Entropy evolution of universes with initial and final de Sitter eras

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Abstract

This brief report studies the behavior of entropy in two recent models of cosmic evolution by J. A. S. Lima, S. Basilakos, and F. E. M. Costa [Phys. Rev. D 86, 103534 (2012)], and J. A. S. Lima, S. Basilakos, and J. Solá [arXiv:1209.2802]. Both start with an initial de Sitter expansion, go through the conventional radiation and matter dominated eras to be followed by a final and everlasting de Sitter expansion. In spite of their outward similarities (from the observational viewpoint they are arbitrary close to the conventional Lambda cold dark matter model), they deeply differ in the physics behind them. Our study reveals that in both cases the Universe approaches thermodynamic equilibrium in the last de Sitter era in the sense that the entropy of the apparent horizon plus that of matter and radiation inside it increases and is concave. Accordingly, they are consistent with thermodynamics. Cosmological models that do not approach equilibrium at the last phase of their evolution appear in conflict with the second law of thermodynamics.

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As daily experience teaches us, macroscopic systems tend spontaneously to thermodynamic equilibrium. This constitutes the empirical basis of the second law of thermodynamics. The latter succinctly formalizes this by establishing that the entropy, $S$, of isolated systems never decreases, $S' \geq 0$, and that it is concave, $S'' < 0$, at least in the last leg of approaching equilibrium (see, e.g., [1]). The prime means derivative with respect the relevant variable. In our view, there is no apparent reason why this should not be applied to cosmic expansion. Before going any further we wish to remark that sometimes the second law is found formulated by stating just the above condition on $S'$ but not on $S''$. While this incomplete version of the law works well for many practical purposes it is not sufficient in general. Otherwise, one would witness systems with an always increasing entropy but never achieving equilibrium, something at stark variance with experience.

In this paper we shall explore whether two recently proposed cosmological models, that start from an initial de Sitter expansion and go through the conventional radiation and matter eras to finally enter a never-ending de Sitter phase, present the right thermodynamic evolution of above. This is to say, we will see whether $S' \geq 0$ at all times and $S'' < 0$ at the transition from the matter era to the final de Sitter one. By $S$ we mean the entropy of the apparent horizon, $S_h = k_B A/(4 \ell_{pl}^2)$, plus the entropy of the radiation, $S_{\gamma}$, and/or pressureless matter, $S_m$, inside it. As usual, $A$ and $\ell_{pl}$ denote the area of the horizon and Planck’s length, respectively.

We shall consider the evolution of the entropy, first in the model of Lima, Basilakos and Costa [2] and then in the model of Lima, Basilakos and Solá [3] (models I and II, respectively). Both assume a spatially flat Friedmann-Robertson-Walker metric, avoid -by construction- the horizon problem and the initial singularity of the big bang cosmology, evolve between an initial and a final de Sitter expansions (the latter being everlasting), and from the observational viewpoint they are very close to the conventional Lambda cold dark matter model.

In spite of these marked coincidences the physics behind the models is deeply different. While model I rests on the production of particles induced by the gravitational field (and
dispenses altogether with dark energy), model II assumes dark energy in the form of a cosmological constant that in reality varies with the Hubble factor in a manner prescribed by quantum field theory.

We shall focus on the transitions from the initial de Sitter expansion to the radiation dominated era, and from the matter era to the final de Sitter expansion. As is well known, in the radiation and matter eras, as well as in the transition from one to another, both $S'$ and $S''$ are positive-definite quantities [4].

As usual, a zero subscript attached to any quantity indicates that it is to be evaluated at the present time.

II. THERMODYNAMIC ANALYSIS OF MODEL I

In model I the present state of cosmic acceleration is achieved not by dark energy or as a result of modified gravity, but simply by the gravitationally induced production of particles [2]. In this scenario the initial phase is a de Sitter expansion which, due to the creation of massless particles becomes unstable whence the Universe enters the conventional radiation dominated era. At this point -as demanded by conformal invariance [6]- the production of these particles ceases whereby the radiation becomes subdominant and the Universe enters a stage dominated by pressureless matter (baryons and cold dark matter). Lastly, the negative creation pressure associated to the production of matter particles accelerates the expansion and ushers the Universe in a never-ending de Sitter era. The model is consistent with the observational tests, including the growth rate of cosmic structures [2].

A. From de Sitter to radiation dominated expansion

The production of massless particles in the first de Sitter era induces a negative creation pressure related to the phenomenological rate of particle production, $\Gamma_r$, given by $p_c = -(1 + w)\rho \Gamma_r/(3H)$ -see [5] for more general treatments of the subject. Here, $\rho$ is the energy density of the fluid (radiation in this case), $w$ its equation of state parameter, and $H = \dot{a}/a$
the Hubble expansion rate. As a consequence, the evolution of the latter is governed by

\[ \dot{H} + \frac{3}{2} (1 + w) H^2 \left( 1 - \frac{\Gamma_r}{3H} \right) = 0, \]  

(1)

cf. equation (7) in [2].

Because the rate must strongly decline when the Universe enters the radiation dominated era, it may be modeled as \( \Gamma_r/(3H) = H/H_I \) with \( H \leq H_I \), being \( H_I \) the initial de Sitter expansion rate. In consequence, for \( w = 1/3 \) (thermal radiation) last equation reduces to

\[ \dot{H} + 2H^2 \left( 1 - \frac{H}{H_I} \right) = 0, \]  

(2)

whose solution in terms of the scale factor reads

\[ H(a) = \frac{H_I}{1 + Da^2} \]  

(3)

with \( D \) a positive-definite integration constant.

The area of the apparent horizon \( A = 4\pi \tilde{r}_A^2 \), where \( \tilde{r}_A = \frac{1}{\sqrt{H^2 + ka}} \) is the radius, trivially reduces, in the case under consideration (a spatially flat universe), to the Hubble length, \( H^{-1} \). Accordingly, the entropy of the apparent horizon, \( S_h = k_B \pi/(\ell_{pl} H)^2 \), as the Universe transits from de Sitter, \( H = H_I \), to a radiation dominated expansion is simply

\[ S_h = \pi k_B \frac{(1 + Da^2)^2}{(\ell_{pl} H_I)^2}. \]  

(4)

It is readily seen that \( S_h \) is a growing, \( S'_h > 0 \), and convex, \( S''_h > 0 \), function of the scale factor (the prime stands for \( d/da \)).

In its turn, the evolution of the entropy of the radiation fluid inside the horizon can be determined with the help of Gibbs equation [3]

\[ T_\gamma dS_\gamma = d \left( \rho_\gamma \frac{4\pi}{3} \tilde{r}_A^3 \right) + p_\gamma d \left( \frac{4\pi}{3} \tilde{r}_A^3 \right), \]  

(5)

where

\[ \rho_\gamma = \rho_I \left[ 1 + \lambda^2 \left( \frac{a}{a_*} \right)^2  \right]^{-2}, \]  

(6)

\( \rho_I \) is the critical energy density of the initial de Sitter phase, \( \lambda^2 = Da_*^2 \), and \( p_\gamma = \rho_\gamma/3 \). In its turn, \( a_* \) denotes the scale factor at the transition from de Sitter to the beginning of the standard radiation epoch.
Likewise, the dependence of the radiation temperature on the scale factor is given by

\[ T_\gamma = T_I \left[ 1 + \lambda^2 \left( \frac{a}{a_*} \right)^2 \right]^{-1/2} \]  

(7)

-cfr. Eq. (13) in [2].

In consequence,

\[ T_\gamma S'_\gamma = \frac{4\pi}{3} \frac{\rho_I D}{H^3} a > 0. \]  

(8)

Accordingly, \( S'_h + S'_\gamma \geq 0 \), i.e., the total entropy -which encompasses the horizon entropy plus the entropy of the fluid in contact with it- does not decrease. In other words, the generalized second law (GSL), first formulated for black holes and their environment [8] and later on extended to the case of cosmic horizons [9], is satisfied.

Let us now discern the sign of \( S''_h + S''_\gamma \). As we have already seen, \( S''_h > 0 \). As for \( S''_\gamma \), we insert \( T_\gamma = T_I (1 + Da^2)^{-1/2} \) into Eq. (8) and obtain

\[ S'_\gamma = \frac{4\pi}{3} \frac{\rho_I D}{H^3 T_I} (1 + Da^2)^{1/2}. \]  

(9)

Thus,

\[ S''_\gamma = \frac{4\pi}{3} \frac{\rho_I D}{H^3 T_I} \left[ \frac{1 + 2Da^2}{(1 + Da^2)^{1/2}} \right] > 0. \]  

(10)

Therefore, \( S''_h + S''_\gamma > 0 \); that is to say, in the transition from the initial de Sitter expansion to radiation domination, the total entropy is a convex function of the scale factor. If it were concave, the Universe could have attained a state of thermodynamic equilibrium and would have not left it unless forced by some “external agent”. The initial de Sitter expansion \((H = H_I, \text{ and no particles})\) was a state of equilibrium, but only a metastable one for the Universe was obliged to leave it by the production of particles which acted as an external agent.

**B. From matter domination to the final de Sitter expansion**

The Hubble function of spatially flat Lambda cold dark matter models obeys

\[ \dot{H} + \frac{3}{2} H^2 \left[ 1 - \left( \frac{H_\infty}{H} \right)^2 \right] = 0, \]  

(11)
where $H_\infty = \text{constant}$ denotes its asymptotic value at the far future.

In a dust filled universe ($w = 0$) with production rate $\Gamma_{dm} \leq 3H$ of pressureless matter the Hubble factor obeys

$$\dot{H} + \frac{3}{2} H^2 \left( 1 - \frac{\Gamma_{dm}}{3H} \right) = 0. \quad (12)$$

Comparison with the previous equation leads to $\Gamma_{dm}/(3H) = (H_\infty/H)^2$, i.e., $\Gamma_{dm} \propto H^{-1}$.

Thus,

$$H^2 = H_0^2 \left[ \tilde{\Omega}_m a^{-3} + \tilde{\Omega}_\Lambda \right], \quad (13)$$

where $\tilde{\Omega}_\Lambda = (H_\infty/H_0)^2 = 1 - \tilde{\Omega}_m = \text{constant} > 0$.

Recalling that $S_h = k_B A/(4 \ell_{pl}^2)$, it follows $S_h' = -2\pi k_B H'/(\ell_{pl}^2 H^3)$ with

$$H' = -\frac{3}{2} H_0 \frac{(1 - \tilde{\Omega}_\Lambda) a^{-4}}{[(1 - \tilde{\Omega}_\Lambda) a^{-3} + \tilde{\Omega}_\Lambda]^{1/2}}. \quad (14)$$

Then

$$S_h' = 6 \pi k_B \ell_{pl}^2 \frac{H_0^2 (1 - \tilde{\Omega}_\Lambda) a^{-4}}{[(1 - \tilde{\Omega}_\Lambda) a^{-3} + \tilde{\Omega}_\Lambda]^2} > 0. \quad (15)$$

As for the entropy of dust matter, it suffices to realize that every single particle contributes to the entropy inside the horizon by a constant bit, say $k_B$. Then, $S_m = k_B \frac{4\pi}{3} \ell_{pl}^2 n$, where the number density of dust particles obeys the conservation equation $n' = (n/(aH)) [\Gamma_{dm} - 3H] < 0$ with $\Gamma_{dm} = 3H_0^2 \tilde{\Omega}_\Lambda/H > 0$.

Thus,

$$S_m' = 4\pi k_B \ell_{pl}^2 \frac{n}{H^4} \left[ \frac{\Gamma_{dm} - 3H}{a} - 3H' \right]. \quad (16)$$

Since $\Gamma_{dm} - 3H < 0$ and $H' < 0$ the sign of $S_m'$ is undecided at this stage. To ascertain it consider the square parenthesis in (16) and multiply it by $aH/3$. One obtains

$$\frac{aH}{3} \left[ \frac{\Gamma_{dm} - 3H}{a} - 3H' \right] = \frac{1}{2} H_0^2 (1 - \tilde{\Omega}_\Lambda) a^{-3} > 0. \quad (17)$$

In consequence, $S_m' > 0$ and the GSL, $S_h' + S_m' \geq 0$, is satisfied also in this case.
Let us now consider the sign of $S''_h + S''_m$ in the limit $a \to \infty$. From $S'_h = -\frac{2\pi k_B}{\ell^2_{pl}} H' H^3$ it follows,

$$S''_h = -\frac{2\pi k_B}{\ell^2_{pl}} \frac{1}{H^4} [HH'' - 3H'^2]. \quad (18)$$

In virtue of (14) we get

$$H H'' = \frac{3}{2} H_0^2 \frac{4(1 - \bar{\Omega}_\Lambda)(1 - \bar{\Omega}_\Lambda)a^{-3} + \bar{\Omega}_\Lambda a^{-5} + (3/2)(1 - \bar{\Omega}_\Lambda)^2 a^{-8}}{(1 - \bar{\Omega}_\Lambda)a^{-3} + \bar{\Omega}_\Lambda}, \quad (19)$$

whence

$$HH'' - 3H'^2 = \frac{3}{2} H_0^2 \left\{ \frac{4(1 - \bar{\Omega}_\Lambda)(1 - \bar{\Omega}_\Lambda)a^{-3} + \bar{\Omega}_\Lambda a^{-5}}{(1 - \bar{\Omega}_\Lambda)a^{-3} + \bar{\Omega}_\Lambda} \right\} > 0. \quad (20)$$

Thereby, in view of (18) we get $S''_h < 0$.

As for the sign of $S''_m$ it suffices to recall, on the one hand, Eq. (16) and realize that $\Gamma_{dm}(a \to \infty) = 3H_0^2 \bar{\Omega}_\Lambda/H_\infty = 3H_\infty$ and that $H'(a \to \infty) \to 0$; then, $S'_m(a \to \infty) = 0$. And, on the other hand, that $S''_m(a < \infty) > 0$. Taken together they imply that $S'_m$ tends to zero from below, i.e., that $S''_m(a \to \infty) < 0$.

Altogether, when $a \to \infty$ one has $S''_h + S''_m < 0$, as expected. Put another way, in the phenomenological model of Lima et al. [2] the Universe behaves as an ordinary macroscopic system [10]; i.e., it eventually tends to thermodynamic equilibrium, in this case characterized by a never-ending de Sitter expansion era with $H_\infty = H_0 \sqrt{\bar{\Omega}_\Lambda} < H_0$.

### III. THERMODYNAMIC ANALYSIS OF MODEL II

The model of Ref. [3] is based on the assumption that in quantum field theory in curved spacetime the cosmological constant is a parameter that runs with the Hubble rate in a specified manner [11, 12]. As in the previous model, the vacuum decays into radiation and nonrelativistic particles while the Universe expands from de Sitter to de Sitter through the intermediate eras of radiation and matter domination. However, as said above, the physics of both models differ drastically from one another. While model I dispenses altogether with dark energy, model II assumes dark energy in the form of cosmological term, $\Lambda$, that evolves with expansion.
According to model II, the running of $\Lambda$ is given by the sum of even powers of the Hubble expansion rate,

$$\Lambda(H) = c_0 + 3\nu H^2 + 3\alpha \frac{H^4}{H_I^2},$$  \(21\)

where $c_0$, $\alpha$ and $\nu$ are constants of the model. The absolute value of the latter is constrained by observation as $|\nu| \sim 10^{-3}$. At early times the last term dominates, and at late times ($H \ll H_I$) it becomes negligible whereby \(21\) reduces to

$$\Lambda(H) = \Lambda_0 + 3\nu (H^2 - H_0^2)$$  \(22\)

with $\Lambda_0 = c_0 + 3\nu H_0^2$.

At the early universe, integration of the field equations gives for the Hubble function, the energy density of radiation and of vacuum, the following expressions \[3\]

$$H(a) = \sqrt{\frac{1 - \nu}{\alpha}} \frac{H_I}{\sqrt{D a^{3\beta} + 1}} \quad (\beta = (1 - \nu)(1 + w)),$$

$$\rho_\gamma = \rho_I \frac{(1 - \nu)^2}{\alpha} D \frac{a^{3\beta}}{(a^{3\beta} + 1)^2}; \quad \text{and} \quad \rho_\Lambda = \frac{\Lambda_0}{8\pi G} = \frac{1 - \nu}{\alpha} \frac{\nu D a^{3\beta}}{(D a^{3\beta} + 1)^2};$$ \(24\)

where $D(>0)$ is an integration constant.

As in model I, at the transition from de Sitter to the radiation era, the first and second derivatives of $S_h$ and $S_\gamma$ (with respect to the scale factor) are all positive. Thus, at this transition the GSL is fulfilled but neither the radiation era nor the subsequent matter era correspond to equilibrium states since at them $S'' > 0$.

By integration of the field equations at late times it is seen that the transition between the stages of matter domination to the second (and final) de Sitter expansion is characterized by

$$H(a) = \frac{H_0}{\sqrt{1 - \nu}} \sqrt{(1 - \Omega_{\Lambda 0}) a^{-3(1-\nu)} + \Omega_{\Lambda 0} - \nu},$$ \(25\)

$$\rho_m = \rho_{m 0} a^{-3(1-\nu)} \quad \text{and} \quad \rho_\Lambda(a) = \rho_{\Lambda 0} + \frac{\nu}{1 - \nu} \rho_{m 0} \left[ a^{-3(1-\nu)} - 1 \right],$$ \(26\)

where $\Omega_{\Lambda 0} = 8\pi G \rho_{\Lambda 0} (3H_0^2)^{-1}$. From the pair of equations \(26\) we learn that dust particles are created out of the vacuum at the rate $\Gamma_{dm} = \nu H$. 

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As in the previous model, $S'_h > 0$ and $S''_h < 0$. However at variance with it, the matter entropy, $S_m = k_B \frac{4\pi}{3} \tilde{r}_A n \propto H^{-3} n$, decreases with expansion and is convex. This is so because, in this case, the rate of particle production, $\Gamma_{dm}$, goes down and cannot compensate for the rate of dilution caused by cosmic expansion. Nevertheless, as it can be easily checked, $S'_h$ and $S''_h$ dominate over $S'_m$ and $S''_m$, respectively, as $a \to \infty$. Thus, as in model I, the total entropy results a growing and concave function of the scale factor, at least at the far future stage. Hence, the Universe gets asymptotically closer and closer to thermodynamic equilibrium.

IV. DISCUSSION AND CONCLUDING REMARKS

The second law of thermodynamics constrains the evolution of macroscopic systems; thus far, all attempts to disprove it by means of “counterexamples” have failed. While it seems reasonable to expect it to be obeyed also by the Universe as a whole, a proof of this on first principles is still lacking. However, persuasive arguments based on the Hubble history, suggesting that, indeed, the Universe behaves as any ordinary macroscopic thermodynamic system (i.e., that it tends to a maximum entropy state), were recently given in [10]. On the other hand, in view of the strong connection between gravitation and thermodynamics -see e.g., [13, 14]- it would be shocking that the Universe behaved otherwise. In this spirit we have considered models [2] and [3], each of them covering the whole cosmic evolution (i.e., the two well-known eras of radiation and matter dominance sandwiched between an initial and a final de Sitter expansions), and consistent with recent observational data. In both models, the entropy, as a function of the scale factor, never decreases and is concave at least at the last stage of evolution, signaling that the Universe is finally approaching thermodynamic equilibrium.

In principle, the initial de Sitter eras should be stable ($H$ and $S$ are constants when $t \to -\infty$) but owing to particle production, which can be viewed as “external” agent acting on the otherwise isolated system, an instability sets in. Once the Universe gets separated from thermodynamic equilibrium it reacts trying to restore it -as ordinary systems do-, only that at lower energy scale. This is finally achieved at the last de Sitter expansion. Given that two de Sitter expansions cannot directly follow one another an intermediate phase (comprised
by the radiation and matter eras) is necessary in between.

As is well known, irreversible particle production, as is the case in models I and II, implies generation of entropy (see e.g., [5], [15]); something rather natural because the new born particles necessarily increase the volume of the phase space. Our analysis takes this into account in an implicit and straightforward manner via the $\Gamma$ rates of particle production. These quantities modify the corresponding expressions for the Hubble factor and hence $S'$ and $S''$. For instance, setting $\Gamma_r$ to zero in Eq. (1) (which would kill model I) leads to $D = 0$ and therefore to $S'_\gamma = S''_\gamma = 0$ (Eqs. (9) and (10), respectively). Likewise if, in the same model, one sets $\Gamma_{dm}$ to zero, then $S'_m$ (Eq. (16)) decreases. Analogous statements can be made about model II if the parameter $\nu$ (that enters the corresponding $\Gamma$ rates) is forced to vanish (again, this would kill the model).

When quantum corrections to Bekenstein-Hawking entropy law are taken into account, the entropy of black hole horizons generalizes to

$$S_h = k_B \left[ \frac{A}{4\ell_{pl}^2} - \frac{1}{2} \ln \left( \frac{A}{\ell_{pl}^2} \right) \right] + \text{higher order terms} \ [16, 17].$$

Assuming this also applies to the cosmic apparent horizon, one may wonder up to what extent this may modify our findings. The answer is that the modifications are negligible whereby our results are robust against quantum corrections to the Bekenstein-Hawking entropy. We illustrate this point by noting that the expression for $S'_h$ of model I in the transition from the initial de Sitter regime to radiation domination presents now the overall multiplying factor $\left\{ 1 - \frac{\ell_{pl}^2 H_I^2}{8\pi(1+Da)^2} \right\}$. In this expression the second term is negligible on account of the quantity $\ell_{pl}^2$ in the numerator. It is noteworthy that the imposing of the condition $S'_h > 0$ leads to $\frac{\ell_{pl}^2 H_I^2}{8\pi(1+Da)^2} < 1$. When this inequality is evaluated in the limit $a \to 0$, the upper bound on the square of the initial Hubble factor, $H_I^2 < 8\pi/\ell_{pl}^2$ follows. Thus, the nice and convincing result, that the initial Hubble factor cannot be arbitrarily large (its square, not much larger than Planck’s curvature) arises straightforwardly from the quantum corrected Bekenstein-Hawking entropy law.

Likewise, a study of $S'_h$ of model II in the transition from de initial Sitter expansion to radiation domination leads, in the same limit of very small $a$, to the upper bound $H_I^2 < 2\pi\alpha/[(1 - \nu)\ell_{pl}^2]$, i.e., to essentially identical result on the maximum permissible value of $H_I$. It is remarkable that in spite of being models I and II so internally different, they
share this bound.

We conclude that models I and II show consistency with thermodynamics, and that their overall behavior (in particular, the reason why they evolve precisely to de Sitter in the long run) can be most easily understood from the thermodynamic perspective. Further, these results remain valid also if quantum corrections to Bekenstein-Hawking entropy law are incorporated.

It would be interesting to explore the possible connection of the second law when applied to expanding universes with the “cosmic no-hair conjecture” [18]. Loosely speaking, the latter asserts that “all expanding-universe models with positive cosmological constant asymptotically approach the de Sitter solution” [19]. There is an ample body of literature on this -see e.g., [20] and [21] and references therein. In the light of the above we may venture to speculate that the said conjecture and the tendency to thermodynamic equilibrium at late times are closely interrelated. Nevertheless, this is by no means the last word as the subject calls for further study.

Before closing, note that the particle production does not vanish in the long run \( \Gamma_{dm}(a \to \infty) = 3H_\infty \), and \( \Gamma_{dm}(a \to \infty) = \frac{\nu}{1-\nu} H_0 \sqrt{\Omega_\Lambda_0 - \nu} \) in models I and II, respectively. Then, the question arises as to whether it will be strong enough to bring instability on the second (an in principle, final) de Sitter expansion. Our tentative answer is in the negative; the reason being that in both cases the expansions tend to strictly de Sitter \( (H = \text{constant} > 0) \). However, a definitive response requires far more consideration and it lies beyond the scope of this work. At any rate, if instability sets in again, one should expect that the whole story repeats itself anew though at a much lower energy.

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