NONLINEAR TRACES

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Abstract. We combine the theory of traces in homotopical algebra with sheaf theory in derived algebraic geometry to deduce general fixed point and character formulas. The formalism of dimension (or Hochschild homology) of a dualizable object in the context of higher algebra provides a unifying framework for classical notions such as Euler characteristics, Chern characters, and characters of group representations. Moreover, the simple functoriality properties of dimensions clarify celebrated identities and extend them to new contexts.

We observe that it is advantageous to calculate dimensions, traces and their functoriality directly in the nonlinear geometric setting of correspondence categories, where they are directly identified with (derived versions of) loop spaces, fixed point loci and loop maps, respectively. This results in universal nonlinear versions of Grothendieck-Riemann-Roch theorems, Atiyah-Bott-Lefschetz trace formulas, and Frobenius-Weyl character formulas. On the one hand, we can then linearize by applying sheaf theories, such as the theories of coherent sheaves and $\mathcal{D}$-modules, developed by Gaitsgory and Rozenblyum, as functors out of correspondence categories (in the spirit of topological field theory). This recovers the familiar classical identities, in families and without any smoothness or transversality assumptions. On the other hand, the formalism also applies to higher categorical settings not captured within a linear framework, such as characters of group actions on categories.

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1. Introduction

This paper is devoted to traces and characters in homotopical algebra and their application to algebraic geometry and representation theory. We observe that many geometric fixed point and trace formulas can be expressed as linearizations of fundamental nonlinear identities, describing dimensions and traces directly in the setting of correspondence categories of varieties or stacks. This gives a simple uniform perspective on (and useful generalizations of) geometric character and fixed point formulas of Grothendieck-Riemann-Roch and Atiyah-Bott-Lefschetz type. In addition, one can also specialize the universal geometric formulas to higher categorical settings not captured within a linear framework, such as characters of group actions on categories.

We present a more detailed introduction below, following the structure of the paper: first, the abstract functoriality of traces in higher category theory; second, their calculation in correspondence categories in derived algebraic geometry; and third, their specialization via sheaf theories, as developed by Gaitsgory and Rozenblyum. We emphasize the formal nature and appealing simplicity of the constructions in any sufficiently derived setting. For example, in the second part, we work within derived algebraic geometry, but the statements and proofs should hold in any setting (for example, derived manifolds) with a suitable notion of fiber product to handle non-transversal intersections. In particular, the main objects appearing in trace formulas are the derived loop space (the self-intersection of the diagonal in its role as the nonlinear trace of the identity map) and more general derived fixed point loci. The importance of a derived setting also appears prominently in the third part, where the sheaf theories we apply must have good functorial properties with respect to fiber products, specifically base change. As a result, the theory of characters in Hochschild and cyclic homology is expressed directly by the geometry, resulting in simpler formulations. For example, the Todd genus in Grothendieck-Riemann-Roch and the denominators in the classical Atiyah-Bott formula arise naturally from derived calculations.

Before proceeding to the rest of the introduction, let us state the most direct generalizations of classical formulas which result from our constructions (while emphasizing that the main contribution of the paper is the simple geometric formalism underlying these formulas). For our general nonlinear results, we need not assume anything about what derived stacks and morphisms we work with. For applications, we will assume all derived stacks and all morphisms are quasi-compact with affine diagonal over some field \( k \) of characteristic zero. In particular, we could restrict to the traditional setting of quasi-compact, quasi-separated schemes. These assumptions are specifically designed to allow us to apply the powerful machinery of sheaf theories in derived algebraic geometry, specifically, ind-coherent sheaves and \( D \)-modules as developed by Gaitsgory and Rozenblyum, in par with Drinfeld, in [G1, DG, GR1, GR2].

Given a derived stack \( \pi_X : X \to \text{Spec} k \), we denote by \( \pi_{LX} : LX = \text{Map}(S^1, X) \to \text{Spec} k \) its derived loop space. In general, the derived loop space is a derived thickening of the inertia stack. For a map \( f : X \to Y \), we will denote by \( Lf : LX \to LY \) the induced map on loops.

**Example 1.1.** For many applications, the following two special cases are noteworthy.

When \( X \) is a smooth scheme, \( LX \simeq T_X[-1] \) is the total space of the shifted tangent complex by the HKR theorem. For \( f : X \to Y \) a map of schemes, \( LF : TX[-1] \to TY[-1] \) is (the shift of) the usual tangent map.

When \( Y = BG \) is a classifying stack, \( LY \simeq G/G \) is the adjoint quotient. For \( X \) a \( G \)-scheme, and \( f : X/G \to BG \) the corresponding classifying map, \( LF : LX(G) \to LBG \simeq G/G \) is the universal family of derived fixed point loci. More precisely, for any element \( g \in G \), the derived fixed point locus \( X^g \subset X \) is precisely the derived fiber \( X^g \simeq LX(G) \times_{G/G} \{ g \} \).
Let $\mathcal{S}(X)$ denote either the stable $\infty$-category of ind-coherent sheaves $\mathcal{Q}^l(X)$ (or equivalently, quasi-coherent sheaves when $X$ is smooth) or $\mathcal{D}$-modules $\mathcal{D}(X)$.

We will make essential use of the following theorem announced by Gaitsgory (building on versions by Gaitsgory and Rozenblyum, in part with Drinfeld, in \cite{G1, DG, GR1, GR2}), establishing the functorial properties of the constructions $\mathcal{Q}^l$ and $\mathcal{D}$:

**Theorem 1.2.** Fix a field $k$ of characteristic zero, and let $\mathcal{S}$ denote either $\mathcal{Q}^l$ or $\mathcal{D}$. Then $\mathcal{S}$ defines a symmetric monoidal functor from the $(\infty, 1)$-category of correspondences of quasi-compact derived stacks with affine diagonal over $k$ to $k$-linear differential graded categories.

Let $\omega_X = \pi_X^* \mathcal{O}_{\mathrm{Spec} k} \in \mathcal{S}(X)$ denote the appropriate dualizing sheaf. Thus for ind-coherent sheaves, $\omega_X \in \mathcal{Q}^l(X)$ is the algebraic dualizing sheaf, and for $\mathcal{D}$-modules, $\omega_X \in \mathcal{D}(X)$ is the Verdier dualizing sheaf. Let $\omega(X) = \pi_X^* \omega_X$ denote the corresponding complex of global volume forms. Thus for ind-coherent sheaves, $\omega(X) \in \mathcal{K}$ consists of algebraic volume forms, and for $\mathcal{D}$-modules, $\omega(X) \in \mathcal{K}$ consists of locally constant distributions (Borel-Moore chains). For a proper map $f : X \to Y$, adjunction provides an integration map $\int_f : \omega(X) \to \omega(Y)$.

**Example 1.3.** Let us continue with the special cases of Example 1.1 and focus in particular on algebraic distributions $\omega_{\mathcal{L}X} \in \mathcal{Q}^l(\mathcal{L}X)$ on the loop space.

When $X$ is a smooth scheme, $\mathcal{L}X \simeq \mathbb{T}_X[-1]$ is naturally Calabi-Yau, and its global volume forms are identified with differential forms $\omega(\mathbb{T}_X[-1]) \simeq \mathcal{O}(\mathbb{T}_X[-1]) \simeq \text{Sym}^*(\Omega_X[1])$. The canonical “volume form” on $\mathcal{L}X$ is given by the Todd genus (as explained by Markarian \cite{Ma}); the resulting integration of functions on $\mathcal{L}X$ differs from the integration of differential forms on $X$ by the Todd genus.

When $Y = BG$ is a classifying stack, $\mathcal{L}Y \simeq G/G$ is naturally Calabi-Yau, and its global volume forms are invariant functions $\omega(G/G) \simeq \mathcal{O}(G/G) \simeq \mathcal{O}(G)^G$. If $G$ is reductive with Cartan subgroup $T \subset G$ and Weyl group $W$, the naive invariants $\mathcal{O}(G)^G \simeq \mathcal{O}(T)^W$ are equivalent to the derived invariants, but in general there may be higher cohomology.

Finally, it is worth recalling that a compact object of $\mathcal{Q}^l(X)$ is a bounded coherent complex of $\mathcal{O}_X$-modules, and when $X$ is a scheme, a compact object of $\mathcal{D}(X)$ is a bounded coherent complex of $\mathcal{D}$-modules. Now we state the most direct generalizations of classical formulas which result from our constructions.

**Theorem 1.4.** Fix a field $k$ of characteristic zero and consider quasi-compact derived stacks with affine diagonal over $k$. Let $\mathcal{S}$ denote the sheaf theory of ind-coherent sheaves or $\mathcal{D}$-modules.

For a derived stack $X$, there is a canonical identification $HH_*(\mathcal{S}(X)) \simeq \omega(\mathcal{L}X)$ of the Hochschild homology of sheaves on $X$ with distributions on the loop space.

**Grothendieck-Riemann-Roch:** For a proper map $f : X \to Y$ and any compact object $M \in \mathcal{S}(X)$ with character $[M] \in HH_*(\mathcal{S}(X)) \simeq \omega(\mathcal{L}X)$, there is a canonical identification

$$[f_*M] \simeq \int_{\mathcal{L}f} [M] \in HH_*(\mathcal{S}(Y)) \simeq \omega(\mathcal{L}Y)$$

In other words, the character of a pushforward along a proper map is the integral of the character along the induced loop map.

**Atiyah-Bott-Lefschetz:** Let $G$ be an affine group, and $X$ a proper $G$-derived stack, so equivalently, a proper map $f : X/G \to BG$. Then for any compact object $M \in \mathcal{S}(X/G)$, and element $g \in G$, there is a canonical identification

$$[f_*M]_g \simeq \int_{\mathcal{L}f} [M]|_{X_g}$$
In other words, under the identification of invariant functions and volume forms on the group, the value of the character of an induced representation at a group element is given by the integral of the original character along the corresponding fixed point locus of the group element.

**Example 1.5.** Here is a reminder of two well-known applications of the Atiyah-Bott-Lefschetz formula in representation theory.

If $G$ is a finite group, and $X = G/K$ is a homogeneous set, and $M = k[G/K]$ the ring of functions, one recovers the Frobenius character formula for the induced representation $k[G/K]$.

If $G$ is a reductive group, $X = G/B$ is the flag variety, $M = L$ is an equivariant line bundle on $G/B$, and $g \in G$ runs over a maximal torus, one recovers the Weyl character formula for the induced representation $H^*(G/B, L)$.

**Remark 1.6.** The reader will note no explicit appearance of the Todd genus in the above formulas. It arises when one unwinds the integration map $\int_{L^f} : \omega(LX) \to \omega(LY)$, given by Grothendieck duality, in terms of functions. In particular, the familiar denominators in the Atiyah-Bott formula, are implicit in the integration measure on the fixed point locus.

For instance, as mentioned above, when $X$ is a smooth scheme, the loop space is the total space of the shifted tangent complex $LX \simeq T_X[-1]$, and global volume forms are canonically functions $\omega(LX) \simeq O(T_X[-1]) \simeq \text{Sym}^*(\Omega_X[1])$. Under this identification (as explained by Markarian [Ma]), the resulting integration of functions on $LX$ differs from the integration of differential forms on $X$ by the Todd genus.

**Remark 1.7.** The reader will note that the theorem treats ind-coherent sheaves and $\mathcal{D}$-modules on equal footing. This reflects the main contribution of this paper: we establish nonlinear versions of character formulas in the setting of derived stacks. Classical formulas and new higher categorical analogues then follow by applying suitable sheaf theories. To recover the classical formulas in their traditional formulations, one can appeal to standard functoriality patterns. They also hold in surprisingly great generality thanks to the powerful mechanism (Theorem 1.2) of sheaf theory developed by Gaitsgory and Rozenblyum, in part with Drinfeld [G1, DG, GR1, GR2].

We are particularly interested in the higher categorical variants where one considers sheaves of categories, in particular Frobenius-Weyl character formulas for group actions on categories. Since the requisite foundations are not yet fully developed, we postpone details of this to future works. Applications include an identification of the character of the category of $\mathcal{D}$-modules on the flag variety with the Grothendieck-Springer sheaf, and of the trace of a Hecke functor on the category of $\mathcal{D}$-modules on the moduli of bundles on a curve with the cohomology of a Hitchin space.

### 1.1. Inspirations and motivations.

This work has many inspirations. First among them is the categorical theory of strong duality, dimensions and traces introduced by Dold and Puppe in [DP] (see [M, PS] for more recent developments) with the express purpose of proving Lefschetz-type formulas. In [DP], dualizability of a space is achieved by linearization (passing to suspension spectra), while our approach is to pass to categories of correspondences (or spans) instead. We were also inspired by the preprint [Ma] and the subsequent work [Cal1, Cal2, Ram, Ram2, Shk]. There have been many recent papers [Pe, Lu, Po, CT] building on related ideas to prove Riemann-Roch and Lefschetz-type theorems in the noncommutative context of differential graded categories and Fourier-Mukai transforms; our work instead places these results in the context of the general formalism of traces in $\infty$-categories, and generalizes them to commutative but nonlinear settings.

The Grothendieck-Riemann-Roch type applications in this paper concern the character map taking coherent sheaves to classes in Hochschild homology (or in a more refined version, to
cyclic homology). This is significantly coarser than the well established theory of Lefschetz-Riemann-Roch theorems valued in Chow groups (see the seminal [Th], the more recent [Jo] and many references therein). Thus for schemes, the quantities compared are Dolbeault (or de Rham) cohomology classes rather than algebraic cycles. The character map factors through algebraic $G$-theory, the $K$-theory of coherent sheaves, though $K$-theoretic considerations play no role in this paper.

Our main influence is the work of Lurie, on the foundations of symmetric monoidal $\infty$-categories [L2], derived algebraic geometry [L5], and the cobordism hypothesis [L3]. Most strikingly, the cobordism hypothesis with singularities provides a powerful unifying tool for higher algebra (as well as a classification of extended topological field theories with all possible defects). It provides a universal refinement of graphical and pictorial calculi for category theory, encoding how higher categories (with appropriate finiteness assumptions) are representations of corresponding cobordism categories. In particular, formal properties of traces are simple instances of the cobordism hypothesis with singularities in dimension one. This can be viewed as a vast generalization of the classical theory Hochschild and cyclic homology and characters therein [Lo], (in particular the natural cyclic symmetry of Hochschild homology is generalized to a circle action on the dimensions of arbitrary dualizable objects). From this perspective, the current paper explores the cobordism hypothesis with singularities on marked intervals and cylinders in the setting of derived algebraic geometry.

The vital link between the abstract geometric formalism of this paper and applications is provided by the work of Gaitsgory and Rozenblyum, in part in collaborations with Drinfeld and Francis [G1, G2, FG, DG, GR1, GR2], as summarized in Theorem 1.2.

Our primary motivation is the development of foundations for “homotopical harmonic analysis” of group actions on categories, aimed at decomposing derived categories of sheaves (rather than classical function spaces) under the actions of natural operators. This undertaking follows the groundbreaking path of Beilinson-Drinfeld within the geometric Langlands program and is consonant with general themes in geometric representation theory. The pursuit of a geometric analogue of the Arthur-Selberg trace formula by Frenkel and Ngô [FN] has also been a source of inspiration and applications. The work of Toën and Vezzosi [TV] on higher Chern characters of sheaves of categories has also profoundly influenced our thinking.

Remark 1.8. A companion paper [BN13] presents an alternative approach to Atiyah-Bott-Lefschetz formulas (and in particular a conjecture of Frenkel-Ngô) as a special case of the “secondary trace formula” identifying trace invariants associated to two commuting endomorphisms of a sufficiently dualizable object. This is also applied to establish the symmetry of the 2-class functions on a group constructed as the 2-characters of categorical representations.

1.2. Traces in category theory. We highlight structures arising in the general theory of dualizable objects in symmetric monoidal higher categories (see also [DP, M, PS]). For legibility, we suppress all $\infty$-categorical notations and complications from the introduction. Since an adequate theory of symmetric monoidal ($\infty, 2$)-categories is not well documented, we will only use general 2-categorical language as a motivating and organizing principle. We make explicit below the precise (and minimal) amount of structure needed, which is readily available in the literature. (We make no claims to originality of this material, but in the absence of an obvious reference provide a detailed exposition for the benefit of the reader.)

The basic notion in the theory is that of dimension of a dualizable object of a symmetric monoidal category $\mathcal{A}$. By definition, for such an object $A$ there exists another $A'$ together with a coevaluation map $\eta_A$ and evaluation map $\epsilon_A$ satisfying standard identities. By definition, the
dimension of \( A \) is the endomorphism of the the unit \( 1_A \) given by the composition

\[
\begin{array}{cccccc}
1_A & \xrightarrow{\eta_A} & A \otimes A^\vee & \xrightarrow{\epsilon_A} & 1_A \\
\downarrow \dim(A) & \nearrow \Phi & & \nearrow \Phi \otimes \text{id}_{A^\vee} & \nearrow \epsilon_A \\
& & A \otimes A^\vee & \xrightarrow{\epsilon_A} & 1_A
\end{array}
\]

**Example 1.9.** For \( V \) a vector space, \( V^\vee = \text{Hom}_k(V,k) \) is the vector space of functionals, \( \epsilon_V : V \otimes V^\vee \to k \) is the usual evaluation of functionals, \( \eta_V : k \to \text{End}(V) \simeq V \otimes V^\vee \) is the identity map (which exists only for \( V \) finite-dimensional), and \( \dim(V) \) can be regarded as an element of the ground field (by evaluating it on the multiplicative unit).

**Remark 1.10 (Duality and naivété in \( \infty \)-categories.)** It is a useful technical observation that the notion of dualizability in the setting of \( \infty \)-categories is a “naive” one: it is a property of an object that can be checked in the underlying homotopy category. As a result, all of the categorical and 2-categorical calculations in this paper are similarly naive and explicit (and analogous to familiar unenriched categorical assertions), involving only small amounts of data that can be checked by hand (rather than requiring higher coherences). We restrict ourselves only to assertions of this naive and accessible nature, specifying all maps that are needed rather than constructing higher coherences (for which we view the cobordism hypothesis with singularities as the proper setting).

The notion of dimension is a special case of the *trace* of an endomorphism \( \Phi \) of a dualizable object \( A \). By definition, the trace of \( \Phi \) is the endomorphism of the unit \( 1_A \) given by the composition

\[
\begin{array}{cccccc}
1_A & \xrightarrow{\eta_A} & A \otimes A^\vee & \xrightarrow{\Phi \otimes \text{id}_{A^\vee}} & A \otimes A^\vee & \xrightarrow{\epsilon_A} & 1_A \\
& & \downarrow \text{Tr}(\Phi) & & \downarrow \Phi \otimes \text{id}_{A^\vee} & & \downarrow \text{Tr}(\Phi) \\
& & 1_A & & 1_A & & 1_A
\end{array}
\]

which recovers the dimension for \( \Phi = \text{id}_A \).

A key feature of dimensions and traces is their *cyclicity*, which at the coarsest level is expressed by a canonical equivalence

\[
m(\Phi, \Psi) : \text{Tr}(\Phi \circ \Psi) \xrightarrow{\sim} \text{Tr}(\Psi \circ \Phi),
\]

see Proposition 2.15. At a much deeper level, an important corollary of the cobordism hypothesis \([L3]\) is the existence of an \( S^1 \)-action on \( \dim(A) \) for any dualizable object \( A \) (and an analogous structure for general traces, see Remark 2.28).

**Remark 1.11 (Dimensions and traces are local).** It is useful for applications to note that the notion of dualizability and the definition of dimension are local in the category \( \mathcal{A} \). Namely, they only require knowledge of the objects \( 1_A, A, A^\vee, A \otimes A^\vee \), the morphisms \( \eta_A, \epsilon_A \), and standard tensor product and composition identities among them. Likewise, the notion of trace only requires the additional endomorphism \( \Phi \) along with a handful of additional identities.

1.2.1. **Functoriality of traces.** Now suppose the ambient symmetric monoidal category \( \mathcal{A} \) underlies a 2-category, so there is the possibility of noninvertible 2-morphisms. This allows for the notion of left and right adjoints to morphisms. Let us say a morphism \( A \to B \) is *continuous*, or *right dualizable*, if it has a right adjoint. (The terminology derives from the setting of presentable categories, where the adjoint functor theorem guarantees the existence of right adjoints for colimit preserving functors.)

Here are natural functoriality properties of dimensions and traces.

**Proposition 1.12.** Let \( A, B \) denote dualizable objects of \( \mathcal{A} \) and \( f_* : A \to B \) a continuous morphism with right adjoint \( f^! \).
There is a canonical map on dimensions
\[
\dim(A) \longrightarrow \text{Tr}(\text{Id}_A) \longrightarrow \text{Tr}(f^* f_*) \sim \text{Tr}(f_* f^*) \longrightarrow \text{Tr}(\text{Id}_B) \longrightarrow \dim(B)
\]
compatible with compositions of continuous morphisms.

Given endomorphisms \( \Phi \in \text{End}(A) \), \( \Psi \in \text{End}(B) \), and a commuting structure
\[
\alpha : f_* \circ \Phi \sim \Psi \circ f_*
\]
there is a canonical map on traces
\[
\text{Tr}(f_*, \alpha) : \text{Tr}(\Phi) \longrightarrow \text{Tr}(\Psi)
\]
compatible with compositions of continuous morphisms with commuting structures.

We refer to the compatibility with compositions stated in the proposition as \textit{abstract Grothendieck-Riemann-Roch}. To see its import more concretely, let us restrict the generality and focus on an object of \( A \) in the sense of a morphism \( V : 1_A \rightarrow A \).

**Corollary 1.13.** Let \( A, B \) denote dualizable objects of \( A \) and \( f : A \rightarrow B \) a continuous morphism. For \( V : 1_A \rightarrow A \) an object of \( A \), we obtain a map on dimensions
\[
\dim(V) : 1_A \rightarrow \dim(A)
\]
called the character of \( V \) and alternatively denoted by \([V]\). It satisfies abstract Grothendieck-Riemann-Roch in the sense that the following diagram commutes
\[
1_A \xrightarrow{[V]} \dim(A) \xrightarrow{\dim(f_*)} \dim(B)
\]

**Remark 1.14.** It follows from the cobordism hypothesis with singularities \textcolor{red}{[L3]} that the morphism \( \dim(f_*) \) is \( S^1 \)-equivariant, and hence the character \([V]\) is \( S^1 \)-invariant, though we will not elaborate on this structure here.

**Remark 1.15 (Functoriality of dimensions and traces is local).** As in Remark\textcolor{red}{[L11]} it is useful to note that the functoriality of dimension is local, depending only on a handful of objects, morphisms and identities, along with the additional adjunction data \((f_*, f^*)\). A similar observation applies to the functoriality of traces.

**Example 1.16.** Let \( \text{dgCat}_k \) denote the symmetric monoidal \( \infty \)-category of \( k \)-linear differential graded categories (or alternatively, stable \( k \)-linear \( \infty \)-categories). In this setting, any compactly generated category \( A \) is dualizable, and its dimension is the Hochschild chain complex \( \dim(A) = \text{HH}_*(A) \). The \( S^1 \)-action on \( \dim(A) \) corresponds to Connes’ cyclic structure on \( \text{HH}_*(A) \), so that in particular, the localized \( S^1 \)-invariants of \( \dim(A) \) form the periodic cyclic homology of \( A \).

More generally, the trace of an endofunctor \( \Phi : A \rightarrow A \) is the Hochschild homology \( \text{Tr}(\Phi) = \text{HH}_*(A, \Phi) \). For example, if \( A = R \)-mod for a dg algebra \( R \), then \( \Phi \) is represented by an \( R \)-bimodule \( M \), and we recover the Hochschild homology \( \text{HH}_*(R, M) \).

Any compact object \( M \in A \) defines a continuous functor
\[
1_{\text{dgCat}_k} = \text{dgVect}_k \xrightarrow{M} A
\]
whose character is a vector
\[
\dim(M) \in \text{HH}_*(A)
\]
in Hochschild homology (with refinement in cyclic homology). The abstract Grothendieck-Riemann-Roch theorem expresses the natural functoriality of characters in Hochschild homology (or their refinement in cyclic homology). In fact, the construction of characters factors through the canonical Dennis trace map

$$A_{cpt} \to K(A) \to HH_*(A)$$

from the space $A_{cpt}$ of compact objects of $A$.

1.3. **Traces in geometry.** To apply the preceding formalism to geometry, it is useful to organize spaces and maps within a suitable categorical framework. We then arrive at loop spaces and fixed point loci as nonlinear expressions of dimensions and traces. This simple observation provides the core of the paper. Throughout the discussion, we continue to suppress all $\infty$-categorical notations and complications. We also continue to use $2$-categorical language only as a motivating and organizing principle. The specific structures we need are modest and can be addressed without a general theory.

To begin, consider the general setup of the symmetric monoidal category $Corr$ of correspondences, where the objects $X \in Corr$ are spaces, the morphisms $Corr(X, Y)$ are correspondences

$$Z \xrightarrow{} X \xrightarrow{} Y$$

the composition of morphisms $Z \in Corr(X, Y)$ and $W \in Corr(Y, U)$ is the derived fiber product

$$Z \times_Y W \xrightarrow{} Z \xrightarrow{} X \xrightarrow{} Y \xrightarrow{} U$$

and the monoidal structure is given by Cartesian product. For the purpose of calculating dimensions and traces, we need not require any further properties of the spaces of $Corr$, since we need only the modest local data discussed in Remarks 1.11 and 1.15. We could very generally proceed in the context of any $\infty$-topos. (See [L3] and [FHLT], where the higher categories $Fam_n$ of iterated correspondences of manifolds are constructed and applied.)

**Remark 1.17 (Correspondences are bimodules).** It is useful to view the correspondence category $Corr$ within the framework of coalgebras in symmetric monoidal categories. The diagonal map $X \to X \times X$ makes any space into a cocommutative coalgebra object with respect to the Cartesian product monoidal structure (or commutative coalgebra in the opposite category). Moreover, a map $Z \to X$ is equivalent to an $X$-comodule structure on $Z$. Thus correspondences from $X$ to $Y$ may be interpreted as $X-Y$-bicomodules, with composition of correspondences given by tensor product of bicomodules.

Furthermore, it is natural to enhance $Corr$ to a 2-category by allowing non-invertible maps between correspondences. This can be viewed as a special case of the Morita category of coalgebras in a symmetric monoidal category (see for example [L3]). The 2-category $Corr$ of spaces, correspondences, and maps of correspondences is the Morita category on spaces regarded as coalgebra objects. (In particular, the cocommutativity of the coalgebra objects implies they are canonically self-dual, and the transpose of a correspondence is the same correspondence
read backwards.) If we further keep track of the $E_n$-coalgebra structure of spaces and consider the corresponding Morita $(n+1)$-category, we recover the $(n+1)$-category of iterated correspondences of correspondences (see for example the category $\text{Fam}_n$ of \text{L3} and \text{PHLT} in the topological setting).

With applications in mind, we will specialize to the correspondence category $\text{Corr}_k$ of derived stacks over a commutative ground ring $k$. It would also be interesting to work with smooth manifolds instead, for example through the theory of $\mathcal{C}^\infty$-stacks \text{J} (see Remark 1.24).

It is natural to enhance $\text{Corr}_k$ to a 2-category by allowing non-invertible maps between correspondences. Our constructions naturally fit into the 2-category $\text{Corr}_k$ with non-invertible 2-morphisms given by maps of correspondences: for objects $X,Y \in \text{Corr}_k$, the morphisms $\text{Corr}_k(X,Y)$ form the category of derived stacks over $X \times Y$.

For the purposes of later applications, it will be convenient to restrict to perfect stacks in the sense of \text{BFN}, and to allow only proper maps between correspondences as 2-morphisms. These restrictions are imposed by wanting to apply sheaf theories (such as coherent sheaves or $\mathcal{D}$-modules) to our stacks, and are independent of the general categorical formalism.

1.3.1. Geometric dimensions and loop spaces. A crucial feature of the category $\text{Corr}_k$ is that any object $X \in \text{Corr}_k$ is dualizable (in fact, canonically self-dual), thanks to the diagonal correspondence.\footnote{Likewise, if we wish to make a space $n$-dualizable for any $n$ we may simply consider it as an object of a higher correspondence category as in Remark 1.17 since $E_n$-(co)algebras are $n+1$-dualizable objects of the corresponding Morita category. In other words, a space $X$ defines a topological field theory of any dimension valued in the appropriate correspondence category.}

We have the following calculations of dimensions and their functoriality. Note that the point $pt = \text{Spec } k$ is the unit of $\text{Corr}_k$. We keep track of properness of maps of correspondences for the later application of sheaf theory.

**Proposition 1.18.** Let $\text{Corr}_k$ be the category of derived stacks and correspondences, and $\text{Corr}_k$ the 2-category of derived stacks, correspondences, and (proper) maps of correspondences.

1. Any derived stack $X$ is dualizable as an object of $\text{Corr}_k$, and its dimension $\dim(X)$ is identified with the loop space

$$\mathcal{L}X = X^{S^1} \simeq X \times_{X \times X} X$$

regarded as a self-correspondence of $pt = \text{Spec } k$.

2. A map $f : X \to Y$ regarded as a correspondence from $X$ to $Y$ is continuous in $\text{Corr}_k$ (if and only if $f$ is proper). Given a (proper) map $f : X \to Y$, its induced map

$$\dim(f) : \dim(X) \longrightarrow \dim(Y)$$

is identified with the loop map

$$\mathcal{L}f : \mathcal{L}X \longrightarrow \mathcal{L}Y$$

**Remark 1.19.** All of the objects and maps of the proposition have natural $S^1$-actions, on the one hand coming from loop rotation, on the other hand coming from the cyclic symmetry of dimensions. One can check that the identifications of the proposition are $S^1$-equivariant (see Remark 3.2).

**Remark 1.20.** Recall \text{To2} \text{BN10b} that for a derived scheme $X$, the loop space $\mathcal{L}X \simeq T_X[-1]$ is the total space of the shifted tangent complex. The action map of the $S^1$-rotation action is encoded by the de Rham differential. For an underived stack $X$, the loop space is a derived enhancement of the inertia stack $IX = \{x \in X, \gamma \in \text{Aut}(x)\}$. The action map of the $S^1$-rotation action is manifested by the “universal automorphism” of any sheaf on $\mathcal{L}X$.\footnote{Likewise, if we wish to make a space $n$-dualizable for any $n$ we may simply consider it as an object of a higher correspondence category as in Remark 1.17 since $E_n$-(co)algebras are $n+1$-dualizable objects of the corresponding Morita category. In other words, a space $X$ defines a topological field theory of any dimension valued in the appropriate correspondence category.}
Example 1.21. Let $G$ denote an algebraic group and $BG = pt/G$ its classifying space. There is a canonical identification $\mathcal{L}BG \simeq G/G$ of the loop space and adjoint quotient.

Suppose we are given a $G$-derived stack $X$, or equivalently a morphism $\pi : X/G \to BG$, from which one recovers $X \simeq X/G \times_{BG} pt$. (Note that if we want $\pi$ proper we should take $X$ itself proper.)

Let us explain how the loop map $\mathcal{L}\pi : \mathcal{L}(X/G) \to \mathcal{L}(BG)$ captures the fixed points of $G$ acting on $X$. For any self-map $g : X \to X$, let us write $X^g$ for the derived fixed point locus given by the derived intersection $X^g = \Gamma_g \times_X X$.

of the graph $\Gamma_g \subset X \times X$ with the diagonal. Then $\mathcal{L}\pi$ map fits into a commutative square

$\begin{array}{ccc}
\mathcal{L}(X/G) & \xrightarrow{\mathcal{L}\pi} & \mathcal{L}(BG) \\
\sim & & \sim \\
\{g \in G, x \in X^g\}/G & \xrightarrow{p} & G/G
\end{array}$

where $p$ projects to the group element.

In particular, fix a group element $g \in G$, with conjugacy class $O_g \subset G$, and centralizer $Z_G(g) \subset G$, so that $O_g/G \simeq BZ_G(g) \subset G/G$. Then the corresponding fiber of $\mathcal{L}\pi$ is the equivariant fixed point locus $X^g_G = X^g/Z_G(g)$, or in other words we have a fiber diagram

$\begin{array}{ccc}
X^g_G & \xrightarrow{O_g/G} & \mathcal{L}(X/G) \\
\downarrow & & \downarrow \\
\mathcal{L}(X/G) & \xrightarrow{\mathcal{L}\pi} & \mathcal{L}(BG)
\end{array}$

Let us specialize to the case of a subgroup $K \subset G$, and the quotient $X = G/K$, so that we have a map of classifying stacks $\pi : BK \simeq G\backslash(G/K) \to BG$. Here the loop map $\mathcal{L}\pi$ realizes the familiar geometry of the Frobenius character formula

$\begin{array}{ccc}
\mathcal{L}(BK) & \xrightarrow{\mathcal{L}\pi} & \mathcal{L}(BG) \\
\sim & & \sim \\
K/K & \xrightarrow{p} & G/G
\end{array}$

The equivariant fixed point loci express the equivariant inclusion of conjugacy classes.

Specializing further, for $G$ a reductive group, $B \subset G$ a Borel subgroup, and $X = G/B$ the flag variety, we recover the group-theoretic Grothendieck-Springer resolution

$\begin{array}{ccc}
\mathcal{L}(BB) & \xrightarrow{\mathcal{L}\pi} & \mathcal{L}(BG) \\
\sim & & \sim \\
B/B & \xrightarrow{p} & G/G
\end{array}$

1.3.2. Geometric traces of correspondences. More generally, we have the following calculations of traces and their functoriality.

Proposition 1.22. Let $\text{Corr}_k$ be the category of derived stacks and correspondences, and $\text{Corr}_k^{\pro}$ the 2-category of derived stacks, correspondences, and proper maps of correspondences.
(1) The trace of a self-correspondence \( Z \in \text{Corr}_k(X, X) \) is its fiber product with the diagonal
\[
\text{Tr}(Z) \simeq Z|_\Delta = Z \times_{X \times X} X \simeq Z \times_X L_X
\]
In particular, for the graph \( \Gamma_f \to X \times X \) of a self-map \( f : X \to X \), its trace is the fixed point locus of the map
\[
\text{Tr}(\Gamma_f) \simeq X^f = \Gamma_f \times_{X \times X} X
\]

(2) Given a proper map \( f : X \to Y \) regarded as a correspondence from \( X \) to \( Y \), and self-correspondences \( Z \in \text{Corr}_k(X, X) \) and \( W \in \text{Corr}_k(Y, Y) \), together with an identification
\[
\alpha : Z \sim X \times_Y W
\]
of correspondences from \( X \) to \( Y \), the induced abstract trace map
\[
\text{Tr}(f, \alpha) : \text{Tr}(Z) \longrightarrow \text{Tr}(W)
\]
is equivalent to the induced geometric map
\[
\tau(f, \alpha) : Z|_{\Delta_X} \longrightarrow W|_{\Delta_Y}
\]

1.4. Trace formulas via sheaf theories. Given any sufficiently functorial method of measuring derived stacks, the preceding calculations of geometric dimensions, traces and their functoriality immediately lead to trace and character formulas. To formalize the functoriality needed, we will use the language of sheaf theories. Broadly speaking, a sheaf theory is a representation (symmetric monoidal functor out) of a correspondence category in the way a topological field theory is a representation of a cobordism category. It provides an approach to encoding the standard operations on coherent sheaves and \( D \)-modules, suggested by Lurie and developed by Gaitsgory and Rozenblyum, in part in collaborations with Drinfeld and Francis \([FG, G1, DG, GR1, GR2]\) (see also \([FHLT]\) for a similar construction in the topological setting). The ultimate statement we will use is Theorem 1.2 announced by Gaitsgory.

For concreteness, we will take the target of our sheaf theories to be the linear setting of the symmetric monoidal \( \infty \)-category \( \mathbf{dgCat}_k \) of \( k \)-linear differential graded categories, though the arguments we use apply quite generally. In practice, natural sheaf theories are usually well-defined on specific subcategories of a full correspondence category, and we will modify notations appropriately.

**Definition 1.23.** A **sheaf theory** is a symmetric monoidal functor
\[
\mathcal{S} : \text{Corr}_k \longrightarrow \mathbf{dgCat}_k
\]
from the correspondence category to \( \mathbf{dgCat} \) categories.

Let us introduce some useful notation associated to a general sheaf theory \( \mathcal{S} \). The graph of a map of derived stacks \( f : X \to Y \) provides a correspondenee from \( X \) to \( Y \) and a correspondence from \( Y \) to \( X \). We denote the respective induced maps by \( f_* : \mathcal{S}(X) \to \mathcal{S}(Y) \) and \( f^! : \mathcal{S}(Y) \to \mathcal{S}(X) \). Observe that the functoriality of \( \mathcal{S} \) concisely encodes base change for \( f_* \) and \( f^! \). For \( \pi : X \to \text{pt} = \text{Spec } k \), we denote by \( \omega_X = \pi^! k \in \mathcal{S}(X) \) the \( \mathcal{S} \)-analogue of the dualizing sheaf, and by \( \omega(X) = \pi_* \omega_X \in \mathcal{S}(\text{pt}) = \mathbf{dgVect}_k \) the \( \mathcal{S} \)-analogue of “global volume forms”. We adopt traditional notations whenever possible, for example writing \( \Gamma(X, F) = \pi_*(F) \), for \( F \in \mathcal{S}(X) \).

We will consider three fundamental examples of sheaf theories. We restrict to the natural class of perfect stacks and morphisms (in the sense of \([BFN]\)), or more generally quasi-compact stacks with affine diagonal (QCA stacks) and such morphisms over a field \( k \) of characteristic zero.

- **Theory \( \mathcal{Q} \):** the theory of quasicoherent sheaves \( \mathcal{Q}(X) \). Assuming \( X \) is perfect, the compact objects of \( \mathcal{Q}(X) \) form the subcategory of perfect complexes \( \text{Perf}(X) \), and we have
$Q(X) = \text{Ind Perf}(X)$. Maps are given by the standard pullback $f^*$ and pushforward $f_*$. The (unfortunately named) $Q$-dualizing sheaf is the structure sheaf $O_X$, and the $Q$-global volume forms are the global functions $R\Gamma(X, O_X)$. The $K$-theory of $Q(X)$ is the usual algebraic $K$-theory $K(X)$.

- Theory $Q^!$: the theory of ind-coherent sheaves $Q^!(X)$ (as developed in [CG], see also [DG, GR2]). This is the “large” version $Q^!(X) = \text{Ind Coh}(X)$ of the category of coherent sheaves, which by definition are the compact objects in $Q^!(X)$. (For smooth $X$, ind-coherent and quasicoherent sheaves are equivalent.) Maps are given by the standard pushforward $f_*$ and exceptional pullback $f^!$. The $Q^!$-dualizing sheaf is the usual dualizing complex $\omega_X$, and (for $X$ proper) the $Q^!$-global volume forms are its sections $R\Gamma(X, \omega_X) = R\Gamma(X, O_X)^\vee$. The $K$-theory of $Q^!(X)$ is algebraic $G$-theory $G(X)$, the homological version of algebraic $K$-theory for potentially singular spaces suited to Grothendieck-Riemann-Roch theorems.

- Theory $D$: the theory of $D$-modules $D(X)$ (as described in [FG], developed in [GR1, DG], and thoroughly studied in [GR2]) with the standard functors $f^*$ and $f_*$. The compact objects are necessarily coherent $D$-modules (this suffices for $X$ a scheme; see [DG] for a characterization in the case of a stack). The $D$-dualizing sheaf is the Verdier dualizing complex $\omega_X$, and the $D$-global volume forms (for $X$ smooth) are the Borel-Moore homology $R\Gamma(X_{dR}, \omega_X) = H_{dR}(X)^\vee$.

Remark 1.24 (Sheaf theories in differential topology and elliptic operators). It is tempting to think of sheaf theories in algebraic geometry as analogues of elliptic operators or complexes in differential topology. In particular, the theory $Q^!(X)$ for a smooth variety $X$ is a natural setting for the study of the Dolbeault $\overline{\partial}$-operator coupled to vector bundles, while the theory $D(X)$ is similarly a natural setting for the study of the de Rham operator $d$ coupled to vector bundles. The pushforward operation is the analogue of the index. In this direction, it would be interesting to develop sheaf theories on derived manifolds, for example $C^\infty$-schemes and stacks. Quasicoherent sheaves in the sense of Joyce [J] are a natural candidate. Another interesting setting is categories of elliptic complexes on manifolds. The general results below would then provide an approach to generalizations of the classical Atiyah-Singer and Atiyah-Bott theorems.

Since a sheaf theory $S$ is symmetric monoidal, it is automatically compatible with dimensions and traces: for any $X \in \text{Corr}_k$, and any endomorphism $Z \in \text{Corr}_k(X, X)$, we have

$$\dim(S(X)) \simeq S(\dim(X)) \quad \text{Tr}(S(Z)) \simeq S(\text{Tr}(Z))$$

Let us combine this with the calculation of the right hand sides and highlight specific examples of interest.

Proposition 1.25. Fix a sheaf theory $S : \text{Corr}_k \to \text{dgCat}_k$.

1. The $S$-dimension $\dim(S(X)) \simeq HH_k(S(X))$ of any $X \in \text{Corr}_k$ is $S^1$-equivariantly equivalent with $S$-global volume forms on the loop space

$$\dim(S(X)) \simeq \omega(LX).$$

In particular, for $G$ an affine algebraic group, characters of $S$-valued $G$-representations are adjoint-equivariant $S$-global volume forms

$$\dim(S(BG)) \simeq \omega(G/G)$$

2. The $S$-trace of any endomorphism $Z \in \text{Corr}_k(X, X)$ is equivalent to $S$-global volume forms on the restriction to the diagonal

$$\text{Tr}(S(Z)) \simeq \omega(Z|_{\Delta})$$
In particular, the $S$-trace of a self-map $f : X \to X$ is equivalent to $S$-volume forms on the $f$-fixed point locus

$$\text{Tr}(f_*) \simeq \omega(X^f)$$

Remark 1.26 (Local sheaf theory). To apply this proposition, far less structure than a full sheaf theory is required. We only need the data of the functor $S$ on the handful of objects and morphisms involved in the construction of dimensions and traces as in Remark 1.11. In particular, we only need base change isomorphisms for pullback and pushforward along specific diagrams, rather than the general base change provided by a functor out of $\text{Corr}_k$. This is often easy to verify in practice, in particular for the examples $Q$, $Q^!$ and $\mathcal{D}$ (see for example [BFN] for the quasicoherent setting).

Let us spell out the main ingredients of the proposition for our three main examples. Recall that for $X$ a smooth scheme, $\mathcal{L}X \simeq \text{Spec}_X \text{Sym}^\bullet(\Omega_X[1])$, and for $BG$ a classifying stack, $\mathcal{L}(BG) \simeq G/G$.

- Theory $Q$: For $X$ a smooth scheme, $Q$-global volume forms on $\mathcal{L}X$ are the usual Hochschild chain complex $\text{dim}(Q(X)) \simeq \Gamma(X, \text{Sym}^\bullet(\Omega_X[1]))$, or more generally, $Q$-global volume forms on $X^f$ are the coherent cohomology $\mathcal{O}(X^f)$. For $BG$ a classifying stack, $Q$-global volume forms on $\mathcal{L}(BG)$ are the coherent cohomology $\mathcal{O}(G/G)$, which for $G$ reductive are the underived invariants $\mathcal{O}(T)^W$.

- Theory $Q^!$: For $X$ smooth, we have $Q(X) \simeq Q^!(X)$, and so we recover the above descriptions. For $X$ proper, $Q$-global volume forms on $\mathcal{L}X$ are the dual of the Hochschild chain complex (see [P]).

- Theory $\mathcal{D}$: For $X$ a smooth scheme, $\mathcal{D}$-global volume forms on $\mathcal{L}X$ are the de Rham cochains $\text{dim}(\mathcal{D}(X)) \simeq C^\text{dR}_*(X)$, or more generally, $\mathcal{D}$-global volume forms on $X^f$ are the de Rham cochains $C^\text{dR}_*(X^f)$, or equivalently those of the underlying underived scheme of $X^f$. For $BG$ a classifying stack, $\mathcal{D}$-global volume forms on $\mathcal{L}(BG)$ are the Borel-Moore homology of $G/G$.

1.4.1. Integration formulas for traces. Now let us turn to the functoriality of dimensions and traces. For the theory $Q$ of quasicoherent sheaves, we expect a contravariant functoriality under arbitrary maps, corresponding to pullback of functions. For the theories $Q^!$ of ind-coherent sheaves and $\mathcal{D}$ of $\mathcal{D}$-modules, we expect a more interesting covariant functoriality under proper maps, corresponding to integration of volume forms.

We will focus on the covariant case where we would like to encode an adjunction $(f_*, f^!)$ for proper maps. We continue with the setting of a sheaf theory $S : \text{Corr}_k \to \mathbf{dgCat}_k$, but now (as suggested by Lurie) enhance it with structures most naturally captured by a symmetric monoidal functor

$$\widetilde{S} : \mathbf{Corr}^{pr}_k \longrightarrow \mathbf{dgCat}_k$$

from the correspondence 2-category with proper maps between correspondences to the 2-category of dg categories. More specifically, for a proper map $f : X \to Y$, we assume the induced maps $f_* : S(X) \to S(Y)$, $f^! : S(Y) \to S(X)$ are equipped with the data of an adjunction

$$S(X) \xrightarrow{f_*} S(Y)$$

$$S(Y) \xleftarrow{f^!} S(X)$$
compatible with compositions and base change. There is a resulting canonical integration map along a proper map \( f : X \to Y \) given by the counit of adjunction

\[
\int_f : \omega(X) \longrightarrow \omega(Y)
\]

We call such an enhanced structure a proper sheaf theory.

**Theorem 1.27.** Fix a proper sheaf theory \( S : \text{Corr}_k \to \text{dgCat}_k \).

1. For any proper map \( f : X \to Y \), the induced map on dimensions

\[
\dim(f^*) : \dim(S(X)) \longrightarrow \dim(S(Y))
\]

is identified (\( S^1 \)-equivariantly) with integration along the loop map

\[
\dim(f^*) \simeq \int_{L_f} : \omega(LX) \longrightarrow \omega(LY)
\]

2. Given a proper map \( f : X \to Y \) regarded as a correspondence from \( X \) to \( Y \), and self-correspondences \( Z \in \text{Corr}_k(X,X) \) and \( W \in \text{Corr}_k(Y,Y) \), together with an identification

\[
\alpha : Z \xrightarrow{\sim} X \times_Y W
\]

of correspondences from \( X \) to \( Y \), the induced trace map is identified with integration along the natural map

\[
\Tr(f_* \alpha) \simeq \int_{\tau(f,s)} : \omega(Z|\Delta_X) \longrightarrow \omega(W|\Delta_Y)
\]

**Remark 1.28.** Similarly, in the case of the theory \( Q \) of quasicoherent sheaves, the standard adjunction \((f^*, f_*)\) leads to an extension of \( Q \) to the \(2\)-category \( \text{Corr}^*_{\omega} \) in which the morphisms \( \text{Corr}^*_\omega(X,Y) \) are the opposite of the category of derived stacks over \( X \times Y \). This results in the evident contravariant functoriality of dimensions under arbitrary maps, given by pullback of functions on loop spaces.

**Remark 1.29 (Local enhanced sheaf theory).** Although it is the natural context for the above discussion, the theory of symmetric monoidal \((\infty,2)\)-categories is not fully documented at this time. In particular, the extensions of \( Q \) and \( D \) to such a setting have not been elaborated. However, in the case of \( D \), as explained in \( [FG] \), proper adjunction is already encoded in the data of the functor \( D : \text{Corr}_k \to \text{dgCat}_k \). This is not the case for other sheaf theories, in particular for \( Q \). Nevertheless, as proposed in \( [G1] \), there is a clever subterfuge: one can extend \( Q \) to an \((\infty,1)\)-category of closed embeddings of stacks. The resulting enhanced sheaf theory (see \( [G1] \) Section 5.4) for details) does indeed uniquely determine the adjunction data for proper maps. Thus for \( Q \) and \( D \), there are sufficient foundations for our applications.

In fact, for our applications, we require far less data. In addition to the data of a sheaf theory, we only need to specify the \((f^*, f_*)\) adjunction for specific proper morphisms. In fact, following Remark 1.15, we need far less than even a sheaf theory. We need only specify the functor \( S \) on a handful of diagrams. This amount of structure is readily accessible for the theories \( Q \) and \( D \) (as well as \( Q \)).

**Remark 1.30 (Categorified version).** For applications to categorical representation theory, in particular the geometric Langlands program, it is interesting to have character formulas for group actions on categories. Such formulas can be formally deduced from our preceding constructions by considering sheaf theories \( S : \text{Corr}_k \to \mathcal{A} \) with values in an \(\infty\)-category \(\mathcal{A}\) other than that of dg categories. Namely, we are interested in categorified analogues of \( D \) and \( Q \), taking values in the \(\infty\)-category \(\text{Pr}^L \) of presentable \(\infty\)-categories, in which we assign to a scheme or stack \( X \) the \(\infty\)-category of quasicoherent sheaves of module categories over \( D \) or \( Q \). Since such theories have not been fully constructed yet, we will only briefly sketch the idea.
For any stack \( X \) and sheaf theory \( S \), the category of sheaves \( S(X) \) is naturally symmetric monoidal, and so we may consider its \( \infty \)-category of (presentable, stable) module categories \( S(X)\text{-mod} \). To obtain a more meaningful geometric theory we should sheafify this construction. For example, strong or Harish-Chandra \( G \)-categories (in other words, module categories over \( D(G) \) with convolution) are identified with sheaves of categories over the de Rham stack of \( BG \).

However, in the quasicoherent case, a recent “affineness” theorem of Gaitsgory \cite{G2} identifies \( Q(X) \)-modules with sheaves of categories over \( X \) for a large class of stacks (specifically, for \( X \) an eventually coconnective quasi-compact algebraic stack of finite type with an affine diagonal over a field of characteristic 0). In particular, \( Q(BG) \)-modules are identified with algebraic \( G \)-categories.

In the quasicoherent case, the general formalism of this paper should provide an \( S^1 \)-equivariant equivalence \( \dim(Q(X)\text{-mod}) = Q(LX) \), identifying the class \([Q(X)]\) of the structure stack with the structure sheaf \( O(LX) \). In particular, the characters of quasicoherent \( G \)-categories are given by \( Q(G/G) \). The induced map on dimensions \( \dim(f_*) : \dim(Q(X)\text{-mod}) \to \dim(Q(Y)\text{-mod}) \) is identified \( S^1 \)-equivariantly with the morphism given by pushforward along the loop map

\[
\dim(f_*) = Lf_* : Q(LX) \longrightarrow Q(LY)
\]

In particular, for an algebraic group \( G \) and \( G \)-space \( X \) with \( \pi : X/G \to BG \), the character of the \( G \)-category \( Q(X/G) \) is given by the pushforward \( L\pi_*O(LX/G) \in Q(G/G) \). Analogous results are expected for strong or Harish-Chandra \( G \)-categories (module categories for \( D(G) \) with convolution) using the sheafification of the theory of \( D(X) \)-module categories. We hope to return to these applications in future works.

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2. Traces in category theory

2.1. Preliminaries. Our working setting is the higher category theory and algebra developed by J. Lurie \cite{L1, L2, L3, L4}.

Throughout what follows, we will fix once and for all a symmetric monoidal \((\infty, 2)\)-category \( \mathcal{A} \) with unit object \( 1_\mathcal{A} \). By forgetting non-invertible 2-morphisms we obtain a symmetric monoidal \((\infty, 1)\)-category \( f(\mathcal{A}) \), which we will abusively refer to as \( \mathcal{A} \) whenever only invertible higher morphisms are involved. Conversely, given a symmetric monoidal \((\infty, 1)\)-category \( \mathcal{C} \), we can always regard it as a symmetric monoidal \((\infty, 2)\)-category \( i(\mathcal{C}) \) with all 2-morphisms invertible. Thus developments for higher \( \infty \)-categories equally well apply to the more familiar \((\infty, 1)\)-categories. In what follows, noninvertible 2-morphisms only play a significant role starting with Section \ref{sec:2.2}.

Moreover, one can rephrase the noninvertible 2-morphisms in terms of more traditional structures.

We will use \( \otimes \) to denote the symmetric monoidal structure of \( \mathcal{A} \). We will write \( \Omega\mathcal{A} = \text{End}_\mathcal{A}(1_\mathcal{A}) \) for the “based loops” in \( \mathcal{A} \), or in other words, the symmetric monoidal \((\infty, 1)\)-category of endomorphisms of the monoidal unit \( 1_\mathcal{A} \). Note that the monoidal unit \( 1_{\Omega\mathcal{A}} \) is nothing more than the identity \( \text{id}_{1_\mathcal{A}} \) of the monoidal unit \( 1_\mathcal{A} \).

\footnote{One can understand the above two operations as forming an adjoint pair \((i, f)\).}
Example 2.1 (Algebras). Fix a symmetric monoidal \((\infty, 1)\)-category \(C\), and let \(\mathcal{A} = \text{Alg}(C)\) denote the Morita \((\infty, 2)\)-category of algebras, bimodules, and intertwiners of bimodules within \(C\). The forgetful map \(\mathcal{A} = \text{Alg}(C) \to C\) is symmetric monoidal, and in particular, the monoidal unit \(1_\mathcal{A}\) is the monoidal unit \(1_C\) equipped with its natural algebra structure. Finally, we have \(\Omega \mathcal{A} \simeq C\).

For a specific example, one could take a commutative ring \(k\) and \(C = k\)\-mod the \((\infty, 1)\)-category of complexes of \(k\)-modules. Then \(\mathcal{A} = \text{Alg}(C)\) is the \((\infty, 2)\)-category of \(k\)-algebras, bimodules, and intertwiners of bimodules.

Example 2.2 (Categories). A natural source of \((\infty, 2)\)-categories is given by various theories of \((\infty, 1)\)-categories. For example, for a commutative ring \(k\), one could consider \(\mathcal{S}^k\), the \((\infty, 2)\)-category of \(k\)-linear stable presentable \(\infty\)-categories, \(k\)-linear continuous functors, and natural transformations.

Observe that \(\text{Alg}(k\text{-mod})\) is a full subcategory of \(\mathcal{S}^k\), via the functor assigning to a \(k\)-algebra its stable presentable \(\infty\)-category of modules. The essential image consists of stable presentable categories admitting a compact generator.

2.2. Dualizability.

Definition 2.3. An object \(A\) of the symmetric monoidal \((\infty, 2)\)-category \(\mathcal{A}\) is said to be dualizable (equivalently, \(A\) is dualizable in the \((\infty, 1)\)-category \(f(\mathcal{A})\)) if it admits a monoidal dual: there is a dual object \(A^\vee \in \mathcal{A}\) and evaluation and coevaluation morphisms

\[
\varepsilon_A : A^\vee \otimes A \to 1_{\mathcal{A}} \quad \eta_A : 1_{\mathcal{A}} \to A \otimes A^\vee
\]

such that the usual compositions are naturally equivalent to the identity morphism

\[
A \xrightarrow{\eta_A \otimes \text{id}_A} A \otimes A^\vee \otimes A \xrightarrow{\text{id}_A \otimes \varepsilon_A} A \xrightarrow{\text{id}_A^\vee \otimes \eta_A} A^\vee \otimes A \otimes A^\vee \xrightarrow{\varepsilon_A \otimes \text{id}_A^\vee} A^\vee
\]

Example 2.4. Any algebra object \(A \in \text{Alg}(C)\) is dualizable with dual the opposite algebra \(A^{op} \in \text{Alg}(C)\). The evaluation morphism

\[
\varepsilon_A : A^{op} \otimes A \to 1_C
\]

is given by \(A\) itself regarded as an \(A\)-bimodule. The coevaluation morphism

\[
\eta_A : 1_C \to A \otimes A^{op}
\]

is also given by \(A\) itself regarded as an \(A\)-bimodule.

2.2.1. Dualizable morphisms. Consider two objects \(A, B \in \mathcal{A}\), and a morphism

\[
\Phi : A \to B.
\]

Example 2.5. If \(\mathcal{A} = \text{Alg}(C)\), then \(\Phi\) is simply an \(A^{op} \otimes B\)-module.

If \(B\) is dualizable with dual \(B^\vee\), we can package \(\Phi\) in the equivalent form of the morphism

\[
e_\Phi : B^\vee \otimes A \to 1_{\mathcal{A}}
\]

defined by

\[
\begin{array}{ccc}
B^\vee \otimes A & \xrightarrow{e_\Phi} & 1_{\mathcal{A}} \\
\downarrow \text{id}_{B^\vee} \otimes \Phi & & \\
B \otimes B^\vee & \xrightarrow{\varepsilon_B} & \\
\end{array}
\]
If \( A \) is dualizable with dual \( A^\vee \), we can package \( \Phi \) in the equivalent form of the morphism

\[
u_\Phi : 1_A \to B \otimes A^\vee
\]
defined by

\[
\begin{array}{c}
A \otimes A^\vee \\
\downarrow \eta_A \\
1_A \\
\end{array}
\to
\begin{array}{c}
B \otimes A^\vee \\
\downarrow \Phi \otimes \text{id}_{A^\vee} \\
1_A \\
\end{array}
\]

If both \( A \) and \( B \) are dualizable, we can also encode \( \Phi \) by its dual morphism

\[
\Phi^\vee : B^\vee \to A^\vee
\]
defined by

\[
\begin{array}{c}
B^\vee \\
\downarrow \text{id}_{B^\vee} \otimes \eta_A \\
B^\vee \otimes A \otimes A^\vee \\
\downarrow \Phi \otimes \text{id}_{A^\vee} \\
B^\vee \otimes A \otimes A^\vee \\
\downarrow \epsilon_{B^\vee} \otimes \text{id}_{A^\vee} \\
A^\vee \\
\end{array}
\]

There is a natural composition identity

\[(\Phi \Psi)^\vee \simeq \Psi^\vee \Phi^\vee\]

Note that for fixed \( A, B \), the construction \( \Phi \mapsto \Phi^\vee \) naturally defines a covariant map

\[(-)^\vee : \text{Hom}(A, B) \to \text{Hom}(B^\vee, A^\vee)\]

and in particular a morphism \( \Phi_1 \to \Phi_2 \) induces a natural morphism \( \Phi_1^\vee \to \Phi_2^\vee \).

Let us record the canonical equivalences encoded by the following commutative diagrams

\[
(2.1)
\]

\[
\begin{array}{c}
\begin{array}{c}
A \otimes A^\vee \\
\downarrow \eta_A \\
1_A \\
\end{array}
\to
\begin{array}{c}
B \otimes A^\vee \\
\downarrow \Phi \otimes \text{id}_{A^\vee} \\
B \otimes B^\vee \\
\downarrow \text{id}_{B} \otimes \Phi^\vee \\
B \otimes B^\vee \\
\end{array}
\end{array}
\]

\[
\begin{array}{c}
\begin{array}{c}
A^\vee \otimes A \\
\downarrow \Phi^\vee \otimes \text{id}_A \\
B^\vee \otimes A \\
\downarrow \text{id}_{B^\vee} \otimes \Phi \\
B \otimes B^\vee \\
\end{array}
\to
\begin{array}{c}
\begin{array}{c}
A^\vee \otimes A \\
\downarrow \epsilon_A^\vee \\
1_A \\
\end{array}
\end{array}
\end{array}
\]

**Example 2.6.** In the setting of algebras, bimodules and intertwiners, the morphisms \( \Phi, u_\Phi, e_\Phi \) and \( \Phi^\vee \) are all different manifestations of the same bimodule \( \Phi \), making their various compatibilities particularly evident.

**Definition 2.7.** (1) A morphism \( \Phi : A \to B \) is said to be **left dualizable** if it admits a left adjoint: there is a morphism \( \Phi^\ell : B \to A \) and unit and counit morphisms

\[
\eta_\Phi : \text{id}_B \to \Phi \circ \Phi^\ell
\]

\[
\epsilon_\Phi : \Phi^\ell \circ \Phi \to \text{id}_A
\]

satisfying the usual identities.

(2) A morphism \( \Phi : A \to B \) is said to be **right dualizable** if it admits a right adjoint: there is a morphism \( \Phi^r : B \to A \) and unit and counit morphisms

\[
\eta_\Phi : \text{id}_A \to \Phi^r \circ \Phi
\]

\[
\epsilon_\Phi : \Phi \circ \Phi^r \to \text{id}_B
\]

satisfying the usual identities.
2.8. Remark. If $A$ and $B$ are dualizable, and $\Phi : A \to B$ is left (resp. right) dualizable, then $\Phi^\vee : B^\vee \to A^\vee$ is right (resp. left) dualizable with right adjoint $(\Phi^\ell)^\vee : A^\vee \to B^\vee$ (resp. left adjoint $(\Phi^r)^\vee : A^\vee \to B^\vee$).

2.3. Traces and dimensions. Let $A \in \mathcal{A}$ be a dualizable object with dual $A^\vee$. Consider an endomorphism

$$\Phi : A \to A$$

Since $A$ is dualizable, $\Phi$ has a trace of $\Phi$ defined as follows.

**Definition 2.9.**

1. The trace of $\Phi : A \to A$ is the object $\text{Tr}(\Phi) \in \Omega \mathcal{A}$ defined by

$$1_A \xrightarrow{\eta_A} A \otimes A^\vee \xrightarrow{\Phi \otimes \text{id}_A} A \otimes A^\vee \xrightarrow{\epsilon_A} 1_A.$$ 

Given a natural transformation $\varphi : \Phi \to \Psi$, we define the induced morphism

$$\text{Tr}(\varphi) : \text{Tr}(\Phi) \to \text{Tr}(\Phi')$$

by applying $\varphi \otimes \text{id}_{A^\vee}$ to the middle arrow above.

2. The dimension (or Hochschild homology) of $A$ is the trace of the identity

$$\text{dim}(A) = \text{Tr}(\text{id}_A) \in \Omega \mathcal{A}$$

or in other words, the object defined by

$$1_A \xrightarrow{\eta_A} A \otimes A^\vee \xrightarrow{\epsilon_A} 1_A$$

**Remark 2.10.** Equivalently, we can describe the trace as the composition

$$1_A \xrightarrow{\Phi} \text{End}(A) \xrightarrow{\sim} A \otimes A^\vee \xrightarrow{\epsilon_A} 1_A$$

where the middle arrow is the identification deduced from the dualizability of $A$.

**Remark 2.11.** Observe that for fixed dualizable $A \in \mathcal{A}$, taking traces gives a functor

$$\text{Tr} : \text{End}(A) \to \Omega \mathcal{A}$$

**Remark 2.12.** Observe that for any dualizable endomorphism $\Phi$, the standard identities encoded by Diagrams 2.1 give rise to an identification

$$\text{Tr}(\Phi) \simeq \text{Tr}(\Phi^\vee)$$

**Example 2.13.** When $A = 1_A$ is the monoidal unit, and $\Phi : 1_A \to 1_A$ is an endomorphism, we have an evident equivalence of endomorphisms

$$\text{Tr}(\Phi) \simeq \Phi$$

**Theorem 2.14 ([L3]).** There is a canonical $S^1$-action on the dimension $\text{dim}(A)$ of any dualizable object $A$ of a symmetric monoidal $\infty$-category $\mathcal{A}$. 
2.3.1. Cyclic symmetry.

**Proposition 2.15.** Given two morphisms

\[
A \xrightarrow{\Phi} B \xleftarrow{\Psi} C
\]

between dualizable objects \(A,B \in \mathcal{A}\), there is a canonical equivalence

\[
m(\Phi, \Psi) : \text{Tr}(\Phi \circ \Psi) \xrightarrow{\sim} \text{Tr}(\Psi \circ \Phi)
\]

functorial in morphisms of both \(\Phi\) and \(\Psi\).

**Proof.** We construct \(m(\Phi, \Psi)\) following the commutative diagram below:

Following the top edge, we find the definition of \(\text{Tr}(\Psi \circ \Phi)\). Following the bottom edge, we find the definition of \(\text{Tr}(\Phi \circ \Psi)\). The identifications filling the left and right diamonds arise from the standard identities encoded by Diagrams 2.1. The identification filling the central square results from the symmetric monoidal structure.

The construction is evidently functorial for morphisms \(\Phi \to \Phi'\). The functoriality for morphisms \(\Psi \to \Psi'\) is similar, once one recalls that the construction \(\Psi \mapsto \Psi^\vee\) is covariantly functorial in morphisms of \(\Psi\).

**Example 2.16.** Taking \(\Phi = \text{id}_A\) yields a canonical equivalence

\[
\gamma' : \text{id}_{\text{Tr}(\Phi')} \xrightarrow{\sim} m(\text{id}_A, \Phi'_A)
\]

and likewise, taking \(\Phi' = \text{id}_A\) yields a canonical equivalence

\[
\gamma : \text{id}_{\text{Tr}(\Phi)} \xrightarrow{\sim} m(\Phi_A, \text{id}_A)
\]

Thus taking \(\Phi = \Phi' = \text{id}_A\) yields an automorphism of the identity of the Hochschild homology

\[
(\gamma')^{-1} \circ \gamma : \text{id}_{\text{Tr}(\text{id}_A)} \xrightarrow{\sim} \text{id}_{\text{Tr}(\text{id}_A)}
\]

called the **BV homotopy**.

**Remark 2.17.** The proposition is only the initial part of the full cyclic symmetry of trace (see Remark 2.22), and the example is the lowest level structure of the \(S^1\)-action on Hochschild homology (see Theorem 2.14) defining cyclic homology.

**Lemma 2.18.** Given morphisms

\[
A \xrightarrow{\Phi} B \xleftarrow{\Psi} C \xrightarrow{\top} A
\]
between dualizable objects $A, B, C \in A$, there is a canonical commutative diagram

$$
\begin{array}{ccc}
\text{Tr}(\Psi \Phi \Upsilon) & \xrightarrow{m(\Psi, \Phi \Upsilon)} & \text{Tr}(\Phi \Upsilon \Psi) \\
& m(\Psi \Phi, \Upsilon) \downarrow & \downarrow m(\Phi, \Upsilon \Psi) \\
& & \text{Tr}(\Upsilon \Phi \Psi)
\end{array}
$$

Proof. We construct the desired equivalence from the following diagram:

The natural transformations $m(\Psi, \Phi \Upsilon)$ and $m(\Phi, \Upsilon \Psi)$ describe passage from the top row to the middle row and from the middle to the bottom, respectively. The transformation $m(\Psi \Phi, \Upsilon)$ can then be identified with the transformation from the top row to the bottom given by inserting the diagonal morphisms $\text{id} \otimes \Phi \Upsilon \circ \Upsilon$ and using standard composition identities. \qed

2.4. Functoriality of dimension. Let $A^{\text{cont}} \subset A$ denote the $(\infty, 2)$-subcategory of dualizable objects and continuous or right dualizable morphisms (morphisms that are left duals).

Definition 2.19. Let $\Psi : A \to B$ denote a morphism in $A^{\text{cont}}$ with right adjoint $\Psi^r : B \to A$. We define the induced morphism of dimensions

$$
\dim(\Psi) : \dim(A) \longrightarrow \dim(B)
$$

to be the composition

$$
\text{Tr}(\text{id}_A) \xrightarrow{\eta_\Psi} \text{Tr}(\Psi^r \circ \Psi) \xrightarrow{m(\Psi^r, \Psi)} \text{Tr}(\Psi \circ \Psi^r) \xrightarrow{\epsilon_\Psi} \text{Tr}(\text{id}_B)
$$

Remark 2.20. In other words, the morphism $\dim(\Psi)$ is defined by the following diagram

Following the top and bottom edge, we find the respective definitions of $\dim(A)$ and $\dim(B)$. The unit $\eta_\Psi$ defines a morphism from the top edge to the top zig-zag. The counit $\epsilon_\Psi$ defines a
morphism from the bottom zig-zag to the bottom edge. The passage from the top to bottom zig-zag is given by the construction \( m(\Psi^r, \Psi) \) and the identification
\[
\text{Tr}(\Psi^r \circ \Psi^r) \simeq \text{Tr}((\Psi \circ \Psi^r)^r) \simeq \text{Tr}(\Psi \circ \Psi^r)
\]

**Proposition 2.21.** For a diagram
\[
A \xrightarrow{\Phi} B \xrightarrow{\Psi} C
\]
within \( \mathcal{A}^{\text{cont}} \), there is a canonical equivalence
\[
\dim(\Psi \circ \Phi) \simeq \dim(\Psi) \circ \dim(\Phi) : \dim(A) \longrightarrow \dim(C)
\]

**Proof.** The equivalence is given by filling in the following diagram
\[
\begin{array}{ccc}
\dim(A) & \xrightarrow{\eta_\Phi} & \text{Tr}(\Phi^r \Phi) & \xrightarrow{m} & \text{Tr}(\Phi \Phi^r) & \xrightarrow{\epsilon_\Phi} & \dim(B) \\
& \searrow & & \downarrow & & \swarrow & \\
\text{Tr}(\Phi^r \Psi^r \Phi \Phi) & \xrightarrow{m} & \text{Tr}(\Phi^r \Psi \Phi \Phi^r) & \xrightarrow{\epsilon_\Psi} & \text{Tr}(\Psi \Phi^r) & \xrightarrow{\epsilon_\Psi} & \dim(C) \\
& \searrow & & \downarrow & & \swarrow & \\
\text{Tr}(\Phi \Phi^r \Psi) & \xrightarrow{\epsilon_\Phi} & \text{Tr}(\Psi \Phi) & & & & \\
\end{array}
\]

Along the three boundary edges, we find the definitions of \( \dim(\Phi) \), \( \dim(\Psi) \) and \( \dim(\Psi \Phi) \) respectively.

The two corner triangles are given by the composition identities for adjoints (for example, at the top left, relating the adjoint of \( \Phi \Psi \) with the composition of adjoints of \( \Psi \) and \( \Phi \)).

The middle triangle is given by the identity of Lemma 2.18.

The top right square is given by taking traces of the evident commutative diagram of endomorphisms
\[
\Phi \Phi^r \otimes \text{Id}_{B^r} \longrightarrow \text{Id}_B \otimes \text{Id}_{B^r}
\]

and using the canonical identification \( \text{Tr}(F) = \text{Tr}(F^r) \) for any dualizable morphism.

Finally, the two remaining commuting squares are given by the functoriality of the cyclic rotation of the trace in its two arguments. For instance, in the top left square, we may either rotate \( \text{Tr}(\Phi^r \circ (\text{Id}_A \circ \Phi)) \) and then apply the unit \( \eta_\Psi : \text{Id}_A \rightarrow \Psi^r \Psi \) or first apply the unit and then rotate.

This concludes the construction. \( \square \)

Since we have an evident equivalence \( \dim(1_A) \simeq 1_A \) for the unit \( 1_A \in A \), we have the following specialization of Proposition 2.21 in which we adopt suggestive notation.
Corollary 2.22 (Abstract Grothendieck-Riemann-Roch). Let \( A, B \in \mathcal{A}_{\text{cont}} \) and \( V : 1_A \to A \) and \( \pi_* : A \to B \) morphisms in \( \mathcal{A}_{\text{cont}} \). Then the following diagram naturally commutes

\[
\begin{array}{ccc}
1_A & \xrightarrow{\dim(V)} & \dim(A) \\
\downarrow{\dim(\pi_* V)} & & \downarrow{\dim(\pi_*)} \\
\dim(B) & & 
\end{array}
\]

Remark 2.23. One can show along the same lines as the proposition that taking dimensions extends to a symmetric monoidal functor

\[ \dim : \mathcal{A}_{\text{cont}} \to \Omega \mathcal{A}. \]

2.5. Functoriality of traces. We would like to capture the functoriality for traces of arbitrary endomorphisms of dualizable objects. For this purpose we define a morphism between pairs

\[ A \in \mathcal{A}_{\text{cont}} \quad \Phi_A \in \text{End}_A(A) \]

of an object and an endomorphism to consist of a pair

\[ \Psi \in \text{Hom}_{\mathcal{A}_{\text{cont}}}(A, B) \quad \psi : \Psi \circ \Phi_A \xrightarrow{\sim} \Phi_B \circ \Psi \]

of a morphism and a commuting structure.

Definition 2.24. For a morphism

\[ (\Psi, \psi) : (A, \Phi_A) \to (B, \Phi_B) \]

as above, we define the induced morphism of traces

\[ \text{Tr}(\Psi, \psi) : \text{Tr}(\Phi_A) \to \text{Tr}(\Phi_B) \]

to be the composition

\[
\begin{array}{ccc}
\text{Tr}(\Phi_A) & \xrightarrow{\eta_\Psi} & \text{Tr}(\Psi^* \Psi \Phi_A) \\
\psi & & \psi \\
\text{Tr}(\Psi^* \Phi_B \Psi) & \xrightarrow{m(\Psi^* \Phi_B \Psi)} & \text{Tr}(\Phi_B \Psi \Psi^*) \\
\epsilon_\Psi & & \epsilon_\Psi \\
\text{Tr}(\Phi_B) & & 
\end{array}
\]

Remark 2.25. Note that we could alternatively define a morphism \( \text{Tr}(\Psi, \psi) \) by applying the unit \( \eta_\Psi \) to the right of \( \Phi_A \), rotating the trace in the opposite direction, and again using the counit on the right. It is elementary to give a natural equivalence of the two constructions using nothing more than the dualizability of \( A \).

Remark 2.26. In parallel with Remark 2.20 about the functoriality of dimensions, it is enlightening to realize the functoriality of traces as a chase through the following diagram.
Proposition 2.27. Suppose given objects \( A, B, C \in A^\text{cont} \), endomorphisms \( \Phi_A, \Phi_B, \Phi_C \), a commutative diagram of continuous morphisms

\[
\begin{array}{ccc}
A & \xrightarrow{\Psi_{AC}} & C \\
\downarrow{\Psi_{AB}} & & \downarrow{\Psi_{BC}} \\
B & \xrightarrow{\Phi_B} & D
\end{array}
\]

and commuting structures

\[
s_{AB} : \Psi_{AB} \Phi_A \sim \Phi_B \Psi_{AB} \quad \quad s_{BC} : \Psi_{BC} \Phi_B \sim \Phi_C \Psi_{BC} \quad \quad s_{AC} : \Psi_{AC} \Phi_A \sim \Phi_C \Psi_{AC}
\]

with an identification \( s_{AC} \simeq s_{BC} s_{AB} \). Then there is a canonical equivalence

\[
\text{Tr}(\Psi_{AC}, s_{AC}) \simeq \text{Tr}(\Psi_{BC}, s_{BC}) \circ \text{Tr}(\Psi_{AB}, s_{AB}) : \text{Tr}(\Phi_A) \rightarrow \text{Tr}(\Phi_C)
\]

Proof. The construction is obtained from following a minor expansion of the diagram proving Proposition 2.21. The additional moves needed are commuting the commuting structures past the symmetry \( m \) of the trace and the unit and counits of the adjunctions. These all follow immediately from the 2-categorical interchange law for natural transformations. \( \Box \)

Remark 2.28. One expects the full functoriality of the trace \( \text{Tr} \) to take the following form. Define the loop category \( \mathcal{L}^{cont}A \) to be the symmetric monoidal \( \infty \)-category with objects consisting of pairs \((A, \Phi_A)\) of a dualizable object \( A \in A \) equipped with a (not necessarily continuous) endomorphism \( \Phi_A \), and morphisms given by pairs \((\Psi, \psi)\) as above with \( \Psi \) continuous. One expects taking traces to extend to a symmetric monoidal functor

\[
\text{Tr} : \mathcal{L}^{cont}A \rightarrow \Omega A
\]

extending the dimension functor

\[
\text{dim} : A^{cont} \rightarrow \Omega A
\]

for constant loops \( \Phi_A = \text{id}_A \), and trivial commuting structures \( \psi = \text{id}_\Psi \).

In order to capture the full cyclic symmetry of the trace \( \text{Tr} \), one should further extend it to a homotopical trace valued in \( \Omega A \), or in other words, to the appropriate full cyclic bar construction (of which the above forms only the one-simplices).

3. Traces in Geometry

3.1. Categories of correspondences. For concreteness, we fix a base commutative ring, and work in the symmetric monoidal \((\infty, 1)\)-category \( \text{Stacks}_k \) of derived stacks over \( \text{Spec} k \). It is worth pointing out that the constructions of this section apply in any presentable \( \infty \)-category with the Cartesian symmetric monoidal structure.

Let \( \text{Corr}_k \) denote the symmetric monoidal \( \infty \)-category of correspondences in \( \text{Stacks}_k \). Thus morphisms are given by the classifying space of correspondences

\[
X \xrightarrow{Z} Y
\]

so all higher morphisms are isomorphisms. Composition of correspondences is given by the derived fiber product. The based loop category

\[
\Omega \text{Corr}_k = \text{End}_{\text{Corr}_k}(\text{Spec} k) \simeq \text{Stacks}_k
\]

is again derived stacks, regarded as self-correspondences of the point \( \text{Spec} k \).
We will also enhance $\text{Corr}_k$ to the symmetric monoidal $(\infty, 2)$-category $\text{Corr}_k$ where we now allow noninvertible maps of correspondences

\[
\begin{array}{c}
Z \\
\downarrow \\
X \\
\downarrow \\
W \\
\end{array}
\quad
\begin{array}{c}
Y \\
\downarrow \\
\end{array}
\]

In other words, the morphisms $\text{Corr}_k(X, Y)$ now form the $\infty$-category $\text{Stacks}_{X \times Y}$ of stacks over $X \times Y$ with arbitrary morphisms rather than isomorphisms as in $\text{Corr}_k(X, Y)$.

We will also have need to restrict the class of morphisms of correspondences to some subcategory of $\text{Stacks}_{X \times Y}$. In particular, we will consider the subcategory $\text{Corr}^{pr}_k$ in which we only allow proper maps of correspondences.

### 3.2. Traces of correspondences

Given a map $Z \to X$, it is convenient to introduce the symmetric presentation of the based loop space

$$L_Z X = Z \times_{Z \times X} Z$$

Note the two natural identification with the traditional based loop space

$$\mathcal{L} X \times_X Z \simeq X \times_{X \times X} Z \xleftarrow{\sim} Z \times_{Z \times X} Z \xrightarrow{\sim} Z \times_{X \times X} X \simeq Z \times X \mathcal{L} X$$

There is a natural rotational equivalence $L_X \times_X Z \simeq Z \times X \mathcal{L} X$ that makes the above two identifications coincide. (It does not preserve base points and is not given by swapping the factors). Thus we can unambiguously identify all of the above versions of the based loop space.

**Proposition 3.1.** (1) Any derived stack $X$ is dualizable as an object of $\text{Corr}_k$, with dual $X^\vee$ identified with $X$ itself, and dimension $\dim(X)$ identified with the loop space

$$\mathcal{L} X = X^{S^1} \simeq X \times_{X \times X} X$$

regarded as a self-correspondence of $pt = \text{Spec} \mathbb{k}$.

(2) The transpose of any correspondence $X \leftarrow Z \to Y$ is identified with the reverse correspondence $Y \leftarrow Z \to X$. The trace of a self-correspondence $X \leftarrow Z \to X$ is identified with the based loop space

$$\text{Tr}(Z) \simeq Z|_{\Delta_X} = Z \times_{X \times X} X \simeq \mathcal{L}_Z X$$

regarded as a self-correspondence of $pt = \text{Spec} \mathbb{k}$.

In particular, the trace of the graph $\Gamma_f \to X \times X$ of a self-map $f : X \to X$ is identified with the fixed point locus

$$\text{Tr}(f) \simeq \Gamma_f|_{\Delta_X} = \Gamma_f \times_{X \times X} X \simeq X^f$$

**Proof.** The evaluation and coevaluation presenting the self-duality of $X$ are both given by $X$ itself as a correspondence between $pt = \text{Spec} \mathbb{k}$ and $X \times X$ via the diagonal map. The standard identities follow from the calculation of the fiber product of the two diagonal maps

$$X_{\Delta_{12}} \times_{X \times X \times X} \Delta_{23} X \simeq X$$
Thus the dimension of $X$ is the loop space

$$
\xymatrix{
\mathcal{L}X & \ar[l]_X \ar[r]^{X} & \ar[d]_{pt} \ar[l]_{X \times X} \ar[r]^{pt} & \ar[d]^{X} \\
Y & \ar[l]_{Y \times Z} \ar[r]^{Z \times X} & \ar[d]_{Y \times X} \ar[l]_{Y \times Z \times X} \ar[r]^{Y \times X} & \ar[d]^{Y \times X} \\
Y & Y \times X \times X & Y \times Y \times X & X}
$$

By definition, the transpose of a correspondence $X \leftarrow Z \rightarrow Y$ is identified with $Y \leftarrow Z \rightarrow X$ by checking the definition

$$
\xymatrix{
Z \times_{Z \times X} Z & \ar[l]_Z \ar[r]^{Z \times X} & \ar[l]_{Z} \ar[r]^{X} & \ar[l]_{pt} \ar[r]^{X \times X} & \ar[l]_{X} \ar[r]^{pt} & \ar[l]_{X}}
$$

The trace of a self-correspondence $X \leftarrow Z \rightarrow X$ is then calculated by the composition

$$
\xymatrix{
Z \times_{Z \times X} Z & \ar[l]_Z \ar[r]^{Z \times X} & \ar[l]_{Z} \ar[r]^{X} & \ar[l]_{pt} \ar[r]^{X \times X} & \ar[l]_{X} \ar[r]^{pt} & \ar[l]_{X}}
$$

Finally, the case of the graph $Z = \Gamma_f$ of a self-map gives the fixed point locus by definition.

\[\square\]

Remark 3.2 (Cyclic version). The identification $\dim(X) \simeq \mathcal{L}X$ above is naturally $S^1$-equivariant for the standard loop rotation on $\mathcal{L}X$ and the cyclic symmetry of $\dim(X)$ provided by the cobordism hypothesis. To see this it is useful to consider $X$ as an $E_{\infty}$-algebra object in $\text{Stacks}^{op}_k$ via the diagonal map (or as an $E_n$-object for any $n$). In other words, for $n = 1$ we identify stacks and correspondences with objects and morphisms in the Morita category $\text{Alg}$(Stacks$^{op}_k$).

It follows from the properties of topological chiral homology [L2, Theorem 5.3.3.8] that for a (constant) commutative algebra $A$ its topological chiral homology over a manifold is given by the tensoring of commutative algebras over simplicial sets $\int_M A = M \otimes A$. In particular (passing back from the opposite category to stacks) we have $\int_{S^1} X = X^{S^1} = \mathcal{L}X$. We also know from [L2, Example 5.3.3.14] or [L3, Example 4.2.2] that the $S^1$-action on the dimension of an associative algebra $A$ is given by the standard cyclic structure on its Hochschild simplicial
object, or equivalently the rotation $S^1$-action on the topological chiral homology $\int_{S^1} A$. In our case this recovers the rotation action on the loop space.

### 3.3. Geometric functoriality of dimension.

**Proposition 3.3.** The graph $X \leftarrow \Gamma_f \rightarrow Y$ of any morphism (respectively, any proper morphism) $f : X \rightarrow Y$ gives a continuous morphism $F : X \rightarrow Y$ in $\text{Corr}_k$ (respectively, in $\text{Corr}^{pr}$), with right adjoint $F^r : Y \rightarrow X$ identified with the opposite correspondence $Y \leftarrow \Gamma_f \rightarrow X$.

**Proof.** We construct the unit and counit of the adjunction as follows. Consider the composition $F^r F : X \rightarrow X$ of correspondences

\[
\begin{array}{ccc}
X \times_Y X & \xrightarrow{\Gamma_f} & X \\
\downarrow & & \downarrow \\
X & \xrightarrow{\eta_f} & Y \\
\end{array}
\]

The unit $\eta_f : \text{id}_X = X \rightarrow F^r F \simeq X \times_Y X$ is given by the relative diagonal map.

Consider the opposite composition of correspondences

\[
\begin{array}{ccc}
X \times_X X & \xleftarrow{\Gamma_f} & X \\
\downarrow & & \downarrow \\
Y & \xleftarrow{\epsilon_f} & X \\
\end{array}
\]

The counit $\epsilon_f : FF^r \simeq X \rightarrow \text{id}_Y = Y$ is given by $f$ itself.

The standard identities are easily verified by identifying the resulting composite map

\[
\begin{array}{ccc}
\Gamma_f & \xrightarrow{\Gamma_f \times_Y \Gamma_f \times_X \Gamma_f} & \Gamma_f \\
\end{array}
\]

of correspondences with the identity. \hfill \Box

**Lemma 3.4.** Let $F_Z : X \rightarrow Y$ and $F_W : Y \rightarrow X$ be morphisms in $\text{Corr}_k$ given by respective correspondences $X \leftarrow Z \rightarrow Y$ and $Y \leftarrow W \rightarrow X$. Then the canonical equivalence

\[
m(F_W, F_Z) : \text{Tr}(F_W \circ F_Z) \xrightarrow{\sim} \text{Tr}(F_Z \circ F_W)
\]

is given by the composition of evident geometric identifications

\[
(Z \times_Y W) \times_{X \times X} X \xrightarrow{\sim} W \times_{X \times Y} Z \xrightarrow{\sim} Z \times_{Y \times X} W \xrightarrow{\sim} (W \times_X Z) \times_{Y \times Y} Y
\]
Proof. Returning to the definition and using our previous identifications, observe that $m(F_Z, F_W)$ is calculated by commutativity of the diagram of correspondences

Following the top edge, we see $\text{Tr}(F_W \circ F_Z) \simeq (Z \times_Y W) \times_{X \times X} X$. Following the bottom edge, we see $\text{Tr}(F_Z \circ F_W) \simeq (W \times_X Z) \times_{Y \times Y} Y$. Moving from the top to bottom edge via the successive equivalences of the three commuting squares, one finds the three successive equivalences in the assertion of the lemma.

**Proposition 3.5.** Suppose $f : X \to Y$ is a morphism (respectively, proper morphism), and $F : X \to Y$ denotes the induced morphism in $\text{Corr}_k$ (respectively, in $\text{Corr}^{pr}_k$) given by the graph $X \leftarrow \Gamma_f \to Y$. Then $\text{dim}(F) : \text{dim}(X) \to \text{dim}(Y)$ is canonically identified with the $S^1$-equivariant morphism $\mathcal{L}f : \mathcal{L}X \to \mathcal{L}Y$.

**Proof.** Denote by $F^\ast : Y \to X$ the right adjoint to $F$. We must calculate

$$
\text{dim}(X) \xrightarrow{\text{Tr}(F^\ast F)} \text{dim}(Y)
$$

We have seen that the first and third morphisms correspond to the natural geometric maps

$$
\mathcal{L}X \simeq X \times_{X \times X} X \xrightarrow{(X \times_Y X) \times_{X \times X} X} X \times_{Y \times Y} Y \xrightarrow{Y \times_{Y \times Y} Y} \mathcal{L}Y
$$

induced by the relative diagonal $X \to X \times_Y X$ and given map $f : X \to Y$ respectively. Furthermore, by Lemma 3.3, the middle map is the natural geometric identification

$$
(X \times_Y X) \times_{X \times X} X \xrightarrow{\sim} X \times_{Y \times Y} Y
$$

Altogether, the composition is easily identified with the loop map $\mathcal{L}f : \mathcal{L}X \to \mathcal{L}Y$.

**Remark 3.6.** It follows from the proposition that the loop map $\mathcal{L}f : \mathcal{L}X \to \mathcal{L}Y$ must be proper when the given map $f : X \to Y$ is proper. Let us note why this is true geometrically from the factorization $\mathcal{L}X \to \mathcal{L}_X Y \to \mathcal{L}Y$ appearing in the proof.

First, the natural morphism $\mathcal{L}X \to \mathcal{L}_X Y$ is the restriction along the diagonal $X \to X \times X$ of the relative diagonal $X \to X \times_Y X$. The relative diagonal is a closed embedding since $f$ is proper, and hence the natural morphism $\mathcal{L}X \to \mathcal{L}_X Y$ is as well. Second, the natural morphism $\mathcal{L}_X Y \to \mathcal{L}Y$ is the restriction along the diagonal $Y \to Y \times Y$ of the proper morphism $f : X \to Y$ and thus is proper as well. Altogether, we see that $\mathcal{L}f : \mathcal{L}X \to \mathcal{L}Y$ is itself proper.

**Remark 3.7.** One can invoke the cobordism hypothesis with singularities to endow the morphism $\text{dim}(F) : \text{dim}(X) \to \text{dim}(Y)$ with a canonical $S^1$-equivariant structure, and it will agree with the canonical geometric $S^1$-equivariant structure on the map $\mathcal{L}f : \mathcal{L}X \to \mathcal{L}Y$ under the identification of the proposition.
3.4. Geometric functoriality of trace. Consider a proper morphism $f : X \to Y$ and endomorphisms $F_Z : X \to X$ and $F_W : Y \to Y$ in $\text{Corr}_k$ given by respective self-correspondences $X \leftarrow Z \to X$ and $Y \leftarrow W \to Y$.

By an $f$-morphism from the pair $(X, F_Z)$ to the pair $(Y, F_W)$, we mean an identification $s : Z \overset{\sim}{\longrightarrow} X \times_Y W$ of correspondences from $X$ to $Y$. This in turn induces an identification of what might be called relative traces $Z \times_{Y \times Y} Y \overset{\sim}{\longrightarrow} X \times_{Y \times Y} W$ generalizing the relative loop space $L_X Y$ from the case of the identity correspondences $Z = X$, $W = Y$. We thus obtain a map of traces

$$\tau(f, s) : Z|_{\Delta_X} \longrightarrow W|_{\Delta_Y}$$

Proposition 3.8. With the preceding setup, the trace map $\text{Tr}(f, s) : \text{Tr}(F_Z) \to \text{Tr}(F_W)$ is canonically identified with the geometric map

$$\tau(f, s) : Z|_{\Delta_X} \longrightarrow W|_{\Delta_Y}$$

Proof. Denote by $F : X \to Y$ the morphism given by the graph $X \leftarrow \Gamma_f \to Y$, and by $F^r : Y \to X$ its right adjoint. We must calculate

$$\text{Tr}(F_Z) \longrightarrow \text{Tr}(F^r F F_Z) \overset{s}{\longrightarrow} \text{Tr}(F^r F W F) \overset{m(F^r, F W F)}{\longrightarrow} \text{Tr}(F W F F^r) \longrightarrow \text{Tr}(F_W)$$

We have seen that the first and fourth morphisms correspond to the natural geometric maps

$$Z|_{\Delta_X} = Z \times_{X \times X} X \longrightarrow Z \times_{X \times X} (X \times_Y X)$$

$$X \times_{Y \times Y} W \longrightarrow Y \times_{Y \times Y} W = W|_{\Delta_Y}$$

induced by the relative diagonal $X \to X \times_Y X$ and given map $f : X \to Y$ respectively.

Using associativity, the second map, induced by $s$, is the natural geometric identification

$$Z \times_{X \times X} (X \times_Y X) \simeq Z \times_{X \times Y} Y \overset{\sim}{\longrightarrow} W \times_{Y \times Y} X$$

By Lemma 3.4 the third map, given by the cyclic symmetry, is nothing more than the natural identification

$$W \times_{Y \times Y} X \simeq X \times_{Y \times Y} W$$

Thus assembling the above maps we arrive at the composition defining $\tau(f, s)$. □

4. Traces for sheaves

In this section, we spell out how to apply the abstract formalism of traces of Section 2 and its geometric incarnation of Section 3 to categories sheaves. As explained in the introduction, the broad idea is as follows. Suppose given a symmetric monoidal functor

$$\mathcal{S} : \text{Corr}_k \longrightarrow \text{dgCat}_k$$

from the correspondence 2-category to dg categories. Applying it to the geometric descriptions of traces of correspondences, one immediately deduces trace formulas for dg categories.

Since the natural setting of 2-categories is not fully mapped in the literature, we work instead with 1-categories and formulate the additional structures needed to deduce the main results.
We adopt the terminology and notation of the introduction: a sheaf theory is a symmetric monoidal functor

$$S : Corr_k \longrightarrow dgCat_k$$

from the correspondence category to dg categories.

The graph of a map of derived stacks $f : X \rightarrow Y$ provides a correspondence from $X$ to $Y$ and a correspondence from $Y$ to $X$. We denote the respective induced maps by $f_\ast : S(X) \rightarrow S(Y)$ and $f^! : S(Y) \rightarrow S(X)$. The functoriality of $S$ concisely encodes base change for $f_\ast$ and $f^!$. For $\pi : X \rightarrow pt = Spec k$, we denote by $\omega_X = \pi^!k \in S(X)$ the $S$-analogue of the dualizing sheaf, and by $\omega(X) = \pi_*\omega_X \in S(pt) = dgVect_k$ the $S$-analogue of “global volume forms”.

Next we will record formal consequences of our prior calculations deduced from the fact that a sheaf theory is symmetric monoidal.

**Proposition 4.1.** Fix a sheaf theory $S : Corr_k \rightarrow dgCat_k$, and $X,Y \in Corr_k$.

1. $S(X) \in dgCat_k$ is canonically self-dual, and for any $f : X \rightarrow Y$, $f^! : S(Y) \rightarrow S(X)$ and $f_\ast : S(X) \rightarrow S(Y)$ are canonically transposes of each other.

2. $S(X)$ is canonically symmetric monoidal with tensor product

$$F \otimes^! G = \Delta_! (\pi_1^! F \otimes \pi_2^! G) \quad F,G \in S(X)$$

3. For any $f : X \rightarrow Y$, the projection formula holds:

$$f_\ast F \otimes^! G \simeq f_\ast (F \otimes^! f^! G) \quad F \in S(X), G \in S(Y)$$

4. There is a canonical equivalence of functors and integral kernels

$$Hom_{dgCat_k}(S(X), S(Y)) \simeq S(X \times Y)$$

5. The functor $q_*p^! : S(X) \rightarrow S(Y)$ associated to a correspondence

$$X \overset{p}{\longrightarrow} Z \overset{q}{\longrightarrow} Y$$

is represented by the integral kernel $(p \times q)_! \omega_Z \in S(X \times Y)$.

**Proof.**

1. Follows immediately from Proposition 3.1.

2. Follows immediately from the commutative algebra structure on $X \in Corr_k$ (or equivalently, commutative coalgebra structure on $X \in Stacks_k$) provided by the diagonal map.

3. Follows from base change for the diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ id \times f \downarrow & & \downarrow \Delta \\ X \times Y & \xrightarrow{f \times id} & Y \times Y \end{array}$$

4. Since $S$ is monoidal, we have

$$S(X) \otimes S(Y) \simeq S(X \times Y)$$

The self-duality of $S(X)$ provides

$$Hom_{dgCat_k}(S(X), S(Y)) \simeq S(X)^\vee \otimes S(Y) \simeq S(X) \otimes S(Y)$$

By construction, the composite identification assigns the functor

$$F_K(F) = \pi_2^!(\pi_1^! F \otimes^! K) \quad K \in S(X \times Y)$$
(5) Follows from the projection formula: consider the diagram

\[
\begin{array}{ccc}
Z & \xrightarrow{p} & X \\
\downarrow{\pi_1} & & \downarrow{\pi_2} \\
X & \xrightarrow{\pi} & X \times Y \\
\downarrow{q} & & \downarrow{}
\end{array}
\]

where \( \Pi = p \times q \). Then we have

\[
q_* p^!(-) \simeq \pi_2_* \Pi_* \Pi^! \pi_1^!(-) \simeq \pi_2_* (\omega_Z \otimes \Pi^! \pi_1^!(-)) \simeq \pi_2_* (\Pi_* \omega_Z \otimes \Pi^! \pi_1^!(-))
\]

\[\square\]

**Proposition 4.2.** Fix a sheaf theory \( S : \text{Corr}_k \to \text{dgCat}_k \).

1. The \( S \)-dimension \( \dim(S(X)) = HH_*(S(X)) \) of any \( X \in \text{Corr}_k \) is \( S^1 \)-equivariantly equivalent with \( S \)-global volume forms on the loop space

\[
\dim(S(X)) \simeq \omega(LX).
\]

In particular, for \( G \) an affine algebraic group, characters of \( S \)-valued \( G \)-representations are adjoint-equivariant \( S \)-global volume forms

\[
\dim(S(BG)) \simeq \omega(G/G)
\]

2. The \( S \)-trace of any endomorphism \( Z \in \text{Corr}_k(X,X) \) is equivalent to \( S \)-global volume forms on the restriction to the diagonal

\[
\text{Tr}(S(Z)) \simeq \omega(Z|\Delta)
\]

In particular, the \( S \)-trace of a self-map \( f : X \to X \) is equivalent to \( S \)-global volume forms on the \( f \)-fixed point locus

\[
\text{Tr}(f_*) \simeq \omega(X^f)
\]

**Proof.** (1) Follows immediately from Proposition 3.3(1). To spell this out, using the previous proposition and base change, \( \dim(S(X)) \) results from applying the composition

\[
\pi_* \Delta^! \Delta_* \pi^! \simeq \pi_* p_{12}^! \pi_{12}^! \simeq \mathcal{L} \pi_* \mathcal{L} \pi^! : \text{dgVect}_k \xrightarrow{\text{equivalent}} \text{dgVect}_k
\]

to the unit \( 1_{\text{dgVect}_k} = k \). Here \( \pi : X \to pt \) and \( \mathcal{L} \pi : \mathcal{L}X \to pt \) are the maps to the point, and \( p_1, p_2 : LX \simeq X \times X \times X \xrightarrow{\text{projections}} X \) are the two natural projections. Thus we find

\[
\dim(S(X)) \simeq \mathcal{L} \pi_* \mathcal{L} \pi^!(k) \simeq \omega(LX).
\]

Furthermore, the \( S^1 \)-equivariance results from the one-dimensional cobordism hypothesis: the one-dimensional topological field theory defined by the dualizable object \( S(X) \in \text{dgCat}_k \) factors through that defined by the dualizable object \( X \in \text{Corr}_k \). Moreover, we identified the \( S^1 \)-action on the dimension \( LX \) with loop rotation.

(2) Similarly follows immediately from Proposition 3.3(2). \[\square\]

**4.1. Integration formulas for traces.** Now we turn to the functoriality of dimensions and traces. We continue with the setting of a sheaf theory \( S : \text{Corr}_k \to \text{dgCat}_k \), but now enhance it with further structure (that would be most naturally formulated by a symmetric monoidal functor \( S : \text{Corr}_k^{op} \to \text{dgCat}_k \) from the correspondence 2-category with proper morphisms).

For a proper map \( f : X \to Y \), we assume the induced maps \( f_* : S(X) \to S(Y), f^! : S(Y) \to S(X) \) are equipped with the data of an adjunction

\[
S(X) \xrightleftharpoons{f_*} f^! S(Y)
\]

\[\text{3One can also check directly that the cyclic structure on the cyclic bar construction of the dg category } S(X) \text{ is induced by the cyclic structure of the loop space } LX \text{ under the identification } \omega(LX) \simeq \dim(S(X)).\]
compatible with compositions and base change as stated below. (We do not specify the evident higher coherences the compatibilities should satisfy, since they are not needed for the constructions we consider. Such higher coherences would be implicit in the 2-categorical setting.)

**Definition 4.3.** A *proper sheaf theory* is a sheaf theory $\mathcal{S} : \text{Corr}_k \to \text{dgCat}_k$ equipped with the following additional data:

1. For $f : X \to Y$ proper, an identification of $f^!$ with the right adjoint of $f_*$
2. Compatibility of the above adjunctions with composition: for $f : X \to Y$ and $g : Y \to Z$ proper, an identification of the adjunction data for the composition $g \circ f$ with the composition of the adjunction data for $g$ and $f$.
3. Compatibility of the above adjunctions with base change: suppose given a fiber square

$$
\begin{array}{ccc}
Z & \xrightarrow{g} & W \\
\downarrow{p} & & \downarrow{q} \\
X & \xrightarrow{f} & Y
\end{array}
$$

with associated base change equivalence

$$
\beta : g_* p^! \sim \eta^q q f_*
$$

If $f$ (hence $g$) is proper, then $\beta$ is induced by the $(f_*, f^!)$ and $(g_*, g^!)$ adjunctions from the equivalence

$$
p^! f^! \sim g^! q^!
$$

Namely, it is equivalent to the composition

$$
g_* p^! \xrightarrow{\eta q} g_* p^! f_* \sim g_* g^! q f_* \xrightarrow{\epsilon q f_*} q f_*
$$

If $q$ (hence $p$) is proper, then $\beta$ is induced by the $(p_*, p^!)$ and $(q_*, q^!)$ adjunctions from the equivalence

$$
q_* g_* \sim f_* p_!
$$

Namely, it is equivalent to the composition

$$
g_* p^! \xrightarrow{\eta q} q_* g_* p^! \sim q^! f_* p^! \sim q f_*
$$

We have the following evident compatibility within the above definition.

**Lemma 4.4.** If in the base change diagram of the above definition all the maps are proper, then the following two diagrams commute

$$
\begin{array}{ccc}
q_* g_* p^! & \xrightarrow{q_*(\beta)} & q_* q^! f_* \\
\sim & \downarrow{\epsilon q f_!} & \downarrow{\epsilon q f_*} \\
q_* g_* p^! & \xrightarrow{f_*(\epsilon p)} & f_*
\end{array}
\quad
\begin{array}{ccc}
g_* p^! f^! & \xrightarrow{\beta \text{id}} & q^! f_* f^! \\
\sim & \downarrow{\epsilon p q^! f} & \downarrow{\epsilon q f_*} \\
g_* p^! f^! & \xrightarrow{\epsilon q f_!} & q^!
\end{array}
$$

Proof. We spell out the verification of the first diagram, the latter is similar.

$$
\begin{array}{ccc}
q_* g_* p^! & \xrightarrow{1 \otimes \eta q} & q_* q^! g_* p^! \\
\sim & \downarrow{\epsilon q \otimes 1} & \downarrow{\epsilon q \otimes 1} \\
q_* g_* p^! & \xrightarrow{\eta p} & q_* q^! f_* p^! \\
\sim & \downarrow{\epsilon q \otimes 1} & \downarrow{\epsilon q \otimes 1} \\
q_* g_* p^! & \sim & f_* p_* p^! \\
\sim & \downarrow{1 \otimes \epsilon p} & \downarrow{1 \otimes \epsilon p} \\
q_* g_* p^! & \sim & f_*
\end{array}
$$
The left triangle expresses the standard composition identity for adjunctions, while the two squares follow from the interchange identity for natural transformations.

### Proposition 4.5

Fix a proper sheaf theory $S : \text{Corr}_k \to \text{dgCat}_k$.

1. A proper map $f : Z \to W$ of correspondences

   \[
   \begin{array}{ccc}
   Z & \xrightarrow{q_Z} & W \\
   X & \xleftarrow{p_Z} & f & \xrightarrow{p_W} & Y \\
   \end{array}
   \]

   induces a canonical integration morphism of integral transforms

   \[
   \int_f : q_Z \cdot p_Z^! \longrightarrow q_W \cdot p_W^!
   \]

   In particular, when $X = Y = \text{pt}$, it induces a map of global volume forms

   \[
   \int_f : \omega(Z) \longrightarrow \omega(W)
   \]

2. There is a canonical composition identity

   \[
   \int_g \circ \int_f \simeq \int_{g \circ f}
   \]

3. For a proper map $f : X \to Y$, the unit and counit of the $(f_*, f^!)$ adjunction are given respectively by integration along the proper maps of self-correspondences $\Delta_{/Y} : X \to X \times_Y X$ of $X$ and $f_{/Y} : X \to Y$ of $Y$.

### Proof

(1) We take the integration map to be the composition

\[
\int_f : q_Z \cdot p_Z^! \xrightarrow{\sim} q_W \cdot f_* f^! p_W^! \xrightarrow{\epsilon_f} q_W \cdot p_W^!
\]

where the second arrow is the counit of the $(f_*, f^!)$ adjunction.

In particular, when $X = Y = \text{pt}$, it takes the form

\[
\int_f : \omega(Z) = \pi_X \cdot \pi_X^! \xrightarrow{\sim} \pi_W \cdot f_* f^! \pi_W^! = \pi_W \cdot \pi_W^! = \omega(W)
\]
(2) The compatibility with composition follows immediately from the assumed compatibility of the adjunction data: we have a commutative diagram

\[
\begin{array}{ccc}
(g \circ f)_* (g \circ f)^! & \xrightarrow{\sim} & g_* f_* f^! \\
\epsilon_{g \circ f} & & \epsilon_f \\
\text{id} & & g_* g^!
\end{array}
\]

(3) The identification of the counit and integration map \( \epsilon_f \simeq \int f \colon f_* f^! \to \text{id}_{S(Y)} \) follows from the definition of the latter:

\[
\int f : f_* f^! \xrightarrow{\sim} \text{id}_{Y^*} f_* f^! \xrightarrow{\epsilon_f} \text{id}_{Y^*} \text{id}_{Y} = \text{id}_{S(Y)}
\]

To identify the unit and integration map \( \eta_f \simeq \int \Delta f : \text{id}_{S(Y)} \to f^! f_* \), we will verify that the latter satisfies the characterizing identities of the former, using the previously checked identity of the counit \( \epsilon_f \simeq \int f \).

First, let us confirm the following composition is the identity

\[
f_* \xrightarrow{\text{id} \circ \int \Delta f} f_* f^! f_* \xrightarrow{\epsilon_f \text{id}} f_*
\]

By Lemma 4.4 we have a commutative diagram

\[
\begin{array}{ccc}
f_* & \xrightarrow{\int \Delta f} & f_* p_2^! p_1^! f_* \\
\cong & & \epsilon_f \text{id}
\end{array}
\]

The composition we are after appears along the top and right edges. We claim the other way around the diagram is the identity. To see this, note that over \( X \times Y \), we have a diagram

\[
\begin{array}{ccc}
X & \xrightarrow{\Delta f} & X \times_Y X \\
\downarrow \text{id}_X & & \downarrow p_1
\end{array}
\]

It thus follows from the functoriality of integration (proved in (2) above) that the above composition is the identity of \( f_* \) as claimed.

One similarly confirms the other composition is the identity

\[
f^! \xrightarrow{\int \Delta f \circ \text{id}} f^! f_* f^! \xrightarrow{\text{id} \circ \epsilon_f} f^!
\]

In particular, one uses the other commutative diagram of Lemma 4.4 similarly specialized to the current situation

\[
\begin{array}{ccc}
f^! & \xrightarrow{\int \Delta f} & f^! f_* f^! \\
\cong & & f^! (\epsilon_f)
\end{array}
\]

\[\square\]
Theorem 4.6. For any proper map $f : X \to Y$, the induced map on dimensions
\[ \dim(f^*) : \dim(S(X)) \to \dim(S(Y)) \]
is identified ($S^1$-equivariantly) with integration along the loop map
\[ \dim(f^*) \simeq \int_{L_f} : \omega(LX) \to \omega(LY) \]

Proof. According to Definition 2.19, we must calculate the composition
\[ \dim(f^*) : \text{Tr}(\text{id}_{S(X)}) \xrightarrow{\text{Tr}(\eta_f)} \text{Tr}(f^!f^*) \xrightarrow{\text{Tr}(\epsilon_f)} \text{Tr}(\text{id}_{S(Y)}) \]
The equivalence of the middle arrow is given by the canonical identifications
\[ \text{Tr}(f^!f^*) \simeq \omega((X \times_Y X) \times_X X X) \simeq \omega(X \times_X Y Y) \simeq \text{Tr}(f^!f^!) \]
By Proposition 4.5, the unit $\eta_f : \text{id}_{S(X)} \to f^!f^*$ is given by the integration morphism
\[ \int_{\Delta_f} : \Delta_f \omega_X \to \omega_{X \times_Y X} \]
and hence its trace $\text{Tr}(\eta_f) : \text{Tr}(\text{id}_{S(X)}) \to \text{Tr}(f^!f^*)$ is given by the induced integration map
\[ \int_{\Delta_f} : \omega(LX) \to \omega((X \times_Y X) \times_X X X) \]
Likewise, the counit $\epsilon_f : f^*f^! \to \text{id}_{S(Y)}$ is given by the integration morphism
\[ \int_f : f^*\omega_X \to \omega_Y \]
and hence its trace $\text{Tr}(\epsilon_f) : \text{Tr}(f^*f^!) \to \text{Tr}(\text{id}_{S(Y)})$ is given by the induced integration map
\[ \int_f : \omega(X \times_X Y Y \times_Y Y) \to \omega(LY) \]
Finally, by Proposition 4.5, their composition is given by the integration map
\[ \int_{L_f} : \omega(LX) \to \omega(LY) \]
\[ \square \]
Finally, we have the functoriality of traces in parallel with the previous theorem on the functoriality of dimensions. Let us recall the relevant setup. Consider a proper morphism $f : X \to Y$ and endomorphisms $F_Z : X \to X$ and $F_W : Y \to Y$ in $\text{Corr}_k$ given by respective self-correspondences $X \leftarrow Z \to X$ and $Y \leftarrow W \to Y$.
By an $f$-morphism from the pair $(X, F_Z)$ to the pair $(Y, F_W)$, we mean an identification
\[ s : Z \leftarrow X \times_Y W \]
of correspondences from $X$ to $Y$. This in turn induces an identification of what might be called relative traces
\[ Z \times_{Y \times Y} Y \leftarrow X \times_{Y \times Y} W \]
generalizing the relative loop space $L_{X}Y$ from the case of the identity correspondences $Z = X$, $W = Y$. We thus obtain a map of traces
\[ \tau(f, s) : Z|_{\Delta_X} = Z \times_{X \times X} X \leftarrow Z \times_{Y \times Y} Y \leftarrow X \times_{Y \times Y} W \leftarrow Y \times_{Y \times Y} W = W|_{\Delta_Y} \]
Theorem 4.7. With the preceding setup, the trace map $\operatorname{Tr}(f_*, s) : \operatorname{Tr}(F_{X*}) \to \operatorname{Tr}(F_{Y*})$ is canonically identified with the integration map
\[ \int_{\tau(f,s)} : \omega(Z|\Delta_X) \longrightarrow \omega(W|\Delta_Y) \]

Proof. The argument is parallel to the proof of Theorem 4.6. One calculates $\operatorname{Tr}(f_*, \alpha)$ from Definition 2.24 using Proposition 3.8 and the compatibility of Propositions 4.1 and 4.5. □

4.2. Classical applications. We now apply the following theorem of Gaitsgory which allows for the concrete application of our results to traditional questions (partial versions of the result, which suffice for the applications, appear in the work of Gaitsgory and Rozenblyum, in part in collaborations with Drinfeld and Francis [G1, FG, DG, GR1, GR2]).

Theorem 4.8. The assignments of ind-coherent sheaves $X \mapsto \mathcal{Q}^!(X)$ and $D$-modules $X \mapsto \mathcal{D}(X)$ extend to define proper sheaf theories on the correspondence $\infty$-category of quasi-compact stacks with affine diagonal in characteristic zero.

For a compact object $M \in S(X)$, regarded as a continuous morphism $dgVect_k \to S(X)$, we denote by $[M] \in \dim S(X)$ its character.

Corollary 4.9. Grothendieck-Riemann-Roch: For a proper map $f : X \to Y$ and any compact object $M \in S(X)$ with character $[M] \in HH_*(S(X)) \simeq \omega(LX)$, there is a canonical identification
\[ [f_*M] \simeq \int_{L_f} [M] \in HH_*(S(Y)) \simeq \omega(LY) \]
In other words, the character of a pushforward along a proper map is the integral of the character along the induced loop map.

Corollary 4.10. Atiyah-Bott-Lefschetz: Let $G$ be an affine group, and $X$ a proper $G$-derived stack, so equivalently, a proper map $f : X/G \to BG$. Then for any compact object $M \in S(X/G)$, and element $g \in G$, there is a canonical identification
\[ [f_*M]|_g \simeq \int_{L_f} [M]|_{X_g} \]
In other words, under the identification of invariant functions and volume forms on the group, the value of the character of an induced representation at a group element is given by the integral of the original character along the corresponding fixed point locus of the group element.

Proof. Follows from Corollary 4.9 and base change. □

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