Vortex energy and vortex bending for a rotating Bose-Einstein condensate

Amandine Aftalion
Laboratoire d’analyse numérique, B.C.187, Université Paris 6, 175 rue du Chevaleret, 75013 Paris, France.

Tristan Riviere
Centre de Mathématiques, Ecole polytechnique, 91128 Palaiseau Cedex, France.

(Dated: November 13, 2018)

Abstract. For a Bose-Einstein condensate placed in a rotating trap, we give a simplified expression of the Gross-Pitaevskii energy in the Thomas-Fermi regime, which only depends on the number and shape of the vortex lines. Then we check numerically that when there is one vortex line, our simplified expression leads to solutions with a bent vortex for a range of rotational velocities and trap parameters which are consistent with the experiments.

PACS numbers: 03.75.Fi,02.70.-c

I. INTRODUCTION

Since the experimental achievement of Bose-Einstein condensates in confined alkali-metal gases in 1995, there has been a huge experimental and theoretical interest in these systems [1, 2, 3, 4, 5, 6, 7, 8, 9, 10]. The study of vortices is one of the key issues. Two different groups have obtained vortices experimentally, the JILA group [8] and the ENS group [7, 5]. In the ENS experiment, a laser beam is imposed on the magnetic trap holding the atoms to create a harmonic anisotropic rotating potential. For sufficiently large angular velocities, vortices are detected in the system. Experimentally, the ENS group [8] has observed that when the vortex is nucleated, the contrast is not 100% which means that the vortex line is not straight but bending. Numerical computations solving the Gross-Pitaevskii equation [11, 12] have shown that there is a range of velocities for which the vortex line is indeed bending. The aim of this paper is to justify these observations theoretically in the Thomas-Fermi regime. We define an asymptotic parameter which is small in the Thomas-Fermi regime. This framework of study has been developed by one of the authors in [13], except that [13] was a two dimensional study for a condensate confined in the $z$ axis. We define the characteristic length $d = (\hbar/m\omega_x)^{1/2}$ and assume $\omega_y = \alpha \omega_x$, $\omega_z = \beta \omega_x$. We set $\varepsilon^2 \sqrt{\frac{\hbar^2}{2Nm}} = \frac{d}{4\pi Na}$, where $g_{3D} = 4\pi\hbar^2/a$. For numerical applications, we are going to use the experimental values of the ENS group $m = 1.445 \times 10^{-26}$ kg, $\alpha = 5.8 \times 10^{-11}m$, $N = 1.4 \times 10^5$ and $\omega_x = 1094s^{-1}$ with $\alpha = 1.06$, $\beta = 0.067$. We find that $\varepsilon = 0.0174$, thus, $\varepsilon$ is small, which will be our asymptotic regime. We re-scale the distance by $R = d/\sqrt{\varepsilon}$ and define $u(r) = R^{3/2} \phi(x)$ where $x = Rr$ and we set $\Omega = \bar{\Omega}/\varepsilon \omega_x$. The velocity $\Omega$ is chosen such that $\Omega < 1/\varepsilon$, that is the trapping potential is stronger than the inertial potential. The energy can be rewritten as:

$$E_{3D}(u) = \int \frac{1}{2} |\nabla u|^2 + \Omega \cdot (iu, \nabla u \times r) + \frac{1}{2\varepsilon^2} (x^2 + \alpha^2 y^2 + \beta^2 z^2) |u|^2 + \frac{1}{4\varepsilon^2} |u|^4.$$ (2)
Due to the constraint $\int |u|^2 = 1$, we can add to $E_{3D}$ any multiple of $\int |u|^2$ so that it is equivalent to minimize

$$\frac{1}{2} \int \nabla u^2 + \Omega \cdot (iu, \nabla u \times r) + \frac{1}{4\varepsilon^2} |u|^4 - \frac{1}{2\varepsilon^2} \rho_{TV}(r)|u|^2$$

where $\rho_{TV}(r) = \rho_0 - (x^2 + \alpha^2 y^2 + \beta^2 z^2)$ for some constant $\rho_0$ to be determined. Let $D$ be the ellipse $\{ \rho_{TV} > 0 \} = \{ x^2 + \alpha^2 y^2 + \beta^2 z^2 < \rho_0 \}$. We impose the following constraint on $\rho_{TV}$:

$$\int_D \rho_{TV}(r) = 1. \quad (3)$$

Indeed, as $\varepsilon$ tends to 0, the minimizer will satisfy that $|u|^2$ will be close to $\rho_{TV}$ so that the constraint will be satisfied automatically by $u$ if we impose (3). In other words, $\rho_{TV}$ is the Thomas Fermi approximation of $u$. Equation (3) leads to

$$\rho_0^{5/2} = 15\alpha\beta/8\pi. \quad (4)$$

To study the problem analytically, it is reasonable to minimize the energy over the domain $D$ with zero boundary data for $u$. Indeed, when $\rho_{TV} \leq 0$, the energy is convex so that the minimizer $u$ goes to zero exponentially at infinity (see the numerical observation in [5] and the analysis on the behavior near the boundary of $D$ as well as the decay at infinity of the order parameter in [15, 16]). We consider the problem

$$\min E_{\varepsilon}(u) \text{ subject to } u \in H^1_0(D), \int_D |u|^2 = 1 \quad (P)$$

where

$$E_{\varepsilon}(u) = \int_D \frac{1}{2} |\nabla u|^2 + \Omega \cdot (iu, \nabla u \times r) + \frac{1}{4\varepsilon^2} (\rho_{TV}(r) - |u|^2)^2. \quad (5)$$

Note that a critical point $u$ of $E_{\varepsilon}$ is a solution of

$$-\Delta u + 2i(\Omega \times r) \cdot \nabla u = \frac{1}{\varepsilon^2} u(\rho_{TV} - |u|^2) + \mu z u \text{ in } D, \quad (6)$$

with $u = 0$ on $\partial D$ and $\mu$ is the Lagrange multiplier. The specific choice of $\rho_0$ in (4) will imply that the term $\mu z u$ is negligible in front of $\rho_{TV}u/\varepsilon^2$.

We have set the framework of study of our energy. In Section 2, we will make an asymptotic development of the energy taking into account that $\varepsilon$ is small (but $\log \varepsilon$ is not big). Then in Section 3, we will check that our approximate energy yields a solution which is consistent with the numerical and experimental observations.

II. ASYMPTOTIC DEVELOPMENT OF THE ENERGY

Our aim is to decouple the energy into 3 terms: a part coming from the solution without vortices, a vortex contribution and a term due to rotation.

A. The solution without vortices

Firstly, we are interested in solutions without vortices, that is $u$ has no zero in the interior of $D$. Thus we consider functions of the form $u = f e^{iS}$, where $\xi$ is in $H^1_0(D)$ and $f$ is real and has no zero in the interior of $D$. We consider first minimizing $E_{\varepsilon}$ over such functions without imposing the constraint that the $L^2$ norm is 1, that is, $f$ and $S$ minimize

$$E_{\varepsilon}(f,S) = \int_D \frac{1}{2} |\nabla f|^2 + \frac{1}{4\varepsilon^2} (\rho_{TV} - f^2)^2$$

$$+ \frac{1}{2} \int f^2 |\nabla S - \Omega \times r|^2 - f^2 \Omega^2 r^2, \quad (7)$$

where $r = xe_x + ye_y$. We have $f_{\varepsilon} = 0$ on $\partial D$ and

$$-\Delta f + f \nabla S \cdot (\nabla S - \Omega \times r) = \frac{1}{\varepsilon^2} f_{\varepsilon}(\rho_{TV} - f_{\varepsilon}^2) \text{ in } D, \quad (8)$$

$$\text{div} (f_{\varepsilon}^2 (\nabla S_{\varepsilon} - \Omega \times r)) = 0. \quad (9)$$

Equation (8) implies that there exists $\xi_{\varepsilon}$ in $H^2(D) \cap H^1_0(D)$ such that

$$f_{\varepsilon}^2 (\nabla S_{\varepsilon} - \Omega \times r) = \Omega \cdot \nabla \xi_{\varepsilon}. \quad (10)$$

So $\xi_{\varepsilon}$ is the unique solution of

$$\text{curl} \left( \frac{1}{f_{\varepsilon}^2} \text{curl} \xi_{\varepsilon} \right) = -2 \text{ in } D, \quad \xi_{\varepsilon} = 0 \text{ on } \partial D. \quad (11)$$

In the special case where the cross section of $D$ is a disc, the minimum of (11) is reached for $\nabla S = 0$ but this is not the case if the cross section is an ellipse and there is a non trivial solution of (11). As $\varepsilon$ tends to 0, since the ellipticity of the cross-section is small, $f_{\varepsilon}^2$ tends to $\rho_{TV}$ in every compact subset of $D$ and the function $\xi_{\varepsilon}$ given by (10) or (11) tends to the unique solution $\xi$ of

$$\text{curl} \left( \frac{1}{\rho_{TV}} \text{curl} \xi \right) = -2 \text{ in } D, \quad \xi = 0 \text{ on } \partial D. \quad (12)$$

One can easily get that

$$\xi(x,y) = -\rho_{TV}^2(x,y)/(2 + 2\alpha^2)e_z. \quad (13)$$

Using (13), we can define $S_0$, the limit of $S_{\varepsilon}$, to be the solution of $\rho_{TV}(\nabla S_0 - \Omega \times r) = \Omega \cdot \nabla \xi$ with zero value at the origin. We have $S_0 = C\Omega \cdot \nabla x$ with $C = (\alpha^2 - 1)/(\alpha^2 + 1)$. We see that $S_0$ vanishes when $\alpha = 1$ that is when the cross-section is a disc. This computation is consistent with the one in [5], though it is derived in a different way.

B. Decoupling the energy

Let $\eta_{\varepsilon} = f_{\varepsilon} e^{iS_{\varepsilon}}$ be the vortex free minimizer of $E_{\varepsilon}$ discussed previously without imposing the constraint on
the norm of $u$. Let $u_\varepsilon$ be a configuration that will minimize $E_\varepsilon$ and let $v_\varepsilon = u_\varepsilon/\eta_\varepsilon$. Since $\eta_\varepsilon$ satisfies the Gross Pitaevskii equation (8)-(9), we have

$$
\int_D \left( |v_\varepsilon|^2 - 1 \right) \left( -\frac{1}{2} \Delta f_\varepsilon - \frac{1}{\varepsilon^2} f_\varepsilon^2 (\rho_{\varepsilon} - f_\varepsilon^2) + |\nabla f_\varepsilon e^{iS_\varepsilon}|^2 - 2f_\varepsilon^2 (\nabla S_\varepsilon \cdot \Omega \times r) \right) = 0.
$$

Using this identity, one can get that the energy $E_\varepsilon(u_\varepsilon)$ decouples as follows

$$
E_\varepsilon(u_\varepsilon) = E_\varepsilon(\eta_\varepsilon) + G_{\eta_\varepsilon}(v_\varepsilon) + I_{\eta_\varepsilon}(v_\varepsilon) \quad (14)
$$

where

$$
G_{\eta_\varepsilon}(v_\varepsilon) = \int_D \frac{1}{2} |\eta_\varepsilon|^2 |\nabla v_\varepsilon|^2 + \frac{|\eta_\varepsilon|^4}{4\varepsilon^2} (1 - |v_\varepsilon|^2)^2,
$$

and

$$
I_{\eta_\varepsilon}(v_\varepsilon) = \int_D |\eta_\varepsilon|^2 (\nabla S_\varepsilon - \Omega \times r) \cdot (iv_\varepsilon, \nabla v_\varepsilon).
$$

The first term in the energy is independent of the solution $u_\varepsilon$, so we have to compute the next two and find for which configuration $u_\varepsilon$ the minimum is achieved. We assume that the solution $u_\varepsilon$ has a vortex line along $\gamma$ and we call $\delta$, the Dirac measure along the line. Our aim is to estimate the energy of $u_\varepsilon$ depending on $\gamma$. A first approximation consists in writing that $|\eta_\varepsilon|^2$ tends to $\rho_{\varepsilon}$ when $\varepsilon$ is small so that we can approximate $G_{\eta_\varepsilon}$ by $G_{\sqrt{\rho_{\varepsilon}}}$ and $I_{\eta_\varepsilon}$ by $I_{\sqrt{\rho_{\varepsilon}}} = I_\varepsilon$.

C. Estimate of $G_\varepsilon(v_\varepsilon)$

We want to estimate

$$
G_\varepsilon(v_\varepsilon) = \int_D \frac{1}{2} |\rho_{\varepsilon}|^2 |\nabla v_\varepsilon|^2 + \frac{|\rho_{\varepsilon}|^4}{4\varepsilon^2} (1 - |v_\varepsilon|^2)^2.
$$

The mathematical techniques to approximate $G_\varepsilon$ have been introduced by [17] in the 2 dimensional case and by [18] in dimension 3, when $\varepsilon$ is very small. The problem here is that $\varepsilon = 0.0174$ so that $|\log \varepsilon|$ is not large and there will be additional terms in the asymptotic expansion. For a minimizing configuration, one can prove that $v_\varepsilon$ is tending to 1 everywhere except close to the vortex line $\gamma$. We define

$$
T_{\lambda \varepsilon} = \{ x \in D \text{ s.t. dist}(x, \gamma) \leq \lambda \varepsilon \}, \quad (15)
$$

and assume that $\lambda \varepsilon$ is small, $\lambda$ being a nondimensional parameter to be fixed later on. Then we split $G_\varepsilon$ into two integrals: one in $T_{\lambda \varepsilon}$ and the other in $D \setminus T_{\lambda \varepsilon}$.

1. Estimate near the vortex core

We are going to estimate $G_\varepsilon$ in $T_{\lambda \varepsilon}$. At each point $\gamma(t)$ of $\gamma$, we define $\Pi^{-1}(\gamma(t))$ to be the plane orthogonal to $\gamma$ at $\gamma(t)$. Since $\lambda \varepsilon$ is small, we assume that $\rho_{\varepsilon}$ is constant in $\Pi^{-1}(\gamma(t)) \cap T_{\lambda \varepsilon}$ and we call the value $\rho_{\varepsilon} = \rho_{\varepsilon}(\gamma(t))$.

We want to compute

$$
\int_{T_{\lambda \varepsilon}} \frac{1}{2} \rho_{\varepsilon} |\nabla v_\varepsilon|^2 + \frac{\rho_{\varepsilon}^2}{4\varepsilon^2} (1 - |v_\varepsilon|^2)^2 
\simeq \int_{\Pi^{-1}(\gamma(t)) \cap T_{\lambda \varepsilon}} |\nabla v_\varepsilon|^2 + \frac{\rho_{\varepsilon}}{2\varepsilon^2} (1 - |v_\varepsilon|^2)^2.
$$

Since $u_\varepsilon$ is a minimizing configuration of $E_\varepsilon$, after scaling by $r \sqrt{\rho_{\varepsilon}}/\varepsilon$, we find that $v_\varepsilon$ is very close to $u_1(r) \rho_{\varepsilon}/\varepsilon$, where $u_1(r, \theta) = f_1(r)e^{i\theta}$ is the solution with a single zero at the origin of

$$
\Delta u + u(1 - |u|^2) = 0 \text{ in } \mathbb{R}^2.
$$

Thus,

$$
\int_{\Pi^{-1}(\gamma(t)) \cap T_{\lambda \varepsilon}} |\nabla v_\varepsilon|^2 + \frac{\rho_{\varepsilon}}{2\varepsilon^2} (1 - |v_\varepsilon|^2)^2 
\simeq \int_{B_{\lambda \varepsilon}\rho_{\varepsilon}} \left| \nabla \left( f_1(r) \left( \frac{\rho_{\varepsilon}}{\varepsilon} \right) e^{i\theta} \right) \right|^2 + \frac{\rho_{\varepsilon}}{2\varepsilon^2} \left( 1 - f_1^2(r) \left( \frac{\rho_{\varepsilon}}{\varepsilon} \right)^2 \right)^2 
= \int_{B_{\lambda \varepsilon}\rho_{\varepsilon}} |\nabla u_1|^2 + \frac{1}{2} (1 - |u_1|^2)^2 
\simeq c_* + 2\pi \log(\lambda \sqrt{\rho_{\varepsilon}}),
$$

where

$$
c_* = \int_{\mathbb{R}^2} f_1^2 + \frac{1}{2} (1 - f_1^2)^2 + \int_{\mathbb{R}^2 \setminus B_1} f_1^2 - \frac{1}{r^2} + \int_{B_1} f_1^2/r^2.
$$

The last approximation is good if $\lambda \sqrt{\rho_{\varepsilon}}$ is large (in fact bigger than 3 is enough). The existence of $\lambda$ is justified by the fact that $\sqrt{\rho_{\varepsilon}}/\varepsilon$ is much bigger than 1, except very close to the boundary.

The final estimate of this section is

$$
G_\varepsilon(v_\varepsilon)|_{T_{\lambda \varepsilon}} \simeq \int_{\gamma} \rho_{\varepsilon} \frac{c_*}{2} + \pi \log(\lambda \sqrt{\rho_{\varepsilon}}) dl \quad (16)
$$

2. Estimate away from the vortex core

We are going to estimate $G_\varepsilon$ in $D \setminus T_{\lambda \varepsilon}$. In this region $v_\varepsilon \simeq 1$, so that only the kinetic energy of the phase has a contribution.

$$
\int_{D \setminus T_{\lambda \varepsilon}} \frac{1}{2} \rho_{\varepsilon} |\nabla v_\varepsilon|^2 + \frac{\rho_{\varepsilon}^2}{4\varepsilon^2} (1 - |v_\varepsilon|^2)^2 
\simeq \int_{D \setminus T_{\lambda \varepsilon}} \frac{1}{2} \rho_{\varepsilon} |\nabla \phi_\varepsilon|^2,
$$

where $\phi_\varepsilon$ is the phase of $v_\varepsilon$. Of course, $\phi_\varepsilon$ is not defined everywhere. But let $\psi$ be such that $\text{div } \psi = 0$ and

$$
\text{curl } \psi = \rho_{\varepsilon} \nabla \phi.
$$
Then $\psi$ is the unique solution of
\[
\text{curl} \left( \frac{1}{\rho_{\text{TF}}} \text{curl} \psi \right) = 2\pi \delta_\gamma, \quad \psi = 0 \text{ on } \partial \mathcal{D},
\]
(17)
where $\delta_\gamma$ is the vectorial dirac measure along $\gamma$ and
\[
\int_{\mathcal{D} \setminus T_\lambda} \frac{1}{2} \rho_{\text{TF}} |\nabla \phi_\epsilon|^2 = \int_{\mathcal{D} \setminus T_\lambda} \frac{1}{2} \rho_{\text{TF}} |\text{curl} \psi|^2
\]
\[
= -\frac{1}{2} \int_{\partial T_\lambda} \psi \cdot \nabla \phi_\epsilon \times \nu
\]
where $\nu$ is the outward unit normal. We will see that $\psi$ is almost constant at a distance $\lambda \varepsilon$ from $\gamma$ and we call this value $\psi_\lambda(\gamma)$. Since the vortex line has a winding number $2\pi$, we locally approximate the curve
\[
\int_{\mathcal{D} \setminus T_\lambda} \frac{1}{2} \rho_{\text{TF}} |\nabla \phi_\epsilon|^2 \simeq \pi \int_{\gamma} \psi_\lambda(\gamma) \cdot dl.
\]
We have to compute $\psi$ on $\partial T_\lambda$. The computation is inspired by the paper of Svidzinsky and Fetter [13]. It follows from (17) that $\psi$ satisfies
\[
-\Delta \psi - \nabla \rho_{\text{TF}} \times \text{curl } \psi = 2\pi \rho_{\text{TF}} \delta_\gamma,
\]
(18)
where $\rho_{\text{TF}}(x^1, x^2) = \rho_{\text{TF}}(x^1, x^2, x^3)$. Let $\Xi = \psi_3/\sqrt{\rho_{\text{TF}}}$. Then it follows from [13] that $\Xi$ satisfies
\[
-\Delta \Xi + \mu \Xi = 2\pi \sqrt{\rho_{\text{TF}}} \delta_\gamma.
\]
(19)
where
\[
\mu = \sqrt{\rho_{\text{TF}}} \Delta \frac{1}{\sqrt{\rho_{\text{TF}}}} = \sqrt{\rho_{\text{TF}}} \Delta \frac{1}{\sqrt{\rho_{\text{TF}}}}.
\]
(20)
Here $\Delta_\perp$ is the laplacian in the plane perpendicular to $e_3 = \hat{\gamma}(x_0)$. If the cross-section is a disc one can compute $\mu$. We denote by $\theta$ the angle of $e_3$ that is $e_3 = \cos \theta e_r + \sin \theta e_\theta$ and $(r, z)$ are the coordinates of $x_0$ in the original frame. Then
\[
\mu = \frac{(1 + \sin^2 \theta) + \beta^2 \cos^2 \theta + 3(r \sin \theta - \beta^2 \tan \theta)^2}{\rho_{\text{TF}}^2}.
\]
(21)
Note that $\mu > 0$. In fact our numerical computations even yield $\mu > 7$. Our aim is now to give an approximate expression for $\Xi$. We locally approximate the curve $\gamma$ near the point $x_0$ by the parabola $z = k z^2/2$, where $k$ is the curvature of $\gamma$ at $x_0$. This is where we use the same ideas as in [13]. Note that in our approximations, we are only taking into account the shape of $\gamma$ close to $x_0$. The justification for this relies on the fact that $\mu > 7$ as our numerics show. Indeed if we solve
\[
-\Delta X + \mu X = f
\]
where $f$ is supported at a distance $d$ of $x_0$. Then using the Green function, we find that
\[
|X| \leq \frac{e^{-\sqrt{\mu d}}}{4\pi \mu^{3/2}}.
\]
In particular, for $d = 0.1$, this gives an error less than $10^{-3}$. This is to be compared to the Euler constant and our approximation is reasonable. We rewrite (19) in local coordinates to get
\[
-\Delta \Xi + k \partial_z \Xi + \mu \Xi = 2\pi \sqrt{\rho_{\text{TF}}(x_0)} \delta_{e_3},
\]
where $\delta_{e_3}$ is the dirac mass supported along the line $e_3$. Thus
\[
-\Delta(e^{-k \varepsilon \Xi}) + \left( \frac{k}{2} \right)^2 + \mu \left( e^{-k \varepsilon \Xi} \right) = 2\pi \sqrt{\rho_{\text{TF}}(x_0)} \delta_{e_3}.
\]
(22)
The solution of this equation is
\[
\sqrt{\rho_{\text{TF}}(x_0)} K_0 \left( \sqrt{\mu + \frac{k^2}{4}} \text{dist}(x, \gamma) \right),
\]
where $K_0$ is a modified Bessel function. In particular, $K_0(x) \sim -\log(e^{C_0} x^2)$ for small $x$ where $C_0 \simeq 0.577$ is the Euler constant. Hence, we deduce
\[
\psi(x) \simeq -\rho_{\text{TF}} \log \left( \frac{e^{C_0}}{2} \sqrt{\mu + \frac{k^2}{4}} \text{dist}(x, \gamma) \right) \hat{\gamma}.
\]
(23)
Thus we conclude by the estimate for $G_x(v_x)$ in $\mathcal{D} \setminus T_\lambda$
\[
G_x(v_x)|_{\mathcal{D} \setminus T_\lambda} \simeq -\pi \int_\gamma \rho_{\text{TF}} \log \left( \frac{e^{C_0}}{2} \sqrt{\mu + \frac{k^2}{4}} \lambda \varepsilon \right) dl.
\]
(24)

D. Estimate of $I_\varepsilon(v_x)$

We want to estimate
\[
I_\varepsilon(v_x) = \int_D \rho_{\text{TF}} (\nabla S_\varepsilon - \Omega \times r) \cdot (iv_x, \nabla v_x).
\]
(25)
Recall that the unique solution of (11) satisfies $\rho_{\text{TF}} (\nabla S_\varepsilon - \Omega \times r) = \Omega \cdot \nabla \phi_\varepsilon$. Hence we integrate by part in (23) to get
\[
I_\varepsilon(v_x) = \Omega \int_D \phi_\varepsilon \cdot \nabla (iv_x, \nabla v_x).
\]
Let $\phi_\varepsilon$ be the phase of $v_x$. Since $v_x$ is tending to one everywhere except on the vortex line, then $(iv_x, \nabla v_x) \sim \nabla \phi_\varepsilon$, hence we can approximate $\nabla (iv_x, \nabla v_x)$ by $2\pi \delta_\gamma$. We use the value of $\Xi$ given by (13) and the fact that $\hat{\gamma}(t) \cdot e_z = dz$ to get
\[
I_\varepsilon(v_x) \simeq -\frac{\Omega \pi}{(1 + \alpha^2)} \int_{\gamma} \rho_{\text{TF}}^2 \frac{dz}{\mu}.
\]
(26)
E. Final estimate for the energy

We use (13), (14), (21), (24) to derive the energy of a solution with a vortex line. Indeed the energy of any solution minus the energy of a solution without vortex is roughly the vortex contribution in the sense:

\[ E_{\varepsilon}(u_\varepsilon) - E_{\varepsilon}(\eta_\varepsilon) \simeq E_\gamma. \]  (27)

We find that the vortex contribution \( E_\gamma \) is

\[ E_\gamma = \int_\gamma \rho_{TV}(\frac{C_2}{2} + \pi \log(\frac{2}{\varepsilon \varepsilon_{C_0}} \sqrt{\frac{\rho_{TV}}{\mu + \frac{k^2}{4}}})) \, dl \\
- \frac{\Omega \pi}{(1 + \alpha^2)} \int_{\gamma} \rho_{TV}^2 \, dz. \]  (28)

Hence if the right-hand-side of (28) is negative, it means that it is energetically favorable to have vortices. Note that in the first integral of \( E_\gamma \), we have \( dl = |\gamma(z)| \, dz \) whereas in the second one, we have \( dz \).

If the vortex line is straight, our computation yields

\[ \frac{\rho_0^{3/2}}{\beta} \left( \frac{2}{3} \left( \frac{C_2}{2} + \pi \log(\frac{\sqrt{2}}{\varepsilon \varepsilon_{C_0}}) \right) + \frac{2\pi}{3} \log \rho_0 \\
+ \pi \left( -\frac{10}{9} + \frac{4}{3} \log 2 \right) - \pi \frac{8\pi \rho_0}{15(1 + \alpha^2)} \right). \]  (29)

This gives a critical angular velocity \( \Omega_1 \) for which a straight vortex has a lower energy than a vortex free solution. With our experimental data, it yields \( \Omega_1 \sim 22.45 \), that is \( \Omega_1/\omega_\varepsilon \sim 0.39 \). We are going to see in the numerical section that for \( \Omega < \Omega_1 \), a bent vortex has a negative energy.

F. Case of several vortices

Let us assume that the solution \( u_\varepsilon \) has \( n \) vortices along the lines \( \gamma_i, 1 \leq i \leq n \). We want to estimate the energy in this case. For each \( \gamma_i \), we define \( T_{\varepsilon, \lambda \varepsilon} \) as in (13).

One can check that the estimates (26) and (16), respectively for \( I_{\varepsilon}(u_\varepsilon) \) and for \( G_{\varepsilon}(u_\varepsilon) \) close to each vortex core, are unchanged if the integral along \( \gamma \) is replaced by the sum of the integrals along \( \gamma_i \). The only difference is for the estimate away from the vortex cores where we have to take into account the interaction between the vortex lines. Let us denote \( D_n = D \setminus \bigcup_i T_{\varepsilon, \lambda \varepsilon} \). We still have

\[ G_{\varepsilon}(u_\varepsilon)|_{D_n} \simeq \int_{D_n} \frac{1}{2\rho_{TV}} |\psi|, \]  (30)

where \( \psi = \sum \psi_i \) and \( \psi_i \) solves (17) with \( \gamma_i \) instead of \( \gamma \). Thus, we need to estimate

\[ \sum \int_{D_n} \frac{1}{2\rho_{TV}} |\psi_i|^2 + \sum \int_{D_n} \frac{1}{2\rho_{TV}} |\psi_k| |\psi_i| \]  (31)

The first integral is estimated as in section C.2 by

\[ \sum_i -\pi \int_{\gamma_i} \rho_{TV} \log \left( \frac{e^{C_0}}{2} \sqrt{\frac{\rho_{TV}}{\mu + \frac{k^2}{4}}} \right) \, dl. \]  (32)

As for the second integral in (31), we integrate it by part to get

\[ \pi \sum_{i \neq k} \int_{\gamma_i} \psi_k \cdot dl. \]  (33)

The computation of \( \psi_k(x) \) from section C.2 is still valid and we have \( \psi_k(x) \simeq -\rho_{TV} K_0(\sqrt{\mu + \frac{k^2}{4}} dist(x, \gamma_k)) \). This yields the contribution of \( n \) vortex lines (to be compared with (28) for 1 vortex)

\[ E_n = \sum \int_{\gamma_i} \rho_{TV} \left( \frac{C_2}{2} + \pi \log(\frac{2}{\varepsilon \varepsilon_{C_0}} \sqrt{\frac{\rho_{TV}}{\mu + \frac{k^2}{4}}}) \right) \, dl \\
- \frac{\Omega \pi}{(1 + \alpha^2)} \int_{\gamma} \rho_{TV}^2 \, dz \]  (34)

\[ -\pi \sum_{i \neq k} \int_{\gamma_i} \rho_{TV} K_0(\sqrt{\mu + \frac{k^2}{4}} dist(x, \gamma_k)) \, dl. \]  (35)

where \( K_0 \) is a modified Bessel function. Note that the curves are going to interact only in the region where they are close to one another.

III. COMPARISON WITH THE NUMERICS

We are interested in the shape of the vortex line that minimizes (28) according to the value of \( \Omega \). We write \( (\gamma(t) = r(t), z(t)) \) and we assume that the vortex line is in the plane \((y,z)\). We will denote \( \rho_{TV}(t) = \rho_0 - \alpha^2 r^2(t) - \beta^2 z^2(t) \) and we define

\[ C(t) = \frac{C_2}{2} + \pi \log(\frac{2}{\varepsilon \varepsilon_{C_0}} \sqrt{\frac{\rho_0 - \alpha^2 r^2(t) - \beta^2 z^2(t)}{\mu + \frac{k^2}{4}}}). \]

Since (28) does not depend on the parametrization \( \gamma(t) \), we choose a special parametrization on the curve such that

\[ C^2(t) \rho_{TV}^2(t)(\dot{r}^2(t) + \dot{z}^2(t)) = 1. \]  (36)

Then it is equivalent to minimize

\[ \int_\gamma C^2(t) \rho_{TV}^2(t)(\dot{r}^2(t) + \dot{z}^2(t)) \, dt - \frac{\Omega \pi}{(1 + \alpha^2)} \int_\gamma \rho_{TV}^2(t) \dot{z}(t) \, dt \]  (37)

under the constraint (28). In our computations below, we will proceed to a minimization of (17) releasing the constraint (36). Indeed, computations show that (30) is true from \( t = 0 \) to \( t^* \) where the shape of the vortex is
determined. Under the assumption that $\mu$ and the curvature do not vary too much along the curve, we derive an equation for the minimum $\gamma$:

$$\frac{d}{dt}(C^2 \rho_{TF} \dot{r}) = -\frac{2\alpha^2 r(t)}{\rho_{TF}(t)} + \frac{2\alpha^2 \Omega}{(1 + \alpha^2)} \rho_{TF} r(t) \dot{z}(t),$$

$$\frac{d}{dt}(C^2 \rho_{TF} \dot{z}) = -\frac{2\beta^2 z(t)}{\rho_{TF}(t)} - \frac{2\alpha^2 \Omega}{(1 + \alpha^2)} \rho_{TF} r(t) \dot{r}(t).$$

Thus, we solve this system with initial conditions $r(0) = r_0$, $\dot{r}_0 = 0$, $z(0) = 0$, $C(0) \rho_{TF}(0) \dot{z}(0) = 1$.

We let $r_0$ vary in order to find the minimizing solution. We have drawn the vortex line for the minimizing solution for some values of $\Omega$ in Figure 1. We find that indeed the vortex line is bending for a range of $\Omega$. The bent vortex starts to exist near the boundary of the ellipse, that is $y = \sqrt{\rho_0/\alpha}$, $z = 0$ for $\Omega_0 = 21.2$, that is $\Omega_0/\omega_x = 0.368$. As $\Omega$ increases, the value of $r_0$ decreases: $r_0 = 0.03$ for $\Omega = 21.8$, $r_0 = 2.9 \times 10^{-5}$ for $\Omega = 25.8$, $r_0 = 10^{-6}$ for $\Omega = 33.1$. As $\Omega$ increases, $r_0$ becomes smaller, the bent vortex gets very close to the straight vortex. The shape of the vortex lines are similar to those obtained in [12] using the full Gross Pitaevskii energy.

![FIG. 1: The vortex line for various values of $\Omega$ in the $z-y$ plane: $\Omega = 21.8$ (straight line), $\Omega = 25.8$ (dotted line), $\Omega = 33.1$ (dashed line).](image)

We plot the energy of the straight vortex line and the bent vortex vs $\Omega$ in Figure 2. One can observe that for $\Omega_c = 21.8$, that is $\Omega_c/\omega_x = 0.38$ in the initial units, the energy of the bent vortex starts to be negative (that is below the energy of a solution without vortex), while the energy of a straight vortex line is positive. For $\Omega = 33.1$, the energy of the bent vortex and of a straight vortex line become equal. These results are consistent with the ones in [11]. They obtain the same value of $\Omega_c$ for which the bent vortex has a negative energy.

Let us point out that the bent vortex is a minimizer even if the cross section is a disc. Nevertheless, when $\epsilon$ is fixed, if $\beta$ gets too big, the straight vortex becomes the minimizer, which is the case for $\beta = 1$. Our analysis could give the critical value of $\beta$ above which the vortex line should be straight.

![FIG. 2: The energy vs. $\Omega$ curves for the solution with a straight vortex (solid line) and a bent vortex (dotted line).](image)

**IV. CONCLUSION**

We have obtained a simplified expression (28) of the energy of a minimizing solution of the Gross Pitaevskii energy with a vortex line $\gamma$ and (34) for a vortex lines $\gamma_i$. This expression depends on the shape of the vortex line. This has allowed us to draw the vortex line for the minimizing solution and compute its energy. We have seen that there is a range of rotational velocities for which a bent vortex line has a lower energy than a straight vortex and a vortex free solution. These computations on the simplified expression of the energy are in agreement with the computations on the full energy [1], [12].

**Acknowledgments**

The authors would like to warmly thank Y.Castin for explaining the work of his team at the ENS and for very interesting and encouraging discussions. The authors are very indebted to him for his critics and remarks on the manuscript. The work has also largely benefited from

\[\frac{d}{dt}(C^2 \rho_{TF} \dot{r}) = -\frac{2\alpha^2 r(t)}{\rho_{TF}(t)} + \frac{2\alpha^2 \Omega}{(1 + \alpha^2)} \rho_{TF} r(t) \dot{z}(t),\]

\[\frac{d}{dt}(C^2 \rho_{TF} \dot{z}) = -\frac{2\beta^2 z(t)}{\rho_{TF}(t)} - \frac{2\alpha^2 \Omega}{(1 + \alpha^2)} \rho_{TF} r(t) \dot{r}(t).\]
discussions with E.Sandier and S.Serfaty. The authors wish to thank them very much.

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