Self-similar collapse of a scalar field in dilaton gravity and critical behaviour

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Abstract

We present new analytical self-similar solutions describing a collapse of a massless scalar field in scalar-tensor theories. The solutions exhibit a type of critical behavior. The black hole mass for the near critical evolution is analytically obtained for several scalar-tensor theories and the critical exponent is calculated. Within the framework of the analytical models we consider it is found that the black hole mass law for some scalar-tensor theories is of the form $M_{BH} = f(p-p_{cr})(p-p_{cr})^\gamma$ which is slightly different from the general relativistic law $M_{BH} = \text{const}(p-p_{cr})^\gamma$.

1 Introduction

Critical phenomena in gravitational collapse have attracted much interest since the pioneering work by Choptuik [1] and they are considered to be one of the greatest successes of the numerical relativity. The essence of the critical phenomena consists in the fact that, just at the threshold of the black hole formation, the field dynamics becomes simple and exhibits discrete or continuous self-similarity despite the nature and non-linearity of the collapsing matter. The critical solution separates the solutions with a black hole formed from those without a formation of a black hole. The mass of a black hole formed in near critical collapse obeys the power law

$$M_{BH} = C(p-p_{cr})^\gamma$$

where the parameter $p$ describes the strength of the initial data and $p_{cr}$ is its critical value. The exponent $\gamma$ is, in general, matter dependent and $C$ is a constant which depends on the initial field configuration. For detail discussion of the critical phenomena we refer the reader to the Gundlach’s review [2] and references therein.

Parallel to the numerical investigations, an analytical approach has been undertaken in order to understand more deeply the critical collapse. Although fascinating,

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this task was proved to be very difficult. The analytical approach is based on approximate (perturbative) techniques or exact toy models which cannot be considered as completely realistic and satisfactory. Nevertheless, the analytical methods, and especially the exact solutions provide valuable independent insight into the subject. It was realized by Brady [4] and Oshiro, Nakamura and Tomimatsu [5] that the one parameter family of exact self-similar real massless scalar-field solutions first discovered by Roberts\(^1\) [3] can be applied to explain some basic features of the critical collapse. Solutions from this family with \( p > 1 \) describe the formation of a black hole and to the leading order around the critical value \( p_{cr} = 1 \) the black hole mass turns out to be \( M \sim (p - 1)^{1/2} \). The critical exponent in this toy model is therefore \( \gamma = 1/2 \) and is slightly different from the numerically calculated value \( \gamma = 0.37 \). In order to gain as much as possible insight into the critical phenomena, some exact toy models in three dimensions were also considered [6].

Critical collapse within the scalar-tensor theories has attracted some interest, too [10], [11]. There were also some analytical works on the critical collapse in Brans-Dicke theory and the theory with conformal coupling [11], [12], [13]. Using conformally transformed Roberts’s solutions the authors were able to find analytically the black hole mass power law and to extract the critical exponent which turns out to be \( \gamma = 1/2 \pm 1/2 \sqrt{2\omega + 3} \) and \( \gamma = 0.21 \) for the Brans-Dicke and the theory with conformal coupling, respectively.

However, there are serious omissions in the works treating analytically critical collapse in scalar-tensor theories. First, only two theories were considered. Second, the authors have considered simplified models where the matter fields are absent (dilaton-vacuum solutions) i.e. the collapsing matter has been taken to be the dilaton field. The aim of the present work is to fill in this gap studying analytically the critical collapse of a massless real scalar field within the scalar-tensor theories of gravity. Our study is based on exact solutions which generalize Roberts’s solution in the presence of a dilaton field.

2 Exact self-similar solutions

The general form of the extended gravitational action in scalar-tensor theories is

\[
S = \frac{1}{16\pi G_\ast} \int d^4 x \sqrt{-\bar{g}} \left( F(\Phi) \bar{R} - Z(\Phi) \bar{g}^{\mu\nu} \partial_\mu \Phi \partial_\nu \Phi - 2U(\Phi) \right) + \int S_m[\Psi_m; \bar{g}_{\mu\nu}] .
\]

(2)

Here, \( G_\ast \) is the bare gravitational constant, \( \bar{R} \) is the Ricci scalar curvature with respect to the space-time metric \( \bar{g}_{\mu\nu} \). The dynamics of the scalar field \( \Phi \) depends on

\(^1\)It turns out that the original Roberts’s solution as presented in [3] solves the field equations only for one nontrivial case and represents just a class of measure zero among the solutions to the spherically symmetric homothetic Einstein-scalar field equations given in [4], [5] and [6]. For detail discussion see [7] where the Roberts’s solution is generalized. I would like to thank L. Burko for pointing me this out.
the functions $F(\Phi)$, $Z(\Phi)$ and $U(\Phi)$. In order for the gravitons to carry positive energy the function $F(\Phi)$ must be positive. The nonnegativity of the energy of the dilaton field requires that $2F(\Phi)Z(\Phi) + 3\left|\frac{dF(\Phi)}{d\Phi}\right|^2 \geq 0$. The action of matter depends on the material fields $\Psi_m$ and the space-time metric $\tilde{g}_{\mu\nu}$.

However, it is much more convenient from a mathematical point of view to analyze the scalar-tensor theories with respect to the conformally related Einstein frame given by the metric:

$$g_{\mu\nu} = F(\Phi)\tilde{g}_{\mu\nu}.$$  \hspace{1cm} (3)

Further, let us introduce the scalar field $\phi$ (the so called dilaton) via the equation

$$\left(\frac{d\phi}{d\Phi}\right)^2 = \frac{3}{4}\left(\frac{d\ln(F(\Phi))}{d\Phi}\right)^2 + \frac{Z(\Phi)}{2F(\Phi)}$$  \hspace{1cm} (4)

and define

$$\mathcal{A}(\phi) = F^{-1/2}(\Phi), \quad 2V(\phi) = U(\Phi)F^{-2}(\Phi).$$  \hspace{1cm} (5)

In the Einstein frame action takes the form

$$S = \frac{1}{16\pi G_*} \int d^4x \sqrt{-g} \left( R - 2g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - 4V(\phi) \right)$$

$$+ S_m[\Psi_m, \mathcal{A}^2(\phi)g_{\mu\nu}]$$  \hspace{1cm} (6)

where $R$ is the Ricci scalar curvature with respect to the Einstein metric $g_{\mu\nu}$.

In what follows we shall consider the case $U(\Phi) = 0$ and the matter source will be a massless real scalar field $\sigma$.

The starting point in our considerations is the Roberts’s solution which is written as

$$ds^2 = -dudv + r^2(u, v)d\Omega^2,$$  \hspace{1cm} (7)

$$\sigma_0 = \pm \frac{1}{2} \ln \left( \frac{u - (1 + p)v}{u - (1 - p)v} \right),$$  \hspace{1cm} (8)

where

$$r^2(u, v) = \frac{1}{4}[u - (1-p)v][u - (1+p)v]$$

and $p > 0$ is an arbitrary constant.

Dilaton-vacuum solutions can be easily obtained via the conformal transformation

$$F^{-1}(\Phi_{de}(u, v)) = \mathcal{A}^2(\phi)|_{\phi = \sigma_0(u, v)},$$  \hspace{1cm} (9)

$$d\tilde{s}^2_{dv} = F^{-1}(\Phi_{de}(u, v))ds^2.$$  \hspace{1cm} (10)

This simple procedure, however, is not applicable when a matter source is present. In order to construct exact scalar-tensor solutions describing a collapsing scalar field
we shall employ the solution generating methods developed in [14]. Roberts’s solution is taken as a seed solution.

The general form of the scalar-tensor solutions is given by

\[ F^{-1}[\Phi(u, v)] = F^{-1}[\Phi(\sigma_0(u, v))] = A^2[\varphi(\sigma_0(u, v))], \tag{11} \]
\[ ds^2 = F^{-1}[\Phi(\sigma_0)]ds^2 = -F^{-1}[\Phi(\sigma_0)]dudv + \tilde{r}^2(u, v)d\Omega^2, \tag{12} \]
\[ \sigma(u, v) = \sigma(\sigma_0(u, v)), \tag{13} \]

where

\[ \tilde{r}^2(u, v) = F^{-1}[\Phi(\sigma_0)]r^2(u, v) \tag{14} \]

and the explicit form of the functions \( F^{-1}(\Phi(\sigma_0)) \) and \( \sigma(\sigma_0) \) depends on the particular scalar-tensor theory. Below we consider several examples of scalar-tensor theories which qualitatively cover the general case and have solutions representable in a closed analytical form.

The solutions above are continuously self-similar since they admit the homothetic Killing vector

\[ \xi = u \frac{\partial}{\partial u} + v \frac{\partial}{\partial v}. \tag{15} \]

The Ricci scalar curvature \( \tilde{R} \) is given by

\[ \tilde{R} = F(\Phi(\sigma_0)) \left( 1 + \frac{3}{2} \frac{d^2}{d\sigma_0^2} \ln[F(\Phi(\sigma_0))] - \frac{3}{4} \left( \frac{d}{d\sigma_0} \ln[F(\Phi(\sigma_0))] \right)^2 \right) R \tag{16} \]

where

\[ R = \frac{p^2uv}{2r^4(u, v)} \tag{17} \]

is the Ricci scalar curvature for the Roberts’s solution.

For all solutions we present here the conformal factor \( F^{-1}(\Phi) \) satisfies \( F^{-1}(\Phi) \geq \lambda^2 > 0 \) where \( \lambda \) is a constant. Therefore, the conformal factor is everywhere regular except at \( r(u, v) = 0 \) where the only singularity\(^2 \) of the Ricci scalar curvature is located. Thus we can conclude that the causal structure of the scalar-tensor solutions is the same as the Roberts’s solution. The singularity \( r(u, v) = 0 \) is given by \( u = (1 - p)v \) and \( u = (1 + p)v \) for \( v > 0 \) and \( v < 0 \), respectively. For \( 0 < p < 1 \) the singularity is timelike while it is null for \( p = 1 \). When \( p > 1 \) the singularity is spacelike in the region \( v > 0 \) and timelike for \( v < 0 \).

In all cases we consider, an apparent horizon exists only for \( p > 1 \) and it surrounds the spacelike singularity while the timelike singularity is naked. In order to get rid of the naked singularity we shall proceed as in the case of the Roberts’s solution. We can replace the region \( v < 0 \) by a flat spacetime smoothly matched to the region \( v > 0 \)

\(^2\)For all scalar-tensor solutions presented here it can be checked that \( \tilde{R} \) could become singular only for \( R \).
across the $v = 0$ line where the energy flux vanishes in both the Jordan and Einstein frame.

The fact that the spacelike singularity is hidden behind and apparent horizon signals the formation of a black hole. That is why we shall consider below the case $p > 1$ in the limit $p \to 1$.

With regard to the case $0 < p < 1$, we can, just as for the Roberts's solution, remove the timelike singularity by replacing the region $u > 0$ and $v > 0$ by a flat spacetime. For $0 < p < 1$ the initial conditions are not strong enough to form a black hole.

The above considerations show that the parameter $p$ is a control parameter and its critical value is $p_{cr} = 1$. Thus the presented solutions can be divided into three classes: subcritical ($0 < p < 1$), critical ($p = 1$) and supercritical ($p > 1$).

### 2.1 Brans-Dicke theory

Brans-Dicke theory is described by the functions $F(\Phi) = \Phi$ and $Z(\Phi) = \omega/\Phi$ where $\omega$ is a constant. Here we shall consider the cases with $\omega > 0$.

Solution generating transformations [14] give the following scalar-tensor solution:

$$
\Phi^{-1}(u, v) = \lambda^2 \cosh^2 \left( \frac{\sigma_0(u, v)}{\sqrt{3 + 2\omega}} \right),
$$

$$
\sigma(u, v) = \frac{\sqrt{3 + 2\omega}}{\lambda} \tanh \left( \frac{\sigma_0(u, v)}{\sqrt{3 + 2\omega}} \right),
$$

where $0 < \lambda^2 < \infty$.

The apparent horizon exist only for $p > 1$ and is located at

$$
u \simeq \frac{2(1 - p)}{1 \pm \frac{1}{\sqrt{3 + 2\omega}} v}.
$$

Here and below, in order to present the results in compact and clear form the location of the apparent horizon is determined for $(p - 1) \ll 1$. The effective gravitational mass inside the apparent horizon is

$$
\tilde{M}_{AH} = \frac{1}{2} \tilde{r}_{AH} \approx \frac{\sqrt{2}}{8} |\lambda| \left( \frac{1}{1 \pm \frac{1}{\sqrt{3 + 2\omega}}} \right)^{1/2} (p - 1)^{1/2 - 1/2\sqrt{3 + 2\omega}} v.
$$

As well know the self-similar spacetimes are not asymptotically flat and the black hole mass diverges when $v \to \infty$. The standard way to get rid of this problem is to truncate the spacetime at some $v_* > 0$ and to add an asymptotically flat region. We shall accept from now on that this has been done.

What is important is that the black hole mass obeys the power law. Hence the analytically calculated critical exponent is found to be $\gamma = \frac{1}{2} - \frac{1}{2\sqrt{3 + 2\omega}}$. In contrast to the dilaton-vacuum case where $\gamma_{dv} = 1 \pm \frac{1}{2\sqrt{3 + 2\omega}}$ [11], [13], in the presence of a massless scalar field as a matter source, only the minus sign is allowed.
2.2 Theory with conformal coupling

This theory is described by the functions $F(\Phi) = 1 - \frac{1}{6} \Phi^2$ and $Z(\Phi) = 1$. The explicit functions in the case under consideration are:

\[
F^{-1}(\Phi(u, v)) = 1 + (1 - \lambda^2) \sinh^2 \left( \frac{\sigma_0(u, v)}{\sqrt{3}} \right),
\]

\[
\sigma(u, v) = \sqrt{3} \tanh^{-1} \left( \lambda \tanh \left( \frac{\sigma_0(u, v)}{\sqrt{3}} \right) \right),
\]

where the parameter $\lambda$ is restricted by $0 < \lambda^2 < 1$.

The apparent horizon exists for $p > 1$ and is located at

\[
u \approx \frac{2(1 - p)}{1 \pm \sqrt{3}} v
\]

The mass inside the apparent horizon is

\[
\tilde{M}_{AH} \approx \frac{\sqrt{2}}{4} \left(1 - \lambda^2\right)^{1/2} \left( \frac{1 \pm \sqrt{3}}{1 \pm \sqrt{3}} \right)^{1/2} (p - 1)^{1/2 - 1/\sqrt{3}} v^*.
\]

The critical exponent is $\gamma = \frac{1}{2} - \frac{1}{2\sqrt{3}}$ and coincides with the previously calculated critical exponent $\gamma_{dv} \approx 0.21$ for the dilaton-vacuum solutions [12].

2.3 $F(\Phi) = \Phi$ and $Z(\Phi) = (\Phi^2 - 3\Phi + 3)/2\Phi(\Phi - 1)$ theory

The explicit form the functions $\Phi(\sigma_0)$ and $\sigma(\sigma_0)$ is given by

\[
\Phi^{-1}(u, v) = \frac{\lambda^2}{\lambda^2 + (1 - \lambda^2) \sin^2 (\lambda \sigma_0(u, v))},
\]

\[
\sigma(u, v) = \frac{1 + \lambda^2}{2\lambda} \sigma_0(u, v) - \frac{1 - \lambda^2}{4\lambda^2} \sin (2\lambda \sigma_0(u, v)).
\]

The range of the parameter $\lambda$ is $0 < \lambda^2 < 1$.

The apparent horizon is located at

\[
u \approx \frac{2(1 - p)}{1 \pm D_\lambda(p)} v
\]

where

\[
D_\lambda(p) = \frac{1}{2} \frac{\lambda(1 - \lambda^2) \sin (\lambda \ln (p - 1))}{\lambda^2 + (1 - \lambda^2) \sin^2 \left( \frac{1}{2} \lambda \ln (p - 1) \right)}
\]

and $D_\lambda^2(p) < 1$.

The mass inside the apparent horizon is given by
\[ \bar{M}_{AH} \approx \frac{2^{1/2}}{4} \frac{|\lambda|}{\sqrt{\lambda^2 + (1 - \lambda^2) \sin^2 \left( \frac{\lambda}{2} \ln(p - 1) \right)}} \left( 1 \pm D_{\lambda}(p) \right)^{1/2} \left( \frac{1}{1 \pm D_{\lambda}(p)} \right)^{1/2} (p - 1)^{1/2} v_* \]  

(30)

We see something interesting. The power law \( M = \text{const} (p - p_{\text{cr}})^\gamma \) observed within the general relativity is slightly modified: \( M = f(p - p_{\text{cr}})(p - p_{\text{cr}})^\gamma \). It is also seen that the black hole mass exhibits "damping oscillations" in the control parameter \( p \) when \( p \to 1 \).

If we define the critical exponent as \( \gamma = \lim_{p \to 1} \ln(M)/\ln(p - 1) \) we obtain \( \gamma = 1/2 \) for the case under consideration.

It is also interesting to find the black hole mass law for the dilaton-vacuum collapse. Using solution generating formulas (9) one can show that

\[ \bar{M}_{AH}^{dv} \approx \frac{1}{\sqrt{2}} \frac{(p - 1)^{1/2}}{|\ln(p - 1)|} v_* \]  

(31)

Hence we find \( \gamma_{dv} = \lim_{p \to 1} \ln(M)/\ln(p - 1) = 1/2 \).

We have checked that the black hole mass for the near critical collapse in the Barker’s theory (\( F(\Phi) = \Phi \) and \( Z(\Phi) = (4 - 3\Phi)/2\Phi(\Phi - 1) \)) exhibits "damping oscillations" in the critical parameter, too. Since the Barker’s case is similar to the case under consideration we do not present detail calculations.

### 2.4 \( F(\Phi) = \Phi \) and \( Z(\Phi) = (1 - 3a^2\Phi)/2a^2\Phi^2 \) theory

Here the parameter \( a \) satisfies \( a > 0 \).

In the case under consideration we have

\[ \Phi^{-1}(u, v) = (1 + a\sigma_0(u, v))^2 + \lambda^2, \]  

(32)

\[ \sigma(u, v) = \frac{1}{a} \arcsin \left( \frac{\lambda}{\sqrt{(1 + a\sigma_0(u, v))^2 + \lambda^2}} \right). \]  

(33)

Here, \( \lambda \) runs \( 0 < \lambda^2 < \infty \).

The location of the apparent horizon is given by

\[ u \approx 2(1 - p)v. \]  

(34)

The mass inside the apparent horizon is

\[ \bar{M}_{AH} \approx \frac{\sqrt{2}}{4} a \left| \ln(p - 1) \right| (p - 1)^{1/2} v_* \]  

(35)

Hence, for the critical exponent we have \( \gamma = \lim_{p \to 1} \ln(M)/\ln(p - 1) = 1/2 \).

Respectively, for the dilaton-vacuum collapse we obtain
and therefore $\gamma_{dv} = \gamma = 1/2$.

3 Conclusion

In the present work we have generalized the general relativistic Roberts-Brady-Oshiro-Nakamura-Tomimatsu (RBONT) model in the case of scalar-tensor theories of gravity. We have presented new exact self-similar solutions describing a collapse of a massless scalar field in scalar-tensor theories. The solutions exhibit a type of critical behavior discussed by Choptuik. Three possible evolutions are distinguished: subcritical, critical and supercritical. For supercritical evolution the black hole mass low has been found and the critical exponent has been extract for several scalar- tensor theories which cover qualitatively the general case. It is interesting to note that for some scalar-tensor theories within the analytical models we consider, the black hole mass law in the near critical collapse is of the form $M = f(p - p_{cr})(p - p_{cr})^\gamma$ where $f(p - p_{cr})$ is a function depending on the particular scalar-tensor theory. One sees that this law is slightly different compared to the general relativistic version $M = const(p - p_{cr})^\gamma$. It is also interesting to note that, in some cases, the black hole mass exhibits ”damping oscillations” in the limit $p \to p_{cr}$.

As the RBONT model our models can not be considered as completely realistic but they are satisfactory enough in explaining analytically some basic features of the critical collapse.

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