EXPLICIT BRACKET IN THE EXCEPTIONAL SIMPLE LIE SUPERALGEBRA cvect(0|3)*

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Abstract. This note is devoted to a more detailed description of one of the five simple exceptional Lie superalgebras of vector fields, cvect(0|3)*, a subalgebra of vect(4|3). We derive differential equations for its elements, and solve these equations. Hence we get an exact form for the elements of cvect(0|3)*. Moreover we realize cvect(0|3)* by "glued" pairs of generating functions on a (3|3)-dimensional periplectic (odd symplectic) supermanifold and describe the bracket explicitly.

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INTRODUCTION

V. Kac [3] classified simple finite-dimensional Lie superalgebras over $\mathbb{C}$. Kac further conjectured [3] that passing to infinite-dimensional simple Lie superalgebras of vector fields with polynomial coefficients we only acquire the straightforward analogues of the four well-known Cartan series: vect($n$), svect($n$), h($2n$) and $\mathfrak{e}(2n+1)$ (of all, divergence-free, Hamiltonian and contact vector fields, respectively, realized on the space of dimension indicated).

It soon became clear [4], [1], [5], [6] that the actual list of simple vectoral Lie superalgebras is much larger. Several new series were found.

Next, exceptional vectoral algebras were discovered [8], [9]; for their detailed description see [10], [2]. All of them are obtained with the help of a Cartan prolongation or a generalized prolongation, cf. [8]. This description is, however, not always satisfactory; a more succinct presentation (similar to the one via generating functions for the elements of $\mathfrak{h}$ and $\mathfrak{e}$) and a more explicit formula for their brackets is desirable.

The purpose of this note is to give a more lucid description of one of these exceptions, cvect(0|3)*. In particular we offer a multiplication table for cvect(0|3)* that is simpler than previous descriptions, by use of "glued" pairs of generating functions for the elements of cvect(0|3)*.

This note can be seen as a supplement to [10]. To be self-contained and to fix notations we introduce some basic notions in section 0.

Throughout, the ground field is $\mathbb{C}$.

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§0. Background

0.1. We recall that a superspace $V$ is a $\mathbb{Z}/2$-graded space; $V = V_0 \oplus V_1$. The elements of $V_0$ are called even, those of $V_1$ odd. When considering an element $x \in V$, we will always assume that $x$ is homogeneous, i.e., $x \in V_0$ or $x \in V_1$. We write $p(x) = i$ if $x \in V_i$. The superdimension of $V$ is $(n|m)$, where $n = \dim(V_0)$ and $m = \dim(V_1)$.

For a superspace $V$, we denote by $\Pi(V)$ the same superspace with the shifted parity, i.e., $\Pi(V_1) = V_{i+1}$.

0.2. Let $x = (u_1, \ldots, u_n, \xi_1, \ldots, \xi_m)$, where $u_1, \ldots, u_n$ are even indeterminants and $\xi_1, \ldots, \xi_m$ odd indeterminates. In the associative algebra $\mathbb{C}[x]$ we have that $x \cdot y = (-1)^{p(x)p(y)} y \cdot x$ (by definition) and hence $\xi_i^2 = 0$ for all $i$.

The derivations $\text{div}(\mathbb{C}[x])$ of $\mathbb{C}[x]$ form a Lie superalgebra; its elements are vector fields. These polynomial vector fields are denoted by $\mathfrak{vect}(n|m)$. Its elements are represented as

$$D = \sum_i f_i \frac{\partial}{\partial u_i} + \sum_j g_j \frac{\partial}{\partial \xi_j}$$

where $f_i \in \mathbb{C}[x]$ and $g_j \in \mathbb{C}[x]$ for all $i, j = 1 \ldots n$. We have $p(D) = p(f_i) + 1$ and the Lie product is given by the commutator

$$[D_1, D_2] = D_1 D_2 - (-1)^{p(D_1)p(D_2)} D_2 D_1.$$

On the vector fields we have a map, $\text{div} : \mathfrak{vect}(n|m) \to \mathbb{C}[x]$, defined by

$$\text{div}D = \text{div}\left(\sum_{i=1}^n f_i \frac{\partial}{\partial u_i} + \sum_{j=1}^n g_j \frac{\partial}{\partial \xi_j}\right) = \sum_{i=1}^n \frac{\partial f_i}{\partial u_i} - (-1)^{p(D)} \sum_{j=1}^n \frac{\partial g_j}{\partial \xi_j}.$$

A vector field $D$ that satisfies $\text{div}D = 0$ is called special. The linear space of special vector fields in $\mathfrak{vect}(n|m)$ forms a Lie superalgebra, denoted by $\mathfrak{svect}(n|m)$.

0.3. Next we discuss the Lie superalgebra of Leitesian vector fields $\mathfrak{le}(n)$. It consists of the elements $D \in \mathfrak{vect}(n|n)$ that annihilate the 2-form $\omega = \sum_i du_i d\xi_i$. Hence $\mathfrak{le}(n)$ is an odd superanalogon of the Hamiltonian vector fields (in which case $\omega = \sum_i dp_i dq_i$). Similar to the Hamiltonian case, there is a map $\text{Le} : \mathbb{C}[x] \to \mathfrak{le}(n)$, with $x = (u_1, \ldots, u_n, \xi_1, \ldots, \xi_n)$:

$$\text{Le}_f = \sum_{i=1}^n \left( \frac{\partial f}{\partial u_i} \frac{\partial}{\partial \xi_i} + (-1)^{p(f)} \frac{\partial f}{\partial \xi_i} \frac{\partial}{\partial u_i} \right)$$

Note that $\text{Le}$ maps odd elements of $\mathbb{C}[x]$ to even elements of $\mathfrak{le}(n)$ and vice versa. Moreover $\text{Ker}(\text{Le}) = \mathbb{C}$. We turn $\mathbb{C}[x]$ (with shifted parity) into a Lie superalgebra with (Buttin) bracket $\{f, g\}$ defined by

$$\text{Le}_{\{f,g\}} = [\text{Le}_f, \text{Le}_g]$$

A straightforward calculation shows that

$$\{f, g\} = \sum_{i=1}^n \left( \frac{\partial f}{\partial u_i} \frac{\partial g}{\partial \xi_i} + (-1)^{p(f)} \frac{\partial f}{\partial \xi_i} \frac{\partial g}{\partial u_i} \right).$$

This way $\mathbb{C}[x] / \mathbb{C} \cdot 1$ is a Lie superalgebra isomorphic to $\mathfrak{le}(n)$. We call $f$ the generating function of $\text{Le}_f$. Here and throughout $p(f)$ will denote the
parity in \(\mathbb{C}[x]\), not in \(\Pi\mathbb{C}[x]\). So \(p(f)\) is the parity of the number of \(\xi\) in a term of \(f\).

0.4. The algebra \(\mathfrak{le}(n)\) contains certain important subalgebras. First of all there is \(\mathfrak{sl}(n)\), the space of special Leitesian vector fields:

\[
\mathfrak{sl}(n) = \mathfrak{le}(n) \cap \mathfrak{svect}(n|n).
\]

We have seen that if \(D \in \mathfrak{le}(n)\) then \(D = \text{Le}_f\) for some \(f \in \mathbb{C}[x]\). Now \(D \in \mathfrak{sl}(n)\) iff \(f\) is harmonic in the following sense

\[
\Delta(f) := \sum_{i=1}^{n} \frac{\partial^2 f}{\partial u_i \partial \xi_i} = 0
\]

Usually we simply say \(f \in \mathfrak{sl}(n)\), identifying \(f\) and \(\text{Le}_f\). This \(\Delta\) satisfies the condition \(\Delta^2 = 0\) and hence \(\Delta : \mathfrak{le}(n) \to \mathfrak{sl}(n)\). The image \(\Delta(\mathfrak{le}(n)) =: \mathfrak{sl}^0(n)\) is an ideal of codimension 1 on \(\mathfrak{sl}(n)\). This ideal, \(\mathfrak{sl}^0(n)\), can also be defined by the exact sequence

\[
0 \to \mathfrak{sl}^0(n) \to \mathfrak{sl}(n) \to \mathbb{C} \cdot \text{Le}_{\xi_1,\ldots,\xi_n} \to 0.
\]

Note that if \(\Phi = \sum u_i \xi_i\) and \(f \in \mathfrak{sl}(n)\), then

\[
\Delta(\Phi f) = (n + \deg u - \deg f) \cdot f
\]

Let \(\nu(f) = n + \deg u - \deg f\). Then \(\nu(f) \neq 0\) iff \(f \in \mathfrak{sl}^0(n)\). So on \(\mathfrak{sl}^0(n)\) we can define the right inverse \(\Delta^{-1}\) to \(\Delta\) by the formula

\[
\Delta^{-1}f = \frac{1}{\nu(f)}(\Phi f).
\]

0.5. Cartan prolongs. We will repeatedly use Cartan prolongation. So let us recall the definition. Let \(g\) be a Lie superalgebra and \(V\) a \(g\)-module. Set \(g_{-1} = V\), \(g_0 = g\) and for \(i > 0\) define the \(i\)-th Cartan prolong \(g_i\) as the space of all \(X \in \text{Hom}(g_{-i}, g_{i-1})\) such that

\[
X(w_0)(w_1, w_2, \ldots, w_i) = (-1)^{p(w_0)p(w_1)} X(w_1)(w_0, w_2, \ldots, w_i)
\]

for all \(w_0, \ldots, w_i \in g_{-1}\).

The Cartan prolong (the result of Cartan’s prolongation) of the pair \((V, g)\) is \((g_{-1}, g_0)_* = \oplus_{i \geq -1} g_i\).

Suppose that the \(g_0\)-module \(g_{-1}\) is faithful. Then

\[
(g_{-1}, g_0)_* \subset \mathfrak{vect}(n|m) = \mathfrak{vt}(\mathbb{C}[x]), \quad \text{where} \ n = \dim(V_0) \text{ and } m = \dim(V_1)
\]

and \(x = (u_1, \ldots, u_n, \xi_1, \ldots, \xi_m)\). We have for \(i \geq 1\)

\[
g_i = \{D \in \mathfrak{vect}(n|m) : \deg D = i, [D, X] \in g_{i-1} \text{ for any } X \in g_{-1}\}.
\]

The Lie superalgebra structure on \(\mathfrak{vect}(n|m)\) induces one on \((g_{-1}, g_0)_*\). This way the commutator of vector fields \([g, v]\), corresponds to the action \(g \cdot v\), \(g \in g\) and \(v \in V\).

We give some examples of Cartan prolongations. Let \(g_{-1} = V\) be an \((n|m)\)-dimensional superspace and \(g_0 = \mathfrak{gl}(n|m)\) the space of all endomorphisms of \(V\). Then \((g_{-1}, g_0)_* = \mathfrak{vect}(n|m)\). If one takes for \(g_0\) only the supertraceless elements \(\mathfrak{sl}(n|m)\), then \((g_{-1}, g_0)_* = \mathfrak{svect}(n|m)\), the algebra of vector fields with divergence 0.
\[ 1.1. \] In this note our primary interest is in a certain Cartan prolongation (denoted by \( \text{vect}(0|3)_+ \)) and the extension \( \text{vect}(0|3)_+ \) thereof. Here we will discuss \( \text{vect}(0|3)_+ \). Now \( \text{vect}(0|3)_+ \) is a short-hand notation for the Cartan prolongation with

\[ V = g_{-1} = \Pi A(\eta_1, \eta_2, \eta_3) / \mathbb{C} \text{ and } g_0 = \partial V \]

So \( V \) is a superspace of dimension \( (4|3) \), with

\[ V_0 = \langle \eta_1 \eta_2 \eta_3, \eta_1, \eta_2, \eta_3 \rangle; \quad V_1 = \langle \eta_2 \eta_3 \eta_1, \eta_1 \eta_2 \rangle \]

and \( \dim g_0 = (12|12) \).

The elements of \( g_{-1} \) and \( g_0 \) can be expressed as vector fields in \( \text{vect}(4|3) \).

Choosing

\[ \eta_1 \eta_2 \eta_3 \simeq -\partial_y; \quad \eta_i \simeq -\partial u_i; \quad \frac{\partial \eta_1 \eta_2 \eta_3}{\partial \eta_i} \simeq -\partial \xi_i. \]

it is subject to straightforward verification that the elements of \( g_0 \), expressed as elements of \( \text{vect}(4|3) \) are of the form:

\[
\begin{align*}
\partial \eta_1 & \simeq -y \partial \xi_1 - \xi_2 \partial u_3 + \xi_3 \partial u_2 \quad \partial \eta_2 \simeq -y \partial \xi_2 - \xi_3 \partial u_1 + \xi_1 \partial u_3 \\
\partial \eta_3 & \simeq -y \partial \xi_3 - \xi_1 \partial u_2 + \xi_2 \partial u_1
\end{align*}
\]

\[
\begin{align*}
\eta_1 \partial \eta_1 & \simeq -u_2 \partial u_3 + \xi_1 \partial \xi_2 \quad \eta_2 \partial \eta_1 \simeq -u_1 \partial u_2 + \xi_2 \partial \xi_1 \\
\eta_3 \partial \eta_1 & \simeq -u_3 \partial u_2 + \xi_3 \partial \xi_1
\end{align*}
\]

1.2. Now we will give a more explicit description of \( \text{vect}(0|3)_+ \). It will turn out that \( \text{vect}(0|3)_+ \) is isomorphic to \( \mathfrak{le}(3) \) as Lie superalgebra; however considered as \( \mathbb{Z} \)-graded algebras we have to define a different grading. The \( \mathbb{Z} \)-graded Lie superalgebra \( \mathfrak{le}(3;3) \) is \( \mathfrak{le}(3) \) as Lie superalgebra with \( \mathbb{Z} \)-degree of \( D \)

\[ D = \sum_i f_i \frac{\partial}{\partial u_i} + \sum_j g_j \frac{\partial}{\partial \xi_j} \]

the \( u \)-degree of \( f_i \) minus 1 (or the \( u \)-degree of \( g_j \), i.e. \( \deg \xi_i = 0 \).

Consider the map \( i_1 : \mathfrak{le}(3;3) \to \text{vect}(4|3) \) given by

a.) If \( f = f(u) \) then

\[ i_1(\text{Le}_f) = \text{Le} \left( \sum_{k=1}^{6} \frac{\partial}{\partial \xi_k} y^k \right) \]

where \( y \) is treated as a parameter and \( (i,j,k) \in A_3 \) (even permutations of \( \{1,2,3\} \)).

b.) If \( f = \sum f_i(u) \xi_i \) then

\[ i_1(\text{Le}_f) = \text{Le}_f - \varphi(u) \sum \xi_i \partial \xi_i + (-\varphi(u)y + \Delta(\varphi(u) \xi_1 \xi_2 \xi_3)) \frac{\partial}{\partial y} \]

where \( \varphi(u) = \Delta(f) \) and \( \Delta \) as given in section 0.4.
Let us describe a general construction, which leads to several new simple Lie superalgebras. Let

\[ u \]

2.2. Definition. The Lie superalgebra \( (\text{subsuperalgebra of } u) \text{ cvect prolongation with } u) \text{ cvect } \]

The map \( i \) preserves the \( \mathbb{Z} \)-degree. We have the following lemma.

1.3. Lemma. The map \( i \) is an isomorphism of \( \mathbb{Z} \)-graded Lie superalgebras between \( \mathfrak{sl}(3; 3) \) and \( \text{vect}(0|3)_* \subset \text{vect}(4|3) \).

Proof. That \( i \) is an embedding can be verified by direct computation. To prove that the image of \( i \) is in \( \text{vect}(0|3)_* \) it is enough to show that this is the case on the components \( \mathfrak{sl}(3; 3)_{-1} \oplus \mathfrak{sl}(3; 3)_0 \), i.e. on functions \( f(u, \xi) \) of degree \( \leq 1 \) with respect to \( u \), as the Cartan prolongation is the biggest subalgebra \( g \) of \( \text{vect}(4|3) \), with given \( g_1 \) and \( g_0 \). The proof that \( i \) is surjective onto \( \text{vect}(0|3)_* \) is given in corollary 4.6.

A generalized version of Lemma 1.3 can be found in [10] and [7]. It states that \( \mathfrak{sl}(n|n) \) and \( \text{vect}(0|n)_* \) are isomorphic for all \( n \geq 1 \).

§2. The construction of \( \text{vect}(0|3)_* \)

2.1. Let us describe a general construction, which leads to several new simple Lie superalgebras. Let \( u = \text{vect}(m|n) \), let \( \mathfrak{g} = (u_{-1}, g_0)_* \) be a simple Lie subsuperalgebra of \( u \). Moreover suppose there exists an element \( d \in u_0 \) that determines an exterior derivation of \( \mathfrak{g} \) and has no kernel on \( u_+ \). Let us study the prolong \( \tilde{\mathfrak{g}} = (g_{-1}, g_0 \oplus \mathbb{C}d)_* \).

Lemma. Either \( \tilde{\mathfrak{g}} \) is simple or \( \tilde{\mathfrak{g}} = \mathfrak{g} \oplus \mathbb{C}d \).

Proof. Let \( I \) be a nonzero graded ideal of \( \tilde{\mathfrak{g}} \). The subsuperspace \( (\text{ad } u_{-1})^k \) of \( u_{-1} \) is nonzero for any nonzero homogeneous element \( a \in u_k \) and \( k \geq 0 \). Since \( g_{-1} = u_{-1} \), the ideal \( I \) contains nonzero elements from \( g_{-1} \); by simplicity of \( \mathfrak{g} \) the ideal \( I \) contains the whole \( \mathfrak{g} \). If, moreover, \( [g_{-1}, \tilde{\mathfrak{g}}] = \mathfrak{g}_0 \), then by definition of the Cartan prolongation \( \tilde{\mathfrak{g}} = \mathfrak{g} \oplus \mathbb{C}d \).

If, instead, \( [g_{-1}, \tilde{\mathfrak{g}}] = \mathfrak{g}_0 \oplus \mathbb{C}d \), then \( d \in I \) and since \( [d, u_+] = u_+ \), we derive that \( I = \tilde{\mathfrak{g}} \). In other words, \( \tilde{\mathfrak{g}} \) is simple.

As an example, take \( \mathfrak{g} = \text{svec}(m|n); g_0 = \mathfrak{sl}(m|n), d = 1_{m|n} \). Then \( (g_{-1}, g_0 \oplus \mathbb{C}d)_* = \text{vect}(m|n) \).

2.2. Definition. The Lie superalgebra \( \text{cvect}(0|3)_* \subset \text{vect}(4|3) \) is the Cartan prolongation with \( \text{cvect}(0|3)_{-1} = \text{vect}(0|3)_{-1} \) and \( \text{cvect}(0|3)_0 = \text{vect}(0|3)_0 \oplus \mathbb{C}d \), with

\[ d = \sum u_i \partial u_i + \sum \xi_i \partial \xi_i + y \partial y. \]

If now

\[ f = \sum_{i=1}^3 \xi_i \partial \xi_i + 2y \partial y, \]

then it is clear that \( f \in \text{vect}(0|3) \oplus \mathbb{C}d \), but \( f \notin \text{vect}(0|3) \).
\textbf{2.3. Theorem.} The Lie superalgebra $\mathfrak{cvect}(0|3)_*$ is simple.

\textit{Proof.} We know that $\mathfrak{vect}(0|3)_* \cong \mathfrak{le}(3; 3)$ is simple. According to Lemma 2.1 it is sufficient to find an element $F \in \mathfrak{cvect}(0|3)_1$, which is not in $\mathfrak{vect}(0|3)_1$. For $F$ one can take

\[ F = y\xi_1\partial_{\xi_1} + y\xi_2\partial_{\xi_2} + y\xi_3\partial_{\xi_3} + y^2\partial_y - \xi_1\xi_2\partial_{u_3} - \xi_3\xi_1\partial_{u_2} - \xi_2\xi_3\partial_{u_1} \]

Indeed, one easily checks that $\partial_y F = f$, while

\[ [\partial_{\xi_i}, F] = -\partial_{u_i} \quad (i = 1, 2, 3), \]

and moreover $[\partial_{u_i}, F] = 0$. This proves the claim. \hfill \Box

Similar constructions are possible for general $n$. For $n = 2$ we obtain $\mathfrak{cvect}(0|2)_* \cong \mathfrak{vect}(2|1)$, while for $n > 3$ one can prove that $\mathfrak{cvect}(0|n)_*$ is not simple. For details, we refer to [10].

\textbf{2.4. Lemma.} A vector field

\[ D = \sum_{i=1}^{3} (P_i\partial_{\xi_i} + Q_i\partial_{u_i}) + R\partial_y \]

in $\mathfrak{vect}(4|3)$ belongs to $\mathfrak{cvect}(0|3)_*$ if and only if it satisfies the following system of equations:

\begin{enumerate}
  \item $\frac{\partial Q_i}{\partial u_j} + (-1)^{p(D)}\frac{\partial P_i}{\partial \xi_i} = 0$ for any $i \neq j$; \hfill (2.1)
  \item $\frac{\partial Q_i}{\partial u_i} + (-1)^{p(D)}\frac{\partial P_i}{\partial \xi_i} = \frac{1}{2} \left( \sum_{1 \leq j \leq 3} \frac{\partial Q_j}{\partial u_j} + \frac{\partial R}{\partial y} \right)$ for $i = 1, 2, 3$; \hfill (2.2)
  \item $\frac{\partial Q_i}{\partial \xi_j} + \frac{\partial Q_j}{\partial \xi_i} = 0$ for any $i, j$; in particular $\frac{\partial Q_i}{\partial \xi_i} = 0$; \hfill (2.3)
  \item $\frac{\partial P_i}{\partial u_j} - \frac{\partial P_j}{\partial u_i} = (-1)^{p(D)}\frac{\partial R}{\partial \xi_k}$ \hfill (2.4)
  \item $\frac{\partial Q_i}{\partial y} = 0$ for $i = 1, 2, 3$; \hfill (2.5)
  \item $\frac{\partial P_k}{\partial y} = (-1)^{p(D)}\frac{1}{2} \left( \frac{\partial Q_i}{\partial \xi_j} - \frac{\partial Q_j}{\partial \xi_i} \right)$ \hfill (2.6)
\end{enumerate}

for any $k$ and for any even permutation $\begin{pmatrix} 1 & 2 & 3 \\ i & j & k \end{pmatrix}$.

\textit{Proof.} Denote by $\mathfrak{g} = \oplus_{i \geq -1} \mathfrak{g}_i$ the superspace of solutions of the system (2.1)–(2.6). Clearly, $\mathfrak{g}_{-1} \cong \mathfrak{cvect}(4|3)_{-1}$. We directly verify that the images of the elements from $\mathfrak{vect}(0|3) \oplus Cd$ satisfy (2.1)–(2.6). Actually, we composed the system of equations (2.1)–(2.6) by looking at these images.

The isomorphism $\mathfrak{g}_0 = \mathfrak{vect}(0|3) \oplus Cd$ follows from dimension considerations.
Set
\[ D_{u_j}(D) = \sum_{i \leq 3} \left( \frac{\partial P_i}{\partial u_j} \frac{\partial}{\partial \xi_i} + \frac{\partial Q_i}{\partial u_j} \frac{\partial}{\partial u_i} + \frac{\partial R}{\partial u_j} \frac{\partial}{\partial y} \right) \]
\[ D_y(D) = \sum_{i \leq 3} \left( \frac{\partial P_i}{\partial y} \frac{\partial}{\partial \xi_i} + \frac{\partial Q_i}{\partial y} \frac{\partial}{\partial u_i} + \frac{\partial R}{\partial y} \frac{\partial}{\partial y} \right) \]
\[ \tilde{D}_{\xi_j}(D) = (-1)^{p(D)} \sum_{i \leq 3} \left( \frac{\partial P_i}{\partial \xi_j} \frac{\partial}{\partial \xi_i} + \frac{\partial Q_i}{\partial \xi_j} \frac{\partial}{\partial u_i} + (-1)^{p(D)} \frac{\partial R}{\partial \xi_j} \frac{\partial}{\partial y} \right) \]

The operators \( D_{u_j}, D_y \), and \( \tilde{D}_{\xi_j} \), clearly, commute with the \( g_{-1} \)-action. Observe: the operators commute, not supercommute.

Since the operators in the equations (2.1)–(2.6) are linear combinations of only these operators \( D_{u_j}, D_y \), and \( \tilde{D}_{\xi_j} \), the definition of Cartan prolongation itself ensures isomorphism of \( g \) with \( \text{cvect}(0|3)_* \).

2.5. Remark. The left hand sides of eqs. (2.1)–(2.6) determine coefficients of the 2-form \( L_D \omega \), where \( L_D \) is the Lie derivative and \( \omega = \sum_{1 \leq i \leq 3} du_i d\xi_i \).

It would be interesting to interpret the right-hand side of these equations in geometrical terms as well.

2.6. Remark. Lemma 2.4 illustrates how \( \text{cvect}(0|3)_* \) can be characterized by a set of first order, constant coefficient, differential operators. This is a general fact of Cartan prolongations; one just replaces the linear constraints on \( g_0 \) by such operators. For example, for \( \text{vect}(0|3)_* \) we have the equations (2.1)–(2.6) and
\[ \frac{\partial R}{\partial y} - \sum_{i=1}^{3} \frac{\partial Q_i}{\partial u_i} = 0 \quad (2.7) \]

Indeed, this equation is satisfied by all elements of \( \text{vect}(0|3)_0 \), see section 1.1, but not by \( d \).

§3. Solution of differential equations (2.1) – (2.6)

Set \( D^3_\xi = \frac{\partial^3}{\partial \xi_1 \partial \xi_2 \partial \xi_3} \).

3.1. Theorem. Every solution of the system (2.1) – (2.6) is of the form:
\[ D = Lf + yA_f - (-1)^{p(f)} \left( y\Delta(f) + y^2 D^3_\xi f \right) \frac{\partial}{\partial y} + A_g - (-1)^{p(g)} \left( \Delta(g) + 2y D^3_\xi g \right) \frac{\partial}{\partial y}, \quad (3.1) \]
where \( f, g \in \mathbb{C}[u, \xi] \) are arbitrary and the operator \( A_f \) is given by the formula:
\[ A_f = \frac{\partial^2 f}{\partial \xi_2 \partial \xi_3} \frac{\partial}{\partial \xi_1} + \frac{\partial^2 f}{\partial \xi_3 \partial \xi_1} \frac{\partial}{\partial \xi_2} + \frac{\partial^2 f}{\partial \xi_1 \partial \xi_2} \frac{\partial}{\partial \xi_3}. \quad (3.2) \]

Proof. First, let us find all solutions of system (2.1)–(2.6) for which \( Q_1 = Q_2 = Q_3 = 0 \). In this case the system takes the form
\[ \frac{\partial P_j}{\partial \xi_i} = 0 \quad \text{for } i \neq j \quad (2.1') \]
\[ (-1)^{p(D)} \frac{\partial P_i}{\partial \xi_i} = \frac{1}{2} \frac{\partial R}{\partial y} \quad \text{for } i = 1, 2, 3 \quad (2.2') \]
Therefore, any vector field $D$ with $Q_1 = Q_2 = Q_3 = 0$ satisfying (2.1) – (2.6) is of the form

$$D = \sum_{i=1}^{3} \Psi_i(u) \frac{\partial}{\partial \xi_i} + \varphi(u) \sum_{i=1}^{3} \xi_i \frac{\partial}{\partial \xi_i} + (-1)^{p(D)} \Psi_0(u) + \Delta (\Psi_1 \xi_2 \xi_3 + \Psi_2 \xi_3 \xi_1 - \Psi_3 \xi_1 \xi_2 - \varphi \xi_1 \xi_2 \xi_3) \partial \xi_1 -$$

$$\left( \frac{\partial \Psi_3}{\partial \xi_1} - \frac{\partial \Psi_1}{\partial \xi_3} \right) \xi_2 - \left( \frac{\partial \Psi_2}{\partial \xi_2} - \frac{\partial \Psi_3}{\partial \xi_3} \right) \xi_3 - \left( \frac{\partial \varphi}{\partial \xi_2} \xi_3 \xi_1 + \frac{\partial \varphi}{\partial \xi_1} \xi_2 \xi_3 + \frac{\partial \varphi}{\partial \xi_3} \xi_1 \xi_2 \right)$$

where, as before,

$$\Delta = \sum_{i=1}^{3} \frac{\partial}{\partial u_i} \frac{\partial}{\partial \xi_i}$$

Set

$$g(u, \xi) = g_0(u, \xi) - \Psi_1 \xi_2 \xi_3 - \Psi_2 \xi_3 \xi_1 - \Psi_3 \xi_1 \xi_2 - \varphi \xi_1 \xi_2 \xi_3,$$

with $\Delta g_0 = \Psi_0$ and $\deg(\xi_0) \leq 1$. Then

$$A_g = \sum_{i=1}^{3} \Psi_i \frac{\partial}{\partial \xi_i} + \varphi \sum_{i=1}^{3} \xi_i \frac{\partial}{\partial \xi_i}; \quad D^3 g = \varphi \quad \text{and} \quad (-1)^{p(D)} = (-1)^{p(g) + 1}$$

for functions $g$ homogeneous with respect to parity. In the end we get:

$$D = A_g + (-1)^{p(D)} (\Delta (g) + 2yD^3 g) \partial y$$

$$= A_g - (-1)^{p(g)} (\Delta (g) + 2yD^3 g) \partial y.$$
Let us return now to the system (2.1) – (2.6). Equations (2.3), (2.5), (2.6)
imply that there exists a function $f(u, \xi)$ (independent of $y$!) such that

$$Q_i = -(-1)^{p(D)} \frac{\partial f}{\partial \xi_i} \quad \text{for} \quad i = 1, 2, 3.$$  

Then (2.1) implies that

$$P_i = \frac{\partial f}{\partial u_i} + f_i(u, \xi, y).$$

From (2.6) it follows that

$$\frac{\partial f_i}{\partial y} = \partial_{\xi_j} \partial_{\xi_k} f \quad \text{for \ even \ permutations \ (i, j, k)}$$
or

$$f_i = y(\partial_{\xi_j} \partial_{\xi_k} f) + \tilde{P}_i(u, \xi_i).$$

Observe that $	ilde{P}_i$ satisfy (2.1') and (2.6'); hence, in view of (2.2), $\frac{\partial \tilde{P}_i}{\partial y}$ does not depend on $i$. Therefore, we can choose $\tilde{R}$ so that $(\tilde{P}_i, \tilde{R})$ satisfy eqs. (2.1'), (2.2'), (2.4'), (2.6'). Thanks to the linearity of system (2.1) – (2.6) the vector field $D$ is then of the form

$$D = D_f + \tilde{D}, \tag{3.4}$$

where $D_f$ and $\tilde{D}$ are solutions of (2.1) – (2.6) such that $\tilde{D} = \sum \tilde{P}_i \partial_{\xi_i} + \tilde{R} \partial_y$ (i.e., $\tilde{D}$ is of the form (3.3)) and

$$D_f = \sum (-(-1)^{p(D)} \frac{\partial f}{\partial \xi_i} \partial_{u_i} + \frac{\partial f}{\partial u_i} \partial_{\xi_i}) + \sum y(\partial_{\xi_j} \partial_{\xi_k} f) \partial_{\xi_i} + R_f \cdot \partial_y$$
$$= L_e f + y A_f + R_f \partial_y.$$

It remains to find $R_f$. Equation (2.2) takes the form

$$(-1)^{p(D)} y D_\xi^2 f = \frac{1}{2} (-(-1)^{p(D)} (\Delta f) + \frac{\partial R_f}{\partial y}).$$

Hence,

$$R_f = (-1)^{p(D)} (y^2 D_\xi^2 f + y \cdot (\Delta f) + R_0(u, \xi)).$$

Then, we can rewrite (2.4) as

$$-y \frac{\partial \Delta f}{\partial \xi_k} + \frac{\partial R_0}{\partial \xi_k} = y \partial_{u_j} \partial_{\xi_j} \partial_{\xi_k} f - y \partial_{u_i} \partial_{\xi_i} \partial_{\xi_k} f.$$  

Observe that the right hand side of the last equation is equal to $-y \frac{\partial \Delta f}{\partial \xi_k}$. This means that $\frac{\partial R_0}{\partial \xi_k} = 0$ or $R_0 = R_0(u)$. Therefore, replacing $\tilde{R}$ with $\tilde{R} + R_0$ we may assume that $R_0 = 0$. Then

$$D_f = L_e f + y A_f + (-1)^{p(D)} (y(\Delta f) + y^2 D_\xi^2 f) \partial_y, \tag{3.5}$$

By uniting (3.3) – (3.5) we get (3.1). \qed
§4 How to generate $\text{cvect}(0|3)_*$ by pairs of functions

We constructed $\text{cvect}(0|3)_*$ as an extension of $\text{vect}(0|3)_* \cong \text{le}(3|3)$, see lemma 1.3. Using the results of section 3, we obtain another embedding $i_2 : \text{le}(3) \to \text{cvect}(0|3)_*$. 

4.1. Lemma. The map

$$i_2 : \text{Le}_f \to \text{Le}_f + yA_f - (-1)^{p(f)} (y\Delta(f) + y^2D^3_\xi) \partial_y$$

(4.1)

determines an embedding of $\text{le}(3)$ into $\text{cvect}(0|3)_*$. This embedding preserves the standard grading of $\text{le}(3)$.

Proof. We have to verify the equality

$$i_2(\text{Le}_{(f,g)}) = [i_2(\text{Le}_f), i_2(\text{Le}_g)].$$

Comparison of coefficients of different powers of $y$ shows that the above equation is equivalent to the following system:

$$\text{Le}_{(f,g)} = [\text{Le}_f, \text{Le}_g].$$

(4.2)

$$A_{(f,g)} = [\text{Le}_f, A_g] + [A_f, \text{Le}_g] - (-1)^{p(f)} (\Delta(f) \cdot A_g + (-1)^{p(f)p(g)} \Delta(g)A_f).$$

(4.3)

$$[A_f, A_g] = (-1)^{p(f)} \left(D^2_\xi f \cdot A_g + (-1)^{p(f)p(g)} D^3_\xi g A_f \right).$$

(4.4)

$$\Delta(\{f, g\}) = \{\Delta f, g\} - (-1)^{p(f)} \{f, \Delta g\}.$$  

(4.5)

$$D^2_\xi \{f, g\} = \{D^3_\xi f, g\} - (-1)^{p(f)} \{f, D^3_\xi g\} - (-1)^{p(f)} (A_f(\Delta g) + (-1)^{p(f)p(g)} A_g(\Delta f) + \Delta f D^3_\xi g - D^3_\xi f \Delta g.$$  

(4.6)

Equation (4.2) is known, see section 0.3. The equalities (4.3)–(4.6) are subject to direct verification.  

We found two embeddings $i_1 : \text{le}(3|3) \to \text{vect}(0|3)_*$ and $i_2 : \text{le}(3) \to \text{cvect}(0|3)_*$. Let us denote

$$\alpha_g = A_g - (-1)^{p(g)} (\Delta g + 2yD^3_\xi g) \partial_y.$$

We want to prove that the sum of the images of $i_1$ and $i_2$ cover the whole $\text{cvect}(0|3)_*$. According to Theorem 3.1, it is sufficient to represent $\alpha_g$ in the form $\alpha_g = i_1g_1 + i_2g_2$. For convenience we simply write $f$ instead of $\text{Le}_f$.

4.2. Lemma. For $\alpha_g$ we have:

$$\alpha_g = \begin{cases} 0 & \text{if } \deg_G g = 0 \\
 i_1(-\Delta g)\xi_1\xi_2\xi_3 & \text{if } \deg_G g = 1 \\
 i_1(g) & \text{if } \deg_G g = 2 \\
 i_1(-\Delta^{-1}(D^3_\xi g)) + i_2(\Delta^{-1}(D^3_\xi g)) & \text{if } \deg_G g = 3. \\
\end{cases}$$

The right inverse $\Delta^{-1}$ of $\Delta$ is given in section 0.4.

The proof of Lemma 4.2 is a direct calculation.
4.3. A wonderful property of $\mathfrak{slc}^o(3)$. In the standard grading of $\mathfrak{g} = \mathfrak{slc}^o(3)$ we have: $\dim \mathfrak{g}_{-1} = (3|3)$, $\mathfrak{g}_0 \cong \mathfrak{spc}(3)$. For the regraded superalgebra $R\mathfrak{g} = \mathfrak{slc}^o(3; 3) \subset \mathfrak{le}(3; 3)$ we have: $\dim R\mathfrak{g}_{-1} = (3|3)$, $R\mathfrak{g}_0 = \mathfrak{svect}(0|3) \cong \mathfrak{spc}(3)$. For the definition of $\mathfrak{spc}(3)$ we refer to [3] or [10]. Therefore, for $\mathfrak{slc}^o(3)$ and only for it among the $\mathfrak{slc}^o(n)$, the regrading $R$ determines a nontrivial automorphism. In terms of generating functions the regrading is determined by the formulas:

1) $\deg_\xi(f) = 0: R(f) = \Delta(f_1\xi_2\xi_3);$
2) $\deg_\xi(f) = 1: R(f) = f;$
3) $\deg_\xi(f) = 2: R(f) = D^2_\xi(\Delta^{-1}f).$

Note that $R^2(f) = (-1)^{p(f)+1}f$. Now we can formulate the following proposition.

4.4. Proposition. The nondirect sum of the images of $i_1$ and $i_2$ covers the whole $\mathfrak{svect}(0|3)_*$, i.e.,

$$i_1(\mathfrak{le}(3; 3)) + i_2(\mathfrak{le}(3)) = (\mathfrak{svect}(0|3))_*.$$

We also have

$$i_1(\mathfrak{le}(3; 3)) \cap i_2(\mathfrak{le}(3)) \cong \mathfrak{slc}^o(3; 3) \cong \mathfrak{slc}^o(3).$$

Proof. The first part follows from Lemma 4.2. The second part follows by direct calculation from solving $i_2(\mathfrak{Le}_f) = i_1(\mathfrak{Le}_g)$. Note that $\mathfrak{Le}_f \in \mathfrak{slc}^o(3)$ iff $\Delta(f) = 0$ and $D^2_\xi f = 0$, and similar for $\mathfrak{Le}_g \in \mathfrak{slc}^o(3; 3)$. The equation $i_2(\mathfrak{Le}_f) = i_1(\mathfrak{Le}_g)$ is only solvable if $f \in \mathfrak{slc}^o(3)$ and $g \in \mathfrak{slc}^o(3; 3)$, and in this case we obtain $g = (-1)^{p(f)+1}Rf$. \qed

Therefore, we can identify the space of the Lie superalgebra $\mathfrak{svect}(0|3)_*$ with the quotient space of $\mathfrak{le}(3; 3) \oplus \mathfrak{le}(3)$ modulo

$$\{(-1)^{p(g)+1}Rg \oplus (-g), g \in \mathfrak{slc}^o(3)\}.$$

In other words, we can represent the elements of $\mathfrak{svect}(0|3)_*$ in the form of the pairs of functions

$$(f, g), \text{ where } f, g \in \Pi \mathbb{C}[u, \xi]/\mathbb{C} \cdot 1 \quad (4.7)$$

subject to identifications

$$(-1)^{p(g)+1}(Rg, 0) \sim (0, g) \quad \text{for any } g \in \mathfrak{slc}^o(3).$$

4.5. Corollary. The map $\varphi$ defined by the formula

$$\varphi|_{i_1(\mathfrak{le}(3; 3))} = \text{sign } i_2^{-1} i_1^{-1}, \quad \varphi|_{i_2(\mathfrak{le}(3))} = i_1 i_2^{-1}$$

is an automorphism of $\mathfrak{svect}(0|3)_*$. Here $\text{sign}(D) = (-1)^{p(D)}D$.

The map $\varphi$ may be represented in inner coordinates of $\mathfrak{vect}(4|3)$ as a regrading by setting $\deg y = -1; \deg u_i = 1; \deg \xi_i = 0$.

In the representation (4.7) we have

$$\varphi(f, g) = (g, (-1)^{p(f)+1}f).$$

Now we can complete the proof of Lemma 1.3.

4.6. Corollary. The embedding $i_1 : \mathfrak{le}(3) \to \mathfrak{svect}(0|3)_*$ is a surjection onto $\mathfrak{vect}(0|3)_*$. 
Proof. By Proposition 4.4 we merely have to prove that $i_2(\text{Le}_f) \in \mathfrak{vect}(0|3)_*$ iff $\Delta f = 0$ and $D_\xi^3 f = 0$. Applying equation (2.7) to $i_2(\text{Le}_f)$, this follows immediately. \qed

§5 The bracket in $\mathfrak{vect}(0|3)_*$

Now we can determine the bracket in $\mathfrak{vect}(0|3)_*$ in terms of representation $(f, g)$ as stated in formula (4.7).

We do this via $\alpha_g$. By Theorem 3.1 any $D \in \mathfrak{vect}(0|3)_*$ is of the form $D = i_2(f) + \alpha_g$ for some generating functions $f$ and $g$. To determine the bracket $[i_2(f), i_1(h)]$, we

1. Compute the brackets $[i_2 f, \alpha_g]$ for any $f, g \in \mathbb{C}[u, \xi]/\mathbb{C} : 1$;
2. Represent $i_1(h)$ in the form $i_1(h) = i_2a(h) + \alpha_{b(h)}$ for any $h \in \mathbb{C}[u, \xi]/\mathbb{C} : 1$;

In Lemma 4.2 we expressed $\alpha_g$ in $i_1$ and $i_2$.

**Remark.** The functions $a(h)$ and $b(h)$ above are not uniquely defined. Any representation will do.

5.1. **Lemma.** For any functions $f, g \in \mathbb{C}[u, \xi]/\mathbb{C} : 1$ the bracket $[i_2 f, \alpha_g]$ is of the form

$$[i_2 f, \alpha_g] = i_2 F + \alpha G,$$

where

$$F = f \cdot D_\xi^3 g - (-1)^{(p(f)+1)(p(g)+1)} A_g f \quad \text{and} \quad G = -f \Delta g.$$

**Proof.** Direct calculation gives that

$$[i_2 f, \alpha_g] = [\text{Le}_f, A_g] + (-1)^{p(f)p(g)+p(f)+1} \Delta g \cdot A_f$$

$$+ y \left( [A_f, A_g] + (-1)^{p(f)p(g)+p(f)+1} \cdot 2 \cdot D_\xi^3 g \cdot A_f \right)$$

$$+ (-1)^{p(g)+1} \left( \{ f, \Delta g \} + (-1)^{p(f)} \Delta f \cdot \Delta g \right) \partial_y$$

$$+ \left( (-1)^{p(f)+1} A_f(\Delta g) + (-1)^{p(f)p(g)+p(g)+1} A_g(\Delta f) + 2 \cdot (-1)^{p(f)+p(g)+1} D_\xi^3 f \cdot \Delta g \right) y \partial_y$$

$$+ (-1)^{p(f)+p(g)+1} 2 \cdot D_\xi^3 f \cdot D_\xi^3 g \cdot y^2 \partial_y.$$

In order to find the functions $F$ and $G$, it suffices to observe that the coefficient of $\partial_y$, non-divisible by $y$, should be equal to $(-1)^{p(f)+1} \Delta G$. This implies the equations:

$$(-1)^{p(f)+1} \Delta G = (-1)^{p(g)+1} \left( \{ f, \Delta g \} + (-1)^{p(f)} \Delta f \cdot \Delta g \right)$$

or

$$(-1)^{p(f)+1} \Delta G = (-1)^{p(f)+p(g)+1} \Delta(f \cdot \Delta g) .$$

Here $p(G) = p(f \cdot \Delta g) = p(f) + p(g) + 1$. Hence, $\Delta G = \Delta( - f \Delta g)$. Since $G$ is defined up to elements from $\text{sl}(\mathfrak{e}(3))$, we can take $G = -f \Delta g$.

The function $F$ to be found is determined from the equation

$$i_2 F = [i_2 f, \alpha_g] - \alpha G .$$
By comparing the coefficients of \( y \partial_y \) in the left and right hand sides of (5.3) we get
\[
(-1)^{p(F)+1} \Delta F = (-1)^{p(g)+1} A_f(\Delta g) + (-1)^{p(f)p(g)+p(f)+1} A_g(\Delta f) \\
+ 2(-1)^{p(g)+1} \{ f, D^3_\xi g \} + (-1)^{p(f)+p(g)+1} 2 \cdot D^3_\xi f \cdot \Delta g \\
- 2 \cdot (-1)^{p(f)+p(g)} D^3_\xi (f \Delta g).
\]
Observe that
\[
D^3_\xi (f \Delta g) = (D^3_\xi f) \Delta g + (-1)^{p(f)} A_f(\Delta g) + \sum_{i=1}^{3} \frac{\partial f}{\partial \xi_i} \cdot (D^3_\xi g) \\
= (D^3_\xi f) \cdot \Delta g + (-1)^{p(f)} A_f(\Delta g) + (-1)^{p(f)} \{ f, D^3_\xi g \}.
\]
Then
\[
(-1)^{p(F)+1} (\Delta F) = (-1)^{p(g)} A_f(\Delta g) + (-1)^{p(f)p(g)+p(f)+1} A_g(\Delta f).
\]
By comparing parities we derive that
\[
p(F) + 1 = p(A_f(\Delta g)) = p(f) + 1 + p(g) + 1 = p(f) + p(g).
\]
It follows that
\[
\Delta F = (-1)^{p(f)} A_f(\Delta g) + (-1)^{p(f)p(g)+p(f)+1} A_g(\Delta f).
\]
Let us transform the right hand side of the equality obtained. The sums over \( i, j, k \) are over \( (i, j, k) \in A_3 \):
\[
(-1)^{p(f)} A_f(\Delta g) + (-1)^{p(f)p(g)+p(f)+1} A_g(\Delta f) \\
= \sum (-1)^{p(f)} \partial_{\xi_j} \partial_{\xi_k} f \cdot \partial_{\xi_i} (\sum_{s=1}^{3} \partial_{u_s} \partial_{\xi_s} g) \\
+ (-1)^{p(f)p(g)+p(f)+1} \sum \partial_{\xi_j} \partial_{\xi_k} g \partial_{\xi_i} (\sum_{s=1}^{3} \partial_{u_s} \partial_{\xi_s} f) \\
= (-1)^{p(f)p(g)+p(f)} \cdot \sum ( \partial_{u_s} \partial_{\xi_j} \partial_{\xi_k} g + \partial_{u_k} \partial_{\xi_j} \partial_{\xi_s} \partial_{\xi_s} g ) \cdot \partial_{\xi_i} \partial_{\xi_s} f \\
- (-1)^{p(f)p(g)+p(f)} \sum ( \partial_{\xi_j} \partial_{\xi_k} g \partial_{\xi_i} \partial_{\xi_s} f + \partial_{u_k} \partial_{\xi_j} \partial_{\xi_s} \partial_{\xi_s} f ) \\
= (-1)^{p(f)p(g)+p(f)} \sum \partial_{u_k} ( \partial_{\xi_j} \partial_{\xi_k} \partial_{\xi_s} f + \partial_{\xi_j} \partial_{\xi_s} \partial_{\xi_k} \partial_{\xi_s} f ) \\
- (-1)^{p(f)p(g)+p(f)+p(g)} \sum_{k=1}^{3} ( \partial_{u_k} \partial_{\xi_k} (A_g f) - \partial_{u_k} D^3_\xi g \cdot \partial_{\xi_k} f ) \\
= - (-1)^{p(f)+1}(p(g)+1) A_g f + (-1)^{p(f)p(g)+p(f)} \Delta (D^3_\xi g \cdot f) \\
= \Delta (f + D^3_\xi g) - (-1)^{p(f)+1}(p(g)+1) A_g f.
\]
Then
\[
F = f \cdot D^3_\xi g - (-1)^{p(f)+1}(p(g)+1) A_g f + F_0, \quad \text{where} \quad \Delta F_0 = 0.
\]
We have shown how to find functions $F$ and $G$. To prove Lemma 5.1 it only remains to compare the elements of the same degree in $y$ in the right-hand and the left-hand side, i.e., to verify the following three equalities:

$$( -1)^{p(F)+1} D^3_x F = 2( -1)^{p(F)+p(g)+1} D^3_x f \cdot D^3_x g$$

$$\text{Le}_F + A_G = \text{[Le}_f, A_g] + ( -1)^{p(F)p(g)+p(f)+1} \Delta g \cdot A_f$$

$$A_F = [A_f, A_g] + 2 \cdot ( -1)^{p(F)p(g)+p(f)+1} D^3_x g \cdot A_f$$

The verification is a direct one. \qed

5.3. Lemma. The representation of $i_1 h$ in the form (5.1) is as follows:

$$i_1 h = \begin{cases} 
i_2(\Delta(h_1 \xi_2 \xi_3)) & \text{if } \deg \xi h = 0, \\
i_2 h + \alpha(\Delta h)\xi_1 \xi_2 \xi_3 & \text{if } \deg \xi h = 1, \\
\alpha h & \text{if } \deg \xi h = 2, \\
\alpha_{\Delta^{-1}(D^3_x h)} & \text{if } \deg \xi h = 3. \end{cases}$$

Proof. It suffices to compare the definition of $\alpha_g$ with the definitions of $i_1$ and $i_2$. If $\deg \xi h = 0$ use the equalities $\sum_{\partial_{\xi h}} \xi_k \xi_k = \Delta(f_1 \xi_2 \xi_3)$ and $A_{\Delta(f_1 \xi_2 \xi_3)} = \text{Le}_f$. In the remaining cases the verification is not difficult. \qed

Making use of the Lemmas 5.1, Lemma 5.2 and Lemma 4.2 we can compute the whole multiplication table of $[i_2 f, i_1 h]$:

- $\deg \xi h = 0$. Then
  $$i_1 h = i_2(\Delta(h_1 \xi_2 \xi_3))$$
  and $[i_2 f, i_1 h] = i_2\{f, \Delta(h_1 \xi_2 \xi_3)\}$.

We also have

$$\{f, \Delta(h_1 \xi_2 \xi_3)\} = \begin{cases} 0 & \text{if } \deg \xi f = 3 \\
\{-\Delta(f, h)\}_1 \xi_2 \xi_3 & \text{if } \deg \xi f = 2. \end{cases}$$

- $\deg \xi h = 1$. Then
  $$[i_2 f, i_1 h] = [i_2 f, i_2 h + \alpha(\Delta h)\xi_1 \xi_2 \xi_3] =$$
  $$i_2\{f, h\} - i_2(\Delta h) + i_2(\Delta h \cdot \sum \xi_i \partial_{\xi_i} f) + \alpha_{-\Delta(\Delta h)}(\xi_1 \xi_2 \xi_3).$$

- $\deg \xi h = 2$. Then
  $$[i_2 f, i_1 h] = [i_2 f, \alpha h] = ( -1)^{p(f)}i_2( A_h f) - \alpha_{(f \Delta h)} =$$
  $$\begin{cases} 
i_1\{\{f, \Delta h\}_1 \xi_2 \xi_3\} & \text{if } \deg \xi f = 0 \\
i_1(\Delta( f h) - f \Delta h) & \text{if } \deg \xi f = 1 \\
i_2( A_h f) - i_2(\Delta^{-1} D^3_x ( f \Delta h)) + i_1(\Delta^{-1} D^3_x ( f \Delta h)) & \text{if } \deg \xi f = 2 \\
- i_2( f D^3_x f) & \text{if } \deg \xi f = 3. \end{cases}$$

- $\deg \xi h = 3$. Then
  $$[i_2 f, i_1 h] = [i_2 f, \alpha_{\Delta^{-1}(D^3_x h)}] = -\alpha_{f D^3_x h} =$$
  $$\begin{cases} 0 & \text{if } \deg \xi f = 0 \\
i_1( -f \Delta h - f \Delta h) & \text{if } \deg \xi f = 1 \\
i_1( -f \cdot D^3_x g) & \text{if } \deg \xi f = 2 \\
i_1(\Delta^{-1}( D^3_x f \cdot D^3_x g)) - i_2(\Delta^{-1}(D^3_x f \cdot D^3_x g)) & \text{if } \deg \xi f = 3. \end{cases}$$

The final result is represented in the following tables.
### The brackets $[i_2 f, i_1 h]$ 

| $\text{deg}_\xi(f)$ | $\text{deg}_\xi(h) = 0$ | $\text{deg}_\xi(h) = 1$ |
|---------------------|-----------------------|-----------------------|
| 0                   | $i_2(\{f, \Delta(h_1 \xi_2 \xi_3)\})$ | $-i_1(\{\Delta(f_1 \xi_2 \xi_3), h\})$ |
| 1                   | $i_2(\{f, \Delta(h_1 \xi_2 \xi_3)\})$ | $i_1(\Delta^{-1}(f, \Delta h)) + i_2(f, h) - \Delta^{-1}(f, \Delta h)$ |
| 2                   | $-i_2(\{\Delta f, h\} \xi_1 \xi_2 \xi_3)$ | $i_2(\{\Delta f, h\} - \Delta(f) \xi_1 \xi_2 \xi_3)$ |
| 3                   | 0                      | $i_2(f \Delta(h) + \Delta(f) h)$ |

| $\text{deg}_\xi(f)$ | $\text{deg}_\xi(h) = 2$ | $\text{deg}_\xi(h) = 3$ |
|---------------------|-----------------------|-----------------------|
| 0                   | $i_1(\{f, \Delta h\} \xi_1 \xi_2 \xi_3)$ | 0                      |
| 1                   | $-i_1(\Delta(f h) + f \Delta h)$ | $i_1(-f \Delta(h) - \Delta(f) h)$ |
| 2                   | $i_1(\Delta^{-1} D_2^2(\Delta f h)) + i_2(A_h f - \Delta^{-1} D_2^2(\Delta f h))$ | $i_1(-f D_2^2 h)$ |
| 3                   | $i_2(-h D_2^2 f)$ | $i_1(\Delta^{-1}(D_2^2 f \cdot D_2^2 h)) - i_2(\Delta^{-1}(D_2^2 f \cdot D_2^2 h))$ |

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