UNIFORM UNLIKELY INTERSECTIONS FOR UNICRITICAL POLYNOMIALS

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Abstract. Fix \( d \geq 2 \) and let \( f_t(z) = z^d + t \) be the family of polynomials parameterized by \( t \in \mathbb{C} \). In this article, we will show that there exists a constant \( C(d) \) such that for any \( a, b \in \mathbb{C} \) with \( a^d \neq b^d \), the number of \( t \in \mathbb{C} \) such that \( a \) and \( b \) are both preperiodic for \( f_t \) is at most \( C(d) \).

1. Introduction

Fix an integer \( d \geq 2 \) and let

\[
f_t(z) = z^d + t
\]

be the family of polynomials parameterized by \( t \in \mathbb{C} \). A point \( a \in \mathbb{C} \) is said to be preperiodic for \( f_t \) if its forward orbit \( O_{f_t}(a) = \{ f_t^n(a) : n \geq 1 \} \) is finite, where \( f_t^n(a) \) is the \( n \)-th iterate of \( a \) under \( f_t \).

Let \( \text{PrePer}(f_t) \) be the set of all preperiodic points of \( f_t \). By [Be, Section 4], [BD1, Theorem 1.2 and Corollary 1.3], and [YZ, Theorem 1.3], \( \text{PrePer}(f_{t_1}) \cap \text{PrePer}(f_{t_2}) \) is finite if and only if \( t_1 \neq t_2 \). Furthermore, when \( d = 2 \), DeMarco, Krieger, and Ye [DKY2, Theorem 1.1] proved that \( \text{PrePer}(f_{t_1}) \cap \text{PrePer}(f_{t_2}) \) is in fact uniformly bounded for any \( t_1 \neq t_2 \). This result is an analogue of [DKY1, Theorem 1.4], which provides a partial solution to the effective finiteness conjecture for elliptic curves proposed in [BFT] and also implies a uniform Manin–Mumford bound under suitable conditions [DKY1, Theorem 1.1].

On the other hand, for any \( a, b \in \mathbb{C} \), we define

\[
S_{a,b} = \{ t \in \mathbb{C} : a \text{ and } b \text{ are both preperiodic for } f_t \}.
\]

Note that \( S_{a,b} \) depends on \( d \) implicitly. Zannier [Za, Section 3.4.7] asked whether \( S_{0,1} \) is finite when \( d = 2 \). Baker and DeMarco [BD1, Theorem 1.1] answered this question affirmatively by showing that \( S_{a,b} \) is finite if and only if \( a^d \neq b^d \). This result is motivated by a theorem of Masser and Zannier. In [MZ1], [MZ2], and [MZ3], they showed that for any \( a \neq b \in \mathbb{C} \setminus \{0, 1\} \), there exist only finitely many \( t \in \mathbb{C} \) such that

\[
(a, \sqrt{a(a-1)(a-t)}) \text{ and } (b, \sqrt{b(b-1)(b-t)})
\]

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are both torsion on the elliptic curve $y^2 = x(x-1)(x-t)$. For further developments on this problem, see also [BD2], [GHT1], and [GHT2].

However, the proof of [BD1, Theorem 1.1] is not effective so that no explicit upper bound for $|S_{a,b}|$ is given. In order to find more information on $S_{a,b}$, Fili [Fi] studied the case $S_{0,1}$ when $d = 2$ and, more importantly, made a key observation [Fi, Theorem 1] which turns out to be crucial for [DKY1] and [DKY2]. In light of [DKY1, Theorem 1.4] and [DKY2, Theorem 1.1], one is led to ask whether $S_{a,b}$ is also uniformly bounded for any $a, b \in \mathbb{C}$ with $a^d \neq b^d$.

In this article, we are able to show that the answer is yes.

**Theorem 1.1.** For any integer $d \geq 2$, there exists a constant $C(d)$ such that $|S_{a,b}| \leq C(d)$ for any $a, b \in \mathbb{C}$ with $a^d \neq b^d$.

Our results and proofs are inspired by [DKY2], and our techniques and strategies also come from their ideas. By a standard specialization argument, it suffices to prove Theorem 1.1 for $a, b \in \overline{\mathbb{Q}}$. Let $K$ be a number field such that $a, b \in K$. For each place $v \in M_K$, we work with the dynamics of $f_t$ on the Berkovich affine line $\mathbb{A}^1_{\mathbb{Q}, v, \mathrm{an}}$. Let $\mu_{a,v}$ (resp.) and $g_{a,v}$ (resp.) be the equilibrium measure and the Green’s function associated to the generalized Mandelbrot set $M_{a,v}$ (resp.). Then the Arakelov–Zhang pairing of $\mu_a = \{\mu_{a,v}\}_{v \in M_K}$ and $\mu_b = \{\mu_{b,v}\}_{v \in M_K}$ is given by

$$\langle \mu_a, \mu_b \rangle = \sum_{v \in M_K} \left[ K_v : \mathbb{Q} \right] \left[ K : \mathbb{Q} \right] \int_{\mathbb{A}^1_{\mathbb{Q}, v, \mathrm{an}}} g_{a,v} d\mu_{b,v}.$$ 

The value of $\langle \mu_a, \mu_b \rangle$ depends on $a$ and $b$ only, and is independent of the choice of $K$. By [FRL, Propositions 2.6 and 4.5] and [BD1, Theorems 1.1 and 3.4], $\langle \mu_a, \mu_b \rangle = \langle \mu_b, \mu_a \rangle \geq 0$ and

$$\langle \mu_a, \mu_b \rangle = 0 \iff \mu_a = \mu_b \iff a^d = b^d \iff |S_{a,b}| = \infty.$$

Given this equivalence relation, it is not surprising that the value of $\langle \mu_a, \mu_b \rangle$ encodes some information about $S_{a,b}$. As in [DKY1] and [DKY2], the main task of this article is to estimate the upper and lower bounds for $\langle \mu_a, \mu_b \rangle$. More precisely, we prove the following counterparts of [DKY1, Theorems 1.5, 1.6, and 1.7] and [DKY2, Theorems 1.6, 1.7, and 1.9].

**Theorem 1.2.** Let $a, b \in \overline{\mathbb{Q}}$ such that $a^d \neq b^d$ and $|S_{a,b}| > 0$. For any $0 < \varepsilon < 4d$, we have

$$\langle \mu_a, \mu_b \rangle \leq \left( \varepsilon + \frac{8d}{|S_{a,b}|} \right) (h(a, b) + 5),$$

where $h$ is the logarithmic Weil height on $\mathbb{A}^2(\overline{\mathbb{Q}})$.

**Theorem 1.3.** For any $a, b \in \overline{\mathbb{Q}}$ such that $a^d \neq b^d$, we have

$$\langle \mu_a, \mu_b \rangle \geq \frac{1}{12d^2} h(a, b) - 1,$$

where $h$ is the logarithmic Weil height on $\mathbb{A}^2(\overline{\mathbb{Q}})$. 
Theorem 1.4. There exists a constant $\delta(d) > 0$ such that $\langle \mu_a, \mu_b \rangle \geq \delta(d)$ for any $a, b \in \bar{\mathbb{Q}}$ with $a^d \neq b^d$.

The main differences between this article and [DKY2] are: (1) Our Theorem 1.1 is valid for any $d \geq 2$, while [DKY2] Theorem 1.1 focuses on $d = 2$. (2) We work with the generalized Mandelbrot sets in place of the Julia sets in [DKY2]. (3) When estimating the lower bounds for $\langle \mu_a, \nu, \mu_b, \nu \rangle$ at the non-Archimedean places $v \in M_0^0$, our computations in Section 4.4 are more simplified than [DKY2] Sections 5 and 6]. This simplification helps us to work with all $d \geq 2$ at the same time.

The plan of this article is as follows: In Section 2 we fix the notation and review the tools we will need. In Section 3 we first estimate the upper bounds for $\langle \mu_a, \mu_b \rangle$ locally, and then combine the local estimates to give the proof of Theorem 1.2. The structure of Section 4 is similar, but this time we estimate the lower bounds for $\langle \mu_a, \mu_b \rangle$ and give the proofs of Theorems 1.3 and 1.4. In Section 5 we show that Theorem 1.1 follows from Theorems 1.2, 1.3 and 1.4.

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2. Notation and Preliminaries

The main references for this section are [BR2], [BD1], [FRL], and [Fi].

Given a number field $K$, let $M_K$ be the set of places, let $M_K^\infty$ be the set of Archimedean places, and let $M_0^0$ be the set of non-Archimedean places. We normalize the absolute values $|\cdot|_v$ on $K$ such that they extend the standard absolute values on $\mathbb{Q}$. For any $a \in K^\times$, we have the product formula

$$\prod_{v \in M_K} |a|_v^{n_v} = 1, \text{ where } n_v = \frac{[K_v : \mathbb{Q}_v]}{[K : \mathbb{Q}]}.$$

2.1. Berkovich Spaces. For each $v \in M_K$, let $K_v$ be the completion of $K$ at $v$, let $\bar{K}_v$ be an algebraic closure of $K_v$, and let $\mathbb{C}_v$ be the completion of $\bar{K}_v$. The Berkovich affine line $\mathbb{A}_v^{1,\text{an}}$ is a locally compact, Hausdorff, path-connected space containing $\mathbb{C}_v$ as a dense subspace. As a topological space, $\mathbb{A}_v^{1,\text{an}}$ is the set of all multiplicative seminorms $[\cdot]_x : \mathbb{C}_v[T] \to \mathbb{R}$ on the polynomial ring $\mathbb{C}_v[T]$ which extend the absolute value $|\cdot|_v$ on $\mathbb{C}_v$, endowed with the weakest topology for which $x \mapsto [f]_x$ is continuous for any $f \in \mathbb{C}_v[T]$. The Berkovich projective line $\mathbb{P}_v^{1,\text{an}}$ can be identified with the one-point compactification of $\mathbb{A}_v^{1,\text{an}}$.

If $v \in M_K^\infty$, then by Gelfand–Mazur theorem, $\mathbb{A}_v^{1,\text{an}}$ is homeomorphic to $\mathbb{C}_v = \mathbb{C}$. If $v \in M_0^0$, then by Berkovich’s classification theorem, each $x \in \mathbb{A}_v^{1,\text{an}}$ corresponds to a decreasing nested
sequence \( \{\overline{\mathcal{D}}(a_n, r_n)\}_{n=1}^{\infty} \) of closed disks on \( \mathbb{C}_v \) such that
\[
[f]_v = \lim_{n \to \infty} \sup_{z \in \overline{\mathcal{D}}(a_n, r_n)} |f(z)|_v.
\]

Based on the nature of \( D = \cap_{n=1}^{\infty} \overline{\mathcal{D}}(a_n, r_n) \), the points of \( \mathbb{A}_{v}^{1, \text{an}} \) can be categorized into four types: (I) \( D \) is a point of \( \mathbb{C}_v \), (II) \( D \) is a closed disk with radius in \( |\mathbb{C}_v^\times| \), (III) \( D \) is a closed disk with radius not in \( |\mathbb{C}_v^\times| \), and (IV) \( D \) is the empty set.

For any \( a \in \mathbb{C}_v \) and any \( r > 0 \), we define \( \mathcal{D}(a, r) \) to be the set of points corresponding to \( \{\overline{\mathcal{D}}(a_n, r_n)\}_{n=1}^{\infty} \) with \( \overline{\mathcal{D}}(a_n, r_n) \subseteq \mathcal{D}(a, r) \), and define \( \zeta_{a, r} \) to be the point corresponding to \( \overline{\mathcal{D}}(a, r) \).

2.2. Potential Theory. When \( v \in M_K^0 \), we introduce the Hsia kernel
\[
\delta_v(x, y) = \limsup_{z, w \to \mathbb{C}_v, z \to x, w \to y} |z - w|_v,
\]
which extends the distance function \( |x - y|_v \) on \( \mathbb{C}_v \) to the entire \( \mathbb{A}_{v}^{1, \text{an}} \). When \( v \in M_K^\infty \), we also write \( \delta_v(x, y) = |x - y|_v \) to unify the notation in the sequel.

Fix \( v \in M_K \) and let \( E \) be a compact subset of \( \mathbb{A}_{v}^{1, \text{an}} \). The logarithmic capacity \( \gamma_v(E) \) of \( E \) is given by
\[
-\log \gamma_v(E) = \inf_{\mu} \int_{E \times E} -\log \delta_v(z, w) d\mu(z) d\mu(w),
\]
where the infimum is taken over all probability measures \( \mu \) supported on \( E \). If \( \gamma_v(E) > 0 \), then there exists a unique probability measure \( \mu_E \), called the equilibrium measure of \( E \), such that the infimum is achieved. The Green’s function of \( E \) is defined by
\[
g_E(z) = -\log \gamma_v(E) + \int_E \log \delta_v(z, w) d\mu_E(w),
\]
which is a non-negative real-valued function on \( \mathbb{A}_{v}^{1, \text{an}} \).

2.3. Generalized Mandelbrot Sets. Now we go back to the dynamics of \( f_t(z) = z^d + t \). For any \( v \in M_K \) and any \( a \in \mathbb{C}_v \), we define the generalized Mandelbrot set by
\[
M_{a, v} = \{ t \in \mathbb{A}_{v}^{1, \text{an}} : \sup_n |f^n_t(a)|_v < \infty \},
\]
where \( f^n_t(a) \) is considered as an element of the polynomial ring \( \mathbb{C}_v[T] \). Note that if \( t \in \mathbb{C}_v \), then \( |f^n_t(a)|_v \) is simply \( |f^n_t(a)| \). It is known that \( M_{a, v} \) is compact. We write \( \mu_{a, v} \) and \( g_{a, v} \) for the equilibrium measure and the Green’s function of \( M_{a, v} \). The following properties are collected from [BD1] Section 3.

**Theorem 2.1.** For any \( v \in M_K \) and any \( a \in \mathbb{C}_v \), we have

1. The logarithmic capacity \( \gamma_v(M_{a, v}) = 1 \).
(2) The Green’s function of $M_{a,v}$ is given by

$$g_{a,v}(t) = \lim_{n \to \infty} \frac{1}{d^n} \log^+ [f_T^{n+1}(a)]_t,$$

where $\log^+ z = \log \max\{z, 1\}$ for any $z \in \mathbb{R}$.

(3) $g_{a,v}(t)$ is continuous on $\mathbb{A}_v^{1,\text{an}}$.

(4) $g_{a,v}(t)$ is harmonic on $\mathbb{A}_v^{1,\text{an}} \setminus M_{a,v}$.

(5) $g_{a,v}(t) = 0$ if and only if $t \in M_{a,v}$.

2.4. Arakelov–Zhang Pairing. As [DKY1, Theorem 1.7] and [DKY2, Theorem 1.9], our Theorem 1.2 also builds on the quantitative equidistribution results of [FRL] and [F].

Let $a \in \mathbb{C}_v$ and $r > 0$. If $v \in M_K^\infty$, we define $m_{a,r,v}$ to be the normalized Haar measure on the circle $\partial D(a, r)$. If $v \in M_K^0$, we define $m_{a,r,v}$ to be the Dirac measure on $\zeta_{a,r}$.

Definition 2.2. [FRL, Définition 1.1] We call $\mu = (\mu_v)_{v \in M_K}$ an adelic measure if

(1) $\mu_v$ is a probability measure on $\mathbb{P}_v^{1,\text{an}}$ for any $v \in M_K$,

(2) $\mu_v = m_{0,1,v}$ for all but finitely many $v \in M_K$,

(3) for any $v \in M_K$, $\mu_v - m_{0,1,v} = \Delta u_v$ for some continuous function $u_v$ on $\mathbb{P}_v^{1,\text{an}}$, where $\Delta$ is the Laplacian on $\mathbb{P}_v^{1,\text{an}}$.

Following [FRL] Sections 2.4 and 4.4), for each $v \in M_K$, we define the mutual energy of two signed measures $\mu_{1,v}$ and $\mu_{2,v}$ on $\mathbb{P}_v^{1,\text{an}}$ by

$$\langle \mu_{1,v}, \mu_{2,v} \rangle_v = \int_{\mathbb{A}_v^{1,\text{an}} \times \mathbb{A}_v^{1,\text{an}} \setminus \text{Diag}_v} - \log \delta_v(z, w) d\mu_{1,v}(z) d\mu_{2,v}(w),$$

where Diag$_v$ is the diagonal on $\mathbb{C}_v \times \mathbb{C}_v$. Suppose $\mu_1$ and $\mu_2$ are adelic measures, then we define their $v$-adic Arakelov–Zhang pairing by

$$\langle \mu_1, \mu_2 \rangle_v = \frac{1}{2} (\mu_1 - \mu_2, \mu_1 - \mu_2)_v,$$

and define their Arakelov–Zhang pairing by

$$\langle \mu_1, \mu_2 \rangle = \sum_{v \in M_K} n_v \langle \mu_1, \mu_2 \rangle_v.$$

Theorem 2.3. [Fi, Theorem 1] The square root of the Arakelov–Zhang pairing $\langle \cdot, \cdot \rangle^{1/2}$ gives a metric on the space of all adelic measures.

Now we go back to the dynamics of $f_t(z) = z^d + t$. Given $a, b \in K$, let $\mu_a = \{\mu_{a,v}\}_{v \in M_K}$ and $\mu_b = \{\mu_{b,v}\}_{v \in M_K}$ be the equilibrium measures defined in Section 2.3. It is known that they are adelic measures and their $v$-adic Arakelov–Zhang pairing can be written as

$$\langle \mu_{a,v}, \mu_{b,v} \rangle_v = \int_{\mathbb{A}_v^{1,\text{an}}} g_{a,v} d\mu_{b,v}.$$
For the Arakelov–Zhang pairing in more general settings, see [Zh], [PST], and [CL2].

3. Upper Bounds

The purpose of this section is to give the proof of Theorem 1.2. In order to do so, we estimate the upper bounds for \( g_{a,v}(s) \) when \( s \) is close to \( M_{a,v} \). We work with \( v \in M^\infty_K \) in Section 3.1 and \( v \in M^0_K \) in Section 3.2. In Section 3.3, we apply the local estimates in Theorem 3.13 to complete the proof of Theorem 1.2.

3.1. Archimedean Estimates. In this section, we assume that \( K \) is a number field, \( a \in K \), and \( v \in M^\infty_K \). Because \( v \) is fixed, we write \( |\cdot|, M_a, \) and \( g_a \) for \( |\cdot|_v, M_{a,v}, \) and \( g_{a,v} \).

From [BD1, Lemma 3.2], we know that \( M_a \) is bounded. Their proof can be modified slightly to give an explicit bound for \( M_a \). We begin with a basic distortion result for univalent maps, which is used in the proof of [BD1, Lemma 3.2] and also in [DKY2, Section 3.1].

**Theorem 3.1.** Let \( U_R = \{ z \in \mathbb{C} : |z| > R \} \). If \( \phi : U_R \to \mathbb{C} \) is analytic, injective, and

\[
\phi(z) = z + \sum_{n=1}^{\infty} a_n z^n,
\]

then \( \phi(U_R) \supseteq U_{2R} \). In particular, \( |\phi(z)| \leq 2|z| \) for any \( z \in U_R \).

**Proof.** The first assertion is [BH] Corollary 3.3. If the second assertion is false, then there exists \( z \in U_R \) such that \( \phi(z) \in U_{2|z|} \subseteq \phi(U_{|z|}) \), which contradicts the injectivity of \( \phi \).

**Proposition 3.2.** If \( |t| > 4^d \max\{|a|, 4\}^d \), then

\[
\log |t| - 1 \leq g_a(t) \leq \log |t| + 1.
\]

In particular, if \( t \in M_a \), then \( |t| \leq 4^d \max\{|a|, 4\}^d \).

**Proof.** For each \( t \in \mathbb{C} \), let

\[
\lambda_t(z) = \lim_{n \to \infty} \frac{1}{d^n} \log^+ |f_t^n(z)|.
\]

There exists an analytic homeomorphism \( \phi_t \), defining the Böttcher coordinate near \( \infty \), which satisfies \( \phi_t(f_t(z)) = \phi_t(z)^d \) and \( \lambda_t(z) = \log |\phi_t(z)| \). The map \( \phi_t \) sends the domain

\[
V_t = \{ z \in \mathbb{C} : \lambda_t(z) > \lambda_t(0) \}
\]

biholomorphically to \( U_{R_t} \) with \( R_t = e^{\lambda_t(0)} \). By Theorem 3.1, \( V_t = \phi_t^{-1}(U_{R_t}) \supseteq U_{2R_t} \). By the proof of [BD1, Lemma 3.2], \( |t| \leq 2^d R_t^d \), and if \( t \) is large enough such that \( R_t^d - 4R_t > 2|a|^d \), then \( a^d + t \in U_{2R_t} \). If \( |t| > 4^d \max\{|a|, 4\}^d \), then we have

\[
R_t \geq \frac{1}{2} |t|^{1/d} > 2 \max\{|a|, 4\}
\]
\[ R^d_t - 4R_e = R_t(R_t^{d-1} - 4) > 2 \max\{|a|, 4\}(2^{d-1} \max\{|a|, 4\}^{d-1} - 4) = 2 \max\{|a|, 4\}^d + ((2^d - 2) \max\{|a|, 4\}^{d-1} - 8) \max\{|a|, 4\} \geq 2|a|^d. \]

Applying Theorem 3.1 to \( \phi_t \), we get
\[ |\phi_t(a^d + t)| \leq 2|a^d + t| \leq 2(|a|^d + |t|) \leq \frac{17}{8} |t|, \]
where the last inequality follows from \( |t| > 4^d \max\{|a|, 4\}^d \geq 16|a|^d \). Applying Theorem 3.1 to \( \phi_t^{-1} \), we get
\[ |\phi_t(a^d + t)| \geq \frac{1}{2} |\phi_t^{-1}(\phi_t(a^d + t))| = \frac{1}{2} |a^d + t| \geq \frac{1}{2} (|t| - |a|^d) \geq \frac{15}{32} |t|. \]

By \( \text{BD1} \) Proposition 3.3, we have
\[ \log |t| - 1 \leq g_a(t) = \log |\phi_t(a^d + t)| \leq \log |t| + 1. \]
In particular, by Theorem 2.1 if \( |t| > 4^d \max\{|a|, 4\}^d \), then \( g_a(t) > 0 \) and \( t \notin M_a \).

By definition, \( O_{f_i}(a) \) is bounded if \( t \in M_a \). The following result shows that \( O_{f_i}(a) \) is in fact uniformly bounded for any \( t \in M_a \).

**Proposition 3.3.** If \( t \in M_a \), then \( |f^n_i(a)| < 8 \max\{|a|, 4\} \) for any \( n \geq 1 \).

**Proof.** By Proposition 3.2 if \( t \in M_a \), then \( |t| \leq 4^d \max\{|a|, 4\}^d \). Suppose there exists \( n \geq 1 \) such that \( |f^n_i(a)| \geq 8 \max\{|a|, 4\} \), then
\[ |f^{n+1}_i(a)| \geq |f^n_i(a)|^d - |t| \geq \left(1 - \frac{1}{2^d}\right)|f^n_i(a)|^d \geq 24|f^n_i(a)|. \]
By induction, \( |f^{n+k}_i(a)| \geq 24^k|f^n_i(a)| \to \infty \) as \( k \to \infty \), a contradiction.

The explicit bound of \( M_a \) given in Proposition 3.2 can be improved as follows.

**Proposition 3.4.** If \( t \in M_a \), then \( |t| < 3 \max\{|a|, 4\}^d \).

**Proof.** If \( t \in M_a \), then by Proposition 3.3
\[ |t| \leq |a^d + t| + |a|^d < 8 \max\{|a|, 4\} + |a|^d \leq \left(\frac{8}{\max\{|a|, 4\}d-1} + 1\right) \max\{|a|, 4\}^d \leq 3 \max\{|a|, 4\}^d. \]

As \( \text{DKY2} \) Section 3.3, we estimate the upper bound for \( g_a(s) \) when \( s \) is close to \( M_a \).

**Proposition 3.5.** If \( t \in M_a \) and \( |s - t| \leq \max\{|a|, 4\} \), then
\[ |f^n_s(a) - f^n_i(a)| \leq |s - t|(18 \max\{|a|, 4\})^{d^n-1}. \]
for any $n \geq 1$.

Proof. Let $A_n = (18 \max\{|a|, 4\})^{d^{n-1}}$ for any $n \geq 1$. We prove the assertion by induction. Because $|f_s(a) - f_t(a)| = |s - t|$, the statement is true for $n = 1$. Assume the statement is true for some $n \geq 1$, then

$$
|f_s^{n+1}(a) - f_t^{n+1}(a)|
= |f_s^n(a)^d + s - f_t^n(a)^d - t|
\leq |f_s^n(a)^d - f_t^n(a)^d| + |s - t|
= \prod_{i=0}^{d-1}|f_s^n(a) - f_t^n(a) + (1 - \zeta_i) f_t^n(a)| + |s - t|
\leq |f_s^n(a) - f_t^n(a)|(|f_s^n(a) - f_t^n(a)| + 2|f_t^n(a)|)^{d-1} + |s - t|
\leq |s - t|A_n(A_n \max\{|a|, 4\} + 16 \max\{|a|, 4\})^{d-1} + |s - t|

(by the assumption, the induction hypothesis, and Proposition 3.3)
= |s - t|(A_n(A_n + 16)^{d-1} \max\{|a|, 4\}^{d-1} + 1)
\leq |s - t|(A_n(A_n + 16)^{d-1} \max\{|a|, 4\}^{d-1} + \max\{|a|, 4\}^{d-1})
= |s - t|(17^{d-1}A_n^d + 1) \max\{|a|, 4\}^{d-1}
\leq |s - t|18^{d-1}A_n^d \max\{|a|, 4\}^{d-1}
= |s - t|A_{n+1}.

This completes the inductive step and hence the proof.

Proposition 3.6. If $t \in M_a$ and

$$
|s - t| \leq \frac{1}{18}(18 \max\{|a|, 4\})^{2^{d^{n-1}}}
$$

for some $n \geq 1$, then

$$
g_a(s) \leq \frac{1}{d^{n-1}} \log(10 \max\{|a|, 4\}).
$$

Proof. Let $A_n = (18 \max\{|a|, 4\})^{d^{n-1}}$ for any $n \geq 1$. By Propositions 3.3 and 3.5

$$
|f_s^n(a)| \leq |f_s^n(a) - f_t^n(a)| + |f_t^n(a)| \leq |s - t|A_n + |f_t^n(a)| \leq 9 \max\{|a|, 4\}.
$$

By Proposition 3.4

$$
|f_s^{n+1}(a)| \leq |f_s^n(a)^d + |t| + |s - t| \leq (9^d + 4) \max\{|a|, 4\}^d.
$$

Let $p(z) = z^d + 4$, then by induction and the following Lemma 3.7

$$
|f_s^{n+k}(a)| \leq p^k(9) \max\{|a|, 4\}^{d^k} \leq (10 \max\{|a|, 4\})^{d^k}
$$

for any $k \geq 1$. Therefore, by Theorem 2.1

$$
g_a(s) = \lim_{k \to \infty} \frac{1}{d^{n+k-1}} \log^+ |f_s^{n+k}(a)| \leq \frac{1}{d^{n-1}} \log(10 \max\{|a|, 4\}).$$
Lemma 3.7. Let $p(z) = z^d + 4$ for some $d \geq 2$. Then $p^n(9) \leq 10^{dn}$ for any $n \geq 1$.

Proof. We prove $q(n) = 10^{dn} - p^n(9) \geq 1$ for any $n \geq 0$ by induction. It is clear that $q(0) = 1$. Assume $q(n) \geq 1$ for some $n \geq 0$, then
\[
q(n+1) = (10^{dn})^d - p^n(9)^d - 4 = q(n)\sum_{i=6}^{d-1}(10^{dn})^i p^n(9)^{d-1-i} - 4 \\
\geq q(n)10^{dn(d-1)} - 4 \geq 1.
\]
This completes the inductive step and hence the proof. ■

3.2. Non-Archimedean Estimates. In this section, we assume that $K$ is a number field, $a \in K$, and $v \in M_K^0$. Because $v$ is fixed, we write $|\cdot|$, $\delta$, $M_a$, and $g_a$ for $|\cdot|_v$, $\delta_v$, $M_{a,v}$, and $g_{a,v}$.

Propositions 3.8 and 3.9 can be seen as the non-Archimedean version of Propositions 3.2, 3.3, and 3.4.

Proposition 3.8. Assume that $|a| \leq 1$. For any $t \in \mathbb{C}_v$, we have
\[
g_a(t) = \log^+ |t|.
\]
In particular, $M_a = D(0,1)$, the closed Berkovich unit disk.

Proof. Let $t \in \mathbb{C}_v$, then for any $n \geq 1$,
\[
|f_t^n(a)| \begin{cases} 
\leq 1, & \text{if } |t| \leq 1, \\
= |t|^{dn-1}, & \text{if } |t| > 1.
\end{cases}
\]
By Theorem 2.1
\[
g_a(t) = \lim_{n \to \infty} \frac{1}{dn-1} \log^+ |f_t^n(a)| = \log^+ |t|.
\]
Therefore, $M_a \cap \mathbb{C}_v = D(0,1)$ and $M_a = \overline{M_a \cap \mathbb{C}_v} = D(0,1)$. ■

Proposition 3.9. Assume that $|a| > 1$. For any $t \in \mathbb{C}_v$, we have
\[
g_a(t) \begin{cases} 
= d \log |a|, & \text{if } |t| < |a|^d, \\
\leq d \log |a|, & \text{if } |t| = |a|^d, \\
= \log |t|, & \text{if } |t| > |a|^d.
\end{cases}
\]
If $t \in M_a \cap \mathbb{C}_v$, then $|t| = |a|^d$ and $|f_t^n(a)| = |a|$ for any $n \geq 1$.

Proof. Let $t \in \mathbb{C}_v$, then for any $n \geq 1$,
\[
|f_t^n(a)| \begin{cases} 
= |a|^{dn}, & \text{if } |t| < |a|^d, \\
\leq |a|^{dn}, & \text{if } |t| = |a|^d, \\
= |t|^{dn-1}, & \text{if } |t| > |a|^d.
\end{cases}
\]
By Theorem 2.1

\[ g_a(t) = \lim_{n \to \infty} \frac{1}{d^{n-1}} \log^+ |f_t^n(a)| \begin{cases} = d \log |a|, & \text{if } |t| < |a|^d, \\ \leq d \log |a|, & \text{if } |t| = |a|^d, \\ = \log |t|, & \text{if } |t| > |a|^d. \end{cases} \]

Therefore, if \( t \in M_a \cap \mathbb{C}_v \), then \( |t| = |a|^d \) and \( t \in M_{f_t^n(a)} \cap \mathbb{C}_v \) for any \( n \geq 1 \). By the same reasoning, we have \( |t| = |f_t^n(a)|^d \) and \( |f_t^n(a)| = |a| \).

The following result will not be used until Section 4. We include it here because it is similar to Proposition 3.9.

**Proposition 3.10.** If \( |a| > 1, t \in \mathbb{C}_v, \) and \( |f_t^n(a)| < |a|^d \) for some \( n \geq 1 \), then \( |t| = |a|^d \) and \( |f_t^k(a)| = |a| \) for any \( 1 \leq k \leq n - 1 \).

**Proof.** For the first assertion, suppose \( |t| \neq |a|^d \), then

\[ |f_t^n(a)| = \max\{|t|, |a|^d\}^d \geq |a|^d. \]

For the second assertion, suppose \( |f_t^k(a)| \neq |a| \) for some \( 1 \leq k \leq n - 1 \), then

\[ |f_t^n(a)| = \max\{|t|, |f_t^k(a)|^d\}^d \geq |t| = |a|^d. \]

Propositions 3.11 and 3.12 can be seen as the non-Archimedean version of Propositions 3.5 and 3.6.

**Proposition 3.11.** If \( |a| > 1, t \in M_a \cap \mathbb{C}_v, s \in \mathbb{C}_v \), and \( |s - t| \leq |a| \), then

\[ |f_s^n(a) - f_t^n(a)| \leq |s - t||a|^{d^{n-1}-1} \]

for any \( n \geq 1 \).

**Proof.** We prove the assertion by induction. Because \( |f_s(a) - f_t(a)| = |s - t| \), the statement is true for \( n = 1 \). Assume the statement is true for some \( n \geq 1 \), then

\[ |f_s^{n+1}(a) - f_t^{n+1}(a)| = |f_s^n(a)^d + s - f_t^n(a)^d - t| \leq \max\{|f_s^n(a)^d - f_t^n(a)^d|, |s - t|\} \]

\[ = \max\{\prod_{i=0}^{d-2}|f_s^n(a) - f_t^n(a) + (1 - \zeta_i)f_t^n(a)|, |s - t|\} \leq \max\{|f_s^n(a) - f_t^n(a)| \max\{|f_s^n(a) - f_t^n(a)|, |f_t^n(a)|\}^{d-1}, |s - t|\} \leq \max\{|s - t||a|^{d^{n-1}-1} \max\{|a|^{d^{n-1}-1}, |a|^d\}^{d-1}, |s - t|\} \]

(by the assumption, the induction hypothesis, and Proposition 3.9)

\[ = |s - t||a|^{d^{n-1}}. \]
This completes the inductive step and hence the proof.

Proposition 3.12. If \( t \in M_a \cap C_v, s \in A_v^{1, \text{an}} \), and 
\[
\delta(s, t) \leq \max\{|a|, 1\}^{2-n-1}
\]
for some \( n \geq 1 \), then
\[
g_a(s) \leq \frac{1}{d^n} \log^+ |a|.
\]

Proof. Let \( A_n = \max\{|a|, 1\}^{2-n-1} \) for any \( n \geq 1 \). By Theorem 2.1 \( g_a \) is continuous. Since \( \overline{D}(t, A_n) \) is dense in \( D(t, A_n) \), it suffices to prove the assertion for \( s \in \overline{D}(t, A_n) \). If \( |a| \leq 1 \), then by Proposition 3.8 \( s \in M_a \) and \( g_a(s) = 0 \). If \( |a| > 1 \), then by Propositions 3.9 and 3.11
\[
|f^n_s(a)| \leq \max\{|f^n_s(a) - f^n_t(a)|, |f^n_t(a)|\} \leq |a|
\]
and
\[
|f^{n+1}_s(a)| \leq \max\{|f^n_s(a)|d, |t|, |s - t|\} \leq |a|^d.
\]
By induction, \( |f^{n+k}_s(a)| \leq |a|^{dk} \) for any \( k \geq 1 \). Therefore, by Theorem 2.1
\[
g_a(s) = \lim_{k \to \infty} \frac{1}{d^{n+k-1}} \log^+ |f^{n+k}_s(a)| \leq \frac{1}{d^{n-1}} \log |a|.
\]

3.3. Proof of Theorem 1.2. Now we are ready to give the proof of Theorem 1.2. Given \( a, b \in \mathbb{Q} \), let \( K \) be a number field such that \( a, b \in K \). As [DKY2, Section 9], we will apply [Fi, Theorem 1] in the following way:
\[
\langle \mu_a, \mu_b \rangle^{1/2} \leq \langle \mu_a, [S]_{\tau} \rangle^{1/2} + \langle \mu_b, [S]_{\tau} \rangle^{1/2},
\]
where \( [S]_{\tau} \) is an adelic measure to be described below.

Let \( S \) be a finite, non-empty, \( \text{Gal}(\overline{K}/K) \)-invariant subset of \( \overline{K} \), and let \( [S] \) be the probability measure supported equally on the elements of \( S \). We call \( \tau = \{\tau_v\}_{v \in \mathcal{M}_K} \) an adelic radius if \( \tau_v > 0 \) for any \( v \in \mathcal{M}_K \), and \( \tau_v = 1 \) for all but finitely many \( v \in \mathcal{M}_K \). Combining the ideas of [FRL, Section 4.6] and [FP, Section 2.1], we define the regularization \( [S]_{\tau} = \{[S]_{\tau_v}\}_{v \in \mathcal{M}_K} \) by
\[
[S]_{\tau_v} = \frac{1}{|S|} \sum_{s \in S} m_{s, \tau_v}.
\]
The following result and the proof of Theorem 1.2 are adapted from [DKY2, Section 9].

Theorem 3.13. Let \( K \) be a number field such that \( a, b \in K \). Then
\[
\langle \mu_a, \mu_b \rangle^{1/2} \leq \sum_{v = a, b} \left( \sum_{v \in \mathcal{M}_K} n_v \left( -\langle \mu_{i,v}, [S]_{\tau_v} \rangle - \frac{\log \tau_v}{2|S|} \right) \right)^{1/2}
\]
for any finite, non-empty, \( \text{Gal}(\overline{K}/K) \)-invariant subset \( S \) of \( K \) and any adelic radius \( \tau \).
Proof. By Theorem 2.1, \((\mu_{i,v}, \mu_{i,v})_v = -\log \gamma_v(M_{i,v}) = 0\) for \(i = a, b\) and any \(v \in M_K\). Then the proof is identical to the proof of [DKY2, Lemma 9.2].

Proof of Theorem 1.2. Let \(K\) be a number field such that \(a, b \in K\), and let \(n \geq -1\) be the integer such that \(d^{-n-1} \leq \varepsilon' = \varepsilon/4 < d^{-n}\). For each \(v \in M_K\), take

\[
\tau_v = (18 \max\{|a|_v, |b|_v, 4\})^{1-d/\varepsilon'} \leq (18 \max\{|a|_v, 4\})^{1-d/\varepsilon'}
\]

\[
\leq \frac{1}{18} (18 \max\{|a|_v, 4\})^{2-d/\varepsilon'} \leq \frac{1}{18} (18 \max\{|a|_v, 4\})^{2-d^{n+1}}.
\]

By Proposition 3.12 if \(|s-t|_v = \tau_v\) for some \(t \in S_{a,b}\), then

\[
g_{a,v}(s) \leq \frac{1}{d^{n+1}} \log(10 \max\{|a|_v, 4\}) \leq \varepsilon' \log(18 \max\{|a|_v, |b|_v, 4\})
\]

Therefore,

\[
- (\mu_{a,v}, [S_{a,b}]_{\tau_v})_v = \frac{\log \tau_v}{2|S_{a,b}|} \leq \int g_{a,v}d[S_{a,b}]_{\tau_v} = \frac{\log \tau_v}{2|S_{a,b}|}
\]

\[
\leq \varepsilon' \log(18 \max\{|a|_v, 4|b|_v, 4\}) - \frac{(1 - d/\varepsilon') \log(18 \max\{|a|_v, |b|_v, 4\})}{2|S_{a,b}|}
\]

\[
\leq \left( \varepsilon' + \frac{d/\varepsilon' - 1}{2|S_{a,b}|} \right) \log(18 \max\{|a|_v, 4|b|_v, 4\})
\]

\[
\leq \left( \varepsilon' + \frac{d/\varepsilon' - 1}{2|S_{a,b}|} \right) \log(\max\{|a|_v, |b|_v, 1\} + 5).
\]

For each \(v \in M_K^0\), take

\[
\tau_v = \max\{|a|_v, |b|_v, 1\}^{1-d/\varepsilon'} \leq \max\{|a|_v, 1\}^{1-d/\varepsilon'}
\]

\[
\leq \max\{|a|_v, 1\}^{2-d/\varepsilon'} \leq \max\{|a|_v, 1\}^{2-d^{n+1}}.
\]

By Proposition 3.12 if \(\delta_v(s, t) = \tau_v\) for some \(t \in S_{a,b}\), then

\[
g_{a,v}(s) \leq \frac{1}{d^{n+1}} \log^+ |a|_v \leq \varepsilon' \log \max\{|a|_v, |b|_v, 1\}
\]

Therefore,

\[
- (\mu_{a,v}, [S_{a,b}]_{\tau_v})_v = \frac{\log \tau_v}{2|S_{a,b}|} \leq \int g_{a,v}d[S_{a,b}]_{\tau_v} = \frac{\log \tau_v}{2|S_{a,b}|}
\]

\[
\leq \varepsilon' \max\{|a|_v, |b|_v, 1\} - \frac{(1 - d/\varepsilon') \log \max\{|a|_v, |b|_v, 1\}}{2|S_{a,b}|}
\]

\[
= \left( \varepsilon' + \frac{d/\varepsilon' - 1}{2|S_{a,b}|} \right) \log \max\{|a|_v, |b|_v, 1\}.
\]

Summing over all \(v \in M_K\), we get

\[
\langle \mu_a, \mu_b \rangle \leq 4 \left( \varepsilon' + \frac{d/\varepsilon' - 1}{2|S_{a,b}|} \right) (h(a,b) + 5) = \left( \varepsilon + \frac{8d/\varepsilon - 2}{|S_{a,b}|} \right) (h(a,b) + 5).
\]

\]
4. Lower Bounds

The purpose of this section is to give the proofs of Theorems 1.3 and 1.4. In order to do so, we estimate the lower bounds for

$$\langle \mu_{a,v}, \mu_{b,v} \rangle_v = \int_{K_v^{1,an}} g_{a,v} d\mu_{b,v}.$$  

In Section 4.1, we review two equidistribution theorems for later usage. As always, we work with $v \in M_K^\infty$ and $v \in M_K^0$ separately. For some technical reasons, we also work with $d = 2$ and $d > 2$ separately when $v \in M_K^\infty$. The local estimates obtained in Sections 4.2, 4.3, and 4.4 are gathered in Section 4.5 to complete the proofs of Theorems 1.3 and 1.4.

4.1. Equidistribution Theorems. In this section, we assume that $K$ is a number field and $a \in K$. We give two equidistribution theorems. Theorem 4.1 will be used in Section 4.4, and Theorem 4.3 will be used in Sections 4.2 and 4.3.

We call $E = \{E_v\}_{v \in M_K}$ an adelic compact set if $E_v$ is a non-empty compact subset of $A_v^{1,an}$ for any $v \in M_K$, and $E_v = D(0,1)$ for all but finitely many $v \in M_K$.

**Theorem 4.1.** [BR2, Theorem 7.52] Let $K$ be a number field, and let $E$ be an adelic compact set with

$$\gamma(E) = \prod_{v \in M_K} \gamma_v(E_v)^{n_v} = 1.$$  

Suppose $S_n$ is a sequence of finite, non-empty, $Gal(\bar{K}/K)$-invariant subsets of $\bar{K}$ such that $|S_n| \to \infty$ and

$$h_E(S_n) = \sum_{v \in M_K} n_v \left( \frac{1}{|S_n|} \sum_{z \in S_n} g_{E_v}(z) \right) \to 0.$$  

Fix $v \in M_K$ and, for any $n \geq 1$, let $[S_n]$ be the probability measure on $A_v^{1,an}$ supported equally on the elements of $S_n$. Then the sequence of measures $[S_n]$ converges weakly to $\mu_{E_v}$ on $A_v^{1,an}$.

For the related results, see also [Bi], [BR1], [CL1], [FRL], and [BR2] Theorem 10.24.

From Theorem 2.1 and Proposition 3.8, we know that $M_a = \{M_{a,v}\}_{v \in M_K}$ is an adelic compact set with $\gamma(M_a) = 1$. Let $S_n$ be the set of all roots of $f^n_T(a)$. The following result shows that $\mu_{a,v}$ can be approximated by $[S_n]$ if we assume that all roots of $f^n_T(a)$ are simple.

**Proposition 4.2.** Assume that, for any $n \geq 1$, all roots of $f^n_T(a)$ are simple. Let $S_n$ be the set of all roots of $f^n_T(a)$. Then for any $v \in M_K$, $[S_n]$ converges weakly to $\mu_{a,v}$ on $A_v^{1,an}$.

**Proof.** By the assumption, we have $|S_n| = d^{n-1} \to \infty$ as $n \to \infty$. To apply Theorem 4.1, it remains to show that $h_{M_a}(S_n) \to 0$ as $n \to \infty$. If $t \in S_n$, then by Theorem 2.1

$$g_{a,v}(t) = \lim_{k \to \infty} \frac{1}{d^{n+k}} \log^+ |f^{n+k+1}(a)|_v = \frac{1}{d^n} \lim_{k \to \infty} \frac{1}{d^k} \log^+ |f^{k+1}(0)|_v = \frac{1}{d^n} g_{0,v}(t).$$
Since \( g_{0,v} \) is continuous, it suffices to show that for any \( v \in M_K \) and any \( t \in S_n \), \(|t|_v \) is bounded by some constant \( C_{a,v} \), where \( C_{a,v} \) is independent of \( n \).

(1) If \( v \in M_K^0 \) and \(|a|_v \leq 1 \), then by Proposition 3.8 \(|t|_v \leq 1 \) for any \( t \in S_n \).

(2) If \( v \in M_K^0 \) and \(|a|_v > 1 \), then by Proposition 3.10 \(|t|_v = |a|_v^d \) for any \( t \in S_n \).

(3) If \( v \in M_K^\infty \), then by Proposition 3.4 the roots of \( f_T^n(a) = a \) are inside

\[
D_{a,v} = D(0, 3 \max\{|a|_v, 4\}^d).
\]

Once we can show that \(|f_T^n(a)|_v > |a|_v \) for any \( t \in \partial D_{a,v} \), we can apply Rouché's theorem to conclude \( S_n \subseteq D_{a,v} \). If \( t \in \partial D_{a,v} \), then

\[
|f_T(a)|_v \geq |t|_v - |a|_v^d = 3 \max\{|a|_v, 4\}^d - |a|_v^d \geq 2 \max\{|a|_v, 4\}^d \geq 8 \max\{|a|_v, 4\}.
\]

Assume that \(|f_T^n(a)|_v \geq 8 \max\{|a|_v, 4\}^d \) for some \( n \geq 1 \), then

\[
|f_T^{n+1}(a)|_v \geq |f_T^n(a)|_v - |t|_v \geq (8^d - 3) \max\{|a|_v, 4\}^d \geq 8 \max\{|a|_v, 4\}.
\]

By induction, \(|f_T^n(a)|_v \geq 8 \max\{|a|_v, 4\} > |a|_v \) for any \( n \geq 1 \).

Proposition 4.10 shows that if \(|a|_v > |d|_v^{-2/(d-1)} \) for some \( v \in M_K^0 \), then all roots of \( f_T^n(a) \) are simple. Since we lack a similar result for \( v \in M_K^\infty \), we need an equidistribution theorem taking the multiplicity into account.

**Theorem 4.3.** Fix \( v \in M_K^\infty \) and a sequence \( 0 \leq k(n) < n \). Let \( \delta_n \) be the discrete probability measure on \( \mathbb{C} \) weighted by multiplicity on the roots of \( f_T^n(a) = f_T^{k(n)}(a) \). Then the sequence of measures \( \delta_n \) converges weakly to \( \mu_{a,v} \) on \( \mathbb{C} \).

**Proof.** See [BD1, Section 4.3] and the proof of [DE, Theorem 1].

**4.2. Archimedean Estimates for \( d = 2 \).** In this section, we assume that \( K \) is a number field, \( a, b \in K \), \( v \in M_K^\infty \), and \( d = 2 \). Because \( v \) is fixed, we write \(|\cdot|_v \), \( M_a \), \( \mu_a \), and \( g_a \) for \(|\cdot|_v, M_{a,v}, \mu_{a,v}, \) and \( g_{a,v} \).

As [DKY2, Section 3.2], we first cover \( M_a \) by some disjoint open disks around the roots of \( f_T^n(a) = f_T(a) \) with \( n = 2, 3 \), and then estimate the lower bound for \( g_a(t) \) when \( t \) is outside the cover. Let

\[
\alpha_1(a) = -a^2 - a, \alpha_2(a) = -a^2 + a, \alpha_3(a) = -a^2 - a - 1, \alpha_4(a) = -a^2 + a - 1
\]

be the roots of \( f_T^3(a) = f_T(a) \). Note that \( \alpha_1(a) \) and \( \alpha_2(a) \) are also roots of \( f_T^2(a) = f_T(a) \).

**Proposition 4.4.** Assume that \(|a| \geq 28 \). Then we have

(1) \( M_a \subseteq \cup_{i=1}^3 D(\alpha_i(a), 10) \).

(2) If \( t \notin \cup_{i=1}^3 D(\alpha_i(a), 10) \), then

\[
g_a(t) \geq \frac{1}{2} \log(13|a|).
\]
(3) \( \mu_a(D(\alpha_i(a), 10)) = 1/2 \) for \( i = 1, 2 \).

**Proof.** For simplicity, let \( \alpha_i = \alpha_i(a) \) and \( D_i = D(\alpha_i, 10) \).

(1) If \( t \in \partial D_1 \), then
\[
|t - \alpha_1| = 10,
|t - \alpha_2| \geq |\alpha_1 - \alpha_2| - |t - \alpha_1| = 2|a| - 10.
\]

When \( |a| \geq 28 \), we have
\[
|f_i^2(a) - f_i(a)| \geq 20|a| - 100 \geq 16|a|.
\]

The same reasoning also works for \( t \in \partial D_2 \). By Rouché’s theorem, for any \( c \) with \( |c| < 16|a| \), the equation \( f_i^2(a) - f_T(a) = c \) has exactly one root in each \( D_i \). By Proposition 3.3
\[
|f_i^2(a) - f_s(a)| \leq |f_i^2(a)| + |f_s(a)| < 16|a|
\]
for any \( s \in M_a \), so we have \( M_a \subseteq \bigcup_{i=1}^2 D_i \).

(2) If \( t \in \partial D_i \) for some \( i = 1, 2 \), then
\[
|t| \leq |t - \alpha_i| + |\alpha_i| \leq 2|a|^2,
|f_i(a)| \leq |t - \alpha_i| + |\alpha_i + a^2| \leq 2|a|,
\]
and
\[
|f_i^2(a)| \geq |f_i^2(a) - f_i(a)| - |f_i(a)| \geq 14|a|.
\]

Let \( p(z) = z^2 - 2 \), then by induction and a similar argument of Lemma 3.7,
\[
|f_i^{n+2}(a)| \geq p^n(14)|a|^{2^n} \geq (13|a|)^{2^n}
\]
for any \( n \geq 1 \). Therefore, by Theorem 2.1
\[
g_a(t) = \lim_{n \to \infty} \frac{1}{2^{n+1}} \log^+ |f_i^{n+2}(a)| \geq \frac{1}{2} \log(13|a|).
\]

Since \( g_a \) is harmonic on \( \mathbb{C} \setminus M_a \), this is true for any \( t \notin \bigcup_{i=1}^2 D_i \).

(3) If \( t \in \partial D_i \) for some \( i = 1, 2 \), then
\[
|f_i(a) \pm a| \leq |f_i(a)| + |a| \leq 3|a|
\]
and
\[
|f_i^n(a) - f_i(a)| \geq |f_i^n(a)| - |f_i(a)| \geq (13|a|)^{2^{n+2}} - 2|a| > 3|a|
\]
for any \( n \geq 2 \). By Rouché’s theorem, \( f_i^n(a) - f_T(a) \) and \( f_i^n(a) \pm a \) have the same number of roots in \( D_i \). Since
\[
f_T^{n+1}(a) - f_T(a) = (f_T^n(a) + a)(f_T^n(a) - a),
\]
by induction each of \( f_T^n(a) - f_T(a) \) and \( f_T^n(a) \pm a \) has \( 2^{n-2} \) roots in \( D_i \). Then the conclusion follows from Theorem 4.3. \( \blacksquare \)
**Proposition 4.5.** Assume that $|a| \geq 28$. Then we have

1. $M_a \subseteq \cup_{i=1}^4 D(\alpha_i(a), 5/|a|)$.
2. If $t \notin \cup_{i=1}^4 D(\alpha_i(a), 5/|a|)$, then
   \[ g_a(t) \geq \frac{1}{4} \log(13|a|). \]
3. $\mu_a(D(\alpha_i(a), 5/|a|)) = 1/4$ for $1 \leq i \leq 4$.

**Proof.** For simplicity, let $\alpha_i = \alpha_i(a)$ and $D_i = D(\alpha_i, 5/|a|)$. If $t \in \partial D_1$, then

- $|t - \alpha_1| = 5/|a|,$
- $|t - \alpha_2| \geq |\alpha_1 - \alpha_2| - |t - \alpha_1| = 2|a| - 5/|a|,$
- $|t - \alpha_3| \geq |\alpha_1 - \alpha_3| - |t - \alpha_1| = 1 - 5/|a|,$
- $|t - \alpha_4| \geq |\alpha_1 - \alpha_4| - |t - \alpha_1| \geq 2|a| - 1 - 5/|a|.$

When $|a| \geq 28$, we have

\[ |f_t^2(a) - f_t(a)| \geq 20|a| - 110 - \frac{50}{|a|} + \frac{525}{|a|^2} - \frac{625}{|a|^4} \geq 16|a|. \]

The rest of the proof is similar to the proof of Proposition 4.4.

As [DKY2, Theorem 4.1], we estimate the complex Arakelov–Zhang pairing as follows.

**Proposition 4.6.** For any $a, b \in K$, we have

\[ \int g_a d\mu_b \geq \frac{1}{8} \log^+ |a^2 - b^2| - \frac{1}{8} \log 5000. \]

Moreover, if $\max\{|a|, |b|\} \geq 50$ and

\[ |a^2 - b^2| \geq \frac{11}{\max\{|a|, |b|\}}, \]

then

\[ \int g_a d\mu_b \geq \frac{1}{16} \log \max\{|a|, |b|\}. \]

**Proof.** Without loss of generality, we assume that $|a| \geq |b|$. We prove the second assertion in parts (1), (2), (3), and prove the first assertion in part (4).

1. **If $t \in M_b$, then by Proposition 3.4.**
   \[ |t| \leq 3 \max\{|b|, 4\}^2 \leq 3 \cdot 28^2 \leq 50(50 - 2) \leq |a|(|a| - 2) = |a|^2 - 2|a| \]
   and
   \[ |f_t(a)| \geq |a|^2 - |t| \geq 2|a|. \]

Let $p(z) = z^2 - 1$, then by induction and a similar argument of Lemma 3.7,

\[ |f_t^{n+1}(a)| \geq p^n(2)|a|^{2^n} \geq |a|^{2^n}. \]
for any \( n \geq 1 \). Therefore, by Theorem 2.1,

\[
g_a(t) = \lim_{n \to \infty} \frac{1}{2^n} \log^+ |f_t^{n+1}(a)| \geq \log |a|
\]

and

\[
\int g_a d\mu_b \geq \log |a|.
\]

(2) \([|a| \geq 50, |b| \geq 28, \text{ and } |a^2 - b^2| \geq 20]\)

Claim: Some \( D(\alpha_i(b), 10) \) is disjoint from \( \bigcup_{i=1}^2 D(\alpha_i(a), 10) \).

Suppose not, then for each \( i \), there exists \( k_i \) such that

\[
|\alpha_{k_i}(a) - \alpha_i(b)| < 20.
\]

If \( k_1 = k_2 \), then

\[
2|b| = |\alpha_1(b) - \alpha_2(b)| \leq \sum_{i=1}^2 |\alpha_{k_i}(a) - \alpha_i(b)| < 40,
\]

a contradiction. If \( k_1 \neq k_2 \), then

\[
2|a^2 - b^2| = \left| \sum_{i=1}^2 \alpha_i(a) - \sum_{i=1}^2 \alpha_i(b) \right| \leq \sum_{i=1}^2 |\alpha_{k_i}(a) - \alpha_i(b)| < 40,
\]

also a contradiction. Therefore, the claim is proved and, by Proposition 4.4,

\[
\int g_a d\mu_b \geq \frac{1}{2} \cdot \frac{1}{2} \log(13|a|) = \frac{1}{4} \log(13|a|).
\]

(3) \([|a| \geq 50, |b| \geq 28, \text{ and } 11/|a| \leq |a^2 - b^2| \leq 20]\)

Claim: Some \( D(\alpha_i(b), 5/|b|) \) is disjoint from \( \bigcup_{i=1}^3 D(\alpha_i(a), 5/|a|) \).

Suppose not, then for each \( i \), there exists \( k_i \) such that

\[
|\alpha_{k_i}(a) - \alpha_i(b)| < 5/|a| + 5/|b|.
\]

If \( k_i = k_j \) for some \( i \neq j \), then

\[
1 \leq |\alpha_i(b) - \alpha_j(b)| \leq |\alpha_{k_i}(a) - \alpha_i(b)| + |\alpha_{k_j}(a) - \alpha_j(b)| < 10/|a| + 10/|b|,
\]

a contradiction. If \( k_i \neq k_j \) for any \( i \neq j \), then

\[
4|a^2 - b^2| = \left| \sum_{i=1}^4 \alpha_i(a) - \sum_{i=1}^4 \alpha_i(b) \right| \leq \sum_{i=1}^4 |\alpha_{k_i}(a) - \alpha_i(b)| < 20/|a| + 20/|b|.
\]

This is also a contradiction because

\[
\frac{|a|}{|b|} \leq \left( \frac{|a^2 - b^2|}{|b|^2} + 1 \right)^{1/2} = \left( \frac{|a^2 - b^2|}{|b|^2} + 1 \right)^{1/2} \leq \left( \frac{20}{28^2} + 1 \right)^{1/2} \leq \frac{6}{5}
\]

and

\[
|a^2 - b^2| \geq 11/|a| \geq 5/|a| + 5/|b|.
\]
Therefore, the claim is proved and, by Proposition 4.5,
\[ \int g_a d\mu_b \geq \frac{1}{4} \int \frac{1}{4} \log(13|a|) = \frac{1}{16} \log(13|a|). \]

(4) Now we prove the first assertion. If \(|a| \leq 50\) or \(|a^2 - b^2| \leq 20\), then
\[ \log^+ |a^2 - b^2| \leq \log \max\{2|a|^2, 20\} \leq \log 5000 \]
and
\[ \int g_a d\mu_b \geq 0 \geq \frac{1}{8} \log^+ |a^2 - b^2| - \frac{1}{8} \log 5000. \]
If \(|a| > 50\) and \(|a^2 - b^2| \geq 20\), then by parts (1) and (2),
\[ \int g_a d\mu_b \geq \frac{1}{4} \log(13|a|) \geq \frac{1}{8} \log(2|a|^2) \geq \frac{1}{8} \log^+ |a^2 - b^2|. \]

4.3. Archimedean Estimates for \(d > 2\). In this section, we assume that \(K\) is a number field, \(a, b \in K\), \(v \in M_K^\infty\), and \(d > 2\). Because \(v\) is fixed, we write \(|\cdot|\), \(M_a\), \(\mu_a\), and \(g_a\) for \(|\cdot|_v\), \(M_{a,v}\), \(\mu_{a,v}\), and \(g_{a,v}\).

We will prove the counterparts of Propositions 4.4, 4.5, and 4.6 for \(d > 2\), but this time we only need to consider the roots of \(f_T^2(a) = f_T(a)\). The reason is as follows: In the equations (4.1) and (4.2), what we really need is
\[ |a^d - b^d| \geq \frac{c_1(d)}{\max\{|a|, |b|\}^{c_2(d)}} \]
for some \(c_1(d), c_2(d) > 0\). When we consider the roots of \(f_T^2(a) = f_T(a)\), we will get \(c_2(d) = d - 2\). This is already good enough for \(d > 2\), so we no longer need to consider the roots of \(f_T^3(a) = f_T(a)\). For \(0 \leq i \leq d - 1\), let
\[ \alpha_i(a) = -a^d + \zeta_d^i a \]
be the roots of \(f_T^2(a) = f_T(a)\).

**Proposition 4.7.** Assume that \(|a| \geq 6\). Then we have

1. \(M_a \subseteq \bigcup_{i=0}^{d-1} D(\alpha_i(a), 12/|a|^{d-2})\).
2. If \(t \notin \bigcup_{i=0}^{d-1} D(\alpha_i(a), 12/|a|^{d-2})\), then
\[ g_a(t) \geq \frac{1}{d} \log(13|a|). \]
3. \(\mu_a(D(\alpha_i(a), 12/|a|^{d-2})) = 1/d\) for \(0 \leq i \leq d - 1\).

**Proof.** For simplicity, let \(\alpha_i = \alpha_i(a)\) and \(D_i = D(\alpha_i, 12/|a|^{d-2})\). If \(t \in \partial D_0\), then
\[ |t - \alpha_0| = 12/|a|^{d-2}, \]
\[ |t - \alpha_i| \geq |\alpha_0 - \alpha_i| - |t - \alpha_0| = |-\zeta_d^i| |a| - 12/|a|^{d-2} \geq (1 - 1/d)|1 - \zeta_d^i| |a|. \]
where the last inequality follows from
\[
\left( \frac{12d}{|1 - \zeta_d^i|} \right)^{\frac{1}{d+1}} \leq \left( \frac{12d^2}{d|1 - \zeta_d|} \right)^{\frac{1}{d}} \leq \left( \frac{4d^2}{\sqrt{3}} \right)^{\frac{1}{d+1}} \leq 2 \cdot 3^{3/4} \leq |a|.
\]

Then we have
\[
|f_t^2(a) - f_t(a)| \geq \frac{12}{|a|^{d-2}} \prod_{i=1}^{d-1} \left( 1 - \frac{1}{d} \right) |1 - \zeta_d^i||a| = 12d \left( 1 - \frac{1}{d} \right)^{d-1} |a| \geq 16|a|.
\]

The rest of the proof is similar to the proof of Proposition 4.4.

**Proposition 4.8.** For any \(a, b \in K\), we have
\[
\int g_d d\mu_b \geq \frac{1}{d^2} \log^+ |a^d - b^d| - \frac{1}{d^2} \log(2 \cdot 9^d).
\]
Moreover, if \(\max\{|a|, |b|\} \geq 9\) and
\[
|a^d - b^d| \geq \frac{25}{\max\{|a|, |b|\}^{d-2}},
\]
then
\[
\int g_d d\mu_b \geq \frac{1}{d^2} \log \max\{|a|, |b|\}.
\]

**Proof.** Without loss of generality, we assume that \(|a| \geq |b|\). We prove the second assertion in parts (1), (2), and prove the first assertion in part (3).

1. \(|a| \geq 9\) and \(|b| \leq 6\): If \(t \in M_b\), then by Proposition 3.4,
\[
|t| \leq 3 \max\{|b|, 4\}^d \leq 3 \cdot 6^d \leq 9(9^{d-1} - 2) \leq |a|(|a|^{d-1} - 2) = |a|^d - 2|a|
\]
and
\[
|f_t(a)| \geq |a|^d - |t| \geq 2|a|.
\]

By a similar argument of the proof of Proposition 4.6 we have
\[
\int g_d d\mu_b \geq \log |a|.
\]

2. \(|a| \geq 9, |b| \geq 6,\) and \(|a^d - b^d| \geq 25/|a|^{d-2}\)

**Claim:** Some \(D(\alpha_i(b), 12/|b|^{d-2})\) is disjoint from \(\cup_{i=0}^{d-1} D(\alpha_i(a), 12/|a|^{d-2})\).

Suppose not, then for each \(i\), there exists \(k_i\) such that
\[
|\alpha_{k_i}(a) - \alpha_i(b)| < 12/|a|^{d-2} + 12/|b|^{d-2}.
\]

If \(k_i = k_j\) for some \(i \neq j\), then
\[
|\zeta_d^i - \zeta_d^j||b| = |\alpha_i(b) - \alpha_j(b)| \leq |\alpha_{k_i}(a) - \alpha_i(b)| + |\alpha_{k_j}(a) - \alpha_j(b)|
\]
\[
< 24/|a|^{d-2} + 24/|b|^{d-2} \leq 48/|b|^{d-2}.
\]
This is a contradiction because
\[
\left( \frac{48}{\alpha_i(a) - \alpha_i(b)} \right)^\frac{1}{d-1} \leq \left( \frac{48d}{|d|1 - \zeta_d} \right)^\frac{1}{d-1} \leq \left( \frac{16d}{\sqrt{3}} \right)^\frac{1}{d-1} \leq 4 \cdot 3^{3/4} \leq |b|.
\]
If \( k_i \neq k_j \) for any \( i \neq j \), then
\[
d|a^d - b^d| = \left| \sum_{i=1}^{d} \alpha_i(a) - \sum_{i=1}^{d} \alpha_i(b) \right| \leq \sum_{i=1}^{d} |\alpha_k(a) - \alpha_i(b)| < d(12/|a|^{d-2} + 12/|b|^{d-2}).
\]
If \(|a^d - b^d| \geq 4\), then
\[
4 \leq |a^d - b^d| < 12/|a|^{d-2} + 12/|b|^{d-2} \leq 24/|b|^{d-2} \leq 4,
\]
a contradiction. If \( 25/|a|^{d-2} \leq |a^d - b^d| \leq 4 \), then
\[
\frac{|a|^{d-2}}{|b|^{d-2}} \leq \frac{|a^d - b^d|}{|b|^d} \leq \frac{|a^d - b^d|}{|b|^d} + 1 \leq \frac{4}{6d} + 1 \leq \frac{13}{12}
\]
and
\[
25/|a|^{d-2} \leq |a^d - b^d| < 12/|a|^{d-2} + 12/|b|^{d-2} \leq 25/|a|^{d-2},
\]
also a contradiction. Therefore, the claim is proved and, by Proposition [4,7],
\[
\int g_a d\mu_b \geq \frac{1}{d} \cdot \frac{1}{d} \log(13|a|) = \frac{1}{d^2} \log(13|a|).
\]
(3) Now we prove the first assertion. If \(|a| \leq 9 \) or \(|a^d - b^d| \leq 3 \), then
\[
\log^+ |a^d - b^d| \leq \log \max\{2|a|^d, 3\} \leq \log(2 \cdot 9^d)
\]
and
\[
\int g_a d\mu_b \geq 0 \geq \frac{1}{d^3} \log^+ |a^d - b^d| - \frac{1}{d^3} \log(2 \cdot 9^d).
\]
If \(|a| \geq 9 \) and \(|a^d - b^d| \geq 3 \geq 25/|a|^{d-2} \), then by parts (1) and (2),
\[
\int g_a d\mu_b \geq \frac{1}{d^2} \log(13|a|) \geq \frac{1}{d^3} \log(2|a|^d) \geq \frac{1}{d^3} \log^+ |a^d - b^d|.
\]

4.4. **Non-Archimedean Estimates.** In this section, we assume that \( K \) is a number field, \( a, b \in K \), and \( v \in M^0_K \). Because \( v \) is fixed, we write \(| \cdot |, \delta, M_a, \mu_a, \) and \( g_a \) for \(| \cdot |_v, \delta_v, M_{a,v}, \mu_{a,v}, \) and \( g_{a,v} \).

As [DKY2 Sections 5.1 and 6.1], we first study the structure of \( M_a \). More precisely, we show that when \( a \) is large enough, \( M_a \) can be described with respect to the roots of \( f^n_T(a) \).

**Proposition 4.9.** Assume that \(|a| > |d|^{-2/(d-1)} \). Fix \( t \in C_v \) such that \(|f^n_T(a)| \leq |a| \) for some \( n \geq 1 \). Let \( s_1, \ldots, s_{d^{n-1}} \) be the roots of \( f^n_T(a) \) such that
\[
|t - s_1| \leq \cdots \leq |t - s_{d^{n-1}}|.
\]
Then we have
\[ |t - s_1| = \frac{|f_t^n(a)|}{(|d||a||d-1|)^{n-1}} \leq \frac{|a|}{(|d||a||d-1|)^{n-1}} < |t - s_2| \leq \frac{|a|}{(|d||a||d-1|)^{n-2}} < |t - s_{d+1}|. \]

Proof. Let \( A_n = |a|/(|d||a||d-1|)^{n-1} \) for any \( n \geq 1 \). By Proposition 3.10, \( |f_t^n(a)| \leq |a| \) implies \( |f_t^k(a)| = |a| \) for any \( 1 \leq k \leq n-1 \). Thus we can prove the assertion by induction. Because \( |t + a^d| = |f_t(a)| \), the statement is true for \( n = 1 \). Assume the statement is true for some \( n \geq 1 \) and fix \( t \in \mathbb{C}_v \) such that \( |f_t^{n+1}(a)| \leq |a| \). Let \( \alpha_1, \ldots, \alpha_d-1 \) be the roots of \( f_t^n(a) \) with \( |t - \alpha_i| \) increasing, and let \( \beta_1, \ldots, \beta_d \) be the roots of \( f_t^{n+1}(a) \) with \( |t - \beta_i| \) increasing, then
\[ \prod_{i=1}^{d^n-1} (t - T - \alpha_i)^d + t - T = f_t^n(a)^d + t - T = f_t^{n+1}(a) = \prod_{i=1}^{d^n} (t - T - \beta_i). \]

Let
\[ \prod_{i=1}^{d^n-1} (t - T - \alpha_i)^d = \sum_{i=0}^{d^n} a_i T^i \text{ and } \prod_{i=1}^{d^n} (t - T - \beta_i) = \sum_{i=0}^{d^n} b_i T^i, \]
then by the assumption \( |a| > |d|^{-2/(d-1)} \) and the induction hypothesis \( |t - \alpha_1| = A_n < |t - \alpha_2| \),
\[
|b_0| = \left| \prod_{i=1}^{d^n} (t - \beta_i) \right| = |f_t^{n+1}(a)|,
\]
\[
|b_1| = |a_1 - 1| = \sum_{j=1}^{d^n-1} |t - \alpha_i - |f_t^n(a)| |d| \text{ for any } 2 \leq i \leq d-1,
\]
\[
|b_i| = |a_i| \leq \frac{|f_t^n(a)|}{A_n} = \frac{|a|^d}{A_n^d} \text{ for any } 2 \leq i \leq d-1,
\]
\[
|b_d| = |a_d| = \frac{|f_t^n(a)|}{A_n} = \frac{|a|^d}{A_n^d}.
\]

Now we consider the Newton polygon of \( f_t^{n+1}(a) \). By the induction hypothesis,
\[
\left( \frac{|b_i|}{|b_d|} \right)^{\frac{1}{d-i}} \leq A_n < \left( \frac{|a_d|}{|a_j|} \right)^{\frac{1}{d-j}} = \left( \frac{|b_d|}{|b_j|} \right)^{\frac{1}{d-j}}
\]
for any \( i < d < j \), so \((d, -\log |b_d|)\) is a vertex of the Newton polygon. By the assumptions,
\[
|b_0| = \frac{|f_t^{n+1}(a)|}{|d||a|^d} \leq \frac{A_n}{|d||a||d-1|} < |d| A_n \leq \left( \frac{|b_1|}{|b_1|} \right)^{\frac{1}{d-1}}
\]
for any \( 2 \leq i \leq d \), so \((1, -\log |b_1|)\) is also a vertex of the Newton polygon. Therefore,
\[
|t - \beta_1| = \frac{|f_t^{n+1}(a)|}{|d||a|^d} \leq \frac{A_n}{|d||a||d-1|} < |t - \beta_2| \leq A_n < |t - \beta_{d+1}|.
\]
This completes the inductive step and hence the proof.
In particular, Proposition 4.9 implies the non-Archimedean version of Propositions 4.4, 4.5, and 4.7 as follows.

**Proposition 4.10.** Assume that \(|a| > |d|^{-2/(d-1)}\). For any \(n \geq 1\), let \(S_n\) be the set of all roots of \(f^n_T(a)\). Then

1. All roots of \(f^n_T(a)\) are simple. Moreover, for any \(s_1, s_2 \in S_n\), we have
   \[|s_1 - s_2| > \frac{|a|}{(|d||a|^{d-1})^{n-1}}.\]
2. \(M_a \subseteq C_v\).
3. For any \(s \in S_n\), we have
   \[
   \mu_a \left( \frac{D(s, \frac{|a|}{(|d||a|^{d-1})^{n-1}})}{d^{n-1}} \right) = \frac{1}{d^{n-1}}.
   \]

**Proof.** For any \(n \geq 1\), let \(A_n = |a|/(|d||a|^{d-1})^{n-1}\). For any \(s \in S_n\), let
\[
T_{n,s} = \{t \in C_v : |s - t| = A_n\}.
\]

1. Fix any \(s \in S_n\) and let \(s_1, \ldots, s_{d^n-1} \in S_n\) such that \(|s - s_i|\) is increasing. By Proposition 4.9, we have \(|s - s_1| = 0 < A_n < |s - s_2|\), so \(s\) is simple.

**Claim 1:** For any \(s \in S_n\), we have \(|S_{n+1} \cap T_{n,s}| = d\).

By Propositions 3.10 and 4.9, we have \(S_{n+1} \subseteq \cup_{s \in S_n} T_{n,s}\). Since \(|S_{n+1}| = d|S_n|\), it suffices to show that \(|S_{n+1} \cap T_{n,s}| \leq d\) for any \(s \in S_n\). Suppose \(\alpha_1, \ldots, \alpha_{d+1} \in S_{n+1} \cap T_{n,s}\), then for any \(2 \leq i \leq d + 1\), we have
\[
|\alpha_1 - \alpha_i| \leq \max\{|s - \alpha_1|, |s - \alpha_i|\} = A_n,
\]
which contradicts Proposition 4.9.

**Claim 2:** If \(t \in T_{n,s}\) for some \(s \in S_n\), then \(|f^n_t(a)| = |a|\).

By Claim 1, there exists \(\alpha \in S_{n+1} \cap T_{n,s}\). By Proposition 4.9, for any \(s' \in S_n \setminus \{s\}\), we have
\[
|s - t| = A_n = |s - \alpha|,
|s' - t| = \max\{|s' - \alpha|, |s - \alpha|, |s - t|\} = |s' - \alpha|.
\]

By Proposition 3.10 we have
\[
|f^n_t(a)| = |s - t| \prod_{s' \in S_n \setminus \{s\}} |s' - t| = |s - \alpha| \prod_{s' \in S_n \setminus \{s\}} |s' - \alpha| = |f^n_\alpha(a)| = |a|.
\]

(2) Let \(T_n = \{t \in C_v : |f^n_t(a)| = |a|\}\), then by Proposition 3.9
\[M_a \cap C_v = \cap_{n=1}^{\infty} T_n \subseteq \cap_{n=1}^{\infty} \overline{T_n}.
\]

By Proposition 4.9 and Claim 2,
\[T_n = \{t \in C_v : |s - t| = A_n\} \text{ for some } s \in S_n\}.
\]
\( \mathcal{T}_n = \{ t \in A_{\nu}^{1,an} : \delta(s, t) = A_n \text{ for some } s \in S_n \} \).

If \( t \in \cap_{n=1}^{\infty} \mathcal{T}_n \), then \( \text{diam}(t) \leq A_n \) for any \( n \geq 1 \). Since \( A_n \to 0 \) as \( n \to \infty \), we have \( t \in \mathcal{C}_\nu \).

Therefore, \( \cap_{n=1}^{\infty} T_n = \cap_{n=1}^{\infty} \mathcal{T}_n \) and \( M_a \cap \mathcal{C}_\nu = \overline{M_a} \cap \mathcal{C}_\nu = M_a \).

(3) By Claim 1 and induction, for any \( s \in S_n \) and any \( k \geq n + 1 \), we have \( |S_k \cap T_{n,s}| = d^{k-n} \).

Then the conclusion follows from Proposition 4.2.

Propositions 4.11 and 4.12 can be seen as the non-Archimedean version of Propositions 4.6 and 4.8.

**Proposition 4.11.** Assume that \( \max\{|a|, |b|\} > |d|^{-2/(d-1)} \). If

\[
|a^d - b^d| > \frac{|d|^{-(n-1)}}{\max\{|a|, |b|\}^{(d-1)(n-1)-1}},
\]

for some \( n \geq 1 \), then

\[
\int g_\alpha d\mu_b \geq \frac{1}{d^{2n-2}} \log \max\{|a|, |b|\}.
\]

**Proof.** If \( |a| \neq |b| \), then by Propositions 3.8 and 3.9

\[
\int g_\alpha d\mu_b = d \log \max\{|a|, |b|, 1\} \geq \frac{1}{d^{2n-2}} \log \max\{|a|, |b|\}.
\]

From now on we assume that \( |a| = |b| \). For any \( n \geq 1 \), let \( A_n = |a|/(|d||a|^{d-1})^{n-1} \), and let \( \alpha_1(a), \ldots, \alpha_{d^n-1}(a) \) be the roots of \( f^{n}_t(a) \). For any \( 1 \leq i \leq d^{n-1} \), let \( D_i(a) = \overline{T}(\alpha_i(a), A_n) \).

Define \( \alpha_i(b) \) and \( D_i(b) \) similarly.

**Claim:** Some \( D_i(b) \) is disjoint from \( \cup_{i=1}^{d^n-1} D_i(a) \).

Suppose not, then for each \( i \), there exists \( k_i \) such that

\[
|\alpha_{k_i}(a) - \alpha_i(b)| \leq A_n.
\]

If \( k_i = k_j \) for some \( i \neq j \), then

\[
|\alpha_i(b) - \alpha_j(b)| \leq \max\{|\alpha_{k_i}(a) - \alpha_i(b)|, |\alpha_{k_j}(a) - \alpha_j(b)|\} \leq A_n,
\]

which contradicts Proposition 4.10. If \( k_i \neq k_j \) for any \( i \neq j \), then

\[
|d^{n-1}|a^d - b^d| = \left| \sum_{i=1}^{d^{n-1}} \alpha_i(a) - \sum_{i=1}^{d^{n-1}} \alpha_i(b) \right| \leq \max\{|\alpha_{k_i}(a) - \alpha_i(b)|\}_{i=1}^{d^{n-1}} \leq A_n,
\]

which contradicts the assumption

\[
|a^d - b^d| > \frac{|d|^{-2(n-1)}}{|a|^{(d-1)(n-1)-1}} = \frac{|a|}{(|d|^2|a|^{d-1})^{n-1}} = \frac{A_n}{|d|^{n-1}}.
\]

Let \( D_i(b) \) be disjoint from \( \cup_{i=1}^{d^n-1} D_i(a) \) and \( t \in D_i(b) \). By Proposition 3.10

\[
|t| = \max\{|t - \alpha_i(b)|, |\alpha_i(b)|\} = |a|^d.
\]
Proposition 4.12. For any \( a, b \in K \), we have
\[
\int g_a \, d\mu_b \geq \frac{1}{d^{n-1}} \cdot \frac{1}{d^{n-1}} \log |a| = \frac{1}{d^{2n-2}} \log |a|.
\]

Proof. If \( \max\{|a|, |b|\} \leq |d|^{-2/(d-1)} \) or \( |a^d - b^d| \leq |d|^{-2/(d-1)} \), then
\[
\log^+ |a^d - b^d| \leq \log \max\{|a|^d, |b|^d, |d|^{-2/(d-1)}\} \leq \log |d|^{-2d/(d-1)}
\]
and
\[
\int g_a \, d\mu_b \geq 0 \geq \frac{1}{d^{3}} \log^+ |a^d - b^d| - \frac{1}{d^{3}} \log |d|^{-2d/(d-1)}.
\]
If \( \max\{|a|, |b|\} > |d|^{-2/(d-1)} \) and
\[
|a^d - b^d| > |d|^{-2/(d-1)} = \frac{|d|^{-2}}{|d|^{-2(d-2)/(d-1)}} \geq \frac{|d|^{-2}}{\max\{|a|, |b|\}^{d-2}},
\]
then by Proposition 4.11
\[
\int g_a \, d\mu_b \geq \frac{1}{d^{3}} \log \max\{|a|, |b|\} = \frac{1}{d^{3}} \log \max\{|a|^d, |b|^d\} \geq \frac{1}{d^{3}} \log^+ |a^d - b^d|.
\]

4.5. Proofs of Theorems 1.3 and 1.4. Now we are ready to give the proofs of Theorems 1.3 and 1.4. Given Propositions 4.6, 4.8, 4.11, and 4.12, our proofs are almost identical to the proofs given in [DKY2], Sections 7 and 8.

Proof of Theorem 1.3. Let \( K \) be a number field such that \( a, b \in K \). Define
\[
(p_v, q_v, m, n) = \begin{cases} 
(50, 11, 1, 4), & \text{if } d = 2 \text{ and } v \in M_K^\infty, \\
(2|v|^2, 2|v|^{-4}, 1, 4), & \text{if } d = 2 \text{ and } v \in M_K^0, \\
(9, 25, d - 2, 2), & \text{if } d > 2 \text{ and } v \in M_K^\infty, \\
(|d|^{-2/(d-1)}, |d|^{-2}, d - 2, 2), & \text{if } d > 2 \text{ and } v \in M_K^0,
\end{cases}
\]
and \( r_v = \max\{|a|_v, |b|_v, p_v\} \) for any \( v \in M_K \). Following [DKY2] Section 8.2, we define
\[
M_{\text{help}} = \left\{ v \in M_K : \max\{|a|_v, |b|_v\} > p_v, |a^d - b^d|_v > \frac{q_v}{\max\{|a|_v, |b|_v\}^m} \right\},
\]
\[
M_{\text{close}} = \left\{ v \in M_K : \max\{|a|_v, |b|_v\} > p_v, |a^d - b^d|_v \leq \frac{q_v}{\max\{|a|_v, |b|_v\}^m} \right\},
\]
\[ M_{\text{bounded}} = \{ v \in M_K : \max\{|a|_v, |b|_v\} \leq p_v \}. \]

Then
\[ 0 = \sum_{v \in M_K} n_v \log |a^d - b^d|_v, \]
\[ \leq \sum_{v \in M_{\text{close}}} n_v \log \frac{q_v}{r^m_v} + \sum_{v \in M_K \setminus M_{\text{close}}} n_v \log (2r^d_v) + \sum_{v \in M_K \setminus M_{\text{close}}} n_v \log r^d_v, \]
\[ = \sum_{v \in M_K} n_v \log q_v - \sum_{v \in M_{\text{close}}} n_v \log r^m_v + \sum_{v \in M_K \setminus M_{\text{close}}} n_v \log r^d_v, \]
\[ = \sum_{v \in M_K} n_v \log q_v - \sum_{v \in M_K} n_v \log r^m_v + \sum_{v \in M_{\text{help}}} n_v \log r^d_v + \sum_{v \in M_{\text{bounded}}} n_v \log r^d_v, \]
\[ \leq (d + m) \sum_{v \in M_{\text{help}}} n_v \log r_v - mh(a, b) + (d + m) \sum_{v \in M_K} n_v \log(p_v q_v). \]

By Propositions 4.6, 4.8 and 4.11,
\[ \langle \mu_a, \mu_b \rangle \geq \sum_{v \in M_{\text{help}}} n_v \int g_{a,v} d\mu_{b,v} \geq \frac{1}{d^n} \sum_{v \in M_{\text{help}}} n_v \log r_v, \]
\[ \geq \frac{1}{d^n} \left( \frac{m}{d + m} h(a, b) - \sum_{v \in M_K} n_v \log(p_v q_v) \right), \]
\[ = \begin{cases} \frac{1}{48} h(a, b) - \frac{1}{16} \log 35200, & \text{if } d = 2, \\ \frac{d-2}{2d^2(d-1)} h(a, b) - \frac{1}{d^2} \log(225d^{2d/(d-1)}), & \text{if } d > 2, \\ \frac{1}{12d^2} h(a, b) - 1. & \end{cases} \]

The proof of Theorem 1.4 relies on the following two results. Theorem 4.13 is adapted from [DKY2, Theorem 7.1], and Proposition 4.14 is adapted from a continuity argument used in the proof of [DKY2, Theorem 1.6].

**Theorem 4.13.** For any \( a, b \in \bar{Q} \), we have
\[ \langle \mu_a, \mu_b \rangle \geq \frac{1}{d^3} h(a^d - b^d) - 2, \]
where \( h \) is the logarithmic Weil height on \( \bar{Q} \).
Proof. Let $K$ be a number field such that $a, b \in K$. Define

$$r_v = \begin{cases} 
5000, & \text{if } d = 2 \text{ and } v \in M_K^\infty, \\
|2|_{v}^{-1}, & \text{if } d = 2 \text{ and } v \in M_{K}^0, \\
2 \cdot 9^d, & \text{if } d > 2 \text{ and } v \in M_K^\infty, \\
|d|_{v}^{-2d/(d-1)}, & \text{if } d > 2 \text{ and } v \in M_K^0. 
\end{cases}$$

By Propositions 4.6, 4.8, and 4.12

$$\langle \mu_a, \mu_b \rangle = \sum_{v \in M_K} n_v \int g_{a,v} d\mu_{b,v} \geq \sum_{v \in M_K} n_v \left( \frac{1}{d^3} \log^+ \left| a^d - b^d \right|_v - \frac{1}{d^3} \log r_v \right)$$

$$= \frac{1}{d^3} h(a^d - b^d) - \frac{1}{d^3} \sum_{v \in M_K} n_v \log r_v$$

$$= \frac{1}{d^3} h(a^d - b^d) - \begin{cases} 
\frac{1}{8} \log 80000, & \text{if } d = 2, \\
\frac{1}{d^3} \log(2 \cdot 9^d d^{2d/(d-1)}), & \text{if } d > 2, 
\end{cases}$$

$$\geq \frac{1}{d^3} h(a^d - b^d) - 2.$$ \hfill \blacksquare

Proposition 4.14. The complex Arakelov–Zhang pairing $\int g_a d\mu_b$ is a continuous function of $(a, b) \in \mathbb{C}^2$.

Proof. Let $a_n \to a$ and $b_n \to b$ as $n \to \infty$, then by Proposition 3.4 there exists $r > 0$ such that for any $n \geq 1$,

$$\{a, b, a_n, b_n\}, M_a, M_b, M_{a_n}, M_{b_n} \subseteq \overline{D}(0, r).$$

By [BH, Proposition 1.2], $g_a(t)$ is a continuous function of $(a, t) \in \mathbb{C}^2$, so $g_a(t)$ is uniformly continuous on the compact set $\overline{D}(0, r) \times \overline{D}(0, r)$. Therefore, as $n \to \infty$,

$$\int g_a d\mu_b - \int g_a d\mu_{b_n} = \int (g_a - g_{a_n}) d\mu_b + \int (g_{b_n} - g_b) d\mu_{a_n} \to 0.$$ \hfill \blacksquare

Proof of Theorem 1.4. Let $K$ be a number field such that $a, b \in K$. Define

$$(p, q, m, n) = \begin{cases} 
(50, 11, 1, 4), & \text{if } d = 2, \\
(9, 25, d - 2, 2), & \text{if } d > 2. 
\end{cases}$$

Let $N$ be a large number to be determined later, and let

$$M_1 = \{v \in M_K^\infty : \max\{|a|_v, |b|_v\} \leq N, |a^d - b^d|_v \geq q/N^m\},$$

$$M_2 = \{v \in M_K^\infty : \max\{|a|_v, |b|_v\} > N, |a^d - b^d|_v \geq q/N^m\},$$

$$M_3 = \{v \in M_K^\infty : |a^d - b^d|_v < q/N^m\}.$$
Since \( \sum_{v \in M^K} n_v = 1 \), we have \( \sum_{v \in M_i} n_v \geq 1/3 \) for some \( 1 \leq i \leq 3 \).

(1) Assume that \( \sum_{v \in M_1} n_v \geq 1/3 \). Let
\[
R_N = \{ (c_1, c_2) \in \mathbb{C}^2 : \max\{|c_1|, |c_2|\} \leq N, |c_1^d - c_2^d| \geq q/N^m \}.
\]
Since \( R_N \) is compact, by [BD1, Lemma 3.4] and Proposition 4.14
\[
r_N = \min_{(c_1, c_2) \in R_N} \int g_{c_1} d\mu_{c_2} > 0.
\]
Therefore,
\[
\langle \mu_a, \mu_b \rangle \geq \sum_{v \in M_1} n_v \int g_{a,v} d\mu_{b,v} \geq \frac{1}{3} r_N.
\]

(2) Assume that \( \sum_{v \in M_2} n_v \geq 1/3 \). If \( v \in M_2 \), then
\[
|a^d - b^d|^v \geq \frac{q}{N^m} \geq \frac{q}{\max\{|a|^v, |b|^v\}^m}.
\]
If \( N \geq p \), then by Propositions 4.6 and 4.8
\[
\int g_{a,v} d\mu_{b,v} \geq \frac{1}{d^m} \log \max\{|a|^v, |b|^v\} \geq \frac{1}{d^m} \log N.
\]
Therefore,
\[
\langle \mu_a, \mu_b \rangle \geq \sum_{v \in M_2} n_v \int g_{a,v} d\mu_{b,v} \geq \frac{1}{3 d^m} \log N.
\]

(3) Assume that \( \sum_{v \in M_3} n_v \geq 1/3 \). By Theorem 4.13
\[
\langle \mu_a, \mu_b \rangle \geq \frac{1}{d^3} h(a^d - b^d) - 2 = \frac{1}{d^3} \sum_{v \in M_K} n_v \log^+ |a^d - b^d|^v - 2
\]
\[
\geq \frac{1}{d^3} \sum_{v \in M_K \setminus M_3} n_v \log^+ |a^d - b^d|^v - 2 \geq \frac{1}{d^3} \sum_{v \in M_K \setminus M_3} n_v \log |a^d - b^d|^v - 2
\]
\[
= \frac{1}{d^3} \sum_{v \in M_3} n_v \log \frac{1}{|a^d - b^d|^v} - 2 \geq \frac{1}{3d^3} \log \frac{N^m}{q} - 2,
\]
which is positive when \( N > q^{1/m} e^{6d^3/m} \).

5. Proof of Theorem 1.1

Finally, we deduce Theorem 1.1 from Theorems 1.2, 1.3, and 1.4.

Proof of Theorem 1.1. (1) Let \( a, b \in \bar{\mathbb{Q}} \) such that \( a^d \neq b^d \) and \( |S_{a,b}| > 0 \). By Theorems 1.2, 1.3, and 1.4 there exist \( c_1, c_2, c_3(\varepsilon), c_4, \delta > 0 \) such that
\[
\max\{c_1 h(a, b) - c_2, \delta\} \leq \langle \mu_a, \mu_b \rangle \leq \left( \varepsilon + \frac{c_3(\varepsilon)}{|S_{a,b}|} \right) (h(a, b) + c_4).
\]
When $\varepsilon$ is small enough, we have

$$
|S_{a,b}| \leq \frac{c_3(\varepsilon)}{\max \left\{ \frac{c_1 h(a,b) - c_2}{h(a,b) + c_4}, \frac{\delta}{h(a,b) + c_4} \right\}} - \varepsilon \leq \frac{c_3(\varepsilon)}{\frac{c_1 \delta}{c_1 c_4 + c_2 + \delta}} - \varepsilon.
$$

(2) Let $a \in \mathbb{C} \setminus \bar{Q}$ and $b \in \mathbb{C}$ such that $a^d \neq b^d$ and $|S_{a,b}| > 0$. Since each $t \in S_{a,b}$ satisfies $f_t^m(a) = f_t^n(a)$ and $f_t^k(b) = f_t^l(b)$ for some $m > n \geq 0$ and $k > l \geq 0$, the field $\mathbb{Q}(a, b, S_{a,b})$ has transcendence degree one over $\mathbb{Q}$. We may view $\mathbb{Q}(a, b, S_{a,b})$ as the function field $K(X)$ of an algebraic curve $X$ defined over a number field $K$. For all but finitely many $x \in X(\bar{\mathbb{Q}})$, the specializations $a(x)^d \neq b(x)^d$ and $t_1(x) \neq t_2(x)$ for any $t_1, t_2 \in S_{a,b}$. Therefore, the uniform bound for the algebraic case also works for the complex case. ■

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