Dirac-Kähler equations on curved spacetimes

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Abstract

A Lagrangian theory giving rise to a version of the Dirac-Kähler equations on curved backgrounds is considered. The principal pieces are the general fields which have values in the algebra of the Dirac matrices and satisfy a Dirac-type equation. Their components are scalar, pseudo-scalar, vector, axial-vector fields and fields strength which satisfy an irreducible systems of first-order Dirac-Kähler equations having remarkable gauge and duality properties similar to those of the flat case. The vector and axial-vector fields are the physical potentials giving rise to the field strength while the scalar fields play an auxiliary role and can be eliminated by fixing a suitable gauge. The chiral components of the field strength are either self-dual or anti self-dual with respect to the Hodge duality.

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1 Introduction

The theory of the relativistic quantum fields focuses especially on the spinor and vector fields describing the fundamental particles (quarks, leptons and gauge bosons) and on the scalar field which is a universal tool or substitute in theories on flat or curved background. The free fields satisfy the well-known Klein-Gordon, Proca and Dirac equations which look like having different origins as long as these are of different orders. This conjecture encouraged many authors to seek for first-order equations, known as Bhabha equations \[1\], even for the fields with integer spins. Successful attempts are the Duffin-Kemmer-Petiau theory \[2]-[4], [5] and some recent generalizations \[6, 7, 8\] based on special algebras that are completely different to that of the Dirac matrices \[9, 10\]. However, the natural generalization the Dirac theory is the Dirac-Kähler approach \[11\] which gives rise to first-order equations for systems of scalar vector and tensor fields \[12\] we call here dual systems.

In the present paper we would like to study the Lagrangian theory of the dual systems on curved manifolds by using a simple algebraic method \[12, 13\] restricting ourselves to consider only free fields minimally coupled to the gravity of the curved spacetime. We exploits the fact that the Dirac matrices, which form a basis of a 16-dimensional algebra over \(\mathbb{C}\), behave as scalars, vectors or skew-symmetric tensors under the transformations of the gauge group of any local-Minkowskian spacetime \[14\]. Therefore, we start with general fields defined on curved manifolds with values in the algebra of the Dirac matrices \[12\]. These fields have to satisfy the usual free Dirac equation (in a matrix version) corresponding to the minimal coupling to gravity. In this manner we obtain the matrix form of the Dirac-Kähler equations \[15\] of the dual systems. The methods of the Dirac theory enable us to study the specific features of this approach in local frames or in natural ones recovering the known properties of the flat case \[12\].

The resulted system of the first-order Dirac-Kähler equations is irreducible and remarkably coherent. The vector and axial-vector fields play the role of potentials giving rise to a skew-symmetric field strength. Other equations couple either the scalar and the vector fields to the field strength or the pseudo-scalar and the axial-vector to the dual field of this field strength (in the sense of the Hodge duality \[16\]). It is interesting that, in an arbitrary gauge, the vector and axial-vector fields satisfy the second-order Proca equation but without to accomplish the Lorentz condition which is mandatory in the Proca theory. This is because of the special position of the scalar fields
which take over the contribution of the divergences of the vector fields being involved in their gauge transformations \[13\]. Under such circumstances, it is obvious that only the vector and axial-vector fields have a specific physical meaning. In addition, these equations have interesting property of chirality and duality resulted from the Dirac-Kähler theory.

This paper is organized as follows. In section 2 we briefly present the Dirac formalism on curved spacetimes pointing out the role of the 16-dimensional algebra of the point-dependent Dirac matrices in local frames. In the next section we define the general fields with values in this algebra and we build the Lagrangian theory generating the Dirac-Kähler equation for the dual system of fields. Herein we derive the second order equation and we study the duality properties using the methods of the Dirac theory. The section 4 is devoted to the covariant formalism in which we present the final form of the Lagrangian density and the resulted Dirac-Kähler field equations whose properties are investigated. In section 5 we discuss the chirality and duality properties pointing out that the chiral components of the field strength are either self-dual or anti self-dual. Finally we present our conclusion while in Appendix we give the algebra we use here.

2 The Dirac formalism

The theory of the fields with half-integer spins on curved local-Mikowskian manifolds, \( (M, g) \), can be formulated in any local chart \( \{x\} \), of coordinates \( x^\mu (\mu, \nu, \ldots = 0, 1, 2, 3) \), but only in local (non-holonomic) frames, \( (e) \), defined by the tetrad fields \( e^\hat{\mu} \) and \( \hat{e}^\mu \). These fields are labeled by local indices, \( \hat{\mu}, \hat{\nu}, \ldots = 0, 1, 2, 3 \), which are raised or lowered by the Minkowski metric \( \eta = \text{diag}(1, -1, -1, -1) \) while for the natural ones we have to use the metric tensor \( g_{\mu\nu} = \eta_{\hat{\alpha}\hat{\beta}} \hat{e}_{\hat{\alpha}}^{\mu} \hat{e}_{\hat{\beta}}^{\nu} \). Therefore, the general geometric context of the theories with spin is given by \( (M, g) \) and \( (e) \) \[14\] \[17\].

The Dirac theory deals with the 16-dimensional algebra \( \mathcal{A}/\mathbb{C} \) of the complex-valued 4 \times 4 matrices where we consider the basis \( \{I, \gamma^5, \gamma^\mu, \gamma^{\hat{\mu}}\gamma^5, S^{\hat{\mu}\nu}\} \). The usual Dirac matrices satisfy \( \{\gamma^\mu, \gamma^\nu\} = 2\eta^{\hat{\mu}\hat{\nu}}I \) where \( I \) is the identity matrix. The matrices \( S^{\hat{\mu}\nu} = \frac{i}{4} \left[\gamma^\mu, \gamma^\nu\right] \) are the basis generators of the spinor representation \( \rho_s = (\frac{1}{2}, 0) \oplus (0, \frac{1}{2}) \) of the subgroup \( \text{Spin}(\eta) = \text{SL}(2, \mathbb{C}) \subset Spin(\tilde{\eta}) \) which is the gauge group of \( (M, g) \). This plays an important role since the covariant fields on \( (M, g) \) transform under isometries according to the representations of the isometry group induced by the finite-dimensional represen-
tations $\rho$ of the gauge group $[18, 19]$. For this reason, the algebra $A$ becomes the principal algebraic ingredient in constructing the tetrad-gauge covariant theory of the fields with spin on $(M, g)$.

The Lagrangian theory of the free Dirac field $\psi$ of mass $m$, minimally coupled to the gravity of $(M, g)$, is based on the action $S_D = \int d^4x \sqrt{g} \mathcal{L}_D$ where we denote $g = |\det(g_{\mu\nu})|$. For given $(M, g)$ and $(e)$, the Lagrangian density reads $[17, 20]$

$$\mathcal{L}_D = \frac{i}{2} [\bar{\psi} \gamma^\alpha \nabla_\alpha \psi - (\nabla_\alpha \bar{\psi}) \gamma^\alpha \psi] - m \bar{\psi} \psi$$

$$\bar{\psi} = \psi^+ \gamma^0.$$  (1)

This depends on the covariant derivatives $\nabla_\alpha = e^\mu_\alpha \nabla_\mu$ whose action on the spinor field, $\nabla_\mu \psi = (\partial_\mu + \Gamma^{\text{spin}}_\mu) \psi$, is given by the spin connections

$$\Gamma^{\text{spin}}_\mu = e^\alpha_\mu \Gamma^{\text{spin}}_\alpha = \frac{i}{2} e^\beta_\nu (e^\alpha_\beta \Gamma^\alpha_\beta \mu - e^\alpha_\mu \Gamma^\alpha_\beta \beta) S^\beta_\alpha.$$  (2)

which satisfy $\Gamma^{\text{spin}}_\mu = -\Gamma^{\text{spin}}_{\mu}$.

The Dirac equation $(i\gamma_\alpha \nabla_\alpha - m) \psi = 0$, resulted from the action $S_D$, is tetrad-gauge invariant since the covariant derivatives assure the covariance of the whole theory under the tetrad-gauge transformations,

$$\hat{e}^\alpha_\mu(x) \rightarrow \hat{e}^{\alpha'}_\mu(x) = \Lambda^\alpha_\beta(x) \hat{e}^{\beta}_{\mu}(x)$$

$$e^\alpha_\mu(x) \rightarrow e^{\mu'}_\alpha(x) = \Lambda_\beta^{\alpha'}(x) e^\beta_\mu(x)$$

$$\psi(x) \rightarrow \psi'(x) = T[A(x)] \psi(x)$$

(3)

determined by the local transformations $A(x) \in SL(2, \mathbb{C})$. The transformation matrices of the spinor representation $\rho_s$ are denoted by $T(A)$ while the notation $\Lambda(A)$ stands for the Lorentz transformations which correspond to the $SL(2, \mathbb{C})$ ones through the canonical homomorphism $[21]$. The matrices $\Lambda(A)$ transform the quantities carrying local indices - for example, the Dirac matrices transform as $T(A) \gamma^\alpha T(A) = \Lambda^{\alpha'}_\beta(A) \gamma^{\beta'}$ (since $T = T^{-1}$).

At this stage it is convenient to introduce in each local frame $(e)$ the point-dependent matrices

$$\gamma^\mu(x) = e^\alpha_\mu(x) \gamma^\alpha, \quad S^{\mu\nu}(x) = e^\alpha_\mu(x) e^\nu_\beta(x) S^{\alpha \beta},$$

(4)

which have the advantage to be covariantly constant (commuting with the covariant derivatives) $[14]$. Without a mathematical close-up view, we note
that, in each point \( x \in (M, g) \), the set \( \{ I, \gamma^5, \gamma^\mu(x), \gamma^\mu(x)\gamma^5, S^{\mu\nu}(x) \} \) defined by Eq. (1) represents a basis of the algebra \( \mathcal{A}[e(x)] \) which obeys the usual algebraic rules but with \( g(x) \) replacing \( \eta \), as presented in Appendix. Based on such properties one derives the commutation relations of the covariant derivatives [14],

\[
[\nabla_\mu, \nabla_\nu] \psi = \frac{1}{4} R_{\alpha\beta\mu\nu} \gamma^\alpha \gamma^\beta \psi,
\]

and the identity \( R_{\alpha\beta\mu\nu} \gamma^\alpha \gamma^\mu \gamma^\nu = -2R_{\alpha\nu} \gamma^\nu \) where \( R_{\alpha\beta\mu\nu} \) is the curvature tensor and \( R_{\alpha\beta} = R_{\alpha\mu\beta\nu} g^{\mu\nu} \). Hereby one deduces the second-order equation,

\[
(i \gamma^\mu \nabla_\mu + m)(i \gamma^\nu \nabla_\nu - m)\psi = - \left( \nabla^2 - \frac{1}{4} R + m^2 \right) \psi = 0
\]

(5)

(6)

(with \( \nabla^2 = g^{\mu\nu} \nabla_\mu \nabla_\nu \) and \( R = R_{\mu\nu} g^{\mu\nu} \) playing the same role as the Klein-Gordon mass condition in special relativity.

3 Dirac-Kähler equations in matrix form

We use the operator formalism of the Dirac theory exploiting the properties of the algebra \( \mathcal{A} \) for constructing the Lagrangian theory of the vector fields on \( (M, g) \). We consider the dual system of complex-valued fields, \((f, h, A, B, F) : (M, g) \rightarrow \mathbb{C}, \) formed by the vector \( A \) and the axial-vector \( B \) which play the role of potentials generating the field strength \( F_{\mu\nu} = -F_{\nu\mu} \). The scalar \( f \) and the pseudo-scalar \( h \) are auxiliary fields involved in the gauge transformations of the fields \( A \) and respectively \( B \). For this reason we say that the pair \((f, A)\) represents the vector sector while the fields \((h, B)\) form the axial sector. This system is called dual since the vector sector will be related to the field \( F \) while the axial sector will couple its dual field, \(*F\), which has the components \(*F_{\alpha\beta} = \frac{1}{2} \varepsilon_{\alpha\beta\mu\nu} F^{\mu\nu} \).

Given a dual system, we define the general field \( W : (M, g) \rightarrow \mathcal{A} \) in each local frame \( (e) \) as

\[
W = i f I + h \gamma^5 + A_\mu \gamma^\mu - i B_\nu \gamma^\nu \gamma^5 + F_{\mu\nu} S^{\mu\nu}.
\]

(7)

As in the flat case [12], the principal piece of our proposal is the action \( S_W = \int d^4x \sqrt{g} \mathcal{L}_W \), of the field \( W \) minimally coupled to the gravity of \((M, g)\), assumed to have the following Lagrangian density,

\[
\mathcal{L}_W = -\frac{1}{4} \text{Tr} \left\{ \frac{i}{2} \left[ W \gamma^\alpha \nabla_\alpha W - (\nabla_\alpha W) \gamma^\alpha W \right] - m WW \right\},
\]

(8)
where $\overline{W} = \gamma^0 W^+ \gamma^0$ is the Dirac adjoint of $W$. The spinor covariant derivatives are defined now by the rule

$$\nabla_\mu W = \partial_\mu W + \left[ \Gamma_{\mu}^{\text{spin}}, W \right], \quad (9)$$

corresponding to the gauge transformations,

$$W(x) \rightarrow W'(x) = T[A(x)]W(x)\overline{T}[A(x)], \quad (10)$$
on the fields $W$ which have two spinor indices (as the matrix $\psi \overline{\psi}$). Furthermore, considering the matrix elements of the fields $W$ and $\overline{W}$ as the canonical variables of the Lagrangian density (8), after a few manipulation, we find the Dirac-Kähler equation in matrix form [15],

$$i\gamma^\alpha \nabla_\alpha W - mW = 0, \quad (11)$$

where the covariant derivatives act as in (9). This is in fact a system of 16 first-order differential equations which are linear independent, determining thus all the 16 components of the dual system $(f, h, A, B, F)$ we consider here. In this approach we have the advantage to find easily the commutator of the covariant derivatives,

$$[\nabla_\mu, \nabla_\nu]W = \frac{i}{4} R_{\alpha\beta\mu\nu} [\gamma^\alpha \gamma^\beta, W], \quad (12)$$

by using Eqs. (5) and (9). Then we can deduce the second-order equations by multiplying Eq. (11) with its Klein-Gordon divisor. We obtain the general second-order equation

$$-(i\gamma^\mu \nabla_\mu + m)(i\gamma^\nu \nabla_\nu - m)W = (\nabla^2 + m^2)W - \frac{i}{2} R_{\alpha\beta\mu\nu} S^{\alpha\beta}[S^\mu\nu, W] = 0 \quad (13)$$

which gives rise to the mass conditions of all the fields $(f, h, A, B, F)$ we present in the next section.

Hence we derived the Dirac-Kähler equations in matrix form which can be studied using the methods of the Dirac theory that help us to investigate the properties of the field equations and the effects of the internal or space-time symmetries. Because of the special structure of $\mathcal{L}_W$ the symmetry transformations may have the general form $W \rightarrow W' = U W \overline{V}$ where $U = U^{-1}$ and $\overline{V} = V^{-1}$. We remind the reader that the internal symmetries preserve the Lagrangian density but the space-time isometries can not do this
leaving merely the action invariant. Obviously, these symmetries depend on the geometry of \((M, g)\) and the couplings of the concrete physical model. Here we consider the simplest example of the \(U(1)\) transformations \(W \rightarrow W e^{i\theta}\) depending on the point-independent parameter \(\theta \in [0, 2\pi)\). As in the case of the Dirac field, the conserved quantity corresponding to this symmetry is the conserved current

\[
J^\mu = \frac{1}{4} \text{Tr} \left( \overline{W} \gamma^\mu W \right), \quad J^\mu_{\;;\mu} = 0.
\]

(14)

Another opportunity of the operator approach is the analysis of the chiral projections of the general fields [13],

\[
W = (P_L + P_R)W(P_L + P_R) = W_{RR} + W_{LL} + W_{RL} + W_{LR},
\]

(15)

obtained with the help of the projection matrices \(P_{L,R} = \frac{1}{2}(I \mp \gamma^5)\). Bearing in mind that the operator \(i\gamma^\mu \nabla_\mu\) has only \(LR\) and \(RL\) blocks we can split the field equation in four chiral projections,

\[
(i\gamma^\mu \nabla_\mu)_{LR}W_{RR} = mW_{LR}, \quad (i\gamma^\mu \nabla_\mu)_{RL}W_{LR} = mW_{RR},
\]

(16)

\[
(i\gamma^\mu \nabla_\mu)_{LR}W_{RL} = mW_{LL}, \quad (i\gamma^\mu \nabla_\mu)_{RL}W_{LL} = mW_{RL},
\]

(17)

which can be helpful in some applications. These projections have different behaviors under duality transformations since the dual field of \(W\) is defined as \(\ast W = iW\gamma^5\) and satisfies

\[
\ast W(f, h, A, B, F) = W(h, -f, B, -A, \ast F).
\]

(18)

Moreover, taking into account that \(\ast(\ast F) = -F\), we deduce a similar property for the general fields, \(\ast(W) = -W\). This conjecture enables us to introduce the duality rotations [16],

\[
W \rightarrow W' = We^{i\theta_{ch}\gamma^5}, \quad \theta_{ch} \in [0, 2\pi),
\]

(19)

whose effect is to multiply Eqs. (16) with \(e^{i\theta_{ch}}\) and Eqs. (17) with \(e^{-i\theta_{ch}}\) but without to change the solutions. Nevertheless, these transformations can not be associated to an internal symmetry since they change the structure of \(\mathcal{L}_W\). For example, taking \(\theta_{ch} = \frac{\pi}{2}\) we have \(W' = \ast W\) and \(\mathcal{L}'_W = -\mathcal{L}_W\) understanding that the fields \(W\) and \(\ast W\) satisfy the same equation (11) but derived from different actions.
4 Lagrangian formalism

The operator approach helped us to find the field equations and to do a rapid inspection of their properties. Now we must rewrite this theory exclusively in the covariant Lagrangian formalism as long as there are only fields carrying natural indices. This can be done (in usual notations, $\nabla_\mu = \partial_\mu$ and $\partial_\mu = f_\mu$) since the covariant derivatives of the general fields,

$$\nabla_\sigma W = i f_\sigma I + h_\sigma \gamma^5 + A_{\mu;\sigma} \gamma^\mu - i B_{\nu;\sigma} \gamma^\nu \gamma^5 + F_{\mu\nu;\sigma} S^{\mu\nu},$$  \hspace{1cm} (20)

may be expressed in terms of the covariant derivatives the fields ($f, h, A, B, F$).

Assuming that the components of these fields and their complex conjugated fields, ($\bar{f}, \bar{h}, \bar{A}, \bar{B}, \bar{F}$) represent the new canonical variables, we replace first Eqs. (7) and (20) in Eq. (8). Then, according to the properties listed in Appendix, we find the definitive form of the Lagrangian density,

$$L_W = m (\bar{f} f - \bar{h} h + \bar{A} A - \bar{B} B + \frac{1}{2} \bar{F} F)$$

First of all, we remark that this is irreducible since the fields of the vector and axial sectors (which appear here with opposite signs) can not be separated among themselves because of the field strength which couples both the vector and axial-vector fields. Nevertheless, a whole sector can be eliminated by dropping out simultaneously both the scalar and vector fields of this sector.

The Lagrangian density (21) gives rise to the following covariant Dirac-Kähler system of first-order equations

$$A_{\mu;\mu} - m f = 0, \quad F_{\mu;\sigma} + f_{\sigma} + m A_{\sigma} = 0, \quad (22)$$

$$B_{\mu;\mu} - m h = 0, \quad *F_{\mu;\sigma} + h_{\sigma} + m B_{\sigma} = 0, \quad (23)$$

$$- \bar{\varepsilon}_{\mu\nu}^{\cdot \cdot \cdot;\sigma} B_{\sigma;\tau} + A_{\mu;\nu} - A_{\nu;\mu} + m F_{\mu\nu} = 0, \quad (24)$$

$$\bar{\varepsilon}_{\mu\nu}^{\cdot \cdot \cdot;\sigma} A_{\sigma;\tau} + B_{\mu;\nu} - B_{\nu;\mu} + m *F_{\mu\nu} = 0, \quad (25)$$

where Eqs. (22), (23) and (24) represent the Euler-Lagrange equations deduced from (21) while Eq. (25) is derived from (24). We specify that Eqs.
are linear combinations of those given by Eq. \((11)\) which means that both these systems are equivalent \([15]\).

The coefficients of Eqs. \((22)-(25)\) are exclusively real numbers thanks to our special parametrization \((7)\). Consequently, there is a particular case in which all our fields can have real-valued components, \(f, h, A, B, F \in \mathbb{R}\). We observe that in this case we must change the Lagrangian density taking only a half of the contribution given by Eq. \((21)\). However, in general, we have to consider complex-valued fields producing the non-vanishing conserved current,

\[
J_\mu = i \left( \bar{A}_\mu f - \bar{f} A_\mu + \bar{A}^\nu F_{\mu\nu} - \bar{F}_{\mu\nu} A^\nu 
- \bar{B}_\mu h + \bar{h} B_\mu - \bar{B}^\nu \ast F_{\mu\nu} + \ast \bar{F}_{\mu\nu} B^\nu \right),
\]

resulted from Eq. \((14)\).

The second-order equations deduced from Eq. \((13)\) or derived directly by using Eqs. \((22)-(25)\) have the form

\[
\left( \nabla^2 - m^2 \right) \begin{pmatrix} f \\ h \end{pmatrix} = 0,
\left( \nabla^2 + m^2 \right) \begin{pmatrix} A_\mu \\ B_\mu \end{pmatrix} - R_{\mu\nu} \begin{pmatrix} A_\nu \\ B_\nu \end{pmatrix} = 0
\]

(27)

The first equations indicate that \(f\) and \(h\) are Klein-Gordon fields of mass \(m\). Moreover, Eqs. \((27b)\) coincide to that of a Proca field \(X\), of the same mass, obeying the mandatory Lorentz condition \(X^{\nu}_{\mu} = 0\). What is new here is that our fields \(A\) and \(B\) satisfy the Proca equation without accomplishing the Lorentz condition. This is possible because of the scalar fields \(f\) and \(h\) which take over the contributions of the divergences \(A^\mu_{\;\nu}\) and \(B^\mu_{\;\nu}\) and couple between themselves the scalar and vector sectors in Eqs. \((22b)\) and \((23b)\).

On the other hand, the scalar fields can be modified according to our current needs since Eqs. \((22)-(25)\) remain invariant under the gauge transformations \([13]\),

\[
f \rightarrow f' = f - \alpha, \quad A_\mu \rightarrow A'_\mu = A_\mu + \frac{1}{m} \partial_\mu \alpha, \quad (28)
\]

\[
h \rightarrow h' = h - \alpha, \quad B_\mu \rightarrow B'_\mu = B_\mu + \frac{1}{m} \partial_\mu \beta, \quad (29)
\]

where the scalar fields \(\alpha, \beta : (M, g) \rightarrow \mathbb{C}\) satisfy the Klein-Gordon equation, \((\nabla^2 + m^2)(\alpha, \beta) = 0\). It is worth pointing out that this gauge affects the scalar fields which can be eliminated by taking \(\alpha = f\) and \(\beta = h\). For example, the
gauge-fixing \( \alpha = f \) drops out the field \( f \) and imposes the Lorentz condition \( A^\mu \, ; \mu = 0 \) so that the vector field \( A \) becomes a genuine complex-valued Proca field but contributing to the field strength given by Eq. (24). However, whether in this gauge we vanish the axial sector taking, in addition, \( h = 0 \) and \( B = 0 \) then we remain only with the Proca field \( A \) and its traditional field strength which satisfy the well-known equations \( A^\mu \, ; \nu - A^\nu \, ; \mu + mF^\mu \nu = 0, \quad F^\mu \cdot \sigma^\nu + mA_{\sigma} = 0 \) and \( *F^\mu \cdot \sigma^\nu = 0 \).

The general conclusion is that only two fields may have a specific physical significance. These are the vector field \( A \) and the axial-vector field \( B \) which play the role of potentials generating the field strength \( F \). The scalar fields can not have an independent meaning since their form depend on the gauge-fixing.

5 Self-duality

The dual system \( (f, h, A, B, F) \) includes pairs of independent scalar and vector fields but with different behaviors under the parity transformation. This situation discourage us to consider arbitrary linear combinations of these fields apart from some special cases of physical (or mathematical) interest as that of the chiral projections defined by Eq. (15). We observe that these can be expressed in a simpler form as,

\[
W_{LL} = i(+)fP_L + \frac{1}{2}(+)F^\mu \cdot S^\mu, \quad W_{RL} = (+)A^\mu \gamma^\mu P_L, \\
W_{RR} = i(-)fP_R + \frac{1}{2}(-)F^\mu \cdot S^\mu, \quad W_{LR} = (-)A^\mu \gamma^\mu P_R,
\]

marking out the new fields

\[
(\pm)f = f \pm ih, \quad (\pm)A = A \pm iB, \quad (\pm)F = F \pm i*F,
\]

which mix the vector and axial sectors. Consequently, the parity transformation has to change the \((\pm)\) components into the \((\mp)\) ones just as in the case of the chiral components of the Dirac field. Therefore, we can say that Eqs. (32) define the chiral components of our dual system.

Changing the parametrization again, we consider the fields (32) and their complex conjugated fields, \((\pm)\bar{f}, \, (\pm)\bar{A}, \, \text{and} \, (\pm)\bar{F}, \) as the new canonical variables of the Lagrangian density (21). Then we find that the system of equations (22)-(25) splits in two independent chiral subsystems, denoted by \((+)\) and
\((\pm)A^\mu_\mu - m (\pm)f = 0\), \((\pm)F^\mu_\sigma;\mu + (\pm)f;_\sigma + m (\pm)A^\sigma_\sigma = 0\), \((\pm)i \varepsilon^{\mu\nu;\sigma} (\pm)A^\sigma_\sigma;\tau\sigma + (\pm)A^\mu_\mu;\nu + (\pm)f;_\nu,\sigma + m (\pm)F^\mu_\nu = 0\) \((33)\)

This splitting is rather formal, at the level of the field equations only, since the Lagrangian density still mixes the components of the chiral subsystems \((\pm)\). For example, the mass term of the scalar fields becomes now

\[
\bar{f}f - \bar{h}h \to \frac{1}{2} \left( (\pm)\bar{f}(\pm)f + (\mp)\bar{f}(\pm)f \right). \quad (35)
\]

We note that Eqs. \((33)\) and \((34)\) can be obtained directly by replacing Eqs. \((30)\) and \((30)\) in Eqs. \((16)\) and \((17)\). Moreover, the second-order equations are similar to Eqs. \((27)\) while the gauge transformations \((28)\) and \((29)\) can be put in the compact form,

\[
(\pm)f \to (\pm)f' = (\pm)f - (\pm)\alpha, \quad (\pm)A_\mu \to (\pm)A'_\mu = (\pm)A_\mu + \frac{1}{m} \partial_\mu (\pm)\alpha, \quad (36)
\]

denoting \((\pm)\alpha = \alpha + i\beta\).

We obtained thus two simple chiral subsystems we expect to have remarkable duality properties. Indeed, bearing in mind that the fields strength and their complex conjugated fields satisfy

\[
{^*}(\pm)F = \mp i (\pm)F, \quad {^*}(\pm)\bar{F} = \pm i (\pm)\bar{F}, \quad (37)
\]

we draw the conclusion that the fields \((\pm)F\) and \((\mp)\bar{F}\) are \textit{self-dual} while \((\pm)\bar{F}\) and \((\mp)F\) are \textit{anti self-dual}. In the particular case of real-valued fields, \(f, h, A, B, F \in \mathbb{R}\), the supplemental rule \((\pm)\bar{F} = (\mp)\bar{F}\) holds for all the fields of the subsystems \((\pm)\) and we are left only with one self-dual field, \((\pm)\bar{F} = (\mp)F\), and one anti-self dual field, \((\mp)F = (\pm)\bar{F}\). We must specify that the structure of Eqs. \((34)\) is crucial in generating the duality properties.

Under such circumstances, the duality rotations \((19)\) transform the fields \((\pm)\) as,

\[
(\pm)f \to (\pm)f e^{\mp i \theta_{ch}}, \quad (\pm)A \to (\pm)A e^{\mp i \theta_{ch}}, \quad (38)
\]

with different phase factors which do not modify the solutions of the systems \((22)-(23)\) but affect the form of \(L_W\). We may verify this fact simply, observing that the mass term \((35)\) transforms as

\[
(\pm)\bar{f}(\pm)f + (\mp)\bar{f}(\mp)f \to e^{2i \theta_{ch}} (\pm)\bar{f}(\pm)f + e^{-2i \theta_{ch}} (\mp)\bar{f}(\mp)f. \quad (39)
\]
Concluding we can say that the chiral subsystems are self-dual since their field strength are either self-dual or anti self-dual and, in addition, the duality rotations do not mix components of different signs ($\pm$).

6 Concluding remarks

The Dirac theory offered us the appropriate framework for analyzing the first-order Dirac-Kähler equations governing the dual systems minimally coupled to the gravity of a curved background. Each system has only two fields of physical relevance, the vector and axial-vector fields. These have the same mass but this does not represent a redundancy as long as these fields play different physical roles. More specific, in a concrete physical model the vector field couples a vector current while the axial-vector field must be coupled to an axial current. We note that these couplings may increase the coherence of the model when the vector and axial-vector fields are related to each other as components of the same dual system [13].

However, before to study the physical behavior of the dual systems, there are many technical problems to be solved. The first one is that of the parametrization for which we do not have many options if we keep the gauge-covariant Dirac-type equations. Nevertheless, a new parametrization must be introduced in the massless case [22] since in the present one the field strength is cut out from its potentials when the mass vanishes. Further investigation may focus on the most interesting part of this theory concerning the structure and properties of the stress-energy tensor and the conserved quantities corresponding to the internal or space-time symmetries. A surprise could be to find that the dual systems on curved backgrounds deal with models of dark matter or energy.

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Appendix: The algebras $A[e(x)]$

For given $(M, g)$ and $(e)$, the matrices $\{I, \gamma^5, \gamma^\mu, \gamma^\mu\gamma^5, S^{\mu\nu}\}$ form a basis of the mapping $A(e) : (M, g) \to A$. In each point $x \in (M, g)$ the algebra
\( \mathcal{A}[e(x)] \) is isomorphic to \( \mathcal{A} \) according to Eqs. (4). This means that any pair of algebras, \( \mathcal{A}[e(x)] \) and \( \mathcal{A}[e'(x')] \), are isomorphic to each other. Fortunately, the algebraic rules of these algebras do not depend on \( (e) \) but only on the metric tensor \( g \). The commutation rules are

\[
[\gamma^\mu, \gamma^\nu] = -4i S^{\mu\nu}, \quad [S^{\mu\nu}, \gamma^5] = 0, \tag{40}
\]

\[
[\gamma^\mu, \gamma^\nu \gamma^5] = 2g^{\mu\nu} \gamma^5, \quad [S^{\mu\nu}, \gamma^\sigma] = i(g^{\sigma\tau} \gamma^\mu - g^{\mu\tau} \gamma^\sigma), \tag{41}
\]

\[
[\gamma^\mu \gamma^5, \gamma^\nu \gamma^5] = 4i S^{\mu\nu}, \quad [S^{\mu\nu}, \gamma^\sigma \gamma^5] = i(g^{\sigma\tau} \gamma^\mu \gamma^5 - g^{\mu\tau} \gamma^\sigma \gamma^5), \tag{42}
\]

\[
[S^{\mu\nu}, S^{\sigma\tau}] = i(g^{\mu\tau} S^{\nu\sigma} - g^{\nu\sigma} S^{\mu\tau} + g^{\nu\tau} S^{\mu\sigma} - g^{\mu\sigma} S^{\nu\tau}), \tag{43}
\]

while the anti-commutation ones read

\[
\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu} I, \quad \{S^{\mu\nu}, \gamma^5\} = -i \varepsilon^{\mu\nu\sigma\tau} S^{\sigma\tau}, \tag{44}
\]

\[
\{\gamma^\mu, \gamma^\nu \gamma^5\} = -2 \varepsilon^{\mu\nu\sigma\tau} S^{\sigma\tau}, \quad \{S^{\mu\nu}, \gamma^\sigma\} = \varepsilon^{\mu\nu\sigma\tau} \gamma^\tau \gamma^5, \tag{45}
\]

\[
\{\gamma^\mu \gamma^5, \gamma^\nu \gamma^5\} = -2g^{\mu\nu} I, \quad \{S^{\mu\nu}, \gamma^\sigma \gamma^5\} = \varepsilon^{\mu\nu\sigma\tau} \gamma^\tau, \tag{46}
\]

\[
\{S^{\mu\nu}, S^{\sigma\tau}\} = \frac{1}{2} (g^{\mu\sigma} \gamma^\nu \gamma^\tau - g^{\mu\tau} \gamma^\nu \gamma^\sigma) I - \frac{i}{2} \varepsilon^{\mu\nu\sigma\tau} \gamma^5, \tag{47}
\]

where we denote \( \varepsilon^{\mu\nu\sigma\tau} = e^\mu_\alpha e^\nu_\beta e^\sigma_\delta e^\tau_\gamma \varepsilon^{\alpha\beta\gamma\delta} \) adopting the convention \( \varepsilon_{0123} = -\varepsilon^{0123} = 1 \) for the usual Levi-Civita symbol \( \varepsilon_{\alpha\beta\gamma\delta} \) carrying local indices. In addition we use the identities \[16\]

\[
\varepsilon^{\mu\nu\sigma\tau} \varepsilon^{\alpha\beta\sigma\tau} = -2 \left( \delta^{\alpha}_\mu \delta^{\beta}_\nu - \delta^{\alpha}_\nu \delta^{\beta}_\mu \right), \quad \varepsilon^{\mu\nu\sigma\tau} \varepsilon^{\alpha\nu\sigma\tau} = -6 \delta^{\alpha}_\mu. \tag{48}
\]

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