Geometric Massive Gravity in Multi-Connection Framework

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What is the right way to interpret a massive graviton? We generalize the kinematical framework of general relativity to multiple connections. The average of the connections is itself a connection and plays the role of the canonical connection in standard General Relativity. At the level of dynamics, the simplest choice of the Einstein-Hilbert action is indistinguishable from the single-connection case. However, inspired by Weyl geometry, we show how one can construct massive gravity to all orders in perturbation theory compatible with the de Rham-Gabadadze-Tolley ghost-free model. We conclude that the mass of the graviton can be interpreted as a geometrical property of spacetime arising from two connections. Furthermore in the multi-connection framework there is no ambiguity in the definition of physical metric and consequently coupling to matter.

I. INTRODUCTION AND MOTIVATIONS

A. Generalization of Kinematics

Einstein general relativity (EGR) has had a lot of successes theoretically and observationally. Einstein (and Hilbert) suggested a dynamical theory for the metric of spacetime. This theory improved our knowledge not only of gravitational force dynamics but also in its geometrical interpretation. At the kinematical level the EGR is based on a manifold with a metric living on it. This metric is responsible for all the geometrical properties of spacetime. It manifests the notion of distance, causal structure and parallel transportation. The latter is realized by the corresponding connection to the given metric i.e. Christoffel symbol. However it should be emphasized that in a general framework the connection can be an independent geometrical object which is responsible for parallel transportation and consequently the geodesic equation and covariant derivative. The Christoffel symbol is a specific kind of connection which satisfies the metric compatibility relation. It is a unique property of Einstein-Hilbert action that imposes a priori independent connection have to be the Christoffel symbol according to the equations of motion.

Though EGR can be interpreted as a triumph in our understanding of gravitational force, there have always been attempts at modifications [1]. One reason for that is the problem of explaining observational data such as dark energy or dark matter [11]. Other reasons include theoretical challenges e.g. cosmological constant problem, singularities and quantum gravity, as well as curiosity of theoretical physicists. For this purpose, one method is a modification at the level of dynamics i.e. the Einstein-Hilbert action e.g. $f(R)$ or $R_{\mu\nu}R^{\mu\nu}$. Such models usually share a kinematic structure with EGR i.e. existence of a metric field and its corresponding Christoffel symbol. However it has been shown that in general, dynamics does not impose metric compatibility. An alternative approach [12] to modify EGR can happen at the level of kinematics e.g. various bi-metric models [2]. Bi-metric models are interesting to study because their richer foundation allows for more opportunities at least theoretically e.g. massive gravity [3]. But existence of more than one metric on the manifold causes some ambiguities. For example our understanding of the measurement of distance, the causal structure of manifolds and the geodesic equation are not uniquely defined in this model. In addition there is another challenge in coupling of matter and gravity which is crucial to understand the behavior of the model in real circumstances.

In this work we introduce a new gravity model with modified kinematics. We do this by allowing the manifold to have more than one connection. We will show how this model has the advantages of its predecessor (i.e. a multi-metric model) but without ambiguity in its physical interpretation. As an example we will study massive gravity in this framework, and so next present a brief review on the status of massive gravity.

B. Massive Gravity

Recently massive gravity has attracted a lot of attention due to a milestone in this topic by de Rham, Gabadadze and Tolley (dRGT) [4]. They could improve existence of a ghost free massive gravity which is an extension of the

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Fierz-Pauli [3] massive gravity to a non-linear regime. In dRGT massive gravity the Boulware-Deser ghost [6] is absent. The dRGT massive gravity needs a fiducial metric in addition to physical metric to construct the mass (potential) term. This fact is easy to understand since the potential in EGR should be a scalar made of just the metric without any derivative operator. The only possibility is the cosmological constant term. So to go further e.g. a mass term, having two metrics seems an essential assumption. In this sense massive gravity can be categorized as a bi-metric model. In addition to the problems of bi-metric model there is a fundamentally important question in massive gravity: What is the geometrical meaning of the mass of graviton? We will try to address this question in the multi-connection framework. But before that let us illustrate why we think this question is important.

After Einstein, not only our knowledge about gravitational force became improved but also it changed our viewpoint on the interpretation of the gravitational force. EGR says very briefly that gravity is geometry which is very profound. EGR formalism behaves with gravity as a field theory same as e.g. electrodynamics. It has been shown that field theoretical viewpoint on EGR results in a massless spin-two particle named graviton which is responsible for gravity force exchange same as photon in electrodynamics. Then the lack of accurate observational data makes it possible to ask if the graviton have a mass? While this can be a quite straightforward question in particle physics, its realization theoretically and its observational consequences should be carefully considered. As we mentioned above the dRGT model has proposed a well-defined massive gravity without any ghosts. In this formalism, as mentioned in [3], all the realizations are equivalent to assuming a bi-metric model in a four dimensional geometry. It is worth mentioning that the bi-metric models have had their own history [2] though became more interesting after establishing their relation with massive gravity. So one can expect that the problems of bi-metric models exist in massive gravity too. In this work we are going to study massive gravity as an example in a multi-connection framework.

II. MULTI-CONNECTION FRAMEWORK

The framework has a lot in common with multi-metric models, but it will be shown that multi-connection framework is more straightforward for physical interpretations. In addition it seems this framework does not suffer from the usual problems of multi-metric models mentioned earlier. To start we study the kinematics of this framework and then consider its dynamics.

A. Kinematics

At the level of kinematics, a connection is responsible for parallel transportation and consequently shows itself in the geodesic equation [13]. The geodesic equation can be written as

$$\frac{d^2x^\mu}{d\lambda^2} = -\Gamma^\mu_{\alpha\beta} \frac{dx^\alpha}{d\lambda} \frac{dx^\beta}{d\lambda},$$

(1)

In the above equation the left hand side is simply the acceleration which appears in the Newton's second law as $a^\mu = \frac{d^2x^\mu}{d\tau^2} = \frac{1}{m} F^\mu$ where $\tau$ is time, $m$ is mass and $F^\mu$ is an external force. By this viewpoint the above equation can be interpreted as the Newton's second law by assuming $F^\mu_{geo.} \propto -\Gamma^\mu_{\alpha\beta} \frac{dx^\alpha}{d\lambda} \frac{dx^\beta}{d\lambda}$ and interpreting $F^\mu_{geo.}$ as a geometrical force. Now it is straightforward to add new connections by reminding what happens to Newton's second law when we have more than one forces. The geodesic equation becomes

$$\frac{d^2x^\mu}{d\lambda^2} = -\sum_{i=1}^{N} \Gamma^\mu_{\alpha\beta} \frac{dx^\alpha}{d\lambda} \frac{dx^\beta}{d\lambda} = -N \left( \frac{1}{N} \sum_{i=1}^{N} \Gamma^\mu_{\alpha\beta} \right) \frac{dx^\alpha}{d\lambda} \frac{dx^\beta}{d\lambda} = -N \gamma^\mu_{\alpha\beta} \frac{dx^\alpha}{d\lambda} \frac{dx^\beta}{d\lambda},$$

(2)

where each connection is labeled by $(i)$ and we define the average connection as $\gamma^\mu_{\alpha\beta} = \frac{1}{N} \sum_{i=1}^{N} \Gamma^\mu_{\alpha\beta}$. It is worth remembering that the average of connections is a connection itself. So effectively the geodesic equation [13] (2) in the presence of many connections is the geodesic equation in [11] where the average connection, $\gamma^\mu_{\alpha\beta}$, plays the role of the canonical connection. As a consequence the covariant derivative should be defined due to average connection $\gamma^\mu_{\alpha\beta}$ to be compatible with the definition of the geodesic equation. So the multi-connection framework works properly at the kinematic level. Now let us turn to dynamics of the model where the average connection will show itself again.

B. Dynamics

Now having reviewed the kinematics of this model, we are next going to study its dynamics. To construct the Lagrangian for this model we need an auxiliary field to make scalars. By having this field $g^\mu_\nu$ [15] and consequently
its inverse $g^{\mu\nu}$ and determinant $g$ we can write the simplest action inspired by Einstein-Hilbert action as

$$S = \int d^4x \sqrt{-g} g^{\mu\nu} \frac{1}{N} \sum_{i=1}^{N} R_{\mu\nu} \left( (i) \Gamma^{\rho}_{\alpha\beta} \right).$$

(3)

For our purpose in this work we restrict the model to a bi-connection model. This assumption is just for simplicity and does not change the physical consequences. The above Lagrangian reduces to

$$S = \int d^4x \mathcal{L} = \int d^4x \sqrt{-g} g^{\mu\nu} \frac{1}{2} \left[ R_{\mu\nu} \left( (1) \Gamma^{\rho}_{\alpha\beta} \right) + R_{\mu\nu} \left( (2) \Gamma^{\rho}_{\alpha\beta} \right) \right]$$

(4)

which can be written as follows

$$\mathcal{L} = \sqrt{-g} g^{\mu\nu} \left[ R_{\mu\nu} \left( \gamma^{\rho}_{\alpha\beta} \right) + \Omega^{\alpha\beta}_{\gamma\lambda} \Omega^{\lambda}_{\nu\mu} - \Omega^{\alpha\rho}_{\nu\mu} \Omega^{\beta\lambda}_{\gamma\lambda} \right]$$

(5)

where $R_{\mu\nu}(\gamma^{\rho}_{\alpha\beta})$ is the Ricci tensor defined by the average connection $\gamma^{\rho}_{\alpha\beta} \equiv \frac{1}{2} \left( (1) \Gamma^{\rho}_{\alpha\beta} + (2) \Gamma^{\rho}_{\alpha\beta} \right)$ and $\Omega^{\rho}_{\alpha\beta} \equiv \frac{1}{2} \left( (1) \Gamma^{\rho}_{\alpha\beta} - (2) \Gamma^{\rho}_{\alpha\beta} \right)$ is a tensor due to transformation rule of connections. Since we assume connections are symmetric in their lower indexes so $\gamma^{\rho}_{\alpha\beta}$ and $\Omega^{\rho}_{\alpha\beta}$ are both symmetric in their lower indexes. The variation of the Lagrangian with respect to $\gamma^{\rho}_{\alpha\beta}$ results in

$$\gamma^{\rho}_{\alpha\beta} \equiv \frac{1}{2} g^{\mu\nu} \left( \partial_{\alpha} g_{\beta\mu} + \partial_{\beta} g_{\alpha\mu} - \partial_{\mu} g_{\alpha\beta} \right),$$

(6)

which means $\gamma^{\rho}_{\alpha\beta}$ is a metric compatible connection. So effectively the above Lagrangian can be written as

$$\mathcal{L} = \sqrt{-g} g^{\mu\nu} \left[ R_{\mu\nu} \left( g_{\alpha\beta} \right) + \Omega^{\alpha\beta}_{\gamma\lambda} \Omega^{\lambda\nu\mu} - \Omega^{\alpha\beta}_{\gamma\lambda} \Omega^{\lambda\nu\mu} \right]$$

(7)

where $g_{\mu\nu}$ and $\Omega^{\alpha}_{\nu\mu}$ can be independent fields. However, we should be very careful about this assumption. If $g_{\mu\nu}$ and $\Omega^{\alpha}_{\nu\mu}$ are totally independent then the equation of motion with respect to $\Omega^{\alpha}_{\nu\mu}$ imposes $\Omega^{\alpha}_{\nu\mu} = 0$. This result can be seen more directly from the Lagrangian (4) by applying Palatini method on both $(1) \Gamma^{\rho}_{\alpha\beta}$ and $(2) \Gamma^{\rho}_{\alpha\beta}$. Then the result says $(1) \Gamma^{\rho}_{\alpha\beta} \equiv (2) \Gamma^{\rho}_{\alpha\beta} = \gamma^{\rho}_{\alpha\beta}$ where $\gamma^{\rho}_{\alpha\beta}$ is the Christoffel symbol (5). Therefore assuming totally independent $g_{\mu\nu}$ and $\Omega^{\alpha}_{\nu\mu}$ reduces the model to pure EGR.

However it is possible to do further analysis by a delicate assumption and a generalization of the Einstein-Hilbert inspired action. They are many possible candidates which all are interesting to study and we mention some of them. I) One can modify the action (4) by assuming $f(R)$ inspired models. In this case the Palatini method does not impose all the connections to be Christoffel symbols and allows the non-vanishing $\Omega^{\alpha}_{\nu\mu}$. II) It is possible to add a kinetic term for $\Omega^{\alpha}_{\nu\mu}$ by hand and consider the model. In principle it is same as EGR in presence of another field e.g. in Brans-Dicke model. III) The other candidate is breaking somehow the independence between $g_{\mu\nu}$ and $\Omega^{\alpha}_{\nu\mu}$. As it is obvious for each candidate there are many kinds of realizations but in this work we consider the last choice and we will see how we can break the mentioned independence employing Weyl geometry.

C. Coupling to Matter

It is always a problem of bi-metric models that what is the role of each metric i.e. which one (or combination) is the physical metric. One way to ask this question is to ask which combination of the given metrics is responsible for coupling to matter. Note that the answer to this question can address the other related problems such as definition of distance and causality in bi-metric models. In the multi-connection framework this problem should exist but there is a unique natural answer to it. This unique candidate is based on assuming indistinguishability of the priory given connections and is very straightforward to see because of the model’s construction. By looking at the Lagrangian (4) it is natural to think that the matter is coupled to $g_{\mu\nu}$ the auxiliary field which is the metric. It is very important to emphasize that this claim is compatible with how the multi-connection framework behaves at the kinematic level. By looking at the geodesic equation in this framework (2) and having in mind the equivalence principle it is easy to say that the matter should see the average connection $\gamma^{\mu\nu}_{\nu\mu}$. It is exactly equivalent to say matter is coupled to the metric $g_{\mu\nu}$ because we have shown that the average connection $\gamma^{\mu\nu}_{\nu\mu}$ is the Christoffel symbol according to the metric $g_{\mu\nu}$. According to above arguments there is just one metric in our model which is responsible for coupling to matter
and so it is in charge for measuring the distance and the causal structure too. It means multi-connection framework has no problem with these issues.

From this natural and unique result in multi-connection framework we can have a hint about multi-metric gravity models. Suppose we start with a bi-metric model then for each metric we can associate a connection (Christoffel symbol) and the average connection can be defined consequently. According to our result this average connection is responsible for the coupling to matter in the geodesic equation. However it is not trivial if is always possible to associate a metric to this average connection.

D. An Example

Now let’s assume a special kind of definition for $\Omega_{\alpha\mu\nu}$ as an example to bridge between multi-connection formalism and massive gravity. This special form is as follows

$$\Omega_{\alpha\mu\nu} \equiv \frac{1}{2}(C_\mu X_{\nu\alpha} + C_\nu X_{\alpha\mu} - C_\alpha X_{\mu\nu}) \tag{8}$$

where $X_{\mu\nu}$ is a symmetric tensor and $C_\alpha$ is a vector. With the above form for $\Omega_{\alpha\mu\nu}$ the second term of the Lagrangian reduces to

$$\mathcal{L}_{MG} = \frac{1}{4} \times [2(X X_{\mu\nu} - X_{\mu\alpha} X_{\nu}^\alpha) C^\mu C^\nu + (X_{\mu\nu} X^{\mu\nu} - X^2) C^2] \tag{9}$$

where $X = g^{\mu\nu} X_{\mu\nu}$. $C^2 = C_\mu C^\mu$ and $g_{\mu\nu}$ is responsible for lowering and raising the indices. For a special case which $C^\mu$ satisfies $C^2 = -m^2$ and $C^\mu X_{\mu\nu} = 0$ the above Lagrangian becomes

$$\mathcal{L}_{MG} = -\frac{m^2}{4} (X_{\mu\nu} X^{\mu\nu} - X^2). \tag{10}$$

The important point about this Lagrangian is the relative coefficient between $X_{\mu\nu} X^{\mu\nu}$ and $X^2$. We emphasize that this relative coefficient is not trivial and is related to the absence of ghosts in massive gravity. We did not say what is the reason for taking the above specific form for $\Omega_{\alpha\mu\nu}$ in (8). In the next section we show how Weyl geometry can inspire us to find a physical meaning for the above form of $\Omega_{\alpha\mu\nu}$ for a specific $X_{\mu\nu}$.

But before that it is worth to mentioning that the above form of the Lagrangian reminds us the dRGT massive gravity. By looking at $X_{\mu\nu} \equiv K_{\mu\nu}$ as the tensor defined in dRGT massive gravity [4] i.e. $K^\mu_\nu = \delta^\mu_\nu - \sqrt{g^\mu} - H^\mu$ where one can define $H_{\mu\nu}$ by using $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} = H_{\mu\nu} + \eta_{\alpha\beta} \partial^\mu \phi^\alpha \partial^\nu \phi^\beta$ where $\phi^\alpha$ are Stukelberg fields. So it can be concluded that the dierence between dierent connections can show itself as a mass term for the graviton. However to make a fully comparison with dRGT model we need to consider higher order terms $O(K^3)$ and $O(K^4)$. We will come back to this issue in the next section.

III. WEYL GEOMETRICAL MASSIVE GRAVITY

In this section we will show how by employing Weyl geometry (WG) in the multi-connection framework we can find a geometrical realization of massive gravity. WG is an important generalization of Riemannian geometry. In a classic work by Ehlers, Pirani and Schild it has been claimed that WG can be deduced from an axiomatical approach to general relativity [8]. In WG in addition to the metric there is another geometrical object which is a vector. This vector changes the amplitude of a given vector due to parallel transportation which is an additional effect to changing in the direction which happens in Riemannian geometry. In WG the connection can be written as

$$\Gamma^\alpha_{\mu\nu} = \{^\alpha_{\mu\nu} \} - \frac{1}{2} g^{\alpha\beta} (g_{\nu\beta} C_\mu + g_{\mu\beta} C_\nu - g_{\mu\nu} C_\beta) \tag{11}$$

where $\Gamma^\alpha_{\mu\nu}$ is the connection and $\{^\alpha_{\mu\nu} \}$ is the Christoffel symbol. This means the metric compatibility relation modifies to $\nabla_\alpha g_{\mu\nu} = C_\alpha g_{\mu\nu}$ where the covariant derivative is due to $\Gamma^\alpha_{\mu\nu}$ and which obviously reduces to Riemannian geometry for $C_\alpha = 0$. However it is easy to show that

$$\Gamma^\alpha_{\mu\nu} = \{^\alpha_{\mu\nu} \} - \frac{1}{2} g^{\alpha\beta} (A_\nu \delta^\alpha_{\mu\beta} + A_\mu \delta^\alpha_{\nu\beta} - A_{\mu\nu} C_\beta) \quad \Rightarrow \quad \nabla_\mu g_{\alpha\beta} = C_\mu A_{\alpha\beta} \tag{12}$$
It can be done by modifying the connections in (13) to our setup. However, it is crucial to show how one can extend the above model for non-linear dRGT massive gravity.

The existence of the mass term shows how far the connections are from the average connection (Christoffel symbol). In other words, the variance (or standard deviation) is represented by the existence of mass term. In a given distribution of connections, the average connection and the difference tensor will be as follows respectively:

\[
\gamma_{\alpha \mu}^\nu = \frac{1}{2} \left( (1) \Gamma_{\alpha \mu}^\nu + (2) \Gamma_{\alpha \mu}^\nu \right) = \{ \alpha \mu \}^\nu,
\]

\[
\Omega_{\alpha \mu}^\nu = (1) \Gamma_{\alpha \mu}^\nu - (2) \Gamma_{\alpha \mu}^\nu = g^{\alpha \beta} (h_{\nu \beta} C_{\mu} + h_{\mu \beta} C_{\nu} - \eta_{\mu \nu} C_{\beta}).
\]

By plugging the above relations into the Lagrangian (5) or equivalently into (7) one gets

\[
\mathcal{L} = h_{\alpha \beta} \epsilon^{\alpha \beta \mu \nu} h_{\mu \nu} - \frac{m^2}{4} \left( h_{\mu \nu} h^{\mu \nu} - h^2 \right)
\]

where \( \epsilon^{\alpha \beta \mu \nu} \) is the EGR kinetic operator, \( h = h^\mu, C^2 = -m^2 \) and \( C^\mu h_{\mu \nu} = 0 \). The above Lagrangian is Fierz-Pauli Lagrangian which is ghost free Lagrangian for massive gravity at linear order (in equations of motion). Mathematically it is ghost free because of relative coefficient between \( h_{\mu \nu} h^{\mu \nu} \) and \( h^2 \) is minus one. This case happens in multi-connection framework automatically and it was not trivial from starting point of this model. In addition, we should say in the multi-connection framework this mass term has a geometrical meaning too. In a given distribution of connections, the variance (or standard deviation) is represented by existence of mass term. In other words, the existence of the mass term shows how far the connections are from the average connection (Christoffel symbol).

### A. Non-Linear Massive Gravity

We could show that the linear (at the level of equation of motion) Fierz-Pauli term can be deduced automatically in setup. However, it is crucial to show how one can extend the above model for non-linear dRGT massive gravity. It can be done by modifying the connections in (13) to

\[
(1) \Gamma_{\alpha \mu}^\nu = \{ \alpha \mu \}^\nu + \frac{1}{2} g^{\alpha \beta} (K_{\nu \beta} C_{\mu} + K_{\mu \beta} C_{\nu} - K_{\mu \nu} C_{\beta}),
\]

\[
(2) \Gamma_{\alpha \mu}^\nu = \{ \alpha \mu \}^\nu - \frac{1}{2} g^{\alpha \beta} (K_{\nu \beta} C_{\mu} + K_{\mu \beta} C_{\nu} - K_{\mu \nu} C_{\beta})
\]

where \( K_{\mu \nu} = g_{\mu \nu} - \sqrt{g_{\mu \nu}} - h_{\mu \nu} \), which reduces to \( K_{\mu \nu} = h_{\mu \nu} \) in the linear order and will produce exactly a branch of the dRGT model for higher order terms. It shows this framework allows to have ghost free massive gravity even at non-linear level. Let us recall that the above forms of connections are respectively equivalent to (1) \( \nabla_{\mu} g_{\alpha \beta} = -C_{\mu} K_{\alpha \beta} \) and (2) \( \nabla_{\mu} g_{\alpha \beta} = +C_{\mu} K_{\alpha \beta} \) and the corresponding Lagrangian will be as following

\[
\mathcal{L}_{MG} = -\frac{m^2}{4} \left( K_{\mu \nu} K^{\mu \nu} - K^2 \right).
\]

Now the natural question is if one can get the higher order dRGT terms i.e. \( \mathcal{O}(K^3) \) and \( \mathcal{O}(K^4) \) terms. To do this we need to generalize (1) \( \nabla_{\mu} g_{\alpha \beta} = -C_{\mu} K_{\alpha \beta} \) and (2) \( \nabla_{\mu} g_{\alpha \beta} = +C_{\mu} K_{\alpha \beta} \) to higher order terms in \( K \)'s. Let us assume

\[
(1) \nabla_{\mu} g_{\alpha \beta} = -C_{\mu} X_{\alpha \beta}
\]

\[
(2) \nabla_{\mu} g_{\alpha \beta} = +C_{\mu} X_{\alpha \beta}
\]

which we know, from previous section, result in a Lagrangian as \( \mathcal{L}_{MG} = -\frac{m^2}{4} \left( X_{\mu \nu} X^{\mu \nu} - X^2 \right) \) for \( C^\nu X_{\mu \nu} = 0 \) and
$C^2 = -m^2$. Now by assuming the following definition for $X_{\mu\nu}$

$$X_{\mu\nu} = K_{\mu\nu}$$

$$+ \alpha \left[ -KK_{\mu\nu} + 2KK^\rho_{\mu\nu} \right]$$

$$+ \left[ -\frac{1}{2} (\beta + \alpha^2) K^{2} K_{\mu\nu} + \beta KK_{\mu\rho} K^\rho_{\nu} - (3\beta + 2\alpha^2) K_{\mu\rho} K^\rho_{\nu} + \frac{1}{2} (3\beta + 4\alpha^2) K_{\rho\nu} K^\rho_{\mu} K_{\nu} \right]$$

the Lagrangian $L_{\text{MG}} = \frac{-m^2}{4}\left( X_{\mu\nu}X^{\mu\nu} - X^2 \right)$ will be

$$L_{\text{dRGT}}^{\text{MG}} = \frac{-m^2}{4}\left( K_{\mu\nu}K^{\mu\nu} - K^2 \right)$$

$$- \frac{m^2}{2} \alpha \left( K^3 - 3KK_{\mu\rho}K^{\mu\rho} + 2K_{\mu\rho}K^{\rho}_{\mu\nu}K^{\nu}\right)$$

$$- \frac{m^2}{4} \beta \left( K^4 - 6K^2K_{\mu\nu}K^{\mu\nu} + 8KK_{\mu\rho}K^{\rho}_{\mu\nu}K^{\nu} + 3(K_{\mu\nu}K^{\mu\nu})^2 - 6K_{\mu\rho}K^{\rho}_{\mu\nu}K^{\nu} \right) + O(K^5)$$

which has exactly the same structure of dRGT terms [4, 5]. Note that $\alpha$ and $\beta$ are two arbitrary parameters of the model.

Although the calculations are algebraic and straightforward but they are not trivial. It is because in above definition of $X_{\mu\nu}$ before fixing the coefficients of each term, for first, second and third order terms we have one, two and four possible terms respectively. But in the Lagrangian for quadratic, cubic and quartic terms we have two, three and five possible terms respectively. It means the number of variables are less than the number of equations which should be satisfied. But it is an astonishment that a solution exists for this over-determined system of equations (see Appendix for details). Hence it can be concluded that Weyl geometrical inspired model introduced in this section in multi-connection framework is consistent with all the dRGT potential terms surprisingly.

**IV. CONCLUSIONS, DISCUSSIONS AND OPEN PROBLEMS**

The multi-connection framework has been introduced and then its kinematics and dynamics have been considered. At the level of kinematics it has been shown that the average connection plays the same role as the connection of usual manifold with a single connection. At the level of dynamics both the average connection and the differences between connection show themselves. The Einstein-Hilbert action imposes that the average connection be the Christoffel symbol. The appearance of the difference tensor needs a generalization of Einstein-Hilbert action. To do this many options exist and what we assumed is the dependence of the given connections. For a specific case we have shown that how Weyl geometry can be used in this framework which not only makes connections dependent but also automatically results in Fierz-Pauli massive gravity at the quadratic level. This can be surprisingly generalized to de Rham-Gabadadze-Tolley massive gravity. This fact is straightforward but not trivial because the system of equations is over-determined. In this specific example the metric compatibility relation is not satisfied by the given connections at the perturbed level. However the average connection is the Christoffel symbol (compatible with the general framework) and consequently it is metric compatible.

It is worth to mention that the multi-connection framework can solve the problems in multi-metric models by construction. In multi-connection framework dynamics, an auxiliary field exist that can be interpreted as the metric. This interpretation is consistent with comparison with Einstein general relativity. Naturally this metric is the physical metric and responsible for coupling to matter. On the other hand it has been shown that the average of given connections is the Christoffel symbol according to this metric. It is important because the average connection appears in geodesic equation. Therefore everything is consistent with the equivalence principle. So in the multi-connection framework there is just one metric which is responsible for coupling to matter and also the measuring distance as well as the causal structure.

**A. Future Perspectives**

It seems that a multi-connection framework is well-defined and does not suffer from ambiguities in physical interpretations. One more deep interpretation of this framework may come from comparing this framework with Feynman path integral. In path integral all the paths between events A and B are allowed by a weight at the level of quantum
mechanics. Then there is a path which is special and it is the classical path. By an analogy to this in multi-connection scenario one can think as follow: for parallel transportation from point A to B on a manifold all the connections are allowed in principle. Then the average of these connections is the Christoffel symbol that appears in the Einstein general relativity. In other words, the Einstein general relativity is the average geometry of multi-connection scenario. We specifically showed all the possible connections can be assumed as representations of an imperfection in metric compatibility relation via Weyl geometry. The challenging question in this viewpoint is that what does play the role of weight [18] in this case? It should be emphasized that this similarity is proposed as a potential clue to understand multi-connection model more deeply. Obviously the energy regimes of Feynman path integral quantization and multi-connection framework are ultra-violet and infra-red scales respectively.

The other interesting topic to consider is the notion of geometrical curvature in the presence of more than one connection [10]. This is out of the scope of this work but it can shed lights on our understanding of geometry of space-time.

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Appendix A: Details on $O(K^3)$ and $O(K^4)$ terms of dRGT model

In this appendix we will show in details the procedure which is used to get all dRGT massive gravity terms. In dRGT massive gravity the coefficients of the mass term are tuned such that the model becomes ghost free. Let us recall 20 i.e. the full dRGT mass term

$$L_{dRGT}^{MG} = -\frac{m^2}{4} (K_{\mu \nu} K^{\mu \nu} - K^2)$$

$$= -\frac{m^2}{2} \alpha \left( K^3 - 3K K_{\mu \nu} K^{\mu \nu} + 2K_{\mu \nu} K_{\rho}^{\mu} K^{\rho \nu} \right)$$

$$= -\frac{m^2}{4} \beta \left( K^4 - 6K^2 K_{\mu \nu} K^{\mu \nu} + 8K K_{\mu \nu} K_{\rho}^{\mu} K^{\rho \nu} + 3(K_{\mu \nu} K^{\mu \nu})^2 - 6K_{\mu \nu} K_{\rho}^{\mu} K_{\sigma}^{\rho} K^{\sigma \nu} \right) + O(K^5)$$

where $\alpha$ and $\beta$ are arbitrary coefficients but as it was mentioned the other coefficients are fixed to make the model ghost free. In Weyl geometrical inspired model in multi-connection framework we have

$$(1) \nabla_{\mu} g_{\alpha \beta} = -C_{\mu \alpha \beta}$$

$$(2) \nabla_{\mu} g_{\alpha \beta} = +C_{\mu \alpha \beta}$$

which results in the potential term as $L_{MG} = -\frac{m^2}{4} (X_{\mu \nu} X^{\mu \nu} - X^2)$. However in our example to make a relationship between our model and dRGT massive gravity we can define $X_{\mu \nu}$ as a function of $K_{\alpha \beta}$ i.e. $X_{\mu \nu}(K_{\alpha \beta})$. Now let us find $X_{\mu \nu}(K_{\alpha \beta})$ order by order to get dRGT terms (A1).

- second order term:
  This one seems obvious since by plugging $X_{\mu \nu} = K_{\mu \nu}$ into $L_{MG} = -\frac{m^2}{4} (X_{\mu \nu} X^{\mu \nu} - X^2)$ what we get is exactly the second order dRGT term i.e. the first line in (A1). However we should mention that at this order we have just one choice for $X_{\mu \nu}$ but we get both terms at the level of the Lagrangian correctly. This means it is not a trivial result though seems very easy. This fact will be clearer in the following when we consider higher order terms.

- third order term:
  To get the third order terms in (A1) we need to assume up to second order term for $X_{\mu \nu}$ as

$$X_{\mu \nu} = K_{\mu \nu} + a K K_{\mu \nu} + b K_{\mu \rho} K^{\rho \nu}$$

(A2)
where $a$ and $b$ are two freedom which we need to fix. If we plug the above $X_{\mu\nu}$ into $\mathcal{L}_{MG}$ we get

$$X_{\mu\nu}X^{\mu\nu} - X^2 = \left(K_{\mu\nu}K^{\mu\nu} - K^2\right) - 2aK^3 + (2a-2b)KK_{\mu\nu}K^{\mu\nu} + 2bK_{\mu\nu}K_{\rho\sigma}K^{\rho\sigma} + O(K^4). \tag{A3}$$

Now by comparing the above result with the third line in (A1) we get five equations for four variables

$$2b = 4a, \quad 2a - 2b = -6\alpha, \quad -2a = 2\alpha. \tag{A4}$$

Though in principle it is an over-determined system of equations but interestingly we can solve all the above equations by just having two variables $a$ and $b$. Then we have

$$X_{\mu\nu} = K_{\mu\nu} + \alpha \left(-KK_{\mu\nu} + 2K_{\mu\rho}K_{\nu}^{\rho}\right). \tag{A5}$$

- fourth order term:

Exactly the procedure is same as before, by taking care of the previous results we need to assume

$$X_{\mu\nu} = K_{\mu\nu} + \alpha \left[-KK_{\mu\nu} + 2K_{\mu\rho}K_{\nu}^{\rho}\right] + \alpha K^2K_{\mu\nu} + bKK_{\mu\rho}K_{\nu}^{\rho} + cK_{\mu\rho}K_{\nu}^{\rho} + dK_{\rho\sigma}K_{\mu\nu}\tag{A6}$$

where $a$, $b$, $c$ and $d$ are four variables which should be fixed. Again by plugging the above relation for $X_{\mu\nu}$ into $\mathcal{L}_{MG}$ we get

$$X_{\mu\nu}X^{\mu\nu} - X^2 = \left(K_{\mu\nu}K^{\mu\nu} - K^2\right) + 2a\left(K^3 - 3KK_{\mu\nu}K^{\mu\nu} + 2K_{\mu\rho}K_{\nu}^{\rho}K^{\mu\rho}\right) \tag{A7}$$

Now by comparing the above result with the third line in (A1) we get five equations for four variables

$$-\alpha^2 - 2a = \beta, \quad 5\alpha^2 + 2a - 2b - 2d = -6\beta, \quad -4\alpha^2 + 2d = 3\beta, \quad -4\alpha^2 + 2b - 2c = 8\beta, \quad 4\alpha^2 + 2c = -6\beta. \tag{A8}$$

which is again an over-determined system of equations. However interestingly one can solve all the above equations by assuming

$$a = -\frac{1}{2}\beta - \frac{1}{2}\alpha^2, \quad b = \beta, \quad c = -3\beta - 2\alpha^2, \quad d = \frac{3}{2}\beta + 2\alpha^2. \tag{A9}$$

So by assuming

$$X_{\mu\nu} = K_{\mu\nu}\tag{A10}$$

$$+ \alpha \left[-KK_{\mu\nu} + 2K_{\mu\rho}K_{\nu}^{\rho}\right]$$

we can get all the dRGT terms correctly. We mention again that it is not a trivial result since the system of equations are over-determined.

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[11] About dark matter there is a very strong alternative from particle physics viewpoint.
[12] It should be emphasized that both approaches result in the same physical consequence which is the additional degrees of freedom. For example see N. Arkani-Hamed’s talk in Prospects in Theoretical Physics (PiTP) 2011.
[13] Note that we do not assume geodesic equation as a result of variation of an action i.e. $S = \int \sqrt{g_{\mu\nu}} dx^\mu dx^\nu$. This assumption says the connection is the Christofell symbol.
[14] Note that the extra $N$ in the last term can be absorbed by rescaling the coordinates as $x^\mu \to N x^\mu$.
[15] We will see this auxiliary field will play the role of the metric.
[16] It is worth to mention that in the Lagrangian the last term i.e. $\Omega^\nu_{\alpha\lambda} \Omega^\lambda_{\nu\mu} - \Omega^\nu_{\nu\lambda} \Omega^\lambda_{\alpha\mu}$ is exactly $\Upsilon_{\nu\mu}$ in bi-metric MOND model. However in this term has been chosen by hand but here this structure appears automatically. We emphasize that bi-metric MOND model is a bi-metric model and fundamentally it is different with our model.
[17] In a special case where $\eta_{\mu\nu}$ is the background metric, one can assume a Gaussian distribution for $C_\mu$ then can use $\langle C_\mu C_\nu \rangle = C^2 \eta_{\mu\nu}$. This fact transforms the Lagrangian to by assuming $C^2 = m^2$.
[18] The weight that appears in path integral approach.