ON MULTIDIMENSIONAL URYSOHN TYPE GENERALIZED
SAMPLING OPERATORS

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ABSTRACT. The concern of this study is to construction of a multidimensional
version of Urysohn type generalized sampling operators, whose one dimensional
case defined and investigated by the author in [28] and [27]. In details, as a
continuation of the studies of the author, the paper centers around to investiga-
tion of some approximation and asymptotic properties of the aforementioned
linear multidimensional Urysohn type generalized sampling operators.

1. Introduction. In the very recent paper [27], to extend the theory of interpo-
lation to operators via generalized sampling series, using an Urysohn-type inter-
polation method, the author introduced an integral operator called Urysohn-type
generalized sampling operators. In the same article, the convergence problem for
this newly defined operators has been studied. Later, the author continued his
research on these operators and obtained their asymptotic properties together with
some Voronovskaya-type theorems in one-dimensional case [28].

The main purpose of this paper consists on the extension of the above results in
the multidimensional frame, whose one dimensional case defined and investigated
by the author in [28], [27]. So, the present study is a continuation and generalization
of the very recent studies of the author (see, e.g., [28], [27], [22]-[26]).

It is important to mention that the theory of the generalized sampling series,
invented and developed at RWTH Aachen by Paul Leo Butzer, is a powerful tool
for investigating and proving the approximation problem on \(\mathbb{R}\). Indeed, from math-
ematical point of view, generalized sampling operators of functions \(f\) defined on the
real axis, which are not necessarily band-limited, were extensively and systematic-
ally studied by Butzer and his collaborators in RWTH Aachen since 1977 (see, e.g.,
[10]-[14], [30]).

Now, we recall some notations and definitions about the multivariate generalized
sampling operators introduced by Butzer, Fischer and Stens [11].

Let \(f\) be a bounded and continuous function defined on \(\mathbb{R}^n\), then multidimen-
sional version of the generalized sampling series \(S_W f\) is given by

\[
(S_W^f)(t) := \frac{1}{(\sqrt{2\pi})^n} \sum_{k \in \mathbb{Z}^n} f \left( \frac{k}{W} \right) \varphi(Wt - k), \quad (t \in \mathbb{R}^n, W > 0),
\]

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where \( \varphi : \mathbb{R}^n \to \mathbb{R} \) is a kernel function satisfying
\[
\varphi \in L^1(\mathbb{R}^n), \quad \sum_{k \in \mathbb{Z}^n} \varphi(u - k) = 1 \text{ for every } u \in \mathbb{R}^n,
\]
(2)
together with
\[
A_\varphi := \sup_{u \in \mathbb{R}^n} \sum_{k \in \mathbb{Z}^n} |\varphi(u - k)| < \infty,
\]
(3)
where the convergence of the series (3) is uniform on each compact subintervals of \( \mathbb{R}^n \).

Especially in the last decade, generalized sampling operators and some of their modifications, such as multivariate versions and Kantorovich type generalizations, have been of great importance in the development of mathematical models for signal and image recovering, as studied by research group (RITA Network) from Perugia led by Bardaro C. and Vinti G. (see, e.g., [1]-[9], [16]-[20]).

Since a detailed model of the one dimensional case of the aforementioned operators has been constructed in [27], now we led to the pass the next step which is to define their multivariate version, and after that investigate approximation and asymptotic properties of these operators. As a continuation of the studies of the author [27] and [28], the present work highlights the construction and importance of multidimensional Urysohn type generalized sampling operators and in modelling the asymptotic approximation properties of functions \( f \) defined on \( \mathbb{R}^n \).

2. Preliminaries and construction of the multidimensional operators.

In this section, we shall introduce some notation and background material used throughout this work.

Throughout this work, we will use the following notations. First of all, the vectors are given in bold-face, namely \( u = (u_1, u_2, \ldots, u_n) \in \mathbb{R}^n \), as we used in the introduction section, and
\[
\|u\|_2 = (u_1^2 + u_2^2 + \ldots + u_n^2)^{1/2}
\]
is the Euclidean norm of this vector on \( \mathbb{R}^n \).

By \( x(s) \) we understand the \( n \)-dimensional vector, namely
\[
x(s) = (x_1(s_1), x_2(s_2), \ldots, x_n(s_n)).
\]

As in the papers [5], [2], [15] and [18], throughout this work, in addition to (2) we assume that the kernel satisfies
\[
\sum_{k \in \mathbb{Z}^n} \varphi(u - k) \equiv \sum_{k_1 \in \mathbb{Z}} \varphi(u_1 - k_1) * \sum_{k_2 \in \mathbb{Z}} \varphi(u_2 - k_2) * \ldots * \sum_{k_n \in \mathbb{Z}} \varphi(u_n - k_n) = 1,
\]
and first two central moments of the multidimensional generalized sampling operators (1) satisfy
\[
m_1(\varphi) : = \sum_{k \in \mathbb{Z}^n} \varphi(u - k)(u - k) = 0,
\]
(4)
\[
m_2(\varphi) : = \sum_{k \in \mathbb{Z}^n} \varphi(u - k)(u - k)^2 = C
\]
for every \( u \in \mathbb{R}^n \) and for a given constant \( C \in \mathbb{R} \).

We also assume that the discrete absolute moment of order \( \beta \) are finite, i.e.,
\[
M_\beta(\varphi) := \sup_{u \in \mathbb{R}^n} \sum_{k \in \mathbb{Z}^n} |\varphi(u - k)| \|u - k\|^\beta_2 < \infty
\]
(5)
for every $u \in \mathbb{R}^n$ and for some $\beta > 0$. Moreover, for every $\delta > 0$ there holds:

$$\lim_{m \to \infty} \sum_{k \in \mathbb{Z}^n, \|mu - k\|_2 > \delta m} |\varphi(mu - k)| \|mu - k\|_2^\beta = 0$$

uniformly with respect to $u \in \mathbb{R}^n$, for a fixed $\beta \in \mathbb{N}$. The formula for $M_\beta(\varphi)$ in case $\beta = 0$ is exactly $A_\varphi$.

Now, we recall the following one dimensional Urysohn integral operators:

$$F(t, x(\cdot)) = \int_0^1 f(t, s, x(s))ds, \quad t \in [0, 1], \quad 0 \leq x(\cdot) \leq 1$$

with unknown kernel $f$, whose properties and values depend on the function $x(\cdot)$ (see, e.g., [21]-[29]).

To define an Urysohn type operator and obtain some positive answers to the approximation problems, in view of the one dimensional case, we consider the multidimensional continuous interpolation conditions hold:

$$F(t, x(\cdot)) = \int_{[0, 1]^n} \cdots \int f(t, s, x(s))ds, \quad t \in [0, 1]^n, \quad -\infty \leq a \leq x(\cdot) \leq b \leq \infty.$$

In view of the multidimensional Urysohn integral operators, we assume that the multidimensional continuous interpolation conditions hold:

$$F(x(t)_i) = \int_{[0, 1]^n} \cdots \int f(t, s, x(s)_i)ds, \quad t \in [0, 1]^n$$

where $x(s)_i = (x_1(s_1)_i, x_2(s_2)_i, ..., x_n(s_n)_i)$ with $-\infty < x(s)_i < \infty$ for each component of $i$, defined as

$$x(s)_i = \frac{i}{m}H(s - \xi); \xi \in [0; 1]^n,$$

for $i = (i_1, i_2, ..., i_n)$ with $i_l = -2, -1, 0, 1, 2, ..., l \in \{1, 2, ..., n\}$. In particular

$$x_1(s_1)_i = \frac{i_1}{m}H(s_1 - \xi_1), \quad x_2(s_2)_i = \frac{i_2}{m}H(s_2 - \xi_2), \quad \vdots$$

$$x_n(s_n)_i = \frac{i_n}{m}H(s_n - \xi_n),$$

for $\xi_1, \xi_2, ..., \xi_n \in [0; 1]$.

Taking into account (7) and (8), by a straightforward calculation the stated identities follow.

$$F\left(\frac{x}{m}H(t - \xi)\right) = \int_{\xi_1}^{1} \int_{\xi_2}^{1} \cdots \int_{\xi_n}^{1} f(t, s, \frac{i_1}{m}H(s_1 - \xi_1), ..., \frac{i_n}{m}H(s_n - \xi_n)ds + \int_{0}^{\xi_1} \int_{\xi_2}^{1} \cdots \int_{\xi_n}^{1} f(t, s, 0, \frac{i_2}{m}H(s_2 - \xi_2), ..., \frac{i_n}{m}H(s_n - \xi_n)ds$$

$$+ \int_{0}^{\xi_1} \int_{\xi_2}^{1} \cdots \int_{\xi_n}^{1} f(t, s, 0, 0, ..., \frac{i_n}{m}H(s_n - \xi_n)ds + \int_{\xi_1}^{1} \int_{0}^{\xi_2} \cdots \int_{\xi_n}^{1} f(t, s, \frac{i_1}{m}H(s_1 - \xi_1), 0, ..., \frac{i_n}{m}H(s_n - \xi_n)ds$$

(9)
and hence, one has from the well-known Leibnitz derivative
\[ \frac{\partial^{\left| n \right|} F \left( \frac{1}{m} H(t - \xi) \right)}{\partial \xi^n} = (-1)^n f(t, \xi_1, ..., \xi_n, \frac{i_1}{m}, \frac{i_2}{m}, ..., \frac{i_n}{m}) + (-1)^{n-1} f(t, \xi_1, ..., \xi_n, 0, \frac{i_2}{m}, ..., \frac{i_n}{m}) + (-1)^{n-2} f(t, \xi_1, \xi_2, ..., \xi_n, 0, 0, ..., 0) \]
where \( m \in \mathbb{N} \), where
\[ f(t, \xi_1, \xi_2, ..., \xi_n, 0, 0, ..., 0) \]
and bounded functions
\[ f(t, \xi_1, ..., \xi_n, \frac{i_1}{m}, \frac{i_2}{m}, ..., \frac{i_n}{m}) \]
are bounded functions.

We therefore introduce the following sequence of multidimensional Urysohn operators of generalized sampling type:

**Definition 1.** Let \( F \) be the Urysohn integral operator of \( f \). Given a continuous and bounded function \( f \) and an arbitrary kernel \( \varphi : \mathbb{R}^n \rightarrow \mathbb{R} \) satisfies (2) and (3). Then the multidimensional Urysohn type generalized sampling operator is defined as:

\[ (US_m F) x(t) := \int_{[0,1]^n} \left[ \sum_{k \in \mathbb{Z}^n} f \left( t, s, \frac{k}{m} \right) \varphi_{k,m} \left( x(s) \right) \right] ds, \] (11)

where \( m \in \mathbb{N} \), \( \varphi_{k,m} (x(s)) = \varphi (mx(s) - k) \) and
\[ \varphi (mx(s) - k) = \varphi (mx_1(s_1) - k_1) \star \varphi (mx_2(s_2) - k_2) \star ... \varphi (mx_n(s_n) - k_n). \] (12)

Here, \( Dom (US_m F) = \mathbb{R} \) and \( Dom (US_m F) \) is the set of all bounded functions \( f : [0,1]^{2n} \times \mathbb{R}^n \rightarrow \mathbb{R}^n \) for which the operator is well defined.

### 3. Convergence property

We now introduce some notations and structural hypotheses, which will be fundamental in proving our convergence theorems.

Let \( C (\mathbb{R}) \) the Banach space of continuous functions \( u : \mathbb{R} \rightarrow \mathbb{R} \) endowed with the norm
\[ \| u \| = \sup \{ |u(x)| : x \in \mathbb{R} \}. \]

**Definition 2.** Let \( f \in C \left( [0,1]^{2n} \times \mathbb{R}^n \right) \) and \( \delta > 0 \) be given. Then the complete multivariate modulus of continuity is given by:

\[ \omega (\delta) = \sup_{\| u - v \|_2 \leq \delta} |f(t, s, u) - f(t, s, v)|. \] (13)

Clearly, if \( \delta = \| u - v \|_2 \), one has
\[ |f(t, s, u) - f(t, s, v)| \leq \omega (\| u - v \|_2) \leq \left( 1 + \frac{\| u - v \|_2}{\delta} \right) \omega (\delta). \] (14)
We are now ready to establish one of the main results of this study:

**Theorem 1.** Let \( F \) be the Urysohn integral operator with \( x(s) = (x_1(s_1), ..., x_n(s_n)) \) with \( -\infty < x_k(s_k) < \infty \) for each component of \( k = 1, 2, ..., n \).

\[
\lim_{m \to \infty} (US_m F)x(t) = Fx(t)
\]

at each point \( x(s) \) of continuity of \( f \).

**Proof.** In view of the definition of the operator (11), by considering (7), (3), (5) and (10), we can prove our theorem as follows.

\[
| (US_m F)x(t) - Fx(t) | = \left| \int_{[0,1]^n} \sum_{k \in \mathbb{Z}^n} \left[ F_1(t, \xi, x(s)) - F_1(t, \xi, \frac{k}{m}) \right] \varphi_{k,m}(x(s)) \, ds \right|.
\]

Let us divide the last term as;

\[
| (US_m F)x(t) - Fx(t) | \leq U_1 + U_2,
\]

where

\[
U_1 = \int_{[0,1]^n} \sum_{k \in \mathbb{Z}^n \mid \|x(s) - \frac{k}{m}\|_2 \leq \delta} \left| F_1(t, \xi, x(s)) - F_1(t, \xi, \frac{k}{m}) \right| \varphi_{k,m}(x(s)) \, ds,
\]

and

\[
U_2 = \int_{[0,1]^n} \sum_{k \in \mathbb{Z}^n \mid \|x(s) - \frac{k}{m}\|_2 > \delta} \left| F_1(t, \xi, x(s)) - F_1(t, \xi, \frac{k}{m}) \right| \varphi_{k,m}(x(s)) \, ds
\]

and \( \delta > 0 \). Since \( x \in C(\mathbb{R}^n) \) is continuity points of \( f \), then there exist \( \delta > 0 \) such that for every \( \epsilon > 0 \)

\[
\left| F_1(t, \xi, x(s)) - F_1(t, \xi, \frac{k}{m}) \right| < \epsilon
\]

holds true when \( \|x(s) - \frac{k}{m}\|_2 \leq \delta \). So one can easily obtain

\[
U_1 < A\varphi \epsilon.
\]

As to the other term

\[
\left| F_1(t, \xi, x(s)) - F_1(t, \xi, \frac{k}{m}) \right| \leq 2M
\]

holds true for some \( M > 0 \), when \( \|x(s) - \frac{k}{m}\|_2 \leq \delta \).

\[
U_2 \leq 2M \int_{[0,1]^n} \sum_{k \in \mathbb{Z}^n \mid \|x(s) - \frac{k}{m}\|_2 > \delta} \varphi_{k,m}(x(s)) \, ds
\]

\[
\leq 2M \frac{M_2(\varphi)}{\delta^2 m^2}.
\]

Collecting these estimates we have

\[
\lim_{m \to \infty} (US_m F)x(t) = Fx(t).
\]

This completes the proof. \( \square \)
Using Weierstrass criterion, as a consequence of the Theorem 1 we can establish the following uniform convergence theorem.

**Theorem 2.** Let $F$ be the Urysohn integral operator of the function $f(t, s, x(s)) \in C\left([0,1]^{2n} \times \mathbb{R}^n\right)$ with $x(s) = (x_1(s_1), \ldots, x_n(s_n))$ satisfying $-\infty < x_k(s_k) < \infty$ for each component of $k = 1, 2, \ldots, n$. Then $(US_m F)$ converges to $F$ uniformly in $x \in C(\mathbb{R}^n)$. That is

$$\lim_{m \to \infty} \left\| (US_m F)x(t) - Fx(t) \right\|_{C(\mathbb{R}^n)} = 0.$$ 

Now, we want to study the order of approximation in $C\left([0,1]^{2n} \times \mathbb{R}^n\right)$ by using the complete multivariate modulus of continuity given in (13).

**Theorem 3.** Let $F$ be the Urysohn integral operator of the function $f(t, s, x(s)) \in C\left([0,1]^{2n} \times \mathbb{R}^n\right)$ with $x(s) = (x_1(s_1), \ldots, x_n(s_n))$ satisfying $-\infty < x_k(s_k) < \infty$ for each component of $k = 1, 2, \ldots, n$. Then

$$\left| (US_m F)x(t) - Fx(t) \right| \leq 2\omega(\delta)$$

holds true, where $\delta = \frac{M_1(\varphi)}{m^2}$.

**Proof.** Clearly one has

$$\left| (US_m F)x(t) - Fx(t) \right| \leq \int \cdots \int_{[0,1]^n} \left| F_1(t, \xi, x(s)) - F_1(t, \xi, \frac{k}{m}) \right| \left| \varphi_{k,m}(x(s)) \right| ds$$

say. Since $x \in C(\mathbb{R}^n)$, in view of (14) we can re-write (15) as follows:

$$\left| (US_m F)x(t) - Fx(t) \right| \leq \int \cdots \int_{[0,1]^n} \left| F_1(t, \xi, x(s)) - F_1(t, \xi, \frac{k}{m}) \right| \left| \varphi_{k,m}(x(s)) \right| ds$$

$$\leq \int \cdots \int_{[0,1]^n} \sum_{k \in \mathbb{Z}^n} \left( 1 + \frac{\|x(s) - \frac{k}{m}\|}{\delta} \right) \omega(\delta) \left| \varphi_{k,m}(x(s)) \right| ds$$

$$\leq \omega(\delta) + \frac{\omega(\delta)}{\delta m^2} M_1(\varphi)$$

$$= 2\omega(\delta).$$

Thus the proof is now complete. \hfill \Box

4. Asymptotic expansion and Voronovskaya-type theorems. This section deals with an asymptotic formula and some Voronovskaya type theorems for the Urysohn type generalized sampling operators.

**Theorem 4.** Let $F$ be the Urysohn integral operator of the bounded function $f : [0,1]^{2n} \times \mathbb{R}^n \to \mathbb{R}$. Moreover, for a certain $r \in \mathbb{N}$, we also assume that $\frac{\partial^{|\alpha|}}{\partial x^{|\alpha|}} f(t, s, x(s))$ exists at a fixed point $x(s)$. Then the following asymptotic formula holds:

$$(US_m F)x(t) = Fx(t) + \sum_{v=1}^{r} \sum_{|\alpha|=v} \frac{\partial^{|\alpha|}}{\partial x^{|\alpha|}} f(t, s, x(s)) \lim_{m \to \infty} m_1(\varphi_{k,m}(x(s))) + o(m^{-r})$$
as $m \to \infty$, where $m_1$ is the $i$-th order moment.

Proof. Since $\frac{\partial^{|I|}}{\partial t^{|I|}} f (t, s, x(s))$ exists at a fixed point $x(s)$, then there exists a bounded function $h$ such that $\lim_{y \to 0} h(t, s, y) = 0$.

By the multivariate version of the local Taylor’s formula, we have

$$f \left( t, s, \frac{k}{m} \right) = f \left( t, s, x(s) \right) + \sum_{v=1}^r \sum_{|i|=v} \frac{\partial^{|I|} f (t, s, x(s))}{i!} \left( \frac{k}{m} - x(s) \right)^i + h \left( t, s, \frac{k}{m} - x(s) \right) \left\| \frac{k}{m} - x(s) \right\|_2^r,$$

where

$$i! := i_1! i_2! \cdots i_n!.$$

In view of (11) and (16), we can write

$$(US_m F) x(t) = \int_{[0,1]^n} \left[ \sum_{k \in \mathbb{Z}^n} f \left( t, s, \frac{k}{m} \right) \varphi_{k,m} (x(s)) \right] ds.$$

$$= \int_{[0,1]^n} \left[ \sum_{k \in \mathbb{Z}^n} \left( f \left( t, s, x(s) \right) + \sum_{v=1}^r \sum_{|i|=v} \frac{\partial^{|I|} f (t, s, x(s))}{i!} \left( \frac{k}{m} - x(s) \right)^i \right) \varphi_{k,m} (x(s)) \right] ds.$$

Let us analyze the terms $I_1$ and $R$, respectively. Let us consider the term $I_1$.

$$I_1 = \int_{[0,1]^n} \left[ \sum_{k \in \mathbb{Z}^n} \left( f \left( t, s, x(s) \right) + \sum_{v=1}^r \sum_{|i|=v} \frac{\partial^{|I|} f (t, s, x(s))}{i!} \left( \frac{k}{m} - x(s) \right)^i \right) \varphi_{k,m} (x(s)) \right] ds.$$

For what concern the remainder term $R$.

Let $C > 0$ be a constant such that $|h(t, s, \frac{k}{m} - x(s))| \leq C$, and let $\varepsilon > 0$ be fixed.

Since $h(t, s, y)$ is a bounded function such that $\lim_{y \to 0} h(t, s, y) = 0$, there exists $\delta > 0$ such that $|h(y)| \leq \varepsilon$ for every $|y| \leq \delta$. In view of the assumptions (5) and
Theorem 6. Let
\[ R = \int_{[0,1]^n} \left( \sum_{k \in \mathbb{Z}^n} h \left( t, s, \frac{k}{m} - x(s) \right) \left\| \frac{k}{m} - x(s) \right\|_2 \varphi_{k,m}(x(s)) \right) ds \]
\[ = \int_{[0,1]^n} \left( \sum_{k \in \mathbb{Z}^n} h \left( t, s, \frac{k}{m} - x(s) \right) \left\| \frac{k}{m} - x(s) \right\|_2 \varphi_{k,m}(x(s)) \right) ds \]
\[ + \int_{[0,1]^n} \left( \sum_{k \in \mathbb{Z}^n} h \left( t, s, \frac{k}{m} - x(s) \right) \left\| \frac{k}{m} - x(s) \right\|_2 \varphi_{k,m}(x(s)) \right) ds \]
\[ \leq \varepsilon m^{-r} M_r(\varphi) + C m^{-r} \varepsilon \]
\[ = o \left( m^{-r} \right). \]

As a consequence of Theorem 4, we can establish the following first and second order Voronovskaya type theorems, respectively.

Theorem 5. Let \( F \) be the Urysohn integral operator of the bounded function \( f : [0,1]^{2n} \times \mathbb{R}^n \rightarrow \mathbb{R} \). Moreover, we also assume that \( \frac{\partial}{\partial x^i} f(t,s,x(s)) \) exists at a fixed point \( x(s) \). Then we have:
\[
\lim_{m \to \infty} m \left[ (US_m F)(t) - Fx(t) \right] = \sum_{|i|=1} \frac{\partial}{\partial x^i} f(t,s,x(s)) m_i (\varphi_{k,m}(x(s))) ,
\]
where \( m_i \) is the \( i \)-th order moment.

Proof. Applying the asymptotic formula of Theorem 4 with \( r = 1 \), and using (16) and assumption (4), we can write:
\[
(US_m F)(t) = Fx(t) + \sum_{i=1}^{2} \sum_{|i|=v} \frac{\partial}{\partial x^i} f(t,s,x(s)) \frac{m_i}{i! m^i} (\varphi_{k,m}(x(s))) + o \left( m^{-1} \right), \quad (m \to \infty).
\]
Then the proof follows by passing to the limit for \( m \to \infty \).

Similarly to above, the following second order Voronovskaya type theorem can be established.

Theorem 6. Let \( F \) be the Urysohn integral operator of the bounded function \( f : [0,1]^{2n} \times \mathbb{R}^n \rightarrow \mathbb{R} \). Moreover, we also assume that \( \frac{\partial^2}{\partial x^i} f(t,s,x(s)) \) exists at a fixed point \( x(s) \). Then we have:
\[
\lim_{m \to \infty} m^2 \left[ (US_m F)(t) - Fx(t) \right] = \sum_{i=1}^{2} \sum_{|i|=v} \frac{\partial}{\partial x^i} f(t,s,x(s)) \frac{m_i}{i! m^i} (\varphi_{k,m}(x(s))) ,
\]
where \( m_i \) is the \( i \)-th order moment.

Proof. As in the proof of Theorem 5, applying the asymptotic formula of Theorem 4 with \( r = 2 \), and using (16) and assumption (4), we can write:
\[
(US_m F)(t) = Fx(t) + \sum_{i=1}^{2} \sum_{|i|=v} \frac{\partial}{\partial x^i} f(t,s,x(s)) \frac{m_i}{i! m^i} (\varphi_{k,m}(x(s))) + o \left( m^{-2} \right), \quad (m \to \infty).
\]
Then the proof follows by passing to the limit for \( m \to \infty \).
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