On Erdős–Szekeres problem and related problems*

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Abstract: Here we give a short survey of our new results. References to the complete proofs can be found in the text of this article and in the literature.

1 Introduction and statement of problems

In 1935 Paul Erdős and George Szekeres formulated the following problem (see [8], [9]).

First Erdős–Szekeres problem. For any integer \( n \geq 3 \), find the minimal positive number \( g(n) \) such that any planar set of points in general position containing at least \( g(n) \) points has a subset of cardinality \( n \) whose elements are the vertices of a convex \( n \)-gon.

In 1978 Erdős suggested the following modification of the first problem (see [10]).

Second Erdős–Szekeres problem. For any integer \( n \geq 3 \), find a minimal positive number \( h(n) \) such that any planar set \( X \) in general position containing at least \( h(n) \) points has a subset of cardinality \( n \) whose elements are the vertices of an empty convex \( n \)-gon, i.e., of an \( n \)-gon containing no other points of \( X \).

Recall that a set of points on the plane is in the general position if any three of its elements do not lie in a straight line.

The above problems are classical in combinatorial geometry and Ramsey theory (see [13], [14], [27], [32]). They can both be generalized as follows.

Third Erdős–Szekeres-type problem. For any integers \( n \geq 3 \) and \( k \geq 0 \), find a minimal positive number \( h(n,k) \) such that any planar set \( X \) in general position containing at least \( h(n,k) \) points has a subset of cardinality \( n \) whose elements are the vertices of convex \( n \)-gon \( C \) with \( |(C \setminus \partial C) \cap X| \leq k \); i.e., the interior of this \( n \)-gon contains at most \( k \) other points of \( X \).

One more generalization was suggested in [3] by Bialostocki, Dierker, and Voxman.

Fourth Erdős–Szekeres-type problem. For any integers \( n \geq 3 \) and \( q \geq 0 \), find a minimal positive number \( h(n \mod q) \) such that any planar set \( X \) in general position containing at least \( h(n \mod q) \) elements has a subset of cardinality \( n \) whose elements are the vertices of convex \( n \)-gon \( C \) with \( |(C \setminus \partial C) \cap X| \equiv 0 \ (\text{mod} q) \); i.e., the interior of this \( n \)-gon contains \( q \) other points from \( X \) and their number is a multiple of \( q \).

One may find more detailed history of Erdős–Szekeres problems, for example, in the following surveys [2], [5], [27].

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2 On the first and second problems

The first problem was considered by Erdős and Szekeres in the article \[8\]. They proved the existence of \(g(n)\) for arbitrary \(n\) by demonstrating the upper bound \(g(n) \leq \binom{2n-4}{n-2} + 1\,\), and they gave the following conjecture:

**Conjecture 1.** \(g(n) = 2^{n-2} + 1\).

This conjecture is proved for \(n \leq 6\). The case \(g(3) = 3\) is obvious here; equality \(g(4) = 5\) was proved by E. Klein in 1935 (see pic. 1 where all three essentially different ways of placing five points on the plane are displayed); expression \(g(5) = 9\) was obtained by E. Makai (see \[8\], \[9\], \[27\]); the fact \(g(6) = 17\) was established rather recently by G. Szekeres, B. McKay and L. Peters in \[35\]. Besides, in 1961 Erdős and Szekeres have also proved the lower bound \(g(n) \geq 2^{n-2} + 1\,\) (see \[9\]).

![Picture 1: Any set of five points contains a convex quadrilateral](image1)

Inequality \(g(n) \leq \binom{2n-4}{n-2} + 1\) was repeatedly improved. The strongest result was obtained in 2005 by G. Toth and P. Valtr: \(g(n) \leq \binom{2n-5}{n-3} + 1\,\) (here \(n \geq 5\); see \[36\]). Thus, the Erdős – Szekeres conjecture is still neither proved nor disproved, and it is only known that

\[
2^{n-2} + 1 \leq g(n) \leq \binom{2n-5}{n-3} + 1.
\]

In connection with bounding \(g(n)\) Erdős and Szekeres introduced the notions of *cup* and *cap*. We assume that a coordinate system \((x, y)\) is fixed in the plane. Let \(\mathcal{X} = \{(x_1, y_1), (x_2, y_2), \ldots, (x_m, y_m)\}\) be a set of points in general position in the plane, with \(x_i \neq x_j\) for all \(i \neq j\). A subset of points \(\{(x_{i_1}, y_{i_1}), (x_{i_2}, y_{i_2}), \ldots, (x_{i_r}, y_{i_r})\}\) is called an \(r\)-cup (see pic. 2) if \(x_{i_1} < x_{i_2} < \ldots < x_{i_r}\) and

\[
\frac{y_{i_1} - y_{i_2}}{x_{i_1} - x_{i_2}} < \frac{y_{i_2} - y_{i_3}}{x_{i_2} - x_{i_3}} < \ldots < \frac{y_{i_{r-1}} - y_{i_r}}{x_{i_{r-1}} - x_{i_r}}.
\]

![Picture 2: cup and cap](image2)
Similarly, the subset is called an \( r\)-cap (see pic. 2) if \( x_{i_1} < x_{i_2} < \ldots < x_{i_r} \) and
\[
\frac{y_{i_1} - y_{i_2}}{x_{i_1} - x_{i_2}} > \frac{y_{i_2} - y_{i_3}}{x_{i_2} - x_{i_3}} > \ldots > \frac{y_{i_{r-1}} - y_{i_r}}{x_{i_{r-1}} - x_{i_r}}.
\]

Define \( f(l, m) \) to be the smallest positive integer for which \( X \) contains an \( l\)-cup or an \( m\)-cap whenever \( X \) has at least \( f(l, m) \) points.

The problem of finding \( f(l, m) \) was completely solved by Erdős and Szekeres (see [8], [9]). They proved that \( f(l, m) = \left(\frac{l+m-4}{2} - \frac{l}{2}\right) + 1 \). Note that the first bound of \( g(n) \) is based on the inequality \( g(n) \leq f(n, n) = \left(\frac{2n-4}{2}\right) + 1 \).

The second problem is more deeply understood. Thus, equalities \( h(3) = 3 \) and \( h(4) = 5 \) for it are obvious (see pic. 1). Expression \( h(5) = 10 \) was obtained by H. Harborth in 1978 (see [15]). And in 1983 J. Horton proved that \( h(n) \) does not exist where \( n \geq 7 \) (see [16]). Actually, Horton proved non-existence of \( h(n, 0) \) where \( n \geq 7 \). The question of existence and exact value of \( h(6) \) (or, which is the same, \( h(6, 0) \)) has been remaining open for a long time. Only in 2006 T. Gerken proved the existence of \( h(6) \), by demonstrating the upper bound \( h(6) \leq g(9) \leq \left(\frac{13}{6}\right) + 1 = 1717 \) (see [12]). Independently of him, C. Nicolas (see [28]) and Váltr (see [37]) presented their proofs, but their upper bounds are worse and equal to, respectively, \( g(25) \) and \( g(15) \). In 2007 the upper bound was improved by the author of this article:

**Theorem 1.** \( h(6) \leq 463 \) (see [19], [20], [24]).

The trivial lower bound \( h(6) \geq g(6) \geq 17 \) is a consequence of one of the Erdős–Szekeres theorems (see [9]). All the other lower bounds for \( h(6) \) were obtained by the computer search. The first one of them was given by D. Rappaport in 1985: \( h(6) \geq 21 \) (see [33]); the second one was done by M. Overmars, B. Scholten and I. Vincent in 1988: \( h(6) \geq 27 \) (see [30]). The best known lower bound was obtained in 2003 by Overmars: \( h(6) \geq 30 \) (see [31]). Thus, for \( h(6) \) estimates \( 30 \leq h(6) \leq 463 \) are proved at present.

### 3 On the third problem

As it is easy to see, for the third problem inequalities \( g(n) \leq h(n, k) \leq h(n) \) are always correct if the appropriate expressions exist. Moreover, \( h(n, 0) \geq h(n, 1) \geq h(n, 2) \geq h(n, 3) \geq \ldots \) and there is a \( k' \) such that \( h(n, k) = g(n) \) for all \( k \geq k' \). For small values of \( n \) the following results are obvious: \( h(3, k) = 3 \), \( h(4, k) = 5 \), \( h(5, 0) = 10 \), \( h(5, 1) = 9 \). The last result follows from the fact that a convex pentagon with two or more points inside always contains a smaller convex pentagon.

Some results relating to the third problem are obtained in an article by Bl. Sendov (see [34]). In this article, with the use of the Horton set (see [16]), through which the non-existence of \( h(7) \) was proved, non-existence of \( h(n, k) \) was proved for certain values of \( k \) where \( n > 7 \). More precisely, \( k \) should be less than or equal to \( (r + 4)2^{m-1} - 4m - r - 1 \), provided \( n + 2 = 4m + r \), where \( m \) is integer and \( r \in \{0, 1, 2, 3\} \). The similar results are obtained in the article by H. Nyklova (see [29]), besides it is proved there that \( h(6, \geq 6) = g(6) \) and the result \( h(6, 5) = 19 \) is presented. Note that Sendov’s and Nyklova’s estimates are asymptotically equal to \( (\sqrt{2} + o(1))^n \).

With respect to the fact that all results for \( g(6) \) and \( h(6) \) were obtained rather recently, the study of the value \( h(6, 1) \) is interesting (values of \( k \), other than 1 may be not so interesting with respect to the conjecture set forth below). We found the upper bound for \( h(6, 1) \) much better than the upper bound for \( h(6, 0) \).
Theorem 2. The inequality holds \( h(6, 1) \leq g(7) \leq 127 \) (see [21]).

Thus, it appears that at present the estimates \( 17 \leq h(6, 1) \leq 127 \) are proved. Note that, if the conjecture [1] of Erdős and Szekeres is true, the equality in Theorem 2 will look as \( h(6, 1) = g(7) = 33 \).

Actually, we suppose that the stronger statement is true:

Conjecture 2. \( h(6, 1) = g(6) = 17 \).

Note that it follows immediately from the conjecture that \( h(6, 1) = h(6, 2) = h(6, 3) = h(6, 4) = h(6, 5) = 17 \). The supposed equality \( h(6, 5) = 17 \) obviously contradicts the result of Nyklova set forth above. The point is that this result was proved inaccurately and there are counterexamples to it.

Now we formulate a new result on the existence of \( h(n, k) \) for all \( n \).

Theorem 3. For odd and for even \( n \) respectively, the following values do not exist \( h(n, (\frac{n-7}{n-7}) - 1), h(n, 2(\frac{n-8}{n-8}) - 1) \) (see [23], [26]).

Note that this theorem gives an asymptotic lower estimate of the form \( 2 + o(1) \) for the maximal value of \( k \) such that \( h(n, k) \) does not exist. This result is much better than the above-mentioned result of Sendov (see [34]). In table 1, we compare maximum values of \( k \) such that \( h(n, k) \) does not exist according to Sendov (see [34]), Nyklova (see [29]), and this author.

| \( n, k \geq \) | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 | 21 | 22 | 23 | 24 | 25 |
|----------------|---|---|---|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|
| Bl. Sendov 1995 | 0 | 1 | 2 | 3 | 6 | 9 | 12 | 15 | 22 | 29 | 36 | 43 | 58 | 73 | 88 | 103 | 134 | 165 | 196 |
| H. Nyklova 2000 | 0 | 1 | 2 | 3 | 6 | 9 | 13 | 19 | 27 | 39 | 51 | 63 | 91 | 119 | 147 | 175 | 238 | 301 | 373 |
| V. Koshelev 2009 | 0 | 1 | 2 | 3 | 6 | 11 | 19 | 39 | 69 | 139 | 251 | 503 | 923 | 1847 | 3431 | 6863 | 12869 | 25739 | 48619 |

Table 1: Comparing lower bounds for \( k \)

It is of interest to find such values of \( k \) that \( h(n, k) = g(n) \) or \( h(n, k) > g(n) \). However, we do not know the exact values of \( g(n) \). We only know the conjecture [1]. So we will prove an estimate concerning the maximum value of \( k \) for which \( h(n, k) > 2^{n-2} + 1 \).

Theorem 4. If \( n \geq 6 \), then \( h\left(n, \left(\frac{n-3}{n-3/2}\right) - \lceil\frac{n}{2}\rceil\right) > 2^{n-2} + 1 \) (see [26]).

We give some similar results for cups and caps. We define \( f(l, m, l_1, m_1) \) to be the smallest positive integer such that any set \( X \) in general position with no two points having the same x-coordinate and of cardinality \( f(l, m, l_1, m_1) \) contains a \( l \)-cup with at most \( l_1 \) points inside or an \( m \)-cap with at most \( m_1 \) points inside.
| \( m \) | 5   | 6   | 7   | 8   | 9   | 10  | 11  | 12  | 13  |
|---------|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| l       |     |     |     |     |     |     |     |     |     |
|         | 0   | 0   | 1   | 04  | 07  | 014 | 021 | 036 | 051 |
| 5       | 0   | 1   | 4   | 2   | 2   | 1   | 1   | 2   | 1   |
| 6       | 1   | 0   | 11  | 4   | 1   | 5   | 2   | 9   | 9   |
| 7       | 4   | 0   | 2   | 5   | 5   | 2   | 1   | 9   | 9   |
| 8       | 7   | 0   | 14  | 0   | 5   | 2   | 4   | 9   | 9   |
| 9       | 14  | 0   | 15  | 1   | 9   | 2   | 8   | 8   |
| 10      | 21  | 0   | 29  | 1   | 9   | 2   | 8   | 8   |
| 11      | 36  | 0   | 44  | 2   | 8   | 3   | 7   | 7   |
| 12      | 51  | 0   | 73  | 1   | 14  | 9   | 9   | 9   |
| 13      | 82  | 0   | 110 | 2   | 14  | 9   | 9   | 9   |

Table 3: Values of \( l_1 \) and \( m_1 \) such that \( f(l, m, l_1, m_1) \) does not exist
Theorem 5. Let \( c(r) = 2^{\left(\frac{r-2}{2}\right)} + 2^{\left(\frac{r-2}{2}\right)} - r - 1 \). If for \( l_0 \) and \( m_0 \) we have \( c(l_0) > 0, c(m_0) > 0 \), then for any \( l \geq 5 \) and \( m \geq 5 \), with \( l \geq l_0 \) and \( m \geq m_0 \), the following value does not exist: \( f(l, m, c(l_0)(l^{-l_0-m_0}) - 1, c(m_0)(l^{-m-m_0}) - 1) \) (see [20]).

Theorem 6. For any \( l \geq 4 \), \( m \geq 4 \) and \( a \geq 0 \) (with all non-negative arguments of \( f \)) the inequalities hold \( f(l, m, (l^{-m-6}) - m + 1, a) > f(l, m), f(l, m, a, (l^{-m-6}) - l + 1) > f(l, m) \) (see [20]).

Tables 3 and 4 consist of numbers illustrating Theorem 5 and Theorem 6.

| \( l \) | \( m \) | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 4 | 0 | 2 | 5 | 9 | 14 | 20 | 27 | 35 | 44 | 54 | 65 | 77 |   |
| 5 | 1 | 16 | 15 | 29 | 49 | 76 | 111 | 155 | 209 | 274 | 351 | 441 |   |
| 6 | 2 | 11 | 30 | 64 | 119 | 202 | 321 | 485 | 704 | 989 | 1352 | 1806 |   |
| 7 | 3 | 17 | 51 | 120 | 245 | 454 | 783 | 1277 | 1991 | 2991 | 4355 | 6174 |   |
| 8 | 4 | 24 | 79 | 204 | 455 | 916 | 1707 | 2993 | 4994 | 7996 | 12363 | 18550 |   |
| 9 | 5 | 32 | 115 | 324 | 785 | 1708 | 3423 | 6425 | 11429 | 19436 | 31811 | 50374 |   |
| 10 | 6 | 41 | 160 | 489 | 1280 | 2995 | 6426 | 12860 | 24299 | 43746 | 75569 | 125956 |   |
| 11 | 7 | 51 | 215 | 709 | 1995 | 4997 | 11431 | 24300 | 48609 | 92366 | 167947 | 293916 |   |
| 12 | 8 | 62 | 281 | 959 | 2996 | 8000 | 19439 | 43748 | 92367 | 184744 | 352703 | 646632 |   |
| 13 | 9 | 74 | 359 | 1359 | 4361 | 12368 | 31815 | 75572 | 167949 | 353713 | 705419 | 1352064 |   |
| 14 | 10 | 87 | 450 | 1814 | 6181 | 18556 | 50379 | 125960 | 293919 | 646634 | 1352065 | 2704142 |   |

Table 4: Values of \( l_1 \) such that \( f(l, m, l_1,?) > f(l, m) \)

4 On the fourth problem

Concerning the fourth problem, Bialostocki, Dierker, and Voxman conjectured that \( h(n, \mod q) \) exists for all \( n \geq 3 \) and \( q \geq 2 \). This conjecture has neither been proved nor disproved thus far. Below, we present the results available and their improvements.

For small values of \( n \) the following results are obvious: \( h(3, \mod q) = 3 \), \( h(4, \mod q) = 5 \), \( h(5, \mod q) = h(5) = 10 \). If \( n = 6 \), then \( h(6, \mod q) = h(6) \) for all \( q \), except finite set of values. Probably \( h(6, \mod 2) = g(6) = 17 \), but for \( q \geq 3 \) it is possible to construct a point set that shows that \( h(n, \mod q) \geq 17 \).

Bialostocki, Dierker, and Voxman proved their conjecture (see [3]) for \( n \geq q + 2 \) and obtained the upper bound \( h(n, \mod q) \leq g \left( R_3 \left( \underbrace{n', \ldots, n'}_{q} \right) \right) \), where \( n' \) is the minimum positive integer satisfying \( n' \geq n \) and \( n' \equiv 2^{(\mod q)} \). Here, \( R_k(l_1, \ldots, l_s) \) is the Ramsey number for complete \( k \)-uniform hypergraphs with edges painted in \( s \) colors in which at least one monochromatic \( i \)-clique with suitable \( i \) is sought (see [13], [14]). In last formula, the Ramsey number has \( q \) arguments with a value of \( n' \). Only astronomical estimates of Ramsey numbers are known. In this case, we have a tower of exponentials.

In 1996, Caro (see [3]) obtained a more general result for points in the plane with assigned values from a finite Abelian group and for convex polygons with a zero inside sum. As applied to the problem under discussion, his theorem gives \( h(n, \mod q) \leq 2^{c(q)n} \).
Here, $c(q)$ is a function independent of $n$ but growing superexponentially in $q$. Thus, we again deal with a multiple exponential.

Of course, last bound has to be refined. This can be done in two directions. On the one hand, it would be desirable to get rid of the superexponential bounds at least for some relations between $n$ and $q$. On the other hand, the constraint $n \geq q + 2$, under which $h(n, \text{mod } q)$ always exists, seems excessive.

The only result in the first direction was obtained in [18], namely, $h(n, \text{mod } q)$ exist for $n \geq 5q/6 + O(1)$, but the upper bound is even worse than in [3]. Note that a similar result with $n \geq 3q/4 + O(1)$ was announced by Valtr, but this result was not published. In the second direction, new results have not been obtained at all. Caro conjectured that $h(n, \text{mod } q) \leq g(c(q) + n)$ with some $c(q)$. We managed to prove the following result.

**Theorem 7.** If $n \geq 2q - 1$, then $h(n, \text{mod } q) \leq g(q(n - 4) + 4)$ (see [22],[23],[25]).

This theorem considerably improves Caro estimate, since $g(q(n - 4) + 4) \leq 2^{2qn+O(1)}$. Thus, we eventually have got rid of the multiple exponentials in the inequalities.

However, the constraint $n \geq 2q - 1$ is somewhat stronger than before, and Caro’s conjecture has not been proved (or disproved). Nevertheless, this is an important step toward the solution of the problem.

Note that Bialostocki—Dierker—Voxman estimate admits a fairly curious refinement, which is weaker than Caro’s result and Theorem 7, but, in our view, deserves to be mentioned.

**Theorem 8.** $h(n, \text{mod } q) \leq R_3(n, n, \ldots, n)$, for even $q$, $h(n, \text{mod } q) \leq R_3(g(n), n, \ldots, n)$, for odd $q$ (see [22],[23],[25]).

The theorem is easy to prove by modifying the original Bialostocki—Dierker—Voxman argument.

5 Chromatic variant of problems

Devillers, Hurtado, Károlyi, and Seara [7] conjectured that every large enough two-colored set of points, with no three points collinear, contains a convex empty monochromatic fourgon. This can be answered in the affirmative if we omit the condition of convexity (see [1]). An example of 18 points with no empty monochromatic convex fourgon from [7] led to the problem of finding the maximum number of two-colored points that do not contain an empty monochromatic convex fourgon. Improved lower bounds were given by Brass — 20 points (see [4]), Friedman — 30 points (see [11]), van Gulik — 32 points (see [38]), and finally Huemer and Seara — 36 points (see [17]). Here we show (see pic. 3) a set of 46 two-colored points, no three points collinear, with no empty monochromatic convex fourgons.
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