On a weighted Trudinger-Moser inequality in $\mathbb{R}^N$

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Abstract

We establish the Trudinger-Moser inequality on weighted Sobolev spaces in the whole space, and for a class of quasilinear elliptic operators in radial form of the type $Lu := -r^{-\theta}(r^\alpha |u'(r)|^\beta u'(r))'$, where $\theta, \beta \geq 0$ and $\alpha > 0$, are constants satisfying some existence conditions. It worth emphasizing that these operators generalize the $p$-Laplacian and $k$-Hessian operators in the radial case. Our results involve fractional dimensions, a new weighted Pólya-Szegö principle, and a bounded value for the optimal constant in a Gagliardo-Nirenberg type inequality.

Key words: weighted Trudinger-Moser inequality, weighted rearrangement, Schwarz symmetrization

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1 Introduction

It is well known the classical Sobolev embedding it holds that the embedding $W^{1,q}_0(\Omega) \hookrightarrow L^p(\Omega)$ is continuous for any $p \leq q \leq Np/(N-p)$, where $p < N$ and $\Omega$ a domain contained in $\mathbb{R}^N$. Although, the embedding $W^{1,N}(\Omega) \hookrightarrow L^q(\Omega)$ is continuous for any $N \leq q < \infty$, $W^{1,N}(\Omega) \not\subset L^\infty(\Omega)$. Motivated by this approach Adams [11] proved that for every $0 < \mu \leq 1$ the Sobolev space $W^{1,N}(\Omega)$ (where $\Omega$ is an unbounded domain in $\mathbb{R}^N$) is embedding in the Orlicz space $L_{\Psi_{\mu,N}}(\Omega)$, where

$$\Psi_{\mu,N}(t) = e^{\mu t^{N-\frac{\mu}{N}} - \sum_{j=0}^{N-2} \frac{\mu^j}{j!} \frac{t^N}{N^j}}.$$

Hempel, Morris and Trudinger [12] showed that the best Orlicz space $L_{\Psi}(\Omega)$ for the embedding of $W^{1,N}_0(\Omega)$ (where $\Omega$ is a bounded domain in $\mathbb{R}^N$) occurs when $\Psi = \phi := e^{\frac{t^N}{N}} - 1$. More precisely, the space $W^{1,N}_0(\Omega)$ may not be continuously imbedding in any Orlicz space $L_{\Psi}(\Omega)$ whose defining function $\Psi$ increases strictly more rapidly than the function $\phi$.

The case when $\Omega$ is a bounded domain was studied by J. Moser in [16], which showed the following sharp result

$$\sup_{u \in W^{1,N}_0(\Omega) \setminus \{0\}} \frac{1}{|\Omega|} \int_{\Omega} e^{\mu \left( \frac{|u|}{\|u\|_{L^p}} \right)^{N/p}} \, dx \begin{cases} \leq C(N, \mu), & \text{if } \mu \leq \mu_N \\ = +\infty, & \text{if } \mu > \mu_N \end{cases}$$

(1)

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where $\mu_N := N\omega_{N-1}$, $|\Omega|$ is a measure of $\Omega$, $\omega_{N-1}$ is the $(N-1)$-dimensional Lebesgue measure of the unit sphere in $\mathbb{R}^N$, and $C(N,\mu)$ is a positive constant depending only on $N$ and $\mu$.

The case $\Omega = \mathbb{R}^N$, was studied by Ruf in [17] for $N = 2$, and Li and Ruf in [14] for $N \geq 3$. In all cases a sharp result as obtained. Namely, there exists $D(N,\mu)$ which depends only on $N$ and $\mu$ satisfying

$$\int_{\mathbb{R}^N} \Psi_{\mu,N}(u)dx \leq D(N,\mu)$$

for all $u \in W^{1,N}(\mathbb{R}^N)$ with $\|u\|_{W^{1,N}(\mathbb{R}^N)} = 1$ and $\mu \leq \mu_N$. Here, the inequality (2) is not valid if $\mu > \mu_N$.

Ishiwata in [10] studied the attainability of the best constant

$$d_{N,\mu} := \sup_{u \in W^{1,N}(\mathbb{R}^N) : \|u\|_{W^{1,N}(\mathbb{R}^N)} = 1} \int_{\mathbb{R}^N} \Psi_{N,\mu}(u)dx,$$

which is associated with (2) [see section 2 Theorem 2.5 and Theorem 2.6]. A similar study was done in [11] for singular weights.

He used a concentration-compactness type argument, proving that the maximizing sequence for (2) are neither vanishing nor concentrating sequence. He also showed that the functional $J(u) := \int_{\mathbb{R}^2} \Psi_{2,\mu}(u)dx$ does not have critical points on $M := \{u \in W^{1,2}(\mathbb{R}^2) : \|u\|_{W^{1,2}(\mathbb{R}^2)} = 1\}$ for $\mu$ sufficiently small, which implies non-existence results in this case.

Our approach for Trudinger-Moser inequality will be done for the class of quasilinear elliptic operators in radial form of the type

$$Lu := -r^{-\theta}(r^{\alpha}|u'(r)|^\beta u'(r))'$$

where $\theta, \beta \geq 0$ and $\alpha > 0$. See [8, 9] for some problems involving the operator $L$. It worth emphasizing that these operators generalize the $p$-Laplacian and $k$-Hessian operators in the radial case, more precisely,

| Case                      | $\alpha$ | $\beta$ |
|---------------------------|----------|---------|
| (i) Laplacian             | $\theta = N-1$ | $\beta = 0$ |
| (ii) $p$-Laplacian ($p \geq 2$) | $\theta = N-1$ | $\beta = p-2$ |
| (iii) $k$-Hessian ($1 \leq k \leq N$) | $\alpha = N-k$, $\theta = N-1$, $\beta = k-1$ |

where these operators act on the weighted Sobolev spaces

$$W^{1,p}_{\alpha,\beta}(0,R) := W^{1,p}((0,R),d\lambda_\alpha,d\lambda_\beta)$$

defined in section 2. The preposition 2.1 in section 2 [see Kufner-Opic [13]] gives us the following Sobolev type continuous embedding for $0 < R < +\infty$

$$W^{1,p}_{\alpha,\beta}(0,R) \hookrightarrow L_q^\alpha(0,R)$$

if $1 \leq q \leq q^*$, $\alpha - \beta - 1 > 0$ and $p := \beta + 2$.

and the number $q^* := \frac{(1+\theta)(\beta+2)}{\alpha-\beta-1}$ is the critical exponent associated with the weighted Sobolev space $W^{1,p}_{\alpha,\beta}(0,R)$. We would like to emphasize that continuity in the above embedding still hold in the following cases $\alpha - \beta - 1 = 0$,

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As in the classical case, a function in $W^{1,p}_{\alpha,\beta}(0,R)$ (when $\alpha - (p-1) = 0$) could have a local singularity, which proves that $W^{1,p}_{p-1,\beta}(0,R) \not\subset L_q^\beta(0,R)$. Motivated by this approach Oliveira and Do Ó [15] studied this embedding, and they proved some results on validity and attainability of the Trudinger-Moser inequality, for bounded domains see section 2 Theorem 2.2 Theorem 2.3 and Theorem 2.4.
Our goal here is twofold: on the one hand, we prove a Trudinger-Moser type inequality for weighted Sobolev spaces involving fractional dimensions in the unbounded case \((0, \infty)\); and on the other hand, we discuss the existence of extremals functions in such inequalities.

We will replace the constant \(c_{\alpha, \theta}\) (which depends on \(\alpha, \theta\) and \(R\)) in Theorem 2.2 by an uniform constant \(d(\alpha, \theta, \mu)\) (which depends on \(\alpha, \theta\) and \(\mu\)), by replacing the Dirichlet norm with weight \(\|u\|_{L_p^\alpha(0, \infty)}\) by the Sobolev norm with weights \(\|u\|_{W^{1, p}_\alpha(0, \infty)}\), in the same spirit of the results stated in [14] and [17]. Furthermore, we investigate the compactness on maximizing sequence for such inequalities in the same sense of the results stablished in [10].

Let

\[
A_{p, \mu}(t) = e^{\mu t^p - t} - \sum_{j=0}^{[p]-1} \frac{\mu^j t^{p-j}}{j!}, \quad \text{with } [p] \text{ the largest integer less than } p.
\]

One of our main results is:

**Theorem 1.1** Let \(p \geq 2\), \(\theta, \alpha \geq 0\) and \(\mu > 0\) be real numbers such that \(\alpha - (p - 1) = 0\) and \(\mu \leq \mu_{\alpha, \theta}\). Then there exists a constant \(D(\theta, \alpha, \mu)\) which depends only on \(\theta, \alpha\) and \(\mu\) such that

\[
\int_0^\infty A_{p, \mu}(|u(x)|)d\lambda_\theta(x) \leq D(\theta, \alpha, \mu)
\]

for all \(u \in W^{1,p}_{\alpha, \theta}(0, \infty)\) with \(\|u\|_{W^{1,p}_{\alpha, \theta}(0, \infty)} = 1\). Furthermore, the inequality (4) fails if \(\mu > \mu_{\alpha, \theta}\), that is, for any \(\mu > \mu_{\alpha, \theta}\) there exists a sequence \((u_j) \subset W^{1,p}_{\alpha, \theta}(0, \infty)\) such that

\[
\int_0^\infty A_{p, \mu} \left( \frac{|u_j(x)|}{\|u_j\|_{W^{1,p}_{\alpha, \theta}(0, \infty)}} \right) d\lambda_\theta(x) \to \infty \quad \text{as} \quad j \to \infty.
\]

To state our next results, we need to define the best constant associated with the inequality (4), namely

\[
d(\theta, \alpha, \mu) := \sup_{0 \neq u \in W^{1,p}_{\alpha, \theta}(0, \infty)} \int_0^\infty A_{p, \mu} \left( \frac{|u(x)|}{\|u\|_{W^{1,p}_{\alpha, \theta}(0, \infty)}} \right) d\lambda_\theta(x),
\]

where \(\alpha - (p - 1) = 0\).

**Theorem 1.2** Under the assumptions of Theorem 1.1, there exists a positive nonincreasing function \(u \in W^{1,p}_{\alpha, \theta}(0, \infty)\) with \(\|u\|_{W^{1,p}_{\alpha, \theta}(0, \infty)} = 1\) such that

\[
d(\theta, \alpha, \mu) = \int_0^\infty A_{p, \mu}(|u(x)|)d\lambda_\theta(x),
\]

in the following cases:

(i) \(p \geq 3\) and \(0 < \mu < \mu_{\alpha, \theta}\),

(ii) \(p = 2\) and \(\frac{2}{B(2, \theta)^{-1}} < \mu < \mu_{\alpha, \theta}\).

where

\[
B(2, \theta)^{-1} := \inf_{0 \neq u \in W^{1,2}_{1, \theta}(0, \infty)} \frac{\|u''\|_{L^2(0, \infty)}^2 \cdot \|u\|_{L^2(0, \infty)}^2}{\|u\|_{L^2(0, \infty)}^4}.
\]
Theorem 1.3 Let \( p = 2, \theta \geq 0 \) and \( \alpha = 1 \). Then there exists \( \mu_0 \) such that \( d(\theta, \alpha, \mu) \) is not achieved for all \( 0 < \mu < \mu_0 \).

To prove (1), Moser [16] used the well known Schwarz Symmetrization arguments, which provides a radially symmetric function \( u^\# \) defined on the ball \( B_R(0) \), where \( \mathcal{L}^N(\Omega) = \mathcal{L}^N(B_R(0)) \) and all the balls \( \{ x \in B_R(0); u^\#(x) > t \} \) has the same \( \mathcal{L}^N \) measure of the sets \( \{ x \in \Omega; u(x) > t \} \). Furthermore, \( u^\# \) satisfies the Pólya-Szegö inequality.

\[
\int_{B_R(0)} |\nabla u^\#|^N dx \leq \int_\Omega |\nabla u|^N dx.
\]

(6)

Thus the proof of (1) was reduced to the subset of radially non-increasing symmetric functions. In our case, Pólya-Szegö inequality for \( W^{1,p}_{\alpha,\theta}(0,\infty) \) was not available. That was one additional difficulty in this type of problem. See, for instance, [15].

In this paper we present the half weighted Schwarz symmetrization with the goal of work around the problem. Thus, we will reduce again the Trudinger-Moser inequality to non-increasing functions.

The paper is organized as follows. In section 2, we define some elements and present some previous results about Trudinger-Moser inequality on \( W^{1,p}_{\alpha,\theta}(0,R) \), where \( R < \infty \). In section 3, we prove a new Pólya-Szegö Principle on \( W^{1,p}_{\alpha,\theta} \) using a new class of isoperimetric inequalities on \( \mathbb{R} \) with respect to weights \( |x|^k \). In section 4, we establish the Trudinger-Moser inequality on \( W^{1,p}_{\alpha,\theta}(0,\infty) \), under the assumptions of Theorem 1.1. In the section 5, we obtain the Theorem 1.2 studying the compactness of a maximizing sequence \( (u_n) \) for (5). In the section 6, we show the Theorem 1.3 proving that the functional \( F(u) = \int_0^\infty A_{2,\mu}(|u(x)|)d\lambda_\theta(x) \) does not have criticals points on \( \{ u \in W^{1,2}_{\alpha,\theta}(0,\infty) : ||u||_{W^{1,2}_{\alpha,\theta}(0,\infty)} = 1 \} \). Finally, in the section 7 we present a brief discourse about Gagliardo-Nirenberg-Sobolev type inequality and we show that \( 2/\beta(2,\theta) < 2\pi(1+\theta) \), thus the case (ii) of the Theorem 1.2 makes sense.

2 Basics definitions and previous results

Let \( 0 < R \leq +\infty, 1 \leq p < +\infty \) and \( \theta \geq 0 \). Let us denote by \( L^p_\theta(0,R) \) the weighed Lebesque space defined as the set of all measurable functions \( u \) on \( (0,R) \) for which

\[
||u||_{L^p_\theta(0,R)} := \left[ \int_0^R |u(x)|^p d\lambda_\theta(x) \right]^{1/p} < \infty,
\]

where

\[
d\lambda_\theta(x) = \omega_\theta x^\theta dx, \ \omega_\theta = \frac{2\pi \Gamma\left(\frac{\theta}{2}\right)}{\Gamma\left(\frac{1+\theta}{2}\right)}, \text{ for all } \theta \geq 0,
\]

with \( \Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt \) the Gamma Function. Besides, we denote by

\[
W^{1,p}_{\alpha,\theta}(0,R) := \left\{ u \in L^p_\theta(0,R); u' \in L^p_\alpha(0,R) \text{ and } \lim_{x \to R^-} u(x) = 0 \right\}
\]

and

\[
||u||_{W^{1,p}_{\alpha,\theta}(0,R)} := \left( ||u'||_{L^p_\alpha(0,R)}^p + ||u||_{L^p_\theta(0,R)}^p \right)^{\frac{1}{p}}.
\]

In the following proposition, see [13] for more details, we collect some embedding results for the weighted spaces \( W^{1,p}_{\alpha,\theta} \), which will be used in this paper.
Proposition 2.1 Let \( u : (0, R] \to \mathbb{R} \) be an absolutely continuous function. If \( R < \infty \), \( u(R) = 0 \) and

1. for \( 1 \leq \beta + 2 < q < \infty \) one has
   \( \alpha > \beta + 1, \theta \geq \frac{\alpha q}{\beta + 2} - q \left( \frac{\beta + 1}{\beta + 2} \right) - 1 \), or
   \( \alpha \leq \beta + 1, \theta > -1 \).

2. for \( 1 \leq q < \beta + 2 < \infty \) one has
   \( \alpha > \beta + 1, \theta > \frac{\alpha q}{\beta + 2} - q \left( \frac{\beta + 1}{\beta + 2} \right) - 1 \), or
   \( \alpha \leq \beta + 1, \theta > -1 \).

then
\[
\left( \int_0^R |u|^q x^\theta \, dx \right)^{\frac{1}{q}} \leq C \left( \int_0^R |u'|^{\beta + 2} x^\alpha \, dx \right)^{\frac{1}{\beta + 2}},
\]
where \( C \) is a constant which does not depend on \( u \).

Next, we present a result due to Oliveira and Do Ó [15].

Theorem 2.2 Let \( \alpha, \theta \geq 0 \) and \( p \geq 2 \) be real numbers such that \( \alpha - (p - 1) = 0 \). Then there exists a constant \( c_{\alpha, \theta} \) depending only on \( \alpha, \theta \) and \( R \) such that

\[
\sup_{u \in W^{1,p}_{\alpha, \theta}(0,R)} \int_0^R e^{\mu(|u|)}|u'|^p \, d\lambda_\theta(r) \begin{cases} \leq c_{\alpha, \theta}, & \text{if } \mu \leq \mu_{\alpha, \theta} := (1 + \theta) \omega_{\alpha, \theta}^{\frac{1}{\theta}} \\
= \infty, & \text{if } \mu > \mu_{\alpha, \theta} \end{cases}
\]

(7)

where \( \|u'\|_{L^p_\alpha} = 1 \).

They also showed the existence of extremal functions for inequality (7) as follows

Theorem 2.3 Under the assumptions of Theorem 2.2 there are extremal functions for \( C_{\alpha, \theta, R}(\mu) \) when \( \mu \leq \mu_{\alpha, \theta} \): that is, there exists \( u \in W^{1,p}_{\alpha, \theta}(0,R) \) such that

\[
C_{\alpha, \theta, R}(\mu) = \int_0^R e^{\mu(|u|)}|u'|^p \, d\lambda_\theta(r),
\]

where

\[
C_{\alpha, \theta, R}(\mu) := \sup_{u \in W^{1,p}_{\alpha, \theta}(0,R): \|u'\|_{L^p_\alpha} = 1} \int_0^R e^{\mu(|u|)}|u'|^p \, d\lambda_\theta(r).
\]

In the same spirit of Adachi and Tanaka (see [2]), Oliveira and Do Ó showed the following result

Theorem 2.4 Let \( \theta, \alpha \geq 0 \) and \( p \geq 2 \) be real numbers such that \( \alpha - (p - 1) = 0 \). Then for any \( \mu \in (0, \mu_{\alpha, \theta}) \) there is a constant \( C_{\mu, p, \theta} \) depending only on \( \mu, p \) and \( \theta \) such that

\[
\int_0^\infty A_p,\mu \left( \frac{|u(r)|}{\|u'\|_{L^p_\alpha(0,\infty)}} \right) d\lambda_\theta(r) \leq C_{\mu, p, \theta} \left( \frac{\|u\|_{L^p_\theta(0,\infty)}}{\|u'\|_{L^p_\alpha(0,\infty)}} \right)^p
\]

(8)
for all \( u \in W^{1,p}_{\alpha,\theta}(0,R) \setminus \{0\} \). Besides that, for any \( \mu \geq \mu_{N,\theta} \) there is a sequence \( (u_j) \subset W^{1,p}_{\alpha,\theta}(0,\infty) \) such that

\[
\|u_j'\|_{L^p_{\alpha}(0,\infty)} = 1 \quad \text{and} \quad \frac{1}{\|u_j'\|_{L^p_{\alpha}(0,\infty)}} \int_0^\infty A_{p,\mu}(|u_j(r)|) \, d\lambda_\theta(r) \to \infty \quad \text{as} \quad j \to \infty.
\]

Where

\[
A_{p,\mu}(t) = e^{\mu t^{\frac{p}{p-1}}} - \sum_{j=0}^{\lfloor p \rfloor} \frac{\mu^j}{j!} t^{\frac{p}{p-1} - j}, \quad \text{with} \quad \lfloor p \rfloor \text{ is the largest integer less than } p.
\]

As mentioned in the Introduction, Ishiwata \cite{10} studied the attainability of \( d_{N,\mu} \) in the classical case. He emphasized the importance of evaluate vanishing behaviour on maximizing sequence in unbounded case. Next, the main results in \cite{10} are presented.

**Theorem 2.5** Let \( N \geq 2 \) and

\[
B_2 := \sup_{0 \neq \psi \in W^{1,2}(\mathbb{R}^2)} \frac{\|\psi\|_{L^4}^4}{\|\nabla \psi\|_{L^2}^2 \|\psi\|_{L^2}^2}.
\]

Then \( d_{N,\mu} \) is attained for \( 0 < \mu < \mu_N \) if \( N \geq 3 \) and for \( 2/B_2 < \mu \leq \mu_2 = 4\pi \) if \( N = 2 \).

**Theorem 2.6** Let \( N = 2 \). If \( \mu \ll 1 \), then \( d_{2,\mu} \) is not attained.

### 3 Pólya-Szegö Principle on \( W^{1,p}_{\alpha,\theta} \)

As mentioned in the introduction, we are going to define a half weighted Schwarz symmetrization to prove a Pólya-Szego Principle, see the inequality (6).

We define the measure \( \mu_t \) by \( d\mu_t(x) = |x|^t \, dx \). Besides, if \( M \subset \mathbb{R} \) is a measurable set with finite \( \mu_t \)-measure, then let \( M^* \) denote the interval \((0, R)\) such that

\[
\mu_t((0, R)) = \mu_t(M).
\]

Further, if \( u : \mathbb{R} \to \mathbb{R} \) is a measurable function such that

\[
\mu_t \left( \{ y \in \mathbb{R}; |u(y)| > t \} \right) < \infty \quad \text{for all} \quad t > 0,
\]

then let \( u^* \) denote the half weighted Schwarcz symmetrization of \( u \), or in short, the half \( \mu_t \)-symmetrization of \( u \), given by

\[
u^*(x) = \sup \left\{ t \geq 0; \mu_t \left( \{ y \in \mathbb{R}; |u(y)| > t \} \right) > \mu_t(0, x) \right\},
\]

for every \( x > 0 \).

**Remark 3.1** The word “half” appears here because our symmetrization is a little bit different in three aspects:

(i) it is defined on \((0, \infty)\);

(ii) we are comparing the distribution \( \rho(t) := \mu_t \left( \{ y \in \mathbb{R}; |u(y)| > t \} \right) \) with the measure of \((0, x)\), instead \( B_{|x|}(0) \);
(iii) the set \( M^* \) is a semi ball with the same measure of \( M \), instead a ball.

We will carry out the proof of the next result based on Isoperimetric Inequality on \( \mathbb{R} \) with weight \( |x|^k \) [see \( \text{[4], Theorem 6.1} \). Besides, it worth noting that the Theorem 8.1 in [4] do not cover the case \( k < l + 1 \) when \( N = 1 \). For negative values of \( k \), the proof is a consequence of the well-known Hardy-Littlewood inequality. See also Cabré and Ros-Oton [5] for monomial weights, and Talenti [13] for some cases when \( N \geq 2 \).

**Theorem 3.2** Let \( k, l \) be real numbers satisfying \( 0 < k \leq l + 1 \). Besides, let \( 1 \leq p < \infty \) and \( m := pk + (1 - p)l \). Then there holds

\[
\int_0^\infty |u'|^p |x|^{pk+(1-p)l} \, dx \geq \int_0^\infty \left( (u^*)' \right)^p |x|^{pk+(1-p)l} \, dx, \tag{9}
\]

for every \( u \in W_{l,m}^{1,p}(0, \infty) \), where \( u^* \) denotes the half \( \mu_1 \)-symmetrization of \( u \).

**Proof.** Observe that it is sufficient to consider \( u \) a non-negative function. Let

\[
I := \int_0^\infty |u'|^p |x|^{pk+(1-p)l} \, dx \quad \text{and} \quad I^* := \int_0^\infty \left( (u^*)' \right)^p |x|^{pk+(1-p)l} \, dx.
\]

The Coarea Formula holds

\[
I := \int_0^\infty \int_{u=t} \left| u' \right|^{p-1} |x|^{pk+(1-p)l} \, d\mathcal{H}^0(x) \, dt \quad \text{and} \quad I^* := \int_0^\infty \int_{u^*=t} \left| (u^*)' \right|^{p-1} |x|^{pk+(1-p)l} \, d\mathcal{H}^0(x) \, dt.
\]

If \( p = 1 \), we get

\[
I := \int_0^\infty \int_{u=t} |x|^k \, d\mathcal{H}^0(x) \, dt \quad \text{and} \quad I^* := \int_0^\infty \int_{u^*=t} |x|^k \, d\mathcal{H}^0(x) \, dt,
\]

hence, we obtain from Isoperimetric Inequality on \( \mathbb{R} \) with weight \( |x|^k \) [see \( \text{[4], Theorem 6.1} \) and definition of \( u^* \) that

\[
\int_{u=t} |x|^k \, d\mathcal{H}^0(x) \geq \int_{u^*=t} |x|^k \, d\mathcal{H}^0(x).
\]

Therefore, \( I \geq I^* \) when \( p = 1 \).

Now, assume that \( 1 < p < \infty \). By Holder’s Inequality we have

\[
\int_{u=t} |x|^k \, d\mathcal{H}^0(x) \leq \left( \int_{u=t} |x|^{kp+(1-p)l} |u'|^{p-1} \, d\mathcal{H}^0(x) \right)^{\frac{1}{p}} \left( \int_{u=t} \left| u' \right|^l \, d\mathcal{H}^0(x) \right)^{\frac{p-1}{p}}
\]

for a.e \( t \in [0, \infty) \), thus we get

\[
I \geq \int_0^\infty \left( \int_{u=t} |x|^k \, d\mathcal{H}^0(x) \right)^{\frac{p}{p-1}} \left( \int_{u=t} \left| u' \right|^l \, d\mathcal{H}^0(x) \right) \, dt. \tag{10}
\]
Since that \(|(u^*)'|\) and \(|x|\) are constants along of \(\{u^* = t\}\), hence, for \(u^*\) we obtain the equality, i.e,
\[
I^* = \int_0^\infty \left( \int_{u^* = t} |x|^k dH^0(x) \right)^p \left( \int_{u^* = t} \frac{|x|^l}{|(u^*)'|} dH^0(x) \right)^{1-p} dt.
\] (11)

In addition, by definition of \(u^*\), we have
\[
\int_{u > t} |x|^l dx = \int_{u^* > t} |x|^l dx,
\]
and as a consequence of Coarea Formula we get
\[
\int_{u = t} |x|^l dH^0(x) = \int_{u^* = t} \frac{|x|^l}{|(u^*)'|} dH^0(x),
\] (12)
for \(a.e\ t \in [0, \infty)\), that is sometimes called Fleming - Rishel’s Formula.

Again, by Isoperimetric Inequality on \(\mathbb{R}\) with weight \(|x|^k\) [see (4),Theorem 6.1] and the definition of \(u^*\) we obtain
\[
\int_{a = t} |x|^k dH^0(x) \geq \int_{a^* = t} |x|^k dH^0(x).
\] (13)

Therefore, from (10), (11), (12), and (13) we have
\[
I \geq I^*,
\]
thus, (9) follows. ■

4 Trudinger-Moser inequality on \(W^{1,p}_{\alpha,\theta}(0, \infty)\)

In this section, we establish a Trudinger-Moser type inequality on \(W^{1,p}_{\alpha,\theta}(0, \infty)\) (Theorem 1.1) via the Pólya-Szegő Principle presented in section 3.

**Lemma 4.1**  
(i) Let \(u\) be a function in \(W^{1,p}_{\alpha,\theta}(0, \infty)\). Then
\[
|u(x)|^p \leq p \omega_\theta^{p-1} \omega_{\alpha}^\frac{1}{p} x^{-\frac{(p-1)\theta + \alpha}{p}} \|u\|_{L^p_{\theta}(0,\infty)}^{p-1} \|u'\|_{L^p_{\theta}(0,\infty)} \quad \text{for all } x > 0.
\] (14)

Consequently, the embedding \(W^{1,p}_{\alpha,\theta}(0, \infty) \hookrightarrow L^q_{\theta}(0, \infty)\) is compact for all \(q\) satisfying
\[
\frac{p^2(1 + \theta)}{(p - 1)\theta + \alpha} \leq q < \frac{p(1 + \theta)}{\alpha - (p - 1)} := p^*,
\]
where \(\alpha \geq (p - 1)\) and \(\alpha \leq p + \theta\).

(ii) Let \(u \in L^p_{\theta}(0, R)\) a nonincreasing function, then
\[
|u(x)| \leq \left( \frac{1 + \theta}{\omega_\theta x^{1+\theta}} \right)^{1/p} \left[ \int_0^R |u(s)|^p d\lambda_\theta(s) \right]^{1/p}, \text{ for all } 0 < x < R.
\] (15)

Hence, if \((u_n) \subset W^{1,p}_{\alpha,\theta}(0, \infty)\) is a nonincreasing sequence converging weakly to \(u\) in \(W^{1,p}_{\alpha,\theta}(0, \infty)\), then \(u_n \to u\) strongly in \(L^p_{\theta}(0, \infty)\), for each \(p < q < p^*\) (\(\alpha \geq p - 1\)).
Proof. It is easy to check out (15) from a nonincreasing function. Then, we will do only (14).

For every $0 < x < y$ we have

$$|u(x)|^p \leq |u(y)|^p + p \int_x^y |u(t)|^{p-1} |u'(t)| \, dt.$$ 

By Holder Inequality and $\lim_{y \to \infty} u(y) = 0$, we get

$$|u(x)|^p \leq p \int_x^\infty |u(t)|^{p-1} |u'(t)| \, dt \leq p \omega_\theta \omega_{\alpha - \frac{1}{p}} \left( \int_0^\infty |u(t)|^p \, d\lambda_\theta(t) \right)^{\frac{p-1}{p}} \left( \int_0^\infty |u'(t)|^p \, d\lambda_\alpha(t) \right)^{\frac{1}{p}},$$

which proves (14). $\blacksquare$

The next remark will be used in the proof of Theorem 1.1.

Remark 4.2 By inequality (14), we have $|u(x)| \leq 1$, for all

$$x \geq \left( \frac{p}{\omega_{\alpha - \frac{1}{p}}} \right)^{\frac{p}{(p-1)(1+\theta)}} := a_0$$

whenever $u \in W^{1,p}_{\alpha,\theta}(0,\infty)$ with $\|u\|_{W^{1,p}_{\alpha,\theta}(0,\infty)} \leq 1$ and $\alpha - (p-1) = 0$. It is worth noting that $a_0$ depends only on $p$ and $\theta$. 

Proof of Theorem 1.1

We can assume by Theorem 3.2 that $u$ is a nonincreasing positive function on $(0,\infty)$.

Let $a \geq a_0$ (see Remark 4.2) to be chosen later. Next, we divide the integral at (4) in two parts, that is,

$$\int_0^\infty A_{p,\mu}(|u(x)|) \, d\lambda_\theta(x) = \int_0^a A_{p,\mu}(|u|) \, d\lambda_\theta(x) + \int_a^\infty A_{p,\mu}(|u|) \, d\lambda_\theta(x).$$

By Lemma 4.1 the second part at (16) can be estimated. Indeed, we have

$$\int_0^\infty A_{p,\mu}(|u|) \, d\lambda_\theta(x) = \sum_{j=|p|}^\infty \frac{\mu^j}{j!} \int_a^\infty |u|^{\frac{p}{p-1}} r^j \, \omega_\theta \, dr.$$ 

We obtain by Lemma 4.1 and Remark 4.2

$$\int_a^\infty A_{p,\mu}(|u|) \, d\lambda_\theta(x) = \sum_{j=|p|}^\infty \frac{\mu^j}{j!} \int_a^\infty |u|^{\frac{p}{p-1}} r^j \, \omega_\theta \, dr$$

$$\leq \omega_\theta \frac{\mu^{|p|}}{|p|!} \int_a^\infty |u|^{\frac{p}{p-1}} r^j \, dr$$

$$+ \omega_\theta \sum_{j=|p|+1}^\infty \frac{\mu^j(1+\theta)^{\frac{j}{p-1}}}{j! \omega_\theta^{\frac{j}{p-1}}} \left[ \omega_\theta \int_0^\infty |u|^{\frac{p}{p-1}} r^{j+\frac{1}{p-1}} \, dr \right] \frac{1}{p-1}$$

$$+ \int_a^\infty \left[ \omega_\theta \int_0^\infty |u|^{\frac{p}{p-1}} r^{j+\frac{1}{p-1}} \, dr \right] \frac{1}{p-1}$$

$$= \frac{\mu^{|p|}}{|p|!} + \sum_{j=|p|+1}^\infty \frac{\mu^j(1+\theta)^{\frac{j}{p-1}}}{j! \omega_\theta^{\frac{j}{p-1}}} \left( (j - (p-1))(1+\theta) \right)$$

$$= \frac{\mu^{|p|}}{|p|!} + \sum_{j=|p|+1}^\infty \frac{\mu^j(1+\theta)^{\frac{j}{p-1}}}{j! \omega_\theta^{\frac{j}{p-1}}} \left( (j - (p-1))(1+\theta) \right)$$

(17)
To estimate the first part at (16), let

\[ v(r) = \begin{cases} 
   u(r) - u(a), & 0 < r \leq a \\
   0, & r \geq a 
\end{cases} \]

Note that if \( 1 < q \leq 2 \) and \( b \geq 0 \), we have \((x + b)^q \leq |x|^q + q b^{q-1} x + b^q \) for all \( x \geq -b \). Then, by Lemma 4.1 we obtain

\[
u(r)^{\frac{p}{p-1}} \leq v(r)^{\frac{p}{p-1}} + \frac{p}{p-1} \frac{v(r)^{\frac{1}{p-1}}}{u(a)} \frac{u(a)^{\frac{p}{p-1}}}{u(a)^{\frac{p}{p-1}}}
\]

\[
\leq v(r)^{\frac{p}{p-1}} + v(r)^{\frac{p}{p-1}} u(a)^{p} + u(a)^{\frac{p}{p-1}} + \frac{1}{(p-1)^{1/p-1}}
\]

\[
\leq v(r)^{\frac{p}{p-1}} \left[ 1 + \frac{1 + \theta}{a^{1+\theta} \omega_\theta} \left( \omega_\theta \int_0^\infty |u|^{p \theta} \, dr \right) \right] + \frac{1 + \theta}{(a^{1+\theta} \omega_\theta)}^{1/p-1} + \frac{1}{(p-1)^{1/p-1}}
\]

\[
:= v(r)^{\frac{p}{p-1}} \left[ 1 + \frac{1 + \theta}{a^{1+\theta} \omega_\theta} \left( \omega_\theta \int_0^\infty |u|^{p \theta} \, dr \right) \right] + d(a).
\] (18)

Hence

\[
u(r) \leq v(r) \left[ 1 + \frac{1 + \theta}{a^{1+\theta} \omega_\theta} \left( \omega_\theta \int_0^\infty |u|^{p \theta} \, dr \right) \right]^{\frac{p-1}{p}} + d(a)^{\frac{p-1}{p}}
\]

\[
:= w(r) + d(a)^{\frac{p-1}{p}},
\]

thus

\[
\omega_\alpha \int_0^a |u'|^{p \alpha} \, dr = \omega_\alpha \int_0^a |u'|^{p} \left[ 1 + \frac{1 + \theta}{a^{1+\theta} \omega_\theta} \left( \omega_\theta \int_0^\infty |u|^{p \theta} \, dr \right) \right]^{p-1} r^\alpha \, dr
\]

\[
= \left[ 1 + \frac{1 + \theta}{a^{1+\theta} \omega_\theta} \left( \omega_\theta \int_0^\infty |u|^{p \theta} \, dr \right) \right]^{p-1} \omega_\alpha \int_0^a |u'|^{p \alpha} \, dr
\]

\[
\leq \left[ 1 + \frac{1 + \theta}{a^{1+\theta} \omega_\theta} \left( \omega_\theta \int_0^\infty |u|^{p \theta} \, dr \right) \right]^{p-1} \left[ 1 - \omega_\theta \int_0^\infty |u|^{p \theta} \, dr \right]
\]

\[
\leq 1
\] (19)

where in the last inequality we used that the function \( f : [0, 1] \to \mathbb{R} \) defined by \( f(t) = (1 + \gamma t)^{p-1} (1 - t) - 1 \) is non-positive for any \( \gamma \) fixed in the interval \((0, 1/(p - 1))\) and consequentely the inequality (19) is valid with

\[
\left( \frac{(p - 1)(1 + \theta)}{\omega_\theta} \right)^{1/(1+\theta)} \leq a < \infty.
\]

Next, from (18) we have

\[
\int_0^a A_{p,\mu}(\|u(x)\|)d\lambda_\theta(x) \leq \omega_\theta \int_0^a e^{\mu|u|^{\frac{p}{p-1}} r^\theta} \, dr
\]

\[
\leq \omega_\theta \int_0^a e^{\mu|u|^{\frac{p}{p-1}} r^\theta} \, dr + \omega_\theta \int_0^a e^{d(a)r^\theta} \, dr.
\] (20)
We combine (17), (19), (20) and Theorem 2.2 to conclude the first part of the proof of the theorem.

For the second part, we are going to do the changing of variable as in [15]. We define \( w(t) = \omega_1 \alpha + \alpha (1 + \theta) u(Re^{-\frac{t}{(1+\theta)}}) \) for all \( u \in W^{1,p}_{\alpha,\theta}(0,R) \), where \( \alpha - (p - 1) = 0 \). Then, we get

\[
\int_0^R |u'(r)|^p d\lambda_{\alpha}(r) = \int_0^\infty |w'(t)|^p dt,
\]

(21)

\[
\int_0^R |u(r)|^p d\lambda_{\theta}(r) = \frac{R^{1+\theta} \omega_{\theta}}{(1+\theta)^p \omega_\alpha} \int_0^\infty |w(t)|^p e^{-t} dt
\]

(22)

and

\[
\int_0^R e^{\frac{|u|^p}{p-1}} d\lambda_{\theta}(r) = \frac{\omega_{\theta} R^{1+\theta}}{1+\theta} \int_0^\infty e^{\frac{\mu}{\mu_{\alpha,\theta}(1+\rho(\alpha,\theta,R)\alpha_j)}} \frac{|u|^p}{p-1} dt
\]

(23)

We consider Moser’s functions

\[
w_j(t) = \begin{cases} \frac{t}{j^{\frac{1}{p-1}}} & 0 \leq t \leq j \\ j^\frac{1}{p-1} & t \geq j \end{cases}
\]

Hence, we obtain from (21), (22) and (23) that

\[
\int_0^R e^{\left( \frac{|u_j|}{\|u_j\|_{W^{1,p}_{\alpha,\theta}(0,R)}} \right)^{\frac{p}{p-1}}} d\lambda_{\theta}(r) = \frac{\omega_{\theta} R^{1+\theta}}{1+\theta} \int_0^\infty e^{\frac{\mu}{\mu_{\alpha,\theta}(1+\rho(\alpha,\theta,R)\alpha_j)}} \frac{|u_j|^p}{p-1} dt
\]

\[
\geq e^{\left( \frac{\mu}{\mu_{\alpha,\theta}(1+\rho(\alpha,\theta,R)\alpha_j)} \right)^{\frac{1}{p-1}}} j
\]

where \( \rho(\alpha,\theta,R) = \frac{R^{1+\theta} \omega_{\theta}}{(1+\theta)^p \omega_\alpha}, \alpha_j = \frac{1}{j} \int_0^j e^{-t^p} dt + j^{p-1} e^{-j} \) and \( w_j(t) = \omega_1 \alpha + \alpha (1 + \theta) u_j(Re^{-\frac{t}{(1+\theta)}}) \). Thus, if \( \mu > \mu_{\alpha,\theta} \)

\[
\lim_{j \to \infty} \int_0^R e^{\left( \frac{|u_j|}{\|u_j\|_{W^{1,p}_{\alpha,\theta}(0,R)}} \right)^{\frac{p}{p-1}}} d\lambda_{\theta}(r) \geq \lim_{j \to \infty} e^{\left( \frac{\mu}{\mu_{\alpha,\theta}(1+\rho(\alpha,\theta,R)\alpha_j)} \right)^{\frac{1}{p-1}}} j
\]

\[
= +\infty.
\]

Which concludes the theorem.

5 Proof of the Theorem 1.2

In this section, we are going to show the Theorem 1.2. To show the attainability, we study the maximizing sequence to (5). Throughout this section we assume (via Lemma 3.2) that \((u_n)\) is a non-increasing positive maximizing sequence to (5). Besides, assume

\[ u_n \rightharpoonup u \text{ in } W^{1,p}_{\alpha,\theta}(0,\infty), \text{ where } \alpha - (p - 1) = 0. \]

We begin with
Lemma 5.1 Let $0 < \mu < \mu_{\alpha,\theta}$. Then, we have

$$\int_{0}^{\infty} A_{p,\mu}(|u_n|) - \frac{\mu p}{[p]!} |u_n|^\frac{p}{p-1} d\lambda_\theta - \int_{0}^{\infty} A_{p,\mu}(|u|) - \frac{\mu p}{[p]!} |u|^\frac{p}{p-1} d\lambda_\theta \to 0$$

as $n \to \infty$. (24)

Proof. We rewritten (24) as follows

$$\int_{0}^{\infty} B_{\lfloor p \rfloor + 1,\mu}(|u_n|) d\lambda_\theta - \int_{0}^{\infty} B_{\lfloor p \rfloor + 1,\mu}(|u|) d\lambda_\theta \to 0$$

as $n \to \infty$, where

$$B_{k,\mu}(t) := \sum_{j=k}^{\infty} \frac{\mu j}{j!} t^{\frac{p}{p-1} - 1}, \text{ where } k \in \mathbb{N} \text{ and } t \in [0, \infty).$$

It follows from Mean Value Theorem and convexity of $B_{\lfloor p \rfloor + 1,\mu}$ that

$$|B_{\lfloor p \rfloor + 1,\mu}(u_n(x)) - B_{\lfloor p \rfloor + 1,\mu}(u(x))|$$

$$\leq \left( B_{\lfloor p \rfloor + 1,\mu}' \right)^{\gamma_n(x) u_n(x) + (1 - \gamma_n(x)) u(x)} \cdot |u_n(x) - u(x)|$$

$$= \mu \frac{p}{p-1} |\gamma_n(x) u_n(x) + (1 - \gamma_n(x)) u(x)|^{\frac{1}{p-1}}$$

$$\cdot B_{\lfloor p \rfloor,\mu}(\gamma_n(x) u_n(x) + (1 - \gamma_n(x)) u(x)) \cdot |u_n(x) - u(x)|$$

$$\leq \mu \frac{p}{p-1} |\gamma_n(x) u_n(x) + (1 - \gamma_n(x)) u(x)|^{\frac{1}{p-1}}$$

$$\cdot \left[ \gamma_n(x) B_{\lfloor p \rfloor,\mu}(u_n(x)) + (1 - \gamma_n(x)) B_{\lfloor p \rfloor,\mu}(u(x)) \right] \cdot |u_n(x) - u(x)|$$

$$\leq \mu \frac{p}{p-1} |\gamma_n(x) u_n(x) + (1 - \gamma_n(x)) u(x)|^{\frac{1}{p-1}} \cdot \left[ A_{p,\mu}(u_n(x)) + A_{p,\mu}(u(x)) \right]$$

$$\cdot |u_n(x) - u(x)|$$

Now, by Hölder’s and Minkowski’s Inequalities, and (25) we get
where \( q, r, t > 1 \) are real numbers satisfying \( \frac{1}{q} + \frac{1}{r} + \frac{1}{t} = 1 \), \( q\mu < \mu_{\alpha, \theta} \), \( \frac{r}{p-1} \geq p \) and \( t > \frac{p^2}{p-1} \). Besides, in the last inequality at (26) we used the following inequality

\[
\left( e^{\mu t \frac{r}{p-1}} - \sum_{j=0}^{\frac{|p|}{|q|}} \frac{\mu^j}{j!} \frac{t^{\frac{r}{p-1}j}}{1} \right)^q \leq e^{q\mu t \frac{r}{p-1}} - \sum_{j=0}^{\frac{|p|}{|q|}} (q\mu)^j \frac{t^{\frac{r}{p-1}j}}{j!}.
\]

Therefore, from (26), Lemma 4.1 and compactness embedding we conclude the proof of the Lemma.

To continue the study of the maximizing sequence \((u_n)\) based on the concentration-compactness type argument, we analyze the possibility of a lack of compactness which is called vanishing.

For this, we will introduce some components as follows

\[
\begin{align*}
\mu_0 &= \lim_{R \to \infty} \lim_{n \to \infty} \left( \int_0^R |u_n(x)|^p \, d\lambda_\theta(x) + \int_0^R |f(x)|^p \, d\lambda_\alpha(x) \right) \\
\mu_\infty &= \lim_{R \to \infty} \lim_{n \to \infty} \left( \int_0^R |u_n(x)|^p \, d\lambda_\theta(x) + \int_0^\infty |f(x)|^p \, d\lambda_\alpha(x) \right) \\
\nu_0 &= \lim_{R \to \infty} \lim_{n \to \infty} \left( \int_0^R A_{p, \mu} \left( |u_n| \right) \, d\lambda_\theta(x) \right) \\
\nu_\infty &= \lim_{R \to \infty} \lim_{n \to \infty} \left( \int_0^\infty A_{p, \mu} \left( |u_n(x)| \right) \, d\lambda_\theta(x) \right) \\
\eta_0 &= \lim_{R \to \infty} \lim_{n \to \infty} \int_0^R |u_n(x)|^\frac{p}{p-1} \, d\lambda_\theta(x) \\
\eta_\infty &= \lim_{R \to \infty} \lim_{n \to \infty} \int_0^\infty |u_n(x)|^\frac{p}{p-1} \, d\lambda_\theta(x)
\end{align*}
\]
taking an appropriate subsequences if necessary. It is easy to see that
\[ \nu_i \geq \mu_0 + \mu_\infty, \quad d(p, \theta, \mu) = \nu_0 + \nu_\infty \quad \text{and} \]
\[ 1 \geq \eta_0 + \eta_\infty \quad \text{(if } p \text{ is an integer),} \]
where \( i = 0 \) or \( i = \infty \).

**Definition 5.2** \((u_n)\) is a normalized vanishing sequence, \((NVS)\) in short, if \((u_n)\) satisfies \( \|u_n\|_{W^{1,p}_0(0,\infty)} = 1 \) (with \( \alpha - (p - 1) = 0 \)), \( u = 0 \) and \( \nu_0 = 0 \).

**Example 5.3** Let \( \phi \) be a smooth nonincreasing function with compact support on \([0, +\infty)\) satisfying \( \|\phi\|_{L^p(0,\infty)} = 1 \). Besides that, we take \( \gamma, \sigma \) positive real numbers such that \( p\gamma - \sigma (1 + \theta) = 0 \). We set
\[ \phi_n(x) := \lambda_n^{\gamma} \phi(\lambda_n^{\gamma} x) \]
\[ (1 + \lambda_n^{\gamma} \lambda_0)^{\frac{1}{p}}, \]
where \( \lambda_0 := \|\phi'\|_{L^p(0,\infty)} \) and \((\lambda_n)\) is a positive sequence such that \( \lambda_n \to 0 \) as \( n \to \infty \). Thus, \( \phi_n \) is a normalized vanishing sequence.

The main aim here it is show that \( d(\alpha, \theta, \mu) \) is greater than the vanishing level, more precisely
\[ d(\alpha, \theta, \mu) > \sup_{\{(u_n) \subset W^{1,p}_0(0,\infty) : (u_n) \text{ is a NVS}} \int_0^\infty A_{p,\mu}(|u_n(x)|) d\lambda_0(x). \]

Thus, we define the normalized vanishing limit as follows

**Definition 5.4** The number
\[ d_{nvl}(\alpha, \theta, \mu) = \sup_{\{(u_n) \subset W^{1,p}_0(0,\infty) : (u_n) \text{ is a NVS}} \int_0^\infty A_{p,\mu}(|u_n(x)|) d\lambda_0(x), \]

is called a normalized vanishing limit.

The **normalized vanishing limit** will depend only on \( \alpha \) and \( \mu \).

Next, we rewrite the elements defined above. Given a real number \( R > 0 \), we take a function \( \phi_R \in C^\infty(\mathbb{R}) \) such that
\[
\begin{align*}
\phi_R(x) &= 1, \quad 0 \leq x < R \\
0 \leq \phi_R(x) &\leq 1, \quad R \leq x \leq R + 1 \\
\phi_R(x) &= 0, \quad R + 1 \leq x \\
|\phi_R'(x)| &\leq 2 \quad x \in \mathbb{R}.
\end{align*}
\]

After that, we define the functions \( \phi^0_R \) and \( \phi^\infty_R \) by
\[ \phi^0_R(x) := \phi_R(x), \quad \phi^\infty_R(x) := 1 - \phi^0_R(x). \]
Lemma 5.5 Let \( u_{n,R}^i = \phi_R^i u_n \) (\( i = 0, \infty \)). We have

\[
\mu_i = \lim_{R \to \infty} \lim_{n \to \infty} \left( \int_0^\infty |u_{n,R}^i(x)|^p d\lambda_\theta(x) + \int_0^\infty |(u_{n,R}^i(x))'|^p d\lambda_\alpha(x) \right) \tag{29}
\]

\[
\nu_i = \lim_{R \to \infty} \lim_{n \to \infty} \int_0^\infty A_{p,\mu} \left( |u_{n,R}^i(x)| \right) d\lambda_\theta(x) \tag{30}
\]

\[
\eta_i = \lim_{R \to \infty} \lim_{n \to \infty} \int_0^\infty |u_{n,R}^i|^\frac{p}{p-1} d\lambda_\theta(x) \tag{31}
\]

Proof. we will prove only (29) with \( i = 0 \). On the one hand,

\[
\int_0^R |u_n|^p d\lambda_\theta \leq \int_0^\infty |\phi_R^0 u_n|^p d\lambda_\theta \leq \int_0^{R+1} |u_n|^p d\lambda_\theta. \tag{32}
\]

On the other hand, from the Mean Value Theorem we obtain

\[
\int_0^\infty |(u_{n,R}^0)|^p d\lambda_\alpha = \int_0^\infty |\phi_R^0 u_n' + (\phi_R^0)' u_n|^p d\lambda_\alpha = \int_0^\infty |\phi_R^0 u_n'|^p d\lambda_\alpha + \rho_{n,R}, \tag{33}
\]

where

\[
\rho_{n,R} = p \int_0^\infty |\phi_R^0 u_n' + t_n(x)(\phi_R^0)' u_n|^{p-2} \phi_R^0 u_n'(\phi_R^0)' u_n d\lambda_\alpha(x) + \int_0^\infty |\phi_R^0 u_n'|^p d\lambda_\alpha + \rho_{n,R},
\]

and \( 0 \leq t_n(x) \leq 1 \).

we get

\[
|\rho_{n,R}| \leq p \left[ \int_0^\infty |\phi_R^0 u_n' + t_n(x)(\phi_R^0)' u_n|^p d\lambda_\alpha \right]^{\frac{p-1}{p}} \left[ \int_0^\infty |(\phi_R^0)' u_n|^p d\lambda_\alpha \right]^{\frac{1}{p}} \leq 2p ||u_n'||_{L^p_\alpha} + 2 ||u_n||_{L^p_\alpha(R,R+1)} \frac{p-1}{p} ||u_n||_{L^p_\alpha(R,R+1)} \leq 2p \left[ 1 + 2 ||u_n||_{L^p_\alpha(R,R+1)} \right]^{p-1} ||u_n||_{L^p_\alpha(R,R+1)}.
\]

From compactness embedding, we have \( \lim_{n \to \infty} ||u_n||_{L^p_\alpha(R,R+1)} = ||u||_{L^p_\alpha(R,R+1)} \). Thus,

\[
\lim_{n \to \infty} \lim_{R \to \infty} \rho_{n,R} = 0. \tag{34}
\]

We conclude (29) (with \( i = 0 \)) from (32), (33) and (34). The others cases follow from similar arguments.

Next, our goal is determining the normalized vanishing limit defined at (28).

Proposition 5.6 It holds that

\[
d_{n,\alpha}(p, \theta, \mu) = \begin{cases} 
\frac{\mu^{p-1}}{(p-1)!}, & \text{if } p \text{ is integer} \\
0, & \text{otherwise}.
\end{cases}
\]

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Proof. Again, we recall that we can suppose that \((u_n)\) is nonincreasing, then by Lemma 4.1

\[
|u_n(x)| \leq \left( \frac{1 + \theta}{\omega_{\theta}} \right)^{\frac{1}{p}} \cdot \frac{1}{x^{\frac{1}{p} - 1}} \left( \int_0^{\infty} |u_n(y)|^p d\lambda_{\theta}(y) \right).
\]

Assume that \(1 \leq R < \infty\), then

\[
\sum_{j=|p|+1}^{\infty} \frac{\mu^j}{j!} \int_{R}^{\infty} |u_n|^{\frac{p}{p-1}} d\lambda_{\theta} \leq \sum_{j=|p|+1}^{\infty} \frac{\mu^j}{j!} \left( \frac{1 + \theta}{\omega_{\theta}} \right)^{\frac{1}{p-1}} \omega_{\theta} \int_{R}^{\infty} x^{\theta - \frac{1}{p-1}} d\lambda_{\theta} \leq \frac{\omega_{\theta}(p-1)}{R^{(1+\theta)\left(\frac{p}{p-1}-1\right)}} \sum_{j=|p|+1}^{\infty} \frac{\mu^j}{j!} \left( \frac{1 + \theta}{\omega_{\theta}} \right)^{\frac{1}{p-1}}.
\]

Thus

\[
\lim_{R \to \infty} \lim_{n \to \infty} \sum_{j=|p|+1}^{\infty} \frac{\mu^j}{j!} \int_{R}^{\infty} |u_n|^{\frac{p}{p-1}} d\lambda_{\theta} = 0. \tag{35}
\]

if \(p\) is not integer, we get

\[
\int_{R}^{\infty} |u_n|^{\frac{p}{p-1}} |p| d\lambda_{\theta} \leq \left( \frac{1 + \theta}{\omega_{\theta}} \right)^{\frac{1}{p-1}} \frac{\omega_{\theta}(p-1)}{(|p| - (p-1)) R^{(1+\theta)\left(\frac{|p|}{p-1}-1\right)}}. \tag{36}
\]

Hence, using (35) and (36), we obtain \(\nu_\infty = 0\), if \(p\) is not integer.

Now, if \(p\) is integer, then \(|p| = p - 1\) and passing to subsequence if necessary, we have

\[
\nu_\infty = \lim_{R \to \infty} \lim_{n \to \infty} \sum_{j=|p|+1}^{\infty} \frac{\mu^j}{j!} \int_{R}^{\infty} |u_n|^{\frac{p}{p-1}} d\lambda_{\theta}
\]

\[
+ \lim_{R \to \infty} \lim_{n \to \infty} \frac{\mu^{p-1}}{(p-1)!} \|u_n\|^p_{L_p^\infty(R,\infty)}
\]

\[
= \frac{\mu^{p-1}}{(p-1)!} \lim_{R \to \infty} \lim_{n \to \infty} \|u_n\|^p_{L_p^\infty(R,\infty)}
\]

\[
\leq \frac{\mu^{p-1}}{(p-1)!}. \tag{37}
\]

Taking \(u_n := \phi_n\) as in the Example 5.3, we obtain (35) as well. Besides, we get

\[
\lim_{R \to \infty} \lim_{n \to \infty} \|u_n\|^p_{L_p^\infty(R,+,\infty)} = \lim_{R \to \infty} \lim_{n \to \infty} \|\phi\|^p_{L_p^\infty(\lambda_{p,R} \infty)} = 1. \tag{38}
\]

From (35), (36), (37) and (38) the proposition follows. \(\blacksquare\)
Proposition 5.7 Let $p \geq 2$ be an integer number. Then
\[
d(p, \theta, \mu) > \begin{cases} 
\frac{\mu^{p-1}}{(p-1)!}, & \text{if } p > 2 \text{ and } \mu \in (0, \mu_{\alpha, \theta}] \\
\frac{2^{p-1}}{(p-1)!}, & \text{if } p = 2 \text{ and } \mu \in \left(\frac{2}{B(2, \theta)}, \mu_{\alpha, \theta}\right]. 
\end{cases}
\]

Proof. Let $\gamma, \sigma$ be positive real numbers such that $\gamma p - \sigma (1 + \theta) = 0$ and let $v \in W^{1,p}_{\alpha, \theta}(0, \infty)$. We set
\[v_t(x) = t^\gamma v(t^\sigma x), \text{ for all } t, x \in (0, \infty).\]
We get
\[
\int_0^\infty A_{p, \mu} \left( \frac{|v_t|}{\|v_t\|_{W^{1,p}_{\alpha, \theta}(0, \infty)}} \right) d\lambda_\theta 
\geq \frac{\mu^{p-1}}{(p-1)!} \left[ \frac{\|v\|^p_{L^p_{\alpha, \theta}}}{\|v\|^p_{L^p_{\alpha, \theta}} + t^{\gamma p} \|v'\|^p_{L^p_{\alpha, \theta}}} + \frac{\mu}{p} \left( \frac{\|v\|^p_{L^p_{\alpha, \theta}}}{\|v\|^p_{L^p_{\alpha, \theta}} + t^{\gamma p} \|v'\|^p_{L^p_{\alpha, \theta}}} \right)^{\frac{p-1}{p}} \right]
:= \frac{\mu^{p-1}}{(p-1)!} h_{p, \theta, \mu}(t).
\]
Note that $\lim_{t \to 0} h_{p, \theta, \mu}(t) = 1$. Thus, it is sufficient to show that $h'_{p, \theta, \mu}(t) > 0$ for $0 < t \ll 1$.
Through straightforward calculation we obtain
\[
h'_{p, \theta, \mu}(t) = \frac{p \gamma t^{p-1}}{\left(\|v\|^p_{L^p_{\alpha, \theta}} + t^{\gamma p} \|v'\|^p_{L^p_{\alpha, \theta}}\right)^2} \cdot \left[ \frac{\mu}{p(p-1)} \|v\|^p_{L^p_{\alpha, \theta}} - \frac{p}{p-1} \|v\|^p_{L^p_{\alpha, \theta}} + t^{\gamma p} \|v'\|^p_{L^p_{\alpha, \theta}} \right]^{\frac{p-2}{p-1}}
- \frac{p}{p-1} \|v\|^p_{L^p_{\alpha, \theta}} + t^{\gamma p} \|v'\|^p_{L^p_{\alpha, \theta}} - \|v\|^p_{L^p_{\alpha, \theta}} + t^{\gamma p} \|v'\|^p_{L^p_{\alpha, \theta}} \right]^{\frac{1}{p-1}}
- \|v\|^p_{L^p_{\alpha, \theta}} + t^{\gamma p} \|v'\|^p_{L^p_{\alpha, \theta}} \right] \right]
\]
Thus we get $h'_{p, \theta, \mu}(t) > 0$ for $0 < t \ll 1$ if $p > 2$. Now, for $p = 2$ is a little bit different, because
\[
h'_{2, \theta, \mu}(t) = \frac{2 \gamma t^{2\gamma - 1}}{\left(\|v\|^2_{L^2_{\alpha, \theta}} + t^{2\gamma} \|v'\|^2_{L^2_{\alpha, \theta}}\right)^2} \cdot \left[ \frac{\mu}{2} \|v\|^4_{L^4_{\alpha, \theta}} - \frac{2 \gamma t^{2\gamma} \|v\|^4_{L^4_{\alpha, \theta}} \|v'\|^2_{L^2_{\alpha, \theta}}}{\left(\|v\|^2_{L^2_{\alpha, \theta}} + t^{2\gamma} \|v'\|^2_{L^2_{\alpha, \theta}}\right)} - \|v\|^2_{L^2_{\alpha, \theta}} \right]
\]

Taking \( v \in W^{1,p}_{\alpha,\theta}(0, \infty) \) such that \( B(2, \theta)^{-1} = B(v)^{-1} \), we obtain \( h_{2,\theta,\mu}(t) > 0 \) for \( 0 < t \ll 1 \), if \( \frac{2}{B(2,\theta)} < \mu \leq 2\pi(1+\theta) \), [see Proposition 7.1].

**Lemma 5.8** Let \( u_i < 1 \ (i = 0, \infty) \) and let \( p \geq 2 \) be an integer. Then we obtain

\[
d(p, \theta, \mu) \| u_{n,R}^i \|_{W^{1,p}_{\alpha,\theta}(0, \infty)}^p \geq \int_0^\infty A_{p,\mu} (\| u_{n,R}^i \|) \, d\lambda_\theta + \left[ \left( \frac{1}{\| u_{n,R}^i \|_{W^{1,p}_{\alpha,\theta}(0, \infty)}} \right) - 1 \right]
\]

\[
\cdot \int_0^\infty A_{p,\mu} (\| u_{n,R}^i \|) - \frac{\mu^{p-1}}{(p-1)!} |u_{n,R}^i|^p \, d\lambda_\theta
\]

whenever \( n \) and \( R \) are sufficiently large.

**Proof.** By definition, we have

\[
d(\alpha, \theta, \mu) \geq \sum_{j=p-1}^{\infty} \frac{\mu^j}{j!} \frac{\| u_{n,R}^i \|_{W^{1,p}_{\alpha,\theta}(0, \infty)}^p}{L_\theta^{\frac{p}{p-1}j}} \geq \left( \frac{1}{\| u_{n,R}^i \|_{W^{1,p}_{\alpha,\theta}(0, \infty)}} \right) \sum_{j=p-1}^{\infty} \frac{\mu^j}{j!} \| u_{n,R}^i \|_{W^{1,p}_{\alpha,\theta}(0, \infty)}^p L_\theta^{\frac{p}{p-1}j}
\]

\[
+ \frac{1}{\| u_{n,R}^i \|_{W^{1,p}_{\alpha,\theta}(0, \infty)}} \sum_{j=p}^{\infty} \left( \frac{1}{\| u_{n,R}^i \|_{W^{1,p}_{\alpha,\theta}(0, \infty)}} - 1 \right) \frac{\mu^j}{j!} \| u_{n,R}^i \|_{W^{1,p}_{\alpha,\theta}(0, \infty)}^p L_\theta^{\frac{p}{p-1}j}
\]

From \( \mu_i < 1 \) and (39) we obtain

\[
d(\alpha, \theta, \mu) \| u_{n,R}^i \|_{W^{1,p}_{\alpha,\theta}(0, \infty)}^p \geq \sum_{j=p-1}^{\infty} \frac{\mu^j}{j!} \| u_{n,R}^i \|_{W^{1,p}_{\alpha,\theta}(0, \infty)}^p L_\theta^{\frac{p}{p-1}j} + \sum_{j=p}^{\infty} \left( \frac{1}{\| u_{n,R}^i \|_{W^{1,p}_{\alpha,\theta}(0, \infty)}} - 1 \right) \frac{\mu^j}{j!} \| u_{n,R}^i \|_{W^{1,p}_{\alpha,\theta}(0, \infty)}^p L_\theta^{\frac{p}{p-1}j}
\]

\[
= \int_0^\infty A_{p,\mu} (\| u_{n,R}^i \|) \, d\lambda_\theta
\]

\[
+ \left( \frac{1}{\| u_{n,R}^i \|_{W^{1,p}_{\alpha,\theta}(0, \infty)}} - 1 \right) \int_0^\infty A_{p,\mu} (\| u_{n,R}^i \|) - \frac{\mu^{p-1}}{(p-1)!} |u_{n,R}^i|^p \, d\lambda_\theta
\]

for large \( R \) and large \( n \).
Proposition 5.9 Let $p \geq 2$ be an integer. Then

$$(\mu_0, \nu_0) = (1, d(p, \theta, \mu)) \text{ and } (\mu_\infty, \nu_\infty) = (0, 0).$$

Proof. By contradiction, suppose that $0 < \mu_0 < 1$. Then $0 < \mu_\infty < 1$, by relation (27). From Lemma 5.5 and Lemma 5.8 we have

$$d(\alpha, \theta, \mu) \geq \nu_i + \left[ \frac{1}{\mu_i^{p-1}} - 1 \right] \left[ \nu_i - \frac{\mu_i^{p-1}}{(p-1)!} \right].$$

By relation (27) and together with (40) we get

$$d(\alpha, \theta, \mu) \geq \nu_i, \text{ for } i = 0, \infty.$$

Thus,

$$d(\alpha, \theta, \mu) = d(\alpha, \theta, \mu)(\mu_0 + \mu_\infty) \geq \nu_0 + \nu_\infty = d(\alpha, \theta, \mu)$$

and consequently

$$d(\alpha, \theta, \mu) \mu_i = \nu_i.$$

From the last relation and (40) we obtain

$$\nu_i \leq \frac{\mu_i^{p-1}}{(p-1)!} \eta_i,$$

whence

$$d(\alpha, \theta, \mu) = \nu_0 + \nu_\infty \leq \frac{\mu_0^{p-1}}{(p-1)!}(\eta_0 + \eta_\infty) \leq \frac{\mu_0^{p-1}}{(p-1)!},$$

which contradicts the Proposition 5.7.

Now, again, by contradiction, suppose that $\mu_0 = 0$. Thus, by Lemma 5.8

$$d(p, \theta, \mu) \frac{\| u_0^n \|}{W^{1,p}(0,\infty)} \geq \frac{1}{2} \int_0^\infty A_{p,\mu}(|u_0^n|) d\lambda_\theta + \frac{1}{2} \int_0^\infty \left( A_{p,\mu}(|u_0^n|) - \frac{\mu_i^{p-1}}{(p-1)!} \right) |u_0^n| d\lambda_\theta.$$

for large $R$ and large $n$.

Taking the double limit in (41), $\lim_{R \to \infty} \lim_{n \to \infty}$, we obtain

$$d(\alpha, \theta, \mu) \mu_0 \geq \nu_0 + \frac{1}{2} \left( \nu_0 - \frac{\mu_0^{p-1}}{(p-1)!} \eta_0 \right) \geq \nu_0,$$

hence $\nu_0 = 0$ from relation (27), and $\mu_0 = 0$, getting a contradiction from Proposition 5.6 relation (27), and Proposition 5.7. Finally, using the same arguments we can get $\nu_\infty = 0$ whenever $\mu_\infty = 0$. Therefore, the proposition follows. ■
Proof of Theorem 1.3.

First of all, we will show that

$$\lim_{n \to \infty} \int_0^\infty |u_n|^{\frac{p}{p-1}} d\lambda \theta = \int_0^\infty |u|^{\frac{p}{p-1}} d\lambda \theta. \quad (42)$$

Indeed, given $R > 0$, note that

$$\left| \int_0^\infty \left( |u_n|^{\frac{p}{p-1}} - |u|^{\frac{p}{p-1}} \right) d\lambda \theta \right| \leq \left| \int_0^R \left( |u_n|^{\frac{p}{p-1}} - |u|^{\frac{p}{p-1}} \right) d\lambda \theta \right| + \int_R^\infty |u_n|^{\frac{p}{p-1}} d\lambda \theta + \int_0^\infty |u|^{\frac{p}{p-1}} dx$$

By compact embedding we have \( \lim_{R \to \infty} \lim_{n \to \infty} I(n, R) = 0 \). From Dominated Convergence Theorem, we obtain \( \lim_{R \to \infty} \lim_{n \to \infty} II(n, R) = 0 \) from \( \mu_\infty = 0 \) (Proposition 5.9). If \( p \notin \mathbb{N} \), we obtain \( \lim_{R \to \infty} \lim_{n \to \infty} II(n, R) = 0 \) from inequality (36). Hence, (42) follows.

Now, assume that either \( p > 2 \) and \( \mu \in (0, \mu_\infty) \) or \( p = 2 \) and \( \alpha \in (2/B(2, \theta), \mu_1, \theta) \). Writing

$$d(p, \theta, \mu) - \int_0^\infty A_{p, \mu}(|u|) d\lambda \theta = \int_0^\infty A_{p, \mu}(|u_n|) d\lambda \theta - \int_0^\infty A_{p, \mu}(|u|) d\lambda \theta + \left( d(p, \theta, \mu) - \int_0^\infty A_{p, \mu}(|u_n|) d\lambda \theta \right)$$

where,

$$V(n) := d(p, \theta, \mu) - \int_0^\infty A_{p, \mu}(|u_n|) d\lambda \theta$$

and

$$IV(n) := \int_0^\infty A_{p, \mu}(|u_n|) d\lambda \theta - \int_0^\infty A_{p, \mu}(|u|) d\lambda \theta.$$

We get, by definition of \( d(\alpha, \theta, \mu) \), that

$$\lim_{R \to \infty} \lim_{n \to \infty} V(n) = 0.$$

Since

$$IV(n) = \int_0^\infty \left( A_{p, \mu}(|u_n|) - \frac{\mu^{[p]}}{[p]!} |u_n|^{\frac{p}{p-1}} \right) d\lambda \theta$$

$$- \int_0^\infty \left( A_{p, \mu}(|u|) - \frac{\mu^{[p]}}{[p]!} |u|^{\frac{p}{p-1}} \right) d\lambda \theta$$

$$+ \frac{\mu^{[p]}}{[p]!} \int_0^\infty \left( |u_n|^{\frac{p}{p-1}} - |u|^{\frac{p}{p-1}} \right) d\lambda \theta.$$
From Lemma 5.1 and relation (42) we obtain

$$\lim_{n \to \infty} IV(n) = 0.$$  

Now, we assert that $$\|u\|_{W^{1,p}_{p-1,\theta}(0,\infty)} = 1.$$ Indeed, on the one hand,

$$\|u\|_{W^{1,p}_{p-1,\theta}(0,\infty)} \leq \liminf_{n \to \infty} \|u_n\|_{W^{1,p}_{p-1,\theta}(0,\infty)} = 1.$$  

On the other hand,

$$d(p, \theta, \mu) \geq \int_{0}^\infty A_{p,\mu} \left( \left( \frac{|u|}{\|u\|_{W^{1,p}_{p-1,\theta}(0,\infty)}} \right) \right) d\lambda_\theta$$

$$= \sum_{j=\lfloor p \rfloor}^{\infty} \frac{\mu^j}{j!} \frac{\|u\|_{W^{1,p}_{p-1,\theta}(0,\infty)}}{L^j_{\theta,\omega}} \cdot \|u\|_{W^{1,p}_{p-1,\theta}(0,\infty)}$$

$$\geq \frac{1}{\|u\|_{W^{1,p}_{p-1,\theta}(0,\infty)}} \sum_{j=\lfloor p \rfloor}^{\infty} \frac{\mu^j}{j!} \|u\|_{W^{1,p}_{p-1,\theta}(0,\infty)}$$

Therefore, $$\|u\|_{W^{1,p}_{p-1,\theta}(0,\infty)} = 1$$ and

$$d(p, \theta, \mu) = \int_{0}^{\infty} A_{p,\mu} \left( \|u\| \right) d\lambda_\theta.$$  

6 Proof of the Theorem 1.3

Throughout this section, we assume that $$p = 2$$ and $$\mu \leq \pi(1 + \theta)/3.$$  

By Theorem 2.4 (inequality (8)), we get

$$\frac{\|u\|^{2j}_{L^2_{\theta}}}{\|u^j\|^{2}_{L^2_{\theta}}} \leq C_{\gamma,2,\theta} \frac{j!}{\gamma^j} \|u^j\|^{2(j-2)}_{L^2_{\theta}}$$  

(43)

for all $$u \in W^{1,2}_{1,\theta}(0, \infty), j \in \mathbb{N},$$ and $$0 < \gamma < (1 + \theta)\omega_1.$$ We are going to use the inequality (43) to prove the Theorem 1.3.

Proof of Theorem 1.3

Let $$S := \{v \in W^{1,2}_{1,\theta}(0, \infty) : ||v||_{W^{1,2}_{1,\theta}(0, \infty)} = 1\}.$$ For each $$u \in S,$$ we define a family of functions by

$$u_t(x) := t^\frac{1}{2} u(t^{1+\theta} x),$$

where $$t > 0$$ is a parameter. Besides, let $$v_t := u_t/||u_t||_{W^{1,2}_{1,\theta}(0, \infty)}.$$ Thus $$v_t$$ is a curve in $$S$$ passing through $$u$$ when $$t = 1.$$ Then it is sufficient to show that
\[ \frac{d}{dt} F(v_t) \big|_{t=1} < 0, \]

where \( F(w) := \int_0^\infty A_{2,\mu}(w(x)) \, d\lambda_\theta(x). \)

Through a direct calculation we have that \( \|u_t\|_{L^2_\theta}^{2j} = t^{j-1} \|u\|_{L^2_\theta}^{2j} \), \( \|(u_t)’\|_{L^2_1}^2 = t \|u’\|_{L^2_1}^2 \) and

\[
F(v_t) = \sum_{j=1}^\infty \frac{\mu^j}{j!} \left( \|u\|_{L^2_\theta}^{2j} + t \|u’\|_{L^2_1}^2 \right)^j. \]

Since

\[
\frac{d}{dt} F(v_t) = \sum_{j=1}^\infty \frac{\mu^j}{j!} \left( (j-1) \|u\|_{L^2_\theta}^{2j} - \|u’\|_{L^2_1}^2 \right) \]

we obtain

\[
\frac{d}{dt} F(v_t) \big|_{t=1} = \sum_{j=1}^\infty \frac{\mu^j}{j!} \left( (j-1) \|u\|_{L^2_\theta}^{2j} - \|u’\|_{L^2_1}^2 \right)
= -\mu \|u\|_{L^2_\theta}^{2j} \|u’\|_{L^2_1}^2 + \sum_{j=2}^\infty \frac{\mu^j}{j!} \left( (j-1) \|u\|_{L^2_\theta}^{2j} - \|u’\|_{L^2_1}^2 \right)
\leq \mu \|u\|_{L^2_\theta}^{2j} \|u’\|_{L^2_1}^2 \left( -1 + \sum_{j=2}^\infty \frac{\mu^j}{(j-1)!} \frac{\|u\|_{L^2_\theta}^{2j}}{\|u’\|_{L^2_1}^2} \right). \tag{44}
\]

From inequality (43) (with \( \gamma := 2\pi(1 + \theta)/3 \)) and (44) we get

\[
\frac{d}{dt} F(v_t) \big|_{t=1}
\leq \mu \|u\|_{L^2_\theta}^{2j} \|u’\|_{L^2_1}^2 \left( -1 + C_2 \frac{2\pi(1 + \theta)}{3} \sum_{j=2}^\infty \frac{\mu^j}{(j-1)!} \left( \frac{3}{2\pi(1 + \theta)} \right)^j \right)
= \mu \|u\|_{L^2_\theta}^{2j} \|u’\|_{L^2_1}^2 \left( -1 + C_2 \frac{2\pi(1 + \theta)}{3} \mu \sum_{j=2}^\infty \frac{\mu^{j-2}}{(j-1)!} \left( \frac{3}{2\pi(1 + \theta)} \right)^{j-2} \right)
\leq \mu \|u\|_{L^2_\theta}^{2j} \|u’\|_{L^2_1}^2 \left( -1 + C_2 \frac{2\pi(1 + \theta)}{3} \mu \sum_{j=2}^\infty j \left( \frac{1}{2} \right)^{j-2} \right). \]

Thus, taking \( \mu_0 := \frac{1}{a C_2 \frac{2\pi(1 + \theta)}{3}} \left( \frac{2\pi(1 + \theta)}{3} \right)^2 \), where \( a := \sum_{j=2}^\infty j \left( \frac{1}{2} \right)^{j-2} \), the proof of the theorem follows.
7 Gagliardo-Nirenberg Inequalities

In this section, we discuss a little bit about the best constant of the Gagliardo-Nirenberg inequality, and we will explore some ideas contained in [3, 6, 7].

It is known the interpolation inequality with weights

\[ \|u\|_{L_p^q(0, \infty)} \leq K(p, q, \alpha, \theta) \|u\|_{L_p^p(0, \infty)}^{\frac{1}{1-\gamma}} \|u\|_{L_p^\infty(0, \infty)}^{\frac{1}{\gamma}}, \tag{45} \]

where \(1 < p \leq q < p^* = \frac{p(1+\theta)}{q-\alpha(p-1)}, \alpha \geq p-1, \theta \geq 0\) and \(1 - \gamma = \frac{\alpha}{q} \cdot \frac{(p^*-q)}{(p^*-p)}\). It is worth noting that when \(\alpha = p - 1\) we have \(1 - \gamma = \frac{p}{q}\).

Throughout this section we will assume that \(\alpha \leq p + \theta\). So, we can computed the optimal \(k = K(p, q, \alpha, \theta)\) in (45) if we determine the explicit solution of the minimization problem

\[ \inf \left\{ E(u) := \frac{1}{p} \int_0^\infty |u'|^p \, d\lambda_\alpha + \frac{1}{p} \int_0^\infty |u|^p \, d\lambda_\theta : \|u\|_{L_p^q(0, \infty)} = 1 \right\}. \tag{46} \]

Indeed, first of all, see Lemma 4.1 and Theorem 3.2 (with \(\alpha = m\), and \(l = \theta\)) for the existence of a minimizer for (46). Now if \(u_\infty\) is a minimizer of the variational problem (46), then

\[ E(u_\infty) \leq E(u) = \frac{1}{p} \|u'|_{L_p^p(0, \infty)}^p + \frac{1}{p} \|u|_{L_p^\infty(0, \infty)}^p \]

for all \(u \in W_{\alpha, \theta}^{1,p}(0, \infty)\) satisfying \(\|u\|_{L_p^q(0, \infty)} = 1\). Thus,

\[ E(u_\infty) \leq \frac{1}{p} \|u'|_{L_p^p(0, \infty)}^p + \frac{1}{p} \|u|_{L_p^\infty(0, \infty)}^p \]

for every \(0 \neq u \in W_{\alpha, \theta}^{1,p}(0, \infty)\). Scaling \(u\) as \(u_t(x) = u(tx)\), we get

\[ E(u_\infty) \leq t^{\theta-(\alpha+1)+\frac{\theta}{q}(1+\theta)} \|u'|_{L_p^p(0, \infty)}^p + t^{(1+\theta)\left(\frac{\alpha}{q} - 1\right)} \|u|_{L_p^\infty(0, \infty)}^p. \]

A direct computation proves that the minimum over \(t\) is achieved at

\[ t = \left[ \frac{(1+\theta)(q-p)}{pq+p(1+\theta)-q(1+\alpha)} \right]^{\frac{1}{p+\theta-\alpha}} \frac{B}{A}, \]

where

\[ A = \frac{\|u'|_{L_p^p(0, \infty)}^p}{p \|u|_{L_p^\infty(0, \infty)}^p} \quad \text{and} \quad B = \frac{\|u|_{L_p^\infty(0, \infty)}^p}{p \|u|_{L_p^\infty(0, \infty)}^p}. \]

Therefore,

\[ E(u_\infty) \leq \left( \frac{(1+\theta)(q-p)}{pq+p(1+\theta)-q(1+\alpha)} \right)^{1-\gamma} + \left( \frac{(1+\theta)(q-p)}{pq+p(1+\theta)-q(1+\alpha)} \right)^{\gamma} \frac{\|u'|_{L_p^p}^p \|u|_{L_p^\infty}^{(1-\gamma)}}{p \|u|_{L_p^\infty}^p} \]

and the equality happen when \(u = u_\infty\).

The next result will be important in the study of attainability in the Trudinger-Moser inequality with weight when \(p = 2\).
Proposition 7.1 If $p = 2$, $\alpha = p - 1$, and $\theta \geq 0$, then the infimum

$$ B(2, \theta)^{-1} := \inf_{0 \neq u \in W^{1,2}_{1,\theta}((0, \infty))} \frac{\|u'\|_{L^2((0, \infty))}^2 \cdot \|u\|_{L^4((0, \infty))}^2}{\|u\|_{L^2((0, \infty))}^4}, $$

is attained by a positive nonincreasing function in $W^{1,2}_{1,\theta}((0, \infty))$. Moreover,

$$ B(2, \theta)^{-1} < \pi(1 + \theta). $$

**Proof.** The first part has been discussed at the beginning of this section. Then, we focus on the second part. Set

$$ B(u)^{-1} := \frac{\|u'\|_{L^2((0, \infty))}^2 \cdot \|u\|_{L^2((0, \infty))}^2}{\|u\|_{L^4((0, \infty))}^4}. $$

Note that it is sufficient to exhibit a function $u \in W^{1,2}_{1,\theta}((0, \infty))$ such that $B(u)^{-1} = \pi(1 + \theta)$ and it is not a solution of

$$ -(u'x)\omega_1 + u x^\theta \omega_\theta - \lambda u^3 x^\theta \omega_\theta = 0, $$

for all $\lambda > 0$.

On the one hand, through a direct calculation we can see that for every positive function $v$ in $W^{1,2}_{1,\theta}((0, \infty))$ of the form

$$ v(x) = a_1(1 + a_2 x^{a_3})^{a_4}, $$

where $a_1, a_2, a_3, a_4$ are real numbers, it is not a solution for (47). On the other hand, choosing

$$ u(x) = \frac{1}{1 + x^{1+\theta}}, $$

then

$$ \|u\|_{L^2((0, \infty))}^2 = \frac{\omega_\theta}{\pi(1+\theta)} $$

$$ \|u\|_{L^4((0, \infty))}^4 = \frac{\omega_\theta}{(1+\theta)^2} $$

$$ \|u'\|_{L^2((0, \infty))}^2 = \frac{\omega_1(1+\theta)}{6} $$

Therefore, $B(u)^{-1} = \pi(1 + \theta)$, and then the proposition follows. \(\square\)

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