Quantum inseparabilities can be classified into three inequivalent forms: entanglement, Einstein-Podolsky-Rosen (EPR) steering, and Bell’s nonlocality. Bell-nonlocal states form a strict subset of EPR steerable states which also form a strict subset of entangled states. Recently, EPR steerable states are shown to be fundamental resources for one-sided device-independent quantum information processing tasks and, hence, identification of EPR steerable states becomes important from foundational as well as informational theoretic perspectives. In the present study we propose a new criteria to detect whether a given two-qubit state is EPR steerable. From an arbitrary given two-qubit state, another two-qubit state is constructed in such a way that the given state is EPR steerable if the new constructed state is entangled. Hence, EPR steerability of an arbitrary two-qubit state can be detected by detecting entanglement of the newly constructed state. Apart from providing a distinctive way to detect EPR steering without using any steering inequality, the novel finding in the present study paves a new direction to avoid locality loophole in EPR steering tests and to reduce the “complexity cost” present in experimentally detecting EPR steering. Finally, we illustrate our result by using our proposed technique to detect EPR steerability of various families of two-qubit mixed states.

I. INTRODUCTION

Einstein-Podolsky-Rosen (EPR) steering is defined as the apparent ability to affect a spatially separated quantum state, which was the central problem in the EPR argument [1] to demonstrate the incompleteness of quantum mechanics. In particular, EPR argument considers an entangled state shared between two spatially separated parties and it implies the possibility to produce different set of states at one party’s end by performing local quantum measurements of any two non-commuting observables on another spatially separated party’s end. This “Spooky action at a distance” motivated Schrodinger to conceive the celebrated concept of ‘EPR steering’ [2]. However, the research field of quantum steering did not progress much until 2007, when Wiseman, Jones, and Doherty (WJD) introduced the concept of EPR steering in the form of a task [3, 4]. The task of quantum steering is that a referee has to determine (using the measurement outcomes communicated classically from the two parties to the referee) whether two spatially separated parties share entanglement, when one of the two parties is untrusted. WJD introduced the notion of EPR steering as the inability to construct a local hidden variable-local hidden state (LHV-LHS) model to explain the joint probabilities of measurement outcomes. Note that in EPR steering scenario the no-signalling condition (the probability of obtaining one party’s outcome does not depend on spatially separated other party’s setting) is always satisfied by the spatially separated two parties.

It is well-known that EPR steering is an intermediate form of quantum inseparabilities in between entanglement [5] and Bell nonlocality [6–8]. Quantum states that demonstrate Bell-nonlocality form a strict subset of quantum states demonstrating EPR steering which also form a strict subset of entangled states [9]. One important point to be stressed here is that EPR steering is inherently asymmetric with respect to the observers unlike quantum nonlocality and entanglement [10]. In this case, the outcome statistics of one subsystem (which is being ‘steered’) is produced due to valid quantum measurements on a valid quantum state. However, there is no such constraint for the other subsystem. In fact it is shown that there exist entangled states which are one-way steerable, i.e., demonstrate steerability from one observer to the other, but not vice-versa [10–12]. Apart from having immense foundational significance, EPR steering has a vast range of information theoretic application in one-sided device-independent scenario where the party, which is being steered, has trust on his/her quantum device but the other party’s device is untrusted. These applications range from one-sided device-independent quantum key distribution [13], advantages in subchannel discrimination [14], secure quantum teleportation [15, 16], quantum communication [15], detecting bound entanglement [17], connections to joint measurability of generalized measurements [18–21], one-sided device-independent randomness generation [22–25], one-sided device-independent self-testing of pure maximally as well as non-maximally entangled states [26–28].

Against this above backdrop, from fundamental viewpoint as well as from information theoretic perspective it is important to detect EPR steerable states. A number of criteria to detect EPR steering have been proposed till date [29–50]. In the present study we provide a completely new elegant criteria to detect EPR steering of an arbitrary two-qubit state. Given an arbitrary two-qubit state, a new two-qubit state is constructed in such a way that EPR-steering of the given state is detected if the new constructed state is entangled. Hence, following Peres-Horodecki criteria [51, 52] we can state that the given state is EPR-steerable if the partial transpose of the new constructed state has at least one negative eigenvalue.

The foundational significance of the present study indicates a deep connection between EPR steering and entanglement. The novelty of the result obtained in the present study is that it presents a method to detect EPR steering without using
any steering inequality. From experimental point of view, the novel result derived in the present study enables one to indirectly test EPR steering of an arbitrary two-qubit state through entanglement witness [53–57] of the new constructed state. Hence, our proposed theorem enables to reduce the “complexity cost” [58] (it quantifies how complex an experiment is in order to determine entanglement, EPR steering, Bell nonlocality) in experimentally determining EPR steering as it has been shown that the “complexity cost” for the least complex demonstration of entanglement is less than the “complexity cost” for the least complex demonstration of EPR steering [58]. Moreover, the present study may be helpful to avoid the locality loophole present in EPR steering test. Because the degree of correlation required for entanglement testing is smaller than that for violation of a steering inequality, it should be correspondingly easier to demonstrate entanglement without making the fair-sampling assumption [59]. Our proposed procedure to test EPR steering through entanglement detection makes experimental demonstration of EPR steering easier since demonstrating EPR steering is strictly harder than demonstrating entanglement as mentioned in Ref. [59]. One important point to be stressed here is that it was shown that Bell nonlocality can be indirectly detected by detecting EPR steering [60]. The present study, therefore, completes demonstrating the connections between three inequivalent forms of quantum inseparabilities.

We organize this paper in the following way. We briefly discuss the concept of EPR steering and entanglement in Section II. In Section III, we present the main result of this paper on detecting EPR steering of an arbitrary two-qubit state indirectly through entanglement detection. We illustrate our result by detecting EPR steerability of various classes of two-qubit mixed states using our proposed technique in Section IV. Finally, in Section V we summarize the results obtained and present the concluding remarks.

II. PRELIMINARIES

Suppose $A \in \mathcal{F}_a$ and $B \in \mathcal{F}_b$ are the possible choices of measurements for two spatially separated observers, say Alice and Bob, with outcomes $a \in \mathcal{G}_a$ and $b \in \mathcal{G}_b$, respectively. Let the state $\rho_{AB}$ is shared between Alice and Bob. After Alice performs arbitrary measurement $A$ with measurement operators $M^A_a$ ($M^A_a \geq 0 \forall A, a$ and $\sum_a M^A_a = I \forall A$) corresponding to the outcome $a$, Bob’s (unnormalized) conditional state becomes

$$\sigma^A_a = \text{Tr}_A[(M^A_a \otimes I)\rho_{AB}], \quad (1)$$

where $I$ is the $2 \times 2$ identity matrix. On this unnormalized conditional state Bob performs measurement $B$ with measurement operators $M^B_b$ corresponding to the outcome $b$ ($M^B_b \geq 0 \forall B, b$ and $\sum_b M^B_b = I \forall B$) to produce the joint probability distribution $P(a, b|A, B, \rho_{AB}) = \text{Tr}[M^B_b \sigma^A_a]$, where $P(a, b|A, B, \rho_{AB})$ denotes the joint probability of obtaining the outcomes $a$ and $b$, when measurements $A$ and $B$ are performed by Alice and Bob locally on state $\rho_{AB}$, respectively.

The bipartite state $\rho_{AB}$ of the system is steerable from Alice to Bob if and only if it is not the case that for all $A \in \mathcal{F}_a$, $B \in \mathcal{F}_b$, $a \in \mathcal{G}_a$, $b \in \mathcal{G}_b$, the joint probability distribution can be written in the form

$$P(a, b|A, B, \rho_{AB}) = \sum_{\lambda} P(\lambda) P(a|A, \lambda) P(b|B, \rho_\lambda), \quad (2)$$

where $P(\lambda)$ is the probability distribution over the local hidden variables (LHV) $\lambda$, $\sum_{\lambda} P(\lambda) = 1$. $P(a|A, \lambda)$ denotes an arbitrary probability distribution and $P(b|B, \rho_\lambda) = \text{Tr}[\rho_\lambda M^B_b]$ denotes the quantum probability of outcome $b$ given measurement $B$ on the local hidden state (LHS) $\rho_\lambda$, $M^B_b$ being the measurement operator of the observable $B$ associated with outcome $b$. In other words, the bipartite state $\rho_{AB}$ is steerable from Alice to Bob if and only if it does not have LHV-LHS model description (2) for arbitrary measurements performed by Alice and Bob.

A bipartite state $\rho_{AB}$ is called separable if and only if the state can be written in the following form

$$\rho_{AB} = \sum_{\lambda} P(\lambda) \rho^A_\lambda \otimes \rho^B_\lambda, \quad (3)$$

where $\sum_{\lambda} P(\lambda) = 1$. A bipartite state, which is not separable, is called entangled. Alternatively, the bipartite state $\rho_{AB}$ of the system is entangled if and only if it is not the case that for all $A \in \mathcal{F}_a$, $B \in \mathcal{F}_b$, $a \in \mathcal{G}_a$, $b \in \mathcal{G}_b$, the joint probability distribution can be written in the form

$$P(a, b|A, B, \rho_{AB}) = \sum_{\lambda} P(\lambda) P_Q(a|A, \rho^A_\lambda) P_Q(b|B, \rho^B_\lambda) \quad (4)$$

where $\sum_{\lambda} P(\lambda) = 1$. $P_Q(a|A, \rho^A_\lambda) = \text{Tr}[\rho^A_\lambda M^A_a]$ denotes the quantum probability of outcome $a$ given measurement $A$ on the local hidden state $\rho^A_\lambda$, $M^A_a$ being the measurement operator of the observable $A$ associated with outcome $a$. $P_Q(b|B, \rho^B_\lambda)$ is similarly defined. In other words, the bipartite state $\rho_{AB}$ is entangled if and only if it does not have LHS-LHS model description (4) for arbitrary measurements performed by Alice and Bob.

III. DETECTING EPR STEERING THROUGH ENTANGLEMENT DETECTION

The main result of this paper is stated in the following theorem.

**Theorem 1.** For any two-qubit state $\rho_{AB}$ shared between Alice and Bob, define two new states $\tau^1_{AB}$ and $\tau^2_{AB}$ given by,

$$\tau^1_{AB} = \mu_1 \rho_{AB} + (1 - \mu_1) \rho_{AB}^1, \quad (5)$$

and

$$\tau^2_{AB} = \mu_2 \rho_{AB} + (1 - \mu_2) \rho_{AB}^2, \quad (6)$$

where $\rho_{AB}^1 = \rho_A \otimes I_2$ with $\rho_A = \text{Tr}_B[\rho_{AB}] = \text{Tr}_B[\tau^1_{AB}]$ being the reduced state at Alice’s side; $\rho_{AB}^2 = I_2 \otimes \rho_B$ with $\rho_B = \text{Tr}_A[\rho_{AB}] = \text{Tr}_A[\tau^2_{AB}]$ being the reduced state at Bob’s side; $\mu_1 \in [0, \frac{1}{\sqrt{3}}]$, $\mu_2 \in [0, \frac{1}{\sqrt{3}}]$. If $\tau^1_{AB}$ is entangled, then $\rho_{AB}$ is EPR steerable from Bob to Alice. On the other hand, if $\tau^2_{AB}$ is entangled, then $\rho_{AB}$ is EPR steerable from Alice to Bob.
Proof. At first we shall prove that if $\tau_{AB}^\dagger$ is entangled, then $\rho_{AB}$ is EPR steerable from Bob to Alice. We shall prove this by proving its converse negative proposition: if $\rho_{AB}$ is not EPR steerable from Bob to Alice, then $\tau_{AB}^\dagger$ is separable.

Let us calculate $P(a, b|A, B, \tau_{AB}^\dagger) = \text{Tr}[M_n^A \varsigma_n^A]$, which is the joint probability of obtaining the outcomes $a$ and $b$, when arbitrary measurements $A$ and $B$ are performed by Alice and Bob locally on state $\tau_{AB}^\dagger$, respectively, where $\varsigma_n^A$ is the (unnormalized) conditional state on Bob’s side when Alice performs measurement $A$ with measurement operators $M_n^A$ corresponding to the outcome $a$.

\[
\varsigma_n^A = \text{Tr}_A \left[ (M_n^A \otimes \mathbb{I}) \tau_{AB}^\dagger \right] \\
= \text{Tr}_A \left[ (M_n^A \otimes \mathbb{I})(\mu_1 \rho_{AB} + (1 - \mu_1)\rho_{AB}^\dagger) \right] \\
= \mu_1 \text{Tr}_A \left[ (M_n^A \otimes \mathbb{I})\rho_{AB} \right] + (1 - \mu_1) \text{Tr}_A \left[ (M_n^A \otimes \mathbb{I})\rho_{AB}^\dagger \right] \\
= \mu_1 \text{Tr}_A \left[ (M_n^A \otimes \mathbb{I})\rho_{AB} \right] + (1 - \mu_1) P(a|A, \rho_{AB}) \frac{1}{2},
\]

where $P(a|A, \rho_{AB})$ denotes the marginal probability of Alice to obtain the outcome $a$ contingent upon performing measurement $A$ on the state $\rho_{AB}$.

Let us assume that

\[
\varsigma_n^A = \begin{pmatrix}
n_1 & n_2 \\
n_3 & n_4
\end{pmatrix} \\
= \frac{n_1 + n_4}{2} + \text{Re}[n_2]|\sigma_x - \text{Im}[n_2]|\sigma_y + \frac{n_1 - n_4}{2}|\sigma_z,
\]

where $n_1, n_4$ are real and $n_2 = \bar{n}_3$ with $\bar{n}_3$ being the complex conjugate of $n_3$ (since $\varsigma_n^A$ is a Hermitian matrix). $\sigma_x, \sigma_y, \sigma_z$ are Pauli matrices. $\text{Re}[n_2]$ and $\text{Im}[n_2]$ denote the real and imaginary part of $n_2$, respectively.

Now let us evaluate the matrix elements of $\varsigma_n^A$ using Eq. (7).

\[
n_1 = \text{Tr} \left[ \pi_+^x \varsigma_n^A \right] \\
= \mu_1 P(a, +|A, \tilde{\rho}_{AB}) + (1 - \mu_1) P(a|A, \rho_{AB}) \frac{1}{2},
\]

where $\pi_+^x$ is the projector onto the eigenstate of $\sigma_x$ corresponding to the eigenvalue $+1$ and it is given by,

\[
\pi_+^x = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.
\]

$P(a, +|A, \tilde{\rho}_{AB})$ is the joint probability of obtaining the outcomes $a$ and $+1$, when measurement $A$ and projective measurement corresponding to the operator $\sigma_x$ are performed by Alice and Bob locally on state $\rho_{AB}$, respectively.

\[
n_4 = \text{Tr} \left[ \pi_-^x \varsigma_n^A \right] \\
= \mu_1 P(a, -|A, \tilde{\rho}_{AB}) + (1 - \mu_1) P(a|A, \rho_{AB}) \frac{1}{2},
\]

where $\pi_-^x$ is the projector onto the eigenstate of $\sigma_x$ corresponding to the eigenvalue $-1$ and it is given by,

\[
\pi_-^x = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.
\]

$P(a, -|A, \tilde{\rho}_{AB})$ is the joint probability of obtaining the outcomes $a$ and $-1$, when measurement $A$ and projective measurement corresponding to the operator $\sigma_x$ are performed by Alice and Bob locally on state $\rho_{AB}$, respectively. Hence, from Eqs. (9) and (11) we get

\[
n_1 + n_4 = P(a|A, \rho_{AB}) \tag{13}
\]

and

\[
n_2 - n_3 = 2\mu_1 P(a, +|A, \tilde{\rho}_{AB}) - \mu_1 P(a|A, \rho_{AB}) \tag{14}
\]

\[
\text{Re}[n_2] = \text{Tr} \left[ \pi_+^y \varsigma_n^A \right] - \frac{1}{2} P(a|A, \rho_{AB}) \\
= \mu_1 P(a, +|A, \tilde{\rho}_{AB}) - \frac{\mu_1}{2} P(a|A, \rho_{AB}), \tag{15}
\]

where $\pi_+^y$ is the projector onto the eigenstate of $\sigma_y$ corresponding to the eigenvalue $+1$ and it is given by,

\[
\pi_+^y = \begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix}.
\]

$P(a, +|A, \tilde{\rho}_{AB})$ is the joint probability of obtaining the outcomes $a$ and $+1$, when measurement $A$ and projective measurement corresponding to the operator $\sigma_y$ are performed by Alice and Bob locally on state $\rho_{AB}$, respectively.

\[
\text{Im}[n_2] = -\text{Tr} \left[ \pi_-^y \varsigma_n^A \right] + \frac{1}{2} P(a|A, \rho_{AB}) \\
= -\mu_1 P(a, +|A, \tilde{\rho}_{AB}) + \mu_1 P(a|A, \rho_{AB}), \tag{17}
\]

where $\pi_-^y$ is the projector onto the eigenstate of $\sigma_y$ corresponding to the eigenvalue $-1$ and it is given by,

\[
\pi_-^y = \begin{pmatrix} 1/2 & -i/2 \\ i/2 & 1/2 \end{pmatrix}.
\]

$P(a, +|A, \tilde{\rho}_{AB})$ is the joint probability of obtaining the outcomes $a$ and $+1$, when measurement $A$ and projective measurement corresponding to the operator $\sigma_z$ are performed by Alice and Bob locally on state $\rho_{AB}$, respectively.

Combining Eqs. (8), (13), (14), (15), (17) we obtain

\[
\varsigma_n^A = P(a|A, \rho_{AB}) \frac{1}{2} \\
+ \left( \mu_1 P(a, +|A, \tilde{\rho}_{AB}) - \frac{\mu_1}{2} P(a|A, \rho_{AB}) \right) \sigma_x \\
+ \left( \mu_1 P(a, +|A, \tilde{\rho}_{AB}) - \frac{\mu_1}{2} P(a|A, \rho_{AB}) \right) \sigma_y \\
+ \left( \mu_1 P(a, +|A, \tilde{\rho}_{AB}) - \frac{\mu_1}{2} P(a|A, \rho_{AB}) \right) \sigma_z. \tag{19}
\]
Hence, from Eq.(19) we get

\[ P(a, b|A, B, \tau_{AB}^1) = \text{Tr}[M_b^B S_a^c \rho_{AB}] \]

\[ = \text{Tr} \left[ M_b^B \left( \sum_a P(\lambda) P_Q(a|A, \rho_{A}^a) \right) \rho(b|B, \lambda) \right] \]  

(21)

where \( \mathbf{\sigma} = (\sigma_x, \sigma_y, \sigma_z) \) is a vector composed of Pauli matrices and

\[ \vec{r}_A = \mu_1 \left( 2P(+|\hat{x}, \lambda) - 1, 2P(+|\hat{y}, \lambda) - 1, 2P(+|\hat{z}, \lambda) - 1 \right) \]  

(28)

Now if \( \rho_{AB} \) is not steerable from Bob to Alice, then for all \( A \in \mathcal{F}_a, B \in \mathcal{F}_b, a \in \mathbb{G}_a, b \in \mathbb{G}_b \), the joint probability distribution can be written in the form

\[ P(a, b|A, B, \rho_{AB}) = \sum_a P(\lambda) P_Q(a|A, \rho_{A}^a) P(b|B, \lambda). \]  

(22)

and

\[ P(\lambda) P_Q(a|A, \rho_{A}^a) = \sum_a P(\lambda) P_Q(a|A, \rho_{A}^a) P(a|A, \rho_{A}^a). \]  

(25)

Now from Eqs. (20), (22), (23), (24) and (25) we get

\[ P(a, b|A, B, \tau_{AB}^1) = \text{Tr} \left[ M_b^B \left( \sum_a P(\lambda) P_Q(a|A, \rho_{A}^a) \right) \rho_{AB} \right] \]

\[ + \mu_1 \left( \sum_a P(\lambda) P_Q(a|A, \rho_{A}^a) \right) \rho_{AB} \]

\[ - \frac{\mu_1}{2} \left( \sum_a P(\lambda) P_Q(a|A, \rho_{A}^a) \right) \rho_{AB} \]

\[ + \mu_1 \left( \sum_a P(\lambda) P_Q(a|A, \rho_{A}^a) \right) \rho_{AB} \]

\[ - \frac{\mu_1}{2} \left( \sum_a P(\lambda) P_Q(a|A, \rho_{A}^a) \right) \rho_{AB} \]

\[ + \mu_1 \left( \sum_a P(\lambda) P_Q(a|A, \rho_{A}^a) \right) \rho_{AB} \]

\[ - \frac{\mu_1}{2} \left( \sum_a P(\lambda) P_Q(a|A, \rho_{A}^a) \right) \rho_{AB} \]

\]  

(26)

Now let us choose

\[ \rho_{A}^a = \frac{\mathbb{I} + \mathbf{\sigma} \cdot \vec{r}_A}{2}, \]  

(27)

where \( \mathbf{\sigma} = (\sigma_x, \sigma_y, \sigma_z) \) is a vector composed of Pauli matrices and

\[ \vec{r}_A = \mu_1 \left( 2P(+|\hat{x}, \lambda) - 1, 2P(+|\hat{y}, \lambda) - 1, 2P(+|\hat{z}, \lambda) - 1 \right) \]  

(28)

Now for \( P(+|\hat{x}, \lambda), P(+|\hat{y}, \lambda), P(+|\hat{z}, \lambda) \in [0, 1] \) it can be easily checked that \( |\vec{r}_A| \leq 1 \) implies that \( \mu_1 \in \left[ 0, \frac{1}{\sqrt{3}} \right] \). Hence, \( \rho_{A}^a \) is a valid quantum state (qubit) for \( \mu_1 \in \left[ 0, \frac{1}{\sqrt{3}} \right] \).

From the above construction one can write down the following for all \( A \in \mathcal{F}_a, B \in \mathcal{F}_b, a \in \mathbb{G}_a, b \in \mathbb{G}_b \)

\[ \sum_a P(\lambda) P_Q(a|A, \rho_{A}^a) P_Q(b|B, \rho_{B}^b) \]

\[ = \sum_a P(\lambda) P_Q(a|A, \rho_{A}^a) \text{Tr} \left[ M_b^B \frac{\mathbb{I} + \mathbf{\sigma} \cdot \vec{r}_A}{2} \right] \]

\[ = \sum_a \left( P(\lambda) P_Q(a|A, \rho_{A}^a) \right) \text{Tr} \left[ M_b^B \frac{\mathbb{I} + \mathbf{\sigma} \cdot \vec{r}_A}{2} \right] \]

\[ \frac{\mu_1}{2} \sigma_x + \mu_1 P(+|\hat{x}, \lambda) \sigma_x - \frac{\mu_1}{2} \sigma_z + \mu_1 P(+|\hat{z}, \lambda) \sigma_z - \frac{\mu_1}{2} \sigma_z \]

\]  

(29)

Comparing Eqs. (26) and (29), we can write

\[ P(a, b|A, B, \tau_{AB}^1) = \sum_a P(\lambda) P_Q(a|A, \rho_{A}^a) P_Q(b|B, \rho_{B}^b) \]  

(30)

Hence, we can conclude that if for arbitrary measurement \( A \in \mathcal{F}_a \) performed by Alice and for arbitrary measurement \( B \in \mathcal{F}_b \) performed by Bob the joint probability distribution obtained from the state \( \rho_{AB} \) can be written in the form given by Eq.(21), then the joint probability distribution obtained from the state \( \tau_{AB}^1 \) can always be written in the form given by Eq.(30). In other words, if \( \rho_{AB} \) is not EPR steerable from Bob to Alice, then \( \tau_{AB}^1 \) is separable.

In a similar way, it can be shown that if \( \rho_{AB} \) is not EPR steerable from Alice to Bob, then \( \tau_{AB}^2 \) is separable. This completes the proof.

According to the famous Peres-Horodecki criteria [51, 52], any given two-qubit state is entangled if and only if the partial transpose of the given state has at least one negative eigenvalue. Hence, the above theorem immediately provides the following important observation.

**Observation 1.** If the partial transpose of the state \( \tau_{AB}^1 \) has at least one negative eigenvalue, then \( \rho_{AB} \) is EPR steerable from Bob to Alice. On the other hand, if the partial transpose of the state \( \tau_{AB}^2 \) has at least one negative eigenvalue, then \( \rho_{AB} \) is EPR steerable from Alice to Bob; where \( \rho_{AB}, \tau_{AB}^1, \tau_{AB}^2 \) are defined in the statement of Theorem 1.

**IV. ILLUSTRATION WITH EXAMPLES**

Note that any two-qubit pure entangled state is EPR-steerable [37]. Hence, any given two-qubit pure state is EPR
steerable if that given state is entangled. However, this is not true for arbitrary mixed two-qubit states, i.e., there exists two-qubit mixed entangled states, which are unsteerable. Hence, for arbitrary two-qubit mixed states EPR steering cannot be detected by detecting entanglement of that state. The novelty of Theorem 1 is that it enables to detect EPR steering of any arbitrary two-qubit state (pure as well as mixed) by detecting entanglement of another constructed two-qubit state. In the following we will detect EPR steering of different families of two-qubit mixed states using our proposed Theorem 1.

- Consider that the following two-qubit Werner state is shared between Alice and Bob.

\[ \rho_{AB} = p|\psi\rangle\langle\psi| + (1-p)\frac{1}{2} \otimes \frac{1}{2}, \]  

(31)

where \(|\psi\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)\) is the singlet state, \(|0\rangle, |1\rangle\) being the orthonormal basis in \(C^2\), \(0 \leq p \leq 1\). We want to test in which range of \(p\) the state \(\rho_{AB}\) given by Eq. (31) is steerable from Bob to Alice or that from Alice to Bob using Theorem 1.

Following Theorem 1, from the given two-qubit state \(\rho_{AB}\) (31) we construct the new two-qubit state given by,

\[ \tau^1_{AB} = \mu_1(\rho_{AB} + (1-\mu_1)\rho_{\text{A}}, \]  

(32)

where \(\rho_{\text{A}} = \rho_{A} \otimes \frac{1}{2}\) with \(\rho_{A} = \text{Tr}_B(\rho_{AB}) = \frac{1}{2}\) being the reduced state at Alice’s side. We choose \(\mu_1 = \frac{1}{\sqrt{3}}\). The newly constructed state \(\tau^1_{AB}\) given by Eq. (32) is entangled for \(p > \frac{1}{\sqrt{3}}\) which can be checked using Peres-Horodecki criteria [51, 52]. Hence, Theorem 1 concludes that the given two-qubit Werner state \(\rho_{AB}\) (31) is steerable from Bob to Alice for \(p > \frac{1}{\sqrt{3}}\).

In a similar way, using Theorem 1 one can show that the given two-qubit Werner state \(\rho_{AB}\) (31) is steerable from Alice to Bob for \(p > \frac{1}{\sqrt{3}}\).

Note that using quantum violation of 2-settings linear steering inequality proposed in [31] (we have used the particular form of this inequality mentioned in [44]), Werner state (31) is steerable (from Alice to Bob and from Bob to Alice) for \(p > \frac{1}{\sqrt{3}}\). On the other hand, Werner state is both-way steerable for \(p > \frac{1}{\sqrt{3}}\) using 3-settings linear steering inequality [31, 44]. Hence, in this case our proposed technique to check EPR steerability through entanglement detection provides advantage with respect to 2-settings linear steering inequality. However, our proposed technique and 3-settings linear steering inequality detect steerability of Werner state in the same region.

One important point to be stressed here is that the converse of Theorem 1 is not always true, i.e., if \(\tau^1_{AB}\) is separable, then \(\rho_{AB}\) may or may not be EPR steerable from Bob to Alice. On the other hand, if \(\tau^2_{AB}\) is separable, then \(\rho_{AB}\) may or may not be EPR steerable from Alice to Bob. From the above example it can be checked using Peres-Horodecki criteria [51, 52] that the newly constructed state \(\tau^2_{AB}\) (32) (with \(\mu_1 = \frac{1}{\sqrt{3}}\)) is separable for \(p \leq \frac{1}{\sqrt{3}}\). However, the given two-qubit Werner state \(\rho_{AB}\) (31) is steerable for \(p > \frac{1}{2}\) [3]. Hence, in the region \(\frac{1}{2} < p \leq \frac{1}{\sqrt{3}}\) the newly constructed two-qubit state \(\tau^1_{AB}\) (32) is separable, but the given two-qubit state \(\rho_{AB}\) (31) is steerable.

- We will now check EPR steerability of a class of maximally entangled mixed state (MEMS) proposed by Munro et al. [61]. MEMS are those states that achieve the greatest possible entanglement for a given mixedness. The MEMS proposed by Munro et al. is given by,

\[ \rho_{\text{munro}} = \begin{pmatrix} h(C) & 0 & 0 & C \frac{2}{3} \\ 0 & 1 - 2h(C) & 0 & 0 \\ 0 & 0 & 0 & h(C) \\ C \frac{2}{3} & 0 & 0 & \frac{1}{3} \end{pmatrix} \]

(33)

where

\[ h(C) = \begin{cases} \frac{1}{3} & \text{if } C < \frac{2}{3}, \\ C & \text{if } C \geq \frac{2}{3}. \end{cases} \]

(34)

with \(C\) denoting the concurrence of the state \(\rho_{\text{munro}}\) (33).

Munro state \(\rho_{\text{munro}}\) (33) demonstrates both-way EPR steerability for \(C > 0.531\) following Theorem 1. On the other hand, Munro state demonstrates both-way steering for \(C > 0.707\) and for \(C > 0.667\) using quantum violations of 2-settings linear steering inequality and 3-settings linear steering inequality, respectively. Hence, in the region \(0.531 < C \leq 0.667\), steerability of Munro state \(\rho_{\text{munro}}\) (33) can be detected using Theorem 1, but not using 2-settings linear steering inequality and 3-settings linear steering inequality.

- We now focus on a class of non-maximally entangled mixed states (NMEMS). The states, which are not MEMS, are called NMEMS. In particular, we investigate EPR steering of the Werner derivative state [62] which can be obtained by applying a nonlocal unitary operator on the Werner state. Werner derivative state is given by,

\[ \rho_{wd} = \frac{1 - F_w}{3} \mathbb{I} \otimes \mathbb{I} + \frac{4F_w - 1}{3} |\phi\rangle\langle\phi|, \]

(35)

where \(|\phi\rangle = \sqrt{a}|00\rangle + \sqrt{1-a}|11\rangle\) with \(\frac{1}{2} \leq a \leq 1\). The state \(\rho_{wd}\) given by Eq. (35) is entangled if and only if [62]

\[ \frac{1}{2} \leq a < \frac{1}{2} + \frac{\sqrt{3}(4F_w^2 - 1)}{4F_w - 1}, \]

(36)

which further gives a restriction on \(F_w\) as \(\frac{1}{2} < F_w \leq 1\). In Fig. 1 we have shown the region of \(F_w\) and \(a\) for which both-way EPR steering of Werner derivative state (35) is detected using Theorem 1, quantum violation of 2-settings linear steering inequality and quantum violation of 3-settings linear steering inequality, respectively. From this Figure it is clear that Theorem 1 detects steerability of Werner derivative state for a larger region of \(F_w\) and \(a\) than 2-settings linear steering inequality and 3-settings linear steering inequality.
ability of ρ given by, violate to the most degree a steering inequality for a given of maximally steerable mixed state (MSMS) (the states that steering inequality. 

hand, EPR steerability of the above state is not detected us-

where 0 ≤ p ≤ 1. Theorem 1 detects both-way EPR steer-

ability of ρ_p given by Eq. (38) for p < 0.073. On the other hand, EPR steerability of the above state is not detected using 2-settings linear steering inequality and 3-settings linear steering inequality.

• Now we will study EPR steering of the following class of maximally steerable mixed state (MSMS) (the states that violate to the most degree a steering inequality for a given mixedness) proposed by Ren et al. [64],

$$\rho_\tau = \frac{1 - \tau}{4} \begin{pmatrix} 0 & 0 & 1 - \tau \\ 0 & 1 + \tau & 1 + \tau \\ 1 - \tau & 0 & 0 \end{pmatrix} \frac{1 - \tau}{4}$$

(38)

where −1 ≤ τ ≤ 1. 2-settings linear steering inequality as well as 3-settings linear steering inequality detect both-way steerability of the state ρ_τ for −1 ≤ τ ≤ 1. Theorem 1 proposed in this study detects both-way steerability of the state ρ_τ for −1 ≤ τ < −0.366 and for 0.366 < τ ≤ 1. Hence, in this case 2-settings linear steering inequality and 3-settings linear steering inequality are more useful in detecting EPR steerability than our proposed theorem.

• Let us consider the following class of state, which demonstrate one-way steerability [10], is shared between Alice and Bob,

$$\rho_\alpha = \alpha |\psi\rangle\langle\psi| + \frac{1 - \alpha}{5} (2|0\rangle\langle0| \otimes \frac{1}{2} + \frac{3}{2} |1\rangle\langle1| \otimes I),$$

(39)

where |ψ⟩ = 1/√2(|01⟩ − |10⟩) and 0 ≤ α ≤ 1. 2-settings linear steering inequality detects both-way EPR steerability of the above state ρ_α for α > 1/3 = 0.707. 3-settings linear steering inequality detects both-way EPR steerability of the above state for α > 1/√3 = 0.577. Our proposed Theorem 1 detects EPR steerability from Bob to Alice of the above state ρ_α for α > 0.577. Interestingly, Theorem 1 detects EPR steerability from Alice to Bob of the state ρ_α (39) for α > 0.566.

V. CONCLUDING DISCUSSIONS

Since EPR steering has immense foundational significance as well as informational theoretic applications, detecting EPR steering is one of the most profound problem in recent times. A number of criteria to detect EPR steering has been proposed till date [29–50]. In the present study we have provided a novel criteria to detect EPR steering of an arbitrary two-qubit state. This criteria enables one to detect EPR steering of the given two-qubit state by detecting entanglement of another constructed two-qubit state. Hence, theoretically one can detect EPR steering of the given state using Peres-Horodecki criteria [51, 52] without using any steering inequality. Besides having immense foundational significance, our proposed technique to detect EPR steering through entanglement detection reduces the “complexity cost” in experimentally determining EPR steering as the “complexity cost” for the least complex demonstration of entanglement is less than the “complexity cost” for the least complex demonstration of EPR steering [58]. Moreover, this study may pave a new way in avoiding locality loophole in detecting EPR steering experimentally.

Note that any quantum state, which is EPR steerable, is entangled as well, since quantum states demonstrating EPR
steering form a strict subset of the entangled states [9]. Hence, entanglement can be detected by detecting EPR steering. However, the converse is not true always as there exist entangled unsteerable states. Hence, in general, detecting entanglement in a quantum state does not guarantee that the state under consideration is EPR steerable. However, the novelty of the present study is that it provides an indirect way to detect EPR steering though entanglement detection. Previously, it was shown that Bell nonlocality can be indirectly detected by detecting EPR steering [60]. Hence, The present study together with the results obtained in [60] demonstrate the deep connections between three inequivalent forms of quantum inseparabilities.

The present study is restricted to two-qubit systems. Whether EPR steering can be detected through entanglement detection in higher dimensional system is worth to be studied in future. Experimental realization of our proposed theorem using known experimental techniques to detect entanglement, e. g., entanglement witness (see, for example, [57] and the references therein) is another area for future research.

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