Spectral problem on graphs and $L$-functions

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Abstract

The scattering process on multiloop infinite $p + 1$-valent graphs (generalized trees) is studied. These graphs are discrete spaces being quotients of the uniform tree over free acting discrete subgroups of the projective group $PGL(2, \mathbb{Q}_p)$. As the homogeneous spaces, they are, in fact, identical to $p$-adic multiloop surfaces. The Ihara–Selberg $L$-function is associated with the finite subgraph—the reduced graph containing all loops of the generalized tree. We study the spectral problem on these graphs, for which we introduce the notion of spherical functions—eigenfunctions of a discrete Laplace operator acting on the graph. We define the $S$-matrix and prove its unitarity. We present a proof of the Hashimoto–Bass theorem expressing $L$-function of any finite (reduced) graph via determinant of a local operator $\Delta(u)$ acting on this graph and relate the $S$-matrix determinant to this $L$-function thus obtaining the analogue of the Selberg trace formula. The discrete spectrum points are also determined and classified by the $L$-function. Numerous examples of $L$-function calculations are presented.
0 Introduction

Two fundamental results relating the $L$-functions (zeta functions) and Laplacian determinants have been obtained respectively by Selberg [32] (developed by Lax and Phillips [23], L. D. Faddeev and B. S. Pavlov [12], A. B. Venkov [36], and others) and Ihara [20], Hashimoto [18], and H. Bass [2]. The first result—the celebrated Selberg trace formula—concerns the zeta functions and Laplacian determinants of compact (or noncompact with finite area) in general multidimensional ($D$-dimensional) manifolds of constant negative curvature, which can be obtained from the $D$-dimensional upper half-space with the hyperbolic metric by factoring it over a discrete acting subgroup of the symmetry group of this space. We briefly expose these results in order to compare them with the second class of results concerning the Ihara–Selberg $L$-functions on finite graphs.

The zeta function associated with a finite graph was introduced by Ihara [20] who proved the structure theorem for torsion-free discrete cocompact subgroups $\Gamma$ of the group $PGL(2, K_p)$; $K_p$ being a $p$-adic number field or a field of power series over a finite field. Then, $\Gamma$ is a free group and the associated zeta function can be constructed as follows. An element $\gamma \in \Gamma$, $\gamma \neq 1$, is a primitive element if it generates its centralizer in $\Gamma$. Define the length $l(\gamma) \equiv \|\lambda_1/\lambda_2\|_p$ of the element where $\| \cdot \|_p$ is the normalized valuation in $K_p$ and $\lambda_1, \lambda_2 \in K_p$ are the eigenvalues of $\gamma$. Let $\mathcal{P}(\Gamma)$ be the set of conjugate classes of primitive elements of $\Gamma$. Then, the Ihara zeta function is

$$Z_\Gamma(u) = \prod_{\gamma \in \mathcal{P}(\Gamma)} (1 - u^{l(\gamma)})^{-1}. \quad (0.1)$$

This definition was extended to finite-dimensional representations of $\Gamma$ in [21]; then, instead of (0.1), we have

$$Z_\Gamma(u, \rho) = \prod_{\gamma \in \mathcal{P}(\Gamma)} \det(\mathbf{I}_1 - u^{l(\gamma)} \rho(\gamma))^{-1}, \quad (0.2)$$

where $\rho(\gamma)$ is the character of the given representation and $\mathbf{I}_1$ is the unit operator. (Original formula (0.1) corresponds to $\rho(\gamma) \equiv 1$.)

Although zeta function (0.2) is an infinite product, it was proved to be a rational function. The Ihara theorem states that

$$Z_\Gamma(u, \rho) = (1 - u^2)^s \det(I_0 - Au + pu^2), \quad (0.3)$$

where $s$ is an integer and $A$ is a matrix acting in a finite-dimensional linear space. Using the rationality formula one can counts the number of conjugate classes of primitive elements of $\Gamma$.

1. In [3, 5] scattering processes on infinite graphs representing spaces of a constant negative curvature were studied. Such are uniform, or univalent graphs, for which a natural number $p \geq 1$ exists such that all vertices of the graph are incident exactly to $p + 1$ ends of edges. For prime $p$, such graphs can be interpreted as homogeneous spaces.
for the $p$-adic projective group $PGL(2, Q_p)$ factorized, first, by its maximal compact subgroup $PGL(2, Z_p)$ and, second, by some discrete, free-acting (Schottky) group $\Gamma$. The homogeneous space $D \simeq PGL(2, Q_p)/PGL(2, Z_p)$ is a uniform infinite tree graph. If we now factorize the tree $D$ by a discrete freely acting finitely generated subgroup $\Gamma_g \subset PGL(2, Q_p)$, where $g$ is the number of the generating elements, then the obtained graph $T_g = D/\Gamma_g$ is again a univalent graph containing $g$ loops and having the tree $D$ as the universal covering.

We present a proper analogue of the Selberg trace formula [32] for such discrete surfaces. These graphs sometimes are referred to as multiloop $p$-adic surfaces because it has been shown in [24], [7] that for $p$ the prime, the discrete Laplacian action on these graphs yields the proper scattering amplitudes of the $p$-adic string.

Note, however, that most of the calculations were performed for trivial (Abelian) representation of the group $\Gamma$ in the $p$-adic string approach.

A natural distance $|P_{x,y}|$ over a path $P_{x,y}$ connecting the points of a graph is merely the number of edges entering (with multiplicities) this path. One can consider linear spaces of functions $C_0$ and $C_1$ depending, respectively, on vertices $x_i$ and oriented edges $\vec{e}_j$ of the graph $T$. A Laplace operator $\Delta$ acts on the space $C_0$ in a standard way,

$$\Delta f(x) = \sum_{|P_{x,x_i}| = 1} f(x_i) - (p + 1) f(x).$$

It is useful to segregate from the graph $T$ its finite “closed” part—the connected reduced graph $F$ containing all internal loops. The valences of $F$ vertices can be arbitrary ($\leq p+1$). In [7], the string theory for such graphs was developed and the proper $p$-adic counterparts of all crucial ingredients of the open string theory, such as prime forms, Schottky groups, etc. were found for the corresponding scattering amplitudes.

2. For the ordinary closed (or, more general, finite area) Riemann surfaces of constant (negative) curvature the Selberg trace formula, which establishes an explicit relation between determinants of the Laplace operators and zeta functions (or, Ihara–Selberg $L$-functions), is indeed close in form to (0.3). The distance $l(\gamma)$ is now the length of a closed geodesics in the constant negative curvature metric. However, no proper analogue of the Selberg formula is known for the essentially noncompact case where the spectrum of the Laplacian contains a continuous part responsible for the scattering processes. Note, however, that zeta function (1.1) is still well defined for such noncompact surfaces; it is the Laplacian spectrum definition that hinders the progress in this direction. One may hope that the graph description of moduli spaces of complex curves with holes [28, 14, 6] related to the geodesic description may help in finding analogous statements for the rationality in the Riemann surface case. We discuss this approach in Sec. 6.

3. We adopt the definition that zeta functions take values in the field $k_p$ (on the definition of the corresponding noncommutative determinants, see [2]), while $L$-functions are assumed to take values in the complex number field $\mathbb{C}$.

In a series of elegant papers by K.-I. Hashimoto [18] and H. Bass [2] on zeta and
functions, the Ihara formula was generalized to the case of an arbitrary (not necessarily univalent) finite graph. In the most general case, it has the form

$$L(u, \rho) = \det^{-1}(I_1 - uT_1), \quad (0.4)$$

where the new operator $T_1$ is the operator acting on the space of functions on oriented edges (simplexes of dimension one), which corresponds to a translation along edges of the graph.

Relation (0.4) is suitable for both zeta and $L$-functions; the difference is only in the definition of determinants (commutative or noncommutative) involved. In this paper, we consider only the case of $L$-functions, which seems to be physically meaningful as pertaining to the spectral and scattering problems on graphs with potentials.

Formula (0.4) is amazingly universal; it expresses the rationality condition, which holds for the zeta functions in a host of cases related to discrete dynamics—one may generalize it to the case of Hermitian (non-Abelian), Schrödinger type potentials on edges [26], to the case of nonzero torsion (potentials depending on “angles” along the path, i.e., on the pairs of consecutive edges $(\vec{e}_i, \vec{e}_{i+1})$) [8], etc. However, in most of these cases, we cannot find a bridge from (0.4) to the Laplacian determinants on nonuniform graphs. Only when the potential $U_{\vec{\mu}}$ on (oriented) edges satisfies the unitarity condition, i.e., $U_{\vec{\mu}} = U_{\vec{\mu}}^{-1}$ for all pairs of oppositely oriented edges, we can express the $L$-function in terms of Laplacian type determinants on arbitrary (not necessarily univalent) graphs,

$$L(u, \rho) = (1 - u^2)^s \det^{-1}(1 - uM_1 + u^2Q) \equiv (1 - u^2)^s \det^{-1}\Delta(u), \quad (0.5)$$

where $M_1$ is the above operator of summation over all neighbors (with probable potential dependence) and the new in comparison with (1.2) element is the operator $Q$, which counts the number of these neighbors: $Qx = qx$ if the vertex $x \in T$ has $q + 1$ neighbors. In the trivial representation case, $s = |V| - |E|$, where $|V|$ and $|E|$ are total number of vertices and (nonoriented) edges of the graph (in a non-Abelian case, $s$ is the differences of total (half)dimensions of linear spaces $v^{|V|}_A$ and $v^{|E|}_A$, where elements of the representation $v_A$ of the group or algebra $A$ dwell on (now oriented) edges and vertices of the graph. However, the operator $\Delta(u)$ becomes the Laplacian only for $u = 1$; these operators even do not commute at different $u$. (For the detailed description of these results and their different applications, see [37].)

4. The graphs $T$ ($p$-adic multiloop surfaces) are noncompact in general, which differs them both from the closed Riemann surfaces and with finite graphs. However, it is possible to use the spherical function technique and describe the problem in physical terms of the scattering theory on graphs in this case. In the original setting, spherical functions $\psi$ are eigenfunctions of the Laplace–Beltrami operator that depend only on the distance to a given point $x_0$ (the center). Because the Laplacian is the second-order operator, we always have two branches of the solution (at a distant point) proportional to $a^d_+$ and $a^d_-$ where $d$ is the distance to the center. Resolving the eigenvalue problem at the central point we fix the ratio of coefficients $a_+$ and $a_-$ standing by these two branches. If we
choose \(\psi(x_0) = 1\), then \(a_+\) and \(a_-\) become Harish-Chandra coefficients and \(c = a_+/a_-\) is a scattering amplitude of the \(s\)-wave. Spherical functions have been found for the scattering on a quantum hyperplane \([35]\) as well as for the scattering on \(p\)-adic hyperbolic plane \([13]\).

The \(S\)-matrices obtained are closely related to the partition functions of the XXZ model and a number of nice but still mysterious relations between them have been obtained by Freund and Zabrodin \([16]\).

Investigating the spectral theory on graphs has already rich history. Results obtained were related to studying eigenvalue problem on finite uniform \((i.e., \text{univalent})\) graphs. Here, it was proved that a deep relationship exists between the modular forms on Teichmüller spaces and finite graphs, namely, the so-called Ramanujan graphs. We do not discuss this interesting approach in this paper and instead refer the reader to the monograph \([31]\) and papers \([29, 10, 37]\).

In \([3]\), an analogue of a spherical function for the multiloop graph was introduced. The main idea is the following. Considering the spectral problem \(L\psi = \lambda\psi\), \(L\) being the Laplacian, we note that any linear superposition of spherical functions with the same eigenvalue \(\lambda\) but different scattering centers is again an eigenfunction. An eigenfunction of the Laplace operator on the factorized tree \(T = X/\Gamma_g\) corresponds to a source distribution function, \(s(x)\), on the tree \(X\) such that for every \(\gamma \in \Gamma_g\) and \(x \in X\), we have \(s(\gamma x) = s(x)\). Then the whole eigenfunction is periodic under the action of \(\Gamma_g\). Moreover, we choose a finite domain (a reduced graph) \(F \subset T\) and consider only \(s(x)\) such that \(\text{supp } s(x) \subseteq F\). Inside \(T\) there is a unique minimal finite connected subgraph containing all loops—the union \(D(\Gamma)\) of invariant axes of all elements of \(\Gamma_g\) factorized over the action of the group \(\Gamma_g\). This graph, \(D(\Gamma)/\Gamma\), contains all information about the “geometrical structure” of \(T\). We always assume \(D(\Gamma)/\Gamma \subseteq F\).

Each eigenfunction \(\psi(x)\) may be presented as a sum of retarded and advanced wave functions:

\[
\psi(x) = A_{adv}(u)\alpha_+^{d(x,u(x))} - A_{ret}(u)\alpha_-^{d(x,u(x))} \equiv \psi_+(x) - \psi_-(x), \tag{0.6}
\]

where \(\alpha_+\) and \(\alpha_-\) are two fixed complex numbers depending only on the eigenvalue \(t\) of the Laplacian \(L \equiv \Delta(1)\), which acts on the whole graph \(T\), \(L\psi = (t - p - 1)\psi\) and on the initial prime number \(p\), \(\alpha_+ = \frac{t}{2p} + \frac{1}{2} - \frac{1}{p}, \alpha_+\alpha_- = 1/p\), and \(u(x)\) is the closest to \(x\) point of the reduced graph \(F\) (for \(x \in F\), \(u(x) = x\)). As in the central symmetric case, \(\psi(x)\) depends only on distance to \(F\) on branches outside \(F\).

Now we can define a scattering matrix \(S\) for such system. As a basis we choose functions \(A^i_{adv}(u)\) and \(A^i_{ret}(u)\) that are nonzero only for some \((i\text{th})\) point of \(F\). Then, determining the asymptotic vectors \(\psi^+_i\) that behave as \(\alpha_+^a\) when going along \(i\text{th}\) branch and are zero otherwise, we can asymptotically expand \(\psi\) in the sum of basis vectors \(\psi^+_i = \sum_j s_{ia,jb} \psi^+_j\), where now \(i_a, j_b\) are multiindices indicating components \(a\) of the representation of \(\psi_\Lambda\) on the \(i\text{th}\) branch. Then, the matrix \(s_{ia,jb}\) has the natural sense of the \(S\)-matrix.

The determinant of the matrix \(S\) depends on the spectral parameter, contains the
information about spectrum, and, moreover, is directly connected with the \( L \)-function of the graph \( T \). (As for eigenfunctions themselves, the Lax–Phillips approach was developed for their description \[20\].)

In order to find the determinant of \( S \) we impose a restriction: \( A_{\text{adv}}(u)/A_{\text{ret}}(u) = \text{const} \) for all points \( u \in F \). Imposing this condition at all—both external and internal points of \( F \), we fix an arbitrariness in the splitting of \( \psi(x) \) into the advanced and retarded waves.

Because the central object—the reduced graph \( F \)—is finite, a finite set of possible eigenvalues of the \( S \)-matrix, \( \{c_i\} \) (the letter “\( c \)” originates from Harish-Chandra \( c \)-function \[19\]) exists. Their product \( C \) is therefore the determinant of the \( S \)-matrix. We also called it the total \( C \)-function. We present the proof of the basic formula establishing the relationship between \( C \) and determinants of a local operator \( \tilde{\Delta}(u) \) acting only on the reduced graph \( F \),

\[
\det S(t) = \left( \frac{\alpha_+}{\alpha_-} \right)^{r_0} \frac{\det \tilde{\Delta}(\alpha_-)}{\det \tilde{\Delta}(\alpha_+)} ,
\]

where \( r_0 \) is the total dimension of linear space of functions at vertices of the reduced graph \( F \), which take values in \( v_A \).

On the contrary, the operator \( \tilde{\Delta}(\alpha_{\pm}) \) in \( \{0.7\} \) is taken from \( \{0.5\} \), \( \tilde{\Delta}(u) = 1 + \tilde{Q}u^2 - u\tilde{M}_1 \) and it is determined completely in terms of the reduced graph itself, not of the whole graph; the only remaining dependence of the “big” graph \( T \) is contained in arguments \( \alpha_+ \) and \( \alpha_- \) of the function \( \tilde{\Delta}(\alpha_{\pm}) \). Comparing \( \{0.7\} \) and \( \{0.3\} \) we obtain the relation between Harish-Chandra total \( C \)-function and the \( L \)-function of the \( p \)-adic curve:

\[
C = \left( \frac{\alpha_+}{\alpha_-} \right)^{r_0 - r_1} \frac{L(\alpha_-)}{L(\alpha_+)} ,
\]

where \( r_0 - r_1 \) is the difference of total dimensions of spaces of functions determined on vertices and (nonoriented) edges; it is equal \((g - 1) \times |v_A|\), where \( g \) is the number of loops (the genus) of the graph and \( |v_A| \) the representation dimension. We also prove using the geometrical setting of \[26\] that the \( S \)-matrix is unitary in the scattering zone.

From another point of view, the idea to consider a proper product of scattering coefficients calculated at different \( p \) is originated from the adelic ideology. Such products in scattering processes was first proposed in \[13\], where the product of \( C \)-functions for scattering on \( p \)-adic hyperplanes taken over all primes \( p \) appeared to be connected with the \( C \)-function of scattering on genus one modular figure considered by L. D. Faddeev and B. S. Pavlov \[12\]. The very general formulas concerning the scattering on symmetrical spaces and gamma-function technique can be found in \[7\]. One could hope to find a proper adelic products of \( L \)-functions appearing in our approach in order to compare them with the ones for Riemann surfaces.

We also discuss the eigenvalue problem as regarding to the discrete part of the Laplacian spectrum and establish two important relations concerning the \( S \)-matrix. The discrete part of spectrum may contain apart of “customary” discrete eigenvalues corresponding to exponentially decreasing in branches eigenfunctions also the so-called exceptional
eigenvalues corresponding to eigenfunctions that vanish identically on all branches. We first prove that poles of $L$-function correspond to normal discrete eigenvalues iff these poles do not cancel each other in (1.8), i.e., if $\alpha_+$ and $\alpha_-$ are not simultaneously the poles of the $L$-function. On the contrary, as soon as such situation takes place, i.e., there are such poles $\alpha_+$ and $\alpha_-$ of the $L$-function that $\alpha_+\alpha_- = 1/p$, the exceptional discrete spectrum appears.

The paper is organized as follows: Section 1 contains definitions, the interpretation of $p$-adic multiloop curves as graphs and the action of the Shottky group on the initial tree graph. In Sec. 2, we describe the automorphic functions and potentials on these graphs and introduce operators acting on the generalized tree simplicial complexes. In Sec. 3, we describe $L$-functions associated with such groups, or, equivalently, with the reduced graphs. The theorem by Hashimoto and Bass is formulated and a proof is presented. In Sec. 4, we consider the spectral problem, introduce the spherical functions on multiloop graphs, define the corresponding $S$-matrix and show that its determinant can be expressed as a ratio of two $L$-functions. We prove the unitarity of the $S$-matrix and describe how to find discrete spectrum eigenvalues using the $L$-function technique. Examples for genus 1 and 2 are presented in Sec. 5 together with a simple algorithm for calculating $L$-functions in lower genera based on the Hashimoto–Bass theorem. Eventually, in Sec. 6, we present a construction of graphs for describing the Teichmüller spaces of complex curves in the Poincaré uniformization picture and set the problem of finding a proper analogue of the Selberg trace formula for open Riemann surfaces.

1 Definitions

1.1 Graphs and trees

Let $p$ be a natural number and $D$ a uniform tree graph of order $p+1$, $V(D)$ and $\tilde{L}(D)$ the sets of its vertices and (oriented) edges. In what follows, we always consider oriented edges, i.e., a two-dimensional subspace corresponds to each (nonoriented) edge. For each two points $x, y$ of the tree, the distance $d(x, y)$ is equal to the length of the unique way connecting these two points.

Definition 1 Let $T$ be a graph with finite number of loops and branches (tails), $V$ the set of its vertices, and $\tilde{L}$ the set of (oriented) edges. The graph $T$ is a coset of its universal covering tree, $D$, over the action of freely acting subgroup $\Gamma$ of the group of motion of the tree, $T = D/\Gamma$. We denote by $\bar{e}$ and $\bar{e}$ two edges from $\tilde{L}$ with the opposite orientations. The modulus $|\cdot|$ of a set is the cardinality (perhaps, infinite) of this set.

Definition 2 Let an oriented path $P_{x,y}$ in a graph $T$ (or $D$) be a (unique) sequence (finite or infinite) $(\bar{e}_1, \ldots, \bar{e}_m)$ of consecutive $(d_0\bar{e}_1 = d_1\bar{e}_{i-1}, 1 < i \leq m$, where $d_0$ and $d_1$ are the respective operators that project an oriented edge to the vertex it starts or
terminates) oriented edges without backtracking (i.e., \( \vec{e}_i \neq \vec{e}_{i-1} \) for \( 1 < i \leq m \)) starting at the vertex \( x = d_0 \vec{e}_1 \) and terminating at the vertex \( y = d_1 \vec{e}_m \). The path length \( |P_{x,y}| \) is the number of edges entering the path. In the tree \( D \), the length of the path \( P_{x,y} \) is always the distance between the vertices \( x \) and \( y \).

The path \( P_{x,y} \) is closed if \( x = y \). The proper closed path is a closed path with \( \vec{e}_1 \neq \vec{e}_m \) (for a path of nonunit length).

Remark 1 The set of proper closed paths with the marked (starting=terminating) points removed (by using the forgetting mapping) is in a one-to-one correspondence with the conjugate classes of the homotopy group \( \pi_1(T) \).

1.2 \( \Gamma \)-action on trees \( D \)

We recall some facts about the construction of the action of finitely generated freely acting group \( \Gamma \) on the tree \( D \).

The tree graph \( D \) possesses a rich group of isometries—the transformations preserving the distance on the graph. Considering a “rigid” graph, i.e., a graph with the fixed ordering of edges in each vertex, and imposing the condition of the ordering preservation under the action of the symmetry group, we reduce this huge group to the group of rotations and translations along the axes (infinite lines) of the tree. (It is possible to interpret the boundary of the tree \( D \) as a \( p \)-adic projective plane \( P_1(\mathbb{Q}_p) \).) Then the full group of motions of the tree \( X \) is the projective group \( PGL(2, \mathbb{Q}_p) \). A rotation element (an analogue of an elliptic element of the projective group) is a tree rotation about a vertex \( x_0 \in D \) through the angle \( 2\pi n/(p+1) \), \( 1 \leq n \leq p \). This is not however the case we consider in this paper. Another type of transformations is provided by elements that have no fixed points inside the tree (free-acting elements).

Without losing the generality, we assume that the group \( \Gamma \) is a discrete free-acting finitely generating subgroup of \( PGL(2, \mathbb{Q}_p) \), that is, no fixed points inside the tree exist for all nonunit elements of this group. (Thus, it is an analogue of a Fuchsian group in ordinary hyperbolic geometry.) We consider in what follows only finitely generating groups \( \Gamma \) and assume that the group has \( g \) generating elements. Each element \( \gamma \in \Gamma \), \( \gamma \neq 1 \), induces a translation of the tree as a whole along the invariant axis \( D(\gamma) \subset D \) of the element. The invariant axis \( D(\gamma) \) is a unique infinite oriented path \( \ldots \vec{e}_{i-1} \vec{e}_0 \vec{e}_1 \vec{e}_2 \ldots \) that maps to itself under the action of \( \gamma \); \( \gamma(\vec{e}_i) = \vec{e}_{i+l} \) shifting in the positive direction w.r.t. its orientation. Here \( l \equiv l(\gamma) \) is the element length, \( l(\gamma) = \inf \{d(x, \gamma(x)) | x \in V(D)\} \), where obviously the minimum is reached on the set \( x \in D(\gamma) \). Explicitly, an element \( \gamma \) defines a translation on the distance \( l(\gamma) \) along the line \( D(\gamma) \). (We assume \( l(\gamma) < \infty \).

\footnotetext{Note that no analogue of parabolic element exists in this geometrical setting.}
A centralizer \(Z(\gamma) \subset \Gamma\) of the element \(\gamma \in \Gamma\) is a cyclic group. If, besides, \(\gamma\) is a generator of \(Z(\gamma)\), then \(\gamma\) is called a \textit{primitive element} of \(\Gamma\). It is useful to define a von Mangoldt function \(\Lambda(\gamma) = l(\varpi)\), where \(\gamma = \varpi^m\) and \(\varpi\) is a primitive element of \(\Gamma\).

Let us consider now the subtree \(D(\Gamma)\), which is the union of all invariant axes of the elements of \(\Gamma\). For an element \(\gamma\) having the length \(l(\gamma)\), the sequence \(\ldots \vec{e}_{-1} \vec{e}_0 \vec{e}_1 \vec{e}_2 \ldots\) has the structure \(\ldots (\vec{e}_1 \vec{e}_2 \ldots \vec{e}_l)(\vec{e}_1' \vec{e}_2' \ldots \vec{e}_l')\ldots\), where \(\vec{e}_i^{(m)}\) are copies of the edge \(\vec{e}_i \in T\). Moreover, the sequence \(\vec{e}_1 \vec{e}_2 \ldots \vec{e}_l\) must be the proper closed path in \(T\) (with possible repetitions, \(i.e.,\) some of \(\vec{e}_i\) may have the same preimages in \(T\)).

Let us consider the set of conjugate classes \(\{\gamma\}\) of the group \(\Gamma\):

\[
\{\gamma\} : \left\{ \bigcup_{\gamma, \omega \in \Gamma} \omega \gamma \omega^{-1} \in \Gamma \right\}. \quad (1.1)
\]

For each element \(\beta \in \{\gamma\}\) and each vertex \(x \in V(D)\), we have \(\beta x = \omega \gamma \omega^{-1} x = y\); hence, \((\omega^{-1} y) = \gamma (\omega^{-1} x)\). Therefore, for each element \(\beta \in \{\gamma\}\), there exists such \(\omega \in \Gamma\) that the invariant axis of \(\beta\) is an \textit{image} of the invariant axis of the generating element \(\gamma\) under the action of \(\omega^{-1}\): \(D(\beta) = \omega^{-1} D(\gamma)\). Thus, there is a one-to-one correspondence between the conjugate classes \(\{\gamma\}\) and a set of all proper closed paths in the graph \(T\).

We call \textit{primitive} conjugate classes \(\{\varpi\}\) such classes \(\{\gamma\}\) that are generated by the primitive elements of \(\Gamma\). Then, the proper closed path corresponding to \(\{\varpi\}\) is the proper closed path in \(T\) that has no subperiods.

All graphs we consider are obtained by factoring a tree \(D\) over the action of a free-acting symmetry group.

### 1.3 Reduced graphs and branches.

**Definition 3** Let the \textit{reduced graph} \(F\) be a finite, necessarily connected subgraph \(F \subset T\) containing all loops of the graph \(T\). Its universal covering, \(D_F\), is a subtree in \(D\).

For the generality sake, we do not demand the reduced graph to be inambiguously determined by the graph \(T\). However, we always can segregate the \textit{minimum} reduced graph, which is the intersection of all reduced graphs admitted by the given graph \(T\). This minimum reduced graph exactly coincides with the union of all \(D(\gamma), \gamma \in \Gamma\), factorized over the action of the group \(\Gamma\). This union is a subtree \(D(\Gamma) \subseteq D\), but neither \(D(\Gamma)\) nor \(F\) must be uniform graphs.

The subgraph \(D(\Gamma)/\Gamma\) and, correspondingly, a reduced graph \(T\) are finite graphs containing exactly \(g\) loops, where \(g\) (the genus) is the number of generating elements of the group \(\Gamma\).

Because a subtree \(D(\Gamma)\) does not coincide in general with the whole tree \(D\), a quotient \(\Gamma \setminus X\) would be an infinite graph in contrast to graphs in paper [20].
The graph $T$ can be presented in the form $T = F \cup B(T)$, where all $g$ loops of the graph $T$ are contained in the reduced graph $F$. Meanwhile, the minimum reduced graph contains no terminal points, i.e., such points that are incident to only one edge in $F$. The complement to $F$, $B(T)$, is a (finite or empty) set of branches growing from the vertices of $F$ in a way to make the total valence of a vertex of $T$ to be $p + 1$. We always assume that the number of branches growing from a vertex of the reduced graph is $p - q$, where $q + 1$ is exactly the incident number of the vertex w.r.t. the edges of the reduced graph. An example of such factorized tree $T$ for $p = 3$ and $g = 1$ is presented in Fig. 1.

**Remark 2** Note that a set of proper closed paths of the graph $T$ has the natural analogue in the continuous projective geometry case—it is a set of closed geodesics on the (open) Riemann surfaces.

For each point $y \in T$, we define its *distance to the reduced graph* $d(y, F)$ by the formula $\inf_{x \in F} d(x, y)$, which we sometimes abridge to $d(y)$. This minimum is between the point $y$ and a unique point $x \in F$, which we call an *image* $t(y) \in F$ of the point $y$. Each branch $B$ can be therefore naturally projected into its summit (in other terms, root, image) $t(y) \in F$.

![Fig. 1. An example of the factorized tree $T$ for $g = 1$, $p = 3$.](image)

Note that the terminology of [26], where the scattering theory on graphs has been explored from the Schrödinger potential standpoint, differs slightly from our notation. The reduced graph is the *basis* graph in Novikov’s notation. Moreover, instead of infinite branches growing from vertices of $F$, half-infinite tails growing from the corresponding points (nests) have been considered. However, this setting is very close to the one used here because we consider (automorphic) functions that depend only on distance to the reduced graph in what follows; in this respect, points of tails just label these distances and the both approaches are equivalent in this respect.

2 Functions, operators, and potentials on graphs

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2An important difference is due to different types of the potential operators considered.
2.1 Automorphic functions and potentials on $T$.

In the original setting by Bass [2], a unitary representation $\rho : \pi_1(T) \to GL(v_{\rho})$ of the group $\Gamma$ was fixed and a linear space $v_{\rho}$ was associated with any vertex (and, in principle, with any oriented edge) of the tree $D$. One can consider the space $L(\chi, D, v_{\rho})$ of automorphic functions on the tree $D$. Those are functions $G(x)$ defined on $V(D)$ (or $\bar{L}(D)$), taking values in $V_{\rho}$, and satisfying the condition $G(\gamma x) = \chi(\gamma) G(x)$ for all $\gamma \in \Gamma$ and $x \in D$. Here $\chi(\gamma)$ is a left-invariant character of the group $\Gamma$. If $Y \subset D$ is a fundamental domain of $\Gamma$ in $D$ then each $G \in L(\chi, D, v_{\rho})$ is completely defined by its values on $Y$.

However, we adopt an equivalent description, which is closer to physically meaningful lattice gauge theories. Namely, instead of considering automorphic functions, we consider periodic functions on $D(F)$, which can be lifted to the graph $T$ (the factorized tree), while nontrivial character set of the group $\Gamma$ is ensured by introducing (unitary) potentials on the edges of the graph $T$.

Let $\mathcal{A}$ be a group (or an algebra) and $v_{\rho}$ the finite-dimensional representation of $\mathcal{A}$.

In [3], the case of the trivial representation $\chi(\gamma) \equiv 1$ was considered; nevertheless, already this simplest example manifests all the main features of the general theory. In the present paper, we demonstrate how the analogous method can be applied to the case of arbitrary unitary (in general, non-Abelian) representation $(v_{\rho}, \chi)$.

Definition 4 Given the group $\mathcal{A}$ and its finite-dimensional representation $v_{\rho}$, we associate the representation space $v$ of the group $\mathcal{A}$ to each vertex and to each oriented edge of the graph $T$. Then, the corresponding linear spaces of functions defined on vertices or edges of a graph and taking values in $v_{\rho}$ are $C_0 = v_{\rho}^{|V|}$ and $C_1 = v_{\rho}^{|\bar{E}|}$ (the power is understood as the tensor product power).

In what follows, we interpret tilde quantities as pertaining to the reduced graph or to its universal covering. For example, the linear spaces $\bar{C}_0$ and $\bar{C}_1$ correspond to vertices and edges of the reduced graph $\bar{F}$. Note that the graph is always assumed to be closed as the simplicial complex, i.e., every edge enters this graph together with its both endpoints (vertices).

We define also the natural bilinear forms on $C_0$ and $C_1$: for $f, g \in C_0$, 

$$\langle f, g \rangle = \sum_{x \in V} f(x)^* g(x), \quad f(x), g(x) \in v_{\rho}, \quad (2.1)$$

and analogously for $f, g \in C_1$.

Definition 5 1. First type of potentials are the unitary potential on edges, $U_{\vec{e}} \in \mathcal{A}$, where $\mathcal{A}$ is a (non-Abelian) group, the unitarity condition implies that $U_{\vec{e}} = U_{\vec{e}}^{-1}$. Those are the potentials one encounters in the Bass case [3].
2. Second type of potential are Hermitian potentials $A_e$ on edges lying in an algebra $\mathcal{A}$ such that $A_e = A_e^* = A_e^+$ (the reality condition) \[^{26}\].

3. Third, we call the nontrivial torsion potentials the potentials $U_{\bar{\mu}\bar{\nu}}$ determined for the pairs of consecutive edges ($\bar{\mu}$ precedes $\bar{\nu}$) such that

$$U_{\bar{\mu}\bar{\nu}}^{-1} U_{\bar{\nu}\bar{\mu}} = I_1$$

(2.2)

Another natural demand, which is often imposed on the set of torsion matrices $U_{\bar{\mu}\bar{\nu}}$ is as follows. Let $\mu_i$, $i = 1, \ldots, n$, be edges coming (in the fixed cyclic ordering) into a vertex of the graph. Then, for all $1 \leq i, j, k \leq n$,

$$U_{\bar{\mu}_i \bar{\mu}_k} U_{\bar{\mu}_k \bar{\mu}_j} = U_{\bar{\mu}_i \bar{\mu}_j}$$

(2.3)

these relations must hold for any vertex and they imply, in particular, that $U_{\bar{e}_i \bar{e}_i} = I_1$ for any edge $\bar{e} \in \bar{L}(D)$.

### 2.2 Hecke operators on graphs

In this section, $C_0$ and $C_1$ are the corresponding spaces of functions for arbitrary graph.

We consider first the operators that act inside linear spaces $C_0$ and $C_1$. All these operators are assumed to act on the corresponding universal covering $D$ (or $D_F$) with the natural identification of (oriented) edges and vertices of $D$ with their preimages in $T$ (or in $F$). We denote by $I_0$ and $I_1$ the identity operators in the respective spaces $C_0$ and $C_1$.

1. We introduce the set of the Hecke operators $M_n : C_0 \to C_0$, $n = 1, \ldots, \infty$, such that their action on basis vectors is

$$M_n v_x = \sum_{y, |P_{x,y}| = n} U_{\bar{\mu}_1} \cdots U_{\bar{\mu}_n} v_y,$$

(2.4)

where the product runs over all oriented edges entering the path $P_{x,y}$ and $x, y \in D$.

For the tilde operators pertaining to simplicial complexes associated with the reduced graph $F$, the corresponding to (2.4) definition is

$$\widetilde{M}_n v_x \equiv \sum_{y, |P_{x,y}^{|F}| = n, P_{x,y} \subset D_F} U_{\bar{\mu}_1} \cdots U_{\bar{\mu}_n} v_y.$$

(2.5)

The action of operators (2.4) and (2.7) (as well as of all other operators) must be continued to the whole space $C_0$ by the linearity property.

2. Next, we have the valency-counting operator $Q : C_0 \to C_0$,

$$Q v_x = (\#\text{neighbors} - 1) v_x.$$

(2.6)

For the uniform tree, the operator $Q$ is merely the identity times $p$. However, for a nonuniform tree, say, for $D_F$, the operator $Q$ possesses some nontrivial properties.
3. The $\Delta$-operators $\Delta(u) : C_0 \to C_0$ and $\hat{\Delta}(u) : \tilde{C}_0 \to \tilde{C}_0$,

$$\Delta(u) = I_0 - uM_1 + u^2Q, \quad \hat{\Delta}(u) = I_0 - u\tilde{M}_1 + u^2Q, \quad u \in C,$$  \hspace{1cm} (2.7)

which have inverse operators (for $|u| < 1/p$ in the trivial character case), play an important role in the Bass construction. We sometimes label their representation dependence writing them as, say, $\Delta_\rho(u)$.

Operators $M_n$ \textcolor{red}{(2.4)} constitute a basis in the center of endomorphism algebra. Their multiplication algebra in the tree $D$ is

$$M_1M_n = M_{n+1} + pM_{n-1}, \quad n \geq 2$$

$$M_1M_1 = M_2 + (p+1)M_0, \quad M_0 \equiv I_0.$$  \hspace{1cm} (2.8)

If $\psi$ is an eigenvector of $M_1$ with the eigenvalue $t$,

$$M_1\psi = t\psi.$$  \hspace{1cm} (2.9)

then it is also an eigenvector of all $M_n$ and

$$M_n\psi = S_n(p,t)\psi$$  \hspace{1cm} (2.10)

where $S_n(p,t)$ is a system of orthogonal polynomials in $t$ with a generating relation $tS_n(p,t) = S_{n+1}(p,t) + pS_{n-1}(p,t)$, $S_1 = t$, $S_2 = t^2 + p + 1$.

In the general case of a nonuniform tree $D_F$, the operator $Q$ obviously does not commute with the Hecke operators, which in turn also become mutually noncommutative. Nevertheless, it is possible to write algebraic relations analogous to (2.8) using the operator $Q$:

$$\tilde{M}_1\tilde{M}_1 = \tilde{M}_2 + (Q+1)\tilde{M}_0, \quad \tilde{M}_0 = \text{id}.$$  \hspace{1cm} (2.11)

Moreover,

$$\sum_{n=1}^{\infty} u^n \tilde{M}_n \hat{\Delta}(u) = (1 - u^2)I_0.$$  \hspace{1cm} (2.12)

Relation (2.12) implies that the operator $\hat{\Delta}(u)$ is the \textit{generating function} for the operators $\tilde{M}_n$.

4. The \textit{Laplacian} of a unitary theory is merely $\Delta(1)$. For this Laplacian to be a Hermitian operator w.r.t. brackets (2.1) we must impose the unitarity condition on the potential,

$$U^+_{\mu} = U^-_{\mu}.$$  \hspace{1cm} (2.13)

In \textcolor{red}{[26]}, the Laplacian was determined as the operator of the Schrödinger type, \textit{i.e.}, it has form (2.7) with the first Hecke operator $M_1$ as in (2.4) but with the \textit{Hermitian
potential $A_{\mu} \equiv A_{\mu^{-}}$ (elements of an algebra) substituted for $U_{\mu}$ (elements of a group) on the edges of a graph. The corresponding Laplacian

$$\Delta v_x = \sum_{y:|P_{x,y}|=1} (A_{\mu_{xy}}v_y - v_x) \quad (2.14)$$

is also Hermitian, but no relations of type (2.11) exist.

5. On the space $C_1$ we first define the inversion map $J : C_1 \to C_1$, which merely changes all orientations of edges,

$$J \vec{e} = U_{\vec{e}} \vec{e}, \quad (2.15)$$

Moreover, there exist the set of Hecke operators for the space $C_1$. They all are generated by a single operator $T \equiv T_1 : C_1 \to C_1$.

**Definition 6** The first Hecke operator acting on the space $C_1$ is

$$T \vec{e} = \sum_{\vec{e}_\mu \text{ following } \vec{e}} A_{\vec{e}} \vec{e}_\mu. \quad (2.16)$$

Then, we can obviously define $T_n$ as the sum over all terminal edges of oriented reduced paths of the length $n + 1$ starting from the given edge,

$$T^n(\vec{e}_0) = \sum_{(\vec{e}_0, \vec{e}_1, ..., \vec{e}_m)_{\text{red}}} \vec{e}_m. \quad (2.17)$$

However, in contrast to the space $C_0$, the relation between $T_n$ and $T_1$ is merely $T_n = T_1^n$ for any graph, and the family of these operators is commutative for every tree, no matter uniform or nonuniform. Then, for any graph, the corresponding generating function is $I_1 - uT_1$:

$$\sum_{n=0}^{\infty} u^n T_n (I_1 - uT_1) = I_1, \quad u \in \mathbb{C}. \quad (2.18)$$

**Definition 6** can be easily generalized to other potentials from Definition 5. For $U_{\vec{e}}$ replaced by the Hermitian matrices $A_{\vec{e}}$, we just obtain the Novikov potential on edges. Important examples are provided by potentials of the third type (with the nontrivial torsion). Then, in the most general case, we can define $T$ as

$$T \vec{e} = \sum_{\vec{e}_\mu \text{ following } \vec{e}} U_{\vec{e}} F_{\vec{e}\vec{e}_\mu} \vec{e}_\mu, \quad (2.19)$$

where neither $U_{\vec{e}}$ nor $F_{\vec{e}\vec{e}_\mu}$ must satisfy the unitarity or Hermiticity conditions; in formula (2.19), the potentials on edges with opposite orientations can be set arbitrary, the $L$-function expression through the operator $T$, which are presented in the following section, remain nevertheless valid.

We note two important examples of (2.13). The graph description of the Teichmüller spaces of complex Riemann surfaces discussed in Sec. 6; the other is the Ising model on a lattice governed by the Kac–Ward operator determinant.
2.3 Intertwining operators between $C_0$ and $C_1$.

Following [2], we introduce a set of operators acting between two spaces $C_0 \leftrightarrow C_1$.

1. First, we have two boundary mappings $\partial_1, \partial_0 : C_1 \rightarrow C_0$,

$$\partial_1 \vec{e} = A_\vec{e} x_1$$

and

$$\partial_0 \vec{e} = x_0,$$

2. We also have the coboundary operator $\sigma_0 : C_0 \rightarrow C_1$,

$$\sigma_0 x = \sum_{|P_{x,y}|=1} \vec{e}_{(x,y)},$$

which set into the correspondence with the point $x$ a linear combination of all edges outgoing from this vertex. (In [2], the operator $\sigma_1$, which sets into the correspondence to the vertex $x$ all incoming edges was defined. However, this definition, which is meaningful for a nonpotential case, becomes ill defined when a potential is introduced. Fortunately, in what follows, we do not need the operator $\sigma_1$.)

2.4 Relations between operators

Let us introduce the total (half-)dimensions of the spaces $C_0$ and $C_1$,

$$r_0 = \text{rank} (C_0), \quad r_1 = \text{rank} (C_1)/2.$$ (2.23)

We also introduce auxiliary operators

$$\partial(u) = \partial_0 u - \partial_1; \quad \sigma(u) = \sigma_0 u.$$ (2.24)

The standard boundary operator is then $\partial \equiv \partial_0 - \partial_1 = \partial(1)$.

Direct calculations show that the above operators satisfy the following relations [2], which hold for any graph:

$$\partial_0 \sigma_0 = Q + 1;$$
$$\partial_1 \sigma_0 = M_1;$$
$$T = \sigma_0 \partial_1 - J;$$
$$\partial(u) \sigma(u) = \Delta(u) - (1 - u^2) I_0;$$
$$\sigma(u) \partial(u) = u(T + J)(uJ - I_1).$$ (2.25)

An important assertion proved by Bass establish a connection between the spectral Laplacian problem and the spectral problem for the Hecke operator $T_1$. 

15
Lemma 1 \[2\] For the potentials of the first type in Definition \[3\], we obtain
\[
\det(C_1 - uT_1) = (1 - u^2)^{r_0 - r_1} \det \Delta(u).
\]

**Proof.** Let us define the matrices acting in $C_0 \oplus C_1$,
\[
L = \begin{bmatrix}
(1 - u^2)I_0 & \partial(u) \\
0 & I_1
\end{bmatrix}, \quad M = \begin{bmatrix}
I_0 & -\partial(u) \\
\sigma(u) & (1 - u^2)I_1
\end{bmatrix}.
\]
Then from \[2.23\], we obtain
\[
LM = \begin{bmatrix}
\Delta(u) & 0 \\
\sigma(u) & (1 - u^2)I_1
\end{bmatrix} \text{ and } ML = \begin{bmatrix}
(1 - u^2)I_0 & 0 \\
\sigma(u)(1 - u^2) & (I_1 - uT)(I_1 - uJ)
\end{bmatrix}.
\]
The assertion of Lemma \[1\] follows from equating the traces of $LM$ and $ML$; we only need to evaluate the determinant of $I_1 - uJ$. It is block–diagonal in the basis of (nonoriented) edges. Each edge admits two orientations, so we have
\[
\det(I_1 - uJ) = \det \left[ \begin{array}{cc}
I_1 & -uU \\
-uU^{-1} & I_1
\end{array} \right] = (1 - u^2)^{r_1}
\]
for any matrix $U$. Note that $r_1$ is exactly the dimension of the representation of the group $U$ times $|E|$, where $|E|$ is the number of unoriented edges. Therefore, $r_0 - r_1 = \dim U \times (|V| - |E|)$ where the difference between the numbers of vertices and (nonoriented) edges is exactly $1 - g$—the Euler characteristic of the graph.

In general, no relation of type of Lemma \[1\] exists for potentials of the second and third type from Definition \[5\]. Therefore, the spectral problems for the operators $\Delta(u)$ and $T(u)$ are not related in these cases. However, for graphs of the third kind from Definition \[5\] with conditions \[2.2\] and \[2.3\] imposed, we can, nevertheless, connect these two operators for the price of “blowing up” the vertices of a graph; this construction is discussed in Sec. 6.

## 3 \ L-function

**Definition 7** \[3\] Let $v_\rho$ be a $C^k$-module with the character $\chi_\rho : T(C[T]) \to C^k$. Then, the Ihara–Selberg $L$-function $L(\rho, u)$ is

\[
L(\rho, u) = \prod_{\{w\} \in \Gamma} \det(I_v - u^{l(\varpi)}\rho(\varpi))^{-1},
\]

where the product ranges all conjugate classes of primitive elements of the group $\Gamma$, or, equivalently, all proper oriented closed paths without periods in the graph $T$, $l_\varpi$ are lengths of these paths, and $\rho(\varpi) = U_{\varpi_1} \ldots U_{\varpi_\ell}$ is the product over the path that corresponds to the element $\varpi \in \Gamma$ of elements of the group $\mathcal{A}$.

We present now the Bass’ formulation and proof \[2\] of the Hashimoto’s theorem \[18\].
Theorem 1 Let $\rho : \Gamma \to GL(V)$ be a finite-dimensional complex representation of $\Gamma$. Then, the $L$-function (3.1) is a rational function in the variable $u$:

$$ L(\rho, u)^{-1} = \det(I_1 - uT_1), \quad (3.2) $$

where $T_1$ is from (2.10).

From Lemma [4] and Theorem [4] the corollary follows

Corollary 1 If the unitarity condition (2.13) is satisfied, then the $L$-function can be expressed through the determinant of the operator $\Delta_\rho(u)$:

$$ L(\rho, u)^{-1} = (1 - u^2)^{r_0 - r_1} \det(\Delta_\rho(u)). \quad (3.3) $$

The product in (3.1) converges absolutely in the domain \{$u : |u| < (q_{\text{max}}|A|)^{-1}$\}, where $q_{\text{max}} \leq p$ is the maximum incident number of the tree $D_F$ and $|A|$ is an absolute value of a maximum element $U_{ij}^\mu$. Also, we have for the logarithmic derivative of $L(\rho, u)$,

$$ \frac{d}{du} \log(L(\rho, u)) = \sum_{\{\gamma\}} \Lambda(\gamma) \operatorname{tr}(\rho(\gamma))u^{l(\gamma)-1}, \quad (3.4) $$

where the sum ranges all conjugate classes \{$\gamma$\} distinct from \{1\}.

Proof of Theorem 1.

Let us choose in the tree $D$ some fundamental domain $D(\Gamma)$ of the symmetry group $\Gamma$. We now consider the contribution to the trace of the operator $T_m$ coming from some edge $\vec{e}_1 \in D(\Gamma)$. Only those $\vec{e}_i \in D$ contribute to $\operatorname{tr}T_m$ that

1. lie on the distance $m$ to the initial edge $\vec{e}_1$ (along some reduced path $\vec{e}_1\vec{e}_2\ldots\vec{e}_{m+1}$);  
2. have the orientation along this path.

The crucial observation follows. Let us consider the action of some element $\gamma \in \Gamma$ on oriented edges of $D$ (Fig. 2). It is easy to see that if the edge does not belong to the invariant axis $D(\gamma)$ of this element (like $\vec{e}_x$ does not on Fig. 2), then the orientation of its image under the action of the element $\gamma$, $\gamma\vec{e}_x$, is opposite to the orientation of the reduced path coming from $\vec{e}_x$ to $\gamma\vec{e}_x$. Therefore, no such edges contribute to the trace! Only the edges that (like $\vec{e}_1$ in Fig. 2) belong to the invariant axis $D(\gamma)$ preserve their orientations toward $D(\gamma)$ under the action of $\gamma$ (the translation along $D(\gamma)$). Say, for $\vec{e}_1$, we have $\vec{e}_{l+1} = \gamma\vec{e}_1$ in Fig. 2.
Fig. 2. The action of the group element $\gamma \in \Gamma$ on the edges of $D$, with the length $l(\gamma) = l$. Note that $\gamma \vec{e}_x \neq T^{m+1} \vec{e}_1$ for all $m$.

Note. This is why we use $\text{tr} T^m$ instead of, say, $M_m$—the $m$th Hecke operator for the space $C_0$—for $\text{tr} M_m$, all pairs of vertices $x, y \in D$, not only those belonging to the invariant axis $D(\gamma)$, contribute to this trace as soon as $d(x, y) = m$ and $y = \gamma x$.

Coming back to the space $C_1$, we fix for a moment the edge $\vec{e}_1 \in D(\Gamma) \subseteq D$. As was mentioned above, for each primitive conjugate class $\varpi$ one can find not necessarily one axis $D(\varpi) \subset D_F$ such that $\vec{e}_1 \in D(\varpi)$. We denote by $\vec{a}_i$ the oriented edges of the reduced graph $F$ itself, thus each $\vec{e}_i \in D_F$ is an image of the edge $\vec{a}_i \in F$ with the orientation naturally preserved. For each element $\gamma \in \Gamma$, we then set into the correspondence the periodic sequence of “letters” (the cyclic word)

$$D(\gamma) \simeq \ldots (\vec{a}_1 \vec{a}_2 \ldots \vec{a}_l)(\vec{a}_1 \vec{a}_2 \ldots \vec{a}_l) \ldots , \quad (3.5)$$

where $l \equiv \Lambda(\gamma) = l(\varpi)$ is the length of the generating element for $\gamma$.

Note that the elements of $\Gamma$ determine the symbolic dynamics of “words” (Lyndon words, see [13]) composed from the “letters” $\vec{a}_i$, which can be noncommutative elements of the group $A$. We even do not need a unitarity condition (2.13) in this case; in particular, Theorem 1 holds even if letters $\vec{a}_i$ and $\vec{a}_i$ are not related. Thus, for the rest of the proof, we merely identify the letters $\vec{a}_i$ with the elements $U_{E_i} \vec{e}_i$ and the proof is valid for any (non-Abelian) group or algebra elements dwelling on the oriented edges of the graph $F$.

It is clear that if $l(\gamma) = m$, then $l(\varpi) = l, l|m$, that is, $l$ is a divisor of $m$. Note again that there can be repeated $\vec{a}_i$ in the cyclic word $\vec{a}_1 \vec{a}_2 \ldots \vec{a}_l (\vec{a}_{l+1} \equiv \vec{a}_1)$, Moreover, this sequence may contain subperiods if $\gamma$ is not a primitive element, but, obviously, the minimum length of this subperiod, $l$, is a divisor, $l|m$.

We now fix the element $\vec{a}_1$—the preimage of $\vec{e}_1$. In order to determine its contribution to $\text{tr} T^m$ we must find all possible different cyclic expressions

$$\vec{a}_1 \vec{a}_2 \ldots \vec{a}_m (\vec{a}_{m+1} = \vec{a}_1) \quad (3.6)$$

that include this element. We now establish a correspondence between sets (3.3) and (3.6).
1. To each finite cyclic sequence $\vec{a}_1 \ldots \vec{a}_m(\vec{a}_1)$ we inambiguously set into the correspondence the infinite periodic sequence $\ldots (\vec{a}_1 \ldots \vec{a}_m)(\vec{a}_1 \ldots \vec{a}_m) \ldots$ corresponding to a unique element of primitive conjugate class $\{\varpi\}$ with $l(\varpi) = l|m$.

2. On the contrary, we now choose an element from $\{\varpi\}$, or, equivalently, from some periodic reduced sequence

$$\ldots (\ldots)(\vec{a}_{i_1} \vec{a}_{i_2} \ldots \vec{a}_{i_l})(\vec{a}_{i_1} \ldots) \ldots$$

(3.7)

with no subperiods and with the minimal period $l = l(\varpi)$. If it contains the (oriented) edge $\vec{a}_1$ $d_1$ times among the edges $\{\vec{a}_{i_1}, \ldots, \vec{a}_{i_l}\}$ and, moreover, $l$ is the divisor of $m$, then there are exactly $d_1$ different sequences $\vec{a}_1 \ldots \vec{a}_m(\vec{a}_1)$ (3.6) containing in (3.7). Eventually, doing the sum over all edges of $F$ (or of the fundamental domain $D(\Gamma) \in D$), i.e., evaluating the trace of the operator $T^m$, we find that the contribution from the sequence $\ldots(\ldots)(\vec{a}_{i_1} \ldots \vec{a}_{i_l})\ldots$ to this trace is

$$\sum_{j=1}^{[\tilde{L}_F]} \# \{\vec{a}_j \text{ in } \{\vec{a}_{i_1}, \ldots, \vec{a}_{i_l}\}\} \equiv l, \quad (3.8)$$

where $l$ is the total length of the primitive element $\varpi$. Therefore, we have the following remarkable formula:

$$\text{tr} T^m = \sum_{l|m} l \cdot \sum_{\{\varpi : l(\varpi) = l\}} \text{tr} [\rho(\varpi)^{m/l}], \quad (3.9)$$

where the sum runs over all primitive conjugate classes of $\Gamma$.

We now obtain from (3.4) the representation for the $L$-function:

$$u \frac{d}{du} \log(L(\rho, u)) = \sum_{\{\varpi\}} \sum_{n=1}^{\infty} l(\varpi) \cdot u^{n(l(\varpi))} \text{tr} [\rho(\varpi)^n]$$

$$= \text{from } (3.9) \sum_{k=1}^{\infty} \text{tr} T^k \cdot u^k = -u \frac{d}{du} \log \det(1 - u \cdot T). \quad (3.10)$$

This completes the proof of the theorem.

### 4 Spectral problem on infinite graphs

We set the spectral problem considering a function $\psi \in C_0$ such that

$$L \psi = \lambda \psi, \quad \lambda \in \mathbb{C}, \quad (4.1)$$

where the Laplacian $L$ is

$$L = M_1 - (Q + 1). \quad (4.2)$$

We first determine the solution of (4.1) on a branch (tail). We assume the graph $T$ to be uniform ($(p + 1)$-valent) but infinite. Then, we consider a branch (tail) of the graph $T$
with the starting vertex \( x_0 \) (the index \( i \) is the distance to \( x_0 \) in the reasoning below). We have \( \psi(x_i^{(s)}) = v_i^{(s)} \); then,

\[
\sum_{s=1}^{p} v_{i+1}^{(s)} + v_{i-1}^{(s)} - (p + 1)v_i = \lambda v_i, \quad i > 0, \tag{4.3}
\]

where \( x_{i-1}^{(s)} \) is a unique vertex (from the \((i - 1)\)th layer) that precedes \( x_i \) (when moving along the branch from its root).

The general solution to (4.3) is

\[
v_i^{(s)} = \sum_{\pm} \alpha_i^{(s)} (A_{i-1}^{(s_i)})^{-1} \cdots (A_1^{(s_1)})^{-1} v_0^{0}, \tag{4.4}
\]

where the product ranges all edges of the path \( P_{x_0, x_i}, v_0^0 \) and \( v_0^{-} \) are constant vectors from \( V \), and \( \alpha_{\pm} \) are the solutions of the equation

\[
p\alpha_{\pm}^2 - (p + 1 + \lambda)\alpha_{\pm} + 1 = 0; \quad t \equiv \frac{p + 1 + \lambda}{2p}, \quad \alpha_{\pm} = t \pm \sqrt{t^2 - 1/p}. \tag{4.5}
\]

The vectors

\[
[v_i^{(s)}]_\rho \equiv \alpha_i^{(s)} (U_{i-1}^{(s_i)})^{-1} \cdots (U_1^{(s_1)})^{-1} (v_0^{0})_\rho, \tag{4.6}
\]

where \((v_0^0)_\rho \) is the \( \rho \)th component of the vector \( v_0^0 \), remain collinear for any \( i \) and their ratio \((\alpha_+ / \alpha_-)^i (v_0^0)_\rho / (v_0^0)_\rho \) depends only on the level \( i \). Therefore, it is natural to relate the data \((v_0^0)_\rho \) and \((v_0^0)_\rho \) with the scattering matrix data on the graph \( T \). For this, we call solution (4.4) with only \( v_0^+ \) nonzero the incoming wave and solution (4.4) with only \( v_0^- \) nonzero—the outgoing wave.

Here it becomes clear why we can relate the branch and tail descriptions. First, we can perform a gauge transformation in order to eliminate the dependence on potentials on edges of branches. For this, we merely set

\[
v_i^{(s)} \rightarrow U_1^{(s_1)} U_2^{(s_2)} \cdots U_{i-1}^{(s_{i-1})} v_i^{(s)}.
\]

No dependence on potentials on edges remains then in the transformed variables, and we obtain merely that

\[
[v_i^{\pm}]_\rho = \alpha_i^{(s)} [v_0^{\pm}]_\rho
\]

for any path of length \( i \) starting from the summit of the branch. In what follows, we set \( U_{\mu} = I_v \) on all external (i.e., not entering the minimum reduced subgraph) edges.

We can now identify all vertices and edges that lie at the same distance from the summit setting

\[
v_s^{(i)} \rightarrow p^{(s-1)/2} v_i \text{ for } s > 0 \tag{4.7}
\]

and erasing the now redundant superscript \((i)\) (see Fig. 3). We thus obtain just the picture of [26] where a finite (probably zero) number of tails (half-axes) begin at a vertex of the basis (or, in our notation, reduced) finite graph. We make transition (4.7) in order to preserve the scalar product (2.1). Simultaneously, we must scale and shift the
corresponding eigenvalue, \( \lambda \rightarrow [\lambda + (\sqrt{p} - 1)^2]/\sqrt{p} \); only after this operation, the answers for two branches of solution (4.7) in our approach and in the approach of [26] do coincide.

![Diagram](image)

**Fig. 3.** The branch → tail transition on the external edges of \( T \).

**Definition 8** Let the **boundary points** \( x_i \in \partial F \) are those points of \( F \) that have **nonzero** number of external tails starting at these points. We also consider the natural embeddings of linear spaces \((\partial F)_{0,1} \subset \tilde{C}_{0,1}\) (recall that \((F)_{0,1} \equiv \tilde{C}_{0,1}\)).

### 4.1 Spherical functions on factorized trees

We now introduce spherical functions for the graph \( T \). First, we consider them on the tree \( D \) itself. Choose the vertex \( x \in V(D) \) and claim it the **center** of the tree. Then the **spherical function** \( \psi(n, x) \) is an eigenvector of \( M_1 \), \( M_1 \psi(n, x) = t \psi(n, x) \) that depends only on the distance in the tree from the point \( x \), i.e., it is constant on each sphere \( S(n, x) \equiv \{ y \in V(D) : |P_{x,y}| = n \} \). In the case of trivial potential (trivial representation of the group \( \Gamma \)), we have

\[
\psi(n, x) = a_+ \alpha_+^n - a_- \alpha_-^n, \tag{4.8}
\]

where (cf. (4.7))

\[
\alpha_+ \alpha_- = 1/p,
\]

and setting \( \psi(0, x) = 1 \), we obtain

\[
a_+ = \frac{p\alpha_+ - \alpha_-}{(p + 1)(\alpha_+ - \alpha_-)}, \quad a_- = \frac{p\alpha_- - \alpha_+}{(p + 1)(\alpha_+ - \alpha_-)}. \tag{4.9}
\]

In order to define a similar object for a general graph \( T \) whose “center” now is the reduced graph \( F \), we consider a **superposition** of solutions (4.8) with the **sources** \( s_y \) placed at the vertices of \( D_F \),

\[
\psi(x) = \sum_{y \in D_F} s_y \psi(d(x, y), y), \tag{4.10}
\]

such that the function \( s_y \) is periodic w.r.t. the action of \( \Gamma \): \( s_{\gamma y} = s_y \) for all \( \gamma \in \Gamma \) and \( y \in D_F \). The behavior of this solution on each branch “growing” from \( \partial F \) is described by (4.8) with the preexponential factors depending only on the point of \( \partial F \) into which this branch can be projected.
Let us introduce a kern function $K$:

$$K(z, y|x) = \sum_{\gamma \in \Gamma} x^{d(z, \gamma(y))} \times \prod_{\bar{\mu}_i \in P_{z, \gamma(y)}} U_{\bar{\mu}_i},$$  \hspace{1cm} (4.11)

where $z, y$ are points of $D_F$. This function is periodic under the action of $\Gamma$ over both its arguments separately. Therefore, it is well-defined on $T$ itself. For the trivial representation $\chi \equiv 1$, this function is also symmetric in $z$ and $y$. Function (4.10) then becomes

$$\psi(x) = \sum_{y \in F} s_y[a_+K(t(x), y|\alpha_+)\alpha^{d(x)}_+ - a_-K(t(x), y|\alpha_-)\alpha^{d(x)}_-],$$  \hspace{1cm} (4.12)

where $t(x) \in F$ is the image of the point $x \in D$. (If $x \in F$, then $t(x) = x$.) We call the part of (4.12) proportional to $\alpha^d_+$ the retarded wave function and the part proportional to $\alpha^d_-$ the advanced wave function. Therefore, $\psi(x)$ has a general form

$$\psi(x) = A_{adv}(t(x))\alpha^d_+ - A_{ret}(t(x))\alpha^d_- \equiv \psi_+(x) - \psi_-(x),$$  \hspace{1cm} (4.13)

### Proposition 1

Given a set of vectors $v^0_+(x_i), \ x_i \in \partial F$, the set of $v^0_+(x_i)$ is uniquely determined for all $t \in \mathbb{C}$ except a finite set of points (the points of the discrete spectrum).

The important case of functions (4.13) is where $\psi_+(x)$ is nonzero on only one branch growing from a single point of $\partial F$. Then, we can define the scattering matrix as follows.

### Definition 9

The scattering matrix (S-matrix) $S(t)$ is the square matrix of the size rank $(\partial F)_0$ with the entries $s_{i_\alpha,j_\beta} \ |\{i_\alpha\} = |\{j_\beta\}| = \text{rank} (\partial F)_0$, $s_{i_\alpha,j_\beta}$ is the value of $(v^0_+)_{j_\beta}(x_j)$ for $(v^0_+)_i(x_k) \equiv \delta_{\alpha_\gamma}\delta_{k_i}$ and $(v^0_-)$ pertain to the solution of spectral problem (4.1) on the whole graph, i.e., entries of the S-matrix correspond to solutions of (4.1) of the form

$$\psi^+_i = \sum_{\{j_\beta\}} s_{i_\alpha,j_\beta}\psi^-_{j_\beta}.$$  \hspace{1cm} (4.14)

We postpone the detailed study of the spectral properties of the S-matrix to the next section and formulate here the main theorem connecting S-matrix and determinants of the operators on the reduced graph.

### Theorem 2

In the case of unitary potentials on edges,

$$\det S(t) = \left(\frac{\alpha_+}{\alpha_-}\right)^{\text{rank}(F)_0} \frac{\det(\Delta_\rho(\alpha_-))}{\det(\Delta_\rho(\alpha_+))},$$

where $(\Delta_\rho(\alpha_+))$ is operator (2.7) in which we explicitly indicate the dependence on the representation $\rho$.  

22
Proof. Each solution of spectral problem (4.1) having form (4.4) outside the reduced graph can be split into the advanced and retarded partial wave functions in the total graph (see (4.13)) where

$$\psi_{\pm}(x) = \sum_{x_\alpha \in F} \sum_{P_{x_\alpha,x}} a_{\pm}^{P_{x_\alpha,x}} \prod_{e_i \in P_{x_\alpha,x}} U_{e_i}^{-1} s(x_\alpha).$$

Here $s(x_\alpha) : F_0 \to V$ is the source function and $a_{\pm}$ are from (4.9).

Note that the “splitting” into the advanced and retarded parts or, equivalently, the source function $s(x_\alpha)$ is inambiguously defined everywhere outside $F$ (the splitting is also inambiguously defined at boundary points $x \in \partial F$), but is ambiguous at the inner points of $F$.

In the space of $s(x)$ (in fact, this space is $\tilde{C}_0$), there exists a linear subspace of source functions that generate one and the same eigenfunction $\psi(x)$. For example, if $x_0 \in F$, $x_0 \notin \partial F$, we can propose two source functions $s_1(x) = \{0, x \neq x_0; 1, x = x_0\}$ and $s_2(x) = \{0, d(x, x_0) \neq 1; 2p/t, d(x, x_0) = 1\}$. Then, these two source functions generate the same eigenfunction $\psi(x)$. Moreover, for almost all $\lambda$ and $s(x) \in \tilde{C}_0$, there exists a unique $\bar{s}(x) \in \tilde{C}_0$ such that both $s(x)$ and $\bar{s}(x)$ generate the same $\psi(x)$ and $\bar{s}(x) \equiv 0$ in all inner points of $F$.

Therefore, the problem is to define a “convenient” splitting (4.13) (an analogue of a finite-dimensional “gauge fixing” in field theory).

Let us consider eigenvalues $c_\alpha$ of $S(t)$. Those are numbers for which there exist solutions of (4.1) with the sets $(v_0^+, x_i)$ and $(v_0^-, x_i)$ proportional to each other:

$$(v_+^0)_{\alpha}(x_i) = c_n (v_-^0)_{\alpha}(x_i) \quad \text{for all } \alpha \text{ and } x_i \in \partial F. \quad (4.16)$$

We now expand the condition (4.16) to all (external as well as internal) points of $F$.

**Definition 10** Let the spherical function on the graph $T$ be the solution of spectral problem (4.1) of form (4.15) such that

$$[\psi_+(x)]_{\alpha}/[\psi_-(x)]_{\alpha} = c_n \left( \frac{\alpha_+}{\alpha_-} \right)^{\text{dist}(x,F)}, \quad (4.17)$$

where $c_n$ is the constant independent on the point $x$ and the representation index $\alpha$ and $\text{dist}(x,F)$ is the well-defined distance between the point $x$ and the reduced graph $F$ (for $x \in F$, $\text{dist}(x,F) = 0$).

**Lemma 2**

$$\det S(t) = \prod_n c_n.$$ 

Actually, condition (4.17) fixes the choice of the admitted source functions $s(x_\alpha)$. At the same time, this is a system of linear homogeneous equations, which has nonzero
solution only for a finite set of $c_n$. Below, we present this system in terms of the Hecke operators (2.5) on the graph $F$.

Considering Eq. (4.17) at the points of the reduced graph, we obtain

$$
\sum_{x, \alpha \in F} a_{+}^{|P_{x, \alpha}|} \left( \prod_{\bar{e}_i \in P_{x, \alpha}} U_{\bar{e}_i}^{-1} \right) s(x, \alpha) = c_n \sum_{x, \alpha \in F} a_{-}^{|P_{x, \alpha}|} \left( \prod_{\bar{e}_i \in P_{x, \alpha}} U_{\bar{e}_i}^{-1} \right) s(x, \alpha),
$$

or

$$
\sum_{y \in F} s_y a_{+} K(z, y|\alpha_{+}) = c_n \sum_{y \in F} s_y a_{-} K(z, y|\alpha_{-}). \quad (4.18)
$$

Then, obviously,

$$
\prod c_n = \frac{\det K(x, y|\alpha_{+})}{\det K(x, y|\alpha_{-})} \left( \frac{a_{+}}{a_{-}} \right)^{\text{rank}(F)_{0}}, \quad (4.19)
$$

where $K(x, y|\alpha)$ is the kernel for the linear operator acting on $\bar{C}_0$ (note that because $F$ contains all loops of the graph $T$ and is connected, then, if $x, y \in F$, then any path $P_{x, y} \subseteq F$):

$$(Ks)(x) = \sum_{y \in F} \sum_{P_{x, y}} \alpha^{|P_{x, y}|} \left( \prod_{\bar{e}_i \in P_{x, y}} U_{\bar{e}_i}^{-1} \right) s(y).$$

Considering the universal covering $D_F$, it is easy to verify that

$$\tilde{\Delta}(\alpha) K(x, y|\alpha) = (1 - \alpha^2) I_0,$$

and, therefore,

$$\det K(x, y|\alpha) = (1 - \alpha^2)^{r_0} \det^{-1} \tilde{\Delta}(\alpha).$$

Now, using the identity

$$\frac{a_{+}(1 - \alpha_{+}^2)}{a_{-}(1 - \alpha_{-}^2)} = \frac{\alpha_{+}}{\alpha_{-}}$$

and formula (4.19), we obtain the assertion of Theorem 2.

Theorems 1 and 2 imply the following correspondence between the scattering matrix determinant and the $L$-function of the reduced graph:

$$
\det S(t) = \left( \frac{\alpha_{+}}{\alpha_{-}} \right)^{r_0} \left( \frac{1 - \alpha_{+}^2}{1 - \alpha_{-}^2} \right)^{r_0 - r_1} \frac{L(p, \alpha_{+})}{L(p, \alpha_{-})}, \quad \alpha_{+}\alpha_{-} = 1/p, \quad (4.20)
$$

where $L(p, \alpha)$ depends only on characters of loops of the reduced graph $F$. Another form of (4.20) is

$$
\det S(t) = \left( \frac{\alpha_{+}}{\alpha_{-}} \right)^{r_0} \frac{\det \tilde{\Delta}(\alpha_{-})}{\det \Delta(\alpha_{+})}. \quad (4.21)
$$

Note that only the first “volume factor” in (4.20) and (4.21) depends on the volume of the graph $F$, all nontrivial factors depend only on loop characteristics of $F$.

Since $\alpha_{-}\alpha_{+} = 1/p$, the combination $(\alpha_{\pm})^{r_0} \det \tilde{\Delta}(\alpha_{\pm})$ acquires the form

$$(\alpha_{\pm})^{r_0} \det \tilde{\Delta}(\alpha_{\pm}) = \det(\alpha_{\pm} \tilde{\Delta}(\alpha_{\pm})) = \det \left( \alpha_{\pm} + \frac{\alpha_{\pm}}{p} \hat{Q} - \frac{1}{p} M_1 \right). \quad (4.22)$$
Remark 3 Returning to (4.18), we see that the determinant of the operator $K(z, y|\alpha_\pm)$ is nondegenerate (everywhere except discrete points of spectrum of $\Delta(u)$). It is in contrast to the already mentioned ambiguity in choice of the source function $s(x)$. Let us now act by the operator $\Delta(\alpha_+)$ on both sides of (4.18). For any interior point of $F$,

$$(\alpha_-\Delta(\alpha_+)s)(x) = (\alpha_+\Delta(\alpha_-)s)(x);$$

therefore, we obtain $s(y) = c_is(y)$, i.e., $s(y) \equiv 0$ in all internal points for all $c_i \neq 1$. Hence, using condition (4.18) we gauge out the whole ambiguity due to the choice of $s(y)$ at almost all values of the spectral parameter $t$. (The cases where such a condition fails just correspond to the exceptional points of the discrete spectrum, see below.)

4.2 Structure of spectrum

We now study the spectral properties of problem (4.1). These properties turn out to be quite similar to the spectral properties of Schrödinger type potentials investigated in [26].

We now consider eigenfunction problem (4.1) as the spectral problem.

Definition 11 The continuous spectrum, or the scattering zone of the spectrum, is the domain $t^2 < 1/p$ where $\alpha_+$ and $\alpha_-$ are complex conjugate to each other and their absolute value is exactly $1/\sqrt{p}$; then,

$$\lambda \in (-p - 1 - 2\sqrt{p}, -p - 1 + 2\sqrt{p}).$$

(4.23)

The points of normal discrete spectrum are eigenvalues of problem (4.1) that lie outside the continuous spectrum domain (4.23) and correspond to real solutions that decay exponentially (as $\alpha^n$) with $|\alpha_+| < 1/\sqrt{p}$ and such that these eigenfunctions are nonzero at least in one branch.

The points of the exceptional discrete spectrum may appear for any real $\lambda$. Those are points such that the corresponding eigenfunctions vanish identically outside the reduced graph.

The first two types of points are customary for a majority of spectral problems. The third type is rather specific for graph problems or, more generally, for discrete spaces (its existence in the problem under consideration was observed in [30]).

Note that points of the exceptional discrete spectrum exist not for all graphs; customarily, it is some graph symmetry (e.g., the $Z_2$ symmetry, see [20]) that is responsible for the appearance of such points. Obviously, for such an exceptional solution to exist, it must vanish not only on all branches, but also at all summits of these branches, i.e., at all boundary points $x \in \partial F$ (and, correspondingly, at all boundary points of the minimum reduced graph). This condition is not easy to formalize; however, in the unitary potential case, we can formulate some regular criterion for the existence of exceptional discrete points in spectrum (see Proposition 3).
If $k$ is the total number of branches growing from the points of $\partial F$, then, for any \( \lambda \) from the scattering zone, considering the spherical function case (the functions that are constant on spherical slices of branches) we always have a \( k \)-dimensional subspace of eigenfunctions corresponding to the given eigenvalue \( \lambda \).

Fixing the spectral parameter \( \lambda \), we can introduce the \( 2k \)-dimensional space \( H^{2k} \) of asymptotic states—the asymptotic spherical function-like local solutions of (4.1), which always has two independent solutions proportional to \( \alpha_n^+ \) and \( \alpha_n^- \) in each branch. These functions, \( \psi_i^{\pm}, \ i=1,\ldots, k \), constitute a basis in \( H^{2k} \). Then, the true scattering problem solutions (4.14), i.e., those that can be continued to the whole graph, span a \( k \)-dimensional subspace \( \Lambda^k \) of \( H^{2k} \).

**Problem 1** Is it possible to develop a Lagrangian plane description for the unitary potential case in analogy with the Schrödinger potential case [26]?

**Proposition 2** The S-matrix (4.14) is unitary in the scattering domain (4.23).

**Proof.** The proof resembles (however, with some important variations) the proof [26] of the analogous statement for the Schrödinger potential scattering case.

As the first step, we define the symmetric function \( W(\phi, \psi) \) (we call this function s-Wronskian by analogy with the antisymmetric Wronskian in [26]), which sets into the correspondence to two functions from \( C_0 \) the function from \( C_1 \) by the formula

\[
W(\phi, \psi) = \sum_{\vec{\mu}_{x,y} \in L} (\phi^*(x)U_{\vec{\mu}_{x,y}}\psi(y) + \psi^*(x)U_{\vec{\mu}_{x,y}}\phi(y)). \tag{4.24}
\]

If \( \phi \) and \( \psi \) are two eigenfunctions from \( \Lambda^k \) (by definition corresponding to the same value of the spectral parameter \( \lambda \)), then the s-Wronskian is a chain (one-cycle with possible open ends), i.e., in the infinite graph, we have

\[ \partial W(\phi, \psi) = 0. \]

This follows from the simple calculation,

\[
\partial W(\phi, \psi)(x) = \phi^*(x)\sum_y U_{\vec{\mu}_{x,y}}\psi(y) + \psi^*(x)\sum_y U_{\vec{\mu}_{x,y}}\phi(y) -
\]

\[
-\phi^*(y)\sum_y U_{\vec{\mu}_{y,x}}\psi(x) - \psi^*(x)\sum_y U_{\vec{\mu}_{y,x}}\phi(x) =
\]

\[
= \phi^*(x)(M_1\psi)(x) - (\phi^*(x)(M_1\psi)(x))^* +
\]

\[
+\psi^*(x)(M_1\phi)(x) - (\psi^*(x)(M_1\phi)(x))^*. \tag{4.25}
\]

Using now the relation \( M_1\phi = (Q+1-\lambda)\phi \) and remembering that \( Q \) and \( \lambda \) are Hermitian, we obtain zero in the r.h.s. of (4.25).

The s-Wronskian of two solutions is therefore a chain. This means in turn that if we now cut the branches of the (infinite) graph at some distance \( n \) from the reduced graph \( F \)
(this distance must not be the same for all branches; it is only preferable to be the same on each branch), thus obtaining some finite graph $T_{\text{cut}}$, then, nevertheless, the total sum vanishes:

$$\sum_{x \in \partial T_{\text{cut}}} \partial W = 0. \quad (4.26)$$

Here the sum ranges all boundary points $x \in \partial T_{\text{cut}}$, and $\partial W$ is restricted to the graph $T_{\text{cut}}$.

It is convenient to introduce the bilinear pairing on $H^{2k}$,

$$\langle\langle \psi, \phi \rangle\rangle \equiv \sum_{x \in \partial T_{\text{cut}}} \partial W(\psi, \phi)$$

on the boundary of $T_{\text{cut}}$ for two arbitrary asymptotic vectors from $H^{2k}$.

The reasoning in the beginning of Sec. 4 show that we can consistently eliminate the gauge potential dependence on all external branches. Then, the $\psi_i^\pm$ functions are mere exponents of $\alpha_\pm$: $\psi_i^\pm = \alpha_i^\pm$ on $i$th branch and zero otherwise. Then, using the identities $\alpha_+ \alpha_- = 1/p$ and $\alpha_- = \alpha_+^*$, which hold in the scattering zone, it is easy to find that

$$\langle\langle \psi_i^\pm, \psi_j^\mp \rangle\rangle = \pm \delta_{i,j} \frac{\alpha_- - \alpha_+}{2p} \equiv \pm \delta_{i,j} \langle\langle \psi_i^+, \psi_j^+ \rangle\rangle$$

and

$$\langle\langle \psi_i^\pm, \psi_j^\mp \rangle\rangle \equiv 0.$$  

Now we remember that for two arbitrary solutions $\psi$ and $\phi$ of spectral problem (4.1) determined on the whole graph, the expression for $\partial W(\psi, \phi)$ exactly coincides with $\langle\langle \psi, \phi \rangle\rangle$, where only asymptotic states must be taken into account. Consider now the eigenfunctions $\Psi_i \equiv \psi_i^+ - \sum_j s_{i,j} \psi_j^-$ (cf. formula (4.14)). Then, for two such functions, we have

$$0 = \langle\langle \Psi_i, \Psi_j \rangle\rangle \equiv \langle\langle \psi_i^+, \psi_j^+ \rangle\rangle \left( \delta_{i,j} - \sum_k s_{i,k}^* s_{j,k} \right). \quad (4.27)$$

Because $\langle\langle \psi_i^+, \psi_j^+ \rangle\rangle \neq 0$, we obtain that the $S$-matrix is unitary. The proposition is therefore proved.\footnote{We use condensed multiindex notation.}

It is interesting that we can deduce the existence of the exceptional discrete spectrum from the $L$-function itself.

**Proposition 3** Exceptional discrete spectrum appears iff $\exists \alpha_+, \alpha_-: \alpha_+ \alpha_- = 1/p$ and the $L$-function has poles at both $\alpha_+$ and $\alpha_-$.\footnote{Considering formula (4.20) we obtain that the $S$-matrix determinant is a unitary function as soon as $\alpha_\pm$ lie in the scattering zone; this is of course in accordance with Proposition 2.}

**Sketch of the proof.** We seek the exponential solution of spectral problem (4.1) such that one of the waves, retarded or advanced, just vanish identically in all vertices of the
graph $T$ (including now also internal points of $F$). This immediately implies that there exists a vector (source function) $f_0 \in \tilde{C}_0$ such that

$$\Delta(\alpha_+)f_0 = 0.$$ 

Then the nonlocal operator $\hat{K}$,

$$\hat{K}f(x) = \sum_{y \in F} K(x, y|\alpha)f(y), \quad f \in \tilde{C}_0,$$

which is inverse to $\hat{\Delta}(\alpha)$, becomes ill defined.

Let $f_0$ be the null vector of $\hat{\Delta}(\alpha)$. This immediately produces the characteristic equation on $\alpha$:

$$\det \hat{\Delta}(\alpha_+)=0,$$

and if simultaneously $\hat{\Delta}(\alpha_-) \neq 0$, then, following Theorem 2, the $S$-matrix develops a singularity (zero or pole), which indicate that for such value of $\lambda$ there exists a solution of spectral problem (4.1) in which one (and only one) of the retarded and advanced waves is nonzero (cf. [30]).

Thus, if no cancellations between numerator and denominator occur in formula (4.20), i.e., the assertion of the proposition is not true, then the discrete spectrum is exhausted by regular eigenfunctions and no exceptional spectrum exists.

On the contrary, when there are such poles of the $L$-function that the assertion of the proposition is satisfied, then the regular discrete spectrum does not provide all possible solutions of the characteristic equation; the solutions that yield zero both in the denominator and in the numerator of (4.20) must therefore correspond to the exceptional discrete eigenvalues.

**Problem 2** To make this sketch of the proof the formal proof.

## 5 Examples. Algorithm for calculating $L(1, u)$.

For simplicity, we assume in what follows that $F = D(\Gamma)/\Gamma$. We omit most of the proofs that are simple exercises in linear algebra.

**Example 1** Graph $T$ with no external branches. Here we consider a limiting case of our construction when the graph $T$ exactly coincides with $F$. In this no-scattering case, $q(z) \equiv p$ and we obtain the inversion relation for $\Delta(u)$. If $\alpha_-\alpha_+ = 1/p$, we have

$$\det^{-1}[(1 + p\alpha_+^2)I - \alpha_+M_1] = \det^{-1}\left[1 + \frac{1}{p\alpha_+^2} - \frac{1}{p\alpha_-}M_1\right]$$

$$= (\alpha_+^2p)^{r_0} \det^{-1}[(1 + p\alpha_-^2)I - \alpha_-M_1]$$

$$= \frac{\alpha_-^{r_0}}{\alpha_+^{r_0}} \det^{-1}[(1 + p\alpha_+^2)I - \alpha_+M_1],$$

(5.1)
and using (4.21) and (5.1), we obtain that the total scattering matrix \( C \equiv 1 \). Therefore, \( C \) is trivial and does not depend at all on the shape of the reduced (or, in this case, total) graph \( T = F \). This is in accordance with the fact that all spectral points are points of the exceptional discrete spectrum in this case.

**Example 2** A one-loop case \((g=1)\). In the case where the graph \( T \) contains a single loop, the \( L \)-function in the case of the trivial character \( \chi(u) = 1 \) is

\[
L(1, u)_{g=1} = \frac{1}{(1 - u^n)^2},
\]

because the group \( \Gamma_1 \) contains only two primitive elements \( \gamma \) and \( \gamma^{-1} \) with the same length \( l(\gamma) = l(\gamma^{-1}) = n \).

### 5.1 Two-loop case. The general structure of \( L(1, u) \).

We present an algorithm for calculating \( L(u) \equiv L(1, u) \) for arbitrary graph. Let \( l_i \) be the lengths of edges of some graph \( F \) and \( a_i \equiv u^{l_i} \). Then, from Theorem 1, we have \( L^{-1}(u) = \det(1 - uT_1) = P_{2r_1} \) — the polynomial of degree \( 2 \sum l_i \) in the variable \( u \). This polynomial can be obviously treated as a polynomial in \( a_i \) (all closed geodesics are built from intervals connecting three- and more-valent vertices of \( F \)) and we may treat \( a_i \) as formal independent variables. Assuming that one of \( l_i \) is greater than the sum of all others, we conclude that the maximum degree of the polynomial \( P_{2r_1}(u) \) in each \( a_i \) is less or equal 2. Thus, \( L^{-1}(u) = P_2(\{a_i\}) \). This polynomial has integer coefficients and from the relation

\[
L^{-1}(u) = \det(1 - uT_1) = (1 - u^2)^{g-1} \det \Delta(u)
\]

it follows that \( L^{-1}(u) \) has zero of order at least \( g - 1 \) at the point \( u = -1 \) and zero of exact order \( g \) at \( u = 1 \). This follows from the relation \( \det \Delta(u)_{|_{u=1}} = 0 \), which holds for arbitrary graph, and from the relation

\[
\frac{\partial}{\partial u} \det \Delta(u)_{|_{u=1}} = \# \text{ maximum trees},
\]

(see [4] where it was noted that the derivative of \( \Delta(u) \) at \( u = 1 \) is expressed by the Kirchhoff formula through the total number of maximum connected trees in the graph \( F \) and is therefore always nonzero).

We turn now to two-loop case. There we have three possibilities (see Fig. 4, a, b, and c).
Fig. 4. Three possible graphs for $g = 2$.

The analysis in \[5\] shows how one can easily calculate the corresponding $L$-functions from the above considerations. The answer is

\[
\begin{align*}
\text{case a} & \quad L_{(a)}^{-1} = \left(1 - \sum_{i<j} a_i a_j - 2a_1 a_2 a_3\right) \left(1 - \sum_{i<j} a_i a_j + 2a_1 a_2 a_3\right), \\
\text{case b} & \quad L_{(b)}^{-1} = (1-a_1)(1-a_2)\left[(1-a_1)(1-a_2) - 4a_1^2 a_2\right], \\
\text{case c} & \quad L_{(c)}^{-1} = (1-a_1)(1-a_2)\left[(1-a_1)(1-a_2) - 4a_1 a_2\right].
\end{align*}
\] (5.3)

(Note that the case c can be obtained either from a or from b case by setting $a_3 = 1$.)

**Example 3** Comparing the Hermitian and unitary potentials. We now consider $L$-functions and the corresponding determinants for two graphs depicted in Fig. 5 endowed with Abelian potentials.

Fig. 5. Examples of reduced graphs with Hermitian and unitary potentials.

We present answers for Hermitian and unitary potentials (where $a_i \rightarrow a_i^{-1}$ for the inversely oriented edges) in case a. For the Hermitian potential, we obtain

\[
\begin{align*}
\det \Delta(u) &= (1 + 2u^2)^2 - s^2 u^2, \quad s \equiv a + b + c, \\
\det(1 - T(u)) &= 1 - 2u^2(ab + bc + ca) + u^4(ab + bc + ca)^2 - 4u^6 a^2 b^2 c^2, \\
\end{align*}
\]

while for the unitary potential, these quantities are, of course, related:

\[
\begin{align*}
\det \Delta(u) &= (1 + 2u^2)^2 - s\tilde{s} u^2, \quad s \equiv a + b + c, \quad \tilde{s} \equiv a^{-1} + b^{-1} + c^{-1}, \\
\end{align*}
\]
\[ \det(1 - T(u)) = (1 - u^2) \det \Delta(u). \]

Case b in Fig. 5 is of interest because it is just the case where the exceptional discrete spectrum is nonempty for \( p = 2 \) and for any \( a \) (it suffices to set equal in the absolute value positive and negative charges at the upper and lower vertices in Fig. 5b; then, the eigenfunction vanishes identically at the middle point). For the \( L \)-function, we obtain

\[ L(u) = (1 + ux + 2u^2)(1 - ux + u^2 - u^3x + 2u^4), \quad x \equiv a + a^{-1}, \ p = 2, \]

i.e., the condition of Proposition 3 is satisfied for any \( a \), because the product of two roots of the first quadratic polynomial is \( 1/2 \equiv 1/p \) in this case.

### 6 Teichmüller spaces via graphs

In this section, we consider Teichmüller spaces \( \mathcal{T}^h \)—the spaces of complex structures on (possibly open) Riemann surfaces \( S \) with holes (punctures) modulo diffeomorphisms homotopy equivalent to identity. In the vicinity of a boundary component, the complex structure can be isomorphic as a complex manifold either to an annulus (hole) or to a punctured disc (puncture).

The graph description following [28] and [14] is suitable for considering the finite covering \( \mathcal{T}^H(S) \) of the Teichmüller space \( \mathcal{T}^h(S) \). A point of \( \mathcal{T}^H(S) \) is determined by a point of \( \mathcal{T}^h(S) \) and by the orientation of all holes of \( S \) that are not punctures. (This covering is obviously ramified over the subspace of surfaces with punctures.)

It is well known that an oriented 2D surface with negative Euler characteristic can be continuously conformally transformed to the constant curvature surface. The Poincaré uniformization theorem claims that any complex surface \( S \) of a constant negative curvature (equal \(-1\) in what follows) is a quotient of the upper half-plane \( \mathbb{H}_+ \) endowed with the hyperbolic metric \( ds^2 = dzd\overline{z}/(3z)^2 \) over the action of a discrete Fuchsian subgroup \( \Delta(S) \) of the automorphism group \( PSL(2, \mathbb{R}) \),

\[ S = \mathbb{H}_+/\Delta(S). \]

In the hyperbolic metric, geodesics are either half circles with endpoints at the real line \( \mathbb{R} \) or vertical half-lines; all points of the boundary \( \mathbb{R} \) are at infinite distance from each other and from any interior point.

Any hyperbolic homotopy class of closed curves \( \gamma \) contains a unique closed geodesic of the length \( l(\gamma) = |\log \lambda_1/\lambda_2| \), where \( \lambda_1 \) and \( \lambda_2 \) are (different) eigenvalues of the element of \( PSL(2, \mathbb{R}) \) that corresponds to \( \gamma \).

Since Strebel [34], the fat, or ribbon, graphs have been used to coordinatize the Teichmüller and moduli space. We use a rather explicit and simple version of this description [14].
Claim 1 For a given three-valent fat graph $T$ of genus $g$ and number of punctures $n$, there exists a one-to-one correspondence between the set of points of $\mathcal{T}^H(S)$ and the set of edges of this graph supplied with real numbers (lengths).

We propose the explicit way how to construct the Fuchsian group $\Delta(S) \subset PSL(2, \mathbb{R})$, which corresponds to a given set of numbers on edges of a graph $T \in T(S)$ such that $S = H_\pm / \Delta(S)$. For this, we must associate an element $P_\gamma \in PSL(2, \mathbb{R})$ to any element of the fundamental group $\gamma \in \pi_1(S)$.

To each edge $\alpha$ we associate the matrix $X_{z_\alpha} \in PSL(2, \mathbb{R})$ of the Möbius transformation

$$X_{z_\alpha} = \begin{pmatrix} 0 & -e^{Z_\alpha/2} \\ e^{-Z_\alpha/2} & 0 \end{pmatrix}.$$  \hfill (6.1)

In order to parameterize a path over edges of the graph, we introduce the matrices of the “right” and “left” turns

$$R = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}, \quad L \equiv R^2 = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}.$$  \hfill (6.2)

The spaces $C_0$ and $C_1$ are spaces of functions that take values in the fundamental two-dimensional representation of the group $PSL(2, \mathbb{R})$.

The “operators of the right and left turns,” $R_z$ and $L_z$, are

$$R_z \equiv RX_z = \begin{pmatrix} e^{-Z/2} & -e^{Z/2} \\ 0 & e^{Z/2} \end{pmatrix},$$
$$L_z \equiv LX_z = \begin{pmatrix} e^{-Z/2} & 0 \\ -e^{-Z/2} & e^{Z/2} \end{pmatrix}.$$  \hfill (6.3, 6.4)

The operator $T(u)$ acts on the space $C_1$ as follows. Let $\vec{e}_R$ (respectively, $\vec{e}_L$) be the oriented edge to which the oriented edge $\vec{e}_z$ is naturally mapped by the right (respectively, left) turn when going along consecutive edges of a path. For $u \in \mathbb{C}$ and $\vec{e} \in \vec{L}$, we obtain

$$T(u)v_{\vec{e}} \cdot \vec{e}_z = v_{\vec{e}_R} \cdot \vec{e}_R + v_{\vec{e}_L} \cdot \vec{e}_L,$$

where

$$v_{\vec{e}_R} = uR_zv_{\vec{e}} \text{ and } v_{\vec{e}_L} = uL_zv_{\vec{e}}.$$  \hfill (6.5)

A geodesic is a closed primitive path in the graph $T$. To each such path we set into the correspondence the product of matrices $P_{\vec{z}_1 \cdots \vec{z}_n} = L_{\vec{z}_n}L_{\vec{z}_{n-1}} \cdots R_{\vec{z}_1}L_{\vec{z}_n}$, where the matrices $L_{\vec{z}_i}$ or $R_{\vec{z}_i}$ are inserted depending on which turn—left or right—the path is going on the corresponding step.

The matrices $L$ and $R$ are torsion potentials from Definition 3. However, we can modify the original graph $T$ at each vertex disconnecting edges at the vertex and connecting them

\footnote{Note that $\pi_1(S)$ is isomorphic to $\pi_1(\Gamma)$.}
by three “short” edges forming a triangle subsequently erasing one (any) of the new edges in order to preserve the number of loops of the graph (see Fig. 6).

![Fig. 6. “Blowing up” vertices of the initial graph.](image)

The matrices $L$ and $R$ then become the potential matrices corresponding to the “short” oriented edges of the resulting graph $\tilde{T}$.

**Proposition 4** [14] There is a one-to-one correspondence between the set of all primitive closed paths $\{P_{z_1\cdots z_n}\}$ in the graph $T$ and closed geodesics $\{\gamma\}$ on the Riemann surface. Moreover, the length $l(\gamma)$ of a geodesic is determined by the relation

$$G(\gamma) \equiv 2 \cosh(l(\gamma)/2) = \text{tr} \, P_{z_1\cdots z_n}. \quad (6.5)$$

**Remark 4** Because both matrices (6.3) and (6.4) have the form $\left\{ \begin{array}{c} (+) \\ (-) \end{array} \right\}$, then any product of such matrices will be of the same form; meanwhile, in a diagonal term of any such product, all summands enter with the plus sign and, in an antidiagonal term, with the minus sign. Then, for closed geodesics around holes (round-the-face geodesics), we obtain $l(\gamma) = |\sum_{i\in I} Z_i|$, where the sum ranges all boundary edges of the face (with the proper multiplicities).

**Definition 12** The Ihara–Selberg $L$-function $L(u)$ for the fat graph $T$ is

$$L(u) = \prod_{\{z\}} \det^{-1}(I - u^n P_{z_1\cdots z_n}) \quad (6.6)$$

(cf. (3.1)) where the product runs over all primitive closed paths, $n$ being the length of a path measured in terms of the distance on the universal covering tree.

Then, from Theorem [4], we obtain that the Ihara–Selberg $L$-function (6.6) is a rational function in variables $e^{z_i/2}$ and $u$, and

$$L(u) = \det^{-1}(I - T(u)) = u^{r_0} \det^{-1} \Delta(u). \quad (6.7)$$

The product (6.6) is absolutely convergent in the circle $|u| < 1/2 \min\{|e^{-z_i}|\}$. However, following Theorem [4], it possesses a unique analytic continuation into the whole $\mathbb{C}$ except a finite number of singular points.
6.1 Selberg trace formula and distribution of geodesics

In the case of closed Riemann surface of constant negative curvature, strong results concerning the distribution of closed geodesics over lengths have been obtained (see [27, 11]).

**Definition 13** The zeta-function for closed geodesics on the (punctured) surface $S$ is

$$
\zeta(s) = \prod_n (1 - e^{-sh_n})^{-1}, \tag{6.8}
$$

where $h$ is a constant that are related to the asymptotic distribution of geodesic lengths.

If $\pi(T)$ is the number of closed (primitive) geodesics with the length at most $T$, then the following assertion holds true (see, e.g., [27, 11]).

**Claim 2** There exists such constant $h$ that

$$
\lim_{T \to \infty} \frac{\pi(T)}{e^{hT}/hT} = 1.
$$

The proof of Claim 2 follows from the $\zeta$-function analyticity properties.

**Proposition 5** [27] $\zeta(s)$ has an extension to $\mathbb{C}$ as a meromorphic function such that:

(a) $\zeta(s)$ has no zeros or poles in $\Re(s) \geq 1$, $s \neq 1$;

(b) $\zeta(s)$ has a simple pole at $s = 1$.

There are two alternative proofs of Proposition 5: one is based on the celebrated Selberg trace formula, which, in its simplest form, can be formulated as follows.

**Proposition 6** (for proofs, see [23, 9]). Given the Laplace–Beltrami operator $\Delta$, which acts on the hyperbolic constant negative curvature metric space and is the linear second-order (unbounded) partial differential operator with the discrete spectrum $0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \ldots$, and the set of closed geodesics with lengths $\{l_n\}$ on the Riemann surface, the Selberg trace formula relates these two sequences of real numbers $\{\lambda_n\}$ and $\{l_n\}$ by the formula

$$
\sum_n \hat{f}( \sqrt{\lambda_n - 1/4} ) + \int f dt = \sum_n c_n(f(l_n)) + 2 \frac{\text{Area}(V)}{4\pi} \int_{-\infty}^\infty r \tanh(r) \hat{f}(r) dr,
$$

where $f : \mathbb{R} \to \mathbb{R}$ is a $C^\infty$ function of compact support, $\hat{f}$ is its Fourier transform, and

$$
c_n(f(l_n)) = \sum_{k=1}^\infty \frac{l_n f(kl_n)}{\sinh(kl_n/2)}.
$$

while $\text{Area}(V)/2\pi = 2(g - 1)$ for a compact surface.
Alternative proof presented in [27] just uses the methods of symbolic dynamics and can be rather close to the combinatorial graph description. Here we formulate two problems.

**Problem 3** To relate the graph description of zeta- (or $L$-)function (3.1) of the Teichmüller space with the standard zeta-function (6.8).

**Problem 4** Few explicit formulas containing the geodesics in the open Riemann surface case are known. Worth mentioning is the paper [25], where the formulas concerning sets of simple (i.e., non-self-intersecting) geodesics were obtained for a punctured torus and for a pair of pants. Are there generalizations of these formulas to a case of surfaces with holes?

### 6.2 Classical projective (modular) transformations

In [14], the projective transformations on graphs that are the mapping class group (modular) transformations were obtained. They correspond to natural operations called the flips, or Whitehead moves, which are elementary transitions between graphs (actually, between neighbor cells of the simplicial complex whose higher dimensional cells label combinatorial types of three-valent fat graphs of the given Riemannian genus $g$ and $n$). The corresponding transformation of the variables $Z_\alpha$ is nonlinear (see [14, 8] for the explicit expressions), but the geodesic length is a modular invariant function.

**Lemma 3** At $u = 1$, the function $L(u)$ is modular invariant, i.e., does not depend on the particular form of the representing graph.

Note, however, that the graph length $n$ of a geodesic varies under the modular transformations, so we are still unable to define a complete analogue of the zeta function (6.8) in terms of graphs. However, constructing $L$-functions (6.6) makes sense in the symbolic dynamics setting where one customarily introduce additional modular-noninvariant partition of a Riemann surface, which would corresponds to a graph decomposition (see Chaps. 5 and 6 in [11]).

**Remark 5** As for the function $L^{-1}(u)$, the determinant in (6.6) is a Laurent polynomial of no more than second order in $e^{\pm z_i/2}$ for each $z_i$. Moreover, from Lemma 2 it follows that the modular-invariant expression for $L(1)$ can depend only on the modular invariants “perimeters” of holes. This imposes severe restrictions on a possible form of $L^{-1}(1)$. Say, for the moduli spaces $M_{g,1}$ of the Riemann surfaces with one puncture (hole), there exists only one such parameter, $p_1$. Taking into account the global symmetry $x_i \to -x_i$, we find that

$$L_{g,1}^{-1}(1) = a \cosh(p_1) + b \cosh(p_1/2) + c, \quad a + b + c = 0,$$

where only the coefficients $a$, $b$, and $c$ depend on the genus of the surface.
**Example 4** 1. For the moduli space $M_{1,1}$ (the torus with one puncture), there are three parameters $x$, $y$, and $z$ and one modular invariant $p_1 = x + y + z$,

$$L_{1,1}^{-1}(1) = -8 \cosh(x + y + z) - 1.$$  

2. The moduli space $M_{0,3}$ (the sphere with three punctures). There are three parameters $x$, $y$, and $z$, and three restrictions $x + y = p_3$, $x + z = p_2$, $y + z = p_1$. Then

$$L_{0,3}^{-1}(1) = -2^5 \left( \cosh \left( \frac{x + y}{2} \right) - 1 \right) \left( \cosh \left( \frac{x + z}{2} \right) - 1 \right) \left( \cosh \left( \frac{y + z}{2} \right) - 1 \right).$$

7 Conclusions

Our main result is the relation (4.20) connecting the $L$-functions with the scattering data. It provides an analogue of the Selberg trace formula for the discrete non-compact case of the graphs.

Spectral properties of the $S$-matrix studied in Sec. 4 deserve further investigation, especially for the case of unitary and Schrödinger type potentials.

One can hope to apply the technique of this paper to the continuous non-compact case in order to obtain analogous formulas for the continuum scattering processes.

Eventually, some $r$-matrix structure must be hidden in these scattering processes. It is interesting how it can be interpreted from the standpoint of the integrable models.

After this paper has been completed we were aware of paper [1], where the similar results concerning magnetic fluxes on graphs have been obtained within the functional integral standpoint.

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