Role Detection in Bicycle-Sharing Networks Using Multilayer Stochastic Block Models

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Abstract

Urban spatial networks are complex systems with interdependent roles of neighborhoods and methods of transportation between them. In this paper, we classify docking stations in bicycle-sharing networks to gain insight into the spatial delineations of three major United States cities from human mobility dynamics. We propose novel time-dependent stochastic block models, with degree-heterogeneous blocks and either mixed or discrete block membership, which (1) detect the roles served by bicycle-sharing docking stations and (2) describe the traffic within and between blocks of stations over the course of a day. Our models produce concise descriptions of daily bicycle-sharing usage patterns in urban environments. They successfully uncover work and home districts, and they also reveal dynamics of such districts that are particular to each city. When we look for more than two roles, we uncover blocks with expected uses, such as leisure activity, as well as previously unknown structures. Our time-dependent SBMs also reveal how the functional roles of bicycle-sharing stations are influenced by surrounding public transportation infrastructure. Our work has direct application to the design and maintenance of bicycle-sharing systems, and it can be applied more broadly to community detection in temporal and multilayer networks.

1 Introduction

It is useful to view cities as large spatial networks under constant evolution, with intermittent large-scale changes [1][2]. Transportation systems and commuting patterns shape and reveal the functional regions in a city [3], and an increasing amount of evidence suggests that polycentric urban structures tend to emerge from classic monocentric structures [4][5][6][7]. The combination of a burst in the development of network-analysis methods and the increasing

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availability of transportation data gives exciting opportunities to improve understanding of urban dynamics. In the present paper, we construct a novel statistical model of networks and aim to uncover the roles of regions in a city through the lens of travel patterns between them.

There is now a deluge of data about transportation and other urban systems, and they offer the potential for numerous fascinating and important insights about cities and human dynamics in urban environments. Local governments release data on many transportation modes, including traffic from buses, trains, and automobiles. Additionally, online databases such as OpenStreetMap [8] and Global Road Network [9] include large-scale road networks with tens of thousands of streets, covering a total length of tens of millions of kilometers. Subway data in many cities are available from electronic ticketing systems [5,10]. However, the infrastructure — such as train lines, bus routes, and highway systems — that underlies these systems adapt very slowly to changing commuting patterns. Pedestrian traffic adapts faster, but it captures only short-distance transportation and is difficult to measure.

Bicycle-sharing systems are an emerging mode of urban transportation that can adapt quickly to the needs of travelers. The number of bicycle-sharing programs worldwide has grown rapidly, from 5 in 2005 to 1571 in 2018 [11]. Over 50 systems were launched in the United States alone from 2010 to 2016, and over 20 bicycle-sharing systems have been launched in France since 2005 [12][13]. Many existing bicycle-sharing systems are also growing. For example, the number of stations in New York City’s “Citi Bike” system has more than doubled since it began in 2013. Docked bicycle-sharing systems follow a general structure: Groups of bikes are parked at “stations” (also called “docks” or “hubs”) throughout a coverage area, and users withdraw and return bikes to these stations on demand, with a cost that depends on usage time. A growing portion of bicycle-share systems are dockless (as are the increasingly prominent e-scooters), so users can park bicycles at any location in a coverage area. In the present paper, we analyze docked systems (but we consider how to adapt our models to dockless systems in Section 6).

Data from bicycle-sharing systems captures commuting behavior [14], including detailed temporal records and GPS-tracked routes in some cases; and they are widely available from many cities throughout the world [15,16,17,18,19]. These properties make it extremely valuable to analyze bicycle-sharing data increase understanding of urban flows and the properties of human commuting. Bicycle-sharing is used often for “last-mile” transportation, bridging the gap between public transportation and a final destination [20]; and insights into the dynamics and function of bicycle-sharing systems can help transit systems evolve to meet the needs of changing cities [21].

In the present paper, we propose two models of temporal network connectivity to capture the functional roles of bicycle-sharing stations. We do this using the lens of mesoscale-structure detection in time-dependent networks [22][23][24]. We examine trip histories from bicycle-sharing systems in the form of multilayer networks [25][26][27] in which each layer is a network of trips in a given hour. Edges in each layer represent the total number of trips from one station to another that begin during that hour, and we do not include any interlayer edges.

We aim to partition a network based on a relational equivalence of nodes (a perspective with a rich history in the social-networks literature [28][29]), rather than on high internal traffic within sets of nodes [16]. Data that has been aggregated over long periods of time
can shed light on “community” structure in the latter sense through a partition of the network into contiguous spatial clusters. However, it ignores how bicycle-sharing usage cycles with travel patterns throughout a day. By contrast, our models are designed to detect functional roles of bicycle-sharing systems based on time-dependent behavior. The models that we introduce in this paper are time-dependent extensions of the stochastic block model (SBM). We include parameters to describe intra-block and inter-block traffic for each hour, but we fix the block assignment of each station over time. That is, we treat a bicycle-sharing network as a temporal multilayer network with fixed node identities across layers. Although our models are inspired by the analysis of bicycle-sharing systems, they are applicable more generally to multilayer networks where nodes belong to fixed classes.

We introduce mixed-membership and discrete-membership versions of our model, where nodes can be members of multiple blocks or exactly one block, respectively. Both versions are degree-corrected, as we parametrize the time-aggregated degree of each node to avoid conflating block divisions with node activity levels. This is especially important for bicycle-sharing networks in which stations have heterogeneous numbers of parking spaces for bicycles and different neighborhoods have different baseline levels of bicycle usage. Our models are applicable to both directed or undirected networks, although we consider only directed examples in the present paper.

Increasing our understanding of the functional roles of docking stations can aid in the design and maintenance of bicycle-sharing networks, and understanding the usage patterns of stations can help inform the forecasting of their usage in unobserved and partially observed. This is valuable for tasks such as the dynamic restocking of stations with bicycles, which is both challenging and expensive, yet vital to the success of bicycle-sharing systems.

A wide variety of community-detection and other clustering methods exist for networks. Such methods include spectral methods, inference using SBMs, optimization of objective functions (such as modularity), local methods based on dynamical processes, and others. They have different strengths and drawbacks, and some of them are more appropriate for some applications than for others. Community detection in multilayer networks is a developing field, with methods applied to diverse applications, such as biology, sociology, and materials science. Several algorithms for community detection have been generalized to multilayer networks; see for examples.

In early work on community detection in multilayer networks, Mucha et al. derived a generalization of modularity optimization for a type of multilayer network known as a “multislice” network. They used it to detect communities that change over time, encompass multiple types of social relationships, or include multiple values of a resolution parameter. Yang et al. introduced a discrete-membership SBM with time-evolving communities and parameters for block-to-block activity that are fixed over time. Both and proposed related models (for unweighted (i.e., binary) and weighted networks, respectively), but they relaxed the assumption of fixed block-to-block activity parameters. Similar in spirit, Zhang et al. developed a time-dependent model with degree correction in which nodes are allowed to switch between blocks which are described by a continuous-time Markov chain. Mixed-membership SBMs with time-evolving communities have also been developed. See for a survey of community-detection methods for time-dependent networks. Based on the classification scheme in that paper, our methods belong to the class with “fixed
memberships, evolving properties”.

Community detection has been applied previously to urban bicycle sharing using various approaches \cite{15,16,43,57,58,59,60,61}. Zhu et al. applied k-means clustering to undirected, time-dependent usage data from bicycle-sharing systems and other urban systems in New York City \cite{10}. Austwick et al. examined modularity optimization with a directed and spatially-corrected null model to identify communities of stations in several cities \cite{15}. However, they detected communities in time-aggregated data, and their discussion pointed out that there are significant limitations to examining community structure while ignoring time-dependent behavior for bicycle-sharing applications. Munoz-Mendez et al. \cite{16} identified communities by hour for bicycle-sharing data from London using an INFOMAP algorithm \cite{58} that respects the directed nature of edges in the underlying trip networks. The changes that they discovered in communities over time highlight the importance of time of day in the usage of bicycle-sharing systems.

Closely related to our work, Matias et al. constructed a time-dependent, discrete-membership SBM with fixed blocks over time and applied it to bicycle-sharing networks in London \cite{62}. They detected some functional blocks, but their approach does not incorporate degree correction. Xie and Wang \cite{59} employed an approach that does not use an SBM directly, yet they were able to successfully partition a bicycle-sharing network to find home and work roles of bicycle-sharing stations. They used the ratio of in-degree to out-degree at different times to discover home–work splits during peak commute times, similar to the results of our paper. A similarity of their approach to ours is that it corrects for degree; a key difference is that they relied on human supervision to determine peak hours, whereas our models implicitly increase the weights of more-active time periods in the likelihood function. Etienne and Latifa clustered bicycle-sharing stations in Paris based on their time-dependent usage profiles using a Poisson mixture model \cite{13}. They were able to capture time-dependent activity for each group, distinguish between incoming and outgoing activity, control for the overall activity level of a given station (via degree correction), and associate identified groups with their role in the city. A key difference between their approach and ours is that we distinguish activity between blocks, which allows us to detect behavior like last-mile commuting that occurs within blocks.

Our paper proceeds as follows. In Section 2 we list our data sources and present basic statistical analysis of the data. In Section 3 we introduce the two versions of our time-dependent SBM — discrete and mixed-membership — and we show that they are equivalent up to a constraint. In Section 4 we describe the estimation algorithms for our discrete and mixed-membership models. In Section 5 we present the results of our models for Los Angeles, San Francisco, and New York City. We discuss the implications of our work for bicycle-sharing systems and suggest areas of further study in Section 6. We show some additional details of our work in an appendix. We include code and data to implement our models and replicate the results in our paper as supplementary material (and also at https://github.com/jcarlen/tdsbm_supplementary_material).
2 Data

We examine United States bicycle-sharing systems in Los Angeles, the Bay Area, and New York City. For Los Angeles, we study only the system’s downtown part, which is self-contained; similarly, in the Bay Area, we consider only the San Francisco network. Our three focal systems vary in size and daily usage. Because of this variation and how the data were reported, we study different time periods for each system. We also selected our time periods to exclude days that are likely to be extremely hot or cold. All of the bicycle-sharing systems that we study have open-data portals, from which we downloaded the following data sets. In summary, after cleaning (see our discussion in the next paragraph), our data consist of the following:

- Downtown Los Angeles: October–December 2016; there are 61 stations and 40,130 trips, of which 73.4% are during weekdays [63].

- San Francisco: September 2015–August 2016; there are 35 stations and 267,412 trips, of which 92.1% are during weekdays [64].

- New York City: October 2016; there are 601 stations and 1,551,692 trips, of which 75.6% are during weekdays [65].

A trip consists of a user checking out a bicycle from a fixed location (a station that includes multiple parking spaces) and returning it to a station. The data for each trip include the starting time; ending time; and starting and ending locations by station ID, latitude, and longitude. Each data set also has a few additional fields about the users; these details include whether they have memberships in the bicycle-sharing system, but we do not use this information in our investigation. We cleaned the data by removing anomalous trips, including extremely short and extremely long trips, and trips to or from a station used for testing or maintenance (as indicated in the data). We also excluded a very small number of stations (two in Los Angeles and six in New York City) that did not have at least one departure and at least one arrival during the given time period. Finally, we excluded one station in New York City that was accessible to other stations only by ferry; it was involved in only nine trips during the given time period. In total, cleaning removed 7.1% of the trips in Los Angeles, 4.5% of the trips in San Francisco, and 1.4% of the trips in New York City. We retain self-edges, which represent trips that start and end at the same station. Although it is common to remove self-edges when analyzing networks [66], self-edges in the present application represent real trips that we expect to have a very similar data-generating mechanism as trips from stations to geographically nearby stations. As in [32], including self-edges also simplifies some elements of parameter estimation.

When fitting our models, we include only weekday trips, as we observe that weekday and weekend activity follow distinct patterns; and weekend activity does not reflect commuting behavior. From the data sets, we construct multilayer networks that are both weighted and directed. In our networks, nodes represent stations, edge values encode the number of

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1We take extremely short trips to be those that last two minutes or less; we take extremely long trips to be those that last 90 minutes or more in Los Angeles and San Francisco and 120 minutes or more in New York City, given that city’s larger coverage area of stations.
directed trips from one station to another that begins in a specified time period, and each of the 24 layers consists of the trips that start in a certain hour.

2.1 Preliminary Data Analysis

Our data show clear patterns of bicycle-sharing usage by time of day and day of the week, including heavier use during commuting hours. In Figure 1, we illustrate usage patterns by plotting the number of trips by starting hour for each city. In New York City and San Francisco, activity spikes during weekday morning and evening commuting hours, whereas weekend trips peak in early afternoon. Similar patterns were observed previously for bicycle-sharing systems in New York and many other cities [10, 13, 15, 59, 67]. By contrast, in Los Angeles, the number of trips by hour has a mid-day peak on weekdays that is nearly as strong as the morning peak. In New York City and Los Angeles, about one quarter of the trips occur on weekends, but only about eight percent of the trips in San Francisco occur then. This suggests that the San Francisco network covers more commercial areas and fewer residential areas.

For individual stations, the morning and evening peaks for in-degree and out-degree are often unbalanced: one direction has a stronger morning peak, and the opposite direction has a stronger evening peak. This is a key motivation for our time-dependent identification of stations into “home” and “work” types.

![Figure 1: Total trips by hour for weekdays, weekends, and overall. Hour 0 designates midnight.](image-url)

To further explore the imbalance between morning and evening activity in each network, we calculate the singular-value decomposition (SVD) of the matrices of in-degree and out-degree for each station by hour. To be explicit, entry $i,j$ of the matrix of in-degrees is equal to the total number of trips that arrive at station $i$ in hour $j$, and the we constructed the matrix of out-degrees analogously for departing trips. We show results for New York City in Figure 2 and for Los Angeles (see Figure 15) and San Francisco (see Figure 16) in the appendix. The first two principle components either strengthen both observed peaks or weaken one peak while strengthening the other. The first two singular vectors explain
at least 88% and as much as 97% percent of the variation in the corresponding matrix, supporting the importance of peak morning and evening commutes for classifying stations.

Another characteristic of our data that we incorporated into the design of our models is the strong positive Pearson correlation coefficient between the total (summed over all time periods) in-degree and out-degree of each station: 0.99 in New York, 0.98 in San Francisco, and 0.91 in Los Angeles. This is an intrinsic feature of docked bicycle-sharing systems, because a bicycle must be returned to a station before a new trip with it can begin. However, the use of trucks to redistribute bicycles in a system can loosen this requirement.

Figure 2: The first two singular vectors from the New York City bicycle-sharing network.

3 Our Stochastic Block Models

In this section, we introduce our time-dependent mixed-membership stochastic block model (TDMM-SBM) and time-dependent discrete stochastic block model (TDD-SBM).

Stochastic block models are a popular class of statistical network model [33]. The motivating principle of SBMs is a type of stochastic equivalence in which edges whose endpoints have the same block membership are identically distributed. It is a standard assumption of SBMs that edge values are independent, given the block membership of nodes. More formally, a binary (Bernoulli) random graph $Y$, with adjacency matrix $A$, from an SBM with $K$ blocks is defined by

$$P(A_{ij} = 1|G) = \eta_{g_i, g_j},$$

where $G$ (with components $g_i \in \{1, 2, \ldots, K\}$) is a vector of block assignments for the nodes of $Y$ and $\eta$ is a $K \times K$ matrix of block-to-block edge probabilities. Note that this definition allows directed graphs, in which $\eta$ can be asymmetric. For early presentations of SBMs, see [30, 31, 68, 69]. More recent advances have added flexibility to SBMs. Examples include the mixed-membership SBMs of [70], models with covariates in [71], the degree-corrected
SBM of [32], and the Bayesian implementations reviewed by [33]. Applications of SBM to longitudinal networks include discrete-membership [49,50] and mixed-membership [54,55] versions where nodes can switch blocks over time, as well discrete-membership versions with fixed blocks over time but without degree-correction [52,62].

3.1 Time-Dependent Mixed-Membership Stochastic Block Model (TDMM-SBM)

We now describe the framework for our mixed-membership SBM. Let \(i, j \in \mathcal{N}\) (with \(|\mathcal{N}| = N\) be nodes, which represent bicycle stations; let \(g, h \in \mathcal{K}\) (with \(|\mathcal{K}| = K\) be blocks. Our data is a three-dimensional array of size \((N, N, T)\), where \(T\) is the number of time slices (i.e., time layers). We consider hourly groupings of the trips based on their starting times. The quantity \(A_{ijt}\) is the observed number of trips from station \(i\) to station \(j\) with starting time greater than or equal to \(t\) and less than \(t + 1\). Let \(\bar{A}_{ij} = \sum_{t=0}^{23} A_{ijt}\) denote the weights of the associated time-aggregated matrix. Our network is a directed multilayer network, so we count each trip that both starts and ends at a node \(i\) during hour \(t\) (i.e., self-edges) exactly once in \(A_{iit}\).

For each node \(i\), there is a length-\(K\) vector of real numbers \(C_{ig} \in [0, 1]\). These numbers represent the mixed-membership block assignment of each node. The block-assignment parameter \(C_{ig}\) indicates the “strength” of node \(i\) in block \(g\). For each ordered pair \(g, h\) of blocks and each time \(t \in \{0, 1, \ldots, 23\}\) (where \(t = 0\) represents the hour that starts at midnight), there is a parameter \(\omega_{ght}\), which we call the “inter-block connectivity” parameter or “block-to-block” parameter, that represents the directed activity from block \(g\) to block \(h\) during hour \(t\). Note that \(\omega_{ght}\) need not be equal to \(\omega_{hgt}\) if the network is directed; this captures any asymmetries in the number of trips with respect to reversing origins and destinations. We also define the notation \(\bar{\omega}_{gh} = \sum_{t=0}^{23} \omega_{ght}\) for the time-aggregated matrix.

For each pair of nodes, \(i\) and \(j\), we assume that the number of trips that depart from \(i\) and arrive at \(j\) at time \(t\) is Poisson-distributed with mean \(\mu_{ijt} = \sum_{g,h} C_{ig}\omega_{ght}C_{jh}\). Our use of the Poisson distribution follows [32] and [33], facilitates computation, and is standard for modeling count data (although overdispersion is a concern).

For identifiability, we apply the constraint \(\sum_i C_{ig} = 1\) for all \(g\). This does not constrain the set of possible models in terms of realizable mean edge activities \(\mu_{ijt}\). Consider a model with unconstrained parameters \(\omega_{ght}\) and \(C_{ig}\). The model with parameters \(\omega'_{ght}\) and \(C'_{ig}\) such that \(C'_{ig} = \frac{C_{ig}}{\sum_j C'_{ij}}\) and \(\omega'_{ght} = \omega_{ght} \left(\sum_j C_{ijg}\right) \left(\sum_j C_{jih}\right)\) is an equivalent model, because the means of the distributions of edge weights are equal to the those of the model with unconstrained parameters. That is, \(\mu'_{ijt} = \sum_{g,h} C'_{ig}\omega'_{ght}C'_{jih} = \sum_{g,h} C_{ig}\omega_{ght}C_{jih} = \mu_{ijt}\).

Because \(\sum_i C_{ig} = 1\), we can think of \(C_{ig}\) as the proportion of the total activity of block \(g\) from the activity of node \(i\); the expected total number of trips at node \(i\) is \(\sum_g C_{ig} \sum_{h,t}(\omega_{ght} + \omega_{hgt})\). In this light, \(\sum_g C_{ig}\) is a measure of the activity of node \(i\) in which we do not weight each \(C_{ig}\) term by the total activity of the corresponding block. We can interpret \(C_{ig}\) relative to \(C_{ih}\) as how strongly block \(g\) is associated with node \(i\) relative to how strongly block \(h\) is associated with node \(i\). We use these quantities when visualizing the TDMM-SBM of our data, because they help ensure that we do not overlook blocks with important usage patterns but relatively lower activity. The parameter \(C_{ig}\) is analogous to the degree-correction parameter for SBMs.
that was introduced in [32], but we apply it to mixed block membership. We elaborate on this connection in Subsection 3.2, where we introduce a model that specifies that nodes have only one block.

We now compute the likelihood function that we will optimize to obtain the maximum-likelihood estimate (MLE). We assume conditional independence between hourly numbers of trips along each edge, given model parameters, so the likelihood of the data is

\[
L(G; \omega, C) = \prod_{t=0}^{23} \prod_{i,j \in \mathcal{N}} \frac{(\mu_{ijt})^{A_{ijt}}}{A_{ijt}!} \exp(-\mu_{ijt}),
\]

where \(\omega\) and \(C\) give the model parameters (i.e., \(\omega = \{\omega_{ght}\}\) and \(C = \{C_{ig}\}\)). Note that \(\mu_{ijt} = \sum_{g,h} C_{ig}\omega_{ght}C_{jh}\) is a function of these parameters, the set \(\mathcal{N}\) of nodes in the network is fixed and pre-determined, and the number \(K\) of blocks is also fixed and pre-determined.

The unnormalized log-likelihood is

\[
\ell(G; \omega, C) = \sum_{t=0}^{23} \sum_{i,j \in \mathcal{N}} [A_{ijt} \log(\mu_{ijt}) - \mu_{ijt}],
\]

although note that we omit the addition of the constant \(-\sum_{i,j,t} \log(A_{ijt})\), because it does not affect maximum of the function.

### 3.2 Time-Dependent Discrete Stochastic Block Model (TDD-SBM)

We derive a single-membership SBM from our mixed-membership SBM by making the extra assumption that, for each node \(i \in \mathcal{N}\), we have that \(C_{ig} > 0\) for only one block \(g \in \mathcal{K}\). (We also call this the “discrete version” of our model.) For our single-membership SBM, we introduce some new notation to aid our description and be consistent with notation in [32, 72]. For a given node \(i\), the block \(g\) for which \(C_{ig} > 0\) is the block \(g_i\) that includes node \(i\). Therefore, we use a single parameter \(\theta_i = C_{ig}\) for each node \(i\) to indicate both the strength of \(i\) in block \(g\) and the membership of node \(i\) in block \(g\). We will show that this term is a multilayer extension of the degree-correction term of [32]. The mean of the Poisson distribution of the value of an edge from node \(i\) to node \(j\) at time \(t\) is \(\theta_i \theta_j \omega_{ght}\). We retain the sum constraints of our mixed-membership model, such that \(\sum_{i \in g} \theta_i = 1\) for all \(g\).

We compute optimal values for the parameters \(\omega\) and \(\theta = \{\theta_i\}_{i \in \mathcal{N}}\). As in the TDMM-SBM, take \(\mathcal{N}\) and \(K\) to be fixed and pre-determined. Again dropping the constant term \(-\sum_{i,j,t} \log(A_{ijt})\), the log-likelihood of our single-membership SBM is

\[
\ell(G; \omega, \theta) = \sum_{t} \sum_{g,h} \sum_{i \in g, j \in h} [A_{ijt} \log(\theta_i) + A_{ijt} \log(\theta_j) + A_{ijt} \log(\omega_{ght}) - \theta_i \theta_j \omega_{ght}].
\]

We find explicit formulas for the MLEs of \(\theta_i\) and \(\omega_{ght}\). In the following calculations, removal of \(t\) from the subscript of a parameter and addition of a tilde designates a sum over all \(t\). Specifically, we define \(\tilde{A}_{ij} = \sum_{t=0}^{23} A_{ijt}\) and \(\tilde{\omega}_{gh} = \sum_{t=0}^{23} \omega_{ght}\). We differentiate \(\ell\) with respect to \(\omega_{ght}\) to yield

\[
\frac{\partial}{\partial \omega_{ght}} \ell = \sum_{i \in g, j \in h} \frac{A_{ijt}}{\omega_{ght}} - 1,
\]
where we have used the block-wise sum constraints on $\theta_i$. Therefore, the MLE for $w_{ght}$ is

$$\hat{\omega}_{ght} = m_{ght},$$

where $m_{ght}$ is the sum of weights of edges from nodes in block $g$ to nodes in block $h$ during hour $t$. That is, $m_{ght} = \sum_{i \in g, j \in h} A_{ijt}$.

We then differentiate $\ell$ with respect to $\theta_i$ to obtain

$$\frac{\partial}{\partial \theta_i} \ell = \frac{\sum_j \tilde{A}_{ij} + \sum_j \tilde{A}_{ji}}{\theta_i} - \sum_h \tilde{\omega}_{gh} - \sum_h \tilde{\omega}_{hg}.$$

At $\hat{\omega}_{ght}$, the MLE for $\theta_i$ is

$$\hat{\theta}_i = \frac{\sum_j \tilde{A}_{ij} + \tilde{A}_{ji}}{\sum_g \tilde{m}_{gh} + \tilde{m}_{hg}} = \frac{k_i}{\kappa_g},$$

where $k_i = \sum_j (\tilde{A}_{ij} + \tilde{A}_{ji})$ is the sum of the in-degree and out-degree of $i$ over all time periods, $\tilde{m}_{gh} = \sum_{t=0}^{23} m_{ght}$, and $\kappa_g = \sum_h (\tilde{m}_{gh} + \tilde{m}_{hg})$ is the sum of the in-degrees and out-degrees of all nodes in block $g$ over all time periods. The term $2\tilde{m}_{gg}$ in the equation for $\kappa_g$ implies that we count each intra-block edge twice: once for emanating from $g$ and once for arriving at $g$. Similarly, $k_i$ includes the term $2\tilde{A}_{ii}$, so we count self-edges twice in this term.

Our computation demonstrates that the MLE of the strength of $i$ in block $g$ is the relative proportion of the strength of node $i$ to the total activity of block $g$. The parameter $\hat{C}_{igi} = \hat{\theta}_i$ in the TDD-SBM for modeling directed, multilayer networks is analogous to the degree-correction parameter in the degree-corrected SBM for undirected networks with one layer. Indeed, the MLE for the degree correction parameter in the latter model is the proportion of number of edges connected to a node to the number of edges connected to its assigned block. Another similarity between a degree-corrected SBM and our TDD-SBM is that in the MLE of the TDD-SBM, the sum over time of the expectation of the degree of a node $i$ is equal to the degree of node $i$ from the observed data. That is, $\sum_t \sum_j (\mu_{ijt} + \mu_{jit})$, the sum of the mean weights of edges connected to $i$, is equal to $k_i$, the degree of node $i$ in the observed data. (See Section 6.2 of the appendix for the proof.) For our mixed-membership SBM, we are not aware of such a precise relationship between the data and the expected value of model statistics, although there does appear to be a positive correlation between the time-aggregated node degrees and the sum of the mixed-membership parameters ($\sum_g C_{igi}$ for all $i$).

We now calculate the MLE of the unnormalized log-likelihood of the TDD-SBM. We obtain

$$\sum_t \left[ \sum_{i,j} \left( A_{ijt} \log \left( \frac{k_i}{\kappa_g} \right) + A_{jit} \log \left( \frac{k_j}{\kappa_g} \right) \right) + \sum_{g,h} m_{ght} \log m_{ght} - \sum_{g,h} m_{ght} \right]$$

$$= \sum_i k_i \log k_i - \sum_i k_i \log \kappa_g + \sum_i \sum_{g,h} m_{ght} \log m_{ght} - \tilde{m},$$
where \( \tilde{m} \) is the total number of edges in the network. By a similar calculation as one in \([32]\), we obtain

\[
\sum_{i} k_{i} \log \kappa_{g_{i}} = \sum_{t} \sum_{g} \sum_{i \in g} k_{i|t} \log \kappa_{g} \\
= \sum_{t} \sum_{g} \sum_{i \in g} (k_{i|t} \log \kappa_{g} + k_{o|t} \log \kappa_{g}) \\
= \sum_{t} \sum_{g} \kappa_{o|t} \log \kappa_{g} + \sum_{t} \sum_{h} \kappa_{i|t} \log \kappa_{h} \\
= \sum_{t} \sum_{g} \sum_{m} m_{ght} \log \kappa_{g} + \sum_{t} \sum_{h} \sum_{m} m_{ght} \log \kappa_{h} \\
= \sum_{t} \sum_{g,h} m_{ght} \log \kappa_{g} \kappa_{h},
\]

where \( k_{i|t} \) and \( k_{o|t} \) are the respective in-degrees and out-degrees of node \( i \) during hour \( t \), the quantity \( \kappa_{i|t} = \sum_{g} k_{i|t} \) is the number of edges going into \( g \) during hour \( t \), and \( \kappa_{o|t} = \sum_{g} k_{o|t} \) is the number of edges that emanate from \( g \) during hour \( t \). Including only the terms that depend on block assignments yields the following objective function:

\[
\sum_{t} \sum_{g,h} m_{ght} \log \left( \frac{m_{ght}}{\kappa_{g} \kappa_{h}} \right).
\]  

(4)

Unlike the directed SBM in \([72]\), we do not have two strength parameters (representing an in-degree strength \( \theta_{i}^{\text{in}} \) and an out-degree strength \( \theta_{i}^{\text{out}} \)) for each station. Nevertheless, our model still captures the directed nature of the data. We can see this by conceptualizing the estimated means as an approximation to the number of trips by hour in both directions at the same time. By representing the 48-dimensional vectors as \( 2 \times 24 \) matrices, we see that

\[
\theta_{i} \theta_{j} \begin{bmatrix} \omega_{g_{i}g_{j}0} & \omega_{g_{i}g_{j}1} & \ldots & \omega_{g_{i}g_{j}23} \\ \omega_{g_{j}g_{i}0} & \omega_{g_{j}g_{i}1} & \ldots & \omega_{g_{j}g_{i}23} \end{bmatrix} \approx \begin{bmatrix} A_{ij0} & A_{ij1} & \ldots & A_{ij23} \\ A_{ji0} & A_{ji1} & \ldots & A_{ji23} \end{bmatrix}.
\]

This perspective also holds for our mixed-membership SBM, for which

\[
\sum_{g,h} C_{ig} C_{jh} \begin{bmatrix} \omega_{gh0} & \omega_{gh1} & \ldots & \omega_{gh23} \\ \omega_{hg0} & \omega_{hg1} & \ldots & \omega_{hg23} \end{bmatrix} \approx \begin{bmatrix} A_{ij0} & A_{ij1} & \ldots & A_{ij23} \\ A_{ji0} & A_{ji1} & \ldots & A_{ji23} \end{bmatrix}.
\]

The validity of this matrix representation depends on there being a large correlation between the time-aggregated in-degrees and out-degrees of nodes. This is related to the fact that in a given 24-hour period, the number of trips from one station to another is predictive of the number of trips in the opposite direction. This matrix representation is related to the fact that the 24-hour time activities for trips between two stations in one direction are predictive of the activities in the opposite direction. The latter observation, in turn, is related to the axiom of human mobility that for each current of travel, there is a counter current \([3]\). We observe (as did \([73]\)), from the above matrix expressions, that maximizing the log-likelihood of both SBMs is equivalent to a form of nonnegative matrix factorization with \( K^{2} \) 48-dimensional basis columns. In a sense, our model is neither an extension of the usual undirected degree-corrected SBM nor one of the usual directed degree-corrected SBMs. Instead, our model’s single-layer network analog is a degree-corrected SBM with parameters \( \theta_{i} \) and \( \omega_{gh} \), except that the \( \omega_{gh} \) are not constrained to be symmetric.
4 Computations

In this section, we describe the algorithms that we use for both the TDMM-SBM and the TDD-SBM.

4.1 Inference using the TDMM-SBM

Let $\omega = \{\omega_{ght}\}$ be the $K \times K \times T$ array that represents the inter-block connectivity parameters, and let $C = \{C_{ig}\}$ be the matrix that represents the collection of node-strength parameters. We estimate the model parameters using a two-step gradient descent. First, we move in the direction of the gradient with respect to $\omega$ and update the inter-block connectivity parameters. Second, we move along the direction of the gradient with respect to $C$ and update the node-strength parameters.

In the description of our algorithm, we let $\omega^{(n)}$ and $C^{(n)}$, respectively, denote the $n^{th}$ update of the inter-block connectivity and node-strength parameters. We initialize the algorithm with random values $\omega^{(0)}$ and $C^{(0)}$ with components distributed according to $\exp(X)$, where $X$ is a Gaussian random variable with mean 0 and variance 1. (That is, we draw random values from a log-normal distribution.) We denote the mean activity along edge $(i, j)$ with initial parameters $\omega^{(0)}$ and $C^{(0)}$ by $\mu_{ij}^{(0)}$. We scale the parameters so that the TDMM-SBM at the starting point of the optimization has the same mean number of trips as the data. Specifically, we multiply the inter-block connectivities $\omega^{(0)}_{ght}$ by $\left(\sum_t \omega^{(0)}_{ght}\right)^{-1} \left(\sum_{i,j,t} A_{ijt}\right) / K^2$ and normalize $C^{0}$ to satisfy the constraint $\sum_i C^{(0)}_{ig} = 1$ for each block $g$. This results in

$$
\omega^{(0)}_{ght} = \sum_{i,j,t} A_{ijt} \sum_{g,h} C^{(0)}_{ig} \omega^{(0)}_{ght} C^{(0)}_{jh} = \sum_{g,h,t} \omega^{(0)}_{ght} = \sum_{i,j,t} A_{ijt} / K^2 = \sum_{i,j,t} A_{ijt},
$$

the total number of edges in the network. Without this scaling, the initial parameters would have very small magnitudes, such that the mean total number of trips from the TDMM-SBM with these initial parameters is much smaller than the total number of trips in the data. Therefore, it is very likely that the randomly chosen initial parameters will have very small log-likelihoods relative to the MLE log likelihood. Early gradient-descent steps might then dramatically increase the magnitude of the parameters while affecting the relative sizes of individual parameters in unpredictable ways.

To ensure that our estimated parameters are nonnegative, we use the following change of variables: $\exp(\tilde{\omega}^{(n)}) = \omega^{(n)}$ and $\exp(\tilde{C}^{(n)}) = C^{(n)}$. We can then write the gradient descent as

$$
\tilde{\omega}^{(n+1)} = \tilde{\omega}^{(n)} + \eta^{(n)} \nabla_{\omega} \ell(C^{(n)}, \omega^{(n)}) \exp(\tilde{\omega}^{(n)}),
$$

$$
\tilde{C}^{(n+1)} = \tilde{C}^{(n)} + h^{(n)} \nabla_{C} \ell(C^{(n)}, \omega^{(n+1)}) \exp(\tilde{C}^{(n)}),
$$

where $h^{(n)}$ and $\eta^{(n)}$ are small positive numbers. From the definitions of $\tilde{C}^{(n)}$ and $\tilde{\omega}^{(n)}$, we write

$$
\omega^{(n+1)} = \omega^{(n)} \exp(\eta^{(n)} \nabla_{\omega} \ell(C^{(n)}, \omega^{(n)})) \omega^{(n)}
$$

$$
C^{(n+1)} = C^{(n)} \exp(h^{(n)} \nabla_{C} \ell(C^{(n)}, \omega^{(n+1)})) C^{(n)}.
$$

Although we are maximizing a function and thus technically performing gradient ascent, we refer to this class of method by its more common monicker of “gradient descent”. 

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We take the exponential of a vector to be the result of applying the exponential to each component of the vector. Let \( h^{(0)} = \eta^{(0)} = \Delta > 0 \) be the fixed initial step size. For our application, we choose \( \Delta = 10^{-4} \). We generate two candidate updates for \( \omega^{(n+1)} \) for the first step in our algorithm using \( h^{(n+1)} = 1.2 h^{(n)} \) and \( h^{(n+1)} = 0.8 h^{(n)} \), and we choose the one that gives a \( \omega^{(n+1)} \) that yields the larger value of \( \ell(C^{(n)}, \omega^{(n+1)}) \). Similarly, we choose the one of \( \eta^{(n+1)} = 1.2 \eta^{(n)} \) or \( \eta^{(n+1)} = 0.8 \eta^{(n)} \) that gives the \( C^{(n+1)} \) with the larger \( \ell(C^{(n+1)}, \omega^{(n+1)}) \).

We compute the gradient of the log-likelihood function \( \ell \) using the chain rule. Recall that we compute the log-likelihood in two parts. One part is the computation of the mean \( \mu_{ijt} = \sum_{g,h} C_{ij} \omega_{gh} C_{jh} \) of the number of trips from node \( i \) to node \( j \) at time \( t \). We then insert the expression for the mean into the function \( \ell = \sum_{i,j} (A_{ijt} \log(\mu_{ijt}) - \mu_{ijt}) \). We compute the derivative of \( \ell \) with respect to \( \mu_{ijt} \) to obtain

\[
\frac{\partial \ell}{\partial \mu_{ijt}} = A_{ijt} \left( \mu_{ijt} - 1 \right).
\]

The derivative of \( \mu_{ijt} \) with respect to \( C_{kg} \) is

\[
\frac{\partial \mu_{ijt}}{\partial C_{kg}} = \delta_{ki} \sum_h \omega_{ght} C_{jh} + \delta_{kj} \sum_h C_{ih} \omega_{ht}.
\]

and the derivative of \( \mu_{ijt} \) with respect to \( \omega_{ght} \) is

\[
\frac{\partial \mu_{ijt}}{\partial \omega_{ght}} = C_{ig} C_{jh}.
\]

Here \( \delta_{ab} \) is the kronecker delta (i.e. \( \delta_{ab} = 1 \) if \( a = b \) and \( \delta_{ab} = 0 \) if \( a \neq b \)). Using the above calculations, we see that the derivatives of \( \ell \) with respect to \( C_{kg} \) and \( \omega_{ght} \) are

\[
\frac{\partial \ell}{\partial C_{kg}} = \sum_{t=0}^{23} \sum_{i,j \in \mathcal{N}} \frac{\partial \ell}{\partial \mu_{ijt}} \frac{\partial \mu_{ijt}}{\partial C_{kg}}
\]

\[
= \sum_{t=0}^{23} \left( \sum_{j \in \mathcal{N}} \left( \frac{A_{ijt}}{\mu_{ijt}} - 1 \right) \sum_h \omega_{ght} C_{jh} + \sum_{i \in \mathcal{N}} \left( \frac{A_{ikt}}{\mu_{ikt}} - 1 \right) \sum_h C_{ih} \omega_{ht} \right),
\]

\[
\frac{\partial \ell}{\partial \omega_{ght}} = \sum_{i,j \in \mathcal{N}} \frac{\partial \ell}{\partial \mu_{ijt}} \frac{\partial \mu_{ijt}}{\partial \omega_{ght}}
\]

\[
= \sum_{i,j \in \mathcal{N}} \left( \frac{A_{ijt}}{\mu_{ijt}} - 1 \right) C_{ig} C_{jh}.
\]

We run the gradient descent until four significant digits of the base-10 floating-point representation of the log-likelihood (3) does not change for 600 steps in a row. For the networks that we examine, this usually takes between 600 and 5000 iterations, with models with more blocks generally needing more iterations to reach this stopping criterion. Because of the non-convexity of the log-likelihood function (3), we are not guaranteed to reach a global optimum. Most of the time, our method converges to an interesting local optimum (which may also be a global optimum), revealing the existence of functional roles (see Section 5).
Our results produce recognizable inter-block connectivity parameters $\omega_{ght}$ (e.g., home–work commute patterns and leisure-usage patterns) and the parameters $C_{ig}$ indicate known spatial divisions of the stations (e.g., residential versus commercial districts). In some cases, however, our algorithm converges to an uninteresting local optimum; one example is when the block-assignment parameters $C_{ig}$ for each station appear as if they are assigned independently at random. To improve our results, we run our algorithm repeatedly (specifically, 10 times for each network) and store the estimate with the largest likelihood. We compare the parameters that we obtain from gradient descent versus those that we obtain by running a Hamiltonian Monte Carlo (HMC) sampling method in a Bayesian framework with weak priors (implemented in Stan \[74,75\]). The log-likelihoods that result from our gradient-descent method are as good or better than those that we obtain with an HMC method. The HMC method is more computationally and memory intensive than our gradient-descent method, although it may be preferable in applications in which one has meaningful prior information about parameters. Improving our optimization method and investigating trade-offs between accuracy and efficiency are worthwhile topics for future work. For instance, it will likely be beneficial to adapt optimization methods for related time-dependent SBMs \[49,52,54,55,62\] to the optimization of our model.

The supplementary material has our Python implementation of our gradient-descent method, as well as code for our inference in R using Stan.

4.2 Inference using the TDD-SBM

To fit our TDD-SBM, we use a Kernighan–Lin-type (KL-type) algorithm \[76\] that we base on the one in \[32\]. Given a number $K$ of blocks, we initialize the algorithm by assigning each node to a block uniformly at random, so each node has a probability of $1/K$ of being assigned to a given block. The algorithm then calculates the best possible block reassignment for any single node with respect to the associated change in log-likelihood (either the largest increase or the smallest decrease). Subsequently, we make the best reassignment for a different node, again chosen uniformly at random, with respect to change in log-likelihood. The algorithm cycles through all nodes; a key feature of the algorithm is that a node that has been reassigned cannot move again until all other nodes have moved. One set of sequential reassignment of all nodes constitutes one step of the algorithm. The algorithm then searches all of the states (with respect to block membership of nodes) that have occurred during the step, and it selects the state with the maximum log-likelihood of any during the step. This state is the starting point for the next step of the algorithm. A single run of the algorithm is completed when a step does not increase the log-likelihood beyond a preset tolerance value near 0. (In practice, we use $1 \times 10^{-4}$.) To find block assignments that are as good as possible, we do many runs of the algorithm for each network. In our examples, we use 50 runs per network. We initialize each run randomly, as described above.

Another key feature of the algorithm is that changes in the block membership of nodes affect only the terms of the objective function that involve the origin and destination blocks of the change. (We see in \[4\] that the objective is a sum over block-pair terms over $T$ time slices.) Consequently, we do not need to recalculate the full objective function at each step.

We implement our KL-type algorithm for TDD-SBM in R using Rcpp \[77,78\]. The back-end calculations are in c++ for speed, and we return results in R to enable visualization and
other analyses. Our implementation can also estimate time-independent SBMs, including
directed and/or degree-corrected ones. This facilitates comparison of the results of inference
from time-dependent and time-independent SBMs. See https://github.com/jcarlen/sbm
for our R package sbmt for parameter estimation for the TDD-SBM. We include code (which
uses the sbmt package) in supplementary material to replicate our examples in Section 5.

5 Results

We apply our models to bicycle-sharing networks in Los Angeles, San Francisco, and New
York City. (See Section 2 for descriptions of these data sets.) The networks that we examine
in downtown Los Angeles and San Francisco are relatively small, with 61 stations and 35
stations, respectively, at the time that we collected our data. The stations in these networks
are concentrated in downtown areas, where high-rise office and residential buildings are
interspersed. The New York City network is much larger than the other two. It includes
about 600 stations at the time of data collection. The stations span most of the lower half
of Manhattan and northwestern Brooklyn. They encompass a range of commercial areas,
residential neighborhoods, parks, and manufacturing areas.

5.1 Downtown Los Angeles

In Figure 3, we show the mixed-membership (TDMM-SBM) and discrete (TDD-SBM) block
assignments of two-block models of the downtown Los Angeles system. For the TDMM-SBM,
the we scale the size of a given node \( i \) in our plots based on \( \sum g C_{ig} \). We refer to these sums as
“\( C \) total” values. These values correlate strongly with node degree (specifically, the sum of
in-degree and out-degree), which is evident in the similarity of node sizes in the left and right
panels of Figure 3. For both models, we observe that home and work blocks are interspersed
geographically. (We will soon describe our method for determining the block labels in Figure
3.) The TDMM-SBM result reveals a group of stations (which we color in gray) in the left
panel of Figure 3 are neither strongly home-identified nor strongly work-identified; instead,
they possess a roughly even mixture of the two types. For this network, the TDD-SBM
output is very similar to what we obtain from a discretization of the TDMM-SBM output
(which we discretize by assigning each node \( i \) to the block with the maximum value of its
\( C_{ig} \) parameter), but this is not true for all of our bicycle-sharing networks.
Our model does not yield “home” and “work” labels for each block on its own, so we use the time-dependent block-to-block parameter estimates $\hat{\omega}_{ght}$ to assign these labels. We assign the labels heuristically under the assumption that the “home” block is the origin of many trips to the work block in the morning and the “work” block is the origin of many trips to the home block in the evening. Figure 4, which shows $\hat{\omega}_{ght}$ for each possible value of $g$ and $h$, with the hour $t$ on the horizontal axis, supports our labeling. Based on our labeling, we observe a clear peak in home-to-work traffic in the mornings and work-to-home traffic in the evenings. We make similar “home” and “work” assignments for San Francisco and New York City. In Los Angeles, the traffic in the work block peaks in the middle of the day. This perhaps represents luncheon errands, leisure activity, or tourist activity, as there are many tourist attractions in the downtown area. The traffic in the home block has a mild evening peak and has by far the least activity overall.
To further validate our block labels, we use the zoning map for downtown Los Angeles from the Los Angeles Department of City Planning [79]. Zoning ordinances determine the allowable uses of city land. They distinguish land that is available for commercial uses, industrial uses, residential uses, park districts, and others. In the background of Figure 5, we show a simplified version of the underlying zoning map (with a grouping of similar designations). The industrial areas house a mixture of manufacturing and commercial uses. Public facilities include government buildings, public schools, parking under freeways, and police and fire stations [80]. In downtown Los Angeles, manufacturing and industrial areas are split cleanly from residential areas, whereas commercial and residential areas are intermixed across the bicycle-sharing system’s coverage area.

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Figure 5: Mixed-membership (TDMM-SBM) assignments of Los Angeles bicycle-share stations overlaid on a simplified LA zoning map. Industrial blocks include manufacturing and commercial areas. As in Figure 3, we scale the area of nodes to the value of $\sum_{i} C_{ig}$.

Figure 5 illustrates that most stations that are strongly home-identified are in or near zones for pure residential use or mixed residential and commercial use. We find that many stations that are not predominantly home-identified or work-identified align with mixed-use commercial/residential zones. The discrete-role plot (see the right panel of Figure 3) has a stripe of “home” stations that cut diagonally through the “work” stations. In Figure 5, we see that this aligns roughly with areas that are zoned for purely residential use. By contrast, industrial and public facility zones tend to host stations that are mostly work-identified (although some of the most strongly work-identified stations are in mixed-used areas).

One station that seems to deviate from the overall pattern is the heavily-trafficked station at Union Station. Although it is adjacent to a public facility zone with many government buildings, it is also strongly home-identified. Although this may seem surprising on its surface, this is consistent with other home-identified stations, because Union Station is a major transit hub for the Los Angeles metropolitan area. Accordingly, many morning trips originate there, as commuters transition from other forms of transportation, and many evening trips conclude there — an activity pattern that is sensibly associated with home-identified stations. Such idiosyncrasies of transit hubs also arise in our results for San Francisco and New York City.
5.2 San Francisco

In Figure 6, we compare the two-block TDMM-SBM and two-block TDD-SBM for San Francisco. As we saw for Los Angeles, the San Francisco blocks are interspersed geographically, and stations vary from strongly home-identified ones to strongly work-identified ones. The most strongly home-identified station is a major transit hub, the San Francisco Caltrain Station on 4th Street.

![San Francisco, Mixed Membership](image1)

![San Francisco, Discrete Membership](image2)

Figure 6: San Francisco bicycle stations classified using (left) a two-block TDMM-SBM and (right) a two-block TDD-SBM. The sizes of the nodes take continuous values. In the left panel, we scale their area based on the value of $\sum_{y} C_{iy}$; in the right panel, we scale them based on the sum of the in-degree and out-degree (divided by the maximum value of that sum).

In Figure 7, we show the estimated traffic between the “home” and “work” blocks for the TDMM-SBM and TDD-SBM. As in downtown Los Angeles, we observe inter-block commuting. However, unlike in downtown LA, using the discrete model (i.e., TDD-SBM), we observe intra-block morning and evening peaks in both the home and work blocks. This may be due to last-mile commuting, such as using bicycle-sharing facilities to get to or from a train station. Recognizing last-mile usage is important for integrating bicycle sharing with nearby public transportation. One possible reason that we do not observe a similar phenomenon in downtown LA is that San Franciscans are more likely than Angelenos (i.e., inhabitants of Los Angeles) to use public transportation [81]. The intra-block morning and evening peaks may also arise from the intermixing of commercial and residential uses of land, such that some travel within blocks may also constitute commuting.
Before presenting our results for New York City, we briefly compare our results from Los Angeles and San Francisco to results for a time-independent SBM fit to these networks, where we have aggregated the data over all time periods. To do this, we calculate the adjacency matrix $\tilde{A}_{ij} = \sum_{t=0}^{T/3} A_{ijt}$ a time-aggregated network. (The time-independent SBM also has two blocks and is both directed and degree-corrected.) For downtown Los Angeles, we observe a clear geographically-based division in the results of the time-independent SBM. For San Francisco, however, the differences between the blocks of the time-dependent SBM and time-independent SBM are less noticeable, although they are still present. This confirms that our time-dependent SBMs are detecting behavior that is not evident in the time-aggregated data.
5.3 New York City

In Figure 9, we compare our results from a three-block TDMM-SBM and a three-block TDD-SBM for New York City. In initial calculations, we found that a two-block TDD-SBM divides the network along the East River into a Manhattan block and Brooklyn block and that the two-block TDMM-SBM divides the network slightly farther north in Lower Manhattan. This suggests a possible limitation to the size of networks for which our time-dependent SBMs can recover functional blocks, as opposed to geographically-based blocks. We will explore this hypothesis further by examining the output of our time-dependent SBMs for the entire New York City bicycle-sharing network and subsequently modeling a subset of the New York City network.
Figure 9: New York City bicycle stations classified using (left) a three-block TDMM-SBM and (right) a three-block TDD-SBM. The sizes of the nodes take continuous values. In the left panel, we scale their area based on the value of $\rho_{ij}$; in the right panel, we scale them based on the sum of the in-degree and out-degree (divided by the maximum value of that sum).

In Figure 10, we compare estimated inter-block traffic, as captured by the values of $\hat{\omega}_{ij}$, for the three-block TDMM-SBM and three-block TDD-SBM. We observe prominently that all intra-block traffic has two peaks and much higher hourly trip counts than inter-block traffic. The double peaks are reminiscent of the overall system activity in Figure 1. This may be due in part to last-mile commuting, as we also suspected in San Francisco. However, for a system that is this large, the double peaks and minimal inter-block traffic suggests that it is useful (and important) to consider each block as its own ecosystem. We also find strong similarity between results from our TDD-SBM and a three-block time-independent SBM for time-aggregated data for New York City (not shown), providing further evidence that our time-dependent SBMs are not capturing time-dependent roles for New York City. Consequently, we choose the labels of these blocks based on the primary borough and zone type of each block’s stations, as indicated in the underlying zoning map for this part of New York City in Figure 11.
Figure 10: Estimated time-dependent block-to-block parameters $\hat{\omega}_{ght}$ for the three-block TDMM-SBM and three-block TDD-SBM for New York City. We use “M” to signify Manhattan and “BK” to signify Brooklyn.

Figure 11: TDD-SBM station roles versus the coverage-area zoning map of New York City.
In Figure 11, we illustrate that there is general overlap, although it is far from perfect, between (1) the Upper Manhattan (“home”) block and residential areas or parks and between (2) the Lower Manhattan (“work”) block and commercial or manufacturing areas. All stations in Brooklyn are in the third block, which contains mostly residential areas. These observations motivate our block labels in this figure, Figure 9, and Figure 10. (Although Figure 11 shows only TDD-SBM-estimated blocks, the same reasoning motivates our labels for the three-block TDMM-SBM.) No block has exclusively commercial or residential areas, reinforcing our conclusion that these blocks represent primarily geographic divisions (with most of the traffic occurring within blocks), as opposed to functionally similar groups of stations.

We examined several time-dependent SBMs for New York City with more than three blocks to try to discover functional blocks, but we found that the blocks were still geographically based. In some cases, TDMM-SBM with larger numbers of blocks were able to find functional divisions within smaller geographic areas (subdividing the blocks in Figure 9), but neither our discrete model nor our mixed model detected system-wide “home” or “work” blocks. See our supplementary material for our code to fit and visualize time-dependent SBMs of the New York City network with a number of blocks other than three.

5.3.1 Manhattan

Figure 12: Comparison of estimated blocks from (left) a five-block TDMM-SBM and (right) a five-block TDD-SBM of the Manhattan (home) block (i.e., the Manhattan subnetwork) of the New York City network (see Figure 11). In the role labels of the TDD-SBM, we use “W” to represent west and “E” to represent east.

To examine the New York City bicycle-sharing network on a smaller scale, we fit models to the subset of stations and trips within the Manhattan (home) block of the three-block TDD-SBM that we identified above (see Figure 11); we refer to this subnetwork as the “Manhattan
subnetwork”. This subnetwork includes 256,840 trips and 166 stations. In Figure 12, we present our results for a five-block TDD-SBM and TDD-SBM applied to the Manhattan subnetwork. The area without stations in the middle of each panel of Figure 12 is Central Park, which has stations on its perimeter but not in its interior.

The estimated blocks of the five-block TDMM-SBM and TDD-SBM (see Figure 12) of the Manhattan subnetwork outline similar subregions. The mixed-membership block assignments also illustrate how the subregions transition into each other. The models return block-membership parameters that capture the residential and commercial sections of the area much better than the three-block TDD-SBM and TDMM-SBM of the full New York City network; one can see this by comparing the five-block subnetwork results with the underlying zoning map for the area in Figure 11. The stations in residential zones generally have larger block-membership parameters for “home” blocks than for “work” blocks, and the opposite is true for stations in commercial zones. We label the five detected blocks as (clockwise from top left) “home (west)”, “park”, “home (east)”, “work”, and “mixed”. We base these labels on the land usage of the underlying areas and the time-dependent block-to-block activity parameters $\hat{\omega}_{ght}$ that we show in Figure 13.

We highlight the appearance of the “park” block, which we have not observed in previous models and has distinctive behavior. The park block is similar to a residential block in terms of its spike in morning traffic to the work block and its spike in evening traffic from the work block, but it has distinct intra-block activity that peaks in the afternoon. The intra-block activity resembles weekend activity in the New York City bicycle-sharing system as a whole (see Figure 1); this reflects leisure use of the bikes. Bicycles near Central Park (which also places them near several major museums) are likely to be used by tourists and other non-commuters during the day for leisure or travel to nearby attractions.

In Figure 13, we show the values of the block-to-block parameters $\hat{\omega}_{ght}$ for the five-block TDD-SBM and TDMM-SBM. Our estimates of $\hat{\omega}_{ght}$ for these models illustrate important differences in the behavior of different blocks that we can observe only with a time-dependent model. We see some overlap in the time-dependent behavior of blocks, evidencing potential overfitting. For example, the home (east), home (west), and mixed block have similar traffic with blocks other than their own. However, models of this subnetwork with fewer than five blocks do not cleanly distinguish the “park” block of stations from other residential stations.

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4 We obtain similar block identifications for this subnetwork using a discrete, directed, degree-corrected, time-independent SBM as we do from a TDD-SBM with the same number of blocks. We do not show the time-independent SBM results, but they can be produced using the code in our supplementary material.
Figure 13: Estimated time-dependent block-to-block parameters $\hat{\omega}_{ght}$ for (left) a TDMM-SBM with five blocks and (right) a TDD-SBM with five blocks of the Manhattan subnetwork of the New York City bicycle-sharing network.

One reason that our time-dependent SBMs of the Manhattan subnetwork of New York City bicycle-sharing network perform better (with respect to detecting functionally meaningful blocks) than any models that we applied to the entire system is the dependence of station-to-station trip counts on the distance between stations. Although our SBMs correct for the overall activity of each station, they do not normalize expected edge values by the distance between stations. In a small geographic area, such as the coverage areas of the Los Angeles and San Francisco networks, this is a reasonable choice, as all stations are within “biking distance” of each other. However, when examining a system as large as New York City’s, the lack of distance correction weakens the functional groupings that we obtain with our time-dependent SBMs. Intra-block trips dwarf inter-block trips (see Figure 10), and it seems more reasonable to construe each block as its own ecosystem.

5.4 Model Selection

Although statistically rigorous model selection is outside the scope of our paper, we briefly compare the number of parameters in our mixed-membership and discrete SBMs. This is valuable for considering model-selection criteria, such as the Akaike information criterion (AIC) and Bayesian information criterion (BIC), that penalize a model based on its number of parameters. For a network with $N$ nodes, $K$ blocks, and $T$ time slices, the number of parameters for the TDMM-SBM is

$$K \times N - K + T \times K^2,$$

and the number of parameters for the TDD-SBM is

$$2 \times N - K + T \times K^2.$$

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The first term of (5) comes from the fact each node in our mixed-membership model has $K$ parameters $(C_{ig},$ with $g \in \{1, \ldots, K\})$ that express the strength of membership in each block. By contrast, each node in our discrete-membership model has one parameter for block membership and one degree-correction parameter. Therefore, given a value of $N,$ the first term in (5) increases linearly with the number of blocks, whereas the corresponding term in (6) is fixed. Otherwise, formulas (5) and (6) are equivalent. The $-K$ term in each formula arises from identifiability constraints for each model. As we described in Section 3), these constraints are $\sum_i C_{ig} = 1$ for all $g$ for the mixed-membership model and $\sum_{i \in g} \theta_i = 1$ for all $g$ for the discrete model. The last term in each formula is the total number of $\omega_{ght}$ terms in the model (see Section 3.1).

In Table 1, we show the unnormalized log-likelihood and number of parameters ($N_p$) for TDMM-SBM and TDD-SBM of the Manhattan subnetwork (which has $N = 166$ nodes) with two, three, four, and five blocks.

| Number of blocks | TDMM-SBM $N_p$ | log-likelihood | TDD-SBM $N_p$ | log-likelihood |
|------------------|----------------|----------------|----------------|----------------|
| 2                | 426            | −260625        | 426            | −270809        |
| 3                | 711            | −235162        | 545            | −254779        |
| 4                | 1044           | −212295        | 712            | −236198        |
| 5                | 1425           | −198489        | 927            | −222539        |
| 6                | 1854           | −189670        | 1190           | −216468        |

Table 1: Comparison of log-likelihood and number of parameters in models of the Manhattan subnetwork, which has $N = 166$ nodes.

In this example, TDMM-SBM outperforms TDD-SBM with respect to log-likelihood when the two models have the same number of parameters. This result makes sense because of the additional constraint of the TDD-SBM that stations must belong to exactly one block.
Calculating AIC, which is given by 

\[ AIC = (2 \times N_p) - (2 \times \text{log-likelihood}) \]

for TDMM-SBM with 2–10 blocks for the Los Angeles bicycle-sharing network selects the TDMM-SBM with the largest number of blocks. The AIC is a cost function for comparing the relative quality of statistical models; one construes the model with a smaller AIC as the “better” model. The AIC takes into account both the likelihood and how description complexity of a model is in its two summands. The negative log-likelihood is smaller for models with higher likelihood, and one measures the complexity of a model based on its number of parameters. From this perspective, a model with a smaller AIC is better at capturing the trends of a data set while avoiding overfitting. In Figure 14, we see that AIC
decreases as we increase the number of blocks for 2–10 blocks. However, the graphs of the MLE values of $\omega_{ght}$ for TDMM-SBM with 7 or more blocks on the Los Angeles network are no more informative than models with fewer blocks. We make this observation for TDMM-SBM with large numbers of blocks using the LA network rather than the Manhattan network, because computing models on data with fewer stations takes less time; and we are confident that the noisy and/or redundant information from using 7 or more blocks on the LA network arise from overfitting.

Although our calculations above are straightforward, choosing appropriate model-selection criteria deserves serious consideration [84,85]. We leave such an investigation for future work.

6 Conclusions and Discussion

We developed time-dependent degree-corrected stochastic block models and used them to analyze daily commute patterns in bicycle-sharing systems in Los Angeles, San Francisco, and New York City. Our SBMs group stations based on their activity over time, allowing us to identify them with home and work roles. Work stations are characterized by inflow from home stations in the morning and outflow to home stations in the afternoon and evening, and residential stations have the opposite characterization. It is also sometimes possible to identify other roles, such as a Manhattan park block that combines residential and leisure/tourist behavior.

We found that many stations in our three focal cities serve a mixture of roles, which we captured with our mixed-membership SBM. However, in some cases, we observed that discrete-membership SBMs that use fewer parameters can provide a clearer picture of the usage patterns that are associated with each. We illustrated through case studies how our discrete and mixed-membership SBMs can provide complementary insights about the bicycle-sharing system behavior. We also demonstrated that applying a time-independent degree-corrected SBM on time-aggregate networks tends to divide stations into contiguous geographic groupings, rather than functional ones.

We evaluated our block labels by comparing them to city zoning maps. The home–work structure that we detected generally aligns well with the underlying zones. However, we found important deviations near major transit hubs. For example, we identified bicycle stations near Union Station in Los Angeles and the San Francisco Caltrain station as home stations because they have high outflow during the mornings and high inflow during the evenings, even though they are not located in residential areas.

It is common to evaluate the results of community detection by comparison with so-called “ground-truth” communities [24,12]. (However, it is crucial to encourage caution with respect to such evaluations [86].) The time-dependent commute flows that we detected with our SBMs enabled us to identify and label the functional roles of our blocks. This, in turn, is useful for revealing functional districts without a corresponding zoning map. In the future, it will be worthwhile to compare our detected traffic flows to activity patterns from other mobility data, such as taxis, subway systems, e-scooters, and geo-tagged mobile-app usage [10].

Developing a deeper understanding of the relationships between station roles and usage patterns throughout a day (and in weekdays versus weekends) can improve the design of
bicycle-sharing systems. For example, it can help determine where to place stations and the appropriate sizes of different stations. It can also provide actionable information for efficient redistribution of bicycles, a question that has received much research attention \[87,88,89,90\].

Although our SBMs revealed work and home blocks across different cities, such roles do not describe identical activity patterns. (Accordingly, our labels of “home” and “work” should not be taken as strict classifications of the trips made between blocks, but rather as indicative of dominant activity patterns in a network.) For example, in downtown Los Angeles, the work-to-work activity peaks in the middle of a day, perhaps suggesting that the bicycle-sharing system is being used for errands or leisure activity. In San Francisco and New York City, home-to-home activity has morning and evening peaks, which may be due to the use of the system for “last-mile” transportation or short commutes within mixed-use areas. Such possibilities are reminiscent of recent work on human mobility motifs \[91,92\]. Using our statistical models, it may be possible to examine footprints of motifs of individual transportation patterns, although it will be necessary for the examined data to include day-by-day movement patterns of individuals to establish a direct connection. Accordingly, we expect it to be fruitful to apply our statistical approaches to the analysis of mobile GPS data.

We now discuss worthwhile future efforts for improving our models and algorithms. In this paper, we presented two types of time-dependent SBMs and used them to reveal interesting urban structures in bicycle-sharing networks. We formulated these SBMs to account for degree heterogeneity and for a balance between cumulative in-degrees and out-degrees of bicycle stations over the course of a day. (The latter feature reflects the classic axiom of Ravenstein \[93\] that every current of human mobility has an associated countercurrent \[3\].) However, there is scope for improving our models, just as there is scope for improving SBMs more generally. One area to improve is the fitting of Poisson random variables to the numbers of trips between pairs of stations. Additionally, although it is a convenient simplification to assume independence of edges conditioned on the latent block structure, it would be nice to relax these assumptions. It would also be useful to examine the assumption that the number of trips during each hour are independent random variables. For example, they are not independent if there are stations that run out of bicycles at some point. As we observed previously, mixed-membership and discrete-membership SBMs can reveal different insights, as can examining SBMs with different numbers of blocks. This helps illustrate why it is important to consider model selection in greater depth.

It would also be useful to explore practical ways that a bicycle-sharing system can exploit our models. Peaks in the computed mean out-degree \( \sum_{ij} \mu_{ijt} \) not only indicate expected usage, but they are also suggestive of the main purpose of a docking station (e.g., home, work, or a mixture of the two). A patron who wants to use a bicycle would be very unsatisfied if they walked to a station to pick up a bicycle to ride to work, but they found that no bicycles were available. This suggests that one potentially viable strategy for maintaining the stock bicycles at a station is to ensure that it is above the expected use that is estimated by our SBMs, as our results from both the TDD-SBM and TDMM-SBM identify the most important ways that people are using these stations.

Another important direction for future work is the exploration of different methods for preprocessing data to include only the most significant edges. The two most apparent ways to do this are (1) eliminating insignificant edges by thresholding and (2) choosing time slices
that reduce variance. The preferential attachment model of \cite{72} gives one possible approach for eliminating insignificant edges. The way that one splits the times of a day can improve both accuracy and computational efficiency by reducing the total number of parameters. For example, in the cities that we studied, bicycle trips occurred sporadically between 1 am and 5 am, so it may be desirable lump all of these time slices into one time interval to decrease the number of parameters by 3 and thereby decrease the variance. There exist methods to find suitable ways to segment time periods \cite{94}, and trying to find the best ones to use in different situations is an active area of research.

Broadening our models to incorporate spatial data is another natural direction to build on our research \cite{1}. The radiation, intervening opportunities, and gravity models have had some success at modeling human mobility over various distances \cite{3}. These models put more weight on longer trips, and some of them take into account opportunities (the so-called intervening ones) that lie between an origin and destination location. Some of these mobility models also possess statistical justification based on entropy arguments, and it is worthwhile to investigate methods to incorporate them into SBMs. Some of these mobility models have already been incorporated into null models in time-dependent modularity objective functions in the work of Sarzynska et al. \cite{95}, who found that radiation and gravity null models perform better than the usual Newman–Girvan null model (which is a variant of a configuration model \cite{96,97}) for spatial networks. Given that there is an equivalence between a SBM and modularity maximization in a planted-partition model \cite{98} (and a generalization of this idea arises in multilayer networks \cite{99}), the stage seems to be set for efforts to incorporate spatial information into SBMs. The value of using spatial null models for bicycle-sharing systems was examined in \cite{15,67}, so this is a very interesting direction to pursue.

It would be interesting to build on our work using urban spatial null models that go beyond distance and incorporate route difficulty due to traffic or terrain. Some bicycle-sharing researchers are harnessing route information from GPS systems to better understand relationships between station usage and availability of bicycling infrastructure (such as bike lanes) \cite{19}. One can also develop models that incorporate station roles. For example, one expects a higher tolerance to distance for traveling between home and work than for a quick bite to eat. Spatial null models can also uncover other types of communities, such as ones that stem from variables (like spoken language and socioeconomic status) that are not directly explainable by spatial data \cite{95,100}.

Our time-dependent SBMs are useful for studying many types of mobility data. For example, it would be interesting to study dockless vehicle-sharing networks, such as e-scooter-sharing programs, using our time-dependent SBMs. If we view the usage of stations as a proxy for a spatial function of demand for bicycles, then data from dockless systems may better approximate such a spatially varying function. One possibility is to partition a city into a grid (including comparing computations that use different levels of granularity) and measure the usage in each region, taking care to recognize irregularities from transit hubs. Depending on how heavily these systems are used in commuting, we may discover primary functional blocks other than “home” and “work”. Looking even further forward, it will also be possible to tailor our methods to analyze multimodal transportation system usage and other urban flows, which are particularly suitable for the setting of multilayer networks \cite{7,21,25,26}. 

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Appendix

6.1 Singular vectors for Los Angeles and San Francisco

We show our singular vectors for the bicycle-sharing networks for downtown Los Angeles in Figure 15 and for San Francisco in Figure 16.

Figure 15: The first two singular vectors of data for the downtown Los Angeles bicycle-sharing network.

Figure 16: The first two singular vectors of the data for the San Francisco bicycle-sharing network.
6.2 Proof that expected node degrees are the same as node degrees in the data generated from our TDD-SBM

Suppose that we are given a network that is generated by our time-dependent discrete-membership stochastic block model (TDD-SBM). We prove that the expected value of the total degree (i.e., the sum of the in-degree and the out-degree) of a node is the same as the total degree of the node in the observed data.

Let $X_{ijt}$ for each node pair $i,j$ and time $t \in \{0,1,\ldots,23\}$ be random edge weights distributed according to the TDD-SBM and inferred from the data $A_{ijt}$. Recall that $g$ denotes block $g$ in the SBM and that $\kappa_g$ is the sum of the in-degrees and out-degrees of all nodes in block $g$ over all time periods. For node $i$, we show that the mean degree of $i$ is equal to the degree of $i$ in the data. That is, $\mathbb{E} \left( \sum_j \sum_{t=0}^{23} X_{ijt} \right) = k_i = \sum_j \sum_{t=0}^{23} A_{ijt}$. We have

$$\mathbb{E} \left( \sum_{t=0}^{23} \sum_j (X_{ijt} + X_{jit}) \right) = \frac{k_i}{\kappa_g} \sum_t \sum_{j \in h} \frac{k_j}{\kappa_h} \left( m_{g,ht} + m_{hg,t} \right)$$

$$= \frac{k_i}{\kappa_g} \sum_h \left( m_{g,ht} + m_{hg,t} \right)$$

$$= k_i.$$

We are not aware of a relationship between the expected degrees and degrees of the observed data for our mixed-membership stochastic block model.

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