Long-range Trap Models on $\mathbb{Z}$ and Quasistable Processes

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Abstract Let $\mathcal{X} = \{X_t : t \geq 0, X_0 = 0\}$ be a mean zero $\beta$-stable random walk on $\mathbb{Z}$ with inhomogeneous jump rates $\{\tau_i^{-1} : i \in \mathbb{Z}\}$, with $\beta \in (1, 2]$ and $\{\tau_i : i \in \mathbb{Z}\}$ a family of independent random variables with common marginal distribution in the basin of attraction of an $\alpha$-stable law, $\alpha \in (0, 1)$. In this paper, we derive results about the long-time behavior of this process, in particular its scaling limit, given by a $\beta$-stable process time changed by the inverse of another process, involving the local time of the $\beta$-stable process and an independent $\alpha$-stable subordinator; we call the resulting process a quasistable process. Another such result concerns aging. We obtain an (integrated) aging result for $\mathcal{X}$.

Keywords Trap model · Stable random walks · Scaling limit · Stable process · Stable subordinator · Aging

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1 Introduction

Trap models have been introduced in the physics literature as simple models of disordered systems where long-time memory effects such as aging and localization can be
established and understood on a rigorous basis. See for instance [7,8] and [10]. Many mathematical papers followed, a few of which we mention below. The derivation of scaling limits of the models is a common theme.

Broadly speaking, a trap model is a continuous time Markov jump process on some regular graph with random transition rates given in terms of heavy-tailed random variables, the trap environment, which give rise to trapping mechanisms leading to the above-mentioned effects. The most studied cases in the mathematics literature involve a jump chain which is independent of the trap environment and spatially homogeneous, and inverse jump rates given by iid heavy-tailed random variables, viewed in this case as trap depths. In these cases, the trap model is thus a time change of the jump chain (which is a discrete time random walk). In this paper, we will be concerned with such a trap model on \( \mathbb{Z} \), so let us discuss the case of \( \mathbb{Z}^d \), \( d \geq 1 \), for a while. (References for the cases of other graphs, such as the complete graph or the hypercube, may be found in the references mentioned below.)

The simple symmetric case in \( d = 1 \) was studied in [11], and a scaling limit was derived, from which aging and localization results followed. The higher dimensional symmetric case was resolved in [4] and [3]. In both cases, the scaling limit is given by a time change of Brownian motion, with the time change dependent of the Brownian motion in \( d = 1 \), but not in \( d \geq 2 \). (The distinction arises as follows: consider the numbers of visits of the jump chain to the deepest traps, which in all cases account for virtually all of the time spent by the continuous time process along its history. In the first case, this numbers are macroscopically correlated with the trajectory of the jump chain, and in the limit, this manifests itself in the representation of the time change in terms of the local time of the scaling limit of the jump chain, as well as in terms of the scaling limit of the deep traps. This mechanism is also at play in the model of this paper. In the second case, those numbers are only weakly correlated with the trajectory of the jump chain, as well as among themselves—the correlations disappear in the scaling limit; this is easy to convince oneself of in the transient case of \( d \geq 3 \), but is also the case in the weak recurrent case of \( d = 2 \). The upshot is that the time change in the limit process is independent of the scaling limit of the jump chain.) Asymptotic aging and localization functions of the trap model are given in terms of the expectations of the scaling limit. A variation of this case is the asymmetric model, a nearest neighbor model, where the transition rates depend on heavy-tailed random variables of both origin and destination sites. In this case, scaling limit and aging results were obtained in [1,2,9] and [16]. The scaling limit is similarly given by the time change of a Brownian motion.

Another variation is in the direction of allowing a generic jump chain/random walk. This includes the case studied in the present paper. Scaling limit and aging results were derived in [12] for the generic case under the validity of a law of large numbers for the range of the jump chain and the slow variation of the tail of the distribution of its return probability. These assumptions include all the random walks in \( d \geq 2 \). The process considered in the latter paper is the trap process, namely the depth of the currently visited trap. The scaling limit (which might not exist in the spatial representation of the process) is given in terms of a subordinator seen at the inverse of another, correlated subordinator.
In this paper, we consider the model on \( \mathbb{Z} \) and assume that the jump chain is a mean zero, \( \beta \)-stable random walk, with \( \beta \in (1,2] \), but otherwise generic. This is outside the assumptions of [12]. The model of [11], where the jump chain is the simple symmetric random walk, is a particular case (of \( \beta = 2 \)). One of our motivations is to close a gap left by the above papers. (Let us point out that the case where \( \beta \in (0,1] \) is included in [12]). We derive the scaling limit of the (spatial version of the) process, given in terms of a time changed \( \beta \)-stable process and then obtain aging results for the trap model in terms of the scaling limit.

We call the limit process (given the proper version of the limit heavy-tailed random variables) a quasistable process, following the terminology of quasidiffusions for the \( \beta = 2 \) case adopted in the literature (see [11] and references therein); see also [13]. Analytical properties of quasidiffusions, such as the existence and continuity of transition density functions, were crucial in the derivation of (nonintegrated) aging results for the simple symmetric case of [11]. The same results can be readily extended to our more general framework for \( \beta = 2 \), but not for \( \beta \in (1,2) \), where the analogous analytical properties for the corresponding quasistable processes seem to be missing in the literature. This point is another of our motivations: to call attention for the lack of analytical results for a class of processes, namely the quasistable processes, which naturally extends a better known subclass, namely the quasidiffusions. Without those results, we may nevertheless obtain integrated aging results, if not ordinary aging results. (See the following discussion on aging and Remark 9 below.)

Let us now briefly discuss aging. Let \( X_t \) be a generic stochastic process, which might be the trap model described above. Consider \( Q(s,t) \) a two-time correlation function of \( X_t \). We call it an aging function. We say that normal aging occurs if there exists a nontrivial function \( Q : \mathbb{R}^+ \to \mathbb{R} \) that is the limit of \( Q(s,t) \) as \( t \) and \( s \) go to infinity proportionally, that is,

\[
\lim_{t,s \to \infty} \frac{Q(s,t)}{t/s} = Q(\theta), \tag{1}
\]

with \( \theta > 0 \). This is the ordinary, nonintegrated case, as opposed to the integrated case, where we introduce a random time \( T \) (independent of \( X_t \)) and consider the aging function given by \( E[Q(\lambda T, \mu T)] \), with \( \mu, \lambda > 0 \), and the expectation taken with respect to \( T \). We then say that integrated normal aging occurs if there exists a nontrivial function \( \tilde{Q} : \mathbb{R}^+ \to \mathbb{R} \) such that

\[
\lim_{\mu, \lambda \to \infty, \mu/\lambda \to \theta} E[Q(\lambda T, \mu T)] = \tilde{Q}(\theta), \tag{2}
\]

for \( \theta > 0 \). Typically, \( Q \) and \( \tilde{Q} \) are decreasing and onto \([0,1] \). In these cases (aging), results such as (1) and (2) may be interpreted as follows, and this explains the terminology: after observing the process at a large time \( t \), the time it takes to get a subsequent (reasonably) decorrelated observation is of the order of \( t \), indicating that an ever increasing slowing down takes place.

The paper is organized as follows. In Sect. 2, we describe our trap model in detail, and its rescaling, and state our scaling limit result (Theorem 1), proved in Sect. 3.
Section 4 is devoted to obtaining an integrated aging result (Theorem 3) for an aging function to be introduced therein. An appendix collects some results on the scaling limit process.

A longer version of this paper, including additional results such as convergence of the trap process and a study of localization, as well as the simpler case $\alpha \geq 1$ (where the scaling limit is an ordinary $\beta$-stable process), can be found at http://arxiv.org/pdf/1302.4758.

2 Model and First Result

Let $\varepsilon = \{\varepsilon_j, j \in \mathbb{N}\}$ be a sequence of iid discrete random variables with distribution function $F$ in the basin of attraction of a stable law with index $\beta \in (1, 2)$, such that $E(\varepsilon_1) = 0$ and $E(e^{it \varepsilon_1}) = 1$ if and only if $t$ is multiple of $2\pi$. The latter assumption is well known to imply that the corresponding random walk is aperiodic.

Let now $X = \{X_i, i \in \mathbb{N} \cup \{0\}\}$ be such that $X_0 = 0$ and for $n \geq 1$

$$X_n = \sum_{j=1}^{n} \varepsilon_j.$$  \hspace{1cm} (3)

This sequence is called a $\beta$-stable random walk (see [15]); this process is also known as long-range random walk.

The object of our study is a continuous time Markov process $X = \{X_t : t \geq 0\}$ on $\mathbb{Z}$ having $X$ as its jump chain, and whose jump rates are given by $\{\tau_i^{-1} : i \in \mathbb{Z}\}$, where $\tau = \{\tau_i : i \in \mathbb{Z}\}$ is a family of iid (strictly) positive random variables in the basin of attraction of a stable law with index $\alpha \in (0, 1)$, independent of $X$. Let us point out that the Markov property of $X$ holds for (almost) every fixed realization of $\tau$. The distribution of $X$ integrated with respect to $\tau$ is not Markovian.

Our first result is a scaling limit of $X$. In order to formulate it, we need to introduce scaling factors, rescaled processes, and limit processes. Let us start with the scaling factors and the rescaled process.

Scaling factors and rescaled process. We recall the well-known fact that the assumption on the jump variables $\varepsilon$ implies the following. If $\beta \in (1, 2)$, then there exist constants $c^- > 0$ and $c^+ > 0$, and a slowly varying function at infinity $h(\cdot)$ such that

$$P(\varepsilon_1 < -x) \sim x^{-\beta}(c^- + o(1))h(x)$$

and

$$P(\varepsilon_1 > x) \sim x^{-\beta}(c^+ + o(1))h(x),$$

where as usual $f_1(x) \sim f_2(x)$ means $\lim_{x \to \infty} f_1(x)/f_2(x) = 1$. If $\beta = 2$, then $H : (0, \infty) \to (0, \infty)$, with $H(z) = \int_{-z}^{z} x^2 dF(x)$, is a slowly varying function at infinity.
It follows that in each case, there exists a positive slowly varying function $v(\cdot)$ such that as $n \to \infty$

$$h(n^{1/\beta} v(n))v^{-\beta}(n) \to 1, \text{ for } \beta \in (1, 2)$$

and

$$H(n^{1/2} v(n))v^{-2}(n) \to 2(c^+ + c^-), \text{ for } \beta = 2.$$ 

The assumption on the inverse rate variables $\tau$ implies that

$$P(\tau_0 > x) \sim x^{-\alpha}(1 + o(1))s(x), \ x \geq 0,$$ 

where $s(\cdot)$ is a slowly varying function at infinity. It follows that there exists a slowly varying function $w(\cdot)$ such that

$$s(n^{1/\alpha} w(n))w^{-\alpha}(n) \to 1$$

as $n \to \infty$.

Let us define the sequences

$$d_n = n^{1/\beta} v(n), \ r_n = nd_n^{-1}, \quad (4)$$

and

$$a_n = r_n b_n, \quad (5)$$

where

$$b_n = d_n^{1/\alpha} w(n).$$

We are now ready to define the rescaled process. Let, for $n \geq 1$, let

$$\chi^{(n)} := \{\chi^{(n)}_t = d_n^{-1} \chi_{a_n t}, t \geq 0\}.$$ 

Remark 1 In the more explicit case where the slowly varying functions entering the distributions of $\varepsilon$ and $\tau$ are asymptotic to constants (say both equal to 1), we get that $a_n = n^{1-\frac{1}{\beta}} + \frac{1}{\alpha \beta}$ and $d_n = n^{\frac{1}{\beta}}$. By taking $m = a_n$ as scaling factor, we find that $b_n = m^{-\frac{1}{\beta} + \frac{1}{\alpha} - 1}$, and we see a slowing down term of $\frac{1}{\alpha} - 1$ appearing due to the traps, as compared with the homogeneous case with no traps, where we would have $b_n = m^{-\frac{1}{\beta}}$. Except for slowly varying corrections, we have the same slowing down term in the general case.

Limit process. An ingredient of the limit process is the stable process $Z = (Z_t)_{t \geq 0}$ with characteristic function given by

$$E(e^{i s Z_t}) = \exp\{-ct|s|^{\beta} [1 + iq \ sgn(s)]\},$$

where $c = -\Gamma(2 - \beta) \frac{\cos(\pi \beta/2)}{\beta - 1}$ and $q = \frac{c - c^+}{c^+ + c^-} \tan(\pi \beta/2).$
Another ingredient is a bilateral $\alpha$-stable process $V = \{V_x : x \in \mathbb{R}, V_0 = 0\}$ independent of $Z$.

Let $\phi(t, x)$ be the local time of $Z_t$, that is, let $\phi : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}^+$ be a random function which is jointly continuous with probability one and satisfies

$$\mathcal{L}(s : Z_s \in A, 0 \leq s \leq t) = \int_A \phi(t, x)dx,$$

for any Borel set $A$, where $\mathcal{L}$ denotes the Lebesgue measure; see [5]. Note that it follows from the fact that $Z$ almost surely does not explode at any finite time that $\phi(t, \cdot)$ is compactly supported for every $t$. Now define

$$S_t = \int_{-\infty}^{\infty} \phi(t, x)dV(x), \quad t \geq 0. \quad (6)$$

The compactness of the support of $\phi(t, \cdot)$, as noted right above, and the local finiteness of $V$ (as a measure) make (6) well defined. From other elementary properties of $\phi$ and $V$, namely $\sup_x \phi(t, x)$ is strictly increasing in $t$ and the support of $V$ is the whole line, we get that $S$ is strictly increasing and continuous. So it has an ordinary inverse $S^{-1}$.

We are now ready to state our scaling limit result.

**Theorem 1** Let $\{a_n : n \in \mathbb{N}\}$ and $\{d_n : n \in \mathbb{N}\}$ be the sequences defined in (5) and (4), respectively. We have that

$$X^{(n)} \Longrightarrow (Z_{S_t^{-1}})_{t \geq 0} \quad (7)$$

as $n \to \infty$, where $\Longrightarrow$ means convergence in distribution on $D([0, \infty), \mathbb{R})$ endowed with the $J_1$-Skorohod topology.

**Remark 2** Given an arbitrary fixed distribution function $F$, we may replace $V$ by $F$ in (6), and then get a process $Z_{S_t^{-1}}$ (in this generality, $S$ may have flat intervals, so $S^{-1}$ may have to be taken as a generalized inverse). In this generality, we call $Z_{S_t^{-1}}$ a quasistable process in an analogy with the term quasidiffusion, used for the case where $Z$ is a Brownian motion. Without this terminology, quasistable processes were introduced in [17]. They are strong Markov processes. Some additional properties are stated and proven in an appendix. Unless otherwise mentioned, we will stick to the $F = V$ case throughout. In the latter context, it is worth emphasizing that the Markov property holds for every fixed realization of $V$, and it does not hold for the process integrated with respect to distribution of $V$.

**Remark 3** In each side of “$\Longrightarrow$” in (7), we have a processes in a random environment. As pointed out in the definition of $X$—see paragraph below (3)—, $X^{(n)}$ is a Markov process given the environment $\tau$. And in the above remark, we have just seen that $(Z_{S_t^{-1}})_{t \geq 0}$ is a Markov process given the environment $V$. The distributions on the
3 Proof of Theorem 1

3.1 Preliminaries

Let \( L(n, x) = \sum_{i=0}^{n} 1 \{ X_i = x \} \) be the local time (occupation time) of the random walk \( X \), that is, the number of times that \( X \) visits the point \( x \) up to time \( n \in \mathbb{N} \cup \{ 0 \} \), and the rescaled local time and rescaled jump chain

\[
\phi_n(t, x) = n^{-1} L([nt], [xd_n]), \quad Z_t^{(n)} = d_n^{-1} X_{[nt]},
\]

for \( t \in [0, \infty) \) and \( x \in \mathbb{R} \). It is well known that the process \( (Z_t^{(n)})_{t \geq 0} \) weakly converges on \( D([0, \infty), \mathbb{R}) \) endowed with the \( J_1 \) topology to \( Z \).

Clock process. A key element of our analysis is the clock process associated with \( X \), defined by \( C_t = (C_t)_{t \geq 0} \), where

\[
C_t = \sum_{i=0}^{[t]} \tau_{X_i} T_i, \ t \geq 0,
\]

where \( \{ T_i : i \in \mathbb{N} \cup \{ 0 \} \} \) is a sequence of iid exponential variables with mean 1 independent of \( X \) and \( \tau \).

Notice that \( X \) may be represented as \( (X C_t^{-1})_{t \geq 0} \), where \( C_t^{-1} \) is the generalized (right continuous) inverse of \( C \).

We have that the clock process (9) is equal in distribution to the process \( \tilde{C} = (\tilde{C}_t)_{t \geq 0} \), where

\[
\tilde{C}_t = \sum_{i \in \mathbb{Z}} \tau_i \sum_{j=1}^{L([t], i)} E_{ij},
\]

and \( E = \{ E_{ij} : i \in \mathbb{Z}, j \in \mathbb{N} \} \) is a family of iid exponential random variables with mean 1 and independent of all random variables defined previously; we here define \( \sum_{j=1}^{0} E_{ij} = 0 \).

We thus have that \( (X \tilde{C}_t^{-1})_{t \geq 0} \) is a version of \( X^{'} \). Furthermore, defining

\[
\tilde{C}_t^{(n)} = n^{-1} \tilde{C}_{nt}, \ t \geq 0,
\]

\[ n \geq 0, \] we may check that

\[
\{ Z_{(\tilde{C}_t^{(n)})^{-1}}(t) : t \geq 0 \} \overset{d}{=} X^{(n)},
\]

where \( \overset{d}{=} \) means equality in distribution.
A convenient version of $\tau$.

The proof of Theorem 1 will involve obtaining the scaling limit of the clock process. Following a strategy used numerously before (for an early reference, see [11], Sect. 3), we will resort to a version of the rescaled trap environment which converges strongly (rather than only weakly, which is the case for the original trap environment). With the new version of $\tau$, we define a new version of the clock process, the rescaling of which we then will later on show converges in distribution for almost every realization of the trap environment. This convergence is a key ingredient of our proof of the scaling limit of $X_t$.

We now present the new version of the environment. Let $V = \{V_x : x \in \mathbb{R}\}$ be a bilateral $\alpha$-stable process independent of $E$ and $X$. Consider a function $G : [0, \infty) \to [0, \infty)$ satisfying $P(V_1 > G(y)) = P(\tau_0 > y)$, $y > 0$, and for $n \geq 0$ let $g_n : \mathbb{R}^+ \to [0, \infty)$ be such that $g_n(y) = b_n^{-1}G^{-1}(d_n^{1/\alpha}y)$. For $n \geq 0$, let

$$\tau_{x}^{(n)} = b_ng_n(V_x+d_n^{-1} - V_x), \ x \in d_n^{-1}\mathbb{Z}.$$ 

One readily checks that $\tau^{(n)} = \{\tau_x^{(n)} : x \in d_n^{-1}\mathbb{Z}\}$ has the same distribution as $\{\tau_i : i \in \mathbb{Z}\}$ for every $n$. It follows that the process (10) follows the same law as that of the following process:

$$\tilde{C}_{i}^{(n)} = \sum_{x \in d_n^{-1}\mathbb{Z}} g_n(V_x+d_n^{-1} - V_x)\phi_n(t, x)\tilde{T}_{x_dn}(nt), \ t \geq 0,$$ 

(11)

where for every $t$ and $i$

$$\tilde{T}_{i}(t) = \begin{cases} \sum_{j=1}^{L([t], i)} E_{ij}, & \text{if } L([t], i) > 0; \\ L([t], i), & \text{otherwise,} \end{cases}$$

and $\phi_n(t, x)$ is the rescaled local time defined in (8).

We thus have that for every $n$

$$\{Z_{\tilde{C}_{i}^{(n)}-1}^{(n)} : t \geq 0\} \overset{d}{=} \chi^{(n)}.$$

We will use the left-hand side version of $\chi^{(n)}$ in the proof of Theorem 1 next.

3.2 Proof of Theorem 1

Path space and topology. Denote by $D([0, T], \mathbb{R})$ and $D([0, T], \mathbb{R}^+)$ the space of the real and nonnegative càdlàg functions on $[0, T]$, where $T > 0$, and let $J_1T$ and $u_T^+$ be the $J_1$-Skorohod and uniform metrics in $D([0, T], \mathbb{R})$ and $D([0, T], \mathbb{R}^+)$, respectively. Further let $d = \sum_{n=1}^{\infty} 2^{-n} \min(J_{1n}, 1)$ and $u_T^+ = \sum_{n=1}^{\infty} 2^{-n} \min(u_{Tn}^+, 1)$. We denote the uniform topology by $U$. 

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An auxiliary result. In order to prove the Theorem 1, we first obtain the joint scaling limit of the clock process and the jump chain \(X\), provided in the next result. This strategy was used, for example, in [3] to find the scaling limit for the trap model on \(\mathbb{Z}^d\) (for \(d \geq 2\)) with nearest neighbors and inverse rates as we consider here. The major work lies in showing the joint convergence of the finite-dimensional distributions of the clock process and the jump chain. In [3], analytical arguments for the joint characteristic function were used. We here consider a different approach based on a probabilistic argument. One important difference between our clock process and that of [3] is that in our case, the scaling limit depends on the scaling limit of the rescaled probabilistic argument. We here consider a different approach based on a strategy was used, for example, in [3] to find the scaling limit for the trap model on \(\mathbb{Z}^d\) (for \(d \geq 2\)).

**Theorem 2** For almost every realization of \(V\), \((Z^{(n)}, \tilde{C}^{(n)}) = \{(Z^{(n)}_t, \tilde{C}^{(n)}_t) : t \geq 0\}\) converges to \((Z, S) = \{(Z_t, S_t) : t \geq 0\}\) in distribution as \(n \to \infty\) on \(D([0, \infty), \mathbb{R}) \times D([0, \infty), \mathbb{R}^+)\) endowed with the \(J_1 \times U\) product topology.

**Remark 4** The limit process of our rescaled clock process is almost surely continuous, while the one considered in [3] is not (there, the limit clock process is an \(\alpha\)-stable subordinator). Another difference between both cases is with respect to the topology where the convergence takes place. It holds with the uniform topology here, while in [3], one needs to use the \(M_1\)-Skorohod topology. Nevertheless, our proof will indeed verify criteria of convergence in the \(J_1 \times M_1\) topology for the bivariate process given in Theorem 2. Since \(S\) is almost surely continuous, the convergence in \(J_1 \times U\) topology follows—see e.g. [18], Subsection 3.3.

**Remark 5** From now on we will denote the conditional distribution and expectation given \(V\) by \(\mathbb{P}() \equiv P(\cdot|V)\) and \(\mathbb{E}() \equiv E(\cdot|V)\), respectively.

**Proof of Theorem 2** By a standard argument, it is enough to show convergence on \(D([0, T], \mathbb{R})\) and \(D([0, T], \mathbb{R}^+)\) equipped with the metrics \(J_1T\) and \(u_1^T\), respectively, with \(T > 0\) arbitrary. We will next prove the convergence of the finite-dimensional distributions of \((Z^{(n)}, \tilde{C}^{(n)})\) and then establish tightness in those path spaces. This then implies the result.

**Convergence of the finite-dimensional distributions of \((Z^{(n)}, \tilde{C}^{(n)})\)**

We resort to a result by Borodin [6], which, under conditions ensured by our assumptions on \(\varepsilon\), guarantees the existence on some probability space of processes \(Z^{(n)} = \{Z^{(n)}_t : t \in [0, T]\}_{n \geq 1}\) and \(Z' = \{Z_t : t \in [0, T]\}\) such that

(a) their finite-dimensional distributions coincide with those of \(Z^{(n)}\) and \(Z\), respectively;

(b) \(Z^{(n)}\) converges almost surely to \(Z'\) on \(D([0, T], \mathbb{R})\) endowed with the \(J_1\)-Skorohod topology;

(c) the local times \(\phi^{(n)}(\cdot, \cdot)\) and \(\phi'(\cdot, \cdot)\) of \(Z^{(n)}\) and \(Z'\), respectively, are such that for any \(T > 0\) and \(\xi > 0\):

\[
\lim_{n \to \infty} P\left( \sup_{(t,x) \in [0,T] \times \mathbb{R}} |\phi^{(n)}(t,x) - \phi'(t,x)| > \xi \right) = 0
\]

(see Theorem 1.1 in [6]).

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Therefore, it is enough to show the convergence of the finite-dimensional distributions of \((Z^n(t), \widehat{C}^{(n)})\) to those of \((Z, S)\), where \(C^{(n)}\) is defined analogously as \((11)\), replacing the quantities that depend on \(Z^{(n)}\) by the corresponding ones depending on \(Z^n(t)\). We will likewise below use the notation \(B'\) when replacing \(Z^{(n)}\) by \(Z^n(t)\) in a quantity \(B\) depending on the former process.

We now define the sets of deep traps. For arbitrary \(\delta > 0\), let

\[
T_\delta = \{ x \in \mathbb{R} : V_x - V_{x-} > \delta \} = \{ \ldots < x_{-1} < x_0 < x_1 < \ldots \}, \\
T_{\delta}^{(n)} = \{ x^{(n)}_j : j \in \mathbb{Z} \}, \ n \geq 1,
\]

where \(x^{(n)}_j = d_n^{-1}[d_n x_j], j \in \mathbb{Z} \). We first show that

\[
\lim_{\delta \downarrow 0} \lim_{n \to \infty} \sup_{x \in d_n^{-1}[\mathbb{Z} \cap (T_{\delta}^{(n)})^c]} g_n(V_{x+d_n^{-1}} - V_x)\phi'_n(t, x)\bar{t}'_{x d_n}(nt) = 0, \tag{13}
\]

in probability, for all \(t \in [0, T]\). This says that the terms of the rescaled clock process that are out of the deep trap set have a negligible contribution to the limit process as \(n \to \infty\) and \(\delta \downarrow 0\).

**Remark 6** Let \(A^{(n, \delta)}\) be a random variable depending on \(\delta, n\) and \(t\). Below, when we say that \(\lim_{\delta \downarrow 0} \lim_{n \to \infty} \sup_{t \in [0, T]} P(A^{(n, \delta)} > \epsilon) = 0\) in probability, that means that for all \(\epsilon > 0\) fixed we have that \(\lim_{\delta \downarrow 0} \lim_{n \to \infty} P(|A^{(n, \delta)}| > \epsilon) = 0\).

Borodin’s result mentioned above then says that for any \(\epsilon > 0\), there exists \(A_\epsilon > 0\) such that

\[
\sup_{n} P\left(\sup_{t \in [0, T]} |Z^n(t)| > A_\epsilon\right) < \epsilon \quad \text{and} \quad P\left(\sup_{t \in [0, T]} |Z^n(t)| > A_\epsilon\right) < \epsilon. \tag{14}
\]

Let \(I_\epsilon = (-A_\epsilon, A_\epsilon)\). Then, for all \(\zeta > 0\), using the above result and the Markov inequality, we get that

\[
\mathbb{P}\left(\sum_{x \in d_n^{-1}[\mathbb{Z} \cap (T_{\delta}^{(n)})^c]} g_n(V_{x+d_n^{-1}} - V_x)\phi'_n(t, x)\bar{t}'_{x d_n}(nt) > \zeta\right) \\
\leq \mathbb{P}\left(\sum_{x \in d_n^{-1}[\mathbb{Z} \cap (T_{\delta}^{(n)})^c \cap I_\epsilon]} g_n(V_{x+d_n^{-1}} - V_x)\phi'_n(t, x)\bar{t}'_{x d_n}(nt) > \zeta\right) + \epsilon \\
\leq \zeta^{-1} \sum_{x \in d_n^{-1}[\mathbb{Z} \cap (T_{\delta}^{(n)})^c \cap I_\epsilon]} g_n(V_{x+d_n^{-1}} - V_x)E(\phi'_n(t, x)) + \epsilon \tag{15}
\]

for almost every realization of \(V\). For all \(M > 0\) integer, we have that

\[
E(\phi^M_n(t, x)) \leq E(\phi^M_n(t, 0)), \quad \lim_{n \to \infty} E(\phi^M_n(t, 0)) = \frac{t^{M(1-1/\beta)}\zeta^\Gamma(1 - 1/\beta)}{\Gamma(2 - 1/\beta)}, \tag{16}
\]
where \( z \) is the value of the density of \( Z_1 \) at zero (see [6], p. 328). From (16) and using the equality of the finite-dimensional distributions of \( \phi_n(t, x) \) and \( \phi'_n(t, x) \), which holds for all \( x \in \mathbb{R} \), we obtain that the sum in (15) is bounded above by constant times

\[
\sum_{x \in d_{n+1}^{-1} \mathbb{Z} \cap (T_{\bar{\delta}}^{(n)}): \bar{\delta} \in I_n} g_n(V_{x+d_{n}^{-1}} - V_x).
\]

Now, arguing as in [11], paragraphs of (3.25) to (3.28), it follows that \( \lim_{\delta \downarrow 0} \limsup_{n \to \infty} \) of the above term vanishes in probability. This and the arbitrariness of \( \varepsilon \) yield (13).

With the above result, we now define the clock process restricted to the deep traps:

\[
\widetilde{C}_t^{(n, \delta)} = \sum_{i \geq 1} g_n(V_{x_{i}^{(n)}+d_{n}^{-1}} - V_{x_{i}^{(n)}})\phi'_n(t, x_{i}^{(n)})\bar{T}'_{x_{i}^{(n)}d_{n}}(nt).
\]  

(17)

Let \( \varepsilon > 0 \) and take \( A_{\varepsilon} \) satisfying (14). Define the set

\[
\Omega_{\varepsilon, n} = \big\{ \sup_{t \in [0, T]} |Z_t^{(i)}| \leq A_{\varepsilon}, 1 \leq i \leq n \big\} \cap \big\{ \sup_{t \in [0, T]} |Z_t| \leq A_{\varepsilon} \big\}.
\]

On \( \Omega_{\varepsilon, n} \), the process given in (17) equals

\[
\widetilde{C}_t^{(n, \delta, \varepsilon)} = \sum_{i = -N_{\delta, \varepsilon}}^{N_{\delta, \varepsilon}} g_n(V_{x_{i}^{(n)}+d_{n}^{-1}} - V_{x_{i}^{(n)}})\phi'_n(t, x_{i}^{(n)})\bar{T}'_{x_{i}^{(n)}d_{n}}(nt),
\]

where \( N_{\delta, \varepsilon} = \max\{|j| \in \mathbb{N} : x_j \in I_\varepsilon\} < \infty \) for almost every \( V \).

Result (12) and the a.s. continuity of \( \phi'(\cdot, \cdot) \) imply that \( \phi'_n(t, x_{i}^{(n)}) \) converges in probability to \( \phi'(t, x_i) \) uniformly in \( i \in [-N_{\delta, \varepsilon}, N_{\delta, \varepsilon}] \) and \( t \in [0, T] \) as \( n \to \infty \). Furthermore, since \( L'([nt], [x_{i}^{(n)}d_{n}]) \xrightarrow{p} \infty \) when \( n \to \infty \) (this follows from the convergence in probability of \( \phi'_n(t, x_{i}^{(n)}) \)), it follows from the Law of the Large Numbers that \( \bar{T}'_{x_{i}^{(n)}d_{n}}(nt) \xrightarrow{p} 1 \) as \( n \to \infty \) for all \( i \) and \( t \). Further, from Proposition 3.1 of [11], it follows that

\[
g_n(V_{x_{i}^{(n)}+d_{n}^{-1}} - V_{x_{i}^{(n)}}) \xrightarrow{a.s.} V_{x_i} - V_{x_{i}^{(n)}},
\]

(18)
as \( n \to \infty \), for almost every \( V \) and for all \( i \).

The convergence in probability of \( \phi'_n(t, x_{i}^{(n)}) \) and \( \bar{T}'_{x_{i}^{(n)}d_{n}}(nt) \) (uniformly in \( i \in [-N_{\delta, \varepsilon}, N_{\delta, \varepsilon}] \)) discussed above, result (18), and the Mapping Theorem imply that for almost every \( V \)

\[
(Z_t^{(n)}, \widetilde{C}_t^{(n, \delta, \varepsilon)}) \xrightarrow{p} (Z'_t, S_t^{(\delta, \varepsilon)}),
\]

(19)
as \( n \to \infty \) for every \( t \), where \( S_t^{(\delta, \epsilon)} = \sum_{x \in T \cap I} (V_x - V_x^-) \phi'(t, x) \). Moreover, we have that
\[
\lim_{\delta \downarrow 0} \lim_{\epsilon \downarrow 0} S_t^{(\delta, \epsilon)} = \int_{-\infty}^{\infty} \phi'(t, x) dV_x, \quad (20)
\]
almost surely. The convergence of the finite-dimensional distributions of \((Z^{(n)}, \tilde{C}^{(n)})\) to those of \((Z, S)\) now follows from (19) and (20).

### Tightness

We now show that the sequence \((Z^{(n)}, \tilde{C}^{(n)})\) is tight on \( D([0, T], \mathbb{R}) \times D([0, T], \mathbb{R}^+) \) endowed with the \( J_1 \times M_1 \) product topology. It is enough to establish tightness of each coordinate. The first coordinate converges in distribution, so it is tight. We are thus left with showing that \((\tilde{C}^{(n)})\) is tight on \( D([0, T], \mathbb{R}^+) \) endowed with the \( M_1 \) topology (and consequently with the \( U \) topology—see Remark 4).

Using the fact that \( \tilde{C}^{(n)} \) is nondecreasing, one readily checks that the oscillation function \( w_{\text{osc}} \) used in the condition (ii) of the Theorem 12.12.3 of [18] equals 0. With this and using the fact that the process is nonnegative, we have that the tightness criteria of the latter theorem are given by

1. \( \lim_{c \to \infty} \limsup_{n \to \infty} P(\tilde{C}_T^{(n)} > c) = 0; \)
2. For each \( \xi > 0 \), \( \lim_{\epsilon \downarrow 0} \limsup_{n \to \infty} P(\max\{\bar{v}(C_n^*, 0, \epsilon), \bar{v}(C_n^*, T, \epsilon)\} > \xi) = 0, \)

where, for \( t \in [0, T] \), we have that
\[
\bar{v}(C_n^*, t, \epsilon) = \sup_{\max\{0, t-\epsilon\} \leq t_1 \leq t_2 \leq \min\{t+\epsilon, T\}} \{\tilde{C}_n^{(T)} - \tilde{C}_n^{(t_1)}\}, \quad (21)
\]

Taking \( \epsilon < T \), we have that, for \( t = 0 \) and \( t = T \), the quantity above reduces to
\[
\bar{v}(C_n^*, 0, \epsilon) = \tilde{C}_n^{(\epsilon)}, \quad (21)
\]
and
\[
\bar{v}(C_n^*, T, \epsilon) = \tilde{C}_n^{(T)} - \tilde{C}_n^{(T-\epsilon)}, \quad (22)
\]
respectively.

We now show that the rescaled clock process satisfies conditions (i) and (ii). From now on, we use the restriction \( \epsilon < T \). Condition (i) follows from the convergence of the finite-dimensional distributions, that is
\[
\lim_{c \to \infty} \limsup_{n \to \infty} P(\tilde{C}_T^{(n)} > c) = \lim_{c \to \infty} P(S_T > c) = 0.
\]

Using (21) and the convergence of the finite-dimensional distributions, we also get that for each \( \xi > 0 \) fixed
\[ \lim \limsup_{\varepsilon \downarrow 0, n \to \infty} \mathbb{P}(\tilde{v}(C^*_n, 0, \varepsilon) > \xi) = \lim_{\varepsilon \downarrow 0} \mathbb{P}(S_\varepsilon > \xi). \]

Since \( S_\varepsilon \) converges to 0 in probability as \( \varepsilon \downarrow 0 \), we find that
\[ \lim \limsup_{\varepsilon \downarrow 0, n \to \infty} \mathbb{P}(\tilde{v}(C^*_n, 0, \varepsilon) > \xi) = 0. \] (23)

Similarly as in the previous case, using (22), for each \( \xi > 0 \) fixed, we have that
\[ \lim \limsup_{\varepsilon \downarrow 0, n \to \infty} \mathbb{P}(\tilde{v}(C^*_n, T, \varepsilon) > \xi) = \lim_{\varepsilon \downarrow 0} \mathbb{P}(S_T - S_{T - \varepsilon} \overset{a.s.}{\longrightarrow} 0 \text{ as } \varepsilon \downarrow 0). \]

Now using the fact that \( S \) is almost surely continuous, we get that \( S_T - S_{T-\varepsilon} \overset{a.s.}{\longrightarrow} 0 \) as \( \varepsilon \downarrow 0 \). With this, we obtain that
\[ \lim \limsup_{\varepsilon \downarrow 0, n \to \infty} \mathbb{P}(\tilde{v}(C^*_n, T, \varepsilon) > \xi) = 0. \] (24)

Results (23) and (24) imply that the condition (ii) is satisfied, and hence, \( \overline{C}^{(n)} \) is tight. \( \square \)

**Proof of Theorem 1** Let us start by defining some subsets of \( D([0, \infty), \mathbb{R}) \) (definitions which hold analogously in the case of \( D([0, \infty), \mathbb{R}^+) \)).

Let \( D_\uparrow([0, \infty), \mathbb{R}) \) be the space of the nonnegative functions in \( D([0, \infty), \mathbb{R}) \) that are nondecreasing. We here denote the space of continuous functions which are strictly increasing by \( C_\uparrow([0, \infty), \mathbb{R}) \). Denote by \( D_u([0, \infty), \mathbb{R}) \) be the space of functions in \( D([0, \infty), \mathbb{R}) \) which are unbounded. Hence, we define \( D_{u, \uparrow}([0, \infty), \mathbb{R}) = D_u([0, \infty), \mathbb{R}) \cap D_\uparrow([0, \infty), \mathbb{R}) \).

Using Theorem 2 and the arguments in its proof, one may check that the finite-dimensional distributions of \( (Z^{(n)}, \overline{C}^{(n)-1}) \) weakly converge to those of \( (Z, S^{-1}) \) as \( n \to \infty \).

Further, we have that \( D_{u, \uparrow}([0, \infty), \mathbb{R}^+) \) endowed with the uniform topology is separable and complete, so by using the weak convergence of \( \overline{C}^{(n)-1}_i \) to \( S^{-1} \) (which follows from Theorem 2 and the Mapping Theorem) and the converse half of Prohorov’s Theorem, we get tightness of \( \overline{C}^{(n)-1} \). Therefore, \( (Z^{(n)}, \overline{C}^{(n)-1}) \Rightarrow (Z, S^{-1}) \) as \( n \to \infty \) on \( D([0, \infty), \mathbb{R}) \times D_{u, \uparrow}([0, \infty), \mathbb{R}^+) \) equipped with the \( J_1 \times U \) product topology.

Now we use Theorem 13.2.2 of [18], which states that the composition map from \( D([0, \infty), \mathbb{R}) \times D_{u, \uparrow}([0, \infty), \mathbb{R}^+) \) to \( D([0, \infty), \mathbb{R}) \) is continuous on \( D([0, \infty), \mathbb{R}) \times C_\uparrow([0, \infty), \mathbb{R}^+) \) equipped with the \( J_1 \) topology. Notice that the trajectories of \( (Z, S^{-1}) \) are in \( D([0, \infty), \mathbb{R}) \times C_\uparrow([0, \infty), \mathbb{R}^+) \) almost surely. The Mapping Theorem and the above results imply that \( X^{(n)} \Rightarrow Z_{S^{-1}} \) on \( D([0, \infty), \mathbb{R}) \) endowed with the \( J_1 \), yielding the result. \( \square \)

We close this section with a result about the self-similarity of the scaling limit. This will prove useful to arguing our aging result. We say that a process \( \{W(t) : t \geq 0\} \) is self-similar of order \( H > 0 \) if \( W(at) \overset{fdd}{=} a^H W(t) \), for all \( a > 0 \), where \( \overset{fdd}{=} \) means equality of all finite-dimensional distributions.

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Proposition 1 The process $\{Z_{S_t^{-1}} : t \geq 0\}$ is self-similar of order $\alpha \beta^2/(1+\alpha(\beta-1))$.

Proof We first claim that $\{S_t : t \geq 0\}$ is self-similar of order $1 - 1/\beta + 1/\alpha \beta$. Let $a > 0$. By using the self-similarity property of the process $Z_t$ (of order $1/\beta$) and the following representation for the local time of the process $Z_t$:

$$
\phi(t, x) = \lim_{\epsilon \downarrow 0} \frac{1}{2\epsilon} \int_0^t \mathbf{1}\{x - \epsilon < Z_s \leq x + \epsilon\} ds,
$$

which holds for $t > 0$ and $x \in \mathbb{R}$, one readily checks that

$$
\phi(at, x) = a^{1-1/\beta} \phi(t, a^{-1/\beta} x).
$$

With this and using the self-similarity (of order $1/\alpha$) of the process $V_t$, we readily justify our first claim by making a straightforward change of variable in the integral defining of $S$ (see (6)). Hence we have that

$$
S_{at}^{-1} f d d = a^{1/(1-1/\beta + 1/\alpha \beta)} S_t^{-1}.
$$

This and the self-similarity of $Z$ imply that $Z_{S_t^{-1}}$ is self-similar of order $\beta/(1 - 1/\beta + 1/\alpha \beta) = \alpha \beta^2/(1+\alpha(\beta-1))$. \hfill \Box

4 Integrated Aging

In the final section, we move our attention to the study of aging for $X$. We prove an integrated aging result, as explained at the introduction. Consider the following integrated aging function for $X$:

$$
\bar{R}(\lambda, \mu) = E[R(\lambda T, \mu T)],
$$

where $\mu, \lambda \geq 0$, $T$ is an absolutely continuous random variable supported on $(0, \infty)$ and independent of all other variables, and

$$
R(s, t) = P(X_t = X_{t+s}),
$$

for $s, t > 0$.

We now state and prove an integrated aging result for (25).

Theorem 3 Let $\mathcal{R} : \mathbb{R}^+ \to [0, 1]$ such that $\mathcal{R}(\theta) = P(Z_{S_1^{-1}} = Z_{S_{1+\theta}^{-1}})$. Then

$$
\lim_{\lambda, \mu \to \infty \atop \lambda/\mu \to \theta} \mathcal{R}(\lambda, \mu) = \mathcal{R}(\theta).
$$
Remark 7  In the appendix, we give an argument for the nontriviality of $\mathcal{R}(\cdot)$. (See Corollary 1.)

Remark 8  In [11], nonintegrated aging results were established for a number of aging functions of $\mathcal{X}$, including the one of Theorem 3, for the case where $\mathcal{X}$ is the simple symmetric random walk in dimension 1. Recall the discussion at the introduction on the limitations on extending this approach to the more general situation of the present paper.

The convergence in distribution stated in Theorem 1 is not sufficient to prove Theorem 3. Additional arguments will be stated and proven in two lemmas which will allow us to replace the process $\mathcal{X}_t$ by a process living in the deep traps. This will lead to the desired result.

For simplicity of notation, we define

$$Y_t^{(n)} = Z_t^{(n)}, \quad Y_t^{(n, \delta)} = Z_t^{(n)} - 1, \quad Y_t^{(\delta)} = Z_{S_t^{(\delta)}} - 1, \quad t \geq 0,$$

where $S_t^{(\delta)}$ is the inverse of $S_t^{(\delta)} = \sum x \in T_\delta \phi(t, x)(V_x - V_{x-})$.

We now introduce an auxiliary process, denoted by $\bar{Y}_t^{(n, \delta)}$, which lives on $\delta$-traps. To define it, let $W_t^{(n, \delta)}$, $i = 0, 1, \ldots$, be the successive $\delta$-traps visited by $Y_t^{(n)}$, with the restriction that $W_t^{(n, \delta)} \neq W_{i+1}^{(n, \delta)}$, and let $\bar{S}_t^{(n, \delta)}$, $i = 0, 1, \ldots$, denote the successive hitting times of those traps by $Y_t^{(n)}$, respectively (so that $Y_t^{(n)} = W_t^{(n, \delta)}$; $Y_t^{(n)} \neq W_t^{(n, \delta)}$). Let us make

$$\bar{Y}_t^{(n, \delta)} = W_t^{(n, \delta)}, \quad \text{if} \quad \bar{S}_t^{(n, \delta)} \leq t < \bar{S}_{i+1}^{(n, \delta)}, \quad i = 0, 1, \ldots.$$

Notice that $Y_t^{(n)} = \bar{Y}_t^{(n, \delta)}$ whenever $Y_t^{(n)}$ is visiting a $\delta$-trap, and different otherwise, and $\bar{Y}_t^{(n, \delta)}$ of course lives on $\delta$-traps.

**Lemma 1**  For any $T > 0$, we have that

$$\lim_{\delta \downarrow 0} \limsup_{n \to \infty} \int_0^T \mathbb{P}(Y_t^{(n)} \neq \bar{Y}_t^{(n, \delta)}) \, dt = 0,$$

for almost every $V$.

Remark 9  A strengthening of this lemma to a nonintegrated version would lead to a nonintegrated version of Theorem 3. As pointed out at the introduction and in comparison with the approach of [11] to obtaining nonintegrated aging results, we lack here analogues of the analytical results for quasistable processes which exist for quasidiffusions.
Proof of Lemma 1  Since \( \phi(t,x) \xrightarrow{a.s.} \infty \) as \( t \to \infty \), for any \( x \in \mathbb{R} \)—see for instance [5]—, we have that \( S_t^{(\delta)} \xrightarrow{a.s.} \infty \) as \( t \to \infty \), for all fixed \( \delta > 0 \). This and the weak convergence \( \lim_{\delta \downarrow 0} \lim_{n \to \infty} \tilde{C}_t^{(n,\delta)} = S_t \), for each \( t \) (which can be easily obtained from the elements of the proof of Theorem 2)
we get that, given \( T, \eta, \delta > 0 \) there exist \( n_0, \Delta > 0 \) such that

\[
P(\tilde{C}_\Delta^{(n)} \leq T) \leq P(\tilde{C}_\Delta^{(n,\delta)} \leq T) \leq \eta, \quad \forall n \geq n_0.
\]

(26)

We then fix \( T, \eta, \delta > 0 \), take \( n_0, \Delta > 0 \) such that the above inequalities hold and obtain that

\[
\int_0^T \mathbb{P}(Y_t^{(n)} \neq \bar{Y}_t^{(n,\delta)})dt = \mathbb{E} \left( \int_0^T 1\{Y_t^{(n)} \neq \bar{Y}_t^{(n,\delta)}\}dt \right) \\
\leq \mathbb{E} \left( \int_0^T 1\{Y_t^{(n)} \neq \bar{Y}_t^{(n,\delta)}, \tilde{C}_\Delta^{(n,\delta)} \geq T\}dt \right) + \eta T \\
\leq \mathbb{E} \left( \sum_{x \in d_n^{-1} \mathbb{Z} \cap \{T_t^{(n)}\}} g_n(V_{x+d_n^{-1}} - V_x) \phi_n(\Delta, x) \bar{T}_{x+d_n}(n\Delta) \right) + \eta T,
\]

The above inequality, (13) and the arbitrariness of \( \eta \) yield the result. \( \square \)

Lemma 2  For almost every realization of \( V \), we have that

\[
\lim_{\delta \downarrow 0} \lim_{n \to \infty} J_1((\bar{Y}_t^{(n,\delta)}), (Y_t^{(n,\delta)})) = 0,
\]

in probability, where \( J_1 \) is the Skorohod metric.

Proof  The processes \( \bar{Y}^{(n,\delta)} = \{\bar{Y}_t^{(n,\delta)} : t \geq 0\} \) and \( Y^{(n,\delta)} = \{Y_t^{(n,\delta)} : t \geq 0\} \) successively visit the same traps but with different sojourn times. So, it is enough to show that the maximum of the differences between the successive sojourn times of the traps visited by both processes within \([0, T]\), respectively, goes to 0 in probability as \( n \to \infty \) and \( \delta \downarrow 0 \). Let \( S_t^{(n,\delta)} \) be the successive jump times of \( Y^{(n,\delta)} \). We then have that

\[
Y_t^{(n,\delta)} = W_t^{(n,\delta)}, \quad \text{if} \quad S_t^{(n,\delta)} \leq t < S_{t+1}^{(n,\delta)},
\]

for \( i \in \mathbb{N} \cup \{0\} \).

Let \( \bar{S}_t^{(n,\delta)} = S_t^{(n,\delta)} - S_{t-1}^{(n,\delta)} \), \( \bar{S}_t^{(n,\delta)} = S_t^{(n,\delta)} - S_{t-1}^{(n,\delta)} \), \( i \geq 1 \), denote the successive sojourn times of \( Y^{(n,\delta)} \) and \( \bar{Y}^{(n,\delta)} \), respectively. We first notice that \( \bar{S}_t^{(n,\delta)} \geq S_t^{(n,\delta)} \) for every \( i \).
Given $T, \eta, \delta > 0$, we take $n_0, \Delta$ satisfying (26) and we may conclude that outside an event of probability at most $\eta$, we have that
\[
\max(\overline{S}_{t}^{(n, \delta)} - \underline{S}_{t}^{(n, \delta)}) \leq \sum_{x \in d_{n}^{-1}Z \cap (T^{(n)}_{\delta})^{c}} g_n(V_{x+d_{n}^{-1}} - V_{x})\phi_n(\Delta, x)\overline{T}, \Delta_{n}(n\Delta),
\]
where the max is taken over all sojourn times of $\delta$-traps visited by $Y_{t}^{(n, \delta)}$ during $[0, T]$.

As seen before, the right side of (27) goes to 0 in probability by first taking $n \to \infty$ and after $\delta \downarrow 0$. This result and the arbitrariness of $\eta$ yield the result. $\square$

Proof of Theorem 3 For simplicity, let us take $\lambda = \theta \mu$ and replace $\mu$ by $a_{n}$ as defined in (5). We have that for every $n \geq 1$ and $T > 0$
\[
\mathcal{R}(\theta a_{n}, a_{n}) = \int_{0}^{\infty} f(t) P(Y_{t}^{(n)} = Y_{t(1+\theta)}) \, dt
\]
\[
= \int_{0}^{T} f(t) P(Y_{t}^{(n)} = Y_{t(1+\theta)}) \, dt + g_n(T) \tag{28}
\]
where $f$ is the probability density function of $\mathbb{T}$ and
\[
g_n(T) = \int_{T}^{\infty} f(t) P(Y_{t}^{(n)} = Y_{t(1+\theta)}) \, dt \leq P(\mathbb{T} > T) \tag{29}
\]
for every $n \geq 1$.

From Lemma 1, it follows that
\[
\lim_{n \to \infty} \int_{0}^{T} f(t) P(Y_{t}^{(n)} = Y_{t(1+\theta)}) \, dt
\]
\[
= \lim_{\delta \downarrow 0} \lim_{n \to \infty} \int_{0}^{T} f(t) P(\mathcal{Y}_{t}^{(n, \delta)} = \mathcal{Y}_{t(1+\theta)}) \, dt.
\]

By using the above result, Lemma 2 and the following weak convergence
\[
\lim_{\delta \downarrow 0} \lim_{n \to \infty} Y_{t}^{(n, \delta)} = Z_{S_{t}^{-1}} \quad \text{(under the } J_1 \text{ metric), we get}
\]
\[
\lim_{n \to \infty} \int_{0}^{T} f(t) P(Y_{t}^{(n)} = Y_{t(1+\theta)}) \, dt = \lim_{n \to \infty} E\left(\int_{0}^{T} f(t) 1\{Y_{t}^{(n)} = Y_{t(1+\theta)}\} \, dt\right)
\]
\[
= E\left(\int_{0}^{T} f(t) 1\{Z_{S_{t}^{-1}} = Z_{S_{t(1+\theta)}^{-1}}\} \, dt\right) = \int_{0}^{T} f(t) P(Z_{S_{t}^{-1}} = Z_{S_{t(1+\theta)}^{-1}}) \, dt
\]
\[
= P(Z_{S_{1}^{-1}} = Z_{S_{1+\theta}^{-1}}) P(\mathbb{T} \leq T), \tag{30}
\]
where the third equality follows from Proposition 1, which in particular implies that $P(Z_{S_t^{-1}} = Z_{S_t^{-1}(1+\theta)})$ does not depend on $t$. The result follows from (28), (29), (30), and the arbitrariness of $T$.

\[ \square \]

Appendix: Results on quasistable processes

We here present some results on quasistable processes, including one that states that the function $R(\cdot)$ is nontrivial. Let us introduce a notation that will be used below. Let $B$ and $C$ be Borel sets on $\mathbb{R}$. We define $B + C = \{ x + y : x \in B, y \in C \}$. Let $Y_t = Z_{S_t^{-1}}, t \geq 0$. For a fixed realization of $V$, define $T = \bigcup_{\delta > 0} T_\delta$, the set of traps.

**Proposition 2** For all $t > 0$, we have that $P(Y_t \in T) = 1$ for almost every $V$.

**Proof** We first show that for any $t > 0$ we have

\[ \int_0^t P(Y_s \in T) \, ds = t. \quad (31) \]

For $\delta > 0$ let $Y^{(\delta)}$ be as in Sect. 4 above (see paragraph right below Remark 8). Arguing similarly as in the proof of Lemma 1, we have that, for fixed $t, \eta, \delta > 0$, there exists $\Delta > 0$ such that

\[ \int_0^t P(Y_s \neq Y_s^{(\delta)}) \, ds \leq \mathbb{E}\left( \sum_{x \in T_{\delta}} (V_x - V_{x-})\phi(\Delta, x) \right) + \eta t. \]

Using the fact that $\sum_{x \in T_{\delta}} (V_x - V_{x-})\phi(\Delta, x) \xrightarrow{\mathbb{P}} 0$ as $\delta \downarrow 0$ and the arbitrariness of $\eta$, we get that

\[ \lim_{\delta \downarrow 0} \int_0^t P(Y_s \neq Y_s^{(\delta)}) \, ds = 0. \]

Hence, it follows that

\[ \int_0^t P(Y_s \in T) \, ds = \lim_{\delta \downarrow 0} \int_0^t P(Y_s \in T_\delta, Y_s = Y_s^{(\delta)}) \, ds \]

\[ = \lim_{\delta \downarrow 0} \int_0^t P(Y_s = Y_s^{(\delta)}) \, ds = t, \]

and (31) is established.
For arbitrary $d > 0$, define the set $B_d = \{s \in (0, d) : \mathbb{P}(Y_s \in T) = 1\}$. From (31), we have that $\mathcal{L}(B_d) = d$ (we recall that $\mathcal{L}$ is the Lebesgue measure).

It can be seen using the Markov property that if $t$ and $s$ belong to $B_d$, then $B_d + B_d \subset B_{2d}$. Since the sum of sets of positive Lebesgue measure contains an interval (see Theorem 4.1 from [14]), we get that $B_{2d}$ contains a subinterval for all $d > 0$. Let $B = \bigcup_{d > 0} B_d$. We have that

1) From the Markov property, $C + D \subset B$ for any subsets $C$ and $D$ from $B$;
2) From the above result, $B$ contains a subinterval of $[0, d]$ for all $d > 0$.

Let $I_1 \equiv [d_1, d_2]$ be a subinterval of $[0, d]$ obtained from 2), with $d_1 < d_2$. From 1), we have that $I_1 + I_1 = [2d_1, 2d_2] \subset B$. So, it follows that $I_2 \equiv I_1 \cup (I_1 + I_1) = [d_1, 2d_2] \subset B$. Inductively, we find that $I_{n+1} \equiv I_1 \cup (I_n + I_1) = [d_1, nd_2] \subset B$ for $n \in \mathbb{N}$. It follows that $[d_1, \infty) \subset B$. Since $d$ is arbitrary, we conclude that $B = (0, \infty)$.

$\square$

**Lemma 3** For all $t > 0$ and $x \in T$, we have that $\mathbb{P}_x(Y_t = x) > 0$ for almost every $V$, where the subscript $x$ in $\mathbb{P}_x(\cdot)$ means that $Y_0 = x$.

**Proof** Let $x \in T$ and $d > 0$. We have that for all $d > 0$

$$
\int_0^d \mathbb{P}_x(Y_s = x)ds = \mathbb{E}_x\left(\int_0^d 1\{Y_s = x\}ds\right)
= \mathbb{E}_x[\phi(x, S_d^{-1})](V_x - V_{x^-}) > 0,
$$

(32)

since $\mathbb{P}_x$-almost surely $S_t > 0$ and $\phi(x, t) > 0$ for all $t > 0$, where $\phi(\cdot)$ is the local time of $Z_t$ (see first result of [17], on p. 632).

Let $F_d = \{s \in (0, d) : \mathbb{P}_x(Y_s = x) > 0\}$. From (32), we get that $\mathcal{L}(F_d) > 0$ for all $d > 0$. Let $F = \bigcup_{d > 0} F_d$. Arguing similarly as in the proof of Proposition 2, we find that $F = (0, \infty)$, so proving the desired result.

$\square$

**Corollary 1** The function $\mathcal{R}(\theta) = P(Z_{S_1^{-1}} = Z_{S_1^{-1} + \theta} = Y_1 = Y_{1+\theta})$ is nontrivial.

**Proof** From Proposition 2 and Lemma 3, we immediately obtain that $\mathbb{P}(Y_1 = Y_{1+\theta}) = \sum_{x \in \mathcal{T}} \mathbb{P}(Y_1 = x) \mathbb{P}_x(Y_{\theta} = x) > 0$ for almost every $V$ and for all $\theta > 0$. Therefore, $\mathcal{R}(\theta) = E[\mathbb{P}(Y_1 = Y_{1+\theta})] > 0$, where the expectation is taken with respect to $V$. That $\mathcal{R}(\theta)$ is not constant may be verified by showing that for almost every $V$ we have that $\mathbb{P}_x(Y_t = y) \rightarrow 0$ as $t \rightarrow \infty$ for every $x, y \in \mathcal{T}$. We leave the details as an exercise.

$\square$

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