Quantum Cloning using Protective Measurement

C S Sudheer Kumar1,*

1NMR Research Center, Dept. of Physics, Indian Institute of Science Education and Research, Pune 411008, India

Here we show that, in principle it is possible to clone (measure) a single arbitrary unknown quantum state of a spin-1/2 particle (an electron) with arbitrary precision and with success probability tending to one, using protective measurement. We first transfer the information from spin to spatial degree of freedom (d.o.f) of system electron, then trap it in a double well potential, and finally measure it protectively using a probe electron which does not get entangled with system electron, but still extracts expectation value of an observable from a single quantum system (system electron) to obtain information about the unknown spin polarization. Nonorthogonal state discrimination being a subclass of cloning, part of the paper (till finding out θm, polar angle corresponding to the unknown spin polarization) is sufficient for discrimination.

I. MOTIVATION

"There is no law other than the law that there is no law...All is mutable" — J A Wheeler [1]. Then how about the ‘no-cloning’ law? Assume there exists a unitary operator U which can clone two non-orthogonal states as follows:

\[ U |0\rangle |0\rangle = |0\rangle |0\rangle, \quad U |+\rangle |0\rangle = |+\rangle |+\rangle \quad (1) \]

where, \(|+\rangle = (|0\rangle + |1\rangle)/\sqrt{2}\) and, \(|0\rangle, |1\rangle\) are eigenkets of z-component of total spin angular momentum (Sz).

Now taking the inner product of two equations in 1, we get

\[ 1 = \sqrt{2}, \] which is wrong. Hence, generalizing, we can say, it is impossible to clone an arbitrary unknown quantum state (which is non-orthogonal to the reference state \(|0\rangle\)), in eq. 1, of course excluding \(|0\rangle\) via a single unitary operation, which is the essence of no-cloning theorem. Now instead consider the following operation:

\[ D |0\rangle |0\rangle = |0\rangle |0\rangle, \quad D |+\rangle |0\rangle = |+\rangle |+\rangle, \quad (2) \]

\[ D = \sum_{k,l=0}^{3} h_{kl} \sigma_k \otimes \sigma_l \quad (3) \]

where, \(D\) is an arbitrary linear operator on the Hilbert space, \(H_2 \otimes H_2\), of two spin-1/2 particles. Operators \(\{1/2 \sigma_k \otimes \sigma_l; k, l = 0, 1, 2, 3\}\), where, \(\sigma_0 = 1\) (2 x 2 identity matrix) and \(\sigma_i\)’s \((i \neq 0)\) are Pauli matrices, form an orthonormal basis in the product Liouville space \(L \otimes L\) of operators on \(H_2 \otimes H_2\) [2]. \(h_{kl}\) are the coefficients of decomposition of \(D\) in the basis \(\{1/2 \sigma_k \otimes \sigma_l\}\). Therefore, we have 16 unknown complex coefficients \(h_{kl}\) i.e., 32 real coefficients and 16 constraint equations (8 real + 8 imaginary, obtained from eq.s in 2). Hence we can express 16 real coefficients in terms of 16 other arbitrary real parameters. This shows there exists infinitely many solutions to equations in 2 (for one such solution see [3]). \(D\) being sum of unitary operators, \(\sigma_k \otimes \sigma_l\), is a non-unitary operator in general and specifically in eq. 2. It may not be possible to directly implement the operator \(D\) in an experiment (however, there might be some indirect way). As there are infinitely many solutions to eqs in 2, many of them may be just mathematical objects with no relevance to physical operations. Now let us generalize the case in eq. 2:

\[ D |0\rangle |0\rangle = |0\rangle |0\rangle, \quad D |\hat{m}\rangle |0\rangle = |\hat{m}\rangle |\hat{m}\rangle \quad (4) \]

\[ D = \sum_{k,l=0}^{3} h_{kl} \sigma_k \otimes \sigma_l \quad (5) \]

where, \(|\hat{m}\rangle\) (given by eq. 8) is the unknown state to be cloned. Again there exists infinitely many solutions to eq.s in 4 (for one such solution see [4]). In eq. 4, if \(|\hat{m}\rangle = |1\rangle\) then, \(D |1\rangle |0\rangle = |1\rangle |1\rangle\). But, \(D\) is a linear operator. Hence,

\[ D |\hat{m}\rangle |0\rangle = \cos \frac{\theta_m}{2} D |0\rangle |0\rangle + \sin \frac{\theta_m}{2} e^{i\phi_m} D |1\rangle |0\rangle \quad (6) \]

\[ = \cos \frac{\theta_m}{2} |0\rangle |0\rangle + \sin \frac{\theta_m}{2} e^{i\phi_m} |1\rangle |1\rangle \neq |\hat{m}\rangle |\hat{m}\rangle \quad (7) \]

Hence, if \(D\) is linear, we cannot clone an arbitrary unknown state [5]. Hence, \(D\) must be a nonlinear operator (i.e., eq. 6 does not hold) apart from being nonunitary, to clone an arbitrary unknown quantum state. The mysterious quantum measurement process, which involves amplifying information to the classical limit, is one such operation which is both nonlinear [6] [7] and nonunitary (as information is irreversibly amplified to the classical limit which increases entropy, eg., electron absorbed by the screen in Stern-Gerlach experiment). Hence, it motivates us to ask the following question: ‘Is it possible to clone an arbitrary unknown quantum state, through a combination of linear unitary and nonlinear nonunitary (measurement) operations ?’. We found the answer to be ‘Yes’. In the protocol that we are going to describe, we carry out many such measurements (nonlinear nonunitary), eg., on probe electrons (which do not get entangled with system electron, but still extracts information from it!), by relaxing both unitarity and linearity constraints, which allows us to clone. In the following protocol, even though operations on system electron are linear unitary, operations on probe electrons and other auxiliary systems are nonlinear nonunitary, and clubbing

* sudheer.kumar@students.iiserpune.ac.in
informations obtained via measurements (which is some kind of sum of unitaries which is nonunitary in general) to prepare another qubit in the state same as that of given qubit, and to get a final state similar to that on RHS of eq. 4, makes the global (system+probes+observer) process nonlinear nonunitary.

Finally we like to mention that, in the cloning protocol that we are going to describe, in principle we can clone a single arbitrary unknown quantum state of a spin-1/2 particle (electron) with success probability tending to one (but not exactly equal to one, and hence there is probability, however small, of failure) and with arbitrary precision (which, strictly speaking, does not correspond to an exact or identical copy, as there is an error, however small). Hence, strictly speaking, it is not perfect cloning (exactly zero error) with success probability exactly equal to one.

II. CLONING PROTOCOL

Protective Measurement was invented by Y Aharonov and L Vaidman in their seminal paper [8]. The cloning protocol that we are going to describe is motivated by an idea proposed by Y Aharonov, J Anandan and L Vaidman in [9]. Let the system be an electron which is a spin-1/2 particle with charge $-e$ and mass $M$. Let its spin be in an arbitrary unknown state:

$$|\hat{m}\rangle = \cos \frac{\theta_m}{2} |0\rangle + \sin \frac{\theta_m}{2} e^{i\phi_m} |1\rangle$$

where $\hat{m}$ is a unit vector along unknown spin polarization given by,

$$\hat{m} = \sin \theta_m \cos \phi_m \hat{i} + \sin \theta_m \sin \phi_m \hat{j} + \cos \theta_m \hat{k}$$

where, $\hat{i}, \hat{j}, \hat{k}$ are unit vectors along positive $x, y, z$ axes respectively of fixed lab frame, $\theta_m$ and $\phi_m$ are polar and azimuthal angles respectively. For convenience, let the system electron’s wave packet be a Gaussian centered at $z_0 = 0$, $p_{0z}$ (average momentum) = 0, which satisfies $\Delta z \Delta p_z = h/2$, say, at time $t = 0$. $\Delta z$ and $\Delta p_z$ are standard deviations in position and momentum respectively. The variance $\Delta p_z^2$, can be Squeezed (at the cost of increasing $\Delta z^2$) and made small enough (but not zero, because, if zero then we have a plane wave, which is not square integrable and hence, rigorously, cannot represent a physical state of system electron (p23 [10])) to be taken as zero for all practical purposes. For eq., if $\Delta z$ is of the order of $10^{-3}m$ then $\Delta p_z$ will be of the order of $10^{-31}kg m s^{-1}$, which can be treated as zero in conjunction with eq. 29, for all practical purposes. Also, in principle we can make $\Delta p_z$ arbitrarily close to zero (but not exactly zero, there by satisfying square integrability requirement). To compensate for the corresponding increase in $\Delta z$, in principle we can make width ‘a’ of potential well (Fig.1), arbitrarily large as explained below eq. 31. Hence, in principle we can push the system electron into one of the eigenstates of $H_{0S}(12)$ or $H_S$ (18), as discussed below eq.s 14 and 28 respectively. As the wave packet evolves, it spreads in position space but not in momentum space. Its average momentum $(p_{0z})$ as well as momentum dispersion $\Delta p_z$, do not change with time. Hence, at $t > 0$, $\Delta z \Delta p_z > h/2$. Because $p_{0z} = 0$, even $z_0$ (center of wave packet in position space) do not change with time (pp 64-65 of [10]). Now an inhomogeneous magnetic field along $z$-axis is switched on for an interval of time $\tau$. System electron evolves under the total Hamiltonian $p_z^2/(2M) + H_{int}$, where, interaction Hamiltonian, $H_{int} = -\mu \cdot B_z k = -\gamma B_z S_z$, $B_z$ is the constant gradient $(Tesla m^{-1})$ in magnetic field along $z$-axis [11], $\gamma$ is the gyromagnetic ratio of electron which is negative. As $p_{0z} = 0$, in the limit $\Delta p_z \rightarrow 0$ (case of our interest, see below eq. 31). $p_z^2/(2M) \rightarrow 0$. However, as it evolves under $H_{int}$, it gains kinetic energy. We can choose static field $B_0$ [11] such that actual interaction energy $H_{int}^\prime \gg p_z^2/(2M)$ for any $B_0$, $\tau$ [12]. Hence, we can neglect the evolution under $p_z^2/(2M)$, and write:

$$|\psi(\tau,p_z)\rangle = \exp(-\frac{i}{\hbar}H_{int}\tau)|\hat{m}\rangle \tilde{\chi}(p_z - 0) =$$

$$\cos \frac{\theta_m}{2} |0\rangle \tilde{\chi}(p_z - \gamma B_1 \hbar/2 \tau) + \sin \frac{\theta_m}{2} e^{i\phi_m} |1\rangle \tilde{\chi}(p_z + \gamma B_1 \hbar/2 \tau)$$

where $\tilde{\chi}(p_z - 0)$ is the wave packet of system electron in momentum space centered at $p_{0z} = 0$ (see eq. 40 for definition of wave packet), which has split and got entangled with its spin d.o.f. Taking its inverse Fourier transform and finding its time evolution, we see that they are still entangled and the split packets are moving in opposite directions. Gradient $B_1$ can be made either negative (p388 of [10]) or positive. Let $B_1 < 0$. Then, system electron is in a superposition state of having momentum $\gamma B_1 \hbar/2 \tau$ (hence moving along positive z-axis) with probability $\cos^2 \frac{\theta_m}{2}$ and having momentum $-\gamma B_1 \hbar/2 \tau$ (hence moving along negative z-axis) with probability $\sin^2 \frac{\theta_m}{2}$. Let’s trap the system electron in a symmetrical double well potential (trapping procedure explained below eq. 31) with the following description(see for eq.,[10, 13]),

$$V_I(z) = \begin{cases} 
0, & \text{for } (b-a/2) < z < (b+a/2), \\
0, & \text{for } -(b+a/2) < z < -(b-a/2) \\
\infty, & \text{everywhere else}
\end{cases}$$

where, $a$ is the length of each potential well and $b$ is the distance from origin of coordinate system to center of potential well(see Fig.1). $V_I(z)$ is the potential experienced by system electron only, but not by any other charges outside the potential well.
Due to entanglement with spin d.o.f. Similarly, which implies $H$ describes the superposition of infinite number of plane waves $\exp(\frac{i}{\hbar} \vec{p} \cdot \vec{r})$ and hence its average kinetic energy is $\frac{p^2}{2M} = \frac{1}{2M}(\gamma B_i \frac{\hbar}{2} \tau)^2$, where all the parameters are known. We can choose the values of free parameters $a, b, B_i$ and $\tau$ (as explained below eq. 14), such that the system electron enters one of the eigenstates of the system Hamiltonian:

$$H_{0S} = \frac{p^2}{2M} + V_I(z)$$

where, $V_I(z)$ is as given in eq. 11. State of the system electron after trapping it in double well potential is given by,

$$|\Phi_{0sn}(z)\rangle = \cos \frac{\theta_m}{2} |0\rangle \left[ \theta(z - (b - \frac{a}{2})) - \theta(z - (b + \frac{a}{2})) \right] \sqrt{\frac{2}{a}} \sin \left( k_{0sn} (b + \frac{a}{2} - z) \right)$$

$$+ \sin \frac{\theta_m}{2} e^{i \phi_m} |1\rangle \left[ \theta(z + (b + \frac{a}{2})) - \theta(z + (b - \frac{a}{2})) \right] \sqrt{\frac{2}{a}} \sin \left( k_{0sn} (b + \frac{a}{2} + z) \right)$$

$$|\Phi_{0n}(z)\rangle = \sin \frac{\theta_m}{2} e^{i \phi_m} |1\rangle \left[ \theta(z + (b + \frac{a}{2})) - \theta(z + (b - \frac{a}{2})) \right] \sqrt{\frac{2}{a}} \sin \left( k_{0sn} (b + \frac{a}{2} + z) \right)$$

(13)

where, $k_{0sn} = \frac{n \pi}{a}$, $n = 1, 2, 3, \ldots$. Unit step function is defined as,

$$\theta(z - z_0) = \begin{cases} 0, & \text{for } z < z_0 \\ 1, & \text{for } z > z_0 \end{cases}$$

(14)

which implies $\frac{d}{dz} \theta(z - z_0) = \delta(z - z_0)$, Dirac-Delta function. We note that, even though $H_{0S}$ commutes with parity operator, its eigenstate $|\Phi_{0sn}(z)\rangle$ is not symmetric due to entanglement with spin d.o.f. Similarly $|\Phi_{0sn}(z)\rangle$ (15) is not antisymmetric. Eq. 13 describes the superposition state of system electron being in upper and lower potential wells, while entangled with its spin degree of freedom. When $\theta_m = 0$ or $\pi$, there is no splitting of the wave packet and eq. 13 reduces to the correct form describing the unsplit wave packet inside the potential well.

Inside the double well potential (excluding the boundary points, as derivative of the wave function does not exist there) $H_{0S} = \frac{p^2}{2M} = -\frac{k^2}{2M} \frac{d^2}{dz^2}$ and it is easy to check, using $\frac{d}{dz} \theta(z - z_0) = \delta(z - z_0)$, that the state $|\Phi_{0sn}(z)\rangle$ given in eq.13 is an eigenket of $H_{0S}$ with eigenvalue $E_n = \frac{k^2}{2M}$.

We can make the system electron to enter the eigenstate $|\Phi_{0sn}(z)\rangle$ of $H_{0S}$, by choosing $E_n = \frac{p^2}{2M} = \frac{1}{2M}(\gamma B_i \frac{\hbar}{2} \tau)^2$ and following an argument similar to that carried out from eq. 28 to 31 with the limit $V_0 \to \infty$ (here we also take the limit $\Delta p_z \to 0$. However, this is not necessary if we follow a different technique as explained in para next to next to eq. 31). Hence, in principle we can push the system electron into one of the eigenstates of $H_{0S}$.

Wavefunction in eq. 13 is Square Integrable unlike the plane waves $\exp(\frac{i}{\tau} \vec{p} \cdot \vec{r})$ and hence represents the actual physical state of system electron. Hence there is no need to construct a wave packet using the eigenkets $\{ |\Phi_{0sn}(z)\rangle \}$, to represent the physical state of system electron inside the potential well, unlike the situation when the electron was free. This justifies our assumption that system electron enters the eigenstate given in eq. 13, inside the potential well. This can be further justified by the fact that, an electron in a bound state of hydrogen atom can exist in one of the eigenstates, even though it was in a superposition of infinite number of plane waves $\exp(\frac{i}{\tau} \vec{p} \cdot \vec{r})$ when it was unbounded.

**Completeness of eigenstates of $H_{0S}$** Using eq.13, taking the inner product of $|\Phi_{0sk}(z)\rangle$ with $|\Phi_{0sn}(z)\rangle$ which involves integration w.r.t $z$ from $-\infty$ to $+\infty$, we get $\langle \Phi_{0sk}(z) | \Phi_{0sn}(z) \rangle = \delta_{kn}$, where $\delta_{kn}$ is the kronecker delta function, and hence orthonormal. In eq. 13 if we introduce a relative phase ($e^{i\tau}$) between two terms (which is equivalent to changing $\theta_m$ to $-\theta_m$, we get a normalized solution which is linearly independent from $|\Phi_{0sn}(z)\rangle$. Then using Gramm-Schmidt orthogonalization, the solution becomes:
\[
|\Phi_{0an}(z)\rangle = \sin \frac{\theta_m}{2} |0\rangle \left[ \theta(z - (b - \frac{a}{2})) - \theta(z - (b + \frac{a}{2})) \right] \sqrt{\frac{2}{a}} \sin \left( k_{0an}(b + \frac{a}{2} - z) \right) \\
- \cos \frac{\theta_m}{2} e^{i\phi_m} |1\rangle \left[ \theta(z + (b + \frac{a}{2})) - \theta(z + (b - \frac{a}{2})) \right] \sqrt{\frac{2}{a}} \sin \left( k_{0an}(b + \frac{a}{2} + z) \right)
\]

where \( k_{0an} = \frac{2\pi n}{a} \), \( n = 1, 2, 3, \ldots \). One can verify that \( |\Phi_{0an}(z)\rangle \) is an eigenket of \( H_{QS} \), with eigenvalue same as that corresponding to the state \( |\Phi_{0an}(z)\rangle \), and hence degenerate with it. One can also verify that:

\[
\sum_{n=1}^{\infty} \sqrt{\frac{2}{a}} \sin \left( \frac{n\pi}{a} z \right) \sqrt{\frac{2}{a}} \sin \left( \frac{n\pi}{a} z' \right) [\theta(z) - \theta(z - a)] [\theta(z') - \theta(z' - a)] = \delta(z' - z) [\theta(z) - \theta(z - a)] [\theta(z') - \theta(z' - a)]
\]

(16)

where, \( \delta(z' - z) \) is the Dirac delta function, one can verify that:

\[
\sum_{n=1}^{\infty} \left( |\Phi_{0an}(z)\rangle \langle \Phi_{0an}(z')| + |\Phi_{0an}(z)\rangle \langle \Phi_{0an}(z')| \right) = |0\rangle \langle 0| \delta(z' - z) \left[ \theta(z - (b - \frac{a}{2})) - \theta(z - (b + \frac{a}{2})) \right] \\
\left[ \theta(z' - (b - \frac{a}{2})) - \theta(z' - (b + \frac{a}{2})) \right] + |1\rangle \langle 1| \delta(z' - z) \left[ \theta(z + (b + \frac{a}{2})) - \theta(z + (b - \frac{a}{2})) \right] \\
\left[ \theta(z' + (b + \frac{a}{2})) - \theta(z' + (b - \frac{a}{2})) \right] \Rightarrow \int_{-\infty}^{\infty} dz' \sum_{n=1}^{\infty} \left( |\Phi_{0an}(z)\rangle \langle \Phi_{0an}(z')| + |\Phi_{0an}(z)\rangle \langle \Phi_{0an}(z')| \right) = \mathbb{1}
\]

(17)

where, \( \mathbb{1} \) is a 2 \times 2 identity matrix, there by satisfying the completeness relation.

To measure a given system protectively, system should be in a non-degenerate eigenstate of system Hamiltonian. But the state \( |\Phi_{0an}(z)\rangle \), which we want to measure protectively, is a degenerate eigenstate of \( H_{QS} \), as discussed in previous paragraph. If we perturb the system by applying a small negative potential in the region \(-(b - \frac{a}{2}) \leq z \leq (b - \frac{a}{2})\), degeneracy will be lifted due to tunneling. Adding a small negative potential to a very large but finite potential(which is infinite for all practical purposes), lowers the net potential slightly, there by allowing tunneling. Instead of solving perturbatively (for perturbative treatment see [10]), we are going to treat the potential in the region \(-(b - \frac{a}{2}) \leq z \leq (b - \frac{a}{2})\) to be \( V_0(>0) \), which is finite and constant, and solve exactly. If necessary, later we can take \( V_0 \) to be very large but finite, to satisfy the condition of small perturbation to \( V_I(z) \) (11). Following treatment is similar to that on pp460-62 of [10]. We are going to find the eigenstates \( |\phi_{an}(z)\rangle, |\phi_{an}(z)\rangle \) of new system Hamiltonian:

\[
H_S = \frac{p^2}{2M} + V_F(z)
\]

(18)

with eigenvalues \( E_{sn} \) and \( E_{an} \) respectively, which are less than \( V_0 \). \( V_F(z) \) is same as \( V_I(z) \) given in eq. 11 but \( V_F(z) = V_0 \) in the region \(-(b - \frac{a}{2}) \leq z \leq (b - \frac{a}{2})\) instead of infinity.

As the system electron is in superposition of being in both wells, it can tunnel from both wells into the region \(-(b - a/2) \leq z \leq (b - a/2)\). As the split wave packets tunnel into the region \(-(b - a/2) < z < (b - a/2)\), they get united and hence disentangles from spin d.o.f. This is justified by the fact that, if we reverse the direction of inhomogeneous magnetic field (see above eq. 49), the split wave packets start moving towards the origin, get united and hence disentangles from spin d.o.f. With this requirement, \( |\Phi_{0an}(z)\rangle \) (13) suggests the following form for one of the eigenstates of \( H_S \) with energy \( E_s < V_0 \):
where, \( q_s = \sqrt{\frac{2M}{\hbar^2}(V_0 - E_s)} = \sqrt{\alpha^2 - k_s^2} > 0 \). We have related energy \( E_s \) of the state with its wave number \( k_s \) by the relation:

\[
E_s = \frac{\hbar^2 k_s^2}{2M} \tag{20}
\]

Exponentially decaying wave \( e^{\psi(z-b)} \) corresponds to tunnelling from upper potential well (i.e., potential well on positive \( z \)-axis) into the region \(-(b - \frac{a}{2}) \leq z \leq (b - \frac{a}{2})\) and \( e^{-\psi(z-(-b))} \) corresponds to that from lower well. Further, \( |\Phi_{0an}(z)\rangle \) (13) suggests to take \( A = A' = A_s \) and \( B = B' = B'_s \) in eq. 19. Evanescent wave reduces to the form: \( B e^{\gamma_s(z-b)} + B' e^{-\gamma_s(z-b)} = B_s \cos(q_s z) \), where \( B_s = 2B_s' e^{-\gamma_bb} \). It is evident that in the limit \( V_0 \) (hence \( q_s \)) going to infinity, both evanescent waves vanish.

and we recover the state in eq. 13 as required. Demanding that the spatial wave function and its first derivative must be continuous at \( z = b - \frac{a}{2} \), we obtain the following constraint equation:

\[
\tan(k_s a) = -\frac{k_s}{\sqrt{\alpha^2 - k_s^2}} \coth \left( \sqrt{\alpha^2 - k_s^2} (b - \frac{a}{2}) \right) \tag{21}
\]

This follows from the fact that \( A_s \) and \( B_s \) cannot vanish simultaneously, else we get trivial solution \( |\phi_s(z)\rangle = 0 \) everywhere. Also in obtaining eq. 21 we dropped unit step function which is justifiable as it is used just for the sake of convenience. Similar joining conditions at \( z = -(b - \frac{a}{2}) \), also gives same constraint eq. 21.

Now, \( |\Phi_{0an}(z)\rangle \) (15) suggests the following form for another eigenstate of \( H_S \) (which is linearly independent from \( |\phi_s(z)\rangle \)) 19 with energy \( E_a < V_0 \):

where, \( q_a = \sqrt{\frac{2M}{\hbar^2}(V_0 - E_a)} = \sqrt{\alpha^2 - k_a^2} > 0 \), \( E_a = \hbar^2 k_a^2/(2M) \). \( |\Phi_{0an}(z)\rangle \) (15) suggests to take \( A' = -A = -A_a \) and \( B' = -B = -B'_a \). With this, evanescent wave takes the form: \( B e^{\gamma_a(z-b)} + B' e^{-\gamma_a(z-(-b))} = B_a \sinh(q_a z), \) where, \( B_a = 2B_a' e^{-\gamma_bb} \). In the limit \( V_0 \to \infty \), \( |\phi_a(z)\rangle \) (22) reduces to \( |\Phi_{0an}(z)\rangle \) (15) as required. Applying joining conditions at \( z = b - \frac{a}{2} \) in a manner similar to previous case, we obtain the following constraint equation:

\[
\tan(k_a a) = -\frac{k_a}{\sqrt{\alpha^2 - k_a^2}} \tanh \left( \sqrt{\alpha^2 - k_a^2} (b - \frac{a}{2}) \right) \tag{23}
\]

Joining conditions at \( z = -(b - \frac{a}{2}) \) also leads to same constraint eq. 23. Equations 21 and 23 can be solved graphically for \( k_s \) and \( k_a \) respectively, to obtain \( k_{sn} \) and \( k_{an} \) as \( n \)th roots, there by quantizing the energy levels.

As constraint eqs. 21 and 23 are not identical in form, \( k_{sn} \), will not be equal to \( k_{an} \). Then, using relation 20 and \( E_a = \hbar^2 k_a^2/(2M) \), we obtain \( E_{sn} \not= E_{an} \), there by lifting degeneracy. In the limit \( V_0 \to \infty \), we obtain from eqs. 21 and 23: \( k_{sn} \to \frac{\pi}{a} \) as required. As \( q_{sn,an} > 0 \) we have \( E_{sn,an} < V_0 \). Eigenstates of \( H_S \), given by eqs. 19 and 22, take the following form after quantization:

\[
|\phi_{sn}(z)\rangle = |\phi_{sn}(z)\rangle + |\hat{m}\rangle B_{sn} \cos(q_{sn} z) \left[ \theta(z - (b - \frac{a}{2})) - \theta(z - (b - \frac{a}{2})) \right],
\]

\[
|\phi_{an}(z)\rangle = |\phi_{an}(z)\rangle + (\sin \frac{\theta_m}{2}(0) + \cos \frac{\theta_m}{2} e^{i\phi_m(1)}) B_{an} \sinh(q_{an} z) \left[ \theta(z + (b - \frac{a}{2})) - \theta(z - (b - \frac{a}{2})) \right] \tag{24}
\]

where, \( |\hat{m}\rangle \) is given by eq. 8, and:
In the limit $V_0 \to \infty$, we obtain $|\phi_{sn}(z)| \to |\phi_{0sn}(z)|$ and $|\phi_{an}(z)| \to |\Phi_{0an}(z)|$ as required. One can verify that $\langle \phi_{ak}(z)|\phi_{sn}(z)\rangle = 0$, for all $k,n$. However, $\langle \phi_{ak}(z)|\phi_{sn}(z)\rangle \neq 0$ and $\langle \phi_{ak}(z)|\phi_{an}(z)\rangle \neq 0$, for $k \neq n$.

We can make them zero via Gram-Schmidt orthogonalization. Arbitrary constants $A_{sn}, A_{an}, B_{sn}$ and $B_{an}$ can be fixed by the requirement that each of the states $\{|\phi_{sn}(z)|, |\phi_{ak}(z)|\}$, for all $s,k$, be normalized and satisfy completeness relation along with the states $\{|\phi_i'(z)|\}$, where $|\phi_i'(z)| = \langle \tilde{n}\rangle \sqrt{\frac{2}{\pi \hbar^2}} \cos(\frac{\pi \tilde{n}}{\hbar r_0} z)$ is an eigenstate of $H_S$ with eigenvalue $E_i > V_0$. Also $\langle \phi_{i}(z)|\phi_{an,n}(z)\rangle \neq 0$. Taking the state of our main interest, $|\phi_{ak}(z)|$, as reference, we can orthogonalize the complete set $\{|\phi_{ak}(z)|, |\phi_{sn}(z)|, |\phi_i'(z)|\}$ via Gram-Schmidt orthogonalization. The set $\{|\phi_{ak}(z)|, |\phi_{an}(z)|, |\phi_i'(z)|\}$, after orthonormalization, should satisfy the completeness relation analogous to that in eq. 17, i.e.,

$$\int_{-\infty}^{\infty} dz' \sum_{k=1}^{M} \left( |\phi_{ak}(z)| \langle \phi_{ak}(z') | + |\phi_{ak}(z') | \langle \phi_{ak}(z') | \right) +$$

$$\sum_{i=2M+1}^{\infty} |\phi_i'(z)| \langle \phi_i'(z') | = \int_{-\infty}^{\infty} dz' \sum_{i=1}^{\infty} |\phi_i'(z)| \langle \phi_i'(z') | = 1 \tag{26}$$

where, $\langle \phi_{ak}(z)| = |\phi_{ak-1}'(z)|$, $|\phi_{ak}(z)| = |\phi_{ak}'(z)|$, $|\phi_i'(z)| = |\phi_i'(z)| \tag{27}$

We can justify this by the fact that, in the limit $V_0 \to \infty$ we recover the completeness relation in eq. 17, which corresponds to $V_f(z)$ (11). By perturbing $V_f$ with a small negative potential, we obtain the case of finite potential, $V_E(z)$ (18). As perturbation does not destroy the completeness property of the set of states being perturbed, completeness relation in eq. 26 is justified.

**Protective Measurement:** An important requirement to do protective measurement is, system should be initially in a nondegenerate eigenstate of system Hamiltonian, $H_S$. Energy of system electron, just before trapping it in double well potential is:

$$\frac{1}{2M} (p_{0z} - \Delta p_z)^2 = \frac{1}{2M} (\gamma B_i \frac{\hbar}{2} \tau \pm \Delta p_z)^2$$

$$= \frac{1}{2M} (\gamma B_i \frac{\hbar}{2} \tau)^2 + \Delta E \tag{28}$$

where, $p_{0z}$ is the average momentum, $\Delta p_z$ is the standard deviation in momentum and

$$\Delta E = \frac{1}{2M} (\Delta p_z^2 \pm 2\gamma B_i \frac{\hbar}{2} \tau \Delta p_z) \tag{29}$$

is the uncertainty in kinetic energy of system electron. Let us consider the ground state $|\phi_{s1}(z)|$ (24) (as $E_{sn}$ is less than $E_{an}$ [10]). From eq. 20, $E_{s1} = \frac{\hbar^2 k_{s1}^2}{2M}$. But eq. 21 says, $k_{s1}$ depends on the potential well parameters $a, b, V_0$. To push the system electron into the state $|\phi_{s1}(z)|$, following condition must be satisfied,

$$E_{s1}(a, b, V_0) = \frac{1}{2M} (\gamma B_i \frac{\hbar}{2} \tau)^2 \tag{30}$$

Even if the values of $a, b, V_0$ in $E_{s1}(a, b, V_0)$ are fixed by some requirements like: compensating increase in $\Delta z$ as $\Delta p_z$ is squeezed (see below eq. 31), etc., constraint eq. 30 can still be satisfied by varying the free parameters $B_i$ and $\tau$. Hence, we can say, energy of system electron, just before trapping is $E_{s1} + \Delta E$. In general, state of system electron after trapping it in double well potential will be,

$$|\phi_{dup}(z)| = c_1 |\phi_{s1}(z)| + \sum_{i \neq 1} c_i |\phi_i'(z)| \tag{31}$$

where, $|\phi_i'(z)|$ is given by eq. 27. However, system electron starts with zero average momentum as mentioned below eq. 9. At $t = 0$, $\Delta z \Delta p_z = \hbar/2$. In principle we can start at $t = 0$ with squeezed wave packet such that $\Delta p_z \to 0$. Switch on the inhomogeneous magnetic field for an interval of time $\tau$ such that it gains an average kinetic energy $E_{s1}(a, b = a/2 + \epsilon, V_0)$ ($\epsilon > 0$, to avoid blowing up of cot hyperbolic function in eq. 21) as given by the constraint eq. 30. Now adiabatically change the Hamiltonian from $p_z^2/(2M)$ to $H_S$ (18) with $b = a/2 + \epsilon$. By this, energy of system electron will not change, as we are introducing potential barriers only. With a’ of well is suitably chosen so as to take care of increase in $\Delta z$ due to squeezing of $\Delta p_z$. In principle a’ can be arbitrarily large. With this we obtain: $\Delta E \to 0$, $|c_1| \to 1$ and $|c_i| \to 0$ ($i \neq 1$) with state of system electron being $|\phi_{s1}(z)|$ corresponding to $b = a/2 + \epsilon$ [14]. Now again adiabatically change the Hamiltonian from $H_S$ with $b = a/2 + \epsilon$ to $H_S$ with $b = a/2 + \epsilon + \mathcal{M}$, where $\mathcal{M}(\geq 0)$ is large but finite i.e., separate the two wells very slowly. Hence, in principle, ‘b’ can also be made arbitrarily large. Here, energy of system electron changes. Hence, by adiabatic principle (see below eq. 36), system
electron will end up in the state $|\phi_{s1}(z)\rangle$ corresponding to $b = a/2 + c + M$. Hence, \textit{in principle, we can push the system electron into the nondegenerate eigenstate, $|\phi_{s1}(z)\rangle$}, of system Hamiltonian $H_S$.

Or, it is sufficient for the system electron to have average kinetic energy (which is $\frac{1}{4\pi\epsilon_0} \langle \gamma B_{\parallel} \rangle^2$) sufficiently greater than the ground state energy $E_{s1}$, which is exactly known. As, all systems have natural tendency to settle down to their ground state, as it is most stable and has least possible energy, if we wait sufficiently long, system electron will settle down to the ground state $|\phi_{s1}(z)\rangle$ \cite{15,16}. Hence, in this method we need not take the limit $\Delta p_z \to 0$, unlike in the previous method. Hence, by either of these two methods, we can push the system electron into a nondegenerate eigenstate $|\phi_{s1}(z)\rangle$ \cite{24} of system Hamiltonian $H_S$ \cite{18}. Even if we trap system electron in the eigenstate $|\phi_{sn}(z)\rangle$, $n > 1$, by satisfying the constraint: $E_{sn} = \frac{1}{4\pi\epsilon_0} \langle \gamma B_{\parallel} \rangle^2$, it won't be stable due to vacuum fluctuations and ultimately decays to ground state, $|\phi_{s1}(z)\rangle$.

We now carry out protective measurement on system electron trapped in the double well potential, using a probe electron which passes in-between the two wells. As it passes, it interacts with system electron through coulomb potential energy. Even though we have trapped system electron in 1-D potential well, it can still interact with probe electron moving in $x$-$z$ plane (let positive $x$-axis be along positive $V_f$-axis in Fig. 1), via coulomb potential energy. Trapped electron is analogous to line charge density (Coulomb per meter), where the potential well is opaque to matter wave (hence charges cannot escape from metal surface) but transparent to electric field and hence they can interact with other free charges outside the well. Only trapped system electron sees infinite potential barrier in between the two wells, but not the probe electron \cite{17}.

As stated in \cite{9}, adiabatic condition required to do protective measurement will be satisfied, assuming system electron is initially in the nondegenerate energy eigenstate $|\phi_{sn}(z)\rangle$, provided the probe electron crosses the potential well in a time $T$, which is large compared to $\hbar/|E_{sn} - E_{an}|$, where, $E_{sn}, E_{an}$ are the energies of the states $|\phi_{sn}(z)\rangle, |\phi_{an}(z)\rangle$ respectively. As the probe electron is moving very slowly, we can neglect the magnetic field induced due to motion of charge. In a more rigorous calculation we can treat it as low energy Quantum Electrodynamical (QED) problem. However, here we are going to approximate it as quantum electrostatic problem. Let the probe electron have an initial constant momentum, $p_{i0}z_k$, where $i$ is an unit vector along $z$-axis. As it measures the system electron protectively, it gains component of momentum along $z$-axis, and hence drifts along, say, positive $z$-axis. As a result, potential energy of interaction, $H_{int}^a = \frac{1}{4\pi\epsilon_0} \frac{e^2}{|\vec{r} - \vec{r}_a(t)|}$, where $\vec{r}$ is the position vector of system electron and $\vec{r}_a(t)$ that of probe electron. However, if the separation between wells, $2(b - a/2)$, is sufficiently large (as justified below eq. \ref{31} and time interval $T$ relatively small (i.e., $T >> \hbar/|E_{sn} - E_{a1}|$ still holds) \cite{18}, we can neglect the change in $H_{int}^a$ during the time interval $T$ and approximate it as:

$$H_{int}^a \approx \frac{1}{4\pi\epsilon_0} \frac{e^2}{|z - z_a|}$$ \hspace{1cm} (32)

with, center of wave packet of probe electron: $z_{a0} = 0$. It is as if probe electron is fixed exactly in between the two potential wells.

As the separation between two wells, $2(b - a/2)$, can be made arbitrarily large as justified below eq. \ref{31}, $1/|z - z_a|$ can be made arbitrarily small for every $z$ such that: $(b - a/2) \leq z \leq (b + a/2)$ and $-(b + a/2) \leq z \leq -(b - a/2)$. Hence $H_{int}^a$ can be made arbitrarily small compared to $|E_{sn} - E_{an}|$ (also see last para of \cite{19}, and second para of \cite{20}), there by satisfying the weak interaction requirement for protective measurement, i.e., $H_{int}^a$ is so weak that it cannot cause transition between the states $|\phi_{sn}\rangle$ and $|\phi_{an}\rangle$. We can vary $|E_{sn} - E_{an}|$ by varying $V_0$.

As we are interested only in the change in $z$-component of probe electron’s momentum, we can neglect its other components. With this and the electrostatic approximation in eq. \ref{32}, we can take free evolution Hamiltonian of probe electron, $H_a = p_{az}^2/(2M) = 0$ \cite{21}. Hence, System and probe electrons evolve under the Hamiltonian:

$$H = H_S + \frac{1}{4\pi\epsilon_0} \frac{e^2}{|z - z_a|}$$ \hspace{1cm} (33)

where, $H_S$ is the system Hamiltonian given in eq. \ref{18}. $z$ is the position coordinate corresponding to system electron and $z_a$ that of probe electron. State of combined system at time $t = T$ is:

$$|\zeta(T)\rangle = \exp \left(-\frac{i}{\hbar} \left(H_S + \frac{1}{4\pi\epsilon_0} \frac{e^2}{|z - z_a|} T \right)\right) |\phi_{s1}(z)\rangle \Theta(p_{az})$$ \hspace{1cm} (34)

where, $|\phi_{s1}(z)\rangle$ is given by eq. \ref{24}, $\Theta(p_{az} - 0)$ is the Gaussian wave packet of probe electron in momentum space centered at $p_{az} = 0$. Following Hari Dass and Tabish \cite{15}, we insert completeness relation similar to that in eq. \ref{26}, to obtain:

$$|\zeta(T)\rangle = \int_{-\infty}^{\infty} dz_a \int_{-\infty}^{\infty} dz'| \sum_{k=1}^\infty \exp \left(-\frac{i}{\hbar} \left(H_S + \frac{1}{4\pi\epsilon_0} \frac{e^2}{|z - z_a|} T \right) \right) \delta(z_a)(|\phi_k(z)\rangle \langle \phi_k(z')|) |\phi_{s1}(z')\rangle \Theta(p_{az})$$ \hspace{1cm} (35)
where, hat on \( z, a \) stands for operator, \( \delta(z_a) \) is the Dirac delta function and \( |\phi_{s1}(z)| \) has been changed to \( |\phi_{s1}(z')| \) because see[22]. If, \( \hat{A}a_i = a_i|a_i| \) then \( f(\hat{A})a_i = f(a_i)|a_i| \). Using this property we can push \( \delta(z_a) \) to the extreme left. Then we obtain:

\[
(\hat{H}_{s} + \frac{e^2}{4\pi\epsilon_0} \frac{1}{|\vec{z} - z_a|}) |\phi_{k}(z)| = \hat{E}_k(z_a) |\phi_{k}(z)|,
\]

where,

\[
\hat{E}_k(z_a) = \langle \hat{H}_{s}\rangle \phi_{k} + \frac{e^2}{4\pi\epsilon_0} \phi_{k}(z) |\frac{1}{|\vec{z} - z_a|}| \phi_{k}(z) \tag{36}
\]

As discussed in the beginning of ‘Protective Measurement’, criteria of adiabaticity and weak interaction between system and probe electrons are satisfied. According to Adiabatic theorem, at \( t = 0 \) if the system electron is in a nondegenerate eigenstate of the system Hamiltonian \( \hat{H}_{s} \) (which we take to be the ground state, \( |\phi_{s1}(z)| \)), then, if \( \hat{H}_{s} \) changes very slowly to \( H = \hat{H}_{s} + \hat{H}_{sa} \) at \( t = T \), such that \( T >> (\hbar/|E_{s1} - E_{a1}|) \) (adiabatic approximation, eq. 10.15 of [23]), then \( |\phi_{s1}(z)| \) changes to the corresponding nondegenerate eigenstate of \( H \) (with \( \hat{z}_a \) replaced by \( z_a \), as in eq. 36) i.e., ground state \( |\phi_{s1}(z)| \)

\[
|\zeta(T)\rangle = \int_{-\infty}^{\infty} dz_a \delta(z_a) \exp \left( -\frac{i}{\hbar} (\hat{H}_{s}\phi_{s1}(z) - T) \right) \exp \left( \frac{-i}{\hbar} \frac{1}{4\pi\epsilon_0} e^2 \phi_{s1}(z) \frac{1}{|\vec{z} - z_a|} \phi_{s1}(z) \right) \Theta(p_{az})
\]

Let, \( f(\hat{z}_a) = \langle \phi_{s1}(z)| \frac{1}{|\vec{z} - z_a|} |\phi_{s1}(z)\rangle \) \tag{39}

Using the definition of wavepacket (p1462 of [10]), we obtain:

\[
\exp \left( -\frac{i}{\hbar} \frac{1}{4\pi\epsilon_0} e^2 f(\hat{z}_a) \right) \Theta(p_{az}) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} dz_a \Theta(z_a) \exp \left( -\frac{i}{\hbar} \frac{1}{4\pi\epsilon_0} e^2 f(z_a) T + p_{az} z_a \right) \tag{40}
\]

Wave packet of probe electron in position space is centered at \( z_0 = 0 \) (see below eq. 32) and center won’t change with time, as \( p_{az} = 0 \) (see above eq. 10). Expanding \( f(z_a) \) around \( z_a = z_0 = 0 \), we obtain, \( f(z_a) = f(0) + f'(0) z_a + O(z_a^2) \), where, \( f'(0) = \frac{d}{dz_a} f(z_a) |_{z_a=0} \). If the spreading of wave packet is small during the time interval \( T \) \[25\] \( \Theta(z_a) \), a Gaussian wave packet, is appreciable only in the interval: \( -\epsilon < z_a < \epsilon, \epsilon > 0 \), such that \( \epsilon^2 \) and higher order terms are negligible i.e., \( \Theta(\epsilon + \delta) \cong 0 \), where, \( \delta > 0 \). Hence in integral (40), \( f(z_a) \) can be approximated as: \( f(z_a) \cong f(0) + f'(0) z_a \). Eq. 40 reduces to \[26\]:

\[
\exp \left( -\frac{i}{\hbar} \frac{1}{4\pi\epsilon_0} e^2 f(\hat{z}_a) \right) \Theta(p_{az}) = \exp \left( -\frac{i}{\hbar} \frac{1}{4\pi\epsilon_0} e^2 f(0) T \right) \Theta(p_{az}) = \exp \left( -\frac{i}{\hbar} \frac{1}{4\pi\epsilon_0} e^2 f(0) T \right) \Theta(p_{az}) + \frac{1}{4\pi\epsilon_0} e^2 f(0) T \tag{41}
\]

Finally we obtain,

\[
|\zeta(T)\rangle = \exp \left( -\frac{i}{\hbar} \left( \hat{H}_{s}\phi_{s1} + \frac{1}{4\pi\epsilon_0} e^2 f(0) T \right) \right) |\phi_{s1}(z)\rangle \Theta(p_{az}) \tag{42}
\]

State of system electron is unaltered while probe electron has gained an average momentum: \( -\frac{1}{4\pi\epsilon_0} e^2 f'(0) T \), and we are now going to show that \( f'(0) \) is proportional to \( \cos \theta_m \). Substituting eq.24 into eq.39, we obtain:

\[
f(z_a) = \langle \phi_{0s1}(z)| \frac{1}{|\vec{z} - z_a|} \langle \phi_{0s1}(z)\rangle \tag{43}
\]

\[
+ \int_{-\delta}^{\delta} dz \cos^2 (q_{s1}z) B_{s1}^2 \frac{1}{|\vec{z} - z_a|} = h(z_a) + g(z_a) \tag{43}
\]
In eqs 43 and 45, integration variable is $z$, and hence $z_a$ is a constant as far as integration is concerned. We want to know the rate at which the function $f(z_a)$ changes, w.r.t $z_a$, at $z_a = 0$. Hence w.r.t $z_a = 0$ (the value of our interest) we can write:

$$|z - z_a| = \begin{cases} z - z_a, & \text{for } z \geq 0 \\ -(z - z_a), & \text{for } z < 0 \end{cases} \quad (44)$$

However, we want $f'(0)$ but not $f(0)$. Hence, we are going to first differentiate $f(z_a)$ w.r.t $z_a$ and then put $z_a = 0$. Using (44) and substituting for $|\phi_{0a}(z)|$ from eq.25 and integrating w.r.t $z$ from $-\infty$ to $+\infty$, we obtain,

$$h(z_a) = \langle \phi_{0a}(z) | \frac{1}{|z - z_a|} | \phi_{0a}(z) \rangle$$

$$= A_s^2 \cos^2 \frac{\theta_m}{2} \int_{b - \frac{q}{2}}^{b + \frac{q}{2}} dz \sin^2(k_1(b + \frac{q}{2} - z))$$

$$+ A_s^2 \sin^2 \frac{\theta_m}{2} \int_{b - \frac{q}{2}}^{b + \frac{q}{2}} dz \sin^2(k_1(b + \frac{q}{2} - z)) \quad (45)$$

Differentiating $h(z_a)$ w.r.t $z_a$ in eq. 45, using Leibnitz rule for differentiation under the integral sign, and then taking $z_a = 0$, we obtain,

$$h'(0) = \cos \theta_m A_s^2 \int_{b - \frac{q}{2}}^{b + \frac{q}{2}} dz \frac{\sin^2(k_1(b + \frac{q}{2} - z))}{z^2} \quad (46)$$

Similarly,

$$g(z_a) = B_s^2 \lim_{\epsilon \to 0} \int dz \left( \frac{\cosh^2(q_1z)}{z + z_a} + \frac{\cosh^2(q_1z)}{z - z_a} \right) \quad (47)$$

To avoid pole at $z = z_a = 0$, we have introduced the limit. As before, differentiating $g(z_a)$ in eq. 47, w.r.t $z_a$, and then taking $z_a = 0$, we get $g'(0) = 0$. Using eq. 43, we obtain $f'(0) = h'(0)$ (46). From eq. 42 we obtain the final average momentum of probe electron (whose initial average momentum was zero) to be:

$$\hat{P}_{dio}^{final} = -\frac{e^2 T A_s^2}{4 \pi \epsilon_0} \cos \theta_m \int_{b - \frac{q}{2}}^{b + \frac{q}{2}} dz \frac{\sin^2(k_1(b + \frac{q}{2} - z))}{z^2} \quad (48)$$

In principle we can measure momentum of probe electron (and hence $\theta_m$) with arbitrary precision [19]. We note that, both $T \to \infty$ and separation between two potential wells tending to infinity are necessary. In the latter limit, the integral in eq. 48 tends to zero (as integration is over only upper well), there by keeping the product, $T \times integral$, finite. We also note that, in an exact calculation (i.e., without doing any approximations), the functional form of eq. 48 may change, but still it will be a function of $\theta_m$. In eq. 48, except $\cos \theta_m$ all other terms are greater than zero. If the unknown $\theta_m$ happens to be zero, there is no splitting of wave packet and the system electron will be in upper well for $B_1 < 0$, according to eq. 10. Hence, probe electron experiences maximum downward force(repulsion) and hence gains maximum negative momentum, which is consistent with $p_{dio}^{final}$ in eq. 48. If $\theta_m = \frac{\pi}{2}$, wave packet splits equally and hence probe electron experiences equal and opposite forces, gaining zero net momentum. $p_{dio}^{final}$ in eq. 48, also gives same result. Finally if $\theta_m = \pi$, again no splitting of wave packet, but this time system electron will be in lower well. As a result, probe electron experiences maximum upward repulsive force, gaining maximum positive momentum. Again $p_{dio}^{final}$ in eq. 48 is consistent with it.

If we just want to discriminate between two nonorthogonal states, say $|0\rangle$ and $(|0\rangle + |1\rangle)/\sqrt{2}$, then we have succeeded in discriminating, as it requires only the knowledge of polar angle $\theta_m$. However, if we want to clone the unknown state completely, we need to still find out its azimuthal angle, $\phi_m$ (8). As the state of system electron is unaltered at the end of protective measurement (42), and because the trapping process was adiabatic (see below eq. 31), it must be reversible, and hence we can get back the state in eq. 10, by adiabatically untrapping (procedure for untrapping is same as for trapping (below eq. 31), but in reverse). Now we can recombine the split wave packets [20] by reversing the direction of inhomogeneous magnetic field which was used initially for splitting i.e., subjecting it to a field: $\vec{B} = B\vec{B} = -B_z x i + (B_0 + B_z) \hat{k}$ (refer [11]), where, $B_1 > 0$ (note that while splitting, $B_1$ was < 0). We recover the initial state, $|\psi(0, p_z)\rangle = |\hat{m}\rangle |\hat{p}_z - 0\rangle$ (10), which had $\Delta p_z \to 0$, but with $\Delta z \Delta p_z > h/2$ [27]. Polarization, $\hat{m}$, of unknown spin state is unaltered w.r.t lab frame. Previously we had applied inhomogeneous magnetic field along positive $z$-axis of lab frame. Now we apply it along a unit vector,

$$\hat{n} = \sin \theta_n \cos \phi_n \hat{i} + \sin \theta_n \sin \phi_n \hat{j} + \cos \theta_n \hat{k} \quad (49)$$

where, $\theta_n \neq 0$ (zero is the value corresponding to previously applied direction of inhomogeneous magnetic field) and $\phi_n \neq 0$. From eqs 49 and 9 we obtain the following constraint equation:

$$\hat{m} \cdot \hat{n} = \cos \theta_{mn} = \cos \theta_m \cos \theta_n + \cos(\phi_m - \phi_n) \sin \theta_m \sin \theta_n \quad (50)$$

Component of unknown spin angular momentum operator along $\hat{n}$ is, $S_n = \hat{S} \cdot \hat{n}$, where $\hat{S} = S_x \hat{i} + S_y \hat{j} + S_z \hat{k} = S_m \hat{m}$, $S_m$ is the component along itself. Eikenkoets of operator $S_n$ are:

$$|+\rangle_n = \cos \frac{\theta_n}{2} |0\rangle + \sin \frac{\theta_n}{2} e^{i\phi_n} |1\rangle$$

and $$|-\rangle_n = -\sin \frac{\theta_n}{2} |0\rangle + \cos \frac{\theta_n}{2} e^{i\phi_n} |1\rangle \quad (51)$$
with eigenvalues $\frac{\hbar}{2}$ and $-\frac{\hbar}{2}$ respectively. In the eigenbasis of $S_n$, unknown spin state $\langle 8 \rangle$ can be decomposed as:

$$|\hat{m}\rangle = \cos \frac{\theta_{mn}}{2} |+\rangle_n + \sin \frac{\theta_{mn}}{2} e^{i\phi_{mn}} |\rangle_n$$

(52)

where, $\theta_{mn}$ is given by eq. 50 and $\phi_{mn}$ is the azimuthal angle of the unit vector $\hat{n}$, in a coordinate system rotated w.r.t the fixed lab frame, in which $\hat{n}$ is along it’s positive z-axis. Inhomogeneous magnetic field along $\hat{n}$ is switched on for an interval of time $\tau$. System electron evolves under the interaction Hamiltonian $H_{int} = -\hat{m}.B_t q_n \hat{n} = -\gamma B_t q_n S_n$, where $q_n$ is the distance measured from origin along the direction $\hat{n}$. State of system electron after interacting with inhomogeneous magnetic field is given by:

$$\psi_{\hat{n}}(\tau, p_n) = \cos \frac{\theta_{mn}}{2} |+\rangle_n \chi(p_n - \gamma B_t \frac{\hbar}{2} \tau) + \sin \frac{\theta_{mn}}{2} e^{i\phi_{mn}} |\rangle_n \chi(p_n + \gamma B_t \frac{\hbar}{2} \tau)$$

(53)

where, $p_n$ is the component along $\hat{n}$ of momentum of system electron i.e., momentum conjugate to $q_n$. As before, trapping the system electron in a double well potential and measuring it protectively with a probe electron, we obtain the value of $\cos \theta_{mn}$. Hence we obtain the constraint eq. 50 with the only unknown $\phi_{mn}$. However we cannot obtain the value of $\phi_{mn}$ unambiguously from a single constraint equation for the following reason: Consider the constraint equation, $\cos \eta = c$, $-1 \leq c \leq 1$. If $\eta$ takes a value in the interval $[0, \pi]$ (just like $\theta_{mn}$), then there is only one value of $\eta$ which satisfies $\cos \eta = c$. On the other hand, if $\eta$ takes a value in the interval $[0, 2\pi]$ (just like $\phi_{mn}$), then there are two values of $\eta$ which satisfies $\cos \eta = c$. Hence we require one more constraint equation independent of the constraint eq. 50, to find out $\phi_{mn}$ unambiguously. This is justified by the fact that, to unambiguously characterize an arbitrary orientation in space, we require three Euler angles, which are linearly independent of each other. Whereas, here we are trying to characterize the unknown direction, $\hat{m}$, with just two parameters $\theta_{mn}$ and $\phi_{mn}$, and hence the ambiguity.

After obtaining the value of $\cos \theta_{mn}$, we shall again recombine the split wave packets, as before. Applying an inhomogeneous magnetic field along the direction $\hat{l}$ given by,

$$\hat{l} = \sin \theta_l \cos \phi_l \hat{i} + \sin \theta_l \sin \phi_l \hat{j} + \cos \theta_l \hat{k}$$

(54)

where, $\theta_l \neq 0$, $\theta_l$ is the interval applied directions of inhomogeneous magnetic field) and $\phi_l \neq 0$. By this we obtain another independent constraint equation:

$$\hat{m} \cdot \hat{l} = \cos \theta_{ml} = \cos \theta_m \cos \theta_l + \cos(\phi_m - \phi_l) \sin \theta_m \sin \theta_l$$

(55)

Through protective measurement we can obtain the value of $\cos \theta_{ml}$. Solving eqs 50 and 55 for $\phi_{ml}$ and picking out the solution common to both of them, we obtain an unambiguous value of $\phi_{mn}$.

### III. MISCELLANEOUS

**Discriminating between $|0\rangle$ and $|+\rangle = (|0\rangle + |1\rangle)/\sqrt{2}$ even when the initial state is in a linear superposition of nondegenerate eigenstates of $H_S$ (18):** Let the initial state of system and probe electron be: $|\zeta(0)\rangle = \sum_i c_i |\phi_i''(z)\rangle \Theta(p_{az})$, where, $|\phi_i''(z)\rangle$ is defined in eq. 27. Hence we need not worry about pushing the system electron into one of the eigenstates of $H_S$, unlike in cloning. Hence discrimination is more feasible than cloning. Allowing $|\zeta(0)\rangle$ to evolve under the Hamiltonian $H$ given in eq. 33, for an interval of time $T$, and inserting completeness relations we obtain:

$$|\zeta'(T)\rangle = \sum_{i=1}^{\infty} c_i \int_{-\infty}^{\infty} dx_{\alpha} \int_{-\infty}^{\infty} dx'_{\alpha} \sum_{k=1}^{\infty} \exp \left( -\frac{i}{\hbar} \left( H_S + \frac{1}{4\pi \epsilon_0} \frac{e^2}{|z - z'|} \right) T \right) \delta(z_{\alpha}) \langle \phi_k(z) | \phi_k(z') \rangle |\phi_i''(z')\rangle \Theta(p_{az})$$

(56)

This is similar to the state in (35) but with an extra summation over the index ‘i’. As we are doing Protective Measurement, criteria of adiabaticity and weak interaction are satisfied and hence as before (below eq. 36) we can take, $|\phi_k(z)\rangle \approx |\phi_{i\alpha}(z)\rangle$. Proceeding in a fashion similar to that from eq. 35 to 48, we obtain,

$$|\zeta'(T)\rangle = \sum_{i=1}^{\infty} c_i \exp \left( -\frac{i}{\hbar} \left( (H_S)_{\phi_i''} + \frac{1}{4\pi \epsilon_0} e^2 f_i(0) T \right) \right) |\phi_i''(z)\rangle \Theta(p_{az} + \frac{1}{4\pi \epsilon_0} e^2 f_i(0) T)$$

(57)
where,
\[
f'_i(0) = \cos \theta_m \ A_i^2 \int_{\theta_m}^{\theta_m + \frac{\pi}{2}} \frac{dz}{2} \sin^2(k_i(b + \frac{a}{2} - z))
\]  

(58)  

If the given state is \( |+\rangle \), then \( \theta_m = \frac{\pi}{2} \) and hence \( f'_i(0) = 0 \) for all \( i \). As a result, there is no entanglement between system electron and probe electron (eq. 57) and probe electron gains zero momentum. This makes sense, because, when \( \theta_m = \frac{\pi}{2} \), wave packet of system electron splits equally. As a result probe electron experiences equal and opposite repulsive forces, gaining no net momentum. However if the state is \( |0\rangle \), then \( \theta_m = 0 \) and hence \( f'_i(0) \) is strictly greater than zero as \( A_i \) being normalization and completeness satisfying constant, cannot be zero and \( k_i \) (20) which corresponds to the energy of the \( i^{th} \) level, cannot also be zero. Hence in this case, probe electron gains non-zero momentum (of course, in this case there will be collapse upon measurement as there is entanglement, but it does not matter as we are interested only in discrimination). Also, wave packet of system electron doesnot split and hence it will be in the upper well.

As a result probe electron experiences downward repulsive force, there by gaining down ward momentum. By measuring the momentum of probe electron (which can be measured as precisely as we want, at the cost of loosing information about its position, which is not required for discrimination, any way) we can always discriminate between the states \( |0\rangle \) and \( |+\rangle \) unambiguously. However, we obtained eq. 58 under the approximation given in expression 32 and a few other approximations. Suppose there is a small correction \( \Delta f'_i(0) \) to \( f'_i(0) \) in eq. 58, under an exact calculation. By choosing \( a \) and \( b \) suitably, we can make the integral along with \( A_i^2 \) in eq. 58 as large as we want. When \( \theta_m = \frac{\pi}{2}, \ f'_i(0) = \Delta f'_i(0) \), where as when \( \theta_m = 0, \ f'_i(0) \gg \Delta f'_i(0) \). Hence, we can still discriminate between \( |0\rangle \) and \( |+\rangle \) always. However, we note that, \( \Delta f'_i(0) \) may also increase as the integral in eq. 58 increases, in which case above argument breaks down. Also it may happen that under an exact calculation, there might be entanglement between system electron and probe electron even for \( \theta_m = \frac{\pi}{2} \), in which case we cannot discriminate, as probe electron also gains non-zero momentum. One has to do exact calculation and see.

How Not to Clone: It’s not possible to clone an arbitrary unknown spin state \( |\tilde{m}\rangle \) (8) by protecting it with a strong homogeneous magnetic field \( (\tilde{B}_0)^2 \). As the direction, \( \tilde{m} \), of unknown spin polarization is not known a priori, we cannot apply \( \tilde{B}_0 \) along \( \tilde{m} \), in general. Hence we choose the direction of \( \tilde{B}_0 \) to be along \( \hat{k} \) (positive z-axis of fixed lab frame) i.e., \( \tilde{B}_0 = \tilde{B}_0 \hat{k} \). Apply an inhomogeneous magnetic field \( \tilde{B}_T \theta \hat{n} \), where, \( \theta \) is the distance from origin measured along \( \hat{n} \) (49), \( \tilde{B}_T \) is the field gradient in Tesla m\(^{-1}\) and in this context, \( T \) is a dimensionless number which controls the gradient strength. As, \( \tilde{B}_0 \) can be arbitrarily large, we can neglect \( p_z^2/(2M) \) compared to \( | - \mu \cdot \tilde{B}_0 \hat{k} | \). Hence, \( |\tilde{m}\rangle \) evolves under the Hamiltonian,

\[
H = -\mu \cdot \tilde{B}_0 \hat{k} - \mu \cdot B_1 \frac{q_n}{T} \tilde{n} = -\mu (B_0' \hat{k} + B_1 \frac{q_n}{T} \tilde{n})
\]

\[
= -\gamma \tilde{S} \cdot \tilde{B} = -\gamma \tilde{S} \cdot \tilde{B}
\]  

(59)  

where, \( \tilde{S} \) is the total spin angular momentum of unknown spin. \( S_B = \tilde{S} \cdot \tilde{B} \), where \( \tilde{B} \) is a unit vector along resultant magnetic field \( \tilde{B} \). In the eigenbasis of the operator \( S_B \), state of unknown spin-\( \frac{1}{2} \) particle at time \( t = T \) is,

\[
|M'(T)\rangle = \cos \frac{\theta B_m}{2} e^{-\frac{\gamma B_n T}{\hbar}} |+\rangle_B \exp(-\frac{i}{\hbar}(-\gamma B) \frac{T}{2} |+\rangle_B B_n |\tilde{m} \rangle + \sin \frac{\theta B_m}{2} e^{\frac{\gamma B_n T}{\hbar}} |+\rangle_B \exp(\frac{i}{\hbar}(-\gamma B) \frac{T}{2} |+\rangle_B B_n |\tilde{m} \rangle)
\]

(60)

where \( p_n \) is the component of momentum of unknown spin-\( \frac{1}{2} \) particle along \( \hat{n} \), \( \cos \theta B_m = B \cdot \tilde{m} \). In the limit \( T \to \infty \), which corresponds to adiabatic and weak interaction limit, we obtain: \( B = |\tilde{B}| \cong B_0 + \frac{\tilde{B}_T}{\hbar} \theta n \cos \theta m \). But \( \cos \theta n = \frac{\tilde{n}}{\tilde{B}}(S_n |0\rangle, \text{ where, } S_n = \tilde{S} \cdot \tilde{n} \text{ and } \tilde{n}) \) is the eigenket of \( S_z \). In this limit, state \( |M'(T)\rangle \) becomes:

\[
|M'(T \to \infty)\rangle = e^{-\frac{\gamma B_n T}{\hbar}} \cos \frac{\theta B_m}{2} |+\rangle_B \chi \left( p_n - \gamma B_1 |S_n \rangle_0 \right)
\]

(61)

In the limit \( T \to \infty \),

\[
\tilde{B} \cdot \tilde{m} = \cos \theta B_m \cong \cos \theta m + \frac{B_1 q_n}{B_0 T} (\cos \theta m - \cos \theta n \cos \theta n)
\]

(62)

where, \( \cos \theta m \) is given by eq. 50. Consider the case where, the direction of unknown spin polarisation, \( \tilde{m} \), happens to accidentally coincide with that of static field direction, \( \hat{k} \), and hence ‘rightly protected’ (this is the case considered in [9] as mentioned by themselves in [28]). In this case \( \theta m = 0 \) and we obtain \( \cos \theta B_m \cong 1 \Rightarrow \theta B_m \cong 0, \ B \cong \tilde{m} \cong \hat{k}, \ |+\rangle_B \cong |\tilde{m} \rangle = |0\rangle \) and the state 61 becomes,

\[
|M'(T \to \infty)\rangle = e^{-\frac{\gamma B_n T}{\hbar}} (0) \chi \left( p_n - \gamma B_1 |S_n \rangle_0 \right)
\]

(63)

There is no entanglement in eq.63 and the unknown spin state is unaltered. Upon measuring the momentum of spin-\( \frac{1}{2} \) particle, we obtain: \( \gamma B_1 |S_n \rangle_0 = \gamma B_1 |S_n \rangle_0 |\tilde{m} \rangle \). Consider now the case where, the direction of unknown spin polarisation, \( \tilde{m} \), donot coincide (which is highly highly probable) with that of static field direction, \( \hat{k} \), and hence ‘wrongly protected’. In this case, \( \cos \theta B_m \neq 1 \) as evident from (62) (this is true even if \( \theta m = 0 \) i.e., \( \tilde{m} \) accidentally coincides with \( \tilde{n} \)) and hence spin and spatial d.o.f are entangled as given by eq. 61. Upon measuring the momentum of particle in momentum basis, we obtain either, momentum \( \gamma B_1 |S_n \rangle_0 \) and spin state col-
lapses to $|+\rangle_B$ with probability $\cos^2 \frac{\theta_m}{2}$, or momentum $-\gamma B_z\langle S_n \rangle|0\rangle$ and spin state collapses to $|−\rangle_B$ with probability $\sin^2 \frac{\theta_m}{2}$. When we obtain the outcome $\gamma B_z\langle S_n \rangle|0\rangle$ there is no way to know if the spin state was rightly protected or wrongly protected, as it is common to both cases. However when we obtain $-\gamma B_z\langle S_n \rangle|0\rangle$, we come to know that the spin state was wrongly protected. In either of the cases we obtain no information about the unknown spin state, $|\hat{m}\rangle$. Hence it’s impossible to clone by this method.

**Cloning in Center of Wave Packet** (CWP) **Frame:** If we can somehow track the CWP of the split wave packets and measure its momentum, then we can perfectly clone an arbitrary unknown spin state, even with von-Neumann impulsive measurement, as then follows: Consider the unknown state $|\hat{m}\rangle$ given in 8. Switch on inhomogeneous magnetic field $B_z\hat{z}\hat{k}$ for an interval of time $\tau$. Spin and spatial states evolve under the interaction Hamiltonian,

$$H_{\text{int}} = -\hat{\mu}_B B_z\hat{k} = -\gamma \hat{S}_z B_z\hat{k} = -\gamma S_m\hat{m} B_z\hat{k}$$

$$=-\gamma B_z \cos \theta_m S_m \hat{z}\rangle = (\hat{m}) \gamma B_z \cos \theta_m \hat{z}\rangle (64)$$

where, $S_m = \hat{S}_z \hat{m}$, and $S_m |\hat{m}\rangle = \frac{\hbar}{m} |\hat{m}\rangle$. We are working in the eigenbasis of the operator, $S_m$. State at time $t = \tau$ is,

$$|\psi(\tau, p_z)\rangle = \exp(\frac{i}{\hbar} \gamma B_z \cos \theta_m S_m \hat{z}\rangle |\hat{m}\rangle \chi(p_z)$$

$$=-|\hat{m}\rangle \chi(p_z - \gamma B_z \cos \theta_m \frac{\hbar}{2} \tau) (65)$$

Hence CWP has gained $z$-component of momentum equal to $\gamma B_z \cos \theta_m \frac{\hbar}{2} \tau$. If we can somehow measure this momentum, we can come to know $\theta_m$ and there is no collapse. By following a procedure similar to that described from eq.s 49 to 55, we can find out $\phi_m$. However if we work in the eigenbasis of the operator $S_z$, we can observe the splitting. $H_{\text{int}}$ can be written in terms of $S_z$ as,

$$H_{\text{int}} = -\hat{\mu}_B B_z\hat{k} = -\gamma \hat{S}_z B_z\hat{k} = -\gamma B_z S_z \hat{z}\rangle (66)$$

and the state at time $t = \tau$ is exactly as given in eq.10, where we can observe splitting and hence entanglement.

**CONCLUSION**

In principle it is possible to do exact calculations without the approximations that we have made. Hence, we conclude that, in principle it is possible to clone a single arbitrary unknown quantum state of a spin-1/2 particle (an electron) with arbitrary precision and with probability tending to one. Of course, in practice we cannot make dimensions of double well potential, $a$ and $b$, and the time interval $T$, arbitrarily large, even though possible in principle. Hence, it reduces the precision and success probability of cloning. However we note that, even a few millimeters and seconds are like infinity on the atomic scale. For eg., in NMR we talk of adiabatic processes even though life time of spins that we prepare is only a few seconds [29]. Hence, to get a quantitative picture of precision and success probability attainable in practice, one has to do thorough calculation considering all practical limitations. Our protocol can be easily generalized to arbitrary charged, spin-1/2 particle. If it is not electrically charged, then one can exploit its gravitational potential energy, in principle, as splitting of wave packet is independent of charge, but depends only on spin. Protocol can also be generalized to higher spin ($> 1/2$) particles. Experimental realization of our protocol is with in the reach of current technology.

Non-orthogonal state discrimination is more feasible than cloning, as it requires only the knowledge of polar angle, $\theta_m$. Also, to discriminate between, say, $|0\rangle$ and $|+\rangle (= (|0\rangle + |1\rangle)/\sqrt{2})$, it is not necessary to find $\theta_m$ with arbitrary precision. So much error is allowed such that we can just unambiguously discriminate between $|0\rangle$ and $|+\rangle$. Here we note that, in spite of all practical constraints on $a$, $b$ etc., if we can still unambiguously discriminate between $|0\rangle$ and $|+\rangle$ in finite time $T$ (such that $T$ is less than time interval corresponding to two space-like separated events) with success probability greater than, say, 0.5, then our protocol may open up the doors to superluminal communication. To check this possibility, one has to do thorough analysis considering all practical constraints. We also note that, the limit $T \to \infty$ corresponds to discriminating between $|0\rangle$ and $|+\rangle$ with success probability tending to one, and it is not possible to do superluminal communication in this limit, as $T$ is greater than any finite time interval corresponding to two space-like separated events. But to communicate superluminally, we donot require success probability tending to one. Hence, there seems to be a possibility still.

**ACKNOWLEDGEMENT**

I am grateful to my guide Dr. T S Mahesh for proposing this idea and giving freedom to work on it. I am also thankful to him for many motivating discussions and ideas. I am also thankful to Prof. N D Hari Dass for introducing to protective measurements. I am grateful to Prof. Masanao Ozawa for valuable comments. I thank Govind Unnikrishnan for pointing to linearity.

[1] J. D. Barrow et al., *Science and Ultimate Reality* (Cambridge University Press, 2004).
[2] J. Audretsch, *Entangled Systems: New Directions in Quantum Physics* (Wiley, 2007).
However small, there may be nonlinear processes explicitly come into picture. There are still many such processes which directly or indirectly involve nonlinear processes (which are also very much part of cloning) like, solving constraint equation 50, 55, and, clubbing the information $\theta_n$, $\phi_n$ and preparing another qudit in the state $|\tilde{m}\rangle$. We also note that, according to Everett’s many-worlds interpretation and Bohm’s causal interpretation of quantum measurement [7], there is no breaking of superposition at the level of consciousness and matter respectively, and hence no nonlinearity. However, we believe that, as we cannot observe superposed states, some where nonlinearity has to come in, either at the level of matter or consciousness.

[7] D. Home, Conceptual Foundations of Quantum Physics chapter 2 (Springer Science and Business media, 1997).

[8] Y. Aharonov and L. Vaidman, Physics Letters A 178, 38 (1993).

[9] Y. Aharonov, J. Anandan, and L. Vaidman, Phys. Rev. A 47, 4616 (1993).

[10] C. Cohen Tannoudji et al., Quantum Mechanics vol. one and two (John Wiley and Sons, 2005).

[11] In an actual Stern Gerlach apparatus both $\vec{v} \times \vec{B} = 0$ and $\vec{v} \times \vec{B} = 0$. The field that satisfies these conditions is: $\vec{B} = BB = -B_0 z + i (B_0 + B_1 z)k$ (80), p388 of [10], where, $B_0 > 0$. For $B_0 > |B_1|$, $\vec{B} \cong k$ and hence spin precesses about $z$-axis with frequency $\Omega \cong -\gamma B_0$. As a result, $x$-component of spin, $S_x$, oscillates in time and hence its average over an interval of time large compared to period of oscillation $2\pi/\Omega$ but small compared to $\tau$, vanishes. Only $S_z$ survives. Hence wave packet splits only along $z$-axis. Due to the presence of static field $B_0 k$, unknown spin precesses about $z$-axis, which can be easily taken into account. Hence we are not going to consider the effect of static field.

[12] From the previous footnote it is evident that actual interaction energy, $H'_{int} = -\vec{p} \cdot \vec{B} = -\gamma B_0 S_z - \gamma B_1 S_x z + \gamma B_0 S_x$. Eigenvalues os $S_z$ are $\pm \hbar/2$. As the reference point can always be shifted, we consider only the difference in energy levels which is: $-\gamma B_0 \hbar > 0$ as $\gamma < 0$. Hence we can write: $H'_{int} = \gamma |B_0| \hbar - |\gamma B_1| |\hbar x| + |\gamma B_0| |\hbar x|$. As $B_0$ can be arbitrarily large, $H'_{int}$ can also be made arbitrarily large for any $B_0/k, x$. Hence, $H'_{int} > \hbar^2/(2M)$ holds for any $B_0, k, \tau$.

[13] L. Hardy, Physics Letters A 167, 11 (1992).

[14] This can also be justified as follows: Consider a Box normalized plane wave: $u_n(z) = L^{-1/2} \exp(i k_n z)$ with periodic boundary condition. It is the spatial wave function of system electron. It is an eigenstate of $p_x^2/(2M)$ with discrete energy $E_{n_z} = h^2 k_n^2 /2m/L$, where, $k_n = 2\pi n_z/L$, $n_z = 0, \pm 1, \pm 2, \ldots$, $L$ is the length of box, which is large but finite. Spacing between the energy levels can be made as small as we want by increasing $L$ [24]. Now switch on the inhomogeneous magnetic field for an interval of time $\tau$, such that $1/2 (\gamma B_0)^2 \tau^2 = E_{n_z} = E_{A}(a, b = a/2 + e, V_0)$. Change the Hamiltonian of system electron from $p_x^2/(2M)$ to $H_S$ with $b = a/2 + e$, in an interval of time $\tau = \hbar/E_{n_{z+1}} - E_{n_z}$. Then, by adiabatic theore (see below eq. 36), system electron will end up in the eigenstate of $H_S$: $|\phi_{A}(z)\rangle \Rightarrow |b = a/2, e\rangle$.

[15] N. D. Hari Dass and T. Qureshi, Phys. Rev. A 59, 2590 (1999).
For ex., imagine a three dimensional (3D) double box potential, where, potential inside the box is zero, while infinite on the six walls of box. Box is opaque to matter wave (as trapped electron cannot escape), while transparent to electromagnetic field. Now, to break the degeneracy, connect the two boxes with a long tube, such that it does not obstruct the motion of probe electron. Inside the tube, potential is finite \((V_0)\), whereas infinite on its walls. This is analogous to the potential \(V_F(z)\) (eq. 18). Our protocol can be generalized to 3D double box potential case.

As a first step, to get an approximate solution, we are assuming that \(T\) is relatively small, so that \(H_{int}\) becomes time independent. In a more accurate solution we can treat \(H_{int}\) as time dependent perturbation or handle it as a quantum electrodynamical problem, and we can handle arbitrarily large \(T\).

Initial state of probe electron (eq. 34) was a Gaussian \(\lambda\). Let \(\phi_1\) (relation 32) be the strength of \(\lambda\) (i.e., \(\Delta z\)). Then, maximum possible strength of \(\phi_1\) is \(1/(b-a/2-\Delta z)\) (upper limit of \(\lambda\)). Requirement that \(H_{int}\) should goto zero, gave: \(b-a/2 = n\Delta z\), \(n > 1\). Equating this with above numerical value of \(\lambda = |E_{11} - E_{a1}|^{m}\), \(m > 1\), we obtain: \(\Delta z = |E_{11} - E_{a1}|^{m} = 1/(n-1)\). This puts a constraint on the rate at which \(\Delta z\) should goto infinity or \(|E_{11} - E_{a1}|^{m}\) should goto zero i.e., the rates should be such that the product \(\Delta z\lambda = |E_{11} - E_{a1}|^{m}\) remains finite (= \(1/(n-1)\)). This can always be satisfied, for eq., taking numerical value of \(\lambda = |E_{11} - E_{a1}|^{m}\) = \(1/(n-1)\).

Similarly, adiabaticity condition: \(T > > h/|E_{11} - E_{a1}|\) (below eq. 46), can also be satisfied even in the limit \(|E_{11} - E_{a1}| > 0\), by choosing \(T\), which can also be arbitrarily large, numerically proportional to \(|E_{11} - E_{a1}|^{m}\), \(n > 1\). Hence, evanescent wave is not a problem while recombining.

Even if we start with probe electron being a Gaussian minimum (i.e., \(\Delta z\alpha\Delta p_{az} = \hbar/2\)) wave packet, with \(p_{0az} = 0\) and \(z_{0} = 0\) and \(\zeta_{0} = 0\). After the interaction with system electron, to measure the momentum gained by probe electron, shine it with a photon of wavelength \(\lambda\). By measuring the change in frequency, due to Doppler effect, of scattered photon we can calculate the velocity of probe electron. The change in momentum of probe electron (= \(\Delta p_{az}\), which is also the error in momentum measurement) due to collision with photon, is inversely proportional to \(\lambda\) (Compton effect [31]). In principle it is possible to take: \(\lambda \rightarrow \infty\). As a result, error: \(\Delta p_{az} \rightarrow 0\) and the measured value of momentum of probe electron tends to \(p_{0az}^{\text{int}}\). Of course, error in position measurement: \(\Delta z_{0}^{\text{int}} \rightarrow 0\) (as \(\Delta z_{0}^{\text{int}}\Delta p_{az}^{\text{int}} \geq \hbar/2\)), in which any way we are not interested. Also, even in the limit \(\Delta p_{az} \rightarrow 0\), it is possible to make \(H_{int}^{\text{a}}\) (32) goto zero as follows: Let the wave packet of probe electron be centered at \(z_{0} = 0\) with standard deviation \(\Delta z_{0}\). Then, maximum possible strength of \(H_{int}^{\text{a}}\) is proportional to \(1/(b-a/2-\Delta z_{0})\). As the separation between two potential wells can be made arbitrarily large, we can take \(b-a/2 = n\Delta z_{0}\), \(n > 1\). Now, in the limit \(\Delta z_{0} \rightarrow \infty\) (i.e., \(\Delta p_{az} \rightarrow 0\), \(H_{int}^{\text{a}}\) (32) demands width \(\alpha\) to be of the order of \(\Delta z\). Hence, \(b \approx \Delta z/2 + n\Delta z_{0}\). In the limit \(\Delta z_{0} \rightarrow \infty\) and \(\Delta z_{0} \rightarrow \infty\), \(b\) also tends to infinity. This can be satisfied as \(b\) can be arbitrarily large.

Let \(\lambda\) be the strength of \(H_{int}^{\text{a}}\) (relation 32). In the non-degenerate perturbation theory, first order correction to \(|\phi_{a1}\rangle >\) goes as \(\lambda/|E_{11} - E_{a1}|\) (p1101 of [10]). In the limit \(|E_{11} - E_{a1}| \rightarrow 0\) (i.e., \(V_0 \rightarrow \infty\)) evanescent wave goes to zero. As explained below relation 32, \(\lambda\) depends on separation between two wells, which can be made arbitrarily large, and hence \(\lambda\) arbitrarily small. We can choose the separation between two wells such that \(\lambda\) is numerically proportional to \(|E_{11} - E_{a1}|^{m}\), \(n > 1\). Now, in the limit \(|E_{11} - E_{a1}| > 0\), both first order correction to \(|\phi_{a1}\rangle >\) and evanescent wave goes to zero. Similarly we can prove all higher order corrections to \(|\phi_{a1}\rangle >\) (p154 of [24]) also goes to zero.

D. J. Griffiths, *Introduction to Quantum Mechanics* (Prentice Hall, 1995).

L. I. Schiff, *Quantum Mechanics* (McGraw-Hill Book Company, 1955).

However we note that, in principle it is possible to take into account, complete spreading of wave packet during
the time interval $T$. For eg., if we treat it as a QED problem, we get exact solution or do the integral (40) to all orders in $\epsilon$. However, to get an approximate solution, we consider spreading of wave packet only to first order in $\epsilon$.

\[ [26] \int_{-\infty}^{\infty} dz_a \Theta(z_a) \exp \left( \frac{-i}{\hbar} \left( \frac{1}{4\pi\epsilon_0} e^2 f(z_a) T + p_{ax} z_a \right) \right) \approx \int_{-\infty}^{\infty} dz_a \Theta(z_a) \exp \left( \frac{-i}{\hbar} \left( \frac{1}{4\pi\epsilon_0} e^2 f(z_a) T + p_{ax} z_a \right) \right) \approx \int_{-\infty}^{\infty} dz_a \Theta(z_a) \exp \left( \frac{-i}{\hbar} \left( \frac{1}{4\pi\epsilon_0} e^2 (f(0) + f'(0) z_a) T + p_{ax} z_a \right) \right). \]

[27] It is not an issue, as increase in $\Delta z$ can be compensated by increasing width ‘a’ of potential well, which can be arbitrarily large in principle.

[28] Y. Aharonov, J. Anandan, and L. Vaidman, Foundations of Physics 26, 117 (1996).

[29] J. A. Jones et al., Nature 403, 869 (2000).

[30] D. E. Platt, American Journal of Physics 60, 306 (1992).

[31] W. Heisenberg in, Quantum Theory and Measurement edited by J A Wheeler and W H Zurek p66 (Princeton university press, 1983).