THE BOLZA CURVE AND SOME ORBIFOLD BALL QUOTIENT SURFACES

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Abstract. We study Deraux’s non-arithmetic orbifold ball quotient surfaces obtained as birational transformations of a quotient $X$ of a particular Abelian surface $A$. Using the fact that $A$ is the Jacobian of the Bolza genus 2 curve, we identify $X$ as the weighted projective plane $\mathbb{P}(1,3,8)$. We compute the equation of the mirror $M$ of the orbifold ball quotient $(X,M)$ and by taking the quotient by an involution, we obtain an orbifold ball quotient surface with mirror birational to an interesting configuration of plane curves of degrees 1, 2 and 3. We also exhibit an arrangement of four conics in the plane which provides the above-mentioned ball quotient orbifold surfaces.

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1. Introduction.

Chern numbers of smooth complex surfaces of general type $X$ satisfy the Bogomolov-Miyaoka-Yau inequality $c_1^2(X) \leq 3c_2(X)$. Surfaces for which the equality is reached are ball quotient surfaces: there exists a cocompact torsion-free lattice $\Gamma$ in the automorphism group $PU(2,1)$ of the ball $B_2$ such that $X = B_2/\Gamma$. This description of ball quotient surfaces by uniformisation is of transcendental nature, and in fact among ball-quotient surfaces, very few are constructed geometrically (e.g. by taking cyclic covers of known surfaces or by explicit equations of an embedding in a projective space).

Among lattices in $PU(2,1)$, only 22 commensurability classes are known to be non-arithmetic. The first examples of such lattices were given by Mostow and Deligne-Mostow (see [22] and [10]), and recently Deraux, Parker and Paupert [12, 13] constructed some more, sometimes related to an earlier work of Couwenberg, Heckman and Looijenga [9].

Being rare and difficult to produce, these examples are particularly interesting and one would like a geometric description of them. To do so, Deraux [14] studies the quotient of the Abelian surface $A = E \times E$, where $E$ is the elliptic curve $E = \mathbb{C}/\mathbb{Z}[i\sqrt{2}]$, by an order 48 automorphism group isomorphic to $GL_2(\mathbb{F}_3)$ that we will denote by $G_{48}$. The ramification locus of the quotient map $A \to A/G_{48}$ is the union of 12 elliptic curves and two orbits of isolated fixed points. The images of these two orbits are singularities of type $A_2$ and $\frac{1}{8}(1,3)$, respectively.

Then Deraux proves that (on some birational transforms) the 1-dimensional branch locus $M_{48}$ of the quotient map $A \to A/G_{48}$ and the two singularities are the support of four ball-quotient orbifold structures, three of these corresponding to non-arithmetic lattices in $PU(2,1)$. Knowing the branch locus $M_{48}$ is therefore important for these ball-quotient orbifolds, since it gives an explicit geometric description of the uniformisation maps from the ball to the surface.

Deraux also remarks in [14] that the invariants of $A/G_{48}$ and its singularities are the same as for the weighted projective plane $\mathbb{P}(1,3,8)$ and, in analogy with cases in [11] and [15] where
weighted projective planes appear in the context of ball-quotient surfaces, he asks whether the two surfaces are isomorphic.

In fact, the quotient $A/G_{48}$ can also be seen as a quotient $\mathbb{C}^2/G$ where $G$ is an affine crystallographic complex reflection group. The Chevalley Theorem assert that if $G'$ is a finite reflection group acting on a space $V$ then the quotient $V/G'$ is a weighted projective space. Using theta functions, Bernstein and Schwarzman [2] observed that for many examples of affine crystallographic complex reflection groups $G$ acting on a space $V$, the quotient $V/G$ is a also weighted projective space. Kaneko, Tokunaga and Yoshida [20] worked out some other cases, and it is believed that this analog of the Chevalley Theorem always happens (see [2, [16, p. 17]), although no general method is known (see also the presentation of the problem given by Deraux in [14], where more details can be found).

In this paper we prove that indeed:

**Theorem A.** The surface $A/G_{48}$ is isomorphic to $\mathbb{P}(1,3,8)$.

We obtain this result by exploiting the fact that $A$ is the Jacobian of a smooth genus 2 curve $\theta$, a curve which was first studied by Bolza [5]. The automorphism group of the curve $\theta$ induces the action of $G_{48}$ on the Jacobian $A$. The main idea to obtain Theorem A is to understand the image of the curve $\theta$ in $A$ by the quotient map $A \to A/G_{48}$ and to prove that its strict transform in the minimal resolution is a $(-1)$-curve.

We then construct birational transformations of $\mathbb{P}(1,3,8)$ to $\mathbb{P}^1 \times \mathbb{P}^1$, $\mathbb{P}^2$ and obtain the equations of the images $M_{\mathbb{P}^1 \times \mathbb{P}^1}$, $M_{\mathbb{P}^2}$ of the branch curve $M_{48}$ in these surfaces (and also $M_{48} \subset \mathbb{P}(1,3,8)$). In particular:

**Theorem B.** In the projective plane, the mirror $M_{\mathbb{P}^2}$ is the quartic curve $(x^2 + xy + y^2 - xz - yz)^2 - 8xy(x + y - z)^2 = 0$.

This curve has two smooth flex points and singular set $a_1 + 2a_2$ (where an $a_k$ singularity has local equation $y^2 - x^{k+1} = 0$). The line $L_0$ through the two residual points of the flex lines $F_1$, $F_2$ contains the node (by flex line we mean the tangent line to a flex point).

The curve $M_{\mathbb{P}^2}$ with the two flex lines $F_1, F_2$ gives rise to the four orbifold ball-quotient surfaces (previously described by Deraux [14]) on suitable birational transformations of the plane. We prove that the configuration of curves described in Theorem B is unique up to projective equivalence.

In [13], Hirzebruch constructed ball quotient surfaces using arrangements of lines and performing Kummer coverings. It is a well-known question whether one can construct other ball quotient surfaces using higher degree curves, the next case being arrangements of conics.

Let $\varphi$ be the Cremona transformation of the plane centered at the three singularities of $M_{\mathbb{P}^2}$. The image by $\varphi$ of the curves $M_{\mathbb{P}^2}$, $F_1$, $F_2$, $L_0$ described in Theorem B is a special arrangement of four plane conics. We remark that by performing birational transforms of $\mathbb{P}^2$ and by taking the images of the 4 conics, one can obtain the orbifold ball-quotients of $[14]$. To our knowledge that gives the first example of orbifold ball quotients obtained from a configuration of conics (ball quotient orbifolds obtained from a configuration of a conic and three tangent lines are studied in [19] and [28]). However we do not know whether one can obtain ball quotient surfaces by performing Kummer coverings branched at these conics.

When preparing this paper, we observed that the mirror $M_{\mathbb{P}^1 \times \mathbb{P}^1}$ and one related orbifold ball quotient surface among the four might be invariant by an order 2 automorphism. Using the equation we have obtained for $M_{\mathbb{P}^2}$, we prove that this is actually the case: there is an involution $\sigma$ on $\mathbb{P}^1 \times \mathbb{P}^1$ with fixed point set a $(1,1)$-curve $D_i$ such that the quotient surface is $\mathbb{P}^2$, moreover the image of $D_i$ is a conic $C_i$ and the image of $M_{\mathbb{P}^2}$ is the unique cuspidal cubic curve $C_u$. In the last section we obtain and describe the following result:
Theorem C. There is an orbifold ball-quotient structure on a surface $W$ birational to $\mathbb{P}^2$ such that the strict transforms on $W$ of $C_0, C_u$ have weights $2, \infty$ respectively.

The paper is structured as follows:
In section 2, we recall some results of Deraux on the quotient surface $A/G_{48}$ and introduce some notation. In section 3, we study properties of the surface $\mathbb{P}(1,3,8)$. In section 4, we introduce the Bolza curve $\theta$ and prove that $A/G_{48}$ is isomorphic to $\mathbb{P}(1,3,8)$. Section 5 is devoted to the equation of the mirror $M_{F_{2}}$. Moreover we describe the four conics configuration. Section 6 deals with Theorem C.

Some of the proofs in sections 5 and 6 use the computational algebra system Magma, version V2.24-5. A text file containing only the Magma code that appear below is available as an auxiliary file on arXiv and at [25].

Along this paper we use intersection theory on normal surfaces as defined by Mumford in [23], Section 2.

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2. Quotient of $A$ by $G_{48}$ and image of the mirrors

2.1. Properties of $A/G_{48}$ and image of the mirrors. In this section, we collect some facts from [14] about the action of the automorphism subgroup $G_{48}$ on the Abelian surface

$$A := \mathbb{C}^2/(\mathbb{Z}[i\sqrt{2}])^2.$$ 

There exists a group $G_{48}$ of order 48 acting on $A$ which is isomorphic to $GL_2(\mathbb{F}_3)$ (see [14], Section 3.1] for generators). The action of $G_{48}$ on $A$ has no global fixed points (in particular some elements have a non-trivial translation part).

The group $G_{48}$ contains 12 order 2 reflections, i.e. their linear parts acting on the tangent space $T_A \simeq \mathbb{C}^2$ are complex order 2 reflections. The fix point set of a reflection being usually called a mirror, we similarly call the fixed point set of a reflection $\tau$ of $G_{48}$ a mirror. The mirror of such a $\tau$ is an elliptic curve on $A$. The group $G_{48}$ acts transitively on the set of the 12 mirrors whose list can be found in [14, Table 1].

We denote by $M$ the union of the mirrors in $A$ and by $M_{48}$ the image of $M$ in the quotient surface $A/G_{48}$. The curve $M_{48}$ is also called the mirror of $A/G_{48}$.

Except the points on $M$, there are two orbits of points in $A$ with non-trivial isotropy, one with isotropy group of order 3 at each point, the other with isotropy group of order 8, see [14, Proposition 4.4]. Correspondingly, the quotient $A/G_{48}$ has two singular points, which are the images of the two special orbits.

Proposition 1. The surface $A/G_{48}$ is rational and its singularities are of type $A_2 + \frac{1}{2}(1,3)$. The minimal resolution $p : X_{48} \to A/G_{48}$ of the surface $A/G_{48}$ has invariants $K_{X_{48}}^2 = 5$ and $c_2(X_{48}) = 7$.

Proof. Let us compute the invariants of $X_{48}$. Let $\pi : A \to A/G_{48}$ be the quotient map. One has

$$\mathcal{O}_A = K_A = \pi^*K_{A/G_{48}} + M,$$

(2.1)
moreover, according to [14] §4, each mirror $M_i$, $i = 1, \ldots, 12$, satisfies $M_i M = 24$, therefore $M^2 = 288$ and 
\[(K_{A/G_{48}})^2 = \frac{1}{48} M^2 = 6.\]

We observe that $M = \pi^* \left( \frac{1}{2} M_{48} \right)$, thus by (2.1), one gets $M_{48} = -2K_{A/G_{48}}$.

The singularities of the quotient surface $A/G_{48}$ are computed in [14] Table 2. Let $C_1, C_2$ be the two $(-3)$-curves above the singularity $\frac{1}{2}(1, 3)$; they are such that $C_1 C_2 = 1$. Since the singularity of type $A_2$ is an $ADE$ singularity, we obtain:

\[K_{X_{48}} = p^* K_{A/G_{48}} - \frac{1}{2} (C_1 + C_2)\]

and $(K_{X_{48}})^2 = 5$.

Let $\tau$ be a reflection in $G_{48}$ and let $G$ be the Klein group of order 4 generated by $\tau$ and the involution $[-1]_A \in G_{48}$. One can check that the quotient surface $A/G$ is rational. Being dominated by the rational surface $A/G$, the surface $A/G_{48}$ is also rational. Thus the second Chern number is $c_2(X_{48}) = 7$ by Noether’s formula. □

The mirror $M_{48}$ (the image of $M$ by the quotient map) does not contain singularities of $A/G_{48}$, moreover:

**Lemma 2.** The pull-back $\tilde{M}_{48}$ of the mirror $M_{48}$ by the resolution map $p : X_{48} \to A/G_{48}$ has self-intersection 24. Its singular set is

\[2a_2 + a_3 + a_5,\]

where $a_k$ denotes a singularity with local equation $y^2 - x^{k+1} = 0$.

**Proof.** The singularities of $\tilde{M}_{48} = p^* M_{48}$ are the same as the singularities of $M_{48}$ since $M_{48}$ is in the smooth locus of $A/G_{48}$. For the computation of the singularities of $M_{48}$, we refer to [14] Table 3, and for the self-intersection of $\tilde{M}_{48}$ (which is the same as the one of $M_{48}$) to [14], §6.2.

□

3. The weighted projective space $\mathbb{P}(1, 3, 8)$.

Since we aim to prove that the quotient surface $A/G_{48}$ is isomorphic to $\mathbb{P}(1, 3, 8)$, one first has to study that weighted projective space: this is the goal of this (technical) section. The reader might at first browse through the main results and notation and proceed to the next section.

3.1. **The surface $\mathbb{P}(1, 3, 8)$ and its minimal resolution.** The weighted projective space $\mathbb{P}(1, 3, 8)$ is the quotient of $\mathbb{P}^2$ by the group $\mathbb{Z}_3 \times \mathbb{Z}_8$ generated by

\[\sigma = \begin{pmatrix} 1 & 0 & 0 \\ 0 & j & 0 \\ 0 & 0 & \zeta \end{pmatrix} \in PGL_3(\mathbb{C}),\]

where $j^2 + j + 1 = 0$ and $\zeta$ is a primitive $8^{th}$ root of unity. The fixed point set of the order 24 element $\sigma$ is $p_1 = (1:0:0)$, $p_2 = (0:1:0)$, $p_3 = (0:0:1)$.

For $i, j \in \{1, 2, 3\}$ with $i \neq j$ let $L_{ij}'$ be the line through $p_i$ and $p_j$. The fixed point set of an order 3 element (e.g. $\sigma^8$) is $p_2$ and the line $L_{12}'$. The fixed point set of an order 8 element (e.g. $\sigma^3$) and its non-trivial powers is $p_3$ and the line $L_{12}'$. Let $\pi : \mathbb{P}^2 \to \mathbb{P}(1, 3, 8)$ be the quotient map: $\pi$ is ramified with order 3 over $L_{13}'$ and with order 8 over $L_{12}'$. The surface $\mathbb{P}(1, 3, 8)$ has two singularities, images of $p_2$ and $p_3$, which are respectively a cusp $A_2$ and a
singularity of type $\frac{1}{3}(1,3)$. We denote by $p : Z \rightarrow \mathbb{P}(1,3,8)$ the minimal desingularization map. The singularity of type $\frac{1}{3}(1,3)$ is resolved by two rational curves $C_1, C_2$ with $C_1C_2 = 1$, $C_1^2 = C_2^2 = -3$, and the singularity $A_2$ is resolved by two rational curves $C_3, C_4$ with $C_3C_4 = 1$, $C_3^2 = C_4^2 = -2$, (see e.g. [1, Chapter III]).

**Lemma 3.** The invariants of the resolution $Z$ are

$$K_Z^2 = 5, \quad c_2(Z) = 7, \quad p_q = q = 0.$$  

**Proof.** We have:

$$K_{\mathbb{P}^2} \equiv \pi^*K_{\mathbb{P}(1,3,8)} + 2L_1' + 7L_2',$$

therefore since $K_{\mathbb{P}^2} \equiv -3L$, we obtain $\pi^*K_{\mathbb{P}(1,3,8)} \equiv -12L$ and

$$(K_{\mathbb{P}(1,3,8)})^2 = \frac{(-12L)^2}{24} = 6.$$  

We have

$$K_Z \equiv p^*K_{\mathbb{P}(1,3,8)} - \sum_{i=1}^{4} a_iC_i$$

where the $a_i$ are rational numbers. The divisor $K_Z$ must satisfy the adjunction formula i.e. one must have $C_iK_Z = -2 - C_i^2$ for $i \in \{1, 2, 3, 4\}$. That gives:

$$K_Z = p^*K_{\mathbb{P}(1,3,8)} - \frac{1}{2}(C_1 + C_2)$$

and therefore $K_Z^2 = 5$. For the Euler number, one may use the formula in [26, Lemma 3]:

$$e(\mathbb{P}(1,3,8)) = \frac{1}{24}(3 + 2(2 - 2) + 7(2 - 2) + 23 \cdot 3) = 3.$$  

Thus $e(Z) = e(\mathbb{P}(1,3,8)) - 2 + 3 + 3 = 7$. Since $\mathbb{P}(1,3,8)$ is dominated by $\mathbb{P}^2$, the surface $Z$ is rational, so that $q = p_g = 0$.  

**3.2. The branch curves in $\mathbb{P}(1,3,8)$ and their pullback in the resolution.** Let $L_{ij}$ be the image of the line $L_{ij}'$ on $\mathbb{P}(1,3,8)$ and let $\tilde{L}_{ij}$ be the strict transform of $L_{ij}$ in $Z$.

**Proposition 4.** We have:

$$\tilde{L}_{23}^2 = -1, \quad \tilde{L}_{23}C_1 = \tilde{L}_{23}C_3 = 1, \quad \tilde{L}_{23}C_2 = \tilde{L}_{23}C_4 = 0,$$

$$\tilde{L}_{13}^2 = 0, \quad \tilde{L}_{13}C_2 = 1, \quad \tilde{L}_{13}C_3 = \tilde{L}_{13}C_4 = 0,$$

$$\tilde{L}_{12}^2 = 2, \quad \tilde{L}_{12}C_1 = \tilde{L}_{12}C_2 = \tilde{L}_{12}C_3 = 0.$$  

**Proof.** On $\mathbb{P}(1,3,8)$ one has $L_{23}^2 = \frac{1}{24}L_{23}^2 = \frac{1}{24}$. Recall that the resolution map is $p : Z \rightarrow \mathbb{P}(1,3,8)$. Let $a_1, \ldots, a_4 \in \mathbb{Q}$ such that

$$\tilde{L}_{23} = p^*L_{23} - \sum_{i=1}^{4} a_iC_i,$$

then $C_i p^*L_{23} = 0$ for $i \in \{1, 2, 3, 4\}$. Let $u_i \in \mathbb{N}$ such that $C_i \tilde{L}_{23} = u_i$. One gets that

$$\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \frac{1}{8} \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \quad \quad \quad \quad \begin{pmatrix} a_3 \\ a_4 \end{pmatrix} = \frac{1}{3} \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} u_3 \\ u_4 \end{pmatrix}.$$  

We have $\pi^*K_{\mathbb{P}(1,3,8)} = -12L_{23}'$, thus

$$K_{\mathbb{P}(1,3,8)}L_{23} = \frac{1}{24}(-12L_{23}' \cdot L_{23}') = -\frac{1}{2}.$$
Figure 3.1. Image of the lines $L'_{ij}$ in the desingularisation of $\mathbb{P}(1,3,8)$

Since $K_Z = p^*K_{\mathbb{P}(1,3,8)} - \frac{1}{2}(C_1 + C_2)$, we get

$$K_Z \bar{L}_{23} = (p^*K_{\mathbb{P}(1,3,8)} - \frac{1}{2}(C_1 + C_2)) \left( p^*L - \sum_{i=1}^4 a_i C_i \right)$$

$$= -\frac{1}{2} - a_1 - a_2 = -\frac{1}{2} (1 + u_1 + u_2),$$

which is in $\mathbb{Z}$, with $u_1, u_2 \in \mathbb{N}$. One computes that

$$\bar{L}_{23}^2 = \frac{1}{24} - \frac{1}{8} (3u_1^2 + 3u_2^2 + 2u_1u_2) - \frac{2}{3} (u_3^2 + u_3u_4 + u_4^2) \in \mathbb{Z}_{\leq 0}.$$

Since $K_Z \bar{L}_{23} + \bar{L}_{23}^2 = -2$, the only possibility is

$$\{u_1, u_2\} = \{0, 1\}, \{u_3, u_4\} = \{0, 1\},$$

which gives the intersection numbers with $\bar{L}_{23}$.

For the curve $L_{13}$, one has $L_{13}K_{\mathbb{P}(1,3,8)} = -\frac{3}{2}$ and $L_{13}^2 = \frac{3}{8}$. Let $u := \bar{L}_{13}C_1 \in \mathbb{N}$, $v := \bar{L}_{13}C_2 \in \mathbb{N}$. Then one similarly computes that

$$\bar{L}_{13}K_Z = -\frac{1}{2}(3 + u + v) \leq -\frac{3}{2}$$

and

$$\bar{L}_{13}^2 = \frac{1}{8}(3 - 3u^2 - 3v^2 - 2uv) \leq \frac{3}{8}.$$

Therefore $\bar{L}_{13}^2 + K_Z \bar{L}_{13} \leq -\frac{9}{8}$ and since $\bar{L}_{13}^2 + K_Z \bar{L}_{13} \geq -2$, the only solution is $\{u, v\} = \{0, 1\}$, thus $\bar{L}_{13}^2 = 0$ and $L_{13}K_Z = -2$.

For the curve $L_{12}$, which does not go through the $\frac{1}{8}(1,3)$ singularity, one has

$$\bar{L}_{12}K_Z = L_{12}K_{\mathbb{P}(1,3,8)} = -4$$

and $L_{12}^2 = \frac{8}{3}$. Let $w := \bar{L}_{12}C_3$, $t := \bar{L}_{12}C_4$. Then

$$\bar{L}_{12}^2 = \frac{1}{3}(8 - 2w^2 - 2t^2 - 2wt) \leq \frac{8}{3}.$$

Therefore $\bar{L}_{12}^2 + K_Z \bar{L}_{12} \leq -\frac{4}{3}$ and the only solution is $\{w, t\} = \{0, 1\}$, thus $\bar{L}_{12}^2 = 2$. \qed
3.3. From $\mathbb{P}(1, 3, 8)$ to the Hirzebruch surface $\mathbb{F}_3$ and back. By contracting the $(-1)$-curve $C_0 := \tilde{L}_{23}$ and then the other $(-1)$-curves appearing from the configuration $C_1, \ldots, C_4, \tilde{L}$, one gets a rational surface with

$$K^2 = 2c_2 = 8$$

containing (depending on the choice of the $(-1)$-curves we contract) a curve which either is a $(−2)$-curve or a $(−3)$-curve. Thus that surface is one of the Hirzebruch surfaces $\mathbb{F}_2$ or $\mathbb{F}_3$. Conversely one can reverse the process and obtain the surface $\mathbb{P}(1, 3, 8)$ by performing a sequence of blow-ups and blow-downs. This process is unique: this follows from the fact that the automorphism group of a Hirzebruch surface $\mathbb{F}_n$, $n \geq 1$ has two orbits, which are the unique $(-n)$-curve and its open complement (see e.g. [4]). In the sequel, only the connection between $\mathbb{P}(1, 3, 8)$ and $\mathbb{F}_3$ will be used.

4. The Bolza genus 2 curve in $A$ and its image by the quotient map

In this section we prove that $A/G_{48}$ is isomorphic to $\mathbb{P}(1, 3, 8)$.

Let us consider the genus 2 curve $\theta$ whose affine model is

$$y^2 = x^5 - x.$$  

It was proved by Bolza [5] that the automorphism group of $\theta$ is $GL_2(\mathbb{F}_3) \simeq G_{48}$ and $\theta$ is the unique genus 2 curve with such an automorphism group.

The automorphisms of $\theta$ are generated by the hyperelliptic involution $\lambda$ and the lift of the automorphism group $G$ of $\mathbb{P}^1$ that preserves the set of 6 branch points $0, \infty, \pm 1, \pm i$ of the canonical map $\theta \to \mathbb{P}^1$ (i.e. the set of points which are fixed by $\lambda$). Note that actually, any map of degree 2 from $\theta$ to $\mathbb{P}^1$ is the composition of this map with an automorphism of $\mathbb{P}^1$.

This is a consequence of the two following facts: on the one hand the 6 ramification points (by the Riemann-Hurwitz formula) of such a map are Weierstrass points, and on the other hand the genus 2 curve $\theta$ has exactly 6 Weierstrass points.

By the universal property of the Abel-Jacobi map, the group $GL_2(\mathbb{F}_3)$ acts naturally on the Jacobian variety $J(\theta)$ of $\theta$, the action on $\theta$ and $J(\theta)$ being equivariant.

There is only one Abelian surface with an action of $GL_2(\mathbb{F}_3)$, which is $A = E \times E$, where $E = \mathbb{C}/\mathbb{Z}[i\sqrt{2}]$ as above (see Fujiki [17] or [3]). We identify $J(\theta)$ with $A$. There are up to conjugation only two possible actions of $GL_2(\mathbb{F}_3)$ on $A$ (see [24]):

a) The action of $G_{48} \simeq GL_2(\mathbb{F}_3)$ which is described in sub-section 2.1: it has no global fixed points;

b) The one obtained by forgetting the translation part of that action. That second action globally fixes the 0 point in $A$.

Let $\alpha : \theta \hookrightarrow J(\theta) = A$ be the embedding of $\theta$ sending the point at infinity of the affine model (4.1) to 0; we identify $\theta$ with its image.

Note that the morphism $\theta \times \theta \to A$, $(x, y) \mapsto [y] - [x] \in \text{Div}_0(\theta) \simeq A$ is onto since $\theta \times \theta$ and $A$ are both two-dimensional. Actually, this map has generic degree 2 and contracts the diagonal. Indeed, assume that $[y] - [x] = [y'] - [x']$ i.e. $[y] + [x'] - [x] - [y'] = 0 \in \text{Div}_0(\theta)$. If $y' = y$ then $x' = x$ (and conversely) because there is no degree 1 map from $\theta$ to $\mathbb{P}^1$. In the same way, $y = x$ iff $y' = x'$. In the remaining cases, there exists a function of degree 2 from $\theta$ to $\mathbb{P}^1$ whose zeroes are $y$ and $x'$ and poles are $x$ and $y'$. But by the remark above, we must have $x' = \lambda(y)$ and $y' = \lambda(x)$. Conversely, by the same argument, it is clear that for all $x$ and $y$ in $\theta$, $[\lambda(y)] - [\lambda(x)] = [x] - [y]$.

This also implies that the points of the type $[y] - [x]$ with $x$ and $y$ being distinct Weierstrass points are exactly the 2-torsion points of $A$. Indeed, since there are 6 Weierstrass points on $\theta$, we have 15 points of that type in $A$ satisfying $[y] - [x] = [\lambda(x)] - [\lambda(y)] = [x] - [y]$ i.e. they are 2-torsion points.
The induced linear action $b)$ is given by $g([y] - [x]) = [g(y)] - [g(x)]$ for which $0 \in \text{Div}_0(\theta)$ is a fixed point.

If we fix the base point $\infty \in \theta$ then for each $y \in \theta$, $\alpha(x) = [x] - [\infty]$. The induced action of $g \in \text{Aut}(\theta)$ on $A$ is then given by $g([y] - [x]) = [g(y)] - [g(x)] + [g(\infty)] - [\infty]$. This is indeed the only action of $\text{Aut}(\theta)$ on $A$ commuting with $\alpha$.

**Lemma 5.** The action of $GL_2(F_3)$ on $A$ inducing the action of $\text{Aut}(\theta)$ on the curve $\theta \hookrightarrow A$ has no global fixed points.

**Proof.** The fixed points on $A$ for the action of the hyperelliptic involution $\lambda$ are its points of 2-torsion (and 0). Indeed, $\lambda([y] - [x]) = [\lambda(y)] - [\lambda(x)] \in \text{Div}_0(\theta)$ since $\infty \in \theta$ is fixed by $\lambda$ and, as a consequence of the discussion above, if $[y] - [x] = [\lambda(y)] - [\lambda(x)]$ then either $y = x$ or $y = \lambda(x)$ i.e. $[y] - [x] = [x] - [y]$ and we saw that this implies that $x$ and $y$ are Weierstrass points.

But for any pair $(x, y)$ of distinct Weierstrass points, it is easy to find $g \in \text{Aut}(\theta)$ (lifting an automorphism of $\mathbb{P}^1$) such that $g(\infty) = \infty$ but $[g(y)] - [g(x)] \neq [y] - [x]$. \hfill $\square$

For $t \in A$, let $\theta_t$ be the curve $\theta_t = t + \theta$. The previous result does not depend on the choice of the embedding $\theta \hookrightarrow A$: indeed the group of automorphisms acting on $A$ and preserving $\theta_t$ is conjugated by the translation $x \mapsto x + t$ to the group of automorphisms acting on $A$ and preserving $\theta$.

We denote by $H_{48}$ the order 48 group acting on $A$ and inducing the automorphism group of the curve $\theta \hookrightarrow A$ by restriction. As a consequence of Lemma 5 we get:

**Corollary 6.** There exists an isomorphism between $H_{48}$ and $G_{48}$. That isomorphism is induced by an automorphism $g$ of the surface $A$ such that $H_{48} = gG_{48}g^{-1}$.

By [6, Theorem (0.3)], the embedding $\alpha : \theta \hookrightarrow A$ is such that the torsion points of $A$ contained in $\theta$ are 16 torsion points of order 6, 5 torsion points of order 2 and the origin, moreover the $x$-coordinates of the 22 torsion points on $\theta$ satisfy

$$x^4 - 4ix^2 - 1 = 0, \quad x^4 + 4ix^2 - 1 = 0$$
$$x^5 - x = 0, \quad x = \infty.$$

**Proposition 7.** (a) These 22 torsion points of $\theta$ are not in the mirror of any of the 12 complex reflections of $H_{48}$;

(b) Each of these 22 points has a non-trivial stabilizer.

**Proof.** Let us prove part (a).

The hyperelliptic involution is given by $(x, y) \mapsto (x, -y)$. By [7], the rational map

$$v : (x, y) \mapsto \left( -\frac{x + i}{ix + 1}, \sqrt{2} \frac{i - 1}{(ix + 1)^3} y \right)$$

defines a non-hyperelliptic involution $v$ on $\theta$. The $x$-coordinates of the fixed point set of $v$ are $x_{\pm} = i(1 \pm \sqrt{2})$. These coordinates $x_{\pm}$ are not among the $x$-coordinates of the 22 torsion points in $\theta$. Let $v$ be the automorphism of $A$ induced by $v$. The fixed point set of $v$ is a smooth genus 1 curve $E_v$ (a mirror) and we have just proved that $E_v$ contains no torsion points of $\theta$. By transitivity of the group $H_{48}$ on its set of 12 non-hyperelliptic involutions, one gets that no mirror contains any of the 22 torsion points.

Let us prove part (b).

The six 2-torsion points are the Weierstrass points of the curve $\theta$, they are fixed by the hyperelliptic involution (whose action on $A$ has only 16 fixed points).
The transformation
\[ w : (x, y) \mapsto \left( \frac{(1 + i)x - (1 + i)}{(1 - i)x + (1 - i)}, \frac{1}{((1 - i)x + (1 - i))^2}y \right) \]
defines an order 3 automorphism of \( \theta \), which acts symplectically on \( A \) and one computes that it fixes a torsion point \( p_0 = (x_0, y_0) \) on \( \theta \) with \( x_0 \) such that \( x_0^4 + 4ix_0^2 - 1 = 0 \), i.e. it is an order 6 torsion point. This torsion point is an isolated fixed point for each non-trivial element of its stabilizer (since by part (a), it is not on a mirror).

Recall that by \([14, Table 2]\), there are exactly two orbits of points of respective orders 6 and 16 with non-trivial stabilizers under \( G_{48} \) which are isolated fixed points of the non-trivial elements of their stabilizer (by a direct computation one can check that these two orbits are 16 points of order 6 and 6 points of order 2). Since \( H_{48} \) is conjugate to \( G_{48} \), the 15 other 6-torsion points on \( \theta \) are also isolated fixed points for each non-trivial element of their stabilizer. □

Since one can change the embedding \( \theta \hookrightarrow A \) by composing with the automorphism \( g \) such that \( H_{48} = gG_{48}g^{-1} \), let us identify \( H_{48} \) with \( G_{48} \).

By sub-section 2.1 (or \([14]\)), the images of the 22 torsion points of \( \theta \) on the quotient surface \( A/G_{48} \) give the singularities \( A_2 \) and \( \frac{1}{8}(1, 3) \).

Let \( m \) be the mirror of one of the 12 complex reflections in \( G_{48} \).

Lemma 8. One has \( \theta \cdot m = 2 \).

Proof. The intersection number \( \theta \cdot m \) is the number of fixed points of the involution \( \iota_m \) with mirror \( m \) restricted to \( \theta \). Since \( \iota_m \) fixes exactly one holomorphic form, the quotient of \( \theta \) by \( \iota_m \) is an elliptic curve, thus by the Hurwitz formula \( \theta \cdot m = 2 \). □

Let \( \theta_{48} \) be the image of \( \theta \) in \( A/G_{48} \). One has:

Proposition 9. The strict transform \( C_0 \) of \( \theta_{48} \) by the resolution \( X_{48} \to A/G_{48} \) is a \((-1)\)-curve and we have \( M_{48}C_0 = 1 \).

Proof. One has
\[ \theta_{48}^2 = \frac{1}{48} \theta^2 = \frac{1}{24}. \]

Let \( \pi : A \to A/G_{48} \) be the quotient map; it is ramified with order 2 on the union \( M \) of the 12 mirrors. One has \( \pi^*(K_A/G_{48} + \frac{1}{2}M_{48}) = K_A = 0 \), thus
\[ K_{A/G_{48}} \theta_{48} = -\frac{1}{48}(M\theta) = -\frac{1}{48} \cdot 2 \cdot 2 = -\frac{1}{2}. \]

The curve \( \theta_{48} \) contains the singularities \( \frac{1}{8}(1, 3) \) and \( A_2 \) (image respectively of the 2-torsion points and the 6-torsion points of \( \theta \)). We are then left with the same combinatorial situation as in the computation of \( L_{23}^2 \) in Proposition 4, thus we conclude that \( C_0^2 = -1 \).

The two intersection points of \( m \) and \( \theta \) in Lemma 8 are permuted by the hyperelliptic involution of \( \theta \) thus \( M_{48}\theta_{48} = 1 \), which implies \( M_{48}C_0 = 1 \). □

We obtain:

Theorem 10. The surface \( A/G_{48} \) is isomorphic to \( \mathbb{P}(1, 3, 8) \).

Proof. Let us denote the resolution map by \( p : X_{48} \to A/G_{48} \). Let \( C_1, C_2 \) be the resolution curves of the singularity \( \frac{1}{8}(1, 3) \), and \( C_3, C_4 \) be the resolution of \( A_2 \). Let \( a \in A \) be an isolated fixed point of an automorphism \( \tau \) of order 3 or 8. The tangent space \( T_{\theta,a} \subset T_{A,a} \) is stable by the action of \( \tau \). Since the local setup is the same, we can reason as in Proposition 4 and we obtain that the curve \( C_0 \) is such that
\[ C_0C_1 = C_0C_3 = 1, \quad C_0C_2 = C_0C_4 = 0. \]
Contracting the curves $C_0, C_1, C_2$, one gets a rational surface with a $(-3)$-curve and with invariants $K^2 = 2c_2 = 8$. This is therefore the Hirzebruch surface $\mathbb{F}_3$. From section 3 we know that reversing the contraction process one gets the weighted projective plane $\mathbb{P}(1,3,8)$ (contracting the curves $C_0, C_1, C_3$, one would have obtained the Hirzebruch surface $\mathbb{F}_2$).

Remark 11. Now we identify $\mathbb{P}(1,3,8)$ with $A/G_{48}$ and we use the notation in section 3. In particular $Z = X_{48}$ is the minimal resolution of $\mathbb{P}(1,3,8)$, the curves $C_1, \ldots, C_4$ are exceptional divisors of the resolution map $Z \to \mathbb{P}(1,3,8)$ and $C_0 = \bar{L}_{23}$ is a $(-1)$-curve in $Z$.

Let us observe that the divisor $\tilde{F} = C_1 + 3C_0 + 2C_3 + C_4$ satisfies

$$\tilde{F}C_1 = \tilde{F}C_0 = \tilde{F}C_3 = \tilde{F}C_4 = 0,$$

thus $\tilde{F}^2 = 0$, moreover $\tilde{F}C_2 = \bar{L}_{13}\tilde{F} = 1$, $\tilde{F}\bar{L}_{13} = 0$ and $\bar{L}_{13}^2 = 0$. This implies that the curves $\tilde{F}$ and $\bar{L}_{13}$ are fibers of the same fibration onto $\mathbb{P}^1$ and $C_2$ is a section of that fibration.

The curves $C_0, \ldots, C_4$ are exceptional divisors or strict transform of generators of the Néron-Severi group of a minimal rational surface. Thus the Néron-Severi group of the rational surface $X_{48}$ is generated by these curves. Knowing the intersection of curves $\bar{L}_{12}$, $\bar{L}_{13}$, $\tilde{M}_{48}$ with these curves (see Propositions 4 and 9) it is easy to obtain their classes in the Néron-Severi group, in particular one gets that $\bar{L}_{12}\tilde{M}_{48} = 8$, $\bar{L}_{13}\tilde{M}_{48} = 3$.

**Figure 4.1.** Configuration of curves $\tilde{M}_{48}$, $\bar{L}_{12}$, $\bar{L}_{13}$ etc... in $X_{48}$ and their intersection numbers

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5. A model of the mirror

5.1. A birational map from $\mathbb{P}(1,3,8)$ to $\mathbb{P}^1 \times \mathbb{P}^1$; images of the mirror.
5.1.1. A rational map $\mathbb{P}(1, 3, 8) \dashrightarrow \mathbb{P}^1 \times \mathbb{P}^1$. As above, we identify $\mathbb{P}(1, 3, 8)$ with $A/G_{48}$; we use the notation of sections 3 and 4.

Take a point $p$ in the Hirzebruch surface $\mathbb{F}_n$ that is not in the negative section. By blowing-up at $p$, and then by blowing-down the strict transform of the fiber through $p$, we get the Hirzebruch surface $\mathbb{F}_{n-1}$. This process is called an elementary transformation.

Recall from sections 3 and 4 that there is a map $\psi : \mathbb{P}(1, 3, 8) \dashrightarrow \mathbb{F}_3$ that contracts the curves $C_0, C_3, C_4$ to a smooth point.

Performing any sequence of three elementary transformations as above, we get a map $\rho : \mathbb{F}_3 \dashrightarrow \mathbb{F}_0 = \mathbb{P}^1 \times \mathbb{P}^1$. This can be seen as a birational transform that, by blowing-up three times at a point $q$ not contained in the negative section, takes the fiber $F_q$ through $q$ to a chain of curves with self intersections $(-1), (-2), (-2), (-1)$, then followed by the contraction of the $(-1), (-2), (-2)$ chain (which contains the strict transform of $F_q$). For our purpose, we choose the three points to blow-up in a specific way, see subsection 5.1.2.

Consider

$$\phi := \rho \circ \psi : \mathbb{P}(1, 3, 8) \dashrightarrow \mathbb{P}^1 \times \mathbb{P}^1.$$ 

We observe that given any two points $t, t' \in \mathbb{P}^1 \times \mathbb{P}^1$ not in a common fiber, the map $\phi$ can be chosen such that the inverse $\phi^{-1}$ is not defined at $t, t'$ and $\phi^{-1}(\mathbb{P}^1 \times \mathbb{P}^1) = \mathbb{P}(1, 3, 8)$.

![Figure 5.1. From $X_{48}$ to $\mathbb{P}^1 \times \mathbb{P}^1$ and back](image)

5.1.2. Image of the mirror $M_{48}$ in $\mathbb{P}^1 \times \mathbb{P}^1$. Let us describe how to choose $\phi$ such that the image $M_{\mathbb{P}^1 \times \mathbb{P}^1}$ of the mirror curve $M_{48}$ is a $(3, 3)$-curve with singularities $a_3 + 2a_2$ and two special fibers tangent to it with multiplicity 3.

The map $\mathbb{P}(1, 3, 8) \dashrightarrow \mathbb{F}_3$ factors through a morphism $\varphi : X_{48} \to \mathbb{F}_3$. Consider the point $t_0 := \varphi(C_0)$. Since $M_{48}C_0 = 1$, then $\varphi(M_{48})$ is a curve which is smooth at $t_0$ and its intersection number with the curve $\varphi(C_1)$ at $t_0$ is 3. The curve $C'_1 := \rho \circ \varphi(C_1)$ is a fiber of $\mathbb{P}^1 \times \mathbb{P}^1$.

Then we choose $q$ to be the $a_5$-singularity of $M_{48}$. The fiber $F_q$ through $q$ cuts $M_{48}$ at $q$ with multiplicity 2 or 3. Suppose that the multiplicity is 3. Then by taking the blow-up at that point and computing the strict transform of the curves $F_q$ and $M_{48}$, one can check that $F_q M_{48} \geq 4$. But $F_q M_{48} = L_{13} M_{48} = 3$ by Remark 11. Therefore the fiber $F_q$ through $q$ cuts $M_{48}$ at $q$ with multiplicity 2, and at another point.

Remark 12. An analogous reasoning gives that the fiber through the $a_3$-singularity has the same property: it is transverse to the tangent of the $a_3$-singularity.

The three successive blow-ups above $q$ are chosen such that they resolve the singularity $a_5$. The three blow-downs we described create a multiplicity 3 tangent point between $M_{\mathbb{P}^1 \times \mathbb{P}^1}$ (the image of $M_{48}$ in $\mathbb{P}^1 \times \mathbb{P}^1$) and the curve $C'_2$ (the image of $C_2$), thus $C'_2 M_{\mathbb{P}^1 \times \mathbb{P}^1} = 3$. Moreover $C'_2 = 0, C'_1 C'_2 = 1$ (see figure 5.1).

The mirror $M_{48}$ does not cut the curves $C_1$ and $C_2$. The transforms of these curves in $\mathbb{P}^1 \times \mathbb{P}^1$ are fibers $C'_1, C'_2$ such that $C'_1$ cuts $M_{\mathbb{P}^1 \times \mathbb{P}^1}$ at one point only, with multiplicity 3.
In particular, the class of $M_{\mathbb{P}^1 \times \mathbb{P}^1}$ in the Néron-Severi group of $\mathbb{P}^1 \times \mathbb{P}^1$ is $3C_1' + 3C_2'$. The singularities of $M_{\mathbb{P}^1 \times \mathbb{P}^1}$ are $a_3 + 2a_2$.

5.1.3. From $\mathbb{P}^1 \times \mathbb{P}^1$ to $\mathbb{P}^2$ and back. Let us recall that the blowup of $\mathbb{P}^1 \times \mathbb{P}^1$ at a point, followed by the blow-down of the strict transform of the two fibers through that point, gives a birational map $\mathbb{P}^1 \times \mathbb{P}^1 \dashrightarrow \mathbb{P}^2$.

We choose to blow-up the point at the $a_3$-singularity $s_0$, so that the strict transform of $M_{\mathbb{P}^1 \times \mathbb{P}^1}$ has a node above $s_0$. The two fibers $F_1, F_2$ of $\mathbb{P}^1 \times \mathbb{P}^1$ passing through $s_0$ cut $M_{\mathbb{P}^1 \times \mathbb{P}^1}$ in two other points respectively $s_1, s_2$ (see Remark 12; the result is preserved through the birational process). The fibers $F_1, F_2$ are contracted into points in $\mathbb{P}^2$ by the rational map $\mathbb{P}^1 \times \mathbb{P}^1 \dashrightarrow \mathbb{P}^2$, the images of $s_1, s_2$ by that map are on the image of the exceptional divisor, which is a line $L_0$ through the node. This implies that the strict transform of $M_{\mathbb{P}^1 \times \mathbb{P}^1}$ is a plane quartic curve $M_{\mathbb{P}^2}$. The process in illustrated in Figure 5.2.

![Figure 5.2](image-url)

The total transform of $M_{\mathbb{P}^1 \times \mathbb{P}^1}$ in $\mathbb{P}^2$ is the union of $2L_0$ with $M_{\mathbb{P}^2}$. This quartic $M_{\mathbb{P}^2}$ has the following properties which follow from its description and the choice of the transformation from $\mathbb{P}^1 \times \mathbb{P}^1$ to $\mathbb{P}^2$:

**Proposition 13.** The singular set of the quartic curve $M_{\mathbb{P}^2}$ is $a_1 + 2a_2$, and the nodal point is contained in the line $L_0$. The curve $M_{\mathbb{P}^2}$ contains two flex points such that each corresponding tangent line meets the quartic at a second point that is contained in the line $L_0$.

5.2. The yoga between the mirrors $M_{\mathbb{P}^2}$ and $M_{48}$. Using the previous description the reader can follow the transformations between the surfaces $\mathbb{P}(1,3,8)$ and the plane. The link between Deraux's ball quotient orbifolds described in [14, Theorem 5] and the quartic $M_{\mathbb{P}^2}$ is as follows:

The singularities $a_1 + 2a_2$ of $M_{\mathbb{P}^2}$ correspond respectively to singularities $a_3 + 2a_2$ of $M_{48}$, so that in order to get the curves $F, G, H$ in [14, Figure 1] one has to blow-up and contract at these 3 points as it is done in [14]. In order to obtain the curve $E$ in [14, Figure 1], one has to blow-up the two flexes three times in order to separate $M_{\mathbb{P}^2}$ and the flex lines. One obtain two chains of $(-1), (-2), (-2)$ curves. Contracting one of the two $(-2), (-2)$ chains one gets an $A_2$-singularity. The curve $E$ is the image by the contraction map of the remaining $(-1)$-curve of the chain. The resolution of the singularity $A_2$ on $\mathbb{P}(1,3,8)$ corresponds to the two $(-2)$-curves on the other chain of $(-1), (-2), (-2)$ curves. After taking the blow-up at the residual intersection of the quartic and the flex lines and after separating the flex lines and the mirror $M_{\mathbb{P}^2}$, one gets two $(-3)$-curves intersecting transversally at one point. In that way the resolution of the singularity $\frac{1}{8}(1,3)$ on $\mathbb{P}(1,3,8)$ by two $(-3)$-curves corresponds to the two flex lines.
5.3. A particular quartic curve in $\mathbb{P}^2$. The aim of this sub-section is to prove the following result:

**Theorem 14.** Up to projective equivalence, there is a unique quartic curve $Q$ in $\mathbb{P}^2$ with distinct points $p_1, \ldots, p_7$ such that:

1. $Q$ has a node at $p_1$ and ordinary cusps at $p_2, p_3$;
2. the points $p_4, p_5$ are flex points of $Q$;
3. the tangent lines to $Q$ at $p_4, p_5$ contain $p_6, p_7$, respectively;
4. the line through $p_6, p_7$ contains $p_1$.

We can assume that $p_1 = [0 : 0 : 1], p_2 = [0 : 1 : 1], p_3 = [1 : 0 : 1]$.

Then the equation of $Q$ is

$$(x^2 + xy + y^2 - xz - yz)^2 - 8xy(x + y - z)^2 = 0,$$

and the points $p_4, p_5$ and $p_6, p_7$ are, respectively,

$\left[ \pm 2\sqrt{-2} + 8 : \mp 2\sqrt{-2} + 8 : 25 \right], \left[ \pm 2\sqrt{-2} : \mp 2\sqrt{-2} : 1 \right]$.

**Corollary 15.** The mirror $M_{\mathbb{P}^2}$ described on sub-section 5.1.3 satisfies the hypothesis of Theorem 14, thus $M_{\mathbb{P}^2}$ is projectively equivalent to the quartic $Q$.

![Figure 5.3. The quartic Q](image)

In order to prove 14, let us first give a criterion for the existence of roots of multiplicity at least 3 on homogeneous quartic polynomials on two variables. We use the computational algebra system Magma; see [25] for a copy-paste ready version of the Magma code.

**Lemma 16.** The polynomial

$$P(x, z) = ax^4 + bx^3z + cx^2z^2 + dxz^3 + ez^4$$

has a root of multiplicity at least 3 if and only if

$$12ae - 3bd + c^2 = 27ad^2 + 27b^2e - 27bcd + 8c^3 = 0.$$ 

**Proof.** The computation below is self-explanatory.
Let us now prove Theorem \[14\].

Proof. We have already chosen 3 points \(p_1, p_2, p_3\) in \(\mathbb{P}^2\). Instead of choosing a fourth point for having a projective base, one can fix two infinitely near points over \(p_2\) and \(p_3\). Indeed the projective transformations that fix points \(p_1, p_2, p_3\) are of the form

\[
\phi : [x : y : z] \mapsto [ax : by : (a - 1)x + (b - 1)y + z]
\]

and these transformations act transitively on the lines through \(p_2\) and \(p_3\). Thus up to projective equivalence, we can fix the tangent cones (which are double lines) of the curve \(Q\) at the cusps \(p_2, p_3\). Let us choose for these cones the lines with equations \(y = z\) and \(x = z\), respectively.

The linear system of quartic curves in \(\mathbb{P}^2\) is 14-dimensional. The imposition of a node and two ordinary cusps (with given tangent cones) corresponds to 13 conditions, thus we get a pencil of curves. We compute that this pencil is generated by the following quartics:

\[
(x^2 + xy + y^2 - xz - yz)^2 + a \cdot xy \cdot (x + y - z)^2 = 0
\]

Notice that, at the points \(p_2, p_3\), the first generator is of multiplicity 2 and the second generator is of multiplicity 3, thus a generic element in the pencil has a cusp singularity at \(p_2, p_3\).

Let us compute the quartic curves \(Q\) satisfying condition (1) to (4) of Theorem \[14\]. The method is to define a scheme by imposing the vanishing of certain polynomials \(P_i = 0\), and the non-vanishing of another ones \(D_i \neq 0\), which is achieved by using an auxiliary parameter \(n\) and imposing \(1 + nD_i = 0\).
M1 := Matrix([[JacobianSequence(F), JacobiSequence(L1)]]);
M1 := Evaluate(M1, [q1, m*q1, 1]);
M2 := Matrix([[JacobianSequence(F), JacobiSequence(L2)]]);
M2 := Evaluate(M2, [q2, m*q2, 1]);

The matrix $M_i$ is of rank 2 if one of its minors is non-zero. Here we make a choice for these minors, but in order to cover all cases the computations must be repeated for all other choices.

D1 := Minors(M1, 2)[1];
D2 := Minors(M2, 2)[1];

Now we intersect the quartic $Q$ with the lines $L_1$, $L_2$:

R1 := Evaluate(F, y, d1*x + (m*q1 - d1*q1)*z);
R2 := Evaluate(F, y, d2*x + (m*q2 - d2*q2)*z);

and we use Lemma 16 to impose that these lines are tangent to $Q$ at flex points of $Q$:

c := Coefficients(R1);
P3 := c[1]*c[5] - 1/4*c[2]*c[4] + 1/12*c[3]^2;
P4 := c[1]*c[4] - 2*c[2]*c[5] - c[3]*c[4] + 8/27*c[3]^3;
c := Coefficients(R2);
P5 := c[1]*c[5] - 1/4*c[2]*c[4] + 1/12*c[3]^2;
P6 := c[1]*c[4] - 2*c[2]*c[5] - c[3]*c[4] + 8/27*c[3]^3;

We note that the lines $L_1$, $L_2$ cannot contain the points $p_2$, $p_3$:

D3 := Evaluate(L1, [0, 1, 1]);
D4 := Evaluate(L1, [0, 1, 1]);
D5 := Evaluate(L2, [0, 1, 1]);
D6 := Evaluate(L2, [0, 1, 1]);

Also the line $L_i$ cannot contain the point $p_1$, $i = 1, 2$:

D7 := (m-d1)*(m-d2);

And it is clear that the following must be non-zero:

D8 := a*q1*q2*(q1-q2);

Finally we define a scheme with all these conditions.

A := AffineSpace(R);
S := Scheme(A, [P1, P2, P3, P4, P5, P6, 1+n*D1*D2*D3*D4*D5*D6*D7*D8]);

We compute (that takes a few hours):

PrimeComponents(S);

and get the unique solution $a = -8$. □

From the equation of the quartic $Q = M_{g2}$, one can compute a degree 24 equation for the mirror $M_{48}$, which is:
(31072410*r+44060139)*x^24+(599304420*r-4660302600)*x^21*y+(-106415505000*r+18054913500)*x^18*y^2+(796474485000*r+363880822500)*x^15*y^3+(-27123660*r-18697014)*x^16*z+(34521715125000*r-31210968093750)*x^12*y^4+(107726220*r+2948918400)*x^13*y*z+(-257483985484500*r-51663217969000)*x^9*y^5+(42798843000*r-32351244300)*x^10*y^2*z+(1544666220033750*r+11942493993804375)*y^8+(-102498120*r-465161400)*x^5*y*z^2+(-319463676000*r+12613760073000)*x*y^5*z+(-2705586000*r+7086771600)*x^2*y^2*z^2+(-712080*r+1186268)*z^3=0

where \( r = \sqrt{-2} \).

### 5.4. A configuration of four plane conics related to the orbifold ball quotient.

In this subsection we describe the configuration of conics which we announced in the introduction.

Let us consider a conic tangent to two lines of a triangle in \( \mathbb{P}^2 \), and going through two points of the remaining line. Performing a Cremona transformation at the three vertices of the triangle one obtains a quartic curve in \( \mathbb{P}^2 \) with singularities \( a_1 + 2a_2 \). Conversely, starting with such a quartic, its image by the Cremona transform at the three singularities is a conic with three lines having the above configuration.

Thus we consider the Cremona transform \( \varphi \) at the three singularities of the quartic \( M_{\mathbb{P}^2} \). Let \( D_1, \ldots, D_4 \) be respectively the images of \( M_{\mathbb{P}^2} \), the line \( L_0 \) through the node and the two residual points of the flex lines, and the two flex lines. Using Magma, we see that these are 4 conics meeting in 10 points, as follows:

|   | \( q_1 \) | \( q_2 \) | \( q_3 \) | \( q_4 \) | \( q_5 \) | \( q_6 \) | \( q_7 \) | \( q_8 \) | \( q_9 \) | \( q_{10} \) |
|---|---|---|---|---|---|---|---|---|---|---|
| \( D_1 \) | 1+ | 1+ | 0 | 0 | 0 | 1 | 1 | 1 | 1 | 1 |
| \( D_2 \) | 1 | 1 | 1 | 1 | 1 | 0 | 0 | 0 | 1 | 1 |
| \( D_3 \) | 0 | 1+ | 1 | 1 | 1 | 1 | 0 | 0 | 0 | 0 |
| \( D_4 \) | 1+ | 0 | 1 | 1 | 1 | 1 | 0 | 1 | 0 | 0 |

Here two \( + \) in the column of \( q_j \) mean that the two curves meet with multiplicity 3 at point \( q_j \). The other intersections are transverse. We see that the various ball-quotient orbifolds that Deraux described in [14] may be obtained from a configuration of conics by performing birational transformations.

### 6. ONE FURTHER QUOTIENT BY AN INVOLUTION

#### 6.1. The quotient morphism \( \mathbb{P}^1 \times \mathbb{P}^1 \rightarrow \mathbb{P}^2 \), image of the mirror as the cuspidal cubic.

Consider the plane quartic curve \( Q \) from Theorem 14. Here we show the existence of a birational map

\[
\rho : \mathbb{P}^2 \rightarrow \mathbb{P}^1 \times \mathbb{P}^1 \subset \mathbb{P}^3
\]

and an involution \( \sigma \) on \( \mathbb{P}^1 \times \mathbb{P}^1 \) that preserves \( \rho(Q) \) and fixes the diagonal \( D \) of \( \mathbb{P}^1 \times \mathbb{P}^1 \) pointwise.

Moreover, we have \( (\mathbb{P}^1 \times \mathbb{P}^1) \cap \sigma = \mathbb{P}^2 \), and the images \( C_u, C_o \) of \( \rho(Q) \), \( D \) are curves of degrees 3, 2, respectively. The curve \( C_u \) has a cusp singularity and intersects \( C_o \) at three points, with intersection multiplicities 4, 1, 1. The map \( \rho \) is the inverse of the birational transform \( \mathbb{P}^1 \times \mathbb{P}^1 \rightarrow \mathbb{P}^2 \) described in sub-section 5.1.3, whose indeterminacy is at the singularity \( a_3 \) of \( M_{\mathbb{P}^1 \times \mathbb{P}^1} \).

K:=Rationals();
R<x>:=PolynomialRing(K);
K:=ext<K|x^2+2>;
P2<x,y,z>:=ProjectiveSpace(K,2);
Q:=Curve(P2,(x^2+x*y+y^2-x*z-y*z)^2-8*x*y*(x+y-z)^2); p6:=P2! [2*r,-2*r,1];
We compute the linear system of conics through the cuspidal points $p_2, p_3$ and take the corresponding map to $\mathbb{P}^3$.

$L := \text{LinearSystem}(\text{LinearSystem}(\mathbb{P}^2, 2), [p_6, p_7])$;

$P^3 < a, b, c, d > := \text{ProjectiveSpace}(K, 3)$;

$\rho := \text{map} < \mathbb{P}^2 \rightarrow \mathbb{P}^3 | \text{Sections}(L) >$;

The image of $\mathbb{P}^2$ is a quadric surface $Q_2$ ($\cong \mathbb{P}^1 \times \mathbb{P}^1$).

$Q_2 := \rho(\mathbb{P}^2);$ $Q_2$;

$C := \rho(Q);$ $C$;

There is an involution preserving both $Q_2$ and the curve $C := \rho(Q)$.

$\sigma := \text{map} < \mathbb{P}^3 \rightarrow \mathbb{P}^3 | [d, b, c, a] >$;

$C := \rho(Q);$ $C$;

$\sigma(Q_2)$ eq $Q_2$;

$\sigma(C)$ eq $C$;

We compute the corresponding map to the quotient. The image of $C$ is a cubic curve, and the image of the diagonal is a conic.

$\psi := \text{map} < \mathbb{P}^3 \rightarrow \mathbb{P}^2 | [a + d, b, c] >$;

$C_u := \psi(C);$ $C_u$;

$C_o := \psi(\text{Scheme}(\rho(\mathbb{P}^2), [a - d]));$

$C_o := \text{Curve}(P^2, \text{DefiningEquations}(C_o))$;

The curve $C_u$ has a cusp singularity:

$\text{Deg}(\text{ReducedSubscheme}(C_o \cap C_u))$ eq 3;

$pt := \text{Points}(C_o \cap C_u) [1]$;

$\text{InterSectionNumber}(C_o, C_u, pt)$ eq 4;

Let $C_1', C_2'$ be the fibers that intersect $M_{\mathbb{P}^1 \times \mathbb{P}^1}$ each at a unique point with multiplicity 3. These fibers are exchanged by the involution $\sigma$ and are sent to a line $F_I$ which cuts the cubic curve $C_u$ at a unique point: this is a flex line. That line $F_I$ also cuts the conic $C_o$ at a unique point.

Conversely, let us start from the data of a conic $C_o$ and a cuspidal cubic $C_u$ intersecting as above, with the flex line (at the smooth flex point) of the cubic tangent to the conic. One can take the double cover of the plane branched over $C_o$, which is $\mathbb{P}^1 \times \mathbb{P}^1$. The pull-back of $C_u$ is then a curve satisfying the properties of Theorem 14 thus the configuration $(C_o, C_u)$ we described is unique in $\mathbb{P}^2$, up to projective automorphisms.

6.2. An orbifold ball-quotient structure from $(\mathbb{P}^2, (C_o, C_u))$. Let $C_u \hookrightarrow \mathbb{P}^2$ be the unique plane cuspidal curve and let $c_1$ be its cuspidal point. Let $F_I$ be the flex line through the unique smooth flex point $c_2$ of $C_u$. By the previous subsection, one has the following result:

**Proposition 17.** There exists a unique conic $C_o \hookrightarrow \mathbb{P}^2$ such that the following holds:

i) $F_I$ is tangent to $C_o$;

ii) $C_o$ cuts $C_u$ at points $c_3, c_4, c_5$ ($\neq c_1, c_2$) with intersection multiplicities 4, 1, 1, respectively.

In this subsection we prove that there is a natural birational transformation $W \rightarrow \mathbb{P}^2$ such that together with the strict transform of the curves $C_o$ and $C_u$ one gets an orbifold ball quotient surface. For definitions and results on orbifold theory, we use [8, 11] and [29].
Let us blow-up over points $c_1, c_2, c_3$ and then contract some divisors as follows (for a pictural description see figure 6.1):

We blow up over $c_1$ three times, the first blow-up resolves the cusp of $C_u$ and the exceptional divisor intersects the strict transform of $C_u$ tangentially, the second blow-up is at that point of tangency and the third blow-up separates the strict transforms of the first exceptional divisor and the curve $C_u$. One obtains in that way a chain of $(-3), (-1)$ and $(-2)$-curves. We then contract the $(-2)$ and $(-3)$-curves obtaining in that way singularities $A_1$ and $\frac{1}{3}(1,1)$. The image of the $(-1)$-curve by that contraction map is denoted by $H$. As an orbifold we put multiplicity $2$ on $H$.

We blow up over $c_2$ (the flex point) three times in order that the strict transform of the curves $F_l$ and $C_u$ get separated over $c_2$. We obtain in that way a chain of $(-1), (-2), (-2)$-curves. We then contract the two $(-2)$-curves and obtain an $A_2$-singularity. The strict transform of the line $F_l$ is a $(-2)$-curve, which we also contract, obtaining in that way an $A_1$-singularity. The contracted curve being tangent to $C_0$, the image $\tilde{C}_0$ has a cusp at the singularity $A_1$.

We moreover blow up over $c_3$ four times, in order that the strict transform of the curves $C_o$ and $C_u$ get separated over $c_3$. We obtain in that way a chain of $(-1), (-2), (-2), (-2)$-curves. We then contract the three $(-2)$-curves and obtain an $A_3$-singularity. The image of the $(-1)$-curve by the contraction map is a curve denoted by $F_d$, we give the weight $2$ to that curve.

**Figure 6.1.** The plane, the surfaces $Z$ and $W$

Let us denote by $W$ the resulting surface. For a curve $D$ on $\mathbb{P}^2$, we denote by $\tilde{D}$ its strict transform on $W$. Let $\mathcal{W}$ be the orbifold with same subjacent topological space, with divisorial part:

$$\Delta = (1 - \frac{1}{\infty})C_u + (1 - \frac{1}{2}) (\tilde{C}_o + F_d + H).$$

The singular points of $W$ are

$$A_1 + A_1 + A_2 + A_3 + \frac{1}{3}(1,1),$$

and they have an isotropy $\beta$ of order $16, 4, 3, 8, 6$ respectively, for $\mathcal{W}$. The computation of the isotropy is immediate, except for the first point (that we shall denote by $r_1$), which is also a cusp on the curve $\tilde{C}_0$ (which has weight $2$). Let $SD_{16}$ be the the semidihedral group of order $16$, generated by the matrices

$$g_1 = \begin{pmatrix} 0 & -\zeta \\ -\zeta^3 & 0 \end{pmatrix},
\quad g_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

where $\zeta$ is a primitive $3$rd root of unity.
where \( \zeta \) is a primitive 8th root of unity. The order 2 elements \( g_2, g_1^{-1}g_2g_1 \) generate an order 8 reflection group \( D_4 \). The quotient of \( \mathbb{C}^2 \) by \( SD_{16} \) has a \( A_1 \) singularity and one computes that the image of the 4 mirrors of \( D_4 \) is a curve with a cusp \( a_2 \) at the \( A_1 \) singularity of \( \mathbb{C}^2/SD_{16} \). The isotropy group of the point \( r_1 \) in the orbifold is therefore the semidihedral group \( SD_{16} \) of order 16. The following proposition is an application of the main result of [21]:

**Proposition 18.** The Chern numbers of the orbifold \( W = (W, \Delta) \) satisfy

\[
c_1^2(W) = 3c_2(W) = \frac{9}{16},
\]

in particular \( W \) is an orbifold ball quotient.

**Proof.** Let us compute the orbifold second Chern number of \( W \). We have (see e.g. [27]):

\[
c_2(W) = e(W) - (1 - \frac{1}{\infty})e(\bar{C}_u \setminus S) + (1 - \frac{1}{2})e(\bar{C}_o \setminus S) + (1 - \frac{1}{2})e(F_d \setminus S) + \sum_{p \in S}(1 - \frac{1}{\beta(p)})
\]

where \( S \) is the union of the singular points of \( W \) with the singular points of the round-up divisor \( [\Delta] \), and where moreover \( \beta(p) \) is the isotropy order of the point \( p \), so that for example for \( p \) on \( C_u \) \( \beta(p) = \infty \) and the unique point \( p \) in \( F_d \) and \( C_o \) has \( \beta(p) = 4 \). Since we have blown-up \( \mathbb{P}^2 \) over 10 points and we have contracted 8 rational curves, we get

\[
e(W) = 3 + 10 - 8 = 5.
\]

We obtain

\[
c_2(W) = 5 - \left( (2 - 4) + \frac{1}{2}(2 - 4) + \frac{1}{2}(2 - 3) + \frac{1}{2}(2 - 3) \right)
- \left( 10 - \frac{1}{16} - \frac{1}{4} - \frac{1}{3} - \frac{1}{8} - \frac{1}{6} - \frac{1}{4} - \frac{1}{\infty} \right),
\]

thus \( c_2(W) = \frac{3}{16} \).

Let us compute \( c_1^2(W) \). One has

\[
c_1^2(W) = (K_W + \Delta)^2,
\]

so that

\[
c_1^2(W) = K_W^2 + 2K_W\bar{C}_u + K_W(\bar{C}_o + F_d + H) + \frac{1}{4}(\bar{C}_o^2 + F_d^2 + H^2) + \bar{C}_u^2
+ \bar{C}_u(\bar{C}_o + F_d + H) + \frac{1}{2}(\bar{C}_oF_d + \bar{C}_oH + F_dH).
\]

Let \( p : Z \to W \) be the surface above \( W \) which resolves \( W \) and is a blow-up of \( \mathbb{P}^2 \). Since \( Z \) is obtained by 10 blow-ups of \( \mathbb{P}^2 \) one has \( K_Z^2 = 9 - 10 = -1 \). Moreover, since all singularities but one are \( ADE \), one has \( K_Z = p^*K_W - \frac{1}{3}D_1 \) where \( D_1 \) is the \((-3)\)-curve on \( Z \) which is contracted to the \( \frac{1}{3}(1,1) \) singularity on \( W \). Since \( p^*K_W \cdot D_1 = 0 \), we obtain

\[
K_W^2 = -\frac{2}{3}.
\]

The curve \( \bar{C}_u \) is a smooth curve of genus 0 on the smooth locus of \( W \). The blow-up at the \( a_2 \)-singularity of the cuspidal cubic decreases the self-intersection by 4, the remaining blow-ups decrease the self-intersection by 1. Since one has \( 4 + 2 + 3 = 9 \) such blow-ups, one gets

\[
\bar{C}_u^2 = 3^2 - 4 - 9 = -4,
\]

and therefore \( K_W\bar{C}_u = 2 \). Let \( \tilde{D} \) be the strict transform on \( Z \) of a curve \( D \) on \( W \) or \( \mathbb{P}^2 \). We have

\[
\bar{C}_o = p^*\bar{C}_o - aF_1.
\]
Since \( \tilde{C}_o F_i = 2 \), then \( a \) is equal to 1. Since moreover \( \tilde{C}_o^2 = 0 \), we get \( 0 = (\tilde{C}_o)^2 = \tilde{C}_o^2 - 2 \), thus \( \tilde{C}_o^2 = 2 \). We have

\[
K_W \tilde{C}_o = (\tilde{C}_o + F_i) \left( K_W + \frac{1}{3} D_1 \right) = -2.
\]

Let \( F_1, F_2, F_3 \subset Z \) be the chain of three \((-2)\)-curves above the \( A_3 \) singularity in \( W \), so that \( \tilde{F}_d F_1 = 1 \). One computes that

\[
\tilde{F}_d = \rho^* F_d - \frac{1}{4} (3F_1 + 2F_2 + F_3)
\]

(it is easy to check that \( \tilde{F}_d F_1 = 1 \), \( \tilde{F}_d F_2 = \tilde{F}_d F_3 = 0 \)). Then

\[
-1 = \tilde{F}_d^2 = F_d^2 - \frac{3}{4}
\]

gives \( F_d^2 = -\frac{1}{4} \). One has

\[
K_W F_d = \left( K_Z + \frac{1}{3} D_1 \right) \left( \tilde{F}_d + \frac{1}{4} (3F_1 + 2F_2 + F_3) \right) = -1.
\]

Let \( D_1, D_2 \) be respectively the \((-3)\) and \((-2)\) curves intersecting \( \tilde{H} \). Since \( \tilde{H} D_1 = \tilde{H} D_2 = 1 \), one has

\[
\tilde{H} = \rho^* H - \frac{1}{3} D_1 - \frac{1}{2} D_2,
\]

thus

\[
-1 = \tilde{H}^2 = H^2 - \frac{1}{3} - \frac{1}{2}
\]

and \( H^2 = -\frac{1}{6} \). Moreover

\[
K_W H = \left( K_Z + \frac{1}{3} D_1 \right) \left( \tilde{H} + \frac{1}{3} D_1 + \frac{1}{2} D_2 \right) = -1 + \frac{1}{3} + \frac{1}{3} - \frac{1}{3} = -\frac{2}{3}.
\]

We compute therefore

\[
c_1^2 (W) = -\frac{2}{3} + 2 \cdot 2 + \left( -2 - 1 - \frac{2}{3} \right) + \frac{1}{4} \left( 2 - \frac{1}{4} - \frac{1}{6} \right) - 4
\]

\[
+ (2 + 1 + 1) + \frac{1}{2} (1 + 0 + 0) = \frac{9}{16},
\]

thus \( c_1^2 (W) = 3c_2 (W) = \frac{9}{16} \). \( \square \)

Remark 19. In [14], Deraux obtains 4 different orbifold ball-quotient structures on surfaces birational to \( A/G_{48} \). Among these, only the fourth one, \( W' \), is invariant by the involution \( \sigma \), the obstruction being the divisor \( E \) in [14] which creates an asymmetry, unless it has weight 1. The orbifold \( W \) we just described can be seen as the quotient of \( W' \) by the involution \( \sigma \).

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