Relic neutrino asymmetry generation from $\nu_\alpha \leftrightarrow \nu_\beta$ oscillations

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Active–sterile neutrino oscillations provide a mechanism by which large differences in the neutrino and antineutrino number densities can be created in the early universe. The quantum kinetic equations are employed in the study of these neutrino asymmetries, which, when solved analytically in the adiabatic limit, generate physically transparent evolution equations that are very useful for the understanding of the nature of the asymmetry growth.

1. Introduction

Active–sterile neutrino oscillations in the early universe can generate large differences in the number densities of relic neutrinos and antineutrinos [1, 2]. For a wide range of small vacuum mixing angles $\theta_0$ and squared mass differences $\Delta m^2$ (provided that $\Delta m^2 < 0$, meaning roughly that $m_\nu_\alpha > m_\nu_\beta$, where $\alpha = e, \mu$ or $\tau$), this is achieved in the temperature range $T \sim 1 \rightarrow O(10)$ MeV (before neutrino decoupling at $T \simeq 1$ MeV) in a $CP$ asymmetric background. The final values of these so-called lepton number asymmetries,

$$L_{\nu_\alpha} \equiv \frac{n_{\nu_\alpha} - n_{\nu_\beta}}{n_\gamma},$$

where $n_\gamma$ is the number density of species $\psi$, are typically of the order 0.1, and have profound cosmological implications: (i) the suppression of $\nu_\beta$ production because of large matter effects [3, 4] and (ii) a modification to primordial $^4$He synthesis if a sufficiently large $L_{\nu_\alpha}$ is created prior to or during big bang nucleosynthesis (BBN) [5]. For a review on asymmetry generation, see Ref. [3].

The study of asymmetry evolution is conducted in the density matrix formalism, and the quantum kinetic equations (QKEs) [5, 6], which quantify the effects of (i) decohering collisions, (ii) matter-affected oscillations, (iii) expansion of the universe, and (iv) repopulation of empty states from the background, are the main tool of trade.

2. Quantum kinetic equations

The properties of a $\nu_\alpha \leftrightarrow \nu_\beta$ system are encoded in the density matrices [5, 6]

$$\rho = \frac{1}{2} (P_0 + P \cdot \sigma),$$

where $P = P_x \hat{\sigma}_x + P_y \hat{\sigma}_y + P_z \hat{\sigma}_z$. All quantities are understood to be functions of temperature $T$ and momentum $p$. The diagonal entries represent, respectively, the $\nu_\alpha$ and $\nu_\beta$ distribution functions:

$$N_{\nu_\alpha} = \frac{1}{2} (P_0 + P_z) N^0_{eq}, \quad N_{\nu_\beta} = \frac{1}{2} (P_0 - P_z) N^0_{eq},$$

in which the reference distribution function $N^0_{eq}$ is of Fermi–Dirac (equilibrium) form,

$$N^\nu_{eq} = \frac{1}{2\pi^2} \frac{p^2}{1 + \exp (\frac{p - \nu}{T})},$$

with chemical potential $\mu$ set to zero. The off-diagonal entries $P_x$ and $P_y$ are the coherences.

The variables $P_0$ and $P$ advance in time according to the quantum kinetic equations (QKEs) [5, 6]

$$\frac{\partial P}{\partial t} = V \times P - D (P_x \hat{\sigma}_x + P_y \hat{\sigma}_y) + R_\alpha \hat{\sigma}_z,$$

$$\frac{\partial P_0}{\partial t} = R_\alpha \simeq \Gamma \left[ \frac{N^\nu_{eq}}{N^0_{eq}} - \frac{1}{2} (P_0 + P_z) \right].$$

Here, the matter potential vector $V = \beta \hat{\sigma}_x + \lambda \hat{\sigma}_y$ governs the coherent matter-affected evolution of the ensemble, with [5, 6]

$$\beta = \frac{\Delta m^2}{2p} \sin 2\theta_0, \quad \lambda = \frac{\Delta m^2}{2p} (b - a - \cos 2\theta_0),$$

where $\Delta m^2$ is the squared mass difference, $b$ and $a$ are the neutrino masses, and $\theta_0$ is the mixing angle.
and
\[ a = -\frac{4\zeta(3)\sqrt{2}G_F L^{(\alpha)}T^4p}{\pi^2\Delta m^2}, \]
\[ b = -\frac{4\zeta(3)\sqrt{2}G_FA_\alpha T^4p^2}{\pi^2\Delta m^2 m_W^2}, \]
where \( G_F \) is the Fermi constant, \( m_W \) the W-boson mass, \( \zeta \) the Riemann zeta function and \( A_\alpha \approx 17, A_{\mu,\tau} \approx 4.9 \). Also present in Eq. (7) is
\[ L^{(\alpha)} = L_{\nu_\alpha} + L_{\nu_e} + L_{\nu_\mu} + L_{\nu_\tau} + \eta, \]
which contains a small term due to the cosmological baryon–antibaryon asymmetry \( \eta \).

The decoherence function \( D \) is related to the total collision rate for \( \nu_\alpha \), \( \Gamma \), via \( D = \Gamma/2 = k_eG_F^2T^4p \), where \( k_e \approx 0.635 \), \( k_{\mu,\tau} \approx 0.46 \), while the repopulation function \( R_\alpha \) determines the rate at which a depleted momentum state in the distribution is refilled from the background plasma so as to restore thermal equilibrium.

A separate set of expressions, denoted with an overhead bar, parameterises the overbar system.

3. Adiabatic limit approximation

By demanding \( a + s \) lepton number conservation, one may extract from the QKEs an exact evolution equation for the asymmetry, that is \[ \frac{dL_{\nu_\alpha}}{dt} = \frac{1}{2m_\gamma} \int \beta(P_y - \overline{P_y})N_{\nu_\alpha}^0 dp. \]
The role of the adiabatic limit approximation \[ \ref{eq:adiabatic_limit} \] is to supply approximate analytical solutions for \( P_y \) and \( \overline{P_y} \) so as to render Eq. (9) a more physically transparent expression, through which the nature of the asymmetry growth may be revealed.

3.1. The Boltzmann limit

The first assumption is a small refilling function \[ \ref{eq:adiabatic_limit}, \] i.e., \( R_\alpha \approx 0 \), so that the QKEs reduce to
\[ \frac{\partial \rho}{\partial t} \approx \mathcal{K} \rho = \begin{pmatrix} -D & -\lambda & 0 \\ \lambda & -D & -\beta \\ 0 & \beta & 0 \end{pmatrix} \begin{pmatrix} P_x \\ P_y \\ P_z \end{pmatrix}. \] 
This may be solved by establishing an “instantaneous diagonal basis” via \( \mathbf{Q} = \mathcal{U} \mathbf{P} \) and \( \mathcal{K}_d = \text{diag}(k_1, k_2, k_3) = \mathcal{U} \mathcal{K} \mathcal{U}^{-1} \), with eigenvalues
\[ k_{1,2} = -D \pm i\omega, \quad k_3 = -\frac{\beta^2 D}{\beta^2 + \omega^2}, \]
where
\[ d = D + \frac{k_3}{2}, \quad \omega^2 = \lambda^2 + \beta^2 + \frac{3}{4}k_3^2, \]
are interpreted as the effective damping factor and oscillation frequency respectively \[ \ref{eq:omega}. \]

In the new basis, Eq. (9) is similar to
\[ \frac{\partial \mathbf{Q}}{\partial t} = \mathcal{K}_d \mathbf{Q} - \mathcal{U} \frac{\partial \mathcal{U}^{-1}}{\partial t} \mathcal{U} \mathbf{Q}, \] 
where the second term is explicitly dependent on the time derivatives of \( D, \lambda \) and \( \beta \). The adiabatic limit is defined by \( \mathcal{U} \frac{\partial \mathcal{U}^{-1}}{\partial t} \approx 0 \), such that
\[ P_3(t) \approx \sum_{i=1}^3 \mathcal{U}^{-1} e^{\int_0^t k_i(t')dt'} \mathcal{U} \rho_i(0) P_i(0), \]
where \( \delta, \epsilon = x, y, z \), and \( i = 1, 2, 3 \), provides an approximate solution to Eq. (9).

Equation (\ref{eq:adiabatic_limit}) simplifies further in the high temperature collision-dominated \( D \gg |\beta| \) limit with the approximation
\[ e^{\int_0^1 k_{1,2}(t')dt'} \rightarrow 0, \]
since, in this limit, \( d \approx D, \omega \approx \lambda \), and thus \( |k_3| \ll |\text{Re}(k_{1,2})| \) by Eqs. (14) and (12). Physically, frequent collisions lead to rapid and complete damping of the system’s innate oscillations. The evolution of the ensemble is dictated instead by a slower rate \( k_3 \), which tends to equalise the \( \nu_\alpha \) and \( \nu_s \) distribution functions \[ \ref{eq:damping_factors}. \]

Specifically, one has, from Eqs. (14) and (13),
\[ P_y(t) \approx \frac{U_{y_3}^{-1}(t)}{U_{z_3}^{-1}(t)} P_z(t) = \frac{k_3}{\beta} P_z(t), \]
which allows Eq. (9) to be approximated as
\[ \frac{dL_{\nu_\alpha}}{dt} \approx \frac{1}{2m_\gamma} \int \left[ k_3(N_{\nu_\alpha} - N_{\nu_s}) - \overline{P_3}(N_{\nu_\alpha} - N_{\nu_s}) \right] dp. \]
Equation (17) is essentially a classical Boltzmann rate equation that is dependent only on the instantaneous \( \nu_\alpha \) and \( \nu_s \) distribution functions. The “rate constant” \( k_3 \) is of particular interest: A matter- and collision-affected mixing angle arises naturally from the QKEs \[ \ref{eq:adiabatic_limit}: \]
\[ \sin^2 2\theta_{m,D} \equiv \frac{\beta^2}{(D + k_3)^2 + \lambda^2 + \beta^2}, \] 
where
\[ d = D + \frac{k_3}{2}, \quad \omega^2 = \lambda^2 + \beta^2 + \frac{3}{4}k_3^2, \] 
are interpreted as the effective damping factor and oscillation frequency respectively \[ \ref{eq:omega}. \]
such that $k_3 = -\sin^2 2\theta_{m,D} \times D$; $k_3$ therefore reflects on the system’s ability to mix and to collide.

After further algebraic manipulations, Eq. (17) shows that, for $\Delta m^2 < 0$, there exists a critical temperature above which $\frac{dl_{mn}}{dt} \propto -L^{(\alpha)}$, and below which $\frac{dl_{mn}}{dt} \propto +L^{(\alpha)}$. The former implies $L_{\nu_\alpha}$ destruction, while the latter causes exponential growth [2]. These features have been confirmed by numerical integration of the exact QKEs.

3.2. Collision-affected MSW effect

Shortly after the initial explosive growth, the system enters a low temperature regime in which coherent Mikheyev–Smirnov–Wolfenstein (MSW) transitions [6] are the dominant asymmetry amplifiers. Subsequent growth follows an approximate power law rate $L_{\nu_\alpha} \propto T^{-4}$, reaching a value of $O(0.1)$ prior to neutrino decoupling [3]. In the following, however, I shall point out another interesting effect relevant for this epoch.

Together with the initial conditions $P_2(0) \simeq P_3(0) \simeq 0$, Eqs. (14) and (15) give, in the low temperature $|\beta| \gg D$ limit, the expression [12]

$$P_2(t) \simeq \cos 2\theta_{m,D}(t) \cos 2\theta_{m,D}(0) F_{\text{eff}} P_2(0), \quad (19)$$

where $F_{\text{eff}} = \exp \int_d^t b_3(t') dt'$ is generally analytically insoluble. However, supposing that the neutrino is created and “measured” well before and after resonance (plus other assumptions), one may write down an asymptotic solution:

$$F_{\text{eff}} = \exp \left[-\frac{\pi \beta D}{\beta^2} \right]_{\text{res}}, \quad (20)$$

so that Eq. (19) becomes, in probability language,

$$\text{Prob}(\nu_\alpha \to \nu_\alpha, \ t \gg t_{\text{res}}) \simeq \frac{1}{2} (1 + F_{\text{eff}}). \quad (21)$$

Physically, as the system crosses a resonance, collisions disrupt the coherent flavour conversion process and weaken the MSW mechanism, the extent of which is parameterised by an efficiency factor $F_{\text{eff}}$. In a collisionless environment, $F_{\text{eff}} = 1$, and Eq. (21) gives $\text{Prob} \simeq 1$ as expected.

The factor $F_{\text{eff}}$ compares the interaction length $\ell_{\text{int}} = 1/D$ with the physical resonance width $\ell_{\text{res}} = |\beta|/\beta^2|$. If $\ell_{\text{res}} > \ell_{\text{int}}$, maximal mixing persists for a sufficiently long time during which substantial collision-induced equilibration may occur. An infinite resonance width means $F_{\text{eff}} \to 0$ and the system emerges from the resonance with equal $\nu_\alpha$ and $\nu_\beta$ distribution functions.

4. Conclusion

Large relic neutrino asymmetries can be generated from active–sterile neutrino oscillations for a wide range of oscillation parameters. These asymmetries may then suppress sterile neutrino production and modify primordial $^4$He synthesis. Approximate evolution equations obtained from the QKEs offer much analytical insight on the nature of the asymmetry growth.

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