Asymptotics of a singularly perturbed GUE partition function

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Abstract

We study the double scaling asymptotic limit for large matrix dimension $N$ of the partition function of the unitary ensemble with weight

$$w(x) := \exp \left( -\frac{z^2}{2x^2} + \frac{t}{x} - \frac{x^2}{2} \right).$$

We derive the asymptotics of the partition function when $z$ and $t$ are of $O(N^{-\frac{1}{2}})$. The results are obtained using the Deift-Zhou steepest descent method and are expressed in terms of a solution of a fourth order nonlinear differential equation. We also compute the asymptotic limit of such a solution when $zN^{\frac{1}{2}} \to 0$. The behavior of this solution, together with fact that the partition function is an odd function in the variable $t$, allows us to reduce such a fourth order differential equation into a second order nonlinear ODE.

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1 Introduction

In a previous article [23] we studied the asymptotics of the partition function

$$E_N(z, t) := \frac{1}{N!} \int_{\mathbb{R}^N} \prod_{j=1}^{N} \exp \left( -\frac{z^2}{2x_j^2} + \frac{t}{x_j} - \frac{x_j^2}{2} \right) \prod_{1 \leq j < k \leq N} |x_k - x_j|^2 d^N x$$ (1.1)

in the range $c_1 N^{-\frac{1}{2}} < z < c_2 N^{\frac{1}{4}}$, where $c_1$ and $c_2$ are two constants independent of $N$. In this paper we extend the results in [23] to cover the case when the parameters $t$ and $z$ are of the following order

$$t = \frac{u_1}{\sqrt{N}} (1 + O(N^{-\frac{1}{2}})), \quad z = \sqrt{\frac{u_2}{N}} (1 + O(N^{-\frac{1}{2}})), \quad \text{as} \ N \to \infty,$$ (1.2)

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where $u_1$ and $u_2$ remain finite.

Partition functions of the form

$$\frac{1}{N!} \int_{J^N} \prod_{j=1}^{N} w(x_j) \prod_{1 \leq j < k \leq N} |x_k - x_j|^2 d^N x,$$  

(1.3)

where $w(x) \geq 0$ is the weight and $J \subseteq \mathbb{R}$ or $J = S^1$, are ubiquitous in Random Matrix Theory because they contain all the information on the correlations of the eigenvalues. The asymptotic properties of the integral (1.3) are determined by the analytic properties of $w(x)$; the weight of the partition function (1.1) has an essential singularity at the origin, which makes its computation particularly difficult.

Besides being an interesting mathematical problem in its own right, the knowledge of $E_N(z, t)$ is important for several reasons. It determines the value distribution of

$$Q_N(x) := \frac{\Lambda_N^2(x)}{\Lambda_N^2(x) - \Lambda_N(x) \Lambda_N'(x)},$$

(1.4)

where

$$\Lambda_N(x) := \prod_{j=1}^{N} (x - x_j)$$

(1.5)

and $x_1, \ldots, x_N$ are real random variables governed by a given joint probability density function (j.p.d.f.). Berry and Shukla [4] discovered that the value distribution of $Q_N$ is a sensitive measure of the rigidity of the zeros of $\Lambda_N(x)$. Furthermore, they showed that all the moments of the distribution of $Q_N$ can be computed from (1.1).

The function (1.4) (or more precisely the denominator of $Q_N$) was introduced by Tuck [25] in a numerical study of the Riemann zeta function. Tuck replaced $\Lambda_N$ by the Hardy function

$$Z(t) := t^{1/4} \exp \left( \frac{\pi}{4} \right) \pi^{-\frac{1}{2}i t} \Gamma \left( \frac{1}{4} + \frac{1}{2}it \right) \zeta \left( \frac{1}{2} + it \right),$$

(1.6)

where $\zeta(z)$ is the Riemann zeta function. He realized that the Riemann hypothesis implies $Q_N > 0$. (Note that the opposite is not true.)

The quantity $E_N(z, t)$ can also be used for other purposes. For example, if we set

$$s = z^2/2 = -t, \quad s > 0,$$  

(1.7)

then $N! E_N(\sqrt{2s}, -s)$ becomes the moment generating function of the singular linear statistics

$$\sum_{j=1}^{N} \left( \frac{1}{x_j^2} + \frac{1}{x_j} \right).$$

(1.8)

The function

$$w(x) = \exp \left( -\frac{z^2}{2x^2} + \frac{t}{x} - \frac{x^2}{2} \right)$$

(1.9)
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can be thought of as a singular perturbation of the GUE weight \( e^{-x^2/2} \), in which an essential singularity emerges. The asymptotics of the partition function with the weight (1.9) can be computed using the method of orthogonal polynomials. Then, one needs to study the asymptotics of the polynomials orthogonal with respect to the weight (1.9). This can be achieved by using the steepest descent method developed by Deift et al in [15], [16] (see also [7]).

When the parameter \( z \), and hence the singular perturbation, is large, the asymptotics of the orthogonal polynomials can be found with the help of the equilibrium measure, which is supported on two disjoint intervals. This was done in [23] (see also [9], [10]). The asymptotics of the orthogonal polynomials can then be used to find the logarithmic derivatives of the partition function with respect to the parameters \( z \) and \( t \). However, as the parameters \( z \) and \( t \) become small, the gap between the two intervals in the support of the equilibrium measure becomes small as the singularity disappears. Therefore, the results in [23] no longer provide an accurate description to the asymptotic behavior of the partition function (1.1). The purpose of the current paper is to obtain the asymptotics of \( E_N(z,t) \) when the parameters \( z \) and \( t \) are small. This requires the use of Painlevé type nonlinear differential equations.

In order to obtain the asymptotics of the orthogonal polynomials with weight (1.9), in §3 we express the orthogonal polynomials as a solution of a Riemann-Hilbert problem. In §4 the equilibrium measure for the GUE is used to modify such a Riemann-Hilbert problem into explicitly solvable ‘model Riemann-Hilbert problems’ in different regions of the complex plane, which allow us to study the asymptotic behavior of their solutions. Since an essential singularity emerges at the origin, the model Riemann-Hilbert problem in a small neighborhood of the origin is solved using a local parametrix constructed from a special solution of a nonlinear differential equation (§5). This technique is widely used in the studies of double scaling limits in Random Matrix Theory (see e.g. [8], [11], [12], [13]).

Usually, the local parametrix is expressed in terms of a special solution of a nonlinear differential equation, which in turn gives the compatibility conditions for the following system of linear ODEs:

\[
\partial_\zeta \Phi(\zeta, u) = A(\zeta, u)\Phi(\zeta, u), \quad \partial_u \Phi(\zeta, u) = B(\zeta, u)\Phi(\zeta, u) = 0, \tag{1.10}
\]

where \( u \) is a parameter measuring the strength of the perturbation from a critical point. In our case the parameter \( u \) is replaced by \( u_1 \) and \( u_2 \) in (1.2). The matrices \( A(\zeta) \) and \( B(\zeta) \) are rational functions of \( \zeta \).

The compatibility conditions for (1.10) are among isomonodromic deformations which include the Painlevé equations. In most of the previous studies of double scaling limits appearing in Random Matrix Theory, one parameter was sufficient to represent the whole family of perturbations from the critical point. Hence the conditions in (1.10) gave rise to a nonlinear ODE describing the double scaling limit. In most cases such double scaling limits are described by the Painlevé equations.

In our study, however, the local parametrix turns out to be more complicated. Indeed compatibility conditions analogous to (1.10) give us a nonlinear PDE instead. By using
the Hamiltonian theory of isomonodromic deformations, we are able to express such a PDE in terms of nonlinear ODEs in the variables $u_1$ and $u_2$. This simplifies the problem considerably from the computational point of view. The ODEs that we obtain are of fourth order, and have not been appeared in the studies of Random Matrix Theory before. The Hamiltonian approach used in this paper can also be adopted in other more general cases of double scaling limits to express local parametrices in terms of solutions of nonlinear ODEs (see Theorem 2.1).

After deriving the nonlinear ODEs describing the local parametrix, in §9 we study its behavior as $u_2 \to 0$ to obtain the initial conditions. Such initial conditions, together with the fact that the partition function $E_N(z, t)$ in (1.1) is odd in $t$, allow us to simplify the nonlinear ODEs further and describe the series expansion of the partition function in $t$ by a solution of a second order nonlinear ODE (11). Since this ODE is a reduction from an isomonodromic deformation that satisfies the Painlevé property, it should satisfy the Painlevé property too. Hence, it should be equivalent to one of the Painlevé equations. However, we have been unable to transform it into a Painlevé equation.

2 Statement of results

The average (1.1) is an entire function of $t$, therefore its Taylor series has an infinite radius of convergence and we can write

$$E_N(z, t) = \sum_{m=0}^{\infty} E_{Nm}(z)t^m. \quad (2.1)$$

In this paper we will assume the parameters $z$ and $t$ are of the order indicated in (1.2) and compute the leading asymptotics of the coefficients $E_{Nm}(z)$ in (2.1). The case when $z$ is of larger order was considered in [23]. When $z$ and $t$ are in the range studied in this article, the ensemble average (1.1) is given in terms of a solution of a fourth order nonlinear differential equation. Such an equation can be expressed in terms of a time-dependent Hamiltonian system and is related to a particular isomonodromic deformation. The Hamiltonian equations are given by the following.
Theorem 2.1. Let $u_1$ and $u_2$ be given by (1.2) and $H_1$, $H_2$, $h_1$ and $h_2$ by

$$H_1 = 2i\frac{P_2}{u_2} + \frac{1}{2}u_1Q_2 - \frac{1}{2}u_2Q_1Q_2 - \frac{1}{4u_2}u_1^2Q_1 + \frac{1}{2}u_1Q_1^2 - \frac{1}{4}u_2Q_1^3$$
$$+ \frac{2}{u_2}P_1P_2Q_2 + \frac{1}{u_2}P_1^2Q_1,$$
$$H_2 = -\frac{iP_1}{u_2} + \frac{iP_2Q_1}{u_2} + \frac{1}{8}u_2Q_2^2Q_2 + \frac{1}{8}u_2Q_1^2 - \frac{1}{2u_2}P_2^2Q_2$$
$$- \frac{1}{2u_2}P_2^2Q_2^2 - \frac{iP_2u_1}{u_2} + \frac{1}{8u_2}u_1^3Q_1 - \frac{1}{4u_2}u_1^2Q_1^2$$
$$+ \frac{1}{8}u_1Q_1^3 - \frac{u_1}{u_2}P_1P_2Q_2 - \frac{u_1}{2u_2}P_1Q_1 - \frac{u_1}{2u_2}P_1,$$
$$h_1 = \frac{P_1}{u_2}, \quad h_2 = -\frac{P_2Q_2}{u_2} - \frac{P_1Q_1}{2u_2} - \frac{u_1}{2u_2}P_1,$$

where the variables $P_1$, $Q_1$, $P_2$ and $Q_2$ satisfy the following time-dependent Hamiltonian equations,

$$\frac{\partial P_k}{\partial u_j} = -\frac{\partial H_j + h_j}{\partial Q_k}, \quad \frac{\partial Q_k}{\partial u_j} = \frac{\partial H_j + h_j}{\partial P_k}, \quad j, k = 1, 2. \quad (2.3)$$

Then the logarithmic derivatives of the average in (1.1) are given by $H_1$ and $H_2$ as $N \to \infty$.

$$\frac{\partial \log E_N}{\partial t} = -\sqrt{N}H_1 + O(N^{-\frac{1}{2}}), \quad (2.4)$$
$$\frac{\partial \log E_N}{\partial z} = -2\sqrt{N}H_2 + O(1).$$

The solution of the Hamiltonian equations (2.3) can be characterized by a Riemann-Hilbert problem which in turn gives the behavior of the canonical variables $P_1$, $Q_1$, $P_2$ and $Q_2$ as $u_2 \to 0$. This is computed in [10] and the asymptotic behavior is given in Theorem 9.1.

It is often useful to write $E_N(z, t)$ as a power series expansion in $t$. For this purpose, it is sufficient to consider the equation (2.3) for $j = 2$ at $u_1 = 0$. In this case, the asymptotic behavior of the canonical variables $P_1$, $Q_1$, $P_2$ and $Q_2$ is much simpler

$$P_1 = i \left( \frac{\sqrt{2}}{2\pi} + \frac{2u_2}{3} \right) + \left( \frac{2\sqrt{2}}{\pi} - \frac{3}{\sqrt{2}\pi} \right) u_2^\frac{1}{2} + \left( \frac{2}{3} + \frac{8}{9\pi} \right) u_2 + O(u_2^\frac{3}{2}),$$
$$Q_1 = i \left( \frac{2}{u_2\pi} - 2 \right) + \left( \frac{2\sqrt{2}}{\pi} + \frac{2\sqrt{2}}{\sqrt{\pi}} - \frac{2\sqrt{2}}{\sqrt{\pi}} \right) u_2^\frac{1}{2} + \left( \frac{16}{\pi^2} - \frac{64}{9\pi} \right) u_2 + O(u_2^\frac{3}{2}),$$
$$P_2 = O(u_2^3),$$
$$Q_2 = -\frac{2\sqrt{2}}{\sqrt{\pi}u_2} + 2 - \frac{4}{\pi} + \left( \frac{2}{\pi} + \frac{2}{3}\pi - \frac{2}{\sqrt{\pi}} \right) u_2^\frac{1}{2} + \left( \frac{32}{9\pi} - \frac{8}{\pi^2} \right) u_2 + O(u_2^3) \quad (2.5)$$
From (1.1), it is easy to see that $E_\ell(z, t)$ is an odd function in $t$. This symmetry condition, together with the relation between $H_1$ and the logarithmic derivative of $E_\ell(z, t)$, allow us to obtain algebraic relations between the canonical variables $P_1, P_2, Q_1$ and $Q_2$.

$$P_2 = 0, \quad Q_2 = -\frac{u_2^2 Q_1^2 - 4 P_1^2}{2u_2^2}. \quad (2.6)$$

As a result, when $u_1 = 0$, the four ODEs (2.3) can be reduced to two first order ODEs.

**Theorem 2.2.** Let $H_0$ and $h_0$ be the following

$$H_0 = -\frac{(u_2^2 Q^2 - 4 P^2)^2}{32 u_2^3} - \frac{i P}{u_2}, \quad h_0 = -\frac{P Q}{2u_2}.$$  

Then as $N \to \infty$, the average (1.1) is given by

$$E_\ell(z, t) = (2\pi)\frac{N}{2} \prod_{j=1}^{N-1} j! \exp \left( \left( - \int_0^{N^2} H_0 du_2 - \sum_{j=1}^{\infty} \frac{\partial^{2j-1} H_1}{\partial u_1^{2j-1}} \bigg|_{u_1=0} \frac{(\sqrt{N^2} t)^2 j}{(2j)!} \right) \left( 1 + O\left(N^{-\frac{1}{2}}\right) \right) \right),$$

where the variables $P, Q$ satisfy the following time-dependent Hamiltonian equation in $u_2$,

$$\frac{\partial P}{\partial u_2} = \frac{\partial (H_0 + h_0)}{\partial Q}, \quad \frac{\partial Q}{\partial u_2} = \frac{\partial (H_0 + h_0)}{\partial P},$$

which is the following

$$\partial_{u_2} P = \frac{Q (u_2^2 Q^2 - 4 P^2)}{8u_2} + \frac{P}{2u_2}, \quad \partial_{u_2} Q = \frac{P (u_2^2 Q^2 - 4 P^2)}{2u_2^2} - \frac{Q}{2u_2} - \frac{i}{u_2}.$$  

The canonical variables $P, Q$ behaves as follows as $u_2 \to 0$.

$$P = i \left( \sqrt{\frac{u_2}{2\pi}} + \frac{2u_2}{\pi} + \left( \frac{2\sqrt{2}}{\pi} - \frac{\sqrt{2}}{\pi^2} - \frac{3}{\sqrt{2}} \right) u_2^\frac{3}{2} + \left( \frac{2}{3} + \frac{8}{\pi^2} - \frac{32}{9\pi} \right) u_2^2 \right) + O(u_2^\frac{5}{2}),$$  

$$Q = i \left( \sqrt{\frac{2}{u_2^2\pi}} - 2 \frac{4}{\pi} + \left( \frac{3\sqrt{2}}{\sqrt{\pi}} + \frac{2\sqrt{2}}{\pi^2} - \frac{3\sqrt{2}}{\pi^2} - \frac{2\sqrt{2}}{\pi^2} \right) u_2^\frac{3}{2} + \left( \frac{16}{\pi^2} - \frac{64}{9\pi} \right) u_2 \right) + O(u_2^\frac{5}{2}).$$

The derivatives $\frac{\partial^{i-1} H_1}{\partial u_1^i}$ in the asymptotic formula of $E_\ell(z, t)$ can be expressed in terms of the canonical variables $P, Q$ by using (2.3) and the substitution $P = P_1, Q = Q_1$, together with the algebraic relations (2.6). For example, we have

$$\left. \frac{\partial H_1}{\partial u_1} \right|_{u_1=0} = \frac{P^2}{u_2^2} - \frac{Q^2}{4},$$  

$$\left. \frac{\partial^3 H_1}{\partial u_1^3} \right|_{u_1=0} = \left( \frac{3}{u_2^2} - \frac{1}{8u_2^4} \right) (u_2^2 Q^2 - 4 P^2)^2 + \frac{2i P}{u_2^2}.$$  

This then gives the leading order asymptotics for the average (1.1).
3 Preliminaries

Let us introduce the variables $u_{1,N}$ and $u_{2,N}$ by

$$u_{1,N} = \sqrt{N} t, \quad u_{2,N} = Nz^2$$

(3.1)

The weight (1.9) then becomes

$$w_N(y) := \exp \left( -N \left( \frac{u_{2,N}^2}{2N^3y_j^2} + \frac{y_j^2}{2} \right) + \frac{u_{1,N}}{Ny_j} \right).$$

(3.2)

We can also define

$$G_N(v_1, v_2) := \frac{1}{N!} \int_{\mathbb{R}^N} \prod_{j=1}^N \exp \left( -N \left( \frac{u_{2,N}^2}{2N^3y_j^2} + \frac{y_j^2}{2} \right) + \frac{u_{1,N}}{Ny_j} \right) \prod_{1 \leq j < k \leq N} |y_k - y_j|^2 d^N y,$$

(3.3)

which is proportional to $E_N(z, t)$, namely

$$E_N(z, t) = N^{N^2/2} G_N(u_{1,N}, u_{2,N}).$$

(3.4)

Let us denote the polynomials orthogonal with respect to $w_N(y)$ by $\pi_j(y)$ and write

$$G_N(u_{1,N}, u_{2,N}) = \det (\mu_{j+k})_{j,k=0}^{N-1} = \prod_{j=0}^{N-1} h_j,$$

(3.5)

where

$$\mu_j := \int_{-\infty}^{\infty} w_N(y)y^j dx, \quad j = 0, 1, \ldots$$

(3.6)

and

$$\int_{-\infty}^{\infty} w_N(y)\pi_j(y)\pi_k(y) dy = h_j \delta_{jk}.$$  

(3.7)

Let us define the matrix valued function

$$Y(y) := \begin{pmatrix} \pi_N(y) \\ \kappa_{N-1}\pi_{N-1}(y) \end{pmatrix} = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\pi_N(s)w_N(s)}{s-y} ds,$$

(3.8)

where $\kappa_{N-1} = -2\pi i h_{N-1}$. It solves the following R-H problem:

1. $Y(y)$ is analytic in $\mathbb{C}/\mathbb{R}$,
2. $Y_+(y) = Y_-(y) \begin{pmatrix} 1 & w_N(y) \\ 0 & 1 \end{pmatrix}$, \quad $y \in \mathbb{R}$
3. $Y(y) = (I + O(y^{-1})) \begin{pmatrix} y^N & 0 \\ 0 & y^{-N} \end{pmatrix}$, \quad $y \to \infty$

(3.9)

where $Y_+(y)$ and $Y_-(y)$ denotes the limiting values of $Y(y)$ as it approaches the left and right hand side of the real axis. It turns out that the partition function $G_N(v_1, v_2)$ can be expressed in terms of $Y(y)$. 


Lemma 3.1 (Bertola, Eynard and Hanard [6]). (See also [23]) The following differential identities hold:

\[
\frac{\partial \log G_N}{\partial u_{1,N}} = - \frac{1}{4\pi iN} \oint_{y=0} \frac{1}{y} \text{Tr} \left( Y^{-1}(y) Y'(y) \sigma_3 \right) dy, \quad (3.10a)
\]

\[
\frac{\partial \log G_N}{\partial u_{2,N}} = \frac{1}{8\pi iN^2} \oint_{y=0} \frac{1}{y^2} \text{Tr} \left( Y^{-1}(y) Y'(y) \sigma_3 \right) dy, \quad (3.10b)
\]

where the contour of integration is a small loop around \(y = 0\) oriented counter-clockwise.

Note that, although the function \(Y(y)\) has a jump discontinuity on the real axis, it has a uniform asymptotic expansion near the origin and the residues in the right hand sides of (3.10) are computed using this uniform asymptotic expansion.

4 Riemann-Hilbert analysis

In this Section we will apply the Deift-Zhou steepest descent analysis to the Riemann-Hilbert problem (3.9).

4.1 First transformation of the Riemann-Hilbert problem

Let us now define the function \(g(y)\) to be

\[
g(y) = \int_{\mathbb{R}} \log(y - s) \rho(s) ds, \quad (4.1)
\]

where \(\rho(s)\) is the equilibrium measure for the potential \(V(y) = \frac{y^2}{2}\):

\[
\rho(s) = \frac{1}{2\pi} \sqrt{4 - s^2}. \quad (4.2)
\]

That is, we have

\[
2 \int_{-\infty}^{\infty} \log |y - s| \rho(s) ds - \frac{y^2}{2} = l, \quad y \in [-2, 2],
\]

\[
2 \int_{-\infty}^{\infty} \log |y - s| \rho(s) ds - \frac{y^2}{2} < l, \quad y \in \mathbb{R} \setminus [-2, 2]. \quad (4.3)
\]

for some constant \(l\). Now let \(T(y), \tilde{g}(y)\) and \(F(y)\) be

\[
T(y) = e^{-\frac{N\sigma_3}{2}} Y(y) e^{-N g(y) \sigma_3} e^{N \sigma_3}, \quad \tilde{g}(y) = \frac{y^2}{4} - g(y),
\]

\[
F(y) = -\frac{1}{2} \left( \frac{u_{2,N}}{2N^2y^2} - \frac{u_{1,N}}{N^2y} \right). \quad (4.4)
\]
Then $T(y)$ is a solution to the following Riemann-Hilbert problem:

1. $T(y)$ is analytic in $\mathbb{C} \setminus \mathbb{R},$
2. $T_+(y) = T_-(y) J_T(y), \quad y \in \mathbb{R},$
3. $T(y) = I + O(y^{-1}), \quad y \to \infty,$

where $J_T(y)$ is the following matrix on $\mathbb{R}$.

$$J_T(y) = \begin{pmatrix} e^{-N(g_+(y) - g_-(y))} & e^{-N(\tilde{g}_+(y) + \tilde{g}_-(y)) + 2F(y)} \\ 0 & e^{N(g_+(y) - g_-(y))} \end{pmatrix}, \quad y \in \mathbb{R}.\quad (4.5)$$

### 4.2 Opening of lens

We now perform a standard technique in the steepest decent method (see [7], [15], [16]). Let $y_0$ be a point $y_0 = O(N^{-1})$ and define the lens contours around $[-2,2]$ as shown in Figure 1. Let the matrix $S(y)$ be

$$S(y) = \begin{cases} T(y), & y \text{ outside the lens}, \\ T(y) \begin{pmatrix} 1 \\ -e^{2N\tilde{g}(y)-2F(y)} \end{pmatrix}, & y \in L_{+,j}, \\ T(y) \begin{pmatrix} 1 \\ e^{2N\tilde{g}(y)-2F(y)} \end{pmatrix}, & y \in L_{-,j}. \end{cases} \quad (4.5)$$

for $j = 1, 2$. Then, the function $S(y)$ satisfies the following Riemann-Hilbert problem:

1. $S(y)$ is analytic in $\mathbb{C} \setminus \mathbb{R};$
2. $S_+(y) = S_-(y) J_S(y), \quad y \in \mathbb{R};$
3. $S(y) = I + O(y^{-1}), \quad y \to \infty.$

(4.6)
where $J_S(y)$ is now defined by the following

$$J_S(y) = \begin{pmatrix} 1 & 0 \\ e^{2N\tilde{g}(y)-2\hat{F}(y)} & 1 \end{pmatrix}, \quad y \in \Xi, \quad j = 1, 2,$$

$$J_S(y) = \begin{pmatrix} 0 & e^{-\frac{2N, N}{2N, N}} + \frac{w_1, N}{N} \\ e^{-\frac{2N, N}{2N, N}} + \frac{w_1, N}{N} & 0 \end{pmatrix}, \quad y \in [-2, -y_0] \cup [y_0, 2], \quad (4.7)$$

$$J_S(y) = \begin{pmatrix} e^{N(\tilde{g}_+(y)-\tilde{g}_-(y))} & e^{-2N\tilde{g}(y)-2\hat{F}(y)} \\ 0 & e^{-N(\tilde{g}_+(y)-\tilde{g}_-(y))} \end{pmatrix}, \quad y \in \mathbb{R} \setminus ([-2, -y_0] \cup [y_0, 2]). \quad (4.8)$$

By the inequalities (4.3), we see that outside of some discs $D_{\pm 2}$ and $D_0$ of radius $\delta$ centered at the points $\pm 2$ and zero, the jump matrix $J_S(y)$ can be approximated by

$$J_S(y) = \begin{pmatrix} 0 & e^{-\frac{2N, N}{2N, N}} + \frac{w_1, N}{N} \\ e^{-\frac{2N, N}{2N, N}} + \frac{w_1, N}{N} & 0 \end{pmatrix} = (I + O(N^{-1})) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad (4.8)$$

This suggests the following approximation to $S(y)$ outside of $D_0$ and $D_{\pm 2}$:

1. $S^\infty(y)$ is analytic in $\mathbb{C} \setminus [-2, 2]$;
2. $S_+^\infty(y) = S_-^\infty(y) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad y \in [-2, 2]; \quad (4.9)$
3. $S^\infty(y) = I + O(y^{-1}), \quad y \to \infty.$

The outer parametrix that solves (4.9) can now be constructed as in [15] and [14]:

$$S^\infty(y) = \begin{pmatrix} \frac{y+2}{y+2} & \frac{y-2}{y+2} \\ \frac{2-y}{2} & \frac{2y-2}{2} \end{pmatrix}, \quad \gamma = \left(\frac{y-2}{y+2}\right)^{1/2}, \quad (4.10)$$

where the branch cut is chosen to be on $[-2, 2]$ and $\gamma \sim 1$ as $y \to \infty.$

## 5 Local parametrix near the origin

Near the origin, the approximation of $S(y)$ by $S^\infty(y)$ fails and we must find an exact solution to the Riemann-Hilbert problem for $S(y)$ and match it with $S^\infty(y)$ as $y$ moves away from the origin. Let $\Xi$ be the union of the lens contours, then inside the neighborhood $D_0$, we want to solve the following Riemann-Hilbert problem:

1. $S^{(0)}(y)$ is analytic in $D_0 \setminus (D_0 \cap (\mathbb{R} \cup \Xi))$,
2. $S^{(0)}_+(y) = S^{(0)}_-(y)J_S(y), \quad y \in D_0 \cap (\mathbb{R} \cup \Xi), \quad (5.1)$
3. $S^{(0)}(y) = (I + O(N^{-1})) S^\infty(y), \quad z \in \partial D_0.$
where the jump matrices are given by

\[
J_S(y) = \begin{pmatrix}
\frac{1}{e^{2N\tilde{g}(y)}} & \frac{u_{2,N}}{2N^2y^2} & \frac{u_{1,N}}{N_y} \\
\frac{u_{2,N}}{2N^2y^2} & \frac{u_{1,N}}{N_y} & 0 \\
\frac{u_{1,N}}{N_y} & 0 & 1
\end{pmatrix}, \quad y \in \Xi_{\pm,j} \cap D_0, \quad j = 1, 2,
\]

\[
J_S(y) = \begin{pmatrix}
0 & e^{-\frac{u_{2,N}}{2N^2y^2} + \frac{u_{1,N}}{N_y}} \\
-e^{-\frac{u_{2,N}}{2N^2y^2} + \frac{u_{1,N}}{N_y}} & e^{-\frac{u_{2,N}}{2N^2y^2} + \frac{u_{1,N}}{N_y}} \\
0 & e^{-\frac{u_{2,N}}{2N^2y^2} + \frac{u_{1,N}}{N_y}}
\end{pmatrix}, \quad y \in (-\delta, -y_0) \cup (y_0, \delta),
\]

\[
J_S(y) = \begin{pmatrix}
e^{2N(\tilde{g}_+(y) - \tilde{g}_+(0))} & e^{-\frac{u_{2,N}}{2N^2y^2} + \frac{u_{1,N}}{N_y}} \\
e^{-2N(\tilde{g}_+(y) - \tilde{g}_+(0))} & e^{-\frac{u_{2,N}}{2N^2y^2} + \frac{u_{1,N}}{N_y}}
\end{pmatrix}, \quad y \in (-y_0, y_0),
\]

5.1 Conformal maps inside $D_0$

Let $\zeta$ be the analytic continuation of the following map in $D_0$:

\[
\zeta = -iN (\tilde{g}_+(y) - \tilde{g}_+(0)) = N \left(y + O(y^3)\right).
\]

Note that $\tilde{g}_+(0) \in i\mathbb{R}$.

For small enough $\delta$, $\zeta$ is a conformal map inside $D_0$ that maps the boundary of $D_0$ into the point $\zeta = \infty$ as $N \to \infty$. Let us define $u_2(y)$ and $u_1(y)$ in $D_0$ to be

\[
u_2(y) = u_{2,N} \frac{\zeta^2}{N^2y^2} = u_{2,N}(1 + O(y^2)), \quad u_1(y) = u_{1,N} \frac{\zeta}{Ny} = u_{1,N}(1 + O(y^2)).
\]

Then, for small enough $\delta$, both these maps are conformal inside $D_0$ and we have

\[
2N\tilde{g}(y) + \frac{u_{2,N}}{2N^2y^2} - \frac{u_{1,N}}{Ny} = \pm 2i\zeta + \frac{u_2(y)}{2\zeta^2} - \frac{u_1(y)}{\zeta} \pm 2N\tilde{g}_+(0), \quad \pm \text{Im}(\zeta) > 0.
\]

Let $\Gamma$ be the union of the contours in the right hand side of Figure 2. Consider the following Riemann-Hilbert problem in the $\zeta$ plane:

1. $P(\zeta)$ is analytic in $\mathbb{C} \setminus \Gamma$,
2. $P_+(\zeta) = P_-(\zeta)J_P(\zeta), \quad \zeta \in \Gamma$,
3. $P(\zeta) = (I + O(\zeta^{-1})) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \text{Im}(\zeta) > 0, \quad \zeta \to \infty.$

\[
P(\zeta) = I + O(\zeta^{-1}), \quad \text{Im}(\zeta) < 0, \quad \zeta \to \infty.
\]
where the jump matrices $J_P(\zeta)$ are given by the following

\[
J_P(\zeta) = \begin{pmatrix}
1 & 0 \\
e^{2i\zeta + \frac{u_2(y)}{2\zeta^2} - \frac{u_1(y)}{\zeta}} & 1
\end{pmatrix}, \quad \zeta \in \Gamma_1 \cup \Gamma_3,
\]

\[
J_P(\zeta) = \begin{pmatrix}
1 & 0 \\
e^{-2i\zeta + \frac{u_2(y)}{2\zeta^2} - \frac{u_1(y)}{\zeta}} & 1
\end{pmatrix}, \quad \zeta \in \Gamma_5 \cup \Gamma_6,
\]

\[
J_P(\zeta) = \begin{pmatrix}
0 & e^{\frac{u_2(y)}{2\zeta^2} + \frac{u_1(y)}{\zeta}} \\
e^{-\frac{u_2(y)}{2\zeta^2} + \frac{u_1(y)}{\zeta}} & 0
\end{pmatrix}, \quad \zeta \in \Gamma_0 \cup \Gamma_4,
\]

\[
J_P(\zeta) = \begin{pmatrix}
e^{2i\zeta} & e^{-\frac{u_2(y)}{2\zeta^2} + \frac{u_1(y)}{\zeta}} \\
e^{-2i\zeta} & e^{-2i\zeta}
\end{pmatrix}, \quad \zeta \in \Gamma_2,
\]

(5.7)

Then the local parametrix in (5.1) can be obtained from the solution of (5.6) by

\[
S^{(0)}(y) = S^{\infty}(y)e^{-N\hat{g}_+(0)\sigma_3} \begin{pmatrix}
0 & -1 \\
1 & 0
\end{pmatrix} P(\zeta)e^{N\hat{g}_+(0)\sigma_3}, \quad \text{Im}(y) > 0,
\]

\[
S^{(0)}(y) = S^{\infty}(y)e^{N\hat{g}_+(0)\sigma_3} P(\zeta)e^{-N\hat{g}_+(0)\sigma_3}, \quad \text{Im}(y) < 0,
\]

as $N\hat{g}_+(0) \in i\mathbb{R}$.

5.2 Existence of the local parametrix

We will now show that the Riemann-Hilbert problem (5.6) has a unique solution. First let us define

\[
\hat{P}(\zeta) = \begin{cases}
P(\zeta) \begin{pmatrix}
0 & -1 \\
1 & 0
\end{pmatrix}, & \text{Im}(\zeta) > 0; \\
P(\zeta), & \text{Im}(\zeta) < 0.
\end{cases}
\]

Then $\hat{P}(\zeta)$ is a solution of the Riemann-Hilbert problem

1. $\hat{P}(\zeta)$ is analytic in $\mathbb{C} \setminus \Gamma$,
2. $\hat{P}_+(\zeta) = \hat{P}_-(\zeta)J_P(\zeta), \quad \zeta \in \Gamma,$
3. $\hat{P}(\zeta) = I + O(\zeta^{-1}), \quad \zeta \to \infty,$

(5.9)
where the jump matrices \( J_\rho(\zeta) \) are given by

\[
J_\rho(\zeta) = \begin{pmatrix} 1 & -e^{2i\zeta + \frac{u_2(y)}{2\zeta^2} - \frac{u_1(y)}{\zeta}} \\ 0 & 1 \end{pmatrix}, \quad \zeta \in \Gamma_1 \cup \Gamma_3,
\]

\[
J_\rho(\zeta) = \begin{pmatrix} 1 & 0 \\ e^{-2i\zeta + \frac{u_2(y)}{2\zeta^2} - \frac{u_1(y)}{\zeta}} & 1 \end{pmatrix}, \quad \zeta \in \Gamma_5 \cup \Gamma_6,
\]

\[
J_\rho(\zeta) = \begin{pmatrix} e^{-\frac{u_2(y)}{2\zeta^2} + \frac{u_1(y)}{\zeta}} & 0 \\ 0 & e^{\frac{u_2(y)}{2\zeta^2} - \frac{u_1(y)}{\zeta}} \end{pmatrix}, \quad \zeta \in \Gamma_0 \cup \Gamma_4,
\]

\[
J_\rho(\zeta) = \begin{pmatrix} e^{-\frac{u_2(y)}{2\zeta^2} + \frac{u_1(y)}{\zeta}} & -e^{2i\zeta} \\ e^{-2i\zeta} & 0 \end{pmatrix}, \quad \zeta \in \Gamma_2.
\]

(5.10)

Since the Riemann-Hilbert problem and the jump matrices have no singularities at the self-intersection points \( \pm y_0 \) of the contours, and since the jump matrices satisfy the cyclic conditions

\[
J_{\rho,2}^{-1} J_{\rho,6} J_{\rho,0} J_{\rho,1} = I,
\]

\[
J_{\rho,2}^{-1} J_{\rho,5} J_{\rho,4} J_{\rho,3} = I;
\]

at the points \( \pm y_0 \), where \( J_{\rho,k} \) is the jump matrix on \( \Gamma_k \), the theory in [26] can be applied to (5.9). In particular, (5.9) has a unique solution if there is no non-trivial function \( \hat{P}_0(\zeta) \) that satisfies 1. and 2. in (5.9) and behaves as \( \hat{P}_0(\zeta) = O(\zeta^{-1}) \) as \( \zeta \to \infty \). Suppose that such a function exists and let the region between \( \Gamma_1 (\Gamma_3) \) and \( \Gamma_0 (\Gamma_4) \) be \( \Omega_{0,+} (\Omega_{4,+}) \) and the region between \( \Gamma_0 (\Gamma_4) \) and \( \Gamma_6 (\Gamma_5) \) be \( \Omega_{0,-} (\Omega_{4,-}) \). Then, define the function \( X(\zeta) \) to be

\[
X(\zeta) = \hat{P}_0(\zeta), \quad \zeta \in \mathbb{C} \setminus \Omega,
\]

\[
X(\zeta) = \hat{P}_0(\zeta) \begin{pmatrix} 1 & -e^{2i\zeta + \frac{u_2(y)}{2\zeta^2} - \frac{u_1(y)}{\zeta}} \\ 0 & 1 \end{pmatrix}, \quad \zeta \in \Omega_{0,+} \cup \Omega_{4,+},
\]

\[
X(\zeta) = \hat{P}_0(\zeta) \begin{pmatrix} 1 & 0 \\ -e^{-2i\zeta + \frac{u_2(y)}{2\zeta^2} - \frac{u_1(y)}{\zeta}} & 1 \end{pmatrix}, \quad \zeta \in \Omega_{0,-} \cup \Omega_{4,-},
\]

where \( \Omega = \Omega_{0,+} \cup \Omega_{0,-} \cup \Omega_{4,+} \cup \Omega_{4,-} \). Then as in [16], \( X(\zeta) \) has the following properties:

1. \( X(\zeta) \) is analytic in \( \mathbb{C} \setminus \mathbb{R} \) and continuous down to \( \mathbb{R} \),

2. \( X_+(\zeta) = X_-(\zeta) \begin{pmatrix} e^{-\frac{u_2(y)}{2\zeta^2} + \frac{u_1(y)}{\zeta}} & -e^{2i\zeta} \\ e^{-2i\zeta} & 0 \end{pmatrix}, \quad \zeta \in \mathbb{R}, \quad (5.11) \)

3. \( X(\zeta) = O(\zeta^{-1}), \quad \zeta \to \infty. \)

The function \( X(\zeta) \) is analytic in \( \mathbb{C}^+ \) and of order \( O(\zeta^{-2}) \) as \( \zeta \to \infty \). Therefore, by Cauchy’s theorem, we have

\[
\int_{\mathbb{R}} X_+(\zeta) \overline{X_-(\zeta)} d\zeta = \int_{\mathbb{R}} X_-(\zeta) \begin{pmatrix} e^{-\frac{u_2(y)}{2\zeta^2} + \frac{u_1(y)}{\zeta}} & -e^{2i\zeta} \\ e^{-2i\zeta} & 0 \end{pmatrix} \overline{X_-(\zeta)} d\zeta = 0.
\]
By adding this matrix to its Hermitian conjugate, we see that

\[ \int_{\mathbb{R}} X_-(-\zeta) \begin{pmatrix} 2e^{-\frac{u_2(y)}{2\zeta^2} + \frac{u_1(y)}{4}} & 0 \\ 0 & 0 \end{pmatrix} X_-(-\zeta) \, d\zeta = 0. \]

This implies that the first column of \( X_-(-\zeta) \) is identically zero. From the jump conditions in (5.11), we see that the second column of \( X_+(-\zeta) \) is also identically zero. Now let the first column of \( X \) be \( X_1 \) and the second column of \( X \) be \( X_2 \) and define the vector \( f(\zeta) \) by

\[ f(\zeta) = \begin{cases} X_2(\zeta), & \text{Im}(\zeta) < 0; \\ X_1e^{2i\zeta}, & \text{Im}(\zeta) > 0. \end{cases} \]

Then from (5.11), we see that \( f(\zeta) \) is analytic in the whole complex plane and behaves as \( \zeta^{-1} \) as \( \zeta \to \infty \). Hence by Liouville’s theorem, we have \( f(\zeta) = 0 \). This shows that there is no non-trivial function \( \hat{P}_0 \) that satisfies 1. and 2. of (5.9) such that \( \hat{P}_0 = O(\zeta^{-2}) \) as \( \zeta \to \infty \). Therefore, the Riemann-Hilbert problem (5.6) is uniquely solvable.

### 5.3 Painlevé type differential equations

We will now transform the Riemann-Hilbert problem (5.6) into a Riemann-Hilbert problem with constant jumps but with essential singularities at \( \zeta = 0 \) and \( \zeta = \infty \). The theory of isomonodromy deformations developed by Jimbo, Miwa, and Ueno can then be applied to derive differential equations that give the solution to (5.6).

Firstly, let us deform the jump contours of (5.6) as in Figure 3. Let the solutions of
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(5.6) in the region $I$ be $P_I(\zeta)$, etc., and define the function $\Phi(\zeta)$ in different regions to be

$$\Phi(\zeta) = P(\zeta) \begin{pmatrix} 0 & -e^{-\frac{w(y)}{2\zeta^2} + \frac{u(y)}{\zeta}} \\ e^{-\frac{w(y)}{2\zeta^2} + \frac{u(y)}{\zeta}} & 0 \end{pmatrix}, \quad \zeta \in I \cup IV$$

$$\Phi(\zeta) = P(\zeta) \begin{pmatrix} 0 & -e^{-\frac{w(y)}{2\zeta^2} + \frac{u(y)}{\zeta}} \\ e^{2i\zeta} & 0 \end{pmatrix}, \quad \zeta \in II,$$

$$\Phi(\zeta) = P(\zeta) e^{-2i\zeta\sigma_3}, \quad \zeta \in III,$$

$$\Phi(\zeta) = P(\zeta), \quad \zeta \in V \cup VIII \cup VII,$$

$$\Phi(\zeta) = P(\zeta) \begin{pmatrix} e^{-2i\zeta + \frac{w(y)}{2\zeta^2} - \frac{u(y)}{\zeta}} & 0 \\ 0 & 1 \end{pmatrix}, \quad \zeta \in VI.$$

Let $\Gamma_{\pm}$ be the semi-circles in Figure 3 in the upper and lower half planes respectively, then one can check that the matrix $\Phi(\zeta)$ satisfies the Riemann-Hilbert problem

1. $\Phi(\zeta)$ is analytic in $\mathbb{C} \setminus (\Gamma_+ \cup \Gamma_- \cup \Gamma_2)$,

2. $\Phi_+(\zeta) = \Phi_-(\zeta) J_\Phi(\zeta), \quad \zeta \in \Gamma_+ \cup \Gamma_- \cup \Gamma_2$, (5.13)

3. $\Phi(\zeta) = I + O(\zeta^{-1}), \quad \zeta \to \infty,$

where $J_\Phi(\zeta)$ are the matrices given by

$$J_\Phi(x) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \zeta \in \Gamma_-,$$

$$J_\Phi(x) = \begin{pmatrix} 1 & 0 \\ -e^{-2i\zeta + \frac{w(y)}{2\zeta^2} - \frac{u(y)}{\zeta}} & e^{2i\zeta - \frac{w(y)}{2\zeta^2} + \frac{u(y)}{\zeta}} \end{pmatrix}, \quad \zeta \in \Gamma_+,$$ (5.14)

$$J_\Phi(x) = \begin{pmatrix} e^{2i\zeta - \frac{w(y)}{2\zeta^2} + \frac{u(y)}{\zeta}} & 0 \\ 1 & 1 \end{pmatrix}, \quad \zeta \in \Gamma_2.$$

The matrix $\Psi(\zeta)$ defined by

$$\Psi(\zeta) = \Phi(\zeta) e^{(i\zeta - \frac{w(y)}{4\zeta^2} + \frac{u(y)}{2\zeta})\sigma_3}$$ (5.15)

satisfies a Riemann-Hilbert problem with constant jumps on $\Gamma_+ \cup \Gamma_- \cup \Gamma_2$:

$$J_\Psi(x) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \zeta \in \Gamma_-,$$

$$J_\Psi(x) = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \quad \zeta \in \Gamma_+,$$ (5.16)

$$J_\Psi(x) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad \zeta \in \Gamma_2.$$
and has essential singularities at the points ζ = 0 and ζ = ∞,

\[ \Psi(ζ) = \hat{Ψ}_0(ζ)e^{-\frac{u_2(y) + u_1(y)}{ζ^2}}σ_3, \quad \zeta \to 0, \]

\[ \Psi(ζ) = \hat{Ψ}_\infty(ζ)e^{iζσ_3}, \quad \zeta \to \infty, \]

where \( \hat{Ψ}_0(ζ) \) and \( \hat{Ψ}_\infty(ζ) \) have the asymptotic expansions

\[ \hat{Ψ}_0(ζ) = Ψ_0 + Ψ_1ζ + ... \]

\[ \hat{Ψ}_\infty(ζ) = I + Ψ_1(∞)ζ^{-1} + ... \]  (5.17)

5.4 Differential equations

From (5.4), we see that \( u_j(y) \) behaves as follows as \( N \to \infty \):

\[ u_j(y) = u_{j,N} \left( 1 + O \left( \frac{ζ^2}{N^2} \right) \right) = u_j \left( 1 + O \left( \frac{ζ^2}{N^2} \right) \right), \quad j = 1, 2, \]

where \( u_j \) are the variables in (1.2). Let \( Φ_c(ζ) \) be the solution of the Riemann-Hilbert problem (5.14) with \( u_j(y) \) replaced by \( u_j \). Since the jump contours of \( Φ(ζ) \) are all bounded, the matrix \( Φ(ζ)Φ_c^{-1}(ζ) \) will have jump discontinuities on the contours \( Γ_2 \) and \( Γ_± \) of order \( I + O(N^{-2}) \) and behave as \( I + O(ζ^{-1}) \) as \( ζ \to \infty \). By the general theory of Riemann-Hilbert problems, (see e.g. [15], §7 and also §7) we have

\[ Φ(ζ)Φ_c^{-1}(ζ) = I + O \left( \frac{1}{N^2(1 + |ζ|)} \right). \]

Hence, the matrix \( Φ_c(ζ) \) approximates \( Φ(ζ) \) up to \( O(N^{-2}) \) as \( N \to \infty \). From now on, we shall replace the functions \( u_j(y) \) by the constants \( u_j \) and denote the solution to (5.14) and (5.15) with \( u_j(y) = u_j \) by the same notation \( Φ(ζ) \) and \( Ψ(ζ) \).

Let us consider the Riemann-Hilbert problem with constants jumps (5.16) and asymptotic behavior (5.17). Since the matrix \( Ψ(ζ) \) has constant jumps on the complex plane, the matrices defined by

\[ \frac{∂Ψ(ζ)}{∂ζ}Ψ^{-1}(ζ) = A(ζ), \quad \frac{∂Ψ(ζ)}{∂u_j}Ψ^{-1}(ζ) = B_j(ζ), \quad j = 1, 2 \]  (5.19)

are rational functions with poles at \( ζ = 0 \) and \( ζ = ∞ \) only. By using (5.17), we see that they are given by

\[ A(ζ) = \left( \frac{1}{2} \left( \frac{u_2}{ζ^3} - \frac{u_1}{ζ^2} \right) \hat{Ψ}_0σ_3\hat{Ψ}_0^{-1} \right)_{pp} + iσ_3 = \sum_{j=1}^{3} \frac{A_j}{ζ^j} + iσ_3, \]

\[ B_1(ζ) = \left( \frac{1}{2ζ} \hat{Ψ}_0σ_3\hat{Ψ}_0^{-1} \right)_{pp} = \frac{A_3}{u_2ζ}, \]

\[ B_2(ζ) = -\left( \frac{1}{4ζ^2} \hat{Ψ}_0σ_3\hat{Ψ}_0^{-1} \right)_{pp} = -\frac{A_3}{2u_2ζ^2} - \frac{1}{2u_2ζ} \left( A_2 + \frac{u_1}{u_2}A_3 \right), \]  (5.20)
where $A_{pp}$ denotes the singular part of $A$ at $\zeta = 0$.

We can write the matrix $A$ as follows.

$$A(\zeta) = \left( \sum_{j=2}^{3} a_j \zeta^{-j} \right) - \left( \sum_{j=2}^{3} b_j \zeta^{-j} \right) + \left( \begin{array}{cc} 0 & b_1 \\ c_1 & 0 \end{array} \right) \zeta^{-1} + i\sigma_3,$$

where the form of the coefficient of $\zeta^{-1}$ can be seen by expanding the first equation of (5.19) near $\zeta = \infty$,

$$\frac{\partial \Psi(\zeta)}{\partial \zeta} = \frac{\partial \Psi_\infty}{\partial \zeta} + \Psi_\infty i\sigma_3 \Psi_\infty^{-1}.$$

From (5.20), we see that the coefficients of $\zeta^{-6}, \zeta^{-5}$ and $\zeta^{-4}$ in the determinant of $A(\zeta)$ coincide with the coefficients in the determinant of $\frac{1}{2} \left( \frac{u_3}{\zeta^2} - \frac{u_1}{\zeta} \right) \sigma_3$. This implies

$$a_3^2 + b_3 c_3 = \frac{1}{4} u_2^2, \quad 2a_2 a_3 + b_2 c_3 + c_2 b_3 = -\frac{1}{2} u_1 u_2,$$

$$b_1 c_3 + c_1 b_3 + b_2 c_2 + a_2^2 = \frac{1}{4} u_1^2.$$

The compatibility of the linear differential systems in (5.19) implies

$$\partial_j A(\zeta) - \partial_\zeta B_j(\zeta) + [A(\zeta), B_j(\zeta)] = 0, \quad j = 1, 2,$$

$$\partial_1 B_2(\zeta) - \partial_2 B_1(\zeta) + [B_2(\zeta), B_1(\zeta)] = 0,$$

where $\partial_j$ is the derivative with respect to $u_j$ and $\partial_\zeta$ the derivative with respect to $\zeta$.

Although these compatibility conditions can readily be expressed as PDEs in the variables $u_1$ and $u_2$, it will be much easier to study the variations if we can write down ODEs in these variables.

### 5.5 Hamiltonian equations and ODE in the variables

We will now express the compatibility conditions (5.23) as a system of Hamiltonian ODEs. First let us define a Poisson bracket on the space of matrices of the form $A(\zeta) = \sum_{j=1}^{3} A_j \zeta^{-j} + i\sigma_3$. Let the loop algebra $\mathfrak{g}$ be the set of smooth maps $f : S^1 \to \mathfrak{sl}_2(\mathbb{C})$. We will split $\mathfrak{g}$ into subalgebras $\mathfrak{g}_+$ and $\mathfrak{g}_-$, where $\mathfrak{g}_+$ ($\mathfrak{g}_-$) is the set of maps that admits holomorphic extension to the inside (outside) of the unit circle and that the maps in $\mathfrak{g}_-$ vanish at infinity. Then by using the pairing $\langle X(\zeta), Y(\zeta) \rangle = \text{Res}_{\zeta=0} \text{Tr}(XY)$, we can identify $\mathfrak{g}_-$ with the dual of $\mathfrak{g}_+$. This gives the Poisson brackets on $\mathfrak{g}_-$ defined by (see, e.g. [17])

$$\{f, g\}(A) = \langle A, [df(A), dg(A)] \rangle,$$

(5.24)
where \( f \) and \( g \) are functions on \( g_- = g^*_+ \). By using the method of the moment map (see [1], [3], [18] and [19]), this Poisson structure can be restricted to the space of matrices of the form 
\[
A(\zeta) = \sum_{j=0}^{3} A_j \zeta^{-j} + i\sigma_3.
\]
The Poisson brackets between the matrix entries \( a_j, b_j \) and \( c_j \) are given by
\[
\{a_j, b_k\} = -b_{j+k-1}, \quad \{a_j, c_k\} = c_{j+k-1}, \quad j + k \leq 4,
\]
\[
\{b_j, c_k\} = 2a_{j+k-1}, \quad j + k - 1 \leq 4
\]
and all the other Poisson brackets vanish.

Remark 1. Note that, although strictly speaking, the matrix 
\[
A(\zeta) = \sum_{j=0}^{3} A_j \zeta^{-j} + i\sigma_3
\]
does not belong to the Lie algebra \( g_- \) because of the \( i\sigma_3 \) term, one can think of the space of matrices \( A \) as being parameterized by the entries \( a_j, b_j \) and \( c_j \) and use (5.25) to define a Poisson bracket on this space. In fact, the original construction in [1] and [3] applies to much more general form of matrices and readers who are interested may consult these references.

The symplectic leaves in this Poisson manifold are the co-adjoint orbits of the matrix \( A \), specified by \( \det(A) = \text{constant} \). A set of canonical coordinates on these symplectic leaves can be found as follows (see [3] and [24]). Let \( \psi(\zeta) \) be an eigenvector of \( A(\zeta) \) with eigenvalue \( \lambda(\zeta) \). Then, the poles \( q_j \) of \( \psi(\zeta) \) and the values \( \lambda(q_j) \) of the eigenvalue \( \lambda \) at \( q_j \) satisfy the relations
\[
\{p_j, q_k\} = \delta_{jk}, \quad \{p_j, p_k\} = \{q_j, q_k\} = 0.
\]
For example, let \( \psi(\zeta) \) be an eigenvector of \( A(\zeta) \) normalized by
\[
(1,0) \cdot \psi(\zeta) = 1.
\]
The poles \( q_j \) of \( \psi(\zeta) \) are then the zeroes of the polynomial
\[
b_1 \zeta^2 + b_2 \zeta + b_3 = 0,
\]
and the eigenvalues at these points are given by
\[
p_j = -3 \sum_{k=2}^{3} a_k q_j^{-k} - i.
\]
Therefore, the symplectic form on these symplectic leaves are given by
\[
\omega = -2 \sum_{j=1}^{3} \sum_{k=2}^{3} da_k q_j^{-k} \wedge dq_j = 2 \sum_{j=1}^{2} \left( \frac{1}{2} da_3 \wedge dq_j^{-2} + da_2 \wedge dq_j^{-1} \right)
\]
\[
= d \left( -a_3 b_2 \over b_3 + a_2 \right) \wedge d \left( -b_2 \over b_3 \right) + da_3 \wedge d \left( -b_1 \over b_3 \right).
\]

Hence, the following gives a set of canonical coordinates on these symplectic leaves
\[
P_1 = -a_3 b_2 \over b_3 + a_2, \quad Q_1 = -b_2 \over b_3, \quad P_2 = a_3, \quad Q_2 = -b_1 \over b_3.
\]
(5.26)
A set of coordinates on the Poisson manifold is then given by \( P_1, Q_1, P_2, Q_2, u_1, u_2 \). The deformations \( \dot{A} = [B, A] \) can be expressed as Hamiltonian flows in this Poisson structure [1], [2], [3].

**Proposition 5.1.** Let \( B = (\zeta^k A)_- \), where \( X_- \) is the projection onto \( g_- \), then the following equation

\[
\dot{A} = [B, A]
\]

is Hamiltonian with respect to the Poisson bracket (5.24). Its Hamiltonian is given by

\[
H_B = \frac{1}{2} \text{Res}_{\zeta=0} \text{Tr} (A^2 \zeta^k)
\]

In particular, the flows \( \dot{A} = [B_j, A] \) have the following Hamiltonians.

\[
H_1 = \frac{1}{2u_2} \text{Res}_{\zeta=0} \text{Tr} (A^2 \zeta^2), \quad H_2 = -\frac{1}{2u_2} \text{Res}_{\zeta=0} \text{Tr} \left( \frac{1}{2} A^2 \zeta \right) - \frac{u_1}{2u_2} H_1.
\]

By using (5.22), we can eliminate the \( c_j \) and express the Hamiltonians in terms of the canonical variables (5.26). They are given by (2.2). Due to the presence of the terms \( \partial_\zeta B_j \) in first equations of (5.23), the deformations of the matrix \( A \) are not of the form \( \dot{A} = [B_j, A] \) and hence the Hamiltonian flows generated by (2.2) are not sufficient to describe the deformations of \( A \). However, by using a technique developed in [22], we can write the first equations in (5.23) as a time-dependent Hamiltonian system. First let us choose a set of coordinates \( \nu_1, \ldots, \nu_4, u_1, u_2 \) on the Poisson manifold such that the partial derivatives \( \partial_{\nu_1} \) and \( \partial_{\nu_2} \) (derivatives that keep \( \nu_1, \ldots, \nu_4 \) fixed) satisfy the following equations:

\[
\begin{align*}
\partial_{\nu_1} a_3 &= \partial_{\nu_2} b_3 = \partial_{\nu_3} c_3 = 0, & \partial_{\nu_1} a_2 &= -\frac{a_3}{u_2}, & \partial_{\nu_1} b_2 &= -\frac{b_3}{u_2}, & \partial_{\nu_1} c_2 &= -\frac{c_3}{u_2}, \\
\partial_{\nu_1} a_1 &= \partial_{\nu_2} b_1 = \partial_{\nu_3} c_1 = 0, \\
\partial_{\nu_2} a_3 &= \frac{a_3}{u_2}, & \partial_{\nu_2} b_3 &= \frac{b_3}{u_2}, & \partial_{\nu_2} c_3 &= \frac{c_3}{u_2}, \\
\partial_{\nu_2} a_2 &= \frac{1}{2u_2} \left( a_2 + \frac{u_1}{u_2} a_3 \right), & \partial_{\nu_2} b_2 &= \frac{1}{2u_2} \left( b_2 + \frac{u_1}{u_2} b_3 \right), \\
\partial_{\nu_2} c_2 &= \frac{1}{2u_2} \left( c_2 + \frac{u_1}{u_2} c_3 \right), & \partial_{\nu_2} a_1 &= \partial_{\nu_2} b_1 = \partial_{\nu_2} c_1 &= 0.
\end{align*}
\]

By using (5.22), one can check that this definition is compatible with \( \partial_{\nu_i} u_i = \delta_{ij} \). Note that the derivatives (5.28) imply the following

\[
\partial_{\nu_1} A = \partial_{\zeta} B_1, \quad \partial_{\nu_2} A = \partial_{\zeta} B_2.
\]

Therefore we can write the equations (5.23) as a time-dependent Hamiltonian system,

\[
\partial_j A = \{H_j, A\} + \partial_{\nu_j} A.
\]
In order to find the evolution equations in the form \( \partial_j P_k = -\frac{\partial H_j}{\partial Q_k}, \partial_j Q_k = \frac{\partial H_j}{\partial P_k} \) for suitable Hamiltonians \( H_j \), we must replace the partial differentiation \( \partial'_j \) by the partial differentiation \( \partial'^{can}_j \) that keeps the canonical coordinates fixed and modify the Hamiltonians \( H_j \) accordingly. We have, by the chain rule,

\[
\partial'_j = \partial'^{can}_j + \sum_{k=1}^{2} \left( \partial'^{can}_j P_k \partial P_k + \partial'^{can}_j Q_k \partial Q_k \right).
\]

From (5.28) and (5.26), we obtain the derivatives \( \partial'^{can}_j P_k \) and \( \partial'^{can}_j Q_k \).

\[
\begin{align*}
\partial'^{can}_1 P_1 &= \partial'^{can}_1 P_2 = \partial'^{can}_1 Q_2 = 0, & \partial'^{can}_1 Q_1 &= \frac{1}{u_2}, \\
\partial'^{can}_2 P_2 &= \frac{P_2}{u_2}, & \partial'^{can}_2 Q_2 &= -\frac{Q_2}{u_2},
\end{align*}
\]

\[
\begin{align*}
\partial'^{can}_1 P_1 &= \frac{1}{2u_2} P_1, & \partial'^{can}_2 Q_1 &= -\frac{1}{2u_2} Q_1 - \frac{u_1}{2u_2}.
\end{align*}
\]

From this and (5.31), we have

\[
\partial'_j A = \partial'^{can}_j A + \{ h_j, A \},
\]

\[
h_1 = \frac{P_1}{u_2}, \quad h_2 = -\frac{P_2 Q_2}{u_2} - \frac{P_1 Q_1}{2u_2}, \quad \frac{u_1}{2u_2}.
\]

Therefore (5.30) becomes

\[
\partial_j A = \{ H_j + h_j, A \} + \partial'^{can}_j A.
\]

In particular, we have the following.

**Theorem 5.1.** The deformations of the matrix \( A \) in (5.21)

\[
\partial_j A(\zeta) - \partial_\zeta B_j(\zeta) + [A(\zeta), B_j(\zeta)] = 0, \quad j = 1, 2,
\]

are equivalent to the time-dependent Hamiltonian equations

\[
\partial_j P_k = -\frac{\partial (H_j + h_j)}{\partial Q_k}, \quad \partial_j Q_k = \frac{\partial (H_j + h_j)}{\partial P_k}, \quad j, k = 1, 2
\]

where \( H_j \) and \( h_j \) are given by (2.2) and (5.33).

These give us ODEs in the variables \( u_1 \) and \( u_2 \). A special solution of these ODEs, specified by the jump conditions (5.16), will give us the local parametrix via (5.19), (5.15), (5.12) and (5.8). In particular, we will show that the leading terms in the logarithmic derivatives of the Hankel determinant \( G_N \) with respect to \( u_1 \) and \( u_2 \) in (3.3) are given by the Hamiltonians \(-H_1\) and \(-H_2\).

### 6 Local parametrices near the points \( \pm 2 \)

Near the points \( \pm 2 \), the approximations of \( S(y) \) by \( S^\infty(y) \) also fail and we must again find exact solutions to the Riemann-Hilbert problem of \( S(y) \) and match them with \( S^\infty(y) \) as \( y \) moves away from these points. Such local parametrices can be constructed using Airy functions as in [8], [14], [15], [16] and we shall not repeat the construction here.
7 The final transformation of the Riemann-Hilbert problem

We now show that the parametrices we constructed in §4.2 and §5 are indeed good approximations to the solution $S(y)$ of the R-H problem (4.6).

Let us define

$$R(y) := \begin{cases} S(y) (S^{(p)}(y))^{-1}, & y \in D_{\pm 2} \cup D_0, \\ S(y) (S^\infty(y))^{-1}, & y \in \mathbb{C} \setminus (D_{\pm 2} \cup D_0), \end{cases}$$

(7.1)

where $S^{(p)}(y)$ are the local parametrices inside the neighborhoods $D_{\pm 2}$ and $D_0$. Then, the function $R(y)$ has jump discontinuities on the contour $\Gamma_R$ shown in Figure 4. In particular,

$$R(y) = I + O(y^{-1}), \quad y \to \infty.$$  

(7.2)

From the definition of $R(y)$ it follows that the jump matrix $J_R(y)$ has the following order of magnitude as $N \to \infty$:

$$J_R(y) = \begin{cases} I + O(N^{-1}), & y \in \partial D_{\pm 2} \cup \partial D_0 \cup \Gamma_0, \\ I + O(e^{-N\eta}), & \text{for some fixed } \eta > 0 \text{ for } y \in \Gamma_R \setminus (\partial D_{\pm 2} \cup \partial D_0 \cup \Gamma_0), \end{cases}$$

(7.3)

where $\Gamma_0$ is the part of $[-2,2]$ that is outside of $D_{\pm 2}$ and $D_0$.

Then, using well established techniques (see, e.g. §7), we have

$$R(y) = I + O\left(\frac{1}{N(|y|+1)}\right),$$

(7.4)

uniformly in $\mathbb{C}$. Therefore, the solution $S(y)$ of the R-H problem (4.6) can be approximated by $S^\infty(y)$ and $S^{(p)}(y)$:

$$S(y) = \begin{cases} (I + O(N^{-1})) S^{(p)}(y), & y \in D_{\pm 2} \cup D_0, \\ (I + O(N^{-1})) S^\infty(y), & y \in \mathbb{C} \setminus (D_{\pm 2} \cup D_0). \end{cases}$$

(7.5)

Combining these expressions with equations (4.5) and (4.4) we obtain an asymptotic formula for the solution of the original R-H problem (3.9).
8 Asymptotics of the Hankel determinant

We will now use Lemma 3.1 to express the logarithmic derivatives of the Hankel determinant $G_N$ in terms of the powers series $\hat{\Phi}_0$ in (5.17). These derivatives will then be identified with the Hamiltonians $H_1$ and $H_2$ in (2.2). Firstly, from the definition (4.4) and (4.5) of $T(y)$ and $S(y)$, we see that

$$Y(y) = e^{\frac{N\eta_3}{2}} S(y) e^{(\frac{\chi - N}{2}) \sigma_3}. \quad (8.1)$$

From (7.5), we have

$$S = e^{\frac{N\eta_3}{2}} R(y) S^\infty (y) e^{-N\hat{\gamma}+(0)\sigma_3} \left( \begin{array}{cc} 0 & -1 \\ 1 & 0 \end{array} \right) P_+ (\zeta) e^{N(\hat{\gamma}+(0)+g_+(y) - \frac{N}{2}) \sigma_3}, \quad \text{Im}(y) > 0,$$

$$S = e^{\frac{N\eta_3}{2}} R(y) S^\infty (y) e^{-N\hat{\gamma}+(0)\sigma_3} P_- (\zeta) e^{N(\hat{\gamma}+(0)+g_-(y) - \frac{N}{2}) \sigma_3}, \quad \text{Im}(y) < 0.$$ 

Now from (5.12) and (5.15), we obtain

$$P_+ (\zeta) = \Psi_+ (\zeta) e^{(\frac{\chi}{2} + \frac{\eta_3}{2} \sigma_3)}.$$ 

Therefore $P(\zeta)$ are related to the expansion $\hat{\Psi}(\zeta)$ by

$$P_+ (\zeta) = \hat{\Psi}_0 (\zeta) e^{\frac{\chi}{2}}.$$ 

We can then use these to compute $\text{Tr} \left( Y^{-1} \frac{dY}{dy} \sigma_3 \right) dy$.

$$\text{Tr} \left( Y^{-1} \frac{dY}{dy} \sigma_3 \right) dy = 2 N g_+(y) dy + 2 i d\zeta + \text{Tr} \left( \hat{\Psi}_0^{-1} (\zeta) \hat{\Psi}_0' (\zeta) \sigma_3 d\zeta \right)$$

$$+ \text{Tr} \left( K_+ (\zeta) (S^\infty_\pm (y))^{-1} (S^\infty_\pm (y))' K_+ (\zeta) \sigma_3 dy \right)$$

$$+ \text{Tr} \left( K_-^{-1} (\zeta) (S^\infty_\pm (y))^{-1} R^{-1} (y) R'(y) S^\infty_\pm (y) K_+ (\zeta) \sigma_3 dy \right),$$

$$K_+ (\zeta) = e^{-N\hat{\gamma}+(0)\sigma_3} \left( \begin{array}{cc} 0 & -1 \\ 1 & 0 \end{array} \right) P_+ (\zeta), \quad \text{K}_- (\zeta) = e^{-N\hat{\gamma}+(0)\sigma_3} P_- (\zeta).$$

From (3.10a), (3.10b) and the relation between $y$ and $\zeta$ (5.3), we obtain

$$\frac{\partial \log G_N}{\partial u_{1,N}} = -\frac{1}{2} \text{Res}_{\zeta=0} \frac{1}{\zeta} \text{Tr} \left( \hat{\Psi}_0^{-1} (\zeta) \hat{\Psi}_0' (\zeta) \sigma_3 d\zeta \right) + O(N^{-1}),$$

$$\frac{\partial \log G_N}{\partial u_{2,N}} = \frac{1}{4} \text{Res}_{\zeta=0} \frac{1}{\zeta^2} \text{Tr} \left( \hat{\Psi}_0^{-1} (\zeta) \hat{\Psi}_0' (\zeta) \sigma_3 d\zeta \right) + O(N^{-1}). \quad (8.2)$$

The leading order terms in these derivatives are the derivatives of the $\tau$-function for the equations (5.23) [21]. Bertola, Harnad, Hurtubise and Pusztai have shown that they are given by the Hamiltonians (7.27).
Proposition 8.1. ([5], [20]) The Hamiltonians in (5.27) can be expressed as
\[
H_1 = \frac{1}{2} \text{Res}_{\zeta=0} \frac{1}{\zeta} \text{Tr} \left( \hat{\Psi}_0^{-1}(\zeta) \hat{\Psi}_0'(\zeta) \sigma_3 d\zeta \right), \\
H_2 = -\frac{1}{4} \text{Res}_{\zeta=0} \frac{1}{\zeta^2} \text{Tr} \left( \hat{\Psi}_0^{-1}(\zeta) \hat{\Psi}_0'(\zeta) \sigma_3 d\zeta \right).
\]

(8.3)

Proof. First let \( \lambda \) be the eigenvalue of the matrix \( A(\zeta) \). Then the expansion of \( \lambda \) near \( \zeta = 0 \) is given by
\[
\lambda = \frac{1}{2} u_2 \zeta^{-3} - \frac{1}{2} u_1 \zeta^{-2} + H_1 - 2H_2 \zeta + O(\zeta^2).
\]

(8.4)

First note that, from (5.19), we have
\[
\hat{\Psi}_0^{-1} \hat{\Psi}_0' + \frac{1}{2} \left( \frac{u_2}{\zeta^3} - \frac{u_1}{\zeta^2} \right) \sigma_3 = \hat{\Psi}_0^{-1} A \hat{\Psi}_0.
\]

Let us now denote \( \hat{\Psi}_0^{-1} A \hat{\Psi}_0 \) by \( \hat{A} \) and write it as
\[
\hat{A} = \frac{1}{2} \left( \frac{u_2}{\zeta^3} - \frac{u_1}{\zeta^2} \right) \sigma_3 + \left( \Pi \ F_{12} \ F_{21} - \Pi \right),
\]

where \( \Pi \) and \( F_{ij} \) are bounded at \( \zeta = 0 \). In particular, we have
\[
\text{Tr} \left( \hat{\Psi}_0^{-1}(\zeta) \hat{\Psi}_0'(\zeta) \sigma_3 \right) = \frac{u_2}{\zeta^3} - \frac{u_1}{\zeta^2} + 2\Pi.
\]

(8.5)

Let \( v_\pm \) be \( v_\pm = \lambda \pm \left( \frac{1}{2} \left( \frac{u_2}{\zeta^3} - \frac{u_1}{\zeta^2} \right) + \Pi \right) \), then since \( \lambda \) is an eigenvalue of \( A \) and \( \hat{A} \) is conjugated to \( A \), we have
\[
\det \left( \lambda - \hat{A} \right) = v_+ v_- - F_{12} F_{21} = 0.
\]

From the expansion of \( \lambda \) in (8.4), we see that \( v_+ \) is of order \( O(\zeta^{-3}) \), while \( F_{12} F_{21} \) is of order \( O(1) \). Therefore, \( v_- \) is of order \( \zeta^3 \). Hence \( \Pi \) behaves as
\[
\Pi = H_1 - 2H_2 \zeta + O(\zeta^2)
\]

This, together with (8.5), imply the proposition. \( \square \)

This proposition and (8.2) implies (2.4) of Theorem 2.1

9 Asymptotics of the canonical coordinates as \( u_2 \rightarrow 0 \)

We will now compute the asymptotic expansion of the solution \( \Phi(\zeta) \) of the R-H problem (5.13) as \( u_2 \rightarrow 0 \). We shall assume the scaling between the variables \( u_1 \) and \( u_2 \) is such that
\[ \tilde{u}_1 = \frac{u_{1}}{\sqrt{u_2}} \] is finite. Let us first divide the \( \zeta \)-plane into 3 regions as in Figure 5, and define the function \( \Phi^{(o)}(\zeta) \) to be

\[ \begin{align*}
\Phi^{(o)}(\zeta) &= I, \quad \zeta \in \Omega_1, \\
\Phi^{(o)}(\zeta) &= \begin{pmatrix} 1 & e^{2i\zeta} \\ -e^{-2i\zeta} & 0 \end{pmatrix}, \quad \zeta \in \Omega_2, \\
\Phi^{(o)}(\zeta) &= \begin{pmatrix} 1 & 0 \\ -e^{-2i\zeta} & 1 \end{pmatrix}, \quad \zeta \in \Omega_3.
\end{align*} \tag{9.1} \]

Now let \( \Delta_0 \) be a small disc around the origin of fixed radius. Let \( \phi(x) \) be

\[ \phi(x) = -\frac{1}{2\pi i} \int_{\mathbb{R}} e^{-\frac{2}{\pi} + \tilde{u}_1 s} s - \frac{x}{2} ds. \tag{9.2} \]

Let \( \xi = u_2^{-\frac{1}{2}} \zeta \) and define \( \Phi^{(p)}(\zeta) \) to be

\[ \begin{align*}
\Phi^{(p)}(\zeta) &= \begin{pmatrix} 1 & e^{2i\zeta} \\ -e^{-2i\zeta} & 0 \end{pmatrix} \begin{pmatrix} (\phi(\xi) - \phi_{+,0}) e^{2i\zeta} \\ 1 \end{pmatrix}, \quad \text{Im}(\zeta) > 0, \\
\Phi^{(p)}(\zeta) &= \begin{pmatrix} 1 & 0 \\ -e^{-2i\zeta} & 1 \end{pmatrix} \begin{pmatrix} (\phi(\xi) - \phi_{-,0}) e^{2i\zeta} \\ 1 \end{pmatrix}, \quad \text{Im}(\zeta) < 0,
\end{align*} \tag{9.3} \]

where \( \phi_{\pm,j} \) are the coefficients in the expansion of \( \phi(x) \) as \( x \to \infty \) in the upper/lower half planes given by

\[ \phi_{\pm,j} = -\frac{1}{2\pi i} \int_{\Gamma_{\pm}} s^{-j-1} e^{-\frac{2}{\pi} + \tilde{u}_1 s} ds. \tag{9.4} \]

Here \( \Gamma_+ (\Gamma_-) \) consists of the two intervals \( (-\infty, -\epsilon) \cup (\epsilon, \infty) \), oriented from \( -\infty \) to \( \infty \), together with the semi circle of radius \( \epsilon \) around the origin in the upper (lower) half plane. The exact value of \( \epsilon \) is not important as the integrand is analytic away from zero and hence by Cauchy’s theorem, integrals with different values of \( \epsilon \) will yield the same result. In particular, by the residue theorem, we have

\[ \sum_{k=0}^{\infty} \phi_{+,k} \frac{u_{2}^{k}}{\zeta^{k}} = \sum_{k=0}^{\infty} \phi_{-,k} \frac{u_{2}^{k}}{\zeta^{k}} + e^{-\frac{u_{2}^{2}}{2\epsilon^2} + \frac{u_{1}}{\zeta}}. \tag{9.5} \]
For the first few coefficients, this gives us

\[ \phi_{+0} - \phi_{-0} = 1, \quad \phi_{+1} - \phi_{-1} = \hat{u}_1, \]

\[ \phi_{+2} - \phi_{-2} = -\frac{1}{2} + \frac{\hat{u}_1^2}{2}, \quad \phi_{+3} - \phi_{-3} = -\frac{1}{2} \hat{u}_1 + \frac{1}{6} \hat{u}_1^3. \] (9.6)

Integrating by parts, we can express the coefficients \( \phi_{\pm,j} \) in terms of \( \phi_{\pm,0} \).

**Proposition 9.1.** The coefficients \( \phi_{\pm,j} \) satisfy the following recursive relation.

\[ \phi_{\pm,j} = \frac{1}{j} (\hat{u}_1 \phi_{\pm,j-1} - \phi_{\pm,j-2}), \quad j \geq 2. \] (9.7)

The first two coefficients \( \phi_{\pm,1} \) and \( \phi_{\pm,0} \) are given by

\[ \phi_{\pm,0} = \pm \frac{1}{2} + \frac{i}{\sqrt{2\pi}} \int_0^{\hat{u}_1} e^{s^2} ds, \quad \phi_{\pm,1} = \pm \frac{\hat{u}_1}{2} + \frac{i}{\sqrt{2\pi}} \left( \hat{u}_1 \int_0^{\hat{u}_1} e^{s^2} ds - e^{\hat{u}_1^2} \right) \] (9.8)

**Proof.** The proof of (9.7) follows from integration by parts immediately.

\[ \phi_{\pm,j} = -\frac{1}{2\pi i} \int_{\Gamma_\pm} s^{-j-1} e^{-s^2 + \hat{u}_1 s} ds = -\frac{1}{2\pi i} \int_{\Gamma_\pm} (\hat{u}_1 - s) s^{-j} e^{-s^2 + \hat{u}_1 s} ds = j^{-1} (\hat{u}_1 \phi_{\pm,j-1} - \phi_{\pm,j-2}) \]

for \( j \geq 2 \). For \( j = 1 \), the above gives

\[ \phi_{\pm,1} = \hat{u}_1 \phi_{\pm,0} + \frac{1}{2\pi i} \int_{\mathbb{R}} e^{-s^2 + \hat{u}_1 s} ds = \hat{u}_1 \phi_{\pm,0} - \frac{i}{\sqrt{2\pi}} e^{\hat{u}_1^2}. \] (9.9)

To compute \( \phi_{\pm,0} \), we expand the factor \( e^{\hat{u}_1 s} \) in a power series:

\[ \phi_{\pm,0} = -\frac{1}{2\pi i} \int_{\Gamma_\pm} s^{-1} e^{-\hat{u}_1 s} \sum_{k=0}^{\infty} \frac{(\hat{u}_1 s)^k}{k!} ds \]

\[ = \frac{i}{2\pi} \int_{\Gamma_\pm} s^{-1} e^{-\hat{u}_1 s} ds + \frac{i}{2\pi} \sum_{k=1}^{\infty} \frac{\hat{u}_1^k}{k!} \int_{\mathbb{R}} s^{k-1} e^{-\hat{u}_1 s} ds \]

\[ = \frac{i}{2\pi} \int_{\Gamma_\pm} s^{-1} e^{-\hat{u}_1 s} ds + \frac{i}{\sqrt{2\pi}} \sum_{k=0}^{\infty} \frac{\hat{u}_1^{2k+1}}{(2k+1)2^k k!} \int_{\mathbb{R}} e^{-\hat{u}_1 s} ds \]

\[ = \frac{i}{2\pi} \int_{\Gamma_\pm} s^{-1} e^{-\hat{u}_1 s} ds + \frac{i}{\sqrt{2\pi}} \int_0^{\hat{u}_1} e^{-\hat{u}_1 s} ds. \] (9.10)

To determine the first integral in the above equation, note that if we change the integration variable from \( s \) to \(-s\), then the path of integration will change from \( \Gamma_\pm \) to \(-\Gamma_\mp \). Therefore we have

\[ \frac{i}{2\pi} \int_{\Gamma_\pm} s^{-1} e^{-\hat{u}_1 s} ds = -\frac{i}{2\pi} \int_{\Gamma_\mp} s^{-1} e^{-\hat{u}_1 s} ds. \] (9.11)
Now, by a residue computation we see that
\[
\frac{i}{2\pi} \int_{\Gamma_{+}} s^{-1} e^{-\frac{s^2}{2}} ds - \frac{i}{2\pi} \int_{\Gamma_{-}} s^{-1} e^{-\frac{s^2}{2}} ds = 1.
\]
From this and (9.11), we see that \(\frac{i}{2\pi} \int_{\Gamma_{\pm}} s^{-1} e^{-\frac{s^2}{2}} ds = \pm \frac{1}{2}\). This, together with (9.10) and (9.9), implies (9.8).

From (9.6), (9.3) and the property of Cauchy transform, one can check that \(\Phi^{(p)}(\zeta)\) satisfies the following jump condition
\[
\Phi_{+}^{(p)}(\zeta) = \Phi_{-}^{(p)}(\zeta) \left( \begin{array}{cc}
1 & e^{2i\zeta - \frac{u_2}{2\zeta} + \frac{u_1}{\zeta}} \\
0 & 1
\end{array} \right).
\]

Hence, if we define \(R_{\Phi}(\zeta)\) to be the matrix
\[
R_{\Phi}(\zeta) := \begin{cases}
\Phi(\zeta) \left( \Phi^{(p)}(\zeta) \right)^{-1}, & \zeta \in \Delta_0, \\
\Phi(\zeta) \left( \Phi^{(o)}(\zeta) \right)^{-1}, & \zeta \in \mathbb{C} \setminus \Delta_0,
\end{cases}
\]
then \(R_{\Phi}(\zeta)\) satisfies the R-H problem
\[
1. \ R_{\Phi}(\zeta) \text{ is analytic in } \mathbb{C} \setminus \Gamma_{R_{\Phi}}, \\
2. \ R_{\Phi,+}(\zeta) = R_{\Phi,-}(\zeta)J_{R_{\Phi}}(\zeta), \quad \zeta \in \Gamma_{R_{\Phi}}, \\
3. \ R_{\Phi}(\zeta) = I + O(\zeta^{-1}), \quad \zeta \to \infty,
\]
where the contour \(\Gamma_{R_{\Phi}}\) is indicated in Figure 6 and the jump matrices are given by
\[
J_{R_{\Phi}} = \begin{pmatrix}
e^{-\frac{u_2}{2\zeta} + \frac{u_1}{\zeta}} & e^{2i\zeta} \left( \frac{1}{e^{-\frac{u_2}{2\zeta} + \frac{u_1}{\zeta}}} - 1 \right) \\
0 & e^{-\frac{u_2}{2\zeta} + \frac{u_1}{\zeta}}
\end{pmatrix}, \quad \zeta \in \Gamma_{+},
\]
\[
J_{R_{\Phi}} = \begin{pmatrix}
e^{-2i\zeta} \left( \frac{1}{e^{-\frac{u_2}{2\zeta} + \frac{u_1}{\zeta}}} - 1 \right) & 0 \\
e^{-\frac{u_2}{2\zeta} + \frac{u_1}{\zeta}} & 1
\end{pmatrix}, \quad \zeta \in \Gamma_{-},
\]
\[
J_{R_{\Phi}} = \begin{pmatrix}
e^{-2i\zeta} \left( 1 - e^{-\frac{u_2}{2\zeta} + \frac{u_1}{\zeta}} \right) & e^{2i\zeta} \left( e^{-\frac{u_2}{2\zeta} + \frac{u_1}{\zeta}} - 1 \right) \\
e^{-2i\zeta} \left( 1 - e^{-\frac{u_2}{2\zeta} + \frac{u_1}{\zeta}} \right) & 2 - e^{-\frac{u_2}{2\zeta} + \frac{u_1}{\zeta}}
\end{pmatrix}, \quad \zeta \in \Gamma_{2},
\]
\[
J_{R_{\Phi}} = I + \sum_{k=1}^{\infty} \phi_{+} u_{\zeta} \left( \frac{1}{e^{-2i\zeta}} \right) \left( e^{2i\zeta} \right), \quad \zeta \in \partial\Delta_0 \cap \mathbb{C}_{\pm}.
\]

With the scaling \(\tilde{u}_1 = \frac{u_1}{\sqrt{u_2}}\) such that \(\tilde{u}_1\) is finite, the jump matrices \(J_{R_{\Phi}}\) are of order \(I + O(\sqrt{u_2})\) and hence, by the same reasoning used in §7, we have
\[
R_{\Phi}(\zeta) = I + O \left( \frac{\sqrt{u_2}}{|\zeta| + 1} \right).
\]
In fact, the matrix $R_{\Phi}(\zeta)$ can be solved as a series expansion in $\sqrt{u_2}$ by using the method in §7 of [15]. In order to find the necessary initial conditions to solve the ODEs in Theorem 5.1 we will need the expansion of $R_{\Phi}(\zeta)$ up to the order $u_2^3$. To do so, let us first simplify the R-H problem of $R_{\Phi}$ with the transformation

$$\tilde{R}(\zeta) = R_{\Phi}(\zeta), \quad \zeta \in \Omega_1 \cup \Delta_0,$$

$$\tilde{R}(\zeta) = R_{\Phi}(\zeta) \left( \begin{array}{cc} e^{u_2^2} & -e^{-2i\zeta} \left( e^{-\frac{u_2^2}{2\zeta^2} + \frac{u_1}{\zeta^2}} - 1 \right) \\ 0 & e^{-\frac{u_2^2}{2\zeta^2} + \frac{u_1}{\zeta^2}} \end{array} \right), \quad \zeta \in \Omega_2 \setminus \Delta_0,$$

$$\tilde{R}(\zeta) = R_{\Phi}(\zeta) \left( \begin{array}{cc} 1 & 0 \\ e^{-2i\zeta} \left( e^{\frac{u_2^2}{2\zeta^2} - \frac{u_1}{\zeta^2}} - 1 \right) & 1 \end{array} \right), \quad \zeta \in \Omega_3 \setminus \Delta_0.$$

Then, the matrix $\tilde{R}(\zeta)$ satisfies the following R-H problem

1. $\tilde{R}(\zeta)$ is analytic in $\mathbb{C} \setminus \partial \Delta_0$,
2. $\tilde{R}_+(\zeta) = \tilde{R}_-(\zeta) J_{\tilde{R}}(\zeta)$, $\zeta \in \partial \Delta_0$ (9.15)
3. $\tilde{R}(\zeta) = I + O(\zeta^{-1})$, $\zeta \to \infty$.

where the jump matrix $J_{\tilde{R}}$ is given by

$$J_{\tilde{R}} = \left( \begin{array}{cc} 1 + e^{-\frac{u_2^2}{2\zeta^2} + \frac{u_1}{\zeta^2}} \sum_{k=1}^{\infty} \phi_{-k} \frac{u_2^2}{\zeta^2} & e^{2i\zeta} \sum_{k=1}^{\infty} \phi_{-k} \frac{u_2^2}{\zeta^2} \\ -e^{-2i\zeta + \frac{u_2^2}{2\zeta^2} - \frac{u_1}{\zeta^2}} \sum_{k=1}^{\infty} \phi_{+k} \frac{u_2^2}{\zeta^2} & 1 - \sum_{k=1}^{\infty} \phi_{-k} \frac{u_2^2}{\zeta^2} \end{array} \right),$$

where we have used (9.5) in obtaining the above equation. Note that $J_{\tilde{R}}$ is of order $I + O(\sqrt{u_2})$.

Let us denote the expansions of $\tilde{R}(\zeta)$ and $J_{\tilde{R}}$ by

$$\tilde{R}(\zeta) = I + \sum_{k=1}^{\infty} u_2^k \tilde{R}_k(\zeta), \quad J_{\tilde{R}} = I + \sum_{k=1}^{\infty} u_2^k J_{\tilde{R},k}(\zeta)$$
By comparing the coefficients of $u_2^k$ in the jump conditions of $\hat{R}$, we obtain the following jump conditions for the coefficients in the expansion of $\hat{R}(\zeta)$.

\[
\begin{align*}
\hat{R}_{+,1} &= \hat{J}_{R,1} + \hat{R}_{1,-}, & \hat{R}_{+,2} &= \left( \hat{R}_{-1} J_{R,1} + J_{R,2} \right) + \hat{R}_{-,2}, \\
\hat{R}_{+,3} &= \left( \hat{R}_{-1} J_{R,2} + \hat{R}_{-1} J_{R,2} + J_{R,3} \right) + \hat{R}_{-,3}.
\end{align*}
\]

(9.16)

This gives the linear jump conditions for the coefficients $\hat{R}_k$. Together with the requirement that $\hat{R}_k = O(\zeta^{-1})$ as $\zeta \to \infty$, we can solve these R-H problems for the coefficients $\hat{R}_k$.

From (9.14) and (9.6), we see that

\[
\begin{align*}
\hat{J}_{R,1} &= \frac{1}{\zeta} \begin{pmatrix} \phi_{-,1} & \phi_{-,1} e^{2i\zeta} \\ -\phi_{+,1} e^{-2i\zeta} & -\phi_{-,1} \end{pmatrix}, & \zeta \in \partial \Delta_0.
\end{align*}
\]

Let $E_j^\pm$ be the following

\[
E_j^\pm(\zeta) = \zeta^{-j-1} \left( \sum_{k=0}^{j} \frac{(\pm 2i\zeta)^k}{k!} - e^{\pm 2i\zeta} \right) = O(1) \text{ in } \zeta.
\]

(9.17)

By using (9.6) and (9.16), it is easy to check that the solution to $R_\Phi^{(1)}$ is

\[
\begin{align*}
\hat{R}_1 &= \begin{pmatrix} 0 & \phi_{-,1} E_0^+ (\zeta) \\ -\phi_{+,1} E_0^- (\zeta) & 0 \end{pmatrix}, & \zeta \in \Delta_0, \\
\hat{R}_1 &= \begin{pmatrix} \frac{\phi_{-,1}}{\zeta} & \frac{\phi_{-,1}}{\zeta} \\ \frac{\phi_{+,1}}{\zeta} & -\frac{\phi_{-,1}}{\zeta} \end{pmatrix}, & \zeta \in \mathbb{C} \setminus \Delta_0.
\end{align*}
\]

(9.18)

Let us now denote the jump matrix of $R_\Phi^{(2)}$ by $\hat{J}_{R,2} = \hat{R}_{-1} J_{R,1} + J_{R,2}$. Then, we have

\[
\hat{J}_{R,2} = \frac{1}{\zeta^2} \left( \begin{pmatrix} \phi_{-,1} (\phi_{-,1} - \phi_{+,1} e^{-2i\zeta}) + \phi_{-,2} & \phi_{+,1} e^{-2i\zeta} - \phi_{-,1} (1 - e^{2i\zeta}) - \phi_{-,2} \\ \phi_{+,1} (\phi_{+,1} e^{-2i\zeta} - \phi_{-,1}) - \phi_{+,2} e^{-2i\zeta} - \phi_{+,1} \phi_{-,1} (1 - e^{2i\zeta}) - \phi_{-,2} \end{pmatrix} \right)
\]

This gives

\[
\begin{align*}
\hat{R}_2 &= \frac{1}{\zeta^2} \left( \begin{pmatrix} -\phi_{+,1} \phi_{-,1} E_1^- (\zeta) & (\phi_{+,1}^2 + \phi_{-,2}) E_1^+ (\zeta) \\ (\phi_{+,1}^2 - \phi_{+,2}) E_1^- (\zeta) & -\phi_{+,1} \phi_{-,1} E_1^+ (\zeta) \end{pmatrix} \right), & \zeta \in \Delta_0, \\
\hat{R}_2 &= \frac{1}{\zeta^2} \left( \begin{pmatrix} \phi_{-,2} - \bar{u}_1 \phi_{-,1} + 2i \phi_{+,1} \phi_{-,1} \zeta & \phi_{-,2} + 2i (\phi_{-,1}^2 + \phi_{-,2}) \zeta \\ \bar{u}_1 \phi_{+,1} - \phi_{+,2} + 2i (\phi_{+,1} - \phi_{+,2}) \zeta & -\phi_{-,2} - 2i \phi_{+,1} \phi_{-,1} \zeta \end{pmatrix} \right), & \zeta \in \mathbb{C} \setminus \Delta_0.
\end{align*}
\]

(9.19)
The entries of the jump matrix \( \tilde{J}_{R,3} = \tilde{R}_{-1}J_{R,2} + \tilde{R}_{-1}J_{R,2} + J_{R,3} \) are given by
\[
(J_{R,3})_{11} = \zeta^{-3}(\iota_{-0} + 2i\zeta\phi_{+1}\phi_{-1}^2 + e^{-2i\zeta}\alpha_0 - 2i\zeta e^{-2i\zeta}\phi_{+1}\alpha_{-1}),
\]
\[
(J_{R,3})_{12} = \zeta^{-3}(\phi_{-1}(\tilde{u}_{1}\phi_{-1} - 2\phi_{-2}) - 2i\zeta\phi_{-1}\alpha_{-1} + e^{2i\zeta}\iota_{-0} + 2i\zeta e^{2i\zeta}\phi_{+1}\phi_{-1}^2),
\]
\[
(J_{R,3})_{21} = \zeta^{-3}(\alpha_0 + 2i\zeta\phi_{-1}\phi_{+1} + e^{-2i\zeta}\iota_{+0} + 2i\zeta e^{-2i\zeta}\phi_{+1}\phi_{-1}^2),
\]
\[
(J_{R,3})_{22} = \zeta^{-3}(-\alpha_0 - \phi_{-3} + 2i\zeta\phi_{+1}\phi_{-1}^2 + e^{2i\zeta}\alpha_0 + 2i\zeta e^{2i\zeta}\phi_{+1}\phi_{-1}^2),
\]
where \( \alpha_0, \phi_{\pm 1} \) and \( \iota_{\pm 0} \) are given by
\[
\alpha_0 = \tilde{u}_{1}\phi_{+1}\phi_{-1} - \phi_{-1}\phi_{+2} - \phi_{+1}\phi_{-2}, \quad \alpha_{\pm 1} = \phi_{\pm 2} \mp \phi_{\pm 1}^2,
\]
\[
\iota_{\pm 0} = 2\phi_{\pm 1}\phi_{\pm 2} \mp \phi_{\pm 3} - \tilde{u}_{1}\phi_{\pm 1}^2.
\]

Inside \( \Delta_0 \) the matrix \( \tilde{R}_3 \)
is
\[
\tilde{R}_3 = \begin{pmatrix}
\alpha_0\mathcal{E}_2^-(\zeta) - 2i\phi_{+1}\alpha_{-1}\mathcal{E}_1^-(\zeta) & \iota_{-0}\mathcal{E}_2^+(\zeta) + 2i\phi_{+1}\phi_{-1}^2\mathcal{E}_1^+(\zeta) \\
\iota_{+0}\mathcal{E}_2^-(\zeta) + 2i\phi_{+1}\phi_{-1}\mathcal{E}_1^-(\zeta) & \alpha_0\mathcal{E}_2^-(\zeta) + 2i\phi_{-1}\alpha_{+1}\mathcal{E}_1^-(\zeta)
\end{pmatrix}
\]
and \( \tilde{R}_3 \) outside of \( \Delta_0 \) is given by the above plus \( J_{R,3} \).

To compute the asymptotics of the coordinate \( Q_2 \), we will need an extra term in \( \tilde{R} \), which can be found in the same way as the other \( \tilde{R}_j \):
\[
\tilde{R}_4 = \begin{pmatrix}
\alpha_{+4}\mathcal{E}_3^- - 2i\alpha_{+3}\mathcal{E}_2^- + \alpha_{+2}\mathcal{E}_1^- & \iota_{+2}\mathcal{E}_3^+ - 2i\alpha_0\phi_{-1}\mathcal{E}_2^+ + \iota_{+1}\mathcal{E}_1^+ \\
\iota_{-2}\mathcal{E}_3^- - 2i\alpha_0\phi_{+1}\mathcal{E}_2^- + \iota_{-1}\mathcal{E}_1^- - \alpha_{-4}\mathcal{E}_3^+ + 2i\alpha_{-3}\mathcal{E}_2^+ + \alpha_{-2}\mathcal{E}_1^+
\end{pmatrix}
\]
\[
\alpha_{\pm 2} = 2(2\phi_{+1}\phi_{+2} \pm \iota_{+0}\phi_{+1}),
\]
\[
\alpha_{\pm 3} = \alpha_{+1}\alpha_{-1} + \phi_{+1}\phi_{-1}^2 \pm \phi_{+1}\iota_{+0},
\]
\[
\alpha_{\pm 4} = \phi_{+1}(\phi_{+1}\phi_{+2} \mp \phi_{+3}) - \phi_{+1}(\iota_{+0} + \alpha_0) \mp \alpha_{+1}\phi_{+2},
\]
\[
\iota_{\pm 1} = 2(2\alpha_{+1}\phi_{+1} + \alpha_0)\phi_{+1},
\]
\[
\iota_{\pm 2} = \pm \phi_{+4} + \phi_{+2}(\phi_{+2} - \tilde{u}_{1}\phi_{+1}) + \phi_{+1}(\phi_{+3} \pm (\iota_{+0} + \alpha_0)),
\]
for \( \zeta \in \Delta_0 \).

We can now use this result and the expansion of \( \Phi^{(p)}(\zeta) \) to compute asymptotics of the matrix \( A(\zeta) \) in (5.20) as \( u_2 \to 0 \). Firstly, note that from (5.15) and (5.17) we have
\[
\hat{\Phi}_0 = R_\Phi(\zeta)\Phi^{(p)}(\zeta)e^{i\zeta\sigma_3}.
\]
One can check that both definitions of \( \Phi^{(p)} \) in (9.3) give the same \( \hat{\Phi}_0 \). From this and (5.20), we can compute the asymptotics of the coefficients of \( A(\zeta) \). The computation is long and tedious but straightforward. They can be obtained via a computer algebra package such as MAPLE. The results are given by the following theorem.
Theorem 9.1. Let $u_1 = \tilde{u}_1 u_2^\frac{1}{3}$ and suppose $\tilde{u}_1$ exists and is finite as $u_2 \to 0$. Then as $u_2 \to 0$, the canonical variables $P_1$, $Q_1$, $P_2$ and $Q_2$ in (2.2) have the following behavior:

$$P_1 = \frac{\phi_1}{2\phi_{-0}} u^{\frac{4}{3}}_2 - i \left( \frac{\phi_1 \phi_{-1}}{\phi^2_{-0}} + 2 \phi_{-0} + 1 \right) u_2 + \left( -\frac{2 \phi_1 \phi^2_{-1}}{\phi^3_{-0}} + \frac{\phi_1 (\phi_{-2} - \phi^2_{-1})}{\phi_{-0}} \right) + \frac{2 \phi_1 (9 \phi^2_{-1} + 9 \phi_1 \phi_{-1} - 17 \phi_{-2})}{\phi_{-0}} - \phi^2_{1} \phi_{-2}$$

$$+ \frac{2 \phi_{-1} (3 \phi_{-1} + \phi_1)}{\phi^2_{-0}} + \frac{10 \phi^2_{-1} - 5 \phi_1 \phi_{-1} + \phi^2_{0}}{3 \phi_{-0}} + 6 \phi^2_{-1} - 4 \phi_1 \phi_{-1} + \frac{1}{3} + 4 \phi^2_{-1} \phi_{-0} \right) u^2 + O(u^2_2),$$

$$Q_1 = \frac{\phi_1}{\phi_{-0}} u^{\frac{4}{3}}_2 - 2 i \left( \frac{\phi_1 \phi_{-1}}{\phi^2_{-0}} + 1 \right) + 2 \left( -\frac{2 \phi_1 \phi^2_{-1}}{\phi^3_{-0}} + \frac{\phi_1 (\phi_{-2} - \phi^2_{-1})}{\phi_{-0}} \right) \phi_{-1} - 2 \phi_{-1}$$

$$+ \frac{i \left( 2 \phi_1 \phi^2_{-1} (19 \phi_{-1} + 17 \phi_1) + 4 \phi_1 \phi_{-1} (9 \phi^3_{-1} + 9 \phi_1 \phi^2_{-1} - \phi_1 \phi_{-2})}{\phi^2_{-0}} + \frac{6 \phi_{-1} (4 \phi_{-1} + 7 \phi_1) + 8 \phi_1 \phi_{-1}}{\phi_{-0}} + \frac{2 \phi_{-0}}{3} + \frac{4 \phi_{-1}}{3} \right) u_2 + O(u_2^2),$$

$$P_2 = \varphi u_2 + i \left( 4 \varphi^2 \tilde{u}_1 - \tilde{u}_1 + 4 \varphi \phi_1 \right) u^{\frac{4}{3}}_2 + \left( \phi_1 \tilde{u}_1 + (3 \tilde{u}_1^2 - 12 \phi^2_1) \varphi - 24 \phi_1 \tilde{u}_1 \varphi^2 - 12 u_1^2 \varphi^3 \right) u_2$$

$$+ i \left( 4 \phi_1 \tilde{u}_1 - \frac{8 \tilde{u}_1^3}{9} + \left( \frac{(140 \tilde{u}_1^2 - 28)}{9} \phi_1 - 32 \phi^3_1 \right) \varphi - \frac{(104 \tilde{u}_1^3}{9} - 96 \phi^3_1 \tilde{u}_1 \right) \varphi^2$$

$$- 96 \phi_1 \tilde{u}_1^2 \varphi^3 - 32 \tilde{u}_1^3 \varphi^4 \right) u^{\frac{4}{3}}_2 + O(u_2^3),$$

$$Q_2 = \frac{2 i \phi_{-1}}{(\phi_{-0})} u^{\frac{4}{3}}_2 + 4 \left( \frac{\phi^2_{-1}}{\phi_{-0}} + \frac{\phi^2_{-1} - \phi_{-1}}{\phi_{-0}} \right) - \frac{2 i \left( 3 \phi_1 \phi^2_{-1} + 3 \phi^2_{-1} + \phi_1 \phi_{-1} \right)}{\phi_{-0}} + \frac{2 \tilde{u}_1 \phi_{-2} + 7 \phi_{-1}}{3 \phi_{-0}}$$

$$+ \frac{4 \phi_{-1}}{3} u^{\frac{4}{3}}_2 - \frac{33 \phi^2_{-1} + 79 \phi_1 \phi_{-1} + 32 \phi^2_1}{\phi_{-0}} + \frac{6 \phi^2_{-1} (11 \phi^2_{-1} + 14 \phi_1 \phi_{-1} + 23 \phi^2_1)}{\phi_{-0}} - 2 \phi^2_{1} \phi_{-2}$$

$$+ \frac{22 \phi_{-1} \phi_{-1} + 15 \phi_1 + 24 \phi_1 \phi^2_{-1}}{3 \phi_{-0}} + \frac{24 \phi^2_{-1} + 52 \phi_1 \phi_{-1}}{3 \phi_{-0}} + \frac{8 \phi^2_{-1} + \frac{5}{3} + \frac{10 \phi_{-0}}{3}}{3 \phi_{-0}} \right) u_2 + O(u_2^3),$$

where $\varphi = \frac{i}{\sqrt{2\pi}} \int_0^\infty e^{-\frac{x^2}{2}} dx$ and the $\phi_{j,-}$ can be computed using Proposition 9.7. In particular, when $\tilde{u}_1 = 0$, this gives (2.7).

10 Reduction to a second order differential equation

From (3.3), we see that the average $G_N(u_{1,N}, u_{2,N})$ is an odd function in $u_{1,N}$ and hence the derivatives $\partial_{u_{1,N}}^{2k+1} \log G_N$ are zero when $u_{1,N}$ is zero. Then from (8.2) and (8.3), we see that

$$\partial_{u_{1}}^{2k} H_1 \big|_{u_{1}=0} = 0, \quad k \in \{0\} \cup \mathbb{N}.$$ (10.1)
This leads to non-trivial algebraic relations in the canonical coordinates that cannot be easily seen from the Hamiltonian equations in Theorem 5.1. This allows us to express the power series expansion of $G_N$ in $u_{1,N}$ in terms of the solution of fewer ODEs.

From the first two equations in (10.1), we obtain the two possible sets of relations between the variables

$$P_2 = 0, \quad Q_2 = -\frac{u_2^2Q_1^2 - 4P_1^2}{2u_2^2} \quad (10.2)$$

or

$$P_2 = -\frac{iu_2^4Q_1^4 - 8iu_2^2P_1^2Q_1^2 + 16iP_1^4 - 8u_2^2P_1}{8u_2^2Q_1}, \quad Q_2 = \frac{4iP_1}{u_2^2Q_1^2 - 4P_1^2}. \quad (10.3)$$

The condition $\partial^4_{u_1} H_1|_{u_1=0}$ does not yield any new constraints and the other equations in (10.1) involve high powers of $P_1$ and $Q_1$ which cannot be solved analytically. The relation (10.2) is compatible with the behavior of the coordinates $P_1$ and $Q_1$ at $u_2 = 0$ (2.5), while the relation (10.3) is not. Therefore at $u_1 = 0$, the Hamiltonian equations in Theorem 5.1 are reduced to the following

$$\partial_{u_2}P = \frac{Q(u_2^2Q^2 - 4P^2)}{8u_2} + \frac{P}{2u_2},$$

$$\partial_{u_2}Q = \frac{P(u_2^2Q^2 - 4P^2)}{2u_2} - \frac{Q}{2u_2} - \frac{i}{u_2}. \quad (10.4)$$

This can be represented as a set of time dependent Hamiltonian equations in the canonical variables $P$ and $Q$ with the Hamiltonian

$$\mathcal{H} = -\frac{(u_2^2Q^2 - 4P^2)^2}{32u_2^3} - \frac{iP}{u_2} - \frac{PQ}{2u_2}.$$ 

Since this system of Hamiltonian equations is a reduction from an integrable systems that satisfy the Painlevé property, it should also satisfies the Painlevé property and therefore be equivalent to one of the Painlevé equations. However, we do not know how to transform it into a Painlevé equations.

References

[1] M. R. Adams, J. Harnad, E. Previato. Isospectral Hamiltonian flows in finite and infinite dimensions. I. Generalized Moser systems and moment maps into loop algebras. *Comm. Math. Phys.*, 117 (1988), no. 3, 451–500.

[2] M. R. Adams, J. Harnad, J. Hurtubise. Isospectral Hamiltonian flows in finite and infinite dimensions. II. Integration of flows. *Comm. Math. Phys.*, 134 (1990), no. 3, 555–585.

[3] M. R. Adams, J. Harnad, J. Hurtubise. Darboux coordinates and Liouville-Arnold integration in loop algebras. *Comm. Math. Phys.*, 155 (1993), no. 2, 385–413.
[4] M. V. Berry and P. Shukla. Tuck’s incompressibility function: statistics for zeta zeros and eigenvalues. *J. Phys. A: Math. Theor.*, **41** (2008), 385202.

[5] M. Bertola, J. Harnad, J. Hurtubise and G. Pusztai. (2004), private communications.

[6] M. Bertola, B. Eynard, J. Harnad. Semiclassical orthogonal polynomials, matrix models and isomonodromic tau functions. *Commun. Math. Phys.*, **263** (2006), 401–437.

[7] P. Bleher and A. Its. Semiclassical asymptotics of orthogonal polynomials, Riemann-Hilbert problem, and universality in the matrix model. *Ann. of Maths. (2)*, **150** (1999), 185–266.

[8] P. Bleher and A. Its. Double scaling limit in the random matrix model: the Riemann-Hilbert approach. *Comm. Pure Appl. Math.*, **56** (2003), no. 4, 433–516.

[9] Chen Y. and A. Its. Painlevé III and a singular linear statistics in Hermitian matrix ensembles I. *Preprint arXiv:0808.3590*

[10] Chen Y. and A. Its. Painlevé III and a singular linear statistics in Hermitian matrix ensembles II. The asymptotic analysis. *Private communication.*

[11] T. Claeys and A. B. J. Kuijlaars. Universality of the double scaling limit in random matrix models. *Comm. Pure Appl. Math.*, **59** (2006), no. 11, 1573–1603.

[12] T. Claeys, A. B. J. Kuijlaars and M. Vanlessen. Multi-critical unitary random matrix ensembles and the general Painlevé II equation. *arXiv:math-ph/0508062*.

[13] T. Claeys and M. Vanlessen. Universality of a double scaling limit near singular edge points in random matrix models. *Comm. Math. Phys.*, **273** (2007), no. 2, 499–532.

[14] P. Deift. *Orthogonal polynomials and random matrices: A Riemann-Hilbert approach.* Courant lecture notes 3. New York University. (1999).

[15] P. Deift, T. Kriecherbauer, K. T. R. McLaughlin and S. Venakides and X. Zhou. Strong asymptotics of orthogonal polynomials with respect to exponential weights. *Comm. Pure Appl. Math.*, **52** (1999), no. 12, 1491–1552.

[16] P. Deift, T. Kriecherbauer, K. T. R. McLaughlin, S. Venakides and X. Zhou. Uniform asymptotics for polynomials orthogonal with respect to varying exponential weights and applications to universality questions in random matrix theory. *Comm. Pure Appl. Math.*, **52** (1999), no. 11, 1335–1425.

[17] L. D. Faddeev and L. A. Takhtajan. *Hamiltonian methods in the theory of solitons*, Springer-Verlag, Heidelberg (1987).

[18] J. Harnad. Dual isomonodromic deformations and moment maps to loop algebras. *Comm. Math. Phys*, **166** (1994) no.2: 337–365.
[19] J. Harnad and M. Routhier. R-matrix construction of electromagnetic models for the Painlevé transcendents. *J. Math. Phys.*, **36**, (1995) 9:4864–4881.

[20] J. Harnad. The Hamiltonian structure of the general rational isomonodromic deformations (2005) Talk at the Colloque international en l’honneur de Pierre van Moerbeke, Poitiers, France. Unpublished.

[21] M. Jimbo, T. Miwa and Ueno. Monodromy preserving deformations of linear ordinary differential equations with rational coefficients I. *Physica 2D*, **2** (1981) 306–352.

[22] M. Mazzocco and M. Mo. The Hamiltonian Structure of the Second Painlevé Hierarchy. *Nonlinearity*, **20**, (2007), 2845-2882.

[23] F. Mezzadri and M. Y. Mo. On an average over the Gaussian Unitary Ensemble. *Int. Math. Res. Notices*, (2009), Article ID rnp062.

[24] A. P. Veselov and S. P. Novikov. Poisson brackets and complex tori. (Russian) Algebraic geometry and its applications. *Trudy Mat. Inst. Steklov.*, **165** (1984), 49–61.

[25] E. O. Tuck. Smallest minima of $(1 - Z(t)Z''(t))/Z'(t)$. http://internal.maths.adelaide.edu.au/people/etuck/ (2008)

[26] X. Zhou. The Riemann-Hilbert problem and inverse scattering. *SIAM J. Math. Anal.*, **20** (1989), 966–986.

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