TOPOLOGICAL EQUIVALENCE TO A PROJECTION

V. V. SHARKO AND YU. YU. SOROKA

Abstract. We present a necessary and sufficient condition for the topological equivalence of a continuous function on a plane to a projection onto one of coordinates.

Let $M$ be a connected surface, i.e. 2-dimensional manifold. Two continuous functions $f, g : M \to \mathbb{R}$ are called topologically equivalent, if there exist two homeomorphisms $h : M \to M$ and $k : \mathbb{R} \to \mathbb{R}$ such that $k \circ f = g \circ h$.

Classification of continuous functions $f : M \to \mathbb{R}$ on surfaces up to topological equivalence was initiated in the works by M. Morse [6], [7], see also [3, 4, 5]. In recent years essential progress in the classification of such functions was made in [8, 10, 11, 1, 9].

Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a continuous function. Assuming that $f$ “has not critical points” we present a necessary and sufficient condition for $f$ to be topologically equivalent to a linear function. First we recall some definitions from W. Kaplan [2].

Definition 1. [2]. A curve in $\mathbb{R}^2$ is a homeomorphic image of the open interval $(0, 1)$. Let $U \subset \mathbb{R}^2$ be an open subset. A family of curves in $U$ is a partition of $U$ whose elements are curves.

A family of curves $\mathcal{S}$ in $U$ is called regular at a point $p \in \mathbb{R}^2$, if there exists an open neighbourhood $U_p$ of $p$ and a homeomorphism $\varphi : (0, 1) \times (0, 1) \to U_p$ such that for every $y \in (0, 1)$ the image $\varphi((0, 1) \times y)$ is an intersection of $U_p$ with some curve from the family $\mathcal{S}$. Such a neighbourhood $U_p$ is called $r$-neighbourhood of $p$.

Thus the curves of regular family are “locally parallel”, however they global behaviour can be more complicated, see Figure 1. We will consider continuous functions $f : \mathbb{R}^2 \to \mathbb{R}$ whose level-sets are “globally parallel”.

One of the basic examples is a projection $g : \mathbb{R}^2 \to \mathbb{R}$ given by $g(x, y) = y$. Its level sets are parallel lines $y = \text{const}$, and in particular they constitute a regular family of curves.

On the other hand, consider the function $f(x, y) = \arctan(y - \tan^2(x))$, see Figure 1. It level sets are not connected, however, the partition into connected components of level sets of $f$ is also a regular family of curves.

Figure 1. Level lines of $f(x, y) = \arctan(y - \tan^2(x))$
The following theorem shows that connectedness of level sets is a characteristic property of a projection.

**Theorem 1.** Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a continuous function and $\mathcal{S} = \{f^{-1}(a) \mid a \in \mathbb{R}^2\}$ be the partition of $\mathbb{R}$ by level sets of $f$. Suppose the following two conditions hold.

1. For each $p \in f(\mathbb{R}^2)$ belonging to the image of $f$, the corresponding level set $f^{-1}(a)$ is a curve. In particular, it is path connected.
2. The family of curves $\mathcal{S}$ is regular.

Then the image $f(\mathbb{R}^2)$ is an open interval $(a, b)$, $a, b \in \mathbb{R} \cup \{\pm \infty\}$, and there is a homeomorphism $\varphi : \mathbb{R} \times (a, b) \to \mathbb{R}^2$ such that $f \circ \varphi(x, y) = y$. In other words, $f$ is topologically equivalent to a projection.

The proof is based on the results of [2].

Let $\mathcal{S}$ be a regular family of curves in $\mathbb{R}^2$. Then by [2, Theorem 16], each curve $C$ of $\mathcal{S}$ is a proper embedding of $\mathbb{R}$, so it has an infinity as a sole limit point. It follows from Jordan’s theorem (applied to the sphere $S^2 = \mathbb{R}^2 \cup \{\infty\}$) that each curve $C$ of $\mathcal{S}$ divides the plane into two distinct regions, having $C$ as the common boundary. This property allows to define the following relation for curves on $\mathcal{S}$.

**Definition 2.** Three distinct curves $K, L, C$ from $\mathcal{S}$ are in the relation $K \cap C \cap L$, if $K$ and $L$ belong to distinct components of $\mathbb{R}^2 \setminus C$.

For an open subset $U \subset \mathbb{R}^2$ let $\mathcal{S}_U$ be the partition of $U$ by connected components of intersections of $U$ with curves from $\mathcal{S}$. Then $\mathcal{S}_U$ is a family of curves in $U$. Notice also that an intersection of $U$ with some curve from $\mathcal{S}$ may have even countable many connected components.

**Definition 3.** Let $p, q \in \mathbb{R}^2$. An arc $[p, q]$, i.e. homeomorphic image of $[0, 1]$, connecting these points, will be called a cross-section relative to $\mathcal{S}$ if there exists an open set $U$ in $\mathbb{R}^2$ containing $[p, q]$ and such that each curve of $\mathcal{S}_U$ meets $[p, q]$ in $U$ at most once.

Evidently, for every $p \in \mathbb{R}^2$, there is an arbitrary small $r$-neighbourhood $V$ of $p$ and a cross-section $[q, s] \subset V$ relative $\mathcal{S}$ passing through $p$.

**Theorem 2.** [2] Let $K, L$ be two distinct curves from a regular family $\mathcal{S}$. Suppose two points $p \in L$ and $q \in K$ can be connected by a cross-section $[p, q]$ and let $S$ be the set of curves crossing $[p, q]$ except for $p$ and $q$. Then $S$ form an open point set and the condition $K \cap C \cap L$ is equivalent to the condition that $C$ is contained in $S$.

Moreover, there is a homeomorphism $\varphi : \mathbb{R} \times [0, 1] \to K \cup S \cup L$ such that $K = \varphi(\mathbb{R} \times 0)$, $L = \varphi(\mathbb{R} \times 1)$, and $\varphi(\mathbb{R} \times t)$ is a curve belonging to $\mathcal{S}$ for all $t \in (0, 1)$.

**Proof.** First we need the following lemma.

**Lemma 1.** Let $[p, q]$ be a cross-section of $\mathcal{S}$. Then the restriction of $f$ to $[p, q]$ is strictly monotone. In particular, $[p, q]$ intersects each curve in $\mathcal{S}$ in at most one point.

**Proof.** Suppose there exists a point $x \in [p, q]$ distinct from $p$ and $q$ and being a local extreme of $f\big|_{[p, q]}$. Let $c = f(x)$. As mentioned above the embedding $f^{-1}(c) \subset \mathbb{R}^2$ is proper, therefore

(i) $f^{-1}(c)$ divides $\mathbb{R}^2$ into two connected components, say $R_1$ and $R_2$ and

(ii) there exists an $r$-neighbourhood $U$ of $x$ relatively to $\mathcal{S}$ such that $U \cap f^{-1}(c)$ is a connected curve dividing $U$ into two components, say $U_1$ and $U_2$, such that $U_1 \subset R_1$ and $U_2 \subset R_2$.

Not losing generality, we can assume that $[p, q] \subset U$ so that $[p, q] \setminus \{x\}$ consists of two half-open arcs $[p, x] \subset U_1$ and $(x, q] \subset U_2$. It follows that $x$ is an isolated local extreme of the restriction of $f\big|_{[p, q]}$, whence there exist $y \in [p, x] \subset R_1$ and $z \in (x, q] \subset R_2$ such that
f(y) = f(z) \neq f(c). Thus y, z \in f^{-1}(f(y)) \subset \mathbb{R}^2 \setminus f^{-1}(c) = R_1 \cup R_2. By (1) \ f^{-1}(f(y)) is connected, and so both y and z belong either to \ R_1 or to \ R_2. This gives a contradiction, whence x is not a local extreme of f. \ \Box

For [c, d] \subset \mathbb{R} denote \ D_{c, d} = f^{-1}[c, d]. Then it follows from Lemma 1 and Theorem 2 that for each cross-section \ [p, q] \ there exists a homeomorphism

\[ \varphi : \mathbb{R} \times [f(p), f(q)] \rightarrow f^{-1}[f(p), f(q)] = D_{f(p), f(q)} \]

such that \ f \circ \varphi(x, y) = y \ for all \ (x, y) \in \mathbb{R} \times [f(p), f(q)].

This also implies that the image \ f(\mathbb{R}^2) \ is an open and path connected subset of \ \mathbb{R}, i.e. an open interval \ (a, b), \ where \ a \ and \ b \ can \ be \ infinite.

Hence we can find a countable strictly increasing sequence \ \{c_i\}_{i \in \mathbb{Z}} \subset \mathbb{R} \ such \ that \ \lim_{k \to -\infty} c_i = a, \ \lim_{k \to +\infty} c_k = b, \ and \ for \ each \ k \in \mathbb{Z} \ a \ homeomorphism

\[ \varphi_k : \mathbb{R} \times [c_k; c_{k+1}] \rightarrow f^{-1}[c_k, c_{k+1}] = D_{c_k, c_{k+1}} \]

satisfying \ f \circ \varphi_k(x, y) = y.

Define a homeomorphism \ \varphi : \mathbb{R} \times (a, b) \rightarrow \mathbb{R}^2 \ as \ follows. Set

\[ \varphi(x, y) = \varphi_0(x, y), \quad (x, y) \in \mathbb{R} \times [c_0, c_1]. \]

Now if \ \varphi \ is defined on \ \mathbb{R} \times [c_{k-1}, c_k] \ for some \ k \geq 1, \ then \ extend \ it \ to \ \mathbb{R} \times [c_k, c_{k+1}] \ by

\[ \varphi(x, y) = \varphi_k(\varphi_k^{-1} \circ \varphi(x, c_k), y), \quad (x, y) \in \mathbb{R} \times [c_k, c_{k+1}]. \]

Similarly, one can extend \ \varphi \ to \ \mathbb{R} \times (a, c_0]. \ It \ easily \ follows \ that \ \varphi \ is \ a \ homeomorphism

satisfying \ f \circ \varphi(x, y) = y, \ (x, y) \in \mathbb{R} \times (a, b). \ \Box

References

1. V. I. Arnold, Topological classification of Morse polynomials. Differential equations and topology II, Tr. Mat. Inst. Steklova. 2010, 40–55.
2. W. Kaplan, Regular curve-families filling the plane I. Duke Math. J. 7 (1941), 154–185.
3. J. A. Jenkins, M. Morse, Contour equivalent pseudoharmonic functions and pseudoconjugates, Amer. J. Math. 74 (1952), 23–51.
4. J. A. Jenkins, M. Morse, Topological methods on Riemann surfaces. Pseudoharmonic functions. Contributions to the theory of Riemann surfaces, pp. 111–139. Annals of Mathematics Studies, no. 30. Princeton University Press, Princeton, N. J., 1953.
5. J. A. Jenkins, M. Morse, Conjugate nets on an open Riemann surface. Lectures on functions of a complex variable, pp. 123–185. The University of Michigan Press, Ann Arbor, 1955.
6. M. Morse, The topology of pseudo-harmonic functions, Duke Math. J. 13 (1946), 21–42.
7. M. Morse, Topological methods in the theory of functions of a complex variable, Annals of Math. Studies, No. 15. Princeton Univ. Press, Princeton, N. J., 1947. 145 pp.
8. A. A. Oshemkov, Morse functions on two-dimensional surfaces. Coding features, Tr. Mat. Inst. Steklova. (1994), 131–140.
9. E. Polulyakh, I. Yurchuk, On the pseudo-harmonic functions defined on a disk. Proceedings of the Institute of Mathematics of NAS of Ukraine 80 (2009), 151 pp.
10. V. V. Sharko, Smooth and topological equivalence of functions on surfaces, Ukr. math. journal 55 (2003), no. 5., 687–700.
11. V. V. Sharko, Topological equivalence of harmonic polynomials, Proceedings of the Institute of Mathematics of NAS of Ukraine 10 (2013), no. 4–5., 542–551.

Institute of Mathematics National Academy of Sciences of Ukraine,
E-mail address: sharko@imath.kiev.ua
Tabas Shevchenko National University of Kiev
E-mail address: soroka.yulya@ukr.net