Generalized Wick Theorems in Conformal Field Theory and the Borcherds Identity

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Abstract

As the missing counterpart of the well-known generalized Wick theorem for interacting fields in two dimensional conformal field theory, we present a new formula for the operator product expansion of a normally ordered operator and a single operator on its right hand. Quite similar to the original Wick theorem for the opposite order operator product, it expresses the contraction i.e. the singular part of the operator product expansion as a contour integral of only two terms, each of which is a product of a contraction and a single operator. We discuss the relation between these formulas and the Borcherds identity satisfied by the quantum fields associated with the theory of vertex algebras. A derivation of these formulas by an analytic method is also presented. The validity of our new formula is illustrated by a few examples including the Sugawara construction of the energy momentum tensor for the quantized currents of affine Lie algebras.

1 Introduction

More than thirty years two dimensional (2D) conformal field theory (CFT) has been widely accepted as a distinguished theory for giving mathematical foundations of string theory and for describing 2D critical phenomena in statistical physics neatly. Among many beautiful formulas in CFT the following one is known as the generalized Wick theorem for interacting fields [2, 4], and has been used extensively in many literatures:

$$\left\langle A(z)(BC)C(w) \right\rangle = \frac{1}{2\pi \sqrt{-1}} \oint_{C_w} \frac{dx}{x-w} \left\{ \hat{A}(x)\hat{B}(x)C(w) + B(x)\hat{A}(z)C(w) \right\}. \quad (1)$$

Here, $A(z)$ etc. are operators for chiral conformal fields, $(BC)(w) = :B(w)C(w)$: denotes the normally ordered product, $A(x)\hat{B}(x)$ denotes the contraction, i.e. the singular part of the operator product expansion (OPE), and $C_w$ is a contour encircling the point $w$ with an infinitesimal radius. Also, the radial orderings of the operators are implicitly assumed.

It is a natural question to ask whether an analogous expression exists if we contract with the normally ordered operator on the left. Actually in [2] they claimed its existence but did not present it there, and as far as the authors know no such expression has been written in any literature. We emphasize that it is a non-trivial question because we are generally dealing with interacting fields rather than free fields. The purpose of this paper is to answer this question by deriving a new formula:

$$\left\langle (AB)(z)C(w) \right\rangle = \frac{1}{2\pi \sqrt{-1}} \oint_{C_w} \frac{dx}{z-x} \left\{ :A(x)\hat{B}(x)C(w): + B(x)\hat{A}(x)C(w) \right\}. \quad (2)$$

Here we note that the expansion coefficients of $\hat{B}(x)C(w)$ and $\hat{A}(x)C(w)$ are fields at the point $w$, hence it makes sense to consider OPEs of $A(x)$ or $B(x)$ and them. The normal ordering : :
for the first term is to subtract the singular terms from such OPEs. We also discuss the relation between these formulas and the Borcherds identity that is an axiom for a vertex algebra [3]. The main results of this paper are Theorems [2] and [3].

This paper is organized as follows: In Sect. 2 we review an algebraic formulation of 2D chiral quantum fields based on Matsuo and Nagatomo [11]. The purpose of this section is to introduce several notions for giving a mathematically rigorous formulation and a proof of the generalized Wick theorems [1] and [2]. The notions include normally ordered product, residue product, OPE, contraction, locality, and so on. In particular we introduce the Borcherds identity satisfied by arbitrary three fields. In Sect. 3 we show that the generalized Wick theorems are equivalent to special cases of the Borcherds identity. So far, the contour integrals of the operator valued functions appearing in [1] and [2] are interpreted as formal symbols for algebraic manipulations, rather than as rigorous analytic calculations. In contrast, in Sect. 4 we consider matrix elements of the product of the operators and interpret these integrals as analytic calculations. The main advantage of this analytic method is that it enables us to derive the Wick theorems in a heuristic way by making use of contour deformations. In Sect. 5 the validity of our new formula is illustrated by a few examples of explicit calculations of OPE. Finally, we give several discussions in Sect. 6.

2 An Algebraic Formulation of Two-dimensional Chiral Quantum Fields and Their Operator Product Expansions

2.1 Formal series, fields and their residue products

In order to express the statement of the Wick theorems in a mathematically rigorous manner, we want to present an algebraic formulation of 2D chiral quantum fields. Though similar formulations are provided in many literatures [6, 7, 8, 12], we adopt the one by Matsuo and Nagatomo [11] which we shall briefly review in this section.

First we introduce the notions of a field and a normally ordered product. Let

$$A(z) = \sum_{n \in \mathbb{Z}} A_n z^{-n-1}$$

be a formal series in which the coefficients $A_n$ are linear transformations on some vector space $M$. The set of all such series is denoted by $(\text{End} M)[[z, z^{-1}]]$. We call $A(z)$ a field (of one variable) if for any $b \in M$ there exists $n_0 \in \mathbb{Z}$ such that $A_n b = 0$ for all $n \geq n_0$. Also for any set of linear transformations $\{A_{p,q}\}_{p,q \in \mathbb{Z}}$ we consider the following formal series

$$A(y, z) = \sum_{p,q \in \mathbb{Z}} A_{p,q} y^{-p-1} z^{-q-1},$$

and denote the set of all such series by $(\text{End} M)[[y, y^{-1}, z, z^{-1}]]$. We call $A(y, z)$ a field of two variables if for any $v \in M$ there exist $p_0, q_0 \in \mathbb{Z}$ such that if $p \geq p_0$ or $q \geq q_0$ then $A_{p,q} v = 0$ (namely, if $p \geq p_0$ then $A_{p,q} v = 0$, and if $q \geq q_0$ then $A_{p,q} v = 0$). If $A(y, z)$ is a field of two variables, then $A(z, z)$ is a field of one variable. Actually we have

$$A(z, z) = \sum_{n \in \mathbb{Z}} \left( \sum_{p \in \mathbb{Z}} A_{p,n-1-p} \right) z^{-n-1},$$

that makes sense as a linear transformation on $M$. This is because for any $v \in M$ we have $A_{p,n-1-p} v = 0$ for large enough $|p|$, and for any $v \in M$ and $p \in \mathbb{Z}$ we have $\sum_{p \in \mathbb{Z}} A_{p,n-1-p} v = 0$ for large enough $n$ because $p \geq p_0$ or $n - 1 - p \geq q_0$ is then satisfied.
Even if both \(A(y)\) and \(B(z)\) are fields, their product \(A(y)B(z)\) is not necessarily a field. However we have:

**Definition 1** Given two series \(A(y)\) and \(B(z)\), their **normally ordered product** is

\[
:A(y)B(z)\:: = A(y)_- B(z) + B(z)A(y)_+ ,
\]

where \(A(y)_+ = \sum_{n \geq 0} A_n y^{-n-1}\), \(A(y)_- = \sum_{n < 0} A_n y^{-n-1}\).

And one easily proves that it is a field of two variables if both \(A(z)\) and \(B(z)\) are fields. Hence

\[
(AB)(z) = :A(z)B(z):,
\]

is a field for any pair of fields \(A(z)\) and \(B(z)\). We call both \(A(y)B(z)\) and \(A(z)B(z)\) normally ordered products.

Now we introduce the notion of a residue product. It is a generalization of the normally ordered product and plays an essential role in this paper.

**Definition 2** (III, Definition 1.4.1.) Given two series \(A(y)\) and \(B(z)\), their **\(m\)-th residue product** \((m \in \mathbb{Z})\) is

\[
A(z)_{(m)}B(z) = \text{Res}_{y=0} A(y)B(z)(y - z)^m |_{|y|>|z|} - \text{Res}_{y=0} B(z)A(y)(y - z)^m |_{|y|<|z|},
\]

where

\[
(y - z)^m |_{|y|>|z|} = \sum_{i=0}^{\infty} (-1)^i \binom{m}{i} y^{m-i} z^i,
\]

\[
(y - z)^m |_{|y|<|z|} = \sum_{i=0}^{\infty} (-1)^{m+i} \binom{m}{i} y^i z^{m-i}.
\]

**Remark 1** The \((y - z)^m |_{|y|>|z|}\) and \((y - z)^m |_{|y|<|z|}\) are formal series obtained by expanding the rational function \((y - z)^m\) into those convergent in the region \(|y| > |z|\) and \(|z| > |y|\) respectively.

One can prove that if both \(A(z)\) and \(B(z)\) are fields, then each term of the right hand side of (6) becomes a field and hence so is \(A(z)_{(m)}B(z)\) [III]. In particular we have

\[
A(z)_{(-1)}B(z) = :A(z)B(z):.
\]

### 2.2 Operator product expansion

Now we present a mathematical formulation of the operator product expansion of the quantum fields. As we will see, the Wick formula [II] is proved to be equivalent to a special case of the Borcherds identity without the assumption of locality (Theorem [II]). So, in this subsection we introduce the notion of OPE without it [II]. When \(m \geq 0\), we have \((y - z)^m |_{|y|>|z|} = (y - z)^m |_{|y|<|z|} = \sum_{i=0}^{m} \binom{m}{i} y^i (-z)^{m-i}\), hence (6) is written as

\[
A(z)_{(m)}B(z) = \text{Res}_{y=0} [A(y), B(z)](y - z)^m
= \sum_{i=0}^{m} \binom{m}{i} \text{Res}_{y=0} A(y)y^i (-z)^{m-i}, B(z)]
= \sum_{i=0}^{m} \binom{m}{i} (-z)^{m-i}[A_i, B(z)],
\]

where \(A_i\) and \(B_i\) are basis of \(\mathcal{H}\) and \(\mathcal{K}\) respectively.
Therefore

\[
\frac{A(z)(m)B(z)}{z^m} = \sum_{i=0}^{m} \binom{m}{i} (-1)^{m-i} \left[ A_i, B(z) \right] \frac{1}{z^i}.
\] (7)

This relation is inverted as (e.g. [13], p. 66)

\[
\frac{[A_m, B(z)]}{z^m} = \sum_{i=0}^{m} \binom{m}{i} \frac{A(z)(i)B(z)}{z^i}.
\] (8)

By this relation we have

\[
[A(y)_+, B(z)] = \sum_{m=0}^{\infty} y^{-m-1} [A_m, B(z)]
\]

\[
= \sum_{m=0}^{\infty} y^{-m-1} \sum_{i=0}^{m} \binom{m}{i} A(z)(i)B(z)z^{m-i}
\]

\[
= \sum_{i=0}^{\infty} A(z)(i)B(z) \sum_{m=i}^{\infty} \binom{m}{i} y^{-m-1} z^{m-i},
\]

in which the inner summation in the last expression can be written as

\[
\sum_{m=i}^{\infty} \binom{m}{i} y^{-m-1} z^{m-i} = \partial_z^{(i)} \sum_{m=0}^{\infty} y^{-m-1} z^m = (y - z)^{-i-1} \left|_{y > |z|} \right.
\]

Here we used the notation \( \partial_z^{(i)} = \frac{1}{i!} \partial_z^i \). Therefore we have:

**Proposition 1** ([11], Remark 2.2.2; [10], Proposition 2.3) For any series \( A(y) \) and \( B(z) \) the following relation holds.

\[
A(y)B(z) = \sum_{i=0}^{\infty} A(z)(i)B(z) (y - z)^{-i-1} \left|_{y > |z|} \right. + :A(y)B(z):
\] (9)

This is an identity in \( (\text{End} \ M)[[y, y^{-1}, z, z^{-1}]] \). Based on this expression it is conventional to introduce the following:

**Definition 3** For any series \( A(y) \) and \( B(z) \) their **contraction** is

\[
\overline{A(y)B(z)} = \overline{A(y)B(z)} = \sum_{i=0}^{\infty} \overline{A(z)(i)B(z)} \frac{1}{(y - z)^{i+1}}.
\] (10)

It is an element of \( (\text{End} \ M)[[z, z^{-1}]][[(y - z)^{-1}]] \). In [7, 11] the correspondence \( A(y)B(z) \sim \overline{A(y)B(z)} \) is called the OPE. Although this is quite an appropriate definition of OPE as a concept of physics, in this paper we use the following:

**Definition 4** Given two series \( A(y) \) and \( B(z) \), their **OPE** is the sum of their contraction and their normally ordered product, which we denote by

\[
R(A(y)B(z)) := \overline{A(y)B(z)} + :A(y)B(z):.
\] (11)

It is an element of \( (\text{End} \ M)[[y, y^{-1}, z, z^{-1}]][[(y - z)^{-1}]] \). The notation \( R \) stands for the radial ordering in the case of mutually local fields. See Remark 2 below.
2.3 Locality

We now review the notion of locality \[11\]. This notion is important for us because the Wick formula \[2\] is proved to be equivalent to a special case of the Borcherds identity with the assumption of locality (Theorem \[2\]). Let \(A(y), B(z)\) be elements of \((\text{End} M)[[y, y^{-1}]])\) and \((\text{End} M)[[z, z^{-1}]])\) respectively. Then their products \(A(y)B(z)\) and \(B(z)A(y)\) are generally two different elements of the set \((\text{End} M)[[y, y^{-1}, z, z^{-1}]])\). In this case, if there exists a non-negative integer \(m\) and the relation

\[A(y)B(z)(y - z)^m = B(z)A(y)(y - z)^m\]

(12)
is satisfied, then \(A(z)\) and \(B(z)\) are called mutually local. If \(m\) is the smallest non-negative integer such that the equality (12) holds, then we say that the order of the locality of \(B\) (Theorem \[2\]). Let the condition is obviously written as \([A(y), B(z)] := A(y)B(z) - B(z)A(y)\), the condition is obviously written as \([A(y), B(z)](y - z)^m = 0\). It should be emphasized that this condition does not necessarily implies \([A(y), B(z)] = 0\) because \(A(y), B(z)\) are not analytic functions but just formal series. However we have the following:

Lemma 1 (\[11\], Lemma 1.1.1.) Let \(a(y, z)\) be a series with only finitely many terms of negative or positive degree in \(y\) or \(z\). If \(a(y, z)\) satisfies \(a(y, z)(y - z)^m = 0\) for some nonnegative integer \(m\), then \(a(y, z) = 0\).

The following proposition on the OPE with the assumption of locality will be used in the arguments in Sect. \[4\].

Proposition 2 (\[11\], Theorem 2.2.1) Let \(A(y)\) and \(B(z)\) be series on a vector space. They are mutually local if and only if both

\[A(y)B(z) = \sum_{i=0}^{m-1} A(z)_{(i)}B(z)(y - z)^{-i-1}|_{|y|>|z|} + :A(y)B(z):,\]

(13)

and

\[B(z)A(y) = \sum_{i=0}^{m-1} A(z)_{(i)}B(z)(y - z)^{-i-1}|_{|y|<|z|} + :A(y)B(z):,\]

(14)

hold for some \(m \in \mathbb{N}\).

Proof. If we multiply \((y - z)^m\) to (13) and (14), then their right hand sides will be equal to each other. Hence if these two equations are satisfied, then \(A(z)\) and \(B(z)\) are mutually local. Conversely, suppose these fields are mutually local, and let \(m\) be the order of locality. Then by the definition of the residue product \[9\] we have \(A(z)_{(i)}B(z) = 0\) for \(i \geq m\). Hence (13) follows from Proposition \[11\].

On the other hand, by the assumption of locality we obtain the following relation:

\[\left\{ [A(y)\_\_+, B(z)] - \sum_{i=0}^{m-1} A(z)_{(i)}B(z)(y - z)^{-i-1} |_{|y|>|z|} \right\}(y - z)^m = \left\{ -[A(y)\_\_-, B(z)] - \sum_{i=0}^{m-1} A(z)_{(i)}B(z)(y - z)^{-i-1} |_{|y|<|z|} \right\}(y - z)^m.\]

(15)

Since (13) has already been proved, one sees that the left hand side of this expression vanishes. Hence (14) follows by using Lemma \[1\] because there is no term of negative degree in \(y\) between the braces in the right hand side of (15).
Remark 2. In this section the operators are defined as formal series and we are not allowed to substitute any real or complex values for the symbols $y$ and $z$. However, in Sect. 4 we will introduce an analytic formulation in which these variables can take values in complex numbers. Then, from Proposition 2 we see that the symbol $R$ in Definition 4 can be interpreted as the radial ordering of the operators. This means that in any correlation functions involving mutually local operators $A(y), B(z)$ we can replace $R(A(y)B(z))$ by $A(y)B(z)$ (resp. $B(z)A(y)$) if the condition $|y| > |z|$ (resp. $|y| < |z|$) is satisfied.

We now present the notion of locality between more than two fields. Say series $A_1(z), \ldots, A_\ell(z)$ are mutually local if all the distinct pairs $A_i(z), A_j(z), (i \neq j)$ are mutually local. The following proposition is known as the Dong’s lemma [7].

Proposition 3 ([11], Proposition 2.1.5.) If $A(z), B(z)$ and $C(z)$ are mutually local fields, then $A(z)^{\langle m \rangle}B(z)$ and $C(z)$ are local.

This and the following results will be used in Sect. 4.

Proposition 4 ([11], Proposition 2.1.6.) Let $A_i(z) = \sum_{n \in \mathbb{Z}} A_{i,n} z^{-n-1}$ be mutually local fields for $1 \leq i \leq \ell$. Then for any $u \in M$ the condition

$$A_{p_1} \cdots A_{p_\ell} u = 0,$$

is satisfied for any sufficiently large $p_1 + \cdots + p_\ell \in \mathbb{Z}$.

2.4 Differentiation and Taylor expansion

We now introduce some formulas on the differentiation of fields. For the series $A(z)$ defined as (3) we define its differentiation $\partial_z A(z)$ as

$$\partial_z A(z) = \sum_{n \in \mathbb{Z}} (n-1)A_{n} z^{-n-2}. \quad (16)$$

If $A(z)$ is a field, so is $\partial_z A(z)$. Also, the differentiation of a field of two variables is given in a similar way. From (16) and (17) one easily proves the following formula

$$(\partial_z A(z))_{(i)} B(z) = -i A(z)^{(i)}_{(i-1)} B(z). \quad (17)$$

By repeated use of (17) we obtain

$$\partial^{(j)} A(z) B(z) = A(z)^{(j)}_{(-j-1)} B(z), \quad (18)$$

for any non-negative integer $j$. Let $I(z)$ be the identity field on $M$: That is an operator whose only non-zero term is the constant term which is the identity operator on $M$. Then by (18) we have:

Proposition 5 ([11], Proposition 1.4.4.)

$$A(z)^{\langle m \rangle} I(z) = \begin{cases} 0, & (m \geq 0), \\ \partial^{-m-1} A(z), & (m \leq -1). \end{cases} \quad (19)$$

This relation will be used to derive the skew symmetry from the Borcherds identity. Also, we have the following Taylor’s formula which will be used in the proof of Theorem 2 as well as in some calculations in Sect. 5.
Proposition 6 ([11], Proposition 2.2.4.) For any field $A(y, z)$ and any positive integer $N$, there exists a unique field $R_N(y, z)$ such that the following relation is satisfied:

$$A(y, z) = \sum_{i=0}^{N-1} \partial^{(i)}_y A(y, z)|_{y=z} (y-z)^i + R_N(y, z)(y-z)^N.$$  

In particular for any field $A(z)$ and any positive integer $N$, there exists a unique field $R_N(y, z)$ such that the following relation is satisfied:

$$A(y) = \sum_{i=0}^{N-1} \partial^{(i)}_z A(z)(y-z)^i + R_N(y, z)(y-z)^N.$$  

Indeed the latter statement follows from the former, since by setting $A_{p,-1} = A_p, A_{p,q}(\neq -1) = 0$ in (4) any field of one variable can be obtained as a special case of a field of two variables.

2.5 Borcherds identity and skew symmetry

Now we introduce a remarkable identity that is relevant for our discussion on the Wick theorems. As one of the main results of Matsuo and Nagatomo [11], they derived an identity that comprises three infinite sums of nested commutators of arbitrary three fields with some binomial expansion factors, which is a consequence of the usual Jacobi identity. By taking the residue on both sides of this identity they proved the following identity satisfied by arbitrary three fields with respect to the residue products.

Proposition 7 ([11], Corollaries 3.2.2. and 3.4.2.) Let $A(w), B(w)$ and $C(w)$ be fields on a vector space. Then, for any $p, r \in \mathbb{N}$ and any $q \in \mathbb{Z}$,

$$\sum_{i=0}^{\infty} \binom{p}{i} (A(w)_{(r+i)} B(w))_{(p+q-i)} C(w) = \sum_{i=0}^{\infty} (-1)^i \binom{p}{i} (A(w)_{(p+r-i)} B(w)_{(q+i)} C(w) - (-1)^r B(w)_{(q+r-i)} (A(w)_{(p+i)} C(w))).$$

Moreover, if $A(w), B(w), C(w)$ are mutually local, then this relation is satisfied for any $p, q, r \in \mathbb{Z}$.

This relation (20) is called the Borcherds identity [7], since it is equivalent to one of the axioms for a vertex algebra introduced by Borcherds [3].

We also introduce an identity that will be used in Sect. 5. As a special case of the Borcherds identity, one can derive the following relation which is known as the skew symmetry.

Proposition 8 ([11], Proposition 3.5.2.)

$$B(z)_{(m)} A(z) = \sum_{i=0}^{\infty} (-1)^{m+i+1} \partial^{(i)}_z (A(z)_{(m+i)} B(z)).$$

Indeed this relation is obtained by setting $p = -1, q = 0, r = m, B(z) = I(z)$ in (20) and by using Proposition 5.
3 Wick Theorems as Special Cases of the Borcherds Identity

3.1 Generalized Wick theorem for $A(z)(BC)(w)$

In this section we show that the Wick theorems (1) and (2) are equivalent to special cases of the Borcherds identity. For this purpose we first prepare some notations. By generalizing our notation in Definition 4 we define the OPEs of an operator and a contraction as

\[
R(A(z)B(x)C(w)) = (A(z)B(x))C(w) + :A(z)B(x)C(w):, \quad (22)
\]

\[
R(B(x)A(z)C(w)) = B(x)(A(z)C(w)) + :B(x)A(z)C(w):, \quad (23)
\]

where

\[
(A(z)B(x))C(w) = \sum_{i=0}^{\infty} \frac{(A(x)B(x))C(w)}{(z-x)^{i+1}} = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \frac{(A(w)(i)B(w)(j))C(w)}{(z-x)^{i+1}(x-w)^{j+1}}, \quad (24)
\]

\[
:B(z)B(x)C(w): = \sum_{i=0}^{\infty} \frac{(A(x)(i)B(x))C(w)}{(z-x)^{i+1}}, \quad (25)
\]

\[
B(x)(A(z)C(w)) = \sum_{i=0}^{\infty} \frac{B(x)(A(w)(i))C(w)}{(z-w)^{i+1}} = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \frac{B(w)(j)(A(w)(i))C(w)}{(x-w)^{j+1}(z-w)^{i+1}}, \quad (26)
\]

\[
:B(x)A(z)C(w): = \sum_{i=0}^{\infty} \frac{B(x)(A(w)(i))C(w)}{(z-w)^{i+1}}. \quad (27)
\]

Before showing our theorems, making a remark on the usage of the contour integrals is in order. So far, the operators are defined as formal series and we are not allowed to substitute any real or complex numbers for the symbols $w, x$ and $z$. Nevertheless, from now on we try to do contour integrals of the operators with respect to these variables. However, this causes no problems because they are simply interpreted as formal symbols for an algebraic manipulation that uses the following formula:

\[
\frac{1}{2\pi \sqrt{-1}} \oint_{C_w} \frac{dx}{(z-x)^{m}(x-w)^{n}} = \left( \frac{m+n-2}{n-1} \right) \frac{1}{(z-w)^{m+n-1}}. \quad (28)
\]

Now we present the well-known generalized Wick theorem in our formulation:

**Theorem 1**

\[
A(z)(BC)(w) = \frac{1}{2\pi \sqrt{-1}} \oint_{C_w} \frac{dx}{x-w} \left\{ R(A(z)B(x)C(w)) + R(B(x)A(z)C(w)) \right\}. \quad (29)
\]

Here $R$ represents the OPE given by (22) and (23).

**Proof.** By using the formula (28) one can rewrite (29) as

\[
\sum_{p=0}^{\infty} \frac{A(w)(i)(B(w)(-1))C(w)}{(z-w)^{p+1}} = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \left( \frac{A(w)(i)(B(w)(j))C(w)}{(z-w)^{i+j+1}} \right)
\]

\[
\sum_{p=0}^{\infty} \frac{B(w)(-1)(A(w)(p))C(w)}{(z-w)^{p+1}},
\]

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which is equivalent to

\[
A(w)(p)(B(w)(-1)C(w)) = \sum_{i=0}^{p} \left( \frac{p!}{i!} \right) (A(w)(i)B(w))(p-i-1)C(w)
\]

\[
+ B(w)(-1)(A(w)(p)C(w)),
\]

for any \( p \in \mathbb{N} \). Now one finds that this is nothing but a special case of the Borcherds identity for non-local fields given by Proposition 7. In fact, by setting \( r = 0 \) and \( q = -1 \) in (20), we obtain the above result. \( \square \)

**Remark 3** In [7] Kac presented a proof of this special case of the Borcherds identity. He called it the non-commutative Wick formula and pointed out its equivalence to the formula in [2].

### 3.2 Generalized Wick theorem for \((AB)(z)C(w)\)

Now we present the main result of this paper in algebraic formulation.

**Theorem 2**

\[
(AB)(z)C(w) = \frac{1}{2\pi \sqrt{-1}} \oint_{C_w} \frac{dx}{z-x} \left\{ :A(x)B(x)C(w): + R(B(x)A(x)C(w)) \right\}.
\]

(31)

Here \( R \) represents the OPE given by (23).

**Proof.** From (27) we have

\[
:A(x)B(x)C(w): = \sum_{i=0}^{\infty} :A(x)(B(w)(i)C(w)): \frac{1}{(x-w)^{i+1}}.
\]

(32)

where we have used Taylor’s formula (Proposition 6) for \( A(x) \) around \( w \), and then used the formula (18). For the sake of simplicity we formally expanded \( A(x) \) up to infinite order, but this does not cause any problems because the regular terms will be dropped by integration. Actually by using (28) we obtain

\[
\frac{1}{2\pi \sqrt{-1}} \oint_{C_w} \frac{dx}{z-x} \left\{ :A(x)B(x)C(w): \right\}
\]

\[
= \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \frac{A(w)(-j-1)(B(w)(i)C(w))}{(z-w)^{i+j+1}},
\]

(33)

From (23) we have

\[
R(B(x)A(x)C(w)) = B(x)(A(x)C(w)) + :B(x)A(x)C(w):,
\]

(34)

where

\[
B(x)(A(x)C(x)) = \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \frac{B(w)(i)(A(w)(j)C(w))}{(x-w)^{i+j+2}}.
\]

(35)
Hence by using (25) we obtain

\[
\frac{1}{2\pi\sqrt{-1}} \oint_{C_w} \frac{dx}{z - x} \left\{ R(B(x)A(x)C(w)) \right\} \\
= \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \frac{B(w)_{(j)}(A(w)_{(j)}C(w))}{(z - w)^{i+j+2}} + \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \frac{B(w)_{(j-1)}(A(w)_{(j)}C(w))}{(z - w)^{i+j+1}} \\
= \sum_{i=0}^{\infty} \sum_{j=-i-1}^{\infty} \frac{B(w)_{(j)}(A(w)_{(j)}C(w))}{(z - w)^{i+j+2}} \\
= \sum_{q=0}^{\infty} \sum_{i=0}^{\infty} \frac{B(w)_{(q-i-1)}(A(w)_{(i)}C(w))}{(z - w)^{q+1}}.
\]  

From (33) and (36) we see that (31) is equivalent to saying that the relation

\[
(A(w)_{(-1)}B(w))_{(q)}C(w) = \sum_{i=0}^{\infty} (A(w)_{(-1)}(B(w)_{(q+i)}C(w))) \\
+ B(w)_{(q-i-1)}(A(w)_{(i)}C(w)))
\]

is satisfied for any \( q \in \mathbb{N} \). Now one finds that this is nothing but a special case of the Borcherds identity for local fields given by Proposition 7. In fact, by setting \( p = 0 \) and \( r = -1 \) in (20), we obtain the above result.

\[ \square \]

4 Derivation of the Formulas by an Analytic Method

4.1 Admissible fields and the restricted dual space

In this section we derive the generalized Wick formulas by an analytic method. Our argument is based on the formulation used in Appendix B of [11], that we shall review briefly in this subsection. Besides the locality, we assume one additional condition of the fields which is called the admissibility.

Let \( M \) be a \( \mathbb{C} \)-vector space and \( M^* \) the dual of \( M \). We denote the canonical paring by

\[ \langle \ , \ \rangle : M^* \times M \to \mathbb{C}. \]

In other words, \( M^* \) is the set of all linear functions on \( M \) which is also a \( \mathbb{C} \)-vector space, and we write \( \varphi(v) = \langle \varphi, v \rangle \) for \( v \in M, \varphi \in M^* \). A subspace \( M^\vee \subset M^* \) is called nondegenerate if there is no non-zero element \( u \in M \) that satisfies the condition \( (M^\vee, u) = 0 \). Then we have:

**Lemma 2** ([11], Lemma B.1.1) Let \( N_m (m \in \mathbb{Z}) \) be subspaces of \( M \) such that \( \cdots \subset N_m \subset N_{m+1} \subset \cdots \) and \( \bigcap_{m \in \mathbb{Z}} \overline{N}_m = \{0\} \), where \( \overline{N}_m \) is the closure of \( N_m \). Then there exists a nondegenerate subspace \( M^\vee \subset M^* \) such that for any \( v^\vee \in M^\vee \), there exists an \( m \in \mathbb{Z} \) such that \( \langle v^\vee, N_m \rangle = 0 \).

**Proof.** Set \( N = \bigcup_{m \in \mathbb{Z}} N_m \) and consider

\[ N_m^\vee = \{ v^\vee \in N^* | \langle v^\vee, N_m \rangle = 0 \} \subset N^*. \]

Then \( N_m^\vee (m \in \mathbb{Z}) \) are subspaces of \( N^* \) such that \( \cdots \subset N_{m+1}^\vee \subset N_m^\vee \subset \cdots \). Take a complement \( P \) of \( N \) in \( M \) so that \( M = N \oplus P \). Since \( (\bigcup_{m \in \mathbb{Z}} N_m^\vee) \) and \( P^* \) are subspaces of \( M^* \), so is their sum space which is defined as

\[ \left( \bigcup_{m \in \mathbb{Z}} N_m^\vee \right) + P^* = \left\{ v_1 + v_2 \in M^* | v_1 \in \left( \bigcup_{m \in \mathbb{Z}} N_m^\vee \right), v_2 \in P^* \right\}. \]
Since \((\bigcup_{m\in\mathbb{Z}} N_m^\vee) \subset N^*\) and \((N \oplus P)^* \cong N^* \oplus P^*\), this sum is actually a direct sum which we denote by

\[ M^\vee = \left( \bigcup_{m\in\mathbb{Z}} N_m^\vee \right) \oplus P^* \subset M^*. \]

Then this \(M^\vee\) has the desired properties as one sees in the follows.

Given any \(v^\vee \in M^\vee\), it can be uniquely written as \(v^\vee = v_1 + v_2\), \(v_1 \in (\bigcup_{m\in\mathbb{Z}} N_m^\vee)\) and \(v_2 \in P^*\). The former implies \(\langle v_1, N_m \rangle = 0\) for some \(m \in \mathbb{Z}\), and the latter implies \(\langle v_2, N_m \rangle = 0\) for any \(m \in \mathbb{Z}\) because \(N \cap P = \{0\}\). Hence for any \(v^\vee \in M^\vee\), there exists an \(m \in \mathbb{Z}\) such that \(\langle v^\vee, N_m \rangle = 0\).

In order to prove the nondegeneracy of \(M^\vee\), suppose \(\langle M^\vee, u \rangle = 0\). Recall that any \(u \in M = N \oplus P\) can be uniquely written as \(u = u_1 + u_2\), \(u_1 \in N\) and \(u_2 \in P\). The nondegeneracy of \(P^*\) with respect to \(P\) requires \(u_2 = 0\), because otherwise there exists \(w \in P^* \subset M^\vee\) such that \(\langle w, u \rangle \neq 0\) which contradicts to the assumption. Now \(\langle M^\vee, u_1 \rangle = 0\) requires \(u_1\) to be an element of \((N_m^\vee)^\vee\) for all \(m \in \mathbb{Z}\) where

\[ (N_m^\vee)^\vee = \{ v \in N | \langle N_m^\vee, v \rangle = 0 \} \subset N. \]

Then, since \((N_m^\vee)^\vee = N_m\) and \(\bigcap_{m\in\mathbb{Z}} N_m = \{0\}\), we have \(u_1 = 0\).

Now we explain the notion of admissibility. Let

\[ A^i(z) = \sum_{n\in\mathbb{Z}} A^i_n z^{-n-1} \quad (1 \leq i \leq \ell) \quad (38) \]

be formal series in \((\text{End } M)[[z, z^{-1}]]\). We denote by \(S_\ell\) the symmetric group acting on \(\{1, \ldots, \ell\}\). For \(u \in M, m \in \mathbb{Z}\) we define

\[ N_{u,m} = \sum_{p_1 + \cdots + p_\ell = -\infty} \sum_{\sigma \in S_\ell} \mathbb{C} A_{p_1}^{\sigma(1)} \cdots A_{p_\ell}^{\sigma(\ell)} u. \]

Here \(\mathbb{C} A_{p_1}^{\sigma(1)} \cdots A_{p_\ell}^{\sigma(\ell)} u\) is the one-dimensional subspace of \(M\), and as a sum of the subspaces of a vector space, so is \(N_{u,m}\). Clearly they satisfy the condition \(\cdots \subset N_{u,m} \subset N_{u,m+1} \subset \cdots\). We say that the series \((38)\) are admissible if the condition \(\bigcap_{m\in\mathbb{Z}} N_{u,m} = \{0\}\) is satisfied for any \(u \in M\).

**Remark 4** In order to assure the soundness of our arguments, we put a slightly stronger assumption and condition than those in \([11]\) in Lemma 2 and in the definition of admissibility.

Suppose that the series \((38)\) are admissible. Then from Lemma 2 we observe that for any \(u \in M\) there exists a nondegenerate subspace \(M_u^\vee \subset M^*\) such that for any \(v^\vee \in M_u^\vee, \sigma \in S_\ell\) the condition

\[ \langle v^\vee, A_{p_1}^{\sigma(1)} \cdots A_{p_\ell}^{\sigma(\ell)} u \rangle = 0, \]

is satisfied for any sufficiently small \(p_1 + \cdots + p_\ell \in \mathbb{Z}\). Such an \(M_u^\vee \subset M^*\) is called a restricted dual space compatible with \(A^1(z), \ldots, A^\ell(z)\) with respect to \(u \in M\).

### 4.2 Consequences of locality and admissibility

In order to derive the generalized Wick formulas analytically, we have to require the matrix elements

\[ \langle v^\vee, A^1(z_1) \cdots A^\ell(z_\ell) u \rangle, \]

which we call the correlation functions, to satisfy a certain property with respect to the permutations of the indices \(1, \ldots, \ell\). Actually only the cases \(\ell = 2, 3\) are relevant for our task. For this purpose we introduce:
**Proposition 9** Suppose \( \ell = 2 \) or 3 and let \( A^1(z), \ldots, A^\ell(z) \) be admissible fields. If they are local, then for any \( u \in M_v \), \( v^\vee \in M_u^* \), and \( \sigma \in S_\ell \), the correlation functions
\[
\langle v^\vee, A^{\sigma(1)}(z_{\sigma(1)}) \cdots A^{\sigma(\ell)}(z_{\sigma(\ell)})u \rangle
\]
are the expansions of a rational function of the form
\[
P(z_1, \ldots, z_\ell) \prod_{i<j} (z_i - z_j)^{n_{ij}}, \quad P(z_1, \ldots, z_\ell) \in \mathbb{C}[z_1, z_1^{-1}, \ldots, z_\ell, z_\ell^{-1}],
\]
that is common to all \( \sigma \in S_\ell \), into the regions satisfying \( |z_{\sigma(1)}| > \cdots > |z_{\sigma(\ell)}| \).

**Proof.** One easily sees that \( \langle v^\vee, A^{\sigma(1)}(z_{\sigma(1)}) \cdots A^{\sigma(\ell)}(z_{\sigma(\ell)})u \rangle \) has only finitely many terms of positive degree in \( z_{\sigma(1)} \) and negative degree in \( z_{\sigma(\ell)} \). This is a consequence of \( A^{\sigma(\ell)}(z_{\sigma(\ell)}) \) being a field, the admissibility, and the locality which implies Proposition 4. Now, take sufficiently large \( n_{ij} \in \mathbb{N}, (i < j) \), and consider the series
\[
\langle v^\vee, A^{\sigma(1)}(z_{\sigma(1)}) \cdots A^{\sigma(\ell)}(z_{\sigma(\ell)})u \rangle \prod_{i<j} (z_i - z_j)^{n_{ij}}.
\]
Then by the locality and the above observation, one sees that they are equal to a Laurent polynomial \( P(z_1, \ldots, z_\ell) \) that is common to all \( \sigma \in S_\ell \).

As an example we consider the case with \( \ell = 3 \) and \( \sigma = \text{Id} \). Let
\[
F_0(z_1, z_2, z_3) = \langle v^\vee, A^1(z_1)A^2(z_2)A^3(z_3)u \rangle - \frac{P(z_1, z_2, z_3)}{\prod_{i<j} (z_i - z_j)^{n_{ij}}}|_{z_1 > z_2 > z_3},
\]
\[
F_1(z_1, z_2, z_3) = F_0(z_1, z_2, z_3)(z_1 - z_3)^{n_{13}},
\]
\[
F_2(z_1, z_2, z_3) = F_1(z_1, z_2, z_3)(z_1 - z_2)^{n_{12}},
\]
\[
F_3(z_1, z_2, z_3) = F_2(z_1, z_2, z_3)(z_2 - z_3)^{n_{23}}.
\]
From the above arguments one sees that \( F_3(z_1, z_2, z_3) = 0 \), and that \( F_i(z_1, z_2, z_3) \) has only finitely many terms of positive degree in \( z_1 \) and negative degree in \( z_3 \) for \( 0 \leq i \leq 2 \). Hence by repeated use of Lemma 4 we obtain the desired result \( F_0(z_1, z_2, z_3) = 0 \). The other cases are proved in a similar way.

**Remark 5** This proposition and its proof is based on Theorem B.2.2 in [17], which claims an analogous result for arbitrary \( \ell \). For the soundness of our arguments we restrict ourselves to the above cases.

### 4.3 Generalized Wick theorem for \( \langle v^\vee, A(z)(BC)(w)u \rangle \)

We now present the well-known theorem (11) under the analytic formulation introduced in Sect. 4.1. Let \( A(z), B(z), C(z) \) be mutually local and admissible fields on a vector space \( M \), and \( M^*_u \subset M^* \) be the restricted dual space compatible with them with respect to \( u \in M \).

**Theorem 3** For any \( u \in M, v^\vee \in M_u^* \) the following relation holds:
\[
\langle v^\vee, A(z)(BC)(w)u \rangle = \frac{1}{2\pi \sqrt{-1}} \oint_{C_w} \frac{dx}{x - w} \langle v^\vee, \left\{ R(A(z)B(x)C(w)) + R(B(x)A(z)C(w)) \right\} u \rangle.
\]
Here \( R \) represents the OPE given by (22) and (24).
Proof. By the definition of the residue product (3), the following relation is satisfied for any $p \in \mathbb{N}$:

$$\langle v^\nu, A(w)(B(w)(-1)C(w))u \rangle = \text{Res}_{y=0} \text{Res}_{x=0} \langle v^\nu, [A(y), [B(x), C(w)]]u \rangle (y - w)^p (x - w)^{-1}.$$  

Here the factor $(x - w)^{-1}$ should be interpreted as its expansion in the region $|x| > |w|$ (resp. $|w| > |x|$) if the order of the product of the operators is $B(x)C(w)$ (resp. $C(w)B(x)$) in each of the four terms in $[A(y), [B(x), C(w)]]$. Let

$$0 < R_1 < R_2 < |w| < R_3 < R_4,$$

$$C_{i, \zeta} = \{ z \in \mathbb{C} | |z - \zeta| = R_i \}, \quad (i = 1, \ldots, 4), \quad (39)$$

and $C_{\zeta}$ be a contour encircling the point $\zeta \in \mathbb{C}$ with an infinitesimal radius. Then we have

$$\text{Res}_{y=0} \text{Res}_{x=0} \langle v^\nu, [A(y), [B(x), C(w)]]u \rangle (y - w)^p (x - w)^{-1}$$

$$= \frac{1}{(2\pi\sqrt{-1})^2} \oint_{C_{1, 0}} dy \oint_{C_{3, 0}} dx \langle v^\nu, A(y)B(x)C(w)u \rangle (y - w)^p (x - w)^{-1}$$

$$- \frac{1}{(2\pi\sqrt{-1})^2} \oint_{C_{2, 0}} dy \oint_{C_{3, 0}} dx \langle v^\nu, A(y)C(w)B(x)u \rangle (y - w)^p (x - w)^{-1}$$

$$- \frac{1}{(2\pi\sqrt{-1})^2} \oint_{C_{1, 0}} dy \oint_{C_{2, 0}} dx \langle v^\nu, B(x)C(w)A(y)u \rangle (y - w)^p (x - w)^{-1}$$

$$+ \frac{1}{(2\pi\sqrt{-1})^2} \oint_{C_{1, 0}} dy \oint_{C_{2, 0}} dx \langle v^\nu, C(w)B(x)A(y)u \rangle (y - w)^p (x - w)^{-1}. \quad (40)$$

According to Proposition (9) we see that $\langle v^\nu, A(y)B(x)C(w)u \rangle$ and its permutations are expansions of the same rational function which we denote by

$$\langle \langle v^\nu, A(y)B(x)C(w)u \rangle \rangle. \quad (41)$$

For instance

$$\langle v^\nu, C(w)B(x)A(y)u \rangle = \langle \langle v^\nu, A(y)B(x)C(w)u \rangle \rangle_{|w|>|x|>|y|},$$

for the last term of (40), where the right hand side is the expansion of (41) into the region $|w| > |x| > |y|$. Since the integration contours for this term ($C_{1, 0}$ for $y$ and $C_{2, 0}$ for $x$) are within this region, we can replace $\langle v^\nu, C(w)B(x)A(y)u \rangle$ by (41). By exactly the same reason we can replace the three-point functions in all of the four terms of (40) by the same rational function (41), and can interpret the factor $(x - w)^{-1}$ as a rational function rather than its expansions. Now all of the four terms of (40) have the same integrand which enables us to deform the contours as

$$\langle v^\nu, A(w)(B(w)(-1)C(w))u \rangle$$

$$= \frac{1}{(2\pi\sqrt{-1})^2} \oint_{C_{1, w}} dy \oint_{C_{w}} dx \langle v^\nu, A(y)B(x)C(w)u \rangle (y - w)^p (x - w)^{-1}$$

$$= \frac{1}{(2\pi\sqrt{-1})^2} \oint_{C_{1, w}} dx \oint_{C_{w}} dx \langle v^\nu, A(y)B(x)C(w)u \rangle (y - w)^p (x - w)^{-1}$$

$$+ \frac{1}{(2\pi\sqrt{-1})^2} \oint_{C_{1, w}} dx \oint_{C_{w}} dx \langle v^\nu, A(y)B(x)C(w)u \rangle (y - w)^p (x - w)^{-1}.$$
Now multiplying by \((z - w)^{-p-1}\) and summing over \(p\) from 0 to infinity, we obtain
\[
\langle v^\vee, A(z)(BC)(w)u \rangle
= \frac{1}{(2\pi\sqrt{-1})^2} \oint_{C_{1,w}} \frac{dx}{x} \oint_{C_x} \frac{dy}{y} \langle \langle v^\vee, A(y)B(x)C(w)u \rangle \rangle (y - z)^{-1}(x - w)^{-1}
+ \frac{1}{(2\pi\sqrt{-1})^2} \oint_{C_{1,w}} \frac{dx}{x} \oint_{C_x} \frac{dy}{y} \langle \langle v^\vee, A(y)B(x)C(w)u \rangle \rangle (y - z)^{-1}(x - w)^{-1}. \tag{42}
\]
For the second term of this expression we consider the integral
\[
I = \frac{1}{2\pi\sqrt{-1}} \oint_{C_w} \frac{dy}{y} \langle \langle v^\vee, A(y)B(x)C(w)u \rangle \rangle (y - z)^{-1}. \tag{43}
\]
Note that the contour \(C_{1,w}\) for \(x\) is expressed as the difference of \(C_{4,0}\) and \(C_{1,0}\). If \(x \in C_{4,0}\) we have
\[
I = \frac{1}{2\pi\sqrt{-1}} \oint_{C_{3,0}} \frac{dy}{y} \langle \langle v^\vee, B(x)A(y)C(w)u \rangle \rangle (y - z)^{-1}
- \frac{1}{2\pi\sqrt{-1}} \oint_{C_{2,0}} \frac{dy}{y} \langle \langle v^\vee, B(x)C(w)A(y)u \rangle \rangle (y - z)^{-1}
= \frac{1}{2\pi\sqrt{-1}} \oint_{C_w} \frac{dy}{y} \langle \langle v^\vee, B(x)R(A(y)C(w))u \rangle \rangle (y - z)^{-1}
= \langle v^\vee, B(x)A(z)C(w)u \rangle.
\]
Here we used Remark \(2\) for mutually local fields \(A(y)\) and \(C(w)\), our definition of OPE \(\text{OPE (11)}\), and the formula \(\text{(28)}\) with \(m = 1\). In the same way if \(x \in C_{1,0}\) we obtain \(I = \langle v^\vee, A(z)C(w)B(x)u \rangle\).

Hence the second term of \(\text{(42)}\) multiplied by \(2\pi\sqrt{-1}\) is written as
\[
\oint_{C_{4,0}} \frac{dx}{x-w} \langle v^\vee, B(x)A(z)C(w)u \rangle - \oint_{C_{1,0}} \frac{dx}{x-w} \langle v^\vee, A(z)C(w)B(x)u \rangle
= \oint_{C_w} \frac{dx}{x-w} \langle v^\vee, R(B(x)A(z)C(w))u \rangle. \tag{44}
\]
Here we used Remark \(2\) for mutually local fields \(B(x)\) and \(A(w)\) which is a consequence of Proposition \(3\) and our definition of OPE \(\text{OPE (23)}\). This gives rise to the second term of the desired formula. In a similar way, one also obtains the first term.

\[\square\]

4.4 Generalized Wick theorem for \(\langle v^\vee, (AB)(z)C(w)u \rangle\)

Now we present the main result of this paper in analytic formulation.

**Theorem 4** For any \(u \in M, v^\vee \in M_u^\vee\) the following relation holds:
\[
\langle v^\vee, (AB)(z)C(w)u \rangle
= \frac{1}{2\pi\sqrt{-1}} \oint_{C_w} \frac{dx}{x-w} \langle v^\vee, \left\{ A(x)B(x)C(w): + R(B(x)A(x)C(w)) \right\} u \rangle.
\]
Here \(R\) represents the OPE given by \(\text{OPE (23)}\).
Proof. We use the same notations in (39) and (41). By the definition of the residue product (6), the following relation is satisfied for any $p \in \mathbb{N}$:

$$\langle v^\gamma, (A(w)(-1)B(w))(y)C(w)u \rangle = \text{Res}_{x=0}\text{Res}_{y=0}\langle v^\gamma, [A(y), B(x)], C(w) \rangle u(y-x)^{-1}(x-w)^p.$$ 

Here the factor $(y-x)^{-1}$ should be interpreted as its expansion in the region $|y| > |x|$ (resp. $|x| > |y|$) if the order of the product of the operators is $A(y)B(x)$ (resp. $B(x)A(y)$) in each of the four terms in $[[A(y), B(x)], C(w)]$. Then we have

$$\text{Res}_{x=0}\text{Res}_{y=0}\langle v^\gamma, [A(y), B(x)], C(w) \rangle u(y-x)^{-1}(x-w)^p$$

$$= \frac{1}{(2\pi \sqrt{-1})^2} \int_{C_{1,0}} dx \int_{C_{1,0}} dy \langle v^\gamma, A(y)B(x)C(w)u \rangle (y-x)^{-1}(x-w)^p$$

$$- \frac{1}{(2\pi \sqrt{-1})^2} \int_{C_{1,0}} dx \int_{C_{1,0}} dy \langle v^\gamma, B(x)A(y)C(w)u \rangle (y-x)^{-1}(x-w)^p$$

$$- \frac{1}{(2\pi \sqrt{-1})^2} \int_{C_{1,0}} dx \int_{C_{1,0}} dy \langle v^\gamma, C(w)A(y)B(x)u \rangle (y-x)^{-1}(x-w)^p$$

$$+ \frac{1}{(2\pi \sqrt{-1})^2} \int_{C_{1,0}} dx \int_{C_{1,0}} dy \langle v^\gamma, C(w)B(x)A(y)u \rangle (y-x)^{-1}(x-w)^p. \quad (45)$$

By exactly the same reason in the proof of Theorem 3, we can replace the three-point functions in all of the four terms of (45) by the same rational function (41), and can interpret the factor $(y-x)^{-1}$ as a rational function rather than its expansions. Now all of the four terms of (45) have the same integrand which enables us to deform the contours as

$$\langle v^\gamma, (A(w)(-1)B(w))(y)C(w)u \rangle$$

$$= \frac{1}{(2\pi \sqrt{-1})^2} \int_{C_{1,w}} dy \int_{C_{w}} dx \langle v^\gamma, A(y)B(x)C(w)u \rangle (y-x)^{-1}(x-w)^p$$

$$- \frac{1}{(2\pi \sqrt{-1})^2} \int_{C_{1,w}} dx \int_{C_{w}} dy \langle v^\gamma, A(y)B(x)C(w)u \rangle (y-x)^{-1}(x-w)^p.$$ 

Here the first term comes from the first and third terms of (45), and the second term comes from its second and fourth terms. Now multiplying by $(z-w)^{-p-1}$ and summing over $p$ from 0 to infinity, we obtain

$$\langle v^\gamma, (AB)(z)C(w)u \rangle$$

$$= \frac{1}{(2\pi \sqrt{-1})^2} \int_{C_{1,w}} dy \int_{C_{w}} dx \langle v^\gamma, A(y)B(x)C(w)u \rangle (y-x)^{-1}(z-x)^{-1}$$

$$- \frac{1}{(2\pi \sqrt{-1})^2} \int_{C_{1,w}} dx \int_{C_{w}} dy \langle v^\gamma, A(y)B(x)C(w)u \rangle (y-x)^{-1}(z-x)^{-1}. \quad (46)$$

In view of the second term of (46), we consider the following integral

$$J = -\frac{1}{2\pi \sqrt{-1}} \int_{C_{w}} dy \langle v^\gamma, A(y)B(x)C(w)u \rangle (y-x)^{-1}.$$ 

Then we see that after replacing $z$ by $x$ in (45) the integral $I$ turns into this $J$. Hence based on the same argument to deriving (41) one sees that the second term of (46) is written as

$$-\frac{1}{2\pi \sqrt{-1}} \int_{C_{w}} \left[ \frac{dx}{z-x} \langle v^\gamma, R(B(x)A(x)C(w))u \rangle \right].$$
Thus we write the generalized Wick theorems simply as (1) and (2), and write the OPE simply.

5.1 The OPE of \((TT)(z)\) and \(T(w)\)

In order to illustrate the validity of our new formula (20), we shall present a few examples for calculations of the OPE. For the sake of simplicity we omit the symbol \(R\) for the radial ordering. Thus we write the generalized Wick theorems simply as (11) and (2), and write the OPE simply as

\[
A(y)B(z) = \overline{A(y)B(z)} + :A(y)B(z):, \tag{47}
\]

instead of (11).

First we consider the OPE of \((TT)(z)\) and \(T(w)\), where \(T\) is the energy momentum tensor satisfying the following OPE

\[
T(x)T(w) = \frac{c/2}{(x-w)^4} + \frac{2T(w)}{(x-w)^2} + \frac{\partial T(w)}{x-w} + :T(x)T(w):. \tag{48}
\]

From this formula and its derivative with respect to \(w\)

\[
T(x)\partial T(w) = \frac{2c}{(x-w)^5} + \frac{4T(w)}{(x-w)^3} + \frac{3\partial T(w)}{(x-w)^2} + \frac{\partial^2 T(w)}{x-w} + :T(x)\partial T(w):, \tag{49}
\]
we have
\[
T(x)T(x)T(w) = T(x) \left\{ \frac{c/2}{(x-w)^4} + \frac{2T(w)}{(x-w)^2} + \frac{\partial T(w)}{x-w} \right\}
\]
\[
= \frac{3c}{(x-w)^6} + \frac{3T(x)}{(x-w)^4} + \frac{cT(x) + 8T(w)}{(x-w)^4} + \frac{5\partial T(w)}{(x-w)^3}
\]
\[
+ \frac{2T(x)T(w): + \partial^2 T(w)}{(x-w)^2} + \frac{3T(x)\partial T(w)}{x-w}.
\] (50)

Then \(T(x)T(x)T(w)\) is obtained from this expression by dropping all the terms that do not contain the field \(T(x)\). Hence we have
\[
: T(x)T(x)T(w) : + T(x)T(x)T(w)
\]
\[
= \frac{3c}{(x-w)^6} + \frac{3T(x)}{(x-w)^4} + \frac{cT(x) + 8T(w)}{(x-w)^4} + \frac{5\partial T(w)}{(x-w)^3}
\]
\[
+ \frac{4T(x)T(w): + \partial^2 T(w)}{(x-w)^2} + \frac{2T(x)\partial T(w)}{x-w}.
\] (51)

By using the Taylor expansion of \(T(x)\) around the point \(w\) we obtain
\[
\text{(RHS of (51))} = \frac{3c}{(x-w)^6} + \frac{(8 + c)T(w)}{(x-w)^4} + \frac{(5 + c)\partial T(w)}{(x-w)^3}
\]
\[
+ \frac{4(TT)(w) + (1 + c/2)\partial^2 T(w)}{(x-w)^2}
\]
\[
+ \frac{(c/6)\partial^3 T(w) + 4\partial T(w)T(w): + 2T(w)\partial T(w)}{x-w}.
\] (52)

Applying (2) to this expression amounts to dropping the regular terms and replacing \(x\) by \(z\). Hence we have
\[
(TT)(z)T(w) = \frac{3c}{(z-w)^6} + \frac{(8 + c)T(w)}{(z-w)^4} + \frac{(5 + c)\partial T(w)}{(z-w)^3}
\]
\[
+ \frac{4(TT)(w) + (1 + c/2)\partial^2 T(w)}{(z-w)^2}
\]
\[
+ \frac{(c/6)\partial^3 T(w) + 4\partial T(w)T(w): + 2T(w)\partial T(w)}{z-w}.
\] (53)

We note that the numerator of the last term can be written as \((c-1)\partial^{(3)} T(w) + 3\partial(TT)(w)\) by the following lemma. Now one observes that our new formula (2) indeed reproduces the known result (equation (6.214) of [4]) for this OPE.

**Lemma 3**

\[ : T(w)\partial T(w): - \partial T(w)T(w): = \partial^{(3)} T(w). \]

**Proof.** From the derivative of \(\partial T(x)T(w)\) with respect to \(x\)
\[
\partial T(x)T(w) = \frac{-2c}{(x-w)^5} - \frac{4T(w)}{(x-w)^3} - \frac{\partial T(w)}{(x-w)^2} + : T(x)T(w):,
\]
one sees that \( \partial T(w)(g_{i-1})T(w) = -2cI(w), -4T(w), -\partial T(w) \) for \( i = 5, 3, 2 \) respectively and zero for the other \( i \in \mathbb{Z}_{>0} \). Then by the skew symmetry \( \square \) we have

\[
: T(w) \partial T(w) : = \sum_{i=0}^{\infty} (-1)^i \partial^{(i)}(\partial T(w)(t_{i-1})T(w)) = : \partial T(w)T(w) : + \partial^{(3)}T(w).
\]

5.2 Sugawara construction for the quantized currents of affine Lie algebras

Let \( G \) be a simple Lie group and \( \mathfrak{g} \) be its Lie algebra. The quantized currents \( J^a(z) (a = 1, \ldots, \text{dim} G) \) associated with \( \mathfrak{g} \) are fields satisfying the following OPE (e.g. \([4, 5]\))

\[
J^a(z)J^b(w) = \frac{(k/2)\delta^{ab}}{(z-w)^2} + \frac{\sqrt{-1}f^{abc}cJ^c(w)}{z-w} + : J^a(z)J^b(w) :. \tag{54}
\]

Here \( k \) is an integer called the level, and \( f^{abc} \) denotes the structure constant of \( \mathfrak{g} \). We adopted the convention of \([5]\) where the indices can be raised and lowered by \( g^{ab} = (1/2)\delta^{ab} \) and \( g_{ab} = 2\delta_{ab} \) respectively. Then the structure constant \( f^{abc} \) is anti-symmetric with respect to any pair of indices. They satisfy the relation \( \sum_{a,b} f^{abc}f_{abc} = 2h_G^2\delta_{ad} \) where \( h_G^2 \) is an integer called the dual Coxeter number. In \([51]\) and hereafter, any repeated indices imply that summations are taken over them.

We consider the OPE of \( (J^bJ^b)(z) \) and \( J^a(w) \). From \([51]\) we have

\[
J^b(x)\overline{J^b(x)J^a(w)} = \overline{J^b(x)} \left\{ \frac{(k/2)\delta^{ba}}{(x-w)^2} + \frac{\sqrt{-1}f^{ba}cJ^c(w)}{x-w} \right\} \\
= \frac{(k/2)J^a(x)}{(x-w)^2} + \frac{\sqrt{-1}f^{ba}c}{x-w} \left\{ \frac{(k/2)\delta^{bc}}{(x-w)^2} + \frac{\sqrt{-1}f^{bc}dJ^d(w)}{x-w} \right\} + :J^b(x)J^c(w) : \\
= \frac{(k/2)J^a(x)}{(x-w)^2} + h_G^2J^a(w) + \frac{\sqrt{-1}f^{ba}c : J^b(x)J^c(w) :}{x-w}.
\]

Here we used \( f^{ba}c\delta^{bc} = 0 \) and \( f^{ba}c_{d}f^{bc}d = -h_G^2\delta_{ad} \). Then \( \overline{J^b(x)J^b(x)J^a(w)} \): is obtained from this expression by dropping the second term. Hence

\[
\overline{J^b(x)J^b(x)J^a(w)} : + \overline{J^b(x)J^b(x)J^a(w)} = \frac{kJ^a(x)}{(x-w)^2} + \frac{h_G^2J^a(w)}{(x-w)^2} + \frac{2\sqrt{-1}f^{ba}cJ^b(x)J^c(w)}{x-w} \\
= \frac{(k+h_G^2)J^a(w)}{(x-w)^2} + \frac{k\partial J^a(w) + 2\sqrt{-1}f^{ba}cJ^b(x)J^c(w)}{x-w} \\
+ \text{(regular terms)}.
\]

Therefore by \( \square \) we have

\[
\overline{(J^bJ^b)(z)J^a(w)} = \frac{(k+h_G^2)J^a(w)}{(z-w)^2} + \frac{k\partial J^a(w) + 2\sqrt{-1}f^{cb}aJ^b(x)J^c(w)}{z-w} \\
= (k+h_G^2) \left\{ \frac{J^a(w)}{(z-w)^2} + \frac{\partial J^a(w)}{z-w} \right\}. \tag{55}
\]

Here we used \( f^{ba}c = f^{cb}a \) and the following:
\textbf{Lemma 4} \(2\sqrt{-1}f^{cb}_a :J^b(w)J^c(w) = h_G^c \partial J^a(w)\).

**Proof.** From (\ref{54}) one sees that \(J^c(w)(-1)J^b(w) = (k/2)\delta^{cb}\sqrt{-1}f^{cb}_a J^d(w)\) for \(i = 2, 1\) respectively and zero for the other \(i \in \mathbb{Z}_{>0}\). Then by the skew symmetry (\ref{21}) we have

\[
:J^b(w)J^c(w): = \sum_{i=0}^{\infty} (-1)^i \partial^{(i)}(J^c(w)(-1)J^b(w))
\]

\[
= :J^c(w)J^b(w): - \sqrt{-1}f^{cb}_a \partial J^d(w).
\]

By multiplying \(\sqrt{-1}f^{cb}_a\) and summing over \(b\) and \(c\) we obtain the desired result. \(\square\)

Therefore we have observed that our new formula (\ref{2}) indeed reproduces the well-known relation (\ref{54}) which implies that if we assume the following

\[
T(z) = \frac{1}{k + h_G^c}(J^b J^b)(z),
\]

to be the energy momentum tensor, then each current \(J^a(w)\) behaves as a primary field of conformal dimension 1.

Next we consider the calculation of the OPE of \(T(z)\) and \(T(w)\) by using our new formula (\ref{2}). If we used the Wick theorem (\ref{1}) instead of (\ref{2}) in the above calculation we would obtain (\ref{2})

\[
J^a(z)T(w) = \frac{J^a(w)}{(z-w)^2}.
\]

On the other hand by setting \(a = b\) in (\ref{54}) and summing over \(a\) from 1 to \(\dim G\) we obtain

\[
J^a(z)J^a(w) = \frac{(k/2)\dim G}{(z-w)^2} + :J^a(z)J^a(w):.
\]

Therefore

\[
J^a(x)J^a(x)T(w) = \frac{(k/2)\dim G}{(x-w)^4} + \frac{\partial J^a(w)J^a(w)}{(x-w)^2}.
\]

Then \(\partial J^a(x)J^a(x)T(w)\): is obtained by dropping its first term. Hence

\[
\partial J^a(x)J^a(x)T(w)\]

\[
= \frac{(k/2)\dim G}{(x-w)^4} + \frac{2\partial J^a(w)J^a(w)}{(x-w)^2} + \frac{2\partial J^a(w)J^a(w)}{x-w} + \text{(regular terms)}.
\]

Thereby by (\ref{2}) we have

\[
\partial J^a(x)J^a(x)T(w) = \frac{(k/2)\dim G}{(x-w)^4} + \frac{2\partial J^a(w)J^a(w)}{(x-w)^2} + \frac{2\partial J^a(w)J^a(w)}{x-w}.
\]

Therefore \(\partial J^a(x)J^a(x)T(w)\) satisfies the correct OPE for the energy momentum tensor (\ref{48}).

Here we introduced the central charge \(c\) defined as \(c = k\dim G/(k + h_G^c)\) and used the relation \(\partial J^a(w)J^a(w) = :J^a(w)\partial J^a(w):\) that is obtained by using the skew symmetry (\ref{21}). Therefore we have observed that our new formula (\ref{2}) indeed reproduces the well-known relation (\ref{54}) which implies that \(T(z)\) in (\ref{54}) satisfies the correct OPE for the energy momentum tensor (\ref{48}).
6 Discussion

In this paper we derived a new formula (2) as a counterpart of the famous generalized Wick theorem (1). Here we want to discuss the significance of our results.

As we have already mentioned, Kac suggested the equivalence of the special case of the Borcherds identity (30) and the generalized Wick theorem (1) in 1996 [7]. However, as far as the authors know, their explicit relation has not been written in any literature. In this paper we have described their explicit relation, which has a certain importance by itself. In addition, we have succeeded to rewrite another specialization of the Borcherds identity (37) into the integral formula (2) which no one had ever imagined. We believe that not only this result but also the detailed description of its derivation will contribute to enhancing our insights into hidden mathematical structures of CFT and the other quantum integrable systems.

From the viewpoint of practical calculations, one may argue that our new formula (2) is not indispensable for evaluating this contraction. In fact, as was explained in [4], one can calculate $(BC)(z)A(w)$ by using (1) in the following way: First you evaluate $A(z)(BC)(w)$ by using (1), then interchange $w \leftrightarrow z$, and finally Taylor expand the fields evaluated at $z$ around the point $w$. However, we believe that the existence of such a simple and nontrivial formula (2) has its own theoretical significance beyond practical usefulness. Obviously, there is still another way of calculating the OPE for normally ordered products, by directly using the special cases of the Borcherds identity (30) and (37) themselves. This way of calculations was actually demonstrated in Sect. 3.6 of [11]. However we want to point out that our generalized Wick theorems are simple enough to be memorized, compared to those special cases of the Borcherds identity.

One may also give an argument that our new formula is certainly new but is almost trivial because there is an identity

$$\frac{1}{2\pi i} \oint_{C_{w}} \frac{dx}{z-x} :A(x)B(x):C(w).$$

In fact, the “integral operator” $\frac{1}{2\pi i} \oint_{C_{w}} \frac{dx}{z-x}$ drops the regular part of the rest of the integrand at the point $x = w$ by Cauchy’s integral theorem, and then replaces $x$ by $z$ in its singular part by the formula (28) with $m = 1$. Then he/she may argue that our formula (2) can be obtained by using the following replacement

$$:A(x)B(x):C(w) \rightarrow :A(x)B(x)C(w): + B(x)A(x)C(w),$$

under the action of this integral operator. However, we emphasize that this replacement is by no means trivial.

What we have done in Sect. 3 implies that the generalized Wick theorems (1) and (2) are generating function versions of the special cases of the Borcherds identity (30) and (37). On the other hand, Kac introduced another type of generating function which he called the $\lambda$-bracket [7]. In this context, our formula (2) should be compared with their right noncommutative Wick formula [1, 9]. We would like to clarify the relation between these formulas in our future works.

In this paper we only dealt with the bosonic cases of integer conformal dimensions. However, the generalization to the cases involving both bosonic and fermionic fields is straightforward [14].

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