On the zero viscosity limit

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Abstract

The question of whether the zero viscosity limit $\nu \rightarrow 0$ is identical to the no viscosity $\nu \equiv 0$ case is investigated in a simple shell (GOY) model with only three shells. We find that it is possible to express two velocities in terms of Bessel functions. The third velocity function acts as a background. The relevant Bessel functions are infinitely oscillating as $\nu \rightarrow 0$ and are not analytic as functions of $\nu$ at the point $\nu = 0$. We also mention a perturbative method which may be used to improve the model.

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1 Introduction

In the Navier-Stokes equation the viscosity $\nu$ plays an important role. It is an interesting question of principle whether the limit of vanishing viscosity is given by the solution to the same equation with $\nu$ identically zero. In other words, is the limit $\nu \to 0$ smooth, or is it non-analytic? An answer to this question may be of interest in various branches of physics, for example in cosmology, where the history due to the expansion of the Universe may contain different viscosities and the behavior during transitions from one $\nu$ to another may not be analytic.

In a numerical study of decaying turbulence some evidence was found in [1] that the small viscosity limit is highly non-trivial and does not conform to the naive expectations. Of course, numerical data do not necessarily allow an extrapolation simulating the limit $\nu \to 0$. Therefore it would be desirable to have some explicit mathematical expression for the velocity field which allow this limit to be investigated.

It is well known that a direct mathematical study the Navier-Stokes equations for high Reynolds numbers is not an easy matter. Therefore there is some motivation for looking for a model of the hydrodynamics equations where an analytic approach may be more hopeful. To this end one may think of the shell (GOY) model of Gledzer, Ohkitani and Yamada[2]. Many properties of turbulence, especially those related to energy transfer and the small intermittency effects, have been understood from the numerical studies of the shell model. For a review of the applications, see ref. [3].

The model is formulated in terms of Fourier space velocity variables $u_n(t)$, and the dynamical equations are given by

$$\left(\frac{d}{dt} + \nu k_n^2\right) u_n^* = -ik_n \left( u_{n+1}u_{n+2} - \frac{\delta}{r} u_{n-1}u_{n+1} - \frac{1 - \delta}{r^2} u_{n-1}u_{n-2} \right) + f\delta_{n,n_0}. \tag{1}$$

Here $k_n = r^n$, and $f$ is an external force acting on shell number $n_0$ and $n$ is less than some maximum number $N$. Usually this equation is studied numerically with a large number of shells. Also, the usual scaling law of Kolmogorov $k^{-1/3}$ appears when the maximum number $N$ of shells go to infinity [3].

There is not much hope that the shell equation (1) can be integrated in terms of standard mathematical functions. Therefore the question of whether the solution of (1) in the limit $\nu \to 0$ is identical to the solution of (1) with $\nu \equiv 0$ would still be subject to numerical extrapolation.

However, the probability of obtaining explicit solutions for the $u_n^*$s may increase if the number of shells is small. In this paper we shall show that a simple truncated three-shell model actually allows a solution in terms of Bessel functions, provided the parameter $\delta$ takes a somewhat special value. These Bessel functions turn out to be infinitely oscillating in the limit $\nu \to 0$, so that as $\nu = 0$ is approached, they can take many different values, and in general the $\nu \to 0$ dynamics does not correspond to the $\nu = 0$ case. In some cases it is possible to “renormalize” the bad behavior so as to obtain a smooth limit for small $\nu$, at the cost of trading a quantity which is infinite at $\nu = 0$ with an initial velocity.

In section 2 we introduce the three-shell model. As a preliminary we solve it for $\nu \equiv 0$. Then in section 3 we include viscosity and solve the model in terms of Bessel functions. In
section 4 the limit \( \nu \to 0 \) is investigated. Section 5 contains a discussion of the large time limit, and section 6 discusses the relatively simple case where the shell distance is very large. The Bessel functions are then approximately replaced by trigonometric functions, which are still oscillating infinitely near \( \nu = 0 \). In section 7 we discuss a possible perturbative improvement of the model, and we conclude in section 8.

2 The three-shell model

We shall now consider a simple GOY-model with only three shells, and with the forcing term acting on the first shell. Furthermore, we consider the case where

\[ \delta = 1. \]

This value means that the dynamics is intermediate between two dimensions, where \( \delta = 1 + 1/r^2 \), and three dimensions, where \( \delta = 1 - 1/r \). The choice of \( \delta \) is connected to invariants like enstrophy or helicity invariants, as was first pointed out by Kadanoff et al. [4]. See also the review [3]. The general equations (1) now reduce to the following three equations

\[
\left( \frac{d}{dt} + \nu k_1^2 \right) u_1^*(t) = -ik_1 u_3(t)u_2(t) + f, \tag{3}
\]

and

\[
\left( \frac{d}{dt} + \nu k_2^2 \right) u_2^*(t) = ik_1 u_3(t)u_1(t), \tag{4}
\]

as well as

\[
\left( \frac{d}{dt} + \nu k_3^2 \right) u_3^*(t) = 0. \tag{5}
\]

The last equation simplifies because of \( \delta = 1 \), since otherwise the right hand side would contain the term \( ik_1 (1 - \delta) u_1 u_2 \).

For future reference we first solve these equations with zero viscosity, \( \nu = 0 \). We impose the boundary conditions

\[ u_1(0) = 0 \quad \text{and} \quad u_2(0) \neq 0. \tag{6} \]

Eq. (5) can be trivially solved,

\[ u_3(t) = C_3, \tag{7} \]

where \( C_3 \) is a (complex) constant. Because of this, the velocity field \( u_3(t) \) will act as a background field for the other velocities \( u_1 \) and \( u_2 \). Using this, eq. (3), which now reads

\[
\frac{du_1^*(t)}{dt} = -ik_1 C_3 u_2(t) + f, \tag{8}
\]

can then be integrated,

\[ u_1^*(t) = ft - ik_1 C_3 \int_0^t dt' u_2(t'). \tag{9} \]
Inserting this solution for \( u_1 \) in eq. (4) with \( u_3 \) replaced by \( C_3 \) we obtain
\[
\frac{d^2 u_2^*(t)}{dt^2} + k_1^2 |C_3|^2 u_2^*(t) = i k_1 C_3 f^*.
\] (10)

This second order differential equation has a simple oscillating solution which can be inserted in eq. (8) in order to obtain \( u_1 \). Using the boundary conditions (6) we simply get
\[
u k_1 |C_3| t, \quad u_2^*(t) = 2 \alpha \cos k_1 |C_3| t + \frac{i C_3 f^*}{k_1 |C_3|^2}, \quad \text{and} \quad u_3^*(t) = C_3^*.
\] (11)

Here \( \alpha \) is an integration constant which is related to \( u_2 \) at \( t = 0 \),
\[
u k_1 |C_3| t, \quad u_2^*(0) = 2 \alpha + \frac{i C_3 f^*}{k_1 |C_3|^2}.
\] (12)

It should be noticed that although the forcing was coupled to the equation (3) for \( u_1 \), in the solutions (11) the constant \( f \) occurs also on the second shell.

3 The three-shell model and viscosity

We now consider the basic equations (3)-(5) with non-vanishing viscosity, \( \nu \neq 0 \). Again, eq. (5) can be solved trivially,
\[
u k_1 |C_3| t, \quad u_3(t) = C_3 e^{-\nu k_3^2 t}.
\] (13)

Again this field will act as a background field for the two other velocities. Inserting eq. (13) in (3) we can integrate to obtain,
\[
u k_1 |C_3| t, \quad u_1^*(t) = e^{-\nu k_1^2 t} \int_0^t dt' e^{\nu k_1^2 t} f(t') - i k_1 C_3 e^{-\nu k_3^2 t} \int_0^t dt' e^{-\nu (k_3^2 - k_1^2) t'} u_2(t').
\] (14)

Inserting this in eq. (4) gives
\[
u k_1 |C_3| t, \quad \left( \frac{d}{dt} + \nu k_2^2 \right) u_2^*(t) = \left( -k_1^2 |C_3|^2 e^{-\nu (k_1^2 + k_2^2) t} \int_0^t dt' e^{-\nu (k_3^2 - k_1^2) t'} u_2(t') + i C_3 k_1 e^{-\nu (k_1^2 + k_3^2) t} \int_0^t dt' e^{\nu k_1^2 t} f^*(t') \right). (15)
\n
and by differentiation we obtain the following second order differential equation,
\[
u k_1 |C_3| t, \quad \frac{d^2 u_2^*}{dt^2} + \nu (k_1^2 + k_2^2 + k_3^2) \frac{du_2^*}{dt} + (\nu^2 k_2^2 (k_1^2 + k_2^2) + \nu k_1^2 |C_3|^2 e^{-2\nu k_3^2 t}) u_2^* = i k_1 C_3 f^*(t) e^{-\nu k_3^2 t}.
\] (16)

It should be noticed that formally eq. (16) reduces to (10) by taking \( \nu = 0 \). The main point of this paper is that this is not true for the \textit{solution} of (16), unless time is very small.

In order to solve eq. (16) it is convenient to introduce the function \( S(t) \),
\[
u k_1 |C_3| t, \quad S(t) = e^{\nu (k_1^2 + k_2^2 + k_3^2) t/2} u_2^*(t).
\] (17)
Eq. (16) then gives
\[
\frac{d^2 S}{dt^2} + \left( -\frac{\nu^2}{4} (k_1^2 + k_2^2 + k_3^2)^2 + \nu^2 k_2^2(k_1^2 + k_3^2) + k_1^2 C_3^2 e^{-2\nu k_3^2 t} \right) S \\
= ik_1 C_3 f^*(t)e^{\pi(k_1^2+k_2^2-k_3^2)t}
\]
(18)
The homogeneous equation corresponding to \( f = 0 \) is of the Bessel type, with the solution
\[
S(t) = J_{\pm a} \left( \frac{k_1 |C_3|}{\nu k_3^2} e^{-\nu k_3^2 t} \right),
\]
(19)
where
\[
a = \frac{1}{2k_3^2} \sqrt{(k_1^2 + k_2^2 + k_3^2)^2 - 4k_2^2(k_1^2 + k_3^2)} = \frac{1}{2k_3^2} (k_2^2 - k_2^2 + k_3^2).
\]
(20)
Since \( k_n = r^n \) we see that for large \( r \) the index \( a \) approaches 1/2.

The inhomogeneous equation (18) can be solved by standard methods from the knowledge of the solution (19) of the homogeneous equation,
\[
S(t) = \frac{\pi}{2\nu k_3^2 \sin \pi a} ik_1 C_3 \left( J_{-a}(z) \int_c^t dt' J_a(z')g(t')f^*(t') + J_a(z) \int_{t'}^d dt' J_{-a}(z')g(t')f^*(t') \right),
\]
(21)
where \( c \) and \( d \) are arbitrary constants, and where
\[
g(t') = e^{-\nu(k_3^2-k_2^2-k_1^2)t'/2}
\]
(22)
and
\[
z = \frac{k_1 |C_3|}{\nu k_3^2} e^{-\nu k_3^2 t},
\]
(23)
and \( z' \) is the same as \( z \) except that \( t \) is replaced by \( t' \).

It should be noticed that the argument of the Bessel function (23) is not nicely behaved as a function of \( \nu \). Therefore the formal coincidence of eqs. (10) and (16) for \( \nu = 0 \) is not reflected in a simple manner in the Bessel solution (19).

4 The limit \( \nu \to 0 \)

In view of the last remarks we shall now study the limit \( \nu \to 0 \). To simplify matters, we start by looking at the homogeneous solution
\[
u_2^*(t) = AJ_a(z) + BJ_{-a}(z),
\]
(24)
where \( A \) and \( B \) are integration constants, and where \( z \) was defined in eq. (23). If we maintain the boundary condition \( u_1(0) = 0 \), it follows from eq. (4) that
\[
\left. \frac{du_2^*}{dt} + \nu k_3^2 u_2^* \right|_{t=0} = 0.
\]
(25)
From the asymptotic form of the Bessel function valid for $\nu \to 0$, i.e. $z \to \infty$,

$$J_{\pm a}(z) \approx \sqrt{\frac{2}{\pi z}} \cos(z \mp a\pi/2 - \pi/4), \quad (26)$$

and from the explicit expression for $z$ in (23) it follows that $u_2^*$ behaves like $\sqrt{\nu}$ times a cosine. The time derivative of $u_2^*$ also behaves like $\sqrt{\nu}$ times a sine, so therefore $\nu k_3^2 u_2^*$ (which behaves like $\nu^{3/2}$) is subdominant relative to the time derivative of $u_2$ when $\nu \to 0$. Therefore we can replace the boundary condition (25) by $du_2/dt = 0$. This allows us to find the following relation between the constants $A$ and $B$ in eq. (24),

$$B = -A \frac{\sin\left(\frac{k_1|C_3|}{\nu k_3^2} - \frac{\pi}{2}a - \frac{\pi}{4}\right)}{\sin\left(\frac{k_1|C_3|}{\nu k_3^2} + \frac{\pi}{2}a + \frac{\pi}{4}\right)}, \quad (27)$$

where we used the asymptotic form (26) for the Bessel functions. Introducing instead of $A$ the initial value $u_2(0)$, which is assumed to be independent of $\nu$, we now get by use of addition theorems for trigonometric functions

$$A = u_2^*(0) \sqrt{\frac{\pi k_1|C_3|}{\nu k_3^2}} \frac{\sin\left(\frac{k_1|C_3|}{\nu k_3^2} + a\pi/2 - \pi/4\right)}{\sin(\pi a)}. \quad (28)$$

This result allows us to trade the integration constant with the initial velocity, and by use of (26) we have

$$u_2^*(t) \approx u_2^*(0) e^{-\nu(k_1^2+k_2^2-k_3^2)t/2} \cos\left(\frac{k_1|C_3|}{\nu k_3^2} \left(1 - e^{-\nu k_3^2 t}\right)\right). \quad (29)$$

Again we used addition theorems for trigonometric functions. This result agrees completely with the result (11) for $\nu = 0$ with $f = 0$ if the expansion $\exp(-\nu k_3^2 t) \approx 1 - \nu k_3^2 t$ is performed.

In this case it is thus possible to absorb the bad behavior for $\nu \to 0$ in the “renormalization” (28) of the constant $A$ in terms of $u_2(0)$. The price to pay for this is that the badly behaved constant $A$ (for $\nu \to 0$ $A$ blows up with infinite oscillations) is traded with an initial velocity assumed to behave nicely as a function of $\nu$. Therefore, if one does numerical simulations it would look as if the limit where $\nu$ is decreased more and more does not lead to definite convergent results, unless the “renormalization” discussed above is performed at each step of the limiting procedure.

The basic reason for the violent behavior of $A$ is that the Bessel function (19) is not analytic as a function of $\nu$ in the point $\nu = 0$. Here the most singular behavior is

$$\frac{\partial J_{\pm a}(z)}{\partial \nu} \propto -\frac{k_1|C_3|}{\nu^{3/2} k_3^2} e^{-\nu k_3^2 t} \cos\left(\frac{k_1|C_3|}{\nu k_3^2} e^{-\nu k_3^2 t} + a\pi/2 - \pi/4\right) + O\left(\frac{1}{\sqrt{\nu}}\right). \quad (30)$$

This shows that the solution of the differential equation (16) does not behave in a simple way for small $\nu$.

It is not always possible to “renormalize” the bad behavior away. The function

$$u_2^*(t) = u_2^*(0) e^{-\nu(k_1^2+k_2^2+k_3^2)t/2} J_a(k_1|C_3|/(\nu k_3^2) e^{-\nu k_3^2 t})/J_a(k_1|C_3|/(\nu k_3^2)) \quad (31)$$

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is a solution which rapidly oscillating for $\nu \to 0$, and no “renormalization” trick can remove this behavior. We thus see that the simple three-shell model can produce a variety of results which look rather non-trivial.

Instead of imposing some initial velocity for the $u_1$ and $u_2$ fields, we can proceed in a more physical way by assuming that initially $u_1 = u_2 = 0$ up to the time $t_0$. Then some external agency applies a force for a limited time from $t_0$ to $t_1$, so after the time $t_1$ the solution of the homogeneous equations of motion emerges, as can be seen from eq. (21) with $f(t)$ inside the integrals. However, if we write this solution as

$$S(t) = A J_a(z) + B J_{-a}(z) \text{ for } t > t_1,$$

then $A$ and $B$ are not arbitrary coefficients to be fixed by some initial velocity of $u_1$ and/or $u_2$. On the contrary, these constants are fixed dynamically by the force and by the initial velocity $C_3$ of $u_3$,

$$A = \frac{-i\pi k_1 C_3}{2 \nu k^2_3 \sin \pi a} \int_{t_0}^{t_1} dt J_{-a}(z) \ g(t) \ f^*(t)B = \frac{i\pi k_1 C_3}{2 \nu k^2_3 \sin \pi a} \int_{t_0}^{t_1} dt J_a(z) \ g(t) \ f^*(t).$$

Here there is no way of “renormalizing” the constants. If the force acts in a short time, the situation is similar to what was discussed above. For example, if we take a delta-function pulse,

$$f(t) = f_0 \delta(t - t_*), \quad (34)$$

we obtain for $\nu$ so small that $g(t) \approx 1$

$$S(t) \approx \frac{-i C_3 f_0^*}{|C_3|} \ \sin \left( \frac{k_1 |C_3|}{\nu k^2_3} (e^{-\nu k^2_3 t} - e^{-\nu k^2_3 t_*}) \right) \theta(t - t_*). \quad (35)$$

In arriving at this result we used standard addition formulas for the trigonometric functions. For $\nu$ small and time not too large this again gives the $\nu = 0$ result, which in the present case is arrived at smoothly.

The situation changes completely if we let the force act for a longer time. Then the integrals in eq. (33) in general involve times where an expansion like $e^{-\nu k^2_3 t} \approx 1 - \nu k^2_3 t$ is not valid inside the integrals. Therefore a result analogous to (35) with a smooth $\nu \to 0$ limit does not appear if the force is allowed to act long enough. As a matter of fact, the integrals determining $A$ and $B$ in (33) are strongly oscillating, as one can see in numerical examples.

5 The large time limit

In the case where $t \to \infty$ in such a way that $1/\nu \ e^{-\nu k^2_3 t} \to 0$ only the lowest order terms in the Bessel functions should be kept, $J_a \propto z^a(1 + O(z^2))$, so for $f = 0$ we have

$$u_2^*(t) \approx L_- e^{-\nu b_- t/2} \left( 1 - \frac{k^2_1 |C_3|^2}{4 \Gamma(2 - a)} e^{-2\nu k^2_3 t} \right) + L_+ e^{-\nu b_+ t/2} \left( 1 - \frac{k^2_1 |C_3|^2}{4 \Gamma(2 + a)} e^{-2\nu k^2_3 t} \right), \quad (36)$$

where the $L$’s are constants and

$$b_\pm = k^2_1 + k^2_2 + k^2_3 \pm 2k^2_3 a, \quad (37)$$
which means
\[ b_+/2 = k_2^2, \quad b_-/2 = k_3^2. \] (38)

The function \( u_1(t) \) can be obtained most simply directly from eq. (4). The result is again expressed in terms of Bessel functions. We can find the asymptotic behavior corresponding to eq. (36) by inserting eq. (36) in eq. (4). From (38) we have for \( t \) large
\[ u_3^2(t) \approx L_- e^{-\nu k_3^2 t} \left( 1 - \frac{k_1^2 |C_3|^2}{4\Gamma(2 - a) e^{-2\nu k_3^2 t}} \right) + L_+ e^{-\nu(k_2^2 + k_3^2) t} \left( 1 - \frac{k_1^2 |C_3|^2}{4\Gamma(2 + a) e^{-2\nu k_3^2 t}} \right). \] (39)

The leading term contains \( e^{-\nu k_3^2 t} \) as one would expect. It is important that this term is annihilated by the operator \( d/dt + \nu k_2^2 \) in eq. (4). By use of eq. (4) the result for \( t \) large is thus
\[ u_1(t) \approx \frac{1}{i k_1 C_3} (-\nu L_+(k_1^2 - k_2^2 + k_3^2) e^{-\nu k_3^2 t} + \frac{\nu L_- k_1^2 k_3^2 |C_3|^2}{2\Gamma(2 - a)} e^{-\nu(k_2^2 + k_3^2) t}) \]
\[ + \frac{L_+ k_1^2 |C_3|^2 (3k_2^2 - k_2^2 + k_3^2)}{4\Gamma(2 + a)} e^{-\nu(k_2^2 + 2k_3^2) t}). \] (40)

As one would expect, the leading term is simply \( e^{-\nu k_3^2 t} \), which is annihilated by the operator \( d/dt + \nu k_2^2 \). This \( u_1 \) can be reinserted in eq. (3) as a check of the self-consistency of eqs. (3) and (4).

6 Simplified results for \( r \gg 1 \)

We shall now mention that in the limit where the shell distance \( r \) is very large, our results simplify considerably. From eq. (20) we obtain \( a \approx 1/2 \), since \( k_1^2 \) and \( k_2^2 \) can be ignored relative to \( k_3^2 \), so the solution can now be expressed in terms of trigonometric functions,
\[ u_3^2(t) = k_3 \sqrt{\frac{\nu}{\pi k_1 |C_3|}} e^{-\nu(k_1^2 + k_3^2) t/2} \left[ A \sin \left( \frac{k_1 |C_3|}{\nu k_3^2} e^{-\nu k_3^2 t} \right) + B \cos \left( \frac{k_1 |C_3|}{\nu k_3^2} e^{-\nu k_3^2 t} \right) \right] \]
\[ + \text{inhomogeneous } f \text{ term}. \] (41)

From eq. (4) we can then find the corresponding field \( u_1(t) \),
\[ u_1(t) = \frac{i k_3}{k_1 C_3} \sqrt{\frac{\nu}{\pi k_1 |C_3|}} e^{\nu(2k_3^2 - k_1^2 - k_2^2)t/2} \]
\[ \times \left[ (-A \nu/2)(k_1^2 - k_2^2) + k_1 |C_3| B e^{-\nu k_3^2 t}) \sin \left( \frac{k_1 |C_3|}{\nu k_3^2} e^{-\nu k_3^2 t} \right) \right. \]
\[ \left. + (-B \nu/2)(k_1^2 - k_2^2) - Ak_1 |C_3| e^{-\nu k_3^2 t} + \nu k_2^2 B) \cos \left( \frac{k_1 |C_3|}{\nu k_3^2} e^{-\nu k_3^2 t} \right) \right] \]
\[ + \text{inhomogeneous } f \text{ term}. \] (42)

Again we can fix the integration constants by suitable boundary conditions. Also, we see that the limit \( \nu \to 0 \) is not well defined due to the non-analytic behavior of the arguments of the sine and cosine, which oscillate violently as \( \nu \) is decreased.
7  Perturbations in $1 - \delta$

To improve the approach presented above one could try a perturbative expansion in $(1 - \delta)$. Denoting the $\delta = 1$ functions found above by $u_1^{(0)}$ and $u_2^{(0)}$ eq. (5) would change into

$$\left( \frac{d}{dt} + \nu k_3^2 \right) u_3(t) = i k_1 (1 - \delta) u_1^{(0)} u_2^{(0)}. \quad (43)$$

This is an equation for $u_3$ with a time-dependent “forcing” term. It can be solved for $u_3$ in terms of the unperturbed functions $u_1^{(0)}$ and $u_2^{(0)}$. This perturbed $u_3$ should then be inserted in eqs. (3) and (4) to give the perturbed $u_1$ and $u_2$. Of course, there is no guarantee that such an expansion is convergent.

From eq. (43) we easily obtain

$$|u_3|^2 \approx e^{-2\nu k_3^2 t} \left[ |C_3|^2 + ik_1 (1 - \delta) \int_0^t dt' e^{2\nu k_3^2 t'} \left( C_3 u_1^{(0)} u_2^{(0)} (t') - C_3^* (u_1^{(0)} (t'))^* (u_2^{(0)} (t'))^* \right) \right]. \quad (44)$$

If we use eq. (3) to express $u_2$ in terms of $u_1$, so we have

$$C_3 u_1^{(0)} u_2^{(0)} (t) - C_3^* (u_1^{(0)} (t))^* (u_2^{(0)} (t))^* = \frac{i e^{2\nu k_3^2 t}}{k_1} \left[ \frac{d|u_1^{(0)}|^2}{dt} + 2\nu k_1^2 |u_1^{(0)}|^2 - f u_1^{(0)} - f^* u_1^{(0)*} \right]. \quad (45)$$

This expression can be inserted in eq. (44) and after a partial integration we obtain

$$|u_3|^2 \approx e^{-2\nu k_3^2 t} \left[ |C_3|^2 - (1 - \delta) \left( e^{2\nu k_3^2 t} |u_1^{(0)} (t)|^2 - |u_1^{(0)} (0)|^2 \right) \right] + (1 - \delta) I(t) + (1 - \delta) J(t), \quad (46)$$

where we defined

$$I(t) = 2\nu (k_3^2 - k_1^2) \int_0^t dt' e^{2\nu k_3^2 t'} |u_1^{(0)} (t')|^2, \quad (47)$$

and

$$J(t) = \int_0^t dt' e^{2\nu k_3^2 t'} \left( f u_1^{(0)} (t') + f^* (u_1^{(0)} (t'))^* \right). \quad (48)$$

It is interesting that the third term in the square bracket in eq. (46) containing the integral $I$ has a definite sign depending on $1 - \delta$: If $1 - \delta > 0$ the sign is positive since $k_3 > k_2$. This $\delta$ corresponds to helicity ($= \sum (-1)^n k_n |u_n|^2$) conservation for $\nu = f = 0$. So through the $I$-term the two first shells give a positive contribution to the energy of the third shell in the lowest order perturbation theory, which means transfer of energy from lower to higher $k$’s. This is precisely what would be expected in three dimensions.

On the other hand, if $1 - \delta < 0$, the $I$-contribution is negative, and the energy in the third shell is decreased by this effect, as expected in two dimensions where the enstrophy ($= \sum k_n^2 |u_n|^2$) is conserved for $\nu = f = 0$.

Of course, in the full expression (46) there are two other terms proportional to $1 - \delta$. For $f = 0$ it should be possible to investigate all the terms in (46) numerically by inserting a Bessel function constructed from the equation of motion (4) and the solution for $u_2$, thereby giving $u_1$. It would be interesting to see if the overall sign is $+(1 - \delta)$ corresponding to the
expected inverse cascade (transfer of energy from shorter to larger scales, i.e. from larger to smaller \( k' \)s) in “two dimensional” systems (\( \delta = 1 + 1/r^2 > 1 \)) and a forward cascade in “three dimensions” (\( \delta = 1 - 1/r < 1 \)) where the energy is transported from smaller to larger \( k' \)s.

It should be noted that if \( r \) is large, \( \delta \) is in both cases close to 1. Consequently, if the three-shell model should attempt to be somewhat similar to the GOY model with many shells, this will work best for large separations between the shells.

8 Conclusions

In the simple three-shell model we have found the velocity functions in terms of Bessel functions with an argument which is not analytic in the viscosity \( \nu \). In some cases it is possible to hide this singular behavior by a suitable “renormalization”. However, this is not true in all cases. So the model may never approach the similar model with no viscosity, \( \nu \equiv 0 \), even if \( \nu \to 0 \)

Of course, the reason for the integrability of the three-shell model is that the third shell’s velocity becomes a fixed background field for \( u_1 \) and \( u_2 \). Thereby the complexity due to the basic non-linearity of the shell model has disappeared or, at best, become rudimentary. However, this does not make the model completely trivial, since there is a non-trivial coupling between \( u_1 \) and \( u_2 \). So different Fourier modes do couple, and the coupling makes transfer of energy between these modes possible.

We also discussed a perturbation approach with an expansion in \( 1 - \delta \) around \( \delta = 1 \) where the three-shell model was originally defined. Although the resulting perturbative change of the energy of the third mode is relatively complicated, we identified one term which has the expected cascade properties between three and four dimensions. We hope to be able to perform an precise analysis of the sign of all the terms later.

In conclusion one can say that although the three-shell model is of course an immense simplification of the GOY model, nevertheless it has features which indicate a non-trivial dynamics.

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