HOMOTOPY COHERENT REPRESENTATIONS

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Abstract. Homotopy coherence has a considerable history, albeit also by other names. For this volume highlighting symmetries, the appropriate use is homotopy coherence of representations, at one time known as representations up to homotopy/homotopy coherent representations. We provide a brief semi-historical survey providing some links that may not be common knowledge.

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1. Background/Motivation: The topological setting ...

Homotopy coherence has a considerable history, albeit also by other names. For this volume highlighting symmetries, the appropriate use is homotopy coherence of representations, at one time known as representations up to homotopy/homotopy coherent representations. Here is a brief semi-historical and idiosyncratic survey providing some links that may not be common knowledge.

In topology, the fundamental theorem of covering spaces asserted that for a ‘nice’ topological space $X$, the functor given by sending a covering space over $X$ to the corresponding representation of its fundamental group $\pi_1(X)$ as permutations of a given discrete fiber over a point is an equivalence. That is, the representation determines a covering space for which the monodromy/holonomy is naturally isomorphic to the original representation.

For fiber bundles and especially for (Hurewicz) fiber spaces, there is a much more subtle correspondence. Consider such a fiber space $F \to E \to B$. One version

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1globally and locally path-connected and semi-locally simply-connected
2Way back when, functors had not yet been named!
of the classification of such bundles/fibrations is by the action of the based loop space $\Omega B$ on the fiber $F$. This began with Hilton showing that there was an action $\Omega B \times F \to F$ which at the level of homotopy classes gave a representation of $\Omega B$ on $F$. In fact, that action gives rise to two maps $\Omega B \times \Omega B \times F \to F$ which are homotopic.

However, by passing to the homotopy classes of based loops in $B$, that information is lost and there was more to be discovered. In those days, $\Omega B$ was the space of based loops which were parameterized in terms of the unit interval and hence associative only up to homotopy. It was Masahiro Sugawara [28, 29] who first studied ‘higher homotopies’, leading to today’s world of $\infty$-structures of various sorts [27, 19, 20]. Although Sugawara did not treat actions up to higher homotopies as such, he did introduce strongly homotopy multiplicative maps of associative H-spaces. With John Moore’s introduction of an associative space of based loops, the adjoint $\Omega B \to F^F$ could be seen as such a map. Considering the action as a representation of $\Omega B$ on $F$, we have perhaps the first example of a homotopy coherent representation.

Remark 1.1. An apparently very different sort of treatment of higher homotopies occurred earlier in a strong form of Borsuk’s shape theory. This was initiated by Christie in [3]. He used homotopy coherence in a truncated form; his work went unnoticed until the 1970s. Borsuk’s form of shape theory did not fully reflect the geometric nature of his intuition. He approximated spaces by polyhedra, up to homotopy, but did not consider the homotopies as part of the structure, just their existence. Various people tried various approaches to give a stronger form, some aspects of which can be found in Mardešić’s book, [22].

The innovation in strong shape theory was, thus, to record the higher homotopies used in the approximation process. For instance, when using the Čech nerve of open covers, choices have to be made of the refinement maps between covers, and whilst these cannot be made to respect the iterated associativity required, ‘on the nose’, so they do not give a commutative diagram indexed by the partially ordered set of open covers, they can be made coherently with respect to further refinement, so recording the higher homotopies in the approximation. One replaces a homotopy approximation by a homotopy coherent one.

The link between that quite geometric topological approach and the notions emerging from Sugawara’s work came from the work on homotopy everything algebraic structures by Boardman and Vogt, [5], and then in Vogt, [31] which gave the link between a detailed ‘geometric’ approach to homotopy coherent diagrams and a homotopical approach related to model category theory. We will see links to Vogt’s results later on.

The associativity being central to Sugawara’s work on homotopy multiplicativity [29] this study, it was not long before maps of spaces with associative structures were generalized to maps of topological categories and homotopy coherent functors, which arise in other contexts involving topological and simplicially enriched categories, [10].

**Definition 1.2.** For topological categories $C$ and $D$, a functor up to strong homotopy $F$, also known as a homotopy coherent functor, consists of maps $F_0 : \text{Ob } C \to$
Ob $D, F_1 : Mor \to Mor$ and maps $F_n : I^{n-1} \times C_n \to D_n$ such that

\[
F_1(x \to y) : F_0(x) \to F_0(y)
\]

\[
F_p(t_1, \ldots, t_{p-1}, c_1, \ldots, c_n) = F_{p-1}(\hat{t}_1, \ldots, \hat{t}_i, \ldots, c_i, c_{i+1}, \ldots) \quad \text{if } t_i = 0
\]

\[
= F_i(t_1, \ldots, t_{i-1}, c_1, \ldots, c_i)F_{p-i}(t_{i+1}, \ldots, c_{i+1}, \ldots, c_p) \quad \text{if } t_i = 1.
\]

Three particularly interesting examples of topological categories and their use are given by

- the path space $B^I$, see section 5;
- the singular complex of $B$, for which see section 7
- and a ‘good’ open cover of $B$, which is in section 8

These also can, and will, be considered for smooth manifolds and smooth maps.

2. Background/Motivation: ... and for dg-categories?

Of course, there are complete analogs of these ideas for categories of (co)chain complexes and, more generally, dg categories. We note that for dg categories, that is categories enriched over some category of chain complexes, a chain analog of cubes applies, so the $I^{n-1}$ in the above is replaced by a chain complex representing it.

For an associative algebra $A$, an $A$-module $M$ can be considered as a representation of $A$, a map $A \otimes M \to M$ or $A \to \text{End}(M)$. In a differential graded context, one again can consider representations up to homotopy of $A$ on $M$. On the chain (dg) level, the corresponding notion is related to that of a twisting cochain, the twisted tensor product differential as introduced in [6], for modeling the chains on the total space of a principal fibration in terms of chains on the base and chains on the fiber.

One can up the ante further by introducing $A_\infty$-spaces, $A_\infty$-algebras and modules and further $A_\infty$-categories, but take care: do you want objects and morphisms to be $A_\infty$ in an appropriate sense or only the morphisms? It is worth pointing out that $A_\infty$-maps between strictly associative dg algebras were studied before [24, 25] by Sugawara as parameterized by cubes and called strong homotopy multiplicative maps. $A_\infty$-maps of $A_\infty$-spaces are parameterized by more complicated polyhedra.

Warning! The strongly related notions of covariant derivative, connection, connection form and parallel transport have distinct definitions, but many authors use the names interchangeably in stating theorems while many proofs depend on a particular definition.

3. Homotopy twisting cochains

The analog of a fiber bundle, $F \to E \to B$, in terms of simplicial sets, $\mathcal{F} \to \mathcal{E} \to \mathcal{B}$, was formalized by Barratt, Gugenheim and Moore [3] using a twisting
function and called a twisted cartesian product, then Ed Brown [6] constructed an algebraic chain model of a twisted cartesian product, a twisted tensor product of chain complexes, \((C_\ast B \otimes C_\ast F, D)\), where the differential \(D\) is \(d_B \otimes 1 + 1 \otimes d_F\) twisted by adding a twisting term \(\tau : C_\ast B \to C_\ast AutF\), that is
\[
D = d_B \otimes 1 + 1 \otimes d_F + \tau
\]
where \(\tau\) corresponds to a representation of the cobar algebra \(\Omega C_\ast B\) on \(F\). Here \(C_\ast(B)\) is regarded as a dg coalgebra and \(C_\ast AutF\) as a dg algebra. The defining relation that \(\tau\) satisfies is
\[
d_F \tau + \tau d_B = \tau \cup \tau.
\]
In turn, Szczarba [30] showed that for a twisted cartesian product base, \(B\), with group \(G\), there is a twisting cochain in \(\text{Hom}(C_\ast(A), C_\ast(G))\). Kadeishvili [17] studied twisting elements in relation to a \(\sim 1\) a in a homotopy Gerstenhaber algebra [13]. Quite recently for G-bundles, Franz [12] proved that the map \(\Omega C_\ast B \to C_\ast G\) is a quasi-isomorphism of dg bi-algebras. That \(\Omega C_\ast B\) is a dg bi-algebra follows from work of Baues [14] and Gerstenhaber-Voronov [13] using the homotopy Gerstenhaber structure of \(C_\ast B\).

For an algebra \(A\), there is a universal twisting morphism \(\tau_A : BA \to A\) such that for any twisting morphism \(\tau : C \to A\), there is a unique morphism \(f_\tau : C \to BA\) of coalgebras with \(\tau = \tau_A \circ f_\tau\) [15].

The above can be generalized to representations up to homotopy/homotopy coherent representations, leading to a notion of a homotopy twisting cochain with values in an \(A_\infty\)-algebra.

At the end of section 10, twisting cochains will be revisited in relation to \(\infty\)-local systems.

4. THE GROTHENDIECK CONSTRUCTION AND HIS ‘PURSUIT OF STACKS’

We mentioned the classical case of covering spaces and the equivalence between such and actions / representations of the fundamental group. This has a purely algebraic aspect as it mirrors the structures evident in Galois theory.

The construction of the covering space from the action is a simple example of a type of semi-direct product. Both Ehresmann and Grothendieck looked at the relevance of this for fiber spaces and fibered categories as Grothendieck had introduced in his work. The notion of a fibred category mimics both topological fibrations and the relationship between the category of all modules (over all rings) and the category of rings itself.

Looking at the analog of local sections in this categorical setting, one finds that a fibred category \(p : \mathcal{E} \to \mathcal{B}\) corresponds to a pseudo-functor from the base category to the category of small categories, picking out the fibers and the actions linking them together. What is a pseudo-functor? It is a functor up to ‘natural equivalence of functors’ and thus up to the relevant notion of ‘homotopy’ in this

They mention the concept had been around for some years. At the time they wrote the nomenclature was semi-simplicial complex or even complete semi-simplicial.

If we write \(D\tau\) for \(d_F \tau + \tau d_B\), we can see this as another manifestation of the Maurer-Cartan principle, which has subsumed the integrability condition in deformation theory and others.
setting. This, thus, is another instance of a homotopy coherent representation, this time of the category $B$ into the 2-category of small categories. Of special interest for us is the case of categories fibred in groupoids. Here the pseudo-functor will take values in the category of small groupoids.

Grothendieck’s construction started with a pseudo-functor and constructed a fibred category. In other words, it started with a homotopy coherent representation of the base category and built a fibred category from it. In his extensive discussion document [14], Grothendieck started exploring the higher homotopy analogues of fibred categories, thus analogues of simplicial fibrations. The fibres were to be models of homotopy types, extending both covering spaces, categories fibred in groupoids, and more generally homotopy $n$-types, for any value of $n$ including $\infty$. These were his $n$-stacks.

Again homotopy coherence of the action, of the representation of the base homotopy type is key to building the fibred structure over that base.

5. The (topological) path space $B^I$

Returning to the topological case, the topological path space $B^I$ can be regarded as the space of morphisms of a topological category with $Ob = B$.

A Hurewicz fiber space $F \rightarrow E \rightarrow B$ is a continuous mapping satisfying the homotopy lifting property with respect to any space: Given a homotopy $\lambda : I \times X \rightarrow B$ and an initial value $X \rightarrow F$, we have $\tau : I \times X \rightarrow E$ so that

$$
\begin{array}{ccc}
I \times X & \xrightarrow{\tau} & E \\
\downarrow & & \downarrow p \\
I \times X & \xrightarrow{\lambda} & B 
\end{array}
$$

is commutative.

The action of the based loop space mentioned above extends (not uniquely) to an action $Maps(I \rightarrow B) \times F_s \rightarrow F_t$ where for a path $\lambda : I \rightarrow B$ denote $F_s$ the fiber over $\lambda(0)$ and $F_t$ the fiber over $\lambda(1)$. Non-traditionally, this might be called parallel transport.

**Definition 5.1.** Higher (parallel) transport for a graded vector bundle $p : E \rightarrow M$ is a homotopy coherent functor $|| : PM \rightarrow End(E)$.

Arias Abad and Schaetz [1] use “higher parallel transport” to refer to the representation up to homotopy/homotopy coherent representation denoting the category of representations up to homotopy of the simplicial set $Simp(M)$ of smooth singular chains on $M$.

6. The (smooth) path space $M^I$

For vector bundles rather than covering spaces, according to n-lab:

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5so of homotopy 1-types

6There is an extensive literature on this, but here is not the place to explore this, although it does relate to homotopy coherent representations.
Apparently one of the oldest occurrences of the idea that a principal bundle with connection over a connected base space may be reconstructed from its holonomies around all smooth loops (for any fixed base point) appeared in or was implied by Kobayashi in 1954.

The covering space classification can be generalized to give an equivalence between representations of $\pi_1(M)$ and vector bundles $V \to E \to M$ with flat connection on $M$. A flat connection determines the lifting $\tau: I \to E$ uniquely, then the holonomy with respect to a curve is given by the evaluation of $\tau$ on the path in $M$. The holonomy group is the image $\tau_*: \pi_1(M) \to GL(V)$ as a subgroup of the structure group of the bundle. The other direction, from a representation $\pi_1(M) \to GL(V)$ to a bundle with flat connection, is achieved by the associated $GL(V)$-bundle construction.

7. The simplicial set/$\infty$-category $\text{Simp}(M)$ of $M$

Just as classical holonomy is given by the evaluation of the lifting $\tau$, consider higher holonomy by lifting $\sigma: \Delta^n \to M$ to $\tau^n: \Delta^n \times V \to E$ and related higher transport, but now with $E \to M$ a graded vector bundle with fiber $V$, a differential graded vector space.

The idea of higher holonomy was introduced by Chen as generalized holonomy, elaborated by Igusa [26] and later related to a notion of representation up to homotopy [1].

The technology of Chen’s iterated integrals suggests a precise and rather natural way to handle generalized holonomy. To capture such ‘higher structure’, Chen used maps of simplices $\sigma: \Delta^k \to M$. Denote the standard ordered $n$-simplex $\Delta^n$ as $< 0, 1, \ldots, n >$ with vertices labelled $0, 1, \ldots, n$. Sub-simplices are denoted $< i_0, i_1, \ldots, i_j >$. The face and degeneracy maps for a simplicial set are:

$$\partial_q < 0, 1, \ldots, n > = < 0, 1, \ldots, q-1, q+1, \ldots, n >$$

$$s_q < 0, 1, \ldots, n > = < 0, 1, \ldots, q, q, \ldots, n >$$

**Definition 7.1.** $\text{Simp}(M)$ is the simplicial set of (smooth) maps of simplices $\sigma: \Delta^k \to M$ For $\sigma: \Delta^k \to M$, we denote by $V_i$ the fibre over the image of the vertex $i \in \Delta$.

**Definition 7.2.** A representation up to homotopy of $\text{Simp}(M)$ on a graded vector space $V$ is a collection $\theta$ of maps $\{ \theta_k \}_{k \geq 0}$ which assign to any $k$-simplex $\sigma: \Delta^k \to M$ a map $\theta_k(\sigma): I^{k-1} \times V \to V$ satisfying, for any $v \in V$, the relations:

$\theta_0$ is the identity on $V$

$\theta_k(\sigma)(t_1, \ldots, t_{k-1}, -): V \to V$ is an isomorphism for any $(t_1, \ldots, t_{k-1}) \in I^{k-1}$.

For any $1 \leq p \leq k-1$ and $v \in V$,

$$\theta_k(\sigma)(\cdots, t_p = 0, \cdots, v) = \theta_{k-1}(\partial_p \sigma)(\cdots, \hat{t}_p, \cdots, v)$$

$$\theta_k(\sigma)(\cdots, t_p = 1, \cdots, v) = \cdots$$

\[7\] Only in [1] does he use the word holonomy.

\[8\] There is a similar but earlier use of that name [26]. Notice the ‘cubical’ nature of the condition.
This collection of maps is coherent in the sense that it respects the facial structure of the cubes and of the simplices.

**Theorem 7.3.** For any flat graded vector bundle \( p : E \to M \) with a flat graded connection, there is a representation up to homotopy \( \theta \) of \( \text{Simp}(M) \) on \( E \).

The desired maps \( \theta_n \) are constructed realizing any simplex \( \Delta^n \) as a family of paths with fixed endpoints 0 and \( n \): \( \gamma_n : I^{n-1} \to \mathcal{P} \Delta^n \) where \( \gamma_1(0) \) is the trivial path, constant at 0 and \( \gamma_2 : I \to \Delta^1 \) is the ‘identity’.

For any \( 1 \leq p \leq k-1 \),

\[
\gamma_k(\cdots, t_p = 0, \cdots) = \gamma_{k-1}(\cdots, \hat{t}_p, \cdots)
\]

and

\[
\gamma_k(\cdots, t_p = 0, \cdots) = \gamma_p(t_1, \cdots, t_{p-1}) \gamma_q(t_{p+1}, \cdots, t_{k-1}).
\]

Such maps were first produced by Adams [2] in the topological context by induction using the contractability of \( \Delta^n \). Later specific formulas were introduced by Chen [7] and then equivalently but more transparently, by Igusa [10].

**Remark 7.4.** Given a category \( C \), Leitch [13] constructs a category with the same objects, but for each morphism \( \phi \) of \( C \), there is a space of morphisms, which he called the derived space of \( \phi \). Considering \( \Delta^n \) as a category, the derived space of \( \Delta^n \) turns out to be \( I^{n-1} \). This was one of the precursors of Cordier’s homotopy coherent nerve construction [9], for which see section 9.

Just as representations of \( \pi_1(M) \) are related to vector bundles with flat connection on \( M \), there is an alternative way to look at homotopy representations in terms of differential forms on \( PM \). Consider a graded vector bundle \( p : E = \prod E^k \to M \). Let \( \text{End}^p(E) \) denote the degree \( p \) part of the endomorphism bundle of \( E \):

\[
\text{End}^p(E) = \prod \text{Hom}(E^k, E^{k+p}).
\]

A \( \mathbb{Z} \)-graded connection form is just the analog of a classical connection form, but with careful attention to grading and signs.

**Definition 7.5.** A \( \mathbb{Z} \)-connection form \( A \) on a graded vector bundle \( p : E \to M \) is a form of total degree 1, i.e. in \( \oplus \Omega^p(M; \text{End}^{1-p}(E)) \).

As such, it corresponds to (a family of) differential forms with values in \( \text{End}(E) \). Let \( \Omega^*(M) \) be the graded algebra of smooth differential forms on \( M \) and let \( \Omega^*(M; E) \) be the graded \( \Omega^*(M) \)-module of forms with values in \( \text{End} E \). As usual, it is useful to describe graded connections locally, so the appropriate covariant derivative can be written: \( d + A \), where \( A = A_0 + A_1 + A_2 + \cdots \) with \( A_p \in \Omega^p(M; E^{1-p}) \)

For every \( t \in I \), consider the evaluation map

\[
ev_t : PM \to M
\]

sending \( \gamma \) to \( \gamma(t) \). Let \( W_t \) be the pull back of \( E \) to \( PM \) along \( ev_t \). That is,

\[
(W_t)_{\gamma} = V_{\gamma(t)}.
\]

For \( 0 \leq s \leq t \leq 1 \), let \( \text{Hom}^q(W_s, W_t) \) be the space of degree \( q \) graded homomorphisms from \( W_s \) to \( W_t \). Define a smooth bundle

\[
\Omega^p(PM, \text{Hom}^q(W_s, W_t)) = \Omega^p(PM) \otimes \text{Hom}^q(W_s, W_t)
\]
whose fiber over $\gamma$ is the vector space of smooth $p$-forms with coefficients in
\[ \mathrm{Hom}^q(W_s, W_t)_{\gamma} = \mathrm{Hom}^q(V_{\gamma(s)}, V_{\gamma(t)}). \]

**Definition 7.6.** A family of *homotopy coherent forms* on $PM$ means a family of forms $\Psi_p(s, t)$ for all $0 \leq s \leq t \leq 1$ for the smooth singular simplicial set $\operatorname{Simp}(M)$ of $M$ such that
\[ \Psi_p(s, t) \in \Omega^p(\operatorname{Mor} U, \operatorname{Hom}^{-p}(W_s, W_t)) \]
satisfying the following at each $\gamma \in PM$:
1. $\Psi_0(s, s)_{\gamma}$ is the identity map in $\mathrm{Hom}^0(V_{\gamma(s)}, V_{\gamma(s)})$.
2. $\Phi(s, t)_{\gamma}$ satisfies a certain first order linear differential equation \[^{[16]}\] :
\[ \Psi_p(\gamma, t, s) \in \Omega^p(\operatorname{Mor} U, \operatorname{Hom}^{-p}(W_s, W_t)). \]

**Theorem 7.1.** \[^{[16]}\]: Given a graded vector bundle $p : E \to M$ with a flat $\mathbb{Z}$-connection form $A$, there is a family of homotopy coherent forms on $V$ of the smooth singular simplicial set $\operatorname{Simp}(M)$ of $M$.

The differential equation determines $\Psi_p(s, t)$ uniquely as a $p$-form on $PM$. Starting with $\Psi_0(s, t)$, one can find $\Psi_p(s, t)$ by induction on $p$ using a version of Chen’s iterated integrals.

8. A ‘good’ open cover $\mathcal{U}$

An open cover $\mathcal{U} = \{U_\alpha\}$ is ‘good’ if all the $U_\alpha$ as well as all their intersections are contractible. If $\{U_\alpha\}$ is the open covering, the disjoint union $\bigsqcup U_\alpha$ can be given a rather innocuous structure of a topological category $U$, i.e., $\operatorname{Ob} U = \bigsqcup U_\alpha$ and $\operatorname{Mor} U = \bigsqcup U_\alpha \cap U_\beta$ that is $x \circ y = x = y$ is defined iff $x \in U_\alpha$, $y \in U_\beta$ and $x = y$. For an ordinary vector bundle with structure group $G \subset \operatorname{GL}(V)$, we have transition functions $g_{\alpha\beta} : U_\alpha \cap U_\beta \to G$ which satisfy the cocycle condition: $g_{\alpha\beta}g_{\beta\gamma} = g_{\alpha\gamma}$, but for fiber spaces life is not so straightforward. The transition functions are defined in terms of trivializations.

A differential graded vector bundle $p : E \to M$ with fiber $V$, a differential graded vector space, is locally trivial over some open covering $\{U_\alpha\}$ with trivializations: $\operatorname{dg}$ maps $h_\alpha$ and $k_\alpha$ such that
\[ h_\alpha : p^{-1}(U_\alpha) \cong U_\alpha \times V : k_\alpha \]
where $h_\alpha$ and $k_\alpha$ are inverse fibrewise equivalences.

The transition functions are again defined by the equation
\[ h_\alpha k_\beta(x, v) = (x, g_{\alpha\beta}(x)(v)), \quad x \in U_\alpha \cap U_\beta, \; v \in V, \]
but now it is not necessarily true that $g_{\alpha\beta}g_{\beta\gamma} = g_{\alpha\gamma}$; rather instead of the cocycle condition, one obtains only that $g_{\alpha\beta}g_{\beta\gamma}$ is homotopic to $g_{\alpha\gamma}$ as a map on $U_\alpha \cap U_\beta \cap U_\gamma$. Moreover, on multiple intersections, higher homotopies arise and constitute a homotopy coherent functor, relevant to classifying the fibration.

The collection of higher homotopies was called a *homotopy transition cocycle* \[^{[33, 32]}\]. They determine a fibration (up to appropriate equivalence) by assembling
pieces $U \times V$, not by gluing them directly to each other, but only after inserting ‘connective’ tissue between those pieces, e.g. consider,

$$U_\alpha \times V \cup (I \times U_\alpha \cap U_\beta) \cup U_\beta \times V.$$  

9. Variations on the theme of a homotopy coherent nerve

A useful tool in handling the properties of homotopy coherent functors is to view the various forms of the homotopy coherent nerve construction.

The definition in probably the most standard form is for a simplicial enriched category, $\mathcal{C}$, in which the hom-sets between the objects are Kan complexes. This fits the cases of $\mathcal{C} = \text{Top}$ or $\text{Kan}$ and can also be adapted to handle the category of chain complexes with a bit of extra work. The basic construction is due to Cordier [9], using ideas developed by Boardman and Vogt, [5].

The idea is that for each ordinal $[n]$, one builds a simplicial enriched category, $S[n]$, that in a certain sense resolves the category $[n]$, and such that $S - \text{Cat}(S[n], \mathcal{C})$ gives the collection of all homotopy coherent diagrams within $\mathcal{C}$ having the form of an $n$-simplex. These collections form a simplicial class, (they will often be too large to be a simplicial set), called the homotopy coherent nerve of the $S$-category $\mathcal{C}$ and denoted $\text{Ner}_{h.c.}(\mathcal{C})$. It then is clear that a homotopy coherent diagram having the form of a small category $A$ is given by a simplicial map from $\text{Ner}(A)$ to $\text{Ner}_{h.c.}(\mathcal{C})$.

Vogt proved, [31], for $\mathcal{C} = \text{Top}$, that there was a category of homotopy coherent diagrams of type $A$ in $\text{Top}$ and that this was equivalent to the category obtained from $\mathcal{C}^A$ by inverting the ‘pointwise’ homotopy equivalences. This gives a number of useful links between the explicit definition of homotopy coherent maps and their interpretation in more homotopical terms.

This homotopy coherent nerve is a quasi-category, so is a model for an $\infty$-category. Given any quasi-category $K$, and thinking of this object as an $\infty$-category, we define a homotopy coherent functor from $K$ to $\mathcal{C}$ to be simply a simplicial map from $K$ to $\text{Ner}_{h.c.}(\mathcal{C})$. In particular for $K = \text{Sing}(X)$, which, as it is a Kan complex is a quasi-category, and is known as the $\infty$-groupoid of $X$, a $\infty$-local system in $\mathcal{C}$ is simply a mapping from $\text{Sing}(X)$ to $\text{Ner}_{h.c.}(\mathcal{C})$. These constructions can be adapted for dg-categories, as was done by Lurie, [21]. It can further be adapted to give a homotopy coherent nerve of an $A_\infty$-category as in Faonte [11], and we meet a variant of this in the next section.

10. $\text{Loc}^\mathcal{C}(\pi_\infty M)$

In a variant of the classical Riemann–Hilbert equivalence, a map

$$\text{Flat}(M) \to \text{Reps}(\pi_1(M))$$

is developed by calculating the holonomy with respect to a flat connection. The holonomy descends to a representation of $\pi_1(M)$ as a result of the flatness. In their generalization of the Riemann–Hilbert correspondence, Block and Smith write:

\footnote{The definition and composition of homotopy coherent morphisms is difficult to describe briefly.}
From the perspective of (smooth) homotopy theory, the manifold $M$ can be replaced by its infinity-groupoid $\pi_\infty M := \text{Sing}^\infty M$ of smooth simplices. Considering the correct notion of a representation of $\pi_\infty M$ will allow us to produce an untruncated Riemann-Hilbert theory.

The notion they define is that of an *infinity-local system* on $M$, denoted $\text{Loc}^\infty(\pi_\infty M)$, which is essentially a homotopy coherent functor from the $\infty$-category $\text{Simp}(M)$ to the category of $\text{Ch}$ of chain complexes.

**Remark 10.1.** Rivera and Zeinalian [23] use both $\infty$-local systems and the realization of a simplex $\Delta^n$ as a family of paths parameterized by $I^{n-1}$ (see 7.2) to shed new light on Brown’s classical result [6].

To generalize the classical Riemann–Hilbert equivalence, Block and Smith invoke the Serre-Swan theorem to change the category of flat graded vector bundles to that of “perfect modules with flat $\mathbb{Z}$-connection. They then write:

"It would be an interesting problem in its own right to define an inverse functor which makes use of a kind of associated bundle construction."

That is work for the future.

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