Quantum Information Processing with Low-Dimensional Systems

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Abstract

A ‘register’ in quantum information processing — is composition of \( k \) quantum systems, ‘qudits’. The dimensions of Hilbert spaces for one qudit and whole quantum register are \( d \) and \( d^k \) respectively, but we should have possibility to prepare arbitrary entangled state of these \( k \) systems. Preparation and arbitrary transformations of states are possible with universal set of quantum gates and for any \( d \) may be suggested such gates acting only on single systems and neighbouring pairs. Here are revisited methods of construction of Hamiltonians for such universal set of gates and as a concrete new example is considered case with qutrits. Quantum tomography is also revisited briefly.

1 Introduction

Discrete quantum variables — are basic resource in quantum computing. A qubit is described by two-dimensional Hilbert space and systems with higher dimensions are also widely used [1].

Quantum mechanics with continuous variables may be more understanding due to a correspondence principle. For example, after change of classical momentum \( q \) and coordinate \( p \) to quantum operators \( \hat{q}, \hat{p} \) in simple Hamiltonians we almost directly may produce correct quantum description.

On the other hand, it is impossible to introduce the \( \hat{p}, \hat{q} \) operators for system with finite-dimensional Hilbert space. Even if for large dimensions \( d \gg 2 \) the continuous case could be used as an approximate model of a discrete system, it does not seem possible for low dimensions.
In 1928 Weyl suggested a method of quantization, appropriate both for finite and infinite-dimensional case [2]. The basic idea — is to use instead of operators of coordinate $\hat{q}$ and momentum $\hat{p}$ they exponents with pure imaginary multipliers and instead of Heisenberg commutation relations to write Weyl system

$$\hat{U} = e^{i\alpha \hat{p}}, \quad \hat{V} = e^{i\beta \hat{q}}, \quad \hat{U} \hat{V} = e^{i\alpha \beta} \hat{V} \hat{U}. \quad (1)$$

An analogue of Weyl-Heisenberg commutation relations Eq. (1) may be written also for discrete quantum variables like qubits. It is recollected in Sec. 2 Eqs. (3,4). Due to relevance of considered scheme for finite-dimensional case with spin-1/2 systems (so-called Jordan-Wigner representation) Weyl wrote:

“Because of these results I feel certain that the general scheme of quantum kinematics formulated above is correct. But the field of discrete groups offers many possibilities which we have not as yet been able to realize in Nature; ” . . .

Nowadays, due to many applications of the Weyl pair Eq. (3) in quantum computations, error correction, cryptography and tomography the note about many possibilities in the field of discrete groups looks quite justified.

In the quantum information processing are used many entangled systems and in Sec. 3 is considered specific constructions with tensor product of Weyl matrices. Due to regular algebraic structure, it is convenient to use such operators for construction of nonlocal Hamiltonians for universal sets of quantum gates in any dimension. In Sec. 4 are presented methods of construction of the sets for any $d \geq 2$ together with an example of Hamiltonians for qutrits.

Quantum tomography describes effective measurement procedures for quantum systems and ensembles. Weyl pair is also useful tool in this area. It is discussed briefly in Sec. 5

\section{2 Pauli and Weyl matrices}

The Pauli matrices $\hat{\sigma}_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $\hat{\sigma}_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$, $\hat{\sigma}_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ with property

$$\hat{\sigma}_\nu \hat{\sigma}_\mu + \hat{\sigma}_\mu \hat{\sigma}_\nu = 2\delta_{\mu\nu}, \quad (2)$$
may be generalized for \(d > 2\) using Weyl pair, i.e., two \(d \times d\) matrices \[2\]

\[
\hat{U} = \begin{pmatrix}
0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
1 & 0 & 0 & \cdots & 0 \\
\end{pmatrix}, \quad \hat{V} = \begin{pmatrix}
1 & 0 & 0 & \cdots & 0 \\
0 & \zeta & 0 & \cdots & 0 \\
0 & 0 & \zeta^2 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & \zeta^{d-1} \\
\end{pmatrix}
\]

with property

\[
\hat{U}\hat{V} = \zeta\hat{V}\hat{U}, \quad \zeta^d = 1, \quad \zeta = e^{2\pi i/d}. \tag{4}
\]

In quantum information processing the matrices Eq. (3) were widely used at first in theory of quantum error correction codes \[1, 3\].

There are different ways to introduce analogues of three Pauli matrices, e.g.,

\[
\hat{X} = \hat{U}, \quad \hat{Y} = \zeta^{(d-1)/2}\hat{U}\hat{V}, \quad \hat{Z} = \hat{V}. \tag{5}
\]

\[
\hat{X}^d = \hat{Y}^d = \hat{Z}^d = \hat{1}, \quad \hat{X}\hat{Y} = \zeta\hat{Y}\hat{X}, \quad \hat{Y}\hat{Z} = \zeta\hat{Z}\hat{Y}, \quad \hat{X}\hat{Z} = \zeta\hat{Z}\hat{X}. \tag{6}
\]

For \(d = 2\) with \(\zeta = \zeta^{-1} = -1\) we have \(\hat{X} = \hat{\sigma}_x, \quad \hat{Y} = \hat{\sigma}_y, \quad \hat{Z} = \hat{\sigma}_z\), and Eq. (6) is reduced to Eq. (2). For \(d > 2\) and \(\zeta \neq \zeta^{-1}\) it is necessary to remember about an order (e.g., \(\hat{Z}\hat{X} = \zeta^{-1}\hat{X}\hat{Z}\)).

### 3 Systems with \(n\) qudits

Hilbert space of the system with \(n\) qudits \((d = \dim\mathcal{H}_d \geq 2)\) is tensor product with \(n\) terms \(\mathcal{H}_d^{\otimes n} = \mathcal{H}_d \otimes \cdots \otimes \mathcal{H}_d\). Let us introduce family with \(2n\) operators

\[
\hat{\xi}_{2k-1} = \underbrace{\hat{Z} \otimes \cdots \otimes \hat{Z}}_{k-1} \otimes \hat{X} \otimes \underbrace{\hat{1} \otimes \cdots \otimes \hat{1}}_{n-k},
\]

\[
\hat{\xi}_{2k} = \underbrace{\hat{Z} \otimes \cdots \otimes \hat{Z}}_{k-1} \otimes \hat{Y} \otimes \underbrace{\hat{1} \otimes \cdots \otimes \hat{1}}_{n-k}. \tag{7}
\]

For any given dimension \(d \geq 2\) operators \(\hat{\xi}_k, \quad k = 1, \ldots, 2n\) have properties

\[
\hat{\xi}_j^d = \hat{1}, \quad \hat{\xi}_j\hat{\xi}_k = \zeta\hat{\xi}_k\hat{\xi}_j, \quad j < k, \quad \zeta = e^{2\pi i/d}. \tag{8}
\]

\[
(a_1\hat{\xi}_1 + a_2\hat{\xi}_2 + \cdots + a_{2n}\hat{\xi}_{2n})^d = a_1^d + a_2^d + \cdots + a_{2n}^d. \tag{9}
\]
For $d = 2$ Eqs. (8,9) define generators of the Clifford algebra $\mathcal{C}l(2n)$ [4]

$$\hat{x}_j \hat{x}_k + \hat{x}_k \hat{x}_j = 2 \delta_{jk}, \quad (a_1 \hat{x}_1 + \cdots + a_{2n} \hat{x}_{2n})^2 = a_1^2 + \cdots + a_{2n}^2. \quad (10)$$

For generalized case $d > 2$ Eq. (8) define an algebra of the quantum plane $A^2n_\zeta$ [5].

4 Universality

The elements described in Sec. 2,3 let construct Hamiltonians for universal set of quantum gates with simple methods of decomposition and useful properties:

1. It is set of one- and two-qudit gates (a gate for given $\hat{H}$ is $\hat{G}^\tau = e^{-i\hat{H}^\tau}$).

2. Two-gates are acting on pairs of neighbouring systems (qudits).

3. Hamiltonians of the two-qudit gates are diagonal.

Basic idea [6,7,8] — is to start with elements $\hat{x}_k \hat{x}_{k+1}^\dagger$ and use them for construction of Hamiltonians of one- and two-qudit gates. In proof of universality are used elements generated via commutators [9,10], but due to Eq. (8) they always have property $[\hat{A}, \hat{B}] = (1 - \zeta^l) \hat{A} \hat{B}$ with an integer $l$ and it produces some simplification.

It is useful also to exchange $\hat{X} \leftrightarrow \hat{Z}^\dagger$ in Eq. (11) and define elements

$$\hat{\delta}_{2k-1}^\dagger = \hat{X}^{\otimes (k-1)} \otimes \hat{Z} \otimes \hat{1}^{\otimes (n-k)}, \quad \hat{\delta}_{2k}^\dagger = \hat{X}^{\otimes (k-1)} \otimes \hat{Y} \otimes \hat{1}^{\otimes (n-k)} \quad (11)$$

to make two-qudit operators, like $\hat{Z}_k^\dagger \hat{Z}_{k+1}$ in Eq. (12) below, diagonal

$$\hat{\delta}_{2k-1} \hat{\delta}_{2k}^\dagger = \hat{X}_k, \quad \hat{\delta}_{2k} \hat{\delta}_{2k+1}^\dagger = \hat{Z}_k^\dagger \hat{Z}_{k+1}. \quad (12)$$

Here is used a brief notation $\hat{X}_k \equiv \hat{1}^{\otimes (k-1)} \otimes \hat{X} \otimes \hat{1}^{\otimes (n-k)}$, etc.

• Qubits. $\hat{X} \equiv \hat{\sigma}_x$, $\hat{Z} \equiv \hat{\sigma}_z$. The elements Eq. (12) are Hermitian and may be used as Hamiltonians. The Hamiltonians generate only subgroup of SU($2^n$) and this subgroup is isomorphic with Spin($2n$) [6], i.e., has only quadratic dimension. It is the demonstration of important class of nonuniversal gates and has analogues both in optical realizations [11] and in “fermionic” implementations [12].
Theorem I Hamiltonians $\hat{X}_k$ of one-qubit gates together with diagonal Hamiltonians $\hat{Z}_k^\dagger \hat{Z}_{k+1}$ of two-qubit gates are not universal: they generate only quadratic subgroup of $\text{SU}(2^n)$ isomorphic to $\text{Spin}(2n)$.

It is enough for universality to add two Hamiltonians of one-qubit gates
\[ \hat{\mathcal{H}}_1 = \hat{Z}_1, \quad \hat{\mathcal{H}}_2 = \hat{Z}_2. \] (13)

Theorem II Hamiltonians $\hat{X}_k, \hat{Z}_1, \hat{Z}_2$ of one-qubit gates together with diagonal Hamiltonians $\hat{Z}_k^\dagger \hat{Z}_{k+1}$ of two-qubit gates generate universal set of quantum gates.

- Qu-dits. For $d > 2$, $\hat{Z}_k$ and $\hat{Z}_k^\dagger$ are not Hermitian, but it is enough to split each term on two Hermitian parts [7].

Theorem III Hamiltonians $\hat{Z}_1 + \hat{Z}_1^\dagger$, $i(\hat{Z}_1 - \hat{Z}_1^\dagger)$, $\hat{X}_k + \hat{X}_k^\dagger$, $i(\hat{X}_k - \hat{X}_k^\dagger)$ of one-qubit gates together with diagonal Hamiltonians $\hat{Z}_k^\dagger \hat{Z}_{k+1}^\dagger + \hat{Z}_{k+1} \hat{Z}_k^\dagger$, $i(\hat{Z}_k^\dagger \hat{Z}_{k+1}^\dagger - \hat{Z}_{k+1} \hat{Z}_k^\dagger)$ of two-qudits gates generate universal set of quantum gates in $\text{SU}(d^n)$.

- Qutrits. Let us consider the Hamiltonians for simplest case of qutrit. Initial (non-Hermitian) matrices here
\[
\hat{X} = \hat{U} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \quad \hat{Z} = \hat{V} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & \omega & 0 \end{pmatrix}, \quad \omega = e^{2\pi i/3}. \] (14)

Let us construct universal set of quantum gates using elements from Th. [III] and they linear combinations. An example of the Hamiltonians for one-qutrit gates:
\[
\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & i & -i \\ -i & 0 & i \\ i & -i & 0 \end{pmatrix}. \] (15)

It is also possible to use only one two-qutrits Hamiltonian
\[
\hat{H}_d = |0\rangle\langle 0| \otimes |0\rangle\langle 0| + |1\rangle\langle 1| \otimes |1\rangle\langle 1| + |2\rangle\langle 2| \otimes |2\rangle\langle 2| \] (16)
for each neighbouring pair together with one-qutrit gate $\hat{X}$ instead of two Hamiltonians $\hat{Z}_k^\dagger \hat{Z}_{k+1}^\dagger + \hat{Z}_{k+1} \hat{Z}_k^\dagger$, $i(\hat{Z}_k^\dagger \hat{Z}_{k+1}^\dagger - \hat{Z}_{k+1} \hat{Z}_k^\dagger)$ from Th. [III].
5 Quantum tomography

It is useful also to recollect briefly methods of quantum tomography related with operators introduced above. Let we have unlimited source of quantum systems with unknown state described by a density matrix \( \hat{\rho} \). A simple set of measurement devices may be described by projectors \( \hat{P}_k = |\phi_k\rangle\langle\phi_k| \). Such device produces “click” with probability

\[
p_k = \text{Tr}(\hat{P}_k\hat{\rho}) = |\langle\phi_k|\hat{\rho}|\phi_k\rangle|.
\] (17)

Which sets of vectors \(|\phi_k\rangle \in \mathcal{H}_d\) are necessary for complete reconstruction of any density matrix, if all probabilities \(p_k\) Eq. (17) are estimated after sufficiently large series of measurements? In general, a density matrix may be described by \(d^2 - 1\) real parameters and it corresponds to minimal amount of such vectors.

For \(d\) is power of prime number \(d = p^m\) exist especial symmetric sets based on \(d + 1\) mutually unbiased bases (MUB) [13]. Such terminology is used because for any two vectors in different bases is true \(|\langle\phi|\varphi\rangle|^2 = 1/d\). The construction for power \(m > 1\) intensively uses theory of Galois fields [13], but if \(d\) itself is prime, there is quite visual model [14]:

**Theorem IV** If dimension \(d\) is prime number, the eigenvectors of \(d + 1\) matrices \(\hat{Z}, \hat{X}, \hat{X}\hat{Z}, \ldots, \hat{X}^{d-1}\hat{Z}\) produce MUB.

Eigenvectors of matrix \(\hat{Z}\) is simply computational basis \(\delta_{kl}\) and \(d^2\) eigenvectors of other \(d\) matrices \(\hat{X}\hat{Z}^n\) have components \(\phi_k = \frac{1}{\sqrt{d}} e^{2\pi i (ak^2 + bk)/d}\), i.e., each such vector is described by two fixed numbers \(a, b = 0, \ldots, d - 1\).

For qutrit it is four matrices \(\hat{Z}, \hat{X}, \hat{X}\hat{Z}, \hat{X}^2\hat{Z}\) with 12 eigenvectors

\[
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}, \frac{1}{\sqrt{3}} \begin{bmatrix}
1 & 1 & 1 \\
1 & \omega & \bar{\omega} \\
1 & \bar{\omega} & \omega
\end{bmatrix}, \frac{1}{\sqrt{3}} \begin{bmatrix}
1 & 1 & 1 \\
\omega & \bar{\omega} & 1 \\
\bar{\omega} & \omega & 1
\end{bmatrix}, \frac{1}{\sqrt{3}} \begin{bmatrix}
1 & 1 & 1 \\
\bar{\omega} & 1 & \omega \\
\omega & 1 & \bar{\omega}
\end{bmatrix}.
\] (18)

For tomography of arbitrary system it is always possible to use representation of Hilbert space as tensor product of prime dimensions, but at least for power of prime such procedure is not optimal [13, 14].

The MUB is yet not maximally symmetric, because scalar product for elements in different bases is nonzero, but in the same basis all vectors are orthogonal and any scalar product is null.
SIC-POVM conjecture [15] suggests existence of other symmetric sets: in any dimension \(d\) exist \(d^2\) vectors with property \(|\langle \phi | \varphi \rangle|^2 = 1/(d + 1)\) for any two vectors and all the vectors may be produced from a single vector \(|\phi \rangle \mapsto \hat{X}^a \hat{Z}^b |\phi \rangle\). It is interesting, that here again is used \(\hat{X}, \hat{Z}\) pair.

6 Conclusion

In quantum information processing together with qubits may be used systems with higher dimension of Hilbert space (qudits) and continuous quantum variables [16]. Any dimension may have specific properties, say for purpose of quantum tomography it is useful to distinguish case of prime dimension \((d = 2, 3, 5, 7, 11, \ldots)\), power of prime \((d = 4, 8, 9, \ldots)\), and “general” case \((d = 6, 10, 12, \ldots)\).

On the other hand, there are general methods discussed above for the work with systems in any dimension. Despite of obvious difference between quantum system with low dimension and continuous limit, there are some tools like Weyl pair, that may provide with useful constructions and hints in many cases.

References

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