A bound on the dissociation number

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Abstract
The dissociation number $\text{diss}(G)$ of a graph $G$ is the maximum order of a set of vertices of $G$ inducing a subgraph that is of maximum degree at most 1. Computing the dissociation number of a given graph is algorithmically hard even when restricted to subcubic bipartite graphs. For a graph $G$ with $n$ vertices, $m$ edges, $k$ components, and $c_1$ induced cycles of length 1 modulo 3, we show $\text{diss}(G) \geq n - \frac{1}{3}(m + k + c_1)$. Furthermore, we characterize the extremal graphs in which every two cycles are vertex-disjoint.

KEYWORDS
cactus, dissociation set

1 | INTRODUCTION

We consider finite, simple, and undirected graphs, and use the standard terminology. A set $D$ of vertices of a graph $G$ is a dissociation set in $G$ if the subgraph $G[D]$ of $G$ induced by $D$ has maximum degree at most 1. The dissociation number $\text{diss}(G)$ of $G$ is the maximum order of a dissociation set in $G$. The dissociation number is algorithmically hard even when restricted, for instance, to subcubic bipartite graphs [3, 9, 13]. Fast exact algorithms [8], (randomized) approximation algorithms [7, 8], and fixed-parameter tractability [11] have been studied for this parameter or its dual, the 3-path (vertex) cover number. Several lower bounds on the dissociation number were proposed: If $G$ is a graph of order $n$ and size $m$, then
The results in the present papers were inspired by bounds in (1).

Our main result is the following.

**Theorem 1.** If $G$ is a graph with $n$ vertices, $m$ edges, $k$ components, and $c_1$ induced cycles of length $1$ modulo $3$, then

\[
diss(G) \geq \begin{cases} 
\frac{n}{2}, & \text{if } G \text{ is outerplanar [4],} \\
\frac{2n}{3}, & \text{if } G \text{ is a tree [4],} \\
\frac{2n}{k+2} - \frac{m}{(k+1)(k+2)}, & \text{if } k = \left\lfloor \frac{m}{n} \right\rfloor - 1 \text{ [5], and} \\
\frac{2n}{3} - \frac{m}{6}, & \text{[4].} 
\end{cases}
\]  

(1)

(2)

Theorem 1 generalizes the lower bound $2n/3$ for trees of order $n$ in (1), strengthens the general lower bound $\frac{2n}{3} - \frac{m}{6}$ in (1) for many graphs, and almost implies the lower bound $n/2$ for subcubic graphs of order $n$, which follows from the first bound in (1). In the proof of Theorem 1, graphs in which all cycles are pairwise vertex-disjoint play an essential role. We call such graphs cycle-disjoint; their components are restricted cactus graphs, where a cactus is a connected graph in which every block is either a $K_2$ or a cycle. As a step towards the understanding of all extremal graphs for Theorem 1, we consider the extremal cycle-disjoint graphs in more detail. While the dissociation number problem is algorithmically hard for the graphs considered in Theorem 1, maximum dissociation sets can be determined efficiently for cactus graphs, because their treewidth is bounded [1]. In particular, the extremal cycle-disjoint graphs can be recognized efficiently. Nevertheless, we aim for a structural description of the extremal cycle-disjoint graphs, because our goal is a structural description of the extremal graphs for Theorem 1.

We propose three extension operations $(O_1)$, $(O_2)$, and $(O_3)$ applicable to a given graph $G'$, attaching a $P_3$ or a cycle of length not $0$ modulo $3$ by a bridge to $G'$, illustrated in Figure 1. It is easy to see that applying one of these operations to a graph that satisfies (2) with equality yields a graph that satisfies (2) with equality. Since $P_3$ and the cycles of lengths not $0$ modulo $3$ satisfy (2) with equality, this already allows one to construct quite a rich family of extremal graphs, yet not all of them.
The two operations \((O_1)\) and \((O_2)\) are sufficient for the constructive characterization of all trees \(T\) of order \(n\) with \(\text{diss}(T) = \frac{2n}{3}\), that is, of all trees that are extremal for the bound from [4] stated in (1). Let \(T\) be the set of all trees that arise from \(P_3\) by repeated applications of the two operations \((O_1)\) and \((O_2)\), attaching a new \(P_3\) by a bridge to trees in \(T\).

**Theorem 2.** For a tree \(T\) of order \(n\), the following statements are equivalent.

(a) \(\text{diss}(T) = \frac{2n}{3}\).
(b) \(T \in T\).
(c) \(n \equiv 0 \mod 3\), and, for every vertex \(y\) of \(T\), at most two components of \(T - y\) have order not \(0\) modulo \(3\).

Next to the three simple operations illustrated in Figure 1, we introduce one slightly more complicated operation involving so-called (very) good spiked cycles: For positive integers \(\ell\) and \(k\) with \(\ell \geq \max\{3, k\}\), and indices \(i_1, ..., i_k \in [\ell]\) with \(i_1 < i_2 < ... < i_k\), a spiked cycle \(C^*\) with \(k\) spikes at \(\{i_1, ..., i_k\}\) arises from the cycle \(C : u_1u_2...u_{\ell-1}u_\ell\) of length \(\ell\) by attaching a new endvertex \(v_{i_j}\) to \(u_{i_j}\) for every \(j \in [k]\). The spiked cycle \(C^*\) is good if either \(k = 1\) and \(\ell \equiv 1 \mod 3\) or \(k \geq 2\),

- \(i_{j+1} - i_j \equiv 2 \mod 3\) for every \(j \in [k - 1]\), and
- \(\ell + i_1 - i_k \equiv 1 \mod 3\),

that is, the \(k\) paths in \(C^*\) between vertices of degree 3 whose internal vertices have degree 2, have lengths 2, ..., 2, and 1 modulo 3. The spiked cycle \(C^*\) is very good if it is good and

- \(\ell \not\equiv 1 \mod 3\),

that is, in particular, \(k \geq 2\). See Figure 2 for an illustration.

Let \(C\) be the set of all graphs that arise from the graphs in

\[C_0 = \{P_3\} \cup \{C_\ell : \ell \in \mathbb{N}, \ell \geq 3, \text{ and } \ell \not\equiv 0 \mod 3\} \cup \{C^* : C^*\text{ is a very good spiked cycle}\}\]

by repeated applications of the three operations \((O_1)\), \((O_2)\), and \((O_3)\), as well as the fourth operation \((O_4)\) of forming the disjoint union of some graph \(G'\) in \(C\) with a very good spiked cycle \(C^*\), and adding a bridge between \(V(G')\) and \(V(C^*)\).

**Lemma 3.** All graphs in \(C\) satisfy (2) with equality. Furthermore, for every vertex \(u\) of every graph \(G\) in \(C\), the graph \(G\) has a maximum dissociation set not containing \(u\).
As our final result, we show that \( \mathcal{C} \) contains all connected cycle-disjoint extremal graphs for Theorem 1. Figure 3 shows two extremal graphs that are not cycle-disjoint.

**Theorem 4.** A connected cycle-disjoint graph satisfies (2) with equality if and only if it belongs to \( \mathcal{C} \).

All proofs are given in Section 2.

2 | PROOFS

*Proof of Theorem 1.* We prove the statement by contradiction, and suppose that \( G \) is a counterexample of minimum order. Clearly, this implies that \( G \) is connected, that is, we have \( k = 1 \).

If \( G \) is not cycle-disjoint, then there is a vertex \( u \) of \( G \) such that \( G' = G - u \) has \( k' \leq d_G(u) - 2 \) components, and the choice of \( G \) implies the contradiction

\[
\text{diss}(G) \geq \text{diss}(G') \geq (n - 1) - \frac{1}{3}((m - d_G(u)) + k' + c_1(G')) \\
\geq n - \frac{1}{3}(m + 1 + c_1),
\]
where \( c_1(G') \) denotes the number of induced cycles of length 1 modulo 3 in \( G' \), and we used the obvious fact that \( c_1(G') \leq c_1 \). Hence, the graph \( G \) is cycle-disjoint.

Using the bound for trees in (1), and \( \text{diss}(C_{\ell}) = \left\lceil \frac{2\ell}{3} \right\rceil \) for every integer \( \ell \geq 3 \), it follows easily that \( G \) is neither a tree nor a cycle. We consider the longest path \( P \), say \( P : BvB'... \), in the block-cutvertex tree \([2]\) of \( G \), that is, \( B \) and \( B' \) are distinct blocks of \( G \), \( v \) is a cutvertex of \( G \) that belongs to \( B \) and \( B' \), and all blocks of \( G \) that contain \( v \)—except for possibly the block \( B' \)—are endblocks. Let \( B \) be the set of all blocks of \( G \) that contain \( v \) and are distinct from \( B' \). Let \( B \) contain \( p \) blocks that are \( K_2 \)s, and \( q \) blocks that are cycles. Since \( G \) is cycle-disjoint, we have \( q \in \{0, 1\} \). The graph \( G' = G - \bigcup_{H \in B} V(H) \) is connected and cycle-disjoint. Since \( B' \) is a \( K_2 \) or a cycle, the number \( d \) of neighbors of \( v \) in \( V(G') \) is 1 or 2. Note that \( c_1(G') \leq c_1 \), and \( c_1(G') \leq c_1 - 1 \) if \( B \) contains a cycle of length 1 modulo 3. See Figure 4 for an illustration.

First, suppose that \( q = 1 \), that is, one block in \( B \) is a cycle \( C_{\ell} \). Since \( G \) is cycle-disjoint, we obtain that \( B' \) is a \( K_2 \), and, hence, \( d = 1 \). Since \( C_{\ell} \) has a maximum dissociation set avoiding \( v \), the choice of \( G \) implies the contradiction

\[
\text{diss}(G) \geq \text{diss}(G') + p + \left\lceil \frac{2\ell}{3} \right\rceil
\]

(3)

\[
\geq \frac{n - p - \ell}{n(G')} - \frac{1}{3} \left( \frac{(m - d - p - \ell) + 1 + c_1(G')}{} \right) + 1 + c_1(G') + \left\lceil \frac{2\ell}{3} \right\rceil
\]

(4)

\[
\geq (n - \ell) - \frac{1}{3} \left( (m - 1 - \ell) + 1 + c_1(G') \right) + \left\lceil \frac{2\ell}{3} \right\rceil
\]

(5)

\[
\geq n - \frac{1}{3} (m + 1 + c_1),
\]

(6)

where the final inequality uses

\[
-\ell + \frac{\ell + 1}{3} - \frac{c_1(G')}{3} + \left\lceil \frac{2\ell}{3} \right\rceil \geq -\frac{c_1}{3},
\]

(7)

\[\text{FIGURE 4} \quad \text{Local configuration within } G \text{ where } p = 4 \text{ and } q = 1.\]
which follows from the relation between $c_1(G')$ and $c_1$ mentioned above. Hence, no block in $B$ is a cycle.

If either $p \geq 2$ or $p = 1$ and $B'$ is a cycle, then $m(G') \leq m - 3$, and, the choice of $G$ implies the contradiction

\[
\text{diss}(G) \geq \text{diss}(G') + p \geq (n - 1 - p) - \frac{1}{3}(m(G') + 1 + c_1(G')) + p \\
\geq n - \frac{1}{3}(m + 1 + c_1).
\]

Hence, we obtain that $p = 1$ and that $B'$ is a $K_2$, which implies that $v$ has degree 2. Let $w$ be the unique neighbor of $v$ that is not an endvertex. The graph $G'' = G - N_G[v] = G' - w$ has $k \leq d_G(w) - 1$ components, and $G''$ has $c_1(G'') \leq c_1$ induced cycles of length 1 modulo 3. Since

\[
m(G'') + k \leq (m - d_G(w) - 1) + (d_G(w) - 1) = m + 1 - 3,
\]

the choice of $G$ implies the contradiction

\[
\text{diss}(G) \geq \text{diss}(G'') + 2 \geq (n - 3) - \frac{1}{3}(m(G'') + k + c_1(G'')) \\
+ 2 \geq n - \frac{1}{3}(m + 1 + c_1),
\]

which completes the proof. \(\square\)

Applied to a subcubic graph, the first reduction considered in the proof of Theorem 1 corresponds to the removal of a vertex of degree 3 that is not a cutvertex. Repeatedly applying this reduction, the set of removed vertices is a nonseparating independent set; a notion that is relevant in the context of feedback vertex sets of subcubic graphs [10, 12].

**Proof of Theorem 2.** (b) \(\Rightarrow\) (a): Clearly, $P_3$ satisfies (a). If $T$ arises from a tree $T'$ that satisfies (a) by applying operation $(O_1)$, then some maximum dissociation set of $T$ consists of $u$, $v$, and some maximum dissociation set of $T'$, which implies that $T$ satisfies (a). Similarly, if $T$ arises from a tree $T''$ that satisfies (a) by applying operation $(O_2)$, then some maximum dissociation set of $T$ consists of $u, u'$, and some maximum dissociation set of $T'$, which implies that $T$ satisfies (a). A simple inductive argument implies that all trees in $\mathcal{T}$ satisfy (a).

(a) \(\Rightarrow\) (c): Let $T$ satisfy (a). By induction on the order $n$ of $T$, we prove (c). Since $P_3$ is the only star that satisfies (a) and $P_3$ satisfies (c), we may assume that $n \geq 4$ and that $T$ has diameter at least 3. Let $P : uvwx \ldots$ be a longest path in $T$. Since

\[
\frac{2n}{3} = \text{diss}(T) \geq |N_T(v) \setminus \{w\}| + \text{diss}(T - (N_T[v] \setminus \{w\})) \\
\xRightarrow{(1)} (d_T(v) - 1) + \frac{2}{3}(n - d_T(v)),
\]
we obtain $d_T(v) \in \{2, 3\}$.

First, suppose that $d_T(v) = 2$. Let $T_1, \ldots, T_p$ be the components of $T - \{u, v, w\}$, and let $n_i$ be the order of $T_i$. Since

$$\frac{2n}{3} = \text{diss}(T) \geq |\{u, v\}| + \sum_{i=1}^{p} \text{diss}(T_i) \geq 2 + \sum_{i=1}^{p} \frac{2n_i}{3} = \frac{2n}{3},$$

equality holds throughout this inequality chain, which implies that each $T_i$ satisfies (a). By induction, each $T_i$ satisfies (c). Now, let $y$ be any vertex of $T$. If $d_T(y) \leq 2$, then $T - y$ has at most two components. Now, let $d_T(y) \geq 3$. If $y \in V(T_j)$, then the order of the component of $T - y$ that contains $w$ is either $3 + \sum_{i \neq j} n_i$ if $y$ is the neighbor of $w$ in $V(T_j)$ or $n(K) + 3 + \sum_{i \neq j} n_i$, where $K$ is the component of $T_i - y$ that contains the neighbor of $w$ in $V(T_i)$. Since each $n_i$ is 0 modulo 3, the term $3 + \sum_{i \neq j} n_i$ is 0 modulo 3. Since $T_i - y$ has at most two components of order not 0 modulo 3, this implies that also $T - y$ has at most two components of order not 0 modulo 3. Finally, if $y = w$, then the only component of $T - y$ of order not 0 modulo 3 consists of $u$ and $v$. Altogether, we obtain that $T$ satisfies (c).

Next, suppose that $d_T(v) = 3$. Since

$$\frac{2n}{3} = \text{diss}(T) \geq |N_T(v)\setminus\{w\}| + \text{diss}(T - (N_T[v]\setminus\{w\})) \geq 2 + \frac{2(n - 3)}{3} = \frac{2n}{3},$$

the tree $T - (N_T[v]\setminus\{w\})$ satisfies (a), and, hence, by induction, also (c). Arguing similarly as above, it follows easily that $T$ satisfies (c).

(c) \Rightarrow (b): Let $T$ satisfy (c). By induction on the order $n$ of $T$, we prove (b). Since $P_3$ is the only star that satisfies (c) and $P_3$ satisfies (b), we may assume that $n \geq 4$ and that $T$ has diameter at least 3. Let $P : uvwx\ldots$ be a longest path in $T$ such that the degree $d_T(v)$ of $v$ in $T$ is as large as possible. Note that all neighbors of $v$ that are distinct from $w$ are endvertices. Since $T - v$ has $d_T(v) - 1$ components of order 1, we obtain, by (c), that $d_T(v) \in \{2, 3\}$. If $d_T(v) = 3$, then it is easy to see that $T' = T - (N_T[v]\setminus\{w\})$ satisfies (c), and, hence, by induction, also (b). Since $T$ arises from $T'$ by applying operation $(O_2)$, it follows in this case that $T$ satisfies (b). Hence, we may assume that $d_T(v) = 2$. If $w$ has a neighbor $v' \neq v$ distinct from $v$ and $x$, then the choice of $P$ implies that either $v'$ is an endvertex or $d_{T(v')}(v') = 2$ and the unique neighbor of $v'$ that is distinct from $w$ is an endvertex. In both cases, the forest $T - w$ has at least three components of order not 0 modulo 3, which is a contradiction. It follows that $d_T(w) = 2$, and, hence, that $T'' = T - \{u, v, w\}$ is connected. Since $T$ satisfies (c), it follows easily that $T''$ satisfies (c), and, hence, by induction, also (b). Since $T$ arises from $T''$ by applying operation $(O_1)$, it follows that $T$ satisfies (b) also in this case, which completes the proof. 

\[\square\]

The trees in $T$ have the following useful property.

**Lemma 5.** For every vertex $u$ of every tree $T$ of order $n$ with $\text{diss}(T) = 2n/3$, the tree $T$ has a maximum dissociation set not containing $u$. 
Proof. The proof is by induction on \( n \). For \( n = 3 \), the statement is obvious. Now, let \( n > 3 \). By Theorem 2, the tree \( T \) arises from the disjoint union of a tree \( T' \) of order \( n' \) with \( \text{diss}(T) = 2n'/3 \) and a copy of \( P_3 \) by adding a bridge between some vertex \( x \) in \( T' \) and some vertex \( y \) in the \( P_3 \). Now, let \( u \) be any vertex of \( T \). If \( u \) is a vertex of \( T' \), then adding the two vertices of the \( P_3 \) that are distinct from \( y \) to a maximum dissociation set of \( T' \) not containing \( u \) yields a maximum dissociation set of \( T \) not containing \( u \). If \( u \) is a vertex of the \( P_3 \), then adding the two vertices of the \( P_3 \) that are distinct from \( u \) to a maximum dissociation set of \( T' \) not containing \( x \) yields a maximum dissociation set of \( T \) not containing \( u \). □

Lemma 6. If \( C^* \) is a spiked cycle with \( k \) spikes of order \( n \), then \( \text{diss}(C^*) \geq \frac{2n-1}{3} \) with equality if and only if \( C^* \) is good. Furthermore, if \( C^* \) is good and \( u \) is a vertex of \( C^* \) such that, for \( k = 1 \), the degree of \( u \) is at least 2, then the good spiked cycle \( C^* \) has a maximum dissociation set not containing \( u \).

Proof. Since all statements are easily verified for \( k = 1 \), we assume now that \( k \geq 2 \). Let \( C^* \) be a spiked cycle with \( k \) spikes at \( \{i_1, ..., i_k\} \), where we use the notation from the definition of spiked cycles. The graph \( T = C^* - \{u_{i_1}, v_{i_1}\} \) is a tree of order \( n - 2 \), and we obtain that

\[
\text{diss}(C^*) \geq \text{diss}(T) + |\{v_{i_1}\}| \geq \frac{2(n-2)}{3} + 1 = \frac{2n-1}{3}. \tag{8}
\]

Now, suppose that (8) holds with equality throughout. This implies that \( n \equiv 2 \mod 3 \) and that \( \text{diss}(T) = \frac{2(n-2)}{3} \). By Theorem 2, the tree \( T \) satisfies (c) of Theorem 2. A path in \( C^* \) between vertices of degree 3 whose internal vertices have degree 2 is called special. If \( i_2 - i_1 \equiv 0 \mod 3 \), then \( T - u_{i_1} \) has three components of order not 0 modulo 3, which is a contradiction. See Figure 5 for an illustration.

Hence, by symmetry, no special path has length 0 modulo 3. If \( i_2 - i_1, i_{j+1} - i_j \equiv 1 \mod 3 \), and \( i_3 - i_2, i_4 - i_3, ..., i_j - i_{j-1} \equiv 2 \mod 3 \) for some \( j \in \{2, ..., k - 1\} \), then \( T - u_{i_{j+1}} \) has three components of order not 0 modulo 3, which is a contradiction. Hence, by symmetry, at most one special path has length 1 modulo 3. Since \( n \equiv 2 \mod 3 \), not all special paths have lengths 2 modulo 3. Altogether, we obtain that exactly one special path has length 1 modulo 3 while the other \( k - 1 \) special paths have lengths 2 modulo 3, that is, the spiked cycle \( C^* \) is good.

**Figure 5** \( T - u_{i_1} = C^* - \{u_{i_1}, v_{i_1}, u_{i_2}\} \).
Next, suppose that the spiked cycle $C^*$ is good. By symmetry, we may assume $i_2 - i_1 \equiv 1 \mod 3$. This easily implies that some maximum dissociation set $D$ of $C^*$ does not contain both $u_{i_1}$ as well as $u_{i_2}$. By symmetry, we may assume that $D$ does not contain $u_{i_1}$. Again, let $T = C^* - \{u_{i_1}, v_{i_1}\}$. In view of $D$, we have $\text{diss}(C^*) = \text{diss}(T) + 1$. Since $C^*$ is good, it is easy to see that $T$ satisfies (c) of Theorem 2. Hence, by Theorem 2, we obtain $\text{diss}(C^*) = \text{diss}(T) + 1 = \frac{2(n-2)}{3} + 1 = \frac{2n-1}{3}$. By Lemma 5, the tree $T$ has a maximum dissociation set avoiding any specified vertex. This easily implies that a maximum dissociation set of $\tilde{C^*}$ is a maximum dissociation set of $C^*$ avoiding $v_{i_1}$. This completes the proof.

Proof of Lemma 3. By Lemma 6, the graphs in $C_0$ satisfy (2) with equality and they have maximum dissociation sets avoiding any specified vertex. Recall that the four operations $(O_j)$ to $(O_4)$ consist in adding a disjoint copy of a graph from $C_0$ to some graph $G'$ and connecting this copy by a bridge to $G'$. It follows that applying one of the four operations to a graph that satisfies (2) with equality yields a graph that satisfies (2) with equality. Now, an inductive argument implies that all graphs in $C$ satisfy (2) with equality. The existence of maximum dissociation sets avoiding specified vertices follows easily by induction arguing as in the proof of Lemma 5 and using Lemma 6.

Proof of Theorem 4. We say that a connected cycle-disjoint graph is extremal if it satisfies (2) with equality. By Lemma 3, all graphs in $C$ are extremal. For the converse, let $G$ be extremal. By induction on the order $n$, we show that $G \in C$. If $G$ is a tree, then Theorem 2 implies $G \in T \subseteq C$. If $G$ is a cycle of length $\ell$, then $\ell$ is not 0 modulo 3 and $G \in C_0 \subseteq C$. If $G$ is a spiked cycle, then Lemma 6 implies that $G$ is good. If $G$ is not very good, then $c_1 = 1$, contradicting the fact that $G$ is extremal. Hence, the graph $G$ is a very good spiked cycle, and $G \in C_0 \subseteq C$.

Now, let $G$ be neither a tree nor a cycle nor a spiked cycle. We choose $P : BvB'..., B, p, q \in \{0, 1\}$, $G'$, $d$, and $c_1(G')$ exactly as in the proof of Theorem 1.

First, we assume that $q = 1$. Since $G$ is cycle-disjoint, this implies that $G$ arises by adding the bridge $B'$ between $\bigcup_{H \in B} H$ and $G'$. Arguing as in (3)–(6) using (7), we obtain that all five inequalities (3)–(7) hold with equality. This implies that $p = 0$, that $G'$ is extremal, and that $\ell \not\equiv 0 \mod 3$. By induction, we obtain that $G' \in C$. Since $G$ is constructed by applying operation $(O_3)$ to $G'$, we obtain $G \in C$. Hence, we may assume that $q = 0$.

Next, we assume that $p \geq 2$. Note that $p + d \geq 3$. By Theorem 1, we obtain

$$\text{diss}(G) \geq p + \text{diss}(G') \geq p + (n - p - 1) - \frac{1}{3}((m - p - d) + 1 + c_1(G')) \geq n - \frac{1}{3}(m + 1 + c_1).$$
Since equality holds throughout this inequality chain, we obtain that $G'$ is extremal, and that $p + d = 3$, which implies $p = 2$ and $d = 1$. By induction, we obtain that $G' \in \mathcal{C}$. Since $G$ is constructed by applying operation $(O_2)$ to $G'$, we obtain $G \in \mathcal{C}$. Hence, we may assume that $p = 1$.

Next, we assume that $v$ does not lie on a cycle, that is, the block $B'$ is a $K_2$ and the degree of $v$ is 2. Let $w$ be the neighbor of $v$ in $B'$. Let $G'' = G - N_G[v] = G' - w$. If $G''$ has $k''$ components, then $k'' \leq d_G(w) - 1$. By Theorem 1, we obtain

$$
\text{diss}(G) \geq 2 + \text{diss}(G'')
\geq 2 + (n - 3) - \frac{1}{3}((m - d_G(w) - 1) + k'' + c_1(G''))
\geq n - \frac{1}{3}(m + 1 + c_1).
$$

Since equality holds throughout this inequality chain, each component of $G''$ is extremal, and $k'' = d_G(w) - 1$, which implies that $w$ is connected by a bridge to each component of $G''$. By induction, each component of $G''$ lies in $\mathcal{C}$. If $k'' = 1$, then $G$ is constructed by applying operation $(O_1)$ to $G''$, and we obtain $G \in \mathcal{C}$. Hence, we may assume that $k'' = 2$. By symmetry, considering an alternative choice for the path $P$, we may assume that one component $K$ of $G''$ has order 2, which contradicts $K \in \mathcal{C}$. Hence, we may assume that $v$ lies on a cycle, that is, the block $B'$ is a cycle.

Since $G$ is not a spiked cycle, it follows, by symmetry, considering alternative choices for the path $P$, that $G$ arises from the disjoint union of

- a spiked cycle $G_0$ of order $n_0$ whose unique cycle is $B'$,
- a connected cycle-disjoint graph $G_1$ of order $n_1$, and
- a set $S$ of $s$ isolated vertices,

with $n_1 + s \geq 2$, by adding all possible 1 + $s$ edges between a vertex $w$ of $G_0$ with $d_{G_0}(w) = 2$ and all 1 + $s$ vertices in $\{x\} \cup S$, where $x$ is some vertex of $G_1$. See Figure 6 for an illustration; the indicated internal structure of $G_1$ is relevant only later. Note that $m = m(G_0) + m(G_1) + s + 1$.

First, we assume that $n_0 \equiv 2 \mod 3$. We now show that $G_0$ has a dissociation set $D_0$ of order $\frac{2n_0 - 1}{3} = n_0 - \frac{1}{3}(m(G_0) + 1)$ that does not contain $w$. If $G_0$ is good, then Lemma 6

![Figure 6](https://example.com/figure6.png)

**Figure 6** Local structure of $G$. 
implies the existence of $D_0$. If $G_0$ is not good, then, by the parity of $n_0$, Lemma 6 implies $\text{diss}(G_0) \geq \frac{2n_0 - 1}{3} + 1$, which also implies the existence of $D_0$. Note that, in the latter case, the set $D_0$ is not a maximum dissociation set of $G_0$. Using $D_0$ and Theorem 1, we obtain

\[
\text{diss}(G) \geq s + \left( n_0 - \frac{1}{3} (m(G_0) + 1) \right) + \text{diss}(G_1)
\]

\[
\geq s + \left( n_0 - \frac{1}{3} (m(G_0) + 1) \right) + \left( n_1 - \frac{1}{3} (m(G_1) + 1 + c_1(G_1)) \right)
\]

\[
\geq n - \frac{1}{3} ((m - s) + 1 + c_1)
\]

\[
\geq n - \frac{1}{3} (m + 1 + c_1).
\]

Since equality holds throughout this inequality chain, we obtain that the graph $G_1$ is extremal, that $s = 0$, and that $c_1(G_1) = c_1$. By induction, we obtain $G_1 \in C$. If $G_0$ is not good, then the union of a maximum dissociation set of $G_0$ and a maximum dissociation set of $G_1$ that does not contain $x$, compare Lemma 3, yields the contradiction that $G$ is not extremal. Hence, the spiked cycle $G_0$ is good, and, since $c_1(G_1) = c_1$, it is very good. Since $G$ is constructed by applying operation $(O_4)$ to $G_1$, and we obtain $G \in C$. Hence, we may assume that $n_0 \not\equiv 2 \mod 3$. If $n_0 \equiv 1 \mod 3$ and $s \geq 1$, then exactly the same argument can be repeated with $G_0$ and $S$ replaced by $G_0'$ and $S'$, where the spiked cycle $G_0'$ with at least two spikes arises from $G_0$ by attaching one vertex from $S$ to $w$, and $S'$ is the set of the remaining $s - 1$ vertices from $S$. Note that $G_0'$ has order $n_0 + 1 \equiv 2 \mod 3$ in that case. Hence, if $n_0 \equiv 1 \mod 3$, then we may assume $s = 0$.

Next, we assume that $n_0 \equiv 0 \mod 3$. The tree $T = G_0 - w$ has a dissociation set of order $\left\lfloor \frac{2n_0 - 1}{3} \right\rfloor = \frac{2n_0}{3} = n_0 - \frac{m(G_0)}{3}$. By Theorem 1, we obtain the contradiction

\[
\text{diss}(G) \geq s + \text{diss}(T) + \text{diss}(G_1)
\]

\[
\geq s + \left( n_0 - \frac{m(G_0)}{3} \right) + \left( n_1 - \frac{1}{3} (m(G_1) + 1 + c_1(G_1)) \right)
\]

\[
> n - \frac{1}{3} (m + 1 + c_1).
\]

Hence, we may assume that $n_0 \equiv 1 \mod 3$, which implies $s = 0$.

Let $G_0' = G_1 - x$ have $r$ components $K_1, \ldots, K_r$; see Figure 6 for an illustration. Clearly, we have $r \leq d_G(x) - 1$. The graph $G_0' = G - V(G_0')$ is a spiked cycle with at least two spikes that arises from $G_0$ by attaching $x$ to $w$. Since the order of $G_0'$ is $n_0 + 1 \equiv 2 \mod 3$, we obtain, similarly as in the case “$n_0 \equiv 2 \mod 3$,” that $G_0'$ has a dissociation set $D_0'$ of order $n(G_0') - \frac{1}{3} (m(G_0') + 1)$ that does not contain $x$. Using $D_0'$ and Theorem 1, we obtain
\[
\text{diss}(G) \geq n\left(G_0'\right) - \frac{1}{3}\left(m\left(G_0'\right) + 1\right) + \text{diss}(K_1) + \cdots + \text{diss}(K_r) \\
\geq n\left(G_0'\right) - \frac{1}{3}\left(m\left(G_0'\right) + 1\right) + \sum_{i=1}^{r}\left(n(K_i) - \frac{1}{3}(m(K_i) + 1 + c_1(K_i))\right) \\
= n - \frac{1}{3}\left((m - d_G(x) + 1) + r + 1 + (c_1(K_i) + \cdots + c(K_r))\right) \\
\geq n - \frac{1}{3}\left(m + 1 + (c_1(K_i) + \cdots + c(K_r))\right) \\
\geq n - \frac{1}{3}(m + 1 + c_1).
\]

Since equality holds throughout this inequality chain, we obtain that \( r = d_G(x) - 1 \), which implies that every component of \( G_1' \) is connected to \( x \) by a bridge, that each \( K_i \) is extremal, which, by induction, implies that \( K_i \in \mathcal{C} \), and that \( c_1(K_i) + \cdots + c(K_r) = c_1 \). If \( G_0' \) is not good, then the union of a maximum dissociation set of \( G_0' \) and maximum dissociation sets of the \( K_i \) that do not contain the neighbors of \( x \), compare Lemma 3, yields the contradiction that \( G \) is not extremal. Hence, the spiked cycle \( G_0' \) is good, and, since \( c_1(K_i) + \cdots + c(K_r) = c_1 \), it is very good. If \( r = 1 \), then \( G \) is constructed by applying operation \((O_1)\) to \( K_1 \) and we obtain \( G \in \mathcal{C} \). Hence, we may assume that \( r \geq 2 \). By symmetry, considering alternative choices for the path \( P \) as well as the previous arguments, we may assume that \( K_r \) is either a \( P_3 \) or a cycle or a very good spiked cycle, and that \( G - V(K_r) \) is in \( \mathcal{C} \). It follows that \( G \) is constructed by applying one of the four operations \((O_1)\) to \((O_4)\) to \( G - V(K_r) \). Hence, we obtain \( G \in \mathcal{C} \), which completes the proof.  

Within our results, the value \( c_1 \) can be replaced by the maximum number of pairwise vertex-disjoint cycles of length 1 modulo 3 [5, 6]. It remains to elucidate the structure of all extremal graphs for Theorem 1.

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