Abstract. We construct a connected graph $H$ such that

1. $\chi(H) = \omega$;
2. $K_{\omega}$, the complete graph on $\omega$ points, is not a minor of $H$.

Therefore Hadwiger’s conjecture does not hold for graphs with infinite coloring number.

1. Notation

In this note we are only concerned with simple undirected graphs $G = (V,E)$ where $V$ is a set and $E \subseteq \mathcal{P}_2(V)$ where

$$\mathcal{P}_2(V) = \{\{x,y\} : x, y \in V \text{ and } x \neq y\}.$$ 

We also require that $V \cap E = \emptyset$ to avoid notational ambiguities. We denote the vertex set of a graph $G$ by $V(G)$ and the edge set by $E(G)$. Moreover, for any cardinal $\alpha$ we denote the complete graph on $\alpha$ points by $K_{\alpha}$.

For any graph $G$, disjoint subsets $S,T \subseteq V(G)$ are said to be connected to each other if there are $s \in S, t \in T$ with $\{s,t\} \in E(G)$. Note that $K_{\alpha}$ is a minor of a graph $G$ if and only if there is a collection $\{S_{\beta} : \beta \in \alpha\}$ of nonempty, connected and pairwise disjoint subsets of $V(G)$ such that for all $\beta, \gamma \in \alpha$ with $\beta \neq \gamma$ the sets $S_{\beta}$ and $S_{\gamma}$ are connected to each other. We will need the following observation later on:

**Fact 1.1.** For any graph $G$, finite or infinite, the following are equivalent:

1. $G$ is connected;
2. if $S,T \subseteq V(G)$ are nonempty and disjoint such that $S \cup T = V(G)$ then $S,T$ are connected to each other.

2. The construction

In [1], Hadwiger formulated his well-known and deep conjecture, linking the chromatic number $\chi(G)$ of a graph $G$ with clique minors. His conjecture can be formulated that $K_{\chi(G)}$ is a minor of $G$ for every graph $G$. In the following we present a connected graph $H$ with chromatic number $\omega$ such that $K_{\omega}$ is not a minor of $H$. Let $\mathbb{N}$ be the set of positive integers. For any $n \in \mathbb{N}$ we let

$$C_n = \{1, \ldots, n\} \times \{n\}$$
and set \( V(H) = \bigcup_{n \in \mathbb{N}} C_n \). As for the edge set of \( H \), we define
\[
E(H) = \{(1, n), (1, n + 1) : n \in \mathbb{N}\} \cup \bigcup_{n \in \mathbb{N}} P_2(C_n).
\]

**Proposition 2.1.** \( \chi(H) = \omega \).

*Proof. Since we have \( \text{card}(V(H)) = \omega \) we get \( \chi(H) \leq \omega \). Moreover, each \( C_n \) is a complete subgraph of \( H \), so \( H \) cannot be colored with finitely many colors. \( \square \)

For the remainder of this note, we assume that \( \{S_n : n \in \omega\} \) is a collection of nonempty, connected, pairwise disjoint subsets of \( H \) such that for \( m \neq n \) the sets \( S_n, S_m \) are connected to each other. Our goal is to show that such a collection cannot exist.

First, we need a simple observation on what a connected subset of \( H \) looks like. If \( S \subseteq V(H) \) we define \( I(S) = \{n \in \mathbb{N} : C_n \cap S \neq \emptyset\} \).

**Lemma 2.2.** Suppose \( S \subseteq V(H) \) is connected and \( m < n \in I(S) \). Then for all \( x \in \mathbb{N} \) with \( m \leq x \leq n \) we have \( (1, x) \in S \).

*Proof. If \((1, m) \notin S\) then \( T = S \cap C_m \) and \( S \setminus T \) are disjoint, nonempty and not connected to each other. By Fact 1.1, \( S \) is not connected, contradicting our assumption. A similar argument shows that \((1, n) \in S\). Suppose there is \( x \) with \( m < x < n \) and \((1, x) \notin S\). Then set \( T = \{(i, j) \in S : j < x\} \). Again, \( T \) and \( S \setminus T \) are nonempty and not connected to each other, so \( S \) is not connected, contradicting our assumption. \( \square \)

If \( \{S_n : n \in \omega\} \) is a collection of subsets of \( V(H) \) as described above, then for every \( k \in \mathbb{N} \) the set of neighbors of \( S_k \), which is denoted by \( N(S_k) \), must be infinite. As the next lemma shows, this implies that \( I(S_k) \) must be infinite for all \( k \in \mathbb{N} \).

**Lemma 2.3.** If \( S \subseteq V(H) \) is such that \( I(S) \) is finite, then \( N(S) \) is finite.

*Proof. Let \( m = \max(I_S) \). Then \( N(S) \subseteq \bigcup_{i=1}^{m+1} C_i \), which is a finite set. \( \square \)

Now we go back to our assumption that \( \{S_n : n \in \omega\} \) is a collection of nonempty, connected, pairwise disjoint subsets of \( H \) such that for \( m \neq n \) the sets \( S_n, S_m \) are connected to each other. We consider just two of these sets, say \( S_0, S_1 \). Because of lemma 2.3 the sets \( I(S_0) \) and \( I(S_1) \) are infinite. For \( k = 0, 1 \) let \( \mu_k = \min(I(S_i)) \). We may assume that \( \mu_0 \leq \mu_1 \). Since \( I(S_0) \) is infinite, there is \( n \in I(S_0) \) with \( n \geq \mu_1 \). So lemma 2.2 implies that \((1, \mu_1) \in S_0 \cap S_1\), contradicting the assumption that the \( S_k \) are pairwise disjoint. So we established:

**Proposition 2.4.** The complete graph \( K_\omega \) is not a minor of \( H \).

**References**

[1] Hadwiger, Hugo, *Über eine Klassifikation der Streckenkomplexe*, Vierteljschr. Naturforsch. Ges. Zürich, 88 (1943), 133–143.