Positive tension 3-branes in an $AdS_5$ bulk

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In this work, we review and extend the so-called consistency conditions for the existence of a braneworld scenario in arbitrary dimensions in the Brans-Dicke (BD) gravitational theory. After that, we consider the particular case of a five-dimensional scenario which seems to have phenomenological interesting implications. We show that, in the BD framework, it is possible to achieve necessary conditions pointing to the possibility of accommodating branes with positive tensions in an AdS bulk by the presence of the additional BD scalar field, avoiding in this way the necessity of including unstable objects in the compactification scheme. Furthermore, in the context of time variable brane tension, it is shown that the brane tension may change its sign, following the bulk cosmological constant sign.

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I. INTRODUCTION

In the basic framework establishment of the string theory, two main outputs are noticeably delineated: the necessity of extra dimensions and the presence of a scalar field (in the low energy limit) sharing the action of the gravitational interaction with the usual rank-two tensorial field \[1\]. The advent of the braneworld scenarios, at least in its modern fashion, may also be partially charged to the string theory developments.

In the Randall-Sundrum braneworld scenario \[2\], which can be understood as an effective Horava-Witten \[3\] compactification model, the branes are performed by two mirror warped domain walls embedded into a five-dimensional bulk spacetime and placed at the end of an orbifold extra dimension. The bulk has a negative cosmological constant (AdS slice) and the two mirror branes have tensions of opposite signs. The brane which mimics our universe is endowed with an exponential warp factor (conformal to the four dimensional Minkowski metric) that is responsible by the hierarchy problem solution.

Most of the characteristics outlined in the previous paragraph may be reproduced by a five-dimensional particularization of a quite ingenious reformulation of the Einstein tensor components, redesigned in order to take the existence of branes and extra dimensions into account \[4, 5\]. Here we shall apply the constraints emerging from that program (the search for the consistency conditions) to the case of three branes embedded into a five-dimensional bulk spacetime within the Brans-Dicke theory. Apart of the mentioned motivation coming from advances in String Theory and other unification programs (such as supergravity and M-theory), there are strong evidences that General Relativity is a natural attractor of more general scalar-tensor gravity theories \[6\], from which the Brans-Dicke case is just the simplest viable example. An important result obtained from such an extension of the consistency conditions is that the Brans-Dicke scalar field relax the obtained constraints, allowing the possibility of an AdS bulk only if the brane tensions are positive. I. e., it seems to be unnecessary to include a negative brane tension even in the five-dimensional bulk case. This is a quite remarkable result, since negative tension branes are unstable objects. Certainly each particular model must be investigated case by case, but the results emerging from the consistency conditions point to scenarios without the need for negative tension branes.

The possibility of a compactification scheme with only positive tension branes is allowed in the General Relativity theory in two cases: 1) more than five dimensions (being six dimensions enough for this purpose) and 2) with the presence of bulk tachyon matter. In the first case, the consistency conditions are relaxed, and the necessity of including a negative tension brane is not
required. In the last case, the presence of a non-canonical scalar field in the bulk modifies the junction conditions, allowing an additional freedom in the brane tensions sign [7].

Roughly speaking, although even in the five-dimensional bulk case, a relaxation of the consistency condition also occurs in Brans-Dicke gravity. In the case studied here, however, the absence of a negative tension brane for a consistent compactification is due to the presence of the Brans-Dicke field, not by some isolated bulk matter. In fact, in our analysis, all extra bulk matter fields will be set to zero. In other words, we show the existence of necessary conditions pointing to the fact that the BD gravity can accommodate only branes with positive tensions in a natural way. The presence of the BD scalar field relax the consistency conditions. Besides, different from the scope analyzed in Ref. [7], our demonstration is completely model independent.

Going further in our analysis we allow the brane tension to be time variable. Recently, the hypothesis that the Universe could be better described by a variable brane tension has been raised [8]. In fact, keeping in mind the cosmological evolution of the Universe, a variable brane tension seems to be an inevitable condition. The application of the consistency conditions to simple cases for variable brane tensions framework can be found in Ref. [9]. We shall implement the tools developed in [9] for the simplest case studied here. As we will see, a dynamical brane tension may change its sign following the bulk cosmological constant sign.

This paper is organized as follows: in the next Section we review and extend the consistency conditions to the case of a Brans-Dicke bulk. Then, in Section III, after particularizing to the phenomenological interesting five-dimensional case, we show how the scalar field acts in order to avoid ill defined scenarios. In Section IV we study the case of time variable brane tensions and in the last Section we summarize our results.

II. CONSISTENCY CONDITIONS WITHIN BRANS-DICKE GRAVITY

In this Section, we establish the basic mathematical tools which will be used throughout this paper. In a sense, the consistency conditions arise from an elegant way of rewriting the Einstein tensor taking into account the presence of embedded branes in a higher dimensional bulk. Their first derivations, employed to specific cases, was developed in [10]. In this paper we use the modern and general fashioned form developed in [4, 5]. The full generalization of the consistency conditions to the Brans-Dicke gravity case was obtained in [11]. We shall outline, following the basic formulation of reference [5], the main steps fixing the notation used here.

We start analyzing a D-dimensional bulk spacetime endowed with a non-factorable geometry,
whose metric is given by

\[ ds^2 = G_{AB}dX^A dX^B = W^2(r)g_{\alpha\beta}dx^\alpha dx^\beta + g_{ab}(r)dr^a dr^b, \]  

where \( W^2(r) \) is the warp factor, assumed to be a smooth integrable function, \( X^A \) denotes the coordinates of the full D-dimensional spacetime, \( x^\alpha \) stands for the \((p + 1)\) non-compact coordinates of the spacetime and \( r^a \) labels the \((D - p - 1)\) directions in the internal compact space\(^1\). Note that this type of metric encodes the possibility of existing q-branes \((q > p)\) \(^5\). In this case, the \((q - p)\) extra dimensions are compactified on the brane and constitute part of the internal space. This possibility is important in the hybrid compactification models context, for instance \(^12\).

It is well known that the D-dimensional spacetime Ricci tensor can be related with the brane Ricci tensor as well as with the internal space partner by the equations \(^4\)

\[ R_{\mu\nu} = \tilde{R}_{\mu\nu} - \frac{g_{\mu\nu}}{(p + 1)W^{p-1}}\nabla^2 W^{p+1}, \]  

and

\[ R_{ab} = \tilde{R}_{ab} - \frac{p + 1}{W}\nabla_a \nabla_b W, \]

where \( \tilde{R}_{ab}, \nabla_a \) and \( \nabla^2 \) are respectively the Ricci tensor, the covariant derivative and the Laplacian operator constructed with the internal space metric \( g_{ab} \). \( \tilde{R}_{\mu\nu} \) is the Ricci tensor derived from \( g_{\mu\nu} \). Denoting the three curvature scalars by \( R = G^{AB}R_{AB}, \tilde{R} = g^{\mu\nu}\tilde{R}_{\mu\nu} \) and \( \bar{R} = g^{ab}\bar{R}_{ab} \), one sees that the traces of equations \(^2\) and \(^3\) are given by

\[ \frac{1}{p + 1}\left(W^{-2}\bar{R} - R_{\mu}^\mu\right) = pW^{-2}\nabla W \cdot \nabla W + W^{-1}\nabla^2 W \]  

and

\[ \frac{1}{p + 1}\left(\bar{R} - R_{a}^a\right) = W^{-1}\nabla^2 W, \]

where \( R_{\mu}^\mu \equiv W^{-2}g^{\mu\nu}R_{\mu\nu} \) and \( R_{a}^a \equiv g^{ab}R_{ab} \) (in such a way that \( R = R_{\mu}^\mu + R_{a}^a \)). Now, being \( \xi \) an arbitrary constant, we have

\[ \nabla \cdot (W^\xi \nabla W) = W^{\xi+1}(\xi W^{-2}\nabla W \cdot \nabla W + W^{-1}\nabla^2 W). \]

Finally, the combination of the Eqs. \(^2\), \(^5\) and \(^6\) leads to

\[ \nabla \cdot (W^\xi \nabla W) = \frac{W^{\xi+1}}{p(p + 1)}[\xi(W^{-2}\bar{R} - R_{\mu}^\mu) + (p - \xi)(\bar{R} - R_{a}^a)]. \]

\(^1\) As an example, if \( D = 5, p = 3 \) and \( W(r) = e^{-2k|r|} \) (being \( k \) a constant factor) we recover the standard Randall-Sundrum model.
This last equation encodes the basic statement of the consistency conditions. The left-hand side (LHS) of Eq. (7) vanishes upon integration over the internal space while its right-hand side (RHS) opens the possibility for the generalization to the Brans-Dicke case since it expresses the geometrical quantities in terms of the scalar field. In fact, if the internal space is compact without boundary or, equivalently, if the physical fields are periodic at the extremities of the internal space interval, the following relation must hold $\oint \nabla \cdot (W \xi \nabla W) = 0$. Therefore, writing the RHS of Eq. (7) in terms of the spacetime stress tensor, one arrives at the one-parameter ($\xi$) family of consistency conditions. Each choice of $\xi$ leads to an additional condition for the consistency of the compactification scheme.

### A. The Brans-Dicke generalization of the consistency conditions

In order to generalize to the Brans-Dicke gravity case we start by remembering that the Einstein-Brans-Dicke bulk field equation is given by

$$R_{MN} - \frac{1}{2}G_{MN}R = \frac{8\pi}{\Phi} T_{MN} + \frac{w}{\Phi^2} \left( \nabla_M \Phi \nabla_N \Phi - \frac{1}{2} \nabla_A \Phi \nabla^A \Phi G_{MN} \right) + \frac{1}{\Phi} \left( \nabla_M \nabla_N \Phi - \frac{8\pi}{(D-1)+(D-2)w} T G_{MN} \right),$$

where $T_{MN}$ is the matter stress-tensor, $T$ is the trace and $w$ the BD parameter. We remark that the scalar part of the BD set of equations, $\Box^2 \Phi = \frac{8\pi}{(D-1)+(D-2)w} T$, was already taken into account in the last term of the RHS of equation (8). We shall use the scalar equation again in the next Section after doing some simplification in the calculations. From (8), calling $T_{\mu}^{\mu} \equiv W^{-2} g_{\mu\nu} T_{\mu\nu}$ ($T = T_{\mu}^{\mu} + T_{m}^{m}$), it is possible to express $R_{\mu}^{\mu}$ and $R_{m}^{m}$ as

$$R_{\mu}^{\mu} = \frac{8\pi}{\Phi} \frac{1}{[(D-1)+(D-2)w]} \left( [D + w(D - p - 3) - 2T_{\mu}^{\mu} - (1 + w)(p + 1)T_{m}^{m}] ight. + \left. \frac{wW^{-2}}{\Phi^2} \nabla^\nu \Phi \nabla_\nu \Phi + \frac{W^{-2}}{\Phi} \nabla^\nu \nabla_\nu \Phi, \right.$$  

and

$$R_{m}^{m} = \frac{8\pi}{\Phi} \frac{1}{[(D-1)+(D-2)w]} \left( [w(p - 1) + p]T_{m}^{m} - (1 + w)(D - p - 1)T_{\mu}^{\mu} \right) + \frac{w}{\Phi^2} \nabla^m \Phi \nabla_m \Phi + \frac{1}{\Phi} \nabla^m \nabla_m \Phi.$$  

It is important to remark that when one takes the limit $w \to \infty$, i.e. $\Phi \to 1/G_N$, the expressions $R_{\mu}^{\mu}$ and $R_{m}^{m}$ recover the case analyzed in the General Relativity theory, as expected.

Replacing the Eqs. (9) and (10) in (7) one finds

$$\nabla \cdot (W^\xi \nabla W) = \frac{W^{\xi+1}}{p(p+1)} \left[ \xi W^{-2} \tilde{R} + (p - \xi) \tilde{R} + \frac{8\pi}{\Phi} \frac{1}{[(D-1)+(D-2)w]} \right]$$
\[ \times \left( T^\mu_\mu[(p - \xi)(w + 1)(D - p - 1) - \xi(D + w(D - p - 3) - 2)] \right. \\
+ T^m_m[\xi(w + 1)(p + 1) - (p - \xi)\{w(p - 1) + p\}] \\
- \frac{w}{\Phi^2}[\xi W^{-2}\nabla^\nu\Phi\nabla_\mu\Phi + (p - \xi)\nabla^m\Phi\nabla_m\Phi] - \frac{1}{\Phi}[\xi W^{-2}\nabla^\nu\nabla_\mu\Phi \\
+ (p - \xi)\nabla^m\nabla_m\Phi] \right). \] (11)

Now, we shall write explicitly the stress tensor partial traces terms \( T^\mu_\mu \) and \( T^m_m \). To do that, we write the bulk general stress tensor in the form \[ T_{MN} = -\Lambda G_{MN} - \sum_i T_q^{(i)} P[G_{MN}]^{(i)} \Delta^{(D-q-1)}(r - r_i) + \tau_{MN}, \] (12)

where \( \Lambda \) is the bulk cosmological constant, \( T_q^{(i)} \) is the \( i^{th} \) q-brane tension with units given by [energy/(length)]\( q \), \( \Delta^{(D-q-1)}(r - r_i) \) is the covariant combination of delta functions which positions the brane\(^2\), \( P[G_{MN}]^{(i)} \) is the pull-back of the bulk metric and any other matter contribution is due to \( \tau_{MN} \). From the Eq. (12) it is simple to get

\[ T^\mu_\mu = -(p + 1)\Lambda + \tau^\mu_\mu - \sum_i T_q^{(i)} \Delta^{(D-q-1)}(r - r_i)(p + 1), \] (13)

and

\[ T^m_m = -(D - p - 1)\Lambda + \tau^m_m - \sum_i T_q^{(i)} \Delta^{(D-q-1)}(r - r_i)(q - p). \] (14)

Inserting Eqs. (13) and (14) in (11) and using the identity \( \oint \nabla \cdot (W^\xi \nabla W) = 0 \) one obtains, after some algebra, the following result

\[ \oint W^{\xi+1} \left[ \xi W^{-2}\tilde{R} + (p - \xi)\tilde{R} - \frac{8\pi}{\Phi} \frac{1}{(D - 1) + w(D - 2)} \left( -\sum_i T_q^{(i)} \Delta^{(D-q-1)}(r - r_i) \right. \\
\times \left[ A(p + 1) + B(q - p) \right] + A\tau^\mu_\mu + B\tau^m_m - \Lambda \left[A(p + 1) + B(D - p - 1)\right] \right) \\
- \xi W^{-2} \left[ \frac{1}{\Phi^2}\nabla^\nu\nabla_\mu\Phi + \frac{w}{\Phi^2}\nabla^\nu\Phi\nabla_\mu\Phi \right] - (p - \xi)\left[\mu \rightarrow m\right] \right] = 0, \] (15)

where

\[ A \equiv (p - \xi)(w + 1)(D - p - 1) - \xi[D + w(D - p - 3) - 2], \] (16)

\(^2\) For a complete discussion about the expression of \( \Delta^{(D-q-1)}(r - r_i) \) we refer the reader to Appendix of the paper [5].
\[ B \equiv \xi(w + 1)(p + 1) - (p - \xi)[w(p - 1) + p] \]  

are constant parameters and the last term in (15) denotes the same parenthesis of the penultimate term, but with summation over the internal space index \( m \). The result encoded in Eq. (15) is exhaustive, in the sense that for every fixed \( \xi \) it gives a constraint that must be obeyed for a consistent compactification scheme in Brans-Dicke theory at a given dimension. The equation (15) itself, however, is not quite useful since it is too general and does not give a specific information about any particular model. Nevertheless, its generality is exactly in what its importance resides: it encapsulates in itself all such possible models.

In the next section we shall explore some possibilities which are contained in the Eq. (15), in particular, a phenomenological interesting five-dimensional case. As we will see, after some simplifications, it is possible to arrive at strong and important constraints for the two branes scenario.

III. FIVE-DIMENSIONAL \textit{AdS} BULK CASE

We shall apply in this Section the main result of the previous section – namely, Eq. (15) – to a particular case. The scenario we would like to explore has the following set up: two 3-branes embedded into a five-dimensional bulk spacetime endowed with non-factorable geometry in Brans-Dicke gravity. This type of scenario may be understood as the scalar-tensor analogous to the Randall-Sundrum (RS) case. In the original RS model, one of the basic, and necessary, characteristic of the brane tensions is that it must be of opposite signs.

The main problem concerning a negative tension object in the compactification scheme is that such a brane is an inherently unstable object\(^3\). In fact the bulk-brane system must be a stable configuration solution of the gravitational field equations, therefore without a negative tension object. The problem with such type of branes may be understood as follows: being the brane a (codimension one) submanifold embedded into a higher dimensional manifold, it is always possible to project the gravitational field equation from the bulk to the brane by, for instance, the well known Gauss-Codazzi formalism \[14\]. The effective gravitational equation brings some specific signatures from the extra dimensions with subtle but important departures from the usual (four-dimensional) case. The effective Newtonian constant on the brane inherits the tension brane sign. This behavior

\(^3\) It is important to remark that some specific frameworks where the negative tension brane is placed at an orientifold plane may circumvent some objections concerning the brane instability \[13\].
is observed in both General Relativity and Brans-Dicke gravities. Therefore, a negative tension brane induces a wrong sign in the brane projected Newtonian constant.

From now on, let us assume a particular case where $D = 5$ and $p = q = 3$ in order to mimic a RS scenario in the framework of the Brans-Dicke gravity. As we will see, the existence of the scalar field points to the possibility of the construction of a bulk-brane structure without the necessity of any negative brane tension. With the chosen dimensionality the constants $A$ and $B$ are simply given by

$$A(D = 5, p = q = 3) = 3(1 + w) - 4\xi, \quad B(D = 5, p = q = 3) = 6w(\xi - 1) + 7\xi - 9. \quad (18)$$

Since this case has codimension one, the scalar of curvature of the internal space (one line in the present case) is zero, i.e., $\tilde{R} = 0$. Then, discarding any additional matter contribution for simplicity ($\tau_{\mu}^\mu = \tau_m^m = 0$)\(^4\), Eq. (15) reads

$$\oint W^{\xi+1}\left(\xi W^{-2}\tilde{R} + \frac{8\pi}{\Phi(4 + 3w)} \left(-4A\sum_{i=1}^{2}T_3^{(i)}(r - r_i) - \Lambda(4A + B)\right) - \xi W^{-2}\left[\frac{1}{\Phi}\nabla^\mu\nabla_\mu\Phi + \frac{w}{\Phi^2}\nabla^\mu\Phi\nabla_\mu\Phi\right] -(3 - \xi)\left[\mu \to m\right]\right) = 0, \quad (19)$$

where $A$ and $B$ are given by Eq. (18). Now, two further simplifications. The first one comes from experience. In trying to reproduce our universe one can set $\tilde{R} = 0$ with an accuracy of $10^{-120}M_{Pl}$, where $M_{Pl}$ is the four-dimensional Planck mass \(^4\). Secondly, we shall look at the consistency conditions focusing on the case where the Brans-Dicke scalar field does not have dynamics on the brane, i.e., $\nabla_\mu\Phi = 0$. It may sounds as an oversimplification. However, it is the most direct way not to contradict the experimental gravitational bounds in the universe performed by a brane \(^4\).

In other words, the simplest model consisting of a Brans-Dicke gravity only in the bulk recovers the (slightly modified) General Relativity on the brane, in such a way that it is compatible with the observational data.

Before reexpressing Eq. (19) in terms of the above simplifications it is worthwhile to mention that when we take the General Relativity limit $w \to \infty$ ($\Phi = \text{cte}$), the $\Lambda$ coefficient $(4A + B)/(4 + 3w)$ reduces to $-2(\xi + 1)$ which factorizes out the cosmological constant from the consistency conditions for the $\xi = -1$ case, in close analogy to the usual RS model presented in Ref. \(^4\).

\(^4\) Note the difference between this approach and the one presented in Ref. \(^7\). In our case we do not use any additional bulk matter, working, instead, within the BD gravity. It is possible to recover the case studied in \(^7\) by setting $w \to \infty$ and $\tau_m^m \neq 0$. As this must be clear from our approach, the presence of additional bulk matter in General Relativity is also responsible for a relaxing in the consistency conditions.
Taking into account the above mentioned simplifications in Eq. (19) we have

\[
\oint W^{\xi+1} \left( \frac{8\pi}{4 + 3w} \left[ -4[3(1 + w) - 4\xi] \sum_{i=1}^{2} T^{(i)}_3 \delta(r - r_i) - 3[2W(\xi + 1) + 1 - 3\xi]\Lambda \right] \right.
\]

\[
- (3 - \xi) \left[ \nabla^m \nabla_m \Phi + \frac{w}{\Phi} \nabla^m \Phi \nabla_m \Phi \right] \bigg) = 0.
\]

This last equation may be expressed in a more suitable way with the aid of the scalar field equation of motion. If the scalar field depends only on the extra dimensional coordinate, then \(
\Box^2 \Phi = \nabla^m \nabla_m \Phi\). Furthermore, the energy-momentum trace is given by \(T = T^\mu_\mu + T^m_m = -4 \sum_{i=1}^{2} T^{(i)}_3 \delta(r - r_i) - 5\Lambda\), in view of equations (13) and (14). Therefore, the scalar equation becomes

\[
\nabla^m \nabla_m \Phi = -\frac{8\pi}{4 + 3w} \left( 4 \sum_{i=1}^{2} T^{(i)}_3 \delta(r - r_i) - 5\Lambda \right).
\]

Inserting the Eq. (21) into (20) we obtain, after a bit of algebra, the following expression

\[
\oint W^{\xi+1} \left( \gamma^2 \sum_{i=1}^{2} W^4_i \delta(r - r_i) + \frac{w(3 - \xi)}{\Phi} \partial^m \Phi \partial_m \Phi \right) = 0,
\]

where \(\gamma\) and \(\eta\) are constants given by

\[
\gamma = \frac{96\pi(\xi - w)}{4 + 3w}, \quad \eta = \frac{16\pi[6 + 2\xi - 3w(\xi + 1)]}{4 + 3w}.
\]

Now we are in a position to study the physical outputs of the consistency conditions, by inserting some key values of the parameter \(\xi\). The most interesting choices are \(\xi = 3\), since it simplifies the consistency condition eliminating the scalar field derivative contributions, and \(\xi = -1\) which eliminates the overall warp factor term.

A. The \(\xi = 3\) case

Suppose \(\xi = 3\). In this case, the equations (22) and (23) together give

\[
(3 - w) \sum_{i=1}^{2} T^{(i)}_3 \oint W^4_4 \delta(r - r_i) + 2(1 - w)\Lambda \oint W^4_4 = 0.
\]

Denoting by \(W(r = r_i) = W_i\) and \(\Phi(r = r_i) = \Phi_i\) the values of the warp factor and the scalar field on the \(i^{th}\) brane, we have

\[
(3 - w) \sum_{i=1}^{2} W^4_i \frac{W^{(i)}_i}{\Phi_i} T^{(i)}_3 + 2(1 - w)\Lambda \oint W^4_4 = 0.
\]
Implementing the AdS bulk case $\Lambda < 0$ in Eq. (25), it simply reads

$$\frac{(3 - w)}{2(1 - w)} \sum_{i=1}^{2} \frac{W_{i}}{\Phi_{i}} T_{3}^{(i)} = |\Lambda| \int \frac{W^{4}}{\Phi}. \quad (26)$$

The remarkable characteristic of the above expression is that it shows a necessary condition pointing to the viability of a consistent compactification scheme with only positive tension branes, avoiding then ill defined scenarios.

The typical functional form of warp factors solving the hierarchy problem are $W \sim \exp(-f(r))$, being $f(r)$ determined by the gravitational field equations. However, as shown in Eq. (26), the important term is given by a “coupling” between the warp factor and the scalar field. In fact, the proposition of braneworld models in the framework of scalar-tensor gravitational theories resides in the fact that the scalar field composes the warp factor. In this sense, it is not surprising that a term as $W^{4}/\Phi$ is determinant to the establishment of the compactification scheme. In other words, the term $W^{4}/\Phi$ can be seen as a composite warp factor, which may act as a tool to solve the hierarchy problem allowing, at the same time, the presence of only positive tension branes. As an aside remark we stress that for flat spacetimes ($W = 1$, everywhere) the presence of negative tension branes may be also unnecessary.

\textbf{B. The $\xi = -1$ case}

Let us now suppose $\xi = -1$. With such specification the $\gamma$ and $\eta$ constants read

$$\gamma = -\frac{96\pi(1 + w)}{4 + 3w}, \quad (27)$$
$$\eta = \frac{64\pi}{4 + 3w}, \quad (28)$$

and Eq. (22) gives

$$\frac{24\pi(1 + w)}{4 + 3w} \sum_{i=1}^{2} \frac{T_{3}^{(i)}}{\Phi_{i}} + \frac{16\pi}{4 + 3w} |\Lambda| \int \frac{1}{\Phi} + w \int g^{rr}[\partial_{r}(\ln \Phi)]^2 = 0. \quad (29)$$

Rewriting the above equation in a more suitable way, we have

$$\int g^{rr}[\partial_{r}(\ln \Phi)]^2 = -\frac{8\pi}{4 + 3w} \left\{ \frac{2|\Lambda|}{w} \int \frac{1}{\Phi} + \frac{3(1 + w)}{w} \sum_{i=1}^{2} \frac{T_{3}^{(i)}}{\Phi_{i}} \right\}. \quad (30)$$

The left-hand side of the equation (30) is positive, therefore the positive tension branes are possible, being $\int 1/\Phi < 0$ and

$$\int \frac{1}{\Phi} > \frac{3(1 + w)}{2|\Lambda|} \sum_{i=1}^{2} \frac{T_{3}^{(i)}}{\Phi_{i}}, \quad (31)$$

which can be satisfied by a typical power-law behavior of the scalar field, i.e., $\Phi \sim r^{-a}$. 
IV. A TIME VARIABLE TENSION CASE

In this Section we shall study an interesting characteristic of a time variable tension toy model. In Ref. [8], a complete solution for a time tension variable braneworld highly compatible with the observable cosmological symmetries is shown. Recently, the time variation of brane tensions was analyzed with the aid of the consistency conditions [9]. In this Section, we shall point out an interesting characteristic arising from a simple toy model based on the time variable tension branes: in an AdS bulk, it is possible for the brane to change the sign of its tension at some point of the time evolution.

In Ref. [9], it is shown that the time variation of the brane tension may be accomplished by the following replacement

\[ T^{(i)}_3 \rightarrow \tilde{T}^{(i)}_3 = T^{(i)}_3 + \kappa^{(i)} \partial_t T^{(i)}_3, \tag{32} \]

in Eq. (22). In (32) \( \kappa^{(i)} \) is a positive constant with units of \((\text{energy})^{-1}\), keeping \( \tilde{T}^{(i)}_3 \) with units of \((\text{energy})/\text{(length)}^3\). This simple prescription ignores the possible high derivative corrections in the time variation and does not concern the mechanism under which the tension becomes variable. However, this toy model does not preclude the existence of non-linear effects in the brane tension. We would like to remark here that it might be possible to achieve a more realistic variation to the brane tension by looking at some cosmological upper bounds of the local rate of change of the gravitational ‘constant’ [15], since both — the brane tension and the gravitational ‘constant’— are related by dimensional reduction of the bulk gravitational equations. Here, we shall keep our analysis to the following toy model, stressing a curious behavior of the brane tension dynamics in an AdS bulk.

The development of the consistency conditions formulae with the replacement given by (32) can be done in the very same way of what we have done in the previous section. From Eq. (22) it is easy to see that

\[ \sum_{i=1}^{2} \frac{T^{(i)}_3 W^{\xi+1}_i}{\Phi_i} = \frac{w(\xi - 3)}{\gamma} \oint W^{\xi+1} g^{rr} [\partial_r (\ln \Phi)]^2 - \frac{\eta \Lambda}{\gamma} \oint W^{\xi+1} / \Phi. \tag{33} \]

For simplicity, we may assume that the tension of the hidden brane behaves independently of the visible one, i.e., that its tension is simply given by \( T^{\text{hid}}_3 \sim \exp(-t/\kappa^{\text{hid}}), \) such that \( \tilde{T}^{\text{hid}}_3 = 0. \) The characteristics which we are going to arrive at do not depend strongly on this assumption. We shall keep it in order to emphasize our point in a clear way. Keeping it in our minds, Eq. (33) results in
the following differential equation
\[ \dot{T}_{3}^{\text{vis}} + \frac{1}{\kappa_{\text{vis}}} T_{3}^{\text{vis}} + \frac{1}{\kappa_{\text{vis}}} F(r) = 0, \] (34)

where \( \dot{T}_{3}^{\text{vis}} = \frac{dT_{3}^{\text{vis}}}{dt} \) and \( F(r) \) is just the right-hand side of Eq. (33) multiplied by the \( \Phi_{\text{vis}}/W_{\text{vis}}^{\xi+1} \) factor. The solution of (34) is given by
\[ T_{3}^{\text{vis}}(t) = Ce^{-t/\kappa_{\text{vis}}} - \frac{1}{\kappa_{\text{vis}}} \frac{\Phi_{\text{vis}}}{W_{\text{vis}}^{\xi+1}} \left[ \frac{w(\xi-3)}{\gamma} \int W^{\xi+1} g^{rr}[\partial_r(ln \Phi)]^2 - \frac{\eta}{\gamma} \Lambda \int \frac{W^{\xi+1}}{\Phi} \right], \] (35)

where \( C \) is a positive integration constant (with units of tension). We remark that taking the limit \( w \to \infty \) (\( \Phi \) constant) one arrives again at the simplest case studied in General Relativity \( \text{(9)} \), as expected. If one is willing to accept the current experimental bound \( \text{(16)} \) of the Brans-Dicke parameter for the bulk, the first term is dominant over the second one, where the cosmological constant is present. Nevertheless it may not be the case, i. e., the Brans-Dicke theory may have another “status” in the bulk. Apart of this, the choices \( \xi \geq 3 \) show that \( T_{3}^{\text{vis}} \) must change its sign at some time, provided that \( \phi_{\text{vis}} \) and \( W_{\text{vis}} \) have the same sign on the brane. Particularly, for \( \xi = 3 \) the relation between the brane tension and the cosmological constant sign appears explicitly. For this case, the Eq. (35) reduces to
\[ T_{3}^{\text{vis}}(t) = Ce^{-t/\kappa_{\text{vis}}} - \frac{2|\Lambda|}{\kappa_{\text{vis}}} \frac{(1-w)}{(3-w)} W_{\text{vis}}^{4} \int \frac{W^{4}}{\Phi}, \] (36)

and the tension changes its sign in the specifical time
\[ \bar{t} = \kappa_{\text{vis}} \ln \left\{ \frac{\kappa_{\text{vis}} C}{2|\Lambda|} \frac{(3-w)}{(1-w)} \Phi_{\text{vis}} \left( \int \frac{W^{4}}{\Phi} \right)^{-1} \right\}. \] (37)

Obviously the value \( \bar{t} \) is lower for bigger absolute values of the bulk cosmological constant.

We shall interpret the general picture encoded in Eq. (35) as follows: although the contribution of the first term of the right-hand side may be positive or negative, the sign of the second one is always negative for an \( AdS_5 \) bulk slice. Therefore (apart of the relative values of the two terms), it acts dynamically pulling the brane’s tension to a negative value. It may be useful to make a rough analogy between the time variation of the brane tension and a classical mechanical system, thinking that \( T_{3}^{\text{vis}}(t) \) is the position of a particle according to the time variation. The equation (34) has a “friction” term given by \( \dot{T}_{3}^{\text{vis}} \) which is responsible for a damping in the \( T_{3}^{\text{vis}} \) curve. It is not difficult to see that the conservative terms of the equation of motion (34) can be derived by the following “Lagrangian” independent of the velocities
\[ L(T_{3}^{\text{vis}}, t) = -\frac{F(r)}{\kappa} T_{3}^{\text{vis}} - \frac{1}{2\kappa} (T_{3}^{\text{vis}})^2, \] (38)
in such a way that

\[
\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{T}^{\text{vis}}_3} \right) - \frac{\partial L}{\partial T^{\text{vis}}_3} = -\dot{T}^{\text{vis}}_3,
\]

(39)

and the equation (34) can be recovered. The energy associated to the Lagrangian is given by

\[
E = T^{\text{vis}}_3 \frac{\partial L}{\partial \dot{T}^{\text{vis}}_3} - L = \frac{T^{\text{vis}}_3}{\kappa} \left( \frac{T^{\text{vis}}_3}{2} + F(r) \right).
\]

(40)

Deriving Eq. (40) with respect to the time and using (34) one gets

\[
\frac{dE}{dt} = -\langle \dot{T}^{\text{vis}}_3 \rangle^2,
\]

(41)

so there is energy loss due to the friction term, as expected. Let us analyze a little further the equation (41) exploring the possibility of a minus sign for the \(F(r)\) term,

\[
E = \frac{T^{\text{vis}}_3}{\kappa} \left( \frac{T^{\text{vis}}_3}{2} - |F(r)| \right).
\]

(42)

Note that if \(T^{\text{vis}}_3/2 > |F(r)|\) the energy remains positive. On the other hand, being \(0 < T^{\text{vis}}_3/2 < |F(r)|\) we have a negative energy \(E < 0\). However, if \(T^{\text{vis}}_3 < 0\) the energy of the mechanical system is always positive. Therefore, a negative \(T^{\text{vis}}_3\) (for \(F(r) < 0\)) is widely consistent with the positivity of the energy; in fact it guarantees a positive energy system. As an aside remark, note that if \(\Lambda = 0\) in Eq. (36) then the energy of the system never becomes negative (see (42)), i.e., the time \(\bar{t}\) above which the tension changes its sign tends to infinity, as can be seen from (37).

This analogy, although useful, should be made with great care. The “Lagrangian” (39) does not define a consistent dynamical system in the strict Hamiltonian sense. It is easy to see that the constraints of the system defined by (39) are such that the “Euler-Lagrange” equation (34) seems to be inconsistent [17]. Note, moreover, that by taking this analogy quite seriously one faces the difficult interpretation of a damped system (due to the friction term) whose Lagrangian does not depend on the velocities (therefore with vanishing acceleration). It is possible that with a more complete replacement in the relation (32), taking into account second order derivatives of \(T^{\text{vis}}_3\), we can set a consistent Hamiltonian dynamical system. We shall investigate it in a future work. With this warning in mind, the rough analogy expressed above is useful to get some physical insight about the shift of the brane tension sign.

V. CONCLUDING REMARKS

In this paper we review and extend the consistency conditions analysis for branes embedded into a bulk respecting Brans-Dicke gravity. It is shown that this scenario allows (by furnishing
necessary conditions) the existence of only positive tension branes in a five-dimensional AdS bulk. This seems to be very attractive, since negative tension branes are inherently unstable objects. This possibility, in practice, arises due to the relaxation of the consistency conditions, coming from the presence of the Brans-Dicke field, i.e., the studied consistency conditions are softened by the addition of the scalar field. Qualitatively, the presence of the scalar field engenders a coupling as $W' / \phi$ (being $r$ a real number) on the brane, which can be interpreted as a new (composite one) warp factor, “cancelling out” the negative contribution of the AdS bulk cosmological constant (see equation (25), for instance).

Going further, we studied a time variable brane tension toy model. It was shown that a dynamical tension brane can change its sign, aligning with the sign of the bulk cosmological constant. This is an example of how interesting the dynamics of the brane tension can be, i.e., even in a simplified toy model, as the present one. Even though simple, it is possible to find out an unusual and rich behavior for the tension time variability.

We would like to remark that the results presented, mainly in Section III, point out to the necessity of including the scalar field in the five dimensional bulk in order to obtain a completely compatible compactification scheme. We can say that this is our main result in this paper since all is based on the fact that gravity is of scalar-tensor nature. Some future research lines may follow from such statement. As stressed in Section III, the Brans-Dicke scalar field depends only on the extra (out of brane) dimension. Perhaps, a more complete scenario may arise by allowing the scalar field to vary with the brane coordinates. In this case, the use of the consistency conditions should be complemented by a more complete set of inputs, approaching a specific model, since the presence of terms accounting for the brane coordinates variation of the scalar field complicates the analysis (see equation (19), for instance). Apart of that, one must be careful in order to not contradict the experimental bounds to the Brans-Dicke parameter.

To summarize, the results presented in this paper suggest that a more complete scenario — contemplating also time variable tension branes — deserves special attention, since it may be the source of unusual and unexplored properties of braneworld scenarios.

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