On normal operator logarithms

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Abstract

Let $X, Y$ be normal bounded operators on a Hilbert space such that $e^X = e^Y$. If the spectra of $X$ and $Y$ are contained in the strip $S$ of the complex plane defined by $|\text{Im}(z)| \leq \pi$, we show that $|X| = |Y|$. If $Y$ is only assumed to be bounded, then $|X|Y = Y|X|$. We give a formula for $X - Y$ in terms of spectral projections of $X$ and $Y$ provided that $X, Y$ are normal and $e^X = e^Y$. If $X$ is an unbounded self-adjoint operator, which does not have $(2k + 1)\pi$, $k \in \mathbb{Z}$, as eigenvalues, and $Y$ is normal with spectrum in $S$ satisfying $e^{iX} = e^{iY}$, then $Y \in \{ e^{iX} \}''$. We give alternative proofs and generalizations of results on normal operator exponentials proved by Ch. Schmoeger.

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1 Introduction

Solutions to the equation $e^X = e^Y$ were studied by E. Hille [1] in the general setting of unital Banach algebras. Under the assumption that the spectrum $\sigma(X)$ of $X$ is incongruent (mod $2\pi i$), which means that $\sigma(X) \cap \sigma(X + 2k\pi i) = \emptyset$ for all $k = \pm 1, \pm 2, \ldots$, he proved that $XY = YX$ and there exist idempotents $E_1, E_2, \ldots, E_n$ commuting with $X$ and $Y$ such that

$$X - Y = 2\pi i \sum_{j=1}^{n} k_j E_j, \quad \sum_{j=1}^{n} E_j = I, \quad E_i E_j = \delta_{ij},$$

where $k_1, k_2, \ldots, k_n$ are different integers. If the hypothesis on the spectrum is removed, it is possible to find non commuting logarithms (see e.g. [1, 6]). In the setting of Hilbert spaces, when $X$ is a normal operator, the above assumption on the spectrum can be weakened. In fact, Ch. Schmoeger [5] proved that $X$ belongs to the double commutant of $Y$ provided that $E_X(\sigma(X) \cap \sigma(X + 2k\pi i)) = 0$, $k = 1, 2, \ldots$, where $E_X$ is the spectral measure of $X$. We also refer to [3] for a generalization of this result by F. C. Paliogiannis.

In this paper, we study the operator equation $e^X = e^Y$ in the setting of Hilbert spaces under the assumption that the spectra of $X$ and $Y$ belong to a non-injective domain of the complex exponential map. Our results include the relation between the modulus of $X$ and $Y$ (Theorem 3.1), a formula for the difference of two normal logarithms in terms of their spectral projections (Theorem 4.1) and commutation relations when $X$ is a skew-adjoint unbounded operator (Theorem 5.1). The proofs of these results are elementary. In fact, they rely on the spectral theorem for normal operators. This approach allows us to give a generalization (Corollary 4.2) and an alternative proof (Corollary 5.2) of two results by Ch. Schmoeger (see [5]).

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2 Notation and preliminaries

Let \((\mathcal{H}, \langle \cdot, \cdot \rangle)\) be a complex Hilbert space and \(\mathcal{B}(\mathcal{H})\) be the algebra of bounded operators on \(\mathcal{H}\). The spectrum of an operator \(X\) is denoted by \(\sigma(X)\), and the set of eigenvalues of \(X\) is denoted by \(\sigma_p(X)\). The real part of \(X \in \mathcal{B}(\mathcal{H})\) is \(\text{Re}(X) = \frac{1}{2}(X + X^*)\) and its imaginary part is \(\text{Im}(X) = \frac{1}{2i}(X - X^*)\).

If \(X\) is a bounded or unbounded normal operator on \(\mathcal{H}\), we denote by \(E_X\) the spectral measure of \(X\). Recall that \(E_X\) is defined on the Borel subsets of \(\sigma(X)\), but we may think that \(E_X\) is defined on all the Borel subsets of \(\mathbb{C}\). Indeed, we can set \(E_X(\Omega) = E_X(\Omega \cap \sigma(X))\) for every Borel set \(\Omega \subseteq \mathbb{C}\). Our first lemma is a generalized version of [4, Ch. XII Ex. 25], where the normal operator can now be unbounded.

**Lemma 2.1.** Let \(X\) be a (possibly unbounded) normal operator on \(\mathcal{H}\) and \(f\) a bounded Borel function on \(\sigma(X)\). Then

\[ E_{f(X)}(\Omega) = E_X(f^{-1}(\Omega)), \]

for every Borel set \(\Omega \subseteq \mathbb{C}\).

**Proof.** We define a spectral measure by \(E'(\Omega) = E_X(f^{-1}(\Omega))\), where \(\Omega\) is any Borel subset of \(\mathbb{C}\). We are going to show that \(E' = E_{f(X)}\). Since \(f\) is bounded, it follows that \(f(X) \in \mathcal{B}(\mathcal{H})\). Moreover, the operator \(f(X)\) is given by

\[(f(X)\xi, \eta) = \int_{\mathbb{C}} f(z) \, dE_X\xi,\eta(z),\]

where \(\xi, \eta \in \mathcal{H}\) and \(E_X\xi,\eta\) is the complex measure defined by \(E_X\xi,\eta(\Omega) = \langle E_X(\Omega)\xi, \eta \rangle\) (see [4, Theorem 12.21]). By the change of measure principle ([4, Theorem 13.28]), we have

\[\int_{\mathbb{C}} z E_{\xi,\eta}'(z) = \int_{\mathbb{C}} f(z) E_X\xi,\eta(z).\]

Therefore \(E'\) satisfies the equation \(\int_{\mathbb{C}} z E_{\xi,\eta}'(z) = \langle f(X)\xi, \eta \rangle\), which uniquely determines the spectral measure of \(f(X)\) (see [4, Theorem 12.23]). Hence \(E' = E_{f(X)}\). \(\Box\)

The following lemma was first proved in [6, Corollary 2]. See also [3, Corollary 3] for another proof. We give below a proof for the sake of completeness, which does not depend on further results of these articles.

**Lemma 2.2.** Let \(X\) and \(Y\) be normal operators in \(\mathcal{B}(\mathcal{H})\). If \(e^X = e^Y\), then \(\text{Re}(X) = \text{Re}(Y)\).

**Proof.** The following computation was done in [4]:

\[e^{X+X^*} = e^X e^{X^*} = e^X (e^X)^* = e^Y (e^Y)^* = e^Y e^{Y^*} = e^{Y+Y^*},\]

where the first and last equalities hold because \(X\) and \(Y\) are normal. Now we may finish the proof in a different fashion: note that the exponential map, restricted to real axis, has an inverse \(\log: \mathbb{R}_+ \to \mathbb{R}\). Since \(\sigma(X + X^*) \subseteq \mathbb{R}\) and \(\sigma(e^{X+X^*}) \subseteq \mathbb{R}_+\), we can use the continuous functional calculus to get \(X + X^* = \log(e^{X+X^*}) = \log(e^{Y+Y^*}) = Y + Y^*\). \(\Box\)

Throughout this paper, we use the following notation for subsets of the complex plane:

- \(\Omega_1 + i\Omega_2 = \{ x + iy : x \in \Omega_1, y \in \Omega_2 \}\), where \(\Omega_i, i = 1, 2\), are subsets of \(\mathbb{R}\).
- For short, we write \(\mathbb{R} + ia\) for the set \(\mathbb{R} + i\{ a \}\).
- We write \(\mathcal{S}\) for the complex strip \(\{ z \in \mathbb{C} : -\pi \leq \text{Im}(z) \leq \pi \}\), and \(\mathcal{S}^o\) for the interior of \(\mathcal{S}\).
Lemma 2.3. Let $X, Y$ be normal operators in $B(H)$ such that $\sigma(X) \subseteq S$ and $\sigma(Y) \subseteq S$. Then $e^X = e^Y$ if and only if the following conditions hold:

i) $E_X(\Omega) = E_Y(\Omega)$ for all Borel subsets $\Omega$ of $S^\circ$.

ii) $\text{Re}(X) = \text{Re}(Y)$.

Proof. Suppose that $e^X = e^Y$. Let $\Omega$ be a Borel measurable subset of $S^\circ$. By the spectral mapping theorem,

$$\sigma(e^X) = \{ e^\lambda : \lambda \in \sigma(X) \} = \{ e^\mu : \mu \in \sigma(Y) \} = \sigma(e^Y).$$

It is well-known that the restriction of the complex exponential map $\exp|_{S^\circ}$ is bijective. Therefore we have $\sigma(X) \cap \Omega = \sigma(Y) \cap \Omega$, and by Lemma 2.1

$$E_X(\Omega) = E_X(\Omega \cap \sigma(X)) = E_X(\exp^{-1}(\exp(\Omega \cap \sigma(X))))$$

$$= E_{e^X}(\exp(\Omega \cap \sigma(X))) = E_{e^Y}(\exp(\Omega \cap \sigma(Y))) = E_Y(\Omega),$$

which proves i). On the other hand, ii) is proved in Lemma 2.2.

To prove the converse assertion, we first note that

$$E_X(\mathbb{R} - i\pi) + E_X(\mathbb{R} + i\pi) = I - E_X(S^\circ) = I - E_Y(S^\circ)$$

$$= E_Y(\mathbb{R} - i\pi) + E_Y(\mathbb{R} + i\pi),$$

since $\sigma(X) \subseteq S$, $\sigma(Y) \subseteq S$ and $E_X(S^\circ) = E_Y(S^\circ)$. Due to the fact that $E_X$ and $E_Y$ coincide on Borel subsets of $S^\circ$, we find that

$$\int_{S^\circ} e^z dE_X(z) = \int_{S^\circ} e^z dE_Y(z).$$

Hence we get

$$e^X = \int_S e^z dE_X(z) = -\int_{\mathbb{R} + i\pi} e^{\text{Re}(z)} dE_X(z) - \int_{\mathbb{R} - i\pi} e^{\text{Re}(z)} dE_X(z) + \int_{S^\circ} e^z dE_X(z)$$

$$= -e^{\text{Re}(X)}(E_X(\mathbb{R} + i\pi) + E_X(\mathbb{R} - i\pi)) + \int_{S^\circ} e^z dE_X(z)$$

$$= -e^{\text{Re}(Y)}(E_Y(\mathbb{R} + i\pi) + E_Y(\mathbb{R} - i\pi)) + \int_{S^\circ} e^z dE_Y(z) = e^Y. \quad \Box$$

Remark 2.4. We have shown that $E_X(\mathbb{R} - i\pi) + E_X(\mathbb{R} + i\pi) = E_Y(\mathbb{R} - i\pi) + E_Y(\mathbb{R} + i\pi)$, whenever $X, Y$ are normal bounded operators such that $\sigma(X) \subseteq S$, $\sigma(Y) \subseteq S$ and $e^X = e^Y$.

Theorem 2.5. (S. Kurepa [2]) Let $X \in B(H)$ such that $e^X = N$ is a normal operator. Then

$$X = N_0 + 2\pi iW,$$

where $N_0 = \log(N)$ and $\log$ is the principal (or any) branch of the logarithm function. The bounded operator $W$ commutes with $N_0$ and there exists a bounded and regular, positive definite self-adjoint operator $Q$ such that $W_0 = Q^{-1}WQ$ is a self-adjoint operator the spectrum of which belongs to the set of all integers.
3 Modulus and square of logarithms

Now we show the relation between the modulus of two normal logarithms with spectra contained in \( S \).

**Theorem 3.1.** Let \( X \) be a normal operator in \( \mathcal{B}(\mathcal{H}) \). Assume that \( \sigma(X) \subseteq S \) and \( e^X = e^Y \).

i) If \( Y \) is normal in \( \mathcal{B}(\mathcal{H}) \) and \( \sigma(Y) \subseteq S \), then \( |X| = |Y| \).

ii) If \( Y \in \mathcal{B}(\mathcal{H}) \), then \( |X|Y = Y|X| \).

**Proof.** i) We will prove that the spectral measures of \( |\text{Im}(X)| \) and \( |\text{Im}(Y)| \) coincide. Let us set \( A = \text{Im}(X) \) and \( B = \text{Im}(Y) \). Given \( \Omega \subseteq [0, \pi), \) put \( \Omega' = \{ x \in \mathbb{R} : |x| \in \Omega \} \). Note that \( \mathbb{R} + i\Omega' \subseteq S^\circ \). As an application of Lemma [2,1] and Lemma [2,3] we see that

\[
E_{|A|}(\Omega) = E_A(\Omega') = E_X(\mathbb{R} + i\Omega') = E_Y(\mathbb{R} + i\Omega') = E_B(\Omega') = E_{|B|}(\Omega).
\]

By Remark [2,4] we have

\[
E_{|A|}(\{ \pi \}) = E_A(\{ -\pi, \pi \}) = E_X(\mathbb{R} - i\pi) + E_X(\mathbb{R} + i\pi) = E_Y(\mathbb{R} - i\pi) + E_Y(\mathbb{R} + i\pi) = E_{|B|}(\{ \pi \}).
\]

Thus, we have proved \( E_{|A|} = E_{|B|} \), which implies that \( |A| = |B| \). On the other hand, by Lemma [2,3] we know that \( \text{Re}(X) = \text{Re}(Y) \). Therefore

\[
|X|^2 = \text{Re}(X)^2 + |A|^2 = \text{Re}(Y)^2 + |B|^2 = |Y|^2.
\]

Hence \( |X| = |Y| \), and the proof is complete.

ii) Since \( X \) is a normal operator, \( e^X = e^Y \) is also a normal operator. Then by a result by S. Kurepa (see Theorem [2,5]), there exist operators \( N_0 \) and \( W \) such that \( N_0 \) is normal, \( e^X = e^{N_0} \), \( W \) commutes with \( N_0 \) and \( Y = N_0 + 2\pi iW \). In fact, \( N_0 \) can be defined using the Borel functional calculus by \( N_0 = \log(e^X) \), where \( \log \) is the principal branch of the logarithm. In particular, this implies that \( \sigma(N_0) \subseteq S \). Now we can apply i) to find that \( |N_0| = |X| \). Since \( N_0W = WN_0 \), we have \( |N_0|W = W|N_0| \), and this gives \( W|X| = |X|W \). Hence \( |X|Y = Y|X| \). \( \square \)

Following similar arguments, we can give an alternative proof of a result by Ch. Schmoeger ([6, Theorem 3]). This result was originally proved using inner derivations. Note that a minor improvement on the assumption on \( \sigma(X) \) over the boundary \( \partial S \) of the strip \( S \) can now be done. Given a set \( \Omega \subseteq \mathbb{C} \), we denote by \( \hat{\Omega} \) the set \( \{ x - iy : x + iy \in \Omega \} \).

**Corollary 3.2.** Let \( X \) be a normal operator in \( \mathcal{B}(\mathcal{H}) \), \( \sigma(X) \subseteq S \), \( Y \in \mathcal{B}(\mathcal{H}) \) and \( e^X = e^Y \). Suppose that for every Borel subset \( \Omega \subseteq \partial S \setminus \{ -\pi, \pi \} \), it holds that \( E_X(\hat{\Omega}) = 0 \), whenever \( E_X(\Omega) \neq 0 \). Then \( X^2Y = YX^2 \).

**Proof.** We will show that \( E_{X^2}(\hat{\Omega}_0) \) commutes with \( Y \) for every Borel subset \( \Omega_0 \subseteq \sigma(X^2) \). From the equation \( e^X = e^Y \), we have \( e^{X^2}Y = Ye^X \), and thus, \( E_{e}(\hat{\Omega})Y = Ye_{e}(\Omega) \) for any Borel set \( \Omega \). Since the set \( \Omega \) is arbitrary, by Lemma [2,1] we get

1. \( E_X(\hat{\Omega}')Y = YE_X(\hat{\Omega}') \) for every subset \( \Omega' \subseteq S^\circ \).

2. \( (E_X(\hat{\Omega}') + E_X(\hat{\Omega}''))Y = Y(E_X(\hat{\Omega}') + E_X(\hat{\Omega}'')) \), whenever \( \Omega' \subseteq \partial S \).

On the other hand, the image of \( S \) by the analytic map \( f(z) = z^2 \) is given by

\[
f(S) = \{ u + i2t \sqrt{u + t^2} : u \in [-\pi^2, \infty), u + t^2 \geq 0 \}.
\]
Let us write \( f^{-1}(\Omega_0) = (-\Omega') \cup \Omega' \), where \( \Omega' \) is a subset of the half-plane \( \text{Re}(z) \geq 0 \) and \( -\Omega' \) denotes the set \( \{ -z : z \in \Omega \} \). We need to consider three cases. In the case in which \( \Omega_0 \subseteq f(S)^c \), then \( f^{-1}((\Omega_0) \subseteq S^c \). It follows that \( E_{X^2}(\Omega_0) = E_X(-\Omega') + E_X(\Omega') \), and by the item 1. above we have \( E_{X^2}(\Omega_0) = Y E_{X^2}(\Omega_0) \). In the case where \( \Omega_0 \subseteq \partial f(S) \setminus \{ \pi^2 \} \), we have either \( E_X(\Omega') = 0 \) or \( E_X(\Omega') = 0 \) by our assumption on the spectral measure of \( X \). Similarly, it must be either \( E(-\Omega') = 0 \) or \( E_X(-\Omega') = 0 \). Therefore item 2. above reduces to the desired conclusion, i.e. \( E_{X^2}(\Omega_0) = Y E_{X^2}(\Omega_0) \). Finally, if \( \Omega_0 = \{ -\pi^2 \} \), then \( E_{X^2}(\Omega_0) = E_X(\{ i\pi \}) + E_X(\{ -i\pi \}) \) commutes with \( Y \) by item 2., and this concludes the proof.

\[ \square \]

4 Difference of logarithms

Let \( X,Y \) be normal operators and \( k \in \mathbb{Z} \). In order to avoid lengthy formulas, let us fix a notation for some special spectral projections of these operators:

- \( P_{2k+1} = E_X(\mathbb{R} + i((2k - 1)\pi, (2k + 1)\pi)) \);
- \( Q_{2k+1} = E_Y(\mathbb{R} + i((2k - 1)\pi, (2k + 1)\pi)) \);
- \( E_{2k+1} = E_X(\mathbb{R} + i(2k + 1)\pi) \);
- \( F_{2k+1} = E_Y(\mathbb{R} + i(2k + 1)\pi) \).

As we have pointed out in the introduction, E. Hille showed that the difference between two logarithms in Banach algebras may be expressed as the sum of multiples of projections (see [1 Theorem 4]). In order to prove that result, the spectrum of one of the logarithms must be incongruent (mod \( 2\pi i \)). In the case where \( X \) and \( Y \) are both normal logarithms on a Hilbert space, the spectral theorem can be used to provide a more general formula.

**Theorem 4.1.** Let \( X,Y \) be normal operators in \( \mathcal{B}(\mathcal{H}) \) such that \( e^X = e^Y \). If \( \sigma(X) \) and \( \sigma(Y) \) are contained in \( \mathbb{R} + i[(2k_0 + 1)\pi, (2k_1 + 1)\pi] \) for some \( k_0, k_1 \in \mathbb{Z} \), then

\[
X - Y = \sum_{k=k_0}^{k_1} 2k\pi i (P_{2k+1} - Q_{2k+1}) + (2k + 1)\pi i (E_{2k+1} - F_{2k+1}).
\]

**Proof.** We first suppose that \( \sigma(X) \) and \( \sigma(Y) \) are contained in the strip \( S \). Then we have

\[
\text{Im}(X) = \text{Im}(X)(E_X(S^c) + E_X(\mathbb{R} + i\pi) + E_X(\mathbb{R} - i\pi)) = \text{Im}(X)P_1 + \pi E_1 - \pi E_{-1}.
\]

Analogously, \( \text{Im}(Y) = \text{Im}(Y)Q_1 + \pi F_1 - \pi F_{-1} \). By Lemma 2.3, we know that \( \text{Re}(X) = \text{Re}(Y) \) and \( E_X(\Omega) = E_Y(\Omega) \) for every Borel subset \( \Omega \) of \( S^c \). It follows that

\[
\text{Im}(X)P_1 = \int_{S^c} \text{Im}(z) dE_X(z) = \int_{S^c} \text{Im}(z) dE_Y(z) = \text{Im}(Y)Q_1,
\]

which implies

\[
X - Y = \pi i (E_1 - F_1) - \pi i (E_{-1} - F_{-1}).
\]

Thus, we have proved the formula in this case. For the general case, without restrictions on spectrum of \( X \) and \( Y \), we need to consider the following Borel measurable function

\[
f(t) = \sum_{k=k_0}^{k_1} (t - 2k\pi) \chi_{((2k-1)\pi, (2k+1)\pi]}(t),\]

where \( \chi_I(t) \) is the characteristic function of the interval \( I \). Set \( A = \text{Im}(X) \) and \( B = \text{Im}(Y) \). By Lemma 2.2, \( \text{Re}(X) = \text{Re}(Y) \), and since the real and imaginary part of \( X \) and \( Y \) commute because \( X \) and \( Y \) are normal, \( e^{iA} = e^{X} e^{-\text{Re}(X)} = e^{Y} e^{-\text{Re}(Y)} = e^{iB} \). The function \( f \) satisfies
\( e^{if(t)} = e^{it} \), which implies that \( e^{if(A)} = e^{iA} = e^{if(B)} = e^{if(B)} \). Since \( \sigma(f(A)) \) and \( \sigma(f(B)) \) are contained in \([-\pi, \pi]\), we can replace in equation (1) to find that

\[
f(A) - f(B) = \pi \left( E_f(A)\{ \{ \pi \} \} - E_f(B)\{ \{ \pi \} \} \right) \\
= \pi \sum_{k=0}^{k_1} E_A(\{ \{ 2k + 1 \} \pi \}) - E_B(\{ \{ 2k + 1 \} \pi \}) \\
= \pi \sum_{k=0}^{k_1} E_{2k+1} - F_{2k+1}.
\]

Here we have used Lemma 2.1 to express \( E_{2k+1} \) in terms of \( E_f(\{ \{ \pi \} \}) \) and \( E_f(\{ \{ -\pi \} \}) \). Therefore

\[
\begin{align*}
1. f(A) &= \sum_{k=0}^{k_1} (A - 2k\pi) \chi_{(2k+1)}(A) = A - \sum_{k=0}^{k_1} 2k\pi (P_{2k+1} + E_{2k+1}), \\
2. f(B) &= B - \sum_{k=0}^{k_1} 2k\pi (Q_{2k+1} + F_{2k+1}).
\end{align*}
\]

Therefore

\[
X - Y = i(A - B) = i(f(A) - f(B)) + \sum_{k=0}^{k_1} 2k\pi i(P_{2k+1} - Q_{2k+1}) + 2k\pi i(E_{2k+1} - F_{2k+1}).
\]

Combining this with the expression in (2), we get the desired formula. \( \square \)

Below we give a generalization of another result due to Ch. Schmoeger (see [6, Theorem 5]). The assumptions on the spectrum of \( X \) and \( Y \) were more restrictive in [6]: \( \|X\| \leq \pi \), \( \|Y\| \leq \pi \) and either \(-i\pi\) or \(i\pi\) does not belong to the point spectrum of one of these operators. However, these hypothesis were necessary to conclude that \(X - Y\) is a multiple of a projection; meanwhile \(XY = YX\) can be obtained under more general assumptions (see [6, Theorem 3], [5, Theorem 1.4] and [3, Theorem 9]).

**Corollary 4.2.** Let \( X, Y \) be normal operators in \( B(H) \). Assume that \( \sigma(X) \subseteq S \), \( \sigma(Y) \subseteq S \) and \( e^X = e^Y \). The following assertions hold:

i) If \( E_1 = 0 \), then \( XY = YX \) and \( X - Y = -2\pi i F_1 \).

ii) If \( E_{-1} = 0 \), then \( XY = YX \) and \( X - Y = 2\pi i F_{-1} \).

iii) If \( E_1 = E_{-1} = 0 \), then \( X = Y \).

**Proof.** i) Under these assumptions on the spectra of \( X \) and \( Y \), we have established that \( E_1 + E_{-1} = F_1 + F_{-1} \) in Remark 2.4. On the other hand, by equation (1) in the proof of Theorem 4.1, we know that \( X - Y = \pi i (E_1 - F_1) - \pi i (E_{-1} - F_{-1}) \). Since \( E_1 = 0 \), we have \( E_{-1} = F_1 + F_{-1} \). It follows that \( X = -2\pi i F_1 \). Hence \( X \) and \( Y \) commute. We can similarly conclude that ii) holds true. To prove iii), note that \( E_1 = E_{-1} = 0 \) implies that \( F_1 + F_{-1} = 0 \), and consequently, \( F_1 = F_{-1} = 0 \). Hence we get \( X = Y \). \( \square \)
5 Unbounded logarithms

Let $X$ be a self-adjoint unbounded operator on $\mathcal{H}$. As before, $E_X$ denotes the spectral measure of $X$. In item $i)$ of our next result, we will give a version of [5, Theorem 1.4] for unbounded operators (see also [3, Theorem 9]). To this end, we extend the definition given in [5] for bounded operators: a self-adjoint unbounded operator $X$ is \textit{generalized $2\pi$-congruence-free} if

$$E_X(\sigma(X) \cap \sigma(X + 2k\pi)) = 0, \quad k = \pm 1, \pm 2, \ldots.$$  

Given $Y \in B(\mathcal{H})$, the commutant of $Y$ is the set

$$\{ Y' = \{ Z \in B(\mathcal{H}) : YZ = ZY \}.$$  

The double commutant of $Y$ is defined by

$$\{ Y'' = \{ W \in B(\mathcal{H}) : WZ = ZW, \text{ for all } Z \in \{ Y' \} \}.$$  

If $X$ is a self-adjoint unbounded operator and $Y \in B(\mathcal{H})$, recall that $XY = YX$, that is $X$ commutes with $Y$, if $YE_X(\Omega) = E_X(\Omega)Y$ for every Borel subset $\Omega \subseteq \mathbb{R}$. Recall that the exponential $e^{iX}$ of a self-adjoint unbounded operator $X$ is a unitary operator, which can be defined via the Borel functional calculus (see e.g. [3]).

**Theorem 5.1.** Let $X$ be a self-adjoint operator on $\mathcal{H}$ and $Y \in B(\mathcal{H})$ such that $e^{iX} = e^{iY}$.

1) If $X$ is generalized $2\pi$-congruence-free, then $E_X(\Omega) = \{ Y' \}$ for all Borel subsets $\Omega$ of $\mathbb{R}$. In particular, $XY = YX$.

2) If $\{(2k + 1)\pi : k \in \mathbb{Z} \} \cap \sigma_p(X)$ has at most one element and $Y$ is normal in $B(\mathcal{H})$ such that $\sigma(Y) \subseteq S$, then $XY = YX$.

3) If $(2k + 1)\pi \notin \sigma_p(X)$ for all $k \in \mathbb{Z}$ and $Y$ is normal in $B(\mathcal{H})$ such that $\sigma(Y) \subseteq S$, then $Y \in \{ e^{iX} \}''$.

**Proof.**

1) Let $Z \in B(\mathcal{H})$ such that $ZY = YZ$. It follows that $Ze^{iY} = e^{iY}Z$. Then we have $Ze^{iX} = e^{iX}Z$, and by Lemma 2.1, $ZE_X(exp^{-1}(\Omega)) = E_X(exp^{-1}(\Omega))Z$ for every $\Omega \subseteq T$. If $\Omega' = exp^{-1}(\Omega) \cap [-\pi, \pi]$, then

$$E_X(exp^{-1}(\Omega)) = \sum_{k \in \mathbb{Z}} E_X(\Omega' + 2k\pi),$$  

where this series converges in the strong operator topology. Suppose now that there is some $k \in \mathbb{Z}$ such that $E_X(\Omega' + 2k\pi) \neq 0$. It follows that $\sigma(X) \cap (\Omega' + 2k\pi) \neq \emptyset$, and $(\Omega' + 2\pi) \cap \sigma(X) \subseteq \sigma(X) \cap \sigma(X + 2(l - k)\pi)$ for all $l \in \mathbb{Z}$. By the assumption on the spectral measure of $X$, $E_X(\Omega' + 2\pi) \leq E_X(\sigma(X) \cap \sigma(X + 2(l - k)\pi)) = 0$ for $l \neq k$. Therefore for each $\Omega$, the above series reduces to only one spectral projection corresponding to a set of the form $\Omega' + 2k\pi$. Hence $Z$ commutes with all the spectral projections of $X$.

2) We need to consider the Borel measurable function $f$ defined in the proof of Theorem 4.1. Since $e^{ix} = e^y$, we have that $e^{i(2k + 1)\pi} = e^y$. Recall that $E_X(\{(2k + 1)\pi \}) \neq 0$ if and only if $(2k + 1)\pi \in \sigma_p(X)$ ([4, Theorem 12.19]). By the hypothesis on the eigenvalues of $X$, there is at most one $n_0 \in \mathbb{Z}$ such that $E_X(\{(2n_0 + 1)\pi \}) \neq 0$. According to Lemma 2.1 we get

$$E_{f(X)}(\{ \pi \}) = \sum_{k \in \mathbb{Z}} E_X(\{(2k + 1)\pi \}) = E_X(\{(2n_0 + 1)\pi \}).$$  

On the other hand, $E_{f(X)}(\{-\pi \}) = 0$ for all $k \in \mathbb{Z}$ by definition of the function $f$. According to Corollary 2.2, it follows that $f(X) = Y + 2\pi i E_{-1}$. By Remark 2.3, we also know that
$E_X(\{ (2n_0+1)\pi \}) = F_{-1} + F_1$. In order to show that $Y$ commutes with all the spectral projections of $X$, we divide into two cases. If $\Omega \subseteq \mathbb{C} \setminus \{ (2k+1)\pi : k \in \mathbb{Z} \}$, note that $E_X(\Omega)F_{-1} = 0$ because $F_{-1} \leq E_X(\{ (2n_0+1)\pi \})$. Hence we get

$$E_X(\Omega)Y = E_X(\Omega)(f(X) - 2\pi i F_{-1}) = E_X(\Omega)f(X) = f(X)E_X(\Omega) = YE_X(\Omega).$$

If $\Omega \subseteq \{ (2k+1)\pi : k \in \mathbb{Z} \}$, we only need to prove that $E_X(\{ (2n_0+1)\pi \})$ commutes with $Y$. This follows immediately, because $E_X(\{ (2n_0+1)\pi \})$ is the sum of two spectral projections of $Y$.

iii) As in the proof of ii), we have $e^{Yf(X)} = e^Y$. Now by the assumption on the eigenvalues of $X$, it follows that

$$E_{f(X)}(\{ -\pi, \pi \}) = \sum_{k \in \mathbb{Z}} E_X(\{ (2k+1)\pi \}) = 0. \quad (3)$$

Applying Corollary 4.2 iii), we get $f(X) = Y$. In particular, $Y$ is a self-adjoint operator such that $\sigma(Y) \subseteq [-\pi, \pi]$.

Let $Z \in B(\mathcal{H})$ such that $Ze^{iX} = e^{iX}Z$. Then we have $ZE_{e^{iX}}(\Omega) = E_{e^{iX}}(\Omega)Z$ for every Borel set $\Omega \subseteq \mathbb{T}$. We are going to show that $ZE_Y(\Omega') = E_Y(\Omega')Z$ for every $\Omega' \subseteq [-\pi, \pi]$.

We need to consider two cases. If $\Omega' \subseteq (-\pi, \pi)$, there exists a unique set $\Omega \subseteq \mathbb{T} \setminus \{ -1 \}$ such that $\exp^{-1}(\Omega) \cap [-\pi, \pi] = \Omega'$. Therefore

$$E_Y(\Omega') = E_{f(X)}(\Omega') = \sum_{k \in \mathbb{Z}} E_X(\Omega' + 2k\pi) = E_X(\exp^{-1}(\Omega)) = E_{e^{iX}}(\Omega).$$

If $\Omega' \subseteq \{ -\pi, \pi \}$, by equation (3) we find that $E_Y(\Omega') = E_{f(X)}(\Omega') = 0$. Hence we obtain that $Z$ commutes with every spectral projection of $Y$. The latter is equivalent to saying that $Z$ commute with $Y$, and this concludes the proof.

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