The Dilating Method for Cayley digraphs on finite Abelian groups *

F. Aguiló\textsuperscript{a}, M.A. Fiol\textsuperscript{b} and S. Pérez\textsuperscript{c}

\textsuperscript{a,\textsuperscript{b,\textsuperscript{c}Departament de Matemàtiques
Universitat Politècnica de Catalunya
Jordi Girona 1-3 , Mòdul C3, Campus Nord
08034 Barcelona, Catalonia (Spain).}
\textsuperscript{bBarcelona Graduate School of Mathematics
Barcelona, Catalonia.}

October 8, 2018

Abstract

A geometric method for obtaining an infinite family of Cayley digraphs of constant density on finite Abelian groups is presented. The method works for any given degree and it is based on consecutive dilates of a minimum distance diagram associated with a given initial Cayley digraph. The method is used to obtain infinite families of dense or asymptotically dense Cayley digraphs. In particular, for degree $d = 3$, an infinite family of maximum known density is proposed.

Keywords. Cayley digraph, Abelian group, Degree/diameter problem, Congruences in $\mathbb{Z}^n$, Smith normal form.

AMS subject classifications. 05012, 05C25.

1 Introduction

Some discrete problems are studied by many authors using bare geometrical methods. Some of them are worth to mention here because of the common used tool. Namely, the so-called minimum distance diagrams (MDD's), which are defined in the next section. Some examples are in the study of numerical semigroups, the Frobenius number, the set of factorizations, and the denumerant \cite{3,7}. Also, some metric properties of Cayley digraphs of Abelian groups have been studied using MDD's, mainly the diameter and the density \cite{2,9,11,18}.

In this work we introduce the Dilating Method that is based on MDD's. This method applies to Cayley digraphs on finite Abelian groups of degree $d \geq 2$. We use it to derive dense

\*Research supported by the “Ministerio de Economía y Competitividad” (Spain) with the European Regional Development Fund under projects MTM2014-60127-P.

E-mails: francesc.aguilo@upc.edu, miguel.angel.fiol@upc.edu, sonia.perez-mansilla@upc.edu
and asymptotically dense families of Cayley digraphs. The plan of the paper is the following:

Section 2 contains main definitions and preliminary results. The Dilating Method is presented in Section 3, and some new dense and asymptotically dense families are constructed in the last section.

2 Preliminaries

Consider an integral matrix $M \in \mathbb{Z}^{n \times n}$ with $N = |\det M|$, and with Smith normal form $S = U M V$, for unimodular matrices $U, V \in \mathbb{Z}^{n \times n}$. Let us consider the Cayley digraph $G_M = \text{Cay}(\mathbb{Z}^n/M\mathbb{Z}^n, E_n)$ where $E_n = \{e_1, \ldots, e_n\}$ is the canonical basis of $\mathbb{Z}^n$.

Let $[r, s) = \{x \in \mathbb{R} : r \leq x < s\}$. Given $a = (a_1, \ldots, a_n) \in \mathbb{Z}^n$, we denote the unitary cube $[a] = [a_1, \ldots, a_n] = [a_1, a_1+1] \times \cdots \times [a_n, a_n+1] \subset \mathbb{R}^n$. In this sense, the cube $[a]$ represents the vertex $(a_1, \ldots, a_n)$ in $G_M$ with the equivalence relation of $\mathbb{Z}^n/M\mathbb{Z}^n$ given by

$$a \equiv b \pmod{M} \iff \exists \lambda \in \mathbb{Z}^n : a - b = M\lambda.$$  

We denote $[a] \sim [b]$ whenever $a \equiv b \pmod{M}$ and $[a] \not\sim [b]$ otherwise. For more details about congruences in $\mathbb{Z}^n$ and their role in the study of Cayley digraphs on Abelian groups, see [10, 12, 13].

For a given pair $a, b \in \mathbb{Z}^n$, we denote by $a \leq b$ when the inequality holds for each coordinate.

Let $N = \mathbb{Z}_{\geq 0}$ denote the set of nonnegative integers. Given $a \in N^n$, consider the set of unitary cubes $\nabla(a) = \{[b] : 0 \leq b \leq a\}$.

**Definition 1 (Hyper-L)** Given a finite Abelian group $\Gamma = \langle \gamma_1, \ldots, \gamma_n \rangle$ of order $N = |\Gamma|$, consider the map $\phi : N^n \rightarrow \Gamma$ given by $\phi(a) = a_1\gamma_1 + \cdots + a_n\gamma_n$. A hyper-L of the Cayley digraph $\text{Cay}(\Gamma, \{\gamma_1, \ldots, \gamma_n\})$, denoted by $L$, is a set of $N$ unitary cubes $L = \{[a_0], \ldots, [a_{N-1}]\}$ such that

(i) $\{\phi(a) : [a] \in L\} = \Gamma$,

(ii) $[a] \in L \Rightarrow \nabla(a) \subset L$.

Given $x \in \mathbb{Z}^n$, let us consider the $\ell_1$ norm $\|x\|_1 = |x_1| + \cdots + |x_n|$. The diameter of the hyper-L $L$ is defined to be $k(L) = \max\{\|a\|_1 : [a] \in L\}$. For the usual definition of the diameter $k(G)$ of a Cayley digraph $G = \text{Cay}(\Gamma, \langle \gamma_1, \ldots, \gamma_n \rangle)$, we have $k(G) \leq k(L)$ for every hyper-L $L$ of $G$.

**Definition 2 (Minimum distance diagram)** A minimum distance diagram $\mathcal{H}$ of the Cayley digraph $G$ is a hyper-L satisfying $\|a\|_1 = \min\{\|x\|_1 : x \in \phi^{-1}(\phi(a))\}$ for all $[a] \in \mathcal{H}$.

It follows that $k(G) = k(\mathcal{H})$ for each minimum distance diagram $\mathcal{H}$ of $G$, and also $k(G) = \min\{k(L) : L$ is a hyper-L of $G\}$. When $\Gamma$ is a cyclic group, Definition 2 is equivalent to [19, Definition 2.1] for multiloop networks.

**Example 1** The Cayley digraph $G_1 = \text{Cay}(\mathbb{Z}_{34}, \{2, 9, 35\})$ has only one minimum distance diagram $\mathcal{H}_1$ which is shown on the left hand side of Figure 7. The diameter is $k(G_1) = k(\mathcal{H}_1) = \|(0, 0, 7)\|_1 = 7$. The cube $[0, 0, 7]$ corresponds to vertex 77 in $G$. There are 13 unit cubes in $\mathcal{H}_1$ with maximum norm. Namely, $[0, 0, 7]$, $[0, 1, 6]$, $[2, 1, 4]$, $[4, 1, 2]$, $[0, 6, 1]$, $[2, 6, 1]$, $[6, 1, 2]$, $[0, 2, 7]$, $[0, 7, 2]$, $[2, 0, 7]$, $[7, 0, 2]$, and $[2, 7, 0]$. For the usual definition of the diameter $k(G)$, we have $k(G) \leq 7$. For the hyper-L $L$ of $G$, we have $k(G) = k(L)$.
Figure 1: Minimum distance diagrams of $G_1 = \text{Cay}(\mathbb{Z}_{84}, \{2, 9, 35\})$ and $G_2 = \text{Cay}(\mathbb{Z}_{84}, \{2, 9, 33\})$.

$[1, 5, 1], [2, 4, 1], [3, 3, 1], [4, 2, 1], [0, 7, 0], [2, 5, 0], [5, 2, 0],$ and $[7, 0, 0]$. These cubes are painted in light color. Each cube $a = [a_1, a_2, a_3]$ in $H_1$ corresponds to the vertex $2a_1 + 9a_2 + 35a_3$ in $\mathbb{Z}_{84}$.

The minimum distance diagram $H_2$ shown on the right hand side of Figure 1 has diameter $k(H_2) = 9$, and corresponds to the digraph $G_2 = \text{Cay}(\mathbb{Z}_{87}, \{2, 9, 33\})$. It has only two cubes with maximum norm: $[2, 6, 1]$ and $[2, 0, 7]$. They correspond to the vertices 7 and 67, respectively. Exhaustive computer search shows that the minimum diameter for a Cayley digraph on a three generated cyclic group with order 87 is 7.

For a Cayley digraph $G$ of an Abelian group, with order $N(G)$, degree $d(G)$, and diameter $k(G)$, Fiduccia, Forcade and Zito [11] defined its density as

$$\delta(G) = \frac{N(G)}{(k(G) + d(G))^d(G)}.$$  \hspace{1cm} (1)

Let us denote

$$\Delta_{d,k} = \max \{\delta(G) : d(G) = d, k(G) = k\}, \text{ and } \Delta_d = \max \{\Delta_{d,k} : d(G) = d\}.$$

Forcade and Lamoreaux proved that $\Delta_2 = 1/3$ by using a generic optimal diagram $H$ in [15, Section 4]. This diagram was already known by Fiol, Yebra, Alegre, and Valero [14] to derive the tight lower bound $k(N) \geq \text{lb}(N) = \lceil \sqrt{3N} \rceil - 2$ for a generic Cayley digraph of degree $d = 2$ and order $N$. The density $\Delta_2$ is attained by the digraphs $G_M = \text{Cay}(\mathbb{Z}^2/M\mathbb{Z}^2, E_2)$, where $M$ is the matrix with rows $(2t, -t)$ and $(-t, 2t)$ or, equivalently, $G_t = \text{Cay}(\mathbb{Z} \oplus \mathbb{Z}_{3t}, \{(1, -1), (0, 1)\})$, with $N_t = 3t^2$ and $k(G_t) = 3t - 2$ for each $t \geq 1$ (see next section). No Cayley digraph on cyclic group $\mathbb{Z}_{3t^2}$ of degree $d = 2$ attains this density for $t > 1$.

As far as we know, $\Delta_d$ remains unknown for $d > 2$. For $d = 3$ there are some main facts: Fiduccia, Forcade, and Zito in [11, Corollary 3.6] proved that $\Delta_3 \leq 3/25 = 0.12$. However, numerical evidences suggest that $\Delta_3$ would be a smaller value. The maximum density attained by known Cayley digraphs is $\delta_0 = 0.084$ and they have been found by computer search. Such digraphs are $F_0$, $F_1$ in [11 Table 1], and $F_1' \cong F_0$, $F_2$, $F_3$ in [9 Table 8.2], where:
for some constant $c$ this work we propose the Dilating Method $\mathcal{D}_G$ from a given initial dense digraph. Notice that a large value of the ratio $N_G : k(G)$ of degree $d$ and diameter $k$, can have. Let us denote $\text{lb}(d, k)$ the lower bound for $N_{d,k}$. Then, from Wong and Coppersmith [20] and Dougherty and Faber [9] Theorem 9.1], it follows that, for $d > 1$,

$$\text{lb}(d, k) = \frac{c}{d(\ln d)^{1+\log_2 c}} \frac{k^d}{d!} + O(k^{d-1}) \leq N_{d,k} < \binom{k+d}{k},$$

(2)

for some constant $c$. As far as we know, no constructions of Cayley digraphs $G$ of order $N(G) \sim \text{lb}(d, k)$ are known. Notice that a large value of the ratio $N(G)/k(G)$ does not guarantee a high density of $G$. In this work we propose the Dilating Method which allows the generation of an infinite family of dense digraphs from a given initial dense digraph.

3 The Dilating Method

For a given finite Abelian Cayley digraph $G = \text{Cay}(\Gamma, \{\gamma_1, \ldots, \gamma_n\})$ of degree $d = n$, with minimum distance diagram $\mathcal{H}$, there is an integral matrix $M \in \mathbb{Z}^{n \times n}$ such that

$$G \cong \text{Cay}(\mathbb{Z}^n/M \mathbb{Z}^n, E_n) \cong \text{Cay}(\mathbb{Z}_{s_1} \oplus \cdots \oplus \mathbb{Z}_{s_n}, \{u_1, \ldots, u_n\}),$$

where $S = \text{diag}(s_1, \ldots, s_n) = UMV$ is the Smith normal form of $M$, $E_n = \{e_1, \ldots, e_n\}$ is the canonical basis of $\mathbb{Z}^n$ and $\{u_1, \ldots, u_n\}$ are the column vectors of the matrix $U$. If the set of column vectors of $M$ is $C_M = \{m_1, \ldots, m_n\}$, then it is well-known that $\mathcal{H}$ tessellates $\mathbb{R}^n$ by translation through the vectors of $C_M$. The map $\psi(x) = Ux$ plays an important rôle because of the equivalence

$$a \equiv b \pmod{M} \iff \psi(a) \equiv \psi(b) \pmod{S}$$

and so, it follows that $[a] \sim [b] \iff \psi(a) \equiv \psi(b) \pmod{S_{s_1} \oplus \cdots \oplus S_{s_n}}$. For more details, see [10] [13].

Example 2 Let us consider again the Cayley digraph $G_1$ and its minimum distance diagram $\mathcal{H}_1$ of Example 7. In this case we have $S = \text{diag}(1, 1, 84)$ and

$$UMV = \begin{pmatrix} 0 & 0 & 1 \\ -2 & 1 & 10 \\ 7 & -3 & -38 \end{pmatrix} \begin{pmatrix} 1 & 2 & -6 \\ 5 & 2 & 4 \\ 2 & -2 & 3 \end{pmatrix} \begin{pmatrix} 2 & 1 & 26 \\ 0 & 1 & 23 \\ -1 & 0 & -2 \end{pmatrix} = S.$$

Thus, $\mathcal{H}_1$ tessellates $\mathbb{R}^3$ through the lattice generated by $\{(1, 5, 2)^T, (2, 2, -2)^T, (-6, 4, 3)^T\}$ and $G_1 \cong \text{Cay}(\mathbb{Z}^3/M \mathbb{Z}^3, E_3)$. It is not difficult to see by computer that $[a] \not\sim [b] \pmod{1 \oplus 5 \oplus -6}$. Hence, $G_1$ is not isomorphic to any other digraph.
Given a unit cube $[a]$, consider the dilate $t[a]$ defined by

$$t[a] = \{t(a + (\alpha_1, \ldots, \alpha_n)) : 0 \leq \alpha_1, \ldots, \alpha_n \leq t - 1\}$$ \hspace{1cm} (3)

for $t \geq 1$, $t \in \mathbb{N}$. Notice that $t[a] \cap t[b] = \emptyset$ whenever $[a] \neq [b]$.

**Definition 3 (The dilation of a hyper-L)** Let $L$ be a hyper-L of some Cayley digraph. The $t$-dilate $tL$ of $L$ is defined by $tL = \{t[a] : [a] \in L\}$.

This definition corresponds to a dilatation of $L$ in such a way that each unit cube $[a]$ in $L$ is dilated to another cube, $t[a]$, of side $t$ in $tL$ according to (3).

**Lemma 1** Given a hyper-L $L$ of the digraph $G_M$, consider the dilate $tL$ for a given integer $t \geq 1$. Then, $x \not\equiv y \pmod{tM}$ for any pair of cubes $[x], [y] \in tL$.

**Proof.** There are three different types of pairs of unit cubes, $[x]$ and $[y]$, in $tL$. That is, for two different cubes $[a]$ and $[b]$ in $L$,

(a) $x = ta$ and $y = tb$,

(b) $x = ta$ and $y = tb + (\beta_1, \ldots, \beta_n)^\top$ with $0 \leq \beta_1, \ldots, \beta_n < t$ and $\beta_1 + \cdots + \beta_n > 0$,

(c) $x = ta + (\alpha_1, \ldots, \alpha_n)^\top$ and $y = tb + (\beta_1, \ldots, \beta_n)^\top$ with $0 \leq \alpha_i, \beta_i < t$ for $1 \leq i \leq n$, $\alpha_1 + \cdots + \alpha_n > 0$ and $\beta_1 + \cdots + \beta_n > 0$.

(a) Let us assume $[ta] \sim [tb]$ in $tL$ for $[a], [b] \in L$. Then, $ta \equiv tb \pmod{tM}$ and $ta = tb + tM\lambda$ for some $\lambda \in \mathbb{Z}^n$. Thus, equality $a = b + M\lambda$ holds and $[a] \sim [b]$ in $L$, a contradiction.

(b) If $[ta] \sim [tb] + (\beta_1, \ldots, \beta_n)^\top$, then $t(a - b) \equiv (\beta_1, \ldots, \beta_n)^\top \pmod{tM}$. Thus, the identity $t(a - b - M\lambda) = (\beta_1, \ldots, \beta_n)^\top$ for some $\lambda \in \mathbb{Z}^n$, leads to contradiction because none of the $\beta_i$'s are multiple of $t$ (and at least one of them does not vanish).

(c) Analogously, we get $t(a - b - M\lambda) = (\beta_1 - \alpha_1, \ldots, \beta_n - \alpha_n)^\top$ for some $\lambda \in \mathbb{Z}^n$. This identity does not hold except for the case $\beta_1 - \alpha_1 = \cdots = \beta_n - \alpha_n = 0$. In this case we have $[a] \sim [b]$ in $L$, which contradicts Definition 3 for $L$ to be a hyper-L. □

**Theorem 1 (The Dilating Method)** For an integer $t \geq 1$,

(a) $L$ is a hyper-L of $G_M \Leftrightarrow tL$ is a hyper-L of $G_{tM}$.

(b) $k(tL) = t(k(L) + n) - n$.

(c) $H$ is an MDD of $G_M \Leftrightarrow tH$ is an MDD of $G_{tM}$.

(d) $k(G_{tM}) = t(k(G_M) + n) - n$.

**Proof.** (a) The set $tL = \{t[a] : [a] \in L\}$ fulfils property (ii) of Definition 3 provided that $L$ is a hyper-L. Now we have to check property (i). Notice that, for the digraph $G_M$, the map $\phi$ is the ‘identity’ map $\phi(x) = x_se_1 + \cdots + x_ne_n \in \mathbb{Z}^n/M\mathbb{Z}^n$ for $x \in \mathbb{N}^n$. By Lemma 1 we have $[x] \not\sim [y]$ for any pair of unit cubes $[x], [y] \in tL$. Assume $N = \text{vol}(L) = |\det M|$. Then,
vol(\(t\mathcal{L}\) = \(t^n N = |\det(tM)|\) and condition (i) holds. Analogous arguments can be used to show that \(\mathcal{L}\) is a hyper-L of \(G_M\) whenever \(t\mathcal{L}\) is a hyper-L of \(G_{tM}\).

(b) Let us assume \(k(\mathcal{L}) = \|a\|_1\) for some \([a] \in \mathcal{L}\). Then, by construction of \(t\mathcal{L}\), we have \(k(t\mathcal{L}) = \|t\mathbf{a} + (t - 1, \ldots, t - 1)^\top\|_1 = \sum_{i=1}^n (tai + t - 1) = t\|a\|_1 + n(t - 1) = (k(\mathcal{L}) + n) - n\).

(c) Assume \(\mathcal{H}\) is an MDD of \(G_M\). From (a), the set of cubes \(t\mathcal{H}\) is a hyper-L of \(G_{tM}\). Now we have to check the optimal property of Definition 2.

\[\|a\|_1 = \min\{\|x\|_1 : x \in \mathbb{N}^n, x \equiv a \, (\text{mod } tM)\}\] for each \([a] \in t\mathcal{H}\).

There are two kinds of unit cubes in \(t\mathcal{H}\). Namely,

(c.1) \([ta]\) with \([a] \in \mathcal{H}\);

(c.2) \([ta + (a_1, \ldots, a_n)^\top]\) with \([a] \in \mathcal{H}\), \(0 \leq a_1, \ldots, a_n < t\), and \(a_1 + \cdots + a_n > 0\).

Let us fix the value \(t\) and consider a type-(c.1) cube \([ta] \in t\mathcal{H}\). Those \(x \in \mathbb{N}^n\) equivalent to \(ta\) in \(\mathbb{Z}^n / tM\mathbb{Z}^n\) are \(x = ta + t\lambda\) for \(\lambda \in \mathbb{Z}^n\). Then,

\[
\min\{\|x\|_1 : x \in \mathbb{N}^n, x \equiv ta \, (\text{mod } tM)\} = \min\{\|ty\|_1 : y \in \mathbb{N}^n, y \equiv a \, (\text{mod } M)\}
= t\min\{\|y\|_1 : y \in \mathbb{N}^n, y \equiv a \, (\text{mod } M)\}
= t\|a\|_1 = \|ta\|_1,
\]

where the last line is a consequence of \(\mathcal{H}\) to be a minimum distance diagram and \([a] \in \mathcal{H}\).

Consider now a type-(c.2) cube \([ta + (a_1, \ldots, a_n)^\top] \in t\mathcal{H}\). Set \(b_t = ta + \alpha\) with \(\alpha = (a_1, \ldots, a_n)^\top\). Assume that there is some \(y_t \equiv b_t \, (\text{mod } tM)\) with \(y_t \in \mathbb{N}^n\) and \(\|y_t\|_1 < \|b_t\|_1\) (and \(y_t \notin t\mathcal{H}\)). Then, \(y_t = b_t + t\lambda\) for some \(\lambda \in \mathbb{Z}^n\) and \(y_t = t(a + M\lambda) + \alpha = c_t + \alpha\) where \(c_t\) must have all its entries positive for being \(y_t \in \mathbb{N}^n\) (all the entries of \(c_t\) are multiple of \(t\) and \(0 \leq a_i < t\) for all \(i\)) and \(c_t \equiv ta \, (\text{mod } tM)\). Then,

\[
\|y_t\|_1 = \|c_t + \alpha\|_1 = \|c_t\|_1 + \|\alpha\|_1 \geq \|ta\|_1 + \|\alpha\|_1 = \|ta + \alpha\|_1 = \|b_t\|_1,
\]

which is a contradiction. Therefore, \(t\mathcal{H}\) is an MDD of \(G_{tM}\).

Assume now \(t\mathcal{H}\) is an MDD of \(G_{tM}\). Thus, \(t\mathcal{H}\) is a hyper-L of \(G_{tM}\) and, by (a), \(\mathcal{H}\) is a hyper-L of \(G_M\). Moreover, from \([ta] \in t\mathcal{H}\) for each \(a \in \mathcal{H}\), we have

\[\|ta\|_1 = \min\{\|x\|_1 : x \in \mathbb{N}, x \equiv ta \, (\text{mod } tM)\} = \min\{\|ty\|_1 : y \in \mathbb{N}, y \equiv a \, (\text{mod } M)\}\]

that is equivalent to \(\|a\|_1 = \min\{\|y\|_1 : y \in \mathbb{N}, y \equiv a \, (\text{mod } M)\}\). So \(\mathcal{H}\) is also an MDD of \(G_M\).

(d) Finally, the equality \(k(G_{tM}) = t(k(G_M) + n) - n\) is a direct consequence of (b) and (c).

Take an initial Cayley digraph \(G_1 = G_M\) of order \(N\), degree \(n\) and diameter \(k\) with minimum distance diagram \(\mathcal{H}\). From Theorem 3 we know that the dilates \(t\mathcal{H}\) are minimum distance diagrams related to the digraphs \(G_t = G_{tM}\) of order \(N_t = t^n N\) and diameter \(k(G_{tM}) = t(k + n) - n\). Thus, their densities are the same for any integral value \(t \geq 1\).

\[
\delta(G_{tM}) = \frac{N_t}{(k(G_{tM}) + n)^n} = \frac{t^n N}{(t(k(G_M) + n))^n} = \frac{N}{(k(G_M) + n)^n} = \delta(G_M).
\]
Now we are ready to use the Dilating Method for obtaining dense families of digraphs. Choose an initial (dense) digraph $G_1$ (perhaps found by computer search). Find any related minimum distance diagram $H$ and the (column) vectors $C_M$ defining the tessellation by $H$. Then, from the digraph isomorphism $G_1 \cong G_M$, it is also obtained an infinite family of (dense) digraphs $G_t = G_{tM}$ (with the same density).

4 New dense families

The criterion for a Cayley digraph $G_M$ to be dense is not established and it is applied in the sense that $\delta(G_M)$ is as large as possible. This criterion is closely related to the degree-diameter problem for these digraphs. The parameter $\alpha = \alpha(G_M)$, where

$$N(G_M) = \alpha(G_M)k(G_M)^d + O(k(G_M)^{d-1}),$$

is also taken into account for a given infinite family of digraphs of degree $d$ and diameter $k(G_M)$. In this context, notice that by (1) $\alpha(G) = \lim_{k \to \infty} \delta(G)$.

In fact, for a fixed degree $d = n$, several authors have proposed some infinite families of digraphs with good related values of $\alpha$, see Table 1. All these proposals are concerned with finite cyclic groups. The value $\alpha = 0.0807$ applies only for diameters $k = 22t + 12$ with $t \not\equiv 2, 7$ (mod 10). The case $\alpha = 0.084$ applies only for diameters $k \equiv 2$ (mod 30).

| Paper                              | $\alpha$   |
|------------------------------------|------------|
| Gómez, Gutiérrez & Ibeas [16] (2007) | 0.0370     |
| Hsu & Jia [17] (1994)              | 0.0620     |
| Aguiló, Fiol & Garcia [2] (1997)   | 0.0740     |
| Chen & Gu [8] (1992)               | 0.0780     |
| Aguiló [1] (1999)                  | 0.0807     |
| Aguiló, Simó & Zaragozá [6] (2001) | 0.0840     |

Table 1: Several proposals for degree $d = 3$

A result of Dougherty and Faber [9, Corollary 8.2] gives the existence of Abelian Cayley digraphs of degree $d = 3$ and order $N = 0.084k^3 + O(k^2)$ for all $k$. However, as far as we know, there is no explicit infinite family satisfying these conditions. It is also worth to mention the work of Rödseth [18] on weighted loop networks, who gave sharp lower bounds for the diameter and mean distance for degree $d = 2$ and general bounds for degree $d = 3$.

The dense family given by Aguiló, Simó, and Zaragozá [6] can be extended to a more general one. Using the same notation as in [6], take the integral matrix $M(m,n)$ given by

$$M(m,n) = \begin{pmatrix}
  n & m & -2m - 2n \\
  3n + m & m & m + 2n \\
  2n & -m & m + n
\end{pmatrix}.$$

Proposition 1 ([6]) Consider the Cayley digraph $G_{m,n} = G_{M(m,n)}$. Then, the order $N_{m,n}$ and diameter $k_{m,n}$ of $G_{m,n}$ are given by

$$N_{m,n} = m^3 + 12m^2n + 14mn^2,$$

$$k_{m,n} \leq \max\{m + 8n - 3, 3m + 4n - 3, 5m - 3\}. \quad (7)$$
The case $M(2,1)$ is given in Example 2 with $N_{2,1} = 84$ and $k_{2,1} \leq 7$. The diameter $k = 7$ is the minimum diameter a digraph $\text{Cay}(\mathbb{Z}_{84}, \{a,b,c\})$ can achieve. In [6, Proposition 3] it is stated that, for $x \equiv 0 \pmod{3}$, the digraph $G_M(2x+1,x)$ is isomorphic to a cyclic Cayley digraph and an explicit family is given. In the following result we extend this family to any value of $x$.

**Proposition 2** The Cayley digraph

$$G_M(2x+1,x) = \text{Cay}(\mathbb{Z}_{84^3 + 74x^2 + 18x + 1}, \{-21x^2 - 15x - 2, 21x^2 + 8x, -42x^2 - 23x - 3\})$$

has diameter $k_{2x+1,x} \leq 10x + 2$, for all integral value $x \geq 1$.

**Proof.** The result follows from the Smith normal form decomposition of the matrix $M(2x+1,x)$ in (6)

$$S_x = \text{diag}(1, 1, 84x^3 + 74x^2 + 18x + 1) = U_x M(2x+1,x) V_x$$

with unimodular matrices

$$U_x = \begin{pmatrix} -1 & 1 & -2 \\ -3x - 1 & 3x & -6x - 1 \\ -21x^2 - 15x - 2 & 21x^2 + 8x & -42x^2 - 23x - 3 \end{pmatrix},$$

$$V_x = \begin{pmatrix} 1 & 1 & -12x^2 - 10x - 2 \\ 0 & -1 & 12x^2 + 6x + 1 \\ 0 & 1 & -12x^2 - 6x \end{pmatrix},$$

and the bounded expression (7) of the diameter in Proposition 1. □

Although Proposition 2 states that the inequality $k_{2x+1,x} \leq 10x + 2$ holds, numerical evidences point out to equality for $x \geq 2$. Thus,

$$N_{2x+1,x} = \frac{1}{1000} (84k_{2x+1,x}^3 + 236k_{2x+1,x}^2 - 152k_{2x+1,x} - 312)$$

and so $N_{2x+1,x} = 0.084k_{2x+1,x}^3 + O(k_{2x+1,x}^2)$ for $x \geq 2$, whence $\alpha(G_M(2x+1,x)) = 0.084$. Although the parameter $\alpha$ of this family attains the value $\alpha_{2x+1,x} = 0.084$, the density of each of its members does not.

Now we give other dense families using the Dilating Method of the previous section.

**Lemma 2** Consider $M \in \mathbb{Z}^{n \times n}$ with Smith normal form $S = U M V$. Then, for an integral value $t \geq 1$, the Smith normal form of $tM$ is $tS = U(tM)V$.

**Proof.** This equality is a direct consequence of the matrix product. □

This lemma states that the set of generators is preserved when applying the Dilating Method. Now we apply this method to several cases, Proposition 3 deals with a dense infinite family of digraphs of degree $d = 3$ whereas Proposition 4 gives an infinite family for any degree $d \geq 2$.

Consider the integral matrices given in Example 2, $S$, $U$, $M$ and $V$. The hyper-L $H$ given in Example 1 is a minimum distance diagram related to the Cayley digraph $G_M$. This fact can be checked by computer. Now we can apply Theorem 1 and Lemma 2 to obtain the following result.
Proposition 3 The family of Abelian Cayley digraphs

\[ G_t = \text{Cay}(\mathbb{Z}_t \oplus \mathbb{Z}_t \oplus \mathbb{Z}_{84t}, \{(1, 10, -38), (0, 1, -3), (0, -2, 7)\}) \]

has order \( N_t = 84t^3 \) and diameter \( k_t = 10t - 3 \) for any integral value \( t \geq 1 \).

Clearly, from (4), the density of this family is the constant value \( \delta_t = 0.084 \) and, from \( t = \frac{k_t + 3}{10} \), in this case, we also obtain the parameter \( \alpha_t = 0.084 \). It can be checked that the known Cayley digraphs of maximum density for degree \( d = 3 \), listed in Section 2, are contained in the previous family. That is, \( F_0 \cong G_1, F_1 \cong G_2, F_2 \cong G_3 \) and \( F_3 \cong G_4 \).

Notice that the Dilating Method generates an infinite family of non-cyclic Cayley digraphs excepting, perhaps, the initial one which depends on the selected digraph to apply the method.

Example 3 An MDD related to \( G_2 \) of Proposition 3 is shown in Figure 2. From the geometrical point of view, this diagram can be seen as the 2-dilate of the diagram \( H_1 \) related to \( G_1 \) of Figure 1, where each unitary cube has been dilated into four regular unitary cubes. The 13 cubes with maximum norm \( 17 = k(G_2) \) are \([1, 1, 15], [1, 3, 13], [5, 3, 9], [9, 3, 5], [1, 13, 3], [3, 11, 3], [5, 9, 3], [7, 7, 3], [9, 5, 3], [1, 15, 1], [5, 11, 1], [11, 7, 1], and [15, 1, 1]\). Each cube
\([a_1, a_2, a_3]\) corresponds to the vertex

\[
\begin{pmatrix}
1 & 0 & 0 \\
10 & 1 & -2 \\
-38 & -3 & 7
\end{pmatrix}
\begin{pmatrix}
a_1 \\
a_2 \\
a_3
\end{pmatrix} \in \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_{168}.
\]

In the following proposition we use the notation \(\mathbb{Z}^{(d-1)}_m = \mathbb{Z}_m \oplus \cdots \oplus \mathbb{Z}_m\).

**Proposition 4** Consider \(A_d = \{(1, 1, \ldots, 1), (1, 2, 1, \ldots, 1), \ldots, (1, 1, \ldots, 1, 2)\} \subset \mathbb{Z}^d\). Then, the Cayley digraph \(G_{d,t} = \text{Cay}(\mathbb{Z}_t \oplus \mathbb{Z}^{(d-1)}_{t(d+1)}, A_d)\) has diameter \(k_{d,t} = t(\binom{d+1}{2}) - d\), for \(d \geq 2\) and \(t \geq 1\).

**Proof.** Consider the family of digraphs given in [4, Theorem 3.2], \(F_d = \text{Cay}(\mathbb{Z}^{(d-1)}_{d+1}, B_d)\) with \(B_d = \{(1, 1, \ldots, 1), (2, 1, \ldots, 1), \ldots, (1, \ldots, 1, 2)\} \subset \mathbb{Z}^{d-1}\) and \(k(F_d) = \frac{d}{2}\), defined for \(d \geq 2\). Use the digraph isomorphism \(F_d \cong G_{d,1}\) and apply the Method to \(G_{d,1}\) for \(t \geq 1\). From Theorem \([d]\) it follows that \(k(G_{d,t}) = t(k(F_d) + d) - d = t\binom{d+1}{2} - d\). \(\square\)

Figure 3: Minimum distance diagrams related to \(G_1 = \text{Cay}(\mathbb{Z}_4 \oplus \mathbb{Z}_4, \{(1, 1), (2, 1), (1, 2)\})\) and \(G_2 = \text{Cay}(\mathbb{Z}_2 \oplus \mathbb{Z}_8 \oplus \mathbb{Z}_8, \{(1, 1, 1), (1, 2, 1), (1, 1, 2)\})\) of Proposition [4]

Notice the identity \(\alpha(G_{d,t}) = \delta(G_{d,t}) = \frac{1}{d+1}(2/d)^d\), for all \(t \geq 1\).

By using the Stirling’s formula, the lower bound expression \(\text{lb}(d, k)\) in [2], gives

\[
\text{lb}(d, k) \sim \frac{c}{\sqrt{2\pi}} e^{d - \frac{3}{2} \ln d - (\ln \ln d)(1 + \log_2 e)} \left(\frac{k}{d}\right)^d + O(k^{d-1}),
\]

with the multiplicative factor of \(\left(\frac{k}{d}\right)^d\) being

\[
\frac{c}{\sqrt{2\pi}} e^{d - \frac{3}{2} \ln d - (\ln \ln d)(1 + \log_2 e)} \sim \frac{c}{\sqrt{2\pi}} e^{d - \frac{3}{2} \ln d}.
\]
Notice that $N(G_{d,t}) = t^d(d + 1)^{d - 1}$ and $k_{d,t} = k(G_{d,t}) = t^{(d + 1)} - d$ hold for all $t$. Thus, as $t = \frac{k_{d,t} + d}{d^2}$, we get

$$N(G_{d,t}) = \frac{2^d}{d + 1} \left( \frac{k_{d,t}}{d} + 1 \right)^d = \frac{2^d}{d + 1} \left( \frac{k_{d,t}}{d} \right)^d + O(k_{d,t}^{d-1}),$$

with the multiplicative factor of $\left( \frac{k_{d,t}}{d} \right)^d$ being $e^{d\ln 2 - \ln(d+1)}$. Then, following [2], this is an asymptotically dense family for large degree $d$.

References

[1] F. Aguiló-Gost, New dense families of triple loop networks, *Discrete Math.*, 197/198 (1999) 15–27.

[2] F. Aguiló, M.A. Fiol, and C. García, Triple loop networks with small transmission delay, *Discrete Math.* 167-168 (1997) 3–16.

[3] F. Aguiló, M.A. Fiol, and S. Pérez, A geometric approach to dense Cayley digraphs of finite Abelian groups, *Electron. Notes Discrete Math.* 54 (2016) 277–282.

[4] F. Aguiló, M.A. Fiol, and S. Pérez, Abelian Cayley digraphs with asymptotically large order for any given degree, *Electron. J. Combin.* 23(2) (2016) #P2.19, 11pp.

[5] F. Aguiló-Gost and P.A. García-Sánchez, Factoring in embedding dimension three numerical semigroups, *Electron. J. Combin.* 17 (2010) #R138, 21pp.

[6] F. Aguiló, E. Simó, and M. Zaragozó, On dense triple-loop networks, *Electron. Notes Discrete Math.* 10 (2001) 261–264.

[7] D. Beihoffer, A. Nijenhuis, J. Hendry, and S. Wagon, Faster algorithms for Frobenius numbers, *J. Number Theory* 12 (2005) #R27, 38pp.

[8] S. Chen and W. Gu, Exact order of subsets of asymptotic bases, *J. Number Theory* 41 (1992) 15–21.

[9] R. Dougherty and V. Faber, The degree-diameter problem for several varietes of Cayley graphs I: The Abelian case, *SIAM J. Discrete Math.* 17 (2004), no. 3, 478–519.

[10] P. Esqué, F. Aguiló, and M.A. Fiol, Double commutative step digraphs with minimum diameters, *Discrete Math.* 114 (1993) 147–157.

[11] C.M. Fiduccia, R.W. Forcade, and J.S. Zito, Geometry and diameter bounds of directed Cayley graphs of Abelian groups, *SIAM J. Discrete Math.* 11 (1998) 157–167.

[12] M.A. Fiol, Congruences in $\mathbb{Z}^n$, finite Abelian groups and the Chinese remainder theorem, *Discrete Math.* 67 (1987) 101–105.

[13] M.A. Fiol, On congruences in $\mathbb{Z}^n$ and the dimension of a multidimensional circulant, *Discrete Math.* 141 (1995) 123–134.
[14] M.A. Fiol, J.L.A. Yebra, I. Alegre, and M. Valero, A discrete optimization problem in local networks and data alignment, IEEE Trans. Comput. C-36 (1987) 702–713.

[15] R. Forcade and J. Lamoreaux, Lattice-simplex coverings and the 84-shape, SIAM J. Discrete Math. 13 (2000), no. 2, 194–201.

[16] D. Gómez, J. Gutiérrez, and Á. Ibeas, Cayley of finite Abelian groups and monomial ideals, SIAM J. Discrete Math. 21 (2007), no. 3, 763–784.

[17] D.F. Hsu and X.D. Jia, Extremal problems in the construction of distributed loop networks, SIAM J. Discrete Math. 7 (1994) 57–71.

[18] Ö.J. Rødseth, Weighted multi-connected loop networks, Discrete Math. 148 (1996) 161–173.

[19] P. Sabariego and F. Santos, Triple-loop networks with arbitrarily many minimum distance diagrams, Discrete Math. 309(6) (2009) 1672–1684.

[20] C.K. Wong and D. Coppersmith, A combinatorial problem related to multimode memory organizations, J. Ass. Comput. Mach. 21 (1974) 392–402.