A critical point theorem on a closed ball and some applications to boundary value problems

Marek Galewski

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Abstract

We consider a functional being a difference of two differentiable convex functionals on a closed ball. Existence and multiplicity of critical points is investigated. Some applications are given.

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1 Introduction

In this paper we are concerned with the existence of a critical point to a differentiable functional on a closed ball. With additional assumptions pertaining to the mountain geometry we are in position to get the existence of another distinct critical point as well. In a finite dimensional setting the third critical point can also be obtained.

Let us introduce the space setting and the structure condition required on the action functional. We assume that

H1 $E$ is a real reflexive Banach space compactly embedded into another reflexive Banach space $Z$;

H2 $\Phi : E \to \mathbb{R}$, $H : Z \to \mathbb{R}$ are Fréchet differentiable convex functionals with derivatives $\varphi : E \to E^*$, $h : Z \to Z^*$ respectively;

H3 there exist constants $\alpha > 1, \gamma > 0$ such that

$$\gamma \|v\|^\alpha \leq \langle \varphi (v), v \rangle_{E^*, E} \text{ for all } v \in E.$$
We denote by $c > 0$ the embedded constant, i.e.
\[ \|v\|_Z \leq c \|v\|_E \] for all $v \in E$
and let $B_\rho \subset E$ be closed ball centered at 0 in $E$ with radius $\rho$. Assuming additionally that functional $x \to \|h(x)\|_{Z^*}$ is bounded from above on $B_\rho$ we will determine such a value $\lambda^* > 0$ that for each $\lambda$ in the interval $(0, \lambda^*]$ the corresponding Euler action functional $J : E \to \mathbb{R}$
\[ J(u) = \Phi(u) - \lambda H(u) \]
has a critical point on $B_\rho$. This implies the solvability of
\[ \varphi(u) = \lambda h(u), \; u \in E \] (1)
which is the Euler-Lagrange equation for $J$. Note that $u$ need not belong to the interior of the ball $B_\rho$ and therefore we cannot use the classical variational tool such as Ekelend’s variational principle in order to demonstrate that the minimizer is a critical point. Such an approach is used in [6] for a functional satisfying the PS-condition and considered on a closed ball. If a minimizer is located in the interior of this ball then it be proved via the Ekelend’s variational principle that it is a critical point. When the PS-condition is not assumed, for example the Fenchel-Young transform can be applied to prove that a minimizer is a critical point. This approach is sketched in [20] and further developed in several papers, see for example [9] and references therein. In the present work we use only basic convexity and concavity to the critical point together with some variational techniques based on the Weierstrass Theorem.

Summarizing our approach: the first critical point (which lies in the ball, perhaps on the boundary of the ball) is obtained through the Weierstrass Theorem, direct method of the calculus of variations and convexity relations, while the second critical point, under assumption that the PS-condition is satisfied, is obtained with the aid of a general type of a Mountain Pass Lemma. In a finite dimensional case, we obtain a third critical point through direct maximization.

We provide applications to elliptic second order partial differential equations and their discrete analogs put in the form of an algebraic system.

For a background on variational methods we refer to [10] for differential equations to [11], [17] while for a background on difference equations to [1].
As for the algebraic systems serving as example in this work, the literature is very rich and we mention the following sources, \[7\], \[19\], \[24\]. The ideas connected with three critical point theorems - different from those used in this work - are to be found for example in \[3\], \[22\]. Let us mention \[21\], \[6\] for some recent results concerning a general type of critical point theorem on a bounded set (with the PS-condition which we do not need), and to \[23\] for some applications of multiple critical point theorem from \[6\] to discrete Neumann anisotropic problems. Note that in \[21\] the bounded critical point theorem due to Schechter is investigated, so the setting is in a Hilbert space, while in \[6\] it is a Banach space. The application of another type of critical point on closed sets has just been developed by Marano, see \[15\], and to \[16\] for applications to differential inclusions, and also some earlier result \[14\].

We provide necessary mathematical prerequisites which are needed for the proof of the main multiplicity result.

Functional \( J : E \to \mathbb{R} \) satisfies the Palais-Smale condition (PS-condition for short) if every sequence \((u_n)\) such that \(\{J(u_n)\}\) is bounded and \(J'(u_n) \to 0\), has a convergent subsequence.

**Lemma 1 (Mountain Pass Lemma, MPL Lemma)** Let \( E \) be a Banach space and assume that \( J \in C^1(E, \mathbb{R}) \) satisfies the PS-condition. Let \( S \) be a closed subset of \( E \) which disconnects \( E \). Let \( x_0 \) and \( x_1 \) be points of \( E \) which are in distinct connected components of \( E \setminus S \). Suppose that \( J \) is bounded below in \( S \), and in fact the following condition is verified for some \( b \)

\[
\inf_{x \in S} J(x) \geq b \quad \text{and} \quad \max\{J(x_0), J(x_1)\} < b. \quad (2)
\]

If we denote by \( \Gamma \) the family of continuous paths \( \gamma : [0, 1] \to E \) joining \( x_0 \) and \( x_1 \), then

\[
c := \inf_{\gamma \in \Gamma} \max_{s \in [0,1]} J(\gamma(s)) \geq \max\{J(x_0), J(x_1)\} > -\infty
\]

is a critical value and \( J \) has a non-zero critical point \( x \) at level \( c \).

**2 A critical point theorem**

We begin with some general result which generalizes the main result from \[9\].
**Theorem 2** Assume that $H1, H2$ are satisfied. Fix some $\lambda^* > 0$ and let $u, v \in E$ be such that

\[ J(u) \leq J(v) \quad \text{and} \quad \varphi(v) = \lambda^* h(u) \]  

(3)

Then $u$ is a critical point to $J$, and thus it solves (1).

**Proof.** The proof follows by simple calculations pertaining to convexity of $\Phi$ and concavity of $-H$. Note that $J(u) \leq J(v)$ implies

\[ \Phi(u) - \Phi(v) \leq -\lambda H(v) - (-\lambda H(u)) \]

so by standard inequalities following from definition of convexity of $\Phi$ at $v$ and concavity of $-H$ at $u$ we obtain

\[ \langle \varphi(v), u - v \rangle \leq \Phi(u) - \Phi(v) \leq -\lambda H(v) - (-\lambda H(u)) \leq \langle \lambda h(u), u - v \rangle. \]

The above and the equality $\varphi(v) = \lambda h(u)$ provide that $\Phi(u) = \langle \varphi(v), u - v \rangle + \Phi(v)$. So from this relation and by convexity again we have

\[ \langle \varphi(v), v - u \rangle = \Phi(v) - \Phi(u) \geq \langle \varphi(u), v - u \rangle. \]

This means that both $\varphi(v)$ and $\varphi(u)$ are the elements of a subdifferential of $\Phi$ at $u$. Since, by differentiability and convexity, this is a singleton, $[17]$, we get that $\varphi(v) = \varphi(u)$. This by the equation in (3) we see that $u$ is a critical point.

However Theorem 2 Some special cases of Theorem 2 can now be stated as follows. We make precise assumptions which lead to have relations (3) satisfied.

**Theorem 3** Let $E$ be a infinite dimensional reflexive Banach space. Assume that $H1$-$H3$ are satisfied. Take any $\rho > 0$.

(i) Assume that functional $x \to \|h(x)\|_Z^*$ is bounded from above on $B_\rho$.

Then there exists $\lambda^* > 0$ such that for each $\lambda \in (0, \lambda^*]$ there exist $u \in B_\rho$ with

\[ J(u) = \inf_{x \in B_\rho} J(x) \]  

(4)

and such that $u$ is a critical point to $J$, and thus it solves (1). If for some $v \in B_\rho$ it holds that $J(v) < 0$, $J(0) = 0$, then $u$ is non-trivial.
(ii) Assume additionally that for all \( \lambda \in (0, \lambda^*] \)
(iii a) \( J \) satisfies the PS-condition,
(iii b) \( J(u) < \inf_{x \in \partial B_{\rho_1}} J(x) \) for some \( \rho_1 > \|u\| \),
(iii c) there exists \( w \in E \) with \( \lim_{t \to \infty} J(tw) = -\infty \).

Then for all \( \lambda \in (0, \lambda^*] \) functional \( J \) has two critical points, namely \( u \) and another non-zero critical point \( z \) different from \( u \).

**Proof.** Denote by \( \beta > 0 \) the upper bound of functional \( x \to \|h(x)\|_{Z^*} \) on \( B_{\rho} \). Put \( \lambda^* = \frac{\|h(u)\|_{Z^*}}{\beta} \) and fix \( \lambda \leq \lambda^* \).

Consider \( J \) on \( B_{\rho} \). Observe that \( J \) is sequentially weakly l.s.c. on \( B_{\rho} \). Indeed, let \( (u_n) \) be a sequence from \( B_{\rho} \). Then we can assume that \( (u_n) \) is weakly convergent in \( E \) and strongly in \( Z \) to some \( u \in B_{\rho} \), so \( H(u_n) \) converges to \( H(u) \). Since \( \Phi \) is weakly l.s.c. as a convex functional, we see that \( J \) is weakly l.s.c. on \( B_{\rho} \). Since \( B_{\rho} \) is weakly compact some \( u \) exists such that (4) holds.

Now consider on \( E \) functional \( J_1 \) given by the formula
\[
J_1(x) = \Phi(x) - \langle h(u), x \rangle_{Z^*,Z}.
\]
Since \( \Phi \) is weakly l.s.c. and is coercive, so is \( J_1 \) and therefore we get the existence of an argument of a minimum to \( J_1 \) over \( E \), which we denote by \( v \). Obviously \( v \) is a critical point to \( J_1 \) and so for any \( x \in E \)
\[
\langle \varphi(v), x \rangle_{E^*,E} - \lambda \langle h(u), x \rangle_{Z^*,Z} = 0.
\]
This means that \( h \) solves
\[
\varphi(v) = \lambda h(u)
\]
in the weak sense. Observe \( v \) belongs to \( B_{\rho} \). Indeed, put \( x = v \) in (6). Thus
\[
\gamma \|v\|^\alpha \leq \langle \varphi(v), v \rangle_{E^*,E} = \lambda \langle h(u), v \rangle_{Z^*,Z} \leq \lambda \|h(u)\|_{Z^*} \|v\|_Z \leq \lambda \beta \|v\|_Z.
\]
Therefore \( \|h\|^\alpha \leq \lambda \beta \leq \rho^\alpha \) and \( h \in B_{\rho} \).

The proof that \( u \) is a critical point follows from Theorem 2 since \( J(u) \leq J(v) \) and since (6) holds. When \( J(v) < 0 \), then also \( J(u) < 0 \) and the assertion that \( u \) is nontrivial follows since \( J(0) = 0 \).
In order to prove part (ii), i.e. in order to get the second critical point, we will use Lemma 1. Since \( \lim_{t \to \infty} J(tw) = -\infty \), so there exists some \( w_1 \) such that

\[
J(w_1) \leq \inf_{x \in B_{\rho}} J(x) < \inf_{x \in \partial B_{\rho_1}} J(x).
\]

Thus we have condition (2) satisfied taking \( x_0 = u \) and \( x_1 = w_1 \). The existence of a second non-zero critical point readily follows. 

We see that when \( J(0) = 0 \) condition (ii b) can be replaced by the following

(ii b) \( \inf_{x \in \partial B_{\rho_1}} J(x) > 0 \) for some \( \rho_1 > 0 \)

In a finite dimensional context, we get easily the existence of a third critical point as follows

**Theorem 4** Let \( E \) be a finite dimensional Banach. Assume that \( H1-H3 \) are satisfied. Take any \( \rho > 0 \). Then there exists \( \lambda^* > 0 \) such that for each \( \lambda \in (0, \lambda^*] \) there exist \( u \in B_{\rho} \) with

\[
J(u) = \inf_{x \in B_{\rho}} J(x)
\]

and such that \( u \) is a critical point to \( J \), and thus it solves (1). If for some \( v \in B_{\rho} \) it holds that \( J(v) < 0, J(0) = 0 \), then \( u \) is non-trivial.

(ii) Assume additionally that

(ii a) \( J \) is anti-coercive

(ii b) \( J(u) < \inf_{x \in \partial B_{\rho_1}} J(x) \) for some \( \rho_1 \geq \rho \),

Then for any \( \lambda \in (0, \lambda^*] \) functional \( J \) has at least three critical points, namely \( u \) and two another critical points one being a Mountain Pass point and the other the argument of a maximum.

**Proof.** We define \( \lambda^* \) as in the proof of Theorem 3. By the Weierstrass Theorem condition (i) holds by continuity. Note that in a finite dimensional setting an anti-coercive functional necessarily satisfies the PS-condition and moreover, there exists \( w \in E \) with \( \lim_{t \to \infty} J(tw) = -\infty \). Thus the existence of two distinct solutions, \( u \) and some \( z \neq 0 \), follows by Theorem 3. Since \( J \) is anti-coercive and continuous it has an argument of a maximum over \( E \) which we denote by \( w \). Since \( J \) is differentiable it follows that \( w \) is a critical point. Since

\[
\max \{ J(z), J(u) \} \leq \sup_{x \in E} J(x)
\]

we see that either \( w \) is a third critical point distinct from the previous ones or else there are infinitely many critical points at the level \( J(z) \).
3 Examples of applications

In this section we apply our abstract results for nonlinear algebraic systems being discretization of some elliptic problem and to their continuous counterpart. The examples show that the discrete case is not only less demanding, but also multiple solutions are obtained in a easier manner. For example we do not need a type of A-R condition. It seems that due to relatively mild conditions required in order to obtain at least one critical point (in fact local growth conditions suffice), Theorem 3 would apply for various boundary value problems. We note that if one wants to obtain a solution of a mountain pass type then the second critical point follows by simple assuming convexity of a potential of a RHS of the equation under consideration which is not very demanding when A-R has been assumed.

3.1 Application to the partial difference equations

We will consider the system

\[
[u(i + 1, j) - 2u(i, j) + u(i - 1, j)] + \frac{u(i, j + 1) - 2u(i, j) + u(i, j - 1)}{2} + \lambda f(i, j) = 0,
\]

for all \(i \in \{1, ..., m\}, j \in \{1, ..., n\}\)

\(u(i, 0) = u(i, n + 1) = 0\) for all \(i \in \{1, ..., m\}\)

\(u(0, j) = u(m + 1, j) = 0\) for all \(j \in \{1, ..., n\}\) \(\tag{7}\)

which serves as the discrete counterpart of the problem

\[
\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \lambda f((x, y), u(x, y)) = 0
\]

\(u(x, 0) = u(x, n + 1) = 0\), for all \(x \in (0, m + 1)\) \(\tag{8}\)

\(u(0, y) = u(m + 1, y) = 0\) for all \(y \in (0, n + 1)\)

Following some ideas from \[13\], we write \(\tag{7}\) as a nonlinear system which
we further investigate. Let

\[
A := \begin{bmatrix}
L & -I_m & 0 & 0 & \ldots & 0 & 0 & 0 & 0 \\
-I_m & L & -I_m & 0 & \ldots & 0 & 0 & 0 & 0 \\
0 & -I_m & L & -I_m & \ldots & 0 & 0 & 0 & 0 \\
0 & 0 & -I_m & L & \ldots & 0 & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \ldots & L & -I_m & 0 & 0 \\
0 & 0 & 0 & 0 & \ldots & -I_m & L & -I_m & 0 \\
0 & 0 & 0 & 0 & \ldots & 0 & -I_m & L & -I_m \\
0 & 0 & 0 & 0 & \ldots & 0 & 0 & -I_m & L \\
\end{bmatrix}
\]

where $I_m$ is identity matrix of order $m$ and $L$ is $m \times m$ matrix defined by

\[
L := \begin{bmatrix}
4 & -1 & 0 & 0 & \ldots & 0 & 0 & 0 & 0 \\
-1 & 4 & -1 & 0 & \ldots & 0 & 0 & 0 & 0 \\
0 & -1 & 4 & -1 & \ldots & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 4 & \ldots & 0 & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & \ldots & 4 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & \ldots & -1 & 4 & -1 & 0 \\
0 & 0 & 0 & 0 & \ldots & 0 & -1 & 4 & -1 \\
0 & 0 & 0 & 0 & \ldots & 0 & 0 & -1 & 4 \\
\end{bmatrix}
\]

Matrix $A$ is positive definite, see [13]. Thus problem (7) can be rewritten as

\[
Au = \lambda f(u),
\]

with

\[
u = (u(1, 1), \ldots, u(m, 1); u(1, 2), \ldots, u(m, 2); u(1, n), \ldots, u(m, n))^T,
\]

\[
f(u) := ((f((1, 1), u(1, 1)), \ldots, f((m, 1), u(m, 1)),
\]

\[
f((1, 2), u(1, 2)), \ldots, f((m, 2), u(m, 2)),
\]

\[
f((1, n), u(1, n)), \ldots, f((m, n), u(m, n))^T.
\]

With $f$ being a continuous function, solutions to (9) correspond in a one to one manner to critical points of a functional $J : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$

\[
J(u) = \frac{1}{2}(u, Au) - \lambda \sum_{i=1}^m \sum_{j=1}^n F((i, j), u(i, j)),
\]

8
where
\[ F((i, j), u(i, j)) := \int_{0}^{u(i, j)} f((i, j), v) dv. \]

By \( \alpha_1, \alpha_2, ..., \alpha_{mn} \) we denote the eigenvalues of \( A \) ordered as
\[ 0 < \alpha_1 < \alpha_2 < ... \leq \alpha_{mn}. \] (10)

The assumptions which we impose read
\[ \textbf{H4} \quad f((i, j), \cdot) : \mathbb{R} \to \mathbb{R} \text{ is continuous for all } i \in \{1, \ldots, m\}, j \in \{1, \ldots, n\} \text{ and there exist constants } \mu > 2, c_1 > 0, c_2 \in \mathbb{R}, d > 0 \]
\[ F((i, j), x) \geq c_1 |x|^\mu + c_2 \]
for all \( i \in \{1, \ldots, m\}, j \in \{1, \ldots, n\} \) and all \( |x| \geq d; \]

\[ \textbf{H5} \quad \text{function } x \to F((i, j), x) \text{ is convex on } \mathbb{R} \text{ for all } i \in \{1, \ldots, m\}, j \in \{1, \ldots, n\}. \]

The assumptions employed here are not very restrictive. There are many functions satisfying both \textbf{H4} and \textbf{H5}. See for example \( F(k, x) = c_1 |x|^\mu + c_2 \) with \( \mu > 2 \) and even. Then we arrive at the following theorem

**Theorem 5** Assume that conditions \textbf{H4}-\textbf{H5} are satisfied. There exists \( \lambda^* > 0 \) such that for all \( 0 < \lambda \leq \lambda^* \) problem (7) has at least three nontrivial solutions.

**Proof.** We see that \( E = Z = \mathbb{R}^m \times \mathbb{R}^n \). In this case \( c = 1 \). Observe that \( \gamma = \alpha_1 \), see (10) and \( \alpha = 2 \) since
\[ (u, Au) \geq \alpha_1 |u|^2. \]

It follows by a direct computation that for any \( \lambda > 0 \) functional \( J \) is anti-coercive, i.e. \( J(x) \to -\infty \) as \( ||x|| \to +\infty \). So there is \( z_1 \in E \) such that \( J(z_1) < 0 \). Take \( \rho \geq |z_1| \). We denote by \( \beta \) the maximal value of a functional \( x \to \sqrt{\sum_{i=1}^{m} \sum_{j=1}^{n} f^2((i, j), x(i, j))} \) on \( B_\rho \) which is finite by a Weierstrass Theorem and we put \( \lambda^* = \frac{\alpha_1 \rho}{\beta} \). Note that condition (ii b) of Theorem 4 follows by anti-coercivity.
3.2 Applications to partial differential equations

In this section we consider problems similar to (8), namely

\[
\begin{cases}
-\Delta u(x) = \lambda f(x, u(x)), & u|_{\partial \Omega} = 0 \\
\quad u \in W^{1,2}_0(\Omega),
\end{cases}
\]

with a numerical parameter \( \lambda > 0 \) and where \( \Omega \subset \mathbb{R}^n, n \geq 2, \Omega \) is a smooth bounded region. Let \( F(x,v) = \int_0^v f(x,\tau)\,d\tau \) for a.e. \( x \in \Omega \). We will assume that

**H6** \( f : \Omega \times \mathbb{R} \to \mathbb{R} \) is a Caratheodory function;

**H7** there exists a constant \( \theta > 2 \) such that for \( v \in \mathbb{R}, v \neq 0 \) and a.e. \( x \in \Omega \)

\[
0 < \theta F(x,v) \leq v F(x,v);
\]

**H8** there exist constants \( \beta_1, \eta > 0, \beta_2 \geq 0 \) with \( \eta > 2 \) and such that for all \( v \in \mathbb{R} \) and a.e. \( x \in \Omega \)

\[
|f(x,v)| \leq \beta_1 |v|^\eta - 1 + \beta_2;
\]

**H9** \( \lim_{v \to 0} \frac{|f(x,v)|}{|v|} = 0 \) uniformly for a.e. \( x \in \Omega \);

**H10** function \( v \to F(x,v) \) is convex on \( \mathbb{R} \) for a.e. \( x \in \Omega \).

We see that the action functional \( J : W^{1,2}_0(\Omega) \to \mathbb{R} \) given by

\[
J(u) = \frac{1}{2} \int_\Omega |\nabla u(x)|^2\,dx - \lambda \int_\Omega F(x,u(x))\,dx
\]

is continuously Gâteaux differentiable. Thus it is a \( C^1 \) functional. Weak solutions to (11) i.e. a functions \( u \) satisfying

\[
\int_\Omega \nabla u(x) \nabla v(x)\,dx = \lambda \int_\Omega f(x,u(x))v(x)\,dx \text{ for all } v \in W^{1,2}_0(\Omega)
\]

are critical points to \( J \). From [12] we get the two lemmas concerning the mountain geometry for (11).
Lemma 6 Suppose that $H6$-$H8$ hold. Then for any $\lambda > 0$ the functional $J$ given by (12) satisfies the PS-condition.

Lemma 7 Suppose that $H6$-$H9$ hold. Then for any $\lambda > 0$ there exist numbers $\kappa, \xi > 0$ such that $J(u) \geq \xi$ for all $u \in W^{1,2}_0(\Omega)$ with $\|u\|_{W^{1,2}_0} = \kappa$. Moreover, there exists an element $z \in W^{1,2}_0(\Omega)$ with $\|z\|_{W^{1,2}_0} > \kappa$ and such that $J(z) < 0$.

Using Mountain Pass Lemma 1 and Lemmas 6 and 7 we get the following

Proposition 8 Suppose that $H6$-$H9$ hold. Then for any $\lambda > 0$ problem (11) has at least one nontrivial solution.

Concerning the multiple solutions we have the main result of this section where we need only assume that $F$ is convex in addition to assumptions leading to a mountain pass solution.

Theorem 9 Assume that conditions $H6$-$H10$ are satisfied. Then there exists $\lambda^* > 0$ such that for all $0 < \lambda \leq \lambda^*$ problem (11) has at least two solutions.

Proof. We put $E = W^{1,2}_0(\Omega)$, $Z = L^2(\Omega)$. Here $\alpha = 2, \gamma = 1$ and the constant $c$ is the best constant in the Poincaré inequality. Condition (i) of Theorem 3 is satisfied by $H8$. Lemmas 6 and 7 provide condition (ii) of Theorem 3 and thus the application of this theorem finishes the proof. ■

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Marek Galewski
Institute of Mathematics,
Technical University of Lodz,
Wolczanska 215, 90-924 Lodz, Poland,
marek.galewski@p.lodz.pl