Multifractal portrayal of the Swiss population

Carmen Delia VEGA OROZCO, Jean GOLAY, Mikhail KANEVSKI

Centre for Research on Terrestrial Environment, Faculty of Geosciences and Environment, University of Lausanne, 1015 Lausanne, Switzerland. Contact: CarmenDelia.VegaOrozco@unil.ch.

Abstract

Fractal geometry is a fundamental approach for describing the complex irregularities of the spatial structure of point patterns. The present research characterizes the spatial structure of the Swiss population distribution in the three Swiss geographical regions (Alps, Plateau and Jura) and at the entire country level. These analyses were carried out using fractal and multifractal measures for point patterns, which enabled the estimation of the spatial degree of clustering of a distribution at different scales. The Swiss population dataset is presented on a grid of points and thus it can be modelled as a "point process" where each point is characterized by its spatial location (geometrical support) and a number of inhabitants (measured variable). The fractal characterization was performed by means of the box-counting dimension and the multifractal analysis was conducted through the Rényi’s generalized dimensions and the multifractal spectrum. Results showed that the four population patterns are all multifractals and present different clustering behaviours. Applying multifractal and fractal methods at different geographical regions and at different scales allowed us to quantify and describe the dissimilarities between the four structures and their underlying processes. This paper is the first Swiss geodemographic study applying multifractal methods using high resolution data.

Keywords: multifractal dimensions, box-counting method, generalized entropy, singularity spectrum, geodemography

1. Introduction

The spatial clustering of real patterns is an important subject in many fields and its characterization can be assessed by an ample number of indices (Cressie, 1993; Kanevski and Maignan, 2004; Illian et al., 2008). Here,
clustering is defined as the spatial non-homogeneity of the way point patterns cover the geographical space in which they are embedded (Tuia and Kanevski [2008]), and variability is related to the variation of the point density. Among these indices, fractal measures are widely developed. Their mathematical framework yields an useful tool for describing the irregularity or complexity of spatial phenomena and imparts great advantages over other traditional statistical methods.

Introduced by Mandelbrot [1967], the word fractal was first coined to describe sets with abrupt and tortuous edges. A set of points whose any scale portion is statistically identical to the original object (statistical self-similarity) is fractal and it can be characterized by a fractal dimension which refers to the invariance of the probability distributions of the set under geometric changes of scale (Rodríguez-Iturbe and Rinaldo [1997]). In the case of multifractal point sets, all the moments of the probability distribution do not scale equivalently and an entire spectrum of generalized fractal dimensions is required (Grassberger and Procaccia [1983]; Hentschel and Procaccia [1983]; Paladin and Vulpiani [1987]; Tél et al. [1989]; Borgani et al. [1993]; Perfect et al. [2006]; Seuront [2010]; Golay et al. [2013]), i.e. the sparser and denser regions of a spatial distribution might have different scaling behaviors.

Many investigations have demonstrated that environmental, ecological and natural data are fractals. Bunde and Havlin [1994] discussed in detail fractals in biology, chemistry and medicine. Burrough [1981] showed that many data of environmental variables and landscapes display a certain degree of statistical self-similarity over many spatial scales. Goodchild and Mark [1987] presented the relevance of fractals to geographic phenomena. Turcotte and Malamud [2004] showed that landslides, forest fires and earthquakes presented a fractal distribution. Telesca et al. [2001, 2004]; Telesca and Lasaponara [2006]; Telesca et al. [2007] characterized the spatial and temporal clustering behaviour of earthquake and forest fire sequences using fractal measures. Frontier [1987] and Seuront [2010] applied fractal theory to ecology and aquatic ecosystems.

In physical and human geography, fractal analyses have been carried out in many cases. A large literature on urban geography mentions the use of fractals to study the geometry and the creation of central places (Arlinghaus [1985]), the town and city systems (François et al. [1995]; Sambrook and Voss [2001]), the irregularities of city morphologies (Batty and Longley [1994]; Frankhauser [1994]), urban growth models (Batty and Lon-
gley, 1986; Batty et al., 1989), intra-urban built-up patterns (Batty and Xie, 1996; Frankhauser, 1998; De Keersmaecker et al., 2003), and the dynamics of population growth (Le Bras, 1998; Ozik et al., 2005). Appleby (1996) applied multifractal methods to characterize the distribution pattern of the human population in the United States and Great Britain. Adjali and Appleby (2001) analysed the multifractal behaviour of the distribution of human population in 10 countries around the world suggesting that the multifractal properties of their population distribution could be related to demographic and economic factors. These works have proved that the implementation of the fractal and multifractal formalism was relevant to urban studies.

In Switzerland, a fractal analysis has been carried out by (1) Frankhauser (2004) who compared the morphology of urban patterns in Europe; (2) Tanvier and Pumain (2013) who applied fractal measures to study the urban space structure and the delimitation of built-up areas in Basle; and (3) Kaiser et al. (2009) who applied the lacunarity index for to study the clustering urban areas at different scales. Nonetheless, there are not known works concerning any structure analysis of the population distribution using multifractal measures at a local level.

Thus, the present research aims at characterizing the spatial distribution pattern of the population in Switzerland at different scales through the fractal and multifractal formalism. The fractal dimension of the Swiss population distribution (SPD) was quantified using the box-counting method, while the multifractal dimensions were estimated using both Rényi’s generalized dimensions and the multifractal spectrum. The population patterns in the three Swiss geographical regions (Alps, Plateau and Jura) were also studied separately and compared. These areas present different topographical features and different clustering behaviour. Therefore, by applying multifractal and fractal methods, we expected to quantify and depict their dissimilarities. Another innovation of this paper lies in the fact that the multifractal analysis of the population distribution was applied to high resolution data scaling from 250 m to 260 km, i.e. from an intra-city level to the country size with no aggregation of data at the city level as it has been done in the literature up to now.

Section 2 describes the dataset and the implemented methodology; section 3 provides the results and section 4 presents the conclusions of our findings.
2. Theory/calculation

2.1. Data

Switzerland is a landlocked country located in Western Europe. It borders France, Germany, Austria, Liechtenstein and Italy and it covers an area of 41,285 km$^2$.

The census of the year 2000 counted 7,351,900 permanent residents and, since then, this amount has increased to 7,954,700 inhabitants [Office, 2010]. The population in Switzerland has more than doubled since the beginning of the 20th century, starting from 3.3 million (1900) to 7.95 million (2011). In the period after World War II (1950-1970), the country underwent an important population growth with an annual average rate of about 1.4%. It slowed down (0.6%) from the 1970s to 1990s as a result of immigration restrictions and because of the economic recession. This growth was mostly concentrated in smaller centres and in agglomeration belts; while some larger urban centres experienced population decline. But, since then, the population growth rate has increased again to 0.8% [Office, 2010] while the population concentration has experienced a reversal trend. Nowadays, Switzerland can be considered as a densely populated country with an average population density of around 193 inhabitants per square kilometre.

Geographically, Switzerland is divided into three main regions: the Swiss Alps, the Plateau and the Jura, (see Figure 1). Each region presents different geological and topographical features, and demographically, they clearly support dissimilar population distributions. Figure 2 displays the SPD of the year 2000 and a 3D visualization of the dataset where an inhomogeneous land-occupation structure is clearly detected with clusters of different sizes.

For instance, while the Alps occupy 60% of the total country territory, only 23% of the population lives in this highly mountainous region (average altitude of 1700 m). The Plateau, which is the economic epicentre, covers 30% of the country’s surface area and it concentrates 2/3 of the total population, most of the Swiss industries and farmlands, as well as the major cities such as Geneva, Bern and Zurich. There are few regions in Europe that are more densely populated than the Plateau (450 people per km$^2$) and, in some areas such as the main cities, the population density can surpass 1000 people per km$^2$ [1]. The Jura constitutes 10% of the country and hosts 9% of the population.

The population database used in the present study is the Swiss census of the year 2000. These high-resolution data are upheld and managed by
the Swiss Federal Statistical Office (Office, 2010) and they can be visualized through the 325,951 nodes (i.e. points) of a grid superimposed onto Switzerland, each of which is associated to the number of people living in an hectare (i.e. 100 x 100 m).

Reference patterns were generated to create confidence intervals of spatial randomness (Illian et al., 2008). This procedure was done by creating a large number of random permutations ($N=999$) of the measured value (i.e. number of inhabitants) in order to destroy the dependence existing between the number of inhabitants and the spatial location of each grid node (or point). For this, the measured values were shuffled and then assigned randomly to the location points (or grid node) and the procedure was iterated $N$ times. See Figure 3. The fractal and multifractal methods were also applied to these simulated samples and their results were compared with the original structure of the Swiss data to evaluate the departure of this raw pattern from a random structure.

2.2. Methodology

A point process is represented as sets of random points (events) generated within a space. Frequently, such processes exhibit a scaling behaviour
indicating a high degree of point clustering over all scales (Lowen and Teich, 1995). Fractal and multifractal tools can be used to characterize the intensity of spatial clustering at a wide range of scales (Lowen and Teich, 1995; Cheng and Agterberg, 1995).

2.2.1. Fractal dimension

According to Lovejoy et al. (1986); Salvadori et al. (1997); Tuia and Kanevski (2008), a fractal dimension can be used to analyse the clustering properties (non-homogeneity) of point process realizations. If the studied distribution is embedded within a 2D space (e.g. geographical space), its fractal dimensions range from 0 (i.e. the topological dimension of a point) to 2 (i.e. the dimension of geographical space). If the points are dispersed or randomly distributed within the 2D study area, the corresponding fractal dimension is equal to 2; but this value decreases as the level of clustering increases and it can reach 0 if all the points are superimposed at one single location. Thus, fractal dimensions allow us to detect the appearance of
Figure 3: The original population distribution in Switzerland (left) and one of the random permutation distribution simulated for comparisons (right)

clustering as a departure from a dispersed or random situation.

A variety of fractal measures have been proposed such as the box-counting method (Russell et al., 1980; Lovejoy et al., 1986; Tuia and Kanevski, 2008), the sandbox-counting method (Grassberger and Procaccia, 1983; Daccord et al., 1986; Feder, 1988; Tél et al., 1989; Vicsek, 1990; Tuia and Kanevski, 2008) and the information dimension (Balatoni and Rényi, 1956; Hentschel and Procaccia, 1983; Seuront, 2010). The fractal dimension of the SPD was estimated by applying the box-counting method.

The Box-Counting Method

In the box-counting method, a regular grid of boxes of size $\delta$ is superimposed on the region under study and the number of boxes, $N(\delta)$, necessary to cover the point set is counted. Then, the linear size $\delta$ of the boxes is reduced and the number of boxes, $N$, is counted again. The algorithm goes on until a minimum size $\delta$ is reached. For a fractal pattern, the scales ($\delta$) and the number of boxes ($N(\delta)$) follow a power law:

$$N(\delta) \propto \delta^{-d_{f_{\text{box}}}}$$  \hspace{1cm} (1)

where $d_{f_{\text{box}}}$ is the fractal dimension measured with the box–counting method (Tuia and Kanevski, 2008). Theoretically, as introduced formerly, self-similarity is a property present over an infinite range of scales, however, in natural fractals, this property is only encountered statistically over a finite range of scales (Abramenko, 2008) for which it is possible to consider $-d_{f_{\text{box}}}$ as the slope of the linear regression fitting the data of the plot which relates $\log(N(\delta))$ to $\log(\delta)$. 

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2.2.2. Multifractality

As stated by Mandelbrot (1988), the notion of self-similarity can be extended to measures (spreading mass or probability) distributed on an Euclidean support (e.g. a point set). In this context, fractal sets might be described by not just one fractal dimension, but rather by a function (Stanley and Meakin 1988) or a spectrum of interlinked fractal dimensions. Such fractal sets are said to be multifractal.

Two different approaches were used to conduct multifractal analysis: (1) Rényi’s generalized dimensions (Hentschel and Procaccia, 1983; Grassberger, 1983; Paladin and Vulpiani, 1987; Tél et al., 1989; Borgani et al., 1993; Perfect et al., 2006; Seuront, 2010) and (2) the multifractal singularity spectrum (Halsey et al., 1986; Meakin et al., 1986; Stanley and Meakin, 1988; Chhabra and Jensen, 1989). These two methods are a generalization of the box-counting method. For both of them, a regular grid of boxes of size $\delta$ is superimposed on the point set and a normalized measure (probability distribution) is computed over all boxes (Lopes and Betrouni, 2009).

Rényi’s Generalized Dimensions

The spectrum of generalized dimensions, $D_q$, is estimated by computing Rényi’s information, $I_q(\delta)$, of $q$th order (Rényi, 1970):

$$I_q(\delta) = \frac{1}{(1-q)} \log(\sum_{i=1}^{N(\delta)} p_i(\delta)^q)$$  \hspace{1cm} (2)

where $p_i = n_i/N$ is the probability mass function in the $i$th box of size $\delta$ and $q \in \mathbb{Z}$. Then, when applied to multifractal sets, $I_q(\delta)$ follows a power law:

$$I_q(\delta) \propto \delta^{-D_q}$$ \hspace{1cm} (3)

Then, Rényi’s generalized dimensions is defined as (Hentschel and Procaccia, 1983; Grassberger, 1983; Paladin and Vulpiani, 1987):

$$D_q = \lim_{\delta \to 0} \frac{I_q(\delta)}{\log(1/\delta)}$$ \hspace{1cm} (4)

The $D_q$ spectrum is obtained by the slope of the plot relating $\log(I_q(\delta))$ to $\log(1/\delta)$. For monofractal sets, $D_q$ is equal for all $q$ order moments, whereas, for multifractal sets, $D_q$ depends on $q$ and decreases as $q$ increases (Hentschel and Procaccia, 1983; Golay et al., 2013) characterizing the variability of the measure ($p_i(\delta)$). For $q = 0$, all non-empty boxes are equally
weighted and, consequently, $D_q$ is equivalent to the box–counting dimension, $d_{\text{box}}$, which corresponds to the dimension of the support. For $q > 0$, the mass within the boxes gradually gains more importance in the overall box contribution to Rényi’s information. As a result, the larger the mass within a box, the higher the weight of the box. Thus, higher $q$ order moments capture the scaling behaviour of regions where the mass is clumped. It is also interesting to notice that $D_1$ and $D_2$ correspond to the information dimension and the correlation dimension respectively (Grassberger and Procaccia 1983; Halsey et al. 1986).

**Multifractal Singularity Spectrum**

The multifractal singularity spectrum describes the scaling behaviours of a measure through an interlinked set of Hausdorff dimensions, $f(\alpha)$, associated to a singularity strength $\alpha$ (Halsey et al. 1986; Meakin et al. 1986; Chhabra and Jensen 1989). Let $p_i(\delta)$ be the probability in the $i_{th}$ box, then, the scales ($\delta$) and $p_i(\delta)$ follow a power law:

$$p_i(\delta) \propto \delta^{\alpha_i} \quad (5)$$

where $\alpha_i$ is the singularity strength of the $i_{th}$ box of size $\delta$. Then, the number of boxes, $N_{\alpha}(\delta)$, having a singularity strength between $\alpha$ and $\alpha + d\alpha$, can be related to the box size, $\delta$, as:

$$N_{\alpha}(\delta) \propto \delta^{-f(\alpha)} \quad (6)$$

where $f(\alpha)$ is the Hausdorff dimension of the set of boxes with singularity strength $\alpha$ (Halsey et al. 1986; Meakin et al. 1986; Chhabra and Jensen 1989). This singularity spectrum can be related to Rényi’s generalized dimensions through a Legendre transform as:

$$(q - 1)D_q = q\alpha(q) - f(\alpha(q)) \quad (7)$$

for more details see (Halsey et al. 1986; Meakin et al. 1986; Mandelbrot 1988; Seuront 2010). Chhabra and Jensen (1989) proposed a method for determining the multifractal singularity spectrum as a function of the $q$ orders without the application of the Legendre transform. Let $\mu(q)$ be the normalized measure of the probabilities in the boxes of size $\delta$, such as:

$$\mu_i(q, \delta) = \frac{[p_i(\delta)]^q}{\sum_i[p_i(\delta)]^q} \quad (8)$$
where, again, $q$ provides a tool for exploring denser and rarer regions
of the singular measure (Seuront, 2010). Then, $\alpha(q)$ and $f(\alpha(q))$ can be
computed as:

$$\alpha(q) = \lim_{\delta \to 0} \frac{\sum_i N(\delta) \mu_i(q, \delta) \log p_i(\delta)}{\log \delta}$$  \hspace{1cm} (9)

and

$$f(\alpha(q)) = \lim_{\delta \to 0} \frac{\sum_i N(\delta) \mu_i(q, \delta) \log \mu_i(q, \delta)}{\log \delta}$$  \hspace{1cm} (10)

The multifractal spectrum is obtained by plotting the singularity spectrum
$f(\alpha(q))$ vs. the singularity exponent $\alpha(q)$. For $q = 0$, $f(\alpha(0))$ takes
its maximum value and is equal to $D_0$ and $d_{f_{\text{box}}}$. For $q = 1$, $f(\alpha(1)) = \alpha(1) = D_1$ is the information dimension.

3. Results and discussion

As it was exposed in section 2, the Swiss population can be divided into
three subsets according to the main geographic regions: the Swiss Alps, the
Plateau and the Jura. In this study, the analysis of the Swiss population
was first conducted from a global perspective and was then narrowed down
to each of the three subsets.

Figure 4 indicates the fractal dimension of the SPD in 2000 in both the
entire country and the three geographic regions. It also exhibits the depen-
dence between the logarithm of the Rényi information and the logarithm
of the box size ($250 \text{ m} \leq \delta \leq 260 \text{ km}$). These results bring to light three
main behaviours characterizing different scale ranges. The first behaviour
is detected at local scales $\leq 1 \text{ km} \ (\log(\delta) = 10)$ and can be interpreted as a
result of the influence of social-economic factors and/or territorial planning
policies. These factors can be directly related to the way people concentrate
in agglomeration areas. The second behaviour concerns scales ranging from
1 to $\sim 4 \text{ km} \ (\log(\delta) = 12)$. It can be related to landuse practices. Finally,
the third behaviour, which the rest of this paper will mainly focus on, is
detected at scales greater than 4 km where topographic factors are likely to
play a major role in constraining the way people appropriate their natural
environment.

The closer to 0 the values of $d_{f_{\text{box}}}$, the stronger the aggregation of the
population, whereas, values of $d_{f_{\text{box}}}$ close to 2 indicate a homogeneous/random
distribution. Comparing the $df_{box}$ values of the four patterns with the simulated samples reveals a significant deviation from an unstructured situation. Nevertheless, Figure 4 does not reveal significant differences between the estimated $df_{box}$ values of the different regions, which leads to the conclusion that a comprehensive analysis of the SPD requires more than a single fractal dimension.

To overcome the limitation of the fractal dimension in differentiating the four studied patterns, a multifractal analysis was carried out by means of Rényi’s generalized dimensions, $D_q$, and the multifractal spectrum, $f(\alpha(q))$.

Figure 5 shows Rényi’s generalized dimensions for the four studied patterns and for $0 \leq q \leq 10$. The dependence between $D_q$ and $q$ is non-linear for each considered SPD, which highlights their multifractal nature. $D_q$ values denote the degree of clustering of the distribution (non-homogeneity), whereas, the range width of the $D_q$ values ($D_{max} - D_{min}$) is an indicator of the variability of the population density.

The $D_q$ curves of the four SPD patterns decline faster than the simulated data, which indicates a substantial departure of the SPD patterns from shuffled/unstructured distributions. Furthermore, the $D_q$ of the SPD of the Alps and Jura regions decrease faster than that of both the Plateau and
the entire country, meaning that the population in these regions (Alps and Jura) are more clustered.

Regarding the range width of the $D_q$ values, the four curves revealed an extensive variability between the high and low densely populated areas for each of the considered regions, although it is more accentuated in the mountainous regions (Alps and Jura). This can also be depicted by comparing $D_0$, $D_1$ and $D_2$ values.

The multifractal spectrum of the four SPD is illustrated in Figure 5. The multifractal spectrum of the four SPD is significantly different from that of the corresponding shuffled patterns. All the original distributions are characterized by an asymmetry skewed to the left with values lower than what is obtained for the shuffled data. This reflects the domination of large densely populated areas and highly clustered distributions.

The computed multifractal spectrum is in agreement with the estimations of Rényi’s generalized dimensions. The SPD of the entire country has a multifractal behaviour similar to that of the population of the Plateau, but its heterogeneity is slightly higher as shown by the width of the $\alpha_{\text{max}}$ and $\alpha_{\text{min}}$ values. This similarity can be interpreted as a strong contribution of the the Plateau to the whole structure when analyses are conducted at entire country level. This follows from the fact that the Plateau concen-
trates two-thirds of the total population as well as the major cities of the country such as Zurich, Geneva, Basel, Bern and Lausanne.

The multifractal spectrum obtained for the Alps and the Jura are more skewed to the left than the previous ones, which is an indicator of highly clustered distributions. This is confirmed by the fact that their $f(\alpha(\delta))$ curves are lower and wider than that of both the Plateau and the entire country. Although, their multifractal natures are quite similar, the spatial structure of the population in the Alps is more heterogeneous than the Jura region.

The significant clustering behaviour observed in the Alps and Jura regions can be mainly explained by the influence of the topography. Topographic features in these mountainous regions are the major factors which constrain land occupation in forcing populations to live in specific areas of the main valleys.

4. Conclusions

A fractal and multifractal analysis of the population distribution in Switzerland was carried out. This investigation is the first Swiss geodemographic analysis which applies the multifractal formalism to high resolution data. This analysis comes out onto the characterization of the spatial
structure of the population distribution for both the entire country and three geographical subregions (e.g. the Jura, the Plateau and the Alps). Rényi’s generalized dimensions and the multifractal spectrum of the four spatial population distributions (SPD) showed different scaling behaviours between the high and low densely populated areas, which led to the differentiation of the four point patterns.

Analyses highlighted that the SPD of the Plateau region was more homogeneous than that of the Alps and Jura regions, especially in the case of densely populated areas. Additionally, the study pointed out the importance of multifractal analyses at different scale levels and in different geographic subregions, which helps discerning between distinct underlying processes. The distribution of the Swiss population has certainly been shaped by the socio-economic history and the complex geomorphology of the country. These factors have rendered a highly clustered, inhomogeneous and very variable population distribution at different spatial scales. Therefore, the analysis of the SPD is an interesting and challenging task and constitutes a fundamental approach for urban studies.

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Notes

1http://www.swissworld.org/en/geography/ consulted in 01/2013

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