GAUSS SUMS OVER SOME MATRIX GROUPS

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Abstract. In this note, we give explicit expressions of Gauss sums for general (resp. special) linear groups over finite fields, which involves Gauss sums (resp. Kloosterman sums). The key ingredient is averaging such sums over Borel subgroups. As applications, we count the number of invertible matrices of zero-trace over finite fields and we also improve two bounds by Ferguson, Hoffman, Luca, Ostafe and Shparlinski in [Some additive combinatorics problems in matrix rings, Rev. Mat. Complut. (23) 2010, 501–513].

1. Introduction

The Gauss sums for classical groups over a finite field have been extensively studied by Kim in several articles [3, 8, 6, 4, 7, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18], and more recently, he also applied the Gauss sum for special linear groups over finite fields to coding theory [19].

Let \( q \) be a power of a prime number \( p \) and \( k = \mathbb{F}_q \) be the finite field with \( q \) elements. Let \( \lambda \) be a fixed nonprincipal additive character of \( \mathbb{F}_q \), e.g., take
\[
\lambda(x) = \exp\left(\frac{2\pi i}{p} \text{tr}_{\mathbb{F}_q/\mathbb{F}_p}(x)\right), \quad \forall x \in \mathbb{F}_q,
\]
and \( \chi \) be a multiplicative character of \( \mathbb{F}_q^* \).

Given two matrices \( U = (u_{ij}), V = (v_{ij}) \in M_n(k) \), their product is defined by
\[
U \cdot V = \sum_{1 \leq i \leq n, 1 \leq j \leq n} u_{ij}v_{ij}.
\]

For \( U \) being a nonzero matrix of \( M_n(k) \), let
\[
G_{GL_n(k)}(U, \chi, \lambda) = \sum_{X \in GL_n(k)} \chi(\det X) \lambda(U \cdot X),
\]
and
\[
G_{SL_n(k)}(U, \lambda) = \sum_{X \in SL_n(\mathbb{F}_q)} \lambda(U \cdot X).
\]

These sums can be viewed as the general linear group and special linear group analogues of classical Gauss sums. For brevity, if \( \chi = 1 \) is trivial, we will write \( G_{GL_n(k)}(U, \lambda) \) instead of \( G_{GL_n(k)}(U, 1, \lambda) \).

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Kim [8] got the formulae of \( G_{GL_n(k)}(I, \chi, \lambda) \) and \( G_{SL_n(k)}(I, \lambda) \) by using the Bruhat decomposition of \( GL_n(k) \) and \( SL_n(k) \). As Kim remarked, these formulae already appeared in the work of Eichler [1] and Lampr echt [22]. (See the introduction of [8]). Fulman [3] also got the same result for \( G_{GL_n(k)}(I, \chi, \lambda) \) by using the technique of generating functions.

In this note, by using orthogonality of characters of finite abelian groups, we present an explicit expressions for the sums (1.2) and (1.3). The main idea in our approach is averaging the sums (1.2) and (1.3) over Borel subgroups, i.e, the group of upper triangular matrices. (See Theorems 2.1 and 2.4 below). As a consequence, we give the upper bounds of the sums (1.2) and (1.3) (in the case \( \chi = 1 \)). (See Corollaries 2.2 and 2.5 below).

By using several results from algebraic geometry, in particular, Sko-robagatov’s estimates of character sums along algebraic varieties [23], Ferguson, Hoffman, Luca, Ostafe and Shparlinski [2] also provided another upper bounds of the sums (1.2) and (1.3) and used them to study some additive combinatorics problems in matrix rings. Their bounds have been applied by us to study some uniform distribution properties of some matrix groups. (See [4]). Our bounds given in this note improve their bounds. (See Remarks 2.3 and 2.6 below).

Finally, as an application, we count the number of invertible matrices of zero-trace over finite fields.

2. Main results

2.1. The case of \( GL_n(k) \). The equation (1.2) can be rewritten as

\[
G_{GL_n(k)}(U, \chi, \lambda) = \sum_{X \in GL_n(k)} \chi(\det X) \lambda(\text{tr} \ U^t X),
\]

where \( U^t \) is the transpose of \( U \) and “\( \text{tr} \)” stands for the trace of the matrix.

Replacing \( U \) by \( PUQ \) in (2.1) with \( P, Q \in GL_n(k) \), we get

\[
G_{GL_n(k)}(PUQ, \chi, \lambda) = \sum_{X \in GL_n(k)} \chi(\det X) \lambda(\text{tr} \ P^t U^t P X^t)
\]

\[
= \chi(\det PQ) \sum_{X \in GL_n(k)} \chi(\det P^t XQ^t) \lambda(\text{tr} \ U^t P X^t)
\]

\[
= \chi(\det PQ) G_{GL_n(k)}(U, \chi, \lambda).
\]

Therefore,

\[
G_{GL_n(k)}(U, \chi, \lambda) = \chi(\det PQ) G_{GL_n(k)}(PUQ, \chi, \lambda).
\]

Let \( u \) be the rank of \( U \). There exist \( P, Q \in GL_n(k) \) such that

\[
PUQ = \begin{pmatrix} I_u & 0 \\ 0 & 0 \end{pmatrix},
\]
where $I_u$ is the $u \times u$ identity matrix. If $u < n$, additionally, we can also require $P, Q \in \text{SL}_n(k)$.

Combining equations (2.3) and (2.4), we get

\begin{equation}
G_{\text{GL}_n(k)}(U, \chi, \lambda) = \begin{cases} 
\bar{\chi}(\det U) \sum_{X \in \text{GL}_n(k)} \chi(\det X) \lambda(\text{tr}_u X) & \text{if } u = n \\
\sum_{X \in \text{GL}_n(k)} \chi(\det X) \lambda(\text{tr}_u X) & \text{if } u < n
\end{cases}
\end{equation}

where

\begin{equation}
\text{tr}_u X = \sum_{i=1}^{u} x_{ii}, \text{ for } X = (x_{ij}) \in \text{M}_n(k).
\end{equation}

So it suffices to calculating the sum

\begin{equation}
\sum_{X \in \text{GL}_n(k)} \chi(\det X) \lambda(\text{tr}_u X), \text{ for } 1 \leq u \leq n.
\end{equation}

Let $B_n(k)$ be the Borel subgroup of $\text{GL}_n(k)$, i.e., the group of upper triangular invertible matrices. The following is the key step of our approach.

\begin{equation}
\sum_{X \in \text{GL}_n(k)} \chi(\det X) \lambda(\text{tr}_u X) = \frac{1}{(q-1)^n q^{n/2}} \sum_{B \in B_n(k)} \sum_{X \in \text{GL}_n(k)} \chi(\det BX) \lambda(\text{tr}_u BX) = \sum_{X \in \text{GL}_n(k)} \chi(\det X) \lambda(\text{tr}_u BX)
\end{equation}

\begin{equation}
= \sum_{(x_{ij}) \in \text{GL}_n(k)} \chi(\det (x_{ij})) \frac{1}{(q-1)^n q^{n/2}} \sum_{b_{ij} \in B_n(k)} \chi(b_{ij}) \lambda(\sum_{i \leq j} b_{ij} x_{ji}) \lambda(\sum_{i \leq j} b_{ij} x_{ji})
\end{equation}

\begin{equation}
= \sum_{(x_{ij}) \in \text{GL}_n(k)} \chi(\det (x_{ij})) \prod_{i=1}^{n} \sum_{b_{ii} \neq 0} \chi(b_{ii}) \lambda(b_{ii} x_{ii}) \cdot \prod_{i=u+1}^{n} \sum_{b_{ii} \neq 0} \chi(b_{ii}) \cdot \prod_{i<j} \sum_{b_{ij} \in k} \lambda(b_{ij} x_{ji}) \cdot \prod_{i>j} \sum_{b_{ij} \in k} \lambda(b_{ij} x_{ji})
\end{equation}

If $\chi$ is not principal and $u < n$, then we have

\begin{equation}
\sum_{X \in \text{GL}_n(k)} \chi(\det X) \lambda(\text{tr}_u X) = 0
\end{equation}

because

\begin{equation}
\sum_{b_{ii} \neq 0} \chi(b_{ii}) = 0, \text{ for } u + 1 \leq i \leq n.
\end{equation}

So we only need to consider the remaining two cases: $u = n$ or $\chi = 1$. 
Firstly, assume $u = n$. The sum (2.8) equals to

\[(2.10) \sum_{(x_{ij}) \in GL_n(k)} \frac{\chi (\text{det} (x_{ij}))}{(q - 1)^n q^{\binom{n}{2}} u} \prod_{i=1}^{n} \sum_{b_{ii} \neq 0} \chi (b_{ii}) \lambda (b_{ii} x_{ii}) \cdot \prod_{i < j, b_{ij} \in k} \sum_{b_{ij} \in k} \chi (b_{ij} x_{ji}) \cdot \prod_{i<j} \sum_{b_{ij} \in k} \lambda (b_{ij} x_{ji}). \]

For $i < j$,

\[\sum_{b_{ij} \in k} \lambda (b_{ij} x_{ji}) \neq 0 \text{ if and only if } x_{ji} = 0.\]

Therefore, (2.10) equals to

\[(2.11) \sum_{(x_{ij}) \in B_n(k)} \frac{1}{(q - 1)^n} \prod_{i=1}^{n} \sum_{b_{ii} \neq 0} \chi (b_{ii}) \lambda (b_{ii} x_{ii})\]

So we get

\[(2.12) \sum_{X \in GL_n(k)} \chi (\text{det} X) \lambda (\text{tr}_u X) = q^{\binom{n}{2}} G(\chi, \lambda)^n,\]

where

\[(2.13) G(\chi, \lambda) = \sum_{x \in k^*} \chi (x) \lambda (x)\]

is the classical Gauss sum for $k = \mathbb{F}_q$.

Secondly, assume $\chi$ is principal. The sum (2.8) equals to

\[(2.14) \sum_{(x_{ij}) \in GL_n(k)} \frac{1}{(q - 1)^u q^{\binom{n}{2}}-(\frac{n-u}{2})} \prod_{i=1}^{n} \sum_{b_{ii} \neq 0} \lambda (b_{ii} x_{ii}) \cdot \prod_{i < j, b_{ij} \in k} \sum_{b_{ij} \in k} \lambda (b_{ij} x_{ji})\]

The terms in the summation (2.14) over $X = (x_{ij}) \in GL_n(k)$ are nonzero if and only if

\[(2.15) X = \begin{pmatrix} A & B \\ 0 & D \end{pmatrix}, \text{ for } A \in B_u(k) \text{ and } D \in GL_{n-u}(k).\]

In that case, they all equal to

\[\frac{(-1)^u}{(q - 1)^u}.\]

Therefore,

\[(2.16) \sum_{X \in GL_n(k)} \chi (\text{det} X) \lambda (\text{tr}_u X)\]

\[= \frac{(-1)^u}{(q - 1)^u} \# \{(A, B, D) \mid A \in B_u(k), B \in M_{u \times (n-u)}(k), D \in GL_{n-u}(k)\}\]

\[= \frac{(-1)^u}{(q - 1)^u} \left((q - 1)^u q^{\binom{n}{2}}-(\frac{n-u}{2})\right) \cdot \prod_{i=0}^{n-u-1} (q^{n-u} - q^i)\]

\[= q^{\binom{n}{2}}(-1)^u \prod_{i=1}^{n-u} (q^i - 1).\]
Putting equations (2.5), (2.9), (2.12) and (2.16) all together, we get

**Theorem 2.1.** Let \( u \) be the rank of \( U \) and \( \lambda \) be nontrivial. Then

\[
G_{\text{GL}_n(k)}(U, \chi, \lambda) = \begin{cases} 
\overline{\chi}(\det U)q^{\binom{n}{2}}G(\chi, \lambda)^n & \text{if } u = n, \\
(-1)^u q^{\binom{n}{2}} \prod_{i=1}^{n-u} (q^i - 1) & \text{if } \chi = 1, \\
0 & \text{if } u < n, \chi \neq 1,
\end{cases}
\]

where \( G(\chi, \lambda) \) is the classical Gauss sum defined in (2.13).

Letting \( u = 1 \) in Theorem 2.1, we get the following corollary.

**Corollary 2.2.** Uniformly over all nonzero matrices \( U \in M_n(\mathbb{F}_q) \) and nontrivial additive characters \( \lambda \), we have

\[
G_{\text{GL}_n(k)}(U, \lambda) = O(q^{n^2-n}),
\]

where the implied constant in the symbol “\( O \)” depends only on \( n \).

**Remark 2.3.** Ferguson, Hoffman, Luca, Ostafe and Shparlinski in [2] obtained \( G_{\text{GL}_n(k)}(U, \lambda) = O(q^{n^2-5/2}) \). (See Lemma 3 of [2]). Corollary 2.2 improves their bounds. Moreover, from our proof, it is easily seen that the bound \( O(q^{n^2-n}) \) can not be improved. So Lemma 3 of [2] should be strengthened as

\[
G_{\text{GL}_n(k)}(U, \lambda) = O(q^{n^2-5/2}), \quad \text{for } n > 2.
\]

### 2.2. The case of \( \text{SL}_n(k) \)

If \( P, Q \in \text{SL}_n(k) \), the same argument as (2.2) shows that

\[
G_{\text{SL}_n(k)}(U, \lambda) = G_{\text{SL}_n(k)}(PUQ, \lambda).
\]

If \( u < n \), from (2.1), we can assume \( P, Q \in \text{SL}_n(k) \), then

\[
G_{\text{SL}_n(k)}(U, \lambda) = \sum_{X \in \text{SL}_n(k)} \lambda(\text{tr}_u X).
\]

Let \( D_h \in \text{GL}_n(k) \) be the diagonal matrix \( \text{Diag}(1, 1, \ldots, 1, h) \), where \( h \in k^* \). Every element \( Y \) of \( \text{GL}_n(k) \) can be uniquely written as \( Y = D_h X \) with \( X \in \text{SL}_n(k) \) and \( h = \det X \). So

\[
\sum_{X \in \text{SL}_n(k)} \lambda(\text{tr}_u X) = \sum_{X \in \text{SL}_n(k)} \frac{1}{q-1} \sum_{h \neq 0} \lambda(\text{tr}_u D_h X) = \frac{1}{q-1} \sum_{Y \in \text{GL}_n(k)} \lambda(\text{tr}_u Y).
\]

Therefore, from Theorem 2.1 we get

\[
G_{\text{SL}_n(k)}(U, \lambda) = \frac{1}{q-1} G_{\text{GL}_n(k)}(U, 1, \lambda) = (-1)^u q^{\binom{n}{2}} \prod_{i=2}^{n-u} (q^i - 1).
\]

Now we consider the case \( u = n \). Let \( B_n(k) \) be the Borel subgroup of \( \text{SL}_n(k) \), i.e., the group of upper triangular matrices with determinate 1.
Using the same method as in the case of $GL_n(k)$, we get

\[(2.20)\]

\[
G_{SL_n}(k)(U, \lambda) = \sum_{X \in SL_n(k)} \lambda(\text{tr } U^t X)
\]

\[
= \sum_{\det X = \det U} \lambda(\text{tr } X)
\]

\[
= \frac{1}{(q-1)^{n-1}q^n} \sum_{B \in B_n(k)} \sum_{X \in SL_n(k) \det X = \det U} \lambda(\text{tr } BX)
\]

\[
= \sum_{\det U} \frac{1}{(q-1)^{n-1}q^n} \sum_{B \in B_n(k)} \lambda(\text{tr } BX)
\]

\[
= \sum_{\det(x_{ij}) = \det U} \frac{1}{(q-1)^{n-1}q^n} \sum_{(b_{ij}) \in B_n(k)} \lambda(\sum_{i \leq j} b_{ij} x_{ji})
\]

\[
= \frac{1}{(q-1)^{n-1}} \sum_{x_{11} \cdots x_{nn} = \det U} \lambda(\sum_{i=1}^n x_{ii}) \prod_{1 \leq j < i \leq k} \lambda(b_{ij} x_{ji})
\]

\[
= q^{\frac{n}{2}} K_n(\lambda, \det U),
\]

where

\[(2.21)\]

\[
K_n(\lambda, y) = \sum_{x_1^2 + \cdots + x_n^2 = y} \lambda(x_1 + x_2 + \cdots + x_n), \text{ for } y \in k^*;
\]

is the Kloosterman sum for $k = \mathbb{F}_q$.

Summing up, we get

**Theorem 2.4.** Let $u$ be the rank of $U$ and $\lambda$ be nontrivial. Then

\[
G_{SL_n}(k)(U, \lambda) = \begin{cases} 
q^{\frac{n}{2}} K_n(\lambda, \det U) & \text{if } u = n, \\
(-1)^u q^{\frac{n}{2}} \prod_{i=2}^{n-u} (q^i - 1) & \text{if } u < n.
\end{cases}
\]

**Corollary 2.5.** Uniformly over all nonzero matrices $U \in M_n(\mathbb{F}_q)$ and nontrivial additive characters $\lambda$, we have

\[
G_{SL_n}(k)(U, \lambda) = \begin{cases} 
O(1) & \text{if } n = 1, \\
O(q^\frac{n}{2}) & \text{if } n = 2, \\
O(q^{n^2-n-1}) & \text{if } n \geq 3.
\end{cases}
\]

where the implied constant in the symbol “$O$” depends only on $n$.

**Proof.** The case: $n = 1$ is trivial. So we assume $n \geq 2$. From Deligne’s bound of Kloosterman sum (see Example 2 in [20]), we get $K_n(\lambda, y) = O(q^{\frac{n}{2}})$. 

where \( y \in k^* \). Therefore, by Theorem 2.4 we get

\[
G_{\text{SL}_n(k)}(U, \lambda) = O(\max\{q^{n^2 - 1}, q^{n^2 - n - 1}\}).
\]

Note that \( n^2 - n - 1 > (n^2 - 1)/2 \) if and only if \( n \geq 3 \). So we get the desired bound.

**Remark 2.6.** Ferguson, Hoffman, Luca, Ostafe and Shparlinski in [2] obtained

\[
G_{\text{SL}_n(k)}(U, \lambda) = O(q^{n^2 - 2}).
\]

(See Lemma 4 of [2]). Corollary 2.5 improves their bounds.

3. Counting Invertible matrices with given trace

For \( \beta \in k \), let

\[
N_\beta = \#\{X \in \text{GL}_n(k) | \text{tr}X = \beta\}.
\]

The usual way of computing \( N_\beta \) involves the Bruhat decomposition of \( \text{GL}_n(k) \), e.g, see [24, Prop. 1.10.15]. In this note, as an application of Theorem 2.1, we can calculate \( N_\beta \) purely by the method of exponential sums.

Since trace is a linear function, we get \( N_\beta = N_{h\beta} \), for \( h, \beta \in k^* \). So \( N_h = N_1 \), for \( h \in k^* \). Then, we have

\[
N_0 + (q - 1)N_1 = \#\text{GL}_n(k) = \prod_{i=0}^{n-1}(q^n - q^i)
\]

and

\[
G_{\text{GL}_n(k)}(I, \lambda) = \sum_{X \in \text{GL}_n(k)} \lambda(\text{tr} X)
\]

\[
= N_0 \lambda(0) + N_1 \sum_{h \in k^*} \lambda(h) = N_0 - N_1.
\]

Combining Theorem 2.1 equations (3.2) and (3.3), we get

**Theorem 3.1.** Let \( h \in k^* \). Then

\[
N_0 = q^{\binom{n}{2}}(q - 1) \left( (-1)^n + \prod_{i=2}^{n} (q^i - 1) \right) / q,
\]

\[
N_h = q^{\binom{n}{2}}(q - 1) \left( (-1)^{n-1}/(q - 1) + \prod_{i=2}^{n} (q^i - 1) \right) / q.
\]

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