On quasisymmetric plasma equilibria sustained by small force

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Abstract. We construct smooth, non-symmetric plasma equilibria which possess closed, nested flux surfaces and solve the Magnetohydrostatic (steady three-dimensional incompressible Euler) equations sustained by a small force. The solutions are also ‘nearly’ quasisymmetric. The primary idea is that, given a desired quasisymmetry direction $\xi$, change the smooth structure on space so that the vector field $\xi$ is Killing for the new metric and construct $\xi$–symmetric solutions of the Magnetohydrostatic equations on that background by solving a generalized Grad-Shafranov equation. If $\xi$ is close to a symmetry of Euclidean space, then these are solutions on flat space up to a small forcing.

1. Introduction

Let $T \subset \mathbb{R}^3$ be a domain with smooth boundary. The three-dimensional Magnetohydrostatic (MHS) equations on $T$ read

$$J \times B = \nabla P + f, \quad \text{in } T, \quad (1.1)$$
$$\nabla \cdot B = 0, \quad \text{in } T, \quad (1.2)$$
$$B \cdot \hat{n} = 0, \quad \text{on } \partial T, \quad (1.3)$$

where $J = \nabla \times B$ is the current, $f$ is an external force and $P$ is the ‘total pressure’, $P = p + \frac{1}{2} |B|^2$. The solution $B$ to (1.1)–(1.3) can be interpreted as either a stationary fluid velocity field which solves the time-independent Euler equation, or as a steady self-supporting magnetic field in a continuous medium with trivial flow velocity. The latter interpretation is robust across a variety of magnetohydrodynamic models (e.g. compressible, incompressible, non-ideal) and makes the system (1.1)–(1.3) central to the study of plasma confinement fusion.

In view of this, there is long standing scientific program to identify and construct magnetohydrostatic equilibria which are effective at confining ions during a nuclear fusion reaction. The most basic requirement for confinement is the existence of a “flux function” $\psi$, whose level sets foliate the domain $T$ and which satisfies $B \cdot \nabla \psi = 0$. To first approximation, ions move along the integral curves of $B$ and so this condition ensures that particle trajectories are approximately constrained to the level sets of $\psi$. For this reason, it is desirable to seek equilibria with nested flux surfaces (isosurfaces of $\psi$) which foliate the plasma domain. When $T$ is the axisymmetric torus, it is natural to look for such solutions in the form of axisymmetric magnetic fields. In the plasma physics literature, this setup is called the tokamak. If $(R, \Phi, Z)$ denote the usual cylindrical coordinates on $\mathbb{R}^3$ and the center line of the torus lies in the $Z = 0$ plane, axisymmetric solutions take the form

$$B_0 = \frac{1}{R^2} \left( C_0(\psi_0) R e_\Phi + R e_\Phi \times \nabla \psi_0 \right), \quad (1.4)$$

with flux function $\psi_0$ and swirl $C_0 = C_0(\psi_0)$. In order for $B_0$ to satisfy (1.1) with $P_0 = P_0(\psi_0)$, taking $f = 0$ momentarily for simplicity and taking $T$ to be the torus with inner radius $R_0 - 1$.
and outer radius \( R_0 + 1 \), say, the flux function needs to satisfy the axisymmetric Grad-Shafranov equation (11, 24)

\[
\frac{\partial^2 R}{\partial Z^2} \psi + \frac{\partial^2 \psi}{\partial R^2} - \frac{1}{R} \frac{\partial R}{\partial R} \psi + R^2 P''(\psi_0) + C_0 C_0'(\psi_0) = 0, \quad \text{in } D_0, \tag{1.5}
\]

\[
\psi = \text{const.} \quad \text{on } \partial D_0, \tag{1.6}
\]

where \( D_0 \) denotes the cross-section of the torus (unit disk) in the \( \Phi = 0 \) half-plane centered at \( R = R_0 \). Conversely, if \( \psi_0 \) is any solution to (1.5) with \( \epsilon_0 \cdot \nabla \psi_0 = 0 \) then the vector field \( B_0 \) defined in (1.4) is divergence-free and satisfies (1.1). If \( \psi_0 \) is taken to be constant on \( \partial T \), \( B_0 \) satisfies (1.3).

Unfortunately, these tokamak equilibria come with a slew of problems from the point of view of plasma confinement fusion [18]. For example, to achieve improved confinement it is desirable for the magnetic field to ‘twist’ as it wraps around the torus and this can only be accomplished in axisymmetry with a large plasma current, \( J \). Such plasma configurations are hard to control in practice. One approach to finding equilibria with better confinement properties is to consider equilibria in geometries which have the desired twist built in. This is the basic design principle behind the stellarator, [10]. It is still desirable for these configurations to possess a form of symmetry, which is known as quasisymmetry.

**Definition 1 (Weak quasisymmetry, [22]).** Let \( \xi \) be a non-vanishing vector field tangent to \( \partial T \). We say that \( \xi \) is a quasisymmetry and the field \( B \) is quasisymmetric with respect to \( \xi \) if

\[
\begin{align*}
\text{div } \xi &= 0, \quad \text{in } T, \tag{1.7} \\
B \times \xi &= -\nabla \psi, \quad \text{in } T, \tag{1.8} \\
\xi \cdot \nabla |B| &= 0, \quad \text{in } T, \tag{1.9}
\end{align*}
\]

for some function \( \psi : T \to \mathbb{R} \).

The significance of the condition (1.8) is that it implies \( B \cdot \nabla \psi = 0 \) and \( \xi \cdot \nabla \psi = 0 \) and so quasisymmetric solutions posses flux functions which are symmetric with respect to \( \xi \). In [22], the authors argue that conditions (1.7)–(1.9) form the minimal requirement that ensures first order (in gyroradius) particle confinement, hence the terminology of weak quasisymmetry. In the confinement fusion literature [18, 3], one encounters the following alternative definition which is actually stronger than the above. It replaces (1.9) with

\[
\xi \times J = \nabla (B \cdot \xi) \quad \text{in } T. \tag{1.10}
\]

We term this set of conditions strong quasisymmetry. When \( f = 0 \) it is this stronger form of quasisymmetry which is equivalent to other definitions in the plasma fusion literature involving Boozer angles, see §8 of [18]. If \( \text{div } B = 0 \) then (1.9) requires only that a single component of (1.10) vanish, \( B \cdot (\xi \times J - \nabla (B \cdot \xi)) = 0 \).

In light of this, the additional content of strong quasisymmetry (1.7), (1.8), and (1.10) is the assumption that the other two components of \( \xi \times J - \nabla (B \cdot \xi) \) vanish. It turns out that when there is no force and the equilibria are toroidal, strong quasisymmetry is equivalent to Definition 1.

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1. We remark that it could be that this equation admits “large” solutions with non-trivial dependence on \( \Phi \). See the work of Garabedian [9].

2. To see this, using standard vector calculus identities, we write

\[
\xi \times \text{curl } B - \nabla (B \cdot \xi) = B \cdot \nabla \xi + \xi \cdot \nabla B + B \times \text{curl } \xi.
\]

Taking the inner product with \( B \) results in \( B \cdot (\xi \times \text{curl } B - \nabla (B \cdot \xi)) = \frac{1}{2} \xi \cdot \nabla |B|^2 + B \cdot \nabla \xi \cdot B \). The argument is completed by using the elementary identity \( \mathcal{L}_B \nabla B = \text{curl}(B \times \xi) + \text{div } B \xi - \text{div } \xi B = \text{curl}(B \times \xi) \). This yields \( B \cdot (\xi \times \text{curl } B - \nabla (B \cdot \xi)) = \xi \cdot \nabla |B|^2 \).

3. M. Landreman, private communication.
From (1.8), if $\xi \cdot B$ is constant on surfaces of constant $\psi$, $\xi \cdot B = C(\psi)$ (by a result in [3], any solution of (1.1) which satisfies (1.7)-(1.9) satisfies this condition), it follows that $B$ is of the form

$$B = \frac{1}{|\xi|^2} \left(C(\psi)\xi + \xi \times \nabla \psi\right),$$

(1.11)

and when $f = 0$, the requirement that (1.1) holds implies that $\psi$ must satisfy the quasisymmetric Grad-Shafranov equation (introduced in [3]) which reads

$$\text{div} \left(\frac{\nabla \psi}{|\xi|^2}\right) - C(\psi) \frac{\xi}{|\xi|^2} \cdot \text{curl} \left(\frac{\xi}{|\xi|^2}\right) + \frac{CC'(\psi)}{|\xi|^2} + P'(\psi) = 0,$$

(1.12)

in $T$, \psi = const. on $\partial T.$

The equations (1.2) and (1.7), (1.9) can be thought of as constraints relating $\psi$ to the deformation tensor of $\xi$, the symmetric two-tensor $L_\xi$ defined by

$$(L_\xi \delta)(X,Y) = \nabla_X \xi \cdot Y + \nabla_Y \xi \cdot X,$$

where $\nabla$ denotes covariant differentiation with respect to the Euclidean metric. Recall that $\xi$ generates an isometry of Euclidean space if and only if $L_\xi \delta = 0$, in which case $\xi$ is a Killing field for the metric $\delta$. Assuming that $\xi \cdot \nabla \psi = 0$ and $\text{div} \xi = 0$, from (1.11) we find

$$\text{div} B = C(\psi)(L_\xi \delta)(\xi,\xi) + (L_\xi \delta)(\xi,\xi \times \nabla \psi),$$

(1.14)

and expanding the condition (1.9) we find

$$\frac{1}{2} L_\xi |B|^2 = (L_\xi)(\xi,\xi) + \frac{2}{C'(\psi)}(L_\xi \delta)(\xi,\xi \times \nabla \psi) + \frac{1}{C(\psi)^2}(L_\xi \delta)(\xi \times \nabla \psi,\xi \times \nabla \psi),$$

(1.15)

see Lemma C.5 of Appendix C. The equation (1.15) is a complicated relationship between $\psi, C(\psi)$ and $\xi$ but notice that it holds trivially (assuming only that $\xi \cdot \nabla \psi = 0$) whenever when $\xi$ is a Killing field. It is well-known that in Euclidean space the only Killing fields are linear combinations of translations and rotations. Therefore, up to a multiplicative constant, the only such field compatible with the geometry of the axisymmetric torus is $\xi = R e_\phi$ and as mentioned above, such solutions are not suitable for confinement. We have arrived at the following problem.

**Problem:** Given a toroidal domain $T$, construct a function $\psi: T \to \mathbb{R}$ with nested flux surfaces and a divergence-free vector field $\xi$ which does not generate an isometry of $\mathbb{R}^3$ and is tangent to $\partial T$, so that (1.13), the nonlinear constraints (1.14), (1.15) and the property $\xi \cdot \nabla \psi = 0$ all hold.

It is not clear that there are any smooth solutions $\psi, \xi$ to the above problem. In fact, in 1967 (long before the above notion of quasisymmetry was introduced), Grad [13,11,12] conjectured that the only smooth solutions to (1.1)-(1.3) possessing a good flux function have a Euclidean symmetry, and this would in particular rule out any solutions of the above type. Since Grad’s work, there have been some constructions of non-symmetric weak solution equilibria with pressure jumps have been rigorously constructed [5] which may have practical implication for the confinement fusion program [14,15,3].

We do not address Grad’s conjecture here and our goal is instead to present a robust method for constructing solutions to (1.1) with small force and which are approximately quasisymmetric with respect to a given vector field $\xi$ (sufficiently close to the axisymmetric vector field $\xi_0 = R e_\phi$), in the sense that (1.8) holds but that (1.9) holds up to a small error.

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4See Lortz [21] for a construction of a non-axisymmetric toroidal equilibrium which nevertheless enjoys plane reflection symmetry (forcing all magnetic field lines to be closed).
In addition to the nontrivial constraint \( \frac{1}{|\xi|^2} \), there are two serious difficulties in constructing solutions to (1.1)–(1.3) of the form (1.11) with given symmetry direction \( \xi \). The first is that by (1.14), unlike in the axisymmetric setting, vector fields of the form (1.11) need not be divergence-free. The second difficulty is that for arbitrary \( \xi \), it is not at all clear that the equation (1.13) admits any solutions with \( \xi \cdot \nabla \psi = 0 \), since the coefficients appearing in (1.13) need not be invariant under \( \xi \). Both of these difficulties can be traced to the fact that \( \xi \) need not be a Killing field with respect to the Euclidean metric. To circumvent these issues, inspired by [20] and [4], we will replace the metric structure of \((\mathbb{R}^3, \delta)\) with \((\mathbb{R}^3, g)\) for a metric \( g \) for which \( \xi \) is a Killing field. The resulting magnetic field will not satisfy the Euclidean MHS equations (1.1) but provided \( \xi \) is sufficiently close to the Euclidean metric, the error will be small. We now explain the idea.

Let us suppose that given \( \xi \), we can find a metric \( g \) on \( \mathbb{R}^3 \) for which \( \mathcal{L}_\xi g = 0 \), that is, for which \( \xi \) generates an isometry (we give an explicit construction of such metrics for a large class of vector fields \( \xi \) after the upcoming statement of Theorem [1.1]). We then consider the following generalization of the ansatz (1.11), introduced in [4]

\[
B_g = \frac{1}{|\xi|^2} \left( C(\psi)\xi + \sqrt{|g|} \xi \times g \nabla \psi \right). \tag{1.16}
\]

Here, \( |\xi|, \times, \nabla \) denote the analogs of the usual Euclidean quantities \( |\xi|, \times, \nabla \) with respect to the metric \( g \) (see Appendix B). In Lemma C.2 we use the fact that \( \mathcal{L}_\xi g = 0 \) to show that vector fields of this form are divergence free assuming only that \( \xi \cdot \nabla \psi = 0 \),

\[
\text{div} B_g = 0, \tag{1.17}
\]

and also that \( \psi \) is a flux function for \( B_g \),

\[
\xi \times \nabla \psi = B_g. \tag{1.18}
\]

We emphasize the somewhat surprising fact that even though the definition (1.16) involves the metric \( g \) in a nontrivial way, it is designed that way so that the identities (1.17)--(1.18) involve only Euclidean quantities. We remark that \( B_g \) will not be divergence-free with respect to the \( g \) metric.

We then seek \( B_g \) of the form (1.16) which satisfy the MHS with respect to the metric \( g \),

\[
\text{curl}_g B_g \times g B_g = \nabla_g P. \tag{1.19}
\]

This ansatz leads to the generalized Grad-Shafranov equation for \( \psi \)

\[
\text{div}_g \left( \sqrt{|g|} \nabla_g \psi \right) - C(\psi) \frac{\xi}{|\xi|^2} \cdot g \text{curl}_g \left( \frac{\xi}{|\xi|^2} \right) + \frac{C(\psi)C'(\psi)}{\sqrt{|g|}|\xi|^2} + \frac{P'(\psi)}{\sqrt{|g|}} = 0, \quad \text{in } T, \tag{1.20}
\]

\[
\psi = (\text{const.}), \quad \text{on } \partial T. \tag{1.21}
\]

where \( |\xi|, \cdot, \times, \text{curl} \) denote the magnitude, dot product and curl with respect to the metric \( g \) (see Appendix B for the definitions and Appendix C for the derivation of (1.20) from (1.16) and (1.19)). Note that (1.20), (1.21) reduces to (1.12)–(1.13) when \( g = \delta \), and when \( g \) is the circle-averaged metric, it agrees with the equation derived in [4].

As another consequence of the fact that \( \mathcal{L}_\xi g = 0 \), the coefficients in the equation (1.20) are invariant under \( \xi \) and so (1.20), unlike (1.13), is consistent with the requirement \( \xi \cdot \nabla \psi = 0 \). The downside is that the equation (1.19) does not agree with (1.1) unless \( g = \delta \) and so \( B_g \) will not satisfy the original MHS equations. However, if we can arrange for the metric \( g \) to be sufficiently close to the Euclidean metric \( \delta \), then \( B_g \) will satisfy the usual MHS equations \( \text{curl} B_g \times g B_g - \nabla P = 0 \) up to a small error. Our approach will be to solve the generalized Grad-Shafranov equation (1.20) by deforming an appropriate solution \( \psi_0 \) of the axisymmetric Grad-Shafranov equation (1.5), using the methods from [6]. In particular, we seek a diffeomorphism \( \gamma : D_\delta \to D \) and requiring that \( \psi = \psi_0 \circ \gamma^{-1} \). It turns out (see section 2) that this reduces to a system of nonlinear elliptic equations for the components of \( \gamma \) which can be solved by an iteration.
In what follows, \( T_0 \) denotes the axisymmetric torus
\[
T_0 = \{(R, \Phi, Z) \mid (R - R_0)^2 + Z^2 \leq a, 0 \leq \Phi \leq 2\pi\},
\]
with thickness \( 0 < a \ll R_0 \). Let \( \xi_0 = \text{Re} \Phi \) be the generator of rotations in the \( Z = 0 \) plane. Let \( D \) be any domain in the half-plane \( \{\Phi = 0\} \) sufficiently close to \( D_0 \). Suppose that \( \xi \) is a vector field which is sufficiently close to the rotation field \( \xi_0 \) with the property that all the orbits of \( \xi \) starting from \( D \) are periodic (with possibly different period \( \tau(p) \)). In this case we define the toroidal domain
\[
T = \{\varphi_s(p) \mid p \in D, \ s \in [0, \tau(p))\},
\]
where \( \varphi_s(p) \) denotes the time-\( s \) flow of \( \xi \) starting from \( p \in D \),
\[
\frac{d}{ds} \varphi_s(p) = \xi(\varphi_s(p)), \quad \varphi_0(p) = p \in D.
\]
In this setting we say that the toroidal domain \( T \) is swept out by \( \xi \) from \( D \).

Figure 1. Tokamak and stellarator geometries.

Our main theorem is that, given a toroidal domain \( T \) sufficiently close to the axisymmetric torus \( T_0 \), we can find magnetic fields with nested flux surfaces and a global symmetry that solve MHS up to a small force whose magnitude is controlled by the deviation of the symmetry from being Euclidean. These fields satisfy two of the three quasisymmetry conditions, the third holding approximately. The proof is constructive and relies on deforming a known axisymmetric steady state satisfying mild conditions \((\text{H1})-(\text{H2})\) stated in \( \S \). In particular, the resulting magnetic field possesses flux surfaces which have the same topology as the axisymmetric base state.

**Theorem 1.1.** Fix \( k \geq 0, \alpha > 0 \) and let \( \xi \in C^{k+2,\alpha}(\mathbb{R}^3) \) be a divergence-free vector field, sufficiently close in \( C^{k+2,\alpha} \) to the rotation vector field \( \xi_0 = \text{Re} \Phi \) and let \( D \) be a domain sufficiently close to \( D_0 \) in \( C^{k+2,\alpha} \) in the sense that \( D = \{(r, \theta) \mid 0 \leq r \leq b(\theta), \theta \in S^1\} \) for a function \( b : S^1 \rightarrow \mathbb{R} \) sufficiently close to 1 in \( C^{k+2,\alpha}(S^1) \). Let \( \psi_0 \in C^{k+2,\alpha}(D_0) \) be a solution of \((1.1)-(1.6)\) with pressure \( P_0 \in C^{k+1,\alpha}(\mathbb{R}) \) and swirl \( C_0 \in C^{k+1,\alpha}(\mathbb{R}) \) satisfying \((\text{H1})-(\text{H2})\).

Let \( \xi \) be a divergence-free non-vanishing vector field with closed integral curves that sweeps out a toroidal domain \( T \) from \( D \). Suppose that there is a metric \( g \in C^{k+2,\alpha}(\mathbb{R}^3) \) with the property \( \mathcal{L}_\xi g = 0 \) which is sufficiently close to the Euclidean metric. Then, for any given swirl \( C \in C^{k+1,\alpha}(\mathbb{R}) \) sufficiently close to \( C_0 \), there is a flux function \( \psi \in C^{k+2,\alpha}(T) \), and a pressure \( P = P(\psi) \) so that the magnetic field \( B \) defined by \((1.16)\) satisfies \( B \cdot \nabla \psi = 0, B \times \xi = \nabla \psi \) as well as MHS \((1.1)-(1.3)\) with a force \( f \) obeying
\[
\|f\|_{C^{k,\alpha}(T)} \leq c\|\delta - g\|_{C^{k+2,\alpha}(T)},
\]
where \( c := c(\|\xi\|_{C^{k+2,\alpha}(T)}, \|\psi\|_{C^{k+2,\alpha}(T)}, \|P_0\|_{C^{k+1,\alpha}(\mathbb{R})}, \|C_0\|_{C^{k+1,\alpha}(\mathbb{R})}). \) Moreover, the flux surfaces of \( B \) (isosurfaces of \( \psi \)) are diffeomorphic to the isosurfaces of \( \psi_0 \).
The point of the bound \(\ref{eq:1.22}\) is that if \(\xi\) is a Killing field for the Euclidean metric \(\delta\) then we can take \(g = \delta\) in the above and by \(\ref{eq:1.22}\), the \(B\) is then an exact solution of the MHS equations \(\ref{eq:1.1}\) with \(f = 0\). In this sense, \(\ref{eq:1.22}\) shows that one can construct approximate solutions to MHS with symmetry direction \(\xi\) with error proportional to how far \(\xi\) is from being a symmetry of \(\mathbb{R}^3\). We remark that the proof is quantitative in that all the small parameters can be explicitly defined in terms of the inputs \(\psi_0, C_0, P_0,\) and \(\xi\).

We now describe how to produce a base state \(\psi_0\) and metric \(g\) which are suitable inputs for Theorem \(\ref{thm1}\). In Appendix \(\ref{appA}\) we provide an example of a base state \(\psi_0\) satisfying \((H1)-(H2)\) living on a large aspect-ratio torus. This is obtained as a perturbation of an explicit profile on an “infinite aspect-ratio” torus. It should be stressed that the conditions \((H1)-(H2)\) are not very stringent and should hold for a wide class of axisymmetric solutions that possess simple nested flux surfaces (which could e.g. be numerically obtained). Next, we describe two large classes of vector fields \(\xi\) and metrics \(g\) satisfying the hypotheses of our theorem.

\textbf{Remark (Designer metrics).} We provide two possible ways of constructing a ‘near’ Euclidean metric given a ‘near’ isometry \(\xi\).

1. **(Deformed metric):** Suppose that the torus \(T\) is given by \(T = f(T_0)\) where \(f\) is a diffeomorphism defined in a neighborhood of \(T_0\) and which is sufficiently close to the identity. Then we can take \(\xi = df(\xi_0)\) where \(df\) denotes the differential and let \(g = f^*\delta\) denote the pullback of the Euclidean metric \(\delta\) by \(f\). Because the Lie derivative is invariant under diffeomorphisms, we have \(\xi\) is a Killing field for \(g\) since \(L_\xi g = f^*(L_{\xi_0}\delta) = 0\).

2. **(Circle averaged metric):** Suppose the orbits of \(\xi\) starting from \(D\) are all \(2\pi\)-periodic. In this case we say that \(\xi\) generates a circle-action. If we define the circle-averaged metric \(g\) by

\[
g = \frac{1}{2\pi} \int_0^{2\pi} \varphi_s^* \delta \, ds,
\]

it follows by a simple computation that \(L_\xi g = 0\). Moreover, when \(\xi\) is a Killing field for Euclidean space, \(g = \delta\). This metric was introduced by [4]. As motivation for the appearance of this particular metric, consider the MHS in terms of one-forms \(L_B(B) = dP\) (see [1]). In this representation, it is clear that the metric appears linearly (in the definition of \(b\)). Therefore, if \(B\) and \(P\) are invariant under the flow of \(\xi\), then one finds \(L_B(B) = dP\) where \(\tilde{5}\) denotes lower the index with the circle average metric. Raising indices with \(g\), we find that any such MHS solution on Euclidean space is also a solution of the circle averaged equation (MHS with respect to the metric \(g\)).

We conclude with some remarks about achieving exact quasisymmetry. By construction, the magnetic field \(B\) from the previous theorem will satisfy \(\ref{eq:1.8}\) but will only approximately satisfy the property \(\ref{eq:1.9}\) of quasisymmetry. Thus, our fields confine particles to zeroth but not first order in the guiding center approximation \(\ref{eq:2.22}\). The error from being an exact weak quasisymmetry can be easily quantified; for a vector field \(B_g\) of the form \(\ref{eq:1.16}\), assuming that \(g\) is such that \(L_\xi g = 0\) the condition \(\ref{eq:1.9}\) reads

\[
\xi \cdot \nabla |B_g| = \frac{C^2(\psi)}{2|\xi|_g^4 |B_g|^3} [(L_\xi \delta)(\xi, \xi) + 2C^{-1}(\psi)(L_\xi \delta)(\xi, \nabla^\perp g^\perp \psi) + C^{-2}(\psi)(L_\xi \delta)(\nabla^\perp g^\perp \psi, \nabla^\perp g^\perp \psi)] . \tag{1.23}
\]

See Lemma \(\ref{lemC.5}\). Since \(\ref{eq:1.23}\) involves the Euclidean deformation tensor alone, it is controlled by the deviation of \(\xi\) from being a Euclidean isometry and our solution will have \(\xi \cdot \nabla |B_g|\) small. The error from being a strong quasisymmetry is also quantifiably small.

It is worth remarking that there are additional freedoms in our construction that could, in principle, be used to further constrain the constructed solution. Specifically, in our theorem, we treat \(\xi\) as a fixed vector field sufficiently close to \(\xi_0\) and we made the somewhat arbitrary choice
that the map $\gamma$ should be volume preserving. The results in [6] actually allow one to construct the map $\gamma$ so that $\det \nabla \gamma := \rho$ is any given function, sufficiently close to one; in fact by iterating that result, one can additionally achieve that $\det \nabla \gamma = X(\phi, \eta, \partial \phi, \partial \eta, \partial \partial_\phi \phi, \partial \partial_\eta \eta)$ for a suitable nonlinearity $X$ sufficiently close to one when $\phi, \eta = 0$. Using this freedom, it is possible to show that, under some (possibly restrictive and undesirable) assumptions on the field $\xi$, the Jacobian $\rho$ can be used to achieve exact quasisymmetry on a slice of the torus (namely on the cross-section $D$). Due to its local character, ensuring this property holds seems of little practical interest for ion confinement in a stellarator. Whether or not a carefully designed field $\xi$, perhaps constructed dynamically alongside the solution, can be used to ensure quasisymmetry exactly on the whole domain remains an open issue.

2. Proof of Theorem 1.1

We start by giving an outline of the arguments used to establish the main theorem. All details can be found in [6].

Let $D_0, D, \psi_0, \xi, g$ be as in the statement of Theorem 1.1. We will start by constructing a solution $\psi$ to the generalized Grad-Shafranov equation (1.20) of the form $\psi = \psi_0 \circ \gamma^{-1}$ for a the unknowns. For simplicity, using the assumption that Vol $\gamma$ is done.

The Grad-Shafranov equation (1.5) is of the form

$$L_0 \psi_0 = F_0(\psi_0) + G_0(r, \theta, \psi_0), \quad \text{in } D_0$$

with nonlinearities $F_0 = F_0', G_0 = \frac{1}{r} F_0 C_0 \psi_0$ and where the operator $L_0$ is elliptic. Similarly, we write the generalized Grad-Shafranov (1.20) in the form

$$L \psi = F(\psi) + G(r, \theta, \psi), \quad \text{in } D.$$  

At this stage, the function $F$ (which is related to the pressure of the solution in our application) is actually undetermined and will be chosen momentarily, while $G$ can be chosen to be any function sufficiently close to $G_0$.

A calculation (see Appendix B of [6]) shows that provided $\det \nabla \gamma^{-1} = 1$, we have

$$(\nabla \psi) \circ \gamma^{-1} = \nabla \psi_0 + \nabla \partial_\phi \phi - \nabla \partial_\phi \phi \cdot \nabla \psi_0 + \nabla \phi \cdot \nabla \psi_0 + \nabla \eta \cdot \nabla \psi_0,$$  

where we have introduced the notation $\partial_\phi = \nabla \psi_0 \cdot \nabla \psi_0$ for the "streamline derivative". Then $\partial_\phi$ is tangent to level sets of $\psi_0$. After a computation, composing both sides of (2.2) with $\gamma^{-1}$ and using (2.3), (2.2) takes the form

$$L \psi_0 \partial_\phi \phi = N_\phi[\phi, \eta] + F(\psi_0) - F_0(\psi_0) + G(r, \theta, \psi_0) - G_0(r, \theta, \psi_0),$$

(2.4)

where $L \psi_0$ is the linearization of $L_0$ around $\psi_0$, for a function $N_\phi[\phi, \eta] = N_\phi(\partial \phi, \partial \eta, \partial \partial_\phi \phi, \partial \partial_\eta \eta)$ (whose explicit form can be found in Appendix B of [6]), which consists of terms which are linear in $\eta$ and its derivatives, and either nonlinear or weakly linear in derivatives of $\phi$, meaning it involves terms which can bounded by $e(\partial \phi)$, for example. This latter point is a consequence of the assumption that $\xi - \xi_0$ is sufficiently small. Notice that this is an equation for $\partial_\phi \phi$ and not $\phi$ itself. In order for this equation to be solvable for $\phi$ at the linear level (given appropriate boundary conditions), there
are two requirements. The first is that $\mathcal{L}_{\psi_0}$ should be invertible. The second is a somewhat subtle condition which is easiest to understand in the simple model case. In order to solve the problem $\Delta \partial_\eta u = f$, in the unit disk, say (with arbitrary boundary conditions), it is clearly necessary that $\int_0^{2\pi} f \, d\theta = 0$. We will now impose a condition on (2.4) which is analogous to this one and which will determine the function $F$ at the linear level. Assume that the Dirichlet problem for $\mathcal{L}_{\psi_0}$,

$$
\mathcal{L}_{\psi_0} u = f, \quad \text{in } D_0,
$$

$$
u = 0, \quad \text{on } \partial D_0,
$$

has a unique solution for $f \in L^2$, say. Writing $\mathcal{L}_{\psi_0}^{-1} f = u$, if we apply $\mathcal{L}_{\psi_0}^{-1}$ to both sides of (2.4) and integrating with respect to $ds$ over the streamline $\{\psi_0 = c\}$ we find

$$
0 = \oint_{\{\psi_0 = c\}} \mathcal{L}_{\psi_0}^{-1} (F(\psi_0) - F_0(\psi_0)) \, ds + \oint_{\{\psi_0 = c\}} \mathcal{L}_{\psi_0}^{-1} (G - G_0 + N) \, ds.
$$

This is an equation which must be solved for $F$. Writing $T(c)q = \oint_{\{\psi_0 = c\}} \mathcal{L}_{\psi_0}^{-1} q \, ds$, given $R$ depending only on the streamline $R = R(c)$, we would like to be able to find $q = q(c)$ with $T(c)q = R$. This is a complicated problem which would be hard to address directly, however in [6], we show that such $q$ can be found, assuming that the following hypotheses hold:

Hypothesis 1 (H1): The operator $\mathcal{L}_{\psi_0}$ is positive definite.

Hypothesis 2 (H2): There exists a constant $C > 0$ such that we have

$$
\mu(c) = \oint_{\{\psi_0 = c\}} \frac{d\ell}{|\nabla \psi_0|} \leq C \quad \text{for } c \in \text{rang}(\psi_0),
$$

where $\ell$ is the arc-length parameter.

Notice that the hypothesis (H1) in particular ensures that the operator $\mathcal{L}_{\psi_0}^{-1}$ is well-defined. This is a condition on $P_0, C_0$. Hypothesis (H2) concerns the travel time $\mu(c)$ for a particle governed by the Hamiltonian system $\dot{x} = \nabla \psi_0(x)$ and moving along the streamline of $\{\psi_0 = c\}$. It is easy to see that it holds provided $\dot{\psi}_0$ has at most one critical point in $D_0 = T_0 \cap \{\varphi = 0\}$ and that it vanishes no faster than to first order there. We remark that (H2) is trivially satisfied if $|\nabla \psi_0|$ is bounded below in the domain $D_0$. This could be accomplished if, for example, one worked on a "hollowed out" toroidal domain.

We now discuss the boundary conditions. Assume that $D$ is the interior of a Jordan curve $B$, $\partial D = \{p \in \mathbb{R}^2 \mid b(p) = 0\}$.

We also write $\partial D_0 = \{p \in \mathbb{R}^2 \mid b_0(r, \theta) = 0\}$ where $b_0$ is chosen with $|\nabla b_0| = 1, \nabla \psi_0 \cdot \nabla b_0 > 0$. We write $\gamma - \text{id} = \nabla \eta + \nabla^\perp \dot{\phi} = \alpha e_x + \beta e_y$ where $(x, y)$ are rectangular coordinates. Using that $\mathcal{B}_0|_{\partial D_0} = 0$, the requirement that $\gamma : \partial D_0 \to \partial D$ can be written as

$$
0 = b \circ \gamma|_{\partial D_0} = b_0 \circ \gamma|_{\partial D_0} + (\delta b) \circ \gamma|_{\partial D_0}
$$

$$
= \alpha \partial_x b_0|_{\partial D_0} + \beta \partial_y b_0|_{\partial D_0} + b_1(\alpha, \beta)|_{\partial D_0}
$$

(2.5)

where $\delta b = b - b_0$, and where the remainder $b_1$ is

$$
b_1(\alpha, \beta, x, y) = b_0 \circ \gamma - b_0 - \alpha \partial_1 b_0 - \beta \partial_2 b_0 + (\delta b) \circ \gamma.
$$

Returning to $\phi, \eta$, we have

$$
\alpha \partial_1 b_0 + \beta \partial_2 b_0 = \nabla \phi \cdot \nabla^\perp b_0 + \nabla \eta \cdot \nabla b_0
$$

(2.6)
By the choice of $b_0$, we have $\nabla \eta \cdot \nabla b_0 = \partial_n \eta$ where $n$ is the outward-facing normal to $\partial D_0$. Additionally using that $\psi_0$ is constant on the boundary we have $\nabla \psi_0 b_0 = \frac{\nabla \psi_0}{|\nabla \psi_0|}$, and using (2.6) and these observations, the formula (2.5) becomes

$$\frac{1}{|\nabla \psi_0|} \partial_n \phi + \partial_n \eta = -b_1(\phi, \eta), \quad \text{on } \partial D_0.$$ 

This is one boundary condition for the two functions $\phi, \eta$. Again we need to ensure that this equation is compatible with the requirement $\int_{\{\psi_0=\psi_0|_{\partial D_0}\}} \partial_n \phi \, ds = 0$. We therefore take $\partial_n \eta$ constant on the boundary and impose the following nonlinear boundary conditions.

$$\partial_n \eta = -\frac{\int_{\partial D_0} b_1(\phi, \eta) \, d\ell}{\text{length}(\partial D_0)} \quad \text{on } \partial D_0, \quad (2.7)$$

$$\partial_s \phi = |\nabla \psi_0| \left( -b_1(\phi, \eta) + \frac{\int_{\partial D_0} b_1(\phi, \eta) \, d\ell}{\text{length}(\partial D_0)} \right) \quad \text{on } \partial D_0, \quad (2.8)$$

We now summarize the result of the above calculation. The function $\psi = \psi_0 \circ \gamma^{-1}$ is a solution of the equation (2.2) in $D$ with constant boundary value provided the diffeomorphism $\gamma$ is of the form $\gamma = \text{id} + \nabla \eta + \nabla \psi_0$ and the functions $\eta, \phi : D_0 \to \mathbb{R}$ satisfy the elliptic equations

$$\Delta \eta = \mathcal{N}_\eta(\phi, \eta) \quad \text{in } D_0,$$

$$\mathcal{L}_\psi \partial_s \phi = \mathcal{N}_\phi(\phi, \eta) + F(\psi_0) - F_0(\psi_0) + G(\psi_0, r, \theta) - G_0(\psi_0, r, \theta), \quad \text{in } D_0,$$

where $F$ is determined by solving

$$\int_{\{\psi_0=c\}} \mathcal{L}_\psi^{-1} F \, ds = \int_{\{\psi_0=c\}} \mathcal{L}_\psi^{-1} (G_0 - G - \mathcal{N}_\phi + F_0) \, ds,$$

and where $\eta, \phi$ satisfy the boundary conditions (2.7)-(2.8).

This nonlinear system can be solved by the following iteration scheme. Given $\eta^{N-1}, \phi^{N-1}$, define $F^N = F^N(c)$ by solving

$$\int_{\{\psi_0=c\}} \mathcal{L}_\psi^{-1} F^N \, ds = \int_{\{\psi_0=c\}} \mathcal{L}_\psi^{-1} (G_0 - G - \mathcal{N}_\phi[F^{N-1}, \eta^{N-1}] + F_0) \, ds,$$

Then solve for $\eta^N, \phi^N$ satisfying

$$\Delta \eta^N = \mathcal{N}_\eta[\phi^{N-1}, \eta^{N-1}] \quad \text{in } D_0, \quad (2.9)$$

$$\mathcal{L}_\psi \phi^N = \mathcal{N}_\phi[\phi^{N-1}, \eta^{N-1}] + F^N - F_0 + G - G_0 \quad \text{in } D_0,$$

with boundary conditions

$$\partial_n \eta^N = \int_{D_0} \mathcal{N}_\eta[\phi^{N-1}, \eta^{N-1}] \, dx, \quad (2.10)$$

$$\phi^N = |\nabla \psi_0| \left( -b_1(\phi^{N-1}, \eta^{N-1}) + \frac{\int_{\partial D_0} b_1(\phi^{N-1}, \eta^{N-1}) \, d\ell}{\text{length}(\partial D_0)} \right) \quad \text{on } \partial D_0.$$ 

The boundary condition for $\eta^N$ has been chosen so that the Neumann problem (2.9)-(2.10) is solvable. Once $\Phi^N$ has been found, as a consequence of the choice of $F^N$ it can be shown that $\int_{\{\psi_0=c\}} \Phi^N \, ds = 0$ for all $c$ and so $\Phi^N = \partial_n \phi^N$ for a function $\phi^N$ which is determined up to a constant, which can be fixed throughout the iteration by requiring that $\int_{D_0} \phi^N = 0$. In [6] we prove that this iteration converges in a suitable topology. We remark that the boundary condition (2.10) is not the same as the boundary condition in (2.7) but as a consequence of $\text{Vol } D_0 = \text{Vol } D$, they agree after taking $N \to \infty$.

We now present the proof of the main result.
PROOF OF THEOREM 1.1. We work in coordinates \((r, \theta, \varphi)\), defined in terms of the usual cylindrical coordinates \((R, Z, \Phi)\) by
\[
R = R_0 + r \cos \theta, \quad Z = r \sin \theta, \quad \Phi = \varphi.
\] (2.11)

In these toroidal coordinates, Grad-Shafranov eqn. (1.5) reads
\[
\partial_r^2 \psi_0 + \frac{1}{r} \partial_r \psi_0 + \frac{1}{r^2} \partial_r^2 \psi_0 - \frac{1}{R} \left( \cos \theta \partial_r \psi_0 - \frac{\sin \theta}{r} \partial_\theta \psi_0 \right) + R^2 P'_0(\psi_0) + C_0 C'_0(\psi_0) = 0
\] (2.12)
where we are writing \(R = R(r, \theta) = R_0 + r \cos \theta\) (see [8]). We write (2.12) in the form (2.1) with
\[
L_0 = \sum_{i,j=1}^2 a_{ij}^r \partial_r \partial_r \psi + \sum_{i=1}^2 b_i^r \partial_r \psi, \quad x^1 = r, \ x^2 = \theta
\]
with coefficients defined by

\[
a_{0r}^r = \frac{1}{R^2}, \quad a_{0r}^\theta = 0, \quad a_{0\theta}^\theta = \frac{1}{r^2 R^2}, \quad b_0^r = \frac{1}{R^2} \left( \frac{1}{r} - \frac{1}{R} \cos \theta \right), \quad b_0^\theta = \frac{1}{R^3} \sin \theta.
\]

By assumption there is a metric \(g\) with \(L_\xi g = 0\). We now express this metric in the toroidal coordinates (2.11),
\[
g = g_{rr} dr^2 + 2g_{r\theta} drd\theta + g_{\theta\theta} d\theta^2,
\]
where \(g_{ij} = g(\partial_i, \partial_j)\) for \(i, j = r, \theta\), and the equation (1.20) can be written in the form
\[
L \psi = F(\psi) + G(x, \psi),
\] (2.13)
with
\[
L \psi = \sum_{i,j=1}^2 a_{ij}^r \partial_r \partial_r \psi + \sum_{i=1}^2 b_i^r \partial_r \psi, \quad x^1 = r, x^2 = \theta
\]
where

\[
a_{rr} = \frac{\sqrt{|g|}}{|\xi|_g^2} g_{rr}, \quad a_{r\theta} = \frac{\sqrt{|g|}}{|\xi|_g^2} g_{r\theta}, \quad a_{\theta\theta} = \frac{\sqrt{|g|}}{|\xi|_g^2} g_{\theta\theta}, \quad b_r = \frac{\sqrt{|g|}}{|\xi|_g^2} g_{rr} \partial_r \left( \frac{\sqrt{|g|}}{|\xi|_g^2} \right) + g_{r\theta} \partial_\theta \left( \frac{\sqrt{|g|}}{|\xi|_g^2} \right).
\]

Here \(g_{ij}\) denote the components of the inverse metric, \(g_{ij} g_{jk} = \delta^i_k\). The nonlinearities \(F, G\) in (2.13) are given by
\[
F(\psi) = \frac{P'(\psi)}{\sqrt{|g|}}, \quad G(x, \psi) = \frac{CC'(\psi)}{\sqrt{|g||\xi|_g^2}} - C(\psi) \frac{\xi}{|\xi|_g^2} \cdot g \text{curl}_g \left( \frac{\xi}{|\xi|_g^2} \right).
\] (2.14)

We now note that since \(L_\xi g = 0\), all the coefficients \(a_{ij}, b^i\) and the coefficients appearing in \(G\) are invariant under \(\xi\) and thus functions of \(r, \theta\) alone.

In order to be able to appeal to the results of [6] we will need that the domain \(D\) is close to \(D_0\). For simplicity, we take the function \(C\) in (2.14) to just be \(C_0\) though this is not essential. From the above explicit formulas we have
\[
\|a_{ij} - \delta^i_j\|_{C^{k,\alpha}} + \|b^i - \delta^i_0\|_{C^{k,\alpha}} + \|G - G_0\|_{C^{k,\alpha}} \leq C \sum_{i,j=1}^3 \|g_{ij} - \delta_{ij}\|_{C^{k+2,\alpha}}
\] (2.15)
where $C$ is a constant depending on $k, \alpha, \sum_{i,j=1}^{3} \|g_{ij}\|C_{k+2,\alpha}$, and $\|\nabla \xi\|C_{k+3,\alpha}$. Here, and in what follows, we are writing $\psi^{k+2,\alpha} = \psi^{k+2,\alpha}(U)$ where $U$ is a domain containing both $D$ and $D_0$. By Theorem 3.1 from [6], there is $\epsilon > 0$ depending on $k, \alpha, \psi_0, D_0$ so that if the following hold,

$$\|a_{ij} - \bar{a}_{ij}\|C_{k,\alpha} + \|b_{ij} - \bar{b}_{ij}\|C_{k,\alpha} + \|G - G_0\|C_{k,\alpha} + \|b - b_0\|C_{k+2,\alpha} \leq \epsilon, \quad (2.16)$$

and the hypotheses (H1) and (H2) hold, there is a function $\psi \in C^{k,\alpha}$ of the form $\psi = \psi_0 \gamma^{-1}$ where $\gamma: D \to D$ is a diffeomorphism and $\psi$ satisfies the generalized Grad-Shafranov equation (1.20) for some pressure profile $P$ close to $P_0$. Using the bound (2.15), we now take $\|\xi - \xi_0\|C_{k+1,\alpha}$ and $\|b - b_0\|C_{k+2,\alpha}$ small enough that (2.16) hold and let $\psi$ be the flux function guaranteed by Theorem 3.1 from [6]. This function is only defined on $D$ but we can extend it to a function $\widetilde{\psi}$ on $T$ by taking $L_\xi \psi = 0$ and $\widetilde{\psi}|_D = \psi$. Since the integral curves of $\xi$ are closed, the resulting $\tilde{\psi}$ is well-defined and as smooth as $\psi_0$. Define $B_\theta$ by (1.16) with $\psi$ replaced by $\psi$. By formula (C.3) in Lemma C.2 using the fact that $L_\xi g = 0$, we have $\text{div} B_\theta = 0$ and from (C.1) we have that (C.8) holds. Writing the generalized Grad-Shafranov equation (1.20) as $G_\xi[\psi] = 0$, by the bounds in (2.16) for the coefficients we have that $G_\xi[\psi, \nabla \psi, \nabla^2 \psi] = E$ where $E := G_\xi[\psi] - G_\xi[\psi_0]$ satisfies (1.22). The vector field $B_\theta$ exactly satisfies the $g$-MHS equations and therefore the satisfies the MHS equation (1.1) with force given by the formula (C.8) which involves the deformation tensor. Note that the force in the direction $\nabla \psi$ is simply $\frac{1}{|\nabla \psi|} E$.

\[\square\]

Appendix A. Flux function satisfying our hypotheses

The purpose of this section is to give a simple example of a flux function satisfying the hypotheses (H1)-(H2). This flux function is not an exact solution of (2.12) but satisfies it when the aspect ratio of the torus is taken to infinity. Using Theorem 3.1 from [6], one can show that there exist solutions on the true axisymmetric torus with large aspect ratio nearby this example. Although they do not have a simple analytical form, they will continue to satisfy (H1)-(H2) as these are open conditions.

We consider the torus where $r$ ranges in $[0, r_0]$ with $0 < r_0 < R_0$ for $R_0 > 1$ and solve the equation (2.12) with the choices

$$\psi_0 (r) = \psi \left(1 - (r/r_0)^2\right), \quad C_0 (\psi) = \bar{c} \sqrt{\psi - \psi + \epsilon}, \quad P_0 (\psi) = \bar{p} (r_0 R_0)^{-2} \psi,$$

for $\epsilon \ll 1$ (this is to regularize the square-root) and for some constants $\bar{\psi}, \bar{c}$ and $\bar{p}$. The functions $C_0$ and $P_0$ are both infinitely differentiable functions of $\psi$. Note that the pressure vanishes at the outer boundary where $\psi_0$ is zero, and so this boundary may be interpreted as vacuum. For special choices of constants, $\psi_0$ solves the “infinite aspect ratio” Grad-Shafranov equation (2.12) as $R_0 \gg 1$)

$$\partial_r^2 \psi_0 + \frac{1}{r^2} \partial_\theta^2 \psi_0 = -R_0^2 P_0'(\psi_0) - C_0 C'_0(\psi_0),$$

since $\partial_r^2 \psi_0 = -2\bar{\psi}/r_0^2, R_0^2 P_0'(\psi_0) = \bar{p}/r_0^2, \text{and} C_0 C'_0(\psi_0) = \bar{c}^2$. Thus $\psi_0$ is a solution if $\bar{p} = 2\bar{\psi} - (\bar{c} r_0)^2$.

Appendix B. Geometric identities

In this section we recall some basic definitions and facts from Riemannian geometry which will be used in the upcoming sections. These are standard and we include the details for the convenience of the reader. Throughout we fix a Riemannian metric $g$. In our applications, we will take either $g = \delta$, the Euclidean metric, or $g$ will be a metric with $L_\xi g = 0$ for a given vector field $\xi$. We let $\flat, \sharp$ denote the usual operations of lowering and raising indices with respect to $g$. If $X = X^i \partial_i$ is a vector field and $\beta = \beta_j dx^j$ is a one-form, where $\{x^i\}_{i=1}^3$ are arbitrary local coordinates, then

$$X^\flat = g_{ij} X^j dx^i, \quad \beta^\sharp = g^{ij} \beta_j \partial_i.$$
We write \( \nabla_g f \) for the gradient of \( f \) with respect to the metric \( g \),
\[
\nabla_g f = (df)^i, \quad (\nabla_g f)^i = g^{ij} \partial_j f. \tag{B.1}
\]

In an arbitrary coordinate system \( \{x^i\}_{i=1}^3 \), if \( X = X^i \partial_i \) is a vector field and \( \beta = \beta_i dx^i \) is a one-form then \( \nabla X, \nabla \beta \) have components
\[
\nabla_i X^j = \frac{\partial}{\partial x^i} X^j + \Gamma^j_{ik} X^k, \quad \nabla_i \beta_j = \frac{\partial}{\partial x^i} \beta_j - \Gamma^k_{ij} \beta_k, \tag{B.2}
\]
where \( \Gamma^i_{jk} \) are the Christoffel symbols in this coordinate system, defined by
\[
\Gamma^k_{ij} = \frac{1}{2} g^{k\ell} (\partial_i g_{\ell j} + \partial_j g_{\ell i} - \partial_{\ell} g_{ij}). \tag{B.3}
\]

Here we are writing \( g_{ij} \) for the components of the metric in this coordinate system and \( g^{ij} \) for the components of the inverse metric. The \( \Gamma \) are symmetric in the lower indices,
\[
\Gamma^k_{ij} = \Gamma^k_{ji}. \tag{B.4}
\]

We also note that covariant differentiation commutes with lowering and raising indices since
\[
\nabla_i g_{jk} = \nabla_j g^{ik} = 0. \tag{B.5}
\]

Let us also recall that the divergence of a vector field can be written as
\[
\text{div}_g X = \nabla_i X^i = \frac{1}{\sqrt{|g|}} \partial_i (\sqrt{|g|} X^i),
\]
where \( |g| = \det g \) denotes the determinant of the matrix with components \( g_{ij} \).

We let \( \mathcal{L}_X \) denote the Lie derivative in the direction \( X \). If \( f \) is a function then \( \mathcal{L}_X f \) is defined by
\[
\mathcal{L}_X f = X^i \partial_i f = Xf.
\]
For a vector field \( Y \), \( \mathcal{L}_X Y \) is the commutator \( \mathcal{L}_X Y = [X,Y] \). In an arbitrary coordinate system, \( \mathcal{L}_X Y = (\mathcal{L}_X Y)^i \partial_i \) with
\[
(\mathcal{L}_X Y)^i = X^j \partial_j Y^i - Y^j \partial_j X^i.
\]

Many of our results will be stated in terms of the deformation tensor of \( X \), denoted \( \mathcal{L}_X g \), which is the \((0,2)\) tensor defined by the formula
\[
X(g(Y,Z)) = (\mathcal{L}_X g)(Y,Z) + g(\mathcal{L}_X Y,Z) + g(Y,\mathcal{L}_X Z) \tag{B.6}
\]
In an arbitrary coordinate system, \( \mathcal{L}_X g = \mathcal{L}_X g_{ij} dx^i dx^j \) and a standard calculation shows that
\[
\mathcal{L}_X g_{ij} = \nabla_i X_j + \nabla_j X_i, \quad X_k = g_{k\ell} X^\ell, \tag{B.7}
\]
where \( \nabla \) denotes covariant differentiation \([B.2]\). We will often abuse notation and write \( \mathcal{L}_X g(Y,\cdot) \) for the vector field with components
\[
(\mathcal{L}_X g(Y,\cdot))^i = g^{ij} (\nabla_j X_k + \nabla_k X_j) Y^k. \tag{B.8}
\]

Let \( *_g \) denote the Hodge star with respect to the Riemannian volume form \( d\mu = \sqrt{|g|} dx^1 \wedge dx^2 \wedge dx^3 \). For the general definition see \([19]\). For our purposes we will only need to compute \( *_g \omega \) when \( \omega \) is a two-form. With \( \epsilon_{ijk} \) denoting the Levi-Civita symbol, so that \( \epsilon_{ijk} \) takes the sign of the permutation taking \((1, 2, 3)\) to \((i, j, k)\), we have
\[
*_g (dx^i \wedge dx^j) = \sqrt{|g|} g^{jk} g^{\ell m} \epsilon_{k\ell m} dx^m.
\]
If \( \beta = \beta_{ij} dx^i \wedge dx^j \) is a two-form then from the above formula,
\[
*_g \beta = \sqrt{|g|} \beta^{jk} \epsilon_{k\ell m} dx^m, \quad \beta^{jk} = g^{jk} g^{\ell m} \beta_{\ell m}.
\]
Let \( d \) denote exterior differentiation. If \( \beta \) is a one-form then \( d\beta \) is defined by
\[
d\beta = \partial_i \beta_j dx^i \wedge dx^j.
\]
We will use the following identity relating \(*_g, d\) and covariant differentiation \(\nabla\). If \(\omega = \omega_{ij}dx^i dx^j\) is a \((0,2)\)-tensor then
\[\ast_g d \ast_g \omega = \delta_g \omega, \quad (\delta_g \omega)_i := g^{kj}\nabla_j \omega_{ik}. \quad (B.9)\]

Given vector fields \(X, Y\), let \(X \times_g Y\) be the vector field
\[X \times_g Y := (\ast_g X^\flat \wedge Y^\flat)^\sharp. \quad (B.10)\]
Explicitly, \(X \times_g Y = (X \times_g Y)^\ell \partial_\ell\) with \((X \times_g Y)^k = \sqrt{|g|}g^{ik}\epsilon_{\ell k\ell}X^\ell Y^j\). The curl of a vector field, \(\text{curl}_g X\), is then defined by
\[\text{curl}_g X = (\ast_g dX)^\flat, \quad (B.11)\]
or, in components,
\[\left(\text{curl}_g X\right)^m = \sqrt{|g|}g^{mn}g^{ik}\epsilon_{\ell k\ell} \partial_\ell X_j = \sqrt{|g|}g^{mn}g^{ik}\epsilon_{k\ell n} \nabla_i X_\ell, \quad (B.12)\]
where the second equality follows from a direct calculation involving the formula for the Christoffel symbols \([B.3]\).

We now collect some basic vector calculus identities.

**Lemma B.1.** Define \(\times_g\) by \([B.10]\), \(\text{curl}_g\) by \([B.11]\), and \(\nabla_g\) by \([B.1]\). Suppose that \(M\) is a subset of \(\mathbb{R}^3\). Let \(\times, \cdot\) denote the usual cross and dot products in Euclidean space. Then we have
\[\begin{align*}
(X \times_g Y) \cdot_g Z &= \sqrt{|g|}X \times Y \cdot Z, \\
\text{curl}_g(fX) &= \nabla_g f \times_g X + f \text{curl}_g X, \\
\text{curl}_g \nabla_g f &= 0, \\
(X \times_g Y) \times_g Z &= (X \cdot Y)Z - (X \cdot Z)Y.
\end{align*}\]

**Proof.** The first three identities are immediate. The last identity is proven by changing coordinates as in the proof of the upcoming identity \([B.3]\) and we omit the proof. \(\square\)

We will also need the following slightly more complicated identities in the next section.

**Lemma B.2.** Let \(\nabla_g\) denote covariant differentiation. For any vector fields \(X, Y\),
\[\begin{align*}
\text{curl}_g(X \times_g Y) &= X \text{div}_g Y - Y \text{div}_g X + \mathcal{L}_Y X, \\
\nabla_g|X|^2_g &= 2\nabla_X X - 2X \times_g \text{curl}_g X, \\
\mathcal{L}_X(X^\flat) &= (\nabla_X X)^\flat + \frac{1}{2} \nabla_g|X|^2_g, \\
\text{div}_g(X \times_g Y) &= Y \cdot_g \text{curl}_g X - X \cdot_g \text{curl}_g Y, \\
X \times_g \text{curl}_g X &= \nabla_g|X|^2 + (\mathcal{L}_g(X, \cdot))^\flat,
\end{align*}\]
where \(\nabla_X := X^i \nabla_i\), and where \(\mathcal{L}_X g\), defined in \([B.7]\), denotes the deformation tensor of \(X\), and we are using the notation \([B.8]\).

**Proof.** We begin by writing
\[\left(\text{curl}_g(X \times_g Y)\right)^\flat = \ast_g d \ast_g (X^\flat \wedge Y^\flat) = \delta_g (X^\flat \wedge Y^\flat).\]
Using \([B.9]\) and writing \(X_i = g_{ij}X^j, Y_k = g_{k\ell}Y^\ell\),
\[\delta_g (X^\flat \wedge Y^\flat)_i = g^{kj}\nabla_j (X_i Y_k - X_k Y_i) = X_i g^{kj} \nabla_j Y_k - Y_i g^{kj} \nabla_j X_k + Y_k g^{kj} \nabla_j X_i - X_k g^{kj} \nabla_j Y_i.\]
The first two terms are \(X_i \text{div}_g Y - Y_i \text{div}_g X\). If we raise the index on the last two terms (using \([B.5]\) and use \([B.4]\) then we see
\[Y_k g^{kj} \nabla_j X^i - X_k g^{kj} \nabla_j Y^i = Y^j \nabla_j X^i - X^j \nabla_j Y^i = Y^i \partial_j X^i - X^i \partial_j Y^i,\]
which gives (B.13). To prove (B.16) we start by computing $X \times g \operatorname{curl}_g X$. Writing $\beta = X^\flat$, a direct calculation using (B.12) shows that
\[
\beta \wedge (*_g d\beta) = \sqrt{|g|} g^{ik} g^{j\ell} \epsilon_{k\ell m} \nabla_i X_j X_n \, dx^n \wedge dx^m
\]
and that
\[
*_g (\beta \wedge (*_g d\beta)) = |g| g^{ik} g^{j\ell} g^{mr} \epsilon_{k\ell m} \epsilon_{rqp} \nabla_i X_j X_n \, dx^p = |g| g^{mq} \epsilon_{k\ell m} \epsilon_{rqp} \nabla^k X^\ell X^r \, dx^p.
\]
The identity (B.16) follows at any given point $P$ after changing coordinates near $P$ so that expressed in these coordinates, the metric is given by diag(1, 1, 1).

To prove (B.17), we write
\[
(L_X X^\flat)_j = X^i \partial_i X_j + X_i \partial_j X^i = X^i \partial_i X_j + \frac{1}{2} \partial_j (g_{i\ell} X^\ell X^i) - \frac{1}{2} (\partial_j g_{i\ell}) X^\ell X^i,
\]
and so it follows from the definition of the covariant derivative that
\[
L_X X_j - \nabla_X X_j - \frac{1}{2} \partial_j |X|^2_g = \Gamma^k_{ij} X^i X_k - \frac{1}{2} (\partial_j g_{i\ell}) X^\ell X^i
\]
and expanding the definition of the Christoffel symbols, the right-hand side is
\[
\Gamma^k_{ij} X^i X_k - \frac{1}{2} (\partial_j g_{i\ell}) X^\ell X^i = \frac{1}{2} X^i X^\ell (\partial_i g_{tj} + \partial_j g_{ti} - \partial_t g_{ij}) - \frac{1}{2} (\partial_j g_{i\ell}) X^\ell X^i = 0.
\]
To prove (B.18), we use (B.5) and write
\[
\operatorname{div}_g (X \times g Y) = \nabla_i (\sqrt{|g|} g^{ij} \epsilon_{jk\ell} X^k Y^\ell) = Y^\ell (\sqrt{|g|} g^{ij} \epsilon_{jk\ell} \nabla_i X^k) + X^k (\sqrt{|g|} g^{ij} \epsilon_{jk\ell} \nabla_i Y^\ell)
\]
\[
= Y^k (\operatorname{curl}_g X)_k - X^k (\operatorname{curl}_g Y)_k.
\]
The final identity (B.19) follows from (B.16). \qed

The following identity involving $\times$ and $\times_g$ is crucial for proving that the vector field $B_g$ defined in (1.16) possesses a flux function. We note that this result is follows directly from standard vector calculus identities when $g = \delta$.

**Lemma B.3.** If $\varphi$ is a function with $L_X \varphi = 0$, then
\[
X \times (X \times_g \nabla_g \varphi) = -\frac{1}{\sqrt{|g|}} |X|^2_g \nabla \varphi,
\]
where $\nabla \varphi$ denotes the Euclidean gradient.

**Proof.** This follows from a straightforward but tedious argument; we include the details for the convenience of the reader. With $\omega$ the quantity on the left-hand side of (B.20), from the definitions we have
\[
\omega^i = \sqrt{|g|} \delta^{ij} g^{\ell m} g_{pq} \epsilon_{jkl} \epsilon_{mnp} X^k X^n \partial_i \varphi.
\]
Fix any $P \in M$ and choose coordinates $(Z^1, Z^2, Z^3)$ near $P$ so that at $P$, we have
\[
g_{ij} \frac{\partial x^i}{\partial Z^\alpha} \frac{\partial x^j}{\partial Z^\beta} = \delta_{\alpha\beta}.
\]
We note the following relation which will be useful in what follows: at $P$, we have
\[
g^{\ell m} = \delta_{\alpha\beta} \frac{\partial x^\ell}{\partial Z^\alpha} \frac{\partial x^m}{\partial Z^\beta}.
\]
Expressing $\omega$ in these coordinates, $\omega = \omega^a \partial_{Z^a}$ with $\omega^a = \frac{\partial Z^a}{\partial x^i} \omega^i$, evaluating at $P$ and using (B.23) to re-write $g^{\ell m}$ and $g^{pq}$, from (B.21) we have

$$
\omega^a = \sqrt{|g|} \frac{\partial Z^a}{\partial x^i} \delta^{ij} g^{\ell m} g^{pq} \epsilon_{jklmnp} X^k X^n \partial_{\varphi}^q
$$

$$
= \sqrt{|g|} \frac{\partial Z^a}{\partial x^i} \frac{\partial x^m}{\partial x^p} \frac{\partial x^n}{\partial x^q} \frac{\partial x^n}{\partial x^q} \frac{\partial x^m}{\partial x^p} \delta_{bc} \delta_{dd'} \epsilon_{jklmnp} X^b X^c \partial_{Z^d} \varphi,
$$

writing e.g. $X^k = \frac{\partial x^k}{\partial Z^b} X^b$. Now we note that

$$
\epsilon_{mnp} \frac{\partial x^m}{\partial Z^c} \frac{\partial x^n}{\partial Z^d} = \epsilon_{cc'd} \det(\partial_Z x), \quad \epsilon_{jklmnp} X^b X^c \partial_{Z^d} \varphi = \epsilon_a b c d \det(\partial_Z x).
$$

Indeed, the quantity on the left-hand side of e.g. the first equality is antisymmetric in all three indices and so is a multiple of $\epsilon_{cc'd}$ and evaluating at $c = 1, c' = 2, d = 3$ gives the result. Therefore (B.24) reads

$$
\omega^a = \sqrt{|g|} \det(\partial_Z x)^2 \delta^{ij} 
\frac{\partial Z^a}{\partial x^i} \frac{\partial Z^c}{\partial x^j} \delta_{bc} \partial_{Z^d} \varphi, \quad \epsilon_{jklmnp} X^b X^c \partial_{Z^d} \varphi = \epsilon_a b c d \det(\partial_Z x),
$$

using a well-known identity for the Levi-Civita symbol. Now we note that

$$
\epsilon^{abcd} \gamma_{cd} X^d = X \cdot \nabla \varphi = 0,
$$

by assumption, and so

$$
\omega^a = -\sqrt{|g|} |(\det \partial_Z x)^2| \delta^{ij} 
\frac{\partial Z^a}{\partial x^i} \frac{\partial Z^c}{\partial x^j} \partial_{Z^d} \varphi X^d = -\sqrt{|g|} |(\det \partial_Z x)^2| \delta^{ij} \frac{\partial Z^a}{\partial x^i} \partial_{x^j} \varphi |X|_g^2.
$$

From (B.22), we have $\sqrt{|g|} |(\det \partial_Z x)^2| = \frac{1}{\sqrt{|g|}}$ and so at $P$ we find

$$
\omega^a = -\frac{1}{\sqrt{|g|}} \delta^{ij} \partial_{x^j} \varphi |X|_g^2,
$$

and since $P$ was arbitrary we get the result. \hfill $\Box$

We finally record some useful formulae involving Lie derivatives along $g$ Killing fields. We have

**Lemma B.4.** Let $X, Y$ be vector fields and let $\xi$ be a Killing vector field for the metric $g$. Then

$$
\mathcal{L}_\xi (X \times_g Y) = \mathcal{L}_\xi X \times_g Y + X \times_g \mathcal{L}_\xi Y,
$$

$$
\mathcal{L}_\xi \text{curl}_g X = \text{curl}_g \mathcal{L}_\xi X.
$$

**Proof.** To prove (B.25), we start from the following fact, which can be found on page 177 of [7]. If $\xi_0$ is a Killing field for a metric $g$, $\mathcal{L}_{\xi_0} g = 0$, then $\mathcal{L}_{\xi_0} *_g \alpha = *_g \mathcal{L}_{\xi_0} \alpha$. Similarly, $\mathcal{L}_\xi X^\flat = (\mathcal{L}_\xi X)_b$, where $b$ denotes lowering indices with $g$. For any vector field $\xi$, if $\Phi_s$ denotes its flow, we have the following identity $\Phi_s^* *_g \alpha = *_g \Phi_s^* \alpha$. If $\xi$ is a Killing field for $g$ then this becomes $\Phi_s^* *_g \alpha = *_g \Phi_s^* \alpha$. Differentiating this at $s = 0$ and using the definition of the lie derivative gives the result. Now, to get the formula for $\mathcal{L}_\xi (X \times_g Y)$ we then recall that $(X \times_g Y)_b = *_g (X^\flat \wedge Y^\flat)$ Using that $\mathcal{L}_\xi$ commutes with $\wedge^2, \wedge^1$, and $*_g$,

$$
\mathcal{L}_\xi (X \times_g Y) = \mathcal{L}_\xi *_g (X^\flat \wedge Y^\flat)^\sharp = *_g (\mathcal{L}_\xi (X^\flat \wedge Y^\flat))^\sharp = *_g ((\mathcal{L}_\xi X^\flat) \wedge Y^\flat + X^\flat \wedge (\mathcal{L}_\xi Y)^\flat)^\sharp = \mathcal{L}_\xi X \times_g Y + X \times_g \mathcal{L}_\xi Y.
$$

To prove (B.26), recall that $\text{curl}_g X = (*_g dX^\flat)^\sharp$, where $\flat$ and $\sharp$ denote lowering and raising the index with $g$. Recall that if $\mathcal{L}_{\xi} g = 0$ then

$$
\mathcal{L}_{\xi} *_g = *_g \mathcal{L}_{\xi}, \quad \mathcal{L}_{\xi} X^\flat = (\mathcal{L}_{\xi} X^\flat), \quad \mathcal{L}_{\xi} \alpha^\sharp = (\mathcal{L}_{\xi} \alpha)^\sharp.
$$

(B.27)
After lowering the index on curl, X and using the fact that Lie derivatives commute with exterior differentiation, \( \mathcal{L}_\xi d = d\mathcal{L}_\xi \), we obtain \( \mathcal{L}_\xi *_g (dX_g) = *_g (\mathcal{L}_\xi X) \). Raising the index with \( g \) and using \( (B.27) \) again we get the result. 

**Appendix C. Generalized quasisymmetric Grad-Shafranov equation**

In this section we summarize the relationship between quasisymmetry and the MHS equation \([1.1]\). Recall that the deformation tensor \( \mathcal{L}_\xi \delta \) is defined by

\[
(\mathcal{L}_\xi \delta)(X,Y) = X \cdot (\nabla \xi + (\nabla \xi)^T) \cdot Y.
\]

We begin by showing that the ansatz \([1.16]\) is automatically Euclidean divergence-free, has flux surfaces and is quasisymmetric until a further conditions.

**Proposition C.1** (Characterization of quasisymmetric \( B_g \) MHS solutions). Let \( \xi \) be a non-vanishing and divergence-free vector field tangent to \( \partial T \) and let \( g \) be any metric with \( \mathcal{L}_\xi g = 0 \). Let \( \psi : T \to \mathbb{R} \) be such that \( \mathcal{L}_\xi \psi = 0 \) and \( |\nabla \psi| > 0 \). Then \( B_g \) given by \([1.16]\) is (Euclidean) divergence-free, satisfies \([1.8]\) and is tangent to \( \partial T \). Moreover \( B_g \) is weakly quasisymmetric if and only if

\[
(\mathcal{L}_\xi \delta)(\xi, \xi) + 2C^{-1}(\psi)(\mathcal{L}_\xi \delta)(\xi, \nabla_g^+ \psi) + C^{-2}(\psi)(\mathcal{L}_\xi \delta)(\nabla_g^+ \psi, \nabla_g^+ \psi) = 0.
\]

The field \( B_g \) additionally solves MHS with forcing \( f \) if and only if \( f \cdot g \nabla_g^+ \psi = f \cdot g \xi = 0 \), and \( \psi \) satisfies the generalized Grad-Shafranov equation

\[
\text{div}_g \left( \frac{\sqrt{|g|}}{|\xi|^2} \nabla_g \psi \right) - C(\psi) \frac{\xi}{|\xi|^2} \cdot g \text{curl}_g \left( \frac{\xi}{|\xi|^2} \cdot g \right) + \frac{C(\psi)C'(\psi)}{\sqrt{|g|}|\xi|^2} \cdot g + \frac{P'(\psi)}{\sqrt{|g|}} = \frac{f \cdot g \nabla_g \psi}{\sqrt{|g|}|\nabla_g \psi|^2}.
\]

This section will build up to the proof of Proposition C.1 by developing the following Lemmas C.3 C.4. The proof is a straightforward combination of these results. First we record some elementary vector identities.

**Lemma C.1.** Fix a metric \( g \). Let \( \xi \) be a vector field with \( |\xi| \neq 0 \) and let \( \psi : T \to \mathbb{R} \) be a function satisfying \( \mathcal{L}_\xi \psi = 0 \). Then we have

\[
\xi \times_g \nabla_g \psi = \nabla_g^+ \psi, \quad \nabla_g \psi \times_g \nabla_g^+ \psi = |\nabla_g \psi|^2 \xi, \quad \nabla_g^+ \psi \times_g \xi = |\xi|^2 \nabla_g \psi.
\]

where we have introduced \( \nabla_g^+ \psi = \xi \times_g \nabla_g \psi \). Thus, the triple \( (\nabla_g \psi, \nabla_g^+ \psi, \xi) \) forms an orthogonal basis of \( \mathbb{R}^3 \) at each \( x \in T \) where \( |\nabla_g \psi|_g > 0 \).

**Proof.** Follows from the identity \([B.14]\). 

The following are the main results in this section and are proved at the end of the section.

**Lemma C.2** (Structural properties of \( B_g \)). Fix a metric \( g \). Let \( \xi : T \to \mathbb{R}^3 \) be a (Euclidean) divergence-free vector field with \( |\xi|_g \neq 0 \) which is tangent to \( \partial T \). Let \( \psi : T \to \mathbb{R} \) be a function satisfying \( \mathcal{L}_\xi \psi = 0 \) which is constant on \( \partial T \). Fix \( C : \mathbb{R} \to \mathbb{R} \). Then \( B_g \) defined in \([1.16]\) satisfies

\[
\xi \times B_g = -\nabla \psi, \quad (C.1)
\]

\[
\text{div} B_g = -\frac{1}{|\xi|^2} (\mathcal{L}_\xi g)(\xi, B_g), \quad (C.2)
\]

\[
\mathcal{L}_\xi B_g = -\frac{1}{|\xi|^2} (\mathcal{L}_\xi g)(\xi, B_g) \xi \quad (C.3)
\]

\[
B_g \cdot \hat{n}|_{\partial T} = 0. \quad (C.4)
\]
For the proof, see Section C.2. The crucial point in the above is despite the fact that $B_g$ is defined in terms of an arbitrary metric $g$, the identities (C.1) and (C.2) involve the Euclidean metric.

We now begin the derivation of the Grad-Shafranov equation (1.20) which involves a somewhat lengthy calculation using the above identities. The most important and complicated ingredient is the following formula, which is a direct consequence of Lemma C.6 below.

**Lemma C.3 (Curl of $B_g$).** Fix a metric $g$. Let $\xi$ be a vector field with $|\xi| \neq 0$ and let $\psi : T \to \mathbb{R}$ be a function satisfying $L_\xi \psi = 0$. Fix a function $C : \mathbb{R} \to \mathbb{R}$. Then $B_g$ defined in (1.16) satisfies

$$\text{curl}_g B_g = F \nabla_g \psi + G \nabla_g^\perp \psi + H \xi,$$

where $\text{curl}_g$ is with respect to the metric $g$, defined in (B.11), and with $F, G, H$ defined by

$$F := -\frac{\sqrt{|g|}}{|\xi|^2_g |\nabla_g \psi|^2_g} \left[ C(\psi) (L_\xi g)(\xi, \nabla_g^\perp \psi) + 2|\xi|^2_g |\nabla_g \psi|^2_g L_\xi \frac{\sqrt{|g|}}{|\xi|^2_g} \nabla_g^\perp \psi - |\xi|^2_g (L_\xi g)(\nabla_g \psi, \nabla_g \psi) - |\nabla_g \psi|^2_g (L_\xi g)(\xi, \xi) \right],$$

$$G := \frac{\sqrt{|g|}}{|\xi|^2_g |\nabla_g \psi|^2_g} \left[ (L_\xi g)(B_g, \nabla_g \psi) - |\nabla_g \psi|^2_g C'(\psi) \frac{\sqrt{|g|}}{|\xi|^2_g} \right],$$

$$H := \text{div}_g \left( \sqrt{|g|} \frac{\nabla_g \psi}{|\xi|^2_g} \right) + \frac{1}{|\xi|^2_g} C(\psi) \xi \cdot g \text{curl}_g \xi + \frac{\sqrt{|g|}}{|\xi|^2_g} (L_\xi g)(\nabla_g \psi, \xi).$$ (C.5)

**Lemma C.4 (MHS for $B_g$).** Fix a metric $g$. Let $\xi$ be a vector field with $|\xi| \neq 0$ and let $\psi : T \to \mathbb{R}$ be a function satisfying $L_\xi \psi = 0$. Fix a function $C : \mathbb{R} \to \mathbb{R}$. Then $B_g$ defined in (1.16) satisfies

$$\text{curl}_g B_g \times_g B_g - \nabla_g P = (C(\psi) G - H - P^*) \nabla_g \psi - \frac{C(\psi)}{|\xi|^2_g} F \nabla_g^\perp \psi + \frac{|\nabla_g \psi|^2_g}{|\xi|^2_g} F \xi,$$ (C.8)

with $F, G$ and $H$ defined by (C.5), (C.6) and (C.7). In particular, if $L_\xi g = 0$ then $B$ satisfies the MHS equation with force $f$.

$$\text{curl}_g B_g \times_g B_g = \nabla_g P + f,$$

if and only if $f \cdot g \nabla_g^\perp \psi = f \cdot \xi = 0$, and $\psi$ satisfies the generalized Grad-Shafranov equation

$$\text{div}_g \left( \sqrt{|g|} \frac{\nabla_g \psi}{|\xi|^2_g} \right) - C(\psi) \frac{\xi}{|\xi|^2_g} \cdot g \text{curl}_g \left( \frac{\xi}{|\xi|^2_g} \right) - \frac{C(\psi) C'(\psi)}{\sqrt{|g|} |\xi|^2_g} + \frac{P'(\psi)}{\sqrt{|g|} |\xi|^2_g} = \frac{f \cdot g \nabla_g^\perp \psi}{\sqrt{|g|} |\nabla_g \psi|^2_g}.$$ (C.9)

**Proof.** Follows from Lemma C.3, (C.1), standard vector identities and $L_\xi \psi = 0$. \qed

The generalized Grad-Shafranov equation (C.9) for vector fields of the form (1.16) was first derived in [4] when $g$ was taken to be the circle-averaged metric.

**Lemma C.5 (Quasisymmetry of $B_g$).** Fix a metric $g$ with $L_\xi g = 0$. Let $\xi$ be a vector field with $|\xi| \neq 0$ and let $\psi : T \to \mathbb{R}$ be a function satisfying $L_\xi \psi = 0$. Fix $C : \mathbb{R} \to \mathbb{R}$. Then $B_g$ satisfies

$$L_\xi |B_g|^2 = \frac{C^2(\psi)}{|\xi|^4_g} \left[ (L_\xi \delta)(\xi, \xi) + 2C^{-1}(\psi)(L_\xi \delta)(\xi, \nabla_g^\perp \psi) + C^{-2}(\psi)(L_\xi \delta)(\nabla_g^\perp \psi, \nabla_g^\perp \psi) \right].$$

**C.1. Auxiliary Lemmas.** We collect some calculations which are useful for the proofs of the other Lemmas in the following statement.
**Lemma C.6.** Fix a metric $g$. Let $\xi$ be a vector field with $\|\xi\| \neq 0$ and let $\psi: T \to \mathbb{R}$ be a function satisfying $L_\xi \psi = 0$. Fix a function $C: \mathbb{R} \to \mathbb{R}$. Then

$$\text{div}_g \left( C(\psi) \frac{\xi}{|\xi|_g^2} \right) = C(\psi) \frac{\text{div}_g \xi - \frac{1}{|\xi|_g^2}(L_\xi g)(\xi, \xi)}{|\xi|_g^2}, \quad \text{(C.10)}$$

$$\text{div}_g \left( \sqrt{|g|} \frac{\nabla^\perp g \psi}{|\xi|_g^2} \right) = -\frac{\sqrt{|g|}}{|\xi|_g^2} (L_\xi g)(\xi, \nabla^\perp g \psi) + \frac{1}{|\xi|_g^2} L_{\nabla^\perp g \psi} \sqrt{|g|}, \quad \text{(C.11)}$$

$$\text{curl}_g \left( C(\psi) \frac{\xi}{|\xi|_g^2} \right) = \frac{1}{|\xi|_g^2} C(\psi) (\xi \cdot \nabla \psi) \xi + \frac{1}{|\xi|_g^2} \chi \left( \nabla^\perp \psi \frac{\xi}{|\xi|_g^2} - C'(\psi) \nabla^\perp \psi \right) \frac{\xi}{|\xi|_g^2} \right)$$

$$- \frac{C(\psi)}{|\xi|_g^4} (L_\xi g)(\xi, \nabla^\perp g \psi) \nabla \psi$$

$$\text{curl}_g \left( \sqrt{|g|} \frac{\nabla^\perp g \psi}{|\xi|_g^2} \right) \left( \text{div}_g \left( \sqrt{|g|} \frac{\nabla^\perp g \psi}{|\xi|_g^2} \right) + \frac{\sqrt{|g|}}{|\xi|_g^2} (L_\xi g)(\nabla \psi, \xi) \right)$$

$$+ \frac{\sqrt{|g|}}{|\xi|_g^2} (L_\xi g)(\nabla \psi, \xi) \nabla \psi$$

$$+ \frac{\sqrt{|g|}}{|\xi|_g^4} (L_\xi g)(\nabla \psi, \xi) \nabla^\perp g \psi.$$ \quad \text{(C.13)}

**Proof.** We will repeatedly use the product rule (B.6) as well as the commutator identity

$$L_\xi \nabla g f = \nabla g L_\xi f - (L_\xi g)(\nabla g f, \cdot). \quad \text{(C.14)}$$

**Step 1: Identity [C.10].** To prove (C.10) we note

$$\text{div}_g \left( C(\psi) \frac{\xi}{|\xi|_g^2} \right) = C(\psi) \frac{\text{div}_g \xi - |\xi|_g^4 C(\psi) L_\xi |\xi|_g^2}{|\xi|_g^2}$$

$$= \frac{C(\psi)}{|\xi|_g^2} \chi \left( \nabla^\perp \psi \frac{\xi}{|\xi|_g^2} - C'(\psi) \nabla^\perp \psi \right) \frac{\xi}{|\xi|_g^2},$$

using the product rule (B.13).

**Step 2: Identity [C.11].** First note that

$$\text{div}_g \left( \sqrt{|g|} \frac{\nabla^\perp g \psi}{|\xi|_g^2} \right) = \sqrt{|g|} \text{div}_g \left( \frac{\nabla^\perp g \psi}{|\xi|_g^2} \right) + \frac{1}{|\xi|_g^2} L_{\nabla^\perp g \psi} \sqrt{|g|}.$$
We now simplify the second term in the above. First note the identity (which follows from (B.19))

\[
\xi \times g \text{curl}_g \xi = \frac{1}{2} \nabla_g |\xi|^2_g - (\xi \cdot g \nabla_g)\xi \\
= \nabla_g |\xi|^2_g - ((\xi \cdot g \nabla_g)\xi + \nabla_g \xi \cdot g \xi) \\
= \nabla_g |\xi|^2_g - (L_{\xi g}) \cdot g \xi.
\]  

so that

\[
\nabla_g |\xi|^2_g \times g \xi = (\xi \times g \text{curl}_g \xi) \times g \xi + ((L_{\xi g}) \cdot g \xi) \times g \xi \\
= |\xi|^2_g \text{curl}_g \xi - (\xi \cdot g \text{curl}_g \xi) \times g \xi,
\]  

where we have used the elementary identity

\[
(\xi \times g \text{curl}_g \xi) \times g \xi = |\xi|^2_g \text{curl}_g \xi - (\xi \cdot g \text{curl}_g \xi) \xi.
\]

Noting finally that

\[
((L_{\xi g}) \cdot g \xi) \times g \xi = (C(\psi) \nabla_g \psi) \times g \xi + (C(\psi) \text{curl}_g \xi = -C'(\psi) \nabla_g \psi + C(\psi) \text{curl}_g \xi.
\]

Using this formula and (C.16), we find that

\[
\text{curl}_g \left( C(\psi) \frac{\xi}{|\xi|^2_g} \right) = \frac{1}{|\xi|^2_g} \left( -C'(\psi) \nabla_g \psi + C(\psi) \text{curl}_g \xi \right) - |\xi|^{-4} C(\psi) \nabla |\xi|^2_g \times g \xi \\
= \frac{1}{|\xi|^2_g} \left( -C'(\psi) \nabla_g \psi + C(\psi) \text{curl}_g \xi \right) \\
- |\xi|^{-4} C(\psi) \left( |\xi|^2_g \text{curl}_g \xi - (\xi \cdot g \text{curl}_g \xi) \xi + (L_{\xi g}) \cdot g \xi \right) \\
= -C'(\psi) \frac{\xi}{|\xi|^2_g} \nabla_g \psi + |\xi|^{-4} C(\psi) \left((\xi \cdot g \text{curl}_g \xi) \xi - (L_{\xi g}) \cdot g \xi \right). (C.18)
\]

Note finally using Lemma C.1 that

\[
(L_{\xi g}) \cdot g \xi \times g \xi = ((L_{\xi g}) \cdot g \xi \times g \xi) \cdot g \nabla_g \psi \nabla_g \psi + ((L_{\xi g}) \cdot g \xi \times g \xi) \cdot g \nabla_g \psi \nabla_g \psi \\
= \frac{1}{|\nabla_g \psi|^2_g} (L_{\xi g})(\xi, \nabla_g \psi) \nabla_g \psi - \frac{1}{|\nabla_g \psi|^2_g} (L_{\xi g})(\xi, \nabla_g \psi) \nabla_g \psi,
\]

where we used the identity (C.17) in passing to the second line together with

\[
((L_{\xi g}) \cdot g \xi) \times g \xi = ((L_{\xi g}) \cdot g \xi) \cdot g (\xi \times g \nabla_g \psi) = -|\xi|^2_g (L_{\xi g})(\xi, \nabla_g \psi).
\]

Combining this with (C.18) gives

\[
\text{curl}_g \left( C(\psi) \frac{\xi}{|\xi|^2_g} \right) = -C'(\psi) \frac{\xi}{|\xi|^2_g} \nabla_g \psi + |\xi|^{-4} C(\psi) (\xi \cdot g \text{curl}_g \xi) \xi \\
+ \frac{1}{|\xi|^2_g} \left( \frac{C(\psi)}{|\xi|^2_g |\nabla_g \psi|^2_g} (L_{\xi g})(\xi, \nabla_g \psi) \nabla_g \psi - \frac{C(\psi)}{|\xi|^2_g |\nabla_g \psi|^2_g} (L_{\xi g})(\xi, \nabla_g \psi) \nabla_g \psi \right).
\]
Rearrangement establishes \((C.12)\).

**Step 4: Identity** \((C.13)\). First note that
\[
curl_g \left( \sqrt{|g|} \frac{\nabla^\perp_g \psi}{|\xi_g|^2} \right) = \sqrt{|g|} \left( \nabla^\perp_g \left( \frac{\nabla^\perp_g \psi}{|\xi_g|^2} \right) + \frac{1}{|\xi_g|^2} \nabla \sqrt{|g|} \times \nabla^\perp_g \psi \right)
\]
\[
= \sqrt{|g|} \left( \nabla^\perp_g \left( \frac{\nabla^\perp_g \psi}{|\xi_g|^2} \right) + \frac{1}{|\xi_g|^2} \right) \left( L \nabla \psi \sqrt{|g|} \right) \xi - \frac{1}{|\xi_g|^2} \left( L \xi \sqrt{|g|} \right) \nabla \psi.
\]
Now, by the identity \((B.15)\),
\[
curl_g \nabla^\perp_g \psi = \nabla \psi \div_g (\xi \times_g \nabla \psi) = \nabla \psi \div_g (\xi - L \xi \nabla \psi)
\]
\[
= \nabla \psi \div_g (\xi - L \xi \nabla \psi) - \xi \Delta \psi - \nabla \psi \div_g \xi + (L \xi \nabla \psi, \cdot),
\]
where we used \((C.14)\) and \((L \xi \nabla \psi, \cdot)\) is defined as in \((B.8)\). Therefore
\[
\sqrt{|g|} \left( \nabla \psi \div_g \left( \frac{\nabla^\perp_g \psi}{|\xi_g|^2} \right) \right) = \sqrt{|g|} \left( \xi \Delta \psi - \nabla \psi \div_g \xi + (L \xi \nabla \psi, \cdot) \right)
\]
\[
+ \sqrt{|g|} \left( L \xi \nabla \psi - (\nabla \psi \div_g \xi - |\xi_g|^2 \div_g \xi) \right)
\]
\[
= \sqrt{|g|} \div_g \left( \frac{\nabla^\perp_g \psi}{|\xi_g|^2} \right) \xi + \sqrt{|g|} \left( L \xi \nabla \psi, \cdot \right) + \sqrt{|g|} \left( L \xi \nabla \psi, \xi \right) - |\xi_g|^2 \div_g \xi \nabla \psi
\]
\[
= \div_g \left( \sqrt{|g|} \frac{\nabla^\perp_g \psi}{|\xi_g|^2} \right) \xi - \frac{1}{|\xi_g|^2} \left( L \nabla \psi \sqrt{|g|} \right) \xi
\]
\[
+ \frac{\sqrt{|g|}}{|\xi_g|^2} \left( L \nabla \psi, \cdot \right) + \frac{1}{|\xi_g|^2} \left( \sqrt{|g|} L \xi \nabla \psi, \xi \right) - |\xi_g|^2 \Delta \psi - \frac{1}{|\xi_g|^2} \nabla \psi
\]
where we \((C.20)\) to say \(\sqrt{|g|} \div_g \xi = L \xi \sqrt{|g|}\) as well as the identity
\[
\nabla \psi |\xi_g|^2 \times \nabla^\perp_g \psi := \nabla \psi |\xi_g|^2 \times (\xi) \times_g \nabla \psi = (\nabla \psi |\xi_g|^2 \times_g \nabla \psi) \xi - (\nabla \psi |\xi_g|^2 \times_g \xi) \nabla \psi.
\]
Finally, note that we can express
\[
(L \xi \nabla \psi, \cdot) = \frac{1}{|\xi_g|^2} \left( L \xi \nabla \psi, \xi \right) + \frac{1}{|\xi_g|^2 |\nabla \psi|_g^2} \left( L \xi \nabla \psi, \nabla^\perp \psi \right) \nabla^\perp \psi
\]
\[
+ \frac{1}{|\nabla \psi|_g^2} \left( L \xi \nabla \psi, \nabla \psi \right) \nabla \psi.
\]
This completes the derivation. \(\square\)

**C.2. Proof of Lemma** \((C.2)\). The result follows from direct computation as follows.

**Step 1: Identity** \((C.1)\). The property of having a flux function \((C.1)\) follows from Lemma \((B.3)\).
Step 2: Identity \[ \text{C.2} \]. For the divergence \( \text{C.2} \), Lemma \( \text{C.3} \) gives
\[
\text{div}_g B_g = \frac{1}{|\xi|_g^2} \left( C(\psi) \text{div}_g \xi - (\mathcal{L}_\xi g)(\xi, B_g) \right) + \frac{1}{|\xi|_g^2} \mathcal{L}_\xi \psi \sqrt{|g|}. \tag{C.19}
\]
Next recall the relation between the divergence on flat and curved backgrounds
\[
\text{div} X = \text{div}_g X - \frac{1}{\sqrt{|g|}} \mathcal{L}_X \sqrt{|g|}. \tag{C.20}
\]
Applying this identity to convert \( \text{C.19} \) to the divergence using the Euclidean metric, we have
\[
\text{div} B_g = \text{div}_g B_g - \frac{1}{\sqrt{|g|}} \mathcal{L}_B g \sqrt{|g|} = \frac{1}{|\xi|_g^2} \left( C(\psi) \text{div} \xi - (\mathcal{L}_\xi g)(\xi, B) \right) - \frac{1}{\sqrt{|g|}} |\xi|_g^2 \mathcal{L}_\xi \sqrt{|g|}. \]
Using \( \text{div} \xi = 0 \) and \( \text{C.20} \) again we find \( \sqrt{|g|} \text{div}_g \xi = \mathcal{L}_\xi \sqrt{|g|} \), and get the claimed result.

Step 3: Identity \[ \text{C.3} \]. We have the identity
\[
\mathcal{L}_\xi B_g := \xi \cdot \nabla B_g - B_g \cdot \nabla \xi = \text{curl}(B_g \times \xi) + (\text{div} B_g) \xi - (\text{div} \xi) B_g = -\frac{1}{|\xi|_g^2} (\mathcal{L}_\xi g)(\xi, B_g) \xi,
\]
and the result follows from \( \text{C.1} \), \( \text{C.2} \) and the assumption \( \text{div} \xi = 0 \).

Step 4: Identity \[ \text{C.4} \]. Let \( \hat{n} \) be the unit outward normal vector to \( \partial T \). Then we have
\[
B_g \cdot \hat{n} = \frac{1}{|\xi|_g^2} \sqrt{|g|} (\xi \times_g \nabla_g \psi) \cdot \hat{n},
\]
since \( \xi \cdot \hat{n} = 0 \) by assumption. Now, for any vector field \( X \) and scalar function \( f \) we have
\[
X \cdot \nabla f = \delta_3 X^i \delta^{jk} \partial_k f = \delta^h_i X^i \partial_h f = g_{im} g^{km} X^i \partial_k f = g_{im} X^i (\nabla_g f)^m = X \cdot_g \nabla_g f.
\]
As a result, since \( \psi \) is assumed constant on the boundary, we can choose \( \hat{n} = \nabla \psi/|\nabla \psi| \) on the boundary and a standard vector identity shows that \( (\xi \times \nabla_g \psi) \cdot \hat{n} = 0 \).

C.3. Proof of Lemma \[ \text{C.5} \].

PROOF. Direct computation shows
\[
|B_g|^2 = \frac{1}{|\xi|_g^2} \left[ C(\psi)|\xi|^2 + 2C(\psi) \xi \cdot \nabla^\perp \psi + |\nabla^\perp \psi|^2 \right].
\]
Since \( \mathcal{L}_\xi g = 0 \), from \( \text{C.3} \) it follows that \( \mathcal{L}_\xi B_g = 0 \). Thus we have
\[
\mathcal{L}_\xi \nabla^\perp \psi = \mathcal{L}_\xi \left( |\xi|_g^2 B_g - C(\psi) \xi \right) = 0.
\]
Using \( \mathcal{L}_\xi |\xi|_g^2 = 0 \), \( \mathcal{L}_\xi \xi = 0 \), \( \mathcal{L}_\xi \psi = 0 \), \( \mathcal{L}_\xi \nabla^\perp \psi = 0 \) and \( \mathcal{L}_\xi |\xi|^2 = (\mathcal{L}_\xi \delta)(\xi, \xi) \) completes the proof. \( \square \)

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