Joint Distribution of Distance and Angles in Finite Wireless Networks

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Abstract—Directional beamforming will play a paramount role in 5G and beyond networks to combat the higher path losses incurred at millimeter wave bands. Appropriate modeling and analysis of the angles and distances between transmitters and receivers in these networks are thus essential to understand performance and limiting factors. Most existing literature considers either infinite and uniform networks, where nodes are drawn according to a Poisson point process, or finite networks with the reference receiver placed at the origin of a disk. Under either of these assumptions, the distance and azimuth angle between transmitter and receiver are independent, and the angle follows a uniform distribution between 0 and 2π. Here, we consider a more realistic case of finite networks where the reference node is placed at an arbitrary location. We obtain the joint distribution between the distance and azimuth angle and demonstrate that these random variables do exhibit certain correlation, which depends on the shape of the region and the location of the reference node. To conduct the analysis, we present a general mathematical framework that is specialized to exemplify the case of a rectangular region. We also derive the statistics for the 3D case where, considering antenna heights, the joint distribution of distance, azimuth, and zenith angles is obtained. Finally, we describe some immediate applications of the present work, including the design of analog codebooks, wireless routing algorithms, and the analysis of directional beamforming, which is illustrated by analyzing the coverage probability of an indoor scenario considering misaligned beams.

Index Terms—Angle distribution, beam alignment, finite networks, millimeter-wave, stochastic geometry.

I. INTRODUCTION

A. Motivation and Scope

The need for greater bandwidths to accommodate the ever-increasing demand of data rates has led to the use of higher frequency bands, e.g., millimeter wave (mmW) bands. The key to compensate for the higher path loss experienced at these bands is directional beamforming, which uses a massive number of antenna elements that can be conveniently packed due to the smaller wavelengths. The enabling role of directional beamforming in the realization of 5G and beyond networks is unquestionable, and appropriate modeling and analysis are needed to identify performance trends, trade-offs, and limiting factors [1], [2], [3]. For that reason, some works address previous problems by analyzing directional beamforming on a broad set of scenarios such as 5G cellular networks [4], [5], [6], vehicular networks [7], [8], [9], device-to-device (D2D) communications [10], [11], [12], unmanned aerial vehicles (UAVs) based networks [13], [14], [15], and wireless communications empowered with reconfigurable intelligent surfaces (RISs) [16], [17], [18]. In these works, assuming a random location of the transmitters/receivers, the analyses typically require the joint statistical distribution of the polar coordinates (distance and azimuth angle) in 2D scenarios; and that of the spherical coordinates, which also include the zenith (elevation) angle in 3D scenarios. The latter case is considered when the height of the nodes is relevant (e.g., UAV scenarios). In 2D scenarios, it is assumed that the distance, R, and azimuth angle, Θ, between the transmitter and receiver are independent random variables (RVs), where Θ follows a uniform distribution between 0 and 2π. Those works investigating 3D scenarios make the same assumptions for the distance and azimuth angle, but they consider the zenith angle, Ψ, correlated with the distance. Such correlation comes from the fact that Ψ can be expressed in simple terms as a trigonometric function of the distance and the height of the antennas.

While these assumptions are valid for infinite networks with uniform node distributions, e.g., governed by a uniform Poisson Point Process (PPP), they do not hold for finite networks. Indeed, as shown in this work, the azimuth angle and distance between a randomly placed node and the reference node are correlated in finite networks. Modeling this correlation is very challenging as it depends on both the shape of the region (network) where the nodes are located and on the position of the reference node. The only exception is the case where the reference node is at the center of a disk; in such case, the azimuth angles are independent of the distance, and follow a uniform distribution within [0, 2π].

B. Related Work

Considering finite networks is crucial to investigate the system-level performance of directional beamforming, which
typically aims at boosting the throughput in limited regions with 
an increased demand for high data rates (hot spots). Nevertheless, 
the analysis of finite networks, where node locations are 
modeled by a binomial point process (BPP), is substantially 
more complex than that of infinite networks modeled by a 
PPP. In particular: i) in finite networks, the distribution of the 
signal to interference plus noise ratio (SINR) depends on 
the position of the reference node and, therefore, the performance 
is location dependent; ii) the distances of all nodes in the network 
towards a reference node are correlated; iii) unlike for PPP-based 
infinite networks, the selection of a randomly placed node in 
finite networks changes the distribution of the underlying point 
process [19].

Despite its relevance, the analysis of directional beamforming 
in finite networks is scarce. Most of the existing works avoid 
the need to compute the joint distribution of distance and angle, 
either because they consider a disk with the receiver placed at 
the center, or because they consider perfect beam-alignment for 
the desired link with sectored antenna patterns. For instance, [20] 
considers that the transmitting nodes are randomly placed within 
disks to analyze the 2D indoor mmW case. The probe receiver is 
placed at a given distance from the center of the disk and perfect 
beam alignment is assumed between the probe transmitter and 
receiver. However, the steering directions of the interfering 
beams are considered uniform and independent, thus ignoring 
the said distance-angle correlation. The uplink of an indoor 
wireless system operating at the terahertz band is analyzed in 
in [21], considering a 3D rectangular region. The probe receiver 
is arbitrarily placed, whereas the interfering transmitters are 
randomly placed within the 3D region. Nevertheless, all transmit 
and receive beams are assumed to be perfectly aligned. Again, 
this avoids the need to compute the joint distribution and neglects 
the said correlation.

In [22], a framework for analyzing single and multi-cluster 
wireless network is presented for the case of omnidirectional 
antennas. For single clustered networks, active transmitters are 
assumed to be placed within a disk of radius $D$ following 
a finite homogeneous Poisson point process (FHP) whereas 
the reference receiver is placed at a distance $d$ from the disk 
center. For multi-cluster networks, a Matern cluster process is 
assumed. Two association strategies are considered for each type 
of network to derive the coverage probability.

An interesting framework is presented in [23], where 2D 
fine mmW systems with sectored antenna patterns are 
investigated. This works assumes a maximum average received 
power association criteria and considers a distance-dependent 
blockage probability model that categorizes the links as LOS and 
NLOS. Again, the transmitters and receivers are assumed to be 
placed within a disk, and the reference receiver is placed at any 
arbitrary location within the disk, thus taking the correlation 
between angle and distance into account. However, perfect 
beam alignment is assumed for the desired link, avoiding the 
need to determine the joint distribution of distance and angle. 
The analysis considers the misalignment between interfering 
transmitters and the reference receiver. However, thanks to the 
assumed sectored pattern, only the probability mass function of 
the beamforming gain is needed, thus circumventing the need to 
compute the said joint distribution.

Moreover, existing results on the distance distributions in 
finite regions do not account for the angle distribution [24], [25], 
[26]

C. Main Contributions

To our knowledge, the azimuth angle distribution has not 
been derived yet for finite networks, despite its relevance to 
model and analyze directional beamforming accurately. This 
issue motivated us to investigate the correlation between the 
distance and azimuth angle. The main contributions of this work 
are:

1) We present a mathematical framework to obtain the joint 
distribution of the distance and azimuth angle in 2D 
arbitrarily shaped networks. We consider the angle and 
distance between an arbitrary (fixed) point and a uniformly 
distributed random point.

2) We particularize the proposed framework to the case of a 
rectangular region, since this kind of region matches many 
practical scenarios of interest. We derive the joint 
cumulative distribution function (CDF) and joint probability 
density function (PDF) of the distance and angle, and 
compute the marginals for the azimuth angle.

3) We extend the obtained results to the case of 3D networks, 
considering both the height of the reference node and that 
of the random node.

4) Finally, we discuss some of the applications of the present 
work, including the analysis of directional beamforming, 
the design of the beam patterns of analog codebooks, and 
the design of wireless routing algorithms. We illustrate 
the usefulness of the expressions by analyzing the coverage 
probability in an indoor scenario and determining the 
optimal AP location and antenna’s bearing angle.

The derivation of the joint distance and azimuth angle dis-
tribution is significantly more challenging than the marginal 
distance distribution considered in the existing literature. For 
the derivation of the marginal distance distribution, say, in a 
regular polygon, one only needs to compute the intersection 
between the polygon (region of interest) and a disk centered 
at the reference node. To derive the joint distribution, however, 
we need to compute the intersection of 3 regions: the region 
of interest, a disk of radius $r$, and a given sector. Besides, 
extending the results to the 3D case requires the inclusion of an 
additional angle, the zenith (elevation) angle, which complicates 
the analysis. Moreover, it should be noted that the analysis of a 
rectangular region is substantially more complex than the case of 
a regular L-polygon or a disk, as rotational symmetry properties 
no longer hold.

D. Notation and Paper Organization

The following notation is used throughout the text. \( \mathbb{Z} \) and \( \mathbb{R} \) 
stand for the set of integers and real numbers, respectively, \( \mathbb{R}^+ \) 
represents the non-negative real numbers, and \( \mathbb{R}^d \) stands for the 
d-dimensional Euclidean space. Other sets are represented with 
fraktur font, e.g., \( \mathcal{M} \subseteq \mathbb{R}^d \), with \( d > 0 \), whereas boolean 
expressions are written with calligraphic font, e.g., \( C = \{x \leq \alpha \} \). The 
Lebesgue measure of the set \( \mathcal{M} \) is denoted by \( |\mathcal{M}| \), which 
represents the area or the volume for \( d = 2 \) or \( d = 3 \), respectively.
The intersection and union of sets are represented with symbols \( \cap \) and \( \cup \), respectively, whereas \( \land \) and \( \lor \) stand for the and or logical operations. The empty set is denoted by \( \emptyset \), whereas \( \overline{C} \) represents the complement of the set \( C \). In addition, the upper bar represents the negation of a logical expression, i.e., \( C \) is true if \( C \) is false. If \( X \) is a RV, then \( F_X(x) = \Pr(X \leq x) \) represents its CDF where \( \Pr(\cdot) \) denotes the probability measure.

II. MATHEMATICAL FRAMEWORK

A. Mathematical Preliminaries and Definitions

We consider points that belong to either the 2D or 3D Euclidean space. They are represented using Cartesian coordinates, in either polar (2D) or spherical (3D) coordinates. Fig. 1 illustrates the spherical and polar coordinate systems, formalized in the following two definitions. Note that 2D points can be seen as projections of 3D points in the \( xy \)-plane.

**Definition 1 (Spherical coordinate system):** An arbitrary point \( u \in \mathbb{R}^3 \) expressed as \( (u_x, u_y, u_z) \) in Cartesian coordinates can be written as \( (d, \theta, \varphi) \), being \( d \) the distance, \( \theta \) the azimuth angle, and \( \varphi \) the zenith angle. The following relations hold:

\[
\begin{align*}
  u_x &= d \cos(\theta) \sin(\varphi); \\
  u_y &= d \sin(\theta) \sin(\varphi); \\
  u_z &= d \cos(\varphi).
\end{align*}
\]  

(1)

**Definition 2 (Polar coordinate system):** An arbitrary point \( u \in \mathbb{R}^2 \) expressed as \( (u_x, u_y) \) in Cartesian coordinates can be written as \( (r, \theta) \), being \( r \) the distance, and \( \theta \) the azimuth angle. The following relations hold:

\[
\begin{align*}
  x &= r \cos(\theta); \\
  y &= r \sin(\theta);
\end{align*}
\]  

(2)

We present some mathematical functions used to analyze joint angle and distance distributions.

**Definition 3 (Dirac Delta function):** The Dirac Delta function, \( \delta(x) \), is described as a generalized function that fulfills these two conditions:

\[
\begin{align*}
  \delta(x) &= \lim_{\epsilon \to 0^+} g(x, \epsilon) = \begin{cases} 
    \infty & \text{if } x = 0 \\
    0 & \text{otherwise}
  \end{cases} \\
  \int_{-\infty}^{\infty} \delta(x) dx &= \lim_{\epsilon \to 0^+} \int_{-\infty}^{\infty} g(x, \epsilon) dx = 1
\end{align*}
\]  

(3)

where \( g(x, \epsilon) \) can be any function satisfying the above equations.

**Definition 4 (Indicator function):** The indicator function evaluated over a set, \( C \subset \mathbb{R}^d \), with \( d > 0 \), is defined as

\[
\mathbb{1}_C(x) = \int_{C} \delta(x-u) du = \begin{cases} 
    1 & \text{if } x \in C \\
    0 & \text{otherwise}
  \end{cases}
\]  

(5)

where \( \delta(x) \) is the Dirac delta function in \( \mathbb{R}^d \).

The expression \( x \in C \) in the above definition can also be viewed as a boolean expression representing an event, \( C \), that is \( \text{true} \) if \( x \) belongs to the set \( C \) and \( \text{false} \) otherwise. Thus, the indicator function can alternatively be written as \( \mathbb{1}_C(x) = 1(C) \) [27], where \( C = \{ x \in \mathbb{R} \} \).

**Definition 5 (Properties of the indicator function):** The logical and \( (\land) \) and or \( (\lor) \) operations on two boolean expressions, \( C_1 \) and \( C_2 \), satisfy the following relations:

\[
\begin{align*}
  1(C_1 \land C_2) &= 1(C_1) \cdot 1(C_2), \\
  1(C_1 \lor C_2) &= 1(C_1) + 1(C_2) - 1(C_1 \land C_2).
\end{align*}
\]  

(6)

Equivalently, the union and intersection of two sets, \( C_1 \) and \( C_2 \), lead to the following equalities:

\[
\begin{align*}
  1_{C_1 \cap C_2}(x) &= 1_{C_1}(x) \cdot 1_{C_2}(x), \\
  1_{C_1 \cup C_2}(x) &= 1_{C_1}(x) + 1_{C_2}(x) - 1_{C_1 \cap C_2}(x).
\end{align*}
\]  

(7)

The following two definitions allow writing the main mathematical expressions obtained in this work in compact form.

**Definition 6 (Positive part operator):** For any real number (or real-valued function) \( x \), the positive part of \( x \) is defined as

\[
(x)^+ = \max\{0, x\} = \begin{cases} x & \text{if } x > 0, \\
0 & \text{otherwise} \end{cases}
\]  

(8)

**Definition 7 (F operator):** For a function \( f \) on the real domain, i.e., \( f : \mathbb{R} \mapsto \mathbb{R} \), the operator \( \mathbb{F}(f; a; b) \) is defined as

\[
\mathbb{F}(f; a; b) = \begin{cases} f(b) - f(a) & \text{if } b > a, \\
0 & \text{otherwise} \end{cases}
\]  

(9)

This definition allows us to write the result of definite integrals in compact form, as illustrated next.

**Proposition 1:** The definite integral \( \int_{b_1(u)}^{b_2(u)} \mathbb{F}(t)g(t)dt \), with \( g(t) = \frac{df(t)}{dt} \) and \( \Omega = [a_1(v), a_2(v)] \) admits

\[
\int_{b_1(u)}^{b_2(u)} \mathbb{F}(t)g(t)dt = \mathbb{F}(f; \max(a_1(u), b_1(u)), \min(a_2(u), b_2(u)))
\]  

(10)

**Proof:** The proof comes after (i) applying the indicator function to the integration limits; (ii) considering that if \( a_1(v) > a_2(v) \), then \( \Omega \) reduces to an empty set and the result of the integral is 0; and (iii) substituting the \( \mathbb{F} \) operator in the resulting expression. \( \blacksquare \)

**Corollary 1:** If \( f(t) \) is a non-decreasing function, then the definite integral in Proposition 1 can be further expressed as

\[
\int_{b_1(u)}^{b_2(u)} \mathbb{F}(t)g(t)dt = (f(\min(a_2(u), b_2(u))) - f(\max(a_1(u), b_1(u))))^+.
\]  

(11)

\(^1\)The indicator function of a subset (or event) maps elements of the subset (i.e., the event that \( x \) falls in the subset) to one and zero otherwise.
Step 1: \((\{\phi \mid \beta_{0}(\phi) < \rho\})\) is the sector spanning \([0, \theta]\) (region in black). The set \(\mathcal{B}(\phi)\) is drawn in orange for the azimuth angles \(\phi_1 \text{ and } \phi_2\).

\[\mathcal{B}(\phi) = (0, \beta_{1}(\phi)] \cap [\beta_{2}(\phi_{1}), \beta_{3}(\phi_{1})];\]

whereas, for \(\phi_{2}, \mathcal{B}(\phi_{2}) = (0, \beta_{1}(\phi_{2})].\)

If \(\Re(\rho = 0)\) is a convex region, we can always write \(\mathcal{B}(\phi) = [0, \beta(\phi)]\), since the line segment between any two points in a convex set will always lie within the set \([28]\). In such case, 

\[\mathcal{Y}(u, r, \theta) = (a) \int_{\phi=0}^{\theta} \int_{\rho=0}^{\min(\rho, \beta(\phi))} \rho \mathrm{d}\rho \mathrm{d}\phi = \frac{1}{2} \int_{\phi=0}^{\theta} \int_{\rho=0}^{\beta(\phi)} \rho \mathrm{d}\rho \mathrm{d}\phi \]

for the boolean expression (or event) \(r \leq \beta(\phi) \text{ and } \mathcal{B}(r, \phi)\) represents its complement, i.e., \(r > \beta(\phi)\).

Note that (14) is significantly more tractable than the initial problem to compute the overlap area \(\mathcal{Y}(u, r, \theta)\).

The computation of (14) requires to express the limits of integration in a tractable form. This can be done in a systematic way by the following step-by-step procedure:

1. **Step 1**: Derive \(\beta(\phi)\) to write the convex set \(\Re(u, \theta) = \{\rho \in \mathbb{R}^+, \phi \in [0, 2\pi] | \mathcal{B}(\rho, \phi) = \{\rho \leq \beta(\phi)\}\}

2. **Step 2**: Solve the inequalities defining the events \(\mathcal{B}(r, \phi)\) and \(\mathcal{B}(r, \phi)\), i.e., \(r \leq \beta(\phi)\) and \(r > \beta(\phi)\) to derive the disjoint sets of azimuth angles which satisfy either of the two events. These sets are written as \(\Xi_{\mathcal{B}}(r) = \{\phi \in [0, 2\pi] | \mathcal{B}(r, \phi) = 1\}\) and \(\Xi_{\mathcal{B}}(r) = \{\phi \in [0, 2\pi] | \mathcal{B}(r, \phi) = 1\}\).

3. **Step 3**: Use \(1(\mathcal{B}(r, \phi)) = 1_{\Xi_{\mathcal{B}}(r)}(\phi)\) and \(1(\mathcal{B}(r, \phi)) = 1_{\Xi_{\mathcal{B}}(r)}(\phi)\) (14) and modify the integration limits according to \(\Xi_{\mathcal{B}}(r)\) and \(\Xi_{\mathcal{B}}(r)\) to solve the integrals.

Note that this procedure is general and applicable to any arbitrarily shaped convex region.

In the next section, we follow the above procedure to derive the joint CDF of distance and angle in the relevant case of a rectangle.

### III. Rectangular Regions

#### A. Angular and Distance Distributions in 2D Networks

Let us assume a rectangular region, \(\Re(\phi)\), centered at the origin \(\omega\), whose side lengths are \(\ell_x\) and \(\ell_y\) on the \(x\) and \(y\) axis, respectively (see Fig. 3). We aim at deriving the joint distribution of the distance and azimuth angle for the link between a random point and a reference point, \(u = (u_x, u_y)\), with \(|u_x| < \ell_x / 2 \text{ and } |u_y| < \ell_y / 2\). As pointed out in Section II-B, this is equivalent to computing the overlap area of a disk and a sector after translating the regions by \(-u\) to that end, we follow the...
where $\chi_{i,j}^{(r)}$, $\mu_{i,j}^{(r)}$, $i = \{1, \ldots, 8\}$, $j = \{1, 2\}$, are given in (22) and (23) shown at the bottom of the next page, and $[h_1, h_2, h_3, h_4, h_5, h_6, h_8] = [h_x-\epsilon, h_x-\epsilon, -h_x+\epsilon, -h_x+\epsilon, h_y+\epsilon, h_y+\epsilon, -h_y-\epsilon, -h_y-\epsilon]$. 

\[ \int_{i=1}^{8} \mathcal{X}_i(r) = \int_{i=1}^{8} \mathcal{M}_i(r) = \emptyset. \] (24)

Each subset is restricted to a given quadrant as follows:

\[ \mathcal{X}_1(r) \subset \mathcal{Q}_1, \quad \mathcal{X}_2(r) \subset \mathcal{Q}_2, \quad \mathcal{X}_3(r) \subset \mathcal{Q}_3, \quad \mathcal{X}_4(r) \subset \mathcal{Q}_4, \quad \mathcal{X}_5(r) \subset \mathcal{Q}_1, \quad \mathcal{X}_6(r) \subset \mathcal{Q}_2, \quad \mathcal{X}_7(r) \subset \mathcal{Q}_3, \quad \mathcal{X}_8(r) \subset \mathcal{Q}_4, \quad \mathcal{M}_1(r) \subset \mathcal{Q}_1, \quad \mathcal{M}_2(r) \subset \mathcal{Q}_2, \quad \mathcal{M}_3(r) \subset \mathcal{Q}_3, \quad \mathcal{M}_4(r) \subset \mathcal{Q}_4, \quad \mathcal{M}_5(r) \subset \mathcal{Q}_1, \quad \mathcal{M}_6(r) \subset \mathcal{Q}_2, \quad \mathcal{M}_7(r) \subset \mathcal{Q}_3, \quad \mathcal{M}_8(r) \subset \mathcal{Q}_4. \] (25)

Proof: See Appendix C.

It only remains to complete Step 3 of the framework to derive the joint CDF of the distance and angle, given next.

**Theorem 1.** The joint CDF of the distance and angle of random points, uniformly distributed in a rectangle $\mathfrak{R}(o, r)$, towards a reference point $u = (u_x, u_y)$, can be written as

\[ F_{R, \Theta}(r, \theta) = \frac{r^2}{2 \ell_x \ell_y} \left[ \sum_{i=1}^{8} \mathbb{I}(r < h_i) \left( \min \left( \theta, \chi_{i,1}^{(r)}(r) \right) - \chi_{i,1}^{(r)}(r) \right) + \mathbb{I}(r \geq h_i) \left( \min \left( \theta, \chi_{i,2}^{(r)}(r) \right) - \chi_{i,1}^{(r)}(r) \right) \right] \]

\[ + \sum_{i=1}^{4} \frac{h_i^2}{2 \ell_x \ell_y} \mathbb{I}(r \geq h_i) F(\tan(\mu_{i,1}(r)); \min(\theta, \mu_{i,2}(r))) \]

\[ - \sum_{i=5}^{8} \frac{h_i^2}{2 \ell_x \ell_y} \mathbb{I}(r \geq h_i) F(\tan^{-1}(\mu_{i,1}(r)); \min(\theta, \mu_{i,2}(r))) \],

(26)

with $h_i$, $\chi_{i,1}^{(r)}(r)$, $\chi_{i,2}^{(r)}(r)$ and $\mu_{i,1}(r)$, $\mu_{i,2}(r)$ given in Lemma 2 with (22) and (23) for $i = \{1, \ldots, 8\}$, $j = \{1, 2\}$.

Proof: See Appendix D.

The joint CDF of Theorem 1 is given in closed-form as the sum of 16 simple terms involving compositions of trigonometric and $\min(x, y)$ functions. We now derive the joint PDF and the marginals for the azimuth angle.

**Corollary 3.** In the settings of Theorem 1, the joint PDF of the distance and angle is given by

\[ f_{R, \Theta}(r, \theta) = \frac{r}{\ell_x \ell_y} \sum_{i=1}^{8} \mathbb{I}(r < h_i) \left( \chi_{i,2}^{(r)}(r) \right) \left( \theta \right) \]

\[ + \sum_{i=1}^{4} \frac{h_i^2}{2 \ell_x \ell_y} \mathbb{I}(r \geq h_i) \left( \chi_{i,2}^{(r)}(r) \right) \left( \theta \right) \]

\[ + \sum_{i=5}^{8} \frac{h_i^2}{2 \ell_x \ell_y} \mathbb{I}(r \geq h_i) \left( \chi_{i,2}^{(r)}(r) \right) \left( \theta \right) \].

(27)
Proof: See Appendix E.

Corollary 4: The marginal CDF of the angle between a reference point at \(u = (u_x, u_y)\) and uniformly distributed random points placed within the rectangle \(\mathcal{R}(\alpha)\) is given by

\[
F_{\Theta}(\theta) = \frac{1}{2\pi x x y y} \left( \sum_{i=1}^{4} h_{i}^{2} F(\tan; \epsilon_{1,1}(\theta); \epsilon_{1,2}(\theta)) - \sum_{i=5}^{8} h_{i}^{2} F(\tan^{-1}; \epsilon_{1,1}(\theta); \epsilon_{1,2}(\theta)) \right),
\]

(28)

where \(h_{i}\) for \(i = \{1, \ldots, 8\}\) is given in Lemma 2 and  
\[
\begin{align*}
\epsilon_{1,1}(\theta), &= 0, \\
\epsilon_{1,2}(\theta) &= \min \left( \theta, \tan \left( \frac{h_{y}^{+}}{h_{x}^{+}} \right) \right), \\
\epsilon_{2,1}(\theta) &= \arctan \left( \frac{h_{y}^{+}}{h_{x}^{+}} \right) + 2\pi, \\
\epsilon_{2,2}(\theta) &= \theta, \\
\epsilon_{3,1}(\theta) &= \pi, \\
\epsilon_{3,2}(\theta) &= \min \left( \theta, \tan \left( \frac{h_{y}^{+}}{h_{x}^{+}} \right) + \pi \right), \\
\epsilon_{4,1}(\theta) &= \arctan \left( \frac{h_{y}^{+}}{h_{x}^{+}} \right) + \pi, \\
\epsilon_{4,2}(\theta) &= \min(\theta, \pi),
\end{align*}
\]

(29)

\[
\begin{align*}
\chi_{1,1}(\theta), &= 0, \\
\chi_{1,2}(\theta) &= \arctan \left( \frac{h_{y}^{+}}{h_{x}^{+}} \right), \\
\chi_{2,1}(\theta) &= \arctan \left( \frac{h_{y}^{+}}{h_{x}^{+}} \right) + 2\pi, \\
\chi_{2,2}(\theta) &= 2\pi, \\
\chi_{3,1}(\theta), &= \pi, \\
\chi_{3,2}(\theta) &= \arctan \left( \frac{h_{y}^{+}}{h_{x}^{+}} \right) + \pi, \\
\chi_{3,3}(\theta) &= 2\pi - \arctan \left( \frac{h_{x}^{+}}{h_{y}^{+}} \right), \\
\chi_{4,1}(\theta) &= \arctan \left( \frac{h_{y}^{+}}{h_{x}^{+}} \right) + \pi, \\
\chi_{4,2}(\theta) &= \arctan \left( \frac{h_{y}^{+}}{h_{x}^{+}} \right), \\
\chi_{5,1}(\theta) &= \arctan \left( \frac{h_{y}^{+}}{h_{x}^{+}} \right), \\
\chi_{5,2}(\theta) &= \frac{\pi}{2}, \\
\chi_{5,3}(\theta) &= \arctan \left( \frac{h_{y}^{+}}{h_{x}^{+}} \right) + \pi, \\
\chi_{6,1}(\theta) &= \frac{\pi}{2}, \\
\chi_{6,2}(\theta) &= \arctan \left( \frac{h_{y}^{+}}{h_{x}^{+}} \right) + \pi, \\
\chi_{6,3}(\theta) &= \arctan \left( \frac{h_{y}^{+}}{h_{x}^{+}} \right) + \pi, \\
\chi_{7,1}(\theta) &= \arctan \left( \frac{h_{y}^{+}}{h_{x}^{+}} \right) + \pi, \\
\chi_{7,2}(\theta) &= \frac{3\pi}{2}, \\
\chi_{7,3}(\theta) &= \arctan \left( \frac{h_{y}^{+}}{h_{x}^{+}} \right) + \pi, \\
\chi_{8,1}(\theta) &= \frac{3\pi}{2}, \\
\chi_{8,2}(\theta) &= \arctan \left( \frac{h_{y}^{+}}{h_{x}^{+}} \right) + 2\pi,
\end{align*}
\]

(30)

\[
\begin{align*}
\mu_{1,1}(r) &= 0, \\
\mu_{1,2}(r) &= \min \left( \arctan \left( \frac{h_{x}^{+}}{h_{y}^{+}} \right), \arctan \left( \frac{h_{y}^{+}}{h_{x}^{+}} \right) \right), \\
\mu_{2,1}(r) &= \max \left( 2\pi - \arctan \left( \frac{h_{x}^{+}}{h_{y}^{+}} \right) + \arctan \left( \frac{h_{y}^{+}}{h_{x}^{+}} \right) + 2\pi \right), \\
\mu_{2,2}(r) &= 2\pi, \\
\mu_{3,1}(r), &= \pi, \\
\mu_{3,2}(r) &= \min \left( 2\pi - \arctan \left( \frac{h_{x}^{+}}{h_{y}^{+}} \right) + \arctan \left( \frac{h_{y}^{+}}{h_{x}^{+}} \right) + \pi \right), \\
\mu_{4,1}(r), &= \max \left( \arctan \left( \frac{h_{x}^{+}}{h_{y}^{+}} \right) + \arctan \left( \frac{h_{x}^{+}}{h_{y}^{+}} \right) \right), \\
\mu_{4,2}(r) &= \pi, \\
\mu_{5,1}(r) &= \max \left( \arctan \left( \frac{h_{x}^{+}}{h_{y}^{+}} \right) + \arctan \left( \frac{h_{x}^{+}}{h_{y}^{+}} \right) \right), \\
\mu_{5,2}(r) &= \frac{\pi}{2}, \\
\mu_{6,1}(r) &= \frac{\pi}{2}, \\
\mu_{6,2}(r) &= \min \left( \arctan \left( \frac{h_{x}^{+}}{h_{y}^{+}} \right) + \arctan \left( \frac{h_{y}^{+}}{h_{x}^{+}} \right) + \pi \right), \\
\mu_{7,1}(r) &= \max \left( \arctan \left( \frac{h_{x}^{+}}{h_{y}^{+}} \right) + \arctan \left( \frac{h_{y}^{+}}{h_{x}^{+}} \right) \right), \\
\mu_{7,2}(r) &= \frac{3\pi}{2}, \\
\mu_{8,1}(r) &= \frac{3\pi}{2}, \\
\mu_{8,2}(r) &= \min \left( 2\pi + \arctan \left( \frac{h_{x}^{+}}{h_{y}^{+}} \right) + \arctan \left( \frac{h_{y}^{+}}{h_{x}^{+}} \right) + 2\pi \right).
\end{align*}
\]

(31)
\[ \epsilon_{5,1}(\theta) = \arctan \left( \frac{h_{y+}}{h_{x+}} \right), \quad \epsilon_{5,2}(\theta) = \min \left( \theta, \frac{\pi}{2} \right), \]

\[ \epsilon_{6,1}(\theta) = \frac{\pi}{2}, \quad \epsilon_{6,2}(\theta) = \arctan \left( \frac{h_{y+}}{h_{x+}} \right) + \pi, \]

\[ \epsilon_{7,1}(\theta) = \arctan \left( \frac{h_{y-}}{h_{x-}} \right) + \pi, \quad \epsilon_{7,2}(\theta) = \min \left( \theta, \frac{3\pi}{2} \right), \]

\[ \epsilon_{8,1}(\theta) = \frac{3\pi}{2}, \quad \epsilon_{8,2}(\theta) = \arctan \left( \frac{h_{y-}}{h_{x+}} \right) + 2\pi. \]

**Proof:** See Appendix E. \[ \square \]

**Corollary 5:** The marginal PDF of the azimuth angle, \( \Theta \), can be expressed as

\[ f_\Theta(\theta) = \frac{1}{2\ell_x\ell_y} \left( \sum_{i=1}^{4} \frac{h_i^2}{\cos^2(\theta)} \mathbb{1}_{[x_{i,1}, x_{i,2}]}(\theta) \right) \]

\[ + \sum_{i=3}^{8} \frac{h_i^2}{\sin^2(\theta)} \mathbb{1}_{[x_{i,1}, x_{i,2}]}(\theta), \quad (29) \]

where \( h_i \) for \( i = \{1, \ldots, 8\} \) are given in Lemma 2 and the terms \( \mathbb{1}_{[x_{i,1}, x_{i,2}]} \) with \( j \in \{1, 2\} \) are given in (22). It is important to remark that the intervals \( [x_{i,1}, x_{i,2}] \) are disjoint sets for \( i \in \{1, \ldots, 8\} \).

**Proof:** The result is readily obtained from the derivative of \( F_\Theta(\theta) \), taking into account Definition 7. \[ \square \]

**B. Asymptotic Case: \( \ell_y \gg \ell_x \)**

In this section we investigate the case when one of the sides of a rectangle is much greater than the other. Without loss of generality, we consider that \( \ell_y \gg \ell_x \) and the area of the rectangle is 1. We formulate mathematically this case as \( \ell_y \to \infty \), with \( \ell_y \ell_x = 1 \), i.e., \( \ell_x \to \ell_y^+ \), and \( |u| < \infty \). The following corollary gives the marginal distribution of the azimuth angle in such case.

**Corollary 6:** The marginal distribution of the azimuth angle, \( \theta \), when \( \ell_y \to \infty \), with \( \ell_y \ell_x = 1 \) can be expressed as follows:

\[ \lim_{\ell_y \to \infty} f_\Theta(\theta) = \frac{1}{2} \left( \delta \left( \theta - \frac{\pi}{2} \right) + \delta \left( \theta - \frac{3\pi}{2} \right) \right), \quad (30) \]

where \( \delta(x) \) is the Dirac delta function.

**Proof:** Coming after realizing that the first 4 terms of the PDF given with Corollary 5 are 0 since \( h_{x+} \) and \( h_{x-} \) are 0 when \( \ell_y \to \infty \), i.e., \( \ell_x \to \ell_y^+ \). Manipulating the remaining terms for \( i = \{5, \ldots, 8\} \), and grouping them leads to:

\[ g(\theta - \pi/2, \ell_y) \]

\[ \lim_{\ell_y \to \infty} \frac{\ell_y^2}{\sin(\theta)^2} \left[ \arctan \left( \frac{h_{y+}}{h_{x+}} \right) \right] + \pi \left( \arctan \left( \frac{h_{y+}}{h_{x+}} \right) + \pi \right) \]  

\[ + \frac{1}{2} \lim_{\ell_y \to \infty} \frac{h_y^2}{\sin(\theta)^2} \left[ \arctan \left( \frac{h_{y+}}{h_{x+}} \right) + \pi, \arctan \left( \frac{h_{y+}}{h_{x+}} \right) + 2\pi \right] \]  

\[ g(\theta - 3\pi/2, \ell_y). \quad (31) \]

The next step is to prove that both terms fulfills the two conditions imposed over the Dirac Delta function as per Definition 3. As \( \ell_y \) increases the indicator function, which can be viewed as a pulse on \( \theta \), gets narrower and centered around \( \{\pi/2, \pi + \pi/2\} \) for the first term, and \( \{3\pi/2, 3\pi + \pi/2\} \), for the second term, when \( \ell_y \to \infty \). On the other hand, the amplitude of the terms \( h_{y+}^2/\sin(\theta)^2 \) and \( h_{y+}^2/\sin(\theta)^2 \) tend to infinity in the limit case. Lastly, identifying the functions \( g(\theta - \pi/2, \ell_y) \) and \( g(\theta - 3\pi/2, \ell_y) \) in (31), and checking that they have unit integral over its domain and tend to infinity on \( \pi/2 \) and \( 3\pi/2 \) while being 0 outside completes the proof. \[ \square \]

The above corollary is a formal proof that the angles of a rectangle with one of the sides much greater than the other tend to be concentrated on two possible values, pointing to the directions of maximal length, i.e., \( \pi/2 \) and \( 3\pi/2 \) when \( \ell_y \to \infty \). To preserve a finite area, increasing one of the sides involves a reduction of the other, and thus, in the limit case with \( \ell_y \to \infty, \ell_x \to 0 \), the nodes are randomly distributed within a 1D line. This motivates the following approximation to the joint PDF of distances and angles.

**Approximation 1:** The joint PDF of distances and angles when \( \ell_y \gg \ell_x \) and the reference node, \( u \), is placed at the center of the rectangle can be approximated as:

\[ f_{R,\Theta}(r, \theta) \approx f_R(r) \left( \frac{1}{2\delta} \left( \theta - \frac{\pi}{2} \right) + \frac{1}{2\delta} \left( \theta - \frac{3\pi}{2} \right) \right), \quad (32) \]

where \( f_R(r) \) stands for the distribution of distances in a 1D line of length \( \ell_y \), which is written as \( f_R(r) = 2/\ell_y, r \leq \ell_y, \) and 0 otherwise.

**C. Extension to 3D Networks: Impact of Height**

For the 3D case, we consider an arbitrary reference point placed at \( u = (u_x, u_y, u_z) \in \mathbb{R}^3 \) and random points \( V = (V_x, V_y, V_z) \) with fixed (deterministic) height \( v_z \). When projected on the \( xy \) plane, the reference point \( u \) falls within a 2D convex region \( \mathcal{R}(o) \), with the center of mass located at the origin. Similarly, the projection (on the \( xy \) plane) of random points \( V \) lies in \( \mathcal{R}(o) \), i.e., \( (V_x, V_y) \in \mathcal{R}(o) \). This is a realistic scenario in many applications. For instance, in terrestrial wireless networks, while users are typically modeled with random locations, their antennas are assumed to have a deterministic (fixed) height—since such height is very similar for all users; at the other end, the height of access points (AP) or base station (BS) antennas is an important design parameter, which is relevant in the design of directional beamforming (e.g., design of analog codebooks) through the joint azimuth and zenith angle dependence. In aerial networks such as UAV-based networks, the height of the UVAs and the BSs—which provide the backhaul links—are considered deterministic [29], and this is a parameter which needs to be optimized in the network design.

We are interested in the joint distribution of the distance, azimuth and zenith angles of the link between \( u \) and \( V \), given in the next theorem.

**Theorem 2:** The joint distribution of distance (\( D \)), azimuth (\( \Theta \)) and zenith (\( \Psi \)) angles for the link between a reference point at \( u = (u_x, u_y, u_z) \in \mathbb{R}^3 \) and random points \( V = (V_x, V_y, V_z) \), with \((V_x, V_y)\) uniformly distributed in a convex region \( \mathcal{R}(o) \subset \mathbb{R}^2 \)
applications. The path loss slope and exponent; \((36)\) are the transmit and receive azimuth angles, \(\Theta^\prime\) and \(\Theta^\prime\) represent the transmit power spectral density (PSD) and noise PSD, respectively. If we consider the downlink, with a BS

\[ \mathbb{R}^2, \text{ and deterministic } v_z, \text{ is expressed as given with (33) shown at the bottom of this page, where } F_{R,\Theta}(r, \theta) \text{ is the joint distribution of distance and azimuth angle in the } xy \text{ plane, given in Theorem 1.} \]

**Proof:** See Appendix F. In Theorem 2, as expected, the CDF is 0 when \(d < |u_z - v_z|\), because the distance between the random nodes and the reference node should be greater than the difference of their antenna heights. This theorem enables the analysis of performance metrics of interest, e.g., the coverage probability, in finite networks where the height of the antennas is relevant. Other joint statistics relating to the angular domain are also instrumental. In particular, the joint distribution of azimuth and zenith angles can be exploited for the design of analog codebooks. We characterize this distribution in the following two corollaries, providing the joint CDF and PDF.

**Corollary 7:** The joint CDF of azimuth (\(\Theta\)) and zenith (\(\Psi\)) angles is given by

\[
F_{\Theta,\Psi}(\theta, \psi) = 1(u_z \geq v_z) \left[ \mathbb{1} \left( \psi \in \left[ \frac{\pi}{2}, \pi \right] \right) (F_{\Theta}(\theta) - F_{R,\Theta}(u_z - v_z \tan(\pi - \psi), \theta))\right] + \mathbb{1} \left( \psi \in \left[ 0, \frac{\pi}{2} \right] \right) F_{\Theta}(\theta) + 1(u_z < v_z) \mathbb{1} \left( \psi \in \left[ \frac{\pi}{2}, \pi \right] \right) F_{R,\Theta}(v_z - u_z \tan(\psi), \theta). \tag{34}
\]

**Proof:** We arrive at the result after computing the limit \(F_{\Theta,\Psi}(\theta, \psi) = \lim_{d \to \infty} F_{D,\Theta,\Psi}(d, \theta, \psi)\), with \(F_{D,\Theta,\Psi}(\theta, \psi)\) given in Theorem 2.\]

**Corollary 8:** The joint PDF of azimuth (\(\Theta\)) and zenith (\(\Psi\)) angles is given by

\[
f_{\Theta,\Psi}(\theta, \psi) = 1(u_z < v_z) \mathbb{1} \left( \psi \in \left[ 0, \frac{\pi}{2} \right] \right) \frac{v_z - u_z}{\cos^2(\psi)} + f_{R,\Theta}(v_z - u_z \tan(\psi), \theta) - 1(u_z \geq v_z) \mathbb{1} \left( \psi \in \left[ \frac{\pi}{2}, \pi \right] \right) \frac{v_z - u_z}{\cos^2(\psi)}. \tag{35}
\]

**Proof:** The result is obtained from the partial derivatives of the joint CDF (given in Corollary 7) with respect to the azimuth and zenith angles, i.e., \(f_{\Theta,\Psi}(\theta, \psi) = \frac{\partial^2 F_{\Theta,\Psi}(\theta, \psi)}{\partial \theta \partial \psi}\).

\[
F_{D,\Theta,\Psi}(d, \theta, \psi) = \mathbb{1}(d > |u_z - v_z| \land u_z \geq v_z) \left[ \mathbb{1}_{\Omega_2}(\psi) \left( F_{R,\Theta} \left( \sqrt{d^2 - (u_z - v_z)^2}, \theta \right) - F_{R,\Theta} \left( (u_z - v_z) \tan(\pi - \psi), \theta \right) \right) \right] + \mathbb{1}_{\Omega_1}(\psi) F_{R,\Theta} \left( \sqrt{d^2 - (u_z - v_z)^2}, \theta \right) + \mathbb{1}(d > |u_z - v_z| \land u_z < v_z) \mathbb{1}_{\Omega_1}(\psi) F_{R,\Theta} \left( \min \left( \sqrt{d^2 - (u_z - v_z)^2}, (v_z - u_z) \tan(\psi) \right), \theta \right) \tag{33}
\]

**IV. APPLICATIONS**

The proposed mathematical framework allows for modeling many scenarios of practical relevance in wireless communications. As illustrated in Fig. 4, a rectangle with \(\ell_x \gg \ell_y\) can model the road to consider the case of vehicular communications. In this case, a mounted AP can be considered the reference node (placed at \(u\)) that communicates with vehicles randomly located along the road. The cases of indoor WiFi-based communications or outdoor cellular communications can also be considered; e.g., for terrestrial communications, a BS with a given antenna height at a reference location and a hot-spot of users with smaller antenna height; in indoor scenarios, the reference node could be an AP mounted on the ceiling. Aerial-to-terrestrial communications (e.g., based on UAVs) can also be considered; for instance, a BS located at a reference position \(u\), providing backhaul access to a network of UAVs that fly at a given altitude, much higher than the BS.

A rectangular region is an appropriate modeling choice for these aforementioned scenarios, where our results can be applied. More specifically, the joint distance and angle distribution given in Theorems 1 and 2 are needed to compute the distribution of the SNR when users are placed within a finite area, and directional radiation patterns are used. In the general 3D case, the SNR for the link between a transmit node placed at \(u\) and a randomly located node can be expressed as

\[
\text{SNR} = \frac{g_t(\Theta_t, \Psi_t) g_r(\Theta_r, \Psi_r) (r D)^{-\alpha} |\beta|^2 P_t}{N_0} \tag{36}
\]

where \(\Theta_t, \Psi_t\) and \(\Theta_r, \Psi_r\) are the transmit and receive azimuth and zenith angles, respectively; \(g_t(\bullet)\) and \(g_r(\bullet)\) represent the transmit and receive antenna gains (radiation patterns); \(|\beta|\) is the fast-fading amplitude; \(\tau, \alpha\) are the path loss slope and exponent; and \(P_t, N_0\) are the transmit power spectral density (PSD) and noise PSD, respectively.
placed at $u$ and randomly positioned nodes, the joint distribution of Theorem 2 would model the transmit angles $\Theta_t, \Psi_t$ and the distance $D$, whereas the receive angles, $\Theta_r, \Psi_r$, would be obtained from the transmit angles after simple trigonometric transformations. Importantly, our results do not make any assumption on the radiation patterns, as opposed to previous works restricted to a sector model, e.g., [23], or assuming perfect beam alignment, e.g., [20], [21]. Our results hold for any radiation patterns $g_t(\bullet)$ and $g_r(\bullet)$, which can be related either to single-element antennas, e.g., horn antennas [30], or antenna arrays like uniform planar arrays (UPAs). For the latter, to compute the SNR in (36), the product of gain patterns, $g_t(\bullet)g_r(\bullet)$, should be replaced by the product of transmit and received beamforming matrices (i.e., $\mathbf{w}_t$ and $\mathbf{w}_r$, respectively) by their array response vectors, $|a_t(\Theta, \Psi)|^2 |a_r(\Theta, \Psi)|^2$. Therefore, our results can be applied to the analysis of emerging techniques related to directional beamforming, such as beam management procedures in 5G and beyond, and the analysis of RIS-empowered networks.

Moreover, our results can be applied to the design and optimization of new emerging techniques. In the 5G 3GPP New Radio (NR) standard, the optimal transmit beam is determined by a process that includes beam-sweeping and beam-refinement, using a pre-defined set of $m$ analog beams that form the analog codebook [32]. The angular distribution of the users, as per Corollary 8, can be exploited to design the optimal set of $m$ beams. Finally, as another example, our results can be applied to the design of wireless routing, where the marginal distribution of azimuth angles of Corollary 5 can be used. In essence, wireless routing aims to transmit a message between two nodes $A$ and $B$ in a wireless multi-hop network [24]. The origin and destination nodes cannot communicate directly due to the limited transmit power that establishes a maximum communicating range, $r_{\text{max}}$. The problem is to find the optimal path that minimizes the number of hops. In this scenario, for each node and its given location $u$, the marginal PDF of azimuth angles can be used to find the optimal transmit direction towards the next node.

As an application example, we will derive the coverage probability, i.e., the CCDF of the SNR for an scenario where the AP is placed at a reference location $u$, and the UEs are randomly placed within a rectangle. The discussions and usefulness of these results are further explored in the numerical results section.

### A. Coverage Probability Analysis of Directional Beamforming in a Noise-Limited Scenario

Let us consider a 2D case, where the AP, placed at $u = (u_x, u_y)$, has a directional antenna gain, $g(\theta)$, whereas the UEs are equipped with omnidirectional antennas. The physical antenna or antenna array has an orientation defined by the bearing angle $\xi$, which points to the direction of maximal gain. With this considerations the SNR of a randomly chosen UE can be expressed as

$$\text{SNR} = \frac{g_t(\Theta - \xi)(\tau R)^{-\alpha}|\beta|^2}{N_0},$$

(37)

where the fast fading is assumed to follow Rayleigh distribution, i.e., $|\beta|^2 \sim \text{Exp}(1)$. Without loss of generality, we consider the antenna gain expressed in decibels as

$$g_t(\theta) = g_{\text{max}} - \min \left( \min \left( \frac{12}{\theta^2}, a_{\text{max}} \right) \right),$$

(38)

where $\theta_{3, \text{dB}}$ is the half power beamwidth (HPBW) in radians, $g_{\text{max}}$ the maximum directional gain, and $a_{\text{max}}$ the side-lobe attenuation factor.

The coverage probability, $\bar{F}_{\text{SNR}}(t)$, is given in the following corollary.

**Corollary 9:** The coverage probability of a UE randomly placed within a rectangle $\mathcal{R}(\theta)$, with omnidirectional antenna pattern, that receives transmission from an AP placed at $u$, with a directional antenna gain $g_t(\theta)$ and bearing angle $\xi$, is given by:

$$\bar{F}_{\text{SNR}}(t) = \frac{1}{\ell_x \ell_y} \sum_{i=1}^{8} \int_{r=0}^{h_i} r s_i^{(1)}(r) \, dr + \int_{r=h_i}^{r_{\text{max}}} r s_i^{(2)}(r) \, dr,$$

(39)

where $r_{\text{max}}$ represents the distance between $u$ and the farthest vertex of $\mathcal{R}(\theta)$, and

$$s_i^{(1)}(r) = \mathbb{P} \left( \chi_{i,1}^2(r) < \chi_{i,2}^2(r) \right) = \int_{r=\chi_{i,1}^2(r)}^{\chi_{i,2}^2(r)} \exp \left( \frac{-N_0 r^2}{N_0 r^2 + 4t} \right) \, d\theta,$$

where $lb = \{<, \geq\}$.

**Proof:** The coverage probability can be expressed as

$$\bar{F}_{\text{SNR}}(t) = \mathbb{P}(\text{SNR} \geq t) = \mathbb{E}_{R, \theta} \left[ \mathbb{P} \left( \frac{|\beta|^2}{\rho g_t(\Theta - \xi)} \geq t \right) \right],$$

(40)

where (a) comes after reordering the expression of the SNR and conditioning over the pair of RVs, $R$ and $\Theta$. Finally, applying the definition of the CCDF of an exponential distribution, expressing the expectation in integral form with the joint PDF of $R$ and $\Theta$ as per Corollary 3 and manipulating the resulting expression to avoid discontinuities over the integration completes the proof.

In the special case where $\ell_y \gg \ell_x$, the CCDF of the SNR can be approximated with the following expression.

**Approximation 2:** The CCDF of the SNR when $\ell_y \gg \ell_x$ and the reference node, $u$ placed at the center of the rectangle can be written as follows:

$$\bar{F}_{\text{SNR}}(t) = \frac{\Gamma(\alpha^{-1}) - \Gamma \left( \alpha^{-1}, 2^{-\alpha}(\pi/2)(\ell_y r)^{\alpha} \right)}{\ell_y \alpha \tau \sqrt{\lambda((\pi/2)}} - \frac{\Gamma(\alpha^{-1}) - \Gamma \left( \alpha^{-1}, 2^{-\alpha}(\pi)(3\pi/2)(\ell_y r)^{\alpha} \right)}{\ell_y \alpha \tau \sqrt{\lambda((3\pi/2)}}),$$

(41)

where $\Gamma(a, z) = \int_{z}^{\infty} t^{a-1}e^{-t} \, dt$ is the upper incomplete gamma function and $\lambda(\phi) = \frac{N_0}{\rho g_t(\Theta - \xi)}$.

**Proof:** Applying Approximation 1 to the expression given in Corollary 9 and solving the integrals completes the proof. $\square$
Fig. 5. Joint CDF of distance in the $xy$-plane, $R$, and azimuth angle, $\Theta$, particularized for angles within the first and second quadrants (a), and third and fourth quadrants (b); and joint CDF of distance, $D$, azimuth and zenith angles, $\Theta$, $\Psi$, particularized for azimuth angles within the first (c), and fourth (d) quadrants.

Fig. 6. Sketch of reference scenarios A, B, C as per Table I: (a) 2D scenario which models a road with 3 lanes (each lane is 3.25 m wide) and a road side unit located close to the edge of the road; (b) indoor office scenario, with a room of $3 \times 5$ m, and an AP placed on the ceiling at the height of 3 m; (c) UAV-based network, with drones flying at the height of 120 m, that communicate with a BS antenna (10 m high). The reference node is drawn in black, whereas the random nodes are drawn in black. All the axes have the same scale.

| Scenario | Shape $[\ell_x, \ell_y]$ | Reference location $u$ | Height $v_z$ |
|----------|-----------------|-----------------|----------|
| $O$      | $\ell_x = 200, \ell_y = 100$ m | $u = (30, 25, 10)$ m | $v_z = 1.5$ m |
| $A$      | $\ell_x = 200, \ell_y = 9.75$ m | $u = (0, 2/3)$ | - |
| $B$      | $\ell_x = 3, \ell_y = 5$ m | $u = (0.5, 1.25, 3)$ m | $v_z = 1.5$ m |
| $C$      | $\ell_x = 200, \ell_y = 100$ m | $u = (30, 25, 10)$ m | $v_z = 120$ m |

TABLE I: MAIN PARAMETERS

V. NUMERICAL RESULTS

A. Validation

We now evaluate the theoretical expressions previously derived for the cases of four exemplary regions that model different scenarios as summarized in Table I.

Throughout this section, theoretical results are validated and double-checked with Monte Carlo (MC) simulations.$^2$ The empirical distributions have been estimated using $10^5$ realizations of random points. We first consider a 2D scenario ($O$ in Table I), for which we represent the joint CDF of the distance on the $xy$ plane, $R$, and the azimuth angle, $\Theta$, given by Theorem 1. The shape of this scenario matches a typical public square in many cities, (e.g., the Dam Square in Amsterdam). We particularize the expression for a set of three angles per quadrant for the variable $\theta = \theta_0$ (see Fig. 5(a) and (b)). For the 3D case, the joint CDF of distance, $D$, azimuth and zenith angles, $\Theta$, $\Psi$ (Theorem 2) is validated in Fig. 5(c) and (d). We observe an excellent match between simulation and theoretical results.

B. Reference Scenarios

After validating the 2D and 3D distributions, we now turn our attention to the scenarios defined in Table I. Fig. 6 illustrates scenarios A, B and C. Scenario A represents a road segment of 200 m with 3 lanes of 3.25 m each, and an RSU on the edge of the road, as shown in Fig. 6(a). The joint PDF of distance and azimuth angle, given in Corollary 3, is numerically evaluated and illustrated in Fig. 7(a). We can see how the joint distribution depends on the shape of the region and the location of the arbitrary node. A high correlation between the distance and azimuth angle is observed. Longer distances, $R$, are associated with azimuth angles close to either $\pi$ or $2\pi$ radians, since those directions point to the two extremes of the road segment. The minimum distance values ($R$) are associated with azimuth angles around $3\pi/2$ radians, since in this direction the reference node points to the perpendicular of the road segment, which has a

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$^2$The code is available at: https://github.com/franmarve/joint_pdf_dist_angles
width of only 9.75 m. It is also observed that the joint PDF is null for azimuth angles between 0 and $\pi$ since those directions point outside the road, and thus they are forbidden locations for the vehicles (i.e., random nodes). This joint PDF identifies the direction (or azimuth angles) where a higher user density is expected, and this can be used in the design of beam patterns.

The joint PDF of azimuth, $\Theta$, and zenith, $\Psi$, angles, given by Corollary 8, is illustrated in Fig. 7(b) for scenario B. This scenario, shown in Fig. 6(b), models an office room of 3 m $\times$ 5 m, with an AP placed on the ceiling at a height of 3 m. The geometry of this scenario sets a range of zenith angles $\Psi \in (0, 0.63\pi, \pi)$, as observed in Fig. 7(b). The highest node density is obtained for $\theta = 1.34\pi$ and $\psi = 0.66\pi$, representing the direction where users are most likely to be found. This information can be exploited for the design of beam patterns and wireless routing as discussed in Section IV.

Lastly, scenario C (cfr. Fig. 6(c)) represents a UAV-based network, with drones flying at a height of 120 m, that communicate with a BS antenna (10 m high). Here, the random nodes are higher (in altitude) than the reference node (BS antenna), resulting in zenith angles within the range $\Psi \in (0, \psi_{\text{max}})$, with $\psi_{\text{max}} = 0.26\pi$. This is observed in Fig. 7(c), which represents the joint PDF of azimuth and zenith angles. In this scenario, the direction of maximum node density is given by $\theta = 1.16\pi$, $\psi = 0.26\pi$. On the other hand, minimal user density is observed at azimuth angles around $\theta = \pi/2$. This is due to the fact that the reference node is close to the edge of the region (on the positive $y$ axis direction) where the drones are.

C. Impact of Shape and Location on the Distribution

Fig. 8(a) shows the marginal PDF of the azimuth angle, given in Corollary 4, for a reference node located at the region’s center of mass. To study the effect of the region’s shape, the PDF is evaluated for different lengths of the $y$-axis side, $\ell_y$, while the other side ($\ell_x$) remains constant. The PDF of a uniform distribution, typically assumed in the literature, is also included for comparison.

We first observe that the uniform assumption is not realistic even if the shape is perfectly regular (i.e., a square $\ell_x = \ell_y$). In this case, the part of the square area exceeding the inscribed circle (close to the vertices and representing $(1 - \pi/4)$ times the total square area) adds peaks to the angle distribution in $\pi/4$ and its odd multiples, whichever the side length. The distribution deviations from the uniform distribution are more pronounced as the shape of the region becomes more irregular, i.e., as one of the sides becomes greater than the other one. As the $\ell_y$ side is reduced, the node density increases in the positive and negative directions of the $x$ axis, which leads to a clear concentration of nodes at azimuth angles close to $\theta = 0$, $\theta = 2\pi$ and $\theta = \pi$.

Finally, the effect of the node location is investigated in Fig. 8(b) for a square region (200 m $\times$ 200 m). Initially, a reference node placed at the center of mass is considered, and then its position is modified to approach the edge of the region in the positive $y$ axis direction. Again, the uniform distribution, i.e., a circular shape with the fixed point at its center, is included for comparison. The distribution becomes more different from the uniform one as the reference node moves away from the center of mass. Additionally, as the reference node approaches the edge in the positive $y$ axis direction, the (random) node density increases in the opposite direction, i.e., $\theta = 3\pi/2$, while it reduces in the direction of movement, i.e., $\theta = \pi/2$. In the extreme case where the reference node is placed at the edge of the region, i.e., $u = (0, 100)$, the marginal PDF is null for angles in the range $(0, \pi)$ since those directions point outside the region.

D. Directional Beamforming in Finite Wireless Networks

In this subsection we investigate the impact of the AP location, $u$, and bearing angle, $\xi$, on the coverage probability, i.e., the CCDF of the SNR, given in Corollary 9. In addition, we highlight...
the usefulness of the obtained results for finite wireless networks design. For this numerical application example we consider an indoor office room, with highly isolated walls (i.e., noise limited communication system) of side length 10 m. We aim to determine the optimal AP location, \( u \), and bearing angle of the AP’s antenna, \( \xi \), that maximize the SNR for 10% of the worst UEs. Thus, this metric is mathematically expressed as the 10%-percentile of the SNR, which can be written as

\[
\text{SNR}_{p} = \left\{ t \in \mathbb{R} | 1 - F_{\text{SNR}}(t) - p = 0 \right\},
\]

where \( p = 0.1 \). For this exercise, we assume that there are some restrictions on the valid AP locations due to implementation issues. More specifically, in this example we assume that the AP must lie at 1 m of the right hand side wall of Fig. 9(a). This involves that the valid locations are expressed as \( u = (u_x, -4) \). Nevertheless, there are no restrictions on the bearing angles in this example, i.e., \( \xi \in [-\pi, \pi] \). Regarding the path loss and antenna gain, we consider the 3GPP indoor office hotspot model as per [33] with a carrier frequency of 2 GHz, a maximal gain of \( g_{\text{max}} = 8 \text{ dBi} \), an HPBW of \( \theta_{3\text{dB}} = 65 \text{ degrees} \), and a side-lobe attenuation factor of \( a_{\text{max}} = -22 \text{ dB} \).

Fig. 9(a) illustrates the considered scenario for 3 different bearing angles. The antenna gain, given by (38), is shown with different colors for \( \xi \in \{0, \pi/4, \pi/2\} \). The CCDF of the SINR is illustrated in Fig. 9(b). A perfect match between simulation and theoretical results is observed. Besides, we can see that AP locations and bearing angles pointing to a greater UE density lead to a higher coverage probability. Finally, the applicability of the obtained expressions to determine the optimal AP and bearing angle is shown in Fig. 9(c). The highest value is obtained for \( u = (-4, -4) \), \( \xi = \pi/4 \). Remark that the difference in SNR at 1 between the worst configuration (−9.27 dB with \( u_x = 0, \xi = 0 \) or \( \xi = \pi/2 \)) and optimal configuration (12.08 dB with \( u_x = -4, \xi = \pi/4 \)) is 21.35 dB, which highlights the importance of a proper selection of the AP location and bearing angle.

Finally, we assess the accuracy of the expression given by Approximation 2 for the case when \( \ell_y \gg \ell_x \), for different side lengths \( \ell_y \in \{10, 100\} \) and bearing angles \( \xi \in \{0, \pi/2\} \). The results are illustrated in Fig. 10 which shows a good match between analytical and simulation results for lengths of the horizontal axis, \( \ell_x \) up to 10% of the vertical axis, \( \ell_y \), while the match between analytical and simulation is perfect when \( \ell_x = 1/\ell_y \).

VI. CONCLUSION

Considering finite wireless networks, we have proven a non-negligible correlation between the distance and azimuth angle that characterize the link between an arbitrarily placed reference node and randomly located nodes. We have first proposed a mathematical framework for analyzing the joint distribution in 2D arbitrarily-shaped regions. We then particularized this framework for the relevant case of a rectangular region. We further extended our results to consider the 3D case where the zenith angle must be considered jointly with the distance and azimuth angle. To illustrate the importance of the proposed framework, a number of relevant applications have been identified. We have presented some numerical results to validate the theoretical expressions and shed light on the dependencies between the distance and angles as well as on the effect of the region’s
shape and the location of the reference node. Finally, we have illustrated the usefulness of the results in finite wireless networks by analyzing the coverage probability of an indoor scenario and determining the optimal AP location and antenna bearing angle.

APPENDIX A
PROOF OF LEMMA 1

The rectangular region can be expressed in Cartesian coordinates as follows

$$\mathcal{R}(-u) = \left\{ (x, y) \in \mathbb{R}^2 \mid -\frac{\ell_x}{2} - u_x \leq x \leq \frac{\ell_x}{2} - u_x \right\} \land -\frac{\ell_y}{2} - u_y \leq y \leq \frac{\ell_y}{2} - u_y \right\}. \quad (41)$$

This set can be written in polar coordinates using (2) in the form of (15), where the condition $\mathcal{B}(\rho, \phi)$ is expressed as

$$\mathcal{B}(\rho, \phi) = -\frac{\ell_x}{2} - u_x \leq \rho \cos(\phi) \leq \frac{\ell_x}{2} - u_x \land -\frac{\ell_y}{2} - u_y \leq \rho \sin(\phi) \leq \frac{\ell_y}{2} - u_y. \quad (42)$$

Then, we isolate the variable $\rho$ from $\mathcal{B}(\rho, \phi)$ as follows

$$\left[\left( \cos(\phi) \geq 0 \right) \land \left( \rho \leq \frac{h_{x+}(\phi)}{\cos(\phi)} \right) \lor \mathcal{A}_x(\phi) \land \left( \rho \leq \frac{h_{y+}(\phi)}{\sin(\phi)} \right) \right] \land \left[\left( \sin(\phi) \geq 0 \right) \land \left( \rho \leq \frac{h_{y+}(\phi)}{\sin(\phi)} \right) \lor \mathcal{A}_y(\phi) \land \left( \rho \leq \frac{h_{y+}(\phi)}{\sin(\phi)} \right) \right]. \quad (43)$$

after some manipulations over the inequalities in (42). It should be noticed that each of the former inequalities yields two inequalities but one of them is always true, and thus it is discarded.

Then, applying the distributive and associative properties of the logical operators, it leads to

$$\mathcal{B}(\rho, \phi) = \mathcal{A}_x(\phi) \land \mathcal{A}_y(\phi) \land \mathcal{A}_{x+}(\rho, \phi) \land \mathcal{A}_{y+}(\rho, \phi) \land \mathcal{A}_{x-}(\rho, \phi) \land \mathcal{A}_{y-}(\rho, \phi) \land \mathcal{A}_l(\phi). \quad (44)$$

It is identified that the and operation of the logical expressions $\mathcal{A}_x(\phi)$ and $\mathcal{A}_y(\phi)$ and its negations are equivalent to checking whether the angle $\phi$ belongs to each of the angular quadrants. For instance, the first term can be expressed as follows: $\mathcal{A}_x(\phi) \land \mathcal{A}_y(\phi) = \phi \in \Omega_1$. The remaining terms are manipulated as follows

$$\mathcal{A}_{x+}(\rho, \phi) \land \mathcal{A}_{y+}(\rho, \phi) = \rho \geq \frac{h_{x+}(\phi)}{\cos(\phi)} \land \rho \leq \frac{h_{y+}(\phi)}{\sin(\phi)} \quad \text{and} \quad \mathcal{A}_{x-}(\rho, \phi) \land \mathcal{A}_{y-}(\rho, \phi) = \rho \geq \frac{h_{y+}(\phi)}{\sin(\phi)} \land \rho \leq \frac{h_{x+}(\phi)}{\cos(\phi)}. \quad (45)$$

The terms $\mathcal{A}_+(\rho, \phi) \land \mathcal{A}_-(\rho, \phi)$ and $\mathcal{A}_+(\rho, \phi) \land \mathcal{A}_-(\rho, \phi)$ lead to similar expressions to (45) but using different combinations of the parameters $h_{x+}$, $h_{y+}$, $h_{x-}$, and $h_{y-}$. Details are omitted for the sake of compactness. Leveraging this fact and substituting the functions $h_{x+}(\phi)$ and $h_{y+}(\phi)$ in the above expression completes the proof.

APPENDIX B
PROOF OF LEMMA 2

The boolean expression $\mathcal{B}(r, \phi) = r \leq \beta(\phi)$ can be expressed as

$$\mathcal{B}(r, \phi) \equiv \begin{cases} (a) & r \leq \frac{h_{x+}(\phi)}{\cos(\phi)} \lor \left(C(\phi) \land \frac{h_{y+}(\phi)}{\sin(\phi)} \right) \\ (b) & \frac{h_{x+}(\phi)}{\cos(\phi)} \geq r \land C(\phi) \lor \left( \frac{h_{y+}(\phi)}{\sin(\phi)} \leq r \land C(\phi) \right) \end{cases} \quad (46)$$

where (a) comes after expressing the min($x, y$) function as the sum of two indicator functions with $C(\phi) = \frac{h_{x+}(\phi)}{\cos(\phi)} \geq h_{y+}(\phi)$ and (b) after reordering and applying the inequality in both terms. Below, we isolate $\phi$ in the boolean expressions $C(\phi), D(\phi), E(\phi)$. The term $C(\phi)$ can be written as

$$C(\phi) = \begin{cases} \mathcal{C}_1(\phi) & \phi \in \left[0, \arctan\left(\frac{h_{y+}(\phi)}{h_{x+}(\phi)}\right)\right) \lor \pi, \arctan\left(\frac{h_{y+}(\phi)}{h_{x+}(\phi)}\right) \\ \mathcal{C}_2(\phi) & \phi \in \left[\arctan\left(\frac{h_{y+}(\phi)}{h_{x+}(\phi)}\right), \pi\right] \end{cases} \quad (47)$$

where the expression of $C(\phi)$ has been manipulated to group the $\sin(x)$ and $\cos(x)$ as a tangent function, and this latter term is isolated by splitting the inequality into the cases where the tangent is positive ($\mathcal{A}_+(\phi) = 1$) and negative ($\mathcal{A}_-(\phi) = 1$). Thus, $\mathcal{A}_+(\phi) = \phi \in \Omega_1 \cup \Omega_3$ and $\mathcal{A}_-(\phi) = \phi \in \Omega_2 \cup \Omega_4$. The solution of the inequality $\mathcal{F}_1(\phi)$ is $\phi \in \left\{-\frac{\pi}{2} + \pi n, \arctan\left(\frac{h_{y+}(\phi)}{h_{x+}(\phi)}\right) + n\pi, n \in \mathbb{Z}\right\}$, whereas the solution of $\mathcal{F}_2(\phi)$ is $\phi \in \left\{\arctan\left(\frac{h_{y+}(\phi)}{h_{x+}(\phi)}\right) + n\pi, \frac{\pi}{2} + n\pi, n \in \mathbb{Z}\right\}$. Thus, the intersection of these solutions with the intervals that define $\mathcal{A}_+(\phi)$ and $\mathcal{A}_-(\phi)$ leads to

$$\mathcal{C}_1(\phi) = \phi \in \left[0, \arctan\left(\frac{h_{y+}(\phi)}{h_{x+}(\phi)}\right)\right) \lor \pi, \arctan\left(\frac{h_{y+}(\phi)}{h_{x+}(\phi)}\right) \quad (48)$$

$$\mathcal{C}_2(\phi) = \phi \in \left[\arctan\left(\frac{h_{y+}(\phi)}{h_{x+}(\phi)}\right), \pi\right] \quad (49)$$
where expressions of \( C(\phi), \bar{C}(\phi), \bar{D}(r, \phi) \) and \( E(r, \phi) \) in (46) are used, and the distributive and associative properties of the logical operators are applied. Now, we identify in (54) the terms ranging from \( X_i(r, \phi) \) up to \( X_8(r, \phi) \). This terms, \( X_i(r, \phi) \forall i \in [1, 8] \subset \mathbb{Z} \), are related to the regions \( X_i(r, \phi) \forall i \in [1, 8] \subset \mathbb{Z} \) given in (20) and (22) as follows:

\[
X_i(r) = \{ \phi \in [0, 2\pi] | X_i(r, \phi) = 1 \}.
\]  

Finally, substituting (48), (51), and (53) in (54) leads to the and operation of 8 terms with the following form:

\[
X(r, \phi) = (r - h_i) \land \phi \in \left[ \frac{\chi_{i,1}^{(\geq)}(r)}{\chi_{i,2}^{(\geq)}(r)} \right] \land \left( r \geq \frac{\chi_{i,1}^{(\geq)}(r)}{\chi_{i,2}^{(\geq)}(r)} \right).
\]  

Identifying the resulting terms \( h_i, \chi_{i,1}^{(\leq)}(r), \chi_{i,2}^{(\leq)}(r), \chi_{i,1}^{(\leq)}(r) \), and \( \chi_{i,2}^{(\leq)}(r) \) on each of the 8 expressions completes the proof for \( \mathcal{B}_{\Theta}(r) \). Following a similar approach for \( \mathcal{B}_{\Theta}(r) \), we can write \( \bar{B}(r, \phi) \) as

\[
\bar{B}(r, \phi) = \bar{D}(r, \phi) \land C(\phi) \land \bar{E}(r, \phi) \land \bar{C}(\phi),
\]  

with \( \bar{D}(r, \phi) = D_1(r, \phi) \lor D_2(r, \phi) \) and \( \bar{E}(r, \phi) = E_3(r, \phi) \lor E_4(r, \phi) \). Then, it can be shown that \( \bar{B}(r, \phi) \) can be expressed as

\[
\bar{B}(r, \phi) = D_1(r, \phi) \land C_1(\phi) \lor \cdots \lor D_4(r, \phi) \land C_4(\phi),
\]  

being \( M_i(r) = \{ \phi \in [0, 2\pi] | M_i(r, \phi) = 1 \} \), where the details are omitted due to space limitations.

\section*{APPENDIX C}

\textbf{Proof of Corollary 2}

The proof comes after realizing that the boolean expression \( C_1(\phi) \) and \( C_2(\phi) \), which are given in (48), are restricted to the events \( \mathcal{A}_1(\phi) \) and \( \mathcal{A}_2(\phi) \), respectively; the expressions \( D_1(\phi) \) and \( D_2(\phi) \) from (51) to the events \( \mathcal{A}_1(\phi) \) and \( \mathcal{A}_2(\phi) \); \( E_1(\phi) \) and \( E_2(\phi) \) from (53) to the events \( \mathcal{A}_3(\phi) \) and \( \mathcal{A}_4(\phi) \); and finally \( C_3(\phi) \) and \( C_4(\phi) \) to \( \mathcal{A}_1(\phi) \) and \( \mathcal{A}_2(\phi) \). Expressing these events as intervals of the angle \( \phi \) and substituting on each of the boolean expressions \( X_i(r, \phi) \) in (54) completes the proof. The same process is followed for the sets \( M_i(r) \) with \( i \in [1, 8] \subset \mathbb{Z} \).

\section*{APPENDIX D}

\textbf{Proof of Theorem 1}

Using Lemmas 1 and 2, the overlap area given by (14) can be written as follows

\[
|\mathcal{B}(u, r, \theta)| = \frac{1}{2} \int_{\phi=0}^{\theta} \mathcal{B}_{\Theta}(r, \phi) \, d\phi \]  

where the equality \( \mathcal{B}(u, r, \phi) = \mathcal{B}_{\Theta}(r, \phi) \) is used, and the term \( \beta(\phi) \) in (14) is expressed as the sum of two indicator functions.
The term \( \mathbb{P}(C(\phi) \land B(r, \phi)) \) can be manipulated as follows

\[
\mathbb{P}(C(\phi) \land B(r, \phi)) = \mathbb{P}(C(\phi) \land (D(r, \phi) \lor C(\phi) \lor E(r, \phi) \lor C(\phi)))
\]

\[
= \mathbb{P}(D_3(r, \phi) \lor D_4(r, \phi) \lor C_1(\phi) \lor C_2(\phi))
\]

\[
= \mathbb{P}\left( \bigvee_{i=1}^{4} M_i(r, \phi) \right) = \sum_{i=1}^{4} \mathbb{P}(M_i(r, \phi)), \tag{61}
\]

where (a) comes after expressing \( B(r, \phi) \) as \( D(r, \phi) \lor C(\phi) \lor E(r, \phi) \lor C(\phi) \); (b) after applying distributive, associative and absorption properties of the logical operators; and (c) after some manipulations, applying Corollary 2 and identifying the terms \( D_3(r, \phi) \lor C_1(\phi) = M_i \) with \( k \in \{3, 4\} \), \( j \in \{1, 2\} \) and \( \ell \in \{1, 2, 3, 4\} \).

Analogously, the term \( \mathbb{P}(\overline{C}(\phi) \land B(r, \phi)) \) can be written as

\[
\mathbb{P}(\overline{C}(\phi) \land B(r, \phi)) = \sum_{i=5}^{8} \mathbb{P}(M_i(r, \phi)). \tag{62}
\]

Finally, substituting (62) and (61) in (59) and (12) and applying Lemma 2, Proposition 1, and Corollary 1 completes the proof after some additional manipulations.

**APPENDIX E**

**PROOF OF COROLLARIES 3 AND 4**

**A. Corollary 3**

The joint PDF of distance and azimuth angle is computed as

\[
f_{R,\Theta}(r, \Theta) = \frac{\partial^2 f_{R,\Theta}(r, \Theta)}{\partial r \partial \Theta}. \tag{56}
\]

It can be noticed that the derivative of the two summations multiplied by \( \frac{h_i^2}{2 \sigma_{\epsilon x y}} \) in (26) are 0 since they do not have terms that depend simultaneously on the \( r \) and \( \Theta \) variables. The partial derivative with respect to \( \Theta \) of the terms \( \min(\theta, \chi_1(r)) - x_1(r)^+ \) can be written as follows

\[
\frac{\partial}{\partial \Theta} \left( \min(\theta, \chi_1(r)) - x_1(r)^+ \right)
\]

\[
= \mathbb{I}(\theta < \chi_1(r)) \mathbb{I}(\min(\theta, \chi_1(r)) > x_1(r))^+, \tag{63}
\]

where the term \( \min(\theta, \chi_1(r)) \) has been expressed as \( \theta \mathbb{I}(\theta < \chi_1(r)) + \chi_1(r) \mathbb{I}(\theta \geq \chi_1(r)) \), it has been computed the derivative with respect to \( \Theta \) and it has been multiplied by \( \mathbb{I}(\min(\theta, \chi_1(r)) > x_1(r))^+ \) as per Definition 6. The superscript \( (\cdot)^+ \) or \( (\cdot)^- \) in \( \chi_1(r) \) and \( \chi_2(r) \) has been omitted to refer to both cases. Finally, multiplying by \( \frac{r^2}{\sigma_{\epsilon x y}} \), deriving with respect to \( r \) and manipulating the resulting expression completes the proof.

**B. Corollary 4**

The marginal CDF is computed as \( F_{E_0}(\theta) = \lim_{r \to \infty} F_{R,\Theta}(r, \Theta) \). It can be shown that the summation multiplied by \( \frac{r^2}{\sigma_{\epsilon x y}} \) in (26) is 0 since the terms multiplied by \( \mathbb{I}(r < h_i) \) are 0 due to the indicator function, and the terms multiplied by \( \mathbb{I}(r \geq h_i) \) are 0 since the argument of the \( (\cdot)^+ \) operator are negative when \( r \to \infty \). This can be checked from (22) since \( \lim_{r \to \infty} \frac{r}{h_1} = \frac{r}{h_2} \) and \( \lim_{r \to \infty} \frac{r}{h_1} \) are 0 and \( \tan(x) \in [0, \pi] \) when \( x \geq 0 \) whereas \( \tan(x) \in [-\pi, 0] \) when \( x < 0 \). Finally, deriving the limit when \( r \to \infty \) of the term \( e_i(\theta) \) and \( \min(\theta, \mu_{i_2}(r)) \) on the two summations multiplied by \( \frac{h_i^2}{2 \sigma_{\epsilon x y}} \) completes the proof.

**APPENDIX F**

**PROOF OF THEOREM 2**

The determination of the distribution of distance, azimuth and zenith angles can be posed as a standard RV transformation problem from the polar coordinates, \( R, \) and \( \Theta \) of the 2D case.

With this approach, the RV transformation is written as follows:

\[
D = f(R, u_z, v_z) = \sqrt{R^2 + (u_z - v_z)^2},
\]

\[
\Theta = \Theta_z,
\]

\[
\Psi = g(R, u_z, v_z) = \begin{cases} \pi - \tan\left(\frac{R}{u_z-v_z}\right) & \text{if } u_z \geq v_z \\ \tan\left(\frac{R}{v_z-u_z}\right) & \text{if } u_z < v_z. \end{cases} \tag{64}
\]

Hence, the CDF of the distance and angle distribution can be expressed as follows

\[
F_{D,\Psi,\Theta}(d, \psi, \Theta) = \mathbb{P}\left( \begin{cases} F(R, u_z, v_z) \leq d, \Theta \leq \Theta_z, \Theta_z \leq \psi \\ F(R, u_z, v_z) \leq \psi \end{cases} \right)
\]

\[
= \int_{\Theta_z}^{\Theta} \int_{d}^{\infty} \mathbb{P}(d) \mathbb{P}(\psi) f_{R,\Theta}(r, \Theta) dr d\Theta. \tag{65}
\]

The next step is to isolate the variable \( R \) on the Boolean expressions \( F(d) \) and \( \psi \). Finally, applying the indicator functions over the limits of the integrals, and identifying the resulting expression in terms of the joint CDF of distance and azimuth angle completes the proof.

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