SPINON DECOMPOSITION AND YANGIAN STRUCTURE OF $\hat{\mathfrak{sl}}_n$ MODULES

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ABSTRACT
We review several fermionic-type character formulae for the characters of the integrable highest weight modules of $\hat{\mathfrak{sl}}_n$ at level $\ell = 1$, and explain how they arise from a spinon basis for these modules. We also review how the Yangian $Y(\mathfrak{sl}_n)$ acts on the integrable $(\hat{\mathfrak{sl}}_n)_1$ modules and we decompose the characters with respect to this $Y(\mathfrak{sl}_n)$ action.

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1. Introduction

Conventionally, the Hilbert space of (rational) two dimensional conformal field theories (RCFT) is described in terms of a chiral algebra that acts on a finite set of fields that are primary with respect to this chiral algebra. This procedure leads to so-called Verma-module bases of RCFT’s, and gives rise to ‘bosonic type’ formulas for the characters (i.e. torus partition functions) of such conformal field theories, reflecting this particular choice for the basis of the Hilbert space. On the other hand, in a conformal field theory (CFT) there are many bases of Hilbert space, none of which is a priori preferred. Which of those different bases is most useful depends on the question that is being asked (see e.g. [1,20]). This is quite different in a massive field theory, where a basis of massive particles is naturally distinguished. In general, considering massive perturbations of CFT’s, and their corresponding particle bases, one may generate, at least conceptually, “quasi-particle” bases of CFT’s, by letting the mass tend to zero. This provides a way to define the notion of a massless “quasi-particle”, which has proven to be very useful in cases where the massive perturbation, used to define such a basis, is Yang-Baxter integrable. The so-generated bases inherit a particle (“Fock-space”) like structure, which is not manifest in the Verma-module type basis of the same CFT. Each basis of the Hilbert space of a given CFT gives rise to a particular way of writing the partition function on a torus. The equality of the different ways to write the torus partition function, in different bases, gives rise to remarkable identities. Moreover, bases of massless quasiparticles appear to be deeply related to Yangian and affine quantum symmetries, that are often present even when conformal invariance is broken. Therefore, we expect that a better understanding of such bases of CFT’s will also provide valuable insights into perturbed CFT’s. Analysis of the thermodynamic Bethe Ansatz, arising from (Yang-Baxter) integrable perturbations of CFT’s, for a variety of models, has led to a wealth of conjectures for so-called quasi-particle (or fermionic) type characters for conformal field theories (see, in particular [30,27]), some of which have been proven through \(q\)-analysis (see e.g. [5,28,21]). Most of these results still lack an interpretation (and/or proof) in terms of a corresponding structure of the Hilbert space of a conformal field theory, i.e. a characterization and/or construction of the corresponding “quasi-particle” basis of the Hilbert space.
A particularly interesting model illustrating the issues above is the so-called \( \mathfrak{sl}_n \) Haldane-Shastry long-range spin chain, which is integrable and has Yangian symmetry (even for finite chains). Its low energy sector is identical to a well-known conformal field theory, namely the \( SU(n) \) level-1 Wess-Zumino-Witten model \([24,7]\). While the description of the Hilbert space of the \( SU(n) \) level-1 WZW model in terms of its chiral algebra, i.e. \( \mathfrak{sl}_n \), is complicated due to the existence of null vectors, it was found that the Hilbert space has a very simple structure \([8,9,11]\), originating from the underlying Yangian symmetry: it may be viewed as a “Fock space” of massless ‘spinon particles’, which satisfy generalized commutation relations \([9]\) (and no other relations for \( \mathfrak{sl}_2 \)). For generalizations of these results to higher level \( \ell > 1 \) we refer to \([10,32,33,4]\).

In this paper we review these new exciting developments from a mathematical point of view.

Let us illustrate the various issues in the case of the affine Lie algebra \( \hat{\mathfrak{sl}}_2 \). The bosonic character formula for the level-1 integrable highest weight modules \( L(\hat{\Lambda}_k) \), \( k = 0, 1 \), reads (see App. B)

\[
\text{ch}_{L(\hat{\Lambda}_k)}(z; q) = \sum_{n \in \mathbb{Z}} \frac{q^{(n+\frac{1}{2})^2}}{(q)_\infty} z^{n+\frac{1}{2}}. \tag{1.1}
\]

There are two, not obviously related, fermionic character formulae

\[
\text{ch}_{L(\hat{\Lambda}_k)}(z; q) = q^{\frac{k}{4}} \sum_{m_1, m_2 \geq 0} \frac{q^{m_1^2 - m_1m_2 + m_2^2 + k(m_1 - m_2)}}{(q)_{m_1}(q)_{m_2}} z^{m_1 - m_2 + \frac{k}{4}}, \tag{1.2}
\]

and

\[
\text{ch}_{L(\hat{\Lambda}_k)}(z; q) = \sum_{m_1, m_2 \geq 0 \mod 2} q^{\frac{1}{2}(m_1 - m_2)^2} \frac{(q)_{m_1}(q)_{m_2}}{(q)_m} z^{\frac{1}{2}(m_1 - m_2)} \tag{1.3}
\]

which are easily shown to be equivalent to (1.1) by using the Durfee square (Lemma D.2). The second equality in (1.3) follows from Lemma D.3 (i). [As compared to the formulae in Appendix B we have defined \( z = \frac{x_1}{x_2} \).]

While (1.2) clearly has an interpretation in terms of two quasi-particles associated to the roots of \( \mathfrak{sl}_2 \), equation (1.3) arises from two quasi-particles associated to the weights
of the fundamental two-dimensional representation of $\mathfrak{sl}_2$ (i.e., the spinor representation of $\mathfrak{so}_3$, hence the name spinon). Moreover, since the quasi-particle operator intertwines between the modules $L(\hat{\Lambda}_0)$ and $L(\hat{\Lambda}_1)$, applying an even or odd number of creation operators to the vacuum results in a vector in $L(\hat{\Lambda}_0)$ or $L(\hat{\Lambda}_1)$, respectively. In this review we will only consider the generalization of (1.3) to $\mathfrak{sl}_n$. For the generalization of (1.2) to $\mathfrak{sl}_n$, see [19,22,23]. The spinon operators are mutually nonlocal. As a consequence they satisfy generalized commutation relations which leads to fractional statistics. Nevertheless, the generalized commutation relations are sufficiently powerful to determine a linearly independent set of basis vectors for the spinon Fock space. Calculating the character of the $(\widehat{\mathfrak{sl}}_2)_1$ integrable highest weight modules using the spinon basis immediately leads to (1.3).

The integrable highest weight modules $L(\hat{\Lambda}_k)$, $k = 0, 1$, of $(\widehat{\mathfrak{sl}}_2)_1$ admit a (semi-simple) action of a certain quantum group, the so-called Yangian $Y(\mathfrak{sl}_2)$. This action is not only most naturally described on the spinon basis but is, in fact, intimately related to the existence of a spinon basis. The 1-spinon states constitute the (two-dimensional) evaluation representation of $Y(\mathfrak{sl}_2)$ with the spinon mode index playing the role of the evaluation parameter, while the action on multi-spinon states corresponds to a (non-standard) comultiplication for $Y(\mathfrak{sl}_2)$.

The modules $L(\hat{\Lambda}_k)$, $k = 0, 1$, can be decomposed with respect to this $Y(\mathfrak{sl}_2)$ action. For $\mathfrak{sl}_2$ this leads to the characterformula

$$
\text{ch}_{L(\hat{\Lambda}_k)}(z;q) = \sum_{N \geq 0} \sum_{\substack{\lambda \in \mathcal{P} \\ l(\lambda) \leq N \equiv k \mod 2}} q^{N^2 + \frac{1}{2} \lambda} \text{ch}^{Y(\mathfrak{sl}_2)}_{\lambda}(z),
$$

(1.4)

where, for each spinon number $N$, the sum is over all partitions (Young diagrams) $\lambda$ of length $l(\lambda) \leq N$ and $\text{ch}^{Y(\mathfrak{sl}_2)}_{\lambda}(z)$ denotes the character of an irreducible finite-dimensional $Y(\mathfrak{sl}_2)$ module $L_\lambda$ labeled by $\lambda$ (see Section 3 for more details). The generalization of this result to $(\widehat{\mathfrak{sl}}_n)_1$ turns out to be slightly more complicated; in that case the Yangian modules that occur are most naturally parametrized in terms of a particular kind of skew Young diagrams, the so-called border strips.

This survey is organized as follows. In Section 2 we prove various fermionic-type formulae for the characters of the integrable modules of $\widehat{\mathfrak{sl}}_n$ at level $\ell = 1$, and explain their origin in terms of a spinon basis for these modules. In Section 3 we explain how the
Yangian $Y(\mathfrak{sl}_n)$ acts on the integrable $\widehat{\mathfrak{sl}}_n$ modules and we decompose the characters with respect to this $Y(\mathfrak{sl}_n)$ action. In order not to deter the reader from the main line of thought we have omitted most of the proofs (which can either be found in the literature or are straightforward) and deferred most of the mathematical prerequisites to appendices. Appendix A is a brief introduction to partitions and symmetric functions. In particular we introduce (skew) Schur functions. Appendix B briefly introduces the Lie algebra $\mathfrak{sl}_n$, its (untwisted) affine extension $\widehat{\mathfrak{sl}}_n$ and some of their modules. In Appendix C we define the Yangian $Y(\mathfrak{sl}_n)$ and discuss its finite-dimensional irreducible representations and, finally, in Appendix D we discuss some $q$-identities necessary to establish the character identities of Section 2.

2. Spinon decomposition of $\widehat{\mathfrak{sl}}_n$ modules

2.1. $N$-spinon cuts of the $\widehat{\mathfrak{sl}}_n$ characters

In this section we prove two fermionic-type character formulae for the characters of the integrable highest weight modules $L(\widehat{\Lambda}_k)$, $k = 0, \ldots, n - 1$, of the affine Lie algebra $\widehat{\mathfrak{sl}}_n$ at level $\ell = 1$. In the next section we will then argue that these formulae find their origin in the existence of a certain basis, the so-called spinon basis, for these modules, where we identify the summation variable $N$ as the spinon number. We refer to Appendix B for definitions and notations regarding $\mathfrak{sl}_n$ and its affinization $\widehat{\mathfrak{sl}}_n$.

Theorem 2.1. For $\widehat{\Lambda} \in P_+^1$ and $\Lambda - \lambda \in Q$ we have the following expressions for the $\widehat{\mathfrak{sl}}_n$ string functions $c^\Lambda_\lambda(q)$ defined by

\[ \text{ch}_{L(\widehat{\Lambda})} = \sum_\lambda c^\Lambda_\lambda(q)e^\lambda. \]  

(2.1)

\[ c^\Lambda_\lambda(q) = \frac{q^{\frac{1}{2}|\lambda|^2}}{(q)_{n-1}} = \sum_{N \geq 0} c^{\Lambda,N}_\lambda(q), \]  

(2.2)
where
\[
c^\Lambda_{\lambda} = q^{\frac{1}{2}|\lambda|^2} \sum_{m_1, \ldots, m_{n-2}} \frac{q^{A_1 m_1 + (A_2 - m_1) m_2 + \ldots + (A_{n-1} - (m_1 + \ldots + m_{n-2})) (A_n - (m_1 + \ldots + m_{n-2}))}}{(q) A_1 (q) A_2 \cdots (q) A_{n-1} - (m_1 + \ldots + m_{n-2}) (q) A_n - (m_1 + \ldots + m_{n-2})} \times \prod_i(q_{A_i})_{m_i}^{-1} = q^{\frac{1}{2}|\lambda|^2} \sum_m (-1)^m \frac{q^{\frac{1}{2}m(m-1)}}{(q)_m} \prod_i(q_{A_i})_{m_i}^{-1},
\]
for those \( N \) such that \( \hat{\Lambda} = \hat{\Lambda}_N \mod n \) and zero otherwise. We have defined
\[
A_i = \frac{N}{n} + (\lambda, \epsilon_i).
\]

Proof: Consider the first expression for \( c^\Lambda_{\lambda} \). Changing \( N \to N + (m_1 + \ldots + m_{n-2})n \) we have
\[
\sum_{N \geq 0} c^\Lambda_{\lambda}(q) = q^{\frac{1}{2}|\lambda|^2} \sum_{N} \sum_{m_1, \ldots, m_{n-2}} \frac{q^{(A_1 + m_1 + \ldots + m_{n-2}) m_1 + (A_2 + m_2 + \ldots + m_{n-2}) m_2 + \ldots + A_{n-1} A_n}}{(q) A_1 + m_1 + \ldots + m_{n-2} (q) A_2 + m_2 + \ldots + m_{n-2} \cdots (q) A_{n-1} (q) A_n} \times \prod_i(q_{A_i})_{m_i}^{-1}
\]
Then apply the Durfee square (Lemma D.2) to, respectively, \( m_1, m_2, \ldots m_{n-2} \) and \( N \) to obtain (2.2). To prove the second expression in (2.3) we successively apply Lemma D.3 (ii) to the right hand side, i.e.
\[
\sum_m (-1)^m \frac{q^{\frac{1}{2}m(m-1)}}{(q)_m} \prod_i(q_{A_i})_{m_i}^{-1} = \sum_{m, m_{n-2}} (-1)^m \frac{q^{\frac{1}{2}m(m-1)}}{(q)_m} \prod_i(q_{A_i})_{m_i}^{-1} = \ldots .
\]
\[
= \sum_{m, m_{n-2}} (-1)^m \frac{q^{\frac{1}{2}m(m-1)}}{(q)_m} \prod_i(q_{A_i})_{m_i}^{-1} = \ldots .
\]
Now change summation variables

\[ m_1 \rightarrow m_1 - m \]
\[ m_2 \rightarrow m_2 + m_1 - m \]
\[ \vdots \]
\[ m_{n-2} \rightarrow m_{n-2} + m_{n-3} + \ldots + m_1 - m \]

and use Lemma D.3 (i) in the form

\[ \sum_m (-1)^m \frac{q^{2m(m-1)}}{(q)_m} \frac{1}{(q)_{A_1-m}(q)_{m_1-m}} = \frac{q^{A_1 m_1}}{(q)_{A_1}(q)_{m_1}}, \]

to obtain (2.3). \( \square \)

Note that, for \( n = 2 \), the formulae (2.3) lead to those of (1.3).

2.2. Generalized commutation relations and spinon basis

The character formulae of Theorem 2.1 find their origin in the existence of a specific basis, the so-called spinon basis, for the integrable highest weight modules of \( \widehat{\mathfrak{sl}}_n \) at level \( \ell = 1 \). We will refer to the intertwiners

\[ \phi^i \left( \begin{array}{c} \Lambda_1 \\ \Lambda_{k+1} \Lambda_k \end{array} \right) (z) : L(\widehat{\Lambda}_k) \rightarrow L(\widehat{\Lambda}_{k+1}), \quad (i = 1, \ldots, n), \]  

transforming in the fundamental \( \mathfrak{sl}_n \) representation \( L(\Lambda_1) \) (see Appendix B for a more precise statement), as the spinon operators of \( \widehat{\mathfrak{sl}}_n \). (Here the subscripts on \( \widehat{\Lambda} \) are taken modulo \( n \).) Their mode expansion is given by

\[ \phi^i \left( \begin{array}{c} \Lambda_1 \\ \Lambda_{k+1} \Lambda_k \end{array} \right) (z) = \sum_{m \in \mathbb{Z}} \phi^i \left( \begin{array}{c} \Lambda_1 \\ \Lambda_{k+1} \Lambda_k \end{array} \right) - \frac{n-2k-1}{2m} \frac{1}{z^{m-k-n}}. \]  

Multiple application of the spinon creation operators to the highest weight state \( |0\rangle \) of \( L(\widehat{\Lambda}_0) \) creates a spinon Fock space. By definition, a state with \( N \) spinon excitations is a vector in \( L(\widehat{\Lambda}_{N \mod n}) \). Hence, it is unambiguous on which space \( L(\widehat{\Lambda}_k) \) the spinon operator is acting and henceforth we will omit the fusion channel and simply write \( \phi^i(z) \).

A general \( N \)-spinon state can be written as a linear combination of vectors

\[ |n_1, \ldots, n_N\rangle \equiv \phi^{i_N} - \frac{n-(2N-1)}{2m} - n_N \ldots \phi^{i_3} - \frac{n-3}{2m} - n_3 \phi^{i_2} - \frac{n-5}{2m} - n_2 \phi^{i_1} - \frac{n-1}{2m} - n_1 |0\rangle. \]
The collection of all the vectors (2.7) forms an (overcomplete) basis for the $(\hat{\mathfrak{sl}}_n)_1$ module \( \oplus_{k=0}^{n-1} L(\hat{\Lambda}_k) \). Note that the energy of the state (2.7) is given by

\[
L_0|n_1, \ldots, n_N\rangle = \left( \frac{N(n - N)}{2n} + \sum_{i=1}^{N} n_i \right) |n_1, \ldots, n_N\rangle.
\] (2.8)

Due to the mutual nonlocality of the spinon operators, the modes (2.6) will not satisfy simple commutation relations. Instead, the modes will satisfy so-called generalized commutation relations. Nevertheless, these generalized commutation relations imply that not all vectors (2.7) are linearly independent. In particular, they allow us to choose an ordering \( 0 \leq n_1 \leq n_2 \leq \ldots \leq n_N \). Let us consider the abstract algebra generated by the spinon modes modulo the generalized commutation relations and the corresponding module, referred to as the spinon Fock space \( \mathcal{F} \), spanned by the vectors (2.7). The problem is to determine a linearly independent set of basis vectors for \( \mathcal{F} \).

Let us illustrate this explicitly for \( \hat{\mathfrak{sl}}_2 \). In this case the generalized commutation relations for the spinon fields are (see [9])

\[
\sum_{l \geq 0} C_l^{(-\frac{1}{2})} \left( \phi_i^{p - \frac{1}{2}l + \frac{1}{2} - l + \frac{1}{2}} \phi_j^{q - \frac{1}{2}l - \frac{1}{2} + l + \frac{1}{2}} - \left( \begin{array}{c} i \leftrightarrow j \\ p \leftrightarrow q \end{array} \right) \right) = (-1)^k \epsilon^{ij} \delta_{p+q+k-1},
\] (2.9)

where \( k = 0, 1 \), depending on whether the left hand side acts on a vector in \( L(\hat{\Lambda}_k), k = 0, 1 \), and the coefficients \( C_l^{(\alpha)} \) are defined by the expansion

\[
(1 - x)^\alpha = \sum_{l \geq 0} C_l^{(\alpha)} x^l.
\] (2.10)

By induction we find the following refinement of the basis (2.7)

**Theorem 2.2** [9]. *The following (linearly independent) set of vectors form a basis for the spinon Fock space \( \mathcal{F} \)*

\[
\phi^{2 \frac{(M_1 + M_2 - 1)}{4} - \frac{1}{2} - m_{M_2}} \cdots \phi^{2 \frac{(M_1 - 1)}{4} - \frac{1}{2} - m_{M_1}} \frac{\phi^{1 \frac{(M_1 - 1)}{4} - \frac{1}{2} - n_{M_1}} \cdots \phi^{1 \frac{(M_2 - 1)}{4} - n_{M_2}} [0]}{4}.
\] (2.11)

The \( L_0 \) eigenvalue of the state (2.11) is given by

\[
\frac{(M_1 + M_2)^2}{4} + \sum_{i=1}^{M_2} m_i + \sum_{i=1}^{M_1} n_i.
\] (2.12)
A priori, the spinon Fock space $F$ could be bigger than $L(\Lambda_0) \oplus L(\Lambda_1)$. In this case, calculating the character of $F$ (using (2.12)) immediately leads to (2.3) (see also (1.3)), proving $F \cong L(\Lambda_0) \oplus L(\Lambda_1)$. For $\widehat{\mathfrak{sl}}_n$, $n > 2$, however, there might be additional relations between the spinon operators beyond those coming from the generalized commutation relations, i.e. there might be null vectors in the Fock module $F$. The details remain to be worked out. In any case, all relations are encoded in the character formulae (2.3).

3. Yangian structure of $\widehat{\mathfrak{sl}}_n$ modules

3.1. Action of $Y(\mathfrak{sl}_n)$ on integrable $(\widehat{\mathfrak{sl}}_n)_1$ modules

Let $x^a$ be an orthonormal basis of $\mathfrak{sl}_n$ and let $f^{abc}$ and $d^{abc}$ be, respectively, the structure constants and 3-index $d$-symbol of $\mathfrak{sl}_n$ normalized such that

$$f^{abc} f^{dbc} = -2n \delta^{ad},$$
$$d^{abc} d^{dbc} = \frac{2(n^2 - 4)}{n} \delta^{ad},$$

i.e., in the fundamental representation $(L(\Lambda_1), \rho)$, we have

$$t^a t^b = \frac{1}{n} \delta^{ab} + \frac{1}{2} f^{abc} t^c + \frac{1}{2} d^{abc} t^c,$$  \hspace{1cm} (3.2)

where $t^a \equiv \rho(x^a)$.

The following theorem was suggested by taking the infinite chain limit of the Yangian generators in the $\mathfrak{sl}_n$ Haldane-Shastry spin chain

**Theorem 3.1** [36]. The following formulae define an (semi-simple) action of $Y(\mathfrak{sl}_n)$, as defined in Definition C.5, on the integrable modules of $\widehat{\mathfrak{sl}}_n$ at level $\ell = 1$

$$x^a = x_0^a,$$
$$J(x^a) = \frac{1}{2} f^{abc} \sum_{m > 0} (x_m^b x_m^c) - \frac{n}{2(n + 2)} W_0^a.$$

where

$$W_0^a = \frac{1}{2} d^{abc} \sum_{m \in \mathbb{Z}} : x_{-m}^b x_m^c :,$$  \hspace{1cm} (3.4)
The proof of Theorem 3.1 is straightforward, albeit cumbersome, and amounts to checking the defining relations (Y1)–(Y3) of Definition C.5.

It is useful to determine the action of the Yangian generators (3.3) on the spinon basis (2.7). To this end, introduce (formal) generating series for the spinon basis,

\[ \Phi_{i_N, \ldots, i_1}(z_N, \ldots, z_1) \equiv \phi_{i_N}(z_N) \cdots \phi_{i_2}(z_2) \phi_{i_1}(z_1)|0\rangle. \] (3.5)

By using the commutation relations between \( x^a_m \) and the spinon operators (see eqn. (B.24)) as well as the null vector structure of the level-1 integrable highest weight modules, one arrives at

**Theorem 3.2** [8,2]. The action of the Yangian generators (3.3) on the spinon basis is given (up to an automorphism of \( Y(\mathfrak{sl}_n) \)) by

\[ x^a \Phi(z_N, \ldots, z_1) = \sum_i t^a_i \Phi(z_N, \ldots, z_1), \]

\[ J(x^a) \Phi(z_N, \ldots, z_1) = \left( -n \sum_i D_i t^a_i + \frac{1}{2} f^{abc} \sum_{i \neq j} \theta_{ij} t^b_i t^c_j \right) \Phi(z_N, \ldots, z_1), \] (3.6)

where \( D_i = z_i \partial_{z_i} \) and \( \theta_{ij} = \frac{z_i}{z_i - z_j} \), and \( t^a_i \) denotes the action of \( t^a \) on the \( i \)-th entry of \( \Phi \).

The differential operators above are precisely the Yangian generators of yet another well-known model with Yangian symmetry, namely the \( \mathfrak{sl}_n \)-spin generalization of the Calogero-Sutherland model (at coupling constant \( \beta = -\frac{1}{2} \)) [7].

### 3.2. Decomposition of characters under \( Y(\mathfrak{sl}_n) \)

After having established that there exists a semi-simple action of the Yangian \( Y(\mathfrak{sl}_n) \) on the integrable highest weight modules of \( \widehat{\mathfrak{sl}_n} \) at level \( \ell = 1 \), it is natural to ask how these modules decompose under \( Y(\mathfrak{sl}_n) \). This is settled by the following theorem

**Theorem 3.3** [11,29]. The character of the level-1 irreducible module \( L(\widehat{\Lambda}_k) \) of \( \widehat{\mathfrak{sl}_n} \) decomposes under the action of \( Y(\mathfrak{sl}_n) \) as

\[ \text{ch}_{L(\widehat{\Lambda}_k)}(x; q) = \sum_{\kappa \in \mathcal{BS}_n, \kappa \equiv k \mod n} q^{E(\kappa)} s_\kappa(x), \] (3.7)
where $\kappa$ runs over the set of border strips of rank $n$ and parametrizes a set of irreducible (tame) $Y(\mathfrak{sl}_n)$ modules with character $s_\kappa(x)$ (Appendix C), and

$$E(\kappa) = \frac{(n-1)|\kappa|^2}{2n} + \sum_{i=1}^{r} (i-r)a_i$$

$$= \frac{|\kappa|(n-|\kappa|)}{2n} + \sum_{i=1}^{s} (s-i)b_i,$$

if $\kappa = (a_1, \ldots, a_r) = [b_1, \ldots, b_s]$ with $b_s < n$.

**Proof:** To prove the character equality observe that, in terms of semi-infinite rapidity sequences (see Appendix A), we can write

$$E(\kappa) = \Delta_k - \sum_i (m_i - m_i^{(k)}).$$

Thus the right hand side of (3.7) can be written as

$$q^\Delta_k \sum_{m \in \mathcal{R}_{n/2}^\infty} q^{-\sum_i (m_i - m_i^{(k)})} s_m(x).$$

which can be shown to equal $\text{ch}_{\mathfrak{sl}_n}(x; q)$ either by invoking the path description of $L(\hat{\Lambda}_k)$ [15] (see also [29]) or, as done in [11], by ‘truncating’ to finite rapidity sequences and using a similar result for the Haldane-Shastry spin chain. Specifically, (3.10) equals

$$\lim_{L \to \infty} q^{\Delta_{\Lambda_L \text{mod } n}} \sum_{m \in \mathcal{R}_L^\infty} q^{-\sum_i (m_i - m_i^{(k)})} s_m(x)$$

$$\equiv \lim_{L \to \infty} \text{ch}_{\text{HS}}^{(L)}(x; q).$$

where we have introduced the character of the $\mathfrak{sl}_n$ Haldane-Shastry spin chain on a chain of length $L$ [25,11]. The proof is then completed by taking the $L \to \infty$ limit of the following

**Lemma 3.4** [11].

$$\text{ch}_{\text{HS}}^{(L)}(x; q) = q^{\frac{n-1}{2n}L^2} H_L^{(n)}(x; q^{-1}),$$

with $H_L^{(n)}(x; q)$ as in (D.5).

In Appendix C (i.e., Theorem C.13) it is shown that the skew Schur function $s_\kappa(x)$ corresponding to some skew Young diagram $\kappa = \lambda/\mu$ can be identified with the $\mathfrak{sl}_n$ character
of an irreducible (finite-dimensional) $Y(\mathfrak{sl}_n)$ module. To prove the remainder of Theorem 3.3, one thus has to construct a $Y(\mathfrak{sl}_n)$ highest weight vector in $L(\widehat{\Lambda}_k)$ for each $\kappa \in B\mathcal{S}_n$. This is most easily done in the spinon basis. For general $n$ the result can be inferred from [37,38]. Here, for simplicity, we only consider $\mathfrak{sl}_2$ [8,26].

We quote the following result for $\widehat{\mathfrak{sl}}_2$

**Theorem 3.5 [8,26]**. In the $N$-spinon sector of the $(\widehat{\mathfrak{sl}}_2)_1$ module $L(\widehat{\Lambda}_0) \oplus L(\widehat{\Lambda}_1)$ there exists a $Y(\mathfrak{sl}_2)$ highest weight vector $\omega_{\lambda,N}$ for every partition $\lambda$ with $l(\lambda) \leq N$. Furthermore,

$$L_0 \omega_{\lambda,N} = \left( |\lambda| + \frac{N^2}{2} \right) \omega_{\lambda,N},$$

and the irreducible $Y(\mathfrak{sl}_2)$ module $V_{\lambda,N}$ generated from $\omega_{\lambda,N}$ has $\mathfrak{sl}_2$ character

$$\text{ch}_{V_{\lambda,N}}(x) = \prod_{i \geq 0} h_{m^i_x}(x),$$

where $m^i_x = \# \{ j : \lambda_j = i \}$ for $i \geq 1$ and $m^0_x \equiv N - \sum_{i \geq 1} m^i_x$.

As we have seen in Theorem 3.2, on the spinon basis the Yangian action is described by the differential operators that generate the Yangian symmetry in the spin Calogero-Sutherland model. For $\mathfrak{sl}_2$, it is well-known (see e.g. [7]) that the corresponding Yangian highest weight vectors are given by Jack polynomials associated to a partition $\lambda$. It remains to determine which Jack polynomials can occur in the $N$-spinon sector (this leads to the restrictions on $\lambda$) and what the irreducible $Y(\mathfrak{sl}_2)$ module is that is generated from this highest weight vector. This last fact is accomplished by determining the Drinfel’d polynomial associated to the highest weight vector (see Appendix C).

To make the comparison of Theorem 3.5 with Theorem 3.3, we have to associate a skew Young diagram $\kappa$ to each partition $\lambda$ above. Let $r \in \mathbb{N}$ be such that $m^i_\lambda = 0$ for $i \geq r$ whilst $m^i_{r-1} \neq 0$. Then define the skew Young diagram $\kappa = \langle a_1, \ldots, a_r \rangle$ by

$$a_i = \begin{cases} 
  m^i_{i-1} + 1 & \text{for } i \in \{1, \ldots, r\}, \\
  m^i_{i-1} + 2 & \text{for } 2 \leq i \leq r - 1.
\end{cases}$$

Then,

$$N = \sum_{i \geq 1} m^i_\lambda = |\kappa| - 2(r-1),$$

$$|\lambda| = \sum_{i \geq 1} i m^i_\lambda = \sum_i i a_i - |\kappa| - (r-1)^2,$$

$$- 11 -$$
such that
\[ |\lambda| + \frac{N^2}{4} = \frac{|\kappa|^2}{4} + \sum_i (i - r) a_i, \quad (3.17) \]
in accordance with (3.8). The equality of the resulting \( Y(\mathfrak{sl}_2) \) characters (see (3.14)) follows from (A.14). \( \square \)

Unfortunately, this simple description of the Yangian highest weight vectors in terms of a single partition \( \lambda \) does not generalize to \( \mathfrak{sl}_n \). To illuminate the connection of Theorem 3.3 to the spinon basis for general \( n \), consider a generic \( M \)-spinon state
\[ |n_1, \ldots, n_M\rangle = \phi^{i_1}_{-\frac{n_1 - (2M - 1)}{2n}} \cdots \phi^{i_3}_{-\frac{n_3 - 5}{2n}} \phi^{i_2}_{-\frac{n_2 - 3}{2n}} \phi^{i_1}_{-\frac{n_1 - 1}{2n}} |0\rangle, \quad (3.18) \]
with \( 0 \leq n_1 \leq n_2 \leq \ldots \leq n_M \). Clearly, the energy of such a state is given by (cf. (2.8))
\[ L_0 |n_1, \ldots, n_M\rangle = \left( \frac{M(n - M)}{2n} + \sum_{i=1}^M n_i \right) |n_1, \ldots, n_M\rangle. \quad (3.19) \]
The similarity with (3.8) is striking. It suggests that to each sequence \( 0 \leq n_1 \leq n_2 \leq \ldots \leq n_M \) can be associated a skew Young diagram \( \kappa = [b_1, \ldots, b_s] \) constructed as follows:
1. Draw a square, call this the 1-st square. Now, by induction, draw the \( i \)-th square on top (resp., to the right) of the \( i - 1 \)-th square if \( n_i = n_{i-1} \) (resp., \( n_i \neq n_{i-1} \)).
2. The resulting skew Young diagram \( \kappa \) is a border strip and, clearly,
\[ |\kappa| = M, \quad \sum_i (s - i) b_i = \sum_i n_i, \quad (3.20) \]
provided \( \#\{i|n_i = k\} \neq 0 \) for all \( 0 \leq k \leq n_M \). This suggests that to each such sequence is associated a \( Y(\mathfrak{sl}_n) \) highest weight vector whose expression in terms of spinon states has a leading term (in general there will be subleading terms with respect to some ordering on \( \{n_i\} \)) given by (3.18). This is indeed the case [11]. The \( \mathfrak{sl}_n \) content of the corresponding irreducible \( Y(\mathfrak{sl}_n) \) module is a subquotient of \( L(\Lambda_1)^{\otimes M} \) which can be understood in terms of a generalized exclusion principle for the spinon \( \phi^i(z) \). Note that to satisfy the condition on the modes \( \{n_i\} \), one has to insert singlet combinations of \( n \) spinons into the ‘gaps’ between the \( n_i \)’s. Hence, the spinon number \( M \) in the description of (3.20) is not the same as the spinon number \( N \) in the corresponding result (3.16) for \( \mathfrak{sl}_2 \). See [11] for more details, in particular on the correspondence between a mode sequence \( \{n_i\} \) in (3.18) and a motif, or equivalently a semi-infinite border strip (see (A.17)), characterizing an irreducible \( Y(\mathfrak{sl}_n) \) highest weight module.
Appendix A. Partitions and symmetric functions

In this appendix we recall the definition of the (skew) Schur functions associated to a partition (see [31] for additional background).

A partition \( \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_r, \ldots) \) is a sequence of non-increasing, non-negative integers \( \lambda_1 \geq \lambda_2 \geq \ldots \geq 0 \) and containing only finitely many non-zero terms. The length \( l(\lambda) \) of \( \lambda \) is the number of non-zero elements in \( \lambda \), and the weight \( |\lambda| \) of \( \lambda \) is defined by \( |\lambda| = \sum_i \lambda_i \). If \( N = |\lambda| \) we say that \( \lambda \) is a partition of \( N \). The set of all partitions of \( N \) is denoted by \( \mathcal{P}_N \), and the set of all partitions by \( \mathcal{P} \). For a partition \( \lambda \in \mathcal{P}_N \), let

\[
m^\lambda_i = \# \{ j : \lambda_j = i \} \tag{A.1}
\]

denote the number of parts of \( \lambda \) that are equal to \( i \). We have

\[
|\lambda| = \sum_{i \geq 1} i m^\lambda_i, \quad l(\lambda) = \sum_{i \geq 1} m^\lambda_i. \tag{A.2}
\]

Sometimes we also denote

\[
\lambda = (1^{m_1}2^{m_2}\ldots r^{m_r}\ldots), \tag{A.3}
\]

where \( m_i \equiv m_i^\lambda \).

As usual, to each partition \( \lambda \) we associate a Young diagram, which we also denote by \( \lambda \). The partition \( \lambda' \), conjugate to \( \lambda \), is then defined by transposing the Young diagram of \( \lambda \) along the main diagonal. Note that

\[
\lambda'_i = \# \{ j : \lambda_j \geq i \}, \tag{A.4}
\]

such that, in particular, \( l(\lambda) = \lambda'_1 \). Moreover

\[
m^\lambda_i = \lambda'_i - \lambda'_{i+1}. \tag{A.5}
\]

For any pair of partitions \( \lambda \) and \( \mu \), we write \( \lambda \supset \mu \) if \( \lambda_i \geq \mu_i \) for all \( i \). If \( \lambda \supset \mu \) then the Young diagram of \( \lambda \) contains the Young diagram of \( \mu \). The skew Young diagram \( \lambda/\mu \) is then obtained as the set-theoretic difference \( \lambda - \mu \). We say that \( \text{rank}(\lambda/\mu) = n \) if the length of any column of \( \lambda/\mu \) does not exceed \( n \), and we put \( |\lambda/\mu| = |\lambda| - |\mu| \).
A path in a skew diagram $\lambda/\mu$ is a sequence $x_0, x_1, \ldots, x_r$ of squares such that $x_{i-1}$ and $x_i$ have a common side, i.e. are adjacent. The diagram $\lambda/\mu$ is said to be connected if any two squares in $\lambda/\mu$ can be connected by a path. Finally, a skew diagram $\lambda/\mu$ is called a border strip if $\lambda/\mu$ is connected and contains no $2 \times 2$ block of squares. We denote the set of border strips of rank $n$ by $\text{BS}_n$.

For given $n \in \mathbb{N}$, a semi-standard tableau of shape $\lambda/\mu$ is an inscription of the numbers $1, 2, \ldots, n$ in each of the boxes of a given skew Young diagram $\lambda/\mu$ such that if $a$ and $b$ are the inscriptions in any pair of adjacent boxes, then

i. $a < b$ if $b$ is lower-adjacent to $a$,

ii. $a \geq b$ if $b$ is left-adjacent to $a$.

We denote the set of semi-standard tableaux of shape $\lambda/\mu$ by $\text{SST}(\lambda/\mu)$. For $T \in \text{SST}(\lambda/\mu)$ we define $m_a(T) = \# \{a : a \in T\}$.

For a skew diagram $\lambda/\mu$ the skew Schur function $s_{\lambda/\mu}(x)$ is now defined as

$$s_{\lambda/\mu}(x) = s_{\lambda/\mu}(x_1, x_2, \ldots, x_n) = \sum_{T \in \text{SST}(\lambda/\mu)} \left( \prod_i x_i^{m_i(T)} \right).$$

(A.6)

Note that the (normal) Schur functions are included in this definition by taking $\mu = \emptyset$.

We have the following expressions for skew Schur functions in terms of the elementary symmetric functions $e_m(x)$ and $h_m(x)$ [31]

$$s_{\lambda/\mu}(x) = \det(h_{\lambda_i - \mu_j - i + j}(x))_{1 \leq i, j \leq r}, \quad \text{(for } r \geq l(\lambda))$$

$$= \det(e_{\lambda'_i - \mu'_j - i + j}(x))_{1 \leq i, j \leq s}, \quad \text{(for } s \geq l(\lambda')).$$  

(A.7)

Note that, in particular,

$$h_m(x) = s_{(m)}(x), \quad e_m(x) = s_{(1^m)}(x).$$  

(A.8)

The skew Schur functions can be expressed in terms of standard Schur functions by means of the Littlewood-Richardson rule

$$s_{\lambda/\mu}(x) = \sum_{\nu} c_{\lambda\mu}^\nu s_{\nu}(x).$$  

(A.9)
Now, let $\lambda/\mu \in BS_n$. Let $r$ and $s$ be the number of rows and columns of $\lambda/\mu$, respectively. Denote the length of the $i$-th row and column by $a_i$ and $b_i$, respectively. Then

$$a_i = \lambda_i - \mu_i,$$

$$b_i = \lambda'_{s-i+1} - \mu'_{s-i+1}.$$  \hfill (A.10)

We denote $\lambda/\mu = \langle a_1, \ldots, a_r \rangle = [b_1, \ldots, b_s]$. Then, (A.7) gives

$$s_{\lambda/\mu}(x) = s_{\langle a_1, \ldots, a_r \rangle}(x) = \sum_{k_1, \ldots, k_n \geq 0} x_1^{k_1} \cdots x_n^{k_n},$$

$$s_{\lambda/\mu}(x) = s_{\langle a_1, \ldots, a_r \rangle}(x) = \sum_{k_1, \ldots, k_n \geq 0} x_1^{k_1} \cdots x_n^{k_n},$$

$$s_{\lambda/\mu}(x) = s_{\langle a_1, \ldots, a_r \rangle}(x) = \sum_{k_1, \ldots, k_n \geq 0} x_1^{k_1} \cdots x_n^{k_n}.$$  \hfill (A.12)

Using that

$$h_a(x) = \sum_{k_1, \ldots, k_n \geq 0, \sum k_i = a} x_1^{k_1} \cdots x_n^{k_n},$$

we find, for $n = 2$ and $a_1, a_2 \geq 1$,

$$h_{a_1}h_{a_2} - h_{a_1+a_2} = h_{a_1-1}h_{a_2-1}.$$  \hfill (A.13)

This gives the following drastic simplification of (A.11) for $n = 2$ (and $a_1, a_r \geq 1$, $a_2, \ldots, a_{r-1} \geq 2$)

$$s_{\langle a_1, \ldots, a_r \rangle} = h_{a_1-1}h_{a_2}h_{a_3} \cdots h_{a_{r-2}}h_{a_{r-1}}.$$  \hfill (A.14)

Note furthermore that, e.g. because of (A.6), we have

$$s_{[b_1, \ldots, b_s, n]}(x) = s_{[b_1, \ldots, b_s]}(x).$$  \hfill (A.15)

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We may thus extend the definition of skew Schur functions to semi-infinite border strips, i.e. border strips stabilizing on columns of length \( n \), by

\[
s_{[b_1, \ldots, b_s, n, n, n, \ldots]}(x) \equiv s_{[b_1, \ldots, b_s]}(x).
\] (A.16)

It is convenient to introduce yet two other parametrizations of \( \text{BS}_n \) which have their origin in integrable spin chain models. The first parametrization is through a semi-infinite sequence of rapidities \( \{m_s\} \), i.e. a sequence of distinct positive integers \( 0 < m_1 < m_2 < m_3 < \ldots \) such that there are no more than \( n - 1 \) consecutive \( m_i \)'s and such that the sequence stabilizes on the sequence \( \{\ldots, a + 1, a + 2, \ldots, a + (n - 1), a + (n + 1), \ldots, a + (2n - 1), a + (2n + 1), \ldots\} \) for some \( a \). Denote the set of such semi-infinite rapidity sequences by \( \mathcal{R}_n^{\infty/2} \). If \( a \equiv k \mod n \) for some \( 0 \leq k \leq n - 1 \) then we have a rapidity sequence of conjugacy class \( k \). The vacuum rapidity sequence \( \{m_s^{(k)}\} \), in conjugacy class \( k \), is given by \( \{m_s^{(k)}\} = \{1, 2, 3, \ldots, k-1, k+1, \ldots, k+n-1, k+n+1, \ldots\} \). The second parametrization is through a semi-infinite motif of rank \( n \), i.e. a sequence \( (d_1, d_2, d_3, \ldots) \) with \( d_i \in \{0, 1\} \), such that there are no more than \( n - 1 \) consecutive 1's and such that the sequence stabilizes on the sequence \( (\ldots, (0, 1, \ldots, 1)^\infty) \). We denote the set of such motifs by \( \mathcal{M}_n^{\infty/2} \). Clearly, we have an isomorphism \( \mathcal{R}_n^{\infty/2} \cong \mathcal{M}_n^{\infty/2} \) by letting \( d_i = 1 \) iff \( i \in \{m_s\} \) and \( d_i = 0 \) otherwise. We also have \( \mathcal{R}_n^{\infty/2} \cong \text{BS}_n \) by associating a semi-infinite border strip \( \langle a_1, a_2, \ldots \rangle \in \text{BS}_n \), defined by

\[
a_i = m_i - m_{i-1}, \quad i = 1, 2, \ldots,
\] (A.17)

where \( m_0 \equiv 0 \) with \( \{m_s\} \in \mathcal{R}_n^{\infty/2} \).

Alternatively, given a motif \( (d_1, d_2, \ldots) \in \mathcal{M}_n^{\infty/2} \), the corresponding border strip \( \kappa \in \text{BS}_n \) is constructed as follows: Write down a square, call this the 0-th square. Now, by induction, write the \( i \)-th square \( (i \in \mathbb{N}) \) under (resp., left to) the \( i - 1 \)-th square if \( d_i = 1 \) (resp., \( d_i = 0 \)).

**Appendix B. The Lie algebras \( \mathfrak{sl}_n \) and \( \widehat{\mathfrak{sl}}_n \)**

In this appendix we collect some results regarding the Lie algebra \( \mathfrak{sl}_n \) and its (untwisted) affinization \( \widehat{\mathfrak{sl}}_n \). We will be rather brief, our main purpose is to explain the notation used throughout the paper.
The Lie algebra \( \mathfrak{gl}_n \) is the algebra of \( n \times n \) (complex) matrices. A basis of \( \mathfrak{gl}_n \) is given by the matrix units \( e_{ij}, 1 \leq i, j \leq n \), with components \( (e_{ij})_{kl} = \delta_{ik}\delta_{jl} \) and commutators

\[
[e_{ij}, e_{kl}] = \delta_{jk}e_{il} - \delta_{li}e_{kj}.
\]

(B.1)

The Cartan subalgebra \( \mathfrak{h} \) of \( \mathfrak{gl}_n \) is spanned by the \( e_{ii}, 1 \leq i \leq n \), i.e. \( h \in \mathfrak{h} \) if and only if \( h = \sum a_i e_{ii} \). Let \( \tilde{\varepsilon}_i, i = 1, \ldots, n \), be the orthonormal basis of \( \mathfrak{h}^* \) defined by \( \tilde{\varepsilon}_i(e_{jj}) = \delta_{ij} \).

Now, let \( V \) be any \( \mathfrak{gl}_n \) module. Since the Cartan subalgebra is semi-simple, it acts diagonally on \( V \). We thus have a decomposition of \( V \) as a direct sum

\[
V \simeq \bigoplus_{\lambda \in \mathfrak{h}^*} V_\lambda,
\]

(B.2)

of eigenspaces \( V_\lambda = \{ v \in V | h \cdot v = \lambda(h)v \} \). We call \( \lambda \) a weight if \( V_\lambda \neq \emptyset \) and we call \( V_\lambda \) a weight space.

The (abstract) character \( \text{ch}_V \) of a finite-dimensional \( \mathfrak{gl}_n \) module \( V \) is defined as

\[
\text{ch}_V = \sum \dim(V_\lambda) e^\lambda.
\]

(B.3)

A vector \( v \in V \) is called a singular vector of weight \( \lambda \) when \( v \in V_\lambda \) and \( e_{ij} \cdot v = 0 \) for all \( i < j \). A highest weight module \( V \) is a module which possesses a singular vector \( v \) such that \( V = U(\mathfrak{gl}_n) \cdot v \). All irreducible highest weight modules are therefore uniquely characterized by the weight \( \lambda \) of the highest weight vector. We will denote the irreducible highest weight module with highest weight \( \lambda \) by \( L(\lambda) \). We can expand \( \lambda = \sum \lambda_i \tilde{\varepsilon}_i \), i.e. \( \lambda_i = (\lambda, \tilde{\varepsilon}_i) \). The module \( L(\lambda) \) is finite-dimensional if and only if \( \lambda_i - \lambda_{i+1} \in \mathbb{Z}_{\geq 0} \) for all \( i = 1, \ldots, n-1 \).

The Weyl group \( W(\mathfrak{gl}_n) \simeq S_n \) of \( \mathfrak{gl}_n \) acts by permutations on the \( \tilde{\varepsilon}_i \) thus, upon defining \( x_i = e^{\tilde{\varepsilon}_i} \), we find that the character \( \text{ch}_V(x) \) is a symmetric function in the \( x_i \).

The Lie algebra \( \mathfrak{sl}_n \) is the subalgebra of \( \mathfrak{gl}_n \) consisting of traceless \( n \times n \) matrices. A Chevalley basis for \( \mathfrak{sl}_n \), i.e. a set of generators satisfying

\[
[h_i, h_j] = 0, \\
[h_i, x^\pm_j] = \pm a_{ij} x^\pm_j, \\
[x^+_i, x^-_j] = \delta_{ij} h_i, \\
(ad x^\pm_i)^{1-a_{ij}} x^\pm_j = 0,
\]

(B.4)
is given by
\[ x_i^+ = e_{ii+1}, \quad x_i^- = e_{i+1i}, \quad h_i = e_{ii} - e_{i+1i+1}, \quad i = 1, \ldots, n-1. \tag{B.5} \]

Here \( a_{ij} = 2\delta_{ij} - \delta_{i-1j} - \delta_{i+1j} \) is the Cartan matrix of \( \mathfrak{sl}_n \).

An (overcomplete) basis for the dual Cartan subalgebra \( \mathfrak{h}^* \) of \( \mathfrak{sl}_n \) is given by \( \epsilon_i \), \( i = 1, \ldots, n \), defined by \( \epsilon_i(\sum_j a_{jj}e_{jj}) = a_{ii} \). They satisfy the constraint \( \sum_i \epsilon_i = 0 \). Note that in terms of the corresponding basis for \( \mathfrak{gl}_n \) we have
\[ \epsilon_i = \tilde{\epsilon}_i - \frac{1}{n} \sum_j \tilde{\epsilon}_j. \tag{B.6} \]

Consequently,
\[ (\epsilon_i, \epsilon_j) = \delta_{ij} - \frac{1}{n}. \tag{B.7} \]

In terms of the \( \epsilon_i \), the simple roots \( \alpha_i \) of \( \mathfrak{sl}_n \) are given by
\[ \alpha_i = \epsilon_i - \epsilon_{i+1}, \quad i = 1, \ldots, n-1, \tag{B.8} \]
while the fundamental weights \( \Lambda_i \), defined by \( (\Lambda_i, \alpha_j) = \delta_{ij} \), are given by
\[ \Lambda_k = \epsilon_1 + \epsilon_2 + \ldots + \epsilon_k, \quad k = 1, \ldots, n-1. \tag{B.9} \]

Furthermore, the inverse Cartan matrix is given by
\[ (a^{-1})_{ij} = (\Lambda_i, \Lambda_j) = \min(i, j) - \frac{1}{n} (ij). \tag{B.10} \]

In particular
\[ \frac{1}{2} |\Lambda_k|^2 = \frac{k(n-k)}{2n}. \tag{B.11} \]

The weight lattice, i.e. the lattice spanned by the fundamental weights \( \Lambda_i \) is denoted by \( P \), while the set of weights in \( P \) of conjugacy class \( k \), i.e. those weights \( \sum m_i \Lambda_i \) such that \( \sum im_i \equiv k \mod n \) is denoted by \( P^{(k)} \). The root lattice \( P^{(0)} \) is also denoted by \( Q \).

The irreducible highest weight modules \( L(\lambda) \) are labeled by a highest weight vector \( \lambda = \sum m_i \Lambda_i \) and are finite-dimensional if and only if \( m_i \in \mathbb{Z}_{\geq 0} \), i.e. iff \( \lambda \) is a dominant integral weight. The set of dominant integral weights is denoted by \( P_+ \). The irreducible
finite-dimensional modules are thus in 1–1 correspondence with Young diagrams of rank $n-1$, i.e. $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_{n-1})$ where $\lambda_i = m_i + m_{i+1} + \ldots + m_{n-1}$. Equivalently, $\lambda_i = (\lambda, \epsilon_i - \epsilon_n)$. Also note that the $\epsilon_i, i = 1, \ldots, n$ are the weights of the $n$-dimensional fundamental irreducible representation $L(\Lambda_1)$ of $\mathfrak{sl}_n$.

Again, upon identifying $x_i = e^{\epsilon_i}$, the character of $L(\lambda)$ becomes a symmetric function in the $x_i$ (here it is implicitly understood that $x_1 \ldots x_n = 1$ because of the constraint $\sum \epsilon_i = 0$). In fact,

$$\text{ch}_{L(\lambda)}(x) = s_\lambda(x). \quad (B.12)$$

In particular we have

$$\text{ch}_{L(\Lambda_m)}(x) = e_m(x), \quad \text{ch}_{L(m\Lambda_1)}(x) = h_m(x). \quad (B.13)$$

For any simple Lie algebra $\mathfrak{g}$ (in this paper we will only consider $\mathfrak{g} \cong \mathfrak{sl}_n$), the untwisted affine Lie algebra $\hat{\mathfrak{g}}$ is defined as a central extension of the loop algebra $\mathfrak{g} \otimes \mathbb{C}[t^{-1}, t]$, i.e.

$$\hat{\mathfrak{g}} = (\mathfrak{g} \otimes \mathbb{C}[t^{-1}, t]) \oplus \mathbb{C} \ell \oplus L_0, \quad (B.14)$$

where $\ell$ is the central element and $L_0$ a derivation. To be precise, for $x, y \in \mathfrak{g}$, $m, n \in \mathbb{Z}$,

$$[x \otimes t^m, y \otimes t^n] = [x, y] \otimes t^{m+n} + m \ell \delta_{m+n,0}(x, y),$$
$$[L_0, x \otimes t^m] = -m(x \otimes t^m),$$
$$[\ell, x \otimes t^m] = [\ell, L_0] = 0. \quad (B.15)$$

Henceforth we will write $x_m$ for the element $x \otimes t^m$ of $\hat{\mathfrak{g}}$. The subalgebra consisting of the elements $x_0 = x \otimes t^0$ is identified with $\mathfrak{g}$. In the remainder we consider $\mathfrak{g} = \mathfrak{sl}_n$. The fundamental weights of $\hat{\mathfrak{sl}}_n$ are denoted by $\hat{\Lambda}_i, i = 0, \ldots, n-1$.

In each irreducible $\hat{\mathfrak{g}}$ module $V$, the central element $\ell$ acts by a constant (also to be denoted by $\ell$) which is referred to as the level of $V$.

In this review we consider two important classes of modules of $(\hat{\mathfrak{sl}}_n)_\ell$. For $\ell = 0$ we have the so-called evaluation modules $L(\Lambda)((z)) \equiv \text{ev}_z^*(L(\Lambda))$ defined by pulling back a finite dimensional irreducible $\mathfrak{sl}_n$ module $L(\Lambda)$ by means of the evaluation homomorphism $\text{ev}_z : U(\mathfrak{sl}_n \otimes \mathbb{C}[t^{-1}, t]) \to U(\mathfrak{sl}_n)$ defined by $\text{ev}_z(x_m) = z^m x$. 


The other important class of modules are the so-called integrable highest weight modules. They exist for \( \ell \in \mathbb{N} \) and highest weights \( \hat{\Lambda} = \sum_{i=0}^{n-1} m_i \hat{\Lambda}_i, m_i \in \mathbb{Z}_{\geq 0} \), such that \( \sum m_i = \ell \). Let us denote the set of such weights by \( \hat{P}^\ell_+ \).

The weight space decomposition of an integrable highest weight module \( V \) with highest weight \( \hat{\Lambda} \in \hat{P}^\ell_+ \) is now defined with respect to \( \mathfrak{h} \), the Cartan subalgebra of \( \mathfrak{sl}_n \), and \( L_0 \). That is, \( V \simeq \oplus V_{\lambda,n} \) where

\[
V_{\lambda,n} = \{ v \in V | h \cdot v = \lambda(h) v, \ L_0 \cdot v = (\Delta(\Lambda) + n)v \}.
\] (B.16)

where

\[
\Delta(\hat{\Lambda}) = \frac{(\Lambda, \Lambda + 2\rho)}{2(\ell + n)}
\] (B.17)

is the conformal dimension of the highest weight vector of \( L(\hat{\Lambda}) \). Here, \( \Lambda \) is the projection of the \( \hat{\mathfrak{sl}}_n \) weight \( \hat{\Lambda} \) onto \( \mathfrak{h}^* \) and \( \rho \) is the Weyl vector of \( \mathfrak{sl}_n \). In particular, for \( \ell = 1 \), we have (cf. (B.11))

\[
\Delta_k \equiv \Delta(\hat{\Lambda}_k) = \frac{k(n-k)}{2n}.
\] (B.18)

The character of the integrable highest weight module \( L(\hat{\Lambda}) \) is defined as

\[
\text{ch}_{L(\hat{\Lambda})} = q^{\Delta(\hat{\Lambda})} \sum_{\lambda,n} \dim(L(\hat{\Lambda})_{\lambda,n}) q^n e^\lambda.
\] (B.19)

The characters of the level-1 integrable highest weight modules \( L(\hat{\Lambda}_k), k = 0, \ldots, n-1 \) of \( \hat{\mathfrak{sl}}_n \) are given by

\[
\text{ch}_{L(\hat{\Lambda}_k)} = \sum_{\lambda \in P(k)} q^{\frac{1}{2}|\lambda|^2} \frac{1}{(q)_{n-1}^{k} e^\lambda},
\] (B.20)

or, more explicitly,

\[
\text{ch}_{L(\hat{\Lambda}_k)}(x; q) = \sum_{k_1, \ldots, k_n \in \mathbb{Z}_{\geq 0}} q^{\frac{1}{2}(k_1^2 + \ldots + k_n^2)} (q)_{n-1}^{k} x_1^{k_1} \ldots x_n^{k_n}.
\] (B.21)

Chiral vertex operators (CVO’s) at level-\( \ell \) are \( \hat{\mathfrak{sl}}_n \) intertwiners

\[
\Phi \left( \begin{array}{c} \lambda_3 \\ \lambda_2 \\ \lambda_1 \end{array} \right) : L(\hat{\Lambda}_1) \otimes L(\lambda_3)(z) \rightarrow L(\hat{\Lambda}_2),
\] (B.22)
where $\lambda_3 \in P_+$ and $\hat{\lambda}_i \in \hat{P}_+^\ell$ for $i = 1, 2$. It is convenient to think of CVO’s as a collection of linear maps

$$\Phi^i\left(\frac{\lambda_3}{\lambda_2 \lambda_1}\right)(z) : L(\hat{\lambda}_1) \to L(\hat{\lambda}_2),$$

(B.23)

transforming under $(\hat{\mathfrak{s}l}_n)_0$ as

$$[x_m, \Phi^i\left(\frac{\lambda_3}{\lambda_2 \lambda_1}\right)(z)] = z^m \rho(x)^i_j \cdot \Phi^j\left(\frac{\lambda_3}{\lambda_2 \lambda_1}\right)(z),$$

(B.24)

where $\rho(x)^i_j$ is the action of $x$ in the $\mathfrak{s}l_n$ irreducible representation $L(\lambda_3)$. A CVO $\Phi\left(\frac{\lambda_3}{\lambda_2 \lambda_1}\right)$ exists if and only if $\lambda_2$ occurs in the fusion rule $\lambda_1 \times \lambda_3$. The CVO’s for $\lambda_3 = \Lambda_1$ will be referred to as spinon operators. Since the level-1 fusion rules read $\Lambda_k \otimes \Lambda_1 = \Lambda_{k+1}$ (here the subscripts are taken modulo $n$) the only non-vanishing spinon operators at $\ell = 1$ are

$$\Phi^i\left(\frac{\Lambda_1}{\Lambda_{k+1} \Lambda_k}\right)(z).$$

(B.25)

Appendix C. Yangians and their representations

Yangians were introduced by Drinfel’d [16,17] (see [18,14] for reviews). Here we will briefly review their definition and some aspects of their representation theory. For definiteness we will restrict ourselves to the algebras $\mathfrak{s}l_n$ and $\mathfrak{gl}_n$, the cases of interest to this paper, but most of what follows applies to arbitrary $\mathfrak{g}$ as well.

The Yangian $Y(\mathfrak{g})$ is defined as the (unique) flat deformation of the loop algebra $U(\mathfrak{g}[t])$ in the class of Hopf algebras. There are three equivalent realizations, each useful for different purposes.

**Definition C.1** [16]. The Yangian $Y(\mathfrak{g}_n)$ is defined to be the Hopf algebra generated by elements $t_{ij}^{(p)}$, $i, j = 1, \ldots, n$, $p \in \mathbb{Z}_{\geq 0}$, with relations

$$[t_{ij}^{(p+1)}, t_{kl}^{(q)}] - [t_{ij}^{(p)}, t_{kl}^{(q+1)}] = -(t_{kj}^{(p)} t_{il}^{(q)} - t_{kj}^{(q)} t_{il}^{(p)}),$$

(C.1)

where we have defined $t_{ij}^{(-1)} = \delta_{ij}$.

Let $t_{ij}(u) \in Y(\mathfrak{g}_n)[[u^{-1}]]$ be defined by $t_{ij}(u) = \delta_{ij} + \sum_{p \geq 0} t_{ij}^{(p)} u^{-p-1}$. Then, after introducing an ‘auxiliary space’ $\mathbb{C}^n$, and writing $t(u) = \sum_{i,j} t_{ij}(u) \otimes e_{ij}$, where $e_{ij}$ is the
matrix with components \( (e_{ij})_{kl} = \delta_{ik}\delta_{jl} \), we can also write (C.1) as

\[
R(u - v)(t(u) \otimes 1)(1 \otimes t(v)) = (1 \otimes t(v))(t(u) \otimes 1)R(u - v),
\]

(C.2)

where \( R(u) = 1 + \frac{1}{u} \sum e_{ij} \otimes e_{ji} \in \text{End}(\mathbb{C}^n \otimes \mathbb{C}^n) \). The matrix \( R(u) \) corresponds to the simplest (rational) solution of the Yang-Baxter equation.

The co-multiplication \( \Delta \), co-unit \( \epsilon \), and anti-pode \( S \) are given by

\[
\Delta(t_{ij}(u)) = \sum_k t_{ik}(u) \otimes t_{kj}(u),
\]
\[
\epsilon(t_{ij}(u)) = \delta_{ij},
\]
\[
S(t(u)) = t(u)^{-1}.
\]

(C.3)

Theorem C.2 [16]. The center \( Z(Y(\mathfrak{gl}_n)) \) of \( Y(\mathfrak{gl}_n) \) is generated by the quantum determinant

\[
det_q(t(u)) = \sum_{\pi \in S_n} \text{sgn}(\pi)t_{1\pi(1)}(u + \frac{n-1}{2}) \ldots t_{n\pi(n)}(u - \frac{n-1}{2})
\]

(C.4)

and

\[
\Delta(det_q(t(u))) = det_q(t(u)) \otimes det_q(t(u)).
\]

(C.5)

The theorem shows that we can define \( Y(\mathfrak{sl}_n) \) as the Hopf subalgebra of \( Y(\mathfrak{gl}_n) \) consisting out of those \( t(u) \) with \( \det_q(t(u)) = 1 \).

Note that we have a family of automorphisms \( \tau_a : Y(\mathfrak{gl}_n) \to Y(\mathfrak{gl}_n) \ (a \in \mathbb{C}) \) defined by

\[
\tau_a(t_{ij}(u)) = t_{ij}(u - a).
\]

(C.6)

The second realization of \( Y(\mathfrak{sl}_n) \) is given by

Definition C.3 [17]. The Yangian \( Y(\mathfrak{sl}_n) \) is defined to be the Hopf algebra with generators \( \{x_{ik}^\pm, h_{ik}\}, i = 1, \ldots, n - 1, k \in \mathbb{Z}_{\geq 0} \), and relations

\[
[h_{ik}, h_{jl}] = 0, \quad [h_{i0}, x_{j}^\pm] = \pm a_{ij} x_{j}^\pm, \quad [x_{ik}^+, x_{jl}^-] = \delta_{ij} h_{ik+l},
\]

(C.7)

\[
[h_{ik+1}, x_{j}^\pm] - [h_{ik}, x_{j}^\pm_{l+1}] = \pm \frac{1}{2} a_{ij}(h_{ik} x_{j}^\pm + x_{j}^\pm h_{ik}),
\]

(C.8)

\[
[x_{ik+1}^\pm, x_{j}^\pm] - [x_{ik}^\pm, x_{j}^\pm_{l+1}] = \pm \frac{1}{2} a_{ij}(x_{ik}^\pm x_{j}^\pm + x_{j}^\pm x_{ik}^\pm),
\]

(C.9)
\[
\sum_{\pi \in S_m} [x_{ik_{\pi(1)}}, [x_{ik_{\pi(2)}}, \ldots [x_{ik_{\pi(m)}}, x_j] \ldots]] = 0, \quad \text{where } m = 1 - a_{ij}, i \neq j. \quad (C.10)
\]

where \(a_{ij}\) is the Cartan matrix of \(\mathfrak{sl}_n\).

Again, it is sometimes convenient to work with the generating series

\[
h_i(u) = 1 + \sum_{k \geq 0} h_{ik} u^{-k-1}, \quad x_i^\pm(u) = \sum_{k \geq 0} x_{ik} u^{-k-1}. \quad (C.11)
\]

For example, a family of Hopf algebra automorphisms \(\tau_a, a \in \mathbb{C}\), is defined by

\[
\tau_a(x_i^+(u)) = x_i^+(u - a), \quad \tau_a(h_i(u)) = h_i(u - a). \quad (C.12)
\]

In order to exhibit the isomorphism between the realizations C.1 and C.3, define

\[
a_i(u) = (\det_q t_{rs}(u))_{r, s \in \{1, \ldots, i\}}, \quad a_0(u) = 1,
\]

\[
b_i(u) = (\det_q t_{rs}(u))_{r \in \{1, \ldots, i\}, s \in \{1, \ldots, i-1, i+1\}},
\]

\[
c_i(u) = (\det_q t_{rs}(u))_{r \in \{1, \ldots, i-1, i+1\}, s \in \{1, \ldots, i\}},
\]

where \(\det_q t(u)\) is the quantum determinant defined in (C.4). Then we have

**Theorem C.4** [17]. A (non-canonical) isomorphism between the definitions C.1 and C.3 of \(Y(\mathfrak{sl}_n)\) is given by

\[
x_i^+(u) = b_i(u)a_i(u)^{-1},
\]

\[
x_i^-(u) = a_i(u)^{-1}c_i(u),
\]

\[
h_i(u) = a_i(u)^{-1}a_i(u + 1)^{-1}a_{i-1}(u + \frac{1}{2})a_{i+1}(u + \frac{1}{2}). \quad (C.14)
\]

Finally, the third realization of \(Y(\mathfrak{sl}_n)\) is

**Definition C.5** [16]. The Yangian \(Y(\mathfrak{sl}_n)\) is defined to be the Hopf algebra generated by \(x, J(x), x \in \mathfrak{sl}_n\), where \(J\) is a linear functional on \(\mathfrak{sl}_n\), with relations

\[
[x, J(y)] = J([x, y]),
\]

\[
[J(x), J([y, z])] + \text{cycl.} = \frac{1}{4} \sum_{a, b, c} ([x, I_a], [[y, I_b], [z, I_c]]) I_a I_b I_c,
\]

\[-23-\]
\[
[[J(x), J(y)], [z, J(w)]] + [[J(z), J(w)], [x, J(y)]]
= \frac{1}{4} \sum_{a,b,c} \left(([[x, I_a], [[y, I_b], [[z, w], I_c]]]) + ([z, I_a], [[w, I_b], [[x, y], I_c]])\right)\{I_a, I_b, J(I_c)\}.
\]

where \{I_a\} denotes an orthonormal basis of \(\mathfrak{sl}_n\) with respect to the Killing form \(\langle , \rangle\), and

\[
\left\{x_1, \ldots, x_n\right\} = \frac{1}{n!} \sum_{\pi \in S_n} x_{\pi(1)} \cdots x_{\pi(n)},
\]

denotes the symmetrizer.

Let us now discuss the representations of \(Y(\mathfrak{sl}_n)\). We will be interested in the finite-dimensional representations in particular.

**Definition C.6.** A representation \(V\) of \(Y(\mathfrak{sl}_n)\) is said to be a highest weight module if it is generated by a vector \(v \in V\), i.e. \(V = Y(\mathfrak{sl}_n) \cdot v\), such that

\[
x_{ik}^+ \cdot v = 0, \quad h_{ik} \cdot v = d_{ik}v,
\]

for all \(i = 1, \ldots, n-1, k \in \mathbb{Z}_{\geq 0}\) and some \(d = d_{ik} \in \mathbb{C}\).

As usual, the Verma module \(M(d)\) is universal in the class of highest weight modules, i.e. every highest weight module is a quotient module of the Verma module, and \(M(d)\) has a unique irreducible quotient \(L(d)\).

For a highest weight module \(V(d)\) define \(d_i(u) \in \mathbb{C}[[u^{-1}]]\) by \(h_i(u) \cdot v = d_i(u)v\), i.e.

\[
d_i(u) = 1 + \sum_{k \geq 0} d_{ik} u^{-k-1}.
\]

The classification of finite-dimensional representations of \(Y(\mathfrak{sl}_n)\) is due to Drinfel’d

**Theorem C.7 [17].**

(i) Every finite-dimensional irreducible representation of \(Y(\mathfrak{sl}_n)\) is a highest weight module.

(ii) The irreducible representation \(L(d)\) is finite-dimensional if and only if there exist polynomials \(P_i(u) \in \mathbb{C}[u]\), \(i = 1, \ldots, n-1\), such that

\[
d_i(u) = \frac{P_i(u+1)}{P_i(u)}.
\]
Corollary C.8. There exists a 1–1 correspondence between finite dimensional irreducible representations of $Y(\mathfrak{sl}_n)$ and sets of monic polynomials $P_i(u) \in \mathbb{C}[u]$, $i = 1, \ldots, n-1$.

Note that for a finite-dimensional irreducible module $V$ associated to Drinfel’d polynomials $P_i(u)$, the module $\tau_a(V)$ is also finite-dimensional, irreducible and the associated polynomials are $P_i(u-a)$ (see (C.12)).

Recall that for $U(\mathfrak{g}[t])$ we can easily construct finite-dimensional irreducible representations by pulling back finite-dimensional irreducible representations $V$ of $U(\mathfrak{g})$ by means of the evaluation homomorphism $ev_a : U(\mathfrak{g}[t]) \to U(\mathfrak{g})$ defined by $ev_a(x \otimes t^k) = a^k x$ for some $a \in \mathbb{C}$. A representation $V(a) = ev_a^*(V)$ of $U(\mathfrak{g}[t])$ is called an evaluation representation. It turns out, moreover, that every finite-dimensional irreducible representation of $U(\mathfrak{g}[t])$ is isomorphic (as a $U(\mathfrak{g})$-module) to a tensor product of evaluation representations [13]. Since $Y(\mathfrak{g})$ is a deformation of $U(\mathfrak{g}[t])$ it is natural to ask whether a similar construction of all finite dimensional irreducibles exists for $Y(\mathfrak{g})$. For general $\mathfrak{g}$, an evaluation homomorphism does not exist and, consequently, finite-dimensional irreducible representations $V$ of $U(\mathfrak{g})$ in general do not extend to representations of $Y(\mathfrak{g})$. It turns out that usually one can extend $V$ by ‘smaller’ representations such that one can define an irreducible action of $Y(\mathfrak{g})$ on the extension. It is an important open problem to find the minimal such extension. For $\mathfrak{g} = \mathfrak{sl}_n$ we have, however,

Theorem C.9. Any finite-dimensional representation $(V, \rho)$ of $\mathfrak{gl}_n$ extends to a representation of $Y(\mathfrak{gl}_n)$. Explicitly,

$$\rho(t_{ij}(u)) = 1 - \frac{1}{u} \rho(e_{ij}). \quad (C.18)$$

On a highest weight vector $v \in V$ of an irreducible $Y(\mathfrak{gl}_n)$ highest weight module, $t(u)$ takes the form

$$t(u) \cdot v = \begin{pmatrix} t_{11}(u) & 0 & 0 & \cdots & 0 \\ * & t_{22}(u) & 0 & \cdots & 0 \\ & \vdots & & \ddots & \vdots \\ * & * & \cdots & t_{nn}(u) \\ & & & & \end{pmatrix} \cdot v. \quad (C.19)$$

Thus, it follows from (C.14) that

$$d_i(u) = \frac{t_{i+1i+1}(u - \frac{i-1}{2})}{t_{ii}(u - \frac{i-1}{2})}, \quad (C.20)$$
which provides the connection of the Drinfel’d polynomials (C.17) with the diagonal eigenvalues of \( t(u) \) on the highest weight vector. In particular, for the irreducible representations \( L(\lambda) (\lambda_i = (\lambda, \tilde{\epsilon}_i)) \), of \( Y(\mathfrak{gl}_n) \) given by Theorem C.9, we find

\[
d_i(u) = \frac{u - \frac{i-1}{2} - \lambda_{i+1}}{u - \frac{i-1}{2} - \lambda_i} = 1 + (\lambda_i - \lambda_{i+1}) \frac{1}{u} + O\left(\frac{1}{u^2}\right),
\]

such that

\[
P_i(u) = \prod_{j=0}^{\lambda_i - \lambda_{i+1} - 1} \left( u - \frac{i - 1}{2} - \lambda_i + j \right)
= \prod_{\lambda_j' = i}^{\lambda_i} \left( u - \frac{1}{2} \lambda_j' - j + \frac{1}{2} \right).
\]  

(C.22)

Note that the roots of the polynomials \( P_i(u) \) for the evaluation representations of Theorem C.9 form a string on the real axis of length \( \lambda_i - \lambda_{i+1} \), i.e. a set of the form \( \{a, a+1, \ldots, a+(n-1)\} \) where \( n = \lambda_i - \lambda_{i+1} \).

For \( Y(\mathfrak{sl}_2) \) we have the following analogue of the abovementioned theorem for \( U(\mathfrak{sl}_2[t]) \)

**Theorem C.10 [12].** Every finite-dimensional representation of \( Y(\mathfrak{sl}_2) \) is isomorphic to a tensor product of evaluation representations \( L(\lambda)(a) \).

In fact, for a given Drinfel’d polynomial \( P(u) \), the corresponding (irreducible) tensor product is easily obtained by collecting the roots of \( P(u) \) in strings and associating an evaluation representation with each string (see [12] for a more precise statement).

For \( Y(\mathfrak{sl}_n) \), \( n \geq 3 \) the situation is not that simple. In fact, the character of the irreducible module associated to an arbitrary set of monic polynomials \( P_i(u), i = 1, \ldots, n-1 \) (see Corollary C.8) is not known in general. Fortunately, for our purposes it suffices to consider a special subclass, the so-called tame modules [34]. They are defined as follows: Consider the canonical filtration

\[
Y(\mathfrak{gl}_1) \subset Y(\mathfrak{gl}_2) \subset \ldots \subset Y(\mathfrak{gl}_n).
\]

(C.23)

The algebra \( A(\mathfrak{gl}_n) \), generated by all the centres of all the algebras in the chain (C.23), is a maximally commutative subalgebra of \( Y(\mathfrak{gl}_n) \), the so-called Gel’fand-Zetlin algebra. A \( Y(\mathfrak{gl}_n) \)-module \( V \) is called tame, if the subalgebra \( A(\mathfrak{gl}_n) \) acts semi-simply on \( V \). In fact,
it turns out that the action of $A(\mathfrak{gl}_n)$ in every irreducible module is simple. The eigenbasis of $A(\mathfrak{gl}_n)$ in such an irreducible module is called a Gel’fand-Zetlin basis.

**Theorem C.11** [35]. Let $N \in \mathbb{Z}_{\geq 0}$ be arbitrary.

i. We have a homomorphism

$$\iota : Y(\mathfrak{gl}_n) \to [U(\mathfrak{gl}_{N+n})]^{\mathfrak{gl}_N},$$

where the right hand side denotes the commutant of $\mathfrak{gl}_N$ in $U(\mathfrak{gl}_{N+n})$. This homomorphism becomes injective for $N \to \infty$.

ii. The image of the homomorphism $\iota$, together with the centre $Z(\mathfrak{gl}_N)$, generates the commutant.

Now, for any dominant integral weights $\lambda$ and $\mu$ of the Lie algebras $\mathfrak{gl}_{N+n}$ and $\mathfrak{gl}_N$, respectively, denote by $L(\lambda, \mu)$ the subspace of all $\mathfrak{gl}_N$ singular vectors of weight $\mu$ in the irreducible $\mathfrak{gl}_{N+n}$ module $L(\lambda)$ of highest weight $\lambda$, i.e. the multiplicity of $L(\mu)$ in $L(\lambda)$ under the canonical embedding $\mathfrak{gl}_N \subset \mathfrak{gl}_{N+n}$. Note that, in particular, $L(\lambda, \mu) \neq \emptyset$ implies $\lambda \supset \mu$ as an inclusion of Young diagrams.

The homomorphism of Theorem C.11 equips $L(\lambda, \mu)$ with the structure of a $Y(\mathfrak{gl}_n)$ module. In fact

**Theorem C.12** [34].

i. $L(\lambda, \mu)$ is an irreducible tame $Y(\mathfrak{gl}_n)$ module.

ii. Every irreducible tame $Y(\mathfrak{gl}_n)$ module splits into a tensor product of modules of the form $L(\lambda, \mu)(a) = \tau_a(L(\lambda, \mu))$.

In fact, it is possible to explicitly describe the action of $Y(\mathfrak{gl}_n)$ on $L(\lambda, \mu)$ on a basis that diagonalizes the GZ-algebra $A(\mathfrak{gl}_n)$. Such a basis is labeled by the GZ-schemes of $L(\lambda, \mu)$. A GZ-scheme $\Lambda$ of $L(\lambda, \mu)$ is an array of (non-negative) integers $\lambda_{m,i}$, $m = 0, \ldots, n$, $i = 1, \ldots, N + m$, i.e.

$$\Lambda = \begin{array}{cccc}
\lambda_{n,1} & \lambda_{n,2} & \cdots & \lambda_{n,N+n} \\
\lambda_{n-1,1} & \cdots & \cdots & \lambda_{n-1,N+n-1} \\
\lambda_{0,1} & \cdots & \lambda_{0,N} \\
\end{array}$$
such that $\lambda_{n,i} = \lambda_i$, $\lambda_{0,i} = \mu_i$ and satisfying the condition $\lambda_{m,i} \geq \lambda_{m-1,i} \geq \lambda_{m,i+1}$ for all $m$ and $i$. Note that, because of these conditions, we have a sequence of partitions

$$\lambda^{(0)} \subset \lambda^{(1)} \subset \ldots \subset \lambda^{(n)}$$

where $\lambda^{(m)} = (\lambda_{m,1}, \lambda_{m,2}, \ldots, \lambda_{m,N+m})$.

The $\mathfrak{sl}_n$ weight of the basis vector labeled by $\Lambda/\mu$ is given by

$$\sum_{m=1}^{n} \left( \sum_{i=1}^{N+m} \lambda_{m,i} - \sum_{i=1}^{N+m-1} \lambda_{m-1,i} \right) \epsilon_m .$$

Thus we easily deduce that there exists a weight-preserving, 1–1 correspondence between the GZ-schemes $\Lambda/\mu$ of $L(\lambda, \mu)$ and the semi-standard tableaux $T \in \text{SST}(\lambda/\mu)$. This proves

**Theorem C.13** [29]. The $\mathfrak{sl}_n$ character of the irreducible $Y(\mathfrak{gl}_n)$ module $L(\lambda, \mu)$ is given by the skew Schur function $s_{\lambda/\mu}(x)$, i.e.

$$\text{ch}_{L(\lambda, \mu)}(x) = s_{\lambda/\mu}(x) .$$

In general the irreducible $Y(\mathfrak{sl}_n)$ module $L(\lambda, \mu)$ is reducible under $\mathfrak{sl}_n \subset Y(\mathfrak{sl}_n)$. The decomposition is given by (A.9). From the explicit action of the Yangian generators [34] one furthermore concludes

**Theorem C.14** [34]. The Drinfel’d polynomials of the irreducible $Y(\mathfrak{sl}_n)$ modules $L(\lambda, \mu)$ are given by

$$P_i(u) = \prod_{\lambda_j' - \mu_j' = i} \left( u - \frac{1}{2} (\lambda_j' + \mu_j') - j + \frac{1}{2} \right) .$$

Note that for $\mu = \emptyset$ we recover (C.22).

**Appendix D. $q$-identities**

In this section we recall some elementary $q$-identities. The proofs can be found in [3].

Recall the definition of the $q$-number

$$(z; q)_N = \prod_{k=1}^{N} (1 - zq^{k-1}) .$$

We will write $(q)_N = (q; q)_N$ for short. We have the following useful expansions

- 28 –
Theorem D.1. Let $N \in \mathbb{N}$, then

i.

$$(z; q)_N = \sum_{n=0}^{N} (-z)^n q^{\frac{1}{2}n(n-1)} \left[ \begin{array}{c} N \\ n \end{array} \right], \quad (D.2)$$

ii.

$$(z; q)_{N-1} = \sum_{n \geq 0} \left[ \begin{array}{c} N + n - 1 \\ n \end{array} \right] z^n. \quad (D.3)$$

Here, the $q$-multinomial is defined by

$$\left[ \begin{array}{c} k_1 + k_2 + \ldots + k_n \\ k_1, k_2, \ldots, k_n \end{array} \right] = \frac{(q)_{k_1+k_2+\ldots+k_n}}{(q)_{k_1}(q)_{k_2} \ldots (q)_{k_n}}, \quad (D.4)$$

and we have put $\left[ \begin{array}{c} N \\ n, N-n \end{array} \right] = \left[ \begin{array}{c} N \\ n \end{array} \right]$ for short. They are the coefficients in the expansion of the generalized Rogers-Szégo polynomial in $n$ variables

$$H_n^{(n)} (x; q) = \sum_{k_1+k_2+\ldots+k_n=N} \left[ \begin{array}{c} k_1 + k_2 + \ldots + k_n \\ k_1, k_2, \ldots, k_n \end{array} \right] x_1^{k_1} \ldots x_n^{k_n}. \quad (D.5)$$

The generating function for the Rogers-Szégo polynomial is given by

$$\sum_{N \geq 0} H_n^{(n)} (x; q) \frac{t^N}{(q)_N} = \frac{1}{(tx_1; q)_{\infty} \ldots (tx_n; q)_{\infty}}. \quad (D.6)$$

Note that for $q = 1$ we can interpret $H_n^{(n)} (x; q)$ as the character of the $\mathfrak{sl}_n$ module $L(\Lambda_1)^\otimes N$ through the identification $x_i = e^{\epsilon_i}$ (see App. B), while $H_n^{(n)} (x; q^{-1})$ for arbitrary $q$ is proportional to the character of the so-called $\mathfrak{sl}_n$ Haldane-Shastry spin chain of length $N$ [25,11].

A very useful lemma is the following

Lemma D.2 (Durfee square). For all $m \in \mathbb{Z}$ we have

$$\sum_{a,b \geq 0 \atop a-b=m} q^{ab} \frac{(q)_a}{(q)_b} = \frac{1}{(q)_{\infty}}. \quad (D.7)$$

Finally, to derive the spinon form of the $\mathfrak{sl}_n$ characters, we need the following lemma
Lemma D.3. For all $M, N \in \mathbb{Z}_{\geq 0}$, we have

i. \[ \sum_{m \geq 0} (-1)^m \frac{q^{\frac{1}{2}m(m-1)}}{(q)_{M-m}(q)_{N-m}(q)_m} = \frac{q^{MN}}{(q)_M(q)_N}. \quad (D.8) \]

ii. \[ \sum_{m \geq 0} \frac{q^{(M-m)(N-m)}}{(q)_{M-m}(q)_{N-m}(q)_m} = \frac{1}{(q)_M(q)_N}. \quad (D.9) \]

Proof:

i. Multiplying the left hand side by $(q)_N z^M$ and summing over $M$, we find

\[
\sum_{M \geq 0} \left( \sum_{m \geq 0} (-1)^m \frac{q^{\frac{1}{2}m(m-1)}}{(q)_{M-m}} [N \atop m] \right) z^M = \left( \sum_{M \geq 0} \frac{z^M}{(q)_M} \right) \left( \sum_{m \geq 0} (-z)^m q^{\frac{1}{2}m(m-1)} [N \atop m] \right)
\]

\[= \frac{(z; q)_N}{(z; q)_{\infty}} = \frac{1}{(zq^N; q)_{\infty}} = \sum_{M \geq 0} \frac{q^{MN} z^M}{(q)_M} \]

ii. Again, multiplying the left hand side by $(q)_N z^M$ and summing over $M$, we find

\[
\sum_{M \geq 0} \left( \sum_{m \geq 0} \frac{q^{(M-m)(N-m)}}{(q)_{M-m}} [N \atop m] \right) z^M = \sum_{M, m \geq 0} \frac{q^{M-Nm}}{(q)_M} [N \atop m] z^{M+m}
\]

\[= \sum_{m \geq 0} \frac{1}{(zq^{N-m}; q)_{\infty}} [N \atop m] z^m = \frac{1}{(z; q)_{\infty}} \sum_{m \geq 0} (z; q)_{N-m} [N \atop m] z^m
\]

\[= \frac{1}{(z; q)_{\infty}} \sum_{m, n \geq 0} (-1)^n q^{\frac{1}{2}n(n-1)} [N \atop m] [N-m \atop n] z^{m+n}
\]

\[= \frac{1}{(z; q)_{\infty}} \sum_{m, n \geq 0} (-1)^n q^{\frac{1}{2}n(n-1)} \frac{(q)_N}{(q)_{m-n}(q)_n(q)_{N-m}} z^m
\]

\[= \frac{1}{(z; q)_{\infty}} \sum_{m \geq 0} (1; q)_m \frac{(q)_N}{(q)_{N-m}(q)_m} z^m = \frac{1}{(z; q)_{\infty}} = \sum_{M \geq 0} \frac{1}{(q)_M} z^M \]

\[\square\]
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