Optimality conditions and local regularity of the value function for the optimal exit time problem

Luong V. Nguyen

We consider the control problem with *exit time*. Unlike the Bolza and Mayer problems, in this problem the terminal time of the trajectories is not fixed, but it is the first time at which they reach a given closed subset - *the target*. The most studied example is the *optimal time problem*, where we want to steer a point to the target in minimal time.

In this section, we first introduce the exit time problem, then we recall the existence of optimal controls, and some regularity results for the value function. We then use a suitable form of the *Pontryagin maximum principle* to study some optimality conditions and sensitivity relations for the exit time problem. The strongest regularity property for the value function that one can expect, in fairly general cases, is semiconcavity. In this case, the value function is twice differentiable almost everywhere. Furthermore, in general, it fails to be differentiable at points where there are multiple optimal trajectories and its differentiability at a point does not guarantee continuous differentiability around this point. In the subsection 0.3 we shown that, under suitable assumptions, the nonemptiness of proximal subdifferential of the value function at a point implies its continuous differentiability on a neighborhood of this point.

0.1 The optimal exit time problem

We assume that a compact nonempty set $U \subset \mathbb{R}^m$ and a continuous function $f : \mathbb{R}^n \times U \rightarrow \mathbb{R}^n$ are given. We consider the *control system*.

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1 A function is semiconcave if it can be written as a sum of a concave function and a $C^2$ function.
\begin{equation}
\begin{cases}
\dot{x}(t) = f(x(t), u(t)), \\
x(0) = x_0 \in \mathbb{R}^n,
\end{cases}
\quad \text{a.e. } t > 0.
\end{equation}

where \( u : \mathbb{R}_+ \to U \) is a measurable function which is called a control for the system (1). The set \( U \) is called the control set. We denote by \( \mathcal{U}_{ad} \) the set of all measurable control functions. We will often require the following assumptions

(A1) There exists \( K_1 > 0 \) such that

\[ |f(x_1, u) - f(x_2, u)| \leq K_1 |x_1 - x_2|, \quad \forall x_1, x_2 \in \mathbb{R}^n, u \in U.\]

(A2) \( D_x f \) exists and is continuous. Moreover, there exists \( K_2 > 0 \) such that

\[ ||D_x f(x_1, u) - D_x f(x_2, u)|| \leq K_2 |x_1 - x_2|, \quad \forall x_1, x_2 \in \mathbb{R}^n, u \in U.\]

It is well known from the ordinal differential equations theory that under assumption (A1), for each \( u \in \mathcal{U}_{ad} \), (1) has a unique solution. In this case, we will denote by \( x^{0,u}(\cdot) \) the solution of (1) and call \( x^{0,u}(\cdot) \) the trajectory (starting at \( x_0 \)) of the control system (1).

We now assume that a closed subset with compact boundary \( \mathcal{X} \) of the state space \( \mathbb{R}^n \) is given and is called the target. For a given trajectory \( x^{0,u}(\cdot) \) of (1), we set

\[ \tau(x_0, u) := \min \{ t \geq 0 : x^{0,u}(t) \in \mathcal{X} \}, \]

with the convention that \( \tau(x_0, u) = +\infty \) if \( x^{0,u}(t) \notin \mathcal{X} \) for all \( t \geq 0 \). Then \( \tau(x_0, u) \) is the time at which the trajectory \( x^{0,u}(\cdot) \) reaches the target for the first time, provided \( \tau(x_0, u) < +\infty \) and we call \( \tau(x_0, u) \) the exit time of the trajectory \( x^{0,u}(\cdot) \). Denote by \( \mathcal{R} \) the set of all \( x_0 \) such that \( \tau(x_0, u) < +\infty \) for some \( u(\cdot) \in \mathcal{U}_{ad} \) and we call \( \mathcal{R} \) the reachable set.

Given two continuous functions \( L : \mathbb{R}^n \times U \to \mathbb{R} \) (called running cost) and \( \psi : \mathbb{R}^n \to \mathbb{R} \) (called terminal cost) with \( L \) positive and \( \psi \) is bounded from below, we consider the functional

\[ J(x_0, u) = \int_0^{\tau(x_0, u)} L(x^{0,u}(s), u(s))ds + \psi(x^{0,u}(\tau(x_0, u))). \]

We are interested in minimizing \( J(x_0, u) \), for \( x_0 \in \mathcal{R} \), over all \( u(\cdot) \in \mathcal{U}_{ad} \). If \( u^*(\cdot) \in \mathcal{U}_{ad} \) is such that

\[ J(x_0, u^*) = \min_{u \in \mathcal{U}_{ad}} J(x_0, u), \]

then we call \( u^*(\cdot) \) an optimal control for \( x_0 \). In this case, \( x^{0,u^*}(\cdot) \) is called an optimal trajectory.

The value function of the optimal exit time problem is defined by

\[ V(x_0) := \inf \{ J(x_0, u) : u(\cdot) \in \mathcal{U}_{ad} \}, \quad x_0 \in \mathcal{R}. \]

From the definition of \( V \), we have the so-called dynamic programming principle...
\[ V(x_0) \leq \int_0^t L(x_0^u(s), u(x)) ds + V(x_0^u(t)) \quad \forall t \in [0, \tau(x_0, u)]. \]

If \( u(\cdot) \) is optimal then the equality holds.

The maximized Hamiltonian associated to the control system is defined by

\[ H(x, p) = \max_{u \in U} \{-p \cdot f(x, u) - L(x, u)\}, \quad (x, p) \in \mathbb{R}^n \times \mathbb{R}^n. \]

It is well-known, under some assumptions (see Theorem 8.18 in [8]), that \( V \) is a viscosity solution of the Hamilton - Jacobi - Bellman equation

\[ H(x, \nabla V(x)) = 0. \]

We now list some more assumptions on the cost functionals and the target which will be used in the sequel.

(A0) For all \( x \in \mathbb{R}^n \), the following set is convex

\[ \mathcal{F}(x) := \{(v, \lambda) \in \mathbb{R}^{n+1} : \exists u \in U \text{ such that } v = f(x, u), \lambda \geq L(x, u)\}. \]

(A3) There exist \( N > 0 \) and \( \alpha > 0 \) such that \( |f(x, u)| \leq N \) and \( L(x, u) \geq \alpha \) for all \( x \in \mathbb{R}^n \) and \( u \in U \).

(A4) The function \( L \) is continuous in both arguments and locally Lipschitz continuous with respect to \( x \), uniformly in \( u \). Moreover, \( L(x, u) \) exists for all \( x, u \) and is locally Lipschitz continuous in \( x \), uniformly in \( u \).

(A5) There exists a neighborhood \( \mathcal{N} \) of \( \text{bdry} \mathcal{K} \) such that \( \psi \) is locally semiconcave and is of class \( C^1 \) in \( \mathcal{N} \). Moreover, denoting by \( G \) the Lipschitz constant of \( \psi \) in \( \mathcal{N} \), we assume

\[ G < \frac{\alpha}{N}. \]

(A6) The boundary of \( \mathcal{K} \) is an \( (n - 1) \)-dimensional manifold of class \( C^{1,1} \) and there exists \( \gamma > 0 \) such that for any \( z \in \text{bdry} \mathcal{K} \), we have

\[ \min_{u \in U} f(z, u)n_z \leq -\gamma, \]

where \( n_z \) denotes the unit outward normal to \( \mathcal{K} \) at \( z \).

Assumption (A0) is a condition to ensure the existence of optimal trajectories. More precisely, one has

**Theorem 1.** [5] [8] Under assumptions (A0) - (A5), there exists a minimizer for optimal control problem for any choice of initial point \( y \in \mathcal{K} \). Moreover, the uniform limit of optimal trajectories is an optimal trajectory; that is, if \( x_k(\cdot) \) are trajectories converging uniformly to \( x(\cdot) \) and every \( x_k(\cdot) \) is optimal for the point \( y_k := x_k(0) \), then \( x(\cdot) \) is optimal for \( y := \lim y_k \).

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2 Throughout the chapter, \( p \) is usually a row vector in \( \mathbb{R}^{n*,} \). However, in this section, \( \mathbb{R}^n \) and its dual space \( \mathbb{R}^{n*,} \) are identical. We also denote by \( a \cdot b \) the inner product between two vectors \( a \) and \( b \).
The condition $G < \alpha / N$ in assumption (A5) can be regarded as a compatibility condition on the terminal cost $\psi$. Together with other assumptions, it ensures the continuity of the value function (see Remark 2.6 in [5] and Proposition IV.3.7 in [1]). Furthermore, we have the following regularity property of the value function.

**Theorem 2.** [5, 8] Under hypothesis (A1)-(A6), the value function $V$ is locally semiconcave in $\mathbb{R} \setminus \mathcal{K}$.

Note that in [5, 8], the semiconcavity result is proved under weaker assumptions on the data. In fact, $\mathcal{K}$ is only assumed to satisfy an interior sphere condition, while $f$, $L$ and $\psi$ are assumed to be semiconcave in the $x$-variable and $L_x$ is only continuous.

For the precise definition, properties and characterizations of semiconcave functions, we refer the reader to [8].

### 0.2 Optimality conditions and sensitivity relations

We present some optimality conditions and sensitivity relations for the optimal exit time problem. One of important tools for our analysis is given by the so-called Pontryagin maximum principle. Before recalling a version of the maximum principle for the optimal control problem under consideration, we need to introduce some notation. For a given subset $A$ of $\mathbb{R}^n$, we denote by $\text{bdry} A$ its boundary, by $A^c$ its complement. The distance function from $A$ is defined for $x \in \mathbb{R}^n$ as

$$d_A(x) := \inf_{y \in A} |x - y|,$$

and the oriented distance function from $A$ is defined by $b_A(x) := d_A(x) - d_{A^c}(x), x \in \mathbb{R}^n$, whenever $A \neq \mathbb{R}^n$.

Let $\Omega$ be an open subset of $\mathbb{R}^n$, $h : \Omega \to \mathbb{R}$ be a lower semicontinuous function and $x \in \Omega$. The **proximal subdifferential** of $h$ at $x$ is the set

$$\partial^p h(x) := \{ v \in \mathbb{R}^n : \text{there exist } c > 0, \rho > 0 \text{ such that }$$

$$h(y) - h(x) - v \cdot (y - x) \geq -c|y - x|^2, \quad \forall y \in B(x, \rho) \}.$$

The **Fréchet subdifferential** of $h$ at $x$ is the set

$$D^- h(x) := \left\{ v \in \mathbb{R}^n : \liminf_{y \to x} \frac{h(y) - h(x) - v \cdot (y - x)}{|y - x|} \geq 0 \right\}.$$

The **Fréchet superdifferential** of $h$ at $x$ is the set

$$D^+ h(x) := \left\{ v \in \mathbb{R}^n : \limsup_{y \to x} \frac{h(y) - h(x) - v \cdot (y - x)}{|y - x|} \leq 0 \right\}.$$

If $h$ is locally Lipschitz, then the reachable gradient of $h$ at $x$ is the set

$$\partial^r h(x) := \{ v \in \mathbb{R}^n : \text{there exist } c > 0, \rho > 0 \text{ such that }$$

$$h(y) - h(x) - v \cdot (y - x) \geq -c|y - x|^2, \quad \forall y \in B(x, \rho) \}.$$
\[ D^* h(x) := \{ v \in \mathbb{R}^n : \exists x_n \to x, \nabla h(x_n) \to v \text{ as } n \to \infty \}. \]

We now start with two technical lemmas.

**Lemma 1.** (see, e.g. [5]) Assume (A1) - (A6). Given \( z \in \text{bdry}\, \mathcal{K} \), let \( \zeta \) be the outer normal to \( \mathcal{K} \) at \( z \). Then there exists a unique \( \mu > 0 \) such that \( H(z, \nabla \psi(z) + \mu \zeta) = 0 \).

Notice that since the boundary of the target \( \mathcal{K} \) is of class \( C^{1,1}_{loc} \), the outer normal to \( \mathcal{K} \) at a point \( z \in \text{bdry}\, \mathcal{K} \) is \( \nabla b_{\mathcal{K}}(z) \). From Lemma 1, the function \( \mu : \text{bdry}\, \mathcal{K} \to \mathbb{R}^+ \) which satisfies \( H(z, \nabla \psi(z) + \mu \nabla b_{\mathcal{K}}(z)) = 0 \) is well-defined. Moreover, we have

**Lemma 2.** [11] Assume (A1) - (A6). The function \( \mu : \text{bdry}\, \mathcal{K} \to \mathbb{R}^+ \) is continuous.

We recall the maximum principle in the following form

**Theorem 3.** Assume (A1) - (A6). Let \( x \in \mathbb{R}^n \setminus \mathcal{K} \) and let \( \bar{u} \) be an optimal control for \( x_0 \). Set for simplicity
\[
    x(t) := x_0 \bar{u}(t), \quad \tau := \tau(x_0, \bar{u}), \quad z := x(\tau).
\]

Let \( p \in W^{1,1}([0, \tau]; \mathbb{R}^n) \) be the solution to the equation
\[
    \dot{p}(t) = D_x f(x(t), \bar{u}(t))^\top p(t) - L_x(x(t), \bar{u}(t)), \tag{2}
\]
with \( p(\tau) = \nabla \psi(z) + \mu(z) \nabla b_{\mathcal{K}}(z) \).

Then \( p \) satisfies
\[
    -p(t). f(x(t), \bar{u}(t)) - L(x(t), \bar{u}(t)) = \mathcal{H}(x(t), p(t)),
\]
for a.e. \( t \in [0, \tau] \)

For the proof of the above maximum principle, we refer the reader to Theorem 4.3 in [5] where the principle is proved under weaker assumptions on \( L \) and \( \psi \).

Given an optimal trajectory \( x(\cdot) \), then, by Lemma 1, there is a unique function \( p(\cdot) \) satisfying the properties of Theorem 3 and we call \( p(\cdot) \) the dual arc associated to the trajectory \( x(\cdot) \). Observe that the dual arc is a nonzero function and satisfies \( p(\tau) = \nabla \psi(x(\tau)) + \mu(x(\tau)) \nabla b_{\mathcal{K}}(x(\tau)) \) where \( \tau \) is the exit time of \( x(\cdot) \). The following theorem gives a connection between the dual arcs and the Fréchet superdifferential of the value function.

**Theorem 4.** [5] Under the assumptions of Theorem 3 the dual arc \( p(\cdot) \) satisfies
\[
    p(t) \in D^+ V(x(t)), \quad \forall t \in [0, \tau).
\]

It is proved in [5, 8] that under the assumptions of Theorem 3 and the following assumption

(H) for any \( x \in \mathbb{R}^n \), if \( \mathcal{H}(x, p) = 0 \) for all \( p \) in a convex set \( C \), then \( C \) is a singleton,
the value function $V$ is differentiable along optimal trajectories except the initial and final point points and therefore, by Theorem \ref{thm:optimal}, if $p(\cdot)$ is the dual arc associated with an optimal trajectory $x_{\text{opt}}(\cdot)$ then $p(t) = \nabla V(x_{\text{opt}}(t))$ for all $t \in (0, \tau(x_0, u))$. This property plays an important role to prove an one-to-one correspondence between the number of optimal trajectories starting at a point $x_0 \in \mathcal{R} \setminus \mathcal{H}$ and the number of elements of the reachable gradient $D^*V(x_0)$ of $V$ at $x_0$. This implies that $V$ is differentiable at $x_0$ iff there is a unique optimal trajectory starting at $x$. The following example shows that without assumption (H), $V$ may not differentiable along optimal trajectories.

**Example 1.** We consider the minimum time problem i.e., $L \equiv 1, g \equiv 0$, for the control system

$$
\begin{align*}
\begin{cases}
\dot{x}_1(t) = u, \\
\dot{x}_2(t) = 0,
\end{cases} 
\quad u \in U := [-1, 1],
\end{align*}
$$

with the initial conditions $x_1(0) = y_1, x_2(0) = y_2$. Define the set

$$
\mathcal{D} = \{(y_1, y_2)^T : 2y_1 - 3y_2 - 2 > 0\} \cap \{(y_1, y_2)^T : 2y_1 + 3y_2 - 2 > 0\} \cap \{(y_1, y_2)^T : 2y_1 + 3y_2 - 14 < 0\} \cap \{(y_1, y_2)^T : 2y_1 - 3y_2 - 14 < 0\}.
$$

The target is the set

$$
\mathcal{H} = \mathbb{R}^2 \setminus \mathcal{D}.
$$

The Hamiltonian is

$$
\mathcal{H}(x, p) = \sup_{u \in U} \left\{-\begin{pmatrix} u \\ 0 \end{pmatrix} \cdot \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} - 1\right\} = |p_1| - 1, \ \forall x \in \mathbb{R}^2, p = (p_1, p_2)^T \in \mathbb{R}^2.
$$

One can easily check that all assumptions of Theorem 3.8 in \cite{5} (which says that the value function is differentiable along optimal trajectories except the starting and the terminal points) are satisfied and assumption (H) is not satisfied. Let $T(\cdot)$ be the minimum time to reach the target.

If $y = (y_1, y_2)^T \in \mathcal{D} \cap \{(y_1, y_2)^T \in \mathbb{R}^2 : y_1 < 4\}$, then $u^*(\cdot) \equiv -1$ is the optimal control for $y$ and

$$
x(t) = \begin{pmatrix} y_1 - t \\ y_2 \end{pmatrix}, \text{ for all } t \in [0, T(y)]
$$

is the optimal trajectory starting at $y$ and we can easily compute that

$$
T(y) = y_1 - \frac{3}{2} |y_2| - 1.
$$

If $y = (y_1, y_2)^T \in \mathcal{D} \cap \{(y_1, y_2)^T \in \mathbb{R}^2 : y_1 > 4\}$, then $u^*(\cdot) \equiv 1$ is the optimal control for $y$, the optimal trajectory is

$$
x(t) = \begin{pmatrix} y_1 + t \\ y_2 \end{pmatrix}, \text{ for all } t \in [0, T(y)],
$$

and the minimum time to reach the target from $y$ is
\[ T(y) = -y_1 - \frac{3}{2} |y_2| + 7. \]

Since \( T \) is not differentiable when \( y_2 = 0 \), \( T \) fails to be differentiable at any point of optimal trajectories starting at \( (y_1, 0) \) ∈ \( \mathcal{D} \).

Later we will see that \( V \) is still differentiable at a point \( x \) iff there is a unique optimal trajectory starting at \( x \) even when (H) is not satisfied. In this case we may not have an one-to-one correspondence between the number of optimal trajectories starting at a point \( x \in \mathcal{R} \setminus \mathcal{H} \) and the number of elements of the reachable gradient \( \mathcal{D}^* V(x) \).

If we assume that

(H1) \( \mathcal{H} \in C^{1,1}_{loc}(\mathbb{R}^n \times (\mathbb{R}^n \setminus \{0\})) \)

then we can compute partial derivatives of the maximized Hamiltonian (see Theorem 7.3.6 and also Remark 8.4.11 in [8]).

**Theorem 5.** If (H1) holds, then for any \((x, p) \in \mathbb{R}^n \times (\mathbb{R}^n \setminus \{0\})\), we have

\[ \mathcal{H}_p(x, p) = -f(x, u^*(x, p)), \]

and

\[ \mathcal{H}_x(x, p) = -D_x f(x, u^*(x, p))^\top p - L_x (x, u^*(x, p)), \]

where \( u^*(x, p) \) is any element of \( U \) such that

\[ -f(x, u^*(x, p)).p - L(x, u^*(x, p)) = \mathcal{H}(x, p). \]

Since we are going to evaluate the Hamiltonian \( \mathcal{H} \) along dual arcs which are nonzero, the lack of differentiability of \( \mathcal{H} \) is not an obstacle. From Theorem 5 and Theorem 5 we have

**Theorem 6.** Assume (A1) - (A6) and (H1). Let \( x(\cdot) \) be an optimal trajectory and let \( p(\cdot) \) be the associated dual arc to \( x(\cdot) \). Then the pair \((x(\cdot), p(\cdot))\) solves the system

\[
\begin{cases}
\dot{x}(t) = -\mathcal{H}_p(x(t), p(t)) \\
\dot{p}(t) = \mathcal{H}_x(x(t), p(t)).
\end{cases}
\]

Consequently \( x(\cdot) \) and \( p(\cdot) \) are of class \( C^1 \).

The next theorem can be seen as a propagation property of the Fréchet subdifferential of the value function forward in time along optimal trajectories.

**Theorem 7.** Assume (A1), (A2) and (A4). Let \( x_0 \in \mathcal{R} \setminus \mathcal{H} \) and let \( \tilde{u}(\cdot) \) be an optimal control for \( x_0 \). Set for simplicity

\[ \tilde{x}(t) := x^{x_0, \tilde{u}}(t), \quad \tau := \tau(x_0, \tilde{u}). \]

Assume that \( D^* V(x_0) \neq \emptyset \) and let \( p \in W^{1,1}([0, \tau]; \mathbb{R}^n) \) be a solution of the equation (2) satisfying \( p(0) \in D^* V(x_0) \). Then \( p(t) \in D^* V(\tilde{x}(t)) \) for all \( t \in [0, \tau] \).
For the proof of the previous Theorem, one can find in [11]. Similarly, one can prove the following propagation result for the proximal subdifferential of the value function which will be used to prove the main results in the next section.

**Theorem 8.** Assume (A1), (A2) and (A4). Let \( x_0 \in \mathcal{R} \setminus \mathcal{K} \) and let \( \bar{u}(\cdot) \) be an optimal control for \( x_0 \). Set for simplicity

\[
\bar{x}(t) := x^{0, \bar{u}}(t), \quad \tau := \tau(x_0, \bar{u}).
\]

Assume that \( \partial^p V(x_0) \neq \emptyset \) and let \( p \in W^{1,1}([0, \tau]; \mathbb{R}^n) \) be a solution of the equation (2) satisfying \( p(0) \in \partial^p V(x_0) \). Then for some \( c > 0 \) and for all \( t \in [0, \tau) \), there exists \( r > 0 \) such that, for every \( z \in B(\bar{x}(t), r) \),

\[
V(z) - V(\bar{x}(t)) \geq p(t).(z - \bar{x}(t)) - c|z - \bar{x}(t)|^2.
\]

Consequently, \( p(t) \in \partial^p V(\bar{x}(t)) \) for all \( t \in [0, \tau) \).

Using above results, we can obtain the following results 9see [11] for the proofs).

**Theorem 9.** Assume (A0) - (A6) and (H1). Let \( x_0 \in \mathcal{R} \setminus \mathcal{K} \) be such that \( V \) is differentiable at \( x_0 \). Consider the solution \( (x(\cdot), p(\cdot)) \) of (3) with initial conditions

\[
\begin{cases}
    x(0) = x_0 \\
p(0) = DV(x_0).
\end{cases}
\]

Then \( x(\cdot) \) is an optimal trajectory for \( x_0 \) and \( p(\cdot) \) is the dual arc associated to \( x(\cdot) \) with \( p(t) = DV(x(t)) \) for all \( t \in [0, \tau) \) where \( \tau \) is the exit time of \( x(\cdot) \). Moreover, \( x(\cdot) \) is the unique optimal trajectory starting at \( x_0 \).

**Theorem 10.** Assume (A0) - (A6) and (H1). Let \( x_0 \in \mathcal{R} \setminus \mathcal{K} \) and \( q \in D^* V(x_0) \). Consider the solution \( (x(\cdot), p(\cdot)) \) of (3) with initial conditions

\[
\begin{cases}
    x(0) = x_0 \\
p(0) = q.
\end{cases}
\]

Then \( x(\cdot) \) is an optimal trajectory for \( x_0 \) and \( p(\cdot) \) is the dual arc associated to \( x(\cdot) \). Moreover \( p(t) \in D^* V(x(t)) \) for all \( t \in [0, \tau) \) where \( \tau \) is the exit time of \( x(\cdot) \).

**Theorem 11.** Assume (A0) - (A6) and (H1). If there is only one optimal trajectory starting at a point \( x \in \mathcal{R} \setminus \mathcal{K} \) then \( V \) is differentiable at \( x \).

From Theorem 9 and Theorem 11 we have

**Corollary 1.** Assume (A0) - (A6) and (H1). The value function \( V \) is differentiable at a point \( x \in \mathcal{R} \setminus \mathcal{K} \) if and only if there exists a unique optimal trajectory starting at \( x \).

In Example 1 the value function is not differentiable at any point \( (y_1, 0)^T \in \mathcal{R} \) although there is a unique optimal trajectory for every \( (y_1, 0)^T \) with \( y_1 \neq 4 \). The reasons are that the maximized Hamiltonian \( \mathcal{K} \) does not belong to \( C^{1,1}_{loc}(\mathbb{R}^2 \times (\mathbb{R}^2 \setminus \mathcal{K})) \)
{0}) and that the target is not smooth. We now give a simple example showing that the value function is differentiable at a point \(x\) although there are multiple optimal trajectories starting at \(x\).

**Example 2.** We consider the minimum time problem for the control system

\[
\begin{aligned}
\dot{y}_1(t) &= u_1, \\
\dot{y}_2(t) &= u_2,
\end{aligned}
\]

with the initial condition \(y_1(0) = x_1, y_2(0) = x_2\). The target is the set

\[
\mathcal{K} = \left\{ (x_1, x_2) \in \mathbb{R}^2 : x_1 \leq 0 \right\} \cap \left\{ (x_1, x_2) : x_2 \leq -\sqrt{-4x_1^2 - 4x_1} \right\}
\]

The Hamiltonian is defined by \(\mathcal{H}(x, p) = |p_1| + |p_2| - 1, \forall x \in \mathbb{R}^2, p = (p_1, p_2)^\top \in \mathbb{R}^2\). We can easily check that assumptions (A0) - (A6) are satisfied and (H1) is not satisfied.

We next give a class of control systems which can be applied Corollary 1.

**Example 3 (see, e.g. Example 4.12 [5]).** We consider the control system with the dynamics given by

\[
f(x, u) = h(x) + \sigma(x)u,
\]

where \(h : \mathbb{R}^n \to \mathbb{R}^n, \sigma : \mathbb{R}^n \to \mathbb{M}^{m \times n}\) and the control set \(U\) is the closed ball of center zero and radius \(R > 0\) in \(\mathbb{R}^n\). We also consider the running cost of the form

\[
L(x, u) = \ell(x) + \frac{1}{2} |u|^2,
\]

where \(\ell : \mathbb{R}^n \to \mathbb{R}\).

Since \(f\) is affine and \(L\) is convex with respect to \(u\) and \(U\) is convex, one can check that assumption (A0) is satisfied. If we assume that \(\sigma, h, \ell\) are of class \(C^{1,1}\), that \(\sigma, h\) are bounded and Lipschitz and that \(\ell\) is bounded below by a positive constant, then assumption (A1) - (A4) are satisfied. The Hamiltonian

\[
\mathcal{H}(x, p) = \max_{u \in U} \left\{ -h(x) + \sigma(x)u, p - \ell(x) - \frac{1}{2}|u|^2 \right\}
\]

\[
= \begin{cases} 
-\ell(x) + \frac{1}{2} |\sigma(x)^\top p|^2 & \text{if } |\sigma(x)^\top p| \leq R \\
-\ell(x) + R |\sigma(x)^\top p| - \frac{R^2}{2} & \text{if } |\sigma(x)^\top p| > R
\end{cases}
\]
satisfies assumption (H1). Then if the final cost function \( \psi \) and the target satisfy assumption (A5) and (A6) then our result can be applied.

### 0.3 Local regularity of the value function

In this section, we provide sufficient conditions which guarantee the continuous differentiability of the value function \( V \) around a given point. Local \( C^1 \) regularity of \( V \) is discussed in the subsection 0.3.1 whereas local \( C^k \) \((k \geq 2)\) regularity of \( V \) is established in the subsection 0.3.2. In both subsections 0.3.1 and 0.3.2, the main condition to ensure the continuous differentiability of \( V \) around a given point \( x \) is the nonemptiness of the proximal subdifferential of \( V \) at \( x \).

#### 0.3.1 Local \( C^1 \) regularity

In addition, we require the following assumptions.

(A7) \( \psi \) is of class \( C^2 \) in a neighborhood of \( \text{bdry} \mathcal{K} \) and \( \text{bdry} \mathcal{K} \) is of class \( C^2 \).

(H2) \( \mathcal{H} \in C^2_{loc}(\mathbb{R}^n \times (\mathbb{R}^n \setminus \{0\})) \).

Below we denote by \( T_{\text{bdry}} \mathcal{K}(z) \) the tangent space to the \((n-1)\)-dimensional \( C^2 \)-manifold \( \text{bdry} \mathcal{K} \) at \( z \in \text{bdry} \mathcal{K} \).

Consider the Hamiltonian system

\[
\begin{align*}
-\dot{x}(t) &= \mathcal{H}_p(x(t), p(t)) \\
\dot{p}(t) &= \mathcal{H}_x(x(t), p(t)),
\end{align*}
\]

on \([0, T]\) for some \( T > 0 \), with the final conditions

\[
\begin{align*}
x(T) &= z \\
p(T) &= \varphi(z),
\end{align*}
\]

where \( z \) is in a neighborhood of \( \text{bdry} \mathcal{K} \) and \( \varphi(z) = \nabla \psi(z) + \mu(z) \nabla b_{\mathcal{H}}(z) \) with \( \mu(\cdot) \) satisfying \( \mathcal{H}(z, \nabla \psi(z) + \mu(z) \nabla b_{\mathcal{H}}(z)) = 0 \). Note that, by (A7), \( \mu(\cdot) \) is of class \( C^1 \) in a neighborhood \( \text{bdry} \mathcal{K} \) (see Proposition 3.2 in [12]) and therefore \( \varphi(\cdot) \) is of class \( C^1 \) in a neighborhood of \( \text{bdry} \mathcal{K} \).

For a given \( z \) in a neighborhood of \( \text{bdry} \mathcal{K} \), let \( (x(\cdot; z), p(\cdot; z)) \) be the solution of (5) - (6) defined on a time interval \([0, T]\) with \( T > 0 \). Consider the so-called variational system

\[
\begin{align*}
-\dot{X} &= \mathcal{H}_{xp}(x(t), p(t))X + \mathcal{H}_{pp}(x(t), p(t))P, & X(T) &= I \\
\dot{P} &= \mathcal{H}_{xx}(x(t), p(t))X + \mathcal{H}_{px}(x(t), p(t))P, & P(T) &= D\varphi(z).
\end{align*}
\]

Then the solution \((X, P)\) of (7) is defined in \([0, T]\) and depends on \( z \). Moreover

\[
X(\cdot; z) = D_x x(\cdot; z) \quad \text{and} \quad P(\cdot; z) = D_x p(\cdot; z), \quad \text{on} \ [0, T].
\]
Definition 1. For $z \in \text{bdry} \mathcal{H}$, the time

$$t_c(z) := \inf\{t \in [0, T] : X(s)\theta \neq 0, \forall 0 \neq \theta \in T_{\text{bdry} \mathcal{H}}(z), \forall s \in [t, T]\}$$

is said to be conjugate-like for $z$ iff there exists $0 \neq \theta \in T_{\text{bdry} \mathcal{H}}(z)$ such that

$$X(t_c(z))\theta = 0.$$ 

In this case, the point $x(t_c(z))$ is called conjugate-like for $z$.

Remark 1. In the classical definition of conjugate point it is required, for some $0 \neq \theta \in \mathbb{R}^N$, $X(t_c(z))\theta = 0$ (see e.g. [7][9][12] and Section 0.5.2). Here, we have narrowed the set of such $\theta$ getting then a stronger result in Theorem 12 below than the one we would have with the classical definition.

Theorem 12. Assume (A0) - (A7) and (H2). Let $x_0 \in \mathbb{R} \setminus \mathcal{H}$ be such that $V$ is differentiable at $x_0$ and $x^{0,u}(\cdot)$ be the optimal trajectory for $x_0$. Set $z = x^{0,u}(\tau(x_0,u))$. If there is no conjugate-like time in $[0, \tau(x_0,u))$ for $z$ then $V$ is of class $C^1$ in a neighborhood of $x_0$.

When the maximized Hamiltonian is strictly convex with respect to the second variable, we can progress as in [2][3] to obtain the following result.

Theorem 13. Assume (A0) - (A7), (H2) and that $\mathcal{H}_{pp}(x, p) > 0$ for all $(x, p) \in \mathbb{R}^n \times (\mathbb{R}^n \setminus \{0\})$. Let $x_0 \in \mathbb{R} \setminus \mathcal{H}$. If $\partial^P V(x_0) \neq \emptyset$, then $V$ is of class $C^1$ in a neighborhood of $x_0$.

Since $V$ is locally semiconcave and $\partial^P V(x_0) \neq \emptyset$, $V$ is differentiable at $x_0$. The idea of the proof is to absent a conjugate time for the final point of the optimal trajectory starting at $x_0$ and then apply Theorem 12.

In Example 3 if $\sigma(x)$ is nonsingular for all $x \in \mathbb{R}^n$ then $\mathcal{H}_{pp}(x, p) > 0$ for all $(x, p) \in \mathbb{R}^n \times (\mathbb{R}^n \setminus \{0\})$. Therefore if $h, \sigma, \ell, \psi$ and $\text{bdry} \mathcal{H}$ are smooth enough then Theorem 13 can be applied.

When the running cost does not depend on $u$, i.e., $L = L(x)$, the maximized Hamiltonian is never strictly convex with respect to the second variable. In this case $0 \neq p \in \ker \mathcal{H}_{pp}(x, p)$ for all $x \in \mathbb{R}^n$ whenever $\mathcal{H}_{pp}(x, p)$ exists. Following the lines for the minimum time problem in [10], we obtain the following particular case.

Theorem 14. Assume that (A0) - (A7), (H2) hold true, the kernel of $\mathcal{H}_{pp}(x, p)$ has the dimension equal to 1 for every $(x, p) \in \mathbb{R}^n \times (\mathbb{R}^n \setminus \{0\})$ and that $L = L(x)$. Let $x_0 \in \mathbb{R} \setminus \mathcal{H}$. If $\partial^P V(x_0) \neq \emptyset$, then $V$ is of class $C^1$ in a neighborhood of $x_0$.

Example 4 (see, e.g., [10]). Consider the control system with the dynamics given by

$$f(x, u) = h(x) + \sigma(x)u.$$
where \( h : \mathbb{R}^n \to \mathbb{R}^n \), \( \sigma : \mathbb{R}^n \to \mathcal{L}(\mathbb{R}^n; \mathbb{R}^n) \) and the control set \( U \) is the closed ball in \( \mathbb{R}^n \) of center zero and radius \( R > 0 \).

Since \( f \) is affine with respect to \( u \), assumption (A0) is verified. Let \( L(x, u) = L(x) \) for all \((x, u) \in \mathbb{R}^n \times U\). The Hamiltonian

\[
\mathcal{H}(x, p) = \max_{u \in U} \{ -(h(x) + \sigma(x)u), p\} - L(x)
\]

\[
= -h(x).p + \max_{u \in U} \{ -u.\sigma(x)\top p \} - L(x)
\]

\[
= -h(x).p + |\sigma(x)\top p| - L(x)
\]

satisfies assumption (H2) whenever \( \sigma(x) \) is also surjective for all \( x \in \mathbb{R}^n \) and \( h, \sigma, L \) are of class \( C^2 \). Furthermore, for all \((x, p) \in \mathbb{R}^n \times (\mathbb{R}^n \setminus \{0\}) \)

\[
\mathcal{H}_p(x, p) = -h(x) + \frac{1}{|\sigma(x)\top p|} \sigma(x)\sigma(x)\top p
\]

and for any \( q \in \mathbb{R}^n \),

\[
\mathcal{H}_pp(x, p)(q, q) = \frac{1}{|\sigma(x)\top p|} |\sigma(x)\top q|^2 - \frac{1}{|\sigma(x)\top p|^3} \left( \sigma(x)\top p.\sigma(x)\top q \right)^2.
\]

Fix any \( q \in \ker \mathcal{H}_pp(x, p) \). Then, from the above equality we get

\[
|\sigma(x)\top p|^2 |\sigma(x)\top q|^2 = \left( \sigma(x)\top p.\sigma(x)\top q \right)^2.
\]  

(8)

On the other hand, if \( \sigma(x)\top q \notin \mathbb{R} (\sigma(x)\top p) \), then

\[
|\sigma(x)\top p.\sigma(x)\top q| < |\sigma(x)\top p||\sigma(x)\top q|.
\]

Hence, by (8), \( \sigma(x)\top q \in \mathbb{R}\sigma(x)\top p \). Let \( \lambda \in \mathbb{R} \) be such that \( \sigma(x)\top q = \lambda \sigma(x)\top p \). Consequently \( \sigma(x)\top (q - \lambda p) = 0 \). Since \( \sigma(x) \) is surjective, we deduce that \( q = \lambda p \) and that \( q \in \mathbb{R} p \).

Using the inclusion \( p \in \ker \mathcal{H}_pp(x, p) \), we deduce that \( \ker \mathcal{H}_pp(x, p) = \mathbb{R} p \) for all \((x, p) \in \mathbb{R}^n \times (\mathbb{R}^n \setminus \{0\}) \), i.e.,

\[
\dim \ker \mathcal{H}_pp(x, p) = 1, \forall (x, p) \in \mathbb{R}^n \times (\mathbb{R}^n \setminus \{0\}) .
\]

So, if the target \( \mathcal{H} \) and \( \psi \) are of class \( C^2 \) and for any \( z \in \text{bdry} \mathcal{H} \), the classical inward pointing condition

\[
\min_{u \in U} (h(z) + \sigma(z)u).n_z < 0
\]

holds true, then Theorem[14] can be applied.
0.3.2 Local $C^k$ regularity

Let $k$ be an integer with $k \geq 2$. In this subsection, we require the following additional assumptions.

(A8) The functions $f$ and $L$ are of class $C^k$ in both arguments and the boundary of the target $\mathcal{K}$ is an $(n-1)$-manifold of class $C^{k+1}$. Moreover, $\psi$ is of class $C^{k+1}$ in a neighborhood $\mathcal{N}$ of $\text{bdry} \, \mathcal{K}$.

(A9) For all $(x,p) \in \mathbb{R}^n \times (\mathbb{R}^n \setminus \{0\})$, there exists a unique $u^* \in U$ such that

$$-f(x,u^*),p - L(x,u^*) = \mathcal{H}(x,p),$$

and the function $u^* : (x,p) \mapsto u^*(x,p)$ is of class $C^k$ in $\mathbb{R}^n \times (\mathbb{R}^n \setminus \{0\})$.

For our analysis, in assumption (A9), we only need that $u^*$ is of class $C^k$ in an open neighborhood of the set $\{(x,p) \in \mathbb{R}^n \times (\mathbb{R}^n \setminus \{0\}) : \mathcal{H}(x,p) = 0\}$. For examples satisfying this condition, one can find in [5, 12]. It follows from (A8) and (A9) that the Hamiltonian satisfies

(H3) $\mathcal{H} \in C^k(\mathbb{R}^n \times (\mathbb{R}^n \setminus \{0\}))$.

We next introduce the definition of conjugate times which is related to the Jacobians of solutions of the backward Hamiltonian system considered in [12]. Given $z \in \text{bdry} \, \mathcal{K}$, we denote by $(y(z,\cdot),q(z,\cdot))$ the solution the backward Hamiltonian system

$$\begin{cases}
\dot{y}(t) = \mathcal{H}_p(y(t),q(t)) \\
\dot{q}(t) = \mathcal{H}_q(y(t),q(t))
\end{cases}$$

with the initial conditions

$$\begin{cases}
y(0) = z \\
q(0) = \varphi(z),
\end{cases}$$

where $\varphi(z) = \nabla \psi(z) + \mu(z)\nabla b_\mathcal{K}(z)$ with $\mu(\cdot)$ satisfying

$$\mathcal{H}(z,\nabla \psi(z) + \mu(z)\nabla b_\mathcal{K}(z)) = 0.$$

Note that $\mathcal{H}(z,\varphi(z)) = 0$ and that under our assumptions the function $\varphi : \text{bdry} \, \mathcal{K} \to \mathbb{R}^n$ is of class $C^k$ (see, e.g., [12]).

As shown in [12], the solution $(y(z,\cdot),q(z,\cdot))$ of (10) - (11) is defined for all $t \in [0,+\infty)$. Moreover, $y(\cdot),q(\cdot)$ are of class $C^k$ on $\text{bdry} \, \mathcal{K} \times [0,+\infty)$.

Now let $Y$ and $Q$ denote respectively the Jacobians of $y(\cdot)$ and $q(\cdot)$ with respect to the pair $(z,t)$ in $\text{bdry} \, \mathcal{K} \times [0,\infty)$ where $(y(z,\cdot),q(z,\cdot))$ solves (10) - (11). Then $(Y,Q)$ is the solution of the system

$$\begin{cases}
\dot{Y} = \mathcal{H}_{yp}(y(z,t),q(z,t))Y + \mathcal{H}_{yp}(y(z,t),q(z,t))Q \\
\dot{Q} = \mathcal{H}_{qq}(y(z,t),q(z,t))Y + \mathcal{H}_{qp}(y(z,t),q(z,t))Q,
\end{cases}$$

with the initial conditions

$$\begin{cases}
Y(z,0) = A(z) \\
Q(z,0) = B(z),
\end{cases}$$

(13)
where $A(z), B(z)$ are square matrices depending smoothly on $z$ which we can compute.

As explained in [12], the Jacobian $Y$ and $Q$ are understood in the following sense. Fixed $z_0 \in \text{bdry} \mathcal{K}$ and $t_0 > 0$. Since $\text{bdry} \mathcal{K}$ is an $(n-1)$-dimensional manifold of class $C^{k+1}$, there exist an open neighborhood $I$ and a parameterized function $\xi : z \in I \mapsto \eta \in \xi(I) \subset \mathbb{R}^{n-1}$ of class $C^{k+1}$ with the inverse $\phi$ of class $C^{k+1}$, where $\psi(I)$ is an open neighborhood of $\eta_0 = \xi(z_0)$. Then $Y(z_0, t_0)$ and $Q(z_0, t_0)$ denote the Jacobians of $Y(\phi(\cdot), \cdot)$ and $Q(\phi(\cdot), \cdot)$ with respect to the coordinates $\eta \in \mathbb{R}^{n-1}$ and the time $t$ at the point $(\eta_0, t_0)$ i.e., $D_Y(\phi(\eta_0), t_0)$ and $D_Q(\phi(\eta_0), t_0)$. In this case,

$$A(z_0) = Y(z_0, 0) = \left( \mathcal{H}_p(z_0, \phi(z_0)), \frac{\partial \phi}{\partial \eta}(\eta_0) \right),$$

and one can compute that $\det A(z_0) = \alpha \mathcal{H}_p(z_0, \phi(z_0)), \phi(z_0)$ for some real constant $\alpha \neq 0$. Therefore, $\det A(z_0) = \det Y(z_0, 0) \neq 0$ (see proof of Lemma 4.2 in [12]). Then

$$\text{rank} \left( \begin{array}{c} Y(z_0, 0) \\ Q(z_0, 0) \end{array} \right) = n$$

and by properties of linear systems, we have

$$\text{rank} \left( \begin{array}{c} Y(z_0, t) \\ Q(z_0, t) \end{array} \right) = \text{rank} \left( \begin{array}{c} Y(z_0, 0) \\ Q(z_0, 0) \end{array} \right) = n, \quad \forall t \in [0, +\infty).$$

Note that this definition of the Jacobian depends on the parameterized function. For our purpose, however, this does not matter because we only focus on the ranks of the matrices $Y$ and $Q$ which are independent of the choice of the parameterized functions.

**Definition 2.** For $z \in \text{bdry} \mathcal{K}$, the time

$$t_c(z) := \inf \{ t \in [0, +\infty) : \det Y(z, s) \neq 0, \forall s \in [0, t] \}$$

is said to be conjugate for $z$ if and only if

$$t_c(z) < +\infty \quad \text{and} \quad \det Y(z, t_c(z)) = 0.$$

Fixed $z_0 \in \text{bdry} \mathcal{K}$ and $T_0 > 0$. Since $Y(z_0, t)$ is invertible for $t$ sufficiently small, if there exists conjugate time $t_c$ for $z_0$ then $t_c > 0$. On the other hand, if there is no conjugate time for $z_0$ in $[0, T_0]$, then by the continuity and the fact that $\det Y(t, z_0) \neq 0$ for all $t \in [0, T_0]$, there exist $\epsilon > 0, \sigma > 0$ such that there is no conjugate time for any $z \in B(z_0, \sigma) \cap \text{bdry} \mathcal{K}$ in $[0, T]$ with $T < T_0 + \epsilon$. In this case $y(\cdot)$ is an one-to-one correspondence in a neighborhood of $(z_0, T_0)$. Using this fact, one can prove the following theorem.

**Theorem 15.** Assume (A0) - (A6) and (A8) - (A9). Let $\bar{x} \in \mathcal{R} \setminus \mathcal{K}$ be such that $V$ is differentiable at $\bar{x}$. Let $x(\cdot)$ be the optimal trajectory starting at $\bar{x}$ and $\tau$ be the exit time of $x(\cdot)$. Set $\bar{z} = x(\tau) \in \text{bdry} \mathcal{K}$. If there is no conjugate time for $\bar{z}$ in $[0, \tau]$ then $V$ is of class $C^k$ on an open neighborhood of $\bar{x}$. 

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Remark 2. Observe that if $V$ is differentiable at a point $\bar{x} \in \mathcal{R} \setminus \mathcal{K}$ then $V$ is differentiable along the optimal trajectory starting at $\bar{x}$ except the final point. Then in Theorem 15 we can conclude that $V$ is of class $C^k$ on an open neighborhood of $x(s)$ for all $s \in [0, \tau)$.

Following the idea used in [4] where the authors study the regularity of the value function for a Mayer optimal control problem, by using Theorem 15 we can prove the following theorem.

**Theorem 16.** Assume (A0) - (A6) and (A8) - (A9). Let $\bar{x} \in \mathcal{R} \setminus \mathcal{K}$. If $\partial^2 P V(\bar{x}) \neq 0$ then $V$ is of class $C^k$ on an open neighborhood of $\bar{x}$.

Remark 3. In the case the running cost does not depend on $u$-variable, i.e., $L = L(x)$, the results of this section still hold true if we replace assumption (A9) by (H3), a weaker assumption.

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