ON INJECTIVE MODULES AND SUPPORT VARIETIES FOR THE SMALL QUANTUM GROUP

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Abstract. Let $u_\zeta(\mathfrak{g})$ denote the “small quantum group” associated to the simple complex Lie algebra $\mathfrak{g}$, with parameter $q$ specialized to a primitive $\ell$-th root of unity $\zeta$ in the field $k$. Generalizing a result of Cline, Parshall and Scott, we show that if $M$ is a finite-dimensional $u_\zeta(\mathfrak{g})$-module admitting a compatible torus action, then the injectivity of $M$ as a module for $u_\zeta(\mathfrak{g})$ can be detected by the restriction of $M$ to certain “root subalgebras” of $u_\zeta(\mathfrak{g})$. If $\text{char}(k) = p > 0$, then this injectivity criterion also holds for the higher Frobenius–Lusztig kernels $U_\zeta(G_r)$ of the quantized enveloping algebra $U_\zeta(\mathfrak{g})$. Now suppose that $M$ lifts to a $U_\zeta(\mathfrak{g})$-module. Using a new rank variety type result for the support varieties of $u_\zeta(\mathfrak{g})$, we prove that the injectivity of $M$ for $u_\zeta(\mathfrak{g})$ can be detected by the restriction of $M$ to a single root subalgebra.

Introduction

Let $k$ be an algebraically closed field of characteristic $p > 0$, and let $G$ be the semisimple, simply-connected algebraic group over $k$ with root system $\Phi$. Fix a maximal torus $T \subset G$, defined and split over the prime field $\mathbb{F}_p$, such that $\Phi$ is the root system of $T$ in $G$. For $\alpha \in \Phi$, let $U_\alpha$ be the corresponding one-dimensional root subgroup in $G$. Set $U = \langle U_\alpha : \alpha \in \Phi^- \rangle$, the unipotent radical of the Borel subgroup $B = TU \subset G$. Let $F : G \to G$ denote the Frobenius morphism, and for $H$ a closed $F$-stable subgroup of $G$, let $H_r$ denote the (scheme-theoretic) kernel of the $r$-th iterate $F^r$ of the Frobenius morphism $F|_H : H \to H$.

In their paper [9], Cline, Parshall and Scott showed that a finite-dimensional $TG_r$-module $M$ is injective if and only if its restriction $M|_{U_{\alpha,r}}$ is injective for each root $\alpha \in \Phi$. They proved this criterion by reducing to the case of a finite-dimensional $TB_r$-module $M$, and then arguing by induction on the dimension of $M$. The induction argument was combinatorial in nature, and relied on a well-chosen ordering for the positive roots in $\Phi$. Later, Friedlander and Parshall [17, 18] deduced a geometric proof of the injectivity criterion in the special case $r = 1$ by studying the support variety $\mathcal{V}_{G_1}(M)$, a conical subvariety of the affine space $N$ of nilpotent elements in $\mathfrak{g} := \text{Lie}(G)$. Their breakthrough was to provide a rank variety type interpretation for $\mathcal{V}_{G_1}(M)$. In particular, they determined that $M|_{U_{\alpha,1}}$ is injective if and only if the corresponding root vector $f_{\alpha} \in \mathfrak{g}$ is not an element of $\mathcal{V}_{U_1}(M)$. Since $\mathcal{V}_{U_1}(M)$ is naturally a $T$-space (by the assumption of the $T$-action on $M$), and since any $T$-stable subvariety of $\mathfrak{g}$ must contain a root vector, this proved the criterion.

Now let $k$ be an algebraically closed field of characteristic $p \neq 2$, and let $\ell$ be an odd positive integer. If $\Phi$ has type $G_2$, then assume also that $p \neq 3$ and that $\ell$ is coprime to 3. Let $U_\zeta(\mathfrak{g})$ be the quantized enveloping algebra (Lusztig form) associated to $\mathfrak{g}$, with parameter $q$ specialized to a primitive $\ell$-th root of unity $\zeta \in k$. Let $u_\zeta(\mathfrak{g})$ be the “small quantum group,” the Hopf algebraic kernel of the quantum Frobenius morphism $F_\zeta : U_\zeta(\mathfrak{g}) \to \text{hy}(G)$. In this paper we generalize the results of [9] and [17, 18] to deduce criteria for detecting the injectivity of modules over $u_\zeta(\mathfrak{g})$ that admit compatible actions by the quantum torus $U_\zeta^0 \subset U_\zeta(\mathfrak{g})$. Our first main result (Theorem 3.1.1) is that a finite-dimensional rational $U_\zeta^0u_\zeta(\mathfrak{g})$-module $M$ is injective if and only if the restriction $M|_{u_\zeta(f_\alpha)}$ is injective for each root subalgebra $u_\zeta(f_\alpha) \subset u_\zeta(\mathfrak{g})$ ($\alpha \in \Phi$). If $p > 0$, then the injectivity

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of a module $M$ for the higher Frobenius–Lusztig kernels $U_\zeta(G_r)$ of $U_\zeta(g)$ can also be detected by the restriction of $M$ to the higher Frobenius–Lusztig kernels $U_\zeta(U_{\alpha,r})$ of the root subalgebras. The problem of identifying subalgebras of a given Hopf algebra that detect injectivity in the above manner is a topic of historical and current interest \cite{31, 19, 9, 32}.

Next, we study cohomological support varieties for $u_\zeta(g)$ and its Borel subalgebras $u_\zeta(b)$ and $u_\zeta(b^+).$ It is known (in characteristic zero) by results of Ginzburg and Kumar \cite{20} and (in positive characteristics) by results of the author \cite{13} that the $u_\zeta(g)$-support variety $V_{u_\zeta(g)}(M)$ of a $u_\zeta(g)$-module $M$ identifies naturally with a conical subvariety of the nullcone $N \subset g,$ and that the $u_\zeta(b)$-support variety $V_{u_\zeta(b)}(M)$ identifies naturally with a conical subvariety of $u := \text{Lie}(U).$ The structure of $V_{u_\zeta(b)}(M)$ is known, for example, if $M = H^0_\zeta(\lambda)$ is an induced module, by work of Ostrik \cite{29} and of Bendel, Nakano, Parshall and Pillen \cite{6}, and if $M = L_\zeta(\lambda)$ is a simple module, by work of the author, Nakano and Parshall \cite{14}. Beyond these special cases few explicit calculations are known. A primary obstruction to computing $V_{u_\zeta(g)}(M)$ for arbitrary $M$ is the lack of a theory of rank varieties for $u_\zeta(g)$-modules, or, more generally, a theory of rank varieties for arbitrary finite-dimensional Hopf algebras. Some partial progress in this direction has been made; see, for example, \cite{7, 30, 32}.

In this paper we contribute to the development of a theory of rank varieties for the Borel subalgebra $u_\zeta(b)$ by proving a partial generalization of Friedlander and Parshall’s result on the support varieties of $p$-unipotent restricted Lie algebras. Specifically, in Section 4 we show that the restriction $M|_{u_\zeta(f_{\alpha})}$ is injective if and only if the root vector $f_{\alpha} \in g$ is not contained in $V_{u_\zeta(b)}(M).$ This enables us in Section 5.1 to provide a second, geometric proof of the $r = 0$ version of Theorem 3.1.1. Finally, in Section 5.2 we consider the case when $M$ is a $U_\zeta(g)$-module. Then $V_{u_\zeta(g)}(M)$ is naturally a $G$-stable subvariety of the nullcone $N.$ In this case, we use the structure of nilpotent orbits in $N$ to deduce that $M$ is injective for $u_\zeta(g)$ if and only if the restriction $M|_{u_\zeta(f_{\alpha h})}$ is injective for the root subalgebra corresponding to the highest long root $\alpha_h \in \Phi^+.$

It is our hope that partial rank variety results in Section 4 could be extended to support varieties for the entire small quantum group $u_\zeta(g).$ We also hope that rank variety results might help answer the question of naturality between support varieties over $u_\zeta(g)$ and $u_\zeta(b),$ that is, given a finite-dimensional $u_\zeta(g)$-module $M,$ whether the intersection $V_{u_\zeta(g)}(M) \cap u$ is equal to the support variety $V_{u_\zeta(b)}(M).$ This is an issue which is easily settled in the affirmative for Frobenius kernels of algebraic groups using rank varieties, but which remains open for small quantum groups.

1. Preliminaries

1.1. Quantized enveloping algebras. Let $\Phi$ be a finite, indecomposable root system. Fix a set of simple roots $\Pi \subset \Phi,$ and let $\Phi^+, \Phi^-$ be the corresponding sets of positive and negative roots in $\Phi.$ Let $\mathbb{Z}\Phi$ denote the root lattice of $\Phi,$ $X$ the weight lattice of $\Phi,$ and $X^+ \subset X$ the subset of dominant weights. Write $X_{p^r \ell} = \{ \lambda \in X^+: (\lambda, \beta^\vee) < p^r \ell \ \forall \beta \in \Pi \}$ for the set of $p^r \ell$-restricted dominant weights. Let $W$ denote the Weyl group of $\Phi.$ It is generated by the simple reflections $\{s_\beta: \beta \in \Pi\}.$

The root system $\Phi$ spans a real vector space $\mathbb{E},$ possessing a positive definite, $W$-invariant inner product $(\cdot, \cdot),$ normalized so that $(\alpha, \alpha) = 2$ if $\alpha \in \Phi$ is a short root.

Let $k$ be a field of characteristic $p \neq 2$ (and $p \neq 3$ if $\Phi$ has type $G_2.$) Let $q$ be an indeterminate over $k.$ Then the quantized enveloping algebra $U_q = U_q(g)$ is the $k(q)$-algebra defined by generators $\{E_\alpha, F_\alpha, K_\alpha, K^{-1}_\alpha: \alpha \in \Pi\}$ and relations as in \cite[Chapter 4]{22}. Multiplication in $U_q$ induces vector space isomorphisms $U_q \cong U_q^- \otimes U_q^0 \otimes U_q^+ \cong U_q^+ \otimes U_q^0 \otimes U_q^- \cong U_q$ (triangular decomposition), where $U_q^-$ (resp. $U_q^0, U_q^+$) is the $k(q)$-subalgebra of $U_q$ generated by $\{F_\alpha: \alpha \in \Pi\}$ (resp. $\{K_\alpha, K^{-1}_\alpha: \alpha \in \Pi\}, \{E_\alpha: \alpha \in \Pi\}).$ The algebra $U_q$ is a Hopf algebra, with Hopf algebra structure maps defined in \cite[§4.8]{22}. The algebras $U_q^+ U_q^0$ and $U_q^- U_q^0$ are Hopf subalgebras of $U_q,$ but $U_q^+$ and $U_q^-$ are not.
Let \( \ell \in \mathbb{N} \) be an odd positive integer, with \( \ell \) coprime to 3 if \( \Phi \) has type \( G_2 \). Fix a primitive \( \ell \)-th root of unity \( \zeta \in k \). Set \( A = k[q, q^{-1}] \). Then \( k \) is naturally an \( A \)-module under the specialization \( q \mapsto \zeta \). For \( n \in \mathbb{N} \), the divided powers \( E_n^{(\alpha)} \), \( F_n^{(\alpha)} \) are defined in \([22, \S 8.6]\). Let \( U_A \) be the \( A \)-subalgebra of \( U_q \) generated by \( \left\{ E_n^{(\alpha)}, F_n^{(\alpha)}, K, K^{-1} : \alpha \in \Pi, n \in \mathbb{N} \right\} \).

Set \( U_k = U_A \otimes_A k \). Then the algebra \( U_\zeta = U_\zeta(g) \) is defined as the quotient of \( U_k \) by the two-sided ideal \( (K_\alpha \otimes 1 - 1 \otimes K_\alpha) : \alpha \in \Pi \). By abuse of notation, we denote the generators \( E_\alpha, F_\alpha, K, K^{-1} \in U_\zeta \) as well as their images in \( U_\zeta \) by the same symbols. The algebra \( U_\zeta \) inherits a triangular decomposition from \( U_q \).

Fix a choice of reduced expression \( w_0 = s_{\beta_1}s_{\beta_2} \cdots s_{\beta_N} \) (\( \beta_i \in \Pi \)) for the longest word \( w_0 \in W \). So \( N = |\Phi^+| \). For \( 1 \leq i \leq N \), set \( w_i = s_{\beta_1} \cdots s_{\beta_{i-1}} \) (so \( w_1 = 1 \)), and set \( \gamma_i = w_i(\beta_i) \). Then \( \{\gamma_1, \ldots, \gamma_N\} \) is a convex ordering of the positive roots in \( \Phi^+ \), that is, if \( \gamma_i + \gamma_j = \gamma_l \) with \( i < j \), then \( i < l < j \).

Now for \( \alpha \in \Pi \), let \( T_\alpha \) denote Lusztig’s automorphism of \( U_q \), as defined in \([22, \text{Chapter 8}]\). Since the \( T_\alpha \) satisfy the braid group relations for \( W \), there exists for each \( w \in W \) a well-defined automorphism \( T_w \) of \( U_q \). For each \( 1 \leq i \leq N \), define the root vetors \( E_{\gamma_i} = T_w(E_{\beta_i}) \) and \( F_{\gamma_i} = T_w(F_{\beta_i}) \). Then the collection of monomials \( \{E_{\gamma_1}^{a_1} \cdots E_{\gamma_N}^{a_N} : a_i \in \mathbb{N}\} \) (resp. \( \{F_{\gamma_1}^{a_1} \cdots F_{\gamma_N}^{a_N} : a_i \in \mathbb{N}\} \)) forms a PBW-type basis for \( U_q^+ \) (resp. \( U_q^- \)). Replacing the root vectors by their divided powers, we obtain \( A \)-bases

\[
(1.1.1) \quad \left\{ E_{\gamma_1}^{(a_1)} \cdots E_{\gamma_N}^{(a_N)} : a_i \in \mathbb{N} \right\} \quad \text{and} \quad \left\{ F_{\gamma_1}^{(a_1)} \cdots F_{\gamma_N}^{(a_N)} : a_i \in \mathbb{N} \right\}
\]

for the algebras \( U_A^+ \) and \( U_A^- \), respectively, which project onto \( k \)-bases for \( U_q^+ \) and \( U_q^- \).

The following lemma, describing “commutation” relations between the root vectors in \( U_q \), generalizes an observation of Levendorskii and Soibelman \([25]\).

**Lemma 1.1.1.** Let \( S \subset A = k[q, q^{-1}] \) be the multiplicatively closed set generated by

\[
\{1\} \quad \text{if} \quad \Phi \text{ has type } ADE,
\]

\[
\{q^2 - q^{-2}\} \quad \text{if} \quad \Phi \text{ has type } BCF,
\]

\[
\{q^2 - q^{-2}, q^3 - q^{-3}\} \quad \text{if} \quad \Phi \text{ has type } G_2.
\]

Set \( \mathcal{A} = S^{-1}A \), and let \( 1 \leq i < j \leq N \). Then in \( U_q \) we have

(a) \( E_{\gamma_i}E_{\gamma_j} = q^{(\gamma_i, \gamma_j)}E_{\gamma_j}E_{\gamma_i} + (*) \), where \( (*) \) is an \( \mathcal{A} \)-linear combination of monomials \( E_{\gamma_1}^{a_1} \cdots E_{\gamma_N}^{a_N} \), with \( a_s = 0 \) unless \( i < s < j \).

(b) \( F_{\gamma_i}F_{\gamma_j} = q^{(\gamma_i, \gamma_j)}F_{\gamma_j}F_{\gamma_i} + (*) \), where \( (*) \) is an \( \mathcal{A} \)-linear combination of monomials \( F_{\gamma_1}^{a_1} \cdots F_{\gamma_N}^{a_N} \), with \( a_s = 0 \) unless \( i < s < j \).

**Proof.** Parts (a) and (b) are equivalent: apply the algebra automorphism \( \omega \) defined below in \((1.2)\). When \( g \) has rank two, part (a) can be verified by direct calculation, using, for example, the QuaGroup package of the computer program GAP (cf. also \([23]\)). From the rank two case the result is deduced for arbitrary \( g \) by the arguments in the proof of \([11, \text{Theorem 9.3}(iv)]\). \( \square \)

**Remarks 1.1.2.** (1) Versions of Lemma 1.1.1 appear in the literature \([12, 11]\) with our choice for the ring \( \mathcal{A} \) replaced by \( \mathbb{Q}[q, q^{-1}] \). Direct calculation in types \( B_2 \) and \( G_2 \) shows that such a formulation is incorrect and that the extra denominators are necessary whenever \( \Phi \) has two root lengths. Since the generators of \( S \) do not map to zero under the specialization \( q \mapsto \zeta \), we get that relations of Lemma 1.1.1 also hold for \( U_\zeta \).

(2) According to \([33, \text{Theorem 2.4}]\), any permutation of the ordering of the root vectors in (1.1.1) also yields an \( A \)-basis for \( U_q^+ \), and similarly for (1.1.2) and \( U_q^- \).
1.2. Frobenius–Lusztig kernels. The elements \( \{E_\alpha, F_\alpha, K_\alpha : \alpha \in \Pi \} \) of \( U_\zeta \) generate a finite-dimensional Hopf subalgebra of \( U_\zeta \), called the small quantum group and denoted by \( u_\zeta = u_\zeta(\mathfrak{g}) \). It is a normal Hopf subalgebra of \( U_\zeta \), and the Hopf algebraic quotient \( U_\zeta//u_\zeta \) is isomorphic to \( \hy(G) \), the hyperalgebra of the semisimple, simply-connected algebraic group \( G \) over \( k \) with root system \( \Phi \). The quotient map \( F_\zeta : U_\zeta \to \hy(G) \) was constructed by Lusztig [20 §8.10–8.16], and is called the quantum Frobenius morphism. For this reason, the algebra \( u_\zeta \) is also called the Frobenius–Lusztig kernel of \( U_\zeta(\mathfrak{g}) \). The subalgebra \( u_\zeta^0 := u_\zeta(\mathfrak{g}) \cap U_\zeta^0 \) is isomorphic to the group ring \( k(\mathbb{Z}_\ell)\rank(\mathfrak{g}) \). Since we assumed \( \zeta \) to be a primitive \( \ell \)-th root of unity in the field \( k \) of characteristic \( p \), we must have \( p \nmid \ell \). Then \( u_\zeta^0 \) is a semisimple algebra.

Fix \( r \in \mathbb{N} \), and suppose \( p > 0 \). Define \( U_\zeta(G_r) \) to be the Hopf-subalgebra of \( U_\zeta \) generated by

\[
\{ E_\alpha, E_\alpha^{(p^\ell)}, F_\alpha, F_\alpha^{(p^\ell)}, K_\alpha : \alpha \in \Pi, 0 \leq i \leq r - 1 \} \subset U_\zeta(\mathfrak{g}).
\]

Then \( U_\zeta(G_r) \) is a finite-dimensional subalgebra of \( U_\zeta \), and \( F_\zeta(U_\zeta(G_r)) = \hy(G_r) \), the hyperalgebra of the \( r \)-th Frobenius kernel of \( G \). We call \( U_\zeta(G_r) \) the \( r \)-th Frobenius–Lusztig kernel of \( U_\zeta \), and collectively we refer to the \( U_\zeta(G_r) \) with \( r \geq 1 \) as the higher Frobenius–Lusztig kernels of \( U_\zeta \). If \( r = 0 \), then \( U_\zeta(G_r) \) reduces to \( u_\zeta \), the small quantum group. The higher Frobenius–Lusztig kernels of \( U_\zeta \) are defined only if \( p = \text{char}(k) > 0 \). Indeed, if \( \text{char}(k) = 0 \), then the algebra generated by the set \( \{ E_\alpha, E_\alpha^{(p^\ell)}, F_\alpha, F_\alpha^{(p^\ell)}, K_\alpha : \alpha \in \Pi, 0 \leq i \leq r - 1 \} \) is all of \( U_\zeta \). Since most of the results presented in this paper are characteristic-independent, we have adopted the position of stating results whenever possible for the higher Frobenius–Lusztig kernels \( U_\zeta(G_r) \) of \( U_\zeta \), with the understanding that the reader should take \( r = 0 \) whenever \( p = 0 \).

We will be concerned with certain distinguished subalgebras of \( U_\zeta \) corresponding to the subgroup schemes \( U_r, B_r \), and \( TG_r \) of \( G \). Define \( U_\zeta(B) = U_\zeta U_\zeta^-, U_\zeta(U_r) = U_\zeta^- \cap U_\zeta(G_r), U_\zeta(B_r) = U_\zeta(B) \cap U_\zeta(G_r), \) and \( U_\zeta(TG_r) = U_\zeta^0 U_\zeta(G_r). \) Then, for example, the algebra \( U_\zeta(U_r) \) admits a basis consisting of all monomials \( \{ U_\zeta \} \) with \( 0 \leq a_i < p^\ell \). If \( r = 0 \), then write \( u_\zeta(\mathfrak{b}), u_\zeta(\mathfrak{u}), \) and \( u_\zeta^0 u_\zeta(\mathfrak{g}) \) for \( U_\zeta(B_r), U_\zeta(U_r), \) and \( U_\zeta(TG_r) \), respectively. We will also be concerned with certain subalgebras of \( U_\zeta(U_r) \) generated by root vectors. For each \( \alpha \in \Phi^+ \), define \( U_\zeta(U_{\alpha,r}) \) to be the subalgebra of \( U_\zeta(U_r) \) generated by the elements

\[
\{ F_\alpha, F_\alpha^{(p^\ell)}, \ldots, F_\alpha^{(p^{\ell - 1})} \} .
\]

Then \( U_\zeta(U_{\alpha,r}) \) admits a basis consisting of all divided powers \( F_\alpha^{(n)} \) with \( 0 \leq n < p^\ell \). As an algebra, \( U_\zeta(U_{\alpha,r}) \) is isomorphic to the truncated polynomial ring

\[
k[Y, X_0, X_1, \ldots, X_{r-1}]/(Y^{p^\ell}, X_0^{p^\ell}, X_1^{p^\ell}, \ldots, X_{r-1}^{p^\ell}).
\]

The isomorphism maps \( F_\alpha \mapsto Y \) and \( F_\alpha^{(p^\ell)} \mapsto X_i \). If \( r = 0 \), then \( U_\zeta(U_{\alpha,r}) \) reduces to the subalgebra \( u_\zeta(f_\alpha) \) of \( u_\zeta \) generated by \( F_\alpha \). Finally, for each \( 1 \leq m \leq N \), define \( U_\zeta(U_{r,m}) \) to be the subspace of \( U_\zeta(U_r) \) spanned by the monomial basis vectors \( \{ U_\zeta \} \) with \( 0 \leq a_i < p^\ell \) and \( a_i = 0 \) for \( i > m \). It follows from Lemma [1.10.4] that \( U_\zeta(U_{r,m}) \) is a normal subalgebra of \( U_\zeta(U_{r,m+1}) \). Of course, \( U_\zeta(U_{r,N}) = U_\zeta(U_r) \).

There exists an involutory \( k(q) \)-algebra automorphism \( \omega \) of \( U_q \) defined by

\[
\omega(E_\alpha) = F_\alpha, \quad \omega(F_\alpha) = E_\alpha, \quad \omega(K_\alpha) = K_\alpha^{-1} \quad (\alpha \in \Pi).
\]

For each \( \gamma \in \Phi^+ \), \( \omega(E_\gamma) = \pm q^a F_\gamma \) for some \( a \in \mathbb{Z} \) (depending on \( \gamma \)) [22 8.14(9)]. The automorphism \( \omega \) descends to an automorphism of \( U_\zeta \) and of its Frobenius–Lusztig kernels. Now define “positive” versions of the distinguished subalgebras of \( U_\zeta \) by setting \( U_\zeta(U_{\alpha,+}) = \omega(U_\zeta(U_{\alpha,r})), U_\zeta(B_{\alpha,+}^\ell) = \omega(U_\zeta(B_r)), U_\zeta(U_{\alpha,r}^+) = \omega(U_\zeta(U_{\alpha,r})) \), and so on. If \( r = 0 \), then denote \( U_\zeta(U_{\alpha,+}) \) by \( u_\zeta(\alpha) \). Collectively, we denote the collection of all (positive and negative) root subalgebras by writing \( U_\zeta(U_{\alpha,r}), \alpha \in \Phi \) (i.e., by not distinguishing between \( \alpha \) being a positive or an arbitrary root).
1.3. Hopf algebra actions on cohomology. Let $H$ be a Hopf algebra with comultiplication $\Delta$ and antipode $S$. Given $h \in H$, write $\Delta(h) = \sum h^{(1)} \otimes h^{(2)}$ (Sweedler notation). Later we may omit the summation symbol, and just write $\Delta(h) = h^{(1)} \otimes h^{(2)}$. The left and right adjoint actions of $H$ on itself are defined for $h, u \in H$ by

$$\Ad_l(h)(u) = \sum h^{(1)} w S(h^{(2)}) \quad \text{and} \quad \Ad_r(h)(u) = \sum S(h^{(1)}) u h^{(2)}.$$  

Now let $V$ be an $H$-module, and let $A$ be an $A$-stable subalgebra of $H$. Let $B_n(A)$ denote the left bar resolution for $A$, defined in [27, X.2]. We denote a typical element of $B_n(A)$ by $a[a_1, \ldots, a_n]$. The right $A$-action of $H$ on $A$ extends diagonally to a right action of $H$ on $B_n(A)$. The Hopf algebra axioms for $H$ guarantee that $B_n(A)$ is thus a complex of right $H$-modules.

The cohomology groups $H^*(A, V)$ can be computed as the homology of the cochain complex $C^*(A, V) := \text{Hom}_A(B_n(A), V)$. Define an action of $H$ on $C^*(A, V)$ by setting, for $h \in H, u \in B_n(A)$, and $f \in C^n(A, V)$,

$$(hf)(u) = \sum h^{(1)} f(u \cdot h^{(2)}).$$

This definition makes sense because $V$ was assumed to be an $H$-module. Using the Hopf algebra axioms for $H$, one can easily show that this action makes $C^*(A, V)$ a complex of left $H$-modules, hence that there exists an induced action of $H$ on $H^*(A, V)$. We call the $H$-action thus obtained the adjoint action of $H$ on $H^*(A, V)$.

Remark 1.3.1. Suppose that $A$ is an $A$-stable subalgebra of $H$. Then $B_n(A)$ is naturally a complex of left $H$-modules, and $\text{Hom}_k(B_n(A), V)$ is a left $H$-module under the “usual” diagonal action of $H$, defined in (2.2.1). This is, however, the wrong approach to take when defining an action of $H$ on $H^*(A, V)$, because it is not clear in general that the usual diagonal action of $H$ on $\text{Hom}_k(B_n(A), V)$ stabilizes the subspace of $A$-homomorphisms, nor that it commutes with the differential of the complex $C^*(A, V)$.

For future reference it will be useful to note:

Lemma 1.3.2. [13 Corollary 3.14] The small quantum group $u_\zeta(\mathfrak{g})$ is stable under the right adjoint action of $U_\zeta$ on itself. The Borel subalgebra $u_\zeta(\mathfrak{b})$ (resp. $u_\zeta(\mathfrak{b}^+)$) is stable under the right adjoint action of $U_\zeta(B)$ (resp. $U_\zeta(B^+)$) on itself.

2. Representation theory

2.1. Ordinary representation theory. Denote the algebra $U_\zeta(U_{r, m})$ defined in [12] by $A_m$. Using Lemma 1.1.1 one can show that for any $A_m$-module $M$, the space of invariants $M^{A_m}$ is non-zero. This implies that, up to isomorphism, there exists a unique irreducible (left or right) $A_m$-module, namely, the trivial module $k$. The space of invariants $(A_m)^{A_m}$ for the (left or right) regular action of $A_m$ on itself is one-dimensional, spanned by the vector

$$f_m := F^{(p^r - 1)}_{r_1} \cdots F^{(p^r - 1)}_{r_m}.$$

(The notation in (2.1.1) is meant to remind the reader of the two-sided integral in the finite-dimensional Hopf algebra $hy(U_r)$.) Then the regular module $A_m$ is indecomposable, hence it is the (left and right) projective cover for the trivial module $k$. Trivially, the dual of every irreducible left (resp. right) $A_m$-module is an irreducible right (resp. left) $A_m$-module, so the (left and right) regular module $A_m$ is also injective [10 Theorem 58.6]. Then the regular module $A_m$ is also the (left and right) injective envelope of the trivial module $k$. From this discussion we conclude:

Lemma 2.1.1. Retain the notation of the previous paragraph.

(a) A finite-dimensional $A_m$-module is projective iff it is injective iff it is free.

(b) There exists an isomorphism of left $A_m$-modules $A_m \cong (A_m)^*$ := $\text{Hom}_k(A_m, k)$, where the left action of $A_m$ on $(A_m)^*$ is induced by the right multiplication of $A_m$ on itself.
The next lemma is similar to [23, II.9.4].

**Lemma 2.1.2.** Let $M$ be a finite-dimensional $U_\zeta(B_r)$-module. The following are equivalent:

1. $M$ is injective for $U_\zeta(B_r)$.
2. $M$ is injective for $U_\zeta(U_r)$.
3. $M$ is projective for $U_\zeta(U_r)$.
4. $M$ is projective for $U_\zeta(B_r)$.

**Proof.** Statements (1) and (4) are equivalent because $U_\zeta(B_r)$ is a finite-dimensional Hopf algebra, cf. [24] and [10] §61–62]. Statements (2) and (3) are equivalent by the $m = N$ case of Lemma 2.1.1. The implication (4) ⇒ (3) follows from the fact that $U_\zeta(U_r)$ is free as a left module over $U_\zeta(U_r)$. Now suppose that $M$ is projective for $U_\zeta(U_r)$, and let $W$ be an arbitrary $U_\zeta(B_r)$-module. Considering the weight space decomposition of $\text{Hom}_{U_\zeta(U_r)}(M, W)$ for the diagonal action of $U_\zeta(T_r) := U_\zeta(B_r) \cap U_\zeta^0$, we get

\[
\text{Hom}_{U_\zeta(U_r)}(M, W) = \bigoplus_{\lambda \in X_{\rho^r}} \text{Hom}_{U_\zeta(U_r)}(M, W)_\lambda \cong \bigoplus_{\lambda \in X_{\rho^r}} \text{Hom}_{U_\zeta(B_r)}(M \otimes \lambda, W).
\]

Since $\text{Hom}_{U_\zeta(U_r)}(M, -)$ is an exact functor, then so must be each $\text{Hom}_{U_\zeta(B_r)}(M \otimes \lambda, -)$. In particular, $\text{Hom}_{U_\zeta(B_r)}(M, -)$ must be exact. This proves the implication (3) ⇒ (4).

**Remark 2.1.3.** A complete description of the indecomposable injective (equivalently, projective) $U_\zeta(B_r)$-modules is provided as follows. Write the identity $1 \in U_\zeta(T_r)$ as a sum of primitive orthogonal idempotents: $1 = \sum_{\lambda \in X_{\rho^r}} e_\lambda$. In the notation of [4, §1.2], $w.e_\lambda = \chi_\lambda(u)e_\lambda$ for all $u \in U_\zeta(T_r)$. Then $\{ke_\lambda : \lambda \in X_{\rho^r}\}$ is a complete set of non-isomorphic simple $U_\zeta(T_r)$-modules. Now write $U_\zeta(B_r) = \bigoplus_{\lambda \in X_{\rho^r}} U_\zeta(U_r)e_\lambda$. Each $U_\zeta(U_r)e_\lambda$ is injective and projective as a module for $U_\zeta(B_r)$ by Lemma 2.1.2. The $U_\zeta(B_r)$-module $U_\zeta(U_r)e_\lambda$ has one-dimensional socle of weight $\lambda - 2(p^r\ell - 1)\rho$, where $\rho = \frac{1}{2} \sum_{\alpha \in \Phi^+} \alpha$, and one-dimensional head of weight $\lambda$. Then, as a $U_\zeta(B_r)$-module, $U_\zeta(U_r)e_\lambda$ is the projective cover of $\lambda$ and the injective hull of $\lambda - 2(p^r\ell - 1)\rho$. Replacing $B_r$ and $U_r$ by $B_r^+$ and $U_r^+$, one obtains a similar description for the indecomposable injective $U_\zeta(B_r^+)$-modules.

2.2. **Rational representation theory.** We presuppose that the reader is familiar with the rational (i.e., integrable) representation theory of quantized enveloping algebras, as developed in the work of Andersen, Polo and Wen [23, 3]. Recall that a $U_\zeta^0$-module $M$ is rational if and only if it admits a weight space decomposition of the form $M = \bigoplus_{\lambda \in X} M_\lambda$. Given $N \in \{B_r, G_r, U_r, m\}$, a $U_\zeta(TN) := U_\zeta^0U_\zeta(N)$-module $M$ is rational if and only if it is rational as a $U_\zeta^0$-module. By definition, all $U_\zeta(N)$-modules are rational. We summarize below certain results on the rational representation theory of the Frobenius–Lusztig kernels that will be needed later in §3.2.

Write $A_m = U_\zeta(U_r, m)$, and write $TA_m$ for the algebra $U_\zeta(TU_r, m) = U_\zeta^0U_\zeta(U_r, m)$.

**Lemma 2.2.1.** Let $M$ be a finite-dimensional rational $TA_m$-module, and suppose that $M$ is free as a module for $A_m$. Then there exists an $A_m$-basis for $M$ consisting of weight vectors for $U_\zeta^0$.

**Proof.** The proof is by induction on the $A_m$-rank of $M$. To begin, suppose $M$ is free of rank one over $A_m$, with basis $\{v\} \subset M$. Write $v = \sum_{\lambda \in X} v_\lambda$, the decomposition of $v$ as a finite sum of $U_\zeta^0$-weight vectors. Since $\int_M v \neq 0$, there exists $\lambda \in X$ with $\int_M v_\lambda \neq 0$. Fix such a $\lambda$. Then $A_m.v_\lambda \cong A_m$ as left $A_m$-modules. Indeed, there exists a natural surjective map $A_m \rightarrow A_m.v_\lambda$, which is nonzero on the one-dimensional socle $\int_M$ of $A_m$, hence must be an isomorphism. Then $A_m.v_\lambda$ is an $A_m$-submodule of $M$ of dimension $\dim A_m = \dim V$, so we must have $M = A_m.v_\lambda$.

Now let $n > 1$, and suppose $M$ is free of rank $n$ over $A_m$ with basis $\{v_1, \ldots, v_n\}$. Then $M = M' \oplus M''$, where $M'$ is the $A_m$-submodule generated by $\{v_1\}$, and $M''$ is the $A_m$-submodule generated by $\{v_2, \ldots, v_n\}$. By induction, there exist $A_m$-bases $S'$ and $S''$ for $M'$ and $M''$, respectively, consisting
of weight vectors for $U^0_\zeta$. Then the union $S = S' \cup S''$ is an $A_m$-basis for $M$ consisting of weight vectors for $U^0_\zeta$.

**Lemma 2.2.2.** The indecomposable rationally injective $TA_m$-modules have the form

$$Y_{\lambda,m} = \text{ind}_{TA_\zeta}^{TA_m} \lambda = \text{Hom}_{TA_\zeta}(TA_m, \lambda) \quad (\lambda \in X).$$

As a module for $A_m$, $Y_{\lambda,m} \cong (A_m)^*$. In particular, $Y_{\lambda,m}$ is a free rank-one $A_m$-module, with basis consisting of a $\lambda$-weight vector for $U^0_\zeta$.

**Proof.** The induction functor takes injective modules to injective modules, and by Frobenius reciprocity, $Y_{\lambda,m}$ has simple socle $\lambda$. This proves that the $Y_{\lambda,m}$ are the indecomposable injective modules for $TA_m$. The last statement follows from Lemmas 2.2.1 and 2.1.1. □

Taking $m = N$ in Lemma 2.2.2, we obtain a description of the indecomposable rationally injective $U_\zeta(TU_r) = U_\zeta(TB_r)$-modules. Now, given $\lambda \in X$, define rational left $U_\zeta(TG_r)$-modules by

$$\hat{Z}_r(\lambda) = U_\zeta(TG_r) \otimes U_\zeta(TB_r \lambda) \quad \text{and}$$

$$\hat{Z}'_r(\lambda) = \text{ind}_{U_\zeta(TB_r)}^{U_\zeta(TG_r)} \lambda = \text{Hom}_{U_\zeta(TB_r)}(U_\zeta(TG_r), \lambda).$$

The left $U_\zeta(TG_r)$-module structure of $\hat{Z}_r(\lambda)$ is induced by the right multiplication of $U_\zeta(TG_r)$ on itself. As a $U_\zeta(U_r)$-module, $\hat{Z}_r(\lambda) \cong U_\zeta(U_r) \otimes \lambda$, and as a $U_\zeta(U_r^+)$-module, $\hat{Z}'_r(\lambda) \cong \text{Hom}_k(U_\zeta(U_r^+), \lambda)$, with the left action induced by the right multiplication of $U_\zeta(U_r^+)$ on itself. Lemma 2.2.2 and its corresponding version for $U_\zeta(TB_r^+)$ now imply the following two lemmas:

**Lemma 2.2.3.** Let $\lambda \in X$.

1. In the category of rational $U_\zeta(TB_r)$-modules, $\hat{Z}_r(\lambda)$ is the projective cover of $\lambda$ and the injective hull of $\lambda - 2(p^r \ell - 1)\rho$.

2. In the category of rational $U_\zeta(TB_r^+)$-modules, $\hat{Z}'_r(\lambda)$ is the projective cover of $\lambda - 2(p^r \ell - 1)\rho$ and the injective hull of $\lambda$.

Recall that, given a Hopf algebra $H$ and left $H$-modules $V$ and $W$, the space $\text{Hom}_k(V, W)$ is made into a left $H$-module by the diagonal action

$$\text{(2.2.1)} \quad (h, f)(v) = \sum h^{(1)} \cdot f(S(h^{(2)})v).$$

In particular, the dual space $V^* = \text{Hom}_k(V, k)$ is made into a left $H$-module.

**Lemma 2.2.4.** Let $\lambda \in X$. Then there exist left $U_\zeta(TG_r)$-module isomorphisms

$$\hat{Z}_r(\lambda)^* \cong Z_r(2(p^r \ell - 1)\rho - \lambda) \quad \text{and}$$

$$\hat{Z}'_r(\lambda)^* \cong Z'_r(2(p^r \ell - 1)\rho - \lambda).$$

**Proof.** The module $\hat{Z}_r(\lambda)^*$ has highest weight $2(p^r \ell - 1)\rho - \lambda$. Then there exists a $U_\zeta(TB_r^+)$-module homomorphism $2(p^r \ell - 1)\rho - \lambda \to Z_r(\lambda)^*$, hence a $U_\zeta(TG_r)$-module homomorphism $\varphi : \hat{Z}_r(2(p^r \ell - 1)\rho - \lambda) \to \hat{Z}_r(\lambda)^*$. Being the dual of an injective $U_\zeta(B_r)$-module, $\hat{Z}_r(\lambda)^*$ is projective for $U_\zeta(B_r)$. It follows that $\hat{Z}_r(\lambda)^*$ is isomorphic to the the projective cover of $2(p^r \ell - 1)\rho - \lambda$, hence that the map $\varphi$ must be surjective. Since the domain and range of $\varphi$ are each of the same finite dimension $(p^r \ell)^N$, $N = |\Phi^+|$, the map $\varphi$ must be an isomorphism of $U_\zeta(G_r)$-modules. Similarly, the natural $U_\zeta(B_r)$-module homomorphism $Z'_r(\lambda)^* \to 2(p^r \ell - 1)\rho - \lambda$ induces an isomorphism of $U_\zeta(G_r)$-modules $Z'_r(\lambda)^* \cong Z'_r(2(p^r \ell - 1)\rho - \lambda)$. □

Let $N \in \{B_r, G_r\}$. We would like to characterize the rationally injective (resp. projective) $U_\zeta(TN)$-modules in terms of their restriction to $U_\zeta(N)$. For this we utilize the Hopf algebra structure of $U_\zeta(TN)$.
Lemma 2.2.5. Let $N \in \{B_r, G_r\}$. Any rationally projective $U_\zeta(TN)$-module is rationally injective.

Proof. The proof is similar to the corresponding result for algebraic groups [23 I.3.18], but some care must be taken owing to the non-cocommutativity of the Hopf algebra $H := U_\zeta(TN)$. In this proof only, we assume that all $U_\zeta(TN)$-modules are rational.

Let $V$ be a finite-dimensional $H$-module, and let $P$ be a projective $H$-module. Then the tensor product $P \otimes V$ is also projective, because for any $H$-module $W$, we have $\text{Hom}_H(P \otimes V, W) \cong \text{Hom}_H(P, W \otimes V^*)$ by [3 Proposition 1.18]. Now, a short exact sequence $0 \to V_1 \to V_2 \to V_3 \to 0$ of finite-dimensional $H$-modules gives rise to the short exact sequence of $H$-modules

$$0 \to P \otimes V_3^* \to P \otimes V_2^* \to P \otimes V_1^* \to 0.$$ 

Each term is projective, so the short exact sequence splits, and the induced map on fixed points $\text{Hom}_H(P \otimes V, W) \to \text{Hom}_H(V, P)$ by [22 §3.10] and [5 Lemma I.4.5] (cf. also [4 Proposition 2.9]). Now let $0 \to V \to W$ be an arbitrary exact sequence of (rational) $H$-modules. Using Zorn’s Lemma and the local finiteness of $W$, it follows that the natural map $\text{Hom}_H(W, P) \to \text{Hom}_H(V, P)$ is surjective, hence that $P$ is an injective $H$-module. □

The next result is similar to Lemma 2.1.2

Lemma 2.2.6. Let $N \in \{B_r, G_r\}$, and let $M$ be a finite-dimensional $U_\zeta(TN)$-module. Then the following statements are equivalent:

1. $M$ is a rationally injective $U_\zeta(TN)$-module.
2. $M$ is an injective $U_\zeta(N)$-module.
3. $M$ is a projective $U_\zeta(N)$-module.
4. $M$ is a rationally projective $U_\zeta(N)$-module.

Proof. Statements (2) and (3) are equivalent because $U_\zeta(N)$ is a finite-dimensional Hopf algebra, while the implication (4) $\Rightarrow$ (1) is just Lemma 2.2.5. The proof of the implication (3) $\Rightarrow$ (4) is essentially the same as the corresponding implication in Lemma 2.1.2 replacing $U_\zeta(U_r)$ by $U_\zeta(N)$, $U_\zeta(TN)$, $U_\zeta(T_r)$ by $U_\zeta^0$, and replacing the index set in (2.1.2) by $p^\ell \cdot X$. Finally, the implication (1) $\Rightarrow$ (2) follows by an argument similar to that used to prove [3 Lemma 4.1(iii)]; the details are left to the reader. □

Having established the above characterization of rationally injective (resp. projective) modules for the algebras $U_\zeta(TB_r)$ and $U_\zeta(TG_r)$, the last two results of this section follow just as in the classical situation for algebraic groups, cf. [23 §II.11.1–11.4]. We leave the details to the reader.

Lemma 2.2.7. For all $\lambda, \mu \in X$, the $U_\zeta(TG_r)$-module $\widehat{Z}_r(\lambda) \otimes \widehat{Z}_r^*(\mu)$ is rationally injective.

Proposition 2.2.8. Let $M$ be a finite-dimensional rational $U_\zeta(TG_r)$-module.

(a) $M$ is injective as a $U_\zeta(B_r)$-module if and only if $M$ admits a filtration by $U_\zeta(TG_r)$-submodules with factors of the form $\widehat{Z}_r(\lambda)$, $\lambda \in X$.

(b) $M$ is injective as a $U_\zeta(B_r^+)$-module if and only if $M$ admits a filtration by $U_\zeta(TG_r)$-submodules with factors of the form $\widehat{Z}_r(\lambda)$, $\lambda \in X$.

2.3. Spectral sequences. In Section 5.2, we will study a certain spectral sequence (2.3.2), which is usually constructed as the spectral sequence associated to the composite of two functors, cf. [20 §3]. Since the arguments in Section 5.2 will require a product structure on (2.3.2) that is not apparent from the Grothendieck construction, as well as knowledge of the edge maps, we present here a construction of (2.3.2) that makes these features more apparent. Our construction mirrors [17 Proposition 1.1].
Let \( V \) be a rational \( U_\zeta \)-module. There exists a Lyndon–Hochschild–Serre spectral sequence

\[
E_2^{ij}(V) = H^i(U_\zeta(B)/u_\zeta(b), H^j(u_\zeta(b), V)) \Rightarrow H^{i+j}(U_\zeta(B), V)
\]

computing the rational \( U_\zeta(B) \)-cohomology of the \( U_\zeta \)-module \( V \). It can be constructed as in \[23\] I.6.6, replacing \( k[G] \) there by \( k[U_\zeta] \); cf. \[3\] §1.34, 2.17. If \( W \) is another rational \( U_\zeta \)-module, then there exists a morphism \( E_r(V) \otimes E_r(W) \to E_r(V \otimes W) \), which on the \( 'E_\infty \)- and \( 'E_\infty \)-pages is just the cup product for rational cohomology. The edge map

\[
H^j(U_\zeta(B), V) \to 'E_2^{0,j}(V) = H^j(u_\zeta(b), V)^{U_\zeta(B)}
\]

is just the restriction map.

Let \( k[G] = \text{ind}^G_\zeta(k) \) be the coordinate ring of \( G \), and let \( k[G]^{(1)} \) denote \( k[G] \) considered as a \( U_\zeta \)-module by pullback along the quantum Frobenius morphism \( F_\zeta : U_\zeta \to \text{hy}(G) \). Note that the map \( a \otimes b \to b \otimes a \) defines a \( U_\zeta \)-module isomorphism \( k[G]^{(1)} \otimes V \cong V \otimes k[G]^{(1)} \). Also, the map \( k[G] \otimes k[G] \cong k[G \times G] \to k[G] \) induced by the diagonal map \( \Delta : G \to G \times G \) is a homomorphism of \( G \)-modules. Now, let \( W \) be another rational \( U_\zeta \)-module. Then there exists a \( U_\zeta \)-module homomorphism

\[
\varphi : (k[G]^{(1)} \otimes V) \otimes (k[G]^{(1)} \otimes W) \cong (k[G] \otimes k[G])^{(1)} \otimes V \otimes W \to k[G]^{(1)} \otimes (V \otimes W).
\]

Next, note that \( k[G]^{(1)} = (\text{ind}^G_\zeta(k))^{(1)} \cong \text{ind}^{U_\zeta}_\zeta(k) \) as \( U_\zeta \)-modules \[8\] Theorem 2.3(ii)]. Then

\[
H^*(u_\zeta(b), k[G]^{(1)} \otimes V) \cong k[G]^{(1)} \otimes H^*(u_\zeta(b), V)
\]

as \( U_\zeta(B) \)-modules. Then \( E_2^{ij}(V) \) can be rewritten as

\[
E_2^{ij} \cong H^i(U_\zeta(B)/u_\zeta(b), k[G]^{(1)} \otimes H^j(u_\zeta(b), V))
\]

\[
\cong H^i(B, k[G] \otimes H^j(u_\zeta(b), V))
\]

\[
\cong R^i \text{ind}^G_U H^j(u_\zeta(b), V) \quad \text{by} \ [23] \ I.4.10\].

Thus, the spectral sequence \( E_r(k[G]^{(1)} \otimes V) \) can be written as:

\[
E_2^{ij}(V) = R^i \text{ind}^G_U H^j(u_\zeta(b), V) \Rightarrow H^{i+j}(u_\zeta(g), V).
\]

Written in this form, the products \( E_r(V) \otimes E_r(W) \to E_r(V \otimes W) \) are induced by the cup products for \( u_\zeta(g) \) and \( u_\zeta(b) \), and the edge map \( H^j(u_\zeta(g), V) \to E_2^{0,j}(V) = \text{ind}^G_U H^j(u_\zeta(b), V) \) is induced by Frobenius reciprocity from the restriction map \( H^*(u_\zeta(g), V) \to H^*(u_\zeta(b), V) \).

3. Injectivity criterion for modules with compatible torus action

3.1. The main result of this section is the following theorem:

**Theorem 3.1.1.** Let \( M \) be a finite-dimensional rational \( U_\zeta(TG_r) \)-module. Then \( M \) is injective for \( U_\zeta(G_r) \) if and only if the restriction \( M|_{U_\zeta(TG_{r\alpha})} \) is injective for each \( \alpha \in \Phi \).
One direction of the theorem is clear: The algebra $U_\zeta(G_r)$ is flat (in fact, free) as a right module over each root subalgebra $U_\zeta(U_{\alpha,r})$ (apply the triangular decomposition for $U_\zeta$, and the explicit description of the PBW bases for $U_\zeta^+$ and $U_\zeta^-$). Then the injectivity of $M$ for $U_\zeta(G_r)$ implies the injectivity of $M$ for each $U_\zeta(U_{\alpha,r})$ by [5, Lemma I.4.3]. This direction of the theorem does not require a compatible $U_\zeta^+$-structure on $M$.

To prove the other direction of the theorem, in Section 3.3 we reduce the problem to the case of a rational $U_\zeta(TG_r)$-module that is injective over the Borel subalgebras $U_\zeta(B_r)$ and $U_\zeta(B_r^+)$.

In Section 3.3.3 we prove that such a module is injective for $U_\zeta(B_r)$ (resp. $U_\zeta(B_r^+)$) if and only if its restriction to each root subalgebra is injective. Our overall strategy is the same as that in [9] for the classical algebraic group situation, but extra care must be taken, especially in Section 3.3, owing to the non-cocommutativity of the Hopf algebras under consideration and the complicated relations between root vectors in $U_\zeta$.

3.2. Reduction to Borel subalgebras.

**Theorem 3.2.1.** Let $M$ be a finite-dimensional rational $U_\zeta(TG_r)$-module. Then $M$ is injective for $U_\zeta(G_r)$ if and only if the restrictions $M|_{U_\zeta(B_r)}$ and $M|_{U_\zeta(B_r^+)}$ are injective.

**Proof.** Suppose that $M$ is injective for $U_\zeta(G_r)$, then $M$ is injective for $U_\zeta(B_r)$ and $U_\zeta(B_r^+)$ by [5, Lemma I.4.3], because $U_\zeta(G_r)$ is free as a right module for either $U_\zeta(B_r)$ or $U_\zeta(B_r^+)$. Now suppose that $M$ is injective as a module for $U_\zeta(B_r)$ and as a module for $U_\zeta(B_r^+)$. Then by Lemma 2.2.4 and Proposition 2.2.8 the $U_\zeta(TG_r)$-module $\text{End}_k(M) \cong M \otimes M^*$ admits a filtration by $U_\zeta(TG_r)$-submodules with factors of the form $Z_\zeta(\lambda) \otimes Z_\zeta(\mu)$ ($\lambda, \mu \in X$). These factors are injective for $U_\zeta(G_r)$ by Lemma 2.2.7, so $\text{End}_k(M)$ must be an injective (equivalently, projective) $U_\zeta(G_r)$-module.

Now, since $\text{End}_k(M)$ is projective for $U_\zeta(G_r)$, then so is $\text{End}_k(M) \otimes M$, cf. the proof of Lemma 2.2.5. The $U_\zeta(G_r)$-module embedding $M \cong (k \cdot \text{id}) \otimes M \hookrightarrow \text{End}_k(M) \otimes M$ splits via the $U_\zeta(G_r)$-module homomorphism $\text{End}_k(M) \otimes M \to M$, $\varphi \otimes m \mapsto \varphi(m)$, so $M$ is isomorphic to a $U_\zeta(G_r)$-direct summand of $\text{End}_k(M) \otimes M$. Then $M$ is projective (equivalently, injective) as a $U_\zeta(G_r)$-module.

3.3. Injectivity for Borel subalgebras. Since we can twist the structure map of any $U_\zeta(B_r^+)$-module $M$ by the automorphism $\omega$ to make it a $U_\zeta(B_r)$-module, to complete the proof of Theorem 3.1.1 it now suffices to prove the following theorem:

**Theorem 3.3.1.** Let $M$ be a rational $U_\zeta(TB_r)$-module. Then $M$ is injective for $U_\zeta(B_r)$ if and only if $M$ is injective for each root subalgebra $U_\zeta(U_{\alpha,r})$, $\alpha \in \Phi^+$.

We will actually show that $M$ is injective for $U_\zeta(U_\alpha)$, but this is equivalent to injectivity for $U_\zeta(B_r)$ by Lemma 2.1.2. The proof is by induction, using the algebras $A_m := U_\zeta(U_{r,m})$ ($1 \leq m \leq N$) defined in Section 1.2. The key to the induction argument is the fact that the convex ordering \{$\gamma_1, \ldots, \gamma_N$\} on $\Phi^+$ defined in Section 1.1 is compatible with a sequence of total orderings $\leq_m$ on the vector space $\mathbb{R}\Phi$.

**Lemma 3.3.2.** (cf. [9, Lemma 3.1]) For each $0 \leq m < N$, there exists an ordering $\leq_m$ on the Euclidean space $\mathbb{R}\Phi$ such that $\gamma_i \succ_m 0$ for $i \leq m$, and $\gamma_i \prec_m 0$ for $i > m$.

**Proof.** The proof goes by induction on $m$. To start, choose $\leq_0$ to be any total ordering on $\mathbb{R}\Phi$ such that $\gamma_i \prec_0 0$ for all $\gamma \in \Phi^+$. Now let $m \geq 0$, and assume by way of induction that there exists a total ordering $\leq_m$ on $\mathbb{R}\Phi$ satisfying the conditions of the lemma. Let $\Phi^+_m$ (resp. $\Phi^-_m$) denote the positive (resp. negative) system of roots determined by $\leq_m$. Then $\gamma_{m+1} \in \Phi^-_m$. We claim that $\gamma_{m+1}$ is simple with respect to the ordering $\leq_m$. Indeed, suppose $\gamma_{m+1} = \alpha_1 + \alpha_2$ for some $\alpha_1, \alpha_2 \in \Phi^+_m$. There are three cases to consider:

1. $\alpha_1 = \gamma_i \in \Phi^-_m$, $\alpha_2 = \gamma_j \in \Phi^-_m$, with $(m + 1) < i < j \leq N$. Then $\gamma_{m+1} = \gamma_i + \gamma_j$, an impossibility because \{$\gamma_1, \ldots, \gamma_N$\} is a convex ordering of $\Phi^+$, and $(m + 1) \notin [i, j]$. 

(2) \( \alpha_1 = -\gamma_i \in \Phi_m^+, \alpha_2 = -\gamma_j \in \Phi_m^+ \), with \( 1 \leq i < j \leq m \). Then \( -\gamma_{m+1} = \gamma_i + \gamma_j \), an impossibility because the sum of two roots in \( \Phi^+ \) is never a root in \( \Phi^- \).

(3) \( \alpha_1 = -\gamma_i \in \Phi_m^+, \alpha_2 = \gamma_j \in \Phi_m^+ \), with \( 1 \leq i < (m+1) \leq N \). Then \( \gamma_i + \gamma_{m+1} = \gamma_j \), an impossibility because \( \{\gamma_1, \ldots, \gamma_N\} \) is a convex ordering of \( \Phi^+ \), and \( j \notin [i, m+1] \).

So \( \gamma_{m+1} \) is simple with respect to the ordering \( \leq_m \).

Now, the Weyl group \( W \) acts transitively on the collection of positive systems in \( \Phi^+ \), and to each positive system \( S \) of roots in \( \Phi \), we can associate a total ordering \( \preceq \) on \( \mathbb{R} \Phi \) such that the positive roots in \( \Phi \) with respect to \( \preceq \) are precisely those in \( S \) (e.g., enumerate a set of simple roots in \( S \), hence an ordered basis for \( \mathbb{R} \Phi \), and then take \( \preceq \) to be the standard lexicographic ordering on \( \mathbb{R} \Phi \) with respect to that basis). Let \( \leq_{m+1} \) be a total ordering on \( \mathbb{R} \Phi \) with associated positive system \( \Phi_{m+1}^+ := s_{\gamma_{m+1}}(\Phi_m^+) \). Since \( \gamma_{m+1} \) is simple with respect to \( \leq_m \), it follows that \( \Phi_{m+1}^+ \cap \Phi^+ = \{\gamma_1, \ldots, \gamma_m, \gamma_{m+1}\} \). Then \( \leq_{m+1} \) satisfies the conditions of the lemma. This completes the induction step, and the proof of the lemma.

Proof of Theorem 3.3.1. Let \( M \) be an injective \( U_\zeta(U_r) \)-module. Then, since for each \( \alpha \in \Phi^+ \) the algebra \( U_\zeta(U_r) \) is free as a right module over \( U_\zeta(U_{r_\alpha}) \), the restriction \( M|_{U_\zeta(U_{r_\alpha})} \) is injective for each positive root \( \alpha \in \Phi^+ \) by [5, Lemma I.4.3]. (This direction of the theorem does not require the action of \( U_\zeta^0 \).) Now let \( M \) be a finite-dimensional rational \( U_\zeta(TB_r) \)-module for which the restricted modules \( M|_{U_\zeta(U_{r_\alpha})} \) are all injective (equivalently, free). We prove, for each \( 1 \leq m \leq N \), that \( M \) is free over \( A_m = U_\zeta(U_{r,m}) \). Taking \( m = N \) then yields the desired result.

By assumption, \( M \) is free as a module over \( A_1 = U_\zeta(U_{r_1}) \), so let \( m \geq 1 \) and assume by induction that \( M \) is free as a module over \( A_m \). We show that \( M \) is free over \( A_{m+1} \). By Lemma 3.3.2 \( M \) admits an \( A_m \)-basis \( S = \{s_1, \ldots, s_r\} \) consisting of weight vectors for \( U_\zeta^0 \). Set \( \alpha = \gamma_{m+1} \). By Lemma 8.3.3 there exists an ordering \( \preceq \) on \( \mathbb{R} \Phi \) with \( \alpha < 0 \), and \( \gamma_i > 0 \) if \( i \leq m \). Choose \( v \in S \) of weight \( \lambda \) minimal with respect to \( \preceq \). Then \( \lambda \) is minimal with respect to \( \preceq \) among all the weights for \( U_\zeta^0 \) in \( M \). Since \( \alpha < 0 \), the minimality of \( \lambda \) implies that \( F_\alpha^{(n)} v = 0 \) for all \( 1 \leq n < p_\ell \). Set \( \int_\alpha = F_\alpha^{(p_\ell-1)} \).

By the freeness of \( M \) over \( U_\zeta(U_{r\alpha}) \), there exists \( w \in M \) with \( \int_\alpha w = v \). Now

\[
\int_{m+1} = F_{\gamma_1}^{(p_\ell-1)} \cdots F_{\gamma_{m}}^{(p_\ell-1)} F_{\gamma_{m+1}}^{(p_\ell-1)} = \int_m \int_\alpha,
\]

so \( \int_{m+1} w = \int_m \int_\alpha w = \int_m v \), and \( \int_m v \neq 0 \) because \( v \) is an \( A_m \)-basis vector for \( M \). Then \( \int_{m+1} w \neq 0 \).

Now write \( w = \sum_{\mu \in X} w_\mu \), the decomposition of \( w \) as a finite sum of \( U_\zeta^0 \)-weight vectors. Since \( \int_{m+1} w \neq 0 \), there exists \( \mu \in X \) with \( \int_{m+1} w_\mu \neq 0 \). It follows that \( M' := A_{m+1}.w_\mu \) is free as a module for \( A_{m+1} \). By Lemma 2.2.2 \( M' \) is injective in the category of rational \( TA_{m+1} \)-modules, so we can decompose \( M \) as a direct sum \( M = M' \oplus M'' \) for some \( TA_{m+1} \)-submodule \( M'' \) of \( M \). By induction on dimension, \( M'' \) is free over \( A_{m+1} \). Then \( M = M' \oplus M'' \) is free for \( A_{m+1} \).

4. Support varieties for Borel subalgebras

4.1. Let \( H \) be a Hopf algebra, and let \( M \) be a left \( H \)-module. Set \( H(H, k) = \mathbb{H}^2(H, k) \), the subring of the cohomology ring \( \mathbb{H}^* \) generated by elements of even degree. Then \( H(H, k) \) is a commutative ring under the cup product [27, Corollary 4.3]. Suppose that \( H(H, k) \) is finitely-generated as an algebra. Then \( V_H(k) := \text{MaxSpec}(H(H, k)) \), the maximal ideal spectrum of \( H(H, k) \), is an affine variety. Define \( J_H(M) \) to be the annihilator for the cup product action of \( H(H, k) \) on \( \text{Ext}_{H}^*(M, M) \cong \text{Ext}_{H}^*(k, \text{Hom}_k(M, M)) \cong \text{Ext}_{H}^*(k, M \otimes M^*) \). Equivalently, \( J_H(M) \) is the kernel of the graded algebra homomorphism \( H(H, k) \hookrightarrow \mathbb{H}^2(H, \text{Hom}_k(M, M)) \) induced by the natural map \( k \to \text{Hom}_k(M, M), 1 \mapsto \text{id} \). Then the support variety \( V_H(M) \) is defined to be the conical subvariety of \( V_H(k) \) determined by the ideal \( J_H(M) \).

We now turn our attention to studying cohomological support varieties for the Borel subalgebras \( u_\zeta(b) \) and \( u_\zeta(b^+) \) of the small quantum group \( u_\zeta(\mathfrak{g}) \). The automorphism \( \omega \) of \( U_\zeta \) restricts to an
isomorphism \( u_\zeta(b) \cong u_\zeta(b^+) \), so any results we prove for \( u_\zeta(b^+) \) can be immediately translated into results for \( u_\zeta(b) \). Thus, in order to simplify some calculations, in this section we choose to work exclusively with the positive Borel subalgebra \( u_\zeta(b^+) \).

In what follows we make the following assumptions:

**Assumption 4.1.1.** Let \( k \) be an algebraically closed field. Assume \( p = \text{char}(k) \) to be odd or zero, and to be good for \( \Phi \). (For the definition of a good prime, see, e.g., [23, II.4.22].) Assume \( \zeta \in k \) to be a primitive \( \ell \)-th root of unity, with \( \ell \in \mathbb{N} \) odd, coprime to three if \( \Phi \) has type \( G_2 \), and \( \ell \geq h, h \) the Coxeter number of \( \Phi \).

Under these assumptions, we have the following descriptions for \( H(u_\zeta(g), k) \) and \( H(u_\zeta(b^+), k) \):

**Theorem 4.1.2.** (cf. [20, Lemma 2.6] and [13, Corollary 4.20]) We have \( H^{\text{odd}}(u_\zeta(b^+), k) = 0 \). There exists a natural \( B^+ \)-algebra isomorphism \( H(u_\zeta(b^+), k) \cong S(u^{++}) \), \( S(u^{++}) \) the symmetric algebra on the dual space \((u^+)^*\), where the action of \( B^+ \) on \( H(u_\zeta(b^+), k) \) is induced by the adjoint action of \( U_\zeta(B^+) \) on \( H^*(u_\zeta(b^+), k) \), and the action of \( B^+ \) on \( S(u^{++}) \) is induced by the coadjoint action of \( B^+ \) on \( u^{++} \).

**Theorem 4.1.3.** (cf. [20, Theorem 3] and [13, Corollary 4.23]) We have \( H^{\text{odd}}(u_\zeta(g), k) = 0 \). There exists a natural graded \( G \)-algebra isomorphism \( H(u_\zeta(g), k) \cong k[N], k[N] \) the coordinate ring of the nullcone \( N \subset g \), where the action of \( G \) on \( H(u_\zeta(g), k) \) is induced by the adjoint action of \( U_\zeta \) on \( H^*(u_\zeta(g), k) \), and the action of \( G \) on \( k[N] \) is induced by the usual adjoint action of \( G \) on \( N \subset g \).

Theorem 4.1.2 implies that for any \( u_\zeta(b^+ \)-module \( M \), the support variety \( V_{u_\zeta(b^+)}(M) \) identifies with a conical (\( B^+ \)-stable) subvariety of \( u^+ = \text{Lie}(U^+) \), and if \( M \) lifts to a \( U_\zeta(B^+) \)-module, then \( V_{u_\zeta(b^+)}(M) \) is even a \( B^+ \)-stable subvariety of \( u^+ \). Similarly, Theorem 4.1.3 implies that for any \( u_\zeta(g) \)-module (resp. \( U_\zeta \)-module) \( M \), the support variety \( V_{u_\zeta(g)}(M) \) identifies with a conical (\( G \)-stable) subvariety of the nullcone \( N \).

In this section we generalize the results of [18] to show that the root vector \( e_\alpha \in u^+ \) is contained in the support variety \( V_{u_\zeta(b^+)}(M) \) of the \( u_\zeta(b^+) \)-module \( M \) if and only if \( M \) is not projective (equivalently, injective) for the root subalgebra \( u_\zeta(e_\alpha) \). The proof is inductive in nature, and utilizes the Lyndon–Hochschild–Serre spectral sequence (4.1.1) in order to show that the action of the coordinate function \( x_\alpha \in S(u^{++}) \cong H(u_\zeta(b^+), k) \) on certain cohomology groups is nilpotent. Since most of the algebras under consideration in the induction argument are not Hopf subalgebras of \( u_\zeta(b^+) \) but are only one-sided coideal subalgebras, we must perform a number of technical calculations that would otherwise be unnecessary when dealing with Hopf algebras. Most of these technical calculations are contained in Section 4.5.

### 4.2. Cohomology products.

Let \( \Lambda \) and \( \Lambda' \) be algebras over \( k \). Let \( V \) and \( W \) be left \( \Lambda \)-modules, and let \( V' \) and \( W' \) be left \( \Lambda' \)-modules. Set \( \Omega = \Lambda \otimes \Lambda' \). Recall that the *wedge product* is a family of \( k \)-bilinear maps

\[
(4.2.1) \quad \vee : \text{Ext}^n_\Lambda(V, W) \otimes \text{Ext}^m_{\Lambda'}(V', W') \to \text{Ext}^{n+m}_{\Omega}(V \otimes V', W \otimes W').
\]

It is defined as follows: Take projective resolutions \( X \to V \) and \( X' \to V' \) by \( \Lambda \)- and \( \Lambda' \)-modules, respectively. Then, for each \( n, m \in \mathbb{N} \), \( X_n \otimes X'_m \) is projective for \( \Omega \), and by the Küneth Theorem, \( X \otimes X' \) is an \( \Omega \)-projective resolution of \( V \otimes V' \). Given \( f \in \text{Hom}_\Lambda(X, W) \) and \( g \in \text{Hom}_{\Lambda'}(X', W') \), define \( f \vee g \in \text{Hom}_\Omega(X \otimes X', W \otimes W') \) by \((f \vee g)(x \otimes x') = f(x) \otimes g(x')\). Then (4.2.1) is the map in cohomology induced by \( \vee : \text{Hom}_\Lambda(X, W) \otimes \text{Hom}_{\Lambda'}(X', W') \to \text{Hom}_\Omega(X \otimes X', W \otimes W') \).

Suppose \( \Lambda \cong k \) and \( \Lambda' \cong k \) are augmented algebras over \( k \), and that \( V = V' = k \). Then we could take \( X = B(\Lambda) \) and \( X' = B(\Lambda') \), the left bar resolutions for \( \Lambda \) and \( \Lambda' \), respectively. Since \( B(\Lambda) \otimes B(\Lambda') \) and \( B(\Omega) = B(\Lambda \otimes \Lambda') \) are both \( \Omega \)-projective resolutions of \( k \cong k \otimes k \), there exists an
$\Omega$-module chain map $\varphi : \mathcal{B}(\Omega) \to \mathcal{B}(\Lambda) \otimes \mathcal{B}(\Lambda^\prime)$, unique up to homotopy, lifting the identity $k \to k$.

An explicit choice for $\varphi$ is given by the following formula (cf. [8, XI.7]):

$$
\varphi([\lambda_1 \otimes \lambda_1', \ldots, \lambda_n \otimes \lambda_n']) = \sum_{i=0}^{n}[\lambda_1, \ldots, \lambda_i] \epsilon(a_{i+1} \cdots a_n) \otimes \lambda'_1 \cdots \lambda'_i[\lambda'_{i+1}, \ldots, \lambda'_n].
$$

Now let $H$ be a Hopf algebra with comultiplication $\Delta : H \to H \otimes H$, and let $V$ and $W$ be left $H$-modules.

Then the cup product

$$
\cup : H^n(H, V) \otimes H^m(H, W) \to H^{n+m}(H, V \otimes W)
$$

is the composite of the wedge product $\vee : \text{Ext}_H^n(k, V) \otimes \text{Ext}_H^m(k, W) \to \text{Ext}_{H \otimes H}^{n+m}(k, V \otimes W)$ with the map $\text{Ext}_H^* \otimes \text{Ext}_H^* \to \text{Ext}_{H \otimes H}^*$ induced by $\Delta$. If $\zeta \in H^n(H, V)$ and $\eta \in H^m(H, W)$ are represented by cocycles $f \in \text{Hom}_H(B_n(H), V)$ and $g \in \text{Hom}_H(B_m(H), W)$, then the cup product $\zeta \cup \eta$ is the cohomology class of the cocycle $f \vee g := f \circ \varphi \circ \Delta \in \text{Hom}_H(B_n+m(H), V \otimes W)$.

Explicitly, writing $\Delta(h_i) = h_i^{(1)} \otimes h_i^{(2)}$ and using the fact that $(\epsilon \otimes 1) \circ \Delta = \text{id}$,

$$
(f \cup g)([h_1, \ldots, h_{n+m}]) = f([h_1^{(1)}, \ldots, h_1^{(n)}]) \otimes h_1^{(2)} \cdots h_1^{(n)} g([h_{n+1}, \ldots, h_{n+m}]),
$$

In particular, if $W$ has trivial $H$-action, then (4.2.3) reduces to

$$
(f \cup g)([h_1, \ldots, h_{n+m}]) = f([h_1, \ldots, h_n]) \otimes g([h_{n+1}, \ldots, h_{n+m}]).
$$

Suppose $A \subseteq H$ is a left coideal subalgebra, that is, $A$ is a subalgebra of $H$ and $\Delta(A) \subseteq H \otimes A$. Then, given an $A$-module $W$, we have the composite

$$
\Delta^* \circ \cup : H^n(H, k) \otimes H^m(A, W) \to \text{Ext}_{H \otimes A}^{n+m}(k, k \otimes W) \to H^{n+m}(A, W).
$$

We call this the cup product action of $H^*(H, k)$ on $H^*(A, W)$. By abuse of notation we also denote it by the symbol $\cup$. Then $\cup$ admits a description at the level of chain complexes by exactly the same formula as (4.2.3), interpreting the $h_i$ now as elements of $A$. By [27, Theorem VIII.4.1], if $\zeta \in H^n(H, k)$ and $\eta \in H^m(A, W)$, then $\Delta \cup \eta = (-1)^{nm} \eta \circ \text{res}_A^H(\zeta)$, where $\text{res}_A^H(\zeta)$ denotes the image of $\zeta$ under the cohomological restriction map $H^*(H, k) \to H^*(A, k)$, and where

$$
\circ : \text{Ext}_A^m(k, W) \otimes \text{Ext}_A^n(k, k) \to \text{Ext}_A^{n+m}(k, W)
$$

denotes the Yoneda composition of extensions. Of course, the Yoneda product (4.2.6) depends only on the ring structure of $A$ (and the $A$-module structure of $W$), so the cup product action of $H^*(H, k)$ on $H^*(A, W)$ is independent of the particular comultiplication map for $H$.

4.3. **Left coideal subalgebras in $U_q^+$**. Let $w \in W$, and suppose that the reduced expression for $w_0$ chosen in §1.1 begins with a reduced expression for $w$, that is, $w_0 = wu'$ with $\ell(w_0) = \ell(w) + \ell(w')$. Here $\ell : W \to \mathbb{N}$ denotes the usual length function on $W$. Set $m = \ell(w)$. Then the algebra $U_q^+[w]$ defined in [22, §8.24] is the subalgebra of $U_q^+$ generated by the root vectors $E_{\gamma_1}, E_{\gamma_2}, \ldots, E_{\gamma_m}$.

Heckenberger and Schneider [21] have shown that every right coideal subalgebra in $U_q(B^+) = U_q^0U_q^+$ containing $U_q^0$ has the form $U_q^0U_q^+[w]$ for some $w \in W$. (Right coideals are defined by the condition $\Delta(A) \subseteq A \otimes H$.) To each $w \in W$ we can also associate a left coideal subalgebra of $U_q(B^+)$, namely, the algebra $T_w(U_q^+[w'])$.

**Lemma 4.3.1.** Let $w \in W$. Write $w_0 = wu'$ with $\ell(w_0) = \ell(w) + \ell(w')$. Then $T_w(U_q^+[w'])$ is a left coideal subalgebra of $U_q(B^+)$. \\

**Proof.** Suppose that the reduced expression for $w_0$ chosen in §1.1 begins with a reduced expression for $w$. Set $m = \ell(w)$. Then $T_w(U_q^+[w'])$ is the subalgebra of $U_q^+$ generated by the root vectors $E_{\gamma_{m+1}}, E_{\gamma_{m+2}}, \ldots, E_{\gamma_{w'}}$. Since $T_w(U_q^+[w']) \subseteq U_q^+$, we know that $\Delta(T_w(U_q^+[w'])) \subseteq U_q(B^+) \otimes U_q^+$. Since $T_w(U_q^+[w']) = \{u \in U_q^+ : T_w^{-1}(u) \in U_q^+\}$, to prove the lemma it suffices to show that

$$(1 \otimes T_w^{-1}) \circ \Delta(T_w(U_q^+[w'])) \subseteq U_q(B^+) \otimes U_q^+.$$
This last claim follows from [1] Proposition C.5(2).

Corollary 4.3.2. Let $w_0 = s_{\beta_1} \cdots s_{\beta_N}$ be an arbitrary reduced expression for $w_0 \in W$, and let \{\gamma_1, \ldots, \gamma_N\} be the corresponding convex ordering of $\Phi^+$. Let $E_{\gamma_m} \in U_q$ be the positive root vector of weight $\gamma_m$ as defined in [17]. Then $\Delta(E_{\gamma_m}) \in V_m \otimes W_m$, where $V_m \subset U_q(B^+)$ is the subalgebra generated by $U_q^0 \cup \{E_{\gamma_1}, \ldots, E_{\gamma_m}\}$, and $W_m \subset U_q^+$ is the subalgebra generated by $\{E_{\gamma_1}, \ldots, E_{\gamma_N}\}$.

Proof. Set $w = s_{\beta_1} \cdots s_{\beta_m}, w' = s_{\beta_1} \cdots s_{\beta_{m-1}}$, and $w'' = s_{\beta_m} \cdots s_{\beta_N}$, so that $w_0 = w'w''$. Now use the fact that $E_{\gamma_m} \in U_q^+[w] \cap T_{w'}(U_q^+[w'])$, $U_q^0U_q^+[w]$ is a right coideal subalgebra of $U_q(B^+)$, and $T_{w'}(U_q^+[w'])$ is a left coideal subalgebra of $U_q^+$.

Let $\tau$ be the anti-automorphism of $U_q$ defined by $\tau(E_\alpha) = E_\alpha$, $\tau(F_\alpha) = F_\alpha$, and $\tau(K_\alpha) = K_\alpha^{-1}$, $\alpha \in \Pi$. We can twist the Hopf algebra structure maps $(\Delta, S, \varepsilon)$ for $U_q$ by $\tau$ to obtain a new set of Hopf algebra structure maps $(\tau \Delta, \tau S, \tau \varepsilon) = ((\tau \circ \Delta) \circ \tau \circ S^{-1} \tau \circ \varepsilon)$ for $U_q$. Considering this new Hopf algebra structure on $U_q$, we get:

Lemma 4.3.3. Let $w \in W$. Write $w_0 = \delta w$ with $\ell(w_0) = \ell(w) + \ell(w')$. Then $U_q^+[w]$ is a left coideal subalgebra and $U_q^0T_{w'}(U_q^+[w'])$ is a right coideal subalgebra for the twisted Hopf algebra structure $(\tau \Delta, \tau S, \tau \varepsilon)$ on $U_q(B^+)$.

Proof sketch. To prove the statement for $U_q^+[w]$, use [1] Proposition C.4] and the identity $\tau \circ T_{\alpha} \circ \tau = T_{\alpha^{-1}}$, $\alpha \in \Pi$. To prove the statement for $U_q^0T_{w'}(U_q^+[w'])$, imitate the proof of Lemma 4.3.1. 

Corollary 4.3.4. Let $w_0 = s_{\beta_1} \cdots s_{\beta_N}$ be an arbitrary reduced expression for $w_0 \in W$, and let \{\gamma_1, \ldots, \gamma_N\} be the corresponding convex ordering of $\Phi^+$. Let $E_{\gamma_m} \in U_q$ be the positive root vector of weight $\gamma_m$ as defined in [17]. Then $\tau \Delta(E_{\gamma_m}) \in V_m' \otimes W_m'$, where $V_m' \subset U_q(B^+)$ is the subalgebra generated by $U_q^0 \cup \{E_{\gamma_1}, \ldots, E_{\gamma_m}\}$, and $W_m \subset U_q^+$ is the subalgebra generated by $\{E_{\gamma_1}, \ldots, E_{\gamma_N}\}$.

4.4. The Lyndon–Hochschild–Serre spectral sequence. Let $A$ be an augmented algebra, and let $B$ be a normal subalgebra of $A$. Write $B_\varepsilon$ for the augmentation ideal of $B$, and set $K = AB_\varepsilon$, the two-sided ideal in $A$ generated by $B_\varepsilon$. Then $A/B = A/K$. Assume that $A$ is flat as a right $B$-module. Then for any $A$-module $V$, there exists a unique natural action of $A$ on the cohomology groups $H^*(B, V)$ extending the action of $A$ on the space of invariants $V^B$. Moreover, there exists a spectral sequence satisfying

$$E_1^{i,j} = E_1^{i,j}(V) = \text{Hom}_A(A/B_\varepsilon(A/B), H^j(B, V)) \Rightarrow H^{i+j}(A, V).$$

This is the Lyndon–Hochschild–Serre (LHS) spectral sequence associated to the algebra extension $0 \to B \to A \to A/B \to 0$ and the $A$-module $V$. Depending on the existence of additional structure on $A$ and $B$, the LHS spectral sequence can be constructed in several equivalent ways, cf. [5] Chapter VIII. We are interested in a construction due to Hochschild and Serre, described in [5] Chapter IV. For this construction we make the additional assumption that $A$ is projective as a right $B$-module. The main points of the construction are summarized below.

Set $C = C^*(A, V) := \text{Hom}_A(B_\varepsilon(A), V)$. Write $\delta$ for the differential on $C$. Define a decreasing filtration $F$ on $C$ by setting $F^n C = C^n, F^{n+1} C = 0$, and for $0 < p \leq n$,

$$F^p C = \{f \in C^n(A, V) : f(a_1, \ldots, a_n) = 0 \text{ if any of } a_{n-p+1}, \ldots, a_n \text{ is in } K\}.$$

(In [5], this filtration is denoted by $F_\ast$.) Then $F$ makes $C$ a filtered differential graded module, and (4.3.1) is the associated spectral sequence. The identification

$$E_1^{i,n-p} \sim \text{Hom}_A(A/B_\varepsilon(A/B), H^{n-p}(B, V))$$

is made as follows: Let $f \in F^p C^n$ be such that $\delta(f) \in F^{p+1} C^{n+1}$. Then $f$ represents a relative cocycle $[f] \in E_0^{p,n-p} = F^p C^n/F^{p+1} C^n$. Given $x_1, \ldots, x_p \in A/B$, define $\eta([f])([x_1, \ldots, x_p]) = [\varphi]$, the cohomology class of $\varphi \in \text{Hom}_B(B_{n-p}(B), V)$, where $\varphi$ is defined by

$$\varphi([b_1, \ldots, b_{n-p}]) := f(b_1, \ldots, b_{n-p}, x_1, \ldots, x_p).$$
That \( \varphi \) is a well-defined cocycle follows from the fact that \( f \) represents a relative cocycle in \( E_0^{p,n-p} \). That \( \eta \) defines an isomorphism follows from [3] Theorem III.1.5 and Lemma IV.3.1].

**Remark 4.4.1.** Suppose \( A \) is a Hopf algebra, and that \( B \) is a Hopf subalgebra of \( A \). Then the LHS spectral sequence \( E_r(k) \) is naturally a spectral sequence of algebras, with products induced by a natural algebra structure on \( C(A,k) \), and \( E_r(V) \) is naturally a module over \( E_r(k) \), with module structure induced by a natural \( C(A,k) \)-action on \( C(A,V) \); see [3] Theorem IV.3.6]. The filtration \( F \) defined here is not compatible with these algebra and module structures, and this is part of the reason for the technical calculations we must conduct in Section II.5 (the other reason for the calculations being that we are not dealing with Hopf algebras in Section II.5). Still, the filtration \( F \) defined here seems to be the most useful for the purposes of our induction argument.

4.5. **Injectivity for root subalgebras.** Let \( \{x_{\alpha} : \alpha \in \Phi^+\} \subset u^+ \) be the dual basis corresponding to the root vector basis \( \{e_{\alpha} : \alpha \in \Phi^+\} \) for \( u^+ \). Then \( H(u_{\zeta}(b^+),k) \cong S(u^{+})\) is the polynomial algebra on the degree two generators \( x_{\alpha}, \alpha \in \Phi^+ \). Before we state the main result of this section, we collect some information on the polynomial generators for \( H(u_{\zeta}(b^+),k) \).

4.5.1. Let \( f \in \text{Hom}_{u_{\zeta}(b^+)}(B_2(u_{\zeta}(b^+)),k) \) be a cocycle representative for \( x_{\alpha} \). Since \( x_{\alpha} \) is a weight vector of weight \(-\alpha \) for \( U_0^\zeta \), we may assume that \( f \) is a weight vector of weight \(-\alpha \) for the adjoint action of \( U_0^\zeta \) on \( C^2(u_{\zeta}(b^+),k) \). Suppose \( \alpha \) is a simple root. Then \( f \) has support in the subspace of \( B_2(u_{\zeta}(b^+)) \) spanned by all vectors of the form \([x_1,x_2]\) with \( x_1,x_2 \in u_{\zeta}^0(e_{\alpha}) \).

4.5.2. Write \( \Phi^+ = \{\gamma_1, \ldots, \gamma_N\} \) with \( \gamma_i = w_i(\beta_i) \) as in \([11]\). Suppose that \( \alpha = \gamma_m \). Set \( \beta = \beta_m \), and set \( w = w_m \). We have \( H(u_{\zeta}(g),k) \cong k[N] \), and the restriction map \( H(u_{\zeta}(g),k) \rightarrow H(u_{\zeta}(b^+),k) \cong S(u^+) \) is just the restriction of functions. For each \( w \in W \), the braid group operator \( T_w \) induces an automorphism of \( u_{\zeta}(g) \), hence an automorphism \( (T_w^{-1})^* \) of \( H(u_{\zeta}(g),k) \). This automorphism maps vectors of weight \( \lambda \) for the adjoint action of \( U_0^\zeta \) to vectors of weight \( w\lambda \). At the level of cochains, \( (T_w^{-1})^* \) is induced by the map \( C(u_{\zeta}(g),k) \rightarrow C(u_{\zeta}(g),k) \), \( f \mapsto f \circ T_w^{-1} \), where we write \( T_w^{-1} : B(u_{\zeta}(g)) \rightarrow B(u_{\zeta}(g)) \) to denote the evident chain map induced by \( T_w^{-1} : u_{\zeta}(g) \rightarrow u_{\zeta}(g) \).

Lifting the coordinate functions \( x_{\alpha},x_{\beta} \in H^2(u_{\zeta}(b^+),k) \) to \( H^2(u_{\zeta}(g),k) \cong g^* \), we get \( (T_w^{-1})^*(x_{\beta}) = x_{w(\beta)} = x_{\alpha} \). Restricting back to \( u_{\zeta}(b^+) \), we now see that we can choose a cocycle representative \( f \in C^2(u_{\zeta}(b^+),k) \) for \( x_{\alpha} \) with support in the subspace of \( B_2(u_{\zeta}(b^+)) \) spanned by all vectors of the form \([x_1,x_2]\) with \( x_1,x_2 \in u_{\zeta}^0(e_{\alpha}) = T_w(u_{\zeta}^0(e_{\beta})) \).

**Proposition 4.5.1.** Let \( V \) be a finite-dimensional \( u_{\zeta}(b^+) \)-module. Let \( \alpha \in \Phi^+ \), and suppose that \( V \) is injective (equivalently, projective) for the root subalgebra \( u_{\zeta}(e_{\alpha}) \). Then under the cup product action of \( H(u_{\zeta}(b^+),k) \) on \( \text{H}^*(u_{\zeta}(b^+),V) \), \( x_{\alpha} \) acts nilpotently on \( \text{H}^*(u_{\zeta}(b^+),V) \), that is, for each fixed \( z \in \text{H}^*(u_{\zeta}(b^+),V) \), \( x_{\alpha}^r z = 0 \) for all \( r \gg 0 \).

**Proof.** First, since \( \text{H}^*(u_{\zeta}(b^+),V) = \text{H}^*(u_{\zeta}(u^+),V)^0 \), it suffices to show that the left cup product action (equivalently, the right Yoneda product action) of \( x_{\alpha} \) on \( \text{H}^*(u_{\zeta}(u^+),V) \) is nilpotent. The proof now breaks down into two cases:

**Case 1.** \( \alpha \) is a simple root. Assume that the reduced expression for \( w_0 \) chosen in \([11]\) begins with the simple reflection \( s_{\alpha} \), so that \( E_{s_{\alpha}} = E_{\alpha} \). (Replacing one reduced expression for \( w_0 \) by another results in a graded \( B^+ \)-automorphism of the ring \( H(u_{\zeta}(b^+),k) \cong S(u^+) \). Any such automorphism must map \( x_{\alpha} \) to a non-zero scalar multiple of itself, so there is no harm in making the above assumption on \( w_0 \).) Since \( V \) is injective for \( u_{\zeta}(e_{\alpha}) \), \( \text{H}^*(u_{\zeta}(e_{\alpha}),V) = \text{H}^0(u_{\zeta}(e_{\alpha}),V) \cong V^*_{u_{\zeta}(e_{\alpha})} \) is finite-dimensional. In particular, the cup product action of \( x_{\alpha} \in H^2(u_{\zeta}(b^+),k) \) on \( \text{H}^*(u_{\zeta}(e_{\alpha}),V) \) must be nilpotent.

Now fix \( 1 \leq m < N \). Let \( A \) be the subalgebra of \( u_{\zeta}(b^+) \) generated by \( \{E_{\gamma_1}, E_{\gamma_2}, \ldots, E_{\gamma_m}\} \), and let \( B \) be the subalgebra of \( u_{\zeta}(b^+) \) generated by \( \{E_{\gamma_1}, \ldots, E_{\gamma_m}\} \). The algebras \( A \) and \( B \) are
left coideal subalgebras for the twisted Hopf algebra structure on $u_\eta(b^+)$ by Corollary 4.3.4 and $B$ is normal in $A$ by Lemma 3.3.1. We prove by induction on $m$ that the cup product action of $x_\alpha$ on $H^\ast(A, V)$ is nilpotent, the case $m = 1$ already having been established. The main tool for the induction argument is the LHS spectral sequence

(4.5.1) \[ E_1^{ij} = \text{Hom}_{A//B}(B_i(A//B), H^j(B, V)) \Rightarrow H^{i+j}(A, V). \]

The algebra $A$ is free as a right $B$-module by the description of the PBW-basis for $u_\eta(u^+)$, so we can use the construction of (4.5.1) presented in 4.3

Fix a cocycle representative $f \in \text{Hom}_{u_\eta(b^+)}(B_2u_\eta(b^+), k)$ for $x_\alpha$ as in (4.5.1). Let $f^{\ast r}$ denote the $r$-fold cup product $f \cup f \cup \cdots \cup f$. To show that the cup product action of $x_\alpha$ on $H^\ast(A, V)$ is nilpotent, it suffices to show that for an arbitrary cocycle $g \in C^n(A, V)$, the iterated cup product $(f^{\ast r}) \cup g \in C^{n+2r}(A, V)$ is a coboundary in $C(A, V)$ for all $r$ sufficiently large. To prove that the cocycle $(f^{\ast r}) \cup g$ is a coboundary, we show that its image in the $E_1$-page of (4.5.1) is zero.

Let $g \in F^pC^n(A, V)$ be a cocycle. Then for all $r \geq 1$, $(f^{\ast r}) \cup g \in F^pC^{n+2r}(A, V)$. It may happen that $(f^{\ast r}) \cup g \in F^sC(A, V)$ for some $s > p$. We claim that for all $r \geq 1$, if $(f^{\ast r}) \cup g \notin F^sC(A, V)$, then $(f^{\ast r}) \cup g \notin F^{s+1}C(A, V)$. In particular, $(f^{\ast r}) \cup g \notin F^sC(A, V)$ for any $s > n$ unless the cocycle is identically zero. Indeed, if $(f^{\ast r}) \cup g \in F^{n+1}C^{n+2r}(A, V)$, then

\[ ((f^{\ast r}) \cup g)((x_1, \ldots, x_{2r}, x_{2r+1}, \ldots, x_{2r+n})) \quad (x_i \in A) \]

must be identically zero whenever one of $x_{2r}, x_{2r+1}, \ldots, x_{2r+n}$ is in $K = ABz$. By (4.2.3),

\[ ((f^{\ast r}) \cup g)((x_1, \ldots, x_{2r+n})) = (f^{\ast r})(x_1^{(1)}, \ldots, x_{2r}^{(1)}): x_1^{(2)} \cdots x_{2r}^{(2)} g(x_{2r+1}, \ldots, x_{2r+n}), \]

where we have used the twisted comultiplication $\tau \Delta$ for $u_\eta(b^+)$. From the description of the cocycle $f$ given in (4.5.1) the term $(f^{\ast r})(x_1^{(1)}, \ldots, x_{2r}^{(1)})$ vanishes identically unless $x_1^{(2)} = u_\eta^0 u_\eta(e_\alpha)$. But by Corollary 4.3.3, $x_1^{(1)} = u_\eta^0 u_\eta(e_\alpha)$ only if $x_{2r} \in K$. This proves that $(f^{\ast r}) \cup g \in F^{n+1}C(A, V)$ if and only if $(f^{\ast r}) \cup g = 0$.

Now, replacing $g$ if necessary by $(f^{\ast s}) \cup g$ for some $s \in \mathbb{N}$, we may assume that $(f^{\ast r}) \cup g$ has nonzero image in $F^pC(A, V)/F^{p+1}C(A, V)$ for all $r \geq 1$. Then for all $r \geq 1$, the image of the cocycle $(f^{\ast r}) \cup g$ in the $E_1$-page of (4.5.1) is the map $\eta([((f^{\ast r}) \cup g)]) \in \text{Hom}_{A//B}(B_p(A//B), H^{n+2r-p}(B, V))$. For fixed $x_1, \ldots, x_p \in A//B$, representative cocycles for

\[ \eta([((f^{\ast r}) \cup g)])(x_1, \ldots, x_p) \in H^{n+2r-p}(B, V) \quad \text{and for} \]

\[ \eta([g]((x_1, \ldots, x_p)) \in H^{n-p}(B, V) \]

are defined by (4.4.3). Comparing the representative cocycles, it is straightforward to check that

\[ \eta([((f^{\ast r}) \cup g)])(x_1, \ldots, x_p) = x_\alpha^p \cdot \eta(g)((x_1, \ldots, x_p)). \]

By induction, $x_\alpha$ acts nilpotently on $H^*(B, V)$. Since $A//B \cong u_\eta(e_{\gamma_m})$ is finite-dimensional, there exists $r \in \mathbb{N}$ such that $\eta([((f^{\ast r}) \cup g)])(x_1, \ldots, x_p) = 0$ for any fixed $x_1, \ldots, x_p \in A//B$. This proves that, for all $r$ sufficiently large, the image of $(f^{\ast r}) \cup g$ in $E_1^{p, n+2r-p}$ is zero, hence that $(f^{\ast r}) \cup g$ is a cocycle in $C^{n+2r}(A, V)$.

Case 2: $\alpha$ is not a simple root. Write $\Phi^+ = \{\gamma_1, \ldots, \gamma_N\}$ with $\gamma_i = w_i(\beta)$ as in (4.1) and suppose that $\alpha = \gamma_m$, $1 < m < N$. Set $w = w_m$. Let $A'$ be the subalgebra of $u_\eta(u^+)$ generated by the root vectors $\{E_{\gamma_1}, \ldots, E_{\gamma_N}\}$. Then $A := T_{w}^{-1}(A')$ is the subalgebra of $u_\eta(u^+)$ generated by the set

\[ S = \{E_{\beta_m} = T_{w_m}^{-1}(E_{\gamma_m}), T_{w_m}^{-1}(E_{\gamma_{m+1}}), \ldots, T_{w_m}^{-1}(E_{\gamma_N})\}. \]

Make $V$ into an $A$-module by defining $a.v = T_{w_m}(a)v$ for all $a \in A$ and $v \in V$. Write $wV$ for $V$ thus considered as an $A$-module. Then $wV$ is projective as a module for $u_\eta(e_{\beta_m})$, the root subalgebra corresponding to the simple root $\beta_m$. 
Note that the elements in the set $S$ are precisely the first $N - m + 1$ root vectors for $U_u^+$ corresponding to the reduced expression for $w_0$ beginning with the reduced word $s_{\beta_m} s_{\beta_{m+1}} \cdots s_{\beta_k}$.

Then by Case 1, the right Yoneda product action of $x_{\beta_m} \in H(u_{\zeta}(b^+), k)$ on $H^\bullet(A, u^V)$ is nilpotent. Applying the map $(T_{w_0}^{-1})^*: H^\bullet(A, u^V) \to H^\bullet(A', V)$, we get that the right Yoneda product action of $(T_{w_0}^{-1})^*(x_{\beta_m}) = x_{\beta_m} = x_{\alpha}$ on $H^\bullet(A', V)$ is nilpotent. The algebra $A'$ is a left coideal subalgebra of $u_{\zeta}(b^+)$ (with respect to the usual Hopf algebra structure) by Corollary 4.3.2 so by the discussion in [4.22] the right Yoneda product action of $H^2(u_{\zeta}(b^+), k)$ on $H^\bullet(A', V)$ is the same as the left cup product action. So the left cup product action of $x_{\alpha} \in H(u_{\zeta}(b^+), k)$ on $H^\bullet(A', V)$ is nilpotent.

The remainder of the proof in Case 2 now proceeds by a spectral sequence induction argument similar to that in Case 1. For the induction argument, the algebras $A$ and $B$ now have the form $A = \langle E_{\gamma_1}, \ldots, E_{\gamma_N} \rangle$, $B = \langle E_{\gamma_j}, \ldots, E_{\gamma_N} \rangle$ with $1 \leq j \leq m$. These algebras are left coideal subalgebras of $u_{\zeta}(b^+)$ by Corollary 4.3.2. Choose a cocycle representative $f$ for $x_{\alpha}$ as in [4.5.2] and then proceed as in Case 1. Details are left to the reader. □

4.6. Support varieties for the small Borel subalgebra. Proposition [4.5.1] can be strengthened by the observation:

**Lemma 4.6.1.** Let $V$ be a finite-dimensional $u_{\zeta}(b^+)$-module. Then the Yoneda product makes $H^\bullet(u_{\zeta}(b^+), V)$ a finitely-generated right $H^\bullet(u_{\zeta}(b^+), k)$-module.

**Proof.** First we prove the corresponding result for $u_{\zeta}(u^+)$-module. Let $V$ be a finite-dimensional $u_{\zeta}(u^+)$-module, and let $V = V_0 \supseteq V_1 \supseteq \cdots \supseteq V_r \supseteq V_{r+1} = 0$ be a composition series for $V$. Then for all $0 \leq i \leq r$, $V_i/V_{i+1} \cong k$, the unique simple $u_{\zeta}(u^+)$-module. Now by a standard argument using induction on the dimension of $V$ and the long exact sequence in cohomology, $H^\bullet(u_{\zeta}(u^+), V)$ is a finite module over $H^\bullet(u_{\zeta}(u^+), k)$.

Now assume that $V$ is a $u_{\zeta}(b^+)$-module. Then $H^\bullet(u_{\zeta}(b^+), V) = H^\bullet(u_{\zeta}(u^+), V)_0^Q$. The space $H^\bullet(u_{\zeta}(u^+), k)$ is a ring under the Yoneda product, and it is finitely-generated over the subring $H^\bullet(u_{\zeta}(u^+), k)_0 \cong H^\bullet(u_{\zeta}(b^+), k)_0$; cf. [20, §2.5]. The commutative ring $u_{\zeta}(u^+)_0$ act compatibly and completely reducibly on both $H^\bullet(u_{\zeta}(u^+), V)$ and the Noetherian ring $H^\bullet(u_{\zeta}(u^+), k)$, hence by [16, Lemma 1.13], $H^\bullet(u_{\zeta}(b^+), V)$ is a finite module over the Noetherian ring $H^\bullet(u_{\zeta}(b^+), k)$. □

**Corollary 4.6.2.** Let $V$ be a finite-dimensional $u_{\zeta}(b^+)$-module. Let $\alpha \in \Phi^+$, and suppose that $V$ is injective for the root subalgebra $u_{\zeta}(e_\alpha)$. Then there exists $r \in \mathbb{N}$ such that the cup product action of $x_\alpha^r \in H^2(u_{\zeta}(b^+), k)$ on $H^\bullet(u_{\zeta}(b^+), V)$ is identically zero.

**Proof.** Apply Proposition 4.5.1 and Lemma 4.6.1 and the commutativity of $H(u_{\zeta}(b^+), k)$. □

Now we state and prove the main theorem of this section:

**Theorem 4.6.3.** Let $M$ be a finite-dimensional $u_{\zeta}(b^+)$-module, and let $\alpha \in \Phi^+$. Then the root vector $e_\alpha \in u^+$ is an element of $\mathcal{V}_{u_{\zeta}(b^+)}(M)$ if and only if $M$ is not projective for $u_{\zeta}(e_\alpha)$.

**Proof.** First suppose that $M$ is projective (equivalently, injective) for the root subalgebra $u_{\zeta}(e_\alpha)$. Then $M$ is projective for the Hopf algebra $u_{\zeta}(b^+_\alpha) := \langle E_\alpha, K_\alpha \rangle \subset u_{\zeta}(b^+)$ by Lemma 2.1.2, hence $\text{Hom}_{u_{\zeta}(b^+)}(M, M) \cong M \otimes M^*$ is also projective for $u_{\zeta}(b^+_\alpha)$. Applying Lemma 2.1.2 again, $\text{Hom}_{u_{\zeta}(b^+)}(M)$ is injective for $u_{\zeta}(e_\alpha)$. Now Corollary 4.6.2 asserts that $x_\alpha^r \in J_{u_{\zeta}(b^+)}(M)$ for some $r \in \mathbb{N}$. It follows that $e_\alpha \notin \mathcal{V}_{u_{\zeta}(b^+)}(M)$.

Now suppose that $e_\alpha \notin \mathcal{V}_{u_{\zeta}(b^+)}(M)$. The restriction map $H(u_{\zeta}(b^+), k) \to H(u_{\zeta}(b^+_\alpha), k) = k[x_\alpha]$ is surjective, and induces a closed embedding

$$\mathcal{V} := \mathcal{V}_{u_{\zeta}(b^+_\alpha)}(M) \hookrightarrow \mathcal{V}_{u_{\zeta}(b^+)}(M).$$
Theorem 5.2.1. Let $\mathcal{V}$ be the variety in $\mathcal{V}_{u_\zeta(b^+)}(M)$ is either zero or the line spanned by the root vector $e_\alpha$. If $e_\alpha \notin \mathcal{V}_{u_\zeta(b^+)}(M)$, then $\mathcal{V} = \{0\}$. By a standard argument (cf. [18 Proposition 1.5]), this implies that $M$ is projective over the Hopf algebra $u_\zeta(b^+)$. □

Corollary 4.6.4. Let $M$ be a finite-dimensional $u_\zeta(b)$-module, and let $\alpha \in \Phi^+$. Then the root vector $f_\alpha \in u$ is an element of $\mathcal{V}_{u_\zeta(b)}(M)$ if and only if $M$ is not projective for $u_\zeta(f_\alpha)$.

Proof. Use the fact that $\omega : u_\zeta(b) \to u_\zeta(b^+)$ is an automorphism. □

Remark 4.6.5. We would like to generalize Corollary 4.6.4 from root vectors to linear combinations of elements of $u$, but it is not clear what the corresponding subalgebras of $u_\zeta(b)$ should be. We would also like to generalize Corollary 4.6.4 from $\mathcal{V}_{u_\zeta(b)}(M)$ to $\mathcal{V}_{u_\zeta(g)}(M)$, but this seems prohibitively difficult under the present strategy, since we have less control over the cocycle representatives for $x_\alpha$ after lifting them to $H(u_\zeta(g), k)$.

5. Applications

5.1. A geometric proof of Theorem 3.1.1. We can now provide a second, geometric proof of the $r = 0$ case of Theorem 3.1.1. The argument is formally similar to that in [18 §3.4]. We continue to impose the conditions of Assumption 4.1.1.

Let $M$ be a finite-dimensional rational $U^0_\zeta u_\zeta(g)$-module, and suppose that $M$ is injective for the Borel subalgebras $u_\zeta(b)$ and $u_\zeta(b^+)$. Then $V := M \otimes M^*$ is injective for $u_\zeta(b)$ and $u_\zeta(b^+)$. By Proposition 2.2.8, the injectivity of $V$ for $u_\zeta(b^+)$ implies that $V$ admits a filtration by $U^0_\zeta u_\zeta(g)$-submodules with factors of the form $\hat{Z}_0^*(\lambda)$, $\lambda \in X$. By Frobenius reciprocity and the exactness of induction from $U^0_\zeta u_\zeta(b)$ to $U^0_\zeta u_\zeta(g)$, $H^*(u_\zeta(g), \hat{Z}_0^*(\lambda)) \cong H^*(u_\zeta(b), \lambda)$. This isomorphism is compatible with the cup product action of $H(u_\zeta(g), k)$. Then for each $\lambda \in X$, $\ker(\text{res}) \subseteq J_{u_\zeta(g)}(\hat{Z}_0^*(\lambda))$, where $\text{res} : H(u_\zeta(g), k) \to H(u_\zeta(b), k)$ denotes the cohomological restriction map. It now follows from the long exact sequence in cohomology and by induction on the number of factors $\hat{Z}_0^*(\lambda)$, $\lambda \in X$, in the $U^0_\zeta u_\zeta(g)$-filtration of $V$ that $\ker(\text{res})$ is contained in the radical of the ideal $J_{u_\zeta(g)}(M)$. This implies that $\mathcal{V}_{u_\zeta(g)}(M) \subset u$. Since $\omega : u_\zeta(b) \to u_\zeta(b^+)$ and $X \circ \omega : H(u_\zeta(b), k) \to H(u_\zeta(b^+), k)$ are isomorphisms, we obtain by symmetry that the injectivity of $V$ for $u_\zeta(b)$ implies that $\mathcal{V}_{u_\zeta(g)}(M) \subset u^+$. Since $u \cap u^+ = \{0\}$, the injectivity of $M$ for $u_\zeta(b)$ and $u_\zeta(b^+)$ implies that $\mathcal{V}_{u_\zeta(g)}(M) = \{0\}$, hence that $M$ is injective for $u_\zeta(g)$.

We have thus geometrically reduced the problem of Theorem 3.1.1 to the statement of Theorem 3.3.1. Now, since $M$ is a rational $U^0_\zeta$-module, the support variety $\mathcal{V}_{u_\zeta(b)}(M)$ is a $T$-stable subvariety of $u$. Similarly, $\mathcal{V}_{u_\zeta(b^+)}(M)$ is a $T$-stable subvariety of $u^+$. By Theorem 4.6.3 and Corollary 4.6.4, neither support variety contains any root vectors. But any non-zero $T$-stable subvariety of $u$ (resp. $u^+$) must contain a root vector. We conclude that $\mathcal{V}_{u_\zeta(g)}(M) = \{0\} = \mathcal{V}_{u_\zeta(b^+)}(M)$, hence that the injectivity of $M$ for each root subalgebra $u_\zeta(e_\alpha)$, $u_\zeta(f_\alpha)$, $\alpha \in \Phi^+$, implies the injectivity of $M$ for $u_\zeta(b)$ and $u_\zeta(b^+)$.

5.2. Support varieties for the small quantum group. We continue to impose the conditions of Assumption 4.1.1. The next result and its proof are a translation to the quantum setting of [17 §1.2].

Theorem 5.2.1. Let $M$ be a finite-dimensional $U_\zeta(g)$-module. Then $\mathcal{V}_{u_\zeta(g)}(M) = G \cdot \mathcal{V}_{u_\zeta(b)}(M)$, where $G \cdot \mathcal{V}_{u_\zeta(b)}(M)$ is the orbit of $\mathcal{V}_{u_\zeta(b)}(M) \subset N$ under the adjoint action of $G$. 
Proof. Make the identifications $H(u_\zeta(g), k) = k[N]$ and $H(u_\zeta(b), k) = S(u^*)$, and define ideals

$$S(u^*) \supset I_M := J_{u_\zeta(b)}(M),$$
$$S(g^*) \supset J_M := J_{u_\zeta(g)}(M),$$
$$S(g^*) \supset K_M := \left\{ f \in S(g^*) : \forall g \in G, (g \cdot f)|_u \in I_M \right\},$$
$$S(g^*) \supset L_M := \left\{ f \in S(g^*) : \forall g \in G, (g \cdot f)|_u \in \sqrt{J_M} \right\}.$$

Then $I_M$ and $J_M$ are the ideals defining $V_{u_\zeta(b)}(M)$ and $V_{u_\zeta(g)}(M)$, respectively, and $L_M$ is the ideal of functions defining the closed subvariety $G \cdot V_{u_\zeta(g)}(M)$ of $N$. To prove the theorem it suffices to show that $L_M = \sqrt{J_M}$. (Note that $L_M$ is already a radical ideal.) The inclusion $\sqrt{J_M} \subseteq L_M$ is easy. Indeed, since the restriction map $H(\cdot, k) \to H(\cdot, k)$ is just the restriction of functions, it is in particular surjective, so $V_{u_\zeta(b)}(M)$ identifies naturally with a closed subvariety of $V_{u_\zeta(g)}(M)$. The support variety $V_{u_\zeta(g)}(M)$ is $G$-stable, so also $G \cdot V_{u_\zeta(b)}(M) \subseteq V_{u_\zeta(g)}(M)$. But this last inclusion is equivalent to the ideal inclusion $\sqrt{J_M} \subseteq L_M$. So it remains to show that $L_M \subseteq \sqrt{J_M}$. We do this by proving the inclusions $L_M \subseteq \sqrt{K_M}$ and $K_M \subseteq \sqrt{J_M}$.

Given $f \in L_M$, let $I_f \subseteq S(g^*)$ be the ideal generated by $\{g \cdot f : g \in G\}$. Since $S(g^*)$ is a Noetherian ring, $I_f$ is finitely generated, say by the set $\{g_1 \cdot f, \ldots, g_j \cdot f\}$. For each $1 \leq i \leq j$, there exists $a_i \in \mathbb{N}$ such that $(g_i \cdot f)^{a_i}|_u \in I_M$. Now fix $A \in \mathbb{N}$ with $A \geq \min \{j \cdot a_i\}$, and let $g \in G$. There exist $\lambda_i \in S(g^*)$ such that $g \cdot (f - \sum_{i=1}^j \lambda_i(x_i \cdot f)) = 0$. Then $g \cdot f^A = \sum_{i=1}^j \lambda_i(x_i \cdot f)^A$.

Expanding the last expression and restricting it to $u$, we see that each summand is an element of $I_M$, because in the expansion of $(\sum_{i=1}^j \lambda_i(x_i \cdot f))^A$, each summand is divisible by $(x_i \cdot f)^{a_i}$ for some $1 \leq i \leq j$. Then $f^A \in K_M$, so $L_M \subseteq \sqrt{J_M}$.

It remains to show $K_M \subseteq \sqrt{J_M}$. By the discussion in [2.3] there exists a commutative diagram

$$
\begin{array}{ccc}
K_M & \longrightarrow & S^*(g^*) \\
\downarrow & & \downarrow \\
\text{ind}^G_B I_M & \longrightarrow & \text{ind}^G_B S^*(u^*) \sim \longrightarrow \text{ind}^G_B H^2(u_\zeta(b), k) \longrightarrow \text{ind}^G_B H^2(u_\zeta(b), \text{Hom}_k(M, M)).
\end{array}
$$

(5.2.1)

Each vertical map is induced by Frobenius reciprocity from the corresponding restriction map: $K_M \to I_M$, $S^*(g^*) \to S^*(u^*)$, etc. The third and fourth vertical maps are the edge maps for the spectral sequences $E_2(k)$ and $E_2(V)$ discussed in [2.3] where $V := \text{Hom}_k(M, M)$. The maps into the right-most column are induced by the map $k \to \text{Hom}_k(M, M), 1 \mapsto 1$.

Composition along the bottom row in (5.2.1) is zero, because the composite map $I_M \to S^*(u^*) \cong H^2(u_\zeta(b), k) \to H^2(u_\zeta(b), V)$ is zero by the definition of $I_M$. This implies that the composite map along the top row and down the right-most column, from $K_M$ to $\text{ind}^G_B H^2(u_\zeta(b), V)$, is zero. Then the image of $K_M$ in $H^2(u_\zeta(g), V)$ must have positive filtration degree with respect to the filtration on $H^*(u_\zeta(g), V)$ coming from the spectral sequence (2.3.2). We have $E_2^{i,j}(V) = 0$ for $i > d := \dim(G/B)$, because $R \text{ind}^B_A(-) = 0$ for $i > d$. Then $E_2^{d,j}(V) = 0$ for $i > d$. Since $E_r(V)$ is a module over $E_r(k)$, we conclude that $(K_M)^{d+1}$ maps to zero in $H^*(u_\zeta(g), V)$, hence that $K_M \subseteq \sqrt{J_M}$.

\[\square\]

**Corollary 5.2.2.** Let $M$ be a finite-dimensional $U_\zeta$-module. Let $\alpha_h \in \Phi^+$ be the highest positive root. Then $M$ is injective for $u_\zeta(g)$ if and only if $M$ is injective for the root subalgebra $u_\zeta(f_{\alpha_h})$.

**Proof.** The surjectivity of the restriction map $H(u_\zeta(g), k) \to H(u_\zeta(b), k)$ implies that there exists a closed embedding $V_{u_\zeta(b)}(M) \to V_{u_\zeta(g)}(M)$. If $M$ is injective for $u_\zeta(g)$, then $V_{u_\zeta(g)}(M) = \{0\}$, so necessarily $V_{u_\zeta(b)}(M) = \{0\}$. Then $M$ is injective for $u_\zeta(f_{\alpha_h})$ by Corollary [1.6.3].

Conversely, suppose $M$ is not injective for $u_\zeta(g)$. Then $V_{u_\zeta(g)}(M) \neq \{0\}$ [15], Proposition 2.4]. (The last statement uses the fact that for every pair of finite-dimensional $u_\zeta(g)$-modules $M$ and $N$,...
the space $\text{Ext}^\bullet(u_\zeta(g), k)(M, N)$ is a finitely-generated $H^\bullet(u_\zeta(g), k)$-module under the cup product, cf. [6] or [13 Theorem 4.24].) By Theorem 5.2.1 this implies that $V_{u_\zeta(b)}(M) \neq \{0\}$. Now, since $M$ is a $U_\zeta(B)$-module, $V_{u_\zeta(b)}(M)$ is a non-zero closed $B$-stable subvariety of $u$. In particular, $V_{u_\zeta(b)}(M)$ is a $T$-stable closed subvariety of $u$, so it must contain a root vector $f_\alpha \in u$, for some $\alpha \in \Phi^+$. If $\beta \in \Phi^+$ and $\alpha + \beta \in \Phi^+$, then $f_{\alpha + \beta} \in U_\beta \cdot f_\alpha$, the $U_\beta$-orbit closure of $f_\alpha$. Since $V_{u_\zeta(b)}(M)$ is closed, it follows that $f_{\alpha h} \in V_{u_\zeta(b)}(M)$, hence that $M$ is not injective for $u_\zeta(f_{\alpha h})$ by Corollary 4.6.4. □

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