PROJECTIVE RESOLUTIONS OF ASSOCIATIVE ALGEBRAS AND AMBIGUITIES

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Abstract. The aim of this article is to give a method to construct bimodule resolutions of associative algebras, generalizing Bardzell’s well-known resolution of monomial algebras. We stress that this method leads to concrete computations, providing thus a useful tool for computing invariants associated to the algebras. We illustrate how to use it giving several examples in the last section of the article. In particular we give necessary and sufficient conditions for noetherian down-up algebras to be 3-Calabi-Yau.

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1. Introduction

The invariants attached to associative algebras and, in particular to finite dimensional algebras, have been widely studied during the last decades. Among others, Hochschild homology and cohomology of different families of algebras have been computed.

The first problem one faces when computing Hochschild (co)homology is to find a convenient projective resolution of the algebra as a bimodule over itself. Of course, the bar resolution is always available but it is almost impossible to perform computations using it.

M. Bardzell provided in [Ba] a bimodule resolution for monomial algebras, that is, algebras of the form $A = kQ/I$ with $k$ a field, $Q$ a finite quiver and $I$ a two-sided ideal which can be generated by monomial relations; in this situation, the class in $A$ of a path in $Q$ is either an element belonging to the basis or just zero. Moreover, this resolution is minimal. A simple proof of the exactness of the complex given by Bardzell has been given by E. Sköldberg in [Sk], where he provided a contracting homotopy. Of course, this resolution does not solve the whole problem, it is just a starting point.

The non monomial case is more difficult, since it involves rewriting the paths in terms of a basis of $A$. Different kinds of resolutions for diverse families of algebras have been given in the literature. For augmented $k$-algebras, Anick constructed in [An] a projective resolution of the ground field $k$. The projective modules in this resolution are constructed in terms of ambiguities (or $n$-chains), and the differentials are not given explicitly. In practice, it is hard make this construction explicit enough in order to compute cohomology. For quotients of path algebras over a quiver $Q$ with a finite number of vertices, Anick and Green exhibited in [AG] a resolution for the simple module associated to each vertex, generalizing the result of [An], which deals with the case where the quiver $Q$ has only one vertex.

One may think that the case of binomial algebras is easier than others, but in fact it is not quite true since it is necessary to keep track of all reductions performed when writing an element in terms of a chosen basis of the algebra as a vector space.

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In this article we construct in an inductive way, given an algebra $A$, a projective bimodule resolution of $A$, which is a kind of deformation of Bardzell’s resolution of a monomial algebra associated to $A$. For this, we use ideas coming from Bergman’s Diamond Lemma and from the theory of Gröbner bases. The resolution we give is not always minimal.

In the context of quotients of path algebras corresponding to a quiver with a finite number of vertices, our method consists on constructing a resolution whose projective bimodules come from ambiguities present in the rewriting system. Of course there are many different ways of choosing a basis, so we must state conditions that assure that the rewriting process ends and that it is efficient.

One of the advantages of doing this is that, once a bimodule resolution is obtained, it is easy to construct starting from it a resolution of any module on one side and, in particular, to recover the resolutions constructed in [An] and [AG] for the case of the simple modules associated to the vertices of the quiver.

To deal with the problem of effective computation of these resolutions, Theorem 4.1 below gives sufficient conditions for a complex defined over these projective bimodules to be exact. We will be, in consequence, able to prove that some complexes are resolutions without following the procedure prescribed in the proof of the existence theorem.

Applying our method we recover a well-known resolution of quantum complete intersections, see for example [BE] and [BGMS]. We also construct a short resolution for down-up algebras which allows us to prove that a noetherian down-up algebra $A(\alpha, \beta, \gamma)$ is 3-Calabi-Yau if and only if $\beta = -1$.

The contents of the article are as follows. In Section 2 we fix notations and prove some preliminary results. In Section 3 we deal with ambiguities. In Section 4 we state the main theorems of this article, namely Theorem 4.1 and Theorem 4.2 after proving some results on orders and differentials. Section 5 is devoted to the proofs of these theorems; it contains several technical lemmas. In Section 6 we construct explicitly the differentials in low degrees and, finally, in Section 7 we give several applications of our results.

We have just seen this week a preprint by Guiraud, Hoffbeck and Malbos [GHM] where they give sufficient conditions for a complex defined over these projective bimodules to be exact. We will be, in consequence, able to prove that some complexes are resolutions without following the procedure prescribed in the proof of the existence theorem.

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2. Preliminaries

In this section we will give some definitions, present some basic constructions and we will also prove results that are necessary in the sequel.

Let $k$ be a field and $Q$ a quiver with a finite set of vertices. Given $n \in \mathbb{N}$, $Q_n$ denotes the set of paths of length $n$ in $Q$ and $Q_{\geq n}$ the set of paths of length at least $n$, that is, $\bigcup_{i \geq n} Q_i$. Whenever $c \in Q_n$, we will write $|c| = n$. If $a, b, p, q \in Q_{\geq 0}$ are such that $q = apb$, we say that $p$ is a divisor of $q$; if, moreover, $a = 1$, we say that $p$ is a left divisor of $q$ and analogously for $b = 1$ and right divisor. We denote $t, s: Q_1 \to Q_0$ the usual source and target functions. Given $s \in Q_{\geq 0}$ and a finite sum $f = \sum_{i} \lambda_i e_i \in kQ$ such that $e_i \in Q_{\geq 0}$ and $t(s) = t(e_i)$, $s(s) = s(e_i)$ for all $i$, we say that $f$ is parallel to $s$. Let $E := kQ_0$ be the subalgebra of the path algebra generated by the vertices of $Q$.

Given a set $X$ and a ring $R$, we shall denote $(X)_R$ the left $R$-module freely spanned by $X$.

Let $I$ be a two sided ideal, $A = kQ/I$ and $\pi: kQ \to A$ the canonical projection. We assume that $\pi(Q_0 \cup Q_1)$ is linearly independent.

We recall some of the terminology in [B], which we will use. A set of pairs $\mathcal{R} = \{(s_i, f_i)\}_{i \in \Gamma}$ where $s_i \in Q_{\geq 0}$, $f_i \in kQ$ is called a reduction system. We will always assume that a reduction...
system $\mathcal{R} = \{(s_i, f_i)\}_{i \in \mathbb{R}}$ satisfies the conditions that $f_i$ is parallel to $s_i$ for all $i$, and that $s_i$ does not divide $s_j$ for $i \neq j$. Given $(s, f) \in \mathcal{R}$ and $a, c \in Q_{\geq 0}$ such that $asc \neq 0$ in $kQ$, we will call the triple $(a, s, c)$ a basic reduction and write it $r_{a,s,c}$. Note that $r_{a,s,c}$ determines an $E$-bimodule endomorphism $r_{a,s,c} : kQ \rightarrow kQ$ such that $r_{a,s,c}(asc) = a fc$ and $r_{a,s,c}(q) = q$ for all $q \neq asc$.

A reduction is an $n$-tuple $(r_n, \ldots, r_1)$ where $n \in \mathbb{N}$ and $r_i$ is a basic reduction for $1 \leq i \leq n$. As before, a reduction $r = (r_n, \ldots, r_1)$ determines an $E$-bimodule endomorphism of $kQ$, the composition of the endomorphisms corresponding to the basic reductions $r_n, \ldots, r_1$.

An element $x \in kQ$ is said to be irreducible if $r(x) = x$ for all basic reductions $r$. A path $p \in Q_{\geq 0}$ will be called reduction-finite if for any infinite sequence of basic reductions $(r_i)_{i \in \mathbb{N}}$, there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$, $r_n \circ \cdots \circ r_1(p) = r_{n_0} \circ \cdots \circ r_1(p)$. Moreover, the path $p$ will be called reduction-unique if it is reduction-finite and for any two reductions $r$ and $r'$ such that $r(p)$ and $r'(p)$ are both irreducible, the equality $r(p) = r'(p)$ holds.

**Definition 2.1.** We say that a reduction system $\mathcal{R}$ satisfies the condition $(\diamondsuit)$ relative to $I$ if

- the ideal $I$ is equal to the two sided ideal generated by the set $\{s - f\}_{(s, f) \in \mathcal{R}}$,
- every path is reduction-unique and
- for each $(s, f) \in \mathcal{R}$, $f$ is irreducible.

There reason why we are interested these reduction systems is the following lemma, which is a restatement of Bergman’s Diamond Lemma.

**Lemma 2.2.** If the reduction system $\mathcal{R}$ satisfies $(\diamondsuit)$ for $I$, then the set $B$ of irreducible paths satisfies the following properties,

(i) $B$ is closed under divisors,
(ii) $\pi(b) \neq \pi(b')$ for all $b, b' \in B$ with $b \neq b'$,
(iii) $\{\pi(b) : b \in B\}$ is a basis of $A$.

**Remark 2.2.1.** In view of this, we can define a $k$-linear map $i : A \rightarrow kQ$ such that $i(\pi(b)) = b$ for all $b \in B$. We denote by $\beta : kQ \rightarrow kQ$ the composition $i \circ \pi$. Notice that if $p$ is a path and $r$ is a reduction such that $r(p)$ is irreducible, then $r(p) = \beta(p)$.

**Definition 2.3.** If $\mathcal{R}$ is a reduction system satisfying $(\diamondsuit)$ for $I$, we define $S := \{s \in Q_{\geq 0} : (s, f) \in \mathcal{R}$ for some $f \in kQ\}$.

**Remark 2.3.1.** Notice that:

1. $S$ is equal to the set $\{p \in Q_{\geq 0} : p \notin B$ and $p' \in B$ for all proper divisor $p'$ of $p\}$.
2. If $s$ and $s'$ are elements of $S$ such that $s$ divides $s'$, then $s = s'$.
3. Given $q \in Q_{\geq 0}$, $q$ is irreducible if and only if there exists no $p \in S$ such that $p$ divides $q$.

**Definition 2.4.** Given a path $p$ and $q = \sum_{i=1}^{n} \lambda_i c_i \in kQ$ with $\lambda_1, \ldots, \lambda_n \in k^\times$ and $c_1, \ldots, c_n \in Q_{\geq 0}$, we write $p \in q$ if $p = c_i$ for some $i$.

Given $p, q \in Q_{\geq 0}$ we write $q \rightsquigarrow p$ if there exist $n \in \mathbb{N}$, basic reductions $r_1, \ldots, r_n$ and paths $p_1, \ldots, p_n$ such that $p_1 = q$, $p_n = p$, and for all $i = 1, \ldots, n-1, p_{i+1} \in r_i(p_i)$.

**Lemma 2.5.** Suppose that every path is reduction-finite with respect to $\mathcal{R}$.

(i) If $p$ is a path and $t$ a reduction such that $p \in t(p)$, then $t(p) = p$.
(ii) The binary relation $\rightsquigarrow$ is an order on the set $Q_{\geq 0}$ which is compatible with concatenation, that is, $\rightsquigarrow$ satisfies that $q \rightsquigarrow p$ implies $apc \rightsquigarrow apc$ for all $a, c \in Q_{\geq 0}$ such that $apc \neq 0$ in $kQ$.
(iii) The binary relation $\rightsquigarrow$ satisfies the descending chain condition.
Proof. (i) The hypothesis is that \( t(p) = \lambda p + x \) with \( \lambda \in k^* \) and \( p \notin x \). Since the sequence of reductions \((t, t, \cdots)\) stabilizes when acting on \( p \), there exists \( k \in \mathbb{N} \) such that \( \lambda^k p + kx = t^k(p) = t^{k+1}(p) = \lambda^{k+1}p + (k+1)x \). As a consequence, \( \lambda = 1 \) and \( x = 0 \).

(ii) It is clear that \( \sim \) is a transitive and reflexive relation and that it is compatible with concatenation. Let us suppose that it is not antisymmetric, so that there exist \( n \in \mathbb{N} \), paths \( p_1, \ldots, p_{n+1} \) and basic reductions \( r_1, \ldots, r_n \) such that \( p_{i+1} \in r_i(p_i) \) for \( 1 \leq i \leq n \) and \( p_{n+1} = p_1 \). Suppose that \( n \) is minimal. There exist \( x_1, \ldots, x_n \in kQ \) and \( \lambda_1, \ldots, \lambda_n \in k^* \) such that \( r_i(p_i) = \lambda_i p_{i+1} + x_i \) with \( p_{i+1} \notin x_i \). Notice that since \( m \) is minimal, \( r_i(p_i) \neq p_i \) and then \( r_i \) act trivially on every path different from \( p_i \), for all \( i \).

Suppose that \( i < j \) and that \( p_i \in x_j \). Let \( m = j - i \), \( u_k = p_{i+k-1} \) and \( t_k = r_{i+k-1} \) for \( 1 \leq k \leq m \) and \( u_{m+1} = p_1 \). Notice that \( u_{k+1} \in t_k(u_k) \) for \( 1 \leq k \leq m \) and \( u_{m+1} = u_1 \). Since \( m < n \) this contradicts the choice of \( n \). It follows that

\[ p_i \notin x_j \text{ for all } i \neq j. \]

This implies that \( r_n \circ \cdots \circ r_1(p_1) = \lambda p_1 + x \) with \( p_1 \notin x \), as one can easily check. Define inductively for all \( i > n \), \( r_i := r_{i-n} \). The sequence \((r_i)_{i \in \mathbb{N}} \) acting on \( p_1 \) never stabilizes: this contradicts the reduction-finiteness of the reduction system \( \mathcal{R} \).

(iii) Suppose not, so that there is a sequence \((p_i)_{i \in \mathbb{N}} \) of paths and a sequence of basic reductions \((t_i)_{i \in \mathbb{N}} \) such that \( p_{i+1} \in t_i(p_i) \). Since \( \sim \) is an antisymmetric relation, \( p_i \neq p_j \) if \( i \neq j \).

Let \( t_1 = 1 \). Notice that \( p_1 \in p_1 \). Suppose that we have constructed \( t_1, \ldots, t_k \) such that the sequence is strictly increasing and that \( p_n \in t_{i_k-1} \circ \cdots \circ t_1(p_1) \) and \( p_j \notin t_{i_k-1} \circ \cdots \circ t_1(p_1) \) for all \( j > i_k \). Set \( X_k = \{ i > i_k : p_i \in t_{i_k-1} \circ \cdots \circ t_1(p_1) \} \). By the inductive hypothesis, there is \( x \in kQ \) and \( \lambda \in k^* \) such that \( t_{i_k-1} \circ \cdots \circ t_1(p_1) = \lambda p_{i_k} + x \) with \( p_{i_k} \notin x \). Since \( p_{i_k+1} \in t_{i_k}(p_{i_k}) \) and \( p_{i_k+1} \notin t_{i_k-1} \circ \cdots \circ t_1(p_1) \), this implies that \( p_{i_k+1} \in t_{i_k}(p_{i_k}) + x = t_{i_k} \circ \cdots \circ t_1(p_1) \) and therefore \( X_k \) is not empty. We may define \( i_{k+1} = \max X_k \), because \( X_k \) is a finite set.

This constructs inductively a strictly increasing sequence of indices \((i_k)_{k \in \mathbb{N}} \) with \( p_{i_k} \in \tilde{p}_{i_k} := t_{i_k-1} \circ \cdots \circ t_1(p_1) \) for all \( k \in \mathbb{N} \). The set \( \{ t_{i_k-1} \circ \cdots \circ t_1(p_1) : k \in \mathbb{N} \} \) is therefore infinite. This contradicts the reduction-finiteness of \( \mathcal{R} \).

From the proof we see that we only need that every path to be reduction-finite with respect to \( \mathcal{R} \) for this Lemma to be true. Moreover, the converse also holds, that is, if \( \mathcal{R} \) is a reduction system for which \( \sim \) is a partial order satisfying the descending chain condition, then every path is reduction-finite. This says that \( \sim \) captures most of the properties we require \( \mathcal{R} \) to verify, and this partial order will be important in the next sections.

The following characterization of the relation \( \sim \) is very useful in practice.

**Lemma 2.6.** If \( p, q \) are paths, then \( q \sim p \) if and only if there exist a reduction \( t \) such that \( p \in t(q) \).

**Proof.** First we prove the necessity of the condition. Let \( n \in \mathbb{N} \), \( r_1, \ldots, r_n \) and \( p_1, \ldots, p_n \) be as in the definition of \( \sim \), and suppose that \( n \) is minimal. Let \( \tilde{p}_1 = p_1 \) and for each \( i = 1, \ldots, n-1 \) put \( \tilde{p}_{i+1} = r_i(\tilde{p}_i) \). Notice that the minimality implies that \( r_i(p_i) \neq p_i \). Let us first show that

\[ (1) \quad \text{if } i > j \text{ then } p_i \notin \tilde{p}_j. \]

Suppose otherwise and let \((i, j)\) be a counterexample with \( j \) minimal. In particular \( p_i \notin \tilde{p}_{j-1} \) and we must have \( p_i \in r_j-1(p_{j-1}) \). Let \( m = n + j - i \), \( t_k = r_k \) and \( u_k = p_k \) if \( k < j - 1 \), and \( t_k = r_{i+k-j} \) and \( u_k = p_{i+k-j} \) if \( j \leq k \leq m \). One can check that \( u_1 = q, u_{n+1-j} = p \) and that \( u_{k+1} \in t_k(u_k) \) for all \( k = 1, \ldots, m - 1 \). Since \( m < n \) this contradicts the choice of \( n \). We thus conclude that \((1)\) holds.

To prove what we want, let us show that for each \( i = 1, \ldots, n \) we have \( p_i \in \tilde{p}_i \); this is enough because we can then take \( t = (r_n, \ldots, r_1) \). By definition \( p_1 \in \tilde{p}_1 \) so we can do an induction.
Suppose $1 \leq i \leq n$ and $p_i \in \hat{p}_i$. Then we have $p_{i+1} \in r_i(p_i)$ and, by equation (1), $p_{i+1} \notin \hat{p}_i$.

Write $\hat{p}_i = \lambda p_i + x$ with $x \in kQ$ and $p_i \notin x$. Since $r_i$ acts nontrivially on $p_i$, it acts trivially on $x$; it follows that $r_i(\hat{p}_i) = \lambda r_i(p_i) + x$ and, in particular, $p_{i+1} \in r_i(\hat{p}_i) = \hat{p}_{i+1}$.

Let us now prove the sufficiency. Let $\ell = (t_m, \ldots, t_1)$ be a reduction such that $p \in \ell(q)$ and $m$ is minimal, and let us proceed by induction on $m$. Notice that if $m = 1$ there is nothing to prove. If $t_i$ is the basic reduction $r_{a_i,s_i,c_i}$, let $p_i = a_i s_i c_i$. Using the same ideas as above one can show that

$$\text{if } u \neq q \text{ and } u \notin t_i(p_i) \text{ for each } 1 \leq i \leq m, \text{ then } u \notin t_1 \circ \cdots \circ t_1(q) \text{ for each } 0 \leq l \leq m.$$ 

Since $p \in \ell(q)$ either $p = q$ or there exists $i \in \{1, \ldots, m\}$ such that $p \in t_i(p_i)$. In the first case $q \sim p$. In the second case, we know that $p_i \sim p$ and we need to prove that $q \sim p_i$. Since $m$ is minimal, $t_i(t_{i-1} \circ \cdots \circ t_1(q)) \neq t_{i-1} \circ \cdots \circ t_1(q)$ and then $p_i \in t_{i-1} \circ \cdots \circ t_1(q)$. The result now follows by induction because $i - 1 < m$.

**Proposition 2.7.** If $I \subseteq kQ$ is an ideal, then there exists a reduction system $R$ which satisfies condition (5).

We will prove this in a series of lemmas.

Let $\leq$ be a well-order on the set $Q_0 \cup Q_1$ such that $e < \alpha$ for all $e \in Q_0$ and $\alpha \in Q_1$. Let $\omega : Q_1 \rightarrow \mathbb{N}$ be a function and extend it to $Q_{\geq 0}$ defining $\omega(e) = 0$ for all $e \in Q_0$ and $\omega(c_n \cdots c_1) = \sum_{i=1}^{n} \omega(c_i)$ if $c_n \cdots c_1$ is a path. Given $c, d \in Q_{\geq 0}$ we write that $c \leq_\omega d$ if

- $\omega(c) < \omega(d)$, or
- $c, d \in Q_0$ and $c < d$, or
- $\omega(c) = \omega(d), c = c_n \cdots c_1, d = d_m \cdots d_1 \in Q_{\geq 1}$ and there exists $j \leq \min(|c|, |d|)$ such that $c_i = d_i$ for all $\{1, \ldots, j-1\}$ and $c_j < d_j$.

Notice that the order $\leq_\omega$ has the following two properties:

1. If $p, q \in Q_{\geq 0}$ and $p \leq_\omega q$, then $\text{cpd} \leq_\omega \text{cpd}$ for all $c, d \in Q_{\geq 0}$ such that $\text{cpd} \neq 0$ and $\text{cpd} \neq 0$ in $kQ$.
2. For all $q \in Q_{\geq 0}$ the set $\{p \in Q_{\geq 0} : p \leq_\omega q\}$ is finite.

It is straightforward to prove the first claim. For the second one, let $\{c^j\}_{i \in \mathbb{N}}$ be a sequence in $Q_{\geq 0}$ such that $c^{j+1} \leq_\omega c^j$ for all $i$. If $c^j \in Q_0$ for some $i$, then it is evident that the sequence stabilizes, so let us suppose that $\{c^j\}_{i \in \mathbb{N}}$ is contained in $Q_{\geq 1}$ and $c^{j+1} \leq_\omega c^j$ for all $i \in \mathbb{N}$. We may also suppose that $\omega(c^j) = \omega(c^j)$ for all $i, j$ and that the lengths of the paths are bounded by some $M \in \mathbb{N}$. By the definition of $\leq_\omega$, we know that the sequence of first arrows of elements of $\{c^j\}_{i \in \mathbb{N}}$ forms a descending sequence in $(Q_1, \leq)$, which must stabilize because $(Q_1, \leq)$ is well-ordered. Let $N \in \mathbb{N}$ be such that the first arrow of $c^j$ equals the first arrow of $c^j$ for all $i, j \geq N$. If $c^j = c^j_n \cdots c^j_1$, and we denote $c^1 = c^1_n \cdots c^1_1$, then $\{c^j\}_{i \geq N}$ is a descending sequence in $(Q_{\geq 0}, \leq_\omega)$ with $|c^1| = M - 1$ for all $i$. Iterating this process we arrive at a contradiction.

**Definition 2.8.** Consider an order $\leq$ on $Q_0 \cup Q_1$ and $\omega : Q_1 \rightarrow \mathbb{N}$ as before, and let $\leq_\omega$ be constructed as above. If $p \in kQ$ and $p = \sum_{i=1}^{n} \lambda_i c_i$ with $\lambda_i \in k^*$, $c_i \in Q_{\geq 0}$ and $c_i <_\omega c_1$ for all $i \neq 1$, we write $\text{tip}(p)$ for $c_1$. If $X \subseteq kQ$, we let $\text{tip}(X) := \{\text{tip}(x) : x \in X \setminus \{0\}\}$.

Consider the set

$$S := \{p \in \text{tip}(I) : p' \notin \text{tip}(I) \text{ for all proper divisors } p' \text{ of } p\}.$$ 

Notice that if $s$ and $s'$ both belong to $S$ and $s \neq s'$, then $s$ does not divide $s'$. For each $s \in S$, choose $f_s \in kQ$ such that $s - f_s \in I$, $f_s <_\omega s$ and $f_s$ is parallel to $s$.

Describing the set $\text{tip}(I)$ is not easy in general. We comment on this problem at the beginning of the last section, where we compute examples.
Lemma 2.9. Let $\leq_\omega$ and $S$ be as before. The ideal $I$ is equal to the two sided ideal generated by the set $\{s - f_s s \in S\}$, which we will denote by $\langle s - f_s s \in S\rangle$.

Proof. It is clear that $\langle s - f_s s \in S\rangle$ is contained in $I$. Choose $x = \sum_{i=1}^{n} \lambda_i c_i \in I$ with $\lambda_i \in k^\times$ and $c_i \in Q_{\geq 0}$. We may suppose that $c_1 = \text{tip}(x)$, so that $c_1 \in \text{tip}(I)$. There is a divisor $s$ of $c_1$ such that $s \in \text{tip}(I)$ and $s' \notin \text{tip}(I)$ for all proper divisor $s'$ of $s$; $s \in S$ by the definition of $S$. Let $a, c \in Q_{\geq 0}$ with $asc = f_1$.

Define $x' := af_sc + \sum_{i=2}^{n} \lambda_i c_i$. We have $x = \lambda_1 c_1 + \sum_{i=2}^{n} \lambda_i c_i = \lambda_1 (s - f_s)c + x'$, so that $x' \in I$ and, by property (1), we see that $c_1 \geq \text{tip}(x')$. We can apply this procedure again to $x'$ and iterate: this process will stop by property (2) and we conclude that $x \in \langle s - f_s s \in S\rangle$.

Lemma 2.10. Let $\leq_\omega$ and $S$ be as before. The set $\mathcal{R} := \langle (s, f_s) s \in S\rangle$ is a reduction system such that every path is reduction-unique.

Proof. Since $s > \text{tip}(f_s)$ for all $s \in S$, properties (1) and (2) guarantee that every path is reduction-finite. We need to prove that every path is reduction-unique. Recall that $\pi$ is the canonical projection $kQ \to kQ/I$. Let $p$ be a path. Since $I = \langle s - f_s s \in S\rangle$, we see that $\pi(r(p)) = \pi(p)$ for any reduction $r$. Let $r$ and $t$ be reductions such that $r(p)$ and $t(p)$ are both irreducible. Then $\pi(r(p) - t(p)) = \pi(r(p) - t(p)) = 0$, so that $r(p) - t(p) \in I$. If this difference is not zero, then the path $d = \text{tip}(r(p) - t(p))$ can be written as $d = asc$ with $a, c$ paths and $s \in S$. It follows that the reduction $r_{a, c, s}$ acts nontrivially on one of $r(p)$ or $t(p)$, and this is a contradiction.

This lemma implies that for each $s \in S$, there exists a reduction $r$ and an irreducible element $f_s$ such that $r(f_s) = f_s$. Consider the reduction system $\mathcal{R}' := \langle (s, f_s) s \in S\rangle$. The set of irreducible paths for $\mathcal{R}$ clearly coincides with the set of irreducible paths for $\mathcal{R}'$ and, since $\pi(s - f_s) = \pi(s - f_s) = 0$, we have that $\langle s - f_s s \in S\rangle \subseteq I$. From Bergman's Diamond Lemma it follows that $I = \langle s - f_s s \in S\rangle$. We can conclude that the reduction system $\mathcal{R}'$ satisfies condition (1), thereby proving Proposition 2.7.

It is important to emphasize that different choices of orders on $Q_{0} \cup Q_1$ and of weights $\omega$ will give very different reduction systems, some of which will better suit our purposes than others. Moreover, there are reduction systems which cannot be obtained by this procedure, as the following example shows.

Example 2.10.1. Consider the algebra

$$A = k\langle x, y, z \rangle/(x^3 + y^3 + z^3 - xyz)$$

and let $\mathcal{R} = \{(xyz, x^3 + y^3 + z^3)\}$. Clearly this reduction system does not come from a monomial order with weights. It is not entirely evident but this reduction system satisfies (1).

Finally, we define a relation $\preceq$ on the set $k^\times Q_{\geq 0} : = \{\lambda p : \lambda \in k^\times, p \in Q_{\geq 0} \} \cup \{0\}$ as the least reflexive and transitive relation such that $\lambda p \preceq \mu q$ whenever there exists a reduction $r$ such that $r(\mu q) = \lambda p + x$ with $p \notin x$. We state $0 \preceq \lambda p$ for all $\lambda p \in k^\times Q_{\geq 0}$.

Lemma 2.11. The binary relation $\preceq$ is an ordering satisfying the descending chain condition and is compatible with concatenation.

Proof. The second claim is clear. Let us prove the first claim. Observe that it is an immediate consequence of Lemma 2.5 that given a path $p$ and a reduction $t$,

(2) \[ t(\lambda_1 p) = \lambda_2 p + x \text{ with } p \notin x, \text{ then } \lambda_1 = \lambda_2. \]

Let $\lambda_1, \ldots, \lambda_{n+1} \in k^\times$, $p_1, \ldots, p_{n+1} \in Q_{\geq 0}$, $x_1, \ldots, x_n \in kQ$ and reductions $t_1, \ldots, t_n$ be such that $t_i(\lambda_i p_i) = \lambda_{i+1} p_{i+1} + x_i, p_{i+1} \notin x_i$ and $\lambda_{n+1} p_{n+1} = \lambda_1 p_1$. This implies that $p_i \sim p_{i+1}$ for each $1 \leq i \leq n$ and $p_{n+1} = p_1$. Since $\sim$ is antisymmetric, it follows that $p_i = p_1$ for all $i$ and (2) implies that $\lambda_i = \lambda_1$ for all $i$. We thus see that $\preceq$ is antisymmetric.
Let now \((\lambda_i p_i)_{i \in \mathbb{N}}\) be a sequence in \(k^\times \mathcal{Q}_{\geq 0}\) and \((t_i)_{i \in \mathbb{N}}\) a sequence of reductions such that \(t_i(\lambda_i p_i) = \lambda_{i+1} p_{i+1} + x_i\) with \(p_{i+1} \not\in x_i\). Then \(p_i \sim p_{i+1}\) for all \(i\) and since \(\sim\) satisfies the descending chain condition there exists \(i_0\) such that \(p_i = p_{i_0}\) for all \(i \geq i_0\). Observation (2) implies then that \(\lambda_i = \lambda_{i_0}\) for all \(i \geq i_0\), so that the sequence \((\lambda_i p_i)_{i \in \mathbb{N}}\) stabilizes. \(\square\)

If \(x = \sum_{i=1}^n \lambda_i p_i \in kQ\) with \(\lambda_i \in k^\times\) and \(\lambda p\) belongs to \(k^\times \mathcal{Q}_{\geq 0}\), we write \(x \preceq \lambda p\) if \(\lambda_i p_i \preceq \lambda p\) for all \(i\). If in addition \(x \neq p\) we also write \(x < p\). The following simple observation is key to everything that follows.

**Corollary 2.12.** For each path \(p\), \(\beta(p) \leq p\). Moreover, \(\beta(p) < p\) if and only if \(p \notin B\).

**Proof.** There is a reduction \(r\) such that \(\beta(p) = r(p) = \sum_{i=1}^n \lambda_i p_i\). It is clear that \(\lambda_i p_i \preceq p\) for all \(i\), so that \(\beta(p) \leq p\). The last claim follows from the fact that \(\beta(p) = p\) if and only if \(p \in B\). \(\square\)

### 3. Ambiguities

There is a monomial algebra associated to \(A\) defined as \(A_S := kQ/(S)\) and equipped with the canonical projection \(\pi' : kQ \to A_S\). The set \(\pi'(B)\) is a \(k\)-basis of \(A_S\).

The modules family if modules \(\{P_i\}_{i \geq 0}\) appearing in the resolution of \(A\) as \(A\)-bimodule will be in bijection with those appearing in Bardzell’s resolution of the monomial algebra \(A_S\). More precisely, we will define \(E\)-bimodules \(kA_i\) for \(i \geq -1\), such that the former will be \(A \otimes E kA_i \otimes E A\) while the latter will be \(A_S \otimes E kA_i \otimes E A_S\). The resolution will start as usual: \(A_{-1} = Q_0\), \(A_0 = Q_1\), \(A_1 = S\).

For \(n \geq 2\), \(A_n\) will be the set of \(n\)-ambiguities of \(R\). We will next recall the definition of \(n\)-ambiguity – or \(n\)-chain according to the terminology used in [Sk], [An], [AG], and to Bardzell’s [Ba] associated sequences of paths, and we will take into account that the sets of left \(n\)-ambiguities and right \(n\)-ambiguities coincide. This fact is proved in [Ba] and also in [Sk].

**Definition 3.1.** Given \(n \geq 2\) and \(p \in Q_{\geq 0}\),

1. the path \(p\) is a **left \(n\)-ambiguity** if there exist \(u_0 \in Q_1\), \(u_1, \ldots, u_n\) irreducible paths such that
   
   (i) \(p = u_0 u_1 \cdots u_n\),
   
   (ii) for all \(i\), \(u_i u_{i+1}\) is reducible but \(u_i d\) is irreducible for any proper left divisor \(d\) of \(u_{i+1}\).

2. the path \(p\) is a **right \(n\)-ambiguity** if there exist \(v_0 \in Q_1\) and \(v_1, \ldots, v_n\) irreducible paths such that
   
   (i) \(p = v_n \cdots v_0\),
   
   (ii) for all \(i\), \(v_{i+1} v_i\) is reducible but \(dv_i\) is irreducible for any proper right divisor of \(v_{i+1}\).

**Proposition 3.2.** Let \(n, m \in \mathbb{N}\), \(p \in Q_{\geq 1}\). If \(u_0, u_1, \ldots, u_n, u_{n+1}, \ldots, u_m\) are paths in \(Q\) such that both \(u_0, \ldots, u_n\) and \(u_{n+1}, \ldots, u_m\) satisfy conditions (1) and (12) of the previous definition for \(p\), then \(n = m\) and \(u_i = \hat{u}_i\) for all \(i\), \(0 \leq i \leq n\).

**Proof.** Suppose \(n \leq m\). It is obvious that \(u_0 = \hat{u}_0\), since both of them are arrows. Notice that \(kQ = T_{kQ_k}kQ_1\), that is the free algebra generated by \(kQ_1\) over \(kQ_0\), which implies that either \(u_0 u_1\) divides \(u_0 \hat{u}_1\) or \(u_0 \hat{u}_1\) divides \(u_0 u_1\), and moreover \(u_0 u_1, u_0 \hat{u}_1 \in A_1 = S\). Remark 2.3.5.1 says that \(u_0 u_1 = \hat{u}_0 \hat{u}_1\). Since \(u_0 = \hat{u}_0\), we must have \(u_1 = \hat{u}_1\). By induction on \(i\), let us suppose that \(u_j = \hat{u}_j\) for \(j \leq i\). As a consequence, \(u_{i+1} \cdots u_n = \hat{u}_{i+1} \cdots \hat{u}_m\).

If \(i + 1 = n\), this reads \(u_n = \hat{u}_n \cdots \hat{u}_m\), and the fact that \(u_n\) is irreducible and \(\hat{u}_j \hat{u}_{j+1}\) is reducible for all \(j < m\) implies that \(m = n\) and \(u_n = \hat{u}_n\). Instead, suppose that \(i < n + 1\). From the equality \(u_{i+1} \cdots u_n = \hat{u}_{i+1} \cdots \hat{u}_m\) we deduce that there exists a path \(d\) such that \(u_{i+1} = \hat{u}_{i+1} d\) or \(\hat{u}_{i+1} = u_{i+1} d\). If \(u_{i+1} = \hat{u}_{i+1} d\) and \(d \in Q_{\geq 1}\), we can write \(d = d_2 d_1\) with
$d_i \in Q_1$. The path $\hat{u}_{i+1}d_2$ is a proper left divisor of $u_{i+1}$ and by condition (iii) we obtain that $u_i\hat{u}_{i+1}d_2$ is irreducible. This is absurd since $u_i\hat{u}_{i+1}d_2 = \hat{u}_i u_{i+1}d_2$ by inductive hypothesis, and the right hand term is reducible by condition (iii). It follows that $d \in Q_0$ and then $u_{i+1} = u_{i+1}$.

The case where $\hat{u}_{i+1} = u_{i+1}d$ is analogous. \hfill $\square$

**Corollary 3.3.** Given $n, m \geq -1$, $A_n \cap A_m = \emptyset$ if $n$ and $m$ are different.

Just to get a flavor of what $A_n$ is one may think about an element of $A_n$ as a minimal proper superposition of $n$ elements of $S$.

We end this section with a proposition that indicates how to compute ambiguities for a particular family of algebras.

**Proposition 3.4.** Suppose $S \subset Q_2$. For all $n \geq 1$,

$$A_n = \{\alpha_0 \cdots \alpha_n \in Q_{n+1} : \alpha_i \in Q_1 \text{ for all } i \text{ and } \alpha_{i-1}\alpha_i \in S\}$$

Moreover, given $p = \alpha_0 \cdots \alpha_n \in A_n$, we can write $p$ as a left ambiguity choosing $u_i = \alpha_i$, for all $i$, and as a right ambiguity choosing $v_i = \alpha_{i-1}$

**Proof.** We proceed by induction on $n$. If $n = 1$ we know that $A_1 = S$ in which case there is nothing to prove. Let $u_0 \cdots u_n u_{n+1} \in A_{n+1}$ and suppose that the result holds for all $p \in A_n$.

Since $u_0 \cdots u_n$ belongs to $A_n$ we only have to prove that $u_{n+1} \in Q_1$ and that $u_n u_{n+1} \in S$. We know that $u_n \in Q_1$, that $u_{n+1}$ is irreducible and that $u_n u_{n+1}$ is reducible. As a consequence, there exist $s \in S$ and $v \in Q_{\geq 0}$ such that $u_n u_{n+1} = sv$. Moreover, $u_n d$ is irreducible for any proper left divisor $d$ of $u_{n+1}$, so the only possibility is $v \in Q_0$. We conclude that $u_n u_{n+1}$ belongs to $S$. Since $S \subset Q_2$ and $u_n \in Q_1$, we deduce that $u_{n+1} \in Q_1$. This proves that $A_{n+1} \subseteq \{\alpha_0 \cdots \alpha_n \in Q_{n+1} : \alpha_i \in Q_1 \text{ for all } i \text{ and } \alpha_{i-1}\alpha_i \in S\}$.

The other inclusion is clear. \hfill $\square$

From now on we fix a reduction system $R$ satisfying condition $(\triangleright)$. Notice that in this situation we can suppose without loss of generality, that $S \subseteq Q_{\geq 2}$.

4. **The resolution**

In this section our purpose is to construct bimodule resolutions of the algebra $A$. We achieve this in Theorems 4.1 and 4.2 in the first one we construct homotopy maps to prove that a given complex is exact, while in the second one we define differentials inductively.

We will make use of differentials of Bardzell’s resolution for monomial algebras, so we begin this section by recalling them. Keeping the notations of the previous section, note that the $kQ$-bimodule $kQ \otimes_E kA_n \otimes_E kQ$ is a $k$-vector space with basis \{a \otimes p \circ c : a, c \in Q_{\geq 0}, p \in A_n, apc \neq 0 \text{ in } kQ\}.

As we have already done for $A$, we define a $k$-linear map $i' : A_S \to kQ$ such that $i'(\pi'(b)) = b$ for all $b \in B$, and we denote by $\beta' : kQ \to kQ$ the composition $i' \circ \pi'$.

Given $n \geq -1$, let us fix notation for the following $k$-linear maps:

$$\pi_n := \pi \otimes \id_{kA_n} \otimes \pi, \quad \pi'_n := \pi' \otimes \id_{kA_n} \otimes \pi',$$

$$i_n := i \otimes \id_{kA_n} \otimes i, \quad i'_n := i' \otimes \id_{kA_n} \otimes i',$$

$$\beta_n := i_n \circ \pi_n, \quad \beta'_n := i'_n \circ \pi'_n.$$ Consider the following sequence of $kQ$-bimodules,

$$\cdots \xrightarrow{f_2} kQ \otimes_E kA_1 \otimes_E kQ \xrightarrow{f_3} kQ \otimes_E kA_0 \otimes_E A \xrightarrow{f_0} kQ \otimes_E kQ \xrightarrow{f_{i-1}} kQ \to 0$$

$$\xrightarrow{\mu} kQ \otimes_E kA_{-1} \otimes_E kQ$$
where

(i) $f_n(a \otimes b) = ab$,
(ii) if $n$ is even, $q \in \mathcal{A}_n$ and $q = u_0 \cdots u_n = v_n \cdots v_0$ are respectively the factorizations of $q$ as a left and right $n$-ambiguity,

$$f_n(1 \otimes q \otimes 1) = v_n \otimes v_{n-1} \cdots v_0 \otimes 1 - 1 \otimes u_0 \cdots u_{n-1} \otimes u_n,$$

(iii) if $n$ is odd and $q \in \mathcal{A}_n$,

$$f_n(1 \otimes q \otimes 1) = \sum_{a \otimes c = q, p \in \mathcal{A}_{n-1}} a \otimes p \otimes c.$$

The maps $f_n$ induce, respectively, $A$-bimodule maps

$$\delta_n : A \otimes_E k\mathcal{A}_n \otimes_E A \to A \otimes_E k\mathcal{A}_{n-1} \otimes_E A$$

where

$$\delta_n := \pi_{n-1} \circ f_n \circ i_n,$$

and $A_S$-bimodule maps

$$\delta'_n : A_S \otimes_E k\mathcal{A}_n \otimes_E A_S \to A_S \otimes_E k\mathcal{A}_{n-1} \otimes_E A_S$$

defined by

$$\delta'_n := \pi'_{n-1} \circ f_n \circ i'_n.$$

Observe that $\delta_{-1}$ is multiplication in $A$.

The algebra $A_S$ is monomial. The following complex provides a projective resolution of $A_S$ as $A_S$-bimodule [23]:

$$\cdots \xrightarrow{\delta'_0} A_S \otimes_E k\mathcal{A}_1 \otimes_E A_S \xrightarrow{\delta'_1} A_S \otimes_E k\mathcal{A}_0 \otimes_E A_S \xrightarrow{\delta'_0} A_S \otimes_E k\mathcal{A}_1 \otimes_E A_S \xrightarrow{\delta'_1} A_S \otimes_E k\mathcal{A}_0 \otimes_E A_S \xrightarrow{\delta'_0} A_S \otimes_E k\mathcal{A}_1 \otimes_E A_S \xrightarrow{\delta'_1} A_S \otimes_E k\mathcal{A}_0 \otimes_E A_S \xrightarrow{\delta'_0} A_S \otimes_E k\mathcal{A}_1 \otimes_E A_S \xrightarrow{\delta'_1} \cdots \rightarrow 0.$$

We will make use of the homotopy that Sköldberg defined in [SK] when proving that this complex is actually exact. We recall it, but we must stress the fact that our signs differ from the ones in [SK] due to the fact that he considers right modules, while we always work with left modules.

Given $n \geq -1$, the morphism of $kQ \otimes E$-bimodules $S_n$ is defined as follows.

For $n = -1$, $S_{-1} : kQ \to kQ \otimes_E k\mathcal{A}_1 \otimes_E kQ$ is the $kQ \otimes E$-bimodule map given by $S_{-1}(a) = a \otimes 1$, for $a \in kQ$.

For $n \in \mathbb{N}_0$, $S_n : kQ \otimes_E k\mathcal{A}_{n-1} \otimes_E kQ \to kQ \otimes_E k\mathcal{A}_n \otimes_E kQ$ is given by

$$S_n(1 \otimes q \otimes b) = (-1)^{n+1} \sum_{a \otimes c \otimes p \otimes q} a \otimes p \otimes c.$$

Let $s'_n = \pi'_n \circ S_n \circ i'_n \circ S_n$. The family of maps $\{s'_n\}_{n \geq -1}$ verify the equalities

$$s'_n \circ \delta'_n + \delta'_{n-1} \circ s'_{n-1} = \text{id}_{A_{S} \otimes_E k\mathcal{A}_n \otimes_E A_{S}}$$

for $n \geq 0$ and $s'_{-1} \circ \delta'_{-1} = \text{id}_{A_{S} \otimes_E k\mathcal{A}_0 \otimes_E A_{S}}$.

Next we define some sets that will be useful in the sequel. For any $n \geq -1$ and $\mu q \in k^\times Q_{\geq 0}$, consider the following subsets of $kQ \otimes_E k\mathcal{A}_n \otimes_E kQ$:

- $\mathcal{L}_{\mathcal{A}}^n(\mu q) := \{\lambda a \otimes p \otimes c : a, c \in Q_{\geq 0}, p \in \mathcal{A}_n, \lambda ap^c \leq \mu q\}$,
- $\mathcal{L}_{\mathcal{A}}^n(\mu q) := \{\lambda a \otimes p \otimes c : a, c \in Q_{\geq 0}, p \in \mathcal{A}_n, \lambda ap^c < \mu q\}$,

and the following subsets of $A \otimes_E k\mathcal{A}_n \otimes_E A$:

- $\mathcal{L}_{\mathcal{A}}^n(\mu q) := \{\lambda \pi(b) \otimes p \otimes \pi(b') : b, b' \in B, p \in \mathcal{A}_n, \lambda bpb' \leq \mu q\}$,
- $\mathcal{L}_{\mathcal{A}}^n(\mu q) := \{\lambda \pi(b) \otimes p \otimes \pi(b') : b, b' \in B, p \in \mathcal{A}_n, \lambda bpb' < \mu q\}$. 
Remark 4.0.1. We observe that

\[ f_{n+1}(x) \in \langle \mathcal{L}_n^\prec(\mu q) \rangle_{Z}, \quad \text{for all } x \in \mathcal{L}_{n+1}^\prec(\mu q), \text{ and} \]

\[ S_n(x) \in \langle \mathcal{L}_n^\prec(\mu q) \rangle_{Z}, \quad \text{for all } x \in \mathcal{L}_{n-1}^\prec(\mu q). \]

Moreover, the only possible coefficients appearing in the linear combinations are +1 and −1.

We will now state the main theorems. Recall that our aim is to construct, for non necessarily monomial algebras, a bimodule resolution starting from a related monomial algebra. The first theorem says that if the difference between its differentials and the monomial differentials can be “controlled”, then we will actually obtain an exact complex. The second theorem says that we can construct the differentials.

**Theorem 4.1.** Set \( d_{-1} := \delta_{-1} \) and \( d_0 := \delta_0 \). Given \( N \in \mathbb{N}_0 \) and morphisms of \( A \)-bimodules \( d_i : A \otimes_E kA_i \otimes_E A \to A \otimes_E kA_{i-1} \otimes_E A \) for \( 1 \leq i \leq N \) such that

(1) \( d_{i-1} \circ d_i = 0 \) for all \( i, 1 \leq i \leq N \),

(2) \( (d_i - \delta_i)(1 \otimes q \otimes 1) \in \langle \mathcal{L}_{i-1}^\prec(q) \rangle_k \) for all \( i \in \{1, \ldots, N\} \) and for all \( q \in A_i \),

the complex

\[ A \otimes_E kA_{N} \otimes_E A \xrightarrow{d_N} \cdots \xrightarrow{d_1} A \otimes_E kA_0 \otimes_E A \xrightarrow{d_0} A \otimes_E A \xrightarrow{d_{-1}} A \longrightarrow 0 \]

is exact.

**Theorem 4.2.** There exist \( A \)-bimodule morphisms \( d_i : A \otimes_E kA_i \otimes_E A \to A \otimes_E kA_{i-1} \otimes_E A \) for \( i \in \mathbb{N}_0 \) and \( d_{-1} : A \otimes_E A \to A \) such that

(1) \( d_{i-1} \circ d_i = 0 \), for all \( i \in \mathbb{N}_0 \),

(2) \( (d_i - \delta_i)(1 \otimes q \otimes 1) \in \langle \mathcal{L}_{i-1}^\prec(q) \rangle_k \) for all \( i \geq -1 \) and \( q \in A_i \).

We will carry out the proofs of these theorems in the following section.

5. PROOFS OF THE THEOREMS

We keep the same notations and conditions of the previous section.

We start by proving some technical lemmas.

**Lemma 5.1.** Given \( n \geq 0 \), the following equalities hold

(1) \( \delta_n \circ \pi_n = \pi_{n-1} \circ f_n \),

(2) \( \delta_n \circ \pi_n = \pi_{n-1} \circ f_n \).

The proof is straightforward, just using the definitions.

Next we prove three lemmas where we study how various maps defined in Section 4 behave with respect to the order.

**Lemma 5.2.** For all \( n \in \mathbb{N}_0 \) and \( \mu q \in k^\times Q_{\geq 0} \), the images by \( \pi_n \) of \( \mathcal{L}_n^\prec(\mu q) \) and of \( \mathcal{L}_n^\succ(\mu q) \) are respectively contained in \( \langle \mathcal{L}_n^\prec(\mu q) \rangle_Z \) and in \( \langle \mathcal{L}_n^\succ(\mu q) \rangle_Z \).

**Proof.** Given \( n \in \mathbb{N}_0 \), \( \mu q \in k^\times Q_{\geq 0} \) and \( x = \lambda a \otimes p \otimes c \in \mathcal{L}_n^\prec(\mu q) \), where \( a, c \in Q_{\geq 0} \) and \( p \in A_n \), suppose \( \beta(a) = \sum_i \lambda_i b_i \) and \( \beta(c) = \sum_j \lambda'_j b'_j \). Since \( \beta(a) \preceq a \) and \( \beta(c) \preceq c \), then \( \lambda_i b_i \preceq a \) and \( \lambda'_j b'_j \preceq c \) for all \( i, j \). This implies

\[ \lambda \lambda_i \lambda'_j b_i b'_j \preceq \lambda a c \preceq \mu q \]

and so \( \lambda \lambda_i \lambda'_j \pi(b_i) \otimes p \otimes \pi(b'_j) \) belong to \( \langle \mathcal{L}_n^\prec(\mu q) \rangle \) for all \( i, j \). The result follows from the equalities

\[ \pi_n(x) = \lambda \pi(a) \otimes p \otimes \pi(c) = \lambda \pi(\beta(a)) \otimes p \otimes \pi(\beta(c)) = \sum_{i,j} \lambda \lambda_i \lambda'_j \pi(b_i) \otimes p \otimes \pi(b'_j). \]
The proof of the second part is analogous. □

**Corollary 5.3.** Let \( n \geq -1 \) and \( \mu q \in k^x Q_{\geq 0} \). Keeping the same notations of the proof of the previous lemma, we conclude that

i) if \( x \in \mathbb{T}_n^x(\mu q) \), then \( \lambda \pi(a) \pi(c) \in (\mathbb{T}_n^x(\mu aqc))_Z \),

ii) if \( x \in \mathbb{T}_n^x(\mu q) \), then \( \lambda \pi(a) \pi(c) \in (\mathbb{T}_n^x(\lambda \mu aqc))_Z \).

**Lemma 5.4.** Given \( n \in \mathbb{N}_0 \) and \( \mu q \in k^x Q_{\geq 0} \), there are inclusions

i) \( \delta_n(\mathbb{T}_n^x(\mu q)) \subseteq (\mathbb{T}_{n-1}^x(\mu q))_Z \),

ii) \( \delta_n(\mathbb{T}_n^x(\mu q)) \subseteq (\mathbb{T}_{n-1}^x(\mu q))_Z \),

iii) \( s_n(\mathbb{T}_n^x(\mu q)) \subseteq (\mathbb{T}_{n-1}^x(\mu q))_Z \),

iv) \( s_n(\mathbb{T}_n^x(\mu q)) \subseteq (\mathbb{T}_{n-1}^x(\mu q))_Z \).

**Proof.** Applying \( i_n \) to \( x = \lambda \pi(b) \otimes p \otimes \pi(b') \in \mathbb{T}_n^x(\mu q) \), with \( b, b' \in B \) and \( p \in A_n \), we get \( i_n(x) = \lambda b \otimes p \otimes b' \). The element \( \lambda b \otimes p \otimes b' \) belongs to \( \mathbb{L}_n^x(\mu q) \) and this implies that \( f_n(\lambda b \otimes p \otimes b') \) belongs to \( (\mathbb{T}_{n-1}^x(\mu q))_Z \), by Remark 4.0.1. As a consequence of Lemma 5.2 we obtain that \( \delta_n(x) = \pi_{n-1}(f_n(\lambda b \otimes p \otimes b')) \) belongs to \( (\mathbb{T}_{n-1}^x(\mu q))_Z \). □

**Lemma 5.5.** Given \( n \geq -1 \), \( \mu q \in k^x Q_{\geq 0} \), if \( x = \lambda a \otimes p \otimes c \in \mathbb{L}_n^x(\mu q) \) is such that \( \pi_n'(x) = 0 \), then

\[ \pi_n(x) = \pi_n(\beta(x)) = \pi_n(\sum_{i,j} \lambda \pi_i \pi_j b_i \otimes p \otimes b'_j) = \sum_{i,j} \lambda \pi_i \pi_j b_i \otimes p \otimes b'_j. \]

\[ \square \]

The importance of the preceding lemmas is that they guarantee how differentials and morphisms used for the homotopy behave with respect to the order. This is stated explicitly in the following corollary.

**Corollary 5.6.** Given \( n \geq 1 \), \( \mu q \in k^x Q_{\geq 0} \) and \( x \in \mathbb{T}_n^x(\mu q) \), the following facts hold:

1) \( \delta_{n-1} \circ \delta_n(x) \in (\mathbb{T}_{n-2}^x(\mu q))_Z \),

2) \( x - \delta_{n+1} \circ s_{n+1}(x) - s_n \circ \delta_n(x) \in (\mathbb{T}_{n+1}^x(\mu q))_Z \).

**Proof.** Let us first write \( x = \lambda \pi(b) \otimes p \otimes \pi(b') \) with \( b, b' \in B \) and \( x' := i_n(x) = \lambda b \otimes p \otimes b' \). Lemma 5.1 implies that

\[ \delta_{n-1} \circ \delta_n(x) = \delta_{n-1} \circ \delta_n(x') = \delta_{n-1} \circ \pi_{n-1} \circ f_n(x') = \pi_{n-2} \circ f_{n-1} \circ f_n(x'). \]

By Remark 4.0.1 \( f_{n-1} \circ f_n(x') \in \mathbb{L}^x_{n-2}(\mu q) \). Next, by Lemma 5.5 in order to prove that \( \delta_{n-1} \circ \delta_n(x) \in \mathbb{T}_{n-2}^x(\mu q)_Z \), it suffices to verify that \( \pi_{n-2} \circ f_{n-1} \circ f_n(x') = 0 \), which is in fact true using Lemma 5.7 and the fact that \( (A_S \otimes E \otimes A_S \otimes E, \delta_n') \) is exact.

In order to prove (2), we first remark that if \( k \in \mathbb{N}_0 \) and \( y \in (\mathbb{T}_n^x(\mu q))_Z \), then \( i_k \circ \pi_k(y) = i_k \circ \pi_k(y) \) is true. Indeed, let us write \( y = \lambda a \otimes p \otimes c \in \mathbb{T}_n^x(\mu q) \). In case \( a \in B \) and \( c \in B \), there are equalities \( i_k \circ \pi_k(y) = y = i_k \circ \pi_k(y) \), and so the difference is zero. If either
a \notin B$ or $c \notin B$, then \( \pi'_1(y) = 0 \) and in this case Lemma 5.5 implies that \( \pi_n(y) \in (\mathcal{L}_k^\infty(\mu q))_Z \). So, $i_n \circ \pi_1(y) \in (\mathcal{L}_k^\infty(\mu q))_Z$ and the difference we are considering belongs to $(\mathcal{L}_k^\infty(\mu q))_Z$.

Fix now $x = \lambda \pi(b) \otimes p \otimes \pi(b')$ and $x' = i_n(x) = \lambda b \otimes p \otimes b'$, with $b, b' \in B$.

Since $x' = i_n(x)$,
\[
x - \delta_{n+1} \otimes s_{n+1}(x) - s_n \circ \delta_n(x) = \pi_n(x') - \pi_n(f_{n+1} \circ i_{n+1} \circ \pi_{n+1} \circ S_{n+1}(x'))
= \pi_n(s_n \circ i_{n-1} \circ \pi_{n-1} \circ f_n(x')).
\]

The previous comments and Remark 4.0.1 allow us to write that
\[
\pi_n \circ f_{n+1} \circ (i_{n+1} \circ \pi' - i_{n+1} \circ \pi_{n+1}) \circ S_{n+1}(x') \in (\mathcal{L}_n^\infty(\mu q))_Z,
\]
\[
\pi_n \circ S_n \circ (i_{n-1} \circ \pi_{n-1} - i_{n-1} \circ \pi_{n-1}) \circ f_n(x') \in (\mathcal{L}_n^\infty(\mu q))_Z.
\]

It is then enough to prove that
\[
\pi_n(x' - f_{n+1} \circ i_{n+1} \circ \pi'_n \circ S_{n+1}(x') - S_n \circ i_{n-1} \circ \pi_{n-1} \circ f_n(x')) \in (\mathcal{L}_n^\infty(\mu q))_Z,
\]
but
\[
\pi_n(x' - f_{n+1} \circ i_{n+1} \circ \pi'_n \circ S_{n+1}(x') - S_n \circ i_{n-1} \circ \pi_{n-1} \circ f_n(x'))
= \pi'_n(x') - \delta_{n+1} \otimes s_{n+1}(\pi_n(x')) - s_n \circ \delta_n'(\pi_n(x'))
= 0.
\]

Finally, we deduce from Lemma 5.5 that
\[
\pi_n(x' - f_{n+1} \circ i_{n+1} \circ \pi'_n \circ S_{n+1}(x') - S_n \circ i_{n-1} \circ \pi_{n-1} \circ f_n(x')) \in (\mathcal{L}_n^\infty(\mu q))_Z.
\]

Next we prove another technical lemma that shows how to control the differentials.

**Lemma 5.7.** Fix $n \in \mathbb{N}_0$, let $R$ be either $k$ or $\mathbb{Z}$.

1. If $d : A \otimes E kA_n \otimes_E A \to A \otimes E kA_{n-1} \otimes_E A$ is a morphism of $A$-bimodules such that $(d - \delta_n)(1 \otimes p \otimes 1) \in (\mathcal{L}_{n-1}^\infty(\mu q))_R$ for all $p \in A_n$, then given $x \in (\mathcal{L}_n^\infty(\mu q))_R$, $(d - \delta_n)(x) \in (\mathcal{L}_n^\infty(\mu q))_R$ for all $\mu q \in k^\infty Q_{\geq 0}$.

2. If $p : A \otimes E kA_n \otimes_E A \to A \otimes E kA_{n+1} \otimes_E A$ is a morphism of $A - E$-bimodules such that $(p - s_n)(1 \otimes p \otimes \pi(b)) \in (\mathcal{L}_{n+1}^\infty(\mu q))_R$, for all $p \in A_n$ and $b \in B$, then for all $x \in (\mathcal{L}_n^\infty(\mu q))_R$, $(p - s_n)(x) \in (\mathcal{L}_n^\infty(\mu q))_R$ for all $\mu q \in k^\infty Q_{\geq 0}$.

**Proof.** Given $\mu q \in k^\infty Q_{\geq 0}$ and $x \in (\mathcal{L}_n^\infty(\mu q))_R$, let us see that $(d - \delta_n)(x) \in (\mathcal{L}_{n-1}^\infty(\mu q))_R$. It suffices to prove the statement for $x = \lambda \pi(b) \otimes p \otimes \pi(b') \in (\mathcal{L}_n^\infty(\mu q))_R$.

By hypothesis, $(d - \delta_n)(1 \otimes p \otimes 1) \in (\mathcal{L}_{n-1}^\infty(\mu q))_R$, so $(d - \delta_n)(x)$ equals $\lambda \pi(b)(d - \delta_n)(1 \otimes p \otimes 1)\pi(b')$ and it belongs to $(\mathcal{L}_{n-1}^\infty(\mu q))_R \subseteq (\mathcal{L}_{n-1}^\infty(\mu q))_R$, using Corollary 5.8.

The second part is analogous. \(\square\)

Next proposition will provide the remaining necessary tools for the proofs of Theorem 1.1 and Theorem 4.2.

**Proposition 5.8.** Fix $n \in \mathbb{N}_0$. Suppose that for each $i \in \{0, \ldots, n\}$ there are morphisms of $A$-bimodules $d_i : A \otimes E kA_i \otimes_E A \to A \otimes E kA_{i-1} \otimes_E A$ for $i \geq 0$, and $d_{-1} : A \otimes E A \to A$, and morphisms of $A - E$-bimodules $\rho_i : A \otimes E kA_i \otimes_E A \to A \otimes E kA_i \otimes_E A$.

If the following conditions hold,

1. $d_{i-1} \circ d_i = 0$ for all $i \in \{0, \ldots, n\}$,
2. $(d_i - \delta_i)(1 \otimes q \otimes 1) \in (\mathcal{L}^\infty_{i-1}(q))_n$ for all $i \in \{-1, \ldots, n\}$ and for all $q \in A_i$,
3. for all $i \in \{-1, \ldots, n-1\}$ and for all $x \in A \otimes E kA_i \otimes_E A$, $x = d_{i+1} \circ \rho_{i+1}(x) + \rho_i \circ d_i(x)$,
Proof. \(\delta \in \) which proves that \(d\) and using Lemma 5.7, \((\text{iv})\) then there exists a morphism of \(A \otimes E\) bimodules \(\rho_{n+1} : A \otimes E kA_n \otimes E A\) such that

(a) \(\mu_{n+1}(1 \otimes q \otimes 1) \in (\mathcal{L}_n^\ast(q))_R\),

then there exists a morphism of \(A \otimes E\) bimodules \(\rho_{n+1} : A \otimes E kA_n \otimes E A \to A \otimes E kA_{n+1} \otimes E A\) such that

(i) \(d_n \circ d_{n+1} = 0\),

(ii) \((d_{n+1} - \delta_{n+1})(1 \otimes q \otimes 1) \in (\mathcal{L}_n^\ast(q))_R\).

(2) there exists a morphism of \(A\)-bimodules \(d_{n+1} : A \otimes E kA_{n+1} \otimes E A \to A \otimes E kA_n \otimes E A\) such that

(i) \(d_n \circ d_{n+1} = 0\),

(ii) \((d_{n+1} - \delta_{n+1})(1 \otimes q \otimes 1) \in (\mathcal{L}_n^\ast(q))_R\).

Proof. In order to prove (2), fix \(q \in A_{n+1}\). By Lemma 5.3, \(\delta_{n+1}(1 \otimes q \otimes 1) \in (\mathcal{L}_n^\ast(q))_R\) and using Lemma 5.7, \((d_n - \delta_n)(1 \otimes q \otimes 1) \in (\mathcal{L}_n^\ast(q))_R\). Corollary 5.6 tells us that \(\delta_n \circ \delta_{n+1}(1 \otimes q \otimes 1) \in \mathcal{L}_n^\ast(q)\). We deduce from the equality

\[
d_n(\delta_{n+1}(1 \otimes q \otimes 1)) = \delta_n \circ \delta_{n+1}(1 \otimes q \otimes 1) + (d_n - \delta_n)(\delta_{n+1}(1 \otimes q \otimes 1))
\]

that \(d_n(\delta_{n+1}(1 \otimes q \otimes 1)) \in (\mathcal{L}_n^\ast(q))_R\).

Let us define \(d_{n+1} : A \otimes kA_{n+1} \otimes A \to A \otimes E kA_n \otimes E A\) by

\[
d_{n+1}(a, q, c) = a\delta_{n+1}(1 \otimes q \otimes 1)c - a\rho_n(d_n(\delta_{n+1}(1 \otimes q \otimes 1)))c,
\]

for \(a, c \in A\), \(q \in A_{n+1}\). The map \(d_{n+1}\) is \(E\)-multilinear and balanced, and it induces a unique map

\[
d_{n+1} : A \otimes kA_{n+1} \otimes E A \to A \otimes E kA_n \otimes E A\.
\]

It is easy to verify that \(d_{n+1}\) is in fact a morphism of \(A\)-bimodules. Putting together the equality \(\rho_n = s_n + (\rho_n - s_n)\) and Lemmas 5.3 and 5.7, we obtain that \((d_{n+1} - \delta_{n+1})(1 \otimes q \otimes 1) = -\rho_n \circ d_n \circ \delta_{n+1}(1 \otimes q \otimes 1) \in (\mathcal{L}_n^\ast(q))_R\). Moreover, given \(x \in A \otimes E kA_{n+1} \otimes E A\), \(x = d_n \circ \rho_n(x) + \rho_{n-1} \circ d_{n-1}(x)\), choosing \(x = d_n(\delta_{n+1}(1 \otimes q \otimes 1))\) yields the equality

\[
d_n \circ \delta_{n+1}(1 \otimes q \otimes 1) = d_n \circ \rho_n \circ d_{n+1}(1 \otimes q \otimes 1)
\]

which proves that \(d_n \circ d_{n+1} = 0\). For the proof of (1), fix \(q \in A_n\) and \(b \in B\). Using Lemmas 5.3 and 5.7, we deduce that the element

\[
1 \otimes q \otimes 1 - \rho_n \circ d_n(1 \otimes q \otimes 1) \in (\mathcal{L}_n^\ast(q))_R\)
\]

differs from \(1 \otimes q \otimes 1 - \rho_n \circ \delta_n(1 \otimes q \otimes 1)\) by elements in \((\mathcal{L}_n^\ast(qb))_R\). We will write that

\[
(id - \rho_n \circ \delta_n)(1 \otimes q \otimes 1) \equiv (id - \rho_n \circ \delta_n)(1 \otimes q \otimes 1)(\mathcal{L}_n^\ast(qb))_R\]

Also,

\[
(id - \rho_n \circ \delta_n)(1 \otimes q \otimes 1) \equiv (id - \rho_n \circ \delta_n)(1 \otimes q \otimes 1)(\mathcal{L}_n^\ast(qb))_R\]

\[
\equiv \delta_{n+1} \circ s_{n+1}(1 \otimes q \otimes 1)(\mathcal{L}_n^\ast(qb))_R\]

\[
\equiv d_{n+1} \circ s_{n+1}(1 \otimes q \otimes 1)(\mathcal{L}_n^\ast(qb))_R.
\]
We deduce from this that there exists a unique \( \xi \in (\mathcal{L}_n^\times (qb))_R \) such that

\[
(id - \rho_n \circ d_n)(1 \otimes q \otimes \pi(b)) = d_{n+1} \circ s_{n+1}(1 \otimes q \otimes \pi(b)) + \xi.
\]

It is evident that \( \xi \) belongs to the kernel of \( d_n \).

The order \( \preceq \) satisfies the descending chain condition, so we can use induction on \( (k^*Q_{\geq 0}, \preceq) \).

If there is no \( \lambda \rho \in k^*Q_{\geq 0} \) such that \( \lambda \rho < qb \), then \( \xi = 0 \) and we define \( \rho_{n+1}(1 \otimes q \otimes \pi(b)) = s_{n+1}(1 \otimes q \otimes \pi(b)) \).

Inductively, suppose that \( \rho_n(\xi) \) is defined. The equality \( d_n(\xi) = 0 \) implies that \( \xi = d_{n+1} \circ \rho_{n+1}(\xi) \) and

\[
(id - \rho_n \circ d_n)(1 \otimes q \otimes \pi(b)) = d_{n+1}(s_{n+1}(1 \otimes q \otimes \pi(b)) + \rho_{n+1}(\xi)).
\]

We define \( \rho_{n+1}(1 \otimes q \otimes \pi(b)) := s_{n+1}(1 \otimes q \otimes \pi(b)) + \rho_{n+1}(\xi) \).

Lemmas 5.4 and 5.7 assure that \( \rho_{n+1}(\xi) \) belongs to \( (\mathcal{L}_{n+1}^\times (qb))_R \), and as a consequence

\[
\rho_{n+1}(1 \otimes q \otimes \pi(b)) - s_{n+1}(1 \otimes q \otimes \pi(b)) \in (\mathcal{L}_{n+1}^\times (qb))_R.
\]

We are now ready to prove the theorems.

**Proof of Theorem 4.1.** We need to prove the existence of an \( A - E \)-bimodule map \( \rho_0 : A \otimes_E kA_{-1} \otimes_E A \rightarrow A \otimes_E kA_0 \otimes_E A \) satisfying \( d_N \circ \rho_0 + \rho_{-1} \circ d_{-1} = id \), where \( d_{-1} = \mu \) and \( \rho_{-1}(a) = s_{-1}(a) = a \otimes 1 \) for all \( a \in A \). Once this achieved, we apply Proposition 5.8 inductively with \( R = k \), for all \( n \) such that \( 0 \leq n \leq N - 1 \), obtaining this way an homotopy retraction of the complex

\[
A \otimes_E kA_N \otimes_E A \xrightarrow{d_N} \cdots \xrightarrow{d_0} A \otimes_E A \xrightarrow{d_{-1}} A \longrightarrow 0
\]

proving thus that it is exact.

Given \( b = b_k \cdots b_1 \in B \), with \( b_i \in Q_1 \), \( 1 \leq i \leq k \),

\[
s_0(1 \otimes \pi(b)) = -\sum_{i=1}^{k} \pi(b_k \cdots b_{k-i+1}) \otimes b_{k-i} \otimes \pi(b_{k-i-1} \cdots b_1).
\]

On one hand \( 1 \otimes \pi(b) - \pi(b) \otimes 1 = 1 \otimes \pi(b) - s_{-1}(d_{-1}(1 \otimes \pi(b))) \) and on the other hand the left hand term equals \( \delta_0(s_0(1 \otimes \pi(b))) \), yielding \( 1 \otimes \pi(b) - s_{-1}(1 \otimes \pi(b)) = \delta_0(s_0(1 \otimes \pi(b))) \). By hypothesis, \( (d_0 - \delta_0)(1 \otimes \pi(b)) \) belongs to \( (\mathcal{L}_{-1}^\times (b))_k \), and so there exists \( \xi \in (\mathcal{L}_{-1}^\times (b))_k \) such that

\[
1 \otimes \pi(b) - s_{-1}(d_{-1}(1 \otimes \pi(b))) = d_0(s_0(1 \otimes \pi(b))) + \xi.
\]

It follows that \( d_{-1}(\xi) = 0 \). Suppose first that there exists no \( \lambda \rho \in k^*Q_{\geq 0} \) such that \( \lambda \rho < b \).

In this case \( \xi = 0 \) and we define \( \rho_0(1 \otimes \pi(b)) = s_0(1 \otimes \pi(b)) \). Inductively, suppose that \( \rho_0(\xi) \) is defined for any \( \xi \) such that \( d_{-1}(\xi) = 0 \). Since in this case \( \xi = d_0(\rho_0(\xi)) \), we set \( \rho_0(1 \otimes \pi(b)) := s_0(1 \otimes \pi(b)) + \rho_0(\xi) \).

**Proof of Theorem 4.2.** It follows from the proof of Theorem 4.1 that

\[
1 \otimes \pi(b) = (s_{-1} \circ d_{-1} + \delta_0 \circ s_0)(1 \otimes \pi(b))
\]

and so \( s_{-1} \circ d_{-1} + \delta_0 \circ s_0 = id_{A_{-1} \otimes_E A} \). Setting \( d_0 := \delta_0 \), the theorem follows applying Proposition 5.8 for \( R = Z \).

We finish this section showing that this construction is a generalization of Bardzell’s resolution for monomial algebras.

**Proposition 5.9.** Given an algebra \( A \), let \( (A \otimes_E kA_n \otimes_E A, d_n) \) be a resolution of \( A \) as \( A \)-bimodule such that \( d_n \) satisfies the hypotheses of Theorem 4.1. If \( p \in A_n \) is such that \( r(p) = 0 \) or \( r(p) = p \), then for all \( a, c \in kQ \),

\[
d_n(\pi(a) \otimes p \otimes \pi(c)) = \delta_n(\pi(a) \otimes p \otimes \pi(c)).
\]
Proof. By hypothesis, there exists no \( \lambda \cdot p' \in k^x Q_{\geq 0} \) such that \( \lambda \cdot p' \prec p \), so \( {\mathcal L}^+_{-1}(p) = \{0\} \) and \( d_n(1 \otimes p \otimes 1) = \delta_n(1 \otimes p \otimes 1) \). Given \( a, c \in kQ \) we deduce from the previous equality that
\[
d_n(\pi(a) \otimes p \otimes \pi(c)) = \delta_n(\pi(a) \otimes p \otimes \pi(c)) = \pi(a)(d_n(1 \otimes p \otimes 1) - \delta_n(1 \otimes p \otimes 1))\pi(c) = 0.
\]

\( \square \)

Corollary 5.10. Suppose the algebra \( A = kQ/I \) has a monomial presentation. Choose a reduction system \( R \) whose pairs have the monomial relations generating the ideal \( I \) as first coordinate and 0 as second coordinate. In this case, the only maps \( d \) verifying the hypotheses of Theorem 4.2 are those in Bardzell's resolution.

6. Morphisms in low degrees

In this section we describe the morphisms appearing in the lower degrees of the resolution.

Let us consider the following data: an algebra \( A = kQ/I \) and a reduction system \( R \) satisfying condition \( \triangleleft \).

We start by recalling the definition of the maps \( \delta_0 \) and \( \delta_{-1} \). For \( a, c \in kQ, \alpha \in Q_1 \),
\[
\delta_0 : A \otimes_E kA_0 \otimes_E A \to A \otimes_E A, \quad \delta_{-1}(\pi(a) \otimes \pi(c)) = \pi(ac)
\]
and
\[
\delta_0 : A \otimes_E kA_0 \otimes_E A \to A \otimes_E A, \quad \delta_0(\pi(a) \otimes \alpha \otimes \pi(c)) = \pi(aa) \otimes \pi(c) - \pi(a) \otimes \pi(ac).
\]

Definition 6.1. We state some definitions

- Let \( \phi_0 : kQ \to A \otimes_E kA_0 \otimes_E A \) be the \( k \)-linear map defined by
\[
\phi_0(c) = \sum_{i=1}^{n} \pi(c_e \cdots c_{i+1}) \otimes c_i \otimes \pi(c_{i-1} \cdots c_1)
\]
for \( c \in Q_{\geq 0}, c = c_n \cdots c_1 \) with \( c_i \in Q_1 \) for all \( i, 1 \leq i \leq n \) and extended by linearity.

- Given a basic reduction \( r = r_{a,s,c} \), let \( \phi_1(r,-) : kQ \to A \otimes_E kA_1 \otimes_E A \) be the unique \( k \)-linear map such that, given \( p \in Q_{\geq 0} \)
\[
\phi_1(r,p) = \begin{cases} 
\pi(a) \otimes s \otimes \pi(c), & \text{if } p = \text{asc}, \\
0 & \text{if not}.
\end{cases}
\]

In case \( r = (r_n, \ldots, r_1) \) is a reduction, where \( r_i \) is a basic reduction for all \( i, 1 \leq i \leq n \), we denote \( r' = (r_n, \ldots, r_2) \) and we define in a recursive way the map \( \phi(r, -) \) as the unique \( k \)-linear map from \( kQ \) to \( A \otimes_E kA_1 \otimes_E A \) such that
\[
\phi_1(r,p) = \phi_1(r_1,p) + \phi_1(r', r_1(p)).
\]

- Finally, we define an \( A \)-bimodule morphism \( d_1 : A \otimes_E kA_1 \otimes_E A \to A \otimes_E kA_0 \otimes_E A \) by the equality
\[
d_1(1 \otimes s \otimes 1) = \phi_0(s) - \phi_0(\beta(s)), \quad \text{for all } s \in A_1.
\]

Next we prove four lemmas necessary to the description of the complex in low degrees.

Lemma 6.2. Let us consider \( p \in Q_{\geq 0} \) and \( x \in kQ \) such that \( x \prec p \). For any reduction \( r \) the element \( \phi_1(r,x) \) belongs to \( \langle \mathcal{L}_1(p) \rangle_z \).

Proof. We will first prove the result for \( x = \mu q \in k^x Q_{\geq 0} \). The general case will then follow by linearity. Fix \( x = \mu q \in k^x Q_{\geq 0} \). We will use an inductive argument on \( (k^x Q_{\geq 0}, \preceq) \).

To start the induction, suppose first that there exists no \( \mu' q' \in k^x Q_{\geq 0} \) and that \( \mu' q' \prec \mu q = x \).

In this case, every basic reduction \( r_{a,s,c} \) satisfies either \( r_{a,s,c}(x) = x \) or \( r_{a,s,c} = 0 \). In the first case, \( asc \neq q \) and so \( \phi_1(r_{a,s,c},x) = 0 \). In the second case, \( asc = q \), so \( \phi_1(r_{a,s,c},x) = \mu(\pi(a) \otimes s \otimes \pi(c)) \).

Given an arbitrary reduction \( r = (r_n, \ldots, r_1) \) with \( r_i \) basic for all \( i \), there are three possible cases.
\( \text{(1) } r_1(x) = x \text{ and } n > 1, \)
\( \text{(2) } r_1(x) = x \text{ and } n = 1, \)
\( \text{(3) } r_1(x) = 0. \)

Denote \( r' = (r_n, \ldots, r_2) \) as before and \( r_1 = r_{a,s,c}. \) In case \( 1, \) \( \phi_1(r, x) = \phi_1(r', x). \) In case \( 2, \) \( \phi_1(r, x) = \phi_1(r_1, x) = 0. \) Finally, in case \( 3, \) \( \phi_1(r, x) = \phi_1(r_1, x) = \mu \pi(a) \otimes s \otimes \pi(c). \) Using Lemma 5.2, we obtain that in all three cases \( \phi_1(r, x) \in (\mathcal{T}_1(p))_{\mathbb{Z}}. \)

Next, suppose that \( x = \mu q \) and that the result holds for \( \mu' q' \in k^2 \mathbb{Q}_{\geq 0} \) such that \( \mu' q' < \mu q = x. \)

Let us consider \( r, r_1 \) and \( r' \) as before. Again, there are three possible cases:

\( \text{(1) } asc = q, \)
\( \text{(2) } asc \neq q \text{ and } n > 1, \)
\( \text{(3) } asc \neq q \text{ and } n = 1. \)

Case \( 3 \) is immediate, since in this situation \( \phi_1(r, x) = 0. \) The second case reduces to the other ones, since \( \phi_1(r, x) = \phi_1(r', x) \) in the first case,

\( \phi_1(r, x) = \mu \pi(a) \otimes s \otimes \pi(c) + \phi_1(r', r_1(x)). \)

We know that \( r_1(x) < x, \) and we may write it as a finite sum \( r_1(x) = \sum_i \mu_i q_i. \) Using the inductive hypothesis, we deduce that \( \phi_1(r, x) \in (\mathcal{T}_1(p))_{\mathbb{Z}}. \) \( \square \)

Lemma 6.3. For all \( x \in A \otimes E kA \otimes E A, x \) belongs to the kernel of \( \delta_0 \circ d_1(x). \)

**Proof.** Since these maps are morphisms of \( A \)-bimodules, we may suppose \( x = 1 \otimes s \otimes 1, \) with \( s \in A_k. \) A direct computation gives

\( \delta_0(d_1(1 \otimes s \otimes 1) = \delta_0(\phi_0(s) - \phi_0(\beta(s))) = \pi(s) \otimes 1 - 1 \otimes \pi(s) - \pi(\beta(s)) \otimes 1 \otimes \pi(\beta(s)) = 0. \)

\( \square \)

Lemma 6.4. Given \( a, c \in \mathbb{Q}_{\geq 0} \) and \( p = \sum_{i=1}^n \lambda_i p_i \in kQ, \) with \( p_i \in \mathbb{Q}_{\geq 0} \) for all \( i, \) we obtain the equality

\( \phi_0(apc) = \phi_0(a)\pi(pc) + \pi(a)\phi_0(p)\pi(c) + \pi(ap)\phi_0(0). \)

The proof is immediate using the definition of \( \phi_0 \) and \( k \)-linearity of \( \phi_0 \) and \( \pi. \)

Next we state the last one of the preparatory lemmas.

Lemma 6.5. Given \( p \in \mathbb{Q}_{\geq 0} \) and a reduction \( r = (r_n, \ldots, r_1), \) with \( r_i \) a basic reduction for all \( i \) such that \( 1 \leq i \leq n, \)

\( d_1(\phi_1(r_1(p))) = \phi_0(p) - \phi_0(r(p)). \)

**Proof.** We will prove the result by induction on \( n. \) We will denote \( r_i = r_{a_i, s_i, c_i}. \)

For \( n = 1, \) there are two cases. The first one is when \( p \neq a_1 s_1 c_1. \) In this situation, \( r(1) = r_1(p) = p, \phi_1(r_1, p) = 0 \) and so the equality is trivially true. In the second case, \( p = a_1 s_1 c_1, \phi_1(r_1, p) = \pi(a_1) \otimes s_1 \otimes \pi(c_1) \) and \( r(p) = r_1(p) = a_1 \beta(s_1) c_1. \) Let us compute \( d_1(\phi_1(r_1, p) + \phi_0(r_1(p))): \)

\( d_1(\phi_1(r_1, p)) + \phi_0(r_1(p)) = d_1(\pi(a_1) \otimes s_1 \otimes \pi(c_1)) + \phi_0(a_1 \beta(s_1) c_1) \)

\( = \pi(a_1) \phi_0(s_1) \pi(c_1) - \pi(a_1) \phi_0(\beta(s_1)) \pi(c_1) + \phi_0(a_1 \beta(s_1) c_1). \)

Using Lemma 6.4 last term equals

\( \phi_0(a_1) \pi(\beta(s_1) c_1) + \pi(a_1) \phi_0(\beta(s_1)) \pi(c_1) + \pi(a_1 \beta(s_1)) \phi_0(c_1), \)

so the whole expression is

\( \pi(a_1) \phi_0(s_1) \pi(c_1) + \phi_0(a_1) \pi(\beta(s_1) c_1) + \pi(a_1 \beta(s_1)) \phi_0(c_1) \)

\( = \pi(a_1) \phi_0(s_1) \pi(c_1) + \phi_0(a_1) \pi(s_1 c_1) + \pi(a_1 s_1) \phi_0(c_1), \)
and again by Lemma [6,3] this equals $\phi_0(p)$. Suppose that the result holds for $n - 1$. As usual, we denote $r' = (r_n, \ldots, r_2)$.

Since $r(p) = r'(r_1(p))$,

$$d_1(\phi_1(r, p)) + \phi_0(r(p)) = d_1(\phi_1(r_1, p)) + d_1(\phi_1(r', r_1(p))) + \phi_0(r'(r_1(p))) = d_1(\phi_1(r_1, p)) + \phi_0(r_1(p)) = \phi_0(p).$$

□

Consider now an element $p \in A_2$. By definition we write $p = u_0u_1u_2 = v_2v_1v_0$ where $u_0u_1$ and $v_1v_0$ are paths in $A_1$ dividing $p$. Suppose $r = r_{a.s.c}$ is a basic reduction such that $r(p) \neq p$. We deduce that either $s = u_0u_1$ or $s = v_1v_0$. For an arbitrary reduction $r = (r_n, \ldots, r_1)$, we will say that $r$ starts on the left of $p$ if $r = r_{a.s.c}$, $s = u_0u_1$ and asc $p$, and we will say that $r$ starts on the right of $p$ if $r_1 = r_{a.s.c}$, $s = v_1v_0$ and asc $p$.

**Proposition 6.6.** Given any $p \in A_2$, let $\{r^p\}_{p \in A_2}$ and $\{t^p\}_{p \in A_2}$ be two sets of reductions such that $r^p(p)$ and $t^p(p)$ belong to $kE$, $r^p$ starts on the left of $p$ and $t^p$ starts on the right of $p$. Consider $d_2 : A \otimes_E kA_2 \otimes_E A \to A \otimes_E kA_1 \otimes_E A$ the map of $A$-bimodules defined by $d_2(1 \otimes p \otimes 1) = \phi_1(t^p, p) - \phi_1(r^p, p)$.

The sequence

$$A \otimes_E kA_2 \otimes_E A \xrightarrow{d_2} A \otimes_E kA_1 \otimes_E A \xrightarrow{d_1} A \otimes_E kA_1 \otimes_E A \xrightarrow{\delta_0} A \otimes_E A \xrightarrow{t - 1} A \xrightarrow{0} 0$$

is exact.

**Proof.** Let us first check that $d_2$ is well defined. Consider the map $\tilde{d}_2 : A \times kA_2 \times A \to A \otimes_E kA_1 \otimes_E A$ defined by $\tilde{d}_2(x, p, y) = x\phi_1(t^p(p)y) - x\phi_1(r^p(p)y)$, for all $x, y \in A$. The map $\tilde{d}_2$ is clearly multilinear, and taking into account the definition of $\phi_1$ it is such that $d_2(x, p, y) = \tilde{d}_2(x, ep, y)$ and $d_2(x, p, y) = \tilde{d}_2(x, p, ey)$ for all $e \in E$, so it induces $d_2$ on $A \otimes_E kA_2 \otimes_E A$.

- $\delta_{-1} \circ \delta_0 = 0$ and $\delta_0 \circ d_1 = 0$ follow from Lemma [6.3].
- Given $p \in A_2$, $d_1(d_2(1 \otimes p \otimes 1)) = d_1(\phi_1(t^p, p) - \phi_1(r^p, p))$. Using Lemma [6.5], this last expression equals $\phi_0(p) - \phi_0(t^p(p)) - \phi_0(p) + \phi_0(r^p(p))$, which is, by Remark [22.1], equal to $-\phi_0(\beta(p)) + \phi_0(\beta(p))$, so $d_1 \circ d_2 = 0$.
- Given $s \in A_1$, $d_1(1 \otimes s \otimes 1) - \delta_1(1 \otimes s \otimes 1)$ belongs to $(\overline{L}_0^<(s))_k$: indeed, notice that $\delta_1(1 \otimes s \otimes 1) = \phi_0(s)$, and $\phi_0(\beta(s))$ belongs to $(\overline{L}_0^<(p))_k$ since $\beta(s) \prec s$. It follows that

$$d_1(1 \otimes s \otimes 1) - \delta_1(1 \otimes s \otimes 1) = -\phi_0(\beta(s)) \in (\overline{L}_0^<(s))_k.$$

- Given $p \in A_2$, we will prove now that $(d_2 - \delta_2)(1 \otimes p \otimes 1)$ belongs to $(\overline{L}_0^<(p))_k$. We may write $p = u_0u_1u_2 = v_2v_1v_0$, as we did just before this proposition and we thus write $\delta_1(1 \otimes p \otimes 1) = \pi(v_2) \otimes v_1v_0 \otimes 1 - 1 \otimes u_0u_1 \otimes \pi(v_2)$. Besides, if $r^p = (r_n, \ldots, r_1)$ and $t^p = (t_m, \ldots, t_1)$ with $t_i$ and $r_j$ basic reductions, the fact that $r^p$ starts on the left of $p$ and $t^p$ starts on the right of $p$ gives

$$(d_2 - \delta_2)(1 \otimes p \otimes 1) = \phi_1(t^p, t_1(p)) - \phi_1(r^p, r_1(p)),$$

where $t^p = (t_m, \ldots, t_2)$ and $r^p = (r_n, \ldots, r_2)$. Since $t_1(p) \prec p$ and $r_1(p) \prec p$, Lemma [6.2] allows us to deduce the result.

Finally, Theorem [4.1] implies that the sequence considered is exact. □
Remark 6.6.1. Given $a \in \mathcal{A}_0 = Q_1$, $\mathcal{L}_{-1}^{-1}(a) = \emptyset$, so for any morphism of $A$-bimodules $d : A \otimes_E k\mathcal{A}_0 \otimes_E A \to A \otimes_E k\mathcal{A}_{-1} \otimes_E A$ such that $(d - \delta_0)(1 \otimes a \otimes 1)$ belongs to $(\mathcal{L}_{-1}^{-1}(a))_e$, we have $d = \delta_0$.

On the other hand, given $s \in \mathcal{A}_1$, write $\beta(s) = \sum \lambda_i b_i$. Let $r = r_{a,s,c}$ be a basic reduction such that $r(s) \neq s$. We must have $s' = s$ and $a, c \in Q_0$ must coincide with the source and target of $s$ respectively. In other words, the only basic reduction such that $r(s) \neq s$ is $r_{a,s,c}$ with $a$ and $c$ as we just said, and in this case $r(s) = \beta(s) \in k\mathcal{B}$.

In this situation
\[ \{ \lambda q \in k^x Q_{\geq 0} : s \prec q \} = \{ \lambda_1 b_1, \ldots, \lambda_m b_m \}, \]
and writing $b_i = b_i^0 \cdots b_i^1$ with $b_i^j \in Q_1$,
\[ \mathcal{L}_{-1}^{-1}(s) = \bigcup_{i=1}^{N} \{ \lambda_i \pi(b_i^0 \cdots b_i^1 \otimes 1, \ldots, \lambda_i \otimes b_i^m \otimes \pi(b_i^{m+1} \cdots b_i^1) \}. \]
If $d : A \otimes_E k\mathcal{A}_1 \otimes_E A \to A \otimes_E k\mathcal{A}_0 \otimes_E A$ verifies $(d - \delta_1)(1 \otimes s \otimes 1) \in \mathcal{L}_{-1}^{-1}(s)$ and $\delta_0 \circ d(s) = 0$ for all $s \in \mathcal{A}_1$, then there exists $\gamma_i^j \in k$ such that
\[ d(1 \otimes s \otimes 1) = \phi_0(s) - \sum_{i=1}^{n} \sum_{j=1}^{m} \gamma_i^j \lambda_i \pi(b_i^{n+1} \cdots b_i^j \cdot \cdots b_i^1 \otimes b_i^j \otimes \pi(b_i^{j+1} \cdots b_i^1)). \]
From this, applying $\delta_0$ and reordering terms we can deduce that $\gamma_i^j = 1$ for all $i, j$. We conclude that the unique morphism with the desired properties is $d_1$.

7. Examples

In this section we construct explicitly projective bimodule resolutions of some algebras using the methods we developed in previous sections.

According to Lemmas 2.9 and 2.10, it is always possible to construct a reduction system $\mathcal{R}$ such that every path is reduction-unique. However, it is not always easy to follow the prescriptions given by these lemmas for a concrete algebra. In fact, describing the set $\text{tip}(I)$ is not in general an easy task.

Bergman’s Diamond Lemma is the tool we use to effectively compute a reduction system in most cases. Next we sketch this procedure, which is also described in [2], Section 5.

The two sided ideal $I$ is usually presented giving a set $\{ x_i \}_i \in \Gamma \subseteq kQ$ of generating relations. If we fix a well order on $Q_0 \cup Q_1$, a function $\omega : Q_1 \to \mathbb{N}$ and consider the total order $\leq_{\omega}$ on $Q_{\geq 0}$, we can easily write $x_i = s_i - f_i$, and we can eventually rescale $x_i$ so that $s_i$ is monic, with $s_i > \omega f_i$ for all $i$ and define the reduction system $\mathcal{R} = \{(s_i, f_i)\} \in \Gamma$. Every path $p$ will be reduction-finite with respect to $\mathcal{R}$. Bergman’s Diamond Lemma says that every path is reduction-unique if and only if for every path $p \in \mathcal{A}_2$ there are reductions $r, t$ with $r$ starting on the left and $t$ starting on the right of $p$ such that $r(p) = t(p)$. This last situation is described by saying that $p$ is resolvable. The set $\mathcal{A}_2$ is usually finite and so there is a finite number of conditions to check.

In case there exists a non resolvable ambiguity $p \in \mathcal{A}_2$, choose any two reductions $r, t$ starting on the left and on the right respectively with $r(p)$ and $t(p)$ both irreducible. The element $r(p) - t(p)$ belongs to $\mathcal{I} \setminus \{ 0 \}$. We can write $r(p) - t(p) = s - f$ with $s < \omega s$ and add the element $(s, f)$ to our reduction system, and so $p$ is now resolvable. New ambiguities may now appear, so it is necessary to iterate this process, which may have infinitely many steps, but we will arrive at a reduction system $\mathcal{R}$ satisfying condition $(\Diamond)$.

We give an example to illustrate this procedure. Consider the algebra of Example 2.10.1. Let $x < y < z$ and $\omega(x) = \omega(y) = \omega(z) = 1$. The ideal $I$ is presented as the two sided ideal generated by the element $x^3 + y^3 + z^3 - xyz$. We see that $z^3 = \text{tip}(z^3 - (xyz - x^3 - y^3))$, so we
start considering the reduction system \( R = \{ (z^3, xyz - x^3 - z^3) \} \). Notice that \( A_2 = \{ z^4 \} \). If we apply the reduction \( r_{2,z^3,z} \) to \( z^4 \) we obtain \( xyz = y^4x^3 - zy^3 \) which is irreducible. On the other side, if we apply the reduction \( r_{1, z, z} \) to \( z^4 \) we obtain \( xyz = x^3z - y^3z^2 \) which is also irreducible and different from the first one. The difference between them is \( y^2z + x^3z^2 - zx^3 - zy^3 \), so we add the element \( (y^2z, x^3z^2 + zy^3 - x^3z^2) \) to the reduction system \( R \). Notice that now the set \( A_2 \) is \( \{ z^4, y^2z^3 \} \). Applying reductions on the left and on the right to the element \( y^3z^3 \) we obtain again two different irreducible elements and, proceeding as before, we see that we have to add the element \( (y^2xyz, x^4yz - xyz^3 + xzyy^3) \) to our reduction system \( R \), and no new ambiguities appear. We have arrived to the reduction system

\[
R_1 = \{ (z^3, xyz - x^3 - z^3), (y^3z, zx^3 + zy^3 - x^3z), (y^2zxyz, x^4yz - xyzx^3 + xzyy^3) \},
\]

which satisfies condition (\( \Diamond \)).

There is another reduction system for this algebra, namely \( R_2 = \{ (xyz, x^3 + y^3 + z^3) \} \). Let us denote \( A^1_n \) and \( A^2_n \) the respective set of \( n \)-ambiguities. Notice that \( z^{2(n+1)} \in A^1_n \) for odd and \( z^{2n+1} \in A^1_n \) for even, so \( A^1_n \) is not empty for all \( n \in \mathbb{N} \). On the other hand, \( A^2_n \) is empty for all \( n \geq 2 \). We conclude that using \( R_2 \) we will obtain a resolution of length 2, with differentials given explicitly by Proposition 6.3 and using \( R_1 \) the resolution obtained will have infinite length. This shows how different can the resolutions for different reduction systems.

Notice that \( R_2 \) cannot be obtained by the procedure described above by any choice of order on \( Q_0 \cup Q_1 \) and weight \( \omega \).

### 7.1. The algebra counterexample to Happel’s question

Let \( \xi \) be an element of the field \( k \) and let \( A \) be the \( k \)-algebra with generators \( x \) and \( y \), subject to the relations \( x^2 = 0 = y^2, yx = \xi xy \). Choose the order \( x < y \) with weights \( \omega(x) = \omega(y) = 1 \) and fix the reduction system \( R = \{ (x^2, \emptyset), (y^2, \emptyset), (yx, \xi xy) \} \). The set \( B \) of irreducible paths is thus \( \{ 1, x, y, xy \} \). It is easy to verify that \( A_2 = \{ x^2, yx^2, y^2x, y^3 \} \) and that all paths in \( A_2 \) are reduction-unique. Bergman’s Diamond Lemma guarantees that \( R \) satisfies (\( \Diamond \)).

The only possible path of length 2 not in \( S \) is \( xy \); Proposition 3.4 implies that for each \( n, A_n \) is the set of paths of length \( n + 1 \) not divisible by \( xy \),

\[
A_n = \{ y^s x^t : s + t = n + 1 \}.
\]

**Lemma 7.1.** The following complex provides the beginning of an \( A \)-bimodule projective resolution of the algebra \( A \)

\[
A \otimes_E kA_2 \otimes_E A \xrightarrow{d_2} A \otimes_E kA_1 \otimes_E A \xrightarrow{d_1} A \otimes_E kA_0 \otimes_E A \xrightarrow{d_0} A \otimes_E A \xrightarrow{d_{-1}} \cdots
\]

where \( d_1 \) is the \( A \)-bimodule map such that

\[
d_1(1 \otimes x^2 \otimes 1) = x \otimes x \otimes 1 - 1 \otimes x \otimes x,
\]

\[
d_1(1 \otimes y^2 \otimes 1) = y \otimes y \otimes 1 - 1 \otimes y \otimes y,
\]

\[
d_1(1 \otimes xy \otimes 1) = y \otimes x \otimes 1 + 1 \otimes y \otimes x - x \otimes y \otimes 1 - 1 \otimes x \otimes y
\]

and \( d_2 \) is the \( A \)-bimodule morphism such that

\[
d_2(1 \otimes y^3 \otimes 1) = y \otimes y^2 \otimes 1 - 1 \otimes y^2 \otimes 1,
\]

\[
d_2(1 \otimes y^2x \otimes 1) = y \otimes yx \otimes 1 + \xi \otimes xy \otimes y + \xi^2 x \otimes y^2 \otimes 1 - 1 \otimes y^2 \otimes x,
\]

\[
d_2(1 \otimes yx^2 \otimes 1) = y \otimes x^2 \otimes 1 - 1 \otimes yx \otimes x - \xi x \otimes yx \otimes 1 - \xi^2 x \otimes y^2 \otimes y
\]

\[
d_2(1 \otimes x^3 \otimes 1) = x \otimes x^2 \otimes 1 - 1 \otimes x^2 \otimes x.
\]
Proof. We apply Proposition 6.6 to the following sets \( \{r^p\}_{p \in A_2} \) of left reductions and \( \{t^p\}_{p \in A_2} \) of right reductions, where

\[
\begin{align*}
    r^{y^3} &= r_{1,y^2,y}, & r^{y^2x} &= r_{1,y^2,x}, \\
    r^{yx^2} &= (r_{1,x^2,y}, r_{x,yx,1}, r_{1,yx,x}), & r^x &= r_{1,x^2,x}.
\end{align*}
\]

One can find an \( A \)-bimodule resolution of \( A \) in [BGMS] and in [BE]; the authors also compute the Hochschild cohomology of \( A \). We recover this resolution with our method.

Given \( q \in A_n \), there are \( s, t \in \mathbb{N} \) such that \( s + t = n + 1 \) and \( q = y^sx^t \). Suppose \( q = apc \) with \( p = y^sx^t \in A_{n-1} \) and \( a, c \in Q_{\geq 0} \). Since \( s + t = n + 1 \) and \( s' + t' = n \), either \( a \) belongs to \( Q_0 \) and \( c = x \) or \( a = y \) and \( c \in Q_0 \). As a consequence of this fact, the maps \( \delta_n : kQ \otimes_E kA_n \otimes_E kQ \to kQ \otimes_E kA_{n-1} \otimes_E A \) are

\[
\delta_n(1 \otimes y^s x^t \otimes 1) = \begin{cases} 
    y \otimes y^{s-1} x^t \otimes 1 + (-1)^{n+1} \otimes y^s x^{t-1} \otimes x & \text{if } s \neq 0 \text{ and } t \neq 0, \\
    y \otimes y^n \otimes 1 + (-1)^{n+1} \otimes y^s \otimes y, & \text{if } t = 0, \\
    x \otimes x^n \otimes 1 + (-1)^{n+1} \otimes x^s \otimes x, & \text{if } s = 0, 
\end{cases}
\]

Moreover, given a basic reduction \( r = r_{s,t,n} \), the fact that \( s \) belongs to \( S \) implies that \( r(y^s x^t) \) is either 0 or \( \xi^{-1} y_{xy} x^{-1} \). Considering the reduction system \( R \), if \( s \neq 0 \) and \( t \neq 0 \), then

\[
\mathcal{T}^\xi_{n-1} (y^s x^t) = \{ \xi \otimes y^s x^{t-1} \otimes 1, \xi \otimes y^{s-1} x^t \otimes y \}.
\]

In case \( s = 0 \) or \( t = 0 \), the set \( \mathcal{T}^\xi_{n-1} (y^s x^t) \) is empty.

The computation of \( d_2 - \delta_2 \) suggests the definition of the maps

\[
d_n : A \otimes_E kA_n \otimes_E A \to A \otimes_E kA_{n-1} \otimes_E A
\]

as follows

\[
d_n(1 \otimes y^s x^t \otimes 1) = \delta_n(1 \otimes y^s x^t \otimes 1) + \epsilon(\xi \otimes y^s x^{t-1} \otimes 1 + \xi \otimes y^{s-1} x^t \otimes y)
\]

where \( \epsilon \) denotes a sign depending on \( s,t,n \). The equality \( d_{n-1} \circ d_n = 0 \) shows that making the choice \( \epsilon = (-1)^s \) does the job.

Finally, Theorem 4.11 shows that the complex

\[
\cdots \to A \otimes_E kA_n \otimes_E A \xrightarrow{d_n} A \otimes_E kA_{n-1} \otimes_E A \to A \otimes_E A \to 0
\]

with

\[
d_n(1 \otimes y^s x^t \otimes 1) = y \otimes y^{s-1} x^t \otimes 1 + (-1)^{n+1} \otimes y^s x^{t-1} \otimes x + (-1)^s \xi \otimes y^{s-1} x^t \otimes y
\]

is a projective bimodule resolution of \( A \).

### 7.2. Quantum complete intersections

These algebras generalize the previous case. Instead of the relations \( x^2 = 0 = y^2 \), \( yx = \xi xy \), we have \( x^n = 0 = y^n \), \( yx = \xi xy \), where \( n \) and \( m \) are fixed positive integers, \( n, m > 1 \).

We still denote the algebra by \( A \). Consider the order \( x < y \) with weights \( \omega(x) = \omega(y) = 1 \). The set of \( 2 \)-ambiguities associated to the reduction system \( R = \{(x^n,0),(y^m,0),(yx,\xi xy)\} \) is \( A_2 = \{y^{m+1},y^m.x,y.x^n,x^{n+1}\} \), and the set of irreducible paths is \( \mathcal{B} = \{x^iy^j \in k\langle x,y \rangle : 0 \leq i \leq n-1, 0 \leq j \leq m-1 \} \). We easily check that every path in \( A_2 \) is reduction-unique and using Bergman’s Diamond Lemma, we conclude that \( R \) satisfies (\( \diamond \)). Also, \( A_1 = S = \{y^m,xy,x^n\} \) and \( A_3 = \{y^m.x,y^m.x^n,y.x^{n+1},x^{2n}\} \).
Denote by \( \varphi : \mathbb{N}_0^2 \rightarrow \mathbb{N}_0 \) the map
\[
\varphi(s, n) = \begin{cases} 
\frac{s}{2} & \text{if } s \text{ is even}, \\
\frac{s - 1}{2} + n + 1 & \text{if } s \text{ is odd}.
\end{cases}
\]

Given \( N \in \mathbb{N} \), the set of \( N \)-ambiguities is \( \mathcal{A}_N = \{ y^{\varphi(s,m)} x^{\varphi(t,n)} : s + t = N + 1 \} \). We will sometimes write \((s,t)\) instead of \( y^{\varphi(s,m)} x^{\varphi(t,n)} \in \mathcal{A}_N \).

We first compute the beginning of the resolution.

**Lemma 7.2.** The following complex provides the beginning of a projective resolution of \( A \) as \( A \)-bimodule:
\[
A \otimes_E kA_2 \otimes_E A \xrightarrow{d_2} A \otimes_E kA_1 \otimes_E A \xrightarrow{d_1} A \otimes_E kA_0 \otimes_E A \xrightarrow{d_0} A \otimes_E E \xrightarrow{\delta_1} A \xrightarrow{\delta_0} 0
\]
where \( d_1 \) and \( d_2 \) are morphisms of \( A \)-bimodules given by the formulas
\[
d_1(1 \otimes x^n \otimes 1) = \sum_{i=0}^{n-1} x^i \otimes x \otimes x^{n-1-i},
\]
\[
d_1(1 \otimes y^n \otimes 1) = \sum_{i=1}^{m-1} y^i \otimes y \otimes y^{m-1-i},
\]
\[
d_2(1 \otimes yx \otimes 1) = 1 \otimes y \otimes x + y \otimes x \otimes 1 - \xi \otimes x \otimes y - \xi x \otimes y \otimes 1
\]
\[
d_1(1 \otimes y^{m+1} \otimes 1) = y \otimes y^m \otimes 1 - 1 \otimes y^m \otimes m,
\]
\[
d_2(1 \otimes y^mx \otimes 1) = \sum_{i=0}^{m-1} \xi y^m y^{-i} y x + \xi^m x \otimes y^m \otimes 1 - 1 \otimes y^m \otimes x
\]
\[
d_2(1 \otimes y^n x \otimes 1) = 1 \otimes x \otimes 1 - 1 \otimes x^n \otimes x.
\]

**Proof.** It is straightforward, using Proposition 6.4 applied to the set \( \{ rP \}_{P \in \mathcal{A}_2} \) of left reductions, where
\[
y^{m+1} = r_{1,y^m}, \quad y^m x = r_{1, y^m},
\]
\[
y^{x^n} = (r_{1,x^n}, \ldots, r_{x,y,x^{n-2}}, r_{1, y^{x,n-1}}) \quad r_{x^{n+1}} = r_{1,x^n},
\]
and the set \( \{ tP \}_{P \in \mathcal{A}_2} \) of right reductions, where
\[
y^{m+1} = r_{y,y^m,1}, \quad y^m x = (r_{x,y^m,1}, \ldots, r_{y^{m-2}y, y}, r_{y^{m-1}, y}, r_{y,x,y})
\]
\[
y^{x^n} = r_{x,y,x^{n-1}}, \quad t^{x^{n+1}} = r_{x,x^n,1}.
\]

Of course we want to construct the rest of the resolution. We will first describe the set \( \mathcal{L}_{N-1}^\infty (s,t) \). There are four cases, depending on the parity of \( s, t \) and \( N \). With this in view, it is useful to make some previous computations that we list below.

1. For \( s \) even, for all \( j \), \( 0 \leq j \leq m - 1 \), \( y^{\varphi(s,m)} = y^{m-1-j} y^{\varphi(s-1,m)} y \).
2. For \( s \) odd, \( y^{\varphi(s,m)} = y y^{\varphi(s-1,m)} = y^{\varphi(s-1,m)} y \).
3. For \( t \) even, for all \( i \), \( 0 \leq i \leq n - 1 \), \( x^{\varphi(t,n)} = x^i x^{\varphi(t-1,n)} x^{n-i-1} \).
4. For \( t \) odd, \( x^{\varphi(t,n)} = x x^{\varphi(t-1,n)} = x^{\varphi(t-1,n)} x \).

There are four different cases to be considered for the description of the set \( \mathcal{L}_{N-1}^\infty (s,t) \) for \((s,t) = y^{\varphi(s,m)} x^{\varphi(t,n)} \in \mathcal{A}_N \).
First case: $N$ even, $s$ even, $t$ odd,
\[ \mathcal{L}^N_{N-1}(s,t) = \{ \xi^{(s,t,n)}_i y^{m-1-j} \otimes (s-1,t) \otimes y_j \}_{j=1}^{m-1} \cup \{ \xi^{(s,m)}_i x \otimes (s,t-1) \otimes 1 \} \]

Second case: $N$ even, $s$ odd, $t$ even,
\[ \mathcal{L}^N_{N-1}(s,t) = \{ \xi^{(s,t,n)}_i y^{m-1-j} \otimes (s-1,t) \otimes y_j \}_{j=1}^{m-1} \cup \{ \xi^{(s,m)}_i x^j \otimes (s,t-1) \otimes x^{n-1-j} \}_{i=1}^{n-1} \]

Third case: $N$ odd, $s$ even, $t$ even,
\[ \mathcal{L}^N_{N-1}(s,t) = \{ \xi^{(s,t,n)}_i y^{m-1-j} \otimes (s-1,t) \otimes y_j \}_{j=1}^{m-1} \cup \{ \xi^{(s,m)}_i x^j \otimes (s,t-1) \otimes x^{n-1-j} \}_{i=1}^{n-1} \]

Fourth case: $N$, $s$ and $t$ odd,
\[ \mathcal{L}^N_{N-1}(s,t) = \{ \xi^{(s,t,n)}_i y^{m-1-j} \otimes (s-1,t) \otimes y_j \}_{j=1}^{m-1} \cup \{ \xi^{(s,m)}_i x^j \otimes (s,t-1) \otimes 1 \} \]

Remark 7.2.1. Looking at what happens for $n = m = 2$, we observe that
\[ (d_1 - \delta_1)(1 \otimes (s,t) \otimes 1) = (-1)^s \sum_{u \in \mathcal{L}_0^{(s,t)}} u, \]
\[ (d_2 - \delta_2)(1 \otimes (s,t) \otimes 1) = (-1)^s \sum_{u \in \mathcal{L}_0^{(s,t)}} u. \]

Proposition 7.3. For $R = \mathbb{Z}$ guarantees that there exist $A$-bimodule maps $d_N : A \otimes E k\mathcal{A}_N \otimes E A \to A \otimes E k\mathcal{A}_{N-1} \otimes E A$ such that $(d_1 - \delta_1)(1 \otimes (s,t) \otimes 1) \in (\mathcal{L}^N_{N-1}(s,t))_z$ and, most important, the complex $(A \otimes E k\mathcal{A}_N \otimes E A, d_\bullet)$ is a projective resolution of $A$ as $A$-bimodule.

We are not yet able at this point to give the explicit formulas of the differentials. In order to illustrate the situation, let us describe what happens for $N = 3$. We know after the mentioned proposition that there exist $t_1, t_2 \in \mathbb{Z}$ such that
\[ d_3(1 \otimes y^{m+1}x \otimes 1) = d_3(1 \otimes (3,1) \otimes 1) = \]
\[ \delta_3(1 \otimes (3,1) \otimes 1) + t_1 \xi \otimes (2,1) \otimes y + t_2 \xi^3 x \otimes (3,0) \otimes 1 \]
\[ = y \otimes y^m x \otimes 1 + 1 \otimes y^{m+1} x \otimes x + t_1 \xi \otimes y^m x \otimes y + t_2 \xi^3 x \otimes y^{m+1} \otimes 1. \]

Of course, $d_2 \circ d_3 = 0$. It follows from this equality that $t_1 = t_2 = -1$. This example motivates the following lemma, stated in terms of the preceding notations.

Lemma 7.3. The $A$-bimodule morphisms $d_N : A \otimes E k\mathcal{A}_N \otimes E A \to A \otimes E k\mathcal{A}_{N-1} \otimes E A$ defined by the formula
\[ d_N(1 \otimes (s,t) \otimes 1) = \delta_N(1 \otimes (s,t) \otimes 1) + (-1)^s \sum_{u \in \mathcal{L}_0^{(s,t)}} u \]
satisfy the hypotheses of Thm. 4.2.

Proof. It is straightforward. \qed

We gather all the information we have obtained about the projective bimodule resolution of $A$ in the following proposition.

Proposition 7.4. The complex of $A$-modules $(A \otimes E k\mathcal{A}_N \otimes E A, d_\bullet)$, with
\[ \mathcal{A}_N = \{ y^{(s,m)}_i x^{(t,n)} : s + t = N + 1 \} \]
and differentials defined as follows is exact.

1. For $N$ even, $s$ even and $t$ odd,
\[ d_N(1 \otimes (s,t) \otimes 1) = y^{m-1} \otimes (s-1,t) \otimes 1 + \sum_{j=1}^{m-1} (-1)^j y^{m-1-j} \otimes (s-1,t) \otimes y^j \]
\[ + (-1)^{N+1} \otimes (s,t-1) \otimes x + (-1)^s y^{(s,m)} x \otimes (s,t-1) \otimes 1. \]
(2) For $N$ even, $s$ odd and $t$ even,
\[ d_N(1 \otimes (s,t) \otimes 1) = y \otimes (s-1,t) \otimes 1 + (-1)^s \xi^{(t,n)} \otimes (s-1,t) \otimes y \]
\[ + (-1)^{N+1} \otimes (s,t-1) \otimes x^{n-1} + \sum_{i=1}^{n-1} (-1)^s \xi^{(s,m)} x^i \otimes (s,t-1) \otimes x^{n-1-i} \]

(3) For $N$ odd, $s$ and $t$ even,
\[ d_N(1 \otimes (s,t) \otimes 1) = y^{m-1} \otimes (s-1,t) \otimes 1 + \sum_{j=1}^{m-1} (-1)^s \xi^{(t,n)} y^{m-j} \otimes (s-1,t) \otimes y^j \]
\[ + (-1)^{N+1} \otimes (s,t-1) \otimes x^{n-1} + \sum_{i=1}^{n-1} (-1)^s \xi^{(s,m)} x^i \otimes (s,t-1) \otimes x^{n-1-i} \]

(4) For $N$, $s$ and $t$ odd,
\[ d_N(1 \otimes (s,t) \otimes 1) = y \otimes (s-1,t) \otimes 1 + (-1)^s \xi^{(t,n)} \otimes (s-1,t) \otimes y \]
\[ + (-1)^{N+1} \otimes (s,t-1) \otimes x + (-1)^s \xi^{(s,m)} x \otimes (s,t-1) \otimes 1. \]

7.3. Down-up algebras. Given $\alpha, \beta, \gamma \in k$, we will denote $A(\alpha, \beta, \gamma)$ the quotient of $k\langle d, u \rangle$ by the two sided ideal $I$ generated by relations
\[ d^2u - \alpha du d - \beta ud^2 - \gamma d = 0, \]
\[ du^2 - \alpha udu - \beta u^2d - \gamma u = 0. \]

Down-up algebras have been deeply studied since they were defined in [BR]. We can mention the articles [CM], [BW], [BG], [CS], [CL], [KK], [KMP], [Ku1], [Ku2], [P1], [P2], [P3], in which the authors prove different properties of down-up algebras. It is well known that they are noetherian if and only if $\beta \neq 0$ [KMP]. They are graded with $dg(d) = 1$, $dg(u) = -1$, and they are filtered if we consider $d$ and $u$ of weight 1. If $\gamma = 0$ they are also graded by this weight.

Down-up algebras are 3-Koszul if $\gamma = 0$, and if $\gamma \neq 0$, they are PBW deformations of 3-Koszul algebras [BG].

Little is known about their Hochschild homology and cohomology, except for the center, described in [Z] and [Ku1]. We apply our methods to construct a projective resolution of $A$ as $A$-bimodule, and use this resolution to compute $H^\bullet(A, A^e)$ and prove that in the noetherian case, $A(\alpha, \beta, \gamma)$ is 3-Calabi-Yau if and only if $\beta = -1$. Moreover, in this situation we exhibit a potential $\Phi(d, u)$ such that the relations are in fact the cyclic derivatives $\partial_q \Phi$ and $\partial_t \Phi$, respectively.

We briefly recall that a $d$-Calab-Yau algebra is an associative algebra such that there is an isomorphism $f$ of $A$-bimodules
\[ \operatorname{Ext}^i_A(A, A^e) \cong \begin{cases} 0 & \text{if } i \neq d, \\ A & \text{if } i = d. \end{cases} \]

where the $A$-bimodule outer structure of $A^e$ is used for the computation of $\operatorname{Ext}^i_A(A, A^e)$, while the isomorphism $f$ takes account of the inner bimodule structure of $A^e$. Bocklandt proved in [BG] that graded Calabi-Yau algebras come from a potential and van den Bergh [VdB] generalized this result.

We fix a lexicographical order such that $d < u$, with weights $\omega(d) = 1 = \omega(u)$. The reduction system $R = \{(d^2u, adud + \beta ud^2 + \gamma d), (du^2, auud + \beta u^2d + \gamma u)\}$ has $B = \{\omega^i(v^j) : i, j \in \mathbb{N}_0\}$ as set of irreducible paths and $A_2 = \{d^2u^2\}$; using Bergman’s Diamond Lemma we see that $R$ satisfies condition $(\diamondsuit)$. Also, $A_0 = \{d, u\}$ and $A_n = 0$ for all $n \geq 3$. The set $B$ is the $k$-basis already considered in [BR].
The reductions $r^2u^2 = (r_u, d^2 u, r_1, d^2 u, u)$ and $t^2u^2 = (t_1, du^2, d, t_d, du^2, u)$ are respectively left and right reductions of $d^2u^2$.

In view of Proposition 6.6 and observing that $\delta_1$ is in fact an epimorphism and that $A_3 = 0$, the following complex gives a free resolution of $A$ as $A$-bimodule:

$$0 \to A \otimes_E kD^2u^2 \otimes_E A \to d_3 A \otimes_E (kD^2u \otimes kudu) \otimes_E A \to d_2 A \otimes_E (kd \otimes ku) \otimes_E A \to d_1 A \otimes_E A \to 0$$

where

$$d_3(1 \otimes d^2 u \otimes 1) = 1 \otimes d \otimes du + d^2 \otimes u \otimes 1 - \alpha (1 \otimes d \otimes ud + d \otimes u \otimes d + du \otimes d \otimes 1)$$

$$- \beta (1 \otimes u \otimes d^2 + u \otimes d \otimes d + ud \otimes d \otimes 1) - \gamma \otimes d \otimes 1,$$

$$d_1(1 \otimes d^2 u \otimes 1) = 1 \otimes d \otimes u^2 + d \otimes u \otimes u + du \otimes u \otimes 1 - \alpha (1 \otimes u \otimes du + u \otimes d \otimes u + ud \otimes u \otimes 1)$$

$$- \beta (1 \otimes u \otimes ud + u \otimes u \otimes d + u^2 \otimes d \otimes 1),$$

and

$$d_2(1 \otimes d^2 u^2 \otimes 1) = d \otimes d^2 u \otimes 1 + \beta \otimes du^2 \otimes d - 1 \otimes d^2 u \otimes u - \beta \otimes du^2 \otimes 1.$$

As we have proved in general, the map $d_2$ takes into account the reductions applied to the ambiguity.

**Proposition 7.5.** Suppose that $\beta \neq 0$. The algebra $A(\alpha, \beta, \gamma)$ is 3-Calabi-Yau if and only if $\beta = -1$.

**Proof.** We need to compute $\text{Ext}^*_A(A, A^e)$. We apply the functor $\text{Hom}_{A^e}(-, A^e)$ to the previous resolution, and we use that for any finite dimensional vector space $V$ which is also an $E$-bimodule, the space $\text{Hom}_{A^e}(A \otimes_E V \otimes_E A, A^e)$ is isomorphic to $\text{Hom}_{E^e}(V, A^e)$, and this last one is, in turn, isomorphic to $A \otimes_E V^* \otimes_E A$. The explicit expression of this last isomorphism is, fixing a $k$-basis $\{v_1, \ldots, v_n\}$ of $V$ and its dual basis $\{\varphi_1, \ldots, \varphi_n\}$ of $V^*$,

$$A \otimes_E V^* \otimes_E A \to \text{Hom}_{E^e}(V, A^e)$$

$$a \otimes \varphi \otimes b \mapsto [v \mapsto \varphi(v)b \otimes a]$$

with inverse $f \mapsto \sum_{i,j} b^i_j \otimes \varphi_i \otimes a^j_i$, where $f(v_i) = \sum_j a^j_i \otimes b^i_j$.

After these identifications, we obtain the following complex of $k$-vector spaces whose homology is $\text{Ext}^*_A(A, A^e)$

$$0 \to A \otimes_E A \to d_0 A \otimes_E (kD \otimes ku) \otimes_E A \to d_1 A \otimes_E (kD^2U \otimes kDU^2) \otimes_E A \to d_2 A \otimes_E kD^2U^2 \otimes_E A \to 0,$$

where we denote by $\{D, U\}$ the dual basis of $\{d, u\}$ and, accordingly, we denote with capital letters the dual bases.

The maps in the complex are, explicitly:

$$\delta_0(1 \otimes 1) = 1 \otimes D \otimes d - 1 \otimes U \otimes u - u \otimes U \otimes 1$$

$$d \delta_1(1 \otimes U \otimes 1) = 1 \otimes D^2U \otimes d^2 - \alpha d \otimes D^2U \otimes d - \beta d^2 \otimes D^2U \otimes 1 + u \otimes DU^2 \otimes d$$

$$+ 1 \otimes DU^2 \otimes du - \alpha du \otimes DU^2 \otimes 1 - \alpha \otimes DU^2 \otimes ud - \beta ud \otimes DU^2 \otimes 1$$

$$- \beta d \otimes DU^2 \otimes u - \gamma \otimes DU^2 \otimes 1,$$

$$d \delta_1(1 \otimes D \otimes 1) = du \otimes D^2U \otimes 1 + u \otimes D^2U \otimes d - \alpha ud \otimes D^2U \otimes 1 - \alpha \otimes D^2U \otimes du$$

$$- \beta d \otimes D^2U \otimes u - \beta \otimes D^2U \otimes ud - \gamma \otimes D^2U \otimes 1 + u^2 \otimes DU^2 \otimes 1$$

$$- \alpha u \otimes DU^2 \otimes u - \beta \otimes DU^2 \otimes u^2,$$

$$d \delta_1(1 \otimes DU^2 \otimes 1) = d \otimes DU^2 \otimes du + \beta d \otimes DU^2 \otimes d^2,$$

$$d \delta_1(1 \otimes DU^2 \otimes 1) = -u \otimes DU^2 \otimes 1 - \beta \otimes DU^2 \otimes u.$$
Consider the following isomorphisms of $A$-bimodules
\[
\psi_0 : A \otimes_E A \rightarrow A \otimes_E kd^2u^2 \otimes_E A,
\psi_0(1 \otimes 1) = 1 \otimes d^2u^2 \otimes 1,
\psi_1 : A \otimes_E (kd^2 \otimes ku) \otimes_E A \rightarrow A \otimes_E (kd^2u \otimes kdu^2) \otimes_E A
\]
\[
\psi_1(1 \otimes D \otimes 1) = 1 \otimes d^2u \otimes 1, \text{ and } \psi_1(1 \otimes U \otimes 1) = 1 \otimes d^2u \otimes 1
\]
\[
\psi_2 : A \otimes_E (kd^2U \otimes kDU^2) \otimes_E A \rightarrow A \otimes_E (kd \otimes ku) \otimes_E A,
\psi_2(1 \otimes D^2U \otimes 1) = 1 \otimes u \otimes 1, \text{ and } \psi_2(1 \otimes DU^2 \otimes 1) = 1 \otimes d \otimes 1
\]
\[
\psi_3 : A \otimes_E kd^2U^2 \otimes_E A \rightarrow A \otimes_E A
\]
\[
\psi_3(1 \otimes D^2U^2 \otimes 1) = 1 \otimes 1.
\]

It is straightforward to verify that the following diagram commutes, thus inducing isomorphisms between the homology spaces of both horizontal sequences:

\[
\begin{array}{c}
0 \rightarrow A \otimes_E A \xrightarrow{d_2^1} A \otimes_E (kd \otimes ku) \otimes_E A \xrightarrow{d_2^1} A \otimes_E (kd^2U \otimes kDU^2) \otimes_E A \xrightarrow{d_2^1} A \otimes_E kd^2U^2 \otimes_E A \rightarrow 0
\\
0 \rightarrow A \otimes_E kd^2u^2 \otimes_E A \xrightarrow{\pi_3} A \otimes_E (kd^2u \otimes kdu^2) \otimes_E A \xrightarrow{\pi_2} A \otimes_E (kd \otimes ku) \otimes_E A \xrightarrow{\pi_1} A \otimes_E A \rightarrow 0
\end{array}
\]

where $\pi_0$ is given by
\[
\pi_0(1 \otimes d^2u^2 \otimes 1) = 1 \otimes d^2u \otimes d - d \otimes d^2u^2 \otimes 1 - u \otimes d^2u \otimes 1 + u \otimes d^2u \otimes u.
\]

$d_1$ is
\[
d_1(1 \otimes d^2u \otimes 1) = 1 \otimes d \otimes d - d \otimes d^2u - ud + d \otimes u \otimes d + d \otimes d \otimes 1
\]

\[
- \beta(-\beta^{-1} u \otimes d - \beta^{-1} u \otimes d \otimes d + ud \otimes d \otimes 1) - \gamma \otimes d \otimes 1
\]

$d_2$ is
\[
d_2(1 \otimes du^2 \otimes 1) = -\beta \otimes d \otimes u^2 - \beta d \otimes u \otimes u + du \otimes u \otimes 1
\]

\[
- \alpha(1 \otimes u \otimes du + u \otimes d \otimes u + u \otimes d \otimes 1)
\]

\[
- \beta(1 \otimes u \otimes ud - \beta^{-1} u \otimes u \otimes d - \beta^{-1} u^2 \otimes d \otimes 1) - \gamma \otimes u \otimes 1
\]

and $d_2$ is
\[
\pi_2(1 \otimes d \otimes 1) = 1 \otimes d + \beta d \otimes 1,
\]

From this we deduce that $HH^3(A, A^e) \cong A \otimes_E A/(\text{Im } \overline{d}_2)$. Let $\sigma$ be the algebra automorphism of $A$ defined by $\sigma(d) = -\beta d$, $\sigma(u) = -\beta^{-1} u$. Recall that $A_\sigma$ is the $A$-bimodule with $A$ as underlying vector space with action of $A \otimes_R A^e$ given by: $(a \otimes b) \cdot x = ax\sigma(b)$, that is, it is twisted on the right by the automorphism $\sigma$.

It is easy to see that if $\beta \neq 0$ then $A_\sigma \cong A \otimes_E A/(\text{Im } \overline{d}_2) \cong HH^3(A, A^e)$ as $A$-bimodules. If $\beta = 0$ then the action on the left by $u$ on $HH^3(A, A^e)$ is zero and then $A \cong HH^3(A, A^e)$ since the action on the left by $u$ on $A$ is injective. We conclude that $HH^3(A, A^e) \cong A$ if and only if $\beta = -1$. Notice that for $\beta = -1$ the complex in the second line of the diagram above is the resolution of $A$. As a consequence, $A$ is 3-Calabi-Yau if and only if $\beta = -1$. In this case the potential $\Phi$ equals $d^2u^2 + \frac{2}{3}dudu + \gamma du$. For $\beta \neq 0, -1$, we shall see in a forthcoming work that $A$ is twisted 3-Calabi-Yau algebra $[BSW]$, coming from a twisted potential.

\[\square\]
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