Integrable quantum chains combining site states in different representations of $su(3)$

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Abstract

The general expression for the local matrix $L(\theta)$ of a quantum chain with the site space in any representation of $su(3)$ is obtained. This is made by generalizing $L(\theta)$ from the fundamental representation and imposing the fulfilment of the Yang-Baxter equation. Then, a non-homogeneous spin chain combining different representations of $su(3)$ is solved by a method inspired in the nested Bethe ansatz. The solution for the eigenvalues of the trace of the monodromy matrix is given as two coupled Bethe equations. A conjecture about the solution of a chain with the site states in different representations of $su(n)$ is presented.
We present the study of solvable alternating chains whose site states of which are a mixture of any two representations of $su(3)$. We made an approach to this problem in our paper \cite{1}, where we solved an alternating chain mixture of the two fundamental representations of $su(3)$ and presented a method, a modification of the nested Bethe ansatz (MNBA), needed to find the Bethe equation (BE) solutions of the problem.

A chain is defined by giving the monodromy matrix that, for a non-homogeneous chain, we call $T^{\text{alt}}$. This matrix is a product of local matrices, generally designed by $L(\theta)$, that are defined in an auxiliary space and a site space \cite{2}. The auxiliary space will be always the $\{3\}$ representation of $su(3)$, whereas for the site space we will take any representation of the same algebra.

We denote a representation of $su(3)$ by the indices of its associated Dynkin diagram $(m_1, m_2)$, where $m_1$ and $m_2$ correspond to the $\{3\}$ and $\{\bar{3}\}$ representations respectively. In the figures a continuous line is used for the fundamental representation $(1,0)$ and a wavy line for any other representation. Thus, the operators $L(\theta)$ are denoted as indicated in figure 1. Besides, in order to simplify the writing of the formulae, we will adopt the following identifications: $L(\theta) \equiv L^{(1,0)(1,0)}(\theta)$ and $L'(\theta) \equiv L^{(1,0)(m_1,m_2)}(\theta)$.

![Figure 1](image_url)

The operator $L(\theta)$ can be written \cite{1}

$$L(\theta) = \begin{pmatrix}
\frac{1}{2}(\lambda^3 q^{-N^\alpha} - \lambda^{-3} q^{N^\alpha}) & \frac{1}{2}\lambda(q^{-1} - q)f_1 & \frac{1}{2}\lambda^{-1}(q^{-1} - q)[f_2, f_1] \\
\frac{1}{2}\lambda^{-1}(q^{-1} - q)e_1 & \frac{1}{2}(\lambda^3 q^{-N^\beta} - \lambda^{-3} q^{N^\beta}) & \frac{1}{2}\lambda(q^{-1} - q)f_2 \\
\frac{1}{2}\lambda(q^{-1} - q)[e_1, e_2] & \frac{1}{2}\lambda^{-1}(q^{-1} - q)e_2 & \frac{1}{2}(\lambda^3 q^{-N^\gamma} - \lambda^{-3} q^{N^\gamma})
\end{pmatrix}, \quad (1)$$

where the parameters $\lambda$ and $q$ have been taken as the functions of $\theta$ and $\gamma$

$$\lambda = e^{\frac{\theta}{2}}, \quad q = e^{-\gamma}, \quad (2)$$
the $N$ matrices are

$$N^\alpha = \frac{2}{3} h_1 + \frac{1}{3} h_2 + \frac{1}{3} I, \quad (3a)$$

$$N^\beta = -\frac{1}{3} h_1 + \frac{1}{3} h_2 + \frac{1}{3} I, \quad (3b)$$

$$N^\gamma = -\frac{1}{3} h_1 - \frac{2}{3} h_2 + \frac{1}{3} I, \quad (3c)$$

and $\{e_i, f_i, q^{\pm h_i}\}, \ i = 1, 2$ are the Cartan generators of the deformed algebra $U_q(sl(3))$.

To obtain the operators $L'(\lambda)$ with the new parameters given in (2), we take (1) as a basis and write

$$L'(\lambda) = \begin{pmatrix}
\frac{1}{2}(\lambda^3 q^{-N^\alpha} - \lambda^{-3} q^{N^\alpha}) & \lambda F_1 & \lambda^{-1} F_3 \\
\lambda^{-1} E_1 & \frac{1}{2}(\lambda^3 q^{-N^\beta} - \lambda^{-3} q^{N^\beta}) & \lambda F_2 \\
\lambda E_3 & \lambda^{-1} E_2 & \frac{1}{2}(\lambda^3 q^{-N^\gamma} - \lambda^{-3} q^{N^\gamma})
\end{pmatrix}, \quad (4)$$

where the operators $\{E_i, F_i\}, \ i = 1, 2, 3$, are unknown and will be determined by imposing the Yan Baxter equation (YBE)

$$R(\lambda/\mu)[L'(\lambda) \otimes L'(\mu)] = [L'(\mu) \otimes L'(\lambda)]R(\lambda/\mu) \quad (5)$$

shown in figure 2. The elements $R^{b,d}_{c,a}(\theta) \equiv [L_{a,b}(\theta)]_{c,d}$ of $R(\lambda/\mu)$ are given 3 by...
\( R(\lambda, \mu) = \begin{pmatrix} a & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & d & 0 & b & 0 & 0 & 0 & 0 \\ 0 & 0 & c & 0 & 0 & 0 & b & 0 \\ 0 & b & 0 & c & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & a & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & d & 0 & b \\ 0 & 0 & b & 0 & 0 & 0 & d & 0 \\ 0 & 0 & 0 & 0 & b & 0 & c & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & a \end{pmatrix} \), \hspace{1cm} (6)

with

\[ a(\lambda, \mu) = \frac{1}{2} (\lambda^3 \mu^{-3} q^{-1} - \lambda^{-3} \mu^3 q), \] \hspace{1cm} (7a)
\[ b(\lambda, \mu) = \frac{1}{2} (\lambda^3 \mu^{-3} - \lambda^{-3} \mu^3), \] \hspace{1cm} (7b)
\[ c(\lambda, \mu) = \frac{1}{2} (q^{-1} - q) \lambda \mu^{-1}, \] \hspace{1cm} (7c)
\[ d(\lambda, \mu) = \frac{1}{2} (q^{-1} - q) \lambda^{-1} \mu. \] \hspace{1cm} (7d)

The relations obtained are

\[ E_1 q^{N_\alpha} = q^{-1} q^{N_\alpha} E_1, \] \hspace{1cm} (8a)
\[ E_1 q^{N_\beta} = q q^{N_\beta} E_1, \] \hspace{1cm} (8b)
\[ F_1 q^{N_\alpha} = q q^{N_\alpha} F_1, \] \hspace{1cm} (8c)
\[ F_1 q^{N_\beta} = q^{-1} q^{N_\beta} F_1, \] \hspace{1cm} (8d)
\[ E_2 q^{N_\alpha} = q q^{N_\alpha} E_2, \] \hspace{1cm} (8e)
\[ E_2 q^{N_\beta} = q^{-1} q^{N_\beta} E_2, \] \hspace{1cm} (8f)
\[ F_2 q^{N_\alpha} = q^{-1} q^{N_\alpha} F_2, \] \hspace{1cm} (8g)
\[ F_2 q^{N_\beta} = q q^{N_\beta} F_2, \] \hspace{1cm} (8h)
\[ [E_1, F_1] = (q^{-1} - q) \left( q^{N_\beta - N_\alpha} - q^{N_\alpha - N_\beta} \right), \] \hspace{1cm} (8i)
\[ [E_2, F_2] = (q^{-1} - q) \left( q^{N_\gamma - N_\beta} - q^{N_\beta - N_\gamma} \right), \] \hspace{1cm} (8j)
\[ E_3 = \frac{1}{(q^{-1} - q)}q^{-N^\beta}[E_1, E_2], \]
\[ F_3 = \frac{1}{(q^{-1} - q)}q^{N^\beta}[F_2, F_1]. \]  

Besides, the modified Serre relations

\[ q^{-1}E_1E_1E_2 - (q + q^{-1})E_1E_2E_1 + qE_2E_1E_1 = 0, \]  
\[ qE_2E_2E_1 - (q + q^{-1})E_2E_1E_2 + q^{-1}E_1E_2E_2 = 0, \]
\[ q^{-1}F_1F_1F_2 - (q + q^{-1})F_1F_2F_1 + qF_2F_1F_1 = 0, \]
\[ qF_2F_2F_1 - (q + q^{-1})F_2F_1F_2 + q^{-1}F_1F_2F_2 = 0, \]

should be verified.

It must be noted that the relations (8) are the usual ones for the quantum group \(U_q(sl(3))\) while the relations (9) are not the usual ones. Since the generators \(e_i\) and \(f_i\) of the deformed algebra do not satisfy the YBE, we must introduce the operators \(E_i\) and \(F_i\) defined by

\[ F_i = \frac{1}{2}(q^{-1} - q)Z_if_i, \]  
\[ E_i = \frac{1}{2}(q^{-1} - q)e_iZ_i^{-1}, \]

where \(e_i\) and \(f_i, i = 1, 2,\) are the generators of \(U_q(sl(3))\) in the representation \((m_1, m_2)\) and \(Z_i\) are two diagonal operators that can be obtained by imposing the verification of the relations (8) and (9). In this way, one obtains the general form of these operators given by

\[ Z_1 = q^{a_1h_1 - \frac{1}{3}h_2 + a_3I}, \]
\[ Z_2 = q^{\frac{1}{3}h_1 + (a_1 + \frac{1}{3})h_2 + b_3I}, \]

where the operators \(h_i, i = 1, 2,\) are the diagonal elements of the algebra \(sl(3),\) and \(a_1, a_3\) and \(b_3\) are free parameters associated with the transformations that leave the YBE invariant.

The knowledge of the operator \(L'\) permits us to build a family of solvable multistate chains, based on the \(su(3)\) algebra, that mixes various representations. The monodromy
operator is built as a product of local operators. For instance, for a chain which alternates
the representations \((1, 0)\) and \((m_1, m_2)\),

\[
T^{(\text{alt})}_{a,b}(\theta) = L^{(1)}_{a,a_1}(\theta)L^{(2)}_{a_1,a_2}(\theta) \ldots L^{(2N-1)}_{a_{2N-2},a_{2N-1}}(\theta)L^{(2N)}_{a_{2N-1},b}(\theta),
\]

that can be represented graphically as shown in figure 3.

![Figure 3](image)

The monodromy operator can be written as a matrix in the auxiliary space whose ele-
ments are operators in the state space of the chain

\[
T^{\text{alt}}(\theta) = \begin{pmatrix}
A(\theta) & B_2(\theta) & B_3(\theta) \\
C_2(\theta) & D_{2,2}(\theta) & D_{2,3}(\theta) \\
C_3(\theta) & D_{3,2}(\theta) & D_{3,3}(\theta)
\end{pmatrix}.
\]

We are interested in finding the eigenvalues of the trace of \(T\).

Due to the fact that in this type of systems one cannot find a pseudovacuum eigenstate
of the diagonal operators \(A\) and \(D_{i,i}\) and such that the action on it of the non-diagonal \(D_{i,j}\)
be null, we must use a modification of the nested Bethe ansatz (MNBA) to find the Bethe
equations solutions of the problem \[4\].

The method in this case has two steps as the usual NBA. In the first step, we start by
building a subspace in the space of states whose elements are eigenstates of \(A\) and stable
under the action of the \(D_{i,j}\) operator. A state of this subspace is taken as pseudovacuum
state. The general state is obtained by applying \(r\) operators \(B_i(\mu_i), i = 1, \ldots, r,\) on this state,
as in the habitual NBA method. In the second step we must repeat the first step with a
two by two monodromy matrix \(T^{(2)}\) that is the product of the \(D\) submatrix of \(T^{(\text{alt})}\) and a
matrix that relates two products of \(r\) operators \(B_i(\mu_i)\) differing in a cyclic permutation. The
usual NBA can be only applied when \(T^{(2)}\) is solely product of cyclic permutation operators.
This new matrix verifies a Yang-Baxter equation and we can repeat the first step, finishing the process. The method is described in ref. [1], where it is applied to non-homogeneous chain that mixes the \{3\} and \{3^*\} representations.

The results depend on the representation of \(su(3)\) in which the states of every site are.

If the site space is in the representation \((m_1, m_2)\), the highest weight state is eigenstate of the diagonal operators in (4),
\[
L_{\lambda,m}^{(i)}(\theta) = L_{\lambda,m}^{(i,0),(m_1,m_2)}(\theta), \quad i = 1, 2, 3,
\]
with eigenvalues
\[
\begin{align*}
  l_{1,1}^{m_1,m_2}(\theta) &= \sinh\left(\frac{3}{2}\theta + \left(\frac{2}{3}m_1 + \frac{1}{3}m_2 + \frac{1}{3}\gamma\right)\right), \\
  l_{2,2}^{m_1,m_2}(\theta) &= \sinh\left(\frac{3}{2}\theta + \left(-\frac{1}{3}m_1 + \frac{1}{3}m_2 + \frac{1}{3}\gamma\right)\right), \\
  l_{3,3}^{m_1,m_2}(\theta) &= \sinh\left(\frac{3}{2}\theta + \left(-\frac{1}{3}m_1 - \frac{2}{3}m_2 + \frac{1}{3}\gamma\right)\right),
\end{align*}
\]

Then, each site introduces the source functions
\[
\begin{align*}
  g_1(\theta) &= \frac{l_{1,1}^{m_1,m_2}(\theta)}{l_{1,1}^{m_1,m_2}(\theta)} = \frac{\sinh\left(\frac{3}{2}\theta + \left(\frac{2}{3}m_1 + \frac{1}{3}m_2 + \frac{1}{3}\gamma\right)\right)}{\sinh\left(\frac{3}{2}\theta + \left(-\frac{1}{3}m_1 + \frac{1}{3}m_2 + \frac{1}{3}\gamma\right)\right)}, \\
  g_2(\theta) &= \frac{l_{3,3}^{m_1,m_2}(\theta)}{l_{2,2}^{m_1,m_2}(\theta)} = \frac{\sinh\left(\frac{3}{2}\theta + \left(-\frac{2}{3}m_1 - \frac{2}{3}m_2 + \frac{1}{3}\gamma\right)\right)}{\sinh\left(\frac{3}{2}\theta + \left(-\frac{1}{3}m_1 + \frac{1}{3}m_2 + \frac{1}{3}\gamma\right)\right)}.
\end{align*}
\]

The Bethe equations for a general chain combining \(N_1\) sites in the \((m_1, m_2)\) representation and \(N_2\) sites in the \((m_1', m_2')\) representation with source functions \(g_i^{(1)}\) and \(g_i^{(2)}\), \(i = 1, 2\), respectively, are
\[
\begin{align*}
  [g_1^{(1)}(\mu_k)]^{N_1} [g_1^{(2)}(\mu_k)]^{N_2} &= \prod_{j=1}^{r} \prod_{j \neq k}^s g(\mu_k - \mu_j) \prod_{i=1}^s g(\lambda_i - \mu_k), \quad k = 1 \cdots r, \\
  [g_2^{(1)}(\lambda_k)]^{N_1} [g_2^{(2)}(\lambda_k)]^{N_2} &= \prod_{j=1}^{r} \prod_{j \neq k}^s g(\lambda_k - \mu_j) \prod_{i=1}^s g(\lambda_k - \lambda_i), \quad k = 1 \cdots s,
\end{align*}
\]
where \(g\) is the \(g_1\) source function of the fundamental representation \(1, 0\)
\[
g(\theta) = \frac{\sinh\left(\frac{3}{2}\theta + \gamma\right)}{\sinh\left(\frac{3}{2}\theta\right)}.
\]

The eigenvalue of the trace of the monodromy matrix is
\[
\Lambda(\theta) = [\rho_1^{(1)}(\theta)]^{N_1} [\rho_1^{(2)}(\theta)]^{N_2} \prod_{j=1}^{r} g(\mu_j - \theta) +
\]

7
\[
+ \prod_{j=1}^{r} g(\theta - \mu_j) \left\{ \left[l_{2,2}^{(1)}(\theta)\right]^{N_1} \left[l_{2,2}^{(2)}(\theta)\right]^{N_2} \prod_{i=1}^{s} g(\lambda_i - \theta) +
\right. \\
+ \left. \left[l_{3,3}^{(1)}(\theta)\right]^{N_1} \left[l_{3,3}^{(2)}(\theta)\right]^{N_2} \prod_{l=1}^{r} \frac{1}{g(\theta - \mu_l)} \prod_{i=1}^{s} g(\theta - \lambda_i) \right\}. \tag{18}
\]

In the light of this, the generalization to the case of mixed chains with more than two different representations seems simple, although the physical models that they represent will be less local and the interaction more complex.

In a non-homogeneous chain combining different representations of \(su(n)\), each representation introduces \((n - 1)\) functions (that we call source functions). Each solution will have \((n - 1)\) sets of equations (the same number of dots in its Dynkin diagram). The left hand side of the equations will be a product of the respective source functions powered to the number of sites of each representation and the right hand side a product of source functions similar to (15).
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