Near-threshold Isospin Violation in the Pion Form Factor from Chiral Perturbation Theory

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Abstract

We examine the isospin violation in the timelike pion form factor near threshold at next-to-leading order in the chiral expansion using the techniques of Chiral Perturbation Theory. This next-to-leading order contribution contains the first nonvanishing isospin violation. This isospin violation is found to be very small near threshold. In particular, the isospin violation at threshold is found to be of order $10^{-4}$, which should be compared with the few percent level seen in the vector meson resonance region.

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1 Introduction

Chiral Perturbation Theory (ChPT) seeks to produce, in a model independent way, a completely general low energy effective hadronic field theory, using as input only the underlying symmetries and pattern of symmetry breaking of the initial QCD Lagrangian. The principle symmetry used in this construction is chiral symmetry (ChS). ChS is the simultaneous requirement of isospin symmetry and helicity conservation, i.e. $\text{SU}(2)_L \otimes \text{SU}(2)_R$. Having $m_u = m_d \neq 0$ violates helicity conservation but not isospin symmetry. In the real world we have $m_u \neq m_d \neq 0$ and both symmetries are broken by nonzero current quark masses. Since $m_u$ and $m_d$ are small on the hadronic scale the violation of chiral symmetry is small. Isospin is also explicitly violated by electromagnetic and weak interactions. The systematic nature of ChPT then provides a model-independent method for examining isospin breaking in the regime of applicability of the method.

The pion form-factor, $F_\pi(q^2)$, was one of the first quantities calculated beyond leading order in the chiral expansion using ChPT [1]. However, this one-loop treatment assumed $m_u = m_d$. In this note, we extend this treatment to the case with $m_u \neq m_d$, following the work of Maltman [2]. The previous calculations of $F_\pi(q^2)$ are briefly reviewed and the isospin-violating calculation is then discussed in detail. It should be noted that the calculation involves simultaneously expanding in three small parameters, $q^2$, $\alpha_{EM}$ and $m_u - m_d$. We shall work to first order in each of these.

2 An introduction to ChPT

Although there are many excellent reviews of ChPT [3], to keep this discussion relatively self contained, we present a short summary of the approach. This will also be useful in showing how we set up the calculation. The basic idea is to take the known symmetries of QCD and reproduce them in a low-energy meson theory. Thus we start with the QCD Lagrangian given by

$$L^{\text{QCD}} = \sum_f \bar{\psi}(x)(i \not\!D - m_f)\psi(x) - \frac{1}{4} F^{a}_{\mu\nu} F^{a\mu\nu}. \quad (1)$$

In the chiral limit the quark masses are zero and the fermion fields, $\psi$, can be split into left and right handed helicity components,

$$\psi_{L,R} = (1 \pm \gamma_5)\psi. \quad (2)$$

These transform independently under the chiral transformation,

$$\psi_{L,R} \rightarrow e^{i\alpha \gamma_5} \psi_{L,R}. \quad (3)$$

\footnote{To work with this, one needs to remove the unphysical gauge degree of freedom which is usually accomplished by adding a gauge fixing term to Eq. (1), however this is not important for our discussion and will be omitted.}
leaving Eq. (1) unchanged. Massless QCD is then said to be *chirally symmetric*. These transformations can then be generalised to separate left and right handed transformations rather than just the single $e^{i\alpha\gamma_5}$ transforming both fields. In this case we have

$$\psi_{L,R} \rightarrow U_{L,R}^\dagger \psi_{L,R}$$

(4)

where $U_L$ and $U_R$ are unitary $N_f \times N_f$ matrices, $N_f$ being the number of flavours. One normally only considers the up, down and strange quarks, for reasons that will become apparent later. The heavier quark flavours play no dynamical role in the region of interest and do not need to be explicitly included. If strange quarks are included the flavour symmetry changes from $SU(2)_{\text{flavour}}$ to $SU(3)_{\text{flavour}}$ and the chiral symmetry group is then $SU(3)_L \otimes SU(3)_R$.

Now, of course, the quarks do have mass, but since the $u$, $d$ and $s$ masses are small, $SU(3)_L \otimes SU(3)_R$ should be an *approximate* symmetry of QCD, and we expect it to have some relevance to the way the theory works, and provide a guide in our construction of a meson theory. To construct this meson theory, we consider the QCD generating functional, in the presence of external left-hand vector, right-hand vector, scalar and pseudoscalar sources $l_\mu, r_\mu, s$ and $p$,

$$\exp[iW[l_\mu, r_\mu, s, p]] = \int [\mathcal{D}\psi][\mathcal{D}\bar{\psi}][\mathcal{D}G_\mu^a] \exp \left[ i \int d^4x \mathcal{L}_{\text{QCD}}(l_\mu, r_\mu, s, p) \right].$$

(5)

The sources for the left and right handed vector currents are decomposed into their flavour octet components via

$$l_\mu \equiv l_\mu^a \lambda^a/2, \quad r_\mu \equiv r_\mu^a \lambda^a/2,$$

(6)

where the $\lambda^a$ are the Gell-Mann matrices that make up the generators of $SU(3)$. The sources for the scalar and pseudoscalar “currents” are similarly decomposed as

$$s = s_0 + s^a_0 \lambda^a/2, \quad p = p_0 + p^a_0 \lambda^a/2,$$

(7)

where, for convenience, one usually includes a source term for the singlet currents. If we define $\mathcal{L}_{\text{QCD}}^0$ to be the massless QCD Lagrangian (Eq. (1) with $m_f = 0$) the full massless QCD Lagrangian, now with sources, can be written,

$$\mathcal{L}_{\text{QCD}}(l_\mu, r_\mu, s, p) = \mathcal{L}_{\text{QCD}}^0 - \bar{q}_L \gamma_\mu l_\mu q_L - \bar{q}_R \gamma_\mu r_\mu q_R - \bar{q}(s - i\gamma_5 p)q.$$

(8)

Defining vector and axial vector current sources through

$$l_\mu^a = v_\mu^a - a_\mu^a, \quad r_\mu^a = v_\mu^a + a_\mu^a,$$

(9)

we can rewrite Eq. (8) in terms of these vector and axial vector sources

$$\mathcal{L}_{\text{QCD}}(v_\mu, a_\mu, s, p) = \mathcal{L}_{\text{QCD}}^0 - \bar{q}((\not{v} + \not{a}) q - \bar{q}(s - i\gamma_5 p)q.$$

(10)

The role of the sources is an important one in the construction of the effective low-energy hadronic theory since it turns out that the QCD Lagrangian in the presence of these
sources has a local symmetry (to be discussed below) which must, therefore, also be
realised in any low-energy effective version of the theory, if that effective theory is to
correctly represent the effects of QCD.

The local symmetry mentioned above consists of the following simultaneous local
transformations of the left- and right-handed quark fields and external sources: if the
left- and right-handed quark fields are transformed via the matrices \( L(x) \) and \( R(x) \),
respectively, then the external left-handed and right-handed vector sources, \( l_\mu \) and \( r_\mu \),
transform as corresponding gauge fields, and the scalar and pseudoscalar sources as

\[
\begin{align*}
(s + ip)(x) & \rightarrow R(x)(s + ip)L^\dagger(x) \\
(s - ip)(x) & \rightarrow L(x)(s - ip)R^\dagger(x).
\end{align*}
\]

Under the above set of transformations, \( \mathcal{L}_{QCD} \) has a larger local \( SU(3)_L \otimes SU(3)_R \) symmetry, which must be also realised in any low-energy effective hadronic field theory pur-
porting to be a representation of QCD.

For the Goldstone boson fields, it has long been known [4] that it is possible to choose
the pseudoscalar fields, \( \pi^a (a = 1, \cdots, 8) \), in such a way that, with \( \lambda^a \) the usual Gell-Mann
matrices and \( \pi \equiv \pi^a \lambda^a \),

\[
\pi \equiv \pi^a \lambda^a = \begin{pmatrix}
\pi^3 + \pi^8/\sqrt{3} & \sqrt{2}\pi^+ & \sqrt{2}K^+ \\
\sqrt{2}\pi^- & -\pi^3 + \pi^8/\sqrt{3} & \sqrt{2}K^0 \\
\sqrt{2}K^- & \sqrt{2}K^0 & -2\pi_8/\sqrt{3} \\
\end{pmatrix},
\]

the matrix variable, \( U = \exp(i\pi/F) \), transforms linearly under the chiral group,

\[
U(x) \rightarrow L(x)U(x)R^\dagger(x).
\]

The low-energy effective theory for the Goldstone boson degrees of freedom is then to
be constructed in such a way that each term appearing in the effective Lagrangian is
invariant under the simultaneous transformation of the external sources described above
and the transformation of the pseudoscalar fields implicit in Eq. (14). When the external
scalar source, \( s \), in the effective theory, is set equal to the quark mass matrix,

\[
s = m \equiv \begin{pmatrix}
m_u & 0 & 0 \\
0 & m_d & 0 \\
0 & 0 & m_s
\end{pmatrix},
\]

then we correctly incorporate the explicit breaking of the chiral symmetries in QCD into
the low-energy effective theory with the same symmetry breaking pattern with which
this breaking occurs in QCD. To say this in another way, the external field method is
essentially a spurion method for incorporating the breaking of chiral symmetry into the
low-energy effective theory. The external left- and right-handed vector, and pseudoscalar
sources are then useful for generating the correct representations of the corresponding
hadronic currents in terms of the Goldstone boson degrees of freedom.
Importantly for our work, the mass matrix in Eq. (15) breaks more than just chiral symmetry, it also breaks isospin symmetry (in the strong interaction itself) if \( m_u \neq m_d \).

We may now proceed to review the construction of the low-energy effective theory for the Goldstone bosons based on the procedure described above. The Lagrangian is written as a series in powers of \( q^2 \) and/or the quark masses, where, owing to the fact that the pseudoscalar squared-masses are linear in the quark masses at leading order, \( m_q \) counts as \( \mathcal{O}(q^2) \). The counting based on this identification defines the so-called “chiral order”. Labelling the terms in the effective Lagrangian by their chiral order one then has

\[
\mathcal{L} = \sum_{n=1}^{\infty} \mathcal{L}_{2n},
\]

where the subscript denotes the chiral order. The construction of \( \mathcal{L}_{2n} \) is, in principle, straightforward. At each order one simply writes down all possible terms invariant under the simultaneous transformations of the matrix variable \( U \) and the external sources, each such term multiplied by a coefficient which is, of course, not fixed by the symmetry arguments alone. These coefficients, called “low-energy constants” (or LEC’s), are to be fixed by comparison with experiment, or estimated in some model-dependent approach (see, for example, Ref. [5]). Although the resulting full effective theory necessarily has an infinite number of terms, and hence is non-renormalisable, to a given order in the chiral series, only a finite number of these terms contribute [6], so that the theory becomes effectively renormalisable.

The most general form of \( \mathcal{L} \), at lowest order in the chiral expansion, is then easily seen to be

\[
\mathcal{L}_2 = \frac{F^2}{4} \langle D_\mu U^\dagger D^\mu U + U^\dagger \chi + \chi^\dagger U \rangle,
\]

where \( F \) is one of the LEC’s mentioned above, which has the dimensions of mass and turns out to be equal to the pion decay constant in the chiral limit, \( \langle A \rangle \) denotes the trace of matrix \( A \), the covariant derivative \( D_\mu U \) is defined by

\[
D_\mu U = \partial_\mu U + i[v_\mu, U] - i\{a_\mu, U\},
\]

and the source, \( \chi \) by,

\[
\chi = 2B_0(s - ip),
\]

where \( B_0 \) is another LEC, whose physical meaning turns out to be that the quark condensate in the chiral limit is \(-B_0F^2\).

The lowest order part of the Lagrangian, Eq. (17), produces the kinetic and mass terms for (say) the pion field, when we set \( s \) in Eq. (19) to the quark mass matrix of Eq. (15). We simply expand the exponential of \( U \) in terms of the pion field to give

\[
\frac{F^2}{4}2B_0\langle m(U^\dagger + U) \rangle = B_0 \left( -\langle m^2 \rangle + \frac{1}{6F^2} \langle m^4 \rangle + \cdots \right).
\]
Making the appropriate identifications gives us the well-known Gell-Mann–Oakes–Renner relation \[7\] between the quark and meson masses (modified to include the leading isospin-breaking contributions \[3\])

\[
\begin{align*}
    m_{\pi^\pm}^2 &= (m_u + m_d)B_0, \\
    m_{\pi^0}^2 &= (m_u + m_d)B_0 - \delta + O(\delta^2) \\
    m_{K^\pm}^2 &= (m_u + m_s)B_0, \\
    m_{K^0}^2 &= (m_d + m_s)B_0
\end{align*}
\]

(21)

where the second-order CSV parameter, \(\delta\), is given by

\[
\delta = \frac{B_0}{4} \frac{(m_u - m_d)^2}{(m_s - m_u - m_d)}.
\]

(22)

One can easily see from this, that if the quark masses vanish, then so do the pseudoscalar meson masses.

Our calculation will incorporate terms at leading and next-to-leading order in the chiral expansion. Following Weinberg’s counting argument \[6\], this means that we must include 1-loop graphs with vertices from \(L_2\) in addition to tree graphs with no more than one vertex from \(L_4\). The terms involving the vertices from \(L_4\) serve to renormalise (at this order) the divergences generated by the loop graphs involving vertices from \(L_2\).

The explicit form of \(L_4\) was worked out by Gasser and Leutwyler \[1\], and is given by

\[
\begin{align*}
    L_4 &= L_1\left(D_\mu D^\mu U^\dagger\right)^2 + L_2\left(D_\mu U D^\nu U^\dagger\right)\left(D^\mu U D^\nu U^\dagger\right) + L_3\left(D_\mu U D^\mu U^\dagger D_\nu U D^\nu U^\dagger\right) \\
    &+ L_4\left(D_\mu U D^\mu U^\dagger\right)\left(\chi U^\dagger + U\chi^\dagger\right) + L_5\left(D_\mu U D^\mu U^\dagger\left(\chi U^\dagger + U\chi^\dagger\right)\right) \\
    &+ L_6\left(\chi U^\dagger + U\chi^\dagger\right)^2 + L_7\left(\chi U^\dagger - U\chi^\dagger\right)^2 + L_8\left(\chi U^\dagger U\chi^\dagger + U\chi^\dagger U\chi^\dagger\right) \\
    &+ i L_9\left(L_{\mu\nu} U R^{\mu\nu} U^\dagger\right) + H_1\left(R_{\mu\nu} R^{\mu\nu} + L_{\mu\nu} L^{\mu\nu}\right) + H_2\left(\chi^\dagger \chi\right).
\end{align*}
\]

(23)

This constitutes the complete set of linearly-independent terms in \(SU(3) \otimes SU(3)\) allowed by the relevant symmetries and of order \(q^4\) in the chiral expansion (remember that \(\chi\), as defined above, is to be considered as \(O(q^2)\) in the chiral counting).

### 3 The standard ChPT treatment of \(F_\pi(q^2)\)

The pion form factor, \(F_\pi(q^2)\), is defined as the strong interaction correction to the naive, electromagnetic prediction of the amplitude for \(e^+e^- \rightarrow \pi^+\pi^-\) \[8\]. So to obtain \(F_\pi(q^2)\) we need some way to incorporate the photon into ChPT. This is straightforward since the low-energy representation of the electromagnetic current may be obtained simply by the variation of \(\mathcal{L}\) with respect to the corresponding external source, \(v^{EM}_\mu = v_3^\mu + v_8^\mu / \sqrt{3}\). Equivalently, one may drop all external sources except for \(s\) (to be set equal to the quark mass matrix \(m\)) and \(v^{EM}_\mu\), and replace \(v^{EM}_\mu\) by the matrix \(B_\mu\) obtained by multiplying the photon field variable \(A_\mu\) by the quark charge matrix, \(Q\),

\[
\begin{align*}
    B_\mu &= B_3^\mu + B_8^\mu, \\
    &= A_\mu Q = A_\mu(x)(Q^3 + Q^8),
\end{align*}
\]

(24)
where
\[ Q = \begin{pmatrix} 2/3 & 0 & 0 \\ 0 & -1/3 & 0 \\ 0 & 0 & -1/3 \end{pmatrix} = \frac{e}{2} \left( \lambda_3 + \frac{1}{\sqrt{3}} \lambda_8 \right) = Q^3 + Q^8, \] (25)
in which case the covariant derivative reduces to
\[ D_\mu U = \partial_\mu U + ie[A_\mu Q, U]. \] (26)
The isospin conserving \((m_u = m_d)\) treatment was first performed by Gasser and Leutwyler \[1\]. The result is
\[ F_{\pi}^{u=d}(q^2) = 1 + q^2 \left[ \frac{2L}{F^2} + \frac{1}{192F^2\pi^2}A \right], \] (27)
where \(F\) is as described above (and can be related to the pion decay constant \(f_\pi = 92.4 MeV\) through the 1-loop expression of Gasser and Leutwyler \[1\]) and
\[
A = 2 \ln(\mu^2/m_\pi^2) + \ln(\mu^2/m_K^2) - B,
\]
\[
B = 1 + 2(1 - 4m_\pi^2/q^2)H(q^2/m_\pi^2) + (1 - 4m_K^2/q^2)H(q^2/m_K^2),
\]
with \(\mu\) the renormalisation scale. (The details are analogous to those for the case \(m_u \neq m_d\), which will be outlined below.) In Eq. (27), \(L\) is the renormalised low energy constant, \(L_\pi^r(\mu)\), (see Eq. (23)). The full result is, of course, independent of \(\mu\), though, in evaluating our result numerically, we will work at \(\mu = m_\rho\), for which
\[ L \equiv L_\pi^r(m_\rho) = (6.9 \pm 0.7) \times 10^{-3}. \] (28)
The quantity \(H\) is defined by
\[
H(q^2/m_\pi^2) = -2 + 2 \sqrt{\frac{4m_\pi^2}{q^2}} - 1 \arccot \left( \frac{4m_\pi^2}{q^2} \right) - 1, \quad 0 < q^2 < 4m_\pi^2,
\]
\[
= -2 + \sqrt{1 - \frac{4m_\pi^2}{q^2}} \left( \ln \left| \frac{1 - \frac{4m_\pi^2}{q^2} + 1}{1 - \frac{4m_\pi^2}{q^2} - 1} \right| + i\pi \right), \quad q^2 > 4m_\pi^2.
\]

4. The \((m_u - m_d)\) contribution

We are now in a position to examine contributions to the \(F_{\pi}(q^2)\) resulting from the quark mass difference. Since it is known that isospin breaking in the isovector component of the pion form factor is \(O \left[(m_d - m_u)^2\right]\) \[1\], the \(O(m_d - m_u)\) contributions are all generated by the isoscalar \((a = 8)\) component of the electromagnetic current. We obtain the low-energy representation of the isosinglet electromagnetic current, \(J_\mu^8\), as usual, by identifying those terms in the effective Lagrangian linear in the external source \(\nu_\mu^8\).

Before writing down the contributions to \(J_\mu^8\), it is useful to have in mind a picture of the graphs that will be relevant to us. These are shown in Fig. \[1\]. We can see from these
figures which pieces of $J^8_\mu$ will be required. The first such contribution is the tree-level piece from $\mathcal{L}_2$, involving $\pi^+\pi^-$ (corresponding to Fig. (a)). Second, for $M$ any meson in Eq. (13) we need terms of the form $\bar{M}M$ in $J^8_\mu$ to give the contribution at the current vertex in Fig (b), and those terms of the form $\bar{M}M\pi^+\pi^-$ arising from the kinetic and mass pieces of $\mathcal{L}_2$ to generate the corresponding strong vertex. Fig (c) is generated by a term in $J^8_\mu$ itself of the form $\bar{M}M\pi^+\pi^-$. The final possible contribution at this order, involving one vertex from $\mathcal{L}_4$, must necessarily correspond to a tree graph and hence is of the form discussed above for Fig (a).

![Figure 1: The chiral contributions to $\gamma \rightarrow \pi^+\pi^-$](image)

It is now a straightforward algebraic exercise to obtain the relevant contributions to the low-energy representation of $J^8_\mu$. From $\mathcal{L}_2$ we find that the terms relevant to our calculation are (the full expression is given in the Appendix)

$$
\left[J^8_\mu\right]^{(2)} = \frac{i\sqrt{3}}{2} \left( \partial_\mu K^0 \bar{K}^0 - \partial_\mu \bar{K}^0 K^0 + \partial_\mu K^+ K^- - \partial_\mu K^- K^+ \right) + \frac{i\sqrt{3}}{4F^2} \left( \partial_\mu \pi^+ \pi^- K^+ K^- - \partial_\mu \mu^+ \mu^- K^+ K^- + \partial_\mu K^+ K^- \pi^+ \pi^- - \partial_\mu K^- K^+ \pi^+ \pi^- + \partial_\mu \pi^+ \pi^- K^0 \bar{K}^0 \right) + \mathcal{O}\left( (\pi a)^6 \right). \tag{29}
$$

We notice in Eq. (29) that there is no tree-level contribution (Fig. (a)) coming from $\mathcal{L}_2$. To calculate the vertices in Fig. (b) we require the $\pi^4$ parts of the kinetic and mass terms of $\mathcal{L}_2$. These are given by (we assume here summation over the Lorentz indices of the partial derivative)

$$
\mathcal{L}^{KE}_{2(\pi^4)} = \frac{1}{6F^2} \left( 2 \partial \pi^+ \pi^- \partial K^+ K^- - \partial \pi^- \partial K^+ K^- - \pi^+ \pi^- \partial K^+ \partial K^- - \partial \pi^+ \partial \pi^- K^+ K^- \right)
$$
\[-\partial \pi^+ \pi^- K^+ \partial K^- + 2 \partial \pi^+ \pi^- K^+ \partial K^- + 2 \partial \pi^+ \pi^- K^0 \partial K^0 - \partial \pi^+ \pi^- \partial K^0 \bar{K}^0
\]
\[-\pi^+ \pi^- K^0 \partial \bar{K}^0 - \partial \pi^+ \pi^- \partial K^0 \bar{K}^0 + 2 \partial \pi^+ \partial \pi^- \partial K^0 \bar{K}^0 - \pi^+ \partial \pi^- \partial K^0 \partial \bar{K}^0\]

\[\mathcal{L}_{2(\pi^+)}^{\text{mass}} = \frac{B_0}{6F_2^2} [(2m_u + m_d + m_s) \pi^+ \pi^- K^+ K^- + (m_u + 2m_d + m_s) \pi^+ \pi^- K^0 \bar{K}^0],\]

where we have written down explicitly only those contributions relevant to the calculation at hand.

This takes care of the contributions from \(\mathcal{L}_2\). We must now go to \(\mathcal{L}_4\), given in Eq. (23). As it turns out, there are no \(\mathcal{L}_4\) contributions to the low-energy representation of \(J_8^8\), though we might have expected a contribution from \(\mathcal{L}_4\) to Fig. 4(a). Usually in ChPT such a term is responsible for removing the divergences (as well as the unphysical dependence on the scale, \(\mu\)) associated with the loops of Fig. 4(b) and (c). Thus, the loop graphs themselves must combine to give a finite answer.

We are now in a position to construct the Feynman amplitudes associated with the graphs of Fig. 4. The problem is completely standard (a good discussion of the relevant loop integrals can be found in, for example Ref. 9). We obtain the amplitude for \(A^8 \to \pi^+ \pi^-\), \(M_\mu\) defining the associated form-factor by

\[M_\mu = -ie(p_+ - p_-)_\mu F_8^8(q^2).\]  

The calculation of the amplitude is described in detail in the appendix, so we merely present the result for the form-factor here

\[F_8^8(q^2) = -\frac{\sqrt{3}}{4F^2} \left[ \frac{1}{96\pi^2} q^2 \ln \frac{m_{K^\pm}^2}{m_{K^0}^2} - \frac{1}{960\pi^2} q^4 \left( \frac{1}{m_{K^\pm}^2} - \frac{1}{m_{K^0}^2} \right) \right].\]  

Using Eq. (21) we can rewrite this form-factor in terms of the quark masses,

\[F_8^8(q^2) = -\frac{\sqrt{3}}{4F^2} \left[ \frac{1}{96\pi^2} q^2 \ln \frac{m_u + m_s}{m_d + m_s} - \frac{1}{960\pi^2} q^4 \left( \frac{m_d - m_u}{B_0(m_u + m_s)(m_d + m_s)} \right) \right].\]  

It is then easily seen that the contribution to the pion form-factor from \(J_\mu^8\) vanishes when \(m_u = m_d\) as required in the isospin limit.

5 Discussion

Setting \(q^2 = 4m_m^2\) in Eq. (21) reveals a surprisingly small \(O(10^{-4})\) isospin violation in the next-to-leading order correction to the leading order expression for \(F_\pi(q^2)\) as given in Eq.(27). Hence we see that in the pion electromagnetic form factor near threshold, the first nonvanishing isospin violation encountered in the chiral expansion is much smaller than the few percent level seen in the vector meson resonance region. The first response to this might be to assume that \(J_\mu^8\) contributes little to the pion form-factor in the low \(q^2\) region relevant to ChPT. While this may well be true, one should bear in mind that certain features of our results imply that the higher order contributions to the isoscalar
form factor might not be negligible. Basically, the low energy constants of Eq. (23) are the result of “integrating out” the heavy resonances in an extended Lagrangian that includes the vector mesons as well as the pseudoscalar octet. Thus, in any calculation where the low energy constants are absent, such as this one, the effects of the vector resonances have not yet been included to the order considered in the chiral expansion. As the isospin violation in $F_\pi(q^2)$, at least in the resonance region, is known to be due largely to the $\omega$ we would expect the corresponding $\omega$ dominated LEC’s to to play an important role in isospin breaking even near threshold. We can compare the situation with that of the decay $\eta \to \pi^0 \gamma \gamma$ where the one loop ChPT prediction is approximately 170 times smaller than the experimental result. The $O(q^6)$ contributions then bring the ChPT result into satisfactory accord with experiment. Maltman finds a similar situation in his calculation for the mixed current correlator $\langle 0 | T(V^3_{\mu}V^8_{\nu}) | 0 \rangle$. Isospin violation is most visible in the pion form-factor data around the $\omega$ pole where we determine that the $\omega$ contributes with a strength $\sim 3\%$ that of the $\rho$. Although one cannot probe the resonance pole region using ChPT, it would thus be very interesting to see a similar two loop study of the pion form-factor including isospin violating effects.

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References

[1] J. Gasser and H. Leutwyler, Ann. of Phys. 158 (1984) 142; Nucl. Phys. B250 (1985) 465.

[2] K. Maltman, Phys. Lett. B351 (1995) 56; Phys. Rev. D 53 (1996) 2563, 2573.

[3] A. Pich, Rept. Prog. Phys. 58 (1995) 563; H. Leutwyler, hep-ph/9406283; J. Bijnens, hep-ph/9502393; G. Ecker, Prog. Part. Nucl. Phys. 36 (1996) 71.

[4] S. Coleman, J. Wess and B. Zumino, Phys. Rev. 177 (1969) 2239; C. Callan, S. Coleman, J. Wess and B. Zumino, Phys. Rev. 177 (1969) 2247.

[5] B.A. Li, hep-ph/9706262.

[6] S. Weinberg, Physica 96A (1979) 327.

[7] M. Gell-Mann, R.J. Oakes and B. Renner, Phys. Rev. 175 (1968) 2195.
The full expression for the current, $J_\mu$, is given by

$$J_\mu^8 = (i/2\sqrt{3}\partial K^0 \bar{K}^0 + i/2\sqrt{3}\partial K^+ K^- - i/2\sqrt{3}\partial K^- K^+)$$

$$- (i\partial K^0 \bar{K}^0 K^- K^+)/\sqrt{3}F^2 - (i\partial K^+ K^+ K^-)/\sqrt{3}F^2$$

$$+ (i\partial K^- K^- K^+ K^+)/\sqrt{3}F^2 - i/2\sqrt{3}\partial K^0 K^0$$

$$- (i\partial K^0 \bar{K}^0 K^- K^+)/\sqrt{3}F^2 - (i\partial K^+ K^0 K^- K^0)/\sqrt{3}F^2$$

$$+ (i\partial K^- \bar{K}^0 K^+ K^0)/\sqrt{3}F^2 + (i\partial \bar{K}^0 K^- K^+ K^0)/\sqrt{3}F^2$$

$$+ (i\partial \bar{K}^0 \bar{K}^0 K^0)/\sqrt{3}F^2$$

$$+ (i/2\sqrt{3}/2\partial \pi_3 \bar{K}^0 K^+ \pi^-)/F^2 - (i/4\sqrt{3}\partial \pi^+ K^- K^+ \pi^-)/F^2$$

$$+ (i/4\sqrt{3}\partial \pi^+ \bar{K}^0 K^0 \pi^-)/F^2 + (i/4\sqrt{3}\partial \pi^- K^- K^+ \pi^+)/F^2$$

$$- (i/4\sqrt{3}/2\partial \pi_3 K^0 K^0 \pi^+)/F^2 - (i/2\sqrt{3}/2\partial \pi_3 K^- K^0 \pi^+)/F^2$$

$$- (i/4\partial K^0 \bar{K}^0 \pi^- \pi^+)/\sqrt{3}F^2 - (i/4\partial K^+ K^- \pi^- \pi^+)/\sqrt{3}F^2$$

$$+ (i/4\partial K^- K^+ \pi^- \pi^+)/\sqrt{3}F^2 + (i/4\partial \bar{K}^0 K^0 \pi^- \pi^+)/\sqrt{3}F^2$$

$$- (i/2\sqrt{3}/2\partial \pi^- \bar{K}^0 K^+ K^- \pi_3)/F^2 + (i/2\sqrt{3}/2\partial \pi^+ K^- K^0 \pi_3)/F^2$$

$$- (i/8\partial K^0 \bar{K}^0 \pi_3^2)/\sqrt{3}F^2 - (i/8\partial K^- K^- \pi_3^2)/\sqrt{3}F^2$$

$$+ (i/8\partial \bar{K}^0 K^0 \pi_3^2)/\sqrt{3}F^2 - (i/2\partial K^+ \bar{K}^0 \pi^- \pi_3)/\sqrt{2}F^2$$

$$+ (i/2\partial \bar{K}^0 K^+ \pi^- \pi_3)/\sqrt{2}F^2 - (i/2\partial K^- K^+ \pi^- \pi_3)/\sqrt{2}F^2$$

$$+ (i/2\partial K^0 K^- \pi^- \pi_3)/\sqrt{2}F^2 + (i/4\partial K^0 K^0 \pi_3 \pi_8)/F^2 -$$

$$- (i/4\partial K^+ K^- \pi_3 \pi_8)/F^2 + (i/4\partial K^- K^+ \pi_3 \pi_8)/F^2 - (i/4\partial \bar{K}^0 K^0 \pi_3 \pi_8)/F^2$$

$$- (i/8\sqrt{3}\partial \bar{K}^0 K^0 \pi_3^2)/F^2$$

$$- (i/8\sqrt{3}\partial K^+ K^- \pi_3^2)/F^2 + (i/8\sqrt{3}\partial K^- K^+ \pi_3^2)/F^2$$

$$+ (i/8\sqrt{3}\partial \bar{K}^0 K^0 \pi_3^2)/F^2 + \cdots$$

(33)

where we have written down explicitly only those terms required in our calculation.
B Useful Integrals

The following integrals are treated in some detail in Refs. [9, 11]. However, Golowich and Kambor expand the expressions in powers of $q^2$ as required for ChPT.

Let us define the one-point integral, in $D = 4 - \epsilon$ dimensions

$$ \frac{i}{16\pi^2} \mu^{D-4} A(m^2) \equiv \int \frac{d^Dk}{(2\pi)^D} \frac{1}{k^2 - m^2}, $$

(34)

where $\mu$ is an arbitrary mass scale required to keep the action ($\int d^Dx L_{\text{int}}$) dimensionless. Evaluating $A(m^2)$ gives us

$$ A = m^2 \left( \Delta - \ln \frac{m^2}{\mu^2} + 1 \right) + O(\epsilon) $$

(35)

where

$$ \Delta = \frac{2}{\epsilon} - \gamma + \ln 4\pi. $$

(36)

For convenience we define the quantity

$$ \sigma = \frac{i}{16\pi^2}. $$

(37)

The higher-point functions are, of course, more complicated, but are related in such a way that one can simplify expressions before calculating them explicitly.

$$ \sigma \mu^{D-4} B(q^2, m^2) \equiv \int \frac{d^Dk}{(2\pi)^D} \frac{1}{(k^2 - m^2)((k + q)^2 - m^2)} $$. (38)

$$ \sigma \mu^{D-4} B_\mu(q^2, m^2) \equiv \int \frac{d^Dk}{(2\pi)^D} \frac{k_\mu}{(k^2 - m^2)((k + q)^2 - m^2)} $$. (39)

$$ \sigma \mu^{D-4} B_{\mu\nu}(q^2, m^2) \equiv \int \frac{d^Dk}{(2\pi)^D} \frac{k_\mu k_\nu}{(k^2 - m^2)((k + q)^2 - m^2)}. $$

(40)

From simple Lorentz covariance, we can rewrite these as,

$$ B_\mu(q^2, m^2) = q_\mu B_1(q^2, m^2) $$

(41)

$$ B_{\mu\nu}(q^2, m^2) = q_\mu q_\nu B_{21}(q^2, m^2) + g_{\mu\nu} B_{22}(q^2, m^2) $$

(42)

The functions $B_{21}$ and $B_{22}$ can be written in terms of $A(m^2)$ and $B(q^2, m^2)$ [9]

$$ B_{21}(q^2, m^2) = \frac{1}{3q^2} \left[ A + (q^2 - m^2)B - m^2 + \frac{q^2}{6} \right] $$

(43)

$$ B_{22}(q^2, m^2) = \frac{1}{6} \left[ A + (2m^2 - \frac{q^2}{2})B + 2m^2 - \frac{q^2}{3} \right]. $$

(44)
$A(m^2)$ is given in Eq. (35) and $B(q^2, m^2)$ is given by

$$B(q^2, m^2) = \Delta - \int_0^1 dx \ln \frac{x(x-1)q^2 + m^2}{\mu^2}. \quad (45)$$

We see from Eq. (36) that $B(q^2, m^2)$ is divergent. Not only that but, as Golowich and Kambor [11] point out, it should be expanded in powers of $q^2$ or otherwise our use of it in ChPT will not be consistent. To do this they define

$$B(q^2, m^2) \equiv B(q^2, m^2) - B(0, m^2) = -\int_0^1 dx \ln \left(1 - x(1-x)\frac{q^2}{m^2}\right) = \frac{1}{6} q^2 + \frac{1}{60} q^4 + ... \quad (46)$$

We note now that

$$B(0, m^2) = \frac{\partial}{\partial m^2} A(m^2) = \frac{A(m^2)}{m^2} - 1 \quad (47)$$

We therefore rewrite Eqs. (43) and (44) using

$$B(q^2, m^2) = \overline{B}(q^2, m^2) + \frac{A(m^2)}{m^2} - 1. \quad (48)$$

We arrive at [11]

$$B_{21}(q^2, m^2) = \frac{1}{3} \left[ \left(1 - \frac{m^2}{q^2}\right) \overline{B} + \frac{A}{m^2} - \frac{5}{6} \right] \quad (49)$$

$$B_{22}(q^2, m^2) = -\frac{q^2}{12} \left[ \left(1 - \frac{4m^2}{q^2}\right) \overline{B} + \frac{A}{m^2} \left(1 - \frac{6m^2}{q^2}\right) - \frac{1}{3} \right]. \quad (50)$$

### C Calculation for $A^8 \to \pi^+ \pi^-$

Now equipped with various ways to handle the integrals appearing in our calculation, we present the relevant details, which would be a distraction in the main body of the text.

We begin by considering Fig. [4]. We split the contributions to the amplitude into $a_\mu$, $b_\mu$ and $c_\mu$ (in an obvious way). The outgoing pions are assigned momenta $p^+$ and $p^-$ and we let $k$ be the loop momentum in amplitudes $b_\mu$ and $c_\mu$. From Eq. (24) we know that $a_\mu = 0$ to this order in the chiral series. So we turn to the $O(\pi^4)$ pieces of $J^8_\mu$ to determine $c_\mu$ which is given by,

$$c_\mu = -\frac{\sqrt{3}}{4F^2} \int_k ((p^+ - p^-)_\mu - 2k_\mu) \frac{1}{k^2 - m^2_{K^+}} - [K^+ \leftrightarrow K^0], \quad (51)$$
where we have used an obvious notation for the integral over \( d^Dk/(2\pi)^D \). In dimensional regularisation, the pieces proportional to \( k_\mu \) form an odd function and vanish upon integration leaving,

\[
c_\mu = -\frac{\sqrt{3}}{4F^2}(p^+ - p^-)_\mu \int_0 k^2 - m^2_{K^+} - [K^+ \leftrightarrow K^0]. \quad (52)
\]

The contribution \( b_\mu \) is slightly more complicated, as we have to consider two ChPT vertices which we shall call \( V_\mu \) (a four-vector) and \( S \) (a scalar). The loop integral now has two propagators,

\[
b_\mu = \int_k V_\mu \frac{1}{(k^2 - m^2_{K^+})((k + q)^2 - m^2_{K^+})} S - [K^+ \leftrightarrow K^0]. \quad (53)
\]

From Eq. (29)

\[
V_\mu = \frac{\sqrt{3}}{2}(2k_\mu + q_\mu). \quad (54)
\]

This deserves a moment’s consideration. If \( S \) had no \( k \) dependence then \( b_\mu \) would be proportional to \( q_\mu(2B_1 + B) \) which vanishes as \( B_1 = -B/2 \). Therefore the only parts of Eq. (53) that will survive are those for which \( S \) contains \( k \). We find that these terms are

\[
S = -\frac{1}{6F^2}(-3p^+ \cdot k + 3p^- \cdot k + q \cdot k + k^2). \quad (55)
\]

We can now write,

\[
b_\mu = \frac{\sqrt{3}}{12F^2} \int_k \frac{(2k_\mu + q_\mu)(3(p^+ - p^-) \cdot k - (q \cdot k + q^2))}{(k^2 - m^2_{K^+})((k + q)^2 - m^2_{K^+})} - [K^+ \leftrightarrow K^0]. \quad (56)
\]

Before attempting to evaluate this, it helps to first consider that, because \( B_1 = -B/2 \), we can add terms independent of \( k \) to the numerator of Eq. (56), hence

\[
\int_k \frac{(2k_\mu + q_\mu)(q \cdot k + q^2)}{(k^2 - m^2)((k + q)^2 - m^2)} = \frac{1}{2} \int_k \frac{(2k_\mu + q_\mu)(k^2 - m^2 + (q + k)^2 - m^2)}{(k^2 - m^2)((k + q)^2 - m^2)}
\]

\[
= \frac{1}{2} \int_k \left( \frac{2k_\mu + q_\mu}{k^2 - m^2} + \frac{2k_\mu + q_\mu}{(q + k)^2 - m^2} \right)
\]

\[
= \frac{1}{2} \int_k \left( \frac{k_\mu - q_\mu}{k^2 - m^2} + \frac{q_\mu}{k^2 - m^2} \right)
\]

\[
= 0.
\]

So the only surviving piece of Eq. (56) is (recalling the factor \( \sigma \) defined in Eq. (37))

\[
b_\mu = \frac{\sqrt{3}}{12F^2} \int_k \frac{3\nu(p^+ - p^-)\nu(2k_\mu + q_\mu)}{(k^2 - m^2_{K^+})((k + q)^2 - m^2_{K^+})} - [K^+ \leftrightarrow K^0]
\]

\[
= \frac{\sqrt{3}}{4F^2} \sigma(p^+ - p^-)\nu(2B_{\mu\nu}(K^+) + q_\mu B_\nu(K^+)) - [K^+ \leftrightarrow K^0]
\]

\[
= \frac{\sqrt{3}}{2F^2} \sigma(p^+ - p^-)B_{22}(K^+) - [K^+ \leftrightarrow K^0], \quad (57)
\]
as \( q \cdot (p^+ - p^-) = 0 \).

We now have the expression for the amplitude,

\[
\mathcal{M}_\mu = b_\mu + c_\mu = \frac{\sqrt{3}}{4F^2} \sigma(p^+ - p^-)_\mu (2B_{22}(K^+) - A(K^+) - [K^+ \leftrightarrow K^0]).
\]

(58)

We now turn to Eqs. (35) and (50) to find expressions for \( A(m^2) \) and \( B_{22}(q^2, m^2) \) respectively. Substituting, we find

\[
2B_{22}(K^+) - A(K^+) - [K^+ \leftrightarrow K^0] = \frac{q^2}{6} \ln \frac{m_{K^+}^2}{m_{K^0}^2} - \frac{q^4}{60} \left( \frac{1}{m_{K^+}^2} - \frac{1}{m_{K^0}^2} \right).
\]

(59)

Eqs. (58) and (59) can then be combined to give us an expression for the form-factor \( F_\pi^8(q^2) \), Eq. (31).