TWO QUASI-LOCAL MASSES EVALUATED ON SURFACES WITH BOUNDARY

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ABSTRACT. We study Hawking mass and the Huisken’s isoperimetric mass evaluated on surfaces with boundary. The convergence to an ADM mass defined on asymptotically flat manifold with a non-compact boundary are proved.

1. INTRODUCTION

ADM mass defined in [ADM60] has deep connections with minimal surface theory and the geometry of scalar curvatures as revealed by the seminal works [SY79, Sch89]. Their theorems states that for an asymptotically flat manifold with appropriate decay rate of the metric, if the manifold is of nonnegative scalar curvature, then the ADM mass is nonnegative. This is extended to an asymptotically flat manifold with a noncompact boundary by Almaraz, Barbosa and de Lima [ABdL16]. First, we recall their definition of an asymptotically flat manifold with a noncompact boundary,

Definition 1. (Asymptotically flat with a noncompact boundary, [ABdL16]) We say that \((M, g)\) is asymptotically flat with decay rate \(\tau > 0\) if there exists a compact subset \(K \subset M\) and a diffeomorphism \(\Psi : M \setminus K \to \mathbb{R}^n_+ \setminus \bar{B}^+_1(0)\) such that the following asymptotics holds as \(r \to +\infty\):

\[
|g_{ij}(x) - \delta_{ij}| + r|g_{i,j,k} + r^2|g_{i,k,j}| = o(r^{-\tau})
\]

where \(\tau > \frac{n-2}{2}\).

Here, \(x = (x_1, \cdots, x_n)\) is the coordinate system induced by the diffeomorphism \(\Psi\), \(r = |x|\), \(g_{ij}\) are the components of \(g\) with respect to \(x\), the comma denotes partial differentiation. We identify \(\mathbb{R}^n_+ = \{x \in \mathbb{R}^n : x_1 \geq 0\}\) and \(\bar{B}^+_1(0) = \{x \in \mathbb{R}^n_+ : |x| \leq 1\}\). In this work, we use the Einstein summation convention with index ranges \(i, j, k = 1, \cdots, n\) and \(a, b, c = 2, \cdots, n\).

Observe that along \(\partial M\), \(\{\partial_a\}\) spans \(T \partial M\) while \(\partial_1\) points inwards of \(M\). See the figure below.

Now we state the definition of the ADM mass for such asymptotically flat manifolds.

Definition 2. (ADM mass, [ABdL16]) The following quantity associated with \((M^n, g)\)

\[
m_{\text{ADM}} := \lim_{r \to +\infty} m(r) = \lim_{r \to +\infty} b_n \left\{ \int_{S^{n-1}_r} (g_{ij,j} - g_{j,ij}) \nu^i d\sigma + \int_{S^{n-1}_r} g_{a\mu} \nu^a d\theta \right\}
\]

defined for an asymptotically flat manifold \((M^n, g)\) with a non-compact boundary is called the ADM mass, \(S^{n-1}_r \subset M\) is a large standard Euclidean coordinate hemisphere of radius \(r\) with outward unit normal \(\nu\), and \(\mu\) is the outward pointing unit co-normal to \(\partial S^{n-1}_r\) in \(\partial M\). If the scalar curvature \(R_g\) is integrable and mean curvature \(H_g\) is integrable on \(\partial M\).
then \( m_{\text{ADM}} \) is well defined. Here \( b_n = \frac{1}{2(n-1)\omega_{n-1}} \) where \( \omega_{n-1} \) is the volume of \((n-1)\)-dimensional standard sphere.

Almaraz, Barbosa and de Lima [ABdL16] established a similar positive mass theorem.

**Theorem 1.** Given an asymptotically flat manifold \((M, g)\) with \( R_g, H_g \geq 0, R_g \in L^1(M)\) and \( H_g \in L^1(\partial M) \). Then \( m_{\text{ADM}} \geq 0 \). The mass \( m_{\text{ADM}} \) is zero if and only if \((M, g)\) is isometric to \((\mathbb{R}^n_+, \delta)\).

The ADM mass are defined for non-compact manifolds, it is then an interesting question to consider quantities for bounded domains. This is the motivation behind the notion of the quasi-local masses. We list a few important quasi-local masses for interested readers: Brown-York mass [BY93], Liu-Yau mass [LY03, LY06], Wang-Yau mass [WY07], Hawking mass [Haw68] and Huisken’s isoperimetric mass [Hui06]. The criterion for a satisfying definition of a quasi-local mass can be found in Liu and Yau’s work [LY06].

All these masses are evaluated on a closed 2-surface. The work [ABdL16] leads us to consider the question:

*Are there similar quantities evaluated on surfaces with boundary?*

We will be concerned with extending the Hawking mass and Huisken’s isoperimetric mass. Our definition of the Hawking mass is the following:

**Definition 3.** (Hawking mass with boundary) Given a 2-surface \( \Sigma \subset M^3 \) with boundary \( \partial \Sigma \neq \emptyset \) intersecting \( \partial M \) orthogonally, the Hawking mass is defined to be

\[
m_H(\Sigma) = \left( \frac{|\Sigma|}{8\pi} \right)^{\frac{1}{2}} \left( \chi(\Sigma) - \frac{1}{8\pi} \int_\Sigma H^2 \right).
\]
In particular, when $\Sigma$ is a disk i.e. $\chi(\Sigma) = 1$, then the mass takes the form

$$m_H(\Sigma) = \left(\frac{|\Sigma|}{8\pi}\right)^{\frac{1}{2}} \left(1 - \frac{1}{8\pi} \int_{\Sigma} H^2\right).$$

Observe that the form of the Hawking mass for a disk resembles the usual Hawking mass for a topological 2-sphere $\Sigma$ which writes as

$$m_H(\Sigma) = \left(\frac{|\Sigma|}{16\pi}\right)^{\frac{1}{2}} \left(1 - \frac{1}{16\pi} \int_{\Sigma} H^2\right).$$

We also have the following isoperimetric mass with boundary.

**Definition 4. (Isoperimetric mass with boundary)** For any asymptotically flat manifold $(M^n, g)$ with a non-compact boundary and an open set $\Omega \subset M^3$ with finite perimeter and $\Omega \cap \partial M$ non-empty, the quantity

$$m_{\text{iso}}(\Omega) = \frac{2}{H^2(\partial^* \Omega \cap \text{int} M)} \left[ V(\Omega) - \sqrt{2} \frac{H^2(\partial^* \Omega \cap \text{int} M)^{3/2}}{6\sqrt{V}(\Omega)} \right]$$

is called the isoperimetric quasi-local mass where $H^2$ is the 2-dimensional measure, $\partial^* \Omega$ is the reduced boundary of $\Omega$ and $V(\Omega)$ is the 3-dimensional Hausdorff measure of $\Omega$. We can take $\Omega$ to be the standard Euclidean hemisphere, let $A(r) = A(S^2_{r,+})$ be the area of the sphere $S^2_{r,+}$ and $V(r)$ is the volume of the region of $M$ bounded to the interior by $S^2_{r,+}$ and $\partial M$. The quantity $m_{\text{ISO}}$ given by

$$m_{\text{ISO}} = \limsup_{r \to \infty} \frac{2}{H^2(\partial^* \Omega \cap \text{int} M)} \left( V(r) - \sqrt{2} \frac{H^2(\partial^* \Omega \cap \text{int} M)^{3/2}}{6\sqrt{V}(r)} \right)$$

is called isoperimetric mass with boundary.

G. Huisken [Hui06] defined an isoperimetric mass for the usual asymptotically flat manifold, see an expression in [FST09, p. 51]. Volkman [Vol14] defined the same isoperimetric mass with boundary which he called relative isoperimetric mass (up to a constant multiple as ours), however, as we will show in the article, his definition is a special case. Also, our proof of convergence to the ADM mass is more direct.

Fan, Shi and Tam [FST09] proved that Huisken’s isoperimetric mass evaluated on standard coordinate spheres with increasing radius approaches the ADM mass, which is the third item of Liu and Yau’s criterion [LY06, Introduction].

In this article, we will show that our boundary version of Hawking mass and isoperimetric mass meet the third requirement of Liu and Yau’s list of criterion of a good quasi-local mass. The article is organized as follows:

In Section 2 we evaluate the ADM mass (1) via Ricci tensor of $M$ and second fundamental form of $\partial M$. In Section 3 we show that the Hawking mass converges to the ADM mass. In Section 4 we show that the isoperimetric quasi-local mass converges to the ADM mass.

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2. Evaluation via Ricci and Second Fundamental Form

Notations We list the notations used in this note. We extend the Euclidean distance $|\cdot|$ given by the diffeomorphism $\Psi$ to all of $M$ by requiring $|\cdot|: K \to [0, 1)$.

- $\Omega$ a domain intersecting $\partial M$, and $\partial \Omega = \Sigma \cup \Pi$
- $\Sigma^+ = \partial \Omega \cap \text{Int} M$
- $\Pi = \partial \Omega \cap \partial M$
- $g_{ij}$ metric on $g$
- $h_{ab}$ induced metric on $\partial M$
- $\nu$ normal of $\partial \Omega$ in $M$
- $\mu$ normal of $\partial M$ in $M$
- $\vartheta$ normal of $\partial \Sigma = \partial \Pi$ in $\Pi$
- $d\sigma$ $(n-1)$-dimensional volume element under metric $g$
- $d\theta$ $(n-2)$-dimensional volume element under metric $g$
- $X$ $X = (x_1, \ldots, x_n)$ is the position vector under $\Psi$
- $D_r = \{ x \in M \cup \partial M : |x| \leq r \}$ for $r > 1$
- $\nabla$ connection on $M$
- $D$ induced connection on $\partial M$
- $A$ second fundamental form of $\partial M$ in $M$
- $H$ mean curvature of $\partial M$ in $M$
- $G$ $G = Rc - \frac{1}{2} Rg$, the Einstein tensor of $M$

The barred quantities are their Euclidean counterparts. $\Omega$ is very often

$$D^+_r = \{ x \in M \cup \partial M : |x| \leq r \}$$

where $r > 1$, in this case we write

$$\Sigma^+_r = \{ x \in \text{Int} M : |x| = r \}$$

and

$$\Pi_r = \{ x \in \partial M : |x| = r \}.$$ 

Obviously, $\Sigma^+_r$ is a standard Euclidean coordinate hemisphere and $\partial D^+_r = \Sigma^+_r \cup \Pi_r$. When there is no ambiguity, we drop the subscript for simplicity.

We have similar expressions as $\text{MTT10}$, (1.3),(1.4) and (1.5) where the usual asymptotically flat case is handled.

**Theorem 2.** Suppose that $(M, g)$ is an asymptotically flat manifold, then

$$m_{\text{ADM}} = -c_n \lim_{r \to \infty} \int_{\Sigma^+_r} G(X, \nu)d\sigma + \int_{\partial \Sigma^+_r} (A - Hg)(X, \vartheta)d\theta,$$

where $c_n = \frac{1}{(n-2)(n-1)\omega_{n-1}}$.

We collect some well known asymptotics in the following lemma:
Lemma 1. We have the following asymptotics:

$$2R_{ij} = g_{kr,kj} + g_{kj,ki} - g_{ij,kk} - g_{kk,ij} + O(r^{-2-\tau});$$

$$R = g_{kr,ik} - g_{kk,ii} + O(r^{-2-\tau});$$

$$d\theta = d\tilde{\theta} + O(r^{-\tau})d\tilde{\theta};$$

$$d\sigma = d\tilde{\sigma} + O(r^{-\tau})d\tilde{\sigma};$$

$$\nu - \tilde{\nu} = O(r^{-\tau});$$

$$\tilde{\theta} - \tilde{\tilde{\theta}} = O(r^{-\tau});$$

$$\mu = -g^{1i}\partial_i/g^{11} = -\partial_1 + O(r^{-\tau});$$

$$A_{ab} = \frac{1}{2}(g_{a1,b} + g_{b1,a} - g_{ab,1}) + O(r^{-1-\tau})$$

$$= O(r^{-1-\tau});$$

$$H = \frac{1}{2}(2g_{a1,a} - g_{aa,1}) + O(r^{-1-\tau})$$

$$= O(r^{-1-\tau}).$$

Proof. The asymptotics of Ricci curvature and scalar curvature are well known, see for example [MT16 (2.2), (2.6)]. Choose $\bar{g}$-orthonormal frame $e_i$, and let $e_1 = \tilde{\nu}$. If $e_i = a^j_i \partial_j$, then we see that $a^j_i$ is an orthogonal matrix. So $g(e_i, e_j) = a^k_j a^l_i g(\partial_k, \partial_l) = \delta_{ij} + O(r^{-\tau})$ and this will give

$$d\sigma = d\tilde{\sigma} + O(r^{-\tau})d\tilde{\sigma}.$$ 

Same reasoning applied on $\partial M$ will give for $d\theta$ the asymptotics

$$d\theta = d\tilde{\theta} + O(r^{-\tau})d\tilde{\theta}.$$

Let $v = v^i e_i$, then $1 = g(v, v) = v^i v^j g(e_i, e_j) = v^i v^j (\delta_{ij} + O(r^{-\tau}))$, we get $\Sigma_i (X^i)^2 = 1 + O(r^{-\tau})$. Similarly, from $g(v, e_a) = 0$, we get $0 = X^a + O(r^{-\tau})$, then

$$v - \tilde{v} = v^a e_a + (v^1 - 1)v_1 = O(r^{-\tau}).$$

For $\theta$, we have as well $\theta - \tilde{\theta} = (r^{-\tau}).$

For the expression of $\mu, A$ and $H$ in in terms of the metric, refer to [ABdL16 (2.12) - (2.16)]. For $\mu$,

$$\mu = -g^{1i}\partial_i/(g^{1k} g^{1j} g_{jk})$$

$$= -g^{1i}\partial_i/g^{11} = -\partial_1 + O(r^{-\tau}).$$
On $\partial M$, we have

$$A_{ab} = -\langle \mu, \nabla_a \partial_b \rangle = (g^{11})^{-1/2} \Gamma^a_{ab}$$

$$= \frac{1}{2} (g_{a1,b} + g_{b1,a} - g_{ab,1}) + O(r^{-1-2\tau})$$

$$= O(r^{-1-\tau}).$$

For $H := H_{\partial M}$,

$$H = h^{ab} A_{ab}$$

$$= \frac{1}{2} (g^{11})^{1/2} (2g_{a1,a} - g_{aa,1})$$

$$= \frac{1}{2} (2g_{a1,a} - g_{aa,1}) + O(r^{-1-2\tau})$$

$$= O(r^{-1-\tau}).$$

That concludes our proof of the asymptotics. \square

**Proof of Theorem 2** We follow [MT16]. Let

$$I_1 = \int_{\Sigma^+ \cup \Pi} (-g_{k_i,k_j} - g_{k_j,k_i} + g_{ij,kk} + g_{kk,ij}) x^i \bar{\nu}^j d\bar{\sigma}$$

and

$$I_2 = \int_{\Pi} (-g_{k_i,k_j} - g_{k_j,k_i} + g_{ij,kk} + g_{kk,ij}) x^i \bar{\nu}^i d\bar{\sigma},$$

then

$$-2 \int_{\Sigma^+} R_{ij} x^i \bar{\nu}^j d\bar{\sigma} = \int_{\Sigma^+} (-g_{k_i,k_j} - g_{k_j,k_i} + g_{ij,kk} + g_{kk,ij}) x^i \bar{\nu}^j d\bar{\sigma} + o(1)$$

$$= I_1 + I_2 + o(1).$$

To facilitate the computation, we can assume that manifold $M$ is diffeomorphic to $\mathbb{R}^n_+$ and extend the metric smoothly to all of $\mathbb{R}^n_+$. Because we are evaluating at infinity, the result will be independent of extensions. We compute $I_1$ first,

$$I_1 = \int_{\Sigma^+ \cup \Pi} (-g_{k_i,k_j} - g_{k_j,k_i} + g_{ij,kk} + g_{kk,ij}) x^i \bar{\nu}^j d\bar{\sigma}$$

$$= (n-2) \int_{\Sigma^+ \cup \Pi} (g_{k_j,k} - g_{kk,j}) \bar{\nu}^i d\bar{\sigma} + \int_{\Sigma^+ \cup \Pi} (-g_{k_j,k} + g_{kk,j}) x^i \bar{\nu}^j d\bar{\sigma}$$

$$= (n-2) \int_{\Sigma^+} (g_{k_j,k} - g_{kk,j}) \bar{\nu}^i d\bar{\sigma} + \int_{\Pi} (-g_{k_j,k} + g_{kk,j}) x^i \bar{\nu}^j d\bar{\sigma}$$

$$- (n-2) \int_{\Pi} (g_{a1,a} - g_{aa,1}) d\bar{\sigma}.$$
We compute $I_2$ using $\vartheta^1 = -1$ again.

\[-I_2 = \int_\Pi (g_{kk,kk} + g_{kk,kl} - g_{kl,kl} - g_{kk,ij})x^i\vartheta^j d\bar{\sigma}\]

\[= \int_\Pi (g_{ba,b1} + g_{b1,ba} - g_{a1,bh} - g_{bh,a1})x^a d\bar{\sigma}\]

\[= \int_{\partial \Pi} g_{ba,1}x^a \vartheta^b d\bar{\sigma} - \int_{\partial \Pi} \frac{1}{x} \frac{\partial x^a}{\partial \vartheta^b} d\bar{\sigma}\]

\[+ \int_{\partial \Pi} g_{b1,ax} \vartheta^a d\bar{\sigma} - \int_{\partial \Pi} \frac{1}{x} \frac{\partial x^a}{\partial \vartheta^b} d\bar{\sigma}\]

\[- \int_{\partial \Pi} g_{a1,xb} \vartheta^b d\bar{\sigma} + \int_{\partial \Pi} g_{ax} \frac{\partial x^a}{\partial \vartheta^b} d\bar{\sigma}\]

\[- \int_{\partial \Pi} g_{bb,1x} \vartheta^a d\bar{\sigma} + \int_{\partial \Pi} \frac{1}{x} \frac{\partial x^a}{\partial \vartheta^b} d\bar{\sigma}\]

\[= \int_{\partial \Pi} g_{ba,1x} \vartheta^a d\bar{\sigma}\]

\[- \int_{\partial \Pi} g_{bb,1x} \vartheta^a d\bar{\sigma} + (n-2) \int_{\partial \Pi} g_{bb,1} d\bar{\sigma}.\]

For the boundary term,

\[2 \int_{\partial \Pi} (A_{ab} - H_{ab})x^a \vartheta^b d\bar{\sigma}\]

\[= \int_{\partial \Pi} (2g_{1a,b} - g_{ab,1})x^a \vartheta^b d\bar{\sigma} - \int_{\partial \Pi} (2g_{1a,a} - g_{aax,1})x^b \vartheta^a d\bar{\sigma} + o(1)\]

\[= - \int_{\partial \Pi} g_{ab,1x} \vartheta^b d\bar{\sigma} + \int_{\partial \Pi} g_{bb,1x} \vartheta^a d\bar{\sigma}\]

\[+ 2 \int_{\partial \Pi} g_{1a,bx} \vartheta^b d\bar{\sigma} - 2 \int_{\partial \Pi} (g_{1a,a} + g_{1a,ab})x^b \vartheta^a d\bar{\sigma} + o(1)\]

\[= - \int_{\partial \Pi} g_{ab,1x} \vartheta^b d\bar{\sigma} + \int_{\partial \Pi} g_{bb,1x} \vartheta^a d\bar{\sigma}\]

\[+ 2 \int_{\partial \Pi} g_{1a,bx} \vartheta^b d\bar{\sigma} - 2 \int_{\partial \Pi} (g_{1a,ab}x^b + g_{1a,b}x^b) \vartheta^a d\bar{\sigma} + o(1)\]

\[= - \int_{\partial \Pi} g_{ab,1x} \vartheta^b d\bar{\sigma} + \int_{\partial \Pi} g_{bb,1x} \vartheta^a d\bar{\sigma} + 2 \int_{\partial \Pi} g_{1a,bx} \vartheta^b d\bar{\sigma}\]

\[2(n-1) \int_{\partial \Pi} g_{1a,ab} d\bar{\sigma}\]

\[- 2(2n-1) \int_{\partial \Pi} g_{1a,bx} \vartheta^a d\bar{\sigma} + o(1)\]

\[= - \int_{\partial \Pi} g_{ab,1x} \vartheta^b d\bar{\sigma} + \int_{\partial \Pi} g_{bb,1x} \vartheta^a d\bar{\sigma} - 2(2n-2) \int_{\partial \Pi} g_{1a,ab} d\bar{\sigma} + o(1).\]

Scalar curvature has asymptotics

\[R = g_{ik,ik} - g_{kk,ii} + o(r^{-2-2r}),\]
hence

\[
\int_{\Sigma^+} (-g_{k,j,k} + g_{kk,jj}) x^i \tilde{\nu}^i d\tilde{\sigma} \\
= - \int_{\Sigma^+} R x^i \tilde{\nu}^i d\tilde{\sigma} + o(1) \\
= - \int_{\Sigma^+} R g(X, \nu) d\sigma + o(1).
\]

Collect all calculations of \(I_1, I_2\) and \(2 \int_{\partial \Sigma^+} (A_{ab} - Hg)x^a \varphi^b d\theta\), we have

\[
\int_{\Sigma^+} (-2Rc + Rg)(X, \nu) d\sigma - 2\int_{\Sigma^+} (A - Hg)(X, \vartheta)d\theta \\
= (n-2)\left[\int_{\Sigma^+} (g_{k,j,k} - g_{kk,j}) \tilde{\nu}^i d\tilde{\sigma} + \int_{\Pi} g_{a1} a d\tilde{\sigma}\right] + o(1) \\
= (n-2)\left[\int_{\Sigma^+} (g_{k,j,k} - g_{kk,j}) \tilde{\nu}^i d\tilde{\sigma} + \int_{\partial \Sigma^+} g_{a1} \tilde{\varphi}^a d\tilde{\theta}\right] + o(1)
\]

and the theorem is proved. \(\square\)

**Remark 1.** The proof of Theorem 2 works for slightly more general bounded open sets. See similar statements of [MT16, Theorem 2.1].

2.1. **Another Proof.** Here we provide an alternate proof of (2). The proof of the standard case is due to Herzlich [Her16], we extend the proof to our settings. We have the following identity.

**Lemma 2.** (Integrated Bianchi identity) Given a Riemannian manifold \((M^n, g)\), we use the same notation with the ones given at the start of this section. \(X\) is a conformal Killing vector field i.e.

\[
\nabla^j X^i = \frac{1}{n} g^{ij} \text{div} X.
\]

\(\Omega \subset M\) is a bounded open set whose \(\partial \Omega\) decomposes as the union of \(\Sigma^+ = \partial \Omega \cap \text{int} M\) and \(\Pi = \partial \Omega \cap \partial M\) who share a common boundary \(\partial \Sigma^+ = \partial \Pi\). \(X\) is tangent to \(\partial M\) and \(X\) is also conformal Killling on \((\Pi, h)\) i.e.

\[
D^a X^b = \frac{1}{n-1} h^{ab} \text{div} \Pi X.
\]

We have then

\[
\int_{\partial \Pi} (A - Hg)(X, \vartheta) + \int_{\Sigma^+} G(X, \nu) \\
= - \frac{n-2}{2n} \int_{\Omega} R \text{div} X - \frac{n-2}{n-1} \int_{\Pi} H \text{div} \Pi X.
\]
Proof. By divergence theorem,

\[
\int_{\partial \Omega} G(X, \nu) = \int_{\Omega} \nabla^i (G_{ij} X^j) \\
= \int_{\Omega} X^i \nabla^i G_{ij} + G_{ij} \nabla^i X^j \\
= \int_{\Omega} G_{ij} \left( \frac{1}{n} \text{div} X g^{ij} \right) \\
= \frac{1}{n} \int_{\Omega} \text{div} X G_{ij} g^{ij} \\
= \frac{2 - n}{2n} \int_{\Omega} R \text{div} X.
\]

Since \( X \) is tangent to \( \partial M \), \( \nu \) is normal to \( \partial M \) along \( \Pi \), we can use the Gauss-Codazzi equation

\[
G(X, \nu) = X^b D^a (A_{ab} - H h_{ab})
\]
to deal with \( G(X, \nu) \) integrated on \( \Pi \), we have

\[
\int_{\Pi} G(X, \nu) = \int_{\Pi} X^b D^a (A_{ab} - H h_{ab}) \\
= \int_{\partial \Pi} (A - H g)(X, \theta) - \int_{\Pi} (A_{ab} - H h_{ab}) D^a X^b \\
= \int_{\partial \Pi} (A - H g)(X, \theta) - \frac{1}{n-1} \int_{\Pi} (A_{ab} - H h_{ab}) \text{div} \Pi X h^{ab} \\
= \int_{\partial \Pi} (A - H g)(X, \theta) + \frac{n-2}{n-1} \int_{\Pi} H \text{div} \Pi X.
\]

Combining the two formulas above, we obtain

\[
- \frac{n-2}{n-1} \int_{\Pi} H \text{div} \Pi X - \frac{n-2}{2n} \int_{\Omega} R \text{div} X = \int_{\Sigma^+} G(X, \nu) + \int_{\partial \Sigma^+} (A - H g)(X, \theta).
\]

Another proof of Theorem 2. For any sufficiently large \( R > 1 \), we define a cutoff function \( \chi_R \) such that it is zero inside the hemisphere of radius \( R/2 \), equals 1 outside the hemisphere of radius \( R/4 \) and it satisfies the following bounds on its derivatives

\[
|\nabla \chi_R| \lesssim CR^{-1}, \quad |\nabla^2 \chi_R| \lesssim CR^{-2}, \quad |\nabla^3 \chi_R| \lesssim CR^{-3}
\]

for some universal constant \( C \) not depending on \( R \). We write in short \( \chi = \chi_R \) when there is no confusion. We then define a half-annulus \( A_R = B^+_R \setminus B^-_{R/4} \) and a metric \( h = g \chi + (1 - \chi) \delta \), here \( \delta \) is the standard Euclidean metric. Now we use subscript or supscript \( e, h, g \) on a term to denote that this term is evaluated under corresponding metric.
Using (3) by assigning $\Omega = A_R$,

$$\int_{\partial \Pi_R} (A - H_g)(X, \vartheta) + \int_{\Sigma_R} G(X, v)$$

$$= \int_{\Omega_R} G_{ij} \nabla^i X^j + \int_{\Pi_R} (A_{ab} - H_h_{ab}) D^a X^b$$

$$= \frac{2 - n}{2n} \int_{\Omega_R} R \text{div} X + \int_{\Omega_R} G_{ij}(\nabla^i X^j)^{\circ}$$

$$+ \frac{2 - n}{n - 1} \int_{\Pi_R} H \text{div}_\Pi X + \int_{\Pi_R} (A_{ab} - H_h_{ab})(D^a X^b)^{\circ}$$

$$=: I_1 + I_2 + J_1 + J_2,$$

where $(\nabla^i X^j)^{\circ} = \nabla^i X^j - \frac{1}{n} g^{ij} \text{div} X$ and $(D^a X^b)^{\circ} = D^a X^b - \frac{1}{n - 1} h^{ab} \text{div}_\Pi X$.

We estimate these terms separately. For $I_1$,

$$\int_{\Omega_R} R \text{div} X d\mathcal{H}^n - \int_{\Omega_R} R \text{div}^\circ X d\mathcal{H}^n$$

$$\leq C \sup(\|\Pi\| X \|\mathcal{H}\| \mathcal{H}^n(\Omega_R))$$

$$\leq CR^{-\tau - 1} R^{n - 2} = o(1);$$

For $J_1$,

$$\int_{\Pi_R} H \text{div}_\Pi X d\mathcal{H}^{n - 1} - \int_{\Pi_R} H \text{div}^\circ_{\Pi} X d\mathcal{H}^{n - 1}$$

$$= \int_{S_R} H(\delta^h) X - \int_{S_R} H(\delta^\circ) X$$

$$\leq C \sup(\|\Pi\| X \|H\|) \mathcal{H}^{n - 1}(S_R)$$

$$\leq CR^{-1 - \tau} R^{n - 1 - \tau} = o(1).$$

It is easy to see that the same reasoning applies to the terms $I_2$ and $J_2$, we get $I_2, J_2 = o(1)$.

We have that

$$\int_{\partial \Pi_R} (A - H_g)(X, \vartheta) + \int_{\Sigma_R} G(X, v)$$

$$= \frac{2 - n}{2n} \int_{\Omega_R} R \text{div} X d\mathcal{H}^n + \frac{2 - n}{n - 1} \int_{S_R} H \text{div}_\Pi X d\mathcal{H}^{n - 1} + o(1)$$

$$= K_1 + K_2 + o(1).$$

Now we estimate $K_1$ and $K_2$. 
\[
K_1 = \int_{\Omega_\rho} R \text{div}_e X d\mathcal{H}^n_c = \int_{\Omega_\rho} \text{div} X Q(e,g) d\mathcal{H}^n_c + n \int_{S_{\rho}^n} \langle U, n \rangle d\mathcal{H}^n_c - n \int_{S_{\rho/4}^n} \langle U, n \rangle d\mathcal{H}^n_c + n \int_{\Pi_{n-\Pi_{\rho/4}}} \langle U, n \rangle d\mathcal{H}^n_c = n \int_{S_{\rho/4}^n} \langle U, n \rangle d\mathcal{H}^n_c + n \int_{\Pi_{n-\Pi_{\rho/4}}} \langle U, n \rangle d\mathcal{H}^n_c + o(1).
\]

And for \(K_2\),
\[
2K_2 = 2 \int_{S_R^c} H \text{div}_1 X d\mathcal{H}^n_c = (n - 1) \int_{S_R^c} 2 H d\mathcal{H}^n_c + o(1) = (n - 1) \int_{S_R^c} (2v_{1a,a} - v_{au,1}) d\mathcal{H}^n_c + o(1).
\]

Along \(S_R\), since \(n = -\partial_1\), we have that
\[
\langle U, n \rangle = v_{ij,j} n^i - v_{jj} n^i = -v_{1j,j} + v_{jj,1} = -v_{1a,a} + v_{au,1}.
\]

So
\[
\int_{\partial \Pi_R} (A - H g)(X, \theta) + \int_{S_{\rho}^n} G(X, \nu) = \frac{2 - n}{2} \int_{S_{\rho/4}^n} \langle U, n \rangle d\mathcal{H}^n_c + \frac{2 - n}{2} \int_{S_{\rho/4}^n} (-v_{1a,a} + v_{au,1}) d\mathcal{H}^n_c = \frac{2 - n}{2} \int_{S_{\rho}^n} (2v_{1a,a} - v_{au,1}) d\mathcal{H}^n_c + o(1) = \frac{2 - n}{2} \int_{S_{\rho}^n} \langle U, n \rangle d\mathcal{H}^n_c + \int_{S_{\rho}^n} g_{1a,a} d\mathcal{H}^n_c + o(1) = \frac{2 - n}{2} \int_{S_{\rho}^n} \langle U, n \rangle d\mathcal{H}^n_c + \int_{S_{\rho}^n} g_{1a,a} d\mathcal{H}^n_c + o(1) = (2 - n)(n - 1)\omega_{n-1} m_{\text{ADM}} + o(1)
\]

thus proving \((7)\). \(\square\)

2.2. **A graphical example.** Motivated by [Lam10], we give a graphical example where \(\partial M\) is given as a graph of a function. Set \(X = (x_1, \ldots, x_n)\) and \(B_{\rho}\) be the ball centered at 0 with radius \(\rho\). Let \(n \geq 2\), given a function \(u : \mathbb{R}^n \setminus B_1 \to \mathbb{R}\), the set
\[
\{(x_0, X) : X \in \mathbb{R}^n \setminus B_1, x_0 \geq u(X)\}
\]
is our \(M \setminus K\). We can extend \(M \setminus K\) arbitrarily. Suppose that \(u\) is bounded and has asymptotics
\[
|X|^{n-1}|Du| + |X|^n |D^2 u| = O(1).
\]
\(\partial M\) contains the set \(\{(u(X), X) : X \in \mathbb{R}^n \setminus B_1\}\), therefore \(M\) has a non-compact boundary. The asymptotic flat structure of \(M\) can be given by the map
\[
\Psi : (x_0, X) \mapsto (x_0 + v(x_0)u(X), X)
\]
where \( v(0) = 1 \) with \( v \) has the same asymptotics of \( u \) and \( M \) carries metric \( g \) induced from this map from \( \mathbb{R}^{n+1} \). Let \( e_i \) where \( i = 0, 1, \ldots, n \) be the standard orthonormal basis of Euclidean space \( \mathbb{R}^{n+1} \) and \((\cdot, \cdot)\) be the standard Euclidean inner product. Let \( \hat{e}_i \) be the vector field induced by the map \( \Psi \) and the coordinate system \((x_0, X)\). It is easy to see that
\[
\hat{e}_0 = (1 + u(X))\partial_0 v)\hat{e}_0, \\
\hat{e}_a = v(x_0)\partial_a u\hat{e}_0 + e_a,
\]
and the metric takes the form
\[
g_{00} = \langle \hat{e}_0, \hat{e}_0 \rangle = 1 + 2u(X)\partial_0 v + u(X)^2|\partial_0 v|^2, \\
g_{aa} = \langle \hat{e}_a, \hat{e}_a \rangle = (\partial_a u + u(X)v(x_0)\partial_v u)^2 + 1.
\]
Here the index \( a \) ranges \( 1, \ldots, n \).

Since when \( x_0^2 + |X|^2 = r \) under standard Euclidean metric, either \( |x_0| \geq r/2 \) or \( |X| \geq r/2 \), considering also that \( u, v \) are bounded, \( |g_{ij} - \delta_{ij}| = O(r^{-n}) \). The decay of first and second order derivatives of the metric follows similarly. So \((M^{n+1}, g)\) is asymptotically flat with a non-compact boundary. The asymptotics specified by (4) is related to complete minimal surfaces regular at infinity, see the work of Schoen [Sch83] and Volkman [Volk14].

Calculating the ADM mass using metric induced from this map is complicated. But we are able to derive a simple formula for the mass in this graphical case.

**Theorem 3.** If \( M \) is an asymptotically flat manifold with a non-compact boundary, \( \partial M \) is given by a function \( u \) on \( \mathbb{R}^n \setminus B_1 \) with asymptotics (4), then the mass has the simpler form
\[
m_{\text{ADM}} = (n - 1) \lim_{\rho \to \infty} \int_{\partial B_\rho} \partial_\rho u d\theta.
\]
where \( \partial_\rho \) is radial unit normal to \( \partial B_\rho \) in \( \partial B_\rho \) with respect to Euclidean metric and \( d\theta \) is the Euclidean \((n - 1)\)-dimensional volume element.

**Proof.** Without loss of generality, we extend \( u \) to all of \( \mathbb{R}^n \) i.e. \( u: \mathbb{R}^n \to \mathbb{R} \). We identify \( B_\rho \) as the set \( \{(0, X): |X| \leq \rho\} \subset \mathbb{R}^{n+1} \), then \( \partial B_\rho = \{(0, X): |X| = \rho\} \).

We have by well known formulas (see for example [Eic09 Section 2])
\[
v = \sqrt{1 + |Du|^2}; \\
A_{ab} = v^{-1} u_{ab} = O(\rho^{-n}); \\
g_{ab} = \delta_{ab} + u_{ab}; \\
g^{ab} = \delta_{ab} - v^{-2} u^a u^b; \\
H = v^{-1} u_{ab} (\delta^{ab} - v^{-2} u^a u^b) \\
= v^{-1} \Delta u - v^{-3} u_{ab} u^a u^b \\
= v^{-1} \Delta u + O(\rho^{-3n+2})
\]
where \( g \) gives the induced metric on the graph. Now we do calculations on \((B_\rho, g)\). The normal \( \partial_\theta \) to \( \partial B_\rho \) in \( B_\rho \) is given by
\[
\partial_\theta = \hat{g}^{bc} \partial_c = \hat{g}^{bc} \frac{x_c}{|X|} = \partial_\rho + O(\rho^{2-2n}).
\]
So $X = \rho \partial_\rho = \rho \tilde{\vartheta} + O(\rho^{3-2n})$ and

$$(A - H g)(X, \tilde{\vartheta}) = \rho A(\partial_\rho, \partial_\rho) - \rho H$$

$$= \rho v^{-1} \text{Hess} u(\partial_\rho, \partial_\rho) - \rho v^{-1} \Delta u + O(\rho^{-3n+3})$$

$$= \rho \text{Hess} u(\partial_\rho, \partial_\rho) - \rho \Delta u + O(\rho^{-3n+3}).$$

On $\partial B_\rho$, we decompose the standard Laplacian of $\mathbb{R}^n$

$$\Delta u = \Delta_{\partial B_\rho} u + \text{Hess} u(\partial_\rho, \partial_\rho) + H_{\partial B_\rho, R} \partial_\rho u$$

where $H_{\partial B_\rho, R} = (n-1)/\rho$ is the mean curvature of a sphere of radius $\rho$ in $\mathbb{R}^n$.

Hence

$$(A - H g)(X, \tilde{\vartheta}) = -(n-1)\partial_\rho u - \rho \Delta_{\partial B_\rho} u + O(\rho^{-3n+3}).$$

Suppose that $\Omega$ is a region intersecting $M$ at $\partial B_\rho$ satisfying requirements of [MT16, Theorem 2.1]. Because $M$ is an unbounded region in $\mathbb{R}^n$, now the mass by Theorem $2$ has only a boundary term, then

$$m_{\text{ADM}} = -\int_{\partial B_\rho} (A - H g)(X, \tilde{\vartheta}) \, d\theta + o(1)$$

$$= \int_{\partial B_\rho} \left[ (n-1)\partial_\rho u + \rho \Delta_{\partial B_\rho} u \right] d\tilde{\theta} + o(1)$$

$$= (n-1) \int_{\partial B_\rho} \partial_\rho u d\tilde{\theta} + o(1).$$

The calculation obviously works for nearly round surfaces (see the definition in [MTX17, Definition 2.1]), we omit the details. \hfill \Box

**Remark 2.** We point out that this expression is the same as [V ol14, Definition 2.5] and we refer readers to the positive mass theorem proved in there.

### 3. HAWKING MASS DERIVATION

We are going to extend Theorem 1.2 in [MTX17] to the boundary case and thus obtain a formula of Hawking type mass. We do not pursue generalities to deal with nearly round surfaces here and we choose coordinate hemispheres as approximating surfaces. We have the following two trivial lemmas.

**Lemma 3.** Let $X = (x^1, \cdots, x^n)$ be the position vector given by coordinates near infinity. For each large $\rho$, on $\Sigma_\rho$

$$|X - \rho \tilde{\nu}| = O(\rho^{1-\tau})$$

and

$$|X - \rho \tilde{\vartheta}| = O(\rho^{1-\tau}) \text{ along } \partial \Sigma_\rho.$$ 

**Lemma 4.** For large $\rho$, the volume $|\Sigma_\rho|$ satisfies

$$\left( \frac{2|\Sigma_\rho|}{\delta_{n-1}} \right)^{1/n} = \rho (1 + O(\rho^{-\tau})).$$
Theorem 4. (Hawking type mass) Let \{\{\Sigma_p\}\} be a family of coordinate hemispheres with radius \(\rho > 1\) in an asymptotically flat manifold \((M, g)\) of dimension \(n \geq 3\). Let \(\mu'\) be normal to \(\partial \Sigma\) in \(\Sigma\). Then

\[
(5) \quad c_n \left( \frac{2|\Sigma|}{\omega_{n-1}} \right)^{1/2} \left[ \int_\Sigma (S - \frac{n-2}{n-1} H_{\Sigma,M}^2) d\sigma + 2 \int_{\partial \Sigma} H_{\Sigma,\Sigma} + (\bar{\partial}, \mu') H_{\partial \Sigma, \partial M} d\theta \right]
\]

converges to the ADM mass \(m_{\text{ADM}}\) as \(\rho \to \infty\) where \(S\) is the scalar curvature of \(\Sigma\).

Proof. By Gauss equation,

\[
G(X, v) = G(\rho v, v) = \rho G(v, v) + O(\rho^{-1-2\tau})
\]

\[
= \frac{1}{2} \rho (H_{\Sigma,M}^2 - |A|^2 - S) + O(\rho^{-1-2\tau})
\]

where \(S\) is the scalar curvature of \(\Sigma\).

On \(\partial \Pi\),

\[
(A - Hg)(X, \vartheta) = (A - Hg)(\rho \bar{\vartheta}, \vartheta)
\]

\[
= \rho (A - Hg)(\bar{\vartheta}, \vartheta) + O(\rho^{-1-2\tau})
\]

\[
= \rho A(\bar{\vartheta}, \vartheta) - \rho H + O(\rho^{-1-2\tau})
\]

\[
= - \sum_i (\nabla_{e_i} \mu, e_i) + O(\rho^{-1-2\tau}).
\]

where \(e_i\)'s are orthonormal basis of \(T\Sigma\). Let \(\mu'\) be the normal to \(\partial \Sigma\) in \(\Sigma\), moreover,

\[
-H_{\partial \Sigma, \Sigma} - [\rho A(\bar{\vartheta}, \vartheta) - \rho H] = \sum (\nabla_{e_i} (\mu - \mu'), e_i).
\]

We note that \(\langle \mu, \mu' \rangle = 1 + O(\rho^{-\tau})\), this implies that \(\mu' := a \bar{\vartheta} + b \mu\) with \(a, b - 1 = O(\rho^{-\tau})\). Then

\[
\sum_i (\nabla_{e_i} (\mu - \mu'), e_i) = \sum a (\nabla_{e_i} \vartheta, e_i) + (b - 1) (\nabla_{e_i} \mu, e_i)
\]

\[
= \sum (\bar{\vartheta}, \mu') H_{\partial \Sigma, \partial M} + O(\rho^{-1-2\tau})
\]

\[
= O(\rho^{-1-\tau}).
\]

So

\[
(A - Hg)(X, \vartheta) = -\rho H_{\partial \Sigma, \Sigma} - \rho (\bar{\vartheta}, \mu') H_{\partial \Sigma, \partial M} + O(\rho^{-1-2\tau}).
\]

Then the theorem is easily obtained by integration and discarding small terms. \(\square\)

Remark 3. Take \(n = 3\) and that \(\Sigma\) meets \(\partial M\) orthogonally. Using as well Gauss-Bonnet theorem for surfaces with boundary, we have that (5) turns into

\[
\frac{1}{8\pi} \left( \frac{2|\Sigma|}{4\pi} \right)^{1/2} (4\pi \chi(\Sigma) - \frac{1}{2} \int_\Sigma H_{\Sigma,M}^2 d\sigma)
\]

which reduces to

\[
\left( \frac{|\Sigma|}{8\pi} \right)^{1/2} (1 - \frac{1}{8\pi} \int_\Sigma H_{\Sigma,M}^2 d\sigma)
\]

when \(\Sigma\) is a standard hemisphere and \(\chi(\Sigma) \equiv 2 - 2g - b = 1\).
4. ISOPERIMETRIC MASS

Notations. We introduce two more notations used in this section.

\[ \sigma_{ij} = g_{ij} - \delta_{ij} \]

\[ \langle \cdot, \cdot \rangle_e \quad \text{inner product in Euclidean metric.} \]

We adapt it to our setting that

**Theorem 5.** \( m_{ISO} = m_{ADM} \).

**Proof.** Step 1, derivative of \( \mathcal{A} \).

Recall that \( d\sigma = (1 + h^i \sigma_{ij} + O(r^{-2\tau})) \frac{1}{2} d\bar{\sigma} \) (see [FST09, (2.7)]), then

\[ \mathcal{A}(r) = 2\pi r^2 + \frac{1}{2} \int_\Sigma h^i \sigma_{ij} d\bar{\sigma} + O(r^{1-2\tau}) \]

and

\[ \mathcal{A}'(r) = 4\pi r + \frac{1}{2} \int_\Sigma \frac{\partial}{\partial r} (h^i \sigma_{ij}) d\bar{\sigma} + \frac{1}{r} \int_\Sigma h^i \sigma_{ij} d\bar{\sigma} + O(r^{1-2\tau}) \]

\[ = 4\pi r + \frac{1}{2} \int_\Sigma \frac{\sigma_{i,kx^k}}{r} d\bar{\sigma} - \frac{1}{2} \int_\Sigma \frac{\sigma_{i,jkx^jx^k}}{r^3} d\bar{\sigma} + \frac{1}{r} \int_\Sigma h^i \sigma_{ij} d\bar{\sigma} + O(r^{1-2\tau}) \]  

(6)

Note that

\[ \int_\Sigma \frac{\sigma_{i,k} x^k x^j x^i}{r^3} d\bar{\sigma} = \int_\Sigma \frac{\partial}{\partial x^k} \left( \frac{\sigma_{i,jx^j}}{r} \right) \frac{x^j x^i}{r} d\bar{\sigma} \]

\[ = - \int_\Sigma \delta_{ik} - \frac{x^k x^i}{r^2} \frac{\partial}{\partial x^k} \left( \frac{\sigma_{i,jx^j}}{r} \right) d\bar{\sigma} + \int_\Sigma \frac{\partial}{\partial x^i} \left( \frac{\sigma_{i,jx^j}}{r} \right) d\bar{\sigma}. \]

Since \( \frac{x^k}{r} \partial_k \) is normal to \( \Sigma \) under Euclidean metric, \( \delta_{ik} - \frac{x^k x^i}{r^2} \) is the Euclidean metric projected to \( S_r \) (or in other words, induced metric), hence letting \( Y_j = \sigma_{kjx^k} \) and \( \bar{\epsilon}_1 = -\frac{\partial}{\partial x^j} \)

we see that by first variation formula under Euclidean metric,

\[ \int_\Sigma \delta_{ik} - \frac{x^k x^i}{r^2} \frac{\partial}{\partial x^k} \left( \frac{\sigma_{i,jx^j}}{r} \right) d\bar{\sigma} \]
\[ = \int_\Sigma \text{div} Y \]
\[ = \int_\Sigma H_c(Y, \tilde{n}) d\bar{\sigma} + \int_{\partial \Sigma} \langle \bar{\epsilon}_1, Y \rangle_c d\tilde{\theta} \]
\[ = \int_\Sigma \frac{2}{r} \sigma_{ij} \frac{x^i x^j}{r} d\bar{\sigma} - \int_{\partial \Sigma} \sigma_{ik} \frac{x^k}{r} d\tilde{\theta} \]
\[ = \int_\Sigma \frac{2 \sigma_{ij} x^i x^j}{r^3} d\bar{\sigma} - \int_{\partial \Sigma} \frac{g_{ij} x^i}{r} d\tilde{\theta}. \]
Hence
\[
\int_{\Sigma} \frac{\sigma_{ij,kl} x^i x^j r^k}{r^3} d\widehat{\sigma} = -2 \int_{\Sigma} \frac{\sigma_{ij} x^j}{r} d\widehat{\sigma} - \int_{\partial \Sigma'} g_{1a} \frac{x^a}{r} d\theta + \int_{\Sigma} \frac{\sigma_{ij} x^j}{r} d\widehat{\sigma} + \int_{\Sigma} \frac{\delta_{ij} - \frac{x^j x^j}{r^2}}{r^3} d\widehat{\sigma} = -2 \int_{\Sigma} \frac{\sigma_{ij} x^j}{r} d\widehat{\sigma} + \int_{\Sigma} \frac{\sigma_{1a}}{r} d\theta
\]
\[+ \int_{\Sigma} \frac{\sigma_{ij} x^j}{r} d\widehat{\sigma} + \int_{\partial \Sigma'} g_{1a} \frac{x^a}{r} d\theta \]
\[= -2 \int_{\Sigma} \frac{\sigma_{ij} x^j}{r} d\widehat{\sigma} + \int_{\Sigma} \frac{1}{r} \int_{\Sigma} h^i \sigma_{ij} d\sigma + O(r^{1-2\tau}). \tag{7}\]

Inserting this back to (6), we obtain
\[
\mathcal{A}'(r) = 4\pi r - \frac{1}{2} \left[ \int_{\Sigma} \left( \frac{\sigma_{ij} x^j}{r} - \frac{\sigma_{ii,kk}}{r^2} \right) d\sigma + \int_{\partial \Sigma'} g_{1a} \frac{x^a}{r} d\theta \right] 
\[+ \frac{1}{2r} \int_{\Sigma} h^i \sigma_{ij} d\sigma + \int_{\Sigma} \frac{\sigma_{ij} x^j}{r^2} d\sigma + O(r^{1-2\tau})
\[= 4\pi r - 4\pi m + \frac{1}{2r} \int_{\Sigma} h^i \sigma_{ij} d\sigma + \int_{\Sigma} \frac{\sigma_{ij} x^j}{r^2} d\sigma + O(r^{1-2\tau})
\[= 2\pi r - 4\pi m + \frac{\mathcal{A}(r)}{r} + \frac{1}{r} \int_{\Sigma} \frac{\sigma_{ij} x^j}{r^2} d\sigma + o(1), \tag{7}\]

where in the last line we used (6). \textbf{Step 2, asymptotics of volume } V(r). \textbf{Recall } \text{[FST09] (2.2)}, \textbf{.}

\[|\nabla r|^2 = 1 - \frac{\sigma_{ij} x^j}{r^2} + O(r^{-2\tau}).\]

By co-area formula,
\[
V'(r) = \int_{\Sigma} \frac{1}{|\nabla r|} d\sigma = \mathcal{A}(r) + \frac{1}{2} \int_{\Sigma} \frac{\sigma_{ij} x^j}{r^2} d\sigma + O(r^{1-2\tau}).
\]

Again by \(d\sigma = (1 + h^i \sigma_{ij} + O(r^{-2\tau}))^\frac{1}{2} d\Theta\), we have that
\[
V'(r) = \int_{\Sigma} \frac{1}{|\nabla r|} d\sigma = \mathcal{A}(r) + \frac{1}{2} \int_{\Sigma} \frac{\sigma_{ij} x^j}{r^2} d\sigma + O(r^{2-2\tau}).
\]

Note that the derivative \(\mathcal{A}'(r)\) in (7), we have that
\[
\mathcal{A}'(r) = 2\pi r - 4\pi m + \frac{\mathcal{A}(r)}{r} + \frac{2}{r} (V'(r) - \mathcal{A}(r)) + o(1)
\]
which gives
\[
(r \mathcal{A}(r))' = 2\pi r^2 - 4\pi mr + 2V'(r) + o(r).
\]

Integration then gives
\[
V(r) = \frac{1}{2} r \mathcal{A}(r) - \frac{1}{3} \pi r^3 + \pi m r^2 + o(r^2).
\]
Step 3, evaluation of isoperimetric mass.

\[
\frac{2}{\mathcal{A}(r)}\left(V(r) - \sqrt{2} \mathcal{A}(r) \sqrt{\frac{3}{3\pi}} \right) = \frac{2}{\mathcal{A}(r)} \left[ \frac{1}{2} r \mathcal{A}(r) - \frac{1}{2} \pi r^3 + \pi m r^2 \right] - \sqrt{2} \mathcal{A}(r) \sqrt{\frac{3}{3\pi}} + o(1) \\
= r + \sqrt{2} \mathcal{A}(r) \left( m - \frac{r}{3} \right) - \frac{2r}{3} \left( \frac{1}{2} \mathcal{A}(r) \right)^{1/2} + o(1) \\
= r + (m - \frac{r}{3}) (1 - \Theta + O(r^{-2\tau})) - \frac{2r}{3} \left( 1 + \frac{1}{2} \Theta + O(r^{-2\tau}) \right) + o(1) \\
= m + o(1),
\]

where we have used that
\[
\Theta := \frac{1}{4\pi r^2} \int_{\Sigma} h^{ij} \sigma_{ij} d\sigma = O(r^{-\tau}).
\]

so that
\[
\mathcal{A}(r) = 2\pi r^2 (1 + \Theta + O(r^{-2\tau})).
\]

\[\Box\]

Remark 4. Because we can also use arbitrary sets of finite perimeter \(\Omega_i\) to define the isoperimetric mass, i.e.

\[
\tilde{m}_{ISO} := \limsup_{\Omega_i \rightarrow M} \frac{2}{\partial^* \Omega_i \cap \partial M} \left( \mathcal{H}^3(\Omega_i) - \sqrt{2} \mathcal{A}(\Omega_i) \mathcal{A}(\partial M) \right) \frac{\sqrt{\mathcal{A}(\partial \Omega_i \cap \partial M)}}{6\sqrt{\pi}}
\]

where \(\partial^* \Omega\) is the reduced boundary of \(\Omega\). We see that \(\tilde{m}_{ISO} \geq m_{ISO} = m_{ADM}\).

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