POINCARÉ SERIES OF RELATIVE SYMMETRIC INVARIANTS FOR 
SLₙ(ℂ)

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ABSTRACT. Let (N, G), where N ⊆ G ⊆ SLₙ(ℂ), be a pair of finite groups and V a finite-dimensional fundamental G-module. We study the G-invariants in the symmetric algebra S(V) = ⊕ₖ≥₀ Sₖ(V) by giving explicit formulas of the Poincaré series for the induced modules and restriction modules. In particular, this provides a uniform formula of the Poincaré series for the symmetric invariants in terms of the McKay-Slodowy correspondence. Moreover, we also derive a global version of the Poincaré series in terms of Chebychev polynomials in the sense that one needs only the dimensions of the subgroups and their group-types to completely determine the Poincaré series.

1. INTRODUCTION

Let V = ℂⁿ be the fundamental module of the special linear group SLₙ(ℂ). The kth symmetric tensor space Sᵏ(V) is a simple SLₙ(ℂ)-module denoted by Γₖ, and the symmetric tensor algebra S(V) = ⊕ₖ=₀ Sₖ(V) is an infinite dimensional SLₙ(ℂ)-module. It is well-known that the set {Γₖ} generates all irreducible finite dimensional representations of SLₙ under the restriction functor according to the Schur-Weyl duality.

Let G be a finite subgroup of SL₂(ℂ). Kostant [17, 18, 19] studied the interesting question on how the restriction Γₖ(G) decomposes itself into simple G-modules and found that the answer relies upon resolution of singularity of certain algebraic surfaces through the McKay correspondence.

The McKay correspondence [20] gives a bijective map between finite subgroups of SL₂(ℂ) and affine Dynkin diagrams of untwisted ADE types. It is known that this correspondence establishes a classification of resolution of singularities of ℂ²/G, where G is a finite subgroup of SL₂(ℂ). Slodowy [22] considered more general minimal resolution of the singularity of ℂ²/N under the action of G/N, where G ≤ SL₂(ℂ). The algebraic counterpart is the so-called McKay-Slodowy correspondence which matches all affine Dynkin diagrams with the pairs N ⊆ G ≤ SL₂(ℂ). For an elementary proof of the McKay-Slodowy correspondence, see [14].

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The goal of this paper is to generalize Kostant’s results in two directions. In the first direction, we consider the general special linear group $\text{SL}_n(\mathbb{C})$. In another aspect, we replace the defining fundamental module $V = \mathbb{C}^n$ by any fundamental irreducible module of the Lie group $\text{SL}_n(\mathbb{C})$. The main results of the paper will show that even in the most general situation of $\text{SL}_n(\mathbb{C})$, one can still obtain similar formulas for the Poincaré series of $G$-invariants in the symmetric tensor algebra, and in the special case of the twisted affine Lie algebras, some of the intrinsic data are also encoded in the Poincaré series of the $(G, N)$-invariants in the symmetric tensor algebra (compare [14]).

Let us now describe the main results in the paper. Let $\mathfrak{h}$ be a Cartan subalgebra of the special linear Lie algebra $\mathfrak{sl}_n(\mathbb{C})$, and let $P$ be the weight lattice spanned by the vectors $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_{n-1}$, where $(\varepsilon_i, \varepsilon_j) = \delta_{ij}$. Denote by $\Gamma(u_1, u_2, \ldots, u_{n-1})$ the finite dimensional irreducible representation of the special linear group $\text{SL}_n(\mathbb{C})$ associated to the dominant weight $\lambda = (u_1 + \ldots + u_{n-1})\varepsilon_1 + (u_2 + \ldots + u_{n-1})\varepsilon_2 + \ldots + u_{n-1}\varepsilon_{n-1}$, where $(u_1, u_2, \ldots, u_{n-1}) \in \mathbb{Z}_{>0}^{n-1}$. Therefore $\Gamma(1,0,\ldots,0)$ is the standard module $V$, and the $r$th fundamental module $\Gamma(0,\ldots,1,0,\ldots,0) = \wedge^r V$.

Let $N$ be a normal subgroup of a finite group $G \leq \text{SL}_n(\mathbb{C})$ and $\{\rho_i|i \in I_G\}$ (resp. $\{\phi_i|i \in I_G\}$) the set of complex finite-dimensional irreducible modules of $G$ (resp. $N$). Let $\{\hat{\rho}_i|i \in \hat{I}\}$ be the set of inequivalent $N$-restriction modules $\text{Res}(\rho_i) := \hat{\rho}_i$. Correspondingly, the set $\{\hat{\phi}_i|i \in \hat{I}\}$ denotes that of inequivalent induced $G$-modules $\text{Ind}(\hat{\phi}_i) =: \rho_i$.

Let $V$ be the natural finite-dimensional $G$-module, denote the $N$-restriction of $V$ by $\hat{V}$. The following tensor products decompose into irreducible components:

$$\wedge^r \hat{V} \otimes \hat{\rho}_j = \bigoplus_{i \in \hat{I}} b_{ji}^{(r)} \hat{\rho}_i$$  \hspace{1cm} and \hspace{1cm} $$\wedge^r V \otimes \rho_j = \bigoplus_{i \in I} d_{ji}^{(r)} \phi_i,$$

where the integral matrices $B_r = (b_{ji}^{(r)})$ and $D_r = (d_{ji}^{(r)})$ are of the same size for $1 \leq r \leq n - 1$ respectively. For $r = 1$, we can construct the representation graph $\mathcal{R}_V(\hat{G})$ (resp. $\mathcal{R}_V(\hat{N})$) by taking the elements of $\hat{I}$ (resp. $I$) as vertices, connecting $i$ and $j$ with $\max(b_{ij}^{(1)}, b_{ji}^{(1)})$ (resp. $\max(d_{ij}^{(1)}, d_{ji}^{(1)})$) edges, and adding an arrow pointing to $i$ if $b_{ij}^{(1)} > 1$ (resp. $d_{ij}^{(1)} > 1$). The digraph $\mathcal{R}_V(\hat{G})$ (resp. $\mathcal{R}_V(\hat{N})$) is called a generalized McKay-Slodowy quiver.

If $N = G \leq \text{SL}_2(\mathbb{C})$, the tensor product between $G$-module $\wedge^r V$ and an irreducible $G$-module $\rho_j$ ($j \in I_G$) come down to

$$\wedge^r V \bigotimes \rho_j = \bigoplus_{i \in I_G} a_{ji}^{(r)} \rho_i,$$

where the matrices $A_r = (a_{ij}^{(r)})$ for $1 \leq r \leq n - 1$. Thus, the generalized McKay quiver $\mathcal{R}_V(\hat{G})$ defined in [13] is just $A_r$ for $r = 1$, whose index set is $I_G$, and there are $a_{ij}^{(1)}$ directed edges from $i$ to $j$, and an undirected edge between $i$ and $j$ represents the pair of arrows between $i$ and $j$.

Let $N \leq G$ be a certain pair of subgroups of $\text{SL}_2(\mathbb{C})$, the McKay-Slodowy quiver $\mathcal{R}_V(\hat{G})$ (resp. $\mathcal{R}_V(\hat{N})$) is the twisted (resp. the non-twisted) multiply laced affine
Dynkin diagram, a detailed description of the McKay-Slodowy correspondence is available in [14]. When $N = G \leq \text{SL}_2(\mathbb{C})$, the McKay-Slodowy correspondence descends to the McKay correspondence, where the McKay quiver $\mathcal{R}_V(G)$ is the simple-laced affine Dynkin diagram. Furthermore, a generalized McKay quiver $\mathcal{R}_V(G)$ had been described in [13] for $G$ is a finite subgroup of $\text{SL}_3(\mathbb{C})$. An impetus to seek a generalized McKay-Slodowy correspondence is that the general formulas of Poincaré series for the relative invariants might be helpful for general minimal resolutions of the singularity in higher dimension.

After giving explicit formulas for the Poincaré series for relative symmetric invariants for the $\text{SL}_n(\mathbb{C})$ in terms of the intrinsic data of the subgroups, we also consider the relation between the Poincaré series of relative symmetric invariants and the finite and affine Coxeter transformation. Tensor invariants and Poincaré series were used by Benkart [1] to realize the exponents of simply laced affine Lie algebras and we have generalized the realization of exponents to all twisted affine Lie algebras except $A^{(1)}_{2n}$ in [14]. Similarly in this paper, we will show that the Poincaré series of symmetric invariants also have very close relation with finite and affine Coxeter transformations, and we also generalize some of these formulae of symmetric invariants to all twisted affine and non-simply laced untwisted affine Lie algebras. Not surprisingly, we are also able to generalize Kostant’s compact formulas of the Poincaré series to all untwisted and twisted affine Dynkin diagrams using intrinsic group data in the context of the McKay-Slodowy correspondence.

We also derive the Poincaré series of symmetric invariants exclusively in terms of Tchebychev polynomials. This implies a surprising beautiful fact about the Poincaré series of invariants that they are completely determined by the types of the distinguished pairs of subgroups and the respective dimensions of the subgroups. Our new formula points out the global picture of the Poincaré series of invariants, for example, one does not need to know the information of exponents or eigenvalues of the Coxeter transformation to determine the Poincaré series. So far this information has only been implicitly available for general symmetric invariants in the literature.

The paper is organized as follows. In section 2, we give the formulas of the Poincaré series of the $N$-restriction modules and induced $G$-modules in symmetric algebra $S(\mathbb{C}^n) = \bigoplus_{k \geq 0} S^k(\mathbb{C}^n)$ respectively. In particular, if $N = G \leq \text{SL}_n(\mathbb{C})$, we get a formula of the Poincaré series for irreducible $G$-modules in symmetric algebra $S(\mathbb{C}^n)$. In Section 3, the Poincaré series of invariants for pairs of finite subgroups (resp. the finite subgroups) of $\text{SL}_2(\mathbb{C})$ are obtained in terms of the quantum affine (resp. finite) Cartan matrices of the affine (resp. finite) Lie algebras. Moreover, the Poincaré polynomial of symmetric invariants exhibit the exponents and Coxeter number of the affine (resp. finite) Lie algebras. In addition, we generalize a classical result of Poincaré series for symmetric $G$-invariants to Poincaré series for $G$-restriction invariants or $N$-induction invariants. In other words, we have provided an unified formula of Poincaré series for symmetric invariants for affine Lie algebras in both untwisted and twisted types. In Section 4, we get the closed-form expressions of the Poincaré series of symmetric invariants for pairs of
subgroups which realize all twisted and untwisted affine Lie algebras. Moreover, the new formulas are of global nature in that they are completely determined by the pairs of subgroups.

2. Poincaré series associated with $\text{SL}_n(\mathbb{C})$

Let $N, G$ be a pair of finite subgroups of $\text{SL}_n(\mathbb{C})$ such that $N \trianglelefteq G$. Let $V = \mathbb{C}^n$ be the standard $\text{SL}_n$-module, also referred as the natural $G$-module. Assume that \{\rho_i\} and \{\phi_i\} are the sets of complex irreducible modules for $N$ and $G$ respectively. As we have remarked that the span of $N$-restriction modules \{\rho_i\} and the span of $G$-induction modules $\phi_i$ have the same dimension, so $|I| = |\bar{I}|$.

Let $\hat{s}_k^j (j \in \bar{I})$ (resp. $\hat{s}_k^j (j \in \bar{I})$) be the multiplicity of the $N$-restriction $\hat{\rho}_j$ in the $k$th symmetric power $S^k(V)$ (resp. induced $G$-module $\hat{\phi}_j$ in $S^k(V)$), namely

$$\hat{s}_k^j = \dim \left( \text{Hom}_N(\hat{\rho}_j, S^k(V)) \right)$$

and

$$\hat{s}_k^j = \dim \left( \text{Hom}_G(\hat{\phi}_j, S^k(V)) \right).$$

Let

$$\hat{s}_k^j(t) = \sum_{k \geq 0} \hat{s}_k^j t^k$$

and

$$\hat{s}_k^j(t) = \sum_{k \geq 0} \hat{s}_k^j t^k$$

be the Poincaré series for the multiplicities of $\hat{\rho}_j$ and $\hat{\phi}_j$ in the symmetric algebra $S(V) = \bigoplus_{k \geq 0} S^k(V)$ respectively.

When $N = G$, we only consider $G$-modules, and let $s_k^j$ be the multiplicity of an irreducible $G$-module $\rho_j$ in the $k$th symmetric power $S^k(V)$ for each $j \in \bar{I}_G$. Accordingly the Poincaré series is then

$$s_k^j(t) = \sum_{k \geq 0} s_k^j t^k = \sum_{k \geq 0} \dim \left( \text{Hom}_G(\rho_j, S^k(V)) \right) t^k.$$

In this section, we will give the general formulas of Poincaré series for arbitrary pair $N \trianglelefteq G \leq \text{SL}_n(\mathbb{C})$ in the symmetric algebra $S(\mathbb{C}^n) = \bigoplus_{k \geq 0} S^k(\mathbb{C}^n)$. Moreover, $N = G \leq \text{SL}_n(\mathbb{C})$ leads to a general formula of Poincaré series of $G$ in $S(\mathbb{C}^n) = \bigoplus_{k \geq 0} S^k(\mathbb{C}^n)$.

2.1. The general formulas of Poincaré series for $\text{SL}_n(\mathbb{C})$. Let $G$ be a finite subgroup of $\text{SL}_n(\mathbb{C})$ and $V = \mathbb{C}^n$ the standard $\text{SL}_n$-module, which is also viewed as the natural $G$-module as $G$-restriction. We will give explicit expression of $S^k(V)$ and establish its relation with character values and adjacency matrices. We first recall some basic results on $\text{SL}_n$-modules.

Lemma 2.1. Pieri Rule [19] Prop. 15.25] Let $\Gamma_{(u_1, \ldots, u_{n-1})}$ be an irreducible representation of $\text{SL}_n(\mathbb{C})$. Then, the tensor product of $\Gamma_{(u_1, \ldots, u_{n-1})}$ with $S^k(V) = \Gamma_{(k, 0, \ldots, 0)}$ decomposes into a direct sum:

$$\Gamma_{(u_1, \ldots, u_{n-1})} \bigotimes \Gamma_{(k, 0, \ldots, 0)} = \bigoplus \Gamma_{(b_1, \ldots, b_{n-1})}$$
where the sum is over all \((b_1, \ldots, b_{n-1})\) for which there are nonnegative integers \(c_1, \ldots, c_n\) whose sum is \(k\), such that \(c_{i+1} \leq u_i\) for \(1 \leq i \leq n-1\) and \(b_i = u_i + c_i - c_{i+1}\) for \(1 \leq i \leq n-1\).

**Lemma 2.2.** \([14, \text{Lem. 3.4}]\) Let \(N\) be a normal subgroup of the finite group \(G\), and \(\{\rho_i| i \in I_G\} \) (resp. \(\{\hat{\phi}_i| i \in I_N\}\)) the set of pairwise inequivalent complex irreducible modules of \(G\) (resp. \(N\)). Let \(\{\hat{\rho}_i| i \in \hat{I}\}\) be the set of mutually inequivalent \(N\)-restrictions of \(\rho_i\)'s such that \(\hat{\rho}_i \cap \hat{\rho}_j = 0\) for \(i, j \in \hat{I}\), and \(\{\hat{\phi}_i| i \in \hat{I}\}\) the set of inequivalent induced \(G\)-modules. Then \(|\{\hat{\rho}_i| i \in \hat{I}\}| = |\{\hat{\phi}_i| i \in \hat{I}\}|\), and the common cardinality is equal to \(|\hat{\Upsilon}(N)|\), where \(\hat{\Upsilon}(N) = \hat{\Upsilon} \cap N\) and \(\hat{\Upsilon}\) is a fixed set of \(G\)-conjugacy class representatives.

**Lemma 2.3.** Let \(V\) be the standard \(\text{SL}_n(\mathbb{C})\)-module. Then for each \(k\) the following relations between symmetric and exterior powers of \(V\) hold:

\[
\bigoplus_{r=0}^{[n/2]} \wedge^{2r}V \bigotimes S^{k-2r}(V) = \bigoplus_{r=0}^{[n/2]} \wedge^{2r+1}V \bigotimes S^{k-2r-1}(V),
\]

where \(S^0(V) = \wedge^0(V) = \mathbb{C}, \wedge^rV = 0\) for \(r > n\) and \(S^r(V) = 0\) for \(r < 0\).

**Proof.** Note that \(\dim V = n\), it follows from the Pieri rule (Lemma 2.1) that

\[
\begin{align*}
V \bigotimes S^{k-1}(V) &= S^k(V) \bigoplus \Gamma_{(k-2,1,0,\ldots,0)}, \\
\wedge^{2r}V \bigotimes S^{k-2r}(V) &= \Gamma_{(k-2,1,0,\ldots,0)} \bigoplus \Gamma_{(k-3,0,1,0,\ldots,0)} \\
&\vdots \\
\wedge^{n-2r}V \bigotimes S^{k-(n-2)}(V) &= \Gamma_{(k-(n-2),0,\ldots,0,1,0)} \bigoplus \Gamma_{(k-(n-1),0,\ldots,0,1)} \\
\wedge^{n-1}V \bigotimes S^{k-(n-1)}(V) &= \Gamma_{(k-(n-1),0,\ldots,0,1)} \bigoplus S^k(V).
\end{align*}
\]

Then in the Grothendieck ring, one sees that (1) holds at the character level, which then implies that the module relations also hold. \(\square\)

In other words, in the Grothendieck ring of \(\text{SL}_n\)-modules one has that

\[
[S^k(V)] = \sum_{r=1}^{n} (-1)^r [\wedge^rV][S^{k-r}(V)].
\]

Let \(N\) be an arbitrary normal group of the finite group \(G\). The decomposition of the tensor product of \(V\) and arbitrary induced (resp. restriction) module of an irreducible \(N\)-module (resp. \(N\)-module) give rise to two adjacency matrices. In \([14]\) we have shown that there is a deep relation between the character values of the group and these adjacency matrices in connection with the McKay-Slodowy correspondence, which generalized Steinberg’s results \([23]\) for the McKay correspondence. In the following we shall generalize our version of the McKay-Slodowy correspondence to all exterior powers of the standard \(\text{SL}_n(\mathbb{C})\)-module.
Here is our group theoretical description of the generalized McKay-Slodowy correspondence which are direct consequence of Lemma 2.2, Lemmas 2.4 and Lemma 2.5.

**Lemma 2.4.** Let $N \trianglelefteq G$ be a pair of finite normal subgroups of $\text{SL}_n(\mathbb{C})$ with \{\(\hat{\rho}_i|i \in I\)\} (resp. \{\(\hat{\phi}_i|i \in \tilde{I}\)\}) be the set of restriction modules (resp. induced $G$-modules). Assume matrix $B_r$ (resp. $D_r$) is afforded by the tensor product of $\wedge^n V$ and $\hat{\rho}_i$ (resp. $\hat{\phi}_i$) for $i \in \tilde{I}$ (resp. $i \in I$) and $1 \leq r \leq n-1$, where $V$ is a natural module. Let \(\chi_{\wedge^n V}\) and \(\chi_{\hat{\rho}_i}\) (resp. \(\chi_{\hat{\phi}_i}\)) be characters of $\wedge^n V$ and $\hat{\rho}_i$ (resp. $\hat{\phi}_i$) respectively. Then,

1. The column vectors \((\chi_{\hat{\rho}_i}(g))_{i \in \tilde{I}}\) and \((\chi_{\hat{\phi}_i}(g))_{i \in I}\) are the eigenvectors of the matrices $\chi_{\wedge^n V}(1)I - B_r^T$ and $\chi_{\wedge^n V}(1)I - D_r^T$ with eigenvalue $\chi_{\wedge^n V}(1) - \chi_{\wedge^n V}(g)$ respectively for $1 \leq r \leq n-1$, where $g$ runs through $\Upsilon(N) = N \cap \Upsilon$, $\Upsilon$ is a set of representatives of conjugacy class of $G$.

2. The column vectors \((\chi_{\hat{\rho}_i}(1))_{i \in \tilde{I}}\) and \((\chi_{\hat{\phi}_i}(1))_{i \in I}\) are eigenvectors of the matrices $\chi_{\wedge^n V}(1)I - B_r^T$ and $\chi_{\wedge^n V}(1)I - D_r^T$ with eigenvalue 0 respectively.

**Lemma 2.5.** Let $G$ be a finite subgroup of $\text{SL}_n(\mathbb{C})$. Let $V$ be the standard $G$-module and \{\(\rho_i|i \in I_G\)\} be the set of irreducible $G$-modules. Assume $A_r = (a_{ij}^{(r)})$ is afforded by the tensor product of $\wedge^n V$ and $\rho_i$ for $i \in I_G$ and $1 \leq r \leq n-1$. Let $\chi_{\wedge^n V}$ and $\chi_{\rho_i}$ be characters of $\wedge^n V$ and $\rho_i$ respectively. Then

1. $A_r = A_{n-r}^T$ for $1 \leq r \leq n-1$.

2. The column vector \((\chi_{\rho_i}(g))_{i \in I_G}\) of the character table of $G$ is an eigenvector of $\chi_{\wedge^n V}(1)I - A_{n-r}$ with eigenvalue $\chi_{\wedge^n V}(1) - \chi_{\wedge^{n-r} V}(g)$ for $1 \leq r \leq n-1$, where $g$ runs over the set $\Upsilon$.

3. The column vector \((\chi_{\rho_i}(1))_{i \in I_G}\) of the degrees of character of $G$ is an eigenvector of matrix $\chi_{\wedge^n V}(1)I - A_{n-r}$ for $1 \leq r \leq n-1$ with eigenvalue 0.

**Proof:** The last two statements are immediate, so we only prove the first one. Note that $\wedge^n V \otimes \rho_j = \bigoplus_{i \in I_G} a_{ij}^{(r)} \rho_i$, then

$$a_{ij}^{(r)} = \dim(\wedge^n V \otimes \rho_j, \rho_i)$$

$$= \dim(\rho_j, (\wedge^n V)^* \otimes \rho_i)$$

$$= \dim(\rho_j, \wedge^{n-r} V \otimes \rho_i) = a_{ji}^{(n-r)},$$

which completes the proof.

For a pair of subgroups $N \trianglelefteq G$ of $\text{SL}_n(\mathbb{C})$ and the standard module $V$, the formulas of the Poincaré series for the multiplicities of each $G$-restriction module or induced $N$-module in the symmetric algebra $S(V) = \bigoplus_{k \geq 0} S^k(V)$ are given in the next two theorems respectively.
Theorem 2.6. Let \( N \vartriangleleft G \) be a pair of finite subgroups of \( \text{SL}_n(\mathbb{C}) \), and \( V \) the standard \( N \)-module. Let \( B_r \) be the matrix given by the tensor product of \( \wedge^r V \) and \( \hat{\rho}_i \) (\( 1 \leq r \leq n - 1 \)), and \( M_i^1 \) the matrix \((1 + (-1)^r t^n)I + \sum_{r=1}^{n-1} (-1)^r B_{n-r} t^r\) with the \( i \)th column replaced by \( \hat{\delta} = (1, 0, \ldots, 0)^T \in \mathbb{N}^{|I|} \). Then the Poincaré series of the multiplicities of \( \hat{\rho}_i \) in \( S^r(\hat{\rho}_i, S^k(V)) \) is

\[
\tilde{s}_i(t) = \frac{\det(M_i^1)}{\det((1 + (-1)^n t^n)I + \sum_{r=1}^{n-1} (-1)^r B_{n-r} t^r)}
\]

\[
= \frac{\det(M_i^1)}{\prod_{g \in \Upsilon(N)} (1 + \sum_{r=1}^{n-1} (-1)^r \chi_{\wedge^n V}(g) t^r + (-1)^n t^n)}
\]

where \( \Upsilon(N) = N \cap \Upsilon \), \( \Upsilon \) is a set of representatives of the conjugacy classes of \( G \).

Proof. By the equality (2) the multiplicity of \( N \)-restriction \( \hat{\rho}_i \) in \( S^k(V) \) is

\[
s_k^i = \dim \left( \text{Hom}_N(\hat{\rho}_i, S^k(V)) \right)
= \sum_{r=1}^{n} (-1)^{r-1} \dim \left( \text{Hom}_N \left( \hat{\rho}_i, \sum_{r=1}^{n} \wedge^r V \bigotimes S^{k-r}(V) \right) \right).
\]

Then the Poincaré series \( \tilde{s}_i(t) \) is computed as follows,

\[
\tilde{s}_i(t) = \sum_{k \geq 0} \dim \left( \text{Hom}_N(\hat{\rho}_i, S^k(V)) \right) t^k
= \delta_{i0} + \sum_{k \geq 1} \sum_{r=1}^{n} (-1)^{r-1} \dim \left( \text{Hom}_N \left( \hat{\rho}_i, \sum_{r=1}^{n} \wedge^r V \bigotimes S^{k-r}(V) \right) \right) t^k
= \delta_{i0} + \sum_{r=1}^{n} (-1)^{r-1} \left( \sum_{k \geq 1} \dim \left( \text{Hom}_N((\wedge^r V)^* \bigotimes \hat{\rho}_i, S^{k-r}(V)) \right) t^k \right)
= \delta_{i0} + \sum_{r=1}^{n} (-1)^{r-1} \left( \sum_{k \geq 1} \dim \left( \text{Hom}_N(\wedge^{n-r} V \bigotimes \hat{\rho}_i, S^{k-r}(V)) \right) t^k \right)
= \delta_{i0} \sum_{r=1}^{n-1} (-1)^{r-1} \left( \sum_{k \geq 1} \dim \left( \text{Hom}_N(b_{ij}^{(n-r)} \hat{\rho}_j, S^{k-r}(V)) \right) t^k \right)
+ (-1)^{n-1} \sum_{k \geq 1} \dim \left( \text{Hom}_N(\hat{\rho}_i, S^{k-n}(V)) \right) t^k
\]
Using the Cramer’s rule, the Poincaré series (4) is then written as the matrix equation

$$
\begin{align*}
\sum_{r=1}^{n-1} (-1)^{r-1} \left( \sum_{j \in I} b_{ij}^{(n-r)} \sum_{k \geq 0} \dim \left( \text{Hom}_{N}(\mathbf{\hat{p}}_{j}, S^k(V)) \right) t^k \right) t^r + (-1)^{n-1} \mathbf{s}(t) t^n.
\end{align*}
$$

Let \( \mathbf{s} = (\mathbf{s}(t))_{t \in \bar{I}} \) be the column vector formed by the Poincaré series, the identity (4) is then written as the matrix equation

$$
\begin{align*}
\left( (1 + (-1)^n t^n) I + \sum_{r=1}^{n-1} (-1)^r B_{n-r} t^r \right) \mathbf{s} = \mathbf{\delta}.
\end{align*}
$$

Using the Cramer’s rule, the Poincaré series \( \mathbf{s}(t) \) is equal to a quotient of two determinants.

Since \( \chi_{\Lambda^r V}(1) - \chi_{\Lambda^r V}(g) \) \((1 \leq r \leq n-1)\) are all eigenvalues of the matrix \( \chi_{\Lambda^r V}(1) I - B_{n-r}^T \), where \( g \) runs through \( \mathcal{T}(N) = \mathcal{N} \cap \mathcal{T} \) of Lemma 2.4. Subsequently \( \sum_{r=1}^{n-1} (-1)^{r-1} (\chi_{\Lambda^r V}(1) I - \chi_{\Lambda^r V}(g)) t^{r-1} \) are all the eigenvalues of the matrix \( \sum_{r=1}^{n-1} (-1)^{r-1} (\chi_{\Lambda^r V}(1) I - B_{n-r}) t^{r-1} \). Thus,

$$
\begin{align*}
det \left( \lambda I - \left( \sum_{r=1}^{n-1} (-1)^{r-1} B_{n-r} t^{r-1} \right) \right) = \prod_{g \in \mathcal{T}(N)} \left( \lambda - \left( \sum_{r=1}^{n-1} (-1)^{r-1} \chi_{\Lambda^r V}(g) t^{r-1} \right) \right).
\end{align*}
$$

Let \( m = |\mathcal{T}(N)| = |\bar{I}| \), we have

$$
\begin{align*}
det \left( (1 + (-1)^n t^n) I + \sum_{r=1}^{n-1} (-1)^r B_{n-r} t^r \right) &= t^m \det \left( (t^{-1} + (-1)^n t^{n-1}) I - \left( \sum_{r=1}^{n-1} (-1)^{r-1} B_{n-r} t^{r-1} \right) \right) \\
&= t^m \prod_{g \in \mathcal{T}(N)} \left( (t^{-1} + (-1)^n t^{n-1}) I - \left( \sum_{r=1}^{n-1} (-1)^{r-1} \chi_{\Lambda^r V}(g) t^{r-1} \right) \right) \\
&= \prod_{g \in \mathcal{T}(N)} \left( 1 + \sum_{r=1}^{n-1} (-1)^r \chi_{\Lambda^r V}(g) t^r + (-1)^n t^n \right).
\end{align*}
$$

This completes the proof. \( \square \)

Using the same method, the Poincaré series of induced \( N \)-modules in the symmetric algebra can also be computed:
Theorem 2.7. Let $N \trianglelefteq G$ be a pair of finite normal subgroups of $\text{SL}_n(\mathbb{C})$ and $\{\hat{\phi}_i|i \in I\}$ be the set of the induced $G$-modules of all irreducible $N$-modules. Assume that $V$ is the standard $G$-module and $D_r$ is the matrix afforded by $\wedge^r V$ tensor $\hat{\phi}_i$ for $1 \leq r \leq n - 1$. Let $M^i_2$ be the matrix $(1 + (-1)^{nt^n})I + \sum_{r=1}^{n-1} (-1)^r D_{n-r}t^r$ with the $i$th column replaced by $\delta = (1, 0, \ldots, 0)^T \in \mathbb{N}^{[1]}$. Then the Poincaré series of $\hat{\phi}_i$ in $S(V) = \bigoplus_{k \geq 0} S^k(V)$ is

$$s^i(t) = \frac{\det(M^i_2)}{\det \left( (1 + (-1)^{nt^n})I + \sum_{r=1}^{n-1} (-1)^r D_{n-r}t^r \right)}$$

$$= \frac{\det(M^i_2)}{\prod_{g \in \Upsilon(N)} \left( 1 + \sum_{r=1}^{n-1} (-1)^r \chi_{\wedge^{n-r} V}(g)t^r + (-1)^{nt^n} \right)}, \tag{5}$$

where $\Upsilon(N) = N \cap \Upsilon$, and $\Upsilon$ is a set of representatives of conjugacy classes of $G$.

If $N = G \leq \text{SL}_n(\mathbb{C})$, then the method of Theorem 2.6 also gives a formula of the Poincaré series for $G$-modules in $S(V) = \bigoplus_{k \geq 0} S^k(V)$ associated with Lemma 2.3 and Lemma 2.5.

Theorem 2.8. Let $G$ be a finite subgroup of $\text{SL}_n(\mathbb{C})$ and $\{\rho_i|i \in I_G\}$ be the set of complex irreducible $G$-modules. Assume that $V$ is the standard $G$-module and $A_r$ is the matrix afforded by $\wedge^r V$ tensor $\rho_i$ for $1 \leq r \leq n - 1$. Let $M^i$ be the matrix $(1 + (-1)^{nt^n})I + \sum_{r=1}^{n-1} (-1)^r A_{n-r}t^r$ with the $i$th column replaced by $\delta = (1, 0, \ldots, 0)^T \in \mathbb{N}^{[1]_G}$. Then the Poincaré series of $\rho_i$ in $S(V) = \bigoplus_{k \geq 0} S^k(V)$ is

$$s^i(t) = \frac{\det(M^i)}{\det \left( (1 + (-1)^{nt^n})I + \sum_{r=1}^{n-1} (-1)^r A_{n-r}t^r \right)}$$

$$= \frac{\det(M^i)}{\prod_{g \in \Upsilon} \left( 1 + \sum_{r=1}^{n-1} (-1)^r \chi_{\wedge^{n-r} V}(g)t^r + (-1)^{nt^n} \right)}, \tag{6}$$

where $\Upsilon$ is a set of conjugacy class representatives of $G$.

2.2. $\text{SL}_3(\mathbb{C})$ and $\text{SL}_4(\mathbb{C})$. The classification of the finite subgroups of the special linear groups $\text{SL}_3(\mathbb{C})$ and $\text{SL}_4(\mathbb{C})$ are given in [26][12]. Let $G$ be a finite subgroup of $\text{SL}_3(\mathbb{C})$ or $\text{SL}_4(\mathbb{C})$ and $V$ the standard $G$-module, we can see the denominator of (6) are connected only with the character of $V$ by applying Lemma 2.5 and Theorem 2.8 to $G$. 
Corollary 2.9. Let $G$ be a finite subgroup of $\text{SL}_3(\mathbb{C})$ and $\{\rho_i | i \in I_G\}$ the set of complex irreducible $G$-modules. Assume that $V$ is the standard $G$-module and $A_1$ is the adjacency matrix of $\mathcal{R}_V(G)$. Let $M^i$ be the matrix $(1 - t^3)I - A_1t + A_1^T t^2$ with the $i$th column replaced by $\hat{\delta} = (1, 0, \ldots, 0)^T \in \mathbb{N}^{1|G|}$. Then the Poincaré series of $\rho_i$ in $S(V) = \bigoplus_{k \geq 0} S^k(V)$ is

$$s^i(t) = \frac{\det(M^i)}{\det((1 - t^3)I - A_1t + A_1^T t^2)} = \frac{\det(M^i)}{\prod_{g \in \Upsilon} \left(1 - \chi_V(g)t + \chi_V(g)t^2 - t^3\right)},$$

where $\chi_V(g)$ is the conjugate of $\chi_V(g)$ and $\Upsilon$ is a set of conjugacy class representatives of $G$.

Corollary 2.10. Let $G$ be a finite subgroup of $\text{SL}_4(\mathbb{C})$ and $\{\rho_i | i \in I_G\}$ the set of complex irreducible $G$-modules. Assume $V$ is the standard $G$-module, $A_1$ and $A_2$ are matrices afforded by $V$ and $\wedge^2 V$ tensored with $\rho_i$ respectively. Let $M^i$ be the matrix $(1 + t^4)I - A_1t + A_2t^2 + A_1^T t^3$ with the $i$th column replaced by $\hat{\delta} = (1, 0, \ldots, 0)^T \in \mathbb{N}^{1|G|}$. Then the Poincaré series of $\rho_i$ in $S(V) = \bigoplus_{k \geq 0} S^k(V)$ is

$$s^i(t) = \frac{\det(M^i)}{\det((1 + t^4)I - A_1t + A_2t^2 - A_1t^3)} = \frac{\det(M^i)}{\prod_{g \in \Upsilon} \left(1 - \chi_V(g)t + \frac{1}{2}(\chi_V(g)^2 - \chi_V(g^2))t^2 - \chi_V(g)t^3 + t^4\right)},$$

where $\chi_V(g)$ is the conjugate of $\chi_V(g)$ and $\Upsilon$ is a set of conjugacy class representatives of $G$.

2.3. The McKay-Slodowy correspondence.

It is well known that the isomorphism classes of the finite subgroups of special linear group $\text{SL}_2(\mathbb{C})$ are a cyclic group $C_n$ of order $n$, a binary dihedral group $D_n$ of order $4n$ and three exceptional polyhedral groups: the binary tetrahedral $T$ of order 24, the binary octahedral group $O$ of order 48, and the binary icosahedral group $I$ of order 120.

The McKay-Slodowy correspondence says that the distinguished pairs $N \triangleleft G$ are $D_{n-1} \triangleleft D_{2(n-1)}$, $C_{2n} \triangleleft D_n$, $C_{2n} \triangleleft D_{2n}$, $T \triangleleft O$, $D_2 \triangleleft T$, $C_2 \triangleleft D_2$ and $V \cong \mathbb{C}^2$, where the corresponding quiver $\mathcal{R}_V(G)$ realizes the twisted multiply laced affine Dynkin diagram of types $A_{2n-1}^{(2)}$, $D_{n+1}^{(2)}$, $A_2^{(2)}$, $E_6^{(2)}$, $D_4^{(3)}$, or $A_2^{(2)}$ respectively, and the quiver $\mathcal{R}_V(N)$ realizes the non-twisted multiply laced affine Dynkin diagram of type $B_n^{(1)}$, $C_n^{(1)}$, $C_n^{(1)}$, $E_6^{(1)}$, $G_2^{(1)}$, or $A_1^{(1)}$ respectively.

Let $X$ be the adjacency matrix of the McKay quiver, then $2I - X^T$ is the corresponding affine Cartan matrix, and denominators $(1 + t^2)I - tX^T$ in (3) or (5) is
called the quantum affine Cartan matrix of the same type [25], as the limit of \( t \to 1 \) is the former.

With respect to the simply or multiply laced affine Dynkin diagrams and quantum affine matrices, the Poincaré series in Section 2.1 can be given as follows.

**Corollary 2.11.** Let \( N \triangleleft G \leq \text{SL}_2(\mathbb{C}) \) and \( \{\bar{\rho}_i | i \in \bar{I}\} \) be the set of \( N \)-restriction modules. Let \( B_1 \) be the adjacency matrix of \( \mathcal{R}_V(\hat{G}) \) and \( M_1^i \) be the transpose of quantum affine matrix \( (1 + t^2)I - tB_1 \) with the \( i \)th column replaced by \( \hat{\mathbf{a}} = (1, 0, \ldots, 0)^T \in \mathbb{N}^{\bar{I}} \). Then the Poincaré series of \( \hat{\rho}_i \) in \( S(V) = \bigoplus_{k \geq 0} S^k(V) \) is

\[
\hat{s}^i(t) = \frac{\det(M_1^i)}{\det((1 + t^2)I - tB_1)} = \prod_{g \in \Upsilon(N)} \frac{\det(M_1^i)}{(1 + t^2 - \chi_V(g)t)},
\]

where \( \Upsilon(N) = N \cap \Upsilon, \Upsilon = \{g \in G | g \) is a set of representatives of conjugacy class of \( G\}. \)

**Corollary 2.12.** Let \( N \triangleleft G \leq \text{SL}_2(\mathbb{C}) \) and \( \hat{\phi}_i \ (i \in \hat{I}) \) be an induced \( G \)-module. Let \( D_1 \) be the adjacency matrix of \( \mathcal{R}_V(\hat{N}) \) and \( M_2^i \) the transpose of quantum affine matrix \( (1 + t^2)I - tD_1 \) with the \( i \)th column replaced by \( \hat{\mathbf{a}} = (1, 0, \ldots, 0)^T \in \mathbb{N}^{\hat{I}} \). Then the Poincaré series of \( \hat{\phi}_i \) in \( S(V) = \bigoplus_{k \geq 0} S^k(V) \) is given by

\[
\hat{s}^i(t) = \frac{\det(M_2^i)}{\det((1 + t^2)I - tD_1)} = \prod_{g \in \Upsilon(N)} \frac{\det(M_2^i)}{(1 + t^2 - \chi_V(g)t)},
\]

where \( \Upsilon(N) = N \cap \Upsilon, \Upsilon \) is a set of conjugacy class representatives of \( G \).

**Corollary 2.13.** Let \( G \leq \text{SL}_2(\mathbb{C}) \) and \( \{\rho_i | i \in I_G\} \) be the set of complex irreducible \( G \)-modules. Let \( A_1 \) be the adjacency matrix of \( \mathcal{R}_V(G) \) and \( M^i \) the quantum affine matrix \( (1 + t^2)I - tA_1 \) with the \( i \)th column replaced by \( \hat{\mathbf{a}} = (1, 0, \ldots, 0)^T \in \mathbb{N}^{\bar{I}} \). Then the Poincaré series of \( \rho_i \) in \( S(V) = \bigoplus_{k \geq 0} S^k(V) \) is

\[
s^i(t) = \frac{\det(M^i)}{\det((1 + t^2)I - tA_1)} = \prod_{g \in \Upsilon} \frac{\det(M^i)}{(1 + t^2 - \chi_V(g)t)},
\]

where \( \Upsilon \) is a set of conjugacy class representatives of \( G \).

Thanks to the close correlation between twisted and non-twisted multiply laced affine Lie algebras, the Poincaré series of affine Lie algebras have the following relations, which is analogous to our results on Poincaré series for restriction and induced modules [14].

**Corollary 2.14.** Let \( N \triangleleft G \) be \( D_{n-1} \triangleleft D_{2(n-1)}, C_{2n} \triangleleft D_n, T \triangleleft O, D_2 \triangleleft T \) in \( \text{SL}_2(\mathbb{C}) \). Then the Poincaré series \( s^i(t) \) and \( \hat{s}^i(t) \) for \( N \)-restriction module \( \hat{\rho}_i \) and
induced $G$-module $\hat{\phi}_t$ in $S(\mathbb{C}^2) = \bigoplus_{k \geq 0} S^k(\mathbb{C}^2)$ have the following proportional relation:

$$\hat{s}^i(t) = \begin{cases} \hat{s}^{i'}(t), & i' \text{ is a long root in } \mathcal{R}_V(\hat{N}) \\ \|G : N\| \hat{s}^{i'}(t), & i' \text{ is a short root in } \mathcal{R}_V(\hat{N}) \end{cases},$$

where $\hat{\rho}_i \leftrightarrow \hat{\phi}_t$ is a bijective map between $\hat{\mathfrak{I}}$ and $\hat{\mathfrak{I}}$. In addition, let $N \triangleleft G$ be $C_{2n} < D_{2n}$ ($n \geq 2$), $C_2 < D_2$, then

$$\hat{s}^i(t) = \begin{cases} \hat{s}^0(t), & i \text{ is the affine vertex of } \mathcal{R}_V(\hat{G}) \text{ corresponding to the trivial module} \\ 2\hat{s}^{i'}(t), & i \text{ (resp. } i') \text{ in the finite Dynkin diagram of } \mathcal{R}_V(\hat{G}) \text{ (resp. } \mathcal{R}_V(\hat{N})) \end{cases}.$$  

3. Poincaré Series of Symmetric Invariants for $SL_2(\mathbb{C})$

Let $N \triangleleft G$ be a pair of finite subgroups of $SL_2(\mathbb{C})$. By removing the affine vertex corresponding to the trivial module, the McKay quivers are the multiply laced Dynkin diagrams corresponding to finite dimensional simple Lie algebras. Thus we denote the quantum (finite dimensional) Cartan matrices of the non-simply laced Dynkin diagrams corresponding to finite dimensional simple Lie algebras. Theorem 3.1. Let $N \triangleleft G \leq SL_2(\mathbb{C})$. Let $(1 + t^2)I - tB_1$ (resp. $(1 + t^2)I - tD_1$) be the transpose of quantum affine Cartan matrix of $\mathcal{R}_V(\hat{G})$ (resp. $\mathcal{R}_V(\hat{N})$), and $(1 + t^2)I - t\tilde{B}_1$ (resp. $(1 + t^2)I - t\tilde{D}_1$) the corresponding quantum Cartan matrix. Then the Poincaré series of the $N$-invariants $S(V)^N$ and $G$-invariants $S(V)^G$ in $S(V) = \bigoplus_{k \geq 0} S^k(V)$ are

$$\hat{s}^0(t) = \frac{\det((1 + t^2)I - t\tilde{B}_1)}{\det((1 + t^2)I - tB_1)} = \frac{\det((1 + t^2)I - t\tilde{D}_1)}{\det((1 + t^2)I - tD_1)}$$

$$\prod_{g \in \Upsilon(N)} (1 + t^2 - \chi_V(g)t),$$

where $\Upsilon(N) = \Upsilon \cap N$ and $\Upsilon$ is a fixed set of representatives of conjugacy class of $G$.

Similarly, we have also the quantum finite Cartan matrix $(1 + t^2)I - tA_1$ for a finite simply laced Lie algebra, where $A_1$ is the adjacency matrix of the simply laced Dynkin diagram corresponding to $\mathcal{R}_V(G)$ for $N = G \leq SL_2$.

Theorem 3.2. Let $G \leq SL_2(\mathbb{C})$. Let $(1 + t^2)I - tA_1$ be the quantum affine Cartan matrix of $\mathcal{R}_V(G)$, and $(1 + t^2)I - t\tilde{A}_1$ be the corresponding quantum finite
Cartan matrix. Then the Poincaré series of $G$-invariants $S(V)^G$ inside $S(V) = \bigoplus_{k \geq 0} S^k(V)$ is

$$s^0(t) = \frac{\det \left( (1 + t^2)I - t\tilde{A}_1 \right)}{\det \left( (1 + t^2)I - tA_1 \right)} = \prod_{g \in \Gamma} \frac{(1 + t^2)I - \chi_V(g)t}{(1 + t^2 - \chi_V(g)t)},$$

where $\Gamma = \{ g \in G \mid g$ is a representative of conjugacy class of $G \}.$

As application of the Poincaré series of symmetric invariants, we now consider their relation with the finite and affine Coxeter transformation. Tensor invariants and Poincaré series have been shown to realize the exponents of simply laced affine Lie algebras by Benkart [1], and we have generalized her result to all types of affine Dynkin diagrams except $A^{(1)}_{2n}$ in [14]. In the following, we will study the connection between the Poincaré series of symmetric invariants and show that they also provide a good setting to give rise to all eigenvalues of the affine and finite Coxeter transformations.

**Table 1** Exponents and Coxeter number.

| Dynkin diagrams | Exponents | Coxeter number |
|-----------------|-----------|---------------|
| $A_n$           | $1, 2, 3, \ldots, n$ | $n + 1$ |
| $B_n$           | $1, 3, 5, \ldots, 2n - 1$ | $2n$ |
| $C_2$           | $1, 3, 5, \ldots, 2n - 1$ | $2n$ |
| $D_n$           | $1, 3, 5, \ldots, 2n - 3, n - 1$ | $2n - 2$ |
| $E_6$           | $1, 4, 5, 7, 8, 11$ | 12 |
| $E_7$           | $1, 5, 7, 9, 11, 13, 17$ | 18 |
| $E_8$           | $1, 7, 11, 13, 17, 19, 23, 29$ | 30 |
| $F_4$           | $1, 5, 7, 11$ | 12 |
| $G_2$           | $1, 5$ | 6 |
| $A^{(1)}_1$     | $0, 1$ | 1 |
| $A^{(1)}_{2\ell+1}$ | $0, 1, 1, \ldots, \ell, \ell, \ell + 1$ | $\ell + 1$ |
| $D^{(1)}_{2\ell+1}$ | $0, 2, \ldots, 2\ell - 2, 2\ell - 1, 2\ell - 1, 2\ell, \ldots, 2(2\ell - 1)$ | $2(2\ell - 1)$ |
| $D^{(1)}_2$     | $0, \ldots, \ell - 2, \ell - 1, \ell - 1, \ell - 1, \ell - 2, \ldots, 2\ell - 2$ | $2\ell - 2$ |
| $E^{(1)}_7$     | $0, 2, 2, 3, 4, 4, 6$ | 6 |
| $E^{(1)}_6$     | $0, 3, 4, 6, 6, 8, 9, 12$ | 12 |
| $E^{(1)}_8$     | $0, 6, 10, 12, 15, 18, 20, 24, 30$ | 30 |
| $A^{(2)}_2$     | $0, 2$ | 2 |
| $A^{(2)}_{2\ell+1}$ | $0, 1, \ldots, \ell$ | $\ell$ |
| $B^{(1)}_{2\ell+1}$, $A^{(2)}_{4\ell+1}$ | $0, 1, \ldots, \ell - 1, \ell, \ell, \ell + 1, \ldots, 2\ell$ | $2\ell$ |
| $B^{(1)}_{2\ell}, A^{(2)}_{2\ell} - 1$ | $0, 2, \ldots, 2\ell - 2, 2\ell - 1, 2\ell, \ldots, 2(2\ell - 1)$ | $2(2\ell - 1)$ |
| $C^{(1)}_2$, $D^{(2)}_{\ell+1}$ | $0, 1, \ldots, \ell$ | $\ell$ |
| $F^{(1)}_4$, $E^{(2)}_6$ | $0, 2, 3, 4, 6$ | 6 |
| $G^{(1)}_2$, $D^{(3)}_4$ | $0, 1, 2$ | 2 |
Theorem 3.3. Let \( N \leq G \leq \text{SL}_2(\mathbb{C}) \) and \((1 + t^2)I - tB_1 \) (resp. \((1 + t^2)I - t\tilde{B}_1\)) be the transpose of quantum affine (resp. finite) Cartan matrix of \( \mathcal{R}_V(G) \) (resp. corresponding finite Dynkin diagram). Let \( \Delta \) (resp. \( \tilde{\Delta} \)) be the set of exponents \( m_i \) (resp. \( \tilde{m}_i \)) of the affine (resp. finite simple) Lie algebra associated with the Dynkin diagram and \( h \) (resp. \( \tilde{h} \)) the affine (resp. finite) Coxeter number. Then the Poincaré series for the \( N \)-invariants and \( G \)-invariants in \( S(V) = \bigoplus_{k \geq 0} S^k(V) \) is

\[
S^0(t) = S^0(t) = \frac{\det((1 + t^2)I - t\tilde{B}_1)}{\det((1 + t^2)I - tB_1)} = \frac{\det((1 + t^2)I - t\tilde{B}_1)}{\prod_{g \in \mathcal{Y}(N)} (1 + t^2 - \chi_V(g)t)} \cdot \prod_{\tilde{m}_i \in \tilde{\Delta}} \frac{1 + t^2 - 2\cos\left(\frac{\tilde{m}_i\pi}{\tilde{h}}\right)t}{(1 + t^2 - 2\cos\left(\frac{m_i\pi}{h}\right)t)},
\]

where \( \mathcal{Y}(N) = N \cap \mathcal{Y} \), and \( \mathcal{Y} \) is a set of conjugacy class representatives of \( G \).

Theorem 3.4. Let \( G \leq \text{SL}_2(\mathbb{C}) \) without \( G \cong C_n \) for \( n \) is odd, and let \((1 + t^2)I - A_1 \) (resp. \((1 + t^2)I - \tilde{A}_1\)) be quantum affine (resp. finite) Cartan matrix of \( \mathcal{R}_V(G) \) (resp. corresponding finite Dynkin diagram). Let \( \Delta \) (resp. \( \tilde{\Delta} \)) be the set of exponents \( m_i \) (resp. \( \tilde{m}_i \)) of the simply laced affine (resp. finite) Lie algebra and \( h \) (resp. \( \tilde{h} \)) the affine (resp. finite) Coxeter number. Then the Poincaré series of \( G \)-invariants \( S(V)^G \) in \( S(V) = \bigoplus_{k \geq 0} S^k(V) \) is

\[
S^0(t) = \frac{\det((1 + t^2)I - \tilde{A}_1t)}{\det((1 + t^2)I - A_1t)} = \frac{\det((1 + t^2)I - \tilde{A}_1t)}{\prod_{g \in \mathcal{Y}} (1 + t^2 - \chi_V(g)t)} \cdot \prod_{\tilde{m}_i \in \tilde{\Delta}} \frac{1 + t^2 - 2\cos\left(\frac{\tilde{m}_i\pi}{\tilde{h}}\right)t}{(1 + t^2 - 2\cos\left(\frac{m_i\pi}{h}\right)t)},
\]

where \( \mathcal{Y} = \{ g \in G | g \) is a representative of conjugacy class of \( G \} \).

Remark 3.5. The proof of the above two theorems are similar to our results in [14] Theorem 4.5. In addition, the conclusions also imply that the character value \( \chi_V(g) \) when \( g \) runs over the set \( \mathcal{Y}(N) \) (resp. \( \mathcal{Y} \)) are equal to \( 2\cos\left(\frac{m_i\pi}{h}\right) \) when \( m_i \) runs through \( \Delta \) the set of exponents of the non-simply laced (resp. simply laced) affine Lie algebra and \( h \) is the corresponding affine Coxeter number [11][14].

Let \( G \) be a subgroup of \( \text{SL}_2(\mathbb{C}) \), then \( G = \langle x, y, z | x^p = y^q = z^r = xz = xyx \rangle \) is a polyhedral group with \( p \geq q \geq 1 \) and \( p = q \) if \( r = 1 \). Assume that \( G \) is not a cyclic group of odd order. The Poincaré series for \( G \)-invariants \( S(C^2)^G \) have been studied extensively for the simply laced types by Gonzalez-Sprinberg and Verdier [11], Knörrer [15] and Kostant [18], who gave a compact formula of the Poincaré
formula using intrinsic data of the finite group $G$. We can generalize the result to all twisted affine Dynkin diagrams as follows.

Let $N \vartriangleleft G \leq \text{SL}_2$ be a distinguished pair of subgroups. Let $\tilde{\rho}_i$ be the $N$-restriction module of the irreducible $G$-module $\rho_i$, and $\phi_i$ is an irreducible $N$-module. We let $h = \sum_{i \in \mathfrak{Y} \cap N} \dim \tilde{\rho}_i$, which is seen the Coxeter number of the finite dimensional Lie algebra associated to the affine Dynkin diagram [14]. Define

$$a = 2\max \{\dim \phi_i | i \in \mathfrak{Y} \}$$

$$b = h + 2 - a.$$

The following result can be proved similarly as in the simply laced type.

**Table 2**

| Dynkin diagrams | $a$ | $b$ | $h$ | $p$ | $q$ | $r$ |
|-----------------|-----|-----|-----|-----|-----|-----|
| $A_{\ell}^{(1)}$ | 2   | $\ell + 1$ | $\ell + 1$ | $\frac{1}{2}(\ell + 1)$ | $\frac{1}{2}(\ell + 1)$ | 1 |
| $D_{\ell}^{(1)}$ | 4   | $2\ell - 4$ | $2\ell - 2$ | $\ell - 2$ | 2 | 2 |
| $E_6^{(1)}$    | 6   | 8   | 12  | 3   | 3   | 2 |
| $E_7^{(1)}$    | 8   | 12  | 18  | 4   | 3   | 2 |
| $E_8^{(1)}$    | 12  | 20  | 30  | 5   | 3   | 2 |
| $A_1^{(1)}$, $A_2^{(2)}$ | 2   | 2   | 2   | 1   | 1   | 1 |
| $A_{2\ell}^{(2)}$ | 2   | $2\ell$ | $2\ell$ | $\ell$ | 1   | 1 |
| $B_1^{(1)}$, $A_{2\ell-1}^{(2)}$ | 4   | $2\ell - 2$ | $2\ell$ | $\ell - 1$ | 2 | 1 |
| $C_\ell^{(1)}$, $D_{\ell+1}^{(2)}$ | 2   | $2\ell$ | $2\ell$ | $\ell$ | 1 | 1 |
| $F_4^{(1)}$, $E_6^{(2)}$ | 6   | 8   | 12  | 3   | 2   | 1 |
| $G_2^{(1)}$, $D_4^{(3)}$ | 4   | 4   | 6   | 2   | 1   | 1 |

**Theorem 3.6.** Let $N \vartriangleleft G \leq \text{SL}_2(\mathbb{C})$ (resp. $G \leq \text{SL}_2(\mathbb{C})$ without $G \cong C_n$ for $n$ is odd) which can realize a pair of multiply (resp. a simply) laced affine Dynkin diagrams. The Poincaré series of invariants $s^0(t)$ and $\hat{s}^0(t)$ (resp. $s^0(t)$) are

\[
1 + \frac{t^b}{(1 - t^a)(1 - t^b)},
\]

where $a, b, h$ are defined conceptually above, also explicitly listed in Table 2.

**Theorem 3.7.** Let $N \vartriangleleft G \leq \text{SL}_2(\mathbb{C})$ (resp. $G \leq \text{SL}_2(\mathbb{C})$ without $G \cong C_n$ for $n$ is odd). Let $X$ be the adjacency matrix of $\mathcal{R}_V(\tilde{G})$ (resp. $\mathcal{R}_V(N)$ or $\mathcal{R}_V(G)$). Then the determinant of quantum affine Cartan matrices are given by

\[
\det((1 + t^2)I - tX) = \frac{(1 - t^{2p})(1 - t^{2q})(1 - t^{2r})}{1 - t^2},
\]

where the parameters $p, q, r$ are given above.

**Remark 3.8.** Using folding symmetry of the Dynkin diagrams, Suter [25] previously obtained some cases of (10) for the types $B_n^{(1)}$, $C_n^{(1)}$, $F_4^{(1)}$ and $G_2^{(1)}$. He also obtained (11) for the simply laced types.
4. The global version of Poincaré series of invariants for $SL_2(\mathbb{C})$

The close relation between the Cartan matrices of the finite Dynkin diagrams and the Tchebychev polynomials have been studied in [6 16], which was further developed to give closed-form expressions of Poincaré series of $G$-invariants in the tensor algebra for $G \leq SU_2$ [11].

In [14] we built upon their relations to compute closed-form expressions of the Poincaré series of invariants in the tensor algebra for all twisted and untwisted Dynkin diagrams. In this section, we will generalize the results to symmetric invariants for all affine Dynkin diagrams. Similar to the simply laced case, our new formula further confirms that the Poincaré series associated to the twisted affine Lie algebras have equally beautiful relation with the character theory of pairs of subgroups of $SL_2(\mathbb{C})$, more importantly the new formulas are global in the sense that the final result depend on the dimension and the type of the pairs of subgroups.

The Tchebychev polynomials of the first kind $T_n(t)$ and the second kind $U_n(t)$ are recursively defined (also see [21]). For $n \geq 1$

(12) $T_0(t) = 1, \quad T_1(t) = t, \quad T_{n+1}(t) = 2tT_n(t) - T_{n-1}(t)$.
(13) $U_0(t) = 1, \quad U_1(t) = 2t, \quad U_{n+1}(t) = 2tU_n(t) - U_{n-1}(t)$.

The polynomials $T_n(t)$ and $U_n(t)$ can be expressed as the additive closed forms

(14) $T_n(t) = \sum_{i=0}^{[n/2]} \binom{n}{2i} t^{n-2i}(t^2 - 1)^i = t^n \sum_{i=0}^{[n/2]} \binom{n}{2i} (1 - t^{-2})^i$, 
(15) $U_n(t) = \sum_{i=0}^{[n/2]} (-1)^i \binom{n - i}{i} (2t)^{n-2i}$,

and their factorization are

(16) $T_n(t) = 2^{n-1} \prod_{i=1}^{n} \left( t - \cos \left( \frac{(2i - 1)\pi}{2n} \right) \right)$,
(17) $U_n(t) = 2^n \prod_{i=1}^{n} \left( t - \cos \left( \frac{\pi i}{n+1} \right) \right)$.

Moreover, they are related by the following relations

(18) $T_n(t) = U_n(t) - tU_{n-1}(t)$
(19) and $2T_n(t) = U_n(t) - U_{n-2}(t)$.

4.1. The pair of subgroups $D_{2(n-1)} \vartriangleleft D_{2(n-1)}$. For $n \geq 3$, let $D_{2(n-1)} = \langle x, y | x^{2(n-1)} = y^2 = -1, yxy^{-1} = x^{-1} \rangle$ be the binary dihedral group of order $8(n-1)$. $D_{2(n-1)}$ is imbedded into $SL_2(\mathbb{C})$ by

(20) $\rho(x) = \begin{pmatrix} \theta_{4(n-1)}^{-1} & 0 \\ 0 & \theta_{4(n-1)} \end{pmatrix}, \quad \rho(y) = \begin{pmatrix} 0 & \sqrt{-1} \\ \sqrt{-1} & 0 \end{pmatrix},$
where \( \theta_{4(n-1)} = e^{2\pi \sqrt{-1}/4(n-1)} \). Since \((x^2, y) = D_{n-1} \triangleleft D_{2(n-1)}, \) the \( D_{n-1} \)-conjugacy classes representatives are \( \Upsilon(D_{n-1}) = \{ \pm 1, x^{2i} (i = 1, \ldots, n - 2) \}, \) and the corresponding character values are \( \chi_V(x^{2i}) = \theta_{4i}^{2} + \theta_{4(n-1)}^{-2i} = 2 \cos(\pi i/(n - 1)), \) and \( \chi_V(y) = 0. \)

Using (7) in Thm. 3.1, we get the determinant of the affine Cartan \( t \)-matrix

\[
\det((1 + t^2)I - tB_1) = \prod_{g \in T(D_{n-1})} (1 + t^2 - \chi_V(g)t)
\]

\[
= (1 + t^2 - 2t)(1 + t^2 + 2t)(1 + t^2) \prod_{i=1}^{n-2} \left( 1 - 2 \cos \left( \frac{\pi i}{n - 1} \right) t \right)
\]

(21)

\[
= (1 - t^2 - t^4 + t^6) \prod_{i=1}^{n-2} \left( 1 - 2 \cos \left( \frac{\pi i}{n - 1} \right) t \right).
\]

It is clear that the twisted affine Dynkin diagram of type \( A_{2n-1}^{(2)} \) is realized by the pair of subgroups \( D_{n-1} \triangleleft D_{2(n-1)} \), there is a multiply laced finite Dynkin diagram of type \( C_n \) by removing the affine vertex of \( A_{2n-1}^{(2)} \). So the adjacency matrix of multiply laced finite Dynkin diagram of type \( C_n \) is

\[
\widetilde{B}_1 = \begin{pmatrix}
0 & 1 & 0 & \cdots & 0 \\
1 & 0 & 1 & \cdots & 0 \\
\vdots & \ddots & \ddots & \ddots & 0 \\
0 & \cdots & 1 & 0 & 2 \\
0 & 0 & \cdots & 1 & 0
\end{pmatrix}.
\]

Let \( c_{n-1}(t) = \det((1 + t^2)I - t\widetilde{B}_1) \). The first two are \( c_0(t) = 1 + t^2, c_1(t) = 1 + t^4 \). By expanding the det of the quantum finite Cartan matrix, we have the recursive relation

\[
c_{n+1}(t) = (1 + t^2)c_n(t) - t^2c_{n-1}(t) \quad \text{for } n \geq 1.
\]

It follows from the definition of Tchebychev polynomial (12) that

\[
c_{n-1}(t) = 2t^n T_n \left( \frac{t + t^{-1}}{2} \right),
\]

where \( T_n(t) \) is the Tchebychev polynomial of the first kind. Therefore

\[
c_{n-1}(t) = 2t^n \left( \frac{t + t^{-1}}{2} \right)^{n} \sum_{i=0}^{[n/2]} \binom{n}{2i} \left( 1 - \left( \frac{t + t^{-1}}{2} \right)^{-2} \right)^i
\]

(22)

\[
= 2^{(1-n)} \sum_{i=0}^{[n/2]} \binom{n}{2i} (1 + t^2)^{n-2i}(1 - t^2)^{2i}
\]

(23)

\[
= \prod_{i=1}^{n} \left( 1 + t^2 - 2 \cos \left( \frac{(2i-1)\pi}{2n} \right) t \right).
\]
Returning to (21), by the Laplace expansion we have that
\[
\det((1 + t^2)I - tB_1) = (1 + t^2)c_{n-1}(t) - (t^2 + t^4)c_{n-3}(t)
\]
\[
= 2t^n(1 + t^2) \left( T_n \left( \frac{t + t^{-1}}{2} \right) - T_{n-2} \left( \frac{t + t^{-1}}{2} \right) \right).
\]

By (18), (13) and (15),
\[
T_n(t) - T_{n-2}(t) = (2t^2 - 2)U_{n-2}(t)
\]
\[
= (2t^2 - 2) \sum_{i=0}^{\lfloor (n-2)/2 \rfloor} (-1)^i \binom{n-2-i}{i} (2t)^{n-2-2i}.
\]

Consequently,
\[
\det((1 + t^2)I - tB_1) = (1 - t^2 - t^4 + t^6) \sum_{i=0}^{\lfloor (n-2)/2 \rfloor} (-1)^i \binom{n-2-i}{i} t^i(1 + t^2)^n - 2i.
\]

Summarizing the above, we have shown the following result.

**Theorem 4.1.** Let \( D_{n-1} \triangleleft D_{2(n-1)} \leq \text{SL}_2(\mathbb{C}) \). The Poincaré series for \( D_{n-1} \)-invariants \( S(\mathbb{C}^2)^{D_{n-1}} \) and \( D_{2(n-1)} \)-invariants \( S(\mathbb{C}^2)^{D_{2(n-1)}} \) inside tensor algebra \( S(\mathbb{C}^2) = \bigoplus_{k \geq 0} S^k(\mathbb{C}^2) \) is
\[
s^0(t) = \sum_{i=1}^{n} \left( 1 + t^2 - 2\cos \left( \frac{(2i-1)\pi}{2n} \right) \right) \\
(1 - t^2 - t^4 + t^6) \prod_{i=1}^{n-2} \left( 1 - 2\cos \left( \frac{\pi i}{n-1} \right) \right) \\
= 2(1-n) \sum_{i=0}^{\lfloor (n-2)/2 \rfloor} (-1)^i \binom{n-2-i}{i} t^i(1 + t^2)^{n-2-2i}.
\]

4.2. The pair of subgroups \( C_{2n} \triangleleft D_n \). For \( n \geq 2 \), let \( D_n = \langle x, y | x^{2n} = y^2 = -1, yxy^{-1} = x^{-1} \rangle \) be the binary dihedral group of order \( 4n \), then the cyclic group \( C_{2n} = \langle x \rangle \) is a normal subgroup with index 2. By the natural imbedding \( \Phi \) of \( D_n \) into \( \text{SL}_2(\mathbb{C}) \), the conjugacy set \( \Upsilon(C_{2n}) = \{ \pm 1, x^i | i = 1, \ldots, n - 1 \} \). The character values are \( \chi_{\Upsilon}(\pm 1) = \pm 2 \) and \( \chi_{\Upsilon}(x^i) = 2\cos(\pi i/n) \), so
\[
\det((1 + t^2)I - tB_1) = \prod_{g \in \Upsilon(C_{2n})} (1 + t^2 - \chi_{\Upsilon}(g)t) \\
= (1 - t^2)^2 \prod_{i=1}^{n-1} \left( 1 - 2\cos \left( \frac{\pi i}{n} \right) t \right).
\]
The twisted affine Dynkin diagram $D^{(2)}_{n+1}$ is realized by the pair of subgroups $C_{2n} \triangleleft D_n$. Removing the affine vertex of $D^{(2)}_{n+1}$ we get the multiply laced finite Dynkin diagram of type $B_n$, the dual of finite Dynkin diagram $C_n$. Let $b_{n-1}(t) := \det((1 + t^2)I - tB_1)$ be the determinant of the quantum finite Cartan matrix of finite Dynkin diagram $B_n$, which satisfies the additive and multiplicative formulas (22) and (23).

Using the Laplace expansion, we have that

$$\det((1 + t^2)I - tB_1) = (1 + t^2) b_{n-1}(t) - 2t^2 b_{n-2}(t)$$

$$= (t^{n-1} + (1 + t^2)^2 - 4t^{n+1}) U_{n-1} \left( \frac{t + t^{-1}}{2} \right)$$

$$= (1 - t^2)^2 \sum_{i=0}^{\lfloor (n-1)/2 \rfloor} (-1)^i \binom{n - 1 - i}{i} t^{2i} (1 + t^2)^{n-1-2i}.$$ 

Therefore, we have the next consequence.

**Theorem 4.2.** Let $C_{2n} \triangleleft D_n \leq \text{SL}_2(\mathbb{C})$. The Poincaré series for $C_{2n}$-invariants and $D_n$-invariants in the symmetric algebra $S(\mathbb{C}^2) = \bigoplus_{k \geq 0} S^k(\mathbb{C}^2)$ is

$$S^0(t) = S^0(t) = \prod_{i=1}^{n} \left( 1 + t^2 - 2\cos \left( \frac{(2i-1)\pi}{2n} \right) t \right)$$

$$= (1 - t^2)^2 \prod_{i=1}^{n-1} \left( 1 - 2 \cos \left( \frac{\pi i}{n} \right) t \right)$$

$$= \frac{2^{(1-n)} \sum_{i=0}^{\lfloor (n-1)/2 \rfloor} \binom{n}{2i} (1 + t^2)^{n-2i} (1 - t^2)^{2i}}{(1 - t^2)^2 \sum_{i=0}^{\lfloor (n-1)/2 \rfloor} (-1)^i \binom{n-1-i}{i} t^{2i} (1 + t^2)^{n-1-2i}}.$$ 

**Remark 4.3.** It is easy to conclude there have the equally Poincaré series of invariants for the two pairs of subgroups $C_{2n} \triangleleft D_n$ and $C_{2n} \triangleleft D_{2n}$ because the same two-dimensional $C_{2n}$-module.

4.3. **The pairs** $T \triangleleft O$, $D_2 \triangleleft T$ and $C_2 \triangleleft D_2$. For the three pairs of subgroups $T \triangleleft O$, $D_2 \triangleleft T$ and $C_2 \triangleleft D_2$ in $\text{SL}_2(\mathbb{C})$, Theorem 3.1 says that the Poincaré polynomials $\tilde{s}^0(t) = \tilde{s}^0(t)$ are expressed as the quotient of the determinants of the quantum affine and finite Cartan matrices, which in turns carried the information on the exponents of the affine Lie algebras.

We list the Poincaré series of the invariants using the parameter $a, b, h$ in Table 2 respectively.
$$s^0(t) = s^0(t) = \frac{1 + t^{12}}{(1 - t^6)(1 - t^8)} = 1 + t^6 + t^8 + 2t^{12} + t^{14} + t^{16} + 2t^{18} + \cdots,$$

$$s^0(t) = s^0(t) = \frac{1 + t^6}{(1 - t^4)(1 - t^4)} = 1 + 2t^4 + 3t^6 + 2t^{10} + 4t^{12} + 3t^{14} + 5t^{16} + 4t^{18} + \cdots,$$

$$s^0(t) = s^0(t) = \frac{1 + t^2}{(1 - t^2)(1 - t^2)} = 1 + 3t^2 + 5t^4 + 7t^6 + 9t^8 + 11t^{10} + 13t^{12} + 15t^{14} + 17t^{16} + 19t^{18} + \cdots.$$

4.4. The groups for simply laced affine Lie algebras. For $n \geq 3$, let $C_n = \langle x | x^n = 1 \rangle$ be the cyclic group of order $n$. The map $\rho(x) = \text{diag}(\theta_n^{-1}, \theta_n)$ ($\theta_n = e^{2\pi \sqrt{-1}/n}$) provides an embedding from $C_n$ into $\text{SL}_2(\mathbb{C})$. Denote by $\xi_i$ ($i = 0, 1, \ldots, n-1$) the $n$ irreducible $C_n$-modules. The module $V \simeq \xi_1 \oplus \xi_{-1}$ and $\chi_V(x^i) = \theta_n^i + \theta_n^{-i}$ indicate the determinant of quantum affine Cartan matrix is

$$\det((1 + t^2)I - t\tilde{A}_1) = \prod_{i=1}^{n-1} \left(1 + t^2 - 2\cos \left(\frac{2\pi i}{n} \right) t \right).$$

The simply laced affine Dynkin diagram $A^{(1)}_{n-1}$ is realized by the group $C_n$. Assume $\tilde{A}_1$ is the adjacency matrix of finite Dynkin diagram $A_{n-1}$ which adding a affine note is the affine Dynkin diagram $A^{(1)}_{n-1}$. Let $a_{n-1}(t) := \det((1+t^2)I - t\tilde{A}_1)$. Set $a_0(t) = 1, a_1(t) = 1 + t^2$. Expanding the determinant $a_{n-1}(t)$ there has an inductive relation

$$a_{n+1}(t) = (1 + t^2)a_n(t) - t^2a_{n-1}(t), \quad \text{for } n \geq 1.$$ 

What is more, for all $n \geq 0$, we have

$$a_n(t) = t^nU_n \left(\frac{t + t^{-1}}{2}\right)$$

$$= \prod_{i=1}^{n} \left(1 + t^2 - 2\cos \left(\frac{\pi i}{n+1} \right) t \right)$$

$$= \sum_{i=0}^{\lfloor n/2 \rfloor} (-1)^i \binom{n-i}{i} t^i (1 + t^2)^{n-2i}. $$
Identities (24), (19) and (14) give rise to

\[
\det((1 + t^2)I - tA_1) = (1 + t^2)a_{n-1}(t) - 2t^2a_{n-2}(t) - 2t^n
\]

\[
= t^n(U_n \left( \frac{t + t^{-1}}{2} \right) - U_{n-2} \left( \frac{t + t^{-1}}{2} \right) - 2)
\]

\[
= t^n(2T_n \left( \frac{t + t^{-1}}{2} \right) - 2)
\]

\[
= 2^{1-n} \sum_{i=0}^{|n/2|} \binom{n}{2i} (1 + t^2)^{n-2i}(1 - t^2)^{2i} - 2t^n.
\]

Subsequently, we have now shown the following result.

**Theorem 4.4.** Let \( C_n \leq \text{SL}_2 \). The Poincaré series \( s^0(t) \) for \( C_n \)-invariants in \( S(\mathbb{C}^2) = \bigoplus_{k \geq 0} S^k(\mathbb{C}^2) \) is

\[
s^0(t) = \frac{\prod_{i=1}^{n-1} (1 + t^2 - 2\cos(\frac{\pi(i)}{n})t)}{\prod_{i=0}^{n-1} (1 + t^2 - 2\cos(\frac{2\pi(i)}{n})t)} \left[ \sum_{i=0}^{(n-1)/2} (-1)^i \binom{n-1-i}{i} t^{2i} (1 + t^2)^{n-1-2i} \right]^{1-n} \sum_{i=0}^{|n/2|} \binom{n}{2i} (1 + t^2)^{n-2i}(1 - t^2)^{2i} - 2t^n.
\]

(27)

**Remark 4.5.** The Poincaré series of symmetric invariants for \( C_n \) of odd order are not included in Theorems 3.4, 3.6 and 3.7. Equality (27) is a uniform formula of the Poincaré series of symmetric \( C_n \)-invariants for arbitrary order.

For the group \( D_n \) (\( n \geq 2 \)), Eqs. (25) and (26) derive the determinant of the quantum affine Cartan matrix in (8) for \( D^{(1)}_{n+2} \):

\[
\det((1 + t^2)I - tA_1) = \prod_{g \in T} (1 + t^2 - \chi_V(g)t)
\]

\[
= (1 + t^2 - 2t)(1 + t^2 + 2t)(1 + t^2)^{n-1} \prod_{i=1}^{n-1} \left( 1 + t^2 - 2\cos\left(\frac{\pi(i)}{n}\right)t \right)
\]

\[
= (1 - t^4)^2 \sum_{i=0}^{(n-1)/2} (-1)^i \binom{n-1-i}{i} t^{2i}(1 + t^2)^{n-1-2i}.
\]

The group \( D_n \) realizes the simply laced affine Dynkin diagram of type \( D^{(1)}_{n+2} \). Set \( d_n(t) = \det((1 + t^2)I - tA_1) \), where \( A_1 \) is the adjacency matrix of finite Dynkin diagram \( D_{n+2} \) which is obtained by removing the affine vertex of affine
Dynkin diagram $D^{(1)}_{n+2}$. Assume $d_0(t) = 1 + 2t^2 + t^4$ and $d_1(t) = 1 + t^2 + t^4 + t^6$. By the Laplace expansion, one has the recursive formula

$$d_{n+1}(t) = (1 + t^2)d_n(t) - t^2d_{n-1}(t), \quad \text{for } n \geq 1.$$ 

In addition, we have

$$d_n(t) = 2t^{n+1}(1 + t^2)\Gamma_{n+1}\left(\frac{t + t^{-1}}{2}\right)$$

$$= (1 + t^2)\prod_{i=1}^{n+1}\left(1 + t^2 - 2\cos\left(\frac{(2i-1)\pi}{2(n+1)}\right)t\right)$$

$$= 2^{-n}\sum_{i=0}^{[(n+1)/2]}\left(\frac{n+1}{2i}\right)(1 + t^2)^{n+2-2i}(1 - t^2)^i.$$ 

Therefore we have that

**Theorem 4.6.** Let $D_n \leq \text{SL}_2(\mathbb{C})$. The Poincaré series $s^0(t)$ for $D_n$ invariants in $S(C^2) = \bigoplus_{k \geq 0} S^k(C^2)$ is given by

$$s^0(t) = \frac{(1 + t^2)\prod_{i=1}^{n+1}\left(1 + t^2 - 2\cos\left(\frac{(2i-1)\pi}{2(n+1)}\right)t\right)}{(1 - t^4)\prod_{i=1}^{n-1}\left(1 + t^2 - 2\cos\left(\frac{\pi i}{n}\right)t\right)} - 2^{-n}\sum_{i=0}^{[(n+1)/2]}\left(\frac{n+1}{2i}\right)(1 + t^2)^{n+2-2i}(1 - t^2)^i.$$ 

For the three exceptional polyhedral groups $T$, $O$, $I$ in $\text{SL}_2(\mathbb{C})$, we list the Poincaré series of symmetric invariants by formula (9) with respect to the exponents and Coxeter numbers in Table 1 respectively.

$$s^0_T(t) = \frac{\prod_{m=1, 4, 5, 7, 8, 11, 13, 17}(1 + t^2 - 2\cos\left(\frac{\pi m}{18}\right)t)}{\prod_{m=0, 2, 3, 4, 6}(1 + t^2 - 2\cos\left(\frac{\pi m}{6}\right)t)} = 1 + t^2 - t^6 + t^{10} + t^{12}$$

$$= 1 + t^6 + t^8 + 2t^{12} + t^{14} + t^{16} + 2t^{18} + \cdots,$$

$$s^0_O(t) = \frac{\prod_{m=1, 5, 7, 9, 11, 13, 17}(1 + t^2 - 2\cos\left(\frac{\pi m}{18}\right)t)}{\prod_{m=0, 3, 4, 6, 8, 9, 12}(1 + t^2 - 2\cos\left(\frac{\pi m}{12}\right)t)} = 1 + t^2 - t^6 - t^8 + 2t^{12} + t^{14}$$

$$= 1 + t^8 + t^{12} + t^{16} + t^{18} + \cdots,$$

$$s^0_I(t) = \frac{\prod_{m=1, 5, 6, 7, 8, 9, 9, 12}(1 + t^2 - 2\cos\left(\frac{2\pi m}{12}\right)t)}{\prod_{m=0, 3, 4, 6, 8, 9, 12}(1 + t^2 - 2\cos\left(\frac{2\pi m}{12}\right)t)} = 1 + t^2 - t^6 - t^8 + t^{10} + t^{14}$$

$$= 1 + t^8 + 2t^{12} + t^{14} + t^{16} + t^{18} + \cdots.$$
\[ s^0(t) = \frac{\prod_{m=1,7,11,13,17,19,23,29} \left( 1 + t^2 - 2 \cos \left( \frac{\pi m}{30} \right) t \right)}{\prod_{m=0,6,10,12,15,18,20,24,30} \left( 1 + t^2 - 2 \cos \left( \frac{\pi m}{30} \right) t \right)} \]

\[ = \frac{1 + t^2 - t^6 - t^8 - t^{10} + t^{14} + t^{16}}{1 + t^2 - t^6 - t^8 - t^{10} - t^{12} + t^{16} + t^{18}} = 1 + t^{12} + t^{20} + \ldots. \]

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