Graph rules for the linked cluster expansion of the Legendre effective action

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Graph rules for the linked cluster expansion of the Legendre effective action $\Gamma[\phi]$ are derived and proven in $D \geq 2$ Euclidean dimensions. A key aspect is the weight assigned to articulation vertices which is itself shown to be computable from labeled tree graphs. The hopping interaction is allowed to be long ranged and scale dependent, thereby producing an in principle exact solution of $\Gamma[\phi]$’s functional renormalization group equation.

I. INTRODUCTION

The Legendre effective action is a central quantity in all areas of many-body quantum physics. In particular, it features prominently in the functional renormalization group approach based on the flow equation

$$\partial_k \Gamma_k = \frac{1}{2} \text{Tr} \{ \partial_k R_k [\Gamma_k^{(2)} + R_k]^{-1} \} ,$$

(1)
describing its response to a modulation of the system’s mode content set by the kernel $R_k$. The flow equation (1) is now being used in fields as diverse as: solid state physics, statistical physics, and quantum gravity, see \cite{1–3} for book-sized accounts. The response (1) is itself kinematical in nature, dynamical information is injected exclusively through initial conditions. As a consequence, the results obtained are only as non-perturbative as the initial conditions are. An especially interesting choice of initial conditions are ultralocal ones as they can in a lattice formulation be be computed exactly from single site integrals \cite{4}. A solution of (1) with such initial data, if feasible, will emulate a linked cluster or hopping expansion but with a scale dependent long-ranged interaction

$$S[\chi] = \sum_x s[\chi_x] + \frac{\kappa}{2} \sum_{x,y} \chi_x \ell_{xy}(k) \chi_y .$$

(2)

For definiteness we consider a self-interacting scalar field theory on a $D$-dimensional hypercubic lattice (identified with $\mathbb{Z}^D$) in a dimensionless formulation. Here, $s : \mathbb{R} \to \mathbb{R}$ is a function bounded from below that collects all terms from the original lattice action referring to a single site. The hopping parameter $\kappa > 0$ arises as a dimensionless combination of the original mass and coupling parameters and the lattice spacing. A fundamental lattice action would only connect nearest neighbors on the lattice through $\ell_{xy}$. In order to obtain a solution of (1) we allow $\ell_{xy}$ to be long-ranged and be modulated by the control parameter $k$. The details of the modulation are inessential in the following as we take $\kappa$ as the control parameter and replace (1) by

$$\partial_k \Gamma_k = \frac{1}{2} \sum_{x,y} \ell_{xy} [\Gamma_k^{(2)} + \kappa \ell_{xy}]^{-1} , \quad \Gamma_k = \Gamma_0 + \sum_{l \geq 2} \kappa^{l} \Gamma_l [\phi] .$$

(3)

Here $\Gamma_0[\phi] = \sum_x \gamma(\phi_x)$, where $\gamma$ and its derivatives $\gamma_l$ are computable at a single site from $s$ only. The $O(\kappa)$ term vanishes, $\Gamma_2[\phi] = - \frac{1}{2} \sum_{x,y} (\ell_{xy})^2 \gamma_2(\phi_x)^{-1} \gamma_2(\phi_y)^{-1}$, and all $\Gamma_l$, $l \geq 3$, are then determined recursively; see (A2) in Appendix A. Importantly the series can be expected to have finite radius...
of convergence $\kappa < \kappa_c$; see the discussion below. Once the series (3) has been constructed, an in principle exact solution of (1), arises simply by substitution, $\Gamma_k = \Gamma_k(\ell \rightarrow \ell(k))$.

The direct iteration (A2) becomes, however, impractical beyond $O(\kappa^6)$ or so (both in manual computations and in automated symbolic implementations). The repeated functional differentiations of $\Gamma_0(\phi)$ leads to site identifications whose combinatorics is best recast in graph theoretical terms. The graph theoretical analysis of hopping expansions of course has a long history, see [13,15] and the references therein. The convergence proofs of generalized Mayer expansions typically rely on tree graph bounds\(^{(9)}\). In the computational uses of linked cluster expansions, the focus is normally on nearest neighbor interactions and quantities of direct interest for critical behavior like generalized Legendre transform of the free energy functional $W_{\kappa}$. Graph theoretical rules for the linked cluster expansion of $W_{\kappa}$ have originally been presented by Wortis\(^{(15)}\) and will be briefly reviewed in Section 2. Graph theoretical expansions for $\Gamma_{\kappa}(\phi)$ have been discussed previously but do not cover the material presented here: the rules and results of [15,16] hinge on specific features of the Ising model which do not generalize. Hybrid perturbative expansions have been considered in [13]. A relevant combinatorial Legendre transform has been studied in [11,20] in a setting that emulates perturbation theory. Some of the results of [11] will reoccur in our analysis of $\Gamma_0$ in Appendix B. There are also abstract variants of a Legendre transform formulated in terms of combinatorial species\(^{(12,20)}\). None of these seem to bear an obvious relation to our result.

We present the solution of the recursion implied by (3) in graph theoretical terms. Let $\mathcal{L}_l$ be the set of one-line irreducible connected graphs with $l$ edges. For any $L = (V,E) \in \mathcal{L}_l$ and any vertex $v \in V$ consider the decomposition of $L$ into one-vertex irreducible subgraphs, $|L(v)|$ of which each contain a copy of $v$. The set $B(v)$ of copies is used to label a class of tree graphs $\mathcal{T}(B(v),n)$, $n = 1, \ldots, |L(v)|$. To each $T \in \mathcal{T}(B(v),n)$ two integers $s(T)$ and $\text{Perm}(B(v))/\text{Sym}(T)$ are assigned, as detailed in Section III. Then:

**Theorem.** For any $l \geq 2$ the exact solution of the recursion implied by (3), i.e. (A2), is given by

\[
\Gamma_l(\phi) = \sum_{L=(V,E) \in \mathcal{L}_l} \frac{(-)^{l+1}}{\text{Sym}(L)} \prod_{e \in E} \phi_\theta(e) \prod_{v \in V} \mu^\Gamma(v|B)
\]

\[
\mu^\Gamma(v|B) = \sum_{n=1}^{\lfloor |L(v)| \rfloor} \sum_{T \in \mathcal{T}(B(v),n)} (-)^{s(T)} \frac{|\text{Perm}(B(v))|}{\text{Sym}(T)} \mu(T).
\]

(4)

In the first line an unconstrained sum over the lattice points associated with the vertices is tacit. Further $E$ is the edge list with $\theta(e)$ the pair of vertices connected by $e$, and $\text{Sym}(L)$ is the symmetry factor of $L$. In the second line, $\mu(T)$ is a weight depending only on the value of $\phi$ at $v$.

The paper is organized as follows. In Section II.A we summarize known graph rules for the free energy functional and set the terminology. A mixed recursion relation (22), equivalent to the one implied by (3) is derived in Section II.B, and used to derive the first line of (4). The relevant class of labeled tree graphs is introduced in Section III.A, the graph rules for $\mu^\Gamma(v|L)$ are formulated and illustrated in Section III.B, and an all-order proof for their validity is given in Section III.C. A simplified version of $\mu^\Gamma(v|B)$ obtained by performing subsums with fixed $\mu(T)$ is derived in Section IV. Appendix A presents explicit and independently computed results for $\Gamma_2, \ldots, \Gamma_5$. Appendix B discusses the single site data and their combinatorics.
II. FROM CONNECTED GRAPHS TO ARTICULATION VERTICES

For convenience we refer to expansions in powers of $\kappa \ell_{xy}$ (with $\ell_{xx} = 0$ but $\ell_{xy} \neq 0$ for $\text{dist}(x,y) \geq 1$) as a long range hopping (LRH) expansion. The graph expansions considered have two main ingredients: first, a class of graphs with some partial order consistent with the order in $\kappa$. Second, a weight function that assigns to each graph of the class a numerical value depending on certain input data. In addition to $\ell_{xy}$ itself, the input data are always the derivatives $\omega_m(h) = \partial^m \omega / \partial h^m$ and/or $\gamma_m(\varphi) = \partial^m \varphi / \partial \varphi^m$ of the single site functions described in Appendix B. The class of graphs and the weight functions will depend on the quantity considered. The goal of this section is to reduce the problem of identifying the graph rules for $\Gamma \kappa$’s LRH expansion to the determination of the weight associated to articulation vertices.

A. Basics

The $\Gamma_k$ flow equation (1) can be obtained as the Legendre transform of a Polchinski-type flow equation for $W_k$, the mode modulated free energy functional. For an action of the form (2) one may again take $\kappa$ as the control parameter to obtain along the familiar lines

$$\partial_\kappa W_k[H] = \frac{1}{2} \sum_{x,y} \ell_{xy} \left\{ \frac{\partial^2 W_k[H]}{\partial H_x \partial H_y} + \frac{\delta W_k[H]}{\partial H_x} \frac{\delta W_k[H]}{\partial H_y} \right\}. \quad (5)$$

Here we impose ultralocal initial data $W_0[H] = \sum \omega(H_x)$, where $\omega(h)$ is determined by the single site action $s$ in (2). The ansatz $W_k[H] = W_0[H] + \sum_{l \geq 1} \kappa^l W_l[H]$ converts (5) into the recursive system

$$W_{l+1}[H] = \frac{1}{2(l+1)} \sum_{x,y} \ell_{xy} \left\{ \frac{\partial^2 W_l[H]}{\partial H_x \partial H_y} + \sum_{k=0}^l \frac{\delta W_k[H]}{\partial H_x} \frac{\delta W_{l-k}[H]}{\partial H_y} \right\}, \quad l \geq 0. \quad (6)$$

Explicitly, the first two orders read

$$W_1[H] = -\frac{1}{2} \sum_{x,y} \ell_{xy} \frac{\partial W_0[H]}{\partial H_x} \frac{\partial W_0[H]}{\partial H_y},$$

$$W_2[H] = \frac{1}{2} \sum_{x,y,z,w} \ell_{xy} \ell_{zw} \frac{\partial^2 W_0[H]}{\partial H_x \partial H_y} \left\{ \frac{1}{2} \frac{\partial^2 W_0[H]}{\partial H_x \partial H_z} + \frac{\delta W_0[H]}{\partial H_x} \frac{\delta W_0[H]}{\partial H_z} \right\}. \quad (7)$$

The repeated $H_x, H_y, \ldots$, functional derivatives of $W_0[H]$ produce point identifications and coefficients that are source-dependent derivatives of the single-site generating function $\omega(h)$. The combinatorics of these point identifications is best formulated in graph theoretical terms. Such rules have been formulated and proven by Wortis\textsuperscript{10} where subject to additional conditions also a convergence proof is given.

Graph rules for W[H]:

(a) At order $l \geq 1$ in $\kappa$ draw all topologically distinct connected graphs $C = (V,E) \in \mathcal{C}_l$ with $l = |E|$ edges connecting $2, \ldots, l+1$ vertices. Assign a dummy label to each vertex.

(b) Divide by the symmetry factor $\text{Sym}(C)$ of the graph.

(c) To each graph a weight $\mu^W(C)$ is assigned as follows: a vertex $i$ of degree $n$ is attributed a weight $\omega_n(H_i)$, an edge connecting $i,j$ is attributed a factor $-\ell_{ij}$.

(d) Embed the graph into the lattice $\mathbb{Z}^D$ by associating each vertex with a unique lattice point, $i \mapsto x_i, i = 1, \ldots, |V|$, the same lattice point may occur several times. Perform an unconstrained sum over all $x_1, x_2, \ldots, x_{|V|}$. 
For illustration consider the graphs in (a) divided by their symmetry factors in (b) to \( O(k^d) \):

\[
W_k[H]^{(a),(b)} = -\frac{1}{2} + \frac{1}{4} \sum + \frac{1}{2} \cdot \\
-\frac{1}{12} -\frac{1}{2} \cdot \cdot -\frac{1}{6} -\frac{1}{2} \cdot -\frac{1}{6} + O(k^d).
\]

Upon application of parts (c),(d) this matches the recursively computed result. Generally, the graph sum is over all connected graphs obtained by differentiation. It is plain from (9) that the cumulants are assigned a graph, where the isomorphisms have to leave the external vertices individually invariant. The edges eventually labeled by \( v \) where the lattice summations from step (d) are tacit and the double product comprises \( \mu^W(C) \). The graph sum is over all connected graphs \( G = (V, E) \) with \( |E| = l \) edges, \( d(v) \) is the degree of the vertex \( v, \theta(e) \) is the pair of vertices \( e \in E \) connects. A recent algorithm that generates these graphs can be found in [14]. The symmetry factor \( \text{Sym}(C) \) of \( C \) is defined below. Since also the graph terminology is not entirely standardized we compiled a brief glossary at the end of this subsection.

Once \( W[H] \) is known to some order, the connected correlation functions (or cumulants) can be obtained by differentiation. It is plain from (9) that the cumulants \( (W(k)[H])_{y_1 \ldots y_k}, y_1 \neq \ldots \neq y_k \) also have a graph expansion and that the contributing graphs are \( k \)-rooted, i.e. have \( k \) external vertices eventually labeled by \( y_1, \ldots, y_k \). The relevant symmetry factor thus is that of the \( k \)-rooted graph, where the isomorphisms have to leave the external vertices individually invariant. The edges are assigned a \( -\ell_{ij} \) factor as before, also for edges where one of the vertices is an external vertex. The vertex weight can always be obtained by differentiation from the \( \omega_{d(v)}(H_v) \) product in (9).

A brief graph glossary:

A graph is a pair \( G = (V, E) \) of nonempty disjoint sets equipped with a map \( \theta \) that associates to each \( e \in E \) an unordered pair \( \theta(e) = \{v, w\}, v, w \in V \). The elements of \( V \) are called vertices (or nodes), those of \( E \) are called edges (or links, or lines). This definition allows for several edges to be mapped into the same unordered pair of vertices, in which case the edges are called multiple edges. Otherwise the graph is called simple, in which case we shall identify \( E \) with a subset of \( V_2 := \{\{v, w\} : v, w \in V\} \). The degree (or valency or number of incident lines) \( d(v) \) of a vertex \( v \in V \) is the cardinality of the set \( \{e \in E : v \in \theta(e)\} \). If \( |V| \) is the cardinality of \( \{v \in V : d(v) = k\} \) one has \( 2|E| = \sum k|V_k| \).

Let \( (V, E) \) be a graph. A trail from \( v \) to \( w \) in \( V, w \in V \) is a sequence \( v_0, e_1, \ldots, e_n, v_w \) with \( v_0 = v, v_n = w \), such that the edges \( e_i \) are distinct and \( \theta(e_i) = \{v_{i-1}, v_i\} \). A graph is connected if for every pair of its vertices \( v, w \) there is a trail from \( v \) to \( w \). A connected component of \( G \) is a maximal connected subgraph of \( G \). A trail from \( v \) to \( w \) such that \( v \) and \( w \) coincide is called a cycle. The cyclomatic number \( c(G) \) is the number of cycles of a graph \( G \). The Euler relation states

\[
c(G) = |E| - |V| + 1.
\]

A tree \( T \) is a connected simple graph without cycles; in particular \( |V| - |E| = 1 \) holds.

Two graphs \( (V, E) \) and \( (V', E') \), with respective maps \( \theta, \theta' \), are called isomorphic (or topologically equivalent) if there exist bijections \( \pi_1 : V \rightarrow V', \pi_2 : E \rightarrow E' \) such that \( \theta(e) = \{v, w\} \) iff \( \theta'(\pi_2(e)) = \{\pi_1(v), \pi_1(w)\} \). These isometries form a group, \( \text{Aut}(G) \), which with the above definition included permutations of multiple edges. The symmetry factor of \( G \) is defined by

\[
\text{Sym}(G) = |\text{Aut}(G)|.
\]

Often the automorphism group refers to the corresponding simple graph only, in which case the permutation of multiple edges occurs as an extra factor in the definition of the symmetry factor [15].

The same notion of isometry applies if the elements of a subset \( R \subset V \), called the rooted vertices, are left individually invariant by the bijection. The elements of \( R \) can be viewed as distinguishable and labeled, \( R = \{r_1, \ldots, r_k\} \), in which case \( G \) is called \( k \)-rooted.
A graph \( G' = (V', E') \) is called a subgraph of \( G = (V, E) \) if \( V' \subset V \) and \( E' \subset E \). For a graph \( G \) let \( G \setminus \{v\} \) be the subgraph obtained by deleting \( v \) and all edges containing \( v \). For a connected graph \( G = (V, E) \) a vertex \( v \in V \) is called an articulation point if the \( G \setminus \{v\} \) is disconnected. A connected graph without articulation points is called one-vertex irreducible (1VI) (or two-connected). For a connected graph \( G \) a block \( G' \) is a maximal 1VI subgraph, i.e., a graph \( G' \subset G \) that is 1VI and such that for any 1VI subgraph \( G'' \subset G \) the inclusion \( G' \subset G'' \) entails \( G' = G'' \). The set of blocks \( \{G_1, \ldots, G_k\} \), \( G_i = (V_i, E_i) \), \( i = 1, \ldots, k \), of a connected graph \( G = (V, E) \) is referred to as \( G \)'s block decomposition\(^{23}\). The blocks induce a partition of the edge set \( E = E_1 \cup \ldots \cup E_k \), with \( E_i \cap E_j = \emptyset \), \( i \neq j \). Each articulation point belongs to more than one \( V_i \) while non-articulation vertices belong to exactly one.

A bridge in a connected graph is an edge whose omission produces a disconnected graph. A one-line irreducible (1LI) graph is a bridgeless connected graph. A one-line irreducible graph may still get disconnected upon removal of a vertex. The block decomposition of 1LI graphs will be central later on.

B. The role of one-line and one-vertex irreducible graphs

Our task will be to convert the above \( W \)-graph rules into ones directly applicable to the \( \Gamma_\kappa \) expansion defined by (3). Both functions are related by the following modified Legendre transform

\[
\Gamma_\kappa[\phi] := \phi \cdot H_0[\phi] - W_0[H_0[\phi]] - \kappa \gamma'[\phi], \quad \frac{\delta W_0}{\delta H} (H_0[\phi]) = \phi, \tag{12}
\]

for a \( \kappa \)-independent mean field \( \phi \). The modification by the \( \gamma'[\phi] := \frac{1}{2} \sum_{l=1}^N \phi \cdot \ell_{xy} \cdot \phi \), term is introduced so as to obtain the closed flow equation (3). Differentiating (12) with respect to \( \kappa \) gives \( \partial_\kappa \Gamma_\kappa = -(\partial_\kappa W_0)[H_0[\phi]] - \gamma'[\phi] \). Inserting the series expansions

\[
W_\kappa[H] = \sum_{l=0}^{\infty} \kappa^l W_l[H], \quad \Gamma_\kappa[\phi] = \sum_{l=0}^{\infty} \kappa^l \Gamma_l[\phi], \quad \Gamma_1[\phi] \equiv 0, \tag{13}
\]

one obtains \( \Gamma_0[\phi] = \phi \cdot H_0[\phi] - W_0[H_0[\phi]] \), \( \Gamma_1[\phi] = -W_1[H_0[\phi]] - \gamma'[\phi] \), and for \( l \geq 2 \)

\[
\Gamma_l[\phi] = -W_l[H_0[\phi]] - \sum_{m=1}^{l-1} \sum_{k=1}^{m} \frac{l-m}{l!} \sum_{m_1 + \ldots + m_k = m} W_{l-m}[H_0[\phi]] \cdot H_{m_1}[\phi] \ldots H_{m_k}[\phi]. \tag{14}
\]

Note that \( W_{l}^{(1)}[H_0[\phi]] = \phi \) still enters \( [14] \) implicitly in defining the \( H_m[\phi] \). Upon expansion one finds \( W_0^{(1)}[H_0[\phi]] = \phi, \ W_0^{(2)}[H_0[\phi]] \cdot H_1[\phi] + W_1^{(1)}[H_0[\phi]] = 0 \), and for \( l \geq 2 \)

\[
H_l[\phi] \cdot W_0^{(2)}[H_0[\phi]] + W_l^{(1)}[H_0[\phi]] + \frac{\delta}{\delta H_0} F_{l-1}[H_0,H_1,\ldots,H_{l-1}] \big|_{H_m = H_m[\phi]} = 0, \tag{15}
\]

where

\[
F_{l-1}[H_0, \ldots, H_{l-1}] := \sum_{m=1}^{l-1} \sum_{k=1}^{m} \bar{B}_{m,k}(H_1, \ldots, H_{m-k+1}) \cdot W_{l-m}[H_0]
\]

\[
+ \sum_{k=2}^{l} \bar{B}_{m,k}(H_1, \ldots, H_{l-1-k}) \cdot W_0^{(k)}[H_0], \tag{16}
\]

and the \( \bar{B}_{m,k} \) are modified Bell polynomials, \( k! \bar{B}_{m,k}(H_1, \ldots, H_{l-1-k}) := \sum_{m_1 + \ldots + m_k = m, m_j \geq 1} H_{m_1} H_{m_2} \ldots H_{m_k} \).

These relations can be solved iteratively for the \( H_l[\phi] \) and also show inductively that \( H_l[\phi] = \Gamma_l^{(1)}[\phi] \).

In \([13], [15]\) similar relations later on there are tacit summations over lattice sites, summarily indicated by a "\( \cdots \)". A contraction of \( \{W_{l-m}[y_1, \ldots, y_k]\} \) may contain subsums where where one or more
lattice points coincide. The graph rules for the cumulants outlined after (9) then change slightly. Since multiple \(h\) derivatives can act on the same \(\omega_{d(v)}(h)\), the number of rooted vertices \(r\) can be \(r = 1, \ldots, k\). The tacit lattice sums ensure that all possible combinations will occur, so that \(W_{l-m}^{(k)}\) expands into a sum of \(l\)-rooted connected graphs with \(l-m\) edges; we write \(C_{l-m}^{(k)}\) for the set of such graphs. The topology of each graph in \(C_{l-m}^{(k)}\) is the same as its counterpart in \(C_{l-m}^{(k)}\), only the rooted vertices have their \(\omega_h\) weight shifted from \(n = d(v)\) to \(n = d(v) + \#\text{of } h\)-derivatives, and the symmetry factor changes. The contracted lattice sums in (14), (15) ensure that each graph in \(C_{l-m}^{(k)}\) is paired with an \(r\)-rooted product of \(H_{m_1}, \ldots, H_{m_r}\)’s graph expansions, such that a term corresponding to an unrooted \(\mathcal{G}_i\) graph arises. This graph expansion of (14)’s right hand side allows for many cancellations. In order to identify the underlying pattern we derive a property of the LRH expansion of the effective action well-known for its perturbative expansion but not limited to it:

**Lemma II.1.** The graphs contributing to \(\Gamma_\kappa[\phi]\)’s LRH expansion are 1LI, i.e. remain \(\ell\)-connected even when any one \(\ell\)-line is cut.

**Proof.** The proof is an adaptation of the argument familiar for the Feynman diagrams occurring in a perturbative expansion. In a first step one computes the linear response of \(\Gamma_\kappa[\phi]\) under a replacement of the hopping matrix

\[
\ell_{xy} \rightarrow \ell_{xy} + \varepsilon e_x e_y,
\]

where \(e_x\) is a vector and \(\varepsilon \geq 0\). We momentarily change notation and write \(W_\varepsilon[H]\), \(\Gamma_\varepsilon[\phi]\) for the functionals obtained by the replacement (17) and \(W[H], \Gamma[\phi]\) for the original ones, without indicating the \(\kappa\)-dependence. Starting from the functional integral realization

\[
\exp W[H] := \int \prod_x d\chi_x \exp \{-S[\chi] + \sum_x H_x \chi_x\}.
\]

\[
S[\chi] = S_0[\chi] + \kappa \varphi[\chi], \quad S_0[\chi] = \sum_x s(\chi_x), \quad \varphi[\chi] = \frac{1}{2} \sum_{x,y} \chi_x \ell_{xy} \chi_y.
\]

and expanding in powers of \(\varepsilon\) one finds to linear order

\[
W_\varepsilon[H] = W[H] - \varepsilon \frac{\kappa}{2} \sum_{x,y} e_x e_y \left( \frac{\delta^2 W}{\delta H_x \delta H_y} + \frac{\delta W}{\delta H_x} \frac{\delta W}{\delta H_y} \right) + O(\varepsilon^2).
\]

For the altered functionals the definition of the modified Legendre transform (12) reads

\[
\Gamma_\varepsilon[\phi] = \phi \cdot H_\varepsilon[\phi] - W_\varepsilon[H_\varepsilon[\phi]] - \frac{\kappa}{2} \phi \cdot (\ell + \varepsilon e \otimes e) \cdot \phi, \quad \frac{\delta W_\varepsilon}{\delta H}(H_\varepsilon[\phi]) = \phi.
\]

Differentiating with respect to \(\varepsilon\) gives \(\partial_\varepsilon \Gamma_\varepsilon[\phi] = \varepsilon \frac{\chi}{2} (e \otimes e) \cdot W(2)(H[\phi]) + O(\varepsilon)\). Since \(W^{(2)}(H[\phi]) = (\Gamma^{(2)} + \kappa \varepsilon)^{-1}\) one obtains

\[
\Gamma_\varepsilon[\phi] = \Gamma[\phi] + \varepsilon \frac{\chi}{2} \sum_{x,y} e_x e_y \left[ \Gamma^{(2)} + \kappa \varepsilon \right]_{xy}^{-1} + O(\varepsilon^2).
\]

The replacement (17) emulates the effect of cutting \(\ell\)-lines and to linear order in \(\varepsilon\) the effect of cutting precisely one \(\ell\)-line is traced. Viewed as a function of \(H\) the response, being proportional to \(W^{(2)}[H]\), expands into \(\ell\)-connected LRH graphs by Section II.A. The recursion (15) shows that the \(\kappa\) expansion of \(H[\phi]\) produces contracted functional derivatives of the \(W_0, W_1, \ldots, W_l\) evaluated at \(H_0[\phi]\) for all \(H_l[\phi]\), \(l \geq 1\). The \(W^{(k)}_{l-m}\) derivatives correspond to \(r \leq k\)-rooted \(\ell\)-connected diagrams and the contractions are pointwise with analogous terms. Hence, also as a functional of \(\phi\) the linear response (21) expands into \(\ell\)-connected LRH graphs only.

The graph expansion of the right hand of (14) contains a large number of terms associated with one-line reducible graphs. By Lemma II.1 these must cancel which allows one to simplify the right hand side considerably. In the sum each \(W_{l-m}^{(k)}\) expands into \(r\)-rooted, \(r = 1, \ldots, k\), connected
diagrams many of which are one-line reducible. The rooted vertices are directly (without extra $\ell_{xy}$ link) attached to (and summed over the lattice point associated with) possibly multiple copies of a 1-rooted graph representing a $\Gamma_{m}^{(1)} + \delta_{m,1} V^{(1)}$. A term occurring in the graph expansion of $W_{r-m}^{(k)}[H_0[\phi]] \cdot H_{m_1}[\phi] \cdots H_{m_\ell}[\phi]$ will be one-line reducible if (i) $m_i = 1$ for one or more $i \in \{1, \ldots, k\}$, since $H_1[\phi] = \sum_x \ell_{xy} \phi_x$. (ii) the $r$-rooted $W$-graph stemming from $W_{r-m}^{(k)}$ is one-line reducible. (iii) if a $W_1^{(k)}$ term enters, as $W_1[H] = -\frac{1}{2} \sum_{x,y} \ell_{xy} \omega(H_x) \omega(H_y)$. All these terms must cancel against the one-line reducible terms in $-W_l$. We write $W_{lf}^{(k)}[\Gamma_0^{(1)}]|_{1LI}$ for the quantity obtained from $W_{lf}^{(k)}[\Gamma_0^{(1)}]$'s graph expansion by omitting all terms corresponding to one-line reducible graphs and $|m|$ for the integer part of $m \in \mathbb{R}^+$. Then $\Gamma_2[\phi] = -W_2[\Gamma_0^{(1)}]|_{1LI}$, $\Gamma_3[\phi] = -W_3[\Gamma_0^{(1)}]|_{1LI}$, and for $l \geq 4$ the following simplified version of (14) holds

$$
\Gamma_l[\phi] = -W_l[\Gamma_0^{(1)}]|_{1LI} - \sum_{m=2}^{l-2} \sum_{k=1}^{\lfloor m/2 \rfloor} \frac{(l-m)}{lk} W_{l-m}^{(k)}[\Gamma_0^{(1)}]|_{1LI} \cdot \Gamma_{m_1}^{(1)} \cdots \Gamma_{m_k}^{(1)}. \tag{22}
$$

An immediate consequence of (22) is:

**Lemma II.2.** Let $L$ be a 1LI graph without articulation points and let $\mu^W(L), \mu^\Gamma(L)$ be the weight (including sign and symmetry factors) with which it occurs in the expansion of $W, \Gamma$, respectively. Then

$$
\mu^\Gamma(L) = -\mu^W(L)|_{H_0=\Gamma_0^{(1)}}. \tag{23}
$$

**Proof.** It suffices to show that all terms in the sum on the right hand side of (22) expand into graphs with articulation points. As seen above, each $W_{l-m}^{(k)}|_{1LI}$ expands into $r$-rooted, $r = 1, \ldots, k$, 1LI graphs that are directly (without extra $\ell_{xy}$ link) attached to an $r$-rooted product $\Gamma_{m_1}^{(1)} \cdots \Gamma_{m_k}^{(1)}$ (with the same $\Gamma_{m_1}^{(1)}, m \geq 2$, possibly occurring several times) where each factor expands into 1-rooted 1LI graphs. Each of the rooted vertices therefore is an articulation point and the graphs contributing to a $W_{l-m}^{(k)}|_{1LI}$ term in the sum have at least one articulation point. \hfill \square

On account of the previous results the problem of finding a graph rule for the LRH expansion of $\Gamma_l$ has been reduced to understanding the weight $\mu^\Gamma(y)$ that ought to be assigned to articulation points: by Lemma II.1 we know that the graphs contributing to $\Gamma_l[\phi]$ are one-line irreducible (1LI). As long as the 1LI graph considered has no articulation points Lemma II.2 straightforwardly provides the weight. The same reasoning shows that the maximal number of articulation vertices in some $L \in Z^I_l$ is $\lfloor (l-2)/2 \rfloor$. One may anticipate a trade-off to occur: the vastly reduced number of graphs to be considered (compared to $W$) will be compensated in part by a more complicated weight assignment for articulation vertices. Overall a very significant simplification is found to occur already at low orders; see Table I.

| $l$ | $|I_l|$ | $|Z_I^l|$ | # art. vert. |
|-----|-------|-------|-------------|
| 3   | 5     | 2     | 0           |
| 4   | 12    | 4     | 1           |
| 5   | 33    | 8     | 2           |
| 6   | 100   | 22    | 8.1         |

**Table I.** Number of connected, one-line irreducible, one-line irreducible graphs with 1, 2, \ldots articulation points, respectively, and $l$ edges.

Up to $l = 3$ all 1LI graphs are also 1VI, so that the graph rules for $W_l$ (with vertex weights $\omega_{a}(\phi) := \omega_{a}(h)|_{h=\phi}, m \geq 1$) gives the correct answer for $l \leq 3$. In the figure below the weights from the $W$ rule match the terms in the directly computed result (A4):

$$
\Gamma_3[\phi] = \frac{1}{4} - \frac{1}{12} + \frac{1}{6} + O(\kappa^4). \tag{24}
$$
For $l = 4$ the same works for all but the second to last term, which corresponds to a “pair of glasses” graph. The vertex in the middle is an articulation point and by inspection of (A4) one reads off the weight that should be attributed to it:

$$\text{Symmetry factor} = 2^3$$
$$\mu^\Gamma(v) = \omega_8 - 2\gamma_2\omega_3\omega_6 - \gamma_2\omega_4\omega_5 + 2\gamma_2^2\omega_3^2\omega_5 + 2\gamma_2\omega_3\omega_3^2 + \gamma_3\omega_3^2\omega_4$$  \hspace{1cm} (25)

In each case we also note the symmetry factor of the full graph next to it. For $l = 5$ there are two graphs with articulation points for which the explicitly computed weights are:

$$\text{Symmetry factor} = 2 \times 3!$$
$$\mu^\Gamma(v) = \omega_5 - \gamma_2\omega_4\omega_3\omega_6 + \gamma_2\omega_4\omega_5\omega_6 + 2\gamma_2^2\omega_3\omega_5\omega_6 + 3\gamma_2\omega_2\omega_3\omega_4\omega_5 + \gamma_3\omega_2\omega_3\omega_4\omega_5$$  \hspace{1cm} (26)

$$\text{Symmetry factor} = 2^2$$
$$\mu^\Gamma(v) = \omega_4\omega_3 - \gamma_2\omega_4\omega_4\omega_5 + \gamma_2\omega_4\omega_4\omega_6 + 2\gamma_2^2\omega_4\omega_4\omega_6 - \gamma_2\omega_4\omega_4\omega_6 + 2\gamma_2\omega_4\omega_4\omega_6 - 3\gamma_2\omega_4\omega_4\omega_6 + \gamma_3\omega_3\omega_4\omega_6$$  \hspace{1cm} (27)

Clearly, the first term in the weight associated to an articulation point is the one expected from the $W$ graph rules; it is the systematics of the additional terms that need to be understood.

C. Recursive computation of the weights of articulation points

Our guiding principle in pinning down these systematics will be the relation (22). It expresses $\Gamma_{\mu}$’s graph expansion in terms of those of $\Gamma_{1}^{(1)}, \ldots, \Gamma_{l-2}^{(1)}$, modulo pieces known from the $W$-graph rules. By construction (22) is equivalent to the closed recursion (A2). In contrast to (A2) the mixed recursion (22) allows one to isolate directly contributions from individual graphs, in particular those with articulation points. For example, for $l = 4$ one has $\Gamma_4 = -W_{41111} - \frac{1}{2}W_{2111111} \cdot \Gamma_2^{(1)}$. Applying the graphical differentiation rules to $W_2$ and $\Gamma_2$ one quickly recovers (25). Similarly, for $l = 5$ one obtains from (22) $\Gamma_5 = -W_{51111} - \frac{1}{3}W_2^{(1)} \cdot \Gamma_2^{(1)} - \frac{3}{5}W_2^{(1)} \cdot \Gamma_3^{(1)} - \frac{1}{3}W_2^{(1)} \cdot \Gamma_3^{(1)}$ and (26), (27) can be confirmed graphically. With $\Gamma_l$, $l = 2, \ldots, 5$, known explicitly from Appendix A the same procedure allows one to obtain the weights of all $l = 6, 7$ graphs with articulation points. At $l = 6$ there are 8 graphs with one articulation vertex and 1 with two articulation vertices, see Table 1. The $l = 6$ graph with two articulation vertices is the “triple bubble” graph and both have the same weight associated to them as $v$ in (25).

More interesting are the $l = 7$ graphs for which we present three examples:

$$\text{Symmetry factor} = 2^3 \times 3!$$
$$\mu^\Gamma(v) = \omega_7 - 2\gamma_2\omega_3\omega_6 - \gamma_2\omega_4\omega_5 + 2\gamma_2^2\omega_3\omega_5 + 2\gamma_2\omega_3\omega_3^2 + \gamma_3\omega_3\omega_4^2$$  \hspace{1cm} (28)

$$\text{Symmetry factor} = 2^3$$
$$\mu^\Gamma(v) = \mu^\Gamma(v') = \omega_4 - \gamma_2\omega_4^3$$  \hspace{1cm} (29)

$$\text{Symmetry factor} = 2^3 \times 3!$$
$$\mu^\Gamma(v) = \mu^\Gamma(v') = \omega_7 - \gamma_2\omega_7\omega_4$$  \hspace{1cm} (30)

Here and below we omit the $\varphi$ arguments of the $\omega_\mu$’s. Note that the weight in (28) is new while those in (29) and (30) are recycled from (25), (27) and (26), respectively.

So far the graph expansion of the explicitly computed $\Gamma_{2}, \ldots, \Gamma_{5}$ from Appendix A could be used as an input to obtain the results for all $l = 6, 7$ graphs. The recursion (22) also allows one compute the weights of individual higher order graphs without knowing the full results for the $\Gamma_{\mu}$’s at lower
orders. We illustrate this with two \( l = 9 \) graphs chosen so that the \( l = 7 \) input graphs are among the ones previously displayed.

\[
\begin{align*}
\text{Symmetry factor} = 2^4 \times 3!; \text{ input } & [28], [30] \\
\mu^\Gamma(v) = \omega_6 - \gamma_2 \omega_4 \omega_4 \\
\mu^\Gamma(v') = \omega_7 - 2 \gamma_2 \omega_6 \omega_6 - \gamma_2 \omega_4 \omega_5 \\
&+ \gamma_2^2 \omega_5^2 \omega_6 + 2 \gamma_2^2 \omega_5 \omega_6^2 + \gamma_2 \omega_2^2 \omega_4
\end{align*}
\]

These examples illustrate a pattern that holds generally. To formulate it we introduce a natural grading for the quantities considered. It is induced by the derivatives of the single site functions \( \omega(h), \gamma(\varphi) \) and their interrelations discussed in Appendix B.

**Lemma II.3.** For a monomial in \( \omega_m, m \geq 3, \omega_2^{-1} \), define its degree by: \( \deg \omega_m = m, \deg \omega_2^{-1} = -2 \), and extended additively to products. Then:

(a) \( \deg \gamma_m = -m, m \geq 2 \).

(b) \( \deg W^{(k)}_l = 2l + k, \deg \Gamma^{(k)}_l = 2l - k, k \geq 0 \), to all orders \( l \geq 1 \) of the LRH expansion.

(c) The weight \( \mu^\Gamma(v) \) assigned to an articulation vertex \( v \) has homogeneous degree \( \deg[\mu^\Gamma(v)] \)
which coincides with its degree in the W-graph rule, i.e. \( \deg[\mu^W(v)] = l \), for an \( l \)-valent vertex.

(d) The weight \( \mu^\Gamma(v) \) assigned to an articulation vertex \( v \) of degree \( d(v) = m \geq 4 \) can be normalized such that

\[
\mu^\Gamma(v) = \omega_m - \sum_{i_3 + \ldots + (m-1)\omega_{m-1} = m+2l_2} d_{i_3 \ldots i_{m-1}} (\omega_2^{i_2}) (\omega_5^{i_5}) \ldots (\omega_{m-1}^{i_{m-1}}),
\]

and analogously in any mixed \( \omega_m, \gamma_m \) form.

**Proof.** (a) manifest from [B4]. (b) \( W^{H}_l \) gives degree \( \omega_l \) to 1, so \( \deg W_l = 2l \), each \( H \) derivative raises the degree by 1, \( \text{deg} \Gamma_2 = 4 \) from [A4], each \( \gamma \) derivative lowers the degree by 1 (as \( \gamma H = \gamma_2 \omega_2 \)), and \( \text{deg} \Gamma_l = 2l \).

Since \( \text{deg} W^{(k)}_l = 2l - k \), \( \text{deg} \Gamma^{(k)}_l = 2l + k \).

compatibility with [A4] is ensured. (c) The weight \( \mu^\Gamma(v) \) is in principle determined by the recursion [14], [22]. By (b) these relations preserve homogeneity which implies \( \deg[\mu^\Gamma(v)] = \deg[\mu^W(v)] \).

(d) is a consequence of (c) and the gross structure of [22].

In summary, let \( \mathcal{L}_l \) be the set of one-line irreducible graphs with \( l = |E| \) links. Then

\[
\Gamma_l[\varphi] = \sum_{L \in \{(V,E) \in \mathcal{L}_l\}} \frac{(-1)^{l+1}}{\text{Sym}(L)} \prod_{e \in L} \varphi(e) \prod_{v \in V} \mu^\Gamma(v|L),
\]

with a tacit unconstrained sum over the lattice points associated with the vertices upon embedding.

Here \( \mu^\Gamma(v|L) \) is as in [33] where only the coefficients \( d_{i_3 \ldots i_{m-1}} \) remain to be determined. These coefficients depend on the 1VI subgraphs that are joined at the articulation vertex, not just on the degree of the vertex; so we write \( \mu^\Gamma(v|L) \) from now on.
For completeness’ sake we justify in detail why the weights \( \mu^\Gamma(v|L) \), \( L \in \mathcal{L}_l \), are determined recursively by (22). For the graphical evaluation of (22), graph rules for \( \Gamma_m^{(1)} \), \( m = 1, \ldots, l-2 \), are needed. Differentiating (34) produces an analogous expansion in terms of 1-rooted one-line irreducible graphs for which we write \( \mathcal{L}^{*1}_m \) at order \( |E| = m \). The product over \( \mu^\Gamma(v|L) \) extends over all but the rooted vertex, where \( \partial \mu^\Gamma/\partial \varphi \) occurs. In the context of (22) the coefficients \( d_{1}^{(1)} \ldots d_{m-1}^{(1)} \) entering the \( \Gamma_m^{(1)} \), \( m = 1, \ldots, l-2 \), are assumed to be known and those for the graphs in \( \mathcal{L}_l \) are to be determined. The additional piece of information entering are the graph rules for \( W_{l-m}^{(k)}|_{1LI} \), \( 1 \leq k \leq [m/2] \). These can be inferred from (9). Since multiple \( h \) derivatives can act on the same \( \omega(d(v))(h) \), the number of rooted vertices \( r \) can be \( r = 1, \ldots, k \). The tacit lattice sums ensure that all possible combinations will occur, so that \( W_{l-m}^{(k)}|_{1LI} \) expands into a sum of \( r \)-rooted 1LI graphs with \( l-m \) edges, the set of which we denote by \( \mathcal{L}_{l-m}^{*r} \). The topology of each graph in \( \mathcal{L}_{l-m}^{*r} \) is the same as its counterpart in \( \mathcal{L}_{l-m} \), only the rooted vertices have their \( \omega_m \) weight shifted from \( m = d(v) \) to \( m = d(v) + \#\text{of } h\text{-derivatives} \), and the symmetry factor changes. Each term in the graph expansion of \( W_{l-m}^{(k)}|_{1LI} \Gamma_0^{(1)} \Gamma_{1}^{(1)} \ldots \Gamma_{m_1}^{(1)} \ldots \Gamma_{m_2}^{(1)} \), then has the rooted subgraphs joined at the roots so that an unrooted graph in \( \mathcal{L}_l \) arises. In any concrete instance the procedure is evident and has been used to work out the previous examples. The formulation of the general evaluation principle for (22)’s right hand side justifies that the recursion works generally and just needs to be ‘solved’.
III. GRAPH IMPLEMENTATION OF THE $\Gamma_{\phi} [\phi]$ LRH EXPANSION

So far each $\Gamma_l$ is known to expand into 1LI irreducible graphs $L$ whose weights in (34) are known modulo the coefficients $d_{i_1 \ldots i_m}$ in (33). These coefficients depend on the decomposition of $L$ into 1VI subgraphs, turn out to be integers, and can be understood in terms a separate set of tree graphs. To preclude a possible confusion let us stress that these tree graphs are conceptually and technically different from the ones governing the interplay between vertex functions and connected correlation functions, see Appendix B for the latter.

A. Labeled tree graphs

We begin by introducing a class of unlabeled tree graphs called ‘dashed’, which get labeled in a second step.

**Definition:** The ‘dashed’ graphs are tree graphs where two types of vertices are connected by dashed lines. The set of “open circle” vertices is denoted by $\nu_0$, the set of “dashed” vertices is denoted by $\nu_1$, and the edge list $\varepsilon \subset (\nu_0 \cup \nu_1)^2$ is constrained as follows. The valency of an open circle vertex is $1, 2, \ldots$, dashed vertices have valency $3, 4, \ldots$, and no two dashed-vertices are connected by a single dashed line. The Euler relation for tree graphs then holds in the form $|\nu_0| + |\nu_1| = |\varepsilon| + 1$. We write $\mathcal{T}_n$ for the set of topologically distinct such graphs with $n = |\nu_0|$ open circle vertices.

For example the graphs in $\mathcal{T}_1, \ldots, \mathcal{T}_4$ are

\[\begin{align*}
\mathcal{T}_1 & \quad \circ \\
\mathcal{T}_2 & \quad \circ \cdots \cdots \cdots \circ \\
\mathcal{T}_3 & \quad \circ \circ \circ \\
\mathcal{T}_4 & \quad \circ \circ \circ \circ \circ \circ \circ \circ \circ \circ \\
\end{align*}\]

The restriction that no two dashed-vertices can be connected by a single dashed line eliminates from consideration graphs of the form

\[\begin{array}{c}
\circ \\
\circ \\
\end{array}\]

The graphs in $\mathcal{T}_{n+1}$ can be obtained from those in $\mathcal{T}_n$ by adding one dashed leg with an open circle in all topologically inequivalent ways to an open circle, a dashed line, or a dashed vertex. Further, the constituents of a dashed graph can be attributed a “dashed degree $d_{\text{deg}}$" as follows:

- $|o|$-valent open circle vertex $o$, $|o| \geq 1$ \hspace{1cm} $d_{\text{deg}} = |o| + d_j$,
- dashed line connecting two open circles $d_{\text{deg}} = -2$,
- $m$-valent dashed vertex, $m \geq 3$ \hspace{1cm} $d_{\text{deg}} = -m$.

Here $d_j, j = 0, \ldots, n-1$, are integers whose significance will become clear shortly. Then:

$$d_{\text{deg}}(t) = \sum_{j=0}^{n-1} d_j, \text{ for any dashed graph } t \in \mathcal{T}_n, n \geq 2.$$
inspection of (35) the assertion holds for \( n = 1, 2, 3 \) and (37) follows. Note that this gives a more fine grained invariant than merely the Euler relation (10) for tree graphs. Instead of viewing the \( d_j \) as parameterizing the ddeg function one may also regard them as labels for the dashed graphs themselves. We then write \( \mathcal{T}_n^D \) for the set of dashed graphs \( \mathcal{T}_n \) with an integer from the \( n \)-tuple \( D = (d_0, d_1, \ldots, d_{n-1}) \) assigned to each open circle vertex. The use of an \( n \)-tuple is natural in the iteration of the map (39) below. Later on we use the same notation \( \mathcal{T}_n^D \) when \( D \) is a multiset of integers of cardinality \( n \).

**Compatibility with differentiation:** Each \( \tau \in \mathcal{T}_n^D \) can be assigned a weight \( \mu(\tau) \) as follows

\[
\mu_{|\omega|+d_j} \quad \text{to an } |\omega|\text{-valent open circle vertex } \omega, \quad |\omega| \geq 1; \\
\gamma_2 \quad \text{to a dashed line connecting two open circles}, \\
\gamma_m \quad \text{to an } m\text{-valent dashed vertex, } m \geq 3.
\]

The degree of each factor in \( \mu(\tau) \) equals the ddeg of the underlying graph, \( d(\omega_{|\omega|+d_j}) = |\omega|+d_j = \text{ddeg}(|\omega|\text{-valent open circle vertex}) \), etc. Hence \( d(\mu(\tau)) = \sum_{j=0}^{n-1} d_j \), for all \( \tau \in \mathcal{T}_n^D \). We write \( \mu(\mathcal{T}_n^D) \) for the span of all \( \mu(\tau), \tau \in \mathcal{T}_n^D \). Augmenting a \((n+1)\)-st integer \( d_n \) we claim that

\[
\omega_{d_n+1} : \mu(\mathcal{T}_n^D) \rightarrow \mu(\mathcal{T}_{n+1}^D), \tag{39}
\]

with the understanding that \( D = (d_0, \ldots, d_{n-1}, d_n) \) in the range. This follows from the basic differentiation rules \( \partial_\phi \omega_m = \gamma_2 \omega_{m+1}, \partial_\phi \gamma_m = \gamma_{m+1} \) and the way \( \omega_{d_n+1} \partial_\phi \) acts on the three types of factors in each \( \mu(\tau) \): acting on \( \omega_{d_j+1} \) the operator produces \( \omega_{d_j+1} \gamma_2 \omega_j \gamma_1 \), equivalent to adding a dashed edge with an open circle vertex to an existing open circle vertex. Acting on \( \gamma_2 \) it produces \( \omega_j \gamma_1 \gamma_2 \), which adds a a dashed edge with an open circle vertex to a dashed line. Finally, acting on \( \gamma_m, m \geq 3 \), gives \( \omega_j \gamma_1 \gamma_{m+1} \), which adds a line with an open circle to an existing dashed vertex. These basic operations are in one-to-one correspondence to those generating the unlabeled graphs \( \mathcal{T}_n+1 \) from \( \mathcal{T}_n \), verifying that (39) has the correct range. Starting at \( n = 1 \) with an \( \omega_{d_0} \) assigned to the open circle vertex, one may verify directly that repeated action of (39) produces a sum of terms whose underlying graphs match those in (35) but with integers \(|\omega|+d_j, j = 0, 1, 2, \ldots, \) assigned to their open circle vertices. The map (39) provides the raison d'être for the dashed graphs.

**Lemma III.1.** The recursion (22) generates only vertex weights \( \mu^\Gamma(v|L) \) in (34) that lie in the \( (v\)-dependent) direct sum of \( \mu(\mathcal{T}_n^D) \), for \( n = 1, \ldots, n_{\text{max}}, n_{\text{max}} \leq d(v) - 3, \) for some integer multiset \( D_n \).

**Proof.** We proceed by induction on \( l \) with \( L \in \mathcal{L} \). The assertion holds by inspection of (25), (26), (27) for \( l = 4, 5 \). For the \( l = 1 \rightarrow l \) step in the recursion (22) we denote by \( L_j \in \mathcal{L}_{m_j} \) one of the I1I graphs in \( \Gamma_{m_j} \)'s graph expansion and by \( L^W \in \mathcal{L}^*_{l-m} \) one of the I1I graphs in \( W_{l-m} \)'s expansion.

We focus on one of the vertices \( v \) where the graphs are joined and write \( v_0 \) for \( v \)'s copy in \( L^W \) and \( v_j \) for \( v \)'s copy in \( L_j \), \( j = 1, \ldots, k \). By the \( W \)-graph rule the structure of \( L^W \) is irrelevant only the weight \( \omega_{d(v_0)}(h)|_{h=H_k} \) (and the inverse symmetric factor \( \text{Sym}(L^W) \) irrelevant here) enters. If \( r \) of the \( k \) functional differentiations with respect to some \( H_i \) act on the chosen \( H_k \) site the weight will be shifted to \( \omega_{d(v_0)+r}(h)|_{h=H_i} \). The associated graph will still be denoted by \( L^W \); it now has one rooted vertex \( v_0 \) to which we attribute multiplicity \( r \). The single differentiation of \( \Gamma_{m_j} \) with respect to some \( \phi_i \) will always produce \( r \)-rooted graphs, and for the ones rooted at \( v_j \) we write \( L^*_{m_j} \in \mathcal{L}^*_{m_j} \).

In any one term contributing to (22) at \( v \), a \( v_0 \) of multiplicity \( r \) will have \( r \) I1I graphs attached, which are selected from the \( L^*_j \in \mathcal{L}^*_{m_j}, j = 1, \ldots, k \). Without loss of generality we take \( L^*_j \in \mathcal{L}_{m_j}, j = 1, \ldots, r \), as the graphs attached to \( v_0 \). For fixed \( r \) the weight associated with \( v \) is by (34)

\[
\omega_{d(v_0)+r} \prod_{j=1}^r \partial_\phi \mu^\Gamma(v_j|L_j).
\]

By the induction hypothesis all \( \mu^\Gamma(v_j|L_j) \) have an expansion in the \( (v_j\)-dependent) direct sum of \( \mu(\mathcal{T}_n^D) \), \( n = 1, \ldots, n_{\text{max}} \). Focus on a term with \( n_j \) open circle vertices in \( \mu^\Gamma(v_j|L_j) \). For \( r = 1 \) the product (40) is directly of the form (39) and the assertion follows. For \( r \geq 2 \) one notices that each
partial derivative of the gluing operation that first creates an external dashed edge and then joins any number of such trees at an extra open circle vertex. As shown in (33) shows that the maximal number of cells that can occur in a normalized weight $\mu^T(v|L)$ is $n_{\text{max}} \leq d(v) - 3$ (while the actual $n_{\text{max}}$ turns out to be much smaller).

It remains to understand the coefficients with which the various dashed graphs occur in $\mu^T(v|L)$. To this end a different type of labeling turns out to be useful.

**Assignment of labels:** The labels are set partitions of vertices as frequently used in other contexts. In the situation at hand, the vertex set $\{b_1, \ldots, b_j\}$ will later be identified with the one associated with an articulation vertex $v$ in the block decomposition (as defined at the end of Section II.A) of the underlying one-line irreducible graph $L$. For now we ignore the origin of the set $B = \{b_1, \ldots, b_j\}$ and consider its set partitions. If all elements of $B$ are distinct, a set partition of $B$ is a set of non-empty disjoint subsets of $B$ whose union is $B$. An element of a partition is called a cell; we write $\mathcal{S}(B, k)$ for the set partitions of $B$ with $k$ cells. The number of partitions of a set $B$ with $l$ distinct elements into $n$ cells is given by $S(l, n)$, the Stirling number of the second kind. The total number of set partitions is given by the Bell number $B(l) = \sum_{n=1}^{l} S(l, n)$. A convenient generating function is $\exp\{y(e^y - 1)\} = \sum_{n,l > 0} S(n, l) y^n x^l / l!$. Generalizations have been considered in [41].

The same concept applies to multisets, i.e. sets of pairs $(b_i, m_i)$ where $m_i \in \mathbb{N}$, specifies the multiplicity with which $b_i$ occurs. We write

$$B = \{b_1^{m_1}, \ldots, b_j^{m_j}\} = \{b_i : i \in I\}, \quad I = \{1, \ldots, 1, \ldots, J, \ldots, J\},$$

(41)

for the multiset with multiplicities $(m_1, \ldots, m_j) \in \mathbb{N}^J$. In the alternative notation with explicitly repeated elements the indexing $I$ is viewed as a multiset. Two multisets are identical if and only if they contain the same elements with the same multiplicities. The partitions of a multiset are defined as the set partitions of $|I| = \sum m_i$ element set where $m_i$ copies of $b_i$ are identified afterwards and ‘duplicates’ are omitted from the list. There are several notions of ‘duplicates’ one can use; we allow both repeated cells and repeated elements within a cell but eliminate duplicates of the same partition. For example, $\mathcal{S}(\{a, b, c^2\}, 3)$ has 4 elements, $\{\{a\}, \{b\}, \{c^2\}\}$, $\{\{a\}, \{b, c\}, \{c\}\}$, $\{\{a, b\}, \{c\}, \{c\}\}$, $\{\{a, c\}, \{b\}, \{c\}\}$, as opposed to $|\mathcal{S}(\{a, b, c^2\}, 3)| = 6$.

Unless specified otherwise we shall use $B$ in the following to be a multiset of the form (41). A partition of $B$ with $n$ cells is then used to label the open circle vertices of a graph in $\mathcal{T}_n$. One may think of each open circle vertex as a ‘bag’ that contains a cell. Technically, the labeling map is for each partition $\pi \in \mathcal{S}(B, n)$, a bijection $v_0 \mapsto v_0^\pi$, of sets of cardinality $n$. While the vertices $v_0$ of the unlabeled graph may be assigned ‘dummy’ labels that can be freely permuted in probing for isomorphisms the elements of $v_0^\pi$ can only be permuted if their labels coincide. Isomorphic labeled graphs are defined as in Section II.A, with $V = v_0^\pi$ as vertex set. We write $\mathcal{T}(B, n)$, $1 \leq n \leq l$, for the set of topologically inequivalent dashed graphs with $n$ open circle vertices labeled by $\mathcal{T}(B, n)$. Further, for some unlabeled $t \in \mathcal{T}_n$ we write $T \in \mathcal{T}(B, n)$ for one of its labeled counterparts.

As an illustration consider $n = 3$. The set partitions of $B = \{b_1, b_2, b_3\}$ are

$$\begin{align*}
\{\{b_1, b_2, b_3\}\}, \\
\{\{b_1\}, \{b_2, b_3\}\}, \\
\{\{b_2\}, \{b_1, b_3\}\}, \\
\{\{b_3\}, \{b_1, b_2\}\}, \\
\{\{b_1\}, \{b_2\}, \{b_3\}\}. \\
\end{align*}$$

(42)
These are then assigned as labels to the open circle vertices of the graphs in $\mathcal{G}_1, \mathcal{G}_2, \mathcal{G}_3$:

\[
\begin{align*}
\{b_1, b_2, b_3\} & \in \emptyset \\
\{b_1\} & \in \emptyset \cdots \cup \{b_2, b_3\}, \quad \{b_2\} & \in \emptyset \cdots \cup \{b_1, b_3\}, \quad \{b_3\} & \in \emptyset \cdots \cup \{b_1, b_2\}, \\
\{b_1\} & \emptyset \cup \{b_3\}, \quad \{b_2\} & \emptyset \cup \{b_3\}, \quad \{b_1\} & \emptyset \cup \{b_2\}, \\
\{b_2\} & \emptyset \\
\{b_1\} & \emptyset \cup \{b_3\}
\end{align*}
\]  
(43)

Clearly, none of the labeled graphs (43) allows for nontrivial automorphisms. This may change when multisets are used to generate the labels.

**Symmetry factors:** Each unlabeled $t \in \mathcal{T}_n$ has an automorphism group which we define in the obvious way: let $V_0$ be the set of open circle vertices, $V_1$ the set of dashed vertices, and $\epsilon \subset (V_0 \cup V_1)$ the edge list, subject to the constraints in the definition. An automorphism of $t$ is a permutation of $V_0 \cup V_1$ that leaves $V_0$, $V_1$ and the edge list separately invariant. These form a group for which we write $\text{Aut}(t)$.

The labeling process described above precludes nontrivial permutations except when $B$ has repeated elements. For a multiset (41) the $m_i$ copies of $b_i$ can be permuted giving rise to a direct product $\text{Perm}(B) := S_{m_1} \times \cdots \times S_{m_n}$ of symmetric groups acting on $B$. For a partition $\pi \in \mathcal{P}(B, n)$ let $V_\pi^C = \{(o_i, c_i) : i = 1, \ldots, n\}$ be the set of $n$ labeled open circle vertices. Note that each dummy labeled $o_i$ is paired with non-dummy $c_i$, the index $i$ merely enumerates the list of pairs. Each cell $c_i, i = 1, \ldots, n$ may again be a multiset $\{b_i^{c_i,1}, \ldots, b_i^{c_i,J}\}$ with multiplicities $c_i,1, \ldots, c_i,J \in \mathbb{N}_0$, where we set the multiplicity to zero if the element does not occur. A subgroup $\text{Perm}(c_i) = S_{c_i,1} \times \cdots \times S_{c_i,J} \subset \text{Perm}(B)$ (with factors absent whose multiplicity is zero) will still permute copies of the same elements that $c_i$ may contain. By the very process of forming set partitions the direct product $\text{fix}(V_0^B) := \text{Perm}(c_1) \times \cdots \times \text{Perm}(c_n) \subset \text{Perm}(B)$ is still a subgroup, with $S_{c_1,1} \times \cdots \times S_{c_n,1} \subset S_{m_1}$, etc. In other words, $\text{fix}(V_0^B)$ is the subgroup of $\text{Perm}(B)$ that maps individual labels of $T \in \mathcal{T}(B, n)$ into themselves. Writing $|\text{fix}(V_0^B)|$ for its order, one has by Lagrange’s theorem $|\text{Perm}(B)|/|\text{fix}(V_0^B)| \in \mathbb{N}$.

An automorphism of $T \in \mathcal{T}(B, n)$ is defined as in the unlabeled case, except that the unlabeled set $V_0$ of $t \in \mathcal{T}_n$ is replaced with the labeled one $V_0^B = \{(o_i, c_i) : i = 1, \ldots, n\}, \pi \in \mathcal{T}(B, n)$. Since the $o_i$ labels are dummy two elements of $V_0^B$ are regarded as equal iff their cells $c_i$ are equal as multisets. The $n$ labeled vertices can thus be grouped into subsets with the same label. An automorphism of the underlying unlabeled graph $t$ that affects only sets of equally labeled vertices is also an automorphism of $T$, and all automorphisms of $T$ arise in that way. They form again a group, denoted by $\text{Aut}(T)$, which is a subgroup of $\text{Aut}(t)$. Finally, the symmetry factor of a labeled tree graph $T \in \mathcal{T}(B, n)$ is defined by

\[
\text{Sym}(T) = |\text{Aut}(T)||\text{fix}(V_0^B)|.
\]  
(44)

As an illustration of these concepts, reconsider the graphs in (43) but now labeled by the set partitions of $\{b, b, b’\}$. The symmetry factors (44) may differ from 1 and are noted to the right of
each graph:

\[
\begin{align*}
\{b, b', b'\} & \in \bigcirc \quad \text{Sym} = 2 \\
\{b\} & \in \bigcirc \quad \bigcirc \ni \{b, b'\} \quad \text{Sym} = 1, \quad \{b'\} \in \bigcirc \quad \bigcirc \ni \{b, b\} \quad \text{Sym} = 2, \\
\{b\} & \quad \bigcirc \ni \quad \bigcirc \ni \{b'\} \quad \text{Sym} = 1, \quad \{b\} \in \bigcirc \quad \bigcirc \ni \{b\} \quad \text{Sym} = 2, \\
\{b\} & \quad \bigcirc \ni \quad \{b'\} \quad \text{Sym} = 2
\end{align*}
\] (45)

We now claim that

\[|\text{Perm}(B)|/\text{Sym}(T) \in \mathbb{N},\] (46)

We present a direct proof on the level of multisets here. In Section IV the result is recovered along different lines. Suppose that \(\bar{\phi}\) of the elements of \(v^b_0\) of \(T\) are equally labeled, and that there is a subgroup \(A\) of \(\text{Aut}(t)\) that acts transitively on \(\{o_1, \ldots, o_p\}\) of \(t\), with \(p \leq \bar{\phi}\). A may act on vertices other than \(\{o_1, \ldots, o_p\}\), once these are labeled only a subgroup \(A_0\) of \(A\) may act on the equally labeled \(\{(a_j, c_j) : j = 1, \ldots, n\}\), \(c_1 = \ldots = c_p\). Note that \(p!/[A_0]\) is an integer. We wish to lift this \(A_0 \subset \text{Aut}(t)\) to an automorphism group of the labeled version \(T\) of \(t\) with the \(p\) equally labeled \(\{(a_j, c_j) : j = 1, \ldots, p\}\). Focus on one of the elements of the \(p\) identical cells \(c_j\) with nonzero multiplicity, say \(b_j^{(j)}\), \(j = 1, \ldots, p\), without loss of generality. Overall these are \(pn_1\) copies of \(b_1\) which arose by distributing the original \(m_1 \geq pn_1\) copies from (41) to the cells under consideration and possibly others. Hence there is a subgroup \(S_{pn1} \subset S_{m1}\) that permutes the copies of \(b_1\) within each cell and mixes the \(b_1\) sectors of different cells. The permutations within each cell are part of \(\text{fix}(v_0^b)\) and have total order \((n_1!)^p\). The other permutations implement the desired automorphisms of \(T\) within \(S_{pn1}\). The relevant ratio thus is \(p!/[A_0] \in \mathbb{N}\) times \((pn_1)!/[p!(n_1!)^p]\). The latter is indeed an integer for all \(n_1 \in \mathbb{N}\). Repeating the argument for all elements of the \(p\) identical cells with nonzero multiplicities one arrives at \(|\text{Perm}(B)|/\text{Sym}(T) \in \mathbb{N}\). This argument does not rely on a group structure of the permutations of elements across cells.

B. Formulation and illustration of the graph rule

We now return to the previous result (34) and provide a graph rule for the missing ingredient \(\mu^L(v|L), \) where \(L \in \mathcal{L}_1\) is a 1LI graph with \(l\) edges and \(v\) is one of its vertices. Recall the notion of a block decomposition from the end of Section II.A. Each \(L \in \mathcal{L}_1\) is either itself 1VI or has a block decomposition \(\{L_1, \ldots, L_N\}\), in terms of maximal 1VI subgraphs \(L_j = (B_j, E_j), j = 1, \ldots, N\), which must also be 1LI. Each articulation vertex occurs in more than one \(B_j\), while non-articulation vertices occur in precisely one \(B_j\). For a fixed articulation vertex \(v\) let \(B(v) = \{L_i(v) = (B_i, E_i), i \in I(v)\}\), with \(2 \leq |I(v)| \leq N\), be the subset of blocks with \(v \in B_i\). Isometric blocks can be permuted, we denote this permutation group by \(|\text{Perm}(B(v))|\) and its order by \(|\text{Perm}(B(v))|\). We write \(b_i(v)\) for the copy of \(v\) in \(B_i\), \(i \in I(v)\), and treat the copies \(b_i(v)\) as identical, \(b_i(v) = b_j(v)\), iff \(L_i(v)\) and \(L_j(v)\) are isomorphic. Viewed as a vertex in \(L_i(v)\) each \(b_i(v)\) has a degree (with respect to the full lines) \(d(b_i(v)) \geq 2\) such that \(\sum_{i \in I(v)} d(b_i(v)) = d(v)\). Non-articulation points can formally be included in this setting by allowing \(|I(v)| = 1\).

With this convention the edge sets are redundant and we also write \(B(v) = \{b_i(v) : i \in I(v)\}\), on which the same permutation group \(|\text{Perm}(B(v))|\) acts. In general \(B(v)\) will be a multiset which use the notations (41). Then \(|\text{Perm}(B(v))|\) is a direct product \(S_{m_1} \times \cdots \times S_{m_j}\) of symmetric groups. Next we generate the set partitions \(\mathscr{P}(B(v), n)\) with \(n = 1, \ldots, |I(v)|\) cells. Each cell \(c_i = \{b_i^{(1)}, \ldots, b_i^{(j)}\}\) may again be a multiset whose multiplicities obey \(\sum_{l=1}^{n} c_{i,l} = m_j\) and 
\(\sum_{l=1}^{n} \sum_{j=1}^{I} c_{i,l} = |I(v)|\). The cells are used to label the open circle vertices of the tree graphs in \(\mathscr{P}_n, 1 \leq n \leq |I(v)|\), as in Section III.A; coinciding labels are allowed and correspond to cells coinciding as multisets. We write
$\mathcal{T}(B(v), n)$ for the set of topologically distinct labeled dashed graphs with $n$ open circle vertices labeled by $\mathcal{T}(B(v), n)$. Individual labeled graphs are denoted by $T \in \mathcal{T}(B(v), n)$, with $\text{Sym}(T)$ the symmetry factor.

**Theorem III.1** (Graph rules for vertex weights). The weights $\mu^T(v|L)$ in (34) depend only on the block decomposition $B(v)$ of $L$ at $v$ and can be obtained by the following graph rule:

(a) A weight $\mu(T)$ is assigned to each labeled graph $T \in \mathcal{T}(B(v), n)$, $1 \leq n \leq |I(v)|$, as follows: for an $|o|$-valent (with respect to the dashed lines) open-circle vertex $o$ labeled by the cell $c_i = \{b_1^i, \ldots, b_m^i\}$ write a factor $\omega^{o|d(c_i)}(\varphi_v)$, $d(c_i) := \sum_{j=1}^m c_i d(b_j)$, for each dashed line connecting two open circle vertices a factor $\gamma^3(\varphi_v)$, and for each vertex with $m \geq 3$ intersecting dashed lines a factor $\gamma^3_m(\varphi_v)$ (the dashed lines that intersect at the dashed-vertex do not contribute a factor). The resulting monomial $\mu(T)$ in $\gamma^3, \omega^3, \gamma^3_m, m \geq 3$, has by (37) degree $\sum_{i=1}^n d(c_i) = \sum_{j=1}^m d(b_j) = d(v)$.

(b) Multiply $\mu(T)$ by

$$(-)^{s(T)} \frac{|\text{Perm}(B(v))|}{\text{Sym}(T)},$$

where $s(T)$ is the sum of the number of dashed lines and the number of dashed vertices. Further, $|\text{Perm}(B(v))|$ is the order of the permutation group of the blocks at $v$, and $\text{Sym}(T)$ is the symmetry factor of the labeled dashed graph as defined in Section III.A.

(c) Sum the contributions from (a),(b) over all $n$ and $T \in \mathcal{T}(B(v), n)$ to obtain $\mu^L(v|L) = \mu^B(v|B)$ as

$$\mu^L(v|B) = \sum_{n=1}^{|I(v)|} \sum_{T \in \mathcal{T}(B(v), n)} (-)^{s(T)} \frac{|\text{Perm}(B(v))|}{\text{Sym}(T)} \mu(T).$$

This is normalized such that $\mu^L(v|L) = \omega^o(\varphi_v)$ for a non-articulation vertex ($|I(v)| = 1$) of degree $d(v)$.

**Illustration of the graph rule:**

(i) The simplest case is the “pair of glasses” graph in (25). It has two isomorphic blocks $\phantom{}$, joined at the articulation point. Hence $|\text{Perm}(B(v))| = 2$. The vertex set $B(v)$ contains two copies of the same element, $\{b, b\}$, say, with $d(b) = 2$. The set partitions of $B(v)$ are $\{(b, b)\}$ and $\{(b)\}$, $\{(b)\}$. Thus the labeled dashed graphs $T \in \mathcal{T}(B(v), 1), \mathcal{T}(B(v), 2)$, are

$$\{b, b\} \in \circ, \quad \{b\} \in \circ \cdots \circ \ni \{b\}. \quad (49)$$

They have each $\text{Sym}(T) = 2$, and contribute $\omega^o(\varphi_v) - \gamma^3(\varphi_v) \omega_3(\varphi_v)^2$, respectively, in the sum (48). This reproduces the weight in (25).

(ii) As a more complicated exemplification consider (31). At $v'$ three block are joined: two copies of $\phantom{}$, and $\phantom{}$. Hence $|\text{Perm}(B(v'))| = 2$. The vertex set $B(v') = \{b, b, b'\}$ with $d(b) = 2, d(b') = 3$, gives rise to the labeled tree graphs presented in (45). The sum (48) evaluates to

$$\mu(v'|L) = \frac{2}{2} \omega^o(\varphi_v) - \frac{2}{2} \gamma^3(\varphi_v) \omega_3(\varphi_v) \omega_3(\varphi_v) - \frac{2}{2} \gamma^3(\varphi_v) \omega_3(\varphi_v) \omega_3(\varphi_v) + \frac{2}{2} \gamma^3(\varphi_v) \omega_3(\varphi_v) \omega_3(\varphi_v) + \frac{2}{2} \gamma^3(\varphi_v) \omega_3(\varphi_v) \omega_3(\varphi_v)$$

in agreement with (31). At $v$ two distinct blocks are joined, $\phantom{}$ and $\phantom{}$. Hence $|\text{Perm}(B(v))| = 1$. The vertex set is $B(v) = \{b, b'\}$, with $d(b) = 2, d(b') = 3$. The labeled tree graphs are as in (49) but with distinct elements $b, b'$. Both have symmetry factor 1 and the sum (48) gives $\mu(v|L) = \omega^o(\varphi_v) - \gamma^3(\varphi_v) \omega_3(\varphi_v) \omega_3(\varphi_v)$, again in agreement with (31).
C. Proof of the graph rule

We first bring into focus what needs to be shown. By Lemma III.1 each weight $\mu^I(v|L)$ lies in the direct sum of $\mu(I^D_n), n = 1, \ldots, n_{\text{max}}, n_{\text{max}} \leq d(v) - 3$, for some integer multiset $D_n$. It is convenient to introduce a projection operation

$$\text{pr} : \mathcal{T}(B(v), n) \rightarrow \mathcal{T}(D(v), n), \quad T \mapsto \text{pr}(T),$$  \hspace{1cm} (51)

where $D(v) = d(B(v))$, as a multiset. Further, each cell $c_i$ labeling $T \in \mathcal{T}(B(v), n)$ is replaced by its integer degree sum $d(c_i)$. The result is an element of $\mathcal{T}_n^D$, where the integers $D_n$ are $d(\pi) := \{d(c_1), \ldots, d(c_n)\}$ (viewed as a multiset), if $\pi = \{c_1, \ldots, c_n\}$ is the partition labeling $T$. Note that each $D_n$ is drawn from the $n$-element partitions of $D(v) = \{d(b_i) : i \in I(v)\}$, with each integer in $D_n$ equal to the sum of the integers in the cell. Since distinct $b_i$'s can have the same degree and many degree sums $d(c_i)$ can be equal as well, the projected label set $\mathcal{T}(D(v), n)$ will in general be of much smaller cardinality than $\mathcal{T}(B(v), n)$. We write $\mathcal{T}(D(v), n)$ for the set of topologically inequivalent labeled dashed graphs with $n$ open circle vertices labeled by some $D_n \in \mathcal{T}(D(v), n)$.

Clearly, $\mu(T) = \mu(\pi)$, for $\pi = \text{pr}T$, if $\mu(\pi)$ is formed according to (38). The graph rule is therefore compatible with Lemma III.1 and the projection (51). What remains to be shown is: $n_{\text{max}} = |I(v)|$, and

$$\text{coefficient of } \mu(T) = \left(-\right)^{v(T)} \frac{|\text{Perm}(B(v))|}{\text{Sym}(T)}, \quad T \in \mathcal{T}(B(v), n), \quad n = 2, \ldots, n_{\text{max}}.$$  \hspace{1cm} (52)

The case $n=1$ is accounted for by (53) and can be omitted. Indeed, for $n=1$ the only $T \in \mathcal{T}(B(v), 1)$ graph is an open circle labeled by $c_1 = B(v)$, the groups $\text{Perm}(B(v))$ and $\text{fix}(v_0^B)$ coincide and the coefficient of $\mu(T) = \omega_{d(v)}$ is 1, in agreement with (53). By Lemma II.2 we can also match the situation where only the $\omega_{d(v)}$ term in (51) is present to graphs $L$ without articulation points, in agreement with the graph rule. The key step is:

Lemma III.2. For 1LI graphs with one articulation point (52) and hence the graph rule (a),(b),(c) is compatible with the recursion (22).

Proof. We proceed by induction in $l$ by assuming that (a), (b), (c) of the graph rule produce the correct vertex expressions for orders $1, \ldots, l-1$. Let $L \in \mathcal{L}_l^I$ be a 1LI graph with a single articulation vertex $v$, and $L_1(v), \ldots, L_l(v)$ is its block decomposition. The $L_i(v)$ are viewed as rooted graphs, with roots $b_i$ regarded as identical iff the $L_i(v)$ are isomorphic as 1VI graphs. With this convention $B(v)$ is a multiset of the form $\{1\}$ which codes the structure of $L$ at $v$. In the recursion (22) the contribution coming from $L$ is reassembled from its block decomposition by gluing together various blocks arising from the graph expansion of the $W_{l-m}^{(1)} \Gamma_0^{(1)} \ldots \Gamma_k^{(1)}$ pieces. The weights associated with non-articulation vertices are known from Lemma II.3, so we can focus on $v$.

At $v$ the structure of the 1LI graph $L^W$ induced by $W_{l-n}^{(k)}$ is irrelevant (as in the proofs of Lemmas III.1 and III.3) and only the shifted weight $\omega_{d_0 + k}$ enters, where $d_0 = d(v_0)$ is the valency (wrt the full lines) of $v$'s copy $v_0$ in the 1LI graph associated with $W_{l-m}$. The shift counts the number of $h$-derivatives acting on $\omega_{d_0}(h)|_{h = H_L}$; since $L$ has by assumption only one articulation point all $k$ derivatives in $W_{l-m}^{(k)}$ must act on the same vertex weight, viz $\omega_{d_0}(h)$. In particular $v_0$ should be viewed a rooted vertex with multiplicity $k$. Attached to $v_0$ will be $k$ 1LI graphs $L_j^W \in \mathcal{L}_{m_j}^{\Gamma}$ that arise from the graph expansion of $\Gamma_j^{(1)}$, respectively, and that are rooted at some $v_j$. Each of the $L_j$ may decompose into several blocks at $v_j$; in the above convention we may write $B(v_j)$ for the set of blocks stemming from $L_j^\Gamma \in \mathcal{L}_{m_j}^{\Gamma}$. Similarly, $L^W$ may decompose into several blocks at $v_0$ and we write $B(v_0)$ for their vertex set. Then $B(v)$ is the union of the $B(v_j), j = 0, 1, \ldots, k$, as $k$ runs through all possible values in (22). For fixed $k$ the weight with $v$ can be written in terms of its copies in $B(v_j)$ as

$$\omega_{d(v_0) + k} \prod_{j=1}^k \partial_{v_j} \mu^W(v_j|B(v_j)), \quad k = 1, \ldots, [m/2],$$  \hspace{1cm} (53)
where by induction hypothesis each $\mu^\Gamma(v_j|B(v_j))$ is given by (48). In particular, each $\mu^\Gamma(v_j|B(v_j))$ expands into contributions associated with dashed graphs in $\mathcal{T}(B(v_j),n_j)$, $n_j = 1,\ldots,|I(v_j)|$, labeled by the set partitions of $B(v_j)$ with $n_j$ cells, $j = 1,\ldots,k$. The minimal number of cells is $k$, the maximal number of cells is $\sum_{j=1}^k |I(v_j)| = |I(v)| - 1$. The blocks in $B(v_0)$ are not subject to the $\Gamma$ graph rule but to the $W$ graph rule and thus only give rise to the $\omega_{d(v_0)}$ factor. In terms of the integer labeled dashed graphs $d(v_0)$ labels a $k$-valent open circle vertex $o \in v_0$ and conversely every $k$-valent open circle vertex in an integer labeled dashed graph can be associated with a $W_{l-m}$ induced piece. Each of the $k$-subtrees joined to it has a dashed edge generated by $\partial_v$ without open circle vertex, while all of its $n_j$ open circle vertices are labeled by the set partitions in $\mathcal{T}(n_j,B(v_j))$, $j = 1,\ldots,k$. The result is a dashed graph $T'$ with $n = 1 + \sum_{j=1}^k n_j \in \{k+1,\ldots,|I(v)|\}$ open circle vertices, all but one of which are labeled by set partitions drawn from those of the individual $B(v_j)$'s. In the setting of (52) this fixes $n_{\max} = |I(v)|$. Subject to the degree constraint $d(v_0) = d(v_0)$ the so far only integer labeled vertex $v_0$ can also be labeled by some vertex set $c_0$.

We are free to postulate that $T'$ ought to be relabeled – while preserving the weight – by the much larger set of set partitions of $B(v)$, viewed as the union of $B(v_j)$, $j = 0,1,\ldots,k$, to produce a graph in $T \in \mathcal{T}(B(v),n)$, $n = k+1,\ldots,|I(v)|$. For later reference we mark the transition from $T'$ to $T$ with $\mu(T') = \mu(T)$ by $(\ast)$. Summing the contributions (53) over all $k = 1,\ldots,[m/2]$, $m = 2,\ldots,l-2$, we know that the result must be of the form (48) but with a yet undetermined coefficient of $\mu(T')$. As noted after (51) the weight only depends on the projection $pr(T)$ of the graph, but we are free to stipulate that the integers occurring are the degree sums of the cell partitions of $B(v)$, i.e. $d_i = d(c_i)$. Then part (a) of the graph rule holds by construction and only the assertion about coefficient (52) needs to be shown.

By the induction hypothesis each the $\Gamma_{m1}$ induced pieces comes with a $1/\text{Sym}(L^F)$ factor where $L^F_j \in \mathcal{L}^{m\bullet}_j$ is a 1-rooted 1LI graph. Similarly, by the $W$ graph rule the $W_{l-m}^{(k)}$ induced piece carries a $1/\text{Sym}(L^W)$ factor, where $L^W$ is also a 1-rooted 1LI graph but with the root attributed multiplicity $k$, as seen above. Since we focus on 1LI graphs that combine to $L$ (and $L$ has only one articulation point) all $L^W$, $L^F_1,\ldots,L^F_k$, have a clover-like block decomposition with the blocks joined at a central vertex $v_0,v_1,\ldots,v_k$, respectively. Each block is treated as 1-rooted and arranged in some lexicographic order that they can be identified with the blocks $L_1,\ldots,L_l$ of $L$ at $v$. The prefactor of a term on the right hand side of (22) contributing to $L$ via the $k+1$ 1LI subgraphs is thus

$$\frac{1}{\text{Sym}(L^W)} \prod_{j=1}^k \frac{1}{\text{Sym}(L^F_j)} = \prod_{j=0}^k \frac{1}{\text{Perm}(B(v_j))} \prod_{i=1}^j \frac{1}{\text{Sym}(L_i)}.$$  

(54)

This collects all the pre-factors arising from (9), (34) and we proceed to the normalized weight for the articulation vertex $v$ obtained from (22).

Each $\partial_v \mu(v|B(v_j))$ in (53) expands into tree graphs $\{T_j\}$ which we regard as 1-rooted, $\{T_j\} \subset \mathcal{T}_{n_j\bullet}$ and the root as the endpoint of a dashed edge without open circle. The normalized weight at $v_j$ carries the coefficient

$$(-)^{|v_1(T_j)|+|e(T_j)|} \frac{\text{Perm}(B(v_j))}{\text{Sym}(T_j)} \quad \{T_j\} \subset \mathcal{T}_{n_j\bullet},$$  

(55)

with the symmetry factor defined in (44). For $j = 1,\ldots,k$ the $|\text{Perm}(B(v_j))|$ cancels against that in (54). Suppose now for fixed $j \in \{1,\ldots,k\}$ there are $k_j$ isomorphic subtrees $T_j$ (not separately named) attached to the $W_{l-m}^{(k)}$ vertex, with $\sum_{j=1}^k k_j = k$. Then accounting for the $1/k!$ in (22) and omitting the $\prod_{i=1}^j 1/\text{Sym}(L_i)$ from (54) we obtain the full prefactor of the choice of $W_{l-m}^{(k)}$ vertex

$$\frac{1}{\text{Perm}(B(v_0))} \prod_{j=1}^k \left(\frac{(-)^{|v_1(T_j)|+|e(T_j)|}}{k_j! \text{Sym}(T_j)^{k_j}}\right).$$  

(56)

Let $T \in \mathcal{T}(B(v),n)$, $n = 1 + \sum_{j=1}^k n_j$, (in the notation introduced after (53)) be the graph reassembled from the rooted subtrees $T_j$ at the vertex $o$ with weight $\omega_{k+d(v_0)}$. The total weight is $\omega_{k+d(v_0)}$ times
the products of the weights of the subtrees, and is of the form $\mu(T)$ in part (a) of the graph rule. The overall sign is $(-)^{|\nu(T)|+|\varepsilon(T)|} = (-)^{\gamma(T)}$. A straightforward application of the orbit stabilizer theorem shows that the modulus of (56) equals

$$\frac{1}{|\text{fix}(v_0^0)||\text{Aut}(T^1)|}$$

(57)

where $T^1$ is $T$ seen as rooted at the open circle vertex $o$. Further, we used that $\text{fix}(v_0^0)$ is a direct product over cells and treated the $\text{Perm}(B(v_0))$ from (56) as the factor in $\text{fix}(v_0^0)$ associated with the $v_0$ cell. There may be several identical choices for the $k$-valent open circle vertex $o$ introduced after (55) and this is just $\text{orb}(o)$, the orbit of $o$ under $\text{Aut}(T)$. Taking into account (54) and the $(l - m)/l$ from (22) the net coefficient is

$$\frac{(-)^{\gamma(T)}}{\text{Sym}(T)} |\text{orb}(o)| \times \frac{l - m}{l}.$$  

(58)

Each of these $k$-valent open circle vertices $o$ is labeled by a cell containing the vertices $b_i(v) \in B_i$ of some subset of blocks $L_i = (B_i, E_i)$, $i \in L_o$, and we usually omit the edge set $E_i$. Restoring some of the information we may attribute to $o$ the total number $l(o) = \sum_{i \in L_o} |E_i|$ of solid lines in the blocks labeling it. Then $\sum_{o \in v_0^0} l(o) = |E|$ is the total number of edges in the original graph with one articulation vertex.

We now apply this to the clover like 1LI graph $L^W$ with $l - m$ edges. After the relabeling $T' \rightarrow T$ described at (*) the relevant vertex set is $v_0^B$, $B = B(v)$, with the blocks attributed to $o$ such that $l(o) = l - m$. The point of the re-interpretation is each orbit in $v_0^B/\text{Aut}(T)$ corresponds to a distinct choice of the $W_i^{(k)}$ piece, and the contribution (58) of $T$ to $\mu(v|L)$ involves

$$\sum_{o \in v_0^B/\text{Aut}(T)} |\text{orb}(o)| l(o) = \sum_{o \in v_0^B} l(o) = l.$$  

(59)

Summing (58) using (59) the coefficient of $\mu(T)$ is $(-)^{\gamma(T)}/\text{Sym}(T)$. Finally, restoring the

$$\prod_{i=1}^l \frac{1}{\text{Sym}(L_i)} = \frac{|\text{Perm}(B)|}{\text{Sym}(L)},$$

(60)

from (54) and the $\mu(T)$ itself the normalized contribution of the articulation vertex $v$ is

$$(-)^{\gamma(T)} \frac{|\text{Perm}(B(v))|}{\text{Sym}(T)} \mu(T),$$

(61)

as claimed by the graph rule. \hfill \square

Next we show a ‘locality’ result which allows one to reduce the case with multiple articulation points to that with just one.

**Lemma III.3.** (Locality) The recursion (22) implies that the weights $\mu^T(v|L)$, $L \in \mathcal{L}_i$, depend only on the block decomposition $B(v) = \{L_i(v) = (B_i, E_i) : i \in I(v)\}$ of $L$ at $v$, symbolically

$$\mu^T(v|L) = \mu^T(v|B),$$

(62)

where on the right hand side $\mu^T(v|\cdot)$ is regarded as a map from $B(v)$ to the smooth functions in $\varphi_v$.

**Proof.** By (53) we know the structure of $\mu^T(v|L)$ but the coefficients could in principle depend on all aspects of the graph $L$ to which $v$ belongs. To exclude this, we retain the notation from the preceding Lemma and trace the changes that occur if the original $L$ has more than one articulation point. We single out one articulation point $v$ write $v_0$ for its copy in the 1LI graph $L^W$. In the paragraph leading to (53) then any number $r = 1, \ldots, k$ of $h$ derivatives can act on the $\theta_{h_0}(h)|_{h = H_i}$,
in which case $v_0$ should be viewed as a root with multiplicity $r$. Attached to $v_0$ will be $r$ 1LI graphs selected from the $L_j^\Gamma \in \mathcal{L}_{m_1}^\Gamma$, $j = 1, \ldots, k$. Without loss of generality we may take $L_j^\Gamma$, $j = 1, \ldots, r$, as the 1LI graphs attached to $v_0$. We write again $B(v_j)$ for the vertex set stemming from $L_j^\Gamma$ and $B(v_0)$ for the vertex set of the $L^W$ blocks at $v_0$. Then $B(v)$ is the union of $B(v_0)$ and $B(v_j)$, $j = 1, \ldots, r$, as $1 \leq r \leq k \leq \lfloor m/2 \rfloor$ runs through all possible values allowed by (22). For any fixed $r$ the weight associated with $v$ is

$$\omega_{d(v_0)} + r \prod_{j=1}^r \partial \phi \mu^\Gamma(v_j|B(v_j)).$$

(63)

With the replacement of $k$ by $r$ the reasoning after (55) carries over and establishes in particular $n_{\text{max}} = |I(v)|$. The relabeling ($*$) from $T'$ to $T$ proceeds as before, with $k$ replaced by $r$. Summing the contributions (63) over all allowed $1 \leq r \leq k \leq \lfloor m/2 \rfloor$ must result in a weight of the form (51) with $n_{\text{max}} = |I(v)|$. The $\mu(T)$ obtained is solely determined by block structure of $L$ at $v$ and adheres to the graph rule. The way the coefficient of $\mu(T)$ is computed by the recursion (22), however, initially refers to pieces of information not localized at $v$.

In pinning down the coefficient of $\mu(T)$ the key difference to the previous Lemma is that the relevant 1-rooted 1LI graphs $L^W$ (multiplicity $r$) and $L_j^\Gamma$, $j = 1, \ldots, r$ (all multiplicity 1) no longer have to have a clover like structure. That is, in addition to articulation vertex $v_0$ and $v_1, \ldots, v_r$ in focus these graphs can have other articulation vertices. This complicates the reduction of the symmetry factors $\text{Sym}(L^W), \text{Sym}(L_j^\Gamma), \ldots, \text{Sym}(L_r^\Gamma)$ to those of the constituent blocks. However (54) remains valid if the $L_i$ on the right hand side are interpreted as the not necessarily 1VI subgraphs that arise by disassembling the $L^W$, $L_j^\Gamma$, $j = 1, \ldots, r$, at $v_0, v_1, \ldots, v_r$, respectively. With this reinterpretation the structure of the $L^W$, $L_j^\Gamma$, $j = 1, \ldots, r$, remains clover-like at the vertex in focus. The line of reasoning from (55) to (58) carries over with $k$ replaced by $r$ and so does the remainder of Lemma III.2. In summary, the pieces of information (54), (58), (60) referring to the global structure of the reassembled graph $L$ cancel out in the final result for the coefficient of $\mu(T)$, which has the form demanded in (52).

Combined, Lemma III.2 and Lemma III.3 imply Theorem III.1
IV. REDUCTION TO INTEGER LABELED TREES

Our formula (48) for the \( \mu^T(v|L) \) weight at \( v \) renders the ‘locality’ of the data \( B(v) \) determining it manifest. The labeling of the tree graphs \( \mathcal{T}(B(v), n), n = 1, \ldots, |v| \), by the set partitions of \( B(v) \) is a convenient way to account for the coefficients with which a certain monomial in \( \mu \in \tau \) with to depend only on these integers, not on the details of the labeling cells \( c_i \) themselves. This turns out to be the case because subsums in (45) with fixed \( \mu(T) \) can be performed and manifestly depend only the unlabeled tree and the \( d(c_i) \). To avoid complications due to accidental degeneracies we sum over a subset of graphs whose defining criterion is sufficient but not necessary for the constancy of \( \mu(T) \).

We return to the projection (51) and note that the weight \( \mu(T) \) depends only on \( \text{pr}(T) = \tau \). Indeed, with \( \mu(\tau) \) formed according to (38), one has \( \mu(T) = \mu(\tau) = m(\tau) \prod_{i=1}^{m} \omega_{\tau_i} \), where \( m(\tau) \) collects the \( \gamma_{\tau_i}, m \geq 2 \), factors that depend only on the unlabeled graph. The integers \( r_i = \deg(\omega_{\tau_i}, d(c_i)) \) lie in the range of the ddeg function (36) and depend only on the integer labeled vertex set \( \text{pr}^T_0 = v_0(\tau), \tau \in \mathcal{S}_n^{\mu^T} \). Here \( d(\tau) \) can be any element of the projected label set \( \mathcal{S}(D(v), n) \) as defined after (51). It is convenient to introduce for given \( t \in \mathcal{B}, D_n \in \mathcal{S}(D(v), n) \), the range of ddeg acting on the vertex set \( v_0(\tau) \) of some \( \tau \in \mathcal{S}_n^D \)

\[
\rho(t, D_n) = \{ \text{ddeg}v_0(\tau) : \tau \in \mathcal{S}_n^D \}.
\]

For \( n \geq 2 \) elements \( \rho_n \) of \( \rho(t, D_n) \) are of the form \( \rho_n = \{ 3 \leq r_i \in \mathbb{N} : i = 1, \ldots, n \} \) and the weight \( \mu(\tau) = \mu(t, \rho_n) \) only depends on \( t \) and \( \rho_n \in \rho(t, D_n) \). We seek to identify labeled graphs \( T \in \mathcal{T}(B(v), n) \) with fixed \( \mu(T) \); by the previous considerations this requires to hold for given \( t, D_n \) the \( \rho_n \in \rho(t, D_n) \) fixed. We thus write \( \mathcal{T}(t, B(v), n) \) for the set of topologically inequivalent dashed graphs that arise by labeling \( t \in \mathcal{B} \) with \( B(v), n \). Defining

\[
\mathcal{S}(B(t, B(v), n) := \{ T \in \mathcal{T}(t, B(v), n) : \text{pr}T = \tau \in \mathcal{S}_n^D, \text{ddeg}(\text{pr}^T_0) = \rho_n \},
\]

all its elements have the same weight \( \mu(t, \rho_n) \). Further, the full set of labeled dashed graphs can be partitioned according to

\[
\mathcal{T}(t, B(v), n) = \bigcup_{D_n \in \mathcal{S}(D(v), n)} \bigcup_{\rho_n \in \rho(t, D_n)} \mathcal{S}(B(t, D_n, \rho_n)).
\]

The image under \( \text{pr} \) of the union of (66) over all \( t \in \mathcal{B} \) is partitioned analogously,

\[
\mathcal{T}(D(v), n) = \bigcup_{D_n \in \mathcal{S}(D(v), n)} \bigcup_{\rho_n \in \rho(t, D_n)} \mathcal{S}(D_n, \rho_n),
\]

with \( \mathcal{S}(D_n, \rho_n) := \{ \tau \in \mathcal{S}_n^D : \text{ddeg}v_0(\tau) = \rho_n \} \). In the graph rule formula (48) the decomposition (66) allows one to ‘pull in’ the sub-sub over \( \mathcal{T}(t, D_n, \rho_n) \). The evaluation of this subsum is the the main result of this section:

**Theorem IV.1.** In the graph rule (48) the sum over graphs \( T \in \mathcal{T}(B(v), n) \) labeled by partitions of the vertex set \( B(v) \) can be replaced with a sum over integer labeled trees. Specifically, \( \mu^T(v|B) = \mu^T(v|D) \), with \( D(v) = d(B(v)) \) and

\[
\mu^T(v|D) = \sum_{D_n \in \mathcal{T}(D(v), n)} \sum_{\rho_n \in \rho(t, D_n)} c(t, D_n, \rho_n) \mu(t, \rho_n),
\]

\[
c(t, D_n, \rho_n) = \sum_{T \in \mathcal{T}(t, D_n, \rho_n)} \text{Perm}(B(v)) \frac{\text{Sym}(T)}{|\text{Aut}(T)|} = \frac{|v_0(D_n, \rho_n)|}{|\text{Aut}(t)|} P(D(v), D_n).
\]
Here \( P(D(v), D_n) \) is the number of partitions of \(|I(v)|\) distinct labels \( \{b_1, \ldots, b_{|I(v)|}\} \) into \( n \) cells such that the sum of the \( d(b_i) \) in the \( i \)-th cell equals the given \( d_i \), and \( |v_0(D_n, \rho_n)| \) is the cardinality of \( \{v_0^\pi : \deg(prv_0^\pi) = \rho_n, \pi \in \mathcal{F}(B(v), D_n)\} \). For the latter one has explicitly

\[
|v_0(D_n, \rho_n)| = \left( \prod_{i=1}^{k} n_i! \right) \prod_{j=1}^{n} \frac{s_j!}{s_{j,1}! \cdots s_{j,k_i}!},
\]

where \( n_1, \ldots, n_k, \sum_{i=1}^{k} n_i = n \), are the numbers of equally valent open circle vertices in \( t \), and \( s_{j,i} \) is the number of equally valent open circle vertices of type \( i \) labeled by \( d_j \), where \( D_n = \{d_1^n, \ldots, d_k^n\} \).

The notational complexity notwithstanding, the formula (69) is in fact a simplification compared to (48) as a much smaller set of labeled trees need to be considered. Before turning to the proof we illustrate the statement in the examples from Sections 2.2 and 2.3. The possible \( D(v) \)'s for the examples considered in Sections 2.2, 2.3 are \( D(v) = \{2, 2\}, \{2, 3\}, \{2, 2, 3\} \). Theorem [IV.1] produces the correct weights for each \( D(v) \) with cardinality 2 by inspection. As a simple illustration of Theorem [IV.1] we detail the constituents for \( D(v) = \{2, 2, 3\} \).

\[
n = 1: D_1 = \{7\}, \quad P(D(v), D_1) = 1.
\]

\[
7 \circ \quad \rho_1 = \{7\} \quad \frac{|v_0(D_1, \rho_1)|}{|\text{Aut}(i)|} = 1 \tag{70}
\]

\[
n = 2: D_2 = \{2, 5\}, \quad P(D(v), D_2) = 2.
\]

\[
2 \circ \quad 5 \quad \rho_2 = \{3, 6\} \quad \frac{|v_0(D_2, \rho_2)|}{|\text{Aut}(i)|} = \frac{2!}{2!} = 1 \tag{71}
\]

\[
D_2 = \{3, 4\}, \quad P(D(v), D_2) = 1.
\]

\[
3 \circ \quad 4 \quad \rho_2 = \{4, 5\} \quad \frac{|v_0(D_2, \rho_2)|}{|\text{Aut}(i)|} = \frac{2!}{2!} = 1 \tag{72}
\]

\[
n = 3: D_3 = \{2, 2, 3\}, \quad P(D(v), D_3) = 1.
\]

\[
3 \quad \begin{array}{c}
2 \circ \quad 2 \\
2 \circ \quad 3 \\
2 \circ \quad 3
\end{array} \quad \rho_3 = \{3, 3, 5\} \quad \frac{|v_0(D_3, \rho_3)|}{|\text{Aut}(i)|} = \frac{2!}{2!} = 1 \\
\rho_3 = \{3, 4, 4\} \quad \frac{|v_0(D_3, \rho_3)|}{|\text{Aut}(i)|} = \frac{2!}{2!} = 2 \\
\rho_3 = \{3, 3, 4\} \quad \frac{|v_0(D_3, \rho_3)|}{|\text{Aut}(i)|} = \frac{3!}{3!} = 1 \tag{73}
\]

This yields \( \omega_i = 2\gamma_2 \omega_2 \omega_8 - \gamma_2 \omega_4 \omega_8 + \gamma_2^2 \omega_2^2 \omega_8 + 2\gamma_2^2 \omega_4 \omega_4^2 + \gamma_4 \omega_4^2 \omega_4 \), in agreement with the result in (25) and (31).

In preparation of the proof of Theorem [IV.1] we note that \( \rho T = \tau \in \mathcal{T}_n \) implies that the set partitions \( \pi \) labeling \( T \in \mathcal{T}(B(v), n) \) are constrained to lie in \( \mathcal{T}(B(v), D_n) \). These are viewed as (constrained) multiset partitions in the sense explicated after (41). For the subsequent proofs a realization of the multiset as sets of distinct elements \( B = \{b_1, \ldots, b_l\} \).
modulo an equivalence relation is convenient (to avoid further complicating the notation we write \(B(v)\) for the multiset and \(B\) for \(\{b_1, \ldots, b_j\}\) equipped with an equivalence relation). For the moment we merely stipulate the existence of an equivalence relation "\(\sim\)" on \(B\) compatible with the degree assignments, i.e. \(b_i \sim b_j\) implies \(d(b_i) = d(b_j)\) but not necessarily vice versa. We denote by \(\text{Perm}(B)\) the subgroup of \(S_B\) that permutes equivalent \(b_i\)’s. In this setting the counterpart of the constrained multiset partitions \(\mathcal{P}(B(v), D_n)\) is \(\mathcal{P}(B, D_n) = \mathcal{P}(B, D_n)/\text{Perm}(B)\), while \(\mathcal{P}(\{b_1, \ldots, b_j\}, D_n)\) does not depend on the equivalence relation and neither does its cardinality \(P(D(v), D_n)\). The counterpart of \(\mathcal{P}^B(t, D_n, \rho_n)\) is \(\mathcal{P}^B(t, D_n, \rho_n)\), the set of labeled graphs obtained by labeling \(t \in \mathcal{P}_n\) with \(C \in \mathcal{P}(B, D_n)\) such that \(d\deg \left(v_0^C\right) = \rho_n\). The latter condition defines the vertex set \(v_0(D_n, \rho_n)\). In this setting we later show:

**Proposition IV.1.** Let \(B = \{b_1, \ldots, b_j\}\) be a set of distinct vertices equipped with an equivalence relation "\(\sim\)" that is compatible with the degrees, i.e. \(b_i \sim b_j\) only if \(d(b_i) = d(b_j)\). Define \(\mathcal{P}(D_n, \rho_n), v_0(D_n, \rho_n)\), and \(P(D(v), D_n)\), as above. Then:

\[
\sum_{T \in \mathcal{P}^B(t, D_n, \rho_n)} \frac{|\text{Perm}(B)|}{|\text{Sym}(T)|} = \frac{P(D(v), D_n)|v_0(D_n, \rho_n)|}{|\text{Aut}(t)|},
\]

(74)

is independent of the equivalence relation on \(B\).

Theorem IV.1 is an easy consequence of Proposition IV.1. Using (66) in (48) (and the fact that \(s(t) = s(t)\) is manifestly labeling independent) one finds (66) with the \(c(t, D_n, \rho_n)\) given by the sum over \(T \in \mathcal{P}^B(t, D_n, \rho_n)\).

\[
\sum_{T \in \mathcal{P}^B(t, D_n, \rho_n)} |\text{Perm}(B)|/|\text{Sym}(T)| = \sum_{T \in \mathcal{P}^B(t, D_n, \rho_n)} |\text{Perm}(B)|/|\text{Sym}(T)|,
\]

the second line of (68) follows from (74). The formula (69) is straightforward combinatorics:

Keeping the integer labels from \(D_n\) fixed, the dummy labels of the equally valent open circle vertices may be permuted while preserving \(\rho_n\). This contributes the factor \(\prod_{i=1}^{k} n_i!\). The remaining factor follows from the number of ways the \(s_j\) labels with degree \(d_j\) can be distributed amongst the \(n_1, \ldots, n_k\) equally labeled vertices. Application of the multinomial theorem gives the contribution \(\prod_{i=1}^{k} s_j!/(s_j)! \cdots (s_j)!\), and hence the result (69).

It remains to establish Proposition IV.1. Its proof is broken up into several lemmas, some of them more general than needed.

**Lemma IV.1.** Let \(B\), "\(\sim\)" , and \(\text{Perm}(B)\) be as in Prop. IV.1 and \(\text{Sym}(T)\) as in (44). Then the ratios \(|\text{Perm}(B)|/|\text{Sym}(T)|\) are integers, and so are the \(d_{i_1, \ldots, i_{m-1}}\) in (33).

**Proof.** The second part follows trivially from the first. The \(\mu(T)\) in (48) refer to a mixed basis of \(\gamma_2, \alpha_m, \gamma_m\) \(m \geq 3\). By Appendix B the transition from \(\alpha_m(\varphi)\) to \(\gamma_m(\varphi)\)’s involves integer coefficients only. That the coefficients in (33) are also integers, \(d_{i_1, \ldots, i_{m-1}} \in \mathbb{Z}\).

For the first part, recall that labels are generated from the set partitions \(\mathcal{P}(B, n)\) of \(B\) into \(n\) cells. Any resulting partition \(\pi = \{c_1, \ldots, c_n\}\) carries an induced equivalence relation defined by \(c_i \sim c_{i'}\) iff there is a (possibly non-unique) \(\sigma \in \text{Perm}(B)\) such that \(\sigma(c_i) = c_{i'}\). This implies that the subset

\[
\text{stab}(\pi) = \{\sigma \in \text{Perm}(B) : \forall i \exists i' \in \{1, \ldots, n\}; \sigma(c_i) = c_{i'}\},
\]

(75)

is a subgroup of \(\text{Perm}(B)\). The case \(i = i'\) in (75) is allowed and gives rise to a subgroup \(\text{fix}(\pi) \subset \text{stab}(\pi)\), which in the multiset formulation corresponds to \(\text{fix}(v_0^B)\). In fact,

\[
\text{fix}(\pi) = \{\sigma \in \text{stab}(\pi) : \forall i \exists i' \in \{1, \ldots, n\} ; \sigma(c_i) = c_{i'}\}
\]

(76)

is a normal subgroup of \(\text{stab}(\pi)\). Recall, \(H \subset G\) is a normal subgroup if \(\forall g \in G, g^{-1}Hg = H\). Here, \(\sigma \in \Phi(\pi), \sigma_1 \in \Phi_1(\pi)\). For each \(i\), \(\sigma(c_i) = c_{i'}\), \(\sigma^{-1}(c_{i'}) = c_i\) with \(c_i \sim c_{i'}\). Note that \(\sigma\) may not be unique but any given \(\sigma\) has a unique inverse. Hence, \(\sigma_1 \sigma(c_i) = c_{i'}\) and \(\sigma^{-1} \sigma_1 \sigma(c_i) = c_i\), valid for all \(\sigma\), implies (76).

Since \(\text{fix}(\pi)\) is a normal subgroup of \(\text{stab}(\pi)\), the quotient group \(\text{stab}(\pi)/\text{fix}(\pi)\) is well defined.

Moreover, as "\(\sim\)" induces an equivalence relation on the partition \(\pi = \{c_1, \ldots, c_n\}\), we may define \(\text{Perm}(\pi)\) as the subgroup of \(S_n\) comprising only those elements that permute equivalent cells. Both groups are naturally isomorphic

\[
\text{stab}(\pi)/\text{fix}(\pi) \cong \text{Perm}(\pi).
\]

(77)
We can set up an isomorphism as follows. Use the fix(\(\pi\)) subgroup to permute in each cell \(c_i\) the equivalent elements it contains into some lexicographic order. Then cells \(c_i, c_i'\) are equivalent iff they contain lexicographically ordered strings of equal cardinalities for each \(\sim\) equivalence class. The quotient group permutes equivalent cells while preserving the lexicographic order of the strings. As such it gives one realization of \(\text{Perm}(\pi)\) and hence \((77)\). Since \(|\text{fix}(\pi)| = |\text{fix}(\nu_0^B)|\) it follows that \(|\text{stab}(\pi)| = |\text{Perm}(\pi)||\text{fix}(\nu_0^B)|\). When treating \(\pi\) as a label set for the graphs \(T \in \mathcal{T}(B(\nu), n)\) the automorphism group \(\text{Aut}(T)\) is a subgroup of \(\text{Perm}(\pi)\). Lagrange’s theorem

\[
\frac{|\text{Perm}(B)|}{|\text{fix}(\nu_0^B)||\text{Aut}(T)|} = \frac{|\text{Perm}(B)|}{|\text{Perm}(\pi)|} = \frac{|\text{Perm}(\pi)|}{|\text{stab}(\pi)|} \in \mathbb{N},
\]

completes the argument.

We proceed with labeling the dashed graphs \(t \in \mathcal{T}_n\) by an abstract \(n\) element label set \(C = \{c_1, \ldots, c_n\}\). Later on the \(c_i\) will be identified with the cells of a set partition in \(\mathcal{T}(B(\nu), n)\), for now the origin of the \(c_i\)’s is irrelevant. In order to model the equivalence of cells we assume that \(C\) carries an equivalence relation \(\sim\) and that a subgroup \(\text{Perm}(C)\) of \(S_n\) acts by permuting equivalent \(c_i\)’s. As before, only the open circle vertices \(v_0\) of \(t \in \mathcal{T}_n\) are labeled, technically via the graph of a bijection \(\sigma : v_0 \to C\). Each graph is referred to as a labeling set (or pairing) and corresponds to a permutation \(\sigma \in S_n\), so for \(v_0 = \{o_1, \ldots, o_n\}\) and \(C = \{c_1, \ldots, c_n\}\) we write a labeling set as \(\nu_0^c = \{(o_i, c_{\sigma(i)}): i = 1, \ldots, n\}\). For fixed \(C\) we now consider the set of all pairings

\[
\nu_0^C = \{\nu_0^c : \sigma \in S_n\}, \quad |\nu_0^C| = n!.
\]

Recall that an unlabeled graph \(t \in \mathcal{T}_n\) may be written as \(t = (v_0 \cup v_1, \varepsilon)\), for one of its labeled counterparts we write \(T = (\nu_0^c \cup v_1, \varepsilon)\). As \(\sigma\) runs through \(S_n\) the set of labeled dashed graphs generated is denoted by \(\mathcal{T}_n^C\).

The product group \(\text{Aut}(t) \times \text{Perm}(C) : \nu_0^C \to \nu_0^C\) acts termwise on the elements of \(\nu_0^C\); for \((g, h) \in \text{Aut}(t) \times \text{Perm}(C)\) and \(\nu_0^C = \{(o_i, c_{\sigma(i)}): i = 1, \ldots, n\}\) define \((g, h)(o_i, c_{\sigma(i)}) := (g(o_i), h(c_{\sigma(i)})\). We note that two distinct labeling sets \(\nu_0^c_1, \nu_0^c_2 \in \nu_0^C\) with \(\sigma_1 \neq \sigma_2\) can correspond to the same labeled \(T \in \mathcal{T}_n^C\). This occurs if there is an element of \(\text{Aut}(t) \times \text{Perm}(C)\) that maps between the labeling sets. As an illustration consider \(t \in \mathcal{T}_3\) with open circle vertex set \(v_0 = \{o_1, o_2, o_3\}\)

![Diagram](image)

With labels \(C = \{c_1, c_2, c_3\}\) two distinct labeling sets are \(\nu_0^c_1 = \{(o_1, c_1), (o_2, c_2), (o_3, c_3)\}\) and \(\nu_0^c_2 = \{(o_1, c_2), (o_2, c_1), (o_3, c_3)\}\), and the resulting labeled graphs \(T_1\) and \(T_2\) are shown in \((81)\). On inspection it is clear that \(T_1\) and \(T_2\) are the same labeled graph as one can be mapped onto the other by interchanging \(o_1\) and \(o_2\).

![Diagram](image)

Generally, labeling sets related by the above action of \(\text{Aut}(t) \times \text{Perm}(C)\) give rise to the same labeled graph. This underlies the following

**Lemma IV.2.** Let \(t \in \mathcal{T}_n\) be an unlabeled graph and \(C = \{c_1, \ldots, c_n\}\) be a set of distinct labels equipped with an equivalence relation \(\sim\). Let \(\text{Perm}(C)\) be the subgroup of \(S_n\) that permutes
equivariant $c_i$’s, and let $\mathcal{T}^C_n(t)$ be the set of all topologically distinct labeled dashed graphs obtained by labeling $t$ with $C$. Then:

$$\sum_{T \in \mathcal{T}^C_n(t)} \frac{|\text{Perm}(C)|}{|\text{Aut}(T)|} = \frac{n!}{|\text{Aut}(t)|},$$

(82)

i.e. this sum is independent of the equivalence relation “\sim” on $C$.

Proof. We consider the orbit $\text{orb}(v_0^\sigma)$ of some $v_0^\sigma \in v_0^C$ under the action of $\text{Aut}(t) \times \text{Perm}(C)$. By the comment after (1) the orbit is the subset of $v_0^C$ whose elements correspond to the same labeled graph $T$. Hence there exists a bijection between labeled graphs in $\mathcal{T}^C_n(t)$ and equivalence classes in $v_0^C/\text{Aut}(t) \times \text{Perm}(C)$, i.e. orbits. The orbits are disjoint and their union is $v_0^C$. A sum over $T \in \mathcal{T}^C_n(t)$ may be reexpressed as a sum over orbits $|v_0^\sigma| \in v_0^C/\text{Aut}(t) \times \text{Perm}(C)$.

Next we claim that $\text{Aut}(T)$ for a labeled $T \in \mathcal{T}^C_n(t)$ is isomorphic to some subgroup $\text{Aut}(t) \times \text{Perm}(C)$. Suppose an element of $\text{Aut}(T)$ permutes two labeled vertices $(o,c)$ and $(\rho, \rho')$ while preserving adjacency. This is possible iff there is a $g \in \text{Aut}(t)$ that exchanges $v, v'$, and there is a $h \in \text{Perm}(C)$ that exchanges $c, c'$. For a labeling set $v_0^\sigma$ corresponding to $T$, then $(g \times h)(v_0^\sigma) = v_0^\sigma$. Conversely, suppose there is an element of $\text{Aut}(t) \times \text{Perm}(C)$ that leaves $v_0^\sigma$ invariant. This is a permutation of the pairs in $v_0^\sigma$ labeling $T$ that preserves adjacency in $t$, and so there is a corresponding element in $\text{Aut}(T)$. Thus $\text{Aut}(T)$ is isomorphic to the subgroup $\text{stab}(v_0^\sigma)$ of $\text{Aut}(t) \times \text{Perm}(C)$ that leaves any labeling set $v_0^\sigma$ corresponding to $T$ invariant.

The stabilizer subgroups of two elements $v_0^\sigma, v_0^\sigma'$ of the same orbit are related by conjugation with the group element linking them. In particular, $|\text{stab}(v_0^\sigma)| = |\text{stab}(v_0^\sigma')|$, for $v_0^\sigma, v_0^\sigma' \in \text{orb}(v_0^\sigma)$. We may write

$$\sum_{T \in \mathcal{T}^C_n(t)} \frac{|\text{Perm}(C)|}{|\text{Aut}(T)|} = \sum_{v_0^\sigma \in v_0^C/\text{Aut}(t) \times \text{Perm}(C)} \frac{|\text{Perm}(C)|}{|\text{stab}(v_0^\sigma)|}. \quad (83)$$

The orbit-stabilizer theorem implies $|\text{Aut}(t) \times \text{Perm}(C)| = |\text{stab}(v_0^\sigma)||\text{orb}(v_0^\sigma)|$, i.e. $|\text{Perm}(C)||/|\text{orb}(v_0^\sigma)| = |\text{stab}(v_0^\sigma)||/|\text{Aut}(t)|$. Thus

$$\sum_{T \in \mathcal{T}^C_n(t)} \frac{|\text{Perm}(C)|}{|\text{Aut}(T)|} \sum_{v_0^\sigma \in v_0^C/\text{Aut}(t) \times \text{Perm}(C)} \frac{|\text{orb}(v_0^\sigma)|}{|\text{Aut}(t)|} = \frac{|v_0^C|}{|\text{Aut}(t)|}, \quad (84)$$

as claimed. \hfill \square

We proceed to a variant of Lemma [192] where the equivalence relation on $C$ is compatible with an integer grading $d : C \to \mathbb{N}^n$. Each element of the label set $C = \{c_1, \ldots, c_n\}$ is assigned an integer $d(c_i) \in \mathbb{N}$. The range $d(C) = \{d(c_1), \ldots, d(c_n)\}$ will in general be a multisets $D_n = \{d_1, \ldots, d_n\}$, with $\sum_{i=1}^n s_i = n$, $s_i \in \mathbb{N}_0$. If $C$ is used to label some $t \in \mathcal{T}_n$, the weight assignment to its open circle vertices will by (36), (38) depend only on the valency of the $o \in v_0$ and some integers which we will now draw from the range $d(C)$. To this end we extend the ddeg function in (36) to the labeled vertices $(o_i, c_{\sigma(i)})$ by $\text{ddeg}(o_i, c_{\sigma(i)}) = |o_i| + d(c_{\sigma(i)})$. In other words, the sum $|o_i| + d(c_{\sigma(i)})$ is viewed as an instance of (36), where the integers arise from the degrees of the labeling set. This carries over to $\text{ddeg} v_0^\sigma := \{\text{ddeg}(o_i, c_{\sigma(i)}): i = 1, \ldots, n\}$ and we define

$$\mathcal{T}^C_n(t) := \{T \in \mathcal{T}^C_n(t): \text{ddeg} v_0^\sigma = \rho_n\},$$

$$v_0(\rho_n) := \{v_0^\sigma \in v_0^C: \text{ddeg} v_0^\sigma = \rho_n\},$$

for some fixed $\rho_n \in \rho(t, d(C))$ in the range of the ddeg function

$$\rho(t, d(C)) := \{\text{ddeg} v_0^\sigma: \sigma \in S_n\}. \quad (85)$$

By the weight assignments (38) all $T \in \mathcal{T}^C_n(\rho_n)$ have the same $\mu(T)$. Equivalently, elements $v_0^\sigma, v_0^\sigma'$ of the same orbit in $v_0^C/\text{Aut}(t) \times \text{Perm}(C)$, have the same $\rho_n$ and hence lie in the same $v_0(\rho_n)$. Clearly, $\mathcal{T}^C_n(t)$ is partitioned by $\mathcal{T}^C_n(\rho_n)$ as $\rho_n$ runs through $\rho(t, d(C))$. \hfill (86)
Lemma IV.3. Let $t \in {\mathcal T}_n$ be an unlabeled graph and $C = \{c_1, \ldots, c_n\}$ be a set of distinct labels equipped with a grading $d : C \to \mathbb{N}^3$ and a compatible equivalence relation "\sim", i.e. $c_i \sim c_j$ only if $d(c_i) = d(c_j)$. Then:

$$
\sum_{T \in {\mathcal T}_n} \frac{|\text{Perm}(C)|}{|\text{Aut}(T)|} = \frac{|V_0(\rho_n)|}{|\text{Aut}(t)|},
$$

(87)

i.e. the sum is independent of the equivalence relation "\sim" on $C$.

Proof. As noted in the proof of Lemma IV.2, there is a bijection between the labeled graphs in ${\mathcal T}_n^C(t)$ and the orbits in $V_0^C/ [\text{Aut}(t) \times \text{Perm}(C)]$. Therefore we may write

$$
\frac{|\text{Perm}(C)|}{|\text{Aut}(T)|} = \frac{\sum_{[\rho] \in V_0^C/[\text{Aut}(t) \times \text{Perm}(C)]} |\text{Perm}(C)|}{|\text{Aut}(t)|} = \frac{\sum_{[\rho] \in V_0^C/[\text{Aut}(t) \times \text{Perm}(C)]} |\text{orb}(V_0^\rho)|}{|\text{Aut}(t)|}.
$$

(88)

In the first identity the constancy of $\rho_n$ within orbits entered, in the second the orbit-stabilizer theorem was used as in the proof of Lemma IV.2. The elements of $V_0(\rho_n)$ depend on the grading but not on the specific equivalence relation "\sim" compatible with it. \qed

We now return to the graph rule, where the label set $C$ originates from partitioning the vertex set $B$ into $n$ cells. We adopt the equivalence class setting from Proposition IV.1, given a vertex set $B = \{b_1, \ldots, b_I\}$ of $I$ distinct elements its $n$-cell set partitions $\{c_1, \ldots, c_n\} \in {\mathcal J}(B, n)$ are formed. The cardinality $|{\mathcal J}(B, n)| = S(I, n)$ is the second Stirling number. We stipulate the existence of an equivalence relation "\sim" on $B$, and take $\text{Perm}(B)$ to permute equivalent elements of $B$. An action $\text{Perm}(B) : {\mathcal J}(B, n) \to {\mathcal J}(B, n)$ is induced, and we write $\text{orb}(\pi)$ for the orbit of $\pi \in {\mathcal J}(B, n)$ under $\text{Perm}(B)$. Observe that for given $\pi \in {\mathcal J}(B, n)$, all elements of $\text{orb}(\pi)$ correspond to the same label set $C$. We omit a formal proof and instead present an illustrative example: let $B = \{b_1, b_2, b_3, b_4, b_5\}$, with $b_1 \sim b_2$. The partitions $\pi_1 = \{\{b_1, b_4\}, \{b_2, b_3, b_5\}\}$ and $\pi_2 = \{\{b_2, b_4\}, \{b_1, b_3, b_5\}\}$ are distinct, but they correspond to the same label set $C$ by virtue of $b_1 \sim b_2$. We define $\tilde{\mathcal J}(B, n) : = {\mathcal J}(B, n)/\text{Perm}(B)$, the set of distinct label sets $C$.

Lemma IV.4. Let $B = \{b_1, \ldots, b_I\}$ be a set of distinct vertices equipped with an equivalence relation "\sim", and let $\text{Perm}(B)$ be the subgroup of $S_I$ that permutes equivalent vertices. For given $t \in {\mathcal T}_n$ let $\tilde{\mathcal T}_a^B(t)$ be the set of topologically distinct labeled dashed graphs obtained by labeling $t$ with $C \in {\mathcal J}(B, n)$. Then:

$$
\sum_{T \in \tilde{\mathcal T}_a^B(t)} \frac{|\text{Perm}(B)|}{|\text{Sym}(T)|} = S(I, n) \cdot \frac{n!}{|\text{Aut}(t)|}.
$$

(89)

Proof. We may trivially rewrite the left hand side of (89)

$$
\sum_{T \in \tilde{\mathcal T}_a^B(t)} \frac{|\text{Perm}(B)|}{|\text{Sym}(T)|} = C = \frac{\sum_{\pi \in {\mathcal J}(B, n)/\text{Perm}(B)} \sum_{T \in \tilde{\mathcal T}_a^B} |\text{Perm}(B)|}{|\text{Sym}(T)|}. 
$$

(90)

From (78) we know

$$
\frac{|\text{Perm}(B)|}{|\text{Sym}(T)|} = \frac{|\text{Perm}(B)|}{|\text{Perm}(C)|} \frac{|\text{Perm}(C)|}{|\text{stab}(\pi)|} \frac{|\text{stab}(\pi)|}{|\text{Aut}(\pi)|},
$$

(91)

where $\text{stab}(\pi)$ is the subgroup of $\text{Perm}(B)$ leaving $\pi \in {\mathcal J}(B, n)$ invariant. It follows from the definition of ${\mathcal J}(B, n)/\text{Perm}(B)$ that if $\pi_1, \pi_2 \in \text{orb}(\pi)$ then $|\text{stab}(\pi_1)| = |\text{stab}(\pi_2)|$. Combining
successively \(\text{Lemma IV.2}\) and the orbit-stabilizer theorem gives the assertion:

\[
\sum_{C = [\pi] \in C(B, n)/\text{Perm}(B)} \sum_{T \in \mathcal{T}_C} \frac{\text{Perm}(B)}{\text{Sym}(T)} = \sum_{C = [\pi] \in C(B, n)/\text{Perm}(B)} \left( \frac{\text{Perm}(B)}{\text{stab}(\pi)} \sum_{T \in \mathcal{T}_C} \frac{\text{Perm}(C)}{|\text{Aut}(T)|} \right)
\]

\[
= \frac{n!}{|\text{Aut}(I)|} \sum_{[\pi] \in C(B, n)/\text{Perm}(B)} |\text{orb}(\pi)| = \frac{n!}{|\text{Aut}(I)|} S(I, n).
\]  

(92)

Note that Lemma IV.4 is the counterpart of Lemma IV.2 for \(C\) induced by set partitions. Similarly, Proposition IV.1 is the counterpart of Lemma IV.3. Instead of holding all \(C \in \mathcal{P}(B, D_n)\), as defined before Proposition IV.1. Indicating all dependencies in the notation we set:

\[
\mathcal{S}^B(t, D_n, \rho_n) := \bigcup_{C \in \mathcal{P}(B, D_n)} \mathcal{S}^C(\rho_n)
\]

\[
= \{ T \in \mathcal{S}^B(t) : \text{ddeg} \nu_0^B = \rho_n, \nu_0^B \in \nu_0^C, C \in \mathcal{P}(B, D_n) \}.
\]  

(93)

**Proof of Proposition IV.1** We begin as in the proof of Lemma IV.4 and rewrite the left hand side of (74) as

\[
\sum_{T \in \mathcal{S}^B(t, D_n, \rho_n)} \frac{|\text{Perm}(B)|}{|\text{Sym}(T)|} = \sum_{C = [\pi] \in \mathcal{P}(B, D_n)/\text{Perm}(B)} \sum_{T \in \mathcal{T}_C(\rho_n)} \left( \frac{|\text{Perm}(B)|}{|\text{stab}(\pi)|} \sum_{T \in \mathcal{T}_C(\rho_n)} \frac{|\text{Perm}(C)|}{|\text{Aut}(T)|} \right).
\]  

(94)

To the subsum in round brackets we apply Lemma IV.5 to obtain

\[
\sum_{T \in \mathcal{S}^B(t, D_n, \rho_n)} \frac{|\text{Perm}(B)|}{|\text{Sym}(T)|} = \frac{|\nu_0(\rho_n)|}{|\text{Aut}(I)|} \sum_{[\pi] \in \mathcal{P}(B, D_n)/\text{Perm}(B)} \frac{|\text{Perm}(B)|}{|\text{stab}(\pi)|},
\]  

(95)

using that \(|\nu_0(\rho_n)|\) is independent of equivalence relation on \(C = [\pi]\). On account of the orbit-stabilizer theorem \(|\text{Perm}(B)|/|\text{stab}(\pi)| = |\text{orb}(\pi)|\) the sum over \([\pi]\) produces the cardinality of the set \(\mathcal{S}^B(\{b_1, \ldots, b_l\}, D_n)\), i.e. \(P(D(v), D_n)\) and establishes (74). Its right hand side is manifestly independent of the (degree compatible) equivalence relation \(\sim\) on \(B\).

\(\square\)

V. CONCLUSIONS

Motivated by the widespread use of the FRG equation \(1\) we formulated a program for its graph theoretical solution. Subject to ultralocal initial conditions \(1\) can be replaced by the iteratively soluble \(3\) producing a long range hopping expansion (LRH) for \(\Gamma_k = \Gamma_0 + \sum_{l \geq 1} \kappa^l \Gamma_l\), from which a solution \(\Gamma_k\) of \(1\) can be obtained by substitution, \(\Gamma_k = \Gamma_k|_{l \to \ell(h)}\). As the iteration of \(1\), or its equivalent mixed form \(22\), is only feasible to moderate orders we formulated graph rules for the direct evaluation of an arbitrary order \(\Gamma_l\). The derivation, computational test, and proof of these graph rules constitute the main result of the paper.

By the results of Section IV the subsums over vertex labeled trees \(T \in \mathcal{S}(B(v), n)\) with fixed weight \(\mu(T)\) have a combinatorial meaning in terms of the number of integer labeled tree graphs of the same topology as \(T\). The graph rule could therefore optimized once explicit results for the number of set partitions \(P(D(v), D_n)\) are available; see \(10\) for some related results.

The construction so far only holds in the formal series sense. Guided by a variety of convergence results for hopping expansions in the literature (see \(9, 21\) and the references therein) we expect that the LRH expansion for \(\Gamma_k\) has finite radius of convergence under natural conditions. From a computational perspective it would also be desirable to identify subclasses of one-line irreducible graphs that can be analytically summed and lead to controlled approximate solutions of \(1\), replacing the traditional ad-hoc Ansätze.
Appendix A: Recursive results to fifth order

Here we present explicit results for $\Gamma_i$, $i = 2, \ldots, 5$, and various checks on them. A closed recursion arises from the expansion of the $\kappa$-flow equation in (3). In preparation we define polynomials $m_l = m_l(u_1, \ldots, u_l)$ in non-commuting variables $u_n$, $n \in \mathbb{N}$, by

$$m_l(u) = \sum_{n=1}^{l} \sum_{n_1 + \ldots + n_m = l} (-)^{n+1} u_{i_1} \ldots u_{i_n}. \tag{A1}$$

At low orders: $m_1 = u_1$, $m_2 = u_2 - u_1^2$, $m_3 = u_3 - u_1 u_2 - u_2 u_1 + u_1^3$. Inserted into (3) one has $\Gamma_1 \equiv 0$ and

$$\Gamma_l = \frac{1}{2l} \sum_{n=1}^{l} (-)^{n} \text{Tr}[u_{i_1} u_{i_2} \ldots u_{i_n}], \quad l \geq 2, \tag{A2}$$

with $\Gamma_0^{(2)} \cdot u_1 = \ell$, $\Gamma_0^{(2)} \cdot u_i = \Gamma_i^{(2)}$, $i \geq 2$. Here $\Gamma_0^{(2)}[\phi]$ is invertible

$$\Gamma_0^{(2)}[\phi]_{x,y} = \gamma_2(\phi_x) \delta_{x,y}, \quad \gamma_2(\phi)^{-1} = \omega_2|_{h=h(\phi)}. \tag{A3}$$

In slight abuse of notation we set $\omega_i(\phi) := \omega_i(h(\phi))$, $\omega_i(h) = \partial^i \omega / \partial h^i$, $i \geq 2$, and find:

$$\Gamma_2[\phi] = -\sum_{x_1,x_2} \frac{1}{4} \omega_2(\phi_{x_1}) \omega_2(\phi_{x_2})(\ell_{x_1 x_2})^2$$

$$\Gamma_3[\phi] = \sum_{x_1,x_2} \frac{1}{12} \omega_3(\phi_{x_1}) \omega_3(\phi_{x_2})(\ell_{x_1 x_2})^3$$

$$+ \sum_{x_1,x_2,x_3} \frac{1}{6} \omega_2(\phi_{x_1}) \omega_2(\phi_{x_2}) \omega_2(\phi_{x_3})(\ell_{x_1 x_2} \ell_{x_2 x_3} \ell_{x_1 x_3})$$

$$\Gamma_4[\phi] = -\sum_{x_1,x_2} \frac{1}{48} \omega_4(\phi_{x_1}) \omega_4(\phi_{x_2})(\ell_{x_1 x_2})^4$$

$$- \sum_{x_1,x_2,x_3} \frac{1}{4} \omega_3(\phi_{x_1}) \omega_3(\phi_{x_2}) \omega_2(\phi_{x_3})(\ell_{x_1 x_2})^2 \ell_{x_1 x_3} \ell_{x_2 x_3}$$

$$- \sum_{x_1,x_2,x_3} \frac{1}{8} \omega_2(\phi_{x_1}) [\omega_4 - \omega_3 \gamma_2 \omega_1](\phi_{x_2}) \omega_2(\phi_{x_3})(\ell_{x_1 x_2})^2 (\ell_{x_2 x_3})^2$$

$$- \sum_{x_1,x_2,x_3,x_4} \frac{1}{8} \omega_2(\phi_{x_1}) \omega_2(\phi_{x_2}) \omega_2(\phi_{x_3}) \omega_2(\phi_{x_4}) \ell_{x_1 x_2} \ell_{x_2 x_3} \ell_{x_3 x_4} \ell_{x_4 x_5}.$$
\[ \Gamma_3[\phi] = \sum_{x_1,x_2} \frac{1}{120} \omega_4(\phi_{x_1}) \omega_5(\phi_{x_2}) (\ell_{x_1 x_2})^5 \]
\[ + \sum_{x_1,x_2,x_3} \frac{1}{12} \omega_4(\phi_{x_1}) \omega_4(\phi_{x_2}) \omega_2(\phi_{x_3}) (\ell_{x_1 x_2})^3 (\ell_{x_1 x_3}) (\ell_{x_2 x_3}) \]
\[ + \sum_{x_1,x_2,x_3} \frac{1}{12} \omega_2(\phi_{x_1}) \left[ \omega_3 - \omega_3 \gamma_2 \omega_4(\phi_{x_2}) \omega_3(\phi_{x_3}) \right] (\ell_{x_1 x_2})^2 (\ell_{x_2 x_3})^3 \]
\[ + \sum_{x_1,x_2,x_3} \frac{1}{8} \omega_3(\phi_{x_1}) \omega_3(\phi_{x_2}) \omega_4(\phi_{x_3}) (\ell_{x_1 x_2})^2 (\ell_{x_1 x_3})^2 (\ell_{x_2 x_3})^2 \]
\[ + \sum_{x_1,x_2,x_3,x_4} \frac{1}{4} \omega_2(\phi_{x_1}) \omega_2(\phi_{x_2}) \omega_3(\phi_{x_3}) \omega_3(\phi_{x_4}) (\ell_{x_1 x_2}) (\ell_{x_2 x_3}) (\ell_{x_3 x_4})^2 (\ell_{x_4 x_1}) \]
\[ + \sum_{x_1,x_2,x_3,x_4} \frac{1}{4} \omega_2(\phi_{x_1}) \omega_2(\phi_{x_2}) \omega_4(\phi_{x_3}) \omega_4(\phi_{x_4}) (\ell_{x_1 x_2}) (\ell_{x_2 x_3}) (\ell_{x_3 x_4}) (\ell_{x_4 x_1}) (\ell_{x_1 x_3}) \]
\[ + \sum_{x_1,x_2,x_3,x_4} \frac{1}{4} \omega_2(\phi_{x_1}) \omega_2(\phi_{x_2}) \omega_2(\phi_{x_3}) \omega_4(\phi_{x_4}) (\ell_{x_1 x_2}) (\ell_{x_2 x_3}) (\ell_{x_3 x_4}) (\ell_{x_4 x_1}) \]
\[ + \sum_{x_1,x_2,x_3,x_4,x_5} \frac{1}{10} \omega_2(\phi_{x_1}) \omega_2(\phi_{x_2}) \omega_2(\phi_{x_3}) \omega_2(\phi_{x_4}) \omega_2(\phi_{x_5}) (\ell_{x_1 x_2}) (\ell_{x_2 x_3}) (\ell_{x_3 x_4}) (\ell_{x_4 x_5}) \]

(A5)

A computational point worth mentioning is that the \((\Gamma^{(2)}_m)_{xy}\), \(m \geq 2\), have in general diagonal elements. In evaluating the traces one has to split off at intermediate steps subsums containing \((\Gamma^{(2)}_m)_{xt}\) contributions. Such contributions combine with others and lead to unrestricted sums in the final result, but with modified coefficients.

The corresponding \(W_j\)'s are readily obtained from the Wortis graph rule and are not displayed explicitly. In line with table 1 the expressions for the \(\Gamma_j\) are (at matching orders) more concise than the \(W_j\)'s and yet code the same information. These results have been tested and compared with partial results in the literature in various ways. (i) \(W_1, \ldots, W_4\) and \(\Gamma_2, \ldots, \Gamma_3\) are related by the mixed recursion (14). (ii) Specialized to the Ising model \(\Gamma_2, \ldots, \Gamma_4\) agree with the results of [5].

The \(W_1, \ldots, W_4\) themselves can be specialized to a nearest neighbor hopping matrix and matched to results in the literature. (iii) for \(H = 0\) the textbook result for the free energy is reproduced. (iv) The 2-point susceptibility \(\chi_2 = \sum_k W^{(2)}_{4,k} |_{H = 0}\) and the 4-point susceptibility \(\chi_4 = \sum_{x_1,x_2} W^{(4)}_{4,x_1,x_2} |_{H = 0}\) match (in \(d = 2\)) the results in [2].
The function $\gamma(\phi)$ entering the ultralocal initial functional $\Gamma_0[\phi] = \sum \gamma(\phi)$ can be characterized by the functional relation

$$\exp -\gamma(\phi) = \int_{-\infty}^{\infty} d\chi \exp \left\{ -s(\chi) + (\chi - \phi) \frac{\partial \gamma}{\partial \phi} \right\}, \quad \text{(B1)}$$

where $s$ is the single site action. By shifting the argument $\phi \mapsto \phi + \alpha$ and expanding in powers of $\alpha$ one can express the derivatives $\gamma_n(\phi) = \partial^n / \partial \phi^n$ in terms of the cumulants $\omega_n(h) = \partial^n / \partial h^n$ of the measure $d\chi e^{-s(\chi)}$. The latter have generating function $e^{\omega(h+k)} = \int_{-\infty}^{\infty} d\chi \exp \left\{ -s(\chi) + (h + k)\chi \right\}$ upon expansion in powers of $k$. To low orders one finds

$$\gamma_2(\phi) = \omega_2^{-1}|_{h=h(\phi)} \quad \gamma_3(\phi) = -\omega_3^{-3} \omega_1|_{h=h(\phi)} \quad \gamma_4(\phi) = \left[ -\omega_4 \omega_2^{-4} + 3 \omega_2 \omega_2^{-5} \right]|_{h=h(\phi)} \quad \text{(B2)}$$

Augmented by $\omega_1|_{h=h(\phi)} = \phi$ and $\gamma(\phi) = h(\phi)$ the inverse relations are obtained by flipping the roles of the $\omega_n$'s and $\gamma_n$'s. This reflects the fact that the generating functions $\omega(h+k)$ and $\gamma(\phi+\alpha)$ are Legendre transforms of each other. In quantum field theoretical terminology the $\gamma(\phi)$ are zero-dimensional vertex functions with non-zero mean field and the $\omega(h)$ are the zero-dimensional cumulants with non-zero source.

The combinatorial patterns arising through the Legendre transform can be analyzed in closed form in the zero dimensional case. The integral realization does not enter, so $\omega : \mathbb{R} \rightarrow \mathbb{R}$, can be any smooth function with nonzero second derivative, $\omega''(h) > 0$, say. We assume that $\omega^{(1)}(h) = \phi$ can be solved for $h(\phi)$, where $h$ is likewise smooth and $h^{(1)}(\phi) > 0$. We define the Legendre transform by $\gamma(\phi) := \phi h(\phi) - \omega(h(\phi))$. Then $\gamma^{(1)}(\phi) = h(\phi)$ and the primary assertion is that $\omega^{(1)}$ and $\gamma^{(1)}$ are compositional inverses of each other

$$\omega^{(1)}(\gamma^{(1)}(\phi)) = \phi \quad \gamma^{(1)}(\omega^{(1)}(h)) = h \quad \text{(B3)}$$

By repeated differentiation of these basic formulas one can generate relations between the derivatives $\gamma^{(1)}(\phi)$ and the $\omega(\phi)$, where the first few relations generated are $^{[2]}$. Since the resulting $\omega(h)$ derivatives are always evaluated at $h = h(\phi)$ it is convenient to set $\omega_n(\phi) := \omega_n(h)|_{h=h(\phi)}$ by slight abuse of notation. In this notation $\partial_\phi$ is a linear derivation acting via $\partial_\phi \omega_n = \omega_{n+1} \partial_\phi \gamma_n$ on the constituents. The differentiation rule implies that the $l$-th order relation has the form

$$\omega_l^{(1)} \gamma = -\omega_l + 2^{l-2} \sum_{i_1 + \cdots + i_l = l} (-1)^{l_1 + \cdots + l_l} (1 \gamma_{i_1} \omega_{i_2}^{-1} \gamma_{i_3} \omega_{i_4} \cdots \gamma_{i_l+1}) \quad l \geq 4 \quad \text{(B4)}$$

for $i_k \in \mathbb{N}$ and integer coefficients $c_{i_1 \cdots i_l} \in \mathbb{N}$. Unless noted otherwise the $\omega_n$ in the following are the $\omega_n(\phi) = \omega_n(h)|_{h=h(\phi)}$ regarded as functions of $\phi$.

The coefficients $c_{i_1 \cdots i_l}$ are indirectly characterized by the duality property $^{[3]}$: solving $^{(B4)}$ recursively for the $\omega_l$ in terms of $\gamma_2, \ldots, \gamma_l$, the same formula arises. An explicit formula for them arises from the known combinatorial expressions for the compositional inverse:

$$c_{i_1 \cdots i_l} = \left( -2 + \sum_{j=3}^{l-1} i_j \right) \prod_{j=3}^{l-1} \frac{1}{(j-1)!^j} \quad \text{(B5)}$$

This has been obtained in $^{[4]}$ in a setting that mimics perturbation theory; the functions are power series in the fields and the numerical coefficients are related as in $^{[5]}$. A little thought shows that the coefficients arising through repeated differentiation of arbitrary smooth functions without setting the argument to zero are the same. For later reference we sketch the argument.

The basic input is an explicit expression for the compositional inverse of a formal power series. There are several variants of such formulas and their tree-graph interpretation; the version most directly leading to $^{[5]}$ is Eq. (4.6) in $^{[6]}$. Given $a(z) = \sum_{n \geq 1} a_n z^n$, $a_1 \neq 0$, the series $b(w) = \sum_{n \geq 1} \frac{b_n}{n!} w^n$
is the compositional inverse of $a$, i.e. $a(b(w)) = w$, iff

$$b_n = \sum_{k_2,k_3,\ldots\geq 0, j=\sum(j-1)k_j=n-1} \frac{(-)^j}{j!} a_j^{-1(1+\Sigma_j j_k)} \prod_j j \frac{a_j^{k_j}}{k_j!}, \quad (B6)$$

where all sums and products range over $j \geq 2$ and are rendered finite by the Euler relation $\sum_j (j-1)k_j = n-1$. For the application here we shift the arguments of $\omega$ and $\gamma$ and re-expand. For the first derivatives this gives

$$\omega^{(1)}(h+k) = \sum_n \frac{k^n}{n!} \omega_{n+1}(h) =: \omega^{(1)}(h;k),$$

$$\gamma^{(1)}(\varphi + \alpha) = \sum_n \frac{\alpha^n}{n!} \gamma_{n+1}(\varphi) =: \gamma^{(1)}(\varphi; \alpha). \quad (B7)$$

By (B3) we require both series to be compositional inverses of each other as series in $k, \alpha$. The result (B5) then follows from (B6).

The formula (B6) has several known combinatorial and tree graph interpretations, see [10][11][12] and the references therein. In the remainder of this appendix we present a graph theoretical interpretation of (B4), (B5) which, together with its proof, mirrors some aspects of its quantum field theoretical counterpart in Section III.

**Graph rules for $\gamma$:**

(i) At order $l \geq 4$ draw all topologically distinct connected tree graphs $t \in \mathcal{T}_l$ with $l = |\mathcal{V}_0|$ external vertices of order 1 and any number $|\mathcal{V}_1|$ of $k$-valent vertices, $k = 3, \ldots, l$, joined by dashed lines. Multiply by $l!/|\text{Aut}(t)| \in \mathbb{N}$, where $\text{Aut}(t)$ is the automorphism group of the graph.

(ii) Attribute to each $t \in \mathcal{T}$ a weight $(-)^{|\mathcal{V}_1|} \mu(t)$ as follows: a factor $\omega_2^{-1}$ to each dashed line, 1 to an 1-valent vertex, and $-\omega_k$ to an $k$-valent vertex, $k \geq 3$.

(iii) Sum over all contributions to obtain

$$\gamma = \sum_{t \in \mathcal{T}_l} (-)^{|\mathcal{V}_1|} \frac{l!}{|\text{Aut}(t)|} \mu(t), \quad \text{i.e.} \quad c_{i_3 \ldots i_{l-1}} = l! \sum_{t \in \mathcal{T}_{i_3 \ldots i_{l-1}}} \frac{1}{|\text{Aut}(t)|}, \quad (B8)$$

where the $c'$s are those in (B4) and the graphs contributing to a fixed $\mu(t)$ with labels $(i_3 \ldots i_{l-1})$ are denoted by $\mathcal{T}_{i_3 \ldots i_{l-1}} \subset \mathcal{T}$.

The automorphism group in (i) is defined as in Section II.A, with $\mathcal{V}_0$ the set of external vertices, $\mathcal{V}_1$ the set of multi-valent vertices, and edge list $e \subset (\mathcal{V}_0 \cup \mathcal{V}_1)_2$. The Euler relations holds in the form $|\mathcal{V}_0| + |\mathcal{V}_1| = |e| + 1$ and $\sum_{k=1}^l k|\mathcal{V}_k| = 2|e|$, where $|\mathcal{V}_k|$ is the number of $k$-valent vertices, $k = 1, 3, 4, \ldots$. The second form ensures that the degrees and signs in (B4) are correctly reproduced by (ii) and only the coefficients $c_{i_3 \ldots i_{l-1}}$ in (iii) need to be understood in graph theoretical terms. Importantly, for $l \geq 6$ several topologically distinct tree graphs contributing to a fixed a $(i_3 \ldots i_{l-1})$ configuration can occur. As an example, the tree graphs contributing to $\gamma_5$ are displayed below

$$\omega_2^{-5} \omega_5 \quad |\text{Aut}(t)| = 5!$$

$$\omega_2^{-6} \omega_3 \omega_4 \quad |\text{Aut}(t)| = 3!2!$$

$$\omega_2^{-7} \omega_3^2 \quad |\text{Aut}(t)| = 2^3 \quad (B9)$$
A proof of (B8) can be based on the known tree graph interpretation of (B6), see [1114] and the references therein. Below we provide an alternative ab-initio proof without reference to the compositional inverse formula. The key ingredient is the following mixed recursion relation

\[ \gamma_i(\varphi) = - \sum_{j=2}^{l} \frac{j!}{k_1 \cdots k_l} \sum \frac{\gamma_{k_1}^{(1)}(\varphi) \cdots \gamma_{k_l}^{(1)}(\varphi)}{k_1! \cdots k_l!}, \quad l \geq 3. \]  

(B10)

This is the zero-dimensional counterpart of the recursions (14), (22) instrumental for our analysis of the \( \Gamma_k \) graph rules. It can be derived along similar lines starting from \( \gamma(\varphi + \alpha) = (\varphi + \alpha)^{\gamma(1)}(\varphi; \alpha) - \omega(\gamma(1)^{(1)}(\varphi; \alpha)) \) and (B7).

**Ab-initio proof of \( \gamma_i \) graph rule based on (B10).** We proceed by induction in \( l \), assuming that (B8) is known to produce the correct coefficients (B3) for \( k = 1, \ldots, l-1 \). To obtain the result at order \( l \) we first note a simple generation recipe (*): the set of tree graphs in \( \mathcal{T}_{k-1} \) can be obtained from those in \( \mathcal{T}_{k-2} \) by insertion of a line in all possible ways either at a multi-valent vertex or in the middle of an existing line. In fact, differentiating a weight of order \( k-2 \) from part (ii) of the graph rule, \( \partial_{\omega} \mu(t) \), produces a sum of terms whose interpretation as order \( k-1 \) tree graphs follows the pattern (*). The terms occur with integer multiplicities which by the origin of (B4) from (B3) must be compatible with (B8).

The recursion (B10) also mirrors the pattern (*). Fix some \( t \in \mathcal{T}_{k} \) generated from order \( l-1 \) graphs as indicated. The contribution of \( t \) to \( \gamma_i/l! \) can be matched to terms on the right hand side of (B10) in the following way. Case 1: any \( 3 \leq j \)-valent internal vertex can be seen as the \( \omega_j \) piece, and the \( j \) subtrees it connects to as the 1-rooted \( \gamma_{k_1}^{(1)}/k_1! \) pieces. Case 2: the middle of an internal line with adjacent vertices of weights \( \omega_{n_1}, \omega_{n_2} \) can be seen as an \( \omega_j \) pseudo-vertex via \( \omega_{n_1} \omega_{n_2} = \partial_{\omega_j} \omega_{n_1} \omega_{n_2} = \partial_{\omega} \mu(t) \), and the two subtrees it connects to as 1-rooted \( \gamma_{k_1}^{(1)}/k_1! \) pieces. The full contribution of \( t \) obtained from (B10) is then the sum of reassembled root graph weights produced by each distinct choice of the \( \omega_j \), \( j \geq 2 \), and \( \gamma_{k_i}^{(1)} \) pieces. Our task is to keep track of the coefficients.

Case 1: \( j \)-valent vertex as \( \omega_j \), \( j \geq 3 \). By induction hypothesis each of the \( \gamma_{k_i} \) has a graph realization in \( \mathcal{T}_{k_i} \) via (i),(ii),(iii). Its derivative \( \gamma_{k_i}^{(1)} \) has the same structure where initially the differentiated weights \( \partial_{\omega} \mu(t) \) occur. By the remark following (*) each \( \partial_{\omega} \mu(t) \) expands into tree graphs of one order higher which we regard as 1-rooted, \( t' \in \mathcal{T}_{k_i+1}^{\bullet} \) (with the rooted vertex always an internal one). The regrouping leads to coefficients of the \( \mu(t') \)'s that must by the differentiation compatibility be given by the graph rule (at lower orders) applied to rooted trees. In summary, each term in the graph expansion of \( \gamma_{k_i}^{(1)}/k_i! \) carries the coefficient

\[ \frac{(-1)^{|V_t(t')|}}{|\text{Aut}(t')|}, \quad t' \in \mathcal{T}_{k_i+1}^{\bullet}. \]  

(B11)

Suppose that there are \( j_i \) isomorphic subtrees \( t'_i, i = 1, \ldots, n \) attached to the \( \omega_j \) vertex. Then, accounting for the \( 1/j! \) in (B10) we obtain the full prefactor for the choice of \( \omega_j \), \( j \geq 3 \), as vertex

\[ \prod_{i=1}^{n} \left( \frac{(-1)^{|V_{t_i}(t'_i)|}}{j_i! |\text{Aut}(t'_i)|} \right). \]  

(B12)

Let \( t \in \mathcal{T}_k \) be the graph reassembled from the rooted subtrees \( t'_i \) at the vertex with weight \( \omega_j \). The total weight is \( \omega_j \) times the product of the weights of the subtrees and is of the form \( \mu(t) \) as in part (ii) of the graph rule. The overall sign \( (-1)^{|V_t(t)|} \), with \( |V_t(t)| \) the number of internal vertices of \( t \). A straightforward application of the orbit stabilizer theorem shows that the modulus of (B12) equals the symmetry factor of \( t \) rooted at our choice of \( \omega_j \) vertex. As an unrooted graph the overall coefficient is \( (-1)^{|V_t(t)|}/|\text{Aut}(t)| \). There may be several choices of \( \omega_j \) vertices contributing equally, so the net coefficient for Case 1 is

\[ \frac{(-1)^{|V_t(t)|}}{|\text{Aut}(t)|} \times \# \text{ of } \omega_j \text{ choices with fixed } t. \]  

(B13)
Case 2: middle of an internal line as $\omega_2$ pseudo-vertex. As before, each of the two subtrees attached to $\omega_2$ contributes with coefficient (B11). While two subtrees may be distinct or identical, their contribution to the overall symmetry factor will be accounted for by the $1/2!$ prefactor in (B10).

Again we write $t \in \mathcal{T}_l$ for the graph obtained by reassembling the two subtrees at the $\omega_2$ pseudo-vertex. The overall symmetry factor obtained is that for $t$ rooted at the two ends of the internal line. When reassembled to $t$ via $\partial_\omega \omega_{l-1} \partial_\phi \omega_{l+1} = \omega_{l_1} \omega_{l_2} \omega_{\ell_3}$ (with $\omega_{l_1}, \omega_{l_2}$ the weights of the rooted vertices) the overall coefficient is $-(-)^{|v_1|}/|\text{Aut}(t)|$. The extra sign accounts for the fact that in the graph rule $\omega_2^{-1}$ carries no sign while in (B10) the $\omega_2$ term does. There may be several equivalent internal lines in $t$ that are reassembled in this way. The net coefficient for Case 2 then is

$$-(-)^{|v_1|}/|\text{Aut}(t)| \times \# \text{ of equivalent internal lines in } t.$$  \hspace{1cm} (B14)

The full contribution to $\gamma_l/l!$ associated with $t$ is obtained from (B10) by summing over the contributions from Case 1 and Case 2 with weight $\mu(t)$ and coefficients (B13), (B14). This gives

$$(-)^{|v_1|}/|\text{Aut}(t)| \times (|v_1| - |e_1|),$$  \hspace{1cm} (B15)

where $|v_1|, |e_1|$ are the total number of internal vertices and internal lines of $t$, respectively. For the tree graphs considered the number of external lines and vertices coincide, $|v_0| = l = |e_0|$, so that the Euler relation reduces to $|v_1| - |e_1| = 1$.

A multi-dimensional version of the above graph rule would similarly relate the vertex functions of a lattice quantum field theory to its connected correlation functions (even at non-zero mean field or source). This is implicit in many textbooks: a proof can be read off from and also the above derivation carries over. We briefly comment here on this multi-dimensional version in order to highlight that the trees invoked are unrelated to those in Section III and IV. We denote the standard (unmodified) Legendre transform by $\Gamma[\phi]:=\phi \cdot H[\phi] - W[H[\phi]]$, with $W(1)[H[\phi]] = \phi$.

Then, $\Gamma(1)[\phi] = H[\phi]$ and the counterpart of (B3) reads

$$W(1)[\Gamma(1)[\phi]] = \phi, \quad \Gamma(1)[W(1)[H]] = H.$$  \hspace{1cm} (B16)

Throughout a superscript $(n)$ denotes $n$-fold differentiation of a functional of one field with respect to its argument. By repeated differentiation with respect to $\phi$ or $H$ one obtains in principle mutually equivalent relations between the $\Gamma^{(l)}[\phi]$ (vertex functions in non-zero mean field) and $W^{(l)}[H]$ (cumulants with non-zero source). These coincide essentially with those in the zero-dimensional case (B4), just that different lattice sum contractions will remove most of the degeneracies that give rise to non-unit coefficients. That is, in the QFT counterpart of (B4) there will be $c_{y_3...y_{l-1}}$ structurally similar terms (with $W^{(m)}[H[\phi]], \Gamma^{(m)}[\phi]$ replacing $\omega_m, \gamma_m$, respectively) where the indices in the lattice sums are contracted differently. The graph rule producing these correctly contracted terms in the $\Gamma^{(l)}[\phi]$ expansion invokes the previous tree graphs $\mathcal{T}_l$, but now labeled by lattice points. The external points $x_1, ..., x_l$ will be taken distinct but the lattice points summed over in the products of $W^{(k)}_{x_1...x_2}, k \geq 3$, vertices may coincide. This may occasionally produce coinciding labels for the internal vertices but the tree structure precludes nontrivial automorphisms. A counterpart of the above graph rule can then easily be formulated, see (B13) for related Hopf algebraic constructions. Despite the occurrence of labeled tree graphs in this context it is $\Gamma^{(l)}[\phi]$, the $l$-th functional derivative of $\Gamma[\phi]$, that is related to its $W^{(k)}$ counterparts, not the order in a $\gamma_\ell$ expansion. Performing a $\kappa$ expansion of both sides of (B4)’s multi-dimensional counterpart is of no immediate help in understanding the graph rule underlying $\Gamma$’s hopping expansion.

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