Operators on random hypergraphs and random simplicial complexes

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Abstract

Random hypergraphs and random simplicial complexes have potential applications in computer science and engineering. Various models of random hypergraphs and random simplicial complexes on \( n \)-points have been studied. Let \( L \) be a simplicial complex. In this paper, we study random sub-hypergraphs and random sub-complexes of \( L \). By considering the minimal complex that a sub-hypergraph can be embedded in and the maximal complex that can be embedded in a sub-hypergraph, we define some operators on the space of probability functions on sub-hypergraphs of \( L \). We study the compositions of these operators as well as their actions on the space of probability functions. As applications in computer science, we give algorithms generating large sparse random hypergraphs and large sparse random simplicial complexes.

1 Introduction

Random graphs, random simplicial complexes and random hypergraphs are all random topological objects. The study of random topological objects is motivated by potential applications in large-data systems in computer science and engineering. Among these random topological objects, random graph is the simplest case. The systematic study of random graph was started by P. Erdős and A. Rényi around 1960.

1.1 Hypergraphs and simplicial complexes

The definitions of hypergraphs and simplicial complexes can be found in [4] and [19] respectively. Let \( n \geq 2 \) be an integer. Let \( V \) be a set of \( n \) points. The power set of \( V \), denoted as \( 2^V \), is the collection of all subsets of \( V \). We assume that the empty set is not in \( 2^V \) if there is no extra claim. A hypergraph \( H \) on \( V \) is a subset of \( 2^V \). In particular, \( 2^V \) is called the complete hypergraph and the empty set \( \emptyset \) is called the empty hypergraph. An element of \( H \) is called a hyperedge, and an element of a hyperedge is called a vertex. The dimension of a hyperedge is the cardinality of the hyperedge minus one. A hyperedge of dimension \( d \) is called a \( d \)-hyperedge for short. The vertex set of \( H \), denoted as \( V_H \), is the subset of \( V \) consisting of all vertices of all hyperedges in \( H \). A hypergraph \( H' \) is said to be a sub-hypergraph of \( H \) if \( H' \subseteq H \). An (abstract) simplicial complex \( K \) on \( V \) is a hypergraph on \( V \) such that for any \( \sigma \in K \) and any nonempty \( \tau \subseteq \sigma \), \( \tau \in K \). The hyperedges of \( K \) are called simplices. A simplicial complex \( K' \) is said to be a sub-complex of \( K \) if \( K' \subseteq K \). Given a sub-complex \( K' \subseteq K \), a \( d \)-clique of \( K' \) in \( K \) is a \( d \)-simplex \( \sigma \in K \) such that for any \( \tau \subseteq \sigma \), \( \tau \subseteq K' \).

\textbf{2010 Mathematics Subject Classification.} 05C80, 05E45, 55U10, 68P05.
\textbf{Keywords and Phrases.} Hypergraphs, Simplicial complexes, Randomness, Probability.
1.2 Random hypergraphs and random simplicial complexes

Let $L$ be a finite simplicial complex. Let $\mathcal{H}(L)$ be the collection of all sub-hypergraphs of $L$. A random sub-hypergraph of $L$ is a probability function on $\mathcal{H}(L)$. Let $D(\mathcal{H}(L))$ be the functional space of all probability functions on $\mathcal{H}(L)$. Let $\text{Map}(\mathcal{H}(L))$ be the group of all self-maps on $\mathcal{H}(L)$. An element $T \in \text{Map}(\mathcal{H}(L))$ induces a self-map $DT$ on $D(\mathcal{H}(L))$ by

$$DT(f)(H) = \sum_{T'H' = H} f(H'),$$

(1.1)

for any $f \in D(\mathcal{H}(L))$ and any $H \in \mathcal{H}(L)$. And a map $F$ from $\mathcal{H}(L)^{\times 2}$ to $\mathcal{H}(L)$ induces a map $DF$ from $D(\mathcal{H}(L))^{\times 2}$ to $D(\mathcal{H}(L))$ by

$$DF(f_1, f_2)(H) = \sum_{F(H_1, H_2) = H} f_1(H_1)f_2(H_2),$$

(1.2)

for any $f_1, f_2 \in D(\mathcal{H}(L))$ and any $H \in \mathcal{H}(L)$. Let $\mathcal{K}(L)$ be the collection of all sub-complexes of $L$. Let $\text{Map}(\mathcal{K}(L))$ be the group of all self-maps on $\mathcal{K}(L)$. A random sub-complex of $L$ is a probability function on $\mathcal{K}(L)$. Let $D(\mathcal{K}(L))$ be the functional space of all probability functions on $\mathcal{K}(L)$. Similar to (1.1), an element of $\text{Map}(\mathcal{K}(L))$ induces a self-map on $D(\mathcal{K}(L))$, and a map from $\mathcal{H}(L)$ to $\mathcal{K}(L)$ induces a map from $D(\mathcal{H}(L))$ to $D(\mathcal{K}(L))$. And similar to (1.2), a map $F$ from $\mathcal{K}(L)^{\times 2}$ to $\mathcal{K}(L)$ induces a map $DF$ from $D(\mathcal{K}(L))^{\times 2}$ to $D(\mathcal{K}(L))$.

Let $0 \leq p \leq 1$. In 1959, P. Erdős and A. Rényi [14] and E.N. Gilbert [16] constructed the Erdős-Rényi model $G(n, p)$ of random graphs by choosing each pair of vertices in $V$ as an edge uniformly and independently at random with probability $p$. In 1960, thresholds for the connectivity of $G(n, p)$ were given in [15]. In recent decades, the clique complex of $G(n, p)$ was studied in [9, 21].

In 2006, N. Linial and R. Meshulam [25] constructed the Linial-Meshulam model $Y_2(n, p)$ of random 2-complexes. They take the complete graph on $V$ and choose each 2-simplex of the complete complex on $V$ uniformly and independently at random with probability $p$. The fundamental group of $Y_2(n, p)$ was studied in [3]. The homology groups of $Y_2(n, p)$ were studied in [5, 6]. The asphericity and the hyperbolicity of $Y_2(n, p)$ were studied in [7, 8].

Let $d$ be a non-negative integer. In 2009, R. Meshulam and N. Wallach [27] generalized $Y_2(n, p)$ and constructed a model $Y_d(n, p)$ of random $d$-complexes. They take the $(d - 1)$-skeleton of the complete complex on $V$, then choose each $d$-simplex of the complete complex on $V$ uniformly and independently at random with probability $p$. The homology groups of $Y_d(n, p)$ were studied in [2, 20, 24]. The cohomology of $Y_d(n, p)$ was studied in [23]. Some thresholds for the homology of $Y_d(n, p)$ were given in [26]. The eigenvalues of the Laplacian on $Y_d(n, p)$ were studied in [17]. The collapsibility property of $Y_d(n, p)$ was studied in [11, 12]. And some sub-structures of $Y_d(n, p)$ were studied in [18].

Let $0 \leq r \leq n - 1$ be an integer. Let $0 \leq p_0, p_1, \ldots, p_{n-1} \leq 1$. Let $\mathbf{p} = (p_0, p_1, \ldots, p_{n-1})$. In 2016, $G(n, p)$, $Y_2(n, p)$, $Y_d(n, p)$ and (the $r$-skeleton of) the clique complex of $G(n, p)$ were generalized universally to a multi-parameter model of random complexes with probability function $P_{n, r, \mathbf{p}}$ by A. Costa and M. Farber [10, 13]. In 2017, the fundamental group of the final-generated complexes has been studied in [11]. The dimension has been studied in [12].
Definition 1. [10, 13] Let $\Delta_n$ denote the complete simplicial complex on $n$ vertices. Let $\Delta_n^{(r)}$ be the $r$-skeleton of $\Delta_n$. An external face of a sub-complex $Y \subseteq \Delta_n$ is a simplex $\sigma \in \Delta_n$ such that $\sigma \notin Y$ but the boundary of $\sigma$ is contained in $Y$. We use $E(Y)$ to denote the set of all external faces of $Y$. Let $p = (p_0, p_1, \ldots, p_r)$ with $0 \leq p_i \leq 1$. We consider the probability space $\mathcal{K}(\Delta_n^{(r)})$. The probability function is

$$P_{n,r,p}(Y) = \prod_{\sigma \in Y, \dim \sigma \leq r} p_{\dim \sigma} \cdot \prod_{\sigma \in E(Y), \dim \sigma \leq r} (1 - p_{\dim \sigma}).$$

The probability function $P_{n,r,p}$ can be obtained as follows (cf. [22, Section 5.5]). (i). We generate the 0-skeleton by choosing each vertex of $\Delta_n$ uniformly and independently at random with probability $p_0$. (ii). For each $0 \leq k \leq r - 2$, suppose the $k$-skeleton is generated. Then we generate the $(k + 1)$-skeleton by choosing each $(k + 1)$-clique of the $k$-skeleton in $\Delta_n^{(r)}$ uniformly and independently at random with probability $p_{k+1}$. The final-generated complexes have probability function $P_{n,r,p}$. Thus

$$\sum_{Y \subseteq \Delta_n^{(r)}} P_{n,r,p}(Y) = 1.$$

Let $p : L \to [0, 1]$ be an arbitrary function. In the next definition, we generalize Definition 1 and give a model of random sub-complex in $L$.

Definition 2 (Generalization of Definition 1). An external face of a sub-complex $Y \subseteq L$ is a simplex $\sigma \in L$ such that $\sigma \notin Y$ but the boundary of $\sigma$ is contained in $Y$. We use $E(Y)$ to denote the set of all external faces of $Y$ in $L$. We consider the probability space $\mathcal{K}(L)$. The probability function is given by

$$P_{L,p}(Y) = \prod_{\sigma \in Y} p(\sigma) \cdot \prod_{\sigma \in E(Y)} (1 - p(\sigma)).$$

In particular, suppose $\dim L = r$ and there exists $0 \leq p_0, p_1, \ldots, p_r \leq 1$ such that for each $\sigma \in L$, $p(\sigma) = p_{\dim \sigma}$. Then we denote $P_{L,p}$ as $P_{L,p}$. We have $P_{\Delta_n^{(r)},p} = P_{n,r,p}$.

The random complex model in Definition 2 can be generated as follows. (i). Choose each vertex $v \in L$ independently at random with probability $p(v)$. (ii). For each $0 \leq k \leq \dim L - 1$, suppose the $k$-skeleton is generated. Then we generate the $(k + 1)$-skeleton by choosing each $(k + 1)$-clique $\sigma$ of the $k$-skeleton in $L$ independently at random with probability $p(\sigma)$. The final-generated complexes have the probability function $P_{L,p}$. Hence

$$\sum_{Y \subseteq L} P_{L,p}(Y) = 1.$$

In the next definition, we consider an analogue of Definition 2 and give a model of random sub-hypergraph in $L$.

Definition 3 (Hypergraphic analogue of Definition 2). We consider the probability space $\mathcal{H}(L)$.
The probability function is given by

\[ \overline{P}_{L,p}(H) = \prod_{\sigma \in H} p(\sigma) \cdot \prod_{\sigma \notin H} (1 - p(\sigma)). \]

In particular, suppose \( \dim L = r \) and there exists \( 0 \leq p_0, p_1, \ldots, p_r \leq 1 \) such that for each \( \sigma \in L \), \( p(\sigma) = p_{\dim \sigma} \). Then we denote \( \overline{P}_{L,p} \) as \( \overline{P}_{L,p} \).

The random hypergraph in Definition \( \overline{3} \) can be generated as follows. We choose each simplex \( \sigma \in L \) independently at random with probability \( p(\sigma) \). We obtain a hypergraph. The probability function of these independent trials is \( \overline{P}_{L,p} \). Therefore,

\[ \sum_{H \subseteq L} \overline{P}_{L,p}(H) = 1. \]

1.3 Our results

Let \( H \in \mathcal{H}(L) \). In this paper, we study the minimal complex \( \Delta H \) that \( H \) can be embedded in, the maximal complex \( \delta H \) that can be embedded in \( H \), and the complement hypergraph \( \gamma H \) in \( L \). By composing \( \Delta, \delta \) and \( \gamma \) iteratively, we obtain a subgroup \( G \) of Map(\( \mathcal{H}(L) \)). And \( G \) induces a group \( DG \) of self-maps on \( D(\mathcal{H}(L)) \). Moreover, by composing \( \Delta \gamma \) and \( \delta \gamma \) iteratively, we obtain a subgroup \( G' \) of Map(\( \mathcal{K}(L) \)). And \( G' \) induces a group \( DG' \) of self-maps on \( D(\mathcal{K}(L)) \). We study the operator algebra acting on \( D(\mathcal{H}(L)) \) induced from \( \Delta, \delta \) and \( \gamma \), and the operator algebra acting on \( D(\mathcal{K}(L)) \) induced from \( \Delta \gamma \) and \( \delta \gamma \). In particular, we give some explicit expressions for the actions of the operator algebra on \( \overline{P}_{L,p} \) and \( P_{L,p} \). As consequences, we give algorithms generating large sparse random hypergraphs with probability function \( \overline{P}_{\Delta_n,p} \), and algorithms generating large sparse random simplicial complexes with probability function \( P_{\Delta_n,p} \).

Let \( \sigma \in L \). The characteristic probability \( \varphi_\sigma \) is the function

\[ \varphi_\sigma(\sigma') = \begin{cases} 0, & \text{if } \sigma' \neq \sigma; \\ 1, & \text{if } \sigma' = \sigma. \end{cases} \]

A path \( s \) in \( L \) is a sequence of simplices \( \sigma_1 \sigma_2 \ldots \sigma_m \) in \( L \) such that the intersection of any two consecutive simplices is nonempty. We call \( m \) the length of \( s \). Given two simplices \( \sigma, \sigma' \in L \), the distance between \( \sigma \) and \( \sigma' \) is

\[ d(\sigma, \sigma') = \min\{m \mid s = \sigma_1 \sigma_2 \ldots \sigma_m \text{ is a path in } L, \sigma_1 = \sigma, \sigma_m = \sigma'\}. \]

The diameter of \( L \) is \( \text{diam} L = \max_{\sigma, \sigma' \in L} d(\sigma, \sigma') \). The first main result of this paper is the next Theorem.

**Theorem 1.1** (Main Result I). Let \( k \) be a non-negative integer. Let \( \text{Ext} = \Delta \gamma \delta \gamma \) and \( \text{Int} = \delta \gamma \Delta \gamma \). Let \( f \in D(\mathcal{H}(L)) \).

(a). If \( k \geq \text{diam} L \), then \( (D\text{Ext})^k(f) = f(\emptyset)\varphi_\emptyset + (1 - f(\emptyset))\varphi_L \).

(b). If \( k \geq \text{diam} L \), then \( (D\text{Int})^k(f) = (1 - f(L))\varphi_\emptyset + f(L)\varphi_L \).
(c) There exists \( f \in \mathcal{H}(L) \) such that

\[
f, (D\text{Ext})(f), (D\text{Ext})^2(f), \ldots, (D\text{Ext})^{\text{diam}L-1}(f), (D\text{Ext})^{\text{diam}L}(f)
\]
are distinct;

(d) There exists \( f \in \mathcal{H}(L) \) such that

\[
f, (D\text{Int})(f), (D\text{Int})^2(f), \ldots, (D\text{Int})^{\text{diam}L-1}(f), (D\text{Int})^{\text{diam}L}(f)
\]
are distinct;

(e) Let \( k \geq 1 \). Then for any probability function \( f \in D(\mathcal{H}(L)) \), the probability that \( \text{Ext}^{k-1}(\gamma H) \subseteq \gamma \text{Int}^k(H) \), the probability that \( \text{Int}^k(H) \subseteq \text{Ext}^{k+1}(\gamma H) \) and the probability that \( \text{Ext} \cap \text{Int}(H) \subseteq \delta H \) are \( 1 \);

(f) For any probability function \( f \in D(\mathcal{H}(L)) \), the probability that \( \Delta H \subseteq \text{Int} \cap \text{Ext}(H) \) is greater than or equal to \( \sum_{\sigma \subseteq H} f(H) \).

Consider the spaces of probability functions

\[
\begin{align*}
F(\mathcal{H}(L)) &= \{ \bar{P}_{L,p} | p : L \rightarrow [0,1] \}, \\
F(\mathcal{K}(L)) &= \{ P_{L,p} | p : L \rightarrow [0,1] \}.
\end{align*}
\]

Then \( F(\mathcal{H}(L)) \) is a subspace of \( D(\mathcal{H}(L)) \) and \( F(\mathcal{K}(L)) \) is a subspace of \( D(\mathcal{K}(L)) \). Let \( \cap \) and \( \cup \) be the intersection and the union of hypergraphs. The second main result of this paper is the next theorem.

**Theorem 1.2 (Main Result II).** The operator \( D\gamma \) maps \( F(\mathcal{H}(L)) \) to itself. The operators \( D\Delta \) and \( D\delta \) map \( F(\mathcal{H}(L)) \) to \( F(\mathcal{K}(L)) \). The operator \( D\cap \) maps \( F(\mathcal{H}(L)) \times 2 \) to \( F(\mathcal{H}(L)) \), and maps \( F(\mathcal{K}(L)) \times 2 \) to \( F(\mathcal{K}(L)) \). And the operator \( D\cup \) maps \( F(\mathcal{H}(L)) \times 2 \) to \( F(\mathcal{H}(L)) \). Precisely,

(a) \( D\gamma \) sends \( \bar{P}_{L,p} \) to \( \bar{P}_{L,1-p} \);

(b) \( D\Delta \) sends \( \bar{P}_{L,p} \) to \( P_{L,p'} \), where \( p' \) is given by

\[
p'(\tau) = 1 - \prod_{\sigma \subseteq L, \tau \subseteq \sigma} (1 - p(\sigma)) \quad (1.3)
\]

for any \( \tau \in L \);

(c) \( D\delta \) sends \( \bar{P}_{L,p} \) to \( P_{L,p''} \), where \( p'' \) is given by

\[
p''(\tau) = \prod_{\sigma \subseteq \tau} p(\sigma) \quad (1.4)
\]

for any \( \tau \in L \);

(d) \( D\cap \) sends the pair \( (P_{L,p'}, P_{L,p''}) \) to \( P_{L,p'p''} \), and sends the pair \( (\bar{P}_{L,p'}, \bar{P}_{L,p''}) \) to \( \bar{P}_{L,p'p''} \);

(e) \( D\cup \) sends the pair \( (\bar{P}_{L,p'}, \bar{P}_{L,p''}) \) to \( \bar{P}_{L,\tau,1-(1-p')(1-p'')} \).
The remaining part of this paper is organized as follows. In Section 2 we study the operator algebra generated by \( \Delta, \delta \) and \( \gamma \). In Section 3 we give some geometric characterizations of certain compositions of \( \Delta, \delta \) and \( \gamma \). In Section 4 we prove Theorem 1.1 and Theorem 1.2. In Section 5 as by-products of Theorem 1.2 we give algorithms generating large sparse random hypergraphs and large sparse random simplicial complexes.

2 Operator algebras on hypergraphs and simplicial complexes

In this section, we study the minimal complex \( \Delta H \) that \( H \) can be embedded in, the maximal complex \( \delta H \) that can be embedded in \( H \), and the complement hypergraph \( \gamma H \) in \( L \). We study the operator algebras of the compositions of \( \Delta, \delta \) and \( \gamma \) as well as the intersections and unions. We also study the restrictions of the compositions of \( \Delta, \delta \) and \( \gamma \) on simplicial complexes.

2.1 The operator algebra on hypergraphs

We consider the operators \( \Delta : \mathcal{H}(L) \rightarrow \mathcal{K}(L) \), \( \delta : \mathcal{H}(L) \rightarrow \mathcal{H}(L) \), and \( \gamma : \mathcal{H}(L) \rightarrow \mathcal{H}(L) \) given by

\[
\begin{align*}
\Delta H &= \{ \sigma \in L \mid \text{there exists } \tau \in H \text{ such that } \sigma \subseteq \tau \}; \\
\delta H &= \{ \sigma \in L \mid \text{for any } \tau \subseteq \sigma, \tau \in H \}; \\
\gamma H &= \{ \sigma \in L \mid \sigma \notin H \}
\end{align*}
\]

for any \( H \in \mathcal{H}(L) \). Then (i). \( \gamma^2 = \text{id} \); (ii). \( \Delta \delta = \delta \); (iii). \( \delta \Delta = \Delta \); (iv). \( \Delta^2 = \Delta \); (v). \( \delta^2 = \delta \); (vi). \( (\Delta \gamma \Delta \gamma)^2 = \Delta \gamma \Delta \gamma \); (vii). \( (\delta \gamma \delta \gamma)^2 = \delta \gamma \delta \gamma \). The equalities (i) - (v) are straight-forward. Let max\((L)\) be the set of all maximal faces of \( L \). We prove (vi) and (vii).

Proof of (vi). Let \( H \in \mathcal{H}(L) \). Then

\[
\Delta \gamma \Delta \gamma H = \Delta \gamma \Delta \{ \sigma \in L \mid \sigma \notin H \} = \Delta \gamma \{ \sigma \in L \mid \text{there exists } \sigma \subseteq \tau \text{ such that } \tau \notin H \} = \Delta \{ \sigma \in L \mid \text{there does not exist any } \sigma \subseteq \tau \text{ such that } \tau \notin H \} = \Delta \{ \sigma \in L \mid \text{for any } \sigma \subseteq \tau, \tau \in H \} = \Delta \{ \sigma \in \text{max}(L) \mid \sigma \in H \} = \Delta (\text{max}(L) \cap H).
\]

Thus

\[
(\Delta \gamma \Delta \gamma)^2 H = \Delta (\text{max}(L) \cap \Delta (\text{max}(L) \cap H)) = \Delta (\text{max}(L) \cap H).
\]

Since \( H \) is arbitrary, we have (vi). \( \square \)
Proof of (iv). Let \( H \in \mathcal{H}(L) \). Then
\[
\delta \gamma \delta \gamma H = \delta \gamma \delta \{ \sigma \in L \mid \sigma \notin H \}
\]
\[
= \delta \gamma \{ \sigma \in L \mid \text{for any } \tau \subseteq \sigma, \tau \notin H \}
\]
\[
= \delta \{ \sigma \in L \mid \text{there exists } \tau \subseteq \sigma, \tau \in H \}
\]
\[
= \{ \sigma \in L \mid \text{for any } \sigma' \subseteq \sigma, \text{there exists } \tau \subseteq \sigma', \tau \in H \}
\]
\[
= \{ \sigma \in L \mid \text{for any vertex } v \text{ of } \sigma, \{v\} \in H \}.
\]
Hence \( \delta \gamma \delta \gamma H \) is the sub-complex of \( L \) spanned by all the 0-hyperedges in \( H \). And \( (\delta \gamma \delta \gamma)^2 H \) is the sub-complex of \( L \) spanned by all the 0-hyperedges in \( \delta \gamma \delta \gamma H \). Since the 0-hyperedges of \( H \) and the 0-hyperedges of \( \delta \gamma \delta \gamma H \) are same, we have \( (\delta \gamma \delta \gamma)^2 H = \delta \gamma \delta \gamma H \). Since \( H \) is arbitrary, we have (iv).

Let \( G \) be the group generated by \( \Delta, \delta, \gamma, \) modulo the relations (i) - (vii). The multiplication of \( G \) is the composition of maps. The unit of \( G \) is \( \text{id} \), the identity map on \( \mathcal{H}(L) \). There are four types of elements in \( G \): (1). \( \gamma x_1 \gamma x_2 \ldots \gamma x_k \gamma \); (2). \( x_1 \gamma x_2 \ldots \gamma x_k \gamma \); (3). \( \gamma x_1 \gamma x_2 \ldots \gamma x_k \gamma \); (4). \( x_1 \gamma x_2 \ldots \gamma x_k \). In (1) - (4), \( k \) is a nonnegative integer, \( x_i = \Delta \) or \( \delta \), and there does not exist any 4 consecutive \( x_i \)'s that take the same value. For any \( w_1, w_2 \in G \) and any \( H_1, H_2 \in \mathcal{H}(L) \), let
\[
(w_1 + w_2)(H_1, H_2) = w_1(H_1) \cup w_2(H_2);
\]
\[
(w_1 \land w_2)(H_1, H_2) = w_1(H_1) \cap w_2(H_2).
\]
For any positive integer \( t \), let
\[
G^t = \{ (\ldots (w_1 \ast w_2) \ast \ldots \ast w_t) \mid \ast = \land \text{ or } +, w_1, w_2, \ldots, w_t \in G \}
\]
with any \( t - 2 \) brackets \( (\cdot) \) giving the order of evaluation.

For any \( W \in G^t \), \( W \) is a map from \( \mathcal{H}(L)^{\times t} \) to \( \mathcal{H}(L) \). Some relations among \( w \in G \), + and \( \land \) are:

(I). \( (w_1 \land w_2) \land w_3 = w_1 \land (w_2 \land w_3) \);

(II). \( (w_1 + w_2) + w_3 = w_1 + (w_2 + w_3) \);

(III). \( \gamma(w_1 + w_2) = (\gamma w_1) \land (\gamma w_2) \), or equivalently, \( \gamma(w_1 \land w_2) = \gamma w_1 + \gamma w_2 \);

(IV). \( \Delta(w_1 + w_2) = (\Delta w_1) + (\Delta w_2) \);

(V). \( \delta(w_1 \land w_2) = \delta w_1 \land \delta w_2 \).

Here \( w_1, w_2, w_3 \in G \), and \( 0 \) is the constant map sending \( \mathcal{H}(L) \) to \( \emptyset \). (I) - (III) are straightforward. Let \( H, H' \in \mathcal{H}(L) \). Let \( H_1 = w_1(H) \) and \( H_2 = w_2(H') \). We prove (IV) and (V).

Proof of (IV). In order to prove (IV), we only need to prove that for any \( H_1, H_2 \in \mathcal{H}(L) \), \( \Delta(H_1 \cup H_2) = \Delta H_1 \cup \Delta H_2 \). Let \( \sigma \in L \). Then \( \sigma \in \Delta(H_1 \cup H_2) \) iff. there exists \( \tau \in H_1 \cup H_2 \) such that \( \sigma \subseteq \tau \). This happens iff. there exists \( \tau \in H_1 \) such that \( \sigma \subseteq \tau \) or there exists \( \tau \in H_1 \) such that \( \sigma \subseteq \tau \). Hence \( \sigma \in \Delta(H_1 \cup H_2) \) iff. \( \sigma \in \Delta H_1 \) or \( \sigma \in \Delta H_2 \), that is, \( \sigma \in \Delta H_1 \cup \Delta H_2 \).
Proof of (IV). In order to prove (IV), we only need to prove that for any \( H_1, H_2 \in \mathcal{H}(L) \), \( \delta(H_1 \cap H_2) = \delta H_1 \cap \delta H_2 \). Let \( \sigma \in L \). Then \( \sigma \in \delta(H_1 \cap H_2) \iff \) for any \( \tau \subseteq \sigma \), \( \tau \in H_1 \cap H_2 \). This happens iff. for any \( \tau \subseteq \sigma \), \( \tau \in H_1 \) and \( \tau \in H_2 \). That is, \( \sigma \in \delta H_1 \cap \delta H_2 \). \[\square\]

Remark 1: Similar to the proofs of (IV) and (V), for any \( H_1, H_2 \in \mathcal{H}(L) \), \( \Delta(H_1 \cap H_2) \subseteq \Delta H_1 \cap \Delta H_2 \) and \( \delta(H_1 \cup H_2) \supseteq \delta H_1 \cup \delta H_2 \). Hence for any \( H, H' \in \mathcal{H}(L) \),

\[
(\Delta(w_1 \land w_2))(H, H') \subseteq (\Delta w_1 \land \Delta w_2)(H, H'),
\]

\[
(\delta(w_1 + w_2))(H, H') \supseteq (\delta w_1 + \delta w_2)(H, H').
\]

2.2 The operator algebra on simplicial complexes

Let \( w \in G \). A subset \( S \) of \( \mathcal{H}(L) \) is called an invariant subspace of \( w \) if for any \( H \in S \), \( w(H) \in S \). We consider \( w \in G \) such that \( K(L) \) is an invariant subspace of \( w \). The collection of all such \( w \) forms a subgroup \( G' \) of \( G \). Since both \( \Delta \) and \( \delta \) act on \( K(L) \) identically, we take an equivalent relation \( \sim \) identifying both \( \Delta \) and \( \delta \) as the unit element of \( G_1 \). We denote the quotient group \( G_1/\sim \) as \( G' \).

Precisely, \( G' \) can be constructed as follows. Let \( \alpha = \Delta \gamma \) and \( \beta = \delta \gamma \). Let \( G' \) be the group generated by \( \alpha \) and \( \beta \) modulo the relations (i)’. \( \alpha^4 = \alpha^2 \); (ii)’. \( \beta^4 = \beta^2 \). The multiplication of \( G' \) is the composition of maps. The unit of \( G' \) is id, the identity map on \( K(L) \). The elements in \( G' \) are: (1)’. \( \alpha m_1 \beta n_1 \ldots \alpha m_k \beta n_k \); (2)’. \( \beta n_1 \alpha m_2 \ldots \alpha m_k \beta n_k \); (3)’. \( \alpha m_1 \beta n_1 \ldots \beta n_{k-1} \alpha m_k \); (4)’. \( \beta m_1 \alpha m_2 \ldots \beta m_{k-1} \alpha m_k \). Here \( k \) is a nonnegative integer and \( 1 \leq m_i, n_i \leq 3 \), \( i = 1, 2, \ldots \) Similar to Subsection 2.1, we define + and \( \land \) for the elements in \( G' \). We construct \( G'' \) for all positive integer \( t \). Each element \( W' \in G'' \) is a map from \( K(L)^{\times t} \) to \( K(L) \).

3 Some characterizations of the operators

Let \( \text{Ext} = \Delta \gamma \delta \gamma \) and \( \text{Int} = \delta \gamma \Delta \gamma \). In this section, we study the properties of the operators \( \text{Ext} \) and \( \text{Int} \). We give some geometric characterizations of \( \text{Ext} \) and \( \text{Int} \). In Subsection 3.1, we use paths in complexes to characterize the powers of \( \text{Ext} \) and \( \text{Int} \). In Subsection 3.2, we use neighborhoods of sub-complexes to study \( \text{Ext} \), \( \text{Int} \) and their compositions.

3.1 Powers of operators and paths

Let \( H \in \mathcal{H}(L) \). Let \( \text{Ext}(H) = \Delta \gamma \delta \gamma(H) \). Then

\[
\text{Ext}(H) = \Delta \{ \tau \in L \mid \text{there exists } \sigma \in H \text{ such that } \sigma \subseteq \tau \} = \{ \tau \in L \mid \text{there exists } \tau \subseteq \tau' \text{ such that there exists } \sigma \in H \text{ with } \sigma \subseteq \tau' \}.
\]

Hence \( \text{Ext}(H) \) is the sub-complex of \( L \) obtained by extending each hyperedge \( \sigma \) of \( H \) to a maximal face of \( L \) containing \( \sigma \). We call \( \text{Ext}(H) \) the extension of \( H \). We notice that every
 maximal face of \( \text{Ext}(H) \) is in \( \max(L) \), and

\[
\text{Ext}(H) = \Delta(\max(L) \cap \text{Ext}(H)) = \Delta \{ \tau_1 \in \max(L) \mid \text{there exists } \sigma \in H \text{ such that } \sigma \subseteq \tau_1 \}. \tag{3.1}
\]

For any \( k \geq 2 \), by an induction on \( k \) and (3.1),

\[
\text{Ext}^k(H) = \Delta \{ \tau_k \in \max(L) \mid \text{there exists } \tau_1, \tau_2, \ldots, \tau_{k-1} \in \max(L) \text{ and } \sigma \in H \\
\text{such that } \tau_i \cap \tau_{i-1} \neq \emptyset \text{ for any } 2 \leq i \leq k \text{ and } \sigma \subseteq \tau_1 \}. \tag{3.2}
\]

A path \( s = \sigma_1 \sigma_2 \ldots \sigma_m \) in \( L \) is called a broad path if for each \( 1 \leq i \leq m, \sigma_i \) is a maximal face of \( L \). For any path \( s \) in \( L \), if we extend each \( \sigma \) of \( s \) to be a maximal face \( \tau \in \max(L) \) such that \( \sigma \subseteq \tau \), then we obtain a broad path \( s' \).

**Lemma 3.1.** Let \( \sigma, \sigma' \in \max(L) \) with \( d(\sigma, \sigma') = n \). Then there exists a broad path of length \( n \) starting from \( \sigma \) and ending at \( \sigma' \).

**Proof.** Since \( d(\sigma, \sigma') = n \), there exists a path \( s = \tau_1 \tau_2 \ldots \tau_n \) in \( L \) such that \( \sigma = \tau_1 \) and \( \sigma' = \tau_n \). For each \( 1 \leq i \leq n \), we extend \( \tau_i \) to be a maximal face \( \tau_i' \) of \( L \). Then we obtain the broad path \( s' = \tau_1' \ldots \tau_n' \).

Let \( \text{Int}(H) = \delta \gamma \Delta \gamma(H) \). Then with the help of (2.1),

\[
\text{Int}(H) = \delta \{ \tau \in L \mid \text{for any } \tau \subseteq \sigma, \sigma \in H \} = \{ \tau \in L \mid \text{for any } \tau' \subseteq \tau \text{ and any } \tau' \subseteq \sigma, \sigma \in H \} = \{ \tau \in L \mid \text{for any } \sigma \text{ with } \sigma \cap \tau = \tau' \neq \emptyset, \sigma \in H \}. \tag{3.3}
\]

Hence \( \text{Int}(H) \) is the sub-complex of \( L \) consisting of all the hyperedges \( \tau \in H \) such that for any \( \sigma \in \gamma H, \sigma \cap \tau \) is empty. We call \( \text{Int}(H) \) the interior of \( H \). It follows from (3.3) that

\[
\text{Int}(H) = \{ \tau \in L \mid \text{for any } \sigma \in \gamma H, \tau \cap \sigma = \emptyset \} = \gamma \{ \tau \in L \mid \text{there exists } \sigma \in \gamma H \text{ such that } \tau \cap \sigma \neq \emptyset \}. \tag{3.4}
\]

For any \( k \geq 1 \), by an induction on \( k \) and (3.4),

\[
\text{Int}^k(H) = \gamma \{ \tau_k \in L \mid \text{there exists } \sigma \in \gamma H \text{ and } \tau_1, \tau_2, \ldots, \tau_{k-1} \in L \\
\text{such that } \tau_i \cap \tau_{i-1} \neq \emptyset \text{ and } \tau_i \cap \tau_{i-1} \neq \emptyset \text{ for any } 2 \leq i \leq k \}. \tag{3.5}
\]

**Lemma 3.2.** Let \( \sigma, \sigma' \) be simplices of \( L \) with \( d(\sigma, \sigma') = n \). If \( s = \sigma_1, \ldots, \sigma_n \) is a path in \( L \) with \( \sigma_1 = \sigma, \sigma_n = \sigma' \) and \( |s| = n \), then for any \( 1 \leq i < j \leq n \), \( d(\sigma_i, \sigma_j) = j - i \).

**Proof.** Suppose to the contrary, there exists \( 1 \leq i < j \leq n \) such that \( d(\sigma_i, \sigma_j) < j - i \). Let \( s_{i,j} \) be the path \( \sigma_i \ldots \sigma_j \) as a subset of \( s \). Then \( |s_{i,j}| = j - i + 1 \). And we can find a path \( s'_{i,j} \) starting from \( \sigma_i \) and ending at \( \sigma_j \) with \( |s'_{i,j}| < j - i + 1 \). Replacing \( s_{i,j} \) with \( s'_{i,j} \), we obtain a new path \( s' \) starting from \( \sigma \) and ending at \( \sigma' \) with \( |s'| < |s| \). This contradicts that \( s \) is the path starting from \( \sigma \) and ending at \( \sigma' \) with the minimal length. \( \square \)
Lemma 3.3. Let $H \in \mathcal{H}(L)$.

(a). Suppose $H \neq \emptyset$. Then for any $k \geq 1$, $\max(L) \cap \text{Ext}^k(H)$ is the union of all the broad paths $s = \tau_1 \ldots \tau_k$ in $L$ such that there exists $\sigma \in H$ with $\sigma \subseteq \tau_1$.

(b). Suppose $H \neq L$. Then for any $k \geq 1$, $\gamma \text{Int}^k(H)$ is the union of all the paths $s' = \tau'_1 \ldots \tau'_k$ in $L$ such that there exists $\sigma' \in \gamma H$ with $\tau'_1 \cap \sigma' \neq \emptyset$.

Proof. Assertion (a) follows from (3.2). Assertion (b) follows from (3.5). \qed

In the next proposition, we list some properties of the powers of Ext and Int.

Proposition 3.4. Let $H \in \mathcal{H}(L)$, $k$ be a nonnegative integer and $n$ be the diameter of $L$.

(a). If $H \neq \emptyset$ and $k \geq n$, then $\text{Ext}^k(H) = L$.

(b). If $H \neq L$ and $k \geq n$, then $\text{Int}^k(H) = \emptyset$.

(c). There exists $H$ such that $H \subseteq \text{Ext}(H) \subseteq \text{Ext}^2(H) \subseteq \ldots \subseteq \text{Ext}^{n-1}(H) \subseteq L$.

(d). There exists $H$ such that $H \supseteq \text{Int}(H) \supseteq \text{Int}^2(H) \supseteq \ldots \supseteq \text{Int}^{n-1}(H) \neq \emptyset$.

(e). Let $k \geq 1$. Then $\text{Ext}^{k-1}(\gamma H) \subseteq \gamma \text{Int}^k(H) \subseteq \text{Ext}^{k+1}(\gamma H)$.

Proof. Let $k \geq n$. Then for any simplices $\sigma_1, \sigma_2 \in L$, there exists a path $s$ of length $n$ starting from $\sigma_1$ and ending at $\sigma_2$.

(a). Suppose $H \neq \emptyset$. Then by Lemma 3.3 (a), any $\sigma \in \max(L)$ is in $\text{Ext}^k(H)$. Thus $\max(L) \cap \text{Ext}^k(H) = L$. Thus $\text{Ext}^k(H) = L$.

(b). Suppose $H \neq L$. Then by Lemma 3.3 (b), any simplex $\sigma \in L$ is in $\gamma \text{Int}^k(H)$. Thus $\text{Int}^k(H) = \emptyset$.

Choose $\sigma, \sigma' \in \max(L)$ such that $d(\sigma, \sigma') = n$. Choose a broad path $s = \sigma_1 \ldots \sigma_n$ in $L$ with $\sigma_1 = \sigma$ and $\sigma_n = \sigma'$.

(c). Let $H = \{\sigma\}$. Then by Lemma 3.2 and Lemma 3.3 (a), for any $0 \leq i \leq n - 1$, $\sigma_j \in \max(L) \cap \text{Ext}^i(H)$ for any $j \leq i + 1$, and $\sigma_j \notin \max(L) \cap \text{Ext}^i(H)$ for any $j \geq i + 2$. Hence for any $0 \leq i \leq n - 1$, $\text{Ext}^i(H) \subseteq \text{Ext}^{i+1}(H)$.

(d). Let $H = \gamma \{\sigma\}$. Then by Lemma 3.2 and Lemma 3.3 (b), for any $0 \leq i \leq n - 1$, $\sigma_j \in \text{Int}^i(H)$ for any $j \geq i + 2$, and $\sigma_j \notin \text{Int}^i(H)$ for any $j \leq i + 1$. Hence for any $0 \leq i \leq n - 1$, $\text{Int}^i(H) \supseteq \text{Int}^{i+1}(H)$.

(e). Let $\tau_{k-1} \in \text{Ext}^{k-1}(\gamma H)$. Then there exists $\sigma_{k-1} \in \max(L) \cap \text{Ext}^{k-1}(\gamma H)$ such that $\tau_{k-1} \subseteq \sigma_{k-1}$. By Lemma 3.3 (a), there exists a broad path $\sigma \ldots \sigma_{k-1}$ of length $k - 1$ and $\tau \in \gamma H$ where $\sigma$ is a maximal face of $L$ such that $\tau \subseteq \sigma$. Consider the path $s = \sigma \ldots \sigma_{k-1} \tau_{k-1}$ of length $k$ in $L$. Since $\sigma \cap \tau = \tau \neq \emptyset$, by Lemma 3.3 (b), $\tau_{k-1} \in \gamma \text{Int}^k(H)$. Hence $\text{Ext}^{k-1}(\gamma H) \subseteq \gamma \text{Int}^k(H)$.

Let $\tau'_k \in \gamma \text{Int}^k(H)$. By Lemma 3.3 (b), there exists a path $s' = \tau'_1 \ldots \tau'_k$ in $L$ and $\sigma' \in \gamma H$ such that $\tau'_1 \cap \sigma' \neq \emptyset$. Since $\Delta \sigma' \subseteq \text{Ext}(\gamma H)$, $\tau'_1 \cap \sigma' \in \text{Ext}(\gamma H)$. Let $\sigma'_i$ be a maximal face of $L$ such that $\tau'_i \subseteq \sigma'_i$, for each $1 \leq i \leq k$. Consider the broad path $s'' = \sigma'_1 \ldots \sigma'_k$ of length $k$. Then since $\tau'_1 \cap \sigma' \subseteq \sigma'_1$, by Lemma 3.3 (a), $\sigma'_k \in \max(L) \cap \text{Ext}^{k+1}(\gamma H)$. Hence $\tau'_k \in \text{Ext}^{k+1}(\gamma H)$. Hence $\gamma \text{Int}^k(H) \subseteq \text{Ext}^{k+1}(\gamma H)$. \qed

The next corollary follows from Proposition 3.4 (e).
Corollary 3.5. Let \( r \) be the smallest integer such that \( \text{Ext}^r(\gamma H) = L \). Let \( t \) be the smallest integer such that \( \text{Int}^t(H) = \emptyset \). Then \( t = r - 1, r \) or \( r + 1 \).

Proof. By Theorem 1.1 (e), for any \( k \geq 1 \), \( \text{Ext}^{k+1}(\gamma H) \neq L \) implies \( \text{Int}^k(H) \neq \emptyset \), and \( \text{Ext}^{k-1}(\gamma H) = L \) implies \( \text{Int}^k(H) = \emptyset \). Let \( k = r - 2 \) and \( k = r + 1 \) respectively, we obtain \( \text{Int}^{r-2}(H) \neq \emptyset \) and \( \text{Int}^{r+1}(H) = \emptyset \). Thus \( t = r - 1, r \) or \( r + 1 \). \( \square \)

The next examples show that all three cases \( t = r - 1, r \) and \( r + 1 \) in Corollary 3.5 could happen. Hence the power-estimation in the inequality Proposition 3.4 (e) is tight.

Example 3.6. Let \( L \) be the 2-complex with vertices \( v_{i,j} \), \( 0 \leq i, j \leq 6, i + j \leq 6 \), given in Figure 1.

1. Let \( H \) be the 2-dimensional sub-complex consisting of all the simplices inside the triangle \([v_{1,2}, v_{3,2}, v_{1,4}]\), including the boundary of \([v_{1,2}, v_{3,2}, v_{1,4}]\). Then \( t = 1, r = 2 \).

2. Let \( H \) be the 2-dimensional sub-complex consisting of all the simplices inside the triangle \([v_{1,1}, v_{4,1}, v_{1,4}]\), including the boundary of \([v_{1,2}, v_{3,2}, v_{1,4}]\). Then \( t = 2, r = 2 \).

3. Let \( H \) be the sub-hypergraph consisting of all the hyperedges inside the triangle \([v_{1,1}, v_{4,1}, v_{1,4}]\), excluding the boundary of \([v_{1,2}, v_{3,2}, v_{1,4}]\). Then \( t = 2, r = 1 \).

3.2 Compositions of operators and neighborhoods

Let \( v \) be a vertex of \( L \). Recall that the closed star of \( v \) in \( L \) is the complex \( \overline{\text{St}}(v, L) = \Delta\{\sigma \in L \mid v \in \sigma\} \). Let \( \tau \in L \). The neighborhood of \( \tau \) in \( L \) is the complex \( \text{Nbd}(\tau) = \cup_{v \in \tau} \overline{\text{St}}(v, L) \). By a straight-forward calculation,

\[
\text{Nbd}(\tau) = \cup_{v \in \tau} \Delta\{\sigma \in L \mid v \in \sigma\} = \Delta \cup_{v \in \tau} \{\sigma \in L \mid v \in \sigma\} = \Delta \{\sigma \in L \mid \text{there exists } v \in \tau \text{ such that } v \in \sigma\} = \Delta \{\sigma \in L \mid \tau \cap \sigma \neq \emptyset\}.
\]

Let \( H \in \mathcal{H}(L) \). The neighborhood of \( H \) in \( L \) is the complex \( \text{Nbd}(H) = \cup_{\tau \in H} \text{Nbd}(\tau) \). The maximal sub-hypergraph in \( L \) whose neighborhood is contained in \( \text{Nbd}(H) \) is \( \text{Nbd}^{-1}(H) = \cup_{\text{Nbd}(H') \subseteq H} H' \).
Proposition 3.7. Let \( H \in \mathcal{H}(L) \). Then

(a). \( \text{Nbd} \circ \text{Nbd}^{-1}(H) \subseteq \delta H \);

(b). \( \Delta H \subseteq \text{Nbd}^{-1} \circ \text{Nbd}(H) \);

(c). \( \text{Nbd}^{-1}(H) = \text{Int}(H) \);

(d). \( \text{Ext}(H) \subseteq \text{Nbd}(H) \), and the equality holds if each vertex of \( H \) is a hyperedge.

Proof. (a) By the definition of neighborhoods, \( \text{Nbd} \circ \text{Nbd}^{-1}(H) \subseteq H \). Moreover, \( \text{Nbd} \circ \text{Nbd}^{-1}(H) \) is a simplicial complex. Hence \( \text{Nbd} \circ \text{Nbd}^{-1}(H) \subseteq \delta H \).

(b) By the definition of neighborhoods, any maximal face of \( \text{Nbd}(H) \) is a maximal face of \( L \). Thus \( \text{Nbd}(H) \) is completely determined by \( \max(L) \cap \text{Nbd}(H) \). In order to prove (b), we only need to show \( \text{Nbd}(\Delta H) = \text{Nbd}(H) \). Since \( H \subseteq \Delta H \), \( \text{Nbd}(H) \subseteq \text{Nbd}(\Delta H) \). Let \( \sigma' \in \max(L) \cap \text{Nbd}(\Delta H) \). Then there exists \( \sigma \in \Delta H \) such that \( \sigma \cap \sigma' \neq \emptyset \). Moreover, there exists \( \tau \in H \) such that \( \sigma \subseteq \tau \). Hence \( \tau \cap \sigma' \neq \emptyset \). Hence \( \sigma' \in \max(L) \cap \text{Nbd}(H) \). Thus \( \max(L) \cap \text{Nbd}(\Delta H) \subseteq \max(L) \cap \text{Nbd}(H) \). Thus \( \text{Nbd}(\Delta H) \subseteq \text{Nbd}(H) \). Therefore, \( \text{Nbd}(\Delta H) = \text{Nbd}(H) \). Consequently, \( \Delta H \subseteq \text{Nbd}^{-1} \circ \text{Nbd}(H) \).

(c) By a straight-forward calculation,

\[
\text{Nbd}^{-1}(H) = \{ \tau \in L \mid \text{Nbd}(\tau) \subseteq H \} \\
= \{ \tau \in L \mid \text{for any } \sigma' \in \gamma H, \sigma' \notin \Delta \{ \sigma \mid \tau \cap \sigma \neq \emptyset } \} \\
= \{ \tau \in L \mid \text{for any } \sigma' \in \gamma H \text{ and any } \sigma \in L \text{ with } \tau \cap \sigma \neq \emptyset, \sigma' \notin \Delta \sigma \} \\
= \{ \tau \in L \mid \text{for any } \sigma' \in \gamma H \text{ and any } \sigma' \subseteq \sigma, \tau \cap \sigma = \emptyset \} \\
= \{ \tau \in L \mid \text{for any } \sigma' \in \gamma H, \tau \cap \sigma' = \emptyset \} \\
= \text{Int}(H). \tag{3.6}
\]

(d) By a straight-forward calculation,

\[
\text{Nbd}(H) = \bigcup_{\tau \in H} \Delta \{ \sigma \in L \mid \tau \cap \sigma \neq \emptyset \} \\
\supseteq \bigcup_{\tau \in H} \Delta \{ \sigma \in L \mid \tau \subseteq \sigma \} \tag{3.7} \\
= \bigcup_{\tau \in H} \Delta \{ \sigma \in \max(L) \mid \tau \subseteq \sigma \} \\
= \text{Ext}(H).
\]

Suppose in addition that each vertex of \( H \) is a hyperedge. Then the equality holds in (3.7). Hence \( \text{Nbd}(H) = \text{Ext}(H) \).

The next corollary follows from Proposition 3.7

Corollary 3.8. Let \( H \in \mathcal{H}(L) \). Then

(a). \( \text{Ext} \circ \text{Int}(H) \subseteq \delta H \);

(b). If each vertex of \( H \) is a hyperedge, then \( \Delta H \subseteq \text{Int} \circ \text{Ext}(H) \).
4 Operator algebras on random hypergraphs and random simplicial complexes

In this section, we study the operators $D\Delta$, $D\delta$ and $D\gamma$ as well as their compositions acting on $D(\mathcal{H}(L))$ and $D(\mathcal{K}(L))$. We prove Theorem 1.1 and Theorem 1.2.

Let $w \in G$. For any $f \in D(\mathcal{H}(L))$ and any $H \in \mathcal{H}(L)$, it follows from (1.1) that

$$Dw(f)(H) = \sum_{w(H') = H} f(H').$$

(4.1)

Let $t$ be a positive integer and let $W \in G^t$. Suppose $W = (\ldots (w_1 * w_2) \ldots * w_t) \in G^t$ with any $(t-2)$-brackets ($\cdot$) giving the order of evaluations, $\ast = \land$ or $+$, and $w_1, w_2, \ldots, w_t \in G$. Then with the help of (1.2), $W$ induces a map

$$DW : D(\mathcal{H}(L))^\times t \rightarrow D(\mathcal{H}(L))$$

(4.2)

given by

$$DW(f_1, f_2, \ldots, f_t)(H) = \sum_{(\ldots (H_1 * H_2) \ldots * H_t) = H} \prod_{i=1}^t (Dw_i(f_i)(H_i))$$

(4.3)

for any $(f_1, \ldots, f_t) \in D(\mathcal{H}(L))^\times t$ and any $H \in \mathcal{H}(L)$. The next lemma shows that (4.3) gives a well-defined map (4.2).

**Lemma 4.1.** For and $t \geq 1$ and any $(f_1, f_2, \ldots, f_t) \in D(\mathcal{H}(L))^\times t$, $DW(f_1, f_2, \ldots, f_t) \in D(\mathcal{H}(L))$.

**Proof.** To prove Lemma 4.1 we need to prove

$$\sum_{H \in \mathcal{H}(L)} DW(f_1, f_2, \ldots, f_t)(H) = 1$$

(4.4)

for any $t \geq 1$. Firstly, we prove (4.4) for $t = 1$. Since $w$ is a self-map on $\mathcal{H}(L)$, for any $f_1 \in D(\mathcal{H}(L))$,

$$\sum_{H_1 \in \mathcal{H}(L)} Dw_1(f_1)(H_1) = \sum_{H_1 \in \mathcal{H}(L)} \sum_{w_1(H'_1) = H_1} f_1(H'_1) = \sum_{H'_1 \in \mathcal{H}(L)} f_1(H'_1).$$

Thus (4.4) holds for $t = 1$. Secondly, we use induction on $t$ and prove (4.4) for $t \geq 2$. By (4.1)
and \((1.3)\), we have
\[
\sum_{H \in \mathcal{H}(L)} DW(f_1, f_2, \ldots, f_t)(H) \\
= \sum_{H_1, H_2, \ldots, H_t \in \mathcal{H}(L)} \prod_{i=1}^{t} \left( \sum_{w_i(H_i') = H_i} f_i(H_i') \right) \\
= \sum_{H_1, H_2, \ldots, H_{t-1} \in \mathcal{H}(L)} \prod_{i=1}^{t-1} \left( \sum_{w_i(H_i') = H_i} f_i(H_i') \right) \left( \sum_{H_t \in \mathcal{H}(L)} \sum_{w_t(H_t') = H_t} f_t(H_t') \right) \\
= \sum_{H_1, H_2, \ldots, H_{t-1} \in \mathcal{H}(L)} \prod_{i=1}^{t-1} \left( \sum_{w_i(H_i') = H_i} f_i(H_i') \right).
\]
By an induction on \(t\), \((4.4)\) follows. \(\square\)

Now we prove Theorem 1.1.

**Proof of Theorem 1.1** Theorem 1.1 (a), (b), (c), (d) follow from Proposition 3.4 (a), (b), (c), (d) respectively. Theorem 1.1 (e) follows from Proposition 3.4 (e) and Corollary 3.8 (a). Theorem 1.1 (f) follows from Corollary 3.8 (b). \(\square\)

Let \(p\) be a function from \(L\) to \([0, 1]\).

**Lemma 4.2.** The operator \(D\Delta\) sends \(\bar{P}_{L,p}\) in Definition \(\mathbb{H}\) to \(P_{L,p'}\) in Definition \(\mathbb{2}\) where \(p'\) is given by \((1.3)\) for any \(\tau \in L\).

**Proof.** Let \(H\) be a sub-hypergraph of \(L\). Let \(\tau \in L\). Then \(\tau\) is a simplex of \(\Delta H\) if and only if there exists a hyperedge \(\sigma\) of \(H\) such that \(\tau\) is a subset of \(\sigma\). The probability that there does not exist any hyperedge \(\sigma\) in \(H\) containing \(\tau\) is \(\prod_{\sigma \subseteq L, \tau \subseteq \sigma}(1 - p(\sigma))\). Hence by taking the complement events, we have that the probability that \(\tau\) is a simplex of \(\Delta H\) is \(p'(\tau)\).

Moreover, if \(\tau\) is a simplex of \(\Delta H\), then all faces of \(\tau\) are simplices of \(\Delta H\). Thus \(p'(\tau)\) is the conditional probability that \(\tau\) is a simplex of \(\Delta H\) under the condition that all \((\dim \tau - 1)\)-faces of \(\tau\) are simplices of \(\Delta H\). Constructing \(\Delta H\) inductively by dimensions, we obtain a sequence of independent trials, which implies that \(\Delta H\) is a random simplicial complex satisfying Definition \(\mathbb{2}\) with probability function \(P_{L,p'}\). \(\square\)

**Lemma 4.3.** The operator \(D\delta\) sends \(\bar{P}_{L,p}\) in Definition \(\mathbb{H}\) to \(P_{L,p''}\) in Definition \(\mathbb{2}\) where \(p''\) is given by \((1.3)\) for any \(\tau \in L\).

**Proof.** Let \(H\) be a hypergraph in \(L\). Let \(\tau \in L\). Then \(\tau\) is a simplex of \(\delta H\) if and only if for any \(\sigma \subseteq \tau\), \(\tau \in H\). Thus the probability that \(\tau\) is a simplex of \(\delta H\) is \(\prod_{\sigma \subseteq \tau} p(\sigma)\). The remaining part is the same as the second paragraph of the proof of Lemma 4.2. \(\square\)

**Lemma 4.4.** The operator \(D\gamma\) sends \(\bar{P}_{L,p}\) to \(P_{L,1-p'}\).

**Proof.** Let \(H\) be a hypergraph in \(L\). Let \(\tau \in L\). Then the probability that \(\tau\) is a hyperedge in \(H\) is \(p(\tau)\). Thus the probability that \(\sigma\) is a hyperedge in \(\gamma H\) is \(1 - p(\tau)\). By the construction of the random hypergraphs in Definition \(\mathbb{H}\) the probability function of \(\gamma H\) is \(\bar{P}_{L,r,1-p'}\). \(\square\)
Let $p', p''$ be functions from $L$ to $[0, 1]$.

**Lemma 4.5.** The operator $D \cap$ sends the pair $(P_{L,p'}, P_{L,p''})$ to $P_{L,p',p''}$. And the operator $D \cup$ sends the pair $(P_{L,p'}, P_{L,p''})$ to $P_{L,1-(1-p')(1-p'')}$. 

**Proof.** We choose hypergraphs $H', H'' \in \mathcal{H}(L)$ independently at random with probability functions $P_{L,p'}$ and $P_{L,p''}$ respectively. In order to prove Lemma 4.5, we need to show

(a). the random hypergraph $H' \cap H''$ satisfies Definition 3 with probability function $P_{L,p',p''}$;

(b). the random hypergraph $H' \cup H''$ satisfies Definition 3 with probability function $P_{L,1-(1-p')(1-p'')}$. 

Let $\sigma \in L$. Consider two independent trials: (a). generate $H'$; (b). generate $H''$.

**Proof of (a).** $\sigma \in H' \cap H''$ if and only if $\sigma \in H'$ in trial (a) and $\sigma \in H''$ in trial (b). Thus $\sigma \in H' \cap H''$ has probability $pp'$. Letting $\sigma$ run over $L$, these trials of $\sigma$'s are independent.

**Proof of (b).** $\sigma \notin H' \cup H''$ if and only if $\sigma \notin H'$ in trial (a) and $\sigma \notin H''$ in trial (b). Thus $\sigma \notin H' \cup H''$ has probability $(1-p')(1-p'')$, and $\sigma \in H' \cap H''$ has probability $1 - (1-p')(1-p'')$. Letting $\sigma$ run over $L$, these trials of $\sigma$'s are independent. \hfill $\Box$

**Proposition 4.6.** We choose complexes $K', K'' \in \mathcal{K}(L)$ independently at random with probability functions $P_{L,p'}$ and $P_{L,p''}$ respectively. Then

(a). the random complex $K' \cap K''$ satisfies Definition 2 with probability function $P_{L,p',p''}$;

(b). the random complex with probability function $P_{L,1-(1-p')(1-p'')}$ can be constructed as follows:

- Take $K' \cup K''$;

- Choosing each external 1-face $\sigma$ of $K' \cup K''$ such that $\sigma$ is not an external face of $K'$ and also not an external face of $K''$ with probability $[1 - (1-p')(1-p'')](\sigma)$, we obtain a complex $Y_1$;

- Choosing each external 2-face $\sigma$ of $Y_1$ such that $\sigma$ is not an external face of $K'$ and also not an external face of $K''$ with probability $[1 - (1-p')(1-p'')](\sigma)$, we obtain a complex $Y_2$;

- ...

- Let $\dim L = r$. Choosing each external $r$-face $\sigma$ of $Y_{r-1}$ such that $\sigma$ is not an external face of $K'$ and also not an external face of $K''$ with probability $[1 - (1-p')(1-p'')](\sigma)$, we obtain a complex $Y_r$;

- The model $Y_r$ satisfies the probability function $P_{L,1-(1-p')(1-p'')}$.

**Proof.** (a). Let $\sigma \in L$. Then $\sigma$ is a face of $K' \cap K''$ iff. $\sigma$ is a face of $K'$ and also a face of $K''$. Moreover, $\sigma$ is an external face of $K' \cap K''$ iff. $\sigma$ is an external face of $K'$ and also an external face of $K''$. By an analogous argument with the assertion (a) in the the proof of Lemma 4.5, we have Proposition 4.6 (a).

(b). Let $\sigma \in L$. Then $\sigma$ is a face of $K' \cup K''$ iff. $\sigma$ is a face of $K'$ or a face of $K''$. Moreover, $\sigma$ is an external face of $K' \cup K''$ iff. one of the following cases happens:

(1). $\sigma$ is an external face of $K'$;
(2). \( \sigma \) is an external face of \( K'' \);

(3). \( \sigma \) is neither an external face of \( K' \) nor an external face of \( K'' \), \( \sigma \notin K' \cup K'' \), and the boundary of \( \sigma \) is a subset of \( K' \cup K'' \).

Let \( d \geq 1 \). In the construction of the random complex with probability function \( P_{L,1-(1-p')(1-p'')} \), once the \( (d-1) \)-skeleton is constructed, the \( d \)-faces are built by choosing each \( d \)-clique \( \sigma \) of the \( (d-1) \)-skeleton with probability \( p(\sigma) \). All the choices of the \( d \)-cliques of \( K' \) and the \( d \)-cliques of \( K'' \) are done in the construction of \( K' \cup K'' \). By choosing the \( d \)-cliques coming out in the construction of \( K' \cup K'' \), we obtain the procedure given (ii).

Now we prove Theorem 1.2.

Proof of Theorem 1.2. Theorem 1.2 (a) follows from Lemma 4.4. Theorem 1.2 (b) follows from Lemma 4.2. Theorem 1.2 (c) follows from Lemma 4.3. Theorem 1.2 (d) follows from the first assertion of Lemma 4.5 and Proposition 4.6 (a). Theorem 1.2 (e) follows from the second assertion of Lemma 4.5.

5 Applications of Theorem 1.2: generating large sparse random hypergraphs and large sparse random simplicial complexes

In computer science, random hypergraphs and random simplicial complexes are used as mathematical models of databases and networks. In order to simulate database and network problems on computers, it is important to generate a hypergraph or a simplicial complex randomly, with certain probability functions, on computers. In this section, as by-products of Theorem 1.2 we give algorithms generating a sparse hypergraph randomly with the probability function \( \tilde{P}_{\Delta_n,p} \), as well as algorithms generating the simplicial-like part the sparse hypergraph. We also give algorithms generating a sparse simplicial complex randomly with probability function \( P_{\Delta_n,p} \).

5.1 An auxiliary lemma

Let \( L \) be a simplicial complex. Let \( m \) be the dimension of \( L \). Let \( 0 \leq p_0, p_1, \ldots, p_m \leq 1 \) and let \( p = (p_0, p_1, \ldots, p_m) \). We introduce some definitions and notations.

- Let \((\Omega_k, \mu_k)\) be a sequence of probability spaces, \( k \in \mathbb{Z}_{\geq 0} \). For each \( k \), let \( \Gamma_k \) be an event in \((\Omega_k, \mu_k)\). Then \( \Gamma_k \) is said to happen asymptotically almost surely (a.a.s.) if

\[
\lim_{k \to \infty} \text{Prob}[\Gamma_k \text{ happens in } (\Omega_k, \mu_k)] = 1.
\]

- Let \( H(L, p) \) be a randomly generated hypergraph with the probability function \( \tilde{P}_{L,p} \). Let \( G(L, p_1) \) be the collection of 1-dimensional hyperedges of \( H(L, p) \). In particular, we write \( H(\Delta_n, p) \) as \( H(n, p) \). And \( G(\Delta_n, p_1) \) is the classical Erdős-Rényi model \( G(n, p_1) \).
For any graph $G$ in $L$, let $X_G$ be the clique complex of $G$. Let $X_{G,L} = X_G \cap L$. In particular, if $n$ is the number of vertices of $L$, then $X_{G,\Delta_n} = X_G$.

The next lemma is a generalization of [9, Corollary 2.6].

**Lemma 5.1.** Let $r \geq 2$ be a fixed integer. Let $L$ be a simplicial complex with $n$ vertices. Suppose both $L$ and $p_1$ depend on $n$. If

$$\lim_{n \to \infty} \frac{p_1}{n^{-2/(r+1)}} = 0,$$

then $\dim X_{G(L,p_1),L} \leq r$ a.a.s. In addition, if

$$\lim_{n \to \infty} \frac{p_1}{n^{-2/r}} = \infty$$

and

$$Sk^r(L) = Sk^r(\Delta_n),$$

then $\dim X_{G(L,p),L} = r$ a.a.s.

**Proof.** For any graph $G$ in $L$, since $L$ is a sub-complex of $\Delta_n$, we have

$$\dim X_{G,L} \leq \dim X_G.$$ (5.4)

Since the choices of distinct edges of $G$ are independent, by the probability law of independent events,

$$\left( \prod_{\sigma \in \Delta_n \setminus L \atop \dim \sigma = 1} (1 - p_1) \right) \text{Prob}[G(L, p_1) = G] = \text{Prob}[G(n, p_1) = G].$$ (5.5)

Therefore,

$$\text{Prob}[\dim X_{G(L,p_1),L} \leq r] = \sum_{\dim X_{G,L} \leq r} \text{Prob}[G(L, p_1) = G]$$

$$\geq \sum_{\dim X_G \leq r} \text{Prob}[G(L, p_1) = G]$$

$$\geq \sum_{\dim X_G \leq r} \text{Prob}[G(n, p_1) = G]$$

$$= \text{Prob}[\dim X_{G(n,p_1)} \leq r].$$ (5.6)

In (5.6), the first inequality follows from (5.4) and the second inequality follows from (5.5).

Suppose (5.1) holds. Then it follows from [9, Corollary 2.6] that as $n \to \infty$, the last row of (5.6) tends to 1. Hence $\text{Prob}[\dim X_{G(L,p),L} \leq r]$ tends to 1 as well. Hence $\dim X_{G(L,p),L} \leq r$ a.a.s.

In addition, assume (5.2) and (5.3) hold. By (5.2) and [9, Corollary 2.6],

$$\lim_{n \to \infty} \text{Prob}[\dim X_{G(n,p_1)} = r] = 1.$$
By (5.3), for any graph \( G \) in \( L \), \( \dim X_G = r \) implies \( \dim X_{G,L} = r \). Moreover, by (5.3) and (5.5), for any graph \( G \) in \( L \),

\[
\text{Prob}[G(L,p_1) = G] = \text{Prob}[G(n,p_1) = G].
\]

Therefore,

\[
1 = \lim_{n \to \infty} \text{Prob}[^{\dim X_{G(n,p_1)} = r}] = \lim_{n \to \infty} \sum_{\dim X_G = r} \text{Prob}[G(n,p_1) = G] = \lim_{n \to \infty} \sum_{\dim X_{G,L} = r} \text{Prob}[G(L,p_1) = G] = \lim_{n \to \infty} \text{Prob}[^{\dim X_{G(L,p_1),L} = r}].
\]

Hence the limit in (5.7) is 1. Therefore, \( \dim X_{G(L,p_1),L} = r \) a.a.s. \( \Box \)

5.2 An algorithm generating large sparse random hypergraphs

Throughout the remaining part of this paper, we suppose \( p_0 = 1 \). Let \( Y(n,p) \) be a randomly generated complex with the probability function \( P_{n,n-1,p} \). For each \( 0 \leq k \leq n-1 \), let

\[
p'_k = 1 - \prod_{i=0}^{n-k-1} (1 - p_{k+i})^{(n-k-1) \choose i}.
\]

Then \( p'_0 = 1 \). Let \( p' = (1, p'_1, \ldots, p'_{n-1}) \). Then by Theorem 1.2 (b),

\[
\Delta H(n,p) = Y(n,p').
\]

The next theorem is a generalization of [9, Corollary 2.6], in the hypergraph context.

**Theorem 5.2.** Let \( r \geq 2 \) be a fixed integer. Suppose

\[
\lim_{n \to \infty} \frac{p'_1}{n^{-2/(r+1)}} = 0
\]

where \( p \) and thus \( p' \) depend on \( n \). Then

(a). The dimension of \( \Delta H(n,p) \) is smaller than or equal to \( r \) a.a.s.

(b). If there exists \( \epsilon > 0 \) such that for any \( n \), \( \prod_{i=2}^{r} p'_i^{(i+1)} \) is bounded below by \( \epsilon \), and

\[
\lim_{n \to \infty} \frac{p'_1}{n^{-2/r}} = \infty,
\]

then the dimension of \( \Delta H(n,p) \) equals to \( r \) a.a.s.
Proof. We first prove (a). For any simplicial complex $K$, we have
\[
\dim K \leq \dim X_{Sk^1(K)}.
\] (5.11)
Hence
\[
\Pr[\dim Y(n, p') > r] = \sum_{\dim K > r} \Pr[Y(n, p') = K] 
\leq \sum_{\dim K > r} \Pr[Sk^1(Y(n, p')) = Sk^1(K)] 
\leq \sum_{\dim X_{Sk^1(K)} > r} \Pr[Sk^1(Y(n, p')) = Sk^1(K)] 
= \sum_{\dim X_{Sk^1(K)} > r} \Pr[X_{Sk^1(Y(n, p'))} = X_{Sk^1(K)}] 
= \Pr[\dim X_{Sk^1(Y(n, p'))} > r].
\]
Since $Sk^1(Y(n, p'))$ is $G(n, p'_1)$, $X_{Sk^1(Y(n, p'))} = X_{G(n, p'_1)}$. Suppose (5.9) holds. Then by Lemma 5.1 as $n \to \infty$, $\Pr[\dim X_{G(n, p'_1)} > r]$ tends to 0. Thus
\[
\lim_{n \to \infty} \Pr[\dim Y(n, p') > r] = 0.
\]
With the help of (5.8), (a) follows.

Before proving (b), we first prove that as $n \to \infty$, the number of $r$-faces of $X_{G(n, p'_1)}$ goes to infinity a.a.s. That is, for any $M > 0$,
\[
\lim_{n \to \infty} \Pr[\text{number of } r\text{-faces of } X_{G(n, p'_1)} \geq M] = 1.
\] (5.12)
For any integer $k \geq 1$, the probability that the number of $r$-faces of $X_{G(n, p'_1)}$ is at most $k$ is smaller than or equal to
\[
\binom{n}{k(r+1)}(1-p'_1)^{\binom{k(r+1)}{2} - \binom{k(r+1)}{2}} \leq \frac{n^{k(r+1)}(1-p'_1)^{n^2/2}}{(k(r+1))!(1-p'_1)^{\binom{k(r+1)}{2}}}. \quad (5.13)
\]
Letting $n \to \infty$, (5.9) implies $p'_1 \to 0$. Thus the denominator of (5.13) converges to a constant $(k(r+1))!$. Moreover, the limit of the numerator of (5.13) satisfies
\[
\lim_{n \to \infty} n^{k(r+1)}(1-q_1)^{n^2/2} \leq \lim_{n \to \infty} n^{k(r+1)}(1-n^{-2/r})^{n^2/2} 
= \lim_{n \to \infty} n^{k(r+1)}(1/e)^{n^{2-2/r}/2} = 0. \quad (5.14)
\]
The last equality of (5.14) follows from the assumption that $r \geq 2$. Consequently, as $n \to \infty$, the limit of (5.13) is 0. Thus (5.12) follows.

Now we prove (b). By the construction of $Y(n, p')$ described in the paragraph after Defini-
tion $[1]$ for any $r$-face $\sigma_r$ of $X_G(n,p'_1)$,

$$\text{Prob}[\sigma_r \in Y(n,p')] = \prod_{i=2}^{r} p_i^{\binom{r+1}{i+1}}.$$  

Suppose in addition, (5.10) holds, and there exists $\epsilon > 0$ such that $\prod_{i=2}^{r} p_i^{\binom{r+1}{i+1}}$ is bounded below by $\epsilon$ for any $n$. Since as $n \to \infty$, the number of the $\sigma_r$’s in $X_G(n,p'_1)$ goes to infinity a.a.s., we obtain that the number of the $\sigma_r$’s that belong to $Y(n,p')$ also goes to infinity a.a.s. With the help of (5.8) and Theorem 5.2 (a), the dimension of $\Delta H(n,p)$ equals to $r$ a.a.s. \hfill \Box

The next algorithm follows from Theorem 5.2.

**Algorithm 1.** Let $r \geq 2$ be a fixed integer. Suppose (5.9) holds. Then for $n$ large enough,

- the sparse random simplicial complex $Y(n,p')$ can be generated approximately by only generating the simplices whose dimensions are smaller than or equal to $r$;
- the sparse random hypergraph $H(n,p)$ can be generated approximately by only generating the hyperedges whose dimensions are smaller than or equal to $r$.

### 5.3 An algorithm generating the simplicial-like parts of large sparse random hypergraphs

Given a hypergraph $H$, we split $H$ into a disjoint union of $\delta H$ and $H \setminus \delta H$. Then $\delta H$ is a simplicial complex contained in $H$, and $H \setminus \delta H$ does not contain any nonempty simplicial complexes. We call $\delta H$ the simplicial-like part of $H$. Let $p''_0 = 1$. For each $1 \leq k \leq n - 1$, let

$$p''_k = \prod_{i=1}^{k} p_i^{\binom{k+1}{i+1}}.$$  

By Theorem 1.2 (c), $\delta H(n,p) = Y(n,p'')$. By a similar proof of Theorem 5.2 the next theorem follows.

**Theorem 5.3.** Let $r \geq 2$ be a fixed integer. Suppose

$$\lim_{n \to \infty} \frac{p''_1}{n^{-2/(r+1)}} = 0$$

(5.15)

where $p$ and $p''$ depend on $n$. Then

(a). The dimension of $\delta H(n,p)$ is smaller than or equal to $r$ a.a.s.

(b). If there exists $\epsilon > 0$ such that for any $n$, $\prod_{i=2}^{r} p_i^{\binom{r+1}{i+1}}$ is bounded below by $\epsilon$, and

$$\lim_{n \to \infty} \frac{p''_1}{n^{-2/r}} = \infty,$$

then the dimension of $\delta H(n,p)$ equals to $r$ a.a.s.
Algorithm 2. Let $r \geq 2$ be a fixed integer. Suppose (5.15) holds. Then for $n$ large enough, the simplicial-like part of the sparse random hypergraph $H(n, p)$ can be generated approximately by only generating the hyperedges whose dimensions are smaller than or equal to $r$.

The next algorithm follows from Theorem 5.3.

Acknowledgements. The project was supported in part by the Singapore Ministry of Education research grant (AcRF Tier 1 WBS No. R-146-000-222-112). The first author was supported in part by the National Research Foundation, Prime Minister’s Office, Singapore under its Campus for Research Excellence and Technological Enterprise (CREATE) programme. The second author was supported in part by the President’s Graduate Fellowship of National University of Singapore. The third author was supported by a grant (No. 11329101) of NSFC of China.

References

[1] L. Aronshtam and N. Linial, The threshold for d-collapsibility in random complexes, Random Struct. Algor. 48 (2016), 260-269.

[2] L. Aronshtam, N. Linial, T. Luczak and R. Meshulam, Collapsibility and vanishing of top homology in random simplicial complexes. Discrete Comput. Geom. 49(2) (2013), 317-334.

[3] E. Babson, C. Hoffman and M. Kahle, The fundamental group of random 2-complexes. J. Amer. Math. Soc. 24(1) (2010), 1-28.

[4] C. Berge, Graphs and hypergraphs. American Elsevier Pub. Co. North-Holland, New York, 1976.

[5] D. Cohen, A. Costa, M. Farber and T. Kappeler, Topology of random 2-complexes. Discrete Comput. Geom. 47 (2012), 117-149.

[6] D. Cohen, A. Costa, M. Farber and T. Kappeler, Correction to Topology of random 2-complexes. Discrete Comput. Geom. 56 (2016), 502-503.

[7] A.E. Costa and M. Farber, The asphericity of random 2-dimensional complexes, Random Struct. Algor. 46 (2015), 261-273.

[8] A.E. Costa and M. Farber, Geometry and topology of random 2-complexes, Israel J. Math. 209 (2015), 883-927.

[9] A. Costa, M. Farber and D. Horak, Fundamental groups of clique complexes of random graphs, Trans. London Math. Soc. 2(1) (2015), 1-32.

[10] A. Costa and M. Farber, Large random simplicial complexes, I, J. Topol. Anal. 8(3) (2016), 399-429.

[11] A. Costa and M. Farber, Large random simplicial complexes, II; the fundamental group, J. Topol. Anal. 9(3) (2017), 441-483.

[12] A. Costa and M. Farber, Large random simplicial complexes, III; the critical dimension, J. Knot Theory Ramifications 26(2) (2017), 1740010-1 - 1740010-26.

[13] A. Costa and M. Farber, Random simplicial complexes, Configuration Spaces 129-153, Springer INdAM Series 14 (2016), Springer, 129-153.

[14] P. Erdös and A. Rényi, On random graphs I, Publ. Math. Debrecen 6 (1959), 290-297.

[15] P. Erdös and A. Rényi, On the evolution of random graphs, Publ. Math. Inst. Hungar. Acad. Sci. 5 (1960), 17-61.

[16] E.N. Gilbert, Random graphs, Ann. Math. Statist. 30(4) (1959), 1141-1144.

[17] A. Gundert, On eigenvalues of random complexes, Israel J. Math. 216 (2016), 545-582.
[18] A. Gundert and U. Wagner, *On topological minors in random simplicial complexes*, Proc. Amer. Math. Soc. 144 (2016), 1815-1828.

[19] A. Hatcher, *Algebraic Topology*. Cambridge University Press, Cambridge, 2002.

[20] C. Hoffman, M. Kahle and E. Paquette, *The threshold for integer homology in random d-complexes*, Discrete Comput. Geom. 57 (2017), 810-823.

[21] M. Kahle, *Topology of random clique complexes*, Discrete Math. 309(6) (2009), 1658-1671.

[22] M. Kahle, *Topology of random simplicial complexes: a survey*, Algebraic Topology: Applications and New Directions, Contem. Math. 620 (2014), 201-221.

[23] M. Kahle and B. Pittle, *Inside the critical window for cohomology of random k-complexes*, Random Struct. Algor. 48 (2016), 102-104.

[24] D. N. Kozlov, *The threshold function for vanishing of the top homology group of random d-complexes*. Proc. Amer. Math. Soc. 138(12) (2010), 4517-4527.

[25] N. Linial and R. Meshulam, *Homological connectivity of random 2-complexes*, Combinatorica 26 (2006), 475-487.

[26] N. Linial and Y. Peled, *On the phase transition in random simplicial complexes*, Ann. Math. 184 (2016), 745-773.

[27] R. Meshulam and N. Wallach, *Homological connectivity of random k-dimensional complexes*, Random Struct. Algor. 34 (2009), 408-417.

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