DOUBLE CUBICS AND DOUBLE QUARTICS

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Abstract. We study a double cover $\psi : X \to V \subset \mathbb{P}^n$ branched over a smooth divisor $R \subset V$ such that $R$ is cut on $V$ by a hypersurface of degree $2(n - \deg(V))$, where $n \geq 8$ and $V$ is a smooth hypersurface of degree 3 or 4. We prove that $X$ is nonrational and birationally superrigid.

1. Introduction.

Let $\psi : X \to V \subset \mathbb{P}^n$ be a double cover branched over a smooth divisor $R \subset V$, where $n \geq 4$ and $V$ is a smooth hypersurface\(^1\). Then $\text{rk} \, \text{Pic}(X) = 1$ (see \([4]\)) and

$$-K_X \sim \psi^*(\mathcal{O}_{\mathbb{P}^n}(d + r - 1 - n)|_V),$$

where $d = \deg V$ and $r$ is a natural number such that $R \sim \mathcal{O}_{\mathbb{P}^n}(2r)|_V$. Therefore $X$ is nonrational in the case when $d + r \geq n + 1$. The variety $X$ is rationally connected if $d + r \leq n$, because it is a smooth Fano variety (see \([3]\)). Moreover, the following result is due to \([11]\).

Theorem 1. The variety $X$ is birationally superrigid\(^2\) if it is general and $d + r = n \geq 5$.

In this paper we prove the following result.

Theorem 2. The variety $X$ is birationally superrigid if $d + r = n \geq 8$ and $d = 3$ or 4.

One can use Theorem 2 to construct explicit examples of nonrational Fano varieties.

Example 3. The complete intersection

$$\sum_{i=0}^{8} x_i^4 = z^2 - x_0^4x_1^4 + x_2^4x_3^4 + x_4^4x_5^4 + x_6^4x_7^4 - 0 \subset \mathbb{P}(1^9, 3) \cong \text{Proj}(\mathbb{C}[x_0, \ldots, x_8, z])$$

is smooth. Hence, it is birationally superrigid and nonrational by Theorem 2.

In the case when $d + r = n \geq 4$ and $d = 1$ or 2 the birational superrigidity of $X$ is proved in \([5]\) and \([10]\). In the case when $d + r = n = 4$ and $d = 3$ the variety $X$ is not birationally superrigid, but it is nonrational (see \([6]\), \([8]\)). In the case when $d + r < n$ the only known way to prove the nonrationality of $X$ is the method of \(3V\) in \([8]\), which implies the following result.

Proposition 4. The variety $X$ is nonrational if it is very general, $n \geq 4$ and $r \geq \frac{d+n+2}{2}$.

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2. Preliminaries.

Let $X$ be a variety and $B_X = \sum_{i=1}^{\ell} a_iB_i$ be a boundary on $X$, where $a_i \in \mathbb{Q}$ and $B_i$ is either a prime divisor on $X$ or a linear system on $X$ having no base components. We say that $B_X$ is effective if every $a_i \geq 0$, we say that $B_X$ is movable if every $B_i$ is a linear system having no fixed components\(^3\). In the rest of the section we we assume that all varieties are $\mathbb{Q}$-factorial.

\(^{1}\)All varieties are assumed to be projective, normal, and defined over $\mathbb{C}$.

\(^{2}\)Namely, we have Bir($X$) = Aut($X$), and $X$ is not birational to the following varieties: a variety $Y$ such that there is a morphism $\tau : Y \to Z$ whose general fiber has negative Kodaira dimension and dim($Y$) $\neq$ dim($Z$) $\neq 0$; a Fano variety of Picard rank 1 having terminal $\mathbb{Q}$-factorial singularities that is not biregular to $X$.

\(^{3}\)Every effective movable log pair can be considered as a usual log pair (see \([1]\)).
Remark 5. We can consider $B_X^2$ as an effective codimension-two cycle if $B_X$ is movable.

The notions such as discrepancies, terminality, canonicity, log terminality and log canonicity can be defined for the log pair $(X, B_X)$ as for usual log pairs (see [2]).

**Definition 6.** The log pair $(X, B_X)$ has canonical (terminal, respectively) singularities if for every birational morphism $f : W \to X$ there is an equivalence

$$K_W + B_W \sim_{Q} f^*(K_X + B_X) + \sum_{i=1}^{n} a(X, B_X, E_i)E_i$$

such that every number $a(X, B_X, E_i)$ is non-negative (positive, respectively), where $B_W$ is a proper transform of $B_X$ on $W$, and $E_i$ is an $f$-exceptional divisor. The number $a(X, B_X, E_i)$ is called the discrepancy of the log pair $(X, B_X)$ in the divisor $E_i$.

The application of Log Minimal Model Program (see [7]) to an effective movable log pair $(X, B_X)$ such that codim$(Z \subset X) = 2$ the inequality $\text{mult}_Z(B_X) \geq 1$ implies $Z \in \mathbb{CS}(X, B_X)$.

In particular, the log pair $(X, B_X)$ has terminal singularities if and only if $\mathbb{CS}(X, B_X) = \emptyset$.

**Remark 8.** Let $H$ be a general hyperplane section of $X$. Then every component of $Z \cap H$ is contained in the set $\mathbb{CS}(H, B_X|_H)$ for every subvariety $Z \subset X$ contained in $\mathbb{CS}(X, B_X)$.

**Remark 9.** Let $Z \subset X$ be a proper irreducible subvariety such that $X$ is smooth at the generic point of $Z$. Suppose that $B_X$ is effective. Then $Z \in \mathbb{CS}(X, B_X)$ implies $\text{mult}_Z(B_X) \geq 1$, but in the case codim$(Z \subset X) = 1$ the inequality $\text{mult}_Z(B_X) \geq 1$ implies $Z \in \mathbb{CS}(X, B_X)$.

The following result is Lemma 3.18 in [1].

**Lemma 10.** Suppose that $X$ is a smooth complete intersection $\cap_{i=1}^{k} G_i \subset \mathbb{P}^n$, and $B_X$ is effective such that $B_X \sim_{Q} rH$ for some $r \in \mathbb{Q}$, where $G_i$ is a hypersurface in $\mathbb{P}^n$, and $H$ is a hyperplane section of $X$. Then $\text{mult}_Z(B_X) \leq r$ for every irreducible subvariety $Z \subset X$ such that $\dim(Z) \geq k$.

The following result is well known (see [2], [3]).

**Theorem 11.** Let $X$ be a Fano variety of Picard rank 1 having terminal $\mathbb{Q}$-factorial singularities that is not birationally superrigid. Then there is a linear system $\mathcal{M}$ on the variety $X$ whose base locus has codimension at least 2 such that the singularities of the log pair $(X, \mu \mathcal{M})$ are not canonical, where $\mu$ is a positive rational number such that $K_X + \mu \mathcal{M} \sim_{Q} 0$.

Let $f : V \to X$ be a birational morphism such that the union of $\cup_{i=1}^{n} f^{-1}(B_i)$ and all $f$-exceptional divisors forms a divisor with simple normal crossing. Then $f$ is called a log resolution of the log pair $(X, B_X)$, and the log pair $(V, B_V^V)$ is called the log pull back of $(X, B_X)$ if

$$B_V^V = f^{-1}(B_X) - \sum_{i=1}^{n} a(X, B_X, E_i)E_i$$

such that $K_V + B_V^V \sim_{Q} f^*(K_X + B_X)$, where $E_i$ is an $f$-exceptional divisor and $a(X, B_X, E_i) \in \mathbb{Q}$.

**Definition 12.** The log canonical singularity subscheme $\mathcal{L}(X, B_X)$ is the subscheme associated to the ideal sheaf $\mathcal{I}(X, B_X) = f_*(\mathcal{O}_V([-B_V^V]))$. A proper irreducible subvariety $Y \subset X$ is called a center of log canonical singularities of the log pair $(X, B_X)$ if there is a divisor $E \subset V$ that is contained in the effective part of the support of $[B_V^V]$ and $f(E) = Y$. The set of all centers of log canonical singularities of $(X, B_X)$ is denoted as $\mathbb{LCS}(X, B_X)$, the set-theoretic union of the elements of $\mathbb{LCS}(X, B_X)$ is denoted as $\mathbb{LCS}(X, B_X)$. 
In particular, we have \( \text{Supp}(\mathcal{L}(X, B_X)) = \text{LCS}(X, B_X) \).

**Remark 13.** Let \( H \) be a general hyperplane section of \( X \) and \( Z \in \text{LCS}(X, B_X) \). Then every component of the intersection \( Z \cap H \) is contained in the set \( \text{LCS}(H, B_X|_H) \).

The following result is Theorem 17.4 in [9].

**Theorem 14.** Let \( g : X \to Z \) be a morphism. Then \( \text{LCS}(X, B_X) \) is connected in a neighborhood of every fiber of the morphism \( g \circ f \) if the following conditions hold:

- the morphism \( g \) has connected fibers;
- the divisor \(-(K_X + B_X)\) is \( g \)-nef and \( g \)-big;
- the inequality \( \text{codim}(g(B_i)) \subset Z) \geq 2 \) holds if \( a_i < 0 \);

The following corollary of Theorem 14 is Theorem 17.6 in [9].

**Theorem 15.** Let \( Z \) be an element of the set \( \text{CS}(X, B_X) \), and \( H \) be an effective Cartier divisor on the variety \( X \). Suppose that the boundary \( B_X \) is effective, the varieties \( X \) and \( H \) are smooth in the generic point of \( Z \) and \( Z \subset H \not\subset \text{Supp}(B_X) \). Then \( \text{LCS}(H, B_X|_H) \neq \emptyset \).

The following result is Theorem 3.1 in [3].

**Theorem 16.** Suppose that \( \dim(X) = 2 \), the boundary \( B_X \) is effective and movable, and there is a smooth point \( O \in X \) such that \( O \in \text{LCS}(X, (1-a_1)\Delta_1 + (1-a_2)\Delta_2 + M_X) \), where \( \Delta_1 \) and \( \Delta_2 \) are smooth curves on \( X \) intersecting normally at \( O \), and \( a_1 \) and \( a_2 \) are arbitrary non-negative rational numbers. Then we have

\[
\text{mult}_O(B_X^2) \geq \begin{cases} 4a_1a_2 & \text{if } a_1 \leq 1 \text{ or } a_2 \leq 1 \\
4(a_1 + a_2 - 1) & \text{if } a_1 > 1 \text{ and } a_2 > 1.
\end{cases}
\]

3. **Main local inequality.**

Let \( X \) be a variety, \( O \) be a smooth point on \( X \), \( f : V \to X \) be a blow up of the point \( O \), \( E \) be an exceptional divisor of \( f \), \( B_X = \sum_{i=1}^e a_iB_i \) be a movable boundary on \( X \), and \( B_V = f^{-1}(B_X) \), where \( a_i \) is a non-negative rational number and \( B_i \) is a linear system on \( X \) having no base components. Suppose that \( O \in \text{CS}(X, B_X) \), but the singularities of \((X, B_X)\) is log terminal in some punctured neighborhood of the point \( O \). The following result is Corollary 3.5 in [3].

**Lemma 17.** Suppose that \( \dim(X) = 3 \) and \( \text{mult}_O(B_X) < 2 \). Then there is a line \( L \subset E \cong \mathbb{P}^2 \) such that \( L \in \text{LCS}(V, B_V + (\text{mult}_O(B_X) - 1)E) \).

Suppose that \( \dim(X) = 4 \) and \( \text{mult}_O(B_X) < 3 \). Then the proof of Lemma 17 and Theorem 14 implies the following result.

**Proposition 18.** One of the following possibilities holds:

- there is a surface \( S \subset E \) such that \( S \in \text{LCS}(V, B_V + (\text{mult}_O(B_X) - 2)E) \);
- there is a line \( L \subset E \cong \mathbb{P}^3 \) such that \( L \in \text{LCS}(V, B_V + (\text{mult}_O(B_X) - 2)E) \).

Now suppose that the set \( \text{LCS}(V, B_V + (\text{mult}_O(B_X) - 2)E) \) does not contain surfaces that are contained in the divisor \( E \) and contains a line \( L \subset E \cong \mathbb{P}^3 \). Let \( g : W \to V \) be a blow up of the variety \( V \) in \( L \), \( F = g^{-1}(L) \), \( E = g^{-1}(E) \), and \( B_W = g^{-1}(B_V) \). Then

\[
B^W = B_W + (\text{mult}_O(B_X) - 3)\bar{E} + (\text{mult}_O(B_X) + \text{mult}_L(B_V) - 5)F.
\]

**Proposition 19.** One of the following possibilities holds:

- the divisor \( F \) is contained in \( \text{LCS}(W, B^W + \bar{E} + 2F) \);
- there is a surface \( Z \subset F \) such that \( Z \in \text{LCS}(W, B^W + \bar{E} + 2F) \) and \( g(Z) = L \).

The following result is implied by Proposition 19.
Theorem 20. Let \( Y \) be a variety, \( \dim(Y) = 4 \), \( \mathcal{M} \) be a linear system on the variety \( Y \) having no base components, \( S_1 \) and \( S_2 \) be sufficiently general divisors in \( \mathcal{M} \), \( P \) be a smooth point on the variety \( Y \) such that \( P \in \text{CS}(Y, \frac{1}{n}\mathcal{M}) \) for \( n \in \mathbb{N} \), but the singularities of \( (Y, \frac{1}{n}\mathcal{M}) \) are canonical in some punctured neighborhood of the point \( P \), \( \pi : \hat{Y} \to Y \) be a blow up of \( P \), and \( \Pi \) be an exceptional divisor of \( \pi \). Then there is a line \( C \subset \Pi \cong \mathbb{P}^3 \) such that the inequality
\[
\text{mult}_P(S_1 \cdot S_2 \cdot \Delta) \geq 8n^2
\]
holds for any divisor \( \Delta \) on \( Y \) such that the following conditions hold:

- the divisor \( \Delta \) contains the point \( P \) and \( \Delta \) is smooth at \( P \);
- the line \( C \subset \Pi \cong \mathbb{P}^3 \) is contained in the divisor \( \pi^{-1}(\Delta) \);
- the divisor \( \Delta \) does not contain subvarieties of dimension 2 contained in \( \text{Bs}(\mathcal{M}) \).

Proof. Let \( \Delta \) be a divisor on \( Y \) such that \( P \in \Delta \), the divisor \( \Delta \) is smooth at \( P \), and \( \Delta \) does not contain any surface that is contained in the base locus of \( \mathcal{M} \). Then the base locus of the linear system \( \mathcal{M}|_\Delta \) has codimension 2 in \( \Delta \). In particular, the intersection \( S_1 \cdot S_2 \cdot \Delta \) is an effective one-cycle. Let \( \hat{S}_1 = S_1|_\Delta \) and \( \hat{S}_2 = S_2|_\Delta \). Then we must prove that the inequality
\[
(\text{mult}_P(S_1 \cdot S_2) \geq 8n^2
\]
holds, perhaps, under certain additional conditions on \( \Delta \). Put \( \hat{\mathcal{M}} = \mathcal{M}|_\Delta \). Then
\[
P \in \text{LCS}(\Delta, \frac{1}{n}\hat{\mathcal{M}})
\]
by Theorem 15. Let \( \bar{\pi} : \hat{\Delta} \to \Delta \) be a blow up of \( P \) and \( \bar{\Pi} = \bar{\pi}^{-1}(P) \). Then the diagram
\[
\begin{array}{ccc}
\hat{\Delta} & \xrightarrow{\bar{\pi}} & \hat{Y} \\
\downarrow & & \downarrow \pi \\
\Delta & \xrightarrow{\pi} & Y \\
\end{array}
\]
is commutative, where \( \hat{\Delta} \) is identified with \( \pi^{-1}(\Delta) \subset \hat{Y} \). We have \( \bar{\Pi} = \Pi \cap \hat{\Delta} \).

Let \( \mathcal{M} = \bar{\pi}^{-1}(\hat{\mathcal{M}}) \). The inequality 21 is obvious if \( \text{mult}_P(\hat{\mathcal{M}}) \geq 3n \). Hence we may assume that \( \text{mult}_P(\hat{\mathcal{M}}) < 3n \). Then
\[
\bar{\Pi} \notin \text{LCS}(\hat{\Delta}, \frac{1}{n}\hat{\mathcal{M}} + (\frac{1}{n}\text{mult}_P(\hat{\mathcal{M}}) - 2)\bar{\Pi}),
\]
which implies the existence of a subvariety \( \Xi \subset \bar{\Pi} \cong \mathbb{P}^2 \) such that \( \Xi \) is a center of log canonical singularities of \( (\hat{\Delta}, \frac{1}{n}\hat{\mathcal{M}} + (\frac{1}{n}\text{mult}_P(\hat{\mathcal{M}}) - 2)\bar{\Pi}) \).

Suppose that \( \Xi \) is a curve. Put \( \hat{S}_i = \bar{\pi}^{-1}(S_i) \). Then
\[
\text{mult}_P(\hat{S}_1 \cdot \hat{S}_2) \geq \text{mult}_P(\hat{\mathcal{M}})^2 + \text{mult}_\Xi(\hat{S}_1 \cdot \hat{S}_2),
\]
buts we can apply Theorem 16 to the log pair \( (\hat{\Delta}, \frac{1}{n}\hat{\mathcal{M}} + (\frac{1}{n}\text{mult}_P(\hat{\mathcal{M}}) - 2)\bar{\Pi}) \) in the generic point of the curve \( \Xi \). The latter implies that the inequality
\[
\text{mult}_\Xi(\hat{S}_1 \cdot \hat{S}_2) \geq 4(3n^2 - n\text{mult}_P(\hat{\mathcal{M}}))
\]
holds. Therefore we have
\[
\text{mult}_P(\hat{S}_1 \cdot \hat{S}_2) \geq \text{mult}_P(\hat{\mathcal{M}})^2 + 4(3n^2 - n\text{mult}_P(\hat{\mathcal{M}})) \geq 8n^2,
\]
which implies the inequality 21.

Suppose now that the subvariety \( \Xi \subset \bar{\Pi} \) is a point. In this case Proposition 18 implies the existence of a line \( C \subset \Pi \cong \mathbb{P}^3 \) such that
\[
C \in \text{LCS}(\hat{Y}, \frac{1}{n}\pi^{-1}(\mathcal{M}) + (\text{mult}_P(\mathcal{M})/n - 2)\bar{\Pi})
\]
and \( \Xi = C \cap \hat{\Delta} \). The line \( C \subset \Pi \) depends only on the properties of the log pair \( (Y, \frac{1}{n}\mathcal{M}) \).
Suppose that initially we take $\Delta$ such that $C \subset \pi^{-1}(\Delta)$. Then we can repeat all the previous steps of our proof. Moreover, the geometrical meaning of Proposition \ref{prop:main} is the following: the condition $C \subset \Delta = \pi^{-1}(\Delta)$ implies that

\[
C \in \text{LCS}(\hat{\Delta}, \frac{1}{n}\hat{\mathcal{M}} + (\text{mult}_P(\hat{\mathcal{M}})/n - 2)\hat{\Pi})
\]

in the case when the set $\text{LCS}(\hat{\Delta}, \frac{1}{n}\hat{\mathcal{M}} + (\frac{1}{n}\text{mult}_P(\mathcal{M}) - 2)\hat{\Pi})$ does not contain any other curve in $\Pi$. Thus we can apply the previous arguments to the divisor $\Delta$ such that $C \subset \hat{\Delta}$ and obtain the proof of the inequality \ref{ineq:main}.

In the rest of the section we prove Proposition \ref{prop:main}. We may assume that $X \cong \mathbb{P}^4$. Let $H$ be a general hyperplane section of $X$ such that $L \subset f^{-1}(H)$, $T = f^{-1}(H)$ and $S = g^{-1}(T)$. Then

\[
K_W + B^W + \tilde{E} + 2F = B_W + (\text{mult}_O(B_X) - 2)\tilde{E} + (\text{mult}_O(B_X) + \text{mult}_L(B_V) - 3)F,
\]

which implies that

\[
F \in \text{LCS}(W, B^W + \tilde{E} + 2F) \iff \text{mult}_O(B_X) + \text{mult}_L(B_V) \geq 4
\]

by Definition \ref{def:condition}. Thus we may assume that $\text{mult}_O(B_X) + \text{mult}_L(B_V) < 4$. We must prove that there is a surface $Z \subset F$ such that $Z \subset \text{LCS}(W, B^W + \tilde{E} + 2F)$ and $g(Z) = L$.

Now let $\hat{H}$ be a sufficiently general hyperplane section of the variety $X$ passing through the point $O, \hat{T} = f^{-1}(\hat{H})$ and $\hat{S} = g^{-1}(\hat{T})$. Then $O \in \text{LCS}(\hat{H}, B_X|_H)$ by Theorem \ref{thm:main} and

\[
K_W + B^W + \tilde{E} + F + \hat{S} \sim_{\hat{Q}} (f \circ g)^*(K_X + B_X + H),
\]

which implies that the log pair $(\hat{S}, (B^W + \tilde{E} + F)|_{\hat{S}})$ is not log terminal. We can apply Theorem \ref{thm:logterminal} to the morphism $f \circ g : S \to H$. Therefore either the locus $\text{LCS}(\hat{S}, (B^W + \tilde{E} + F)|_{\hat{S}})$ consists of a single isolated point in the fiber of the morphism $g|_F : F \to L$ over the point $\hat{T} \cap L$ or it contains a curve in the fiber of the morphism $g|_F : F \to L$ over the point $\hat{T} \cap L$.

Remark 23. Every element of the set $\text{LCS}(\hat{S}, (B^W + \tilde{E} + F)|_{\hat{S}})$ that is contained in the fiber of the $\mathbb{P}^2$-bundle $g|_F : F \to L$ over the point $\hat{T} \cap L$ is an intersection of $\hat{S}$ with some element of the set $\text{LCS}(W, B^W + \tilde{E} + F)$ due to the generality in the choice of $\hat{H}$.

Therefore the generality of $\hat{H}$ implies that either $\text{LCS}(W, B^W + \tilde{E} + F)$ contains a surface in the divisor $F$ dominating the curve $L$ or the only center of log canonical singularities of the log pair $(W, B^W + \tilde{E} + F)$ that is contained in the divisor $F$ and dominates the curve $L$ is a section of the $\mathbb{P}^2$-bundle $g|_F : F \to L$. On the other hand, we have

\[
\text{LCS}(W, B^W + \tilde{E} + F) \subseteq \text{LCS}(W, B^W + \tilde{E} + 2F),
\]

which implies that in order to prove Proposition \ref{prop:main} we may assume that the divisor $F$ contains a curve $C$ such that the following conditions hold:

- the curve $C$ is a section of the $\mathbb{P}^2$-bundle $g|_F : F \to L$;
- the curve $C$ is the unique element of the set $\text{LCS}(W, B^W + \tilde{E} + F)$ that is contained in the $g$-exceptional divisor $F$ and dominates the curve $L$;
- the curve $C$ is the unique element of the set $\text{LCS}(W, B^W + \tilde{E} + F)$ that is contained in the $g$-exceptional divisor $F$ and dominates the curve $L$.

We have $O \in \text{LCS}(H, M_X|_H)$ by Theorem \ref{thm:main}, but $\text{LCS}(S, (B^W + \tilde{E} + 2F)|_S) \neq \emptyset$, where $S$ is the proper transform of $H$ on $W$. We can apply Theorem \ref{thm:logterminal} to the log pair $(S, (B^W + \tilde{E} + 2F)|_S)$ and the birational morphism $f \circ g|_S : S \to H$, which implies that one of the following holds:

- the locus $\text{LCS}(S, (B^W + \tilde{E} + 2F)|_S)$ consists of a single point;
- the locus $\text{LCS}(S, (B^W + \tilde{E} + 2F)|_S)$ contains a curve $C$.

Corollary 24. Either $C \subset S$ or $S \cap C$ consists of a single point.
By construction we have $L \cong C \cong \mathbb{P}^1$ and 

$$F \cong \text{Proj}(O_L(-1) \oplus O_L(1) \oplus O_L(1))$$

and $S|_F \sim B + D$, where $B$ is the tautological line bundle on $F$ and $D$ is a fiber of the natural projection $g|_F : F \to L \cong \mathbb{P}^1$.

**Lemma 25.** The group $H^1(O_W(S - F))$ vanishes.

**Proof.** The intersection of the divisor $-g^*(E) - F$ with every curve that is contained in the divisor $E$ is non-negative and $(-g^*(E) - F)|_F \sim B + D$. Hence $-4g^*(E) - 4F$ is $h$-big and $h$-nef, where $h = f \circ g$. However, we have $X \cong \mathbb{C}^4$ and 

$$K_W - 4g^*(E) - 4F = S - F,$$

which implies $H^1(O_W(S - F)) = 0$ by the Kawamata–Viehweg vanishing (see [7]). $\square$

Thus the restriction map 

$$H^0(O_W(S)) \to H^0(O_F(S|_F))$$

is surjective, but $|S|_F$ has no base points (see §2.8 in [12]).

**Corollary 26.** The curve $C$ is not contained in $S$.

Let $\tau = g|_F$ and $\mathcal{I}_C$ be an ideal sheaf of $C$ on $F$. Then $R^1 \tau_*(B \otimes \mathcal{I}_C) = 0$ and the map 

$$\pi : O_L(-1) \oplus O_L(1) \oplus O_L(1) \to O_L(k)$$

is surjective, where $k = B \cdot C$. The map $\pi$ is given by a an element of the group 

$$H^0(O_L(k + 1)) \oplus H^0(O_L(k - 1)) \oplus H^0(O_L(k - 1)),$$

which implies $k \geq -1$.

**Lemma 27.** The equality $k = 0$ is impossible.

**Proof.** Suppose $k = 0$. Then the map $\pi$ is given by matrix $(ax + by, 0, 0)$, where $a$ and $b$ are complex numbers and $(x : y)$ are homogeneous coordinates on $L \cong \mathbb{P}^1$. Thus the map $\pi$ is not surjective over the point of $L$ at which $ax + by$ vanishes. $\square$

Therefore the divisor $B$ can not have trivial intersection with $C$. Hence the intersection of the divisor $S$ with the curve $C$ is either trivial or consists of more than one point, but we already proved that $S \cap C$ consists of one point. The obtained contradiction proves Proposition [19]

The following result is a generalization of Theorem 20.

**Theorem 28.** Let $Y$ be a variety of dimension $r \geq 5$, $\mathcal{M}$ be a linear system on $Y$ having no base components, $S_1$ and $S_2$ be general divisors in the linear system $\mathcal{M}$, $P$ be a smooth point of the variety $Y$ such that $P \in \text{CS}(Y, \frac{1}{n}\mathcal{M})$ for some natural number $n$, but the singularities of the log pair $(Y, \frac{1}{n}\mathcal{M})$ are canonical in some punctured neighborhood of $P$, $\pi : \hat{Y} \to Y$ be a blow up of the point $P$, and $\Pi$ be a $\pi$-exceptional divisor. Then there is a linear subspace $C \subset \Pi \cong \mathbb{P}^{r-1}$ having codimension 2 such that $\text{mult}_P(S_1 \cdot S_2 \cdot \Delta) > 8n^2$, where $\Delta$ is a divisor on $Y$ passing through $P$ such that $\Delta$ is smooth at $P$, the divisor $\pi^{-1}(\Delta)$ contains $C$, the divisor $\Delta$ does not contain any subvarieties of $Y$ of codimension 2 that are contained in the base locus of $\mathcal{M}$.

**Proof.** We consider only the case $r = 5$. Let $H_1, H_2, H_3$ be general hyperplane sections of the variety $Y$ passing through $P$. Put $\tilde{Y} = \cap_{i=1}^3 H_i$ and $\mathcal{M} = \mathcal{M}|_{\tilde{Y}}$. Then $\tilde{Y}$ is a surface, which is smooth at $P$, and $P \in \text{LCS}(\tilde{Y}, \frac{1}{n}\mathcal{M})$ by Theorem 15. Let $\pi : Y \to \tilde{Y}$ be a blow up of $P$, $\Pi$ be an exceptional divisor of $\pi$, and $\mathcal{M} = \pi^{-1}(\mathcal{M})$. Then the set 

$$\text{LCS}(\tilde{Y}, \frac{1}{n}\mathcal{M} + (\text{mult}_P(\mathcal{M})/n - 2)\Pi)$$

contains a subvariety $Z \subset \Pi$ such that $\dim(Z) \geq 2$. 

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In the case dim($Z$) = 4 the claim is obvious. In the case dim($Z$) = 3 we can proceed as in the proof of Theorem 2. Suppose that dim($Z$) = 3, and consider the following.

**Lemma 30.** The inequality dim($Z$) $\neq 0$ holds.

**Proof.** Suppose that $Z$ is a point. Let $S_1$ and $S_2$ be sufficiently general divisors in the linear system $M$, $f : U \to X$ be a blow up of $Z$, and $E$ be an $f$-exceptional divisor. Then Theorem 2 implies the existence of a linear system $M$ of codimension 2 such that

$$\text{mult}_Z(S_1 \cdot S_2, D) > 8m^2$$

holds for any $D \in |-K_X|$ such that $\Pi \subset f^{-1}(D)$, the divisor $D$ is smooth at $Z$, and $D$ does not contain any subvariety of $X$ of codimension 2 that is contained in the base locus of $M$.

Let $H$ be a linear system of hyperplane sections of the hypersurface $V$ such that $H \in \mathcal{H}$ if and only if $\Pi \subset (\psi \circ f)^{-1}(H)$. Then there is a linear subspace $\Sigma \subset \mathbb{P}^n$ of dimension $n - 3$ such that the divisors in the linear system $\mathcal{H}$ is cut on $V$ by the hyperplanes in $\mathbb{P}^n$ that contains the linear subspace $\Sigma$. Hence the base locus of the linear system $\mathcal{H}$ consists of the intersection $\Sigma \cap V$, but we have $\Sigma \not\subset V$ by the Lefschetz theorem. In particular, dim($\Sigma \cap V$) = $n - 4$. 

4. **Birational superrigidity.**

In this section we prove Theorem 2. Let $\psi : X \to V \subset \mathbb{P}^n$ be a double cover branched over a smooth divisor $R \subset V$ such that $n \geq 7$. Then $\psi \circ \mathcal{O}_{\mathbb{P}^n}(2r)|_V$ for some $r \in \mathbb{N}$, and

$$-K_X \sim \psi^*(\mathcal{O}_{\mathbb{P}^n}(d + r - 1 - n)|_V),$$

where $d = \deg V$. Suppose that $d + r = n$ and $d = 3$ or 4. Then the group Pic$(X)$ is generated by the divisor $-K_X$, and $(-K_X)^2 = 2d \leq 8$. Suppose that $X$ is not birationally superrigid. Then Theorem 2 implies the existence of a linear system $M$ whose base locus has codimension at least 2 and the singularities of the log pair $(X, \frac{1}{m}M)$ are not canonical, where $m$ is a natural number such that the equivalence $M \sim -mK_X$ holds. Hence the set $\text{CS}(X, \frac{1}{m}M)$ contains a proper irreducible subvariety $Z \subset X$ such that $Z \subset \text{CS}(X, \mu M)$ for some positive rational number $\mu < 1/m$.

**Corollary 29.** For a general $S \in M$ the inequality mult$_Z(S) > m$ holds.

A priori we have dim($Z$) $\leq$ dim($X$) $- 2 = n - 3$. We may assume that $Z$ has maximal dimension among subvarieties of $X$ such that the singularities of the log pair $(X, \frac{1}{m}M)$ are not canonical in their generic points. 

**Lemma 30.** The inequality dim($Z$) $\neq 0$ holds.

**Proof.** Suppose that $Z$ is a point. Let $S_1$ and $S_2$ be sufficiently general divisors in the linear system $M$, $f : U \to X$ be a blow up of $Z$, and $E$ be an $f$-exceptional divisor. Then Theorem 2 implies the existence of a linear subspace $\Pi \subset E \cong \mathbb{P}^{n-2}$ of codimension 2 such that

$$\text{mult}_Z(S_1 \cdot S_2, D) > 8m^2$$

holds for any $D \in |-K_X|$ such that $\Pi \subset f^{-1}(D)$, the divisor $D$ is smooth at $Z$, and $D$ does not contain any subvariety of $X$ of codimension 2 that is contained in the base locus of $M$. 


Let $H$ be a general divisor in $\mathcal{H}$ and $D = \psi^{-1}(H)$. Then $\Pi \subset f^{-1}(D)$, and $D$ is smooth at the point $Z$. Moreover, the divisor $D$ does not contain any subvariety $\Gamma \subset X$ of codimension 2 that is contained in the base locus of $\mathcal{M}$, because otherwise $\psi(\Gamma) \subset \Sigma \cap V$, but $\dim(\psi(\Gamma)) = n - 3$ and $\dim(\Sigma \cap V) = n - 4$. Let $H_1, H_2, \ldots, H_k$ be general divisors in $|-K_X|$ passing through the point $Z$, where $k = \dim(Z) - 3$. Then we have

$$2dm^2 = H_1 \cdots H_k \cdot S_1 \cdot S_2 \cdot D \geq \text{mult}_Z(S_1 \cdot S_2 \cdot D) > 8m^2,$$

which is a contradiction. \qed

Lemma 31. The inequality $\dim(Z) \geq \dim(X) - 4$ holds.

Proof. Suppose that $\dim(Z) \leq \dim(X) - 5$. Let $H_1, H_2, \ldots, H_k$ be sufficiently general hyperplane sections of the hypersurface $V \subset \mathbb{P}^n$, where $k = \dim(Z) > 0$. Put

$$\bar{V} = \cap_{i=1}^k H_i, \quad \bar{X} = \psi^{-1}(\bar{V}), \quad \bar{\psi} = \psi|_{\bar{X}} : \bar{X} \to \bar{V},$$

and $\mathcal{M} = \mathcal{M}|_{\bar{X}}$. Then $V$ is a smooth hypersurface of degree $d$ in $\mathbb{P}^{n-k}$, $\bar{\psi}$ is a double cover branched over a smooth divisor $R \cap \bar{V}$, $\mathcal{M}$ has no base components, and $\bar{V}$ does not contain linear subspaces of $\mathbb{P}^{n-k}$ of dimension $n - k - 3$ by the Lefschetz theorem. Let $P$ be any point of the intersection $\bar{Z} \cap \bar{X}$. Then $P \in CS(\bar{X}, \frac{1}{m}\mathcal{M})$ and we can repeat the proof of Lemma 30 to get a contradiction. \qed

Lemma 32. The inequality $\dim(Z) \neq \dim(X) - 2$ holds.

Proof. Suppose that $\dim(Z) = \dim(X) - 2$. Let $S_1$ and $S_2$ be sufficiently general divisors in the linear system $\mathcal{M}$, and $H_1, H_2, \ldots, H_{n-3}$ be general divisors in $|-K_X|$. Then

$$2dm^2 = H_1 \cdots H_{n-3} \cdot S_1 \cdot S_2 \geq \text{mult}_Z(S_1)\text{mult}_Z(S_2)(-K_X)^{n-3} \cdot Z > m^2(-K_X)^{n-3} \cdot Z,$$

because $\text{mult}_Z(\mathcal{M}) > m$. Therefore $(-K_X)^{n-3} \cdot Z < 2d$. On the other hand, we have

$$(K_X)^{n-3} \cdot Z = \begin{cases} \deg(\psi(Z) \subset \mathbb{P}^n) \text{ when } \psi|_Z \text{ is birational}, \\ 2\deg(\psi(Z) \subset \mathbb{P}^n) \text{ when } \psi|_Z \text{ is not birational}. \end{cases}$$

The Lefschetz theorem implies that $\deg(\psi(Z))$ is a multiple of $d$. Therefore $\psi|_Z$ is a birational morphism and $\deg(\psi(Z)) = d$. Hence either $\psi(Z)$ is contained in $R$, or the scheme-theoretic intersection $\psi(Z) \cap R$ is singular in every point. However, we can apply the Lefschetz theorem to the smooth complete intersection $R \subset \mathbb{P}^n$, which gives a contradiction. \qed

Lemma 33. The inequality $\dim(Z) \leq \dim(X) - 5$ holds.

Proof. Suppose that $\dim(Z) \geq \dim(X) - 4 \geq 3$. Let $S$ be a sufficiently general divisor in the linear system $\mathcal{M}$, $\hat{S} = \psi(S \cap R)$ and $\hat{Z} = \psi(Z \cap R)$. Then $\hat{S}$ is a divisor on the complete intersection $R \subset \mathbb{P}^n$ such that $\text{mult}_{\hat{Z}}(\hat{S}) > m$ and $\hat{S} \sim O_{\mathbb{P}^n}(m)|_R$, because $R$ is a ramification divisor of $\psi$. Hence, the inequality $\dim(\hat{Z}) \geq 2$ is impossible by Lemma 10. \qed

Therefore Theorem 2 is proved.

5. Reduction into characteristic $2$.

In this section we prove Proposition 1. The following result is Theorem 5.12 in §V of [5].

Theorem 34. Let $f : X \to S$ be a proper and flat morphism having irreducible and reduced fibers, $g : Z \to T$ be a proper and flat morphism having reduced fibers, where $S$ is irreducible scheme, and $T$ is a spectrum of discrete valuation ring with closed point $O$. Suppose that some component of the fiber $g^{-1}(O)$ is not geometrically ruled and the generic fiber of $g$ is birational to a fiber of the morphism $f$. Then there are countably many closed subvarieties $S_i \subset S$ such that for any closed point $s \in S$ the fiber $f^{-1}(s)$ is geometrically ruled $\iff s \in \cup S_i$.

Let $Y$ be a scheme, $L$ be a line bundle on the scheme $Y$, and $s$ be a global section of the line bundle $L^k$ for some $k \in \mathbb{N}$. Let us construct a $k : 1$ cover $Y^k_{s,L}$ of $Y$ ramified along the zeroes of the section $s$ as follows:
• let $U$ be a total space of $L$ with a natural projection $\pi : U \to Y$;
• we have $\pi_*(\mathcal{O}_U) = \bigoplus_{i \geq 0} L^{-i}$ and $\pi_*(\pi^*(L)) = \bigoplus_{i \geq -1} L^{-i}$;
• there is a canonical section $y$ of $\pi^*(L)$ that corresponds to $1 \in H^0(\mathcal{O}_Y)$;
• both $y$ and $s$ can be viewed as a section of $\pi^*((L^k))$ since $\pi_*(\pi^*((L^k))) = \bigoplus_{i \geq -k} L^{-i}$;
• let $y^k = s$ be an equation of $Y^k_{s,L}$ in $U$;
• there is a natural projection $\pi|_{Y^k_{s,L}} : Y^k_{s,L} \to Y$;
• the morphism $\pi|_{Y^k_{s,L}}$ is a $k : 1$ cover ramified along the zeroes of the section $s$.

Example 35. Let $Y = \mathbb{P}^n$ considered as a scheme over $\mathbb{Z}$, $L = \mathcal{O}_{\mathbb{P}^n}(r)$ for some $r \in \mathbb{N}$, and $s$ be a global section of $\mathcal{O}_{\mathbb{P}^n}(2r)$. Consider the weighted projective space

$$\mathbb{P}(1, \ldots, 1, r) = \text{Proj}(\mathbb{Z}[x_0, \ldots, x_n, y])$$

where $\text{wt}(y) = r$ and $\text{wt}(x_i) = 1$. Then $Y^k_{s,L} \cong V(y^2 - s) \subset \mathbb{P}(1, \ldots, 1, r)$.

The following result is Theorem 5.11 in §V of [8].

Theorem 36. Let $Y$ be a smooth projective variety over an algebraically closed field of characteristic $p$, $L$ be a line bundle on $Y$, $s$ be a general global section of $L^p$ such that $\dim(Y) \geq 3$, the divisor $L^p \otimes K_Y$ is ample and the restriction map $H^0(Y, L^p) \to (\mathcal{O}_Y/m_Y^2) \otimes L^p$ is surjective for every point $x \in Y$. Then $Y^p_{s,L}$ is not separably uniruled.

Let $Y$ be a smooth hypersurface in $\mathbb{P}^n$ of degree $d$ defined over an algebraically closed field of characteristic $2$. Let $L = \mathcal{O}_{\mathbb{P}^n}(r)|_V$ for some $r \in \mathbb{N}$ and $s$ be a sufficiently general global section of the line bundle $\mathcal{O}_{\mathbb{P}^n}(2r)|_V$. Then $Y^2_{s,L}$ is not ruled if $r \geq \frac{d+n+2}{2}$ and $n \geq 4$ by Theorem 36; therefore, Theorem 36 implies Proposition 4.

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