THE ARITHMETIC VOLUME OF THE MODULI SPACE OF ABELIAN SURFACES

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Abstract. Let $\mathcal{A}_g$ denote the moduli stack of principally polarized abelian varieties of dimension $g$. The arithmetic height, or arithmetic volume, of $\mathcal{A}_g$, is defined to be the arithmetic degree of the metrized Hodge bundle $\omega_g$ on $\mathcal{A}_g$. In 1999, Kühn proved a formula for the arithmetic volume of $\mathcal{A}_1$ in terms of special values of the Riemann zeta function. In this article, we generalize his result to the case $g = 2$.

1. Introduction

1.1. Arithmetic volumes. Let $\mathcal{A}_g$ denote the moduli stack of principally polarized abelian varieties of dimension $g$. In 1943, Siegel [19] inductively computed the geometric volume, i.e., the degree of the associated Hodge bundle $\omega_g$, as

$$\text{vol}(\mathcal{A}_g) := \text{deg}(\omega_g) = (g-1)!\pi^{-g}\zeta(2g)\text{vol}(\mathcal{A}_{g-1}),$$

where $\zeta(\cdot)$ denotes the Riemann $\zeta$-function.

A model for the moduli stack $\mathcal{A}_g$ over $\text{Spec}(\mathbb{Z})$ was constructed by Mumford et al. in [16]. Later, Faltings and Chai [6] were able to construct a model for toroidal compactifications $\overline{\mathcal{A}}_g$ over $\text{Spec}(\mathbb{Z})$. An arithmetic intersection theory, i.e., an intersection theory for varieties over $\text{Spec}(\mathbb{Z})$, or, more generally, over the spectra of arithmetic rings $A$, was first approached by Arakelov [1] (and later extended by Deligne [5]) in the case of relative dimension 1. By “compactifying” the variety over $A$ by a complex fibre induced by the complex embeddings of $A$, the authors were able to define a good notion of intersection, considering the additional datum of Green functions induced by holomorphic vector bundles with smooth Hermitian metrics on the complex fibre. Soon after, this approach was generalized by Gillet and Soulé for arbitrary relative dimension.

However, this theory is limited to vector bundles with smooth metrics and it can thus not be applied to the case of the metrized Hodge bundle $\overline{\omega}_g$ on $\overline{\mathcal{A}}_g$, as the $L^2$-metric acquires logarithmic singularities on the toroidal boundary. This fact was one of the motivations for Burgos, Kramer, and Kühn [4] to come up with a more flexible concept of arithmetic Chow groups that can be applied to the situation of metrics having mild singularities. Laying the foundation for this generalized intersection theory, in 1999, Kühn [14] computed the arithmetic height, or arithmetic volume, of $\mathcal{A}_1$, which is by definition the arithmetic degree of the metrized Hodge bundle $\overline{\omega}_1$ on $\overline{\mathcal{A}}_1$, to equal

$$\widetilde{\text{vol}}(\mathcal{A}_1) = \widetilde{\text{deg}}(\overline{\omega}_1) = \zeta(-1)\left(\frac{\zeta'(-1)}{\zeta(-1)} + \frac{1}{2}\right).$$

Further results by Bruinier, Burgos and Kühn [2] include the case of Hilbert modular surfaces. A result connecting integrals over Borcherds forms with special values of Eisenstein series by Kudla [13] is a main ingredient for the result of this article. Moreover, an inductive approach for Shimura varieties of orthogonal type was given by Hörmann in [9], which is also applicable to the problem tackled here. More general conjectures about arithmetic intersection numbers by Kramer, Maillot–Roessler [15], and by the so-called Kudla-Program are stating connections between Fourier coefficients of certain modular forms and classes of algebraic cycles in the arithmetic Chow groups. In particular, the arithmetic degree of the bundle of modular forms
is conjectured to be a rational linear combination of logarithmic derivatives of the \( \zeta \)-function evaluated at negative odd integers.

1.2. Main result. In this article, we consider the case \( g = 2 \) and we obtain a formula for the arithmetic height, or arithmetic volume, of \( \mathcal{A}_2 \), by computing the degree of the metrized Hodge bundle \( \omega_2 \) on \( \mathcal{A}_2 \) as (see Theorem 18)

\[
\hat{\text{vol}}(\mathcal{A}_2) = \hat{\text{deg}}(\omega_2) = \zeta(-3)\zeta(-1) \left( 2\frac{\zeta'(-3)}{\zeta(-3)} + 2\frac{\zeta'(-1)}{\zeta(-1)} + \frac{17}{6} \right) + c_2 \log 2 + c_3 \log 3,
\]

with constants \( c_2, c_3 \in \mathbb{Q} \) arising from the intersection number at the finite places 2 and 3. We are taking an explicit approach, identifying the Hodge bundle with the bundle of modular forms, choosing certain well-investigated sections of this bundle and tracing the value of the arithmetic volume back to the results mentioned above by giving a recursive formula of integrals over cycles.

1.3. Outline of the article. The paper is organized as follows. In Section 2, we begin by collecting background information. In particular, we state an identity that allows us to decompose integrals over *-products of Green currents into computable parts (see Proposition 2). In Section 3, we apply Proposition 2 inductively to obtain an explicit expression for the contribution of the arithmetic volume in question arising from the complex fibre (see Proposition 4). In Section 4, we treat the term \((B)\) arising in this expression, in particular the integrals over the positive codimensional cycles of \( \mathcal{A}_2 \). In Section 5, we treat the term \((A)\) arising in the above mentioned expression. In Section 6, we prove the main result of the paper (see Theorem 18).

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2. Background material

In this section, we will give some background theory. First, we will introduce the moduli space \( \mathcal{A}_2 \) and its toroidal compactification. Then, we will give formulas for some particular Siegel modular forms living on it. Finally, we will define the notion of its arithmetic volume and state a result that will be used for its explicit computation.

2.1. The moduli space \( \mathcal{A}_2 \). The Siegel upper half-space \( \mathbb{H}_2 \) is the set

\[
\mathbb{H}_2 := \{ \tau = x + iy \in \text{Sym}_2(\mathbb{C}) \mid x, y \in \text{Sym}_2(\mathbb{R}), y > 0 \},
\]

where the notation \( y > 0 \) denotes that the matrix \( y = \text{im}(\tau) \) is positive definite. As an open submanifold of \( \text{Sym}_2(\mathbb{C}) \), it has dimension 3. We denote the coordinates on \( \mathbb{H}_2 \) by

\[
\tau := \begin{pmatrix} \tau_1 & \tau_{12} \\ \tau_{12} & \tau_2 \end{pmatrix} = x + iy = \begin{pmatrix} x_1 & x_{12} \\ x_{12} & x_2 \end{pmatrix} + i \begin{pmatrix} y_1 & y_{12} \\ y_{12} & y_2 \end{pmatrix},
\]

Setting

\[
J = \begin{pmatrix} 0_2 & \mathbb{1}_2 \\ -\mathbb{1}_2 & 0_2 \end{pmatrix},
\]

the symplectic group \( \text{Sp}_4(\mathbb{Z}) \) is the subgroup of \( \text{GL}_4(\mathbb{Z}) \) defined via

\[
\text{Sp}_4(\mathbb{Z}) := \{ M \in \text{Mat}_4(\mathbb{Z}) \mid M^tJM = J \}.
\]
From now on, we will shortly write \( \Gamma_2 := \text{Sp}_4(\mathbb{Z}) \). The action of the symplectic group on the Siegel upper half-space \( \mathbb{H}_2 \) is given by the prescription

\[
M \tau := (A \tau + B)(C \tau + D)^{-1} \quad \text{with} \quad M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_2, \ \tau \in \mathbb{H}_2.
\]

The quotient \( \mathcal{A}_2 := \Gamma_2 \backslash \mathbb{H}_2 \) is the moduli space of principally polarized abelian surfaces. We denote the quotient morphism by \( \pi_2 : \mathbb{H}_2 \rightarrow \mathcal{A}_2 \).

A particular divisor on \( \mathcal{A}_2 \) appearing later is the Humbert surface given as

\[
H := \pi_2 \left( \left\{ \left( \begin{array}{c} \tau_1 \\ 0 \\ \tau_2 \end{array} \right) \mid \tau_1, \tau_2 \in \mathbb{H}_1 \right\} \right) \cong \text{Sym}^2(\mathcal{A}_1) = \mathcal{A}_1 \times \mathcal{A}_1 / ( (\tau_1, \tau_2) \sim (\tau_2, \tau_1)),
\]

Due to the isomorphism \( \mathcal{H} \cong \text{Sym}^2(\mathcal{A}_1) \), we will denote a subset of \( \mathcal{H} \) that is a product of two subsets \( S_1, S_2 \) of \( \mathcal{A}_1 \) by

\[
S_1 \times_{\pi_2} S_2 := \pi_2 \left( \left\{ \left( \begin{array}{c} \tau_1 \\ 0 \\ \tau_2 \end{array} \right) \mid \tau_1 \in S_1, \tau_2 \in S_2 \right\} \right).
\]

A smooth compactification \( \overline{\mathcal{A}_2} \) of (a cover of) \( \mathcal{A}_2 \) such that the boundary is a normal crossing divisor can be achieved by a suitable toroidal compactification. We recall that the boundary divisor of a toroidal compactification of \( \mathcal{A}_2 \) is isomorphic to a compactification of the universal family over \( \mathcal{A}_1 \), and, therefore, the boundary is of codimension 1, i.e., we have

\[
\partial \mathcal{A}_2 = \overline{\mathcal{A}_2} \setminus \mathcal{A}_2 \cong (\mathbb{Z}^2 \times \Gamma_1) \setminus (\mathbb{C}^2 \times \mathbb{H}_1)^{+}.
\]

Local coordinates in a neighbourhood of \( \partial \mathcal{A}_2 \) are given by

\[
t := \exp(2\pi i \tau_1), \ \tau_2 = x_2 + iy_2, \ \tau_{12} = x_{12} + iy_{12}.
\]

### 2.2. Siegel modular forms.

A holomorphic function \( f : \mathbb{H}_2 \rightarrow \mathbb{C} \) is called \textit{Siegel modular form of degree 2 and weight} \( k \in \mathbb{N} \) for \( \Gamma_2 \) if the following conditions are satisfied:

(i) \( f(M \tau) = \det(C \tau + D)^k f(\tau) \quad (\tau \in \mathbb{H}_2, \ M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_2) \),

(ii) \( f \) is bounded on all subsets of the form \( \{ \tau = x + iy \in \mathbb{H}_2 \mid y - y_0 > 0 \} \subseteq \mathbb{H}_2 \) with \( y_0 \in \text{Sym}^2(\mathbb{R}) \) positive definite.

Condition (ii) implies that \( f \) is holomorphic at the boundary of \( \mathbb{H}_2 \). Here, it actually follows automatically from the holomorphicity of \( f \) and condition (i) by the Koecher principle. Siegel modular forms of degree 2 and weight \( k \) obviously form a vector space, which we will denote by \( M_k(\Gamma_2) \). Due to their transformation properties with respect to \( \Gamma_2 \), one can see that they are global sections of a line bundle \( \mathcal{M}_k(\Gamma_2) \) on \( \mathcal{A}_2 \).

With the above observation, one obtains the identification

\[
H^0(\mathcal{A}_2, \omega_2^{\otimes k}) \cong M_k(\Gamma_2),
\]

where \( \omega_2 \) denotes the Hodge bundle on \( \mathcal{A}_2 \). The \( L^2 \)-metric on the Hodge bundle induces the so-called Petersson metric on the line bundle \( \mathcal{M}_k(\Gamma_2) \), defined by

\[
\| f(\tau) \|^2_{\text{Pet}} := |f(\tau)|^2 ((4\pi)^2 \det(\text{Im}(\tau)))^k,
\]

for \( f \) a global section of \( \mathcal{M}_k(\Gamma_2) \).

The Siegel modular forms in question will now be constructed by means of \( \vartheta \)-series. The \( \vartheta \)-series of a vector \( (a, b) \in (\mathbb{Z}/2\mathbb{Z})^2 \times (\mathbb{Z}/2\mathbb{Z})^2 \) is given by the equality

\[
\vartheta_{a,b}(\tau) := \sum_{n \in \mathbb{Z}^2} e^{2\pi i \left( \frac{a}{4} \left( n + \frac{a}{2} \right)^2 + \frac{b}{4} \left( n + \frac{b}{2} \right)^2 \right)}.
\]
In the coordinates of \( \mathbb{H}_2 \), and with \( n = (n_1, n_2)^t, a = (a_1, a_2)^t, b = (b_1, b_2)^t \), this can be rewritten as
\[
\vartheta_{a,b}(\tau) = \sum_{n_1, n_2 \in \mathbb{Z}} e^{\pi i \left( \frac{n_1 + a_1}{12} \right)^2 \tau_1 + 2 \left( \frac{n_1 + a_1}{12} \right) \left( \frac{n_2 + a_2}{12} \right) \tau_2 + \left( \frac{n_1 + a_1}{12} \right) b_1 + \left( \frac{n_2 + a_2}{12} \right) b_2 \}.
\]

We note that \( \vartheta_{a,b} \) is non-trivial if and only if \( a^t b \) is even; we then call \( \vartheta_{a,b} \) even. There are 10 even \( \vartheta \)-series of degree 2.

The first Siegel modular form we will use is the cusp form \( \chi_{10} \) of weight 10. It is given (up to normalization) by the product of the squares of the 10 even \( \vartheta \)-series
\[
\chi_{10}(\tau) := \frac{1}{2^{12}} \prod_{(a,b) \text{ even}} \vartheta_{a,b}^2(\tau) = e^{2\pi i (\tau_1 + \tau_2 + \tau_3)} \prod_{n,l,m \in \mathbb{Z}, (n,l,m) > 0} \left( 1 - e^{2\pi i (n\tau_1 + l\tau_2 + m\tau_3)} \right)^{2f(n,m,l)}.
\]

Here, \((n,m,l) > 0\) means that \( m,n \geq 0 \), and \( l = -1 \) for \( m = n = 0 \). The exponents \( f(n,m,l) \) of the product expansion are given by the Fourier coefficients of a weak Jacobi form, see [8], Chapter 4. The notation \((n,m,l) > 0\) means that \( m,n \geq 0 \), and \( l = -1 \) for \( m = n = 0 \). Note that the normalization differs from the convention in the literature by the factor 4, and rather corresponds to the one of \( \chi_{10} \) in [11]. This is done in order to have integer coprime Fourier coefficients, see also [17]. It is a well-known result, following from the product expansion, that the divisor of \( \chi_{10} \) is given by
\[
\text{div}(\chi_{10}) = \partial \mathcal{A}_2 + \overline{\mathcal{H}}.
\]
with \( \partial \mathcal{A}_2 \) given by (12) and \( \mathcal{H} \) as in (11).

We will further consider the following three modular forms of degree 2 given by
\[
E_4(\tau) := \frac{1}{4} \sum_{(a,b) \text{ even}} \vartheta_{a,b}^8(\tau), \quad E_6(\tau) := \frac{1}{4} \sum \pm \left( \vartheta_{a_1,b_1}(\tau) \vartheta_{a_2,b_2}(\tau) \vartheta_{a_3,b_3}(\tau) \right)^4, \quad \chi_{12}(\tau) := \frac{1}{2^{15}} \sum \left( \vartheta_{a_1,b_1}(\tau) \cdots \vartheta_{a_6,b_6}(\tau) \right)^4.
\]
The second sum runs over all syzygous triples \((a_j, b_j)\) \((j = 1, 2, 3)\), and the third sum runs over the complements of the syzygous quadruples as described in [10], Chapter 4. A triple \((m_1, m_2, m_3)\) is called syzygous if \( m_1 + m_2 + m_3 \) is even. A quadruple \((m_1, m_2, m_3, m_4)\) is called syzygous if any triple \((m_j, m_k, m_l)\) \((1 \leq j < k < l \leq 4)\) is syzygous. The signs in the second sum arise from a symmetrization process, making it a modular form for \( \Gamma_2 \). For details, see [11]. Note that the normalization of \( \chi_{12} \) again differs from the convention in the literature by a factor 12, and rather corresponds to the one of \( X_{12} \) in [11], in order to have integer coprime Fourier coefficients, see again also [17].

If \( \tau_{12} = 0 \), i.e., if
\[
\tau = \begin{pmatrix} \tau_1 & 0 \\ 0 & \tau_2 \end{pmatrix},
\]
we see from [11] that the \( \vartheta \)-series \( \vartheta_{a,b} \) with \( a = (a_1, a_2)^t \) and \( b = (b_1, b_2)^t \) decomposes as
\[
\vartheta_{a,b}(\tau) = \vartheta_{a_1,b_1}(\tau_1) \vartheta_{a_2,b_2}(\tau_2).
\]

Applying this decomposition to the formulas for the Siegel modular forms \( E_4, E_6, \chi_{12} \), their restrictions to \( \overline{\mathcal{H}} \) decompose as
\[
E_4(\tau) = E_4(\tau_1) E_4(\tau_2), \quad E_6(\tau) = E_6(\tau_1) E_6(\tau_2), \quad \chi_{12}(\tau) = 12 \Delta(\tau_1) \Delta(\tau_2),
\]
with the degree 1 modular forms

\[ E_4(\tau_1) = \frac{1}{2} \left( \vartheta_{00}^4(\tau_1) + \vartheta_{01}^8(\tau_1) + \vartheta_{10}^8(\tau_1) \right), \]

\[ E_6(\tau_1) = \frac{1}{2} \left( \vartheta_{00}^6(\tau_1) + \vartheta_{01}^4(\tau_1) \right) \left( \vartheta_{01}^4(\tau_1) + \vartheta_{10}^4(\tau_1) \right) \left( \vartheta_{01}^4(\tau_1) - \vartheta_{10}^4(\tau_1) \right), \]

\[ \Delta(\tau_1) = \frac{1}{24} \left( \vartheta_{00}(\tau_1) \vartheta_{01}(\tau_1) \vartheta_{10}(\tau_1) \right)^8 = \exp(2\pi i \tau_1) \prod_{n \in \mathbb{Z}} (1 - e^{2\pi in \tau_1})^{24}. \]

Note that with this normalization we have

(8) \[ E_4^3(\tau_1) - E_6^2(\tau_1) = 12^{\frac{3}{2}} \Delta(\tau_1). \]

Similarly, one sees that for \( \tau = \left( \frac{\tau_1}{\tau_2}, \frac{\tau_4}{\tau_2} \right) \in \partial A_2 \), the \( \vartheta \)-series reduce to

\[ \vartheta_{a,b}(\tau) = \vartheta_{a_2,b_4}(\tau_2), \quad \text{if} \ (a_1, b_1) = (0, b_1), \]

\[ \vartheta_{a,b}(\tau) = 0, \quad \text{if} \ (a_1, b_1) = (1, b_1). \]

Hence, on \( \partial A_2 \), the Eisenstein series restrict to

(9) \[ E_4(\tau)|_{\partial A_2} = E_4(\tau_2), \quad E_6(\tau)|_{\partial A_2} = E_6(\tau_2). \]

Note that, with the above normalizations, the Fourier coefficients around the cusp \( \tau_1 = i\infty \) of the modular forms \( E_4, E_6, \chi_{10}, \chi_{12} \) are all integer and coprime. Therefore, the forms are defined over \( \text{Spec}(\mathbb{Z}) \) by the \( q \)-expansion principle, see, e.g., [12].

2.3. The arithmetic volume. We will define the arithmetic volume of \( \overline{A}_2 \) as the highest self-intersection number of the Hodge bundle on \( \overline{A}_2 \) over \( \text{Spec}(\mathbb{Z}) \). In order to do so, we first introduce the notion of a partition of unity adapted to two cycles.

We consider a proper smooth real variety \( X \), and \( Y, Z \) be two cycles on \( X \) of codimension \( p, q \) with support \( |Y|, |Z| \), respectively. We assume that neither \( |Y| \) nor \( |Z| \) are contained in the intersection \( |Y| \cap |Z| \). We define a partition of unity \( \{ \sigma_{YZ}, \sigma_{ZY} \} \) as follows: By a result of [3] there exists a resolution \( \pi: \overline{X}_R \to X_R \) of singularities of \( |Y| \cup |Z| \) which factors through embedded resolutions of \( |Y|, |Z|, |Y| \cap |Z| \). Denote by \( \overline{Y} \) the normal crossing divisor formed by the components of \( \pi^{-1}(|Y|) \) that are not contained in \( \pi^{-1}(|Y| \cap |Z|) \). Analogously, denote by \( \overline{Z} \) the normal crossing divisor formed by the components of \( \pi^{-1}(|Z|) \) that are not contained in \( \pi^{-1}(|Y| \cap |Z|) \). Hence, \( \overline{Y} \) and \( \overline{Z} \) are closed subsets of \( \overline{X} \) that do not meet. Therefore, there exist two smooth, \( F_{\infty} \)-invariant functions \( \sigma_{YZ} \) and \( \sigma_{ZY} \) satisfying \( 0 \leq \sigma_{YZ}, \sigma_{ZY} \leq 1, \sigma_{YZ} + \sigma_{ZY} = 1 \) with \( \sigma_{YZ} = 1 \) in a neighborhood of \( \overline{Y} \) and \( \sigma_{ZY} = 1 \) in a neighborhood of \( \overline{Z} \).

Remark 1. In the following, \( \{ \sigma_{4,6}, \sigma_{6,4} \} \) will always denote a partition of unity adapted to the cycles \( \text{div}(E_4) \) and \( \text{div}(E_6) \), restricted to \( X \subseteq \overline{A}_2 \) arising from the context.

The toroidally compactified moduli stack \( \overline{A}_2 \) is the complex fibre for an arithmetic variety over \( \text{Spec}(\mathbb{Z}) \), constructed by Faltings and Chai [4]. The Hodge bundle on \( \overline{A}_2 \) is the complex bundle associated to the line bundle of invariant 2-differentials on the universal semi-abelian scheme also constructed in [6]. The arithmetic degree of the Hodge bundle over \( \text{Spec}(\mathbb{Z}) \), i.e., its arithmetic self intersection number, can then be interpreted as the arithmetic volume of \( \overline{A}_2 \) (as was done with the geometric volume in the introduction). As the Petersson metric has logarithmic singularities at the boundary \( \partial A_2 \), one has to apply the arithmetic intersection theory established in [4] by Burgos, Kramer and Kühn, generalizing the intersection theory for
smooth metrics by Gillet and Soulé \[7\], to compute the self intersection number of the Hodge bundle over \(\text{Spec}(\mathbb{Z})\). By this theory, we can write the arithmetic volume \(\widehat{\text{vol}}(\mathcal{A}_2)\) as a sum
\[
\widehat{\text{vol}}(\mathcal{A}_2) = \widehat{\text{vol}}(\mathcal{A}_2)_{\text{fin}} + \widehat{\text{vol}}(\mathcal{A}_2)_{\infty},
\]
with \(\widehat{\text{vol}}(\mathcal{A}_2)_{\text{fin}}\) the self intersection number of the Hodge bundle over \(\text{Spec}(\mathbb{Z})\) in the sense of Serre, and the contribution of the volume coming from the complex fibre
\[
\widehat{\text{vol}}(\mathcal{A}_2)_{\infty} = \frac{1}{(2\pi i)^d} \int_{\mathcal{A}_2} g_1 * g_2 * g_3 * g_4.
\]
Here, the \(g_j = \log \|f_j\|_{\text{Pet}}\) are Green currents corresponding to the divisors \(D_1, \ldots, D_4\) four sections \(f_1, \ldots, f_4\) of the Hodge bundle on \(\mathcal{A}_2\) that intersect successively properly. In the case of logarithmically singular metrics, the \(*\)-product of two Green currents \(g_Y, g_Z\) corresponding to cycles \(Y, Z\) is given by
\[
(10) \quad g_Y * g_Z = 4\pi i(\sigma_{YZ} g_Y) \wedge dd^c g_Z + 4\pi i dd^c (\sigma_{YZ} g_Y) \wedge g_Z,
\]
see \[3\], Section 6.1 for details.

Subsequently, we will denote by \(g_4, g_6, g_{10}, g_{12}\) the Green currents corresponding to \(E_4, E_6, \chi_{10}, \chi_{12}\), respectively. Furthermore, we will use the notation \(\omega_k = 4\pi i dd^c g_k\) \((k = 4, 6, 10, 12)\). The following proposition gives an explicit way to decompose the integral over \(*\)-products of Green currents into computable parts.

**Proposition 2.** Let \(X\) be a proper smooth real variety, and let \(D\) be a normal crossing divisor on \(X\). Assume that \(D = D_1 \cup D_2\), where \(D_1\) and \(D_2\) are normal crossing divisors of \(X\) satisfying \(D_1 \cap D_2 = \emptyset\). Let \(Y\) and \(Z\) be cycles of codimension \(p\) and \(q\) with corresponding Green objects \(g_Y = (\omega_Y, g_Y)\) and \(g_Z = (\omega_Z, g_Z)\), respectively. Assume that \(p + q = d + 1\), and that \(Y\) and \(Z\) intersect properly, i.e., \(|Y| \cap |Z| = \emptyset\). Furthermore, assume \(|Y| \cap D_2 = \emptyset\) and \(|Z| \cap D_1 = \emptyset\). Then, we have the decomposition
\[
\frac{1}{(2\pi i)^d} \int_X g_Y * g_Z = \lim_{\varepsilon \to 0} \left( \frac{1}{(2\pi i)^d} \int_{X \setminus B_\varepsilon(D)} g_Y \wedge \omega_Z - \frac{2}{(2\pi i)^{d-1}} \int_{\partial B_\varepsilon(D_1)} [g_Z \wedge d^c g_Y - g_Y \wedge d^c g_Z] \right) + \frac{1}{(2\pi i)^{d-1}} \int_{Y \setminus Y \cap D_1} g_Z.
\]

**Proof.** See \[2\], Theorem 1.14. \(\square\)

### 3. A FORMULA FOR THE COMPLEX CONTRIBUTION

In this section, we will apply Proposition \[2\] inductively to obtain an explicit formula for the complex contribution of the arithmetic volume. First, in Lemma \[5\] we will check the necessary prerequisites of Proposition \[2\] are satisfied by showing that the divisors of the modular forms in question intersect successively properly. Then, in Proposition \[4\] we show that the complex contribution decomposes into two terms \((A)\) and \((B)\), where \((A)\) is the limit term arising from the first induction step, and \((B)\) is essentially the Faltings height of the Humbert surface.
Lemma 3. In the notation of Section 2 we have the following intersections
\[
\text{div}(\chi_{10}) \cdot \text{div}(E_6) = \partial A_2 \cdot \text{div}(E_6) + \frac{1}{2} \left( \{i\} \times_{\mathcal{H}} A_1 \right),
\]
\[
\text{div}(\chi_{10}) \cdot \text{div}(E_6) \cdot \text{div}(E_4) = \frac{1}{6} \left( \{i\} \times_{\mathcal{H}} \{\omega\} \right),
\]
\[
\text{div}(\chi_{10}) \cdot \text{div}(E_6) \cdot \text{div}(E_4) \cdot \text{div}(\chi_{12}) = 0.
\]
Hence, in particular, the divisors corresponding to the modular forms $\chi_{10}$, $E_6$, $E_4$, $\chi_{12}$ intersect successively properly.

Proof. Restricting $E_6$ to $\overline{\mathcal{H}}$ and $\partial A_2$, one obtains by means of (7) and (9) that
\[
\text{div}(E_6) \cdot \text{div}(\chi_{10}) = \pi_2 \left( \left\{ \left( \frac{i}{2}, \frac{\tau_{12}}{i} \right) \mid \tau_{12} \in \mathbb{C} \right\} \right) + \frac{1}{2} \left( \{i\} \times_{\mathcal{H}} A_1 \right),
\]
as $E_6(\tau_2)$ has a zero of order $1/2$ for $\tau_2 = i$. Restricting $E_4$ to $\text{div}(E_6) \cdot \text{div}(\chi_{10})$ and noting that $E_4(\tau_2)$ has a zero of order $1/3$ at $\tau_2 = \omega$, one obtains by means of (7) and (9) that
\[
\text{div}(E_4) \cdot \text{div}(E_6) \cdot \text{div}(\chi_{10}) = \frac{1}{6} \left( \{i\} \times_{\mathcal{H}} \{\omega\} \right).
\]
Finally, as $\chi_{12}$ vanishes only at the boundary $\partial A_2$, one sees that
\[
\text{div}(\chi_{12}) \cdot \text{div}(E_4) \cdot \text{div}(E_6) \cdot \text{div}(\chi_{10}) = 0,
\]
so the divisors intersect properly.

Proposition 4. The contribution of the arithmetic volume arising from the complex fibre can be computed as
\[
\widehat{\text{vol}}(A_2)_{\infty} = \frac{1}{(2\pi i)^3} \int_{\mathcal{A}_2 \setminus \partial A_2} g_{10} \wedge \omega_6 \wedge \omega_4 \wedge \omega_{12} = (A) + (B),
\]
where we have set
\[
(A) := \lim_{\varepsilon \to 0} \left( \frac{1}{(2\pi i)^3} \int_{\mathcal{A}_2 \setminus B_\varepsilon(\partial A_2)} g_{10} \wedge \omega_6 \wedge \omega_4 \wedge \omega_{12} \right.
\]
\[
- \frac{2}{(2\pi i)^2} \int_{\partial B_\varepsilon(\partial A_2)} \left[ (g_{10} \wedge \omega_{12}) \wedge d^c g_{10} - g_{10} \wedge d^c (g_{10} \wedge \omega_{12}) \right]
\]
\[
(B) := - \lim_{\varepsilon \to 0} \left( \frac{2}{2\pi i} \int_{\partial B_\varepsilon(\partial \mathcal{H})} \left[ g_4 \wedge d^c (\sigma_{6,4} g_6) \wedge \omega_{12} - (\sigma_{6,4} g_6) \wedge d^c g_4 \wedge \omega_{12} \right] \right)
\]
\[
+ \frac{1}{(2\pi i)^2} \int_{\mathcal{H}} g_6 \wedge \omega_4 \wedge \omega_{12} + \frac{1}{4\pi i} \int_{\{i\} \times_{\mathcal{H}} A_3} g_4 \wedge \omega_{12} + \frac{1}{6} \int_{\{i\} \times_{\mathcal{H}} \{\omega\}} g_{12}.
\]

Proof. We start by applying Proposition 2 with $X = \mathcal{A}_2$ and $D = D_1 \cup D_2$, where $D_1 = \partial A_2$ and $D_2 = \emptyset$. We consider the cycles
\[
Y = \text{div}(\chi_{10}) = \partial A_2 + \overline{\mathcal{H}}, \quad Z = \text{div}(E_6) \cdot \text{div}(E_4) \cdot \text{div}(\chi_{12}) = 0,
\]
having codimension 1 resp. 3, so \( d = 3 \). By Lemma \( 8 \) \( Y \) and \( Z \) intersect properly. Clearly, \(|Y| \cap D_2 = \emptyset\), and \(|Z| \cap D_1 = \emptyset\) holds, as \( D_1 \) is a component of \( Y \). Thus, by Proposition \( 2 \) we obtain the equality

\[
\frac{1}{(2\pi i)^2} \int_{\mathcal{A}_2} g_{10} \ast (g_6 \ast g_4 \ast g_{12}) = (A) + \frac{1}{(2\pi i)^2} \int_{\mathcal{P}} g_6 \ast g_4 \ast g_{12},
\]

observing that \( \overline{Y} \setminus Y \cap D_1 = \overline{D} \). We will later see that the first integral in the decomposition converges for \( \varepsilon \) approaching 0 and can therefore be taken out of the limit. To treat the second term, we will next apply Proposition \( 2 \) with \( X = \overline{D} \) and \( D = D_1 \cup D_2 \), where \( D_1 = \emptyset \) and \( D_2 = \partial \mathcal{H} \). We consider the cycles

\[
Y = \text{div}(E_6) \cdot \text{div}(E_4) \cdot \overline{D} = \frac{1}{6} (\{i\} \times_\mathcal{H} \{\omega\}), \quad Z = \text{div}(\chi_2). \overline{D} = \{i\infty\} \times_\mathcal{H} \mathcal{A}_1,
\]

having codimension 2 and 1, so \( d = 2 \). Hence, \( Y \) and \( Z \) intersect properly, and clearly, \(|Y| \cap D_2 = \emptyset\) and \(|Z| \cap D_1 = \emptyset\). By Proposition \( 2 \) we then obtain

\[
\frac{1}{(2\pi i)^2} \int_{\mathcal{P}} (g_6 \ast g_4) \ast g_{12} = \lim_{\varepsilon \to 0} \left( \frac{1}{(2\pi i)^2} \int_{\mathcal{P} \setminus B_\varepsilon(\partial \mathcal{H})} (g_6 \ast g_4) \wedge \omega_{12} \right) + \frac{1}{6} \int_{\{i\} \times_\mathcal{H} \{\omega\}} g_{12},
\]

observing that \( \overline{Y} \setminus Y \cap D_1 = \overline{Y} \). To treat the first term, we now rewrite \( g_6 \ast g_4 \), using \( (10) \), and obtain

\[
g_6 \ast g_4 = 4\pi i \left( (\sigma_{1,6} g_6) \text{d}^c g_4 + \text{d}^c (\sigma_{6,4} g_6) \wedge g_4 \right)
\]

\[
= 4\pi i \left( g_6 \wedge \text{d}^c g_4 - (\sigma_{6,4} g_6) \wedge \text{d}^c g_4 + (\text{d}(\text{d}^c (\sigma_{6,4} g_6) \wedge g_4) - \text{d}^c (\sigma_{6,4} g_6) \wedge \text{d} g_4) \right)
\]

\[
= 4\pi i \left( g_6 \wedge \text{d}^c g_4 + \text{d}(\text{d}^c (\sigma_{6,4} g_6) \wedge g_4 - (\sigma_{6,4} g_6) \wedge \text{d}^c g_4) \right),
\]

with \( \{\sigma_{1,6}, \sigma_{6,4}\} \) a partition of unity adapted to \( \text{div}(E_4) \) and \( \text{div}(E_6) \). Note that this expression is singular along \( \text{div}(E_6) \cdot \overline{D} = \frac{1}{2} (\{i\} \times_\mathcal{H} \mathcal{A}_1) \). We therefore set \( \mathcal{H}_\varepsilon := \overline{D} \setminus B_\varepsilon(\partial \mathcal{H} \cup \{i\} \times_\mathcal{H} \mathcal{A}_1) \) and obtain

\[
\frac{1}{(2\pi i)^2} \int_{\mathcal{H}_\varepsilon} (g_6 \ast g_4) \wedge \omega_{12} = \frac{1}{(2\pi i)^2} \int_{\mathcal{H}_\varepsilon} g_6 \wedge \omega_4 \wedge \omega_{12}
\]

\[
+ \frac{4\pi i}{(2\pi i)^2} \int_{\mathcal{H}_\varepsilon} \text{d}(g_4 \wedge \text{d}^c (\sigma_{6,4} g_6) \wedge \omega_{12} - (\sigma_{6,4} g_6) \wedge \text{d}^c g_4 \wedge \omega_{12}).
\]

Applying Stokes’ theorem and taking the limit \( \varepsilon \to 0 \), we obtain

\[
\lim_{\varepsilon \to 0} \frac{1}{(2\pi i)^2} \int_{\mathcal{H}_\varepsilon} (g_6 \ast g_4) \wedge \omega_{12} = \frac{1}{(2\pi i)^2} \int_{\mathcal{P}} g_6 \wedge \omega_4 \wedge \omega_{12} + \frac{1}{4\pi i} \int_{\{i\} \times_\mathcal{H} \mathcal{A}_1} g_4 \wedge \omega_{12}
\]

\[
- \lim_{\varepsilon \to 0} \frac{2}{2\pi i} \int_{\partial B_\varepsilon(\partial \mathcal{H})} [g_4 \wedge \text{d}^c (\sigma_{6,4} g_6) \wedge \omega_{12} - (\sigma_{6,4} g_6) \wedge \text{d}^c g_4 \wedge \omega_{12}] .
\]

This completes the proof of the theorem. \( \square \)

4. THE FALTINGS HEIGHT OF THE Humbert SURFACE

In this section, we treat the term \( (B) \) arising in Proposition \( 3 \) of Section \( 3 \). First, using a well-known result by Rohrlich, we are able to explicitly compute the integrals in the line of \( (B) \), i.e. over the positive codimensional cycles of \( \mathcal{A}_2 \). Then, we show the vanishing of the boundary term, by explicit estimates. We start by recalling Rohrlich’s result.
Lemma 5. Let \( f \) be a modular form of weight \( k \) with \( f(i\infty) = 1 \). Then, we have the equality

\[
\int_{A_1} \log \| f(\tau_1) \|_{\text{Pet}} \frac{dx_1 dy_1}{4\pi y_1^2} = -k \left( \frac{1}{2} \zeta(-1) + \zeta'(-1) \right) - \frac{1}{12} \sum_{\tau_0 \in \text{div}(f)} \ord_{\tau_0}(f) \log \| \Delta(\tau_0) \|_{\text{Pet}}.
\]

Here, \( \tau_1 = x_1 + iy_1 \) is the coordinate on \( A_1 \), and the term \( \| \Delta(\tau_0) \|_{\text{Pet}} \) denotes the quantity \((4\pi)^6 |\Delta(\tau_0)y_0^6|\).

Proof. The lemma follows from a theorem of Rohrlich, see [18]. For the computations leading to the stated version of the formula, see the proof of Theorem 1.6.1 in [14]. \( \square \)

Proposition 6. The second line in (B), see Proposition 4, has the value

\[
\frac{1}{(2\pi i)^2} \int_{\Pi} g_6 \wedge \omega_4 \wedge \omega_{12} + \frac{1}{2\pi i} \int_{\{i\times H A_1} g_4 \wedge \omega_{12} + \frac{1}{6} \int_{(i\times H)(\omega)} g_{12}
\]

\[
= -6 \left( \frac{1}{2} + \frac{\zeta'(-1)}{\zeta(-1)} \right) - \frac{4}{3} \log 2 - \frac{2}{3} \log 3.
\]

Proof. We will show the equality

\[
\frac{1}{(2\pi i)^2} \int_{\Pi} g_6 \wedge \omega_{12} \wedge \omega_4 = 48 \left( \frac{1}{2} \zeta(-1) + \zeta'(-1) \right) + \frac{1}{3} \log \| \Delta(i) \|_{\text{Pet}}.
\]

The other terms can be computed in a similar way. Noting that \( \tau_{12} = 0 \) on \( H \), we see that the Chern forms \( \omega_{12} \) and \( \omega_4 \) reduce to

\[
\omega_{12} = 6 \cdot 4\pi i \left( \frac{dx_1 dy_1}{4\pi y_1^2} + \frac{dx_2 dy_2}{4\pi y_2^2} \right), \quad \omega_4 = 2 \cdot 4\pi i \left( \frac{dx_1 dy_1}{4\pi y_1^2} + \frac{dx_2 dy_2}{4\pi y_2^2} \right)
\]

on \( H \). For their product \( \omega_4 \wedge \omega_{12} \), one obtains

\[
\omega_4 \wedge \omega_{12} = 24(4\pi i)^2 \frac{dx_1 dy_1}{4\pi y_1^2} \wedge \frac{dx_2 dy_2}{4\pi y_2^2}.
\]

Recalling the isomorphism \( H \cong \text{Sym}_2(A_1) \) (insert in Section 3), we note that \( A_1 \times A_1 \) is a double cover of \( H \). We obtain

\[
\frac{1}{(2\pi i)^2} \int_{\Pi} g_6 \wedge \omega_4 \wedge \omega_{12} = 24 \left( \frac{4\pi i}{(2\pi i)^2} \right) \int_{\Pi} g_{E_6} \wedge \frac{dx_1 dy_1}{4\pi y_1^2} \wedge \frac{dx_2 dy_2}{4\pi y_2^2}
\]

\[
= 96 \int_{\Pi} g_6 \wedge \frac{dx_1 dy_1}{4\pi y_1^2} \wedge \frac{dx_2 dy_2}{4\pi y_2^2} = 48 \int_{A_1 \times A_1} g_6 \wedge \frac{dx_1 dy_1}{4\pi y_1^2} \wedge \frac{dx_2 dy_2}{4\pi y_2^2}.
\]

With \( g_6 = -\log \| E_6 \|_{\text{Pet}} \) and the decomposition of \( E_6 \) on \( H \) as \( E_6(\tau) = E_6(\tau_1)E_6(\tau_2) \), the latter integral writes as

\[
48 \int_{A_1 \times A_1} g_6 \wedge \frac{dx_1 dy_1}{4\pi y_1^2} \wedge \frac{dx_2 dy_2}{4\pi y_2^2}
\]

\[
= 48 \int_{A_1 \times A_1} -\log \| E_6(\tau) \|_{\text{Pet}} \wedge \frac{dx_1 dy_1}{4\pi y_1^2} \wedge \frac{dx_2 dy_2}{4\pi y_2^2}
\]

\[
= 48 \int_{A_1 \times A_1} [ -\log \| E_6(\tau_1) \|_{\text{Pet}} - \log \| E_6(\tau_2) \|_{\text{Pet}} ] \frac{dx_1 dy_1}{4\pi y_1^2} \frac{dx_2 dy_2}{4\pi y_2^2}.
\]
With respect to the volume form $\frac{dx dy}{4\pi y^2}$, the volume of $A_1$ equals $\frac{1}{12}$. Hence, we can simplify the last integral to
\[
48 \int\int_{A_1} \left[ -\log \|E_6(\tau_1)\|_{\text{Pet}} - \log \|E_6(\tau_2)\|_{\text{Pet}} \right] \frac{d\tau_1 d\tau_2}{4\pi y^2} = 48 \left( \frac{1}{12} \int_{A_1} -\log \|E_6(\tau_1)\|_{\text{Pet}} \frac{d\tau_1}{4\pi y^2} + \frac{1}{12} \int_{A_1} -\log \|E_6(\tau_2)\|_{\text{Pet}} \frac{d\tau_2}{4\pi y^2} \right) = -8 \int_{A_1} \log \|E_6(\tau)\|_{\text{Pet}} \frac{d\tau}{4\pi y^2}.
\]

Noting that $\text{ord}_i(E_6) = \frac{1}{7}$, Lemma 5 yields
\[
\int_{A_1} \log \|E_6\|_{\text{Pet}} \frac{d\tau}{4\pi y^2} = -6 \left( \frac{1}{2} \zeta(-1) + \zeta'(-1) \right) - \frac{1}{24} \log \|\Delta_i\|_{\text{Pet}}.
\]

Plugging (12) into (11), proves the claim of the proposition. \(\square\)

**Proposition 7.** The integral in the limit in (B)
\[
\frac{4}{2\pi i} \int_{\partial B_{\varepsilon}(\partial \mathcal{H})} \left[ g_4 \wedge d^c (\sigma_{6,4} g_6) \wedge \omega_{12} - (\sigma_{6,4} g_6) \wedge d^c g_4 \wedge \omega_{12} \right]
\]
converges absolutely, and its value tends to 0 for $\varepsilon$ approaching 0.

**Proof.** Noting that $\tau_{12} = 0$ and $y_1 = M$ are constant on $\partial B_{\varepsilon}(\partial \mathcal{H})$, the form $\omega_{12}$ reduces to
\[
\omega_{12} = 4\pi i dd^c g_{12} = 6i \frac{d\tau_1 d\tau_2}{y_2^2}
\]
on $\partial B_{\varepsilon}(\partial \mathcal{H})$. As $\sigma_{6,4}$ depends only on the coordinate $\tau_2$, the form $d^c \sigma_{6,4} \wedge \omega_{12}$ vanishes. Estimating $d^c g_4$ and $d^c g_6$, we can bound the integrand by
\[
|g_4 \wedge d^c (\sigma_{6,4} g_6) \wedge \omega_{12} - (\sigma_{6,4} g_6) \wedge d^c g_4 \wedge \omega_{12}| < (|g_4| + |g_6|) \frac{1}{M} d\theta \frac{d\tau_1 d\tau_2}{y_2^2}.
\]
Furthermore, note that on $\partial B_{\varepsilon}(\partial \mathcal{H})$, we have the equality $\tau_1 = x_1 + iM$, with $M = -\log \varepsilon/2\pi$. The equality $E_k(\tau) = E_k(\tau_1)E_k(\tau_2)$ on $\mathcal{H}$ ($k = 4, 6$), see (insert in Section 3), induces the decomposition of the Green forms $g_4(\tau)$ and $g_6(\tau)$ as
\[
g_k(\tau) = g_k(\tau_1) + g_k(\tau_2) \quad (k = 4, 6)
\]
on $\partial B_{\varepsilon}(\partial \mathcal{H})$. As the degree 1 Eisenstein series $E_4(\tau_1)$ and $E_6(\tau_1)$ do not vanish for $M$ approaching $\infty$, the term
\[
g_k(\tau_1) = -\log \|E_k(\tau_1)\|_{\text{Pet}} = -|E_k(\tau_1)| - k/2 \log(4\pi) - k/2 \log M \quad (k = 4, 6)
\]
is of order $\log M$ for $\tau_1 = x_1 + iM$. Therefore, on $\partial B_{\varepsilon}(\partial \mathcal{H})$, we obtain
\[
|g_k(\tau)| < \log M + |g_k(\tau_2)| \quad (k = 4, 6).
\]
The domain of integration $\partial B_{\varepsilon}(\partial \mathcal{H}) \subseteq \mathcal{H}$ is of the form $\partial B_{\varepsilon}(i\infty) \times_{\mathcal{H}} A_1$.\]
Hence, we can estimate
\[
\left| \frac{4}{2\pi i} \int_{\partial B_c(\partial\mathcal{H})} \left[ g_4 \wedge d^c(\sigma_{6,4} g_6) \wedge \omega_{12} - (\sigma_{6,4} g_6) \wedge d^c g_4 \wedge \omega_{12} \right] \right| \\
\lesssim \frac{\log M}{M} + \frac{1}{M} \int_0^{2\pi} \int_{A_1} \left| g_4(\tau_2) \right| + \left| g_6(\tau_2) \right| \frac{dx_2 dy_2}{y_2^2} d\theta.
\]
As the integrals of \( g_4(\tau_2) \) and \( g_6(\tau_2) \) over \( A_1 \) converge absolutely, the last sum vanishes for \( \varepsilon \) approaching 0, and, hence, \( M \) approaching \( \infty \), and the claim follows. \( \square \)

The two previous theorems imply the following result.

**Theorem 8.** The term \((B)\) in Proposition 4 converges absolutely, and its value tends to 0 for \( \varepsilon \) approaching 0.

5. A BOUNDARY INTEGRAL

In this section, we treat the term \((A)\) arising in Proposition 4 of Section 4. First, we are able to trace back one part of \((A)\) to a deep result by Kudla. The main part of this section is then devoted to proof of the vanishing of remaining boundary integral.

**Proposition 9.** The limit for the first term in \((A)\) in Proposition 4 exists and has the value
\[
\frac{1}{(2\pi i)^3} \int \frac{g_{10} \wedge \omega_6 \wedge \omega_4 \wedge \omega_{12}}{x_2^3} = 10 \cdot 6 \cdot 4 \cdot 12 \zeta(-3) \zeta(-1) \left( \frac{4}{3} + 2 \frac{\zeta'(-3)}{\zeta(-3)} - \frac{\zeta'(-1)}{\zeta(-1)} + \frac{6}{5} \log 2 \right).
\]

**Proof.** This follows from a result of Kudla, see [13]. \( \square \)

In the following, we will prove the vanishing of term \((B)\) in Proposition 4. The proof of the vanishing of the first integral in \((B)\) requires several steps, simplifying the integrand and dividing the domain of integration. We will prepare the proof with some preliminary lemmata.

**Lemma 10.** On \( \partial B_c(\partial\mathcal{A}_2) \), we have the equality
\[
\frac{1}{(4\pi i)^2} (g_6 * g_4 * g_{12}) \wedge d^c g_{10} = (\sigma_{4,6} g_6) \wedge dd^c g_4 \wedge dd^c g_{12} \wedge d^c g_{10} + dd^c(\sigma_{6,4} g_6) \wedge g_4 \wedge dd^c g_{12} \wedge d^c g_{10}.
\]

**Proof.** To compute the form \( g_6 * g_4 * g_{12} \), we first consider the non-intersecting cycles \( \text{div}(E_6) \cdot \text{div}(E_4) \) and \( \text{div}(\chi_{12}) = \partial\mathcal{A}_2 \). Let now \( \{\sigma_{(6,4),12}, \sigma_{12,(6,4)}\} \) be a partition of unity adapted to \( \text{div}(E_6) \cdot \text{div}(E_4) \) and \( \text{div}(\chi_{12}) \). By the definition of the \(*\)-product [10], we obtain
\[
g_6 * g_4 * g_{12} = 4\pi i \left( (\sigma_{12,(6,4)}(g_6 * g_4)) \wedge dd^c g_{12} + dd^c(\sigma_{6,4}(g_6 * g_4)) \wedge g_{12} \right) .
\]
In a neighbourhood of \( \text{div}(\chi_{12}) = \partial\mathcal{A}_2 \), we have \( \sigma_{(6,4),12} = 0 \) and \( \sigma_{12,(6,4)} = 1 \). For \( \varepsilon \) small, we therefore obtain on \( \partial B_c(\partial\mathcal{A}_2) \) the equality
\[
g_6 * g_4 * g_{12} = 4\pi i (g_6 * g_4) \wedge dd^c g_{12} .
\]
Using the definition of the \(*\)-product to compute \( g_6 * g_4 \), we then obtain on \( \partial B_c(\partial\mathcal{A}_2) \) the equality
\[
g_6 * g_4 * g_{12} = (4\pi i)^2 \left( (\sigma_{4,6} g_6) \wedge dd^c g_4 \wedge dd^c g_{12} + dd^c(\sigma_{6,4} g_6) \wedge g_4 \wedge dd^c g_{12} \right) .
\]
with \( \{\sigma_{4,6}, \sigma_{6,4}\} \) a partition of unity adapted to \( \text{div}(E_4) \) and \( \text{div}(E_6) \). This proves the claim of the lemma. \( \square \)
Lemma 11. On \( \partial B_c(\partial A_2) \), we have the equality
\[
\frac{1}{(4\pi i)^2} g_{10} \wedge d^c(g_6 \ast g_4 \ast g_{12}) = g_{10} \wedge d^c(\sigma_{4,6}g_6) \wedge dd^c g_4 \wedge dd^c g_{12} + g_{10} \wedge dd^c(\sigma_{6,4}g_6) \wedge d^c g_4 \wedge dd^c g_{12}.
\]

Proof. The claim follows by a straightforward computation, as in the proof of Lemma 11. \( \square \)

To give estimates for the integrand \((g_6 \ast g_4 \ast g_{12}) \wedge d^c g_{10} - g_{10} \wedge d^c(g_6 \ast g_4 \ast g_{12})\), we will embed the domain of integration \(\partial B_c(\partial A_2)\) into a fundamental domain for \(\Gamma_2 \backslash \mathbb{H}_2\).

Lemma 12. Let \( M = -\log \varepsilon/2\pi \). There is a fundamental domain \( F \subseteq \mathbb{H}_2 \) for the action of \( \Gamma_2 \) on \( \mathbb{H}_2 \) such that the preimage of \( \partial B_c(\partial A_2) \) under the quotient morphism \( \pi_2: \mathbb{H}_2 \rightarrow A_2 \) restricted to \( F \) is contained in the set \( S_\varepsilon \) given by restricting the local coordinates under consideration as follows:
\[
y_1 = M, \ y_2 \in [1/2, M], \ y_{12} \in [0, y_2/2], \ \theta \in [0, 2\pi), \ x_2, x_{12} \in [-1/2, 1/2].
\]

Proof. Recall that a fundamental domain \( F_2 \) for the action of \( \Gamma_2 \) on \( \mathbb{H}_2 \) is given by the following Minkowski conditions on \( \tau = x + iy \in \mathbb{H}_2 \):

(i) For all \( \gamma = (\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}) \in \Gamma_2 \) the inequality \( |\det(C\tau + D)| \geq 1 \) holds.

(ii) The matrix \( y \) is Minkowski reduced, i.e., for all \( l \in \mathbb{Z}^2 \) such that the last \( g - k + 1 = 3 - k \) entries are relatively prime, we have \( y_k \leq l'y_l \) \((k = 1, 2)\); furthermore, \( y_{12} \geq 0 \) holds.

(iii) The matrix \( x \) satisfies \( |x_k| \leq \frac{1}{2} \) \((k = 1, 2, 12)\).

Applying condition (ii) with \( k = 1 \) and \( l = (\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}) \), one obtains \( y_1 \leq y_2 \). For \( k = 2 \) and \( l = (\begin{smallmatrix} 1 \\ -1 \end{smallmatrix}) \), one obtains
\[
y_2 \leq \begin{pmatrix} 1 \\ -1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_{12} \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = y_1 + y_2 - 2y_{12},
\]
and the condition \( 2y_{12} \leq y_1 \) follows. We immediately deduce the inequality \( \det y \leq y_1 y_2 \leq 2 \det y \). Furthermore, applying condition (i) for \( C = (\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}) \) and \( D = (\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}) \), together with condition (iii), one gets \( \tau_1 = 1 \in \mathcal{F}_1 \), where \( \mathcal{F}_1 \) is the standard fundamental domain for the action of \( \Gamma_1 \) on \( \mathbb{H}_1 \). Hence, we can assume \( y_1 \geq \sqrt{3}/2 > 1/2 \). Interchanging the roles of \( \tau_1 \) and \( \tau_2 \) by translating \( \mathcal{F}_2 \) employing the action of the matrix
\[
S := \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \in \Gamma_2.
\]
on \( \mathbb{H}_2 \), we obtain a new fundamental domain \( F = S \mathcal{F}_2 \) such that \( \partial B_c(\partial A_2) \) lies in the set \( S_\varepsilon \) as claimed. \( \square \)

Remark 13. From now on, we will identify \( \partial B_c(\partial A_2) \) with its preimage in the fundamental domain \( \mathcal{F} \) introduced in Lemma 12.

Lemma 14. The partition of unity \( \{\sigma_{4,6}, \sigma_{6,4}\} \) adapted to the divisors \( \text{div}(E_4) \) and \( \text{div}(E_6) \) can be chosen in a way such that in a small neighbourhood of \( \partial A_2 \), and hence on \( \partial B_c(\partial A_2) \) (for small \( \varepsilon > 0 \)), the following properties hold:

(i) The value of \( \sigma_{4,6} \) and \( \sigma_{6,4} \) depends only on the value of the coordinate \( \tau_2 \), i.e., \( \sigma_{4,6}(\tau) = \sigma_{4,6}(\tau_2) \) and \( \sigma_{6,4}(\tau) = \sigma_{6,4}(\tau_2) \).

(ii) We have \( \sigma_{4,6}(\tau_2) = \sigma_{6,4}(\tau_2) = 1/2 \) for \( y_2 > 2 \).

Proof. By (to be inserted), the value of \( E_4 \) and \( E_6 \) restricted to \( \partial A_2 \) only depends on the coordinate \( \tau_2 \). Hence, we can assume the same for the partition of unity in a small neighbourhood of \( \partial A_2 \). Furthermore, we note that the cycles \( \text{div}(E_4) \cdot \partial A_2 \) and \( \text{div}(E_6) \cdot \partial A_2 \) are supported in the open set defined by the condition \( y_2 < 2 \), as \( \text{Im}(i), \text{Im}(\omega) < 2 \). Therefore, we can choose \( \sigma_{4,6}(\tau_2) \) and \( \sigma(\tau_2) \) to equal 1/2 outside this range. \( \square \)
Lemma 15. For any \( \vartheta \)-series \( \vartheta_{a,b} \) as in (4), we have the bound

\[
\left| \frac{\partial}{\partial x_1} \vartheta_{a,b}(\tau) \right| = 2\pi \left| \frac{\partial}{\partial \vartheta} \vartheta_{a,b}(\tau) \right| \approx \varepsilon \frac{1}{\pi}
\]

for \( \tau \in S_\varepsilon \) defined in Lemma 12.

Proof. On \( \partial A_2 \), the coordinate \( \tau_1 \) is constant, equal to \( i\infty \). Hence, the partial derivative \( \frac{\partial}{\partial x_1} \vartheta_{a,b} \) vanishes on \( \partial A_2 \). To determine its vanishing order, we note that for a matrix \( \tau \in S_\varepsilon \) the inequalities \( y_{12} \leq y_2/2 \) and \( y_2 < y_1 \) hold by definition of \( S_\varepsilon \). We deduce the inequality \( (y_{12}/2)^2 - y_{12}^2 > 0 \). Therefore, the matrix

\[
\tau' = \left( \begin{array}{cc}
\tau_1 - \frac{y_2}{2} & \tau_{12} \\
\tau_{12} & \tau_2
\end{array} \right)
\]

obtained from \( \tau \) by replacing the coordinate \( y_1 \) by \( y_{12}/2 \), lies in \( \mathbb{H}_2 \). Letting \( n = (n_1, n_2) \in \mathbb{Z}^2 \) and \( a = (a_1, a_2)^t, b = (b_1, b_2)^t \in (\mathbb{Z}/2\mathbb{Z})^2 \), we expand \( \vartheta_{a,b} \) as

\[
\vartheta_{a,b}(\tau) = \sum_{n \in \mathbb{Z}^2} e^{2\pi i \left( \frac{1}{2} (n + \frac{a}{2})^t \tau (n + \frac{a}{2}) + (n + \frac{a}{2})^t \frac{b}{2} \right)}
= \sum_{n \in \mathbb{Z}^2} e^{\pi \left( (n + \frac{a_1}{2})^2 \tau_1 + 2(n + \frac{a_2}{2})(n_2 + \frac{a_2}{2}) \tau_{12} + (n_2 + \frac{a_2}{2})^2 \tau_2 + (n_1 + \frac{a_1}{2}) b_1 + (n_2 + \frac{a_2}{2}) b_2 \right)}
= \sum_{n \in \mathbb{Z}^2} e^{-\pi (n + \frac{a_1}{2})^2 \frac{y_2}{2}} e^{2\pi i \left( \frac{1}{2} (n + \frac{a}{2})^t \tau (n + \frac{a}{2}) + (n + \frac{a}{2})^t \frac{b}{2} \right)}.
\]

For \( n_1 \neq 0 \) or \( a_1 \neq 0 \), we deduce with \( y_1 = -\log \varepsilon/2\pi \) that

\[
e^{-\pi (n + \frac{a_1}{2})^2 \frac{y_1}{2}} \leq e^{\varepsilon \frac{1}{\pi} \log \varepsilon} = \varepsilon \frac{1}{\pi}.
\]

For the partial derivative \( \frac{\partial}{\partial x_1} \vartheta_{a,b} \), we now obtain

\[
\left| \frac{\partial}{\partial x_1} \vartheta_{a,b}(\tau) \right| = \sum_{n \in \mathbb{Z}^2} e^{-\pi (n + \frac{a_1}{2})^2 \frac{y_2}{2}} \pi i \left( n_1 + \frac{a_1}{2} \right)^2 e^{2\pi i \left( \frac{1}{2} (n + \frac{a}{2})^t \tau (n + \frac{a}{2}) + (n + \frac{a}{2})^t \frac{b}{2} \right)} \leq \varepsilon \frac{1}{\pi} \sum_{n \in \mathbb{Z}^2} \left| \pi i \left( n_1 + \frac{a_1}{2} \right)^2 e^{2\pi i \left( \frac{1}{2} (n + \frac{a}{2})^t \tau (n + \frac{a}{2}) + (n + \frac{a}{2})^t \frac{b}{2} \right)} \right|,
\]

and the latter sum converges, as the sum

\[
\sum_{n \in \mathbb{Z}^2} \pi i \left( n_1 + \frac{a_1}{2} \right)^2 e^{2\pi i \left( \frac{1}{2} (n + \frac{a}{2})^t \tau (n + \frac{a}{2}) + (n + \frac{a}{2})^t \frac{b}{2} \right)} = \frac{\partial}{\partial x_1} \vartheta_{a,b}(\tau')
\]

converges absolutely for \( \tau' \in \mathbb{H}_2 \). This proves the claim. \( \square \)

We will now prove the vanishing of the integral along \( \partial B_\varepsilon(\partial A_2) \) by suitably subdividing the integration domain.

Theorem 16. The integral

\[
\frac{1}{(4\pi i)^2} \int_{\partial B_\varepsilon(\partial A_2)} \left[ (g_6 * g_4 * g_{12}) \land d^c g_{10} - g_{10} \land d^c (g_6 * g_4 * g_{12}) \right]
\]

converges absolutely, and its value tends to 0 for \( \varepsilon \) approaching 0.
Proof. Let \( \{\sigma_{4,6}, \sigma_{6,4} \} \) be a partition of unity adapted to the divisors \( \text{div}(E_4) \) and \( \text{div}(E_6) \). Let \( N_4 \subseteq \partial B_\varepsilon(\partial A_2) \) be a neighbourhood of \( |\text{div}(E_4)| \cap \partial B_\varepsilon(\partial A_2) \) such that \( \sigma_{4,6} = 1 \) on \( N_4 \), and let \( N_6 \subseteq \partial B_\varepsilon(\partial A_2) \) be a neighbourhood of \( |\text{div}(E_6)| \cap \partial B_\varepsilon(\partial A_2) \) such that \( \sigma_{6,4} = 1 \) on \( N_6 \). We define an open subset \( U \) of \( \partial B_\varepsilon(\partial A_2) \) by setting

\[
U := N_4 \cup N_6 \cup (\partial B_\varepsilon(\partial A_2) \cap \{\tau \in \mathcal{F} \mid y_2 > 2\}) \subseteq \partial B_\varepsilon(\partial A_2).
\]

We note that this is a disjoint union. We first show that for \( U \subseteq \partial B_\varepsilon(\partial A_2) \) as before, the integral

\[
\frac{1}{(4\pi i)^2} \int_U [(g_6 * g_4 * g_{12}) \wedge d^c g_{10} - g_{10} \wedge d^c (g_6 * g_4 * g_{12})]
\]

converges absolutely, and its value tends to 0 for \( \varepsilon \) approaching 0. Applying Lemmas 10 and 11, we see that the form

\[
\frac{1}{(4\pi i)^2} \left( (g_6 * g_4 * g_{12}) \wedge d^c g_{10} - g_{10} \wedge d^c (g_6 * g_4 * g_{12}) \right)
\]

restricts on \( N_4 \) to

\[
g_6 \wedge dd^c g_4 \wedge dd^c g_{12} \wedge d^c g_{10} - g_{10} \wedge d^c g_6 \wedge dd^c g_4 \wedge dd^c g_{12},
\]

on \( N_6 \) to

\[
(dd^c g_6 \wedge g_4 \wedge dd^c g_{12} \wedge d^c g_{10} - g_{10} \wedge dd^c g_6 \wedge d^c g_4 \wedge dd^c g_{12},
\]

and on \( \partial B_\varepsilon(\partial A_2) \cap \{\tau \in \mathcal{F} \mid y_2 > 2\} \) to

\[
\frac{1}{2} (g_6 \wedge dd^c g_4 \wedge dd^c g_{12} \wedge d^c g_{10} - g_{10} \wedge d^c g_6 \wedge dd^c g_4 \wedge dd^c g_{12}
\]

\[
+ dd^c g_6 \wedge g_4 \wedge dd^c g_{12} \wedge d^c g_{10} - g_{10} \wedge dd^c g_6 \wedge d^c g_4 \wedge dd^c g_{12}),
\]

as here \( \sigma_{4,6} = \sigma_{6,4} = 1/2 \) holds. With the explicit formulas for \( g_4, g_6, g_{10}, g_{12} \) (to be inserted in Section 2), we can explicitely express the integrand on \( U \) in the form

\[
f(\tau)d\theta dx_2 dy_2 dx_{12} dy_{12},
\]
where \( f(\tau) \) is a smooth function depending on \( \varepsilon \). We will first bound \( f(\tau) \) on \( N_4 \), and then deduce that the same bound holds for the whole \( U \). With \( M = -\log \varepsilon/2\pi \), one finds

\[
f(\tau) = \frac{3}{16\pi^3} \left( -\log |E_6| - 6\log(4\pi) - 3\log (M_{y_2} - y_{12}^2) \right) \times
\]

\[
\left( \frac{4M^2}{(M_{y_2} - y_{12}^2)^3} \left( -\varepsilon \frac{1}{|\chi_{10}|} \frac{\partial}{\partial r} |\chi_{10}| + \frac{5}{2\pi} \frac{y_2}{M_{y_2} - y_{12}^2} \right) + \frac{4y_{12}^2}{(M_{y_2} - y_{12}^2)^3} 2\pi \left( \frac{1}{|\chi_{10}|} \frac{\partial}{\partial y_2} |\chi_{10}| + \frac{5M}{M_{y_2} - y_{12}^2} \right) + \frac{4M_{y_{12}}}{(M_{y_2} - y_{12}^2)^3} 2\pi \left( \frac{1}{|\chi_{10}|} \frac{\partial}{\partial y_{12}} |\chi_{10}| - \frac{10y_{12}}{M_{y_2} - y_{12}^2} \right) \right) \]

\( (17) \)

The modular form \( \chi_{10} \) has a simple zero along \( \partial A_2 \), and a simple zero along \( \mathcal{H} \), given by the equality \( \tau_{12}^2 = 0 \) for the local coordinate \( \tau_{12} \) around \( \mathcal{H} \). Therefore, it decomposes as \( \chi_{10} = r\tau_{12}^2\phi_{10}(\tau) \), with \( \phi_{10}(\tau) \) a non-zero smooth function in a neighbourhood of \( \partial A_2 \). Hence, the terms

\[
\varepsilon \frac{1}{|\chi_{10}|} \frac{\partial}{\partial r} |\chi_{10}|, \quad \frac{1}{|\chi_{10}|} \frac{\partial}{\partial y_2} |\chi_{10}|, \quad \frac{y_{12}}{|\chi_{10}|} \frac{\partial}{\partial y_{12}} |\chi_{10}|
\]

are bounded from above on \( \partial B_{\varepsilon}(\partial A_2) \), and the bound is independent of \( \varepsilon \). Note that, as we are outside \( N_6 \), the term \( \log |E_6| \) and its partial derivatives occurring in \( (17) \) are bounded independently of \( \varepsilon \) as well. Applying these considerations and noting that the term \( \log |\chi_{10}| \) is of order \( M \), we can bound the absolute value of \( f(\tau) \) on \( N_4 \) by

\[
|f(\tau)| \leq \frac{\log (M_{y_2} - y_{12}^2)}{(M_{y_2} - y_{12}^2)^3} \left( M^2 \left( 1 + \frac{y_2}{M_{y_2} - y_{12}^2} \right) + y_{12}^2 \left( 1 + \frac{M}{M_{y_2} - y_{12}^2} \right) + M \left( 1 + \frac{M_{y_{12}}}{M_{y_2} - y_{12}^2} \right) \right) + \frac{M}{(M_{y_2} - y_{12}^2)^3} \left( M^2 \left( \varepsilon + \frac{y_2}{M_{y_2} - y_{12}^2} \right) + y_{12}^2 \left( 1 + \frac{M}{M_{y_2} - y_{12}^2} \right) + M_{y_{12}} \left( 1 + \frac{M_{y_{12}}}{M_{y_2} - y_{12}^2} \right) \right).
\]

With Lemma 12, one obtains the estimates

\[
\frac{1}{2} \leq y_2 \leq M, \quad 0 \leq y_{12} \leq y_2, \quad \frac{1}{M_{y_2}} \leq \frac{1}{M_{y_2} - y_{12}^2} \leq \frac{2}{M_{y_2}}.
\]
Moreover, \( \varepsilon \leq 1/M \) for small \( \varepsilon > 0 \). Hence, we can further bound \( |f(\tau)| \) as

\[
|f(\tau)| \lesssim \log \frac{M}{(My)^3} \cdot \left( M^2 \left( 1 + \frac{y_2}{My} \right) + y_2^2 \left( 1 + \frac{M}{My} \right) + M \left( 1 + \frac{y_2^3}{My} \right) \right) + \frac{M}{(My)^3} \cdot \left( M^2 \left( \varepsilon + \frac{y_2}{My} \right) + y_2^2 \left( 1 + \frac{M}{My} \right) + My \left( 1 + \frac{y_2}{My} \right) \right) \lesssim \log M \left( \frac{1}{My^2} + \frac{1}{M^3y_2} + \frac{1}{My^2y_2} \right) + \frac{1}{My^2} + \frac{1}{M^2y_2} + \frac{1}{My^2}.
\]

By analogous computations, one easily sees that the same bound for \( |f(\tau)| \) holds on the whole of \( U \). As, again by Lemma 12, the domain of integration is contained in the set \( S_\varepsilon \) given by the restrictions

\[
y_1 = M, \ y_2 \in \left[ \frac{1}{2}, M \right], \ y_{12} \in [0, y_2/2], \ \theta \in [0, 2\pi), \ x_2, x_{12} \in [-1/2, 1/2],
\]

the value of the integral of \( [(g_6 * g_4 * g_{12}) \wedge d^c g_10 - g_10 \wedge d^c(g_6 * g_4 * g_{12})] \) over the open set \( U \) can be bounded from above by the integral of the estimate for \( |f(\tau)|d\theta dx_2dy_2dx_{12}dy_{12} \) given in (18) over the set \( S_\varepsilon \). One obtains

\[
\left| \frac{1}{(4\pi i)^2} \int_\Omega [(g_6 * g_4 * g_{12}) \wedge d^c g_10 - g_10 \wedge d^c(g_6 * g_4 * g_{12})] \right| \lesssim \int_{S_\varepsilon} \left( \log \frac{M}{My^2} + \frac{1}{My^2} \right)d\theta dx_2dy_2dx_{12}dy_{12} \lesssim M \int_{1/2} \left( \log \frac{M}{My^2} + \frac{1}{My^2} \right)dy_2 \lesssim \frac{\log M}{M},
\]

as integrating over \( \theta \) and \( x_j \) (\( j = 1, 12, 2 \)) gives a factor \( 2\pi \), and integration over \( y_{12} \) multiplies the integrand by \( y_{12}/2 \). As \( \log M/M \) tends to 0 for \( \varepsilon \) approaching 0, the claim follows.

We will now show that for \( U \subseteq \partial B_\varepsilon(\partial A_2) \) as before, the integral

\[
\frac{1}{(4\pi i)^2} \int_{\partial B_\varepsilon(\partial A_2) \setminus U} [(g_6 * g_4 * g_{12}) \wedge d^c g_10 - g_10 \wedge d^c(g_6 * g_4 * g_{12})]
\]

converges absolutely, and its value tends to 0 for \( \varepsilon \) approaching 0. By Lemmas 10 and 11, the integrand has the form

\[
(\sigma_{4,6}g_6) \wedge dd^c g_4 \wedge dd^c g_{12} \wedge dd^c g_{10} + dd^c(\sigma_{6,4}g_6) \wedge g_4 \wedge dd^c g_{12} \wedge dd^c g_{10} - g_{10} \wedge dd^c(\sigma_{6,4}g_6) \wedge dd^c g_{12} \wedge dd^c g_{4} \wedge dd^c g_{12}
\]

on \( \partial B_\varepsilon(\partial A_2) \setminus U \), with \( \{\sigma_{4,6}, \sigma_{6,4}\} \) a partition of unity adapted to \( \text{div}(E_4) \) and \( \text{div}(E_6) \). In the following, we will give bounds for the forms occurring in (19). For forms \( \alpha, \beta \) on \( \partial B_\varepsilon(\partial A_2) \), we will use the notation \( \alpha \prec \beta \) if there exists a positive real constant \( C \) such that \( \int_V \alpha \leq C \int_V \beta \).
for all closed subsets $V \subseteq \partial B_\varepsilon(\partial A_2)$ whenever the integrals are defined. We compute
\[
dg_6 = -\frac{1}{|E_6|} \frac{\partial}{\partial \theta} |E_6| d\theta - \frac{1}{|E_6|} \frac{\partial}{\partial x_2} |E_6| dx_2 - \frac{1}{|E_6|} \frac{\partial}{\partial x_{12}} |E_6| dx_{12}
- \left( \frac{1}{|E_6|} \frac{\partial}{\partial y_2} |E_6| + 3 \frac{M}{M y_2 - y_{12}^2} \right) dy_2
- \left( \frac{1}{|E_6|} \frac{\partial}{\partial y_{12}} |E_6| - 3 \frac{2y_{12}}{M y_2 - y_{12}^2} \right) dy_{12}.
\]

We can apply Lemma 15 and obtain the bound
\[
\left| \frac{\partial}{\partial x_1} E_6 \right| = \frac{1}{2\pi} \left| \frac{\partial}{\partial \theta} E_6 \right| < \varepsilon \frac{M}{M} < \frac{1}{M}
\]
on $S_\varepsilon$ for $\varepsilon$ small.

Applying the conditions $1/2 < y_2 < 2$ and $y_{12} \leq y_2$ to the coordinate expansions of $d^c g_k$ and $dd^c g_k$ ($k = 4, 6, 12$), and noting that $\sigma_{6,4}$ only depends on the local coordinate $\tau_2$, we obtain the estimates
\[
d\sigma_{6,4}, d^c \sigma_{6,4} \prec dx_2 + dy_2,
\]
\[
dd^c \sigma_{6,4} \prec dx_2 dy_2,
\]
\[
d^c g_6, dg_6, d^c g_4 = \frac{1}{M} d\theta + dx_2 + dy_2 + dx_{12} + dy_{12},
\]
\[
d^c g_{10} \prec d\theta + dx_2 + dy_2 + dx_{12} + dy_{12}
\]
and
\[
\dd^c g_4, \dd^c g_6, \dd^c g_{12} \prec \frac{1}{M^2}(d\theta dy_2 + d\theta dy_{12}) + \frac{1}{M}(dx_2 dy_{12} + dx_{12} dy_2 + dx_{12} dy_{12}) + dx_2 dy_2
\]
on $\partial B_\varepsilon(\partial A_2) \setminus U$.

Using these estimates to bound the summands in (19) on $\partial B_\varepsilon(\partial A_2) \setminus U$, one obtains the bound $\frac{\log(M^2)}{M}$ for the integrand and the theorem follows.

6. An Explicit Formula for the Arithmetic Volume

**Proposition 17.** The contribution $\widehat{vol}(A_2)_0$ for the arithmetic self intersection number coming from the finite fibres is a rational linear combination of $\log 2$ and $\log 3$.

**Proof.** The statement can be verified by considering $\vartheta$-embeddings for $A_3(2)$ into projective space, see, e.g., [11], for the defining equations for its image in $\mathbb{P}^4$. These are only defined over $\mathbb{Z}[\frac{1}{2}]$. Setting
\[
t = -\vartheta_2^{20110},
\]
\[
x_{12} = -i\vartheta_2^{20000},
\]
\[
x_{21} = i\vartheta_2^{00001},
\]
\[
x_{23} = i\vartheta_2^{01100},
\]
\[
x_{32} = -i\vartheta_2^{00111},
\]
the map $\tau \mapsto (t^2(\tau) : x_{11}^2(\tau) : \ldots : x_{33}^2(\tau))$ gives an embedding of $A_2$ into $\mathbb{P}^9$. The $\vartheta$-relations translate to
\[
\sum_{k=1,2,3} x_{j_1 k} x_{j_2 k} - \delta_{j_1 j_2} t^2, \quad \sum_{j=1,2,3} x_{j k_1} x_{j k_2} - \delta_{k_1 k_2} t^2
\]
for $j_1, j_2, j_3, k_1, k_2, k_3 \in \{1, 2, 3\}$. With these relations, one finds dependencies of the 10 coordinates above and can replace them by

$$y_0 = \partial^4_{110}, \ y_1 = \partial^4_{010}, \ y_2 = \partial^4_{000}, \ y_3 = -\partial^4_{100} - \partial^4_{0110}, \ y_4 = -\partial^4_{110} - \partial^4_{0110},$$

so we see that $\text{proj}(A_2)$ is a quartic hypersurface in $\mathbb{P}^4$, defined by the equation

$$(y_0 y_1 + y_0 y_2 + y_1 y_2 - y_3 y_4)^2 - 4y_0 y_1 y_2 (y_0 + y_1 + y_2 + y_3 + y_4) = 0.$$  

In these coordinates, the modular forms become

$$\chi_{10} = y_0 y_1 y_2 (-y_2 - y_4)(y_0 + y_1 + y_2 + y_3 + y_4)(-y_2 - y_3)(y_0 + y_3)(-y_1 - y_3)$$

$$(21) \quad E_4 = y_0^2 + y_1^2 + y_2^2 + (-y_2 - y_4)^2 + (y_0 + y_1 + y_2 + y_3 + y_4)^2 + (-y_2 - y_3)^2$$

$$(22) \quad E_6 = (-y_1 - y_4)y_2(-y_2 - y_3) + (-y_1 - y_4)y_2y_0 + \ldots$$

$$(23) \quad E_6 = (-y_1 - y_4)y_2(-y_2 - y_3) + (-y_1 - y_4)y_2y_0 + \ldots$$

$$(24) \quad E_12 = y_0 y_1 (y_0 + y_3)(-y_1 - y_3)(y_0 + y_4)(-y_1 - y_4) + \ldots$$

Note that these equations are symmetric in $y_0, y_1, y_2$ and $y_3, y_4$. One of the factors in equation (21) has to vanish. Consider the case $y_0 = 0$. With $y_0 = 0$, equation (20) becomes $y_1 y_2 - y_2 y_3 = 0$. Plugging this into equation (23), one obtains

$$6y_2^2 y_4^2(y_1 + y_2 + y_3 + y_4)^2.$$ 

For $p \neq 2, 3$ assume $y_3 = 0$ and, therefore, $y_1 = 0$. The remaining two equations (22) and (21) then give $y_2 = y_4 = 0$. The other cases can be treated equivalently and one sees that the modular forms have empty intersection in the finite fibres, except for $p = 3$. For $p = 3$, the system of equations has the six solutions

$$(0 : 0 : 1 : 0 : 1), (0 : 0 : 1 : 1 : 0), (0 : 1 : 0 : 0 : 1),$$

$$(0 : 1 : 0 : 1 : 0), (1 : 0 : 0 : 0 : 1), (1 : 0 : 0 : 1 : 0).$$

An element $M \in \text{Sp}_4(\mathbb{Z}/2\mathbb{Z}))$ acts on the $\vartheta$-functions via

$$\vartheta_{M \cdot m} = c(M, m) \det(C \tau + D)^{1/4} \vartheta_m \quad (m \in (\mathbb{Z}/2\mathbb{Z})^4)$$

with $c(M, m)$ an 8th root of unity. We find that under this action, all above points are equivalent. \hfill \square

**Main Theorem 18.** The arithmetic self intersection number, i.e., the arithmetic degree of the line bundle $\mathcal{M}_k(\Gamma_2)$ of modular forms of weight $k$ on $\mathbb{A}_2$, equipped with the Petersson metric, is given as

$$\widehat{\text{deg}}(\mathcal{M}_k(\Gamma_2), \| \cdot \|_{\text{Pet}}) = k^4 \left( \zeta(-3) \zeta(-1) \left( 2 \frac{\zeta'(-3)}{\zeta(-3)} + 2 \frac{\zeta'(-1)}{\zeta(-1)} + \frac{17}{6} \right) + c_2 \log 2 + c_3 \log 3 \right),$$

with $c_2, c_3 \in \mathbb{Q}$.

**Proof.** The main theorem follows from computing the complex contribution by adding the terms $(A)$ and $(B)$ from Proposition 4 and considerations about the finite contribution. As the boundary integral along $\partial B_\varepsilon(\partial A_2)$ vanishes in the limit $\varepsilon \to 0$ by Theorem 16, the value of $(A)$ is given by

$$(A) = 10 \cdot 6 \cdot 4 \cdot 12 \zeta(-3) \zeta(-1) \left( \frac{4}{3} + 2 \frac{\zeta'(-3)}{\zeta(-3)} - \frac{\zeta'(-1)}{\zeta(-1)} + \frac{6}{5} \log 2 \right),$$

according to Proposition 10. The boundary integral along $\partial B_\varepsilon(\partial \mathcal{H})$ vanishes by Proposition 7; therefore, the value of $B$ is given by

$$B = -6 \left( \frac{1}{2} + \frac{\zeta'(-1)}{\zeta(-1)} \right) - \frac{4}{3} \log 2 - \frac{2}{3} \log 3.$$
according to Proposition 6. We obtain
\[
\hat{\text{vol}}(A_2) = \frac{1}{10 \cdot 6 \cdot 4 \cdot 12}((A) + (B))
\]
\[
= \frac{1}{10} \cdot 6 \cdot 4 \cdot 12 \left( \frac{17}{6} + 2 \frac{\zeta(-3)}{\zeta(-3)} + 2 \frac{\zeta'(-1)}{\zeta(-1)} \right) - \frac{56}{15} \log 2 - \frac{2}{3} \log 3,
\]
as claimed. □

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