Clock Rigidity and Joint Position-Clock Estimation in Ultrawideband Sensor Networks

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Abstract—Joint position and clock estimation are crucial in many wireless sensor network applications, especially in distance-based estimation with time-of-arrival (TOA) measurements. In this work, we study the graph properties under which an ultrawideband sensor network is localizable and clock synchronizable with one round TOA-based timestamp measurements. A novel clock rigidity theory is proposed and its topological condition is proved to have a close relationship to bearing rigidity. Based on clock rigidity, a graphical approach for analyzing the joint position and clock problem is investigated. It is shown that a position-clock framework with certain graph properties can be uniquely determined up to some trivial variations corresponding to both position and clock parameters. Simulation results are presented to demonstrate how clock rigidity theory is exercised in clock estimation and joint position-clock estimation.

Index Terms—Graph rigidity, joint position and clock estimation, network localization, wireless sensor network.

I. INTRODUCTION

POSITION estimation of multiagent systems, also referred to as network localization, is a critical aspect for a variety of robotic applications such as vehicle tracking and industrial process monitoring [1]. The global positioning system (GPS) is widely used for robot position estimation, but it lacks precision and can fail entirely in a GPS-denied environment, such as inside buildings and underground locations. With the development of low-cost, low-power, and multifunctional sensors, many research works focus on position estimation in wireless sensor networks, in which specialized sensors are mounted on robots and positions are estimated by using knowledge of the absolute positions of a limited number of sensors (called anchors) and intersensor measurements such as distance and bearing [2].

A related problem to multiagent position estimation is whether an anchor-free sensor network’s position information can be uniquely determined up to some trivial motions such as translation, rotation, and scaling. The problem can be equivalently stated as whether the shape of the network can be uniquely determined. Graph theory, particularly graph rigidity, is a useful tool for analyzing and solving this problem. The application of graph rigidity to position estimation is investigated and demonstrated in recent literature [3]–[10], where different intersensor measurements are considered, e.g., distance [3]–[5]; bearing [6], [7]; angle [8]; ratio of distance [9]; hybrid measurements [10], etc.

Clock estimation, also referred to as clock synchronization, is another critical requirement for data fusion, coordinated communication, and time-of-arrival (TOA) ranging methods in wireless sensor networks. The purpose of clock estimation is to estimate the clock parameters of each node in the network, enabling the node to convert the timestamp of its own clock to that of a common clock. One approach is to designate one node as a root node (or anchored node), then build a spanning tree from the root node to obtain clock synchronization over the network [11], [12]. Another approach, without relying on a root node, is to synchronize the network clocks to a common virtual clock based on distributed-consensus algorithms with requirements on graph connectivity [13]–[16].

The close relation between position and clock estimation in TOA-based ranging methods necessitates a joint estimation approach. Several research works [17]–[20] have studied the simultaneous estimation of network position and clock information, which are mostly tied from a statistical signal-processing perspective and use redundant communication and known position or clock information of some sensors (called anchors) to acquire a unique estimation result. The relationship between the graph properties of an anchor-free network and the network’s capability to estimate its position and clock by one round timestamp measurements is still an open problem.

Ultrawideband (UWB) is a short range, high-bandwidth radio technology. It uses a broad range of frequencies to generate energy pulses with sharp rising edges, which allow for highly precise signal sending and receiving timestamp measurements [21]. UWB sensors are widely used for distributed sensing in position estimation due to their high accuracy, low price, and low computation complexity.

In this article, we consider TOA-based UWB sensor networks in which both sending and receiving timestamps are accurately...
measured. Analogous to the distance (bearing) rigidity theory in position estimation [4], [7], we show that given timestamp measurements, the clock estimation, and joint position and clock estimation, problem can be also studied in a rigidity perspective. The contributions are stated as follows.

1) A TOA-based clock rigidity theory is first proposed in this article. It is shown that an infinitesimally clock rigid framework can be uniquely determined up to some trivial variations (a shift on clock offset and a skew on all clock parameters), which is able to synchronize the network clocks to a common virtual clock.

2) The connection between clock rigidity and bearing rigidity theory is established, providing a simple method for examining and constructing a clock rigid graph. It is shown that a clock framework is infinitesimally clock rigid if and only if its underlying graph is generically bearing rigid in \( \mathbb{R}^2 \) with at least one redundant edge.

3) Based on the proposed on the rigidity theory, a graph-theoretic approach for studying joint position-clock estimation problem is investigated. It is shown that certain graph rigidity properties can guarantee the estimates to be a trivial variation of the true clock and position, which is able to simultaneously determine the shape of the network and synchronize the network clocks to a common virtual clock.

4) This work is the first to study the graph topological conditions under which an anchor-free network with one round TOA-based timestamp measurements can be localizable and clock synchronizable.

The rest of this article is organized as follows. Section II provides preliminaries. Section III presents the TOA-based clock rigidity theory under the bidirectional communication assumption. Section IV establishes the connection between clock rigidity and bearing rigidity theory. Section V analyzes the joint position and clock problem based on the combination of clock rigidity theory and distance rigidity theory. Section VI applies this new theory to the clock estimation and the position-clock estimation problems using a gradient-descent method. Finally, Section VII concludes this article.

II. PRELIMINARIES

A. Notation

Matrices are denoted by capital letters (e.g., \( A \)). The rank and null space of a matrix \( A \) are denoted by \( \text{rank}(A) \) and \( \text{Null}(A) \), respectively. A diagonal matrix with diagonal entries \( d_1, \ldots, d_n \) is denoted as \( \text{diag}(d_i) \). A matrix or a vector that consists of all zero entries is denoted by \( \mathbf{0} \). An elemental rotation in \( d \)-dimensional space is a rotation about \( \partial \mathbf{p} \). The identity matrix in \( \mathbb{R}^{n \times n} \) is denoted by \( \mathbf{I}_n \). The Kronecker product of two matrices (vectors) \( A \) and \( B \) is written as \( \otimes \) and \( \| \cdot \| \) denotes the Euclidean norm of a vector. An elemental rotation in \( d \)-dimensional space is a rotation about the \( (d-2) \) dimensional subspace containing a set of \( (d-2) \) vectors in the standard basis. Matrix \( J_d \) denotes the infinitesimal generator of the \( i \)th elemental rotation in \( d \)-dimensional space, where \( i \in \{1, 2, \ldots, d(d-1)/2 \} \). For example, for \( d = 2 \) and \( d = 3 \),

\[
J_2 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad J_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}, \quad J_3 = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.
\]

An undirected graph, denoted as \( G = (V, \mathcal{E}) \), consists of a vertex set \( V = \{v_1, \ldots, v_n\} \) and an edge set \( \mathcal{E} \subseteq V \times V \) with cardinality \( |V| = n \) and \( |\mathcal{E}| = m \). Two vertices \( v_i \) and \( v_j \) are called neighbors when \( \{v_i, v_j\} \in \mathcal{E} \). The set of neighbors of vertex \( v_i \) is denoted as \( \mathcal{N}_i \). A directed graph is denoted as \( D = (V, \mathcal{E}_D) \), where \( \mathcal{E}_D \) is a directed edge set. In some circumstances, we use an edge index \( k \) to represent the node pair \( \{v_i, v_j\} \in \mathcal{E} \) for some edge ordering \( k \in \{1, \ldots, m\} \), denoted as \( k \sim \{v_i, v_j\} \).

Given a network of \( n \) nodes in \( \mathbb{R}^d \), where \( n \geq 2, d \geq 2 \), let the position of node \( i \) be \( p_i \in \mathbb{R}^d \) and the position configuration be \( p = [p_1^T, \ldots, p_n^T]^T \in \mathbb{R}^{dn} \). A position configuration, denoted as \( (G, p) \), is a combination of an undirected graph \( G \) and a position configuration \( p \).

B. Rigidity Theory

Consider a position framework \( (G, p) \), where the inter-neighbor distances bearings can be expressed as \( f_{d,k}(p) = \|b_k(p)\| \) for all \( k \sim \{v_i, v_j\} \in \mathcal{E} \). Let \( \delta p \) be a variation of the position configuration \( p \).

1) **Distance Rigidity:** If the framework is distance-preserved, we have constraints \( f_{d,k}(p) = d_k \in \mathbb{R} \) for all \( k \in \{1, \ldots, m\} \), where \( d_k \) is a constant. The **distance rigidity matrix** is defined as the Jacobian of the distance function with respect to \( p \)

\[
R_d(p) \triangleq \frac{\partial F_d(p)}{\partial p} \in \mathbb{R}^{m \times dn}
\]

where \( F_d(p) \triangleq [f_{d,1}(p), \ldots, f_{d,m}(p)]^T \in \mathbb{R}^m \). If \( R_d(p)\delta p = 0 \), then \( \delta p \) is called an infinitesimal distance motion of \( (G, p) \). An infinitesimal distance motion is called trivial if it only corresponds to a translation and a scaling of the framework.

**Definition 1 ([4]):** A position framework is **infinitesimally distance rigid** if all the infinitesimal distance motions are trivial.

2) **Bearing Rigidity:** If the framework is bearing-preserved, we have constraints \( f_{b,k}(p) = b_k \) for all \( k \in \{1, \ldots, m\} \), where \( b_k \in \mathbb{R}^d \) is a constant. The **bearing rigidity matrix** is defined as the Jacobian of the bearing function with respect to \( p \)

\[
R_b(p) \triangleq \frac{\partial F_b(p)}{\partial p} \in \mathbb{R}^{dm \times dn}
\]

where \( F_b(p) \triangleq [f_{b,1}(p), \ldots, f_{b,m}(p)]^T \in \mathbb{R}^{dm} \). If \( R_b(p)\delta p = 0 \), then \( \delta p \) is called an infinitesimal bearing motion of \( (G, p) \). An infinitesimal bearing motion is called trivial if it only corresponds to a translation and a scaling of the entire framework.

**Definition 2 ([7]):** A position framework is **infinitesimally bearing rigid** if all the infinitesimal bearing motions are trivial.
Define $d_{ij}$ as the distance between the $i$th and $j$th node, the sending timestamp at the local time coordinate of node $i$ is denoted as $T^i_{(i,j)}$, and the receiving timestamp at the local time coordinate of node $j$ is denoted as $T^j_{(i,j)}$.

The communication behind (1) is directed. In order to consider the network problem in an undirected way, we have the following assumption.

**Assumption 1 (Bidirectional communication):** The interode communication is bidirectional, i.e., the $i$th node can receive a ranging signal from the $j$th node if and only if the $j$th node can receive a ranging signal from the $i$th node.

Assumption 1 assumes the symmetric visibility between neighbors, which is trivially satisfied when all nodes share a common communication range. It simplifies the network problem from a directed graph to an undirected graph, and also provides a useful equivalent relation between the neighbor nodes, which will be shown later.

Now we define some necessary notation. Given a UWB sensor network of $n$ nodes, under Assumption 1, it can be represented by an undirected graph $G = (\mathcal{V}, \mathcal{E})$, where each vertex $v_i$ in the vertex set $\mathcal{V}$ is associated with $i$th sensor node and each edge $\{v_i, v_j\}$ in the edge set $\mathcal{E}$ corresponds to a sensor node pair which has bidirectional communication. A clock configuration is denoted as $\varphi = [\varphi^T_1, \ldots, \varphi^T_n] \in \mathbb{R}^{2n}$, where $\varphi_i = [\alpha_i, \beta_i]^T$.

We also define a clock skew configuration $\alpha = [\alpha^T_1, \ldots, \alpha^T_n]$ and a clock offset configuration $\beta = [\beta^T_1, \ldots, \beta^T_n]$. A clock framework, denoted as $(G, \varphi)$, is a combination of an undirected graph $G$ and a clock configuration $\varphi$, which provides a mapping from vertex $v_i \in \mathcal{V}$ to the parameter $\varphi_i$. Note that for a static UWB sensor network, we assume that $d_{ij}$ is fixed and the sending timestamp $T^i_{(i,j)}$ and $T^j_{(j,i)}$ are known for all $\{v_i, v_j\} \in \mathcal{E}$.

So the receiving timestamp measurement $T^j_{(i,j)}$ and $T^i_{(j,i)}$ can be uniquely determined by $\varphi_i$ and $\varphi_j$.

Under Assumption 1 and the distance relation (1), TOA ranging measurements from both ends of one edge are available and equal, i.e., $d_{ij} = d_{ji}$. Rewriting this distance equivalence with the timestamp notation, we have

$$\alpha_i T^i_{(i,j)} + \beta_i - \alpha_i T^i_{(i,j)} - \beta_i = \alpha_j T^j_{(j,i)} + \beta_j - \alpha_j T^j_{(j,i)} - \beta_j.$$  

which can be written as follows for every $\{v_i, v_j\} \in \mathcal{E}$:

$$\alpha_i T^i_{(i,j)} + \beta_i - \alpha_j T^j_{(j,i)} - \beta_j = 0 \quad (2)$$

where

$$\overline{T^i_{ij}} = \frac{T^i_{(i,j)} + T^j_{(j,i)}}{2} \quad \text{and} \quad \overline{T^j_{ij}} = \frac{T^j_{(i,j)} + T^i_{(j,i)}}{2}.$$  

Define the edge clock function $f_{c,ij} : \mathbb{R}^{2n} \rightarrow \mathbb{R}$ as

$$f_{c,ij}(s_{ij}(\varphi), \varphi) = s_{ij}(\varphi)^T \varphi$$

where

$$s_{ij}(\varphi) = \left[ \begin{array}{c} 0^T \overline{T^i_{ij}} \vdots \overline{T^j_{ij}} \end{array} \right]. \quad (3)$$

Equation (2) can be equivalently written as

$$f_{c,ij}(s_{ij}(\varphi), \varphi) = 0.$$  

![Fig. 1. TOA measurement and local timestamp notation.](image-url)
We are now ready to define the fundamental concepts in clock rigidity. These concepts are defined analogously to those in distance rigidity [4] and bearing rigidity [7].

**Definition 5:** Clock frameworks \((G, \varphi)\) and \((G, \varphi')\) are clock equivalent if \(f_{c,ij}(s_{ij}(\varphi), \varphi') = 0\) for all \(\{v_i, v_j\} \in E\).

**Definition 6:** Clock frameworks \((G, \varphi)\) and \((G, \varphi')\) are clock congruent if \(f_{c,ij}(s_{ij}(\varphi), \varphi') = 0\) for all \(v_i, v_j \in V\).

**Definition 7:** A clock framework \((G, \varphi)\) is clock rigid if there exists a constant \(\epsilon > 0\) such that any clock framework \((G, \varphi')\) that is clock equivalent to \((G, \varphi)\) and satisfies \(||\varphi' - \varphi|| < \epsilon\) is also clock congruent to \((G, \varphi)\).

**Definition 8:** A clock framework \((G, \varphi)\) is globally clock rigid if an arbitrary clock framework that is clock equivalent to \((G, \varphi)\) is also clock congruent to \((G, \varphi)\).

We next define infinitesimal clock rigidity, which is one of the most important concepts in the clock rigidity theory.

For convenience, we also reference the edge clock functions \(f_{c,ij}\) and \(s_{ij}\) using an edge index \(k\) rather than node pair \(\{v_i, v_j\} \in E\) for some edge ordering \(k \in \{1, \ldots, m\}\) as

\[ f_{c,k} \triangleq f_{c,ij}, \quad s_k \triangleq s_{ij} \]

and define the matrix

\[ S(\varphi) = [s_1(\varphi), \ldots, s_m(\varphi)]^T. \]

Now, we define the clock function \(F_c : \mathbb{R}^{2n} \rightarrow \mathbb{R}^m\) as

\[ F_c(S(\varphi), \varphi) \triangleq [f_{c,1}(s_1(\varphi)), \ldots, f_{c,m}(s_m(\varphi), \varphi)]^T, \]

so the constraint (2) can be written as

\[ F_c(S(\varphi), \varphi) = S(\varphi)\varphi = 0. \]  

(5)

The clock function describes all the clock constraints in the clock framework. We define the clock rigidity matrix as the Jacobian of the clock function

\[ R_c(\varphi) \triangleq \frac{\partial F_c(S(\varphi), \varphi)}{\partial \varphi} \in \mathbb{R}^{m \times 2n}. \]

(6)

Let \(\delta \varphi\) be a variation of the configuration \(\varphi\). If \(R_c(\varphi)\delta \varphi = 0\), then \(\delta \varphi\) is called an infinitesimal clock variation of \((G, \varphi)\). This is analogous to infinitesimal motions in distance rigidity [4] and bearing rigidity [7]. Distance preserving motions of a framework include translations and rotations. Bearing preserving motions include translations and scalings. For a clock framework, time-constituent preserving variations include translations (a common shift) on the clock offset configuration \(\beta\) and scalings (a common skew) of the entire clock framework. An infinitesimal clock variation is called trivial if it corresponds to a translation of the clock offset configuration \(\beta\) and a scaling of the entire clock framework. See Fig. 2.

**Definition 9:** A clock framework is infinitesimally clock rigid if all the infinitesimal clock variations are trivial.

Up to this point, we have introduced the fundamental concepts in clock rigidity theory. We next connect these concepts using the clock rigidity matrix.

**Lemma 10:** For a clock framework \((G, \varphi)\), the clock rigidity matrix in (6) can be expressed as \(R_c(\varphi) = S(\varphi)\) and satisfies \(R_c(\varphi)\varphi = 0\).

**Proof:** It follows from the definition of the clock function and clock rigidity matrix in (4) and (6). The \(k\)th row of the clock rigidity matrix can be written as

\[ \frac{\partial f_{c,k}(s_k(\varphi), \varphi)}{\partial \varphi} = s_k(\varphi) \]

for all \(k \in \{1, \ldots, m\}\). So \(R_c(\varphi) = [s_1(\varphi), \ldots, s_m(\varphi)]^T = S(\varphi)\). Following from (5), \(R_c(\varphi)\varphi = S(\varphi)\varphi = 0\).

**Lemma 11:** A clock framework \((G, \varphi)\) satisfies span\(\{1_n \otimes [0, 1]^T, \varphi\} \subseteq \text{Null}(R_c(\varphi))\) and rank\((R_c(\varphi)) \leq 2n - 2\).

**Proof:** By Lemma 10, it is clear that \(\varphi \subseteq \text{Null}(R_c(\varphi))\). The expression of \(s_k\) shown in (3) indicates that \(1_n \otimes [0, 1]^T \subseteq \text{Null}(R_c(\varphi))\). Considering the assumption that \(\alpha_i \in \mathbb{R}^+\), for all \(i \in \{1, \ldots, n\}\), the vector \(\varphi\) and \(1_n \otimes [0, 1]^T\) are linearly independent. The inequality \(\text{rank}(R_c(\varphi)) \leq 2n - 2\) follows immediately from \(\text{span} \{1_n \otimes [0, 1]^T, \varphi\} \subseteq \text{Null}(R_c(\varphi))\).

For any undirected graph \(G = (V, E)\), denote \(K_n\) as the n-node complete graph over the same vertex set \(V\), and \(R^K_c(\varphi)\) as the clock rigidity matrix of the clock framework \((K_n, \varphi)\). The next result gives the necessary and sufficient conditions for clock equivariance and clock congruency.

**Theorem 12:** Two clock frameworks \((G, \varphi)\) and \((G, \varphi')\) are clock equivalent if and only if \(R_c(\varphi)\varphi' = 0\), and clock congruent if and only if \(R^K_c(\varphi)\varphi' = 0\).

**Proof:** By Lemma 10, we have

\[ R_c(\varphi)\varphi' = 0 \leftrightarrow F_c(S(\varphi), \varphi') = 0 \]

so \(f_{c,k}(s_k(\varphi), \varphi') = 0\), for all \(k \in \{1, \ldots, m\}\). By Definition 5, the two clock frameworks are clock equivalent if and only if \(R_c(\varphi)\varphi' = 0\). By Definition 6, it can be similarly shown that clock frameworks are clock congruent if and only if \(R^K_c(\varphi)\varphi' = 0\).

Since any infinitesimal variation \(\delta \varphi\) is in \(\text{Null}(R_c(\varphi))\), Theorem 12 implies that \(R_c(\varphi)(\varphi + \delta \varphi) = 0\) and hence \((G, \varphi + \delta \varphi)\) is clock equivalent to \((G, \varphi)\).

It is worth noting that Theorem 12 for clock rigidity has an analogous expression in bearing rigidity theory; see [7, Th. 1]. It indicates a possible relationship between clock rigidity and bearing rigidity, which will be further discussed later. Here, we provide Theorems 13–17 as a straightforward extension of Theorem 12 following the corresponding bearing rigidity proofs in [7, Th. 2–6]. We refer the reader to this work for detailed proofs.

**Theorem 13:** A clock framework \((G, \varphi)\) is globally clock rigid if and only if \(\text{Null}(R^K_c(\varphi)) = \text{Null}(R_c(\varphi))\) or equivalently \(\text{rank}(R^K_c(\varphi)) = \text{rank}(R_c(\varphi))\).
**Theorem 14:** A clock framework \((G, \varphi)\) is clock rigid if and only if it is globally clock rigid.

**Theorem 15:** For a clock framework \((G, \varphi)\), the following statements are equivalent:
1) \((G, \varphi)\) is infinitesimally clock rigid;
2) \(\text{rank}(R_\varphi) = 2n - 2\);
3) \(\text{Null}(R_\varphi) = \{1_n \otimes [0, 1]^T, \varphi\}\).

**Theorem 16:** Infinitesimal clock rigidity implies global clock rigidity.

**Theorem 17:** An infinitesimally clock rigid framework can be uniquely determined up to a translation of clock offset \(\beta\) and a scaling of the entire clock framework.

Similar to bearing rigidity, the clock rigidity of a clock framework is also a generic property, which depends on its graph rather than the clock configuration. However, bearing measurements in bearing rigidity are only determined by the position configuration, whereas timestamp measurements in clock rigidity are not uniquely determined for a given clock configuration; it also depends on the distance between sensor nodes and the sending timestamp. We define a vector pair \((d_n, T_s)\) where \(d_n = \{d_1, \ldots, d_n\}^T\) denotes the internode distances and \(T_s = \{T^i_{(ij)}, T^j_{(ij)}, \ldots\}^T\) for all \(\{v_i, v_j\} \in E\) denotes the sending timestamps. Given \((d_n, T_s)\), we can define a generically clock rigid graph and a generic clock configuration paralleling the structure and analysis of bearing rigidity [22], then show the following result to infinitesimal clock rigidity.

**Definition 18:** Given \((d_n, T_s)\), a graph \(G\) is **generically clock rigid** if there exists at least one clock configuration \(\varphi\) such that \((G, \varphi)\) is infinitesimally clock rigid.

**Definition 19:** A clock configuration \(\varphi\) is **generic** for graph \(G\) if \((G, \varphi)\) is infinitesimally clock rigid.

**Lemma 20:** Given \((d_n, T_s)\), if \(G\) is generically clock rigid, then \((G, \varphi)\) is infinitesimally clock rigid for almost all \(\varphi\).

**Proof:** The lemma proof follows a similar structure as [22, Lemma 2].

### IV. CONNECTION TO BEARING RIGIDITY

The clock rigidity theory studies whether a clock framework can be uniquely determined by the interneighbor TOA timestamp measurements, which follows from research directions in the distance and bearing rigidity theory [4], [7]. In this section, we establish the connection between clock rigidity and bearing rigidity and prove that a clock framework is infinitesimally clock rigid if and only if its graph is generically bearing rigid in \(\mathbb{R}^2\) with at least one redundant edge. The main result is shown in Theorem 28. Fig. 7 presents how the following theorems and lemmas contribute to the proof of Theorem 28.

To establish this connection, first we introduce the dummy variable \(\gamma = [\gamma_1, \ldots, \gamma_m]^T\) to rewrite (2) for every \(\{v_i, v_j\} \in E\) with a corresponding fixed edge index \(k\) in the following form:

\[
\begin{align}
T^i_{ij} \alpha_i + \beta_i - \gamma_k &= 0 \quad (7a) \\
T^j_{ij} \alpha_j + \beta_j - \gamma_k &= 0 \quad (7b)
\end{align}
\]

![Fig. 3. Clock constraint visualization. The constraint (7) can be comprehended as two straight lines with the same vertical intercept in a 2-D coordinate, passing through the points \((\alpha_i, \beta_i)\) and \((\alpha_j, \beta_j)\), respectively. Since the gradient of the straight line is constant, the constraint (8) follows.](image)

which can be also written as

\[
\begin{align}
[\begin{array}{c}
T^i_{ij}, 1 \\
T^j_{ij}, 1
\end{array}] [\begin{array}{c}
\alpha_i, \beta_i - \gamma_k \\
\alpha_j, \beta_j - \gamma_k
\end{array}]^T &= 0 \\
[1, -T^i_{ij}]^T [1, -T^j_{ij}]^T &= 0.
\end{align}
\]

The vectors \([\alpha_i, \beta_i - \gamma_k]^T\) and \([\alpha_j, \beta_j - \gamma_k]^T\) are orthogonal to the vectors \([1, -T^i_{ij}]^T\) and \([1, -T^j_{ij}]^T\), respectively. Since \(\alpha_i, \alpha_j > 0\), for every \(\{v_i, v_j\} \in E\) we have

\[
\begin{align}
\left\| [\alpha_i, \beta_i - \gamma_k]^T \right\| &= 1, -T^i_{ij} \\
\left\| [\alpha_j, \beta_j - \gamma_k]^T \right\| &= 1, -T^j_{ij}.
\end{align}
\]

Fig. 3 shows the clock constraint (7) in a 2-D coordinate. Intuitively, in order to maintain the constraint (7), the gradient of the line which passes through the points \((\alpha_i, \beta_i)\) [or \((\alpha_j, \beta_j)\)] and \((0, \gamma_k)\) should be unchanged, which is equivalent to maintaining a constant bearing constraint in \(\mathbb{R}^2\). We take the partial derivative of the left-hand side of (8) with respect to \(\zeta = [\gamma^T, \gamma^T]^T \in \mathbb{R}^{2n+m}\) for every \(\{v_i, v_j\} \in E\) and define the resulting Jacobian with respect to \(\zeta\) as \(S'\zeta(\zeta) \in \mathbb{R}^{4m \times (2n+m)}\).

Then, we can give the following lemma.

**Lemma 21:** Given a clock framework \((G, \varphi)\) and a variation of the clock configuration \(\delta \varphi\), \(\delta \varphi \in \text{Null}(R_\varphi(\varphi))\) if and only if there exists a vector \(\delta \gamma \in \mathbb{R}^m\) such that \(\delta \zeta = [\delta \varphi^T, \delta \gamma^T]^T \in \text{Null}(S'\zeta(\zeta))\).

**Proof:** By writing (7) for every \(\{v_i, v_j\} \in E\) as a linear matrix equation, we have \(S'\zeta(\zeta) \zeta = 0\), where \(S'\zeta(\zeta) \in \mathbb{R}^{2m \times (2n+m)}\).

We define extended clock function \(F_1: \mathbb{R}^{2n+m} \rightarrow \mathbb{R}^{2m}\) as \(F_1(S'\zeta(\zeta)) \triangleq S'\zeta(\zeta)\). It is clear that the Jacobian with respect to \(\zeta\) of the extended clock function is \(S'\zeta(\zeta)\). Equations (7) and (8) are equivalent, so the infinitesimal variations which preserve the equality in (8) also preserve the equality in (7), i.e., \(\text{Null}(S'\zeta(\zeta)) = \text{Null}(S'\zeta(\zeta))\).

By elementary row operations which preserve the null space of matrix \(S'\zeta(\zeta)\), the matrix \(S'\zeta(\zeta)\) can be written in the following form:

\[
T_2T_3S'\zeta(\zeta) = \begin{bmatrix}
S(\varphi) & 0 \\
X & -I_m
\end{bmatrix}
\]
where $T_1$ is a row switching elementary matrix so that the resulting matrix $T_1S_1^\prime$ has the $k$th row and $(m+k)$th row corresponding to the $k$th edge. Further, $T_2$ is a row addition elementary matrix, which has the form

$$
T_2 = \begin{bmatrix}
I_m & -I_m \\
0 & I_m
\end{bmatrix}.
$$

Each row of the matrix $X$ corresponds to only one of the constraints (7) of an edge $\{v_i, v_j\} \in E$, e.g.,

$$
\begin{bmatrix}
0^T & 0^T & T_{ij}^T & 1 & 0^T & -1 & 0^T
\end{bmatrix}.
$$

Since the identity matrix $I_m$ is full rank and $R_\gamma (\varphi) = S(\varphi)$, we have $R_\gamma (\varphi) \delta \varphi = 0$ if and only if there exists $\delta \zeta = [\delta \varphi^T, \delta \gamma^T]^T$ such that $S_\gamma' (\zeta) \delta \zeta = 0$. 

It can be observed that the left-hand side of (8) is the expression of the bearing between the point $(\alpha_i, \beta_i)$ or $(\alpha_j, \beta_j)$ and $(0, \gamma_k)$ and the right-hand side is a constant. Therefore, the property of the clock constraints. We next define this new clock framework and explore how it relates to the original framework.

Define a new graph $G' = (V \cup \mathcal{V}', E_1 \cup E_2)$, where $\mathcal{V}'$ is constructed according to the edge in $G$, i.e., $\mathcal{V}' = \{v'_1, \ldots, v'_m\}$ and $v'_k$ corresponds to the $k$th edge $\{v_i, v_j\} \in E$. The edge set $E_1$ is constructed so that for every $\{v_i, v_j\} \in E$, we have $\{v_i, v'_k\}, \{v_j, v'_k\} \in E_1$. The edge set $E_2$ is constructed so that $(\mathcal{V}', E_2)$ is a spanning tree of the line graph of $G$, i.e., the adjacent vertices in $(\mathcal{V}', E_2)$ imply the adjacent edges in $G$. See Fig. 4. The line graph of any connected graph is connected [23], so there always exists a spanning tree. We denote $\eta = [\eta_1, \ldots, \eta_m]^T$, where $\eta_k = [0, \gamma_k]^T$ and define a clock framework $(G', [\varphi^T, \eta'^T]^T)$. So the configuration $[\varphi^T, \eta'^T]^T$ provides a mapping from vertex $v_i \in V$ to the points $\varphi_i$, and from $v'_k \in \mathcal{V}'$ to the points $\eta_k$. See Fig. 5. Given $(d_m, T_0), (G', [\varphi^T, \eta'^T]^T)$ can be uniquely decided by $(G, \varphi)$.

Now, we are ready to prove the relation between graph $G$ and $G'$. First we give the definition of redundant edge.

**Definition 22:** An edge in a generically rigid graph $G$ is a **redundant edge** if $G$ is still generically rigid after removing this edge.

The following lemma gives the necessary and sufficient condition between the bearing rigidity of graph $G$ and $G'$.

**Lemma 23:** The graph $G'$ is generically bearing rigid in $\mathbb{R}^2$ if and only if $G$ is generically bearing rigid in $\mathbb{R}^2$ with at least one redundant edge.

**Proof:** The Henneberg operation is a useful method for adding new vertices into a graph while preserving its rigidity. For a graph $G = (V, E)$, the vertex addition operation in $\mathbb{R}^d$ adds a new vertex with $d$ new incident edges. The edge splitting operation in $\mathbb{R}^d$ removes an existing edge and adds a new vertex with $d+1$ new incident edges. Both operations preserve the bearing rigidity of graphs in $\mathbb{R}^2$ [22].

Suppose that $G$ is generically bearing rigid. We denote a spanning tree of the line graph of $G$ as $G_s$, so the 4th vertex of $G_s$ corresponds to the $k$th edge of $G$. Define a set $M = \emptyset$. Then, as shown in Fig. 6, we do the following steps to get a graph $G'$ from $G$:

**Step 1:** Remove a redundant edge $k_1 \sim \{v_p, v_q\} \in G$. By definition, the resulting graph is still generically bearing rigid. Then, add a new vertex $v'_{k_1}$ with two new edges $\{v_p, v'_{k_1}\}$ and $\{v_q, v'_{k_1}\}$ to the graph $G$. Add the integer $k_1$ into the set $M$, i.e., $M = \{k_1\}$ (vertex addition).

**Step 2:** Select a remaining edge $k \sim \{v_i, v_j\} \in E$ where the $k$th vertex in $G_s$ is adjacent to the $k$th vertex and $k_2 \in M$. Remove the edge $k$ in $G$ and add a new vertex $v'_{k_2}$. Then, add three new edges $\{v_i, v'_{k_1}\}, \{v_j, v'_{k_1}\}$, and $\{v_i, v'_{k_2}\}$, where the first two edges are in $E_1$ and the last one is in $E_2$. Add the integer $k$ into the set $M$. Repeat this step for the remaining edges in $E$ until all the edges in $E$ are removed (edge splitting).

Since all of the above steps are rigidity-preserving, $G'$ is generically bearing rigid. The corresponding deconstruction steps are also rigidity-preserving, so the proof is reversible. 

Lemma 23 establishes the rigidity relation between the original graph $G$ and the new graph $G'$. Recall, the Jacobian of the left-hand side of (8) is $S_\gamma' (\zeta)$. Next, we will prove the relation between the bearing rigidity of the new clock framework $(G', [\varphi^T, \eta'^T]^T)$ and the rank of the matrix $S_\gamma'(\zeta)$.

Before proceeding, we need to introduce Assumption 2 (Device assumption), which can be trivially satisfied in a typical UWB implementation.

**Assumption 2a (Single antenna):** A UWB sensor has only a single antenna, so it cannot receive more than one message at a time, i.e., there exists no $T^a,(a,i) = T^b,(b,i)$ for all $a \neq b$. 

![Original clock framework $(G, \varphi)$ and new clock framework $(G', [\varphi^T, \eta'^T]^T)$](Image 5.Original clock framework $(G, \varphi)$ and new clock framework $(G', [\varphi^T, \eta'^T]^T)$). The solid blue vertices correspond to $V'$ and the blue edges correspond to $E_2$. 

![Line graph of $G$](Image 4. Line graph of $G$) 

**Fig. 4.** Original graph $G$ to new graph $G'$. The solid blue vertices correspond to $V'$ and the blue edges correspond to $E_2$, which is a spanning tree of the line graph of $G$. 

**Fig. 5.** Original clock framework $(G, \varphi)$ and new clock framework $(G', [\varphi^T, \eta'^T]^T)$. The solid blue vertices correspond to $V'$ and the blue edges correspond to $E_2$. 

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where $A$ is a submatrix of $R_b$, whose columns correspond to the $x$-coordinate of points in $\mathcal{V}'$ and rows correspond to the edges in $\mathcal{E}_1$. The matrix $B = QH \otimes [1,0]^T \in \mathbb{R}^{2(m-1) \times m}$, where $Q = \text{diag}(|\gamma_k - \gamma_i|^{-1})$ for all $\{v'_k,v'_e\} \in \mathcal{E}_2$ and $H$ is the incidence matrix of $(\mathcal{V}',\mathcal{E}_2)$. Lemma 24 guarantees that $\gamma_k \neq \gamma_i$ in the matrix $Q$. The matrix $P_1$ permutes rows of $R_b$ so that the first $4m$ rows correspond to the edges in $\mathcal{E}_1$ and the last $2(m-1)$ rows correspond to the edges in $\mathcal{E}_2$. The matrix $P_2$ permutes columns of $R_b$ so that the first $2n + m$ columns correspond to $[\varphi^T, \gamma^T]^T$, i.e., the coordinates of points in $\mathcal{V}'$ and the $y$-coordinate of points in $\mathcal{V}'$. The last $m$ columns after permutation correspond to the $x$-coordinate of points in $\mathcal{V}'$.

(Necessity) Suppose that $([G', \varphi^T, \eta^T]^T)$ is infinitesimally bearing rigid. The permuted bearing rigidity matrix $P_1R_bP_2$ has the same rank as $R_b$, i.e., $\text{rank}(P_1R_bP_2) = 2(n + m) - 3$ and $\text{Null}(P_1R_bP_2) = \{u_1, u_2, u_3\}$, where $u_1 = [\varphi^T, \gamma^T, \mathbf{0}^T]^T$, $u_2 = [1^n, \mathbf{0}^T, \mathbf{0}^T]^T$, and $u_3 = [1^n \otimes [1,0], \mathbf{0}^T, 1^m_1]^T$, $\mathbf{0}^T$.

Suppose that a vector $w \in \text{Null}(S'' \otimes [1,0])$, then $w \in \text{Null}(P_1R_bP_2)$. Hence, $w$ must be a linear combination of $u_1, u_2,$ and $u_3$, i.e., for some scalar $a, b, c$, not all zero, $w = au_1 + bu_2 + cu_3$, and the last $m$ rows gives $0 = au_1 + bu_2 + cu_3$, where $u_1 = 0$, $u_2 = 0$, and $u_3 = 1^m_1$. So $c = 0$ and $a, b$ are not both zero. For the remaining $2n + m$ rows, $w = au_1 + bu_2$, where $u_1 = [\varphi^T, \gamma^T]^T$ and $u_2 = [1^n_1 \otimes [1,0], 1^m_1]^T$. Since it holds for any $w \in \text{Null}(S'')$, $\text{rank}(S'') = 2n + m - 2$ and $\text{Null}(S'') = \{u_1, u_2\}$.

(Sufficiency) Suppose that $\text{Null}(S'') = \{u_1, u_2\}$. The matrix $B = QH \otimes [1,0]^T$, where $Q$ is a diagonal matrix and $H$ is the incidence matrix of $(\mathcal{V}',\mathcal{E}_2)$, hence $\text{Null}(B) = \text{span}(\mathbf{1}_n)$. Thus, $\text{Null}(P_1R_bP_2) = \text{span}(u_1, v_0^T, 0^T, [u_2^T, 0^T, [u_3^T, 1^m_1]^T])$, where $S''v_0^T + A = 1^m_1 = 0$. It follows immediately that $\text{Null}(P_1R_bP_2) = \text{span}(u_1, u_2, u_3)$ and hence the clock framework $(G', [\varphi^T, \gamma^T]^T)$ is infinitesimally bearing rigid.

The following theorem states the relation between the clock rigidity of clock framework $(G, \varphi)$ and the bearing rigidity of clock framework $(G', [\varphi^T, \gamma^T]^T)$.

**Theorem 26:** Under Assumption 2, a clock framework $(G, \varphi)$ is infinitesimally bearing rigid if and only if $(G', [\varphi^T, \gamma^T]^T)$ is infinitesimally bearing rigid.

**Proof:** We first prove the necessity. Suppose $(G, \varphi)$ is infinitesimally bearing rigid. By Theorem 15, we have $\text{Null}(R_c(\varphi)) = \text{span}(\mathbf{1}_n \otimes [0,1]^T, \varphi^T)$, so Lemma 21 gives that $\text{rank}(S'' \otimes [0,1]^T, \varphi^T, 1^m_1)^T = \text{span}(\{u_1, \mathbf{0}^T, [u_2^T, 0^T, [u_3^T, 1^m_1]^T\}$). By Lemma 25, $(G', [\varphi^T, \gamma^T]^T)$ is infinitesimally bearing rigid. Since Theorem 15 and Lemmas 21 and 25 all state necessary and sufficient conditions, the proof for the sufficiency of this theorem also holds.

**Corollary 27:** Under Assumption 2, a clock configuration $[\varphi^T, \eta^T]^T$ is generic for bearing rigidity of graph $G'$ if and only if the clock configuration $\varphi$ is generic for clock rigidity of graph $G$.

Now, we are ready to prove the following theorem, which gives a sufficient and necessary graph property for establishing infinitesimal clock rigidity.
Theorem 28 (Main result): Under Assumption 2, for any generic clock configuration \( \varphi \), a clock framework \((G, \varphi)\) is infinitesimally clock rigid if and only if \(G\) is generically bearing rigid in \(\mathbb{R}^2\) with at least one redundant edge.

Proof: (Necessity) Suppose \((G, \varphi)\) is infinitesimally clock rigid. By Theorem 26, \((G', [\varphi^T, \eta^T]^T)\) is infinitesimally bearing rigid and hence \(G'\) is generically bearing rigid. Then, by Lemma 23, \(G'\) is generically bearing rigid in \(\mathbb{R}^2\) with at least one redundant edge.

(Sufficiency) Suppose \(G\) is generically bearing rigid in \(\mathbb{R}^2\) with at least one redundant edge. By Lemma 23, \(G'\) is generically bearing rigid in \(\mathbb{R}^2\). Since \(\varphi\) is a generic configuration for \(G\), by Corollary 27, \((G', [\varphi^T, \eta^T]^T)\) is infinitesimally bearing rigid. By Theorem 26, \((G, \varphi)\) is infinitesimally clock rigid.

The theorems and lemmas contributing to this proof are shown in Fig. 7.

Theorem 28 suggests that the infinitesimal clock rigidity can be determined by checking if the underlying graph is generically bearing rigid with at least one redundant edge. Up to this point there is no existing direct graph based method to check clock rigidity. Theorem 28 can be used to provide a topological method to establish infinitesimal clock rigidity based on Laman graphs; defined here.

Definition 29 ([4], [25]): A graph \(G = (V,E)\) is Laman if \(|E| = 2|V| - 3\) and every subset of \(k \geq 2\) vertices spans at most \(2k - 3\) edges.

The topological result is stated in the subsequent corollary and draws on the following theorem.

Theorem 30 ([22]): A graph \(G\) is generically bearing rigid in \(\mathbb{R}^2\) if and only if it contains a Laman spanning subgraph.

Corollary 31: Under Assumption 2, for any generic clock configuration \(\varphi\), a clock framework \((G, \varphi)\) is infinitesimally clock rigid if and only if \(G\) contains a Laman spanning subgraph \(G'\) and \(G' \neq G\).

Proof: The result follows directly from the statements of Theorem 28 and Theorem 30.

The following example exercises Corollary 31 for the clock framework \((K_3, \varphi)\) and \((K_4, \varphi)\).

Example 32: The complete graph \(K_3\) is generically bearing rigid in \(\mathbb{R}^2\) since it contains a Laman spanning subgraph \(G\) as shown in Fig. 8, but \(G' = G\), i.e., there is no redundant edge for its rigidity. Following from Corollary 31, \((K_3, \varphi)\) is not infinitesimally clock rigid for any generic configuration \(\varphi\).

The complete graph \(K_4\) contains a Laman spanning subgraph \(G_{\ell}\) and \(G_{\ell} \neq G\), so \((K_4, \varphi)\) is infinitesimally clock rigid following from Corollary 31. Note that \(K_4\) is also the minimal graph which establishes infinitesimal clock rigidity property.

Theorem 28 and Corollary 31 also show great value in the joint position and clock problem, which is further discussed in Sections V and VI.

V. JOINT RIGIDITY

It is studied in clock rigidity theory whether a clock framework with certain graph property can be determined up to some trivial variations given the TOA timestamp measurements between neighbors. The close relation between distance and time in TOA measurements also provides an invariant equality involving both position and clock information. In this section, we combine clock rigidity theory and distance rigidity theory to analyze the joint position and clock problem. We explore the conditions under which the position and clock information in a framework can be uniquely and simultaneously determined up to some trivial variations.

Consider a TOA-based UWB sensor network. Define a position configuration \(p = [p_1^T, \ldots, p_n^T]^T \in \mathbb{R}^{n^d}\) and a clock configuration \(\varphi = [\varphi_1^T, \ldots, \varphi_n^T]^T \in \mathbb{R}^{2n}\). We next take both position and clock into account and define a position-clock framework \((D, \sigma)\), where \(D = (V, E_D)\) is a directed graph and \(\sigma = [\varphi^T, \eta^T]^T \in \mathbb{R}^{n(d+2)}\) is a position-clock configuration. The position-clock configuration provides a mapping from \(v_i \in V\) to \(\sigma_i = [\varphi_i^T, \eta_i^T]^T \in \mathbb{R}^{d+2}\), including a position \(p_i \in \mathbb{R}^d\) and a clock \(\varphi_i \in \mathbb{R}^2\).

Based on (1), the position-clock framework satisfies the following constraint for every edge \((v_i, v_j) \in E_D\):

\[
||p_i - p_j||^2 - c^2(\alpha_j T_{(i,j)}^2 + \beta_j - \alpha_i T_{(i,j)}^2 - \beta_i)^2 = 0
\]

where \(c\) is the speed of light. We take the partial derivative of the left-hand side of (9) with respect to \(\sigma\) for every \((v_i, v_j) \in E_D\) and call the resulting Jacobian \(\frac{\partial R_{cd}(\sigma)}{\partial \sigma}\) with respect to \(\sigma\) the joint rigidity matrix since it includes the information about both clock rigidity and distance rigidity. Each row of \(\frac{\partial R_{cd}(\sigma)}{\partial \sigma}\) corresponds to an edge \((v_i, v_j)\), which has the form

\[
\begin{bmatrix}
0^T & (p_i - p_j)^T & 0^T & (p_j - p_i)^T & 0^T & cd_{ij} T_{(i,j)} & cd_{ij} \\
\cline{1-7}
\end{bmatrix} 
\]

\[
\begin{bmatrix}
0^T & cd_{ij} T_{(i,j)}^2 & -cd_{ij} \\
\cline{1-2}
\end{bmatrix} (v_i) 
\begin{bmatrix}
0^T \\
\cline{1-1}
\end{bmatrix} (v_j) 
\in \mathbb{R}^{(d+2)n}
\]

Fig. 7. Diagram of the statements contributing to the proof of Theorem 28.

Fig. 8. Complete graph \(K_3\) and \(K_4\). The blue vertices and edges show the Laman spanning subgraph. The black edge in \(K_3\) is the redundant edge for its bearing rigidity. (a) Non-infinitesimally clock rigid. (b) Infinitesimally clock rigid.
Given a position-clock framework is infinitesimally joint rigid if and only if rank$(R_{cd}(\sigma)) = (d + 2)n - d(d + 1)/2 - 2$.

Let $\delta \sigma$ be a joint variation of the configuration $\sigma$. If $R_{cd}(\sigma)\delta \sigma = 0$, then $\delta \sigma$ is called an infinitesimal joint variation of $(D, \sigma)$. Infinitesimal joint variations preserve timestamp measurements. An infinitesimal joint variation is called trivial if it corresponds to a translation and a rotation of position configuration $p$, a translation of the clock offset configuration $\beta$ and a scaling of the entire position-clock framework. See Fig. 9. Analogous to infinitesimal distance (bearing) rigidity, we define infinitesimal joint rigidity.

**Definition 33:** A position-clock framework is infinitesimally joint rigid if all the infinitesimal joint variations are trivial.

We next give a necessary and sufficient condition of infinitesimal joint rigidity.

**Theorem 34:** A position-clock framework $(D, \sigma)$ in $\mathbb{R}^{d+2}$ is infinitesimally joint rigid if and only if rank$(R_{cd}(\sigma)) = (d + 2)n - d(d + 1)/2 - 2$.

**Proof:** The expression of joint rigidity matrix $R_{cd}$ in (10) shows that span$(\{[J_1 \otimes J_2]_T^T, \phi^T(I_n \otimes J_1^d)_T^T, \phi^T(0^T)T, \ldots, [p^T(I_n \otimes J_1^d(2d-1))/2)_T^T, \phi^T(0^T)T\}) \subset \text{Null}(R_{cd})$. It can be observed that these vectors correspond to translations and rotations of position configuration $p$ in $d$-dimensional space. We also have $\{[0^T, 1^T_d \otimes [0, 1]]^T \subset \text{Null}(R_{cd}),$ corresponding to a translation of the clock offset configuration $\beta$.

Since $\sigma = [p^T, \phi^T]_T$ and $R_{cd}(\sigma)\sigma = 0$ satisfies the constraints in (9), span$\{\sigma\} \subset \text{Null}(R_{cd})$. The configuration $\sigma$ corresponds to a scaling of the position-clock framework. Following from Definition 33, $(D, \sigma)$ is infinitesimally joint rigid if and only if dim$(\text{Null}(R_{cd})) = d(d + 1)/2 + 2$, i.e., rank$(R_{cd}(\sigma)) = (d + 2)n - d(d + 1)/2 - 2$.

A consequence of Theorem 34 is a graph $D$ must be sufficiently connected to be infinitesimally joint rigid.

**Corollary 35:** An infinitesimally joint rigid position-clock framework $(D, \sigma)$ in $\mathbb{R}^{d+2}$ only if it has at least $(d + 2)n - d(d + 1)/2 - 2$ edges.

Fig. 10 shows some examples of infinitesimal joint rigidity and flexibility in $\mathbb{R}^{2+2}$. Fig. 10(a) and (b) are infinitesimally joint rigid since the rank of their joint rigidity matrix equals to $4n - 5$. They also exhibit the minimum number of directed edges for joint rigidity on five and six node graphs. Fig. 10(c) has the same number of edges as Fig. 10(b), but it is noninfinitesimally joint rigid, showing that the necessary condition in Corollary 35 is not sufficient.

We denote the disoriented graph of $D = (V, E_D)$ as $G = (V, E)$, which is the undirected graph obtained after removing the orientation of the directed edges of $D$. In a directed graph $D$, it is challenging to conclude generic graph properties from the joint rigidity matrix $R_{cd}(\sigma)$ due to its nontrivial expression. As an example, Fig. 10(b) and (c) shows that even with the same disoriented graph and same number of edges, the joint rigidity of the position-clock framework is indeterminable. We next leverage Assumption 1 (bidirectional communication) to simplify the problem from a directed graph to an undirected graph and study the graph property for realizing infinitesimal joint rigidity.

Under Assumption 1, assume that $|E_D| = 2m$, so $|E| = m$. We define $d_k \triangleq d_{ij} = c(\alpha_j T_{(i,j)} + \beta_j - \alpha_i T_{(i,j)} - \beta_i)$ for all $k \sim \{v_i, v_j\} \in E$. The next lemma shows the characteristics of the joint rigidity matrix under Assumption 1.

**Lemma 36:** Given a position-clock framework $(D, \sigma)$ and its disoriented graph $G$ in $\mathbb{R}^{d+2}$ under Assumption 1, the joint rigidity matrix $R_{cd}(\sigma)$ has the following form after elementary row operations:

$$ TR_{cd}(\sigma) = \begin{bmatrix} R_d(p) & Y \\ 0 & 2c DR_{c}(\varphi) \end{bmatrix} $$

where $T$ represents the corresponding elementary row operations, $R_d(p)$ is the distance rigidity matrix of $(G, p)$, $R_c(\varphi)$ is the clock rigidity matrix of $(G, \varphi)$, $D = \text{diag}(\{d_k\})$, and $Y$ is a submatrix of $R_{cd}$, whose columns correspond to $\varphi$ and rows correspond to one of the directed edges between each neighboring node pair.

**Proof:** Under Assumption 1, $(v_i, v_j) \in E_D$ with $\sigma = [p^T, \varphi^T]_T$ if and only if $(v_i, v_j) \in E_P$. The measured distances satisfy $d_{ij} = d_{ij}$, as discussed in Section III. The elementary row operation is $T = T_1T_i$, where $T_1$ is a row switching elementary matrix so that the $k$th row and $(m + k)$th row of $T_1R_{cd}$ correspond to edges $(v_i, v_j)$ and $(v_j, v_i)$, respectively, and $T_3$ is a row addition...

---

**Fig. 9.** Basic trivial infinitesimal joint variations in $\mathbb{R}^{2+2}$. Position variations (red arrows) and clock variations (blue arrows) are shown in the $xy$-coordinate plane and the $\alpha\beta$-coordinate plane, respectively. (a) Translation and rotation of position configuration $p$. (b) Translation of $\beta$ (c) Scaling of the entire framework.

**Fig. 10.** Examples of position-clock frameworks in $\mathbb{R}^{2+2}$. (a) Infinitesimally joint rigid $(|V| = 5, |E| = 15, \text{rank}(R_{cd}) = 15)$. (b) Infinitesimally joint rigid $(|V| = 6, |E| = 19, \text{rank}(R_{cd}) = 19)$. (c) Non-infinitesimally joint rigid $(|V| = 6, |E| = 19, \text{rank}(R_{cd}) = 18)$.
Theorem 37: Under Assumption 1, a position-clock framework \((D, \sigma)\) in \(\mathbb{R}^{d+2}\) with \(\sigma = [p^T, \varphi^T]^T\) is infinitesimally joint rigid if the position framework \((G, p)\) in \(\mathbb{R}^d\) is infinitesimally distance rigid and the clock framework \((G, \varphi)\) is infinitesimally clock rigid where \(G\) is the disoriented graph of \(D\).

Proof: Suppose that a position framework \((G, p)\) in \(\mathbb{R}^d\) is infinitesimally distance rigid and a clock framework \((G, \varphi)\) is infinitesimally clock rigid. By Theorem 15 and the distance rigidity theory in [4], we have \(\text{Null}(R_d(p)) = \text{span}\{1_n \otimes I_d, (I_n \otimes J^T_2)p, \ldots, (I_n \otimes J^{d(d-1)/2})p\}\) and \(\text{Null}(R_c(\varphi)) = \text{span}\{1_n \otimes [0, 1]^T, \varphi\}\).

By Lemma 36, from the expression of \(Y\) and (9), we have \(R_d(p) 0 + Y(1_n \otimes [0, 1]^T) = 0\) and \(R_d(p)Y + Y = 0\). So \(\text{rank}(R_d(\sigma)) = \text{rank}(R_d(p))\) if and only if \(\text{rank}(R_c(\varphi)) = \text{rank}(R_c(\varphi))\) if and only if \(\text{rank}(R_c(\varphi)) = \text{rank}(R_c(\varphi))\) if and only if \(\text{rank}(R_c(\varphi)) = \text{rank}(R_c(\varphi))\). By Theorem 34, \((D, \sigma)\) in \(\mathbb{R}^{d+2}\) is infinitesimally joint rigid.

Theorem 38: Under Assumptions 1 and 2, for a position-clock framework \((D, \sigma)\) in \(\mathbb{R}^{d+2}\) with \(\sigma = [p^T, \varphi^T]^T\) and corresponding disoriented graph \(G\), the following statements are equivalent for generic position configuration \(p\) and generic clock configuration \(\varphi\):

a) the position-clock framework \((D, \sigma)\) in \(\mathbb{R}^{d+2}\) is infinitesimally joint rigid;

b) the clock framework \((G, \varphi)\) in \(\mathbb{R}^d\) is infinitesimally clock rigid;

c) the position framework \((G, p)\) in \(\mathbb{R}^d\) is infinitesimally distance rigid with at least one redundant edge.

Proof: Since infinitesimal distance rigidity is equivalent to infinitesimal bearing rigidity in \(\mathbb{R}^2\), (c) implies that \(G\) is generically bearing rigid with at least one redundant edge. So by Theorem 28, (b) is equivalent to (c). Then, it follows immediately from Theorem 37 that (c) implies (a). We next prove that (a) implies (c).

Suppose the position-clock framework \((D, \sigma)\) in \(\mathbb{R}^{d+2}\) is infinitesimally joint rigid. By Theorem 34, \(\text{rank}(R_d(\sigma)) = 4n - 5\) and \(\text{Null}(R_d(\sigma)) = \text{span}\{[x_1^T, 0^T]^T, [x_2^T, 0^T]^T, [x_3^T, 0^T]^T, [0^T, y_1^T]^T, [0^T, y_2^T]^T, [p^T, \varphi^T]^T\}\), where \(x_1 = 1_n \otimes [0, 1]^T\), \(x_2 = y_1 = 1_n \otimes [0, 1]^T\), and \(x_3 = (I_n \otimes J^T_2)p\). Since elementary row operations preserve the null space, \(\text{Null}(R_d(\sigma)) = \text{Null}(TR_{cd})\).

Consider a nonzero vector \(w \in \text{Null}(R_d(p))\). Let \(u = [u^T, 0^T]^T \in \mathbb{R}^{4n}\), then by Lemma 36, \(u \in \text{Null}(TR_{cd}(\sigma))\). Hence \(u\) must be a linear combination of \(u_1, u_2, u_3, u_4, u_5\), and \(u_5\), i.e., for some scalar \(a_1, a_2, a_3, a_4, a_5\), not all zero, \(u = a_1u_1 + a_2u_2 + a_3u_3 + a_4u_4 + a_5u_5\). Examining the last 2\(n\) rows of \(u\) gives \(0 = a_4y_4 + a_5\varphi\). Vectors \(y_4\) and \(\varphi\) are linearly independent, hence \(a_4 = a_5 = 0\) and \(a_1, a_2, a_3\) are not all zero. For the remaining 2\(n\) rows, \(w = a_1x_1 + a_2x_2 + a_3x_3\). Since it holds for any \(w \in \text{Null}(R_d(p))\), \(1_n \otimes [0, 1]^T\), and \(x_3\) are linearly independent, then \(\text{rank}(R_d) = 2n - 3\) and \(\text{Null}(R_d) = \text{span}\{x_1, x_2, x_3\}\), i.e., the position framework \((G, p)\) is infinitesimally distance rigid.

Under Assumption 1, the number of edges in \(D\) must be even. Since \(\text{rank}(R_d(\sigma)) = 4n - 5\), the joint rigidity matrix \(R_d\) should have at least \((4n - 4)\) rows. So the distance rigidity matrix \(R_d(p)\) has at least \((2n - 2)\) rows, i.e., there are at least \((2n - 2)\) edges in \(G\). As only \((2n - 3)\) edges are necessary for infinitesimal distance rigidity, then the position framework \((G, p)\) is infinitesimally distance rigid with at least one redundant edge, so (a) implies (c).

Theorem 38 proves the equivalence between clock rigidity and joint rigidity for \(d = 2\). So the topological method to establish clock rigidity in Corollary 31 is also applicable to joint rigidity.

The necessary and sufficient condition in Theorem 38 cannot be extended to \(d = 3\). This follows as that statement (b) \(\Rightarrow\) (c) does not hold for \(d = 3\) since distance rigidity in \(\mathbb{R}^3\) with at least one redundant edge (whose minimum edge number is \(3n - 5\)) is not a necessary condition for bearing rigidity in \(\mathbb{R}^2\) with at least one redundant edge (whose minimum edge number is \(2n - 2\)). It can be seen that for \(n \geq 4, 2n - 2 < 3n - 5\). A cardinality argument can be used to show the contradiction. For \(3D\) applications, we establish the following theorem, which shows a sufficient condition for infinitesimal joint rigidity in \(\mathbb{R}^{3+2}\).

Theorem 39: Under Assumptions 1 and 2, for any generic clock configuration \(\varphi\), a position-clock framework \((D, \sigma)\) in \(\mathbb{R}^{3+2}\) with \(\sigma = [p^T, \varphi^T]^T\) is infinitesimally joint rigid if the position framework \((G, p)\) in \(\mathbb{R}^3\) is infinitesimally distance rigid and \(n \geq 4\), where \(G\) is the disoriented graph of \(D\).

Proof: By Theorem 37, we only need to show that \((G, \varphi)\) is infinitesimally clock rigid.

Since \((G, p)\) is infinitesimally distance rigid in \(\mathbb{R}^3\), there must exist a subgraph \(G_s = (V, E_s)\) with \(|E_s| = 3|V| - 6\), such that every subset of \(k \geq 3\) vertices spans at most \(3k - 6\) edges [4]. In other words, \(G_s\) can be formed from a triangle graph by Henneberg construction in \(\mathbb{R}^3\), including vertex addition in \(\mathbb{R}^3\) which adds a new vertex with three new incident edges, and edge splitting in \(\mathbb{R}^3\) which removes an existing edge and adds a new vertex with four new incident edges.

Now we can reconstruct the graph from a triangle graph by replacing the Henneberg operations in \(\mathbb{R}^3\) by the corresponding Henneberg operations in \(\mathbb{R}^2\), i.e., add one less edge in every operation (add two less edges if the edge to be removed does not exist). Then, for \(n \geq 4\), the resulting graph \(G'_s \subset G_s \subset G\) and \(G'_s\) is a Laman graph.
TABLE I
SUMMARY OF DIFFERENT RIGIDITY THEORIES

| Framework                      | Distance rigidity | Bearing rigidity | Clock rigidity | Joint rigidity |
|--------------------------------|-------------------|-----------------|----------------|---------------|
| \( (G, \varphi), \varphi_{i} \in \mathbb{R}^{d} \) | \( (G, \varphi), \varphi_{i} \in \mathbb{R}^{2} \) | \( (G, \varphi), \varphi_{i} \in \mathbb{R}^{d+2} \) |
| Measurement                    | Distance \( d_{ij} \) | Bearing \( b_{ij} \) | Timestamps \( T_{ij}^{L} \), \( T_{ij}^{C} \) | Timestamps \( T_{ij}^{L} \), \( T_{ij}^{C} \) |
| Invariant equality             | \( \|p_{i} - p_{j}\| = d_{ij} \) | \( \|p_{i} - p_{j}\| = b_{ij} \) | Equation (2) | Equation (9) |
| Rigidity matrix                | \( R_{d} \) | \( R_{b} \) | \( R_{c} \) | \( R_{c} \) |
| Trivial infinitesimal variations| Translations, rotations | Translations, scaling | Translation of \( \beta \), scaling | Translations of \( p \), rotations of \( p \), Translation of \( \beta \), scaling of \( \sigma \) |
| Infinitesimal rigidity         | \( \text{rank}(R_{d}) = \frac{dn}{2} \) | \( \text{rank}(R_{b}) = \frac{dn}{2} - 1 \) | \( \text{rank}(R_{c}) = \frac{d(n-1)}{2} \) | \( \text{rank}(R_{c}) = \frac{d(n-1)}{2} \) |

Minimum rigid graph for \( d = 2 \)  
Laman graph  
Laman graph with one redundant edge

With generic \( \varphi \), a clock framework \( (G, \varphi) \) is infinitesimally distance rigid if and only if a subgraph of \( G \) is a Laman graph [25]. So \( (G, \varphi) \) is infinitesimally distance rigid. Since for \( n \geq 4, G_{n} \subset G \), there must exist at least one redundant edge for distance rigidity. By Theorem 28, the clock framework \( (G, \varphi) \) is infinitesimally clock rigid.

The analysis and results of joint rigidity share many similarities with distance, bearing, and clock rigidity theory. Their parallels are summarized in Table I.

VI. JOINT POSITION AND CLOCK ESTIMATION

Clock (Joint) rigidity theory shows the graph properties with which a TOA-based UWB sensor network can be determined up to some trivial variations, i.e., localizable and clock synchronizable. This section studies how these theories are exercised in the corresponding estimation problems and demonstrates through simulations.

A. Clock Estimation

Clock estimation aims to provide clock parameter estimates for converting the time of a local clock to that of a common clock. Let \( \hat{\varphi} \) be an estimation of the true clock configuration \( \varphi \). We consider the estimation error \( e_{c}(\hat{\varphi}, \varphi) = F_{c}(S(\varphi), \hat{\varphi}) - F_{c}(S(\varphi), \varphi) \), where \( F_{c} \) is the clock function defined in (4). Since the clock configuration of the network is assumed to be constant, by (5), \( F_{c}(S(\varphi), \varphi) = 0 \). We write the estimation error as \( e_{c}(\hat{\varphi}) = F_{c}(S(\varphi), \hat{\varphi}) \) for simplicity.

The objective of the clock estimation can be stated as the minimization of the following function:

\[
P_{1}(\hat{\varphi}) = \frac{1}{2} \| e_{c}(\hat{\varphi}) \|^{2} = \frac{1}{2} \sum_{\{v_{i}, v_{j}\} \in \mathcal{E}} e_{c_{ij}}(\hat{\varphi})^{2} \tag{11}
\]

where \( e_{c_{ij}}(\hat{\varphi}) = \hat{\alpha}_{ij}T_{ij}^{L} + \hat{\beta}_{ij} - \hat{\alpha}_{ij}T_{ij}^{C} - \hat{\beta}_{ij} \). The minimization of (11) can be obtained by the gradient descent method

\[
\dot{\hat{\varphi}} = -k_{g} \frac{\partial P_{1}(\hat{\varphi})}{\partial \hat{\varphi}} = -k_{g} F_{c}(\hat{\varphi})^{T} e_{c}(\hat{\varphi})
\]

where \( k_{g} \) is a positive gain. The clock estimator at the \( i \)-th node is

\[
\dot{\hat{\varphi}}_{i} = -k_{g} \sum_{v_{j} \in \mathcal{N}_{i}} \left( \hat{\alpha}_{ij}T_{ij}^{L} + \hat{\beta}_{ij} - \hat{\alpha}_{ij}T_{ij}^{C} - \hat{\beta}_{ij} \right) \left[ T_{ij}^{L} \right]^{T}
\]

which is distributed around \( \hat{\alpha}_{i}, \hat{\beta}_{i}, \) and \( T_{ij}^{L} \) are locally available or measured at node \( i \) and \( \hat{\alpha}_{j}, \hat{\beta}_{j}, \) and \( T_{ij}^{C} \) can be obtained at node \( j \), then transmitted using one-hop communication for every \( v_{i} \in \mathcal{N}_{i} \).

If a clock framework \( (G, \varphi) \) is infinitesimally clock rigid, then for any sufficiently small neighborhood around the true clock configuration \( \varphi \), the clock estimate \( \hat{\varphi} \) converges to a set where \( P_{1}(\hat{\varphi}) = 0 \). This follows from LaSalle’s invariance principle and the semidefiniteness of \( R_{c}(\hat{\varphi})^{T} R_{c}(\hat{\varphi}) \). So the clock estimate \( \hat{\varphi} \) must be a trivial variation of the true configuration (a translation of clock offset configuration \( \beta \) and a scaling of the entire clock framework), i.e., the estimated \( \hat{\varphi} \) should reach a clock configuration such that

\[
\hat{\varphi} = k_{s} \varphi + 1_{n} \otimes [0,k_{\beta}]^{T}
\]

where \( k_{s} \) is the scaling factor and \( k_{\beta} \) is the translation factor of \( \beta \). At global time \( t \), the estimated \( \hat{\varphi} \) converts the time \( t_{i} \) of the local clock to the time \( \hat{t} \) of a common virtual clock for any \( v_{i} \in \mathcal{V} \) since

\[
\hat{t} = \hat{\alpha}_{i}t_{i} + \hat{\beta}_{i} = k_{s} \alpha_{i}t_{i} + k_{s} \beta_{i} + k_{\beta} = k_{s}t + k_{\beta} \tag{12}
\]

which realizes clock synchronization over the network. The clock skew and clock offset of the common virtual clock with respect to the global clock are determined by the clock estimate’s scaling and translation factors, respectively.

Simulation results are shown in Fig. 11. The clock framework \( (K_{4}, \varphi) \) is infinitesimally clock rigid by Example 32. Given a
random initial configuration in the sufficiently small neighborhood of true configuration, the estimated clock configuration is a trivial variation of the true clock parameters.

**B. Joint Position and Clock Estimation**

Joint position and clock estimation in an anchor-free network aims to simultaneously determine the shape of the network and synchronize the network clocks to a common clock. Let \( \hat{\sigma} = [\hat{p}^T, \hat{\varphi}^T]^T \) be an estimation of the true position-clock configuration \( \sigma = [p^T, \varphi^T]^T \) and denote the estimation error as \( e(\hat{\sigma}) \) with elements \( e_{ij}(\hat{\sigma}) = \|\hat{p}_i - \hat{p}_j\|^2 - \sigma_{ij}^2(\hat{\alpha}_{ij}T_{(i,j)}^j + \hat{\beta}_j - \alpha_{ij}T_{(i,j)}^i - \hat{\beta}_i)^2 \) for all \( (i, j) \in E_D \). Following from (9), the estimation objective function is

\[
P_2(\hat{\sigma}) = \frac{1}{4}\|e(\hat{\sigma})\|^2 = \frac{1}{4} \sum_{(i, j) \in E_D} e_{ij}(\hat{\sigma})^2.
\]  

(13)

The objective function is equal to zero if and only if \( e_{ij}(\hat{\sigma}) = 0 \) for all \( (i, j) \in E_D \). The minimization of (13) can be obtained by the gradient descent method

\[
\dot{\hat{\sigma}} = -K_g \frac{\partial P_2(\hat{\sigma})}{\partial \hat{\sigma}} = -K_g R_{cd}(\hat{\sigma})^T e(\hat{\sigma})
\]

where \( K_g \) is a diagonal matrix whose diagonal entries are positive gains for the position and clock variables, respectively. The position and clock are usually at different measurement scales. In order to determine a suitable step size for the gradient descent method, we use a diagonal matrix \( K_g \) to choose appropriate gains for position and clock terms. The joint estimator at the \( i \)th node is

\[
\dot{\hat{\sigma}}_i = -K_{g,i} \sum_{(i, j) \in E_D} e_{ij}(\hat{\sigma}) \left[(\hat{p}_i - \hat{p}_j)^T, c\hat{d}_{ij}T_{(i,j)}^i, c\hat{d}_{ij} \right]^T
\]

\[
-\hat{K}_{g,i} \sum_{(i, j) \in E_D} e_{ji}(\hat{\sigma}) \left[(\hat{p}_i - \hat{p}_j)^T, -c\hat{d}_{ji}T_{(i,j)}^j, -c\hat{d}_{ji} \right]^T
\]

where \( K_{g,i} \) is a diagonal submatrix of \( K_g \) and \( \hat{d}_{ij} = c(\hat{\alpha}_{ij}T_{(i,j)}^i + \hat{\beta}_j - \alpha_{ij}T_{(i,j)}^i - \hat{\beta}_i) \). It is distributed since each node only requires the estimates from its neighbors and the timestamp measurements from itself and its neighbors.

If a position-clock framework \((D, \sigma)\) is infinitesimally joint rigid then in any sufficiently small neighborhood of \( \sigma \), the position-clock estimate \( \hat{\sigma} \) converges to a set where \( P_2(\hat{\sigma}) = 0 \) following from LaSalle’s invariance principle and the semidefiniteness of \( R_{cd}(\hat{\sigma})^T R_{cd}(\hat{\sigma}) \). The position-clock estimate \( \hat{\varphi} \) must be a trivial variation of the true position-clock configuration (translation and rotation of \( \rho \), translation of \( \beta \), and a scaling of entire position-clock framework). Therefore, in 2-D space, for example, the estimated \( \hat{\sigma} = [\hat{p}^T, \hat{\varphi}^T]^T \) should reach a position-clock configuration such that

\[
\hat{\rho} = k_\rho p + k_r(I_n \otimes J_2)p + 1_n \otimes k_i
\]

\[
\hat{\varphi} = k_\varphi + 1_n \otimes [0, k_\beta]^T
\]

(14)

where \( k_\rho \) is the scaling factor, \( k_r \) is the rotation factor, \( k_i \in \mathbb{R}^2 \) is the translation factor of \( p \), and \( k_\beta \) is the translation factor of \( \beta \). The estimate \( \hat{\varphi} \) uniquely determines the shape of the network since the position variations only include translation, rotation, and scaling. The estimate \( \hat{\varphi} \) enables the network to synchronize to a common virtual clock due to (12).

Under Assumption 1, by Theorem 38, the estimated \( \hat{\sigma} \) in 2-D space follows (14) if and only if the disoriented graph of \( D \) is generically distance rigid with at least one redundant edge. Simulation results are shown in Fig. 12. The position-clock framework \((K_4, \sigma)\) is infinitesimally joint rigid in \( \mathbb{R}^{2+2} \) and the variation of the resulting estimate is a combination of translation and rotation of \( \rho \), translation of \( \beta \), and scaling of the entire framework, where the scaling is obvious for clock in Fig. 12(b), whereas not obvious for position in Fig. 12(a) due to the figure scale.

**VII. Conclusion**

In this article, we proposed a clock rigidity theory for TOA-based UWB sensor network, showing that a clock framework with certain graph properties can be uniquely determined up to some trivial variations (a shift on clock offset and a skew on all clock parameters) given the TOA timestamp measurements. We also showed that a clock framework is infinitesimally clock rigid if and only if its underlying graph is generically bearing rigid in \( \mathbb{R}^2 \) with at least one redundant edge, providing a topological method for the analysis of clock rigidity.
Building on the proposed clock rigidity theory, we studied the joint position and clock estimation problem. We similarly proved that a position-clock framework with certain graph properties can be uniquely determined up to some trivial variations corresponding to both position and clock. Clock estimation and joint position-clock estimation method have been proposed and validated in the simulations.

The estimation considered in this article is anchor free. To uniquely determine a position-clock framework without any trivial variation, in the future work, we will formulate the problem in the presence of anchors (absolute position and/or clock information) and investigate how to select necessary anchors based on joint rigidity theory. The proposed gradient-descent method forms a distributed estimator, but exhibits slow convergence. Faster distributed estimation methods is another meaningful direction to be studied in the future.

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