Boxing with Neutrino Oscillations

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We develop a model-independent “box” parameterization of neutrino oscillations. Oscillation probabilities are linear in these new parameters, so measurements can straightforwardly determine the box parameters which can then be manipulated to yield magnitudes of mixing matrix elements. We examine the effects of unitarity on the box parameters and reduce the number of parameters to the minimum set. Using the box algebra, we show that CP-violation may be inferred from measurements of neutrino flavor mixing even when the oscillatory factor has averaged. The framework presented here will facilitate general analyses of neutrino oscillations among $n \geq 3$ flavors.

I. INTRODUCTION

If neutrinos have mass, then the states of definite mass, $\nu_i$, may be distinct from the states of definite flavor, $\nu_\alpha$. (We will use latin indices $i, j, k, \ldots$ to refer to mass states, and greek indices $\alpha, \beta, \gamma, \ldots$ to refer to flavor states.) If the eigenvalues of the neutrino mass matrix are non-degenerate, then neutrinos may change flavors, or oscillate, as they propagate. The long-standing solar neutrino deficit \cite{1}, the atmospheric neutrino anomaly \cite{2}, and the recent results from the LSND experiment \cite{3} can all be understood in terms of oscillations between neutrinos. Traditional oscillation analyses generally assume no more than three generations of neutrinos. But resonant oscillations for the sun, oscillations for the atmosphere, and the LSND data each require a different neutrino mass-squared difference in neutrino oscillations are to account for all features of the data \cite{4}. Since three-neutrino models can have at most two independent mass-squared differences, a sterile neutrino is apparently needed to reconcile all the data while retaining consistency with LEP measurements of $Z \rightarrow \nu\bar{\nu}$ \cite{5}. Several four-neutrino analyses appear in the literature \cite{4,6}. If, however, statistical or systematic errors in the data evolve in the future, or if some data turns out to have an explanation other than neutrino oscillations, then three-neutrino oscillations may be sufficient. One recent analysis of the complete set of data in the three-neutrino framework is given in reference \cite{7}.

Oscillation probabilities depend on products of four mixing-matrix elements. Several parameterizations of the mix-
ing matrix in terms of rotation angles have been introduced, beginning with the pioneering work of Kobayashi and Maskawa [8]. When three or more neutrino generations are included in the oscillations analysis, the oscillation probabilities become complicated functions of the neutrino mixing angles. But oscillations are observable and therefore parameterization-invariant. One must ask if there is not a better description of oscillations which avoids the arbitrariness of angular-parameterization schemes. In this paper, we introduce a new “box” parameterization of neutrino mixing valid for any number of neutrino generations. Oscillation probabilities are linear in the boxes, enabling a straightforward analysis of oscillation data. In what follows, we develop the algebra of the boxes and the unitarity constraints on that algebra. To conclude we illustrate the boxes’ reduction to a basis in the case of three generations. A phenomenological analysis of existing oscillation data will be performed in a future publication.

II. THE STANDARD FORMULATION OF NEUTRINO OSCILLATIONS

The probability for a neutrino to oscillate from \( \nu_\alpha \) to \( \nu_\beta \) is given by the square of the transition amplitude:

\[
P_{\nu_\alpha \rightarrow \nu_\beta} (x) = \left| \sum_{i=1}^{n} V_{\alpha i} V_{\beta i}^* e^{-i\phi_i} \right|^2 = \sum_{i=1}^{n} \sum_{j=1}^{n}(V_{\alpha i} V_{\beta i}^* V_{\alpha j} V_{\beta j}^*) e^{-i(\Phi_{ij})},
\]

where \( n \) is the number of neutrino generations and

\[
\Phi_{ij} \equiv \frac{1}{2} (\phi_i - \phi_j) = \frac{1}{2} (E_i t_i - p_i x_i - E_j t_j + p_j x_j).
\]

(1)

For relativistic neutrinos, \( \Phi_{ij} \) is given by

\[
\Phi_{ij} \approx \frac{\Delta m_{ij}^2}{4p}, \quad \text{where} \quad \Delta m_{ij}^2 \equiv m_i^2 - m_j^2.
\]

(2)

With a little bit of algebra, the oscillation probability may be brought into the form

\[
P_{\nu_\alpha \rightarrow \nu_\beta} (x) = \sum_{i=1}^{n} |V_{\alpha i}|^2 |V_{\beta i}|^2 + 2 \text{Re} \left[ \sum_{i=1}^{n} \sum_{j \neq i} V_{\alpha j} V_{\beta j}^* V_{\alpha i} V_{\beta i}^* e^{-i\Phi_{ij}} \right],
\]

or equivalently, the form

\[
P_{\nu_\alpha \rightarrow \nu_\beta} (x) = -2 \sum_i \sum_{j \neq i} \text{Re}(V_{\alpha i} V_{\beta j}^* V_{\alpha j}^* V_{\beta j}) \sin^2 (\Phi_{ij})
\]

\[+ \sum_i \sum_{j \neq i} \text{Im}(V_{\alpha i} V_{\beta j}^* V_{\alpha j}^* V_{\beta j}) \sin (2\Phi_{ij}) + \delta_{\alpha\beta}.
\]

(4)

The probability for an antineutrino to oscillate from \( \bar{\nu}_\alpha \) to \( \bar{\nu}_\beta \) is obtained from \( P_{\nu_\alpha \rightarrow \nu_\beta} (x) \) by replacing \( V \) with \( V^* \). This is equivalent to changing the sign of \( \Phi_{ij} \). \( P_{\bar{\nu}_\alpha \rightarrow \bar{\nu}_\beta} (x) \) therefore differs from \( P_{\nu_\alpha \rightarrow \nu_\beta} (x) \) merely in the sign of the second term in equation (3) or of the exponent in equation (4). Note that \( P_{\nu_\alpha \rightarrow \bar{\nu}_\beta} (x) = P_{\bar{\nu}_\alpha \rightarrow \nu_\beta} (x) \), as required by CPT-invariance.
Let us first make contact with the familiar case of two neutrino flavors, \( n = 2 \). An arbitrary \( 2 \times 2 \) unitary matrix
will have one rotation angle parameterizing the real degree of freedom, and three phases. But all three phases may be
absorbed into the definitions of Dirac fermion fields (or will cancel in oscillation probabilities for Majorana neutrinos
\[10,11\]), so a \( 2 \times 2 \) mixing matrix \( V \) has the simple form of a rotation matrix:
\[
V = \begin{pmatrix}
\cos \theta & -\sin \theta \\
\sin \theta & \cos \theta
\end{pmatrix}.
\]
(6)

Since the matrix is explicitly real, the term in equation (5) involving imaginary matrix elements disappears, and the
oscillation probability in the two-flavor case is
\[
P_{\nu_\alpha \rightarrow \nu_\beta}(x) = \sin^2 2\theta \sin^2 \left( \frac{\Delta m^2_{12}}{4p} x \right) + \delta_{\alpha\beta}, \quad n = 2.
\]
(7)

This result is simple, and the mixing-angle parameterization is a natural choice in the two-flavor situation.

The math becomes more complicated when we include three generations. An arbitrary \( 3 \times 3 \) unitary matrix has
three real degrees of freedom and six phases. \( 2n - 1 = 5 \) phases are not observable \[10,12\], since this is the number of
phase differences among the fermion fields which may be absorbed into field redefinitions. The three-generation mixing
matrix is therefore described by three mixing angles and one phase. The original choice of these four parameters, by
Kobayashi and Maskawa to describe quark mixing, is perhaps the best known parameterization. Their choice of \( V \) is
\[
\begin{pmatrix}
c_1 & s_1c_3 & s_1s_3 \\
-s_1c_2 & c_1c_2c_3 - s_2s_3\text{e}^{i\delta} & c_1c_2s_3 + s_2c_3\text{e}^{i\delta} \\
-s_1s_2 & c_1s_2c_3 + c_2s_3\text{e}^{i\delta} & c_1s_2s_3 - c_2c_3\text{e}^{i\delta}
\end{pmatrix},
\]
(8)
where \( c_a \equiv \cos \theta_a \), and \( s_a \equiv \sin \theta_a \). We will refer to this particular choice of angles as the “standard” or “KM”
parameterization.

In the standard KM parameterization, the phase only appears in the lower right-hand sub-block of the matrix.

There is arbitrariness associated with the placement of the phase, since we absorb five relative phases into the field
definitions. Clearly the location of the phase cannot be measurable. Indeed, the CP-violating effects of the phase are
contained in a single function of the phase called the Jarlskog invariant \( J \), which in the standard KM parameterization
has the form \[13\] \( J = c_1s_1^2c_2s_2c_3s_3 \sin \delta \). Because of the arbitrariness of the phase convention, the phases of individual
matrix elements are not observable; only the magnitudes of matrix elements are observable.
The observable oscillation probabilities are quite complicated functions of the mixing angles in angle-based parameterizations. As an example, consider the product \( V_{22}^* V_{32}^* V_{33} \) appearing in the \( \nu_\mu \rightarrow \nu_\tau \) oscillation probability. In the standard KM mixing matrix, this product is given by

\[
V_{22} V_{32}^* V_{33} = c_3^2 s_2^2 [s_2^2 c_2^2 (s_1^4 + 6 c_1^2 + 2 c_1^2 \cos 2\delta) - c_1^2] \\
+ \frac{J}{s_1} (1 + c_1^2) (c_2^2 - s_2^2) (s_3^2 - c_3^2) \cot \delta + iJ, \quad n = 3.
\]  

Besides being ugly, thereby motivating the often-made two-generation approximation, the expression (but not its value) on the right-hand side of equation (9) is convention-dependent. Other parameterizations of the unitary mixing matrix involving three angles and a phase are equally valid and yield similarly complex oscillations probabilities. Our development of a model-independent parameterization has been motivated by the arbitrariness and complexity of the traditional approach.

III. A NEW PARAMETERIZATION

A. Boxes Defined, and Their Symmetries

The immeasurability of the mixing matrix elements in the quark sector has been addressed by numerous authors, such as those of references [13–17]. Measurable quantities include only the magnitudes of mixing matrix elements, the products of four mixing-matrix elements appearing in the oscillation probabilities, and particular higher-order functions of mixing-matrix elements [15,18]. As evidenced in equations (1), (4), and (5), neutrino oscillation probabilities depend linearly on the fourth-order objects,

\[
^{\alpha_1 \square \beta_j} V^{\alpha_1 \beta_j} = V_{\alpha_1 \alpha_j} V_{\beta j}^* V_{\beta j}^* V_{\beta j},
\]  

which we call “boxes” since each contains as factors the corners of a submatrix, or “box,” of the mixing matrix. For example, the upper left 2 \( \times \) 2 submatrix elements produce the box

\[
^{11 \square 22} V^{11 \square 22} = V_{11} V_{12}^* V_{21} V_{22}.
\]  

These boxes are the neutrino equivalent of the “plaques” used by Bjorken and Dunietz for another purpose in the quark sector in reference [10]. The name “box” also seems appropriate in light of the Feynman diagram which describes the oscillation process (Figure 1).
FIG. 1. Interference diagram contributing to the box $\alpha_i \square_{\mu j}$, which appears in the $\nu_e \rightarrow \nu_\mu$ oscillation probability, as the neutrino travels a distance $D$. Arrows on the diagram represent fermion number, not momentum.

In general, each of the box indices $i, j, \alpha,$ and $\beta$ may be any number between 1 and $n$, the number of neutrino flavors. We therefore initially have $n^4$ possible boxes. Examination of equation (10), however, reveals a few symmetries in the indexing:

\[
\alpha_i \square_{\beta j} = \beta_j \square_{\alpha i} = \beta_i \square_{\alpha^* j} = \alpha_j \square_{\beta^* i}.
\]  
(12)

If the order of either set of indices is reversed (\textit{id est}, $j \leftrightarrow i$ or $\beta \leftrightarrow \alpha$), the box turns into its complex conjugate; if both sets of indices are reversed, the box returns to its original value [14]. And if $V$ is replaced by $V^\dagger$, then $\alpha_i \square_{\beta j} \rightarrow i\alpha_{\sqrt{}}\beta_j$.

B. Degenerate and Nondegenerate Boxes

Equation (12) demonstrates that boxes with $\alpha = \beta$ or $i = j$, are real. Indeed, these are given from equation (10) as

\[
\alpha_i \square_{\alpha j} = |V_{\alpha i}|^2 |V_{\alpha j}|^2, \quad \text{and} \quad \alpha_i \square_{\beta i} = |V_{\alpha i}|^2 |V_{\beta i}|^2.
\]  
(13)

Those boxes with both sets of indices equal are

\[
\alpha_i \square_{\alpha i} = |V_{\alpha i}|^4.
\]  
(14)

We call boxes with one and two repeated indices “singly-degenerate” and “doubly-degenerate,” respectively. Boxes with $\alpha \neq \beta$ and $i \neq j$ are called nondegenerate. As can be seen from equation (5), singly-degenerate boxes with
repeated flavor indices enter into the formulae for flavor-conserving survival probabilities, but not for flavor-changing transition probabilities. Degenerate boxes with repeated mass indices (including the doubly-degenerate boxes) do not appear in any oscillation formula. Degenerate boxes may be expressed in terms of the nondegenerate boxes, as will be shown shortly. This possibility and the symmetries expressed in equation (12) allow us to express combinations of boxes in terms of only the nondegenerate “ordered” boxes for which \( \alpha < \beta \) and \( i < j \).

The number of flavor-index pairs satisfying \( \alpha < \beta \) ordering is \( N \equiv \binom{n}{2} = \frac{n(n-1)}{2} \). \( N \) mass-index pairs similarly satisfy \( i < j \) ordering. Thus, the number of ordered nondegenerate boxes is \( N^2 \). The number of doubly-degenerate boxes is \( n^2 \). The number of ordered singly-degenerate boxes is \( \frac{1}{2} n^2 (n-1) = nN \) for flavor-degenerate boxes, and the same for mass-degenerate boxes, yielding \( 2nN \) total singly-degenerate ordered boxes. For \( n = 3 \), \( N = 3 \) too, and we have nine ordered nondegenerate boxes, nine ordered singly-degenerate boxes, and nine doubly-degenerate boxes. For \( n > 3 \), matrix elements \( V_{\alpha i} \) with \( \alpha > 3 \) are not easily accessible since \( \alpha > 3 \) labels heavy charged leptons beyond the usual \( e, \mu, \tau \). Therefore, in the absence of processes with sufficient energy to excite the heavy charged lepton degree of freedom, boxes with flavor indices exceeding 3 are not physically accessible either. The number of accessible ordered nondegenerate boxes is \( 3N \). The number of accessible ordered singly-degenerate boxes is \( 3N \) with flavor degeneracy and \( 3n \) with mass degeneracy, yielding \( 3(n + N) \) for the total. These counts are recapped later in Table I, along with other box counts.

**C. Oscillation Probabilities in Terms of Boxes**

Using the symmetries expressed in equation (12), equation (5) becomes

\[
P_{\nu_{\alpha} \rightarrow \nu_{\beta}}(x) = -2 \sum_{i=1}^{n} \sum_{j>i} \left[ 2 \Re[\beta_{ij}] \sin^2 \Phi_{ij} - \Im[\beta_{ij}] \sin 2\Phi_{ij} \right] + \delta_{\alpha\beta},
\]

where we have defined the shorthand \( \Re[\beta_{ij}] \equiv \Re \left( \alpha_{i} \square \beta_{j} \right) \) and \( \Im[\beta_{ij}] \equiv \Im \left( \alpha_{i} \square \beta_{j} \right) \). When individual oscillations cannot be resolved due to uncertainties in energy or position, the \( \sin^2 \Phi_{ij} \) and \( \sin 2\Phi_{ij} \) are effectively averaged, giving the transition probabilities

\[
\langle P_{\nu_{\alpha} \rightarrow \nu_{\beta}}(x) \rangle = -2 \sum_{i=1}^{n} \sum_{j>i} \Re[\beta_{ij}],
\]

(16)

These probabilities are equivalently given by

\[
\langle P_{\nu_{\alpha} \rightarrow \nu_{\beta}}(x) \rangle = \sum_{i=1}^{n} |V_{\alpha i}|^2 |V_{\beta i}|^2 = \sum_{i=1}^{n} \alpha_{i} \square \beta_{i},
\]

(17)
which may be a more familiar expression; the equivalence of these expressions will be shown later in this paper.

The survival probabilities $P_{\nu_{\alpha} \rightarrow \nu_\alpha}(x)$ may be found from the transition probabilities $P_{\nu_{\alpha} \rightarrow \nu_\beta}(x)$ by

$$P_{\nu_{\alpha} \rightarrow \nu_\alpha}(x) = 1 - \sum_{\beta \neq \alpha} P_{\nu_{\alpha} \rightarrow \nu_\beta}(x). \quad (18)$$

Survival probabilities are more simply expressed in terms of degenerate boxes, or $|V|s$, rather than nondegenerate boxes. They are

$$P_{\nu_{\alpha} \rightarrow \nu_\alpha}(x) = 1 - 4 \sum_{i=1}^{n} \sum_{j>i}^{n} \alpha_i \square \alpha_j \sin^2 \Phi_{ij} = 1 - 4 \sum_{i=1}^{n} \sum_{j>i}^{n} |V_{\alpha i}|^2 |V_{\alpha j}|^2 \sin^2 \Phi_{ij}. \quad (19)$$

If oscillations are averaged, the survival probability is

$$\langle P_{\nu_{\alpha} \rightarrow \nu_\alpha}(x) \rangle = 1 - 2 \sum_{i=1}^{n} \sum_{j>i}^{n} |V_{\alpha i}|^2 |V_{\alpha j}|^2 = 1 - \sum_{i=1}^{n} \sum_{j \neq i}^{n} |V_{\alpha i}|^2 |V_{\alpha j}|^2 = \sum_{i=1}^{n} |V_{\alpha i}|^4 = \sum_{i=1}^{n} \alpha_i \square \alpha_i. \quad (20)$$

The third equality here results from unitarity of $V$.

Interchanging $\alpha \leftrightarrow \beta$ in equation (15) gives the time-reversed reactions $P_{\nu_\beta \rightarrow \nu_\alpha}(x)$:

$$P_{\nu_\beta \rightarrow \nu_\alpha}(x) = -2 \sum_{i=1}^{n} \sum_{j>i}^{n} \left[ 2 R_{\beta j} \sin^2 \Phi_{ij} + J_{\beta j} \sin 2\Phi_{ij} \right] + \delta_{\alpha \beta}, \quad (21)$$

so a measure of T-violation, or equivalently CP-violation, in the neutrino sector is

$$P_{\nu_{\alpha} \rightarrow \nu_\beta}(x) - P_{\nu_{\beta} \rightarrow \nu_\alpha}(x) = P_{\nu_{\alpha} \rightarrow \nu_\beta}(x) - P_{\nu_{\beta} \rightarrow \nu_\alpha}(x) = 4 \sum_{i=1}^{n} \sum_{j>i}^{n} \beta_{\beta j} \sin 2\Phi_{ij}. \quad (22)$$

Ignoring possible CP-violating phases in the mixing matrix, there are $\binom{n}{2} = N$ real parameters determining $V$. Determining these $N$ parameters determines the complete mixing matrix. Conveniently, there are $N$ transition probabilities $P_{\nu_{\alpha} \rightarrow \nu_\beta}(x) = P_{\nu_{\beta} \rightarrow \nu_\alpha}(x)$. Thus, all of the information in the mixing matrix is contained in the $N$ transition probabilities. In this sense, they form a convenient basis for determining all oscillation parameters. Of course, if the same transition probability is measured at two or more different distances, then all $N$ transition probabilities may not be needed to determine $V$.

Allowing CP-violation in the mixing matrix, there are $N$ real parameters and $\frac{1}{2}(n-1)(n-2)$ phases, for a total of $(n-1)^2$ parameters. With CP-violation, however, there are $2N = n(n-1)$ independent transition probabilities $P_{\nu_{\alpha} \rightarrow \nu_\beta}(x)$. The number of transition probabilities exceeds the number of independent parameters, so they again form a convenient basis for determining the mixing matrix. In reality, only the three flavor indices $e, \mu, \tau$ are easily accessible, as alluded to at the end of the previous section. Moreover, some of the $N$ parameters in the mixing matrix, namely those which rotate sterile states for $n \geq 5$, are not accessible. The inaccessibility issue complicates the counting. We do not pursue it further here.
The transition probabilities for which \( \alpha \neq \beta \) in equation (15) may be expressed in matrix form. The matrix of boxes will necessarily be an \( N \times N \) matrix since the vector on the left-hand side and the sum on the right-hand side of equation (15) each contain \( N \) components or terms. For three flavors, we have

\[
\mathcal{P}(n = 3) \equiv \begin{pmatrix}
P_{\nu_e \rightarrow \nu_\mu} (x) \\
P_{\nu_\mu \rightarrow \nu_e} (x) \\
P_{\nu_e \rightarrow \nu_\tau} (x)
\end{pmatrix}
= -4 \text{Re}(\mathcal{B}) \mathcal{S}^2(\Phi) + 2 \text{Im}(\mathcal{B}) \mathcal{S}(2\Phi),
\]

(23)

where

\[
\mathcal{B} \equiv \begin{pmatrix}
e_1 \square_{\mu 2} & e_2 \square_{\mu 3} & e_1 \square_{\mu 3} \\
m_1 \square_{\tau 2} & m_2 \square_{\tau 3} & m_1 \square_{\tau 3} \\
e_1 \square_{\tau 2} & e_2 \square_{\tau 3} & e_1 \square_{\tau 3}
\end{pmatrix}, \quad \text{and} \quad \mathcal{S}^k(\Phi) \equiv \begin{pmatrix}
\sin^k \Phi_{12} \\
\sin^k \Phi_{23} \\
\sin^k \Phi_{13}
\end{pmatrix}, \quad n = 3.
\]

(24)

For the time-reversed channels, or for the antineutrino channels, the sign of the \( \text{Im}(\mathcal{B}) \) term is reversed. The definitions of \( \mathcal{P} \), \( \mathcal{B} \) and \( \mathcal{S}^k(\Phi) \) may be extended to any number of flavors. Adding a fourth, perhaps sterile, neutrino flavor increases the dimensions of \( \mathcal{B} \) to 6 \( \times \) 6, since there are now six independent transition probabilities and six mass differences. The number of ordered nondegenerate boxes increases from nine to thirty-six, but only eighteen of them are accessible. For the six flavors of neutrinos in mirror-symmetric schemes, \( \mathcal{B} \) is a fifteen-by-fifteen matrix, and we have two hundred twenty-five ordered nondegenerate boxes. But the number of accessible boxes is just forty-five.

Our parameterization is especially well-suited for considering higher numbers of generations. While using boxes in \( n > 3 \) situations may be cumbersome, it is much less so than extending a mixing-angle parameterization to higher generations. Our matrix \( \mathcal{B} \) of boxes merely acquires extra columns when new flavors are introduced; extra rows are not accessible at energies below new charged-lepton thresholds. Furthermore, oscillation probabilities are linear in boxes, no matter how many generations.

**D. A Return to Two Generations**

The boxes may be used to illustrate when the two-flavor oscillation formula (1) is a meaningful approximation and when it is not. For this, we assume ordering of the mass indices \( m_1 < m_2 < \cdots < m_n \). If the transit distance \( L \) of the neutrino is just large enough to impact upon the shortest oscillations length \( \frac{4\pi}{\Delta m^2_{\text{min}}} \), but not upon the next-shortest, then

\[
\sin^2 2\theta^{\text{eff}}_{\alpha \beta} = -4 \Re_{\beta n}
\]

(25)
is meaningful. If one amplitude $\mathcal{R}_{\beta j}$ among those associated with $\Delta m_{ij}^2$ satisfying $\Delta m_{ij}^2 \gtrsim \frac{4\pi p}{L}$ dominates the others, then

$$\sin^2 2\theta_{\alpha\beta}^{eff} = -4 \mathcal{R}_{\beta j}$$

is a meaningful approximation. If there is a dominant mass $m_n \gg m_{n-1}, \cdots, m_1$, then for $\frac{4\pi p}{m_n} \lesssim L \ll \frac{4\pi p}{\Delta m_{n-1,1}}$, \[ \sin^2 2\theta_{\alpha\beta}^{eff} = -4 \sum_{i=1}^{n-1} \mathcal{R}_{\beta n} \] is meaningful. We will show later (equation (28)) that the right-hand side of equation (27) may be written as $4 |V_{\alpha n}|^2 |V_{\beta n}|^2$. If the largest masses are nearly degenerate, with degeneracy $q$, then the above becomes \[ \sin^2 2\theta_{\alpha\beta}^{eff} = -4 \sum_{i=1}^{n-q} \sum_{j=n-q+1}^{n} \mathcal{R}_{\beta j}, \] again meaningful. However, if two or more well-separated $\Delta m_{ij}^2$ s exceed $\frac{4\pi p}{L}$, then the two-flavor formula is inapplicable.

E. Relations between Mixing Matrix Elements and Boxes

Neutrino oscillation experiments will directly measure the boxes in equation (15), not the individual mixing matrix elements, $V_{\alpha i}$. But one would like to obtain the fundamental $V_{\alpha i}$ from the measured boxes. We develop here an algebra relating boxes and mixing matrix elements.

Some tautologous relationships between the degenerate and nondegenerate boxes are easily confirmed using equation (10); they hold for any number of generations:

$$|V_{\alpha i}|^2 |V_{\alpha j}|^2 = \frac{\alpha_i \alpha_j}{\eta_i \lambda_j}, \quad (\eta \neq \lambda \neq \alpha),$$

$$|V_{\alpha i}|^2 |V_{\beta i}|^2 = \frac{\alpha_i \alpha_j}{\alpha x \beta y}, \quad (x \neq y \neq i), \quad \text{and} \quad (30)$$

$$\frac{|V_{\alpha i}|^2}{|V_{\beta j}|^2} = \frac{\alpha_i \alpha_j}{\alpha x \beta y \eta \lambda}, \quad (\eta \neq \alpha \neq \beta, \text{ and } x \neq i \neq j).$$

In these equations and what follows, $\alpha, \beta, \gamma, i, j$, and $k$ will usually be reserved for indices that are chosen at the start of a calculation, while other indices such as $x, y, \eta$, and $\lambda$ primarily represent “dummy” indices which are chosen arbitrarily except to respect the inequality constraints following equations such as equation (31).
The tautologies (29) to (31) become evident not only by algebraically using the definitions of the boxes, but also by considering a graphical representation we have developed. This representation is discussed in detail in Appendix A, and it proves quite useful when considering relationships between boxes. Figure 2 illustrates the representations of $11\square_{22}$ and $(11\square_{22})^{-1}$.

![Figure 2](image)

FIG. 2. The graphical representation for (a) the box $11\square_{22}$, and (b) its inverse $(11\square_{22})^{-1}$. Vertical arrows point from the matrix elements which are not complex conjugated in the box to the complex-conjugated elements. While shown here in three generations, this representation may be extended to higher generations merely by adding more dots.

To illustrate an example of equation (29), the degenerate box $12\square_{13}$ is “created” in Figure 3 when one uncanceled vertical arrow enters and another leaves each of the matrix elements $V_{12}$ and $V_{13}$, while arrows at all other points cancel. Those arrows are the result of the combination of ordered boxes given in (29), as shown in Figure 3b.

![Figure 3](image)

FIG. 3. The graphical representation in three generations for (a) the degenerate box $12\square_{13}$, and (b) the combination of nondegenerate boxes which “create” it.

Equations (29) and (30) are themselves special cases of the more general

$$\alpha_i\square_{\beta j} \gamma_i\square_{\delta j} = \left[V_{\alpha i} V_{\alpha j}^* V_{\beta j} V_{\delta i}^* \right] \left[V_{\gamma i} V_{\gamma j}^* V_{\delta j} V_{\beta i}^* \right]$$

$$= \left[V_{\alpha i} V_{\alpha j}^* V_{\delta j} V_{\beta i}^* \right] \left[V_{\gamma i} V_{\gamma j}^* V_{\beta j} V_{\delta i}^* \right] = \alpha_i\square_{\delta j} \gamma_i\square_{\beta j},$$

(32)
and the analogous relation

\[ \alpha_i \beta_j \alpha_k \beta_l = \alpha_i \beta_l \alpha_k \beta_j. \] (33)

These relations hold for both degenerate boxes and nondegenerate boxes. Including disordered boxes adds no new information, so we will as usual consider only ordered boxes.

This index rearrangement of equations (32) and (33) may straightforwardly be extended to obtain higher-order relationships (trilinear, quadrilinear, et cetera) in the boxes. These higher-order relationships do not enter into oscillation probabilities, so we will not examine them here.

We may express \(|V_{\alpha i}| = (\alpha_i \square \alpha_i)^4\) in terms of three singly-degenerate boxes by setting \(\alpha = \beta\) in equation (30). Then, using equation (29) to substitute for the singly-degenerate boxes yields an expression for the doubly-degenerate box in terms of nine nondegenerate boxes:

\[ |V_{\alpha i}|^4 = \alpha_i \square \alpha_i = \frac{\alpha_i \square \alpha_x \alpha_i \square \alpha_y}{\alpha_x \square \alpha_y} = \frac{\alpha_x \square \tau \alpha_i \square \alpha_x \alpha_y \square \alpha_i \square \zeta \omega_x \square \mu_y}{\tau \square \omega_x \zeta \square \mu_y}, \]

\[ \begin{array}{l}
\tau \neq \sigma \neq \alpha \\
\zeta \neq \rho \neq \alpha \\
\mu \neq \omega \neq \alpha \\
x \neq y \neq i
\end{array}. \] (34)

As noted earlier, flavor-degenerate boxes give the probabilities for flavor-conserving oscillations measured in disappearance experiments, while nondegenerate boxes give the probabilities for flavor-changing oscillations measured in appearance experiments. Since the degenerate and nondegenerate boxes are related, individual \(|V_{\alpha i}|\) may be deduced from an appropriate set of measurements of either kind.

One can obtain a relationship similar to equation (34) by setting \(i = j\) in equation (29) and then using equation (30) to substitute for the singly-degenerate boxes:

\[ |V_{\alpha i}|^4 = \alpha_i \square \alpha_i = \frac{\alpha_i \square \lambda_i \alpha_i \square \eta_i}{\lambda_i \square \eta_i} = \frac{\alpha_i \square \lambda_{\alpha p} \alpha_i \square \eta_{\alpha q} \alpha_i \square \eta_{\alpha s} \alpha_i \square \lambda_{\alpha u}}{\lambda_{\alpha p} \square \eta_{\alpha q} \eta_{\alpha s} \lambda_{\alpha u} \beta_{\alpha u}}, \]

\[ \begin{array}{l}
n \neq p \neq i \\
r \neq s \neq i \\
t \neq u \neq i \\
\lambda \neq \eta \neq \alpha
\end{array}. \] (35)

In the three-generation case, equations (34) and (35) are uniquely specified by the index constraints in brackets, and they are equivalent to each other since all nine boxes are used in both equations. For example,

\[ |V_{11}|^4 = \frac{11 \square 11^* 11 \square 11^* 11 \square 11^* 11 \square 11^* 22 \square 33}{12 \square 12^* 12 \square 23^* 21 \square 33^* 21 \square 33^* 21 \square 33^*}. \] (36)
with $\alpha = i = 1$ holds with any number of generations, but it is the unique 5 on 4 box representation of $|V_{11}|^4$ in three generations. Another example in three generations is

$$|V_{21}|^4 = \frac{21 \Box_{32} 11 \Box_{22} 21 \Box_{33} 11 \Box_{23} 12 \Box_{33}}{11 \Box_{32} 11 \Box_{33} 22 \Box_{33} 12 \Box_{23}},$$

(37)

again true for all $n$ and unique in $n = 3$. The equalities in equations (36) and (37) are illustrated in Figure 4.

FIG. 4. The graphical representation for (a) $|V_{11}|^4$, and (b) $|V_{21}|^4$. Notice that each matrix point except the one being represented has an equal number of vertical arrows as horizontal arrows entering and leaving. The element being represented has two vertical arrows leaving (representing $V^2_{\alpha i}$) and two entering (representing $V^{*2}_{\alpha i}$) to produce $|V_{\alpha i}|^2$. Appendix A provides more details.

If $n > 3$, equations (34) and (35) still hold, but the indices of each are not specified uniquely. In higher generations the indices may take on values larger than 3; if $n = 6$, equation (34) may contain up to six different flavor indices but only three mass indices, while equation (35) may contain more mass indices and only three flavor indices.

Before continuing, we should note that all of the relationships in this section follow from the definitions of the boxes in equation (10) and so are valid for any matrix, unitary or otherwise. The constraints of unitarity, to which we will turn our attention after we count the number of independent boxes, will provide us with expressions for $|V_{\alpha i}|^4$ (found in equations (38) and (39)) which are easier to manage than the expressions (34) and (35) above.

**F. The Number of Independent Boxes**

An $n \times n$ arbitrary matrix has $n^2$ elements, which may be complex, leading to $2n^2$ parameters. The neutrino mixing matrix, however, is unitary, and has $n^2$ unitarity conditions as constraints. Of the $n^2$ remaining parameters, $\frac{1}{2}n(n - 1)$ (the number of parameters in a real unitary, or orthogonal, matrix) may be taken to be real rotation angles, and the remaining $\frac{1}{2}n(n + 1)$ are phases making the matrix complex. We may redefine the $2n$ neutrino and charged lepton
fields to remove $2n - 1$ relative phases from the mixing matrix, leaving $\frac{1}{2}n(n+1) - (2n - 1) = \frac{1}{2}(n-1)(n-2)$ phases measurable in oscillations. The number of independent real and imaginary box parameters should be the same as the number of real and imaginary CKM parameters $[18]$. As we will see, unitarity relates the imaginary and real box parameters, so one may choose more real boxes and fewer imaginary boxes as the independent basis of $(n-1)^2$ elements. In fact, one may choose a basis of only real boxes. For $n > 3$, not all of the boxes are generally accessible (and not all of the $(n-1)^2$ mixing matrix parameters are accessible), so the basis of boxes may be smaller. The parameter counts for the mixing matrix and the boxes are summarized in Table I.
TABLE I. Parameter counting for the mixing matrix and boxes. $N$ is shorthand for the frequently-appearing combination \(\binom{n}{2} \equiv \frac{1}{2}n(n-1)\). $n_f$ represents the number of active neutrino flavors, not to exceed 3, and $N_f \equiv \binom{n_f}{2}$.

|                       | $n$ | 2   | 3   | 4   | 6   |
|-----------------------|-----|-----|-----|-----|-----|
| **Number of generations** |     |     |     |     |     |
| Params. for arbitrary matrix | $2n^2$ | 8   | 18  | 32  | 72  |
| Unitarity constraints   | $n^2$ | 4   | 9   | 16  | 36  |
| Relative phases         | $2n-1$ | 3   | 5   | 7   | 11  |
| Real params for unitary V | $N$ | 1   | 3   | 6   | 15  |
| Remaining phases in V   | $\frac{1}{2}(n-1)(n-2)$ | 0   | 1   | 3   | 10  |
| Physical params in V    | $(n-1)^2$ | 1   | 4   | 9   | 25  |
| **Initial boxes**       | $n^4$ | 16  | 81  | 256 | 1296 |
| Doubly-degenerate boxes | $n^2$ | 4   | 9   | 16  | 36  |
| Ordered singly-degenerate boxes | $2nN$ | 4   | 18  | 48  | 180 |
| Ordered nondegenerate boxes | $N^2$ | 1   | 9   | 36  | 225 |
| Independent Re$(\alpha_i \beta_j)$'s | $N$ | 1   | 3   | 6   | 15  |
| Independent Im$(\alpha_i \beta_j)$'s | $\frac{1}{2}(n-1)(n-2)$ | 0   | 1   | 3   | 10  |
| Accessible boxes        | $n_f^2n^2$ | 16  | 81  | 144 | 324 |
| Accessible doubly-deg. boxes | $n_fn$ | 4   | 9   | 12  | 18  |
| Accessible ordered singly-deg. boxes | $n_FN + N_fn$ | 4   | 18  | 30  | 63  |
| Accessible ordered nondeg. boxes | $N_fn$ | 1   | 9   | 18  | 45  |
IV. UNITARITY RELATIONS AMONG THE BOXES

Unitarity requires that

\[ \sum_{\eta=1}^{n} V_{\eta i} V_{\eta j}^* = \delta_{ij}, \quad \text{and} \]

\[ \sum_{y=1}^{n} V_{\eta y} V_{\beta y}^* = \delta_{\alpha \beta}. \]  

These equations are not independent sets of constraints at this point: equation (38) states \( V V^\dagger = 1 \), and equation (39) states \( V^\dagger V = 1 \). We can, however, use these equivalent equations to obtain two separate sets of constraints on boxes by multiplying equation (38) by \( V_{\lambda i}^* \) and equation (39) by \( V_{\alpha x}^* \):

\[ \sum_{\eta=1}^{n} \sqrt{\lambda_i \lambda_i} \delta_{ij}, \quad \text{and} \]

\[ \sum_{y=1}^{n} \sqrt{\alpha_x \alpha_x} \delta_{\alpha \beta}. \]  

Isolating the manifestly degenerate boxes from the nondegenerate boxes, equation (40) becomes

\[ \sum_{\eta \neq \lambda} \sqrt{\lambda_i \lambda_i} \delta_{ij} = \sqrt{\lambda_i \lambda_i} \delta_{ij} - \sqrt{\lambda_i \lambda_j}, \]  

and equation (41) becomes

\[ \sum_{y \neq x} \sqrt{\alpha_x \alpha_x} \delta_{\alpha \beta} = \sqrt{\alpha_x \alpha_x} \delta_{\alpha \beta} - \sqrt{\alpha_x \beta_x}. \]  

One can show that equations (42) and (43), although distinct, are related by the symmetry that takes \( \alpha_i \beta_j \rightarrow i \alpha_i \beta_j \), which we identified earlier as the symmetry \( V \rightarrow V^\dagger \). We will find several pairs of equations related by this symmetry which differ only in whether the sum is over a flavor index or a mass index.

Summing equation (40) over \( \lambda \) in the \( i \neq j \) case, we find

\[ 0 = \sum_{\lambda=1}^{n} \sum_{\eta=1}^{n} \sqrt{\lambda_i \lambda_i} \delta_{ij} = \sum_{\lambda=1}^{n} \lambda_i \lambda_j + 2 \sum_{\lambda=1}^{n} \sum_{\eta<\lambda} \eta \lambda_j, \]  

Comparison of equation (44) with equations (24) and (29) reveals an interesting property of the matrix \( B \):
with constant $\alpha$ and $\beta$ specifying the row of $\mathbf{B}$ to be summed. Sums over rows are preferred to sums over columns phenomenologically when $n > 3$, because complete rows are accessible whereas complete columns are not.

For $n \geq 3$, some $R$s will be positive while others are negative. An upper bound on the values of the positive $R$s may be established from the Schwarz inequality for two-component complex “vectors” $\mathbf{V}_\alpha \equiv (V_{\alpha i}, V_{\alpha j})$ and $\mathbf{V}_\beta \equiv (V_{\beta i}, V_{\beta j})$:

$$|V_{\alpha i}V_{\beta j} + V_{\alpha j}V_{\beta i}^*|^2 \leq \left[|V_{\alpha i}|^2 + |V_{\alpha j}|^2\right] \left[|V_{\beta i}|^2 + |V_{\beta j}|^2\right].$$

A bit of algebra gives the desired result:

$$\Re_{ij} \leq \frac{1}{2} \left[|V_{\alpha i}|^2 |V_{\beta j}|^2 + |V_{\alpha j}|^2 |V_{\beta i}|^2\right].$$

Later we will actually derive an equality in three generations for $\Re_{ij}$ in terms of the four $|V|$s appearing in this equation (found in equation (47)).

An alternative way to obtain constraints from unitarity is to start with the definition of the boxes (10) and use the unitarity of the mixing matrix:

$$\alpha_i \Box_{\beta j} = (V_{\alpha i}V_{\alpha j}^*) (V_{\beta j}V_{\beta i}^*) = \left(\delta_{ij} - \sum_{\eta \neq \alpha} V_{\eta i}V_{\eta j}^*\right) \left(\delta_{ij} - \sum_{\lambda \neq \beta} V_{\lambda i}V_{\lambda j}^*\right).$$

After a bit of algebra, this becomes:

$$\sum_{\eta \neq \alpha} \sum_{\lambda \neq \beta} \eta_i \Box_{\lambda j} = \alpha_i \Box_{\beta j} - \delta_{ij} (-1 + |V_{\alpha i}|^2 + |V_{\beta i}|^2).$$

The number of unitarity constraints from equations (40) and (41), both real and imaginary, grows as $n^3$, while the number of ordered boxes grows as $n^4$, so additional relationships between ordered boxes must exist to identify the $(n - 1)^2$ independent $J$s and $R$s. These additional identities will come from the definitions of the boxes, and the resulting tautologies (29) and (30), each of which grows in number as $n^6$, implying a high degree of redundancy. Some of these relationships are explored below in equations (74) to (81).

1 This relation also follows from equation (46):

$$\sum_{\lambda \neq \beta} \sum_{\eta \neq \alpha} \eta_i \Box_{\lambda j} = \sum_{\lambda \neq \beta} \left(\sum_{\eta = 1}^n \eta_i \Box_{\lambda j} - \alpha_i \Box_{\lambda j}\right) = \sum_{\lambda \neq \beta} (|V_{\lambda i}|^2 \delta_{ij} - \alpha_i \Box_{\lambda j})$$

$$= \delta_{ij} \sum_{\lambda \neq \beta} |V_{\lambda i}|^2 - \left(\sum_{\lambda = 1}^n \alpha_i \Box_{\lambda j} - \alpha_i \Box_{\beta j}\right)$$

$$= \delta_{ij} \left(1 - |V_{\beta i}|^2\right) - |V_{\alpha i}|^2 \delta_{ij} + \alpha_i \Box_{\beta j}$$

\[50\]
The constraints (40) and (41) hold independently for the real and imaginary parts of each sum. We will first explore the implications of these constraints for the imaginary parts of boxes, before turning to the implications of these constraints for the real parts.

A. Unitarity Constraints on the Imaginary Parts of Boxes

Because the right-hand sides of the equations are manifestly real, as are terms on the left-hand side involving degenerate boxes, the sums of nondegenerate boxes in equations (40) and (41) must be real. This leads to imaginary constraints of the form

$$\sum_{\eta \neq \lambda} \eta \lambda J = 0, \text{ and}$$

$$\sum_{\eta \neq \lambda} \eta \lambda J = 0.$$  \hspace{1cm} (54)

$$\sum_{y \neq x} \alpha J_{\beta x} = 0.$$  \hspace{1cm} (55)

In three generations, each sum in equation (54) or (55) contains only two terms, leading to the equality (up to a sign) of two \( J \)s. For example, choosing \( \lambda = 1 \) in three generations yields

$$1_j J_{2i} = -1_j J_{3i}, \ n = 3,$$  \hspace{1cm} (56)

so

$$1_j J_{22} = -1_j J_{32}, \ n = 3.$$  \hspace{1cm} (57)

We may continue in a similar manner and show that every \( J \) is related to \( J \equiv 1_j J_{22} \) \[13\], giving

$$\text{Im}(\mathcal{B}) = \begin{pmatrix} J & J & -J \\ J & J & -J \\ -J & -J & J \end{pmatrix}, \ n = 3.$$  \hspace{1cm} (58)

One consequence of the equality up to a sign of all \( J \)s in three generations is that if any one \( V_{\alpha i} \) is zero, then all \( \alpha J_{\beta j} \) vanish and there can be no CP-violation.

\(^2\) The constraints (54) and (55) may be written exclusively in terms of ordered boxes as

$$\sum_{\eta < \lambda} \eta \lambda J - \sum_{\eta > \lambda} \eta \lambda J = 0, \text{ and}$$

$$\sum_{y < x} \alpha J_{\beta x} - \sum_{y > x} \alpha J_{\beta y} = 0.$$  \hspace{1cm} (52) \hspace{1cm} (53)

While expressing everything in terms of ordered boxes is our goal, we will find it more convenient to use the complete sums of equations (54) and (55) in mathematical manipulations and switch to ordered boxes at the end rather than to deal with the two separate terms in equations (52) and (53).
For $n > 3$ generations, the sums in equations (54) and (55) contain more than two terms, so individual $J$s become equal to sums of other $J$s rather than just equal to another individual $J$. In four generations, we know from counting phases in the mixing matrix, or more rigorously from the work in reference [18] that we should find three independent $J$s out of the 36 ordered $J$s. This reduction, unlike the reduction in three generations, is not achieved solely by equations of the forms (54) and (55). No more than 27 independent constraints may be derived from those equations, so at least 6 constraints on the imaginary parts of boxes in four generations will come from the expressions derived below which relate real and imaginary parameters.

### B. Unitarity Constraints on the Real Parts of Boxes

We now consider the real parts of the constraints (40) and (41), considering first the homogeneous constraints for which the Kronecker delta is zero. These relationships give the singly-degenerate boxes as sums of ordered boxes:

$$|V_{\lambda i}|^2 |V_{\lambda j}|^2 = \lambda_i \square \lambda_j = - \sum_{\eta \neq \lambda} \eta_i \square \lambda_j , \quad i \neq j, \quad \text{and} \tag{59}$$

$$|V_{\alpha x}|^2 |V_{\beta x}|^2 = \alpha_x \square \beta_x = - \sum_{y \neq x} \alpha_y \square \beta_x , \quad \alpha \neq \beta. \tag{60}$$

For three generations, each of the sums contains two terms, allowing us to express the singly-degenerate boxes in terms of two nondegenerate boxes measurable in neutrino appearance oscillation experiments.

For fixed $(i, j)$ in equation (59), $\lambda$ can take $n$ possible values, implying $n$ constraint equations. $N$ ordered non-degenerate boxes appear in these $n$ equations. Thus, for $N \leq n$, which is true for $n \leq 3$, the unitarity constraint (54) and (60) may be inverted to find a nondegenerate box in terms of singly-degenerate boxes. Adding the sums in equation (60) for $x = i$ and $x = j$, then subtracting the $x = k$ sum yields the desired expression:

$$R_{\beta j} = -\frac{1}{2} (|V_{\alpha i}|^2 |V_{\beta i}|^2 + |V_{\alpha j}|^2 |V_{\beta j}|^2 - |V_{\alpha k}|^2 |V_{\beta k}|^2) , \quad n = 3, \tag{61}$$

where we have used the property that $R$s are real and therefore unchanged under mass (or flavor) index interchanges. Many sources, such as reference [18], use these sums as the coefficients of the oscillatory terms in three-flavor oscillation probabilities to write

$$P_{\nu_{\alpha} \rightarrow \nu_{\beta}} (x) = 2 \left( \sin^2 \Phi_{12} + \sin^2 \Phi_{13} - \sin^2 \Phi_{23} \right) |V_{\alpha 1}|^2 |V_{\beta 1}|^2$$

$$+ 2 \left( \sin^2 \Phi_{12} - \sin^2 \Phi_{13} + \sin^2 \Phi_{23} \right) |V_{\alpha 2}|^2 |V_{\beta 2}|^2$$

$$+ 2 \left( - \sin^2 \Phi_{12} + \sin^2 \Phi_{13} + \sin^2 \Phi_{23} \right) |V_{\alpha 3}|^2 |V_{\beta 3}|^2 , \quad n = 3. \tag{62}$$
A simpler derivation of equations (61) and (63) is available. Three-generation unitarity gives

\[ V_{\alpha i} V_{\beta j}^* + V_{\alpha j} V_{\beta i}^* = -V_{\alpha k} V_{\beta k}^*. \]  

Squaring both sides of this equation then leads directly to equation (63). Generalizing this derivation to \( n > 3 \), we see why a single nondegenerate box cannot be expressed as a sum of degenerate boxes for \( n > 3 \), but we also obtain some new relations among boxes. For example, with \( n = 4 \), we may square both sides of

\[ V_{\alpha 1} V_{\beta 1}^* + V_{\alpha 2} V_{\beta 2}^* = -V_{\alpha 1} V_{\beta 1}^* - V_{\alpha 1} V_{\beta 1}^*. \]  

This has an interesting appearance, for it expresses the real part of the box \( \text{Re} \left[ V_{\alpha i} V_{\alpha j}^* V_{\beta j} V_{\beta i}^* \right] \) in terms of the magnitudes of the four complex \( V \)s which define the box.

The unitarity constraints (59) and (60) greatly simplify our expressions for a doubly-degenerate box \( \alpha \square \alpha_i = |V_{\alpha i}|^2 \):

\[
\begin{align*}
\alpha_i \square \alpha_i &= \frac{\alpha_i \square_{\alpha x} \alpha_i \square_{\alpha y}}{\alpha_x \square_{\alpha y}} = \frac{-\sum_{\eta \neq \alpha} \Re(\eta x) \left( -\sum_{\lambda \neq \alpha} \Re(\lambda y) \right)}{-\sum_{\eta \neq \alpha} \Re(\eta y)}, \quad x \neq y \neq i, \quad \text{and} \\
\alpha_i \square \alpha_i &= \frac{\alpha_i \square_{\lambda x} \alpha_i \square_{\eta i}}{\alpha_x \square_{\eta i}} = \frac{-\sum_{\sigma \neq i} \Re(\lambda x) \left( -\sum_{\eta \neq \alpha} \Re(\eta i) \right)}{-\sum_{\sigma \neq i} \Re(\eta i)}, \quad \lambda \neq \eta \neq \alpha,
\end{align*}
\]

where the first equalities are due to equations (29) and (30). Applying equation (68) to three generations, one finds that doubly-degenerate boxes are expressible in terms of the real parts of six ordered boxes, rather than the nine complex boxes used in equations (34) and (35). For example,

\[ |V_{11}|^4 = 11 \square_{11} = -\left( \frac{\Re_{22} + \Re_{32}}{\Re_{23} + \Re_{33}} \right), \quad n = 3. \]
When considering \( n > 3 \), each sum has more terms, but all terms in the numerators in equations (68) and (69) always contain \( R_s \) to the second order, while the denominator terms contain only the first order of \( R_s \). Thus these expressions will be much more manageable than equations (34) and (35) which exhibit the fifth order of boxes in the numerator and the fourth order in the denominator.

Summing equation (59) over \( j \neq i \) yields another expression for \( |V_{\lambda i}|^2 \) in terms of nondegenerate boxes:

\[
|V_{\lambda i}|^2 \sum_{j \neq i} |V_{\lambda j}|^2 = |V_{\lambda i}|^2 (1 - |V_{\lambda i}|^2) = - \sum_{j \neq i} \sum_{\eta \neq \lambda} \mathbb{H}_{\lambda j}.
\] (71)

The same result is obtained by summing equation (40) over all \( j \). The explicit solution or the above equation, valid for any number of generations, has a two-fold ambiguity:

\[
|V_{\lambda i}|^2 = \frac{1}{2} \left[ 1 \pm \sqrt{1 + 4 \sum_{j \neq i} \sum_{\eta \neq \lambda} \mathbb{H}_{\lambda j}} \right].
\] (72)

When the double sum is small, an approximation to the exact equation (72) is

\[
|V_{\lambda i}|^2 \approx \left( - \sum_{j \neq i} \sum_{\eta \neq \lambda} \mathbb{H}_{\lambda j}, 1 + \sum_{j \neq i} \sum_{\eta \neq \lambda} \mathbb{H}_{\lambda j} \right).
\] (73)

For three generations, this approximation yields \( |V_{\lambda i}|^2 \) as a linear equation of four nondegenerate boxes. For example,

\[
|V_{12}|^2 = \frac{1}{2} \left[ 1 \pm \sqrt{1 + 4 \left( \mathbb{H}_{22} + \mathbb{H}_{32} + \mathbb{H}_{23} + \mathbb{H}_{33} \right)} \right]
\approx \left( - (\mathbb{H}_{22} + \mathbb{H}_{32} + \mathbb{H}_{23} + \mathbb{H}_{33}), 1 + (\mathbb{H}_{22} + \mathbb{H}_{32} + \mathbb{H}_{23} + \mathbb{H}_{33}) \right), \quad n = 3.
\]

C. Unitarity Relationships Relating Real Parts of Boxes with Imaginary Parts of Boxes

We may use the homogeneous unitarity conditions (59) and (60), along with the tautologies (29) and (30) to obtain constraints between nondegenerate boxes, thereby reducing the number of real degrees of freedom. Recognizing that the tautology (29) gives

\[
\alpha_i \square_{\alpha j} = \text{Re} \left( \frac{\alpha_i \square_{\eta j} \alpha_i \square_{\lambda j}}{\eta \square_{\lambda j}} \right) = \frac{\mathbb{R}_{\eta j} \mathbb{R}_{\lambda j} \mathbb{R}_{\lambda j} + \mathbb{F}_{\eta j} \mathbb{F}_{\lambda j} \mathbb{R}_{\lambda j} - \mathbb{F}_{\eta j} \mathbb{F}_{\lambda j} \mathbb{R}_{\lambda j} + \mathbb{R}_{\eta j} \mathbb{F}_{\lambda j} \mathbb{F}_{\lambda j}}{(\mathbb{R}_{\lambda j})^2 + (\mathbb{F}_{\lambda j})^2},
\] (74)

the unitarity constraint (59) becomes

\[
\mathbb{R}_{\eta j} \mathbb{R}_{\lambda j} \mathbb{R}_{\lambda j} + \mathbb{F}_{\eta j} \mathbb{F}_{\lambda j} \mathbb{R}_{\lambda j} - \mathbb{F}_{\eta j} \mathbb{F}_{\lambda j} \mathbb{R}_{\lambda j} + \mathbb{R}_{\eta j} \mathbb{F}_{\lambda j} \mathbb{F}_{\lambda j} = - \left( (\mathbb{R}_{\lambda j})^2 + (\mathbb{F}_{\lambda j})^2 \right) \sum_{\tau \neq \alpha} \mathbb{R}_{\tau j},
\] (75)
with the usual inequalities $\eta \neq \lambda \neq \alpha, i \neq j$ satisfied. A similar treatment may be applied to equation (34) to arrive at the relation

$$
\mathcal{R}_{\beta x} \mathcal{R}_{\beta y} + \mathcal{F}_{\beta x} \mathcal{F}_{\beta y} \mathcal{R}_{\beta y} - \mathcal{F}_{\beta x} \mathcal{R}_{\beta y} \mathcal{F}_{\beta y} + \mathcal{R}_{\beta x} \mathcal{F}_{\beta y} \mathcal{F}_{\beta y} = - \left( (\mathcal{R}_{\beta y})^2 + (\mathcal{F}_{\beta y})^2 \right) \sum_{z \neq i} \mathcal{R}_{\beta z},
$$

(76)

with $\alpha \neq \beta$, and $x \neq y \neq i$. These constraints interrelate imaginary and real parts of boxes through unitarity for any number of generations.

We may infer constraints similar to equations (75) and (76) by setting the imaginary part of a singly-degenerate box to zero:

$$
\text{Im} \left( \alpha^i \Box_{\alpha j} \right) = \text{Im} \left( \frac{\alpha^i \Box_{\eta j}}{\eta^i \Box_{\lambda j}} \right) = \frac{\mathcal{R}_{\eta j} \mathcal{F}_{\lambda j} \mathcal{R}_{\lambda j} - \mathcal{F}_{\eta j} \mathcal{R}_{\lambda j} + \mathcal{R}_{\eta j} \mathcal{F}_{\lambda j} - \mathcal{F}_{\eta j} \mathcal{F}_{\lambda j}}{\mathcal{R}_{\lambda j} + \mathcal{F}_{\lambda j}} = 0,
$$

(77)

($\eta \neq \lambda \neq \alpha, i \neq j$), and

$$
\text{Im} \left( \alpha^i \Box_{\beta i} \right) = \text{Im} \left( \frac{\alpha^i \Box_{\beta j}}{\alpha^i \Box_{\beta y}} \right) = \frac{\mathcal{R}_{\beta x} \mathcal{F}_{\beta y} + \mathcal{R}_{\beta y} \mathcal{F}_{\beta y} - \mathcal{R}_{\beta x} \mathcal{F}_{\beta y} - \mathcal{F}_{\beta x} \mathcal{F}_{\beta y} \mathcal{F}_{\beta y}}{\mathcal{F}_{\beta y} + \mathcal{F}_{\beta y}} = 0,
$$

(78)

($\alpha \neq \beta$, and $x \neq y \neq i$), so

$$
\mathcal{R}_{\eta j} \mathcal{F}_{\lambda j} \mathcal{R}_{\lambda j} - \mathcal{F}_{\eta j} \mathcal{R}_{\lambda j} + \mathcal{R}_{\eta j} \mathcal{F}_{\lambda j} - \mathcal{F}_{\eta j} \mathcal{F}_{\lambda j} = 0,
$$

(79)

($\eta \neq \lambda \neq \alpha, i \neq j$), and

$$
\mathcal{R}_{\beta x} \mathcal{F}_{\beta y} + \mathcal{R}_{\beta y} \mathcal{F}_{\beta y} - \mathcal{R}_{\beta x} \mathcal{F}_{\beta y} - \mathcal{F}_{\beta x} \mathcal{F}_{\beta y} \mathcal{F}_{\beta y} = 0.
$$

(80)

($\alpha \neq \beta$, and $x \neq y \neq i$). With a bit of index switching, these constraints lead to the isolation of a single $R$:

$$
\mathcal{R}_{\beta j} = \frac{\mathcal{F}_{\beta j} \mathcal{R}_{\lambda j} + \mathcal{F}_{\beta j} \mathcal{F}_{\lambda j} \mathcal{F}_{\lambda j}}{\mathcal{R}_{\lambda j} - \mathcal{F}_{\lambda j} \mathcal{F}_{\lambda j}} = \frac{\mathcal{F}_{\beta j} \mathcal{R}_{\lambda j} + \mathcal{F}_{\beta j} \mathcal{F}_{\lambda j} \mathcal{F}_{\lambda j}}{\mathcal{F}_{\lambda j} \mathcal{R}_{\lambda j} - \mathcal{F}_{\beta j} \mathcal{F}_{\lambda j} \mathcal{F}_{\lambda j}},
$$

(81)

with $\eta \neq \lambda \neq \alpha, i \neq j$ in the first expression, and $\alpha \neq \beta$, and $x \neq y \neq i$ in the second.

### D. Indirect Measurement of CP-Violation

Suppose CP is conserved. Then $\mathcal{F}_{\beta j} = 0$ for all index choices. The implication for equations (75) and (76) are

$$
\mathcal{R}_{\eta j} \mathcal{R}_{\lambda j} + \mathcal{R}_{\lambda j} \sum_{i \neq \alpha} \mathcal{R}_{\tau j} = 0, \quad (\eta \neq \lambda \neq \alpha, \text{ and } i \neq j),
$$

(82)
and
\[ R_{\beta x} R_{\beta y} + R_{\beta y} \sum_{z \neq 1} R_{\beta z} = 0, \quad (x \neq y \neq z, \text{ and } \alpha \neq \beta). \] (83)

If either of these relations is violated, then so is CP. An attempt to determine the validity of these relations may be facilitated by substituting
\[ -\frac{1}{2} \sum_{\tau=1}^{n} |V_{\alpha \tau}|^2 |V_{\alpha \tau}|^2 \]
for the sum over \( \tau \) and
\[ -\frac{1}{2} \sum_{i=1}^{n} |V_{\alpha i}|^2 |V_{\beta i}|^2 \]
for the sum over \( \tau \), as given in equations (53) and (54).

For three generations, \( \tau \) must equal \( \lambda \) in equation (73), \( \mathcal{Y}_{\beta j} = -\mathcal{Y}_{\lambda j} = \mathcal{J} \) by equation (54), and equation (73) may be solved for \( \mathcal{J}^2 \) directly:
\[ \mathcal{J}^2 = R_{\beta j} R_{\lambda j} + R_{\beta j} R_{\lambda j} + R_{\lambda j} R_{\lambda j}, \quad n = 3, \] (84)
which expresses \( \mathcal{J} \) in terms of 3 \( R \)s summed in pairs. Similarly, in equation (76), \( z = y \), and \( \mathcal{Y}_{\beta j} = -\mathcal{Y}_{\beta y} = \mathcal{Y}_{\beta y} \) by equation (55), leading to another expression for \( \mathcal{J}^2 \):
\[ \mathcal{J}^2 = R_{\beta j} R_{\beta y} + R_{\beta j} R_{\beta y} + R_{\beta y} R_{\beta y}, \quad n = 3. \] (85)

These relationships between the \( R \)s and \( J \)s in three generations exhibit a simple parameter symmetry: \( \mathcal{J}^2 \) in equation (84) equals \( R_{\beta j} R_{\beta y} + \text{terms cyclic in } (\alpha, \beta, \lambda) \); in equation (85), it equals \( R_{\beta j} R_{\beta y} + \text{terms cyclic in } (i, j, y) \). Equations (84) and (85) say that the three real elements in any row or column of the matrix \( B \) may be summed in their three pairwise products to yield the CP-violating invariant \( \mathcal{J}^2 \). Note that if CP is conserved and \( \mathcal{J} \) is zero, then equations (84) and (85) also tell us that all three \( R \)s in any column or row cannot have the same sign.

If CP is violated, \( \mathcal{J} \neq 0 \), so the combination of real parts on the right-hand sides of equations (84) and (85), measurable with CP-conserving averaged neutrino oscillations, cannot be zero. Thus, even if CP violating asymmetries are not directly observable in an experiment, the effects of CP violation may be seen through the relationships among the real parts of different boxes! A related observation has been made for the quark sector by Hamzaoui in reference [19].

For three generations, unitarity enforces the simple relations (54) and (57) among the \( J \)s in equation (81), which then reproduces the expressions (84) and (85). For \( n > 3 \), however, equation (81) is different from the latter equations, since the sums \( \sum_{\tau \neq \alpha, i} \) and \( \sum_{z \neq i, j} \) in equations (73) and (74) ensure that boxes with all \( n \) flavor or mass indices enter into the equations replacing (84) or (85), respectively. Equation (81) involves only three such indices for any \( n \), so it will be different from the unitarity constraints in higher generations, providing additional identities among the real parts of boxes.
E. Inhomogeneous Unitarity Constraints

The homogeneous unitarity constraints cannot provide the desired normalization of the \( V_{\alpha i} \) or the boxes; we need the inhomogeneous unitarity constraints arising when the Kronecker delta in equations (40) and (41) is not zero. We now develop these inhomogeneous constraints\(^3\) which are functions of degenerate boxes and therefore purely real:

\[
\alpha_i \square_{\alpha i} + \sum_{\eta \neq \alpha} \alpha_i \square_{\eta i} = \sqrt{\alpha_i \square_{\alpha i}}, \quad \text{and} \quad (87)
\]

\[
\alpha_i \square_{\alpha i} + \sum_{z \neq i} \alpha_i \square_{\alpha z} = \sqrt{\alpha_i \square_{\alpha i}}. \quad (88)
\]

Parenthetically, we note by comparing equations (87) and (88) that a sum over mass-degenerate boxes equals a sum over flavor-degenerate boxes:

\[
\sum_{\eta \neq \alpha} \alpha_i \square_{\eta i} = \sum_{z \neq i} \alpha_i \square_{\alpha z}. \quad (89)
\]

Equations (87) and (88) can be rewritten strictly in terms of nondegenerate boxes by using the homogeneous unitarity constraints (59) and (60). We find

\[
\left( -\sum_{\lambda \neq \alpha} H_{\lambda x} \right) \left( -\sum_{\sigma \neq \alpha} H_{\sigma y} \right) \left( -\sum_{\tau \neq \alpha} R_{\tau y} \right) - \sqrt{\left( -\sum_{\lambda \neq \alpha} H_{\lambda x} \right) \left( -\sum_{\sigma \neq \alpha} H_{\sigma y} \right) \left( -\sum_{\tau \neq \alpha} R_{\tau y} \right) + \sum_{\eta \neq \alpha} \left( -\sum_{z \neq i} \check{R}_{\eta i} \right) } = 0, \quad (90)
\]

with \( x \neq y \neq i \), and

\[
\left( -\sum_{\sigma \neq i} R_{\sigma i} \right) \left( -\sum_{y \neq i} \check{R}_{\eta i} \right) \left( -\sum_{t \neq i} \check{H}_{\eta i} \right) - \sqrt{\left( -\sum_{\sigma \neq i} R_{\sigma i} \right) \left( -\sum_{y \neq i} \check{R}_{\eta i} \right) \left( -\sum_{t \neq i} \check{H}_{\eta i} \right) + \sum_{z \neq i} \left( -\sum_{\eta \neq \alpha} \check{R}_{\eta i} \right) } = 0, \quad (91)
\]

with \( \lambda \neq \eta \neq \alpha \). Note that these inhomogeneous unitarity constraints do not involve the \( J \)s.

Isolating the square root, squaring the equation, and multiplying through by the resulting denominator, we get quartic equations, each relating \( n(n-1) \) \( R \)s:

\[
|V_{\alpha i}|^4 + \sum_{\eta \neq \alpha} |V_{\alpha i}|^2 |V_{\eta i}|^2 = |V_{\alpha i}|^2, \quad \text{or} \quad \sum_{\eta=1}^{n} |V_{\eta i}|^2 = 1. \quad (86)
\]

\(^3\) In terms of mixing-matrix elements, equations (87) and (88) are trivial. For example, equation (87) is

\[
|V_{\alpha i}|^4 + \sum_{\eta \neq \alpha} |V_{\alpha i}|^2 |V_{\alpha i}|^2 = |V_{\alpha i}|^2, \quad \text{or} \quad \sum_{\eta=1}^{n} |V_{\eta i}|^2 = 1. \quad (86)
\]
\[
\left( \sum_{\lambda \neq \alpha} \mathcal{H}_{\lambda \alpha} \right) \left( \sum_{\sigma \neq \alpha} \mathcal{H}_{\sigma \alpha} \right) \left( \sum_{\tau \neq \alpha} \mathcal{H}_{\tau \alpha} \right) \left[ 1 + 2 \left( \sum_{\eta \neq \alpha} \sum_{z \neq i} \mathcal{H}_{\eta z} \right) \right] + \\
\left( \sum_{\lambda \neq \alpha} \mathcal{H}_{\lambda \alpha} \right)^2 \left( \sum_{\sigma \neq \alpha} \mathcal{H}_{\sigma \alpha} \right)^2 + \left( \sum_{\tau \neq \alpha} \mathcal{H}_{\tau \alpha} \right)^2 \left( \sum_{\eta \neq \alpha} \sum_{z \neq i} \mathcal{H}_{\eta z} \right)^2 = 0,
\]

(92)

\((x \neq y \neq i)\), and

\[
\left( \sum_{x \neq i} \mathcal{H}_{\lambda x} \right) \left( \sum_{y \neq i} \mathcal{H}_{\eta y} \right) \left( \sum_{t \neq i} \mathcal{H}_{\eta t} \right) \left[ 1 + 2 \left( \sum_{\eta \neq \alpha} \sum_{z \neq i} \mathcal{H}_{\eta z} \right) \right] + \\
\left( \sum_{x \neq i} \mathcal{H}_{\lambda x} \right)^2 \left( \sum_{y \neq i} \mathcal{H}_{\eta y} \right)^2 + \left( \sum_{t \neq i} \mathcal{H}_{\eta t} \right)^2 \left( \sum_{\eta \neq \alpha} \sum_{z \neq i} \mathcal{H}_{\eta z} \right)^2 = 0
\]

(93)

for \(\lambda \neq \eta \neq \alpha\). In three generations, each of the sums in equations (92) and (93) has only two terms, which is not so formidable. For example, for \(\alpha = 2\) and \(i = 1\), we have

\[
\left( \mathcal{H}_{22} + \mathcal{H}_{32} \right) \left( \mathcal{H}_{23} + \mathcal{H}_{33} \right) \left( \mathcal{H}_{23} + \mathcal{H}_{33} \right) \left[ 1 + 2 \left( \mathcal{H}_{22} + \mathcal{H}_{32} + \mathcal{H}_{23} + \mathcal{H}_{33} \right) \right] + \\
\left( \mathcal{H}_{22} + \mathcal{H}_{32} \right)^2 \left( \mathcal{H}_{23} + \mathcal{H}_{33} \right)^2 + \left( \mathcal{H}_{22} + \mathcal{H}_{32} + \mathcal{H}_{23} + \mathcal{H}_{33} \right)^2 \left( \mathcal{H}_{23} + \mathcal{H}_{33} \right)^2 = 0, \quad n = 3,
\]

(94)

and

\[
\left( \mathcal{H}_{22} + \mathcal{H}_{23} \right) \left( \mathcal{H}_{32} + \mathcal{H}_{33} \right) \left( \mathcal{H}_{32} + \mathcal{H}_{33} \right) \left[ 1 + 2 \left( \mathcal{H}_{22} + \mathcal{H}_{23} + \mathcal{H}_{32} + \mathcal{H}_{33} \right) \right] + \\
\left( \mathcal{H}_{22} + \mathcal{H}_{23} \right)^2 \left( \mathcal{H}_{32} + \mathcal{H}_{33} \right)^2 + \left( \mathcal{H}_{22} + \mathcal{H}_{23} + \mathcal{H}_{32} + \mathcal{H}_{33} \right)^2 \left( \mathcal{H}_{32} + \mathcal{H}_{33} \right)^2 = 0, \quad n = 3.
\]

(95)

V. REDUCTION TO A BASIS

A. General Algorithm

As with the constraints on the imaginary parts of boxes, many of the constraints (74) to (93) are redundant, but an independent set may be used to reduce the number of box parameters to a basis. The homogeneous constraints (73), (74), (81) are much simpler than the inhomogeneous constraints (72) and (73), so it is advantageous to use as many homogeneous constraints as possible to construct the basis. Still, some inhomogeneous constraints must be invoked if the boxes and matrix elements of \(V\) are to be normalized.

Notice that with only fixed \(i\) and \(j\) mass indices and a flavor sum over the entire \((i, j)\) column of \(B\), in equation (73), and only fixed \(\alpha\) and \(\beta\) flavor indices and a mass sum over the entire \((\alpha, \beta)\) row of \(B\) in equation (74), these equations express relationships among all the boxes within single columns and rows, respectively, of the matrix \(B\) defined in equation (24). For each choice for the set of indices, equations (73) and (74) may be used to express one \(R\) or \(J\) in terms of the others. This suggests a general algorithm for constructing a basis set of boxes. One \(R\) from each row and column may be taken as dependent using the homogeneous unitarity constraints (75) and (76), or (81). This leaves \((n - 1)^2\) \(Rs\) and \(q\) \(Js\) in the independent set, where \(q\) is the number of \(Js\) which are still independent after
the application of the homogeneous constraints equations (54) and (55). The $n \times n$ mixing matrix is parameterized by $(n - 1)^2$ parameters, so we seek a basis set of $(n - 1)^2$ box parameters. The inhomogeneous unitarity equations, which we will address shortly, must supply the remaining constraints. The number of inhomogeneous constraints to be used is apparently equal to $q$. We have seen that with three flavors this is just one. The end result is a basis of $(n - 1)^2 R$s and $J$s with no more than $n - 1$ $R$s coming from any row or any column. If one restricts the bases and counting to accessible boxes, the algorithm may differ for $n > 3$. We do not pursue this here.

B. Three Generations

We provide here an example of a basis construction for three generations obtained by substituting in the unitarity equations derived above. Rearranging the three-generation equations (84) and (85) yields expressions for one $R$ in terms of two other $R$s and $J$:

$$\mathcal{H}_{\beta j} = \frac{J^2 - \mathcal{H}_{\lambda j} \mathcal{H}_{\lambda j}}{\mathcal{H}_{\lambda j} + \mathcal{H}_{\beta j}}$$  
$$= \frac{J^2 - \mathcal{H}_{\beta y} \mathcal{H}_{\beta y}}{\mathcal{H}_{\beta y} + \mathcal{H}_{\beta y}}$$  

(96)

(97)

We may then eliminate $\mathcal{H}_{23}$ by either equation (96) or equation (97):

$$\mathcal{H}_{23} = \frac{\mathcal{H}_{33} \mathcal{H}_{33} - J^2}{-\mathcal{H}_{33} - \mathcal{H}_{33}} = \frac{\mathcal{H}_{23} \mathcal{H}_{23} - J^2}{-\mathcal{H}_{23} - \mathcal{H}_{23}}.$$  

(98)

We may similarly eliminate $\mathcal{H}_{32}$ and $\mathcal{H}_{33}$

$$\mathcal{H}_{32} = \frac{\mathcal{H}_{32} \mathcal{H}_{32} - J^2}{-\mathcal{H}_{32} - \mathcal{H}_{32}} = \frac{\mathcal{H}_{33} \mathcal{H}_{33} - J^2}{-\mathcal{H}_{33} - \mathcal{H}_{33}},$$  

(99)

and

$$\mathcal{H}_{33} = \frac{\mathcal{H}_{23} \mathcal{H}_{33} - J^2}{-\mathcal{H}_{23} - \mathcal{H}_{33}} = \frac{\mathcal{H}_{32} \mathcal{H}_{33} - J^2}{-\mathcal{H}_{32} - \mathcal{H}_{33}},$$  

(100)

$\mathcal{H}_{33}$ may be eliminated from equation (98), and $\mathcal{H}_{33}$ may be eliminated from equation (100):

$$\mathcal{H}_{33} = \frac{\mathcal{H}_{33} \mathcal{H}_{22} \mathcal{H}_{23} + J^2 (\mathcal{H}_{23} + \mathcal{H}_{22} - \mathcal{H}_{33})}{\mathcal{H}_{33} \mathcal{H}_{23} + \mathcal{H}_{33} \mathcal{H}_{22} - \mathcal{H}_{23} \mathcal{H}_{22} + J^2},$$  

and

$$\mathcal{H}_{33} = \frac{\mathcal{H}_{33} \mathcal{H}_{23} \mathcal{H}_{32} + J^2 (\mathcal{H}_{32} + \mathcal{H}_{33} - \mathcal{H}_{23})}{\mathcal{H}_{23} \mathcal{H}_{32} + \mathcal{H}_{33} \mathcal{H}_{23} - \mathcal{H}_{32} \mathcal{H}_{32} + J^2}.$$  

(101)

(102)

Equation (99) will not provide an additional constraint; it is redundant to the other two. As expected, we must turn to the inhomogeneous constraints to eliminate the last degree of freedom. We will here choose the constraint (94),
since its expression in terms of our four remaining boxes is the least complicated. We may substitute the second
equality from equation (100) for $R_{33}$ and the second equality of equation (98) for $R_{23}$ into equation (94).

Multiplying through to place all of the terms in the numerator, we are left with a constraint which contains the
five parameters $R_{22}, R_{23}, R_{32}, R_{33},$ and $J^2$:

$$0 = (R_{22} + R_{23})^2 (R_{22} + R_{32})^2 \left[-R_{32} R_{33} + R_{23} (R_{32} + R_{33}) + J^2\right]^2$$

$$+ \left[(R_{32})^2 + (R_{21} + R_{22}) (R_{32} + R_{33}) + J^2\right]^2$$

$$\times \left[R_{23} R_{33} + R_{22} (-R_{23} + R_{33}) J^2\right]^2$$

$$- (R_{22} + R_{32}) (R_{22} + R_{23}) \left[R_{22} (R_{23} - R_{33}) - R_{23} R_{33} - J^2\right]$$

$$\times \left[R_{32} + 2 R_{22} R_{32} + 2 (R_{32})^2 + 2 R_{32} R_{23}
+ R_{33} + 2 R_{22} R_{33} + 2 R_{23} R_{33} + 2 J^2\right] \left[R_{23} (R_{32} + R_{33}) - R_{32} R_{33} + J^2\right].$$

This equation is quartic in all five parameters. We may eliminate any one by either algebraic or numeric means,
leaving us with the desired four parameters as the basis.

VI. SUMMARY

Neutrino physics has entered a golden age of research. New experiments all over the globe promise an unequaled
amount of data from the sun, the atmosphere, accelerators, supernovae, and other cosmic sources. The latest data
suggests that more than three neutrino flavors may participate in neutrino oscillations [1]. Analyzing such refined
data requires a consistent, model-independent approach which may be easily applied, and easily extended to higher
generations. Our work presented here offers such an approach. One result which we view as particularly noteworthy
is that high-statistics data on averaged oscillations is sufficient to determine the conservation or non-conservation of
CP in the lepton mixing matrix; this indirect test of CP is a consequence of the unitarity of the mixing matrix.

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APPENDIX A: GRAPHICAL REPRESENTATION OF BOX RELATIONS

Many of the relationships between boxes developed in Section III E can be derived using a graphical representation. In this method, boxes in the numerator of a product are represented by two vertical lines; boxes in the denominator are represented by two horizontal lines. Lines exit the locations of the matrix elements which are not complex-conjugated and enter the locations of the complex-conjugated matrix elements. For example, the box \( 11 \boxtimes 22 = V_{11} V_{12}^* V_{22} V_{21}^* \) is represented by a vertical line pointing from \( V_{11} \) to \( V_{21} \) and a vertical line pointing from \( V_{22} \) to \( V_{12} \), as shown in Figure 5a. The inverse box \( \frac{1}{11 \boxtimes 22} = \left( 11 \boxtimes 22 \right)^{-1} \) is represented by a horizontal line pointing from \( V_{11} \) to \( V_{12} \) and one pointing from \( V_{22} \) to \( V_{21} \), as shown in Figure 5b. The complex-conjugated box \( \overline{11 \boxtimes 22} \) is equal to \( 12 \boxtimes 21 \), so one just reverses the arrows to complex conjugate a box, as shown in Figure 5c.

![Figure 5](image)

To multiply boxes together graphically, one merely draws the lines corresponding to each factor on the same grid. Horizontal arrows entering or leaving a point cancel out vertical arrows entering or leaving, respectively, that point. Uncanceled arrows entering a point signify the survival of the complex-conjugated matrix element associated with that point. Those leaving a point signify the survival of the ordinary matrix element of that point. Figure 5a represents the product \( 11 \boxtimes 22 \times 12 \boxtimes 33 \times \left( 23 \boxtimes 32 \right)^{-1} \). The horizontal arrows do not cancel vertical arrows at any point, so no simplification may occur. The upward arrow at \( V_{22} \) represents that element in the numerator. The horizontal arrow represents the element \( V_{22}^* \) in the denominator, which does not cancel. Counting off the arrows at each vertex, we find the expression

\[
11 \boxtimes 22 \times 12 \boxtimes 33 \times \left( 23 \boxtimes 32 \right)^{-1} = \frac{V_{11} V_{12} V_{13}^* V_{12}^* V_{22} V_{23} V_{33}^* V_{32}^* V_{33}^*}{V_{22} V_{23} V_{32} V_{33}^*}, \tag{A1}
\]

which agrees with the definitions of boxes.
FIG. 6. The graphical representation for the products a) $^{11} \Box^{22} \ 12 \Box^{33} \ (^{23} \Box^{32})^{-1}$, and b) $^{11} \Box^{22} \ 12 \Box^{33} \ (^{22} \Box^{33})^{-1}$.

Figure (b) represents $^{11} \Box^{22} \ 12 \Box^{33} \ (^{22} \Box^{33})^{-1}$, a product in which some canceling does occur. Picking out the uncanceled arrows at each vertex, we are left with

$$^{11} \Box^{22} \ 12 \Box^{33} \ (^{22} \Box^{33})^{-1} = \frac{V_{11} V_{12} V_{13} V_{21}}{V_{23}}. \quad (A2)$$

The graphical method is a powerful tool for finding relationships among boxes. For example, trying to obtain the equations (34) and (35) for $\alpha_i \Box \alpha_i = |V_{\alpha i}|^4$ of Section III E without graphs involved quite a few false starts. Using the graphical method, we need only find a series of arrows which cancel for every point except $(\alpha, i)$ and leave two incoming and two outgoing vertical arrows at that point. Consider $|V_{21}|^4$ as an example. We choose to draw all of the arrows involving $V_{21}$ pointing downward, as shown in Figure (a). These arrows must be part of boxes, so in Figure (b) we add the arrows to finish those boxes. Next we draw two horizontal boxes in Figure (c) to cancel the extra arrows in the first column of the matrix. This still leaves $V_{22}$ and $V_{23}$ with two sets of uncanceled arrows apiece. In Figure (d) we draw two more horizontal boxes to compensate. This adds arrows to our previously clean $V_{12}$, $V_{13}$, $V_{32}$, and $V_{33}$. Drawing the final vertical box in Figure (e) cancels those. Recapping what we have done, we see that step (b) completes $^{11} \Box^{22}$, $^{11} \Box^{23}$, $^{21} \Box^{32}$, and $^{21} \Box^{33}$ in the numerator. Step (c) divides by $^{11} \Box^{32}$ and $^{11} \Box^{33}$, and step (d) divides by $^{12} \Box^{23}$ and $^{22} \Box^{33}$. Step (e) multiplies by $^{12} \Box^{33}$, leaving only the point $V_{21}$ with uncanceled arrows. It has two vertical arrows coming in and two leaving, so our graph represents the equation

$$|V_{21}|^4 = \frac{^{21} \Box^{32} \ ^{11} \Box^{22} \ ^{21} \Box^{33} \ ^{11} \Box^{23} \ ^{12} \Box^{33}}{^{11} \Box^{32} \ ^{11} \Box^{33} \ ^{22} \Box^{33} \ ^{12} \Box^{23}}. \quad (37)$$

of Section III E. Other examples of this representation are included in that section.
FIG. 7. The steps to obtaining $|V_{21}|^4$ as a function of ordered, non-degenerate boxes. The additions in each step are designated by the thicker arrows.

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