Phase Structure of the Gauged Yukawa Model

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Abstract

Based on the ladder Schwinger-Dyson equation, we investigate phase structure of the gauged Yukawa model possessing a global $SU(2)_L \times SU(2)_R$ symmetry and an unbroken (vector-like) gauge symmetry. We show that even when we tune the squared mass of the scalar boson in the Lagrangian to be positive, there still exists the dynamical chiral symmetry breaking due to fermion pair condensate (VEV of the composite scalar) triggered by the strong Yukawa coupling larger than a certain critical value. We find a “nontrivial ultraviolet fixed line” and “renormalized trajectories” in the three-dimensional coupling space of the Yukawa coupling, the gauge coupling and the “hopping parameter” of the elementary scalar field. Presence of the gauge coupling is crucial to existence of the fixed line. Implications of the result for the lattice calculation and the top quark condensate model are also discussed.
1 Introduction

As it turned out, the standard model of modern particle physics is extremely successful. However, the central mystery of this model is the two missing ingredients, the Higgs boson and the top quark: Why are they so heavy? The ever increasing experimental bound of the top quark mass is now getting closer to the weak scale 246 GeV (the present LEP constraint is $164\text{GeV} \pm 27\text{GeV}$, while the CDF bound is $>113\text{GeV}$ [1]). This seems to suggest a special role of the top quark in the electroweak symmetry breaking and hence a strong connection with yet another missing ingredient, the Higgs boson. In contrast to the passive role of Yukawa couplings simply picking up the already tuned Higgs vacuum expectation value (VEV) to give mass to the known fermions, such a strong Yukawa coupling of the top quark may affect the entire dynamical picture (phase structure) of the standard Higgs sector.

This situation can be most naturally understood by the Top Quark Condensate Model (Top Mode Standard Model)[2] which was proposed by Miransky, Tanabashi and Yamawaki (MTY)[3] and by Nambu[4] independently. This model entirely replaces the standard Higgs doublet by the composite one formed by the strongly coupled short range dynamics (four-fermion interaction) which is responsible for the top quark condensate. The Higgs boson emerges as a $t\bar{t}$ bound state and hence is deeply connected with the top quark itself. Actually, based on the explicit solution of the ladder Schwinger-Dyson (SD) equation of the gauged Nambu-Jona-Lasinio (NJL) model (QCD plus four-fermion interaction), MTY[3] predicted the top quark mass to be about 250GeV (for the Planck scale cutoff), which is just the order of the electroweak symmetry breaking scale. This model was further formulated in an elegant fashion by Bardeen, Hill and Lindner[5] in the standard model language: They incorporated the composite Higgs loop effects, which turned out to reduce the above MTY value down to 220GeV, a somewhat smaller value but still on the order of the weak scale. Although the prediction appears to be substantially higher than the present experimental bound mentioned above, there still remains a possibility that (at least) the essential feature
of the top quark condensate idea may eventually survive.

What is the origin of the top mode four-fermion interaction, then? This question was first addressed in a concrete manner by Kondo, Tanabashi and Yamawaki (KTY)[6] (See also Ref.[7]), who suggested a heavy spinless boson exchange model with the strong Yukawa coupling ("Yukawa-driven Top Mode Model (YTMM)") in the framework of the SD equation. The idea was further discussed by Clague and Ross[8] in a slightly different framework. One might be tempted to consider an alternative, the heavy spin 1 boson exchange model. However, as was already pointed out[6], this kind of model does not give rise to the desired four-fermion interaction, $g^{(2)}$ term in Ref.[3], which communicates the top quark condensate to the bottom quark mass. Hence it has no chance to give mass to the bottom quark without suffering from the axion problem (See the second paper of Ref.[3], and also Refs.[2, 9, 10]). This difficulty also applies to the more recently proposed models[11, 12] of heavy spin 1 boson exchange.

In YTMM[6, 7, 8] the top quark condensate is triggered by the strong Yukawa interaction. Even when we start with the same Lagrangian as the standard model (with opposite sign of the squared scalar mass and hence without Higgs VEV), we have a quite different dynamical picture: The electroweak symmetry breaking takes place mainly due to the top quark condensate instead of the ad hoc tuned Higgs VEV. Thus the principal role of the "elementary" Higgs is not to break the electroweak symmetry through its VEV but to supply a strong attractive force between the top and anti-top through the Yukawa coupling so as to trigger the top quark condensate or the composite Higgs VEV.\(^1\)

It should be noted that YTMM[6, 7, 8] yields potentially strong higher dimensional operators in addition to the original four-fermion interaction[3, 5] relevant to the top quark condensate. Actually, it was first pointed out by Suzuki[14] that inclusion of

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\(^1\)This picture is contrasted with the Nambu’s bootstrap[13] which implies identification of the “elementary” Higgs with the composite one, while we here distinguish between the two Higgses, i.e., one (elementary) with the GUT or Planck scale mass and the other (composite) with the weak scale mass whose VEV makes a dominant contribution to the $W/Z$ masses.
such higher dimensional operators may reduce the top mass prediction of the top mode standard model. The role of higher dimensional operators was further clarified by Hasenfratz et al.[15] (see also Ref.[16]). Once we specify a possible underlying theory, we can estimate the effects of these higher dimensional operators on the top mass prediction. For example, the spin 1 boson exchange model does not seem to yield large coefficients for the higher dimensional operators [12]. If, on the other hand, there exist large effects of the higher dimensional operators in YTMM, then this model may have a chance to predict a smaller top mass to be consistent with even the present LEP experiments mentioned above.  

Apart from the top quark condensate, the strongly coupled Yukawa model may be applied to other models beyond the standard model. An interesting example is a “Heavy Scalar Technicolor Model”[17] in which the role of the ETC gauge bosons in the ordinary technicolor scenario is replaced by the heavy weak doublet scalar boson exchange which communicates the technifermion condensate to the ordinary fermion mass. This is actually described by the gauged Yukawa model, with the gauge interaction being the technicolor instead of the QCD. Such a model may have a “walking/standing technicolor”[18] gauge coupling. When the Yukawa coupling becomes strong, the dynamical feature of this model is expected to be somewhat similar to the “strong ETC model”[19] based on the gauged NJL model, but may provide a rather different picture to be testable in the future experiments.

In order to draw a definite conclusion on the above problems, however, we need to solve nonperturbative dynamics of the strong Yukawa coupling. This is a very difficult task, however, and cannot be done at once. Actually, as the first step KTY[6] studied the phase structure of the pure Yukawa theory (without gauge coupling) with a global $SU(2)_L \times SU(2)_R$ symmetry within the framework of the ladder SD equation. (The extension has also been made[7] to include the $SU(2)_L \times U(1)_Y$-invariant Yukawa model,

\footnote{Although the YTMM might then loose predictive power for the top mass itself, it might maintain predictability of the top to Higgs mass ratio which would be testable in the future experiments.}
which corresponds to the realistic cases of the top quark condensate and the heavy scalar technicolor models.) It was shown that without tadpole, we have a clear signal of the dynamical symmetry breaking; a vanishing scalar VEV and nonzero fermionic condensate (fermion dynamical mass $M \neq 0$) for the strong Yukawa coupling larger than the critical coupling. However, inclusion of the tadpole correlates the both order parameters and makes the concept of “dynamical symmetry breaking” somewhat ambiguous, even if we have a nonzero fermionic condensate for the strong coupling region (Actually, there exists a critical coupling.). KTY proposed a possible criterion for the “dynamical symmetry breaking” that the fermionic current contribution dominates the scalar one to the decay constant $F_π$ of the Nambu-Goldstone (NG) bosons in such a way that they are mostly the composite Higgs with a small admixture of the elementary one. Such a situation in fact turned out to be the case in a wide range of the parameter space in the $SU(2)_L \times SU(2)_R$ Yukawa model.

In this paper we shall extend the previous analysis[6] so as to include a vector-like gauge coupling ("gauged Yukawa model" in the same sense as the gauged NJL model) and an analog of the hopping parameter $Z_φ$ of the elementary scalar field $φ$ whose kinetic term is parameterized as $\frac{Z_φ}{2} (\partial_μ φ)^2$. Based on the SD equation with the standing gauge coupling (running effect ignored), we study the phase structure of the model in the three-dimensional parameter space (Yukawa coupling, gauge coupling, $Z_φ$) compared with the previous analysis in one-dimensional space of the Yukawa coupling[6].

Inclusion of the “hopping parameter” $Z_φ$ is of course vital to our analysis, since this is the very parameter that characterizes the deviation, though not all, from the gauged NJL model$^3$ ($Z_φ = 0$) and hence from the original top quark condensate model. Actually, we find that the fermion mass is lowered due to $Z_φ > 0$ when compared with the gauged NJL model.

$^3$For extensive study of the gauged NJL model with standing gauge coupling, see Ref.[21] and references cited therein. As to the model with running coupling, see Ref.[22].
Inclusion of the gauge coupling is motivated by the realistic situation of the top quark condensate model where the QCD coupling makes a significant contribution to the top mass prediction [2]. Actually, the gauge coupling drastically changes the phase structure: We discover the “fixed line” and the “renormalized trajectories” which can only be revealed by the presence of the (standing) gauge coupling. This we believe is a novel feature of the present model and is one of the major achievements of this paper. Roughly speaking, the renormalized trajectories correspond to the boundary which separates the region of the “dynamical symmetry breaking” (fermionic dominance) from that of the “non-dynamical symmetry breaking” (fermionic non-dominance) in the sense of Ref.[6] mentioned above.

The paper is organized as follows. In Section 2 we set up our machinery, the ladder SD equation for the gauged Yukawa model with a standing gauge coupling and a global $SU(2)_L \times SU(2)_R$ symmetry. We include a tadpole contribution. In Section 3 we present a new formula (“generalized Pagels-Stokar (PS) formula”) for the decay constant $F_\pi$ of the NG boson in this model. $F_\pi^2$ consists of two parts one from the scalar current, and the other from the fermionic one, the latter being given by the PS formula[20] originally developed in the theory without scalar field. The very structure of our formula dictates that the fermion mass must be smaller in the gauged Yukawa model than in the gauged NJL model. Then in Section 4 numerical analysis of the phase structure of this model is given. We identify the line of constant $M$ (fermion dynamical mass) and constant $F_\pi$ with the “renormalization-group (RG) flow”. We discover a “fixed line” and “renormalized trajectories” in the three-dimensional parameter space of (Yukawa coupling, gauge coupling, $Z_\phi$). Section 5 is the analytical study of the model which reproduces the numerical one in Section 4 and further provides the scaling relation near the phase transition points (critical surface). We find an analytical expression for the whole fixed line. Section 6 is devoted to the conclusion and discussions. Possible implications for the lattice studies and the top quark condensate model are discussed. It is suggested that the top quark mass prediction of YTMM may
be lower than the original top mode prediction[3, 5] based on the gauged NJL model.

2 Schwinger-Dyson equation

We start with the Lagrangian of the gauged Yukawa model (Yukawa model coupled to a vector-like \( SU(N_c) \) gauge interaction) with a global \( SU(2)_L \times SU(2)_R \) symmetry:

\[
\mathcal{L} = \bar{\psi} \left( i \partial \! \! \! / + \frac{g}{\sqrt{N_c}} G \right) \psi - \tilde{g}_y \left[ \bar{\psi}_L \Phi \psi_R + \bar{\psi}_R \Phi \psi_L \right] - \frac{1}{4} G^{\alpha \beta} G_{\alpha \beta} \\
+ \frac{1}{4} \text{tr} \left( \partial_{\mu} \Phi^\dagger \partial^\mu \Phi \right) - m_\phi^2 \frac{1}{4} \text{tr} (\Phi^\dagger \Phi) - \frac{\tilde{\lambda}_\phi}{16} \left( \text{tr}(\Phi^\dagger \Phi) \right)^2, \tag{2.1}
\]

where \( \Phi \) is a \( 2 \times 2 \) matrix, \( \Phi = \sigma + i \tau^a \pi^a \) with Pauli matrices \( \tau^a (a = 1, 2, 3) \), \( \psi \) is a fermion doublet field, \( G_{\mu} := G^\alpha_{\mu} T^\alpha (\alpha = 1, \ldots, N^2_c - 1) \) are \( SU(N_c) \) gauge fields, and \( \psi_{L/R} := P_{L/R} \psi \), with \( P_{L/R} := (1 \mp \gamma^5) / 2 \) being chiral projection operators. We take \( m_\phi^2 > 0 \).

It is well known that this system with negative \( m_\phi^2 \) causes the spontaneous chiral symmetry breaking already at the tree level and the fermion acquires its mass in a passive manner from the VEV of the scalar field. However, this picture is not appropriate for the strong coupling Yukawa region where the feedback from the fermion sector to the scalar potential is significant as stressed before. In this region we expect the fermion determinant plays a more active role in the spontaneous chiral symmetry breaking and we need to treat the dynamics in a nonperturbative manner. Such a nonperturbative dynamics may be investigated by using a truncated set (“ladder”) of the Schwinger-Dyson (SD) self-consistency equations as the first approximation.

Since a concept of renormalized parameters becomes somewhat obscure in this kind of nonperturbative analysis due to the ambiguity of renormalization schemes, we study the physical quantities (e.g., the fermion dynamical mass \( M \), the NG boson decay constant \( F_\pi \), etc.) directly as functions of bare parameters (e.g., \( m_\phi, \tilde{g}_y \), etc.) and the ultraviolet (UV) cutoff \( \Lambda \). Once these physical quantities are calculated, we can determine the
scaling properties of those bare parameters ("renormalization group (RG)"") in such a way as to fix the physical quantities on the variation of the cutoff $\Lambda$. Of course the RG functions determined in this manner are not unique and depend on the choice of the "physical parameters" utilized in this procedure. It should be emphasized, however, that the qualitative feature of the phase diagram is expected not to change depending upon the choice of the parameters, as far as such a choice covers (at least) minimal set of possible relevant operators. In this paper, we choose the dynamical mass of the fermion $M$ and the NG boson decay constant $F_\pi$ for such physical parameters.

Now, we turn to the ladder SD equation of the gauged Yukawa model. It is convenient to rewrite the Lagrangian Eq.(2.1) into

$$
\mathcal{L} = \bar{\psi} \left( i \partial + \frac{g}{\sqrt{N_c}} \mathcal{G} \right) \psi - g_y \left[ \bar{\psi}_L \Phi^\dagger \psi_R + \bar{\psi}_R \Phi \psi_L \right] - \frac{1}{4} G_{\mu\nu}^a G^{a\mu\nu} \\
+ \frac{Z_\phi N_c}{4} \text{tr} \left( \partial_\mu \Phi^\dagger \partial^\mu \Phi \right) - \frac{M_\phi^2 N_c}{4} \text{tr} \left( \Phi^\dagger \Phi \right) - \frac{\lambda_\phi N_c}{16} \left( \text{tr} (\Phi^\dagger \Phi) \right)^2,
$$

(2.2)

where we have rescaled the scalar field $\Phi$: $\Phi \to \sqrt{Z_\phi N_c} \Phi$ and defined parameters: $g_y^2 := N_c Z_\phi \bar{g}_y^2$, $M_\phi^2 := Z_\phi m_\phi^2$ and $\lambda_\phi := N_c Z_\phi^2 \bar{\lambda}_\phi$. Again note that we take $M_\phi^2 > 0$, since we are interested in the symmetry breaking due to the strong Yukawa coupling. Here we take $Z_\phi > 0$ in order to guarantee the stability of the vacuum. $Z_\phi$ plays a role of the "hopping parameter" in the lattice formulation (See the discussion in Section 6). It should be noted here that our new parameterization of the gauged Yukawa model, Eq.(2.2), contains a redundant parameter compared with the original one Eq.(2.1). We shall return to this point later. It should also be emphasized that the Eq.(2.2) is appropriate for the comparison to the previous analysis of the gauged NJL model ($Z_\phi = \lambda_\phi = 0$).

The SD equations for the VEV of the scalar field $\langle \sigma \rangle$ and the fermion propagator $S(p)$ are given by[6]

$$
M_\phi^2 \langle \sigma \rangle + \lambda_\phi \langle \sigma (\sigma^2 + \vec{\pi}^2) \rangle = - \frac{g_y}{N_c} \langle \bar{\psi} \psi \rangle,
$$

(2.3)
\[ iS^{-1}(p) = p - g_y \langle \sigma \rangle + \frac{g}{\sqrt{N_c}} \int \frac{d^4k}{(2\pi)^4} \gamma^\mu T^\alpha S(p-k) \Gamma^\nu(p-k,p) D_{\mu\nu}(k) \]

\[ + g_y \int \frac{d^4k}{(2\pi)^4} D^\sigma(k) S(p-k) \Gamma^\sigma(p-k,p) \]

\[ + g_y \int \frac{d^4k}{(2\pi)^4} D^\pi_{ab}(k) i\gamma^5 \tau^b S(p-k) \Gamma^\pi_a(p-k,p), \tag{2.4} \]

respectively, with \( D_{\mu\nu}(k) \) being the gauge filed propagator, \( D^\sigma(k) \) and \( D^\pi_{ab}(k) \) the scalar and the pseudoscalar propagators, \( \Gamma^\nu \), \( \Gamma^\sigma \) and \( \Gamma^\pi_a \) the vertex functions. It is hopeless to solve a full set of the SD equations because of the lack of knowledge of the scalar/gauge boson propagators and the vertex functions. Here we take the ladder approximation in the sense that we take bare vertices and bare boson propagators instead of full vertices and full propagators:

\[ N_c D^\sigma(k) = \frac{i}{Z_\phi k^2 - M_\phi^2}, \quad \Gamma^\sigma = -ig_y, \quad (2.5a) \]

\[ N_c D^\pi_{ab}(k) = \frac{i}{Z_\phi k^2 - M_\phi^2} \delta_{ab}, \quad \Gamma^{\pi a} = -ig_y i\gamma^5 \tau^a, \quad (2.5b) \]

\[ N_c D_{\mu\nu}(k) = \frac{-i}{k^2} (g_{\mu\nu} - (1 - \xi) \frac{k_\mu k_\nu}{k^2}), \quad \Gamma^\mu = -ig \sqrt{N_c} \gamma^\mu T_\alpha, \quad (2.5c) \]

with \( \xi \) being the gauge fixing parameter. Though there is no solid reason to justify this approximation, the ladder approximation becomes plausible, when the gauge coupling runs slowly ("walking") or does not run at all ("standing") [18], and the effects of the scalar/pseudoscalar boson propagators are small. Actually, the ladder approximation yields a successful phenomenology even for the QCD running coupling used in the SD equation.[24]

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\[ \text{The effects of the scalar/pseudoscalar boson propagators ("rainbow" graph) are actually suppressed by } 1/N_c \text{ compared with those of the gauge boson propagators (gauge boson "rainbow") and the VEV of the scalar field ("tadpole"). (See Eqs.(2.8a)-(2.8b).) The } N_c \to \infty \text{ limit yields qualitatively the same phase structure as } N_c = 1 \text{ case (see Section 5).} \]
We substitute
\[ iS(p)^{-1} = A(-p^2)\dot{\phi} - B(-p^2) \] (2.6)
into Eq.(2.4) and introduce the UV cutoff \( \Lambda^2 \) for \( p^2 \) after Wick rotation. In the ladder approximation, we can perform the angular integration of Eq.(2.4), which yields (see, e.g., Ref.[23])

\[ B(x) = g_y\langle \sigma \rangle + \int_0^{\Lambda^2} dy K_B(x, y) \frac{yB(y)}{A(y)^2 y + B(y)^2}, \] (2.7a)
\[ A(x) = 1 + \int_0^{\Lambda^2} dy K_A(x, y) \frac{A(y)}{A(y)^2 y + B(y)^2}. \] (2.7b)

The integral kernels are given by

\[ K_B(x, y) = \frac{h}{N_c Z_\phi} K_B(x, y; \frac{M_\phi^2}{Z_\phi}) + \lambda(1 + \xi/3) K_B(x, y; 0), \] (2.8a)
\[ K_A(x, y) = \frac{y}{x} \left\{ \frac{h}{N_c Z_\phi} K_A(x, y; \frac{M_\phi^2}{Z_\phi}) + \frac{2}{3} \xi \lambda K_A(x, y; 0) \right\}, \] (2.8b)

where

\[ K_B(x, y; m^2) := \frac{2}{\pi} \int_0^\pi d\theta \frac{\sin^2 \theta}{x + y - 2\sqrt{xy} \cos \theta + m^2} \]
\[ = \frac{2}{x + y + m^2 + \sqrt{(x + y + m^2)^2 - 4xy}}, \] (2.9a)
\[ K_A(x, y; m^2) := \frac{4}{\pi} \int_0^\pi d\theta \frac{\sqrt{xy} \cos \theta \sin^2 \theta}{x + y - 2\sqrt{xy} \cos \theta + m^2} \]
\[ = \frac{4xy}{[x + y + m^2 + \sqrt{(x + y + m^2)^2 - 4xy}]^2}, \] (2.9b)

and \( x, y, h \) and \( \lambda \) are defined as \( x := -p^2, y := -q^2, \)
\[ h := \frac{g_\sigma^2}{8\pi^2}, \quad \lambda := \frac{3g_\sigma^2 C_F}{16\pi^2 N_c}, \] (2.10)

with \( C_F := \sum_\alpha T^\alpha T^\alpha = (N_c^2 - 1)/2N_c \) being the quadratic Casimir of the fermion.
The VEV of the scalar field \( \langle \sigma \rangle \) is determined from Eq.(2.3) under factorizability assumption, which reads[6]

\[
\langle \sigma \rangle = -\frac{g_y}{N_c} \frac{\langle \bar{\psi} \psi \rangle}{M_\phi^2} + \frac{g_y^3}{N_c^3 M_\phi^8} \langle \langle \bar{\psi} \psi \rangle \rangle^3 + \cdots ,
\]

(2.11)

where the fermion condensate \( \langle \bar{\psi} \psi \rangle \) is given by

\[
\langle \bar{\psi} \psi \rangle = -\int \frac{d^4q}{(2\pi)^4} \text{tr}S(q) = -\frac{N_c}{2\pi^2} \int_0^{\Lambda_0^2} dy \frac{yB(y)}{A(y)^2 + B(y)}.
\]

(2.12)

At this stage, the VEV \( \langle \sigma \rangle \), the chiral condensate \( \langle \bar{\psi} \psi \rangle \) and the fermion mass function \( B(x) \) are inter-related among each other and are responsible for the chiral symmetry breaking. The \( O(\lambda_\phi) \) or higher terms in Eq.(2.11) do not produce significant effect near the critical point \( |\langle \bar{\psi} \psi \rangle| \ll M_\phi^3 \). We thus disregard those effects in the following calculations. Then \( \langle \sigma \rangle \) is given by the “tadpole” contribution alone:

\[
\langle \sigma \rangle = -\frac{g_y}{N_c} \frac{\langle \bar{\psi} \psi \rangle}{M_\phi^2} = \frac{g_y}{2\pi^2 M_\phi^2} \int_0^{\Lambda_0^2} dy \frac{yB(y)}{A(y)^2 + B(y)}.
\]

(2.13)

Combining Eq.(2.7a) and Eq.(2.13), we finally obtain the closed SD equation for the fermion propagator:

\[
B(x) = \frac{4h}{M_\phi^2} \int_0^{\Lambda^2} dy \frac{yB(y)}{yA(y)^2 + B(y)^2} + \int_0^{\Lambda^2} dy \mathcal{K}_B(x, y) \frac{yB(y)}{yA(y)^2 + B(y)^2} ,
\]

(2.14a)

\[
A(x) = 1 + \int_0^{\Lambda^2} dy \mathcal{K}_A(x, y) \frac{A(y)}{yA(y)^2 + B(y)^2}.
\]

(2.14b)

In this paper we take the Landau gauge (\( \xi = 0 \)). The Landau gauge is required to be consistent with the bare vertex function approximation at least in the pure gauge theory due to the Ward-Takahashi identity, since the part of the kernel \( \mathcal{K}_A \)
from the gauge interaction is identically zero in this gauge and then the SD equation reduces to \( A(-p^2) \equiv 1 \) (see, e.g., Ref.[23]). For simplicity of calculation we take a further approximation for the integral kernel from the Yukawa interaction such that \( \mathcal{K}_A = 0 \). For this approximation to be consistent with the bare vertex approximation, the solution of the coupled SD equations must lead to the result: \( A(-p^2) \approx 1 \) for \( A(-p^2) \), namely, the deviation of \( A(-p^2) \) from 1 must be small. Indeed this has been confirmed by the previous work[6] on the Yukawa model and also in the massive vector boson model[25]. Thus in what follows we put

\[
A(-p^2) \equiv 1, \tag{2.15}
\]

which implies no wave function renormalization for the fermion. Then we only have to solve the single integral equation for the fermion mass function \( B(x) \):

\[
B(x) = \frac{4h}{M_\phi^2} \int_0^{\Lambda^2} dy \frac{yB(y)}{y + B(y)^2} + \int_0^{\Lambda^2} dy \mathcal{K}_B(x, y) \frac{yB(y)}{y + B(y)^2}. \tag{2.16}
\]

This is our basic equation.

Although Eq.(2.16) can be solved numerically, here we solve it by converting the integral equation into a more tractable differential equation plus boundary conditions. This can be done by adopting an approximation[26] for the kernel:

\[
K_B(x, y; m^2) \simeq \theta(x - y) \frac{1}{x + m^2} + \theta(y - x) \frac{1}{y + m^2}, \tag{2.17}
\]

where \( \theta(x) \) is the step function. Then the SD equation Eq.(2.14a) is reduced to

\[
B(x) = \frac{4h}{M_\phi^2} \int_0^{\Lambda^2} dy \frac{yB(y)}{y + B(y)^2} \\
+ \int_0^{\Lambda^2} dy \left[ \theta(x - y) \left( \frac{\lambda}{x} + \frac{h/N_c}{Z_\phi x + M_\phi^2} \right) + \theta(y - x) \left( \frac{\lambda}{y} + \frac{h/N_c}{Z_\phi y + M_\phi^2} \right) \right] \frac{yB(y)}{y + B(y)^2}. \tag{2.18}
\]
This technical simplification does not change the qualitative structure of the SD equation [23].

Eq. (2.18) is readily converted into a set of differential equations;

\[
\frac{d}{dx} B(x) = - \left( \frac{\lambda}{x} + \frac{h}{N_c (Z_\phi x + M_\phi^2)^2} \right) V(x), \tag{2.19a}
\]

\[
\frac{d}{dx} (xV(x)) = \frac{xB}{x + B(x)^2}. \tag{2.19b}
\]

plus infrared (IR) and UV boundary conditions (BC);

\[
V(0) = 0 \quad \text{(IRBC)} \tag{2.20a}
\]

\[
\left( \frac{\Lambda^2}{M_\phi^2} + \frac{1}{N_c Z_\phi \Lambda^2 + M_\phi^2} \right) h = \left( \frac{B(\Lambda^2)}{V(\Lambda^2)} - \lambda \right) \quad \text{(UVBC)}, \tag{2.20b}
\]

with the “condensation function” \( V(x) \) being given by

\[
V(x) = \frac{1}{x} \int_0^x dy \frac{yB(y)}{y + B^2(y)}. \tag{2.21}
\]

Eqs. (2.19a)–(2.20b) are the equations that our analysis is actually based on. Note here that the fermion pair condensate is related to \( V(x) \) as

\[
\langle \bar{\psi} \psi \rangle = - \frac{N_c}{2\pi^2} \Lambda^2 V(\Lambda^2). \tag{2.22}
\]

3 Nambu-Goldstone boson decay constant \( F_\pi \)

As we have mentioned before, we need to calculate physical quantity other than the dynamical mass of the fermion to determine the “RG” flow of the bare parameters. Though the mass of the physical scalar boson \( m_\phi^{\text{phys}} \) would be the first candidate for this purpose, we need to solve the SD equation for the scalar boson propagator in order to calculate \( m_\phi^{\text{phys}} \), which is far beyond the scope of the present paper. We therefore
choose the decay constant of the NG boson $F_\pi$.

One might suspect that the $F_\pi$ is already calculated as the VEV of the elementary scalar field $\langle \sigma \rangle$ through Eq.(2.13). This is not true in this model, however, because of the presence of the mixing of $\pi$ with the composite pseudoscalar field. On might also think that the Pagels-Stokar (PS) formula[20] of the (composite) pion decay constant in QCD is applicable to this problem. However, it cannot be used as it stands due to the mixing with the elementary pseudoscalar field. Actually, the wave function of the real NG boson contains the fermion composite part $\langle 0| T\psi(x)\bar{\psi}(y) |NG \rangle$ as well as the elementary pseudoscalar part $\langle 0| \Phi(x) |NG \rangle$ in the gauged Yukawa model.

In this section, we incorporate the mixing effect into the PS formula for $F_\pi$ and obtain a generalized expression for $F_\pi$ (“generalized PS formula”) in such a model. We find that square of the decay constant $F_\pi^2$ of the diagonalized NG state is divided into two parts: $F_\pi^2 = F_b^2 + F_f^2$, with $F_b$ and $F_f$ being the bosonic part and the fermionic part, respectively. The fermion part is evaluated by a certain integral formula of the fermion mass function $B(x)$. This result is actually in accord with the well-known result in the two (elementary) doublet model: the square of the decay constant $F_\pi^2$ is expressed by the sum of the (VEV)$^2$ of each doublet.

The NG boson decay constant $F_\pi$ is defined by

$$\langle 0| J_5^{a\mu}(x) |NG^b(q) \rangle = -iq^\mu F_\pi e^{iq \cdot x} \delta^{ab}, \quad (3.1)$$

with $|NG^b(q)\rangle$ being the (diagonalized) NG boson state with momentum $q$ ($q^2 = 0$) and isospin $b$. The Noether current $J_5^{a\mu}$ is given by

$$J_5^{a\mu} = -i N_c Z_\phi \frac{1}{4} \text{tr}(\tau^a \Phi \partial^\mu \Phi - \tau^a \Phi \partial^\mu \Phi^\dagger) - \bar{\psi} \gamma^\mu \gamma_5 \frac{\tau^a}{2} \psi. \quad (3.2)$$

The calculation of $F_\pi$ becomes straightforward, if we know exactly the wave function
of the NG boson:

\[ \chi_{P\Phi}^a := \langle 0|\Phi(0)|NG^a(q) \rangle, \]  

\[ S(p)\chi_{P\bar{\psi}\psi}(p,q)S(p-q) := \int d^4x e^{-ip\cdot x} \langle 0|T\psi(x)\bar{\psi}(0)|NG^a(q) \rangle, \]  

where \( S(p) \) is the fermion propagator and \( q^2 = 0 \). The decay constant \( F_\pi \) may be expressed in terms of the wave function as (see Fig.1):

\[ F_\pi q^\mu \delta^{ab} = \frac{N_c Z_\phi}{4} i q^\mu \text{tr}\{\tau^a, \langle \Phi \rangle \} \chi_{P\Phi}^b \]

\[ - \int \frac{d^4q}{(2\pi)^4i} \text{tr}(\gamma^\mu \gamma^5 \tau^a) S(p)\chi_{P\bar{\psi}\psi}(p,q)S(p-q)). \]  

Generally speaking, however, it is quite difficult problem to solve the wave function of composite particles (Bethe-Salpeter amplitude). Fortunately, this problem is simplified for the NG boson state, thanks to the Ward-Takahashi identities of the broken symmetry:

\[ \partial^a_{\mu}\langle T J_5^{\alpha\mu}(x)\Phi(y) \rangle \]

\[ = \{\frac{\tau^a}{2}, \langle \Phi(y) \rangle \} \delta^{(4)}(x-y), \]  

\[ \partial_{\mu}^a \langle T J_5^{\alpha\mu}(x)\psi(y)\bar{\psi}(z) \rangle \]

\[ = \frac{\tau^a}{2} \gamma_5 \langle T\psi(y)\bar{\psi}(z) \rangle \delta^{(4)}(x-y) + \langle T\psi(y)\bar{\psi}(z) \rangle \frac{\tau^a}{2} \gamma_5 \delta^{(4)}(x-z). \]  

The spectral representation of (3.5a) and (3.5b) is written as

\[ \sum_n \langle 0|\Phi(0)|n(q) \rangle \frac{q^\mu}{q^2 - m_n^2} \langle n(q)|J_5^{\alpha\mu}(0)\rangle \]

\[ = \{\frac{\tau^a}{2}, \langle \Phi \rangle \}, \]  

\[ \int d^4y e^{-ip\cdot y} \sum_n \langle 0|T\psi(y)\bar{\psi}(0)|n(q) \rangle \frac{q^\mu}{q^2 - m_n^2} \langle n(q)|J_5^{\alpha\mu}(0)\rangle \]

\[ = S(p) \left( S^{-1}(p) \frac{\tau^a}{2} \gamma_5 - \frac{\tau^a}{2} \gamma_5 S^{-1}(p-q) \right) S(p-q). \]
Among $|n^a(q)\rangle$ the only state which survives in $q_\mu = 0$ limit is the NG boson state $|NG^a(q)\rangle$ ($m_{NG}^2 = 0$). We therefore obtain the wave functions $\chi_{\rho\Phi}^a$, $\chi_{\rho\bar{\psi}\psi}^a(p, q = 0)$ which are completely determined at zero-momentum $q = 0$ by the VEV of scalar field and the fermion propagator, respectively:

$$\chi_{\rho\Phi}^a = -i \frac{F_\pi}{2} \{\tau^a, \langle\Phi\rangle\}, \quad (3.7a)$$
$$\chi_{\rho\bar{\psi}\psi}^a(p, q = 0) = -i \frac{F_\pi}{2} \{\tau^a, \gamma_5, S^{-1}(p)\}. \quad (3.7b)$$

Following the paper of Pagels and Stokar[20, 9], we approximate the fermionic wave function

$$\chi_{\rho\bar{\psi}\psi}^a(p, q) \simeq \chi_{\rho\bar{\psi}\psi}^a(p, q = 0). \quad (3.8)$$

Now, it is straightforward to calculate the decay constant $F_\pi$ by plugging the wave function Eqs.(3.7a)–(3.7b) into Eq.(3.4). We find

$$F_\pi^2 = F_b^2 + F_f^2, \quad (3.9a)$$
$$F_b^2 = N_c Z_\phi \langle\sigma\rangle^2, \quad (3.9b)$$

$$F_f^2 = \frac{N_c}{4\pi^2} \int_0^{\Lambda^2} dx \frac{B_2(x) - \frac{x}{2} \frac{d}{dx} B_2(x)}{(x + B_2(x))^2}, \quad (3.9c)$$

where $F_b^2$ comes from the elementary scalar wave function Eq.(3.7a) and $F_f^2$ from the fermion composite wave function Eq.(3.7b).

Remarkably enough, this generalized PS formula definitely predicts that the fermion mass gets lower in the gauged Yukawa model ($Z_\phi > 0$) than in the gauged NJL model ($Z_\phi = 0$) for the same $F_\pi$. Actually it dictates $F_f^2 = F_\pi^2 - F_b^2 < F_\pi^2$ for $F_b^2 = N_c Z_\phi \langle\sigma\rangle^2 > 0$. Now, the fermion mass $M^2$ is determined by the formula for $F_f^2$, Eq.(3.9c), as an increasing function of $F_f^2$, which was the essence of the top quark mass determination in the top quark condensate model [3, 2]. Thus we have $M^2$(gauged Yukawa) < $M^2$(gauged NJL), since $F_f^2$(gauged Yukawa) < $F_\pi^2$ =
Having set up the machineries Eqs.(2.19a)–(2.20b) and Eqs.(3.9a)–(3.9c), we are now ready to perform numerical analysis of the phase and “RG” structure in the gauged Yukawa model. As we mentioned before, we take $M_φ^2 > 0$ so that the chiral symmetry breaking can only be caused by the fermion dynamics. Because of the redundancy of the parameters in the Lagrangian Eq.(2.2), we may fix one parameter $M_φ = Λ$ for positive $M_φ^2$ without loss of generality.

The goal of the numerical analysis here is to determine the bare parameters $(h, λ, Z_φ)$ as functions of the ultraviolet cutoff Λ and the “physical quantities” $(M, F_π, λ_R)$:

$$h(Λ, M, F_π, λ_R), \quad λ(Λ, M, F_π, λ_R), \quad Z_φ(Λ, M, F_π, λ_R), \quad (4.1)$$

with $M := B(0)$ being the dynamical mass of the fermion, and $λ_R$ the renormalized gauge coupling. Dimensional analysis of Eq.(4.1) shows that $h(Λ, M, F_π, λ_R) = h(Λ/M, 1, F_π/M, λ_R)$ and $Z_φ(Λ, M, F_π, λ_R) = Z_φ(Λ/M, 1, F_π/M, λ_R)$. Thus $M$ can be set unity, $M = 1$, without loss of generality. Since we restrict ourselves to the standing gauge coupling, 5 we do not consider the RG flow to the gauge coupling $λ$ direction: $λ_R = λ$. We also disregard the RG flow to the $λ_φ$ direction, since feedback of running of $λ_φ$ is expected to be small as we already argued in Section 2. We thus obtain the “RG” flows in $(h, λ, Z_φ)$ space (more precisely, $(h, Z_φ)$ plane sliced for each $λ$) as we vary Λ while keeping $M$ and $F_π$ fixed in Eq.(4.1).

Numerical calculations are performed as follows in our program. For given gauge

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5It is rather straightforward to incorporate running effect of the gauge coupling as was done in the gauged NJL model [22].
coupling $\lambda$, we solve the differential equations Eqs.(2.19a)–(2.19b);

$$
\begin{align*}
\frac{d}{dx} B(x) &= - \left( \frac{\lambda}{x} + \frac{\hat{h}}{N_c} \left( \frac{Z_\phi x}{x} + \Lambda^2 \right) \right) V(x), \\
\frac{d}{dx} V(x) &= - \frac{V(x)}{x} + \frac{B(x)}{x + B^2(x)},
\end{align*}
$$

(4.2)

Together with the differential form of $F_\tau^2 = F_\tau^2(\Lambda^2)$ Eq.(3.9c);

$$
\frac{d}{dx} F_\tau^2(x) = \frac{N_c}{4\pi^2} \frac{xB(x)[B(x) - \frac{x}{2} B'(x)]}{(x + B^2(x))^2}
$$

(4.3)

plus the respective IRBC’s;

$$
B(0) = 1, \quad V(0) = 0, \quad F_\tau^2(0) = 0,
$$

(4.4)

with $(\hat{h}, \hat{Z}_\phi)$ being trial parameters. Then we obtain the next trial parameters $(\hat{h}', \hat{Z}'_\phi)$ from Eqs.(2.20b),(3.9a)–(3.9c):

$$
\begin{align*}
\hat{h}' &= \left( 4 + \frac{1}{N_c} \frac{1}{Z_\phi + 1} \right)^{-1} \left( \frac{B(\Lambda^2)}{V(\Lambda^2)} - \lambda \right), \\
\hat{Z}'_\phi &= \frac{\pi^2}{2N_c} \frac{F_\pi^2 - F_\tau^2(\Lambda^2)}{\hat{h} V^2(\Lambda^2)}.
\end{align*}
$$

(4.5)

We repeat the above steps after substituting the trial parameters $(\hat{h}', \hat{Z}'_\phi)$ into $(\hat{h}, \hat{Z}_\phi)$, and so on. After several iterations, we obtain the function Eq.(4.1) with sufficient accuracy.

Let us now turn to the result of our numerical calculations. We here present numerical result only for the $U(1)$ case, i.e., $C_F = 1$ and $N_c = 1$ in Eqs.(2.10), (3.9b)–(3.9c). Actually, the $N_c$ dependence is not significant for the essential feature of the phase diagram as we will demonstrate by the analytical study in the next section. When we discuss the $N_c > 1$ effect, we also have to include the running effects of the gauge coupling. Numerical study including both effects will be given in a separate paper.
We first discuss $\lambda = 0$ case (pure Yukawa model with vanishing gauge coupling). In Fig. 2 we show the critical line in the space $(h, Z_\phi)$, which separates the chiral symmetric phase ($M = 0$) at $h < h_c$ and the spontaneously broken phase ($M \neq 0$) at $h > h_c$, with $h_c = h_c(Z_\phi)$ being a critical coupling for each $Z_\phi$. This is confirmed by the analytical study in the next section. Above the critical coupling $h > h_c$, the chiral condensate $\langle \bar{\psi} \psi \rangle$ exhibits a characteristic behavior shown in Fig. 3, which was first obtained in Ref.[6]. Note that the critical Yukawa coupling $h_c$ monotonically increases in $Z_\phi$ and the dependence of $h_c$ on $Z_\phi$ is very small. This also agrees with the analytical study.

Next we consider another limit, $Z_\phi = 0$, which corresponds to the gauged NJL model. Actually, lines of the equi-correlation length $\xi_f = \Lambda/M$ in the $Z_\phi = 0$ plane are depicted in Fig. 4 for $N_c = 1$, which converge in the $\xi_f \to \infty$ limit toward the critical line (bold line in Fig. 4). This critical line is nothing but the well-known critical line[27] of the gauged NJL model written in the space $(h, \lambda)$:

$$
\left(4 + \frac{1}{N_c}\right) h = \frac{1}{4(1 + \sqrt{1 - \lambda/\lambda_c})^2}
$$

for $\lambda < \lambda_c = 1/4$. (This analytical expression will be derived in the next section.)

Then in Fig.5 the surface of the equi-$\xi_f$ is depicted in the three-dimensional bare parameter space $(h, \lambda, Z_\phi)$ for the Yukawa coupling $h$, the gauge coupling $\lambda$ and the “hopping parameter” of the scalar field $Z_\phi$. The $\xi_f \to \infty$ limit is the critical surface $h = h_c(\lambda, Z_\phi)$ separating the chiral symmetric phase ($h < h_c$) and the spontaneous symmetry breaking phase ($h > h_c$), which is shown by the surface with mark “×” in Fig. 5. Below the critical surface ($h < h_c$), there is no nontrivial solution for the SD equation ($M = 0$).

Now to the RG flow, which is obtained by fixing the NG boson decay constant $F_\pi$ as

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6In the previous work[6] the case of $Z_\phi = 1$ was investigated by varying the scalar mass $M_\phi$. It was shown (see Fig. 3 of Ref.[6]) that there exists a critical Yukawa coupling $h_c$ and the dependence of $h_c$ on $M_\phi$ is very small.
well as $M$. Given the value of $F_\pi$, we obtain the $F_\pi$-constant surface (two-dimensional manifold) in the space $(h, \lambda, Z_\phi)$, which is then sliced by the $\lambda$-fixed plane in accord with our setting of the standing gauge coupling. RG flows are obtained as the intersection of the $F_\pi$-constant surface with the $\lambda$-fixed planes, see Fig. 6. Imagine the situation that each RG flow passes across the equi-$\xi_f$ surfaces of Fig. 5. Then each flow curve may be considered to be parameterized with respect to the value of $\xi_f$. We may interpret that the crossed points of a RG flow with the equi-$\xi_f$ surfaces correspond to the result of successive steps of the RG transformation.\footnote{Our analysis based on the SD equation can determine the flow only in the spontaneous chiral symmetry breaking phase where the nontrivial solution ($M \neq 0, F_\pi \neq 0$) exists. Thus the renormalized trajectory cannot be drawn as extended to the symmetric phase, which of course is an “artifact” of this framework. We may improve this situation by calculating effective potential (or $m_\phi^{\text{phys}}$) as we did in the gauged NJL model [21].}

All the flows in each figure of Fig. 6 are for the same $F_\pi$ and for different values of $\lambda$. Most of the flows either run to $Z_\phi \to \infty$ or $Z_\phi \to 0$ for $\xi_f := \Lambda/M \gg 1$, but there is an indication of a single flow (“renormalized trajectory”) terminating at a finite value of $Z_\phi$ and $\lambda$ (“(nontrivial) fixed point”) on the critical surface. We can draw a similar figure for a different $F_\pi$, which indicates a different fixed point and the renormalized trajectory (compare (a) and (b) in Fig. 6). A set of such fixed points form a line (“fixed line”) which is depicted by the dotted dashed line in Fig. 6. (The precise location of the fixed line is determined by the analytical study in the next section where we find that the fixed line extends from $(h, \lambda, Z_\phi) = (-1/20, 1/4, 0)$ to $(1/4, 0, \infty)$ for $N_c = 1$.)

We may take another look at this phase diagram by Fig. 2 ($\lambda = 0$) and Fig. 7 ($\lambda > 0$), where the flows are depicted for different values of $F_\pi$ in the fixed-$\lambda$ plane $(h, Z_\phi)$. In Fig. 7 ($\lambda > 0$) there is again an indication that a single flow (renormalized trajectory) terminates at a finite $Z_\phi$ (fixed point) on the critical surface, while all others run to either $Z_\phi \to \infty$ or $Z_\phi \to 0$ for $\xi_f \gg 1$. This is sharply contrasted with the pure Yukawa model ($\lambda = 0$) whose RG flows are shown in Fig. 2 (solid lines). There is no indication of the existence of the “(nontrivial) fixed point”. This is confirmed by the analytical study in the next section. This also agrees quite well with the result of the
lattice Yukawa model (see Section 6). Thus we found that a fixed line exists in the gauged Yukawa model solely due to the presence of the gauge coupling. This is our main result.

The RG flow shows that there is a “critical value” for the decay constant $F_\pi$ which separates the three-dimensional space (above the critical surface $h > h_c$) into two regions:

(I) $F_\pi < F_\infty^f$ ($F_\infty^f := F_f(\Lambda = \infty)$); RG flows extend to $Z_\phi = 0$ for $\xi_f \gg 1$,

(II) $F_\pi > F_\infty^f$; RG flows run away to $Z_\phi \to \infty$.

The surface $F_\pi = F_\infty^f$ is the boundary of the region (I) and (II) which consists of the renormalized trajectories. The intersection of the critical surface and the renormalized trajectories forms the fixed line.

Finally, we consider the ratio $F_b^2/F_f^2$ for the NG boson decay constant $F_\pi^2$, where $F_b^2$ and $F_f^2$ are the bosonic part and the fermionic part of $F_\pi^2$, respectively, as defined by Eqs. (3.9b)–(3.9c). In Fig. 8 the ratio $F_b^2/F_f^2$ is shown along the RG flow (solid line) in the fixed-$\lambda$ plane and the broken line passing through the origin denotes the fixed-cutoff $\Lambda$ line where the line with the steeper slope corresponds to the smaller cutoff.

In the presence of gauge coupling, $F_f$ is finite even in the infinite cutoff limit ($F_\infty^f < \infty$), because the solution $B(-p^2)$ is damping in the asymptotic UV region (see Eqs. (5.22b), (5.23b)). The RG flow in the region (II) runs away to $Z_\phi \to \infty$ as $\Lambda \to \infty$. However, even in the $Z_\phi \to \infty$ limit, the ratio

$$
\frac{F_b^2}{F_f^2} = \frac{F_\pi^2}{F_f^2} - 1
$$

(4.7)

takes a finite non-zero value depending on the initially specified value of $F_\pi$ for the RG flow. As $F_\pi$ approaches to $F_f$ from the above, the ratio decreases and finally becomes zero at $\Lambda = \infty$ just for $F_\pi = F_\infty^f$. In the region (I) including the $F_\pi = F_f(\Lambda^2)$ surface, therefore, we can always make the contribution from the bosonic part $F_b^2$ to $F_\pi^2$ arbitrarily small by choosing large cutoff or small $Z_\phi$. This implies that the fermionic
part $F_\phi^2$ is dominant in $F_\pi^2$ in the small $Z_\phi$ region for large $\Lambda$, as was already noted\cite{6} in the pure Yukawa model which has actually only the region (I)\cite{6}.

Comparing two points on the fixed-$\Lambda$ line, we find a larger $F_\pi^2$ by increasing $Z_\phi$. This implies that the contribution of the fermion dynamical mass $M$ to $F_\pi$ gets smaller by increasing $Z_\phi$, since we fixed the fermion mass $M = B(0) = 1$ in our numerical calculation. Here we notice again that the gauged Yukawa model reduces to the gauged NJL model in the limit $Z_\phi = 0$. Therefore this result can be rephrased as follows; by including the Yukawa interaction due to non-zero $Z_\phi$, the fermion dynamical mass $M$ can be reduced for a fixed value of $F_\pi$. This is actually a direct consequence of the salient feature of our generalized PS formula Eqs.(3.9a)–(3.9c), as we emphasized in the end of Section 3. This mechanism may be applied to a scenario of the top quark condensate ("Yukawa-driven Top Mode Model")\cite{6, 7, 8} to lower the mass of the top quark.

5 Analytical study

In the previous section, we have solved the nonlinear SD equation Eq.(2.18) numerically. We solve here the SD equation Eq.(2.18) in a analytical manner by making use of the linearization (bifurcation)\cite{28, 23}, which is valid near the critical point ($M \ll \Lambda$). We thus obtain here an analytical expressions of the critical surface and the fixed line. These results actually confirm the numerical calculations in the previous section.

Let us start with the bifurcation approximation of Eq.(2.18). Near the critical point $B(x) \ll \Lambda$, the integrand can be safely replaced by a linearized expression, $xB(x)/(x + B(x)^2) \rightarrow B(x)$ for dominant region of the integral $x \gtrsim M^2$ with $M = B(M^2)$. Small but still important nonlinear effect from the region $x \lesssim M^2$ can be properly taken into account by introducing the infrared cutoff $M$ in Eq.(2.18) except for some subtlety to be discussed later in the pure NJL limit.

Under this approximation, the differential equation Eq.(2.19a) and UVBC Eq.(2.20b)
remain the same, while the Eq.(2.19b) and IRBC Eq.(2.20a) are changed into
\[
\frac{d}{dx} (xV(x)) = B(x),
\]  
(5.1a)
\[
V(M^2) = 0, \quad B(M^2) = M,
\]  
(5.1b)

respectively. From equation Eq.(5.1a) and Eq.(2.19a) we obtain the differential equation for \(V(x)\):
\[
\frac{d^2}{dx^2} (xV(x)) + \left(\frac{\lambda}{x} + \frac{h}{N_c (Z\phi x + M^2\phi)^2}\right) V(x) = 0.
\]  
(5.2)

Before starting the full analysis of Eq.(2.18) with bifurcation approximation, we briefly describe the result in the \(1/N_c \to 0\) limit, in which the analytical calculation is greatly simplified without changing the qualitative feature. In this limit, the differential equation Eq.(5.2) and its boundary conditions are simplified:
\[
\frac{d^2}{dx^2} (xV(x)) + \frac{\lambda}{x} V(x) = 0, \quad B(x) = \frac{d}{dx}(xV(x)),
\]
\[
V(M^2) = 0, \quad B(M^2) = M, \quad 4\frac{\Lambda^2}{M^2} h = \frac{B(\Lambda^2)}{V(\Lambda^2)} - \lambda.
\]  
(5.3)

The solution of Eq.(5.3) is given by
\[
V(x) = \frac{M}{2\mu} \left[ \left(\frac{x}{M^2}\right)^{-1/2+\mu} - \left(\frac{x}{M^2}\right)^{-1/2-\mu} \right],
\]  
(5.4a)
\[
B(x) = \frac{M}{2\mu} \left[ \left(\frac{1}{2} + \mu\right) \left(\frac{x}{M^2}\right)^{-1/2+\mu} - \left(\frac{1}{2} - \mu\right) \left(\frac{x}{M^2}\right)^{-1/2-\mu} \right],
\]  
(5.4b)

where \(2\mu := \sqrt{1 - 4\lambda}\) and \(M/\Lambda (= \xi_f^{-1})\) is calculated as
\[
\left(\frac{1}{\xi_f}\right)^{4\mu} = \left(\frac{M}{\Lambda}\right)^{4\mu} = \frac{4h - (1/2 + \mu)^2 r_\phi}{4h - (1/2 - \mu)^2 r_\phi},
\]  
(5.5)

with \(r_\phi := M^2\phi/\Lambda^2\). From Eq.(5.5) we obtain the critical surface
\[
4h = (1/2 + \mu)^2 r_\phi,
\]  
(5.6)
which is independent of $Z_\phi$. Eq.(5.6) coincides with the critical line of the gauged NJL model[27].

We next discuss the RG flow at the $1/N_c$ leading order. The function $h(\Lambda, M, F_\pi, \lambda_R = \lambda)$ of Eq.(4.1) can be read from Eq.(5.5):

$$h(\Lambda, M, F_\pi, \lambda) = \frac{r_\phi (1/2 + \mu)^2 - (1/2 - \mu)^2 (M/\Lambda)^{4\mu}}{1 - (M/\Lambda)^{4\mu}}.$$  \hspace{1cm} (5.7)

From Eq.(3.9a)–(3.9b) the function $Z_\phi(\Lambda, M, F_\pi, \lambda)$ is given by

$$Z_\phi(\Lambda, M, F_\pi, \lambda) = \frac{\pi^2}{2N_c} \frac{F_\pi^2 - F_j^2(\Lambda^2)}{h(\Lambda, M, F_\pi, \lambda)V^2(\Lambda^2)},$$  \hspace{1cm} (5.8)

where the IR cutoff $M^2$ was introduced:

$$F_j^2 = \frac{N_c}{4\pi^2} \int_{M^2}^{\Lambda^2} \frac{d^2}{dx} \frac{B^2(x) - \frac{1}{2} B^2(x)}{(x + B^2(x))^2},$$  \hspace{1cm} (5.9)

since the mass function $B(x)$ is determined only for $x > M^2$ in the bifurcation approximation. Among these “RG” flows for various $F_\pi$ and $M$, the “renormalized trajectory” possesses a distinguished property: The functions $h$ and $Z_\phi$ remain finite in the limit $\Lambda/M \to \infty$ (fixed point):

$$h(\Lambda, M, F_\pi, \lambda) = h(\Lambda/M, 1, F_\pi/M, \lambda) \to h(\infty, 1, F_\pi/M, \lambda), \hspace{1cm} (5.10a)$$

$$Z_\phi(\Lambda, M, F_\pi, \lambda) = Z_\phi(\Lambda/M, 1, F_\pi/M, \lambda) \to Z_\phi(\infty, 1, F_\pi/M, \lambda). \hspace{1cm} (5.10b)$$

Noting that $V(\Lambda^2) \to 0$ for $\Lambda/M \to \infty$, we can easily see from Eq.(5.7) and Eq.(5.8) that this condition can be satisfied only for

$$F_\pi^2 = F_j^2(\Lambda^2 = \infty), \hspace{1cm} (5.11)$$

otherwise $Z_\phi \to \pm \infty$ for $\Lambda/M \to \infty$.

The mass function $B(x)$ is independent of $(h, Z_\phi)$ which depends on $\Lambda$ through
Eq.(5.7) and Eq.(5.8). Thus, it is also independent of the cutoff \( \Lambda \) and we can easily calculate the function \( \bar{F}_f^2(\Lambda^2) := F_f^2(\infty) - F_f^2(\Lambda^2) \) by

\[
\bar{F}_f^2(\Lambda^2) = \frac{N_c}{4\pi^2} \int_{\Lambda^2}^\infty dx \frac{B^2(x) - \frac{x}{4} \frac{d}{dx} B^2(x)}{(x + B^2(x))^2} \approx \frac{N_c}{4\pi^2} \int_{\Lambda^2}^\infty dx \frac{B^2(x) - \frac{x}{4} \frac{d}{dx} B^2(x)}{x} ,
\]

(5.12)

where we have made an approximation \( B(x > \Lambda^2) = (M/2\mu)(1/2 + \mu)(x/M^2)^{-1/2+\mu} \).

Plugging Eq.(5.11) and Eq.(5.12) into Eq.(5.7) and Eq.(5.8), we obtain the explicit form of the RG flow on the renormalized trajectory for \( M \ll \Lambda \) in the \( 1/N_c \to 0 \) limit:

\[
h(\Lambda, M, F_f(\infty), \lambda) = \frac{r_\phi}{4} (1/2 + \mu)^2 \left( 1 + \frac{8\mu}{(1 + 2\mu)^2} \left( \frac{M}{\Lambda} \right)^{4\mu} \right) ,
\]

(5.13a)

\[
Z_\phi(\Lambda, M, F_f(\infty), \lambda) h(\Lambda, M, F_f(\infty), \lambda) = \frac{r_\phi^2 (5/4 - \mu/2)(1/2 + \mu)^2}{8 (1 - 2\mu)} ,
\]

(5.13b)

where we estimated \( V(\Lambda^2) \approx (M/2\mu)(M/\Lambda)^{1-2\mu} \). It should be emphasized that the disappearance of the “renormalized trajectory” at \( \lambda = 0 \) can be already observed at this stage: \( Z_\phi \to \infty \) as \( \lambda \to 0 \).

Now we return to the analysis of Eq.(2.18) for finite \( N_c \). Rewriting Eq.(5.2) in terms of a new variable \( \eta := Z_\phi x/M_\phi^2 \) for \( v(\eta) \):

\[
V(x) = \eta^{-1/2+\mu}(1 + \eta)^{1/2+\nu} v(\eta) ,
\]

(5.14)

we obtain a differential equation for \( v \):

\[
\eta(1 + \eta)v''(\eta) + (1 + 2\mu + 2(1 + \mu + \nu)\eta)v' + \frac{1}{2}(1 + 2\mu)(1 + 2\nu)v = 0 ,
\]

(5.15)

where the prime denotes the differentiation with respect to \( \eta \), \( v' := \frac{d}{d\eta} v \), and \( \mu \) and \( \nu \) are defined by

\[
\mu := \frac{1}{2} \sqrt{1 - 4\lambda} , \quad \nu := \frac{1}{2} \sqrt{1 - 4h/Z_\phi N_c} .
\]

(5.16)
The general solution of Eq.(5.15) is obtained as a linear combination of two hypergeometric functions. The ratio of the two coefficients of the linear combination of the hypergeometric functions is determined from the IRBC Eq.(5.1b). Then we obtain the solution of Eq.(5.2):

\[ V(\eta) = \kappa [V_+(\eta) - V_-(\eta)], \quad (5.17) \]

where \( \kappa \) is a normalization constant and \( V_\pm(\eta) \) is defined as

\[ V_\pm(\eta) := \left( \frac{\eta}{\eta_M} \right)^{-1/2+\mu} \left( \frac{1+\eta}{1+\eta_M} \right)^{1/2+\nu} F\left( \frac{1}{2} \pm \mu \pm \nu + \omega, \frac{1}{2} \pm \mu \pm \nu - \omega, 1 \pm 2\mu; -\eta \right) \frac{1}{F\left( \frac{1}{2} \pm \mu \pm \nu + \omega, \frac{1}{2} \pm \mu \pm \nu - \omega, 1 \pm 2\mu; -\eta_M \right)}, \quad (5.18) \]

with \( \eta_M := Z_\phi M^2 / M_\phi^2 \) and

\[ \omega := \frac{1}{2} \sqrt{1 - 4\lambda - 4h/Z_\phi N_c}. \quad (5.19) \]

Then the mass function \( B(x) \) is obtained from Eq.(5.1a) as follows:

\[
B(x) = \frac{d}{d\eta} (\eta V(\eta)) = \kappa \left[ \left( \frac{1}{2} + \mu + \left( \frac{1}{2} + \nu \right) \frac{\eta}{1+\eta} - \tilde{V}_+(\eta) \right) V_+(\eta) - \left( \frac{1}{2} - \mu + \left( \frac{1}{2} - \nu \right) \frac{\eta}{1+\eta} - \tilde{V}_-(\eta) \right) V_-(\eta) \right], \quad (5.20) \]

where \( \tilde{V}_\pm \) are defined by

\[ \tilde{V}_\pm(\eta) := (1/2 \pm \nu) \eta F\left( \frac{3}{2} \pm \mu \pm \nu + \omega, \frac{3}{2} \pm \mu \pm \nu - \omega, 2 \pm 2\mu; -\eta \right) \frac{1}{F\left( \frac{1}{2} \pm \mu \pm \nu + \omega, \frac{1}{2} \pm \mu \pm \nu - \omega, 1 \pm 2\mu; -\eta \right)}. \quad (5.21) \]

The coefficient \( \kappa \) is determined by the normalization condition \( B(M^2) = M \).

It is worth remarking that the asymptotic form of the solution for \( V(x) \) and \( B(x) \) are given as follows:

\[ V(\eta) = C_1 \eta^{-\frac{1}{2}+\mu} + C_2 \eta^{-\frac{1}{2}-\mu}, \quad (5.22a) \]
in the small $Z_\phi$ region, and

\begin{align}
V(\eta) &= C'_1 \eta^{-\frac{1}{2}+\omega} + C'_2 \eta^{-\frac{1}{2}-\omega}, \\
B(x) &= C'_1 \left(1 + \frac{1}{2} + \omega \right) \eta^{-\frac{1}{2}+\omega} + C'_2 \left(1 - \omega \right) \eta^{-\frac{1}{2}-\omega}
\end{align}

(5.23b)

in the large $Z_\phi$ region, where $C_1$, $C_2$, $C'_1$ and $C'_2$ are constants determined by the IRBC

Eq.(5.1b) and the normalization condition $B(M^2) = M$.

From the UVBC Eq.(2.20b) we obtain the scaling relation, equi-$\xi_f$ ($:= \Lambda/M$) surface,
of the fermion dynamical mass for three parameters $(h, \lambda, Z_\phi)$:

\begin{align}
\left( \frac{M}{\Lambda} \right)^{4\mu} \left( \frac{Z_\phi \Lambda^2 + r_\phi}{Z_\phi + r_\phi} \right)^{2\nu} P\left( \frac{Z_\phi M^2}{r_\phi \Lambda^2} \right) \\
= \frac{(4h/r_\phi - (1/2 + \mu)^2) - Z_\phi (1/2 + \nu)^2 + \tilde{V}_+(Z_\phi/r_\phi) P\left( \frac{Z_\phi}{r_\phi} \right)}{(4h/r_\phi - (1/2 - \mu)^2) - Z_\phi (1/2 - \nu)^2 + \tilde{V}_-(Z_\phi/r_\phi) P\left( \frac{Z_\phi}{r_\phi} \right)},
\end{align}

(5.24)

where we have defined $r_\phi := M_\phi^2/\Lambda^2$ and

\begin{align}
P(y) := \frac{F\left(\frac{1}{2} + \mu + \nu + \omega, \frac{1}{2} + \mu + \nu - \omega, 1 + 2\mu; -y \right)}{F\left(\frac{1}{2} - \mu - \nu + \omega, \frac{1}{2} - \mu - \nu - \omega, 1 - 2\mu; -y \right)}.
\end{align}

(5.25)

When we take the limit $Z_\phi \to 0$, we obtain the scaling relation for the gauged NJL model[27]:

\begin{align}
\left( \frac{M}{\Lambda} \right)^{4\mu} = \frac{(4 + 1/N_c)h - r_\phi(1/2 + \mu)^2}{(4 + 1/N_c)h - r_\phi(1/2 - \mu)^2},
\end{align}

(5.26)

since $P(0) = 1$ and $\tilde{V}_\pm(0) = 0$. This implies the critical line:

\begin{align}
h = h_c(\lambda) = \frac{r_\phi(1/2 + \mu)^2}{4 + 1/N_c},
\end{align}

(5.27)
which yields the critical point of the pure NJL model in the limit $\lambda \to 0$:

$$h_c(0) = \frac{r_\phi}{4 + 1/N_c}. \quad (5.28)$$

The critical exponent $\nu_M$ for the fermion mass defined by

$$\frac{M}{\Lambda} = (h - h_c(\lambda))^{\nu_M} \quad (5.29)$$

takes the continuously changing value (for $\lambda < 1/4$):

$$\nu_M(\lambda) = \frac{1}{4\mu} = \frac{1}{2\sqrt{1 - 4\lambda}}. \quad (5.30)$$

For further results on other critical exponents, see, e.g., Ref.[23].

In the vanishing gauge coupling limit, $\lambda \to 0$, we obtain the scaling relations for the pure Yukawa model:

$$\frac{Z_\phi M^2}{Z_\phi + r_\phi} = \begin{cases} 
\left[ \frac{(4r_\phi + 4Z_\phi + 1/N_c)h - (1/2 + \nu)Z_\phi}{(4r_\phi + 4Z_\phi + 1/N_c)h - (1/2 - \nu)Z_\phi} \right]^{1/2\nu} \\
\exp \left[ -\frac{2\tanh^{-1}\frac{1}{2}Z_\phi \sqrt{1 - \frac{4Z_\phi h}{Z_\phi N_c}}}{\sqrt{1 - \frac{4h}{Z_\phi N_c}}} \right] \quad (Z_\phi > 4h/N_c) \\
\exp \left[ -\frac{4}{4Z_\phi + 4r_\phi - 1/N_c} \right] \quad (Z_\phi = 4h/N_c) \\
\exp \left[ -\frac{2n\pi + 2\tan^{-1}\frac{1}{2}Z_\phi \sqrt{1 - \frac{4h}{Z_\phi N_c}} - 1}{\sqrt{1 - \frac{4h}{Z_\phi N_c}}} \right] \quad (0 < Z_\phi < 4h/N_c)
\end{cases} \quad (5.31)$$

where $n$ is an integer ($n = 0, 1, 2, ...$). This scaling relation is not the same as the previous result[6] on the pure Yukawa model which does not include the tadpole contribution. Without tadpole we have the solution only for $Z_\phi < 4h/N_c$. 

In the $Z_\phi \to \infty$ limit, Eq.(5.31) reduces to

$$\frac{M^2}{\Lambda^2} = \frac{4h - 1}{4h} = \exp \left[ -2 \tanh^{-1} \left( \frac{1}{1 + 8(h - 1/4)} \right) \right],$$  \hspace{1cm} (5.32)

which implies the critical point $h_c = 1/4$. On the other hand, the $Z_\phi \to 0$ limit reproduces Eq.(5.28) for $n = 0$. Hence the critical line $h = \tilde{h}_c(Z_\phi)$ extends from $(h, \lambda, Z_\phi) = (r_\phi/(4 + 1/N_c), 0, 0)$ to $(1/4, 0, \infty)$ in the $\lambda = 0$ plane.

It is worth remarking that the scaling relations Eq.(5.26) and Eq.(5.31) do not correctly reproduce the scaling relation of the pure NJL model by taking both limits $Z_\phi \to 0$ and $\lambda \to 0$ [29], although the gauged Yukawa model reduces to the pure NJL model in this limit. This is because the bifurcation technique cannot be applied to the pure NJL model in this limit where the mass function has no $p^2$ dependence. Note that the above scaling relations obtained in the bifurcation technique give a meaningful result only in the neighborhood of the critical point $M/\Lambda = 0$ which is not sensitive to the nonlinearity of the SD equation. Therefore the $Z_\phi \to 0$ limit of the scaling Eq.(5.26) and the $\lambda \to 0$ limit of Eq.(5.31) cannot give the correct scaling relation, apart from the critical point and the critical exponent.\(^8\)

Now let us consider the $\xi_f := \Lambda/M \to \infty$ limit of the scaling relation Eq.(5.24), namely, the critical surface $h = h_c(\lambda, Z_\phi)$ in three-dimensional parameter space $(h, \lambda, Z_\phi)$. Since the left-hand side of Eq.(5.24) vanishes in the limit for $0 < \lambda < \lambda_c = 1/4$ ($\mu > 0$), the critical surface obeys the equation

$$[4h/r_\phi - (1/2 + \mu)^2 - \frac{Z_\phi}{r_\phi + Z_\phi}(1/2 + \nu)^2 + \bar{V}_+(Z_\phi/r_\phi)]P(Z_\phi/r_\phi) = 0, \hspace{1cm} (5.33)$$

\(^8\)In the pure Yukawa model the critical exponent $\tilde{\nu}_M$ defined by $\frac{M}{\Lambda} = (h - \tilde{h}_c(Z_\phi))\tilde{\nu}_M$ changes continuously depending on the value $Z_\phi$ in the case $r_\phi \to 0$, while $\tilde{\nu}_M$ takes, irrespective of $Z_\phi$, the mean-field value: $\tilde{\nu}_M = 1/2$, as long as $r_\phi \neq 0$ in any cutoff $\Lambda$, as was pointed out in Ref.[25]. In the latter case, all the critical exponents take their mean field values. Actually the chiral condensate takes the critical exponent $1/2$, with which the numerical calculation shown in Fig.2 is consistent.
which implies
\[
\left[ 4h/r_\phi - \left( \frac{1}{2} + \mu \right)^2 - \frac{Z_\phi}{r_\phi + Z_\phi} \left( \frac{1}{2} + \nu \right)^2 \right. \\
+ \left. \left( \frac{1}{2} + \nu \right) \frac{Z_\phi}{r_\phi} \frac{F\left( \frac{3}{2} + \mu + \nu + \omega, \frac{3}{2} + \mu + \nu - \omega, 2 + 2\mu; -Z_\phi/r_\phi \right)}{F\left( \frac{1}{2} + \mu + \nu + \omega, \frac{1}{2} + \mu + \nu - \omega, 1 + 2\mu; -Z_\phi/r_\phi \right)} \right] \\
\times \frac{F\left( \frac{1}{2} + \mu + \nu + \omega, \frac{1}{2} + \mu + \nu - \omega, 1 + 2\mu; -Z_\phi/r_\phi \right)}{F\left( \frac{1}{2} - \mu - \nu + \omega, \frac{1}{2} - \mu - \nu - \omega, 1 - 2\mu; -Z_\phi/r_\phi \right)} = 0. \tag{5.34}
\]

This is the exact form of the critical surface of the gauged Yukawa model. By solving this equation, we obtain the following critical surface:
\[
h = h_c(\lambda, Z_\phi) = r_\phi \left( \frac{1}{2} + \mu \right)^2 \frac{1 + Z_\phi/r_\phi}{4 + 1/N_c + 4Z_\phi/r_\phi + \frac{1/N_c Z_\phi}{2+2\mu}} \quad \text{for } Z_\phi \ll 1. \tag{5.35}
\]

The above equation shows the deviation from the critical line Eq.(5.27) of the gauged NJL model ($Z_\phi = 0$) due to the presence of $Z_\phi$.

Next we consider the decay constant $F_\pi$. Using Eq.(2.13) and Eq.(2.22) with the IR cutoff $M$, we may rewrite Eq.(3.9a)–(3.9c) as
\[
Z_\phi = r_\phi^2 \frac{\pi^2}{2N_c} \frac{F_\pi^2 - \tilde{F}_f^2(M^2)}{hV^2(\Lambda^2)} + r_\phi^2 \frac{\pi^2}{2N_c} \frac{\tilde{F}_f^2(\Lambda^2)}{hV^2(\Lambda^2)}, \tag{5.36}
\]
with the function $\tilde{F}_f(x)$ being defined by
\[
\tilde{F}_f^2(x) := \frac{N_c}{4\pi^2} \int_x^\infty dy \frac{B(y)(B(y) - \frac{d}{dy}B(y))}{(y + B^2(y))^2}, \tag{5.37}
\]
which is finite (for $0 < \lambda < \lambda_c = 1/4$) from the asymptotic form Eqs.(5.22b),(5.23b) of the solution $B(x)$. $\tilde{F}_f^2(M^2)$ in the bifurcation plays the role of $F_\pi^{\infty 2}(:= F_\pi^2(\Lambda^2 = \infty))$ in the full nonlinear numerical analysis in Section 4.

Let us now consider the curve (RG flow) on which both the decay constant $F_\pi$ and the fermion mass $M := B(M^2)$ are fixed, in the three-dimensional parameter space $(h, \lambda, Z_\phi)$. There are three types of flows corresponding to the sign of $F_\pi^2 - \tilde{F}_f^2(M^2)$. 

For the case (II) $F_\pi^2 > \bar{F}_f^2(M^2)$ we have $Z_\phi \to \infty$, because the condensation function $V(\Lambda^2)$ goes to zero as $\xi_f := \Lambda/M \to \infty$. In the case (I) $F_\pi^2 < \bar{F}_f^2(M^2)$ we can have $Z_\phi = 0$. This is consistent with the numerical result.

However, for $F_\pi^2 = \bar{F}_f^2(M^2)$ we obtain a finite value for $Z_\phi$, since the first term of Eq.(5.36) is identically zero and the second term converges according to the asymptotic form of $B(x)$ and $V(x)$, Eqs.(5.22a)–(5.23b). Thus the surface $F_\pi^2 = \bar{F}_f^2(M^2)$ yields renormalized trajectories. Then we calculate the fixed line, i.e., the intersection of the surface of the renormalized trajectories $F_\pi^2 = \bar{F}_f^2(M^2)$ and the critical surface Eq.(5.34). On the critical surface ($\Lambda/M \to \infty$), the renormalized trajectories (Eq.(5.36) for $F_\pi^2 = \bar{F}_f^2(M^2)$) take the form

$$Z_\phi = r_\phi^2 \frac{\pi^2}{2N_c} \frac{\frac{d}{dx} \bar{F}_f^2(x)}{h \frac{d}{dx} V(x)} \bigg|_{x=\Lambda^2 \to \infty} = -\frac{r_\phi^2}{16h} \frac{B(x) \left[ B(x) + \frac{1}{2} \left( \frac{Z_\phi h/N_c}{(r_\phi + Z_\phi)^2} + \lambda \right) V(x) \right]}{V(x)[B(x) - V(x)]} \bigg|_{x=\Lambda^2 \to \infty},$$

(5.38)

where the differentiation of $B(x)$ and $V(x)$ are rewritten by using the differential equation Eqs.(5.1a),(2.19b).

Substituting the UVBC (2.20b) into Eq.(5.38) we obtain a relation among the parameters $(h, \lambda, Z_\phi)$:

$$r_\phi^2 \left( \frac{4Z_\phi/r_\phi + 4 + 1/N_c}{Z_\phi + r_\phi} h + \lambda \right) \left( \frac{4Z_\phi/r_\phi + 4 + 1/N_c}{Z_\phi + r_\phi} h + \frac{1}{2} \frac{Z_\phi/N_c}{(r_\phi + Z_\phi)^2} h + \frac{3}{2} \lambda \right) + 16hZ_\phi \left( \frac{4Z_\phi/r_\phi + 4 + 1/N_c}{Z_\phi + r_\phi} h + \lambda - 1 \right) = 0.$$

(5.39)

Note that Eq.(5.39) is not the renormalized trajectories unless the parameters $(h, \lambda, Z_\phi)$ lie on the critical surface, since $\Lambda/M \to \infty$ is already taken in Eq.(5.38). By requiring both Eq.(5.39) and Eq.(5.34) to be satisfied simultaneously, we finally obtain the fixed line which is shown in Fig. 9. Specifically, we find that the fixed line extends from $(h, \lambda, Z_\phi) = \left(-\frac{1}{16+4/N_c}, 1/4, 0\right)$ to $(1/4, 0, \infty)$ when $r_\phi = 1$. This is one of the main results of this paper.
Now we state the result on the fixed line. The fixed line on the critical surface separates $F_\pi = \bar{F}_f(M^2)$ surface into two regions, one of which extends to $Z_\phi \to \infty$ as $\Lambda/M \to \infty$ and another to $Z_\phi \to 0$ for $\Lambda/M \gg 1$. The surface $F_\pi = \bar{F}_f(M^2)$ may be identified with the surface of renormalized trajectories. So we can imagine the renormalized trajectories lying on the surface $F^2_\pi = \bar{F}^2_f(M^2)$ terminate at the fixed line on the critical surface.

This picture is consistent with the numerical result in Section 4. Thus the analytical study confirm the existence of the renormalized trajectories and the fixed line. It is again emphasize that $Z_\phi \to \infty$ as $\lambda \to 0$, i.e., no (nontrivial) fixed point in the pure Yukawa model. The fixed line is only revealed by presence of the gauge coupling.

6 Conclusion and discussion

In this paper we have investigated phase structure of the $(SU(N_c), U(1)−)$ gauged Yukawa model with a global symmetry $SU(2)_L \times SU(2)_R$, using the ladder SD equation with “standing” gauge coupling, Eq.(2.7a), and the newly derived “generalized Pagels–Stokar (PS) formula” Eqs.(3.9a)–(3.9c). In the three-dimensional space of bare parameters $(h, \lambda, Z_\phi)$, we have obtained the $\xi_f$-constant surface ($\xi_f := \Lambda/M$, fermion correlation length), or the scaling relation (see Fig.5, Eq.(5.24)). Then, as a limit of $\xi_f \to \infty$ or the fermion dynamical mass $M \to 0$ ($\Lambda$-fixed), we obtained the critical surface (Fig.5, Eq.(5.34)) which separates the spontaneous chiral symmetry breaking phase ($M \neq 0$) and the chiral symmetric phase ($M = 0$).

We have further obtained the “renormalization-group (RG) flows” by requiring the NG boson decay constant $F_\pi$ as well as $M$ be fixed as we change the cutoff $\Lambda$ (Figs.6,7). In the three-dimensional space each RG flow is a cross section of the equi-$F_\pi$ surface sliced by the $\lambda$-constant plane. Among such RG flows we have discovered the “renormalized trajectories” which terminate on the critical surface (Figs.6,7). The intersection of a set of renormalized trajectories with the critical surface may be identified
with a set of UV fixed points, the UV fixed line (Fig.9), which exists only for nonzero
gauge coupling ($\lambda > 0$). In the $N_c \to \infty$ limit we obtained an explicit analytical
form of the renormalized trajectory as well as the fixed line, Eqs.(5.13a)–(5.13b). The
analytical study has shown that the qualitative feature of the phase structure does
not depend on $N_c$. The fixed line extends connecting $(h, \lambda, Z_{\phi}) = (-\frac{1}{16+4/N_c}, 1/4, 0)$
and $(1/4, 0, \infty)$. We thus have established existence of the fixed line in the gauged
Yukawa model ($\lambda > 0$) and non-existence of the (nontrivial) fixed point in the pure
Yukawa model ($\lambda = 0$) through the analytical study and the numerical one within the
framework of the ladder SD equation. This strongly suggests that although the pure
Yukawa model might be “trivial”, the gauged Yukawa model may be a “nontrivial”
(interacting) continuum field theory thanks to the presence of the gauge coupling.

The above fixed line should be compared with that in the gauged NJL model. The
$Z_{\phi} \to 0$ limit of the gauged Yukawa model is equivalent to the gauged NJL model with
appropriate rescaling of the Yukawa coupling (see Eq.(5.27)). Now in the gauged NJL
model with the standing gauge coupling, the fixed line is identified with the critical
line itself[27, 29, 21], while the fixed line in the gauged Yukawa model deviates from
that of the gauged NJL model ($Z_{\phi} = 0$) and drifts into the direction of $Z_{\phi} > 0$ on the
critical surface in the three-dimensional bare parameter space (Fig.9). Actually, the
projection of the fixed line onto the $Z_{\phi} = 0$ plane exactly coincides with the critical line
of the gauged NJL model. This result is due to the propagating (“hopping”) degree
of freedom of the scalar field, since the corresponding scalar field in the gauged NJL
model is merely an auxiliary one and has no kinetic term.

It was pointed out[22] that, within the gauged NJL model, the line of $M$-constant
does not exactly coincide with that of $F_\pi$-constant particularly in the weak gauge
coupling region. Actually, $M$ is a slowly decreasing function of the cutoff $\Lambda$ along
the $F_\pi$-constant line as was demonstrated in the top quark condensate model[3, 2].

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9This is not a problem of the top quark condensate model as a phenomenological model with an
explicit cutoff. This in fact was the very predictability of the top quark mass in that model. Actually,
we can even take the formal limit $\Lambda \to \infty$ of the model and obtain a finite (“nontrivial”) continuum
This suggests that we need (at least one) other relevant operators to obtain a sensible continuum field theory. Having introduced a $Z_\phi$ degree of freedom, we can explicitly see how this problem is resolved in this enlarged coupling space. Let us discuss the small $Z_\phi(>0)$ in the region (I): $F_\pi < F_f^\infty$, which is bounded between the surface of renormalized trajectories ending at the fixed line ($F_\pi = F_f^\infty$ surface) and the $Z_\phi = 0$ plane (see Fig.7). If we specify a point on the RG flow and perform one step of RG (by decreasing the cutoff to the smaller $\xi_f$), the point moves to another point on the same flow with a larger $Z_\phi$. Thus the requirement of both $F_\pi$ and $M$ to be constant can only be met by the change in $Z_\phi$ direction and the continuum theory is obtained at the fixed line with $Z_\phi > 0$.

Here we should remark that our results (the existence of the critical surface, the fixed line and the renormalized trajectory) are based on the “standing” ansatz of the gauge coupling. The RG evolution of the scalar quartic coupling $\lambda_\phi$ is also disregarded in the present paper. In principle, it is possible to take these effects into account in our analysis by fixing yet another “physical quantities”. The mass of the physical scalar boson $m_\phi^{\text{phys}}$ and the (infrared) scale of $SU(N_c)$ gauge interaction $\Lambda_{\text{gauge}}$ would suit such a purpose. We then need to estimate the possible feedback from the running of those parameters to the restricted analysis here within the essentially two parameter space $(h, Z_\phi)$.

As for the running of $\lambda_\phi$, we need to modify two aspects so as to take account of the feedback in the SD equation utilized in this paper. One is the inter-relation among $\langle \bar{\psi} \psi \rangle$ and $\langle \sigma \rangle$ which should be changed with respect to the quartic coupling $\lambda_\phi$ (see Eq.(2.11)). This effect is, however, suppressed by $\langle \bar{\psi} \psi \rangle / \Lambda^3$ and thus we hope can be disregarded near the critical point as far as the phase transition is the second order. Another is the use of the $\lambda_\phi$ dependent dressed propagators of the scalar/pseudoscalar field in the rainbow graph of the scalar/pseudoscalar. This is also suppressed by $1/N_c$ and is expected to give only a tiny effect for large $N_c$.

\[ \text{theory}\{22, 2\}, \text{although such a limit is peculiar in the sense we just mentioned.} \]
It is a rather delicate problem, however, to estimate the feedback from the (asymptotically free) running of the gauge coupling $\lambda$. It is straightforward to include such a running effect (at one-loop) into our analysis as we did in the gauged NJL model ($Z_\phi = 0$)[22] where almost all the RG flows tend to the pure NJL point $(h, \lambda) = (h_c, 0)$ as $\Lambda \to \infty$; namely, the fixed line shrinks into a single point $(h, \lambda) = (h_c, 0)$, the UV fixed point, due to the one-loop running effect of the gauge coupling[22]. Thus the fixed line of the gauged Yukawa model in $(h, \lambda, Z_\phi)$ space may also shrink into $\lambda \to 0$ for the running gauge coupling, since its projection onto the $(h, \lambda)$ plane is again expected to be that of the gauged NJL model as in the case of the standing gauge coupling discussed above. Now, this limit is rather subtle, since the fixed line extends into the region $Z_\phi \to \infty$ as $\lambda \to 0$, see Fig.9. Thus even the qualitative structure of the RG seems to depend critically on the speed of running of the gauge coupling, or on the number of quark flavors for the QCD coupling.[22] Similarly, in the perturbative RG analysis of the standard model[30], the qualitative structure of the RG flow of the Yukawa coupling in fact depends on the speed of the running of the QCD coupling. This problem will be dealt with in the forthcoming paper.

The phase structure obtained in this paper may be compared with results of the lattice theory. If we restrict the model to the pure Yukawa case ($\lambda = 0$), our results on the phase structure are consistent with those of the lattice Yukawa model [31]. We here predict the phase structure of the gauged Yukawa model on the lattice using the parameterization which is similar to the lattice theory. After the transformation of the parameters ($d$: dimensionality of the space-time, $a$: lattice spacing):

$$\kappa_H = \frac{4Z_\phi}{2dZ_\phi + a^2 M^2_\phi}, \quad g^2_Y = \frac{1}{N_c} \frac{g^2_y}{2dZ_\phi + a^2 M^2_\phi},$$

(6.1)

our Yukawa model (excluding the gauge part) can be cast into the lattice version with

\[^{10}\text{In Ref.[22] we found another UV fixed point } (h, \lambda) = (0, 0), \text{ the "pure QCD point", which has no correspondence to the fixed point of the model with the standing gauge coupling discussed in this paper.}\]
the lattice action:

\[
S = -\kappa_H \sum_{x,\mu} a^{d-2} \varphi_\alpha(x)[\varphi_\alpha(x + \mu) + \varphi_\alpha(x - \mu)] + 4 \sum_x a^{d-2} \varphi_\alpha^2(x) \\
+ \frac{1}{2} \sum_{x, z, \mu} a^{d-1} \bar{\psi}_i(x) \gamma_\mu [\delta_{x+z, \mu} - \delta_{x-z, \mu}] \psi_i(z) \\
+ \sqrt{8} g_Y \sum_x a^d \bar{\psi}_i(x) [\varphi_0(x) + i\gamma_5 \vec{\tau} \cdot \vec{\varphi}(x)] \psi_i(x),
\]

with \( i = 1, \ldots, N_c \) and \( \alpha = 0, \ldots, 3 \) being indices of the \( SU(N_c) \) fundamental representation and the vector representation of \( O(4) = SU(2)_L \times SU(2)_R \), respectively. The phase structure of the gauged Yukawa model as well as the pure Yukawa model in this parameterization is shown in Fig.10. Our claim on the existence of the fixed line in the presence of the gauge coupling is based on the nonperturbative analysis through the SD equation. Actually, such a remarkable feature does not appear by perturbatively including the effect of the gauge coupling [32]. In view of this there exists, to our knowledge, no available data from the lattice theory which are comparable with our result. Our finding in this paper will be sufficient to urge the lattice people to try to perform a full study of the gauged Yukawa model on the lattice, which will really confirm whether our claim is right or not.

In this paper we have assumed \( Z_\phi > 0 \), since otherwise the vacuum would be unstable already in the scalar/pseudoscalar sector (there appear tachyon ghosts). In fact the ladder SD equation has a tachyon pole and cannot be Wick-rotated for \( Z_\phi < -r_\phi \). Furthermore the RG flow becomes singular at \( Z_\phi = -r_\phi [1 - (\sqrt{4N_c + 1} - 1)/(4N_c)] \) \((> -r_\phi)\), which can be seen from the UVBC Eq.(2.20b) and the generalized Pagels-Stokar formula Eqs.(3.9a)-(3.9c). However, we can formally extend our RG flow to the negative value \( 0 > Z_\phi > -r_\phi [1 - (\sqrt{4N_c + 1} - 1)/(4N_c)] \) in our analysis. Although this \( Z_\phi \) region might also be pathological, it is amusing to compare our RG flow with the lattice Yukawa model. By the above correspondence we can extend the RG flow of Fig.10a (no gauge coupling) into the negative \( \kappa_H \) region, which indicates no sign of fixed point. This is consistent with the recent lattice analysis of the pure Yukawa
model.

Finally, besides the RG analysis, we consider a physical application of the above information on the phase structure. Here we keep the UV cutoff $\Lambda$ fixed. This is actually the case for the Yukawa-driven top mode model$^[6, 8]$ and the heavy scalar technicolor model$^[17]$. First we discuss the criterion whether the symmetry breaking is “dynamical” or not. This is not so clear in the system having the elementary scalar from the onset, since it always mixes with the composite one. However, as we can see from our phase diagram, we can discriminate two distinct regions separated by the renormalized trajectories: region (I) ($F_\pi < F_\infty^f$) and region (II) ($F_\pi > F_\infty^f$). In region (I) we can always arrange the fermionic contribution $F^2_f$ to dominate the bosonic one $F^2_\phi$ in the the NG boson decay constant $F^2_\pi$, particularly for a small $Z_\phi$ or a large $\Lambda$ (see Fig. 8). This is contrasted with the region (II) where $Z_\phi$ is relatively large and the bosonic contribution persists even for a large $\Lambda$. We may call such a fermionic dominance of the $F_\phi$ in region (I) the “dynamical”, in contrast with the fermionic non-dominance in region (II) which may be called the “non-dynamical”. Although the nomenclature is somewhat obscure, our definition is based on the phase diagram and hence is without ambiguity. This concept may be useful for analyzing the unified models based on the strong fermion dynamics involving elementary scalar field.

Next we discuss the dependence of the fermion dynamical mass $M$ on $Z_\phi$. For a given value of $F_\pi$, we have shown that $M$ decreases as $Z_\phi$ increases along the $\Lambda$-fixed line, see Fig. 8. This implies that $M$ in the gauged Yukawa model ($Z_\phi > 0$) is always smaller than that in the gauged NJL model ($Z_\phi = 0$) for the same $F_\pi$ value, which is actually derived more generally as a direct consequence of our generalized PS formula Eqs.(3.9a)–(3.9c) as was explained in Section 3. This might appear rather peculiar from the general view point of a “generalized NJL model” (without gauge coupling) of Hasenfratz et al.$^[15]$ presuming arbitrary higher dimensional operators. Such operators yield a priori an equal potentiality either to raise or lower the fermion dynamical mass compared with the NJL model. On the contrary, we have specified a model, the gauged
Yukawa model, for the higher dimensional operators instead of introducing arbitrary such operators. Therefore we have obtained a definite answer that the fermion mass in this model must be lower than in the gauged NJL model. If this conclusion in the $SU(2)_L \times SU(2)_R$-invariant gauged Yukawa model persists in the $SU(2)_L \times U(1)_Y$-invariant gauged Yukawa model, then the Yukawa-driven top quark condensate based on the latter model will give a smaller top quark mass prediction than the original one\[3, 5\] based on the $SU(2)_L \times U(1)_Y$-invariant gauged NJL model. The analysis along this direction including the running effects of QCD coupling is under way.

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Figure Captions

Figure 1:   
Graphical representation for the NG boson decay constant $F_\pi$, Eq.(3.4).

Figure 2:   
Critical line of the pure Yukawa model ($\lambda = 0$). The critical line is drawn by the bold line and the solid lines denote the renormalization-group flows. Each (upper) flow corresponds to a different (larger) value of $F_\pi$. The dotted lines are the equi-$\xi_f$ (correlation length) lines.

Figure 3:   
Yukawa coupling dependence of the chiral condensate. The solid line, broken line, dashed line and dotted line correspond to $Z_\phi = 0.01, 0.1, 1, 10$, respectively.

Figure 4:   
Critical line (bold line) and equi-$\xi_f$ lines for $Z_\phi = 0$ (gauged NJL model).

Figure 5:   
Equi-$\xi_f$ (correlation length) surface in the space ($h, \lambda, Z_\phi$). The framed surfaces from top to bottom corresponds to $\xi_f = \Lambda/M = \exp(t/2)$ for $t = 1,3,5,7,15$. The final surface is very close to the critical surface ($\xi_f = \infty$).

Figure 6:   
Renormalization-group flows for fixed $F_\pi$, which are shown as the intersection of the equi-$F_\pi$ surface with the fixed-$\lambda$ (gauge coupling) plane. (a) $F_\pi^2/M^2 = 0.142$, (b) $F_\pi^2/M^2 = 0.086$. 


Figure 7:
Critical line (bold line) and renormalization-group flows (solid line) for the gauged Yukawa model ($\lambda > 0$). They are depicted in the fixed-$\lambda$ plane. Each (upper) flow corresponds to a different (larger) value of $F_{\pi}$. The dotted lines denote the equi-$\xi_f$ lines. (a) $\lambda = 0.05$, (b) $\lambda = 0.15$.

Figure 8:
The ratio $F_\phi^2/F_f^2$ along the renormalization-group flow in the fixed-$\lambda$ plane. The line passing through the origin denotes the fixed-cutoff $\Lambda$ line. (a) $\lambda = 0.02$, (b) $\lambda = 0.08$.

Figure 9:
The mesh denoted by solid line shows the critical surface obtained from the analytical solution of the SD equation Eq.(2.19a) and Eq.(5.1a) in the limit of $\Lambda/M \to \infty$. The fixed line on the critical surface is plotted by bold line. The projection of the fixed line to the $Z_\phi$-$h$ plane (a) and the $\lambda$-$Z_\phi$ plane (b) are plotted by solid line. The equi-$\lambda$ lines (a) and equi-$h$ lines (b) are shown by the lines corresponding to different values of $\lambda$ and $h$ (see the upper right corner), respectively.

Figure 10:
Critical line (bold line) and renormalization-group flows (solid line) in the fixed-$\lambda$ plane. The parameters are translated by using Eq.(6.1), and we choose the lattice spacing as $a^2 M_\phi^2 = 50$ and $N_c = 1$. The dotted line denotes the equi-$\xi_f$ line. (a) $\lambda = 0$ (pure Yukawa model), (b) $\lambda = 0.1$. 
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Fig. 2
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Fig. 3

\[-2\pi^2 \langle \dot{\psi} \rangle / \Lambda^2\]
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Fig. 6a
Fig. 6b
Fig. 7a
Fig. 7b
Fig. 8b
Fig. 9a
Fig. 9b
Fig. 10a
Fig. 10b
Fig. 1

\[ J^a_{\mu} \quad + \quad J^a_{\mu} \]

\[ \chi_p \quad \Phi \quad \chi_b \]

\[ \psi \quad \psi \quad \psi \]

\[ p-q \]

\[ q \quad q \]

\[ p \quad \psi \quad \psi \]