On the maximum number of edges in plane graph with fixed exterior face degree

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Abstract

A well known Euler’s formula consequence’s corollary in graph theory states that: For a connected simple planar graph with \( n \) vertices and \( m \) edges, and girth \( g \), we have \( m \leq \frac{g}{g-2}(n-2) \). We show that a connected simple plane graph with \( n \) vertices and girth \( g \), and exterior face of degree \( h \) has at most \( \frac{g}{g-2}(n-2) - \frac{1}{g-2}(h-g) \) edges. A convex hull \( g \)-angulation is a connected plane graph in which the exterior face is a simple \( h \)-cycle and all inner faces are \( g \)-cycles. For a given set \( S \) of \( n \) point in the plane having \( h \) points in the boundary of its convex hull, we present the necessary and sufficient condition to obtain a convex hull \( g \)-angulation on \( S \). We also determine the number of edges and inner faces in the convex hull \( g \)-angulation.

1 Introduction

A topological graph \( G \) is a graph drawn in the plane, that is, its vertex set, \( V(G) \), is a set of distinct points, and its edge set, \( E(G) \), is a set of Jordan arcs. A topological graph is simple if there is no loop nor parallel edges. The girth of a graph \( G \) is the length of a shortest cycle (if any) in \( G \). A graph is planar if it can be embedded in the plane; a plane graph has already been embedded in the plane. We will refer to the regions defined by a plane graph as its faces, the unbounded region being called the exterior face. The number of edges bordering a particular face is called the degree of the face. To talk about the number of edges in plane graph we deal with girth \( g \geq 3 \) and simple graphs (otherwise, \( g \leq 2 \) and self-loop or multiple edges are available, and then there is no limit on the number of edges).
The Euler formula for polyhedra is one of the classical results of mathematics. Euler Polyhedron Formula states: For any spherical polyhedron with $n$ vertices, $m$ edges, and $f$ faces, $n - m + f = 2$ (Harary’s 11.1 [2]).

Euler’s formula has many consequence’s corollaries. In particular, the following result is well known in graph theory: For a connected simple planar graph $G$ with $n$ vertices and $m$ edges, and girth $g$, we have $m \leq \frac{g}{g-2}(n - 2)$ (Jungnickel’s 1.5.3 [3]). For a connected simple planar graph with $n$ vertices and $m$ edges, and exterior face of degree $h \geq g$, we determine the maximum number of edges to be $\frac{g}{g-2}(n - 2) - \frac{1}{g-2}(h - g)$ (Theorem 1). That is, if $G$ contains $m$ edges where $\frac{g}{g-2}(n - 2) - \frac{1}{g-2}(h - g) < m \leq \frac{g}{g-2}(n - 2)$ for a given fixed $h > g$, then $G$ may be planar graph but it can not be embedded in the plane with an exterior face of degree $h$.

Another Euler’s formula consequence’s corollary states: If $G$ is a connected simple plane graph with $n$ vertices and $m$ edges in which every face is a $g$-cycle, then $m = \frac{g}{g-2}(n - 2)$ (Harary’s 11.1(a) [2]). Determination of the number of edges of a connected simple plane graph in which every inner face is a $g$-cycle while its exterior face is a simple cycle of degree $h \geq g$ is given by Theorem 2.

These results are then applied to present new proofs for well known results in graph theory.

Throughout this paper, a connected plane graph with $n$ vertices and girth $g$, and exterior face of degree $h$ will be denoted by $G_{g,h}$. A convex hull $g$-angulation, denoted by $H_{g,h}$, is a connected plane graph with $n$ vertices in which the exterior face is a simple cycle of degree $h$ and all inner faces are $g$-cycles.

### 2 Main Result

The number of edges in any simple plane graph depends on the number of vertices that are on its exterior face. This is made precise in next result.

**Theorem 1** A connected simple plane graph with $n$ vertices and girth $g$, and exterior face of degree $h \geq g$ has $m \leq \frac{g}{g-2}(n - 2) - \frac{1}{g-2}(h - g)$ edges.

**Proof:** Let $G_{g,h}$ be a connected simple plane graph with $n$ vertices and girth $g$, and exterior face of degree $h \geq g$. Then $h$ is the number of boundary edges of the exterior face. Let $m$ and $f$ be the number of edges and faces of $G_{g,h}$, respectively. Farther let $f_{g'}$ be the number of inner faces of degree $g'$. Clearly $1 + \sum_{g'} f_{g'} = f$. 2
Counting the bounding edges of all the faces we count every edge exactly twice (since each edge belongs to exactly two faces) and this follows \(2m = h + \sum_{g'} f_{g'}\). Now since \(g = \min\{g'\}\), that yields \(2m \geq h + g \sum_{g'} f_{g'} = h + g(f - 1)\). Since \(G_{g,h}\) is connected, we can apply Euler’s formula, \(n - m + f = 2\). Substituting for \(f - 1\) in the inequality yields 
\[-2m \leq -h - g(1 - n + m) = -h - g + ng - mg\]
and then \(mg - 2m \leq g(n - 2) - h + g\). Thus, \(m \leq \frac{g}{g-2}(n - 2) - \frac{h-g}{g-2}\).

**Definition 1**

(i) A convex hull \(g\)-angulation, \(H_{g,h}\), is a connected simple plane graph with \(n\) vertices in which the exterior face is a simple \(h\)-cycle and all inner faces are \(g\)-cycles.

(ii) A convex \(g\)-angulation, \(H_{g,n}\), is a connected simple plane graph with \(n\) vertices in which the exterior face is a simple \(n\)-cycle and all inner faces are \(g\)-cycles.

(iii) A \(g\)-angulation, \(H_{g,g}\), is a connected simple plane graph with \(n\) vertices in which all faces are \(g\)-cycles.

**Theorem 2** Let \(S\) be a set of \(n\) points in the plane having \(h\) vertices in the convex hull and let \(g \geq 3\). A simple plane graph on \(S\) with girth \(g\) and exterior face of degree \(h \geq g\), is a connected convex hull \(g\)-angulation \(H_{g,h}\) if and only if \(2n - h - g\) is divisible by \(g - 2\) and \(m = \frac{g}{g-2}(n - 2) - \frac{1}{g-2}(h - g)\). Moreover, \(H_{g,h}\) has exactly \(m = n + t\) edges and \(t + 1\) inner faces where \(t = \frac{2n - h - g}{g-2}\).

**Proof:** Let \(S\) be a set of \(n\) points in the plane having \(h\) vertices in the boundary of its convex hull and let \(g \geq 3\).

Assume that \(G_{g,h}\) be a connected simple plane graph on \(S\) with \(m = \frac{g}{g-2}(n - 2) - \frac{1}{g-2}(h - g)\) edges and girth \(g\), and exterior face of degree \(h \geq g\). Assume further that \(2n - h - g\) is divisible by \(g - 2\).

Suppose on the contrary that there is an inner face of degree \(g' > g\). Assume without loss of generality that \(g' = g + 1\). By repeating the same argument of previous proof, we see that \(2m = g(f - 2) + h + g + 1\). Since \(G_{g,h}\) is connected, we can apply Euler’s formula, \(n - m + f = 2\). Substituting for \(f - 2\) yields 
\(2m = g(m - n) + h + g + 1\), and then \(mg - 2m = mg(n - 2) - h + g - 1\) which follows \(m = \frac{g}{g-2}(n - 2) - \frac{1}{g-2}(h - g) - \frac{1}{g-2}\), a contradiction. Hence, \(G_{g,h}\) is a \(g\)-angulation.

Assume that, \(H_{g,h}\) is a convex hull \(g\)-angulation. Since each inner face of \(H_{g,h}\) is of degree \(g\), then by repeating the same argument of previous proof, we see
that $2m = h + gf_g = h + g(f - 1)$. Substituting for $f - 1$ of Eulers Formula $(n - m + f = 2)$ yields $m = \frac{2}{g-2}(n - 2) - \frac{1}{g-2}(h - g)$. Now, since $m$ is a natural number then $g(n - 2) - (h - g) = 2n - h - g + n(g - 2)$ is divisible by $g - 2$. Hence, $2n - h - g$ is divisible by $g - 2$. Let $t = \frac{2n-h-g}{g-2}$. Thus, $m = n + t$.

Now, by (Euler Polyhedron Formula), the number of inner faces is $f - 1 = m - n + 1 = t + 1$.

As a direct consequence of Theorem 1 and Theorem 2, we have the following Corollaries.

**Corollary 1** A convex connected simple plane graph $G_{g,n}$ with $n$ vertices and girth $g$ has $m \leq \frac{(g-1)n-g}{g-2}$ edges.

**Proof:** By applying Theorem 1 with inserting $h = n$. □

**Corollary 2** A convex $g$-angulation $H_{g,n}$ with $n$ vertices where $n = g + t(g - 2)$ for some integer $t \geq 0$ has $m = \frac{(g-1)n-g}{g-2}$ edges. Moreover, $m = n + t$ and the number of inner faces is $t + 1$.

**Proof:** Assume that $H_{g,n}$ is a convex $g$-angulation with $n = g + t(g - 2)$ vertices for some integer $t \geq 0$. By applying Theorem 2 with substituting for $h = n$ yields $m = \frac{(g-1)n-g}{g-2}$ and $\frac{2n-h-g}{g-2} = \frac{n-g}{g-2} = t$. Hence, $m = n + t$ and $f - 1 = t + 1$. □

An important result in planarity that any outerplane graph (or convex connected simple plane graph) $G_{3,n}$ of girth 3 has $m \leq 2n - 3$ edges and the equality is achieved when the graph is a maximal outerplane (or a convex triangulation) $H_{3,n}$ can be proved by applying Corollary 1 and Corollary 2 with substituting for $g = 3$.

For following well known results in graph theory, we present new proofs depending on Theorem 1 and Theorem 2.

**Theorem 3** (9.1 [7]) Any convex hull triangulation, of $n$ vertices having exterior face of degree $h$, has $3n - 3 - h$ edges and $2n - 2 - h$ inner triangles.

**Proof:** By applying Theorem 2 with substituting for $g = 3$ yields $t = 2n - 3 - h$. Hence, $m = n + t = 3n - 3 - h$ and the number of inner faces is $t + 1 = 2n - 2 - h$. □
Theorem 4 (Jungnickel’s 1.5.3 [3]) A connected simple plane graph $G_{g,g}$ with $n$ vertices and girth $g$, and exterior face of degree $g$ has $m \leq \frac{g}{g-2}(n-2)$ edges.

Proof: By applying Theorem 1 with substituting for $h = g$. □

Corollary 3 (Harary’s 11.1(a) [2]) A $g$-angulation $G_{g,g}$ with $n$ vertices, where $n = g + t'(g-2)$ for some integer $t' \geq 0$, has $m = \frac{g}{g-2}(n-2)$ edges. Moreover, $m = n + 2t'$ and the number of inner faces is $2t' + 1$.

Proof: Assume that $G_{g,g}$ is a $g$-angulation with $n = g + t'(g-2)$ vertices for some integer $t' \geq 0$. By applying Theorem 2 with substituting for $h = g$ yields $m = \frac{g}{g-2}(n-2)$ and $t = \frac{2n-h-g}{g-2} = 2\frac{n-g}{g-2} = 2t'$. Hence, $m = n + 2t'$ and the number of inner faces is $f - 1 = 2t' + 1$. □

References

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