Holographic Superconductors in Quasi-topological Gravity

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Abstract

In this paper we study (3+1) dimensional holographic superconductors in quasi-topological gravity which is recently proposed by R. Myers et.al.. Through both analytical and numerical analysis, we find in general the condensation becomes harder with the increase of coupling parameters of higher curvature terms. In particular, comparing with those in ordinary Gauss-Bonnet gravity, we find that positive cubic corrections in quasi-topological gravity suppress the condensation while negative cubic terms make it easier. We also calculate the conductivity numerically for various coupling parameters. It turns out that the universal relation of $\omega_g/T_c \simeq 8$ is unstable and this ratio becomes larger with the increase of the coupling parameters. A brief discussion on the condensation from the CFT side is also presented.
I. INTRODUCTION

The AdS/CFT correspondence links a \((d + 1)\) dimensional gravity theory in the bulk to a dual \(d\) dimensional quantum field theory on the boundary\([1–3]\). Recent progress indicates that this duality plays a significant role in studying various phenomena in condensed matter physics\([4–9]\), since it provides a powerful tool to describe some strongly correlated behaviors in the vicinity of quantum critical point which has not been well understood in conventional field theory. Particularly, in \([10, 11]\) Gubser shows the possibility that there is a spontaneous breaking of U(1) symmetry near the AdS black hole horizon in the bulk when the Higgs model couples to gravity. This mechanism leads to a phenomenon that AdS black holes can be surrounded by a nonvanishing scalar field \(\psi\) when the temperature is low enough, which calls for a re-understanding of the ‘no hair’ theorem of black hole. More importantly, based on gauge/gravity correspondence such a process of forming scalar hairs strongly implies a second-order phase transition occurring on the CFT side. Inspired by this observation, Hartnoll et al. constructed a holographic superconductor model exhibiting such a phase transition\([12]\). It is remarkable that in this model some basic features of superconductors can be accomplished below a critical temperature. Stimulated by this work, various holographic superconductors have been constructed in other gravity theories, for instance the Gauss-Bonnet (GB) gravity, M theory and Hořava-Lifshitz theory etc.\([13–23]\).

Based on AdS/CFT correspondence, the presence of higher order derivatives in AdS gravity means new couplings among operators in the dual CFT. So various higher curvature couplings will lead to more classes of dual field theories. However, in ordinary GB gravity only one quadratic coupling term involves, which greatly limits the range of dual field theories. In order to extend holographic studies to new classes of CFTs, we may consider introducing new terms with higher order derivatives into gravity, for instance a curvature-cubed term. A straightforward way is to add the cubic term in Lovelock gravity. Unfortunately, such a term is topological and has effects on equations of motion (EOM) only in very high dimensions. Until recently, Myers et al. construct a new higher derivative theory of gravity in 5 dimensional spacetime which contains not only the Gauss-Bonnet term but also a curvature-cubed interaction\([24]\).\(^1\) Unlike Lovelock gravity, this cubic term is not purely topological but

\(^1\) It is worthwhile to point out that before their work some important insights into the curvature-cubed interaction has been given in \([25]\). Recent relevant work can also been found in \([26, 29]\).
dynamically contributive to the evolution of fields in the bulk, thus it may be useful for us to apply the holographic studies to wider classes of field theories in ordinary dimension. In gauge/gravity duality, quasi-topological gravity theory is thought to be dual to the large $N$ limit of some conformal field theory without supersymmetry. In this theory equations of motion are the fourth order in derivatives in general backgrounds. However, the linearized equations reduce to the second order in AdS case, which implies the holographic description in this model is still under control. Moreover, exact black hole solutions with an asymptotically AdS behavior have been found and many interesting features have been revealed when different coupling parameters are chosen [24]. The holographic studies for these black hole solutions have been started and some recipes of AdS/CFT dictionary for this duality have been presented in [30]. Applying this to the hydrodynamics of dual fields, they found that the curvature-cubed term suppresses the ratio bound of the shear viscosity to entropy density to $\eta/s \approx \frac{0.1440}{4\pi}$, which is conjectured to be $\frac{1}{4\pi}$ in Einstein’s gravity. In this paper, we intend to continue the holographic studies on quasi-topological gravity by constructing a holographic model of superconductors in the probe limit. Employing both analytical and numerical methods, we obtain the critical temperature $T_c$ of the superconductors, and find that in general the condensation becomes harder with the increase of coupling parameters of higher curvature terms. In particular, comparing with those in ordinary Gauss-Bonnet gravity, we find that positive cubic corrections in quasi-topological gravity suppress the occurrence of phase transitions while negative cubic terms make it easier. Furthermore, we calculate the conductivity $\sigma$ numerically and find the universal relation of $\omega_g/T_c \approx 8$ is unstable and the ratio becomes larger with the increase of coupling parameters. According to the gauge/gravity duality, we find the dual description with direct parameters in the field theory on the boundary.

The paper is organized as follows. In section II, we quickly review the black hole physics in quasi-topological gravity. Then focusing on the black hole solution with positive coupling parameters we construct a holographic superconductor model and study the condensation of the scalar field analytically and numerically. In section III, the conductivity is calculated numerically and the ratio of the frequency gap to the critical temperature is discussed. We turn to investigate the holographic superconductivity for black hole solutions with negative coupling parameters in section IV. Its dual picture on the CFT side is also briefly discussed. Our conclusions and discussions are presented in the last section.
II. QUASI-TOPOLOGICAL HOLOGRAPHIC SUPERCONDUCTORS

In quasi-topological gravity, the bulk action of gravity in five-dimensional spacetime is given by

\[ S_g = \frac{1}{16\pi G_5} \int d^5x \sqrt{-g} \left[ R + \frac{12}{L^2} + \frac{\alpha L^2}{2} \mathcal{X}_4 + \frac{7\beta L^4}{8} \mathcal{Z}_5 \right], \tag{1} \]

where \( \alpha \) and \( \beta \) are Gauss-Bonnet coupling parameter and curvature-cubed interaction parameter, respectively. Here \( \mathcal{X}_4 \) is defined as

\[ \mathcal{X}_4 = R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} - 4 R_{\mu\nu} R^{\mu\nu} + R^2, \tag{2} \]

and \( \mathcal{Z}_5 \) is a curvature-cubed term with the form

\[ \mathcal{Z}_5 = R_{\mu\nu} R^{\mu\nu} \alpha_3 R_{\alpha\beta}^{\mu\nu} R_{\mu\nu} R^{\alpha\beta} \alpha \right] R^\alpha + 144 R_{\mu\nu} R^{\mu\nu} R^{\sigma} R^\sigma + 128 R_{\mu}^{\nu} R_{\rho}^{\mu} R_{\sigma}^{\nu} - 108 R_{\mu}^{\nu} R_{\nu}^{\mu} R_{\sigma}^{\nu} + 11 R^3. \tag{3} \]

The planar black hole solution with an asymptotically AdS behavior in quasi-topological gravity has been obtained in \[24\] as well. It is given as

\[ ds^2 = \frac{r^2}{L^2} (-N(r)^2 f(r) dt^2 + dx^2 + dy^2 + dz^2) + \frac{L^2}{r^2 f(r)} dr^2, \tag{4} \]

where \( N(r) \) is the lapse function. Inserting this metric into (1) and taking the variation \( \delta N \), one finds that \( f(r) \) should satisfy the following equation

\[ 1 - f(r) + \alpha f(r)^2 + \beta f(r)^3 = \frac{\eta^4}{r^4}, \tag{5} \]

where \( \eta \) is a constant and \( f(r) \) vanishes on the horizon of the black hole \( r_H \), namely \( f(r_H) = 0 \). Following Ref. \[24\], we choose \( \eta = r_H \) and \( N^2 = 1/f_\infty \), where \( f_\infty \) satisfies

\[ 1 - f_\infty + \alpha f_\infty^2 + \beta f_\infty^3 = 0. \tag{6} \]

Usually a black hole horizon exists if \( f(r \neq r_H) > 0 \), so with positive \( \alpha \) and \( \beta \) we have

\[ f_\infty = 1 + \alpha f_\infty^2 + \beta f_\infty^3 > 1, \tag{7} \]

which means the lapse function \( N \) is always smaller than one in this case. With the standard approach, one can find the Hawking temperature of the black hole is given by

\[ T = \frac{N}{4\pi} f'(r)|_{r=r_H} = \frac{N r_H}{\pi L^2}. \tag{8} \]
It will also be viewed as the temperature of the dual CFT on the boundary.

In the bulk of this black hole background, the action of matter is proposed as

\[
S_m = \int d^5x \sqrt{-g} \left[ -\frac{1}{4} F_{\mu\nu} F_{\mu\nu} - |\nabla \psi - iqA\psi|^2 - m^2 |\psi|^2 \right],
\]

(9)

where \( F_{\mu\nu} \) is the strength of a gauge field with U(1) symmetry and \( \psi \) is a charged scalar field. Through this paper we only consider the probe limit, which means the backreaction of matter fields on the background can be neglected if the charge is large enough. Taking the radial ansatz, \( \psi = \psi(r) \) and \( A_\mu = (\phi(r),0,0,0) \) and rescaling two fields by a factor \( q^{-2} \), we obtain the EOM of \( \psi \) and \( \phi \) as

\[
\psi'' + \left( \frac{g'}{g} + \frac{3}{r} \right) \psi' + \left( \frac{\phi^2}{N^2 g^2} - \frac{m^2}{L^2 g} \right) \psi = 0
\]

(10)

and

\[
\phi'' + \frac{3}{r} \phi' - \frac{2\psi^2}{g} \phi = 0,
\]

(11)

where \( g(r) = \frac{r^2 f(r)}{L^2} \) and the prime denotes a derivative with respect to \( r \). In the following analysis, we choose the mass term \( m^2 = -3/L^2 \) which is above the Breitenlohner-Freedman bound for initial stability [31]. In order to solve these equations we need to give some boundary conditions at the horizon and the asymptotically AdS region (\( r \to \infty \)):

- The regularity conditions at the horizon \( (r = r_H) \) give rise to

\[
\phi(r_H) = 0, \quad \psi(r_H) = \frac{g'}{m^2} r_H \psi'(r_H).
\]

(12)

- Near the AdS boundary \( (r \to \infty) \), the asymptotical behaviors of fields are like

\[
\phi(r) = \mu - \frac{\rho}{r^2}, \quad \psi = \frac{C_-}{r^{\lambda_-}} + \frac{C_+}{r^{\lambda_+}}.
\]

(13)

where \( \lambda_\pm = 2 \pm \sqrt{4 - 3\frac{L^2}{L^2}} \) with an effective AdS radius \( \tilde{L} = L/\sqrt{f_\infty} = LN \). In the above equation \( \mu \) and \( \rho \) are understood as the chemical potential and the charge density on the boundary respectively. From Eq. (7), it is easy to find that the value of \( \lambda_+ \) in quasi-topological gravity is larger than that in standard Einstein and Gauss-Bonnet theory.

Next, we need to find non-trivial solutions to the scalar field \( \psi \) in the bulk. Such kind of black hole hair is possible since gravity has a negative contribution to the effective mass term which leads to a breaking of the U(1) gauge symmetry. As the scalar field changes from \( \psi = 0 \) to \( \psi \neq 0 \) in the bulk, correspondingly one thinks that a phase transition occurs
around a critical temperature $T_c$ for the dual conformal field theory on the boundary. More explicitly, according to the recipe of AdS/CFT dictionary, the non-zero scalar field breaking the local U(1) symmetry in the bulk is actually dual to a condensation operator (order parameter) $O$ which breaks the global U(1) symmetry in the large $N$ field on the boundary, and the crucial relations are $\langle O_+ \rangle \sim C_+$ and $\langle O_- \rangle \sim C_-$, respectively. Here $C_-$ and $C_+$ can also be considered as the source and the vacuum expectation values of the operator. Therefore, a non-vanishing $\psi$ implies the occurrence of the condensation in CFT below the critical temperature. Next we set $C_- = 0$ and investigate the condensation of $\langle O_+ \rangle$ from analytical and numerical aspect separately. We remark that the other case of setting $C_+ = 0$ can be worked out in a similar manner.

A. Analytical calculation

At first we will analytically calculate the critical temperature for the phase transition of the holographic superconductors, and then demonstrate how the critical temperature behaves as coupling parameters $\alpha$ and $\beta$ change.

In this section we focus our discussion on positive coupling parameters. As pointed out in Ref. [24], if both coupling parameters $\alpha$ and $\beta$ are positive, only one real root of $f_3$ leads to a stable AdS black hole solution. It has the form as

$$f_3(r) = -\frac{1}{2}(u + v) - i\frac{\sqrt{3}}{2}(u - v) - \frac{\alpha}{3\beta}, \quad (14)$$

where

$$u = (q + \sqrt{q^2 - p^3})^{1/3}, \quad v = (q - \sqrt{q^2 - p^3})^{1/3}, \quad (15)$$

and

$$p = \frac{3\beta + \alpha^2}{9\beta^2}, \quad q = -\frac{2\alpha^3 + 9\alpha\beta + 27\beta^2(1 - \frac{r^4}{r^4_H})}{54\beta^3}. \quad (16)$$

To maintain this AdS solution stable, one more condition should be satisfied, which is $D_\infty = (q^2 - p^3)|_{r \to \infty} < 0$. This condition confines the values of $\alpha$ and $\beta$ to be chosen. In FIG. 1 we illustrate a border line for $\alpha$ vs. $\beta$, only below which this black hole solution makes sense. From this figure, we also notice that the larger the value of $\alpha$ is fixed, the smaller
FIG. 1: Below the border line is the range of $\alpha$ and $\beta$ which maintains the black hole stable.

the range of $\beta$ could take, and vice versa. Especially, if we let $\beta = 0$, the largest value of $\alpha$ would be 0.25, which is nothing but the Chern-Simons limit in GB gravity. Moreover, there also exists a limit $\beta = \frac{4}{27}$ when we take $\alpha = 0$. Keep this in mind, we will make all the following analysis in an appropriate range which is below the upper bound of $\alpha$ and $\beta$.

For simplicity, we define a dimensionless variable $z = \frac{r_H}{r}$. Then the EOM of two fields can be rewritten as

$$
\psi'' + \left(\frac{g'}{g} - \frac{1}{z}\right)\psi' + \frac{r_H^2}{z^4} \left(\frac{\phi^2}{N^2 g^2} + \frac{3}{L^2 g}\right) = 0 \quad (17)
$$

and

$$
\phi'' - \frac{1}{z}\phi' - \frac{r_H^2}{z^4} \frac{2\psi^2}{g} \phi = 0, \quad (18)
$$

where the prime denotes a derivative with respect to $z$ now. The corresponding boundary conditions can be expressed as follows.

- At the horizon ($z = 1$)

$$
\phi(1) = 0, \quad \psi(1) = \frac{g'|_{z=1}}{3} \psi'(1). \quad (19)
$$

- Near the boundary ($z \to 0$)

$$
\phi = \mu - \rho \frac{z^2}{r_H^2} \equiv \mu - q z^2, \quad \psi = C_+ z^{\lambda_+} + C_- z^{\lambda_-}, \quad (20)
$$

We take the Taylor expansion of the fields near the horizon as:

$$
\phi(z) = \phi(1) - \phi'(1)(1 - z) + \frac{1}{2} \phi''(1)(1 - z)^2 + \cdots, \quad (21)
$$

$$
\psi(z) = \psi(1) - \psi'(1)(1 - z) + \frac{1}{2} \psi''(1)(1 - z)^2 + \cdots. \quad (22)
$$
Then expanding (10) and (11) near the horizon gives

\[ \phi''(1) \approx \left(1 - \frac{L^2}{2} \psi(1)^2\right) \phi'(1), \quad (23) \]

\[ \psi''(1) \approx \left(-\frac{15}{8} \psi'(1) + 4\alpha\right) \psi'(1) - \frac{L^4}{32N^2r_H^2} \phi'(1)^2 \psi(1). \quad (24) \]

We obtain the following important expressions by inserting (23), (24) into (21) and (22), respectively

\[ \phi(z) \approx -\phi'(1)(1 - z) + \frac{1}{2} \left(1 - \frac{L^2}{2} \psi(1)^2\right) \phi'(1)(1 - z)^2 + \cdots, \quad (25) \]

\[ \psi(z) \approx \psi(1) - \frac{3}{4} \psi(1)(1 - z) + \left(-\frac{15}{64} + \frac{3\alpha}{2} - \frac{L^4}{64N^2r_H^2} \phi'(1)^2\right) \psi(1)(1 - z)^2 + \cdots. \quad (26) \]

Now, matching the solutions (20), (25) and (26) at any point \( z_m \) in the region \( z \in [0, 1] \), for instance \( z_m = \frac{1}{2} \), we obtain four continuity equations of wave functions

\[ \mu - \frac{1}{4}q = \frac{1}{2} \left(1 - \frac{L^2}{2} a^2\right), \quad (27) \]

\[ -q = b - \frac{1}{2} \left(1 - \frac{L^2}{2} a^2\right), \quad (28) \]

\[ C_+ \left(\frac{1}{2}\right)^{\lambda_+} = \frac{5}{8} a + \frac{1}{4} a \left(-\frac{15}{64} + \frac{3\alpha}{2} - \frac{L^4}{64N^2r_H^2} b^2\right), \quad (29) \]

\[ 2\lambda_+ C_+ \left(\frac{1}{2}\right)^{\lambda_+} = \frac{3}{4} a - a \left(-\frac{15}{64} + \frac{3\alpha}{2} - \frac{L^4}{64N^2r_H^2} b^2\right), \quad (30) \]

where we have defined \( \psi(1) = a \) and \(-\phi'(1) = b\) for simplicity. Combining (29) and (30), we obtain

\[ C_+ = \frac{13}{8} \frac{2\lambda_+}{\lambda_+ + 2} a, \quad (31) \]

where \( a \) can be solved from (28)

\[ a^2 = \frac{2q}{L^2 b} \left(1 - \frac{b}{2q}\right). \quad (32) \]

Moreover, combining (29) and (31) we get

\[ b = 8 \frac{Nr_H}{L^2} \sqrt{\frac{5\lambda_+ - 3}{2(\lambda_+ + 2)} - \frac{15}{64} + \frac{3\alpha}{2}}, \quad (33) \]

Then (32) can be rewritten as

\[ a^2 = \frac{2}{L^2 T_c^3} \frac{T_c^3}{T_c^3} \left(1 - \frac{T_c^3}{T_c^3}\right), \quad (34) \]
where the critical temperature $T_c$ is defined as

$$T_c = \left[ \frac{\rho}{4L} \frac{1}{\sqrt{\frac{5\lambda_+ - 3}{2(\lambda_+ + 2)} - \frac{15}{64} + \frac{3\alpha}{2}}} \right]^{\frac{1}{4}} \frac{N^{\frac{3}{2}}}{\pi L}. \quad (35)$$

Since $N$ is smaller than one and $\frac{5\lambda_+ - 3}{2(\lambda_+ + 2)}$ is a monotonic increasing function of $\lambda_+$, we find the value of $T_c$ depends on couplings and decreases as the value of $\alpha$ or $\beta$ increases. Thus we may conclude that the presence of the positive curvature-cubed term makes the condensation more difficult. Moreover, based on AdS/CFT dictionary $\langle O_+ \rangle \equiv LC_+ r_+^\lambda H_+ L^{-2\lambda}$, we express the condensation as

$$\frac{\langle O_+ \rangle}{T_c} = \frac{2\pi}{N} \left( \frac{13}{8} \frac{\sqrt{2}}{\lambda_+ + 2} \right)^{\frac{1}{\lambda_+}} \frac{T}{T_c} \left[ \frac{T^3}{T_c^3} \left( 1 - \frac{3}{T^3} \right) \right]^\frac{1}{2\lambda_+}. \quad (36)$$

Therefore the condensation $\langle O_+ \rangle \neq 0$ only if $T < T_c$, and its critical behavior $(1 - T^3/T_c^3)^{1/2}$ implies the phase transition is the second order. To present a more qualitative analysis about the impact of the curvature-cubed term on the condensation, we compare critical temperatures of condensation with those in standard Einstein gravity and GB gravity by substituting specific values of $\alpha$ and $\beta$ into (35).

In Einstein’s gravity, one has $T_c = 0.2017\rho^{1/3}/L$ by analytical calculation. In GB gravity if one sets $\alpha = 0.01$, the critical temperature $T_c = 0.2004\rho^{1/3}/L$. While in quasi-topological gravity, once the Gauss-Bonnet constant $\alpha$ is fixed, the critical temperature goes down as the curvature-cubed constant $\beta$ increases. For instance if we fix $\alpha = 0.01$, but increase $\beta$ from 0.001 to 0.01 and 0.1, $T_c$ decreases from $T_c = 0.2003\rho^{1/3}$ to 0.1995$\rho^{1/3}$ and 0.1889$\rho^{1/3}$. Similarly, fixing $\beta = 0.001$ but changing $\alpha$ from 0.0001 to 0.1 and 0.2, one has $T_c$ from 0.2016$\rho^{1/3}$ to 0.1880$\rho^{1/3}$ and 0.1653$\rho^{1/3}$. The explicit dependence of $T_c$ on coupling parameters $\alpha$ and $\beta$ is shown in FIG.2 and FIG.3. In FIG.3, we notice that $T_c$ experiences a quick-fall at the right-up corner when $\alpha$ and $\beta$ are large enough, implying that the analytical result (35) is no longer stable in this region, which is consistent with our previous discussion about the range of $\alpha$ and $\beta$ as shown in FIG.1.

At the end of this subsection we should mention that the analytical approximation given above works well only when the temperature is not too far from $T_c$. In order to have a

\footnote{Different from Ref.\[17\], we have set the lapse function $N = 1/\sqrt{f_{\infty}}$ rather than one, so that in our case the induced (3+1) metric on the boundary is conformal to the Minkowski space. Therefore the value of $T_c$ obtained here is a little bit smaller than that obtained in \[17\].}
FIG. 2: Relations between $T_c$ and the coupling parameters $\alpha$ and $\beta$, respectively. In the left figure the line is for $\alpha = 0.01$, while in the right figure the line is for $\beta = 0.001$.

FIG. 3: The critical temperature $T_c$ as a function of $\alpha$ and $\beta$.

cOMPLETE UNDERSTANDING ON THE RELATION BETWEEN THE CONDENSATION AND TEMPERATURE WE NEED TO SOLVE EQUATIONS OF MOTION IN A NUMERICAL WAY. THAT IS WHAT WE INTEND TO DO IN THE NEXT SUBSECTION.

B. Numerical result

We solve equations of motion (10) and (11) numerically and plot FIG.4 to demonstrate the condensation as a function of temperature. Numerically, we obtain the condensation of $\langle \mathcal{O}_+ \rangle$ with various positive values of coupling parameters $\alpha$ and $\beta$. We find that the critical temperature goes down as either the Gauss-Bonnet constant $\alpha$ or the curvature-cubed constant $\beta$ increases. Specifically, if we set $\alpha = 0.01$, but change $\beta$ from 0.001 to 0.01
FIG. 4: The condensation as a function of temperature with different values of coupling parameters. In the left figure, the Gauss-Bonnet parameter $\alpha$ is fixed at 0.01, while the curvature-cubed constant $\beta$ runs from 0.001 (red line), 0.01 (black line) to 0.1 (blue line) respectively. In the right figure $\beta$ is fixed at 0.001 but $\alpha$ varies from 0.0001 (blue line), 0.1 (red line) to 0.2 (black line). In both cases the condensation tends to increase with the coupling parameters.

and 0.1, numerical analysis shows that the critical temperature $T_c$ decreases from 0.1953$/rho^{1/3}$ to 0.1942$/rho^{1/3}$ and 0.1739$/rho^{1/3}$, respectively. On the other hand, when $\beta$ is fixed at 0.001 but $\alpha$ increases from 0.0001 to 0.1 and 0.2, $T_c$ decreases from 0.1977$/rho^{1/3}$ to 0.1755$/rho^{1/3}$ and 0.1531$/rho^{1/3}$, which is the same phenomenon as found in Gauss-Bonnet gravity. Therefore, we conclude that a phase transition occurs in the dual field theory, but both the curvature-cubed term and the Gauss-Bonnet term with positive couplings make the condensation harder in the sense that the critical temperature goes down as the coupling parameters increase. This conclusion agrees to our previous analytical results as well. However, it is worthwhile to point out that $\langle O_+ \rangle^{1/\lambda_+}/T_c$ has larger values when couplings increase.

III. ELECTRICAL CONDUCTIVITY

According to the recipe of AdS/CFT dictionary, Maxwell field $A_\mu$ in the bulk corresponds to a 4-electrical current density $J_\mu$ on the boundary. In order to calculate the conductivity $\sigma$ in the boundary theory, we firstly introduce perturbations of the Maxwell field $\delta A_x$, then consider the linear response to such perturbations. For simplicity let us suppose the vector potential is radially symmetric and time dependent as $\delta A_x(t,r) = A_x(r)e^{-i\omega t}dx$, which
satisfies the EOM as

\[ A''_x + \left( \frac{g'}{g} + \frac{1}{r} \right) A'_x + \left( \frac{\omega^2}{N^2 g^2} - \frac{2\psi^2}{g} \right) A_x = 0. \]  

(37)

We solve this equation under the incoming boundary condition near the horizon, namely

\[ A_x(r) \sim g(r)^{-i\frac{\omega}{4\pi r H}}, \]  

(38)

since this choice gives rise to the retarded Green function of the system. In the asymptotically AdS region \((r \to \infty)\), the general solution behaves as

\[ A_x = A^{(0)} + \frac{A^{(2)}}{r^2} + \frac{A^{(0)}\omega^2 N^4 \log \Lambda}{2 r^2}, \]  

(39)

where \(A^{(0)}, A^{(2)}\) and \(\Lambda\) are integration constants. As pointed out in [33], the logarithmic term in the solution leads to a divergence in the Green function but this can be removed by introducing an appropriate counter-term in the action. Taking into account the results obtained in the previous section and the incoming boundary condition, we can numerically solve the equation of motion (37) and obtain the values of factors \(A^{(0)}\) and \(A^{(2)}\). Moreover, the conductivity can be obtained by means of the retarded Green function

\[ \sigma(\omega) = \frac{1}{i\omega} G^R(\omega), \]  

(40)

where \(G^R\) can be calculated through the AdS/CFT dictionary as [32]

\[ G^R = - \lim_{r \to \infty} Ng(r)r A_x A'_x. \]  

(41)

Here we have normalized \(A_x\) on the boundary. Substituting the asymptotical solution into the above equations, we can finally obtain the relation between the conductivity and the frequency as

\[ \sigma = \frac{2A^{(2)}}{iN\omega A^{(0)}} + \frac{iN^3\omega}{2}. \]  

(42)

The numerical results with different couplings are illustrated in FIG. 5 and FIG. 6, where red lines and blue lines represent the real part and imaginary part of \(\sigma\), respectively. In FIG. 5, we fix \(\alpha = 0.01\) but set \(\beta\) to be 0.001, 0.01, 0.1 and 0.14, respectively. All these lines are plotted at temperature \(T/T_c = 0.153\). On the other hand, in FIG. 6 we fix \(\beta = 0.001\) but change \(\alpha\) from 0.0001, 0.1, 0.2 to 0.245 at temperature \(T/T_c = 0.158\). From the real
part of the conductivity, we notice a gap in the conductivity with the frequency $\omega_g$. In Einstein’s case, it has been found that there is a remarkable universal relation $\frac{\omega_g}{T_c} \simeq 8$ for various condensations [33]. This ratio is more than twice of the value in weakly coupled BCS theory, implying that the holographic superconductor is strongly coupled. In our case as both coupling parameters are small, the relation $\frac{\omega_g}{T_c} \simeq 8$ holds indeed. However, when quadratic and cubic corrections become strong, $\frac{\omega_g}{T_c}$ is no longer fixed but running with the values of coupling parameters. From these figures, we easily notice that this ratio enlarges with the increase of $\alpha$ or $\beta$, until it reaches a maximal value when coupling parameters take values on the border line as shown in FIG.1. This is quite similar to what happens in GB gravity[17]. But different from GB theory, the cubic term will intensively suppress the imaginary part of the conductivity when $\beta$ is large enough. For the imaginary part

FIG. 5: Conductivity for superconductors with fixed $\alpha = 0.01$ for different values of $\beta$ at the same temperature $T/T_c = 0.153$. 
FIG. 6: Conductivity for superconductors with fixed $\beta = 0.001$ for different values of $\alpha$ at the same temperature $T/T_c = 0.158$.

of the conductivity, there also exists a pole at $\omega = 0$, indicating that the real part of the conductivity contains a delta function according to the Kramers-Kronig relation.

IV. GENERAL HOLOGRAPHIC SUPERCONDUCTORS WITH NON-POSITIVE COUPLINGS

In previous sections we have studied the holographic superconductivity for the case that both coupling parameters $\alpha$, $\beta$ are positive. In this section we intend to study the holographic superconductor in quasi-topological gravity with non-positive couplings in a parallel way. The main difference is that for these coupling parameters, one need consider different kinds of AdS black hole backgrounds, which comes from the fact that there exist three distinct
solutions to Eq.\ref{5}. Besides $f_3$, the other two solutions are

$$f_1(r) = u + v - \frac{\alpha}{3\beta}, \quad (43)$$

$$f_2(r) = -\frac{1}{2}(u + v) + i\frac{\sqrt{3}}{2}(u - v) - \frac{\alpha}{3\beta}, \quad (44)$$

where $u$ and $v$ are defined in Eq.\ref{15}. As illustrated in figure one in \cite{24}, for different regions in parameter space, we need choose the appropriate solution to obtain stable AdS background. Next we focus our discussion on the cases that $\alpha$ and $\beta$ are not both positive. Without loss of generality, we may pick up some points in different regions of the parameter space and obtain the corresponding critical temperatures numerically. The results are summarized as TABLE \ref{table1}.

|   | $f_1$    | $f_2$    | $f_3$    |
|---|----------|----------|----------|
| $\alpha$ | $-0.25$  | $-0.1$   | $-0.05$  | $0.24$  | $0.12$  | $0.04$  | $-0.10$ | $-0.10$ | $-0.35$ |
| $\beta$  | $-0.02$  | $-0.0002$| $-0.0002$| $-0.0020$| $-0.0020$| $-0.0004$| $0.1000$| $0.0005$| $0.1000$|
| $T_c/\rho^{1/3}$ | $0.2438$ | $0.2152$ | $0.2067$ | $0.1384$ | $0.1750$ | $0.1908$ | $0.1986$ | $0.2151$ | $0.2446$ |

TABLE I: The dependence of the critical temperature on two couplings for different black hole backgrounds.

From this table, one finds a general rule that for smaller coupling parameters the value of the critical temperature becomes larger. More explicitly, as one of the parameters is fixed, we find the condensation is always becoming harder with the increase of the other parameter. This observation is consistent with our results obtained in previous sections. As a result, when the coupling parameters take negative values, it is possible to construct a superconductor model with a critical temperature below the one in Einstein’s gravity theory, which is $T_{co} \equiv 0.198\rho^{1/3}$. As a matter of fact, when both parameters are negative, our numerical results show that the critical temperature is always above $0.198\rho^{1/3}$, which is in contrast to our previous results where the temperature is always below this value for both positive parameters. A more delicate situation occurs when one parameter is positive while the other is negative. Whether the critical temperature is above or below $T_{co}$ depends on the competition of the fluctuation effects due to these two couplings. One special case is that these two effects are cancelled out such that the critical temperature is equal to $T_{co}$. 

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FIG. 7: The trajectory of $T_c = T_{co} = 0.198 \rho^{1/3}$ in parameter space. Comparing with that in Einstein’s theory, the condensation becomes harder in the white region but easier in the purple region.

![Diagram showing the trajectory of $T_c = T_{co}$]

FIG. 8: The condensations for $\alpha > 0$ and $\beta < 0$ in the left figure, $\alpha < 0$ and $\beta > 0$ in the middle figure, and $\alpha < 0$ and $\beta < 0$ in the right one. Notice that the corresponding black hole backgrounds are specified by $f_2$, $f_3$ and $f_1$, respectively.

![Three diagrams showing different condensation scenarios]

For explicitness we plot the trajectory of $T_c = T_{co}$ numerically in the parameter space and the final result is illustrated in FIG.7. In this figure it is manifest that in the white region $T_c < T_{co}$ such that the condensation is suppressed, while in the purple region $T_c > T_{co}$ and the condensation is easier comparing with that in Einstein’s gravity theory.

Next we consider the electrical conductivity for non-positive coupling parameters. Following the algebra in previous section, it is straightforward to obtain the numerical results of the conductivity for different coupling parameters. For convenience we choose those values in TABLE II for $\alpha$ and $\beta$ and show the numerical results in FIG.8-FIG.11 separately. From these figures, we notice that the ratio $\omega_g/T_c$ is unstable which is similar to the case presented
FIG. 9: The conductivities for $\alpha > 0$ and $\beta < 0$. With the increase of $\alpha$ and the decrease of $\beta$, the value of $\omega_g/T_c$ runs from about 8 to more than 12.

FIG. 10: The conductivities for $\alpha < 0$ and $\beta > 0$. A spike exists in the rightmost figure.

in section III. One difference is that this ratio may shift to the region less than 8 when the coupling parameters are negative enough. Moreover, we notice that there exist extra spikes that appear inside the gap as shown in FIG.10 and FIG.11. These spikes emerge only when coupling terms are negative enough, and such a phenomenon has also been discussed in other holographic superconductor models when the mass term is close to the BF bound[34–36].

FIG. 11: The conductivities for $\alpha < 0$ and $\beta < 0$. A spike also emerges in the rightmost figure.
FIG. 12: The critical temperature changes with $\delta$ and $t4$

By now we have investigated the superconductivity from the side of bulk gravity. In the end of this section we remark that perhaps it is instructive to understand this phenomenon from the side of conformal field theory. Thanks to the AdS/CFT dictionary, the correspondence between the couplings in the bulk and the couplings on the boundary is manifest in quasi-topological gravity\[^{30}\]. On the boundary, the dual CFT is characterized by central charges, $c$ and $a$, and flux parameters, $t2$ and $t4$. Explicitly, these parameters can be related to the coupling constants in the bulk as follows

$$\delta = \frac{c - a}{c} = \frac{4f_\infty(\alpha - 3\beta f_\infty)}{1 - 2\alpha f_\infty - 3\beta f_\infty^2}, \quad (45)$$

$$t2 = \frac{24f_\infty(\alpha - 87f_\infty\beta)}{1 - 2\alpha f_\infty - 3\beta f_\infty^2} \quad (46)$$

$$t4 = \frac{3780f_\infty^2\beta}{1 - 2\alpha f_\infty - 3\beta f_\infty^2} \quad (47)$$

where $f_\infty$ satisfies Eq.(7). Therefore, we may establish a direct relation between the critical temperature of the phase transition and the parameters in strongly coupled system, namely $a$, $c$, $t2$ and $t4$. From Eq.$(45)$\[^{47}\] and $(46)$, we know any change of the coupling parameters in the gravity theory will lead to a change of parameters $a$, $c$ and $t2$, $t4$ on the CFT side. However, since there are only two free couplings in the gravity theory, two free parameters is enough to describe the correlations on the CFT side. For simplicity, we take $\delta$ and $t4$ as the free parameters and make a 3D plot to demonstrate how $T_c$ changes with the running of $\delta$ and $t4$, which is illustrated in FIG. 12. From this figure, we notice $T_c$ depends on $\delta$ more sensitively, but in general we find that the dependence of $T_c$ on the parameters is not so simple as that
on the gravity side. Thus a clear understanding on this numerical result from the CFT side is still missing. We expect further investigation would disclose this with more details.

V. DISCUSSION AND CONCLUSIONS

In this paper we have constructed a (3+1) holographic superconductor model in quasi-topological gravity which contains a curvature-cubed interaction term. Firstly, we study the condensation of the scalar field in the probe limit, and obtain the relation between the condensation and the temperature. The results obtained through an analytical approximation are compatible with those through numerical analysis as well. Secondly, we find higher curvature terms with both positive coupling parameters suppress the phase transition such that the condensation becomes harder comparing with those in ordinary Einstein’s gravity theory. From a physical point of view, this is because the positive correction terms in the quasi-topological gravity which describe quantum fluctuations in the bulk will lead to thermal fluctuations of the order parameter in the dual field theory and thus suppress the occurrence of the spontaneous breaking of the U(1) symmetry. Nevertheless, since we study the holographic superconductor in a higher dimensional spacetime, these quantum fluctuations are not strong enough to destroy the condensation, in contrast to the (2+1) holographic superconductors where the Coleman-Mermin-Wagner theorem is applicable [38].

Thirdly, we calculate the conductivity of holographic superconductors numerically and find a running ratio $\omega_g/T_c$ which becomes larger with the increase of couplings parameters. Moreover, we find the cubic interaction term suppresses the imaginary part. Especially, when the cubic coupling $\beta$ is large enough, the line of imaginary part of the conductivity is always below the one of real part. Finally, we extend our discussion to the general case where the coupling parameters may be non-positive. A general rule we found is that no matter couplings are positive or negative, the critical temperature always decreases when the couplings increase. However, on the side of CFT it seems hard to figure out any general rule which can describe its dependence on central charges and flux parameters on the boundary. In addition, the ratio $\omega_g/T_c$ is always unstable but increasing when the coupling parameters become larger.

Through this paper we have only considered the simplest s-wave superconductors. Recent progress have shown that there also exist holographic p-wave and d-wave superconductors [39, 40]. In these models the order parameters are no longer scalar and the phase transition...
corresponds to the symmetry breaking due to a condensation of non-Abelian gauge fields. Some non-conventional features which are different from s-wave ones have already been disclosed. Our investigation on p-wave superconductor in quasi-topological gravity is under progress and will be presented elsewhere[41].

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