STAFNEY’S LEMMA HOLDS FOR SEVERAL “CLASSICAL” INTERPOLATION METHODS

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Abstract. Let \((B_0, B_1)\) be a Banach pair. Stafney showed that one can replace the space \(\mathcal{F}(B_0, B_1)\) by its dense subspace \(\mathcal{G}(B_0, B_1)\) in the definition of the norm in the Calderón complex interpolation method on the strip if the element belongs to the intersection of the spaces \(B_i\). We shall extend this result to a more general setting, which contains well-known interpolation methods: the Calderón complex interpolation method on the annulus, the Lions-Peetre real method (with several different choices of norms), and the Peetre “\(\pm\)” method.

1. Introduction

Stafney showed in his paper [17] (Lemma 2.5, p. 335) that one can replace the space \(\mathcal{F}(B_0, B_1)\) by its dense subspace \(\mathcal{G}(B_0, B_1)\) in the definition of the norm of an element in the Calderón complex interpolation space \([B_0, B_1]_\theta\) if the element belongs to the intersection of the two Banach spaces. Among the various applications of Stafney’s result, we mention that it can be used to give an apparently simpler proof of part of Calderón’s duality theorem, namely that, for each \(\theta \in (0, 1)\), \([B_0^\theta, B_1^\theta]_{\theta} \subset ([B_0, B_1])^*\) holds if \((B_0, B_1)\) is a regular Banach couple ([9], pp. 20-1).

We shall obtain a version of Stafney’s lemma in the general setting of pseudolattices, which, for appropriate selections of the parameters, will give us analogues of this lemma for the Calderón complex interpolation method on the annulus, for the “discrete definition” of the Lions-Peetre real method and for the Peetre “\(\pm\)” method. In the case of the Lions-Peetre method we can work either with the norm defined via the \(J\)-functional or with an earlier used variant of that norm introduced in [14]. The formulation of our version of Stafney’s lemma for these methods can be found in Remark 21.

There is some overlap of our result for the case of the Lions-Peetre method with a result in Appendix 3 on pp. 47–48 of [9]. The latter result applies to more general versions of interpolation spaces defined by the \(K\)-functional, but, on the other hand, the relevant norm estimate there is only to within equivalence of norms. See also the “Note added in proof” on
Stafney’s lemma cannot be extended to all interpolation methods where there are natural analogues of the space \( \mathcal{F} \) and its dense subspace \( \mathcal{G} \). For example, in \([4]\) an analogue of Calderón’s complex interpolation space generated by an \( n \)-tuple of Banach spaces, rather than just a couple, is introduced. The example in Appendix 1 on pp. 223–6 of \([4]\) shows that, perhaps surprisingly, for \( n \geq 3 \), the expected analogue of Stafney’s lemma can fail to hold. In fact, in that setting the natural analogues of the two quantities which appear below in the formula (3.1) in Theorem \([18]\) may even fail to be equivalent.

2. Preliminaries and examples

Before stating our main result we need to provide a number of definitions and examples, most of which are from \([8]\).

**Definition 1.** Let \( Ban \) be the class of all Banach spaces over the complex numbers. A mapping \( \mathcal{X} : Ban \rightarrow Ban \) will be called a pseudolattice if

(i) for each \( B \in Ban \) the space \( \mathcal{X}(B) \) consists of \( B \) valued sequences \( \{b_n\}_{n \in \mathbb{Z}} \),

(ii) whenever \( A \) is a closed subspace of \( B \) it follows that \( \mathcal{X}(A) \) is a closed subspace of \( \mathcal{X}(B) \), and

(iii) there exists a positive constant \( C = C(\mathcal{X}) \) such that, for all \( A, B \in Ban \) and all bounded linear operators \( T : A \rightarrow B \) and every sequence \( \{a_n\}_{n \in \mathbb{Z}} \in \mathcal{X}(A) \), the sequence \( \{Ta_n\}_{n \in \mathbb{Z}} \in \mathcal{X}(B) \) and satisfies the estimate

\[
\|\{Ta_n\}_{n \in \mathbb{Z}}\|_{\mathcal{X}(B)} \leq C(\mathcal{X}) \|T\|_{A \rightarrow B} \|\{a_n\}_{n \in \mathbb{Z}}\|_{\mathcal{X}(A)}.
\]

The following examples will be relevant for our applications.

**Example 2.** Let \( X \) be a Banach lattice of real valued functions defined on \( \mathbb{Z} \). We will use the notation \( X = X \) to mean that, for each \( B \in Ban \), \( \mathcal{X}(B) \) is the space, usually denoted by \( X(B) \), consisting of all \( B \) valued sequences \( \{b_n\}_{n \in \mathbb{Z}} \) such that \( \{\|b_n\|_B\}_{n \in \mathbb{Z}} \in X \). It is normed by \( \|\{b_n\}_{n \in \mathbb{Z}}\|_{\mathcal{X}(B)} = \|\{\|b_n\|_B\}_{n \in \mathbb{Z}}\|_X \). In particular, we shall be interested in the choices \( X = \ell^p \) for \( p \in [1, \infty) \) and \( X = c_0 \).

**Example 3.** For each \( B \in Ban \) let \( FC(B) \) be the space of all \( B \) valued sequences \( \{b_n\}_{n \in \mathbb{Z}} \) such that \( b_n = \frac{1}{2\pi} \int_0^{2\pi} e^{-int} f(e^{it}) \, dt \) for all \( n \) and some continuous function \( f : \mathbb{T} \rightarrow B \). \( FC(B) \) is normed by \( \|\{b_n\}_{n \in \mathbb{Z}}\|_{FC(B)} = \sup_{t \in [0,2\pi]} \|f(e^{it})\|_B \). The notation \( X = FC \) will mean that \( \mathcal{X}(B) = FC(B) \) for each \( B \).
Example 4. We shall use the notation \( \mathcal{X} = UC \), when \( \mathcal{X}(B) = UC(B) \) for every \( B \in \text{Ban} \), where \( UC(B) \) denotes the Banach space of all \( B \) valued sequences \( \{b_n\}_{n \in \mathbb{Z}} \) such that \( \sum_{n \in \mathbb{Z}} \lambda_n b_n \) converges for all complex sequences \( \{\lambda_n\}_{n \in \mathbb{Z}} \) satisfying \( |\lambda_n| \leq 1 \) for each \( n \in \mathbb{Z} \) (i.e. such that the sequence \( b_n \) is unconditionally convergent), and \( UC(B) \) is normed by \( \|\{b_n\}_{n \in \mathbb{Z}}\|_{UC(B)} = \sup \{\|\sum_{n \in F} \lambda_n b_n\|_B\} \), where the supremum is taken over all finite subsets \( F \) of \( \mathbb{Z} \) and all sequences \( \{\lambda_n\}_{n \in \mathbb{Z}} \) which satisfy \( |\lambda_n| \leq 1 \) for all \( n \) (see [16] pp. 174-5). (Note that, as was pointed out on p. 58 of [11], it suffices to consider sequences \( \{\lambda_n\}_{n \in \mathbb{Z}} \) with \( \lambda_n = \pm 1 \), since this yields the same space to within equivalence of norms.)

Analogously, we shall use the notation \( \mathcal{X} = WUC \), when \( \mathcal{X}(B) = WUC(B) \) for every \( B \in \text{Ban} \), where \( WUC(B) \) denotes the space consisting of all \( B \) valued sequences \( \{b_n\}_{n \in \mathbb{Z}} \) for which the above norm \( \|\{b_n\}_{n \in \mathbb{Z}}\|_{UC(B)} \) is finite, but for which the unconditional convergence of the sequence \( b_n \) is not required. Such sequences are said to be weakly unconditionally convergent. (See e.g. p. 58 of [11] or pp. 99 of [13] for an equivalent definition.)

Let \( \overline{B} = (B_0, B_1) \) be a Banach pair (i.e. \( B_0 \) and \( B_1 \) are two Banach spaces which are continuously embedded in some Hausdorff topological vector space). Let \( \mathcal{X}_0 \) and \( \mathcal{X}_1 \) be any two pseudolattices. We consider them as a pair, which we denote by \( X = \{\mathcal{X}_0, \mathcal{X}_1\} \).

Definition 5. For each Banach pair \( B \) and pseudolattice pair \( X \) we define \( J(X, \overline{B}) \) to be the space of all \( B_0 \cap B_1 \) valued sequences \( \{b_n\}_{n \in \mathbb{Z}} \) for which the sequence \( \{e^{jn}b_n\}_{n \in \mathbb{Z}} \) is in \( \mathcal{X}_j(B_j) \) for \( j = 0, 1 \). This space is normed by

\[
\|\{b_n\}_{n \in \mathbb{Z}}\|_{J(X, \overline{B})} = \max_{j = 0, 1} \|\{e^{jn}b_n\}_{n \in \mathbb{Z}}\|_{\mathcal{X}_j(B_j)}.
\]

Definition 6. Let \( A \) denote the annulus \( \{z \in \mathbb{C} : 1 \leq |z| \leq e\} \) and let \( A^\circ \) denote its interior. We shall say that the pseudolattice pair \( X \) is nontrivial if, for the special one-dimensional Banach pair \( B = (\mathbb{C}, \mathbb{C}) \) and each \( s \in A^\circ \), there exists a sequence \( \{b_n\}_{n \in \mathbb{Z}} \in J(X, \overline{B}) \) such that the limit

\[
\lim_{M,N \to +\infty} \sum_{n=-M}^{N} s^n b_n \exists \text{ and is finite and non zero.}
\]

(Note that here our notation differs slightly from that in [8], but is the same as that in [7].)

Definition 7. We shall say that the pseudolattice pair \( X \) is Laurent compatible if it is nontrivial and if for every Banach pair \( \overline{B} \), every sequence \( \{b_n\}_{n \in \mathbb{Z}} \) in \( J(X, \overline{B}) \) and every fixed \( z \) in the open annulus \( A^\circ \), the Laurent series \( \sum_{n \in \mathbb{Z}} z^n b_n \) converges in \( B_0 + B_1 \) and \( \|\sum_{n \in \mathbb{Z}} z^n b_n\|_{B_0 + B_1} \leq C \|\{b_n\}_{n \in \mathbb{Z}}\|_{J(X, \overline{B})} \) for some constant \( C = C(z) \) independent of the choice of \( \{b_n\}_{n \in \mathbb{Z}} \).
Remark 8. If $X$ is Laurent compatible, then the $B_0 + B_1$ valued function $f(z) = \sum_{n \in \mathbb{Z}} z^n b_n$ is analytic in $\mathbb{A}^\circ$.

Definition 9. For each Banach pair $\vec{B}$, each Laurent compatible pair $X$ and each fixed $s \in \mathbb{A}^\circ$ we define the space $\vec{B}_{X,s}$ to consist of all the elements of the form $b = \sum_{n \in \mathbb{Z}} s^n b_n$ where $\{b_n\}_{n \in \mathbb{Z}} \in \mathcal{J}(X, \vec{B})$, with the norm $\|b\|_{\vec{B}_{X,s}} = \inf \{ \| \{b_n\}_{n \in \mathbb{Z}} \|_{\mathcal{J}(X, \vec{B})} : b = \sum_{n \in \mathbb{Z}} s^n b_n \}$.

Remark 10. $\vec{B}_{X,s}$ equipped with this norm is a Banach space.

Remark 11. As remarked on p. 251 of [8], in elaboration of a point of view going back to [16], the space $\vec{B}_{X,s}$ coincides with various known interpolation spaces for appropriate choices of $\mathcal{X}_0$, $\mathcal{X}_1$ and $s$. In each of the three following examples we set $s = e^\theta$ for some $\theta \in (0, 1)$.

(i) If $\mathcal{X}_0 = \mathcal{X}_1 = FC$, the space $\vec{B}_{X,s}$ coincides isometrically with the variant of Calderón’s complex interpolation space obtained when an annulus is used instead of a strip in the interpolation method. (The proof of this last claim is rather straightforward, but, since it is not given explicitly in [16] or [8], we provide it for the reader’s convenience in Section 4 together with the relevant definitions.) Note that this variant of Calderón’s complex interpolation space, which was apparently first considered in [16], coincides with $[B_0, B_1]_{\theta}$ to within equivalence of norms, as was shown on pp. 1007-9 of [8].

(ii) If $\mathcal{X}_0 = \mathcal{X}_1 = \ell^p$, then $\vec{B}_{X,s}$ is the Lions-Peetre real method space $\vec{B}_{\theta,p} = \langle B_0, B_1 \rangle_{\theta,p}$. In fact the norm that we obtain here is exactly the norm introduced in formula (1.3) on p. 17 of [14] for suitable choices of the parameters $p_0, p_1, \xi_0$ and $\xi_1$. In [14] this space is denoted by $s(p_0, \xi_0, B_0; p_1, \xi_1, B_1)$, a notation which is now rarely used. It has become more customary to use other equivalent norms on $\langle B_0, B_1 \rangle_{\theta,p}$ which are defined via the Peetre $J$-functional or $K$-functional. For example, in terms of the $J$-functional (i.e. $J(t, x; B_0, B_1) = \max_{j=0,1} t^j \|x\|_{B_j}$ for $x \in B_0 \cap B_1$), one can use the norm $\|x\| = \inf \{ e^{-\theta n J(e_n, c_n; B_0, B_1)} \}_{n \in \mathbb{Z}} \|_{\ell^p}$, where the infimum is taken over all representations $x = \sum_{n \in \mathbb{Z}} c_n$ (with convergence in $B_0 + B_1$) with $\{c_n\}_{n \in \mathbb{Z}} \in \mathcal{J}(\{\ell^p, \ell^p\}, \vec{B})$.

(iii) If $\mathcal{X}_0 = \mathcal{X}_1 = UC$, then $\vec{B}_{X,s}$ is an appropriate slight modification (using powers of $e$ instead of powers of 2) of the interpolation space $\vec{B}_{\theta} = \langle B_0, B_1 \rangle_{\theta}$ introduced by Peetre on p. 175-6 of [16], for the function parameter $\rho(t) = t^\theta$, and if $\mathcal{X}_0 = \mathcal{X}_1 = WUC$, then $\vec{B}_{X,s}$ is (a modification, again with powers of $e$ in place of 2 of) the Gustavsson-Peetre variant.
of $\langle B_0, B_1 \rangle_\theta$ which is denoted by $\langle \bar{B}, \rho \rangle$ (see p. 45 of \cite{10}). In fact, for these two choices of the pseudolattice pair, the spaces $B_{X,s}$ will coincide exactly with the spaces $\langle B_0, B_1 \rangle_\theta$ and $\langle \bar{B}, \rho \rangle$ respectively if we perform the following rather obvious changes in their construction: replace powers of $e$ by powers of 2 in Definition \ref{def:5} replace the annulus $\mathbb{A}$ by the annulus \{ $z \in \mathbb{C} : 1 \leq |z| \leq 2$ \} in Definition \ref{def:9} and set $s = 2^\theta$ for some $\theta \in (0, 1)$.

(The method which yields the spaces $\langle B_0, B_1 \rangle_\theta$ is sometimes referred to as the “±” method, since, as mentioned above, in the definition of unconditional convergence it suffices to consider sequences $\lambda_n$ whose values are 1 and $-1$.)

**Definition 12.** Let $X = \{ X_0, X_1 \}$ be a pair of pseudolattices. We shall say that $X$ admits differentiation if it is Laurent compatible and, for each complex Banach space $B$,

(i) for each $r \in (0, 1)$, each element $\{ b_n \}_{n \in \mathbb{Z}} \in X_0 (B)$ satisfies

$$\lim_{k \to \infty} r^{-k} \| b_k \|_B = 0$$

and each element $\{ b_n \}_{n \in \mathbb{Z}} \in X_1 (B)$ satisfies

$$\lim_{k \to \infty} r^k \| b_k \|_B = 0,$$

(ii) for every complex number $\rho$ satisfying $0 < |\rho| < 1$ and for every sequence $\{ b_n \}_{n \in \mathbb{Z}} \in X_0 (B) \cap X_1 (B)$, the new sequence $\{ b_n^1 \}_{n \in \mathbb{Z}}$ is also in $X_j (B)$ for $j = 0, 1$, where $\{ b_0^1 \}_{n \in \mathbb{Z}}$ and $\{ b_1^1 \}_{n \in \mathbb{Z}}$ are defined by setting $b_0^1 = \sum_{k < 0} \rho^{-k} b_{n+k+1}$ and $b_1^1 = \sum_{k \geq 0} \rho^k b_{n+k+1}$ (where the convergence of these sums in $B$ is guaranteed by condition (i)), and if also

(iii) for $j = 0, 1$ and each $\rho$ as above, the linear map $D_{j, \rho}$ defined on $X_j (B)$ by setting $D_{j, \rho} (\{ b_n \}_{n \in \mathbb{Z}}) = \{ b_n^1 \}_{n \in \mathbb{Z}}$ maps $X_j (B)$ boundedly into itself.

**Remark 13.** The pair $X = \{ X_0, X_1 \}$ admits differentiation whenever $X_0$ and $X_1$ are each chosen to be any of $\ell^p$ \footnote{\cite{10}}, $c_0$, $FC$, $UC$ or $WUC$ (see p. 256 of \cite{3}).

The property of admitting differentiation has the following consequence (which also explains the choice of terminology for this property).

**Lemma 14.** Let $X$ be a pair of pseudolattices which admits differentiation and let $\bar{B}$ be a Banach pair. Let the sequence $\{ f_n \}_{n \in \mathbb{Z}}$ be an element of $\mathcal{J} (X, \bar{B})$ and let $f : \mathbb{A}^\circ \to B_0 + B_1$ be the analytic function defined by $f (z) = \sum_{n \in \mathbb{Z}} z^n f_n$. Suppose that $f (s) = 0$ for some point $s \in \mathbb{A}^\circ$ and let $g : \mathbb{A}^\circ \to B_0 + B_1$ be the analytic function obtained by setting $g (s) = f' (s)$ and $g (z) = \frac{1}{z-s} f (z)$ for all $z \in \mathbb{A}^\circ \setminus \{ s \}$. Let $\{ g_n \}_{n \in \mathbb{Z}}$ be the sequence of coefficients in the Laurent expansion $g (z) = \sum_{n \in \mathbb{Z}} z^n g_n$ of $g$ in $\mathbb{A}^\circ$. Then $\{ g_n \}_{n \in \mathbb{Z}}$ is also an element of $\mathcal{J} (X, \bar{B})$. 
For the proof we refer the reader to pp. 258-9 of [8]. This lemma will also hold if we replace $e$ by some $r > 1$ in the definitions of $\mathcal{A}$ and $\mathcal{J}(X, B)$.

We conclude this section with two more definitions of notions which will appear explicitly in our main theorem.

**Definition 15.** For each Banach pair $B$ we define $\mathcal{J}_0(B)$ to be the space of all $B_0 \cap B_1$ valued sequences $\{b_n\}_{n \in \mathbb{Z}}$ with finite support.

**Remark 16.** For every Banach pair $\bar{B}$, we obviously have that $\mathcal{J}_0(\bar{B}) \subset \mathcal{J}(\{X_0, X_1\}, \bar{B})$ whenever $X_0$ and $X_1$ are chosen to be any of the pseudolattices $FC, UC, WUC, \ell^p$ for $p \in [1, \infty]$ or $c_0$. Furthermore, one can verify that $\mathcal{J}_0(\bar{B})$ is dense in $\mathcal{J}(\{FC, FC\}, \bar{B})$ and also dense in $\mathcal{J}(\{X_0, X_1\}, \bar{B})$ for $X_i \in \{UC, c_0, \ell^p\}, 1 \leq p < \infty, i = 0, 1$. But, in general, $\mathcal{J}_0(\bar{B})$ is not dense in $\mathcal{J}(\{WUC, WUC\}, \bar{B})$ and, except for trivial Banach spaces $B_0, B_1$, is never dense in $\mathcal{J}(\{\ell^\infty, \ell^\infty\}, \bar{B})$.

**Definition 17.** Let $S$ denote the right-shift operator on two-sided sequences defined by $S(\{b_n\}_{n \in \mathbb{Z}}) = \{b_n - 1\}_{n \in \mathbb{Z}}$.

3. The main theorem

We can now state and prove our main theorem. The steps of the proof parallel the steps of Stafney’s proof on p. 335 of [17].

**Theorem 18.** Let $X$ be a pair of pseudolattices which admits differentiation and let $\bar{B}$ be a Banach pair. Suppose that

(i) $\mathcal{J}_0(\bar{B}) \subset \mathcal{J}(X, \bar{B})$ and $\mathcal{J}_0(\bar{B})$ is dense in $\mathcal{J}(X, \bar{B})$ and that

(ii) The right-shift operator $S$ maps $X_j(B_j)$ boundedly into itself for $j = 0, 1$.

Then, for each $x \in B_0 \cap B_1$ and $s \in \mathbb{N}^\infty$,

\[
\|x\|_{\bar{B}_\infty} = \inf \left\{ \left\| \{b_n\}_{n \in \mathbb{Z}} \right\|_{\mathcal{J}(X, \bar{B})} : \sum_{n \in \mathbb{Z}} s^n b_n = x, \ \{b_n\}_{n \in \mathbb{Z}} \in \mathcal{J}_0(\bar{B}) \right\}.
\]

**Remark 19.** Our proof will also hold if we replace the norm $\|\cdot\|_{\mathcal{J}(X, \bar{B})}$ in both Definition 9 and Equation (3.1) by an equivalent one.

**Remark 20.** As hinted in part (iii) of Remark 11 and in the remark following the statement of Lemma 14, if we replace $e$ in our definitions by any positive number greater than 1, then we can obtain an appropriate reformulation of Theorem 18.
Remark 21. By Remarks 11, 13, and 16 we can obtain an appropriate formulation of Theorem 18 for the “annulus” variant of Calderón’s complex interpolation method space, for the Lions-Peetre real method space $(B_0, B_1)_{\theta,p}$ for $1 \leq p < \infty$ and for the Peetre interpolation space $\bar{B}_{\theta} = (B_0, B_1)_{\theta}$ for $\theta \in (0,1)$. (Of course condition (ii) of Theorem 18 obviously holds in these cases and in fact $S$ is even an isometry.)

Our theorem in the case of the “annulus” variant of Calderón’s complex interpolation method space can also be obtained by an alternative argument similar to Stafney’s proof on p. 335 of [17], if one replaces Calderón’s space $\mathcal{G} (B_0, B_1)$ by the space of all Laurent polynomials with coefficients in $B_0 \cap B_1$ and if $\mathcal{F} (B_0, B_1)$ is replaced by its counterpart for the annulus (see also Section 4 and the paragraph which precedes Definition 4.1 on p. 80 of [17]).

For the Lions-Peetre real method space $(B_0, B_1)_{\theta,p}$ for $1 \leq p < \infty$, our theorem shows that if $x \in B_0 \cap B_1$, $\theta \in (0,1)$ and $p \in [1, \infty)$, then

$$
\| x \|_{(B_0, B_1)_{\theta,p}} = \inf \left\{ \max_{j=0,1} \left\| \left\{ e^{(j-\theta)n}c_n \right\}_{n \in \mathbb{Z}} \right\|_{\ell^p(B_j)} : x = \sum_{n \in \mathbb{Z}} c_n, \{c_n\}_{n \in \mathbb{Z}} \in \mathcal{J}_0 (\bar{B}) \right\}.
$$

By Remark 19, if we equip the space $(B_0, B_1)_{\theta,p}$ with the norm $\| x \| = \inf \left\| \left\{ e^{-\theta n}J(e^n, c_n; B_0, B_1) \right\}_{n \in \mathbb{Z}} \right\|_{\ell^p}$, where the infimum is taken over all representations $x = \sum_{n \in \mathbb{Z}} c_n$ (with convergence in $B_0 + B_1$) with $\{c_n\}_{n \in \mathbb{Z}} \in \mathcal{J} (\{\ell^p, \ell^p\}, \bar{B})$, then our theorem shows that for every $x \in B_0 \cap B_1$, $\theta \in (0,1)$ and $p \in [1, \infty)$, we have that

$$
\| x \| = \inf \left\{ \left\| \left\{ e^{-\theta n}J(e^n, c_n; B_0, B_1) \right\}_{n \in \mathbb{Z}} \right\|_{\ell^p} : x = \sum_{n \in \mathbb{Z}} c_n, \{c_n\}_{n \in \mathbb{Z}} \in \mathcal{J}_0 (\bar{B}) \right\}.
$$

By Remark 20, we can also obtain a version of our theorem if, for instance, we equip the space $(B_0, B_1)_{\theta,p}$ with the $J$-functional norm $\| x \| = \inf \left\| \left\{ 2^{-\theta n}J(2^n, c_n; B_0, B_1) \right\}_{n \in \mathbb{Z}} \right\|_{\ell^p}$, where the infimum is taken over all representations $x = \sum_{n \in \mathbb{Z}} c_n$ (with convergence in $B_0 + B_1$) with $\{c_n\}_{n \in \mathbb{Z}}$ belonging to the variant of $\mathcal{J} (\{\ell^p, \ell^p\}, \bar{B})$ obtained by replacing powers of $e$ by powers of 2 (this norm appears on p. 43 of [11]).

If we choose this norm, then for every $x \in B_0 \cap B_1$, $\theta \in (0,1)$ and $p \in [1, \infty)$ we obtain that

$$
\| x \| = \inf \left\{ \left\| \left\{ 2^{-\theta n}J(2^n, c_n; B_0, B_1) \right\}_{n \in \mathbb{Z}} \right\|_{\ell^p} : x = \sum_{n \in \mathbb{Z}} c_n, \{c_n\}_{n \in \mathbb{Z}} \in \mathcal{J}_0 (\bar{B}) \right\}.
$$
By part (iii) of Remark 11 and by Remark 20 for the Peetre interpolation space $B(\theta) = (B_0, B_1)_\theta$ for $\theta \in (0, 1)$, our theorem shows that if $x \in B_0 \cap B_1$, then

$$
\|x\|_{(B_0, B_1)_\theta} = \inf \left\{ \max_{j=0,1} \left\| \left\{ 2^{j-\theta} c_n \right\}_{n \in \mathbb{Z}} \right\|_{UC(B_j)} : x = \sum_{n \in \mathbb{Z}} c_n, \left\{ c_n \right\}_{n \in \mathbb{Z}} \in \mathcal{J}_0 \left( \bar{B} \right) \right\}.
$$

Remark 22. We can in fact also obtain a version of our theorem for a discrete version of the (generalised) $J$-method which is discussed (for example) on p. 381 of [12] and apparently originated in the work of Peetre in [15]. We shall recall its definition.

Let $\Phi$ be a Banach lattice of two sided sequences satisfying $\{0\} \neq \Phi \subset \ell_1 + \ell_1$. Here $\ell_1$ denotes the space of all real valued sequences $\{b_n\}_{n \in \mathbb{Z}}$ such that the sum $\sum_{n \in \mathbb{Z}} 2^{-n} |b_n|$ is finite with the norm $\left\| \{b_n\}_{n \in \mathbb{Z}} \right\|_{\ell_1} = \sum_{n \in \mathbb{Z}} 2^{-n} |b_n|$ for $j = 0, 1$.

We define $\|x\|_{J^d(B)} = \inf \left\{ \|\{J(2^n, x_n; B_0, B_1)\}_{n \in \mathbb{Z}}\|_{\Phi} : x = \sum_{n \in \mathbb{Z}} x_n \right\}$ where the infimum is taken over all representations $x = \sum_{n \in \mathbb{Z}} x_n$ (with convergence in $B_0 + B_1$) with $x_n \in B_0 \cap B_1$.

We shall now show what is required in order to obtain our theorem for the discrete $J$-method. Set $x_n = 2^{n\theta} c_n$. Defining $\|\{b_n\}_{n \in \mathbb{Z}}\|_X = \left\| \{2^{n\theta} b_n\}_{n \in \mathbb{Z}} \right\|_{\Phi}$ yields $\|x\|_{J^d(B)} = \inf \left\{ \|\{J(2^n, c_n; B_0, B_1)\}_{n \in \mathbb{Z}}\|_X \right\}$.

Since $\|\{J(2^n, c_n; B_0, B_1)\}_{n \in \mathbb{Z}}\|_X$ and

$$
\max_{j=0,1} \left\| \left\{ 2^{nj} c_n \right\}_{n \in \mathbb{Z}} \right\|_{X(B_j)} = \max_{j=0,1} \left\| \left\{ 2^{nj} c_n \right\}_{n \in \mathbb{Z}} \right\|_{X(B_j)}
$$

are equivalent, by Remarks 19 and 20 we can obtain a version of Stafney’s lemma in this case if the pseudolattice pair $\{\mathcal{X}_0, \mathcal{X}_1\} = \{X, X\}$ satisfies the conditions of our main theorem. (Here, of course, we must replace $e$ by $2$ in the appropriate definitions.)

Remark 23. Janson showed that if we equip $B_0 \cap B_1$ with the norms of $\langle \bar{B}, \rho_0 \rangle$ and $\bar{B}(\theta)$, we obtain two normed spaces with equivalent norms (see pp. 59-60 of [11]), and thus a weaker version of (3.1), i.e. that the left and right sides are equivalent, can also be obtained for $\mathcal{X}_0 = \mathcal{X}_1 = WUC$, even though, as pointed out in Remark 16 condition (i) fails to hold in this case.

Proof of the theorem. Let $x$ be in $B_0 \cap B_1$, $s$ in $\mathbb{A}_s$ and $\varepsilon$ an arbitrary positive number. The sequence $\{b_n\}_{n \in \mathbb{Z}}$ defined by setting $b_0 = x$ and $b_n = 0$ for $n \neq 0$ is in $\mathcal{J}_0 (\bar{B})$ and satisfies $\sum_{n \in \mathbb{Z}} s^n b_n = x$. It is clear from the definition of the norm $\|\cdot\|_{B(\theta)}$ that, for some $\{c_n\}_{n \in \mathbb{Z}}$ in the subspace $\mathcal{N}_s (\mathbb{X}, \bar{B})$
of \( \mathcal{J}(X, B) \), consisting of all sequences \( \{d_n\}_{n \in \mathbb{Z}} \) such that \( \sum_{n \in \mathbb{Z}} s^n d_n = 0 \), 
\[ \|\{b_n\}_{n \in \mathbb{Z}} - \{c_n\}_{n \in \mathbb{Z}}\|_{\mathcal{J}(X, B)} < \|x\|_{\mathcal{B}_{X, s}} + \varepsilon/2. \] We need the following proposition:

**Proposition.** \( \mathcal{J}_0(B) \cap \mathcal{N}_s(X, B) \) is dense in \( \mathcal{N}_s(X, B) \) with respect to the norm of \( \mathcal{J}(X, B) \) restricted to \( \mathcal{N}_s(X, B) \).

We will first give a proof of the proposition and then continue with the proof of the theorem. Let \( \{c_n\}_{n \in \mathbb{Z}} \) be in \( \mathcal{N}_s(X, B) \). Set \( f(z) = \sum_{n \in \mathbb{Z}} z^n c_n \).

Then, by Lemma 14, the function \( g : \mathbb{A}^n \rightarrow B_0 + B_1 \) defined by setting \( g(s) = f'(s) \) and \( g(z) = \frac{1}{z-s}f(z) \) for all \( z \in \mathbb{A}^n \backslash \{s\} \) has a Laurent expansion \( g(z) = \sum_{n \in \mathbb{Z}} z^n g_n \) with \( \{g_n\}_{n \in \mathbb{Z}} \in \mathcal{J}(X, B) \). Set \( C = \max_{j=0,1} ||S||_{X_j(B_j) \rightarrow X_j(B_j)} \). Since \( \mathcal{J}_0(B) \) is dense in \( \mathcal{J}(X, B) \) we can find some \( \{h_n\}_{n \in \mathbb{Z}} \in \mathcal{J}_0(B) \) such that

\[
(3.2) \quad \|\{h_n\}_{n \in \mathbb{Z}} - \{g_n\}_{n \in \mathbb{Z}}\|_{\mathcal{J}(X, B)} < \frac{\varepsilon}{e(1+C)}.
\]

For every analytic function \( f : \mathbb{A}^n \rightarrow B_0 + B_1 \) with a Laurent expansion \( f(z) = \sum_{n \in \mathbb{Z}} z^n b_n \) with \( \{b_n\}_{n \in \mathbb{Z}} \in \mathcal{J}(X, B) \) we shall define

\[
\|f\|_{\mathcal{J}, B} := \|\{b_n\}_{n \in \mathbb{Z}}\|_{\mathcal{J}(X, B)}.
\]

(This is well-defined due to the uniqueness of the Laurent expansion in the annulus.) Set \( h(z) = \sum_{n \in \mathbb{Z}} z^n h_n \).

Note that for every element \( \{k_n\}_{n \in \mathbb{Z}} \in \mathcal{J}(X, B) \) if \( k(z) = \sum_{n \in \mathbb{Z}} z^n k_n \) and if \( r(z) = z-s \) then \( (rk)(z) = \sum_{n \in \mathbb{Z}} z^n (k_{n-1} - sk_n) \) and thus

\[
\|rk\|_{\mathcal{J}, B} = \|\{k_{n-1} - sk_n\}_{n \in \mathbb{Z}}\|_{\mathcal{J}(X, B)}
\leq \|\{k_{n-1}\}_{n \in \mathbb{Z}}\|_{\mathcal{J}(X, B)} + |s| \|\{k_n\}_{n \in \mathbb{Z}}\|_{\mathcal{J}(X, B)}.
\]

Assumption (ii) of the theorem yields that

\[
\|\{k_{n-1}\}_{n \in \mathbb{Z}}\|_{\mathcal{J}(X, B)} = \max \left\{ \|\{k_{n-1}\}_{n \in \mathbb{Z}}\|_{X_0(B_0)}, \|\{e^n k_{n-1}\}_{n \in \mathbb{Z}}\|_{X_1(B_1)} \right\}
\leq C \max \left\{ \|\{k_n\}_{n \in \mathbb{Z}}\|_{X_0(B_0)} e, \|\{e^n k_n\}_{n \in \mathbb{Z}}\|_{X_1(B_1)} \right\}
\leq eC \|\{k_n\}_{n \in \mathbb{Z}}\|_{\mathcal{J}(X, B)}
\]

and thus \( \|rk\|_{\mathcal{J}, B} \leq e(1+C) \|k\|_{\mathcal{J}, B} \). The preceding calculation, along with Equation (3.2), shows, in particular, that

\[
\|\{h_{n-1} - sh_n\}_{n \in \mathbb{Z}} - \{c_n\}_{n \in \mathbb{Z}}\|_{\mathcal{J}(X, B)} = \|rh - f\|_{\mathcal{J}, B}
\leq \|rh - rg\|_{\mathcal{J}, B}
\leq e(1+C) \|h - g\|_{\mathcal{J}, B} < \varepsilon.
\]

Since \( \{h_{n-1} - sh_n\}_{n \in \mathbb{Z}} \) is in \( \mathcal{J}_0(B) \cap \mathcal{N}_s(X, B) \), the proposition follows.
Continuing with the proof of the theorem, we choose an element \( \{u_n\}_{n \in \mathbb{Z}} \) in \( J_0(B) \cap \mathcal{N}_\varepsilon(X, B) \) such that \( \|\{c_n\}_{n \in \mathbb{Z}} - \{u_n\}_{n \in \mathbb{Z}\} \|_{J(X, B)} < \varepsilon/2. \)

We have that \( \{b_n - u_n\}_{n \in \mathbb{Z}} \in J_0(\bar{B}) \) and \( \sum_{n \in \mathbb{Z}} s^n (b_n - u_n) = x. \) Furthermore,

\[
\|\{b_n - u_n\}_{n \in \mathbb{Z}}\|_{J(X, \bar{B})} \\
\leq \|\{b_n\}_{n \in \mathbb{Z}} - \{c_n\}_{n \in \mathbb{Z}\} \|_{J(X, \bar{B})} + \|\{c_n\}_{n \in \mathbb{Z}} - \{u_n\}_{n \in \mathbb{Z}\} \|_{J(X, \bar{B})} \\
< \|x\|_{\bar{B}_X, s} + \varepsilon.
\]

So the proof of the theorem is complete. \( \square \)

4. ADDITIONAL REMARKS REGARDING COMPLEX INTERPOLATION

We begin by explicitly recalling the definition of complex interpolation spaces on the annulus.

Let \( B = (B_0, B_1) \) be a Banach pair. Let \( \mathcal{F}_h(B) \) be the space of all continuous functions \( f : \mathbb{A} \to B_0 + B_1 \) such that \( f \) is analytic in \( A^o \) and for \( j = 0, 1 \) the restriction of \( f \) to the circle \( e^{jT} \) is a continuous map of \( e^{jT} \) into \( B_j \). We norm \( \mathcal{F}_h(B) \) by \( \|f\|_{\mathcal{F}_h(B)} = \sup \{\|f(e^{j+it})\|_{B_j} : t \in [0, 2\pi], \ j = 0, 1\} \). For each \( \theta \in (0, 1) \), let \( [B]_{\theta, h} \) denote the space of all elements in \( B_0 + B_1 \) of the form \( b = f(e^{\theta}) \) for some \( f \in \mathcal{F}_h(\bar{B}) \). It is normed by

\[
\|b\|_{[B]}_{\theta, h} = \inf \{\|f\|_{\mathcal{F}_h(\bar{B})} : f \in \mathcal{F}_h(\bar{B}), b = f(e^{\theta})\}.
\]

Here, as promised above, we give a detailed proof that

\[
\bar{B}_{\{FC, FC\}, e^\theta} = [B]_{\theta, h}
\]

with equality of norms, for each \( \theta \in (0, 1) \) as was stated in [3]. Some parts of the proof can also be found on pp. 78-9 of [7]. Related ideas appear already in [3].

Let \( b \in [B]_{\theta, h} \) and let \( \varepsilon \) be an arbitrary positive number. We can find some \( f \in \mathcal{F}_h(\bar{B}) \) which satisfies \( b = f(e^\theta) \) and \( \|f\|_{\mathcal{F}_h(B)} < \|b\|_{[B]}_{\theta, h} + \varepsilon. \)

As shown on pp. 78-9 of [7], \( f(z) = \sum_{n \in \mathbb{Z}} z^n \hat{f}(n) \) for every \( z \in A^o \), where

\[
\hat{f}(n) = \frac{1}{2\pi} \int_0^{2\pi} e^{-nit} f(e^{it}) dt = \frac{1}{2\pi} \int_0^{2\pi} e^{-n(1+it)} f(e^{1+it}) dt,
\]
from which we obtain that \( b = \sum_{n \in \mathbb{Z}} e^{\theta n} \hat{f}(n) \), and thus \( b \in \tilde{B}_{(FC, FC), e^\theta} \) and 
\([\tilde{B}]_{\theta, A} \subset B_{(FC, FC), e^\theta} \). Furthermore
\[
\|b\|_{[\tilde{B}]_{\theta, A} + \varepsilon} > \sup \left\{ \|f(e^{j+it})\|_{B_j} : t \in [0, 2\pi], j = 0, 1 \right\}
\]
\[
= \max_{j=0,1} \left\| \left\{ e^{jn} \hat{f}(n) \right\}_{n \in \mathbb{Z}} \right\|_{(FC, B_j)}
\]
\[
= \left\| \left\{ \hat{f}(n) \right\}_{n \in \mathbb{Z}} \right\|_{(FC, \tilde{B})}
\]
\[
\geq \|b\|_{\tilde{B}_{(FC, FC), e^\theta}}.
\]

It follows that \( \|b\|_{[\tilde{B}]_{\theta, A}} \geq \|b\|_{\tilde{B}_{(FC, FC), e^\theta}} \) for all \( b \in [\tilde{B}]_{\theta, A} \).

Now, for the reverse inclusion and norm inequality, let \( b \in \tilde{B}_{(FC, FC), e^\theta} \) and let \( \varepsilon \) be an arbitrary positive number. We can find some \( \{b_n\}_{n \in \mathbb{Z}} \in J(\{FC, FC\}, \tilde{B}) \) such that
\[
(4.1) \quad b = \sum_{n \in \mathbb{Z}} e^{\theta n} b_n
\]
and \( \left\| \{b_n\}_{n \in \mathbb{Z}} \right\|_{J(\{FC, FC\}, \tilde{B})} \leq \|b\|_{\tilde{B}_{(FC, FC), e^\theta}} + \varepsilon \). By the definition of \( FC \), we can find for \( j = 0, 1 \) continuous functions \( f_j : \mathbb{T} \to \mathbb{B}_j \) such that \( e^{jn} b_n = \frac{1}{2\pi} \int_0^{2\pi} e^{-int} f_j(e^{it}) dt \) for all \( n \). We shall define a sequence of functions \( g_N : \mathbb{A} \to \mathbb{B}_0 \cap \mathbb{B}_1 \) by
\[
g_N(z) = \sum_{n=-N}^{N} z^n \left( 1 - \frac{|n|}{N+1} \right) b_n,
\]
and we shall first show that \( g_N(z) \) converges in \( \mathbb{B}_0 + \mathbb{B}_1 \) for each \( z \in \mathbb{A} \).

If \( z \in \mathbb{A}^0 \) then, by Remark [8] the sequence of Laurent polynomials \( S_N(z) := \sum_{n=-N}^{N} z^n b_n \) converges in \( \mathbb{B}_0 + \mathbb{B}_1 \), and thus, since in fact \( g_N(z) = \frac{1}{N+1} \sum_{n=0}^{N} S_N(z) \), it follows, by standard arguments, that \( g_N(z) \) also converges in \( \mathbb{B}_0 + \mathbb{B}_1 \) to the same limit \( \sum_{n \in \mathbb{Z}} z^n b_n \).

If we apply the lemma on pp. 10–11 of [12] with the Fejér summability kernel \( K_n(t) = \sum_{j=-n}^{n} \left( 1 - \frac{|j|}{n+1} \right) e^{i(j-t)} \) and with \( \varphi(\tau) = f_j(e^{it}) \) and \( B = B_j \) for \( j = 0, 1 \), then we obtain that
\[
(4.2) \quad \lim_{N \to \infty} \|g_N(e^{j+it}) - f_j(e^{it})\|_{B_j} = 0
\]
for each \( t \in [0, 2\pi) \) and thus \( g_N(z) \) also converges for \( z \in \mathbb{A} \setminus \mathbb{A}^0 \). Moreover, since
\[
\limsup_{\tau \to 0} \left\{ \|f_j(e^{it}) - f_j(e^{it})\|_{B_j} : t \in [0, 2\pi), j = 0, 1 \right\} = 0,
\]
we can in fact obtain, by making a slight modification to the proof of the lemma in [12] for our particular case, that

\[
\lim_{N \to \infty} \sup \left\{ \| g_N(e^{j+it}) - f_j(e^{it}) \|_{B_j} : t \in [0, 2\pi], \ j = 0, 1 \right\} = 0.
\]

We shall denote the pointwise limit of \(g_N\) in \(B_0 + B_1\) by \(g\). By Equation (4.2), we obtain that \(g(e^{j+it}) = f_j(e^{it})\) for each \(t \in [0, 2\pi]\) and \(j = 0, 1\), and thus the restriction of \(g\) to the circle \(e^{jT}\) is a continuous map of \(e^{jT}\) into \(B_j\). Since \(g(z) = \sum_{n \in \mathbb{Z}} z^n b_n\) for every \(z \in \mathbb{A}^\circ\), by Remarks 8 and 13 and Equation (4.1), \(g\) is an analytic function on \(\mathbb{A}^\circ\) which satisfies

\[
g(e^\theta) = b.
\]

By Equation (4.3) and the maximum principle, the sequence of continuous functions \(g_N\) converges in \(B_0 + B_1\) uniformly on \(\mathbb{A}\) and consequently its limit is also a continuous \(B_0 + B_1\) valued function. Thus, \(g \in \mathcal{F}_\mathbb{A}(\tilde{B})\) and so, by Equation (4.4), \(b \in [\tilde{B}]_{\theta, \mathbb{A}}\) and \(\tilde{B}_{\{FC, FC\}, e^\theta} \subset [\tilde{B}]_{\theta, \mathbb{A}}\). Furthermore, the preceding calculations show that

\[
\left\| \{b_n\}_{n \in \mathbb{Z}} \right\|_{\mathcal{F}(\{FC, FC\}, \tilde{B})} = \max_{j=0,1} \sup_{t \in [0, 2\pi]} \left\| f_j(e^{it}) \right\|_{B_j} = \max_{j=0,1} \sup_{t \in [0, 2\pi]} \left\| g(e^{j+it}) \right\|_{B_j} = \| g \|_{\mathcal{F}_\mathbb{A}(\tilde{B})} \geq \| b \|_{[\tilde{B}]_{\theta, \mathbb{A}}}.
\]

Therefore, \(\| b \|_{[\tilde{B}]_{\theta, \mathbb{A}}} \leq \| b \|_{\tilde{B}_{\{FC, FC\}, e^\theta}}\) for all \(b \in \tilde{B}_{\{FC, FC\}, e^\theta}\). This completes the proof.

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