A Dyson-Schwinger model beyond isospin limit

prepared for investigating $U_A(1)$-breaking temperature dependence

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Abstract. Motivated by our earlier findings of sensitive quark-flavor dependence of QCD topological susceptibility on products of current quark masses and corresponding condensates, we allow the breaking of isospin symmetry. For the purpose of future investigations of $U_A(1)$ symmetry breaking and restoration at $T > 0$, we perform (at $T = 0$) refitting of the quark-mass parameters of a phenomenologically successful effective model of low-energy QCD. It belongs to the class of separable-interaction models within the Dyson-Schwinger approach to the quark-antiquark substructure of mesons.

1 Introduction and survey

In the realm of nonperturbative strong interactions, *ab initio* calculations are often of prohibitive difficulty. The usage of simplified models mimicking the underlying fundamental theory of QCD, is still often unavoidable in its low-energy, nonperturbative regime. This holds especially in applications at high temperature around or above the (pseudo-)critical temperature ($T \gtrsim T_{\text{crit}}$) and/or density, such as heavy-ion collisions, as well as in astrophysical and cosmological applications. The more convoluted a context of applications happens to be, the stronger simplifications in modeling of dynamics are needed in the quest for tractability, as long as some crucial properties of the underlying QCD are reproduced. Obviously, favored are models which are as simple as possible while achieving as much as possible. The case in point is the Dyson-Schwinger (DS) separable-interaction model of Blaschke et al. [1].

On the one hand, it makes predictions for so much low-energy phenomenology that it is competitive, in the low-energy and momentum regime, with some more elaborate model interactions (incorporating also the high-energy part, see corresponding predictions in, e.g., [2–6]), while on the other hand, it reproduces the proper chiral behavior of QCD. The latter quality, shared with other consistently applied DS approaches to QCD but very rare or even absent in other quark-bound-state approaches to hadrons,

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is usually even more important than the former. It is certainly so when dealing with light pseudoscalar mesons and issues concerning the dynamically broken $SU_A(N_f)$ chiral symmetry and the related $U_A(1)$ symmetry and anomaly.

Thus, chirally correct DS models exhibiting (at high $T$, but at vanishing and low density) a second order transition in the chiral limit and a crossover for realistic $u$, $d$, $s$-quark masses (i.e., $N_f = 2 + 1$) are well suited for modeling of low-energy QCD. The presently used model, stemming from Blaschke et al. [1], served very well for the task of extending our DS-approach studies of $U_A(1)$ breaking through $\eta'$-$\eta$ complex at $T = 0$ [2,4–8] to nonvanishing temperatures [9–14]. In the next section, we present the results of the re-fitting, out of the isospin-symmetric limit, of the quark mass parameters in the model’s so-called rank-2 variant, used in [9–14].

Of course, the isospin symmetry holds very well for almost all purposes in the context of hadronic physics. Let us, however, observe that this led, in general, to a somewhat cavalier attitude among hadronic model users, where the usual traditional practice has been to fit the experimental charged pion and kaon masses $M_{\pi^\pm,K^\pm}^{\exp}$ in models mimicking only QCD, even though the QED contributions can be taken as vanishing only for neutral pions and kaons due to Dashen’s theorem [15], which is not violated much [16]. Then, the electromagnetic contribution would be there only for charged pions (around 4.5–4.7 MeV [16–18]) and charged kaons (around 1.3–2.5 MeV [16–18]). Hence, if a model only mimics QCD, one should fit it to hadron masses out of which the electromagnetic contribution has been taken out. (Let us distinguish such masses by carets: $\hat{M}_{\pi^\pm}, \hat{M}_{K^\pm}$, etc.).

If one abandons the isospin symmetry, one should find the three different light quark masses $m_u$, $m_d$ and $m_s$ by fitting the three meson masses. These are the masses (1) of $\pi^+$, $K^+$ and the neutral kaon $K^0$, out of which the electromagnetic contributions were taken out, because the model is of QCD only. The appropriate results are given by the FLAG collaboration [16] in their equation (11) for the pion and kaon masses occurring in just the QCD sector of the Standard Model [16], i.e., with QED turned off:

$$\hat{M}_{\pi^+} = 134.8 \pm 0.3 \text{ MeV}, \quad \hat{M}_{K^+} = 491.2 \pm 0.5 \text{ MeV}, \quad \hat{M}_{K^0} = 497.2 \pm 0.4 \text{ MeV.} \quad (1)$$

By contrast, the traditional fit would be to the corresponding experimental values (rounded to one decimal place):

$$M_{\pi^+}^{\exp} = 139.6 \text{ MeV}, \quad M_{K^+}^{\exp} = 493.7 \text{ MeV}, \quad M_{K^0}^{\exp} = 497.6 \text{ MeV.} \quad (2)$$

For the both cases of refitting, in the next section we predict the concrete model mass of $\pi^0$, along with the masses of the other two flavorless pseudoscalars $\eta'$ and $\eta$, after we incorporate the anomalous $U_A(1)$ breaking.

Concerning the concrete presently used model, the reference [1] had pertained only to the non-strange sector, and upon including $s$-quarks starting with reference [9] some re-fitting was already done. The model details including the parameter values we were using earlier [9–14], are listed in the Appendix of our reference [14]. Everything so far has been in the isospin limit of equal $u$- and $d$-quark masses, which is usually completely adequate in hadronic physics. However, extending to $T > 0$ our treatment the $U_A(1)$ anomaly contribution (needed for $T$-dependence of $\eta'$ and $\eta$ mesons [13] and of axions [14]), led us to expressions involving the harmonic average of the products of light quark masses and condensates: the light-quark expression for QCD topological susceptibility and the anomalous contribution of the masses in the $\eta'$-$\eta$ complex (respectively, Eqs. (19) and (18) in reference [13], as well as (9) and (8) below). The harmonic average is dominated by the lightest flavor to such an extent that in reference [13] we concluded one should check whether the isospin breaking
between $u$- and $d$-quark masses can significantly affect the $T$-dependence of the $U_A(1)$ anomaly mass contribution. This, and not a search for a better description of the light pseudoscalar nonet masses, is the reason that as the first step (at $T = 0$) we perform the re-fitting of the quark mass parameters of our Dyson-Schwinger model of choice, but without the constraint of the isospin symmetry.

The model calculation and its results at $T = 0$ are presented in the next section.

## 2 The model calculation and the present re-fitting

All model details, except of course the quark-mass parameter values, can be found in one place – in the Appendix of our reference [14]. From there, we adopt as the interaction model the effective gluon propagator in a Feynman-like gauge and in the separable form:

$$ g^2 D_{\mu\nu}^{ab}(p - \ell)_{\text{eff}} = \delta^{ab} g^2 D_{\mu\nu}^{\text{eff}}(p - \ell) \rightarrow \delta_{\mu\nu} D(p^2, \ell^2, p \cdot \ell) \delta^{ab}. \quad (3) $$

We choose the separable interaction variant called rank-2,

$$ D(p^2, \ell^2, p \cdot \ell) = D_0 \mathcal{F}_0(p^2) \mathcal{F}_0(\ell^2) + D_1 \mathcal{F}_1(p^2) (p \cdot \ell) \mathcal{F}_1(\ell^2), \quad (4) $$

because it was used also in reference [13] on $T$-dependence of $\eta'$ and $\eta$ mesons. The momentum-dependent functions $\mathcal{F}_0(p^2)$ and $\mathcal{F}_1(p^2)$ [9,19] are

$$ \mathcal{F}_0(p^2) = \exp(-p^2/\Lambda_0^2) \quad \text{and} \quad \mathcal{F}_1(p^2) = \frac{1 + \exp(-p_0^2/\Lambda_0^2)}{1 + \exp((p^2 - p_0^2)/\Lambda_1^2)}, \quad (5) $$

where $D_0 \Lambda_0^2 = 219$, $D_1 \Lambda_0^4 = 40$, $\Lambda_0 = 0.758$ GeV, $\Lambda_1 = 0.961$ GeV and $p_0 = 0.6$ GeV. These values of the parameters of the interaction in the present work are the same as in references [13,14], because in the present paper we vary only the quark mass parameters $m_q$ ($q = u, d, s$) away from the isosymmetric values $m_u = m_d \equiv m_t = 5.49$ MeV and $m_s = 115$ MeV, used in references [13,14].

All calculations are done as in reference [13] up to the point where the isospin symmetry leads to simplifications due to $m_u = m_d$, from which we now refrain and take $m_u \neq m_d$.

Re-fitting is an arduous procedure, but it is in principle straightforward to vary values of $m_q$’s into the gap equations for dressed quark propagators of different flavors and then, in turn, into the consistent Bethe-Salpeter equations for $q' \bar{q}$ ($q, q' = u, d, s$) pseudoscalar bound-state vertices and masses $M_{q'q}$. Through the solutions of these DS equations, varying values of $m_q$’s affect all calculated quantities calculated in reference [13], notably in the condensates $\langle \bar{q}q \rangle$, which now all differ for different flavors $q$ (all are flavor-nonuniversal now, while before, $\langle \bar{u}u \rangle = \langle \bar{d}d \rangle$ [13,14]).

We obtain the chiral-limit-vanishing bound-state masses $M_{q'q}$ ($q', q = u, d, s$) by solving consistent DS gap and Bethe-Salpeter equations in the rainbow-ladder approximation, and this cannot capture effects of the $U_A(1)$ anomaly. But in the flavorless, or hidden-flavor sector (where $q' = q$), the non-anomalous masses $M_{q'q}$ cannot provide the whole story on the masses of flavorless pseudoscalars, since the $U_A(1)$ anomaly contributes through the flavor-changing transitions $|q \bar{q}\rangle \rightarrow |q' \bar{q}'\rangle$, like those depicted schematically in Figure 1. The famous example of the relatively very heavy $\eta'$ meson shows it is essential to include the anomalous $U_A(1)$ symmetry breaking at least at the level of the masses. We do it as described in references [2,4,6–8], i.e., relying on the $U_A(1)$ anomaly being suppressed in the limit of large number of QCD colors $N_c$ [20,21]. Thanks to this, the anomaly contribution $M_A^2$ to the total mass matrix
Fig. 1. Flavor-changing, axial anomaly-driven transitions of quark-antiquark pseudoscalars $P = q\bar{q}$ to $P' = q'\bar{q}'$ comprising the pseudoscalar mesons in the hidden-flavor sector. All quark and gluon lines and vertices are dressed nonperturbatively. The gray oval and three dots stand for infinity of all intermediate gluon states enabling such transitions. (The number of gluons must be an even [4] number; the simplest case is when this figure reduces to the “diamond graph”, with no oval blob and just two gluons, albeit dressed nonperturbatively.).

(squared) $M^2$ of the hidden-flavor complex $\eta' -$ $\eta -$ $\pi^0$, can be treated formally as a perturbation. In the lowest order, it is simply added [2,4,6] to the non-anomalous mass matrix (squared), $M^2_{NA}$, made of the $M_{q'q}$ contributions: $M^2 = M^2_{NA} + M^2_A$.

The non-anomalous part of the mass matrix (in the basis $|q\bar{q}\rangle$) is still $M^2_{NA} = \text{diag}[M^2_{uu}, M^2_{dd}, M^2_{ss}]$, but now $M_{uu} \neq M_{dd} \neq M_{ud} = M_{\pi\pm}$.

What is nevertheless qualitatively different, is that since the isospin symmetry is not enforced, it no longer precludes a contribution to the neutral pion due to the $U_A(1)$ anomaly. It will contribute, albeit quantitatively very little, to $M_{\pi^0}$ and to the mass difference between $\pi^0$ and $\pi^\pm$, so that $M^2_{\pi^0}$ will not be exactly $\frac{1}{2}(M^2_{uu} + M^2_{dd})$. This is because $M^2_A$, the anomalous part of the mass matrix (squared) cannot any longer be reduced to the $2 \times 2$ matrix of the isoscalar subspace of the $\eta' -$ $\eta$ complex.

In the hidden-flavor sector, the flavor-changing transitions due to the $U_A(1)$ anomaly, $|q\bar{q}\rangle \rightarrow |q'\bar{q}'\rangle$, depicted schematically in Figure 1, yield the matrix elements of the anomalous mass matrix (squared):

$$\langle q\bar{q}|M^2_A|q'\bar{q}'\rangle = b_q b_{q'},$$

where $b_q \equiv \sqrt{\beta}$ for $q = u$. But, the amplitudes for the transitions from, and into, lightest $u\bar{u}$ pairs are larger than those for the significantly more massive $s\bar{s}$ pairs. Thus, as in our earlier papers, we allow for the effects of the breaking of the SU(3) flavor symmetry for $q, q' = s$ by $b_s = X\sqrt{\beta}$, where $X = f_{u\bar{u}}/f_{s\bar{s}}$ [13]. However, now we do it also for $q, q' = d$; namely, $b_d = Y\sqrt{\beta}$ and $Y = f_{d\bar{d}}/f_{s\bar{s}}$, even though it is clear that here in the mass matrix at $T = 0$, the effect of the isospin breaking is small, due to $f_{u\bar{u}} \approx f_{d\bar{d}} \approx f_\pi$. (Actually, here we define $f_\pi = f_{\pi^0} = \frac{1}{2}(f_{u\bar{u}} + f_{d\bar{d}})$, since here we consider the anomaly only on the level of the masses and neglect its possible influence on meson decay constants.).

The total mass matrix of the hidden-flavor sector in the flavor basis $|q\bar{q}\rangle$ is

$$M^2 = M^2_{NA} + M^2_A = \begin{bmatrix} M^2_{uu} + \beta & \beta Y & \beta X \\ \beta Y & M^2_{dd} + \beta Y^2 & \beta XY \\ \beta X & \beta XY & M^2_{ss} + \beta X^2 \end{bmatrix},$$

(7)
where, as in references \[8,13\],

\[
\beta = \frac{2A}{f_{\pi}^2} \quad \text{and} \quad A = \frac{1}{1 + \chi\left(\frac{1}{m_u \langle \bar{u}u \rangle} + \frac{1}{m_d \langle \bar{d}d \rangle} + \frac{1}{m_s \langle \bar{s}s \rangle}\right)}, \tag{8}
\]

in line with Shore’s generalization \[22\] of Witten-Veneziano relation \[20,21\], where \(A\) is the full QCD topological charge parameter and where the QCD topological susceptibility \(\chi\) is for light flavors given by the current masses \(m_q\) multiplied by respective condensates \(\langle \bar{q}q \rangle\) realistically away from the chiral limit (which gives the crossover behavior at large \(T\)):

\[
\chi = \frac{-1}{m_u \langle \bar{u}u \rangle} + \frac{1}{m_d \langle \bar{d}d \rangle} + \frac{1}{m_s \langle \bar{s}s \rangle} + C_m. \tag{9}
\]

As before \[8,12,13\], the small-magnitude and necessarily negative correction term \(C_m\) (higher order in small quark masses) is found by using (as well as Shore \[22\]) the \(1/N_c\) approximation \(A = \chi_{YM}\) valid at \(T = 0\). As before, we adopt the lattice result \(\chi_{YM} = (191 \text{ MeV})^4\) \[23\].

With these ingredients, the pseudoscalar meson states and masses containing the influence of the \(U_A(1)\) anomaly are readily obtained by diagonalization of the complete mass matrix of the hidden-flavor sector, equation (7). The eigenvalues of this matrix are the present model predictions for the squared physical masses of \(\eta', \eta,\) and \(\pi^0\), given in Table 1 in the next subsection.

### 2.1 Results out of the isospin symmetry limit

The values of the model quark mass parameters \(m_q\) \((q = u, d, s)\) are obtained by fitting mass eigenvalues of Bethe-Salpeter equations to various values \(M_{\pi^\pm}^{\text{fit}}, M_{K^\pm}^{\text{fit}}\) and \(M_{K^0}^{\text{fit}}\), assigned to \(\pi^\pm, K^\pm\) and \(K^0\) meson masses in three different ways described below.

In each of the tables presenting our model results, the first row corresponds to the isosymmetric case \(m_u = m_d\). It is here for comparison because it just repeats what we already had (at \(T = 0\)) in our previous references \[7–14\] employing this model, where the parameters (including \(m_u = m_d = 5.49\) MeV and \(m_s = 115.12\) MeV) were obtained through the fit of the pion and kaon masses in the isospin limit, so that all pion masses, including \(M_{\pi^\pm}^{\text{fit}}\), were taken equal (140 MeV), and all kaon masses were taken equal (495 MeV), including \(M_{K^\pm}^{\text{fit}}\) and \(M_{K^0}^{\text{fit}}\).

The next two rows in each of the tables correspond to the results of two different fits out of the isospin limit, \(m_u \neq m_d\), but fitting somewhat differently defined pion and kaon masses. The second row in every table is labeled by the superscript \(\text{(2)}\), because this corresponds to fitting the values in equation (2). This is the traditional fit to the experimental values (here, rounded to the first decimal place). It yields \(m_u = 4.37\) MeV, \(m_d = 6.55\) MeV and \(m_s = 115.34\) MeV.

In the same way, the third row in every table starts with the superscript label \(\text{(1)}\), because it results from the fit aiming at the masses in equation (1). It gives the FLAG \[16\] values for the \(\pi^\pm, K^\pm\) and \(K^0\) just-QCD masses (i.e., with the corresponding electromagnetic contributions removed). This fit yields \(m_u = 3.40\) MeV, \(m_d = 6.80\) MeV = 2\(m_u\) and \(m_s = 115.61\) MeV, but the minimization procedure did not reproduce the values in equation (1) exactly. Still, the difference between \(M_{\pi^\pm}^{\text{fit}}, M_{K^\pm}^{\text{fit}}\) in the row \(\text{(1)}\) and equation (1) – stemming overwhelmingly from the kaon sector – is satisfactorily small considering the rather large difference between
Table 1. The first three rows of numbers represent our three fits. The first block of columns are the masses to which the model quark-mass parameters $m_q$ were fitted. Next are the predicted observables: charged pion and kaon decay constants ($f_{\pi^+} = f_{ud}$ and $f_{K^+} = f_{us}$, respectively), and the masses of the flavorless pseudoscalars $\pi^0$, $\eta$ and $\eta'$. The last row gives the experimental values of all these quantities. All values are in MeV.

| Type of fit | $M_{\pi^\pm}$ | $M_{K^\pm}$ | $M_{K^0}$ | $f_{\pi^+}$ | $f_{K^+}$ | $M_{\pi^0}$ | $M_\eta$ | $M_{\eta'}$ |
|-------------|----------------|--------------|-----------|-------------|-----------|-------------|---------|------------|
| $m_u = m_d$ | 140.0          | 495.0        | 495.0     | 92.0        | 108.8     | 140.0       | 554.0   | 997.0      |
| $m_u = 0.67m_d$ | 139.6         | 493.7        | 497.6     | 92.0        | 108.5     | 139.6       | 554.5   | 995.5      |
| $m_u = 0.5m_d$ | 134.9         | 492.6        | 498.7     | 91.8        | 108.6     | 134.9       | 554.6   | 994.2      |
| experiment  | 139.57        | 493.68       | 497.61    | 92.1        | 110.1     | 134.98      | 547.86  | 957.78     |

Table 2. For our three fits, the results for unphysical pseudoscalar bound states $|u\bar{u}\rangle$, $|d\bar{d}\rangle$, and $|s\bar{s}\rangle$: their masses and decay constants (and also $f_{\pi^0} = (f_{u\bar{u}} + f_{d\bar{d}})/2$), and the $SU(3)$ flavor and isospin symmetry breaking parameters $X \equiv f_{u\bar{u}}/f_{s\bar{s}}$ and $Y \equiv f_{u\bar{u}}/f_{d\bar{d}}$. All values are in MeV.

| Type of fit | $M_{u\bar{u}}$ | $M_{d\bar{d}}$ | $M_{s\bar{s}}$ | $f_{u\bar{u}}$ | $f_{d\bar{d}}$ | $f_{s\bar{s}}$ | $f_{\pi^0}$ | $X$ | $Y$ |
|-------------|----------------|----------------|---------------|-------------|-----------|-------------|-----------|-----|-----|
| $m_u = m_d$ | 140.1          | 140.0          | 685.0         | 92.0        | 92.0      | 119.0       | 92.0      | 0.773 | 1.0 |
| $m_u = 0.67m_d$ | 124.8          | 153.0          | 684.9         | 91.5        | 92.4      | 118.7       | 92.0      | 0.771 | 0.991 |
| $m_u = 0.5m_d$ | 110.0          | 155.9          | 684.9         | 91.1        | 92.5      | 118.7       | 91.8      | 0.768 | 0.986 |

the model quark mass parameter $m_s \approx 115$ MeV and the QCD $s$-quark current mass parameter $m_s = 93^{+11}_{-5}$ MeV [24].

For the three just described fittings, the first block of Table 1 gives the corresponding versions of the three pion and kaon masses which are protected by charge and/or strangeness from any influence of the $U_A(1)$ anomaly even out of the isospin symmetry limit, and to which we fitted the three parameters $m_q$.

The last block of Table 1 gives, for our three fits, the predictions of our chosen DS separable model for the observable masses of the flavorless light pseudoscalar mesons $\pi^0$, $\eta$ and $\eta'$. Out of the isospin limit, the neutral pion is not protected from the $U_A(1)$ anomaly, but its contribution, as well as the related admixture of the $s\bar{s}$ pseudoscalar bound state to $\pi^0$, is of course small, since the isospin symmetry is very close to reality.

The middle block of Table 1 also contains our prediction of observables for the above fits, namely the decay constants of $\pi^+$ and $K^+$.

Table 2 gives mostly the quantities which, except the pion decay constant $f_\pi$, are not strictly observable. They are nevertheless presented, since they are illustrative for the calculations outlined in the text above, especially of $M_{N^2_A}$, the non-anomalous part of the mass matrix.

Table 3 gives quantities which enter into the calculation of $M_{N_A}^2$, the anomalous part of the mass matrix, since the products $m_q \langle \bar{q}q \rangle$ ($q = u, d, s$) determine the QCD topological susceptibility $\chi$ (9) and topological charge parameter $A$ (8). Their behavior at $T > 0$ will determine the fate of $U_A(1)$ symmetry breaking and restoration in the future investigations in the present model.

3 Summary

A well-tried DS effective model intended for investigations at $T > 0$ has been refitted at $T = 0$ by allowing its quark mass parameters $m_q$ to take values out of the
Table 3. For the old isosymmetric fit and the new fits, (2) and (1), with broken isospin symmetry, $m_u \neq m_d$, the three sets of values of the model quark mass parameters $m_q$ ($q = u, d, s$), are related to the model results for topological susceptibility $\chi$ and “massive”, i.e., flavor-nonuniversal condensates $\langle \bar{u}u \rangle$, $\langle \bar{d}d \rangle$ and $\langle \bar{s}s \rangle$. Our model predictions for the topological susceptibility $\chi$ are evaluated from these condensates and the corresponding $m_q$’s. These same sets of $m_q$’s yield the corresponding values of the topological susceptibility $\chi_0$ when one uses the flavor-universal or “massless”, i.e., chiral-limit condensate $\langle \bar{q}q \rangle_0 = -217^{+3}_{-5}$ MeV$^3$. All values are in MeV (or indicated powers of MeV).

| Type of fit | $m_u$ | $m_d$ | $m_s$ | $\chi_0$ | $\langle \bar{u}u \rangle$ | $\langle \bar{d}d \rangle$ | $\langle \bar{s}s \rangle$ | $\chi$ |
|------------|-------|-------|-------|---------|----------------------|----------------------|----------------------|-------|
| $m_u = m_d$ | 5.49  | 5.49  | 115.12| 72.18$^4$| $-219^3$ | $-219^3$ | $-239^3$ | 72.73$^4$ |
| (2)$m_u = 0.67 m_d$ | 4.37  | 6.55  | 115.34| 71.56$^4$| $-219^3$ | $-220^3$ | $-239^3$ | 72.22$^4$ |
| (1)$m_u = 0.50 m_d$ | 3.40  | 6.80  | 115.61| 69.09$^4$| $-219^3$ | $-220^3$ | $-239^3$ | 69.61$^4$ |

isospin limit. Potentially the most significant improvement of the ensuing model parametrization is lowering the quark mass parameter of the lightest flavor to $m_u = m_d/2 = 3.40$ MeV, as explained in the rest of the text.

Namely, the isospin symmetry is mostly a very accurate approximation to reality, so that (as already hinted in the Introduction), our aim of relaxing the isospin limit is not a better description of the masses, decay constants and other observables at $T = 0$. (But it is good to check just in case, that nothing is spoiled by such refitting.) Indeed, relaxing the isospin limit did not bring significant changes in directly observable quantities at $T = 0$. For example, even beyond the precision displayed in our Tables, the calculated values of the two pion decay constants remained unique, $f_{\pi^+} = f_{\pi^0}$, even for the fit yielding the larger difference between the two lightest flavors. Some marginal improvement is seen in the two last columns of Table 1, i.e., the masses of the isoscalar mesons $\eta$ and $\eta'$. The largest improvement in predicting observable masses is the mass of the neutral pion, but just relaxing the isosymmetric requirement $m_u = m_d$ is of course not sufficient for that. This is illustrated by the difference between our two fits beyond the isosymmetric limit. Fitting cavalierly the empirical masses of the charged pseudoscalar mesons imposes unjustifiably their electromagnetic contributions on the predicted neutral ones. If a model interaction mimics only the QCD one, fully consistent out-of-isospin-limit fits should be to masses from which the electromagnetic contributions have been taken out, such as equation (1).

Before explaining the potential importance of lowering the quark mass parameters of the model, let us remark that we are of course aware that as parameters of a phenomenological model, our $m_q$’s cannot be related quite unambiguously and precisely to the still somewhat lower values of the fundamental QCD current quark masses $m_u = 2.16^{+0.40}_{-0.20}$ MeV, $m_d = 4.67^{+0.18}_{-0.17}$ MeV and $m_s = 93^{+11}_{-5}$ MeV [24]. The relationship of ratios is better defined, since differences of various schemes tend to cancel in them. Hence, our second ratio $m_u/m_d = 0.5$, within errors of the ratio of the lightest QCD current masses, $m_u/m_d = 0.47^{+0.06}_{-0.07}$ [24], provides a better-defined connection between our model parameters and current quark masses of QCD.

Now, since the QCD topological susceptibility $\chi$ (9) and topological charge parameter $A$ (8) depend on the products of quark masses and the corresponding condensates, $m_q \langle \bar{q}q \rangle$ ($q = u, d, s$), as their harmonic averages, the lightest flavor is dominant. While the absolute values of condensates fall rather slowly with the mass of their corresponding quark flavor towards the saturation at their limiting, chiral-symmetric value $\langle \bar{q}q \rangle_0$, their temperature dependence is a different story. The crossover fall of a condensate with $T$ quickly gets increasingly steeper for smaller values of $m_q$. So, the steepest falling condensate is multiplied by the smallest quark mass. In conjunction with the harmonic average type of dependence in equations (8) and (9), it is likely that reducing $m_u$ below its isospin partner $m_u$ will significantly
influence, e.g., $T$-dependence of the $\eta'-\eta$ complex studied in reference [13], and analogously various other cases of $U_A(1)$ restoration. This is why it is important to have models capable of investigating such situations also beyond the isospin limit.

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**References**

1. D. Blaschke, G. Burau, Y.L. Kalinovsky, P. Maris, P.C. Tandy, Int. J. Mod. Phys. A **16**, 2267 (2001)
2. D. Klabučar, D. Kekez, Phys. Rev. D **58**, 096003 (1998)
3. D. Kekez, D. Klabučar, Phys. Lett. B **457**, 359 (1999)
4. D. Kekez, D. Klabučar, M.D. Scadron, J. Phys. G **26**, 1335 (2000)
5. D. Kekez, D. Klabučar, Phys. Rev. D **71**, 014004 (2005)
6. D. Kekez, D. Klabučar, Phys. Rev. D **73**, 036002 (2006)
7. D. Horvatić, D. Blaschke, Y. Kalinovsky, D. Kekez, D. Klabučar, Eur. Phys. J. A **38**, 257 (2008)
8. S. Benić, D. Horvatić, D. Kekez, D. Klabučar, Phys. Lett. B **738**, 113 (2014)
9. D. Horvatić, D. Blaschke, D. Klabučar, A.E. Radzhabov, Phys. Part. Nucl. **39**, 1033 (2008)
10. D. Horvatić, D. Klabučar, A.E. Radzhabov, Phys. Rev. D **76**, 096009 (2007)
11. D. Horvatić, D. Blaschke, D. Klabučar, O. Kaczmarek, Phys. Rev. D **84**, 016005 (2011)
12. S. Benić, D. Horvatić, D. Kekez, D. Klabučar, Phys. Rev. D **84**, 016006 (2011)
13. D. Horvatić, D. Kekez, D. Klabučar, Phys. Rev. D **99**, 014007 (2019)
14. D. Horvatić, D. Kekez, D. Klabučar, Universe **5**, 208 (2019)
15. R.F. Dashen, Phys. Rev. **183**, 1245 (1969)
16. S. Aoki et al., Eur. Phys. J. C **74**, 2890 (2014)
17. J. Gasser, H. Leutwyler, Phys. Rep. **87**, 77 (1982)
18. J.F. Donoghue, A.F. Perez, Phys. Rev. D **55**, 7075 (1997)
19. D. Blaschke, D. Horvatic, D. Klubucar, A.E. Radzhabov, [hep-ph/0703188] [HEP-PH]
20. E. Witten, Nucl. Phys. B **156**, 269 (1979)
21. G. Veneziano, Nucl. Phys. B **159**, 213 (1979)
22. G.M. Shore, Nucl. Phys. B **744**, 34 (2006)
23. L. Del Debbio, L. Giusti, C. Pica, Phys. Rev. Lett. **94**, 032003 (2005)
24. M. Tanabashi et al. [Particle Data Group], Phys. Rev. D **98**, 030001 (2018) and 2019 update