Expansion of eigenvalues of the perturbed discrete bilaplacian

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Abstract
We consider the family
\[
\hat{H}_\mu := \hat{\Delta} \hat{\Delta} - \mu \hat{V}, \quad \mu \in \mathbb{R},
\]
of discrete Schrödinger-type operators in \(d\)-dimensional lattice \(\mathbb{Z}^d\), where \(\hat{\Delta}\) is the discrete Laplacian and \(\hat{V}\) is of rank-one. We prove that there exist coupling constant thresholds \(\mu_o, \mu^o \geq 0\) such that for any \(\mu \in [-\mu^o, \mu_o]\) the discrete spectrum of \(\hat{H}_\mu\) is empty and for any \(\mu \in \mathbb{R} \setminus [-\mu^o, \mu_o]\) the discrete spectrum of \(\hat{H}_\mu\) is a singleton \(\{e(\mu)\}\), and \(e(\mu) < 0\) for \(\mu > \mu_o\) and \(e(\mu) > 4d^2\) for \(\mu < -\mu^o\). Moreover, we study the asymptotics of \(e(\mu)\) as \(\mu \searrow \mu_o\) and \(\mu \nearrow -\mu^o\) as well as \(\mu \to \pm \infty\). The asymptotics highly depends on \(d\) and \(\hat{V}\).

Keywords Discrete bilaplacian · Essential spectrum · Discrete spectrum · Eigenvalues · Asymptotics · Expansion

Mathematics Subject Classification 47A10 · 47A55 · 47A75 · 41A60
1 Introduction

In this paper we investigate the spectral properties of the perturbed discrete biharmonic operator

$$\hat{H}_\mu := \hat{\Delta} \hat{\Delta} - \mu \hat{V}, \quad \mu \in \mathbb{R},$$

in the $d$-dimensional cubical lattice $\mathbb{Z}^d$, where $\hat{\Delta}$ is the discrete Laplacian and $\hat{V}$ is a is rank-one potential with a generating potential $\hat{v}$. This model is associated to a one-particle system in $\mathbb{Z}^d$ with a potential field $\hat{v}$, in which the particle freely “jumps” from a node $X$ of the lattice not only to one of its nearest neighbors $Y$ (similar to the discrete Laplacian case), but also to the nearest neighbors of the node $Y$. From the mathematical point of view, the discrete bilaplacian represents a discrete Schrödinger operator with a degenerate bottom, i.e., $\hat{\Delta} \hat{\Delta}$ is unitarily equivalent to a multiplication operator by a function $e$ which behaves as $o(|p - p_0|^2)$ close to its minimum point $p_0$.

The spectral properties of discrete Schrödinger operators with non-degenerate bottom (i.e., $e$ behaves as $O(|p - p_0|^2)$ close to its minimum point $p_0$), in particular with discrete Laplacian, have been extensively studied in recent years (see e.g. [1, 2, 7, 8, 10, 11, 20, 21, 23, 26, 28] and references therein) because of their applications in the theory of ultracold atoms in optical lattices [16, 24, 35, 36]. In particular, it is well-known that the existence of the discrete spectrum is strongly connected to the threshold phenomenon [18, 20–22], which plays an role in the existence the Efimov effect in three-body systems [31, 32, 34]: if any two-body subsystem in a three-body system has no bound state below its essential spectrum and at least two two-body subsystem has a zero-energy resonance, then the corresponding three-body system has infinitely many bound states whose energies accumulate at the lower edge of the three-body essential spectrum.

Recall that the Efimov effect may appear only for certain attractive systems of particles [29]. However, recent experimental results in the theory of ultracold atoms in an optical lattice have shown that two-particle systems can have repulsive bound states and resonances (see e.g. [36]), thus, one expects the Efimov effect to hold also for some repulsive three-particle systems in $\mathbb{Z}^3$.

The strict mathematical justification of the Effect effect including the asymptotics for the number of negative eigenvalues of the three-body Hamiltonian has been successfully established in 3-space dimensions (for both $\mathbb{R}^3$ and $\mathbb{Z}^3$) see e.g., [1, 4, 13, 19, 29, 31, 32, 34] and the references therein. In particular, the non-degeneracy of the bottom of the (reduced) one-particle Schrödinger operator played an important role in the study of resonance states of the associated two-body system [1, 31]. Another keypoint in the proof of the Efimov effect in $\mathbb{Z}^3$ was the asymptotics of the (unique) smallest eigenvalue of the (reduced) one-particle discrete Schrödinger operator which creates a singularity in the kernel of a Birman-Schwinger-type operator which used to obtain an asymptotics to the number of three-body bound states.

To the best of our knowledge, there are no published results related to the Efimov effect in lattice three-body systems in which associated (reduced) one-body Schrödinger operator has degenerate bottom.
We also recall that fourth order elliptic operators in $\mathbb{R}^d$ in particular, the biharmonic operator, play also a central role in a wide class of physical models such as linear elasticity theory, rigidity problems (for instance, construction of suspension bridges) and in streamfunction formulation of Stokes flows (see e.g. [9, 25, 27] and references therein). Moreover, recent investigations have shown that the Laplace and biharmonic operators have high potential in image compression with the optimized and sufficiently sparse stored data [15]. The need for corresponding numerical simulations has led to a vast literature devoted to a variety of discrete approximations to the solutions of fourth order equations [5, 12, 33]. The question of stability of such models is basically related to their spectral properties and therefore, numerous studies have been dedicated to the numerical evaluation of the eigenvalues [3, 6, 30].

The aim of the present paper is the study of the existence and asymptotics of eigenvalues as well as threshold resonance and bound states of $\hat{H}_\mu$ defined in (1.1), which corresponds to the one-body Schrödinger operator with degenerate bottom. Namely, we study the discrete spectrum of $\hat{H}_\mu$ depending on $\mu$ and on $\vec{v}$. For simplicity we assume the generator $\vec{v}$ of $\hat{V}$ to decay exponentially at infinity, however, we urge that our methods can also be adjusted to less regular cases (see Remark 2.6). Since the generator $\vec{v}$ and $\hat{V}$, the associated asymptotics are highly dependent not only on the dimension $d$ of the lattice (as in the discrete Laplacian case [20, 21]), but also values on the multiplicity $2n_o$ and $2n^o$ of $0$ and $\pi$ (if $\nu(0) = 0$ and $\pi \in \{v = 0\}$ (if $\nu(\pi) = 0$), respectively. More precisely, depending on $d$ and $n_o$, $e(\mu)$ has a convergent expansion

- in $(\mu - \mu_o)^{1/3}$ for $2n_o + d = 1, 7$;
- in $\mu - \mu_o$ for $2n_o + d = 3, 5$;
- in $(\mu - \mu_o)^{1/4}$ for $2n_o + d \geq 9$ with $d$ odd;
- in $\mu - \mu_o$ and $-(\mu - \mu_o) \ln(\mu - \mu_o)$ for $2n_o + d = 2, 6$;
- in $\mu - \mu_o$ and $e^{-1/(\mu - \mu_o)}$ for $2n_o + d = 4$;
- in $(\mu - \mu_o)^{1/2}$, $-(\mu - \mu_o) \ln(\mu - \mu_o)$, $(-1/\ln(\mu - \mu_o))^{1/2}$ and $-\ln(\mu - \mu_o)^{-1}$ for $2n_o + d = 8$;
- in $(\mu - \mu_o)^{1/2}$ and $-(\mu - \mu_o)^{1/2} \ln(\mu - \mu_o)$ for $2n^o + d \geq 10$ with $d$ even

(see Theorem 2.4). Moreover, resonance states of 0-energy, i.e. non-zero solutions $f$ of $\hat{H}_{\mu_o} f = 0$ not belonging to $l^2(\mathbb{Z}^d)$ appear if and only if $2n_o + d \in \{5, 6, 7, 8\}$. Recall that the emergence of 0-energy resonances in more lattice dimensions could allow the Efimov effect to be observed in other dimensions than $d = 3$.

Furthermore, observing that the top $e(\vec{\pi}) = 4d^2$ of the essential spectrum is non-degenerate, one expects the asymptotics of $e(\mu)$ as $\mu \to -\mu^o$ to be similar as in the discrete Laplacian case [20, 21]; more precisely, depending on $d$ and $n^o$, $e(\mu)$ has a convergent expansion

- in $\mu + \mu^o$ for $2n_o + d = 1, 3$;
– in \((\mu + \mu_o)^{1/2}\) for \(2n_o + d \geq 5\) with \(d\) odd;
– in \(\mu + \mu_o\) and \(e^{-1/(\mu+\mu_o)}\) for \(2n_o + d = 2\);
– in \(\mu + \mu_o\), \(-\frac{1}{\ln(\mu+\mu_o)}\) and \(-\frac{\ln_n(\mu+\mu_o)^{-1}}{\ln(\mu+\mu_o)}\) for \(2n_o + d = 4\);
– in \(\mu + \mu_o\) and \(-(\mu + \mu_o)\ln(\mu + \mu_o)\) for \(2n_o + d \geq 6\) with \(d\) even

(see Theorem 2.5). Moreover, the resonance states of energy \(4d^2\), i.e. non-zero solutions \(f\) of \(\hat{H}_{-\mu_o} f = 4d^2 f\) not belonging to \(\ell^2(\mathbb{Z}^d)\) appear if and only if \(2n_o + d = 3, 4\).

The threshold analysis for more general class of nonlocal discrete Schrödinger operators with \(\delta\)-potential of type

\[
\hat{H}_{\mu} = \Psi(-\hat{\Delta}) + \mu \delta_{x_0},
\]

can be found in [14], where \(\Psi\) is some strictly increasing \(C^1\)-function and \(\delta_{x_0}\) is the Dirac’s delta-function supported at 0. Besides the existence of eigenvalues, authors of [14] classify (embedded) threshold resonances and threshold eigenvalues depending on the behaviour of \(\Psi\) at the edges of the essential spectrum of \(-\hat{\Delta}\) and on the lattice dimension \(d\). The eigenvalue expansions for the discrete bilaplacian with \(\delta\)-perturbation have been established in [17] for \(d = 1\) using the complex analytic methods.

The paper is organized as follows. In Sect. 2 after introducing some preliminaries we state the main results of the paper. In Theorem 2.2 we establish necessary and sufficient conditions for non-emptiness of the discrete spectrum of \(\hat{H}_{\mu}\), and in case of existence, we study the location and the uniqueness, analiticity, monotonicity and convexity properties of eigenvalues \(e(\mu)\) as a function of \(\mu\). In particular, we study the asymptotics of \(e(\mu)\) as \(\mu \to \mu_o\) and \(\mu \to -\mu_o\) as well as \(\mu \to \pm \infty\). As discussed above in Theorems 2.4 and 2.5 we obtain expansions of \(e(\mu)\) for small and positive \(\mu - \mu_o\) and \(\mu + \mu_o\). In Sect. 3 we prove the main results. The main idea of the proof is to obtain a nonlinear equation \(\Delta(\mu; z) = 0\) with respect to the eigenvalue \(z = e(\mu)\) of \(\hat{H}_{\mu}\) and then study properties of \(\Delta(\mu; z)\). Finally, in appendix Section A we obtain the asymptotics of certain integrals related to \(\Delta(\mu; z)\) which will be used in the proofs of main results.

Data availability statement

We confirm that the current manuscript has no associated data.

2 Preliminary and main results

Let \(\mathbb{Z}^d\) be the \(d\)-dimensional lattice and \(\ell^2(\mathbb{Z}^d)\) be the Hilbert space of square-summable functions on \(\mathbb{Z}^d\). Consider the family

\[
\hat{H}_{\mu} := \hat{H}_0 - \mu \hat{V}, \quad \mu \geq 0,
\]
of self-adjoint bounded discrete Schrödinger operators in $\ell^2(\mathbb{Z}^d)$. Here $\hat{H}_0 := \hat{\Delta}\hat{\Delta}$ is discrete bilaplacian, where

$$\hat{\Delta} f(x) = \frac{1}{2} \sum_{|s|=1} [f(x) - f(x + s)], \quad f \in \ell^2(\mathbb{Z}^d),$$

is the discrete Laplacian, and $\hat{\mathcal{V}}$ is a rank-one operator

$$\hat{\mathcal{V}} \hat{f}(x) = \hat{v}(x) \sum_{y \in \mathbb{Z}^d} \hat{v}(y) f(y),$$

where $\hat{v} \in \ell^2(\mathbb{Z}^d) \setminus \{0\}$ is a given real-valued function.

Let $\mathbb{T}^d$ be the $d$-dimensional torus equipped with the Haar measure and $L^2(\mathbb{T}^d)$ be the Hilbert space of square-integrable functions on $\mathbb{T}^d$. By $F$ we denote the standard Fourier transform

$$F : \ell^2(\mathbb{Z}^d) \to L^2(\mathbb{T}^d), \quad F \hat{f}(p) = \frac{1}{(2\pi)^{d/2}} \sum_{x \in \mathbb{Z}^d} \hat{f}(x) e^{ix \cdot p}.$$ 

Further we always assume that $\hat{v}$ and its Fourier image $v(p) := F \hat{v}(p) = \frac{1}{(2\pi)^{d/2}} \sum_{x \in \mathbb{Z}^d} \hat{v}(x) e^{ix \cdot p}$ satisfy the following assumptions:

There exist reals $C, a > 0$ and nonnegative integers $n_0, n_0' \geq 0$ such that

$$(H1) \quad |\hat{v}(x)| \leq Ce^{-a|x|} \quad \text{for all } x \in \mathbb{Z}^d,$$

$$(H2) \quad |v(0)|^2 = D^2 |v(0)|^2 = \ldots = D^{2n_0-2} |v(0)|^2 = 0, \quad D^{2n_0} |v(0)|^2 \neq 0,$$

$$(H3) \quad |v(\vec{\pi})|^2 = D^2 |v(\vec{\pi})|^2 = \ldots = D^{2n_0-2} |v(\vec{\pi})|^2 = 0, \quad D^{2n_0} |v(\vec{\pi})|^2 \neq 0,$$

Here $D^j f(p)$ is the $j$-th order differential of $f$ at $p$, i.e. the $j$-th order symmetric tensor

$$D^j f(p)[w, \ldots, w] = \sum_{j \text{-times}} \frac{\partial^j f(p)}{\partial p_1^{i_1} \ldots \partial p_d^{i_d}} w_1^{i_1} \ldots w_d^{i_d},$$

and $\vec{\pi} = (\pi, \ldots, \pi) \in \mathbb{T}^d$. Notice that under assumption $(H1)$, $v$ is analytic on $\mathbb{T}^d$.

Recall that $\sigma(\hat{\Delta}) = \sigma_{\text{ess}}(\hat{\Delta}) = [0, 2d]$ (see e.g. [1]). Hence, $\sigma(\hat{H}_0) = \sigma_{\text{ess}}(\hat{H}_0) = [0, 4d^2]$, and by the compactness of $\hat{\mathcal{V}}$ and Weyl’s Theorem,

$$\sigma_{\text{ess}}(\hat{\mathcal{H}}_\mu) = \sigma_{\text{ess}}(\hat{H}_0) = [0, 4d^2].$$
for any $\mu \in \mathbb{R}$.

Before stating the main results let us introduce the constants

$$
\mu_o := \left( \int_{T^d} \frac{|v(q)|^2dq}{\epsilon(q)} \right)^{-1}, \quad \mu^o := \left( \int_{T^d} \frac{|v(q)|^2dq}{4d^2 - \epsilon(q)} \right)^{-1},
$$

(2.2)

$$
\hat{c}_v := \int_{T^d} \frac{|v(q)|^2dq}{\epsilon(q)^2}, \quad \hat{C}_v := \int_{T^d} \frac{|v(q)|^2dq}{(4d^2 - \epsilon(q))^2},
$$

(2.3)

and

$$
c_v := \frac{2n_o + d}{(2n_o)!} \int_{S^{d-1}} D^{2n_o} |v(0)|^2[w, \ldots, w] d\mathcal{H}^{d-1}(w),
$$

(2.4)

$$
C_v := \frac{2n_o + d - 1}{(8d) n_o + d/2 (2n_o)!} \int_{S^{d-1}} D^{2n_o} |v(\pi)|^2[w, \ldots, w] d\mathcal{H}^{d-1}(w),
$$

(2.5)

where $S^{d-1}$ is the unit sphere in $\mathbb{R}^d$ and

$$
\epsilon(q) := \left( \sum_{i=1}^d (1 - \cos q_i) \right)^2.
$$

**Remark 2.1** Under assumptions (H1)-(H3), $\mu_o, \mu^o \geq 0$, $c_v, C_v > 0$, and $\hat{c}_v, \hat{C}_v \in (0, +\infty]$. Moreover, by Propositions A.1 and A.2:

- $\mu_o = 0$ (resp. $\mu^o = 0$) if and only if $2n_o + d \leq 4$ (resp. $2n^o + d \leq 2$);
- $\hat{c}_v < \infty$ (resp. $\hat{C}_v < \infty$) if $2n_o + d \geq 9$ (resp. $2n^o + d \geq 5$).

### 2.1 Main results

First we concern with the existence of the discrete spectrum of $\hat{H}_\mu$.

**Theorem 2.2** Let $\mu_o, \mu^o \geq 0$ be given by (2.2). Then $\sigma_{\text{disc}}(\hat{H}_\mu) = \emptyset$ for any $\mu \in [-\mu^o, \mu_o]$ and $\sigma_{\text{disc}}(\hat{H}_\mu)$ is a singleton $\{e(\mu)\}$ for any $\mu \in \mathbb{R} \setminus [-\mu^o, \mu_o]$. Moreover, the associated eigenfunction $\hat{f}_\mu$ to $e(\mu)$ is given by $\hat{f}_\mu := \mathcal{F}^* f_\mu$, where

$$
f_\mu(p) = \frac{v(p)}{\epsilon(p) - e(\mu)}.
$$

Furthermore, if $\mu < -\mu^o$ (resp. $\mu > \mu_o$), then $e(\mu) > 4d^2$ (resp. $e(\mu) < 0$). Moreover, the function $\mu \in \mathbb{R} \setminus [-\mu^o, \mu_o] \mapsto e(\mu)$ is real-analytic strictly decreasing, convex in $(-\infty, -\mu^o)$ and concave in $(\mu_o, +\infty)$, and satisfies

$$
\lim_{\mu \searrow \mu_o} e(\mu) = 0 \quad \text{and} \quad \lim_{\mu \nearrow \mu^o} e(\mu) = 4d^2
$$

(2.6)
and
\[ \lim_{\mu \to \pm \infty} \frac{e(\mu)}{\mu} = -\int_{\mathbb{T}^d} |v(q)|^2 dq. \] (2.7)

Next we study the threshold resonances of \( \hat{H}_\mu \).

**Theorem 2.3** Let \( n_o, n^o \geq 0 \) be given by (H2)–(H3).
(a) Let \( 2n_o + d \geq 5 \). Then \( \hat{f} := F^* f \in c_0(\mathbb{Z}^d) \), i.e., \( \hat{f}(x) \to 0 \) as \( |x| \to +\infty \), where
\[
\hat{f}(p) = \frac{v(p)}{\varepsilon(p)} \in L^1(\mathbb{T}^d).
\]
Moreover, \( \hat{f} \in c_0(\mathbb{Z}^d) \setminus \ell^2(\mathbb{Z}^d) \) for \( 2n_o + d \in \{5, 6, 7, 8\} \), \( \hat{f} \in \ell^2(\mathbb{Z}^d) \) for \( 2n_o + d \geq 9 \), and \( \hat{f} \) solves the equation \( \hat{H}_{\mu_o} f = 0 \).
(b) Let \( 2n^o + d \geq 3 \). Then \( \hat{g} := F^* g \in \ell^0(\mathbb{Z}^d) \), where
\[
g(p) = \frac{v(p)}{4d^2 - \varepsilon(p)}.
\]
Moreover, \( \hat{g} \in \ell^0(\mathbb{Z}^d) \setminus \ell^2(\mathbb{Z}^d) \) for \( 2n^o + d \in \{3, 4\} \), \( \hat{g} \in \ell^2(\mathbb{Z}^d) \) for \( 2n^o + d \geq 5 \), and \( \hat{g} \) solves the equation \( \hat{H}_{-\mu^o} f = 4d^2 f \).

We recall that in the literature the non-zero solutions of equations \( \hat{H}_{\mu} \hat{f} = 0 \) and \( \hat{H}_{\mu} \hat{g} = 4d^2 \hat{g} \) not belonging to \( \ell^2(\mathbb{Z}^d) \) are called the resonance states [1, 2].

Now we study the rate of the convergences in (2.6).

**Theorem 2.4** (Expansions of \( e(\mu) \) at \( \mu = \mu_o \)) For \( \mu > \mu_o \) let \( e(\mu) < 0 \) be the eigenvalue of \( \hat{H}_\mu \).
(a) Suppose that \( d \) is odd:
(a1) if \( 2n_o + d = 1, 3 \), then \( \mu_o = 0 \) and for sufficiently small and positive \( \mu \),
\[
(-e(\mu))^{1/4} = \begin{cases} \left( \frac{\pi c_e}{4} \right)^{1/3} \mu^{1/3} + \sum_{n \geq 1} c_{1,n} \mu^{n+1}, & 2n_o + d = 1, \\ \frac{\pi c_e}{8} \mu + \sum_{n \geq 1} c_{3,n} \mu^{n+1}, & 2n_o + d = 3, \end{cases}
\]
where \( \{c_{1,n}\} \) and \( \{c_{3,n}\} \) are some real coefficients;
(a2) if \( 2n_o + d = 5, 7 \), then \( \mu_o > 0 \) and for sufficiently small and positive \( \mu - \mu_o \),
\[
(-e(\mu))^{1/4} = \begin{cases} \frac{8}{\pi c_e \mu_o} (\mu - \mu_o) + \sum_{n \geq 1} c_{5,n} (\mu - \mu_o)^{n+1}, & 2n_o + d = 5, \\ \left( \frac{8}{\pi c_e \mu_o} \right)^{1/3} (\mu - \mu_o)^{1/3} + \sum_{n \geq 1} c_{7,n} (\mu - \mu_o)^{n+1/3}, & 2n_o + d = 7, \end{cases}
\]
where \( \{c_{5,n}\} \) and \( \{c_{7,n}\} \) are some real coefficients;
(a3) If $2n_o + d \geq 9$, then $\mu_o > 0$ and for sufficiently small and positive $\mu - \mu_o$,
\[
(-e(\mu))^{1/4} = (\mu_o^{2c_v})^{-1/4} (\mu - \mu_o)^{1/4} + \sum_{n \geq 1} c_{9,n}(\mu - \mu_o)^{n/4},
\]
where $\{c_{9,n}\}$ are some real coefficients.

(b) Suppose that $d$ is even:

(b1) If $2n_o + d = 2, 4$,
\[
\mu_o = 0
\]
and for sufficiently small and positive $\mu$,
\[
(-e(\mu))^{1/2} = \begin{cases} 
\pi c_v \mu + \sum_{n+m \geq 1, n,m \geq 0} c_{2,nm} \mu^{n+1} (-\mu \ln \mu)^m, & 2n_o + d = 2, \\
\left( \frac{8}{c_v \mu_o^2} \right)^{1/2} \tau^2 + \sum_{n+m \geq 1, n,m \geq 0} c_{4,nm} \tau^{2n+2}\theta^m, & 2n_o + d = 4,
\end{cases}
\]
where $\{c_{2,nm}\}$ and $\{c_{4,nm}\}$ are some real coefficients and $c > 0$;

(b2) If $2n_o + d = 6, 8$,
\[
\mu_o > 0
\]
and for sufficiently small and positive $\mu - \mu_o$,
\[
(-e(\mu))^{1/2} = \begin{cases} 
\pi c_v \mu + \sum_{n+m \geq 1, n,m \geq 0} c_{6,nm} \tau^{2n+2}\theta^m, & 2n_o + d = 6, \\
\left( \frac{8}{c_v \mu_o^2} \right)^{1/2} \tau^2 + \sum_{n+m+k \geq 1, n,m,k \geq 0} c_{8,nmk} \tau^{n+1}\sigma^{m+1}\eta^k, & 2n_o + d = 8,
\end{cases}
\]
where $\{c_{4,nm}\}$ and $\{c_{8,nmk}\}$ are some real coefficients and
\[
\tau := (\mu - \mu_o)^{1/2}, \ \theta := -\tau^2 \ln \tau, \ \sigma := \left( -\frac{1}{\ln \tau} \right)^{1/2}, \ \eta := -\frac{\ln \ln \tau^{1-1}}{\ln \tau},
\]
where $\{c_{10,nm}\}$ are some real coefficients.

Here $c_v > 0$ and $\tilde{c}_v > 0$ are given by (2.4) and (2.3), respectively.

**Theorem 2.5** (Expansions of $e(\mu)$ at $\mu = -\mu^o$) For let $\mu < -\mu^o$ let $e(\mu) > 4d^2$ be the eigenvalue of $\tilde{H}_\mu$.

(a) Suppose that $d$ is odd:
(a1) if $2n_0 + d = 1$, then $\mu^o = 0$ and for sufficiently small and negative $\mu$,

$$(e(\mu) - 4d^2)^{1/2} = -\pi C_v \mu + \sum_{n \geq 1} C_{1,n} \mu^{n+1},$$

where $\{C_{1,n}\}$ are some real coefficients;

(a2) if $2n_0 + d = 3$, then $\mu^o > 0$ and for sufficiently small and positive $\mu + \mu^o$,

$$(e(\mu) - 4d^2)^{1/2} = (\pi C_v \mu^{o^2})^{-1} (\mu + \mu^o) + \sum_{n \geq 1} C_{3,n} (\mu + \mu^o)^{n+1},$$

where $\{C_{3,n}\}$ and $\{C_{7,n}\}$ are some real coefficients;

(a3) if $2n_0 + d \geq 5$, then $\mu^o > 0$ and for sufficiently small and positive $\mu + \mu^o$,

$$(e(\mu) - 4d^2)^{1/2} = (\hat{C}_v \mu^{o^2})^{-1/2} (\mu + \mu^o)^{1/2} + \sum_{n \geq 1} C_{5,n} (\mu + \mu^o)^{(n+1)/2},$$

where $\{C_{5,n}\}$ are some real coefficients.

(b) Suppose that $d$ is even:

(b1) if $2n_0 + d = 2$, then $\mu_0 = 0$ and for sufficiently small and negative $\mu$,

$$e(\mu) - 4d^2 = C e^{1/2} \mu + \sum_{n+m \geq 1, n,m \geq 0} C_{2,nm} \mu^{n+1} \left( -\frac{1}{\mu} e^{1/2} \right)^{m+1},$$

where $\{C_{2,nm}\}$ are some real coefficients and $C > 0$;

(b2) if $2n_0 + d = 4$, then $\mu^o > 0$ and for sufficiently small and positive $\mu + \mu^o$,

$$e(\mu) - 4d^2 = (C_v \mu^{o^2})^{-1} \mu \sigma + \sum_{n+m+k \geq 1, n,m,k \geq 0} C_{4,nmk} \tau^{n+1} \sigma^{m+1} \eta^k,$$

where $\{C_{4,nm}\}$ are some real coefficients and

$$\tau := \mu + \mu^o, \quad \sigma := -\frac{1}{\ln \tau}, \quad \eta := -\frac{\ln \ln \tau^{-1}}{\ln \tau};$$

(b3) if $2n_0 + d \geq 6$, then $\mu^o > 0$ and for sufficiently small and positive $\mu + \mu^o$,

$$e(\mu) - 4d^2 = (\hat{C}_v \mu^{o^2})^{-1} (\mu + \mu^o)$$

$$+ \sum_{n+m \geq 1, n,m \geq 0} C_{6,nm} (\mu + \mu^o)^{n+1} [-(\mu + \mu^o) \ln(\mu + \mu^o)]^m,$$

where $\{C_{6,nm}\}$ are some real coefficients.

Here $C_v$ and $\hat{C}_v$ are given by (2.5) and (2.3), respectively.
Remark 2.6 Few comments on the main results are in order.
1. The assertions of Theorem 2.2 hold in fact for any $\hat{v} \in \ell^2(\mathbb{Z}^d)$ (see Remark 3.2);
2. Similar expansions of $e(\mu)$ in Theorems 2.4 and 2.5 at $\mu = \mu_o$ and $\mu = -\mu_o$, respectively, still hold for any exponentially decaying $\hat{v} : \mathbb{Z}^d \rightarrow \mathbb{C}$ (see Remark 3.3);
3. If $\hat{v}$ decays at most polynomially at infinity, i.e. $\hat{v}(x) = O(|x|^{-\alpha})$ for some $\alpha > 0$, then instead of the expansions in Theorem 2.4 and 2.5 we obtain only asymptotics of $e(\mu)$ (see Remark 3.4).

3 Proof of main results

In this section we prove the main results. By the Birman-Schwinger principle and the Fredholm Theorem we have

Lemma 3.1 A complex number $\mu \in \mathbb{C} \setminus [0, 4d^2]$ is an eigenvalue of $\hat{H}_\mu$ if and only if

$$\Delta(\mu; z) := 1 - \mu \int_{\mathbb{T}^d} \frac{|v(q)|^2 dq}{\epsilon(q) - z} = 0.$$ 

Proof of Theorem 2.2 By the definition of $\mu_o$, for any $\mu < -\mu_o$ :

$$\lim_{z \uparrow -\mu_o} \Delta(\mu; z) = 1 + \frac{\mu}{\mu_o} < 0, \quad \lim_{z \to +\infty} \Delta(\mu; z) = 1.$$ 

Since $\Delta(\mu; z) > 1$ for $z < 0$ and $\mu > -\mu_o$, in view of the strict monotonicity $\Delta(\mu; \cdot)$ in $(4d^2, \infty)$, for any $\mu < -\mu_o$ there exists a unique $e(\mu) \in (4d^2, +\infty)$ such that $\Delta(\mu; e(\mu)) = 0$. Analogously, for any $\mu > \mu_o$ there exists a unique $e(\mu) \in (-\infty, 0)$ such that $\Delta(\mu; e(\mu)) = 0$. By the Implicit Function Theorem the function $\mu \in \mathbb{R} \setminus [-\mu_o, \mu_o] \mapsto e(\mu)$ is real-analytic. Moreover, computing the derivatives of the implicit function $e(\mu)$ we find:

$$e'(\mu) = -\frac{1}{\mu} \int_{\mathbb{T}^d} \frac{|v(q)|^2 dq}{\epsilon(q) - e(\mu)} \left( \int_{\mathbb{T}^d} \frac{|v(q)|^2 dq}{(\epsilon(q) - e(\mu))^2} \right)^{-1}, \quad \mu \neq 0, \quad (3.1)$$

thus, using $\mu(\epsilon(q) - e(\mu)) > 0$ we get $e'(\mu) < 0$, i.e. $e(\cdot)$ is strictly decreasing in $\mathbb{R} \setminus \{0\}$. Differentiating (3.1) one more time we get

$$e''(\mu) = \frac{2e'(\mu)}{\mu} \left( 1 - \mu e'(\mu) \int_{\mathbb{T}^d} \frac{|v(q)|^2 dq}{(\epsilon(q) - e(\mu))^3} \left( \int_{\mathbb{T}^d} \frac{|v(q)|^2 dq}{(\epsilon(q) - e(\mu))^2} \right)^{-1} \right).$$

Therefore, $e''(\mu) > 0$ (i.e. $e(\cdot)$ is strictly convex) for $\mu < 0$ and $e''(\mu) < 0$ (i.e. $e(\cdot)$ is strictly concave) for $\mu > 0$.

To prove (2.7), first we let $\mu \rightarrow \pm \infty$ in

$$1 = \mu \int_{\mathbb{T}^d} \frac{|v(q)|^2 dq}{\epsilon(q) - e(\mu)} \quad (3.2)$$
and find \( \lim_{\mu \to \pm \infty} e(\mu) = \mp \infty \). In particular, if \( |\mu| \) is sufficiently large, \( \frac{e(q)}{e(\mu)} < \frac{1}{2} \) and hence, by (3.2) and the Dominated Convergence Theorem,

\[
\lim_{\mu \to \pm \infty} \frac{e(\mu)}{\mu} = - \lim_{\mu \to \pm \infty} \int_{\mathbb{T}^d} \frac{|v(q)|^2 dq}{1 - \frac{e(q)}{e(\mu)}} = - \int_{\mathbb{T}^d} |v(q)|^2 dq.
\]

To prove that \( \hat{f}_\mu \) solves \( \hat{H}_\mu \hat{f}_\mu = e(\mu) \hat{f}_\mu \) we consider the equivalent equality \( \mathcal{F} \hat{H}_\mu \mathcal{F}^* f_\mu = e(\mu) f_\mu \), which is easily reduced to the equality \( \Delta(\mu; e(\mu)) = 0 \).

\[\text{Remark 3.2} \quad \text{In view of Lemma 3.1 and the proof of Theorem 2.2, their assertions still hold for any } v \in l^2(\mathbb{Z}^d).\]

**Proof of Theorem 2.3** We prove only (a), the proof of (b) being similar. Repeating the proof of the continuity (resp. differentiability) of \( I_f \) at \( z = 0 \) in Proposition A.1 one can show that \( f \in L^1(\mathbb{T}^d) \setminus L^2(\mathbb{T}^d) \) for \( 2n_o + d \in \{5, 6, 7, 8\} \) and \( f \in L^2(\mathbb{T}^d) \) for \( 2n_o + d \geq 9 \). Thus, by the Riemann–Lebesgue Lemma, \( \hat{f} \in l^0(\mathbb{Z}^d) \). To show that \( \hat{H}_\mu \hat{f} = 0 \) it suffices to observe that \( \mathcal{F} \hat{H}_\mu \mathcal{F}^* f = 0 \).

**Proof of Theorem 2.4** Since

\[
|v(p)|^2 = (2\pi)^{-d} \left( \sum_{x \in \mathbb{Z}^d} \hat{v}(x) \cos p \cdot x \right)^2 + (2\pi)^{-d} \left( \sum_{x \in \mathbb{Z}^d} \hat{v}(x) \sin p \cdot x \right)^2,
\]

the function \( p \in \mathbb{T}^d \mapsto |v(p)|^2 \) is nonnegative even real-analytic function. Notice also that if \( n_o \geq 1 \), then by the nonnegativity of \(|v|^2\), \( p = 0 \) is a global minimum for \(|v|^2\). Therefore, the tensor \( D^{2n_o} |v(0)|^2 \) is positively definite and

\[
c_v := \frac{2^{2n_o + d}}{(2n_o)!} \int_{S^{d-1}} D^{2n_o} |v(0)|^2 [w, \ldots, w] dH^{d-1} > 0.
\]

Note that

\[
\hat{c}_v = \hat{l}_v |v|^2(0) = \int_{\mathbb{T}^d} \frac{|v(q)|^2 dq}{e(q)^2},
\]

where \( l_f \) is defined in (A.1). By Proposition A.1, \( f(p) = \frac{v(p)}{e(p)} \in L^2(\mathbb{T}^d) \) if and only if \( 2n_o + d \geq 9 \). Moreover, by definition, \( \mu_o > 0 \) and \( \Delta(\mu_o; 0) = 0 \) for \( 2n_o + d \geq 5 \), and hence, as in the proof of Lemma 3.1 for such \( d \) one can show that \( H_{\mu_o} f = 0 \).

In view of the strict monotonicity and (2.6) there exists a unique \( \mu_1 > 0 \) such that \( e(\mu) \in (-\frac{1}{128}, 0) \) for any \( \mu \in (0, \mu_1) \). Since

\[
\mu = (l_{|v|^2}(e(\mu)))^{-1},
\]

(3.4)
we can use Proposition A.1 with \( f = |v|^2 \) and \( e := e(\mu) \), to find the expansions of the inverse function \( \mu := \mu(e) \). Then applying the appropriate versions of the Implicit Function Theorem in analytical case we get the expansions of \( e = e(\mu) \). Notice that from (A.3) and (A.4) as well as (3.5) it follows that \( \mu_o = 0 \) for \( 2n_o + d \leq 4 \) and \( \mu_o = \left( \int_{\mathbb{T}^d} \frac{|v(q)|^2 dq}{e(q)} \right)^{-1} > 0 \) for \( 2n_o + d \geq 5 \).

(a) Suppose that \( d \) is odd. In view of the expansions (A.3) of \( f \), in this case, (3.4) is reduced to the inverting the equation

\[
\mu = g(\alpha),
\tag{3.5}
\]

where \( \alpha := (-e)^{1/4} \) and \( g \) is an analytic function around \( \alpha = 0 \).

Case \( 2n_o + d = 1 \). In this case by (A.3),

\[
g(\alpha) := \frac{\alpha^3}{c_1 + \sum_{n \geq 1} a_n \alpha^n},
\]

where \( \{a_n\} \subset \mathbb{R} \) and \( c_1 := (\pi c_v/4)^{1/3} \) and (3.5) is equivalently represented as

\[
\alpha = \mu \left( c_1^3 + \sum_{n \geq 1} a_n \alpha^n \right)^{1/3},
\tag{3.6}
\]

where \( \mu = \mu^{1/3} \). Now setting

\[
\alpha = \mu (c_1 + u),
\tag{3.7}
\]

and using the Taylor series of \( (c_1^3 + x)^{1/3} \), for \( \mu \) and \( u \) sufficiently small we rewrite (3.6) as

\[
F(u, \mu) := u - \sum_{n \geq 1} \tilde{a}_n \mu^n (c_1 + u)^n = 0,
\tag{3.8}
\]

where \( F(\cdot, \cdot) \) is analytic at \((0, 0)\), \( F(0, 0) = 0 \) and \( F_u(0, 0) = 1 \). Hence, by the Implicit Function Theorem, there exists \( \gamma_1 > 0 \) such that for \( |\mu| < \gamma_1 \), (3.8) has a unique real-analytic solution \( u = u(\mu) \) which can be represented as an absolutely convergent series \( u = \sum_{n \geq 1} b_n \mu^n \). Putting this in (3.7) and recalling the definitions of \( \alpha \) and \( \mu \) we get the expansion of \((-e(\mu))^{1/4}\) for \( \mu > 0 \) small.

Case \( 2n_o + d = 3 \). By (A.3),

\[
g(\alpha) = \alpha \left( c_3 + \sum_{n \geq 1} a_n \alpha^n \right)^{-1},
\tag{3.9}
\]
where \( \{a_n\} \subset \mathbb{R} \) and \( c_3 := \pi c_v/8 \), and hence, (3.5) is represented as

\[
\alpha = \mu \left( c_3 + \sum_{n \geq 1} a_n \alpha^n \right). 
\]

Then setting \( \alpha = \mu (c_3 + u) \) we rewrite (3.9) in the form (3.8), and as in the case of \( 2n_o + d = 1 \), we get the expansion of \( (-e(\mu))^{1/4} \).

**Case 2** \( n_o + d = 5 \). In this case by (A.3)

\[
g(\alpha) = \left( \frac{1}{\mu} - \frac{\pi c_v \alpha}{8} \left( 1 + \sum_{n \geq 1} a_n \alpha^n \right) \right)^{-1},
\]

where \( \{a_n\} \subset \mathbb{R} \), and hence, by (3.5),

\[
\frac{\mu - \mu_o}{\mu \mu_o} = \frac{\pi c_v \alpha}{8} \left( 1 + \sum_{n \geq 1} a_n \alpha^n \right). \tag{3.10}
\]

Note that if \( |\mu - \mu_o| < \mu_o \), then

\[
\frac{\mu - \mu_o}{\mu \mu_o} = \frac{\mu - \mu_o}{\mu_o^2 + \mu_o (\mu - \mu_o)} = \frac{\mu - \mu_o}{\mu_o^2} \sum_{n \geq 0} \left( \frac{\mu - \mu_o}{\mu_o} \right)^n, \tag{3.11}
\]

thus from (3.10) we get

\[
\alpha = (\mu - \mu_o) \left( c_5 + c_5 \sum_{n \geq 1} \mu_o^{-n}(\mu - \mu_o)^n \right) \left( 1 + \sum_{n \geq 1} a_n \alpha^n \right)^{-1}. 
\]

and \( c_5 := 8/(\pi c_v \mu_o^2) \). Now setting \( \alpha = (\mu - \mu_o) (c_5 + u) \) for sufficiently small and positive \( \mu - \mu_o \) we get

\[
u = \sum_{n,m \geq 1} \tilde{c}_{n,m}(\mu - \mu_o)^n(c_5 + u)^m,
\]

where \( \tilde{c}_{n,m} \subset \mathbb{R} \). By the Implicit Function Theorem, for sufficiently small \( \mu - \mu_o \) there exists a unique real-analytic function \( u = u(\mu) \) given by the absolutely convergent series \( u(\mu) = \sum_{n \geq 1} b_n(\mu - \mu_o)^n \). By the definition of \( \alpha \), this implies the expansion of \( (-e(\mu))^{1/4} \).
Case $2n_o + d = 7$. As the previous case, by (A.3) and (3.11), the equation (3.5) is represented as

$$
(\mu - \mu_o) \left( c_7^3 + c_7^3 \sum_{n \geq 1} \mu_o^{-n} (\mu - \mu_o)^n \right) = \alpha^3 \left( 1 + \sum_{n \geq 1} a_n \alpha^n \right),
$$

(3.12)

where $\{a_n\} \subset \mathbb{R}$ and $c_7 := \left[ 8/(\pi c_v \mu_0^2) \right]^{1/3}$. When $\mu - \mu_o > 0$ is small enough, by the Taylor series of $(1 + x)^{\pm 1/3}$ at $x = 0$, (3.12) is equivalently rewritten as

$$
\alpha = (\mu - \mu_o)^{1/3} \left( c_7 + \sum_{n \geq 1} \tilde{c}_n (\mu - \mu_o)^n \right) \left( 1 + \sum_{n \geq 1} \tilde{a}_n \alpha^n \right),
$$

(3.13)

Thus, for $\rho = (\mu - \mu_o)^{1/3}$, setting $\alpha = \rho (c_7 + u)$ in (3.13), for sufficiently small and positive $\rho$ we get

$$
u = \sum_{n,m \geq 1} \tilde{c}_{n,m} \rho^n (c_7 + u)^m.
$$

By the Implicit Function Theorem, this equation has a unique real-analytic solution $u = u(\rho)$ given by the absolutely convergent series $u = \sum_{n \geq 1} b_n \rho^n$. This, definitions of $\alpha$ and $\rho$ imply the expansion of $(-e(\mu))^{1/4}$.

Case $2n_o + d = 9$. In this case by (A.3) and (3.11)

$$
(\mu - \mu_o) \left( c_9^4 + c_9^4 \sum_{n \geq 1} \mu_o^{-n} (\mu - \mu_o)^n \right) = \alpha^4 \left( 1 + \sum_{n \geq 1} a_n \alpha^n \right),
$$

(3.14)

where $\{a_n\} \subset \mathbb{R}$ and $c_9 := (\mu_o^2 c_v)^{-1/4}$. Thus, for sufficiently small and positive $\mu - \mu_o$ using the Taylor series of $(1 + x)^{\pm 1/4}$ at $x = 0$, this equation can also be represented as

$$
\alpha = \rho \left( c_9 + \sum_{n \geq 1} \tilde{b}_n \rho^{4n} \right) \left( 1 + \sum_{n \geq 1} \tilde{a}_n \alpha^n \right),
$$

where $\rho := (\mu - \mu_o)^{1/4}$. Now setting $\alpha = \rho (c_9 + u)$ in (3.14) we get

$$
u = \sum_{n,m \geq 1} \tilde{c}_{n,m} \rho^n (c_9 + u)^m,
$$

and the expansion of $(-e(\mu))^{1/4}$ follows as in the case of $2n_o + d = 7$. 
(b) Suppose that \( d \) is even. In view of the expansion (A.3) of \( l_f \), in this case, (3.4) is reduced to the inverting the equation

\[
\mu = \frac{\alpha^l}{g(\alpha) + h(\alpha) \ln \alpha},
\]  

(3.15)

where \( \alpha := (-e)^{1/2}, l \in \mathbb{N}_0 \), and \( g \) and \( h \) are analytic around \( \alpha = 0 \). Presence of \( \ln \alpha \) implies that unlike the case of odd dimensions, \( \alpha \) is not necessarily analytic with respect to \( \mu^s \). Therefore, we need to introduce new variables dependent on \( \ln \mu \) to reduce the problem to the Implicit Function Theorem.

**Case** \( 2n_0 + d = 2 \). By (A.4), in this case for \( c_2 := \pi c_v / 8 \)

\[
l = 1, \quad g(\alpha) = c_2 + \sum_{n \geq 1} a_n \alpha^n, \quad h(\alpha) = \sum_{n \geq 1} b_n \alpha^{2n}.
\]

Hence, setting

\[
\alpha = \mu(c_2 + u)
\]

(3.16)

and \( \tau = -\mu \ln \mu \) we represent (3.15) as

\[
F(u, \mu, \tau) := u - \sum_{n \geq 1} a^n \mu^n (c_2 + u)^n + \ln(c_2 + u) \sum_{n \geq 1} b^n \mu^n (c_2 + u)^n
\]

\[
- \tau \sum_{n \geq 1} b^n \mu^{n-1} (c_2 + u)^n = 0,
\]

where \( F \) is analytic around \( (0, 0, 0) \), \( F(0, 0, 0) = 0, F_u(0, 0, 0) = 1 \). Hence, by the Implicit Function Theorem, there exists a unique real-analytic function \( u = u(\mu, \tau) \) given by the convergent series \( u(\mu, \tau) = \sum_{n+m \geq 1, n, m \geq 0} \xi_{n,m} \mu^n \tau^m \) for sufficiently small \( |\mu| \) and \( |\tau| \), which satisfies \( F(u(\mu, \tau), \mu, \tau) = 0 \). Inserting \( u \) in (3.16) we get the expansion of \( \alpha = (-e)^{1/2} \).

**Case** \( 2n_0 + d = 4 \). In this case, by (A.4) for \( c_4 := 8 / c_v \)

\[
l = 0, \quad g(\alpha) = \sum_{n \geq 0} a_n \alpha^n, \quad h(\alpha) = -c_4 + \sum_{n \geq 1} b_n \alpha^{2n}.
\]

Letting \( \alpha = e^{-\frac{1}{c_4 \mu}} (c + u) \), where \( c = e^{\theta_0 / c_4} > 0 \), we represent (3.15) as

\[
\ln(c + u) - b_0 = \frac{1}{\mu} e^{-\frac{1}{c_4 \mu}} \sum_{n \geq 1} a^n e^{-\frac{n-1}{c_4 \mu}} (c + u)^n
\]

\[
+ \ln(c + u) \sum_{n \geq 1} b^n e^{-\frac{n}{c_4 \mu}} (c + u)^n - \sum_{n \geq 1} a^n e^{-\frac{n}{c_4 \mu}} (c + u)^n = 0.
\]

(3.17)
Writing \( \tau := \frac{1}{\mu} e^{-\frac{1}{c_4 \mu}} \) so that \( e^{-\frac{1}{c_4 \mu}} = \mu \tau \), (3.17) is represented as

\[
F(u, \mu, \tau) := \ln(c + u) - b_0 - \mu \sum_{n \geq 1} a_n \mu^{n-1} \tau^{n-1} (c + u)^n \\
- \ln(c + u) \sum_{n \geq 1} b_n \mu^n \tau^n (c + u)^n + \sum_{n \geq 1} a_n \mu^n \tau^n (c + u)^n = 0,
\]

where \( F \) is analytic around \((0, 0, 0), F(0, 0, 0) = 0, \) and \( F_u(0, 0, 0) = \frac{1}{c} > 0. \) Thus, by the Implicit Function Theorem, for \(|\mu|, |\tau| \) and \(|u| \) small there exists a unique real analytic function \( u = u(\mu, \tau) \) given by the convergent series

\[
u = \sum_{n+m \geq 1, n, m \geq 0} c_{n,m} \mu^n \tau^m \text{ such that } F(u(\mu, \tau), \mu, \tau) \equiv 0.
\]

Since \( \tau = \frac{1}{\mu} e^{-\frac{1}{c_4 \mu}} \), this implies

\[
\alpha = e^{-\frac{1}{c_4 \mu}} (c + u) = ce^{-\frac{1}{c_4 \mu}} + \sum_{n+m \geq 1, n, m \geq 0} c_{n,m} \mu^{n+1} \left( \frac{1}{\mu} e^{-\frac{1}{c_4 \mu}} \right)^{m+1}.
\]

**Case 2n_o + d = 6.** In this case, by (A.4), for \( c_6 := 8/(\pi c_4 \mu_o^2) \)

\[
l = 0, \quad g(\alpha) = \frac{1}{\mu_o} - \frac{1}{c_6 \mu_o^2} \left( \alpha + \sum_{n \geq 2} a_n \alpha^n \right), \quad h(\alpha) = \frac{1}{c_6 \mu_o^2} \sum_{n \geq 1} b_n \alpha^{2n},
\]

and hence, (3.15) is represented as

\[
\frac{1}{\mu} - \frac{1}{\mu_o} = \frac{1}{c_6 \mu_o^2} \left( \alpha + \sum_{n \geq 2} a_n \alpha^n + \ln \alpha \sum_{n \geq 1} b_n \alpha^{2n} \right),
\]

or equivalently, by (3.11),

\[
\alpha = c_6 (\mu - \mu_o) \sum_{n \geq 0} \left( \frac{\mu - \mu_o}{\mu_o} \right)^n - \sum_{n \geq 2} a_n \alpha^n - \ln \alpha \sum_{n \geq 1} b_n \alpha^{2n}. \tag{3.18}
\]

Recalling the definitions of \( \tau \) and \( \theta \) in (2.8), setting \( \alpha = \tau^2 (c_6 + u) \), we represent (3.18) as

\[
F(u, \tau, \theta) := u - c_6 \sum_{n \geq 1} \frac{\tau^{2n}}{\mu_o^n} - \sum_{n \geq 2} a_n \tau^{2n-2} (c_6 + u)^n \\
- \ln (c_6 + u) \sum_{n \geq 1} b_n \tau^{4n} (c_6 + u)^{2n} - \theta \sum_{n \geq 1} b_n \tau^{4n-4} (c_6 + u)^{2n} = 0.
\]
where $F$ is real-analytic around $(0, 0, 0)$, $F(0, 0, 0) = 0$ and $F_u(0, 0, 0) = 1$, and $F$ is even in $\tau$. Thus, by the Implicit Function Theorem, for $|u|$, $|\tau|$ and $|\theta|$ small there exists a unique real analytic function $u = u(\tau, \theta)$, even in $\tau$, given by the convergent series

$$u = \sum_{n+m\geq 1,n,m \geq 0} \tilde{c}_{n,m} \tau^n \theta^m$$

such that $F(u(\tau, \theta), \tau, \theta) = 0$. Thus,

$$\alpha = \tau^2 (c_6 + u) = c_6 \sigma + \sum_{n+m \geq 1,n,m \geq 0} \tilde{c}_{n,m} \tau^{2n+2} \theta^m.$$

Case $2n_o + d = 8$. By (A.4), for $c_8 := [8/c_v \mu_o^2]^{-1/2}$,

$$l = 0, \quad g(\alpha) = \frac{1}{\mu_o^2 c_8^2} \sum_{n \geq 2} a_n \alpha^n, \quad h(\alpha) = \frac{1}{\mu_o^2 c_8^2} \left( \alpha^2 + \sum_{n \geq 2} b_n \alpha^{2n} \right),$$

thus, as in the case of $2n_o + d = 6$, (3.15) is represented as

$$c_8^2 (\mu - \mu_o) \sum_{n \geq 0} \left( \frac{\mu - \mu_o}{\mu_o} \right)^n = \alpha^2 \ln \alpha + \ln \alpha \sum_{n \geq 2} b_n \alpha^{2n} + \sum_{n \geq 2} a_n \alpha^n. \quad (3.19)$$

For $\tau$, $\sigma$ and $\eta$ given in (2.8) set $\alpha = \tau \sigma (c_8 + u)$ and represent (3.19) as

$$2c_8 u + u^2 = c_8^2 \sum_{n \geq 1} \frac{\tau^{2n}}{\mu_o^n} + \sum_{n \geq 2} a_n \tau^{n-1} \sigma^{n+1} (c_8 + u)^{n+2}$$

$$- \sum_{n \geq 2} b_n (\tau \sigma)^{2n-2} (c_8 + u)^{2n+2}$$

$$+ \left( \sigma^2 \ln (c_8 + u) - \frac{\eta}{2} \right) \left( (c_8 + u)^2 + \sum_{n \geq 2} b_n (\tau \sigma)^{2n-2} (c_8 + u)^{2n+2} \right).$$

This equation is represented as $F(u, \tau, \sigma, \eta) = 0$, where $F$ is real-analytic in a neighborhood of $(0, 0, 0, 0)$, $F(0, 0, 0, 0) = 0$ and $F_u(0, 0, 0, 0) = 2c_8 > 0$. Hence, for $|u|$, $|\tau|$, $|\sigma|$ and $|\eta|$ small, by the Implicit Function Theorem, there exists a unique real-analytic function $u = u(\tau, \sigma, \eta)$ given by the convergent series

$$u = \sum_{n+m+k \geq 1,n,m,k \geq 0} \tilde{c}_{n,m,k} \tau^n \sigma^m \mu^k$$

such that $F(u(\tau, \sigma, \eta), \tau, \sigma, \eta) = 0$. Thus,

$$\alpha = \tau \sigma (c_8 + u) = c_8 \tau \sigma + \sum_{n+m+k \geq 1,n,m,k \geq 0} \tilde{c}_{n,m,k} \tau^{n+1} \sigma^{m+1} \mu^k.$$

Case $2n_o + d \geq 10$. By (A.4) for $c_{10} := (\mu_o^2 \tilde{c}_v)^{-1/2}$,

$$l = 0, \quad g(\alpha) = \frac{1}{\mu_o} + \tilde{c}_v \alpha^2 + \sum_{n \geq 2} a_n \alpha^{n+2}, \quad h(\alpha) = \sum_{n \geq 2} b_n \alpha^{2n},$$
and as in the case of $2n_o + d = 6$, (3.15) is represented as

$$\frac{\mu - \mu_o}{\mu_o^2} \sum_{n \geq 0} \left( \frac{\mu - \mu_o}{\mu_o} \right)^n = \tilde{c}_0 \alpha^2 + \sum_{n \geq 2} a_n \alpha^{n+2} + \ln \alpha \sum_{n \geq 2} b_n \alpha^{2n}. \quad (3.20)$$

Recalling the definitions of $\tau$ and $\theta$ in (2.8), we set $\alpha = \tau(c_{10} + u)$. Then (3.20) is represented as

$$F(u, \tau, \theta) := 2c_{10}u + u^2 - c_{10}^2 \sum_{n \geq 1} \frac{\tau^{2n}}{\mu_o^n} + \sum_{n \geq 2} a_n \tau^n (c_{10} + u)^{n+2} - \theta \sum_{n \geq 2} b_n \tau^{2n-4} (c_8 + u)^{2n} + \ln(c_{10} + u) \sum_{n \geq 2} b_n \tau^{2n-2} (c_8 + u)^{2n} = 0,$$

where $F$ is analytic at $(0, 0, 0)$, $F(0, 0, 0) = 0$ and $F_u(0, 0, 0) = 2c_{10} > 0$. Thus, by the Implicit Function Theorem, for $|u|$, $|\tau|$ and $|\theta|$ small there exists a unique real-analytic function $u = u(\tau, \theta)$ given by the convergent series $u = \sum_{n+m \geq 1, n,m \geq 0} \tilde{c}_{n,m} \tau^n \theta^m$ such that $F(u(\tau, \theta), \tau, \theta) \equiv 0$. Then

$$\alpha = \mu(c_{10} + u) = c_{10} \mu + \sum_{n+m \geq 1, n,m \geq 0} \tilde{c}_{n,m} \mu^{n+1} \theta^n.$$

Theorem is proved. \(\square\)

**Proof of Theorem 2.5** From (3.3) it follows that the map $p \in \mathbb{T}^d \mapsto |v|^2(\tilde{\pi} + p)$ is even. Now the expansions of $e(\mu)$ at $\mu = -\mu_o$ can be proven along the same lines of Theorem 2.4 using Proposition A.2 with $f = |v|^2$. \(\square\)

**Remark 3.3** Let $\tilde{v} : \mathbb{Z}^d \to \mathbb{C}$ satisfy (H1). Since $\epsilon(\cdot)$ is even,

$$\int_{\mathbb{T}^d} \frac{|v(p)|^2 dp}{\epsilon(p) - z} = \frac{1}{(2\pi)^d} \int_{\mathbb{T}^d} \frac{f(p) dp}{\epsilon(p) - z},$$

where

$$f(p) := \left( \sum_{x \in \mathbb{Z}^d} \tilde{v}_1(x) \cos p \cdot x \right)^2 + \left( \sum_{x \in \mathbb{Z}^d} \tilde{v}_2(x) \cos p \cdot x \right)^2$$

$$+ \left( \sum_{x \in \mathbb{Z}^d} \tilde{v}_1(x) \sin p \cdot x \right)^2 + \left( \sum_{x \in \mathbb{Z}^d} \tilde{v}_2(x) \sin p \cdot x \right)^2.$$
Expansion of eigenvalues of the perturbed discrete bilaplacian

and \( \hat{v} = \hat{v}_1 + i \hat{v}_2 \) for some \( \hat{v}_1, \hat{v}_2 : \mathbb{Z}^d \to \mathbb{R} \). By Lemma 3.1, the unique eigenvalue \( e(\mu) \) of \( H_\mu \) solves

\[
1 - \mu \int_{\mathbb{T}^d} \frac{f(p)dp}{\varepsilon(p) - e(\mu)} = 0.
\]

Since both \( p \in \mathbb{T}^d \mapsto f(p) \) and \( p \in \mathbb{T}^d \mapsto f(\tilde{\pi} + p) \) are even analytic functions, we can still apply Propositions A.1 and A.2 to find the expansions of \( z \mapsto \int_{\mathbb{T}^d} \frac{f(p)dp}{\varepsilon(p) - z} \) and thus, repeating the same arguments of the proofs of Theorems 2.4 and 2.5 one can obtain the corresponding expansions of \( e(\mu) \).

**Remark 3.4** When

\[
|\hat{v}(x)| = O(|x|^{2n_0 + d + 1}) \quad \text{as } |x| \to \infty
\]

for some \( n_0 \geq 1 \), in view of Remark A.3, we need to solve equation (3.4) with respect to \( \mu \) using only that left-hand side is an asymptotic sum (not a convergent series). This still can be done using appropriate modification of the Implicit Function Theorem for differentiable functions. As a result, we obtain only (Taylor-type) asymptotics of \( e(\mu) \).

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**Appendix A. Asymptotics of some integrals**

In this section we study the behaviour of the integral

\[
I_f(z) := \int_{\mathbb{T}^d} \frac{f(q)dq}{\varepsilon(q) - z}, \quad z \in \mathbb{C} \setminus [0, 4d^2], \tag{A.1}
\]

as \( z \to 0 \) and \( z \to 4d^2 \), where \( f : \mathbb{T}^d \to \mathbb{R} \) is a real-analytic even function on \( \mathbb{T}^d \). Further we denote by \( W_r(\xi) \subset \mathbb{C} \) the complex disc of radius \( r > 0 \) centered at \( \xi \in \mathbb{C} \).

**Proposition A.1** Let \( f : \mathbb{T}^d \to \mathbb{R} \) be a real-analytic even function such that

\[
f(0) = D^2 f(0) = \ldots = D^{2n_0 - 2} f(0) = 0, \quad D^{2n_0}(0) \neq 0 \tag{A.2}
\]
for some $n_o \geq 0$. Then:

- $f$ is continuous at $0$ if and only if $2n_o + d \geq 5$;
- $f$ is continuously differentiable at $0$ if and only if $2n + d \geq 9$, in this case,

$$l'_f(0) := \int_{T^d} \frac{f(q) dq}{(e(q))^2} = \lim_{x \to 0} \int_{T^d} \frac{f(q) dq}{(e(q) - z)^2}.$$ 

Moreover, for any $z \in (-\frac{1}{64}, 0)$:

(a) if $d$ is odd, then

$$l_f(z) = \begin{cases} 
\frac{\pi}{4(-z)^{3/4}} \left( c_f + \sum_{n \geq 1} a_n^d (-z)^{n/4} \right), & 2n_o + d = 1, \\
\frac{\pi}{8(-z)^{1/4}} \left( c_f + \sum_{n \geq 1} a_n^d (-z)^{n/4} \right), & 2n_o + d = 3, \\
l_f(z) = \begin{cases} 
n_f(0) - \frac{\pi(-z)^{1/4}}{8} \left( c_f + \sum_{n \geq 1} a_n^d (-z)^{n/4} \right), & 2n_o + d = 5, \quad (A.3) \\
n_f(0) - \frac{\pi(-z)^{3/4}}{8} \left( c_f + \sum_{n \geq 1} a_n^d (-z)^{n/4} \right), & 2n_o + d = 7, \\
n_f(0) + z \left( l'_f(0) + \sum_{n \geq 1} a_n^d (-z)^{n/4} \right), & 2n_o + d \geq 9, 
\end{cases}
\end{cases}$$

(b) if $d$ is even, then

$$l_f(z) = \begin{cases} 
\frac{\pi}{8(-z)^{1/2}} \left( c_f + \sum_{n \geq 1} b_n^d (-z)^{n/2} \right) - \frac{1}{16} \ln(-z) \sum_{n \geq 0} c_n^d z^n, & 2n_o + d = 2, \\
- \frac{1}{16} \ln(-z) \left( c_f + \sum_{n \geq 1} c_n^d z^n \right) + \sum_{n \geq 0} b_n^d (-z)^{n/2}, & 2n_o + d = 4, \\
l_f(z) = \begin{cases} 
n_f(0) - \frac{\pi(-z)^{1/2}}{8} \left( c_f + \sum_{n \geq 1} b_n^d (-z)^{n/2} \right) + z \ln(-z) \sum_{n \geq 0} c_n^d z^n, & 2n_o + d = 6, \\
n_f(0) - \frac{\pi(-z)^{1/2}}{8} \left( c_f + \sum_{n \geq 1} c_n^d z^n \right) + \sum_{n \geq 2} b_n^d (-z)^{n/2}, & 2n_o + d = 8, \\
n_f(0) + z \left( l'_f(0) + \sum_{n \geq 1} b_n^d (-z)^{n/2} \right) + z^2 \ln(-z) \sum_{n \geq 0} c_n^d z^n, & 2n_o + d \geq 10, 
\end{cases}
\end{cases}$$

(A.4)

where $\{c_n^d\}$, $\{b_n^d\}$ and $\{c_n^d\}$ are some real coefficients,

$$c_f := \frac{2^{2n_o+d}}{(2n_o)!} \int_{S^{d-1}} D^{2n_o} f(0)[w, \ldots, w] d\nu^{d-1};$$

(A.5)

and all series in (A.3) and (A.4) converge absolutely for $z \in W_{1/64}(0) \subset \mathbb{C}$. 

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Proof: Given $\gamma \in (0, \frac{1}{\sqrt{2}}]$, let $\varphi : B_{\gamma}(0) \subset \mathbb{R}^d \to \varphi(B_{\gamma}(0)) \subset \mathbb{R}^d$ be the smooth diffeomorphism

$$\varphi_i(y) = 2 \arcsin y_i, \quad i = 1, \ldots, d.$$  

Note that

$$\epsilon(\varphi(y)) = \left( \sum_{i=1}^{d} (1 - \cos(2 \arcsin(y_i))) \right)^2 = 4 \left( \sum_{i=1}^{d} y_i^2 \right)^2 = 4y^4, \quad (A.6)$$

due to

$$\epsilon(q) \geq 4\gamma^4 \quad \text{for any } q \in T^d \setminus \varphi(B_{\gamma}). \quad (A.7)$$

We rewrite $l_f(z)$ as

$$l_f(z) := \int_{\varphi(B_{\gamma}(0))} \frac{f(q)dq}{\epsilon(q) - z} + \int_{T^d \setminus \varphi(B_{\gamma}(0))} \frac{f(q)dq}{\epsilon(q) - z} = l^*(z) + l^{**}(z).$$

By virtue of (A.7),

$$l^{**}(z) = \int_{T^d \setminus \varphi(B_{\gamma}(0))} \frac{f(q)dq}{\epsilon(q)} \left(1 - \frac{z}{\epsilon(q)}\right)^{-1} dq = \sum_{n \geq 0} z^n \int_{T^d \setminus \varphi(B_{\gamma}(0))} \frac{f(q)dq}{(\epsilon(q))^{n+1}}, \quad (A.8)$$

i.e. $l^{**}(\cdot)$ is analytic in $W_{2\gamma^4}(0)$. In $l^*$ making the change of variables $q = \varphi(y)$ and using (A.6) we get

$$l^*(z) = \int_{B_{\gamma}(0)} \frac{f(\varphi(y)) J(\varphi(y)) dy}{4y^4 - z}, \quad (A.9)$$

where $y^4 := (y^2)^2$ with $y^2 := \sum_{i=1}^{d} y_i^2$, and

$$J(\varphi(y)) = \prod_{i=1}^{d} \frac{2}{\sqrt{1 - y_i^2}} \quad (A.10)$$

is the Jacobian of $\varphi$. Since $f$ is an even analytic function satisfying (A.2), even each coordinate, from the Taylor series for $f$ it follows that

$$f(p) = \sum_{n \geq n_0} \frac{1}{(2n)!} D^{2n} f(0)[p, \ldots, p], \quad (A.11)$$
and by the analyticity of \( f \) in \( B_{\pi}(0) \subset \mathbb{R}^d \), the series converges absolutely in \( p \in B_{\pi}(0) \). By the definition of \( \varphi \), \( \varphi(rw) \subset B_{\pi}(0) \) for any \( r \in (0, \gamma) \) and \( w = (w_1, \ldots, w_d) \in S^{d-1} \), where \( S^{d-1} \) is the unit sphere in \( \mathbb{R}^d \). Then letting \( p = \varphi(rw) \) and using the Taylor series

\[
\varphi_i(rw) = 2rw_i + \frac{r^3w_i^3}{3} + \sum_{n \geq 3} \tilde{c}_n r^{2n-1} w_i^{2n-1}
\]

of \( 2 \arcsin(\cdot) \), which is absolutely convergent for \( |rw_i| < 1 \), from (A.11) we obtain

\[
f(\varphi(rw)) = \sum_{n \geq n_o} \tilde{C}_n(w) r^{2n}, \tag{A.12}
\]

where \( \tilde{C}_n : S^{d-1} \to \mathbb{R} \) is a homogeneous polynomial of \( w \in S^{d-1} \) of degree \( 2n \), and

\[
\tilde{C}_{n_o}(w) = \frac{2^{2n_o}}{(2n_o)!} D^{2n_o} f(0) \begin{bmatrix} w, \ldots, w \end{bmatrix}\text{2n_o-times}
\]

Next consider \( J(\varphi(y)) \). Inserting the Taylor series of \( (1 - t)^{-1/2} \) into (A.10) we obtain

\[
J(\varphi(rw)) = 2^d \left( 1 + \sum_{n \geq 1} \hat{C}_n(w) r^{2n} \right), \tag{A.13}
\]

where \( \hat{C}_n : S^{d-1} \to \mathbb{R} \) is a homogeneous symmetric polynomial of \( w \in S^{d-1} \) of degree \( 2n \), and the series converges absolutely.

Now passing to polar coordinates by \( y = rw \) in (A.9) and using (A.12) and (A.13) as well as the absolute convergence of the series we get

\[
\iota^*(z) = 2^d \int_0^\gamma \int_{S^{d-1}} r^{d-1} \left( \sum_{n \geq n_o} \int_{S^{d-1}} C_n(w) r^{2n} \right) d\mathcal{H}^{d-1} dr = \sum_{n \geq n_o} \tilde{c}_n \int_0^\gamma \frac{r^{2n+d-1} dr}{4r^4 - z}, \tag{A.14}
\]

where \( C_n : S^{d-1} \to \mathbb{R} \) is a homogeneous polynomial of \( w \in S^{d-1} \) of degree \( 2n \) and

\[
\tilde{c}_n := 2^d \int_{S^{d-1}} C_n(w) d\mathcal{H}^{d-1}.
\]

Note that \( \tilde{c}_{n_o} = c_f \), where \( c_f \) is given by (A.5) and the last series in (A.14) uniformly converges in any compact subset of \( \mathbb{C} \setminus [0, 4] \) since \( \iota^* \) and

\[
z \in \mathbb{C} \setminus [0, 4] \mapsto j_{2n+d-1}(z) := \int_0^\gamma \frac{r^{2n+d-1} dr}{4r^4 - z}
\]
are analytic functions in $\mathbb{C} \setminus [0, 4]$ and all series in (A.14) converge pointwise\(^1\). Note that for any $m \geq 0$, there exist $c_m \in \mathbb{R}$ and an analytic function $f_m$ in the ball $W_{\gamma^4}(0) \subset \mathbb{C}$ such that for any $z \in (-\gamma^4, 0)$,

\[
j_m(z) = z^n j^0_l(z) + c_m + z^\nu f_m((-z)^{1/2}),
\]

where $n := \lceil \frac{m}{4} \rceil$, $l := m - 4n \in \{0, 1, 2, 3\}$, $\nu = \frac{1}{2}$ for $m = 0, 2$ and $\nu = 1$ for $m = 1, 3$ or $m \geq 4$, and

\[
j^0_l(z) := \begin{cases} 
\frac{\pi}{4} (-z)^{-3/4} & \text{if } l = 0, \\
\frac{\pi}{8} (-z)^{-1/2} & \text{if } l = 1, \\
\frac{\pi}{4} (-z)^{-1/4} & \text{if } l = 2, \\
-\frac{1}{16} \ln(-z) & \text{if } l = 3.
\end{cases}
\]

Inserting (A.15) into (A.14) we obtain

\[
\Gamma^*(z) = \sum_{n \geq n_0} \hat{c}_n \left( z^{\lfloor \frac{2n+d-1}{4} \rfloor} j^0_{2n+d-1-\lfloor \frac{2n+d-1}{4} \rfloor}(z) 
+ c_{2n+d-1} + \hat{c}_n (-z)^\nu f_{2n+d-1}((-z)^{1/2}) \right),
\]

where $\{c_{2n+d-1}\} \subset \mathbb{R}$ and $\{f_{2n+d-1}\}$ is a sequence of analytic functions in $W_{\gamma^4}(0)$ and

\[
\nu_n := \begin{cases} 
\frac{1}{2}, & 2n + d = 1, 3, \\
1, & \text{otherwise}.
\end{cases}
\]

Since (A.14) converges locally uniformly in $\mathbb{C} \setminus [0, 4]$, $C := \sum_{n \geq n_0} \hat{c}_n c_{2n+d-1}$ is finite and

\[
\sum_{n \geq n_0} \hat{c}_n (-z)^\nu f_{2n+d-1}((-z)^{1/2}) = (-z)^\nu g((-z)^{1/2}),
\]

where $g$ is analytic in $W_{\gamma^2}(0)$ and $\nu = \frac{1}{2}$ for $2n_0 + d = 1, 3$ and $\nu = 1$ otherwise. Hence,

\[
\Gamma^*(z) = C + (-z)^\nu g((-z)^{1/2}) + \sum_{n \geq n_0} \hat{c}_n z^{\lfloor \frac{2n+d-1}{4} \rfloor} j^0_{2n+d-1-\lfloor \frac{2n+d-1}{4} \rfloor}(z), \quad (A.16)
\]

---

\(^1\) If $\{h_n\}$ is an equi-bounded sequence of analytic functions in a connected open set $\Omega \subset \mathbb{C}$ converging pointwise to a function $h : \Omega \rightarrow \mathbb{C}$, then $h$ is analytic and $h_n$ converges uniformly to $h$ in compact subsets of $\Omega$. 

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If $0 \leq 2n_o + d - 1 \leq 3$, then by (A.16),

$$l^*(z) = C + (-z)^{\nu} g((-z)^{1/2}) + \hat{c}_{n_o} j^o_{2n_o+d-1}(z) + \sum_{n \geq n_o + 1} \hat{c}_n z^{[\frac{2n+d-1}{4}]} j^o_{2n+d-1-4\lfloor \frac{2n+d-1}{4} \rfloor}(z).$$  (A.17)

In view of (A.8) and the definition of $j^o_l$, from (A.17) we obtain the expansions (A.3) and (A.4) of $l_f$ for $2n_o + d \leq 4$. In particular, since $[\frac{2n+d-1}{4}] \geq 1$ for any $n \geq n_o + 1$, letting $z \to 0$ in (A.17) we get

$$\lim_{z \to 0} l^*(z) = +\infty.$$  (A.18)

If $2n_o + d - 1 \geq 4$, then $[\frac{2n+d-1}{4}] \geq 1$ for any $n \geq n_o$. Therefore, by (A.16), $l^*(0) := \lim_{z \to 0} l^*(z)$ exists and equals to $C$. In particular, for $2n_o + d - 1 \leq 7$, one has

$$l^*(z) = l^*(0) - zg((-z)^{1/2}) + \hat{c}_{n_o} z^{j^o_{2n_o+d-1}}(z) + \sum_{n \geq n_o + 1} \hat{c}_n z^{[\frac{2n+d-1}{4}]} j^o_{2n+d-1-4\lfloor \frac{2n+d-1}{4} \rfloor}(z),$$  (A.19)

from which and (A.8) we deduce the expansions (A.3) and (A.4) of $l_f$ for $5 \leq 2n_o+d \leq 8$. In particular, by virtue of (A.18) and analyticity of $l^{**}$ at $z = 0$, $l_f$ is continuous at 0 if and only if $2n_o + d \geq 5$. Notice also by (A.19)

$$\lim_{z \to 0} \frac{l^*(z) - l^*(0)}{z} = +\infty,$$  (A.20)

i.e. $l^*$ (and hence $l_f$) is not differentiable at $z = 0$.

Finally, if $2n_o + d - 1 \geq 8$, then $[\frac{2n+d-1}{4}] \geq 2$ for any $n \geq n_o$. Therefore, by (A.16) there exists

$$l^*(0) := \lim_{z \to 0} \frac{l^*(z) - l^*(0)}{z} = -g(0).$$

Now using the Taylor series of $g$ at 0 we get

$$z g((-z)^{1/2}) = l^{**}(0) z + z \sum_{n \geq 1} \frac{g^{(n)}(0)}{n!} (-z)^{n/2}.$$  

Inserting this in (A.16), using the definition of $j^o_l$ and the analyticity of $l^{**}$ we get the expansions (A.3) and (A.4) of $l_f$ for $2n_o + d \geq 9$.

By (A.18) and (A.20), $l_f$ is continuously differentiable at 0 if and only if $2n_o + d \geq 9$.

Now the choice $\gamma = \frac{1}{\sqrt{2}}$ completes the proof. \hfill \Box
Proposition A.2 Let $f: \mathbb{T}^d \to \mathbb{R}$ be a real-analytic function such that $q \in \mathbb{T}^d \mapsto f(\vec{x} + q)$ is even and

$$f(\vec{x}) = D^2 f(\vec{x}) = \ldots = D^{2n_o - 2} f(\vec{x}) = 0, \quad D^{2n_o}(\vec{x}) \neq 0$$

for some $n_o \in \mathbb{N}_0$. Then:

- $\mathcal{I}_f$ is continuous at $z = 4d^2$ if and only if for $2n_o + d \geq 3$,
- $\mathcal{I}_f$ is continuously differentiable at $z = 4d^2$ if and only if for $2n_o + d \geq 5$, in this case

$$l_f'(4d^2) := \int_{\mathbb{T}^d} \frac{f(q) dq}{(\varepsilon(q) - 4d^2)^2} = \lim_{z \searrow 4d^2} \int_{\mathbb{T}^d} \frac{f(q) dq}{(\varepsilon(q) - z)^2}$$

exists.

Moreover, if $z - 4d^2 \in (0, \frac{1}{16})$, $l_f(z)$ is represented as:

(a) if $d$ is odd, then

$$l_f(z) = \begin{cases} 
- \frac{\pi C_f}{\sqrt{z - 4d^2}} + \sum_{k \geq 0} a_k^d (z - 4d^2)^k/2, & 2n_o + d = 1, \\
l_f(4d^2) + \pi C_f \sqrt{z - 4d^2} + \sum_{k \geq 2} a_k^d (z - 4d^2)^k/2, & 2n_o + d = 3, \\
l_f(4d^2) + l_f'(4d^2) (z - 4d^2) + \sum_{k \geq 3} a_k^d (z - 4d^2)^k/2, & 2n_o + d \geq 5;
\end{cases}$$

(A.21)

(b) if $d$ is even, then

$$l_f(z) = \begin{cases} 
C_f \ln \alpha + \ln \alpha \sum_{k \geq 1} b_k^d \alpha_k + \sum_{k \geq 0} c_k^d \alpha_k, & 2n_o + d = 2, \\
l_f(4d^2) - C_f \alpha \ln \alpha + \ln \alpha \sum_{k \geq 2} b_k^d \alpha_k + \sum_{k \geq 1} c_k^d \alpha_k, & 2n_o + d = 4, \\
l_f(4d^2) + l_f'(4d^2) \alpha + \ln \alpha \sum_{k \geq 2} b_k^d \alpha_k + \sum_{k \geq 2} c_k^d \alpha_k, & 2n_o + d \geq 6;
\end{cases}$$

(A.22)

where $\alpha := z - 4d^2$, $\{a_k^d\}, \{b_k^d\}, \{c_k^d\} \subset \mathbb{R}$ and

$$C_f := \frac{2^{2n_o + d - 1}}{(8d)^{n_o + d/2} (2n_o)!} \int_{\mathbb{S}^{d-1}} D^{2n_o} f(\vec{x})[w, \ldots, w] d\mathcal{H}^{d-1}.$$

Proof Since $4d^2 - \varepsilon(\cdot)$ has a unique non-degenerate minimum at $\vec{x}$, the asymptotics of $l_f(z)$ as $z \searrow 4d^2$ can be done along the lines of, for instance, [22, Lemma 4.1], hence, we skip the proof. \qed
Remark A.3 When

\[ |\hat{v}(x)| = O(|x|^{2n_0+d+1}) \quad \text{as } |x| \rightarrow \infty \]

for some \( n_0 \geq 1 \), one has \( v \in C^{2n_0}(\mathbb{R}^d) \). In this case the Taylor series of \( f \) becomes only asymptotics of order \( 2n_0 - 1 \) and thus, instead of expansions (A.3)-(A.4) and (A.21)-(A.22) of \( I_f \) one has only asymptotics up to order \( 2n_0 - 1 \).

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