Superintegrable systems with spin and second-order tensor and pseudo-tensor integrals of motion

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Abstract

We investigate a quantum non-relativistic system describing the interaction of two particles with spin \(\frac{1}{2}\) and spin 0, respectively. Assuming that the Hamiltonian is rotationally invariant and parity conserving we identify all such systems which allow additional tensor and pseudo-tensor integrals of motion that are second-order matrix polynomials in the momenta. Previously we found all the scalar, pseudo-scalar, vector and axial vector integrals of motion. No non-obvious tensor integrals exist. However, nontrivial pseudo-tensor integrals do exist. Together with our earlier results we give a complete list of such superintegrable Hamiltonian systems allowing second-order integrals of motion.

Keywords: superintegrable systems, spin, tensor integrals of motion, pseudo-tensor integrals of motion

1. Introduction

The purpose of this article is to report on a research program which investigates the superintegrability properties of systems involving particles with spin. This time the contribution is given by analyzing the second-order tensor and pseudo-tensor integrals of motion. In this program we consider two non-relativistic particles with spin \(s = \frac{1}{2}\) and \(s = 0\), respectively, which can be interpreted e.g. as a nucleon–pion interaction or an electron–\(\alpha\) particle one. An earlier
article [1] was devoted to the investigation of the second-order scalar, pseudo-scalar, vector
and axial vector integrals of motion for those Hamiltonian systems in a real three-dimensional
Euclidean space. Hence, this work completes the full classification problem for those systems
with second-order integrals of motion. The other articles in this program were devoted to the
investigation of the same problem with first-order integrals of motion in \( E_2 \) and \( E_3 \) [2–4] and
second-order integrals of motion in \( E_2 \) [5].

Let us first remind the definitions of integrability and superintegrability that we have used. In
classical mechanics, integrability (in the Liouville sense, of course) is defined by the existence
of \( n \) functionally independent integrals of motion for a Hamiltonian system with \( n \) degrees of
freedom. These integrals, including the Hamiltonian itself, must be well-defined functions on
phase space and be in involution. On the other hand, superintegrability is defined by requiring
the existence of at least one and at most \( n-1 \) (in order to have dynamics in the system)
additional integrals of motion. The total set of integrals of motion must be functionally inde-
pendent, however, the additional ones are not necessarily in involution among themselves, nor
with the already existing \( n \) integrals of motion (except the Hamiltonian itself). In quantum
mechanics, similar definitions are used with well-defined linear integrals of motion operators
which are now supposed to be algebraically independent.

The best known superintegrable systems are: Kepler, or Coulomb system (where the poten-
tial is \( \frac{1}{r} \)) and the Harmonic oscillator (where the potential is \( \omega r^2 \)). These are characterized
by the fact that all finite classical trajectories in these systems are periodic. Indeed, due to
Bertrand’s theorem [6] these are the only two spherically symmetric potentials in which all
bounded trajectories are closed. Releasing the restriction of spherical symmetry for the poten-
tial can lead to many new possibilities e.g. the anisotropic harmonic oscillator with rational
ratio of frequencies [7].

A systematic search for the properties of superintegrable systems was started quite
some time ago [8–10]. Originally the approach concentrated on the natural Hamiltonians
of type

\[
H = -\frac{1}{2} \Delta + V(\vec{r}),
\]

(1.1)

with integrals of motion that are second-order polynomials in the momenta and directly
related with the multiseparability in two- and three-dimensional Euclidean spaces [8, 11–13].
This relationship between integrability and separability breaks down in other cases. For
example, for natural Hamiltonians (1.1), the existence of third-order integrals does not lead
to the separation of variables [14–16]. Furthermore, if we consider velocity dependent
potentials

\[
H = -\frac{1}{2} \Delta + V(\vec{r}) + (\vec{A}, \vec{p}),
\]

(1.2)

then quadratic integrability no longer implies the separation of variables [17, 18]. Here and
throughout the text \((\cdot, \cdot)\) denotes the inner product in real three-dimensional Euclidean space
\(E_3\).

Second-order superintegrability has also been studied in two- and three-dimensional spaces
of constant and nonconstant curvature [19–22], [23–30] and also in \( n \) dimensions [31–33].

After the discovery of infinite families of classical and quantum systems with integrals of
arbitrary order [34], the direction of the research has been shifted to higher-order integrabil-
ity/superintegrability [35–50]. Recently extended reviews have been published describing the
current status of the subject [48, 49].
More recently, exotic and standard potentials appearing in classical and quantum superintegrable systems have been studied both in Cartesian and polar coordinates [51–60].

Other studies of superintegrable systems with spin include [61–70].

The outline of the paper is as follows. In the next section, we introduce the formulation of the problem. Then, in section 3, we give the most general second-order tensor integrals of motion, obtain the determining equations and solve them in order to find all the second-order tensor integrals of motion. In section 4, we repeat the similar analysis for pseudo-tensor integrals of motion. Finally, in the last section we give a complete list of second-order integrals of motion and conclude about our results.

2. Formulation of the problem

We consider the following Hamiltonian

\[ H = -\frac{\hbar^2}{2} \Delta + V_0(r) + V_1(r) (\vec{\sigma}, \vec{L}), \] (2.1)

in real three-dimensional Euclidean space \( E_3 \) with the standard notations, where

\[ p_k = -i\hbar \partial_{x_k}, \quad L_k = -i\hbar \epsilon_{klm} x_l \partial_{x_m}, \] (2.2)

are the linear and angular momentum respectively and

\[ \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \] (2.3)

are the Pauli matrices. Here \( H \) is a matrix operator acting on a two-component spinor and we decompose it in terms of the \( 2 \times 2 \) identity matrix \( I \) and Pauli matrices (we drop the matrix \( I \) whenever this does not cause confusion). We assume that the scalar potential \( V_0(r) \) and the spin–orbital one \( V_1(r) \) depend on the (scalar) distance \( r \) only. Hence, the considered Hermitian Hamiltonian is rotationally invariant and also satisfies the requirements of parity (i.e. sending \( \vec{x} \rightarrow -\vec{x} \)) invariance. Our aim is to find all the superintegrable potentials \( V_0 \) and \( V_1 \) that admit additional integrals of motion which are second-order matrix polynomials in the momenta. In order to achieve this goal we require the vanishing of the commutators of those integrals of motion with the Hamiltonian (2.1). This requirement leads to the so called determining equations whose complete solution would provide the desired results. However, taking the advantage of a rotationally invariant Hamiltonian we decompose the space of integrals of motion into subspaces transforming under irreducible representations of \( O(3) \). Furthermore, invariance of the Hamiltonian (2.1) under reflections in \( E_3 \), provides us to choose those subspaces with definite behavior under the parity operator.

Since we have two vectors \( \vec{x}, \vec{p} \) and one axial vector \( \vec{\sigma} \), we construct scalars, pseudo-scalars, vectors, axial vectors and symmetric two component tensors and pseudo-tensors in the direct product space of these vectors \( (\vec{x}, \vec{p}, \vec{\sigma}) \). Let us remind here that pseudo-scalars, axial vectors and pseudo-tensors transform with an additional sign change under parity transformation and otherwise transform as scalars, vectors and tensors, respectively. Searching for second-order superintegrability the integrals that we should consider can involve at most second-order powers of \( \vec{p} \) and first-order powers of \( \vec{\sigma} \), but arbitrary powers of \( \vec{x} \).

The same system (2.1) was already considered in [1, 4]. In [1] the search for superintegrable systems was restricted to second-order scalar, pseudo-scalar, vector and axial vector integrals of motion and in [4] the search was restricted to first-order integrals. Here we will investigate
the second-order tensor and pseudo-tensor integrals of motion which are second-order matrix polynomials in the momenta.

As in every classification problem we should rule out the trivial cases from the beginning. Thus, first we impose that the spin–orbital interaction be present \((V_1 \neq 0)\). Secondly, having spherically symmetric potentials \(V_0(r)\) and \(V_1(r)\) and bearing in mind that \((\vec{\sigma}, \vec{L})\) is a scalar, we see that the Hamiltonian \(H\), given in (2.1) commutes with the total angular momentum

\[ \vec{J} = \vec{L} + \frac{\hbar}{2} \vec{\sigma}. \]  

Thus, the system (2.1) is integrable (in the sense of the definition given in the introduction) by construction. Being an integrable system it admits complete set of quantum numbers that characterize the system. However, we shall continue to search for superintegrability because it entails exact solvability; meaning that (i) calculation of the energy levels algebraically and (ii) expressing the wave functions in terms of polynomials.

Thirdly, (2.1) admits some other integrals which are present for arbitrary functions \(V_0(r)\) and \(V_1(r)\), e.g.

\[ H, \quad J, \quad (\vec{\sigma}, \vec{L}), \quad \vec{L}^2. \]  

We call them trivial integrals. Such kind of integrals will not lead to any new superintegrable system. On the other hand the products of lower order integrals will also provide us with further integrals which are nontrivial, but obvious. For example, in [4] we gave integrals of order 0 or 1 in the momenta for some potentials \(V_0(r)\) and \(V_1(r)\). The product of those integrals with themselves or with the above trivial integrals gives us obvious integrals. Since they may be useful for solving the corresponding superintegrable systems we will mention them whenever they occur.

Finally, we should consider the allowed gauge transformations that leave the Hamiltonian (2.1) form invariant. In [4], it was explicitly shown that starting from a scalar Hamiltonian (in which \(V_1 \equiv 0\), a spin–orbit term could be gauge induced. More specifically, we had the following transformation

\[ \tilde{H} = U^{-1} \left( -\frac{\hbar^2}{2} \Delta + V_0(\vec{x}) \right) U = -\frac{\hbar^2}{2} \Delta + V_0(\vec{x}) + \frac{\hbar^2}{r^2} + \frac{\hbar}{r^2} \vec{\sigma} \cdot \vec{L}, \]  

where the transformation matrix was given as

\[ U = e^{i\beta_4} \begin{pmatrix} e^{i\beta_1} \cos(\beta_3) & e^{i\beta_2} \sin(\beta_3) \\ -e^{-i\beta_2} \sin(\beta_3) & e^{-i\beta_1} \cos(\beta_3) \end{pmatrix}, \]

with

\[ \beta_1 = \varphi \pm \pi + c, \quad \beta_2 = c, \quad \beta_3 = -\theta \pm \frac{\pi}{2}, \quad \beta_4 = 0, \quad 0 \leq \theta \leq \pi, \quad 0 \leq \varphi < 2\pi, \]

where \(c\) is a constant.

Thus for the potentials

\[ V_1(r) = \frac{\hbar}{r^2}, \quad V_0(r) \text{ arbitrary}, \]

(2.7)
the Hamiltonian (2.1) allows 2 first-order axial vector integrals of motion. They are \( \vec{J} \) and

\[
\vec{S} = -\frac{\hbar}{2} \vec{\sigma} + \frac{\hbar}{r^2} (\vec{x} \times \vec{\sigma}).
\]  

(2.8)

For potentials

\[
V_1(r) = \frac{\hbar}{r^2}, \quad V_0(r) = \frac{\hbar^2}{2r^2},
\]

(2.9)

it allows 2 first-order axial vector integrals and 1 first-order vector integral. They are \( \vec{J}, \vec{S} \) and

\[
\vec{\Pi} = \vec{p} - \frac{\hbar}{r^2} (\vec{x} \times \vec{\sigma}).
\]  

(2.10)

These two systems are first-order superintegrable and the term \( V_1(r) = \frac{\hbar}{r^2} \) can be induced from a scalar Hamiltonian \( (V_1(r) \equiv 0) \) by a gauge transformation. For the gauge induced potentials given in (2.7), the integrals of motion are either just the gauge transforms of the terms obtained from \( \vec{L}, \vec{\sigma} \), or else live in the enveloping algebra of a direct sum of the algebra \( o(3) \) with itself \( o(3) \oplus o(3) = \{ \vec{J} - \vec{S} \} \oplus \{ \vec{S} \} \). For the gauge induced potentials given in (2.9), such integrals of motion are either the gauge transforms of the terms obtained from \( \vec{L}, \vec{p}, \vec{\sigma} \), or else live in the enveloping algebra of a direct sum of the Euclidean Lie algebra \( e(3) \) with the algebra \( o(3); e(3) \oplus o(3) = \{ \vec{J} - \vec{S}, \vec{\Pi} \} \oplus \{ \vec{S} \} \). (For details; see [4].) Hence, in the subsequent analysis we exclude the cases when the spin–orbital potential is a gauged induced one. However, we will present those tensor and pseudo-tensor integrals of motion corresponding to gauge induced potentials in the appendix in order to explicitly verify that they reduce to the well-known results in the absence of the spin–orbit interaction.

As we already mentioned we decompose the space of integrals of motion into subspaces transforming under reducible representations of \( O(3) \). In this article, we shall construct symmetric two-component tensors and pseudo-tensors in the direct product space of the vectors \( (\vec{x}, \vec{p}, \vec{\sigma}) \). The quantities \( \vec{x}, \vec{p} \) and \( \vec{\sigma} \) allow us to define six independent ‘directions’, namely \( (\vec{x}, \vec{p}, \vec{L} = \vec{x} \times \vec{p}, \vec{\sigma} \wedge \vec{x}, \vec{\sigma} \wedge \vec{p}) \) in this space and any \( O(3) \) tensor can be expressed in terms of these. Those tensor integrals of motion can allow arbitrary powers of \( \vec{x} \) but can involve at most second-order powers of \( \vec{p} \) and first-order powers of \( \vec{\sigma} \). Also any scalar in \( \vec{x} \) space will be written as \( f(r) \), where \( f \) is an arbitrary function of \( r = \sqrt{x^2 + y^2 + z^2} \). Under these circumstances we can form 28 two-component symmetric tensors and 26 symmetric pseudo-tensors. In the following sections we will separately and explicitly give the form of these integral of motion.

3. Tensor integrals of motion

Two index tensors are expressed as follows:

\[
T_1^{ij} = x^i x^j, \quad T_2^{ij} = (\vec{x}, \vec{p}) x^i x^j, \quad T_3^{ij} = (\vec{\sigma}, \vec{L}) x^i x^j, \quad T_4^{ij} = x^i p^j + p^j x^i, \\
T_5^{ij} = x^i (\vec{x} \wedge \vec{\sigma})^j + (\vec{x} \wedge \vec{x} \wedge \vec{\sigma}^j x^i, \quad T_6^{ij} = (\vec{x}, \vec{p}) x^i (\vec{x} \wedge \vec{\sigma})^j + (\vec{x} \wedge \vec{\sigma})^j x^i, \\
T_7^{ij} = L^i \sigma^j + \sigma^i L^j, \quad T_8^{ij} = p^i (\vec{x} \wedge \vec{\sigma})^j + (\vec{x} \wedge \vec{\sigma})^j p^i, \\
T_9^{ij} = \vec{p}^2 x^i x^j, \quad T_{10}^{ij} = \vec{L}^2 x^i x^j, \quad T_{11}^{ij} = (\vec{x}, \vec{p} (\vec{\sigma}, \vec{L}) x^i x^j, \\
T_{12}^{ij} = L^i L^j + L^j L^i, \quad T_{13}^{ij} = (\vec{x}, \vec{L}) (x^i p^j + p^j x^i).
\]
Each of the quantities in (3.1) can be multiplied by a scalar $f(r)$ without changing its properties under rotations or reflections.

It can be shown that we have three linear relations

$$T_{20}^{ij} = -T_{17}^{ij} + T_{18}^{ij}, \quad T_{21}^{ij} = -T_{16}^{ij} + T_{18}^{ij}, \quad T_{22}^{ij} = T_{14}^{ij} - T_{16}^{ij} + T_{13}^{ij}. \quad (3.2)$$

Additionally, we find the following linear relations with coefficients depending on the distance $r$:

$$T_{22}^{ij} = \frac{1}{2}(\overrightarrow{r}^{2}T_{17}^{ij} + T_{12}^{ij} + T_{9}^{ij}),$$

$$T_{23}^{ij} = \overrightarrow{r}^{2}T_{9}^{ij} - T_{10}^{ij},$$

$$T_{24}^{ij} = \overrightarrow{r}^{2}T_{14}^{ij} - T_{15}^{ij},$$

$$T_{25}^{ij} = -\overrightarrow{r}^{2}T_{7}^{ij} + T_{6}^{ij} + T_{3}^{ij},$$

$$T_{26}^{ij} = -\overrightarrow{r}^{2}T_{16}^{ij} + T_{24}^{ij} + T_{11}^{ij},$$

$$T_{27}^{ij} = -\overrightarrow{r}^{2}T_{19}^{ij} + T_{18}^{ij} + T_{15}^{ij}. \quad (3.3)$$

We use relations (3.2) and (3.3) to remove the left-hand sides of these equations from the analysis completely.

### 3.1. The commutativity condition and determining equations

In this subsection, we take the linear combinations of all the two-component tensors and then fully symmetrize them in order to obtain the determining equations from the commutativity condition.

Let us take the linear combination of the independent tensors given in (3.1) as

$$X_{T}^{ij} = \sum_{k=1}^{19} f_{k}(r)T_{k}^{ij},$$

which can be fully symmetrized as described in [1]. In the commutativity relation $[H, X_{T}^{ij}] = 0$ it is enough to consider only two indices, say $i = 1, j = 2$ since the others then necessarily commute due to the rotations. In the analysis we consider the full symmetric form of $X_{T}^{12}$.
which, however, is rather long to be presented here. From the requirement $[H, X_j^{ij}] = 0$, we obtain the determining equations.

The determining equations obtained by equating the coefficients of third-order terms to zero in the commutativity equation give us

\begin{align*}
 f_{10} &= 0, \quad f_9 = 0, \quad f_{12} = c_1, \quad f_{17} = c_2, \quad f_{11} = -f_{15}, \quad (3.4) \\
 f_{16} &= -f_{14}, \quad f_{19} = -r^2 f_{18} + c_3, \quad f_{13} = r^2 f_{15} + f_{14} + f_{18}, \quad f_{14} = c_3, \quad (3.5) \\
 2rf_{15}V_1 + hf'_1 &= 0, \quad (3.6) \\
 2c_3V_1 - h(f_{15} + rf'_{15}) &= 0, \quad (3.7) \\
 2r(c_3 + r^2 f_{15} - f_{18})V_1 - h f'_{18} &= 0, \quad (3.8) \\
 hf_{18} + 2c_4V_1 + h(c_3 + rf'_{18}) &= 0, \quad (3.9)
\end{align*}

where $c_i (i = 1, 2, 3, 4)$ are integration constants.

Introducing the relations given in (3.4) and (3.5) into the determining equations obtained by equating the coefficients of the second-order terms to zero in the commutativity equation, we get

\begin{align*}
 f_3 &= f_6 + c_5, \quad f_7 = r^2 f_6 + c_6, \quad (3.10) \\
 rf_{15}V_1 - f'_2 &= 0, \quad (3.11) \\
 4rf_2 + r(c_3 + 2r^2 f_{15} + f_{18})V_1 + f'_4 - 2c_4V_1 &= 0, \quad (3.12) \\
 2c_3hr + 2rf_6(-h + r^2 V_1) - h(f'_8 + 2c_2V'_1) &= 0, \quad (3.13) \\
 2rf_4V_1 + hf'_6 &= 0, \quad (3.14) \\
 -3hrf_6 - 2r(c_6 - c_1h + f_8)V_1 - hf'_8 &= 0, \quad (3.15) \\
 2rf_2 + r(-3c_3 - 2r^2 f_{15} + 3f_{18})V_1 + f'_4 &= 0, \quad (3.16) \\
 f_4 + (3c_4 + c_3)r^2 + r^4 f_{15} - 2r^2 f_{18}V_1 + c_4rV'_1 &= 0, \quad (3.17) \\
 2c_3hr + hf_6 + 2r(c_6 - c_1h + r^2 f_6 + f_8)V_1 - 2c_2hV'_1 &= 0 \quad (3.18) \\
 -2c_6V_1 + h(f_6 + 2c_1V_1 + rf'_{6}) &= 0, \quad (3.19) \\
 f_8 - 2c_2V_1 + r(3rf_6 + r^2 f'_6 + f'_8) &= 0, \quad (3.20)
\end{align*}

where, again, $c_5$ and $c_6$ are integration constants.

Now, introducing the relations given in (3.4), (3.5) and (3.10) into the determining equations obtained by equating the coefficients of the first-order terms to zero in the commutativity equation yields

\begin{align*}
 r(2c_5 - f_6)V_1 - f'_1 + (r^2 f_6 + f_8)V'_1 &= 0, \quad (3.21) \\
 4c_2V'_0 - 2r(2f_1 + (2r^2(c_5 + f_6) + 3f_8)V_1 + r(r^2 f_6 + f_8)V'_1) &= 0, \quad (3.22) \\
 12hrf_{15}V_1 + 8h^2 f'_{15} - 2hf_{18}V_1 + 4f'_5 - 4c_3hV'_1 - 4hf_{18}V'_1 + 4f_4V'_1 \\
 + 4rf_2(-2V_1 + rV'_1) + h^2 r f''_{15} &= 0, \quad (3.23) \\
 - 4f_5 + 6hrf_{18}V_1 + 4rf_4V_1 - 2hrf_{15}(h - 5r^2 V_1) + 8h^2 r f'_{15} + 10h^2 f'_{18} \\
 - 2hr^2 f'_{18}V_1 + 4c_4V'_0 + 4c_3hV'_1 + h^2 r f''_{15} + 2h^2 r f''_{18} &= 0. \quad (3.24)
\end{align*}
Finally, introducing the relations given in (3.4), (3.5) and (3.10) into the determining equations obtained by equating the coefficients of the zeroth-order terms to zero in the commutativity equation, we obtain

\[-2c_2V_0' + r \left( -r \left( 6f_1' + 3V_1(-2c_5r + 3rf_6 + r^2f_6' + f_6'' + r f_6') \right) \right) + 2c_2V_0'' = 0, \tag{3.25}\]

\[12r^2(-rf_5V_1 + 3hf_2') + 4r^2(r^2f_2 + f_4)V_0' + h^2(-4f_4' + r\left( 4f_4'' + r(13rf_4' + r f_4'') + 2c_4V_0'' \right)) = 0, \tag{3.26}\]

\[10hr^2f_2V_1 - 2r^2f_2V_1 + h(-2c_4V_0' + r(r(2V_1(r^2f_2' + f_2'') - 6f_2') + (c_3 + f_1)hV_0' - rf_3'') + 2c_4V_0'') = 0, \tag{3.27}\]

\[24hr^2f_6' + 2r^2V_1(4rf_1 + h(-2c_5r + 3rf_6 + r^2f_6' + f_6'')) + 4r^2(r^2f_6 + f_8)V_0' + h^2\left( -4f_8' + r\left( 4f_8'' + r(11rf_8' + r f_8'') + f_8'' \right) \right) = 0. \tag{3.28}\]

We notice that there are four types of determining equations:

(a) Those that are independent of the potentials $V_1$ and $V_0$, namely (3.4), (3.5) and (3.10).

(b) Equations involving only $V_1$ but not its derivative (3.6)–(3.9), (3.11), (3.14)–(3.16), (3.19) and (3.20).

(c) Those involving $V_1$ and $V_1'$ which are (3.12), (3.13), (3.17), (3.21) and (3.23).

(d) Those involving $V_0$, $V_1$ and their derivatives.

\subsection*{3.2. Solutions of the determining equations}

In this subsection, we find the solutions of the determining equations coming from the tensors and then obtain the tensor integrals of motion. We present a complete classification by following the above four classes of determining equations sequentially. As indicated before, the first type of determining equations had already been introduced into the others to eliminate or simplify them. So, we start the analysis with ten determining equations of second type. These equations can be viewed as linear algebraic equations for $V_1$. They must all be satisfied simultaneously. Thus they must all be multiples of just one equation. Its solution determines $V_1$ uniquely (though there may be several parametric versions of it depending on the coefficients $f_1, c_3$).

Let us first consider (3.6). Assuming $f_{15} \neq 0$, we obtain

\[V_1 = -\frac{hf_{15}'}{2rf_{15}}. \tag{3.29}\]

This equation can be introduced into (3.7) to give

\[f_{15} = -c_3 + \epsilon \frac{c_3^2 + \gamma rf_{15}^2}{r^2}. \tag{3.30}\]
where \(c_3\) and \(\gamma_1\) are real constants. First, it would be better to see what happens when \(c_3 = 0\). The reason for this will become clear shortly. In this case, we immediately find
\[
f_{15} = \frac{\alpha}{r},
\] (3.31)
where \(\alpha\) is an integration constant. Introducing this into (3.6), we find
\[
2\alpha V_1 - \frac{\alpha \hbar}{r^2} = 0,
\] (3.32)
which gives us two options, either \(\alpha = 0\) or \(V_1 = \frac{\hbar}{2r^2}\). But, the first one yields \(f_{15} = 0\) which is undesirable due to the assumption. Then, let us take \(V_1 = \frac{\hbar}{2r^2}\). If this potential is considered in (3.8) and (3.9), one can easily get \(c_3 = 0\) and \(\alpha = 0\), which again yields \(f_{15} = 0\). So, we need to take \(c_3 \neq 0\) which lead to (3.30) to be rewritten as
\[
f_{15} = -\frac{c_3(1 + \sqrt{1 + \beta r^2})}{r^2},
\] (3.33)
where \(\epsilon^2 = 1\) and \(\beta\) is a real constant. From (3.29) and (3.33) we get the following potential:
\[
V_1 = \frac{\hbar}{2r^2} \left(1 + \frac{\epsilon}{\sqrt{1 + \beta r^2}}\right).
\] (3.34)
It is obvious that \(V_1 = \frac{\hbar}{2r^2}\) and \(V_1 = \frac{\hbar}{2r^2}\) are special solutions for \((\epsilon, \beta) = (1, 0)\) and \((1, \infty)\), respectively. The first special potential is a gauge induced one and had already been thoroughly considered in [4]. Hence, we exclude the analysis of this option which means that we will assume \(\beta \neq 0\).

By considering the above relations (3.33) and (3.34) in the equations (3.8) and (3.9), one can obtain
\[
f_{18} = -\frac{c_3 r^2(1 + \beta r^2) + c_4(1 + \epsilon \sqrt{1 + \beta r^2})}{r^2}.
\] (3.35)
Substituting this back into (3.9) yields \(\beta = 0\) or \(c_3 = 0\). The case \(\beta = 0\) gives either \(V_1 = 0\) or \(V_1 = \frac{\hbar}{2r^2}\) both of which are excluded. On the other hand, from (3.33) the case \(c_3 = 0\) means that \(f_{15} = 0\) which contradicts with the assumption \(f_{15} \neq 0\).

The above discussion states that \(f_{15}\) must vanish. So, (3.7) directly yields \(c_3 = 0\). Keeping in mind these two facts, let us continue the analysis by considering (3.8). Assume that \(f_{18} \neq 0\). Then, one can easily get
\[
V_1 = -\frac{\hbar f_{18}'}{2rf_{18}},
\] (3.36)
and introducing this into (3.9) we obtain
\[
f_{18} = \frac{c_4 + \epsilon \sqrt{c_4^2 + \gamma_2 r^2}}{r^2},
\] (3.37)
where \(c_4\) and \(\gamma_2\) are real constants. Similar to the above, it can be easily checked that the case \(c_4 = 0\) will eventually cause \(f_{18} = 0\). Hence, we take \(c_4 \neq 0\) which allows us to rewrite (3.37) as
\[
f_{18} = \frac{c_4(1 + \epsilon \sqrt{1 + \beta r^2})}{r^2}, \quad \epsilon^2 = 1.
\] (3.38)
Then, from the last two relations (3.33) and (3.34) we again have the potential
\[ V_1 = \frac{\hbar}{2r^2} \left( 1 + \frac{\epsilon}{\sqrt{1 + \beta r^2}} \right). \] (3.39)

Then, by introducing these into (3.14), (3.15), (3.19) and (3.20) one can find
\[ f_6 = \frac{(c_1 h - c_0 (1 + \epsilon \sqrt{1 + \beta r^2})}{r^2}, \] (3.40)
\[ f_8 = (c_0 - c_1 h)(1 + \beta r^2 + \epsilon \sqrt{1 + \beta r^2}) - \frac{\hbar c_2 (1 + \epsilon \sqrt{1 + \beta r^2})}{r^2}. \] (3.41)

Substituting these relations into (3.15) gives \( \beta (c_6 - c_1 h) = 0 \). Hence, we directly have \( c_6 = c_1 h \) since the case \( \beta = 0 \) is excluded.

The remaining two determining equations of second type, namely (3.11) and (3.16), give
\[ f_2 = d_1, \quad f_4 = -d_1 r^2 + \frac{3c_4 h (1 + \epsilon \sqrt{1 + \beta r^2})}{2r^2} + d_2, \] (3.42)
where \( d_1 \) and \( d_2 \) are real constants. Then, we see that for the relations (3.38), (3.40), (3.42) and the potential given in (3.39) all of the determining equations of second type are satisfied. Now, we need to continue the analysis with the determining equations of third type. However, one can easily check that introducing these relations into (3.12) immediately yields either \( (d_1, \beta) = (0, 0) \) or \( (d_1, c_4) = (0, 0) \). We already know that the first case is excluded. The second case gives us a contradiction. This means that we do not need to examine the case \( f_{18} \neq 0 \) further.

Hence, \( f_{18} \) must vanish too, which immediately gives us \( c_4 = 0 \). We continue the analysis with the relations \( f_{15} = 0, c_3 = 0, f_{18} = 0 \) and \( c_4 = 0 \). By introducing these relations into (3.11) and (3.16) we find
\[ f_2 = d_1, \quad f_4 = -d_1 r^2 + d_2, \] (3.43)
where \( d_1 \) and \( d_2 \) are integration constants.

Assuming \( f_6 \neq 0 \) in (3.14) we obtain
\[ V_1 = -\frac{\hbar f_6'}{2rf_6}, \] (3.44)
and introducing this into (3.19) yields
\[ f_6 = -(c_6 - c_1 h) + \epsilon \sqrt{(c_6 - c_1 h)^2 + \gamma r^2}, \] (3.45)
where \( c_1, c_6 \) and \( \gamma_3 \) are real constants. From the remaining determining equations, one can see that the special case when \( c_6 = c_1 h \) will give rise to \( f_6 = 0 \). Therefore, we need to assume \( c_6 \neq c_1 h \) which gives
\[ f_6 = \frac{(c_1 h - c_6)(1 + \epsilon \sqrt{1 + \beta r^2})}{r^2}, \quad \epsilon^2 = 1. \] (3.46)

Then, the last two relations once again give the potential
\[ V_1 = \frac{\hbar}{2r^2} \left( 1 + \frac{\epsilon}{\sqrt{1 + \beta r^2}} \right). \] (3.47)
One can easily check that substituting these relations into (3.15) and (3.19) yields 
\((c_6 - c_1 \hbar) \beta = 0\). Thus, we have \(c_6 = c_1 \hbar\) due to the fact that \(\beta \neq 0\). However, this contradicts with the assumption \(f_6 \neq 0\). Therefore, we need to take \(f_6 = 0\) which directly yields 
\(c_6 = c_1 \hbar\) by (3.19).

The above discussion implies that two determining equations of second type remain:

\[
2rf_8 V_1 + h f'_8 = 0, \quad f_8 + rf'_8 - 2c_2 V_1 = 0. \tag{3.48}
\]

Let us first assume \(f_8 \neq 0\). In this case, it is rather easy to see that \(c_2\) cannot vanish for this case. So, similar steps to the previous cases yield the relation

\[
f_8 = -\frac{c_2 \hbar (1 + \epsilon \sqrt{1 + \beta r^2})}{r^2}, \quad \epsilon^2 = 1, \tag{3.49}
\]

and the potential

\[
V_1 = \frac{\hbar}{2r^2} \left(1 + \frac{\epsilon}{\sqrt{1 + \beta r^2}}\right). \tag{3.50}
\]

So, all of the determining equations of second type are satisfied. But, by introducing these relations into (3.13) we see that either \((c_5, \beta) = (0,0)\) or \((c_5, c_1) = (0,0)\). Again, the first case is excluded and the second one causes a contradiction.

Eventually, by solving the determining equations of second type we just find the following relations for an arbitrary spin–orbital potential:

\[
f_{15} = 0, \quad f_{18} = 0, \quad f_6 = 0, \quad f_8 = 0, \quad c_3 = 0, \quad c_4 = 0, \quad c_6 = c_1 \hbar, \quad c_2 = 0. \tag{3.51}
\]

Let us examine the determining equations of third type. Introducing the relations in (3.51) into the determining equations (3.12), (3.13), (3.17), (3.21) and (3.23), one can easily obtain

\[
f_2 = 0, \quad f_4 = 0, \quad f_1 = k_1, \quad c_5 = 0, \tag{3.52}
\]

where \(k_1\) is an integration constant. By these relations, all of the determining equations of third type are satisfied. Considering (3.51) and (3.52) in the remaining equations, one can see that \(f_1 = 0\) and those equations are satisfied for arbitrary scalar potential \(V_0\). Hence, we have just two functions \(f_{12}\) and \(f_7\) that do not vanish.

Finally, all the determining equations are satisfied for arbitrary potentials \(V_0 = V_0(r)\) and \(V_1 = V_1(r)\). Having left with just one arbitrary constant \(c_1\), we have the following trivial integral of motion

\[
\mathcal{T}_{1}^{ij} = \hbar (L^i \sigma^j + \sigma^i L^j) + L L^i + L^i L = \{J^i, J^j\} - \delta^{ij} \frac{\hbar}{2}, \tag{3.53}
\]

where \(\{\cdot, \cdot\}\) stands for the anti-commutator.

4. Pseudo-tensor integrals of motion

Two index pseudo-tensors are expressed as follows:

\[
Y_{1}^{ij} = x^i \sigma^j + \sigma^i x^j, \quad Y_{2}^{ij} = (\vec{r}, \vec{p})(x^i \sigma^j + \sigma^i x^j), \quad Y_{3}^{ij} = p^i \sigma^j + \sigma^i p^j,
\]
Here, we take the linear combinations of all the two-component pseudo-tensors and then determine equations (4.1).

We use relations (4.2) and (4.3) to remove the left-hand sides of these equations from the analysis completely.

4.1. The commutativity condition and determining equations

Here, we take the linear combinations of all the two-component pseudo-tensors and then fully symmetrize them in order to obtain the determining equations from the commutativity condition.

Let us take the linear combination of the independent pseudo-tensors given in (4.1) as

\[ X^i_Y = \sum_{k=1}^{19} f_k(r) Y^i_k, \]
which can be fully symmetrized as described in [1]. Similar to the tensors it is sufficient to consider the full symmetric form of \(X_{ij}^{kl}\). From the requirement \([H, X_{ij}^{kl}] = 0\), we obtain the determining equations.

The determining equations obtained by equating the coefficients of third-order terms to zero in the commutativity equation, are given as follows:

\[
\begin{align*}
&f_{17} = 0, \quad f_{10} = c_1, \quad f_{10} = f_{12} - f_{18}, \quad f_9 = -f_{11}, \\
&f_{13} = f_{11} - r^2 f_{12} + c_2, \quad f_{11} = r^2 f_{15} + c_3, \quad f_{16} = -r^2 f_{15} + \frac{c_4}{r}, \\
2rf_{14}V_1 + hf_{14}' &= 0, \\
hf_{12} + 2c_2V_1 + hrf_{12}' &= 0, \\
hrf_{14} + 2rf_{10}V_1 + h(f_{12} + r^2 f_{14}') &= 0, \\
2rf_{12}V_1 + hf_{18}' &= 0, \\
2hc_4 - r^3(f_{15} + f_{18}) - 4r^2(c_2 + c_3)r - r^3 f_{12}V_1 &= 0, \\
h(f_{12} + 2r^2 f_{14} + f_{15} - f_{18}) - 2(c_3 + r^2(-f_{12} + f_{15} + f_{18}))V_1 &= 0, \\
-2c_3V_1 + f_{12}(h + 4r^2 V_1) + h(f_{15} - f_{18} + r(f_{12} + f_{15} + f_{18})) &= 0,
\end{align*}
\]

where \(c_i (i = 1, 2, 3, 4)\) are integration constants.

Introducing the relations given in (4.4) and (4.5) into the determining equations obtained by equating the coefficients of the second-order terms to zero in the commutativity equation, we get

\[
\begin{align*}
f_2' + c_1 V_1' &= 0, \\
f_2 - c_1V_1 + rf_2' &= 0, \\
r f_2' + c_1rV_1 + f_3' &= 0, \\
hf_2 + 2(-c_1h + f_3)V_1 &= 0, \\
2rf_3 + 2hr^2f_6 + hf_3 + 2c_1hV_1 + 2r^2 f_2 V_1 - 2f_3 V_1 + hr f_3' &= 0, \\
-2hrf_6 - 2r(f_2 + f_7)V_1 - hf_6' + c_1hV_1' &= 0, \\
2rf_6 V_1 + hf_6' &= 0, \\
4hrf_6 + 2r(f_2 + 2f_5 + f_7)V_1 + h(-f_2' + f_5') &= 0, \\
h(f_2 + 2f_5 + f_7) + (c_1h - 2(f_3 + r^2 f_7))V_1 &= 0, \\
2r(-f_2 + f_7)V_1 + 2rf_6(3h - 2r^2 V_1) + h(f_2' + 2f_5' + f_7') &= 0, \\
4hrf_6 + 2r(f_2 + 2f_5 + f_7)V_1 + h(f_7' + c_1 V_1') &= 0, \\
2hrf_6 - 2rf_2 V_1 + h(f_5' + r^2 f_5') &= 0, \\
r f_8 + (2c_4 + 3(c_2 + c_3)r + r^3(-3f_{12} + f_{15} + f_{18}))V_1 + c_2r^2 V_1' &= 0, \\
2hrf_6 + 2r(f_2 + f_7)V_1 + h(-2f_2' + f_7') - 3c_1hV_1' &= 0, \\
2r(f_2 + 2f_5 - f_7)V_1 + 4rf_6(h + r^2 V_1) - h(2f_5' + 2f_5' + f_7') + c_1 h V_1' &= 0.
\end{align*}
\]
In like manner, introducing the relations given in (4.4) and (4.5) into the determining equations obtained by equating the coefficients of the first-order terms to zero in the commutativity equation yields

\[ 4r(r(2f_5 - f_7)V_1 + c_1 V_0') + 2r(-c_5 + 2c_2 r^2 + 2f_3)V_1' = 0, \]  
\[ 6c_4 h^3 - 4hr^3 f_4 + 8h^2 r^2 f_{14}(h + 2r^2 V_1) + r^2 (2V_1 (3c_4 h^2 \nonumber \] 
\[ + r^3 (2f_1 + h (2f_5 + h(f_{15} + f_{18} + f'_{13}))) + hr (12h^2 r f'_{12} + 20h^2 r^3 f'_{14} + 14h^2 r f'_{15} + 4c_4 V_0^2 + 2c_2 r V_0') + 2c_1 V_0 + 2c_2 V_1' + 3c_2 h V_1' - c_1 h V_1' - 2hr f_{15} V_1' + 2h^2 r^2 f''_{12} \nonumber \] 
\[ + 2h^2 r^2 f''_{14} + 2h^2 r^2 f''_{15}))) = 0, \]  
\[ 4c_4 h^2 + 4r^2 f_1 - 8hr^3 f_{14}(h - r^2 V_1) + r^2 (2(h(-4c_4 + c_2 r - c_3 r) \nonumber \] 
\[ + r^3 (5hf_{12} + hf_{18} + 2f_8)) V_1 + 2hr f_{15}(-2h + 5r^2 V_1) \nonumber \] 
\[ + r^2 (4c_2 V_0^2 + h (4c_2 V_1' + h (6f'_{12} + 6f'_{15} + 6f_{18} \nonumber \] 
\[ + r(f'_{12} + f'_{15} + f'_{18})))) = 0, \]  
\[ 2(-2c_4 h + r^3 (13hf_{12} + 7hf_{15} - 7hf_{18} + 2f_8)) V_1 - 40hr^3 f_{14}(h - r^2 V_1) \nonumber \] 
\[ + r (-4rf_1' - 6h^2 r f'_{12} + 10h^2 r f'_{15} + 18h^2 r f'_{18} + 4c_4 r V_0 + 8c_4 h V_1') \nonumber \] 
\[ + 4hr(c_2 + c_3)V_1' + h^2 r^2 (-f''_{12} + f''_{15} + 3f''_{18})) = 0, \]  
\[ 12c_4 h^3 + 8h^2 r f_{14}(h - 15r^2 V_1) + 16r^2 f_{14}(h - r^2 V_1) + r^2 (V_1 (8c_4 h^2 \nonumber \] 
\[ + 2r^3 (-4f_5 + h (2f_8 + h(-27f_{12} - 13f_{15} + 17f_{18} + 2f'_{13})))) + hr (4r f_1' + 30h^2 r f'_{12} - 80h^2 r f'_{15} - 42h^2 r f'_{18} - 8c_4 V_0) \nonumber \] 
\[ + 4c_3 V_0 + 8r^3 f_{15} V_0 - 4c_4 h V_1' - 6c_2 h V_1' - 14c_3 h V_1' + 4hr^2 f_{15} V_1' \nonumber \] 
\[ + 5h^2 r^2 f''_{12} - 8h^2 r^2 f''_{15} + 15h^2 r^2 f''_{18} + 8c_4 h V_1' + 2h^2 r^2 f''_{15}))) = 0. \]  

Finally, introducing the relations given in (4.4) and (4.5) into the determining equations obtained by equating the coefficients of the zeroth-order terms to zero in the commutativity equation, we obtain

\[ 8r^4 f_4 V_1 - h(4c_4 V_0' + r^3 (8f_4' - 4f_{15} V_0' + r f''_0) - 2c_4 r V_0'')) = 0, \]  
\[ 2r^2 f_1 V_1 + 4r^2 f_4 (-h + r^2 V_1) - h (4r f_1' + (-2c_4 - c_2 r + c_3 r + 2r^3 f_{15}) V_0' + r^3 f''_0) = 0, \]  
\[ 2h(-3h + 4r^2 V_1) f_4' + 56h^2 r^2 f_6 + 8hr^3 V_1 f_6' - 6h^2 f_5' + 8hr^2 V_1 f_7' \nonumber \] 
\[ + 4r^2 V_0 (f_3 + f_7) + 4r^3 f_6 (12h V_1 + r V_0') + 6h^2 r f''_0 + 16h^2 r f''_6 \nonumber \] 
\[ + 6h^2 r f''_7 + h^2 r^2 (f''_5 + r^2 f''_6 + f''_7)) = 0. \]
\[ 2c_3 \hbar^2 + 24r^4 f_6(h - \hbar^2 V_1) + r \left( -2\hbar^2 f'_3 - 2hrV_1 \left( -c_5 + 4c_6r^2 \right) \right. \]
\[ + r \left( f'_3 + r \left( 2f_3 + f_7 + 2rf'_5 + r^2f'_6 + f'_7 \right) \right) + r \left( 12\hbar^2 rf'_3 + 4\hbar^2 r^3 f'_6 \right. \]
\[ + 8\hbar^2 rf'_7 - 2c_3rV'_0 + 4c_6r^3V_0 + 4rf_3V'_0 + 2\hbar^2 f'_5 + 2\hbar^2 r^2 f'_7 \]
\[ + \left. \hbar^2 r f''_7 + \hbar^2 r f''_3(\cdot) \right) = 0. \tag{4.36} \]

Similar to the situation with tensors, we classify the above determining equations into four types:

(a) Those that are independent of the potentials \( V_1 \) and \( V_0 \), namely (4.4) and (4.5). These relations are substituted into the other determining equations. This eliminates some of the equations involving the potential and we have presented only the remaining ones.

(b) Those involving only \( V_1 \) but not its derivative (4.6)--(4.12), (4.14)--(4.17), (4.19)--(4.22), (4.24).

(c) Those involving \( V_1 \) and \( V'_1 \) which are (4.13), (4.18), (4.23), (4.25)--(4.27).

(d) Those involving \( V_0, V_1 \) and their derivatives (4.28)--(4.36).

### 4.2. Solutions of the determining equations

The aim of this subsection is to present the solutions of the determining equations given in section 4.1 and then reveal the pseudo-tensor integrals of motion. Similar to the tensors, we shall analyze the determining equations by following the above four classes and again the determining equations of first type had already been substituted to the others to eliminate or simplify them. Hence, we start with the determining equations of second type. But first, one can see that integrating (4.13) allows us to consider it as a determining equation of second type:

\[ f_2 + c_1V_1 + c_5 = 0, \tag{4.37} \]

where \( c_1 \) and \( c_5 \) are real constants. This equation will simplify the analysis.

Adding (4.37) to (4.13) yields

\[ rf'_2 + 2f_2 + c_5 = 0, \tag{4.38} \]

which can be integrated to give

\[ f_2 = -\frac{c_5}{2} + \frac{c_6}{r^2}, \tag{4.39} \]

where \( c_6 \) is an integration constant. Introducing (4.38) into (4.14), we obtain

\[ c_1V_1 + \frac{c_6}{r^2} + \frac{c_5}{2} = 0. \tag{4.40} \]

To continue analyzing, it is better to consider the following cases depending upon the constant \( c_1 \):

- **Case 1.** \( c_1 \neq 0 \).

  In this case, we immediately get a potential as

  \[ V_1 = -\frac{c_6}{c_1r^2} - \frac{c_5}{2c_1}. \tag{4.41} \]
Introducing this equation into (4.15) and integrating, we get
\[ f_3 = \frac{cs^2}{2} + c_7, \tag{4.41} \]
where \(c_7\) is an integration constant. Considering (4.40) and (4.41) in equation (4.16), we obtain
\[ f_7 = \frac{(2c_1 - 2c_1\hbar + csr^2)(2c_6 + csr^2)}{c_1\hbar r^2}. \tag{4.42} \]
On the other hand, if we introduce (4.37), (4.40)–(4.42) into the equation (4.17), we find
\[ f_6 = \frac{4c_6^2 - 4c_6c_7 + cr^2(-4c_6 + 4c_1\hbar - 5csr^2) - 4c_1\hbar r^2f_5}{4c_1\hbar r^4}, \tag{4.43} \]
which can be considered in (4.24) to deduce
\[ c_5 = 0. \tag{4.44} \]
If we introduce (4.38), (4.40), (4.42), (4.43) and (4.44) into (4.20), we obtain
\[ \frac{(c_6 + c_1\hbar)(-2c_6c_7 + 3c_1c_6\hbar - 2c_1\hbar r^2f_5)}{c_1^2r^6} = 0. \tag{4.45} \]
Obviously, we encounter two different cases here. However, the case \(c_6 = -c_1\hbar\) gives the potential \(V_1 = \frac{\hbar}{r^2}\) which can be induced by a gauge transformation and had been considered thoroughly in [4]. So, we take
\[ f_5 = \frac{c_6(-2c_7 + 3c_1\hbar)}{2c_1\hbar r^2}, \tag{4.46} \]
which together with (4.43) can be considered in (4.19) to give
\[ c_6(c_6 + 2c_1\hbar)(-2c_6 - 2c_7 + 3c_1\hbar) \tag{4.47} \]
Then we have three subcases. The first one is the case \(c_6 = 0\). However, it leads to \(V_1 = 0\). Therefore, we analyze the following two subcases.

**Subcase 1.** \(c_6 = -2c_1\hbar\).

From (4.40) and (4.44), we immediately obtain
\[ V_1 = \frac{2\hbar}{r^2}. \tag{4.48} \]
Then, from (4.21) it is easy to show that
\[ c_7 = \frac{3c_1\hbar}{2}. \tag{4.49} \]
By considering the above spin–orbital potential, let us solve all the determining equations of second type. Integrating (4.6) and (4.7), we find
\[ f_{14} = \frac{k_1}{r^4}, \quad f_{12} = \frac{4c_2 + rk_2}{r^2}. \tag{4.50} \]
where $k_1$ and $k_2$ are integration constants. Then, from (4.8) we get

\[
f_{18} = \frac{8c_2 + 3k_1 + k_2 r}{4r^2}.
\quad (4.51)
\]

Introducing (4.50) and (4.51) into (4.9) yields

\[
k_1 = 8c_2, \quad k_2 = 0.
\quad (4.52)
\]

By substituting (4.50)–(4.52) into the equations (4.10)–(4.12), one can conclude that

\[
f_{15} = -\frac{4(c_2 + c_3)}{3r^2}, \quad c_3 = 2c_2, \quad c_4 = 0.
\quad (4.53)
\]

Hence, all the determining equations of second type are satisfied. Thus, we continue the analysis with the determining equations of third type.

Substituting (4.48), (4.50), (4.51) and (4.53) into (4.25) gives us

\[
f_8 = \frac{2c_2 \hbar}{r^2},
\quad (4.54)
\]

which guarantees that the determining equations of third type are also satisfied. Introducing the above relations into the remaining determining equations yields

\[
V_0 = \frac{3h^2}{r^2}, \quad f_1 = -\frac{9c_2 h^2}{r^2}, \quad f_4 = \frac{24c_2 h^2}{r^4},
\quad (4.55)
\]

where a redundant additive constant in $V_0$ is omitted.

Hence, all determining equations are satisfied. For this case, we have two arbitrary constants $c_1$ and $c_2$. The two integrals of motion are:

\[
\mathcal{Y}_{1j}^{ij} = l'L^j + l' p^j - \frac{2h}{r^2} (i\hbar + (\vec{x}, \vec{p})) (x'\sigma^j + \sigma' x^j)
\]
\[+ \frac{3h}{2} (p'\sigma^j + \sigma' p^j) + \frac{4h}{r^2} (3i\hbar + 2(\vec{x}, \vec{p})) (\vec{\sigma}, \vec{x}) x' x^j
\]
\[+ \frac{2h}{r^2} (\vec{\sigma}, \vec{x})(x' p^j + p' x^j),
\quad (4.56)
\]

\[
\mathcal{Y}_{2j}^{ij} = -2 \left( \frac{2h^2}{r^2} - \frac{2i\hbar}{r^2} (\vec{x}, \vec{p}) - \vec{p}^2 + \frac{2}{r^2} \vec{L}^2 \right) (x'\sigma^j + \sigma' x^j)
\]
\[+ (4i\hbar + 2(\vec{x}, \vec{p}) (p'\sigma^j + \sigma' p^j) - \frac{8}{r^2} (i\hbar (\vec{x}, \vec{p}) (\vec{\sigma}, \vec{x}) )
\]
\[+ \frac{2h^2}{r^2} (\vec{\sigma}, \vec{x}) + i\hbar (\vec{\sigma}, \vec{p}) - \frac{2}{r^2} (\vec{\sigma}, \vec{x} \vec{L}^2) \right) x' x^j + 8(h \vec{\sigma}, \vec{x}) p' p^j
\]
\[+ \frac{4}{r^2} (\vec{x}, \vec{p} + (\vec{x} \vec{\sigma}, \vec{L}^j + \vec{L}^j \vec{\sigma})) - 5 ((\vec{p} \vec{\sigma}) \vec{L}^j
\]
\[+ \vec{L}^j (\vec{p} \vec{\sigma})) - \frac{4}{r^2} (\vec{x}, \vec{p} + (\vec{x} \vec{\sigma}, \vec{L}^j + \vec{L}^j \vec{\sigma})) x' p^j + p' x^j
\]
\[+ \frac{2}{r^2} (h + 4(\vec{\sigma}, \vec{L}^j)) (x' L^j + L^j x^j).
\quad (4.57)
\]
Remark 4.1. Note that only the off-diagonal elements of the above pseudo-tensors commute with the Hamiltonian. Observe that they are not traceless and none of the diagonal elements are integrals of motion. However, we know that traces of such integrals of motion appear separately as scalars and pseudo-scalars. Indeed, if the diagonal elements are redefined as \( Y_{ii} = \frac{1}{2} \text{tr}(\mathcal{Y}^{ij}) \), then these new elements commute with the Hamiltonian as well. For example, the trace of \( \mathcal{Y}_{1}^{ij} \) is the pseudo-scalar \( 3(\vec{\sigma} \cdot \vec{p}) \) and so \( \mathcal{Y}_{1}^{\alpha} - (\vec{\sigma}, \vec{p}) \) commutes with the Hamiltonian. Note that in the rest of the paper only one pseudo-tensor integral of motion \( \mathcal{Y}_{6}^{ij} \) will be traceless.

Subcase 2. \(-2c_{6} - 2c_{7} + 3c_{1}h = 0,\)

By using the relation \( c_{7} = -c_{6} + \frac{3c_{1}h}{2} \) in (4.21), one can directly obtain

\[
\frac{c_{6}^{2}}{c_{1}^{2}} - \frac{h^{2}}{r^{2}} = 0,
\]

which gives either \( c_{6} = -c_{1}h \) or \( c_{6} = c_{1}h \). However, the first one is excluded due to the same reason mentioned before. Then, we continue to analyze by taking \( c_{6} = c_{1}h \), which in the vicinity of (4.40) gives

\[
V_{1} = -\frac{h}{r^{2}}.
\]

By following very similar steps to subcase 1, one can obtain the relations given as:

\[
f_{14} = 0, \quad f_{12} = -\frac{2c_{2}}{r^{2}}, \quad f_{18} = \frac{2c_{2}}{r^{2}}, \quad c_{3} = -2c_{2}, \quad c_{4} = 0, \quad f_{15} = 0.
\]

With these relations all the determining equations of second type are satisfied. On the other hand, introducing the relations (4.58) and (4.59) into (4.25) we get

\[
f_{8} = \frac{3c_{2}h}{r^{2}}.
\]

Then, the determining equations of third type are also satisfied. Finally, considering the above relations in the remaining determining equations give us

\[
V_{0} = \alpha, \quad f_{1} = -\frac{3c_{1}h^{2}}{2r^{2}}, \quad f_{4} = 0,
\]

where \( \alpha \) is a real constant. Therefore, all the determining equations are satisfied for the spin–orbital potential \( V_{1} = -\frac{h}{r^{2}} \) and the scalar one \( V_{0} = \alpha \). Having left with two arbitrary constants \( c_{1} \) and \( c_{2} \), there exist two integrals of motion which read:

\[
\mathcal{Y}_{3}^{ij} = p' L' + L' p' + h\frac{1}{r^{2}} (i\hbar + (\vec{x}, \vec{p})) (x'^{i} \sigma^{j} + \sigma^{j} x'^{i}) + \frac{h}{2} (\sigma' \sigma^{j} + \sigma^{j} \sigma')
\]

\[
- \frac{h}{r^{2}} (\vec{\sigma}, \vec{x}) (x'^{i} p^{j} + p^{j} x'^{i}) + \frac{2h}{r^{2}} (\vec{\sigma}, \vec{p}) x^{j},
\]

\[
\mathcal{Y}_{4}^{ij} = \frac{3h^{2}}{2r^{2}} - \frac{2i\hbar}{r^{2}} (\vec{x}, \vec{p}) + 2\vec{p}^{2} - \frac{4}{r^{2}} L^2 (x'^{i} \sigma^{j} + \sigma^{j} x'^{i})
\]

\[
+ 2(i\hbar - (\vec{x}, \vec{p}))(p' \sigma^{j} + \sigma' p')
\]

\[
+ \frac{2i\hbar}{r^{2}} (2(\vec{\sigma}, \vec{p}) x^{j} - (\vec{\sigma}, \vec{x}) (x'^{i} p^{j} + p^{j} x'^{i})) - \frac{2}{r^{2}} (\vec{x}, \vec{p}) ((\vec{x} \wedge \vec{\sigma})^{i}.
\]
\[
\times L^i + L(\vec{x} \wedge \vec{\sigma} y^i) + (\vec{p} \wedge \vec{\sigma} y^i)L^i + L(\vec{p} \wedge \vec{\sigma} y^i)
+ \frac{1}{r^2}(2(\vec{\sigma}, \vec{D}) + 3\hbar)(x^i L^i + L^i x^i).
\] (4.61)

\textbf{Case 2.} \ c_1 = 0, \ V_1 \text{ unspecified.}
In this case, (4.14) together with (4.38) yield
\[
c_5 = 0, \quad c_6 = 0.
\] (4.62)
Introducing these into (4.15) and using (4.38) give
\[
f_3 = c_8, \quad c_8 \in \mathbb{R}.
\] (4.63)
By considering these relations in (4.24) and then integrating, we find
\[
f_5 = -r^2 f_6 + c_9,
\] (4.64)
where \(c_9\) is an integration constant. Introducing this equation into the sum of (4.16) and (4.17), we obtain
\[
f_7 = -c_9 + \frac{c_{10}}{r^2}, \quad c_{10} \in \mathbb{R},
\] (4.65)
which can be substituted back into (4.16) to obtain
\[
2c_8 V_1 + \frac{h c_{10}}{r^2} - h c_9 = 0.
\] (4.66)
If \(c_8 \neq 0\), we have
\[
V_1 = \frac{h(c_9 r^2 - c_{10})}{2 c_8 r^2}.
\] (4.67)
Introducing (4.62), (4.64), (4.65) and (4.67) into (4.20) and (4.22) and after making some computations we obtain
\[
c_9 = 0, \quad f_6 = 0, \quad c_{10}(2c_8 + c_{10}) = 0,
\]
which implies the existence of two subcases, namely \(c_{10} = 0\) or \(c_{10} = -2c_8\) to consider. However, the former yields \(V_1 = 0\) and the latter gives the gauge induced potential \(V_1 = \frac{\hbar}{r^2}\).
Thus, we take \(c_8 = 0\) which immediately yields \(c_9 = 0\) and \(c_{10} = 0\). Then, by (4.20) we find \(f_6 = 0\). We continue the analysis by considering the determining equation (4.7) which now reads
\[
h f_{12} + 2c_2 V_1 + hr f'_1 = 0.
\] (4.68)
Let us examine the following subcases based upon the constant \(c_2\).
\textbf{Subcase 1.} \(c_2 = 0\).
This case directly yields
\[
f_{12} = \frac{c_{11}}{r},
\] (4.69)
where \( c_{11} \) is a real constant. One can show after a routine computation that the case \( c_{11} = 0 \) gives either \( V_1 = 0 \) or \( V_1 = \frac{\hbar}{2r} \). Hence, we assume \( c_{11} \neq 0 \). Then, from (4.9) we obtain
\[
V_1 = -\frac{\hbar f_{18}'}{2c_{11}}. \tag{4.70}
\]
Introducing these into (4.12), (4.10) and (4.8), respectively, we find
\[
f_{18} = \frac{c_{12} - rc_{11}f_{15}}{c_3 - rc_{11}},
\]
\[
f_{15} = -\frac{c_3c_{11}(c_4 + rc_{11}) - r(c_4(c_{11}^2 + c_{12}) + rc_{11}^2c_{11})}{r^2c_{11}(c_4 + r(c_3 - rc_{11}))}, \tag{4.71}
\]
\[
f_{14} = -\frac{(c_3c_{11} - r(c_{11}^2 + c_{12}) - c_1(c_{11}c_{12}))(2c_4c_{11} + r(c_3c_{11} - rc_{11}^2 + rc_{12} + c_1c_{11})))}{2r^2c_{11}(c_4 + r(c_3 - rc_{11}))^2} + \frac{c_{14}}{r}. \tag{4.72}
\]
where \( c_i (i = 12, 13, 14) \) are integration constants. If the relations (4.69)–(4.72) are introduced into the remaining determining equations of second type, the following relations are obtained:
\[
c_{14} = 0, \quad (3c_{11}^2 + c_{12})(c_{11}^2 + c_{12})(c_{11}^2 - c_{12}) = 0. \tag{4.73}
\]
Therefore, we have the following three possibilities:
\[\text{I.} \quad c_{12} = -3c_{11}^2.\]
This immediately gives us the potential
\[
V_1 = \frac{3\hbar}{2r^2}.
\]
Then, we obtain
\[
c_{13} = 0, \quad c_3 = 0, \quad c_4 = 0. \tag{4.74}
\]
Hence, the determining equations of second type are satisfied. By introducing these relations into the remaining equations, we get
\[
f_8 = 0, \quad f_1 = \frac{-6\hbar^2c_{11}}{r}, \quad f_4 = \frac{15\hbar^2c_{11}}{r^3}.
\]
Now, all the determining equations are satisfied for any \( V_0 = V_0(r) \). Having just one arbitrary constant \( c_{11} \) we have the following integral of motion:
\[
\mathcal{Y}_{\delta}^{ij} = -r ((\vec{\sigma} \wedge \vec{\partial}) L^j + L^i(\vec{\sigma} \wedge \vec{\partial})) + \frac{\hbar(r^2 + r^2)}{r} \left( \frac{\sigma, \vec{\partial}}{r} \right)^2 \left( 4L^2 + \hbar^2 \right)
\]
\[
- \frac{2}{r} \left( -2L^2 + \hbar(r, \vec{\sigma}) \right) x^i x^j + \frac{1}{r} \left( -2L^2 + \hbar(r, \vec{\sigma}) - \frac{3\hbar^2}{2} \right)(x^i \sigma^j + \sigma^j x^i). \tag{4.75}
\]
II. \( c_{12} = -c_{11}^2 \).
In this case, we directly find
\[
V_1 = \frac{\hbar}{2r^2}.
\]
Similar to the above case, introducing these into the remaining determining equations yields
\[
c_{13} = 0, \quad c_{3} = 0, \quad c_{4} = 0, \quad f_{8} = \frac{\hbar c_{11}}{r}, \quad f_{1} = 0, \quad f_{4} = 0.
\]
So, all the determining equations are satisfied for arbitrary scalar potential \( V_0 = V_0(r) \). We have just one arbitrary constant \( c_{11} \) which yields the following integral of motion:
\[
Y_{ij}^{(6)} = -r \left( \vec{p} \times \vec{\sigma} \right) L' \left( \vec{p} \times \vec{\sigma} \right) L' \left( \vec{x} \times \vec{\sigma} \right) L' \left( \vec{x} \times \vec{\sigma} \right)
+ \frac{1}{2} \left( \vec{\sigma} \times \vec{\sigma} \right) \left( x^i L' + x^i L' \right) - \frac{\hbar r}{2} \left( \vec{\sigma} \times \vec{\sigma} \right) \left( x^i L' + x^i L' \right)
+ \frac{\hbar}{2} \left( \vec{x} \times \vec{p} \right) \left( x^i L' + x^i L' \right) \left( \vec{x} \times \vec{p} \right)
+ \frac{\hbar}{2} \left( \vec{x} \times \vec{p} \right) \left( x^i L' + x^i L' \right) \left( \vec{x} \times \vec{p} \right).
\]
However, the above pseudo-tensor integral of motion can be written as
\[
Y_{ij}^{(6)} = \{ X^i_V, J^j \} + \delta_{ij} \frac{\hbar^2 X_P}{2},
\]
where
\[
X_P = \frac{\left( \vec{\sigma} \times \vec{x} \right)}{r}
\]
is a first-order pseudo-scalar integral of motion and
\[
\vec{X}_V = \{ \vec{J}, X_P \}
\]
is a first-order vector integral of motion for the case \( V_1 = \frac{\hbar}{2r^2} \) and \( V_0 = V_0(r) \), (for details see; [1, 4]). Hence, we say that \( Y_{ij}^{(6)} \) is an obvious integral of motion.
III. \( c_{12} = -c_{11}^2 \).
This case yields the potential
\[
V_1 = -\frac{\hbar}{2r^2}.
\]
By following similar steps to the above cases, we find the following relations:
\[
c_{13} = 0, \quad c_{3} = 0, \quad c_{4} = 0, \quad f_{8} = -\frac{2\hbar c_{11}}{r}, \quad f_{1} = 0, \quad f_{4} = 0.
\]
Again, all the determining equations are satisfied for arbitrary scalar potential \( V_0 = V_0(r) \) and having just one arbitrary constant \( c_{11} \), we obtain the following integral of motion:
\[
Y_{ij}^{(7)} = -r \left( \vec{p} \times \vec{\sigma} \right) L' \left( \vec{p} \times \vec{\sigma} \right) L' \left( \vec{x} \times \vec{\sigma} \right) L' \left( \vec{x} \times \vec{\sigma} \right)
+ \frac{\hbar c_{11}}{r} \left( \vec{x} \times \vec{p} \right) \left( \vec{x} \times \vec{p} \right) \left( \vec{x} \times \vec{p} \right).
\]
\[- \frac{1}{r} \left( \vec{\sigma} \cdot \vec{L} + 2\hbar \right) (\chi' L' + L' \chi') + i\hbar \left( \frac{\vec{\sigma} \cdot \vec{x}}{r} - r \right) \]
\times \left( p' \sigma^j + \sigma^j p' \right) - \frac{2i\hbar}{r} (\vec{\sigma} \cdot \vec{p}) \chi' x' + \frac{1}{r} \left( 2\vec{L}^2 + i\hbar (\vec{x}, \vec{p}) - \frac{3\hbar^2}{2} \right) \]
\times (x' \sigma^j + \sigma^j x').  

(4.77)

**Subcase 2.** $c_2 \neq 0$.

With this choice, the equation (4.68) directly gives

\[ V_1 = -\frac{\hbar (f_{12} + rf'_{12})}{2c_2}, \]

and upon introducing this into (4.9), we get

\[ f_{18} = \frac{r^2 f_{12}^2}{2c_2} + c_{15}, \]

where $c_{15}$ is an integration constant. Using (4.6), (4.8) and (4.12), we obtain

\[ f_{15} = \frac{2c_2 (rc_{15} + c_{16}) - 2(c_2 + c_3) r f_{12} + r^3 f_{12}^3}{2c_2 r}, \]

\[ f_{14} = \frac{2c_2 r c_{15} f_{12} + r^3 f_{12}^3 - 2c_2 (c_2 - r c_{15}) f_{12}' + r^3 f_{12}' f_{12}'}{2c_2 r (c_2 + r^2 f_{12} + r^3 f_{12}')}, \]

where $c_{16}$ is a real constant. Note that the denominator of (4.80) is assumed to be nonvanishing. Otherwise it leads to $f_{12} = \frac{c_3}{r^2} + \frac{c_3}{r}$ for a constant $c_3$ and gives the potential $V_1 = \frac{\hbar}{2c_2}$. In order to satisfy the other determining equations we need to set $c_2 = 0$ which of course contradicts with the beginning assumption that $c_2 \neq 0$.

By substituting (4.78) and (4.79) in the (4.10), we find

\[ f_{12} = \frac{c_4 + (c_2 + c_3) r + \epsilon \sqrt{c_4^2 + 2c_4 c_3 r + r^2 (c_3^2 + c_3^2 + 2c_2 (c_4 - r (c_{15} + c_{16})) + c_{17})}}{r^3}, \]

(4.81)

where $c_{17}$ is a real constant and $\epsilon^2 = 1$. After introducing these relations into the remaining determining equations of second type, we obtain the following relations:

\[ c_4 = 0, \quad c_{16} = 0, \quad c_{15} (c_2^3 - c_3^3) (2c_2 c_3 + c_3^2 + c_{17}) = 0. \]

(4.82)

Here, we have three cases to consider.

The case $c_{15} = 0$ yields $c_{17} = -c_3^2 - 2c_2 c_3 - c_3^2$ by the help of (4.11). This implies that $f_{12} = \frac{c_2 + c_3}{r^2}$ and $f_{14} = -\frac{c_2^3 + c_3^3}{2c_2 r^4}$. If we introduce these relations into the other determining equations, we find $c_3 = \pm c_2$. Then, it is easy to see that the option $c_3 = c_2$ gives $V_1 = \frac{\hbar}{2c_2}$ while the other one $c_3 = -c_2$ causes the potential $V_1$ to be vanished.

The case $c_3^2 - c_2^3 = 0$ gives $c_{15} = 0$. Then, we are back in the above case.

Finally, for the possibility $2c_2 c_3 + c_3^2 + c_{17} = 0$ we easily find that either $c_3 = 0$ or $c_3 = \pm 2c_2$. If $c_3 = 0$, then we get $c_{15} = 0$ which yields either $f_{12} = 0$ or $f_{12} = \frac{2\hbar}{r^2}$, both of which give the excluded potentials. If $c_3 = 2c_2$, then we obtain $c_{15} = 0$. This directly yields either $f_{12} = \frac{2\hbar}{r^2}$ or $f_{12} = \frac{4\hbar}{r^2}$. However, we do not need to continue the analysis here since the first
case yields a known potential $V_1 = \hbar r^2$ and the second one gives $V_1 = 2\hbar r^2$ which has already been investigated in subcase 1 of case 1. Similarly, the case $c_3 = -2c_2$ yields either $V_1 = 0$ or $V_1 = -\frac{\hbar}{r^2}$ which has been analyzed in subcase 2 of case 1.

The above facts state that no new pseudo-tensor integrals of motion are found in this case.

5. Conclusions

A classification of superintegrable systems preserving rotational invariance, involving spin interaction and admitting second-order integrals of motion was pursued in [1]. However, the results presented in [1] were restricted to scalar, pseudo-scalar, vector and axial vector integrals of motion. This paper serves as a continuation of [1] by considering integrals that are two index tensors and pseudo-tensors. Hence, the classification of such superintegrable systems is completed.

After a complete analysis, we found six different tensor integrals which can be seen in (3.53), (A.1)–(A.5). However, the first one exists for all $V_0(r)$ and $V_1(r)$ whereas the others correspond to the gauge induced potential $V_1 = \hbar r^2$. Among the tensor integrals for the gauge induced potential, (A.1), (A.3) and (A.5) exist for any scalar potential $V_0(r)$. However, the integrals (A.2) and (A.4) exist only for the scalar potential $V_0(r) = \alpha r^2 + \beta r^2$. Such a system can be viewed as a deformation of the harmonic oscillator ($V_0(r) = \alpha r^2$). This naturally should bring to mind a well-known integral of motion, the quadrupole tensor [7] (also known as the Fradkin tensor [71]) for the harmonic oscillator. Notice that in the limit $\hbar \to 0$ the tensor integral (A.2) gives the Fradkin tensor for the spinless case whereas (A.4) vanishes. We conclude that no nontrivial second-order tensor integrals of motion exist for the system (2.1).

On the other hand, 12 different pseudo-tensor integrals of motion have been obtained, seven of which are nontrivial (see (4.56), (4.57), (4.60), (4.61), (4.75)–(4.77)). However, it has been demonstrated that the integral given in (4.76) can be obtained from a first-order pseudo-scalar integral and a first-order vector integral, so it is an obvious integral of motion.

The other five integrals given in (A.6)–(A.10) correspond to the gauge induced potential. Among these integrals, (A.6)–(A.9) do exist for $V_0 = \frac{\hbar}{r^2}$. There are no pseudo-tensor integrals of motion for arbitrary scalar potential $V_0(r)$. In the case of $V_0 = \frac{\hbar}{r^2} + \alpha r^2$, we have the pseudo-tensor integral (A.10). This superintegrable system represents a deformation of the Kepler–Coulomb one $V_0 = \frac{\hbar}{r^2}$. It is easy to see that (A.10) reduces to zero for the spinless case.

All the results presented in [1] and obtained in this paper are summed up in table 1 which represents the complete list of superintegrable potentials and their integrals of motion for the current Hamiltonian system under investigation. The second column gives the spin–orbital potentials arising from the analysis. In the third column, the scalar potentials are given where $V_0(r)$ means an arbitrary potential. The results obtained in [1] are listed in the 4th, 5th and 6th columns that correspond to pseudo-scalars, vectors and axial vectors, respectively. However, we leave blank some entries corresponding to the gauge induced potential in these columns since this potential was excluded from the analysis in [1]. As expected there may be some scalar, pseudo-scalar, vector and axial vector integrals of motion for these cases. But it is not necessary to present them here since the corresponding systems are already superintegrable with or without the existence of such integrals. Finally, integrals of motion corresponding to superintegrable potentials obtained in this paper are presented in columns 7 and 8. The real constants $\alpha$ and $\beta$ are arbitrary and $\epsilon = \pm 1$. One can conclude from the table that we have four new nontrivial superintegrable potentials given in No. 3, 4, 5 and 6. The remaining one given
Table 1. Complete list of superintegrable potentials and their integrals of motion.

| No | $V_1$ | $V_0$ | Pseudo-scalars | Vectors | Axial Vectors | Tensors | Pseudo-Tensors |
|----|-------|-------|----------------|---------|---------------|---------|----------------|
| 1  | $\frac{\hbar}{r}$ | $V_0(r)$ | (2.8) | (A.1), (A.3), (A.5) | — | — | — |
|    | $\frac{\hbar^2}{r^2}$ | $\frac{\hbar^2}{r^2} + \alpha r^2$ | (2.10) | (A.1), (A.3), (A.5) | (A.6)–(A.9) | — | — |
|    | $\frac{\hbar}{r^2} - \frac{\hbar}{r}$ | — | — | — | — | — | — |
| 2  | $\frac{\hbar}{2r^2}$ | $V_0(r)$ | (B.1), (B.2) | (B.10), (B.11) | — | — | (4.76) |
|    | $\frac{\hbar^2}{2r^2} - \frac{\hbar}{r}$ | (B.1), (B.2) | (B.8)–(B.11) | — | — | — | — |
| 3  | $\frac{\hbar}{2r^2}$ | $V_0(r)$ | — | — | — | — | — |
| 4  | $\frac{3\hbar}{2r^2}$ | $\frac{\hbar^2}{2r^2} + \alpha r^2$ | (B.3) | — | — | — | — |
|    | $\frac{3\hbar}{2r^2}$ | $V_0(r)$ | — | — | — | — | (4.75) |
| 5  | $\frac{\hbar}{r^2}$ | $\alpha$ | — | — | — | — | — |
|    | $\frac{\hbar^2}{2r^2} + \alpha r^2$ | (B.4) | — | — | — | — | — |
| 6  | $\frac{\hbar}{r^2}$ | $\alpha r^2$ | — | — | — | (B.15) | — | — |
|    | $\frac{\hbar^2}{2r^2} + \alpha r^2$ | — | — | — | — | (4.56), (4.57) | — | — |
| 7  | $\frac{\hbar}{2r^2} + \beta$ | — | — | (B.12), (B.13) | — | — | — |
| 8  | $\frac{\hbar}{2r^2}$ | $1 + \frac{\hbar}{\sqrt{1+\beta r^2}}$ | $\frac{\hbar^2}{2r^2} \left( 1 + \frac{\hbar}{\sqrt{1+\beta r^2}} \right)$ | (B.5), (B.6) | (B.14) | — | — | — |
| 9  | $\frac{\hbar}{2r^2}$ | $1 + \frac{\hbar}{\sqrt{1+\beta r^2}}$ | (5.1) | (B.7) | — | — | — | — |
| 10 | $\frac{\hbar}{2r^2}$ | $r^2 \sqrt{1+\beta r^2}$ | (5.2) | (B.17) | — | — | — | — |
| 11 | $\frac{\hbar}{2r^2}$ | $1 + \frac{\hbar}{\sqrt{1+\beta r^2}}$ | (5.3) | — | — | (B.18) | — | — |
in No. 2 has been previously found in [1] for pseudo-scalars and vectors. Here, we have shown that there is also an obvious pseudo-tensor integral of motion in addition to those for this case. We give the scalar potentials coming from [1] in the rows 9, 10 and 11 of table 1 below since they do not fit into the table:

\[ V_0 = \frac{\hbar^2}{8r^2(1 + \beta r^2)^2} \left( 7 + 10r^2 \beta + 8\epsilon \left( 1 + \beta r^2 \right)^{3/2} \right) - \frac{\alpha}{4 \beta (1 + \beta r^2)}, \quad (5.1) \]

\[ V_0 = \frac{\hbar^2}{8r^2(1 + \beta r^2)^2} \left( 4 + 6\beta r^2 - r^4 \beta^2 + 4\epsilon \left( 1 + \beta r^2 \right)^2 \right) + \frac{\alpha}{1 + \beta r^2}, \quad (5.2) \]

\[ V_0 = \frac{3\hbar^2 \left( 4 + 5\beta r^2 + 4\epsilon \left( 1 + \beta r^2 \right)^3 \right)}{8r^2(1 + \beta r^2)^2} - \frac{\alpha}{2\beta (1 + \beta r^2)}. \quad (5.3) \]

A study of the algebras of the integrals of motion presented in this paper is in progress and will be presented in a future article.

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**Data availability statement**

All data that support the findings of this study are included within the article (and any supplementary files).

**Appendix A. Integrals of motion for the gauge induced potential**

In the previous sections, it has been pointed out that we excluded the special case \( V_1 = \frac{\hbar}{r^2} \) since it is a gauge induced potential. Nevertheless, we present the tensor and pseudo-tensor integrals of motion appeared in such cases.

For the gauge induced potential \( V_1 = \frac{\hbar}{r^2} \), we have the following five tensor integrals of motion:

\[ \mathcal{T}_2^{ij} = (L^i L^j + L^j L^i) + 2h(L^i \sigma^j + \sigma^i L^j) + \frac{2h}{r^2} \left( 2(\vec{\sigma}, \vec{L}) + \hbar \right) x^i x^j \]

\[ - 2h \left( p^i (\vec{x} \wedge \vec{\sigma})^j + (\vec{x} \wedge \vec{\sigma}) p^j \right) + \frac{2h}{r^2} ((\vec{x}, \vec{p}) + i\hbar) \]

\[ \times \left( x^i (\vec{x} \wedge \vec{\sigma})^j + (\vec{x} \wedge \vec{\sigma}) x^i \right), \quad (A.1) \]
\[ T^i_j = 2(p'p^j + 2\alpha x^j') - \frac{2\hbar}{r^2} \left( \frac{\hbar}{r^2} x^i x^j + (p'(\vec{x} \wedge \vec{\sigma})') + (\vec{x} \wedge \vec{\sigma})' p^i \right) \\
+ \frac{i\hbar}{r^2} (x^i (\vec{x} \wedge \vec{\sigma})' + (\vec{x} \wedge \vec{\sigma})' x^i) \right), \] (A.2)

\[ T^i_j = i\hbar (L' + \alpha') \left( (x^i p'^j + p'^i x^j) - (\vec{x}, \vec{p}) (x^i (\vec{x} \wedge \vec{\sigma})') \right) + \frac{2\hbar}{r^2} \left( \frac{\hbar^2}{r^2} (x^i (\vec{x} \wedge \vec{\sigma})' + (\vec{x} \wedge \vec{\sigma})' x^i) \right) \right), \] (A.3)

\[ T^i_j = \frac{(3\hbar + 2(\vec{\sigma}, \vec{\pi}))}{r^2} \left( \frac{2i\hbar}{r^2} x^i x^j + (x^i p'^j + p'^i x^j) \right) - (p'(\vec{\pi} \wedge \vec{\sigma})')' \]
\[ + (\vec{\pi} \wedge \vec{\sigma})' p^i) + \frac{2\hbar}{r^2} ((\vec{x}, \vec{p}) + i\hbar) (p'(\vec{x} \wedge \vec{\sigma})') + (\vec{x} \wedge \vec{\sigma})' p^i) \]
\[ + \left( \frac{2\alpha - \frac{\hbar}{r^2} (3\hbar - 2i(x, \vec{p})) \right) (x^i (\vec{x} \wedge \vec{\sigma})' + (\vec{x} \wedge \vec{\sigma})' x^i), \] (A.4)

\[ T^i_j = - (L' \sigma'^j + \alpha' L') - \frac{2\hbar}{r^2} \left( 2(\vec{\sigma}, \vec{L}) + \hbar \right) x^i x^j + 2(p'(\vec{x} \wedge \vec{\sigma})') + (\vec{x} \wedge \vec{\sigma})' p^i) \]
\[ - \frac{2\hbar}{r^2} ((\vec{x}, \vec{p}) + i\hbar) (x^i (\vec{x} \wedge \vec{\sigma})' + (\vec{x} \wedge \vec{\sigma})' x^i). \] (A.5)

We observe two scalar potentials for the gauge induced potential which are \( V_0 = \frac{\hbar^2}{2r^2} \) and \( V_0 = \frac{\hbar^2}{2r^2} + \alpha r^2 \). We see that the integrals (A.1), (A.3) and (A.5) exist for any \( V_0(r) \) or \( V_0 = \frac{\hbar^2}{2r^2} \).

On the other hand, (A.2) and (A.4) exist only for \( V_0 = \frac{\hbar^2}{2r^2} + \alpha r^2 \).

For the gauge induced potential \( V_1 = \frac{\hbar^2}{2r^2} \), we have the following pseudo-tensor integrals of motion:

\[ \mathcal{Y}^i_j = \frac{ bo}{r^2} (x^i p'^j + p'^i x^j) - \frac{ 2(\vec{x}, \vec{p}) + \hbar}{r^2} (x^i \sigma'^j + \sigma'^i x^j) \]
\[ \times (x^i p'^j + p'^i x^j) + \frac{ 2(\vec{x}, \vec{p}) + \hbar}{r^2} (x^i \sigma'^j + \sigma'^i x^j) \right) x^i x^j + \frac{ h}{2} (p' \sigma'^j + \sigma'^i p^i), \] (A.6)

\[ \mathcal{Y}^i_j = \frac{ 2(\vec{x}, \vec{p})}{r^2} (x^i p'^j + p'^i x^j) - \frac{ 2(\vec{x}, \vec{p}) + \hbar}{r^2} (x^i \sigma'^j + \sigma'^i x^j) \right) x^i x^j + \frac{ h}{2} (p' \sigma'^j + \sigma'^i p^i), \] (A.7)

\[ \mathcal{Y}^i_j = \frac{ 2(\vec{x}, \vec{p})}{r^2} (x^i p'^j + p'^i x^j - \frac{ 2(\vec{x}, \vec{p})}{r^2} (x^i \sigma'^j + \sigma'^i x^j)) \]
\[ \right) x^i x^j + \frac{ h}{2} (2(\vec{x}, \vec{p}) + \hbar) (x^i \sigma'^j + \sigma'^i x^j) + \frac{ 1}{r^2} \left( 2(\vec{x}, \vec{p}) + \hbar \right) (x^i L^I + L' x^i). \] (A.7)
X_{11}^{ij} = (p' \sigma' + \sigma' p') - \frac{2}{r^2} (\sigma, \bar{\sigma})(x^i p^j + p^i x^j) - \frac{4i \hbar}{r} (\sigma, \bar{\sigma}) x^i x^j,
(A.9)

\begin{align*}
X_{12}^{ij} &= \frac{2}{r^2} \left( -2i \hbar \left( (\bar{\sigma}, \bar{p}) + \frac{1}{r^2} (\bar{x}, \bar{p})(\bar{\sigma}, \bar{x}) \right) + \left( \frac{\alpha}{r} + \frac{\hbar^2}{r^2} \right) (\bar{\sigma}, \bar{x}) \right) x^i x^j \\
&\quad - (2i \hbar + (\bar{x}, \bar{p})) (p' \sigma' + \sigma' p') - \frac{2}{r^2} (\bar{x}, \bar{p})(\bar{\sigma}, \bar{x})(x^i p^j + p^i x^j) \\
&\quad + 4(\bar{\sigma}, \bar{x}) p' p^i - 2 \left( (\bar{p} \land \bar{\sigma}) L^j + L'(\bar{p} \land \bar{\sigma}) \right) + \frac{2}{r^2} (\bar{x}, \bar{p}) (x^i \bar{x} \land \bar{\sigma}) L^j \\
&\quad + L'(\bar{x} \land \bar{\sigma}) + \frac{2i \hbar}{r^2} (\bar{x}, \bar{p}) + \bar{\beta}^2 + \left( \frac{\alpha}{r} + \frac{3 \hbar^2}{r^2} \right) (x^i \sigma' + \sigma' x^i) \\
&\quad + \frac{2}{r^2} (\bar{\sigma}, \bar{L} + \hbar) (x^i L^j + L' x^j). 
(A.10)
\end{align*}

The last pseudo-tensor integral of motion can also be represented as

\begin{align*}
X_{12}^{ij} &= X_{10}^{ij} + \frac{\alpha}{r} \left( \frac{2}{r^2} (\sigma, \bar{x}) x^i x^j - (x^i \sigma' + \sigma' x^i) \right). 
(A.11)
\end{align*}

There are two scalar potentials for this case that are
\( V_0 = \frac{\hbar^2}{r^2} \) and \( V_0 = \frac{\hbar^2}{r^2} - \frac{\alpha}{r} \). The integrals
\( (A.6)-(A.9) \) exist for \( V_0 = \frac{\hbar^2}{r^2} \) whereas the other one \( (A.10) \) exists for the potential \( V_0 = \frac{\hbar^2}{r^2} - \frac{\alpha}{r} \).

**Appendix B. Pseudo-scalar, vector and axial vector integrals of motion**

In this appendix, we remind the pseudo-scalar, vector and axial vector integrals of motion presented in [1]. No nontrivial scalar integrals of motion exist.

**Pseudo-scalars:**

\begin{align*}
X^1_p &= \frac{(\bar{\sigma}, \bar{x}) r}{r}, 
(B.1)
\end{align*}

\begin{align*}
X^2_p &= -r(\bar{\sigma}, \bar{p}) + \frac{1}{r} (\bar{\sigma}, \bar{x}) (\bar{x}, \bar{p}) - \frac{i \hbar}{r} (\bar{\sigma}, \bar{x}), 
(B.2)
\end{align*}

\begin{align*}
X^3_p &= \frac{(\bar{\sigma}, \bar{x})}{r} \left( \frac{3 \hbar^2}{2 r^2} + 4 \alpha r^2 - 2(\bar{p}, \bar{p}) \right) + \frac{4}{r} ((\bar{x}, \bar{p}) - i \hbar) (\bar{\sigma}, \bar{p}), 
(B.3)
\end{align*}

\begin{align*}
X^4_p &= \frac{(\bar{\sigma}, \bar{x})}{r} \left( -\frac{5 \hbar^2}{2 r^2} + 4 \alpha r^2 - 2 \frac{2 i \hbar}{r^2} (\bar{x}, \bar{p}) - 2(\bar{p}, \bar{p}) + \frac{8}{r^2} (\bar{x}, \bar{p}) (\bar{p}, \bar{p}) \right) \\
&\quad - \frac{4}{r} ((\bar{x}, \bar{p}) - 2i \hbar) (\bar{\sigma}, \bar{p}), 
(B.4)
\end{align*}

\begin{align*}
X^5_p &= -\frac{1}{\beta} \sqrt{1 + \beta r^2} (\bar{\sigma}, \bar{p}) + \frac{(\bar{\sigma}, \bar{x})}{\epsilon + \sqrt{1 + \beta r^2}} ((\bar{x}, \bar{p}) - i \hbar), 
(B.5)
\end{align*}

\begin{align*}
X^6_p &= \frac{2}{r^2} \left( 1 + \epsilon \sqrt{1 + \beta r^2} \right) (\bar{\sigma}, \bar{x}) (\hbar^2 + 3i \hbar (\bar{x}, \bar{p}) - (\bar{x}, (\bar{x}, \bar{p}) \bar{p})). 
(B.6)
\end{align*}
\[
\begin{align*}
\vec{X}_p^2 &= \frac{2 r^4 \alpha (1 + \beta r^2) - \hbar^2 (1 + 2 \beta r^2) \left( 1 + 4 \epsilon \sqrt{1 + \beta r^2} + 4 \beta \left( 1 + \epsilon \sqrt{1 + \beta r^2} \right) \right)}{2 r^3 (1 + \beta r^2)^2} (\vec{\sigma}, \vec{\bar{x}}) \\
&- \frac{2 i \hbar}{r} \left( 5 + 3 \beta r^2 - \frac{\epsilon}{\sqrt{1 + \beta r^2}} + 6 \epsilon \sqrt{1 + \beta r^2} \right) (\vec{\sigma}, \vec{x})(\vec{\bar{p}}, \vec{\bar{p}}) \\
&+ \frac{2 i \hbar}{r} \left( 1 + \beta r^2 - \frac{\epsilon}{\sqrt{1 + \beta r^2}} + 4 \epsilon \sqrt{1 + \beta r^2} \right) (\vec{\sigma}, \vec{p}) \\
&- \frac{2 r}{1 + \beta r^2} (\vec{\sigma}, \vec{x})(\vec{\bar{p}}, \vec{\bar{p}}) + \frac{2 r}{r} \left( 2 + \beta r^2 + 2 \epsilon \sqrt{1 + \beta r^2} \right) (\vec{\sigma}, \vec{x}) \\
&\times (\vec{x}, (\vec{\bar{x}}, \vec{\bar{p}}))(\vec{\sigma}, \vec{\bar{p}}). \quad (B.7)
\end{align*}
\]

**Vectors:**

\[
\begin{align*}
\vec{X}_\nu &= - \left( \vec{\sigma}, \vec{L} \right) \vec{p} + \frac{3 \hbar}{2} \vec{p} - \frac{\hbar \vec{x}}{2r^2} (\vec{x}, \vec{p}) + \frac{i \hbar}{2} (\vec{\sigma} \wedge \vec{p}) + \frac{\hbar^2}{2} \left( \frac{\vec{x}}{2r^2} \right) \\
&+ \frac{4 \alpha \tau - \frac{\hbar^2}{2}}{4 r^2} (\vec{x} \wedge \vec{\sigma}), \quad (B.8)
\end{align*}
\]

\[
\begin{align*}
\vec{X}_\nu &= 2 \vec{x} \vec{p}^2 - 2 (\vec{x}, \vec{p}) \vec{p} - \frac{\hbar \vec{x}}{r^2} \left( \vec{\sigma}, \vec{L} \right) + 2 i \hbar \vec{p} - h (\vec{\sigma} \wedge \vec{p}) + \frac{\hbar^2}{2} \left( \frac{\vec{x} \wedge \vec{\sigma}}{2r^2} \right) \\
&+ \frac{\vec{x}}{r^2} (\vec{\sigma} \wedge \vec{x}), \quad (B.9)
\end{align*}
\]

\[
\begin{align*}
\vec{X}_\nu &= (\vec{\sigma}, \vec{\bar{p}}) \vec{L} + \frac{\hbar}{2 r} (\vec{x} - i (\vec{x} \wedge \vec{\sigma})), \quad (B.10)
\end{align*}
\]

\[
\begin{align*}
\vec{X}_\nu &= r \vec{L}(\vec{\sigma}, \vec{\bar{p}}) + \frac{\hbar r}{2} \vec{p} - \frac{i \hbar r}{2} (\vec{\sigma} \wedge \vec{p}) + \vec{X}_\nu \left( \vec{x} \right) (\vec{\bar{p}} - (\vec{x}, \vec{p})) \vec{p} - h (\vec{\sigma} \wedge \vec{\bar{p}}), \quad (B.11)
\end{align*}
\]

\[
\begin{align*}
\vec{X}_\nu &= \vec{x} \left( 2 \vec{p}^2 - \left( \frac{2 \beta r^2 - \alpha}{r^2} \right) \right) \left( \left( \vec{\sigma}, \vec{L} \right) + h \right) + 2 \left( \vec{x} \right) (\vec{\sigma} \wedge \vec{\bar{p}}) \vec{p} - h (\vec{\sigma} \wedge \vec{\bar{p}}) \\
&+ i \hbar \left( \frac{2 \beta r^2 - \alpha}{2 r^2} \right) (\vec{\sigma} \wedge \vec{x}), \quad (B.12)
\end{align*}
\]

\[
\begin{align*}
\vec{X}_\nu &= \left( \frac{1}{2} (2h + \alpha - 2 \beta r^2) + \left( \vec{\sigma}, \vec{L} \right) \right) \vec{p} + \vec{x} \frac{2 \beta r^2 - \alpha}{2 r^2} \left( \vec{x}, \vec{p} \right) - i \hbar \\
&+ \frac{i \hbar}{2} (\vec{\sigma} \wedge \vec{\bar{p}}) - \frac{h}{4} \left( \frac{2 \beta r^2 - \alpha}{r^2} \right) (\vec{\sigma} \wedge \vec{x}), \quad (B.13)
\end{align*}
\]

\[
\begin{align*}
\vec{X}_\nu &= \{ \vec{X}_\nu, \vec{J} \}. \quad (B.14)
\end{align*}
\]
Axial vectors:
\[
\tilde{X}^1_A = - \left( 2 \alpha r^2 + \tilde{\beta}^2 \right) \hat{\sigma} + 2 (\hat{\sigma}, \bar{\hat{p}}) \bar{\hat{p}} + \frac{2}{r^2} (\hat{x}, (\hat{\sigma}, \bar{\hat{x}}) \bar{\hat{x}}^2 + 2 \hbar (\hat{\sigma}, \bar{\hat{p}})
- 2(\bar{\hat{x}}, \bar{\hat{p}}) (\hat{\sigma}, \bar{\hat{p}}) + i \hbar (\hat{\sigma}, \bar{\hat{p}}) - (\hat{\sigma}, \bar{\hat{x}}) \bar{\hat{x}})) \right), \\
\tilde{X}^2_A = \left( 3 \tilde{\beta}^2 - 2 \alpha r^2 + \frac{4}{r^2} (i \hbar, \bar{\hat{x}}) \bar{\hat{x}}^2 - (\hat{x}, (\bar{\hat{x}}, \bar{\hat{p}} \bar{\hat{p}})) \right) \hat{\sigma} - \frac{2}{r^2} \left( \hbar + (\hat{L}, \hat{\sigma}) \right) \hat{L}
- 2 \left( (\hat{\sigma}, \bar{\hat{p}}) - \frac{3 \hbar (\hat{\sigma}, \bar{\hat{x}})}{r^2} \right) \bar{\hat{p}} + \frac{2}{r^2} \left( 3 \hbar r^2 (\hat{\sigma}, \bar{\hat{p}}) - (\hat{\sigma}, \bar{\hat{x}}) (3 \hbar^2 - 2r^4 \alpha
+ 2r^2 \tilde{\beta}^2 + 12i \hbar (\hat{x}, \bar{\hat{p}})) - 4(\hat{x}, (\bar{\hat{x}}, \bar{\hat{p}}) \bar{\hat{p}}) \right),
\]
\[
\tilde{X}^3_A = \left( 2 \hbar (\hat{\sigma}, \bar{\hat{x}}) \tilde{Q} - 4 \varepsilon \sqrt{1 + \beta r^2} (\hat{\sigma}, \bar{\hat{p}}) \right) \bar{\hat{p}} - \frac{4}{r^2} \left( \hbar + 2 \alpha r^2
+ \frac{\varepsilon \hbar \sqrt{1 + \beta r^2}}{\sqrt{1 + \beta r^2}} + q(\hat{\sigma}, \tilde{L}) \right) \tilde{L}
+ \frac{2}{r^2} \left( 2i \hbar q(\hat{x}, \bar{\hat{p}}) - \tilde{Y} - 2q(\hat{x}, (\bar{\hat{x}}, \bar{\hat{p}}) \bar{\hat{p}}) + r^2 (2q - 1) \bar{\hat{p}}^2 \right) + \frac{2}{r^2} \left( i \hbar \tilde{Q} \right)
+ 4 \varepsilon \sqrt{1 + \beta r^2} (\hat{x}, \bar{\hat{p}}) \left( \frac{4q(\hat{x}, (\bar{\hat{x}}, \bar{\hat{p}}) \bar{\hat{p}}) - 2r^2 \left( 2q - 1 - \varepsilon \sqrt{1 + \beta r^2} \right) \bar{\hat{p}}^2
- Z - 4i \hbar W(\hat{x}, \bar{\hat{p}}) + i \hbar \tilde{Q}(\hat{\sigma}, \bar{\hat{p}}) \right),
\]

where \( Q_\pm, q, \tilde{Q}, \tilde{Y}, Z \) and \( W \) are given by the following relations
\[
\begin{align*}
Q_\pm &= 1 + \frac{\beta}{2} r^2 \pm \frac{3 + 4 \beta r^2}{\sqrt{1 + \beta r^2}}, \\
q &= 1 + \frac{\beta}{2} r^2 + \varepsilon \sqrt{1 + \beta r^2},
\end{align*}
\[
\begin{align*}
Y &= \frac{-4 \hbar^2 - 8r^2 \alpha - 6 \hbar^2 r^2 \beta - 8r^4 \alpha \beta + \hbar^2 r^4 \beta^2}{(1 + \beta r^2)^2} = \frac{-4 \varepsilon \hbar^2}{\sqrt{1 + \beta r^2}}, \\
\tilde{Q} &= 3 + \frac{5 \beta}{2} r^2 + \frac{3 + 4 \beta r^2}{\sqrt{1 + \beta r^2}}, \\
W &= 3 + 2 \beta r^2 + \varepsilon \frac{6 + 7 \beta r^2}{2 \sqrt{1 + \beta r^2}}, \\
\tilde{Y} &= \frac{\hbar^2 r^4 (4 \alpha - 6 \beta^2) + \hbar^2 r^6 (3 \beta^3 + 4 \beta^2 \alpha - 6 \beta^2 \alpha \beta ^2)}{4(1 + \beta r^2)^2} + \frac{2 \varepsilon \hbar^2 \beta r^2}{\sqrt{1 + \beta r^2}}, \\
Z &= \frac{-4r^4 \alpha (1 + \beta r^2) + 3 \hbar^2 (2 + \beta r^2 (7 + 6 \beta r^2))}{2(1 + \beta r^2)^2} + \frac{\varepsilon \hbar^2 (1 + 2 \beta r^2)}{\sqrt{1 + \beta r^2}}.
\end{align*}
\]
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