Sylvester-type quaternion matrix equations with arbitrary equations and arbitrary unknowns

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Abstract: In this paper, we prove a conjecture which was presented in a recent paper [Linear Algebra Appl. 2016; 496: 549–593]. We derive some practical necessary and sufficient conditions for the existence of a solution to a system of coupled two-sided Sylvester-type quaternion matrix equations with arbitrary equations and arbitrary unknowns $A_i X_i B_i + C_i X_{i+1} D_i = E_i$, $i = 1, \ldots, k$. As an application, we give some practical necessary and sufficient conditions for the existence of an $\eta$-Hermitian solution to the system of quaternion matrix equations $A_i X_i A_i^\eta + C_i X_{i+1} C_i^\eta = E_i$ in terms of ranks, $i = 1, \ldots, k$.

Keywords: Sylvester equation; Solvability Quaternion; $\eta$-Hermitian

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1. Introduction

In this paper, we consider the the system of coupled two-sided Sylvester-type quaternion matrix equations with $k$ equations and $k + 1$ unknowns

$$
\begin{align*}
A_1 X_1 B_1 + C_1 X_2 D_1 &= E_1, \\
A_2 X_2 B_2 + C_2 X_3 D_2 &= E_2, \\
A_3 X_3 B_3 + C_3 X_4 D_3 &= E_3, \\
&\quad \vdots \\
A_k X_k B_k + C_k X_{k+1} D_k &= E_k,
\end{align*}
$$

(1.1)

where $A_i, B_i, C_i, D_i,$ and $E_i$ are given matrices, $X_i$ are unknowns. Since Baksalary [1] first studied the system (1.1) for the case $k = 1$ over the field in 1980, there have been many papers to consider the case $k = 1$ (e.g., [2], [10]). In 2016, He et.al [4] investigated a simultaneous decomposition to consider the system (1.1) for the case $k = 3$. They established some necessary and sufficient conditions for the existence of a solution to the system (1.1) in terms of ranks of the matrices involved [4]. At the end of the paper [4], they gave a conjecture on the solvability condition to the system (1.1) in terms of ranks for the case $k \geq 4$, see Theorem 2.1. Notice however, it is hard to solve the conjecture using the approach presented in [4].

In this paper, we use another approach to prove Theorem 2.1, i.e., the conjecture which proposed in [4]. We then consider solvability conditions to the system of quaternion matrix
equations involving $\eta$-Hermicity

$$
\begin{align*}
A_1X_1A_1^{\eta} + C_1X_2C_1^{\eta} &= E_1, \\
A_2X_2A_2^{\eta} + C_2X_3C_2^{\eta} &= E_2, \\
A_3X_3A_3^{\eta} + C_3X_4C_3^{\eta} &= E_3, \\
& \quad \ldots \\
A_kX_kA_k^{\eta} + C_kX_{k+1}C_k^{\eta} &= E_k,
\end{align*}
$$

(1.2)

The remainder of this paper is organized as follows. In Section 2, we give the main result of this paper. We derive some practical necessary and sufficient conditions for the existence of a solution to the system (1.1). In Section 3, we give the proof of Theorem 2.1. In Section 4, we derive some practical necessary and sufficient conditions for the existence of an $\eta$-Hermitian solution to the system (1.2) in terms of ranks, see Theorem 4.1.

Let $\mathbb{R}$ and $\mathbb{H}^{m \times n}$ stand, respectively, for the real field and the space of all $m \times n$ matrices over the real quaternion algebra

$$
\mathbb{H} = \{a_0 + a_1i + a_2j + a_3k \mid i^2 = j^2 = k^2 = ijk = -1, a_0, a_1, a_2, a_3 \in \mathbb{R}\}.
$$

The symbols $r(A)$ and $A^*$ stand for the rank of a given quaternion matrix $A$ and the conjugate transpose of $A$ and the transposed of $A$, respectively. $I$ and $0$ are the identity matrix and zero matrix with appropriate sizes, respectively. The Moore-Penrose inverse $A^\dagger$ of a quaternion matrix $A$, is defined to be the unique matrix $A^\dagger$, such that

(i) $AA^\dagger A = A$, (ii) $A^\dagger AA^\dagger = A^\dagger$, (iii) $(AA^\dagger)^* = AA^\dagger$, (iv) $(A^\dagger A)^* = A^\dagger A$.

Furthermore, $L_A$ and $R_A$ stand for the projectors $L_A = I - A^\dagger A$ and $R_A = I - AA^\dagger$ induced by $A$, respectively. For more definitions and properties of quaternions, we refer the reader to the book [7].

2. The main result

In this section, we give some practical necessary and sufficient conditions for the existence of a solution to the system (1.1) in terms of ranks.

**Theorem 2.1.** The system (1.1) is consistent if and only if the following $2k(k+1)$ rank equalities hold for all $i = 1, \ldots, k$ and $1 \leq m < n \leq k$

$$
r \begin{pmatrix} A_i & E_i & C_i \end{pmatrix} = r \begin{pmatrix} A_i & C_i \end{pmatrix}, \quad r \begin{pmatrix} B_i \\ E_i \\ D_i \end{pmatrix} = r \begin{pmatrix} B_i \\ D_i \end{pmatrix},
$$

(2.1)

$$
r \begin{pmatrix} A_i & E_i \\ 0 & D_i \end{pmatrix} = r(A_i) + r(D_i), \quad r \begin{pmatrix} B_i & 0 \\ E_i & C_i \end{pmatrix} = r(B_i) + r(C_i),
$$

(2.2)
\[ r \begin{pmatrix} A_m & E_m & C_m \\ D_m & A_{m+1} - E_{m+1} & C_{m+1} \\ & \ddots & \ddots \\ & & A_n - (-1)^{n-m} E_n & C_n \end{pmatrix} + r \begin{pmatrix} D_m & B_{m+1} & D_{m+2} \\ D_m & D_{m+1} & D_{m+2} \\ & \ddots & \ddots \\ & & D_{n-1} & B_n \end{pmatrix}, \quad (2.3) \]

\[ r \begin{pmatrix} B_m & E_m & C_m \\ D_m & A_{m+1} - E_{m+1} & \ddots \\ & \ddots & B_n \\ & & \ddots & \ddots \\ & & & A_n - (-1)^{n-m} E_n \end{pmatrix} + r \begin{pmatrix} D_m & B_{m+1} \\ D_m & D_{m+1} \end{pmatrix}, \quad (2.4) \]

\[ r \begin{pmatrix} A_m & E_m & C_m \\ D_m & A_{m+1} - E_{m+1} & C_{m+1} \\ & \ddots & \ddots \\ & & A_n - (-1)^{n-m} E_n & C_n \end{pmatrix} + r \begin{pmatrix} D_m & B_{m+1} & D_{m+2} \\ D_m & D_{m+1} & D_{m+2} \\ & \ddots & \ddots \\ & & D_{n-1} & B_n \end{pmatrix}, \quad (2.5) \]

where the blank entries in (2.3)-(2.6) are all zeros.

3. Proof of Theorem 2.1

In this section, we give the proof of Theorem 2.1. We proceed with the proof by induction. Lemma 3.1 proves Theorem 2.1 for the case \( k = 1 \).
Lemma 3.1. ([1], [9], [10]) Consider the quaternion matrix equation
\[ A_1X_1B_1 + C_1X_2D_1 = E_1. \]  
Let \( M_1 = R_{A_1}C_1, N_1 = D_1L_{B_1}, S_1 = C_1L_{M_1}. \) Then the following statements are equivalent:
(1) Equation (3.1) is consistent.
(2) \( R_{M_1}R_{A_1}E_1 = 0, \ EL_{B_1}L_{N_1} = 0, \ R_{A_1}E_1L_{D_1} = 0, \ R_{C_1}E_1L_{B_1} = 0. \)  
(3) \( r(A_1 E_1 C_1) = r(A_1 C_1), \ r(B_1 E_1) = r(B_1), \)  
\( r(A_1 E_1) = r(A_1) + r(D_1), \ r(B_1 E_1 C_1) = r(B_1) + r(C_1). \)  
In this case, the general solution to (3.1) can be expressed as
\[ X_1 = A_1^†E_1B_1^† - A_1^†C_1M_1^†E_1B_1^† - A_1^†S_1C_1E_1N_1^†D_1B_1^† - A_1^†S_1Y_1R_{N_1}D_1B_1^† + L_{A_1}Y_2 + Y_3R_{B_1}, \]  
\[ X_2 = M_1^†E_1D_1^† + S_1^†S_1C_1^†E_1N_1^† + L_{M_1}L_{S_1}Y_4 + Y_5R_{D_1} + L_{M_1}Y_1R_{N_1}. \]  
where \( Y_1, Y_2, Y_3, Y_4, Y_5 \) are arbitrary matrices over \( H \) with appropriate sizes. As a special case of (3.1), the matrix equation
\[ A_1X_1 + X_2D_1 = E_1 \]  
is consistent if and only if
\[ R_{A_1}E_1L_{D_1} = 0. \]  

To simplify the ranks of the proof of Theorem 2.1 we need the following lemma.

Lemma 3.2. ([9]). Let \( A \in H^{m \times n}, B \in H^{m \times k}, \) and \( C \in H^{l \times n} \) be given. Then
(1) \( r(A) + r(R_A B) = r(B) + r(R_B A) = r \left( \begin{array}{cc} A & B \end{array} \right). \)
(2) \( r(A) + r(CL_A) = r(C) + r(AL_C) = r \left( \begin{array}{c} A \\ C \end{array} \right). \)

Now we give the proof of Theorem 2.1

Proof of Theorem 2.1. We proceed with the proof by induction. If \( k = 1, \) then the system (1.1) becomes the equation (3.1) and the rank equalities (2.1)-(2.6) become (3.3)-(3.4). Hence, the statement is true if \( k = 1. \)

Suppose the statement is true when the number of the equations is \( k - 1. \) We show by induction that it is true for the number of the equations \( k. \)
We separate the system (1.1) into $k$ parts

$$A_1 X_1 B_1 + C_1 X_2 D_1 = E_1,$$  \hspace{1cm} (3.9)

$$A_2 X_2 B_2 + C_2 X_3 D_2 = E_2,$$  \hspace{1cm} (3.10)

$$A_3 X_3 B_3 + C_3 X_4 D_3 = E_3,$$  \hspace{1cm} (3.11)

$$\vdots$$

$$A_k X_k B_k + C_k X_{k+1} D_k = E_k.$$  \hspace{1cm} (3.12)

It follows from Lemma 3.1 that the equations (3.9) - (3.12) are consistent if and only if the rank equalities (2.1) and (2.2) hold. The general solution to the equation

$$A_i X_i B_i + C_i X_{i+1} D_i = E_i$$  \hspace{1cm} (3.13)

can be expressed as

$$X_i = A_i^\dagger E_i B_i^\dagger - A_i^\dagger C_i M_i^\dagger E_i B_i^\dagger - A_i^\dagger S_i C_i^\dagger E_i N_i^\dagger D_i B_i^\dagger - A_i^\dagger S_i Y_i R_i N_i D_i B_i^\dagger + L_{A_i} Z_1^{(i)} + Z_2^{(i)} R_{B_i},$$

$$X_{i+1} = M_i^\dagger E_i D_i^\dagger + S_i^\dagger S_i C_i^\dagger E_i N_i^\dagger + L_{M_i} L_{S_i} Z_3^{(i)} + Z_4^{(i)} R_{D_i} + L_{M_i} Y_i R_{N_i},$$

where

$$M_i = R_{A_i} C_i, \quad N_i = D_i L_{B_i}, \quad S_i = C_i L_{M_i},$$

and $Y_i, Z_1^{(i)}, Z_2^{(i)}, Z_3^{(i)}, Z_4^{(i)}$ are arbitrary matrices over $\mathbb{H}$ with appropriate sizes.

Let $X_{i+1}$ in the $i$th equation be equal to $X_{i+1}$ in the $(i+1)$th equation for every $i = 1, \ldots, k-1$. Then, we have the following system

$$
\begin{pmatrix}
L_{M_1} L_{S_1} & -L_{A_1} & Z_1^{(1)} \\
L_{M_2} L_{S_2} & -L_{A_2} & Z_1^{(2)} \\
\vdots & \vdots & \vdots \\
L_{M_k} L_{S_k} & -L_{A_{k-1}} & Z_1^{(k)}
\end{pmatrix}
\begin{pmatrix}
Z_2^{(1)} \\
Z_2^{(2)} \\
\vdots \\
Z_2^{(k)}
\end{pmatrix}
\begin{pmatrix}
R_{D_1} \\
R_{D_2} \\
\vdots \\
R_{D_{k-1}}
\end{pmatrix}
= F_1 - L_{M_1} Y_1 R_{N_1} - A_1^\dagger S_1 Y_2 R_{N_1} D_2 B_2^\dagger,
$$

$$
\begin{pmatrix}
L_{M_2} L_{S_2} & -L_{A_3} & Z_3^{(2)} \\
L_{M_3} L_{S_3} & -L_{A_4} & Z_3^{(3)} \\
\vdots & \vdots & \vdots \\
L_{M_k} L_{S_k} & -L_{A_{k-1}} & Z_3^{(k)}
\end{pmatrix}
\begin{pmatrix}
Z_4^{(1)} \\
Z_4^{(2)} \\
\vdots \\
Z_4^{(k)}
\end{pmatrix}
\begin{pmatrix}
R_{D_2} \\
R_{D_3} \\
\vdots \\
R_{D_{k-1}}
\end{pmatrix}
= F_2 - L_{M_2} Y_2 R_{N_2} - A_3^\dagger S_3 Y_3 R_{N_2} D_3 B_3^\dagger,
$$

$$
\vdots
$$

$$
\begin{pmatrix}
L_{M_k} L_{S_k} & -L_{A_{k-1}} & Z_3^{(k)} \\
L_{M_k} L_{S_{k-1}} & -L_{A_k} & Z_3^{(k-1)} \\
\vdots & \vdots & \vdots \\
L_{M_{k-1}} L_{S_{k-1}} & -L_{A_{k-1}} & Z_3^{(k-1)}
\end{pmatrix}
\begin{pmatrix}
Z_4^{(k)} \\
Z_4^{(k-1)} \\
\vdots \\
Z_4^{(k-1)}
\end{pmatrix}
\begin{pmatrix}
R_{D_{k-1}} \\
R_{D_{k-1}} \\
\vdots \\
R_{D_{k-1}}
\end{pmatrix}
= F_{k-1} - L_{M_{k-1}} Y_{k-1} R_{N_{k-1}} - A_k^\dagger S_k Y_k R_{N_k} D_k B_k^\dagger,
$$

where $j = 1, k-1$, and

$$F_j = A_{j+1}^\dagger E_{j+1} B_{j+1}^\dagger - A_{j+1}^\dagger C_{j+1} M_{j+1}^\dagger E_{j+1} B_{j+1}^\dagger - A_{j+1}^\dagger S_{j+1} C_{j+1}^\dagger E_{j+1} N_{j+1}^\dagger D_{j+1} B_{j+1}^\dagger - (M_j^\dagger E_j B_j^\dagger + S_j^\dagger S_j C_j^\dagger E_j N_j^\dagger),$$

Hence, the system (1.1) is consistent if and only if (2.1) and (2.2) hold and the system (3.17) is consistent. We now turn our attention to the solvability conditions to the system (3.17). Observe
that each equation in the system \((3.17)\) has the form of \((3.7)\). It follows from the condition \((3.8)\) in Lemma \((3.1)\) that the equation

\[
\left( L_{M_j}L_{S_j} - L_{A_{j+1}} \right) \left( \begin{array}{c} Z_3^{(j)} \\ Z_4^{(j+1)} \end{array} \right) + \left( \begin{array}{c} Z_2^{(j)} \\ Z_2^{(j+1)} \end{array} \right) \left( \begin{array}{c} R_{D_j} \\ -R_{B_{j+1}} \end{array} \right) = F_j - L_{M_j}Y_jR_{N_j} - A_{j+1}^\dagger S_{j+1}Y_{j+1}R_{N_{j+1}}D_{j+1}B_{j+1}^\dagger
\]

is consistent if and only if

\[
R\left( L_{M_j}L_{S_j} - L_{A_{j+1}} \right) \left( F_j - L_{M_j}Y_jR_{N_j} - A_{j+1}^\dagger S_{j+1}Y_{j+1}R_{N_{j+1}}D_{j+1}B_{j+1}^\dagger \right) L \left( \begin{array}{c} R_{D_j} \\ -R_{B_{j+1}} \end{array} \right) = 0.
\]

Put

\[
\hat{A}_j = R\left( L_{M_j}L_{S_j} - L_{A_{j+1}} \right) L_{M_j}, \quad \hat{B}_j = R_{N_j} L \left( \begin{array}{c} R_{D_j} \\ -R_{B_{j+1}} \end{array} \right),
\]

\[
\hat{C}_j = R\left( L_{M_j}L_{S_j} - L_{A_{j+1}} \right) A_{j+1}^\dagger S_{j+1}, \quad \hat{D}_j = R_{N_{j+1}} D_{j+1} B_{j+1}^\dagger L \left( \begin{array}{c} R_{D_j} \\ -R_{B_{j+1}} \end{array} \right),
\]

\[
\hat{E}_j = R\left( L_{M_j}L_{S_j} - L_{A_{j+1}} \right) F_j L \left( \begin{array}{c} R_{D_j} \\ -R_{B_{j+1}} \end{array} \right).
\]

Then the equation \((3.20)\) becomes the following form

\[
\hat{A}_j Y_j \hat{B}_j + \hat{C}_j Y_{j+1} \hat{D}_j = \hat{E}_j.
\]

Thus, the system \((3.17)\) is consistent if and only if the following system is consistent

\[
\begin{align*}
\hat{A}_1 Y_1 \hat{B}_1 + \hat{C}_1 Y_2 \hat{D}_1 &= \hat{E}_1, \\
\hat{A}_2 Y_2 \hat{B}_2 + \hat{C}_2 Y_3 \hat{D}_2 &= \hat{E}_2, \\
&\vdots \\
\hat{A}_{k-1} Y_{k-1} \hat{B}_{k-1} + \hat{C}_{k-1} Y_k \hat{D}_{k-1} &= \hat{E}_{k-1},
\end{align*}
\]

where \(\hat{A}_j, \hat{B}_j, \hat{C}_j, \hat{D}_j, \hat{E}_j\) are defined in \((3.21)-(3.23)\). Note that the form of the system \((3.25)\) is same as the main system \((1.1)\) and the number of the equations in \((3.25)\) is \(k-1\). Applying the induction hypothesis on the system \((3.25)\), we have that the system \((3.25)\) is consistent if and only if the following \(2(k-1)k\) rank equalities hold for all \(j = 1, \ldots, k-1\) and \(1 \leq m < n \leq k-1\)

\[
r\left( \begin{array}{cc} \hat{A}_j & \hat{E}_j \\ \hat{B}_j & \hat{D}_j \end{array} \right) = r\left( \begin{array}{cc} \hat{A}_j & \hat{C}_j \end{array} \right),
\]

\[
r\left( \begin{array}{cc} \hat{B}_j & \hat{E}_j \\ \hat{B}_j & \hat{D}_j \end{array} \right) = r\left( \begin{array}{cc} \hat{B}_j & \hat{D}_j \end{array} \right),
\]

\[
r\left( \begin{array}{cc} \hat{A}_j & \hat{E}_j \\ 0 & \hat{D}_j \end{array} \right) = r(\hat{A}_j) + r(\hat{D}_j),
\]
\[ r \begin{pmatrix} \hat{B}_j & 0 \\ \hat{E}_j & \hat{C}_j \end{pmatrix} = r(\hat{B}_j) + r(\hat{C}_j), \] (3.29)

\[ r \begin{pmatrix} A_m & E_m & \hat{C}_m \\ \hat{D}_m & B_{m+1} & \hat{C}_m \\ \hat{A}_{m+1} & -E_{m+1} & \hat{C}_m \\ \vdots & \vdots & \vdots \\ \hat{A}_n & (\hat{A}_n) & (-1)^{m-n} \hat{E}_n \hat{C}_n \end{pmatrix} + r \begin{pmatrix} D_m & \hat{B}_{m+1} & \hat{B}_{m+2} \\ \hat{D}_m & \hat{B}_{m+1} & \hat{B}_{m+2} \\ \hat{D}_{m+1} & \hat{B}_{m+2} & \hat{B}_{m+3} \\ \vdots & \vdots & \vdots \end{pmatrix}, \] (3.30)

\[ r \begin{pmatrix} \hat{B}_m & E_m & \hat{C}_m \\ \hat{D}_m & \hat{B}_{m+1} & \hat{C}_m \\ \hat{A}_{m+1} & -E_{m+1} & \hat{B}_{m+1} \\ \vdots & \vdots & \vdots \\ \hat{B}_n & (\hat{B}_n) & (-1)^{n-m} \hat{E}_n \hat{B}_n \end{pmatrix} + r \begin{pmatrix} \hat{C}_m & \hat{C}_{m+1} & \hat{C}_{m+1} \\ \hat{C}_{m+1} & \hat{C}_{m+2} & \hat{C}_{m+2} \\ \hat{C}_{m+2} & \hat{C}_{m+3} & \hat{C}_{m+3} \\ \vdots & \vdots & \vdots \end{pmatrix}, \] (3.31)

\[ r \begin{pmatrix} A_m & E_m & \hat{C}_m \\ \hat{D}_m & B_{m+1} & \hat{C}_m \\ \hat{A}_{m+1} & -E_{m+1} & \hat{B}_{m+1} \\ \vdots & \vdots & \vdots \end{pmatrix} + r \begin{pmatrix} D_m & B_{m+1} & \hat{B}_{m+2} \\ \hat{D}_m & B_{m+1} & \hat{B}_{m+2} \\ \hat{D}_{m+1} & B_{m+2} & \hat{B}_{m+3} \\ \vdots & \vdots & \vdots \end{pmatrix}. \] (3.32)
We establish some useful facts that will be used throughout this part.

Next we will prove that the rank equalities (3.26)-(3.33) are equivalent with the rank equalities (2.3)-(2.6). We establish some useful facts that will be used throughout this part.

**Fact 1:** The expression of $F_j$ in (3.13): Since

\[ X_{j+1}^2 := A_{j+1}^\dagger E_{j+1} B_{j+1} - A_{j+1}^\dagger C_{j+1} M_{j+1}^\dagger E_{j+1} B_{j+1} - A_{j+1}^\dagger S_{j+1} C_{j+1}^\dagger E_{j+1} N_{j+1}^\dagger D_{j+1} B_{j+1} \]

(3.34)

and

\[ X_{j+1}^1 := M_{j+1}^\dagger E_{j+1} D_{j+1} - S_{j+1}^\dagger S_{j+1} C_{j+1}^\dagger E_{j+1} N_{j+1}^\dagger \]

(3.35)

are special solutions to equations

\[ A_{j+1} X_{j+1} B_{j+1} + C_{j+1} X_{j+2} D_{j+1} = E_{j+1} \]

(3.36)

and

\[ A_j X_j B_j + C_j X_{j+1} D_j = E_j, \]

(3.37)

respectively, under the rank equalities (2.1) and (2.2). Hence,

\[ F_j = X_{j+1}^2 - X_{j+1}^1, \]

(3.38)

where $X_{j+1}^2$ and $X_{j+1}^1$ satisfy the equations

\[ A_{j+1} X_{j+1}^2 B_{j+1} + C_{j+1} X_{j+2} D_{j+1} = E_{j+1} \]

(3.39)

and

\[ A_j X_j^2 B_j + C_j X_{j+1}^1 D_j = E_j. \]

(3.40)

**Fact 2:** Formulas about $S_{j+1}$: From

\[ S_{j+1} - A_{j+1} A_{j+1}^\dagger S_{j+1} = R_{A_{j+1} S_{j+1}} = R_{A_{j+1} C_{j+1} L_{M_{j+1}}} = M_{j+1} L_{M_{j+1}} = 0, \]

(3.41)

we infer that

\[ A_{j+1} A_{j+1}^\dagger S_{j+1} = S_{j+1}. \]

(3.42)
Fact 3: The ranks of \( \begin{pmatrix} R_{N_j} \\ R_{D_j} \end{pmatrix} \) and \( R_{N_j} \): Applying Lemma 3.2 to \( r \left( \begin{pmatrix} R_{N_j} \\ R_{D_j} \end{pmatrix} \right) - r(R_{N_j}) \) gives

\[
r \left( \begin{pmatrix} N_j \\ D_j \end{pmatrix} \right) = r \left( \begin{pmatrix} I & N_j \\ I & 0 \end{pmatrix} \right) - r \left( \begin{pmatrix} 0 & 0 \\ N_j & 0 \end{pmatrix} \right) = r \left( \begin{pmatrix} N_j & 0 \\ D_j & 0 \end{pmatrix} \right) = 0.
\]

Hence, we have

\[
r \left( \begin{pmatrix} R_{N_j} \\ R_{D_j} \end{pmatrix} \right) = r(R_{N_j}),
\]

eq (3.43)

i.e.,

\[
R_{D_j} = T_j R_{N_j},
\]

(3.44)

where \( T_j \) is a matrix.

Fact 4: Formulas about \( R_{N_{j+1}} D_{j+1} B_{j+1}^{\dagger} \): Note that

\[
R_{N_{j+1}} D_{j+1} - R_{N_{j+1}} D_{j+1} B_{j+1}^{\dagger} B_{j+1} = R_{N_{j+1}} D_{j+1} L_{B_{j+1}} = R_{N_{j+1}} N_{j+1} = 0.
\]

Hence, we have

\[
R_{N_{j+1}} D_{j+1} B_{j+1}^{\dagger} B_{j+1} = R_{N_{j+1}} D_{j+1}.
\]

(3.45)

We show that (3.26)-(3.33) are equivalent with (2.3)-(2.6) through the following three steps.

Step 1. We show that the rank equality (3.26) is equivalent with (2.3) for the case \( n - m = 1 \).

It follows from Lemma 3.2 that

\[
r \left( \begin{pmatrix} I & F_j \\ M_j & 0 \end{pmatrix} \left( \begin{pmatrix} A_{j+1}^{\dagger} S_{j+1} \\ L_{M_j} \end{pmatrix} \right) L_{M_j} \right) = r \left( \begin{pmatrix} 0 & 0 \\ R_{D_j} & 0 \end{pmatrix} \right) \left( \begin{pmatrix} A_{j+1}^{\dagger} S_{j+1} \\ L_{A_{j+1}} \end{pmatrix} \right)
\]

Add \( \left( \begin{pmatrix} L_{M_j} \end{pmatrix} \right) \) to both sides

\[
= r \left( \begin{pmatrix} I & F_j \\ M_j & 0 \end{pmatrix} \right) \left( \begin{pmatrix} A_{j+1}^{\dagger} S_{j+1} \\ L_{M_j} \end{pmatrix} \right) = r \left( \begin{pmatrix} I & A_{j+1}^{\dagger} S_{j+1} \\ M_j & 0 \end{pmatrix} \right) + r \left( \begin{pmatrix} R_{D_j} \\ R_{B_{j+1}} \end{pmatrix} \right)
\]

Add \( A_{j+1} \) to both sides

\[
= r \left( \begin{pmatrix} I & A_{j+1}^{\dagger} S_{j+1} \\ M_j & 0 \end{pmatrix} \right) + r \left( \begin{pmatrix} R_{D_j} \\ R_{B_{j+1}} \end{pmatrix} \right)
\]
\[
\begin{pmatrix}
I & F_j & A_j^\dagger S_{j+1} & L_{M_j} L_{S_j} & I \\
M_j & 0 & 0 & 0 & 0 \\
0 & R_{D_j} & 0 & 0 & 0 \\
0 & R_{B_{j+1}} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & A_{j+1}
\end{pmatrix}
= r
\begin{pmatrix}
I & A_j^\dagger S_{j+1} & L_{M_j} L_{S_j} & I \\
M_j & 0 & 0 & 0 \\
0 & 0 & 0 & A_{j+1}
\end{pmatrix}
+ r
\begin{pmatrix}
R_{D_j} \\
R_{B_{j+1}}
\end{pmatrix}
\]

Use (4.42) and elementary operations

\[
\begin{pmatrix}
I & F_j & 0 & I \\
M_j & 0 & 0 & 0 \\
0 & R_{D_j} & 0 & 0 \\
0 & R_{B_{j+1}} & 0 & 0 \\
0 & 0 & S_{j+1} & A_{j+1}
\end{pmatrix}
= r
\begin{pmatrix}
I & 0 & I \\
M_j & 0 & 0 \\
0 & S_{j+1} & A_{j+1}
\end{pmatrix}
+ r
\begin{pmatrix}
R_{D_j} \\
R_{B_{j+1}}
\end{pmatrix}
\]

Add \( A_j, D_j, B_{j+1}, M_{j+1} \) to both sides

\[
\begin{pmatrix}
I & F_j & 0 & I & 0 & 0 & 0 \\
C_j & 0 & 0 & 0 & A_j & 0 & 0 \\
0 & I & 0 & 0 & 0 & D_j & 0 \\
0 & I & 0 & 0 & 0 & 0 & B_{j+1} \\
0 & 0 & C_{j+1} & A_{j+1} & 0 & 0 & 0 \\
0 & 0 & M_{j+1} & 0 & 0 & 0 & 0
\end{pmatrix}
= r
\begin{pmatrix}
I & 0 & I & 0 \\
C_j & 0 & 0 & A_j \\
0 & C_{j+1} & A_{j+1} & 0 \\
0 & M_{j+1} & 0 & 0 & 0
\end{pmatrix}
+ r
\begin{pmatrix}
I & D_j & 0 \\
I & 0 & B_{j+1}
\end{pmatrix}
\]

Use \( M_{j+1} = R_{A_{j+1}} C_{j+1} \), and elementary operations

\[
\begin{pmatrix}
I & F_j & 0 & I & 0 & 0 & 0 \\
C_j & 0 & 0 & 0 & A_j & 0 & 0 \\
0 & I & 0 & 0 & 0 & D_j & 0 \\
0 & I & 0 & 0 & 0 & 0 & B_{j+1} \\
0 & 0 & C_{j+1} & A_{j+1} & 0 & 0 & 0 \\
0 & 0 & M_{j+1} & 0 & 0 & 0 & 0
\end{pmatrix}
= r
\begin{pmatrix}
I & 0 & I & 0 \\
C_j & 0 & 0 & A_j \\
0 & C_{j+1} & A_{j+1} & 0 \\
0 & M_{j+1} & 0 & 0 & 0
\end{pmatrix}
+ r
\begin{pmatrix}
I & D_j & 0 \\
I & 0 & B_{j+1}
\end{pmatrix}
\]

Use (4.38)
We have showed that the rank equality (3.26) is equivalent with (2.3) when $n - m = 1$.

**Step 2.** Now we will prove that the rank equality (3.27) is equivalent with (2.4) when $n - m = 1$. Applying Lemma 3.2 to (3.27) yields

$$r \begin{pmatrix} A_j & E_j & C_j \\ D_j & A_{j+1} & B_{j+1} \\ A_{j+1} & -E_{j+1} & C_{j+1} \end{pmatrix} = r \begin{pmatrix} A_j & C_j \\ A_{j+1} & C_{j+1} \end{pmatrix} + r \begin{pmatrix} D_j & B_{j+1} \end{pmatrix} \quad \text{Put } m = j \text{ and } n = j+1 \text{ in (2.3), (2.4).}
$$

We have showed that the rank equality (3.26) is equivalent with (2.3) when $n - m = 1$. Applying Lemma 3.2 to (3.27) yields

$$r \begin{pmatrix} B_j \\ E_j \\ D_j \end{pmatrix} = r \begin{pmatrix} B_j \\ D_j \end{pmatrix} \iff
$$

$$r \begin{pmatrix} R_{N_j} L \begin{pmatrix} R_{D_j} \\ -R_{B_{j+1}} \end{pmatrix} \\ R \left( L_{M_j} L_{S_j} - L_{A_{j+1}} \right) F_j L \begin{pmatrix} R_{D_j} \\ -R_{B_{j+1}} \end{pmatrix} \\ R_{N_{j+1}} D_{j+1} B_{j+1}^t L \begin{pmatrix} R_{D_j} \\ -R_{B_{j+1}} \end{pmatrix} \end{pmatrix} = r \begin{pmatrix} R_{N_j} L \begin{pmatrix} R_{D_j} \\ -R_{B_{j+1}} \end{pmatrix} \\ R_{N_{j+1}} D_{j+1} B_{j+1}^t L \begin{pmatrix} R_{D_j} \\ -R_{B_{j+1}} \end{pmatrix} \end{pmatrix}
$$

Add $\left( L_{M_j} L_{S_j} - L_{A_{j+1}} \right)$, $\begin{pmatrix} R_{D_j} \\ -R_{B_{j+1}} \end{pmatrix}$

to both sides

$$r \begin{pmatrix} F_j \\ R_{N_j} \\ R_{N_{j+1}} D_{j+1} B_{j+1}^t \\ R_{D_j} \\ R_{B_{j+1}} \end{pmatrix} = r \begin{pmatrix} L_{M_j} L_{S_j} - L_{A_{j+1}} \end{pmatrix} + r \begin{pmatrix} R_{N_j} \\ R_{N_{j+1}} D_{j+1} B_{j+1}^t \\ R_{D_j} \\ R_{B_{j+1}} \end{pmatrix}
$$

Use (3.33) and (3.44)
\[
\begin{pmatrix}
F_j & L_{M_j} & L_{S_j} & L_{A_{j+1}} \\
R_{N_{j+1}} & 0 & 0 & 0 \\
R_{N_{j+1}} D_{j+1} B_{j+1}^\dagger & 0 & 0 & 0 \\
R_{B_{j+1}} & 0 & 0 & 0
\end{pmatrix}
= r \begin{pmatrix}
L_{M_j} & L_{S_j} & L_{A_{j+1}} \\
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}
+ r \begin{pmatrix}
R_{N_{j+1}} D_{j+1} B_{j+1}^\dagger \\
R_{B_{j+1}}
\end{pmatrix}
\]

Add \( A_{j+1}, B_{j+1}, S_i, N_i \) to both sides

\[
\begin{pmatrix}
F_j & L_{M_j} & I & 0 & 0 \\
I & 0 & 0 & N_j & 0 \\
0 & 0 & 0 & R_{N_{j+1}} D_{j+1} B_{j+1}^\dagger & 0 \\
I & 0 & 0 & 0 & B_{j+1} \\
0 & C_j L_{M_j} & 0 & 0 & 0 \\
0 & 0 & A_{j+1} & 0 & 0
\end{pmatrix}
= r \begin{pmatrix}
L_{M_j} & I \\
I & 0
\end{pmatrix}
+ r \begin{pmatrix}
I & N_j & 0 \\
0 & 0 & R_{N_{j+1}} D_{j+1}
\end{pmatrix}
\]

Use (3.43) and elementary operations

Add \( M_j \) to both sides

\[
\begin{pmatrix}
F_j & I & I & 0 & 0 \\
I & 0 & 0 & D_j L_{B_j} & 0 \\
0 & 0 & 0 & R_{N_{j+1}} D_{j+1} B_{j+1}^\dagger & 0 \\
I & 0 & 0 & 0 & B_{j+1} \\
0 & C_j & 0 & 0 & 0 \\
0 & 0 & A_{j+1} & 0 & 0
\end{pmatrix}
= r \begin{pmatrix}
I & I \\
I & 0
\end{pmatrix}
+ r \begin{pmatrix}
I & D_j L_{B_j} & 0 \\
0 & 0 & R_{N_{j+1}} D_{j+1}
\end{pmatrix}
\]

Add \( N_{j+1} \) and \( B_i \) to both sides

\[
\begin{pmatrix}
F_j & I & I & 0 & 0 \\
I & 0 & 0 & D_j & 0 \\
0 & 0 & 0 & 0 & D_{j+1} \\
I & 0 & 0 & 0 & B_{j+1} \\
0 & C_j & 0 & 0 & 0 \\
0 & 0 & A_{j+1} & 0 & 0 \\
0 & 0 & 0 & B_j & 0
\end{pmatrix}
= r \begin{pmatrix}
I & I \\
I & 0
\end{pmatrix}
+ r \begin{pmatrix}
I & D_j & 0 \\
0 & 0 & D_{j+1}
\end{pmatrix}
\]

Use (3.38)
Similarly, it can be found that

\[
\begin{pmatrix}
X_{j+1}^2 - X_{j+1}^1 & I & I & 0 & 0 \\
I & 0 & 0 & D_j & 0 \\
0 & 0 & 0 & D_{j+1} & 0 \\
I & 0 & 0 & 0 & B_{j+1} \\
0 & C_j & 0 & 0 & 0 \\
0 & 0 & A_{j+1} & 0 & 0 \\
0 & 0 & 0 & B_j & 0
\end{pmatrix}
= r
\begin{pmatrix}
I & I \\
C_j & 0 \\
0 & A_{j+1} \\
0 & B_j
\end{pmatrix}
+ r
\begin{pmatrix}
I & D_j & 0 \\
0 & 0 & D_{j+1} \\
I & 0 & B_{j+1} \\
0 & B_j & 0
\end{pmatrix}
\]

Use elementary operations

\[
\begin{pmatrix}
B_j \\
E_j \\
D_j \\
A_{j+1}
\end{pmatrix}
= r
\begin{pmatrix}
C_j \\
B_{j+1} \\
-A_{j+1} \\
D_{j+1}
\end{pmatrix}
+ r
\begin{pmatrix}
B_j \\
D_j \\
B_{j+1} \\
D_{j+1}
\end{pmatrix}
\]

Put \( m = j \) and \( n = j + 1 \) in (2.4).

Similarly, it can be found that

\[
(3.28) \iff (2.6), \quad (m = j, \; n = j + 1),
\]

\[
(3.29) \iff (2.7), \quad (m = j, \; n = j + 1).
\]

**Step 3.** We will prove that \((3.30) \iff (2.3)\) for the case \( n - m > 1 \). First, we only deal with \( \widehat{A}_m, \widehat{C}_m, \; \widehat{A}_{m+1}, \; \widehat{D}_m, \; \widehat{B}_{m+1}, \; \widehat{E}_m \in (3.30) \). We want to find some rules. Applying Lemma 3.2 to \( \widehat{A}_m, \; \widehat{C}_m, \; \widehat{A}_{m+1}, \; \widehat{D}_m, \; \widehat{B}_{m+1}, \; \widehat{E}_m \in (3.30) \) that

\[
\begin{pmatrix}
\widehat{A}_m & \widehat{E}_m & \widehat{C}_m \\
\widehat{D}_m & \widehat{A}_{m+1} & \widehat{B}_{m+1} \\
\widehat{A}_{m+1} & \widehat{E}_{m+1} & \widehat{C}_{m+1} \\
\vdots & \vdots & \vdots \\
\widehat{A}_n & (-1)^{n-m}\widehat{E}_n & \widehat{C}_n
\end{pmatrix}
= r
\begin{pmatrix}
\widehat{A}_m & \widehat{C}_m \\
\widehat{A}_{m+1} & \widehat{C}_{m+1} \\
\vdots & \vdots \\
\widehat{A}_n & \widehat{C}_n
\end{pmatrix}
+ r
\begin{pmatrix}
\widehat{D}_m & \widehat{B}_{m+1} \\
\widehat{D}_{m+1} & \widehat{B}_{m+2} \\
\vdots & \vdots \\
\widehat{D}_{n-1} & \widehat{B}_n
\end{pmatrix}
\]

Replace \( \widehat{A}_m, \; \widehat{C}_m, \; \widehat{A}_{m+1}, \; \widehat{D}_m, \; \widehat{B}_{m+1}, \; \widehat{E}_m \) by \( 3.21 - 3.23 \).
Add $M_m$, $\left( L_{m+1} S_{m} - L_{A_{m+1}} \right)$ and $\left( R_{Dm} - R_{Bm+1} \right)$ to both sides

Use elementary operations
\[
\begin{pmatrix}
I & F_m & A_{m+1}^t S_{m+1} & L_{A_{m+1}} & 0 & 0 \\
0 & R_{N_{m+1}} D_{m+1} B_{m+1}^t & 0 & 0 & R_{N_{m+1}} L & 0 \\
0 & R_{D_m} & 0 & 0 & 0 & 0 \\
0 & R_{B_{m+1}} & 0 & 0 & 0 & 0 \\
M_m & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & R \left( L_{M_{m+1}} L_{S_{m+1}} - L_{A_{m+2}} \right) L_{M_{m+1}} & 0 & -E_{m+1} & C_{m+1} \\
\end{pmatrix}
\]

\[
= r \begin{pmatrix}
I & A_{m+1}^t S_{m+1} & L_{A_{m+1}} & 0 & 0 & 0 \\
M_m & 0 & 0 & 0 & 0 & 0 \\
0 & R \left( L_{M_{m+1}} L_{S_{m+1}} - L_{A_{m+2}} \right) L_{M_{m+1}} & 0 & -E_{m+1} & C_{m+1} \\
\end{pmatrix}
\]

Add \( A_{m+1}, A_m, D_m, \) and \( B_{m+1} \) to both sides

\[
\begin{pmatrix}
I & F_m & A_{m+1}^t S_{m+1} & I & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & R_{N_{m+1}} D_{m+1} B_{m+1}^t & 0 & 0 & I & 0 & 0 & 0 & 0 & 0 \\
0 & I & 0 & 0 & 0 & D_m & 0 & 0 & 0 & 0 \\
0 & I & 0 & 0 & 0 & 0 & B_{m+1} & 0 & 0 & 0 \\
0 & I & 0 & 0 & 0 & 0 & 0 & A_m & 0 & 0 \\
0 & 0 & 0 & 0 & R \left( L_{M_{m+1}} L_{S_{m+1}} - L_{A_{m+2}} \right) L_{M_{m+1}} & 0 & 0 & 0 & -E_{m+1} & C_{m+1} \\
\end{pmatrix}
\]

\[
= r \begin{pmatrix}
I & A_{m+1}^t S_{m+1} & I & 0 & 0 \\
C_m & 0 & 0 & A_m & 0 \\
0 & R \left( L_{M_{m+1}} L_{S_{m+1}} - L_{A_{m+2}} \right) L_{M_{m+1}} & 0 & 0 & C_{m+1} \\
\end{pmatrix}
\]

\[
+ r \begin{pmatrix}
I & D_m & 0 & 0 & 0 \\
I & 0 & B_{m+1} & 0 & 0 \\
\end{pmatrix}
\]

Use \(3.42\) and \(3.45\)
Add $N_{m+1}$ and $M_{m+1}$ to both sides

Use elementary operations
Use elementary operations, (3.39) and (3.40)
\[
\begin{pmatrix}
I & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -D_{m+1} & 0 & L \\
0 & I & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -C_m & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -C_{m+1} & A_{m+1} & 0 \\
0 & 0 & R \left( L_{M_{m+1}} L_{S_{m+1}} - L_{A_{m+2}} \right) & 0 & 0 & 0 & 0 & -E_{m+1} \\
\end{pmatrix}
\]

\[
\begin{pmatrix}
I & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -D_{m+1} & 0 & L \left/ \left( R_{D_{m+1}} \right) \right. \\
0 & -C_m & A_m & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -C_{m+1} & A_{m+1} & 0 \\
0 & 0 & R \left( L_{M_{m+1}} L_{S_{m+1}} - L_{A_{m+2}} \right) & 0 & 0 & 0 & 0 & -E_{m+1} \\
\end{pmatrix} =
\begin{pmatrix}
I & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -D_{m+1} & 0 & L \left/ \left( R_{D_{m+1}} \right) \right. \\
0 & -C_m & A_m & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -C_{m+1} & A_{m+1} & 0 \\
0 & 0 & R \left( L_{M_{m+1}} L_{S_{m+1}} - L_{A_{m+2}} \right) & 0 & 0 & 0 & 0 & -E_{m+1} \\
\end{pmatrix}
\]

\[
A_m & E_m & C_m \\
D_m & B_{m+1} & -A_{m+1} X^2_{m+1} B_{m+1} & C_{m+1} \\
D_{m+1} & -A_{m+1} X^2_{m+1} B_{m+1} & C_{m+1} \\
D_{m+1} & -L \left/ \left( R_{D_{m+1}} \right) \right. \\
D_{m+1} & -E_{m+1} & C_{m+1} \\
D_{m+1} & -E_{m+1} & C_{m+1} \\
\end{pmatrix}
\]

Continuing in this way, we obtain that

\[
(3.46) \iff
\]
Add \((L_{Mn}L_{Sn} - L_{n+1})\) and \((-R_{Dn}
abla - R_{Bn+1})\) to both sides.

Add \(d_n\) and \(b_{n+1}\) to both sides and use (3.42), (3.39) and (3.40).
Use elementary operations

\[ \begin{pmatrix} A_m & E_m & C_m \\ \vdots & \ddots & \ddots \\ D_{n-1} & A_n & (-1)^{n-m} A_n X_n^2 B_n & C_n \\ D_n & I & (-1)^{n-m+1} F_n & I \\ I & I & D_n & D_{n+1} \\ \end{pmatrix} \]

\[= \begin{pmatrix} A_m & C_m \\ A_{m+1} & C_{m+1} \\ \vdots & \vdots \\ A_n & C_n \\ I & I \\ I & A_{n+1} & I & A_{n+1} \\ \end{pmatrix} \]

\[+ r \begin{pmatrix} D_m & B_{m+1} \\ D_{m+1} & B_{m+2} \\ \vdots & \vdots \\ D_n & B_{n+1} \\ \end{pmatrix} \]

\[\iff (2.3), \ (n - m > 1). \]

Similarly, it can be found that

\[ \begin{align*}
\[3.31\] & \iff (2.4), \ (n - m > 1), \\
\[3.32\] & \iff (2.5), \ (n - m > 1), \\
\[3.33\] & \iff (2.6), \ (n - m > 1).
\end{align*} \]
As special cases of Theorem 2.1, solvability conditions to the following systems of one-sided Sylvester-type quaternion matrix equations can be given

\[
\begin{align*}
    A_1X_1 + X_2D_1 &= E_1, \\
    A_2X_2 + X_3D_2 &= E_2, \\
    A_3X_3 + X_4D_3 &= E_3, \\
    &\vdots \\
    A_kX_k + X_{k+1}D_k &= E_k,
\end{align*}
\]  

(3.47)

\[
\begin{align*}
    A_1X_1 + X_2D_1 &= E_1, \\
    A_2X_2 + X_3D_2 &= E_2, \\
    A_3X_3 + X_4D_3 &= E_3, \\
    &\vdots \\
    A_kX_{2k+1} + X_{2k}D_k &= E_{2k},
\end{align*}
\]  

(3.48)

\[
A_iX_k - X_jB_i = C_i, \quad i = 1, \ldots, n, \quad k \neq j, \quad k, j \in \{i, i+1\}. \tag{3.49}
\]

Some authors have considered the solvability conditions to one-sided Sylvester-type matrix equations (e.g., [2], [3], [8], [11]).

4. Solvability conditions to the system (1.2)

In this section, we use Theorem 2.1 to give some solvability conditions to the system of quaternion matrix equations involving \(\eta\)-Hermiticity

\[
\begin{align*}
    A_1X_1A_1^{\eta^*} + C_1X_2C_1^{\eta^*} &= E_1, \\
    A_2X_2A_2^{\eta^*} + C_2X_3C_2^{\eta^*} &= E_2, \\
    A_3X_3A_3^{\eta^*} + C_3X_4C_3^{\eta^*} &= E_3, \\
    &\vdots \\
    A_kX_kA_k^{\eta^*} + C_kX_{k+1}C_k^{\eta^*} &= E_k,
\end{align*}
\]  

(4.1)

At first, we give the definition of \(\eta\)-Hermitian quaternion matrix.

**Definition 4.1 (\(\eta\)-Hermitian Matrix).** [9] For \(\eta \in \{i, j, k\}\), a quaternion matrix \(A\) is said to be \(\eta\)-Hermitian if \(A = A^{\eta^*}\), where \(A^{\eta^*} = -\eta A^*\eta\).

**Theorem 4.1.** The system (4.1) has an \(\eta\)-Hermitian solution if and only if the following \(k(k+1)\) rank equalities hold for all \(i = 1, \ldots, k\) and \(1 \leq m < n \leq k\)

\[
r \begin{pmatrix} A_i & E_i & C_i \end{pmatrix} = r \begin{pmatrix} A_i & C_i \end{pmatrix}, \quad r \begin{pmatrix} A_i & E_i & C_i^{\eta^*} \end{pmatrix} = r(A_i) + r(C_i), \tag{4.2}
\]
The solvability conditions (4.2)-(4.4) can be obtained by using Theorem 2.1.

Proof. The system (4.1) has an $\eta$-Hermitian solution if and only if the following system is consistent

\[
\begin{align*}
A_1Y_1A_1^{\eta*} + C_1Y_2C_1^{\eta*} &= E_1, \\
A_2Y_2A_2^{\eta*} + C_2Y_3C_2^{\eta*} &= E_2, \\
A_3Y_3A_3^{\eta*} + C_3Y_4C_3^{\eta*} &= E_3, \\
&\quad \ddots \quad \ddots \\
A_kY_kA_k^{\eta*} + C_kY_{k+1}C_k^{\eta*} &= E_k. 
\end{align*}
\]

In this case, the general $\eta$-Hermitian solution to the system (4.1) can be expressed as

\[X_i = \frac{Y_i + Y_i^{\eta*}}{2}.
\]

The solvability conditions (4.2)-(4.4) can be obtained by using Theorem 2.1.

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