Analytic Study of Rotating Black-Hole Quasinormal Modes

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(Dated: February 1, 2008)

A Bohr-Sommerfeld equation is derived for the highly-damped quasinormal mode frequencies \( \omega(n \gg 1) \) of rotating black holes. It may be written as

\[
2 \int_C (p_r + i p_\theta) \, dr = (n + 1/2) \hbar,
\]

where \( p_r \) is the canonical momentum conjugate to the radial coordinate \( r \) along a null geodesic of energy \( \hbar \omega \) and angular momentum \( \hbar m \), and the contour \( C \) connects two complex turning points of \( p_r \). The solutions are

\[
\omega(n) = -m \Delta - i(\phi + \bar{\phi}) + m \phi, \quad \Delta, \phi > 0
\]

are functions of the black-hole parameters alone. Some physical implications are discussed.

We analytically derive \( \bar{\omega} \) for rotating black holes in a method similar to the spherical black-hole analysis of \( \bar{\omega} \), by analytically continuing the relevant solution of Teukolsky’s radial equation \( \bar{\omega} \) to the complex plane, and matching the monodromy of the wave-function along two different contours. Our analytical results confirm and generalize the numerical results of \( \bar{\omega} \), as well as admit a physical interpretation. In this Rapid Communication we outline the derivation and present the main results, deferring a more elaborate description of the analysis to a future, detailed paper.

Teukolsky’s equation. — Linear, massless field perturbations of a neutral, rotating black hole are described by Teukolsky’s equation. For a scalar field, this equation can be generalized to accommodate electrically charged black holes \( \bar{\omega} \); in what follows, \( Q \neq 0 \) is understood to apply only to such fields. The wave-function is separated into two ordinary differential equations using

\[
\psi(x) = e^{i(m \phi - \omega t)} S_l(m) R_l(m), \quad x = (t, r, \theta, \phi)
\]

are Boyer-Lindquist coordinates. This yields radial and angular equations coupled by a separation constant \( A \), where \( A_l(m \phi - \omega t) = i A_1 \omega + (A_0 + m^2 - Q^2 \omega^2) \), with \( A_1 \in \mathbb{R} \). The radial equation then becomes

\[
\left[ \frac{\partial^2}{\partial r^2} + \frac{q_0(r) \omega^2 + q_1(r) \omega + q_2(r)}{\Delta^2} \right] \tilde{R}_{l(m)} = 0,
\]

where

\[
\tilde{R}_{l(m)} \equiv \Delta^{(s+1)/2} R_{l(m)}, \quad \Delta \equiv r^2 - 2Mr + a^2 + Q^2, \quad a \equiv J/M, \quad \text{and we have defined}
\]

\[
q_0 \equiv (r^2 + a^2)^2 - a^2 \Delta, \quad q_1 \equiv -2am(2Mr - Q^2) - iaA_1 \Delta + 2is[r(\Delta + Q^2) - M(r^2 - a^2)],
\]

and

\[
q_2 \equiv -m^2(\Delta - a^2) - \Delta(s + A_0) + M^2 - a^2 - Q^2 - s(M - r)[2iam + s(M - r)].
\]
The spin-weight parameter $s$ specifies the equation to gravitational ($s = -2$), electromagnetic ($s = -1$), scalar ($s = 0$), or two-component neutrino ($s = -1/2$) fields. For physical boundary conditions of purely outgoing waves at both spatial infinity and the event horizon (i.e., crossing the horizon into the black hole), Eq. (11) admits solutions only for a discrete set of QNM frequencies $\omega(n)$, where $\omega_l < 0$ (time decay) diverges as $n \to \infty$.

**Analysis.**—By defining $z \equiv \int V(r') dr'$, with $V \equiv \Delta^{-1}(q_0 + \omega^{-1} q_1)^{1/2}$, Eq. (11) becomes

$$
\left( -\frac{\partial^2}{\partial z^2} + V_1 - \omega^2 \right) \hat{R} = 0 ,
$$

where $\hat{R} = V^{1/2} \hat{R}$ and $V_1 = V''/(2V^3) - 3(V')^2/(4V^4) - q_2/(V\Delta)^2$. A nonconventional tortoise coordinate $z$ was defined such that the effective potential $V_1 = O(\omega^0)$. The boundary condition at the horizon becomes $\hat{R}(r \to r_+) \sim \exp(-i\omega z) \propto (r - r_+)^{-i\omega \sigma^+}$, where

$$
\omega \sigma^+ = \omega \text{Res}_{r=r_+} (V) = (\omega - m\Omega) - \frac{is}{2} + O(|\omega|^{-1}) .
$$

Here, $\Omega \equiv a/(r_+^2 + a^2)$ is the angular velocity of the event horizon, $\beta \equiv h/(4\pi T) = (r_+^2 + a^2)/(r_+ - r_-)$, $T$ is the Bekenstein-Hawking temperature, $r_\pm = M \pm (M^2 - a^2 - Q^2)^{1/2}$ are the outer and inner horizon radii, and the tilde in $\hat{w}$ is omitted unless necessary (henceforth). $\hat{R}(r \simeq r_+)$ is multivalued, such that a clockwise rotation around $r_+$ multiplies $\hat{R}$ by a factor $\Phi_1 = \exp(-2\pi i\omega \sigma^+)$.

Let $r_1$ and $r_2$ be the two complex conjugate roots of $q_0(r)$ lying in the fourth and in the first quadrants, respectively. Denote $t_1$ and $t_2$ as the turning points of $V$ [defined by $V(r = t_i) = 0$] which lie near (a factor $\sim |\omega|^{-1}$ away from) $r_1$ and $r_2$, respectively (see Figure 1). The monodromy $\Phi_2$ of $\hat{R}$ along a clockwise contour $C$, which passes through $t_1$ and $t_2$ and encloses $r_+$, is used to determine $\omega$ by demanding $\Phi_1 = \Phi_2$, as in [3]. A reader uninterested in details of the derivation may skip directly to the result, Eq. (3).

Near the turning points, $(z - z_i) \propto (r - t_i)^{-3/2}$, where $z_i \equiv z(t_i)$. Therefore three anti-Stokes lines, defined by $\Im(i\omega z) = 0$, emanate from $t_i$. Two anti-Stokes lines connect $t_1$ to $t_2$; one (denoted $l_2$) crosses the real axis between $r_-$ and $r_+$, while the other crosses it at $r > r_+$. The third anti-Stokes line ($l_1$) emanating from $t_1$ extends to $P_1$, where $|P_1| \to \infty$ and $\text{arg}(P_1) = -\pi/2$. A similar line ($l_3$) runs from $t_2$ to $P_2$, with $|P_2| \to \infty$ and $\text{arg}(P_2) = +\pi/2$. A Stokes line, defined by $\Im(i\omega z) = 0$, emanates between every two anti-Stokes lines of $t_i$. Let $C$ be the closed, clockwise contour running from $P_1$ to $P_2$ along the anti-Stokes lines $l_1$, $l_2$, and $l_3$, and closing back on $P_1$ through the large semicircle $l_{\infty}$, where $|r| \to \infty$ and $-\pi/2 < \text{arg}(r) < \pi/2$. The turning points $t_1$ and $t_2$ are excluded from $C$ by partially rotating around them counterclockwise. Figure [1] illustrates these features in the $r$-plane.

Along anti-Stokes lines, the WKB approximation $\hat{R}(z, z_0) \simeq c_+ \exp[+i\omega(z - z_0)] + c_- \exp[-i\omega(z - z_0)]$ holds. Off the lines, this may also be written as $c_df_d + c_sf_s$, where $f_d$ is exponentially large (dominant) and $f_s$ is exponentially small (subdominant). For $\omega_R < 0$, the boundary condition at spatial infinity can be analytically continued to $P_1$, such that $\hat{R}(P_1) \sim \exp(-i\omega z)$, i.e., $\{c_+, c_-; z_0\} = \{1, 0; z_1\}$ up to a multiplicative factor. This remains invariant along $C_1$ till the vicinity of $t_1$, so we denote $\hat{R}(l_1) = \{1, 0; z_1\}$. When an anti-Stokes line is crossed, the dominant and subdominant parts exchange roles. When a Stokes line is crossed while circling a regular turning point, $c_df_d + c_sf_s$ becomes $c_df_d + (c_s \pm ic_d)f_s$, where the positive (negative) sign corresponds to a counterclockwise (clockwise) rotation. This so-called Stokes phenomenon [12] implies that after rotating around $t_1$ from $t_1$ to $t_2$, thus crossing two Stokes lines and the anti-Stokes line between them, $\hat{R}(l_2) = \{0, i; z_1\} = \{0, i\exp(-i\omega \delta); z_2\}$, where

$$
\delta \equiv z_2 - z_1 = \int_{l_2} V \, dr .
$$

Similarly, after rotating from $l_2$ to $l_3$, $\hat{R}(l_3) = \{- \exp(-2i\omega \delta), 0; z_1\}$. Finally, along $l_{\infty}$ the coefficient of the dominant part of the solution $c_+$ remains invariant till $P_1$. In addition to the above changes in $c_+$, it accumulates a phase $e^{+2i\pi \omega \sigma^+}$ due to the (only) singularity at $r_+$ enclosed by $C$. Thus, the total phase accumulated by $\hat{R}$ along $C$ is $\Phi_2 = -\exp(-2i\omega \delta + 2\pi i \omega \sigma^+)$. For $\omega_R > 0$, the boundary condition at spatial infinity is continued to $P_2$ and the two contours are chosen counterclockwise, such that the resulting equation $\Phi_1 = \Phi_2$ is unchanged.

The constraint $\Phi_1 = \Phi_2$ finally yields the highly-
damped QNM equation \[ e^{-2\pi\sigma_+} = -e^{-2\pi\delta + 2\pi\sigma_+}. \] 

Explicitly, to order \( O(\omega^{-1}) \) this may be written as

\[ 4\pi\beta (\omega - m\Omega) - 2\pi is = 2i\omega \int_{C_{t,i}} V dr - \pi i(2n+1), \]  

or in a more compact form as

\[ 2\omega \int_{C_{t,o}} V dr = 2\pi \left( n + \frac{1}{2} \right), \]

where \( n \in \mathbb{Z} \). Here, \( C_{t,i} (C_{t,o}) \) is a complex-plane contour running from \( t_1 \) to \( t_2 \), crossing the real axis in (out) of the event horizon, at some point \( r_- < r < r_+ (r > r_+) \).

Before solving for \( \bar{\omega} \), note that in the highly-damped limit the real and the imaginary contributions to the integrals of Eqs. (7)-(10) are easily separated. For example, the real part of Eq. (9) may be written in the form \[ 4\pi\beta(\omega_R - m\Omega) = \Re \left( 2i \int_{C_{t,i}} \omega V_R dr \right), \]  

where the complex potential \( V_R \) is given by

\[ (\omega V_R)^2 = \frac{q_0\omega^2 - 2am(2Mr - Q^2)\omega - m^2(\Delta - a^2)}{\Delta^2}. \]

The last term \( \propto \omega^0 \), taken from \( q_2 \), was added to \( V_R \) for future use and has no effect in the highly-damped limit. An equation analogous to Eq. (11) is found for the imaginary part \( 4\pi\beta\omega_I - 2\pi s \).

**QNM frequencies.** — In order to obtain a closed-form expression for \( \omega \), expand \( 2\delta - 4\pi\sigma + \delta_0 + (m\delta_m + is\delta_s + iA_1\delta_A)\omega^{-1} + O(\omega^{-2}) \). Here

\[ \delta_j = 2i \int_{C_{r,o}} V_j dr, \]

with \( V_0 = q_0^{1/2}\Delta^{-1/2}, V_m = -a(2Mr - Q^2)\Delta^{-1/2}q_0^{-1/2}, V_s = [r(\Delta + Q^2) - M(r^2 - a^2)]\Delta^{-1/2}q_0^{-1/2}, \) and \( V_A = -q_0^{-1/2}a/2 \). The integration contour \( C_{r,o} \) runs from \( r_1 \) to \( r_2 \), crossing the real axis outside the event horizon. Since \( r_2 = r_1^2 \), \( \{\delta_0, \delta_s, \delta_A, \delta_m\} \) are all real. Analytic expressions for these \( \delta_j \) functions are readily found in terms of elliptic integrals.

With the above definitions we finally obtain

\[ \omega = -m\bar{\omega} - i(\hat{\omega} + \bar{\omega}) \]

where \( \bar{\omega} = \delta_m/\delta_0, \hat{\omega} = 2\pi/\delta_0, \) and \( \hat{\phi} = (s\delta_s + A_1\delta_A - \pi)/\delta_0 \). As shown in Figures 2 and 3, these analytic results agree with the numerical calculations of Fig. 2.

Eq. (14) yields one branch of solutions \( \omega_m(n) \) in the asymptotic limit. Interestingly, in the low-n regime (and in spherically-symmetric black holes) two branches of solutions are identified, for given field and black-hole parameters [13].

The asymptotic QNMs are not continuous at \( a = 0 \) [18]. For \( Q = 0 \), \( \bar{\omega}(a \to 0) \propto a^{1/3} \to 0 \), whereas \( \omega_R(a = 0) = (8\pi M)^{-1}\ln 3 \). Such discontinuous behavior sometimes occurs in the Schwarzschild limit, for example in the inner structure of the black hole [12]. Note that the level spacing \( \delta \) does continuously asymptote to the Schwarzschild result \( \Delta\omega = 2\pi T/\hbar \) as \( \{a, Q\} \to 0 \).

**Discussion.** — We have analytically studied the highly-damped QNM frequencies \( \omega(n) \) of a rotating black hole. A Bohr-Sommerfeld-like equation for \( \omega \) was derived [Eqs. (4)-(10)], analytically solved [Eq. (14)], and shown to agree and generalize previous numerical results [8] (Figures 2 and 3).

It is instructive to quantize the linear field perturbations described by the QNM [19]. A quantum of complex energy \( \hbar\omega(n) \) and angular momentum \( \hbar\sigma(n) \) may thus be associated with the highly-damped QNM frequency \( \omega_m(n) \). Multiplying Eq. (14) by \( \hbar \) yields

\[ 2 \int_{C_{t,o}} p dr = \left( n + \frac{1}{2} \right) \hbar, \]

where \( p = \hbar\omega V \). This equation strongly resembles the Bohr-Sommerfeld quantization rule \( \oint p dq = (n + 1/2)\hbar \).
where \( p \) is the canonical momentum conjugate to some coordinate \( q \), and the integration is carried out along a closed orbit. To elucidate the connection, recall that the covariant radial momentum \( p_r \) for geodesic motion of a neutral, massless particle of energy \( E \) and angular momentum \( p_\phi \), is given by

\[
(p_r \Delta)^2 = [(r^2 + a^2)^2 - a^2 \Delta^2]E^2 - 2a(2Mr - Q^2)Ep_\phi - (\Delta - a^2)p_\phi^2 - QC\Delta ,
\]

(16)

where \( QC \) is Carter’s (fourth) constant of motion \[15\]. Comparing this with Eq. (12) indicates that \( V_R \approx p_r \), provided that \( E = \hbar \omega \), \( p_\phi = \hbar m \), and \( QC = O(E^0) \). Hence, up to an \( O(\omega^0) \) term which leads to an imaginary offset in \( \omega(n) \), the integrand in Eq. (15) truly is of the form \( pdq \) for the above QNM quantization. The implied physical content of Eq. (15) suggests that the full QNM spectrum may be determined by a generalized Bohr-Sommerfeld equation, which reduces to Eq. (15) as \( \omega \to -\infty \). The general form of \( p \) is not uniquely determined by our highly-damped analysis. Up to \( O(|\omega|^{-1}) \) corrections, we may write

\[
p = p_r + i\hbar sV_s + i\hbar A_1 V_A .
\]

(17)

The preceding discussion implies that Eq. (15) can be interpreted as a complex version of the Bohr-Sommerfeld quantization rule. This rule was used in (the old) quantum mechanics to determine the quantum-mechanically allowed trajectories, as well as the quantized values of the associated constants of motion. Realizing the full meaning of Eq. (15) may well require a quantum theory of gravity. Conversely, this equation can possibly be used to constrain and shed light on the theory.

The quantum manifestation of a QNM may be complicated. A simple example is motivated by the outgoing boundary conditions of the QNMs and the symmetry of their frequencies \( \omega_{-m} = -\omega_m \) \[13\], evident in Eq. (14). These suggest that a quantum pair of opposite angular momentum may fundamentally correspond to a QNM; a positive energy quantum escaping to infinity and a negative energy quantum falling into the black hole, in resemblance of Hawking’s semiclassical radiation. Under such circumstances, a quantum process corresponding to a QNM changes the black-hole mass by \( \Delta M = \hbar \omega_R \) and its angular momentum by \( \Delta J = \hbar m \). For such small changes in the black-hole parameters, the corresponding change in its entropy, \( \Delta S = T^{-1}(\Delta M - \Omega \Delta J) \), is given directly by Eq. (11), which we may now write as

\[
\hbar \Delta S = \Delta A/4 = \Re \left( 2i \int_{C_{t,i}} p_r \, dr \right) .
\]

(18)

This is another indication of the adiabatic invariance of the area/entropy \[3\].

We thank A. Neitzke, J. Maldacena, P. Goldreich and J. Bekenstein for helpful discussions. U.K. is supported by the NSF (grant PHY-0503584).

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[16] Eq. (3) can also be derived as in Ref. \[3\], by solving for \( \hat R \) near the turning points where \( V_1 \simeq -(5/36)(z-z_1)^-2 \).
[17] Using \( \int_{t_1}^{t_2} i|f|dr \in R \). The integration endpoints \( \{t_1\} \) and \( \{r_1\} \) may be used interchangeably, as \( q_0(t_1) = 0 \) ensures that the resulting \( O(|\omega|^{-1}) \) correction terms vanish.
[18] The analysis is valid only for \( 0 < a^2 < M^2 - Q^2 \). It does not apply for \( a = 0 \), where \( r_1 \) and \( r_2 \) coalesce to 0, nor in the extremal case \( M^2 - a^2 - Q^2 = 0 \), where \( r_- \) and \( r_+ \) merge to cut off the anti-Stokes line \( l_z \). It does apply in the extremal limit, where numerical calculations fail and we find \( \bar{\omega}(a \to M) \simeq 0.051704/M \).
[19] The analysis can alternatively proceed in the geometrical optics approximation, where radiation follows null geodesics.