Hardness of exact distance queries in sparse graphs through hub labeling

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Shortest-path oracle

What is the shortest path from A to B?
Distance oracle

What is the distance from A to B?

Trade-off data-structure size vs query time.

Fastest oracles in road networks use hub labeling [Abraham, Delling, Fiat, Goldberg, Werneck 2016]

Huge gap between lower and upper bounds for sparse graphs.

This talk: better understand why.
Distance oracle

What is the distance from A to B?

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Distance oracles
Distance labelings
Motivation: understand gaps for sparse graphs

Sparse: \( m = O(n) \) or \( \Delta = O(1) \)

1/ \( \Omega(\sqrt{n}) \leq \text{DistLab}(n) \leq O \left( n \frac{\log \log n}{\log n} \right) \)

2/ \( \Omega \left( \frac{\sqrt{n}}{\log n} \right) \leq \text{HubLab}(n) \leq O \left( \frac{n}{\log n} \right) \)

[Gavoille, Peleg, Pérennes, Raz 2004]
[Alstrup, Dahlgaard, Bæk, Knudsen 2016]
[Gawrychowski, Kosovski, Uznański 2016]
Our results ($\Delta = O(1)$)

1/ \[ \frac{1}{2^O(\sqrt{\log n})} \text{SumIndex}(n) \leq \text{DistLab}(n) \]

\[ \Omega(\sqrt{n}) \leq \text{SumIndex}(n) \leq \tilde{O}\left(\frac{n}{2\sqrt{\log n}}\right) \]

[Nisan, Wigderson 1993]
[Babai, Gal, Kimmel, Lokan 1995, 2003]
[Pudlak, Rodl, Sgall 1997]

2/ \[ \frac{n}{2^O(\sqrt{\log n})} \leq \text{HubLab}(n) \leq O\left(\frac{n}{\text{RS}(n)^{1/7}}\right) \]

\[ 2^{\Omega(\log^* n)} \leq \text{RS}(n) \leq 2^O(\sqrt{\log n}) \]

[Ruzsa, Szemerédi 1978]
[Behrand 1946] [Elkin 2010] [Fox 2011]
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A hard instance: $2\ell + 1$ grids of dim. $\ell = \sqrt{\log n}$
Connection with Ruzsa-Szemerédi

**RS-graph** : can be decomposed into $n$ induced matchings.\\[\frac{n^2}{RS(n)}\] is the maximum number of edges in an RS-graph.
\[ G^D_y = \{ x_0 z_{2\ell} \mid y = \frac{x+z}{2} \text{ and } d_G(x, z) = D \} \quad \exists D \text{s.t. } \bigcup_y G^D_y \geq \frac{n^2}{2^{O(\sqrt{\log n})}} \]
Converse

Any cst. deg. graph $G$ has hub sets of av. size $O\left(\frac{n}{RS(n)^{1/7}}\right)$.

Idea: use a vertex cover of each $G^D_y$ (VC $\leq$ 2MM).
Connection with SumIndex

\[ \text{SumIndex}(n) = \min_{\text{Encoder}} \max_X |M_A| + |M_B| \]
\( G_X = G \setminus \{ y_\ell \mid X_y = 0 \}, \) send \( x = 2a, L_{x_0}, z = 2b, L_{z_{2\ell}}, \) check \( d(x_0, z_{2\ell}). \)
Thanks