Information inequalities and a dependent
Central Limit Theorem

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Abstract

We adapt arguments concerning information-theoretic convergence in the Central Limit Theorem to the case of dependent random variables under Rosenblatt mixing conditions. The key is to work with random variables perturbed by the addition of a normal random variable, giving us good control of the joint density and the mixing coefficient. We strengthen results of Takano and of Carlen and Soffer to provide entropy-theoretic, not weak convergence.

1 Introduction and notation

Under a variance constraint, entropy is maximised by the Gaussian. It is natural to consider whether entropy converges to this maximum in the Central Limit Theorem regime. This is a strong sense of convergence, and is discussed by Brown [3], Barron [1] and Johnson [7]. These papers only deal with the case of independent random variables, [3] and [1] in the case of identically distributed variables, and [7] for non-identical variables satisfying a Lindeberg-like condition. This paper extends these techniques to weakly dependent random variables.

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Takano [12], [13] considers the entropy of convolutions of dependent random variables, though he imposes a strong $\delta_4$-mixing condition (see Definition 2.3). Carlen and Soffer [4] also use entropy-theoretic methods in the dependent case, though the conditions which they impose are not transparent. Takano, in common with Carlen and Soffer, does not prove convergence in relative entropy of the full sequence of random variables, but rather convergence of the ‘rooms’ (in Bernstein’s terminology), equivalent to weak convergence of the original variables. Our conclusion is stronger. In a previous paper [8], we used similar techniques to establish entropy-theoretic convergence for FKG systems, which whilst providing a natural physical model, restrict us to the case of positive correlation.

We will consider a doubly infinite stationary collection of random variables $\ldots, X_{-1}, X_0, X_1, X_2, \ldots$, with mean zero and finite variance. We write $v_n$ for $\text{Var}(\sum_{i=1}^n X_i)$ and $U_n = (\sum_{i=1}^n X_i)/\sqrt{n}$. We will consider perturbed random variables $V_n^{(\tau)} = (\sum_{i=1}^n X_i + Z_i^{(\tau)})/\sqrt{n} \sim U_n + Z^{(\tau)}$, for $Z_i^{(\tau)}$ a sequence of $N(0, \tau)$ independent of $X_i$ and each other. In general, $Z^{(s)}$ will be a $N(0, s)$. If the limit $\sum_{j=-\infty}^{\infty} \text{Cov}(X_0, X_j)$ exists then we denote it by $v$.

**Definition 1.1** Given two random variables $S, T$, the $\alpha$-mixing coefficient is defined to be:

$$\alpha(S, T) = \sup_{A,B} |\mathbb{P}((S \in A) \cap (T \in B)) - \mathbb{P}(S \in A)\mathbb{P}(T \in B)|.$$

If $\Sigma_a$ is the $\sigma$-field generated by $X_a, X_{a+1}, \ldots, X_b$ (where $a$ or $b$ can be infinite), then for each $t$, define:

$$\alpha(t) = \sup \left\{ \alpha(S, T) : S \in \Sigma_{-\infty}^0, T \in \Sigma_t^\infty \right\},$$

and define the process to be $\alpha$-mixing if $\alpha(t) \to 0$ as $t \to \infty$.

See Bradley [2] for a discussion of the properties and alternative definitions of mixing coefficients. Note that $\alpha$-mixing is sometimes referred to as strong mixing, and is implied by uniform mixing (control of $|P(A|B) - P(A)|$, equivalent to the Doeblin condition for Markov chains). All $m$-dependent processes are $\alpha$-mixing, as well as any stationary, real aperiodic Harris chain (which includes every finite state irreducible aperiodic Markov chain).

**Definition 1.2** For a random variable $U$ with smooth density $p$, we consider the score function $\rho(u) = p'(u)/p(u)$, the Fisher information $J(U) = \mathbb{E}\rho^2(U)$, and the standardised Fisher information $J_{st}(U) = \sigma_U^2 J(U) - 1.$
We continue the technique used to prove convergence in relative entropy first developed by Barron [1], and later adapted to the non-identical case by Johnson [7]. That is, we use de Bruijn’s identity:

**Lemma 1.3** If $U$ is a random variable with density $f$ and variance 1, and $Z^{(\tau)}$ is a sequence of normals independent of $U$, then the relative entropy distance $D$ between $f$ and the standard Gaussian density $\phi$ is given by:

$$D(f\|\phi) = \frac{1}{2} \int_{0}^{\infty} \left( J(U + Z^{(\tau)}) - \frac{1}{1 + \tau} \right) d\tau.$$  

Our main theorems concerning strong mixing variables are as follows:

**Theorem 1.4** Consider a stationary collection of random variables $X_i$, with finite $(2 + \delta)$th moment. If $\sum_{j=1}^{\infty} \alpha(j)^{\delta/(2+\delta)} < \infty$, then for any $\tau > 0$:

$$\lim_{n \to \infty} J_{st}(V_n^{(\tau)}) \to 0.$$  

Note that the condition on the $\alpha(j)$ implies that $v_n/n \to v < \infty$ (see Lemma 2.7). In the next theorem, we have to distinguish two cases, where $v = 0$ and where $v > 0$. For example, if $Y_j$ are IID, and $X_j = Y_j - Y_{j+1}$ then $U_n = (Y_1 - Y_{n+1})/\sqrt{n} \to \delta_0$. However, since we make a normal perturbation, we know that $J_{st}(V_n^{(\tau)}) = (v_n/n + \tau)J(V_n^{(\tau)}) - 1 \leq (v_n/n + \tau)J(Z^{(\tau)}) - 1 = v_n/n\tau$, so the case $v = 0$ automatically works in Theorem 1.4.

We can provide a corresponding result for convergence in relative entropy, with some extra conditions:

**Theorem 1.5** Consider a stationary collection of random variables $X_i$, with finite $(2 + \delta)$th moment. If

1. $\sum_{j=1}^{\infty} \alpha(j)^{\delta/(2+\delta)} < \infty$
2. $v = \sum_{j=-\infty}^{\infty} \text{Cov}(X_0, X_j) > 0$
3. $f_N(\tau) = \sup_{n \geq N} \left( \frac{nJ_{st}(V_n^{(\tau)})}{v_n + n\tau} \right)$, for some $N$, $\int f_N(\tau)d\tau < \infty$
then writing $g_n$ for the density of $(\sum_{i=1}^{n} X_i)/\sqrt{v_n}$ then:

$$\lim_{n \to \infty} D(g_n\|\phi) \to 0.$$  

**Proof** Follows from Theorem 1.4 by a dominated convergence argument using de Bruijn’s identity, Lemma 1.3.

Note that convergence in relative entropy is a strong result and implies convergence in $L^1$ and hence weak convergence of the original variables.

Convergence of Fisher information, Theorem 1.4, is actually implied by Ibragimov’s classical weak convergence result. This follows since the density of $V_n^{(r)}$ (and its derivative) can be expressed as expectations of a continuous bounded function of $U_n$. Shimizu discusses this technique, which can only work for random variables perturbed by a normal. We hope our method may be extended to the general case, since results such as Proposition 3.2 do not need the random variables to be in this smoothed form. For example in the independent case, we show in a forthcoming paper that $J_{st}(U_n) \to 0$, if $J(U_m)$ is finite for some $m$, and if $U_{2k}$ is unimodal for infinitely many $k$ (no normal perturbation is necessary). In any case, we feel there is independent interest in seeing why the normal distribution is the limit of convolutions, as the score function becomes closer to the linear case which characterises the Gaussian.

## 2 Fisher Information and convolution

**Definition 2.1** For random variables $X, Y$ with score functions $\rho_X, \rho_Y$, for any $\beta$, we define $\tilde{\rho}$ for the score function of $\sqrt{\beta}X + \sqrt{1-\beta}Y$ and then:

$$\Delta(X,Y,\beta) = \mathbb{E} \left( \sqrt{\beta}\rho_X(X) + \sqrt{1-\beta}\rho_Y(Y) - \tilde{\rho} \left( \sqrt{\beta}X + \sqrt{1-\beta}Y \right) \right)^2.$$  

Firstly, we provide a theorem which tells us how Fisher information changes on the addition of two random variables which are nearly independent.

**Theorem 2.2** Let $S$ and $T$ be random variables, with max$(\text{Var } S, \text{Var } T) \leq K\tau$. Define $X = S + Z_S^{(r)}$ and $Y = T + Z_T^{(r)}$ (for $Z_S^{(r)}$ and $Z_T^{(r)}$ normal $N(0, \tau)$
independent of $S$, $T$ and each other), with score functions $\rho_X$ and $\rho_Y$. There exists a constant $C = C(K, \tau, \epsilon)$ such that:

$$\beta J(X) + (1-\beta)J(Y) - J\left(\sqrt{\beta X} + \sqrt{1-\beta Y}\right) + C\alpha(S,T)^{1/3-\epsilon} \geq \Delta(X,Y,\beta).$$

If $S,T$ have bounded $k$th moment, we can replace $1/3$ by $k/(k+4)$. The proof requires some involved analysis, and is deferred to Section 3.

In comparison, Takano [12], [13] produces bounds which depend on $\delta_4(S,T)$, where:

**Definition 2.3** For random variables $S,T$ with joint density $p_{S,T}(s,t)$ and marginal densities $p_S(s)$ and $p_T(t)$, define the $\delta_n$ coefficient to be:

$$\delta_n(S,T) = \left( \int p_S(s)p_T(t) \left| \frac{p_{S,T}(s,t)}{p_S(s)p_T(t)} - 1 \right|^n dsdt \right)^{1/n}.$$

In the case where $S,T$ have a continuous joint density, it is clear that Takano’s condition is more restrictive, and lies between two more standard measures of dependence:

$$4\alpha(S,T) \leq \delta_4(S,T) \leq \delta_\infty(S,T) = \psi(S,T) = \sup_{A,B} \left| \frac{P(A \cap B)}{P(A)P(B)} - 1 \right|.$$

(as before see Bradley [2] for a discussion of different mixing conditions).

Another use of the smoothing of the variables allows us to control the mixing coefficients themselves:

**Theorem 2.4** For $S$ and $T$, define $X = S + Z_\tau(S)$ and $Y = T + Z_\tau(T)$, where $\max(\text{Var } S, \text{Var } T) \leq K\tau$. If $Z$ has variance $\epsilon$, then there exists a function $f_K$ such that

$$\alpha(X + Z, Y) \leq \alpha(X, Y) + f_K(\epsilon),$$

where $f_K(\epsilon) \to 0$ as $\epsilon \to 0$.

**Proof** See Section 4.

To complete our analysis, we need lower bounds on the term $\Delta(X,Y,\beta)$. For independent $X, Y$ it equals zero exactly when $\rho_X$ and $\rho_Y$ are linear, and if it is small then $\rho_X$ and $\rho_Y$ are close to linear. Indeed, in [7] we make two definitions:
Definition 2.5 For a function $\psi$, define the class of random variables $X$ with variance $v_X$ such that:

$$C_\psi = \{ X : \mathbb{E}X^2 \mathbb{I}(|X| \geq R\sqrt{v_X}) \leq v_X \psi(R) \}.$$  

Further, define a semi-norm $\| \cdot \|_{\Theta}$ on functions via:

$$\| f \|_{\Theta}^2 = \inf_{a,b} \mathbb{E} \left( f(Z(\tau/2)) - aZ(\tau/2) - b \right)^2.$$  

Combining results from previous papers we obtain:

Proposition 2.6 For $S$ and $T$ with $\max(\text{Var} S, \text{Var} T) \leq K\tau$, define $X = S + Z(\tau)S$, $Y = T + Z(\tau)T$. For any $\psi, \delta > 0$, there exists a function $\nu = \nu_{\psi, \delta, K, \tau}$, with $\nu(\epsilon) \to 0$ as $\epsilon \to 0$, such that if $X, Y \in C_\psi$, and $\beta \in (\delta, 1 - \delta)$ then

$$J_{\text{st}}(X) \leq \nu(\Delta(X,Y,\beta)).$$

Proof We reproduce the proof of Lemma 3.1 of Johnson and Suhov [9], which implies $p(x,y) \geq (\exp(-4K)/4)\phi_{\tau/2}(x)\phi_{\tau/2}(y)$. This follows since by Chebyshev $\int \mathbb{I}(s^2+t^2 \leq 4K\tau) dF_{S,T}(s,t) \geq 1/2$, and since $(x-s)^2 \leq 2x^2+2s^2$:

$$p(x,y) = \int \phi_{\tau}(x-s)\phi_{\tau}(y-t)dF_{S,T}(s,t)$$

$$\geq \frac{1}{2} \min \{ \phi_{\tau}(x-s)\phi_{\tau}(y-t) : s^2 + t^2 \leq 4K\tau \}$$

$$= \frac{\phi_{\tau/2}(x)\phi_{\tau/2}(y)}{4} \exp \left( \min_{s^2+t^2 \leq 4K\tau} \left\{ -\frac{s^2-t^2}{\tau} \right\} \right)$$

$$\geq \frac{1}{4} \exp(-4K)\phi_{\tau/2}(x)\phi_{\tau/2}(y)$$

Hence writing $h(x,y) = \sqrt{\beta}\rho_{X}(x) + \sqrt{1-\beta}\rho_{Y}(y) - \tilde{\rho}(\sqrt{\beta}x + \sqrt{1-\beta}y)$, then:

$$\Delta(X,Y,\beta) = \int p(x,y)h(x,y)^2dxdy$$

$$\geq \frac{\exp(-8K)}{16} \int \phi_{\tau/2}(x)\phi_{\tau/2}(y)h(x,y)^2dxdy$$

$$\geq \frac{\beta(1-\beta)\exp(-8K)}{32}(\|\rho_{X}\|_{\Theta}^2 + \|\rho_{Y}\|_{\Theta}^2).$$
by Proposition 3.2 of Johnson [7]. The crucial result of [7] implies that for fixed \( \psi \), if the sequence \( X_n \in C_\psi \) have score functions \( \rho_n \), then \( \| \rho_n \|_\Theta \to 0 \) implies that \( J_{st}(X_n) \to 0 \).

We therefore concentrate on random processes such that the sums \( X_1 + X_2 + \ldots X_m \) have uniformly decaying tails:

Lemma 2.7 (Ibragimov, [6]) If \( \{X_j\} \) are stationary with \( E|X|^{2+\delta} < \infty \) for some \( \delta > 0 \) and \( \sum_{j=1}^{\infty} \alpha(j)^{\delta+2} < \infty \), then

1. \( (X_1 + \ldots X_m) \) belong to some class \( C_\psi \), uniformly in \( m \).
2. \( v_n/n \to v = \sum_{j=-\infty}^{\infty} \text{Cov}(X_0, X_j) < \infty \).

We are able to complete the proof of the CLT, under strong mixing conditions.

Proof of Theorem 1.4 Combining Theorems 2.2 and 2.4 and defining \( \tilde{V}_n^{(\tau)} = (\sum_{i=m+1}^{n} X_i + Z_i^{(\tau)})/\sqrt{n} \), we obtain that for \( m \geq n \),

\[
J_{st}(V_{m+n}^{(\tau)}) \leq \frac{m}{m+n} J_{st}(V_m^{(\tau)}) + \frac{n}{m+n} J_{st}(V_n^{(\tau)}) + c(m) - \Delta(V_m^{(\tau)}, \tilde{V}_n^{(\tau)}, \frac{m}{m+n}),
\]

where \( c(m) \to 0 \) as \( m \to \infty \). We show this using the idea of ‘rooms and corridors’ – that the sum can be decomposed into sums over blocks which are large, but separated, and so close to independence. For example, writing \( W_n^{(\tau/2)} = (\sum_{i=m+1}^{n} X_i)/\sqrt{n} + Z_{m/2}^{(\tau/2)} \), Theorem 2.4 shows that

\[
\alpha(V_m^{(\tau/2)}, W_n^{(\tau/2)}) \leq \alpha(V_{m/2}^{(\tau/2)}, W_n^{(\tau/2)}) + f_k(1/\sqrt{m}) = \alpha(\sqrt{m}) + f_k(1/\sqrt{m}).
\]

In the notation of Theorem 2.2 \( c(m) = C(K, \tau/2, \epsilon)(\alpha(\sqrt{m}) + f_k(1/\sqrt{m}))^{1/3-\epsilon} \).

We first establish convergence along the ‘powers of 2 subsequence’ \( S_k = V_{2^k}^{(\tau)} \), writing \( \tilde{S}_k \) for \( (\sum_{i=2^k}^{2^{k+1}} X_i + Z_i^{(\tau)})/\sqrt{2^k} \), since

\[
J_{st}(S_{k+1}) \leq J_{st}(S_k) + c(k) - \Delta(S_k, \tilde{S}_k, 1/2)
\]

where \( c(k) \to 0 \). Then use an argument structured like Linnik’s proof [10]. Given \( \epsilon \), we can find \( K \) such that \( c(k) \leq \epsilon/2 \), for all \( k \geq K \). Now

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1. either for all \( k \geq K \), \( 2c(k) \leq \Delta(S_k, \tilde{S}_k, 1/2) \), and so
\[
J_{st}(S_k) - J_{st}(S_{k+1}) \geq \Delta(S_k, \tilde{S}_k, 1/2) / 2,
\]
so summing the telescoping sum, we deduce that \( \sum_k \Delta(S_k, \tilde{S}_k, 1/2) \) is finite, and hence there exists \( L \) such that \( \Delta(S_L, \tilde{S}_L, 1/2) \leq \epsilon \).

2. or for some \( L \geq K \), \( 2c(L) \geq \Delta(S_L, \tilde{S}_L, 1/2) \), then \( \Delta(S_L, \tilde{S}_L, 1/2) \leq \epsilon \).

Thus, in either case, there exists \( L \) such that \( \Delta(S_L, \tilde{S}_L, 1/2) \leq \epsilon \), and hence by Proposition 2.6 \( J_{st}(S_L) \leq \nu(\epsilon) \).

Now, for any \( k \geq L \), either \( J_{st}(S_{k+1}) \leq J_{st}(S_k) \), or \( \Delta(S_k, \tilde{S}_k, 1/2) \leq c(k) \leq \epsilon \). In the second case, \( J_{st}(S_k) \leq \nu(\epsilon) \), so that \( J_{st}(S_{k+1}) \leq \nu(\epsilon) + \epsilon \). In either case, we prove by induction that for all \( k \geq L \), that \( J_{st}(S_{k+1}) \leq \nu(\epsilon) + \epsilon \).

We can fill in the gaps to gain control of the whole sequence, adapting the proof of the standard sub-additive inequality, using the methods described in Appendix 2 of Grimmett [5].

\[\square\]

### 3 Proof of sub-additive relations

This is the key part of the argument, proving the bounds at the heart of the limit theorems. However, although the analysis is somewhat involved, it is not technically difficult.

We introduce notation where it will be clear whether densities and score functions are associated with joint or marginal distributions, by their number of arguments: \( \rho_X(x) \) will be the score function of \( X \), and \( p'_x(x) \) the derivative of its density. For joint densities \( p_{X,Y}(x,y) \), \( p^{(1)}_{X,Y}(x,y) \) will be the derivative of the density with respect to the first argument and \( p^{(1)}_{X,Y}(x,y) = p^{(1)}_{X,Y}(x,y) / p_{X,Y}(x,y) \), and so on.

Note that a similar equation to the independent case tells us about the behaviour of Fisher Information of sums:

**Lemma 3.1** If \( X, Y \) are random variables, with joint density \( p(x,y) \), and score functions \( \rho^{(1)}_{X,Y} \) and \( \rho^{(2)}_{X,Y} \) then \( X + Y \) has score function \( \tilde{\rho} \) given by
\[
\tilde{\rho}(z) = \mathbb{E} \left[ \rho^{(1)}_{X,Y}(X,Y) \mid X + Y = z \right] = \mathbb{E} \left[ \rho^{(2)}_{X,Y}(X,Y) \mid X + Y = z \right].
\]
Proof Since $X + Y$ has density $r(z) = \int p_{X,Y}(z - y, dy)$, then:

$$r'(z) = \int p_{X,Y}^{(1)}(z - y, dy).$$

Hence dividing, we obtain that:

$$\tilde{\rho}(z) = \frac{r'(z)}{r(z)} = \int p_{X,Y}^{(1)}(z - y, y) \frac{p_{X,Y}(z - y, y)}{r(z)} dy,$$

as claimed.

For given $a, b$, define the function $M(x, y) = M_{a,b}(x, y)$ by:

$$M(x, y) = a \left( \rho_{1,X,Y}^{(1)}(x, y) - \rho_X(x) \right) + b \left( \rho_{2,X,Y}^{(2)}(x, y) - \rho_Y(y) \right),$$

which is zero if $X$ and $Y$ are independent. Using properties of the perturbed density, we will show that if $\alpha(S, T)$ is small, then $M$ is close to zero.

**Proposition 3.2** If $X, Y$ are random variables, with marginal score functions $\rho_X, \rho_Y$, and if the sum $\sqrt{\beta X} + \sqrt{1 - \beta Y}$ has score function $\tilde{\rho}$ then

$$\beta J(X) + (1 - \beta) J(Y) - J \left( \sqrt{\beta X} + \sqrt{1 - \beta Y} \right) + 2\sqrt{\beta(1 - \beta)} \mathbb{E} \rho_X(X) \rho_Y(Y) + 2 \mathbb{E} M_{\sqrt{\beta X}, \sqrt{1 - \beta Y}}(X, Y) \tilde{\rho}(X + Y)$$

$$= \mathbb{E} \left( \sqrt{\beta} \rho_X(X) + \sqrt{1 - \beta} \rho_Y(Y) - \tilde{\rho} \left( \sqrt{\beta X} + \sqrt{1 - \beta Y} \right) \right)^2$$

**Proof** By the two-dimensional version of Stein’s equation, for any function $f(x, y)$ and for $i = 1, 2$:

$$\mathbb{E} \rho^{(i)}_{X,Y}(X, Y) f(X, Y) = -\mathbb{E} f^{(i)}(X, Y).$$

Hence, we know that taking $f(x, y) = \tilde{\rho}(x + y)$, for any $a, b$:

$$\mathbb{E} (a \rho_X(X) + b \rho_Y(Y)) \tilde{\rho}(X + Y) = (a + b) J(X + Y) - \mathbb{E} M_{a,b}(X, Y) \tilde{\rho}(X + Y).$$

By considering $\int p(x, y) (a \rho_X(x) + b \rho_Y(y) - (a + b) \tilde{\rho}(x + y))^2 dxdy$, dealing with the cross term with the expression above, we deduce that:

$$a^2 J(X) + b^2 J(Y) - (a + b)^2 J(X + Y) + 2ab \mathbb{E} \rho_X(X) \rho_Y(Y) + 2(a + b) \mathbb{E} M_{a,b}(X, Y) \tilde{\rho}(X + Y)$$

$$= \mathbb{E} (a \rho_X(X) + b \rho_Y(Y) - (a + b) \tilde{\rho}(X + Y))^2 \geq 0.$$
As in the independent case, we can rescale, and consider \( X' = \sqrt{\beta}X, Y' = \sqrt{1-\beta}Y \), and take \( a = \beta, b = 1-\beta \). Note that \( \sqrt{\beta}\rho_X(u) = \rho_X(u/\sqrt{\beta}) \), \( \sqrt{1-\beta}\rho_Y(v) = \rho_Y(v/\sqrt{1-\beta}) \).

Next, we require an extension of Lemma 3 of Barron [1] applied to single and bivariate random variables:

**Lemma 3.3** For any \( S,T \), define \((X,Y) = (S + Z_{S}^{(\tau)}, T + Z_{T}^{(\tau)}) \) and define \( p^{(2\tau)} \) for the density of \((S + Z_{S}^{(2\tau)}, T + Z_{T}^{(2\tau)}) \). There exists a constant \( c_{r,k} = \sqrt{2}(2k/\tau e)^{k/2} \) such that for all \( x,y \):

\[
\begin{align*}
p_X^{(\tau)}(x)|\rho_X(x)|^k & \leq c_{r,k} p^{(2\tau)}(x) \\
p^{(\tau)}(x,y)|\rho_{X,Y}^{(1)}(x,y)|^k & \leq c_{r,k} p^{(2\tau)}(x,y) \\
p^{(\tau)}(x,y)|\rho_{X,Y}^{(2)}(x,y)|^k & \leq c_{r,k} p^{(2\tau)}(x,y)
\end{align*}
\]

and hence

\[
(\mathbb{E}|\rho_X(X)|^k)^{1/k} \leq \sqrt{2^{1/k}2k/\tau e}.
\]

**Proof** We adapt Barron’s proof, using Hölder’s inequality and the bound;

\[
(u/\tau)^k \phi_{r}(u) \leq c_{r,k} \phi_{2\tau}(u)
\]

for all \( u \).

\[
p_X'(x)^k = \left( \mathbb{E} \left( \frac{x-S}{\tau} \phi_{r}(x-S) \right)^k \right)
\leq \left( \mathbb{E} \left( \frac{x-S}{\tau} \right)^k \phi_{r}(x-S) \right) (\mathbb{E} \phi_{r}(x-S))^{k-1}
\leq c_{r,k} (\mathbb{E} \phi_{2\tau}(x-S)) p_X(x)^{k-1}
\]

A similar argument gives the other bounds. \( \square \)

Now, the normal perturbation ensures that the density doesn’t decrease too large, and so the modulus of the score function can’t grow too fast.

**Lemma 3.4** Consider \( X \) of the form \( X = S + Z_{S}^{(\tau)} \), where \( \text{Var} S \leq K\tau \). If \( X \) has score function \( \rho \), then for \( B > 1 \):

\[
\int_{-B\sqrt{\tau}}^{B\sqrt{\tau}} \rho(u)^2 du \leq \frac{8B^3}{\sqrt{\tau}} (3 + 2K).
\]
Proof  As in Proposition 2.6, \( p(u) \geq (2 \exp 2K)^{-1} \phi_{\tau/2}(u) \), so that for \( u \in (-B\sqrt{\tau}, B\sqrt{\tau}) \), \((B\sqrt{\tau}p(u))^{-1} \leq 2\sqrt{\tau} \exp(B^2 + 2K) / B \leq 2\sqrt{\tau} \exp(B^2 + 2K)\). Hence for any \( k \geq 1 \), by Hölder’s inequality:

\[
\int_{-B\sqrt{\tau}}^{B\sqrt{\tau}} \rho(u)^2 du \leq \left( \int_{-B\sqrt{\tau}}^{B\sqrt{\tau}} |\rho(u)|^{2k} du \right)^{1/k} (2B\sqrt{\tau})^{1-1/k} \\
\leq \left( \int_{-B\sqrt{\tau}}^{B\sqrt{\tau}} p(u)|\rho(u)|^{2k} du \right)^{1/k} (2B\sqrt{\tau}) \\
\leq \left( \frac{8B}{\sqrt{\tau}} \right)^k \left( 2\sqrt{2\pi} \exp(B^2 + 2K) \right)^{1/k} \exp(-1).
\]

Since we have a free choice of \( k \geq 1 \) to maximise \( k \exp(v/k) \), choosing \( k = v \geq 1 \) means that \( k \exp(v/k) \exp(-1) = v \). Hence we obtain a bound of

\[
\int_{-B\sqrt{\tau}}^{B\sqrt{\tau}} \rho(u)^2 du \leq \frac{8B}{\sqrt{\tau}} \left( B^2 + 2K + \log(2\sqrt{2\pi}) \right) \leq \frac{8B^3}{\sqrt{\tau}} (3 + 2K).
\]

By considering \( S \) normal, so that \( \rho \) grows linearly with \( u \), we know that the \( B^3 \) rate of growth is a sharp bound.

Lemma 3.5  For random variables \( S, T \), let \( X = S + Z_S^{(\tau)} \) and \( Y = Y + Z_T^{(\tau)} \), define \( L_B = \{ |x| \leq B\sqrt{\tau}, |y| \leq B\sqrt{\tau} \} \). If \( \max(\text{Var} S, \text{Var} T) \leq K\tau \) then there exists a function \( f_1(K, \tau) \) such that for \( B \geq 1 \):

\[
\mathbb{E} M_{a,b}(X, Y) \widetilde{\rho}(X + Y) \mathbb{I}((X, Y) \in L_B) \leq \alpha(S, T) B^4(a + b) f_1(K, \tau).
\]

Proof  Lemma 1.2 of Ibragimov [6] states that if \( \xi, \nu \) are random variables measurable with respect to \( \mathcal{A}, \mathcal{B} \) respectively, with \( |\xi| \leq C_1 \) and \( |\nu| \leq C_2 \) then:

\[
|\text{Cov}(\xi, \nu)| \leq 4C_1C_2\alpha(\mathcal{A}, \mathcal{B}).
\]

Now since \( |\phi_{\tau}(u)| \leq 1/\sqrt{2\pi \tau} \), and \( |u\phi_{\tau}(u)/\tau| \leq \exp(-1/2)/\sqrt{2\pi \tau^2} \), we deduce that:

\[
|p_{X,Y}(x, y) - p_X(x)p_Y(y)| = |\text{Cov}(\phi_{\tau}(x - S), \phi_{\tau}(y - T))| \leq \frac{2}{\pi \tau} \alpha(S, T).
\]
Similarly:

\[ |p_{X,Y}^{(1)}(x,y) - p'_X(x)p_Y(y)| = \left| \text{Cov} \left( \left( \frac{x-S}{\tau} \right) \phi_\tau(x-S), \phi_\tau(y-T) \right) \right| \leq 4 \left( \frac{\exp(-1/2)}{\sqrt{2\pi\tau^2}} \frac{1}{\sqrt{2\pi\tau}} \right) \alpha(S,T). \]

By rearranging \( M_{a,b} \), we obtain:

\[ p_{X,Y}(x,y)|M_{a,b}(x,y)| \leq 2 \alpha(S,T) \pi \tau (a + b) \left( \frac{4\sqrt{2} \tau \sqrt{3}}{\tau e} + \sqrt{16B^4(3 + 2K)} \right). \]

By Cauchy-Schwarz:

\[
\int p_{X,Y}(x,y)M_{a,b}(x,y)\tilde{\rho}(x+y)\mathbb{I}((x,y) \in L_B)dxdy \\
\leq \left( \frac{2\alpha(S,T)}{\pi \tau} \right) \sqrt{32B^4(3 + 2K)} (a + b) \left( \frac{\sqrt{4B^2\tau}}{\sqrt{\tau e}} + \sqrt{16B^4(3 + 2K)} \right) \\
\leq \alpha(S,T)B^4(a + b) \left( \frac{40\sqrt{2}(3 + 2K)}{\tau} \right). 
\]

This follows firstly since by Lemma 3.4

\[
\int \rho_X(x)^2 \mathbb{I}((x,y) \in L_B)dxdy \leq (2B\sqrt{\tau}) \int_{-B\sqrt{\tau}}^{B\sqrt{\tau}} \rho_X(x)^2 dx \leq 16B^4(3 + 2K)
\]

and by Lemma 3.4

\[
\int \tilde{\rho}(x+y)^2 \mathbb{I}((x,y) \in L_B)dxdy \\
\leq \int \tilde{\rho}(x+y)^2 \mathbb{I}(|x+y| \leq 2B\sqrt{\tau}) \mathbb{I}(|y| \leq B\sqrt{\tau})dxdy \\
\leq 2B\sqrt{\tau} \int_{-2B\sqrt{\tau}}^{2B\sqrt{\tau}} \tilde{\rho}(z)^2 dz \leq 32B^4(3 + 2K).
\]

Now uniform decay of the tails gives us control everywhere else:
Lemma 3.6  For \( S,T \) with mean zero and variance \( \leq K \tau \), let \( X = S + Z_S^{(r)} \) and \( Y = T + Z_T^{(r)} \). There exists a function \( f_2(\tau, K, \epsilon) \) such that:

\[
\mathbb{E}M_{a,b}(X,Y)\tilde{\rho}(X+Y)\mathbb{I}((X,Y) \notin L_B)dx\,dy \leq (a+b)\frac{f_2(\tau, K, \epsilon)}{B^{2-\epsilon}}.
\]

For \( S,T \) with \( k \)th moment \( (k \geq 2) \) bounded above, we can achieve a rate of decay of \( 1/B^{k-\epsilon} \).

**Proof**  By Chebyshev \( \mathbb{P}\left((S + Z_S^{(2r)}, T + Z_T^{(2r)}) \notin L_B\right) \leq \int p^{(2r)}(x,y)(x^2 + y^2)/(2B^2)\,dx\,dy \leq (K+2)/B^2 \) so by Hölder-Minkowski for \( 1/p + 1/q = 1 \):

\[
\mathbb{E}\rho_{X,Y}^{(1)}(X,Y)\tilde{\rho}(X+Y)\mathbb{I}((X,Y) \notin L_B)
\leq \left( \mathbb{E}|\rho_{X,Y}^{(1)}(X,Y)|^p\mathbb{I}((X,Y) \notin L_B) \right)^{1/p} \left( \mathbb{E}|\tilde{\rho}(X+Y)|^q \right)^{1/q}
\leq c_{r,p}^1 c_{r,q}^{1/q} \mathbb{P}\left((S + Z_S^{(2r)}, T + Z_T^{(2r)}) \notin L_B\right)^{1/p}
\leq \frac{2\sqrt{2} \exp(-1)}{\tau} \sqrt{pq} \left( \frac{K+2}{B^2} \right)^{1/p}
\]

By choosing \( p \) arbitrarily close to 1, we can obtain the required expression. The other terms work in a similar way. \( \Box \)

Similarly we bound the remaining product term:

Lemma 3.7  For random variables \( S,T \) with mean zero and variances satisfying \( \max(\text{Var } S, \text{Var } T) \leq K \tau \), let \( X = S + Z_S^{(r)} \) and \( Y = T + Z_T^{(r)} \). There exist functions \( f_3(\tau, K) \) and \( f_4(\tau, K) \) such that

\[
\mathbb{E}\rho_X(X)\rho_Y(Y) \leq f_3(\tau, K)B^4\alpha(S,T) + f_4(\tau, K)/B^2.
\]

**Proof**  Using part of Lemma 3.5, we know that \( p_{X,Y}(x,y) - p_X(x)p_Y(y) \leq 2\alpha(S,T)/(\pi \tau) \). Hence by an argument similar to that of Lemmas 3.6, we
obtain that:

$$\mathbb{E} \rho_X(X) \rho_Y(Y) = \int (p_{X,Y}(x,y) - p_X(x)p_Y(y)) \rho_X(x)\rho_Y(y) dxdy$$

$$\leq \frac{2\alpha(S,T)}{\pi \tau} \int |\rho_X(x)||\rho_Y(y)| \mathbb{I}((x,y) \in L_B) dxdy$$

$$+ \int p(x,y) |\rho_X(x)||\rho_Y(y)| \mathbb{I}((x,y) \notin L_B) dxdy$$

$$+ \int p(x)p(y) |\rho_X(x)||\rho_Y(y)| \mathbb{I}((x,y) \notin L_B) dxdy$$

$$\leq \frac{2\alpha(S,T)}{\pi \tau} \left( \int_{-B}^{B} |\rho_X(x)|^2 dx \right)^2$$

$$+ 2 \left( \int p_{X,Y}(x,y) |\rho_X(x)|^2 \mathbb{I}((x,y) \notin L_B) dxdy \right).$$

as required. \(\square\)

**Proof of Theorem 2.2** Combining Lemmas 3.5, 3.6 and 3.7, we obtain for given \(K, \tau, \epsilon\) that there exist constants \(C_1, C_2\) such that

$$\mathbb{E} M_{\sqrt{\beta}, \sqrt{1-\beta}} \rho + \sqrt{\beta(1-\beta)} \mathbb{E} \rho_X \rho_Y \leq C_1 \alpha(S,T)B^4 + C_2/B^{2-\epsilon},$$

so choosing \(B = (1/4\alpha(S,T))^{1/6} > 1\), we obtain a bound of \(C\alpha(S,T)^{1/3-\epsilon}\).

By Lemma 3.6, note that if \(X, Y\) have bounded \(k\)th moment, then we obtain decay at the rate \(C_1 \alpha(S,T)B^4 + C_2/B^{k'}\), for any \(k' < k\). Choosing \(B = \alpha(S,T)^{-1/(k'+4)}\), we obtain a rate of \(\alpha(S,T)^{k'/(k'+4)}\). \(\square\)

### 4 Control of the mixing coefficients

To control \(\alpha(X+Z, Y)\) and to prove Theorem 2.4, we use truncation, smoothing and triangle inequality arguments similar to those of the previous section. Write \(W\) for \(X + Z\), \(L_B = \{(x,y) : |x| \leq B\sqrt{\tau}, |y| \leq B\sqrt{\tau}\}\), and \(R\) for \(R \cap (-B\sqrt{\tau}, B\sqrt{\tau})\). Note that by Chebyshev, \(\mathbb{P}(|W| \in L_B) \leq \mathbb{P}(|W| \geq B\sqrt{\tau}) + \mathbb{P}(|Y| \geq B\sqrt{\tau}) \leq 2(K+1)/B^2\). Hence by the triangle inequality, for
any sets $S, T$:

$$\left| \mathbb{P}( (W, Y) \in (S, T) ) - \mathbb{P}(W \in S) \mathbb{P}(Y \in T) \right|$$

$$\leq \left| \mathbb{P}( (W, Y) \in (S, T) \cap L_B ) - \mathbb{P}(W \in \overline{S}) \mathbb{P}(Y \in \overline{T}) \right|$$

$$+ \mathbb{P}( (W, Y) \in L_B^c ) + \mathbb{P}( |W| \geq B\sqrt{\tau}) \mathbb{P}(|Y| \geq B\sqrt{\tau})$$

$$\leq \left| \mathbb{P}( (W, Y) \in (\overline{S}, \overline{T}) ) - \mathbb{P}(X, Y) \in (\overline{S}, \overline{T}) \right|$$

$$+ \left| \mathbb{P}(X \in \overline{S}) - \mathbb{P}(W \in \overline{S}) \mathbb{P}(Y \in \overline{T}) \right|$$

$$+ \frac{4(K + 1)}{B^2}$$

$$\leq \int |p_{W,Y}(w, y) - p_{X,Y}(w, y)| \mathbb{I}( (w, y) \in L_B ) dw dy + \alpha(X, Y)$$

$$+ \int |p_X(w) - p_W(w)| \mathbb{I}( |w| \leq B\sqrt{\tau}) dw + \frac{4(K + 1)}{B^2}$$

Here, the first inequality follows on splitting $\mathbb{R}^2$ into $L_B$ and $L_B^c$, the second by repeated application of the triangle inequality, and the third by expanding out probabilities using the densities. Now the key result is that:

**Proposition 4.1** For $S$ and $T$, define $X = S + Z_S^{(\tau)}$ and $Y = T + Z_T^{(\tau)}$, where $\max(\text{Var } S, \text{Var } T) \leq K\tau$. If $Z$ has variance $\epsilon$, then there exists a constant $C = C(B, K, \tau)$ such that:

$$\int |p_{W}(w) - p_{X}(w)| \mathbb{I}( |w| \leq B\sqrt{\tau}) dw \leq (\exp(C\epsilon^{1/5}) - 1) + 2\epsilon^{1/5}.$$ 

**Proof** We can show that for $|z| \leq \delta^2$ and $|x| \leq B\sqrt{\tau}$:

$$\frac{p_{X,Z}(x - z, z)}{p_{X,Z}(x, z)} = \exp \left( \int_{x-z}^{x} \rho_{X,Z}^{(1)}(u, z) du \right)$$

$$\leq \exp \left( \left( \int_{-2B\sqrt{\tau}}^{-2B\sqrt{\tau}} \rho_{X,Z}^{(1)}(u, z)^2 du \right)^{1/2} \delta \right)$$

$$\leq \exp C\delta,$$
by adapting Lemma 3.4 to cover bivariate random variables. Hence we know that:

\[
\int |p_W(w) - p_X(w)| \mathbb{1}(|w| \leq B\sqrt{\tau})dw \\
\leq \int |p_{X,Z}(w-z, z) - p_{X,Z}(w, z)| \mathbb{1}(|z| \leq \delta^2, |w| \leq B\sqrt{\tau})dzdw \\
+ \int |p_{X,Z}(w-z, z) - p_{X,Z}(w, z)| \mathbb{1}(|z| \geq \delta^2)dzdw \\
\leq \int p_{X,Z}(w, z)(\exp C\delta - 1)dwdz + 2\mathbb{P}(|Z| \geq \delta^2) \\
\leq (\exp C\delta - 1) + 2\mathbb{P}(|Z| \geq \delta^2)
\]

Thus choosing \( \delta = \epsilon^{1/5} \), the result follows. \( \square \)

Similar analysis allows us to control

\[
\int |p_{W,Y}(w, y) - p_{X,Y}(w, y)| \mathbb{1}((w, y) \in L_B)dwdy.
\]

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