 4, 16, 64.

1. Introduction

The aim of this paper is to prove a spectral decomposition formula for the average

\[ \sum_{l=1}^{\infty} \omega(l) \mathcal{L}_{n^2 - 4l^2}(s), \]

where \( \omega \) is a suitable test function and the L-series is defined as

\[ \mathcal{L}_n(s) = \frac{\zeta(2s)}{\zeta(s)} \sum_{q=1}^{\infty} \frac{b_q(n)}{q^s} \]

for \( \Re s > 1 \) and can be meromorphically continued to the whole complex plane, see [21, Proposition 3]. Here \( \zeta(s) \) denotes the Riemann zeta function and

\[ b_q(n) := \# \{ x \pmod{2q} : x^2 \equiv n \pmod{4q} \}. \]

The dual problem of investigating the average over \( n \)

\[ \sum_{n=1}^{\infty} \omega(n) \mathcal{L}_{n^2 - 4l^2}(s) \]

was studied in [3] in connection with the prime geodesic theorem. See also [4], [5], [7], [19], [20] for related results. Furthermore, sums of the form (1.2) appear in the explicit formulas for the first moments of symmetric square L-functions associated to holomorphic (see [21], [2]) or Maass (see [1]) cusp forms. These explicit formulas can serve

2010 Mathematics Subject Classification. Primary: 11F12; 11F30; 11M99.

Key words and phrases. L-functions; Gauss sums; Eisenstein series; Kuznetsov trace formula.
as a starting point for analyzing second moments of symmetric square $L$-functions. For example, if we take the first moment of Maass form symmetric square $L$-functions for $SL_2(\mathbb{Z})$ twisted by the Fourier coefficient $\rho_j(l^2)$

$$\sum_j \rho_j(l^2) L(s, \text{sym}^2 u_j),$$

multiply it by $\zeta(2s)l^{-s}$ and sum over $l$ from 1 to $\infty$, we obtain the second moment

$$\sum_j L(s, \text{sym}^2 u_j)^2.$$ 

Applying these manipulations to the explicit formula proved in [1], we discover expressions of the following form

$$(1.3) \quad \sum_{l=1}^{\infty} \sum_{n=1}^{\infty} \mathcal{L}_{n^2-4l^2}(s) f(n, l; s)$$

on the right-hand side of the explicit formula for the second moment. A possible approach to evaluating (1.3) consists in using spectral methods. However, spectral decomposition for the inner sum over $n$ results in a loop bringing us back to the moment we started from. To avoid this problem, we can change the order of summation and investigate the average over $l$ first. This is the main reason behind our interest in a spectral decomposition formula for (1.1).

Even though, the averages (1.1) and (1.2) look similar, there are some important differences in approach. Spectral decomposition of (1.2) is a relatively straightforward application of the Kuznetsov trace formula to the generalized Kloosterman sums, while in case of (1.1) our method relies heavily on various properties of Gauss sums. It is interesting to note that Gauss sums occur naturally in various papers on second moments of symmetric square $L$-functions, see [6], [11], [12]. In our case, we express the average (1.1) in terms of sums of a product of two Gauss sums. Evaluating these Gauss sums, we obtain sums of Kloosterman sums for $\Gamma_0(N)$ with $N = 4, 16, 64$ at various cusps and twisted by $\chi_4$ (non-trivial Dirichlet character modulo 4). Finally, applying the Kuznetsov trace formula we derive a spectral decomposition formula for (1.1) which contains moments of symmetric square $L$-functions for $\Gamma_0(N)$ with $N = 4, 16, 64$ twisted by Fourier coefficients at the cusps 0 and $\infty$.

To state the main results rigorously, we introduce the function

$$\psi(x) = \psi(x; n; s) = \frac{2}{\sqrt{\pi}} \left( \frac{x}{n} \right)^s \int_0^\infty \omega(y) \cos \left( \frac{2xy}{n} \right) dy$$
and denote by $\psi_H(x)$ and $\psi_D(x)$ the Bessel integral transforms of $\psi(x)$ appearing in the Kuznetsov trace formula, see (2.14) and (2.15). These transforms can be expressed in terms of the Gauss hypergeometric function as shown in Lemmas 4.1 and 4.2. Let $\omega \in C^\infty$ be a function of compact support on $[a_1, a_2]$ for some $0 < a_1 < a_2 < \infty$ and let $\hat{\omega}$ stand for its Mellin transform. It is also required to define the generalized divisor function

$$
\sigma_s(\chi; n) := \sum_{d|n} \chi(d) d^s.
$$

For a cusp \(a\) of $\Gamma_0(N)$, let us introduce the following notation

(1.5) \[ M_a(n, N, s) = M_{a}^{\text{hol}}(n, N, s) + M_{a}^{\text{disc}}(n, N, s), \]

where

$$
M_{a}^{\text{hol}}(n, N, s) := \sum_{k > 1 \atop k \text{ odd}} \psi_H(k) \Gamma(k) \sum_{f \in H_k(N, \chi_4)} \rho_{fa}(n) \overline{L(s, \text{sym}^2 f_\infty)},
$$

$$
M_{a}^{\text{disc}}(n, N, s) := \sum_{f \in H(N, \chi_4)} \frac{\psi_D(t_f)}{\cosh(\pi t_f)} \rho_{fa}(n) \overline{L(s, \text{sym}^2 f_\infty)}
$$

are the moments of symmetric square $L$-functions associated to holomorphic and Maass cusp forms of level $N$ with nebentypus $\chi_4$ twisted by the Fourier coefficient $\rho_{fa}(n)$ of $f$ at a cusp $a$. The bar over $L$-functions means complex conjugation and $f_\infty$ means that the $L$-functions are formed using the Fourier coefficients of $f$ around the cusp $\infty$.

**Theorem 1.1.** Assume that $n$ is even. For $0 < \Re{s} < 1$ the following explicit formula holds

$$
\sum_{l=1}^{\infty} \omega(l) \mathcal{L}_{n^2-4l^2}(s) = M_{\text{even}}^D(n, s) + M_{\text{even}}^C(n, s) + C(n, s)
$$

$$
- \frac{2^{1-s} \pi^{1/2-s-i}}{1-2^{-2s}} M_{\infty}(n^2/4, 4, s) + \frac{2\pi^{1/2-s}}{1-2^{-2s}} M_{0}(n^2/4, 4, s),
$$

where

(1.6) \[ M_{\text{even}}^D(n, s) = \frac{\hat{\omega}(1) \zeta(2s)}{L(\chi_4, 1+s)} \left[ n^{-2s} \sigma_s(\chi_4; n^2) + \sigma_{-s}(\chi_4; n^2) \right]. \]
\[ M^C(n, s) = \frac{\Gamma(s - 1/2)}{2^{s-1} \pi^{-s-1/2}} \left( \sigma_{s-1}(\chi_4; n^2) + \frac{\sigma_{1-s}(\chi_4; n^2)}{n^{2-2s}} \right) \]

\[
\times \frac{\zeta(2s - 1)}{L(\chi_4, 2 - s)} \left( \sin(\pi s/2) \int_0^{n/2} \omega(y) \left( \frac{n^2}{4} - y^2 \right)^{1/2-s} dy 
+ \cos(\pi s/2) \int_{n/2}^{\infty} \omega(y) \left( y^2 - \frac{n^2}{4} \right)^{1/2-s} dy \right),
\]

\[ \mathcal{E}(n, s) = \frac{L(\chi_4, s)}{4\pi^{s-1/2}} \frac{1}{2\pi i} \int_{-\infty}^{\infty} \psi_D(t) \sinh(\pi t) \frac{\zeta(s + 2it)\zeta(s - 2it)}{L(\chi_4, 1 + 2it)L(\chi_4, 1 - 2it)} dt. \]

**Theorem 1.2.** Assume that \( n \) is odd. For \( 0 < \Re s < 1 \) the following explicit formula holds

\[
\sum_{l=1}^{\infty} \omega(l) \mathcal{L}_{n^2 - 4l^2}(s) = M^D_{\text{odd}}(n, s) + \frac{1}{2} M^C(n, s) + \frac{1}{2} \mathcal{E}(n, s)
+ \frac{8\pi^{1/2-s}}{1 - 2^{2-2s}} \mathcal{M}_0(n^2, 64, s) + \frac{4\pi^{1/2-s}}{1 + 2^{-s}} \mathcal{M}_0(n^2, 16, s),
\]

where

\[ M^D_{\text{odd}}(n, s) = \frac{\tilde{\omega}(1)\zeta(2s)}{L(\chi_4, 1 + s)} \sigma_{-s}(\chi_4; n^2), \]

\( M^C(n, s) \) is defined by (1.7) and \( \mathcal{E}(n, s) \) by (1.8).

**Remark 1.3.** Note that the main terms

\[ M^D_{\text{even}}(n, s) + M^C(n, s), \quad M^D_{\text{odd}}(n, s) + \frac{1}{2} M^C(n, s) \]

are holomorphic at the central point \( s = 1/2 \). See Section 8.2 for details.

The paper is organized as follows. In Section 2 we collect all required tools and preliminary results. In Section 3, assuming that \( \Re s \) is sufficiently large, we isolate the diagonal and non-diagonal terms for (1.1), compute the diagonal term explicitly, and prove an expression for the non-diagonal term which is suitable for application of the Kuznetsov trace formula. Section 4 is devoted to the analysis of the Bessel integral transforms \( \psi_H(x) \) and \( \psi_D(x) \) appearing after the Kuznetsov trace formula is applied. More precisely, we show how to express \( \psi_H(x) \) and \( \psi_D(x) \) in terms of the Gauss hypergeometric functions. Sections 5 and 6 are concerned with evaluation of the continuous spectrum, while the
holomorphic and discrete spectra are studied in Section 7. Finally, in Section 8 we complete the proof of Theorems 1.1 and 1.2 and compute the main terms at the central point.

2. Preliminaries

2.1. Generalized divisor function. In this subsection we collect various results related to the function $\sigma_s(\chi; n)$ defined by (1.4).

Note that the derivative of $\sigma_s(\chi; n)$ with respect to $s$ is equal to

$$\sigma'_s(\chi; n) = \sum_{d|n} \chi(d) d^s \log d.$$ 

Let $\chi_4$ be a non-trivial Dirichlet character modulo 4, so that $\chi_4(1) = 1$, $\chi_4(3) = -1$, and

$$\sigma_s\left(\chi_4; \left(\frac{n}{2}\right)^2\right) = \sigma_s(\chi_4; n^2).$$

Lemma 2.1. For odd $n$ the following identity holds

$$\sigma_{1/2-u}(\chi_4; n^2) = n^{1-2u} \sigma_{-1/2+u}(\chi_4; n^2).$$

Proof. Let $n^2 = bd$. Since $n$ is odd

$$1 = \chi_4(n^2) = \chi_4(bd).$$

Thus $\chi_4(b) = \chi_4(d)$. Consequently,

$$\sigma_{1/2-u}(\chi_4; n^2) = \sum_{d|n^2} d^{1/2-u} \chi_4(d) = \sum_{bd=n^2} \left(\frac{n^2}{b}\right)^{1/2-u} \chi_4(d)$$

$$= n^{1-2u} \sum_{bd=n^2} b^{-1/2+u} \chi_4(b) = n^{1-2u} \sigma_{-1/2+u}(\chi_4; n^2).$$

Consider the Dirichlet series:

$$Z(z, s) := \sum_{n=1}^{\infty} \frac{\sigma_s(\chi_4; n^2)}{n^z}.$$ 

Lemma 2.2. We have

$$Z(z, s) = \frac{1 - 2^{2s-z} L(\chi_4, z - s) \zeta(z) \zeta(z - 2s)}{1 - 2^{2s-2z} \zeta(2z - 2s)}.$$
Proof. First, assume that $\Re z > 1 + 2\Re s$. The Euler product for $Z(z, s)$ is equal to

(2.4) \[ Z(z, s) = \prod_p \left( 1 + \frac{\sigma_s(\chi_4; p^2)}{p^z} + \frac{\sigma_s(\chi_4; p^4)}{p^{2z}} + \ldots \right) \]

\[ = \left( 1 + \sum_{k=1}^{\infty} \frac{\sigma_s(\chi_4; 2^{2k})}{2^{kz}} \right) \prod_{p > 2} \left( 1 + \sum_{k=1}^{\infty} \frac{\sigma_s(\chi_4; p^{2k})}{p^{kz}} \right). \]

Note that $\chi_4(d) = 0$ for even $d$, and therefore,

\[ \sigma_s(\chi_4; 2^{2k}) = \sum_{d|2^{2k}} d'' \chi_4(d) = 1. \]

Consequently,

(2.5) \[ 1 + \sum_{k=1}^{\infty} \frac{\sigma_s(\chi_4; 2^{2k})}{2^{kz}} = \frac{1}{1 - 2^{-z}}. \]

Next, we evaluate the second multiple on the right hand side of (2.4). Since $p^2 \equiv 1 \pmod{4}$ we have

$\chi_4(p^{2m}) = 1$, $\chi_4(p^{2m+1}) = \chi_4(p)$ for any $m \in \mathbb{N}$.

Therefore,

$\sigma_s(\chi_4; p^{2k}) = \frac{(p^{2s})^{k+1} - 1}{p^{2s} - 1} + \frac{(p^{2s})^k - 1}{p^{2s} - 1} \chi_4(p)p^s.$

This implies that

\[ \sum_{k=1}^{\infty} \frac{\sigma_s(\chi_4; p^{2k})}{p^{kz}} = \frac{p^{2s} + \chi_4(p)p^s}{(p^{2s} - 1)(p^{z-2s} - 1)} - \frac{1 + \chi_4(p)p^s}{(p^{2s} - 1)(p^z - 1)}. \]

Using the property $\chi_4^2(p) = 1$ we infer

(2.6) \[ 1 + \sum_{k=1}^{\infty} \frac{\sigma_s(\chi_4; p^{2k})}{p^{kz}} = \frac{1 + \chi_4(p)/p^{z-s}}{(1 - 1/p^{z-2s})(1 - 1/p^z)} \]

\[ = \frac{1 - 1/p^{2(z-s)}}{(1 - 1/p^{z-2s})(1 - 1/p^z)(1 - \chi_4(p)/p^{z-s})}. \]

Finally, substituting (2.5) and (2.6) in (2.4) we prove the lemma. \[ \square \]
2.2. Cusps and Kloosterman sums. For a positive integer \( N \), let \( \Gamma = \Gamma_0(N) \) denote the Hecke congruence subgroup of level \( N \).

The stabilizer of the cusp \( a \) in \( \Gamma \) is defined by
\[
\Gamma_a := \{ \gamma \in \Gamma : \gamma a = a \}.
\]

A scaling matrix for the cusp \( a \) is a matrix \( \sigma_a \in \text{SL}_2(\mathbb{R}) \) such that
\[
\sigma_a \infty = a, \quad \sigma_a^{-1} \Gamma_a \sigma_a = \{ \pm \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} : n \in \mathbb{Z} \} := B.
\]

Note that the choice of scaling matrix is not unique.

Let \( \chi \) be a Dirichlet character modulo \( N \). This can be extended to \( \Gamma \) as follows:
\[
\chi(\gamma) = \chi(d), \quad \gamma = \begin{pmatrix} a & b \\ cN & d \end{pmatrix} \in \Gamma.
\]

Let \( \lambda_a \) be defined by \( \sigma_a^{-1} \lambda_a \sigma_a = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \). The cusp \( a \) is called singular for \( \chi \) if \( \chi(\lambda_a) = 1 \).

Suppose that \( N = rs, (r, s) = 1 \). Then a cusp of the form \( a = 1/r \) is called an Atkin-Lehner cusp. Note that Atkin-Lehner cusps are singular with respect to any Dirichlet character modulo \( N \), see [13, page 395].

Let \( \kappa \) be defined by \( \chi(-1) = (-1)^\kappa \) and let \( a, b \) be two singular cusps for \( \chi \) with corresponding scaling matrices \( \sigma_a, \sigma_b \).

Similarly to [13, Eq. 2.3], we define the Kloosterman sum associated to \( a, b \) as
\[
S_{ab}(m, n; c; \chi) := \sum_{\gamma = (a b) \in \Gamma_\infty \setminus \sigma_a^{-1} \Gamma_a \sigma_b / \Gamma_\infty} \chi(\text{sgn}(c)) \overline{\chi(\sigma_a \gamma \sigma_b^{-1})} e\left(\frac{am + dn}{c}\right).
\]

The set of allowed moduli is given by
\[
C_{a,b}(N) = \{ \gamma > 0 \text{ such that } \begin{pmatrix} * & * \\ c & d \end{pmatrix} \in \sigma_a^{-1} \Gamma \sigma_b \}.
\]

2.3. Holomorphic and Maass cusp forms. Let \( H_k(N, \chi) \) be an orthonormal basis of holomorphic cusp forms of weight \( k > 0 \), \( k \equiv \kappa \) (mod 2), level \( N \) and nebentypus \( \chi \). The Fourier expansion of \( f \in H_k(N, \chi) \) around a singular cusp \( a \) with a scaling matrix \( \sigma_a \) is given by
\[
f(\sigma_a z) = \sum_{m \geq 1} \frac{\rho_f(m)}{\sqrt{m}} (4\pi m)^{k/2} e(mz),
\]
where \( i(\sigma_a, z) := cz + d \) for \( \sigma_a = \begin{pmatrix} * & * \\ c & d \end{pmatrix} \).
Let $H(N, \chi)$ be an orthonormal basis of the space of Maass cusp forms of weight $\kappa \in \{0, 1\}$. For the function $f \in H(N, \chi)$ (which is an eigenfunction of the Laplace-Beltrami operator with eigenvalue $1/4 + t_f^2$), the following Fourier-Whittaker expansion holds around the cusp $\alpha$ with scaling matrix $\sigma_\alpha$

$$f(\sigma_\alpha z) e^{-\kappa \arg(\sigma_\alpha z)} = \sum_{m \neq 0} \frac{\rho_{f_\alpha}(m)}{m} W_{\frac{\kappa}{2}, m} (4\pi |m| y) e(mx),$$

where $z = x + iy$ and the Whittaker function $W_{\lambda, \mu}(z)$ is defined in [10, Section 9.22].

For $f \in H_k(N, \chi)$ or $f \in H(N, \chi)$, we define

$$(2.8) \quad L(s, \text{sym}^2 f_\infty) = \zeta(N)(2s) \sum_{l=1}^{\infty} \frac{\rho_{f_\infty}(l^2)}{l^s}, \quad \Re s > 1,$$

where the superscript in $\zeta(N)(2s)$ means that Euler factors at primes dividing $N$ have been removed. Shimura [18] proved an analytic continuation and a functional equation for (2.8).

### 2.4. Eisenstein series

Fix $\kappa = 1$. For $\Gamma = \Gamma_0(N)$ the Eisenstein series associated to a singular cusp $c$ for the nebentypus $\chi$ is defined as

$$E_c(z, s) := \sum_{\gamma \in \Gamma_c \setminus \Gamma} \chi(\gamma) j_{\sigma_c^{-1}\gamma}(z)^{-1} \left( \Im(\sigma_c^{-1}\gamma z) \right)^s,$$

where $\sigma_c$ is a scaling matrix for $c$ and

$$j_\gamma(z) := \frac{cz + d}{|cz + d|} = e^{i\arg(cz + d)}, \quad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$
\[
\phi_{a,c}(m, s, \chi) = \sum_{\gamma = \left( \begin{array}{rr} a & b \\ c & d \end{array} \right) \in \Gamma_\infty \setminus \sigma_a^{-1} \Gamma \sigma_a / \Gamma_\infty} \overline{\chi} (\sigma_c \gamma \sigma_a^{-1}) \frac{e(m d/c)}{c^{2s}} \\
= \sum_{c \in C, a(N)} S_{a}(0, m; c; \chi) \frac{1}{c^{2s}}.
\]

**Proof.** Consider the Eisenstein series

\[
E_c(\sigma_a z, s) = \sum_{\gamma \in \Gamma_1 \setminus \Gamma} \overline{\chi}(\gamma) j_{\gamma^{-1}}(\sigma_a z)^{-1} (\Im(\sigma_c \gamma \sigma_a^{-1} z))^{s}.
\]

Making the change of variables \( \tau := \sigma_c^{-1} \gamma \sigma_a \) (so that \( \tau \in B \setminus \sigma_c^{-1} \Gamma \sigma_a \)) we infer

\[
E_c(\sigma_a z, s) = \sum_{\tau \in B \setminus \sigma_c^{-1} \Gamma \sigma_a} \overline{\chi}(\sigma_c \tau \sigma_a^{-1}) j_{\tau \sigma_a^{-1}}(\sigma_a z)^{-1} (\Im(\tau z))^{s}.
\]

Using the property

\[
j_{\tau \sigma_a^{-1}}(\sigma_a z)^{-1} j_{\sigma_a}(z)^{-1} = j_{\tau}(z)^{-1}
\]

we find that

\[
E_c(\sigma_a z, s) j_{\sigma_a}(z)^{-1} = \sum_{\tau \in B \setminus \sigma_c^{-1} \Gamma \sigma_a} \overline{\chi}(\sigma_c \tau \sigma_a^{-1}) j_{\tau}(z)^{-1} (\Im(\tau z))^{s}.
\]

Using the property

\[
j_{\tau \sigma_a^{-1}}(\sigma_a z)^{-1} j_{\sigma_a}(z)^{-1} = j_{\tau}(z)^{-1}
\]

we find that

\[
E_c(\sigma_a z, s) l_{\sigma_a}(z)^{-1} = \sum_{\tau \in B \setminus \sigma_c^{-1} \Gamma \sigma_a} \overline{\chi}(\sigma_c \tau \sigma_a^{-1}) l_{\tau}(z)^{-1} (\Im(\tau z))^{s}.
\]

Note that \( \overline{\chi}(\sigma_c \gamma \sigma_a^{-1}) = \overline{\chi}(\sigma_c \gamma \sigma_a^{-1}) \) since \( a \) is singular. Furthermore, taking \( \gamma = \left( \begin{array}{rr} a & b \\ c & d \end{array} \right) \) and \( \tau = \left( \begin{array}{rr} 1 & n \\ 0 & 1 \end{array} \right) \), we obtain

\[
\gamma \tau z = \frac{a}{c} - \frac{1}{c(c(z + n) + d)}, \quad l_{\gamma \tau} = \frac{cz + cn + d}{|cz + cn + d|}.
\]

Consequently, for \( z = x + iy \) we have

\[
\Im(\gamma \tau z) = \frac{y}{c^2 (x + n + d/c)^2 + y^2}
\]

and

\[
E_c(\sigma_a z, s) l_{\sigma_a}(z)^{-1} = \delta_{ac} y^s + \sum_{\gamma \in B \setminus \sigma_c^{-1} \Gamma \sigma_a / B} \overline{\chi}(\sigma_c \gamma \sigma_a^{-1})
\]

\[
\times \sum_{n \in \mathbb{Z}} \left( \frac{cz + cn + d}{|cz + cn + d|} \right)^{-1} \left( \frac{y}{c^2 (x + n + d/c)^2 + y^2} \right)^s.
\]
In order to evaluate the sum over $n$ we apply the Poisson summation formula, showing that

$$
\sum_{n \in \mathbb{Z}} \left( \frac{cz + cn + d}{|cz + cn + d|} \right)^{-1} \left( \frac{y}{c^2 (x + n + d/c)^2 + y^2} \right)^s
= \sum_{m \in \mathbb{Z}} \int_{-\infty}^{\infty} \left( \frac{c(z + v) + d}{|c(z + v) + d|} \right)^{-1} \frac{(yc^{-2})^s e(-mv)}{((x + d/c + v) + y^2)^s} dv.
$$

Making the change of variables $t := x + d/c + v$, this is equal to

$$
\sum_{m \in \mathbb{Z}} e \left( mx + \frac{md}{c} \right) \int_{-\infty}^{\infty} \left( \frac{t + iy}{t^2 + y^2} \right)^{-1} \left( \frac{yc^{-2}}{t^2 + y^2} \right)^s e(-mt) dt.
$$

Let us assume first that $m = 0$. Then

$$
\int_{-\infty}^{\infty} \frac{|t + iy| (yc^{-2})^s}{t + iy (t^2 + y^2)^s} dt = \frac{1}{i} \int_{-\infty}^{\infty} \frac{(yc^{-2})^s dt}{(y + it)^{s-1/2} (y - it)^{s+1/2}} = \frac{2\pi (2y)^{1-2s} \Gamma(2s)}{(2s - 1) \Gamma(s - 1/2) \Gamma(s + 1/2)} = -\sqrt{\pi} i \Gamma(s) \frac{y^{1-s}}{\Gamma(s + 1/2) c^{2s}},
$$

where we used [10] Eq. 8.381.1 to evaluate the integral. If $m \neq 0$ the integral in (2.12) can be computed using [10] 3.384.9 as follows

$$
\frac{(yc^{-2})^s}{i} \int_{-\infty}^{\infty} (y + it)^{1/2-s} (y - it)^{-1/2-s} e(-mt) dt
= \frac{(2\pi)^{s-2-s} |m|^{s-1}}{ic^{2s} \Gamma(1/2 + s)} \frac{W_{|m| 2^{1/2-s}} (4\pi y |m|).}
$$

Consequently, (2.12) is equal to

$$
\sum_{m \neq 0} e \left( mx + \frac{md}{c} \right) \pi^{1/2} i \Gamma(s) \frac{y^{1-s}}{\Gamma(s + 1/2) c^{2s}}
+ \frac{\pi^s}{i \Gamma(1/2 + s) c^{2s}} \sum_{m \neq 0} |m|^{s-1} e(mx + md/c)W_{|m| 2^{1/2-s}} (4\pi y |m|).
$$

The statement follows by replacing the sum over $n$ in (2.11) by (2.13). □

2.5. Kuznetsov trace formula. In this section we state the Kuznetsov trace formula for Dirichlet multiplier system and general cusps. To this end, we follow [8], [13] Section 3.3 and [9] Section 4.1.3 assuming that $\kappa = 1$ (i.e. $\chi(1) = -1$).

Consider the function $\psi \in C^\infty$ such that

$$
\psi(0) = \psi'(0) = 0, \quad \psi^{(j)}(x) \ll (1 + x)^{-2-\eta}, \quad j = 0, 1, 2, 3.
$$
for some $\eta > 0$. It is also required to introduce the following transforms:

\begin{equation}
\psi_H(k) := 4i^k \int_0^\infty J_{k-1}(x) \psi(x) \frac{dx}{x},
\end{equation}

\begin{equation}
\psi_D(t) := \frac{2\pi it}{\sinh(\pi t)} \int_0^\infty (J_{2it}(x) + J_{-2it}(x)) \psi(x) \frac{dx}{x},
\end{equation}

where $J_\alpha$ denotes the $J$-Bessel function of order $\alpha$.

For $m, n \geq 1$

\begin{equation}
\sum_{c \in C_{a,b}(N)} S_{ab}(m, n; c; \chi) \sum_{\rho} \psi_H(k) \Gamma(k) \rho_f(m) \rho_f(n),
\end{equation}

\begin{equation}
H := \sum_{k>1 \ (\text{mod} \ 2)} \sum_{f \in H_k(N, \chi)} \psi_H(k) \Gamma(k) \rho_f(m) \rho_f(n),
\end{equation}

\begin{equation}
D := \sum_{f \in H(N, \chi)} \psi_D(t_f) \cosh(\pi t_f) \rho_f(m) \rho_f(n),
\end{equation}

\begin{equation}
C := \sum_{c \in \text{sing.}} \frac{\sqrt{mn}}{4\pi} \int_{-\infty}^{\infty} \psi_D(t) \cosh(\pi t) \rho_{a,c}(m, 1/2 + it) \rho_{b,c}(n, 1/2 + it) dt.
\end{equation}

According to (2.9) the continuous part can be written as

\begin{equation}
C = \sum_{c \in \text{sing.}} \frac{1}{4\pi} \int_{-\infty}^{\infty} \psi_D(t) \frac{\pi}{\cosh(\pi t) \Gamma(1 + it)^2} m^{-it} n^{-it} \times \phi_{a,c}(m, 1/2 + it, \chi) \phi_{b,c}(n, 1/2 + it, \chi) dt.
\end{equation}

Using the identity (see [17, Eq. 5.4.3])

\[ \Gamma(1 + it) \Gamma(1 - it) = \frac{\pi t}{\sinh(\pi t)} \]

we find that

\begin{equation}
C = \sum_{c \in \text{sing.}} \frac{1}{4\pi} \int_{-\infty}^{\infty} \psi_D(t) \frac{\sinh(\pi t)}{t \cosh(\pi t)} m^{-ut} n^{-ut} \times \phi_{a,c}(m, 1/2 + it, \chi) \phi_{b,c}(n, 1/2 + it, \chi) dt.
\end{equation}
2.6. **Gauss sums.** For a Dirichlet character \( \chi \) modulo \( q \), we define the Gauss sum of \( \chi \) by

\[
(2.18) \quad g(\chi; q; m) := \sum_{u \pmod{q}, (u,q)=1} \chi(u) e \left( \frac{mu}{q} \right), \quad \tau(\chi) := g(\chi; q; 1).
\]

Let \( \chi \) be a character modulo \( q \) induced from a primitive character \( \chi^* \) modulo \( q^* \). Then according to [16, Lemma 3.1.3 (2)] we have

\[
(2.19) \quad g(\chi; q; m) = \tau(\chi^*) \sum_{d \mid (m,q/q^*)} d\chi^* \left( \frac{q}{q^*d} \right) \chi^* \left( \frac{m}{d} \right) \mu \left( \frac{q}{q^*d} \right).
\]

The generalized quadratic Gauss sums is given by

\[
G(a, n; q) := \sum_{x \pmod{q}, (a,q)=1} e \left( \frac{ax^2 + nx}{q} \right), \quad (a, q) = 1.
\]

Let \( G(a; q) := G(a, 0; q) \).

The notation \( \overline{a}_q \) means that \( \overline{a}_q a \equiv 1 \pmod{q} \).

**Lemma 2.4.** For \((a, q) = 1\) the following identity holds

\[
G^2(a; q) = \begin{cases} 
q\chi_4(q), & q \text{ is odd} \\
0, & q \equiv 2 \pmod{4} \\
2qi\chi_4(a), & q \equiv 0 \pmod{4}.
\end{cases}
\]

**Proof.** If \( q \) is odd we have by \([15, \text{Eq. 23}]\)

\[
G^2(a; q) = \left( \left( \frac{a}{q} \right) i^{(\frac{a-1}{2})^2} \sqrt{q} \right)^2 = q(-1)^{(\frac{a-1}{2})^2} = q\chi_4(q).
\]

If \( q \equiv 2 \pmod{4} \), then using \([15, \text{Eq. 25}]\) we find that \( G(a; q) = 0 \). Finally, if \( q \equiv 0 \pmod{4} \), then \([15, \text{Eq. 25}]\) implies that

\[
G^2(a; q) = q(1 + ia)^2 = 2qi^2 = 2qi\chi_4(a).
\]

\(\square\)

**Lemma 2.5.** If \( n \) is even and \( q \) is odd, then

\[
(2.20) \quad G(a, n; q) = e \left( -\frac{\overline{n}(n/2)^2}{q} \right) G(a; q) = e \left( -\frac{(4a)_q n^2}{q} \right) G(a; q).
\]

**Proof.** See \([15, \text{Eq. (26)}]\).

\(\square\)

**Lemma 2.6.** Suppose that \( n \) and \( q \) are even.

If \( q \equiv 2 \pmod{4} \), then \( G(a, n; q) = 0 \).
If \( q \equiv 0 \pmod{4} \), then

\[
G(a, n; q) = e \left( -\frac{\overline{\alpha}_q(n/2)^2}{q} \right) G(a; q).
\]

Proof. The relation (2.21) follows from [15, Eq. (26)] for any even \( n \) and \( q \). However, \( G(a; q) = 0 \) for \( q \equiv 2 \pmod{4} \) by [15, Eq. (25)]. \( \square \)

**Lemma 2.7.** Suppose that \( n \) and \( q \) are odd. Then

\[
G(a, n; q) = e \left( -\frac{(4a)_q n^2}{q} \right) G(a; q).
\]

Proof. In this case according to [15, Eq. (27)] we have

\[
G(a, n; q) = e \left( -\frac{\overline{\alpha}_q (n+q/2)^2}{q} \right) G(a; q).
\]

Note that it follows from the relation

\[
\overline{\alpha}_q n^2 \equiv (n + q)^2 \pmod{q}
\]

that

\[
\overline{(4a)_q n^2} \equiv \overline{\alpha}_q \left( \frac{n + q}{2} \right)^2 \pmod{q}.
\]

This implies the statement. \( \square \)

**Lemma 2.8.** Suppose that \( n \) is odd and \( q \) is even. If \( q \equiv 0 \pmod{4} \), then \( G(a, n; q) = 0 \). Otherwise, we can write \( q = 2r \) with \( r \) odd.

\[
G(a, n; q) = 2e \left( -\frac{(8a)_r n^2}{r} \right) G(2a; r).
\]

Proof. If \( n \) is odd and \( q \equiv 0 \pmod{4} \), then \( G(a, n; q) = 0 \) by [15, Eq. (28)]. Assume that \( q = 2r \) with \( r \) odd. Using the twisted multiplicativity of Gauss sums, we find

\[
G(a, n; 2r) = G(a \overline{\alpha}_r, n \overline{\alpha}_r; r) G(a \overline{\alpha}_2, n \overline{\alpha}_2; 2).
\]

By direct computations \( G(a \overline{\alpha}_2, n \overline{\alpha}_2; 2) = 2 \). Finally,

\[
G(a \overline{\alpha}_r, n \overline{\alpha}_r; r) = G(2a, n; r) = e \left( -\frac{(8a)_r n^2}{r} \right) G(2a; r),
\]

where we used (2.22) to evaluate \( G(2a, n; r) \). \( \square \)
3. DIAGONAL AND NON-DIAGONAL TERMS

Assuming that $s$ is sufficiently large, we prove in this section an explicit formula for

$$\sum_{l=1}^{\infty} \omega(l) \mathcal{L}_{n^2-4l^2}(s)$$

with diagonal and non-diagonal terms. Applying the results of Section 2.6, we compute the diagonal term explicitly and prove an expression for the non-diagonal term in terms of sums of Kloosterman sums suitable for application of the Kuznetsov trace formula.

Lemma 3.1. For $\Re s > 3/2$ the following formula holds

$$\sum_{l=1}^{\infty} \omega(l) \mathcal{L}_{n^2-4l^2}(s) =$$

$$\tilde{\omega}(1) \zeta(2s) \sum_{q=1}^{\infty} \frac{1}{q^{2+s}} \sum_{c,d \pmod{q}} S(d^2, c^2; q)e \left( \frac{nc}{q} \right)$$

$$+ \frac{\zeta(2s)}{\pi^{s-1/2}} \sum_{l=1}^{\infty} \frac{1}{l^s} \sum_{q=1}^{\infty} \frac{f(\omega, s; 4\pi nl/q)}{q^2} \sum_{c,d \pmod{q}} S(d^2, c^2; q)e \left( \frac{nc + ld}{q} \right),$$

where $S(d, c; q)$ is the ordinary Kloosterman sum and for $a < 1$

$$f(\omega, s; x) := \frac{1}{2\pi i} \int_{(a)} \tilde{\omega}(\alpha) \frac{\Gamma(1/2 - \alpha/2)}{\Gamma(\alpha/2)} \left( \frac{x}{4n} \right)^{\alpha-s-1} d\alpha.$$

Proof. Using the Mellin transform for the function $\omega$ we have

$$\sum_{l=1}^{\infty} \omega(l) \mathcal{L}_{n^2-4l^2}(s) = \frac{1}{2\pi i} \int_{(a)} \tilde{\omega}(\alpha) \sum_{l=1}^{\infty} \mathcal{L}_{n^2-4l^2}(s ) l^\alpha d\alpha,$$

where $a > 1$. According to [2, Eq. 4-9], the function $\mathcal{L}_{n^2-4l^2}(s)$ can be written in terms of sums of Kloosterman sums

$$\mathcal{L}_{n^2-4l^2}(s) = \zeta(2s) \sum_{q=1}^{\infty} \frac{1}{q^{1+s}} \sum_{c \pmod{q}} S(l^2, c^2; q)e \left( \frac{nc}{q} \right).$$

Substituting this expression into (3.3) we can change the order of summation, summing with respect to $q$ first and over $l$ second, as long as $\Re s > 3/2$. Furthermore, dividing the range of summation for $l$ into
arithmetic progressions, we obtain

\begin{align*}
(3.4) \quad \sum_{l=1}^\infty \omega(l) \mathcal{L}_{n^2-d^2}(s) &= \frac{\zeta(2s)}{2\pi i} \int_{(\alpha)} \tilde{\omega}(\alpha) \sum_{q=1}^\infty \frac{1}{q^{1+s}} \\
&\quad \times \sum_{c,d \pmod{q}} S(d^2, c^2; q) e \left( \frac{nc}{q} \right) \sum_{l \geq 1 \pmod{d}} \frac{1}{l^{\alpha}} d\alpha.
\end{align*}

The inner sum on the right-hand side of (3.4) can be written in terms of the Lerch zeta function \(\zeta(a, b; \alpha)\) with the following functional equation

\begin{align*}
(3.5) \quad \sum_{l \geq 1 \pmod{d}} \frac{1}{l^{\alpha}} &= \frac{1}{q^{\alpha}} \sum_{l=1}^\infty \frac{1}{(l+d/q)^{\alpha}} = \frac{\zeta(d/q, 0; \alpha)}{q^{\alpha}} = \\
&= \frac{\Gamma(1-\alpha)}{q^{\alpha}(2\pi)^{1-\alpha}} \left[ -ie(\alpha/4) \sum_{l=1}^\infty \frac{e(ld/q)}{l^{1-\alpha}} + i(-\alpha/4) \sum_{l=1}^\infty \frac{e(-ld/q)}{l^{1-\alpha}} \right].
\end{align*}

In order to apply this functional equation, we move the contour of integration in (3.4) to \(\Re \alpha < 0\), crossing a simple pole of \(\zeta(d/q, 0; \alpha)\) at \(\alpha = 1\). The contribution of this pole is equal to

\begin{align*}
(3.6) \quad \tilde{\omega}(1) \zeta(2s) \sum_{q=1}^\infty \frac{1}{q^{2+s}} \sum_{c,d \pmod{q}} S(d^2, c^2; q) e \left( \frac{nc}{q} \right).
\end{align*}

Next, applying the functional equation (3.5) and using the fact that

\[ i e(-\alpha/4) - i e(\alpha/4) = 2 \sin \left( \frac{\pi \alpha}{2} \right), \]

we find that (3.4) is equal to (3.6) plus

\begin{align*}
(3.7) \quad \frac{\zeta(2s)}{2\pi i} \int_{(\alpha)} \tilde{\omega}(\alpha) \sum_{q=1}^\infty \frac{1}{q^{1+s+\alpha}} \sum_{l=1}^\infty \frac{1}{l^{1-\alpha}} \\
&\quad \times \sum_{c,d \pmod{q}} S(d^2, c^2; q) e \left( \frac{nc + ld}{q} \right) \frac{2\Gamma(1-\alpha) \sin(\pi \alpha/2)}{(2\pi)^{1-\alpha}} d\alpha, \quad \alpha < 0.
\end{align*}

It follows from [17, Eqs. 5.5.5, 5.5.3] that

\[ \Gamma(1-\alpha) \sin(\pi \alpha/2) = \pi^{1/2-\alpha} \frac{\Gamma(1/2 - \alpha/2)}{\Gamma(\alpha/2)}. \]

Substituting this into (3.7), we prove the lemma. \(\square\)
3.1. The inner sum. In order to evaluate the inner sum in (3.1) we express it as a sum of a product of two Gauss sums:

\( K(n, l; q) := \sum_{c,d \pmod q} S(c^2, d^2, q) e \left( \frac{nc + ld}{q} \right) \)

\[ = \sum_{a,b \pmod q} \sum_{c,d \pmod q} e \left( \frac{ac^2 + db^2}{q} \right) e \left( \frac{nc + ld}{q} \right) \]

\[ = \sum_{a,b \pmod q \atop ab \equiv 1} G(a, n; q) G(b, l; q). \]

It is required to examine different cases depending on the even-odd parity of the parameters \( n, l, \) and \( q. \) To this end, we use the results of Section 2.6.

**Lemma 3.2.** Suppose that \( q \) is odd. Then

\[ K(n, l; q) = q \chi_4(q) \sum_{a,b \pmod q \atop ab \equiv 1} e \left( -\frac{4qbn^2 + 4qal^2}{q} \right). \]

**Proof.** There are four different cases to consider depending on the parity of \( n \) and \( l. \) Assume first that \( n \) and \( l \) are even. Using (3.8) and (2.20), we have

\[ K(n, l; q) = \sum_{a,b \pmod q \atop ab \equiv 1} e \left( -\frac{4qbn^2 + 4qal^2}{q} \right) G(a; q) G(b; q). \]

Note that \( G(a; q) = G(b; q) \) since \( ab \equiv 1 \pmod q. \) Then Lemma 2.4 implies the statement when \( n \) and \( l \) are even. Other three cases can be treated similarly by evaluating \( G(a, n; q) \) using (2.20) if \( n \) is even and (2.22) if \( n \) is odd. \( \square \)

**Lemma 3.3.** Suppose that \( q \) is even and \( n + l \) is odd. Then

\[ K(n, l; q) = 0. \]

**Proof.** According to (3.8) we have

\[ K(n, l; q) = \sum_{a,b \pmod q \atop ab \equiv 1} G(a, n; q) G(b, l; q), \]

where one of the parameters \( n \) and \( l \) is even and another one is odd. Without loss of generality, we can assume that \( n \) is even and \( l \) is
odd. Then Lemma 2.6 implies that $G(a, n; q)$ is nonzero only if $q \equiv 0 \pmod{4}$, but in that case $G(b, l; q) = 0$ by Lemma 2.8.

Lemma 3.4. Suppose that $q$, $n$ and $l$ are even. Then $K(n, l; q) = 0$ if $q \equiv 2 \pmod{4}$, and otherwise

(3.9) $K(n, l; q) = 2i q \sum_{a, b \equiv 1 \pmod{q}} \chi_4(a) e\left(-\frac{a(l/2)^2 + b(n/2)^2}{q}\right)$.

Proof. It follows directly from (3.8) and Lemma 2.6 that $K(n, l; q) \neq 0$ only if $q \equiv 0 \pmod{4}$. In that case, (3.9) is a consequence of Lemmas 2.6 and 2.4.

Lemma 3.5. Suppose that $q$ is even and $n$, $l$ are odd. Then $K(n, l; q) = 0$ if $q \equiv 0 \pmod{4}$. If $q \equiv 2 \pmod{4}$, then $r := q/2$ is odd and

$K(n, l; q) = 2q \chi_4(r) \sum_{a, b \equiv 1 \pmod{q}} e\left(-\frac{(8a)_r n^2 + (8b)_r l^2}{r}\right)$,

where $r := q/2$ is odd. This is equal to

$K(n, l; q) = 2q \chi_4(r) S\left((8)_r r^2 n^2, (8)_r r^2 l^2; r\right)$.

Writing $q = 2r$ and applying the multiplicity property of Kloosterman sums, we infer

$K(n, l; q) = 2q \chi_4(r) S\left((8)_r r^2 n^2, (8)_r r^2 l^2; q\right)$.

Then the assertion follows by using the fact that

$S\left(2(8)_r r^2 n^2, 2(8)_r r^2 l^2; 2\right) = 1$.

3.2. The diagonal term. Now we are ready to evaluate the diagonal main term in (3.1), namely

(3.10) $M^D(n, s) := \hat{\omega}(1) \zeta(2s) \sum_{q=1}^{\infty} \frac{1}{q^{2+s}} K(n, 0; q)$.

We use the subscripts “even” and “odd” in $M^D_{even}(n, s)$ and $M^D_{odd}(n, s)$ to mark the parity of $n$. 
Lemma 3.6. If $n$ is even, then

\begin{equation}
M_{\text{even}}^D(n,s) = \frac{\hat{\omega}(1)\zeta(2s)}{L(\chi_4, 1+s)} \left( n^{-2s} \sigma_s(\chi_4;n^2) + \sigma_{-s}(\chi_4;n^2) \right) .
\end{equation}

If $n$ is odd, then

\begin{equation}
M_{\text{odd}}^D(n,s) = \frac{\hat{\omega}(1)\zeta(2s)}{L(\chi_4, 1+s)} \sigma_{-s}(\chi_4;n^2) .
\end{equation}

Proof. Assume first that $n$ is even. We can split the sum over $q$ in (3.10) as follows

\begin{equation}
\sum_{q=1}^{\infty} = \sum_{q \equiv 0 \pmod{2}} + \sum_{q \equiv 1 \pmod{2}} .
\end{equation}

For $n$ and $q$ even, we have $K(n,0;q) = 0$ unless $q \equiv 0 \pmod{4}$. If $q \equiv 0 \pmod{4}$ the following identity holds $\chi_4(a) = \chi_4(b)$ for $ab \equiv 1 \pmod{q}$. Then by Lemma 3.4

\begin{equation}
\frac{K(n,0;q)}{2iq} = \sum_{b \pmod{q}} \chi_4(b)e \left( -\frac{b(n/2)^2}{q} \right) = \chi_4(-1)g(\chi_4;q,(n/2)^2) ,
\end{equation}

where the star over the sum above means that we are summing over $b \pmod{q}$ such that $(b,q) = 1$. Consequently, using (2.19) we find that

\begin{equation}
\sum_{q \equiv 0 \pmod{2}} \frac{K(n,0;q)}{q^{2+s}} = 2i\chi_4(-1) \sum_{q \equiv 0 \pmod{4}} \frac{g(\chi_4;q,(n/2)^2)}{q^{1+s}}
\end{equation}

\begin{equation}
= \frac{2i\chi_4(-1)}{4^{1+s}} \tau(\chi_4) \sum_{q=1}^{\infty} \frac{1}{q^{1+s}} \sum_{d\mid((n/2)^2,q)} d\chi_4 \left( \frac{q}{d} \right) \chi_4 \left( \frac{(n/2)^2}{d} \right) \mu \left( \frac{q}{d} \right) .
\end{equation}

Computing the sum over $q$, we obtain

\begin{equation}
\sum_{q \equiv 0 \pmod{2}} \frac{K(n,0;q)}{q^{2+s}} = \frac{2i\chi_4(-1)\tau(\chi_4)}{4^{1+s}L(\chi_4, 1+s)} \sum_{d\mid(n/2)^2} d^{-s} \chi_4 \left( \frac{(n/2)^2}{d} \right)
\end{equation}

\begin{equation}
= \frac{2i\chi_4(-1)\tau(\chi_4)}{4^{1+s}L(\chi_4, 1+s)} \left( \frac{n}{2} \right)^{-2s} \sigma_s(\chi_4;(n/2)^2) = \frac{\sigma_s(\chi_4;(n/2)^2)}{L(\chi_4, 1+s)} n^{-2s} .
\end{equation}

Now consider the sum over odd $q$ in (3.13). Applying Lemma 3.2 to compute $K(n,0;q)$ and making the change of variables $-\frac{1}{4}q \rightarrow b$, we
evaluate the second sum

\[ \sum_{q \equiv 1 \pmod{2}} \frac{K(n, 0; q)}{q^{2+s}} = \sum_{q=1}^{\infty} \frac{\chi_4(q)}{q^{1+s}} \sum_{b \pmod{q}} e \left( \frac{bn^2}{q} \right) = \sum_{q=1}^{\infty} \frac{\chi_4(q)}{q^{1+s}} \sum_{d|q, n^2} d \mu \left( \frac{q}{d} \right) = \sum_{d|n^2} d \sum_{q \equiv 0 \pmod{d}} \frac{\chi_4(q) \mu(q/d)}{q^{1+s}} = \frac{\sigma_s(\chi_4; n^2)}{L(\chi_4, 1+s)}. \]

Combining (3.15) and (3.14) and using the fact that \(\sigma_s(\chi_4; (n/2)^2) = \sigma_s(\chi_4; n^2)\), we prove (3.11).

Similarly, for odd \(n\) the identity (3.12) follows from

\[ \sum_{q=1}^{\infty} \frac{K(n, 0; q)}{q^{2+s}} = \frac{\sigma_s(\chi_4; n^2)}{L(\chi_4, 1+s)}. \]

\[ \square \]

3.3. The non-diagonal term. In this subsection, we study the non-diagonal term in (3.1), namely

\[ M_{\text{ND}} := \frac{\zeta(2s)}{\pi^{s-1/2}} \sum_{l=1}^{\infty} \frac{1}{l^s} \sum_{q=1}^{\infty} \frac{f(\omega, s; 4\pi nl/q)}{q^2} K(n, l; q). \]

For simplicity, let \(\psi(x) := f(\omega, s; 4x)\). Consider the following two Kloosterman sums (see, for example, [13 Eqs. (2.20), (2.23)]):

\[ S_{\infty,0}(m, n; c\sqrt{N}; \chi) = \overline{\chi}(c) S(Nm, n; c), \quad (c, N) = 1, \]

\[ S_{\infty,\infty}(m, n; c; \chi) = \sum_{ab \equiv 1 \pmod{c}} e \left( \frac{am + bn}{c} \right) \overline{\chi}(b). \]

Using (2.7) and [13 Eq. 2.15] we find that

\[ C_{\infty,\infty}(4) = \{ \gamma = q > 0, \quad q \equiv 0 \pmod{4} \}, \]

\[ C_{\infty,0}(4) = \{ \gamma = 2q > 0, (q, 4) = 1 \}, \]

\[ C_{\infty,\infty}(16) = \{ \gamma = 4q > 0, \quad (q, 2) = 1 \}, \]

\[ C_{\infty,0}(64) = \{ \gamma = 8s > 0, \quad (s, 2) = 1 \}. \]
Lemma 3.7. If $n$ is even (let $n_1 := n/2$), then

\[ \begin{align*}
M^{ND} &= -\frac{2i\zeta(2s)}{2^s\pi^{s-1/2}} \sum_{l=1}^{\infty} \frac{1}{l^s} \sum_{\gamma \in C_{\infty,\infty}(4)} \frac{1}{\gamma} \psi \left( \frac{4\pi ln_1}{\gamma} \right) S_{\infty \infty} \left( l^2, n_1^2; \frac{\gamma}{4}; \chi_4 \right) \\
&\quad + \frac{2\zeta(2s)}{\pi^{s-1/2}} \sum_{l=1}^{\infty} \frac{1}{l^s} \sum_{\gamma \in C_{\infty,0}(4)} \frac{1}{\gamma} \psi \left( \frac{4\pi ln_1}{\gamma} \right) S_{\infty 0} \left( l^2, n_1^2; \frac{\gamma}{4}; \chi_4 \right).
\end{align*} \]

If $n$ is odd, then

\[ \begin{align*}
M^{ND} &= \frac{8\zeta(2s)}{\pi^{s-1/2}} \sum_{l=1}^{\infty} \frac{1}{l^s} \sum_{\gamma \in C_{\infty,0}(64)} \frac{1}{\gamma} \psi \left( \frac{4\pi ln_1}{\gamma} \right) S_{\infty 0} \left( l^2, n_1^2; \frac{\gamma}{4}; \chi_4 \right) \\
&\quad + (1 - 2^{-s}) \frac{4\zeta(2s)}{\pi^{s-1/2}} \sum_{l=1}^{\infty} \frac{1}{l^s} \sum_{\gamma \in C_{\infty,0}(16)} \frac{1}{\gamma} \psi \left( \frac{4\pi ln_1}{\gamma} \right) S_{\infty 0} \left( l^2, n_1^2; \frac{\gamma}{4}; \chi_4 \right).
\end{align*} \]

Proof. First, assume that $n$ is even. Then applying Lemmas 3.2, 3.3, 3.4 we infer

\[ \begin{align*}
M^{ND} &= \frac{\zeta(2s)}{\pi^{s-1/2}} \sum_{l=0}^{\infty} \frac{1}{l^s} \sum_{\gamma \equiv 0 \pmod{2}} \frac{1}{\gamma} \psi \left( \frac{4\pi ln_1}{\gamma} \right) S_{\infty \infty} \left( l^2, n_1^2; \frac{\gamma}{4}; \chi_4 \right) \\
&\quad \times \sum_{a,b \equiv 1 \pmod{q}} \chi_4(a) e \left( -\frac{a(l/2)^2 + b(n/2)^2}{q} \right) \\
&\quad + \frac{\zeta(2s)}{\pi^{s-1/2}} \sum_{l=1}^{\infty} \frac{1}{l^s} \sum_{\gamma \equiv 0 \pmod{4}} \frac{1}{\gamma} \psi \left( \frac{4\pi ln_1}{\gamma} \right) S_{\infty 0} \left( l^2, n_1^2; \frac{\gamma}{4}; \chi_4 \right) \\
&\quad \times \sum_{a,b \equiv 1 \pmod{q}} \chi_4(q) f \left( \frac{4\pi ln_1}{q} \right) e \left( \frac{\overline{a}bn^2 + \overline{a}al^2}{q} \right).
\end{align*} \]

Since $\chi_4(a) = \chi_4(b)$ for $q \equiv 0 \pmod{4}$, the first summand in (3.19) is equal to

\[ \begin{align*}
-\frac{2i\zeta(2s)}{\pi^{s-1/2}} \sum_{l=0}^{\infty} \frac{1}{l^s} \sum_{\gamma \in C_{\infty,\infty}(4)} \frac{1}{\gamma} \psi \left( \frac{\pi ln_1}{\gamma} \right) S_{\infty \infty} \left( l^2, n_1^2; \frac{\gamma}{4}; \chi_4 \right) \\
&\quad \times \sum_{a,b \equiv 1 \pmod{q}} \chi_4(a) e \left( -\frac{a(l/2)^2 + b(n/2)^2}{q} \right) \\
&\quad + \frac{\zeta(2s)}{\pi^{s-1/2}} \sum_{l=1}^{\infty} \frac{1}{l^s} \sum_{\gamma \equiv 0 \pmod{4}} \frac{1}{\gamma} \psi \left( \frac{4\pi ln_1}{\gamma} \right) S_{\infty 0} \left( l^2, n_1^2; \frac{\gamma}{4}; \chi_4 \right) \\
&\quad \times \sum_{a,b \equiv 1 \pmod{q}} \chi_4(q) f \left( \frac{4\pi ln_1}{q} \right) e \left( \frac{\overline{a}bn^2 + \overline{a}al^2}{q} \right).
\end{align*} \]

Using (3.16), we obtain

\[ \begin{align*}
\chi_4(q) \sum_{a,b \equiv 1 \pmod{q}} e \left( \frac{\overline{a}bn^2 + \overline{a}al^2}{q} \right) &= S_{\infty 0} \left( l^2, n_1^2; q\sqrt{4}; \chi_4 \right).
\end{align*} \]
Therefore, the second summand in (3.19) is equal to
\[
\frac{2\zeta(2s)}{\pi^{s-1/2}} \sum_{l=1}^{\infty} l^s \sum_{\gamma \in C_{\infty,0}(4)} \frac{1}{\gamma} \psi \left( \frac{4\pi l n_1}{\gamma} \right) S_{\infty 0}(l^2, n_1^2; \gamma; \chi_4).
\]

This completes the proof of (3.17).

Now we assume that \( n \) is odd. Using Lemmas 3.2, 3.3, 3.5 we obtain

\[
M^{ND} = \frac{\zeta(2s)}{\pi^{s-1/2}} \sum_{l(2)=1}^{\infty} \frac{1}{l^s} \sum_{q \equiv 2 \pmod 4} \frac{2\chi_4(q/2)}{q} f \left( \frac{4\pi n l}{q} \right) \times \sum_{a,b \equiv 1 \pmod r} e \left( \frac{aS_r n^2 + bS_r l^2}{r} \right)
\]

\[+ \frac{\zeta(2s)}{\pi^{s-1/2}} \sum_{l=1}^{\infty} \frac{1}{l^s} \sum_{q=1}^{\infty} \chi_4(q) f \left( \frac{4\pi n l}{q} \right) \sum_{a,b \equiv 1 \pmod q} e \left( \frac{\overline{\chi}_4 b n^2 + \overline{\chi}_4 a l^2}{q} \right),
\]

where \( r = q/2 \). Note that \( \chi_4 \) can be extended to \( \chi_{16} \) as follows
\[
\chi_{16}(q) = \begin{cases} 
\chi_4(q) & \text{if } (q, 16) = 1 \\
0 & \text{otherwise.}
\end{cases}
\]

Consequently, \( \chi_4(q) = \chi_{16}(q) \) for all \( (q, 2) = 1 \). Therefore, the second summand in (3.20) is equal to

\[
\frac{4\zeta(2s)}{\pi^{s-1/2}} \sum_{l=1}^{\infty} \frac{1}{l^s} \sum_{\gamma \in C_{\infty,0}(16)} \frac{1}{\gamma} \psi \left( \frac{4\pi l n}{\gamma} \right) S_{\infty 0}(l^2, n^2; \gamma, \chi_4).
\]

In order to evaluate the first summand in (3.20), we split the sum over \( l \) as

\[
\sum_{l(2)=1}^{\infty} = \sum_{l=1}^{\infty} - \sum_{l \equiv 0 \pmod 2},
\]

which yields

\[
\frac{8\zeta(2s)}{\pi^{s-1/2}} \sum_{l=1}^{\infty} \frac{1}{l^s} \sum_{\gamma \in C_{\infty,0}(64)} \frac{1}{\gamma} \psi \left( \frac{4\pi l n}{\gamma} \right) S_{\infty 0}(l^2, n^2; \gamma, \chi_4)
\]

\[= - \frac{4\zeta(2s)}{2\pi^{s-1/2}} \sum_{l=1}^{\infty} \frac{1}{l^s} \sum_{\gamma \in C_{\infty,0}(16)} \frac{1}{\gamma} \psi \left( \frac{4\pi l n}{\gamma} \right) S_{\infty 0}(l^2, n^2; \gamma, \chi_4).
\]

This completes the proof.
4. Special functions

In this section, we study the Bessel transforms $\psi_H(k)$ and $\psi_D(t)$ defined by (2.14) and (2.15) respectively. Our goal is to prove integral representations for these functions in terms of the Gauss hypergeometric function.

First, recall that in our case $\kappa = 1$ and $k \equiv \kappa \pmod{2}$. Therefore, we can write $k = 2m + 1$ for $m \in \mathbb{N}$ and

\begin{align*}
\psi_H(k) &= 4i^{2m+1} \int_0^\infty J_{2m}(x)\psi(x)\frac{dx}{x}, \\
\psi_D(t) &= \frac{2\pi it}{\sinh(\pi t)} \int_0^\infty (J_{2it}(x) + J_{-2it}(x))\psi(x)\frac{dx}{x},
\end{align*}

where for $a < 0$ (see (3.2))

$$\psi(x) = f(\omega, s; 4x) = \frac{1}{2\pi i} \int_{(a)} \frac{\Gamma(1/2 - \alpha/2)}{\Gamma(\alpha/2)} \hat{\omega}(\alpha) \left(\frac{x}{n}\right)^{\alpha+s-1} d\alpha.$$ 

The function $\psi(x)$ has another integral representation:

\begin{equation}
\psi(x) = \frac{2}{\sqrt{\pi}} \left(\frac{x}{n}\right)^s \int_0^\infty \omega(y) \cos \left(\frac{2xy}{n}\right) dy,
\end{equation}

see [3, Eq. (1.7)] and [3, p. 1984] for the proof.

**Lemma 4.1.** The following identity holds

\begin{align*}
\psi_H(k) &= 8i^{2m+1} \sqrt{\pi n} s \int_0^{n/2} \omega(y) F \left( \frac{s}{2} + m, \frac{s}{2} - m; \frac{1}{2}; \left(\frac{2y}{n}\right)^2 \right) dy + \frac{\Gamma(m + s/2 + 1/2)}{\Gamma(2m + 1)} \times \\
&\quad \cos \left(\frac{\pi s}{2}\right) \frac{n^{2m}}{2^{2m}} \int_{n/2}^\infty \omega(y) F \left( m + \frac{s}{2}, m + \frac{s + 1}{2}; 2m + 1; \left(\frac{n}{2y}\right)^2 \right) dy.
\end{align*}

**Proof.** Substituting (4.3) into (4.1) we have

$$\psi_H(k) = \frac{8i^{2m+1}}{\sqrt{\pi n} s} \int_0^\infty \omega(y) \int_0^\infty J_{2m}(x)x^{s-1} \cos \left(\frac{2y}{n}\right) dxdy.$$ 

The outer integral over $y$ can be split in two parts: $\int_0^\infty = \int_0^{n/2} + \int_{n/2}^\infty$. For each of these parts we evaluate the inner integral over $x$. When
If $2y/n < 1$ we apply \[10\] 6.699 (2) and Euler’s reflection formula, so that

$$
\int_0^\infty J_{2m}(x)x^{s-1}\cos\left(x\frac{2y}{n}\right)dx = \frac{2^{s-1}(-1)^m\sin(\pi s/2)}{\pi} \\
\times \Gamma(s/2 + m)\Gamma(s/2 - m)F\left(\frac{s}{2} + m, \frac{s}{2} - m; \frac{1}{2}; \left(\frac{2y}{n}\right)^2\right).
$$

This gives the first summand in (4.4). The second summand is obtained similarly using \[10\] 6.699 (2) and the fact that $2y/n \geq 1$. □

**Lemma 4.2.** The following identity holds

(4.5) \[\psi_D(t) = \frac{2\pi it}{\sinh(\pi t)}(h_1(t) + h_1(-t) + h_2(t)),\]

where

(4.6) \[h_1(t) = \frac{\cos(\pi(s/2 + it))\Gamma(s/2 + it)\Gamma(s/2 + 1/2 + it)}{\pi \Gamma(1 + 2it)} \\
\times \int_{n/2}^\infty \omega(y) \left(\frac{2y}{n}\right)^{-2it} F\left(\frac{s}{2} + it, \frac{s + 1}{2} + it; 1 + 2it; \left(\frac{n}{2y}\right)^2\right)dy,
\]

(4.7) \[h_2(t) = \frac{2\cosh(\pi t)\sin(\pi s/2)\Gamma(s/2 + it)\Gamma(s/2 - it)}{\pi \Gamma(1/2)} \\
\times \int_0^{n/2} \omega(y) F\left(\frac{s}{2} + it, \frac{s}{2} - it; \frac{1}{2}; \left(\frac{2y}{n}\right)^2\right)dy.
\]

**Proof.** Substituting (4.3) into (4.2) we show that

$$
\psi_D(t) = \frac{2\pi it}{\sinh(\pi t)}(h_1(t) + h_1(-t) + h_2(t)),
$$

where

$$
h_1(t) = \frac{2}{\sqrt{\pi} n^s} \int_{n/2}^\infty \omega(y) \int_0^\infty J_{2it}(x)x^{s-1}\cos\left(x\frac{2y}{n}\right)dx\,dy,
$$

$$
h_2(t) = \frac{2}{\sqrt{\pi} n^s} \int_0^{n/2} \omega(y) \int_0^\infty (J_{2it}(x) + J_{-2it}(x))x^{s-1}\cos\left(x\frac{2y}{n}\right)dx\,dy.
$$

Evaluating the inner integrals with respect to $x$ in $h_1(t)$ and $h_2(t)$ using \[10\] Eq. 6.699 (2), we complete the proof. □
Lemma 4.3. We have

\( h_1 \left( \frac{1 - s}{2i} \right) = 0, \)

\( h_1 \left( \frac{s - 1}{2i} \right) = \frac{\Gamma(s - 1/2) \sin(\pi s)}{\pi (n/2)^{1-s}} \int_{n/2}^{\infty} \omega(y)(y^2 - n^2/4)^{1/2-s} dy, \)

\( h_2 \left( \frac{1 - s}{2i} \right) = h_2 \left( \frac{s - 1}{2i} \right) \)
\[ = \frac{2}{\pi} \sin^2(\pi s/2) \frac{\Gamma(s - 1/2)}{(n/2)^{1-s}} \int_0^{n/2} \omega(y)(n^2/4 - y^2)^{1/2-s} dy. \]

Proof. The identity (4.8) follows directly from (4.7). In order to prove (4.9) and (4.10), we apply [17, Eq. 15.4.6]. Accordingly,

\[ F \left( s - \frac{1}{2}, s, s; \left( \frac{n}{2y} \right)^2 \right) = \frac{(y^2 - n^2/4)^{1/2-s}}{y^{1-2s}}, \]

\[ F \left( \frac{1}{2}, s - \frac{1}{2}, \frac{1}{2}, \frac{2y}{n}; \left( \frac{2y}{n} \right)^2 \right) = \frac{(n^2/4 - y^2)^{1/2-s}}{(n/2)^{1-2s}}. \]

The statement follows. \( \square \)

Corollary 4.4. The following identity holds

\( \psi_D \left( \frac{1-s}{2i} \right) \frac{\sinh \left( \frac{\pi - s}{2i} \right)}{(1 - s) \cosh \left( \frac{\pi - s}{2i} \right)} = \)
\[ 2\frac{\Gamma(s - 1/2)}{(n/2)^{1-s}} \left( \sin(\pi s/2) \int_0^{n/2} \omega(y) \left( \frac{n^2}{4} - y^2 \right)^{1/2-s} dy \right. \]
\[ \left. + \cos(\pi s/2) \int_{n/2}^{\infty} \omega(y) \left( y^2 - \frac{n^2}{4} \right)^{1/2-s} dy \right). \]

Proof. This is a consequence of Lemma 4.3 and (4.5). \( \square \)

5. Fourier coefficients of Eisenstein series

As a preliminary step towards understanding the continuous spectrum arising from the Kuznetsov trace formula, we compute explicitly some Fourier coefficients of Eisenstein series for the groups \( \Gamma_0(4), \Gamma_0(16) \) and \( \Gamma_0(64) \). To this end, it is required to determine a list of singular cusps for the considered groups and to compute various characters appearing as a part of Fourier coefficients.
5.1. **Singular cusps for $\Gamma_0(4)$, $\Gamma_0(16)$ and $\Gamma_0(64)$**. Let $\sigma_a$ be a scaling matrix for a cusp $a$ and let $\lambda_a$ be defined by $\sigma_a^{-1}\lambda_a\sigma_a = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.

Recall that $a$ is called singular for $\chi$ if $\chi(\lambda_a) = 1$. It follows from [13, Proposition 3.3] that if $c = 1/\omega$ is a cusp of $\Gamma = \Gamma_0(N)$ and $N = (N,\omega)N'$, $\omega = (N,\omega)'$, $N' = (N',\omega)N''$, then we have

$$\lambda_{1/\omega} = \begin{pmatrix} 1 - \omega N'' & N'' \\ -\omega^2 N'' & 1 + \omega N'' \end{pmatrix}.$$  

**Lemma 5.1.** The following cusps are singular for $\Gamma_0(4)$:

$0, \infty$.

The following cusps are singular for $\Gamma_0(16)$:

$0, 1/2, 1/4, 1/8, 1/12, \infty$.

The following cusps are singular for $\Gamma_0(64)$:

$0, 1/2, 1/4, 1/8, 1/12, 1/16, 1/24, 1/32, 1/40, 1/48, 1/56, \infty$.

**Proof.** According to [13, Corollary 3.2], a complete set of representatives for the set of inequivalent cusps of $\Gamma = \Gamma_0(N)$ is given by $\frac{1}{w} = \frac{1}{uf}$, where $f$ runs over divisors of $N$ and $u$ runs modulo $(f,N/f)$ such that $u$ is coprime to $(f,N/f)$, where $u$ is chosen so that $(u,N) = 1$ after adding a suitable multiple of $(f,N/f)$.

Consequently, the group $\Gamma_0(4)$ has the following inequivalent cusps:

$$\begin{array}{c}
\frac{1}{1} \sim 0, \frac{1}{2}, \frac{1}{4} \sim \infty.
\end{array}$$

For the group $\Gamma_0(16)$ there are six inequivalent cusps:

$$\begin{array}{c}
\frac{1}{1} \sim 0, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{12}, \frac{1}{16} \sim \infty,
\end{array}$$

and for the group $\Gamma_0(64)$ twelve:

$$\begin{array}{c}
\frac{1}{1} \sim 0, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{12}, \frac{1}{16}, \frac{1}{24}, \frac{1}{32}, \frac{1}{40}, \frac{1}{48}, \frac{1}{56}, \frac{1}{64} \sim \infty.
\end{array}$$

In order to check which of these cusps are singular we use (5.1). Accordingly, the cusp is singular if

$$\chi_4(1 + wN'') = 1.$$ 

Verifying this condition, we find that for $\Gamma_0(16)$ and $\Gamma_0(64)$ all the cusps listed above are singular. For $\Gamma_0(4)$ the cusps $0, \infty$ are singular, but $1/2$ is not because in this case $\chi_4(1 + wN'') = \chi_4(3) = -1$. \hfill $\square$
5.2. Computations with characters.

Lemma 5.2. Let $c = 1/w$ be any cusp and $a = 1/r$ be an Atkin-Lehner cusp of $\Gamma_0(N)$. Let us choose the scaling matrices as follows:

$$
\sigma_c = \left( \begin{array}{cc} 1 & 0 \\ w & 1 \end{array} \right) \left( \begin{array}{cc} \sqrt{N''} & 0 \\ 0 & 1/\sqrt{N''} \end{array} \right),
$$

$$
\sigma_a = \left( \begin{array}{cc} 1 & (ss - 1)/r \\ r & ss \end{array} \right) \left( \begin{array}{cc} \sqrt{s} & 0 \\ 0 & 1/\sqrt{s} \end{array} \right).
$$

Then

$$
\sigma_c^{-1} \Gamma \sigma_a = \left\{ \begin{array}{l} \left( \begin{array}{cc} \frac{A}{N''} \sqrt{N''} & \frac{B}{N''} \sqrt{\frac{N''}{s}} \\ C \sqrt{N''} & D \sqrt{\frac{N''}{s}} \end{array} \right) : C \equiv -wA \pmod{r} \\ D \equiv -wB \pmod{s} \\ AD - BC = 1 \end{array} \right\}
$$

and for $\rho \in \sigma_c^{-1} \Gamma \sigma_a$ we have

$$
\chi_4(\sigma_c \rho \sigma_a^{-1}) = \chi_4 \left( (wA + C) \frac{1 - ss}{r} + wB + D \right).
$$

In particular,

$$
\chi_4(\sigma_c \rho \sigma_a^{-1}) = \chi_4(wA + C),
$$

$$
\chi_4(\sigma_c \rho \sigma_0^{-1}) = \chi_4(wA + C).
$$

Proof. For the proof of (5.2) see [13, Lemma 3.5]. By direct computations we find that

$$
\sigma_c \rho \sigma_a^{-1} = \left( \begin{array}{cc} * & * \\ * & (wA + C) \frac{1 - ss}{r} + wB + D \end{array} \right).
$$

This implies (5.3). If $a = \infty$, this formula can be simplified further by noting that in this case $r = N$, $s = 1$, and therefore,

$$
\frac{1 - ss}{r} = 0.
$$

Similarly, for $a = \infty$, we have $r = 1$, $s = N$ and

$$
\frac{1 - ss}{r} = 1 - N\overline{N} = 1 - N.
$$

Then using the relation $wB + D \equiv 0 \pmod{N}$ in (5.2) we prove (5.5). \qed

Corollary 5.3. For the group $\Gamma_0(4)$ we have

$$
\chi_4(\sigma_\infty \rho \sigma_\infty^{-1}) = \chi_4(\rho), \quad \chi_4(\sigma_0 \rho \sigma_\infty^{-1}) = \chi_4(-C),
$$

$$
\chi_4(\sigma_\infty \rho \sigma_0^{-1}) = \chi_4(C), \quad \chi_4(\sigma_0 \rho \sigma_0^{-1}) = \chi_4(D).
$$
Proof. If \( a = c = \infty \), then \( w = N = 4 \) and by (5.4) we have \( \chi_4(\sigma c_0 \rho \sigma_a^{-1}) = \chi_4(D) \).

If \( a = \infty, c = 0 \), then \( w = 1 \) and by (5.4) we have \( \chi_4(\sigma c_0 \rho \sigma_a^{-1}) = \chi_4(B + D) \). According to (5.2)
\[
\begin{align*}
AD - BC &= 1 \\
C &\equiv -A \pmod{N}.
\end{align*}
\]

Therefore, \( A(B + D) \equiv 1 \pmod{N} \) and \( B + D \equiv -A \pmod{N} \). Consequently, \( \chi_4(\sigma c_0 \rho \sigma_a^{-1}) = \chi_4(-C) = \chi_4(-C) \).

If \( a = 0, c = \infty \), then \( w = N = 4 \) and by (5.5) we have \( \chi_4(\sigma c_0 \rho \sigma_a^{-1}) = \chi_4(B + D) \). According to (5.2)
\[
\begin{align*}
AD - BC &= 1 \\
C &\equiv -A \pmod{N}.
\end{align*}
\]

Therefore, \( A(B + D) \equiv 1 \pmod{N} \) and \( \chi_4(\sigma c_0 \rho \sigma_a^{-1}) = \chi_4(D) \).

Corollary 5.4. For the groups \( \Gamma_0(16) \) and \( \Gamma_0(64) \) we have
\[
\chi_4(\sigma_0 \rho \sigma_0^{-1}) = \chi_4(D), \quad \chi_4(\sigma_0 \rho \sigma_0^{-1}) = \chi_4(-C),
\]
\[
\chi_4(\sigma_1/2 \rho \sigma_0^{-1}) = \chi_4(-C/2), \quad \chi_4(\sigma_1/2 \rho \sigma_0^{-1}) = \chi_4(D/2).
\]

For all other singular cusps \( c \) of \( \Gamma_0(16) \) and \( \Gamma_0(64) \) the following holds
\[
\chi_4(\sigma c_0 \rho \sigma_a^{-1}) = \chi_4(D), \quad \chi_4(\sigma c_0 \rho \sigma_a^{-1}) = \chi_4(C).
\]

Proof. Note that if \( w \equiv 0 \pmod{4} \), then
\[
\chi_4(\sigma c_0 \rho \sigma_a^{-1}) = \chi_4(D), \quad \chi_4(\sigma c_0 \rho \sigma_a^{-1}) = \chi_4(C).
\]
This is the case for all singular cusps of \( \Gamma_0(16) \) and \( \Gamma_0(64) \) except \( c = 1/1 \sim 0 \) and \( c = 1/2 \). For \( c = 1/1 \sim 0 \) all computations are exactly the same as in Corollary 5.3.

Consider \( c = 1/2 \). It follows from (5.4) that
\[
\chi_4(\sigma_1/2 \rho \sigma_0^{-1}) = \chi_4(2B + D).
\]

According to (5.2)
\[
\begin{align*}
AD - BC &= 1 \\
C &\equiv -2A \pmod{N},
\end{align*}
\]
which implies that \( A(D + 2B) \equiv 1 \pmod{N} \) and \( \chi_4(\sigma_{1/2}^0 \sigma_{-1}^{-1}) = \chi_4(A) \). Since \( A \overline{A} \equiv 1 \pmod{4} \) and \( A \equiv -C/2 \pmod{N/2} \) we have 
\[
\chi_4(\sigma_{1/2}^0 \sigma_{-1}^{-1}) = \chi_4(A) = \chi_4(A) = \chi_4(-C/2).
\]
As a consequence of (5.3) we find that 
\[
\chi_4(\sigma_{1/2}^0 \sigma_{-1}^{-1}) = \chi_4(2A + C).
\]
According to (5.2) 
\[
\begin{cases}
AD - BC = 1 \\
D \equiv -2B \pmod{N}.
\end{cases}
\]
Therefore, 
\[
-B(2A + C) \equiv 1 \pmod{N}
\]
and 
\[
\chi_4(\sigma_{1/2}^0 \sigma_{-1}^{-1}) = \chi_4(-B) = \chi_4(-B) = \chi_4(D/2).
\]

5.3. Fourier coefficients. In this section we assume that \( m \) is positive. For the sake of brevity, we introduce the notation:
\[
\delta_n(m) := \begin{cases} 
1 & \text{if } n|m \\
0 & \text{otherwise},
\end{cases}
\]
(5.6)
\[
s(m) := \frac{\sigma_{1-2s}(\chi_4;m)}{L(\chi_4,2s)},
\]
(5.7)
\[
t(m) := \frac{\tau(\chi_4)\sigma_{2s-1}(\chi_4;m)m^{1-2s}}{L(\chi_4,2s)},
\]
\[
\sum_{a \pmod{m}}^* := \sum_{a \pmod{m}} \sum_{(a,m)=1}^*.
\]

In this section, we will frequently use the following lemma, which follows directly from the Chinese remainder theorem.

Lemma 5.5. For \((m,n) = 1\) we have
\[
\sum_{c \pmod{mn}}^* f(c) = \sum_{a \pmod{m}}^* \sum_{b \pmod{n}}^* f(an\overline{m}_m + bm\overline{m}_n),
\]
where \(n\overline{m}_m \equiv 1 \pmod{m}\) and \(m\overline{m}_n \equiv 1 \pmod{n}\).

Now we are ready to compute the Fourier coefficients (2.10) for \(N = 4, 16, 64\) and \(a = 0, \infty\). We provide a complete proof for the case \(N = 64\) and \(a = \infty\). The other cases can be proved by the same manner.
Lemma 5.6. Let $N = 64$ and $a = \infty$. Then
\[
\phi_{\infty,0}(m, s, \chi_4) = \chi_4(-1) \frac{s(m)}{8^2s},
\]
\[
\phi_{\infty,1/2}(m, s, \chi_4) = \chi_4(-1)e\left(\frac{m}{2}\right) \frac{s(m)}{8^2s},
\]
\[
\phi_{\infty,1/4s}(m, s, \chi_4) = \chi_4(-u)e\left(-\frac{mu}{4}\right) \frac{s(m)}{8^2s}, \quad u = 1, 3,
\]
(5.8)
\[
\phi_{\infty,1/8s}(m, s, \chi_4) = \chi_4(-u)e\left(-\frac{mu}{8}\right) \frac{s(m)}{8^2s}, \quad u = 1, 3, 5, 7,
\]
(5.9)
\[
\phi_{\infty,1/16s}(m, s, \chi_4) = \chi_4(-u)4\delta_4(m)e\left(-\frac{mu}{16}\right) \frac{s(m)}{16^2s}, \quad u = 1, 3,
\]
(5.10)
\[
\phi_{\infty,1/32s}(m, s, \chi_4) = \frac{8}{(32)^2s}\delta_8(m)t\left(\frac{m}{8}\right) - \frac{16}{(64)^2s}\delta_{16}(m)t\left(\frac{m}{16}\right),
\]
(5.11)
\[
\phi_{\infty,1}(m, s, \chi_4) = \frac{16}{(64)^2s}\delta_{16}(m)t\left(\frac{m}{16}\right).
\]

Proof. We need to evaluate (2.10) for $a = \infty$. Consequently, [13, Eq. 3.20] can be simplified as follows for $r = N = 64$:
\[
\Gamma_\infty \backslash \sigma_\infty^{f-1} \Gamma_\sigma_\infty / \Gamma_\infty =
\left\{ \left( \begin{array}{c}
\ast \\
C\sqrt{N'}
\end{array} \right) \left( \begin{array}{c}
\ast \\
D\sqrt{N''}
\end{array} \right) : 
\begin{array}{l}
D \equiv \frac{-(C/f)u}{(f,N/f)} \mod 1 \\
(D,C) = 1, \quad (C,N) = f
\end{array} \right\}.
\]

If $c = 0$, then $\chi_4(\sigma_0\rho\sigma_\infty^{-1}) = \chi_4(-C)$ by Corollary 5.4 and
\[
f = 1, \quad (f, N/f) = 1, \quad N'' = 64, \quad D \equiv -C \mod 1.
\]

As a result, we conclude that
\[
\phi_{\infty,0}(m, s, \chi_4) = \sum_{(C,2)=1} \frac{\chi_4(-C)}{(8C)^2s} \sum_{(D \mod C)}^\ast e\left(\frac{mD}{C}\right).
\]
The condition $(C, 2) = 1$ can be omitted since $\chi_4(-C) = 0$ if this doesn’t hold. Furthermore, we use the following identity for the inner sum
\[
\sum_{D \mod C}^\ast e\left(\frac{mD}{C}\right) = \sum_{d | (m,C)} d\mu(C/d),
\]
and interchange the order of summations, getting

$$
\phi_{\infty,0}(m, s, \chi_4) = \frac{\chi_4(-1)}{8^{2s}} \sum_{d|m} d \sum_{C \equiv 0 (\mod d)} \frac{\chi_4(C)}{C^{2s}} \mu(C/d)
\phantom{=} \chi_4(-1) \sum_{d|m} (C/d) = \chi_4(-1) \frac{s(m)}{8^{2s}}.
$$

If $c = \frac{1}{2}$, then $\chi_4(\frac{\sigma_1/2}{\rho_\sigma \infty}) = \chi_4(-C/2)$ by Corollary 5.4, and

$$f = 2, \quad (f, N/f) = 2, \quad N'' = 16, \quad D \equiv -C/2 \pmod{2}.$$

This yields that

$$
\phi_{\infty,\frac{1}{2}}(m, s, \chi_4) = \sum_{(C,2)=1} \frac{\chi_4(-C)}{(8C)^{2s}} \sum_{d\mid(2C)}^* e \left( \frac{mD}{2C} \right).
$$

By Lemma 5.5 we have

$$
\sum_{D \equiv \frac{md_2}{C} (\mod 2C)}^* e \left( \frac{mD}{2C} \right) = e \left( \frac{m}{2} \right) \sum_{d_2(C)}^* e \left( \frac{md_2}{C} \right).
$$

Consequently,

$$\phi_{\infty,\frac{1}{2}}(m, s, \chi_4) = \chi_4(-1)e \left( \frac{m}{2} \right) \frac{s(m)}{8^{2s}}.$$

If $c = \frac{1}{4u}$ with $u = 1, 3$, then $\chi_4(\frac{\sigma_0/\rho_\sigma \infty}{\sigma_\infty}) = \chi_4(D)$ by Corollary 5.4, and

$$f = 4, \quad (f, N/f) = 4, \quad N'' = 4, \quad D \equiv -uC/4 \pmod{4}.$$

Therefore,

$$
\phi_{\infty,\frac{1}{4u}}(m, s, \chi_4) = \sum_{(C,2)=1} \frac{\chi_4(-uC)}{(8C)^{2s}} \sum_{D \equiv \frac{md_2}{C} (\mod 4C)}^* e \left( \frac{mD}{4C} \right).
$$
Applying Lemma 5.5 we infer

$$\sum_{D \equiv -uC \pmod{4C}}^* e\left(\frac{mD}{4C}\right) = \sum_{d_1 \equiv -u \pmod{4}} e\left(\frac{md_1C_4}{4}\right)$$

$$\times \sum_{d_2 \pmod{C}} e\left(\frac{md_2}{C}\right) = e\left(-\frac{mu}{4}\right) \sum_{d|C} d\mu(C/d).$$

This implies that for $u = 1, 3$ we have

$$\phi_{\infty, \frac{1}{8u}}(m, s, \chi_4) = \chi_4(-u)e\left(-\frac{mu}{8}\right)s(m) \frac{s(m)}{8^{2s}}.$$
The inner sum can be evaluated by applying Lemma 5.5 as follows

\[
\sum_{\substack{D \equiv -uC \pmod{16} \\ D \equiv -uC \pmod{4}}} e\left(\frac{mD}{16C}\right) = \sum_{d_1 \equiv -uC_4 \pmod{16}} \sum_{d_2 \equiv -u \pmod{C}} e\left(\frac{md_1C_{16}}{16}\right) e\left(\frac{md_2C_{16}}{C}\right).
\]

Note that \(C_4 \equiv C \pmod{4}\), and therefore, the condition \(Cd_1 \equiv -uC_4 \pmod{4}\) can be replaced by \(d_1 \equiv -u \pmod{4}\).

Making the change of variables \(d_3 := -u + 4d_3 \pmod{4}\) we infer

\[
\sum_{d_1 \equiv -u \pmod{16}} e\left(\frac{md_1}{16}\right) = \sum_{d_3 \equiv -u \pmod{4}} e\left(-\frac{mu}{16}\right) e\left(\frac{md_3}{4}\right) = 4\delta_4(m)e\left(-\frac{mu}{16}\right).
\]

Consequently,

\[
\phi_\infty \frac{1}{16m}(m, s, \chi_4) = \chi_4(-u)4\delta_4(m)e\left(-\frac{mu}{16}\right) s(m)\frac{1}{16^{2s}}.
\]

If \(\epsilon = \frac{1}{32}\), then \(\chi_4(\sigma_0 \rho \sigma_\infty^{-1}) = \chi_4(D)\) by Corollary 5.4 and

\[
f = 32, \quad (f, N/f) = 2, \quad N'' = 1, \quad D \equiv -uC/32 \pmod{2}.
\]

In these settings

\[
\phi_\infty \frac{1}{16}(m, s, \chi_4) = \sum_{(C, 2) = 1} \frac{1}{(32C)^{2s}} \sum_{\substack{D \equiv -u \pmod{32C} \\ D \equiv -u \pmod{C}}} \chi_4(D)e\left(\frac{mD}{32C}\right).
\]
where the inner sum is the Gauss sum $g(\chi_4; 32C; m)$ defined by (2.18).

Then it follows from (2.19) that

$$\phi_{\infty, \frac{1}{16}}(m, s, \chi_4) = \frac{\tau(\chi_4)}{(32)^2 s} \sum_{d|m} d \chi_4 \left( \frac{m}{d} \right) \sum_{\substack{C \equiv 0 \pmod{d/(8,d)} \atop (C,2)=1}} \chi_4 \left( \frac{8C}{d} \right) \mu \left( \frac{8C}{d} \right) C^{2s}$$

$$= \frac{\tau(\chi_4)}{(32)^2 s} \sum_{d|m \atop (d/(8,d),2)=1} d \chi_4 \left( \frac{m}{d} \right) \left( \frac{(8,d)}{d} \right)^2 s \sum_{(C,2)=1} \chi_4 \left( \frac{8C}{(8,d)} \right) \mu \left( \frac{8C}{(8,d)} \right) C^{2s}.$$

Since $\chi_4 \left( \frac{8C}{(8,d)} \right) = 0$ unless $(8,d) = 8$, the expression above simplifies to

$$\phi_{\infty, \frac{1}{16}}(m, s, \chi_4) = \frac{\tau(\chi_4)}{(32)^2 s} \sum_{d|m \atop (d \equiv 0 \pmod{8}) \atop (d/8,2)=1} d \chi_4 \left( \frac{m}{d} \right) \left( \frac{8}{d} \right)^{2s} \sum_{(C,2)=1} \chi_4 \left( \frac{C}{(8,d)} \right) \mu \left( \frac{C}{(8,d)} \right) C^{2s}.$$ 

If $m/8$ is odd then the condition $(d,2) = 1$ can be removed, and if $m/8$ is even we have $\chi_4 \left( \frac{m/8}{d} \right) = 0$ for $d$ odd. Consequently,

$$\phi_{\infty, \frac{1}{16}}(m, s, \chi_4) = \frac{8}{(32)^2 s} \delta_8(m)(1 - \delta_{16}(m)) t \left( \frac{m}{8} \right). \quad (5.12)$$

For $m \equiv 0 \pmod{16}$ the following identity holds

$$\left( \frac{m}{8} \right)^{1-2s} \sigma_{2s-1}(\chi_4; m/8) = 2^{1-2s} \left( \frac{m}{16} \right)^{1-2s} \sigma_{2s-1}(\chi_4; m/16),$$

and therefore,

$$\phi_{\infty, \frac{1}{16}}(m, s, \chi_4) = \frac{8}{(32)^2 s} \delta_8(m) t \left( \frac{m}{8} \right) - \frac{16}{(64)^{2s}} \delta_{16}(m) t \left( \frac{m}{16} \right).$$

If $\epsilon = \frac{1}{64}$, then $\chi_4(\sigma_0 \rho \sigma_0^{-1}) = \chi_4(D)$ by Corollary 5.4 and

$$f = 64, \quad (f, N/f) = 1, \quad N'' = 1, \quad D \equiv -uC/64 \pmod{1}. \quad \text{(mod 1)}.$$

This implies that

$$\phi_{\infty, \infty}(m, s, \chi_4) = \sum_{C} \frac{1}{(64C)^{2s}} \sum_{D \pmod{64C}} \chi_4(D) e \left( \frac{mD}{64C} \right).$$
The inner sum is the Gauss sum \( g(\chi_4; 64C; m) \) defined by (2.18). Applying the representation (2.19) for this sum, we have

\[
\phi_{\infty, \infty}(m, s, \chi_4) = \frac{\tau(\chi_4)}{(64)^{2s}} \sum_{C=1}^{\infty} \frac{1}{C^{2s}} \sum_{d | (m, 16C)} d \chi_4 \left( \frac{16C}{d} \right) \chi_4 \left( \frac{n}{d} \right) \mu \left( \frac{16C}{q} \right)
\]

\[= \frac{\tau(\chi_4)}{(64)^{2s}} \sum_{d | m, (mod\ 16)} d \chi_4 \left( \frac{m}{d} \right) \left( \frac{16, d}{d} \right) 2^s \sum_{C=1}^{\infty} \frac{\chi_4(C) \mu(C)}{C^{2s}} \chi_4 \left( \frac{16C}{16, d} \right).
\]

We remark that \( \chi_4 \left( \frac{16C}{16, d} \right) = 0 \) unless \( \frac{(16, d)}{d} = 1 \). Therefore, we can assume that \( d \equiv 0 \ (mod\ 16) \) and

\[
\phi_{\infty, \infty}(m, s, \chi_4) = \frac{\tau(\chi_4)}{(64)^{2s}} \sum_{d \equiv 0 (mod\ 16)} d \chi_4 \left( \frac{m}{d} \right) \left( \frac{16, d}{d} \right) 2^s \sum_{C=1}^{\infty} \frac{\chi_4(C) \mu(C)}{C^{2s}}
\]

\[= \frac{\tau(\chi_4)}{(64)^{2s}} \delta_{16}(m) \sum_{d \equiv 0 (mod\ 16)} d \chi_4 \left( \frac{m}{16d} \right) \frac{16d^{-2s+1}}{L(\chi_4, 2s)} = \frac{16}{(64)^{2s}} \delta_{16}(m) t \left( \frac{m}{16} \right).
\]

\[\square\]

**Lemma 5.7.** Let \( N = 64 \) and \( a = 0 \). Then

\[
\phi_{0, 0}(m, s, \chi_4) = \frac{16}{(64)^{2s}} \delta_{16}(m) t \left( \frac{m}{16} \right),
\]

(5.13) \[
\phi_{0, 1/2}(m, s, \chi_4) = \frac{8}{(32)^{2s}} \delta_8(m) t \left( \frac{m}{8} \right) - \frac{16}{(64)^{2s}} \delta_{16}(m) t \left( \frac{m}{16} \right),
\]

\[
\phi_{0, 4/3}(m, s, \chi_4) = 4 \delta_4(m) e \left( \frac{mu}{16} \right) \frac{s(m)}{16^{2s}}, \ u = 1, 3,
\]

(5.14) \[
\phi_{0, 2/3}(m, s, \chi_4) = e \left( \frac{mu}{8} \right) \frac{s(m)}{8^{2s}}, \ u = 1, 3, 5, 7,
\]

(5.15) \[
\phi_{0, 1/3}(m, s, \chi_4) = e \left( \frac{mu}{4} \right) \frac{s(m)}{8^{2s}}, \ u = 1, 3,
\]

\[
\phi_{0, 1/2}(m, s, \chi_4) = e \left( \frac{m}{2} \right) \frac{s(m)}{8^{2s}};
\]

(5.16) \[
\phi_{0, \infty}(m, s, \chi_4) = \frac{s(m)}{8^{2s}}.
\]
Lemma 5.8. Let $N = 16$ and $\alpha = 0$. Then
\begin{align*}
\phi_{0,0}(m, s, \chi_4) &= \frac{4}{(16)^{2s}} \delta_4(m) t \left( \frac{m}{4} \right), \\
\phi_{0, \frac{1}{2}}(m, s, \chi_4) &= \frac{2}{8^{2s}} \delta_2(m) t \left( \frac{m}{2} \right) - \frac{4}{(16)^{2s}} \delta_4(m) t \left( \frac{m}{4} \right), \\
\phi_{0, \frac{1}{4u}}(m, s, \chi_4) &= e \left( \frac{mu}{4} \right) \frac{s(m)}{4^{2s}}, \quad u = 1, 3, \\
\phi_{0, \frac{1}{4}}(m, s, \chi_4) &= e \left( \frac{m}{2} \right) \frac{s(m)}{4^{2s}},
\end{align*}

(5.17) \quad (5.18) \quad (5.19)

Lemma 5.9. Let $N = 16$ and $\alpha = \infty$. Then
\begin{align*}
\phi_{\infty,0}(m, s, \chi_4) &= \chi_4(-1) \phi_{0, \infty}(m, s, \chi_4) = \chi_4(-1) \frac{s(m)}{4^{2s}}, \\
\phi_{\infty, \frac{1}{2}}(m, s, \chi_4) &= \chi_4(-1) \phi_{0, \frac{1}{2}}(m, s, \chi_4) = \chi_4(-1) e \left( \frac{m}{2} \right) \frac{s(m)}{4^{2s}}, \\
\phi_{\infty, \frac{1}{4u}}(m, s, \chi_4) &= \chi_4(-u) \phi_{0, \frac{1}{4u}}(m, s, \chi_4) \\
&= \chi_4(-u) e \left( -\frac{mu}{4} \right) \frac{s(m)}{4^{2s}}, \quad u = 1, 3, \\
\phi_{\infty, \frac{1}{4}}(m, s, \chi_4) &= \phi_{0, \frac{1}{2}}(m, s, \chi_4) \\
&= \frac{2}{8^{2s}} \delta_2(m) t \left( \frac{m}{2} \right) - \frac{4}{(16)^{2s}} \delta_4(m) t \left( \frac{m}{4} \right), \\
\phi_{\infty, \infty}(m, s, \chi_4) &= \phi_{0,0}(m, s, \chi_4) = \frac{4}{(16)^{2s}} \delta_4(m) t \left( \frac{m}{4} \right).
\end{align*}

(5.20) \quad (5.21) \quad (5.22) \quad (5.23)

Lemma 5.10. Let $N = 4$. Then
\begin{align*}
\phi_{0,0}(m, s, \chi_4) &= t(m) \frac{s(m)}{4^{2s}}, \quad \phi_{0, \infty}(m, s, \chi_4) = \frac{s(m)}{2^{2s}}, \\
\phi_{\infty,0}(m, s, \chi_4) &= \chi_4(-1) \phi_{0, \infty}(m, s, \chi_4), \\
\phi_{\infty, \infty}(m, s, \chi_4) &= \phi_{0,0}(m, s, \chi_4).
\end{align*}

(5.24) \quad (5.25) \quad (5.26)
Corollary 5.11. For $N = 64$ we have
\begin{equation}
(5.27) \quad \phi_{\infty, \frac{1}{32}}(m^2, s, \chi_4) = \phi_{0, \frac{1}{32}}(m^2, s, \chi_4) = 0.
\end{equation}

For $N = 16$ we have
\begin{equation}
(5.28) \quad \phi_{\infty, \frac{1}{2}}(m^2, s, \chi_4) = \phi_{0, \frac{1}{2}}(m^2, s, \chi_4) = 0.
\end{equation}

Proof. The identity (5.27) is a consequence of (5.12), (5.10) and (5.13). Similarly, (5.28) follows from (5.22) and (5.18). □

6. Contribution of the continuous spectrum

Applying the Kuznetsov trace formula (2.16) to the sums of Kloosterman sums in (3.17), we find that for even $n$ the continuous spectrum (2.17) can be written as a sum of
\begin{equation}
(6.1) \quad C^1_{\text{even}} := -\frac{2i\zeta(2s)}{2^s\pi^{s-1/2}} \sum_{l=1}^{\infty} \frac{1}{l^s} \sum_{c \text{ sing. } \Gamma_0(4)} \frac{1}{4\pi} \int_{-\infty}^{\infty} \psi_D(t) \sinh(\pi t) \frac{\phi_{\infty, c}(l, 1/2 + it, \chi_4)}{t \cosh(\pi t)} dt
\end{equation}
and
\begin{equation}
(6.2) \quad C^2_{\text{even}} := \frac{2\zeta(2s)}{\pi^{s-1/2}} \sum_{l=1}^{\infty} \frac{1}{l^s} \sum_{c \text{ sing. } \Gamma_0(4)} \frac{1}{4\pi} \int_{-\infty}^{\infty} \psi_D(t) \sinh(\pi t) \frac{\phi_{\infty, c}(l, 1/2 + it, \chi_4)}{t \cosh(\pi t)} dt.
\end{equation}

Similarly, applying (3.18) for odd $n$ it is required to investigate
\begin{equation}
(6.3) \quad C^1_{\text{odd}} := \frac{2\zeta(2s)}{\pi^{s-1/2}} \sum_{l=1}^{\infty} \frac{1}{l^s} \sum_{c \text{ sing. } \Gamma_0(64)} \frac{1}{4\pi} \int_{-\infty}^{\infty} \psi_D(t) \sinh(\pi t) \frac{\phi_{\infty, c}(l, 1/2 + it, \chi_4)}{t \cosh(\pi t)} dt
\end{equation}
and
\begin{equation}
(6.4) \quad C^2_{\text{odd}} := \frac{4\zeta(2s)}{\pi^{s-1/2}} \sum_{l=1}^{\infty} \frac{1}{l^s} \sum_{c \text{ sing. } \Gamma_0(16)} \frac{1}{4\pi} \int_{-\infty}^{\infty} \psi_D(t) \sinh(\pi t) \frac{\phi_{\infty, c}(l, 1/2 + it, \chi_4)}{t \cosh(\pi t)} dt
\end{equation}
\begin{equation}
\times (1 - 2^{-s}) l^{-2it} \phi_{\infty, c}(l^2, 1/2 + it, \chi_4) n^{2it} \phi_{0, c}(n^2, 1/2 + it, \chi_4) dt.
\end{equation}

In order to compute the sums over $l$ and $c$ in the expressions above we use the results of the previous section. Even and odd cases require separate treatment.
6.1. Even case.

Lemma 6.1. For $\Re s > 1$ we have

$$C_{even}^1 = \frac{L(\chi_4, s)}{4\pi^{s-1/2}2^s(1 - 2^{-2s})} \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\psi_D(t) \sinh(\pi t)}{t \cosh(\pi t)}$$

$$\times \frac{\zeta(s + 2it) \zeta(s - 2it)}{L(\chi_4, 1 + 2it)L(\chi_4, 1 - 2it)} \left( (1 - 2^{-2it-s})n_1^{2it} \sigma_{-2it}(\chi_4; n_1^2) \right)$$

$$+ (1 - 2^{-2it-s})n_1^{-2it} \sigma_{2it}(\chi_4; n_1^2) dt.$$ 

Proof. There are two nonequivalent singular cusps for $\Gamma_0(4)$: 0 and $\infty$.

Consider first $c = \infty$. Applying (5.26), (5.7), (2.3) with $z := s - 2it$ and $s := -2it$, we compute the sum over $l$ in (6.1)

$$\sum_{l=1}^{\infty} \frac{1}{l} \left( \frac{\zeta(s + 2it) \zeta(s - 2it)}{L(\chi_4, s)} \frac{n_1^{-2it} \sigma_{2it}(\chi_4; n_1^2)}{\zeta(2s) L(\chi_4, 1 + 2it)L(\chi_4, 1 - 2it)} \right).$$

Now let us consider the case $c = 0$. Using (5.25), (5.6), (2.3) with $z := s + 2it$ and $s := 2it$, we infer

$$\sum_{l=1}^{\infty} \frac{1}{l} \left( \frac{\zeta(s + 2it) \zeta(s - 2it)}{L(\chi_4, s)} \frac{n_1^{-2it} \sigma_{-2it}(\chi_4; n_1^2)}{\zeta(2s) L(\chi_4, 1 + 2it)L(\chi_4, 1 - 2it)} \right).$$

The assertion follows by summing the last two expressions and using the fact that $|\tau(\chi_4)|^2 = 4$.

□

Lemma 6.2. For $\Re s > 1$ the following identity holds

$$C_{even}^2 = \frac{L(\chi_4, s)}{4\pi^{s-1/2}2^s(1 - 2^{-2s})} \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\psi_D(t) \sinh(\pi t)}{t \cosh(\pi t)}$$

$$\times \frac{\zeta(s + 2it) \zeta(s - 2it)}{L(\chi_4, 1 + 2it)L(\chi_4, 1 - 2it)} \left( (1 - 2^{-2it-s})n_1^{2it} \sigma_{-2it}(\chi_4; n_1^2) \right)$$

$$+ (1 - 2^{2it-s})n_1^{-2it} \sigma_{2it}(\chi_4; n_1^2) dt.$$ 

Proof. Similarly to the previous lemma, the sum over $c$ in (6.2) contains only two summands: 0 and $\infty$. 

Let us start with $c = \infty$. Applying (5.26), (5.24), (5.7), (5.6), (2.3) with $z := s - 2it$ and $s := -2it$, we obtain

$$\sum_{l=1}^{\infty} \frac{1}{l^s} \phi_{\infty, \infty}(l^2, 1/2 + it, \chi_4) n_1^{2it} \phi_{0, \infty}(n_1^2, 1/2 + it, \chi_4)$$

$$= \frac{\tau(\chi_4) 1 - 2^{-2it - s}}{2^{-2it}} \frac{1 - 2^{-2s}}{1 - 2^{-2s}} n_1^{2it} \sigma_{-2it}(\chi_4; n_1^2) \frac{L(\chi_4, s) \zeta(s + 2it) \zeta(s - 2it)}{\zeta(2s) L(\chi_4, 1 + 2it) L(\chi_4, 1 - 2it)}.$$ 

Next let us consider $c = 0$. Applying (5.25), (5.24), (5.7), (5.6), (2.3) with $z := s + 2it$ and $s := 2it$, we find that

$$\sum_{l=1}^{\infty} \frac{1}{l^s} \phi_{\infty, 0}(l^2, 1/2 + it, \chi_4) n_1^{2it} \phi_{0, 0}(n_1^2, 1/2 + it, \chi_4)$$

$$= \frac{\tau(\chi_4) \chi_4(-1) 1 - 2^{2it - s}}{2^{2it}} \frac{1 - 2^{-2s}}{1 - 2^{-2s}} n_1^{2it} \sigma_{2it}(\chi_4; n_1^2) \frac{L(\chi_4, s) \zeta(s + 2it) \zeta(s - 2it)}{\zeta(2s) L(\chi_4, 1 + 2it) L(\chi_4, 1 - 2it)}.$$ 

The assertion follows by summing the last two expressions and by noting that $\tau(\chi_4) = 2t$ and $\chi_4(-1) = -1$. 

Using (2.1), as a direct consequence of Lemmas 6.1 and 6.2 we obtain the following corollary.

**Corollary 6.3.** For $\Re s > 1$ we have

$$C^1_{\text{even}} + C^2_{\text{even}} = \frac{L(\chi_4, s)}{4\pi^{3-1/2}} \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\psi_D(t) \sinh(\pi t)}{t \cosh(\pi t)}$$

$$\times \frac{\zeta(s + 2it) \zeta(s - 2it)}{L(\chi_4, 1 + 2it) L(\chi_4, 1 - 2it)} \left( n_1^{2it} \sigma_{-2it}(\chi_4; n_1^2) + n_1^{-2it} \sigma_{2it}(\chi_4; n_1^2) \right) dt.$$ 

Finally, we extend this result to the critical strip using the following lemma.

**Lemma 6.4.** Suppose that the function $F(s)$ is defined for $\Re s > 1$ by

$$F(s) = \frac{1}{2\pi i} \int_{(0)} f(s, z) dz,$$

where $f(s, z)$ has two simple poles at the points $z_1 = 1 - s$ and $z_2 = s - 1$. Then for $\Re s < 1$ we have

$$F(s) = \frac{1}{2\pi i} \int_{(0)} f(s, z) dz + \text{Res}_{z_1} f(s, z) - \text{Res}_{z_2} f(s, z).$$
Lemma 6.5. For completes the proof. □

Proof. Assume that $1 < \Re s < 1 + \epsilon/2$ and $\Im s > 0$. Consider the the new contour of integration as follows

$$
\gamma_1 = (-i\infty, -i\Im s - i\epsilon) \cup C^-_\epsilon \cup (-i\Im s + i\epsilon, i\Im s - i\epsilon) \cup C^+_\epsilon \cup (i\Im s + i\epsilon, i\infty),
$$

where $C^-_\epsilon$ is a semicircle in the left half-plane of radius $\epsilon$ and $C^+_\epsilon$ is a semicircle in the right half-plane of radius $\epsilon$. While changing the contour of integration to $\gamma_1$ we cross poles at the points $z_1, z_2$. Therefore,

$$
F(s) = \frac{1}{2\pi i} \int_{(\gamma_1)} f(s, z) dz + \text{Res}_{z_1} f(s, z) - \text{Res}_{z_2} f(s, z).
$$

Now if $\Re s < 1$, we can change the contour back to $\Re z = 0$. This completes the proof. □

Lemma 6.6. For $0 < \Re s < 1, s \neq 1/2$ we have

$$
(6.6) \quad C_{\text{even}}^1 + C_{\text{even}}^2 = M^C(n, s) + \frac{L(\chi_4, s)}{4\pi s - 1/2} \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\psi_D(t) \sinh(\pi t)}{t \cosh(\pi t)}
$$

$$
\times \frac{\zeta(s + 2it)\zeta(s - 2it)}{L(\chi_4, 1 + 2it) L(\chi_4, 1 - 2it)} \left( n^{2it} \sigma_{-2it}(\chi_4; n^2) + n^{-2it} \sigma_{2it}(\chi_4; n^2) \right) dt,
$$

where $M^C(n, s)$ is defined by (1.7).

Proof. Making the change of variables $z := 2it$ in (6.5), we have

$$
C_{\text{even}}^1 + C_{\text{even}}^2 = \frac{L(\chi_4, s)}{4\pi s - 1/2} \frac{1}{2\pi i} \int_{(0)} \frac{\psi_D\left(\pi \frac{z}{2}\right)}{z \cosh\left(\pi \frac{z}{2}\right)}
$$

$$
\times \frac{\zeta(s + z)\zeta(s - z)}{L(\chi_4, 1 + z)L(\chi_4, 1 - z)} \left( n^z \sigma_z(\chi_4; n^2) + n^{-z} \sigma_z(\chi_4; n^2) \right) dt.
$$

Then (6.6) follows from Lemma 6.4 and (4.11). □

6.2. Odd case.

Lemma 6.6. For $\Re s > 1$ we have

$$
C_{\text{odd}}^1 = \frac{L(\chi_4, s)}{4\pi s - 1/2 (1 - 2^{-2s})} \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\psi_D(t) \sinh(\pi t)}{t \cosh(\pi t)}
$$

$$
\times \frac{\zeta(s + 2it)\zeta(s - 2it)}{L(\chi_4, 1 + 2it) L(\chi_4, 1 - 2it)} n^{2it} \sigma_{-2it}(\chi_4; n^2)
$$

$$
\times \left( (1 - 2^{2it-s})(1 - 2^{-2it-s}) + \frac{2^{-2it} + 2^{2it} - 2^{1-s}}{2^{2s}} \right) dt.
$$
Proof. Let us decompose the sum over \( l \) in (6.3) as follows
\[
\sum_{l=1}^{\infty} = \sum_{l \equiv 1 \pmod{2}} + \sum_{l \equiv 0 \pmod{2}},
\]
so that \( C_{\text{odd}}^{1,1} = C_{\text{odd}}^{1,1} + C_{\text{odd}}^{1,2} \).

Consider first \( C_{\text{odd}}^{1,1} \). By Lemma 5.7 we obtain for odd \( n \) that
\[
(6.7) \quad \phi_{0, c}(n^2, 1/2 + it, \chi_4) = 0, \quad c = 0, 1/4, 1/12.
\]

If \( l \) is odd, it follows from Lemma 5.6 that
\[
\phi_{\infty, c}(l^2, 1/2 + it, \chi_4) = 0, \quad c = \infty, 1/32, 1/16, 1/48.
\]

It is left to consider the case \( c = 1/8, u = 1, 3, 5, 7 \). Note that for odd \( n \) we have \( e(n^2 u/8) = e(u/8) \). Consequently, applying (5.6), (5.8) and (5.14) we obtain
\[
(6.8) \quad \sum_{l=1}^{\infty} \frac{1}{2s + 2it} \phi_{\infty, c}(l^2, 1/2 + it, \chi_4) n^{2it} \phi_{0, c}(n^2, 1/2 + it, \chi_4) = \frac{n^{2it} \sigma_{-2it}(\chi_4; n^2) \chi_4(-u) e(u/4)}{64 L(\chi_4, 1 + 2it) L(\chi_4, 1 - 2it)} \sum_{l=1}^{\infty} \frac{\sigma_{2it}(\chi_4; l^2)}{l^{s + 2it}}.
\]

Using (2.3) with \( z := s + 2it, s := 2it \) we show that
\[
(6.9) \quad \sum_{l=1}^{\infty} \frac{\sigma_{2it}(\chi_4; l^2)}{l^{s + 2it}} = (1 - 2^{-s - 2it}) \sum_{l=1}^{\infty} \frac{\sigma_{2it}(\chi_4; l^2)}{l^{s + 2it}} = (1 - 2^{-s - 2it}) \frac{1 - 2^{2it-s} L(\chi_4, s) \zeta(s + 2it) \zeta(s - 2it)}{1 - 2^{-2s}} \zeta(2s).
\]

Substituting (6.9) into (6.8), summing over \( u = 1, 3, 5, 7 \) and using the identity
\[
\sum_{u=1,3,5,7} \chi_4(-u) e(u/4) = -4i,
\]
we infer
\[
(6.10) \quad C_{\text{odd}}^{1,1} = \frac{L(\chi_4, s)}{4\pi s - 2(1 - 2^{-2s})} \frac{1}{2\pi i} \int_{-\infty}^{\infty} \psi_p(t) \sinh(\pi t) \frac{\sigma_{-2it}(\chi_4; n^2)}{t \cosh(\pi t)} \frac{\zeta(s + 2it) \zeta(s - 2it)}{L(\chi_4, 1 + 2it) L(\chi_4, 1 - 2it)} (1 - 2^{2it-s})(1 - 2^{-2it-s}) dt.
\]
Now let us evaluate $C_{c,0}^{1,2}$. Using (6.7) we conclude that it is left to consider the following cases: $c = \frac{1}{8u}$, $u = 1, 3, 5, 7$ and $c = \frac{1}{16}, \frac{1}{48}$, $c = \frac{1}{32}$, $c = \infty$.

Assume that $c = \frac{1}{8u}$. Using the fact that $e(n^2u/8) = e(u/8)$ for odd $n$, and applying (5.6), (5.8) and (5.14) we obtain

$$
\sum_{l=0}^{\infty} \frac{1}{l+2it} \phi_{\infty,1/(8u)}(l^2, 1/2 + it, \chi_4)n^{2it} \phi_{\infty,1/(8u)}(n^2, 1/2 + it, \chi_4)
= \frac{n^{2it} \sigma_{-2it}(\chi_4; n^2) \chi_4(-u)e(u/8)}{64L(\chi_4, 1 + 2it)L(\chi_4, 1 - 2it)} \sum_{l=0}^{\infty} \frac{e(l^2u/8)\sigma_{2it}(\chi_4; l^2)}{l+2it}.
$$

Note that $e(m^2u/2) = 1$ if $m$ is even and $e(m^2u/2) = -1$ if $m$ is odd. Therefore,

$$
\sum_{l=0}^{\infty} \frac{e(l^2u/8)\sigma_{2it}(\chi_4; l^2)}{l+2it} = \frac{1}{2s+2it} \sum_{l=1}^{\infty} e(l^2u/2)\sigma_{2it}(\chi_4; l^2)
= \frac{1}{2s+2it} \sum_{l=0}^{\infty} (\sigma_{2it}(\chi_4; l^2)) - \frac{1}{2s+2it} \sum_{l=1}^{\infty} (\sigma_{2it}(\chi_4; l^2)).
$$

Since

$$
\sum_{u=1,3,5,7} \chi_4(-u)e(u/8) = 0,
$$

the contribution of the part with $c = \frac{1}{8u}$, $u = 1, 3, 5, 7$ to $C_{c,0}^{1,2}$ is zero.

Now assume that $c = \frac{1}{16u}$, $u = 1, 3$. Applying (5.6), (5.9) and (5.15), we have

$$
(6.11) \quad \sum_{l=0}^{(l,2)=1} n^{2it} \phi_{\infty,1/(16u)}(l^2, 1/2 + it, \chi_4) \phi_{\infty,1/(16u)}(n^2, 1/2 + it, \chi_4)
= \frac{n^{2it} \sigma_{-2it}(\chi_4; n^2) \chi_4(-u)e(u/4)}{2^{5-2it}L(\chi_4, 1 + 2it)L(\chi_4, 1 - 2it)} \sum_{l=0}^{\infty} \frac{e(l^2u/16)\sigma_{2it}(\chi_4; l^2)}{l+2it}.
$$

Using (2.4) we infer

$$
\sum_{l=0}^{\infty} \frac{e(l^2u/16)\sigma_{2it}(\chi_4; l^2)}{l+2it} = \frac{1}{2s+2it} \sum_{l=1}^{\infty} e(l^2u/4)\sigma_{2it}(\chi_4; l^2)
= \frac{1}{4s+2it} \sum_{l=1}^{\infty} \sigma_{2it}(\chi_4; l^2) + \frac{e(u/4)}{2s+2it} \sum_{l=0}^{(l,2)=1} \sigma_{2it}(\chi_4; l^2)\phi_{\infty,1/(16u)}(l^2, 1/2 + it, \chi_4)\phi_{\infty,1/(16u)}(n^2, 1/2 + it, \chi_4)\phi_{\infty,1/(16u)}(\chi_4, 1 + 2it, \chi_4)\phi_{\infty,1/(16u)}(\chi_4, 1 - 2it, \chi_4).
$$
Rewriting the sum over $l$ as

$$
\sum_{(l,2)=1}^{\infty} = \sum_{l=1}^{\infty} - \sum_{l=0 \mod 2}^{\infty}
$$

and applying (2.3) with $z := s + 2it$, $s := 2it$, we show that

$$
(6.12) \quad \sum_{l=0 \mod 2}^{\infty} \frac{e(l^2u/16)\sigma_{2it}(\chi_4; l^2)}{l^{s+2it}} = \frac{1 - 2^{2it-s}}{2^{s+2it}(1 - 2^{-2s})} \times \frac{L(\chi_4, s)\zeta(s+2it)\zeta(s-2it)}{\zeta(2s)} (2^{-s-2it} + e(u/4)(1 - 2^{-s-2it}))
$$

Substituting (6.12) into (6.11) and summing over $u = 1, 2$ we conclude that the contribution of the part with $c = \frac{1}{16u}$, $u = 1, 2$ to $C_{1,2}^{1,2}$ is equal to

$$
(6.13) \quad \frac{n^{2it} \sigma_{-2it}(\chi_4; n^2)}{16i} \frac{1 - 2^{2it-s}}{2^{2s+2it}(1 - 2^{-2s})} \times \frac{L(\chi_4, s)\zeta(s+2it)\zeta(s-2it)}{\zeta(2s)L(\chi_4, 1 + 2it)L(\chi_4, 1 - 2it)}
$$

Next, assume that $c = \frac{1}{32}$. By Corollary 5.11 we have

$$
\phi_{\infty,1/(32)}(l^2, 1/2 + it, \chi_4) = 0.
$$

Finally, assume that $c = \infty$. As a consequence of (5.11), (5.16), (5.6), (5.7), (2.3), we obtain

$$
(6.14) \quad \sum_{l=0 \mod 2}^{\infty} \frac{n^{2it}}{l^{s+2it}} \phi_{\infty,\infty}(l^2, 1/2 + it, \chi_4)\phi_{0,\infty}(n^2, 1/2 + it, \chi_4) = \frac{n^{2it} \sigma_{-2it}(\chi_4; n^2)}{16i} \frac{1 - 2^{-2it-s}}{2^{2s-2it}(1 - 2^{-2s})} \frac{L(\chi_4, s)\zeta(s+2it)\zeta(s-2it)}{\zeta(2s)L(\chi_4, 1 + 2it)L(\chi_4, 1 - 2it)}.
$$

Combining (6.3), (6.13), (6.14) we find that

$$
(6.15) \quad C_{1,2}^{1,2} = \frac{L(\chi_4, s)}{4\pi^{s-1/2}(1 - 2^{-2s})} \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\psi_D(t) \sinh(\pi t)}{t \cosh(\pi t)} \sigma_{-2it}(\chi_4; n^2) \times \frac{\zeta(s+2it)\zeta(s-2it)}{L(\chi_4, 1 + 2it)L(\chi_4, 1 - 2it)} \left(\frac{2^{-2it} + 2^{2it} - 2^{-1-s}}{2^{2s}}\right) dt.
$$

The statement follows by summing (6.10) and (6.15). \qed
Lemma 6.7. For $\Re s > 1$ the following identity holds

$$C_{\text{odd}}^2 = \frac{L(\chi_4, s)(1 - 2^{-s})}{4\pi^{s-1/2}(1 - 2^{-2s})} \frac{1}{2\pi i} \int_{-\infty}^{\infty} \psi_D(t) \sinh(\pi t) \frac{2\pi i}{t \cosh(\pi t)} \sigma_{-2it}(\chi_4; n^2)$$

$$\times n^{2it} \frac{\zeta(s + 2it)\zeta(s - 2it)}{L(\chi_4, 1 + 2it)L(\chi_4, 1 - 2it)} \frac{2^{-2it} + 2^{2it} - 2^{1-s}}{2^s - 2} \cdot$$

Proof. The list of singular cusps for $\Gamma_0(16)$ is given in Lemma 5.1. Consider $c = 0$ and $c = 1/2$. As a consequence of (5.17) and Corollary 5.11 we obtain that for odd $n$

$$\phi_{0,0}(n^2; 1/2 + it, \chi_4) = \phi_{0,1/2}(n^2, 1/2 + it, \chi_4) = 0.$$

Consider $c = 1/4$. By Corollary 5.11 we have

$$\phi_{\infty,1/8}(l^2; 1/2 + it, \chi_4) = 0.$$

Consider $c = 1/4$, $u = 1, 3$. Using (5.21), (5.6), we compute the sum over $l$ in (6.14)

$$\sum_{l=1}^{\infty} \frac{\phi_{\infty,1/(4u)}(l^2, 1/2 + it, \chi_4)}{l^{s+2it}} = \frac{\chi_4(-u)}{4^{1-2it}L(\chi_4, 1 - 2it)} \sum_{l=1}^{\infty} \frac{\sigma_{2it}(\chi_4; l^2) e(l^2u/4)}{l^{s+2it}}.$$

The last sum can be split into two parts

$$\sum_{l=1}^{\infty} \frac{\sigma_{2it}(\chi_4; l^2) e(l^2u/4)}{l^{s+2it}} = \frac{1}{2^{s+2it}} \sum_{l=1}^{\infty} \frac{\sigma_{2it}(\chi_4; 4l^2)}{l^{s+2it}} + e(u/4) \sum_{l \equiv 1 \mod 2} \frac{\sigma_{2it}(\chi_4; 2l^2)}{l^{s+2it}}.$$

Furthermore,

$$\sum_{l \equiv 1 \mod 2} \frac{\sigma_{2it}(\chi_4; l^2)}{l^{s+2it}} = \sum_{l=1}^{\infty} \frac{\sigma_{2it}(\chi_4; l^2)}{l^{s+2it}} - \frac{1}{2^{s+2it}} \sum_{l=1}^{\infty} \frac{\sigma_{2it}(\chi_4; 4l^2)}{l^{s+2it}}.$$

Note that $\sigma_{2it}(\chi_4; 4l^2) = \sigma_{2it}(\chi_4; l^2)$, and therefore,

$$\sum_{l=1}^{\infty} \frac{\sigma_{2it}(\chi_4; l^2) e(l^2u/4)}{l^{s+2it}} = \left( \frac{1}{2^{s+2it}} + e(u/4)(1 - 2^{-s-2it}) \right) \sum_{l=1}^{\infty} \frac{\sigma_{2it}(\chi_4; 4l^2)}{l^{s+2it}}.$$
Substituting (6.17) into (6.16) and applying (2.3) with \( z := s + 2it \) and \( s := 2it \) yields

\[
\sum_{l=1}^{\infty} \frac{\phi_{\infty,1/4u}(l^2, 1/2 + it, \chi_4)}{l^{s+2it}} = \left( \frac{1}{2s+2it} + e(u/4)(1 - 2^{-s-2it}) \right) \times \frac{\chi_4(-u)}{4^{1-2it}} \frac{1 - 2^{2it-s} L(\chi_4, s)\zeta(s+2it)\zeta(s-2it)}{1 - 2^{-2s} \zeta(2s)}.
\]

Using (5.19), (5.6), and the fact that \( n \) is odd, we infer

\[
(6.18) \quad \sum_{l=1}^{\infty} \frac{\phi_{\infty,1/4u}(l^2, 1/2 + it, \chi_4)}{l^{s+2it}} n^{2it} \phi_{0,1/4u}(n^2, 1/2 + it, \chi_4)
\]

\[
= \left( \frac{1}{2s+2it} + e(u/4)(1 - 2^{-s-2it}) \right) n^{2it} \zeta(2s) L(\chi_4, 1 - 2it) L(\chi_4, 1 + 2it).
\]

Consider \( \kappa = \infty \). In order to evaluate the sum over \( l \) in this case we apply (5.23), (5.7), and (2.3) with \( z := s - 2it \), \( s := -2it \). Furthermore, using (5.20), (5.6) and the fact that \( \tau(\chi_4) = 2i \), we obtain

\[
(6.19) \quad \sum_{l=1}^{\infty} \frac{1}{l^{s+2it}} \phi_{\infty,\infty}(l^2, 1/2 + it, \chi_4) n^{2it} \phi_{0,\infty}(n^2, 1/2 + it, \chi_4)
\]

\[
= \frac{-i}{2^{3s+2it}} \frac{1 - 2^{-2it-s}}{1 - 2^{-2s}} n^{2it} \zeta(2s) L(\chi_4, 1 + 2it) L(\chi_4, 1 - 2it).
\]

Summing (6.19), (6.18) for \( u = 1, 3 \) and noting that

\[
\sum_{u=1,3} \chi_4(-u) = 0, \quad \sum_{u=1,3} \chi_4(-u)e(u/4) = -2i,
\]

we prove the lemma.

We combine the previous two lemmas, we prove the following result.

**Corollary 6.8.** For \( \Re s > 1 \) we have

\[
C_{odd}^1 + C_{odd}^2 = \frac{L(\chi_4, s)}{4\pi^{s-1/2}} \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\psi_D(t) \sinh(\pi t)}{t \cosh(\pi t)} \frac{\zeta(s+2it)\zeta(s-2it)}{L(\chi_4, 1 + 2it) L(\chi_4, 1 - 2it)} n^{2it} \zeta(2s) L(\chi_4, 1 + 2it) L(\chi_4, 1 - 2it) dt.
\]
Lemma 6.9. For $0 < \Re s < 1$, $s \neq 1/2$ the following holds

\[ C_{\text{odd}}^1 + C_{\text{odd}}^2 = \frac{1}{2} M^C(n, s) + \frac{L(\chi_4, s)}{4\pi^{s-1/2}} \int_{-\infty}^{\infty} \frac{\psi_D(t) \sinh(\pi t)}{t \cosh(\pi t)} \]
\[ \times \frac{\zeta(s+2it)\zeta(s-2it)}{L(\chi_4, 1+2it)L(\chi_4, 1-2it)} n^{2it} \sigma_{-2it}(\chi_4; n^2) dt, \]

where $M^C(n, s)$ is defined by (1.7).

Proof. The proof is the same as the one of Lemma 6.5. \qed

7. Contribution of the discrete and homomorphic spectra

Applying the Kuznetsov trace formula (2.16) to the sums of Kloosterman sums in (3.17) for even $n$, we find that the discrete spectrum is a sum of two twisted moments of symmetric square $L$-functions associated to Maass cusp forms of level 4 with nebentypus $\chi_4$, namely

\[ D_{\text{even}}^1 = -\frac{2^{1-s} \pi^{1/2-s} i}{1 - 2^{-2s}} \sum_{f \in H(4, \chi_4)} \frac{\psi_D(t_f)}{\cosh(\pi t_f)} \rho_{f_{\infty}}(n^2/4) L(s, \text{sym}^2 f_{\infty}) \]

and

\[ D_{\text{even}}^2 = \frac{2\pi^{1/2-s}}{1 - 2^{-2s}} \sum_{f \in H(4, \chi_4)} \frac{\psi_D(t_f)}{\cosh(\pi t_f)} \rho_{f_0}(n^2/4) L(s, \text{sym}^2 f_{\infty}). \]

Similarly, the holomorphic spectrum is a sum of two twisted moments of symmetric square $L$-functions associated to holomorphic cusp forms of level 4 with nebentypus $\chi_4$ given by

\[ H_{\text{even}}^1 = -\frac{2^{1-s} \pi^{1/2-s} i}{1 - 2^{-2s}} \sum_{k > 1 \atop \text{k odd}} \psi_H(k) \Gamma(k) \sum_{f \in H_k(4, \chi_4)} \rho_{f_{\infty}} \left( \frac{n^2}{4} \right) L(s, \text{sym}^2 f_{\infty}) \]

and

\[ H_{\text{even}}^2 = \frac{2\pi^{1/2-s}}{1 - 2^{-2s}} \sum_{k > 1 \atop \text{k odd}} \psi_H(k) \Gamma(k) \sum_{f \in H_k(4, \chi_4)} \rho_{f_0} \left( \frac{n^2}{4} \right) L(s, \text{sym}^2 f_{\infty}). \]

Next let us consider the case of odd $n$. The main difference with the previous case is that now we obtain moments of $L$-functions associated to forms of levels 64 and 16. Indeed, applying the Kuznetsov trace
formula (2.16) to the sums of Kloosterman sums in (3.18), we infer that the discrete spectrum consists of

\[ D_{\text{odd}} = \frac{8\pi^{1/2-s}}{1 - 2^{-2s}} \sum_{f \in H(64, \chi_4)} \frac{\psi_D(t_f)}{\cosh(\pi t_f)} \rho_{f_0}(n^2) \overline{L(s, \text{sym}^2 f_{\infty})} \]

and

\[ D_{\text{odd}} = \frac{4\pi^{1/2-s}}{1 + 2^{-s}} \sum_{f \in H(16, \chi_4)} \frac{\psi_D(t_f)}{\cosh(\pi t_f)} \rho_{f_0}(n^2) \overline{L(s, \text{sym}^2 f_{\infty})}. \]

Similarly, the holomorphic spectrum is a sum of these two parts:

\[ H_{\text{odd}}^{1} = \frac{8\pi^{1/2-s}}{1 - 2^{-2s}} \sum_{\substack{k > 1 \\ k \text{ odd}}} \psi_{H}(k) \Gamma(k) \sum_{f \in H_{k}(64, \chi_4)} \rho_{f_0}(n^2) \overline{L(s, \text{sym}^2 f_{\infty})}, \]

\[ H_{\text{odd}}^{2} = \frac{4\pi^{1/2-s}}{1 + 2^{-s}} \sum_{\substack{k > 1 \\ k \text{ odd}}} \psi_{H}(k) \Gamma(k) \sum_{f \in H_{k}(16, \chi_4)} \rho_{f_0}(n^2) \overline{L(s, \text{sym}^2 f_{\infty})}. \]

8. Proof of main theorems

8.1. Proof of Theorems 1.1 and 1.2. The diagonal main terms in Theorems 1.1 and 1.2 are computed in Lemma 3.6. In order to evaluate the non-diagonal part, we apply the Kuznetsov trace formula (2.16) to the sums of Kloosterman sums in (3.17) and (3.18). It follows from Lemmas 6.5 and 6.9 that the contribution of the continuous spectrum is equal to \( M^C(n, s) + C(n, s) \) in case of even \( n \) and to \( 1/2M^C(n, s) + 1/2C(n, s) \) in case of odd \( n \), where

\[ C(n, s) = C_{\text{even}}^1 + C_{\text{even}}^2 - M^C(n, s). \]

Finally, the contribution of the discrete and holomorphic spectra is given in Section 7. For the sake of brevity, we express the final result in terms of sums of moments (1.5) using the following identities:

\[ D_{\text{even}}^1 + H_{\text{even}}^1 = -\frac{2^{1-s}\pi^{1/2-s}i}{1 - 2^{-2s}} \mathfrak{m}_{\infty}(n^2/4, 4, s), \]

\[ D_{\text{even}}^2 + H_{\text{even}}^2 = \frac{2\pi^{1/2-s}}{1 - 2^{-2s}} \mathfrak{m}_0(n^2/4, 4, s), \]

\[ D_{\text{odd}}^1 + H_{\text{odd}}^1 = \frac{8\pi^{1/2-s}}{1 - 2^{-2s}} \mathfrak{m}_0(n^2, 64, s), \]
\[ D_{\text{odd}}^2 + H_{\text{odd}}^2 = \frac{4\pi^{1/2-s}}{1+2-s} \mathcal{M}_0(n^2, 16, s). \]

8.2. **Main terms at the central point.** As the final step we show that the main terms in Theorems 1.1 and 1.2 are holomorphic at the central point.

**Lemma 8.1.** For \( n \) even the following identity holds

\[
M^C(n, 1/2) + M^D_{\text{even}}(n, 1/2) = \frac{\sigma_{-1/2}(\chi_4; n^2) + n^{-1} \sigma_{1/2}(\chi_4; n^2)}{2L(\chi_4, 3/2)} \int_0^\infty \omega(y) \, dy \\
\times \left( \log |y^2 - n^2/4| + \frac{\pi}{2} \text{sgn}(y-n/2) - 2 \frac{L'(\chi_4, 3/2)}{L(\chi_4, 3/2)} - \log(2\pi) + 3\gamma \right) \, dy \\
- \frac{\sigma'_{-1/2}(\chi_4; n^2) - n^{-1} \sigma'_{1/2}(\chi_4; n^2) + 2 \log(n)n^{-1} \sigma_{1/2}(\chi_4; n^2)}{2L(\chi_4, 3/2)} \int_0^\infty \omega(y) \, dy.
\]

**Proof.** As a consequence of the functional equations for the Riemann zeta function and for the Gamma function, we obtain

\[ (8.1) \quad \zeta(2u) \Gamma(u) = (2\pi)^{2u} \frac{\Gamma(1-2u)}{\Gamma(1-u)} \zeta(1-2u). \]

Combining (1.6), (1.7) and (8.1) we conclude that

\[
M^C(n, 1/2 + u) + M^D_{\text{even}}(n, 1/2 + u) = \zeta(1+2u) \frac{\sigma_{-1/2}(\chi_4; n^2) + n^{-1-2u} \sigma_{1/2+u}(\chi_4; n^2)}{L(\chi_4, 3/2 + u)} \int_0^\infty \omega(y) \, dy \\
+ \zeta(1-2u) (2\pi)^u \frac{\Gamma(1-2u) \sigma_{-1/2+u}(\chi_4; n^2) + n^{-1+2u} \sigma_{1/2-u}(\chi_4; n^2)}{\Gamma(1-u) L(\chi_4, 3/2 - u)} \\
\times \sqrt{2} \left( \sin(\pi/4 + \pi u/2) \int_0^{n/2} \omega(y) \left( \frac{n^2}{4} - y^2 \right)^{-u} \, dy \right. \\
+ \left. \cos(\pi/4 + \pi u/2) \int_{n/2}^\infty \omega(y) \left( y^2 - \frac{n^2}{4} \right)^{-u} \, dy \right).
\]

The expression above is holomorphic at \( u = 0 \). Consequently, letting \( u \) tend to zero and applying the L’Hôpital rule, we prove the lemma. \( \square \)
Lemma 8.2. For \( n \) odd the following identity holds
\[
\frac{1}{2} M^C(n, 1/2) + M^D_{\text{odd}}(n, 1/2) = \frac{\sigma_{-1/2}(\chi_4; n^2)}{2L(\chi_4, 3/2)} \int_0^\infty \omega(y) \times \\
\left( \log |y^2 - n^2/4| + \frac{\pi}{2} \text{sgn}(y - n/2) - \frac{L'(\chi_4, 3/2)}{L(\chi_4, 3/2)} - \log(2\pi) + 3\gamma \right) dy \\
- \frac{\sigma'_{-1/2}(\chi_4; n^2)}{2L(\chi_4, 3/2)} \int_0^\infty \omega(y) dy.
\]

Proof. Using (1.7), (1.9), (2.2) and (8.1) we obtain
\[
\frac{1}{2} M^C(n, 1/2 + u) + M^D_{\text{odd}}(n, 1/2 + u) = \\
\zeta(1 + 2u) \frac{\sigma_{-1/2-u}(\chi_4; n^2)}{L(\chi_4, 3/2 + u)} \int_0^\infty \omega(y) dy \\
+ \zeta(1 - 2u)(2\pi)^u \frac{\Gamma(1 - 2u)}{\Gamma(1 - u)} \frac{\sigma_{-1/2+u}(\chi_4; n^2)}{L(\chi_4, 3/2 - u)} \\
\times \sqrt{2} \left( \sin(\pi/4 + \pi u/2) \int_{n/2}^{n/2} \omega(y) \left( \frac{n^2}{4} - y^2 \right)^{-u} dy \\
+ \cos(\pi/4 + \pi u/2) \int_{n/2}^{n/2} \omega(y) \left( y^2 - \frac{n^2}{4} \right)^{-u} dy \right).
\]
The expression above is holomorphic at \( u = 0 \). Therefore, the assertion follows by letting \( u \) tend to zero and applying the L'Hospital rule. □

Acknowledgement

The reported study was funded by RFBR, project number 19-31-60029.

References

[1] O. Balkanova, The first moment of symmetric square L-functions, Ramanujan J (2020), https://doi.org/10.1007/s11139-020-00272-z.
[2] O. Balkanova, D. Frolenkov, The mean value of symmetric square L-functions, Algebra Number theory 12:1 (2018), 35–59.
[3] O. Balkanova, D. Frolenkov, Convolution formula for the sums of generalized Dirichlet L-functions, Rev. Mat. Iberoam. 35 (2019), no. 7, 1973–1995.
[4] O. Balkanova, D. Frolenkov, M. S. Risager, Prime geodesics and averages of the Zagier L-series. arXiv:1912.05277 [math.NT].
[5] A. Balog, A. Biro, G. Cherubini, N. Laaksonen, Bykovskii-type theorem for the Picard manifold, Int. Math. Res. Not., https://doi.org/10.1093/imrn/rnaa128.
[6] V. Blomer, On the central value of symmetric square L-functions, Math. Z. 260:4 (2008), 755–777.
[7] V.A. Bykovskii, Density theorems and the mean value of arithmetic functions on short intervals. (Russian) Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI) 212 (1994), Anal. Teor. Chisel i Teor. Funktsii. 12, 56–70, 196; translation in J. Math. Sci. (New York) 83 (1997), no. 6, 720–730.

[8] J.-M. Deshouillers and H. Iwaniec, Kloosterman sums and Fourier coefficients of cusp forms, Invent. Math. 70 (1982), no. 2, 219–288.

[9] S. Drappeau, Sums of Kloosterman sums in arithmetic progressions, and the error term in the dispersion method, Proc. Lond. Math. Soc. (3) 114 (2017), no. 4, 684–732.

[10] I. S. Gradshteyn and I. M. Ryzhik, Table of Integrals, Series, and Products. Edited by A. Jeffrey and D. Zwillinger. Academic Press, New York, 7th edition, 2007.

[11] H. Iwaniec and P. Michel, The second moment of the symmetric square \( L \)-functions, Annales Acad. Sci. Fenn. Mathematica, Volume 26 (2001), 465–482.

[12] R. Khan and M. P. Young, Moments and hybrid subconvexity for symmetric-square \( L \)-functions, arXiv:2009.08419 [math.NT].

[13] E.M. Kıral and M.P. Young, Kloosterman sums and Fourier coefficients of Eisenstein series, Ramanujan J 49 (2019), 391–409.

[14] E.M. Kıral and M.P. Young, The fifth moment of Modular \( L \)-functions, J. Eur. Math. Soc. (JEMS), DOI: 10.4171/JEMS/1011.

[15] A.V. Malyshev, On the representation of integers by positive quadratic forms, in Russian, Trudy Mat. Inst. Steklov 65 (1962), 3–212.

[16] T. Miyake, Modular forms. Springer Monographs in Mathematics. Springer-Verlag, Berlin, english edition, 2006. Translated from the 1976 Japanese original by Yoshitaka Maeda.

[17] F.W.J. Olver, D.W. Lozier, R.F. Boisvert and C.W. Clarke, NIST Handbook of Mathematical Functions, Cambridge University Press, Cambridge (2010).

[18] G. Shimura, On the holomorphy of certain Dirichlet series, Proc. London Math. Soc. (3) 31 (1975), 79–98.

[19] K. Soundararajan and M. P. Young, The prime geodesic theorem, J. Reine Angew. Math. 676 (2013), 105–120.

[20] H. Wu and G. Zabradi, On Kuznetsov–Bykovskii’s formula of counting prime geodesics, arXiv:1901.03824 [math.NT].

[21] D. Zagier, Modular forms whose Fourier coefficients involve zeta-functions of quadratic fields. Modular functions of one variable, VI (Proc. Second Internat. Conf., Univ. Bonn, Bonn, 1976), pp. 105–169. Lecture Notes in Math., Vol. 627, Springer, Berlin, 1977.