Self-adjoint extensions of network Laplacians and applications to resistance metrics

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Abstract
Let \((G, c)\) be an infinite network, and let \(E\) be the canonical energy form. Let \(\Delta_2\) be the Laplace operator with dense domain in \(\ell^2(G)\) and let \(\Delta_E\) be the Laplace operator with dense domain in the Hilbert space \(H_E\) of finite energy functions on \(G\). It is known that \(\Delta_2\) is essentially self-adjoint, but that \(\Delta_E\) is not. In this paper, we characterize the Friedrichs extension of \(\Delta_E\) in terms of \(\Delta_2\) and show that the spectral measures of the two operators are mutually absolutely continuous with Radon-Nikodym derivative \(\lambda\) (the spectral parameter), in the complement of \(\lambda = 0\). We also give applications to the effective resistance on \((G, c)\). For transient networks, the Dirac measure at \(\lambda = 0\) contributes to the spectral resolution of the Friedrichs extension of \(\Delta_E\) but not to that of the self-adjoint \(\ell^2\) Laplacian.

Keywords: Graph energy, discrete potential theory, graph Laplacian, spectral graph theory, resistance network, effective resistance, Hilbert space, reproducing kernel, unbounded linear operator, self-adjoint extension, essentially self-adjoint, spectral resolution, defect indices, Sturm-Liouville, limit-point, limit-circle, Dirichlet, Neumann.

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1. Introduction

We study Laplace operators on infinite networks, and their self-adjoint extensions. Here, a network is just an connected undirected weighted graph \((G, c)\); see Definition 2.1. The associated network Laplacian \(\Delta\) acts on functions \(u : G \to \mathbb{R}\); see Definition 2.2. We restrict attention to the case when the network is transient\(^1\).

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\(^1\)This equivalent to assuming the existence of monopoles; see Definition 2.11 and Remark 2.12.
and we are particularly interested in the case when \( \Delta \) is unbounded, in which case some care must be taken with the domains. We consider \( \Delta \) separately as an operator on \( \mathcal{H}_E \), the Hilbert space of finite energy functions on \( G \) and as an operator on \( \ell^2(G) \). Although the two operators agree formally, their spectral theoretic properties are quite different. The space \( \mathcal{H}_E \) is defined in terms of the quadratic form \( E \), which gives the Dirichlet energy of a function \( u \); see Definition 2.4. By \( \ell^2(G) \), we mean the unweighted space of square-summable functions on \( G \) under counting measure; see Definition 2.18.

Neither of the two Hilbert spaces is contained in the other, and the two Hilbert norms do not compare. It follows that, these two incarnations of the Laplacian may have quite different spectral theory. Common to the two is that \( \Delta \) is defined on its natural dense domain in each of the Hilbert spaces (these domains are given in Definition 2.16 and Definition 2.19), and in each case it is a Hermitian and non-negative operator. However, it is known from \([\text{Woj07}, \text{JP09e}, \text{KL09}, \text{KL10}]\) that \( \Delta \) is essentially self-adjoint on its natural domain in \( \ell^2(G) \) but in \([\text{JP09e}]\) it is shown that \( \Delta \) is not essentially self-adjoint on its natural domain in \( \mathcal{H}_E \) (see Definition 2.16). Nonetheless, we prove that the Friedrich extension of the latter has a spectral theory that can be compared with the former.

Theorem 4.18 is our first main result, and it characterizes the Friedrichs extension (see Definition 4.12) of the Laplacian on \( \mathcal{H}_E \) in terms of the Laplacian on \( \ell^2(G) \). Theorem 5.2 is our other main result, and it shows that the spectral measures of the Laplacian on \( \ell^2(G) \) and the Friedrichs extension on \( \mathcal{H}_E \) are mutually absolutely continuous with Radon-Nikodym derivative \( \lambda \) (the spectral parameter). We use Theorem 5.2 to derive a number of spectral-theoretic conclusions. In particular, Corollary 6.8 gives a formula for the (effective) wired resistance metric on \( (G, c) \) in terms of the spectral resolution of \( \Delta \) on \( \ell^2(G) \), and Corollary 6.9 shows that a spectral gap for \( \Delta \) on \( \ell^2(G) \) implies a bound on the wired resistance. Resistance is a natural metric on networks and has been considered previously in many contexts; see \([\text{Kig01}, \text{Kig03}, \text{LP}, \text{Soa94}, \text{Woe09}, \text{DS84}, \text{Tho90}]\). Also see Definition 6.1 and the ensuing discussion; some examples are explored in \( \S7 \).

We are also able to use spectral methods to recover some classical results for the integer lattices \( (\mathbb{Z}^d, 1) \) in \( \S7 \). It turns out that it is the spectral theory of \( \Delta \) as an operator in \( \mathcal{H}_E \) (as opposed to \( \ell^2(G) \)) which reflects important properties of the network, including certain precise notions of metric and boundary. The Friedrichs extension of the Laplacian arises naturally in this context as it corresponds to a limiting case of Dirichlet boundary conditions for the Laplacian, and hence to wired resistance metric; see \([\text{JP10a}, \text{Rem. 2.22}]\).

To make this paper accessible to diverse audiences, we have included a number of definitions we shall need from the theory of (i) infinite networks, and (ii) the use of unbounded operators on Hilbert space in discrete contexts. Some useful background references for the first are \([\text{Soa94}]\) and \([\text{Woe09}]\) and the multifarious references cited therein; see also \([\text{Yam79, Zem91, HK10, KY89}])
KY84, KY82, MYY94, vBL09, DJ10]. For the second, see [DS88] (especially Ch. 12) and [vN32, Sto90, DJ06, BB09]. For relevant background on reproducing kernels, see e.g., [PS72, Ar050, MYY94, Kal70]. In our first section below, we have recorded some lemmas from [JP09b, JP10a, JP10c, JP09e, JP09c, JP10d, JP09a, JP10b, JP10e, JP09d] in the form in which they will be needed in the rest of the paper. Some of these results are folkloric or well known in the literature; in such cases, we refer to our own papers only for convenience.

2. Basic terms and previous results

We now proceed to introduce the key notions used throughout this paper: resistance networks, the energy form $E$, the Laplace operator $\Delta$, and their elementary properties.

**Definition 2.1.** A (resistance) network $(G, c)$ consists of a connected undirected graph $G$ and a symmetric conductance function $c : G \times G \to [0, \infty)$. We write $x, y \in G$ to indicate that $x$ and $y$ are vertices of the graph. The conductance function defines the adjacency relation as follows: $x$ and $y$ are neighbours (i.e., there is an edge connecting $x$ and $y$) iff $c_{xy} > 0$, in which case the nonnegative number $c_{xy} = c_{yx}$ is the weight (conductance, or reciprocal resistance) associated to this edge.

We make the standing assumption that $(G, c)$ is locally finite. This means that every vertex has finite degree, i.e., for any fixed $x \in G$ there are only finitely many $y \in G$ for which $c_{xy} > 0$. We denote the net conductance at a vertex by

$$c(x) := \sum_{y \sim x} c_{xy}. \quad (2.1)$$

In this paper, connected means simply that for any $x, y \in G$, there is a finite sequence $\{x_i\}_{i=0}^n$ with $x = x_0$, $y = x_n$, and $c_{x_{i-1}x_i} > 0$, $i = 1, \ldots, n$.

For any network, one can fix a reference vertex, which we shall denote by $o$ (for “origin”). It will always be apparent that our calculations depend in no way on the choice of $o$.

**Definition 2.2.** The Laplacian on $G$ is the linear difference operator which acts on a function $u : G \to \mathbb{R}$ by

$$(\Delta u)(x) := \sum_{y \sim x} c_{xy}(u(x) - u(y)). \quad (2.2)$$

A function $u : G \to \mathbb{R}$ is harmonic iff $\Delta u(x) = 0$ for each $x \in G$.

The domain of $\Delta$, considered as an operator on $\mathcal{H}_E$ or $\ell^2(G)$, is given in Definition 2.16 and Definition 2.19.
2.1. The energy space $H_E$

**Definition 2.3.** The energy form is the (closed, bilinear) Dirichlet form

$$E(u, v) := \frac{1}{2} \sum_{x, y \in G} c_{xy}(u(x) - u(y))(v(x) - v(y)),$$  \hspace{1cm} (2.3)

which is defined whenever the functions $u$ and $v$ lie in the domain

$$\text{dom } E = \{ u : G \to \mathbb{R} : E(u, u) < \infty \}. \hspace{1cm} (2.4)$$

Hereafter, we write the energy of $u$ as $E(u) := E(u, u)$. Note that $E(u)$ is a sum of nonnegative terms and hence converges iff it converges absolutely.

The energy form $E$ is sesquilinear and conjugate symmetric on $\text{dom } E$ and would be an inner product if it were positive definite. Let 1 denote the constant function with value 1 and observe that $\ker E = \mathbb{R}1$. One can show that $\text{dom } E/\mathbb{R}1$ is complete and that $E$ is closed; see [JP09b, JP09d, Kat95], or [FÔT94].

**Definition 2.4.** The energy (Hilbert) space is $H_E := \text{dom } E/\mathbb{R}1$. The inner product and corresponding norm are denoted by

$$\langle u, v \rangle_E := E(u, v) \quad \text{and} \quad \|u\|_E := E(u, u)^{1/2}. \hspace{1cm} (2.5)$$

It is shown in [JP09b, Lem. 2.5] that the evaluation functionals $L_x u = u(x) - u(o)$ are continuous, and hence correspond to elements of $H_E$ by Riesz duality (see also [JP09b, Cor. 2.6]).

**Definition 2.5.** Let $v_x$ be defined to be the unique element of $H_E$ for which

$$\langle v_x, u \rangle_E = u(x) - u(o), \quad \text{for every } u \in H_E. \hspace{1cm} (2.6)$$

Note that $v_o$ corresponds to a constant function, since $\langle v_o, u \rangle_E = 0$ for every $u \in H_E$. Therefore, $v_o$ may be safely omitted in some calculations.

Equation (2.6) means that the collection $\{v_x\}_{x \in G}$ forms a reproducing kernel for $H_E$ and thus has dense span in $H_E$. We call $\{v_x\}_{x \in G}$ the energy kernel.

**Remark 2.6 (Diﬀerences and representatives).** Equation (2.6) is independent of the choice of representative of $u$ because the right-hand side is a difference: if $u$ and $u'$ are both representatives of the same element of $H_E$, then $u' = u + k$ for some $k \in \mathbb{R}$ and $u'(x) - u'(o) = (u(x) + k) - (u(o) + k) = u(x) - u(o)$. By the same token, the formula for $\Delta$ given in (2.2) describes unambiguously the action of $\Delta$ on equivalence classes $u \in H_E$. Indeed, formula (2.2) defines a function $\Delta u : G \to \mathbb{R}$ but we may also interpret $\Delta u$ as the class containing this representative.
Definition 2.7. Let \( \delta_x \in \ell^2(G) \) denote the Dirac mass at \( x \), i.e., the characteristic function of the singleton \{x\} and let \( \delta_x \in \mathcal{H}_E \) denote the element of \( \mathcal{H}_E \) which has \( \delta_x \in \ell^2(G) \) as a representative. The context will make it clear which meaning is intended. Observe that \( E(\delta_x) = c(x) < \infty \) is immediate from (2.3), and hence one always has \( \delta_x \in \mathcal{H}_E \) (recall that \( c(x) \) is the total conductance at \( x \); see (2.1)).

Definition 2.8. For \( v \in \mathcal{H}_E \), one says that \( v \) has finite support iff there is a finite set \( F \subseteq G \) such that \( v(x) = k \in \mathbb{C} \) for all \( x \notin F \). Equivalently, the set of functions of finite support in \( \mathcal{H}_E \) is

\[
\text{span}\{\delta_x\} = \{u \in \text{dom } E : u(x) = k \text{ for all } x \notin F\},
\]

for some finite \( F \subseteq G \). Define \( \text{Fin} \) to be the \( E \)-closure of \( \text{span}\{\delta_x\} \).

Definition 2.9. The set of harmonic functions of finite energy is denoted

\[
\mathcal{H}_{\text{arm}} := \{v \in \mathcal{H}_E : \Delta v(x) = 0, \text{ for all } x \in G\}.
\]

The following result is well known; see [Soa94, §VI], [LP, §9.3], [JP09b, Thm. 2.15], or the original [Yam79, Thm. 4.1].

Theorem 2.10 (Royden Decomposition). \( \mathcal{H}_E = \text{Fin} \oplus \mathcal{H}_{\text{arm}} \).

Definition 2.11. A monopole is any \( w \in \mathcal{H}_E \) satisfying the pointwise identity \( \Delta w = \delta_x \) (in either sense of Remark 2.6) for some vertex \( x \in G \). A dipole is any \( v \in \mathcal{H}_E \) satisfying the pointwise identity \( \Delta v = \delta_x - \delta_y \) for some \( x, y \in G \).

Remark 2.12. It is easy to see from the definitions (or [JP09b, Lemma 2.13]) that energy kernel elements are dipoles, i.e., that \( \Delta v_x = \delta_x - \delta_o \), and that one can therefore always find a dipole for any given pair of vertices \( x, y \in G \), namely, \( v_x - v_y \). On the other hand, monopoles exist if and only if the network is transient (see [Woe00, Thm. 2.12] or [JP09b, Rem. 3.5]).

Remark 2.13. Denote the unique energy-minimizing monopole at \( o \) by \( w_o \); the existence of such an object is explained in [JP09b, §3.1]. We will be interested in the family of monopoles defined by

\[
w^o_x := w_o + v_x, \quad x \neq o.
\]

In §3.1 (see Lemmas 3.1–3.2) we use the representatives specified by

\[
w^o_x(x) = E(w^o_x), \quad \text{and} \quad v_x(o) = 0.
\]

When \( \mathcal{H}_{\text{arm}} = 0 \), \( E(w^o_x) \) is the capacity of \( x \); see, e.g., [Woe09, §4.D].

Lemma 2.14 ([JP09b, Lem. 2.11]). For \( x \in G \) and \( u \in \mathcal{H}_E \), \( \langle \delta_x, u \rangle_E = \Delta u(x) \).
Proof. Compute \( \langle \delta_x, u \rangle_E = \mathcal{E}(\delta_x, u) \) directly from formula (2.3). \( \square \)

Lemma 2.15. For any \( x, y \in G \),

\[
\Delta w^v_x(y) = \Delta w^v_y(x) = \langle w^v_x, \Delta w^v_y \rangle_E = \langle \Delta w^v_y, w^v_y \rangle_E = \delta_{xy},
\]

(2.11)

where \( \delta_{xy} \) is the Kronecker delta.

Proof. First, note that \( \Delta w^v_x(y) = \delta_{xy} = \Delta w^v_y(x) \) as functions, immediately from the definition of monopole. Then Lemma 2.14 gives \( \langle w^v_x, \Delta w^v_y \rangle_E = \langle w^v_y, \delta_y \rangle_E = \Delta w^v_x(y) \) since \( \Delta w^v_y = \delta_y \) and \( \langle u, \delta_y \rangle_E = \Delta u(y) \), and similarly for the other identity. \( \square \)

Definition 2.16. On \( \mathcal{H}_E \), start with \( \Delta \) defined on span\( \{w^v_x\}_{x \in G} \) pointwise by (2.2), and then obtain the closed operator \( \Delta_E \) by taking the graph closure; the following lemma shows that this is justified.

Lemma 2.17. \( \Delta_E \) is well-defined and non-negative (hence also closed and Hermitian).

Proof. Let \( \xi \in \text{dom} \Delta_E \) with spt \( \xi \) contained in some finite set \( F \subseteq G \). By (2.11),

\[
\langle u, \Delta u \rangle_E = \sum_{x, y \in F} \xi_x \xi_y \langle w^v_x, \Delta w^v_y \rangle_E = \sum_{x, y \in F} \xi_x \xi_y \delta_{xy} = \sum_{x \in F} |\xi_x|^2 \geq 0.
\]

(2.12)

The closure of any semibounded operator is semibounded. This implies \( \Delta_E \) is Hermitian and hence contained in its adjoint. Since every adjoint operator is closed, \( \Delta_E \) is closable. \( \square \)

2.2. The Hilbert space \( \ell^2(G) \)

As there are many uses of the notation \( \ell^2(G) \), we provide the following elementary definitions to clarify our conventions.

Definition 2.18. For functions \( u, v : G \to \mathbb{R} \), define the inner product

\[
\langle u, v \rangle_2 := \sum_{x \in G} u(x)v(x).
\]

(2.13)

Definition 2.19. On \( \ell^2(G) \), we begin with \( \Delta \) defined pointwise by (2.2) on \( \text{span}\{\delta_x\}_{x \in G} \), the subspace of (finite) linear combinations of point masses, and then obtain the closed operator \( \Delta_2 \) by taking the graph closure (see Remark 2.20).

Remark 2.20. [JP09e, Lem. 2.7 and Thm. 2.8] states that \( \Delta_2 \) is semibounded and essentially self-adjoint. It follows that \( \Delta_2 \) is closable by the same arguments as in the end of the proof of Lemma 2.17. See also [Woj07, KL09, KL10].
3. Some properties of the Laplacian and the monopoles

**Lemma 3.1.** For any \( x \in G \),
\[
\delta_x = c(x)w_x^v - \sum_{y \sim x} c_{xy}w_y^v. \tag{3.1}
\]

**Proof.** For any \( z \in G \), formulas (2.10), (2.11) and (2.2) give
\[
\delta_x(z) = \delta_z(x) = \Delta w_x^v(x) = c(x)w_x^v(x) - \sum_{y \sim x} c_{xy}w_y^v(y) = c(x)w_x^v(z) - \sum_{y \sim x} c_{xy}w_y^v(z). \tag*{□}
\]

**Lemma 3.2.** For any \( x, y \in G \), if \( \delta_{xy} \) is the Kronecker delta and \( \Delta_x \) denotes the Laplacian taken with respect to the \( x \) variable, then
\[
\Delta_x \langle w_x^v, w_y^v \rangle_E = \langle \Delta w_x^v, w_y^v \rangle_E = \delta_{xy}. \tag{3.2}
\]

**Proof.** Using (2.2), we have
\[
\Delta_x \langle w_x^v, w_y^v \rangle_E = c(x)\langle w_x^v, w_y^v \rangle_E - \sum_{z \sim x} c_{xz} \langle w_z^v, w_y^v \rangle_E = \left( c(x)w_x^v - \sum_{z \sim x} c_{xz}w_z^v, w_y^v \right)_E,
\]
whence the result follows by applying (3.1) and then (2.11).

### 3.1. The transformation \( \Phi : \delta_x \mapsto w_x^v \) and the matrix \( M \)

**Definition 3.3.** Define \( \Phi : \ell^2(G) \to \mathcal{H}_E \) on \( \text{dom} \Phi = \text{span} \{ \delta_x \}_{x \in G} \) by \( \Phi \delta_x = w_x^v \).

Note that \( \text{ran} \Phi \) is dense in \( \mathcal{H}_E \) because it contains \( \text{span} \{ v_x \}_{x \in G} \); see (2.9).

**Remark 3.4.** The operator \( \Phi \) may not be closable. This necessitates some care in the formulation of the Friedrichs extension in Definition 4.8.

**Definition 3.5.** Let \( M \) be the (infinite) matrix with entries \( M_{xy} = \langle w_x^v, w_y^v \rangle_E \).

**Lemma 3.6.** For all \( \xi \in \text{dom} \Phi \), one has \( \langle \xi, M\xi \rangle_{\ell^2} = \| \Phi(\xi) \|_E^2 \).

**Proof.** The computation is immediate:
\[
\| \Phi(\xi) \|_E^2 = \left( \sum_{x \in F} \xi(x)w_x^v, \sum_{y \in F} \xi(y)w_y^v \right)_E = \sum_{x \in F} \sum_{y \in F} \xi(x)\xi(y)M_{x,y} = \langle \xi, M\xi \rangle_{\ell^2}. \tag*{□}
\]

Lemma 3.6 shows that the matrix \( M \) plays the role of the formal expression \( \Phi^*\Phi \). We will need a couple of lemmas relating \( \Delta_2 \) and \( \Delta_E \) to \( \Phi \). Lemma 3.7 relates the inner product of \( \mathcal{H}_E \) to the inner product on \( \ell^2(G) \), and shows that they differ “by a Laplacian”; see also [JP09e, Lem. 5.30].
Lemma 3.7. For all $\xi, \eta \in \text{dom } \Phi$, one has $\langle \Phi \xi, \Delta \Phi \eta \rangle_E = \langle \xi, \eta \rangle_2$.

Proof. Letting $\xi = \sum_{x \in F} \xi_x \delta_x$ and $\eta = \sum_{x \in F} \eta_x \delta_x$ (for some finite $F \subseteq G$) and arguing as in the proof of Lemma 2.17, we have

$$\langle \Phi \xi, \Delta \Phi \eta \rangle_E = \sum_{x, y \in F} \xi_x \eta_y \langle w^v_x, \Delta w^v_y \rangle_E = \sum_{x, y \in F} \xi_x \eta_y \delta_{xy} = \langle \xi, \eta \rangle_2. \quad \square$$

Corollary 3.8. For $\xi \in \text{dom } \Phi$, one has $\Delta \Phi \xi = \Phi \Delta^2 \xi = \sum_{x \in F} \xi_x \delta_x$.

Note that Corollary 3.8 is an identity in $H_E$, and for this reason we have written $\sum_{x \in F} \xi_x \delta_x$ and not $\xi$, so as to account for the possible projection to $\text{Fin}$.

4. Characterization of the Friedrichs extension

It is known from [JP09e, Prop. 4.9] that $\Delta_E$ may fail to be essentially self-adjoint. Therefore, we construct a canonical self-adjoint extension of $\Delta_E$, following the methods of Friedrichs (and von Neumann); see [DS88, §XII.5] for background.

Remark 4.1. Recall from Remark 2.20 that $\Delta_2$ is essentially self-adjoint and hence has a unique and well-defined spectral representation. We henceforth assume without loss of generality that $\Delta_2$ is self-adjoint.

In this section, we relate the domains given in Definition 2.16 and Definition 2.19. This will entail comparing a quadratic form $q_M$ from $\ell^2(G)$ with a quadratic form $q_\Delta$ from $H_E$. We will have occasion to use the following result, which is a special case of [Kat95, Ch. VI, Thm. 2.1 and Thm. 2.6].

Theorem 4.2 (Kato’s Theorem). Let $q$ be a densely defined, closed, symmetric sesquilinear form in a Hilbert space $H$ which satisfies $\inf\{q(u) : u \in \text{dom } q\} = 0$, for $q(u) = q(u, u)$. Then there is a unique self-adjoint operator $T$ with $\inf\{\langle u, Tu \rangle : u \in \text{dom } T\} = \inf \text{spec } T = 0$ satisfying

(i) $\text{dom } T \subseteq \text{dom } T^{1/2} = \text{dom } q$.
(ii) $q(u, v) = \langle Tu, v \rangle$ for any $u \in \text{dom } T$ and $v \in \text{dom } q$.
(iii) If $u \in \text{dom } q, w \in H$, and $q(u, v) = \langle w, v \rangle$ for every $v \in \text{dom } T$, then $u \in \text{dom } T$ and $Tu = w$.

The next step is to extend the mapping $\Phi : \delta_x \to w^v_x$ from Definition 3.3 to functions which do not have finite support in Definition 4.8; this requires some further development of $M$ from Definition 3.5.
Definition 4.3. The real Hermitian (symmetric) matrix $M$ defines a quadratic form with dense domain in $\ell^2(G)$. Define $q_M$ to be the closure of this form; note that this is justified by Kato’s theorem because Lemma 3.6 shows that $M$ is non-negative.

Definition 4.4. Let $M$ denote the self-adjoint operator corresponding to the quadratic form $q_M$ by Kato’s theorem.

Remark 4.5. Lemma 4.6 is a renormalized version (or a symmetrized version; see Remark 2.13) of the standard identity that the Laplacian and Green operator are inverses. In this context, the proof comes by comparing quadratic forms associated to $M$ and to $\Delta$. In a different context, the question of comparing two quadratic forms, and closability, comes up in the study of Gaussian stochastic processes, see [AJL11, §5] and [AJ11].

Lemma 4.6. $\Delta_2$ and $M$ are inverses of each other:

(i) $M\Delta_2\eta = \eta$, for any $\eta \in \text{dom}\Delta_2$, and (ii) $\Delta_2M\xi = \xi$, for any $\xi \in \text{dom}\ M$.

Proof. Since $\Delta_2$ is a self-adjoint operator in $\ell^2(G)$ and $\text{span}\{\delta_x\}$ is contained in $\text{dom}\Delta_n^2$ for any $n \geq 1$, the matrix of $\Delta_2$ relative to the o.n.b $\text{span}\{\delta_x\}$ is

$$\tilde{\Delta}_{x,y} := \langle \delta_x, \Delta_2\delta_y \rangle_2 = \begin{cases} c(x), & y = x, \\ -c_{xy}, & y \sim x, \\ 0, & \text{else}. \end{cases}$$

The following matrix multiplication uses Lemma 2.15 to show that $\tilde{\Delta}M = I$:

$$\tilde{\Delta}_{x,y}M_{z,y} = \sum_{z \in X} \tilde{\Delta}_{x,z}M_{z,y} = \sum_{z \in X} \tilde{\Delta}_{x,z}(w_z^x, w_y^z)E = \Delta_x(w_x^y, w_y^x)E = \delta_{xy}. \quad (4.1)$$

Note that the summation over $z$ is finite because $\tilde{\Delta}$ is banded; see Remark 4.7. The computation for $M\tilde{\Delta} = I$ is identical by the symmetry of the matrices. Therefore, $\Delta_2$ and $M$ are inverses on a formal level.

For (i), let $\eta \in \text{dom}\Delta_2$. Since $\Delta_2$ is banded, the following double sum is finite:

$$q_M(\Delta_2(\eta_n - \eta_m)) = \sum_{x,y} (\tilde{\Delta}(\eta_n - \eta_m))(x)M_{x,y}\tilde{\Delta}(\eta_n - \eta_m)(y)$$

$$= \sum_x \tilde{\Delta}(\eta_n - \eta_m)(x)(\eta_n - \eta_m)(x) \quad \text{by (4.1).}$$

This is the inner product of two Cauchy sequences tending to 0 (by choice of $\eta$), and hence tends to 0. Since $\Delta_2$ and $M$ are both self-adjoint, (ii) now follows from the spectral theorem. \qed
Remark 4.7. In general, it is difficult to determine spectral properties of operators in Hilbert space from a representation of the operators in the form of an infinite matrix. This was noted by von Neumann in [vN43, vN51]. However, restriction to the class of banded operators allows one to obtain many explicit results; see [Jør78], for example. An infinite matrix is banded iff each row and each column has only finitely many nonzero entries. In the present context, the assumption of local finiteness of the network is equivalent to the bandedness of the Laplacian (on $\mathcal{H}_E$ or on $\ell^2(G)$); this hypothesis is used only for Lemma 3.8. It is quite possible that there may exist an alternative proof, in which case this hypothesis may turn out to be unnecessary.

**Definition 4.8.** Define

$$\tilde{\Phi}(\xi) := \lim_{n \to \infty} \Phi(\xi_n), \quad \text{for any } \xi \in \text{dom } q_M, \quad (4.2)$$

where $$(\xi_n)_{n=1}^\infty \subseteq \text{dom } \Phi$$ is any sequence for which $$\lim_{n \to \infty} q_M(\xi_n - \xi) = 0.$$

**Lemma 4.9.** The operator $\tilde{\Phi}$ is well-defined.

**Proof.** Let $\xi \in \text{dom } q_M$, and let $$(\xi_n)_{n=1}^\infty$$ be a sequence of finitely supported functions for which $$\lim_{n \to \infty} q_M(\xi_n - \xi) = 0.$$ Then Lemma 3.6 gives

$$\|\Phi(\xi_n - \xi_m)\|_E^2 = \langle \xi_n - \xi_m, M(\xi_n - \xi_m) \rangle_2 = q_M(\xi_n - \xi_m),$$

which converges because $q_M$ is closed. $\square$

Remark 4.10. Note that $\Phi$ is an isometry from $\text{span}\{\delta_x\}$ (equipped with the $q_M$-norm) into $\mathcal{H}_E$; Lemma 4.9 just emphasizes that this isometry is maintained under completion.

**Definition 4.11.** For $u \in \text{dom } \Delta_E$, define the quadratic form $r(u) := \langle u, \Delta u \rangle_E + \|u\|_E^2$ and denote the closure of this form (and its domain) by $q_\Delta$.

**Definition 4.12** (Friedrichs extension). The Friedrichs extension $\Delta_F$ is the unique self-adjoint and non-negative operator (with greatest lower bound 0) associated to $q_\Delta$ by Kato’s theorem.

**Remark 4.13.** Kato’s theorem (Theorem 4.2) gives $\text{dom } \Delta_F^{1/2} = \text{dom } q_\Delta$ and

$$q_\Delta(u) = \|\Delta_F^{1/2} u\|_E^2 + \|u\|_{E'}^2 \quad \text{for } u \in \text{dom } q_\Delta. \quad (4.3)$$

**Lemma 4.14.** The domain of the Friedrichs extension $\Delta_F$ is

$$\text{dom } \Delta_F = (\text{dom } \Delta_E^*) \cap (\text{dom } q_\Delta). \quad (4.4)$$
Proof. This follows from [Kat95, IV.3] or [DS88, §XII.5]. Recall that convergence in energy implies pointwise convergence. \[\]  

**Definition 4.15.** Since \( \Delta_2 \) is self-adjoint (see Remark 4.1), we define the operator  
\[
\Delta_2^{-1/2} := \pi^{-1/2} \int_0^\infty t^{-1/2} e^{-t\Delta_2} dt
\]
for those functions \( \xi \) lying in the domain  
\[
\text{dom} \Delta_2^{-1/2} := \left\{ \xi : \left( \pi^{-1/2} \int_0^\infty t^{-1/2} e^{-t\Delta_2} dt \right)(\xi) \in L^2(G) \right\}.
\]
Observe that this integral converges because 0 is not an eigenvalue of \( \Delta_2 \); recall that we consider only infinite networks. See [DS88, Ch. XII] or [Nel69, Ch.6–7].

The characterization of the Friedrichs domain extension given in Theorem 4.18 will require the following two lemmas.

**Lemma 4.16.** \( \text{dom} \Delta_2^{1/2} = \tilde{\Phi}(\text{dom} \Delta_2^{-1/2}) \).

*Proof.* In light of Remark 4.13, it suffices to show \( \text{dom} q_\Delta = \tilde{\Phi}(\text{dom} \Delta_2^{-1/2}) \). For any \( u \in \text{dom} q_\Delta \), one can find \( (u_n)_n \) with \( \lim q_\Delta(u_n - u_m) = 0 \) and \( u_n = \Phi \xi_n \) for \( \xi_n \in \text{dom} \Phi \). Then  
\[
q_\Delta(u_n - u_m) = \langle \Phi(\xi_n - \xi_m), \Delta \Phi(\xi_n - \xi_m) \rangle + ||\Phi(\xi_n - \xi_m)||^2_{\tilde{\Phi}},
\]
by Lemma 3.7 and Lemma 3.6. Now by Lemma 4.6 and Lemma 4.9, the convergence of (4.7) is equivalent to both \( (\xi_n)_n \) and \( (\Delta_2^{-1/2} \xi_n)_n \) being Cauchy in \( L^2(G) \), but this means precisely that \( \xi := \lim \xi_n \in \text{dom} \Delta_2^{-1/2} \).

Conversely, if \( \xi \) is the limit in \( L^2(G) \) of a sequence \( (\xi_n)_n \subseteq \text{dom} \Phi \), then observe that for \( u_n = \Phi \xi_n \), the same identity follows by Definition 4.11. \[\Box\]

**Lemma 4.17.** \( \text{dom} \Delta_2^{\ast} = \tilde{\Phi}(\text{dom} \Delta_2^{-1/2}) \).

*Proof.* To begin, we show that any element \( u \in \text{dom} \Delta_2^{\ast} \) can be written as \( u = \tilde{\Phi}(\xi) \) for some \( \xi \in \text{dom} \tilde{\Phi} \). By Definition 2.16 and Definition 4.8, \( u \in \text{dom} \Delta_2^{\ast} \) iff there exists a \( C < \infty \) for which  
\[
||\langle \Delta_2 \tilde{\Phi} \eta, u \rangle ||_{\tilde{\Phi}}^2 \leq C ||\tilde{\Phi} \eta||_{\tilde{\Phi}}^2, \quad \text{for all } \eta \in \text{dom} \tilde{\Phi}.
\]
Combining Lemma 3.6, Lemma 4.9, and Definition 4.3 gives \( ||\tilde{\Phi} \eta||_{\tilde{\Phi}}^2 = ||\Delta_2^{-1/2} \eta||_2^2 \), so that (4.8) is equivalent to  
\[
||\langle \Delta_2 \tilde{\Phi} \eta, u \rangle ||_{\tilde{\Phi}}^2 \leq C ||\Delta_2^{-1/2} \eta||_2^2, \quad \text{for all } \eta \in \text{dom} \tilde{\Phi}.
\]

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Now Riesz duality gives a $\xi \in \ell^2(G)$ for which $\langle \Delta_2^{-1/2}\eta, \xi \rangle_{\mathcal{C}} = \langle \Delta_2 \Phi \eta, u \rangle_{\mathcal{C}}$, so with $\rho = \Delta_2^{-1/2}\eta$, we can rewrite (4.9) as

$$\left| \langle \rho, \xi \rangle_{\mathcal{C}} \right|^2 \leq C \|\rho\|^2_{L_2}, \quad \text{for all } \rho \in \Delta_2^{-1/2}(\text{dom } q_M),$$

which means $\xi \in \text{dom}(\Delta_2^{1/2})^* = \text{dom} \Delta_2^{1/2}$, since $\Delta_2$ is self-adjoint by [JP09e, Lem. 2.7 and Thm. 2.8]; see Remark 4.1. \hfill $\Box$

**Theorem 4.18** (Friedrichs characterisation). The Friedrichs extension is given by

$$\text{dom } \Delta_F = \Phi(\text{dom } \Delta_2^{1/2} \cap \text{dom } \Delta_2^{-1/2}). \quad (4.10)$$

**Proof.** Starting with (4.4), applying Lemma 4.16 and Lemma 4.17 gives

$$\text{dom } \Delta_F = \text{dom } \Delta_2^* \cap \text{dom } q_\Lambda = \Phi(\text{dom } \Delta_2^{1/2}) \cap \Phi(\text{dom } \Delta_2^{-1/2}) \quad (4.11)$$

Suppose that $u \in \text{dom } \Delta_F$ can be written as $u = \Phi(\xi)$ for $\xi \in \text{dom } \Delta_2^{1/2}$ and as $u = \Phi(\eta)$ for $\eta \in \text{dom } \Delta_2^{-1/2}$. Then

$$\langle \delta_x, \Phi \xi \rangle_{\mathcal{E}} = \sum_{y \in G} \xi_y \langle \delta_x, w_y \rangle_{\mathcal{E}} = \sum_{y \in G} \eta_y \langle \delta_x, w_y \rangle_{\mathcal{E}} = \langle \delta_x, \Phi \eta \rangle_{\mathcal{E}}$$

Therefore, $\xi = \eta$ and $\Phi$ preserves intersections, so (4.11) is equal to (4.10). \hfill $\Box$

### 5. Relating the Friedrichs extension $\Delta_F$ on $\mathcal{H}_E$ to $\Delta_2$

The main result of this section is Theorem 5.2, in which we show that the spectral measures of $\Delta_F$ and $\Delta_2$ are mutually absolutely continuous in the complement of $\lambda = 0$ and compute the Radon-Nikodym derivative.

**Definition 5.1.** For $u \in \text{dom } \Delta_F$, let $\mu_u^F$ denote the spectral measure in the spectral resolution of $\Delta_F$, and for $\xi \in \text{dom } \Delta_2$, again let $\mu_\xi^E$ denote the spectral measure in the spectral resolution of $\Delta_2$.

**Theorem 5.2.** For $\xi \in \text{dom } \Phi$, the spectral measures of $\Delta_F$ and $\Delta_2$ are related by

$$\lambda d\mu_{\Phi \xi}^F = d\mu_\xi^E, \quad (5.1)$$

where $\lambda$ is the spectral parameter. In particular, $d\mu_{\Phi \xi}^F$ and $d\mu_\xi^E$ are mutually absolutely continuous on $(0, \infty)$ with Radon-Nikodym derivative $\lambda$.  

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Proof. For $\xi \in \text{dom } \Phi$,

$$\langle \Phi \xi, \Delta_n \Phi \xi \rangle_E = \langle \Phi \xi, \Delta \Phi \Delta_n \Phi \xi \rangle_E = \langle \xi, \Delta_n \Phi \xi \rangle_2$$

follows by applying Lemma 3.8 and then Lemma 3.7 (with $\eta = \Delta_n \xi$). Note that $\xi$ has finite support, and therefore so does $\Delta_n \xi$ for any $n$; see the proof of Lemma 4.6. This identity also holds for operators with larger domains, so

$$\langle \Phi \xi, \Delta_{n+1} \Phi \xi \rangle_E = \langle \xi, \Delta_n \xi \rangle_2$$ \quad \text{for all } \xi \in \text{dom } \Delta_n.

(5.2)

If $P_2$ denotes the projection-valued measure in the spectral resolution of $\Delta_2$ and $P_F$ denotes the projection-valued measure in the spectral resolution of $\Delta_F$, then the spectral theorem gives

$$\int_0^\infty \lambda^{n+1} \|P_F(d\lambda)\Phi \xi\|^2_E = \int_0^\infty \lambda^n \|P_2(d\lambda)\xi\|^2_E. \quad (5.3)$$

Considering the above as an identity for monomials $\lambda^n$, it is clear the measures

$$\lambda d\mu^F_{\Phi \xi} (\lambda) := \lambda \|P_F(d\lambda)\Phi \xi\|^2_E \quad \text{and} \quad d\mu^\ell_2 (\lambda) := \|P_2(d\lambda)\xi\|^2_E$$

have the same moments. Since $\Delta_2$ is essentially self-adjoint, the corresponding moment problem is determinate, i.e., these moments determine the measures uniquely (see, e.g., [Akh65, AG81, Fug83]). Consequently, we have

$$\int_0^\infty \lambda f(\lambda) d\mu^F_{\Phi \xi} (\lambda) = \int_0^\infty f(\lambda) d\mu^\ell_2 (\lambda)$$

for any bounded Borel function $f$. (Note that $f$ is required to be bounded because the operators $\Delta_2, \Delta_F$ may not be.) This completes the proof of (5.1). \qed

Remark 5.3. Note that we are only considering infinite networks here, so 0 is never an eigenvalue of $\Delta_2$. Thus, (5.1) states that the spectra of $\mu^F_{\Phi \xi}$ and $\mu^\ell_2$ must agree on the support of the Radon-Nikodym derivative, i.e., up to an eigenvalue at 0 for $\mu^F_{\Phi \xi}$ (which is present precisely in the case $\text{Harm} \neq 0$). Several useful facts follow immediately from spec$_{\mu^F_{\Phi \xi}} \Delta_F \setminus \{0\} = \text{spec}_\ell \Delta_2$ and (5.1). For example,

$$\|\Phi \xi\|^2_E = \|\Delta_2^{-1/2} \xi\|^2_2, \quad \text{for all } \xi \in \text{dom } \Delta_2^{-1/2}, \quad (5.4)$$

which allows us to prove transience of the integer lattice networks in Theorem 7.9. Corollary 5.4 is another useful consequence of this fact.

It also follows that for transient networks, the Dirac measure at $\lambda = 0$ contributes to the spectral resolution of the Friedrichs extension of $\Delta_E$ but not to that of the self-adjoint $\ell^2$ Laplacian.
Corollary 5.4. $\Delta_E$ is bounded if and only if $\Delta_2$ is bounded.

Proof. Note that $\Delta_F$ is bounded if and only if $\|\Delta_F\| = \sup \text{spec} \Delta_F < \infty$. It is immediate from (5.1) that $\sup \text{spec} \Delta_F = \sup \text{spec} \Delta_2$, and hence $\|\Delta_F\| < \infty$ is equivalent to $\|\Delta_2\| = \sup \text{spec} \Delta_2 < \infty$. Since $\Delta_F$ is an extension of $\Delta_E$ (which coincides with $\Delta_E$ whenever $\Delta_E$ is bounded), the result is immediate. □

Corollary 5.5. Let $\xi \in \ell^2(G)$. Then $\Phi(\xi) \in H_E$ if and only if $\xi \in \text{ran} \Delta_{1/2}^{-1} \Delta_2^{-1/2}$.

Proof. By the spectral theorem,

$$\|\Phi \xi\|_E^2 = \int_0^\infty \|P_\xi(d\lambda)\Phi \xi\|_E^2 = \int_0^\infty \frac{1}{\lambda} \|P_\xi(d\lambda)\xi\|_2^2 = \|\Delta_2^{-1/2} \xi\|_2^2, \quad (5.5)$$

where the second equality comes by (5.1), since $\frac{1}{\lambda} = (\lambda^{-1/2})^2$. Then (5.5) is finite if and only if $\eta = \Delta_2^{-1/2} \xi \in \ell^2(G)$, that is, $\xi \in \text{ran} \Delta_{1/2}^{-1} \Delta_2^{-1/2}$. □

6. Applications to effective resistance

The main result of this section is Theorem 6.8, a corollary to Theorem 5.2 which may be compared with Remark 6.2. Recall the monopole notation $w^v_x$ from Remark ???. We use $P_{fin}$ to denote the orthogonal projection of $v_x$ to $Fin$ with respect to $\langle \cdot, \cdot \rangle_E$; see Theorem 2.10. Thus $f_x := P_{fin}v_x$ and $w^f_x := f_x + w_0 = P_{fin}w^v_x$.

Definition 6.1. Denote the free effective resistance between $x$ and $y$ by

$$R^F(x, y) := \mathcal{E}(w^v_x - w^v_y) = \mathcal{E}(v_x - v_y), \quad (6.1)$$

for $w^v_x$ as in (2.9). Denote the wired effective resistance between $x$ and $y$ by

$$R^W(x, y) := \mathcal{E}(f_x - f_y) = \mathcal{E}(w^f_x - w^f_y). \quad (6.2)$$

Remark 6.2. Several alternative and equivalent formulations of the free and wired resistances are collected in [JP10a]. It turns out that $R^F$ and $R^W$ are metrics on $G$; for details, see [JP10a, LP, Kig03]. Note that $R^F(x, y) \geq R^W(x, y)$ in general, and that strict inequality holds if and only if $\text{Harm} \neq 0$.

The following lemma combines results from [JP09c, Lem. 6.9] and [JP09b, Lem. 2.22] and will be useful in the sequel.

Lemma 6.3. Every $w^f_x$ is $\mathbb{R}$-valued, with $w_x(y) - w_x(o) > 0$ for all $y \neq o$. Moreover, every $w_x$ is bounded, with $\|w_x\|_\infty \leq R^F(x, o)$ (see (6.1)).
**Definition 6.4.** The probabilities $p(x, y) := c_{xy}/c(x)$ define a random walk $(X_n)_{n=0}^\infty$ on the network by $\mathbb{P}[X_{n+1} = y | X_n = x] = p(x, y)$. Here $X_n$ is a $G$-valued random variable giving the location of the random walker at time $n$. Then let

$$\mathbb{P}[x \to y] := \mathbb{P}_x(\tau_y < \tau_x^+)$$  \hspace{1cm} (6.3)

be the probability that the random walk started at $x$ reaches $y$ before returning to $x$. In (6.3), $\tau_z := \min\{n \geq 0 : X_n = z\}$ is the hitting time of $z$ and $\tau_x^+ := \max\{\tau_z, 1\}$.

**Corollary 6.5 ([JP10a, Cor. 3.13 and Cor. 3.15]).** For any $x \neq y$, one has

$$R^F(x, o) = \frac{1}{c(o)\mathbb{P}[o \to x]}.$$  \hspace{1cm} (6.4)

**Lemma 6.6.** If $\Delta E$ is bounded on $\mathcal{H}_E$, then $(G, R^F)$ is uniformly discrete, i.e.,

$$R^F(x, y) \geq \frac{2}{\|\Delta E\|}.$$  \hspace{1cm} (6.5)

**Proof.** Since the inequality $\langle u, \Delta u \rangle_E \leq \|\Delta E\| \cdot \|u\|^2_E$ holds for all $u \in \text{dom } \Delta E = \mathcal{H}_E$, apply it to $u = w_x^v - w_y^v$ (with $x \neq y$) to obtain

$$\langle w_x^v - w_y^v, \Delta(w_x^v - w_y^v) \rangle_E \leq \|\Delta E\| \cdot \|w_x^v - w_y^v\|^2_E.$$  \hspace{1cm} (6.6)

Note also that Lemma 2.14 gives

$$\langle w_x^v - w_y^v, \Delta(w_x^v - w_y^v) \rangle_E = \langle w_x^v - w_y^v, \delta_x - \delta_y \rangle_E$$

$$= \langle w_x^v, \delta_x \rangle_E - \langle w_x^v, \delta_y \rangle_E - \langle w_y^v, \delta_x \rangle_E + \langle w_y^v, \delta_y \rangle_E$$

$$= \Delta w_x^v(x) - \Delta w_x^v(y) - \Delta w_y^v(x) + \Delta w_y^v(y)$$

$$= 1 - 0 - 0 + 1 = 2.$$  \hspace{1cm} (6.7)

Combining (6.6), (6.7), and (6.1) gives (6.5). \hspace{1cm} $\square$

**Remark 6.7.** Note that $0$ is never an eigenvalue of $\Delta_2$.\footnote{It is well-known that; the only harmonic function in $\ell^2(G)$ on an infinite network $G$ is the constant function 0; see [LP, Woe09, Soa94, JP09b] and elsewhere.} It follows that we can only apply (5.1) in the orthogonal complement of $\mathcal{H}_{\text{Harm}}$ (since the formula may not hold at $\lambda = 0$), so there is no analogue of Theorem 6.8 for $R^F$.

**Theorem 6.8.** For an infinite network $G$, the wired effective resistance $R^W(x, y)$ is

$$R^W(x, y) = \int_0^\infty \frac{1}{\lambda} ||P_2(d\lambda)(\delta_x - \delta_y)||^2_2$$  \hspace{1cm} (6.8)
Proof. If \( R^W(x, y) = \|w_x - w_y\|^2 \), we have
\[
R^W(x, y) = \int_0^\infty \|P\mathcal{F}(d\lambda)(\delta_x - \delta_y)\|^2_E = \int_0^\infty \frac{1}{\lambda}\|P_2(d\lambda)(\delta_x - \delta_y)\|^2_E.
\]
Note that \( P\mathcal{F}_{\text{fin}} \) is the projection to the orthocomplement of \( \ker \Delta_F \), so we can remove it:
\[
R^W(x, y) = \int_0^\infty \|P\mathcal{F}(d\lambda)\Phi(\delta_x - \delta_y)\|^2_E = \int_0^\infty \frac{1}{\lambda}\|P_2(d\lambda)(\delta_x - \delta_y)\|^2_E.
\]
where the last equality follows by Theorem 5.2.

\[\square\]

**Corollary 6.9.** On an infinite network \( G \), if \( \Delta_2 \) has a spectral gap, then the wired effective resistance is bounded. More precisely, if \( \text{spec} \Delta_2 \subseteq [\gamma, \infty) \), then
\[
R^W(x, y) \leq \frac{2}{\gamma}, \quad \text{for all } x, y \in G. \tag{6.9}
\]

Proof. In this case, (6.8) gives
\[
R^W(x, y) = \int_\gamma^\infty \frac{1}{\lambda}\|P_2(d\lambda)(\delta_x - \delta_y)\|^2_E \leq \frac{1}{\gamma} \int_\gamma^\infty \|P_2(d\lambda)(\delta_x - \delta_y)\|^2_E = \frac{1}{\gamma}\|\delta_x - \delta_y\|^2_E,
\]
and \( \|\delta_x - \delta_y\|^2 \leq 2 \) with strict inequality if \( x = y \), in which case \( R(x, y) = 0. \) \( \square \)

7. Examples

In this section, we apply our results to obtain concrete formulas for some common and well-studied examples, including trees and integer lattices.

7.1. The binary tree

Consider the network \((T_2, 1)\), the binary tree with all edges having conductance 1. The vertices of this network can be labeled with finite words on the symbol set \( \{0, 1\} \), using \( \emptyset \) to denote the empty word, which we take as the origin, i.e. \( o = \emptyset \). For a vertex of the tree \( x \in \{0, 1\}^k \), let \( |x| := k \) (with \( |\emptyset| = 0 \)). Using symmetry and elementary calculations, it is easy to check that \( w(x) := 2^{-|x|} \) is the unique energy-minimizing monopole on \( T_2 \). On any tree with conductance function \( c = 1 \), it is straightforward to see that the free resistance \( R^F(x, y) \) coincides with combinatorial (shortest-path) distance, and is hence unbounded. One way to see this is to show that \( v_x - v_y \) has a representative \( u \) defined as follows:

---

\(^3\)Let \( T = T^* \) be a self-adjoint operator densely defined on the Hilbert space \( \mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1 \), where \( \mathcal{H}_0 = \ker T \). If \( P_1 \) is projection to \( \mathcal{H}_1 \), then \( T P_1 = T \).
let $\gamma$ be the shortest path from $y$ to $x$, and for $s \in \gamma$, define $u(s)$ to be the number of edges between $s$ and $y$. The, for $s \notin \gamma$, let $s_\gamma$ be the unique closest point of $\gamma$ to $s$, and define $u(s) = u(s_\gamma)$. Now $u$ increases by 1 with each step along $\gamma$ and is locally constant outside of $\gamma$. One can check that $u$ satisfies the reproducing property required by $v_x - v_y$, and it is immediate that $E(u) < \infty$. However, in [DJ10], the spectral gap for $\Delta_2$ on this example is computed to be $3 - \sqrt{2}$, and so it follows from Corollary 6.9 that

$$R(x, y) \leq \frac{2}{3 - 2\sqrt{2}}, \quad \text{for all } x, y \in G. \quad (7.1)$$

This result appears to be new to the literature.

7.2. Expansive networks for which $\mathcal{H}_{\text{arm}} = 0$

Suppose we now consider the network $(T_2 \times \mathbf{Z}, 1)$ formed by taking the Cartesian product of $(T_2, 1)$ with the 1-dimensional integer lattice $(\mathbf{Z}, 1)$.

**Definition 7.1.** The network $(G, c)$ satisfies a **strong isoperimetric inequality** iff there exists $\delta > 0$ such that for any finite vertex subset $S$, one has

$$\frac{|\partial S|}{|S|} \geq \delta > 0, \quad \text{where } |S| = \sum_{x \in S} 1 \quad \text{and } |\partial S| = \sum_{(xy) \in \partial S} c_{xy}, \quad (7.2)$$

and $\partial S$ is the set of edges with exactly one end (vertex) in $S$. The infimum of $\frac{|\partial S|}{|S|}$ (taken over all nonempty finite subsets $S \subseteq G$) is called the **expansion constant**.

It is well-known that $\Delta_2$ has a spectral gap (i.e., $\inf \{\text{spec} \Delta_2\} > 0$ if and only if (7.2) is satisfied. Networks satisfying these equivalent properties are called **expanders**.

To see that $T_2 \times \mathbf{Z}$ satisfies (7.2), first observe that any regular tree of degree $d > 2$ satisfies (7.2) with $\delta = d - 1$. Next, use the fact that the expansion constant for a Cartesian product is the sum of the expansion constants for the two factor networks. Furthermore, it follows from [LP, Ex. 9.7] that $\mathcal{H}_{\text{arm}} = 0$ for $(T_2 \times \mathbf{Z}, 1)$, and so one has $R^F(x, y) = R^W(x, y) \leq \frac{2}{\delta}$. It is clear that these considerations hold more generally than in this example.

**Lemma 7.2.** Let $(G, c)$ be an infinite network which satisfies a strong isoperimetric inequality and for which $\mathcal{H}_{\text{arm}} = 0$. Then the (necessarily unique) effective resistance on $(G, c)$ is bounded.

---

4For more information on this equivalence and its connections to mixing times of Markov chains, Kazhdan’s property $T$ (or more precisely, property $\tau$), and other fascinating topics, we refer the reader to [Chu96, §6] or [Woe09, LP, HLW06]; see also the various works of Lubotzky, Zü"uk, Bourgain, Gamburd, and Sarnak.
For the example of the binary tree \((T_2, 1)\), note that \(\mathcal{H}_{\text{arm}} \neq 0\) and \(R^f\) is unbounded, but also that \(R^W\) is bounded as in (7.1).

**Remark 7.3 (Gel’fand spaces, and the 1-point compactification of \((G, R^W)\)).** Denote the collection of bounded functions of finite energy by

\[
\mathcal{A}_E := \{u \in \mathcal{H}_E : u \text{ is bounded}\}. \tag{7.3}
\]

Define multiplication on \(\mathcal{A}_E\) by the pointwise product \((u_1 u_2)(x) := u_1(x) u_2(x)\), and let the norm on \(\mathcal{A}_E\) be given by \(\|u\|_{\mathcal{A}} := \|u\|_\infty + \|u\|_E\). With these definitions, it is shown in [JP10c, Lem. 5.5] that \((\mathcal{A}_E, \|\cdot\|_{\mathcal{A}})\) is a Banach algebra.

For a Banach algebra \(\mathcal{A}\), the associated Gel’fand space is the spectrum \(\text{spec}(\mathcal{A})\) realized as either the collection of maximal ideals of \(\mathcal{A}\) or as the collection of multiplicative linear functionals on \(\mathcal{A}\). See [Arv02, Arv76]. In [JP10c, Thm. 5.12], it is shown that if \(\mathcal{H}_{\text{arm}} = 0\), then the 1-point compactification of \((G, R^W)\) coincides with the Gel’fand space of \(\mathcal{A}_E\).

### 7.3. The integer lattices \(\mathbb{Z}^d\)

Consider the \(d\)-dimensional integer lattice network \((\mathbb{Z}^d, 1)\) with vertices

\[
\mathbb{Z}^d = \{x = (x_1, \ldots, x_d) : x_i \in \mathbb{Z}, i = 1, \ldots, d\} \tag{7.4}
\]

and with unit-conductance edges between nearest neighbours, that is,

\[
\epsilon_{xy} = \begin{cases} 
1, & y = x + \epsilon_k \text{ for some } k = 1, \ldots, d, \\
0, & \text{else,}
\end{cases} \tag{7.5}
\]

where \(\epsilon_k = [0, \ldots, 0, 1, 0, \ldots, 0]\) has the 1 in the \(k\)th slot. Let \(o = 0 = (0, \ldots, 0)\).

The following result is well-known; see [Soa94], for example.

**Lemma 7.4.** On the network \((\mathbb{Z}^d, 1)\), the Fourier transform of \(\Delta\) is multiplication by

\[
S(t) = S(t_1, \ldots, t_d) = 4 \sum_{k=1}^{d} \sin^2 \left( \frac{t_k}{2} \right). \tag{7.6}
\]

**Lemma 7.5.** Let \(\{v_x\}_{x \in \mathbb{Z}^d}\) be the energy kernel on the integer lattice \(\mathbb{Z}^d\) with \(c = 1\). Then for \(y \in \mathbb{Z}^d\),

\[
v_x(y) = \frac{1}{(2\pi)^d} \int_{\mathbb{T}^d} \frac{\cos((x - y) \cdot t) - \cos(y \cdot t)}{S(t)} \, dt, \tag{7.7}
\]

where \(dt\) is Haar measure on the \(d\)-torus \(\mathbb{T}^d\).
Proof. Under the Fourier transform, Lemma 7.4 indicates that the equation \( \Delta v_x = \delta_x - \delta_o \) becomes \( S(t) \hat{\delta}_x = e^{dx} - 1 \), whence

\[
v_x(y) = \frac{1}{(2\pi)^d} \int_{\mathbb{T}^d} e^{-iy \cdot \xi} \frac{e^{dx} - 1}{S(t)} \, dt.
\] 

(7.8)

Since \( v_x \) is \( \mathbb{R} \)-valued, the result follows.

\( \square \)

Remark 7.6. It is known that no nonconstant harmonic functions of finite energy exist on the integer lattices, and hence the free and wired resistance metrics on \((\mathbb{Z}^d, 1)\) coincide \([\text{Woe00, Woe09, LP, JP10a}]\); see also Definition 6.1 and Remark 6.2. Hence, we write \( R(x, y) \) for \( R^W(x, y) = R^F(x, y) \) in the Theorem 7.7.

Theorem 7.7. Resistance distance on the integer lattice \((\mathbb{Z}^d, 1)\) is given by

\[
R(x, y) = \frac{1}{(2\pi)^d} \int_{\mathbb{T}^d} \sin^2((x - y) \cdot \frac{\xi}{2}) S(t) \, dt,
\]

(7.9)

where \( S(t) = 4 \sum_{k=1}^d \sin^2 \left( \frac{t_k}{2} \right) \) as in (7.6).

Proof. We compute the resistance distance via \( R(x, y) = v_x(x) + v_y(y) - v_x(y) - v_y(x) \). Using \( e_x = e^{dx} \), substitute in the terms from (7.8) of Lemma 7.5:

\[
R(x, y) = \frac{1}{(2\pi)^d} \int_{\mathbb{T}^d} \frac{e_x(e_x - 1) + e_y(e_y - 1) - e_x(e_y - 1) - e_y(e_x - 1)}{S(t)} \, dt
\]

(7.10)

and the formula follows by the half-angle identity and other algebra.

\( \square \)

Remark 7.8. The fact that \((\mathbb{Z}^d, 1)\) is transient if and only if \( d \geq 3 \) was first discovered by Polya \([\text{P6l21}]\), and so Theorem 7.9 is a result which is well-known in the literature (cf. \([\text{Soa94, Thm. 5.11}]\) and \([\text{DS84, NW59}]\), e.g.). We include this result here because the present context allows for a brief proof which offers some intuition for this startling dichotomy.

Theorem 7.9. The network \((\mathbb{Z}^d, 1)\) has a monopole

\[
w_o(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{T}^d} \frac{\cos(x \cdot t)}{S(t)} dt
\]

(7.11)

if and only if \( d \geq 3 \), in which case the monopole at \( o \) is unique.

Proof. As in the proof of Lemma 7.5, we use the Fourier transform to solve \( \Delta w_o = \delta_o \) by converting it into \( S(t) \hat{\delta}_o(t) = 1 \). This gives (7.11), in which the integral converges because \( \frac{\cos t}{S(t)} \approx \frac{1}{S(t)} \in L^1(\mathbb{T}^d) \) iff \( d \geq 3 \). To see this, note that
upon switching to spherical coordinates, \( \frac{1}{S(\rho)} = O(\rho^{-2}) \), as \( \rho \to 0 \), and one requires

\[
\frac{1}{S(t)} \in L^1(T^d) \iff \int_0^1 |\rho^{-2}|\rho^{d-1} \, dS_{d-1} < \infty, \tag{7.12}
\]

where \( dS_{d-1} \) is the usual \((d-1)\)-dimensional spherical measure. Of course, (7.12) holds precisely when \(-2 + d - 1 > -1\), i.e., when \( d \geq 3 \). It remains to check that \( w_o \in H_\xi \). Applying (5.4) with \( \xi = \delta_o \), we obtain

\[
\|w_o\|_{\xi} = \|\Phi_\delta_o\|_{\xi} = \|\Delta^{-1/2}\delta_o\|_2 = \frac{1}{\int_{T^d} \frac{1}{S(t)} \, dt} < \infty, \tag{7.13}
\]

by (7.12) again (and \( \hat{\delta}_o = 1 \), as noted above). To see uniqueness, suppose \( w' \) were another monopole. Then \( \Delta(w_o - w') = \delta_o - \delta_o = 0 \) and \( w_o - w' \) is harmonic. By Remark 7.6, the only finite-energy harmonic functions on \((\mathbb{Z}^d, 1)\) are constant. \( \Box \)

**Remark 7.10.** Comparing (7.11) to (7.7) gives a heuristic as to why all networks support finite-energy dipoles, but not all support monopoles: the numerator in the integral for the monopole is \( o(1) \) as \( t \to 0 \), while the corresponding numerator for the dipole is \( o(t) \) as \( t \to 0 \).

**Corollary 7.11.** For \((\mathbb{Z}^d, 1)\), one has \( v_x \in \ell^2(\mathbb{Z}^d) \) if and only if \( d \geq 3 \).

**Proof.** One can see that in absolute values, the integrand \(|(e^{i\xi \cdot t} - 1)/S(t)|\) of (7.8) is in \( L^2(T^d) \) if and only if \( d \geq 3 \) (one only needs to check for \( t \approx 0 \), which is easy in spherical coordinates), in which case Parseval’s theorem applies. \( \Box \)

Corollary 7.11 is a result comparable to [CW92, Prop. 2], but for a situation in which the isoperimetric inequality is not satisfied; see (7.2).

**Corollary 7.12.** For \((\mathbb{Z}^d, 1)\), the monopoles \( w_x \) lie in \( \ell^2(\mathbb{Z}^d) \) if and only if \( d \geq 5 \).

**Proof.** The proof is the same as in Corollary 7.11, except that the integrand is \( 1/S(t) \), which is in \( L^2(T^d) \) if and only if \( d \geq 5 \). \( \Box \)

We now apply some of our results from previous sections to obtain an explicit formula for the spectral resolution of \( \Delta_2 \) and \( \Delta_\gamma \), for the example \((\mathbb{Z}^d, 1)\).

**Lemma 7.13.** For \((\mathbb{Z}^d, 1)\), if \( \xi = \delta_x - \delta_y \) for any fixed \( x, y \in G \), then the corresponding spectral measure of \( \Delta_2 \) is

\[
\lambda \left( \int_{S(t) = \lambda} (1 - \cos((x - y) \cdot t)) \, dt \right) d\mu_{\xi}^2 = \frac{1}{(2\pi)^d} \gamma \circ S^{-1}, \tag{7.14}
\]

where \( S(t) = 4 \sum_{k=1}^d \sin^2 \left( \frac{t_k}{2} \right) \) as in (7.6).
Proof. This is a standard result from spectral theory, but we include it because explicit formulas are not commonplace in this subject. Since $\Delta_2$ corresponds to multiplication by $\lambda$ on the spectral side and multiplication by $S(t)$ on the Fourier side, it is easiest to see (7.14) from

$$\int_0^{\infty} \frac{1}{\lambda} \|P_2(d\lambda)(\delta_x - \delta_y)\|^2_2 = \frac{1}{(2\pi)^d} \int_{T^d} \frac{\sin^2((x - y) \cdot \frac{t}{2})}{S(t)} \, dt,$$

which follows by comparing the two expressions for $\|\Delta_2^{-1/2}(\delta_x - \delta_y)\|^2_2$ found in Corollary 6.8 and in Theorem 7.7.

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