COMPARISON RESULTS FOR UNBOUNDED SOLUTIONS FOR A PARABOLIC
CAUCHY-DIRICHLET PROBLEM WITH SUPERLINEAR GRADIENT GROWTH

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Abstract. In this paper we deal with uniqueness of solutions to the following problem
\[
\begin{aligned}
&u_t - \Delta_p u = H(t, x, \nabla u) &\quad &\text{in } Q_T, \\
u(t, x) = 0 &\quad &\text{on } (0, T) \times \partial \Omega, \\
u(0, x) = u_0(x) &\quad &\text{in } \Omega,
\end{aligned}
\]
where \(Q_T = (0, T) \times \Omega\) is the parabolic cylinder, \(\Omega\) is an open subset of \(\mathbb{R}^N\), \(N \geq 2\), \(1 < p < N\), and the right hand side \(H(t, x, \xi) : (0, T) \times \Omega \times \mathbb{R}^N \to \mathbb{R}\) exhibits a superlinear growth with respect to the gradient term.

1. Introduction

The present paper is devoted to the study of the uniqueness and, more in general, to the comparison principle between sub and supersolutions of nonlinear parabolic problems with lower order terms that have at most a power growth with respect to the gradient. More specifically, we set \(\Omega\) a bounded open subset of \(\mathbb{R}^N\), with \(N \geq 3\), and \(T > 0\). We consider a Cauchy–Dirichlet problem of the type
\[
\begin{aligned}
&u_t - \Delta_p u = h(t, x, \nabla u) + f(t, x) &\quad &\text{in } Q_T, \\
u(t, x) = 0 &\quad &\text{on } (0, T) \times \partial \Omega, \\
u(0, x) = u_0(x) &\quad &\text{in } \Omega,
\end{aligned}
\]
where \(Q_T = (0, T) \times \Omega\) denotes the parabolic cylinder, \(-\Delta_p\) is the usual \(p\)-Laplacian with \(p > 1\), the functions \(u_0\) and \(f\) belong to suitable Lebesgue spaces and \(h(t, x, \xi)\) is a Cartheodory function that has (at most) \(q\)-growth with respect to the last variable, being \(q\) “superlinear” and smaller than \(p\).

The model equation we have in mind is the following
\[
\begin{aligned}
&u_t - \Delta_p u = |\nabla u|^q + f(t, x) &\quad &\text{in } Q_T, \\
u(t, x) = 0 &\quad &\text{on } (0, T) \times \partial \Omega, \\
u(0, x) = u_0(x) &\quad &\text{in } \Omega,
\end{aligned}
\]
for \(1 < q < p\), \(f \in L^r(0, T; L^m(\Omega))\), for some \(r, m \geq 1\), and \(u_0 \in L^s(\Omega)\), for some \(s \geq 1\).

The literature about comparison principles for weak sub/super solutions of (1.2) is mainly devoted to cases in which solutions are smooth (say for instance continuous), the equation is exactly the one in (1.2) or the growth of the nonlinear term is “sublinear”.

Our aim is to generalize this kind of results to the case of unbounded solutions and non regular data (both the initial datum and the forcing term), dealing with sub/supersolutions in a suitable class.

Let us mention that in the elliptic framework such a kind of results have been studied in several papers using different techniques. Let us recall the papers [3], [4] [10], [11], [12], [24] (and references cited therein) where unbounded solutions for quasilinear equations have been treated. We want also to highlight the results of [28] (see also[6], [22] and [21]) that have inspired our work, where the comparison principle among unbounded sub/supersolutions has been proved, for sub/supersolutions that have a suitable power that belongs to the energy space.

Let us also mention that, as well explained in [1] (see also [2] for the parabolic counterpart) things change drastically when one deals with the so called natural growth (i.e. \(q = p\) in (1.2)), since in this case the right class in which looking for uniqueness involves a suitable exponential of the solution (one can convince himself just by performing the Hopf-Cole transformation to the equation in (1.2)).

The literature is much poorer in the parabolic case, especially when unbounded solutions are considered. Let us mention the results in [18],[17] where nonlinear problem of the type (1.1) are considered where \(h(t, x, \xi)\) has a sublinear (in the sense of the \(p\)-Laplacian type operators, see [23] for more details about such a threshold) growth with respect to the last variable.

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In order to prove the comparison principle (that has the uniqueness has byproduct), several techniques have been developed. Let us mention, among the others, the results that have been proved by using the monotone rearrangement technique (see for instance [10] and references cited therein) and by means of viscosity solutions (see for instance [16], [5] and references cited therein).

Our choice, that has been mainly inspired by [28], uses both an argument via linearization and a method that exploits a sort of convexity of the hamiltonian term with respect to the gradient. These two approaches are, in some sense, complementary since the first one (the linearization) works in the case $1 < p \leq 2$ while the second one (the “convex” one) deals with $p \geq 2$. Of course, the only case in which both of them are in force is when $p = 2$.

Since we want to deal with unbounded solutions and irregular data, the way of defining properly the sub/supersolutions is through the renormalized formulation (see [13], [25], [14] and [26]).

The renormalized formulation, that is the most natural one in this framework, is helpful in order to face the first difficulty of our problem, that is the unboundedness of the sub/supersolutions. Indeed we can decompose the sub/supersolutions into their bounded part plus a reminder that can be estimated, using the uniqueness class we are working in.

According with the results in the stationary case, we prove that the uniqueness class (i.e. the class of functions for which we can prove the comparison principle, and uniqueness as a byproduct) is the set of functions whose a suitable power $\gamma$ (that depends only on $q$, $p$ and $N$) belongs to the energy space. Let us recall (see [23]) that such a class is also the right one in order to have existence of solutions.

Even more, we show, through a counterexample, that at least for $p = 2$, the class of uniqueness is the right one, adapting an argument of [7]-[8] to our case.

We first consider the case with $1 < p \leq 2$, and we look for an inequality solved by the difference between the bounded parts of the sub and supersolutions, using the linearization of the lower order term. Let us recall that this is the typical approach for singular (i.e. $p \leq 2$) operators, that has extensively used in several previous papers (see for instance [18] and [3] and references cited therein). In this case we are allowed to deal with general Leray-Lions operators, even if, due to a lack of regularity of the the sub/supersolutions, we cannot cover all the superlinear and subnatural growths.

The second part of the paper is devoted to the case $p \geq 2$ that is, in some way, more complicated, due to the degenerate nature of the operator. In fact, we need to straight the hypotheses on the differential operator considering a perturbation (through a matrix with bounded coefficients) of the standard $p$-Laplacian.

Here the idea is to perturb the difference between the bounded parts of the sub and supersolutions and to exploit the convexity of the lower order term with respect to the gradient (at least in the case $p = 2$, otherwise the general hypothesis is more involved).

The plan of the paper is the following: in Section 2 we collect all the statement of our results, while Section 3 is devoted to some technical results. The proofs of the main results are set in Section 4, if $1 < p \leq 2$ and in Section 5 if $p \geq 2$.

Finally in the Appendix there is an example that shows that the uniqueness class is the right one, at least for $p = 2$ and $1 < q < 2$.

2. Assumptions and Statements of the Results

As already explained in the Introduction, we deal with the following Cauchy-Dirichlet problem:

$$
\begin{cases}
  u_t - \text{div } a(t,x,\nabla u) = H(t,x,\nabla u) & \text{in } Q_T, \\
  u(t,x) = 0 & \text{on } (0,T) \times \partial \Omega, \\
  u(0,x) = u_0(x) & \text{in } \Omega,
\end{cases}
$$

with $u_0 \in L^1(\Omega)$.

The main assumptions on the functions involved in (2.1) are the following: the vector valued function $a(t,x,\xi) : (0,T) \times \Omega \times \mathbb{R}^N \rightarrow \mathbb{R}^N$ is a Carathéodory function such that

$$
\exists \alpha > 0 : \quad (a(t,x,\xi) - a(t,x,\eta)) \cdot (\xi - \eta) \geq \alpha (|\xi|^2 + |\eta|^2)^{\frac{1}{2}} |\xi - \eta|^2, \quad (2.2)
$$

$$
\exists \beta > 0 : \quad |a(t,x,\xi)| \leq \beta \left[ f(t,x) + |\xi|^{p-1} \right], \quad \ell \in L^p(Q_T), \quad (2.3)
$$

$$
\begin{array}{l}
  a(t,x,0) = 0.
\end{array}
$$

(2.4)

for a.e. $(t,x) \in Q_T$, $\forall \eta, \xi \in \mathbb{R}^N$, with $1 < p < N$.

As far as the lower order term is concerned, we suppose that $H(t,x,\xi) : (0,T) \times \Omega \times \mathbb{R}^N \rightarrow \mathbb{R}^N$ is a Carathéodory function that satisfies the following growth condition:

$$
\exists c_1 > 0 : \quad |H(t,x,\xi)| \leq c_1 |\xi|^q + f \quad \text{with} \quad \max \left\{ \frac{p}{2}, \frac{p(N+1)-N}{N+2} \right\} < q < p, \quad (2.5)
$$

with $f \in L^{ \frac{2}{p-2}}(Q_T)$.
for a.e. \((t,x) \in Q_T, \forall \xi \in \mathbb{R}^N\) and with \(f = f(t,x)\) belonging to some Lebesgue space.

First of all we need to determine the meaning of sub/supersolutions we want to deal with. Since we are interested in possibly irregular data and, in general, in unbounded solutions, the most natural way to mean sub/supersolutions is through the renormalized formulation. In order to introduce such an issue, we first need to define a natural space where such sub/supersolutions are defined: taking inspiration from Lemma \(3.4\), we set

\[
\mathcal{T}^1_{0,p}(Q_T) = \left\{ u : Q_T \to \mathbb{R} \quad \text{a.e. finite : } T_k(u) \in L^p(0,T;W^{1,p}_0(\Omega)) \right\}, \quad \text{for } p \geq 1,
\]

where \(T_k(s) = \max\{-k, \min\{k,s\}\}, \text{ for } k \geq 0 \text{ and } s \in \mathbb{R}^N\).

Now we are ready to define the renormalized sub/super solutions to \((2.1)\).

**Definition 2.1.** We say that a function \(u \in \mathcal{T}^1_{0,p}(Q_T)\) is a renormalized subsolution (respectively a supersolution) of \((2.1)\) if

\[
H(t,x,\nabla u) \in L^1(Q_T), \quad u \in C([0,T]; L^1(\Omega))
\]

and it satisfies:

\[
- \int_{\Omega} S(u_0)\varphi(0) \, dx + \int_{Q_T} \left[ -S(u)\varphi_t + S'(u)\varphi a(t,x,\nabla u) \cdot \nabla \varphi + S''(u) a(t,x,\nabla u) \cdot \nabla u \varphi \right] \, dx \, dt \leq \left( \geq \right) \int_{Q_T} H(t,x,\nabla u)\varphi \, dx \, dt,
\]

with

\[
u(t,x)|_{t=0} = \begin{cases} \geq u_0(x) & \text{in } L^1(\Omega), \\
0 & \text{for every } S \in W^{2,\infty}(\mathbb{R}) \text{ such that } S'(\cdot) \text{ is nonnegative, compactly supported and for every } 0 \leq \varphi \in L^\infty(Q_T) \cap L^p(0,T; W^{1,p}_0(\Omega)), \varphi_t \in L^1(Q_T) + L^p(0,T; W^{-1,p}(\Omega)), \varphi(T,x) = 0.
\end{cases}
\]

Note that a subsolution (a supersolution) on \(Q_T\) for \((t,x,\xi)\) is deduced directly by the renormalized formulation, via a trace result (see Lemma \(3.4\)). More specifically, we have to impose such a condition only in the case in which we deal with \(L^1\)-data and with “low” values of \(q\).

**Remark 2.2.** Let us observe that usually the renormalized formulation is equipped with an additional condition about the asymptotic behavior of the energy, i.e.

\[
\lim_{n \to \infty} \frac{1}{n} \int_{\{n < |u| < 2n\}} |\nabla u|^p \, dx \, dt = \lim_{n \to \infty} \frac{1}{n} \int_{\{n < |v| < 2n\}} |\nabla v|^p \, dx \, dt = 0.
\]

Such a condition is required in order to guarantee that renormalized solution are, in fact, distributional ones. In our case we do not have to ask, in general, \((2.7)\) to hold since it is a consequence of the class of uniqueness that we consider (see Lemma \(3.4\)). More specifically, we have to impose such a condition only in the case in which we deal with \(L^1\)-data and with “low” values of \(q\).

**Remark 2.3.**

i) Note that a subsolution (a supersolution) on \(Q_T\) turns out to be a subsolution (a supersolution) on \(Q_t\) for any \(0 < t \leq T\). Thus, with an abuse of notation, we refer to Definition 2.1 even if we take into account \((2.6)\) evaluated over \(Q_t\), with \(0 < t \leq T\).

ii) For renormalized solutions of an equation of the type \((1.2)\), the regularity \(u \in C([0,T]; L^1(\Omega))\) is deduced directly by the renormalized formulation, via a trace result (see \([27]\)). However, since we are dealing with sub/supersolutions, we need to add it to the definition.

### 2.1. Assumptions for \(p = 2\)

As already announced in the Introduction, for problem \((2.1)\) with \(p = 2\) we can use both the approach by linearization and the one by “convexity”.

The first approach we want to deal with is the one by linearization. Hence we assume that \(a(t,x,\xi) : (0,T) \times \Omega \times \mathbb{R}^N \to \mathbb{R}^N\) satisfies \((2.2)-(2.4)\), that in this particular case read as:

\[
3\alpha > 0 : \quad a(t,x,\xi - \eta) \cdot (\xi - \eta) \geq \alpha |\xi - \eta|^2 \quad (2.8)
\]

\[
3\beta > 0 : \quad |a(t,x,\xi)| \leq \beta |a(t,x)| + |\xi| \quad \text{for } \ell \in L^2(Q_T),
\]

\[
a(t,x,0) = 0, \quad (2.9)
\]

a.e. \((t,x) \in Q_T, \forall \xi, \eta \in \mathbb{R}^N\).

Moreover we assume the growth assumption

\[
3\gamma > 0 : \quad |H(t,x,\xi)| \leq c_1 |\xi|^q + f(t,x) \quad \text{for } 1 \leq q < 2,
\]

a.e. in \((t,x) \in Q_T, \forall \xi \in \mathbb{R}^N\). In addition we suppose the following locally (weighted) Lipschitz condition

\[
3\gamma > 0 : \quad |H(t,x,\xi) - H(t,x,\eta)| \leq c_2 |\xi - \eta| \left[ g(t,x) + |\xi|^{q-1} + |\eta|^{q-1} \right] \quad (2.12)
\]

is in force, for some function \(g(t,x)\) belonging to a suitable Lebesgue space we specify later.
Before stating our comparison results, we need to introduce the class of uniqueness. As for the elliptic case (see [28, 6] and also [19]–[20]), the right framework is the set of sub/supersolutions \( u, v \) whose power \( \gamma = \gamma(q) \) belongs to the energy space for a suitable choice of \( \gamma \). Moreover we consider the initial data \( u_0, v_0 \) belonging to \( L^\sigma(\Omega) \) for some \( \sigma \geq 1 \). More precisely, we consider sub/supersolutions satisfying

\[
u, v \in C([0,T];L^\sigma(\Omega)) \quad \text{with} \quad \sigma = \frac{N(q-1)}{2-q} \tag{2.13}
\]

and

\[
(1 + |u|)^{\frac{2}{q}-1} u, (1 + |v|)^{\frac{2}{q}-1} v \in L^2(0,T;H^1_0(\Omega)). \tag{2.14}
\]

Such a class of uniqueness makes sense whenever \( 1 \leq \sigma \), i.e. if \( q \geq 2 - \frac{N}{N+1} \).

**Remark 2.4.** One can convince himself that the uniqueness class is the right one just by constructing a counterexample of a problem of the type (2.1) that admits (at least) two solutions, whose just one belongs to the right class. The construction of such a pair of solutions is a bit involved and we left it to the Appendix A.

The assumptions about the data are strictly related to the value of the superlinearity \( q \). For this reason, we split the superlinear growth of the gradient term into two subintervals for which, in turn, we require two different compatibility conditions on the data and two different class of uniqueness.

We first consider superlinear rates belonging to the range

\[
2 - \frac{N}{N+1} < q \leq 2 - \frac{N}{N+2} \tag{2.15}
\]

that correspond to the case \( 1 < \sigma \leq 2 \), and that allows us to consider \( f(t,x) \in L^\sigma(0,T;L^m(\Omega)) \) in (2.11) that verifies

\[
m \neq \infty, r \neq \infty \quad \text{s.t.} \quad \frac{N\sigma}{m} + \frac{2\sigma}{r} \leq N+2 \sigma \tag{2.16}
\]

while \( g(t,x) \in L^d(Q_T) \) in (2.12) satisfies

\[
g \in L^d(Q_T) \quad \text{with} \quad d = N+2. \tag{2.17}
\]

Our first result is the following.

**Theorem 2.5.** Assume that \( a(t,x,\xi) \) satisfies (2.8)–(2.10), \( H(t,x,\xi) \) satisfies (2.11) (2.12) and that (2.15)–(2.17) hold true. Let \( u \) and \( v \) be a renormalized subsolution and a supersolution of (2.1), respectively, satisfying (2.13), (2.14) and let \( u_0, v_0 \in L^\sigma(\Omega) \) such that \( u_0 \leq v_0 \). Then \( u \leq v \) in \( Q_T \).

**Remark 2.6.** As far as the limit the case \( q = 2 - \frac{N}{N+1} \) is concerned, we observe that the result of Theorem 2.5 still holds true assuming the data

\[
u_0 \in L^{1+\omega}(\Omega), f \in L^{1+\omega}(Q_T), \quad \forall \omega > 0,
\]

for sub/supersolutions \( u, v \) that belong to the class

\[
(1 + |u|)^{\frac{2}{q}-1} u, (1 + |v|)^{\frac{2}{q}-1} v \in L^2(0,T;H^1_0(\Omega)).
\]

The proof follows as the one of Theorem 2.5 with minor changes, so we omit it.

Secondly consider the range given by

\[
1 \leq q < 2 - \frac{N}{N+1} \tag{2.18}
\]

that correspond to \( \sigma < 1 \), and we require that the functions \( f \) and \( g \) satisfy

\[
f \in L^1(Q_T) \tag{2.19}
\]

and

\[
g \in L^d(Q_T) \quad \text{with} \quad d > N+2. \tag{2.20}
\]

Thus our result in this framework is the following.

**Theorem 2.7.** Assume that \( a(t,x,\xi) \) satisfies (2.8)–(2.10), \( H(t,x,\xi) \) satisfies (2.11) (2.12) and that (2.18)–(2.20) hold true. Let \( u \) and \( v \) be a renormalized subsolution and a supersolution of (2.1), respectively, satisfying (2.7) and let \( u_0, v_0 \in L^1(\Omega) \) such that \( u_0 \leq v_0 \). Then \( u \leq v \) in \( Q_T \).
Let us observe that, in fact, our results do not cover all the interval $1 \leq q < 2$. This is due to a lack of regularity of the sub/super solutions (see Remark 2.12 below).

The second approach to the comparison principle deals with a trick that uses the convexity of the lower order term. Such a method is not as robust as the linearization one, so we need to strengthen the hypotheses on the differential operator.

We consider here the following problem

$$
\begin{align*}
\begin{cases}
  u_t - \text{div} \left( A(t,x) \nabla u \right) = H(t,x,\nabla u) & \text{in } Q_T, \\
  u(t,x) = 0 & \text{on } (0,T) \times \partial \Omega, \\
  u(0,x) = u_0(x) & \text{in } \Omega.
\end{cases}
\end{align*}
$$

(2.21)

We assume that $A : (0,T) \times \Omega \to \mathbb{R}^{N \times N}$ is a bounded and uniformly elliptic matrix with measurable coefficients, i.e.

$$
A(t,x) = \begin{pmatrix} a_{ij}(t,x) \end{pmatrix}_{i,j=1}^N \quad \text{with} \quad a_{ij} \in L^\infty(Q_T) \quad \forall 1 \leq i,j \leq N,
$$

such that

$$
\exists \alpha, \beta : 0 < \alpha \leq \beta \quad \text{and} \quad \alpha |\xi|^2 \leq A(t,x) \xi \cdot \xi \leq \beta |\xi|^2,
$$

(2.22)

for almost every $(t,x) \in Q_T$ and for every $\xi \in \mathbb{R}^N$.

As far as the lower order term is concerned, we suppose that the nonlinear term $H(t,x,\xi)$ satisfies (2.11) with $1 < q < 2$ and it can be decomposed as

$$
H(t,x,\xi) = H_1(t,x,\xi) + H_2(t,x,\xi)
$$

(2.23)

where, for a.e. $(t,x) \in Q_T$ and for every $\xi, \eta \in \mathbb{R}^N$, the functions $H_1(t,x,\xi)$ and $H_2(t,x,\xi)$ verify:

- $H_1(t,x,\xi) : (0,T) \times \Omega \times \mathbb{R}^N \to \mathbb{R}$ is a convex function with respect to the $\xi$ variable, i.e.
  $$
  \forall \varepsilon \in (0,1) \; \; H_1(t,x,\xi \pm \varepsilon |\eta|) \leq \varepsilon H_1(t,x,\xi) + (1-\varepsilon)H_1(t,x,\eta) \; \forall \xi, \eta \in \mathbb{R}^N
  $$
  (2.24)

- $H_2(t,x,\xi) : (0,T) \times \Omega \times \mathbb{R}^N \to \mathbb{R}$ is a Lipschitz function with respect to $\xi$, namely
  $$
  \exists c_2 > 0 : |H_2(t,x,\xi) - H_2(t,x,\eta)| \leq c_2 |\xi - \eta|
  $$
  (2.25)

that satisfies the following inequality for sufficiently small $\varepsilon > 0$

$$
H_2(t,x,(1-\varepsilon)\xi) - (1-\varepsilon)H_2(t,x,\xi) \leq 0
$$

(2.26)

for almost every $(t,x) \in Q_T$ and for all $\xi, \eta \in \mathbb{R}^N$.

As for the approach by linearization, we have two types of results, depending on the regularity of the sub/supersolutions under consideration.

First we deal with solutions in the class (2.13)–(2.14): in this case we consider lower order terms whose growth with respect to the gradient is at most a power $q$ in the range

$$
2 - \frac{N}{N+1} < q < 2,
$$

(2.27)

and we assume that $f$ in (2.11) belongs to $L^r(0,T;L^m(\Omega))$ with $(m,r)$ such that (2.16) holds true.

Hence we have the following result.

**Theorem 2.8.** Assume that $A(t,x)$ satisfies (2.22) and $H(t,x,\xi)$ (2.11) together with (2.23)–(2.26), (2.27) and (2.16), and let $u$ and $v$ be, respectively, a renormalized subsolution and a supersolution of (2.21) satisfying (2.13)–(2.14). and let $u_0, v_0 \in L^q(\Omega)$ be such that $u_0 \leq v_0$. Then we have that $u \leq v$ in $Q_T$.

As far as the low values of $q$ are considered, we deal with the same range considered in (2.18) and $L^1$ data.

**Theorem 2.9.** Assume that $A(t,x)$ satisfies (2.22) and $H(t,x,\xi)$ (2.11) together with (2.23)–(2.26), (2.18) and (2.19). Let $u$ and $v$ be a renormalized subsolution and a supersolution of (2.21), respectively, satisfying (2.7) and let $u_0, v_0 \in L^1(\Omega)$ with $u_0 \leq v_0$. Then we have that $u \leq v$ in $Q_T$. 

2.2. **Assumptions for** $1 < p < 2$. Le us now go back to our original problem (2.1) and let us assume that the vector valued function $a(t, x, \xi)$ satisfies (2.2)–(2.4), with $1 < p < 2$.

As far as the lower order term is concerned, we suppose that it satisfies the growth condition (2.5) and we assume that a suitable weighted Lipschitz assumption with respect to the last variable is in force, i.e.

$$
\exists c_2 > 0 : \quad |H(t, x, \xi) - H(t, x, \eta)| \leq c_2 |\xi - \eta|^q [g + |\xi|^{q-1} + |\eta|^{q-1}]
$$

(2.28)

for a.e. $(t, x) \in Q_T$, for all $\xi, \eta \in \mathbb{R}^N$ and for some measurable function $g = g(t, x)$ belonging to $L^d(Q_T)$, for a suitable choice of $d \geq 1$.

In this setting, we determine two ranges of $q$ each of them giving a different type of result in function of the required class of uniqueness (and the regularity) of the solutions.

We start by considering

$$
p - \frac{N}{N+1} < q \leq p - \frac{N}{N+2} \quad \text{for} \quad 1 + \frac{N}{N+1} < p < 2
$$

and

$$
1 \leq q < p - \frac{N}{N+2} \quad \text{for} \quad 1 + \frac{N}{N+2} \leq p < 1 + \frac{N}{N+1}
$$

(2.29)

together with

$$
g \in L^d(Q_T) \quad \text{with} \quad d = \frac{N(g - (p - 1)) - p + 2q}{q - 1}.
$$

(2.30)

Moreover we assume that

$$
f \in L^r(0, T; L^m(\Omega)) \quad \text{with} \quad (m, r) \quad \text{such that}
$$

$$
m \neq \infty, \ r \neq \infty, \quad \frac{N\sigma}{m} + \frac{N(p-2) + p\sigma}{r} \leq N(p-1) + p\sigma.
$$

(2.31)

When we deal with this range, we assume the continuity regularity

$$
u, \ v \in C([0, T]; L^\sigma(\Omega)) \quad \text{with} \quad \sigma = \frac{N(g - (p - 1))}{p - q}
$$

(2.32)

and we deal with the uniqueness class

$$(1 + |u|)^{\gamma-1} u, (1 + |v|)^{\gamma-1} v \in L^p(0, T; W^{1,p}_0(\Omega)) \quad \text{with} \quad \gamma = \frac{\sigma + p - 2}{p}.
$$

(2.33)

In this ranges of values of $q$ we have that $\sigma \in (1, 2)$.

**Theorem 2.10.** Assume (2.2)–(2.4), and that $H(t, x, \xi)$ satisfies (2.5), (2.28) with (2.29)–(2.31). Let $u$ and $v$ be a subsolution and a supersolution of (2.1), respectively, satisfying (2.32)–(2.33). Then $u \leq v$ in $Q_T$.

The next range corresponds to the case of lower values of $q$. Namely, we consider

$$
1 \leq q < p - \frac{N}{N+1} \quad \text{and} \quad 1 + \frac{N}{N+1} < p < 2
$$

(2.34)

and that $g$ fulfils

$$
g \in L^d(Q_T) \quad \text{with} \quad d > \frac{p(N+1) - N}{p(N+1) - (2N + 1)} \quad (d = \infty \text{ if } p = 1 + \frac{N}{N+1}).
$$

(2.35)

As far as the source term $f$ is concerned, we suppose

$$
f \in L^1(Q_T).
$$

(2.36)

Thus the result is the following.

**Theorem 2.11.** Assume (2.2)–(2.3) and that $H(t, x, \xi)$ satisfies (2.5) and (2.28) with (2.34)–(2.36). Let $u$ and $v$ be a renormalized subsolution and a renormalized supersolution of (2.1), respectively, such that (2.7) holds true. Then, we have that $u \leq v$ in $Q_T$.

**Remark 2.12.** Let us mention some peculiarity of our results, for $1 < p \leq 2$.

i) It is worth pointing out that the case $p - \frac{N}{N+2} < q < p$ is not considered here. Indeed, as already observed in the elliptic case (see [28, Remark 3.5]), we would need to require more regularity on the gradient of the sub/supersolutions, in order to apply the linearization technique, which turns out to be unnatural in our framework. Indeed we should require the sub/supersolution to belong to the space

$L^\mu(0, T; W^{1,\mu}_0(\Omega)) \quad \text{with} \quad \mu = (\sigma - 1)(p - q) + q = N(q - (p - 1)) + 2q - p$
As far as the Hamiltonian term is concerned, we suppose, in addition to (a superlinear growth condition with respect to the gradient.

\[ \exists \alpha, \beta : 0 < \alpha \leq \beta \quad \text{and} \quad \alpha |\xi|^2 \leq A(x)|\xi|^2 \leq \beta |\xi|^2 \quad \text{a.e.} \ x \in \Omega, \ \forall \xi \in \mathbb{R}^N. \]

As far as the Hamiltonian term is concerned, we suppose, in addition to (2.5), that \( \exists M > 0 \), such that for any \( \varepsilon \in (0, 1) \)

\[ H(t, x, \xi) - (1 - \varepsilon)^{p-1} H \left( (1 - \varepsilon)^{p-2} t, x, \frac{\eta}{1 - \varepsilon} \right) \leq c_2 \varepsilon^{1-q} |\xi - \eta|^q \left( |\xi|^\frac{p+2}{q} + |\eta|^\frac{p+2}{q} \right) + \varepsilon M \]

\[ \text{with} \quad p - 1 < q < p, \]

\[ \text{a.e.} \ (t, x) \in Q_T, \text{ for all } \xi, \eta \in \mathbb{R}^N \text{ with } \xi \neq \eta \text{ and } \varepsilon \in (0, 1). \]

The above hypothesis seems to be quite technical, since it combines several properties of the nonlinear lower order term. It is not so hard to see that, for example, the model Hamiltonian

\[ H(t, x, \xi) = |\xi|^q + f_0(t, x) \]

satisfies (2.39) for \( p - 1 < q < p \) and for a function \( f_0 \) bounded above and not increasing with respect to the \( t \) variable. Let us underline that also some perturbations, through locally Lipschitz function, weighted with the \( (p - 2)/2 \) power of the gradient of such a model Hamiltonian still fulfill hypothesis (2.39).

As for the previous results, we have two regimes depending on the values of \( q \).

We first deal with the range

\[ p - \frac{N}{N+1} < q < p \]  

(2.40)

that let us considering solutions in the class (2.32)–(2.33).

**Theorem 2.13.** Assume (2.38) and that \( H(t, x, \xi) \) satisfies (2.5) together with (2.31), (2.39) and (2.40). Let \( u \) and \( v \) be a subsolution and a supersolution of (2.37), respectively, satisfying (2.32) and (2.33). Assume that \( u_0 \leq v_0 \leq w_0 < \infty \) with \( u_0, v_0 \in L^p(\Omega) \). Then we have that \( u \leq v \) in \( Q_T \).

Next, we consider the last case, that is the range

\[ p - 1 < q < p - \frac{N}{N+1} \]  

(2.41)

and we have the following result.

**Theorem 2.14.** Assume (2.38) and that \( H(t, x, \xi) \) satisfies (2.5) together with (2.36), (2.39) and (2.41). Let \( u \) and \( v \) be a subsolution and a supersolution of (2.37), respectively, such that (2.7) holds. Assume that \( u_0 \leq v_0 \leq w_0 < \infty \) with \( u_0, v_0 \in L^p(\Omega) \). Then we have that \( u \leq v \) in \( Q_T \).

3. Notation and basic tools

With the purpose of dealing with the bounded part of the sub/supersolutions considered during the paper, we here introduce a smooth approximation of the classical truncation function \( T_n(s) = \max\{-n, \min\{n, s\}\}, \)
We define the smoothed truncation function $S_n(\cdot)$ and $\theta_n(\cdot)$ as follows:

$$S_n(z) = \int_0^z \theta_n(\tau) \, d\tau,$$

while

$$\theta_n(z) = \begin{cases} 
\frac{1}{n} & \text{if } |z| \leq n, \\
\frac{2n - |z|}{n} & \text{if } n < |z| \leq 2n, \\
0 & \text{if } |z| > 2n.
\end{cases}$$

Moreover, here and in all the paper, we denote by $G_k(z)$ the function

$$G_k(z) = (z - k)^+$$

for every $z \in \mathbb{R}$ and for any $k \geq 0$.

Here we recall a classical parabolic regularity result that we use systematically in the following.

**Theorem 3.1** (Gagliardo-Nirenberg). Let $\Omega \subset \mathbb{R}^N$ be a bounded and open subset and $T > 0$; if

$$w \in L^\infty(0, T; L^h(\Omega)) \cap L^h(0, T; W_0^{1, \eta}(\Omega))$$

with

$$h, \eta \geq 1, \eta < N \quad \text{and} \quad h \leq \eta^*,$$

then we have that

$$w \in L^\nu(0, T; L^j(\Omega))$$

where the pair $(j, y)$ fulfills

$$0 \leq j \leq \nu^*, \eta \leq y \leq \infty \quad \text{and} \quad \frac{Nh}{j} + \frac{N(\eta - h) + \eta y}{y} = N.$$

Moreover there exists a positive constant $c(N, \eta, h)$ such that the following inequality holds true:

$$\int_0^T \|w(t)\|_{L^j(\Omega)}^y \, dt \leq c(N, \eta, h)\|w\|_{L^\infty(0, T; L^h(\Omega))}^{\frac{\nu}{\eta}} \int_0^T \|\nabla w(t)\|_{L^\nu(\Omega)}^\eta \, dt.$$  

(3.2)

Next we state two useful Lemmata that we will use in the sequel in order to conclude the proofs of our results.

**Lemma 3.2.** Let $w \in L^\rho(0, T; W_0^{1, \rho}(\Omega)) \cap L^\infty(0, T; L^{\rho^*}(\Omega))$, with $\rho, \nu \geq 1$ satisfy

$$\exists \xi_0 > 0, \eta \geq 1 \quad \text{and} \quad m > 0 : \quad \|G_k(w)\|_X \leq c_0 \|G_k(w)\|^m_X \left( \int_{E_k} B \, dx \right)^m$$

for some $B \in L^1(Q_T)$, where $X$ is a Banach space, and $E_k = \{(t, x) \in Q_T : w > k, |\nabla w| > 0\}$. Then $w \equiv 0$ in $Q_T$.

**Proof.** The proof follows directly from the one of Lemma 2.1 in [28].

The next Lemma is a sort of parabolic version of the above one.

**Lemma 3.3.** Let $w \in L^\rho(0, T; W_0^{1, \rho}(\Omega)) \cap C([0, T]; L^{\rho^*}(\Omega))$ be a function satisfying the following inequality:

$$\sup_{s \in (0, t)} \|w(s)\|_{L^{\rho^*}(\Omega)}^\eta + \|w\|_{L^\rho(0, t; W_0^{1, \rho}(\Omega))}^p \leq c_0 \sup_{s \in (0, t)} \|w(s)\|_{L^{\rho^*}(\Omega)}^\eta \|w\|_{L^\rho(0, t; W_0^{1, \rho}(\Omega))}^m$$

(3.3)

with $0 \leq t \leq T$ and for some $c_0 > 0, \eta > 0, m \geq p$ and with $\|w(0)\|_{L^{\rho^*}(\Omega)} = 0$. Then $w \equiv 0$ in $Q_T$.

**Proof.** Without loss of generality we take into account the case $m = p$. Then, since $\|w(0)\|_{L^{\rho^*}(\Omega)} = 0$ and by the continuity assumption we define

$$T^* = \sup \left\{ \tau > 0 : \|w(s)\|_{L^{\rho^*}(\Omega)} \leq \frac{1}{(2c_0)^\frac{1}{\eta}} \forall s \leq \tau \right\} > 0$$

which implies that, at least for $s \leq T^*$, we get

$$\sup_{s \in (0, t)} \|w(s)\|_{L^{\rho^*}(\Omega)}^\eta + \frac{1}{(2c_0)^\frac{1}{\eta}} \|w\|_{L^\rho(0, t; W_0^{1, \rho}(\Omega))}^p \leq 0$$

(3.4)

from (3.3). Now, let us suppose by contradiction that $T^* < T$. Then, if $t = T^*$ and by definition of $T^*$ we would find $\|w(T^*)\|_{L^{\rho^*}(\Omega)} \leq \frac{1}{(2c_0)^\frac{1}{\eta}} \leq 0$ which is in contrast with the assumption $c_0 > 0$. We thus deduce that (3.4) holds for all $t \leq T$ and, in particular, we conclude that $\|w(t)\|_{L^{\rho^*}(\Omega)} \equiv 0$ for every $t \in [0, T]$.

During the proof of our main results, we need some regularity results, as the next two Lemmata.
Lemma 3.4. Let $u \in C([0, T]; L^p(\Omega))$ be such that (2.33) holds true. Then
\[ |\nabla u| \in L^{N(q - (p - 1)) + 2q - p}(Q_T) \quad \text{with} \quad 1 \leq \sigma \leq 2 \quad \text{\big(i.e.,} \quad \frac{N}{N + 2} \leq p - q \leq \frac{N}{N + 1} \big) \] (3.5)
and
\[ \frac{1}{n} \int_{(n < |u| < 2n)} |\nabla u|^p \, dx \, dt = o(n^{-(\sigma - 1)}) \quad \text{as} \quad n \to \infty. \] (3.6)

Proof. Let us start with the proof of the regularity in (3.5); using (2.33) and Theorem 3.1 with $\eta = p$ and $k = \frac{2}{\gamma}$ we deduce that $u \in L^{\frac{N}{N - 1} + \nu}(Q_T)$. Then, by Young’s inequality, we get
\[ |\nabla u|^\frac{N}{N + \nu} \leq c |\nabla u|^p + c(1 + u)^{\frac{N}{N + \nu}} \]
and (3.5) follows by definitions of $\sigma$ and $\gamma$.

As far as (3.6) is concerned, we have the inequality
\[ \frac{n^{\sigma - 1}}{n} \int_{(n < |u| < 2n)} |\nabla u|^p \, dx \, dt \leq c \gamma \int_{(n < |u| < 2n)} |\nabla u|^\gamma \, dx \, dt. \]
Then, since $n \geq 1$ and consequently dealing with $|u| > 1$, it implies that we can employ (2.33) and thus mean$\{(t, x) \in (0, T) \times \Omega : n < |u| \leq 2n\} \to 0$ as $n \to \infty$ and thus (3.6) follows.

Lemma 3.5. Let $u, v$ respectively be a renormalized subsolution and a renormalized supersolution of (2.1) such that (2.7), (2.5) with (2.3)) hold. Then
\[ |\nabla u_+|, |\nabla v_-| \in L^r(Q_T) \quad \text{for} \quad 1 \leq r < p - \frac{N}{N + 1}. \] (3.7)

Proof. We only deal with the case of subsolution $u$, since having $v$ be a supersolution implies that $-v$ is a subsolution.

In fact, (3.7) is a consequence of Corollary 4.7 (see also Remark 4.2) applied to the positive part of $u$, that yields to the following inequality
\[ \int_{\Omega} \Theta(u_+(T)) \, dx + \alpha \int_{Q_T} |\nabla T_k(u_+)|^p \, dx \, dt \leq k \left[ \|H(t, x, \nabla u)\|_{L^1(Q_T)} + \|u_0\|_{L^1(\Omega)} \right]. \]
By the standard results on the regularity, we deduce that $u_+ \in L^r(0, T; W^{1, r}(\Omega))$ for every $r < p - \frac{N}{N + 1}$ (see [29] and [15]).

We conclude this Section with another useful result.

Lemma 3.6. Let $\rho, m \geq 1$ and $w \in T_0^{1, \rho}(Q_T) \cap L^\infty(0, T; L^m(\Omega))$ satisfy
\[ \|w\|^\rho \in L^\infty(0, T; L^\infty(\Omega)) \quad \text{and} \quad \int_{Q_T} |\nabla w|^\rho (|w| + \mu)^\gamma \, dx \, dt \leq M \mu^{-\nu} \] (3.8)
where $L, M, \mu, \nu > 0$ and $0 < \gamma < \rho$. Then
\[ \exists \varepsilon = c(\rho, \gamma, N, \mu, m) : \quad \|\nabla w|^\rho \in L^1(Q_T) \quad \leq c \left( L^{\frac{N}{N + m}} M \right) \frac{M^b(N + m) + 1}{M^{b(N + m) + 1}} \quad \text{with} \quad b = \frac{N(\rho - \gamma) + m\rho}{N + m}. \] (3.9)

Proof. The above assumptions on $w$ imply that we can apply Theorem 3.1 to the function
\[ \left( (|T_k(w)| + \mu)^{\frac{\rho}{\rho - \gamma}} - \mu^{\frac{\rho}{\rho - \gamma}} \right) \in L^\infty(0, T; L^\infty(\Omega)) \cap L^\rho(0, T; W^{1, \rho}_0(\Omega)) \]
and get the regularity estimate
\[ \int_{Q_T} \left( (|T_k(w)| + \mu)^{\frac{\rho}{\rho - \gamma}} - \mu^{\frac{\rho}{\rho - \gamma}} \right)^{\frac{N + m}{\rho - \gamma}} \, dx \, dt \]
\[ \leq \int_{L^\infty(0, T; L^\infty(\Omega))} \int_{Q_T} |\nabla T_k(w)|^\rho (|w| + \mu)^\gamma \, dx \, dt. \] (3.10)
Then, since the inequality $(a + b)^\alpha \leq a^\alpha + b^\alpha$ holds for $a, b > 0, 0 < \alpha < 1$ and $\frac{\rho - \alpha}{\alpha - \gamma} < 1$, we have by (3.10), combined with (3.8), and Fatou’s Lemma
\[ \int_{Q_T} \left( (|w| + \mu)^{\frac{\rho}{\rho - \gamma}} \right)^{\frac{N + m}{\rho - \gamma}} \, dx \, dt \leq c \left( \mu^\gamma M^b(N + m) + L^\infty M^{-\nu} \right). \]
Minimizing the right hand side above with respect to \( \mu \), we get that the minimum is achieved at \( \mu = c \left( \frac{L}{N} \right)^{\frac{p}{p-\rho+b}} \) for a constant \( c \) that depends only on \( \rho, \gamma, N, \nu, m \).

We are now ready to prove (3.9): we use the Hölder’s inequality with \( \left( \frac{p}{\sigma}, \frac{\rho}{\rho-b} \right) \) in order to get
\[
\iint_{Q_T} \left| \nabla u \right|^b \, dx \, dt \leq \left( \iint_{Q_T} \left| \nabla u \right|^p \, dx \, dt \right)^{\frac{b}{p}} \left( \iint_{Q_T} \left( |w| + \mu \right)^{\frac{\rho}{\rho-b}} \, dx \, dt \right)^{\frac{p-b}{p}}
\]
and we use that \( b \) satisfies
\[
\frac{\rho}{\rho-b} = \frac{N(\sigma-\gamma)+mp}{N},
\]
so that (3.9) follows from the choice of \( \mu \).

\[ \square \]

4. Proofs in the case \( 1 < p \leq 2 \)

We start by proving Theorem 2.10 and Theorem 2.5.

**Proof of Theorem 2.10 and of Theorem 2.5.** As anticipated, we need to rewrite the inequalities satisfied by sub/supersolutions in terms of their bounded parts plus some (quantified) reminder. To this aim, we set
\[
u_n = S_n(u) \quad \text{and} \quad v_n = S_n(v)
\]
where \( S_n(\cdot) \) has been defined in (3.1). We consider the renormalized formulation in (2.6) with \( S(u) = \theta_n(u) \) so we obtain
\[
\int_{\Omega} u_n(t)\varphi(t) \, dx + \int_{Q_T} a(s, x, \nabla u)\theta_n(u) \cdot \nabla \varphi + a(s, x, \nabla u) \cdot \nabla u \theta'_n(u)\varphi \, dx \, ds
\]
\[
\leq \int_{Q_T} H(s, x, \nabla u)\theta_n(u)\varphi \, dx \, ds + \int_{\Omega} u_n(0)\varphi(0) \, dx.
\]
Reasoning in the same way on the supersolution \( v \) - of course, with \( S(v) = \theta_n(v) \) - and considering the difference between the above inequalities, we get \( \forall 0 < t \leq T, \)
\[
\int_{\Omega} (u_n(t) - v_n(t))\varphi(t) \, dx
\]
\[
+ \int_{Q_T} [a(s, x, \nabla u)\theta_n(u) - a(s, x, \nabla v)\theta_n(v)] \cdot \nabla \varphi + [a(s, x, \nabla u) \cdot \nabla u \theta'_n(u) - a(s, x, \nabla v) \cdot \nabla v \theta'_n(v)] \varphi \, dx \, ds
\]
\[
\leq \int_{Q_T} [H(s, x, \nabla u)\theta_n(u) - H(s, x, \nabla v)\theta_n(v)] \varphi \, dx \, ds + \int_{\Omega} [u_n(0) - v_n(0)]\varphi(0) \, dx.
\]
We now define
\[
z_n = u_n - v_n
\]
and rewrite the above inequality as
\[
\int_{\Omega} z_n(t)\varphi(t) \, dx + \int_{Q_T} [a(s, x, \nabla u_n) - a(s, x, \nabla v_n)] \cdot \nabla \varphi \, dx \, ds
\]
\[
\leq \int_{Q_T} [H(s, x, \nabla u_n) - H(s, x, \nabla v_n)] \varphi \, dx \, ds + R_n
\]
where
\[
R_n = \int_{Q_T} [a(s, x, \nabla u_n) - a(s, x, \nabla v_n) - a(s, x, \nabla u)\theta_n(u) + a(s, x, \nabla v)\theta_n(v)] \cdot \nabla \varphi \, dx \, ds
\]
\[
- \int_{Q_T} [a(s, x, \nabla u) \cdot \nabla u \theta'_n(u) - a(s, x, \nabla v) \cdot \nabla v \theta'_n(v)] \varphi \, dx \, ds
\]
\[
+ \int_{Q_T} [H(s, x, \nabla u)\theta_n(u) - H(s, x, \nabla v)\theta_n(v) - H(s, x, \nabla u_n) + H(s, x, \nabla v_n)] \varphi \, dx \, ds
\]
\[
+ \int_{\Omega} z_n(0)\varphi(0) \, dx
\]
In virtue of the density result [26, Proposition 4.2], we are allowed to take
\[
\varphi(z_n) = \left[ \left( G_k(z_n) + \mu \right)^{\sigma} \right] - \mu^{\sigma-1}, \quad \mu > 0,
\]
in the inequality in (4.3) and then, thanks also to (2.2) and (2.28), we obtain
\[
\int_{\Omega} \Phi_k(z_n(t)) \, dx + \alpha(\sigma - 1) \int_{Q_T} \left[ |\nabla u_n|^2 + |\nabla v_n|^2 \right] \frac{|\nabla G_k(z_n)|^2}{(G_k(z_n) + \mu)^{2-\sigma}} \, dx \, ds
\]
\[
\leq c_1 \int_{Q_T} \left[ |\nabla G_k(z_n)|^q + |\nabla v|^q \right] \left[ \left( (G_k(z_n) + \mu)^{\sigma-1} \right) - \mu^{\sigma-1} \right] \, dx \, ds + R_n
\]
where \( \Phi_k(z) \, dx = \int_0^{G_k(z)} \varphi(\tau) \, d\tau. \)
We want to prove now that \( \lim_{n \to \infty} R_n = 0 \). From now on, we denote by \( \omega_n \) any quantity that vanishes as \( n \) diverges, and we set
\[
R_n = I_1 + I_2 + I_3 + I_4
\]
where
\[
I_1 = \int_Q [a(s,x,\nabla u_n) - a(s,x,\nabla v_n)] - [\theta_n(u) a(s,x,\nabla u) - \theta_n(v) a(s,x,\nabla v)] \cdot \nabla z_n \varphi'(z_n) \, dx \, ds,
\]
\[
I_2 = -\int_Q [\theta_n'(u) a(s,x,\nabla u) \cdot \nabla u - \theta_n'(v) a(s,x,\nabla v) \cdot \nabla v] \varphi(z_n) \, dx \, ds,
\]
\[
I_3 = \int_Q [\theta_n(u) H(s,x,\nabla u) - \theta_n(v) H(s,x,\nabla v)] - [H(s,x,\nabla u_n) - H(s,x,\nabla v_n)] \varphi(z_n) \, dx \, ds
\]
\[
I_4 = \int_\Omega z_n(0) \varphi(0) \, dx.
\]
Let us start by studying \( I_1 \). The definition of \( \theta_n(\cdot) \) and (2.4) imply that
\[
I_1 = \int_{\{n<|u|<2n\}} [a(s,x,\nabla u_n) - \theta_n(u) a(s,x,\nabla u)] \cdot \nabla (u_n - v_n) \varphi'(u_n - v_n) \, dx \, ds
\]
being \( a(s,x,\nabla u)\theta_n(u) - a(s,x,\nabla u_n) \equiv 0 \) when \( |u| \leq n \) and since \( \nabla u_n \equiv 0 \) when \( |u| \geq 2n \) (and the same holds for \( v_n \)). We just prove that the first integral in (4.7) behaves as \( \omega_n \) since the second one can be dealt in the same way. The definition of \( \varphi(\cdot) \) and (2.3) allow us to deduce that
\[
\int_{\{n<|u|<2n\}} [a(s,x,\nabla u_n) - \theta_n(u) a(s,x,\nabla u)] \cdot \nabla (u_n - v_n) \varphi'(u_n - v_n) \, dx \, ds
\]
\[
\leq c n^{-2-\sigma} \left[ \int_{\{n<|u|<2n\}} |\nabla u|^p \, dx \, ds + \int_{\{n<|u|<2n\}} |\nabla u|p^\gamma dx \right]
\]
\[
+ \int_{\{n<|u|<2n, \, |v|<2n\}} |\nabla u|p^{-1} |\nabla v| \, dx \, ds + \int_{\{n<|v|<2n\}} |\nabla v|\ell \, dx \, ds
\]
and thanks to (2.33) we estimate the first integral in the right hand side above, since
\[
n^{-2-\sigma} \int_{\{n<|u|<2n\}} |\nabla u|^p \, dx \, ds \leq c \int_{\{n<|u|<2n\}} |\nabla u|\gamma|^p \, dx \, ds = \omega_n,
\]
using that \( |(t,x) \in (0,T) \times \Omega : n < |u| < 2n| \to 0 \) as \( n \to \infty \). As far as the third integral is concerned, Hölder’s inequality with \((p,p')\) implies
\[
n^{-2-\sigma} \int_{\{n<|u|<2n, \, |v|<2n\}} |\nabla u|^{p-1} |\nabla v| \, dx \, ds
\]
\[
\leq c n^{-2-\sigma} \left[ \int_{\{n<|u|<2n\}} |\nabla u|^p \, dx \, ds \right]^\frac{1}{p} \left( \int_{\{|v|<2n\}} |\nabla v|^p \, dx \, ds \right)^{\frac{1}{p'}}
\]
\[
\leq \left( \int_{\{n<|u|<2n\}} |\nabla u|^\gamma \, dx \, ds \right)^\frac{1}{p} \left( \int_{\{Q_T\}} |\nabla v|\gamma \, dx \, ds \right)^{\frac{1}{p'}} = \omega_n,
\]
thanks again to (2.33). Finally, we deal with the second term in (4.8) (the fourth is treated in the same way) applying again Hölder’s inequality with \((p,p')\) and so obtaining
\[
c n^{-2-\sigma} \int_{\{n<|u|<2n\}} |\nabla u|\ell \, dx \, ds
\]
\[
\leq c n^{-2-\sigma} \left[ \int_{\{n<|u|<2n\}} |\nabla u|^p \, dx \, ds \right]^\frac{1}{p} \left( \int_{\{n<|u|<2n\}} |\ell|^p' \, dx \, ds \right)^{\frac{1}{p'}}
\]
\[
\leq \left( \int_{\{n<|u|<2n\}} |\nabla u|\gamma \, dx \, ds \right)^\frac{1}{p} \left( n^{-2-\sigma} \int_{\{Q_T\}} |\ell|\gamma' \, dx \, ds \right)^{\frac{1}{p'}} = \omega_n.
\]
A similar argument can be done for the last term in (4.7), so that we have that \( I_1 = \omega_n \).
By the definition of $\theta_n(\cdot)$ and since $|\varphi(u_n - v_n)| \leq c n^{\sigma - 1}$, $I_2$ can be estimated as

$$I_2 \leq \frac{c}{n^{2-\sigma}} \left[ \int_{\{n < |u| < 2n\}} |a(s, x, \nabla u)||\nabla u||dx| ds + \int_{\{n < |v| < 2n\}} |a(s, x, \nabla v)||\nabla v||dx| ds \right],$$

and $I_2 = \omega_n$ thanks to (2.3) and Lemma 3.4 (see (3.6)).

As far as $I_3$ is concerned, we have that

$$|I_3| \leq c n^{\sigma - 1} \left[ \int_{\{n < |u| < 2n\}} |\theta_n(u)H(s, x, \nabla u) - H(s, x, \nabla u_n)|\,dx| ds \right]$$

and

$$\int_{\{n < |v| < 2n\}} |\theta_n(v)H(s, x, \nabla v) - H(s, x, \nabla u_n)|\,dx| ds \right],$$

by definition of $\theta_n(\cdot)$: indeed, $\theta_n(u)H(s, x, \nabla u) - H(s, x, \nabla u_n) = 0$ if $|u| \leq n$ and $|\nabla u_n| = 0$ if $|u| \geq 2n$. We only consider the first term in the inequality above since the second one can be dealt with in the same way. Thanks to the growth assumption (2.11), the desired convergence of the first term follows once we prove that

$$cn^{\sigma - 1} \int_{\{n < |u| < 2n\}} |\theta_n(u)H(s, x, \nabla u) - H(s, x, \nabla u_n)|\,dx| ds \leq c \int_{\{n < |u| < 2n\}} |\nabla u|^q |u|^{\sigma - 1}\,dx| ds + \int_{\{n < |v| < 2n\}} |f||u|^\sigma\,dx| ds = \omega_n.$$  

An application of Hölder’s inequality with indices $(\frac{p}{q}, \frac{p}{p-q})$, Sobolev’s embedding and Theorem 3.1 (see (3.2)) lead us to

$$\int_{\{n < |u| < 2n\}} |\nabla u|^q |u|^{\sigma - 1}\,dx| ds \leq \int_0^t \left( \int_{\{n < |u(s)| \leq 2n\}} |\nabla u(s)|^p\,dx \right)^{\frac{q}{p}} \left( \int_{\{n < |u(s)| \leq 2n\}} |u(s)|^{\sigma - 1 + \frac{q}{p}\gamma}\,dx \right)^{\frac{q}{p}\gamma} \,ds$$

$$\leq c \sup_{s \in [0,t]} \left( \int_{\Omega} |u(s)|^\gamma\,dx \right)^{\frac{p}{q\gamma}} \int_{\{n < |u| < 2n\}} |\nabla u|^\gamma |u|^\sigma\,dx| ds = \omega_n$$

for the same reasons given above. As far as the integral involving the forcing term is concerned, we observe that, applying the Hölder inequality with indices $(m, m')$ and $(r, r')$ as in (2.31), we get

$$\int_{\{n < |u| < 2n\}} |f||u|^\sigma\,dx| ds \leq \|f\| L^{r'}(0,T; L^m(\Omega)) \left[ \int_0^t \left( \int_{\{n < |u(s)| \leq 2n\}} |u(s)|^{m'\gamma - 1}\,dx \right)^{\frac{r'}{\gamma}} \,ds \right].$$

We go further invoking Theorem 3.1 with $w = |u|^\gamma$ and, in particular, the inequality in (3.2) becomes

$$\int_0^t \left( \int_{\{n < |u(s)| \leq 2n\}} |u(s)|^\gamma m\,dx \right)^{\frac{1}{\gamma}} \,ds \leq c_{GN} \|u\| L^\gamma(0,T; L^\infty(\Omega)) \int_{\{n < |u| < 2n\}} |\nabla u|^\gamma |u|^\sigma\,dx| ds.$$  

We observe that since the couple $(m, r)$ satisfies (2.31), we have that the pair $(j, y)$ fulfills

$$j \geq m'\gamma - 1 \quad \text{and} \quad y \geq r'^{-}\gamma - 1.$$  

We thus proceed through Lebesgue spaces inclusion and we deduce

$$\|f\| L^{r'}(0,T; L^m(\Omega)) \left[ \int_0^t \left( \int_{\{n < |u(s)| \leq 2n\}} |u(s)|^\gamma m\,dx \right)^{\frac{1}{\gamma}} \,ds \right]^{\frac{\gamma}{\gamma - 1}}$$

$$\leq c_{GN} \|u\| L^\gamma(0,T; L^\infty(\Omega)) \left( \int_{\{n < |u| < 2n\}} |\nabla u|^\gamma |u|^\sigma\,dx| ds \right)^{\frac{\gamma}{\gamma - 1}} = \omega_n,$$

that implies $I_3 = \omega_n$. Finally, recalling that $u_0 \leq v_0$ and the definition of $\theta_n(\cdot)$, we conclude that also $I_4 \leq \omega_n$, so that that $R_n = \omega_n$.

We now get into the main step of the proof. Let $E_{n,k}$ be the subset of $Q_t$ defined by

$$E_{n,k} = \{(t, x) \in Q_t : z_n > k \quad \text{and} \quad |\nabla z_n| > 0\},$$

and we set

$$E_{n,1} = [g + |\nabla u_n|^\gamma - 1 + |\nabla v_n|^\gamma - 1] \chi_{\{u_n > v_n\}}, \quad E_{n,2} = [\nabla u_n]^2 + |\nabla v_n|^2]^{\frac{\gamma}{\gamma - 1}} \chi_{\{u_n > v_n\}},$$

(4.11)
where the parameter $a \leq p \leq 2$ has to be fixed. An application of Hölder’s inequality with indices $(\frac{2}{a}, \frac{2}{2-\sigma})$ and the inequality in (4.4) provide with the following estimate:

$$
\int_{Q_t} \frac{|\nabla G_k(z_n)|^a}{(G_k(z_n) + \mu)^{\frac{a}{2}}(2-\sigma)} \, dx \, ds \\
\leq \left( \int_{E_{n,k}} B_{n,2} \, dx \, ds \right)^{\frac{2}{2-a}} \left( \int_{E_{n,k}} \frac{|\nabla G_k(z_n)|^2}{(G_k(z_n) + \mu)^{2-\sigma}} \left[ |\nabla u_n|^2 + |\nabla v_n|^2 \right] \, dx \, ds \right)^{\frac{2}{2}} \\
\leq c \left( \int_{E_{n,k}} B_{n,2} \, dx \, ds \right)^{\frac{2}{2-a}} \left( \int_{Q_t} |\nabla G_k(z_n)| B_{n,1} \left[ (G_k(z_n) + \mu)^{\sigma-1} - \mu^{\sigma-1} \right] \, dx \, ds + \omega_n \right)^{\frac{2}{2}}.
$$

(4.13)

We recall (2.30) (i.e. we know that $g \in L^2(\Omega)$ for $d = \frac{N(q-(p-1))}{q-1}-p+2q$) and we set $a$ such that $\frac{a(2-p)}{2-a} = d(q-1) = N(q-(p-1)) + 2q - p$, namely

$$
a = \frac{2d(q-1)}{2 - p + d(q-1)} = \frac{2(N(q-(p-1)) - p + 2q)}{(N+2)(q-(p-1))}.
$$

Then, the gradient regularity (3.5) contained in Lemma 3.4 applied on both $|\nabla u_n|$ and $|\nabla v_n|$ (we recall also (2.33)) implies that the first integral in the right hand side of (4.13) is finite. Now, let us focus on the second one. Hölder’s inequality with indices $\left(a, d, \frac{2p(N\gamma+\sigma)}{N} \right)$ (indeed, $1 - \frac{1}{a} - \frac{1}{d} = \frac{1}{2} \frac{\sigma(p-q)}{(p-\sigma)-p+2q} = \frac{2}{\sigma} \frac{p(N\gamma+\sigma)}{N}$ by definitions of $\sigma$ and $\gamma$) yields to

$$
\int_{Q_t} |\nabla G_k(z_n)| B_{n,1} \left[ (G_k(z_n) + \mu)^{\sigma-1} - \mu^{\sigma-1} \right] \, dx \, ds \\
\leq \int_{Q_t} \left[ |\nabla G_k(z_n)| (G_k(z_n) + \mu)^{\frac{\sigma}{2}} B_{n,1} \left[ (G_k(z_n) + \mu)^{\sigma-1} - \mu^{\sigma-1} \right] \right] \, dx \, ds \\
\leq \left( \int_{Q_t} \frac{|\nabla G_k(z_n)|^a}{(G_k(z_n) + \mu)^{\frac{a}{2}}(2-\sigma)} \, dx \, ds \right)^{\frac{1}{2}} \left( \int_{E_{n,k}} B_{n,1}^d \, dx \, ds \right)^{\frac{1}{2}} \times \\
\times \left( \int_{Q_t} \left[ (G_k(z_n) + \mu)^{\sigma-1} - \mu^{\sigma-1} \right] \left( G_k(z_n) + \mu \right)^{\frac{\sigma}{2}} \right)^{\frac{2p(N\gamma+\sigma)}{Np\sigma}} \, dx \, ds \right) + \omega_n.
$$

(4.14)

and then we take advantage of (4.14) in (4.13), we obtain

$$
\int_{Q_t} |\nabla G_k(z_n)| B_{n,1} \left[ (G_k(z_n) + \mu)^{\sigma-1} - \mu^{\sigma-1} \right] \, dx \, ds \\
\leq c \left( \int_{E_{n,k}} B_{n,1}^d \, dx \, ds \right)^{\frac{2}{2-a}} \left( \int_{E_{n,k}} B_{n,2} \, dx \, ds \right)^{\frac{2}{2-a}} \times \\
\times \left( \int_{Q_t} \left[ (G_k(z_n) + \mu)^{\sigma-1} - \mu^{\sigma-1} \right] \left( G_k(z_n) + \mu \right)^{\frac{\sigma}{2}} \right)^{\frac{2p(N\gamma+\sigma)}{Np\sigma}} \, dx \, ds \right) + \omega_n.
$$

(4.15)

We observe that the first two integrals in the right hand side above are bounded thanks to (2.30), the definition of $a$ and Lemma 3.4. Moreover, using (4.15) in (4.14) leads to

$$
\int_{Q_t} |\nabla G_k(z_n)| B_{n,1} \left[ (G_k(z_n) + \mu)^{\sigma-1} - \mu^{\sigma-1} \right] \, dx \, ds \\
\leq c \left( \int_{E_{n,k}} B_{n,1}^d \, dx \, ds \right)^{\frac{2}{2-a}} \left( \int_{E_{n,k}} B_{n,2} \, dx \, ds \right)^{\frac{2}{2-a}} \times \\
\times \left( \int_{Q_t} \left[ (G_k(z_n) + \mu)^{\sigma-1} - \mu^{\sigma-1} \right] \left( G_k(z_n) + \mu \right)^{\frac{\sigma}{2}} \right)^{\frac{2p(N\gamma+\sigma)}{Np\sigma}} \, dx \, ds \right) + \omega_n.
$$

(4.16)

and the uniform estimate of the right hand side in (4.4) is closed.

Next, we observe that the definitions of $a, \gamma$ and $\sigma$ imply that

$$
\frac{2}{\sigma} \frac{p(N\gamma+\sigma)}{N} = a \frac{N + 2}{N}
$$

\frac{a(2-p)}{2-a} = d(q-1) = N(q-(p-1)) + 2q - p.

where \( a \frac{N+2}{N} \) is the Gagliardo-Nirenberg regularity exponent applied with spaces
\[ L^\infty(0, T; L^2(\Omega)) \cap L^a(0, T; W^{1,a}_0(\Omega)). \]
In particular, the inequality in (3.2) applied to the function
\[ \left( G_k(z_n) + \mu \right)^{\sigma-1} - \mu^{\sigma-1} \]
gives
\[ \int_{Q_t} \left( \left( G_k(z_n) + \mu \right)^{\sigma-1} - \mu^{\sigma-1} \right)^{\frac{1}{\sigma}} (G_k(z_n) + \mu)^{\frac{1}{\sigma}} \frac{a \frac{N+2}{N}}{d} \ dx \ ds \]
(4.17)
\[ \leq c_{GN} \| G_k(z_n) + \mu \|_{L^\infty(0, t; L^a(\Omega))} \int_{Q_t} \frac{|\nabla(G_k(z_n))|^a}{(G_k(z_n) + \mu)^{\frac{a}{p} - \sigma}} \ dx \ ds. \]
So far, we know that the second integral in (4.17) is bounded thanks to (4.15). Furthermore, it holds from (4.4) and (4.16) that
\[ \| G_k(z_n) + \mu \|_{L^\infty(0, t; L^a(\Omega))} \leq \int_\Omega \Phi_k(z_n(t)) \ dx \]
\[ \leq c \left( \int_{E_{n,k}} B_{d,1}^d \ dx \ ds \right)^{\frac{2}{d}} \left( \int_{E_{n,k}} B_{n,2}^d \ dx \ ds \right)^{\frac{2}{d} - \frac{\sigma}{a}} \times \]
\[ \times \left( \int_{Q_t} \left( \left( G_k(z_n) + \mu \right)^{\sigma-1} - \mu^{\sigma-1} \right)^{\frac{1}{\sigma}} (G_k(z_n) + \mu)^{\frac{1}{\sigma}} \frac{a \frac{N+2}{N}}{d} \ dx \ ds \right) + \omega_n \]
and then, using (4.15) and (4.17), it follows that
\[ \int_{Q_t} \left( \left( G_k(z_n) + \mu \right)^{\sigma-1} - \mu^{\sigma-1} \right)^{\frac{1}{\sigma}} (G_k(z_n) + \mu)^{\frac{1}{\sigma}} \frac{a \frac{N+2}{N}}{d} \ dx \ ds \]
\[ \leq c \left( \int_{E_{n,k}} B_{d,1}^d \ dx \ ds \right)^{\frac{2}{d}} \left( \int_{E_{n,k}} B_{n,2}^d \ dx \ ds \right)^{\frac{2}{d} - \frac{\sigma}{a}} \times \]
\[ \times \left( \int_{Q_t} \left( \left( G_k(z_n) + \mu \right)^{\sigma-1} - \mu^{\sigma-1} \right)^{\frac{1}{\sigma}} (G_k(z_n) + \mu)^{\frac{1}{\sigma}} \frac{a \frac{N+2}{N}}{d} \ dx \ ds \right) + \omega_n \]
Finally, passing to the limit first with respect to \( n \) (we apply Lebesgue Theorem on the right hand side) and subsequently as \( \mu \to 0 \), we obtain:
\[ \int_{Q_t} G_k(u - v) \frac{a \frac{N+2}{N}}{d} \ dx \ ds \leq c \left( \int_{E_k} B_1^d \ dx \ ds \right)^{\frac{2}{d}} \left( \int_{E_k} B_2^d \ dx \ ds \right)^{\frac{2}{d} - \frac{\sigma}{a}} \left( \int_{Q_t} G_k(u - v) \frac{a \frac{N+2}{N}}{d} \ dx \ ds \right) \]
(4.18)
where
\[ E_k = \{(t, x) \in Q_t : \ |u - v > k \ and \ |\nabla(u - v)| > 0\} \]
while
\[ B_1 = \left[ g + |\nabla u|^q + |\nabla v|^q \right] \chi_{\{u > v\}} \quad \text{and} \quad B_2 = \left[ |\nabla u|^q + |\nabla v|^q \right]^{\frac{N(q-(p-1))}{N+(p-1)}} \chi_{\{u > v\}}. \]
We conclude the proof applying Lemma 3.2 with \( \rho = a, \nu = 2 \) to the function \( G_k(u - v) \).

Using the same ideas, adapted to the case of \( L^1 \) data and low values of \( q \), we prove now Theorem 2.11 and Theorem 2.7.

Proof of Theorem 2.11 and Theorem 2.7. Before getting into the real proof, we recall (2.35) and observe that
\[ \left[ g + |\nabla u|^q + |\nabla v|^q \right] \chi_{\{u > v\}} \in L^d(Q_T). \]
Indeed, Lemma 3.5 provides us that \( |\nabla u|, |\nabla v| \in L^d(Q_T) \) for every \( q < 2 - \frac{N}{N-1} \). So \( |\nabla u|^q \) and \( |\nabla v|^q \) belong to \( L^q(Q_T) \) for every \( r > N + 2 \). Furthermore, being \( (u - v)_+ \) a subsolution itself and reasoning as in the just mentioned Lemma with \( T_k(z_n) \), then \( (u - v)_+ \) inherits the \( L^q(0, T; W^{1,q}_0(\Omega)) \) regularity for every \( q < 2 - \frac{N}{N-1} \). We underline that we use the fact that \( \omega_n \) is assumed to be uniformly bounded in \( n \) - it will soon proved - and also converging to 0. Then, the regularity in (4.19) follows since
\[ |\nabla v| \chi_{\{u > v\}} \leq |\nabla v| \chi_{\{v < 0\}} + |\nabla v| \chi_{\{0 < v < u\}} \leq |\nabla v| + |\nabla u| + |\nabla(u - v)|, \]
and
\[ |\nabla u| \chi_{\{u > v\}} \leq |\nabla v| + |\nabla u| + |\nabla(u - v)|. \]
We use the same notation of Theorem 2.10 (see (4.1) and (4.2)); in particular we take again \( S(u) = S_n(u) \) in (4.3) and \( S(v) = S_n(v) \) in the formulation of the supersolution, and we set
\[
\varphi = \varphi(z_n) = \frac{1}{\mu^\lambda} - \frac{1}{(G_k(z_n) + \mu)^\lambda} \quad \text{with} \quad \lambda = \frac{N+1}{N}(p-q) - 1 \quad \text{and} \quad \mu > 0
\]
getting
\[
\int_{\Omega} \Phi_k(z_n(t)) \, dx + \lambda \int \left( \int_{Q_t} \frac{|\nabla (G_k(z_n))|^2}{(G_k(z_n) + \mu)^{\lambda+1}} \left[ |\nabla u_n|^2 + |\nabla v_n|^2 \right]^{\frac{\gamma}{\gamma-2}} \, dx \, ds \right) \leq \frac{c_1}{\mu^\lambda} \int \left( \int_{Q_t} |\nabla G_k(z_n)| \, B_{n,1} \, dx \, ds \right. + \frac{R_n}{\mu^\lambda} \right)
\]
where \( B_{n,1} \) and \( R_n \) have been defined in (4.12) and (4.6), respectively.

Once again, we have that \( \lim_{n \to \infty} R_n = 0 \); the proof of this fact is quite similar to the one contained in Theorem 2.10. We just recall the decompositions (4.5)–(4.6) and that, using the current choice of \( \varphi(\cdot) \) and the asymptotic energy condition (2.7), we deduce that \( I_1 + I_2 = \omega_n \). The terms \( I_3 \) and \( I_4 \) follow as in Theorem 2.10, just observing that now \( f \) only belong to \( L^1(Q_T) \), using again (2.7).

The uniform boundedness of the right hand side in (4.20) follows from the above remark by (4.19).

Now, let the parameter \( a \leq p \) be such that \( \frac{a(2-p)}{2-a} = q \), i.e. \( a = \frac{2q}{q+p+2} \). Then, recalling the inequality in (4.20), we obtain
\[
\int_{Q_t} \frac{|\nabla G_k(z_n)|^a}{(G_k(z_n) + \mu)^{\lambda+1}} \, dx \, ds \leq \frac{c}{\mu^\lambda} \left( \int_{E_{n,k}} |\nabla G_k(z_n)| \, B_{n,1} \, dx \, ds + \omega_n \right)^\frac{q}{p} \left( \int_{E_{n,k}} B_{n,2} \, dx \, ds \right)^\frac{2-p}{p} \tag{4.21}
\]
for \( B_{n,1}, B_{n,2} \) as in (4.12), \( E_{n,k} \) as in (4.11). In particular, (4.20) provides us with
\[
\int_{\Omega} \Phi_k(z_n(t)) \, dx \leq \frac{c}{\mu^\lambda} \int_{E_{n,k}} |\nabla G_k(z_n)| \, B_{n,1} \, dx \, ds + \frac{\omega_n}{\mu^\lambda}. \tag{4.22}
\]
Furthermore, the definition of \( \Phi_k(\cdot) \) implies
\[
\Phi_k(w) \geq \frac{c}{\mu^\lambda} G_k(w) + c, \tag{4.23}
\]
with \( c \) independent from \( \mu \), so that
\[
\int_{\Omega} \Phi_k(z_n(t)) \, dx \geq \frac{c}{\mu^\lambda} \int_{\Omega} G_k(z_n(t)) \, dx + c.
\]
We use (4.23) in (4.22), so that
\[
\int_{\Omega} G_k(z_n(t)) \, dx \leq c \int_{E_{n,k}} |\nabla G_k(z_n)| \, B_{n,1} \, dx \, ds + \omega_n + c\mu^\lambda
\]
that becomes, taking the limits as \( \mu \to 0 \), and then as \( n \to +\infty \),
\[
\int G_k(u(t) - v(t)) \, dx \leq c \int_{E_k} |\nabla G_k(u-v)| \, B_1 \, dx \, ds,
\]
where the last convergence follows thanks to Lemma 3.5.

Furthermore, letting \( n \to \infty \) also in (4.21), we get
\[
\int_{Q_t} \frac{|\nabla G_k(u-v)|^a}{[G_k(u-v) + \mu]^{\lambda+1}} \, dx \, ds \leq \frac{c}{\mu^\lambda} \left( \int_{Q_t} |\nabla G_k(u-v)| \, B_1 \, dx \, ds \right)^\frac{q}{p} \left( \int_{E_k} B_2 \, dx \, ds \right)^{\frac{2-p}{p}}
\]
with \( E_k \) as in (4.18) and
\[
B_1 = \left[ g + |\nabla u|^{q-1} + |\nabla v|^{q-1} \right] \chi_{(u>v)}, \quad B_2 = \left[ |\nabla u|^2 + |\nabla v|^2 \right]^{\frac{q}{2}} \chi_{(u>v)}.
\]
We now apply Lemma 3.6 with
\[
\rho = a, \quad \nu = \frac{\lambda a}{2}, \quad m = 1, \quad \gamma = a \frac{1 + \lambda}{2}
\]
and
\[
M = c \left( \int_{Q_t} |\nabla G_k(u-v)| \, B_1 \, dx \, ds \right)^\frac{q}{p} \left( \int_{E_k} B_2 \, dx \, ds \right)^{\frac{2-p}{p}}, \quad L = c \int_{Q_t} |\nabla G_k(u-v)| \, B_1 \, dx \, ds.
\]
In particular, the estimate in (3.9) holds with
\[
b = a \frac{N(1-\lambda) + 2}{2(N+1)} = q
\]
by the definitions of \( a, \lambda \) and
\[
\| \nabla G_k(u - v) \|_{L^q(C, \mathbb{R}^d)} \leq c \left( \int_{Q_t} \| \nabla G_k(u - v) \|_{L^p} dx \right) \left( \int_{E_k} B_{2q} dx ds \right)^{(2 - \gamma)/(N + 1)}.
\]

We apply Hölder’s inequality with \((q, q')\) obtaining
\[
\| \nabla G_k(u - v) \|_{L^q(C, \mathbb{R}^d)} \leq c \| \nabla G_k(u - v) \|_{L^q(C, \mathbb{R}^d)} \left( \int_{Q_t} B_{2q}^{1 - \gamma} dx \right) \left( \int_{E_k} B_{2q} dx ds \right)^{(2 - \gamma)/(N + 1)}.
\]

Observe that, since \( \frac{p}{q} \leq \frac{N}{N + 1} \) as \( q \not> \frac{N}{N + 1} \), the integral involving \( B_1 \) is bounded (up to choose \( q \) closer to the threshold). Then we conclude by applying Lemma 3.2 with \( \rho = q \).

## 5. Proofs in the case \( 2 \leq p < N \)

We start by proving the results for \( p = 2 \), since their proofs are different from those of the case \( p > 2 \).

### 5.1. The case \( p = 2 \)

**Proof of Theorem 2.8.** We follow the same notation that we have used for the proof of Theorem 2.10, by defining \( u_n, v_n \) as in (4.1). Thus we consider the inequalities in (2.6) satisfied by the sub/supersolutions with \( S(u) = u_n \) and \( S(v) = v_n \) respectively so that, we have
\[
\int_{\Omega} u_n(t) \varphi(t) dx + \int_{Q_t} A(s, x) \nabla u_n \cdot \nabla \varphi + A(s, x) \nabla u \theta'_n(u) \varphi dx ds \\
\leq \int_{Q_t} H(t, x, \nabla \theta_n(u)) \varphi dx ds + \int_{\Omega} u_n(0) \varphi(0) dx
\]
and
\[
\int_{\Omega} (1 - \varepsilon) v_n(t) \varphi(t) dx + \int_{Q_t} A(s, x) \nabla ((1 - \varepsilon) v_n) \cdot \nabla \varphi + A(s, x) \nabla ((1 - \varepsilon) v) \cdot \nabla \theta'_n(v) \varphi dx ds \\
\geq (1 - \varepsilon) \int_{Q_t} H(t, x, \nabla v) \theta_n(v) \varphi dx ds + (1 - \varepsilon) \int_{\Omega} v_n(0) \varphi(0) dx
\]
where the inequality related to the supersolution has been multiplied by \( (1 - \varepsilon) \), for \( \varepsilon \in (0, 1) \). Then, taking into account the difference between the inequalities above, we get
\[
\int_{\Omega} \left[ u_n(t) - (1 - \varepsilon) v_n(t) \right] \varphi(t) dx + \int_{Q_t} A(s, x) \nabla \left( u_n - (1 - \varepsilon) v_n \right) \cdot \nabla \varphi dx ds \\
+ \int_{Q_t} \left[ A(s, x) \nabla u \cdot \nabla \theta'_n(u) - A(s, x) \nabla ((1 - \varepsilon) v) \cdot \nabla \theta'_n(v) \right] \varphi dx ds \\
\leq \int_{Q_t} \left[ H(t, x, \nabla \theta_n(u)) - (1 - \varepsilon) H(t, x, \nabla \theta_n(v)) \right] \varphi dx ds + \int_{\Omega} z^*_n(0) \varphi(0) dx.
\]
We use the hypothesis (2.23) in order to rewrite the above inequality as
\[
\int_{\Omega} \left( u_n(t) - (1 - \varepsilon) v_n(t) \right) \varphi(t) dx + \int_{Q_t} A(s, x) \nabla \left( u_n - (1 - \varepsilon) v_n \right) \cdot \nabla \varphi dx ds \\
\leq \int_{Q_t} \left[ H_1(s, x, \nabla u_n) - (1 - \varepsilon) H_1(s, x, \nabla v_n) \right] \varphi + [H_2(s, x, \nabla u_n) - (1 - \varepsilon) H_2(s, x, \nabla v_n)] \varphi dx ds + R_n,
\]
where
\[
R_n = \int_{Q_t} \left[ A(s, x) \nabla u \cdot \nabla \theta'_n(u) - A(s, x) \nabla ((1 - \varepsilon) v) \cdot \nabla \theta'_n(v) \right] \varphi dx ds \\
+ \int_{Q_t} \left[ H(s, x, \nabla u) \theta_n(u) - (1 - \varepsilon) H(s, x, \nabla v) \theta_n(v) - H(s, x, \nabla u_n) + (1 - \varepsilon) H(s, x, \nabla v_n) \right] \varphi dx ds \\
+ \int_{\Omega} z^*_n(0) \varphi(0) dx.
\]
Let us observe that the conditions (2.25) and (2.26) imply that
\[
H_2(s, x, \nabla u_n) - (1 - \varepsilon)H_2(s, x, \nabla v_n)
\]
\[
\leq |H_2(s, x, \nabla u_n) - H_2(s, x, (1 - \varepsilon)\nabla v_n)| + H_2(s, x, (1 - \varepsilon)\nabla v_n) - (1 - \varepsilon)H_2(s, x, \nabla v_n)
\]
\[
\leq c_2|\nabla (u_n - (1 - \varepsilon)v_n)| \leq c_2 \left[ \frac{|\nabla (u_n - (1 - \varepsilon)v_n)|^q}{\varepsilon^q} + 1 \right],
\]
by Young’s inequality. On the other hand, since \( H_1(t, x, \xi) \) satisfies (2.24) (i.e. the convexity assumption with respect to \( \xi \)), we have
\[
H_1(s, x, \nabla u_n) - (1 - \varepsilon)H_1(s, x, \nabla v_n) \leq \varepsilon H_1 \left( t, x, \frac{\nabla (u_n - (1 - \varepsilon)v_n)}{\varepsilon} \right).
\]
Finally, the growth assumption on \( H_1(s, x, \xi) \) contained in (2.11) allows us to improve (5.1) as
\[
\int_{Q_t} (u_n(t) - (1 - \varepsilon)v_n(t)) \varphi(t) \, dx + \int_{Q_t} A(s, x) \nabla (u_n - (1 - \varepsilon)v_n) \cdot \nabla \varphi \, dx \, ds
\]
\[
\leq \varepsilon \int_{Q_t} \left( c_1 |\nabla (u_n - (1 - \varepsilon)v_n)|^q + \tilde{f} \right) \, dx \, ds + R_n
\]
where \( \tilde{f} = f + c \). In particular, the inequality above can be written in terms of the function
\[
z_n^\varepsilon(t, x) = \frac{e^{-t/\varepsilon}}{\varepsilon} (u_n(t, x) - (1 - \varepsilon)v_n(t, x))
\]
as
\[
\int_{Q_t} z_n^\varepsilon(t) \varphi(t) \, dx + \int_{Q_t} z_n^\varepsilon \varphi \, dx \, ds + \int_{Q_t} A(s, x) \nabla z_n^\varepsilon \cdot \nabla \varphi \, dx \, ds
\]
\[
\leq \int_{Q_t} \left( c_1 |\nabla z_n^\varepsilon|^q + \tilde{f} \right) \varphi \, dx \, ds + \frac{R_n}{\varepsilon}.
\]
We consider the inequality in (5.3) with
\[
\varphi(z_n^\varepsilon) = \int_0^{G_k(z_n^\varepsilon)} (\mu + |w|)^{\sigma-2} \, dw, \quad \mu > 0,
\]
(again, we recall the density results in [26, Proposition 4.2]) getting
\[
\int_{Q_t} \Phi_k(z_n^\varepsilon(t)) \, dx + \int_{Q_t} z_n^\varepsilon \varphi \, dx \, ds + \alpha \int_{Q_t} |\nabla \Psi_k(G_k(z_n^\varepsilon))|^2 \, dx \, ds
\]
\[
\leq c_1 \int_{Q_t} |\nabla G_k(z_n^\varepsilon)|^q \left( \int_0^{G_k(z_n^\varepsilon)} (\mu + |w|)^{\sigma-2} \, dw \right) \, dx \, ds
\]
\[
+ \int_{Q_t} |\tilde{f}| \chi_{\{\tilde{f} \geq k\}} \left( \int_0^{G_k(z_n^\varepsilon)} (\mu + |w|)^{\sigma-2} \, dw \right) \varphi \, dx \, ds + \int_{Q_t} |\tilde{f}| \chi_{\{|\tilde{f}| \leq k\}} \varphi \, dx \, ds + \frac{R_n}{\varepsilon},
\]
with
\[
\Psi_k(G_k(z_n^\varepsilon)) = \int_0^{G_k(z_n^\varepsilon)} (\mu + |w|)^{\frac{n-2}{2}} \, dw \quad \text{and} \quad \Phi_k(z_n^\varepsilon(t)) = \int_0^{G_k(z_n^\varepsilon(t))} \varphi(w) \, dw.
\]
Observe that since \( \varphi \) is supported where \( z_n^\varepsilon \geq k \), then
\[
\int_{Q_t} z_n^\varepsilon \varphi \, dx \, ds - \int_{Q_t} |\tilde{f}| \chi_{\{|\tilde{f}| \leq k\}} \varphi \, dx \, ds \geq k \int_{Q_t} \varphi \, dx \, ds - k \int_{Q_t} \varphi \, dx \, ds \geq 0,
\]
and we are reduced to study
\[
\int_{Q_t} \Phi_k(z_n^\varepsilon(t)) \, dx + \alpha \int_{Q_t} |\nabla \Psi_k(G_k(z_n^\varepsilon))|^2 \, dx \, ds
\]
\[
\leq c_1 \int_{Q_t} |\nabla G_k(z_n^\varepsilon)|^q \left( \int_0^{G_k(z_n^\varepsilon)} (\mu + |w|)^{\sigma-2} \, dw \right) \, dx \, ds + \int_{Q_t} |\tilde{f}| \chi_{\{|\tilde{f}| > k\}} \left( \int_0^{G_k(z_n^\varepsilon)} (\mu + |w|)^{\sigma-2} \, dw \right) \, dx \, ds + \frac{R_n}{\varepsilon}.
\]
We just note that, by definitions of \( \theta_n(\cdot) \), \( \varphi(\cdot) \) and thanks to (2.22), (2.33), the proof of \( \lim_{n \to \infty} R_n = 0 \) follows reasoning as in Theorem 2.10.
We observe that the definition of \( \Psi_k(\cdot) \) combined with Hölder’s inequality with indices \( \left( \frac{1}{2-q}, \frac{1}{q-1} \right) \) and also an application Young inequality with \( \left( \frac{2}{q}, \frac{2}{q-1} \right) \) leads to

\[
\frac{1}{\sigma} \int_{Q_1} |\nabla G_k(z_n^\varepsilon)|^q \left( \int_0^{G_k(z_n^\varepsilon)} (\mu + |w|)^{(\sigma-2)} \, dw \right) \, dx \, ds \\
\leq c \frac{1}{\sigma} \int_{Q_1} |\nabla \Psi_k(G_k(z_n^\varepsilon))|^q \left( \int_0^{G_k(z_n^\varepsilon)} (\mu + |w|)^{(\sigma-2)} \, dw \right) \, dx \, ds \\
\leq c \frac{1}{\sigma} \int_{Q_1} |\nabla \Psi_k(G_k(z_n^\varepsilon))|^q |\Psi_k(G_k(z_n^\varepsilon))|^{2-q} G_k(z_n^\varepsilon)^{q-1} \, dx \, ds \\
\leq c \frac{1}{\sigma^2} \int_{Q_1} |\nabla \Psi_k(G_k(z_n^\varepsilon))|^2 dx \, ds + c \frac{1}{\sigma^2} \int_{Q_1} |\Psi_k(G_k(z_n^\varepsilon))|^2 G_k(z_n^\varepsilon) \frac{2(q-1)}{(2-q-1)} \, dx \, ds.
\]

We now focus on the term involving the source \( f \). We recall the estimate in Theorem 2.10 and apply Hölder’s inequality with \((m, m')\) and \((r, r')\) getting

\[
\frac{1}{\sigma} \int_{Q_1} |\tilde{f}(\chi_{\{\tilde{f} > k\}}) \left( \int_0^{G_k(z_n^\varepsilon)} (\mu + |w|)^{(\sigma-2)} \, dw \right) \, dx \, ds \\
\leq c \frac{1}{\sigma} \int_{Q_1} |\tilde{f}(\chi_{\{\tilde{f} > k\}}) \left( \int_0^{G_k(z_n^\varepsilon)} (\mu + |w|)^{(\sigma-2)} \, dw \right) \, dx \, ds \\
\leq c \|\tilde{f}(\chi_{\{\tilde{f} > k\}})\|_{L^r(0, T, L^{m'}(\Omega))} \|\Psi_k(G_k(z_n^\varepsilon))\|_{L^r(\Omega)}^2 \frac{2(q-1)}{(2-q-1)} \, dx \, ds
\]

where \( r, m \) verifies (2.31). Finally, we gather together the estimates above and find that

\[
\int_\Omega \Phi_k(z_n^\varepsilon(t)) \, dx + \frac{\alpha}{2} \int_{Q_1} |\nabla \Psi_k(G_k(z_n^\varepsilon))|^2 \, dx \, ds \leq c \int_{Q_1} |\Psi_k(G_k(z_n^\varepsilon))|^2 G_k(z_n^\varepsilon) \frac{2(q-1)}{(2-q-1)} \, dx \, ds + c \|\tilde{f}(\chi_{\{\tilde{f} > k\}})\|_{L^r(0, T, L^{m'}(\Omega))} \|\Psi_k(G_k(z_n^\varepsilon))\|_{L^r(\Omega)}^2 \, dx \, ds
\]

and, reasoning as in (4.10), we get

\[
\|\Psi_k(G_k(z_n^\varepsilon))\|_{L^2(\Omega)}^2 \leq c \|G_k(z_n^\varepsilon)\|_{L^{q-1}(0, T, L^\infty(\Omega))} \|\nabla \Psi_k(G_k(z_n^\varepsilon)(s))\|_{L^2(\Omega)}^2 ds,
\]

so that the inequality (5.4) reads as

\[
\int_\Omega \Phi_k(z_n^\varepsilon(t)) \, dx + \frac{\alpha}{2} \int_{Q_1} |\nabla \mu_k(G_k(z_n^\varepsilon))|^2 \, dx \, ds \\
\leq C_1 \left[ \|G_k(z_n^\varepsilon)\|_{L^{q-1}(0, T, L^\infty(\Omega))} + \|\Psi_k(G_k(z_n^\varepsilon))\|_{L^{q-1}(0, T, L^\infty(\Omega))} \|\Psi_k(G_k(z_n^\varepsilon)(s))\|_{L^2(\Omega)}^2 ds \right] \\
+ C_2 \|\tilde{f}(\chi_{\{\tilde{f} > k\}})\|_{L^r(0, T, L^{m'}(\Omega))} + \int_\Omega \Phi_k(v_0) \, dx + \omega_n
\]

for any \( \mu > 0 \), where we have used that \( z_n^\varepsilon(0) \leq v_0 \).

Now, reasoning as in the proof of the a priori estimates contained in [23]. We fix a value \( \delta_0 \) such that

\[
\max \left\{ \delta_0 \varepsilon, \delta_0 \frac{1}{q-1} \right\} = \frac{\alpha}{2} \frac{\delta_0}{2} \quad \text{and we take} \quad k \geq k_0 \quad \text{such that}
\]

\[
\int_\Omega G_k(v_0)^\sigma \, dx + C_2 \|\tilde{f}(\chi_{\{\tilde{f} > k\}})\|_{L^r(0, T, L^{m'}(\Omega))} < \delta_0
\]

for any \( k \geq k_0 \). We also define

\[
T^* = \sup \left\{ \tau > 0 : \|G_k(z_n^\varepsilon(s))\|_{L^{\infty}(\Omega)} \leq \delta_0 \quad \forall s \leq \tau \right\}
\]

which is strictly positive by (2.13) and since \( u_0 \leq v_0 \) in \( \Omega \). Note also that \( T^* \) continuously depends on \( n \) by (2.13).
Then, for $k \geq k_0$ and for any $t \leq T^*$ we have
\[ \sup_{t \in [0, T^*]} \int_{\Omega} \Phi_k(z_n^\varepsilon(t)) \, dx < \delta_0 \]
and, letting $\mu \to 0$, we deduce
\[ \sup_{t \in [0, T^*]} \int_{\Omega} G_k(z_n^\varepsilon(t))^\sigma \, dx < \delta_0. \]  
(5.5)

Now, if $T^* < T$, then (5.5) would be in contrast with both the continuity regularity in (2.13) and the definition of $T^*$, so (5.5) holds up to $T$.

We thus deduce a bound, uniform in $\varepsilon$, for the function $z_n^\varepsilon(t)$ in $L^\sigma(\Omega)$. Indeed, we have
\[ \sup_{t \in [0, T]} \int_{\Omega} G_k(z_n^\varepsilon(t))^\sigma \, dx \leq \delta_0 + k_0^\sigma T|\Omega| \]
and then, letting $n \to \infty$ and recalling the definition of $z_n$ in (5.2), leads to
\[ \int_{\Omega} [u(t) - (1 - \varepsilon)v(t)]^\sigma \, dx \leq \varepsilon^\sigma c \]
which, letting $\varepsilon \to 0$, implies $u \leq v$ in $Q_T$ and thus the assertion is proved. \(\square\)

Next we prove Theorem 2.9.

Proof of Theorem 2.9. We start recalling the inequality in (5.3), with $z_n^\varepsilon$ defined in (5.2), and we set
\[ \varphi(z_n^\varepsilon) = 1 - \frac{1}{(1 + G_k(z_n^\varepsilon))^\mu}, \quad \text{with} \quad \mu = \frac{N + 1}{N} (2 - q) - 1 < 1, \]
so we get
\[
\int_{\Omega} \Phi(z_n^\varepsilon(t)) \, dx + \mu \alpha \int_{Q_T} \frac{|\nabla G_k(z_n^\varepsilon)|^q}{(1 + G_k(z_n^\varepsilon))^\mu} \, dx \, ds + \int_{Q_T} z_n^\varepsilon \left[ 1 - \frac{1}{(1 + G_k(z_n^\varepsilon))^\mu} \right] \, dx \, ds \\
\leq c_1 e^{(q-1)T} \int_{Q_T} |\nabla G_k(z_n^\varepsilon)|^q \left[ 1 - \frac{1}{(1 + G_k(z_n^\varepsilon))^\mu} \right] \, dx \, ds + \int_{Q_T} |\tilde{f}| |\chi| |f| \, dx \, ds + \int_{Q_T} \varphi(z_n^\varepsilon(0)) \, dx + R_n, \\
\]
where $\Phi(u) = \int_0^u \varphi(y) \, dy$. Again, we observe that
\[
\int_{Q_T} z_n^\varepsilon \left[ 1 - \frac{1}{(1 + G_k(z_n^\varepsilon))^\mu} \right] \, dx \, ds - \int_{Q_T} |\tilde{f}| |\chi| \, dx \, ds \geq 0 \\
\]
so we drop it, and we just deal with
\[
\int_{\Omega} \Phi(z_n^\varepsilon(t)) \, dx + \mu \alpha \int_{Q_T} \frac{|\nabla G_k(z_n^\varepsilon)|^q}{(1 + G_k(z_n^\varepsilon))^\mu} \, dx \, ds \\
\leq c_1 e^{(q-1)T} \int_{Q_T} |\nabla G_k(z_n^\varepsilon)|^q \left[ 1 - \frac{1}{(1 + G_k(z_n^\varepsilon))^\mu} \right] \, dx \, ds + \int_{Q_T} |\tilde{f}| |\chi| |f| \, dx \, ds + \int_{\Omega} \varphi(z_n^\varepsilon(0)) \, dx + R_n. \\
\]

Our purpose is to recover an a priori estimate for $z_n^\varepsilon$. We begin applying Young’s inequality with \( \left( \frac{2}{q^*}, \frac{2}{2-q^*} \right) \) to the first integral in the right hand side of (5.6) obtaining
\[
c_1 e^{(q-1)T} \int_{Q_T} |\nabla G_k(z_n^\varepsilon)|^q \left[ 1 - \frac{1}{(1 + G_k(z_n^\varepsilon))^\mu} \right] \, dx \, ds \\
\leq \frac{\mu \alpha}{2} \int_{Q_T} \frac{|\nabla G_k(z_n^\varepsilon)|^2}{(1 + G_k(z_n^\varepsilon))^\mu+1} \, dx \, ds + c \int_{Q_T} (1 + G_k(z_n^\varepsilon))^{\frac{\mu+1}{2-q^*}} \left[ 1 - \frac{1}{(1 + G_k(z_n^\varepsilon))^\mu} \right] \frac{2}{q^*} \, dx \, ds. \\
\]
We observe that, since \( \mu < 1 \), it holds that \( 1 - \frac{1}{1 + G_k(z_n^\ast)} \lesssim \frac{G_k(z_n^\ast)}{1 + G_k(z_n^\ast)} \), and since \( \frac{2}{2-q} > 1 \), it follows
\[
\left( \frac{G_k(z_n^\ast)}{1 + G_k(z_n^\ast)} \right)^{\frac{2}{2-q}} \leq G_k(z_n^\ast)
\]
and thus we deduce the uniform boundedness
\[
\int_\Omega \Phi(z_n^\ast(t)) \, dx + \mu \int_{Q_T} |\nabla G_k(z_n^\ast)|^2 \, dx \, ds 
\]
\[
\leq c e^{(q-1)T} \int_{Q_T} |G_k(z_n^\ast)| \, dx \, ds + \int_{Q_T} |\tilde{f}(x,\cdot)|^2 \, dx \, ds + \int_\Omega \Phi(z_n^\ast(0)) \, dx + R_n ,
\]
since, as already observed, the asymptotic condition (2.7) takes the place of (2.33) in the proof that \( \lim_{n \to \infty} R_n = 0 \).

In particular, this means that the energy term above is uniformly bounded in \( n \).

Furthermore, since \( q \frac{2}{2-q} = q + \frac{N + 1}{N} \) is the Gagliardo-Nirenberg exponent associated to the spaces \( L^2(0,T; H^1(\Omega)) \cap L^\infty(0,T; L^{\frac{2}{2-q}}(\Omega)) \), (see Theorem 3.1) we have that
\[
\int_{Q_T} (1 + G_k(z_n^\ast))^{\frac{u(p+1)}{2-q}} \left[ 1 - \frac{1}{1 + G_k(z_n^\ast)} \right] \frac{2}{2-q} \, dx \, dt \leq \int_{Q_T} (1 + G_k(z_n^\ast))^{\frac{u(p+1)}{2-q}} G_k(z_n^\ast) \, dx \, dt 
\]
\[
\leq c G_N \|G_k(z_n^\ast)\|_{L^\infty(0,T; L^{1}(\Omega))} \int_{Q_T} \frac{|\nabla G_k(z_n^\ast)|^2}{(1 + G_k(z_n^\ast))^{\mu+1}} \, dx \, dt .
\]

Then (5.6) becomes
\[
\int_\Omega \Phi(z_n^\ast(t)) \, dx + \mu \int_{Q_T} |\nabla G_k(z_n^\ast)|^2 \, dx \, ds 
\]
\[
\leq c \|G_k(z_n^\ast)\|_{L^\infty(0,T; L^{1}(\Omega))} \int_{Q_T} |\nabla G_k(z_n^\ast)|^2 \, dx \, ds + \int_{Q_T} |\tilde{f}(x,\cdot)|^2 \, dx \, ds + \int_\Omega G_k(v_0) \, dx + \omega_n .
\]

We observe that the function \( \Phi(z) \) can be estimated from below as \( \Phi(w) > C_1 \min\{ G_k(w), G_k(w)^2 \} \), for a certain \( C_1 > 0 \), from which
\[
\int_\Omega G_k(z_n^\ast(t)) \, dx \leq \int_{\{G_k(z_n^\ast(t)) > 1\}} G_k(z_n^\ast(t)) \, dx + |\Omega|^\frac{1}{2} \left( \int_{\{G_k(z_n^\ast(t)) \leq 1\}} G_k(z_n^\ast(t))^2 \, dx \right)^\frac{1}{2} 
\]
\[
\leq \frac{1}{C_1} \int_\Omega \Phi(z_n^\ast(t)) \, dx + \left( \frac{|\Omega|}{C_1} \right)^{\frac{1}{2}} \left( \int_\Omega \Phi(z_n^\ast(t)) \, dx \right)^{\frac{1}{2}} .
\]

We fix a small value \( \delta_0 \) so that the equality \( \frac{\mu \alpha}{2} = c(C_0 \delta_0^\frac{1}{2})^{\frac{1}{\alpha}} \) holds for \( C_0 = 2 \max \left\{ \frac{1}{C_1}, \left( \frac{|\Omega|}{C_1} \right)^{\frac{1}{2}} \right\} \). Moreover, for \( \delta < \delta_0 \), we let \( k_0 \) large enough so that
\[
\int_\Omega G_k(v_0) \, dx + \int_{Q_T} |\tilde{f}(x,\cdot)|^2 \, dx \, ds < \delta \quad \forall k \geq k_0
\]
and we define
\[
T^* := \sup \{ \tau > 0 : \|G_k(z_n^\ast(s))\|_{L^1(\Omega)} \leq C_0 \delta^\frac{1}{2}, \forall s \leq \tau \} \quad \forall k \geq k_0.
\]
Notice that \( T^* > 0 \) thanks to the regularity \( u, v \in C([0,T];L^1(\Omega)) \). We underline again that such a continuity regularity implies that \( T^* \) continuously depends on \( n \).

The above choice of \( \delta_0 \) and (5.8) imply
\[
\int_\Omega \Phi(z^\ast(t)) \, dx \leq \int_\Omega G_k(v_0) \, dx + \int_{Q_T} |\tilde{f}(x,\cdot)|^2 \, dx \, ds < \delta \quad \forall k \geq k_0, \forall t \leq T^* .
\]

Therefore, by definition of \( C_1 \) and \( C_0 \) and thanks to (5.7), we obtain
\[
\int_\Omega G_k(z_n^\ast(t)) \, dx < C_0 \delta^\frac{1}{2} \quad \forall t \leq T^* .
\]

By the continuity regularity \( u, v \in C([0,T];L^1(\Omega)) \) and (5.10) we deduce that \( T^* = T \), since if \( T^* < T \) then (5.10) would be in contrast with the definition of \( T^* \) and since \( u, v \in C([0,T];L^1(\Omega)) \).

Once we have (5.10) for \( T = T^* \) then we have
\[
\int_\Omega z_n^\ast(t) \, dx \leq \int_\Omega G_k(z_n^\ast(t)) \, dx + \int_\Omega T_k(z_n^\ast(t)) \, dx \leq C_0 \delta^\frac{1}{2} + k
\]
which, letting \( n \to \infty \) and recalling the definition of \( z_n \) in (5.2), leads to
\[
\int_\Omega \left( u(t) - (1 - \varepsilon)v(t) \right) \, dx \leq \varepsilon c' \left( C_0 \delta^\frac{1}{2} + k \right)
\]
and the proof follows once we let $\varepsilon \to 0$. □

5.2. The case $2 < p < N$. Here we prove our results via the “convexity” method.

Proof of Theorem 2.13. We want to follow the first part of Theorem 2.8. In order to do it, we recall the definitions of $u_n$, $v_n$ in (4.1) and consider the renormalized formulations in (2.6). We focus on the one related to the supersolution $v$: we consider $S(v) = v_n$ and multiply its inequality by $(1 - \varepsilon)^{p-1}$, we get

$$
\int_\Omega (1 - \varepsilon)^{p-1} v_n(t) \varphi(t) dx + \int_0^T \int_{Q_t} A(x) |\nabla(1 - \varepsilon) v_n|^{p-2} \nabla((1 - \varepsilon) v_n) \cdot \nabla \varphi dx ds
\geq (1 - \varepsilon)^{p-1} \int_0^T \int_{Q_t} H(s, x, \nabla v_n) \varphi dx ds
- \int_0^T \int_{Q_t} A(x)(\theta_n(v)) |\nabla((1 - \varepsilon) v)|^{p-2} \nabla((1 - \varepsilon) v) - |\nabla((1 - \varepsilon) v_n)|^{p-2} \nabla((1 - \varepsilon) v_n)| \cdot \nabla \varphi dx ds
- (1 - \varepsilon)^{p-1} \int_0^T \int_{Q_t} A(x) \nabla v \cdot \nabla \theta_n(v) \varphi dx ds + (1 - \varepsilon)^{p-1} \int_0^T \int_{Q_t} [H(s, x, \nabla v) \theta_n(v) - H(s, x, \nabla v_n)] \varphi dx ds
+ (1 - \varepsilon)^{p-1} \int_\Omega v_n(0) \varphi(0) dx,
$$

Then, rescaling in time this last inequality and defining $v_n^\varepsilon$ as

$$
v_n^\varepsilon(t, x) = (1 - \varepsilon)v_n((1 - \varepsilon)^{p-2} t, x),
$$

we obtain

$$
\int_\Omega v_n^{\varepsilon}(t) \varphi(t) dx + \int_0^T \int_{Q_t} A(x) |\nabla v_n^{\varepsilon}|^{p-2} \nabla v_n^{\varepsilon} \cdot \nabla \varphi dx ds \geq (1 - \varepsilon)^{p-1} \int_0^T \int_{Q_t} H \left( (1 - \varepsilon)^{p-2} s, x, \frac{\nabla v_n^{\varepsilon}}{1 - \varepsilon} \right) \varphi dx ds
- \int_0^T \int_{Q_t} A(x)(\theta(v)) |\nabla((1 - \varepsilon) v)|^{p-2} \nabla((1 - \varepsilon) v) - |\nabla((1 - \varepsilon) v_n)|^{p-2} \nabla((1 - \varepsilon) v_n)| \cdot \nabla \varphi dx ds
- (1 - \varepsilon)^{p-1} \int_0^T \int_{Q_t} A(x) \nabla v \cdot \nabla \theta(v) \varphi dx ds + (1 - \varepsilon)^{p-1} \int_0^T \int_{Q_t} [H(s, x, \nabla v) \theta(v) - H(s, x, \nabla v_n)] \varphi dx ds
+ \int_\Omega v_n^\varepsilon(0) \varphi(0) dx,
$$

where $Q_t^\varepsilon = \left(0, \frac{1}{(1-\varepsilon)^{p-2}}\right) \times \Omega$ for $0 \leq t \leq T$. We want to take into account the difference between

$$
\int_\Omega u_n(t) \varphi(t) dx + \int_0^T \int_{Q_t} A(x) |\nabla u_n|^{p-2} \nabla u_n \cdot \nabla \varphi dx ds \leq \int_0^T \int_{Q_t} \left[ H(s, x, \nabla u_n) \varphi dx ds
- \int_0^T \int_{Q_t} A(x) \theta(u) |\nabla u|^{p-2} \nabla u - |\nabla u_n|^{p-2} \nabla u_n| \cdot \nabla \varphi dx ds
+ \int_0^T \int_{Q_t} [H(s, x, \nabla u) \theta(u) - H(s, x, \nabla u_n)] \varphi dx ds + \int_\Omega u_n(0) \varphi(0) dx
\right]
$$

and (5.11): to this aim, we restrict the integrals to the time interval $0 \leq t \leq T$ since (5.11) holds in $Q_t^\varepsilon \subset Q_t$.

As in the previous cases, our aim is to write an inequality solved by the following function

$$
z_n^\varepsilon(t, x) = \frac{e^{-M}}{\varepsilon} (u_n(t, x) - v_n^\varepsilon(t, x)) - e^{-M} \bar{v}_0 - M t
$$

where $M > 0$ has been defined in (2.39), $\lambda > 0$ to be fixed and $\bar{v}_0$ (the upper bound of $v_0$) is assumed, without loss of generality, to be positive

$$
\int_\Omega z_n^\varepsilon(t) \varphi(t) dx + \int_0^T \int_{Q_t} (\lambda z_n^\varepsilon + M) \varphi dx ds + \int_0^T \int_{Q_t} \frac{e^{-M}}{\varepsilon} A(x) \left[ |\nabla u_n|^{p-2} \nabla u_n - |\nabla v_n^\varepsilon|^{p-2} \nabla v_n^\varepsilon \right] \cdot \nabla \varphi dx ds
\leq \int_0^T \int_{Q_t} \frac{e^{-M}}{\varepsilon} \left[ H(s, x, \nabla u_n) - (1 - \varepsilon)^{p-1} H \left( (1 - \varepsilon)^{p-2} s, x, \frac{\nabla v_n^\varepsilon}{1 - \varepsilon} \right) \right] \varphi dx ds + \int_\Omega z_n^\varepsilon(0) \varphi(0) dx + R_n
$$

(5.13)
and
\[
R_n = -\int_{Q_t} e^{-\frac{\lambda t}{\varepsilon}} A(x) [\theta_n(u)|\nabla u|^p - 2|\nabla u_n|^p - 2\nabla u_n] \cdot \nabla \varphi \, dx \, ds \\
- \int_{Q_t} \theta_n'(u) A(x) |\nabla u|^p - 2\nabla u \cdot \nabla \varphi \, dx \, ds + \int_{Q_t} [H(s, x, \nabla u) \theta_n(u) - H(s, x, \nabla u_n)] \varphi \, dx \, ds \\
- \int_{Q_t} e^{-\frac{\lambda t}{\varepsilon}} A(x) [\theta_n(v((1-\varepsilon)^p-2, t, x)) |\nabla v|^p - 2|\nabla v_n|^p - 2\nabla v_n] \cdot \nabla \varphi \, dx \, ds \\
- (1-\varepsilon)^{-1} \int_{Q_t} \theta_n'(v((1-\varepsilon)^p-2, t, x)) A(x) |\nabla v|^p - 2\nabla v \cdot \nabla \varphi \, dx \, ds \\
+ \int_{Q_t} [H((1-\varepsilon)^p-2, s, x, \nabla v)] \varphi \, dx \, ds.
\]

Let us define, for any $\mu > 0$ and $a \geq 1$, the following function:
\[
\Psi_{a, \mu}(v) = \int_0^{G_k(v)} (\mu + |w|)^{\frac{\sigma-2}{2}} \, dw.
\] (5.14)

Then, we set
\[
\varphi(z_n^\varepsilon) = \Psi_{1, \mu}(z_n^\varepsilon) = \int_0^{G_k(z_n^\varepsilon)} (\mu + |w|)^{\sigma-2} \, dw
\]
in (5.13), with $k \geq \tilde{v}_0$ (in order to have $G_k(z_n^\varepsilon(0, x)) \equiv 0$ in $\Omega$) in (5.14) so that, thanks to (2.39), we get
\[
\int_\Omega \Phi(z_n^\varepsilon(t)) \, dx + \lambda \int_{Q_t} G_k(z_n^\varepsilon) \left( \int_0^{G_k(z_n^\varepsilon)} (\mu + |w|)^{\sigma-2} \, dw \right) \, dx \, ds + \frac{\alpha}{2} \varepsilon^{-p-2} \int_{Q_t} |\nabla \Psi_{\mu, \nu}(G_k(z_n^\varepsilon))|^p \, dx \, ds \\
\leq c_1 e^{\lambda(1-q)T} \int_{Q_t} |\nabla G_k(z_n^\varepsilon)|^q \left( \int_0^{G_k(z_n^\varepsilon)} (\mu + |w|)^{\sigma-2} \, dw \right) \, dx \, ds \\
+ c_2 e^{\frac{\alpha - p}{2} \lambda T} \int_{Q_t} |\nabla G_k(z_n^\varepsilon)| \left[|\nabla u_n|^\frac{p-2}{2} + |\nabla v_n|^\frac{p-2}{2}\right] \left( \int_0^{G_k(z_n^\varepsilon)} (\mu + |w|)^{\sigma-2} \, dw \right) \, dx \, ds + R_n.
\] (5.15)

The proof $\lim_{n \to \infty} R_n = 0$ follows using the analogous one contained in Theorem 2.10 with $\ell = 0$ (see (2.38)), changing (2.14) with (2.33) and taking advantage of the definition of $\varphi(\cdot)$.

We first deal with the latter integral in the right hand side. The definition of $|\nabla \Psi_{\mu, \nu}(G_k(z_n^\varepsilon))|$ and Young’s inequality yield to
\[
c_2 e^{\frac{\alpha - p}{2} \lambda T} \int_{Q_t} |\nabla G_k(z_n^\varepsilon)| \left[|\nabla u_n|^\frac{p-2}{2} + |\nabla v_n|^\frac{p-2}{2}\right] \left( \int_0^{G_k(z_n^\varepsilon)} (\mu + |w|)^{\sigma-2} \, dw \right) \, dx \, ds \\
\leq c_2 e^{\frac{\alpha - p}{2} \lambda T} \int_{Q_t} |\nabla \Psi_{\mu, \nu}(G_k(z_n^\varepsilon))| \left[|\nabla u_n|^\frac{p-2}{2} + |\nabla v_n|^\frac{p-2}{2}\right] \Psi_{\mu, \nu}(G_k(z_n^\varepsilon)) \, dx \, ds \\
\leq \frac{\alpha}{2} \int_{Q_t} |\nabla \Psi_{\mu, \nu}(G_k(z_n^\varepsilon))|^2 \left[|\nabla u_n|^\frac{p-2}{2} + |\nabla v_n|^\frac{p-2}{2}\right] \, dx \, ds + c \int_{Q_t} \Psi_{\mu, \nu}(G_k(z_n^\varepsilon))^2 \, dx \, ds \\
\leq \frac{\alpha}{2} \int_{Q_t} |\nabla \Psi_{\mu, \nu}(G_k(z_n^\varepsilon))|^2 \left[|\nabla u_n|^\frac{p-2}{2} + |\nabla v_n|^\frac{p-2}{2}\right] \, dx \, ds + c \int_{Q_t} G_k(z_n^\varepsilon) \left( \int_0^{G_k(z_n^\varepsilon)} (\mu + |w|)^{\sigma-2} \, dw \right) \, dx \, ds.
\]

Then, we use such an estimate to prove that, having also (2.33), provides us with the uniform bound in $n$ of $(1 + G_k(z_n^\varepsilon))^{-1} G_k(z_n^\varepsilon)$ in $L^p(0, T; W^{1, p}(\Omega))$. Indeed, (5.15) becomes
\[
\int_\Omega \Phi(z_n^\varepsilon(t)) \, dx + \frac{\alpha}{2} \varepsilon^{-p-2} \int_{Q_t} |\nabla \Psi_{\mu, \nu}(G_k(z_n^\varepsilon))|^p \, dx \, ds \\
\leq c_1 e^{(1-q)T} \int_{Q_t} |\nabla G_k(z_n^\varepsilon)|^q \left( \int_0^{G_k(z_n^\varepsilon)} (\mu + |w|)^{\sigma-2} \, dw \right) \, dx \, ds + \omega_n
\]
provided \( \lambda \geq c \), and since
\[
\int_{Q_t} |\nabla G(z^\varepsilon_n)|^q \left( \int_0^{G(z^\varepsilon_n)} (\mu + |w|)^{-2} \, dw \right) \, dx \, ds
\]
\[
\leq \int_{Q_t} |\nabla \Psi_{p,\mu}(G(z^\varepsilon_n))|^q \left( \int_0^{G(z^\varepsilon_n)} (\mu + |w|)^{-(2-q)} \, dw \right) \, dx \, ds
\]
\[
\leq \int_{Q_t} |\nabla \Psi_{p,\mu}(G(z^\varepsilon_n))|^q |\Psi_{p,\mu}(G(z^\varepsilon_n))|^{p-q} G(z^\varepsilon_n)^{q-(p-1)} \, dx \, ds
\]  
(5.16)
by definition of \( \Psi_{p,\mu}(\cdot) \) and Hölder’s inequality with indices \( \left( \frac{1}{p-q}, \frac{1}{q-(p-1)} \right) \), we have that
\[
\int_\Omega \Phi(z^\varepsilon_n(t)) \, dx + \frac{\alpha}{2} \varepsilon^{p-2} \int_0^t \int_{Q_t} |\nabla \Psi_{p,\mu}(G(z^\varepsilon_n))|^p \, dx \, ds 
\leq c_1 \int_{Q_t} |\nabla \Psi_{p,\mu}(G(z^\varepsilon_n))|^q |\Psi_{p,\mu}(G(z^\varepsilon_n))|^{p-q} G(z^\varepsilon_n)^{q-(p-1)} \, dx \, ds + \omega_n. 
\]  
(5.17)
An application of Young’s inequality with \( \left( \frac{p}{q}, \frac{p}{p-q} \right) \) implies
\[
\int_\Omega \Phi(z^\varepsilon_n(t)) \, dx + \frac{\alpha}{4} \varepsilon^{p-2} \int_0^t \int_{Q_t} |\nabla \Psi_{p,\mu}(G(z^\varepsilon_n))|^p \, dx \, ds 
\leq c \int_{Q_t} |\Psi_{p,\mu}(G(z^\varepsilon_n))|^p G(z^\varepsilon_n)^{\frac{q-(p-1)}{p-q}} \, dx \, ds + \omega_n.
\]
The uniform boundedness of the right hand side above is due to the fact that \( p\gamma + \frac{p(q-(p-1))}{p-q} = p\frac{N\beta+\sigma}{N} \) and that \( (u-v) \in L^{\frac{N\beta+\sigma}{N}}(Q_T) \) by (2.33).
We continue applying once more the Hölder inequality with indices \( \left( \frac{p}{q}, \frac{p}{p-q} \right) \) and also Sobolev’s embedding, so we finally get
\[
\int_{Q_t} |\nabla \Psi_{p,\mu}(G(z^\varepsilon_n))|^q |\Psi_{p,\mu}(G(z^\varepsilon_n))|^{p-q} G(z^\varepsilon_n)^{q-(p-1)} \, dx \, ds 
\leq c \sup_{s \in (0,t)} \|G(z^\varepsilon_n(s))\|_{L^p(\Omega)}^{q-p+1} \int_0^t \|\nabla \Psi_{p,\mu}(G(z^\varepsilon_n(s)))\|_{L^p(\Omega)}^p \, ds
\]
where \( c = c(\gamma, N, q, T) \) and finally deduce that
\[
\int_\Omega \Phi(z^\varepsilon_n(t)) \, dx + \frac{\alpha}{2} \varepsilon^{p-2} \int_0^t \int_{Q_t} |\nabla \Psi_{p,\mu}(G(z^\varepsilon_n))|^p \, dx \, ds 
\leq c \sup_{s \in (0,t)} \|G(z^\varepsilon_n(s))\|_{L^p(\Omega)}^{q-p+1} \int_0^t \|\nabla \Psi_{p,\mu}(G(z^\varepsilon_n(s)))\|_{L^p(\Omega)}^p \, ds + \omega_n.
\]

Then, the above uniform boundedness in \( n \) on the difference between sub/supersolutions allow us to let \( n \to \infty \) getting
\[
\int_\Omega \Phi(z^\varepsilon(t)) \, dx + \frac{\alpha}{2} \varepsilon^{p-2} \int_0^t \int_{Q_t} |\nabla \Psi_{p,\mu}(G(z^\varepsilon))|^p \, dx \, ds 
\leq c \sup_{s \in (0,t)} \|G(z^\varepsilon(s))\|_{L^p(\Omega)}^{q-p+1} \int_0^t \|\nabla \Psi_{p,\mu}(G(z^\varepsilon(s)))\|_{L^p(\Omega)}^p \, ds.
\]
We now reason as in Theorem 2.8 and, being \( \Phi(w) \to \frac{|w|^\gamma}{\gamma(\sigma-1)} \) as \( \mu \to 0 \) and thanks to Lemma 3.3, we have that \( G_k(z^\varepsilon) \equiv 0 \) in \( Q_T \). In particular, this means that
\[
e^{-Mt \left( u(t,x) - (1-\varepsilon)v((1-\varepsilon)^p-2t,x) - \varepsilon \bar{v}_0 \right)} - \varepsilon Mt \leq \varepsilon k
\]
and letting \( \varepsilon \) vanishes we deduce that \( u(t,x) \leq v(t,x) \) in \( Q_T \), as desired.
\[\square\]

**Proof of Theorem 2.14.** We set \( \lambda = 1 \) in (5.12) and take
\[
\varphi(z^\varepsilon_n) = 1 - \frac{1}{(1 + G(z^\varepsilon_n))^\mu} \quad \text{with} \quad \mu = (p-q) \frac{N+1}{N} - 1 > p-1
\]
in (5.13), so that

\[
\int_\Omega \Phi(z_n^e(t)) dx + \frac{\mu \alpha}{2} \int Q_\nu \frac{|\nabla G_k(z_n^e)|^p}{(1 + G_k(z_n^e))^{\mu+1}} dx ds
\]

\[+ \frac{\mu}{2} \int Q_\nu \frac{|\nabla G_k(z_n^e)|^2}{(1 + G_k(z_n^e))^{\mu+1}} \left[|\nabla u_n|^{p-2} + |\nabla v_n|^{p-2}\right] dx ds
\]

\[\leq c_1 e^{(q-1)\lambda T} \int Q_\nu |\nabla G_k(z_n^e)|^q \left[1 - \frac{1}{(1 + G_k(z_n^e))^{\mu}}\right] dx ds
\]

\[+ c_2 e^{\frac{p^2}{2} \lambda T} \int Q_\nu |\nabla G_k(z_n^e)| \left[|\nabla u_n|^{p-2} + |\nabla v_n|^{p-2}\right] \left[1 - \frac{1}{(1 + G_k(z_n^e))^{\mu}}\right] dx ds + R_n
\]

(5.18)

thanks to (2.39).

The proof that \(\lim_{n \to \infty} R_n = 0\) follows reasoning as in Theorem 2.11 (see also Theorem 2.9), so we skip it.

We start estimating the first integral in the right hand side above through Young’s inequality with indices \((\frac{q}{q-1}, \frac{p}{p-1})\), getting

\[c_1 e^{(q-1)\lambda T} \int Q_\nu |\nabla G_k(z_n^e)|^q \left[1 - \frac{1}{(1 + G_k(z_n^e))^{\mu}}\right] dx ds
\]

\[\leq \frac{\mu \alpha}{4} \int Q_\nu \frac{|\nabla G_k(z_n^e)|^p}{(1 + G_k(z_n^e))^{\mu+1}} dx ds + c \int Q_\nu \frac{(1 + G_k(z_n^e))^{\frac{p(q+1)}{q}}}{(1 + G_k(z_n^e))^{\mu+1}} dx ds
\]

As far as the second integral in the right hand side of (5.18), applying again Young’s inequality with indices \((2, 2)\) we get

\[c_2 e^{\frac{p^2}{2} \lambda T} \int Q_\nu |\nabla G_k(z_n^e)| \left[|\nabla u_n|^{p-2} + |\nabla v_n|^{p-2}\right] \left[1 - \frac{1}{(1 + G_k(z_n^e))^{\mu}}\right] dx ds
\]

\[\leq \frac{\mu}{2} \int Q_\nu \frac{|\nabla G_k(z_n^e)|^2}{(1 + G_k(z_n^e))^{\mu+1}} \left[|\nabla u_n|^{p-2} + |\nabla v_n|^{p-2}\right] dx ds + c \int Q_\nu \frac{(1 + G_k(z_n^e))^{\mu+1}}{(1 + G_k(z_n^e))^{\mu+1}} dx ds
\]

(5.19)

Using the previous estimates into (5.18) we obtain

\[
\int_\Omega \Phi(z_n^e(t)) dx + \frac{\mu \alpha}{4} \int Q_\nu \frac{|\nabla G_k(z_n^e)|^p}{(1 + G_k(z_n^e))^{\mu+1}} dx ds
\]

\[\leq c \left[\int Q_\nu \frac{(1 + G_k(z_n^e))^{\frac{p(q+1)}{q}}}{(1 + G_k(z_n^e))^{\mu+1}} dx ds + \int Q_\nu \frac{(1 + G_k(z_n^e))^{\mu+1}}{(1 + G_k(z_n^e))^{\mu+1}} dx ds\right] + \omega_n.
\]

Then, since having \(p > 2\) and \(q > \frac{p(N+1)-N}{N+2}\) imply that both \(\frac{p(q+1)}{q}\) are \(\mu + 1\) smaller than \(q^{N+1}\) (see Lemma 3.5 and Theorem 3.1), the right hand side in the above inequality is uniformly bounded with respect to \(n\).

Now, let us focus on the right hand side of (5.18). We apply Hölder’s inequality with \((\frac{q}{q-1}, \frac{p}{p-1})\) on the integral involving the \(q\) power of the gradient, getting

\[c_1 e^{(q-1)\lambda T} \int Q_\nu |\nabla G_k(z_n^e)|^q \left[1 - \frac{1}{(1 + G_k(z_n^e))^{\mu}}\right] dx ds
\]

\[\leq c \left(\int Q_\nu \frac{(1 + G_k(z_n^e))^{\frac{p(q+1)}{q}}}{(1 + G_k(z_n^e))^{\mu+1}} dx ds\right)^{\frac{p}{q}} \left(\int Q_\nu \frac{|\nabla G_k(z_n^e)|^p}{(1 + G_k(z_n^e))^{\mu+1}} dx ds\right)^{\frac{q}{p}}.
\]

We observe that the value \(\nu = \frac{N+1}{p(N+1)+p-N}\) corresponds to the Gagliardo-Nirenberg regularity exponent for the spaces

\[L^\infty(0, T; L^{\frac{p}{p-1}}(\Omega)) \cap L^p(0, T; W^{1,p}_0(\Omega)).\]

We justify such a gradient regularity recalling Lemma 3.5. Then, we have

\[
\int_{Q_T} (1 + G_k(z_n^e))^{\frac{p(q+1)}{q}} \left[1 - \frac{1}{(1 + G_k(z_n^e))^{\mu}}\right]^{\frac{p}{q}} dx dt \leq \int_0^T \|\nabla G_k(z_n^e)\|^p_{L^p(\Omega)} dt
\]

\[\leq c_{GN} \|G_k(z_n^e)\|_{L^\infty(0, T; L^1(\Omega))} \int_{Q_T} \frac{|\nabla G_k(z_n^e)|^p}{(1 + G_k(z_n^e))^{\mu+1}} dx dt
\]
thanks to (3.2). Furthermore, since the last integral in (5.19) can be estimated by
\[
\iint_{Q_T} (1 + G_k(z_n^e))^{\mu} G_k(z_n^e) \, dx \, ds \leq \int_{Q_T} (1 + G_k(z_n^e))^q \int_{Q_T} \frac{\|\nabla G_k(z_n^e)\|_{L^\infty(0,T;L^1(\Omega))}^{p}}{(1 + G_k(z_n^e))^{\mu+1}} \, dx \, dt
\]
we are left with the study of
\[
\int_{\Omega} \Phi(z_n^e(t)) \, dx + \frac{\mu_0 c}{2} e^{p-2} \iint_{Q_T} \frac{\|\nabla (G_k(z_n^e))\|_{L^\infty(0,T;L^1(\Omega))}^{p}}{(1 + G_k(z_n^e))^{\mu+1}} \, dx \, ds \leq c\|G_k(z_n^e)\|_{L^\infty(0,T;L^1(\Omega))}^{\mu} \int_{Q_T} \frac{\|\nabla G_k(z_n^e)\|_{L^\infty(0,T;L^1(\Omega))}^{p}}{(1 + G_k(z_n^e))^{\mu+1}} \, dx \, dt + \omega_n.
\]
Then, letting \( n \to \infty \) in the inequality above yields to
\[
\int_{\Omega} \Phi(z^e(t)) \, dx + \frac{\mu_0 c}{2} e^{p-2} \iint_{Q_T} \frac{\|\nabla (G_k(z^e))\|_{L^\infty(0,T;L^1(\Omega))}^{p}}{(1 + G_k(z^e))^{\mu+1}} \, dx \, ds \leq c\|G_k(z^e)\|_{L^\infty(0,T;L^1(\Omega))}^{\mu} \int_{Q_T} \frac{\|\nabla G_k(z^e)\|_{L^\infty(0,T;L^1(\Omega))}^{p}}{(1 + G_k(z^e))^{\mu+1}} \, dx \, ds
\]
We conclude reasoning as in the proof of Theorem 2.9, recalling Lemma 3.3 and letting \( \epsilon \to 0 \).

\( \square \)

**Appendix A.**

Our current goal is proving that one needs to consider sub/super solutions belonging to the regularity class \((2.13)-(2.14)\) in order to have a uniqueness result for problems of \((1.2)\) type.

Here we use a result contained in \([7, \text{Section 3}]\) (see also \([8]\)), where it is proved that the Cauchy problem
\begin{equation}
\begin{aligned}
\begin{cases}
u_t - \Delta u &= c_1 |\nabla u|^q \quad \text{in } (0, T) \times \mathbb{R}^N, \\
u(0, x) &= 0 \quad \text{in } \mathbb{R}^N,
\end{cases}
\end{aligned}
\end{equation}
with \( c_1 > 0 \) and \( q > 1 \) and \( N \geq 2 \) admits at least two solutions.

More precisely they prove that there exists a value \( \alpha_0 > 0 \) and a (unique) solution \( U \in C^2(0, \infty) \cap C^1[0, \infty) \) of the following Cauchy problem:
\begin{equation}
\begin{aligned}
\begin{cases}
U'' + \left( \frac{N}{q} - \frac{2}{q} \right) U' + kU + c_1 |U'|^q &= 0 \quad \text{for } 0 < y < \infty, \\
U'(0) &= 0, \\
U(0) &= \alpha_0
\end{cases}
\end{aligned}
\end{equation}
that satisfies
\begin{equation}
\begin{aligned}
U(y) &= ce^{-\frac{y^2}{2}} y^{-\frac{N}{q}} (1 + o(y^{-2})) \quad \text{as } y \to \infty, \quad \text{with } \sigma = \frac{N(q-1)}{2-q},
\end{aligned}
\end{equation}
and
\begin{equation}
\begin{aligned}
U'(y) &= -\frac{y}{2} U(y) (1 + o(1)) \quad \text{as } y \to \infty
\end{aligned}
\end{equation}
\begin{equation}
\begin{aligned}
U(y) \in L^j([0, \infty); y^N dy) \quad \text{for any } 1 \leq j < \infty.
\end{aligned}
\end{equation}
Consequently it is easy to see that both \( u_1 \equiv 0 \) and \( u_2(t, x) = t^{-\frac{N}{2q}} U(|x|/\sqrt{t}) \) solve (A1).

We use such a result in order to show that the class of uniqueness \((2.13)-(2.14)\) in the right one in order to have comparison (and thus uniqueness). Indeed we construct, for the following problem
\[\begin{aligned}
\begin{cases}
u_t - \Delta u &= |\nabla u|^q + f(t, x) \quad \text{in } (0, T) \times B_R(0), \\
u(t, x) &= 0 \quad \text{on } (0, T) \times \partial B_R(0), \\
u(0, x) &= 0 \quad \text{in } B_R(0),
\end{cases}\end{aligned}\]
for a suitable choice of \( f \) smooth, a pair of solutions, whose only one belong to the class \((2.13)-(2.14)\), while the other one is not regular enough.

In fact we have the following result.

**Theorem A.1.** Let \( 2 - \frac{N}{N+1} < q < 2, R > 0 \) and let \( U \) be the positive solution of (A.2). Then the Cauchy-
Dirichlet problem
\begin{equation}
\begin{aligned}
\begin{cases}
u_t - \Delta v &= c_1 |\nabla v|^q + \left(t^{-\frac{N}{2q}} U(R/\sqrt{t})\right)' \quad \text{in } (0, T) \times B_R(0), \\
v &= 0 \quad \text{on } (0, T) \times \partial B_R(0), \\
v(0, x) &= 0 \quad \text{in } B_R(0),
\end{cases}\end{aligned}
\end{equation}
admits at least two solutions \( v_{1,2} \) such that:

- \( v_1 \in C([0, T]; L^\infty(B_R(0))) \) and \( |v_1|^q \in L^2(0, T; H^1_0(B_R(0))) \), with \( \sigma = \frac{N(q-1)}{2-q}; \)
\[ v_2(t, x) = t^{-\frac{n}{2}}(U(|x|/\sqrt{t}) - U(R/\sqrt{t})), \] and it satisfies
\[ v_2 \in C([0, T]; L^p(B_R(0))) \] for any \( 1 \leq \mu < \sigma \) but \( v_2 \notin C([0, T]; L^\sigma(B_R(0))) \) (A.7)

and
\[ |v_2|^{\beta} \in L^2(0, T; H^0(B_R(0))), \] for any \( \beta < \frac{\sigma}{2} \), but \( |v_2|^{\frac{\sigma}{2}} \notin L^2(0, T; H^0(B_R(0))) \). (A.8)

**Proof.** We proceed observing that, thanks to (A.3)-(A.4), then \( \left(t^{-\frac{n}{2}}U\left(\frac{1}{\sqrt{t}}\right)\right)^t \in C^1([0, T]) \) and thus (A.6) admits a solution \( v_1 \) such that \( v_1 \in C([0, T]; L^\sigma(\Omega)) \) and \( |v_1|^{\frac{\sigma}{2}} \in L^2(0, T; H^0(B_R(0))) \) (see [23]).

Then, we are left with the proofs of (A.7)-(A.8).

In order to prove that \( ||v_2(t)||_{L^\sigma(B_R(0))} \to 0 \) as \( t \to 0^+ \), for \( \mu < \sigma \), we compute
\[
\int_{B_R(0)} |v_2(t, x)|^\mu \, dx = t^{-\frac{n}{2}} \int_0^R |U(r/\sqrt{t}) - U(R/\sqrt{t})|^\mu r^{N-1} \, dr = t^{\frac{N}{2}(1-\frac{\mu}{\sigma})} \int_0^\infty |U(y) - U(R/\sqrt{t})|^\mu y^{N-1} \, dy
\]
\[
\leq c t^{\frac{N}{2}(1-\frac{\mu}{\sigma})} \int_0^\infty |U(y) - U(R/\sqrt{t})|^\mu y^{N-1} \, dy \leq ct^{\frac{N}{2}(1-\frac{\mu}{\sigma})}
\]
where the last inequality follows from (A.5). Then, if \( \mu < \sigma \), we have \( \frac{N}{2}(1-\frac{\mu}{\sigma}) > 0 \) which implies that the right hand side of the above inequality vanishes as \( t \to 0^+ \). If, on the contrary, we set \( \sigma = \mu \) then the integral above becomes
\[
\int_{B_R(0)} |v_2(t, x)|^\sigma \, dx = \int_0^\infty |U(y) - U(R/\sqrt{t})|^\sigma y^{N-1} \, dy
\]
which is bounded from below, thanks to (A.5), by a positive constant.

In order to prove (A.8), we observe that
\[
\int_0^T \int_{B_R(0)} |\nabla v_2|^2 \, dx \, dt = \int_0^T \int_0^R |\nabla t^{-\frac{n}{2}} U(r/\sqrt{t})|^2 r^{N-1} \, dr \, dt
\]
\[
= \beta^2 \int_0^T \int_0^R t^{-\frac{n}{2}\beta-1} U(r/\sqrt{t})^{2(\beta-1)} U'(r/\sqrt{t})^2 r^{N-1} \, dr \, dt
\]
\[
= \beta^2 \int_0^T t^{-\frac{n}{2}\beta-1+\frac{2}{n}} \int_0^\infty U(y)^{2(\beta-1)} U'(y)^2 y^{N-1} \, dy \, dt.
\]

Then, recalling (A.3)-(A.4), we get
\[
\int_0^T \int_{B_R(0)} |\nabla v_2|^2 \, dx \, dt \leq c \int_0^T t^{-\frac{n}{2}\beta-1+\frac{2}{n}} \int_0^\infty U(y)^{2(\beta-1)} U'(y)^2 y^{N-1} \, dy \, dt
\]
\[
\leq c \int_0^T t^{-\frac{n}{2}\beta-1+\frac{2}{n}} \int_0^\infty e^{-\frac{\beta}{2}y^{2N(\frac{\beta}{2}-1)+N+1}} \, dy \, dt \leq c \int_0^T t^{-\frac{n}{2}\beta-1+\frac{2}{n}} \, dt,
\]
which is finite if \( \beta < \frac{\sigma}{2} \). On the other hand, if we suppose that \( \beta = \frac{\sigma}{2} \), then (A.9) becomes
\[
\int_0^T \int_{B_R(0)} |\nabla v_2|^2 \, dx \, dt = \frac{\sigma^2}{4} \int_0^T t^{-1} \int_0^\infty U(y)^{\sigma-2} U'(y)^2 y^{N-1} \, dy \, dt
\]
\[
\geq \frac{\sigma^2}{4} \int_0^1 t^{-1} \int_0^R U(y)^{\sigma-2} U'(y)^2 y^{N-1} \, dy \, dt \geq c \int_0^1 t^{-1} \, dt = +\infty,
\]
and the assertion follows. \qed

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