GREEN’S FUNCTIONS FOR FAR-SIDE SEISMIC IMAGES: A POLAR EXPANSION APPROACH

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ABSTRACT

We have computed seismic images of magnetic activity on the far surface of the Sun by using a seismic-holography technique. As in previous works, the method is based on the comparison of waves going in and out of a particular point in the Sun but we have computed here the Green’s functions from a spherical polar expansion of the adiabatic wave equations in the Cowling approximation instead of using the ray-path approximation previously used in the far-side holography. A comparison between the results obtained using the ray theory and the spherical polar expansion is shown. We use the gravito-acoustic wave equation in the local plane-parallel limit in both cases and for the latter we take the asymptotic approximation for the radial dependencies of the Green’s function. As a result, improved images of the far-side can be obtained from the polar-expansion approximation, especially when combining the Green’s functions corresponding to two and three skips. We also show that the phase corrections in the Green’s functions due to the incorrect modeling of the uppermost layers of the Sun can be estimated from the eigenfrequencies of the normal modes of oscillation.

Subject headings: Local helioseismology, magnetic activity

1. INTRODUCTION

The idea that the observed wavefield at the solar surface can be used to estimate the wavefield at any other location in the Sun has resulted in the development of several local helioseismology methods such as the seismic-holography technique (Lindsey & Braun 1997). Although the technique has been used to characterize flows in the solar subphotosphere (Braun et al. 2004) and to investigate seismic sources from flares (Donea et al. 2006), its application to map areas of strong magnetic field (Lindsey & Braun 1998) has remained a major focus. In particular, the mapping of active regions in the non-visible (or far side) of the Sun (Lindsey & Braun 1990, 2000a, González Hernández et al. 2009) has proven to be a powerful tool for space weather forecasting. A general review on local helioseismology can be found in Gizon & Birch (2005).

The use of phase-sensitive seismic holography to map areas of strong magnetic field is based on the fact that there is a travel delay or phase shift between waves entering and exiting an active region (Braun et al. 1992, Duvall et al. 1996). The technique compares theoretical waves going in and out of a particular point in the Sun (the focus) with the observed wavefield, \( \tilde{\Psi}_{\text{obs}}(\Omega, t) \), in some portion of the solar surface (the pupil \( \mathcal{P}_\pm \)), by computing the ingression and egression functions given by,

\[
\tilde{H}_\pm(r_0, t_0) = \int \int_{\mathcal{P}_\pm} d\Omega \tilde{G}_\pm(r_0, t_0; R, \Omega, t) \tilde{\Psi}_{\text{obs}}(\Omega, t),
\]

where \( r_0 \) is the location of the focus, \( R \) the solar radius, \( \Omega = (\theta, \varphi) \), \( \tilde{G}_+ \) is a Green’s function that represents the theoretical disturbance at \((R, \Omega, t)\) resulting from a unit impulse originating at \((r_0, t_0)\) and \( \tilde{G}_- \) is the time-reverse counterpart (Lindsey & Braun 2000b, 2004).

In the quiet Sun, the wave field can be well represented by a superposition of gravito-acoustic Green’s functions of a standard reference model and the ingress and egression functions are very similar. However, when the focus is located in an area of high magnetic activity, the measured signal deviates from that, effectively introducing a phase shift between the waves going in and out. This can be detected by computing the correlation that in the Fourier domain becomes the product of the Fourier transforms, \( H_+ \) and \( H_- \),

\[
C(r_0) = \int_{\omega_1}^{\omega_2} H_+(r_0, \omega)H_-^*(r_0, \omega)d\omega.
\]

The phase of the correlation, \( \phi(r_0) = \arg C(r_0) \), is related to the perturbed travel time by \( \phi \simeq \omega \Delta \tau \). The seismic-holography maps are representations of \( \phi(r_0) \) as a measurement of the disturbance of the wave field due to the presence of magnetic regions.

Up to the present, far-side seismic holography has used Green’s functions calculated using the (acoustic) ray theory, e.g. Lindsey & Braun (2000b). Formally, this approximation is only valid for high frequencies and high angular degrees. In this paper, we develop the formalism to determine the Green’s functions through their polar angular degrees. Although the method allows a full numerical computation both in the radial and angular variables, in the present work we have used the asymptotic approximation for the radial dependencies of the Green’s function. On the other hand, the angular factors are com-
computed without any asymptotic approximation and hence the Green’s functions are expected to be more accurate for low and intermediate degrees.

The outline of the paper is as follow. In section 2 we describe the basic wave equations. Section 3 presents the calculation of the Green’s functions using the gravito-acoustic ray-path approximation and section 4 the Green’s functions using the spherical polar expansion. The phase shift introduced by the non-adiabatic sub-surface layers of the Sun is reviewed in section 5. In section 6, we present an example of the improvement achieved by using the new Green’s functions to calculate far-side maps of solar activity.

2. BASIC EQUATIONS

We start the analysis with the adiabatic wave equation in the Cowling and local plane-parallel approximations. In terms of the scalar field \( \Psi(r,t) = -\rho^{-1/2}\delta p \) where \( \rho \) is the density of the background state and \( \delta p(t,r) \) the Lagrangian pressure fluctuation, this equation is given by [see Gongh (1993)]

\[
c^{-2} \left( \frac{\partial^2}{\partial t^2} + \frac{\omega^2}{c^2} \right) \frac{\partial^2 \Psi}{\partial r^2} - \frac{\partial^2}{\partial r^2} \nabla^2 \Psi - N^2 \nabla^2 \Psi = 0
\]

(3)

where \( c \) and \( N \) are the adiabatic sound speed and buoyancy frequency, defined as usual, \( \nabla^2 \) represent the terms involving horizontal derivatives in the \( \nabla^2 \) operator and the critical acoustic frequency is given by

\[
\omega_c = \frac{c}{2H} \left( 1 - 2 \hat{r} \cdot \nabla H \right)^{1/2}.
\]

(4)

Here \( H \) is the density-scale height and \( \hat{r} \) is an upward directed unit vector.

If the background state does not depend on time, as we will assume, the scalar field can be Fourier decomposed such that the wave field is a superposition of monochromatic waves given by \( \Psi(r,t) = \Psi(r,\omega) \exp(-i\omega t) \) and Eq. (3) becomes

\[
\nabla^2 \Psi - \frac{N^2}{\omega^2} \nabla^2 \Psi + \frac{1}{c^2} \left( \omega^2 - \omega_c^2 \right) \Psi = 0.
\]

(5)

We define the operator

\[
\mathcal{L}_0 = \nabla^2 - \frac{N^2}{\omega^2} \nabla^2 + \frac{1}{c^2} \left( \omega^2 - \omega_c^2 \right),
\]

(6)

where \( N_0, c_0 \) and \( \omega_c \) are the buoyancy frequency, sound speed and cut-off frequency of a spherically-symmetric reference model. In this work we will use Model S from Christensen-Dalsgaard et al. (1996). We assume that the actual solar oscillations satisfy a wave equation of the form

\[
\mathcal{L}_0 \Psi + \mathcal{L}_0' \Psi = 0,
\]

(7)

with \( \mathcal{L}_0' \Psi \) accounting for all departures from Eq. (6) (e.g. non adiabatic effects) and from the reference model (e.g. non spherical perturbations to the sound speed).

2.1. Integral representation

Let us take a target point \( r_0 \), the focus, anywhere in the Sun. We introduce Green’s functions, \( G(r,r_0;\omega) \), that satisfy the source-point wave equation for the standard solar model,

\[
\mathcal{L}_0 G(r,r_0;\omega) = -\delta(r - r_0).
\]

(8)

Multiplying Eq. (7) by \( G \), Eq. (8) by \( \Psi \), subtracting both, integrating over a volume \( V \) that containing the focus \( r_0 \) and, finally, using the Green theorem, the above equation can be written as

\[
\int_S [G \nabla \Psi - \Psi \nabla G] \cdot dS = \Psi(r_0,\omega) + \int_V \mathcal{L}_0' \Psi dV,
\]

(9)

where \( S \) is the surface enclosing \( V \), \( dS \) is directed outwards and \( V = \nabla - (N_0^2/\omega^2) \nabla_h \). If \( S \) is the solar surface with radius \( R \), the expression is simplified to

\[
R^2 \int_S \left( G \frac{\partial \Psi}{\partial r} - \Psi \frac{\partial G}{\partial r} \right) d\Omega = \Psi(r_0,\omega) + \int_V \mathcal{L}_0' \Psi dV.
\]

(10)

A filtered version of the left-hand side of Eq. (10) can be obtained from the observed wavefield and a reference model. Moreover, if we compare the left-hand side of Eq. (10) obtained from two sets of Green’s functions, the differences will be due to the term with \( \mathcal{L}_0' \). This includes what can be called the “anomalies” (e.g. magnetic activity), but also systematic errors in the wave operator \( \mathcal{L}_0 \).

As indicated in the introduction we will consider Green’s functions \( G_{\pm} \) corresponding to outgoing and ingoing waves from the focus. In this case, Eq. (10) is a Kirchhoff representation for the wave field \( \Psi \) in the frequency domain. We compute the egression and ingress functions as

\[
H_{\pm}(r_0,\omega) = \int_{P_{\pm}} \Psi_{\text{obs}}(\Omega,\omega) \frac{\partial G_{\pm}}{\partial r}(\Omega,\omega; r_0) d\Omega,
\]

(11)

where \( \Psi_{\text{obs}}(\Omega,\omega) \) is the observed signal (Doppler shift velocity in the frequency domain) and the integral is over a limited region of the solar surface, the pupil \( (P_{\pm}) \). Correlating the ingress and egression functions one can expect to obtain a filtered version of the second term on the right-hand side of Eq. (10).

3. GRAVITO-ACOUSTIC RAY APPROXIMATION

We will compare our results with those obtained from the ray theory. Rather than the acoustic approximation used by Lindsey & Braun (2000a), we will consider a more general approximation.

For high frequencies, the general solution of \( \mathcal{L}_0 \Psi = 0 \) can be expressed as a superposition of “rays” of the form

\[
\Psi(r) = A(r) \exp \left( \pm i \int k dt \right)
\]

(12)

where we have introduced the local wave number, \( k \). The integration is along a ray path and we have explicitly indicated the two possible directions for each path. In this equation and hereafter we will removed the subindex 0 for the reference model.

In the ray theory, the ray path and the local wave vector, \( k(r) = k_r \hat{r} + k_h \hat{k}_h \), are determined from a first order 3D Liouville-Green expansion (a “WKBJ asymptotic approximation”). Specifically the dispersion relation is found to be given by

\[
\frac{1}{c^2} \left( \omega^2 - \omega_c^2 \right) - k^2 + \frac{N^2}{\omega^2} k_h^2 = 0.
\]

(13)

If the background state is spherically symmetric, \( k_h^2 = L^2/r^2 \), where \( L \) is a constant.
The ray path is determined from the group velocity by
\[ \frac{d\theta}{dr} = \frac{v_\theta}{r v_r}. \tag{14} \]

Here \( r \) and \( \theta \) are the usual polar coordinates and the group velocity \( v = v_r \hat{r} + v_\theta \hat{\theta} \) has components
\[ v_r = \frac{\partial \omega}{\partial k_r} = \frac{k_r \omega^3 c^2}{\omega^4 - k_h^2 c^2 N^2} \tag{15} \]
\[ v_\theta = \frac{\partial \omega}{\partial k_\theta} = k_h \omega^2 c^2 \left( \frac{\omega^2 - N^2}{\omega^4 - k_h^2 c^2 N^2} \right). \tag{16} \]

The complex amplitude \( A(r) \) is obtained from a second order Liouville-Green expansion, called the transport equation. In our case this equation reduces to
\[ \nabla \cdot \left( A^2 \left( k - \frac{N^2}{\omega^2} k_h \right) \right) = 0. \tag{17} \]

Integrating this equation over the volume of a ray tube, applying the Gauss theorem and taking \( \Delta S \to 0 \) for the tube section, one gets
\[ A^2 \left( k - \frac{N^2}{\omega^2} k_h \right) \Delta S \cdot \hat{n} \simeq \text{constant}, \tag{18} \]
where \( \hat{n} \) is a unit vector normal to the section. A representation of a ray tube similar to that considered here can be found in Figure 7 of \cite{Lindsey & Braun 2000}. If we take the section such that \( \hat{n} \) is in the radial direction, the amplitude can be determined by
\[ A(r, k_r \sin \theta \Delta \theta)^{1/2} = \text{constant}, \tag{19} \]
where \( \Delta \theta \) is the angular size of the tube of section \( \Delta S \to 0 \).

3.1. Green’s functions

The Green’s functions defined by Eq. \eqref{eq:G} have also a solution of the form \eqref{eq:A} but with prescribed initial conditions, corresponding to a Dirac-delta impulse at the focus. Thus, the (complex) constant in Eq. \eqref{eq:A} is fixed. Specifically, the amplitude for a Green’s function at any point \( r \) far from the focus is given by
\[ A_{\pm}(r|r_0) = \frac{\exp \left( \pm ik_0 |r - r_0| \right) r_+}{4\pi |r_+ - r_0|} \left( \frac{k_+ r_+ \sin \theta}{k_+ r \sin \theta} \right)^{1/2} \left( \frac{\Delta \theta_{+}}{\Delta \theta} \right)^{1/2}. \tag{20} \]

Here \( r_+ \) is a point close to \( r_0 \) where the integration along the ray path in \eqref{eq:A} starts up. For the Green’s functions, here and in \eqref{eq:A}, the + sign corresponds to waves moving out of the focus \( r_0 \) (amplitude \( A_+ \)) and the – sign to waves moving into the focus (amplitude \( A_- \)).

We are interested in computing outward and inward Green’s functions but, to be more general, after a given number of bounces at the surface and the interior. At the turning points, \( k_r = 0 \) and the amplitude cannot longer be computed by means of Eq.\eqref{eq:A}. In fact, we know that at the turning points \( A(r) \) is just retarded by \( \pi/2 \). Since these Green’s functions are defined only within the resonance cavity, both \( r \) and \( r_0 \) must be inside the cavity. In particular, at least formally, the Green’s functions cannot be computed in the photosphere or above. Thus for a ray that initially goes inwards and then it is observed near the surface at its first arrival, the phase shift is \(-\pi/2\). For any additional pair of inner and outer reflections the phase shift is \(-\pi\). So after \( s \) inner and \( s - 1 \) surface bounces (hereafter \( s \) skips), the phase shift is \(-(2s - 1)\pi/2\) and the inward and outward Green’s functions at a point \( r \) are given by
\[ G_\pm(r|r_0) = A_\pm(r|r_0) \exp \left( \pm i \frac{(2s - 1)\pi}{2} \right) \times \exp \left( \pm i \int_{r_0 \to r} kdl + (s - 1) \int_{r_1 \to r_2} kdl + \int_{r_1 \to r} kdl \right) \tag{21} \]
where \( r_1 \) and \( r_2 \) are the inner and upper turning points and \( A_\pm(r|r_0) \) is given by \eqref{eq:A}. The limits on the integrals indicate the interval over the ray path to be considered.

4. SPHERICAL POLAR EXPANSION: ASYMPTOTIC APPROXIMATION

For a spherically symmetric reference model, Eq. \eqref{eq:G} can be solved by expanding the solution in terms of spherical harmonics in the form
\[ G(r|r_0) = \sum_{\ell=0}^{\infty} g^{\ell}(|r| r_0) \sum_{m=-\ell}^{\ell} Y_{\ell m}(\Omega) Y_{\ell m}(\Omega) = \frac{1}{4\pi} \sum_{\ell=0}^{\infty} g^{\ell}(|r| r_0)(2\ell + 1)P_{\ell}(\mu) \tag{22} \]
where \( P_\ell(\mu) \) is a Legendre polynomial and \( \mu = \cos \theta \cos \phi_0 + \sin \theta \sin \phi_0 \cos (\phi - \phi_0) \). Here \( r, \theta, \) and \( \phi \) are standard spherical coordinates and \( \Omega = (\theta, \phi) \). Since we are considering a spherically symmetric reference model, we can choose, without losing generality, the polar axis in such a way that \( \theta_0 = 0 \) and \( \phi_0 = 0 \) for the focus. In this way \( \mu = \cos \theta \) and the Green’s functions have not dependence on \( \phi \).

Inserting the decomposition given by \eqref{eq:G} into Eq. \eqref{eq:G}, it is found that the radial factors \( g_\ell \) satisfy the equation
\[ \frac{d^2 g_\ell}{dr^2} + k^2 g_\ell = 0 \quad \text{at} \quad r \neq r_0 \tag{23} \]
where \( g_\ell = r g_\ell \) and \( k_\ell \) is given by Eq. \eqref{eq:k} with \( L^2 = l(l + 1) \). The Dirac-delta term is transformed into the following inhomogeneous initial conditions:
\[ g_\ell(r_0^+) |r_0) = \tilde{g}_\ell(r_0^+) |r_0) \tag{24} \]
and
\[ \frac{d}{dr} (r_0^+) |r_0) \right) - \frac{d}{dr} (r_0^-) |r_0) \right) = -1 \frac{1}{r_0}. \tag{25} \]

We also recall that the angular factors, given by the Legendre polynomials, satisfies the equation
\[ \frac{d^2 Q_\ell}{d\theta^2} + k^2_\ell Q_\ell = 0 \tag{26} \]
where \( P_\ell(\cos \theta) = (\csc^{1/2} \theta) Q_\ell(\theta) \) and
\[ k^2_\ell = \left( \ell + \frac{1}{2} \right)^2 + \frac{1}{4} \sin^2 \theta. \tag{27} \]
4.1. Asymptotic inwards and outwards Green’s functions

We are interested in keeping the same kind of ingoing and outgoing Green’s functions than in the ray theory, since this will allow us to do a straightforward comparison. Since we have done a separation of radial and angular variables, we need to search for inwards and outwards radial solutions and prograde and retrograde angular solutions and combine both in a suitable way. This kind of “travel-wave” solutions are well defined in the asymptotic limit and we start with that.

In a first order Liouville-Green expansion the general solution of the angular factors can be written as \( Q_\ell(\theta) = \beta_\ell^+ B_\ell^+(\theta) + \beta_\ell^- B_\ell^-(\theta) \), where

\[ B_\ell^\pm(\theta) = \kappa_\ell^{-1/2} \exp \left( \pm i \int_0^\theta \kappa_\ell \, d\theta \right) \]  

and \( \beta_\ell^+ \) and \( \beta_\ell^- \) are constants. The value \( \theta_0 \) will be taken at the focus. In a similar way the solution of the radial factors can be written as \( \tilde{g}_\ell(r) = \alpha_\ell^+ C_\ell^+(r) + \alpha_\ell^- C_\ell^-(r) \) where \( \alpha_\ell^+ \) and \( \alpha_\ell^- \) are constants and

\[ C_\ell^{\pm}(r) = k_r^{-1/2} \exp \left( \pm i \int_{r_1}^r k_r \, dr \right). \]  

Here \( r_1 \) is the inner turning point. The constants \( \alpha_\ell^+ \) and \( \alpha_\ell^- \) can be obtained by introducing the equation \( \tilde{g}_\ell(r) = \alpha_\ell^+ C_\ell^+(r) + \alpha_\ell^- C_\ell^-(r) \) into the initial conditions, and \((\ref{24})\). One gets

\[ \alpha_\ell^\pm = \frac{i}{2r_0} \frac{1}{\sqrt{k_r(r_0)}} \exp \left( \mp i \int_{r_1}^{r_0} k_r \, dr \right). \]  

In principle, we can take out the outwards prograde part of the solution, proportional to \( \alpha_\ell^+ C_\ell^+(r) \beta_\ell^+ B_\ell^+(\theta) \), as the Green’s function going out of the focus. However, the solution \( \alpha_\ell^+ C_\ell^+(r) \) corresponds to an outgoing wave emanating from the focus \( r_0 \) and reaching the surface without any reflection. Since we are interested in the outgoing Green’s function after the focus \( r_0 \), we need to define new inwards and outwards asymptotic solutions. Let us call \( C_\ell^* \) to the wave emanating inwards from the focus and now going outwards after \( s \) skips. This function and his inwards counterparts are:

\[ C_\ell^{s*} = C_\ell^{s} \exp \left[ \pm i \left\{ (s - 1) \left( \int_{r_1}^{r_2} k_r \, dr - \frac{\pi}{2} \right) - \frac{\pi}{2} \right\} \right], \]  

where we have taken into account a phase jump of \(-\pi/2\) every time the wave reach a turning point.

The angular factors has no turning points, so we can use the asymptotic functions defined in Eq. \((\ref{24})\). Then, the Green’s function for a given degree \( l \) and frequency \( \omega \) going out of the focus, proportional to \( \alpha_\ell^+ C_\ell^+(r) \beta_\ell^+ B_\ell^+(\theta) \), is

\[ G_\ell^+(\vec{r}|r_0, \omega) = \frac{i(2l + 1)}{8\pi r_0 \sqrt{k_r(r_0)}} r^{-1} k_r^{-1/2} \kappa_\ell^{1/2} \csc^{1/2} \theta \times \exp \left\{ \left( \int_{\theta_0}^{\theta} \kappa_\ell \, d\theta + \int_{r_1}^{r_2} k_r \, dr \right) \right\} \times \exp \left\{ i \left( (s - 1) \left( \int_{r_1}^{r_2} k_r \, dr - \frac{\pi}{2} \right) - \frac{\pi}{2} + \int_{r_1}^{r_0} k_r \, dr \right) \right\}, \]  

The ingoing Green’s function is the complex conjugate, \( G_\ell^- = G_\ell^* \).

4.2. Non-asymptotic inwards and outwards Green’s functions

Let us start with the angular functions. This corresponds to Eq. \((\ref{25})\) that of course has the solution \( Q_\ell = \sqrt{\sin \theta} P_l(\cos \theta) \). To split it in prograde and retrograde parts we written the solution as

\[ Q_\ell(\theta) = \beta_\ell^+(\theta) B_\ell^+(\theta) + \beta_\ell^-(\theta) B_\ell^-(\theta), \]  

where \( B_\ell^\pm \) are still given by Eq. \((\ref{25})\). In the asymptotic approximation \( \beta_\ell^\pm \) were taken to be constants but now they are functions that change smoothly, at least where the asymptotic approximation is expected to work. Thus we replace the constants by osculating parameters that satisfies the additional condition

\[ Q_\ell'(\theta) = \beta_\ell^+(\theta) B_\ell'^+(\theta) + \beta_\ell^-(\theta) B_\ell'^-(\theta) \]  

where the primes means derivatives respect to \( \theta \). With this definition it is follow that

\[ \beta_\ell^+ = \frac{i}{2} (Q_\ell B_\ell^-' - Q_\ell' B_\ell^-) \quad \text{and} \quad \beta_\ell^- = -\frac{i}{2} (Q_\ell B_\ell'^- - Q_\ell' B_\ell^-). \]  

Since \( Q_\ell \) is known, \( \beta_\ell^\pm \) and hence the prograde \( \beta_\ell^+ B_\ell^+ \) and retrograde parts \( \beta_\ell^- B_\ell^- \) are obtained.

For reasons explained later, for the applications shown in this work we will use the asymptotic approximation for the radial factors. However, for completeness, we explain here how to obtain the non-asymptotic radial Green’s functions. The general solution of Eq. \((\ref{25})\) can be written as

\[ \tilde{g}_\ell(r|\theta_0) = \alpha_\ell^+(r) C_\ell^+(r) + \alpha_\ell^-(r) C_\ell^-(r). \]  

where \( C_\ell^* \) are given by \((\ref{21})\). Here again, to fix the osculating parameters we add the condition

\[ \tilde{g}_\ell'(r|\theta_0) = \alpha_\ell^+(r) C_\ell'^+(r) + \alpha_\ell^-(r) C_\ell'^-(r), \]  

from which it is found that

\[ \alpha_\ell^+ = \frac{i}{2} (\tilde{g}_\ell(C_\ell^*)^* - \tilde{g}_\ell C_\ell^{*\dagger}) \quad \text{and} \quad \alpha_\ell^- = -\frac{i}{2} (\tilde{g}_\ell(C_\ell^*)^* - \tilde{g}_\ell C_\ell^{*\dagger}). \]  

Thus, for a given solution \( \tilde{g}_\ell(r|\theta_0) \), the functions \( \alpha_\ell^+(r) \) and \( \alpha_\ell^-(r) \) can be computed. Note that \( \tilde{g}_\ell(r|\theta_0) \) satisfies Eq. \((\ref{23})\) and since we are dealing with waves reflected in the interior, the solution must be regular at the center. With this condition alone, \( \tilde{g}_\ell(r|\theta_0) \) is determined up to a constant factor. On the other hand \( \tilde{g}_\ell(r|\theta_0) \) can be taken as the one given by the asymptotic solution, namely \( \tilde{g}_\ell(r) = \alpha_\ell^-(r) C_\ell^-(r) + \alpha_\ell^+(r) C_\ell^+(r) \), with \( \alpha_\ell^\pm(r_0) \) given by Eq.\((\ref{30})\) and \( C_\ell^{s*}\) by Eq.\((\ref{31})\). This fixes \( \tilde{g}_\ell(r|\theta_0) \) completely and hence the inward and outwards solutions, \( \alpha_\ell^-(r) C_\ell^-(r) \) and \( \alpha_\ell^+(r) C_\ell^+(r) \) respectively. Combining the outward solution with the prograde solution of the angular part, the outgoing wave is given by

\[ G_\ell^+(\vec{r}|r_0) = \frac{2l + 1}{16\pi r} \csc^{1/2} \theta \times \left\{ Q_\ell(\theta) B_\ell'^-(\theta) - Q_\ell'(\theta) B_\ell^-(\theta) \right\} B_\ell^+(\theta) \times \left\{ \tilde{g}_\ell(r) C_\ell'^-(r) - \tilde{g}_\ell(r) C_\ell^-(r) \right\} C_\ell^+(r). \]  

The ingoing Green’s function is the complex conjugate, \( G_\ell^- = G_\ell^{*\dagger} \).
served frequencies into the asymptotic expression
\[ \int_{r_1}^{r_2} k_r \, dr = \pi \left( n - \frac{1}{2} - \delta\alpha \right), \]  
(41)

where \( n \) is an integer, the radial order of the mode. In our case the wave number is given by Eq. (40) and hence
\[ \int_{r_1}^{r_2} \left[ \frac{1}{c^2} (\omega^2 - \omega_r^2) - \left( 1 - \frac{N^2}{\omega_r^2} \right) \frac{L^2}{r^2} \right] \, dr = \pi \left( n - \frac{1}{2} - \delta\alpha \right). \]  
(42)

To obtain \( \delta\alpha(\omega) \), eigenfrequencies from \( \nu = 2500 \mu\text{Hz} \) to \( \nu = 4500 \mu\text{Hz} \) and degrees \( \ell = 20 - 80 \) were taken from MDI data. Modes of lower degrees are rejected because they do not satisfy the Cowling approximation as good as intermediate and high-degree modes. In addition, Model S from Christensen-Dalsgaard et al. (1996) was used. A fitted polynomial of degree 10 to Eq. (42) was considered. The result is presented in Fig. 1. Notice that \( \pi\delta\alpha \) changes by more than 1.5 rad in the frequency range of interest.

6. NUMERICAL RESULTS

To validate the Green’s functions calculated using this new approach, we apply them to create seismic maps of active regions at the far side of the Sun. In the case of far-side seismic holography, the focus is located at (or just below) the surface of the far hemisphere of the Sun and the pupil is an annulus surrounding the antipodes of the focus. Although seismic maps of the full far-side hemisphere can be calculated using waves following one, two and three-skip ray paths before arriving at the pupil, for simplicity, here we will concentrate only on the central part of the maps. Traditionally, this is calculated considering waves following a 2-skip ray path, as described by Lindsey & Braun (2000b).

6.1. Comparison of ray theory and polar expansion

As it has been done in previous work, we normalize the Green’s functions at each frequency to its mean absolute value. Fig. 2 shows these normalized Green’s functions against the polar angle for different frequencies. Results for the polar expansion and the gravito-acoustic rays are compared. Eq. (39) has been used for the polar expansion, but with the asymptotic approximation for the radial factors, while for the ray theory Eq. (24) has been considered. In both cases the surface phase shift shown in Fig. 1 has been added according to Eq. (10). Finally, what we actually show are the derivatives of the Green’s functions that we will use to compute the ingression and egression fields following Eq. (11). The focus is located at \( r_0 = 0.9995R_\odot \), \( \theta_0 = 0 \) and the Green functions are shown at the upper turning point after two skips.

In the ray theory, for a given number of skips, and a given frequency \( \omega \), each point in the pupil corresponds to a given value of the horizontal wave number at the focus, \( k_h(r_0) \) and hence of \( L = \ell (\ell + 1) \). Here \( \ell \) is a real constant. Our pupil goes from \( \theta_{\text{min}} = 2 \) to \( \theta_{\text{max}} = 3 \) rad; this corresponds to and interval in “degrees” from \( \ell_{\text{min}} \simeq 15 \) to \( \ell_{\text{max}} \simeq 28 \) for \( \nu = 2500 \mu\text{Hz} \) and \( \ell_{\text{min}} \simeq 27 \) to \( \ell_{\text{max}} \simeq 50 \) for \( \nu = 4500 \mu\text{Hz} \). In the polar expansion there is not such relation, in fact in Fig. 2 Green’s functions for degrees up to \( l = 60 \) have been added up, the same for all the frequencies.
Fig. 2.—Comparison of Green’s functions at selected frequencies. The black continuous lines correspond to the Green’s functions calculated using the Legendre polynomial decomposition and the red dashed ones to those calculated using a gravito-acoustic ray path approximation.

As it can be seen in Fig. 2, there is a systematic shift between the Green’s functions computed with the two different approaches, but the solutions are closer at low angles; in fact in the ray theory these angles correspond to higher $l$ where this approximation is expected to work better. On the other hand the solutions of the polar expansion approximation look more irregular at low frequencies; this is related to the fact that at these frequencies the upper turning point $r_2$ is too close to the focus $r_0$.

We use both sets of Green’s functions described in this paper, as well as the traditional ones (acoustic ray approximation), to calculate maps of magnetic activity at the non-visible hemisphere of the Sun. Fig. 3 presents far-side maps calculated using Global Oscillation Network Group (GONG, Harvey et al. (1996)) data for September 1 and 2 2005 for the three sets of Green’s functions. The maps clearly show the signature of active region NOAA-10808 as a large negative phase shift. The Green’s functions used for the analysis include frequencies computed from $\nu = 2500 \mu\text{Hz}$ up to $4500 \mu\text{Hz}$ with a step of $\sim 8 \mu\text{Hz}$. Angles go from $\theta = 2$ to $3\text{rad}$ with a step of 0.0016 rad. Active region NOAA 10808 appeared at the East limb of the Sun September 7 2005.

The signature of the active region is very similar in the maps for all the cases. However, while the noise level is similar in all three cases, the spatial distribution is only correlated between the maps calculated using the two types of Green’s functions described in this paper. This may be due to the different approaches used for the sub-surface phase shift correction, explained in section 5. However more research is necessary in order to understand the different contributors to the noise.

6.2. Green’s functions combining different number of bounces

As mentioned before, in the polar expansion approximation there is not a one to one relation between the angle in the pupil and the mode degree. Thus, even for the same pupil and the same angular degrees it is possible to combine the Green’s functions from different bounces. In Fig. 4 we show the Green’s functions from the polar expansion against the polar angle. In this case we have used degrees from $\ell = 15$ to $\ell = 100$. The Green’s functions are shown for $s = 2$, $s = 3$ and the sum of both (with the normalization at each frequency done after the sum). We do not consider the $s = 1$ case since most of the signal for the first bounce in our pupil comes from very low degrees for which the Cowling approximation does not work properly. As it can be seen, the maximum amplitude of these Green functions change both with $s$ and frequency.

We calculate a series of far-side maps using the polar-expansion Green’s functions that combine two and three skips. In Fig. 5 we compare these maps with those using Green’s functions that include only $s = 2$. A sequence of three days spanning November 27 to 29 2006 showing a weaker active region just south of the equator an
skips before calculating the map in seismic holography we improve the signal to noise in a similar way. Since the spherical geometry of the observational pupil makes the calculation of the ingression and egression functions for far-side maps computationally expensive, the solution presented here is an ideal way of improving the signal to noise without the need for increasing the computation time. Our approach is also formally different, since we sum the Green’s functions from different bounces without a previous normalization.

There are some previous works where Green’s functions beyond the ray approximation were used, for instance in Lindsey & Braun (2004), but there a completely plano-parallel approximation was considered and hence the technique was limited to focuses in the front-side and close to the surface. On the other hand for our Green’s functions, although we have shown here the application to the particular case of far-side seismic holography, the technique is general enough to be applied in the front side and for focuses under the surface, even below the convection zone, at least formally. In that scenario, the signal-to-noise increases considerably and more work would be required to test the inferences.

An interesting result of our work is the fact that the sub-surface phase shift, that has been recognized as a fundamental correction to any Green’s function in the past, can be computed by using the observed frequencies of the normal modes of oscillations. This opens a link between local and global helioseismology and perhaps the issues of the local approach can be used to learn something about the global one and vice versa.

7. CONCLUSIONS

We have implemented a holographic, or migration technique, that includes a factorization of the Green’s functions in angular and radial parts. This technique is more general than the one provided by the ray theory. We show examples of how these new Green’s functions can be used to improve far-side seismic images by combining the Green’s functions corresponding to travel waves with two and three skips or inner reflections. Zhao (2007) and Ilonidis et al. (2009) using a time-distance technique have demonstrated that adding more skips to the calculation of far-side maps improves the signal-to-noise. However, in their analysis, they combine seismic images obtained from different skip ray paths after they are calculated. Here, we show that by combining the Green’s functions associated to two and three skips before calculating the map in seismic holography we improve the signal to noise in a similar way. Since the spherical geometry of the observational pupil makes the calculation of the ingression and egression functions for far-side maps computationally expensive, the solution presented here is an ideal way of improving the signal to noise without the need for increasing the computation time. Our approach is also formally different, since we sum the Green’s functions from different bounces without a previous normalization.

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