Skyrmions with arbitrary topological charges in spinor Bose–Einstein condensates

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Abstract

Topological Skyrmion defects with arbitrary topological charges are shown to appear in spinor \( F = 1 \) Bose–Einstein condensates, under the presence of external gauge or magnetic fields with cylindrical symmetry. We show that, depending on the magnetic field boundary conditions, the spin texture, at the planes perpendicular to the cylindrical axis, can be forced to show Skyrmions with topological charge zero, half-integer, or any desired arbitrary value between \(-1/2\) and \(1/2\). Our findings are obtained by numerically solving the corresponding fully coupled Gross–Pitaevskii equations without any symmetry assumptions and by analyzing the Skyrmion topological charge. We study, both, polar \(^{23}\text{Na}\) and ferromagnetic \(^{85}\text{Rb}\) condensates.

Keywords: spinor Bose–Einstein condensates, topological excitations, ultracold gases

(Some figures may appear in colour only in the online journal)

1. Introduction

The concept of Skyrmions, while originating in the modeling of atomic nuclei [1, 2], has nowadays permeated distinct physical systems ranging from nuclear physics [3–5], particle physics [6–8], superconductivity [9], magnetic solid state physics [10, 11], liquid crystals [12], string theory [13, 14], and Bose–Einstein condensates (BECs), thus promising to be a topological unifying framework for seemingly different physical phenomena [5, 15]. In BEC systems, early proposals [16–20] to create and observe topological defects, to recent ones [21–25], have already led to their experimental realization [26–31]. In particular, for the spinor \( F = 1 \) BEC, that concerns us here, an intense recent activity on Skyrmions has developed [25, 27, 32, 33].

Skyrmions are topological defects of a given spatial vector field or order-parameter of the system in question, that can be classified in terms of homotopy groups of the corresponding field [34–37]. There are versions in two- and three-dimensions and, typically, the Skyrmion topological charge is either an integer \((0, \pm 1)\) or a half-integer \((\pm 1/2)\) [16, 38, 39]. For a three-dimensional spinor \( F = 1 \) BEC, the field that develops the Skyrmion defects is the spin texture, a real measurable quantity. In this article we analyze different phases in the mentioned spinor BEC, that show the same two-dimensional Skyrmion topological defect in all planes perpendicular to a privileged axis, similarly to a so-called ‘baby-Skyrmion’ [40]. While this system has been the focus of great attention [19, 25, 31–33, 41–43], we report here an interesting and novel aspect regarding the value of the acquired Skyrmion charge. That is, we show that depending on the boundary conditions of the external magnetic or gauge field that nucleates the vortices and the Skyrmions in the spinor BEC, the 2D Skyrmion charge in each plane may take any desired value. Although we believe that those defects are of a true topological nature—since their charge is independent of the location of the defect and the total charge is the sum of the charges of the individual defects present—their dependence on details of the external fields may suggest reservations on the use of the precise value of the topological charge as a proper variable to classify the ensuing condensate states.

Our study is based on the full numerical solution of the 3D spinor \( F = 1 \) Gross–Pitaevskii (GP) equations, without assuming any symmetry of the solution [44]. We analyze both
polar and ferromagnetic condensates, with parameters corresponding to actual values of $^{87}$Rb and $^{23}$Na [45]. In addition to the numerical analysis, we discuss analytic results to support our discussion. Regarding the numerical study, we search for stationary states of $N$ confined, weakly interacting bosons of spin $F = 1$, in the presence of an actual external magnetic field, or arising from a gauge field in general, and whose energy functional is,

$$
E[\Psi] = \int d^3r \left\{ \frac{\hbar^2}{2m} \nabla^{2}\Psi^{*}_f \nabla \Psi_f + V_{\text{ext}}(\vec{F}) \Psi^{*}_f \Psi_f \\
+ \frac{\omega_1^2}{2} \Psi^{*}_f \Psi^{*}_{f'} \Psi_{f'} \Psi_f + \frac{\omega_2^2}{2} \Psi^{*}_f \vec{F}_k \cdot \vec{F}_{k'} \Psi_{f'} + \mu \vec{B} \cdot \vec{\Psi}_f \vec{\Psi}_f \right\}
$$

The conﬁning potential, $V_{\text{ext}}(\vec{F}) = m\omega^{2}(x^2 + y^2 + z^2)/2$ is an isotropic harmonic optical trap; $c_0$ and $c_2$ are the usual two-body interaction parameters as defined by Ho [46], with $c_2 < 0$ for ferromagnetic phases and $c_2 > 0$ for polar ones, in the absence of the external field $\vec{B}$. The explicit form of these coefficients is given by $c_0 = (g_0 + 2g_2)/3$ and $c_2 = (g_0 - g_2)/3$, where $g_0 = 4\pi\hbar^2a^2_{2\text{F}}/m$ are the coupling strengths of the binary collisions in the channels of total spin $F_1 = 0, 2$, and $a_{2\text{F}}$ the corresponding scattering lengths. We use the masses for $^{87}$Rb and $^{23}$Na, and the scattering lengths $g_0 = 101.80\text{nm}$, $g_2 = 100.42\text{nm}$ for rubidium and $g_0 = 50\text{nm}$, $g_2 = 55\text{nm}$ for sodium [23, 45], with $a_{2\text{F}}$ the Bohr radius. We also use typical experimental values for the number of atoms, $N = 5 \times 10^{13}$, the trap frequency, $\omega = 2\pi \times 130\text{Hz}$, and the linear Zeeman coupling $p = -\mu_B/2$, with $\mu_B$ the Bohr magneton. The vector $\vec{F}$ are the $F = 1$ angular momentum matrices. The Latin subindices run over the three components of spin $F = 1$, and we use the sum convention over repeated indices. In our numerical calculations we use dimensionless units $\hbar = m = \omega = 1$.

The external $\vec{B}$ fields that we consider are of cylindrical symmetry along an axis, parallel to the $z$-axis, but located at any arbitrary point $(x_0, y_0)$ on the $xy$-plane,

$$
\vec{B}(x, y) = B_0((x - x_0)x - (y - y_0)y) + B_z(\rho)z.
$$

The variable $\rho = ((x - x_0)^2 + (y - y_0)^2)^{1/2}$ is the distance to the symmetry axis of the field, and $B_0 = B_z(\rho)$ of the condensate system is defined by the minimum of the external (harmonic) confining potential $V_{\text{ext}}(\vec{F})$; $x$, $y$ and $z$ denote the corresponding Cartesian unit vectors. Because of the arbitrariness of the planar point $(x_0, y_0)$, the $z$-axes of the spatial symmetries of the external field $\vec{B}$ and of the confining potential $V_{\text{ext}}$ need not coincide. As has been already studied [30, 47–49], and as we review below, an external field of this type with $B_z(\rho) = 0$, nucleates quantized vortices and 2D Skyrmions around the $z$-axis of the $\vec{B}$ field. The vortices can have topological charges $(0, \pm 1, \pm 2)$ and the Skyrmions $(0, \pm 1/2)$. We recall that in the absence of an external field $\vec{B}$, the stationary states do not show any topological defect, being those states either of a polar or of a ferromagnetic character, as described by Ho [46], depending on the sign of $c_2$. Thus, the presence of the field $\vec{B}$ is necessary for the emergence of the topological defects. The question we address here is how the defects and their charges depend on the details of the applied external field. We analyze two simple types of modifications of the field $\vec{B}$, that preserve its cylindrical symmetry, by considering $B_r = \text{constant}$ and $B_r = B_0\rho$, with $B_0$ a constant. The purpose of adding either $B_r$ component of the field is the modification of the boundary values of the $\vec{B}$ field as $\rho \rightarrow \infty$. We will show that, in turn, this modifies the Skyrmion charge. We point out here that if $B_r = \text{constant}$, the $\vec{B}$ field lies always on the $xy$-plane as $\rho \rightarrow \infty$. However, for the case $B_r = B_0\rho$, the asymptotic direction of the $\vec{B}$ field no longer points on the plane, but in a direction defined by the values of $B_0$ and $B_r$.

$$
\frac{\vec{B}}{|\vec{B}|} = \frac{B_0(\mathbf{x}\cos \phi - \mathbf{y}\sin \phi)}{(B_0^2 + B_r^2)^{1/2}} + B_rz \quad \text{as} \quad \rho \rightarrow \infty.
$$

Our numerical analysis shows that in the presence of the $\vec{B}$ field, the polar and ferromagnetic character of the state of the condensate is lost as $\rho \rightarrow \infty$, the gas behaving as ‘para-magnetic’ with the spin texture pointing along the direction of the field. Thus, the case $B_r = \text{constant}$ yields Skyrmions with charges 0 and $\pm 1/2$ always, while the other one, $B_r = B_0\rho$, gives rise to Skyrmions whose charges are different from 0 and $\pm 1/2$ and that depend on the precise boundary value of the $\vec{B}$ field. In both cases, however, because $\vec{B}$ is zero along the $z$-line at $(x_0, y_0)$, there appear well defined vortices with charges 0, $\pm 1$, and $\pm 2$ always [48].

In section 2 we discuss and show how the vortices and the Skyrmions appear and how they are connected. In section 3, we present the analysis concerning the Skyrmions topological charges. We conclude with some remarks in section 4, discussing, in particular, the stability of the stationary states used throughout the article, and the possibilities for experimental realizations of the analyzed Skyrmion structures.

2. Vortices and skyrmions in spinor condensates

As pointed out above, magnetic fields of the type here considered generate quantized vortices on the spin condensate components, along the $z$-line centered at $(x_0, y_0)$. As it is shown in [48], since the line or lines of zero field are at our disposal, one can create vortices of the Mermin–Ho type [38] at arbitrary locations. Here, we show that a Skyrmion-string is also created at the same lines of zero field and, hence, that they can also be externally created and controlled on demand.

The condensate stationary solution $\Psi_\rho(\vec{r})$, with the projections $n = +1, 0, -1$ along the $z$-direction of the external $\vec{B}$-field, minimize the energy functional $E[\Psi]$ given in equation (1), subject to the condition that the number of particles equals $N$. This yields the following set of three coupled GP equations,

$$
- \frac{\hbar^2}{2m} \nabla^2 \Psi_\rho + V_{\text{ext}}(\Psi_\rho + c_0 \Psi_0^2 \Psi_\rho) \Psi_\rho \\
+ c_2 \Psi_0^2 \Psi_0 \vec{F}_k \Psi_\rho + \mu \vec{B} \cdot \vec{\Psi}_\rho \vec{\Psi}_\rho = 0.
$$
with $\mu$ the chemical potential, which adjusts itself to yield the solution to the above GP equations [48].

Our numerical solutions of the corresponding GP equations, given by equation (4), indicate the existence of two or three stationary states, that may be called ‘ground’ and ‘excited’ states, depending on their corresponding chemical potential values. In order to classify those states we use the notation for Mermin–Ho vortices, namely, we find states $\{1, 0, -1\}$, $\{0, -1, -2\}$ and $\{+2, +1, 0\}$, whose notation $(l, m, n)$ represents vortices of charge $l$, $m$ and $n$ in the spinor components $n = +1, 0, -1$ of the condensate spinor $\Psi$. Figure 1 shows the observed vortex-Skyrmion phases, in terms of the equilibrium chemical potential $\mu$ as a function of $z$-component of the external fields analyzed. The left panel of figure 1 refer to the case of $B_z = \text{constant}$, while the right one to $B_z = B_z \rho$, and in both cases we show the results for ferromagnetic $^{87}\text{Rb}$ and polar $^{23}\text{Na}$. The labels $\{+2, +1, 0\}$, $\{+1, 0, -1\}$ and $\{0, -1, -2\}$, indicate the three different vortex phases. These states are numerically stable, see text.

Figure 1. Chemical potential $\mu$ as a function of $B_z = \text{constant}$, left panel, and as a function of $B_z$, right panel, for the case $B_z = B_z \rho$, both for a ferromagnetic condensate $^{87}\text{Rb}$ and a polar one $^{23}\text{Na}$. The labels $\{+2, +1, 0\}$, $\{+1, 0, -1\}$ and $\{0, -1, -2\}$, indicate the three different vortex states. These states are numerically stable, see text.

The solution of the three GP equations for a $F = 1$ BEC can be written, in general, as

$$\hat{\Psi}(\vec{r}) = \sqrt{\varphi}(\vec{r}) \begin{pmatrix} \zeta_1(\vec{r}) e^{i\theta_0(\vec{r})} \\ \zeta_0(\vec{r}) e^{i\theta_0(\vec{r})} \\ \zeta_{-1}(\vec{r}) e^{i\theta_0(\vec{r})} \end{pmatrix},$$

where $\varphi(\vec{r}) = \hat{\Psi}^\dagger(\vec{r}) \hat{\Psi}(\vec{r})$ is the total particle density of the condensate. Please note that we use the Greek letters $\varphi$ for the particle density and $\rho$ for the orthogonal distance to the $z$-axis of the $\vec{B}$-field. The amplitudes $\zeta_0(\vec{r})$ are real functions, obeying $\zeta_1^2(\vec{r}) + \zeta_0^2(\vec{r}) + \zeta_{-1}^2(\vec{r}) = 1$ everywhere. An analysis of the vortex solutions, if they exist and if single valuedness is imposed, leads to the following general results [48]: (1) only two components can show a vortex, say $\zeta_0 \to 0$ and $\zeta_\beta \to 0$ as $\rho \to 0$, with $\theta_0 = 0$ and $\theta_\beta = 0$, while the third one, $\zeta_\alpha \to 1$ as $\rho \to 0$, and $\theta_\alpha = 0$. And, (2) the phases differences obey $\theta_\alpha - \theta_{m-1} = \phi$, with $\phi$ the polar angle referred to the $z$-axis located at $(\alpha, \rho)$. These two conditions, for the field $\vec{B}$ given in equation (2), imply the appearance of three vortex solutions, with boundary conditions at $\rho \to 0$ and with charges $Q_m$ given by $\theta_m = Q_m \phi$. The following table summarizes the three type of vortex solutions:

| $Q_m$ | $Q_{-m}$ | $\zeta_0$ | $\zeta_1$ | $\zeta_{-1}$ |
|-------|-----------|-----------|-----------|-------------|
| $+1$  | $-1$      | $0$       | $1$       | $0$         |
| $0$   | $0$       | $0$       | $1$       | $0$         |
| $-1$  | $+1$      | $0$       | $1$       | $0$         |

2 To perform our calculations we use state of the art graphics processing unit (GPU) programming via the PyCUDA library in a GPU with 2304 CUDA cores. We do calculations with double and single precision on 128$^3$ and 256$^3$ mesh points. We point out that the structure of the stationary states can be numerically ensured with 128$^3$ mesh points, within single precision calculations.
The vortex topological charges have the same value regardless of the position $\rho \rightarrow 0$. For the other cases, $B_i = B_i(\rho)$, these vortex structures remain very similar to figure 2. Since we can tailor the external $B$ field, we can produce fields with more than one axis at which the field vanishes, thus creating states with two or three vortices of the same or of different type, as exemplified in figure 3. Certainly, for each case of two or three vortices there is a 'phase diagram', $\mu$ versus $B_i$, similar to the one in figure 1; those details are beyond the scope of this article and will be reported elsewhere. Nevertheless, calculating the circulation $C$ of the velocity field $\mathbf{v}$, with $C$ a circuit enclosing all vortices, indeed yields that the total topological charge is the sum of charges of the individual vortices. This exercise illustrates the topological nature of the vortices; the same will be concluded for the Skyrmion defects, see section 3.1 below. We now turn to the spin texture description. For this, it first appears convenient to factorize the phase of the 0-component of the full solution $\hat{\Psi}(\mathbf{r})$, equation (5), and write

$$C = \oint_C \mathbf{v} \cdot d\mathbf{l}$$

$$\hat{\Psi}(\mathbf{r}) = \sqrt{\rho}(\mathbf{r}) e^{i \theta(\mathbf{r})} \hat{\zeta}(\mathbf{r}),$$

where

$$\hat{\zeta}(\mathbf{r}) = \begin{pmatrix} \zeta_{+1}(\mathbf{r}) e^{i \varphi_{+1}(\mathbf{r})} \\ \zeta_0(\mathbf{r}) \\ \zeta_{-1}(\mathbf{r}) e^{i \varphi_{-1}(\mathbf{r})} \end{pmatrix}$$
with $\delta_{m} = \theta_{m} - \theta_{0}$. We shall leave the $\vec{f}$ dependence implicit in the foregoing analysis. As it is common, a 3D complex vector $\vec{a} = (a_{x}, a_{y}, a_{z})$ can be introduced, with $a_{x} = \frac{1}{\sqrt{2}}(\zeta_{-1}e^{i\phi} - \zeta_{1}e^{-i\phi})$, $a_{y} = \frac{1}{\sqrt{2}}(\zeta_{-1}e^{i\phi} + \zeta_{1}e^{-i\phi})$, and $a_{z} = \zeta_{0}$, obeying $\vec{a} \cdot \vec{a}^{*} = 1$. This vector can be further decomposed in its real and imaginary parts, $\vec{a} = \vec{a}_{R} + i \vec{a}_{I}$.

The magnetization, a physical observable, is given by $\vec{\Psi} \vec{F} \vec{\Psi} = g \vec{f}$, with $\vec{f} = \frac{\zeta}{\zeta^{*}} \zeta^{*}$ the spin texture. Using the above decomposition of the state, the spin texture can be expressed as $\vec{f} = 2\vec{a}_{R} \times \vec{a}_{I}$. Let us analyze this expression for different cases. The simplest one is when the full Zeeman contribution is zero, namely $\vec{B} = 0$ in equation (1). The solution, as shown by Ho [46], is that for the polar case, $c_{2} < 0$, $\vec{a}$ is real and $\vec{f} = 0$. For the ferromagnetic case, $c_{2} > 0$, $\vec{f} = \vec{f}_{0}$, a constant vector everywhere, which by an appropriate rotation can be brought to the case $\zeta_{1} = 1$, $\zeta_{0} = \zeta_{-1} = 0$, namely $\vec{f} = \vec{z}$. For $\vec{B} = 0$, the polar and ferromagnetic characters are overridden and the three types of quantum-vortex phases appear.

The above spinor vortex structures have an additional associated topological defect that one may call a 'string-Skyrmion' due to the presence of the external magnetic field. That is, while the trap imposes its spherical symmetry on the total density $\rho(\vec{r})$, the external magnetic field imposes its additional cylindrical symmetry on the texture field $\vec{f}$. Hence, the spin texture does not depend on the $z$ coordinate, $\vec{f} = \vec{f}(x, y)$, and furthermore, it shows cylindrical symmetry around the location of the zero line (or lines) of the $\vec{B}$ field, at $(x_{0}, y_{0})$, namely, $\vec{f} = \vec{f}(\rho)$. Thus the 'string' qualifier. In other words, the spin texture shows the same geometric and topological structure in all $z$-planes. Regarding our numerical calculations, we point out that we perform them in a Cartesian mesh, with Cartesian coordinates, without any assumption on its symmetry. This is, in fact, what allows us to tailor the external fields as desired.

The connection between the vortex solutions and the Skyrmions can be found as follows. An important step lies in the restriction on the phases of the vortex structure, $\theta_{m} - \theta_{0} = \phi$. This restriction yields the phase requirement $\theta_{1} + \theta_{-1} = 2\theta_{0} = 0$, which in turn, and only in this case, makes the set $(\vec{a}_{R}, \vec{a}_{I})$ an orthogonal triad (group O(3)). Their explicit form is

$$\vec{a}_{R} = \frac{\zeta_{-1} - \zeta_{1}^{*}}{\sqrt{2}} \rho + \zeta_{0} \vec{z},$$

$$\vec{a}_{I} = -\frac{\zeta_{1} - \zeta_{-1}^{*}}{\sqrt{2}} \delta,$$

and, hence,

$$\vec{f} = (\zeta_{-1} - \zeta_{1}^{*}) \rho + \sqrt{2} \zeta_{0} (\zeta_{1} + \zeta_{-1}) \delta.$$ (14)

The unit vectors are $\rho = x \cos \delta - y \sin \delta$ and $\delta = x \sin \delta + y \cos \delta$, with $\delta = \delta_{0} = -\delta_{1}$, with $\delta_{0} = \theta_{0} - \theta_{0}$. The angle $\delta$ spans $0 \rightarrow 2\pi$, since it is actually the phase of the wavefunctions that yield the vortex charges, however, it is not the polar angle $\tan \phi = y/x$. Nevertheless, the triad unit vector $(\rho, \delta, \vec{z})$ together with $\rho = ((x - x_{0})^{2} + (y - y_{0})^{2})^{1/2}$ and $\vec{z}$ span the space with cylindrical symmetry. This is important, as we shall return below, since far from the singularity, $\rho$ tends to align to the in-plane component of the $\vec{B}$-field, namely, to $(x \cos \phi - y \sin \phi)$. It is also important to mention that while $\vec{a}_{R}$ and $\vec{a}_{I}$ depend on the choice of the global phase, as shown in equation (10), $\vec{f}$ does not. This is because $\vec{f}$ is an observable and $\vec{a}_{R}$ and $\vec{a}_{I}$ are not.

Since the (real) spinor components depend only on $\rho$, $\zeta_{0} = \zeta_{0}(\rho)$, the spin texture can be written as $\vec{f} = f_{x}(\rho) \vec{z} + f_{y}(\rho) \rho \vec{y}$, and the components $f_{x}$ and $f_{y}$ can be read off of equation (14). With these, the existence or not of an Skyrmion structure in any plane $z = \text{constant}$, can be checked. To this end one recalls that the 2D Skyrmion charge is given by

$$Q_{\text{sky}}^{2D} = \frac{1}{4\pi} \int \int f_{x}(\rho) \frac{\partial f_{y}(\rho)}{\partial \rho} - f_{y}(\rho) \frac{\partial f_{x}(\rho)}{\partial \rho} \, d\rho \, d\phi,$$ (15)

which, with the cylindrical symmetry of $\vec{f}$, can be cast as,

$$Q_{\text{sky}}^{2D} = \frac{1}{2} \int_{0}^{\infty} f_{x}(\rho) \frac{df_{x}}{d\rho} - f_{y}(\rho) \frac{df_{y}}{d\rho} \, d\rho.$$ (16)

There exists, however, an additional important constraint for the cases studied in this work, except for the polar vortex (+1, 0, −1), namely, that $\vec{f} \cdot \vec{f} = 1$ everywhere. Although we have not been able to prove this constraint rigorously, our numerical solutions demonstrate it. This restriction may be written as $f_{x}^{2} + f_{y}^{2} = 1$, which further implies that the Skyrmion charge, equation (16), can be cast as,

$$Q_{\text{sky}}^{2D} = \frac{1}{2} (f_{x}(\infty) - f_{x}(0)) \text{ if } \vec{f} \cdot \vec{f} = 1.$$ (17)

That is, if $\vec{f} \cdot \vec{f} = 1$ everywhere, the Skyrmion charge is given solely by (one-half) the difference of the boundary values of the $z$-component of the spin texture. For the polar vortex (+1, 0, −1) the simple formula given above, equation (17), does not hold, yet we can still calculate its Skyrmion charge using equation (16). The boundary value at $\rho = 0$ can always be found, as can be seen from equations (6)–(8) and (14), however, the boundary value at $\rho \rightarrow \infty$ cannot be directly accessed since the confining harmonic trap allows for calculations up to the Thomas–Fermi radius of the atomic cloud only. Below, along the presentation of our results, we show how we circumvent this difficulty and how we find trustable values of the topological charges.

3. Skyrmions in magnetic fields with different boundary conditions

As discussed in the previous section, the vortex structures (+2, +1, 0), (+1, 0, −1) and (0, −1, −2) each have associated their own spin textures and Skyrmion defects. Moreover, although the spin texture of the vortex (+1, 0, −1) does not satisfy $\vec{f} \cdot \vec{f} = 1$ everywhere, because $\vec{f} \rightarrow 0$ as $\rho \rightarrow 0$, it is still clear that the boundary conditions are essential to determine the Skyrmion charges. The boundary conditions at $\rho = 0$ are completely determined by the corresponding vortex.
structure, as shown by equations (6)–(8). However, supported by our calculations, the boundary values at $\rho \to \infty$ depend on the external $\vec{B}$-field. We find that far from the vortex singularity, $\vec{f}$ tends to align to the direction of the magnetic field $\vec{B}$. This can be further understood by looking at the expression for the energy $E[\Psi]$, equation (1), in which the Zeeman term can be written as $p\vec{B} \cdot \psi^* \frac{\partial \psi}{\partial \rho} \psi = p\rho|\vec{B}|(\vec{\nabla} / |\vec{B}|) \cdot \vec{f}$. Since $p < 0$ in our calculations, this term tends to minimize the energy when $\vec{B} / |\vec{B}|$ and $\vec{f}$ are parallel. We discuss separately the three general cases. Figure 4 illustrates the spin texture $\vec{f}$ for several typical Skyrmions.

3.1. $B_z = 0$

First, for completeness, we review the case in the absence of a $z$-component of the $\vec{B}$-field, namely, $B_z = 0$. For the vortex solution $(+1, 0, -1)$, by symmetry (and numerically verified) $\zeta_{-1} = \zeta_{-1}^2$ for all $\rho$, hence, $f_z = 0$, see equation (14). That is, the vector $\vec{f}$ not only lies on the $xy$-plane, it becomes zero as $\rho \to 0$. Its Skyrmion charge is thus zero, $Q^\text{3D}_{\text{Sky}}(+1, 0, -1) = 0$. This is the so-called polar coreless vortex [47]. For the vortices $(+2, +1, 0)$ and $(0, -1, -2)$, equations (7), (8) and (14), show that $f_z \to -1$ and $f_z \to +1$, respectively, as $\rho \to 0$. The numerical solutions further show that $\zeta_{-1}^2 = \zeta_{-1}^2$ as $\rho \to \infty$, namely $f_z \to 0$ as $\rho \to \infty$ for both cases. See figure 4 as an example of this case. Hence, one finds that the vortex $(+2, +1, 0)$ has an associated Skyrmion charge $Q^\text{3D}_{\text{Sky}}(+2, +1, 0) = +1/2$ and, analogously, the vortex $(0, -1, -2)$ has $Q^\text{3D}_{\text{Sky}}(0, -1, -2) = -1/2$. We also calculated these charge values by integrating directly $Q^\text{3D}_{\text{Sky}}$ using the full equation (15), finding values very close to $\pm 1/2$. In figure 3 we show configurations of two and three vortices, by tailoring the external field $\vec{B}$ with $B_z = 0$. The total Skyrmion charge of each configuration, calculated with equation (15), indeed yields the sum of the known individual ones. This adds further support to the topological nature of the Skyrmion defects.

3.2. $B_z = \text{constant}$

We now consider a constant $z$-component of the external $\vec{B}$-field, but different from zero. If $B_z < 0$, the vortex structure remains the same as before but, as seen in figure 1, the most stable case is now $(+2, +1, 0)$. For $B_z > 0$, the situation is reversed and the stable phase is $(0, -1, -2)$. A direct calculation of $Q^\text{3D}_{\text{Sky}}$ using the definition given by equation (15) or equation (16), shows that the charge is apparently not zero for the polar Skyrmion $(+1, 0, -1)$ and different from $\pm 1/2$ for $(+2, +1, 0)$ and $(0, -1, -2)$ (results not shown here). However, we show now evidence that this a numerical artifact, caused by the fact that the condensate cloud reaches out...
Figure 5. Spin texture component \( f_\rho \) as a function of \( \rho \), for different values of \( B_z \) = constant, for the vortex structure (+2, +1, 0), upper panel, and (+1, 0, −1), lower panel. The dotted lines corresponds to a magnetic field centered at \((x_0 = 0, y_0 = 0)\), while the continuous ones to \((x_0 = 0, y_0 = 2)\). This constitutes an indication that \( f_\rho \rightarrow 0 \) as \( \rho \rightarrow \infty \) for \( B_z \) = constant. Both examples correspond to a \(^{87}\)Rb condensate.

essentially up to the Thomas–Fermi radius, with its density becoming numerically negligible beyond it. That is, if we take the case \((x_0, y_0) = (0, 0)\) for instance, the Skyrmion charge should directly be numerically calculated using,

\[
Q_{\text{sky}}^{2D} \approx \frac{1}{2} \int_0^{R_{\text{TF}}} f_\rho \left( \frac{\partial f_\rho}{\partial \rho} - f_\rho \frac{\partial f_\rho}{\partial \rho} \right) d\rho,
\]

where \( R_{\text{TF}} \) is the Thomas–Fermi radius, defined as the radius at which the condensate density \( \rho(\vec{r}) \) becomes negligible due to the external trap \( V_{\text{ext}}(\vec{r}) \). The point we make is that, notwithstanding that the structure of the spin texture \( \vec{f} \) is modified by the \( B_z \neq 0 \) component, we claim that the topological charges remain 0 for the polar Skyrmion (+1, 0, −1) and \( \pm 1/2 \) for (+2, +1, 0) and (0, −1, −2), even if the charge given by equation (18) yields a different value. To verify this, we calculated the spin texture for two (or more) cases: one where the \( \vec{B} \)-field has its zero line at \((x_0 = 0, y_0 = 0)\) and others where the line is at \((x_0 = 0, y_0 = 0)\). The resulting spin textures can then be superimposed in a single graph; figure 5 shows the result for \( f_\rho(\rho) \) when \( \rho = (x^2 + y^2)^{1/2} \) (dotted lines) and when \( \rho = (x^2 + (y - 2)^2)^{1/2} \) (continuous lines), for different values of \( B_z \). Then, with this procedure we can observe, and calculate, the Skyrmion structure beyond the Thomas–Fermi Radius, which is \( R_{\text{TF}} \approx 10 \) for most of our calculations. This operation can be repeated for other values of \((x_0, y_0)\) to reach even more further than \( R_{\text{TF}} \). The conclusion is that, as \( \rho \) grows, \( f_\rho \) keeps diminishing with no bound. That is, the numerical evidence is that \( f_\rho(\rho) \rightarrow 0 \) as \( \rho \rightarrow \infty \), for all three vortex states (+2, +1, 0), (0, −1, −2) and (+1, 0, −1). Therefore, for the two former cases, we conclude that the Skyrmion charges are \( Q_{\text{sky}}^{2D}(+2, +1, 0) = 1/2 \) and \( Q_{\text{sky}}^{2D}(0, −1, −2) = −1/2 \). For the case (+1, 0, −1), lower panel in figure 5, we see that the spin texture structure changes strongly from \( B_z \) = 0 to \( B_z \neq 0 \). However, since the calculation of the charge keeps diminishing as we extend the range of \( f_\rho \) and \( f_\rho \), the value of its topological charge \( Q_{\text{sky}}^{2D}(+1, 0, −1) \) remains zero. We conclude this from a direct calculation of the Skyrmion charge for larger values of the cutoff of the radial integral equation (16). The explanation that as \( \rho \rightarrow \infty \), \( f_\rho(\rho) \rightarrow 0 \), for all these cases, thus yielding charges 0 or \( \pm 1/2 \) independently of the value of \( B_z \) = constant, is that far from the defect location \((x_0, y_0)\), the spin texture \( \vec{f} \) becomes paramagnetic in the sense that it gets aligned to the direction of the external \( \vec{B} \)-field. Since this field lies along the \( xy \)-plane as \( \rho \rightarrow \infty \), hence \( f_\rho(\rho) \rightarrow 0 \). This insight suggests to consider the case where the \( \vec{B} \)-field can point at asymptotic arbitrary directions, such that the boundary value of \( \vec{f} \), as \( \rho \rightarrow \infty \), can be also made to point along the direction of the \( \vec{B} \)-field. For completeness, and to reinforce the topological nature of the defects, we plot in figure 6 the profile of two Skyrmion defects of the same type in the cloud, one case with the same sign of the charges and the other with opposite signs, thus yielding the double of the charge and zero, respectively.

3.3. \( B_z = B_x \rho \)

We turn our attention now to the case with \( B_z = B_x \rho \), which implies that, as \( \rho \rightarrow \infty \), the \( \vec{B} \)-field can be made to point at any arbitrary direction, depending on the values of \( B_x \) and \( B_z \), see equation (3). A word of caution may be in order here. We notice that with this form of the \( B_z \) component, the \( \vec{B} \) field has no longer zero curl, indicating that a local current may be needed to produce it. This apparent technical difficulty should not prevent us from considering such a field, first, since this is an in principle study and, second, because the \( \vec{B} \)-field needs not to be a true magnetic one, it could be obtained from an artificial gauge field, for instance, with an external optical coupling to other atomic states not participating in the BEC.
phenomenon, that would couple the spinor components similarly to a Zeeman term, in a fashion similar to the ones used in [50–52]. We defer further discussion of the experimental possibilities to the Final Comments. The purpose of using this particular field is simply to have one example that preserves the cylindrical symmetry but that its boundary value does not point in a direction along the xy-plane, but in an arbitrary one out of such a plane. As we show now, this boundary effect has a determinant effect in the topological Skyrmion charge. Figures 7 shows typical results of this case. The first refers to vortices (+2, +1, 0) and the second one to (+1, 0, −1), for ⁸⁷Rb; very similar graphs are obtained for ²³Na. Similarly to figure 5, in figure 7 we have superimposed the Skyrmion component fₓ(ρ) of two cases, one created with a B field whose z-axis is located at (x₀ = 0, y₀ = 0), ρ = (x² + y²)¹/² (dotted lines), and the other with a B field with the z-axis at (x₀ = 0, y₀ = 2), ρ = (x² + (y − 2)²)¹/² (continuous lines); for several values of the amplitude Bₓ. Again, the idea is to observe the Skyrmion structure beyond the Thomas–Fermi Radius; incidentally, it turns out that for this type of B field the asymptotic value of fₓ is reached within the cloud, namely, within RTF. For all Bₓ values that we checked, we found that as ρ → ∞, fₓ(ρ) ≠ 0 and, therefore, the asymptotic spin texture no longer lies on the xy-plane. The explanation of this overall behavior is that the spin texture fₓ aligns to the corresponding asymptotic direction of the B field. This yields Skyrmion charges of arbitrary values. Figure 8 summarizes the topological charges Q_{sky} obtained for all cases, as the amplitude Bₓ is varied. For the (+2, +1, 0) and (0, −1 − 2) states, we observe that the Skyrmion charges can take values larger (smaller) than 1/2 (−1/2), for the corresponding numerically stable cases Bₓ > 0 and Bₓ < 0. On the other hand, those charges tend to zero as the z-component amplitude B_z becomes larger; this can already be inferred from figure 7 in which we see that fₓ(ρ) becomes essentially constant for all values of ρ. We calculated the charges for the (+2, +1, 0) and (0, −1 − 2) states, using both formulae, equations (15) and (17), yielding the same value. For the polar Skyrmion (+1, 0, −1), the situation is similar to the previous ones in the sense that the charges Q_{sky} yield continuous values as Bₓ is varied. However, important differences arise. First, since the spin texture fₓ → 0 as ρ → 0, we cannot use the formula given in equation (17) to predict the value of the charge. For this case, we must use the formula in equation (15), which is an integral that involves both spin texture components fₓ and fᵧ. Second, from figure 8, we observe that the Skyrmion charge for (+1, 0, −1) levels off at an asymptotic value |Q_{sky}| ≈ 0.2. We lack an explanation for this particular asymptotic value. Once more, the topological nature can again be assured from the standpoint of view that the charge does not depend on the spatial location of the defect and that the charges of several defects add up.

4. Final comments

To the best of our knowledge, there are no previous reports on Skyrmions with arbitrary topological charges, thus, there remains as a task the full elucidation of whether this property can be fully considered as of topological nature. While such a problem is beyond the scope of the present study, our analysis.

Figure 7. Spin texture component fₓ as a function of ρ, for different values of Bₓ = Bₓρ, for the vortex structure (+2, +1, 0), upper panel, and (+1, 0, −1), lower panel. The dotted lines corresponds to a magnetic field centered at (x₀ = 0, y₀ = 0), while the continuous ones to a (x₀ = 0, y₀ = 2). This constitutes an indication that fₓ ≠ 0 as ρ → ∞ for Bₓ = Bₓρ. These cases are for a ⁸⁷Rb condensate.

Figure 8. Skyrmion topological charges Q_{sky} as a function of the amplitude Bₓ of the component Bₓ = Bₓρ of the external magnetic field. Upper panel corresponds to ⁸⁷Rb, lower one to ²³Na. vortex structures (+2, +1, 0), (+1, 0, −1), (0, −1 − 2) are indicated.
shows that the values of the topological charges are independent of the location of the singularities and, when there is more than one defect, the charges add up. On the other hand, as mentioned in the introduction, the use of topological concepts, beyond its intrinsic interest, may become an important and useful tool to classify and describe a complicated spinor condensate state given by its full wavefunction. It does, therefore, appear as an extra complication that the cated spinor condensate state given by its full wavefunction.

However, taken that this is the way these defects manifest themselves, we recall that, at least for the cases where the spin texture obeys $\vec{f} \cdot \vec{f} = 1$ everywhere, which is the case for the states $(0, -1, -2)$ and $(+2, +1, 0)$, the value of the Skyrmion topological charge $Q_\text{sky}$ can be found directly from the very simple expression given in equation (17), which indicates explicitly its dependence on the boundary values of $f_i(\rho)$. On the other hand, for the state $(+1, 0, -1)$, which should be expected to yield $Q_\text{sky} = 0$, one also finds that the charge changes depending on the boundary value of $f_i(\rho)$, yet we do not have an explicit expression to predict its value. Putting aside preconceived topological concepts, a fairly straightforward explanation for all cases is that the spin texture $\vec{f}$, away from the location of the $\vec{B}$-field symmetry axis, behaves as the expected magnetization of a magnetic gas, aligning parallel to the external field $\vec{B}$. This indicates that the boundary values of $\vec{f}$, and thus its Skyrmion charge, can be adjusted by appropriately tailoring the external magnetic field $\vec{B}$. In this regard, the ‘arbitrary’ value of this topological charge is also reminiscent of the ‘arbitrariness’ of Berry phases in spin systems.

An important question left to be addressed is the stability of the states here considered. We have numerically tested this stability criterion as follows. First, we recall that the states are found through a minimization numerical procedure of the energy functional given by equation (1), which is equivalent to solving the time-independent GP equation. Part of this procedure allows to finding the corresponding chemical potential $\mu$. If the state is stationary, it should not evolve under the propagation of the time-dependent GP,

$$i\hbar \frac{\partial \Phi_n}{\partial t} = \frac{\hbar^2}{2m} \nabla^2 \Phi_n + V_{\text{ext}} \Phi_n + c_0 \Phi_n^* \Phi_k \Phi_k + c_2 \Phi_n^* \vec{F}_0 \cdot \vec{F}_0 \Phi_n + \mu\vec{B} \cdot \vec{F}_G \Phi_n,$$

(19)

with initial condition $\Phi_n(\vec{r}, 0) = \Psi_n(\vec{r})$. Indeed, the states found were evolved for more than 100 time units, approximately 50 ms for the systems here considered, and they remained completely stable. That is, the time evolved states are simply given by $\Phi_n(\vec{r}, t) = \Psi_n(\vec{r}) e^{-i\omega t/\hbar}$. Some of the states, as mentioned above, have already been shown to be stable in [47]. Moreover, we also expect the topological defects to be quite stable, as shown in [53], where it is observed that non-stationary vortex states evolve in time, with the vortices precessing in the cloud, without dissipating into other structures.

Finally, we discuss the feasibility of experimentally observing these defects. First of all, as mentioned in the introduction, there is current theoretical and experimental interest in these topological defects [21–33]. In particular, there are already reported experimental observations of Skyrmions that concerns the present study, starting with the formation of coreless vortices in an $F = 1$ spinor BEC of $^{23}$Na [30], where a magnetic Ioffe–Pritchard trap was used, then with the observation of 2D Skyrmions also in $^{23}$Na, within an optical trap and a 3D quadrupole magnetic field [27, 31], and the creation of Skyrmions and half-Skyrmions in a $F = 2$ spinor BEC in $^{87}$Rb, where the spin texture is prepared with properly polarized Raman beams. [26] In those studies, the obtained Skyrmions are of integer of half-integer charge, since the effective Zeeman fields have the corresponding boundary condition, as discussed in this paper. Therefore, in order to observe arbitrary Skyrmion charges, there remains the challenge of producing Zeeman fields with different boundary condition, as for example given in equation (3). As already indicated, a direct implementation of such a field may need the presence of an external electrical current within the condensate, an apparent current technical difficulty. However, as we have shown, the appearance of defects with arbitrary topological charges depends not only on the Zeeman being cylindrically symmetric, but more importantly in having arbitrary boundary condition far from the condensate. That is, the external magnetic field proposed here may not need to be the only solution. An alternative may be to use optical means to create an inhomogeneous coupling to spinor components, simulating a Zeeman term, similarly to the production of artificial gauge fields in Bose condensates, such as in [50–52]. It remains as a further study to establish the precise form to implement it experimentally.

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