Convex bodies of states and maps

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Abstract

We give a general solution to the question of when the convex hulls of orbits of quantum states on a finite-dimensional Hilbert space under unitary actions of a compact group have a non-empty interior in the surrounding space of all density operators. The same approach can be applied to study convex combinations of quantum channels. The importance of both problems stems from the fact that, usually, only sets with non-vanishing volumes in the embedding spaces of all states or channels are of practical importance. For the group of local transformations on a bipartite system we characterize maximally entangled states by the properties of a convex hull of orbits through them. We also compare two partial characteristics of convex bodies in terms of the largest balls and maximum volume ellipsoids contained in them and show that, in general, they do not coincide. Separable states, mixed-unitary channels and \(k\)-entangled states are also considered as examples of our techniques.

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(Some figures may appear in colour only in the online journal)

1. Introduction

In many issues of quantum information theory and the geometry of quantum states, one is confronted with the problem of whether some subset of states or quantum channels is ‘large enough’ to be of significance in applications. On a qualitative level the problem can be reduced

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to the question of whether the considered set contains an open subset (as a subset of the set of all states/channels). If the answer is affirmative, one can ask more quantitative questions about the relative volume of such sets, about some estimates of their volumes, or the radiiuses of the maximal balls they contain. In concrete applications such quantitative answers depend on a particular choice of the volume measure appropriate for the considered problems. Thus, for example, in the case of density matrices one can choose from several ‘natural’ measures (see, e.g. [1, 2]). The choice may be dictated by the demand for special properties of the measure, like monotonicity or the fact that it comes from a Riemannian metric, or by the requirement that it agrees with commonly used measures on the subset of pure states. In this sense it is hard to distinguish the most ‘natural’ measure on the set of mixed quantum states (density operators), but it should be clear that a ‘natural’ measure should not vanish only on subsets containing non-empty sets in their interiors. Many questions of this type can be regarded as instances of the following general problem (see e.g. [3]).

**Problem 1.1.** Let \( V \) be a Euclidean space, i.e. a finite-dimensional real vector space equipped with a scalar product \( \langle \cdot | \cdot \rangle_V \) and let \( K \) be a compact group acting on \( V \) by orthogonal transformations, \( K \times V \ni (U,x) \mapsto U \cdot x \in V \). Given a \( K \)-invariant affine subspace \( A \) of \( V \) and a vector \( x_0 \in A \), decide whether the convex hull \( \text{Conv}(K \cdot x_0) \) of the orbit \( K \cdot x_0 \) (the convexed orbit) is a convex body in \( A \), i.e. whether the interior of \( \text{Conv}(K \cdot x_0) \) is a non-empty open set of \( A \). This is the same as deciding whether the volume of \( \text{Conv}(K \cdot x_0) \) is positive in \( A \).

Let us recall that affine subspaces in \( V \) are exactly subsets closed with respect to affine combinations,

\[
a_1, a_2 \in A \Rightarrow \forall t \in \mathbb{R} \ [ta_1 + (1-t)a_2 \in A],
\]

and that the differences \( a_1 - a_2 \) of the points of \( A \) form a real vector subspace \( V(A) \) of \( V \) called the linear part of \( A \). Convex combinations are those affine combinations \( ta_1 + (1-t)a_2 \) for which \( 0 \leq t \leq 1 \). After recalling in section 2 some elementary notions, we present in section 3 several examples to which our analysis can be applied, both in the cases of states and channels. Some of them concern problems for which the answer to the posed question is known, but they provide a perfect insight into the unifying power of our approach. The full answer to the above-stated problem 1.1 is given in section 4. In section 5 we show how to apply the obtained result to the examples of section 3. In the case of maximally entangled states our approach leads to a unique characterization of such states in terms of the properties of convexed orbits through them. In section 7 we compare characterizations of convex bodies of states in terms of the largest ball which can be inscribed within the body in question and the so-called maximal volume ellipsoid of that body. In principle, the latter notion is an affine one, whereas the former bears a metric nature. However, in some important cases both notions coincide (e.g. for the set of all density operators), in others (e.g. for convexed local orbits of pure states in composite systems) this is no longer true.

### 2. Notations and conventions

Let \( \mathcal{H} \) be an \( n \)-dimensional Hilbert space with a Hermitian product \( \langle x,y \rangle_{\mathcal{H}} \) being, by convention, \( \mathbb{C} \)-linear with respect to \( y \) and anti-linear with respect to \( x \). Let \( gl(\mathcal{H}) \) be the complex vector space of all complex linear operators on \( \mathcal{H} \). It is also canonically a Hilbert space with the Hermitian product

\[
\langle A, B \rangle_{\mathcal{H}} = \text{tr}(A^\dagger B),
\]

(1)
where $A^\dagger$ is the Hermitian conjugate of $A$, i.e., $\langle Ax, y \rangle_\mathcal{H} = \langle x, A^\dagger y \rangle_\mathcal{H}$. The unitary group $U(\mathcal{H})$ consists of those complex linear operators $U \in gl(\mathcal{H})$ on $\mathcal{H}$ which satisfy $UU^\dagger = I$. It acts canonically on $\mathcal{H}$ preserving the Hermitian product. Fixing an orthonormal basis $(e_i)$ of $\mathcal{H}$ allows us to identify the Hermitian product $\langle x, y \rangle_\mathcal{H}$ on $\mathcal{H}$ with the canonical Hermitian product on $\mathbb{C}^n$ of the form $\langle a, b \rangle_{\mathbb{C}^n} = \sum_{i=1}^n \bar{a}_i b_i$, the group $U(\mathcal{H})$ of unitary transformations of $\mathcal{H}$ with $U(n)$, its Lie algebra $u(\mathcal{H})$ with $u(n)$, etc. In this picture, $(a_i b_i)^\dagger = (\bar{a}_i)$. One important convention we want to introduce is that we identify the (real) vector space of Hermitian operators with the dual operators of rank $1$. In the standard terminology, $\mathcal{D}^1(\mathcal{H})$ is the set of pure states, i.e. the set of one-dimensional orthogonal projectors $|\psi\rangle\langle\psi|$, where $\|\psi\|^2 = \langle\psi|\psi\rangle = 1$. It is known that the set of extreme points of $\mathcal{D}(\mathcal{H})$ coincides with the set $\mathcal{D}^1(\mathcal{H})$ of pure states. Hence, every element of $\mathcal{D}(\mathcal{H})$ is a convex combination of points from $\mathcal{D}^1(\mathcal{H})$. The space $\mathcal{D}^1(\mathcal{H})$ of pure states can be identified with the complex projective space $\mathbb{P}(\mathcal{H}) \cong \mathbb{CP}^{n-1}$ via the projection

$$\mathcal{H} \setminus \{0\} \ni \psi \mapsto P_\psi = \frac{|\psi\rangle\langle\psi|}{\|\psi\|^2} \in \mathcal{D}^1(\mathcal{H})$$

which identifies the points of the orbits of the $\mathbb{C} \setminus \{0\}$-group action by complex homotheties. Actually, due to the probabilistic interpretation, a pure quantum state is a point in this projective space $\mathbb{P}(\mathcal{H}) \cong \mathcal{D}^1(\mathcal{H})$ rather than a vector in $\mathcal{H}$. The unitary group $K = U(\mathcal{H})$ acts canonically and orthogonally on the Euclidean space $\mathcal{V} = u^*(\mathcal{H})$ by

$$A \mapsto U A U^\dagger = U A U^{-1},$$

and the orbits of this action are distinguished by the spectrum of the Hermitian operator $A$. Of course, we can consider the $U(\mathcal{H})$-action on the Hilbert space $gl(\mathcal{H})$ as the complexification of the orthogonal action on $u^*(\mathcal{H})$, since

$$gl(\mathcal{H}) = \mathbb{C} \otimes u^*(\mathcal{H}) = u^*(\mathcal{H}) \oplus iu^*(\mathcal{H}) = u^*(\mathcal{H}) \oplus u(\mathcal{H}).$$

All operators proportional to the identity, $\lambda I$, are fixed points of this action. It is also easy to see that the trace is preserved, so that the affine spaces $u^*_1(\mathcal{H}) = \{A \in u^*(\mathcal{H}) : \text{tr} A = \lambda\}$ are invariant under the $U(\mathcal{H})$-action. In particular, for any $|\psi\rangle \in \mathcal{H}$, $|\psi\rangle \neq 0$, the orbit $U(\mathcal{H}), P_\psi$ is a minimal orbit of $U(\mathcal{H})$ in $\mathcal{A} = u^*_1(\mathcal{H})$ which coincides with the set $\mathcal{D}^1(\mathcal{H})$ of pure states.

3. Examples

3.1. Density operators

The space of all non-negatively defined operators, i.e. of those $\rho \in gl(\mathcal{H})$ which can be written in the form $\rho = T^\dagger T$ for a certain $T \in gl(\mathcal{H})$, is denoted by $\mathcal{P}(\mathcal{H})$. It is a convex cone in the Euclidean space $\mathcal{V} = u^*(\mathcal{H})$. The set of density operators, $\mathcal{D}(\mathcal{H})$, is distinguished in the cone $\mathcal{P}(\mathcal{H})$ by the equation $\text{tr}(\rho) = 1$, so it is a convex subset in the affine subspace $\mathcal{A} = u^*_1(\mathcal{H}) \subset u^*(\mathcal{H})$ of trace 1 Hermitian operators. The linear part of $u^*_1(\mathcal{H})$ is the subspace $u^*_0(\mathcal{H})$ of Hermitian operators with trace $0$. Denote by $\mathcal{D}^k(\mathcal{H})$ the set of all density operators of rank $k$. In the standard terminology, $\mathcal{D}^1(\mathcal{H})$ is the space of pure states, i.e. the set of density operators $\mathcal{D}(\mathcal{H})$. Hence, every element of $\mathcal{D}(\mathcal{H})$ is a convex combination of points from $\mathcal{D}^1(\mathcal{H})$. The space $\mathcal{D}^1(\mathcal{H})$ of pure states can be identified with the complex projective space $\mathbb{P}(\mathcal{H}) \cong \mathbb{CP}^{n-1}$ via the projection

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and whose convex hull $\text{Conv}(U(\mathcal{H}), P_\phi)$ is the convex set $\mathcal{D}(\mathcal{H})$ of all (mixed) states. It is well known that $\mathcal{D}(\mathcal{H})$ is canonically a Kähler manifold with respect to the metric induced from $u^*(\mathcal{H})$, the Fubini–Study metric, and the symplectic form of a coadjoint orbit of $U(\mathcal{H})$ (see [4]).

3.2. States of composite systems

The Hilbert space of a bipartite composite system is the tensor product of subsystem Hilbert spaces,

$$\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2.$$  \hspace{1cm} (4)

A pure state in $\mathcal{H}$ is separable if it corresponds to a simple tensor,

$$|\psi\rangle = |\phi^1\rangle \otimes |\phi^2\rangle.$$  \hspace{1cm} (5)

As such, it can be identified with the rank-one projection,

$$P_\phi = \frac{|\psi\rangle \langle \psi|}{|\langle \psi|\psi\rangle|}.$$ \hspace{1cm} (6)

Denote the set of separable pure states with $S^1(\mathcal{H}) = S^1(\mathcal{H}_1 \otimes \mathcal{H}_2)$ (this depends on the decomposition of $\mathcal{H}$ into the tensor product). It is easy to see that it is a single minimal orbit $\mathcal{O}_{P_\phi \otimes P_\phi}$ of the obvious orthogonal action of $K = U(\mathcal{H}_1) \times U(\mathcal{H}_2) \subset U(\mathcal{H}_1 \otimes \mathcal{H}_2)$ on the Euclidean space

$$\mathcal{V} = u^*(\mathcal{H}_1 \otimes \mathcal{H}_2) = u^*(\mathcal{H}_1) \otimes u^*(\mathcal{H}_2)$$

going through the point $P_\phi \otimes P_\phi$, for some (arbitrary) $|\phi^{1,2}\rangle \in \mathcal{H}_{1,2}$,

$$S^1(\mathcal{H}) = \{(U_1 P_\phi U_1^* \otimes (U_2 P_\phi U_2^*) : U_i \in U(\mathcal{H}_i), \; i = 1, 2\}. \hspace{1cm} (7)$$

A mixed state $\rho$ is, by definition, separable if it belongs to the convex hull of this orbit, i.e. it is a convex combination of pure separable states,

$$\rho = \sum_{k=1}^n p_k P_{\phi_k} \otimes P_{\phi_k}, \quad p_k \geq 0, \quad \sum_{k=1}^n p_k = 1, \hspace{1cm} (8)$$

for some $\phi^1, \ldots, \phi^n \in \mathcal{H}_i, i = 1, 2$. The other states are called entangled states. The problem of whether the set $\bar{S}(\mathcal{H})$ of mixed separable states possesses a nonzero volume, (see [5]), reduces to the question of whether $\text{Conv}(\bar{S}(\mathcal{H}))$ contains a non-trivial open subset of $\mathcal{A} = u^*_1(\mathcal{H}_1 \otimes \mathcal{H}_2) \subset \mathcal{V}$. It is known that any element $|\psi\rangle \in \mathcal{H}_1 \otimes \mathcal{H}_2$ admits a Schmidt decomposition

$$|\psi\rangle = \sum_{j=1}^r \lambda_j \cdot |\phi^1_j\rangle \otimes |\phi^2_j\rangle, \hspace{1cm} (9)$$

with $|\langle \phi^1_j|\rangle$ and $|\langle \phi^2_j|\rangle$ being (not necessarily complete) orthonormal sets, and $\lambda_j$ being positive real numbers. The number $r$ of summands in this decomposition we call the Schmidt rank of $|\psi\rangle$ and denote $\text{Sr}(\psi)$. Directly by definition, a pure state $P_\phi = |\psi\rangle \langle \psi|$ on $\mathcal{H}_1 \otimes \mathcal{H}_2$ is separable if and only if the Schmidt rank of $|\psi\rangle$ is 1. This easy characterization of separable pure states has been used by Terhal and Horodecki [6] to develop the concept of the Schmidt number of an arbitrary density operator $\rho$ (quantum state in finite dimensions). This number characterizes the minimum Schmidt rank of the pure states that are needed to construct such a density matrix. The Schmidt number is non-increasing under local operations and classical communications, i.e. it provides a legitimate entanglement measure. We can construct an entanglement measure, the Schmidt measure $\mu_S$, which is additionally convex, using the
convex roof construction (see e.g. [7]). This construction, proposed as a general tool for entanglement measures (see e.g. [4, 8, 9]), can be repeated in infinite dimensions as

\[ \mu_S(\rho) = \inf \left\{ \sum_j p_j \text{Tr}(\psi_j) \right\}, \tag{10} \]

where the infimum is taken over all possible realizations of \( \rho \) as infinite-convex combinations \( \rho = \sum_j p_j |\psi_j\rangle \langle \psi_j| \) with \( 0 \leq p_j \leq 1, \sum_j p_j = 1 \) and \( |\psi_j\rangle \in \mathcal{H}_1 \otimes \mathcal{H}_2 \). Every quantum state admits such a realization and a reasoning analogous to the one in [4] shows that \( \mu_S \) is infinite-convex, non-negative, and vanishes exactly on separable states. The Schmidt rank can be conveniently expressed in terms of the Jamiołkowski isomorphism

\[ \tilde{J} : \mathcal{L}(gl(\mathcal{H}_2), gl(\mathcal{H}_1)) \rightarrow gl(\mathcal{H}_1 \otimes \mathcal{H}_2), \]

identifying linear maps on \( \mathcal{H}_1 \otimes \mathcal{H}_2 \) with the space \( \mathcal{L}(gl(\mathcal{H}_2), gl(\mathcal{H}_1)) \) of linear maps \( \Phi : gl(\mathcal{H}_2) \rightarrow gl(\mathcal{H}_1) \) as follows.

**Theorem 3.1.** The Schmidt rank of \( |\psi\rangle \) is \( r \) if and only if \( \tilde{J}^{-1}(P_\psi) : gl(\mathcal{H}_2) \rightarrow gl(\mathcal{H}_1) \) is a linear operator of rank \( r^2 \) [10]. In particular, \( P_\psi \) is separable if and only if \( \tilde{J}^{-1}(P_\psi) \) is of rank 1.

Recall that a pure state \( P_\psi \) is called \( k \)-entangled if the Schmidt rank of \( |\psi\rangle \) is \( \leq k \). Denote the family of all such states with \( \mathcal{E}_k(\mathcal{H}_1 \otimes \mathcal{H}_2) \). This concept emerged from the study of a duality for \( k \)-positive maps [6, 11, 12]. According to the above theorem, \( P_\psi \in \mathcal{E}_k(\mathcal{H}_1 \otimes \mathcal{H}_2) \) if and only if \( \tilde{J}^{-1}(P_\psi) : gl(\mathcal{H}_2) \rightarrow gl(\mathcal{H}_1) \) is a linear operator of rank \( \leq k^2 \). A mixed state \( \rho \) on \( \mathcal{H}_1 \otimes \mathcal{H}_2 \) is called \( k \)-entangled if it belongs to the convex hull \( \text{Conv}(\mathcal{E}_k(\mathcal{H}_1 \otimes \mathcal{H}_2)) \). Note that 1-entangled states are exactly separable states.

### 3.3. Maximally entangled states

If we assume that \( \dim(\mathcal{H}_1) \geq \dim(\mathcal{H}_2) = m \), then the Schmidt rank of any \( |\psi\rangle \in \mathcal{H}_1 \otimes \mathcal{H}_2 \) is not bigger than \( m \). Moreover, \( \text{sr}(\psi) = m \) if and only if

\[ |\psi\rangle = \sum_{j=1}^m \lambda_j |\phi_j^1\rangle \otimes |\phi_j^2\rangle, \]

where \( \{ |\phi_j^1\rangle \} \) is an orthonormal basis in \( \mathcal{H}_2 \), \( \{ |\phi_j^2\rangle \} \) is an orthonormal system in \( \mathcal{H}_1 \), and \( \lambda_j > 0, j = 1, \ldots, m \). The corresponding pure state \( P_\psi \) is called maximally entangled if all \( \lambda_j \) are equal, i.e. for normalized \( |\psi\rangle \), \( \lambda_j = 1/\sqrt{m}, j = 1, \ldots, m \). Since for normalized \( |\psi\rangle \), \( 1 = \langle \psi|\psi \rangle = \sum_j \lambda_j^2 \),

\[ P_\psi = |\psi\rangle \langle \psi | = \sum_{i,j=1}^m \lambda_{ij} |\phi_i^1\rangle |\phi_j^2\rangle \langle \phi_i^1| \langle \phi_j^2|, \]

the (obviously defined) partial traces are

\[ \text{tr}_1 P_\psi = \sum_{j=1}^m \lambda_j^2 |\phi_j^1\rangle |\phi_j^2\rangle, \quad \text{tr}_2 P_\psi = \sum_{j=1}^m \lambda_j^2 |\phi_j^1\rangle |\phi_j^1\rangle. \tag{11} \]

It follows that \( P_\psi \) is maximally entangled if and only if \( \text{tr}_1 P_\psi \) is proportional to the identity operator \( I_{\mathcal{H}_2} \) on \( \mathcal{H}_2 \),

\[ \text{tr}_1 P_\psi = I_{\mathcal{H}_2}. \tag{12} \]
where \( \|\mathcal{H}\| = \frac{1}{\dim(\mathcal{H})} \). Moreover, \( \text{tr}_1 P_\varphi = \|\mathcal{H}\| \) if and only if \( \dim(\mathcal{H}_1) = \dim(\mathcal{H}_2) \) and \( \text{tr}_1 P_\psi = \|\mathcal{H}\| \). Conversely, if (12) is satisfied, then in view of (11),

\[
\sum_{j=1}^{m} \lambda_j^2 |\phi_j^2\rangle = \frac{1}{m} \mathcal{H},
\]

so \( \lambda_1 = \cdots = \lambda_m = \frac{1}{\sqrt{m}} \) and we get the following.

**Proposition 3.1.** A pure state \( P_\psi \) on \( \mathcal{H}_1 \otimes \mathcal{H}_2 \), \( \dim(\mathcal{H}_1) \geq \dim(\mathcal{H}_2) > 1 \) is maximally entangled if and only if \( \text{tr}_1 P_\psi = \|\mathcal{H}\| \). Moreover, \( \text{tr}_2 P_\psi = \|\mathcal{H}\| \) if and only if \( \dim(\mathcal{H}_1) = \dim(\mathcal{H}_2) \) and \( P_\psi \) is maximally entangled.

From the above it is clear that the \( K = U(\mathcal{H}_1) \times U(\mathcal{H}_2) \)-orbit \( O_{\rho_{\text{max}}} \) through a maximally entangled state \( \rho_{\text{max}} \),

\[
O_{\rho_{\text{max}}} = \{ (U_1 \otimes U_2) \circ \rho_{\text{max}} \circ (U_1^\dagger \otimes U_2^\dagger) : U_i \in U(\mathcal{H}_i), \ i = 1, 2 \},
\]

consists of all maximally entangled pure states. We can ask whether the convex hull of this orbit is a convex body in the affine space \( A = u_1(\mathcal{H}_1 \otimes \mathcal{H}_2) \). Although the problem might not be of particular interest per se, it is closely related (by the Jamiołkowski isomorphism) to that of example 3.4 below, which draws considerable attention.

### 3.4. Mixed-unitary channels

As before, let \( \mathcal{H} \) be a finite-dimensional Hilbert space. In the simplest setting, a *quantum channel* or a *stochastic map* is a completely positive, trace preserving map \( A : gl(\mathcal{H}) \rightarrow gl(\mathcal{H}) \).

According to Choi’s theorem, any completely positive map can be written in the form of a *Kraus map*

\[
A(\rho) = \sum_k X_k \rho X_k^\dagger,
\]

for some \( X_k \in gl(\mathcal{H}) \). To ensure trace preserving, they have to fulfil

\[
\sum_k X_k^\dagger X_k = I_{\mathcal{H}}.
\]

One considers also *doubly stochastic channels* for which not only the trace but also the identity is preserved, \( A(I_{\mathcal{H}}) = I_{\mathcal{H}} \), i.e.,

\[
\sum_k X_k^\dagger X_k = I_{\mathcal{H}}.
\]

Let us point out that the \( \mathbb{R} \)-linear span of Kraus maps is the space \( HP(gl(\mathcal{H})) \) of *Hermiticity preserving* operators \( A : gl(\mathcal{H}) \rightarrow gl(\mathcal{H}) \). On this space there are two natural maps \( T_1, T_2 : HP(gl(\mathcal{H})) \rightarrow u^*(\mathcal{H}) \) defined on Kraus maps (14) by

\[
T_1 (A) = \sum_k X_k^\dagger X_k, \quad T_2 (A) = \sum_k X_k X_k^\dagger,
\]

so that a doubly stochastic channel is a completely positive map \( A \) satisfying \( T_1 (A) = T_2 (A) = I_{\mathcal{H}} \). The set we want to investigate from our general point of view is the set \( C_{\text{MUC}} \) of *mixed-unitary channels* [13], consisting of doubly stochastic channels of the form

\[
A(\rho) = \sum_k p_k U_k \rho U_k^\dagger, \quad U_k U_k^\dagger = I_{\mathcal{H}}, \quad p_k > 0, \quad \sum_k p_k = 1.
\]
This is clearly the convex hull of the set of doubly stochastic channels \( \{ \rho \to U\rho U^\dagger : U \in U(\mathcal{H}) \} \) which can be interpreted also as the orbit \( O_{\text{MUC}} \) of the identity channel \( I_{\mathcal{H}}(\rho) = \rho \) under the group \( K = U(\mathcal{H}) \times U(\mathcal{H}) \) acting on the Hilbert space \( g\ell(\mathcal{H}) \) by
\[
((U_1, U_2) A)(\rho) = U_1 A (U_2 \rho U_2^\dagger) U_1^\dagger.
\] (19)

Under the identification via the Jamiołkowski isomorphism \([10, 14, 15]\)
\[ \tilde{J} : g\ell(\mathcal{H}) \to g\ell(\mathcal{H} \otimes \mathcal{H}), \]
(20)
the real vector space \( HP(g\ell(\mathcal{H})) \) of Hermiticity preserving maps corresponds to the Euclidean space \( \mathcal{V} = v^\ast(\mathcal{H} \otimes \mathcal{H}) = v^\ast(\mathcal{H}) \otimes u^\ast(\mathcal{H}) \) of Hermitian operators on \( \mathcal{H} \otimes \mathcal{H} \), and completely positive maps to non-negatively defined operators. With this identification, the \( K = U(\mathcal{H}) \times U(\mathcal{H}) \)-action (19) goes to the obvious tensor product \( K \)-action,
\[
(U_1, U_2)(X_1 \otimes X_2) = (U_1 X_1 U_1^\dagger) \otimes (U_2 X_2 U_2^\dagger).
\] (21)
The question of how big is \( C_{\text{MUC}} \) is therefore equivalent to the question of how big is \( \text{Conv}(O) \) for the orbit \( O = K, \tilde{J}(I_{\mathcal{H}}) \). We will come back to this problem in section 5.4.

4. Characterizing convex bodies

4.1. A solution

An answer to problem 1.1 is given by the following.

**Theorem 4.1.** Under the assumptions of problem 1.1, the convex hull \( \text{Conv}(K \cdot x_0) \) has an empty interior in \( \mathcal{A} \) if and only if there exists a proper invariant subspace \( W \) of the linear part \( \mathcal{X} = \mathcal{V}(\mathcal{A}) \) such that \( x_0 \in \mathcal{V}_K + W \), where \( \mathcal{V}_K = \{ x \in \mathcal{V} : K x = x \} \) is the subspace of \( K \)-stationary points.

**Proof.** Let us assume that \( \text{Conv}(K \cdot x_0) \) has an empty interior in \( \mathcal{A} \). It means that it is contained in a proper affine subspace \( \mathcal{A}_0 \) of \( \mathcal{A} \); \( \mathcal{A}_0 \) is the affine span of \( \text{Conv}(K \cdot x_0) \), \( \mathcal{A}_0 = \text{Aff}(K \cdot x_0) \). The affine subspace \( \mathcal{A}_0 \) is invariant with respect to the action of \( K, K \cdot \mathcal{A}_0 \subset \mathcal{A}_0 \), and the same is true for its linear part \( V(\mathcal{A}_0) = \mathcal{X} \). Since the action of \( K \) is orthogonal, the orthogonal complement \( \mathcal{X}^\perp \) in \( \mathcal{A} \) is invariant as well. As a consequence of its definition the subspace \( \mathcal{X}^\perp \) intersects \( \mathcal{X} \) at a single point, hence the same is true for the intersection of \( \mathcal{X}^\perp \) and \( \mathcal{A}_0 \), i.e. they meet at a single point \( v \in \mathcal{X}^\perp \cap \mathcal{A}_0 \). Since both \( \mathcal{X}^\perp \) and \( \mathcal{A}_0 \) are \( K \)-invariant, it follows that \( K v = v \), i.e., \( v \in \mathcal{V}_K \). But then \( x_0 = v + x' \) for some \( x' \in \mathcal{X} \), so we can take \( \mathcal{V} = \mathcal{X} \). Let us now assume that \( x_0 = v + w \) for some \( v \in \mathcal{V}_K \) and \( w \in W \), where \( \mathcal{V} \) is a proper invariant subspace of \( \mathcal{V} \). Then, the orbit \( K x_0 = v + K w \) is contained in the proper affine subspace \( v + \mathcal{W} \) of \( \mathcal{A} \), hence it has an empty interior in \( \mathcal{A} \) (see figure 1 for an illustration). \( \Box \)

**Corollary 4.1.** The convex hull \( \text{Conv}(K \cdot x_0) \) is a convex body in \( \mathcal{A} \) if and only if the image of \( x_0 \) under the orthogonal projection \( \pi : \mathcal{V} \to V(\mathcal{A}) \) does not belong to a proper \( K \)-invariant subspace of \( V(\mathcal{A}) \). In particular, if \( V(\mathcal{A}) \) is irreducible and \( \pi(x_0) \neq 0 \), then \( \text{Conv}(K \cdot x_0) \) is a convex body in \( \mathcal{A} \).

**Proof.** There is a unique vector \( v \in V(\mathcal{A})^\perp \) such that \( \mathcal{A} = v + V(\mathcal{A}) \), thus \( v \in \mathcal{V}_K \) due to the invariance of \( \mathcal{A} \) and \( V(\mathcal{A}) \). Hence, \( \pi(x_0) \) belongs to a proper invariant \( \mathcal{V} \subset V(\mathcal{A}) \) if and only if \( x_0 \in v + \mathcal{W} \), so \( \text{Conv}(K \cdot x_0) \) has an empty interior in \( \mathcal{A} \) due to the above theorem. \( \Box \)
4.2. Convexity and coadjoint orbits

A particular instance, that permeates most of the results exhibited in the rest of the paper, happens when the linear space \( V \) is the dual of the Lie algebra \( \mathfrak{h}^* \) of a Lie group \( H \) and the orthogonal action of \( H \) in \( \mathfrak{h}^* \) (with respect to a given invariant metric) is the coadjoint action of \( H \) on \( \mathfrak{h}^* \). We will denote by \( \mathcal{O}_{x_0} = H \cdot x_0 \) the coadjoint orbit of \( H \) passing through \( x_0 \in \mathfrak{h}^* \). Let us recall that \( \mathcal{O}_{x_0} \) carries a canonical symplectic structure. Suppose now that \( K \subset H \) is a compact subgroup of \( H \), then the restriction to the coadjoint orbit \( \mathcal{O}_{x_0} = H \cdot x_0 \) of the canonical projection \( \pi : \mathfrak{h}^* \to \mathfrak{k}^* \) is the momentum map of the action of \( K \) in the symplectic manifold \( \mathcal{O}_{x_0} \).

First, we can make a few simple remarks concerning the convex hull of the coadjoint orbit \( H \cdot x_0 = \mathcal{O}_{x_0} \) and the range \( \pi(\mathcal{O}_{x_0}) \) of the momentum map.

**Lemma 4.1.** Under the assumptions above:

(i) \( \pi(\text{Conv}(\mathcal{O}_{x_0})) = \text{Conv}(\pi(\mathcal{O}_{x_0})) \).

(ii) If \( \text{Conv}(H \cdot x_0) \) is a convex body, so is \( \pi(\text{Conv}(\mathcal{O}_{x_0})) \).

**Proof.**

(i) Because \( \pi \) is a linear map, then \( \pi(x) \in \text{Conv}(\pi(\mathcal{O}_{x_0})) \) for all \( x \in \text{Conv}(\mathcal{O}_{x_0}) \), hence \( \pi(\text{Conv}(\mathcal{O}_{x_0})) \subseteq \text{Conv}(\pi(\mathcal{O}_{x_0})) \) and \( \pi(\text{Conv}(\mathcal{O}_{x_0})) \) is a convex set. Hence \( \pi(\text{Conv}(\mathcal{O}_{x_0})) = \text{Conv}(\pi(\mathcal{O}_{x_0})) \).

(ii) Suppose that \( \pi(\text{Conv}(\mathcal{O}_{x_0})) \) is not a convex body, hence because of theorem 4.1 there exist a fixed point \( x \) and a proper subspace \( W \) of \( \mathfrak{k}^* \) such that \( \pi(\text{Conv}(\mathcal{O}_{x_0})) \subset x + W \). Hence if we consider \( y \in \text{Conv}(\mathcal{O}_{x_0}) \) such that \( \pi(y) = x \) and \( \tilde{W} = \pi^{-1}(W^\perp) \) we have that...
Conv(O_{x_0}) \subset y + \tilde{W}, and \tilde{W} is a proper subspace of \mathfrak{h}^*, hence because of theorem 4.1, Conv(\mathcal{H} \cdot x_0) cannot be a convex body.

In particular we may choose \( K \subset H \) to be a maximal torus \( T \), then the fixed points of the action of \( T \) are linearly independent.

\[ \text{Corollary 4.2.} \quad \text{If the convex hull Conv}(\mathcal{O}_{x_0}) \text{ is a convex polytope whose vertices are the fixed points of the action of } T \text{ in } \mathcal{O}_{x_0}. \]

\[ \text{Proof.} \quad \text{As was previously indicated, because of Kostant and Atiyah’s convexity theorem [17], the image of the momentum map } \pi : O_{x_0} \to R^n \text{ is a convex polytope whose vertices are the projections of the fixed points } x_i \text{ of the action of } T \text{ on } O_{x_0}. \text{ Then, because of lemma 4.1 we have that the convex hull Conv}(K \cdot x_0) \text{ of the coadjoint orbit } O_{x_0} \text{ is just } \pi(O_{x_0}) \text{ and it is a convex body if Conv}(K \cdot x_0) \text{ is.} \]

By using Atiyah’s convexity theorem [17] as indicated in the proof of the previous theorem, or rather the extension of the theorem as proved by Guillemin–Sternberg [18] and Kirwan [19] we can extend the result in theorem 4.2 as follows. Let \( M \) be a compact symplectic manifold and \( H \) a compact Lie group acting on it. Let \( J : M \to \mathfrak{h}^* \) be the corresponding momentum map and \( J(M)\mathfrak{h}^* \) its range. Clearly \( J(M) \) is a collection of coadjoint orbits of \( H \). Consider the convex hull Conv(J(M)) of the range of the momentum map. We can characterize if it will be a convex body by again using a maximal Abelian subgroup \( T \subset H \). Consider now \( t^* \) embedded in \( \mathfrak{h}^* \) by using an invariant metric, then consider the intersection of \( J(M) \) with the positive Weyl chamber \( t^*_+ \). According to Guillemin–Sternberg–Kirwan’s theorem, \( J(M) \cap t^*_+ \) is a convex polytope whose vertices are the fixed points of the action of \( T \) [18]. Hence we get the following.

\[ \text{Corollary 4.2.} \quad \text{If the convex hull Conv}(J(M)) \text{ of the family of coadjoint orbits } J(M) \text{ is a convex body then the fixed points of the action of } T \text{ are linearly independent.} \]

5. Applications to examples

5.1. Convex body of density operators

To show how corollary 4.1 can be applied in order to see that mixed states form a convex body in \( \mathcal{A} = u_*^* (\mathcal{H}) \), consider first the orthogonal action of the unitary group \( K = U(\mathcal{H}) \) on the Euclidean space \( \mathcal{Y} = u^* (\mathcal{H}) \) of Hermitian operators on a \( d \)-dimensional Hilbert space \( \mathcal{H}, \) \( d > 1 \), by

\[ U.A = UAU^+ . \]  

Proposition 5.1. The representation (22) of \( U(\mathcal{H}) \) in \( u^* (\mathcal{H}) \) has two irreducible components: the space \( \mathcal{L}_H \), spanned by the trace-normalized identity map

\[ \mathcal{L}_H = \frac{1}{d} I_{\mathcal{H}}, \]

and the subspace \( su^* (\mathcal{H}) \) consisting of all Hermitian operators with trace 0,

\[ u^* (\mathcal{H}) = \mathcal{L}_H \oplus su^* (\mathcal{H}). \]
Hermitian operators with the spectrum (diagonal form) corresponds to a pure separable state $P$. The corresponding representation of the Lie algebra $\mathfrak{su}(\mathcal{H})$ is irreducible, as every invariant subspace corresponds, via the multiplication by $i$, to a Lie ideal in the Lie algebra $\mathfrak{su}(\mathcal{H})$ which is known to be simple.

If now $|\psi\rangle \in \mathcal{H}$ is a nonzero vector, then the 1-dimensional projector $P_\psi$ splits, according to (23), as

$$P_\psi = I_\mathcal{H} + (P_\psi - I_\mathcal{H}).$$

Since the projection $\pi(P_\psi)$ onto $V(A) = \mathfrak{su}^*(\mathcal{H})$ is $P_\psi - I_\mathcal{H} \neq 0$ and $\mathfrak{su}^*(\mathcal{H})$ is irreducible, the set $D(\mathcal{H}) = \text{Conv}(U(\mathcal{H}), P_\psi)$ is a convex body in $\mathfrak{u}_1^*(\mathcal{H})$. Of course, the above constatation is well known and it is taken here to show how corollary 4.1 works. Actually, more geometrical information is known in this case. For instance, the radius of the largest ball $B$ contained in $D(\mathcal{H})$ and centered at $I_\mathcal{H}$ is known (see [20] or [9, corollary 3]) to be

$$r = \frac{1}{\sqrt{d(d-1)}}. \quad (24)$$

This ball touches the boundary of $D(\mathcal{H})$ at points of the $U(\mathcal{H})$-orbit consisting of Hermitian operators with the spectrum (diagonal form)

$$\left(0, \frac{1}{d-1}, \ldots, \frac{1}{d-1}\right).$$

5.2. Convex body of separable states

Let Hilbert spaces $\mathcal{H}_1, \mathcal{H}_2$ have dimensions $d_1, d_2 > 1$. A simple tensor

$$|\psi\rangle = |\phi^1\rangle \otimes |\phi^2\rangle \in \mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$$

corresponds to a pure separable state $P_\psi = P_{\phi^1} \otimes P_{\phi^2}$: whose K-orbit under the obvious action of $K = U(\mathcal{H}_1) \times U(\mathcal{H}_2) \subset U(\mathcal{H})$ consists of all pure separable states, $K P_\psi = S^1(\mathcal{H})$. Its convex hull is, by definition, the set $S(\mathcal{H})$ of all (mixed) separable states, contained in the affine subspace $A = \mathfrak{u}_1^*(\mathcal{H}_1 \otimes \mathcal{H}_2)$ of $\mathcal{V} = \mathfrak{u}^*(\mathcal{H}_1 \otimes \mathcal{H}_2) = \mathfrak{u}^*(\mathcal{H}_1) \otimes \mathfrak{u}^*(\mathcal{H}_2)$. According to proposition 5.1, the decomposition of $\mathcal{V}$ into irreducible parts is

$$\mathcal{V} = ([I_{\mathcal{H}_1} \otimes I_{\mathcal{H}_2}] \oplus ([I_{\mathcal{H}_1} \otimes \mathfrak{u}^*(\mathcal{H}_2)]) \oplus ([\mathfrak{u}^*(\mathcal{H}_1) \otimes I_{\mathcal{H}_2}]) \oplus ([\mathfrak{u}^*(\mathcal{H}_1) \otimes \mathfrak{u}^*(\mathcal{H}_2)]), \quad (25)$$

where $I_j$ denotes $I_{\mathcal{H}_j}, j = 1, 2$. Here,

$$[I_{\mathcal{H}_1} \otimes \mathfrak{u}^*(\mathcal{H}_2)] \oplus [\mathfrak{u}^*(\mathcal{H}_1) \otimes I_{\mathcal{H}_2}] = \mathfrak{u}^*(\mathcal{H}_1 \otimes \mathcal{H}_2). \quad (26)$$

The projection $\pi(P_\psi) = P_\psi - I_1 \otimes I_2$ of $P_\psi$ on $\mathfrak{u}^*(\mathcal{H}_1 \otimes \mathcal{H}_2)$ decomposes as

$$I_1 \otimes (P_\psi - I_2) + (P_\psi - I_1) \otimes I_2 + (P_\psi - I_1) \otimes (P_\psi - I_2), \quad (27)$$

so all components in irreducible parts are non-trivial if $d_1, d_2 > 1$. Hence, according to corollary 4.1, $\text{Conv}(K P_\psi) = S(\mathcal{H}_1 \otimes \mathcal{H}_2)$ is a convex body in $\mathfrak{u}_1^*(\mathcal{H})$. Here also more is known about the radius of the largest inscribed ball [21].

5.3. Orbits of maximally entangled pure states

For the composite system as above, assume that $d_1 \geq d_2 > 1$ and take a unit vector $|\psi\rangle \in \mathcal{H}$. Decompose the projection

$$\pi(P_\psi) = P_\psi - I_1 \otimes I_2 \in \mathfrak{u}^*(\mathcal{H}_1 \otimes \mathcal{H}_2)$$

into

$$\pi(P_\psi) = I_1 \otimes P_{\psi}^{10} + P_{\psi}^{01} \otimes I_2 + P_{\psi}^{00}, \quad (28)$$
according to the decomposition (26) into irreducible parts. Then,
\[ \text{tr}_1 P_\psi - \mathbb{1}_2 = P_\psi^0, \]
as \( \text{tr}_1 (P_\psi^0 \otimes \mathbb{1}_2 + P_\psi^0) \) is clearly 0. If \( P_\psi \) is maximally entangled, then \( \text{tr}_1 P_\psi - \mathbb{1}_2 = 0 \), thus \( P_\psi \) is a proper \( K \)-invariant subspace and the convexed orbit \( \text{Conv}(K P_\psi) \) of the \( K = U(\mathcal{H}_1) \otimes U(\mathcal{H}_2) \)-action has an empty interior in \( \mathcal{A} = \text{su}_2^*(\mathcal{H}_1 \otimes \mathcal{H}_2) \). Conversely, if the convexed orbit \( \text{Conv}(K P_\psi) \) of the \( K = U(\mathcal{H}_1) \otimes U(\mathcal{H}_2) \)-action has an empty interior in \( \mathcal{A} = \text{su}_2^*(\mathcal{H}_1 \otimes \mathcal{H}_2) \), then \( \pi(P_\psi) \) belongs to a proper \( K \)-invariant subspace, so at least one of \( P_\psi^0, P_\psi^1, P_\psi^2 \) is 0. Observe first that \( P_\psi^0 \neq 0 \) only if \( d_2 > 1 \). Indeed, in this case we can find orthogonal \( e_1, e_2 \in \mathcal{H}_1 \) and \( f_1, f_2 \in \mathcal{H}_2 \) such that \( \langle e_1 \otimes f_1 | \psi \rangle_{\mathcal{H}} \neq 0, \langle e_2 \otimes f_2 | \psi \rangle_{\mathcal{H}} \neq 0 \). But then
\[ A = |e_1 \otimes f_1| e_2 \otimes f_2 | = |e_1| \otimes |f_1| \]
begins to \( \text{su}_2^*(\mathcal{H}_1) \otimes \text{su}_2^*(\mathcal{H}_2) \), so
\[ \langle P_\psi^0 | A \rangle = \langle P_\psi | A \rangle = \langle \psi | e_1 \otimes f_1 \rangle_{\mathcal{H}} \langle e_1 \otimes f_1 | \psi \rangle_{\mathcal{H}} \neq 0. \]
If \( P_\psi^0 = 0 \), then \( \text{tr}_1 P_\psi = \mathbb{1}_2 \) and, according to proposition 3.1, \( P_\psi \) is maximally entangled. Finally, \( P_\psi^0 = 0 \) gives \( \text{tr}_1 P_\psi = \mathbb{1}_2 \) and, again, \( P_\psi \) is maximally entangled. On the other hand, as \( K \)-action on \( \mathcal{V}_0 = \text{su}_2^*(\mathcal{H}_1) \otimes \text{su}_2^*(\mathcal{H}_2) \) is irreducible, the orbit of a maximally entangled state is a convex body in \( \mathbb{1}_{\mathcal{H}} + \mathcal{V}_0 \). This proves the following characterization of maximally entangled states.

**Theorem 5.1.** A pure state \( P_\psi \) on \( \mathcal{H}_1 \otimes \mathcal{H}_2 \) is maximally entangled if and only if the convexed orbit \( \text{Conv}(K P_\psi) \) of the canonical action of the group \( K = U(\mathcal{H}_1) \times U(\mathcal{H}_2) \) in the space of Hermitian operators on \( \mathcal{H}_1 \otimes \mathcal{H}_2 \) has an empty interior in \( \text{su}_2^*(\mathcal{H}_1 \otimes \mathcal{H}_2) \) (so its volume in the convex body of density operators on \( \mathcal{H}_1 \otimes \mathcal{H}_2 \) is zero). In the latter case, however, \( \text{Conv}(K P_\psi) \) is a convex body in the affine space \( \mathbb{1}_{\mathcal{H}} + \text{su}_2^*(\mathcal{H}_1) \otimes \text{su}_2^*(\mathcal{H}_2) \).

### 5.4. The convex body of mixed-unitary channels

As already mentioned in section 3.4, the set \( C_{\text{MUC}} \) of mixed-unitary channels is the convex hull of the orbit \( O_{\text{MUC}} \) of the channel \( I_\mathcal{H} \) under the \( K = U(\mathcal{H}) \times U(\mathcal{H}) \)-action (19). We will show that this picture is related, via the Jamiołkowski isomorphism (20), to that in the previous section. It is well known that Hermiticity preserving operators correspond, via the Jamiołkowski isomorphism, to Hermitian operators on \( \mathcal{H} \otimes \mathcal{H} \). A convenient definition of \( \tilde{J} \) is given, in the tensorial notation [10], by
\[ \langle x_i \otimes \tilde{x}_j | A(x_k \otimes \tilde{x}_k) \rangle = \langle x_i \otimes x_k | \tilde{J}(A)(x_k \otimes x_k) \rangle. \]

(29)

Here, \( x_i, x_j, x_k, x_l \) are arbitrary vectors in \( \mathcal{H} \) and \( x_i \otimes \tilde{x}_j \) is the tensorial notation for Dirac’s \( |x_i| \otimes |x_j| \). A direct description in terms of a mixed tensorial-Dirac notation is as follows:
\[ \tilde{J}(x_i \otimes \tilde{x}_j)(x_k \otimes \tilde{x}_k) = |x_i \otimes x_k |(x_k \otimes x_k). \]

(30)

Here, \( A = |x_i \otimes \tilde{x}_j | (x_k \otimes \tilde{x}_k) \) represents
\[ A(\rho) = (x_i \otimes \tilde{x}_j) \rho (x_k \otimes \tilde{x}_k)^\dagger = (x_i \otimes \tilde{x}_j) \rho (x_k \otimes \tilde{x}_k). \]

(31)

From (29) one sees immediately that \( A \) preserves the positivity if and only if \( \tilde{J}(A) \) is positively defined:
\[ \langle x_i \otimes \tilde{x}_j | A(x_k \otimes \tilde{x}_k) \rangle \geq 0 \iff \langle x_i \otimes x_k | \tilde{J}(A)(x_k \otimes x_k) \rangle \geq 0. \]

The additional doubly stochastic conditions (15) and (16) for (14) correspond to the following conditions for partial traces:
\[ \text{tr}_1 \tilde{J}(A) = I_{\mathcal{H}}, \quad \text{tr}_2 \tilde{J}(A) = I_{\mathcal{H}}. \]

(32)
Indeed, if \((e_i)\) is an orthonormal basis in \(\mathcal{H}\), then
\[
T_1((e_i \otimes \tilde{e}_j)(e_k \otimes \tilde{e}_l)) = (e_k \otimes \tilde{e}_l)^\dagger \circ (e_i \otimes \tilde{e}_j) = (e_i \otimes \tilde{e}_j) \circ (e_j \otimes \tilde{e}_k) = \delta^j_k \cdot (e_l \otimes \tilde{e}_j),
\]
which coincides with
\[
\text{tr}_1(\tilde{\mathcal{J}}((e_i \otimes \tilde{e}_j)(e_k \otimes \tilde{e}_l))) = \text{tr}_1((e_i \otimes e_j)(e_k \otimes e_l)) = \delta^j_k \cdot (|e_i\rangle\langle e_j|).
\]
Similarly,
\[
T_2(A) = \text{tr}_2(\tilde{\mathcal{J}}(A)). \tag{33}
\]
This means that \(\tilde{\mathcal{J}}\) establishes an isomorphism between the convex set of doubly stochastic operators and the convex set of those non-negatively defined operators on \(\mathcal{H} \otimes \mathcal{H}\), whose partial traces both equal \(I_{\mathcal{H}}\). Another important observation is that \(\tilde{\mathcal{J}}\) intertwines the \(\mathcal{K} = U(\mathcal{H}) \times U(\mathcal{H})\)-action (21) on \(u^*(\mathcal{H} \otimes \mathcal{H}) = u^*(\mathcal{H}) \otimes u^*(\mathcal{H})\). Indeed, for \(A\) as in (31), it is easy to see that
\[
((U_1, U_2).A)(\rho) = U_1A(U_2\rho U_2^\dagger)U_1^\dagger = U_1 \circ (x_i \otimes \tilde{x}_j) \circ U_2 \circ \rho \circ U_2^\dagger \circ (x_i \otimes \tilde{x}_j) \circ U_1^\dagger
\]
so that
\[
\tilde{\mathcal{J}}((U_1, U_2).(|x_i \otimes \tilde{x}_j)|x_k \otimes \tilde{x}_l)\rangle = \tilde{\mathcal{J}}((|x_i \otimes \tilde{x}_j)|x_k \otimes \tilde{x}_l\rangle)\rangle
\]
\[
= |U_1x_i \otimes U_2x_j|U_1x_k \otimes U_2x_l\rangle = (U_1 \circ |x_i\rangle|x_k\rangle \circ U_1^\dagger) \otimes (U_2 \circ |x_j\rangle|x_l\rangle \circ U_2^\dagger). \tag{34}
\]
All this implies that our convex set \(\mathcal{C}_{\text{MUC}}\) is Jamiołkowski-equivalent to the convex hull \(\text{Conv}(\mathcal{O})\) of the orbit \(\mathcal{O} = \mathcal{K} \tilde{\mathcal{J}}(I_\mathcal{H})\). But,
\[
\tilde{\mathcal{J}}(I_\mathcal{H}) = \tilde{\mathcal{J}}(\sum_{i,j} |e_i \otimes \tilde{e}_j\rangle\langle e_j \otimes \tilde{e}_j|)
\]
where \((e_i)\) is an orthonormal basis in \(\mathcal{H}\). The latter, however, is proportional to a maximally entangled pure state \(P_\psi\) associated with the normalized vector
\[
|\psi\rangle = \frac{1}{\sqrt{\dim(\mathcal{H})}} \sum_i e_i \otimes e_i.
\]
More precisely,
\[
\tilde{\mathcal{J}}(I_\mathcal{H}) = \dim(\mathcal{H}) \cdot P_\psi. \tag{35}
\]
Now, we are in the situation of the previous section; the only difference is that all is rescaled by \(\dim(\mathcal{H})\). In view of theorem 5.1, the convex hull of the \(\mathcal{K}\)-orbit of \(\tilde{\mathcal{J}}(I_\mathcal{H})\) is then a convex body in the affine space
\[
\mathcal{A} = \dim(\mathcal{H}) \cdot \mathbb{I}_{\mathcal{H} \otimes \mathcal{H}} + su^*(\mathcal{H}) \otimes su^*(\mathcal{H}).
\]
As a consequence, \(\mathcal{C}_{\text{MUC}}\) is a convex body inside the set of doubly stochastic channels. The convex body \(\mathcal{C}_{\text{MUC}}\) is clearly centered at
\[
\Omega = \dim(\mathcal{H}) \cdot \tilde{\mathcal{J}}^{-1}(\mathbb{I}_{\mathcal{H} \otimes \mathcal{H}}) = \frac{1}{\dim(\mathcal{H})} \cdot \tilde{\mathcal{J}}^{-1}(\mathbb{I}_{\mathcal{H} \otimes \mathcal{H}}).
\]
However, according to (29),
\[
|e_i \otimes \tilde{e}_j, \Omega(e_k \otimes \tilde{e}_l)|\rangle = \frac{1}{\dim(\mathcal{H})} \cdot \delta^j_k \delta^i_l,
\]
which immediately implies that
\[
\Omega(X) = \frac{\text{tr}(X)}{\dim(\mathcal{H})} I_{\mathcal{H}} = \text{tr}(X) \mathbb{I}_{\mathcal{H}}. \tag{36}
\]
The mixed-unitary channel $\Omega$ is called the completely depolarizing channel. One can find $\Omega$ easily without the use of the Jamiołkowski isomorphism. It is clear that

$$\Omega(X) = \int_{U(H)} UXU^\dagger d\mu(U),$$

(37)

where $\mu$ is the probabilistic Haar measure on $U(H)$. Since $\Omega$ is stabilized by $U(H)$,

$$U\Omega(X)U^\dagger = \Omega(X),$$

for any Hermitian $X$ and any $U \in U(H)$. This implies that $\Omega(X)$ is proportional to $I_H$, i.e.,

$$\Omega(X) = \text{tr}(X_0 X) I_H,$$

(38)

for a certain $X_0 \in \omega^*(H)$. On the other hand, any left-invariant Haar measure on $U(H)$ is automatically right-invariant, so

$$\Omega(UXU^\dagger) = \Omega(X)$$

and thus $\text{tr}(X_0 UXU^\dagger) = \text{tr}(X_0 X)$ for all $X$ and all $U$. Hence, $X_0$ is proportional to the identity, $X_0 = c \cdot I_H$ and

$$\Omega(X) = c \cdot \text{tr}(X) I_H.$$

Finally, as $\Omega(I_H) = I_H$, we get $1 = c \cdot \dim(H)$, and thus (36). We can summarize as follows.

**Theorem 5.2.** Any doubly stochastic channel in a neighborhood of the completely depolarizing channel $\Omega$ is mixed-unitary.

This is clearly a slightly weaker version of a recent result of Watrous [13].

5.5. The largest balls of $k$-entangled states

Consider again a bipartite Hilbert space $H = H_1 \otimes H_2$ of the total dimension $d = d_1 d_2$, where $d_1 = \dim(H_1) \geq \dim(H_2) = d_2$, and consider the convex sets $\mathcal{E}_k(H_1 \otimes H_2)$ of $k$-entangled states, $k = 1, 2, \ldots, d_2$. It is known [21] that the radius of the largest ball contained in $S(H_1 \otimes H_2)$ and centered at $I_H = I_1 \otimes I_2$ is $r = \frac{1}{\sqrt{d_1 d_2 - 1}}$, with $d = d_1 d_2$. This is exactly the same ball as the largest ball $B$ (24) contained in the (bigger) convex body $\mathcal{D}(H)$ of all mixed states (see [9, 20]). In other words, $I_H + A$ is separable for all $A$ with $\|A\|_w \leq a$ if and only if $a \leq \frac{1}{\sqrt{d_1 d_2 - 1}}$. This observation, however, implies immediately that the largest ball $B_1$, centered at $I$ and contained in $\text{Conv}(\mathcal{E}_k(H_1 \otimes H_2))$, must be the same, since

$$S_1^k(H_1 \otimes H_2) = \mathcal{E}_1(H_1 \otimes H_2) \subset \mathcal{E}_k(H_1 \otimes H_2) \subset \mathcal{D}_1^k(H).$$

**Proposition 5.2.** The largest ball $B_k$, centered at $I$ and contained in $\text{Conv}(\mathcal{E}_k(H_1 \otimes H_2))$ has radius (24) and coincides with the largest ball $B$ contained in the convex body $\mathcal{D}(H)$ of all density operators, for all $k = 1, 2, \ldots, d_2$. In particular, $I_H + A$ is $k$-entangled for all $A$ with $\|A\|_w \leq \frac{1}{\sqrt{d(d - 1)}}$.  

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6. Convexed local orbits

As the $k$-entangled states are convex hulls of families of orbits, in spite of the above proposition, looking for single orbits of a particular pure bipartite state is still an interesting problem.

Let $|\psi\rangle \in \mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$ be a nonzero vector, $k = \text{Str}(\psi)$ be its Schmidt rank, and $P_\psi = \frac{|\psi\rangle\langle\psi|}{\text{tr}(\psi)}$ be the corresponding pure state. With $C_\psi$ we will denote the \textit{convexed local orbit} of $\rho = P_\psi$, i.e. the convex hull of the orbit $O_\psi = K.P_\psi$ of the pure state $P_\psi$ under the unitary action $\rho \mapsto U.\rho = U \rho U^\dagger$ of the group $K = U(\mathcal{H}_1) \times U(\mathcal{H}_2)$, where $U$ runs over all local unitary operators $U \in U(\mathcal{H}_1) \times U(\mathcal{H}_2)$ represented by the tensor products $U_1 \otimes U_2$, $U_i \in U(\mathcal{H}_i)$, $i = 1, 2$. According to the Schmidt decomposition (9) and the form of the partial trace (11), elements $\rho$ in the orbit $O_\psi$ are determined by the spectrum $(\lambda_1^2, \ldots, \lambda_k^2)$ of their partial trace $\text{tr}_1 \rho$. Indeed, the spectrum determines $\lambda_1, \ldots, \lambda_k > 0$ and thus the Schmidt decomposition (9) which identifies the pure state up to a local unitary transformation.

**Theorem 6.1.** The convexed local orbit $C_\psi$ is a $K$-invariant subset of $u_1^*(\mathcal{H})$ centered at $\mathbb{I}_\mathcal{H}$ and contained in the convex body $\mathcal{D}(\mathcal{H})$ of all density operators. Moreover, $C_\psi$ is itself a convex body unless $|\psi\rangle$ is maximally entangled.

**Proof.** In view of theorem 5.1, it is enough to show that $\mathbb{I}_\mathcal{H} \in C_\psi$. Take the probabilistic Haar measure $\mu$ on $K = U(\mathcal{H}_1) \times U(\mathcal{H}_2)$ and consider

$$\rho_0 = \int_K U P_\psi U^\dagger \, d\mu(U) \in u_1^*(\mathcal{H}).$$

By construction, $\rho_0$ is a $K$-invariant element in $C_\psi$. It is easy to see that $\rho_0 = \mathbb{I}_\mathcal{H}$. Indeed, using decomposition (28), we get

$$\rho_0 - I_1 \otimes I_2 = \int_K U. (I_1 \otimes P_\psi^{10} + P_\psi^{01} \otimes I_2 + P_\psi^{00}) \, d\mu(U) = 0,$$

since the latter integral reduces to

$$\int_{U(\mathcal{H}_1)} U_2 P_\psi^{10} U_1^\dagger \, d\mu_2(U_2) + \int_{U(\mathcal{H}_1)} U_1 P_\psi^{01} U_1^\dagger \, d\mu_1(U_1) \otimes I_2$$

$$+ \sum_j \left( \int_{U(\mathcal{H}_1)} U_1 P_j^{10} U_1^\dagger \, d\mu_1(U_1) \otimes \int_{U(\mathcal{H}_2)} U_2 P_j^{00} U_2^\dagger \, d\mu_2(U_2) \right)$$

and the only $U(\mathcal{H}_i)$-invariant element in $u_1^*(\mathcal{H}_i)$ is 0. Here, $\mu_i$ is the probabilistic Haar measure on $U(\mathcal{H}_i)$, $i = 1, 2$, and

$$P_\psi^{00} = \sum_j (P_j^1 \otimes P_j^2).$$



7. Maximum volume ellipsoids

Let us recall that among all ellipsoids contained in a convex body $C$ there is a unique ellipsoid $E_{\text{max}}(C)$ of the maximum volume, which we call the \textit{maximum volume ellipsoid of $C$} and which is also called the \textit{John ellipsoid} of $C$ [22]. Actually, $E_{\text{max}}(C)$ does not depend on the choice of a Euclidean metric in $C$, so it is determined completely by the affine (and convex) structure. On the other hand, it is clear that $E_{\text{max}}(C)$ may be larger than the largest ball $B(C)$ contained in $C$, since the latter clearly depends strongly on the metric. However, in many important cases of convex bodies in Euclidean spaces the maximal ellipsoids are the largest balls. This is the case, for instance, of the convex body $\mathcal{D}(\mathcal{H})$ of all density operators that easily follows from the following observation.
Proposition 7.1. If a compact group $K$ acts irreducibly on a Euclidean space $V$ by orthogonal transformations, then the maximum volume ellipsoid contained in the convex hull $C = \text{Conv}(K \cdot x_0)$ of any $K$-orbit is a ball, $E_{\text{max}}(C) = B(C)$.

Proof. We may assume that $x_0 \neq 0$, so that $C$ is a convex body in $V$ centered at 0. We will show that the largest ball $B$ centered at 0 and contained in $C$ coincides with $E_{\text{max}}$. Indeed, $B \subset E_{\text{max}}$ and it suffices to show that all principal axes of $E_{\text{max}}$ are equal. Suppose the contrary and let $v \in V$ be the direction of the largest axis. Let $V_0$ be the orthogonal completion of $v$. As the boundary of $B$ intersects the boundary of $E_{\text{max}}$ in $V_0$, the only points at which $B$ touches the boundary of $C$ must lie in $V_0$. But these points form a $K$-invariant subspace in $V$, in contradiction with the irreducibility. \hfill \square

As $C = D(H) - 1_H$ is the convex hull of an orbit of $U(H)$-action on $su^*(H)$, $E_{\text{max}}(D(H)) = B(D(H))$. This is, however, no longer true for convexed local orbits $C_{\psi}$ of pure states in $H = H_1 \otimes H_2$. Let us consider the simple case of a two-qubit system: $\dim(H_1) = \dim(H_2) = 2$. Suppose that a normalized vector $|\phi\rangle \in H$ has a Schur-like decomposition

$$|\phi(\lambda)\rangle = \lambda \cdot e_1 \otimes f_1 + \sqrt{1 - \lambda^2} \cdot e_2 \otimes f_2,$$

with $0 \leq \lambda^2 \leq 1$. Here, $(e_1, e_2)$ and $(f_1, f_2)$ are orthonormal bases in $H_1$ and $H_2$, respectively. If $\lambda^2$ varies from 1 to $1/2$ (or from 0 to 1/2), then $P_{\phi(\lambda)}$ varies from a separable state to the maximally entangled pure state $P_{\psi} = P_{\phi(\pm 1/\sqrt{2})}$ associated with

$$|\psi\rangle = \frac{1}{\sqrt{2}}(\pm e_1 \otimes f_1 + e_2 \otimes f_2).$$

Let $R(\lambda)$ be the radius of the largest ball $B(\lambda)$ centered at $1_H$ and contained in $C(\lambda) = \text{Conv}(K P_{\phi(\lambda)}), \ K = U(H_1) \times U(H_2)$.

According to theorem 6.1, $C(\lambda)$ is a convex body in $u_q^*(H)$ if and only if $\lambda \neq \frac{1}{\sqrt{2}}$. If $\lambda = \frac{1}{\sqrt{2}}$, then $C(\lambda) - 1_H$ flattens to a convex body in the irreducible subspace $su^*(H_1) \otimes su^*(H_2)$.

In view of proposition 7.1, the largest ball $B(1/\sqrt{2})$ is the maximal volume ellipsoid. We will show that this is not true in general, i.e., $E_{\text{max}}(C(\lambda))$ differs from $B(\lambda)$ for $\lambda^2$ close to $1/2, \lambda^2 \neq 1/2$. The partial traces of $P_{\phi(\lambda)}$ are:

$$\text{tr}_1 P_{\phi(\lambda)} = \lambda^2 P_{f_1} + (1 - \lambda^2) P_{f_2}, \quad \text{tr}_2 P_{\phi(\lambda)} = \lambda^2 P_{e_1} + (1 - \lambda^2) P_{e_2},$$

so that, in the decomposition (28),

$$P_{\phi(\lambda)}^{10} = \left(\lambda^2 - \frac{1}{2}\right) P_{f_1} - \left(\lambda^2 - \frac{1}{2}\right) P_{f_2}, \quad P_{\phi(\lambda)}^{01} = \left(\lambda^2 - \frac{1}{2}\right) P_{e_1} - \left(\lambda^2 - \frac{1}{2}\right) P_{e_2}.$$

This implies that the orthogonal projection of $C(\lambda) - I$, and thus of $B(\lambda) - I$, onto the subspace $(I_1) \otimes su^*(H_2)$ lies in the ball of the radius

$$r(\lambda) \leq \|\text{tr}_1 P_{\phi(\lambda)} - 1_2\| = \lambda^2 - \frac{1}{2}.$$

Hence, $R(\lambda) \leq r(\lambda)$. Since $B(\lambda) \subset B(D(H))$ and the latter have the radius $\frac{1}{\sqrt{2}}$ (see (24)), we get the following.

Proposition 7.2. The radius of $B(\lambda)$ can be estimated by

$$R(\lambda) \leq \min \left\{ \lambda^2 - \frac{1}{2}, \frac{1}{\sqrt{12}} \right\}.$$

In particular, $R(\lambda) \rightarrow 0$ as $\lambda^2 \rightarrow \frac{1}{2}$. 

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Let us note that, given $\lambda$, both states $P_{\phi(\pm \lambda)}$ belong to the same $K$-orbit, so $C(\lambda) = C(-\lambda)$ and

$$\rho_0 = \frac{1}{2} (P_0 \otimes P_0 + P_2 \otimes P_2) = \frac{1}{2} (P_{\phi(\lambda)} + P_{\phi(-\lambda)})$$

(39)

belongs to $C(\lambda)$ for all $-1 \leq \lambda \leq 1$. In particular, $\rho_0$ lies in the convexed orbit of maximally entangled states, so that $\rho_0 - I \in su^*(H_1) \otimes su^*(H_2)$ and we get the following.

**Proposition 7.3.** The convexed $K$-orbit $C_0 = \text{Conv}(K, \rho_0)$ is a convex body in the affine space $A_0 = I + su^*(H_1) \otimes su^*(H_2)$, contained in $C(\lambda)$ for any $-1 \leq \lambda \leq 1$.

If now $B_0 = B(C_0)$ is the largest ball in $C_0$ and $r_0$ is the radius of $B_0$, then $B(\lambda), B_0 \subset C(\lambda)$. Hence, $B(\lambda)/2 + B_0/2 \subset C(\lambda)$. In particular,

$$I + \rho^{10} + \rho^{01} + \rho^{00} \in I + (I_1 \otimes su^*(H_2)) \oplus (su^*(H_1) \otimes I_2) \oplus (su^*(H_1) \otimes su^*(H_2))$$

belongs to $C(\lambda)$ if only $\|\rho^{10} + \rho^{01}\| \leq \frac{R(\lambda)}{2}$ and $\|\rho^{00}\| \leq r_0/2$. This implies the following.

**Theorem 7.1.** The ellipsoid

$$E(\lambda) = \left\{ I + \rho^{10} + \rho^{01} + \rho^{00} : \frac{\|\rho^{10}\|^2}{R(\lambda)} + \frac{\|\rho^{01}\|^2}{R(\lambda)} + \frac{\|\rho^{00}\|^2}{r_0^2} \leq \frac{1}{4} \right\}$$

is contained in $C(\lambda)$. The fraction of volumes,

$$\frac{\text{vol}(B(\lambda))}{\text{vol}(E(\lambda))} = \left( \frac{R(\lambda)}{r_0} \right)^9 \leq \left( \frac{\lambda^2 - \frac{1}{2}}{r_0} \right)^9,$$

tends to 0 as $\lambda^2 \to 1/2$. In particular, $B(\lambda) \neq E_{\max}(\lambda)$ for $\lambda^2$ close to $1/2$, $\lambda^2 \neq 1/2$.

8. Summary

Numerous problems of quantum information theory involve convex combinations of linear operators taken from a prescribed set. The most prominent example is that of mixed separable density operators which, by definition, are convex combinations of pure separable states, i.e. simple tensor products of projections on one-dimensional subspaces of the underlying Hilbert space. In processes of transformation and transmission of quantum information, one is often confronted with the possibility of applying several quantum channels with some prescribed probabilities, which again leads to convex combinations of operators representing channels. Usually, convex sets obtained in this manner are of practical importance only if they constitute a significant part of the whole set of states or channels, i.e. when they form a convex body in these sets, or in other words, contain an open subset of the set of all states or channels. In the paper, we gave a unifying way of deciding whether this is the case when the set in question is a convexed orbit of some symmetry group through some distinguished state(s) or channel(s). This is a fairly general situation, since usually we have at our disposal the local symmetry group consisting of invertible quantum operations applied individually to components of a composite quantum system. The general problem (see problem 1.1) of whether the convex hull of an orbit is a convex body is answered by theorem 4.1, which is then applied to various cases involving states and channels. In particular, we gave a unique characteristic of maximally entangled states (see theorem 5.1) in terms of the convexed orbits of the local group through them. A state is maximally entangled if the convex hull of the orbit has an empty interior in the space of all density operators of a composite system. The characterization of orbits whose convex hulls are convex bodies provided by theorem 4.1 combined with the Atiyah–Guillemin–Sternberg–Kirwan theorem on the convexity properties of the momentum map is
applied to the study of the convex hull of coadjoint orbits, showing that they are convex bodies if an independence property of the fixed points of the action of a Cartan subgroup is satisfied.

Convex bodies can be partially characterized by the largest balls around some distinguished ‘center’ contained in the body. Such a characterization is useful when analyzing how strong we may perturb the distinguished (e.g. maximally separable) state without losing a desired property (e.g. separability). Such a characterization depends on the metric used. There exists another way of portraying a convex body in an approximate way in terms of the maximum volume ellipsoid contained in it, which is actually independent on the choice of metric, thus bearing a purely affine character. The largest ball and the maximal ellipsoid can, however, coincide in some cases (for a particular choice of a ‘natural’ metric) and differ in other cases. We showed that the former situation occurs for the convex body of states embedded in the space of trace-one operators, whereas the latter takes place for convexed local orbits through pure states of bipartite systems. In both cases the natural metric is the Hilbert–Schmidt one.

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