Theory of two-parameter Markov chain with an application in warranty study.

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ABSTRACT: In this paper we present the classical results of Kolmogorov’s backward and forward equations to the case of a two-parameter Markov process. These equations relates the infinitesimal transition matrix of the two-parameter Markov process. However, solving these equations is not possible and we require a numerical procedure. In this paper, we give an alternative method by use of double Laplace transform of the transition probability matrix and of the infinitesimal transition matrix of the process. An illustrative example is presented for the method proposed. In this example, we consider a two-parameter warranty model, in which a system can be any of these states: working, failure. We calculate the transition density matrix of these states and also the cost of the warranty for the proposed model.

Key Words: Markov chain homogeneous with continuous two-dimensional parameter, the infinitesimal transition matrix, Kolmogorov equations for homogeneous two-parameter Markov chain.

1 Introduction

One of the most important elements when we develop the theory about Markov processes with continuous time parameter and an enumerable space states, is the characterization
that they have with their infinitesimal transition matrix. It is possible to find the matrix of
transitions probabilities trough the system of differential equations of Kolmogorov. The main
purpose of this work is to extend the concepts and results to the case of Markov processes
with continuous two-dimensional parameters (which are usually the time and usage) and
with an enumerable space states. For this purpose it is necessary to start introducing a
concept similar to the infinitesimal transitions rates between states, which are defined in the
case of Markov processes with a continuous-time parameter. In the case of continuous two
dimensional parameters, the definition of the infinitesimal transition rates is that, in this case,
they are not the derivatives of the transitions probabilities at time zero. Here it is required
to work with the second derivatives of the transitions probabilities with respect to each of
the parameters involved in the two-dimensional point \((0,0)\). Then, with these infinitesimal
transition rates, new equations are proposed as backward and forward Kolmogorov equations.

The partial differential equations that are generated, in general, are not easy to solve,
we use the double Laplace transform to present a simple result. This result however is not
easy to invert. So, it requires the use of numerical methods for its inversion. After that, we
present a result to find the distributions of the waiting region and these distributions can be
used to calculate costs for a previously chosen warranty policy.

# Preliminaries

**Definition 1 (MCCTP).** A two-parameter stochastic process \( \{X(t,u) \mid (t,u) \in [0,\infty)^2\} \)
with discrete state space \( S \subset \mathbb{Z}^+ \), is a Markov Chain with continuous two-dimensional
parameter (MCCTP), if for all \( i,j,i_0,\ldots,i_k \in S \), for all \( t_0 < t_1 < \cdots < t_k < s < s + t \) and
for all \( u_0 < u_1 < \cdots < u_k < w < w + u \), the following equality holds:

\[
P \{X(t + s, u + w) = j \mid X(s, w) = i, X(t_0, u_0) = i_0, \ldots, X(t_k, u_k) = i_k\} = \nonumber
\]

\[
P \{X(t + s, u + w) = j \mid X(s, w) = i\}, \quad (2.1)
\]

**Definition 2 (MCHCTP).** The Markov Chain with continuous two dimensional parameter
\( \{ X(t,u) \mid (t,u) \in [0,\infty]^2 \} \) is homogeneous (MCHCTP), if

\[
p_{ij}(t,u) := P \{ X(s+t,w+u) = j \mid X(s,w) = i \} = P \{ X(t,u) = j \mid X(0,0) = i \}, \quad \forall s,w \geq 0 \quad (2.2)
\]

**Remark 1.** Note that

\[
0 \leq p_{ij}(t,u) \leq 1, \quad \forall i,j \in S, \forall t,u \geq 0 \quad \text{and} \quad \sum_{j \in S} p_{ij}(t,u) = 1, \quad \forall i \in S, \forall t,u \geq 0 \quad (2.3)
\]

**Remark 2.** Let \( t, u \geq 0 \) and \( j \in S \), then,

\[
\pi_j(t,u) := P \{ X(t,u) = j \}
= \sum_{i \in S} P \{ X(t,u) = j, X(0,0) = i \}
= \sum_{i \in S} P \{ X(t,u) = j \mid X(0,0) = i \} \cdot P \{ X(0,0) = i \}
\]

So,

\[
\pi_j(t,u) = \sum_{i \in S} p_{ij}(t,u) \cdot \pi_i(0,0) \quad (2.4)
\]

Also,

\[
\sum_{j \in S} \pi_j(t,u) = 1 \quad (2.5)
\]

**Definition 3 (The transition probability matrix).** Let \( t, u \geq 0 \), let us definite the transition probability matrix by \( P(t,u) = (p_{ij}(t,u))_{i,j \in S} \) and the probability row vector by \( \pi(t,u) = (\pi_j(t,u))_{j \in S} \). Then, (2.4) can be written as:

\[
\pi(t,u) = \pi(0,0) \cdot P(t,u). \quad (2.6)
\]

**Remark 3.** Note that \( P(0,0) = I \), where \( I \) is of identity matrix.

**Definition 4 (Initial probability vector).** The vector \( \pi(0,0) \) is called the initial probability vector.
3 The waiting region for a change of state

Definition 5 (The waiting region). Suppose that \( \{X(t, u)\} \) is a MCHCTP and that in the point \( (t_0, u_0) = (0, 0) \), the state of the process \( X(t_0, u_0) = X(0, 0) = i \), is known. The time-use taken for a change of state from state \( i \), is a random vector, say \( (\tau_i, \gamma_i) \) which is called the waiting region for a change of state from state \( i \).

Notation 1 (cdf). If \( i \in S \), and \( t, u \geq 0 \), we write the cumulative distribution function of the waiting region \( (\tau_i, \gamma_i) \) as:

\[
F_i(t, u) := P(\tau_i \leq t, \gamma_i \leq u \mid X(0, 0) = 0).
\]

Also, we write

\[
\overline{F}_i(t, u) := P(\tau_i > t, \gamma_i > u \mid X(0, 0) = 0).
\]

Proposition 1. Suppose that \( X(t; u) \) is a MCHCTP. If \( i \in S \), and \( s, t, w, u \geq 0 \), then,

\[
\overline{F}_i(s + t, w + u) = \overline{F}_i(t, u) \cdot \overline{F}_i(s, w)
\] (3.1)

Proof. First of all, let us realize that

\[
P \{\tau_i > s + t, \gamma_i > w + u \mid X(0, 0) = i\}
\]

\[
= P \{\tau_i > s + t, \gamma_i > w + u \mid X(0, 0) = i, \tau_i > s, \gamma_i > w\}. 
\]

\[
P \ {\tau_i > s, \gamma_i > w \mid X(0, 0) = i} \ (3.2)
\]

Now, since \( \{X(t, u)\} \) is homogeneous, then

\[
P \ {\tau_i > s + t, \gamma_i > w + u \mid X(0, 0) = i, \tau_i > s, \gamma_i > w}\]

\[
= P \ {\tau_i > s + t, \gamma_i > w + u \mid X(s, w) = i}\]

\[
= P \ {\tau_i > s + t, \gamma_i > w + u, X(s + t, w + u) = i \mid X(s, w) = i}\]

\[
= P \ {\tau_i > t, \gamma_i > u, X(t, u) = i \mid X(0, 0) = i}\]

\[
= P \ {\tau_i > t, \gamma_i > u \mid X(0, 0) = i}\] (3.3)

then by \(3.3\), the equality \(3.2\) can be written as \(3.1\).
Remark 4. In the paper [2] of Marshall and Olkin, we may see that if $X$ and $Y$ are two random variables, such that

$$P(X > s + t, Y > w + u) = P(X > t, Y > u) \cdot P(X > s, Y > w)$$

(3.4)

for all $s, t, w, u \geq 0$, then:

$$P(X > t, Y > u) = \exp \left\{ -\lambda_1 t - \lambda_2 u \right\}, \quad u, t \geq 0$$

(3.5)

for some $\lambda_1, \lambda_2 > 0$.

Proof. In univariate distribution theory is known that if $Y$ is a positive random variable, then $Y \sim \exp(\lambda)$, for some $\lambda > 0$, is equivalent to

$$F_Y(s + t) = F_Y(s) F_Y(t), \quad \text{for all } s, t \geq 0,$$

(3.6)

where $F_Y(s) := P(Y > s)$.

Now, let $s, t, w, u \geq 0$ and suppose (3.4), then:

$$F(s, 0) = P(X > s, Y > 0)$$
$$= P(X > s)$$
$$= F_X(s)$$

(3.7)

So,

$$F_X(s + t) = F(s + t, 0)$$
$$= F(s, 0) F(t, 0)$$
$$= F_X(s) F_X(t)$$

by (3.7).

Therefore, $X \sim \exp(\lambda_1)$, for some $\lambda_1 > 0$. Similarly, $F(0, u) = F_Y(u), F_Y(w + u) =$
Finally, \( \mathcal{F}(t, u) = \mathcal{F}(t + 0, 0 + u) = \mathcal{F}(t, 0) \mathcal{F}(0, u) = \mathcal{F}(t, 0) \mathcal{F}_Y(u) = e^{-\lambda_1 t - \lambda_2 u} \) by (3.4).

**Proposition 2.** Suppose that \( X(t; u) \) is a MCHCTP. If \( i \in S \), then

\[
\mathcal{F}_i(t, u) = \exp\left\{ -\lambda_{1i} t - \lambda_{2i} u \right\}, \quad t, u \geq 0 \tag{3.8}
\]

for some \( \lambda_{1i}, \lambda_{2i} \geq 0 \).

By using the proposition [square] and the remark [square] the result is immediately.

### 4 Kolmogorov Equations

Next it is going to enunciate and demonstrate the important result of Chapman-Kolmogorov, for the case of Markov chains with two parameters.

**Theorem 1** (Chapman Kolmogorov equations). If \( \{X(t, u)\} \) is a MCHCTP and let be \( i, j \in S, t, s, w, u \geq 0 \), then:

\[
p_{ij}(t + s, u + w) = \sum_{k \in S} p_{ik}(t, u) \cdot p_{kj}(s, w) \tag{4.1}
\]

or in matrix notation,

\[
P(t + s, u + w) = P(t, u) \cdot P(s, w) \tag{4.2}
\]
Proof. We have that,

\[ p_{ij}(s + t, w + u) = P_r \{ X(s + t, w + u) = j \mid X(0, 0) = i \} \]

\[ = \sum_{k \in S} P_r \{ X(s + t, w + u) = j, X(t, u) = k \mid X(0, 0) = i \} \]

\[ = \sum_{k \in S} P_r \{ X(s + t, w + u) = j, X(t, u) = k \} \cdot P_r \{ X(t, u) = k \mid X(0, 0) = i \} \]

(because \( \{ X(t, u) \} \) is a Markov chain)

\[ = \sum_{k \in S} P_r \{ X(s, w) = j, X(t, u) = k \} \cdot P_r \{ X(t, u) = k \mid X(0, 0) = i \} \]

(because \( \{ X(t, u) \} \) is a homogeneous chain)

So (4.1) is obtained and (4.2) results then immediatly. \[ \square \]

Definition 6 (Infinitesimal transition between states). If we suppose that for \( t = 0 \) or \( u = 0 \), \( P(t, u) = I \), then we define the infinitesimal transition from state \( i \) to state \( j \), as \( a_{ij} = \frac{\partial^2 p_{ij}}{\partial t \partial u} (0, 0) \). Also, it is defined the infinitesimal transition matrix as the matrix \( A = (a_{ij})_{i,j \in S} \).

Remark 5. Note that:

\[ a_{ij} = \frac{\partial^2 p_{ij}}{\partial t \partial u} (0, 0) \]

\[ = \lim_{h \to 0^+, k \to 0^+} \frac{1}{h k} [p_{ij}(h, k) - p_{ij}(h, 0) - p_{ij}(0, k) + p_{ij}(0, 0)] \]

So, if \( i \neq j \), \( a_{ij} = \lim_{h \to 0^+, k \to 0^+} \frac{p_{ij}(h, k)}{h k} \) and \( a_{ii} = \lim_{h \to 0^+, k \to 0^+} \frac{p_{ii}(h, k) - 1}{h k} \)

These relations show that \( a_{ij} \geq 0 \) if \( i \neq j \), and than \( a_{ii} \leq 0 \). Therefore,

\( \text{If } i \neq j, \quad p_{ij}(h, k) = h k a_{ij} + o(h) o(k) \)

and \( p_{ii}(h, k) = 1 + h k a_{ij} + o(h) o(k) \)
Moreover:

\[
\sum_{j \in S} a_{ij} = \sum_{j \in S} \left[ \lim_{h \to 0^+} \lim_{k \to 0^+} \frac{p_{ij}(h, k) - p_{ij}(h, 0) - p_{ij}(0, k) + p_{ij}(0, 0)}{h k} \right]
\]

\[
= \lim_{h \to 0^+} \lim_{k \to 0^+} \frac{1}{h k} \left[ \sum_{j \in S} p_{ij}(h, k) - \sum_{j \in S} p_{ij}(h, 0) - \sum_{j \in S} p_{ij}(0, k) + \sum_{j \in S} p_{ij}(0, 0) \right]
\]

\[
= \lim_{h \to 0^+} \lim_{k \to 0^+} \frac{1}{h k} [1 - 1 - 1 + 1] = 0, \text{ by (2.3)}
\]

So,

\[a_{ii} = - \sum_{j \in S, j \neq i} a_{ij}, \quad \forall i \in S. \quad (4.3)\]

**Theorem 2** (Backward Kolmogorov equations). If \(\{X(t, u)\}\) is a MCHCTP, \(i, j \in S, t, u \geq 0\), \(P(t, u) = \left(p_{ij}(t, u)\right)_{i,j \in S}\) is the transition probability matrix and \(A = \left(a_{ij}\right)_{i,j \in S}\) is the infinitesimal transition matrix, then:

\[
\frac{\partial^2 p_{ij}(t, u)}{\partial t \partial u} = \sum_{k \in S} a_{ik} p_{kj}(t, u), \quad (4.4)
\]

or in matrix form:

\[
\frac{\partial^2 P(t, u)}{\partial t \partial u} = A \cdot P(t, u). \quad (4.5)
\]

**Proof.** Recall the Chapman Kolmogorov equation (4.1), we have that:

\[p_{ij}(t + s, u + w) = \sum_{k \in S} p_{ik}(t, u) \cdot p_{kj}(s, w)\]

If \(x(t, s) = t + s\) and \(y(u, w) = u + w\); and differenting both sides of the before equation with respect to \(t\), it is obtained:

\[
\frac{\partial p_{ij}(x, y)}{\partial x} \left(\frac{\partial x}{\partial t}\right)(t, s) + \frac{\partial p_{ij}(x, y)}{\partial y} \left(\frac{\partial y}{\partial t}\right)(u, w) = \sum_{k \in S} \frac{\partial p_{ik}(t, u)}{\partial t} \cdot p_{kj}(s, w)
\]

So,

\[
\frac{\partial p_{ij}(x, y)}{\partial x} = \sum_{k \in S} \frac{\partial p_{ik}(t, u)}{\partial t} \cdot p_{kj}(s, w)
\]
Now differenting both sides of the before equation with respect to $u$, it is obtained:

$$\frac{\partial^2 p_{ij}}{\partial x \partial x}(x, y) \cdot \frac{\partial x}{\partial u}(t, s) + \frac{\partial^2 p_{ij}}{\partial y \partial x}(x, y) \cdot \frac{\partial y}{\partial u}(u, w) = \sum_{k \in S} \frac{\partial^2 p_{ik}}{\partial u \partial t}(t, u) \cdot p_{kj}(s, w)$$

Therefore,

$$\frac{\partial^2 p_{ij}}{\partial y \partial x}(x, y) = \sum_{k \in S} \frac{\partial^2 p_{ik}}{\partial u \partial t}(t, u) \cdot p_{kj}(s, w)$$

But, taking $t = 0$ and $u = 0$, then

$$\frac{\partial^2 p_{ij}}{\partial w \partial s}(s, w) = \sum_{k \in S} \frac{\partial^2 p_{ik}}{\partial u \partial t}(0, 0) \cdot p_{kj}(s, w) = \sum_{k \in S} a_{ik} \cdot p_{kj}(s, w)$$

and thus the theorem has been proved. \(\Box\)

Similarly can be proved the next theorem:

**Theorem 3 (Forward Kolmogorov equations).** If \(\{X(t, u)\}\) is a MCHCTP, \(i, j \in S, t, u \geq 0\), \(P(t, u) = \left(p_{ij}(t, u)\right)_{i, j \in S}\) is the transition probability matrix and \(A = \left(a_{ij}\right)_{i, j \in S}\) is the infinitesimal transition matrix, then

$$\frac{\partial^2 p_{ij}(t, u)}{\partial t \partial u} = \sum_{k \in S} p_{ik}(t, u) a_{kj}, \quad (4.6)$$

or in a matrix form:

$$\frac{\partial^2 P(t, u)}{\partial t \partial u} = P(t, u) \cdot A \quad (4.7)$$

**Notation 2 (Laplace transform.)**. If \(g(x), x \geq 0\), is any function for which exists its Laplace Transform, then we shall write \(g^*(s) = \mathcal{L}(g(x)) = \int_0^\infty e^{-sx} g(x) \, dx\) if \(s \geq 0\) for the Laplace Transform of \(g(x)\). Similarly, if \(g(x) = \left(g_{ij}(x)\right)\) is a matrix function for which all its components have Laplace Transform, then we shall write \(g^*(s) = \mathcal{L}(g(x)) = \left(g_{ij}^*(s)\right)\) for the Laplace Transform of \(g(x)\).
If \( k(x, y), x \geq 0, y \geq 0 \) is any bivariate function for which exists its bivariate Laplace Transform, then we shall write \( k^{**}(s_1, s_2) = \mathcal{L}^2(k(x, y)) = \int_0^\infty \int_0^\infty e^{-s_1 x - s_2 y} k(x, y) \, dx \, dy \) if \( s_1, s_2 \geq 0 \) for the \textit{bivariate Laplace Transform} of \( k(x, y) \). Similarly, if \( k(x, y) = \left(k_{ij}(x, y)\right) \) is a matrix function for which all its components have Laplace Transform, then we shall write \( k^{**}(s_1, s_2) = \mathcal{L}^2(k(x, y))) = \left(k^{**}_{ij}(s_1, s_2)\right) \) for the \textit{bivariate Laplace Transform} of \( k(x, y) \).

Now, we are going to give a relation, between \( A \) and \( P \), by using the double Laplace transform.

**Theorem 4.** If \( \{X(t, u)\} \) is a MCHCTP, and let be \( s_1, s_2 \geq 0 \), \( P(t, u) = \left(p_{ij}(t, u)\right)_{i,j \in S} \) the transition probability matrix and \( A = \left(a_{ij}\right)_{i,j \in S} \) the infinitesimal transition matrix, then

\[
P^{**}(s_1, s_2) = \left(s_1 s_2 I - A\right)^{-1}
\]  

(4.8)

where \( I \) is the identity matrix.

**Proof.** Let be \( t, u \geq 0 \). Remember that in definition 6 we had supposed that \( P(t, u) = I \) if \( t = 0 \) or \( u = 0 \). Now, let us call \( H(t, u) = \frac{\partial P}{\partial u}(t, u) \). then (4.5) can be written as

\[
\frac{\partial H}{\partial t}(t, u) = A \cdot P(t, u)
\]

and taking Laplace Transform in both sides, with respect to variable \( t \),

\[
s_1 H^*(s_1, u) - H(0, u) = A \cdot P^*(s_1, u)
\]

(4.9)

But,

\[
H(0, u) = \frac{\partial P}{\partial u}(0, u) = \lim_{k \to 0} \frac{1}{k} \left[P(0, u + k) - P(0, u)\right]
\]

\[
= \lim_{k \to 0} \frac{1}{k} \left[I - I\right] = 0
\]

where \( 0 \) is the null matrix. So (4.9) can be written as:

\[
s_1 H^*(s_1, u) = A \cdot P^*(s_1, u)
\]

(4.10)
Now, we observe that:

\[ H^*(s_1, u) = \int_0^\infty e^{-t s_1} H(t, u) \, dt = \int_0^\infty e^{-t s_1} \frac{\partial}{\partial u} P(t, u) \, dt \]

\[ = \frac{\partial}{\partial u} \int_0^\infty e^{-t s_1} P(t, u) \, dt = \frac{\partial}{\partial u} P^*(s_1, u) \]

Then (4.10) can be written as:

\[ s_1 \frac{\partial}{\partial u} P^*(s_1, u) = A \cdot P^*(s_1, u) \]

And taking Laplace Transform in both sides with respect to variable \( u \):

\[ s_1 \left[ s_2 P^{**}(s_1, s_2) - P^*(s_1, 0) \right] = A \cdot P^{**}(s_1, s_2) \quad (4.11) \]

But,

\[ P^*(s_1, 0) = \int_0^\infty e^{-t s_1} P(t, 0) \, dt = \int_0^\infty e^{-t s_1} I \, dt \]

\[ = \left( \int_0^\infty e^{-t s_1} \, dt \right) \cdot I = \frac{1}{s_1} \cdot I \]

So (4.11) can be written as:

\[ s_1 \left[ s_2 P^{**}(s_1, s_2) - \frac{1}{s_1} \cdot I \right] = A \cdot P^{**}(s_1, s_2) \]

and solving this equation for \( P^{**}(s_1, s_2) \), we obtain (4.8). \( \square \)

5 Application

A high technology machine to produce juice has several identical components. It can work if at least one of its components is in a good condition. However, just damaged one of its components, it is removed and repaired. When one or more of its components are being repaired, the machine does not allow a full work. Once it is fixed, it is placed back into the machine. In this sense, the probability that all the components simultaneously are damaged, is practically zero. Moreover, not all the time, the machine has the same amount of work.
Therefore, the machine has a measurer that records the amount of work done by the machine. For this reason, the warranty policy takes into account both the time since the machine is running, and the amount of work done. For the model that is being analyzed, we suppose that \( X(t, u) \) is a MCHCTP and that its states space is \( S = \{0, 1\} \).

It is interpreted as follows: the state 1 means that the machine is working full capacity, that is, all its components are in good condition. The state 0 means that at least one of the components is being repaired and the machine is not working full capacity. The parameter \( t \) represents the total time (in years) since the machine was started. The parameter \( u \) represents the amount of work (in million of juice liters) that the machine has performed.

The infinitesimal transition matrix of the process \( \{X(t, u)\} \) is

\[
A = \begin{bmatrix} -2 & 2 \\ 0.6 & -0.6 \end{bmatrix}.
\]

Therefore, by (4.8),

\[
P^{**}(s_1, s_2) = \frac{1}{s_1 s_2 (5s_1 s_2 + 13)} \begin{bmatrix} 5s_1 s_2 + 3 & 10 \\ 3 & 5(s_1 s_2 + 2) \end{bmatrix},
\]

So, by using the method for to invert the double Laplace Transform that was showed in [4] by Moorthy, for instance, we have found the next results, which have relative errors less that the 4%:

\[
P(0.2, 0.6) = \begin{bmatrix} 0.7781 & 0.2219 \\ 0.0666 & 0.9334 \end{bmatrix}
\]

and

\[
P(2.0, 2.0) = \begin{bmatrix} 0.4272 & 0.5754 \\ 0.1726 & 0.8300 \end{bmatrix}.
\]
In this application, we suppose that the initial probability vector is \( \pi(0, 0) = [0, 1] \) and then, we obtain that,

\[
\pi(0.2, 0.6) = [0.0666, 0.9334],
\]

and

\[
\pi(2.0, 2.0) = [0.1726, 0.8300].
\]

Suppose that the state of the process when \( t = 0 \) and \( u = 0 \) is \( X(0, 0) = i \). Let \( (\tau_i, \gamma_i) \) the waiting region for a change of state from state \( i \) and let:

\[
F(t, u) = Pr (\tau_0 \leq t, \gamma_0 \leq u)
\]

\[
G(t, u) = Pr (\tau_1 \leq t, \gamma_1 \leq u)
\]

\[
\overline{F}(t, u) = Pr (\tau_0 > t, \gamma_0 > u)
\]

\[
\overline{G}(t, u) = Pr (\tau_1 > t, \gamma_1 > u)
\]

\[
f(t, u) = \frac{\partial^2 F}{\partial t \partial u}(t, u)
\]

and

\[
g(t, u) = \frac{\partial^2 G}{\partial t \partial u}(t, u).
\]

Then, we can write the next integral equations:

\[
p_{00}(t, u) = \overline{F}(t, u) + \int_0^u \int_0^t p_{10}(t - \xi, u - \omega) dF(\xi, \omega) \quad (5.1)
\]

\[
p_{01}(t, u) = \int_0^u \int_0^t p_{11}(t - \xi, u - \omega) dF(\xi, \omega) \quad (5.2)
\]

\[
p_{10}(t, u) = \int_0^u \int_0^t p_{00}(t - \xi, u - \omega) dG(\xi, \omega) \quad (5.3)
\]

\[
p_{11}(t, u) = \overline{G}(t, u) + \int_0^u \int_0^t p_{01}(t - \xi, u - \omega) dG(\xi, \omega) \quad (5.4)
\]
By taking double Laplace Transform on (5.2) and (5.3), we obtain:

\[ p_{01}^{**}(s_1, s_2) = f^{**}(s_1, s_2) p_{11}^{**}(s_1, s_2) \]  
(5.5)

\[ p_{10}^{**}(s_1, s_2) = g^{**}(s_1, s_2) p_{00}^{**}(s_1, s_2) \]  
(5.6)

Since \( P^{**}(s_1, s_2) \) is known already, then by (5.5) and (5.6), we deduce that

\[ f^{**}(s_1, s_2) = \frac{10}{5s_1 s_2 + 10} \] and \( g^{**}(s_1, s_2) = \frac{3}{5s_1 s_2 + 3} \)

and so

\[ F^{**}(s_1, s_2) = \frac{10}{s_1 s_2 (5s_1 s_2 + 10)} \] and \( G^{**}(s_1, s_2) = \frac{3}{s_1 s_2 (5s_1 s_2 + 3)} \).

Again, we use [4] for to invert these double Laplace transforms.

The warranty conditions are as follows: If damage in one of the machine components occurs, within the first six months after putting the machine in use and before it produces 200,000 liters of juice, the machine provider changes the item immediately by a new one whose components have been fully checked prior to installation and they meet the quality requirements (ie better than new). However, if damage occurs outside the above ranges, but during the first year of operation of the machine and before the machine produces 300,000 liters of juice, the machine provider agrees to make the change of the just damage component immediately and to make a general revision of the machine to state it better than new. Once one of these has been done, the machine provider does not offer more warranty service. Suppose that the cost of the machine is \( C \) and the cost of the change of one of the components and the general revision is \( 1 = 10C \). Then, by using the results proposed by Dimitrov et al [1], the expected warranty expense is:

\[ EW E = C \cdot G(0.5, 0.2) + \frac{C}{10} \cdot [G(1, 0.3) - G(0.5, 0.2)] \]

\[ = 0.0591C + 0.1130 \frac{C}{10} = 0.0704C \]
Conclusion

It is known that the Markov chains with a single continuous parameter are characterized by the infinitesimal transition matrix. This study concluded that Markov chains with two continuous parameters are characterized by a matrix, that it’s also called “infinitesimal transition matrix”, although its structure is different to the case of a single parameter. In addition, this Markov process can be used in a particular type of two-dimensional warranty policies problems.

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