ISOTROPIZATION OF SLOWLY EXPANDING SPACETIMES

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Abstract
We show that the homogeneous, massless Einstein-Vlasov system with toroidal spatial topology for initial data close to isotropic data isotropizes towards the future and in particular asymptotes to a radiative Einstein-deSitter model.

1. Introduction
Determining the asymptotic behaviour for cosmological models is a fundamental objective of mathematical cosmology. For the Einstein-Vlasov system, which models universes containing ensembles of self-gravitating collisionless particles [Re08], this program is quite advanced for the class of spatially homogeneous spacetimes, i.e., for the Bianchi models (with dynamical systems approach initiated by Rendall [Re96]; e.g., CH09 CH10 CH11 He12 RT99 RU00 RU06 PH19, and also with small data approach [Nu10 Nu12 Nu13 Nu14]), but still incomplete. In particular, for the corresponding system with massless particles, which behaves substantially different from the massive case for various spatial topologies, the problem of determining the future asymptotic behaviour is generally open. We resolve the problem of determining the future asymptotics for Bianchi I models with massless Vlasov matter for initial data close to the well-known radiative Einstein-deSitter model by showing that the latter is in fact an attractor within the Bianchi I class.

The radiative EdS model. The radiative Einstein-deSitter model (cf. e.g. [We08])

\[ ((0, \infty) \times T^3, -dt^2 + t \cdot \gamma), \]

where \((T^3, \gamma)\) is a flat torus, is a solution to the Einstein equations coupled to a radiation fluid. With a scale factor \(a(t) = \sqrt{t}\) it expands significantly slower than the related FLRW vacuum solution on hyperbolic spatial topologies (the Milne model) with \(a(t) = t^2\) and also slower than the corresponding solution on \(T^3\) for dust, the Einstein-deSitter model, with \(a(t) = t^{4/3}\) [Re08]. The Einstein-deSitter models pose interesting examples of matter-dominated cosmological spacetimes, i.e., spacetimes whose asymptotic behaviour is altered by the presence of matter. While for initial data close to the Milne geometry on hyperbolic spatial manifolds vacuum and non-vacuum future asymptotics are similar [AF17], on toroidal spatial topologies vacuum asymptotics deviate drastically from the matter dominated regime of the Einstein-deSitter models [Re08]. It is of essential interest to investigate the stability properties of those model solutions in order to understand whether their behaviour is representative for generic spacetimes with similar initial data. The stability properties of Einstein-deSitter models are unknown except in the homogeneous context, i.e., for Bianchi type I models. In that case it has been shown by Nungesser that the massive Einstein-deSitter model is a future attractor of the Einstein-Vlasov system with massive particles [Nu10]. The analogous problem for the radiative Einstein-deSitter model, which concerns massless particles (or radiation) is addressed for the massless Einstein-Vlasov system in the present paper. We show that Bianchi type I initial data sufficiently close to an isotropic state for the...
massless Einstein-Vlasov system isotropizes towards the future and asymptotes towards a member of the family of radiative EdS models.

The slower expansion rate for the radiative case makes it a priori more difficult to establish sufficiently strong decay estimates for perturbations. We point out that nonlinear stability results are established for exponential scale factors \([\text{Ri13]}\) or polynomial scale factors with significantly higher exponents \([\text{AF17]}\). Indeed, our analysis requires a more careful treatment of the evolution equation for the shear tensor, which keeps track of the sign of the relevant matter term, to establish the required decay estimates.

The theorem assures that the radiative Einstein-deSitter model is an attractor in the restricted class of Bianchi-i symmetric solutions to the massless Einstein-Vlasov system. In how far stability holds in less restricted sets of solutions as for instance the set of surface-symmetric or \(T^2\)-symmetric solutions is an open problem that can be addressed using the framework of previous works as for instance \([\text{ARW04]}\) and will be subject of future studies.

2. The massless Einstein-Vlasov system in Bianchi type I symmetry

Bianchi spacetimes admit a Lie algebra of Killing vector fields \(K_1, K_2, \text{and } K_3\) which are tangent to the orbits of the group which is identified with the universal covering space of Bianchi models. These orbits are called surface of homogeneity. Moreover, the Killing vector fields satisfy the commutation relation

\[
[K_i, K_j] = C_{ij}^k K_k,
\]

where \(C_{ij}^k\) are structure constants. Note that in Bianchi I models \(C_{ij}^k = 0\). Choosing a unit vector field \(n\) normal to the group orbits, one has natural choice for the time coordinate. Therefore, one can choose a basis \(\{E_i\}\) of the surfaces of homogeneity such that they commute with the Killing vector fields. In this way, we construct the so-called left-invariant frame \(\{n, E_i\}\) which is generated by the right-invariant Killing vector fields. We now consider general Bianchi I spacetimes of the form

\[
g = -dt^2 + g, \quad g = g_{ab}(t)W^a \otimes W^b, \quad \text{where } W^i \text{ denote the dual one-forms to the left-invariant basis } E_i.
\]

We denote by \(k_{ab}\) the second fundamental form and decompose via

\[
k_{ab} = \sigma_{ab} - Hg_{ab}, \quad \text{where } H = -\frac{1}{3}\text{tr}_g k \quad \text{and } \sigma \text{ is the trace-free part of } k.
\]

For technical simplicity we choose \(t_0 = (2H(t_0))^{-1}\) which does not restrict the generality. Moreover, we define the rescaled trace-free part by

\[
(2.1) \quad \Sigma^b_a := H^{-1}\sigma^b_a
\]

and its square by

\[
(2.2) \quad F := \Sigma^b_a \Sigma^a_b.
\]

The energy-momentum tensor of massless Vlasov matter is determined by the distribution function \(f\) which solves the transport equation

\[
(2.3) \quad \partial_t f + 2k^a_{\mu} p^b \partial_{\nu} f = 0,
\]

since we require that the distribution function is compatible with the Bianchi I symmetry, i.e., it does not depend on the position; hence \(f = f(t, p)\). The relevant components of the energy momentum tensor entering the Einstein equations are the energy density and the spatial part of the energy momentum tensor, i.e., (cf. \([\text{Re08]}\))

\[
(2.4) \quad \rho = \int_{\mathbb{R}^3 \setminus \{0\}} f(t, p) \sqrt{g_{ab}p^ap^b} \sqrt{\text{det } g} dp,
\]

\[
S_{ab} = \int_{\mathbb{R}^3 \setminus \{0\}} f(t, p) \frac{p_ap_b}{\sqrt{g_{ab}p^ap^b}} \sqrt{\text{det } g} dp,
\]
where both integrals exclude the element \( p = 0 \) to assure regularity of the integrand. Here \( dp := dp^1 dp^2 dp^3 \). Note, that in the massless case \( \tau_0 S = \rho \). In this terminology the massless Einstein-Vlasov system in Bianchi type I symmetry takes the following form (cf. [Nug10]).

\[
\begin{align*}
(2.5a) \quad 16\pi \rho &= H^2 (6 - F), \\
(2.5b) \quad \partial_t g_{ab} &= -2(\sigma_{ab} - H g_{ab}), \\
(2.5c) \quad \partial_t (H^{-1}) &= 2 + \frac{1}{6} F, \\
(2.5d) \quad \dot{k}_{ab} &= k k_{ab} - 2k_{ac}k^c_b - 8\pi S_{ab}, \\
(2.5e) \quad \dot{k}^a_b &= k k^a_b - 8\pi S^a_b, \\
(2.5f) \quad \dot{\Sigma}^a_b &= -H \left[ \Sigma^a_b \left( 3 + \frac{\dot{H}}{H^2} \right) + \frac{8\pi S^a_b}{H^2} \right]; \quad a \neq b, \\
(2.5g) \quad 0 &= \partial_t f + 2k^a_b \partial^b \rho f.
\end{align*}
\]

The main theorem of this paper, concerning the preceding system, is the following. Note that \( C \) denotes some positive constant that can change from line to line throughout the present paper.

**Theorem 2.6.** Consider initial data for the massless Einstein-Vlasov system with Bianchi I symmetry, \( (g_0, H_0, F_0, \rho(f_0)) \) at \( t_0 = (2H(t_0))^{-1} \). There exists an \( \varepsilon > 0 \) such that \( F_0 + |S(f_0)|^{2}_{g_0} < \varepsilon \) implies the following future asymptotics for a constant \( C > 0 \),

\[
\begin{align*}
(2.7) & \quad F(t) \leq C\varepsilon t^{-1/2} \ln t, \\
& \quad 2t \leq H^{-1}(t) \leq 2t(1 + C\varepsilon t^{-1/2} \ln t).
\end{align*}
\]

and

\[
(2.8) \quad tg^{ij} \rightarrow g^{ij}_\infty \text{ as } t \rightarrow \infty \text{ with } |g^{ij}_\infty - t_0 g^{ij}_0| < C\sqrt{\varepsilon}.
\]

In particular, the rescaled square of the shear-tensor, \( F = |\Sigma|_g^2 \) vanishes asymptotically, i.e., the spacetime isotropizes. Moreover, the rescaled spatial metric \( t^{-1}g(t) \) converges to a limit metric \( g_\infty \), which remains \( \varepsilon \)-close to the initial metric \( g_0 \). In result, the massless EdS model is orbitally stable in the set of solution to the massless Einstein-Vlasov system with Bianchi I symmetry.

**Remark 2.9.** It is important to emphasize that the initial data we consider concerns the regime where \( \rho(t_0) \) is large while we require \( S_{ij}(t_0) \) to be small in comparison. Note that we require \( \rho(t_0) \) to be large indirectly via the Hamiltonian constraint (2.5a) by choosing \( F(t_0) \) small. This is possible since we consider small perturbations of the euclidean metric \( g_0 \approx \delta \) and moreover we can choose \( f_0 \) almost symmetric in \( p^i \) such that \( f(t_0, p) \approx f(t_0, -p) \).

### 3. Proof of the main theorem

We consider initial data at \( t_0 > 0 \) in the form \( (g_0, H_0, \Sigma_0, f_0) \). We then make the following bootstrap assumptions on a finite interval \([t_0, t_1)\)

\[
(3.1) \quad F(t) \leq C_F \varepsilon (1 + t)^{-\alpha}, \\
|tg^{ij}| \leq |t_0 g^{ij}(t_0)| + C,
\]

where \( C_F < 6(1 + t_0) \) and \( 0 < \alpha < 1/2 \). Based on this assumption and the evolution equation for \( H \) we obtain

\[
(3.2) \quad 2 \leq \partial_t (H^{-1}) \leq 2 + \frac{1}{6} \varepsilon C_F (1 + t)^{-\alpha}
\]
and in turn
\begin{equation}
2t \leq H^{-1}(t) \leq 2t + \frac{\varepsilon C_F}{6(1-\alpha)} t^{1-\alpha}.
\end{equation}

The essential dynamical quantity is the square of the trace-free part of the second fundamental form, rescaled by \( H(t) \) which obeys the following evolution equation
\begin{equation}
\dot{F} = 2HF(-1 + \frac{1}{6} F) - \frac{16\pi}{H^2} \sigma_{ab} S^{ab}.
\end{equation}

The first term induces decay and the second is decaying fast since it is quadratic in \( F \). The problematic term is in fact the third term on the right-hand side. A rough estimate based on the decay of momentum variables does not lead to a sufficient decay estimate for that term. Therefore we need to proceed differently. The crucial observation here is the fact that the sign of this term is relevant. We distinguish two cases. We denote \( \langle \sigma, S \rangle = \sigma_{ab} S^{ab} \). If \( \langle \sigma, S \rangle \geq 0 \) the term can be neglected in the estimate for an upper bound on \( \dot{F} \). As we want to show the decay of \( F \) to zero and \( F > 0 \) a negative term on the right-hand side can be ignored. We can therefore restrict our analysis to the case \( \langle \sigma, S \rangle < 0 \). A straightforward computation yields with \( A = -H^{-2} \langle \sigma, S \rangle \)
\begin{equation}
\frac{d}{dt} A = (-3 + \frac{1}{3} F) HA - H^2 \left[ \frac{1}{6} \frac{1}{16\pi} (6 - F)^2 + 2\Sigma_{c} \Sigma_{d} S_{c}^{b} S_{d}^{a} H^{-2} - 8\pi H^{-4} |S|_g^2 - H^{-4} \sigma_{ab} W^{ab} \right].
\end{equation}

The first terms on the right-hand side yield decay. As \( A \) is positive, the third term, in brackets, can be ignored in a decay estimate if \( (\ast) < 0 \) holds. In the following we show that under the imposed smallness assumptions this bound holds. For sufficiently large \( t, 6 - F > 5 \) by the bootstrap assumption.

We now provide an estimate for the metric which follows from
\begin{equation}
\dot{g}^{ij} = 2H \Sigma_{k} g^{kij} - 2H g^{ij}
\end{equation}
and in turn
\begin{equation}
\frac{d}{dt} (tg^{ij}) = 2H \Sigma_{k} (tg^{kij}) + (t^{-1} - 2H) tg^{ij}.
\end{equation}

This implies, based on the bootstrap assumption and the estimate for \( H \) that
\begin{equation}
|tg^{ij} - t_0 g^{ij}(t_0)| < C \sqrt{\frac{C_F \varepsilon}{\alpha}} t_0^{-\alpha/2}.
\end{equation}

For sufficiently small \( \varepsilon \) this improves the bootstrap assumption on \( g^{ij} \). We turn to the estimate for \( S_{ab} \) based on the equation
\begin{equation}
\dot{S}_{ab} = -2H S_{ab} - W_{ab},
\end{equation}
which follows from the equations \( (2.5a) \) and \( (2.5g) \), where
\begin{equation}
W_{ab} := \int \frac{p_a p_b}{|p|} \frac{\sigma_{ij} p_i p_j}{|p|_g^2} \sqrt{g} \, d^3 p.
\end{equation}

From the form of \( W_{ab} \) we obtain the immediate estimate
\begin{equation}
|W_{ab}| \leq C H \sqrt{F} \rho.
\end{equation}

Using the Hamiltonian constraint we can deduce
\begin{equation}
\rho \leq \frac{3}{8\pi} H^2.
\end{equation}
This yields
\begin{equation}
\frac{d}{dt}|S_{ab}| \leq -2H|S_{ab}| + CH^2 \sqrt{F}
\end{equation}
and in particular, using Grönwall’s inequality,
\begin{equation}
|S_{ab}(t)| \leq t^{-1} \left( t_0 |S_{ab}(t_0)| + C \sqrt{C_F \sqrt{\varepsilon \alpha^{-1} t_0^{-\alpha/2}} } \right).
\end{equation}
In combination with the estimate for \( g^{-1} \) this yields
\begin{equation}
|\mathcal{S}_a^b(t)| \leq |\mathcal{S}_a(t)g^{ab}(t)| \leq t^{-2} \left( t_0 |S_{ai}(t_0)| + C \sqrt{C_F \sqrt{\varepsilon \alpha^{-1} t_0^{-\alpha/2}} } \right) \left[ t_0 g^{ib}(t_0) + \frac{C \sqrt{C_F \varepsilon}}{\alpha t_0^{\alpha/2}} \right] \left( 2 + \frac{C_F \varepsilon}{6(1-\alpha)} t^{-2} \right).
\end{equation}
Combining this with estimates on \( H \) and \( F \) this provides
\begin{equation}
H^{-4} \sigma_{ab} W^{ab} \leq H^{-3} \Sigma_b W^b \leq CH^{-3} \sqrt{F} H \sqrt{F_\rho} \leq CF,
\end{equation}
where we used \eqref{3.12}. This implies, using the bootstrap assumptions
\begin{equation}
H^{-4} \sigma_{ab} W^{ab} \leq CC_F \varepsilon (1 + t)^{-\alpha}.
\end{equation}
In conclusion, choosing \( \varepsilon \) and \( S_{ij}(t_0) \) sufficiently small we can show \( \ast \) > 0 and in particular the following estimate holds
\begin{equation}
\frac{d}{dt} A \leq (-3 + \frac{1}{3} F)HA.
\end{equation}
Invoking the estimates for \( H \) and the bootstrap assumption for \( F \) yields
\begin{equation}
\frac{d}{dt} (t^{3/2} A) \leq C_F \varepsilon/12 \left( 2 + \frac{3}{2(1-\alpha)} \right) t^{-1-\alpha} (t^{3/2} A).
\end{equation}
This yields
\begin{equation}
A(t) \leq t^{-3/2} CC_F \varepsilon.
\end{equation}
Based on this estimate we obtain the following for \( F \)
\begin{equation}
\frac{d}{dt} (\sqrt{t} F) \leq \frac{1}{2} t^{-1} (\sqrt{t}) F + 2H(\sqrt{t} F)(-1 + \frac{1}{6} F) + 16\pi C_F \varepsilon t^{-1}.
\end{equation}
Using the bootstrap assumption on \( F \) and the estimate for \( H \) we identify the last term as the dominant one yielding and estimate of the form
\begin{equation}
F(t) \leq t^{-1/2} (F(t_0) \sqrt{t_0} + C \varepsilon \ln t) C.
\end{equation}
We conclude that there exists a constant $C > 0$ such that
\begin{equation}
F(t) \leq C \varepsilon t^{-1/2}(\sqrt{t_0} + \ln t).
\end{equation}
For sufficiently large times, this improves the bootstrap assumption on $F$. The estimates for $H$ and $g$ when these asymptotics for $F$ are combined with the respective evolution equation.

\section*{References}

[AF17] Andersson, L.; Fajman, D.: Nonlinear stability of the Milne model with matter \textit{arXiv:1709.00267}, 2017

[ARW04] Andréasson, H.; Rendall, A.; Weaver, M.: Existence of constant areal time foliations in $T^2$-symmetric spacetimes with Vlasov matter \textit{Comm. Partial Diff. Eq.} \textbf{29} 237-262, 2004

[CH09] Calogero, S.; Heinzle, M.: Dynamics of Bianchi Type I Solutions of the Einstein Equations with Anisotropic Matter \textit{Ann. Henri Poincaré} \textbf{10}, 225274, 2009

[CH10] Calogero, S.; Heinzle, M.: Oscillations toward the singularity of locally rotationally symmetric Bianchi type IX cosmological models with Vlasov matter \textit{SIAM J. Appl. Dyn. Syst.} \textbf{9}, 1244-1262, 2010

[CH11] Calogero, S.; Heinzle, M.: Bianchi cosmologies with anisotropic matter: Locally rotationally symmetric models \textit{Physica D: Nonlinear Phenomena} \textbf{240}, 636669, 2011

[FH19] Fajman, D.; Heiβel, G.: Kantowski-Sachs cosmology with Vlasov matter \textit{arXiv:1902.02871v1}, 2019

[He12] Heißel, G.: Dynamics of locally rotationally symmetric Bianchi type VIII cosmologies with anisotropic matter \textit{Gen. Relativity and Gravitation} \textbf{44}, 29012922, 2012

[Nu10] Nungesser, E.: Isotropization of non-diagonal Bianchi I spacetimes with collisionless matter at late times assuming small data \textit{Class. Quant. Grav.} \textbf{27}, 235025, 2010

[Nu12] Nungesser, E.: Future non-linear stability for solutions of the Einstein-Vlasov system of Bianchi types II and IV\textsubscript{0} \textit{J. Math. Phys.} \textbf{53}, 102503, 2012

[Nu13] Nungesser, E.: Future non-linear stability for reflection symmetric solutions of the Einstein-Vlasov system of Bianchi types II and IV\textsubscript{0} \textit{Ann. Henri Poincaré} \textbf{14}, 967-999, 2013

[Nu14] Nungesser, E.; Andersson, L.; Coley, A.: Isotropization of solutions of the Einstein-Vlasov system with Bianchi V symmetry \textit{Ann. Henri Poincaré} \textbf{14}, 967-999, 2013

[Re06] Rendall, A.: The initial singularity in solutions of the Einstein-Vlasov system of Bianchi type I \textit{J. Math. Phys.} \textbf{37}, 438-451, 1996

[Re08] Rendall, A.: Partial Differential Equations in General Relativity Oxford Graduate Texts in Mathematics, Oxford University Press, 2008

[Re02] Rendall, A.: Cosmological Models and Centre Manifold Theory \textit{Gen. Relativity and Gravitation} \textbf{34}, 12771294, 2002

[RT99] Rendall, A.; Tod, P.: Dynamics of spatially homogeneous solutions of the Einstein-Vlasov equations which are locally rotationally symmetric \textit{Class. Quant. Grav.} \textbf{16}, 17051726, 1999

[RU00] Rendall, A.; Uggla, C.: Dynamics of spatially homogeneous locally rotationally symmetric solutions of the Einstein-Vlasov equations \textit{Class. Quant. Grav.} \textbf{17}, 46974713, 2000

[RU06] Rendall, A.; Uggla, C.: Dynamics of the spatially homogeneous Bianchi type I Einstein–Vlasov equations \textit{Class. Quant. Grav.} \textbf{23}, 34633489, 2006

[Ri13] Ringström, H.: On the Topology and Future Stability of the Universe Oxford University Press, 2013

[We08] Weinberg, S.: Cosmology Oxford University Press, 2008

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