Optimal Exponent for Coalescence of Finite Geodesics in Exponential Last Passage Percolation

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Abstract

In this note, we study the model of directed last passage percolation on \( \mathbb{Z}^2 \), with i.i.d. exponential weight. We consider the maximum paths from vertices \((0, \lfloor k^{2/3} \rfloor)\) and \((\lfloor k^{2/3} \rfloor, 0)\) to \((n, n)\), respectively. For the coalescing point of these paths, we show that the probability for it being \( > kR \) far away from the origin is in the order of \( R^{-2/3} \). This is motivated by a recent work of Basu, Sarkar, and Sly [BSS19], where the same estimate was obtained for semi-infinite geodesics, and the optimal exponent for the finite case was left open.

1 Introduction

As a model of fluid flow in a random medium, first passage percolation (FPP) has been studied by probabilists for more than fifty years, while many predictions about its geometric structure remain unsettled. One major prediction is that the planar FPP belongs to the so-called KPZ universality class, proposed by Kardar, Parisi, and Zhang in their seminal work [KPZ86]. Planar growth models in the KPZ universality class are predicted to have length fluctuation exponent of \( \frac{1}{3} \) and transversal fluctuation exponent of \( \frac{2}{3} \). While little progress has been made to rigorously establish this prediction for the planar FPP, similar results are known for some exactly solvable directed last passage percolation (DLPP) models, where there are exact distributional formulas, obtained from combinatorics, representation theory, or random matrix theory. The first such result was given by Baik, Deift and Johansson [BDJ99], where they studied length of the longest increasing path from \((0, 0)\) to \((n, n)\) in a homogeneous Poissonian field on \( \mathbb{R}^2 \), proving that the fluctuation has an exponent of \( \frac{1}{3} \), with GUE Tracy-Widom scaling limit. Since then much progress has been made in understanding these exactly solvable models, see e.g. [Cor12][QR14] for surveys in this direction.

For general planar FPP, due to the absence of such formulas, its study relies more on understanding of the geodesics. In particular, coalescence of geodesics has been wildly used in obtaining geometric information of the model. The study in this direction was initiated by Newman and co-authors, see e.g. [New95], where certain coalescence results are established under curvature assumptions on the limit shape. A breakthrough was then made by Hoffman [Hol08], where he used Busemann functions to study infinite geodesics. These techniques then led to more progress in the geometric structure of geodesics, see e.g. [AH16][DH14][DH17].

For DLPP models, in recent years there are also much interest in studying the coalescence of geodesics (maximal paths instead), see e.g. [Cou11][FP05]; and some of these results have been proved beyond exactly solvable models, see [GRAS17][GRAS17b]. There are several motivations to study the geometry of geodesics in exactly solvable DLPP models. For example, for models obtained by adding local defects to exactly solvable ones, the integrable structure are destroyed,
thus the geometric properties of the geodesics play an important role in the study of such models; see e.g. [BSS14], where the authors settled the “slow bond problem” in Totally Asymmetric Simple Exclusion Process (TASEP), corresponding to adding extra weights to the diagonal in the exactly solvable DLPP models. Besides, the coalescence of geodesics in exactly solvable models would help to understand the geometry of the scaling limiting objects of the KPZ universality class, e.g. the coalescence structure in Brownian LPP is used towards understanding Brownian regularity of the Airy process [Ham19][Ham20].

In this note, we study the distribution of the coalescing location of two finite geodesics, for DLPP with exponential weights. For two semi-infinite geodesics, a lower bound of the tail of the distribution was obtained by Pimentel in [Pim16], using a duality argument, and a corresponding upper bound was also conjectured there. This was settled by Basu, Sarkar, and Sly in [BSS19]. In the same paper the authors also studied the finite case, where for two distinct points, consider the geodesics from them to the same finite point in the (1, 1) direction. In [BSS19], the tail of the distribution of the coalescing location was conjectured to have the same order as that of semi-infinite geodesics; and the authors also gave an upper bound of polynomial decay. In this note we settle this problem, by providing matching upper and lower bounds, up to a constant factor. The ideas are inspired by the proof of the case of semi-infinite geodesics in [BSS19], and we use more geometric understanding of the geodesics and the environment.

We also mention that, as in [BSS19], our arguments should work equally well for some other exactly solvable models, including Poissonian DLPP in continuum, and DLPP with geometric weights. This is because the results we use only rely on the Tracy-Widom limit and one point upper and lower tail moderate deviation estimates for the last passage times. Such results can be found in [Joh00] [BSS14] (see also [BFP14]) for exponential DLPP, in [LM01] [LMR02] for Poissonian DLPP, and in [BDM+01] [CLW16] for geometric DLPP.

1.1 Notations and statement of main result

We set up notations for the model and formally state our results here.

Consider the 2D lattice $\mathbb{Z}^2$. For each $v \in \mathbb{Z}^2$ we associate $\xi_v$, which is distributed as $\text{Exp}(1)$ and are independent from each other. For any upper-right oriented path $\gamma$ in $\mathbb{Z}^2$, we define the passage time of the path to be

$$X(\gamma) := \sum_{v \in \gamma} \xi_v. \tag{1.1}$$

For any $u, v \in \mathbb{Z}^2$, denote $u < v$, if $u \neq v$ and $u$ is less or equal to $v$ in each coordinate. For any $u < v \in \mathbb{Z}^2$, there are finitely many upper-right paths from $u$ to $v$. Almost surely, there is a unique one $\gamma$ with the largest $X(\gamma)$. We denote it to be the geodesic $\Gamma_{u,v}$, and $X_{u,v} := X(\gamma)$ to be the passage time from $u$ to $v$. For any $u = (u_1, u_2) \in \mathbb{Z}^2$, we denote $d(u) := u_1 + u_2$.

For any $u, v < w \in \mathbb{Z}^2$, we denote $\mathcal{C}_{u,v,w} = (C_{1,u,v,w}, C_{2,u,v,w}) \subseteq \mathbb{Z}^2$ to be the first coalescing point of $\Gamma_{u,v}$ and $\Gamma_{v,w}$, i.e., $C_{u,v,w} \in \Gamma_{u,v} \cap \Gamma_{v,w}$ with the smallest $d(C_{u,v,w})$.

For any $n, k \in \mathbb{Z}_+$, we denote $\mathbf{n} := (n, n)$, $\mathbf{k}^{(1)} := ([k^{2/3}], 0)$, $\mathbf{k}^{(2)} := (0, [k^{2/3}])$. In particular, we let $0 := (0, 0)$.

Our result is about the location of the first coalescing point.

**Theorem 1.1.** There exists universal constants $C_1, C_2, R_0 > 0$, such that for any $k \in \mathbb{Z}_+$, $R > R_0$, and $n > Rk$, we have

$$C_1 R^{-2/3} < \mathbb{P} \left[ d \left( \mathbf{C}^{(k^{(1)}, k^{(2)}); \mathbf{n}} \right) > Rk \right] < C_2 R^{-2/3}. \tag{1.2}$$
In [BSS19, Theorem 1] the problem was presented in a slightly different setting, where the first coordinate of $C^{0, k(2), n}$ was studied; and it was shown there that $\mathbb{P}[c_{1, k}^{0, k(2), n} > Rk] < CR^{-c}$, for some constants $C, c > 0$. Our result confirms that the optimal $c$ is $\frac{2}{3}$.

**Corollary 1.2.** There exists universal constants $C_1', C_2', R_0 > 0$, such that for any $k \in \mathbb{Z}_+, R > R_0$, and $n > 4Rk$, we have

$$C_1' R^{-2/3} < \mathbb{P}[c_{1, k}^{0, k(2), n} > Rk] < C_2' R^{-2/3}. \quad (1.3)$$

**Proof.** For the upper bound, applying Theorem [1.1] we have

$$\mathbb{P}[C_{1}^{0, k(2), n} > Rk] \leq \mathbb{P}[d(C^{0, k(2), n}) > Rk] \leq \mathbb{P}[d(C^{0, k(1), k(2), n}) > Rk] < C_2' R^{-2/3}. \quad (1.4)$$

For the lower bound, first, by reflection symmetry of the model and Theorem [1.1] we have

$$\frac{1}{8} C_1 R^{-2/3} < \mathbb{P}[d(C^{0, k(1), k(2), n}) > 4Rk] \leq \mathbb{P}[d(C^{0, k(2), n}) > 4Rk] + \mathbb{P}[d(C^{0, k(1), n}) > 4Rk] = 2 \mathbb{P}[d(C^{0, k(2), n}) > 4Rk]. \quad (1.5)$$

Second, using bounds on transversal fluctuations, i.e. Lemma [2.3] below, we have

$$\mathbb{P}[d(C^{0, k(2), n}) > 4Rk, C_{1}^{0, k(2), n} \leq Rk] \leq \mathbb{P}[f_0 < Rk - [2Rk]] < \exp(-cR^{2/3}k^{2/3}), \quad (1.6)$$

where $f_0 \in \mathbb{Z}$ such that $([2Rk] + f_0, [2Rk] - f_0) = \Gamma_{0, n}$, and $c$ is an absolute constant. Thus in conclusion, we have

$$\mathbb{P}[c_{1, k}^{0, k(2), n} > Rk] \geq \mathbb{P}[d(C^{0, k(2), n}) > 4Rk] - \mathbb{P}[d(C^{0, k(2), n}) > 4Rk, C_{1}^{0, k(2), n} \leq Rk] \geq \frac{1}{16} C_1 R^{-2/3} - \exp(-cR^{2/3}k^{2/3}), \quad (1.7)$$

and when $R_0$ is large enough the lower bound follows. \hfill \Box

The proof of Theorem [1.1] is based on geometric operations of the model, and relies on its invariance under translations and rotation by $\pi$. We translate the event $d(C^{k(1), k(2), n}) > Rk$ by $i(k^{(1)} - k^{(2)})$, for $i \in \mathbb{Z}_+$, $i \lesssim R^{2/3}$. Thus we consider $R^{2/3}$ “parallel” geodesics, and their intersections with the anti-diagonal $\{v \in \mathbb{Z}_+: d(v) = |kR|\}$. We show that the expected number of different intersecting points is both upper and lower bounded by constants; and on the other hand, it is also approximately $R^{2/3}$ times the probability where coalescing point of two paths are $kR$ away.

A similar argument of exploiting translation invariance was used in [BSS19] in proving the same bounds for semi-infinite geodesics. The main difference is that, in their case, all the semi-infinite geodesics are in the same direction, thus are parallel; while for a finite end point $n$, geodesics from different starting points have different slopes. We resolve this by considering the intersection of the geodesics $\Gamma_{k(2), n}$ and $\Gamma_{k(1), n+k(1) - k(2)}$ instead, and using invariance of this model under rotation by $\pi$. In doing this rotation we require $n > Rk$.

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2 Proof of the main result: coalescing of parallel geodesics

We start by setting up some notations to be used throughout this section.

For each $r \in \mathbb{Z}$, we denote $\mathbb{L}_r$ to be the line $\{v \in \mathbb{Z}^2 : d(v) = r\}$.

Take $a, b \in \mathbb{Z}$ and $m, d, s \in \mathbb{Z}_+$. We define two sequences of points: $u_0, u_1, \ldots, u_m \in \mathbb{L}_0$ and $v_0, v_1, \ldots, v_n \in \mathbb{L}_s$, where $u_i := (a + id, -a - id)$, and $v_i := (\lfloor s/2 \rfloor + b + id, \lfloor s/2 \rfloor - b - id)$ for each $1 \leq i \leq m$.

We consider a family of geodesics $\{\Gamma_{u_i,v_i}\}_{i=1}^{m}$, and study the number of intersections of them with $\mathbb{L}_r$, for some $0 < r < s$. We will show that when $md \sim r^{2/3}$, its expectation is lower and upper bounded by constants. These bounds are given in the next two subsections, as Proposition 2.1 and 2.5 respectively. In the last subsection we use them to deduce Theorem 1.1.

2.1 Intersection with parallel geodesics: lower bound

The goal of this subsection is to prove the following estimate.

**Proposition 2.1.** There exists $M, r_0 \in \mathbb{R}_+$, such that whenever $md > Mr^{2/3}$, $r_0 < r < s$, and $|a|, |b| < r^{2/3}$, we have

$$\mathbb{E} \left[ \left| \mathbb{L}_r \cap \bigcup_{i=1}^{m} \Gamma_{u_i,v_i} \right| \right] > \frac{3}{2}, \quad (2.1)$$

We need an estimate about spatial fluctuation of geodesics.

**Proposition 2.2.** [BSS19] Theorem 3] Fix $L > 0$. There exists $n_0, \ell_0, t_0, c \in \mathbb{R}_+$, relying only on $L$, such that for any $n, \ell, k, k' \in \mathbb{Z}$, $t \in \mathbb{R}$, such that $n \geq n_0$, $t \geq t_0$, $\ell \geq \ell_0$, $|k|, |k'| \leq Ln^{2/3}$, we have

$$\mathbb{P} \left[ |\Gamma_{(0,k'),(n,n+k)}(\ell) - \ell| \geq tL^{2/3} \right] \leq \exp(-ct^2);$$

$$\mathbb{P} \left[ |\Gamma^{-1}_{(0,k'),(n,n+k)}(\ell) - \ell| \geq tL^{2/3} \right] \leq \exp(-ct^2), \quad (2.2)$$

where $\Gamma_{(0,k'),(n,n+k)}(\ell)$ is the maximum number such that $(\ell, \Gamma_{(0,k'),(n,n+k)}(\ell)) \in \Gamma_{(0,k'),(n,n+k)}$, and $\Gamma^{-1}_{(0,k'),(n,n+k)}(\ell)$ is the maximum number such that $(\Gamma^{-1}_{(0,k'),(n,n+k)}(\ell), \ell) \in \Gamma_{(0,k'),(n,n+k)}$.

We translate these bounds to be more convenient to use in our case.

**Lemma 2.3.** There are absolute constants $c, r_0, t_0 \in \mathbb{R}_+$ such that the following is true. Take $f_0 \in \mathbb{Z}$ such that $(\lfloor \frac{r}{2} \rfloor + f_0, \lfloor \frac{r}{2} \rfloor - f_0) = \Gamma_{u_0,v_0} \cap \mathbb{L}_r$. If $r_0 \leq r < s$, $|a|, |b| < r^{2/3}$, for any $t > t_0$ we have

$$\mathbb{P}[|f_0| > tr^{2/3}] < \exp(-ct^2).$$

Proof. Take $y \in \mathbb{Z}$ to be the maximum number such that $(\lfloor \frac{r}{2} \rfloor, y) \in \Gamma_{u_0,v_0}$. Since $|a|, |b| < r^{2/3}$, by Proposition 2.2,

$$\mathbb{P} \left[ y - \frac{r}{2} > t \frac{r}{2}^{2/3} \right] < \exp(-ct^2). \quad (2.3)$$

As both $(\lfloor \frac{r}{2} \rfloor, y)$ and $(\lfloor \frac{r}{2} \rfloor + f_0, \lfloor \frac{r}{2} \rfloor - f_0)$ are in the same geodesic $\Gamma_{u_0,v_0}$, we must have both $\lfloor \frac{r}{2} \rfloor \geq \lfloor \frac{r}{2} \rfloor + f_0$ and $y \geq \lfloor \frac{r}{2} \rfloor - f_0$, or both $\lfloor \frac{r}{2} \rfloor \leq \lfloor \frac{r}{2} \rfloor + f_0$ and $y \leq \lfloor \frac{r}{2} \rfloor - f_0$. This means that $1 + [y - \lfloor \frac{r}{2} \rfloor] \geq |f_0|$, so

$$\mathbb{P}[|f_0| > tr^{2/3}] \leq \mathbb{P}[y - \frac{r}{2} > 4t \frac{r}{2}^{2/3} - 1] \leq \mathbb{P}[y - \frac{r}{2} > t \frac{r}{2}^{2/3}] < \exp(-ct^2), \quad (2.4)$$

when $r \geq 2$ and $t$ large enough, and our conclusion follows. □
Proof of Proposition 2.4. It suffices to consider two geodesics \( \Gamma_{u_0,v_0} \) and \( \Gamma_{u_m,v_m} \).

Take \( f_0, f_m \in \mathbb{Z} \) such that \( (\lfloor \frac{n}{2} \rfloor + f_0, \lfloor \frac{n}{2} \rfloor) = \Gamma_{u_0,v_0} \cap \mathbb{L}_r \), and \( (\lfloor \frac{n}{2} \rfloor + f_m, \lfloor \frac{n}{2} \rfloor - f_m) = \Gamma_{u_r,v_r} \cap \mathbb{L}_r \). By Lemma 2.3 and translation invariance, we can find constant \( p \in \mathbb{R}_+ \) such that

\[
\mathbb{P}[|f_0| > pr^{2/3}] \leq \mathbb{P}[f_m - md > pr^{2/3}] < \frac{1}{4}.
\]

(2.5)

By taking \( M > 2p \), we have \( md > 2pr^{2/3} \), and \( \mathbb{P}[f_0 = f_m] < \frac{1}{2} \). Then

\[
\mathbb{E} \left[ |\mathbb{L}_r \cap \bigcup_{i=1}^m \Gamma_{u_i,v_i}| \right] \geq \mathbb{E} [|\mathbb{L}_r \cap (\Gamma_{u_0,v_0} \cup \Gamma_{u_m,v_m})|] \geq 1 + \mathbb{P} [\mathbb{L}_r \cap \Gamma_{u_0,v_0} \neq \mathbb{L}_r \cap \Gamma_{u_m,v_m}] \geq \frac{3}{2},
\]

(2.6)

and our conclusion follows.

\[ \square \]

2.2 Intersection with parallel geodesics: upper bound

For the upper bound we also use the estimate on spatial fluctuation of geodesics (i.e. Lemma 2.3); and in addition, we need the following estimate on the number of non-coalescing geodesics.

**Proposition 2.4.** [BHIST] Corollary 2.7. There exists \( n_0, \ell_0, c \in \mathbb{R}_+ \) such that the following is true.

Take any \( n, \ell \in \mathbb{Z}_+ \), \( n > n_0 \), \( n^{0.01} > \ell \geq \ell_0 \). Let \( A_{\ell,n} := \{(i,-i) : i \in \mathbb{Z}, |i| < 11/16n^{2/3}\} \subset \mathbb{L}_0 \), and \( B_{\ell,n} := \{(n+i, n-i) : i \in \mathbb{Z}, |i| < 11/16n^{2/3}\} \subset \mathbb{L}_2n \). Let \( \mathcal{E}_{\ell,n} \) be the event that there exists \( \tilde{u}_1, \ldots, \tilde{u}_\ell \in A_{\ell,n} \) and \( \tilde{v}_1, \ldots, \tilde{v}_\ell \in B_{\ell,n} \), such that the geodesics \( \{\Gamma_{\tilde{u}_i,\tilde{v}_i}\}_{i=1}^\ell \) are mutually disjoint. Then \( \mathbb{P}[\mathcal{E}_{\ell,n}] \leq \exp(-c\ell^{1/4}) \).

Now we can establish the upper bound.

**Proposition 2.5.** There exists constants \( C, r_0 \in \mathbb{R}_+ \), such that if \( r > r_0 \), \( s > \frac{3r}{2} \), and \( md, |a|, |b| < r^{2/3} \), we have

\[
\mathbb{E} \left[ |\mathbb{L}_r \cap \bigcup_{i=0}^m \Gamma_{u_i,v_i}| \right] < C.
\]

(2.7)

**Proof.** We can assume that \( r > 10^{1000} \), since otherwise the result follows by taking \( C \) large enough.

Take an absolute constant \( \tau_0 := \max\{10^4, 2\ell_0 + 1\} \), where \( \ell_0 \) is from Proposition 2.3. For any \( \tau \in \mathbb{Z}_+ \), \( \tau < \tau < r^{0.01} \), we wish to bound the probability of the event:

\[
|\mathbb{L}_r \cap \bigcup_{i=0}^m \Gamma_{u_i,v_i}| > \tau.
\]

(2.8)

We denote \( w_i := \Gamma_{u_i,v_i} \cap \mathbb{L}_r \), for each \( i = 0, \ldots, m \). For any \( 0 \leq i < j \leq m \), if \( w_i \neq w_j \), then at least one of the following two events happen:

1. \( \Gamma_{u_i,w_i} \cap \Gamma_{u_j,w_j} = \emptyset \);
2. \( \Gamma_{w_i,v_i} \cap \Gamma_{w_j,v_j} = \emptyset \).

Otherwise we take \( u' \in \Gamma_{u_i,w_i} \cap \Gamma_{u_j,w_j} \) and \( v' \in \Gamma_{w_i,v_i} \cap \Gamma_{w_j,v_j} \), then \( u', v' \in \Gamma_{u_i,v_i} \cap \Gamma_{u_j,v_j} \); and this implies that

\[
w_i = \Gamma_{u_i,v_i} \cap \mathbb{L}_r = \Gamma_{u',v'} \cap \mathbb{L}_r = \Gamma_{u_j,v_j} \cap \mathbb{L}_r = w_j.
\]

(2.9)

which contradicts.
Now we denote
\[ I := \{ i \in \{0, \ldots, m - 1 \} : w_i \neq w_{i+1} \}, \]
\[ I_1 := \{ i \in \{0, \ldots, m - 1 \} : \Gamma_{u_i, u_{i+1}} \cap \Gamma_{u_{i+1}, u_{i+1}} = \emptyset \}, \]
\[ I_2 := \{ i \in \{0, \ldots, m - 1 \} : \Gamma_{u_i, v_i} \cap \Gamma_{u_i, u_{i+1}} = \emptyset \}. \]

From this definition, and the discussion above, we have that \( I = I_1 \cup I_2 \). We also have that \( \{ \Gamma_{u_i, w_i} \}_{i \in I_1} \) are mutually disjoint, and that \( \{ \Gamma_{u_i, v_i} \}_{i \in I_2} \) are mutually disjoint.

We let \( f_0, f_m \in \mathbb{Z} \) such that \( (\frac{r}{2^2} + f_0, \frac{r}{2^2} - f_0) = w_0 \), and \( (\frac{r}{2^2} + f_m, \frac{r}{2^2} - f_m) = w_m \). By Lemma \( \ref{lemma:translation} \) and translation invariance, for some constant \( c_1 \in \mathbb{R}_+ \) we have
\[
\mathbb{P} \left[ |f_0| > \frac{1}{2} \left| \frac{r}{2^2} \right|^{1/16} \left| \frac{r}{4} \right|^{2/3} \right], \quad \mathbb{P} \left[ |f_m - md| > \frac{1}{2} \left| \frac{r}{2^2} \right|^{1/16} \left| \frac{r}{4} \right|^{2/3} \right] < \exp(-c_1 \tau^{1/8}). \tag{2.11}
\]

Since \( r \) is taken large enough, \( md < r^{2/3} \) and \( \tau > 10^{10} \), we further have that
\[
\mathbb{P} \left[ \max \{|f_0|, |f_m|\} > \left| \frac{r}{2^2} \right|^{1/16} \left| \frac{r}{4} \right|^{2/3} - 2 \right] < \exp(-c_1 \tau^{1/8}). \tag{2.12}
\]

Now we assume that \( \ref{eq:case5} \) happens. Then \( |I| \geq \tau \), and either \( |I_1| \geq \tau/2 \) or \( |I_2| \geq \tau/2 \). Since \( s > \frac{3}{2^2} \), we have \( \frac{\tau}{2^2} > \frac{\tau}{2} - 1 \). This implies that if \( |f_0|, |f_m| < \left| \frac{r}{2^2} \right|^{1/16} \left| \frac{r}{4} \right|^{2/3} - 1 \), either the event \( \mathcal{E}_{\frac{r}{2^2}, \frac{r}{2}} \) happens, or the event \( \mathcal{E}_{\frac{r}{2^2}, \frac{\tau}{2^2} - \frac{r}{2}} \) translated by \( (\frac{r}{2^2}, \frac{r}{2}) \) happens. Thus
\[
\mathbb{P} \left[ \mathbb{L}_r \cap \bigcup_{i=0}^m \Gamma_{u_i, v_i} > \tau \right] \leq \mathbb{P} \left[ \max \{|f_0|, |f_m|\} > \left| \frac{r}{2^2} \right|^{1/16} \left| \frac{r}{4} \right|^{2/3} - 2 \right] + \mathbb{P} \left[ \mathcal{E}_{\frac{r}{2^2}, \frac{r}{2}} \right] + \mathbb{P} \left[ \mathcal{E}_{\frac{r}{2^2}, \frac{\tau}{2^2} - \frac{r}{2}} \right]. \tag{2.13}
\]

Note that \( \ell_0 < \left| \frac{r}{2^2} \right| < \left| \frac{r}{4} \right|^{0.01} \leq \left| \frac{r}{2^2} - \frac{r}{4} \right|^{0.01} \), so by Proposition \( \ref{prop:prob} \) we have
\[
\mathbb{P} \left[ \mathcal{E}_{\frac{r}{2^2}, \frac{r}{2}} \right] + \mathbb{P} \left[ \mathcal{E}_{\frac{r}{2^2}, \frac{\tau}{2^2} - \frac{r}{2}} \right] < \exp(-c_2 \tau^{1/4}), \tag{2.14}
\]
for some constant \( c_2 \in \mathbb{R}_+ \). Using this and \( \ref{eq:case5} \), we have that
\[
\mathbb{P} \left[ \mathbb{L}_r \cap \bigcup_{i=0}^m \Gamma_{u_i, v_i} > \tau \right] < 2 \exp(-c_1 \tau^{1/8}) + 2 \exp(-c_2 \tau^{1/4}). \tag{2.15}
\]
Finally, note that \(|\mathbb{L}_r \cap \bigcup_{i=0}^m \Gamma_{u_i, v_i}| \leq m + 1 < r^{2/3} + 1\), then we have
\[
\mathbb{E} \left[ \left| \mathbb{L}_r \cap \bigcup_{i=0}^m \Gamma_{u_i, v_i} \right| \right] < \tau_0 + \tau_0 \sum_{\tau = [\tau_0]}^{r^{0.01}} \mathbb{P} \left[ \mathbb{L}_r \cap \bigcup_{i=0}^m \Gamma_{u_i, v_i} > \tau \right] + (r^{2/3} + 1) \mathbb{P} \left[ \mathbb{L}_r \cap \bigcup_{i=0}^m \Gamma_{u_i, v_i} > r^{0.01} \right] \]
\[
< \tau_0 + \tau_0 \sum_{\tau = [\tau_0]}^{r^{0.01}} 2 \exp(-16c_1 \tau^{1/8}) + 2 \exp(-c_2 \tau^{1/4}) \]
\[
+ (r^{2/3} + 1) \left( 2 \exp(-16c_1 (r^{0.01} - 1)^{1/8}) + 2 \exp(-c_2 (r^{0.01} - 1)^{1/4}) \right), \tag{2.16}
\]
and this is upper bounded by a constant. \( \square \)
2.3 Invariance under translation and rotation

To use the bounds on parallel geodesics we obtained above, we need to convert the probability considered in Theorem \ref{t:bounds} to the probability of coalescing of two parallel geodesics. This is done by rotation invariance of the model as following.

**Lemma 2.6.** Take any \( k, n \in \mathbb{Z}_+ \), \( R > 10 \), with \( n > kR \). We have

\[
\frac{1}{2} \mathbb{P} \left[ \Gamma_{k(2) \cdot n} \cap \mathbb{L}[kR] \neq \Gamma_{k(1) \cdot n} \cap \mathbb{L}[kR] \right] \leq \mathbb{P} \left[ d \left( \mathcal{O}^{(1)}k(2) \cdot n \right) > kR \right] \leq \mathbb{P} \left[ \Gamma_{k(2) \cdot n} \cap \mathbb{L}[kR] \neq \Gamma_{k(1) \cdot n} \cap \mathbb{L}[kR] \right].
\]

(2.17)

**Proof.** Denote \( \mathcal{E}_1 \) to be the event where

\[
\Gamma_{k(1) \cdot n} \cap \mathbb{L}[kR] \neq \Gamma_{k(2) \cdot n} \cap \mathbb{L}[kR],
\]

(2.18)

Then we note that the event \( d \left( \mathcal{O}^{(1)}k(2) \cdot n \right) > kR \) is equivalent to \( \mathcal{E}_1 \). Also, denote \( \mathcal{E}_2 \) to be the event where

\[
\Gamma_{k(1) \cdot n + k(1) \cdot (k(2) - k(2)) \cap \mathbb{L}[kR] \neq \Gamma_{k(1) \cdot n} \cap \mathbb{L}[kR],
\]

(2.19)

and \( \mathcal{E}_3 \) to be the event where

\[
\Gamma_{k(2) \cdot n} \cap \mathbb{L}[kR] \neq \Gamma_{k(1) \cdot n} \cap \mathbb{L}[kR].
\]

(2.20)

Now (2.17) is equivalent to \( \frac{1}{2} \mathbb{P}[\mathcal{E}_3] \leq \mathbb{P}[\mathcal{E}_1] \leq \mathbb{P}[\mathcal{E}_3] \).

We define \( \mathcal{E}'_1 \) as \( \mathcal{E}_1 \) rotated by \( \pi \) around \( \left( \frac{k^{2/3}}{2} + n, \frac{a}{2} \right) \), i.e.

\[
\Gamma_{k(1) \cdot n + k(1) \cdot (k(2) - k(2)) \cap \mathbb{L}[kR] \neq \Gamma_{k(1) \cdot n} \cap \mathbb{L}[kR].
\]

(2.21)

Since \( n > kR \), we have \( \left( k^{2/3} \right) + 2n - kR \) \( kR \), so \( \mathcal{E}_2 \) implies \( \mathcal{E}'_1 \). We also have that \( \mathcal{E}_3 = \mathcal{E}_1 \cup \mathcal{E}_2 \).

Thus

\[
\mathbb{P}[\mathcal{E}_3] \leq \mathbb{P}[\mathcal{E}_1] + \mathbb{P}[\mathcal{E}_2] \leq \mathbb{P}[\mathcal{E}_1] + \mathbb{P}[\mathcal{E}'_1] = 2\mathbb{P}[\mathcal{E}_1] \leq 2\mathbb{P}[\mathcal{E}_3],
\]

(2.22)

and our conclusion follows.

We use Proposition \ref{p:invariance} and \ref{p:rotation} to bound the probability where two parallel geodesics coalesce, using translation invariance.

**Proof of Theorem \ref{t:bounds}** Using the notations defined at the beginning of this section, we take \( r = kR \), \( s = 2n - [k^{2/3}] \), \( a = 0 \), \( b = n - \left( \frac{a}{2} \right) \), \( d = [k^{2/3}] \). We leave \( m \in \mathbb{Z}_+ \) to be determined.

By letting \( R \) large, we can let \( r \) large, and \( |a|, |b| < r^{2/3} \), \( s > \frac{3}{2r} \) (since we require that \( n > kR \)).

By translation invariance, for each \( 0 \leq i \leq m - 1 \), we have

\[
\mathbb{P} \left[ \Gamma_{u_0, v_0} \cap \mathbb{L}_r \neq \Gamma_{u_1, v_1} \cap \mathbb{L}_r \right] = \mathbb{P} \left[ \Gamma_{u_i, v_i} \cap \mathbb{L}_r \neq \Gamma_{u_{i+1}, v_{i+1}} \cap \mathbb{L}_r \right].
\]

(2.23)

This implies that

\[
\mathbb{E} \left[ \mathbb{L}_r \cap \bigcup_{i=0}^{m} \Gamma_{u_i, v_i} \right] = 1 + m \mathbb{P} \left[ \Gamma_{u_0, v_0} \cap \mathbb{L}_r \neq \Gamma_{u_1, v_1} \cap \mathbb{L}_r \right].
\]

(2.24)

First, we take \( m = \left[ 2MR^{2/3} \right] \), where \( M \) is from Proposition \ref{p:invariance}. By taking \( R \) large enough, we have \( m < 3MR^{2/3} \). Also, \( md > 2MR^{2/3} \left( k^{2/3} \right) > Mr^{2/3} \), then by Proposition \ref{p:rotation} we have

\[
\mathbb{P} \left[ \Gamma_{u_0, v_0} \cap \mathbb{L}_r \neq \Gamma_{u_1, v_1} \cap \mathbb{L}_r \right] > \frac{1}{2m} \geq (6M)^{-1} R^{-2/3}.
\]

(2.25)
Second, we take \( m = \lfloor R^{2/3}/2 \rfloor \), and we have \( m > R^{2/3}/3 \) and \( md \leq (kR)^{2/3}/2 < r^{2/3} \) when \( R \) is large enough. By Proposition \([2,5]\) we have that
\[
\mathbb{P}[\Gamma_{u_0,v_0} \cap L_r \neq \Gamma_{u_1,v_1} \cap L_r] < \frac{C}{m} < \frac{C}{3} R^{-2/3}.
\] (2.26)

Finally, by Lemma \([2,6]\)
\[
(6M)^{-1} R^{-2/3} < \frac{1}{2} \mathbb{P}[\Gamma_{u_0,v_0} \cap L_r \neq \Gamma_{u_1,v_1} \cap L_r] \leq \mathbb{P}\left[d\left(C^{k(1)},k^{(2)},n\right) > kR\right] \leq \mathbb{P}[\Gamma_{u_0,v_0} \cap L_r \neq \Gamma_{u_1,v_1} \cap L_r] < \frac{C}{3} R^{-2/3},
\] (2.27)
and our conclusion follows.

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