SLICE REGULAR BESOV SPACES OF HYPERHOLOMORPHIC
FUNCTIONS AND COMPOSITION OPERATORS

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Abstract. In this paper we investigate some basic results on the slice regular Besov
spaces of hyperholomorphic functions on the unit ball $\mathbb{B}$. We also characterize the boundedness, compactness and find the essential norm estimates of composition operators between these spaces.

1. Introduction

In the last ten years the theory of slice regular functions is developed systemically in
the papers [22, 30, 31, 32, 34, 35, 36, 37, 38, 39, 40, 41]. Slice hyperholomorphic functions
when defined and takes values in quaternions are called slice regular, see [5, 6, 28]. In case
they are defined on the Euclidean space $\mathbb{R}^{N+1}$ and takes values in the Clifford algebra $\mathbb{R}_\omega$
they are called slice monogenic functions, see [15, 16]. Several function spaces of the slice
hyperholomorphic functions are studied. The quaternionic Hardy spaces are studied in [6, 7, 8, 12, 13, 14, 51]. The Bergman spaces of slice hyperholomorphic functions are invesigated in [19, 20, 21]. For Fock spaces in the slice hyperholomorphic settings, see [5]. Further, weighted Bergman spaces, Bloch, Besov and Dirichlet spaces of slice hyperholomorphic functions on
the unit ball are considered in [47]. D. Alpay etc.al studied Schur analysis in the slice
hyperholomorphic setting see e.g., [1, 2, 6, 8] and references therein. The study of slice
hyperholomorphic functions have wide range of applications. For complete discussion of
slice regular functions and their applications, we refer the book [33] and a recent survey [18].

For each $q \in \mathbb{H}$, we can write $\mathbb{H} = \{q = x_0 + ix_1 + jx_2 + kx_3, \text{ for all } x_1, x_2, x_3 \in \mathbb{R}\}$, where
$\{1, i, j, k\}$ form the basis of quaternions with imaginary units $i, j, k$ such that $i^2 = j^2 = k^2 = -1, ij = -ji = k, jk = -kj = i, ki = -ik = j$. The Euclidean norm on $\mathbb{H}$ is given by $|q| = \sqrt{\overline{q}q} = \sqrt{x_0^2 + x_1^2 + x_2^2 + x_3^2}$, where $\overline{q} = \text{Re}(q) - \text{Im}(q) = x_0 - (ix_1 + jx_2 + kx_3)$ represents the conjugate of q with $\text{Re}(q) = x_0, \text{Im}(q) = ix_1 + jx_2 + kx_3$ and the multiplicative inverse $q^{-1}$ of non-zero quaternion $q$ is given by $\overline{q}/|q|^2$. An element $q$ in $\mathbb{H}$ can be also written as linear combination of two complex numbers $q = (x_0 + ix_1) + (x_2 + ix_3)j$. By symbol $\mathbb{S}$ we denote the two-dimensional unit sphere of purely imaginary quaternions i.e, $\mathbb{S} = \{q = ix_1 + jx_2 + kx_3 \text{ such that } x_1^2 + x_2^2 + x_3^2 = 1\}$. If $I \in \mathbb{S}$ then $I^2 = -1$. Let $\mathbb{C}(i)$ be

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the space generated by \( \{1, i\} \). For any \( i \in \mathbb{S} \), let \( \Omega_i = \Omega \cap \mathbb{C}_i \), for some subset (domain) \( \Omega \) of \( \mathbb{H} \). For a nonreal quaternion \( q \) we can write \( q = x + Im(q) = x + I_q|Im(q)| = x + I_qy \), where \( x = x_0 \), \( y = ix_1 + jx_2 + kx_3 \) with \( I_q = \frac{Im(q)}{|Im(q)|} \), so it lies in the complex plane \( \mathbb{C}(i) \).

We define the slice regular functions on the open ball \( \mathbb{B}(0, 1) = \mathbb{B} = \{q \in \mathbb{H} : |q| < 1\} \) in \( \mathbb{H} \) and \( \mathbb{B} \cap \mathbb{C}(i) = \mathbb{B}_i \) denote the unit disk in the complex plane, for \( i \in \mathbb{S} \). The study of slice holomorphic functions is now an active area of research and lot of work is being done in this direction. For slice holomorphic functions we refer to [23, 26, 40, 47] and references therein. Here, we collect some basic definitions and basic results already obtained in the quaternionic-valued slice regular functions.

**Definition 1.1.** Let \( \Omega \) be an open set in \( \mathbb{H} \). A real differential function \( f : \Omega \to \mathbb{H} \) is said to be (left) slice regular or slice hyperholomorphic on \( \Omega_i \), if for every \( i \in \mathbb{S} \),

\[
\frac{\partial}{\partial x} f_i(x + iy) + i \frac{\partial}{\partial y} f_i(x + iy) = 0,
\]

where \( f_i \) denote the restriction of \( f \) to \( \Omega \cap \mathbb{C}(i) \). The class of slice regular function on \( \Omega \) is denoted by \( \text{SR}(\Omega) \).

**Lemma 1.1.** [26] Lemma 4.1.7 (Splitting Lemma) If \( f \) is a slice regular function on the domain \( \Omega \), then for any \( i, j \in \mathbb{S} \), with \( i \perp j \) there exists two holomorphic functions \( f_1, f_2 : \Omega_i \to \mathbb{C}(i) \) such that

\[
(1) \quad f_i(z) = f_1(z) + f_2(z)j; \text{ for any } z = x + iy \in \Omega_i.
\]

One of the most important property of the slice regular functions is their Representation Formula. It only holds on the open sets which are stated below.

**Definition 1.2.** Let \( \Omega \) be an open set in \( \mathbb{H} \). We say \( \Omega \) is Axially symmetric if for any \( q = x + I_qy \in \Omega \), all the elements \( x + iy \) is contained in \( \Omega \), for all \( i \in \mathbb{S} \) and \( \Omega \) is said to be slice domain if \( \Omega \cap \mathbb{R} \) is non empty and \( \Omega \cap \mathbb{C}(i) \) is a domain in \( \mathbb{C}(i) \) for all \( i \in \mathbb{S} \).

**Theorem 1.3.** [26] Theorem 4.3.2 (Representation Formula) Let \( f \) be a slice regular function in the symmetric slice domain \( \Omega \subseteq \mathbb{H} \) and let \( j \in \mathbb{S} \). Then for all \( z = x + iy \in \Omega \) with \( i \in \mathbb{S} \), the following equality holds

\[
f(x + iy) = \frac{1}{2} \left\{ (1 - ij) f(x + jy) + (1 + ij) f(x - jy) \right\}.
\]

**Remark 1.4.** Let \( i, j \) be orthogonal imaginary units in \( S \) and \( \Omega \) be an axillay symmetric slice domain. Then the Splitting Lemma and the Representation formula generate a class of operators on the slice regular functions as follows:

\[
Q_1 : \text{SR}(\Omega) \to \text{hol}(\Omega_i) + \text{hol}(\Omega_i)j
\]

\[
Q_i : f \mapsto f_1 + f_2j
\]

\[
P_i : \text{hol}(\Omega_i) + \text{hol}(\Omega_i)j \to \text{SR}(\Omega)
\]
\[ P_i[f](q) = P_i[f](x + i_qy) = \frac{1}{2}[(1 - i_q) f(x + iy) + (1 + i_q) f(x - iy)]. \]

Also,
\[ P_i \circ Q_i = I_{SR(\Omega)} \text{ and } Q_i \circ P_i = I_{SR(hol(\Omega) + hol(\Omega))}, \]
where \( I \) is an identity operator.

Since pointwise product of functions does not preserve slice regularity, a new multiplication operation for regular functions is defined. In the special case of power series, the regular product (or \( \star \)-product) of \( f(q) = \sum_{n=0}^{\infty} q^n a_n \) and \( g(q) = \sum_{n=0}^{\infty} q^n b_n \) is
\[ f \star g(q) = \sum_{n \geq 0} q^n \sum_{k=0}^{n} a_k b_{n-k}. \]

The \( \star \)-product is related to the standard pointwise product by the following formula.

**Theorem 1.5.** [13] Proposition 2.4] Let \( f, g \) be regular functions on \( \mathbb{B} \). Then \( f \star g(q) = 0 \) if \( f(q) = 0 \) and \( f(q) g(f(q)^{-1} q f(q)) \) if \( f(q) \neq 0 \). The reciprocal \( f^{-\star} \) of a regular function \( f(q) = \sum_{n=0}^{\infty} q^n a_n \) with respect to the \( \star \)-product is
\[ f^{-\star}(q) = \frac{1}{f \star f^{-\star}(q)} f^c(q), \]
where \( f^c(q) = \sum_{n=0}^{\infty} q^n \overline{a_n} \) is the regular conjugate of \( f \). The function \( f^{-\star} \) is regular on \( \mathbb{B} \setminus (q \in \mathbb{B} | f \star f^{-\star}(q) = 0) \) and \( f \star f^{-\star} = 1 \) there.

2. Besov spaces

Now we define Besov space of slice hyperholomorphic functions on the unit ball \( \mathbb{B} \). Let \( \mathbb{D} \) be a unit disk in the complex plane \( \mathbb{C} \) and \( dA \) be the normalized area measure on \( \mathbb{D} \). For \( 1 < p < \infty \), a holomorphic function \( f : \mathbb{D} \to \mathbb{C} \) is said to be in Besov space \( \mathfrak{B}_{p,\mathbb{C}}(\mathbb{D}) \) if
\[ \int_{\mathbb{D}} |(1 - |z|^2) f'(z)|^p d\lambda(z) < \infty, \]
where \( d\lambda(z) = \frac{dA(z)}{(1 - |z|^2)^2} \) and is Möbius invariant measure on \( \mathbb{D} \). The space \( \mathfrak{B}_{p,\mathbb{C}} \) is a Banach space under the norm
\[ ||f||_{\mathfrak{B}_{p,\mathbb{C}}} = |f(0)| + \left( \int_{\mathbb{D}} |(1 - |z|^2) f'(z)|^p d\lambda(z) \right)^{\frac{1}{p}}. \]
For details on the Besov space of the unit disk one can refer to [11] [58] [59] and references therein.

**Definition 2.1.** Let \( p > 1 \) and let \( i \in \mathbb{S} \). The quaternionic right linear space of slice regular functions \( f \) is said to be the quaternionic slice regular Besov space on the unit ball \( \mathbb{B} \), if
\[ \sup_{i \in \mathbb{S}} \int_{\mathfrak{B}_i} \left| (1 - |q|^2) \frac{\partial f}{\partial x_0}(q) \right|^p d\lambda_i(q) < \infty, \]
that is,
\[ \mathfrak{B}_p = \{ f \in SR(\mathbb{B}) : \sup_{i \in \mathbb{S}} \int_{\mathfrak{B}_i} \left| (1 - |q|^2) \frac{\partial f}{\partial x_0}(q) \right|^p d\lambda_i(q) < \infty \}. \]
where \( d\lambda_i(q) = \frac{dA_i(q)}{(1 - |q|^2)^2} \) and is Möbius invariant measure on \( \mathbb{B} \). The space \( \mathfrak{B}_p \) is a Banach space under the norm
\[
\|f\|_{\mathfrak{B}_p} = |f(0)| + \left( \sup_{i\in\mathbb{S}} \int_{\mathbb{B}_i} \left(1 - |q|^2\right)^\frac{p}{2} \left| \frac{\partial f}{\partial x_0}(q) \right|^p \, d\lambda_i(q) \right)^{\frac{1}{p}}.
\]

For details on Besov spaces of quaternions holomorphic functions one can refer to [17]. By space \( \mathfrak{B}_{p,i}, p > 1 \), we means the quaternionic right linear space of slice regular functions on the unit ball \( \mathbb{B} \) such that
\[
\int_{\mathbb{B}_i} \left| (1 - |z|^2)Q_i[f]^p(z) \right| d\lambda_i(z) < \infty,
\]
and the norm of this space is given by
\[
\|f\|_{\mathfrak{B}_{p,i}} = |f(0)| + \left( \int_{\mathbb{B}_i} \left| (1 - |z|^2)Q_i[f]^p(z) \right| d\lambda_i(z) \right)^{\frac{1}{p}}
\]
where \( Q_i[f]^p(z) = \frac{\partial Q_i[f]}{\partial x_0}(z) \) is a holomorphic map of complex variable \( z = x_0 + iy \) and \( i \in \mathbb{S} \).

**Remark 2.2.** Let \( j \in \mathbb{S} \) be such that \( j \perp i \). Then there exist holomorphic functions \( f_1, f_2 : \mathbb{B}_i \to \mathbb{C}(i) \) such that \( Q_i[f] = f_1 + f_2j \) and so \( \frac{\partial f}{\partial x_0}(z) = f_1^p(z) + f_2^p(z) \) for some \( z \in \mathbb{B}_i \). Thus, for \( z \in \mathbb{B}_i \), it follows that
\[
|f_1^p(z)| \leq \left| \frac{\partial f}{\partial x_0}(z) \right|^p \leq 2^{\max(0,p-1)} \left( |f_1^p(z)| + |f_2^p(z)| \right)^p, \quad l = 1, 2.
\]
Thus, the function \( f \in \mathfrak{B}_{p,i} \) if and only if \( f_1, f_2 \in \mathfrak{B}_{p,C} \) on \( \mathbb{B}_i \), (see [17] Remark 4.3).}

The proof of the following proposition is analogous to [17] Proposition 2.6.

**Proposition 2.3.** Let \( i \in \mathbb{S} \), then \( f \in \mathfrak{B}_{p,i}, p > 1 \) if and only if \( f \in \mathfrak{B}_p \). Moreover, the spaces \( (\mathfrak{B}_{p,i}, \|\|_{\mathfrak{B}_{p,i}}) \) and \( (\mathfrak{B}_p, \|\|_{\mathfrak{B}_p}) \) have equivalent norms. More precisely, one has
\[
\|f\|_{\mathfrak{B}_{p,i}} \leq \|f\|_{\mathfrak{B}_p} \leq 2^p\|f\|_{\mathfrak{B}_{p,i}}.
\]
For all \( z, w \in \mathbb{D} \), Bergman metric on the unit disc \( \mathbb{D} \) in the complex plane \( \mathbb{C} \) is given by
\[
\beta(z, w) = \frac{1}{2} \log \frac{1 + \rho(z, w)}{1 - \rho(z, w)},
\]
where \( \rho(z, w) = \frac{|z-w|}{1-z\bar{w}} \).

**Definition 2.4.** [17]. For \( i \in \mathbb{S} \) and all \( z, w \in \mathbb{B}_i \), we define
\[
\beta_i(z, w) = \frac{1}{2} \log \left( \frac{1 + \frac{|z-w|}{|1-z\bar{w}|}}{1 - \frac{|z-w|}{|1-z\bar{w}|}} \right).
\]
Proposition 2.5. For $1 < p, t < \infty$, with $\frac{1}{p} + \frac{1}{t} = 1$. Let $f \in B_p$ and $i \in S$ be fixed. Then for all $q, w \in B_i$, there exists a constant $M_p > 0$ such that

$$|f(q) - f(w)| \leq 2M_p\|f\|_{B_p}\beta_i(q, w)^t,$$

where

$$\beta_i(q, w) = \frac{1}{2}\log\left(\frac{1 + |q-w|}{1 - |q-w|}\right).$$

Proof. By Lemma 1.1 there exist two holomorphic functions $f_1, f_2 : B_i \rightarrow \mathbb{C}(i)$ such that $Q_i[f] = f_1 + f_2j$, where $j \perp i$. Moreover, the functions $f_i \in B_{p, C} ; l = 1, 2$ on $B_i$. Furthermore, $\|f_i\|_{B_{p, C}}^p \leq \|f\|_{B_{p, C}}^p$; $l = 1, 2$ and $p > 1$. Therefore, from [58, Theorem 9], it follows that for all $q, w \in B_i$ in the complex plane $\mathbb{C}(i)$, one has

$$|f(q) - f(w)|^p \leq 2^{p-1}(|f_1(q) - f_1(w)|^p + |f_2(q) - f_2(w)|^p) \leq 2^{p-1}M_p\left(\|f_1\|_{B_{p, C}}^p\beta_i(q, w)^t + \|f_2\|_{B_{p, C}}^p\beta_i(q, w)^t\right) \leq 2^{p-1}M_p\|f\|_{B_{p, C}}^p\beta_i(q, w)^t \leq 2^{p-1}M_p\|f\|_{B_{p, C}}^p\beta_i(q, w)^t.$$

The following proposition on Besov spaces over the unit disk was proved in [58, Theorem 8] and for its proof on Bloch spaces of slice holomorphic functions one can refer to [47, Theorem 2.20].

Proposition 2.6. Let $f \in B_p, p > 1$ and \( \{a_n\}_{n \in \mathbb{N}} \subset \mathbb{H} \) be a sequence of quaternions such that

$$f(q) = \sum_{n=0}^{\infty} q^n a_n \text{ for } q \in B.$$

Then there exists a constant $K_p > 0$ such that

$$|a_n|^p \leq 2^p K_p\|f\|_{B_p}^p \text{ for } n \in \mathbb{N} \cup \{0\}.$$

Proof. Let $i, j \in S$ be such that $j \perp i$. On applying Splitting Lemma 1.1, we can restrict $f$ on $B_i$ such that $Q_i[f] = f_1 + f_2j$, for some holomorphic functions $f_1, f_2 : B_i \rightarrow \mathbb{C}(i)$ in the complex Besov space $B_{p, C}$ on $B_i$. Furthermore, for any $z \in B_i$ and $p > 1$, we have

$$|f(z)|^p \leq 2^{p-1}(|f_1(z)|^p + |f_2(z)|^p).$$

Now for any $n \in \mathbb{N} \cup \{0\}$, let $a_{1, n}, a_{2, n} \in \mathbb{C}(i)$ such that $a_n = a_{1, n} + a_{2, n}j$. Thus we have

$$|f(z)|^p = \left(\sum_{n=0}^{\infty} z^n a_n\right)^p \leq 2^{p-1}\left(\sum_{n=0}^{\infty} z^n a_{1, n}\right)^p + \left(\sum_{n=0}^{\infty} z^n a_{2, n}\right)^p = 2^{p-1}(|f_1(z)|^p + |f_2(z)|^p).$$

Therefore, from [58, Theorem 8 (1)], it follows that for any $n \in \mathbb{N}$, we have

$$|a_{1, n}| \leq \frac{K_p}{n^p}\|f_1\|_{B_{p, C}} ; l = 1, 2.$$
and so $\|f_l\|_{B_{p, c}} \leq \|f\|_{B_{p, j}} : l = 1, 2$. Then, one has
\[
|a_n|^p = 2^{p-1}(|a_{1,n}|^p + |a_{2,n}|^p) \\
\leq 2^{p-1} K_p \left(\|f_1\|_{B_{p, c}}^p + \|f_2\|_{B_{p, c}}^p\right) \\
\leq 2^{p-1} 2^n \|f\|_{B_{p, j}}^p \\
\leq \frac{2^p K_p}{n} \|f\|_{B_{p, j}}^p.
\]

\[\square\]

**Remark 2.7.** Let $L^p(\mathbb{B}_i, d\lambda_i, \mathbb{H})$, $1 \leq p < \infty$ denote the space of quaternionic valued equivalence classes of measurable functions $g : \mathbb{B}_i \to \mathbb{H}$ such that
\[
\int_{\mathbb{B}_i} |g(w)|^p d\lambda_i(w) < \infty.
\]
Furthermore, for any $j \in \mathbb{S}$ with $j \perp i$ and $g = g_1 + g_2 j$ where $g_1, g_2$ are holomorphic functions in complex plane $\mathbb{C}(i)$, then, $g \in L^p(\mathbb{B}_i, d\lambda_i, \mathbb{H})$ if and only if $g_l \in L^p(\mathbb{B}_i, d\lambda_i, \mathbb{C}(i))$, $l = 1, 2$, the usual $L^p$-space of complex valued measurable functions on $\mathbb{B}_i$.

Now we define the bounded mean oscillation of the slice regular functions.

**Definition 2.8.** For any $z \in \mathbb{B}_i$, let $\Delta_i(z, r) = \{w \in \mathbb{B}_i : \beta_i(z, w) < r\} \subset \mathbb{B}_i$ for some $r > 0$, be the Euclidean disk. Let $f_{r,i}^*(z) = \frac{1}{2\pi} \int_{\Delta_i(z, r)} f(w) dA_i(w)$, for some arbitrary $i \in \mathbb{S}$.

A slice regular function $f$ is said to be in $BMO(\mathbb{B}_i)$ if
\[
\sup_{z \in \mathbb{B}_i} \left(\frac{1}{2\pi} \int_{\Delta_i(z, r)} |f(w) - f_{r,i}^*(z)|^p dA_i(w)\right)^{\frac{1}{p}} < \infty,
\]
with norm defined by
\[
\|f\|_{BMO(\mathbb{B}_i)} = \sup_{z \in \mathbb{B}_i} \left(\frac{1}{2\pi} \int_{\Delta_i(z, r)} |f(w) - f_{r,i}^*(z)|^p dA_i(w)\right)^{\frac{1}{p}}.
\]

We say function $f \in BMO(\mathbb{B})$ if
\[
\|f\|_{BMO(\mathbb{B})} := \sup_{i \in \mathbb{S}} \|f\|_{BMO(\mathbb{B}_i)} := \sup_{i \in \mathbb{S}} \Lambda_{r,i}(f) < \infty,
\]
where
\[
\Lambda_{r,i}(f)(z) = \sup\{\|f(z) - f(w)\| : w \in \Delta_i(z, r)\} \text{ for some } i \in \mathbb{S}.
\]

**Proposition 2.9.** Let $p > 1$ and $i, j \in \mathbb{S}$. Then $f \in BMO(\mathbb{B}_i)$ if and only if $f \in BMO(\mathbb{B}_j)$.

**Proof.** Let $f \in SR(\mathbb{B})$ and choose $w = x + y j \in \mathbb{B}_j$ and $z = x + y i \in \mathbb{B}_i$. Then by Representation formula, we have
\[
|f(w)| = \frac{1}{2} |(1 - ji)f(z) + (1 + ji)f(\bar{z})| \leq |f(z)| + |f(\bar{z})|.
\]
Therefore
\[ \frac{1}{2\pi} \int_{\Delta_j(z,r)} |f(w) - f^*_r(z)|^p dA_j(w) \leq 2^{\max\{p-1,0\}} \frac{1}{2\pi} \int_{\Delta_i(w,r)} |f(z) - f^*_r(w)|^p dA_i(z) + 2^{\max\{p-1,0\}} \frac{1}{2\pi} \int_{\Delta_i(w,r)} |f(\bar{z}) - f^*_r(\bar{w})|^p dA_i(\bar{z}). \]

On changing \( \bar{z} \rightarrow z \) and \( \bar{w} \rightarrow w \), we have
\[ \frac{1}{2\pi} \int_{\Delta_j(z,r)} |f(w) - f^*_r(z)|^p dA_j(w) \leq 2^{\max\{p,1\}} \frac{1}{2\pi} \int_{\Delta_i(w,r)} |f(z) - f^*_r(w)|^p dA_i(z). \]

Thus, we conclude that for any \( f \in BMO(\mathbb{B}_i) \) implies \( f \in BMO(\mathbb{B}_j) \). Finally, on interchanging the role of \( i \) and \( j \), we get the remaining one.

\[ \square \]

Proposition 2.10. For \( p > 1 \) and \( \alpha > -1 \), let \( f \in \mathfrak{B}_p \). Then \( f \in BMO(\mathbb{B}) \) if and only if \( f \in BMO(\mathbb{B}_i) \), for some \( i \in S \).

**Proof.** Since the direct part is obvious, so we only remains to prove the converse part. Suppose \( f \in BMO(\mathbb{B}_i) \), for some arbitrary imaginary unit \( i \) in \( S \). Therefore by Representation formula, we have
\[ \frac{1}{2\pi} \int_{\Delta_j(z,r)} |f(w) - f^*_r(z)|^p dA_j(w) \leq 2^{p-1} \frac{1}{2\pi} \left( \int_{\Delta_i(w,r)} |f(z) - f^*_r(w)|^p dA_i(z) \right) + 2^{p-1} \frac{1}{2\pi} \left( \int_{\Delta_i(w,r)} |f(\bar{z}) - f^*_r(\bar{w})|^p dA_i(\bar{z}) \right). \]

On taking supremum over all \( z \in \mathbb{B}_i \), we have
\[ \|f\|_{BMO(\mathbb{B}_i)} \leq \sup_{z \in \Delta_i(w,r)} 2^{p-1} \frac{1}{2\pi} \left( \int_{\Delta_i(w,r)} |f(z) - f^*_r(w)|^p dA_i(z) \right) + 2^{p-1} \frac{1}{2\pi} \left( \int_{\Delta_i(w,r)} |f(\bar{z}) - f^*_r(\bar{w})|^p dA_i(\bar{z}) \right) \leq 2^{p-1}\|f\|_{BMO(\mathbb{B}_i)} < \infty. \]

Since \( j \) is arbitrary, so we get the desired result.

\[ \square \]

By previous proposition we conclude the following inequality
\[ \|f\|_{BMO(\mathbb{B}_i)}^p \leq \|f\|_{BMO(\mathbb{B})}^p \leq 2\|f\|_{BMO(\mathbb{B}_i)}^p. \]

Proposition 2.11. For \( p > 1 \), let \( f \) be a slice regular function. Then \( f \in \mathfrak{B}_p \) if and only if \( \Lambda_r(f) \in L^p(\mathbb{B}_i, d\lambda_i, \mathbb{H}) \), for some \( i \in S \).

**Proof.** Suppose \( f \in \mathfrak{B}_p \) implies \( f \in \mathfrak{B}_p \). Let \( j \in S \) be such that \( j \perp i \). By Splitting Lemma \([11]\), we can restrict \( f \) on \( \mathbb{B}_i \) with respect to \( j \), as \( \Lambda_i[f](z) = f_1(z) + f_2(z)j \), for some holomorphic functions \( f_1, f_2 \in C(i) \). If we decompose \( \Lambda_r(f) \) on \( \mathbb{B}_i \) as \( \Lambda_r(f) = \Lambda_{r,1}(f_1) + \Lambda_{r,2}(f_2)j \), for some complex oscillation functions \( \Lambda_{r,1}(f_1) \) and \( \Lambda_{r,2}(f_2) \). Then one can see directly from the complex result (see \([58\text{, Theorem 6}]\) ) and Remark 2.7 that the
functions $\Lambda_{r,l}(f_l); \ l = 1, 2$ lie in the usual $L^p$-space of complex valued measurable functions on $B_i$ if and only if $\Lambda_r(f) \in L^p(B_i, d\lambda_i, \mathbb{H})$.

Conversely, assume $\Lambda_r(f) \in L^p(B_i, d\lambda_i, \mathbb{H})$. So we can write

$$\Lambda_{r,1}(f_1) + \Lambda_{r,2}(f_2) = \Lambda_r(f)$$

$$= \sup_{i \in \mathbb{S}} \sup \{|f_1(z) - f_1(w)| : w \in \Delta_i(z, r) \subset B_i\}$$

$$+ \sup_{i \in \mathbb{S}} \sup \{|f_2(z) - f_2(w)| : w \in \Delta_i(z, r) \subset B_i\}.$$ 

This implies

$$\Lambda_{r,i}(f_i) = \sup_{i \in \mathbb{S}} \{|f_i(z) - f_i(w)| : w \in \Delta_i(z, r) \subset B_i\} \in L^p(B_i, d\lambda_i, \mathbb{C}(i)),\ \text{for} \ l = 1, 2.$$ 

Again thanks to above classical result, we conclude that both $f_1$ and $f_2$ belong to complex Besov space $B_{p, C}$ on $B_i$ which is equivalent to $f \in B_{p, i}(B_i)$ and so $f \in B_p(B)$.

**Proposition 2.12.** For $p > 1$, let $f \in SR(B)$. Then $f \in B_p$ if and only if

$$BMO(f) \in L^p(B_i, d\lambda_i, \mathbb{H}), \ \text{for some} \ i \in \mathbb{S}.$$ 

**Proof.** Let $f \in B_p$. Then $f \in B_{p,i}$. Let $j \in \mathbb{S}$ with $j \perp i$. According to Lemma (1.1), any $f \in SR(B)$ can be restricted to $B_i$ decomposes as $Q_i[f](z) = f_1(z) + f_2(z)j$, for some $z \in B_i$ and holomorphic functions $f_1, f_2 \in B_i$. Thus, the condition (2) holds if and only if $BMO(f_i) \in L^p(B_i, d\lambda_i, \mathbb{C}(i)), \ \text{for some} \ i \in \mathbb{S}, \ l = 1, 2.$

Now, by [58, Theorem 7], it follows that the above condition holds if and only if $f_1, f_2$ lie in the complex Besov space $B_{p, C}$ on $B_i$ which is same as $f \in B_{p,i}$ and so $f \in B_p.$

3. **Composition operators on Besov spaces**

3.1. **Boundedness and Compactness.** In this section, we characterize boundedness and compactness of composition operators on Besov spaces of the slice holomorphic functions.

**Definition 3.1.** Let $0 < p < \infty$ and let $\Phi : B \to B$ be a slice hyperholomorphic map such that $\Phi(B_i) \subset B_i$ for some $i \in \mathbb{S}$. Then the composition operator $C_\Phi$ on $B_p$ on the unit ball $B$ induced by $\Phi$ is defined by

$$C_\Phi f = f \circ \Phi, \ \text{for all} \ f \in B_p.$$ 

Composition operators are extensively studied on various holomorphic function spaces of different domains in $\mathbb{C}$ or $\mathbb{C}^n$. For a study of composition operators on spaces of holomorphic functions, one can refer to [27] and [52]. For composition operators on Besov spaces see, [11].

A study of composition operators on Hardy spaces of slice holomorphic functions is initiated in [49]. Recently, Carleson measures for Hardy and Bergman spaces in the quaternionic unit ball are characterized in [50]. In [13], Hankel operators are studied on Hardy spaces via Carleson measures in a quaternionic variables.
The following theorem characterize bounded composition operators on the slice regular Besov spaces $\mathfrak{B}_p$.

**Theorem 3.2.** Let $\Phi$ be a slice holomorphic map on $\mathfrak{B}$ such that $\Phi(\mathfrak{B}_i) \subset \mathfrak{B}_i$ for some $i \in \mathbb{S}$. For all $q \in \mathfrak{B}$ and $a \in \mathfrak{B}_i$, let $\sigma_a(q) = (1 - qa)(a - q)$ be slice regular Möbius transformation. Then the composition operator $C_\Phi$ is bounded on Besov space $\mathfrak{B}_p$, $1 < p < \infty$ if and only if

$$\sup \| C_\Phi \sigma_a \|_{\mathfrak{B}_p} < \infty.$$  

**Proof.** Since the slice regular Möbius transformation on $\mathfrak{B}_i$ coincides with the usual one dimensional complex Möbius transformation, so assume $\sigma_a \in \mathfrak{B}_{p,i}$. Let $j \in \mathbb{S}$ with $j \perp i$. So we can write $\sigma_a = \sigma_{a,1} + \sigma_{a,2} j$, for each one dimensional complex Möbius transformation $\sigma_{a,l} \in \mathfrak{B}_{p,c}$, $l = 1, 2$.

Therefore, from [10, Theorem 13], we have

$$\sup_{i \in \mathbb{S}} \int_{\mathfrak{B}_i} \left| 1 - |z|^2 \right|^p \left| \frac{\partial C_\Phi \sigma_a}{\partial x_0}(z) \right|^p \, d\lambda_i(z) \leq 2^{p-1} \sup_{i \in \mathbb{S}} \int_{\mathfrak{B}_i} \left| 1 - |z|^2 \right|^p \left| \frac{\partial C_\Phi \sigma_{a,1}}{\partial x_0}(z) \right|^p \, d\lambda_i(z) + 2^{p-1} \sup_{i \in \mathbb{S}} \int_{\mathfrak{B}_i} \left| 1 - |z|^2 \right|^p \left| \frac{\partial C_\Phi \sigma_{a,2}}{\partial x_0}(z) \right|^p \, d\lambda_i(z) = 2^{p-1} (\| C_\Phi \sigma_{a,1} \|_{\mathfrak{B}_{p,c}}^p + \| C_\Phi \sigma_{a,2} \|_{\mathfrak{B}_{p,c}}^p) \leq 2^p \| C_\Phi \sigma_a \|_{\mathfrak{B}_{p,i}}^p. \tag{4}$$

Now, let $q = x_0 + Iy \in \mathfrak{B}$ for some $I \in \mathbb{S}$. Then by Theorem [13], it follows that

$$\left| \frac{\partial C_\Phi \sigma_a}{\partial x_0}(q) \right| = \left| \frac{1}{2} (1 - I) \frac{\partial C_\Phi \sigma_a}{\partial x_0}(z) + \frac{1}{2} (1 + I) \frac{\partial C_\Phi \sigma_a}{\partial x_0}(\bar{z}) \right|.$$  

Since, $|q| = |z| = |\bar{z}|$, on applying triangle inequality, we have

$$\left| (1 - |q|^2) \frac{\partial C_\Phi \sigma_a}{\partial x_0}(q) \right| \leq \left| (1 - |z|^2) \frac{\partial C_\Phi \sigma_a}{\partial x_0}(z) \right| + \left| (1 - |\bar{z}|^2) \frac{\partial C_\Phi \sigma_a}{\partial x_0}(\bar{z}) \right|.$$  

On taking integral over $\mathfrak{B}_i$ on both sides of the above inequality and for $p > 1$, we see

$$\sup_{q \in \mathfrak{B}} \int_{\mathfrak{B}_i} \left| (1 - |q|^2) \frac{\partial C_\Phi \sigma_a}{\partial x_0}(q) \right|^p \, d\lambda_i(q) \leq \sup_{i \in \mathbb{S}} \sup_{z \in \mathfrak{B}_i} \int_{\mathfrak{B}_i} \left| (1 - |z|^2) \frac{\partial C_\Phi \sigma_a}{\partial x_0}(z) \right|^p \, d\lambda_i(z) + \sup_{i \in \mathbb{S}} \sup_{z \in \mathfrak{B}_i} \int_{\mathfrak{B}_i} \left| (1 - |\bar{z}|^2) \frac{\partial C_\Phi \sigma_a}{\partial x_0}(\bar{z}) \right|^p \, d\lambda_i(\bar{z}) \leq 2 \sup_{i \in \mathbb{S}} \sup_{z \in \mathfrak{B}_i} \int_{\mathfrak{B}_i} \left| (1 - |z|^2) \frac{\partial C_\Phi \sigma_a}{\partial x_0}(z) \right|^p \, d\lambda_i(z). \tag{5}$$

Thus, by using (4) in (3), we have

$$\sup \| C_\Phi \sigma_a \|_{\mathfrak{B}_p}^p = \sup \sup_{i \in \mathbb{S}} \sup_{q \in \mathfrak{B}_i} \int_{\mathfrak{B}_i} \left| (1 - |q|^2) \frac{\partial C_\Phi \sigma_a}{\partial x_0}(q) \right|^p \, d\lambda_i(q) \leq 2 \sup \sup_{i \in \mathbb{S}} \sup_{z \in \mathfrak{B}_i} \int_{\mathfrak{B}_i} \left| (1 - |z|^2) \frac{\partial C_\Phi \sigma_a}{\partial x_0}(z) \right|^p \, d\lambda_i(z) \leq 2^{p+1} \| C_\Phi \sigma_a \|_{\mathfrak{B}_{p,i}}^p.$$
Since \( C_\Phi \) is bounded operator on the complex Besov space on \( \mathbb{B}_i \), therefore \( \| C_\Phi \sigma_a \|_{\mathcal{B}_p}^p < \infty \).

Now suppose condition (3) holds. Then by [11] Theorem 13, it holds if and only if \( C_\Phi \) is bounded operator on the complex Besov space which is equivalent to the boundedness of \( C_\Phi \) on \( \mathcal{B}_{p,i} \) and so \( C_\Phi \) is bounded on \( \mathcal{B}_p \).

By using Splitting Lemma, Remark 2.2 and [54] Lemma 3.6, the proof of the following lemma follows easily.

**Lemma 3.1.** For \( p \geq 1 \), let \( \mathcal{B}_p \) be a slice regular Besov space on the unit ball \( \mathbb{B} \). Then the following condition holds:

1. every slice regular bounded sequence \( \{ f_m \}_{m \in \mathbb{N}} \) in \( \mathcal{B}_p \) on compact sets is uniformly bounded;
2. for any slice regular sequence \( \{ f_m \}_{m \in \mathbb{N}} \) in \( \mathcal{B}_p \) such that \( \| f_n \|_{\mathcal{B}_p} \to 0 \), \( f_n - f_n(0) \to 0 \) uniformly on the compact sets.

The next result is essential for the proof of Proposition 3.3.

**Lemma 3.2.** [54] Lemma 3.7 | Let \( X, Y \) be two Banach spaces of analytic functions on the unit disk \( \mathbb{D} \). Suppose

1. the point evaluation functionals on \( X \) are continuous;
2. the closed unit ball in \( X \) is a compact subset of \( X \) in the topology of uniform convergence on compact sets;
3. \( T : X \to Y \) is continuous, where \( X \) and \( Y \) are equipped with the topology of uniform convergence on compact sets. Then \( T \) is a compact operator if and only if given a bounded sequence \( \{ f_n \} \) in \( X \) such that \( f_n \to 0 \) uniformly on compact sets, then the sequence \( \{ T f_n \} \) converges to zero in the norm of \( Y \).

The following proposition gives the characterization for compact composition operators.

**Proposition 3.3.** For \( p > 1 \), let \( \mathcal{B}_p \) be a slice regular Besov space on the unit ball \( \mathbb{B} \). Let \( \Phi \) be a slice holomorphic map on \( \mathbb{B} \) such that \( \Phi(\mathbb{B}_i) \subset \mathbb{B}_i \) for some \( i \in \mathcal{S} \). Then \( C_\Phi : \mathcal{B}_p \to \mathcal{B}_p \) is compact if and only if for any bounded sequence \( \{ f_m \}_{m \in \mathbb{N}} \) in \( \mathcal{B}_p \) with \( f_m \to 0 \) as \( m \to \infty \) on compact sets, \( \| C_\Phi f_m \|_{\mathcal{B}_p} \to 0 \).

**Proof.** The proof of the theorem is established if we prove the condition of Lemma 3.2. As a consequence of Lemma 3.1, we see that conditions (1) and (3) holds. Now, it remains to prove the condition (2). For this, let \( \{ f_m \} \) be a slice regular bounded sequence in \( \mathcal{B}_p \). Then by Lemma 3.1 \( \{ f_m \} \) is uniformly bounded on the compact sets. Consider \( \{ f_{m_k} \} \) a subsequence of \( \{ f_m \} \) in \( \mathcal{B}_p \) such that \( f_{m_k} \) converges uniformly to \( h \) on the compact sets, for some \( h \in \text{SR}(\mathbb{B}) \). Let \( j \in \mathbb{S} \) with \( j \perp i \). Then by Lemma 1.1 there exist holomorphic functions \( f_{1,m_k}, f_{2,m_k} : \mathbb{B}_i \to \mathbb{C}(i) \) such that \( Q_j[f_{m_k}](z) = f_{1,m_k}(z) + f_{2,m_k}(z)j \), for some \( z \in \mathbb{B}_i \). Furthermore, \( f_{1,m_k} \to h_1 \) and \( f_{2,m_k} \to h_2 \) uniformly on the compact sets, where \( h_l \in \mathbb{C}(i), l = 1, 2 \) with \( Q_j[h_l] = h_1 + h_2j \). From Remark 2.2 we conclude that \( f_{1,m_k} \) and \( f_{1,m_k} \) belong to the complex Besov space \( \mathcal{B}_{p,C}(\mathbb{B}_i) \). Thus, from [54] Lemma 3.8 and by
applying Minkowski’s inequality and Fatou’s Theorem, for $p > 1$, we have
\[
\left( \int_{B_i} \left( \frac{\partial h}{\partial x_0}(z) \right)^p (1 - |z|^2) d\lambda_i(z) \right)^{\frac{1}{p}} \leq \left( \int_{B_i} 2^{p-1} |(h'_1(z) + h'_2(z)f)(1 - |z|^2)|^p d\lambda_i(z) \right)^{\frac{1}{p}}
\]
\[
\leq \left( \int_{B_i} 2^{p-1} |(h'_1(z))|^{p} (1 - |z|^2)^{p} d\lambda_i(z) \right)^{\frac{1}{p}}
\]
\[
+ \left( \int_{B_i} 2^{p-1} |(h'_2(z))|^{p} (1 - |z|^2)^{p} d\lambda_i(z) \right)^{\frac{1}{p}}
\]
\[
= 2 \frac{p-1}{p} \left( \int_{B_i} \lim_{k \to \infty} |f'_{1,m_k}(z)(1 - |z|^2)|^{p} d\lambda_i(z) \right)^{\frac{1}{p}}
\]
\[
+ 2 \frac{p-1}{p} \left( \int_{B_i} \lim_{k \to \infty} |f'_{2,m_k}(z)(1 - |z|^2)|^{p} d\lambda_i(z) \right)^{\frac{1}{p}}
\]
\[
\leq 2 \frac{p-1}{p} \lim_{k \to \infty} \left( \int_{B_i} |f'_{1,m_k}(z)(1 - |z|^2)|^{p} d\lambda_i(z) \right)^{\frac{1}{p}}
\]
\[
+ 2 \frac{p-1}{p} \lim_{k \to \infty} \left( \int_{B_i} |f'_{2,m_k}(z)(1 - |z|^2)|^{p} d\lambda_i(z) \right)^{\frac{1}{p}}
\]
\[
= 2 \frac{p-1}{p} \lim_{k \to \infty} \inf \{ \lim_{k \to \infty} \|f_{1,m_k}\|_{B_{p,c}} + \lim_{k \to \infty} \|f_{2,m_k}\|_{B_{p,c}} \}
\]
\[
\leq 2 \frac{p-1}{p} \lim_{k \to \infty} \inf \{ \|f_{m_k}\|_{B_{p,i}} \}
\]
\[
< \infty.
\]

The next result is the immediate consequence of Proposition 3.3.

**Corollary 3.4.** For $1 < p < \infty$, let $\Phi$ be a slice holomorphic map such that $\Phi(B_i) \subset B_i$ for some $i \in \mathbb{S}$. If $\|\Phi\|_{\infty} < 1$, then $C_\Phi : B_p \to B_p$ is compact.

**Proof.** Let $\{f_n\}$ be a bounded sequence in $B_p$. Then $f_n \in B_{p,i}$ such that $f_n \to 0$ uniformly on the compact subsets of $B_i$ for some $i \in \mathbb{S}$. Let $j \in \mathbb{S}$ be such that $j \perp i$. Let $f_{1,n}, f_{2,n} : B_i \to C(i)$ be holomorphic functions such that $Q_i(f)(z) = f_{1,n}(z) + f_{2,n}(z)j$, for some $z = x_0 + iy \in B_i$. By Remark 2.2, we have $f_{1,n}, f_{2,n}$ lie in the complex Besov space $B_{p,c}$ on $B_i$, where $B_i$ is identified with $\mathbb{D} \subset C(i)$. Therefore,
\[
\sup_{i \in \mathbb{S}} \int_{B_i} \left( 1 - |z|^2 \right)^{\frac{1}{p}} \frac{\partial C_\Phi f_n}{\partial x_0}(z) d\lambda_i(z) \leq 2^{p-1} \sup_{i \in \mathbb{S}} \int_{B_i} \left( 1 - |z|^2 \right)^{\frac{1}{p}} \frac{\partial C_\Phi f_{1,n}}{\partial x_0}(z) d\lambda_i(z)
\]
\[
+ 2^{p-1} \sup_{i \in \mathbb{S}} \int_{B_i} \left( 1 - |z|^2 \right)^{\frac{1}{p}} \frac{\partial C_\Phi f_{2,n}}{\partial x_0}(z) d\lambda_i(z)
\]
\[
= 2^{p-1}(\|C_\Phi f_{1,n}\|_{B_{p,c}^i}^p + \|C_\Phi f_{2,n}\|_{B_{p,c}^i}^p)
\]
\[
\leq 2^p \|C_\Phi f_n\|_{B_{p,i}^i}^p.
\]

Now, appealing to Theorem 1.3 and the fact that $|q| = |\bar{z}| = |z|$, equation (6) and Corollary 2.12, it follows that
\[
\sup_{q \in \mathcal{B}_1} \int_{|q| = 1} \left(1 - |q|^2\right) \frac{\partial C_\Phi f_m}{\partial x_0}(q) \, d\lambda_i(q) \leq \sup_{i \in S} \sup_{z \in \mathcal{B}_i} \int_{|z| = 1} \left(1 - |z|^2\right) \frac{\partial C_\Phi f_m}{\partial x_0}(z) \, d\lambda_i(z) \\
\quad \quad + \sup_{i \in S} \sup_{z \in \mathcal{B}_i} \int_{|z| = 1} \left(1 - |z|^2\right) \frac{\partial C_\Phi f_m}{\partial x_0}(z) \, d\lambda_i(z) \\
\leq 2^p \sup_{i \in S} \sup_{z \in \mathcal{B}_i} \int_{|z| = 1} \left|\frac{\partial f_m}{\partial x_0}(\Phi(z))\right|^p \left(1 - |z|^2\right)^{p} \\
\quad \quad \cdot \left|\frac{\partial \Phi}{\partial x_0}(z)\right| \, d\lambda_i(z) \\
\leq 2^{p+1} \sup_{i \in S} \sup_{z \in \mathcal{B}_i} \int_{|z| = 1} \left|\frac{\partial f_m}{\partial x_0}(\Phi(z))\right|^p \left(1 - |z|^2\right)^{p} \\
\quad \quad \cdot \left|\frac{\partial \Phi}{\partial x_0}(z)\right| \, d\lambda_i(z) \\
\leq 2^{p+1} \varepsilon.
\]
Therefore, \(\|C_\Phi f_m\|_{\mathcal{B}_p} \to 0\) as \(n \to \infty\). Hence the result.

The following proposition gives the compactness between Besov and Bloch spaces of slice regular functions.

**Proposition 3.5.** For \(p > 1\), let \(\Phi\) be a slice holomorphic map on \(\mathcal{B}\) such that \(\Phi(\mathcal{B}_i) \subset \mathcal{B}_i\), for some \(i \in S\). Then \(C_\Phi : \mathcal{B}_p \to \mathcal{B}\) is compact if and only if

\[
(7) \quad \|C_\Phi \sigma_a\|_\mathcal{B} \to 0, \text{ as } |a| \to 1,
\]
where \(\sigma_a(q) = (1 - qa)^* (a - q)\), \(q \in \mathcal{B}\) and \(\mathcal{B}\) is a slice regular Bloch space on the unit ball \(\mathcal{B}\). Here \(*\) denotes the slice regular product.

**Proof.** Let \(\{\sigma_a : a \in \mathcal{B}\}\) be a set in \(\mathcal{B}_p\) such that \(\sigma_a - a \to 0\) as \(|a| \to 1\). Suppose \(C_\Phi\) is compact operator. Then by Lemma \[53\] \(\{\sigma_a\}\) is a bounded set in \(\mathcal{B}_p\). Therefore, \(\|C_\Phi \sigma_a\|_\mathcal{B} = 0\). Suppose condition (7) holds. Let \(f_m\) be a bounded sequence in \(\mathcal{B}_{p,i}\) such that \(f_m \to 0\) uniformly on the compact sets as \(m \to \infty\). We claim \(C_\Phi : \mathcal{B}_p \to \mathcal{B}\) is compact. For this, take \(j \in S\) with \(j \cdot i\). Let \(f_{1,m}, f_{2,m}\) be holomorphic functions such that \(Q_{i,m}(f_m) = f_{1,m}(z) + f_{2,m}(z)\), for some \(z = x_0 + iy \in \mathcal{B}_i\). By Remark \[22\] we have \(f_{1,m}, f_{2,m}\) lie in the complex Besov space \(\mathcal{B}_{p,c}(\mathcal{B}_i)\). Therefore, from \[53\] Theorem 4.1] and as \(\|f_i\|_{\mathcal{B}_{p,c}} \leq \|f\|_{\mathcal{B}_{p,i}}\), we have

\[
\|C_\Phi f_m\|_\mathcal{B} = \sup_{i \in S} \sup_{z \in \mathcal{B}_i} \left\{ (1 - |z|^2) \left| \frac{\partial C_\Phi \left( f_{1,m} + f_{2,m} \right)}{\partial x_0}(z) \right| \right\} \\
\quad \quad = \sup_{i \in S} \sup_{z \in \mathcal{B}_i} \left\{ (1 - |z|^2) \left| \frac{\partial C_\Phi f_{1,m}}{\partial x_0}(z) + \frac{\partial C_\Phi f_{2,m}}{\partial x_0}(z) \right| \right\} \\
\quad \quad \leq \sup_{i \in S} \sup_{z \in \mathcal{B}_i} \left\{ (1 - |z|^2) \left| \frac{\partial C_\Phi f_{1,m}}{\partial x_0}(z) \right| \right\} + \sup_{i \in S} \sup_{z \in \mathcal{B}_i} \left\{ (1 - |z|^2) \left| \frac{\partial C_\Phi f_{2,m}}{\partial x_0}(z) \right| \right\} \\
\quad \quad \leq 2 \sup_{i \in S} \sup_{z \in \mathcal{B}_i} \left\{ (1 - |z|^2) \left| \frac{\partial \Phi}{\partial x_0}(z) \right| \right\} \sup_{z \in \mathcal{B}_i} \left\{ (1 - |z|^2) \left| \frac{\partial f_m}{\partial x_0}(\Phi(z)) \right| \right\} \\
\quad \quad \leq 2 \sup_{i \in S} \sup_{z \in \mathcal{B}_i} \left\{ (1 - |z|^2) \left| \frac{\partial \Phi}{\partial x_0}(z) \right| \right\} \|f_m\|_\mathcal{B}, \\
\quad \quad \leq 2 \sup_{i \in S} \sup_{z \in \mathcal{B}_i} \left\{ (1 - |z|^2) \left| \frac{\partial \Phi}{\partial x_0}(z) \right| \right\} \|f_m\|_{\mathcal{B}_{p,i}}.
\]
Since \( \{ f_m \} \) is bounded in \( \mathcal{B}_{p,i} \), so \( \| C_\Phi f_m \|_{\mathcal{B}_{p,i}} \to 0 \) as \( m \to \infty \). Thus, \( \| C_\Phi \|_\mathcal{B} \to 0 \) as \( m \to \infty \). Hence by Lemma 3.3, \( C_\Phi : \mathcal{B}_p \to \mathcal{B} \) is compact.

4. Essential norm

In this section, we find some estimates for the essential norm of composition operators on the slice regular Besov space. Firstly, we define Carelson measure.

**Definition 4.1.** For \( 1 < p < \infty \), let \( \mathcal{B}_p \) be a slice regular Besov space. Let \( \mu \) be a \( \mathbb{H} \)-valued positive measure on \( \mathcal{B}_i \). Then \( \mu \) is said to be \( \mathbb{H} \)-valued \( p \)-Carleson measure on \( \mathcal{B}_i \) if there is a constant \( M > 0 \) such that

\[
\int_{\mathcal{B}_i} \left| \frac{\partial f}{\partial x_0}(q) \right|^p \mu(q) \leq M \| f \|^p_{\mathcal{B}_p},
\]

for all \( f \in \mathcal{B}_p(\mathbb{B}) \).

**Theorem 4.2.** Let \( f \in \text{SR}(\mathbb{B}) \). If \( \mu = \mu_1 + \mu_2 j \) for some \( i \in \mathbb{S} \). Then \( \mu \) is \( \mathbb{H} \)-valued \( p \)-carleson measure on the slice regular Besov space if and only if \( \mu_1, \mu_2 \) are \( p \)-Carleson measure on the complex Besov space \( \mathcal{B}_{p,C}, 1 < p < \infty \) in \( \mathbb{B}_i \).

**Proof.** Let \( j \in \mathbb{S} \) be such that \( i \perp j \). Then for any \( f \in \mathcal{B}_{p,i} \) there exist holomorphic functions \( f_1, f_2 : \mathbb{B}_i \to \mathbb{C}(i) \) such that \( Q_i[f] = f_1(z) + f_2(z) j \), for some \( z = x_0 + iy \in \mathbb{B}_i \). Now, \( \mu \) is \( \mathbb{H} \)-valued \( p \)-Carleson measure on \( \mathcal{B}_p \) if and only if \( \mu \) is \( \mathbb{H} \)-valued \( p \)-Carleson measure on \( \mathcal{B}_{p,i} \) if and only if

\[
\int_{\mathcal{B}_i} \left| \frac{\partial f}{\partial x_0}(q) \right|^p \mu(q) \leq M \| f \|^p_{\mathcal{B}_{p,i}},
\]

if and only if \( \mu_1, \mu_2 \) for \( l = 1, 2 \) is \( p \)-Carleson measure on \( \mathcal{B}_{p,C}(\mathbb{B}_i) \).

**Definition 4.3.** [26, 47] The slice regular Möbius transformation \( \sigma_a \) for every \( a \in \mathbb{B} \) is define as

\[
\sigma_a(q) = (1 - qa)^-* (a - q), \quad \text{for } q \in \mathbb{B},
\]

where \( * \) is slice regular product.

The slice regular Möbius transformation \( \sigma_a \) satisfies the following conditions:

(i) \( \sigma_a : \mathbb{B} \to \mathbb{B} \) is a bijective mapping,

(ii) For all \( z \in \mathbb{B}_i \), \( \sigma_a(z) = \frac{a - z}{1 - az} \).
(iii) For all \( q \in \mathbb{B}, \) \( \sigma_a(a) = 0, \sigma_a(0) = a \) and \( \sigma_a \circ \sigma_a(q) = q. \)

Now we give the definition of essential norm.

**Definition 4.4.** The essential norm of a continuous linear operator \( T \) between the normed linear spaces \( X \) and \( Y \) is its distance from the compact operator \( K, \) that is

\[
\|T\|_{e}^{X \rightarrow Y} = \inf \{\|T - K\|^{X \rightarrow Y} : K \text{ is compact operator}\},
\]

where \( \|\cdot\|^{X \rightarrow Y} \) denotes the operator norm and \( \|\cdot\|_{e}^{X \rightarrow Y} \) is the essential norm.

Here, we give an essential norm estimate for composition operators on the slice regular Besov space \( \mathcal{B}_p. \)

**Theorem 4.5.** For \( 1 < p < \infty \) and \( \alpha > -1, \) let \( \Phi \) be a slice holomorphic map such that \( \Phi(\mathbb{B}_i) \subset \mathbb{B}_i, \) for some \( i \in \mathbb{S}. \) Suppose the composition operator \( C_\Phi : \mathcal{B}_p \rightarrow \mathcal{B}_p \) is bounded. Then there is an absolute constant \( M \geq 1 \) such that

\[
\lim_{|a| \to 1} \sup_{i \in \mathbb{S}} \int_{\mathbb{B}_i} \frac{(1 - |a|^2)^{2+\alpha}}{|1 - \bar{a}q|^{2(2+\alpha)}} d\mu_p(q) \leq \|C_\Phi\|_e \leq M2^p \lim_{|a| \to 1} \sup_{i \in \mathbb{S}} \int_{\mathbb{B}_i} \frac{(1 - |a|^2)^{2+\alpha}}{|1 - \bar{a}q|^{2(2+\alpha)}} d\mu_p(q).
\]

**Proof.** Let \( f = \sum_{k=0}^{\infty} q^k a_k \in \mathcal{B}_{p,i}, \) for some \( i \in \mathbb{S}. \) For \( 0 < r < 1, \) denote \( \mathbb{B}_r = \{|z| < r\} \) in the complex plane \( \mathbb{C}(i). \) Consider an operator \( R_n f(q) = \sum_{k=n+1}^{\infty} q^k a_k, \) for some integer \( n. \) Suppose \( j \in \mathbb{S} \) with \( j \perp i. \) Then there exists holomorphic functions \( f_1, f_2 : \mathbb{B}_1 \rightarrow \mathbb{C}(i) \) such that \( Q_i[f] = f_1(z) + f_2(z)j, \) for some \( z = x_0 + iy \in \mathbb{B}_i. \) By Remark 2.2, we have

\[
f_l = \sum_{k=0}^{\infty} q^k a_{l,k} \in \mathcal{B}_{p,c}(\mathbb{B}_i), \text{ and } R_{l,n} f_l(q) = \sum_{k=n+1}^{\infty} q^k a_{l,k}, \text{ for some integer } n \text{ and } l = 1, 2.
\]

Therefore, we have

\[
\|C_\Phi\|_e \leq \lim_{n \to \infty} \inf \|C_\Phi R_n\|_{\mathcal{B}_p} \leq \lim_{n \to \infty} \inf \|f\|_{\mathcal{B}_p} \sup \|C_\Phi R_n\|_{\mathcal{B}_p}.
\]

Now,

\[
\|C_\Phi R_n\|_{\mathcal{B}_p}^p = \left( \left| R_{l,n} f_1(\Phi(0)) \right| + \sup_{i \in \mathbb{S}} \int_{\mathbb{B}_i} \frac{(1 - |\zeta|^2) \partial(C_\Phi R_{l,n}) f_1(\Phi(\zeta))}{\partial x_0} \frac{d\lambda_i(\zeta)}{\partial \lambda_i} \right)^p + \left( \left| R_{l,n} f_2(\Phi(0)) \right| + \sup_{i \in \mathbb{S}} \int_{\mathbb{B}_i} \frac{(1 - |\zeta|^2) \partial(C_\Phi R_{l,n}) f_2(\Phi(\zeta))}{\partial x_0} \frac{d\lambda_i(\zeta)}{\partial \lambda_i} \frac{j}{\partial \lambda_i} \right)^p.
\]

Let \( \mu = \mu_{1,p} + \mu_{2,p}j, \) where \( \mu_{1,p} \) and \( \mu_{2,p} \) are two \( p-\)Carleson measure on \( \mathbb{B}_i \) with the values in \( \mathbb{C}(i). \) Then again thanks to Remark 2.2 and Theorem 3.4, we have

\[
\sup_{i \in \mathbb{S}} \int_{\mathbb{B}_i} \left| (1 - |\zeta|^2) \frac{\partial(C_\Phi R_{l,n}) f_1(\Phi(\zeta))}{\partial x_0} \right|^p \frac{d\lambda_i(\zeta)}{\partial \lambda_i} \leq 2^{p-1} \sup_{i \in \mathbb{S}} \int_{\mathbb{B}_i} \left| (1 - |\zeta|^2) \frac{\partial(C_\Phi R_{l,n}) f_1(\Phi(\zeta))}{\partial x_0} \right|^p \frac{d\lambda_i(\zeta)}{\partial \lambda_i}.
\]
\[ \int (1 - |z|^2) \frac{\partial (C_\Phi R_{2,n}) f_2}{\partial x_0} (\Phi(z)) \, d\lambda_i(z). \]

Hence the desired result.

Now, let

\[ \int_{\mathbb{S}_1} \frac{\partial (C_\Phi R_{1,n}) f_1}{\partial x_0} (q) \, d\mu_{1,p}(z) \]

Thus, we have the upper bound.

\[ \int_{\mathbb{S}_1} \frac{\partial (C_\Phi R_{2,n}) f_2}{\partial x_0} (q) \, d\mu_{2,p}(z). \]

Again by [42, Theorem 3.4], for some fixed \( r \), we have

\[ 2^p \sup_{p,i} \int_{\mathbb{B}_r} \left| \frac{\partial (C_\Phi R_n) f}{\partial x_0} (q) \right|^p \, d\mu_p(z) \to 0 \text{ as } n \to \infty \]

and

\[ \int_{\mathbb{B}_r \setminus \mathbb{B}_r} \left| \frac{\partial (C_\Phi R_n) f}{\partial x_0} (q) \right|^p \, d\mu_{p,r}(z) \leq C_1 C_2 \| \mu_p \|_{r}^*, \]

for some absolute constants \( C_1, C_2 \) and \( \| \mu_p \|_{r}^* = \lim_{\| f \|_{r} \to 0} \sup_{a \to 1} \left| \frac{\partial \sigma_a(q)}{\partial x_0} \right|^p \, d\mu_p(q) \), for \( 0 < r < 1 \) and \( p > 1 \). Let \( \mu_{p,r} \) be the restriction of measure \( \mu_p \) to the set \( \mathbb{B}_r \setminus \mathbb{B}_r \). Thus,

\[ \| C_\Phi \|^p \leq \lim_{n \to \infty} \inf_{\| f \|_{p} \leq 1} \| (C_\Phi R_n) f \|^p_{2^p} \]

\[ \leq 2^p C_1 C_2 \lim_{r \to 1} \| \mu_p \|_{r}^* \]

\[ = 2^p C_1 C_2 \lim_{r \to 1} \sup_{\| f \|_{p} \leq 1} \int_{\mathbb{B}_r} \left| \frac{\partial \sigma_a(q)}{\partial x_0} \right|^p \, d\mu_p(q) \]

\[ = 2^p \lim_{\| f \|_{p} \leq 1} \int_{\mathbb{B}_r} \left( \frac{1 - |q|^2}{|1 - \bar{a}q|^2(2 + \alpha)} \right) \, d\mu_{p,r}(q). \]

Thus, we have the upper bound.

Now, let \( \sigma_a(z) = \frac{a - z}{1 - \bar{a}z} \) be the complex Möbius transformation on \( \mathbb{B}_r \), associated with \( a \). Clearly \( \sigma_a \) is bounded in \( \mathbb{B}_r \). Also \( \sigma_a(a) \to 0 \text{ as } |a| \to 1 \) uniformly on the compact subsets of \( \mathbb{B}_i \) and \( |\sigma_a(z) - a| = |z| \frac{1 - |a|^2}{|1 - \bar{a}z|^2}. \) Furthermore, \( \| K(\sigma_a - a) \|_{2_{p,i}} \to 0 \text{ as } |a| \to 1 \) for some compact operator \( K \) on \( 2_{p,i} \). Therefore,

\[ \| C_\Phi \|^p \geq \| C_\Phi - K \|^p_{2_{p,i}} \geq \| C_\Phi - K \|^p_{2_{p,i}} \]

\[ \geq \lim_{|a| \to 1} \| (C_\Phi - K) \sigma_a \|^p_{2_{p,i}} \]

\[ \geq \limsup_{|a| \to 1} \| C_\Phi \sigma_a \|^p_{2_{p,i}} - \limsup_{|a| \to 1} \| K \sigma_a \|^p_{2_{p,i}} \]

\[ = \limsup_{|a| \to 1} \sup_{i \in \mathbb{S}_1} \int_{\mathbb{B}_r} \left( \frac{1 - |a|^2}{|1 - \bar{a}q|^2(2 + \alpha)} \right) \, d\mu_{p,i}(q). \]

Hence the desired result. \( \Box \)
Remark 4.6. By using Splitting Lemma 1.1 and Representation Theorem 1.2 it can be proved that the composition operators on spaces of slice holomorphic will be bounded if and only if the corresponding composition operators are bounded on classical holomorphic function spaces.

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