Physical Consonance Law of Sound Waves

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Abstract

Sound consonance is the reason why it is possible to exist music in our life. However, rules of consonance between sounds had been found quite subjectively, just by hearing. To care for, the proposal is to establish a sound consonance law on the basis of mathematical and physical foundations. Nevertheless, the sensibility of the human auditory system to the audible range of frequencies is individual and depends on a several factors such as the age or the health in a such way that the human perception of the consonance as the pleasant sensation it produces, while reinforced by an exact physical relation, may involves as well the individual subjective feeling.

1 Introduction

Sound consonance is one of the main reason why it is possible to exist music in our life. However, rules of sound consonance had been found quite subjectively, just by hearing [1]. It sounds good, after all music is art! But physics challenge is to discover laws wherever they are [2]. To care for, it is proposed, here, a sound consonance law on the basis of mathematical and physical foundations.

As we know, Occidental musics are based in the so called Just Intonation Scale, built with a set of musical notes which frequencies are related between
them in the interval of frequencies from some $f_0$ to $f_1 = 2f_0$ that defines the octave. Human audible sound frequencies comprehend from about 20Hz to 20,000Hz, and a typical piano keyboard covers 7 octaves from notes $A_0$ to $C_8$, with frequencies 27.5Hz and 4,224Hz, respectively, using the standard tuning up frequency attributed to the note $A_4$ (440Hz) [3], [4].

The origin of actual musical scale remits us to the Greek mathematician Pythagoras. Using a monochord, a vibrating string with a movable bridge that transforms into a two vibrating strings with different but related lengths and frequencies, he found that combinations of two sounds with frequency relations 2 : 1, 3 : 2 and 4 : 3 are particularly pleasant, while many other arbitrary combinations are unpleasant. Two sounds getting a pleasant combined sound are called consonants, otherwise they are dissonant. This set of relations encompassed into the frequencies interval of one octave defines the Pythagorean Scale, which is shown in Table 1, with notes and frequency relations of each note compared to the first one, $C_i$. Table 2 shows the frequency relations of the two adjacent notes, with a tone given by 9/8 and a semitone given by 256/243.

| $C_i$ | D | E | F | G | A | B | $C_f$ |
|-------|---|---|---|---|---|---|-------|
| 1     | 9/8 | 81/64 | 4/3 | 3/2 | 27/16 | 243/128 | 2 |

Table 1: notes and frequency relations compared with the first note $C_i$, in the Pythagorean scale.

| D/$C_i$ | E/$D$ | F/$E$ | G/$F$ | A/$G$ | B/$A$ | $C_f/B$ |
|--------|-------|-------|-------|-------|-------|---------|
| 9/8    | 9/8   | 256/243 | 9/8  | 9/8  | 9/8  | 256/243 |

Table 2: notes and frequency relations of the two consecutive notes, in the Pythagorean scale.

Consonance and dissonance are not absolute concepts, and the Greek astronomer Ptolomy added to the Pythagorean consonant relations 3 : 2 : 1 another set of relations who considered as well as consonant, 4 : 5 : 6. This enlarged set is the base of the so called Just Intonation Scale, which is shown in table 3, with notes and frequency relations of each note compared to the first one, $C_i$. Table 4 shows the frequency relations of the two adjacent notes, with a major tone given by 9/8, the minor tone by 10/9 and a semitone by 16/15.
Actually, in terms of the consonant frequency relations, the Just Intonation Scale is richer than the Pythagorean Scale. It can be understood why it is just so attempting to the physical condition to the sound waves consonance we are going to establish in next section.

Notice that all natural sound can be treated as composed by a linear combination of harmonic waves with frequencies related to the fundamental one, \( f_0 \), as \( f_n = n f_0 \) for integer \( n (= 1, 2, 3, ...) \). All these harmonic components are considered as consonant with the fundamental and the relative weight of these harmonics is a characteristic of the sound source, which frequencies spectrum defines one of the most important sound quality, the timbre.

Natural musical scale as the Just Intonation Scale implies some practical problems due to the unequal frequency relations that define one tone or half tone related notes. To avoid such trouble, it was created the scale of equal temperament (chromatic scale), composed with 12 notes with equal frequency relation \( r \) between adjacent notes from a fundamental \( f_0 \) to the octave above \( f_{12} = r^{12} f_0 = 2 f_0 \) frequencies,

\[
  f_0, f_1 = r f_0, f_2 = r f_1 = r^2 f_0, f_3 = r^3 f_0, \ldots, f_{12} = r^{12} f_0 = 2 f_0 ,
\]

such that

\[
  r = 12 \sqrt{2} \simeq 1, 0594631 ,
\]

there is no exact frequency relation (??), but it is closely approximated, in a compromise to favor the practice. The scale of equal temperament is widely used as an universal tuning up of mostly popular musical instruments, with a few exceptions as the violin or the singing natural human voice.
2 Consonance Law of Sound Waves

For the purpose to establish the law of sound consonance, the essential thing is to know how two sound waves with different frequencies, $f_1$ and $f_2$, combine when produced simultaneously (harmony) or in a quick time sequence (melody). Our daily hearing experience suggests us that the quality of the sounds composition is essentially due to the frequencies combination of the sounds, irrespective to their phases and amplitudes. In this sense, it is sufficient, at least at the first sight, to examine just the time oscillation, given by the trigonometric relation [5]

$$\cos 2\pi f_1 t + \cos 2\pi f_2 t = 2 \cos 2\pi \frac{|f_1 - f_2|}{2} t \cos 2\pi \frac{(f_1 + f_2)}{2} t ,$$  \hspace{1cm} (2)

which shows a main wave with mean frequency

$$\overline{f} = f_+ = \frac{(f_1 + f_2)}{2} \hspace{1cm} (3)$$

modulated by the beat frequency

$$f_- = \frac{(f_2 - f_1)}{2} .$$ \hspace{1cm} (4)

Without loss of generality, we are going to suppose $f_2 > f_1$.

We can see that, if

$$(f_2 + f_1) = n(f_2 - f_1) ,$$  \hspace{1cm} (5)

that is,

$$\frac{f_2}{f_1} = \frac{(n + 1)}{(n - 1)} , \hspace{1cm} (6)$$

for integers $n > 1$, the resulting wave is, yet, a regular and periodic wave, behaving like a sound wave composed with harmonic sound components, as it is in fact.

It can be seen inverting the trigonometric relation (2) above using the frequency relation (6), which leads to

$$2 \cos 2\pi \frac{(f_2 - f_1)}{2} t \cos 2\pi \frac{(f_1 + f_2)}{2} t = \cos 2\pi(n-1)f_0 t + \cos 2\pi(n+1)f_0 t \hspace{1cm} (7)$$

for some frequency

$$f_0 = f_2/(n + 1) = f_1/(n - 1) \hspace{1cm} (8)$$
used as the fundamental one. It is all we need to have an harmonious or consonant combinations of sound waves. If the frequency relation (6) is an integer, the two sounds are harmonically related and trivially consonant sounds. Otherwise, it defines the frequency relations in the range of an octave, the frequency interval that comprises a musical scale, that is, \( f_1 < f_2 < 2f_1 \), such that the relation (6) must be a rational number limited by

\[
1 < \frac{(n+1)}{(n-1)} < 2 ,
\]

for an integer \( n > 3 \), remembering that \( n = 2 \) and \( n = 3 \) implies the harmonic relations \( f_2 = 3f_1 \) and \( f_2 = 2f_1 \), respectively.

In equations (7) and (8), it is assumed that

\[
f_2 = (n + 1)f_0
\]

and

\[
f_1 = (n - 1)f_0 ,
\]

which can be used in equations of the beat and the mean frequencies, (4) and (3), respectively, resulting

\[
f_- = \frac{(f_2 - f_1)}{2} = \frac{(n + 1)f_0 - (n - 1)f_0}{2} = f_0
\]

and

\[
\overline{f} = f_+ = \frac{(f_1 + f_2)}{2} = nf_0 ,
\]

showing that, taking into account the physical consonance condition (5), all the relevant frequencies are related harmonically to a fundamental one, \( f_0 \), which can assume values in the range

\[
0 < f_0 < f_1.
\]

It is easy to handle the situation when the primary sound waves has different amplitude. Equation (2) must be replaced by

\[
A_1 \cos 2\pi f_1 t + A_2 \cos 2\pi f_2 t = A_1 \left[ \cos 2\pi f_1 t + \cos 2\pi f_2 t \right] + (A_2 - A_1) \cos 2\pi f_2 t
\]

and the condition of consonance applied to the equal amplitude terms, taken aside the last term. Then, using the inverse relation (7), the last term can be reincorporated, resulting the more general formula
\[ A_1 \cos 2\pi f_1 t + A_2 \cos 2\pi f_2 t = A_1 \cos 2\pi (n-1)f_0 t + A_2 \cos 2\pi (n+1)f_0 t , \quad (15) \]

getting more confidence considering that in a real world a fine control of the sound intensity is not a simple task.

An ultimate generalization is need to take into account the phase difference between the primary sound waves. Such phase difference can be originated due to the small, uncontrollable, time difference the primary sounds are produced. Then, the left side of the sum (15) is better to be rewritten as

\[ A_1 \cos 2\pi f_1 (t - t_1) + A_2 \cos 2\pi f_2 (t - t_2) = W [\cos] + W [\sin] , \quad (16) \]

where

\[ W [\cos] = B_1 \cos 2\pi f_1 t + B_2 \cos 2\pi f_2 t \quad (17) \]

and

\[ W [\sin] = C_1 \sin 2\pi f_1 t + C_2 \sin 2\pi f_2 t \quad (18) \]

are the cosine and the sine wave components, respectively, with coefficients

\[ B_1 = A_1 \cos 2\pi f_1 t_1, \quad B_2 = A_2 \cos 2\pi f_2 t_2 \quad (19) \]

of the cosine component and

\[ C_1 = A_1 \sin 2\pi f_1 t_1, \quad C_2 = A_2 \sin 2\pi f_2 t_2 \quad (20) \]

of the sine component. Applying the consonance condition (5), the cosine component is just given by the equation (15),

\[ W [\cos] = B_1 \cos 2\pi (n-1)f_0 t + B_2 \cos 2\pi (n+1)f_0 t . \quad (21) \]

Using the trigonometric relation

\[ \sin 2\pi f_1 t + \sin 2\pi f_2 t = 2 \cos 2\pi \frac{(f_2 - f_1)}{2} t \sin 2\pi \frac{(f_2 + f_1)}{2} t , \]

the same consonance condition (5) works for the sine waves, resulting

\[ W [\sin] = C_1 \sin 2\pi (n-1)f_0 t + C_2 \sin 2\pi (n+1)f_0 t . \quad (22) \]
It is possible to reverse all these proceeding, getting the quite general expression

\[ A_1 \cos 2\pi f_1 (t - t_1) + A_2 \cos 2\pi f_2 (t - t_2) \]

\[ = A_1 \cos 2\pi (n - 1) f_0 (t - t_1) + A_2 \cos 2\pi (n + 1) f_0 (t - t_2) t , \quad (23) \]

a guarantee that the consonance condition (5) works in a real situation.

It is easy to verify that the frequency relations as 3 : 2 : 1 and 4 : 5 : 6 satisfy, all of than, the condition (5) or (6). For example, in the Pythagorean frequency relations, we have

\[ \frac{3}{2} = \frac{6}{4} = \frac{5 + 1}{5 - 1} \quad \frac{3}{1} = \frac{2 + 1}{2 - 1} \quad \frac{2}{1} = \frac{3 + 1}{3 - 1} \quad (24) \]

and, in the Ptolomyan frequency relations,

\[ \frac{6}{5} = \frac{12}{10} = \frac{11 + 1}{11 - 1} \quad \frac{6}{4} = \frac{5 + 1}{5 - 1} \quad \frac{5}{4} = \frac{10}{8} = \frac{9 + 1}{9 - 1} . \quad (25) \]

In Pythagorean Scale, in the table 1 there are 4 consonant relations (9/8, 4/3, 3/2 and 2) and in the table 2 there are 5 consonant relations 9/8. In the Just Intonation Scale, in the table 3 there are 6 consonant relations (9/8, 5/4, 4/3, 3/2, 5/3, 2) and in the table 4 all of the 7 frequency relations are consonant. It is the reason why the Just Intonation Scale is better than the Pythagorean Scale.

Sound sources are vibrating systems, and produce sounds that are combinations of a fundamental and its harmonics, with a particular combination defining the timbre of the sound. So, to the consonance condition being consistent, frequency relations like (5) must be valid simultaneously to all, the fundamental and its harmonic frequencies. Fortunately, it is so, as we can see easily. Really, taking the compositions of all oscillating modes of the two sound sources with fundamental frequencies \( f_1 \) and \( f_2 \),

\[ u_1(t) = \sum_{k=1}^{\infty} A_k \cos 2\pi k f_1 t \quad (26) \]

and

\[ u_2(t) = \sum_{k=1}^{\infty} B_k \cos 2\pi k f_2 t , \quad (27) \]
respectively, the combination of these two composed sounds (considering at a moment the equal amplitudes $B_k = A_k$) becomes, from (2),

$$u(t) = \sum_{k=1}^{\infty} A_k \left( \cos 2\pi k f_1 t + \cos 2\pi k f_2 t \right)$$

$$= 2 \sum_{k=1}^{\infty} A_k \cos 2\pi k \frac{|f_1 - f_2|}{2} t \cos 2\pi k \frac{(f_1 + f_2)}{2} t . \quad (28)$$

Applying the consonance condition (6), supposing $f_1 < f_2$, we obtain the inverse trigonometric expansion given by (7) in a general form (15),

$$u(t) = \sum_{k=1}^{\infty} A_k \cos 2\pi k (n - 1) f_0 t + B_k \cos 2\pi k (n + 1) f_0 t , \quad (29)$$

which is an harmonic series. Again, it is all we need to have an harmonious or consonant combinations of sound waves.

### 3 Hearing Dissonance

The physical consonance condition given by equation (5) assures the harmonic structure of the sound waves combination such that it behaves like a natural sound waves with an enriched timbre. However, we have to take into account the hearing sensibility of the human auditory system [6] and [7], able to recognize sounds at frequencies range from about $20\text{Hz}$ to $20,000\text{Hz}$. Figure 1 shows a mathematical representation of an artificial computer generated pure sound frequencies (a) $440\text{Hz}$ of the musical scale standard $A_4$ note, (b) $264\text{Hz}$ corresponding to the note $C_4$, (c) $495\text{Hz}$ of the note $B_4$ and (d) $20\text{Hz}$, the low audio frequency threshold, in the time interval $0 < t < 0.05s$.

If the beat frequency is below the low audio frequency threshold,

$$f_0 = f_- \lesssim 20\text{Hz} , \quad (30)$$

Figure 1
that occurs when the primary frequencies $f_2$ and $f_1$ are close, which implies big $n$, the fundamental frequency $f_0$ is going to be missing for our audition. In this situation, the sound composition given by (2) cannot be heard as an harmonically related sounds (7), but instead, it is listen as

$$\cos 2\pi f_1 t + \cos 2\pi f_2 t = (2 \cos 2\pi f_0 t) \cos 2\pi n f_0 t ,$$

(31)
an unique sound with frequency $nf_0$ modulated by an inaudible beat frequency $f_- = f_0$. This modulation, for a very close frequencies, leads to the well known beat phenomenon. Beat frequency near the transition region between audible and inaudible frequencies might be confusing to the auditory system like an antenna trying to tune in a signal with frequency in the border of its work range frequencies. In such a way, the missing of the fundamental frequency, while of physiological nature, can breaks down the consonance condition of the primary frequencies. So, even satisfying the physical consonance condition given by equation (5), our audition will not going to perceive as consonant. In the central octave frequencies range, taking the first note, $C_4$, with frequency $f_1 = 264Hz$, the low frequency limit (30) occurs at the condition $n \gtrsim 12$, above which the auditory perception of consonance is going to be broken. As a result, the major second one tone interval as the $C_4 - D_4$, related by frequencies ratio $9/8$, which corresponds to $n = 17$, is not considered as consonant. With primary frequencies $264Hz$ and $297Hz$ of the notes $C_4$ and $D_4$, the resultant mean and the beat frequencies are $f_+ = 280.5Hz$ and $f_- = 16.5Hz$, respectively. This beat frequency is out of the audible frequencies range and, even satisfying the physical consonance condition (5), this is not considered as consonant. It is shown in the figure 2a, with the presence of a characteristic wave modulation. In sequence, figure 2b is an illustration of the sound composition with frequencies ratio $16/15$, the half tone interval, satisfying the consonance condition with $n = 30$. Frequencies considered are $264Hz$ and $281.6Hz$ of the notes $C_4$ of the sharp $\#C_4$, resulting a sound with the mean frequency $f_+ = 272.8Hz$ modulated by the beat frequency $f_- = 6.6Hz$. Figure 2c is a composition of the notes $C_4$ (264.0Hz) and $B_4$ (495.0Hz), with frequencies ratio $15/8$, a clearly non consonant relation. Figure 2d is a simple example of a non consonant sound combination, with arbitrary frequencies, actually $264Hz$ ($C_4$) and $340Hz$, mean and beat frequencies $f_+ = 302Hz$ and $f_- = 38Hz$, respectively. Figures 2a and 2b, while satisfying the physical consonance, has a well defined periodicity commanded by their beat frequencies, but figures 2b and 2d, do not satisfying the
consonance condition, show a clearly non periodic time evolution, resulting an undefined pitch of the resultant sounds.

Figure 2

The sensibility of the human auditory system to the audio frequency range is individual and depends on a several factors such as the age or the health, to cite some of them. As a consequence, the human perception of the consonance as the pleasant sensation it produces, while reinforced by a physical condition as the equation (5), involves as well the individual subjective feeling.

Also, the human hearing perception of consonance or dissonance is not absolute, and a slight deviation around the consonance condition (5) is not perceived by the human auditory system. It is the reason why the musical scale of equal temperament is acceptable.

In sequence, the table 5 shows the notes and respective frequencies of the central octave in the Just Intonation Scale using the standard frequency \( f(A_4) = 440\, \text{Hz} \) and the table 6 shows the beat frequencies of the two adjacent notes.

| C_4  | D_4  | E_4  | F_4  | G_4  | A_4  | B_4  | C_5  |
|------|------|------|------|------|------|------|------|
| 264.0| 297.0| 330.0| 352.0| 396.0| 440  | 495  | 528  |

Table 5: notes and frequencies of the central octave for the standard \( f(A_4) = 440\, \text{Hz} \), in the Just Intonation Scale.

| D_4 – C_4 | E_4 – D_4 | F_4 – E_4 | G_4 – F_4 | A_4 – G_4 | B_4 – A_4 | C_5 – B_4 |
|-----------|-----------|-----------|-----------|-----------|-----------|-----------|
| 16.5      | 16.5      | 11.0      | 22.0      | 22.0      | 27.5      | 16.5      |

Table 6: beat frequencies of the pairs of adjacent notes of the central octave, in the Just Intonation Scale.

Figure 3 contains typical consonant sound combinations, represented by the \( C_4 – E_4 \) (major third) and the \( C_4 – G_4 \) (perfect fifth) intervals, both satisfying the physical consonance condition (5). In (a), a sound composition of the notes \( C_4 \) (264Hz) and \( E_4 \) (330Hz) frequencies ratio \( 5/4 \), satisfying the consonance condition with \( n = 9 \). The resultant mean and beat frequencies are \( f_+ = 297\, \text{Hz} \) and \( f_- = 33\, \text{Hz} \), respectively. In (b), sound composition of the notes \( C_4 \) (264Hz) and \( G_4 \) (396Hz), frequencies relation \( 3/2 \), \( n = 5 \).
The resultant mean and beat frequencies are \( f_+ = 330 \text{Hz} \) and \( f_- = 66 \text{Hz} \), respectively. In (c) and (d), the same major third \( C_4 - E_4 \) and the perfect fifth \( C_4 - G_4 \), but with different relative phases. A principal characteristic of a consonant combination of sounds is the well-defined periodicity of the resultant wave, a clear consequence of the condition of consonance (5). Another important feature is that the consonance condition is not affected by changing the relative phases or the relative amplitudes of the composing sound waves. An important consequence is that the consonance condition works as well as for harmony and melody.

4 Conclusions

An exact mathematical frequencies relation is presented to define a physical consonance law of sound waves. It assures an harmonic structure of the sound waves combination such that it behaves like a natural sound waves with an enriched timbre, the beat frequency working as the fundamental one. It is not affected by changing the relative phases or the relative amplitudes of the primary sound waves and, as a consequence, the consonance condition is valid for harmonic and melodic sound composition. Nevertheless, in situation where the beat frequency is out of the audible range of frequencies encompassed by the human auditory system, the consonance perception is going to be broken, even the physical condition is satisfied. As a consequence, the human perception of the consonance as the pleasant sensation it produces, while reinforced by an exact physical relation, may involve as well the individual subjective feeling that depends on a several factors such as the age or the health, for example.

A principal characteristic of the consonance condition is the well defined periodicity of the resultant wave. The dissonance is characterized by the absence of periodicity in such a way that there is no defined pitch. It suggests that the consonance condition should be released to a more weak form of the consonance condition,

\[
(f_2 + f_1) = \frac{n}{m} (f_2 - f_1) ,
\]

(32)
which implies
\[ \frac{f_2}{f_1} = \frac{(n + m)}{(n - m)}, \]  
(33)
for integers \(n\) and \(m < n\), instead the more restrictive condition given by equations (5) and (6), from now on should be referred as the strong form \((m = 1)\) of the consonance condition. In this released form, any rational number is going to satisfy it for some integer \(m\), but now the beat frequency does not work as the fundamental one; they are related by \(f_0 = f_{\text{beat}}/m\) and the chance of the long time periodicity given by \(1/f_0\) to be out of the hearing perception increases together \(m\).

Anyway, first of all, sound consonance or dissonance is an human conception, related to pleasant or unpleasant hearing sensation, which depends on the physiology and the consequent acuity of the human auditory system and might be strongly influenced by the cultural environment.

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Figure captions

Figure 1: Mathematical representation of pure sound waves with frequencies (a) 440 Hz of the musical scale standard A₄ note, (b) 264 Hz corresponding to the note C₄, (c) 495 Hz of the note B₄ and (d) 20 Hz, the low audio frequency threshold.

Figure 2: Examples of non consonant sound combination. In (a), characteristic beat modulation given by the one tone, major second, interval C₄ − D₄. (b) 16/15 half tone C₄ (264 Hz) and sharp #C₄ (281.6 Hz) interval, satisfying the consonance condition with n = 30. The result is a sound with the mean frequency \( f_+ = 272.8\text{Hz} \) modulated by the beat frequency \( f_- = 6.6\text{Hz} \). (c) C₄ (264.0 Hz) and B₄ (495.0 Hz) composition, with frequencies ratio 15/8, a clearly non consonant relation. (d) Simple example of a non consonant sound combination, with arbitrary frequencies, actually 264 Hz (C₄) and 340 Hz, mean and beat frequencies \( f_+ = 302\text{Hz} \) and \( f_- = 38\text{Hz} \), respectively.

Figure 3: Typical examples of consonant sound waves combination. In (a), major third C₄ − E₄, with frequencies ratio 5/4, n = 9. In (b), perfect fifth C₄ − G₄, frequencies ratio 3/2, n = 5. In (c) and (d), the same major third C₄ − E₄ and the perfect fifth C₄ − G₄, but with different relative phases.
