Coherence and Josephson oscillations between two tunnel-coupled one-dimensional atomic quasicondensates at finite temperature

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We revisit the theory of tunnel-coupled atomic quasicondensates in double-well elongated traps at finite temperatures. Using the functional integral approach, we calculate the relative phase correlation function beyond the harmonic limit of small fluctuations of the relative phase and its conjugate relative-density variable. We show that the thermal fluctuations of the relative phase between the two quasicondensates decrease the frequency of Josephson oscillations and even wash out these oscillations for small values of the tunnel coupling.

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I. INTRODUCTION

Systems of ultracold bosonic atoms in two parallel atomic waveguides mutually coupled via quantum tunneling (so-called extended bosonic Josephson junctions) have been a subject of intensive theoretical \cite{10,11,12} and experimental \cite{8} studies. The finite spatial extension of these systems provides much richer physics compared to the case of a point-like bosonic Josephson junction \cite{9}. The novel features arise due to the enhanced role of noise and correlations in low-dimensional ultracold atomic systems.

Before discussing the effects of tunneling, we recall the basic properties of a bosonic system in an isolated waveguide \cite{10,12}. This system is effectively one-dimensional (1D), if the interaction energy per atom (we assume interatomic repulsion characterized by the effective 1D coupling strength \( g > 0 \)) and the temperature are well below the spacing between the discrete energy levels of the potential of tight radial confinement. In this case quantum degeneracy does not lead to establishment of the long-range order; instead, atoms form a quasicondensate, i.e., a system describable by a macroscopic wave function with strong phase fluctuations. The characteristic length of the phase coherence in a quasicondensate at finite temperature \( T \) is \( \lambda_T = 2\hbar^2 n_{1D} / (mk_B T) \), where \( m \) is the atomic mass, and \( n_{1D} \) is the mean linear density of atoms \cite{10} (we assume an infinite system; thermodynamic limit implies constant \( n_{1D} = N / L \) while both the atom number \( N \) and the quantization length \( L \) tend to infinity). The power-law decrease of the single-particle correlation function takes place only at \( T = 0 \).

If two waveguides are tunnel-coupled, the system is described by the generalized Hamiltonian

\[
\hat{H} = \int dz \left[ \sum_{j=1}^{2} \left( \frac{\hbar^2}{2m} \frac{\partial^2 \hat{q}_j}{\partial z^2} + \frac{g}{2} \frac{\partial \hat{q}_j}{\partial z} \hat{q}_j \hat{p}_j - \mu \hat{p}_j \hat{q}_j \right) - \hbar J \left( \hat{q}_1 \hat{p}_2 + \hat{q}_2 \hat{p}_1 \right) \right],
\]

(1)

where \( \hat{q}_j \) is the atomic annihilation operator for the \( j \)th waveguide \((j = 1, 2)\), \( \mu = \hbar gn_{1D} - \hbar J \) is the chemical potential and \( 2J \) is the tunnel splitting (in frequency units), i.e., the frequency interval between the two lowest eigenstates of the radial trapping Hamiltonian (the antisymmetric and symmetric superpositions of the single-atom states localized in either \( j = 1 \) or \( j = 2 \) wells of the double-well Hamiltonian). In this case the situation changes qualitatively: the tunnel coupling mutually locks phase fluctuations in the two quasicondensates \cite{1}. Phase locking (as we shall quantify later, in Sec. III) means that the distribution of the relative phase between the two quasicondensates becomes peaked around zero, while the local phase of an individual \((j = 1 \text{ or } 2)\) quasicondensate remains fully random (the phase-density representation for quasicondensates will be discussed in Sec. III). In the spatial correlation of the local relative phase between two quasicondensates a new length parameter appears \cite{1,3},

\[
l_J = \sqrt{\hbar /(4mJ)}.
\]

(2)

The length \( l_J \) sets the scale of restoration of the inter-waveguide coherence due to finite tunnel-coupling strength \( J \). The tunnel-coupling strength is usually estimated from the single-particle energy (kinetic and potential) and the overlap in the potential barrier region of the wave functions for a particle localized in the 1st and 2nd waveguide. However, it is also possible to take into account atomic interactions, see Ref. \cite{6} and references therein.

Experimentally, the interwell coherence can be observed by releasing the two quasicondensates from the trap and measuring locally the contrast and the phase of their interference pattern after time of flight \cite{13,14}. Up to now, only the theory based on linearization of the Hamiltonian \cite{11} has been developed \cite{11} and applied to the analysis of the experimental data \cite{8}. Our work is aimed to develop a model of the steady-state thermal noise in tunnel-coupled quasicondensates beyond the harmonic approximation as well as to quantify the influence of the thermal noise to the macroscopic coherent dynamics of the system (Josephson oscillations).

Our paper is organized as follows. In Sec. III we summarize the harmonic approach of Ref. \cite{11}. Section III is...
II. HARMONIC APPROXIMATION

Following the standard procedure [11], we represent our atomic field operators through the phase \(\hat{\theta}_j (z)\) and density \(\hat{\rho}_j (z)\) operators, obeying the commutation relation \([\hat{\theta}_j (z), \hat{\rho}_j (z')] = -i\delta (z - z') \delta_{jj'}\), as

\[\hat{\psi}_j (z) = \exp [i \hat{\theta}_j (z)] \sqrt{\hat{\rho}_j (z)}, \quad j = 1, 2.\]  

A discussion of the way to introduce the phase operator for quasicondensates by coarse graining a lattice model on length scales containing sufficiently many atoms can be found in Ref. [11]. The density operator can be represented as \(\hat{\rho}_j (z) = n_{1D} + \hat{\rho}_j (z)\). Since for quantum gases with repulsive atomic interactions density fluctuations are suppressed, we can always consider the corresponding operator \(\hat{\rho}_j (z)\) as a small correction. However, the same is not always true for the phase fluctuations.

Whitlock and Bouchoule [11] from the very beginning assumed the phase fluctuations to be small and thus linearized the Hamiltonian Eq. (1) reducing it to \(\hat{H} \approx \hat{H}_{lin}\),

\[\hat{H}_{lin} = \int dz \left[ \frac{\hbar^2 n_{1D}}{m} \left( \frac{\partial \hat{\theta}_s}{\partial z} \right)^2 + \frac{\hbar^2}{16 m n_{1D}} \left( \frac{\partial \hat{\rho}_s}{\partial z} \right)^2 + \right.\]

\[\frac{g_2}{4} \hat{\rho}_s^2 + \frac{\hbar^2 n_{1D}}{4m} \left( \frac{\partial \hat{\theta}_a}{\partial z} \right)^2 + \frac{\hbar^2}{4 m n_{1D}} \left( \frac{\partial \hat{\rho}_a}{\partial z} \right)^2 + \]

\[g \hat{\rho}_s^2 + \hbar J n_{1D} \hat{\rho}_a^2 \].  

Here the symmetric (s) and antisymmetric (a) variables are introduced via canonical transformation

\[\hat{\phi}_s (z) = \hat{\phi}_1 (z) + \hat{\phi}_2 (z), \quad \hat{\phi}_a (z) = [\hat{\theta}_1 (z) + \hat{\theta}_2 (z)]/2, \]

\[\hat{\phi}_a (z) = [\hat{\phi}_1 (z) - \hat{\phi}_2 (z)]/2, \quad \hat{\theta}_a (z) = \hat{\theta}_1 (z) - \hat{\theta}_2 (z).\]

Diagonalization of the Hamiltonian [11] is based on the Fourier transform \(\hat{\theta}_s(a)(z) = L^{-1/2} \sum_{k \neq 0} \hat{\phi}_s(a)(k)e^{ikz}\),

\[\hat{\theta}_s(a)(z) = L^{-1/2} \sum_{k \neq 0} \hat{\phi}_s(a)(k)e^{ikz}.\]  

The frequencies \(\omega_s(a)(k)\) of the symmetric and antisymmetric modes with the momentum \(\hbar k\) are given by the dispersion relations

\[\omega_s^2 (k) = \frac{\hbar k^2}{2m} \left( \frac{\hbar k^2}{2m} + 2 g n_{1D} \right),\]

\[\omega_a^2 (k) = \frac{\hbar k^2}{2m} + 2J \left( \frac{\hbar k^2}{2m} + 2 J + 2 g n_{1D} \right).\]  

Correlations in two tunnel-coupled quasicondensates are experimentally accessible via the two-point correlation function \(g^2 (z - z') = n_{1D} \langle \hat{\psi}_1 (z) \hat{\psi}_2 (z') \hat{\psi}_2 (z) \hat{\psi}_1 (z') \rangle\).

Since the system described by the Hamiltonian [11] is translationally invariant, \(g^2\) depends only on the difference of the two co-ordinates. The symbol \(\langle \cdot \hat{O} \cdot \rangle\) denotes the average of the normal ordered (with respect to the atomic operators \(\hat{\psi}, \hat{\psi}^\dagger\)) form of the operator \(\hat{O}\). In what follows, we omit the normal ordering notation, thus neglecting the atomic shot noise.

Since the density fluctuations for \(|k| \lesssim \xi^{-1}, \xi = \hbar/\sqrt{mg n_{1D}} = \hbar/(mc)\) being the healing length, are suppressed by the atomic repulsion [10, 11], the main contribution to this correlation function is given by the phase fluctuations, \(g^2 (z - z') \approx \langle \exp [i \hat{\theta}_s (z') - i \hat{\theta}_s (z)] \rangle\).

The experimentally accessible length scale cannot be shorter than the optical resolution length \(\Delta_{zopt}\). On this scale the shot noise yields the quantum uncertainty of the relative phase, coarse grained over the distance \(\Delta_{zopt}\), of the order of \(1/\sqrt{2 n_{1D} \Delta_{zopt}}\). For \(\Delta_{zopt} \gtrsim 3 \mu m\) and \(n_{1D} \gtrsim 30 \mu m^{-1}\) the shot-noise induced phase uncertainty does not exceed 0.075 rad. This relatively small value can be always kept in mind when comparing theoretical predictions to measurement results. However, for the sake of simplicity, in what follows we assume \(\langle \exp [i \hat{\theta}_s (z') - i \hat{\theta}_s (z)] \rangle \approx \langle \exp [i \hat{\theta}_s (z') - i \hat{\theta}_s (z)] \rangle\) and so on.

Another point related to the use of the fully classical approximation is the substitution of the Bose-Einstein statistics of the elementary excitations by its classical limit,

\[\frac{1}{\exp [\hbar \omega_s (k)/(k_B T)] - 1} \approx \frac{k_B T}{\hbar \omega_s (k)}.\]

One obtains strong deviations from Eq. (4) for \(\hbar \omega_s (k) \gtrsim k_B T\), which corresponds, under typical experimental conditions, to the range of wave lengths shorter than \(\Delta_{zopt}\), i.e., not resolvable optically.

These considerations justify our method based on genuinely classical statistics.

In the harmonic approximations fluctuations are Gaussian, hence, \(\langle \exp [i \hat{\theta}_s (z') - i \hat{\theta}_s (z)] \rangle = \exp \left\{ - \frac{1}{4} \langle [\hat{\theta}_s (z') - \hat{\theta}_s (z)]^2 \rangle \right\}\). Expressing \(\hat{\theta}_s\) through creation and annihilation operators of the elementary excitations and calculating thermal populations of the elementary modes using
Each of the waveguides are

\[ \langle \hat{\theta}_a(z') - i \hat{\theta}_a(z) \rangle = \exp \left[ \frac{-2J}{\lambda_T} (1 - e^{-|z-z'|/\lambda_T}) \right] \]

(8)

From this expression we can see that tunnel coupling locks the relative phase between two quasicondensates. This locking means that the relative-phase correlation function \( \delta \rho \) does not decrease to zero, but even at \( |z-z'| \to \infty \) has a finite value, corresponding to \( \langle \hat{\theta}_a^2(z) \rangle = 2J/\lambda_T \). On the contrary, the phase correlations in each of the waveguides are \( \langle i \hat{\theta}_j(z') - i \hat{\theta}_j(z) \rangle = \langle i \hat{\theta}_j(z') + \frac{1}{2} \hat{\theta}_a(z') - \hat{\theta}_a(z) + \frac{1}{2} \hat{\theta}_a(z) \rangle \), the upper and lower signs corresponding to \( j = 1 \) and \( j = 2 \), respectively. We can evaluate them using the statistical independence of noise in the symmetric and antisymmetric modes.

The result

\[ \langle i \hat{\theta}_j(z') - i \hat{\theta}_j(z) \rangle = \exp \left\{ -\frac{1}{2} \langle \hat{\theta}_j(z') - \hat{\theta}_a(z) \rangle^2 - \frac{1}{8} \langle \hat{\theta}_a(z') - \hat{\theta}_a(z) \rangle^2 \right\} \]

\[ \exp \left[ -\frac{|z-z'|}{2\lambda_T} - \frac{J}{2\lambda_T} (1 - e^{-|z-z'|/\lambda_T}) \right] \]

(9)

decreases \( \propto \exp[|z-z'|/(2\lambda_T)] \) at \( |z-z'| \to \infty \) because of the unlimited growth of the fluctuations of the symmetric component of the phase along the z-direction. The correlation properties of the symmetric mode can be experimentally measured using the density-density correlations of the ultracold gas in a time-of-flight experiment [13], however, this subject is beyond the scope of our present paper.

The phase locking of the relative phase becomes most apparent if we treat the evolution of the relative phase along z in the harmonic approximation as the Ornstein-Uhlenbeck stochastic process [6]: while thermal excitations result in the relative phase diffusion, with the diffusion coefficient proportional to \( \lambda_T^{-1} \), the tunnel coupling gives rise to the “friction” force that tends to restore a small (ultimately zero) local phase difference between the two quasicondensates.

III. CORRELATION FUNCTIONS AND THE INTERWELL COHERENCE BEYOND THE HARMONIC APPROXIMATION

A. Equilibrium theory

In the present work we make a step further with respect to the theory of Ref. [1] and abandon the assumption of small phase fluctuations (but still consider small density fluctuations, which is a reasonable approximation for quasicondensates with repulsive interactions). We evaluate the partition function

\[ Z = \int \mathcal{D} \delta \rho \int \mathcal{D} \theta_a \int \mathcal{D} \delta \rho \int \mathcal{D} \theta_a \exp[\{-H/(k_B T)\}] \]

(10)

where

\[ H = \int dz \left\{ \frac{\hbar^2 n_{1D}}{m} \left( \frac{\partial \theta_a}{\partial z} \right)^2 + \frac{g}{4} \delta \rho^2 + \frac{\hbar^2 n_{1D}}{4m} \left( \frac{\partial \theta_a}{\partial z} \right)^2 + g \delta \rho^2 + 2\hbar J n_{1D} (1 - \cos \theta_a) \right\} \]

(11)

is the Hamiltonian expressed through the classical fields \( \delta \rho \), \( \theta_a \) (in the co-ordinate representation), over which the functional integrals are taken. For the sake of simplicity, we write the Hamiltonian [1] in the phononic limit, where the fluctuation wavelengths are long compared to the healing length of the quasicondensate and Eqs. [5, 6] are reduced to \( \omega^2(k) \approx c^2 k^2 \) and \( \omega^2(k) \approx c^2 k^2 + 4J n_{1D}/\hbar \), where \( c = \sqrt{\hbar n_{1D}/m} \) is the speed of sound. Of course, the phase-density description can be extended into short-wavelength excitation range [16, 11], bringing about the Hamiltonian terms \( \propto \langle \partial \phi_{\theta, a}/\partial z \rangle^2 \) and thus revealing the full Bogoliubov-like spectra [5, 6]. However, we are not interested in the short-wavelength limit, since the respective length scales cannot be resolved by optical imaging systems [8, 13, 14].

The system’s description by Eq. [11] is fully consistent with Haldane’s bosonization method [16]. The relative phase \( \theta_a \) is accessible through interference patterns observed in time-of-flight experiments [8, 13, 14]. We develop here the way to evaluate its correlation properties. Since the density fluctuations are small, we can decouple symmetric and antisymmetric modes [17] and integrate out the variables of the symmetric mode. The absence of cross-terms containing both \( \delta \rho \) and \( \theta_a \) in Eq. [11] allows us to integrate out \( \delta \rho \) as well and to obtain, as an intermediate result, the partition function in the form

\[ Z = \text{const} \int \mathcal{D} \theta_a \exp \left\{ -\int dz \left[ \frac{\hbar^2 n_{1D}}{4m k_B T} \left( \frac{\partial \theta_a}{\partial z} \right)^2 + \frac{2\hbar J n_{1D}}{k_B T} (1 - \cos \theta_a) \right] \right\} \]

(12)

that was considered long ago [18, 19] in the context of the statistical mechanics of systems describable by the sine-Gordon equation, which is known to adequately account for the low-energy physics of tunnel-coupled 1D ultracold atomic systems [17].

Note that anharmonic Hamiltonian terms, which depend on the density fluctuations neglected in our present theory, do not affect much the static properties of the quasicondensate [11]. One needs to take them into account in the analysis [20] of a slow process of the system’s relaxation towards equilibrium starting from a nonequilibrium, pre-thermalized initial state [21], characterized by two different temperatures \( T_a \) and \( T_c \ll T_a \) for the symmetric and antisymmetric modes, respectively.
The applicability range of our fully classical approach can be determined as follows. Consider, for the sake of simplicity, distances shorter than \( l_J \). The effects of tunnel coupling can be neglected at such short length scales, and the fully classical correlation function can be estimated \(^{[1]}\) as \( \langle \exp[i\theta_n(z') - i\theta_n(z)] \rangle \approx \exp(-2|z - z'|/\lambda_T) \). We have to compare this result to the power-law decay of correlations due to quantum effects, which is obtained in the limit \( T \to 0 \) \(^{[10, 11]}\). Neglecting, as previously, the contribution of the density fluctuations, we can write \( \lim_{T \to 0} \langle \exp[i\theta_n(z') - i\theta_n(z)] \rangle \approx \lim_{T \to 0} \langle |\psi_1(z')\psi_1(z)\rangle \rangle \psi_2(z') \rangle \rangle \), and finally,

\[
\lim_{T \to 0} \langle \exp[i\theta_n(z') - i\theta_n(z)] \rangle \approx \left( \frac{\Lambda_{UV}}{|z - z'|} \right)^{1/\kappa},
\]

where the quantum mechanical average over the ground state is taken, \( \kappa = \pi \hbar \sqrt{n_{1D}/(mg)} \) is the Luttinger liquid parameter (for quasicondensates, which are weakly interacting systems, \( \kappa \gg 1 \)), and \( \Lambda_{UV} \) is the ultraviolet cutoff of the theory. Eq. \(^{13}\) is valid if

\[
|z - z'| \gg \Lambda_{UV}.
\]

The estimation by Popov \(^{[22]}\) yields \( \Lambda_{UV} \sim \xi \).

We can fully neglect quantum fluctuations if their contribution to the decay of correlations is small, compared to the contribution of the thermal noise, on a given length scale. The correlation decay is dominated by the thermal noise if the classical formula \( \exp(-2|z - z'|/\lambda_T) \) yields stronger decay of correlations than the quantum limit \(^{[14]}\), i.e., if

\[
2|z - z'|/\lambda_T \gg \kappa^{-1} \ln(|z - z'|/\xi).
\]

The experimentally relevant range of \( |z - z'| \) is bound from below by \( \Delta z_{opt} \), as we discussed in Sec. \(^{[11]}\) and \( \Delta z_{opt} \gg \xi \) in a typical example \(^{[3]}\). Therefore the use of the fully classical approach is reasonable for

\[
k_B T \gtrsim mc^2 \xi \ln(\Delta z_{opt}/\xi)/\pi \Delta z_{opt}.
\]

We can evaluate the partition function \(^{[12]}\) using the transfer operator technique \(^{[13, 14, 23]}\). First of all, we evaluate the phase-correlation function as (see Appendix \(^{A}\) for the sketch of derivation)

\[
\langle \exp[i\theta_n(z') - i\theta_n(z)] \rangle = 
\sum_{n=0}^{\infty} \langle |n|e^{i\theta}|0 \rangle |^2 \exp[-(\epsilon_n - \epsilon_0)|z - z'|],
\]

where

\[
\langle |n|e^{i\theta}|0 \rangle = \int_{-\pi}^{\pi} d\theta \Psi_n^{*}(\theta)e^{i\theta}\Psi_0(\theta),
\]

\( \Psi_n(\theta) \) is the eigenfunction (normalized to 1) of the auxiliary Schrödinger-type equation

\[
\left[ -\frac{2}{\lambda_T} \frac{\partial^2}{\partial \theta^2} - \frac{\lambda_T}{4l_J^2} (\cos \theta - 1) \right] \Psi_n(\theta) = \epsilon_n \Psi_n(\theta),
\]

and \( \epsilon_n, n = 0, 1, 2, \ldots \) is the respective eigenvalue. For simplicity, we set periodic (and not quasiperiodic) boundary conditions to Eq. \(^{[19]}\) with the period \( 2\pi \), thus neglecting the band structure of its spectrum, since the zero-quasimomentum solutions define all the system properties \(^{[13]}\), which are relevant to our present work.

In the limit of strong tunnel coupling, \( l_J \ll \lambda_T \), the operator in the left-hand-side of Eq. \(^{[19]}\) can be approximated by the harmonic oscillator Hamiltonian (in proper units), and \( \epsilon_n = l_J^{-1}(n + 1/2) \), \( n = 0, 1, 2, \ldots \). In this limit Eq. \(^{17}\) reproduces the result \(^{[8]}\) that holds for small phase fluctuations.

In the opposite limit, Eq. \(^{19}\) can be solved perturbatively, and we obtain

\[
\langle \exp[i\theta_n(z') - i\theta_n(z)] \rangle \approx \left( \frac{\lambda_T^2}{8l_J^2} \right)^2 + \left[ 1 - \left( \frac{\lambda_T^2}{8l_J^2} \right)^2 \right] \exp\left( -\frac{2|z - z'|}{\lambda_T} \right), \quad l_J \gg \lambda_T.
\]

In what follows, we will be interested in calculating the value of

\[
\langle \cos \theta_n \rangle = \langle 0 \rangle \langle \cos \theta |0 \rangle,
\]

which can be viewed as the mean interwell coherence. This expression can be derived in different ways, e.g., from Eq. \(^{17}\) by employing the statistical independence of phase fluctuations at two very distant points, \( |z - z'| \to \infty \), and recalling that \( \langle \sin \theta_n \rangle = 0 \). In a general case, Eq. \(^{21}\) can be evaluated from the lowest-energy solution of the Mathieu equation \(^{[24]}\). In the two limiting cases we obtain the asymptotics

\[
\langle \cos \theta_n \rangle \approx \left\{ \begin{array}{ll}
\exp(-l_J/\lambda_T), & l_J \ll \lambda_T \\
\lambda_T^2/(8l_J^2), & l_J \gg \lambda_T.
\end{array} \right.
\]

A possible physical explanation of the fact that the mean interwell coherence decreases at \( l_J/\lambda_T \to \infty \) much slower that the harmonic approximation \(^{[1]}\) predicts, is the large probability of thermal excitation of a soliton in this limit. Each emerging soliton decreases the number of phononic states by 1 \(^{[19]}\), and the phononic density of states is reduced mostly in the long-wavelength range (for phonon momenta less than or of the order of \( h/l_J \)), which gives the main contribution to the long-distance behavior of the correlation function \(^{[7]}\) and, hence, to \( \langle \cos \theta_n \rangle \).

B. Relaxation to the equilibrium after a quench

The results of Sec. \(^{[11, 15]}\) are obtained at the equilibrium. However, it is interesting to investigate also the process of equilibration in the system of two 1D quasi-condensates after a quench. The study of this dynamical problem is motivated by our recent numerical results \(^{[25]}\), related to thermalization in a single 1D quasi-condensate. In Ref. \(^{[25]}\) we found that, despite the numerically confirmed integrability of the system, phononic
(low-momentum) modes rapidly relaxed from their initial non-equilibrium state towards a final equilibrium state; particle-like (large-momentum) excitations, on the contrary, exhibited almost no relaxation. The equilibrium ensemble of phonons was different from the classical limit of equipartition of the thermal energy between all the degrees of freedom and was quite close to the Bose-Einstein distribution with the temperature $T_{\text{eff}}$ determined by the total excitation energy of the initial non-equilibrium state. Observed fluctuations around this equilibrium state were due to the finite size of the system inherent to numerical modeling. Remarkably, the correlations observed at the length scales, which are large compared to the healing length to the healing length, as well as to the wavelength of an elementary excitation with the energy equal to $k_{B}T_{\text{eff}}$, were well described by classical expressions. Note that the main contribution to the noise on these length scales stems from the low-energy excitations, which approximately exhibit classical equipartition of energy.

The need to extend the numerical approach of Ref. [25] to tunnel-coupled 1D quasicondensates can also be seen from the following considerations. Our aim is to numerically check the theoretically predicted correlations of two tunnel-coupled quasicondensates at equilibrium. This equilibrium state can be viewed as a result of the system’s relaxation from its initial non-equilibrium state. Moreover, the available analytic theory predicts only averages; unlike the case of harmonic approximation, there is no way yet to generate individual realizations of the phase, obeying the necessary statistics, without simulating numerically the equilibration process. The most obvious way to obtain numerically the equilibrium solution is to observe the numerical relaxation after a quench and wait until a steady-state regime establishes. Particular type of the quench and the corresponding initial conditions are, up to a certain degree, arbitrary, as long as the system exhibits true relaxational dynamics.

Motivated by these considerations, we performed numerical modeling of the thermal equilibrium values of $\langle \cos \theta_{a} \rangle$ after the dynamical process of relaxation in our system after a quench. We simulated the time evolution of two coupled Gross-Pitaevskii equations using the split-step method [26] previously used by us [25] to simulate the dynamics of a single quasicondensate and now extended to the case of tunnel-coupled systems. As the initial conditions we took two independent quasicondensates with phonon modes populated randomly according to the Bose-Einstein thermal distribution. At $t=0$ we quenched the system by switching on the tunnel coupling between them. We solved this coupled system for a time long enough to provide equilibration.

To juxtapose the input parameters of our numerical simulations to typical parameters of modern atom-chip experiments [8, 13, 14], we give the system parameters used in our simulations first in dimensional units, but later show them also in dimensionless form. The linear density for a single quasicondensate $n_{1D} = 30 \mu m^{-1}$ and the interaction constant $g = 2\hbar\omega a_s$ with the radial trapping frequency $\omega = 2\pi \times 3 \, kHz$ and the s-wave scattering length $a_s = 5.3 \, nm$ for $^{87}\text{Rb}$ yields the healing length $\xi \approx 0.35 \, \mu m$ and the Luttinger liquid parameter $K \approx 33$. The periodic boundary conditions were set at an interval of the length $L = 100 \, \mu m \approx 290\xi$. The maximum integration time was $t_{\text{max}} = 0.8 \, s$. After few hundreds milliseconds some kind of equilibrium was obtained. The total energy of the system was conserved in our numerical simulations with a good ($\approx 10^{-3}$) accuracy, however, it was constantly redistributed in an oscillatory manner between different low-frequency elementary modes, including Josephson oscillations. The nonlinear interaction between different modes (see Section [X]) lead to excitation of Josephson oscillations of the total number imbalance $(N_1 - N_2)/2$, where $N_j$ is the integral of the density in the $j$th quasicondensate over the whole length $L$, i.e., the number of atoms in this quasicondensate, $N_1 + N_2 = 2N$. In general, the numerical stability of our split-step method was controlled using the criteria of Ref. [27]. The thermal coherence length was determined from the phase-correlation functions for each of the two quasicondensates taken separately by comparison of the numerically obtained value of $\langle \exp[i\theta_j(z) - i\theta_j(z')] \rangle$, $j = 1, 2$, with its theoretical value $\exp(-|z-z'|/\Lambda_T)$ for $|z-z'| < \Lambda_T$ [10, 11] (if we trace out the phase and density variables of one of the two tunnel-coupled quasicondensates, the properties of its remaining counterpart will be described by the same temperature as of the whole system at equilibrium). The averaging is performed over statistically uncorrelated (separated by sufficiently large distances) intervals of the whole length.

![FIG. 1: (Color online) Mean interwell contrast as a function of the ratio of the length scales $\lambda_T$ and $l_f$. Solid line: exact theory given by Eq. (21). Dashed line: small-fluctuations approximation $\langle \cos \theta_{a} \rangle = \exp(-\lambda_T/\lambda_T')$ following from the linearized theory [3]. Dots: results of the numerical simulations of the equilibration dynamics of two coupled condensates. Units on the axes are dimensionless. Inset: Magnified part of the main plot for small $\lambda_T/l_f$, illustrating the high-temperature asymptotics of Eq. (21) in comparison to the linearized theory result.](image-url)
L for \(|z - z'| \ll \lambda_T\). We never obtain complete equilibration. In each realization, the correlation length \(\lambda_T\) obtained in such a way oscillates around certain mean value, and so does the value of \(\langle \cos \theta_a \rangle\) (averaged over the length \(L\)). Typically, \(\lambda_T \approx 8 \mu m\), which corresponds to \(T \approx 40 nK\).

We present the results of our numerical simulations in Fig. IV. Dots represent mean values of \(\langle \cos \theta_a \rangle\) obtained by averaging over both the time (on the quasi-equilibration stage of the system evolution) and the ensemble of realizations. The error bars in Fig. IV show the standard deviations of \(\langle \cos \theta_a \rangle\) and \(\lambda_T\). These error bars indicate slow, quasiperiodic variations of \(\langle \cos \theta_a \rangle\) and \(\lambda_T\) detected in our simulations. The range of \(\lambda_T/L_J\) shown in Fig. IV corresponds to \(J\) increasing from \(2\pi \times 0.1\) Hz up to \(2\pi \times 8\) Hz.

To summarize the results of the present Section, we can state that we developed a theory describing the static correlation properties more precisely than the harmonic model [1]. Our approach is based on consideration of the classical partition function for the antisymmetric mode of our problem (describable by the sine-Gordon model) and application of the well-known transfer operator technique [18, 19]. As one can see from Fig. IV the difference between our results and those of Ref. [1] is most apparent for intermediate and small values of \(\lambda_T/L_J\) (intermediate and weak tunnel coupling).

IV. JOSEPHSON OSCILLATIONS IN A NOISY EXTENDED JUNCTION

The thermal noise effects considered in Sec. III reduce the frequency of Josephson oscillations.

Consider the absolute number imbalance between two wells, \(N_{12} \equiv (N_1 - N_2)/2\), and its canonically conjugate variable, the overall phase difference \(\Phi\) between two quasi-condensates. In the limit of the atomic repulsion energy dominating over the tunneling, \(g_{n1D} \equiv gN/L \gg \hbar J\), and for small-amplitude oscillations, \(|N_1 - N_2| \ll N\), the evolution of these “global” variables is described by the set of equations (see Appendix II)

\[
\frac{d}{dt} \Phi = -\frac{2gN_{12}}{L\hbar},
\]

\[
\frac{d}{dt} N_{12} = 2Jn_{1D} \int_0^L dz \sin \theta_a,
\]

which is reduced, after elimination of the number-difference variable, to

\[
\frac{d^2}{dt^2} \Phi = -\omega^2 J_0 \int_0^L dz \sin \theta_a,
\]

where

\[
\omega_{J0} = \sqrt{4Jg_{n1D}/\hbar}
\]

is the frequency of the Josephson oscillations for bosonic junction unaffected by thermal noise. At zero temperature, when the thermal noise is absent, and for \(\ln(L/\xi) \ll K\), when the quantum noise can be neglected, spatial extension of the ultracold-atomic Josephson junction plays no role and we can derive Eq. (26) from the results of Ref. 8. In the case of small-amplitude Josephson oscillations, the statistical properties of \(\cos \theta_a\) and \(\cos(\theta_a - \Phi)\) do not differ significantly, in particular, \(\langle \cos \theta_a \rangle \approx \langle \cos(\theta_a - \Phi) \rangle\), i.e., the quadratic in \(\Phi\) correction is negligible, and Eq. (26) reduces to

\[
\frac{d^2}{dt^2} \Phi + [\omega^2_J + \delta \omega^2_J(t)] \Phi = \zeta(t),
\]

where

\[
\omega^2_J = \omega^2_{J0} \langle \cos \theta_a \rangle.
\]

In Eq. (27) we explicitly indicate the time argument of the random driving force

\[
\zeta(t) = \omega^2_{J0} \frac{1}{L} \int_0^L dz \sin(\theta_a - \Phi)
\]

and the term

\[
\delta \omega^2_J(t) = \omega^2_{J0} \frac{1}{L} \int_0^L dz (\cos \theta_a - \langle \cos \theta_a \rangle)
\]
i.e., with the frequency reduced by $\sqrt{\langle \cos \theta_a \rangle}$ compared to the noise-free case of Eq. (26).

The presence of the noise broadens the power spectrum of Josephson oscillations

$$S(\omega) = \frac{1}{\tau} \int_{t_{\text{max}}}^{t_{\text{max}}-\tau} dt e^{i\omega t} \eta(t) \bigg|^2 \quad (31)$$

where $\eta = (N_1 - N_2)/(2N)$ is the relative number imbalance. The integration in Eq. (31) is taken over the time interval $\tau$ when the system has already reached its nearly-equilibrium state (typically, $\tau \approx 0.65$ s). If $\omega_1$ is high enough, the theory predicts $S(\omega)$ to be a peaked function, centered at $\omega_1$ and having the half-width at the half-maximum of the peak height $\gamma = [\hbar L/(8g_k b_T)] \Re \int_0^{\infty} dt' \langle \zeta(t) \zeta(t+t') \rangle \exp(i\omega_1 t')$. The latter expression, roughly evaluated as $\gamma \sim \frac{1}{2} k_b T/(\hbar k_c \langle \cos \theta_a \rangle^2)$, correctly describes the order of magnitude of the bandwidth $\Delta \omega/(2\pi) \sim 10$ Hz of the numerically obtained spectra $S(\omega)$.

The presence of the random driving force is the source of excitation of Josephson oscillations in the course of the system’s evolution, even if initially at $t = 0$, $\Phi = 0$ and $\eta \propto \frac{1}{\sqrt{\tau}} \Phi = 0$. Note, that all the elementary excitations with nonzero momenta in the antisymmetric mode have frequencies larger than $\omega_1$. The energy transfer between nonzero-momentum excitations and Josephson mode is thus an essentially nonlinear process. The nonlinear structure of the right-hand-side of Eq. (24) provides the presence of the frequency $\omega_1$ in the spectrum $\int_{-\infty}^{\infty} dt' \langle \zeta(t) \zeta(t+t') \rangle \exp(i\omega t')$ of the driving force and thus ensures the parametric excitation of the Josephson oscillations.

We confirmed our analytic estimations by the numerical simulations of two coupled 1D Gross-Pitaevskii equations already described in Subsection III B. An example of a sharp-peaked power spectrum of relative number imbalance is given in Fig. 4 together with an example of time dependence of $\eta$.

If, on the contrary, $\omega_1 \ll \omega_T$, where $\omega_T = 2c/\lambda_T$ is the typical time scale of fluctuations of $\zeta(t)$, then the behavior of $\eta(t)$ becomes irregular and $S(\omega)$ does not exhibit a peak at $\omega \approx \omega_1$ any more (see Fig. 3).

The results of numerical simulations shown in Figs. 2 and 3 demonstrate certain energy exchange, but no full equilibration between the Josephson oscillations and phononic modes. If we set $\Phi|_{t=0} = 0$ and $\eta|_{t=0} = 0$ for $J/(2\pi) = 8$ Hz (or 0.1 Hz), then at times $t$ between 650 ms and 1 s the mean energy of Josephson oscillations is by an order of magnitude (or by 1.5 orders of magnitude, respectively) less than $k_b T$, where temperature $T$ is determined from the phase-correlation function for a single quasicondensate and is thus associated with the phononic modes. This may indicate an extremely long thermalization time for Josephson oscillations.

We selected our simulations that display a pronounced narrow peak of $S(\omega)$ far from zero frequency (which was the case for $J > 2\pi \times 0.7$ Hz), estimated the Josephson
frequency $\omega_j$ and analyzed the dependence of $\omega_j^2$ on the mean interwell coherence. The resulting values are in a good agreement with our theoretical prediction given by Eq. (28), as can be seen from Fig. 4.

V. CONCLUSION

To conclude, we applied the transfer-operator technique to evaluate coherence and correlation properties of two tunnel-coupled 1D weakly-interacting, ultracold systems (quasicondensates) of bosonic atoms. These properties are determined by the ratio of the two length scales: $\lambda_T$ that describes the spatial scale of the loss of correlations between two points and $l_j$ that describes the scale for the phase-locking between two quasicondensates due to interwell tunneling. In the limit $l_j \lesssim \lambda_T$ the fluctuations of the relative phase are small and we reproduce the results of the linearized theory of Ref. [1]. In the opposite case, we found the mean interwell coherence to decrease much slower ($\propto \lambda_T^2/l_j^2$) than the exponential law predicted by the linearized theory. We interpret these results as a signature of thermal creation of sine-Gordon solitons, which provide a shift of the relative phase by $2\pi$ and thus do not contribute to the coherence loss, and the corresponding decrease of the density of states for phonons (the excitations responsible for the coherence loss at large distances).

Our analytic estimations are confirmed by numerical modeling of the equilibrium state as a final state of the system’s relaxational evolution after a quench. This task is solved by extending our numerical method [25] to integration of two coupled 1D Gross-Pitaevskii equations.

We demonstrate, both analytically and numerically, that thermal fluctuations of the relative phase between two quasicondensates reduce the frequency of Josephson oscillations in proportion to $\sqrt{\cos\theta_n}$ and broaden their spectrum. If the theoretically predicted value of $\omega_j$ is much less than the bandwidth of the thermal fluctuation (which is of the order of the speed of sound divided by $\lambda_T$), regular Josephson oscillations are not observed.

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Appendix A: Derivation of Eq. (17)

We briefly recall here the basics of the transfer operator technique, following Refs. [18, 19, 23]. We introduce a lattice with the step $\Delta z = L/M$, $M$ being the number of sites. We assume cyclic boundary conditions,

$$\theta_{M+1} = \theta_1.$$  \hspace{1cm} (A1)

Then the partition function [12] can be written as

$$Z = \prod_{j=1}^{M} \exp[-f(\theta_{a,j}, \theta_{a,j+1})],$$  \hspace{1cm} (A2)

where

$$f(\theta_{a,j}, \theta_{a,j+1}) = \frac{\hbar^2 n_{1D}}{4mk_B T \Delta z} (\theta_{a,j} - \theta_{a,j+1})^2 + \frac{\hbar J n_{1D} \Delta z}{k_B T} (2 - \cos \theta_{a,j} - \cos \theta_{a,j+1})$$  \hspace{1cm} (A3)

and integrals in our case are taken from $-\pi$ to $\pi$. We omit the constant prefactor in Eq. (A2) for the sake of simplicity. Assume that eigenfunctions $\Psi_n(\theta)$ of the transfer operator

$$\int d\theta \Psi_n^*(\theta) \Psi_n(\theta_{a,j}) = e^{-\epsilon_n \Delta z} \Psi_n(\theta_{a,j+1})$$  \hspace{1cm} (A4)

form a set, which is complete, orthogonal, and normalized to unity, namely

$$\int d\theta \Psi_n^*(\theta) \Psi_n(\theta) = \delta_n n, \hspace{1cm} (A5)$$

$$\sum_n \Psi_n^*(\theta') \Psi_n(\theta) = \delta(\theta' - \theta). \hspace{1cm} (A6)$$

Substituting Eq. (A0) into Eq. (A2) and using Eq. (A4), we obtain

$$Z = \sum_n \exp(-\epsilon_n L).$$  \hspace{1cm} (A7)

The eigenvalues $\epsilon_n$ are positive; in the thermodynamic limit the partition function (A2) is dominated by the lowest eigenvalue $\epsilon_0$,

$$Z \approx \exp(-\epsilon_0 L), \hspace{1cm} L \to \infty.$$  \hspace{1cm} (A8)

In the continuous limit $\Delta z \to 0$ Eq. (A7) is equivalent to the Schrödinger-type equation [19]. Strictly speaking, the spectrum of Eq. (A4) is shifted with respect to the spectrum of Eq. [12] by a common offset $\omega_0$, which is related to normalization of the eigenfunctions. Since $\omega_0$ does not depend on $n$, we neglect it in our calculations.

To calculate correlation functions, in particular, Eq. (17), we note that $e^{i \theta_n(z)}$ and $e^{-i \theta_n(z)}$ act on $\Psi_0$ like quantum-mechanical perturbations, coupling $\Psi_0$ to the whole spectrum of eigenfunctions with the matrix elements given by Eq. (13). Therefore the leading term for $\langle \exp[i \theta_n(z) - i \theta_n(z')]\rangle$ in the limit of $L \to \infty$ is the second-order perturbative correction to the propagator for the ground state (with $L$ playing the role of imaginary time), and we obtain thus Eq. (17).

Appendix B: Derivation of Eqs. (23, 24)

We begin with the lattice version of the classical sine-Gordon Hamiltonian that describes the dynamics of the
antisymmetric mode of our system:

\[ H_a = \sum_{j=1}^{M} \left[ \frac{\hbar^2 n_{1D}}{4m\Delta z} (\theta_{a,j} - \theta_{a,j+1})^2 + \frac{g}{\Delta z} \delta N_{a,j}^2 + 2\hbar J n_{1D} \Delta z (1 - \cos \theta_{a,j}) \right], \]  

(B1)

where the \( j \)th generalized co-ordinate \( \delta N_{a,j} = \delta \phi_a \Delta z \) is the half-difference of the atomic numbers in the 1st and 2nd quasicondensates at the \( j \)th site, i.e., the variable canonically conjugate to the local phase difference \( \theta_{a,j} \) (the \( j \)th generalized momentum). Here we neglect the nonlinear coupling between the symmetric and antisymmetric modes, like in Eq. (11) in the continuous limit.

For the sake of simplicity, we assume an odd number of sites in the lattice, \( M = 2M_0 + 1 \), where \( M_0 \) is a positive integer. Then we do a canonical transformation

\[ \delta N_{a,j} = \sum_{\ell = -M_0}^{M_0} \delta \tilde{N}_{a}(\ell) \eta(\ell, j), \quad \theta_{a,j} = \sum_{\ell = -M_0}^{M_0} \tilde{\theta}_{a}(\ell) \eta(\ell, j), \]  

(B2)

where

\[ \eta(\ell, j) = \begin{cases} \sqrt{2/M} \cos(2\pi \ell j/M), & \ell = -1, -2, \ldots, -M_0 \\ 1/\sqrt{M}, & \ell = 0 \\ \sqrt{2/M} \sin(2\pi \ell j/M), & \ell = 1, 2, \ldots, M_0 \end{cases} . \]  

(B3)

Then the Hamiltonian (13) reads

\[ H_a = \sum_{\ell = -M_0}^{M_0} \left[ \frac{\hbar^2 n_{1D}}{2m\Delta z} [1 - \cos(2\pi \ell j/M)] \tilde{\theta}_{a}^2(\ell) + \frac{g}{\Delta z} \delta \tilde{N}_{a}^2(\ell) \right] + 2\hbar J n_{1D} \Delta z \sum_{j=1}^{M} \left[ 1 - \cos \left( \sum_{\ell = -M_0}^{M_0} \tilde{\theta}_{a}(\ell) \eta(\ell, j) \right) \right] . \]  

(B4)

From the Hamiltonian equations

\[ \frac{d}{dt} \delta \tilde{N}_{a}(\ell) = \frac{\partial H_a}{\partial \theta_{a,j}(\ell)}, \quad \frac{d}{dt} \tilde{\theta}_{a}(\ell) = -\frac{\partial H_a}{\partial \delta \tilde{N}_{a}(\ell)} \]  

(B5)

we find, in particular,

\[ \frac{d}{dt} \tilde{\theta}_{a}(0) = -\frac{2g\delta \tilde{N}_{a}(0)}{\Delta z} . \]  

(B6)

In the limit of \( \Delta z \to 0 \) the sums over \( j \) converge to integrals over \( z \). Taking into account that \( N_{12} = \int dz \delta \phi_a = \sum_{j=1}^{M} \delta N_{a,j} = \sqrt{M} \delta \tilde{N}_{a}(0) \), identifying the generalized momentum conjugate to \( \Phi = \tilde{\theta}_{a}(0)/\sqrt{M} = (1/M) \sum_{j=1}^{M} \theta_{a,j} \) and recalling that \( L = M \Delta z \), we obtain Eqs. (23, 24). The spatially fluctuating part of the phase is then \( \theta_{a,j} - \Phi \).

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