Integral representation for the eigenfunctions of quantum periodic Toda chain

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Abstract

Integral representation for the eigenfunctions of quantum periodic Toda chain is constructed for \(N\)-particle case. The multiple integral is calculated using the Cauchy residue formula. This gives the representation which reproduces the particular results obtained by Gutzwiller for \(N = 2, 3\) and \(4\)-particle chain. Our method to solve the problem combines the ideas of Gutzwiller and \(R\)-matrix approach of Sklyanin with the classical results in the theory of the Whittaker functions. In particular, we calculate Sklyanin’s invariant scalar product from the Plancherel formula for the Whittaker functions derived by Semenov-Tian-Shansky thus obtaining the natural interpretation of the Sklyanin measure in terms of the Harish-Chandra function.

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1 Introduction

This paper is devoted to construction of the integral representation for the eigenfunctions of the Hamiltonians for the quantum periodic Toda chain. Our work was initiated by the recent results of Smirnov [1] concerning the structure of matrix elements in periodic Toda chain. The key point of this approach is Sklyanin’s invariant scalar product [2] and we tried to interpret it as a part of the Plancherel formula which relates the scalar product of the eigenfunctions in original coordinate representation with those arising in the space of the separated variables. Such interpretation forces to combine the analytic method of Gutzwiller [3] with the algebraic approach of Sklyanin [2].

To be more precise, Gutzwiller [3] discovered an interesting phenomenon which understood nowadays as quite universal one. If one tries to find the eigenfunctions of the periodic Toda chain in terms of a formal series in (auxiliary) eigenfunctions of the open chain with unknown coefficients labelled by multiple index, then the multi-dimensional difference equation on the coefficients thus obtained can be factorized to the one-dimensional Baxter equations. This is what is called the quantum separation of variables. In this way Gutzwiller obtained the solution to the eigenvalue problem for the $N = 2, 3$ and 4-particle cases only due to the lack of appropriate explicit solution for an arbitrary open Toda chain. The next important step was done by Sklyanin [2]. He applied the R-matrix formalism to the present problem and simplified drastically the derivation of the Baxter equation to an arbitrary number of particles. The evident advantages of Sklyanin’s method are a wide area of application and effective way to investigate different structures, for example, the invariant scalar products. But, unfortunately, his purely algebraic method had no deal with the analytical properties of auxiliary eigenfunctions of the specific model and, therefore, some important information is inevitably lost to compare with the original Gutzwiller’s approach. For example, as a price for efficiency the measure is defined up to an arbitrary $\hbar$-periodic function in general and there is no natural way to fix this freedom uniquely; there is a fundamental question what is the underlying principle to determine the asymptotics and analytical properties of the solution to Baxter equation and so on. Moreover, the transition from initial coordinate-dependent wave function to those in separated variables...
is somewhat obscured. The advantage of Gutzwiller approach consists of explicit information concerning the analytical properties of auxiliary eigenfunctions. As a consequence, it gives the possibility to calculate many fine tuning quantities, for example the numerical coefficient in Baxter equation, the asymptotics of its solutions etc. Therefore, there is a natural problem to generalize the Gutzwiller results to an arbitrary $N$-particle Toda chain and to extend them to other models. In particular, the investigation of algebraic origin of the auxiliary functions arising in other integrable models, is of higher importance.

We consider the periodic Toda chain as a test example to future generalization in these directions. This is the main motivation of the present paper.

We start from well-known observation that the Whittaker functions [4]-[6] give the solution of the eigenvalue problem for the general open Toda chain [7, 8]. So it is naturally to use them to construct the wave functions for the periodic Toda chain in the spirit of Gutzwiller. As the main result we obtain the integral representation for the eigen functions of periodic Toda chain using intensively the analytical properties of Whittaker functions investigated in [4]-[6]. The second important counterpart of this representation is the explicit solution of the Baxter equation constructed by Pasquier and Gaudin [9]. We calculate the multiple integral by taking the residues thus obtaining another representation for the wave function which reproduces the particular results of Gutzwiller by putting $N = 2, 3$ and $N = 4$ in our general formula. We observe also that the Plancherel formula obtained by Semenov-Tian-Shansky [8] for the wavelets of Whittaker functions produce the natural way to calculate Sklyanin’s scalar product. It turns out that the Sklyanin measure is deeply connected with the Harish-Chandra function arising in the theory of Whittaker functions.

The paper is organized as follows.

In Section 2 we briefly describe $N$-particle quantum periodic Toda chain in the framework of $R$-matrix approach and formulate the spectral problem to be solved. The material here is quite standard.

The solution of this problem is described by two theorems stated in Section 3. This are the main results of the paper.

The next four sections are devoted to the building blocks which allows to prove the above theorems.

The solution for the open $N-1$-particle Toda chain in terms of Whittaker functions [4]-[8] is discussed in Section 4. We introduce Weyl invariant Whittaker functions by simple renormalization of the original ones and describe their properties using the classical results [4]-[6]. We calculate also the scalar product of such functions using the Plancherel formula derived in [8].

In Section 5 we follow the Gutzwiller approach [3] to construct the auxiliary functions which play the same role for the periodic Toda chain as the exponentials for the standard Fourier transform. We generalize the Gutzwiller solutions to an arbitrary $(N-1)$-particle solution in terms of Weyl invariant Whittaker functions and calculate the action of the diagonal operators of the monodromy matrix on the auxiliary functions thus obtaining the functional relations which generalize the corresponding relations for the Macdonald function.

In Section 6 we reformulate the spectral problem using the (generalized) Fourier transform to the so-called $\gamma$-representation which is an analytic version of the algebraic approach by Sklyanin [2]. The analytical properties of the auxiliary function play a crucial role here. We show that the wave function in $\gamma$-representation is an entire function with definite asymptotics and it satisfies to
multi-dimensional Baxter equation. We derive the Plancherel formula thus rigorously obtaining the scalar product in $\gamma$-variables introduced in [2]. As by-product, we obtain the connection of the Harish-Chandra function [4]–[6] with the Sklyanin measure.

The solution to the Baxter equation is outlined in Section 7. We essentially use the approach by Gutzwiller [3] and especially those by Pasquier and Gaudin [9].

In Section 8 the proofs of the theorems stated in Section 3 are presented.

2 Periodic Toda chain (description of the model)

The quantum $N$-periodic Toda chain is a multi-dimensional eigenvalue problem with $N$ mutually commuting Hamiltonians $H_k(x_0, p_0; \ldots; x_{N-1}, p_{N-1})$, $(k = 1, \ldots, N)$ where the simplest Hamiltonians have the form

$$H_1 = \sum_{k=0}^{N-1} p_k$$

$$H_2 = \sum_{k<m} p_k p_m - \sum_{k=0}^{N-1} e^{x_k-x_{k+1}}$$

$$H_3 = \sum_{k<m<n} p_k p_m p_n + \ldots$$

etc. and the phase variables $x_k, p_k$ satisfy to commutation relations $[x_k, p_m] = i\hbar \delta_{km}$. This system can be nicely described using the $R$-matrix approach [2]. It is well known that the Lax operator

$$L_n(\lambda) = \begin{pmatrix} \lambda - p_n & e^{-x_n} \\ -e^{x_n} & 0 \end{pmatrix}$$

satisfies the commutation relations

$$R(\lambda - \mu)(L_n(\lambda)) \otimes I)(I \otimes L_n(\mu)) = (I \otimes L_n(\mu))(L_n(\lambda) \otimes I)R(\lambda - \mu)$$

where

$$R(\lambda) = I \otimes I + \frac{i\hbar}{\lambda} P$$

is a rational $R$-matrix. The monodromy matrix

$$T_N(\lambda) \overset{\text{def}}{=} L_{N-1}(\lambda) \ldots L_0(\lambda) \equiv \begin{pmatrix} A(\lambda) & B(\lambda) \\ C(\lambda) & D(\lambda) \end{pmatrix}$$

satisfies the analogous equation

$$R(\lambda - \mu)(T(\lambda) \otimes I)(I \otimes T(\mu)) = (I \otimes T(\mu))(T(\lambda) \otimes I)R(\lambda - \mu)$$

From (2.6) it can be easily shown that the trace of the monodromy matrix

$$\hat{t}(\lambda) = A(\lambda) + D(\lambda)$$

(2.7)
satisfies to commutation relations $\hat{t}(\lambda), \hat{t}(\mu)] = 0$ and is a generating function for the Hamiltonians of the periodic Toda chain:

$$\hat{t}(\lambda) = \sum_{k=0}^{N} (-1)^k \lambda^{N-k} H_k$$

(2.8)

Let us consider the spectral problem

$$H_k \Psi_E = E_k \Psi_E \quad k = 1, \ldots, N$$

(2.9)

where $E \equiv (E_1, \ldots, E_N)$. In other words, $\Psi_E$ is an eigenfunction of the operator $\hat{t}(\lambda)$:

$$\hat{t}(\lambda) \Psi_E = t(\lambda; E) \Psi_E$$

(2.10)

where

$$t(\lambda; E) = \sum_{k=0}^{N} (-1)^k \lambda^{N-k} E_k$$

(2.11)

For the future convenience we shall denote the coordinate dependence of the solutions as $\Psi_E(x_0, x)$ thus selecting $x_0$ from all other coordinates $x = (x_1, \ldots, x_{N-1})$. The reason for doing this will be clear below. Evidently, the wave function has the following structure:

$$\Psi_E(x_0, x) = \tilde{\Psi}_E(x_0 - x_1, \ldots, x_{N-2} - x_{N-1}) \exp \left\{ \frac{i}{\hbar} \sum_{k=0}^{N-1} x_k \right\}$$

(2.12)

The main goal is to find the solution to (2.10) such that $\tilde{\Psi}_E \in L^2(\mathbb{R}^{N-1})$. In equivalent terms, we impose the requirement

$$\int f(E_1) \Psi_E(x_0, x) dE_1 \in L^2(\mathbb{R}^{N})$$

(2.13)

for any smooth function $f(y), (y \in \mathbb{R})$ with finite support.

### 3 Main results

**Theorem 3.1** The solution to the spectral problem (2.10), (2.13) can be represented as the integral over real variables $\gamma = (\gamma_1, \ldots, \gamma_{N-1})$ in the following form:

$$\Psi_E(x_0, x) = \frac{(2\pi \hbar)^{-N}}{(N-1)!} \int_{\mathbb{R}^{N-1}} \mu^{-1}(\gamma) C(\gamma; E) \Psi_{\gamma,E_1}(x_0, x) d\gamma$$

(3.1)

where

(i) The function $\mu(\gamma)$ is defined by the formula

$$\mu(\gamma) = \prod_{j<k} \left| \Gamma\left(\frac{\gamma_j - \gamma_k}{i\hbar}\right) \right|^2$$

(3.2)
(ii) The function $C(\gamma; E)$ is the solution of multi-dimensional Baxter equation in the Pasquier-Gaudin form \[ C(\gamma; E) = \prod_{j=1}^{N-1} \frac{c_+^{(\gamma_j; E)} - \xi(E)c_-^{(\gamma_j; E)}}{\prod_{k=1}^{N} \sinh \frac{\pi}{\hbar}(\gamma_j - \delta_k(E))} \] where the entire functions $c_\pm^{(\gamma)}$ are two Gutzwiller solutions \[ t(\gamma; E)c^{(\gamma; E)} = i^{-N}c(\gamma + i\hbar; E) + i^Nc(\gamma - i\hbar; E) \] and the parameters $\xi(E), \delta = (\delta_1(E), \ldots, \delta_N(E))$ satisfy the Gutzwiller conditions (the energy quantization) \[ \delta_j(E) \neq \delta_k(E), \quad \delta_j(E) \neq \delta_k(E) \] \[ (3.3) \]

(iii) The wave function $\Psi_{\gamma,E_1}(x_0, x)$ is defined in terms of Whittaker function $w(x; \gamma)$ \[ \Psi_{\gamma,E_1}(x_0, x) = \hbar^{-2i(\gamma, \rho)/\hbar} \prod_{j<k} \pi^{-1/2} \Gamma\left(\frac{\gamma_j - \gamma_k}{i\hbar} + \frac{1}{2}\right) w(x; \gamma) \exp\left\{ \frac{i}{\hbar}\left(E_1 - \sum_{k=1}^{N-1} \gamma_k\right)x_0 \right\} \] \[ (3.5) \]

The function (3.5) is symmetric under the permutation of $\gamma$-variables 1.

The multiple integral (3.1) can be explicitly evaluated. Let $y = (y_1, \ldots, y_N)$ be an arbitrary vector in $\mathbb{R}^N$. We denote $y^{(s)} \equiv (y_1, \ldots, y_{s-1}, y_{s+1}, \ldots, y_N) \in \mathbb{R}^{N-1}$

**Theorem 3.2** Assuming that $\delta_j(E) \neq \delta_k(E)$, the solution (3.1) can be written (up to unessential numerical factor) in equivalent form

$$
\Psi_E(x_0, x) = \sum_{s=1}^{N} (-1)^{N-s} \sum_{n^{(s)} \in \mathbb{Z}^{N-1}} \Delta(\delta^{(s)} + i\hbar n^{(s)}) C_+^{(\delta^{(s)} + i\hbar n^{(s)}, E_1)}(x_0, x) \tag{3.6}
$$

where

$$
C_+^{(\gamma)} = \prod_{j=1}^{N-1} c_+^{(\gamma_j; E)} \tag{3.7}
$$

and $\Delta(\gamma) = \prod_{j>k} (\gamma_j - \gamma_k)$ is the Vandermonde determinant.

**Remark 3.1** For $N = 2, 3$ and $N = 4$ particle Toda chain the formula (3.6) reproduces the results obtained by Gutzwiller [3].

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1 In (3.5) we use the notation $(\gamma, \rho) \equiv \frac{1}{2} \sum_{k=1}^{N-1} (N-2k)\gamma_k$.
4  GL(\(N-1, \mathbb{R}\)) Toda chain and Whittaker function

It can be easily shown that the operator

\[
B(\lambda) = e^{-x_0} \sum_{k=0}^{N-1} (-1)^k \lambda^{N-k-1} h_k(x_1, p_1; \ldots; x_{N-1}, p_{N-1}) \tag{4.1}
\]

is the generating function for the Hamiltonians \(h_k\) of GL(\(N-1, \mathbb{R}\)) Toda chain \(^2\). In particular, the simplest Hamiltonians have the form

\[
h_1 = \sum_{k=1}^{N-1} p_k \tag{4.2}
\]

\[
h_2 = \sum_{j<k} p_j p_k - \sum_{k=1}^{N-2} e^{x_k-x_{k+1}}
\]

Let \(\gamma = (\gamma_1, \ldots, \gamma_{N-1})\) be the set of real parameters. We consider the spectral problem for GL(\(N-1, \mathbb{R}\)) Toda chain

\[
h_k \psi_\gamma(x) = \sigma_k(\gamma) \psi_\gamma(x) \quad k = 1, \ldots, N-1 \tag{4.3}
\]

where \(\sigma_k(\gamma)\) are elementary symmetric functions. In equivalent terms, \(\psi_\gamma(x)\) satisfies to the equation

\[
B(\lambda) \psi_\gamma(x) = e^{-x_0} \prod_{m=1}^{N-1} (\lambda - \gamma_m) \psi_\gamma(x) \tag{4.4}
\]

As a trivial corollary,

\[
B(\gamma_j) \psi_\gamma(x) = 0 \quad \forall \gamma_j \in \gamma \tag{4.5}
\]

**Remark 4.1** The equations (4.3) are an analog of the notion of the "operator zeros" introduced by Sklyanin \(^3\).

Obviously, in the asymptotic region \(x_{k+1} \gg x_k\), \((k = 1, \ldots, N-2)\) all potentials vanish and the solution to (4.3) is a superposition of plane waves. The main goal is to find the solution to (4.3) satisfying the following properties:

(i) The solution vanishes very rapidly \(^3\) as \(x_k - x_{k+1} \to \infty\) for any given \(k\).

(ii) The function \(\psi_\gamma\) is Weyl-invariant, i.e. it is symmetric under any permutation

\[
\psi_{\gamma_1 \ldots \gamma_{k-1} \gamma_k \ldots} = \psi_{\gamma_1 \ldots \gamma_{k-1} \ldots \gamma_k} \tag{4.6}
\]

\(^2\)The commutativity of the operators \(h_k\) follows from the commutation relation \([B(\lambda), B(\mu)] = 0\) which is encoded in (2.6).

\(^3\) More precisely, \(\psi_\gamma(x) \sim \exp\{-\frac{2}{\hbar} e^{(x_k-x_{k+1})/2}\} \) as \(x_k - x_{k+1} \to \infty\).
The property (i) defines the unique solution (up to common $\gamma$-dependent factor). It is called the Whittaker function $w(x;\gamma)$ whose analytical and invariant properties were studied in [4]-[6]. In the context of the open Toda chain the Whittaker functions have been appeared for the first time in [7, 8]. The requirement of Weyl invariance (4.6) is very convenient for our purposes. It turns out that symmetric function $\psi_\gamma$ is most suitable to construct the solution for the periodic Toda chain.

For $GL(N-1,\mathbb{R})$ Toda chain the Whittaker function has the following integral representation

$$w(x;\gamma) = e^{\frac{1}{h}(\gamma,x)} \int_{-\infty}^{\infty} dz \frac{\prod_{k=1}^{N-2} \Delta_k(\gamma_{k+1} - \gamma_k - \frac{1}{2})}{\prod_{m=1}^{N-1} \Delta_m(\gamma_{m+1} - \gamma_m - \frac{1}{2})} \exp \left\{ \frac{2i}{h} \sum_{k=1}^{N-2} e^{(x_k-x_{k+1})/2} \right\}$$

(4.7)

where the integration is taken over the upper triangular $(N-1) \times (N-1)$ matrix $z = ||z_{jk}||$ with unit diagonal and $\Delta_k(z)$ are the principal minors of the matrix $zz^t$. Finally, $(\ldots)$ is the standard scalar product in $\mathbb{R}^{N-1}$.

In particular, taking $N = 3$ and using the integral representation for the Macdonald function $K_\nu(y)$, one obtains the appropriate solution

$$w(x;\gamma) = 2\sqrt{\pi} \frac{\frac{h}{\gamma_1-\gamma_2}}{\Gamma\left(\frac{h}{\gamma_1-\gamma_2} + \frac{1}{2}\right)} e^{\frac{h}{\gamma_1-\gamma_2}(\gamma_1+\gamma_2)(x_1+x_2)} K_{\frac{h}{\gamma_1-\gamma_2}}\left(\frac{2}{\gamma_1-\gamma_2}(x_1-x_2)/2\right)$$

(4.8)

of the spectral problem

$$(p_1 + p_2)w = (\gamma_1 + \gamma_2)w$$

$$\left\{p_1p_2 - e^{x_1-x_2}\right\}w = \gamma_1\gamma_2 w$$

(4.9)

Indeed, $K_\nu(y) \sim y^{-1/2}e^{-y}$ as $y \to \infty$ and property (i) is fulfilled.

All essential theorems for the classical Whittaker function [4]-[6] hold in the present case as well. We represent four lemmas below without proof.

**Lemma 4.1** For any $s \in W$

$$w(x; s\gamma) = \mathcal{M}(s; \gamma)w(x; \gamma)$$

(4.10)

where for the permutation $s_{jk} : \gamma_j \leftrightarrow \gamma_k (j < k)$

$$\mathcal{M}(s_{jk}; \gamma) = \prod_{m=j+1}^{k} \frac{h^{2(\gamma_m-\gamma_j)/h}}{\Gamma\left(\gamma_m-\gamma_j + \frac{1}{2}\right)\Gamma\left(\frac{h}{\gamma_m-\gamma_j} + \frac{1}{2}\right)} \prod_{m=j+1}^{k} \frac{h^{2(\gamma_m-\gamma_k)/h}}{\Gamma\left(\gamma_m-\gamma_k + \frac{1}{2}\right)\Gamma\left(\frac{h}{\gamma_m-\gamma_k} + \frac{1}{2}\right)}$$

(4.11)

4 The function (4.7) differs from the corresponding solution for the $SL(N-1,\mathbb{R})$ case by the factor

$$\exp \left\{ \frac{i}{(N-1)h} \sum_{j=1}^{N-1} \gamma_j \cdot \sum_{k=1}^{N-1} x_k \right\}$$

5 The notations and results are extracted from the papers [6].
The coefficients \( \mathcal{M}(s; \gamma) \) satisfy to relations
\[
\mathcal{M}(s_1 s_2; \gamma) = \mathcal{M}(s_1; s_2 \gamma) \mathcal{M}(s_2; \gamma) \tag{4.12}
\]
\[
\mathcal{M}(s; \gamma) \overline{\mathcal{M}(s; \gamma)} = 1 \quad \text{Im} \gamma_k = 0 \tag{4.13}
\]
(where \( \overline{z} \) always denote the complex conjugation of \( z \)).

\[\tag{4.12}\]

**Lemma 4.2** The Whittaker function \( w(x; \gamma) \) can be analytically continued to an entire function of \( \gamma \in \mathbb{C}^{N-1} \).

**Lemma 4.3** There is a unique solution \( v(x; \gamma) \) to (4.3) with the asymptotics
\[
v(x; \gamma) = e^{i \overline{\gamma} x} + O \left( \max \left\{ e^{x_{k+1} - x_k} \right\}^{N-2} \right) \tag{4.14}
\]
in the region \( x_{k+1} \gg x_k, \ (k = 1, \ldots, N-2) \). Let \( W \) be the Weyl group (the permutation group of the variables \( \gamma \)). The functions \( v(x; s \gamma), \ (s \in W) \) form a basis of the solutions to the spectral problem (4.3).

**Lemma 4.4** The Whittaker function (4.7) has the following expansion in terms of basis functions \( v(x; s \gamma), \ (s \in W) \):
\[
w(x; \gamma) = \sum_{s \in W} c(s \gamma) \mathcal{M}^{-1}(s; \gamma) v(x; s \gamma) \tag{4.15}
\]
where
\[
c(\gamma) = \prod_{j<k} B \left( \frac{\gamma_j - \gamma_k}{i \hbar}, \frac{1}{2} \right) \tag{4.16}
\]
is (non-normalized) Harish-Chandra function [6].

Now we are able to define the Weyl symmetric solution and to describe its properties. Let \( \rho \) is a half-sum of positive roots of \( sl(N-1, \mathbb{R}) \) written in standard basis \( \{ e_k \} \in \mathbb{R}^{N-1} \):
\[
\rho = \sum_{k=1}^{N-1} \rho_k e_k; \quad \rho_k = \frac{1}{2} (N-2k) \tag{4.17}
\]

**Lemma 4.5** The function
\[
\psi(\gamma)(x) = \hbar^{-\frac{2i}{(\gamma, \rho)}/h} \prod_{j<k} \pi^{-1/2} \Gamma \left( \frac{\gamma_j - \gamma_k}{i \hbar} + \frac{1}{2} \right) w(x; \gamma) \tag{4.18}
\]

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6 In (4.10) \( B(z, w) \equiv \frac{\Gamma(z) \Gamma(w)}{\Gamma(z + w)} \).
(i) is Weyl invariant.

(ii) it can be analytically continued to an entire function of $\gamma \in \mathbb{C}^{N-1}$.

(iii) in terms of the basis $v(x; s\gamma)$, $(s \in W)$

$$
\psi_\gamma(x) = \sum_{s \in W} N(s\gamma) a(s\gamma) v(x; s\gamma)
$$

where $N(\gamma) = \hbar^{-2i(\gamma, \rho)/\hbar}$ and

$$
a(\gamma) = \prod_{j<k} \Gamma\left(\frac{\gamma_j - \gamma_k}{i\hbar} + \frac{1}{2}\right)
$$

(iv) for real $\gamma$ variables the function $\psi_\gamma$ obeys the asymptotics

$$
\psi_\gamma(x) \sim |\gamma_k|^{\frac{2-N}{2}} \exp \left\{ - \frac{\pi}{2\hbar} (N-2)|\gamma_k| \right\}
$$

as $|\gamma_k| \to \infty$.

**Proof.** The Weyl invariance is a direct corollary of the formulae (4.10), (4.11).

To prove (ii), one should note that the poles coming from $\Gamma$ functions in the r.h.s. of (4.18) are cancelled by the corresponding zeros of the Whittaker function $w(x; \gamma)$. Indeed, the function $\mathcal{M}(s_{jk}; \gamma)$ has the poles at $\gamma_j - \gamma_k = -i\hbar(n + \frac{1}{2})$, $(n \in \mathbb{N})$. But the function $w(x; s_{jk} \gamma)$ is an entire function according to Lemma 4.2. Hence, from (4.10) it is clear that $w(x; \gamma)$ vanishes exactly at the same points.

To prove (iii), let

$$
b(\gamma) \equiv \prod_{j<k} \pi^{-1/2} \Gamma\left(\frac{\gamma_j - \gamma_k}{i\hbar} + \frac{1}{2}\right)
$$

for brevity. Then the Harish-Chandra function (4.16) is written identically as a ratio

$$
c(\gamma) = \frac{a(\gamma)}{b(\gamma)}
$$

while the definition (4.18) reads

$$
\psi_\gamma(x) \equiv N(\gamma)b(\gamma)w(x; \gamma)
$$

Using the Weyl invariance $\psi_{s\gamma} = \psi_\gamma$, one easily finds from (4.10) and (4.24)

$$
\mathcal{M}(s; \gamma) = \frac{N(\gamma)}{N(s\gamma)} \frac{b(\gamma)}{b(s\gamma)}
$$

Hence, the expansion (4.19) is a simple corollary of (4.13) and definition (4.24).

The proof of (iv) follows from the Stirling formula for $\Gamma$-functions and from the calculation of the integrals in (4.7) by the stationary phase method.

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7Actually, one can prove the exponential decreasing $\psi_\gamma \sim \exp \left\{ - \frac{\pi}{2\hbar} (N-2)|\text{Re}\gamma_k| \right\}$ as $|\text{Re}\gamma_k| \to \infty$ in any finite strip of the complex plane.
Lemma 4.6 Let $\gamma = (\gamma_1, \ldots, \gamma_{N-1})$ and $\gamma' = (\gamma'_1, \ldots, \gamma'_{N-1})$ be the real parameters. Then
\[
\int_{-\infty}^{\infty} \psi_{\gamma'}(x) \overline{\psi}_{\gamma}(x) dx = (2\pi \hbar)^{N-1} |a(\gamma)|^2 \sum_{s \in W} \delta(s \gamma - \gamma') \tag{4.26}
\]

Proof. This is a consequence of the Plancherel formula proved in [8] for the $SL(n, \mathbb{R})$ case. The scalar product (4.26) results from Semenov-Tian-Shansky formula taking into account the relations (4.12), (4.13) and (4.25).

Conjecture 4.1 The Whittaker functions obey the completeness condition
\[
\int_{-\infty}^{\infty} |a(\gamma)|^{-2} \psi_{\gamma}(x) \overline{\psi}_{\gamma}(y) d\gamma = (N-1)! (2\pi \hbar)^{N-1} \delta(x - y) \tag{4.27}
\]

5 Auxiliary functions

Let us introduce the auxiliary function
\[
\Psi_{\gamma, \epsilon}(x_0, x) \overset{\text{def}}{=} e^{i \frac{\bar{\hbar}}{\hbar} x_0 (\epsilon - \sum_{m=1}^{N-1} \gamma_m)} \psi_{\gamma}(x) \tag{5.1}
\]
where $\epsilon$ is an arbitrary real parameter. Obviously, this function is the solutions of (4.3) or, in equivalent terms,
\[
B(\lambda) \Psi_{\gamma, \epsilon}(x_0, x) = e^{-x_0} \prod_{m=1}^{N-1} (\lambda - \gamma_m) \Psi_{\gamma, \epsilon}(x_0, x) \tag{5.2}
\]
In particular,
\[
B(\gamma_j) \Psi_{\gamma, \epsilon}(x_0, x) = 0 \quad \forall \gamma_j \in \gamma \tag{5.3}
\]
Further, by construction the function $\Psi_{\gamma, \epsilon}$ satisfies to the equation
\[
H_1 \Psi_{\gamma, \epsilon} = \epsilon \Psi_{\gamma, \epsilon} \tag{5.4}
\]
The scalar product of the auxiliary functions follows from (4.26):
\[
\int_{-\infty}^{\infty} \overline{\Psi}_{\gamma', \epsilon'}(x_0, x) \Psi_{\gamma, \epsilon}(x_0, x) dx_0 dx = (2\pi \hbar)^N \mu(\gamma) \delta(\epsilon - \epsilon') \sum_{s \in W} \delta(s \gamma - \gamma') \tag{5.5}
\]
where
\[
\mu(\gamma) = \prod_{j < k} \left| \Gamma \left( \frac{\gamma_j - \gamma_k}{\bar{\hbar}} \right) \right|^2 \tag{5.6}
\]
Moreover, the auxiliary functions satisfy to completeness condition
\[
\int_{-\infty}^{\infty} \mu^{-1}(\gamma) \Psi_{\gamma, \epsilon}(x_0, x) \overline{\Psi}_{\gamma, \epsilon}(y_0, y) d\gamma de = (N-1)! (2\pi \hbar)^N \delta(x_0 - y_0) \delta(x - y) \tag{5.7}
\]

\footnote{We confirm this conjecture by explicit calculations for $GL(2, \mathbb{R})$ chain. The general proof will be published elsewhere.}
provided the conjecture \((4.27)\) holds.

Certainly, the function \((5.1)\) does not satisfy the whole system \((2.10)\). Nevertheless, it possesses some remarkable properties which allow to construct the eigenfunctions of the periodic Toda chain.

First of all, the functions \(\Psi_{\gamma,\epsilon}\) and \(\Psi_{\gamma,\epsilon}^{\overline{\gamma}}\) can be extended to the entire functions of the complex variables \(\gamma \in \mathbb{C}^{N-1}\). This is an evident consequence of Lemma 4.5. We shall denote the corresponding analytic continuations by the same letters. Further, the following statement will be of importance below:

**Lemma 5.1** The action of the diagonal operators of the monodromy matrix \((2.3)\) on auxiliary function \(\Psi_{\gamma,\epsilon}(\gamma \in \mathbb{C}^{N-1})\) is defined by the formulae

\[
A(\gamma_j)\Psi_{\gamma,\epsilon} = i^N \Psi_{\gamma+i\hbar e_j,\epsilon} \\
D(\gamma_j)\Psi_{\gamma,\epsilon} = i^{-N} \Psi_{\gamma-i\hbar e_j,\epsilon}
\]

**Proof.** We consider the following commutation relations encoded in \((2.6)\):

\[
(\lambda - \mu + i\hbar)D(\mu)B(\lambda) = (\lambda - \mu)B(\lambda)D(\mu) + i\hbar D(\lambda)B(\mu) \\
(\lambda - \mu + i\hbar)A(\lambda)B(\mu) = (\lambda - \mu)B(\mu)A(\lambda) + i\hbar A(\mu)B(\lambda)
\]

(5.9)

(5.10)

Taking \(\mu = \gamma_j\) in \((5.9)\) and applying this operator identity to the function \(\psi_\gamma(x)\), one arrives to the relation

\[
B(\lambda)D(\gamma_j)\psi_\gamma(x) = e^{-x_0} (\lambda - \gamma_j + i\hbar) \prod_{m \neq j} (\lambda - \gamma_m) D(\gamma_j)\psi_\gamma(x)
\]

(5.11)

since \((1.5)\) holds \[7\]. Hence, \(D(\gamma_j)\psi_\gamma(x)\) is a solution to the spectral problem \((1.4)\) with the set of eigenvalues

\[
\gamma \rightarrow (\gamma_1, \ldots, \gamma_{j-1}, \gamma_j - i\hbar, \gamma_{j+1}, \ldots, \gamma_{N-1})
\]

(5.12)

i.e. one obtains the expansion

\[
D(\gamma_j)\psi_\gamma(x) = \sum_{s \in W} d_j(s; \gamma) v(x; s(\gamma - i\hbar e_j))
\]

(5.13)

due to completeness of the basis \(v(x; \gamma)\) \((s \in W)\). On the other hand, the function \(D(\gamma_j)\psi_\gamma(x)\) vanishes as \(x_k - x_{k+1} \rightarrow \infty\) for any given \(k\). Hence, the r.h.s. in \((5.13)\) is proportional to \(\psi_{\gamma-i\hbar e_j}\) because of uniqueness of the Whittaker function. Thus,

\[
D(\gamma_j)\psi_\gamma(x) = d_j(\gamma) \psi_{\gamma-i\hbar e_j}
\]

(5.14)

To find the coefficients \(d_j(\gamma)\), we consider the asymptotics in the region \(x_{k+1} \gg x_k\), \((k = 1, \ldots N - 2)\) where \((4.14)\) holds. Note that in this limit

\[
D(\lambda) = -e^{x_{N-1}-x_0} \left\{ \prod_{m=1}^{N-2} (\lambda - p_m) + O(\max\left\{e^{x_k-x_{k+1}}\right\}_{k=1}^{N-2}) \right\}
\]

(5.15)
Let $\gamma$ be the complex variables with $\text{Im} \gamma_k \leq 0$. Then it is easy to find that asymptotically

$$D(\gamma_j)\psi_\gamma(x) \sim i^{-N} e^{-x_0} \frac{\hat{\Psi}(\gamma_j - \lambda)x_{N-1}}{\hbar} \prod_{m \neq j} \Gamma\left(\frac{\gamma_m - \gamma_j + i\hbar}{i\hbar}\right) \times \psi_{\gamma_1, \ldots, \gamma_{j-1}, \gamma_{j+1}, \ldots, \gamma_{N-1}}(x_1, \ldots, x_{N-2})$$

(5.16)

where $\tilde{\psi}$ is the asymptotical value of $GL(N-2, \mathbb{R})$ Whittaker function. On the other hand, under above conditions

$$\psi_{\gamma - i\hbar e_j} \sim e^{\frac{\hat{\Psi}(\gamma_j - \lambda)x_{N-1}}{\hbar} - 2i(\gamma_j - \lambda)p_{N-1} / \hbar} \prod_{m \neq j} \Gamma\left(\frac{\gamma_m - \gamma_j + i\hbar}{i\hbar}\right) \times \psi_{\gamma_1, \ldots, \gamma_{j-1}, \gamma_{j+1}, \ldots, \gamma_{N-1}}(x_1, \ldots, x_{N-2})$$

(5.17)

The comparison of (5.16) and (5.17) gives $d_j(\gamma) = i^{-N} e^{-x_0}$. Hence, in the region $\text{Im} \gamma_k \leq 0$

$$D(\gamma_j)\psi_\gamma(x) = i^{-N} e^{-x_0} \psi_{\gamma - i\hbar e_j}(x)$$

(5.18)

and this relation can be obviously continued to the whole complex region. The formula (5.8b) follows then from the definition (5.1).

Of course, it is possible to prove (5.8a) according to the same reasoning. Actually, the relation (5.8a) is a corollary of (5.8b) since the quantum determinant of the monodromy matrix (2.5) is equal to 1. Applying the operator identity

$$A(\gamma_j - i\hbar)D(\gamma_j) - C(\gamma_j - i\hbar)B(\gamma_j) = 1$$

(5.19)

to the function $\Psi_{\gamma, \epsilon}$ and using the relations (5.3) and (5.8b), one arrives to (5.8a).

As a direct consequence of (5.8) and definition (2.7), for any $\gamma \in \mathbb{C}^{N-1}$

$$\hat{t}(\gamma_j)\Psi_{\gamma, \epsilon} = i^N \Psi_{\gamma + i\hbar e_j, \epsilon} + i^{-N} \Psi_{\gamma - i\hbar e_j, \epsilon}$$

$$\hat{t}^\dagger(\gamma_j)\Psi_{\gamma, \epsilon} = i^N \Psi_{\gamma + i\hbar e_j, \epsilon} + i^{-N} \Psi_{\gamma - i\hbar e_j, \epsilon}$$

(5.20)

where $\hat{t}(\lambda; x_0, p_0; \ldots; x_{N-1}, p_{N-1}) \equiv \hat{t}(\lambda; x_0, -p_0; \ldots; x_{N-1}, -p_{N-1}), (\lambda \in \mathbb{C})$ and the analytic continuation of the functions is understood.

Remark 5.1 The formula (5.2b) has been derived by Gutzwiller [5] using cumbersome explicit calculations for $n = 2$ and $n = 3$-particle open Toda chain.

To prove Theorem 5.1 one needs to determine the action of the operator $\hat{t}(\lambda) (\lambda \in \mathbb{C})$ on the function $\Psi_{\gamma, \epsilon}$.

Lemma 5.2

$$\hat{t}(\lambda)\Psi_{\gamma, \epsilon} =$$

$$= \left(\lambda - \epsilon + \sum_{j=1}^{N-1} \gamma_j \right) \prod_{m=1}^{N-1} (\lambda - \gamma_m) \Psi_{\gamma, \epsilon} + \sum_{j=1}^{N-1} \prod_{m \neq j} (\lambda - \gamma_m) \left(i^N \Psi_{\gamma + i\hbar e_j, \epsilon} + i^{-N} \Psi_{\gamma - i\hbar e_j, \epsilon}\right)$$

(5.21)
Proof. The function \( \hat{t}(\lambda)\Psi_{\gamma,\epsilon} \) is a polynomial in \( \lambda \) of order \( N \) with the leading terms
\[
\hat{t}(\lambda)\Psi_{\gamma,\epsilon} = (\lambda^N - \lambda^{N-1}\epsilon)\Psi_{\gamma,\epsilon} + O(\lambda^{N-2})
\]  
according to (5.4). Moreover, this polynomial is defined at the points \( \lambda = \gamma_j \), \( (j = 1, \ldots, N - 1) \) due to (5.20). Using the standard interpolation formula, one easily obtains (5.21). \( \blacksquare \)

Remark 5.2 The formula (5.21) is a coordinate version of the operator interpolation formula obtained in [2].

6 Periodic Toda chain in \( \gamma \)-representation

Let \( \Psi_E(x_0, x) \) be the fast decreasing solution of the problem (2.10). We define the function \( C(\gamma) \) by generalized Fourier transform:
\[
\delta(E_1 - \epsilon) C(\gamma; E) = \int_{-\infty}^{\infty} \Psi_E(x_0, x) \overline{\Psi}_{\gamma,\epsilon}(x_0, x) dx_0 dx
\]  
(6.1)

Theorem 6.1 The function \( C(\gamma) \) possesses the following properties:

(i) It is a symmetric function with respect to \( \gamma \)-variables.

(ii) It is an entire function of the variables \( \gamma \in \mathbb{C}^{N-1} \).

(iii) The function \( C(\gamma) \) obeys the asymptotics
\[
C(\gamma) \sim |\gamma_k|^{-N/2} e^{-\frac{\pi N|\gamma_k|}{2\hbar}}
\]  
as \( |\gamma_k| \to \infty \) along the real axes.

(iv) The function \( C(\gamma) \) satisfies the multi-dimensional Baxter equation
\[
t(\gamma_j; E)C(\gamma; E) = i^N C(\gamma + i\hbar e_j; E) + i^{-N} C(\gamma - i\hbar e_j; E)
\]  
where \( t(\gamma; E) \) is defined by (2.11).

Proof. The symmetricity of the function \( C(\gamma) \) is obvious. We present here only the sketch of the proof concerning the statements (ii) and (iii).

The statement (ii) follows from the assertion that the auxiliary function \( \Psi_{\gamma,\epsilon} \) is an entire one while the solution of the periodic chain vanishes very rapidly as \( |x_k - x_{k+1}| \to \infty \).

(iii) The asymptotics (6.2) is a combination of two factors. The first one comes from the asymptotics (4.21) which results from the Stirling formula for the \( \Gamma \)-functions in the definition (4.18).

10 Actually, the boundary conditions have the same importance here as the requirement of compact support in the theory of the analytic continuation for the usual Fourier transform.
The second factor $\sim |\gamma_k|^{-1}\exp\{-\pi|\gamma_k|/\hbar\}$ results from the stationary phase method while calculating the multiple integral including the Whittaker function (4.7). The calculation is based heavily on the exact asymptotics of the function $\Psi_{E}(x_0, x)$ as $|x_k - x_{k+1}| \to \infty$.

The proof of (iv) is simple. Using the definition (2.10) and integrating by parts (evidently, boundary terms vanish), one obtains

$$
\delta(E_1 - \epsilon) t(\gamma; E)C(\gamma) \equiv \int_{-\infty}^{\infty} \{\hat{t}(\gamma)\Psi_{E}(x_0, x)\} \overline{\Psi}_{\gamma,\epsilon}(x_0, x)dx_0dx =
$$

$$
= \int_{-\infty}^{\infty} \Psi_{E}(x_0, x) \{\hat{t}^+(\gamma)\overline{\Psi}_{\gamma,\epsilon}(x_0, x)\}dx_0dx
$$

(6.4)

Taking into account the relation (5.21), the Baxter equation (6.3) follows from definition (6.1).

Let us consider the following function

$$
\Psi_{E}(x_0, x) = \frac{(2\pi\hbar)^{-N}}{(N-1)!} \int_{-\infty}^{\infty} \mu^{-1}(\gamma) C(\gamma; E) \Psi_{\gamma,E}(x_0, x)d\gamma
$$

(6.5)

The integral (6.3) integral is correctly defined. Indeed, the measure

$$
\mu^{-1}(\gamma) = \prod_{j<k} \left|\Gamma(\frac{\gamma_j - \gamma_k}{i\hbar})\right|^{-2} =
$$

$$
= (\pi\hbar)^{-(N-1)(N-2)/2} \prod_{j<k} (\gamma_j - \gamma_k) \sinh \frac{\pi}{\hbar}(\gamma_j - \gamma_k)
$$

(6.6)

is an entire function. Therefore, there are no poles in the integrand. Moreover,

$$
\mu^{-1}(\gamma) \sim |\gamma_k|^{N-2} \exp\left\{\frac{\pi}{\hbar}(N-2)|\gamma_k|\right\}
$$

(6.7)

as $|\gamma_k| \to \infty$. Taking into account the asymptotics (1.21) and (6.2) one concludes that the integrand has the behavior $\sim |\gamma_k|^{-1}\exp\{-\pi|\gamma_k|/\hbar\}$ as $|\gamma_k| \to \infty$. Therefore, the integral (6.5) is convergent.

Using the scalar product (5.3), one can write the Plancherel formula

$$
(2\pi\hbar)^N \int_{-\infty}^{\infty} \overline{\Psi}_{E'}(x_0, x)\Psi_{E}(x_0, x)dx_0dx = \frac{1}{(N-1)!} \delta(E_1 - E'_1) \int_{-\infty}^{\infty} \mu^{-1}(\gamma) \overline{C}(\gamma; E')C(\gamma; E)d\gamma
$$

(6.8)

The integral in the r.h.s. of (6.3) is absolutely convergent due to asymptotics (1.22) and (6.7). Hence, the norm $||\Psi_{E}||$ is finite modulo $GL(1)$ $\delta$-function $\delta(E_1 - E'_1)$ (see corresponding factor in (2.12) which results to this function).

11The expression (6.3) is an inversion formula to (6.1) assuming the completeness condition (5.7). Otherwise one can only conclude that (6.1) is a corollary of (6.3) due to the scalar product (5.3).

12Let us stress again that (6.8) is deeply connected with the Plancherel formula for $SL(n, \mathbb{R})$ Toda chain derived in [8].
Remark 6.1 The scalar product in $\gamma$-variables appears for the first time in [4]. We see that the Sklyanin measure $\mu^{-1}(\gamma)$ is naturally connected with the Harish-Chandra function (4.23).

To prove that the function (6.5) satisfies to the spectral problem (2.10), one needs more details information concerning the asymptotics of function $C(\gamma; E)$: it should exponentially decrease in any finite strip of the complex plane $\gamma_k \in \mathbb{C}$ as $|\text{Re } \gamma_k| \to \infty$. Then (6.5) is a solution if $C(\gamma; E)$ satisfies the Baxter equation (6.3) (see the proof of Theorem 3.1 below). Up to now we can prove only the asymptotics (6.2) starting from the definition (6.1) (though the analytic continuation seems almost evident). Therefore, in the next section we describe the explicit solution to equation (6.3) [3, 9] which obviously possesses the required asymptotics.

7 Solution of Baxter equation

In this section the Pasquier-Gaudin solution [9] to the Baxter equation (6.3) is described which leads to refined derivation of Gutzwiller quantization condition [3].

We solve the equation (6.3) in the separated form

$$C(\gamma; E) = \prod_{j=1}^{N-1} c(\gamma_j; E)$$

(7.1)

Obviously, the "one particle" functions $c(\gamma)$ satisfy to the one-dimensional Baxter equation

$$t(\gamma; E)c(\gamma; E) = i^N c(\gamma + i\hbar; E) + i^{-N} c(\gamma - i\hbar; E)$$

(7.2)

where $t(\gamma; E)$ is, by definition, a polynomial $\prod_{k=1}^{N} (\gamma - \lambda_k(E))$ . We assume that the roots $\lambda_k(E)$ are real for the solutions with the finite norm [3]. We shall oftenly do not write the explicit dependence on energies for brevity.

Lemma 7.1 [3] The equation (7.2) admits two fundamental solutions

$$\tilde{c}_\pm(\gamma) = e^{-\frac{\pi N\gamma}{\hbar}} c_\pm(\gamma)$$

(7.3)

where

$$c_\pm(\gamma) = \frac{K_\pm(\gamma)}{\prod_{k=1}^{N} \left(1 + \frac{i}{\hbar}(\gamma - \lambda_k(E))\right)}$$

(7.4)

and $K_\pm(\gamma)$ be the following $N \times N$ determinants:

$$K_+(\gamma) = \begin{vmatrix}
1 & \frac{1}{t(\gamma)} & 0 & \cdots & \cdots \\
\frac{1}{t(\gamma+i\hbar)} & 1 & \frac{1}{t(\gamma+i\hbar)} & 0 & \cdots \\
0 & \frac{1}{t(\gamma+2i\hbar)} & 1 & \frac{1}{t(\gamma+2i\hbar)} & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots
\end{vmatrix}$$

(7.5a)

This can be demonstrated directly for $N = 2, 3$ particle chain.
\[ K^{-} (\gamma) \equiv \begin{vmatrix} \cdots & \cdots & \cdots & \cdots \\ \cdots & 0 & \frac{1}{t(\gamma-3i\hbar)} & \frac{1}{t(\gamma-3i\hbar)} & 0 \\ \cdots & \frac{1}{t(\gamma-2i\hbar)} & 1 & \frac{1}{t(\gamma-2i\hbar)} \\ \cdots & 0 & \frac{1}{t(\gamma-i\hbar)} & 1 \end{vmatrix} \quad (7.5b) \]

**Proof.** The determinants \( K_{\pm}(\gamma) \) are correctly defined \([10]\) and satisfy to the equations

\[
K_{+}(\gamma - i\hbar) = K_{+}(\gamma) - \frac{1}{t(\gamma)t(\gamma + i\hbar)}K_{+}(\gamma + i\hbar)
\]

\[
K_{-}(\gamma + i\hbar) = K_{-}(\gamma) - \frac{1}{t(\gamma)t(\gamma - i\hbar)}K_{-}(\gamma - i\hbar)
\]

(7.6)

This can be easily proved by expansion of these determinant with respect to the first row. The rest is trivial. Note that solutions (7.4) are determined up to arbitrary \( i\hbar \)-periodic function. □

**Remark 7.1** The function \( c_{\pm}(\gamma) \) are the solutions of the Baxter equation

\[ t(\gamma)c(\gamma) = i^{-N}c(\gamma + i\hbar) + i^{N}c(\gamma - i\hbar) \quad (7.7) \]

**Lemma 7.2** The functions \( c_{\pm}(\gamma) \) possess the following properties:

(i) \( c_{\pm}(\gamma) \) are the entire functions in \( \gamma \).

(ii) Let \( \gamma \) be real. Then

\[
c_{\pm}(\gamma) \sim \begin{cases} 
|\gamma|^{-N/2} e^{\frac{\pi N |\gamma|}{2\hbar}} \exp \left\{ \pm \frac{iN\gamma}{\hbar} \log \frac{|\gamma|}{e} \pm \frac{\pi iN}{4} \right\} & \gamma \to +\infty \\
|\gamma|^{-N/2} e^{\frac{\pi N |\gamma|}{2\hbar}} \exp \left\{ \pm \frac{iN\gamma}{\hbar} \log \frac{|\gamma|}{e} \mp \frac{\pi iN}{4} \right\} & \gamma \to -\infty
\end{cases} \quad (7.8)
\]

(iii)

\[
c_{+}(\gamma) \sim \exp \left\{ -\frac{N|\gamma|}{\hbar} \log |\gamma| \right\} & \gamma \to +i\infty \\
c_{-}(\gamma) \sim \exp \left\{ -\frac{N|\gamma|}{\hbar} \log |\gamma| \right\} & \gamma \to -i\infty
\]

(7.9)

**Proof.** (i) It is easy to see that the poles of \( K_{\pm}(\gamma) \) are cancelled by the corresponding poles coming from the product of \( \Gamma \)-functions in (7.4). Hence, \( c_{\pm}(\gamma) \) are an entire functions.

(ii) It is clear that \( \lim_{|\gamma| \to \infty} K_{\pm}(\gamma) = 1 \) for real \( \gamma \). Similarly, \( K_{\pm}(\gamma) \to 1 \) as \( \gamma \to \pm i\infty \). The asymptotics follow from the Stirling formula for \( \Gamma \)-functions. Note that the Stirling formula gives even more detailed asymptotics than (7.8); actually \( c_{\pm}(\gamma) \sim \exp\left( \frac{\pi N}{2\hbar} |\text{Re} \gamma| \right) \) as \( |\text{Re} \gamma| \to \infty \) in any finite strip \( |\text{Im} \gamma| \leq \epsilon, \ (\epsilon \geq 0) \) . □
In order to construct the solution with the asymptotics (6.2), one should recall that the solutions (7.3) are defined up to any $\bar{h}$-periodic factor. Clearly, the factor

$$\prod_{k=1}^{N} e^{\frac{\pi \gamma}{\bar{h}} \sinh \frac{\pi}{\bar{h}} (\gamma - \delta_k)}$$

(where $\delta_k$ are an arbitrary parameters) does not spoil the asymptotics of the functions (7.4) as $\gamma \to +\infty$ along the real axis while giving the correcting factor $e^{-\frac{2\pi|\gamma|}{\bar{h}}}$ as $\gamma \to -\infty$. Hence, one can consider the following solution of the Baxter equation (7.2):

$$c(\gamma) = \frac{c_+(\gamma) - \xi c_-(\gamma)}{\prod_{k=1}^{N} \sinh \frac{\pi}{\bar{h}} (\gamma - \delta_k)}$$

(7.11)

where $\xi$ is a arbitrary constant. Due to (7.8) this solution has prescribed asymptotics

$$c(\gamma) \sim |\gamma|^{-N/2} e^{-\frac{\pi N|\gamma|}{2\bar{h}}}$$

as $\gamma \to \pm \infty$. On the other hand, the denominator in (7.11) has the infinite number of poles at $\gamma = \delta_k + i\bar{h}n_k$, $n_k \in \mathbb{Z}$, $k = 1, \ldots, N$ and the solution (7.11) is not an entire function in general. The poles are cancelled only if the following conditions hold:

$$c_+(\delta_k + i\bar{h}n_k) = \xi c_-(\delta_k + i\bar{h}n_k)$$

(7.13)

In turn, this means that the Wronskian

$$W(\gamma) = c_+(\gamma) c_-(\gamma + i\bar{h}) - c_+(\gamma + i\bar{h}) c_-(\gamma)$$

vanishes at $\gamma = \delta_k + i\bar{h}n_k$. Due to Baxter equation (7.2)

$$W(\gamma + i\bar{h}) = (-1)^N W(\gamma)$$

(7.15)

Substitution of (7.4) to (7.14) gives

$$W(\gamma) = i^{-N} \mathcal{H}(\gamma) \prod_{k=1}^{N} \frac{\pi}{\bar{h}} \sinh \frac{\pi}{\bar{h}} (\gamma - \lambda_k)$$

(7.16)

where the function

$$\mathcal{H}(\gamma) \equiv K_+(\gamma) K_-(\gamma + i\bar{h}) - \frac{K_+(\gamma + i\bar{h}) K_-(\gamma)}{t(\gamma) t(\gamma + i\bar{h})}$$

(7.17)

is called the Hill determinant [10].

**Lemma 7.3** [10], [3] The function $\mathcal{H}(\gamma)$ has the following three-diagonal determinant form:

$$\mathcal{H}(\gamma) = \begin{vmatrix} \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ \ldots & 1 & \frac{1}{t(\gamma - i\bar{h})} & 0 & \ldots & \ldots & \ldots \\ \ldots & \frac{1}{t(\gamma)} & 1 & \frac{1}{t(\gamma)} & 0 & \ldots & \ldots \\ \ldots & 0 & \frac{1}{t(\gamma + i\bar{h})} & 1 & \frac{1}{t(\gamma + i\bar{h})} & 0 & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \end{vmatrix}$$

(7.18)
This infinite determinant is absolutely convergent; \( \mathcal{H}(\gamma) \) is a meromorphic function with the poles at the points \( \lambda_k + in_k h \), \( (n_k \in \mathbb{Z}, k = 1, \ldots, N) \). The Hill determinant is \( h \)-periodic functions with the asymptotics \( \mathcal{H}(\gamma) = 1 + O(\gamma^{-1}) \) as \( \gamma \to \pm \infty \). The following representation is valid:

\[
\mathcal{H}(\gamma) = 1 + \sum_{k=1}^{N} \alpha_k(E) \coth \frac{\pi}{h} (\gamma - \lambda_k(E))
\]  

(7.19)

where \( \alpha_k(E) \) are some definite functions depending on the parameters \( E \) only. These functions satisfy to the constraint

\[\sum_{k=1}^{N} \alpha_k(E) = 0 \]  

(7.20)

The equation \( \mathcal{H}(\gamma) = 0 \) has exactly \( N \) real solutions \( \delta_k(E), (k = 1, \ldots, N) \) subjected to the constraint

\[\sum_{k=1}^{N} \delta_k(E) = \sum_{k=1}^{N} \lambda_k(E) \]  

(7.21)

Due to its periodicity, \( \mathcal{H}(\gamma) = 0 \) at any point \( \delta_k(E) + i nh_k \) \( (n_k \in \mathbb{Z}) \). Hence, the solution (7.11) has no poles if one takes \( \delta_k = \delta_k(E) \) provided that the constant \( \xi \) is chosen in such a way that

\[\xi = \frac{c_+ (\delta_1)}{c_- (\gamma)} \bigg|_{\gamma=\delta_k(E)} \]  

(7.22)

Hence, one arrives to the following

**Lemma 7.4** [9]  

The function

\[C(\gamma; E) = \prod_{j=1}^{N-1} \frac{c_+ (\gamma_j; E) - \xi (E) c_- (\gamma_j; E)}{\prod_{k=1}^{N} \sinh \frac{\pi}{h} (\gamma_j - \delta_k(E))}\]  

(7.23)

where \( \delta_k(E) \) are the zeros of the Hill determinant (7.18) and the constant \( \xi \) is chosen according to (7.22), satisfies to conditions of Theorem 7.1.

The quantization conditions

\[\frac{c_+ (\delta_1)}{c_- (\delta_1)} = \ldots = \frac{c_+ (\delta_N)}{c_- (\delta_N)} \]  

(7.24)

determine the energy spectrum of the problem. These have been obtained for the first time by Gutzwiller [3] using quite different method.

**8 Proof of the main theorems**

**Proof of Theorem 3.1.** We have proved already that the function \( \Psi_E \) is integrable in a sense of (6.8), (2.13). Now we prove that the function (6.3) satisfies to the spectral problem (2.10) if the entire solution \( C(\gamma; E) \) to the Baxter equation (6.3) obeys the asymptotics \( \sim \exp \{- \frac{\pi}{2h} |\text{Re} \gamma_j| \} \)
Due to (5.21) one obtains

\[
\hat{t}(\lambda)\Psi_E = \int_{-\infty}^{\infty} d\gamma \, \mu^{-1}(\gamma) C(\gamma) \times \\
\left\{ (\lambda - E_1 + \sum_{j=1}^{N-1} \gamma_j) \prod_{m=1}^{N-1} (\lambda - \gamma_m) \psi_{\gamma,E_1} + \sum_{j=1, m \neq j}^{N-1} \prod_{m \neq j} (\lambda - \gamma_m) \right. \\
\left. + \left( i^N \psi_{\gamma+ihe_j,E_1} + i^{-N} \psi_{\gamma-ihe_j,E_1} \right) \right\} 
\]

(8.1)

Changing \( \gamma_j \to \gamma_j \pm i\hbar \) in appropriate parts of the integrand and noting that the measure \( (8.6) \) satisfies to the difference equation

\[
\mu^{-1}(\gamma + i\hbar \delta_j) = (-1)^N \mu^{-1}(\gamma) \prod_{m \neq j} \frac{\gamma_j - \gamma_m + i\hbar}{\gamma_j - \gamma_m}
\]

(8.2)

one arrives to the expression

\[
\hat{t}(\lambda)\Psi_E = \int_{-\infty}^{\infty} d\gamma \, \mu^{-1}(\gamma) C(\gamma) \left( \lambda - E_1 + \sum_{j=1}^{N-1} \gamma_j \right) \prod_{m=1}^{N-1} (\lambda - \gamma_m) \psi_{\gamma,E_1} + \\
+ i^N \sum_{j=1}^{N-1} \int_{-\infty-i\hbar}^{\infty} d\gamma_1 \cdots \int_{-\infty-i\hbar}^{\gamma_j} \cdots \int_{-\infty-i\hbar}^{\gamma_j} \cdots \int_{-\infty-i\hbar}^{\gamma_j} \mu^{-1}(\gamma) C(\gamma + ihe_j) \psi_{\gamma,E_1} \prod_{m \neq j} \frac{\lambda - \gamma_m}{\gamma_j - \gamma_m} + \\
+ i^{-N} \sum_{j=1}^{N-1} \int_{-\infty+i\hbar}^{\infty} d\gamma_1 \cdots \int_{-\infty+i\hbar}^{\gamma_j} \cdots \int_{-\infty+i\hbar}^{\gamma_j} \cdots \int_{-\infty+i\hbar}^{\gamma_j} \mu^{-1}(\gamma) C(\gamma - ihe_j) \psi_{\gamma,E_1} \prod_{m \neq j} \frac{\lambda - \gamma_m}{\gamma_j - \gamma_m}
\]

(8.3)

The integration contours \([-\infty \pm i\hbar, \infty \pm i\hbar]\) can be deformed to the standard one \([-\infty, \infty]\). Indeed, the corresponding integrands are the entire functions (all possible poles are cancelled by appropriate zeros of the measure \( \mu^{-1}(\gamma) \), see (6.6)). Moreover, it is not hard to see that the integrand is exponentially decreasing in the strip \(-i\hbar \leq \text{Im} \gamma_j \leq i\hbar\) as \( \text{Re} \gamma_j \to \infty \). Therefore, the contour integral around the strip vanishes and all integrals in (8.3) can be deformed to the standard ones. Finally, using the Baxter equation (8.3), one obtains from (8.3)

\[
\hat{t}(\lambda)\Psi_E = \\
\int_{-\infty}^{\infty} d\gamma \, \mu^{-1}(\gamma) C(\gamma) \psi_{\gamma,E_1} \left\{ (\lambda - E_1 + \sum_{j=1}^{N-1} \gamma_j) \prod_{m=1}^{N-1} (\lambda - \gamma_m) + \sum_{j=1}^{N-1} t(\gamma_j; E) \prod_{m \neq j} \frac{\lambda - \gamma_m}{\gamma_j - \gamma_m} \right\}
\]

(8.4)

The expression in curly brackets is nothing but \( t(\lambda; E) \) according to interpolation formula and, therefore, the function \( \Psi_E \) is a solution to the spectral problem (2.10). Note that the explicit form of \( C(\gamma; E) \) is not required here.

\[\text{We discard unessential numerical factor in (6.3).}\]
Proof of Theorem 3.2. The proof is heavily based on conjecture that all the zeros of the Hill determinant are simple \cite{3}. To derive the formula (3.6) from (3.4) let us consider the integral over $\gamma_1$ first selecting the expression

$$c_+(\gamma_1) - \xi c_-(\gamma_1) \prod_{k=1}^{N} \sinh \frac{\pi}{\hbar}(\gamma_1 - \delta_k)$$

(8.5)

By construction, it is an entire function but the expressions

$$\frac{c_+(\gamma_1)}{\prod_{k=1}^{N} \sinh \frac{\pi}{\hbar}(\gamma_1 - \delta_k)}$$

(8.6)

has an infinite number of poles. Clearly, the original integral can be written as follows while selecting the variable $\gamma_1$ and denoting the rest of variables as $\gamma'$:

$$\int_{-\infty}^{\infty} d\gamma_1 d\gamma' \frac{c_+(\gamma_1) - \xi c_-(\gamma_1)}{\prod_{k=1}^{N} \sinh \frac{\pi}{\hbar}(\gamma_1 - \delta_k)} \mu^{-1}(\gamma) C(\gamma') \Psi_\gamma =$$

$$= \int_{C_+} d\gamma_1 d\gamma' \frac{c_+(\gamma_1)\mu^{-1}(\gamma)}{\prod_{k=1}^{N} \sinh \frac{\pi}{\hbar}(\gamma_1 - \delta_k)} C(\gamma') \Psi_\gamma + \xi \int_{C_-} d\gamma_1 d\gamma' \frac{c_-(\gamma_1)\mu^{-1}(\gamma)}{\prod_{k=1}^{N} \sinh \frac{\pi}{\hbar}(\gamma_1 - \delta_k)} C(\gamma') \Psi_\gamma$$

(8.7)

where the contour $C_+$ encloses the poles $\delta_k + i\hbar n_k$, ($n_k \geq 0$, $k = 1, \ldots, N$) while the contour $C_-$ encloses the poles $\delta_k + i\hbar n_k$, ($n_k < 0$, $k = 1, \ldots, N$). Such deformations of contours are possible since the integrands in the r.h.s of (8.7) vanish exponentially on large upper and lower semi-circles respectively; this follows from the asymptotics (7.8) and (7.9). Now the integrals can be calculated using the residue formula. Note that $\xi c_-(\delta_k + i\hbar n_k) = c_+(\delta_k + i\hbar n_k)$ according to (7.13). Hence, the final expression can be written in terms of the function $c_+$ only. The same is true for the rest of integration.

Let us consider the contribution coming from the residues $\delta_1 + i\hbar n_1$ while integrating over $\gamma_1$. Using the explicit expression for the measure

$$\mu^{-1}(\gamma) \sim \prod_{j>k}(\gamma_j - \gamma_k) \sinh \frac{\pi}{\hbar}(\gamma_j - \gamma_k)$$

(8.8)

one obtains the following contribution:

$$\sim \sum_{n_1 \in \mathbb{Z}} c_+(\delta_1 + i\hbar n_1) \int d\gamma \prod_{k=2}^{N-1} \int_{-\infty}^{\infty} d\gamma_k (\gamma_k - \delta_1 - i\hbar n_1) \sinh \frac{\pi}{\hbar}(\gamma_k - \delta_1) \mu^{-1}(\gamma') C(\gamma') \Psi_{\delta_1 + i\hbar n_1, \gamma_2, \ldots, \gamma_{N-1}}$$

(8.9)

where we drop the common factor

$$\left\{(N-1)! \prod_{k=2}^{N} \sinh \frac{\pi}{\hbar}(\delta_1 - \delta_k)\right\}^{-1}$$

(8.10)

for a moment. Now we consider the analogous integration over variable $\gamma_2$. It is easy to see that there are no poles at $\gamma_2 = \delta_1 + i\hbar n_k$ because of the factor $\sinh \frac{\pi}{\hbar}(\gamma_2 - \delta_1)$ in the integrand. \footnote{Up to now we are unable to prove this conjecture though it seems very plausible.}
in (8.9). Quite similarly, while calculating the residues at \( \gamma_2 = \delta_2 + i h n_2 \), one concludes that the integration over \( \gamma_3 \) does not include the contribution from the points \( \delta_1 + i h n_k \) and \( \delta_2 + i h s_k \) and so on. Therefore, the successive evaluation of the residues at “well arranged” points \( \gamma_k = \delta_k + i h n_k, \ k = 1, \ldots, N - 1 \) gives the contribution

\[
\sim \sum_{n_1, \ldots, n_{N-1} \in \mathbb{Z}} \prod_{k=1}^{N-1} c_+(\delta_k + i h n_k) \Delta(\delta^{(N)} + i h n^{(N)}) \Psi_{\delta_1 + i h n_1, \ldots, \delta_{N-1} + i h n_{N-1}} (8.11)
\]

where \( \Delta(\gamma) \) is the Vandermonde determinant and the notation \( y^{(s)} \) means the \( N - 1 \) dimensional vector obtained from the corresponding \( N \) dimensional one by cancellation of the \( s \)-th component, i.e. \( y^{(s)} \equiv (y_1, \ldots, y_{s-1}, y_{s+1}, \ldots, y_N) \). After a little algebra one finds that the common factor to (8.11) is

\[
(-1)^{N(N-1)/2} \left\{ (N-1)! \prod_{j>k} \sinh \frac{\pi}{h} (\delta_j - \delta_k) \right\}^{-1} (8.12)
\]

By construction the integrand in (3.1) is symmetric under the permutation of \( \gamma \)-variables. Therefore, one obtains the same answer (8.11) while calculating all other possible residues from the set \( \delta^{(N)} + i h n^{(N)} \). This gives the additional factor \( (N-1)! \).

The last step is to take into account the poles at \( \delta_N + i h n_N \). One can easy to show that the calculation of the corresponding residues while integrating over \( \gamma_s \) gives the additional factor \( (-1)^{N-s} \) to compare with (8.12). Hence, one obtains the final answer (up to unessential numerical constant)

\[
\Psi_E(x_0, \mathbf{x}) = \frac{1}{\prod_{j>k} \sinh \frac{\pi}{h} (\delta_j - \delta_k)} \times \sum_{s=1}^N (-1)^{N-s} \sum_{n^{(s)} \in \mathbb{Z}^{N-1}} \Delta(\delta^{(s)} + i h n^{(s)}) C_+(\delta^{(s)} + i h n^{(s)}) \Psi_{\delta^{(s)} + i h n^{(s)}, E_1}(x_0, \mathbf{x}) (8.13)
\]

where

\[
C_+(\gamma) \equiv \prod_{k=1}^{N-1} c_+(\gamma_k) (8.14)
\]

and Theorem 3.2 is proved.

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