The Fekete–Szegö Coefficient Inequality For a New Class of m-Fold Symmetric Bi-Univalent Functions Satisfying Subordination Condition

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Abstract

In this paper, we investigate a new subclass $S_{\varphi,\lambda}^{m}$ of $\Sigma_{m}$ consisting of analytic and $m$-fold symmetric bi-univalent functions satisfying subordination in the open unit disk $U$. We consider the Fekete-Szegö inequalities for this class. Also, we establish estimates for the coefficients for this subclass and several related classes are also considered and connections to earlier known results are made.

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1 Introduction and Definitions

Let $A$ denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n,$$

which are analytic in the open unit disk $U = \{ z : |z| < 1 \}$, and let $S$ be the subclass of $A$ consisting of the form (1) which are also univalent in $U$.

The Koebe one-quarter theorem [7] states that the image of $U$ under every function $f$ from $S$ contains a disk of radius $\frac{1}{4}$. Thus every such univalent function has an inverse $f^{-1}$ which satisfies

$$f^{-1} (f(z)) = z \quad (z \in U)$$
and
\[ f \left( f^{-1} (w) \right) = w \left( |w| < r_0 (f) \land r_0 (f) \geq \frac{1}{4} \right), \]

where
\[ f^{-1} (w) = w - a_2 w^2 + \left(2a_2^2 - a_3\right) w^3 - \left(5a_2^3 - 5a_2 a_3 + a_4\right) w^4 + \cdots. \]  (2)

A function \( f \in A \) is said to be bi-univalent in \( U \) if both \( f \) and \( f^{-1} \) are univalent in \( U \). Let \( \Sigma \) denote the class of bi-univalent functions defined in the unit disk \( U \).

For a brief history and interesting examples in the class \( \Sigma \), see [23]. Although, the familiar Koebe function is not in the class of \( \Sigma \), there are some examples of functions member of \( \Sigma \), such as \( z - z^2 \) and \( -\log(1 - z) \) and so on. Other common examples of functions in \( S \) for example
\[ z - \frac{z^2}{2} \text{ and } \frac{z}{1 - z^2} \]
are also not members of \( \Sigma \) (see [23]).

An analytic function \( f \) is said to be subordinate to another analytic function \( g \), written
\[ f(z) \prec g(z), \]  (3)
provided that there is an analytic function \( w \) defined on \( U \) with
\[ w(0) = 0 \text{ and } |w(z)| < 1 \]
satisfying the following condition:
\[ f(z) = g(w(z)). \]

Lewin [13] is the first mathematician studied the class of bi-univalent functions, obtaining the bound 1.51 for modulus of the second coefficient \( |a_2| \). Subsequently, Brannan and Clunie [3] conjectured that \( |a_2| \leq \sqrt{2} \) for \( f \in \Sigma \) and Netanyahu [20] showed that \( \max |a_2| = \frac{4}{3} \). Brannan and Taha [?] introduced certain subclasses of the bi-univalent function class \( \Sigma \) similar to the familiar subclasses. \( S^* (\beta) \) and \( K (\beta) \) of starlike and convex function of order \( \beta \) \((0 \leq \beta < 1)\) in turn (see [20]). The classes \( S^*_\Sigma (\alpha) \) and \( K_\Sigma (\alpha) \) of bi-starlike functions of order \( \alpha \) and bi-convex functions of order \( \alpha \), corresponding to the function classes \( S^* (\alpha) \) and \( K (\alpha) \), were also introduced analogously. For each of the function classes \( S^*_\Sigma (\alpha) \) and \( K_\Sigma (\alpha) \), they found non-sharp estimates on the initial coefficients. In fact, the beforementioned work of Srivastava et al. [23] fundamentally revived the investigation of diversified subclasses of the bi-univalent function class \( \Sigma \) in recent times. Recently, many authors searched bounds for various
subclasses of bi-univalent functions ([1], [2], [8], [16], [23], [27], [30]). Not much is known about the bounds on the general coefficient $|a_n|$ for $n \geq 4$. In the literature, the only few works determining the general coefficient bounds $|a_n|$ for the analytic bi-univalent functions ([5], [9], [10]). The coefficient estimate problem for each of $|a_n|$ ($n \in \mathbb{N} \setminus \{1, 2\}$; $\mathbb{N} = \{1, 2, 3, \ldots\}$) is still an open problem.

For each function $f \in S$, the function

$$h(z) = \sqrt[m]{f(z^m)} \quad (z \in U, \ m \in \mathbb{N})$$

is univalent and maps the unit disk $U$ into a region with $m$-fold symmetry. A function is said to be $m$-fold symmetric (see [12], [21]) if it has the following normalized form:

$$f(z) = z + \sum_{k=1}^{\infty} a_{mk+1} z^{mk+1} \quad (z \in U, \ m \in \mathbb{N}).$$

We symbolize by $S_m$ the class of $m$-fold symmetric univalent functions in $U$, which are normalized by the series expansion (5). Indeed, the functions in the class $S$ are one-fold symmetric.

Similar to the concept of $m$-fold symmetric univalent functions, here, in this work, we introduced the concept of $m$-fold symmetric bi-univalent functions. Each function $f \in \Sigma$ generates an $m$-fold symmetric bi-univalent function for each integer $m \in \mathbb{N}$. The normalized form of $f$ is given by (5) and the series expansion for $f^{-1}$ is given by below:

$$g(w) = w - a_{m+1} w^{m+1} + \left[ (m+1) a_{m+1}^2 - a_{2m+1} \right] w^{2m+1}$$

$$- \left[ \frac{1}{2} (m+1)(3m+2) a_{m+1}^3 - (3m+2) a_{m+1} a_{2m+1} + a_{3m+1} \right] w^{3m+1} + \cdots.$$

where $f^{-1} = g$. We denote by $\Sigma_m$ the class of $m$-fold symmetric bi-univalent functions in $U$. Taylor Maclaurin series expansion of the inverse function of $f^{-1}$ has been recently proven by Srivastava et al. [28]. For $m = 1$, the formula (6) induces the formula (2) of the class $\Sigma$. Some examples of $m$-fold symmetric bi-univalent functions are given here below:

$$\left( \frac{z^m}{1-z^m} \right)^{1 \over m}, \quad [- \log(1-z^m)]^{1 \over m}, \quad \left[ \frac{1}{2} \log \left( \frac{1+z^m}{1-z^m} \right) \right]^{1 \over m}.$$

In this work, the class of analytic functions of the form is

$$p(z) = 1 + p_1 z + p_2 z^2 + p_3 z^3 + \cdots$$

such that

$$R(p(z)) > 0 \quad (z \in U)$$

holds and this class is denoted by $P$. 

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In the work of Pommerenke \[21\], the m-fold symmetric function $p$ in the class $P$ is given of the form:

$$p(z) = 1 + p_m z + p_{2m} z^{2m} + p_{3m} z^{3m} + \cdots \quad (7)$$

Throughout this study, $\varphi$ will be assumed as an analytic function with positive real part in the unit disk $U$ such that

$$\varphi(0) = 1 \quad \text{and} \quad \varphi(0) > 0.$$ 

and $\varphi(U)$ is symmetric with respect to the real axis. The function $\varphi$ has a series expansion of the form:

$$\varphi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + \cdots (B_1 > 0). \quad (8)$$

Let $u(z)$ and $v(z)$ be two analytic functions in the unit disk $U$ with

$$u(0) = v(0) = 0 \quad \text{and} \quad \max\{|u(z)|, |v(z)|\} < 1.$$ 

We observe that

$$u(z) = b_m z^m + b_{2m} z^{2m} + b_{3m} z^{3m} + \cdots \quad (9)$$

and

$$v(w) = c_m w^m + c_{2m} w^{2m} + c_{3m} w^{3m} + \cdots. \quad (10)$$

Also we assume that

$$|b_m| \leq 1, \quad |b_{2m}| \leq 1 - |b_m|^2, \quad |c_m| \leq 1, \quad |c_{2m}| \leq 1 - |c_m|^2 \quad (11)$$

Making some simple calculations we can notice that

$$\varphi(u(z)) = 1 + B_1 b_m z^m + (B_1 b_{2m} + B_2 b_m^2) z^{2m} + \cdots (|z| < 1) \quad (12)$$

and

$$\varphi(v(w)) = 1 + B_1 c_m w^m + (B_1 c_{2m} + B_2 c_m^2) w^{2m} + \cdots (|w| < 1). \quad (13)$$

In this study, derived substantially by the work of Ma and Minda \[15\] and \[28\], we introduce some new subclasses of m-fold symmetric bi-univalent functions and obtain bounds for the Taylor-Maclaurin coefficients $|a_{m+1}|$ and $|a_{2m+1}|$ and Fekete-Szegö functional problems for functions in these new classes.
Definition 1 A function \( f \in \Sigma_m \) is said to be in the class \( S^\lambda_{\Sigma_m}(\phi) \) if the following conditions are satisfied:

\[
f \in \Sigma_m, \quad \frac{zf'(z)}{(1 - \lambda)f(z) + \lambda z f'(z)} \prec \phi(z) \quad (0 \leq \lambda < 1, \ z \in U) \tag{14}
\]

and

\[
\frac{\lambda g'(w)}{(1 - \lambda)g(w) + \lambda wg'(w)} \prec \phi(w) \quad (0 \leq \lambda < 1, \ w \in U) \tag{15}
\]

where the function \( g = f^{-1} \), given by the (6).

Remark 2 For the case of one fold symmetric functions the class \( S^\lambda_{\Sigma_m}(\phi) \) reduces to the following classes:

1. In the case of \( m = 1 \) in Definition 1 we have the class

\[
S^\lambda_{\Sigma_1}(\phi) = \mathcal{G}^{\phi \circ \phi}(\gamma)
\]

investigated by Magesh and Yamini [17] defined by requiring that

\[
\frac{zf'(z)}{(1 - \lambda)f(z) + \lambda z f'(z)} \prec \phi(z) \quad (0 \leq \lambda < 1, \ z \in U)
\]

and

\[
\frac{\lambda g'(w)}{(1 - \lambda)g(w) + \lambda wg'(w)} \prec \phi(w) \quad (0 \leq \lambda < 1, \ w \in U),
\]

where the function \( g = f^{-1} \) given by the equation (6).

2. In the case of \( m = 1 \) and \( \lambda = 0 \) in Definition 1 then we have the class \( S^0_{\Sigma_1}(\phi) \) which is the class of Ma Minda starlike functions, introduced by Ma and Minda [15]. This class consists of the functions

\[
\frac{zf'(z)}{f(z)} \prec \phi(z)
\]

3. In the case of \( m = 1 \) in Definition 1 for the different choices of the function \( \phi(z) \), we obtain interesting known subclasses of analytic function class.

For example, If we let

\[
\phi(z) = \frac{1 + (1 - 2\beta)z}{1 - z} \quad \text{or} \quad \phi(z) = \frac{1 - (1 - 2\beta)z}{1 - z}, (0 \leq \beta < 1, \ z \in U)
\]
then the class \( S^\lambda_\Sigma(\varphi) \) reduces to the class \( S^0_\Sigma(\varphi) = S\mathcal{S}^\lambda_\Sigma(\beta, \lambda) \). This class contains the functions satisfying the conditions

\[
\left| \arg \left( \frac{zf'(z)}{(1-\lambda)f(z) + \lambda zf'(z)} \right) \right| < \frac{\alpha \pi}{2} \quad (0 < \alpha \leq 1, 0 \leq \lambda < 1, \ z \in U)
\]

and

\[
\left| \arg \left( \frac{\lambda g'(w)}{(1-\lambda)g(w) + \lambda wg'(w)} \right) \right| < \frac{\alpha \pi}{2} \quad (0 < \alpha \leq 1, 0 \leq \lambda < 1, \ w \in U).
\]

Similarly, if we let

\[
\varphi(z) = \left( \frac{1+z}{1-z} \right)^\alpha \quad \text{or} \quad \varphi(z) = \left( \frac{1-z}{1+z} \right)^\alpha \quad (0 < \alpha \leq 1, \ z \in U)
\]

then the class \( S^\lambda_\Sigma(\varphi) \) reduces to the class \( S^0_\Sigma(\varphi) = S\mathcal{S}^\lambda_\Sigma(\alpha, \lambda) \). This class contains the functions satisfying the conditions

\[
f \in \Sigma, \quad \text{Re} \left( \frac{zf'(z)}{(1-\lambda)f(z) + \lambda zf'(z)} \right) > \beta \quad (0 \leq \beta < 1, 0 \leq \lambda < 1, \ z \in U)
\]

and

\[
\text{Re} \left( \frac{wg'(w)}{(1-\lambda)g(w) + \lambda wg'(w)} \right) > \beta \quad (0 \leq \beta < 1, 0 \leq \lambda < 1, \ w \in U).
\]

The classes \( \mathcal{G}_\Sigma(\beta, \lambda) \) and \( \mathcal{G}_\Sigma(\alpha, \lambda) \) were introduced and studied by Murugusundaramoorthy et al. (see Definition 1 and Definition 2 in [18]). Also if we choose \( \lambda = 0 \), \( \mathcal{G}_\Sigma(\beta, 0) : \mathcal{G}^\beta_\Sigma \) and \( \mathcal{G}_\Sigma(\alpha, 0) : \mathcal{G}^\alpha_\Sigma \). These classes are called bi-starlike functions of order \( \beta \) and strongly bi-starlike functions of order \( \alpha \), respectively. The classes \( \mathcal{G}_\Sigma(\beta) \) and \( \mathcal{G}_\Sigma(\alpha) \) were investigated and studied by Brannan and Taha [19] (see Definition 1.1 and Definition 1.2).

If we set \( \lambda = 0 \) in Definition 1, then the class \( S^\lambda_\Sigma(\varphi) \) reduces to the class \( S^0_\Sigma(\varphi) \) defined by below:

**Definition 3** A function \( f \in \Sigma_m \) is said to be in the class \( S^\lambda_{\Sigma_m}(\varphi) \) if the following conditions are satisfied:

\[
f \in \Sigma_m, \quad \frac{zf'(z)}{f(z)} < \varphi(z) \quad (0 \leq \lambda < 1, \ z \in U)
\]

and

\[
\frac{wg'(w)}{g(w)} < \varphi(w) \quad (0 \leq \lambda < 1, \ w \in U)
\]

where the function \( g = f^{-1} \) given by (7).
Remark 4 In the case of $m = 1$ in Definition 3, it is interesting that, also for $\lambda = 0$, the classes $S_{\Sigma_1}^\alpha \left(\frac{1+1-2\beta}{1-z}\right)$ and $S_{\Sigma_1}^\lambda \left(\frac{1+1-2\beta}{1-z}\right)$ lead the class $\delta_{\Sigma}^\alpha(\beta)$ of bi-starlike functions of order $\alpha$ and $\delta_{\Sigma}^\lambda(\beta)$ of bi-starlike functions of order $\beta$, respectively.

For $m$-fold symmetric analytic and bi-univalent functions, Altınkaya and Yalçın [2] defined and investigated the function classes $S_{\Sigma_m}^{\alpha, \lambda}(\beta, \lambda)$ and $S_{\Sigma_m}^{\alpha, \lambda}(\beta, \lambda)$ as following.

A function $f \in \Sigma_m$ is said to be in the class $S_{\Sigma_m}^{\alpha, \lambda}(\beta, \lambda)$ if the following conditions are satisfied:

$$\left| \arg \left( \frac{zf'(z)}{(1-\lambda)f(z) + \lambda zf'(z)} \right) \right| < \frac{\alpha \pi}{2} \quad (0 < \alpha \leq 1, 0 \leq \lambda < 1, \ z \in U)$$

and

$$\left| \arg \left( \frac{\lambda g'(w)}{(1-\lambda)g(w) + \lambda wg'(w)} \right) \right| < \frac{\alpha \pi}{2} \quad (0 < \alpha \leq 1, 0 \leq \lambda < 1, \ w \in U).$$

In the same way, the function $f \in \Sigma_m$ is said to be in the class $S_{\Sigma_m}^{\alpha, \lambda}(\beta, \lambda)$ if the following conditions are satisfied:

$$f \in \Sigma, \quad \text{Re} \left( \frac{zf'(z)}{(1-\lambda)f(z) + \lambda zf'(z)} \right) > \beta \quad (0 \leq \beta < 1, 0 \leq \lambda < 1, \ z \in U)$$

and

$$\text{Re} \left( \frac{wg'(w)}{(1-\lambda)g(w) + \lambda wg'(w)} \right) > \beta \quad (0 \leq \beta < 1, 0 \leq \lambda < 1, \ w \in U)$$

where the function $g = f^{-1}$ given by (2).

Theorem 5 [2] Let $f$ given by (4) be in the class $S_{\Sigma_m}^{\alpha, \lambda}(\alpha, \lambda)$, $0 < \alpha \leq 1$. Then

$$|a_{m+1}| \leq \frac{2\alpha}{m(1-\lambda)\sqrt{\alpha + 1}}$$

and

$$|a_{2m+1}| \leq \frac{\alpha}{m(1-\lambda)} + \frac{2(m+1)\alpha^2}{m^2(1-\lambda)^2}.$$

Theorem 6 [2] Let $f$ given by (4) be in the class $S_{\Sigma_m}^{\beta, \lambda}(\beta, \lambda)$, $0 \leq \beta < 1$. Then

$$|a_{m+1}| \leq \frac{\sqrt{2(1-\beta)}}{m(1-\lambda)}$$

and

$$|a_{2m+1}| \leq \frac{(1-\beta)}{m(1-\lambda)} + \frac{2(m+1)\alpha^2}{m^2(1-\lambda)^2}.$$
2 The Main Results and Their Consequences

Theorem 7 Let \( f \) given by (5) be in the class \( S_{\Sigma_m}^\lambda (\varphi) \). Then

\[ |a_{m+1}| \leq \frac{B_1 \sqrt{B_1}}{m(1-\lambda) \sqrt{B_1^2 - 2B_2} + B_1} \]  \hspace{1cm} (16)

and

\[ |a_{2m+1}| \leq \begin{cases} \frac{(m + 1 - \frac{m(1-\lambda)}{B_1}) B_1^3}{2m^2(1-\lambda)^2|B_1 + |B_1 - 2B_2|^2|} + \frac{B_1}{2m(1-\lambda)^2} , & B_1 \geq \frac{m(1-\lambda)}{m+1} \\ \frac{B_1}{2m(1-\lambda)} , & B_1 < \frac{m(1-\lambda)}{m+1} \end{cases} \]  \hspace{1cm} (17)

Proof. Let \( f \in S_{\Sigma_m}^\lambda (\varphi) \). Then there are analytic functions \( u : U \to U \) and \( v : U \to U \), with

\[ u(0) = v(0) = 0, \]

satisfying the following conditions:

\[ \frac{zf'(z)}{(1-\lambda)f(z) + \lambda zf'(z)} = \varphi(u(z)) \]

and

\[ \frac{\lambda g'(w)}{(1-\lambda)g(w) + \lambda wg'(w)} = \varphi(v(w)). \]  \hspace{1cm} (18)

Using the equalities (12), (13) in (18) and comparing the coefficient of (18), we have

\[ m(1-\lambda)a_{m+1} = B_1 b_m, \]  \hspace{1cm} (19)

\[ m(1-\lambda) \left[ 2a_{2m+1} - (\lambda m + 1)a_{m+1}^2 \right] = B_1 b_{2m} + B_2 b_m^2, \]  \hspace{1cm} (20)

and

\[ -m(1-\lambda)a_{m+1} = B_1 c_m, \]  \hspace{1cm} (21)

\[ m(1-\lambda) \left[ (1 + m(2 - \lambda))a_{m+1}^2 - 2a_{2m+1} \right] = B_1 c_{2m} + B_2 c_m. \]  \hspace{1cm} (22)

From (19) and (21) we find that

\[ b_m = -c_m. \]  \hspace{1cm} (23)

Adding (20) and (22), we get

\[ 2m^2(1-\lambda)^2a_{m+1}^2 = B_1 (b_{2m} + c_{2m}) + B_2 (b_m^2 + c_m^2). \]  \hspace{1cm} (24)

and using the relation (19) and (23) in (24), we have

\[ 2m^2(1-\lambda)^2a_{m+1}^2 = B_1 (b_{2m} + c_{2m}) + \frac{2B_2m^2(1-\lambda)^2}{B_1^2} a_{m+1}. \]
Therefore, by a simple calculation we get
\[ 2m^2(1 - \lambda)^2 \left(B_1^2 - 2B_2\right) a_{m+1}^2 = B_1^3 \left(b_{2m} + c_{2m}\right) \quad (25) \]
By using the inequalities given by (11) in (25) for the coefficients \(b_{2m}\) and \(c_{2m}\), we obtain
\[ |2m^2(1 - \lambda)^2 \left(B_1^2 - 2B_2\right) a_{m+1}^2| \leq 2B_1^3 \left(1 - |b_m^2|\right). \quad (26) \]
Also by using (19) in (26) we have
\[ |a_{m+1}|^2 \leq \frac{B_1^3}{m^2(1 - \lambda)^2 \left(|B_1^2 - 2B_2| + B_1\right)} \]
which implies the assertion (16).

Next, in order to find the bound on \(|a_{2m+1}|\), by subtracting (22) from (20), we obtain
\[ 4m(1 - \lambda)a_{2m+1} = 2m(m + 1)(1 - \lambda)a_{m+1} + B_1 \left(b_{2m} - c_{2m}\right). \quad (27) \]
Then, in view of (19) , (23) and (27), applying the inequalities in (11) for the coefficients \(p_{2m}, p_m, q_m\) and \(q_{2m}\), we have
\[ |a_{2m+1}| \leq \left(\frac{m + 1 - \frac{m(1 - \lambda)}{B_1}}{2m^2 \left(B_3 + |B_1^2 - 2B_2|\right)} + \frac{B_1}{2m(1 - \lambda)}\right) B_1^3 \]
which implies the assertion (17). This completes the proof of Theorem 7.

For \(\lambda = 0\), we can state the following corollary:

**Corollary 8** Let \(f\) given by (3) be in the class \(S_{\Sigma_m}(\varphi)\). Then
\[ |a_{m+1}| \leq \frac{B_1 \sqrt{B_1}}{m \sqrt{|B_1^2 - 2B_2|} + B_1} \quad (28) \]
and
\[ |a_{2m+1}| \begin{cases} \leq \left(\frac{m(1 - \lambda)}{B_1} \right) \frac{B_1^3}{2m^2 \left[B_1 + |B_1^2 - 2B_2|\right]} + \frac{B_1}{2m(1 - \lambda)} & B_1 \geq \frac{m}{m+1} \\ \leq \frac{B_1}{2m}, & B_1 < \frac{m}{m+1} \end{cases} \quad (29) \]

For one fold symmetric functions, we obtain the following corollaries:

**Corollary 9** If the function \(f \in \Sigma\) is in the class of \(S_{\Sigma_1}^\lambda(\varphi) = \mathcal{G}_{\Sigma}^{\varphi,\psi}(\gamma)\), then
\[ |a_2| \leq \frac{B_1 \sqrt{B_1}}{(1 - \lambda) \sqrt{|B_1^2 - 2B_2|} + B_1} \quad (30) \]
and
\[ |a_3| \begin{cases} \leq \left(\frac{2 - (1 - \lambda)}{B_1} \right) \frac{B_1^3}{2(1 - \lambda)^2 + B_1 |B_1^2 - 2B_2|} + \frac{B_1}{2(1 - \lambda)} & B_1 \geq \frac{1 - \lambda}{2} \\ \leq \frac{B_1}{2(1-\lambda)}, & B_1 < \frac{1 - \lambda}{2} \end{cases} \quad (31) \]
For the case of one-fold symmetric functions and for $\varphi(z) = (1+z)^{\alpha} = 1 + 2\alpha z + 2\alpha^2 z^2 + ...$, Theorem 1 reduces to the following result:

**Corollary 10** If the function $f \in \Sigma$ is in the class of $S_{1,0}^\lambda((1+z)^{\alpha})$, then

$$|a_2| \leq \frac{2\alpha}{1-\lambda}$$

and

$$|a_3| \leq \begin{cases} \frac{4\alpha^2}{(1-\lambda)^2}, & \text{for } \alpha \geq \frac{1-\lambda}{4} \\ \frac{4\alpha^2}{(1-\lambda)^2}, & \text{for } \alpha < \frac{1-\lambda}{4} \end{cases}.$$  

The estimates for $|a_2|$ and $|a_3|$ asserted by Corollary 10 are more accurate than those given by Corollary 1 in Magesh and Yamini.

For the case of one-fold symmetric functions and for $\varphi(z) = (1+(1-2\beta)z + (1-\beta)z^2 + ...$, Theorem 1 reduces to the following result:

**Corollary 11** If the function $f \in \Sigma$ is in the class of $S_{1,0}^\lambda((1+(1-2\beta)z)$, then

$$|a_2| \leq \frac{2(1-\beta)}{(1-\lambda)\sqrt{2\beta + 1}}$$

and

$$|a_3| \leq \begin{cases} \frac{4\alpha^2}{(1-\lambda)^2}, & \text{for } \alpha \geq \frac{1-\lambda}{4} \\ \frac{4\alpha^2}{(1-\lambda)^2}, & \text{for } \alpha < \frac{1-\lambda}{4} \end{cases}.$$  

The estimates for $|a_2|$ and $|a_3|$ asserted by Corollary 11 are more accurate than those given by Corollary 2 in Magesh and Yamini.

Also, if we choose $\lambda = 0$ in Corollary 7, we have the following corollary:

**Corollary 12** If the function $f \in \Sigma$ is in the class of $S_{1,1}^\lambda$ then

$$|a_2| \leq \frac{B_1 \sqrt{B_1}}{\sqrt{|B_1^2 - 2B_2| + B_1}}$$

and

$$|a_3| \leq \begin{cases} \left(2 - \frac{1}{\alpha}\right) \frac{B_3}{2 + B_1 |B_1^2 - 2B_2|^2} + \frac{B_1}{B_1^2}, & \text{for } B_1 \geq \frac{1}{2} \\ \frac{B_1}{B_1^2}, & \text{for } B_1 < \frac{1}{2} \end{cases}.$$  

The estimates for $|a_2|$ and $|a_3|$ asserted by Corollary 12 are more accurate than those given by Corollary 3 in Magesh and Yamini.

For one-fold symmetric functions, if we choose the function $\varphi(z)$ in different forms, then we have the following corollaries. named Corollary 13 and Corollary 14:
Corollary 13 Let the function \( f(z) \) given by the equality (1) be in the class \( SS_{\Sigma}^{2}(\beta, \lambda), 0 \leq \beta < 1 \) and \( 0 \leq \lambda < 1 \). Then

\[
|a_2| \leq \frac{2\sqrt{(1 - \beta)}}{(1 - \lambda)}
\]

and

\[
|a_3| \leq \frac{4(1 - \beta)^2}{(1 - \lambda)^2} + \frac{(1 - \beta)}{(1 - \lambda)}.
\]

Corollary 14 Let the function \( f(z) \) given by the equality (1) be in the class \( SS_{\Sigma}^{2}(\alpha, \lambda), 0 < \alpha \leq 1 \) and \( 0 \leq \lambda < 1 \). Then

\[
|a_2| \leq \frac{2\alpha}{(1 - \lambda)\sqrt{1 + \alpha}}
\]

and

\[
|a_3| \leq \frac{4\alpha^2}{(1 - \lambda)^2} + \frac{\alpha}{(1 - \lambda)}.
\]

For one-fold symmetric bi-univalent functions and \( \lambda = 0 \), Theorem 5 reduces to Corollary which were proven earlier by Murugunsundaramoorthy et al. [18]

Corollary 15 Let \( f \) given by (4) be in the class \( S_{\Sigma}^{2}(\alpha) \ (0 < \alpha \leq 1) \). Then

\[
|a_2| \leq \frac{2\alpha}{\sqrt{\alpha + 1}}
\]

and

\[
|a_3| \leq 4\alpha^2 + \alpha.
\]

Here, in this study, we will specify the theorem concerning the Fekete-Szegö inequality for the class \( S_{\Sigma_m}^{2}(\alpha) \). To improve the result, especially Theorem 2.1, we consider Fekete-Szegö inequality for the class \( S_{\Sigma_m}^{2}(\alpha) \). This kind of studies has been made by many authors. The results regarding this problem are given in the works of [6], [11], [14], [22]. The conclusions given in the study are not sharp, but, unfortunately, there isn’t any method giving sharp results as regards these problems.

Theorem 16 Let \( f \) given by (4) be in the class \( S_{\Sigma_m}^{2}(\alpha) \). Then
\[ |a_{2m+1} - \gamma a_{m+1}^2| \leq \left\{ \begin{array}{ll} \frac{b_1}{2m(1-\lambda)} & \text{for } 0 \leq |h(\gamma)| < \frac{1}{4m(1-\lambda)} \\ 2B_1 |h(\gamma)| & \text{for } |h(\gamma)| \geq \frac{1}{4m(1-\lambda)} \end{array} \right. \] (38)

\[ h(\gamma) = \left( \frac{m+1-2\gamma}{2} \right) \frac{B_1^2}{2m^2(1-\lambda)^2 (B_1^2-2B_2)} \]

**Proof.** From the equations (25) and (27),

\[ a_{2m+1} = \frac{B_1^3 (b_{2m} + c_{2m})}{2m^2(1-\lambda)^2 (B_1^2-2B_2)} \] (39)

and

\[ a_{2m+1} = \frac{m+1}{2} a_{m+1} - \frac{B_1 (b_{2m} - c_{2m})}{4m(1-\lambda)} \] (40)

By using the equalities (39) and (40), we have

\[ a_{2m+1} - \gamma a_{m+1}^2 = B_1 \left[ \left( h(\gamma) + \frac{1}{4m(1-\lambda)} \right) b_{2m} + \left( h(\gamma) - \frac{1}{4m(1-\lambda)} \right) c_{2m} \right] \]

where

\[ h(\gamma) = \left( \frac{m+1-2\gamma}{2} \right) \frac{B_1^2}{2m^2(1-\lambda)^2 (B_1^2-2B_2)} \]

Due to the fact that all \( B_i \) are real and \( B_1 > 0 \), which holds the assertion (38), the proof of the theorem is completed.

For \( m \)-fold symmetric functions, if we choose \( \lambda = 0 \) in the Theorem 16, we obtain the following corollary:

**Corollary 17** Let \( f \) given by (4) be in the class \( S^\Sigma_m (\varphi) \). Then

\[ |a_{2m+1} - \gamma a_{m+1}^2| \leq \left\{ \begin{array}{ll} \frac{b_1}{2m} & \text{for } 0 \leq |h(\gamma)| < \frac{1}{4m} \\ 2B_1 |h(\gamma)| & \text{for } |h(\gamma)| \geq \frac{1}{4m} \end{array} \right. \] (41)

where

\[ h(\gamma) = \left( \frac{m+1-2\gamma}{2} \right) \frac{B_1^2}{2m^2(B_1^2-2B_2)} \]

For one-fold symmetric functions, we can state the Fekete-Szegő inequality for the class \( S^\Sigma_1 (\varphi) \) as following:
Corollary 18 If the function $f \in \Sigma$ is in the class of $S_{\Sigma_m}^{\lambda}(\varphi) = G_{\Sigma_m}^{\phi, \lambda}(\gamma)$, then we get

$$|a_3 - \gamma a_2^2| \leq \begin{cases} \frac{B_1}{4B_1 |h(\gamma)|} & \text{for } 0 \leq |h(\gamma)| < \frac{1}{4(1-\lambda)} \\ \frac{B_1^2}{4B_1 |h(\gamma)|} & \text{for } |h(\gamma)| \geq \frac{1}{4(1-\lambda)} \end{cases}.$$  

where

$$h(\gamma) = (1-\gamma) \frac{B_1^2}{2(1-\lambda)^2 (B_1^2 - 2B_2)}.$$

If we choose $\lambda = 0$ in Corollary 15, then we have the following corollary:

Corollary 19 If the function $f \in \Sigma$ is in the class of $S_{\Sigma_1}^{0}(\varphi)$ then

$$|a_3 - \gamma a_2^2| \leq \begin{cases} \frac{B_1}{4B_1 |h(\gamma)|} & \text{for } 0 \leq |h(\gamma)| < \frac{1}{4} \\ \frac{B_1^2}{4B_1 |h(\gamma)|} & \text{for } |h(\gamma)| \geq \frac{1}{4} \end{cases}.$$  

where

$$h(\gamma) = (1-\gamma) \frac{B_1^2}{2(1-\lambda)^2 (B_1^2 - 2B_2)}.$$

Choosing $\gamma = 1$ and $\gamma = 0$ in Theorem 13, we obtain following corollary:

Corollary 20 Let $f$ given by (5) be in the class $S_{\Sigma_m}^{\lambda, \varphi}$. Then

$$|a_{2m+1} - \gamma a_{m+1}^2| \leq \begin{cases} \frac{B_1}{4m(1-\lambda)} & \text{for } 0 \leq |h(\gamma)| < \frac{1}{4m(1-\lambda)} \\ \frac{B_1}{4B_1 |h(\gamma)|} & \text{for } |h(\gamma)| \geq \frac{1}{4m(1-\lambda)} \end{cases}.$$  

Choosing $\gamma = 1$ and $m = 1$ in Corollary 17, we have the following corollary:

Corollary 21 Let $f$ given by (7) be in the class $S_{\Sigma_1}^{\lambda, \varphi}$. Then

$$|a_3 - \gamma a_2^2| \leq \frac{B_1}{4(1-\lambda)}.$$  

Also, if we choose $\lambda = 0$, then we have

$$|a_3 - \gamma a_2^2| \leq \frac{B_1}{4}.$$  

Conclusion
In this study, we have composed of several new subclasses of m-fold symmetric bi-univalent analytic functions by means of subordination. For functions belonging to the classes introduced here, we have obtained inequalities on the Taylor Maclaurin coefficients $|a_{m+1}|$ and $|a_{2m+1}|$. Also, for functions contained these classes, we find Fekete-Szegő inequalities and we have made some connections to some of earlier known results.

Author’s contributions
All author worked on the results and read and approved the final manuscript.

Competing interests
The authors declare that they have no competing interests.

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References

[1] A. Akgül, On the coefficient estimates of analytic and bi-univalent m-fold symmetric functions, Mathematica Aeterna, 7 (3), 2017, 253 - 260

[2] Ş. Altınkaya and S. Yalçın, Coefficient Bounds for Certain Subclasses of m-Fold Symmetric Bi-univalent Functions, Journal of Mathematics, Volume 2015, (2015), 5 pages.

[3] D. A. Brannan and J. Clunie, Aspects of contemporary complex analysis, Proceedings of the NATO Advanced Study Institute Held at University of Durham, New York: Academic Press, (1979).

[4] D. A. Brannan and T. S. Taha, On some classes of bi-univalent functions, in Mathematical Analysis and Its Applications (S. M. Mazhar, A. Hamoui and N. S. Faour, Editors) (Kuwait; February 18–21, 1985), KFAS Proceedings Series, Vol. 3, Pergamon Press (Elsevier Science Limited), Oxford, 1988, pp. 53–60; see also Studia Univ. Babes-Bolyai Math. 31 (2) (1986), 70–77.

[5] S. Bulut, Coefficient estimates for a new subclass of analytic and bi-univalent functions, Annals of the Alexandru Ioan Cuza University—Mathematics, Tomul LXII, 2016, f. 2, vol. 1

[6] J.H.Choi, Y. C. Kim, and T. Sugawa, “A general approach to the Fekete-Szegő problem,” Journal of the Mathematical Society of Japan, vol. 59, no. 3, pp. 707–727, 2007.
[7] P. L. Duren, Univalent Functions, Grundlehren der Mathematischen Wis- 
senschaften, Springer, New York, USA 259 (1983).

[8] B. A. Frasin and M. K. Aouf, New subclasses of bi-univalent functions, 
Applied Mathematics Letters, 24 (2011) 1569-1573.

[9] S. G. Hamidi and J. M. Jahangiri, Faber polynomial coefficient estimates 
for analytic bi-close-to-convex functions, C. R. Acad. Sci. Paris, Ser.I, 352 
(1) (2014) 17–20.

[10] J. M. Jahangiri and S. G. Hamidi, Coefficient estimates for certain classes 
of bi-univalent functions, International Journal of Mathematics and Math- 
ematical Sciences, Article ID 190560, (2013) 4 p.

[11] S. Kanas, “An unified approach to the Fekete-Szegő problem,” Applied 
Mathematics and Computation, vol. 218, no. 17, pp. 8453–8461, 2012.

[12] W. Koepf, Coefficient of symmetric functions of bounded boundary rota-
tions, Proc. Amer. Math. Soc., 105 (1989) 324-329.

[13] M. Lewin, On a coefficient problem for bi-univalent functions, Proc. Amer. 
Math. Soc., 18 (1967) 63-68.

[14] R. R. London, “Fekete-Szegő inequalities for close-to-convex functions,” 
Proceedings of the American Mathematical Society, vol. 117, no. 4 , pp. 
947–950, 1993.

[15] W. C. Ma and D. Minda, A unified treatment of some special classes of 
univalent functions, in Proceedings of the Conference on Complex Analysis 
(Tianjin: June 19–23, 1992) (Zhong Li, Fuyao Ren, Lo Yang and Shunyan 
Zhang, Editors), Conference Proceedings and Lecture Notes in Analysis, 
Vol. I, International Press, Cambridge, Massachusetts, 1994, pp. 157–169.

[16] N. Magesh and J. Yamini, Coefficient bounds for a certain subclass of 
bi-univalent functions, International Mathematical Forum, 8 (27) (2013) 
1337-1344.

[17] N. Magesh and J. Yamini, Coefficient Estimates for a Certain General 
Subclass of Analytic and Bi-Univalent Functions, Applied Mathematics, 
(5), (2014), 1047-1052

[18] G. Murugusundaramoorthy, N. Magesh and V. Prameela, Coefficient bounds 
for certain classes of bi-univalent function, Abstract and Applied Analysis, 
Article ID 573017, (2013) 3 pp.

[19] G. Murugusundaramoorthy, Subclasses of Bi-Univalent Functions of Com-
plex Order Based On Subordination Conditions Involving Wright Hyper-
geometric Functions, J. Math. Fund. Sci., 47, (1), (2015), 60-75
[20] E. Netanyahu, The minimal distance of the image boundary from the origin and the second coefficient of a univalent function in $|z| < 1$, Archive for Rational Mechanics and Analysis, 32 (1969) 100-112.

[21] C. Pommerenke, Univalent Functions, Vandenhoeck & Ruprecht, Göttingen, Germany, 1975.

[22] H. M. Srivastava, A. K. Mishra, and M. K. Das, “The Fekete-Szego problem for a subclass of close-to-convex functions,” Complex Variables, Theory and Application, vol. 44, pp. 145–163, 2001.

[23] H. M. Srivastava, A. K. Mishra and P. Gochhayat, Certain subclasses of analytic and bi-univalent functions, Applied Mathematics Letters, 23 (10) (2010) 1188-1192.

[24] H.M. Srivastava, S. Sivasubramanian, and R. Sivakumar, Initial coefficient bounds for a subclass of m-fold symmetric bi-univalent functions, Tbilisi Mathematical Journal, vol. 7, no. 2, pp. 1–10, 2014.

[25] H. M. Srivastava, S. Gaboury and F. Ghanim, Coefficient estimates for a general subclass of analytic and bi-univalent functions of the Ma–Minda type, RACSAM DOI 10.1007/s13398-017-0416-5

[26] Q.-H. Xu, Y.-C. Gui, and H. M. Srivastava, “Coefficient estimates for a certain subclass of analytic and bi-univalent functions,” Applied Mathematics Letters, 25 (6), (2012) 990–994.

[27] H. M. Srivastava, S. Bulut, M. Çağlar and N. Yaşmur, Coefficient estimates for a general subclass of analytic and bi-univalent functions, Filomat 27 (5) (2013) 831-842.

[28] H. M. Srivastava, S. Sivasubramanian and R. Sivakumar, Initial coefficient bounds for a subclass of m-fold symmetric bi-univalent functions, Tbilisi Math. J. 7 (2) (2014) 1-10.

[29] H. M. Srivastava, S. Gaboury and F. Ghanim, Coefficient estimates for a general subclass of analytic and bi-univalent functions of the Ma–Minda type, RACSAM DOI 10.1007/s13398-017-0416-5

[30] Q. H. Xu, Y. C. Gui and H. M. Srivastava, Coefficient estimates for a certain subclass of analytic and bi-univalent functions, Applied Mathematics Letters, 25 (2012) 990-994.