HURWITZ SPACES OF QUADRUPLE COVERINGS OF ELLIPTIC CURVES AND THE MODULI SPACE OF ABELIAN THREEFOLDS $A_3(1,1,4)$

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Abstract. We prove that the moduli space $A_3(1,1,4)$ of polarized abelian threefolds with polarization of type $(1,1,4)$ is unirational. By a result of Birkenhake and Lange this implies the unirationality of the isomorphic moduli space $A_3(1,4,4)$. The result is based on the study the Hurwitz space $H_{d,n}(Y)$ of quadruple coverings of an elliptic curve $Y$ simply branched in $n \geq 2$ points. We prove the unirationality of its codimension one subvariety $H_{0,d,A}(Y)$ which parametrizes quadruple coverings $\pi: X \to Y$ with Tschirnhausen modules isomorphic to $A^{-1}$, where $A \in \text{Pic}^{n/2}Y$, and for which $\pi^*: J(Y) \to J(X)$ is injective. This is an analog of the result of Arbarello and Cornalba that the Hurwitz space $H_{d,n}(\mathbb{P}^1)$ is unirational.

Introduction

In the present paper we study the Hurwitz space $H_{d,n}(Y)$ which parametrizes simple quadruple coverings of an elliptic curve $Y$ branched in $n \geq 2$ points. There is a canonical smooth fibration $h: H_{d,n}(Y) \to \text{Pic}^{n/2}Y$ (see (2.2)). The fiber over $A$, which we denote by $H_{d,A}(Y)$, parametrizes the coverings whose Tschirnhausen module has determinant isomorphic to $A^{-1}$. When studying coverings of an elliptic curve of non-prime degree it is natural to consider coverings which satisfy the condition that $\pi_*: H_1(X,\mathbb{Z}) \to H_1(Y,\mathbb{Z})$ is surjective, or equivalently that $\pi^*: J(Y) \to J(X)$ is injective, since coverings for which $\text{Coker } \pi_* \neq 0$ are reduced to coverings of smaller degree via certain étale coverings $\tilde{Y} \to Y$. We denote by $H_{0,d,n}(Y)$ and $H_{0,d,A}(Y)$ the corresponding Hurwitz spaces.

One of the two main results of the paper states that $H_{0,4,n}(Y)$ is connected and $H_{0,4,A}(Y)$ is connected and unirational (Theorem 2.16). We mention that $H_{4,n}(Y)$ has three other connected components (see Remark 1 following Theorem 2.16). In our previous paper [Ka] we proved that $H_{d,A}(Y)$ is unirational for $d \leq 3$. We notice the analogy with the well-known result of Arbarello-Cornalba [AC] which states that $H_{d,n}(\mathbb{P}^1)$ is unirational if $d \leq 5$. O. Schreyer gave in [Sch] an alternative proof of the unirationality of $H_{d,n}(\mathbb{P}^1)$ for $d \leq 5$ by a method which was then developed by Casnati

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and Ekedahl in \[\text{CE}\] (for \(d = 4\)) and Casnati in \[\text{Ca}\] (for \(d = 5\)) and yields a description of Gorenstein coverings of degree 4 and 5 by means of a pair of vector bundles over the base, a connection analogous to the one found by Miranda for triple coverings \[\text{Mi}\]. In our proof of the unirationality of \(\mathcal{H}_{4,A}^0(Y)\) we use the result of \[\text{CE}\]. Atiya’s theory of vector bundles over elliptic curves \[\text{At2}\] is heavily used.

The way we prove the unirationality of \(\mathcal{H}_{4,A}^0(Y)\) suggests a construction by which we are able to prove the unirationality of \(A_3(1, 1, 4)\), the moduli space of abelian threefolds with polarization of type \((1, 1, 4)\), and this is our second main result (Theorem 3.5). We follow the same pattern by which in \[\text{Ka}\] we proved the unirationality of \(A_3(1, 1, d)\) for \(d \leq 3\). We consider a smooth elliptic fibration \(q : Y \to B \subset \mathbb{P}^1\) and construct a pair of vector bundles \(F\) and \(E\) over \(Y\) by which we construct a family of quadruple coverings with a rational base \(T\). One associates with such a family the Prym mapping \(\Phi : T \to A_3(1, 1, 4)\) and we deduce the unirationality of \(A_3(1, 1, 4)\) by proving that \(\Phi\) is dominant. According to a theorem of Birkenhake-Lange \[\text{BL}\] \(A_3(1, 1, 4) \cong A_3(1, 4, 4)\). Thus our result establishes the unirationality of \(A_3(1, 4, 4)\) as well. We refer to the introduction of \[\text{GP}\] the reader who may be interested about other results on the unirationality of moduli spaces of abelian varieties.

**Notation and conventions.** A morphism (or holomorphic mapping) \(\pi : X \to Y\) is called covering if it is finite, surjective and flat. Unless otherwise specified we make distinction between locally free sheaves and vector bundles and we denote differently their projectivizations. If \(E\) is a locally free sheaf of \(Y\) and if \(E\) is the corresponding vector bundle, i.e. \(E \cong \mathcal{O}_Y(E)\), then \(\mathbb{P}(E) := \text{Proj}(S(E)) \cong \mathbb{P}(E^*)\). Unless otherwise specified or clear from the context a curve is assumed to be irreducible. A covering of projective curves \(\pi : X \to Y\) of degree \(d\) is called simple if \(X\) and \(Y\) are smooth and for each \(y \in Y\) one has \(d - 1 \leq \# \pi^{-1}(y) \leq d\). If \(E\) is a locally free sheaf of a smooth curve \(Y\) we denote the rank, the degree and the slope of \(E\) by \(r(E), d(E)\) and \(\mu(E) = d(E)/r(E)\). If \(Y\) is an elliptic curve there is a unique, up to isomorphism, indecomposable locally free sheaf of rank \(r\) and degree \(0\) which has nontrivial sections \[\text{At2}\]. We denote it by \(F_r\). Unless otherwise specified we assume the base field \(k = \mathbb{C}\).

1. Preliminaries

Let \(\pi : X \to Y\) be a finite covering of smooth, projective curves of degree \(d \geq 2\), suppose \(g(Y) \geq 1\). Let \(P = \ker(Nm_\pi : J(X) \to J(Y))^0\) be the Prym variety of the covering. Let \(\Theta\) be the canonical polarization of \(J(X)\) and let \(\Theta_P\) be its restriction on \(P\). The following statement is proved in \[\text{Ko}\] Lemma 1.1.

**Proposition 1.1.** The following three conditions are equivalent: \(\pi_* : H_1(X, \mathbb{Z}) \to H_1(Y, \mathbb{Z})\) is surjective; \(\pi^* : J(Y) \to J(X)\) is injective; \(\ker Nm_\pi\) is connected. Suppose these conditions hold and let \(P = \ker Nm_\pi\). Then
the polarization $\Theta_P$ is of type $(1, \ldots, 1, d, \ldots, d)$ where the $d$'s are repeated $g(Y)$ times.

1.2. Let $\pi : X \to Y$ be a covering as above, suppose $Y$ is an elliptic curve and $g(X) \geq 3$. Let $d_2 = |H_1(Y, \mathbb{Z}) : \pi_*H_1(X, \mathbb{Z})|$. Then $\pi$ may be decomposed as $X \xrightarrow{\tilde{\pi}_1} \tilde{Y} \xrightarrow{\tilde{\pi}_2} Y$ where $\tilde{\pi}_2$ is an isogeny of degree $d_2$ and $\deg \pi_1 = d_1 = \frac{d_2}{d}$. According to the preceding proposition the type of the polarization $\Theta_P$ is $(1, \ldots, 1, d_1)$. In particular if $d = 4$ one obtains polarization of type $(1, \ldots, 1, 4)$ when $\pi_* : H_1(X, \mathbb{Z}) \to H_1(Y, \mathbb{Z})$ is surjective and $(1, \ldots, 1, 2)$ when it is not. Counting parameters we see that one might obtain a general abelian variety $A$ with polarization of type $(1, \ldots, 1, 4)$ as Prym variety of a covering $\pi : X \to Y$ only if $\dim A = 2$ or $\dim A = 3$.

1.3. We will need some facts about vector bundles over elliptic curves, a theory due to Atiyah [At2]. In this section we will make the customary identification between vector bundles and locally free sheaves. For generalities on vector bundles over smooth, projective curves we refer the reader to the survey articles [Oc] and [Br]. The following facts about vector bundles over an elliptic curve are known (see e.g. [Br] p.87). Every indecomposable vector bundle is semistable. A vector bundle is semistable if and only if it is a direct sum of indecomposable vector bundles of the same slope. A vector bundle of degree $d$ and rank $r$ is stable if and only if it is indecomposable and $(d, r) = 1$. If $G$ is a semistable vector bundle over an elliptic curve, then $h^1(G) = 0$ if $\mu(G) = \frac{d(G)}{r(G)} > 0$ and $G$ is generated by its global sections if $\mu(G) > 1$ (see e.g. [Oc] p.39). Let $r \geq 1$ and $d \in \mathbb{Z}$. The isomorphism classes of indecomposable vector bundles of rank $r$ and degree $d$ over an elliptic curve $Y$ depend on one parameter and are parametrized by $\text{Pic}^0(Y) \cong Y$. In fact if one fixes one such $E$, then all others are obtained as $E \otimes L$ for some $L \in \text{Pic}^0(Y) = J$. Furthermore $E \otimes L \cong E \otimes L'$ if and only if $L' \otimes L^{-1}$ is a point of order $r' = r/h$ of $\text{Pic}^0(Y)$ where $h = (r, d)$ (cf. [At2] Theorem 10). So, the indecomposable vector bundles of rank $r$ and degree $d$ are parametrized by $J/J'$, $J' \cong J$.

One may construct a Poincaré vector bundle as follows. Let us fix a point $y_0 \in Y$. Let $\mathcal{L}$ be the Poincaré line bundle on $Y \times J$ normalized by $\mathcal{L}|_{y_0 \times J} \cong \mathcal{O}_J$. First suppose $(r, d) = 1$. Then the general theory of stable vector bundles yields a Poincaré vector bundle $\mathcal{E}(r, d)$ on $Y \times U(r, d)$ where $U(r, d)$ is the fine moduli space of stable vector bundles ([Ne] Ch.5 §5). By [At2] p. 434 the morphism $U(r, d) \to J$ given by $u \mapsto \det \mathcal{E}(r, d)_u \otimes \mathcal{O}_Y(-d y_0)$ is an isomorphism. So we may replace $U(r, d)$ by $J$. If $r = r' h, d = d' h$ where $(r', d') = 1$ we let $\mathcal{E}(r, d) = \mathcal{E}(r', d') \otimes \pi^{-1}_1 F_h$. According to [At2] Lemma 24 and Theorem 10 this family has the property that each indecomposable locally free sheaf $E$ of rank $r$ and degree $d$ is isomorphic to $\mathcal{E}(r, d)_u$ for some $u \in J$ and if $u \neq u'$, then $\mathcal{E}(r, d)_u \not\cong \mathcal{E}(r, d)_{u'}$. Furthermore the invertible sheaf $\det \mathcal{E}(r, d)$ determines a morphism $J \to J$ given by $u \mapsto \det \mathcal{E}(r, d)_u \otimes \mathcal{O}_Y(-d y_0)$ and this morphism coincides with $h \cdot id_J : J \to J$.

The following statement is proved in [Te] Lemma 2.3.
Proposition 1.4. Let $E$ and $G$ be semistable vector bundles over an elliptic curve. Then $E \otimes G$ is semistable of slope $\mu(E \otimes G) = \mu(E) + \mu(G)$.

Corollary 1.5. Let $E_1, E_2, E_3, \ldots$ be semistable vector bundles over an elliptic curve. Then

$$S^{k_1} E_1 \otimes \wedge^{k_2} E_2 \otimes S^{k_3} E_3^* \otimes \wedge^{k_4} E_4^* \otimes \cdots$$

is semistable of slope $\mu = k_1 \mu(E_1) + k_2 \mu(E_2) - k_3 \mu(E_3) - k_4 \mu(E_4) + \cdots$

Proof. By Proposition 1.3, the tensor product $E_1 \otimes \cdots \otimes E_n$ is semistable, so it is a direct sum of indecomposable vector bundles of slope $\mu$. The vector bundle $E_1 \otimes \cdots \otimes E_n$ is its direct summand, so the corollary follows from an analog of the Krull-Schmidt theorem due to Atiyah \[\text{[At1]}\]. \qed

If $E$ is a vector bundle of rank 2 and even degree over an elliptic curve the symmetric powers $S^n E$ were calculated by Atiyah: if $E \cong F_2 \otimes L$, then $S^n (F_2 \otimes L) \cong F_n \otimes L^n$ (cf. \[\text{[At2]}\] p.438). In the case of rank 2 vector bundles of odd degree the following statement holds.

Proposition 1.6. Let $E$ be a vector bundle over an elliptic curve of rank 2 and odd degree. Then for every $k \geq 1$ one has

(i) $S^{2k-1} E \cong (\Lambda^2 E)^{k-1} \otimes (E^k)\cap$,

(ii) $S^{2k} E \cong (\Lambda^2 E)^k \otimes \left[ I^{(2k+1-3\lfloor \frac{k}{2} \rfloor)} \oplus (\oplus_{i=1}^{3} L_i)^{\lfloor \frac{k}{2} \rfloor} \right]$

where $L_i = I, L_i \not\cong I$.

Proof. We prove the formula for $S^n E$ by induction on $n$. Let $n = 2$. Let $A$ be a fixed line bundle of degree 1 as in \[\text{[At2]}\]. One has $E \cong E_A(2,1) \otimes M$ for some line bundle $M$. According to Lemma 22 (ibid.) $E \otimes E^* \cong \oplus_{i=0}^{3} L_i$, where $L_0 = I$. Since $E \cong E^* \otimes \Lambda^2 E$ one obtains $E \otimes E \cong \Lambda^2 E \otimes (\oplus_{i=1}^{3} L_i)$. On the other hand $E \otimes E \cong S^2 E \otimes \Lambda^2 E$. So $S^2 E \cong \Lambda^2 E \otimes \sum_{i=1}^{3} L_i$ by \[\text{[At1]}\].

Let us prove the formula for $2k + 1, 2k + 2$ assuming it holds for $n \leq 2k$. By the Clebsch-Gordan formula $S^{2k} E \otimes E \cong S^{2k+1} E \otimes (\Lambda^2 E \otimes S^{2k-1} E)$ which is isomorphic to $S^{2k+1} E \oplus ((\Lambda^2 E)^k \otimes E^k)$ by the induction hypothesis applied to $n = 2k - 1$. One has $E \otimes L_i \cong E$ (cf. \[\text{[At2]}\] p.434), so by the induction hypothesis applied to $n = 2k$ one has $S^{2k} E \otimes E \cong (\Lambda^2 E)^k \otimes E^{\oplus (2k+1)}$. Using \[\text{[At1]}\] one obtains $S^{2k+1} E \cong (\Lambda^2 E)^k \otimes E^{\oplus (k+1)}$. Let $a_k = 2k + 1 - 3\lfloor \frac{k}{2} \rfloor$, $b_k = \lfloor \frac{k}{2} \rfloor$. One has

$$S^{2k+1} E \otimes E \cong S^{2k+2} E \oplus (\Lambda^2 E \otimes S^{2k} E)$$

$$\cong S^{2k+2} E \oplus (\Lambda^2 E)^{k+1} \left[ I^{a_k} \oplus (\sum_{i=1}^{3} L_i)^{b_k} \right].$$
On the other hand
\[ S^{2k+1}E \otimes E \cong ((\wedge^{2}E)^{k} \otimes E^{(k+1)}) \otimes E \]
\[ \cong (\wedge^{2}E)^{k} \otimes (E \otimes E)^{(k+1)} \cong (\wedge^{2}E)^{k+1} \otimes \left[ I^{k+1} \oplus (\sum_{i=1}^{3} L_{i})^{k+1} \right]. \]
By [At1] we may cancel and obtain
\[ S^{2k+2}E \cong (\wedge^{2}E)^{k+1} \otimes \left[ I^{k+1-a_{k}} \oplus (\sum_{i=1}^{3} L_{i})^{k+1-b_{k}} \right]. \]
That \( k+1 - a_{k} = a_{k+1} \) and \( k+1 - b_{k} = b_{k+1} \) is clear. \( \square \)

2. Hurwitz spaces of quadruple coverings of elliptic curves

2.1. We study the Hurwitz spaces of quadruple coverings by means of a result of Casnati and Ekedahl [CE] which describes such coverings in terms of a pair of vector bundles on the base. We recall their construction in the special case we need. Let \( \pi : X \to Y \) be a covering of smooth projective curves of degree \( d \). The Tschirnhausen module of the covering is the quotient sheaf \( E^{\vee} \) defined by the exact sequence
\[ 0 \to \mathcal{O}_{Y} \xrightarrow{\pi^{\#}} \pi_{*}\mathcal{O}_{X} \to E^{\vee} \to 0. \]
One has \( E^{\vee} \cong \text{Ker}(Tr_{\pi} : \pi_{*}\mathcal{O}_{X} \to \mathcal{O}_{Y}) \) and this is a locally free sheaf of rank \( d - 1 \). There is a canonical embedding \( i : X \to \mathbf{P}(E) \) defined by the relative dualizing sheaf \( \omega_{X/Y} \cong \omega_{X} \otimes (\pi^{*}\omega_{Y})^{-1} \) and satisfying \( i^{*}\mathcal{O}_{\mathbf{P}(E)}(1) \cong \omega_{X/Y} \). When \( d = 4 \) every fiber \( X_{y} \) is an intersection of two conics in \( \mathbf{P}(E)_{y} \). This globalizes as follows. There is a locally free sheaf \( F \) of rank 2 on \( Y \) such that if \( \rho : \mathbf{P}(E) \to Y \) is the canonical fibration and \( N = \rho^{*}F \), then the resolution of \( \mathcal{O}_{X} \) is given by
\[ 0 \to \rho^{*}\text{det}E(-4) \to N(-2) \xrightarrow{\delta} \mathcal{O}_{\mathbf{P}(E)} \to \mathcal{O}_{X} \to 0. \]
By the results of Casnati and Ekedahl one has \( \text{det}F \cong \text{det}E \) and given such a pair of locally free sheaves on \( Y \) a quadruple covering is determined uniquely by the homomorphism \( \delta \in H^{0}(\mathbf{P}(E), \bar{N}(2)). \) Let \( \phi : H^{0}(Y, \bar{F} \otimes S^{2}E) \to H^{0}(\mathbf{P}(E), \bar{N}(2)) \) be the canonical isomorphism and let \( \delta = \phi(\eta) \).
Then \( \eta \) satisfies the following property. For every \( y \in Y \) the value \( \eta(y) : F(y) \to S^{2}E(y) \) determines a pencil of conics in \( \mathbb{P}^{2} = \mathbf{P}(E)_{y} \) whose base locus is of dimension 0. According to [CE] Definition 4.2 the sections \( \eta \) with this property are called of right codimension for every \( y \in Y \). The described construction is valid for every integral scheme \( Y \) and yields a bijective correspondence between the following data ([CE] Theorem 4.4.)

(A) Finite, flat Gorenstein coverings \( \pi : X \to Y \) of degree 4.
(B) Locally free sheaves $F$ of rank 2 and $E$ of rank 3 on $Y$ such that $\det F \cong \det E$, and a section $\eta \in H^0(Y, \tilde{F} \otimes S^2E)$ which has right codimension at every $y \in Y$. The section is determined uniquely up to multiplication by scalars.

Given $F$ and $E$ with $\det F \cong \det E$ one can prove that the subset of $\mathbb{P}H^0(Y, \tilde{F} \otimes S^2E)$ which parametrizes the elements $\langle \eta \rangle$ which have right codimension at every $y \in Y$ is Zariski open (cf. Lemma 2.14). In case it is not empty, by [CE] Theorem 4.4, the group $PGL(F) \times PGL(E)$ acts faithfully on this Zariski open set. The orbits of this action correspond bijectively to the equivalence classes of Gorenstein coverings of degree 4 of $Y$ whose canonically associated pair of locally free sheaves is isomorphic to the pair $(E, F)$.

2.2. Let $Y$ be a smooth, projective curve. Let $\mathcal{H}_{d,n}(Y)$ be the Hurwitz space which parametrizes the simple coverings of $Y$ of degree $d$ branched in $n$ points. We denote by $\mathcal{H}^0_{d,n}(Y)$ the subset of the Hurwitz space $\mathcal{H}_{d,n}(Y)$ whose points correspond to coverings $\pi : X \to Y$ with the property that $\pi_* : H_1(X, \mathbb{Z}) \to H_1(Y, \mathbb{Z})$ is surjective. This property is preserved under deformation, so $\mathcal{H}^0_{d,n}(Y)$ is a union of connected components of $\mathcal{H}_{d,n}(Y)$.

Given $A \in \text{Pic}^{n/2}(Y)$ we denote by $\mathcal{H}_{d,A}(Y)$ the closed reduced subscheme whose points correspond to coverings having a Tschirnhausen module with determinant isomorphic to $A^{-1}$ (cf. [Ka] (2.4) and Lemma 2.5). We denote by $\mathcal{H}^0_{d,A}(Y)$ the intersection $\mathcal{H}^0_{d,n}(Y) \cap \mathcal{H}_{d,A}(Y)$. In this paper we study $\mathcal{H}^0_{4,n}(Y)$ and $\mathcal{H}^0_{4,A}(Y)$ where $Y$ is elliptic curve and $n \geq 2$. Notice that both Hurwitz spaces are nonempty. Indeed, if $\pi_* : H_1(X, \mathbb{Z}) \to H_1(Y, \mathbb{Z})$ is not surjective, then $\pi$ may be decomposed as $X \to \tilde{Y} \to Y$, where $\tilde{Y} \to Y$ is an unramified double covering. Hence the monodromy group of $\pi : X \to Y$ is different from $S_4$. Simple branched coverings with monodromy group $S_4$ are easily constructed (see e.g. [Ka] Lemma 2.1), so $\mathcal{H}^0_{4,n}(Y) \neq \emptyset$ for every pair $n \geq 2$. Using Lemma 2.5 of [Ka] one concludes that $\mathcal{H}^0_{4,A}(Y) \neq \emptyset$ for every $A \in \text{Pic}^{n/2}Y$.

The unirationality results proved in the paper are based on the following theorem.

**Theorem 2.3.** Let $Y$ be an elliptic curve. Let $n = 2e \geq 2$. There is a Zariski open, dense subset $U \subset \mathcal{H}^0_{4,n}(Y)$ such that for every $[X \to Y] \in U$ one has for the associated pair of locally free sheaves $(E, F)$:

1. If $e \not\equiv 0(\text{mod}3)$ then $E$ is indecomposable of degree $e$.
2. If $e \equiv 0(\text{mod}3)$, then $E$ is isomorphic to $L_1 \oplus L_2 \oplus L_3$ where $L_i \in \text{Pic}^eY$ and $L_i \not\cong L_j$ for $i \neq j$.
3. If $e \equiv 1(\text{mod}2)$ then $F$ is indecomposable of degree $e$.
4. If $e \equiv 0(\text{mod}2)$, then $F \cong M_1 \oplus M_2$ where $M_i \in \text{Pic}^eY$ and $M_1 \not\cong M_2$. 


The proof of the theorem is rather long, it occupies (2.4) – (2.15), and we indicate first the main steps. We use the result of Casnati and Ekedahl [CE] by which quadruple coverings of an elliptic curve are described in terms of a pair of vector bundles \((E, F)\) of ranks 3 and 2 respectively. Our first task is to bound from above the number of parameters on which the coverings depend when the pair varies within a family of a given type. This quite long calculation is done in 7 steps according to the polygons of the Harder-Narasimhan filtrations of the vector bundles \(E\) and \(F\). The outcome is that only the types specified in the theorem may give sufficient number of moduli. This work is done in (2.4) – (2.13). The remaining part of the proof is devoted to show that for each of the occurring types of \((E, F)\) one may construct a family of quadruple coverings \(X \to Y \times T\) such that when applying the universal property of the Hurwitz space on one obtains a morphism \(f : T \to \mathcal{H}_{4,n}(Y)\) such that the dimension of the image of \(T\) equals the number of parameters calculated in the first part of the proof. Here the technical result of Lemma 2.14 is used. This is also applied later in connection with the moduli space \(A_3(1,1,4)\). This lemma together with the calculation of the number of parameters permits to conclude in (2.15) that every sufficiently general quadruple covering of \(\mathcal{H}_{4,n}(Y)\) has a pair \((E, F)\) of the type specified in the theorem.

**Lemma 2.4.** Let \(E\) and \(F\) be locally free sheaves over an elliptic curve of ranks \(r(E) = 3, r(F) = 2\). Let \(\deg E = \deg F = e\). Then

\[
\chi(\bar{F} \otimes S^2E) = h^0(\bar{F} \otimes S^2E) - h^1(\bar{F} \otimes S^2E) = 2e
\]

Proof. By Riemann-Roch \(\chi(\bar{F} \otimes S^2E) = \deg(\bar{F} \otimes S^2E)\). One has \(c_1(\bar{F} \otimes S^2E) = 8c_1(E) - 6c_1(F)\). Thus \(\deg(\bar{F} \otimes S^2E) = 2e\). \(\square\)

**2.5.** For the various types of \((E, F)\) considered below the following numbers are constant: \(h^0(\text{End } E), h^0(\text{End } F), h^0(\bar{F} \otimes S^2E)\). Specifying the type of a pair \((E, F)\) means generally specifying the decomposition types of \(E\) and \(F\), the possibility about isomorphy or non isomorphy of various summands that appear, and when \(h^0(\bar{F} \otimes S^2E)\) may jump, imposing additional conditions on \(F\) and \(E\) in order that \(h^0(\bar{F} \otimes S^2E)\) stays fixed. By (2.1) given a pair \((E, F)\), if there are Gorenstein coverings of degree 4 whose associated pair is isomorphic to the given one, then such coverings depend on the following number of parameters:

\[
\dim \mathbb{P}H^0(\bar{F} \otimes S^2E) - \dim \text{PGL}(E) - \dim \text{PGL}(F)
\]

\[
= h^0(\bar{F} \otimes S^2E) - h^0(\text{End } E) - h^0(\text{End } F) + 1
\]

Varying \((E, F)\) in a family of a given type one obtains

\[\text{(2) \ # moduli } [X \to Y] = \]

\[\text{ # moduli } (E, F) + h^0(\bar{F} \otimes S^2E) - h^0(\text{End } E) - h^0(\text{End } F) + 1.\]
Let us consider the Hurwitz space $\mathcal{H}_{4,n}(Y)$ parametrizing equivalence classes of simple coverings of an elliptic curve $\pi : X \to Y$ branched in $n$ points. Here the Tschirnhausen modules have degree $-e$ where $n = 2e$ ([K2, Lemma 2.3]). Hence by Lemma [2.4] for the pairs $(E, F)$ associated to the points of $\mathcal{H}_{4,n}(Y)$ one has $h^0(F \otimes S^2E) = n + h^1(F \otimes S^2E)$. Taking into account that $h^0(\text{End} \, E) = h^1(\text{End} \, E)$, $h^0(\text{End} \, F) = h^1(\text{End} \, F)$ we obtain the following criterion:

Pairs of locally free sheaves $(E, F)$ of a given type yield quadruple coverings $\pi : X \to Y$ with insufficient number of moduli, i.e. $\# \text{moduli} [X \to Y] < n = \dim \mathcal{H}_{4,n}(Y)$ if

$$\# \text{moduli} (E, F) < h^1(\text{End} \, E) + h^1(\text{End} \, F) - 1 - h^1(F \otimes S^2E) = h^0(\text{End} \, E) + h^0(\text{End} \, F) - 1 - h^1(F \otimes S^2E)$$

2.6. Case 1. $E$ and $F$ are semistable locally free sheaves. According to Corollary [1.3] $F \otimes S^2E$ is semistable. Its slope $\mu(F \otimes S^2E) = -\mu(F) + 2\mu(E) = -\frac{e}{2} + 2 \cdot \frac{e}{2} = \frac{e}{2} > 0$. Hence $h^1(F \otimes S^2E) = 0$. Let us decompose in a direct sum of indecomposable locally free sheaves: $E = E_1 \oplus \cdots \oplus E_k$, $F = G_1 \oplus \cdots \oplus G_m$. One has $\mu(E_i) = \mu(E)$, $\mu(G_j) = \mu(F)$ for every $i, j$. One obtains various possible types of the pairs $(E, F)$ by: a) fixing the ranks of $E_i$ and $G_j$; b) requiring that certain direct summands of $E$ are isomorphic to each other and similarly for $F$. We have $\# \text{moduli} (E, F) \leq \ell + m - 1$. Here subtracting 1 is for $\det E \cong \det F$. One has a strict inequality if one considers a type where certain direct summands are isomorphic to each other. So, in this case the inequality [3] holds since $\ell \leq h^1(\text{End} \, E)$ and $m \leq h^1(\text{End} \, F)$. Suppose one considers a type of $(E, F)$ where one of the direct summands $E_i$ is indecomposable of rank $r' h$ and degree $d' h$ where $h > 1$ and $(r', d') = 1$. Then $h^1(\text{End} \, E_i) = h$ (cf. the proof of [A2, Lemma 23]). Therefore $\ell < h^1(\text{End} \, E)$ and [3] holds. A similar argument may be applied to $F$. If every direct summand $E_i$, $G_j$ has degree and rank prime to each other, then $r(E_i) = r(E_j)$ and $r(G_i) = r(G_j)$ for $i \neq j$. Indeed $\mu(E_i) = \mu(E_j)$ implies $d(E_i) = d(E_j) = d(E_i) = d(E_j) = d(E_i)$. This implies $r(E_i) = r(E_j)$, $d(E_i) = d(E_j)$. We conclude that the only possible types of $(E, F)$ which might give $\# \text{moduli} [X \to Y] = n$ are the types $E \cong \oplus_{i=1}^k E_i$, $F \cong \oplus_{j=1}^m G_j$ where each $E_i$ or $G_j$ is indecomposable with rank prime to its degree, $r(E_i) = r(E_j)$, $r(G_i) = r(G_j)$ and furthermore $E_i \not\cong E_j$ and $G_i \not\cong G_j$ for $i \neq j$. When $n = 2e$ is fixed there is only one such type and this is the type of Theorem [2.5]. All other possible types with semistable $E$ and $F$ yield $\# \text{moduli} [X \to Y] < n$ according to the criterion of [2.5].

2.7. We need an explicit form of $X \subset \mathbf{P}(E)$ in order to exclude some types of $(E, F)$. Let $U \subset Y$ be a Zariski open subset such that $E|_U$ and $F|_U$ are trivial. Let $E|_U = \mathcal{O}_U e_1 \oplus \mathcal{O}_U e_2 \oplus \mathcal{O}_U e_3$, $F|_U = \mathcal{O}_U f_1 \oplus \mathcal{O}_U f_2$. One has
\[ S^2 E|_U = \sum_{i \leq j} O_a e_i e_j. \]

If \( \eta \in H^0(\tilde{\mathcal{F}} \otimes S^2 E) = \text{Hom}_Y(F, S^2 E) \), then
\[
\eta(\alpha_1 f_1 + \alpha_2 f_2) = \alpha_1 \eta(f_1) + \alpha_2 \eta(f_2)
= \alpha_1 \sum_{i \leq j} a_{ij,1} e_i e_j + \alpha_2 \sum_{i \leq j} a_{ij,2} e_i e_j.
\]

If \( f^1, f^2 \) are the sections of \( \tilde{\mathcal{F}}|_U \) dual to \( f_1, f_2 \), i.e. \( \langle f_i, f^j \rangle = \delta_{ij} \), then \( \eta \in H^0(\tilde{\mathcal{F}} \otimes S^2 E) \) is locally given by
\[
\eta|_U = \sum_{i \leq j} a_{ij,1} f^1 \otimes e_i e_j + \sum_{i \leq j} a_{ij,2} f^2 \otimes e_i e_j.
\]

For every \( y \in U \) the fiber \( \mathbb{P}(E)_y \approx \mathbb{P}^2 \) has coordinates \( e_1(y), e_2(y), e_3(y) \). Every pair \( \xi_1, \xi_2 \in \mathbb{C} \) with \( (\xi_1, \xi_2) \neq (0, 0) \) defines a conic with equation
\[
\sum_{i \leq j} (\xi_1 a_{ij,1}(y) + \xi_2 a_{ij,2}(y)) e_i(y) e_j(y) = 0.
\]

Varying \( (\xi_1, \xi_2) \) one obtains a pencil of conics whose base locus is \( X_y \subset \mathbb{P}(E)_y \).

**2.8. Case 2.** \( F \) is semistable, \( E = E_1 \oplus E_2 \), where \( E_i \) are semistable of slopes \( \mu(E_1) = \mu_1 > \mu(E_2) = \mu_2 \). This case is subdivided further according to \( r(E_1) \), but we treat the two subcases simultaneously whenever possible.

We have \( S^2 E = S^2 E_1 \oplus (E_1 \otimes E_2) \oplus S^2 E_2 \). Here \( S^2 E_2 = E_2^2 \) if \( r(E_2) = 1 \) and \( S^2 E_1 = E_1^2 \) if \( r(E_1) = 1 \).

**Subcase 2A.** \( \mu(F \otimes S^2 E_2) = -\frac{\mu_1}{2} + 2\mu_2 > 0 \), or \( \mu(\tilde{\mathcal{F}} \otimes S^2 E_2) = 0 \) and \( h^1(\tilde{\mathcal{F}} \otimes S^2 E_2) = h^0(\tilde{\mathcal{F}} \otimes S^2 E_2) = 0 \). Here
\[
\mu(F \otimes S^2 E_1) > \mu(F \otimes E_1 \otimes E_2) > \mu(\tilde{\mathcal{F}} \otimes S^2 E_2) \geq 0
\]
so we obtain \( h^1(\tilde{\mathcal{F}} \otimes S^2 E) = 0 \). It is clear that
\[
\# \text{ moduli } (E, F) \leq h^1(\text{End } E_1) + h^1(\text{End } E_2) + h^1(\text{End } F) - 1
\]
We have \( h^1(\text{End } E) = h^1(\text{End } E_1) + h^1(\text{End } E_2) + h^1(E^*_1 \otimes E_2) \) and \( h^1(E^*_1 \otimes E_2) = h^0(E_1 \otimes E_2) = 2(\mu_1 - \mu_2) > 0 \). We conclude inequality \([3]\) holds whatever type of \((E, F)\) satisfying the conditions of this subcase is considered.

**Subcase 2B.** \( \mu(F \otimes S^2 E_2) = -\frac{\mu_1}{2} + 2\mu_2 < 0 \). Here \( H^0(\tilde{\mathcal{F}} \otimes S^2 E_2) = 0 \). If \( r(E_2) = 1 \) this means that if we choose in \([2.7]\) the frame of \( E|_U \) so that \( e_1, e_2 \) generate \( E_1|_U \) and \( e_3 \) generates \( E_2|_U \), then \( a_{33,1} = a_{33,2} = 0 \). This implies that in each fiber \( \mathbb{P}(E)_y \), \( y \in U \) the point \( e_1(y) = e_2(y) = 0 \) belongs to the pencil of conics defined by \([5]\). This means that the section of \( \mathbb{P}(E) \) defined by \( E \rightarrow E_1 \rightarrow 0 \) is a component of the Gorenstein covering \( \pi : X \rightarrow Y \) determined by any \( \eta \in H^0(\tilde{\mathcal{F}} \otimes S^2 E) \). We may thus exclude from consideration such type, since no irreducible, smooth, quadruple cover \( X \) may be obtained from such a pair \((E, F)\). If \( r(E_1) = 1, r(E_2) = 2 \) choosing \( e_i \) in \([2.7]\) so that \( e_1 \) generates \( E_1|_U \) and \( e_2, e_3 \) generate \( E_2|_U \) we see that for \( \forall y \in U \) all conics of \([5]\) must contain the line \( \{e_1(y) = 0\} \). So no pair
(E, F) with this property could be associated with a Gorenstein covering X → Y.

**Subcase 2C.** \( \mu(\tilde{F} \otimes S^2E_2) = 0, \ h^0(\tilde{F} \otimes S^2E) > 0. \) Here \( h^1(\tilde{F} \otimes S^2E) = h^0(\tilde{F} \otimes S^2E_2). \) We have \(-\mu(F) + 2\mu(E_2) = 0, \) so \( \mu(F) \) is an integer and \( e = d(F) = 2\mu(F) \) is even. One has two possibilities. Either \( F \) is indecomposable of even degree or \( F \cong L_1 \oplus L_2 \) where \( \deg L_i = \frac{e}{2}. \) We recall \( F_r \) denotes the unique (up to isomorphism) locally free sheaf on \( Y \) of rank \( r \) and degree 0 with \( h^0(F_r) \geq 1. \)

**Subcase 2C’.** Assume \( F \) is indecomposable, \( F \cong F_2 \otimes L \) with \( \deg L = e/2. \) In this case the equality \( \det F = \det E \) determines \( F \) by \( E \) up to tensoring by an element of \( (Ku^0Y)_2. \) Thus \# moduli \( (E, F) = \# moduli (E). \)

One has \( h^1(\text{End } F) = h^0(\text{End } F_2) = h^0(F_1 \oplus F_2) = 2 \) (cf. \([At2]\) p.437) and furthermore

\[
\text{(6)} \quad \# \text{ moduli } (E) \leq h^1(\text{End } E_1) + h^1(\text{End } E_2)
\]

So one would have the inequality \( (\text{3}) \) for any type where it holds

\[
\text{(7)} \quad 1 + h^0(E_1 \otimes E_2^*) - h^0(\tilde{F} \otimes S^2E) > 0
\]

We have \( \mu(E_1 \otimes E_2^*) = \mu_1 - \mu_2 > 0. \) So \( h^0(E_1 \otimes E_2^*) = 2(\mu_1 - \mu_2) \geq 1. \) If \( E_2 \) is of rank 2 and decomposable then \( \mu_2 \) is an integer, so \( h^0(E_1 \otimes E_2^*) \geq 2. \) If \( E_2 \) is of rank 1, then \( \tilde{F} \otimes S^2E_2 = \tilde{F} \otimes E_2^2 \) is indecomposable of degree 0, so \( h^0(\tilde{F} \otimes E_2^2) \leq 1. \) This shows \( (\text{7}) \) holds. If \( E_2 \) is of rank 2 one has the following cases.

**Subcase 2C(i).** \( E_2 \) is indecomposable of odd degree. Then according to Proposition \( 1.6 \) one has \( S^2E_2 \cong \bigwedge^2E_2 \otimes (\eta_1 \oplus \eta_2 \oplus \eta_3) \) where \( \eta_i^2 \cong O_Y, \eta_i \not\cong O_Y. \) Let \( F \cong F_2 \otimes L. \) Then \( \tilde{F} \otimes S^2E_2 \) is a direct sum of \( F_2 \otimes L^{-1} \otimes \bigwedge^2E_2 \otimes \eta_i, \) \( i = 1, 2, 3. \) According to \([At2]\) Theorem 5 only one of these locally free sheaves might be isomorphic to \( F_2. \) Thus \( h^0(\tilde{F} \otimes S^2E_2) \leq 1 \) and therefore \( (\text{7}) \) holds.

**Subcase 2C(ii).** \( E_2 \) is indecomposable of even degree. Then \( E_2 \cong F_2 \otimes M \) for some invertible sheaf \( M. \) Here \( S^2E_2 \cong F_3 \otimes M^2 \) and \( \tilde{F} \otimes S^2E_2 \cong F_2 \otimes F_3 \otimes L^{-1}M^2 \cong (F_2 \oplus F_4) \otimes L^{-1}M^2 \) (cf. \([At2]\) p.437). It holds \( h^0(\tilde{F} \otimes S^2E_2) \leq 2, \) so one has only \( \geq \) in \( (\text{7}). \) However here \( \# \text{ moduli } (E) \leq 2, \) while \( h^1(\text{End } E_1) = 1, \) \( h^1(\text{End } E_2) = h^0(\text{End } F_2) = 2. \) Thus one has strict inequality in \( (\text{6}) \) and \( (\text{3}) \) holds.

**Subcase 2C(iii).** \( E_2 \cong M_1 \oplus M_2, \ M_1 \not\cong M_2. \) Here \( h^0(E_1 \otimes E_2^*) \geq 2 \) and \( S^2E_2 \cong M_1^2 \oplus M_1M_2 \oplus M_2^2. \) Only two of these invertible sheaves might be isomorphic to each other, so \( h^0(\tilde{F} \otimes S^2E_2) \leq 2. \) Thus \( (\text{7}) \) holds.

**Subcase 2C(iv).** \( E_2 \cong M \oplus M. \) Here \( h^0(\tilde{F} \otimes S^2E_2) \leq 3, \) \( h^1(\text{End } E_1) = 1, \) \( h^1(\text{End } E_2) = 4, \) \( h^0(E_1 \otimes E_2^*) \geq 2 \) and \( \# \text{ moduli } (E) = \# \text{ moduli } (E) \leq 2. \) Thus we have \( \geq 0 \) in \( (\text{7}) \) and \( < 0 \) in \( (\text{6}) \) so the inequality \( (\text{3}) \) holds.

Under the condition of Subcase 2C we now assume
Subcase 2C". \( F \) is decomposable, \( F \cong L_1 \oplus L_2 \), \( \deg L_i = e/2 \). The inequality (8) which we want to verify reads here as follows:

\[
\# \text{moduli}(E,F) < h^1(End E_1) + h^1(End E_2)
\]

\[
+ h^1(End F) - 1 + h^0(E_1 \otimes E_2^*) - h^0(\hat{F} \otimes S^2 E_2)
\]

Subcase 2C"(i). Let \( r(E_2) = 1 \). Here \( \det F \cong \det E \) yields \( L_1 \otimes L_2 \cong \det E_1 \otimes E_2 \). One has \( \hat{F} \otimes S^2 E_2 \cong (L_1^{-1} \otimes E_2^2) \oplus (L_2^{-1} \otimes E_2^2) \). So in order that \( h^0(\hat{F} \otimes S^2 E_2) > 0 \) it should hold \( L_i \cong E_2^2 \) for \( i = 1 \) or \( i = 2 \). We conclude \( F \) is determined uniquely by \( E \), so \( \# \text{moduli}(E,F) = \# \text{moduli}(E) \). We have

\[
\# \text{moduli}(E) \leq h^1(End E_1) + h^1(End E_2)
\]

so proving that

\[
h^0(End F) - 1 + h^0(E_1 \otimes E_2^*) - h^0(\hat{F} \otimes S^2 E_2) > 0
\]

would imply (8). We have

\[
h^0(End F) - 1 = \begin{cases} 
1 & \text{if } F \cong L_1 \oplus L_2, \ L_1 \not\cong L_2 \\
3 & \text{if } F \cong L \oplus L
\end{cases}
\]

and \( h^0(E_1 \otimes E_2^*) = 2(\mu_1 - \mu_2) \geq 1 \). If \( F \cong L_1 \oplus L_2, \ L_1 \not\cong L_2 \), then \( h^0(\hat{F} \otimes E_2^2) \leq 1 \) and if \( F \cong L \oplus L \) then \( h^0(\hat{F} \otimes E_2^2) \leq 2 \). In both cases (10) holds.

Subcase 2C"(ii). \( r(E_2) = 2 \). Here \( \hat{F} \otimes S^2 E_2 \) is a direct sum of indecomposable locally free sheaves of degree 0. An easy check of the various cases for \( E_2 \) shows that the conditions \( h^0(\hat{F} \otimes S^2 E_2) > 0 \) and \( \det F \cong \det E \) determine \( F \) by \( E \). We have again the inequality (9) and it suffices to prove (10). We proceed similarly to Subcases 2C(i) - 2C(iv).

If \( E_2 \) is indecomposable of odd degree, then \( S^2 E_2 \cong \wedge^2 E_2 \otimes (\eta_1 \oplus \eta_2 \oplus \eta_3) \) where \( \eta_i \) are the three points of order 2 of \( Pu^0 Y \). If \( F \cong L \oplus L \) then \( h^0(\hat{F} \otimes S^2 E_2) \leq 2 \). Since \( h^0(End F) = 4 \) we obtain (10). If \( F \cong L_1 \oplus L_2, \ L_1 \not\cong L_2 \) then in case \( h^0(\hat{F} \otimes S^2 E_2) \leq 1 \) the inequality (10) holds. The only other possibility might be \( h^0(\hat{F} \otimes S^2 E_2) = 2 \) when \( L_1 \cong \wedge^2 E_2 \otimes \eta_i, \ L_2 \cong \wedge^2 E_2 \otimes \eta_j \) for some pair \( i, j \). Here we can only claim \( \geq 0 \) in (10). Now the isomorphism \( L_1 L_2 \cong \det F \cong \det E \cong E_1 \otimes \det E_2 \) yields \( E_1 \cong \wedge^2 E_2 \otimes \eta_i \eta_j \). Thus \( \# \text{moduli}(E) \leq 1 \). In this case the inequality (9) is strict and again we see (8) holds.

If \( E_2 \) is indecomposable of even degree, \( E_2 \cong F_2 \otimes M \), then \( S^2 E_2 \cong F_3 \otimes M^2 \), so if \( F \cong L_1 \oplus L_2 \) with \( L_1 \not\cong L_2 \) then \( h^0(\hat{F} \otimes S^2 E_2) \leq 2 \). In both cases (10) holds.

If \( E_2 \) is decomposable, \( E_2 \cong M_1 \oplus M_2 \), then \( h^0(E_1 \otimes E_2^*) \geq 2 \). \( S^2 E_2 \cong M_1^2 \oplus M_2^2 \). If \( h^0(\hat{F} \otimes S^2 E_2) \leq 2 \) then (10) holds. One might have \( h^0(\hat{F} \otimes S^2 E_2) \geq 3 \) only if two of the summands of \( S^2 E_2 \) are isomorphic to each other, i.e. either \( M_1 \not\cong M_2 \), \( M_1^2 \cong M_2^2 \) or \( M_1 \cong M_2 \cong M \). In the former case one has \( h^0(\hat{F} \otimes S^2 E_2) \geq 3 \) either if \( L_1 \cong L_2 \cong M_1^2 \cong M_2^2 \) and then \( h^0(\hat{F} \otimes S^2 E_2) = 4 \) and (10) holds or if \( L_1 \cong M_1^2 \cong M_2^2, \ L_2 \cong M_1 M_2 \) and then
$h^0(\mathcal{F} \otimes S^2E_2) = 3$. In this case the left-hand side of (10) is $\geq 0$. Here one has $L_1L_2 \cong \det E \cong E_1M_1M_2$. Thus $E_1 \cong M_1^2$ and $\# \text{moduli (E)} \leq 1$ since $M_1^2 \cong M_2^2$. We have that $h^1(\text{End } E_1) + h^1(\text{End } E_2) = 3$, so (12) holds. For $E_2$ it remains the case $E_2 \cong M \oplus M$. One might have $h^0(\mathcal{F} \otimes S^2E_2) \geq 3$ in two cases. Either $F \cong L_1 \oplus L_2$, $L_1 \not\cong L_2$, $L_1 \cong M^2$ and then $h^0(\mathcal{F} \otimes S^2E_2) = 3$ or $F \cong L \oplus L$, $L \cong M^2$. Thus $h^0(\mathcal{F} \otimes S^2E_2) = 6$. In both cases the left-hand side of (10) is $\geq -1$ while $\# \text{moduli (E, F)} = \# \text{moduli (E)} \leq 2$ and $h^1(\text{End } E_1) = 1$, $h^1(\text{End } E_2) = 4$. Thus (13) holds. Case 2 is completed.

2.9. Case 3. $F$ is semistable, $E = E_1 \oplus E_2 \oplus E_3$ with $d(E_1) > d(E_2) > d(E_3)$. Let $d_i = d(E_i)$. We have $h^1(\text{End } E) = \sum h^1(\text{End } E_i) + \sum_{i,j} h^0(\mathcal{E}_i \otimes E_j^*) = 3 + 2(d_1 - d_3)$, so the inequality (14) that we want to verify for different types of $(E, F)$ within this case becomes

$$\# \text{moduli (E, F)} < 2 + h^1(\text{End } F) + 2(d_1 - d_3) - h^1(\mathcal{F} \otimes S^2E)$$

**Subcase 3A.** $\mu(\mathcal{F} \otimes E_3^*) = -\frac{e}{2} + 2d_3 > 0$ or $\mu(\mathcal{F} \otimes E_3^*) = 0$ and $h^0(\mathcal{F} \otimes E_3^*) = 0$. Here we have $h^1(\mathcal{F} \otimes S^2E) = 0$, $\# \text{moduli (E, F)} \leq 3 + h^1(\text{End } F) - 1$ and $d_1 - d_3 \geq 2$, thus (12) holds.

**Subcase 3B.** $\mu(\mathcal{F} \otimes E_3^*) < 0$. The same argument as in Subcase 2B of (2.8) shows that such types of $(E, F)$ cannot occur in the case of pair associated with an irreducible, Gorenstein quadruple cover $X$.

**Subcase 3C.** $\mu(\mathcal{F} \otimes E_3^*) = 0$, $h^0(\mathcal{F} \otimes E_3^*) > 0$. Here one has $h^1(\mathcal{F} \otimes S^2E) = h^1(\mathcal{F} \otimes E_3^*) = h^0(\mathcal{F} \otimes E_3^*) \leq 2$. Since $d_1 - d_3 \geq 2$ we conclude (12) holds.

We thus examined in (2.6) - (2.9) all cases for $(E, F)$ with semistable $F$. The remaining types to be considered are with $F \cong L_1 \oplus L_2$ where $d(L_1) > d(L_2)$. Let $d(L_i) = \lambda_i$. Here $h^1(\text{End } F) = 2 + h^0(L_1L_2^{-1}) = 2 + \lambda_1 - \lambda_2$. A possible type for $(E, F)$ would yield insufficient number of moduli if it holds the inequality (cf. (13))

$$\# \text{moduli (E, F)} < h^1(\text{End } E) + 1 + (\lambda_1 - \lambda_2) - h^1(\mathcal{F} \otimes S^2E)$$

It obviously holds $\# \text{moduli (E, F)} \leq h^1(\text{End } E) + 1$

2.10. Case 4. $F \cong L_1 \oplus L_2$, $E$ is semistable. Since $\det E \cong \det F$ we have $e = \lambda_1 + \lambda_2 = 3\mu(E)$ and $\lambda_1 > \frac{e}{3} > \lambda_2$. One has $h^1(\mathcal{F} \otimes S^2E) = h^1(L_1^{-1} \otimes S^2E) + h^1(L_2^{-1} \otimes S^2E)$ and $\mu(L_2^{-1} \otimes S^2E) = -\lambda_2 + \frac{2e}{3} > \frac{e}{6} > 0$. Thus $h^1(L_2^{-1} \otimes S^2E) = 0$ and the inequality (13) we aim to prove becomes

$$\# \text{moduli (E, F)} < h^1(\text{End } E) + 1 + (\lambda_1 - \lambda_2) - h^1(L_1^{-1} \otimes S^2E)$$

**Subcase 4A.** $\mu(L_1^{-1} \otimes S^2E) > 0$. Here $h^1(L_1^{-1} \otimes S^2E) = 0$, therefore $h^1(\mathcal{F} \otimes S^2E) = 0$ and (14) holds.

**Subcase 4B.** $\mu(L_1^{-1} \otimes S^2E) \leq 0$ or $\mu(L_1^{-1} \otimes S^2E) = 0$ and $h^0(L_1^{-1} \otimes S^2E) = 0$. Here we have $\text{Hom}_Y(L_1, S^2E) = 0$. This is impossible for a pair associated to a Gorenstein quadruple covering of $Y$. Indeed, the associated
relative pencil of conics is given by a monomorphism $\eta : F \to S^2E$ (cf. [CE] pp.450,451), so $\eta|L_1$ must be a nonzero element for such a pair.

**Subcase 4C.** $\mu(L_1^{-1} \otimes S^2E) = 0$ and $h^0(L_1^{-1} \otimes S^2E) > 0$. Here $d(L_1^{-1} \otimes S^2E) = 0$, so $h^1(L_1^{-1} \otimes S^2E) = 0$. Furthermore $\lambda_1 = \mu(L_1) = \mu(S^2E) = \frac{2}{3}$, so in this subcase $e \equiv 0 \pmod{3}$.

**Subcase 4C(i).** $E$ is indecomposable. Here $E \cong F_3 \otimes M$ since $3|e$. We have $S^2E = S^2F_3 \otimes M^2$ and we claim $S^2F_3 \cong F_1 \oplus F_3$. Indeed, $F_3 \otimes F_3 \cong F_1 \oplus F_3 \oplus F_5$ by [AV2] p.438. One has $F_3 \otimes F_3 \cong S^2F_3 \oplus \wedge^2F_3$. The pairing $\wedge^2F_3 \times F_3 \to \wedge^3F_3 \cong \mathcal{O}_Y$ is nondegenerate, so $\wedge^2F_3 \cong F_3 \otimes F_3$. We conclude by [At1] that $S^2F_3 \cong F_1 \oplus F_5$. Calculating the various entries in $\mathbf{[14]}$ we obtain: $h^0(L_1^{-1} \otimes S^2E) \leq 2$, $h^1(End E) = 3$, so the right-hand side is $\geq 3$. The condition $\det E \cong \det F$ reads as $M^3 \cong L_1L_2$, so $E$ is determined by $F$ up to tensoring by a point of order $3$ in $Pic^0Y$. Hence $\# \text{ moduli } (E,F) = \# \text{ moduli } (F) \leq 2$ and the inequality $\mathbf{[13]}$ holds.

**Subcase 4C(ii).** $E \cong E_1 \oplus E_2$ where $E_1$ is indecomposable of rank 2. We have from semistability $\mu(E) = \mu(E_1) = \mu(E_2)$, thus $\mu(E_1) = \frac{2}{3}$. Since $3|e$ we conclude $d(E_1)$ is even, so $E_1 \cong F_2 \otimes M_1$. We have $h^1(End E) = h^0(End F_2) + h^0(End E_2) + h^0(F_2 \otimes M_1E_2^{-1})$, so $h^1(End E) = 3$ if $M_1 \not\cong E_2$ and $h^1(End E) = 4$ if $M_1 \cong E_2$. We have $S^2E \cong S^2E_1 \oplus (E_1 \oplus E_2) \oplus E_2^2$, so

$$L_1^{-1} \otimes S^2E \cong (F_3 \otimes L_1^{-1}M_1^2) \oplus (F_2 \otimes L_1^{-1}M_1E_2) \oplus L_1^{-1}E_2^2$$

We have now various cases:

If $M_1^2, M_1E_2$ and $E_2^2$ are not isomorphic to each other then $h^0(L_1^{-1} \otimes S^2E) \leq 1$. So the right-hand side of $\mathbf{[14]}$ is $\geq 4$ while $\# \text{ moduli } (E,F) \leq 2 + 2 - 1 = 3$, thus $\mathbf{[13]}$ holds.

If $M_1^2 \cong E_2^2$ but $M_1 \not\cong E_2$ then $h^0(L_1^{-1} \otimes S^2E) \leq 2$, while $\# \text{ moduli } (E,F) \leq 2$, so again $\mathbf{[14]}$ holds.

If $M_1 \cong E_2$, then $h^0(L_1^{-1} \otimes S^2E) \leq 3$, $h^1(End E) = 4$ and $\# \text{ moduli } (E,F) \leq 2$, so $\mathbf{[13]}$ holds.

**Subcase 4C(iii).** $E \cong M_1 \oplus M_2 \oplus M_3$ where $\deg M_i = \frac{2}{3}$. Here one has $h^1(End E) = 3 + 2\sum_{i<j} h^0(M_iM_j^{-1})$ and $S^2E \cong \oplus_{i<j} M_iM_j$. The conditions $h^0(L_1^{-1} \otimes S^2E) \geq 1$ and $\det F \cong \det E$ determine $F$ from $E$. Hence $\# \text{ moduli } (E,F) = \# \text{ moduli } (E) \leq 3 \leq h^1(End E)$. If $h^0(L_1^{-1} \otimes S^2E) = 1$ we see $\mathbf{[14]}$ holds. One has the inequality $h^0(L_1^{-1} \otimes S^2E) \geq 2$ in the following cases.

a) Up to reordering $M_1 \cong M_2, M_3 \not\cong M_1$. Then $S^2E \cong (M_1^2) \oplus (M_1M_2) \oplus (M_1M_3) \oplus (M_2^2) \oplus M_3^2$. Here $h^0(L_1^{-1} \otimes S^2E) \leq 3, h^1(End E) = 5$, so $\mathbf{[13]}$ holds.

b) The sheaves $M_1, M_2, M_3$ are pairwise non-isomorphic, two of the summands of $S^2E$ are isomorphic to each other, $L_1$ is isomorphic to this summand and no three summands of $S^2E$ are isomorphic to each other. Up to reordering this might happen if $M_1^2 \cong M_2^2 \cong L_1$ or $M_2^2 \cong M_2M_3 \cong L_1$. Then $h^0(L_1^{-1} \otimes S^2E) = 2$, $\# \text{ moduli } (E) \leq 2$ thus $\mathbf{[14]}$ holds as well.
c) The sheaves \(M_1, M_2, M_3\) are pairwise non-isomorphic, \(M_1 \cong M_2 \cong M_3 \cong L_1\). Here \(h^0(L_1^{-1} \otimes S^2E) = 3\), \# moduli \((E) \leq 1\), thus (14) holds.

d) \(M_1 \cong M_2 \cong M_3\), \(L_1 \cong M_2^2\). Here \(h^0(L_1^{-1} \otimes S^2E) = 6\), \(h^1(\text{End } E) = 9\) and \# moduli \((E) \leq 1\) thus (14) holds as well. Case 4 is completed.

2.11. Case 5. \(F \cong L_1 \oplus L_2\), \(E \cong E_1 \oplus E_2\) where \(E_1\) is semistable of rank 2 and \(\mu(E_1) = \mu_1 > \mu_2 = d(E_2)\). Here \(h^1(\text{End } E) = h^1(\text{End } E_1) + 1 + 2(\mu_1 - \mu_2)\). So, a possible type of \((E, F)\) of this class would be excluded if one shows that

\[
\# \text{ moduli } (E, F) < h^1(\text{End } E_1) + 2
\]

(15)

\[
+ (\lambda_1 - \lambda_2) + 2(\mu_1 - \mu_2) - h^1(F \otimes S^2E).
\]

The condition \(\det E \cong \det E_1 \oplus E_2 \cong \det F \cong L_1L_2\) determines \(E_2\) from \(E_1\) and \(F\), so

\[
\# \text{ moduli } (E, F) \leq h^1(\text{End } E_1) + 2
\]

(16)

We see (15) would hold in cases when

\[
\lambda_1 - \lambda_2 + 2(\mu_1 - \mu_2) - h^1(F \otimes S^2E) > 0.
\]

(17)

One has

\[
h^1(F \otimes S^2E) = h^1(L_1^{-1} \otimes S^2E_1) + h^1(L_1^{-1} \otimes E_1 \otimes E_2) + h^1(L_1^{-1} \otimes E_2^2)
\]

\[
+ h^1(L_2^{-1} \otimes S^2E_1) + h^1(L_2^{-1} \otimes E_1 \otimes E_2) + h^1(L_2^{-1} \otimes E_2^2).
\]

The direct summands \(L_1^{-1} \otimes S^2E_1, \ldots, L_2^{-1} \otimes E_2^2\) are semistable with the following slopes

\[
- \lambda_1 - 2\mu_1 > -\lambda_1 + \mu_1 + \mu_2 > -\lambda_1 + 2\mu_2
\]

(18)

\[
- \lambda_2 + 2\mu_1 > -\lambda_2 + \mu_1 + \mu_2 > -\lambda_2 + 2\mu_2
\]

where in each column the upper number is smaller then the lower one. If \(-\lambda_1 + 2\mu_2 < 0\) or if \(-\lambda_2 + 2\mu_2 = 0\) and \(h^0(L_2^{-1} \otimes E_2^2) = 0\) then \(h^0(F \otimes E_2^2) = 0\) and the same argument as in Subcase 2B of (2.8) shows that such case is impossible for a pair obtained from an irreducible cover, so it is excluded. If \(-\lambda_1 + \mu_1 + \mu_2 < 0\) or if \(-\lambda_1 + \mu_1 + \mu_2 = 0\) and \(h^1(L_1^{-1} \otimes E_1 \otimes E_2) = 0\) then \(H^0(L_1^{-1} \otimes E_1 \otimes E_2) = 0 = H^0(L_1^{-1} \otimes E_2^2)\) or equivalently \(H^0(L_1^{-1}E_2 \otimes E) = 0\). Suppose that in the explicit representation of (2.7) the sections \(c_1, c_2\) form a frame of \(E_1|U\) and \(c_3\) generates \(E_2|U\). Then representing \(\eta\) by (11) we see that for each \(y \in U\) the quadratic polinomial \(\sum_{i \leq j} a_{ij}e_i(y)e_j(y)\) has no monomials which contain \(e_3(y)\). Hence the corresonding conic is degenerate. If \(\eta \in H^0(F \otimes S^2E) = \text{Hom}_Y(L_1 \oplus L_2, S^2E)\) is obtained from a reduced cover \(X\), then \(\eta|_{L_1}\) yields a conic bundle over \(Y\) whose general fibre is a union of two distinct lines. Let \(\pi_* : \tilde{Y} \to Y\) be the associated double covering. We obtain that \(\tilde{\pi} : X \to \tilde{Y}\) may be decomposed as \(X \to \tilde{Y} \to Y\). We consider only simple coverings, so only the case of étale \(\pi_2 : \tilde{Y} \to Y\) is of interest to us. Now, such coverings are excluded from the hypothesis of Theorem (2.3), namely from the condition that \(\pi_* : H_1(X, \mathbb{Z}) \to H_1(Y, \mathbb{Z})\) is
surjective. We conclude that pairs \((E, F)\) of Case 5 may yield quadruple coverings \(\pi : X \to Y\) from \(\mathcal{H}_{4,n}^0(Y)\) only if

- \(-\lambda_2 + 2\mu_2 \geq 0\) and if it is 0 then \(h^0(L^{-1}_2 \otimes E^3_2) \geq 1\).
- \(-\lambda_1 + \mu_1 + \mu_2 \geq 0\) and if it is 0 then \(h^0(L^{-1}_1 \otimes E_1 \otimes E_2) \geq 1\).

Looking at \((18)\) we obtain

\[
h^1(\bar{F} \otimes S^2E) = h^1(L^{-1}_1 \otimes E_1 \otimes E_2) + h^1(L^{-1}_1 \otimes E^3_2) + h^1(L^{-1}_2 \otimes E^2_2).
\]

Notice that \(-\lambda_2 + 2\mu_1 > -\lambda_1 + 2\mu_1 > -\lambda_1 + \mu_1 + \mu_2 \geq 0\). Since \(2\mu_1 \in \mathbb{Z}\) we conclude \(-\lambda_2 + 2\mu_1 \geq 2\).

**Subcase 5A.** Assume \(-\lambda_2 + 2\mu_2 > 0\) and \(-\lambda_1 + \mu_1 + \mu_2 > 0\). Then \(h^1(\bar{F} \otimes S^2E) = h^1(L^{-1}_1 \otimes E^2_2)\). The inequality \((17)\) holds obviously if \(-\lambda_1 + 2\mu_2 \geq 0\). If \(-\lambda_1 + 2\mu_2 < 0\) then \(h^1(L^{-1}_1 \otimes E^2_2) = \lambda_1 - 2\mu_2\) and we have

\[
\lambda_1 - \lambda_2 + 2(\mu_1 - \mu_2) - (\lambda_1 - 2\mu_2) = -\lambda_2 + 2\mu_1 \geq 2
\]

Therefore \((17)\) holds.

**Subcase 5B.** Assume \(-\lambda_1 + \mu_1 + \mu_2 > 0\, \text{and} \, -\lambda_2 + 2\mu_2 = 0\) and \(h^0(L^{-1}_2 \otimes E^3_2) \geq 1\). Then \(h^1(\bar{F} \otimes S^2E) = \lambda_1 - 2\mu_2 + 1\). As in the preceding case we obtain \((17)\) holds.

**Subcase 5C.** Assume \(-\lambda_2 + 2\mu_2 > 0\, \text{and} \, -\lambda_1 + \mu_1 + \mu_2 = 0\) and \(h^0(L^{-1}_1 \otimes E_1 \otimes E_2) \geq 1\). Then \(h^1(\bar{F} \otimes S^2E) = h^0(L^{-1}_1 \otimes E_1 \otimes E_2) + \lambda_1 - 2\mu_2\). If \(h^0(L^{-1}_1 \otimes E_1 \otimes E_2) \leq 1\) we conclude as in the preceding case. The only other possibility might be \(E_1 \cong M \oplus M\) and \(h^0(L^{-1}_1 \otimes E_1 \otimes E_2) = 2\). However then the inequality \((16)\) is strict since \(h^1(\text{End} E_1) = 4\). Thus \((15)\) holds.

**Subcase 5D.** Assume \(-\lambda_2 + 2\mu_2 = 0\, \text{and} \, h^0(L^{-1}_2 \otimes E^3_2) \geq 1\) and \(-\lambda_1 + \mu_1 + \mu_2 = 0\). \(h^0(L^{-1}_1 \otimes E_1 \otimes E_2) \geq 1\). Then

\[
h^1(\bar{F} \otimes S^2E) = h^0(L^{-1}_1 \otimes E_1 \otimes E_2) + h^0(L^{-1}_1 \otimes E^3_2) + \lambda_1 - 2\mu_2
\]

The right-hand side of \((15)\) becomes

\[
h^1(\text{End} E_1) + 2 + (-\lambda_2 + 2\mu_1) - (h^0(L^{-1}_1 \otimes E_1 \otimes E_2) + h^0(L^{-1}_1 \otimes E^3_2))
\]

As we saw \(-\lambda_2 + 2\mu_1 \geq 2\), while \(2 \leq h^0(L^{-1}_1 \otimes E_1 \otimes E_2) + h^0(L^{-1}_1 \otimes E^3_2) \leq 3\).

**Subcase 5D(i)** \(h^0(L^{-1}_1 \otimes E^3_2) = 1\, \text{and} \, h^0(L^{-1}_1 \otimes E_1 \otimes E_2) = 1\). Here from the first equality \(L_2 \cong E^3_2\) and from \(\det F \cong \det E\) it follows \(L_2 \cong \det E_1 \otimes E^{-1}_2\). Thus \# moduli \((E, F) \leq h^1(\text{End} E_1) + 1\) and therefore \((15)\) holds.

**Subcase 5D(ii)** \(h^0(L^{-1}_1 \otimes E^3_2) = 1\, \text{and} \, h^0(L^{-1}_1 \otimes E_1 \otimes E_2) = 2\). With respect to the previous case we have in addition that \(E_1 \cong M \oplus M\), so \(h^1(\text{End} E_1) = 4\) and \((15)\) is fulfilled. Case 5 is completed.

**2.12. Case 6.** \(F \cong L_1 \oplus L_2, E \cong E_1 \oplus E_2\) where \(E_2\) is semistable of rank 2 and \(\mu_1 = d(E_1) > \mu_2 = \mu(E_2)\). A calculation similar to the one in \((2.11)\) shows that a type of \((E, F)\) in this class would be excluded if one shows that
\[ \# \text{moduli } (E, F) < h^1(\text{End} E_2) + 2 \]
\[ + (\lambda_1 - \lambda_2) + 2(\mu_1 - \mu_2) - h^1(\bar{F} \otimes S^2E) \]

Furthermore it holds \# moduli \((E, F) \leq h^1(\text{End} E_2) + 2\), so \((20)\) would hold if one shows that
\[ (\lambda_1 - \lambda_2) + 2(\mu_1 - \mu_2) - h^1(\bar{F} \otimes S^2E) > 0 \]

One has
\[ h^1(\bar{F} \otimes S^2E) = h^1(L_1^{-1} \otimes E_1^2) + h^1(L_1^{-1} \otimes E_1 \otimes E_2) + h^1(L_1^{-1} \otimes S^2 E_2) \]
\[ + h^1(L_2^{-1} \otimes E_1^2) + h^1(L_2^{-1} \otimes E_1 \otimes E_2) + h^1(L_2^{-1} \otimes S^2 E_2) \]

where for the slopes of the semistable direct summands of \(\bar{F} \otimes S^2E\) one has the same table as that of \((18)\). If \(-\lambda_1 + 2\mu_2 < 0\) or \(-\lambda_1 + 2\mu_2 = 0\) and \(h^0(L_1^{-1} \otimes S^2 E_2) = 0\) then one would have \(H^0(L_1^{-1} \otimes S^2 E_2) = 0\). We may choose in \((27)\) \(e_1\) to generate \(E_1|_U\) and \(e_1, e_2\) to form a frame of \(E_2|_U\). Then every \(\eta \in H^0(\bar{F} \otimes S^2E) = \text{Hom}_Y(F, S^2E)\) has the property that in the explicit representation of \((1)\) all monomials of \(\eta|_{L_1}\) contain \(e_1(y)\) as a factor. This means that the conic bundle corresponding to \(\eta|_{L_1}\) is reducible and one of its components is the ruled surface corresponding to the epimorphism \(E \to E_1 \to 0\). This implies that the quadruple cover \(X\) is reducible, which is excluded from the hypothesis of Theorem 2.3.

We may thus assume that \(-\lambda_1 + 2\mu_2 \geq 0\) and if \(-\lambda_1 + 2\mu_2 = 0\) then \(h^0(L_1^{-1} \otimes S^2 E_2) \geq 1\). Looking at the table \((18)\) we see that \(h^1(F \otimes S^2E) = h^1(L_1^{-1} \otimes S^2 E_2)\). If \(-\lambda_1 + 2\mu_2 > 0\) then \(h^1(L_1^{-1} \otimes S^2 E_2) = 0\), so \((21)\) holds. Assume \(-\lambda_1 + 2\mu_2 = 0\). Then \(h^1(L_1^{-1} \otimes S^2 E_2) = h^0(L_1^{-1} \otimes S^2 E_2)\). Since \(L_1^{-1} \otimes S^2 E_2\) is semistable of rank 3 and slope 0 one has \(h^0(L_1^{-1} \otimes S^2 E_2) \leq 3\). If \(h^0(L_1^{-1} \otimes S^2 E_2) = 1\) then \((21)\) holds since \((\lambda_1 - \lambda_2) + 2(\mu_1 - \mu_2) \geq 2\). This case happens if \(E_2\) is indecomposable. Indeed, in this case if \(d(E_2)\) is even, then \(S^2 E_2 \cong E_3 \otimes M^2\), so by \((12)\)
\[ h^0(F_3 \otimes M^2 L_1^{-1}) \leq 1. \]
If \(d(E_2)\) is odd then by Proposition \((16)\) one has \(S^2 E_2 \cong \wedge^2 E_2 \otimes (\eta_1 \oplus \eta_2 \oplus \eta_3)\) where \(\eta_i\) are the three points of order 2 in \(\text{Pic}^0 Y\), so again \(h^0(L_1^{-1} \otimes S^2 E_2) \leq 1\). So, cases with \(h^0(L_1^{-1} \otimes S^2 E_2) = 2\) could occur only if \(E_2\) is decomposable. Then \(\mu_2 = \mu(E_2)\) is an integer, \(\mu_1 > \mu_2\), so \((\lambda_1 - \lambda_2) + 2(\mu_1 - \mu_2) \geq 3\). If \(E_2\) is decomposable and \(h^0(L_1^{-1} \otimes S^2 E_2) \leq 2\) the inequality \((21)\) holds. The only remaining subcase is \(h^0(L_1^{-1} \otimes S^2 E_2) = 3\) which is possible if \(E_2 \cong M \oplus M\), \(L_1 \cong M^2\). Then \# moduli \((E, F) \leq 2\) while \(h^1(\text{End} E_2) = 4\). Thus \((20)\) holds. All possible types of Case 6 are thus excluded.

2.13. Case 7. \(F \cong L_1 \oplus L_2\), \(E \cong E_1 \oplus E_2 \oplus E_3\) where \(d(E_i) = \mu_i\), \(\mu_1 > \mu_2 > \mu_3\). Here \(h^1(\text{End} F) = 2 + \lambda_1 - \lambda_2\), \(h^1(\text{End} E) = 3 + 2(\mu_1 - \mu_3)\). So, a possible type for \((E, F)\) would be excluded if one shows that \((cf. 13)\)
\[ \# \text{moduli } (E, F) < 4 + (\lambda_1 - \lambda_2) + 2(\mu_1 - \mu_3) - h^1(\bar{F} \otimes S^2E) \]
Since \( \# \text{ moduli} (E, F) \leq 4 \) this would be the case if the following inequality holds.

\[
(\lambda_1 - \lambda_2) + 2(\mu_1 - \mu_3) - h^1(\tilde{F} \otimes S^2 E) > 0
\]

We have

\[
h^1(\tilde{F} \otimes S^2 E) = \sum_{i=1}^{2} \sum_{j \leq k} h^1(L^{-1}_i E_j E_k)
\]

and the direct summands of \( \tilde{F} \otimes S^2 E \) have degrees

\[
-\lambda_1 + 2\mu_1 > -\lambda_1 + \mu_1 + \mu_2 > d_1 > -\lambda_1 + \mu_2 + \mu_3 > -\lambda_1 + 2\mu_3
\]

\[
-\lambda_2 + 2\mu_1 > -\lambda_2 + \mu_1 + \mu_2 > d_2 > -\lambda_2 + \mu_2 + \mu_3 > -\lambda_2 + 2\mu_3.
\]

Here \( d_i \) stays for either \( -\lambda_1 + 2\mu_2 \) or \( -\lambda_1 + \mu_1 + \mu_3 \) and in each column the upper number is smaller then the respective lower one. If \( -\lambda_2 + 2\mu_3 < 0 \) then \( H^0(\tilde{F} \otimes E^3_2) = 0 \). If \( -\lambda_1 + 2\mu_2 < 0 \) then \( H^0(L^{-1}_1 \otimes S^2(E_2 \oplus E_3)) = 0 \). If \( -\lambda_1 + \mu_1 + \mu_3 < 0 \) then \( H^0(L^{-1}_1 E_3 \otimes E) = 0 \). In each case the same arguments as those in Subcase 2B, Case 6 or Case 5 respectively show that such possibilities are excluded from the assumption that the quadruple cover is irreducible and cannot be decomposed through a double covering. So we may assume \( -\lambda_2 + 2\mu_3 \geq 0, -\lambda_1 + 2\mu_2 \geq 0 \) and \( -\lambda_1 + \mu_1 + \mu_3 \geq 0 \). We thus have

\[
h^1(\tilde{F} \otimes S^2 E) = h^1(L^{-1}_1 E_1 E_3) + h^1(L^{-1}_1 E_2^2)
\]

\[
+ h^1(L^{-1}_1 E_2 E_3) + h^1(L^{-1}_1 E_3^2) + h^1(L^{-1}_2 E_3^2)
\]

We now consider various cases. If \( -\lambda_1 + 2\mu_3 \geq 0 \) then all summands in (24) are zero except possibly \( h^1(L^{-1}_1 E_3^2) \) which is \( \leq 1 \). So (23) holds. If \( -\lambda_1 + 2\mu_3 < 0 \) and \( -\lambda_1 + \mu_2 + \mu_3 \geq 0 \) then \( h^1(\tilde{F} \otimes S^2 E) = \epsilon_1 + (\lambda_1 - 2\mu_3) + \epsilon_2 \) where \( 0 \leq \epsilon_i \leq 1 \). The left-hand side of (23)

\[
(-\lambda_2 + 2\mu_3) + (-\lambda_1 + 2\mu_1) + (\mu_2 - \mu_3) - \epsilon_1 - \epsilon_2 - \epsilon_3
\]

We have \( -\lambda_2 + 2\mu_3 \geq 0, -\lambda_1 + 2\mu_1 > -\lambda_1 + \mu_1 + \mu_2 > -\lambda_1 + 2\mu_2 > 0, \) so \( -\lambda_1 + 2\mu_1 \geq 2 \). Moreover \( \mu_2 - \mu_3 \geq 1 \). Hence inequality (23) holds when at least one of \( \epsilon_i \) is zero. If \( \epsilon_i = 1 \) for \( \forall i \) then we would have \( L_1 \cong E_1^2, L_1 \cong E_1 E_3, L_2 \cong E_3^2 \). In this case \( \# \text{ moduli} (E, F) \leq 2 \) therefore (22) holds. All possible types of Case 7 are thus excluded.

We thus verified that except for the types of \((E, F)\) specified in the theorem and considered in Case 1 all other types yield number of parameters for the equivalence classes of coverings \( X \to Y \) less then \( n \). In order to complete the proof of the theorem we need a lemma which is related to Theorem 4.5 of [CE] and is an analog of Lemma 2.8 of [Ka]. We state and prove a more general result than we actually need for the proof of Theorem 2.8.
Lemma 2.14. Assume the base field $k$ is algebraically closed and $\text{char}(k) = 0$. Let $q: \mathcal{Y} \to Z$ be a smooth, proper morphism with connected fibers, where $Z$ is smooth. Let $\mathcal{E}$ and $\mathcal{F}$ be locally free sheaves on $\mathcal{Y}$ of ranks 3 and 2 respectively, such that $\det \mathcal{E}_z \cong \det \mathcal{F}_z$ for every $z \in Z$. Suppose $H^0(\mathcal{Y}_z, \mathcal{F}_z \otimes S^2\mathcal{E}_z)$ is independent of $z \in Z$ and is $\neq 0$. Consider the locally free sheaf $\mathcal{H} = q_* (\mathcal{F} \otimes S^2\mathcal{E})$ on $Z$. Let $f: \mathbb{H} \to Z$ be the associated vector bundle with fibers $\mathbb{H}_z = H^0(\mathcal{Y}_z, \mathcal{F}_z \otimes S^2\mathcal{E}_z)$. Then the subset $\mathbb{H}_0 \subset \mathbb{H}$ consisting of $\eta$ which satisfy the following three conditions is Zariski open in $\mathbb{H}$.

(a) If $f(\eta) = z$ then $\eta$ is of right codimension for every $y \in \mathcal{Y}_z$.

(b) Assuming (a), if $\pi_\eta: X_\eta \to \mathcal{Y}_z$, $X_\eta \subset \mathbb{P}(\mathcal{E}_z)$ is the Gorenstein quadruple covering determined by $\eta$, then $X_\eta$ is smooth and irreducible.

(c) Assuming (a) and (b) the discriminant scheme of $\pi_\eta: X_\eta \to \mathcal{Y}_z$ is a smooth subscheme of $\mathcal{Y}_z$.

Suppose $\mathbb{H}_0 \neq \emptyset$. Consider the base change $\mathcal{Y}' = \mathcal{Y} \times_Z \mathbb{H}_0$ and let $\mathcal{E}' = \pi_1^* \mathcal{E}$, $\mathcal{F}' = \pi_1^* \mathcal{F}$. Then every $\eta_0 \in \mathbb{H}_0$ has a neighborhood $U = f^{-1}(V) \cap \mathbb{H}_0$, where $V$ is a Zariski open subset of $Z$, such that it exists a smooth quadruple covering $X'_U \to \mathcal{Y}'_U = \mathcal{Y} \times_Z U$ with the property that for every $\eta \in U$ with $f(\eta) = z$ the fiber $X'_\eta \to \mathcal{Y}'_\eta$ is equivalent to $\pi_\eta: X_\eta \to \mathcal{Y}_z$.

Proof. The statement is local with respect to $Z$. According to [Ha] Ch.III Ex.12.4 there is an invertible sheaf $\mathcal{L}$ on $Z$ such that $\det \mathcal{E} \cong \det \mathcal{F} \otimes q^* \mathcal{L}$. So we may assume that $Z$ is irreducible and $\det \mathcal{E} \cong \det \mathcal{F}$ on $\mathcal{Y}$. The proof then proceeds similarly to that of Lemma 2.8 of [Ka] and uses Theorem 4.4 of [CE]. If $\mathbb{H}_0$ is empty there is nothing to prove. Suppose $\mathbb{H}_0 \neq \emptyset$.

Step 1. Let $\mathbb{H}'$ be the set of $\eta \in \mathbb{H}$ for which (a) holds. We claim $\mathbb{H}'$ is Zariski open in $\mathbb{H}$. Let $\rho: \mathbb{P}(\mathcal{E}) \to \mathcal{Y}$ be the projectivization and let $\mathcal{N} = \rho^* \mathcal{F}$. One has an isomorphism $\Phi: q_* (\mathcal{F} \otimes S^2\mathcal{E}) \to (q \circ \rho)_* \mathcal{N}(2)$. Every $\eta \in \mathbb{H}_z$ determines a section $\Phi_\eta(\mathcal{N}(2))$ with zero set $D(\Phi_\eta(\mathcal{N}(2))) \subset \mathbb{P}(\mathcal{E})_\eta$. We consider the incidence correspondence $\Gamma \subset \mathbb{P}(\mathcal{E}) \times_Z \mathbb{H}$ defined as follows.

$$\Gamma = \{(x, \eta)| x \in D(\Phi_\eta(\mathcal{N}(2))) \text{ where } x \in \mathbb{P}(\mathcal{E})_\eta, \eta \in \mathbb{H}_z, y \in \mathcal{Y}_z\}.$$ 

Consider the projection $\varepsilon: \Gamma \to \mathcal{Y} \times_Z \mathbb{H}$, $\varepsilon(x, \eta) = (y, \eta)$. An element $\eta \in \mathbb{H}_z$ fails to be of right codimension in $y \in \mathcal{Y}_z$ if and only if $\Phi_\eta(\mathcal{N}(2)) \subset H^0(\mathcal{Y} \times_Z \mathbb{H}, \mathcal{N}(2)) \cong H^0(\mathbb{P}^2, O_{\mathbb{P}^2}(2) \oplus O_{\mathbb{P}^2}(2))$ determines two degenerate conics in $\mathbb{P}^2$ with a common line. This happens if and only if the zero set of the section $\Phi_\eta(\mathcal{N}(2))$ has dimension $\geq 1$. Equivalently $(y, \eta) \in \Sigma \subset \mathcal{Y} \times_Z \mathbb{H}$ where $\Sigma$ is the subset of points for which $\dim \varepsilon^{-1}(y, \eta) \geq 1$. Since $\Sigma$ is closed in $\mathcal{Y} \times_Z \mathbb{H}$ and since $\mathcal{Y} \times_Z \mathbb{H} \to \mathcal{Y}$ is proper, the projection of $\Sigma$ in $\mathbb{H}$ is closed. Thus $\mathbb{H}'$ is open in $\mathbb{H}$.

The proof of (b) and (c) is similar to that of [Ka] Lemma 2.8. Namely if $\mathcal{E}_\mathbb{H}$ and $\mathcal{F}_\mathbb{H}$ are the pull-backs via the projection $\mathcal{Y} \times_Z \mathbb{H} \to \mathcal{Y}$ one constructs a tautological section $N \in H^0(\mathcal{Y} \times_Z \mathbb{H}, \mathcal{F}_\mathbb{H} \otimes S^2\mathcal{E}_\mathbb{H})$ and further proceeds as in Step 2 and Step 3 of the proof of Lemma 2.8 of [Ka].
2.15. End of the proof of Theorem 2.3. Given \( n = 2e \geq 2 \) it suffices to prove the following property for each of the types of \((E, F)\) considered in (2.6) – (2.13). Unless this is the type specified in the theorem the set of equivalence classes \([X \to Y] \in \mathcal{H}_{4,n}(Y)\) with associated pair of the given type is either empty or if nonempty it is contained in a closed subscheme of \(\mathcal{H}_{4,n}(Y)\) of codimension \(\geq 1\). In order to prove such a statement we use Lemma 2.14 and the calculation of the \# moduli \([X \to Y]\) made in (2.6) – (2.13). Let us analyze one case, all others being similar. Consider the possible types with indecomposable \(F\) of degree \(e\), \(E \cong E_1 \oplus E_2\) where \(E_1\) is indecomposable of rank 2 and degree \(e_1\) and \(E_1\) is of rank 1 and degree \(e_2\), \(e = e_1 + e_2\). Types which satisfy this condition occur as subcases of Case 1 (cf. (2.10)) and Case 2 (cf. (2.11)). Let \(E(r, d)\) be the Poincaré locally free sheaf on \(Y \times J\) defined in (1.3) and parametrizing the indecomposable locally free sheaves on \(Y\) of rank \(r\) and degree \(d\). We consider \(E' = p_{12}^*E(2, e_1) \oplus p_{13}^*E(1, e_2)\) defined on \(Y \times J \times J\) and \(F' = E(2, e)\) defined on \(Y \times J\). The invertible sheaves \(det E'\) and \(det F'\) induce respectively morphisms

\[
\begin{align*}
h_1 : J \times J & \to Pic^e Y \to J, \\
h_2 : J & \to Pic^e Y \to J,
\end{align*}
\]

where

\[
\begin{align*}
h_1(u_1, u_2) &= \det E(2, e_1)_{u_1} \otimes E(1, e_2)_{u_2} \otimes O_Y(-ey_0), \\
h_2(u) &= \det E(2, e)_u \otimes O_Y(-ey_0).
\end{align*}
\]

Let \(Z\) be the fibre product

\[
\begin{array}{ccc}
Z & \xrightarrow{f_1} & J \times J \\
\downarrow{f_2} & & \downarrow{h_1} \\
J & \xrightarrow{h_2} & J
\end{array}
\]

Let \(E = (id \times f_1)^*E'\) and let \(F = (id \times f_2)^*F'\). Since \(h_1\) and \(h_2\) are smooth, surjective morphisms the fibre product \(Z\) is smooth and by construction \(det E_z \cong det F_z\) for every \(z \in Z\). If we specify the pair \((E, F)\) to be of one of the types considered in Case 1 or Case 2, e.g. \(E\) has even degree, \(F\) has odd degree etc., then in particular \(h^0(Y, F_z \otimes O_Z)\), \(h^0(Y, End E_z)\), \(h^0(Y, End F_z)\) are independent of \(z \in Z\). We may now apply Lemma 2.14 with \(Y = Y \times Z\). If there exists \([X \to Y] \in \mathcal{H}_{4,n}(Y)\) which has a pair \((E, F)\) of the considered type then \(\mathbb{H}_0 \neq \emptyset\). Let us denote by \(\mathbb{H}_0^{epi}\) the union of connected components of \(\mathbb{H}_0\) which correspond to \(\pi : X \to Y\) with surjective \(\pi_* : H_1(X, \mathbb{Z}) \to H_1(Y, \mathbb{Z})\). For every of the open sets \(U\) defined in Lemma 2.14 consider the family of quadruple coverings \(\mathcal{H}_{4,n}(Y)\). These morphisms may be glued. Taking the quotient by \(\mathbb{C}^*\) we obtain a morphism \(f : \mathbb{H}_0^{epi} \to \mathcal{H}_{4,n}(Y)\). Every \([X \to Y] \in \mathcal{H}_{4,n}(Y)\) with a pair of the given type belongs to \(f(\mathbb{H}_0^{epi})\). We may now give the precise meaning of Formula (2). \# moduli \([X \to Y] = \dim f(\mathbb{H}_0^{epi}), \# moduli(E, F) = \dim Z\), the fibers of \(\mathbb{H}_0^{epi} \to Z\) have dimension \(h^0(\mathcal{F} \otimes S^2E) - 1\), the fibers of
$f : \mathbb{P}H^{p_0}_{0} \to \mathcal{H}_{4,n}^{0}(Y)$ have dimension $h^0(End E) - 1 + h^0(End F) - 1$. The calculations of (2.6) and (2.8) show that the right-hand side of (2) is $< n$. Therefore $\text{codim} f(\mathbb{P}H^{p_0}_{0}) \geq 1$. In a similar manner for each of the types considered in Cases 1 – 7 one constructs a smooth $Z$ of dimension $\# \text{moduli}(E, F)$ and locally free sheaves $\mathcal{E}$ and $\mathcal{F}$ over $Y \times Z$ and then applies Lemma 2.14. The calculations made in (2.6) – (2.13) show that all possible types except the one specified in the theorem yield closed subschemes of $\mathcal{H}_{4,n}(Y)$ of codimension $\geq 1$. Theorem 2.3 is proved.

**Theorem 2.16.** Let $Y$ be an elliptic curve. Let $n$ be a pair integer $n = 2e \geq 2$. Let $A \in \text{Pic}^{e}Y$. The Hurwitz spaces $\mathcal{H}_{4,n}^{0}(Y)$ and $\mathcal{H}_{4,A}^{0}(Y)$ (cf. 2.3) are irreducible. The variety $\mathcal{H}_{4,A}^{0}(Y)$ is unirational. It is rational if $(e, 6) = 1$.

**Proof.** Let $y_0 \in Y$ be a fixed point. The morphism $h : \mathcal{H}_{4,n}^{0}(Y) \to \text{Pic}^{e}Y$ defined by $h([X \to Y]) = \det E$ is surjective with fibers $\mathcal{H}_{4,A}^{0}(Y)$ which are isomorphic to each other (cf. [Ka] Lemma 2.5). Hence it suffices to prove the statements for $\mathcal{H}_{4,A}^{0}(Y)$ with $A = \mathcal{O}_Y(ey_0)$. We have four cases according to Theorem 2.3.

Case 1. $e \not\equiv 0(\text{mod} 3), e \equiv 1(\text{mod} 2)$, i.e. $(e, 6) = 1$. According to Atiyah’s results [A12] up to isomorphism there are unique indecomposable locally free sheaves $E$ and $F$ of ranks 3 and 2 respectively with $\det E \cong A \cong \det F$. Let us apply Lemma 2.14 with $Z = \{\ast\}, Y = Y \times Z, \mathcal{E} = E, \mathcal{F} = F$. The subset $\mathbb{H}_0 \subset H^0(Y, \mathcal{F} \otimes S^2E)$ of that lemma is not empty since by Theorem 2.3 every sufficiently general $[X \to Y] \in \mathcal{H}_{4,n}^{0}(Y)$ has an associated pair of locally free sheaves isomorphic to $(E, F)$. Clearly $\mathbb{H}_0$ is invariant with respect to multiplication by constants in $\mathbb{C}^*$. Using the universal property of the Hurwitz space $\mathcal{H}_{4,n}(Y)$ one obtains a morphism $f : \mathbb{P}\mathbb{H}_0 \to \mathcal{H}_{4,n}(Y)$. Since $\mathbb{P}\mathbb{H}_0$ is irreducible its image belongs to $\mathcal{H}_{4,A}^{0}(Y)$. The morphism $f : \mathbb{P}\mathbb{H}_0 \to \mathcal{H}_{4,n}(Y)$ is dominant by Theorem 2.3 and injective since $h^0(End E) = h^0(End F) = 1$ (cf. 2.1). Hence $\mathcal{H}_{4,A}^{0}(Y)$ is irreducible and rational.

Case 2. $e \equiv 0(\text{mod} 3), e \equiv 1(\text{mod} 2)$. Let $F$ be indecomposable of rank 2 with $\det F \cong A$. In order to define $Z$ and $\mathcal{E}$ we recall a construction due to Friedman, Morgan and Witten [FMW]. Let $m \geq 1$, let $\mathbb{P}^{m-1} = |my_0|$ and let $|my_0|_{s}$ be the open subset consisting of simple divisors. In Theorem 2.1 (ibid) it is constructed a locally free sheaf $U(m)$ of rank $m$ over $Y \times |my_0|$ with the property that for every $z \in |my_0|_{s}, z = y_1 + \cdots + y_m, y_i \neq y_j$, the restriction $U(m)|_{Y \times \{z\}} \cong \mathcal{O}_Y(y_1 - y_0) \oplus \cdots \oplus \mathcal{O}_Y(y_m - y_0)$. For Case 2 we need $m = 3$. We let $Z = |my_0|_{s}$ and apply Lemma 2.14 with $Y = Y \times Z, q = p_2 : Y \times Z \to Z, \mathcal{E} = U(3) \otimes p_1^*\mathcal{O}_Y(y_0)$ and $\mathcal{F} = p_1^*F$. As in Case 1 we obtain a dominant morphism $f : \mathbb{P}\mathbb{H}_0 \to \mathcal{H}_{4,A}^{0}(Y)$. Since $\mathbb{P}\mathbb{H}_0$ is Zariski open subset in the projectivization of a vector bundle over $Z$ we conclude $\mathcal{H}_{4,A}^{0}(Y)$ is irreducible and unirational.
Case 3. \( e \neq 0 \mod 3, e \equiv 0 \mod 2 \). This case is similar to the preceding one. We let \( E \) be an indecomposable locally free sheaf of rank 3 with \( \det E \cong A \). We consider \( U(2) \) over \( Y \times |2y_0| \), let \( Z = |2y_0|,s \) and apply Lemma 2.14 with \( Y = Y \times Z, \mathcal{E} = p_1^0 E, \mathcal{F} = U(2) \otimes p_1^1 \mathcal{O}_Y(\frac{2}{5}y_0) \).

Case 4. \( e \equiv 0 \mod 6 \). Here we consider \( U(3) \) over \( Y \times |3y_0| \) and \( U(2) \) over \( Y \times |2y_0| \). We let \( Z = |3y_0|,s \times |2y_0|,s \) and applying Lemma 2.14 to \( Y = Y \times Z, \mathcal{E} = p_1^0 U(3) \otimes p_1^1 \mathcal{O}_Y(\frac{3}{5}y_0) \) and \( \mathcal{F} = p_1^0 U(2) \otimes p_1^1 \mathcal{O}_Y(\frac{2}{5}y_0) \) we obtain a dominant morphism \( f : \mathbb{P} \mathbb{H}_0 \to \mathcal{H}_{4,A}^0(Y) \). This shows \( \mathcal{H}_{4,A}^0(Y) \) is irreducible and unirational.

Remark 1. We showed in the preceding theorem that \( \mathcal{H}_{4,n}^0(Y) \) is a connected component of \( \mathcal{H}_{4,n}(Y) \). The Hurwitz space \( \mathcal{H}_{4,n}(Y) \) has three other connected components which correspond to quadruple coverings \( \pi : X \to Y \) such that \( |H_1(Y,\mathbb{Z}) : \pi_*H_1(X,\mathbb{Z})| = 2 \). Namely one fixes an étale covering \( \pi_2 : \bar{Y} \to Y \) of degree 2. Then every double covering \( \pi_1 : X \to \bar{Y} \) branched in \( n \) points which belong to different fibers of \( \pi_2 \) yields a simple quadruple covering \( \pi = \pi_2 \circ \pi_1 : X \to Y \). One obtains in this way a connected component of \( \mathcal{H}_{4,n}(Y) \) isomorphic to a Zariski open subset of \( \mathcal{H}_{2,n}(\bar{Y}) \).

Remark 2. Graber, Harris and Starr proved \cite{GHS} the irreducibility of the space \( \mathcal{H}_{d,n}^S(Y) \) parameterizing simple coverings with monodromy group \( S_d \) for any \( Y \) of positive genus when \( n \geq 2d \).

2.17. We considered so far families of quadruple coverings over a fixed elliptic curve \( Y \). In order to treat the problem of unirationality of \( \mathcal{A}_3(1,1,4) \) we need to vary \( Y \). Simple quadruple coverings \( \pi : X \to Y \) which have 3-dimensional Prym varieties are branched in 6 points and according to Theorem 2.3 a general \( [X \to Y] \) in \( \mathcal{H}_{4,6}^0(Y) \) is associated to: a pair \( E = M_1 \oplus M_2 \oplus M_3 \) with \( \deg M_i = 1 \); an indecomposable \( F \) with \( \det F \cong \det E \) and an element \( \eta \in H^0(Y,\mathcal{F} \otimes S^2E) \). Let \( q : \mathcal{Y} \to B \) be a smooth family of elliptic curves obtained from a general pencil of cubic curves in \( \mathbb{P}^2 \) by blowing-up the nine base points and discarding the singular fibers. \( B \) is an open subset of \( \mathbb{P}^1 \). Let \( \sigma : B \to \mathcal{Y} \) be a section of the family, \( \sigma(B) = D \). One constructs by extension as in \cite{Ka} (2.13) a rank 2 locally free sheaf \( F \) on \( \mathcal{Y} \) such that for each \( b \in B \) the restriction \( F_b = F|_{\mathcal{Y}_b} \) is indecomposable with \( \det F_b \cong \mathcal{O}_b(3\sigma(b)) \). Following Section 4 of \cite{FMW} let \( \mathcal{V}_3 = q_* \mathcal{O}_\mathcal{Y}(3D) \). The projective bundle \( \mathbf{P}\mathcal{V}_3 \to B \) has fiber over \( b \in B \) equal to \( |3\sigma(b)| \subset (\mathcal{V}_b)^{(3)} \). Let us consider the locally free sheaf \( U_0 \) over \( \mathcal{Y} \times_B \mathbf{P}\mathcal{V}_3 \) defined just after \cite{FMW} Theorem 4.11. It has the property that if \( z = y_1 + y_2 + y_3 \in |\mathcal{O}_b(3\sigma(b))| \) is a simple divisor, then \( U_0|_{\mathcal{Y}_b \times \{z\}} \cong \mathcal{O}_b(y_1 - \sigma(b_1)) \oplus \mathcal{O}_b(y_2 - \sigma(b_2)) \oplus \mathcal{O}_b(y_3 - \sigma(b_3)) \). Let \( Z \subset \mathbf{P}\mathcal{V}_3 \) be the open subset consisting of the simple divisors in the fibers of \( \mathbf{P}\mathcal{V}_3 \), let \( q' : \mathcal{Y}' = \mathcal{Y} \times_B Z \to Z \) be the projection, let \( \mathcal{E} = U_0 \otimes p_1^0 \mathcal{O}_Y(D) \) and let \( \mathcal{F} = p_1^0 F \). Applying Lemma 2.14 and letting \( \mathbb{P}\mathbb{H}_0 = T \) we obtain a
commutative diagram

Letting \( Y_T = Y' \times_Z T = Y \times_B T \) one obtains a family of quadruple coverings over \( T \):

\[
\begin{array}{ccc}
X & \xrightarrow{p} & Y_T \\
\downarrow & & \downarrow \\
T & \xrightarrow{} & B
\end{array}
\]

(25)

**Proposition 2.18.** The constructed family of quadruple coverings has the following properties.

(a) Every sufficiently general elliptic curve is isomorphic to a fiber of \( Y \to B \).

(b) Let \( b \in B \). The fibers \( X_\eta \to Y_b \) with \( \eta \in T \), \( \eta \mapsto b \) correspond to a Zariski open nonempty subset of the Hurwitz space \( H^0_{4,A}(Y_b) \) with \( A = \mathcal{O}_{Y_b}(3\sigma(b)) \).

(c) \( T \) is a rational variety of dimension 8.

**Proof.** The statements follow from Lemma and Theorem □

### 3. The Prym mapping

**3.1.** A family of quadruple coverings of elliptic curves is given by a commutative triangle

\[
\begin{array}{ccc}
X & \xrightarrow{p} & Y \\
\downarrow & & \downarrow \\
T & \xrightarrow{f} & \xrightarrow{q}
\end{array}
\]

(26)

where \( X, Y \) and \( T \) are smooth, connected, \( f \) and \( q \) are smooth, proper of relative dimension 1 with connected fibers, \( g(X_s) = g(Y_s) = 1 \) and \( p \) is finite, surjective (and therefore flat) of degree 4. We will work both in the algebraic and the complex analytic category. Assume \( (p_s)_s : H_1(X_s, \mathbb{Z}) \to H_1(Y_s, \mathbb{Z}) \) is surjective for some (and thus for all) \( s \in T \). Define the Prym mapping using Proposition and Proposition 3.14 by \( \Phi : T \to A_{g-1}(1, \ldots, 1, 4) \) where \( \Phi(s) = [\text{Ker}(Nm_s)] \). The Prym mapping is holomorphic and furthermore if the family is algebraic then \( \Phi \) is an algebraic morphism. We aim to prove the unirationality of \( A_3(1, 1, 4) \) by proving that the family yields a dominant morphism. For this we need to verify that the differential of the corresponding Prym mapping is generically surjective.
3.2. Given a simple quadruple covering \( \pi : X \to Y \) of an elliptic curve such that \( \pi_* : H_1(X, \mathbb{Z}) \to H_1(Y, \mathbb{Z}) \) is surjective we consider as in \([Ka]\) (4.2) a commutative diagram of holomorphic mappings

\[
\begin{array}{ccc}
X & \xrightarrow{p} & Y \\
\downarrow f & & \downarrow q \\
N \times H & \xrightarrow{\pi_1} & N
\end{array}
\]

where \( q : Y \to N = \Delta \) is a minimal versal deformation of \( Y, H \cong \Delta^n \). Here \( \Delta \) is a unit disk. The family of coverings is versal for deformations of \( \pi : X \to Y \) in the sense of Horikawa \([Hor]\) (cf. \([Ka]\) Proposition 4.3). The Prym mapping may be lifted to a holomorphic period mapping into the Siegel upper half space

\[
\tilde{\Phi} : N \times H \to \tilde{A}_{g-1}(D)
\]

where \( D = (1, \ldots, 1, d) \). Given a covering of an elliptic curve there is a canonically defined point \( q^- \in |\omega_X|^* = \mathbb{P}^{g-1} \) which corresponds to the hyperplane \( H^0(X, \omega_X)^- \) of differentials which have trace 0. Let \( s_0 \in N \times H \) be the reference point which corresponds to \( \pi : X \to Y \). In \([Ka]\) Proposition 4.16 the following criterion is proved:

Suppose \( X \) is not hyperelliptic and \( g(X) \geq 4 \). Then \( \dim \ker d\tilde{\Phi}(s_0) = 1 \) (the minimal possible dimension) if and only if the point \( q^- \in |\omega_X|^* \) does not belong to the intersection of quadrics which contain \( \phi_K(X) \).

In particular if \( g(X) = 4 \) the condition is \( q^- \notin Q \) where \( Q \) is the unique quadric which contains \( \phi_K(X) \).

Proposition 3.3. Let \( Y \) be an elliptic curve. Then every sufficiently general simple quadruple covering \( \pi : X \to Y \) such that \( g(X) = 4 \) and \( \pi_* : H_1(X, \mathbb{Z}) \to H_1(Y, \mathbb{Z}) \) is surjective satisfies the conditions of the above criterion: \( X \) is not hyperelliptic and \( q^- \notin Q \).

Proof. That a general quadruple cover \( X \) of genus 4 is not hyperelliptic is a particular case of \([Ka]\) Proposition 4.14. We prove that \( q^- \notin Q \) for a general \([X \to Y] \in \mathcal{H}^0_{4,6}(Y) \) by way of degeneration in a manner similar to the case of triple coverings (cf. \([Ka]\) (4.19)-(4.22)).

Step 1. Choose three points \( \{b_1, b_2, b_3\} \subset Y \). Let \( C_1 = Y, p_1 : C_1 \to Y \) be the identity mapping and let \( x_i = p_1^{-1}(b_i) \). Let \( p_2 : C_2 \to Y \) be a cyclic unramified covering of degree 3. Let \( p_2^{-1}(b_i) = \{y_i, y'_i, y''_i\} \). We consider the following quadruple covering of \( Y \):

\[
X' = C_1 \cup C_2 / \{x_i \sim y_i\}_{i=1}^3, \quad \pi' = p_1 \cup p_2 : X' \to Y.
\]
The same argument as that of the proof of [Ka] Proposition 4.20 shows that \( \phi_K(X') \) is contained in a unique quadric \( Q \) which is reducible and \( q^- \notin Q \).

**Step 2.** We construct curves \( X' \) of the considered type on an elliptic ruled surface. Let \( \eta \) be a point of order 3 in \( Pic^0 Y \). Let \( G = \mathcal{O}_Y \oplus \eta \). Consider the ruled surface \( W = P(\mathcal{O}_Y \oplus \eta) \), \( \rho : W \to Y \). Let \( Y_0 \) be the section corresponding to \( \mathcal{O}_Y \oplus \eta \to \eta \to 0 \). We have \( Y_0 \in |\mathcal{O}_{P(G)}(1)| \). Let \( y \in Y \).

Using the notation of [Ha] Ch.V Proposition 2.9, one has \( h^1(D) = 0, h^0(D) = 2, D^2 = 2 \). Let \( \epsilon \) be a divisor on \( Y \) defined by \( \eta = \wedge^2 G \cong L(\epsilon) \). Consider the section \( Y_\infty \) which corresponds to the second normalization \( G \oplus \eta^{-1} \) or equivalently to \( G \to \mathcal{O}_Y \to 0 \). Then \( Y_\infty \sim Y_0 - cf, Y_0 \cdot Y_\infty = 0 \) (cf. [Ha] Ch.V Proposition 2.9) and if \( y_1 \sim y - \epsilon \) one has \( Y_0 + yf \sim Y_\infty + y_1 f \). This shows that the pencil \( |D| = [Y_0 + yf] \) has two base points, \( Bs|D| = \{P_1, P_2\} \) where \( P_1 = Y_\infty \cap \rho^{-1}(y_1) \), \( P_2 = Y_0 \cap \rho^{-1}(y_1) \). From Bertini’s theorem it is clear that the general member of \( |D| \) is irreducible and nonsingular.

Now, consider \( |3Y_0| \). We have \( \rho_* \mathcal{O}_{P(G)}(3) = \mathcal{O}_Y \oplus \eta \oplus \eta^2 \oplus \mathcal{O}_Y \). Thus \( h^0(W, \mathcal{O}_W(3Y_0)) = 2 \). Since \( \eta \cong L(\epsilon) \) we have \( 3\epsilon \sim 0 \). Thus \( 3Y_0 \sim 3Y_\infty \). We conclude \( |3Y_0| \) is a pencil without base points. Let \( \varphi = \varphi|_{3Y_0} : W \to \mathbb{P}^1 \). Let \( W \to Z \twoheadrightarrow \mathbb{P}^1 \) be the Stein decomposition. Every fiber of \( \rho : W \to \mathbb{P}^1 \) maps surjectively to \( Z \), so \( Z \cong \mathbb{P}^1 \). Since \( 3Y_0 \sim 3Y_\infty \), if \( \deg(g) > 1 \) then \( \deg(g) = 3 \) and \( g : Z \to \mathbb{P}^1 \) would have total ramification at the points corresponding to \( 3Y_0 \) and \( 3Y_\infty \). Since \( Z \cong \mathbb{P}^1 \) this would imply \( Y_0 \sim Y_\infty \) which is absurd.

We conclude \( |3Y_0| \) is a pencil without base points whose general member is irreducible and nonsingular. The arithmetic genus \( p_a(3Y_0) = 1 \), so every sufficiently general curve \( C \in |3Y_0| \) is an elliptic curve which is étale triple covering of \( Y \). Let \( C_1 \in |Y_0 + yf| \) and \( C_2 \in |3Y_0| \) be sufficiently general. Then \( C_1 \cdot C_2 = 3 \) and the intersection points belong to different fibers of \( C_2 \to Y \) since \( C_1 \cdot f = 1 \). Furthermore one may choose \( C_1 \) and \( C_2 \) in such a way that \( C_1 \cap C_2 \) does not have points in common with \( Bs|Y_0 + yf| \). We see that \( X' = C_1 \cup C_2 \) is a curve of the type considered in Step 1. It is easy to show by Bertini’s theorem that \( X' \) is smoothable. However we need a stronger statement.

**Step 3.** \( X' \) is strongly smoothable (cf. [HH] p.100). This means that there exist smooth, connected \( X \) and \( B \), \( \dim X = 2 \), \( \dim B = 1 \) and an embedding \( X \subset W \times B \) such that the second projection \( \pi_2 : X \to B \) is proper, one of its fibers \( \pi_2^{-1}(b_0) \cong X' \) and all other fibers are smooth curves in \( W \). We notice that the standard criterion for such smoothing \( K_W \cdot C_i < 0 \) for every \( i \) (cf. [N1]) cannot be applied here since \( K_W \cdot C_2 = 0 \). We use the smoothing technique of Hartshorne and Hirschowitz [HH]. In their paper it is stated for curves in \( \mathbb{P}^3 \), however the arguments can be easily extended to curves lying in arbitrary smooth projective variety, in particular to the simple case of curves on a surface. Let \( N_X' \) be the normal sheaf of \( X' = C_1 \cup C_2 \) and let \( T^1_X \) be the \( T^1 \) functor of Lichtenbaum-Schlessinger [LS]. Since \( X' \) is a curve
whose singular points \( P \in S = Sing X \) are nodes one has \( T^1_X, \cong \oplus_{P \in T^1_P} \) where \( T^1_P \cong C_P \). According to \[\text{(III)}\] Proposition 1.1 we have to prove two things: a) \( H^1(N_X) = 0 \) and b) the natural map \( H^0(N_X) \to H^0(T^1_P) \) is surjective for every node \( P \). According to \[\text{(III)}\] Corollary 3.2 for \( i = 1, 2 \) one has an exact sequence

\[ O \to N_{C_1} \to N_{X'_i} \to T^1_i \to 0. \]

Hence \( N_{X'_i}|_{C_1} \cong N_{C_1}(x_1 + x_2 + x_3), N_{X'_i}|_{C_2} \cong N_{C_2}(y_1 + y_2 + y_3) \). We have \( \deg N_{C_1} = 2, N_{C_2} \cong O_{C_2} \). We now apply the analog of \[\text{(III)}\] Theorem 4.1. The conditions that we have to check in our case are: a) \( H^1(C_2, N_{C_2}(y_i + y_j)) = 0 \) for \( 1 \leq i < j \leq 3 \) and b) \( H^1(C_1, N_{C_1}) = 0 \). Both are obviously satisfied. The same proof as that of \[\text{(III)}\] Theorem 4.1 then shows \( X' \) is strongly smoothable.

**Step 4.** Let \( p : X \to Y \times B \) be the composition of \( X \xhookrightarrow{i} W \times B \) with \( \rho \times id : W \times B \to Y \times B \). We claim that replacing \( B \) by a neighborhood of \( b_0 \) we may assume that for each \( b \in B - \{b_0\} \) the covering \( p_b : X_b \to Y \times \{b\} \) is simple and \( (p_b)_* : H_1(X_b, \mathbb{Z}) \to H_1(Y, \mathbb{Z}) \) is surjective. The first statement is clear since \( \pi' : X' \to Y \) has the property that each fiber \( (\pi')^{-1}(y) \) has at least 3 elements. The second property is topological, so it suffices to prove it replacing \( B \) by a small disk \( \Delta \) of \( b_0 \). Let \( b \in \Delta - \{b_0\} \). We have a commutative diagram

\[
\begin{array}{ccc}
X_b & \xrightarrow{i} & X \\
\downarrow & & \downarrow \\
Y \times \{b\} & \xhookrightarrow{} & Y \times \Delta & \xhookrightarrow{} & Y \times \{b\}
\end{array}
\]

It is well-known that \( X \) is a deformation retraction of \( X_{b_0} \) and \( i_* : H_1(X_b, \mathbb{Z}) \to H_1(X, \mathbb{Z}) \) is surjective [\text{CI}][\text{H8}]. By the Mayer-Vietoris sequence it is clear that \( \pi'_* : H_1(X_{b_0}, \mathbb{Z}) \to H_1(Y, \mathbb{Z}) \) is surjective. Thus \( (p_b)_* : H_1(X_b, \mathbb{Z}) \to H_1(Y, \mathbb{Z}) \) is surjective.

**Step 5.** Replacing \( B \) by a neighborhood of \( b_0 \) we may achieve that for each \( b \in B - \{b_0\} \) the canonical map \( \phi_K : X_b \to \mathbb{P}^3 \) is an embedding and the point \( q^{-1}(b) \) corresponding to \( H^0(X_b, \omega_{X_b})^{-} \) does not belong to the unique quadric which contains \( \phi_K(X_b) \). This statement is proved by the same argument used in [\text{K}a Proposition 4.22].

We may now conclude the proof of the proposition. Let \( T = \mathcal{H}^0_{4,6}(Y) \) and let (abusing the notation) \( p : X \to Y \times T \) be the corresponding universal family of quadruple coverings. We showed in Steps 4 and 5 that there is an \( s \in T \) such that the statement of the proposition holds for \( p_s : X_s \to Y \times \{s\} \). Applying again to this family the argument of [\text{K}a Proposition 4.22] and using the fact that \( T \) is irreducible (Theorem 2.10) we conclude that the property \( q^{-} \notin Q \) holds for a Zariski open dense subset of \( \mathcal{H}^0_{4,6}(Y) \).
Our approach yields an alternative proof of a result due to Birkenhake, Lange and van Straten \cite{BLvS}.

**Theorem 3.4.** The moduli space of polarized abelian surfaces $A_2(1,4)$ is unirational.

Proof. This is proved in the same way as \cite{Ka} Theorem 5.1 by fixing $A \in \text{Pic}^2 Y$, proving that the Prym mapping $\Phi: H^0_{A,Y} \to A_2(1,4)$ is dominant and thus deducing the unirationality of $A_2(1,4)$ from the unirationality of $H^0_{A,Y}$ proved in Theorem 2.16. □

**Theorem 3.5.** The moduli spaces of polarized abelian threefolds $A_3(1,1,4)$ and $A_3(1,4,4)$ are unirational.

Proof. By a result of Birkenhake and Lange \cite{BL} the moduli spaces $A_3(1,1,4)$ and $A_3(1,4,4)$ are isomorphic to each other. So it suffices to prove that $A_3(1,1,4)$ is unirational. The proof is analogous to the proof of the unirationality of $A_3(1,1,3)$ (cf. \cite{Ka} Theorem 5.3). Using the family of Proposition 2.18 one obtains the Prym morphism $\Phi: T \to A_3(1,1,4)$. We wish to prove $\Phi$ is dominant. If $s_0 \in T$ is general enough then using Proposition 3.3 and the same argument as in \cite{Ka} Theorem 5.3 one obtains a lifting of $\Phi$, $\tilde{\Phi}' : S \to \mathfrak{s}_3$ in a complex neighborhood $S$ of $s_0$ and a commutative diagram of holomorphic mappings

\[
\begin{array}{ccc}
S & \xrightarrow{\tilde{\Phi}'} & \mathfrak{s}_3 \\
\mu & \nearrow & \\
N \times H & \xrightarrow{\Phi} & \\
\end{array}
\]

such that a neighborhood of $\tilde{\Phi}'(s_0)$ is contained in $\tilde{\Phi}(N \times H)$. Consider the family of quadruple coverings $\mathcal{X} \to \mathcal{Y} \times_N (N \times H)$ induced from (27). Let $\mathcal{E}$ and $\mathcal{F}$ be the associated locally free sheaves of rank 3 and 2. The set of $u \in N \times H$ such that $\mathcal{F}_u$ is stable and $\mathcal{E}_u$ is semistable is open in $N \times H$ (cf. \cite{Ka} Appendix B). Moreover it is open the set of $u \in N \times H$ such that $\mathcal{E}_u$ is regular polystable (ibid.) Since $\deg \mathcal{E}_u = 3 = rk \mathcal{E}_u$, being regular polystable means that $\mathcal{E}_u$ is isomorphic to a direct sum $M_1 \oplus M_2 \oplus M_3$, where $M_i$ are invertible sheaves which are pairwise non-isomorphic to each other.

So for every $u$ in a neighborhood of $\mu(s_0)$ the pair $(\mathcal{E}_u, \mathcal{F}_u)$ is of the type specified in Theorem 2.3. Composing a covering of an elliptic curve with its translations one obtains coverings which have the same kernel of the norm map of the Jacobians while the determinants of the Tschirnhausen modules vary over the whole Picard group of invertible sheaves of degree $-e$, in our case $e = 3$. By the construction of the family of Proposition 2.18 this shows that $\tilde{\Phi}'(S)$ contains the image by $\tilde{\Phi}$ of a certain neighborhood of $\mu(s_0)$ in $N \times H$. Therefore $\tilde{\Phi}'(S)$ contains a neighborhood of $\tilde{\Phi}(s_0)$ in $\mathfrak{s}_3$. This shows $\Phi : T \to A_3(1,1,4)$ is dominant. Therefore $A_3(1,1,4)$ is unirational. □
The proof of the theorem as well as [Ka] Proposition 3.14 yield the following corollary.

**Corollary 3.6.** Every sufficiently general abelian threefold with polarization of type $(1, 1, 4)$ is isomorphic to the Prym variety of a simple quadruple covering of an elliptic curve branched in 6 points. Every sufficiently general abelian threefold with polarization of type $(1, 4, 4)$ is isomorphic to $\text{Pic}^0 X/\pi^* \text{Pic}^0 Y$ for a certain quadruple covering $\pi : X \to Y$ as above.

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