A new bound for the large sieve inequality with power moduli

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Abstract

We give a new bound for the large sieve inequality with power moduli \( q^k \) that is uniform in \( k \). The proof uses a new theorem due to T. Wooley from his work on efficient congruencing.

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1 Introduction

Let \( \{v_n\} \) denote a sequence of complex numbers, \( M, N, k \in \mathbb{N} \), and let \( Q \) be a real number \( \geq 1 \). We write \( e(\alpha) := \exp(2\pi i \alpha) \) for \( \alpha \in \mathbb{R} \).

The large sieve inequality with power moduli aims to give upper bounds for the sum

\[
\Sigma_{Q,N,k} := \sum_{q \leq Q} \sum_{1 \leq a \leq q^k \ \gcd(a,q) = 1} \sum_{M < n \leq M+N} v_n \left( \frac{a}{q^k n} \right)^2.
\]

It is known that an application of the standard large sieve inequality gives the upper bounds

\[
\Sigma_{Q,N,k} \ll_k (N + Q^{2k}) |v|^2 \quad \text{and} \quad \Sigma_{Q,N,k} \ll_k (Q N + Q^{k+1}) |v|^2,
\]

where \( |v|^2 := \sum_{M < n \leq M+N} |v_n|^2 \), and it is conjectured by L. Zhao in [5] that the upper bound

\[
\Sigma_{Q,N,k} \ll_{k,\varepsilon} |v|^2 (N + Q^{k+1})(NQ)^\varepsilon
\]

The bounds (1) verify the conjecture for $Q \leq N^{1/(2k)}$ and $Q \geq N^{1/k}$, so the problem is to prove it in the range

$$N^{1/(2k)} \leq Q \leq N^{1/k} \iff Q^k \leq N \leq Q^{2k}.$$  

Especially the cases for small $k$, namely $k = 2, 3$ are of interest and were considered in the papers [1], [2] and [5]. In this paper we investigate the problem uniform in $k$. The following nontrivial bounds are known in this case.

L. Zhao showed in [5] the bound

$$\sum_{Q,N,k} \ll_{k,\varepsilon} |v|^2(Q^{k+1} + (NQ^{1-1/\kappa} + N^{1-1/\kappa}Q^{1+k/\kappa})N^{\varepsilon}),$$  

(3)

where $\kappa := 2k - 1$.

In [2], it was shown by S. Baier and L. Zhao that

$$\sum_{Q,N,k} \ll_{k,\varepsilon} |v|^2(Q^{k+1} + N + N^{1/2+\varepsilon}Q^k)(\log \log 10NQ)^{k+1}$$  

(4)

holds, which improves Zhao’s bound (3) for $Q \ll N^{(\kappa-2)/(2(k-1)\kappa-2k)-\varepsilon}$.

In this paper we prove the following result:

**Theorem 1.** Let $\delta := (2k(k-1))^{-1}$. Then we have the bound

$$\sum_{Q,N,k} \ll_{k,\varepsilon} |v|^2(NQ)^{\varepsilon}(Q^{k+1} + Q^{1-\delta}N + Q^{1+k\delta}N^{1-\delta}).$$

This bound improves the bound (3) for all $k$ sufficiently large, and the bound (1) for $Q^k \leq N \leq Q^{2k-2+2\delta}$ and all $k \geq 3$, but it does not confirm any case of Zhao’s conjecture (2), too. Further, the result is not sufficient to give an improvement of the bound in [2] for $k = 3$, but comes near to it.

**Notation.** In the following, we suppress the dependence of the implicit constants on $k$ or $\varepsilon$ in our estimates and write simply $\ll$ for $\ll_{k,\varepsilon}$. The small value $\varepsilon > 0$ may depend on $k$ and may change its value during calculation. The symbol $\|\alpha\|$ means the distance of $\alpha$ to the nearest integer, and by $\{\alpha\} := \alpha - \lfloor\alpha\rfloor$ we denote the fractional part of $\alpha$, and by $\lfloor\alpha\rfloor$ the largest integer smaller or equal to $\alpha$. 

2
2 Lemmas

We make use of the following version of the large sieve inequality.

**Lemma 1.** Let $S$ denote a finite set of positive integers, $M, N \in \mathbb{Z}$ and let \( \{v_n\} \) be a complex sequence. Further let

\[
F := \{(a, q) \in \mathbb{Z}^2; \ q \in S, \ 0 < a < q, \ \gcd(a, q) = 1\}.
\]

Then

\[
\sum_{(a, q) \in F} \left| \sum_{M < n \leq M+N} v_n e\left(\frac{a}{q} n\right) \right|^2 \leq \left|v\right|^2 \left( 4 \sum_{q \in S} q + \max_{(b, r) \in F} \int_{1/N}^{1/2} \#F_{b, r}(x) \frac{dx}{x^2} \right), \tag{5}
\]

where

\[
F_{b, r}(x) := \{(a, q) \in F; \ \left| \frac{a}{q} - \frac{b}{r} \right| \leq x \}.
\]

**Proof:** We use Halasz-Montgomery’s inequality

\[
\sum_{r \leq R} |\langle v, \varphi_r \rangle|^2 \leq |v|^2 \cdot \max_{r \leq R} \sum_{s \leq R} |\langle \varphi_r, \varphi_s \rangle| \tag{6}
\]

that holds for any sequence \( \{\varphi_r\} \) of vectors of \( \mathbb{C}^N \), and where \(|v|^2 = \langle v, v \rangle\), and \( \langle \cdot, \cdot \rangle \) is the standard scalar product on \( \mathbb{C}^N \).

So the left hand side of (5) is

\[
\sum_{(a, q) \in F} \left| \sum_{M < n \leq M+N} v_n e\left(\frac{a}{q} n\right) \right|^2 \leq |v|^2 \max_{(b, r) \in F} \sum_{(a, q) \in F} \left| \sum_{M < n \leq M+N} e\left(\frac{a}{q} n\right) e\left(-\frac{b}{r} n\right) \right| \leq |v|^2 \max_{(b, r) \in F} \sum_{(a, q) \in F} \min \left( N, \left\| \frac{a}{q} - \frac{b}{r} \right\|^{-1} \right).
\]

Now we have to estimate

\[
\max_{(b, r) \in F} \sum_{(a, q) \in F} \min \left( N, \left\| \frac{a}{q} - \frac{b}{r} \right\|^{-1} \right). \tag{6}
\]
For this, fix \((b, r) \in \mathcal{F}\). For \(\Delta > 0\) write
\[
P(\Delta) := \#\mathcal{F}_{b, r}(\Delta).
\]
Let \(\Delta_0 := \frac{1}{N}\) and for \(L \in \mathbb{N}\) let \(h := (\frac{1}{2} - \frac{1}{N})L^{-1}\). Now let \(\Delta_i := \frac{1}{N} + hi\), so \(\Delta_L = \frac{1}{2}\). Since \(\|\alpha\| = \min\{\|\alpha\|, 1 - |\alpha|\}\) for \(-1 < \alpha < 1\), we have
\[
\sum \min \left( N, \left\| \frac{a}{q} - \frac{b}{r} \right\|^{-1} \right) 
\leq 2NP \left( \frac{1}{N} \right) + 2 \sum_{0 \leq i < L} \sum_{(a, q) \in \mathcal{F}: \Delta_i < |a/q - b/r| \leq \Delta_{i+1}} \frac{1}{\Delta_i}
\]
\[
= 2NP \left( \frac{1}{N} \right) + 2 \sum_{0 \leq i < L} \Delta_i \left( P(\Delta_{i+1}) - P(\Delta_i) \right)
\]
\[
= 2 \sum_{0 \leq i < L} \left( \frac{1}{\Delta_i} - \frac{1}{\Delta_{i+1}} \right) P(\Delta_{i+1}) + \frac{2}{\Delta_L} P(\Delta_L).
\]

The last summand is \(\leq 4 \sum_{q \in S} q\), and the sum over \(i\) approximates the Riemann-Stieltjes-integral
\[
\int_{1/N}^{1/2} P(x)dg(x) \text{ with } g(x) = -\frac{1}{x}, \quad (7)
\]
if \(L \to \infty\). Therefore the sum over \((a, q) \in \mathcal{F}\) in (6) is at most as large as the integral (7), plus \(4 \sum_{q \in S} q\).

Since \(g\) is continuously differentiable on \([\frac{1}{N}, \frac{1}{2}]\) and since \(P\) is Riemann-integrable, the integral (7) equals
\[
\int_{1/N}^{1/2} P(x)g'(x)dx = \int_{1/N}^{1/2} P(x)\frac{dx}{x^2} = \int_{1/N}^{1/2} \#\mathcal{F}_{b, r}(x)\frac{dx}{x^2}.
\]

This was to be shown. \(\square\)

Further we use the following estimate for the exponential sum occurring in the proof of Theorem [1].
Lemma 2. Let \( f(x) := \alpha x^k \in \mathbb{R}[x] \) be a monomial of degree \( k \geq 2 \), and \( S_Q := \sum_{Q < q \leq 2Q} e(f(q)) \), \( \delta := (2k(k-1))^{-1} \). Then
\[
S_Q \ll Q^{1+\varepsilon} \left( Q^{-1} + Q^{-k} \sum_{1 \leq v \leq Q} \min(Q^kv^{-1}, \|v\alpha\|^{-1}) \right)^\delta.
\]

Proof:

Suppose that \( a, q \in \mathbb{Z} \) with \((a, q) = 1\) and \( |q\alpha - a| \leq q^{-1}\).

We apply Theorem 1.5 in T. Wooley’s article \[5\] on efficient congruencing and obtain
\[
S_Q \ll Q^{1+\varepsilon}(q^{-1} + Q^{-1} + qQ^{-k})^\delta.
\]

By a standard transference principle (see Ex. 2 of section 2.8 in Vaughan’s book \[3\]), this implies that
\[
S_Q \ll Q^{1+\varepsilon}\left( (v + Q^k|v\alpha - u|)^{-1} + Q^{-1} + (v + Q^k|v\alpha - u|)Q^{-k} \right)^\delta (8)
\]
for any integers \( u, v \in \mathbb{Z} \) with \((u, v) = 1\) and \( |v\alpha - u| \leq v^{-1}\).

Now by Dirichlet’s Approximation Theorem, there exist such integers \( u, v \) with \( 1 \leq v \leq Q^{k-1} \) and \( |v\alpha - u| \leq Q^{1-k} \), for these
\[
(v + Q^k|v\alpha - u|)Q^{-k} \ll (Q^{-k} + Q)Q^{-k} \ll Q^{-1}
\]
holds. Further we get
\[
(v + Q^k|v\alpha - u|)^{-1} \ll Q^{-k} \min(Q^kv^{-1}, |v\alpha - u|^{-1}).
\]

Now if \( v > Q \), this expression is again \( \ll Q^{-1} \). If otherwise \( 1 \leq v \leq Q \), it is bounded by
\[
Q^{-k} \sum_{1 \leq v \leq Q} \min(Q^kv^{-1}, \|v\alpha\|^{-1}),
\]

since \( |v\alpha - u| \geq \|v\alpha\| \).

Hence, these estimates included in (8) show the assertion.

Lemma 3. Let \( X, Y, \alpha \in \mathbb{R}, X, Y \geq 1, \) and \( a, q \in \mathbb{Z}, \gcd(a, q) = 1, \) with \( |q\alpha - a| \leq q^{-1} \). Then
\[
\sum_{v \leq X} \min \left( XYv^{-1}, \|\alpha v\|^{-1} \right) \ll XY(q^{-1} + Y^{-1} + q(XY)^{-1}) \log(2Xq).
\]

This is Lemma 2.2 of \[3\].
3 Proof of Theorem \[\text{(1)}\]

Let \( k \in \mathbb{N} \) with \( k \geq 2 \), let \( Q \geq 1 \) and assume that the integer \( N \) is in the range \( Q^k \leq N \leq Q^{2k} \).

We apply Lemma \[\text{(1)}\] with

\[ \mathcal{F} := \{(a, q^k) \in \mathbb{Z}^2; Q < q \leq 2Q, 0 < a < q^k, \gcd(a, q) = 1\}, \]

which shows that

\[ \Sigma_{Q,N,k} \ll |v|^2 Q^e \left( Q^{k+1} + \max_{(b,r^k) \in \mathcal{F}} \int_{1/N}^{1/2} \# \mathcal{F}_{b,r^k}(x) \frac{dx}{x^2} \right), \]

since we have the admissible error \( \sum_{q \leq Q} q^k \ll Q^{k+1} \).

Now we aim to give an estimate for

\[ \max_{(b,r^k) \in \mathcal{F}} \int_{1/N}^{1/2} \# \mathcal{F}_{b,r^k}(x) \frac{dx}{x^2} \]

The integrand counts for fixed \((b, r^k) \in \mathcal{F}\) all \((a, q^k) \in \mathcal{F}\) with

\[ \left| \frac{a}{q^k} - \frac{b}{r^k} \right| \leq x. \]

So for fixed \( Q < q \leq 2Q \), we count every \( a \) with

\[ \frac{|ar^k - bq^k|}{2r^kQ^kx} \leq \frac{1}{2}. \]

Now we use the Fourier analytic method from the papers \[\text{(1), (2) and (5)}\] by Baier and Zhao. For this, consider the function

\[ \phi(x) := \left( \frac{\sin \pi x}{2x} \right)^2, \quad \phi(0) := \lim_{x \to 0} \phi(x) = \frac{\pi^2}{4}. \]

Then \( \phi(x) \geq 1 \) for \( |x| \leq 1/2 \), and the Fourier transform of \( \phi \) is

\[ \hat{\phi}(s) = \frac{\pi^2}{4} \max\{1 - |s|, 0\}. \]

For fixed \( q \), we get for the number of corresponding \( a \) the estimate

\[ \sum_{a, (a,q) \in \mathcal{A}_{b,r^k}(x)} 1 \leq \sum_{a \in \mathbb{Z}} \phi \left( \frac{ar^k - bq^k}{2r^kQ^kx} \right) = \sum_{a \in \mathbb{Z}} \int_{-\infty}^{\infty} \phi \left( \frac{sr^k - bq^k}{2r^kQ^kx} \right) e(as) ds, \]
where we applied in the last step Poisson’s summation formula. Summing up over $q$ and a linear transformation gives

$$\sum_{Q < q \leq 2Q} \sum_{a \in \mathbb{Z}} \int_{-\infty}^{\infty} \phi(v) e\left(ab \frac{q^k}{r_k}\right) e(2Qk x av) 2Q^k x dv$$

$$= \sum_{|a| \leq B} \hat{\phi}\left(\frac{a}{B}\right) B^{-1} \sum_{Q < q \leq 2Q} e\left(ab \frac{q^k}{r_k}\right),$$

where we have set $B := (2Qk x)^{-1}$, and we may assume w.l.o.g. that $B \geq 1$.

We separate the summand with $a = 0$ and get

$$\ll Q^{k+1} x + B^{-1} \sum_{1 \leq a \leq B} \left| \sum_{Q < q \leq 2Q} e\left(ab \frac{q^k}{r_k}\right) \right|.$$

The separated term $Q^{k+1} x$ leads again to the admissible contribution

$$\int_{1/N}^{1/2} \frac{Q^{k+1} dx}{x} \ll Q^{k+1+\varepsilon}.$$

Consider the monomial $f(q) := \frac{ab}{r_k} q^k$ of degree $k$ in $q$ and coefficient $\alpha := \frac{ab}{r_k} \neq 0$. It remains to give a good upper bound for the expression

$$\int_{1/N}^{1/2} B^{-1} \sum_{1 \leq a \leq B} \left| \sum_{Q \leq q < 2Q} e(f(q)) \right| \frac{dx}{x^2}. \quad (9)$$

Denote by $S_Q$ the occurring exponential sum

$$S_Q := \sum_{Q < q \leq 2Q} e(f(q)).$$

By Lemma we have

$$S_Q \ll Q^{1+\varepsilon} \left(Q^{-1} + Q^{-k} \sum_{1 \leq v \leq Q} \min(Q^k v^{-1}, \|v\alpha\|^{-1})\right)^\delta.$$

The summand $Q^{-1}$ in big parantheses provides already the contribution

$$\int_{1/N}^{1/2} Q^{1-\delta+\varepsilon} \frac{dx}{x^2} \ll Q^{1-\delta+\varepsilon} N \quad (10)$$
to (9), and it remains to consider the term with the sum over \( v \).

We estimate its contribution to \( S_Q \) as follows using Hölder’s inequality and Lemma 3. We have

\[
Q^{1+\varepsilon-k\delta} \sum_{a \leq B} \left( \sum_{v \leq Q} \min \left( Q^k v^{-1}, \left\lVert \frac{ab}{r^k v} \right\rVert^{-1} \right) \right)^{\delta}
\ll Q^{1+\varepsilon-k\delta} B^{1-\delta} \left( \sum_{\ell \leq BQ} d(\ell) \min \left( BQ^k \ell^{-1}, \left\lVert \frac{b}{r^k \ell} \right\rVert^{-1} \right) \right)^{\delta}
\ll Q^{1+\varepsilon-k\delta} B^{1-\delta} \left( (BQ^k)^{1+\varepsilon} (r^{-k} + Q^{1-k} + r^k (BQ^k)^{-1}) \right)^{\delta}
\ll BQ^{1+\varepsilon} (Q^{1-k} + B^{-1})^{\delta}.
\]

The contribution to (9) becomes

\[
\ll Q^{1+\varepsilon+(1-k)\delta} N + Q^{1+\varepsilon} \int_{1/N}^{1/2} B^{-\delta} \frac{dx}{x^2}
\ll Q^{1-(k-1)\delta+\varepsilon} N + Q^{1+\varepsilon} \int_{1/N}^{1/2} Q^k x^\delta \frac{dx}{x^2}
\ll Q^{1-(k-1)\delta+\varepsilon} N + Q^{1+k\delta+\varepsilon} N^{1-\delta}.
\]

The first term can be estimated by the bound (10), since \( k \geq 2 \). We obtain the stated bound of Theorem 1.

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