On measures strongly log-concave on a subspace

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Abstract

In this work we study the concentration properties of log-concave measures that are curved only on a subspace of directions. Proofs uses an adapted version of the stochastic localization process.

1 Introduction

Let $V : \mathbb{R}^n \to \mathbb{R}$ be a $C^2$ convex function, such that $d\mu(x) = e^{-V(x)}dx$ is a log-concave probability measure. It is well-known that if $\mu$ is $t$-strongly log-concave, that is $V$ satisfies the Bakry-Émery condition:

$$\nabla^2 V \geq t$$

for some $t > 0$, where $\nabla^2$ stands for the Hessian, it has good isoperimetric properties. In particular, its Poincaré constant is at most $\frac{1}{t}$. Recall that the measure $\mu$ is said to satisfy a Poincaré inequality with constant $c$ if for all locally Lipschitz function $f$ we have:

$$\text{Var}_\mu(f) \leq c^2 \mathbb{E}_\mu(\|\nabla f\|^2)$$

where here and in the sequel, $\|\cdot\|$ stands for the Euclidean norm. The best such constant is denoted by $c_P(\mu)$, the Poincaré constant of $\mu$. The KLS conjecture [7] proposes that when $\mu$ is log-concave, its Poincaré constant is, up to a universal constant, less than the operator norm of its covariance matrix. Since Poincaré inequalities are homogeneous, we can state the conjecture only for normalized measures, without loss of generality. A measure $\mu$ is called isotropic if it is centered and its covariance matrix is the identity. Introduce

$$\Psi_n = \sup_{\mu} c_P(\mu),$$

where the supremum runs over all isotropic log-concave measures of $\mathbb{R}^n$. The KLS conjecture then reads:

$$\Psi_n \leq c$$

for some universal constant $c > 0$.

A related property of strongly log-concave probabilities is that they exhibit good concentration function. Recall that the concentration function of a measure $\mu$ is the function $\alpha_\mu : \mathbb{R}^+ \to [0, 1/2]$ defined by:

$$\alpha_\mu(r) = \sup_{\{S, \mu(S) = 1/2\}} \mu(S_r^c)$$
where $S_r = \{ x \in \mathbb{R}^n \mid d(x, S) \leq r \}$ and $d(x, S)$ is the Euclidean distance between $x$ and $S$. It follows from the Prékopa-Leindler inequality that if $\mu$ is $t$-strongly log-concave, then for all measurable sets $S$,

$$\mu(S_r^c) \leq \frac{1}{\mu(S)} \exp \left( -\frac{tr^2}{4} \right).$$

In particular, it has a Gaussian-type concentration function:

$$\alpha_\mu(r) \leq 2 \exp \left( -\frac{tr^2}{4} \right),$$

see for instance [4] Proposition 2.6 and its proof. It was first observed by Gromov and Milman ([6] see also [10] Corollary 3.2 for a better constant) that a Poincaré inequality implies exponential concentration, that is:

$$\alpha_\mu(r) \leq \exp \left( -\frac{r}{3c_P(\mu)} \right).$$

The converse implication has been established in the log-concave case by E.Milman [12] where he shows that when $\mu$ is log-concave,

$$c_P(\mu) \lesssim \alpha_\mu^{-1}(1/4)$$

where for two expressions $a, b$ depending on parameters, $a \lesssim b$ means there is a universal constant $c > 0$ such that $a \leq cb$. We also write $a \simeq b$ when $a \lesssim b$ and $b \lesssim a$.

In an attempt to tackle the KLS conjecture, Eldan [4] introduced a stochastic process, known as stochastic localization, which, roughly, decomposes, for all time $t \geq 0$, a log-concave measure $\mu$ into an average of measures $\mu_t(\omega)$ which are $t$-strongly log-concave. This strategy enabled Eldan to relate the KLS conjecture to the a priori weaker Variance conjecture, then Lee and Vempala [11] to obtain the then better bound on $\Psi_n : \Psi_n \lesssim n^{1/4}$. Recently, Chen obtained that $\Psi_n = o(n^{\alpha})$ for every $\alpha > 0$, [2], and very recently, Klartag and Lehec obtained $\Psi_n = O(\log(n)^5)$ [9].

In this note, we propose a slight generalization of the criterion (1) allowing the potential to be flat in some directions. The observation is that stochastic localization behaves well when restrained to a subspace.

Our main result is the following:

**Theorem 1.1.** Let $V : \mathbb{R}^n \to \mathbb{R}$ be a $C^2$ convex potential such that $d\mu(x) = e^{-V(x)}dx$ is a probability measure. Suppose that there is $1 \leq k \leq n$, a subspace $E$ of codimension $k$ and $\eta > 0$ such that

$$\nabla^2 V \geq \eta P_E$$

where $P_E$ is the orthogonal projector onto $E$. Let $K$ be the covariance matrix of $\mu$. Define $Q = P_{E^\perp}KP_{E^\perp}$

$(i)$ $c_P(\mu) \lesssim \max \left( \frac{1}{\sqrt{\eta}}, \|Q\|_{op}^{1/2} \Psi_k \max(\log(k), 1) \right)$

$(ii)$ There is a universal constant $c > 0$ such that for every $A$ such that $\mu(A) = \frac{1}{2}$,

$$\mu(A_r^c) \leq \exp \left( -c \min \left( \frac{r}{\|Q\|_{op}^{1/2}}, r^2 \min \left( \frac{1}{\Psi_k \max(\log(k), 1) \|Q\|_{op}}, \frac{\eta}{\max(\log(k), 1) \|Q\|_{op}} \right) \right) \right)$$

In the particular case $E = \{0\}$, inequality $(ii)$ implies a new bound for the concentration function of log-concave measures, which we state, without loss of generality, in the isotropic case.
Corollary 1.2. For any isotropic log-concave measure $\mu$ and any $r > 0$, we have

$$\alpha_{\mu}(r) \lesssim \exp \left( -c \min \left( r, \frac{r^2}{\psi_n^2 \log(n)} \right) \right)$$

Remark 1. Note that (ii) implies (i). Indeed, choosing $r = c' \max \left( \frac{1}{\sqrt{n}} \|Q\|_n^{1/2} \Psi_k \sqrt{\max(\log(k), 1)} \right)$, for an appropriate choice of constant $c' > 0$, we get that $\mu(A_r^c) \leq \frac{1}{2}$. By (5), this implies (i). On the other hand it is easy to check that the exponential concentration obtained by combining (i) with (4) is weaker than (ii).

Remark 2. The idea of evaluating concentration functions with stochastic localization already appears in the work of Lee and Vempala ([11], Theorem 16). To improve the Paouris deviation inequality for the Euclidean norm ([14]), they develop a more refined analysis of the process, using the so-called Stieltjes potential. They prove that for any $L$-Lipschitz function $g$, and any isotropic log-concave probability measure $\mu$ one has:

$$\forall t \geq 0 \quad \mathbb{P}(|g(X) - \bar{g}(X)| \geq Lt) \leq \exp \left( -\frac{ct^2}{t + \sqrt{n}} \right)$$

where $X \sim \mu$ and $\bar{g}(X)$ is the median or mean of $g(X)$. Notice that when $g$ is the Euclidean norm, then by Borell’s Lemma [1], $\mathbb{E}_\mu(|x|) \simeq \mathbb{E}_\mu(|x|^2)^{1/2} = \sqrt{n}$ since $\mu$ is isotropic. Plugging this into (6) yields

$$\forall t \geq 0 \quad \mathbb{P}(|X| \geq t \sqrt{n}) \leq \exp(-c \min(t, t^2) \sqrt{n})$$

However, thanks to the new estimate of Chen, $\Psi_k = o(n^k)$ for every $\alpha > 0$, we can obtain this result directly from Corollary 1.2. Indeed, for a general isotropic log-concave probability measure $\mu$ it asserts that for all measurable $A$ such that $\mu(A) = 1/2$ and all $r > 0$,

$$\mu(A_r^c) \lesssim \exp \left( -c \min \left( r, \frac{r^2}{\psi_n^2 \log(n)} \right) \right).$$

Let $g$ be a $L$-Lipschitz function, and let $A = \{ x \in \mathbb{R}^n, g(x) \leq \bar{g}(X) \}$, by definition of the median, $\mu(A) = 1/2$. Now set $G_r = \{ x \in \mathbb{R}^n, g(x) \leq \bar{g}(X) + Lr \}$, then because $g$ is $L$-Lipschitz, $A_r \subset G_r$, where $A_r$ is the $r$-extension of $A$. We get that

$$\mu(G_r^c) \leq \mu(A_r^c) \lesssim \exp \left( -c \min \left( r, \frac{r^2}{\psi_n^2 \log(n)} \right) \right),$$

For the Euclidean norm, which is $1$-Lipschitz, this yields

$$\mathbb{P}(|X| \geq r \sqrt{n}) \lesssim \exp \left( -c \min \left( r \sqrt{n}, \frac{r^2 n}{\psi_n^2 \log(n)} \right) \right) \lesssim \exp \left( -c \min(r, r^2) \sqrt{n} \right),$$

where we used the fact that $\psi_n^2 \log(n) = o(\sqrt{n})$ thanks to Chen’s estimate. Notice that using the Lee-Vempala estimate $\psi_n^2 = O(\sqrt{n})$ would lead to an extra logarithmic factor in the deviation estimate whose removal was the object of their work with the Stieltjes potential.

Lemma 1.3. It is enough to prove Theorem 1.1 when $Q = I_k$.

Proof. Let $d\mu(x) = e^{-V(x)}dx$ be a measure satisfying the hypothesis of Theorem 1.1 and let $X$ be a random vector whose law is $\mu$. Set $S = \left[ \begin{array}{cc} \|Q\|_n^{1/2} I_{n-k} & 0 \\ 0 & Q^{1/2} \end{array} \right]$ where the matrix is expressed in a
basis adapted to the splitting $\mathbb{R}^n = E \oplus E^\perp$. Define the random vector $\tilde{X} = S^{-1}X$, whose law is $d\tilde{\mu}(x) = e^{-V(x)}dx = |\det S| e^{-V(Sx)}dx$ and covariance matrix

$$\tilde{K} = S^{-1}KS^{-1}.$$ 

For a symmetric $n \times n$ matrix $M$, we denote by $\lambda_1(M) \geq \cdots \geq \lambda_n(M)$ its ordered eigenvalues. It is classical and easy to check that for every $r > 0$ one has

$$\alpha_\mu(r) \leq \alpha_{\tilde{\mu}} \left( \frac{r}{\lambda_1(S)} \right) = \alpha_{\tilde{\mu}} \left( \frac{r}{\|Q\|^{1/2}_{op}} \right)$$

However, with this choice of $S$, $\tilde{\mu}$ satisfies:

$$\lambda_{n-k}(P_E\nabla^2 \tilde{V} P_E) \geq \tilde{\eta} = \|Q\|_{op} \tilde{\eta} \quad \text{and} \quad \tilde{Q} = P_{E^\perp} \tilde{K} P_{E^\perp} = I_k.$$ 

We can then apply Theorem 1.1 to $\tilde{\mu}$ which, combined with (8), yields the result. $\square$

We conclude this introduction with a classical inequality, which essentially goes back to Freedman [5], that we will use for controlling deviation of martingales in the sequel.

**Lemma 1.4.** Let $M_t$ be a continuous local martingale starting from 0.

$$\forall T > 0 \quad \mathbb{P}(M_T \geq a, [M]_T \leq b) \leq \exp(-\frac{a^2}{2b}).$$

**Proof.** For all $\lambda \in \mathbb{R}$, define the process $E(\lambda M)$ by

$$E(\lambda M)_t = \exp \left( \lambda M_t - \frac{\lambda^2}{2}[M]_t \right).$$

Elementary Itô calculus shows that $E(\lambda M)$ is a local martingale. Moreover, it is positive, so by Fatou’s lemma it is a supermartingale. In particular, for all $t \geq 0$, $\mathbb{E} E(\lambda M)_t \leq E(\lambda M)_0 = 1$, that is:

$$\forall t \geq 0, \quad \mathbb{E} \exp \left( \lambda M_t - \frac{\lambda^2}{2}[M]_t \right) \leq 1$$

Now, assume that $[M]_T \leq b$ almost surely. Then,

$$\mathbb{P}(M_T \geq a) = \mathbb{P}(E(\lambda M)_t \geq e^{\lambda a - \frac{\lambda^2}{2}[M]_T})$$

$$\leq \mathbb{P}(E(\lambda M)_t \geq e^{\lambda a - \frac{\lambda^2}{2}b})$$

$$\leq \mathbb{E}(E(\lambda M)_t) \cdot e^{\frac{\lambda^2}{2}b - \lambda a}$$

Choosing the optimal $\lambda = \frac{a}{b}$ yields:

$$\mathbb{P}(M_T \geq a) \leq e^{-\frac{a^2}{2b}}.$$ 

The proof follows from applying this argument to the local martingale $M^\tau_t = M_{t \wedge \tau}$, where $\tau = \inf\{t \geq 0, [M]_t \geq b\}$ is a stopping time. Indeed, remark that $[M_{t \wedge \tau}]_t \leq b$ almost surely, and that

$$\mathbb{P}(M_T \geq a, [M]_T \leq b) \leq \mathbb{P}(M^\tau_T \geq a) \leq e^{-\frac{a^2}{2b}}.$$ 

$\square$
2 Restricted stochastic localization

Let $\mu$ be a log-concave measure satisfying the hypothesis of Theorem 1.1 with $Q = I_k$. We denote by $P : \mathbb{R}^n \mapsto \mathbb{R}^n$ the orthogonal projection onto the $k$-dimensional subspace $E^\perp$. In the following we work in an orthonormal basis such that this subspace is spanned by the $k$ first basis vectors.

Let $f$ be the density of $\mu$, for all $x \in \mathbb{R}^n$, consider the following stochastic differential equations :

$$df_t(x) = (x - a_t)^TPdB_t f_t(x) \quad ; \quad f_0(x) = f(x) \quad (9)$$

where $a_t = \int_{\mathbb{R}^n} xf_t(x)dx$ is the barycenter of the measure $\mu_t$, which we define here as having density $f_t$, and $(B_t)_{t \geq 0}$ is a standard Brownian motion on $\mathbb{R}^n$.

This system of equation is the same as the usual stochastic localization, except for the addition of the matrix $P$ which projects the random direction given by the Brownian onto the subspace where we need to bend the potential. The idea of adding a projector first appears in a paper of Klartag [8] for other purposes. The following facts and computations are very standard, and we refer the reader to [4] and [11] for a more detailed exposition. In particular, we need to assume that the support of $\mu$ is bounded to grant the existence and well-definedness of the process for all time $t \geq 0$ and then extend the result to arbitrary $\mu$ by approximation; we again refer to [4].

**Proposition 2.1.**

- Equation (9) defines a function-valued martingale $f_t$ in the sense that for any continuous and compactly supported function $\phi$ :

$$\int_{\mathbb{R}^n} \phi(x)f_t(x)dx \quad \text{is a martingale} \quad (10)$$

- $f_t$ is a density and for all $x \in \mathbb{R}^n$,

$$f_t(x) = \frac{1}{Z_t} e^{-\frac{1}{2}x^TPx+c_t \cdot x} f(x) := e^{-V_t(x)} \quad (11)$$

where $c_t$ is the solution of :

$$c_0 = 0, \quad dc_t = PdB_t + Pa_t dt \quad (12)$$

in particular we see that $\nabla^2 V_t \geq \min(\eta, t) Id$

*Proof.* For the existence and well-definedness of the process, see the remark below. While it is possible to check that $f_t$ as defined by (11) satisfy (9), we sketch a different proof to lighten the exposition. Let $m_t = \int_{\mathbb{R}^n} f_t(x)dx$ be the total mass at time $t$. Recall that $a_t = \frac{1}{m_t} \int_{\mathbb{R}^n} xf_t(x)dx$ is the barycenter of $f_t$. Then, by (9),

$$dm_t = \left( P \int_{\mathbb{R}^n} (x - a_t)f_t(x)dx \right) dB_t$$

$$= (Pa_t(m_t - 1)) dB_t.$$

It is easy to check that this simple stochastic differential equation admits a unique solution (see
It is given by $m_t = 1$. To establish (11), we use (9) to compute:

$$d \log f_t(x) = \frac{df_t(x)}{f_t(x)} - \frac{1}{2} \frac{d[f(x)]_t}{f_t(x)^2}$$

$$= (P(x - a_t)) \cdot dB_t - \frac{1}{2} (x - a_t)^T P(x - a_t) dt$$

$$= x \cdot (P dB_t + Pa_t dt) - \frac{1}{2} x^T P x dt + d z_t$$

$$= x \cdot dc_t - \frac{1}{2} x^T P x dt + d z_t$$

where $d z_t$ regroups the terms that do not depend on $x$. It encodes the normalizing factor $Z_t$. The expression (11) together with the proof that $m_t = 1$ ensures that $f_t$ is a density. The martingale property (10) is straightforward since, for any compactly supported $\phi$,

$$d \int_{\mathbb{R}^n} \phi(x) f_t(x) dx = \left( \int_{\mathbb{R}^n} \phi(x) P(x - a_t) f_t(x) dx \right) \cdot dB_t.$$  

Finally, the lower-bound on the Hessian of $V_t$ is a direct consequence of (11).

Remark 3. Equation (9) defines an infinite system of stochastic differential equations. It is therefore a priori unclear whether a solution exists. However there is a simpler, although arguably less intuitive, way of defining the process. First notice that, given the initial data $f$, $a_t$ is but a function of $t$ and $c_t$ defined as the barycenter of the density $f_t$ (11). Hence, we can first define $c_t$ by equation (12) and then $f_t$ by equation (11), and only then compute $df_t(x)$. The next two lemmas are standard and straightforward computations in stochastic localization which are obtained using Itô calculus. See [4] and ([11], Lemma 20). We denote by $K_t$ the covariance matrix of $\mu_t$. Since the computation for its infinitesimal change $dK_t$ is a bit tedious, we omit it to lighten the exposition.

**Lemma 2.2.** $da_t = K_t P dB_t$

**Proof.** By Itô calculus and (9),

$$da_t = d \int_{\mathbb{R}^n} x f_t(x) dx = \int_{\mathbb{R}^n} x(x - a_t)^T P f_t(x) dx dB_t$$

$$= \int_{\mathbb{R}^n} x(x - a_t)^T f_t(x) dx P dB_t$$

$$= \int_{\mathbb{R}^n} (x - a_t)(x - a_t)^T f_t(x) dx P dB_t = K_t P dB_t$$

**Lemma 2.3.** $dK_t = \int_{\mathbb{R}^n} (x - a_t)(x - a_t)^T P(x - a_t)^T dB_t f_t(x) dx - K_t PK_t dt$

Now we want to have an estimate of the concentration function of $\mu$. We first need to understand how the measure of a set evolves along the process.

**Lemma 2.4.** Let $S \subset \mathbb{R}^n$ be a measurable set and define $s_t = \mu_t(S)$, then:

$$d[s]_t \leq (\|PK_t P\|_{op}) dt$$
Proof.

\[ ds_t = \int_S df_t(x) dx = \langle \int_S P(x - a_t) f_t(x) dx, dB_t \rangle \]

So the quadratic variation is

\[ d[s]_t = \max_{|\xi| \leq 1} \left( \left( \int_S \xi^T P(x - a_t) f_t(x) dx \right)^2 \right) dt \]
\[ \leq \max_{|\xi| \leq 1} \left( \left( \int_S (\xi^T P(x - a_t))^2 f_t(x) dx \right) \left( \int_S f_t(x) dx \right) \right) dt \]
\[ \leq \max_{|\xi| \leq 1} (\xi^T PK_t P\xi) dt \leq (\|PK_t P\|_{op}) dt \]

\[ \square \]

To control the above quadratic variation, we need to control the norm of \( Q_t = PK_t P \). This is the purpose of the next section.

3 Control of the covariance matrix

We will see that the matrix \( Q_t \), seen as a \( k \times k \) matrix, follows the same dynamics as the covariance matrix of the standard stochastic localization in \( \mathbb{R}^k \). To be more precise, it is the covariance matrix of the marginal density, which follows a stochastic localization dynamics. Hence, to control its operator norm, we use the same strategy as Eldan.

Lemma 3.1. Define \( g_t(y) = \int_{\mathbb{R}^n-k} f_t(y, x) dx \) the marginal density of the vector \( Y_t = PX_t \), where \( X_t \) is the random vector with density \( f_t \). The barycenter of \( Y_t \) is \( b_t = Pa_t \). Then,

\[ dg_t(y) = (y - b_t)^T dW_t g_t(y) \]

where \( W_t \) is a standard Brownian in \( \mathbb{R}^k \). Moreover \( Q_t \) is the covariance matrix of \( Y_t \) and

\[ dQ_t = \int_{\mathbb{R}^k} (y - b_t)(y - b_t)^T dW_t g_t(y) \]
\[ dy - Q_t^2 dt \]

Proof. The lemma follows from straightforward computations.

Remark 4. Equation (13) is the definition of the stochastic localization process used by Lee and Vampala [11] and Klartag and Lehec [9]. It is also the process used by Chen [2] when the initial measure is isotropic. Eldan [4] has a slightly different definition, even if most of the ideas used to analyze one process transfer to the other.

From now, the main purpose of this section is to show that the operator norm of \( Q_t \) is bounded by a constant up to time \( T = \frac{c_0}{\Psi_{\max(k,1)}}, \) with \( c_0 \) a universal constant, see Lemma 3.6 below. This result essentially goes back to Eldan [4], in a slightly different setting, and further appears in Lee-Vampala ([11], Lemma 58) and Chen ([2], Lemma 7). We provide a simplified exposition of the proof of Chen. Following Eldan, we use the potential \( \Gamma_t = \text{tr}(Q_t^p) = \sum_{i=1}^k \lambda_i^p \) for some \( p \geq 1 \) where \( \lambda_1 \geq \cdots \geq \lambda_k \) are the eigenvalues of \( Q_t \). In the following, we denote by \( (e_1, \ldots, e_k) \) a basis of eigenvectors of \( Q_t \), where the dependence on \( t \) and \( \omega \) is implicit.
Lemma 3.2.

\[
d\Gamma_t = \sum_i p\lambda_i^{p-1}u_{ii} \cdot dW - \sum_i p\lambda_i^p dt + \sum_i p\lambda_i^p |u_{ij}|^2 dt + \sum_i \frac{p(p-1)}{2}\lambda_i^{p-2} u_{ii}^2 dt \tag{15}\]

where for all \(i, j\), \(u_{ij} = \int_{\mathbb{R}^k} (y - b_t) \cdot e_i (y - b_t) \cdot e_j (y - b_t) g_t(y) dy\)

Proof. The functional \(\Phi: M \mapsto \text{tr}(M^p)\) defined on symmetric matrices is \(C^\infty\). On the dense open set \(U\) of matrices whose eigenvalues are pairwise distinct, the functionals \(M \mapsto \lambda_i(M)\) are smooth by implicit value theorem. Let \(Q \in U\), with eigenvalues \(\lambda_1, ..., \lambda_k\) and eigenvectors \(e_1, ..., e_k\) and let \(q_{i,j}\) be the entries of \(Q\) in the basis \(e\). The following computations are standard, see ([3], Lemma 1.4.8)

\[
\nabla \lambda_i(Q) = e_i e_i^T.
\]

For the second derivative, the only non-zero terms are

\[
\frac{\partial^2 \lambda_i}{\partial q_{i,j}^2} = \frac{2}{\lambda_i - \lambda_j}.
\]

Combining this with (14) proves the result when \(Q_t\) belongs to \(U\), it is easy to see that it extends to the general case. \(\square\)

Lemma 3.3.

\[
d(\Gamma_t^{1/p}) = v_t \cdot dW_t + \delta_t dt \tag{16}\]

where

\[
v_t = \left( \sum_i \lambda_i^p \right)^{\frac{1}{p} - 1} \left( \sum_i \lambda_i^{p-1} u_{ii} \right) \tag{17}\]

and

\[
\delta_t \leq (p-1) \left( \sum_i \lambda_i^p \right)^{\frac{1}{p} - 1} \sum_{i,j} \lambda_i^{p-2} |u_{ij}|^2 \tag{18}\]

Proof. By Ito calculus, \(d(\Gamma_t^{1/p}) = \frac{1}{p} \Gamma_t^{\frac{1}{p} - 1} d\Gamma_t + \text{Itô term}\). But \(x \mapsto x^{1/p}\) is concave, so the Itô term is negative. Injecting equation (15) yields

\[
v_t = \left( \sum_i \lambda_i^p \right)^{\frac{1}{p} - 1} \left( \sum_i \lambda_i^{p-1} u_{ii} \right)
\]

and

\[
\delta_t \leq \frac{p-1}{2} \left( \sum_i \lambda_i^p \right)^{\frac{1}{p} - 1} \sum_i \lambda_i^{p-2} |u_{ii}|^2 + \left( \sum_i \lambda_i^p \right)^{\frac{1}{p} - 1} \sum_{i \neq j} \lambda_i^{p-1} \lambda_j |u_{ij}|^2.
\]
Now, notice that \( u_{ij} = u_{ji} \) so that
\[
\sum_{i \neq j} \frac{\lambda_i^{p-1}}{\lambda_i - \lambda_j} |u_{ij}|^2 = \frac{1}{2} \sum_{i \neq j} \frac{\lambda_i^{p-1} - \lambda_j^{p-1}}{\lambda_i - \lambda_j} |u_{ij}|^2 \\
\leq \frac{1}{2} \sum_{i \neq j} (p-1) \max(\lambda_i, \lambda_j)^{p-2} |u_{ij}|^2 \\
\leq \frac{p-1}{2} \sum_{i \neq j} (\lambda_i^{p-2} + \lambda_j^{p-2}) |u_{ij}|^2 \\
\leq (p-1) \sum_{i \neq j} \lambda_i^{p-2} |u_{ij}|^2
\]

which proves the lemma. \( \square \)

In the next two lemmas, we bound \(|v_t|\) and \(\delta_t\) in terms of \(\Gamma_t^{\frac{1}{p}}\) in order to apply a Gronwall-type argument.

**Lemma 3.4.** There is a universal constant \( c > 0 \) such that for all \( t \geq 0 \),
\[
|v_t| \leq c \left( \Gamma_t^{\frac{1}{p}} \right)^{3/2} \text{ a.s.}
\]

**Proof.** Let \( \tilde{Y} = Y_t - b_t \) be distributed according to \( g_t(y - b_t)dt \), where we drop the dependence in \( t \) for readability. Let \( \tilde{Y}_1, \ldots, \tilde{Y}_k \) its coordinates in the basis \( e_1, \ldots, e_k \). \( \tilde{Y} \) is a centered log-concave vector of \( \mathbb{R}^k \) of covariance \( Q_t \) and for all \( 1 \leq i \leq k \), \( \mathbb{E} \tilde{Y}_i^2 = \lambda_i \). Note that for all \( 1 \leq i \leq k \), \( u_{ii} = \mathbb{E} \left[ \tilde{Y}_i^2 \tilde{Y} \right] \). Then, for all \( \theta \in S^{k-1} \),
\[
|u_{ii} \cdot \theta| = |\mathbb{E} \left[ \tilde{Y}_i^2 \tilde{Y} \cdot \theta \right] | \\
\leq \mathbb{E} \left[ \tilde{Y}_i^4 \right]^{1/2} \mathbb{E} \left[ (\tilde{Y} \cdot \theta)^2 \right]^{1/2} \\
\leq \mathbb{E} \left[ \tilde{Y}_i^2 \right] \|Q_t\|_{op}^{1/2} \\
\leq \lambda_i \Gamma_t^{1/2p}
\]

where in the second inequality we used Borell’s lemma (\([1]\)). This proves the lemma. \( \square \)

**Lemma 3.5.** For all \( t \geq 0 \),
\[
\delta_t \leq 4 p \Gamma_t^{2/p} \Psi_k^2
\]

**Proof.** With the same notations as in the previous lemma, for all \( 1 \leq i, j \leq k \), \( u_{ij} = \mathbb{E} \left[ \tilde{Y}_i \tilde{Y}_j \tilde{Y} \right] \).

For all \( 1 \leq i \leq k \), we define the matrix \( \Delta_i = \mathbb{E} \left[ \tilde{Y}_i \tilde{Y} \tilde{Y}^T \right] \). Following Chen \([2]\), we compute:
\[
\sum_{i,j} \lambda_i^{p-2} |u_{ij}|^2 = \sum_{i,j,k} \lambda_i^{p-2} \mathbb{E}(\tilde{Y}_i \tilde{Y}_j \tilde{Y}_k)^2 \\
= \sum_i \lambda_i^{p-2} \text{tr}(\Delta_i^2) \\
= \sum_i \lambda_i^{p-2} \text{tr}(\Delta_i \mathbb{E} \tilde{Y}_i \tilde{Y} \tilde{Y}^T) \\
= \sum_i \lambda_i^{p-2} \mathbb{E} \left( \tilde{Y}_i \tilde{Y}^T \Delta_i \tilde{Y} \right) \\
\leq \sum_i \lambda_i^{p-2} \mathbb{E}(\tilde{Y}_i^2)^{1/2} \text{Var}(\tilde{Y}^T \Delta_i \tilde{Y})^{1/2} \\
\leq \sum_i \lambda_i^{p-2} \lambda_i^{1/2} c_P(\tilde{Y}) \left( 4 \mathbb{E} \left[ |\Delta_i \tilde{Y}|^2 \right] \right)^{1/2} \\
= 2c_P(\tilde{Y}) \sum_i \lambda_i^{p-3/2} \text{tr}(\Delta_i Q_t \Delta_i)^{1/2} \\
\leq 2c_P(\tilde{Y}) \left( \sum_i \lambda_i^p \right)^{1/2} \left( \sum_i \lambda_i^{p-3} \text{tr}(\Delta_i^2 Q_t) \right)^{1/2}
\]

Now,

\[
\sum_i \lambda_i^{p-3} \text{tr}(\Delta_i^2 Q_t) = \sum_i \lambda_i^{p-3} \sum_{j,k} \lambda_j (\Delta_i)_{j,k}^2 \\
= \sum_{i,j,k} \lambda_i^{p-3} \lambda_j (\Delta_i)_{j,k}^2 \\
= \sum_{i,j,k} \lambda_i^{p-3} \lambda_j (\tilde{Y}_i \tilde{Y}_j \tilde{Y}_k)^2 \\
\leq \sum_{i,j,k} \lambda_i^{p-2} \mathbb{E}(\tilde{Y}_i \tilde{Y}_j \tilde{Y}_k)^2
\]

where in the last inequality, we used the convexity inequality : \( \lambda_i^{p-3} \lambda_j \leq \frac{p-3}{p-2} \lambda_i^{p-2} + \frac{1}{p-2} \lambda_j^{p-2} \).

Plugging this into the inequality above yields :

\[
\sum_{i,j} \lambda_i^{p-2} |u_{ij}|^2 \leq 2c_P(\tilde{Y}) \left( \sum_i \lambda_i^p \right)^{1/2} \left( \sum_{i,j} \lambda_i^{p-2} |u_{ij}|^2 \right)^{1/2}
\]

which implies

\[
\sum_{i,j} \lambda_i^{p-2} |u_{ij}|^2 \leq 4c_P(\tilde{Y})^2 \left( \sum_i \lambda_i^p \right).
\]

Plugging this into (18) remarking that \( c_P(\tilde{Y}) = c_P(Y_t) \) yields :

\[
\delta_t \leq 4p \Gamma_t^{1/p} c_P(Y_t)^2 \\
\leq 4p \Gamma_t^{1/p} \|Q_t\|_{op} \Psi_k^2 \\
\leq 4p \Gamma_t^{2/p} \Psi_k^2.
\]

\( \square \)
We are now in position to control the growth of $\Gamma_t$ by a Gronwall-type argument.

**Lemma 3.6.** There are constants $c_0, c_1 > 0$ such that for any $t \leq T = \frac{c_0}{\Psi_k^2 \max(\log(k), 1)}$, we have:

$$
\mathbb{P} \left( \max_{s \in [0,t]} \|Q_s\|_{op} \geq 10 \right) \leq \exp \left( -\frac{c_1}{t} \right).
$$

As a consequence, for any measurable set $S \subset \mathbb{R}^n$ of measure $\mu(S) = 1/2$, setting $s_t = \mu_t(S)$, we have:

$$
\mathbb{P}(|s_t| \geq 10t) \leq \exp \left( -\frac{c_1}{t} \right)
$$

**Proof.** Set $p = \max(\log(k), 1)$, so that

$$
\Gamma_t^{1/p} \leq e
$$

as we will use repeatedly in the proof. Recall that

$$
d(\Gamma_t^{1/p}) = v_t \cdot dW_t + \delta_t dt
$$

and define the stopping time $\tau = \inf \{ t \geq 0 \mid \Gamma_t^{1/p} \geq 3\Gamma_0^{1/p} \}$. We denote by $M_t$ the martingale term $M_t = \int_0^t v_s \cdot dW_s$. For all $t \geq 0$ we have:

$$
\Gamma_t^{1/p} = \Gamma_0^{1/p} + M_{t \wedge \tau} + \int_0^{t \wedge \tau} \delta_s ds
$$

where in the first inequality we used Lemma 3.5. By Lemma 3.5,

$$
\Gamma_t^{1/p} \leq 2\Gamma_0^{1/p} + M_{t \wedge \tau}.
$$

Consequently, for all such $t$,

$$
\mathbb{P}(\tau \leq t) = \mathbb{P}(\Gamma_{t \wedge \tau} = \Gamma_t)
\leq \mathbb{P} \left( M_{t \wedge \tau} \geq \Gamma_0^{1/p} \right).
$$

Now, $\tau$ being a stopping time, $M_{t \wedge \tau}$ is a martingale, whose quadratic variation is

$$
[M]_{t \wedge \tau} = \int_0^{t \wedge \tau} |v_s|^2 ds \leq c \int_0^{t \wedge \tau} 3 \left( \Gamma_0^{1/p} \right)^{3/2} ds \leq 3ce^{3/2} t = \tilde{c}_1 t
$$

where in the first inequality we used Lemma 3.4. By Lemma 1.4 we get:

$$
\mathbb{P}(\tau \leq t) \leq \mathbb{P} \left( M_{t \wedge \tau} \geq \Gamma_0^{1/p} \right) = \mathbb{P} \left( M_{t \wedge \tau} \geq \Gamma_0^{1/p} , [M]_{t \wedge \tau} \leq \tilde{c}_1 t \right)
\leq \exp \left( -\frac{\Gamma_0^{2/p}}{2\tilde{c}_1 t} \right)
\leq \exp \left( -\frac{c_1}{t} \right).
$$

With $c_1 = \frac{1}{2\tilde{c}_1}$. Now notice that $3\Gamma_0^{1/p} \leq 3e \leq 10$ which proves the first statement. The second statement follows from Lemma 2.4.
4 Proof of the main theorem

Take a subset $S$ of measure $1/2$ and $r > 0$, for $t \leq T = \frac{c_0}{\Psi_k \max\{\log(k), 1\}}$ we have :

$$
\mu(S^c_r) = \mathbb{E} \mu_t(S^c_r) \leq \mathbb{E} \left[ \mu_t(S^c_r) \mathbbm{1}_{\mu_t(S) \geq \frac{1}{4}} \right] + \mathbb{P} \left( \mu_t(S) \leq \frac{1}{4} \right) \\
\leq 4 \exp\left( -\frac{1}{4} \min(\eta, t) r^2 \right) + \mathbb{P} (s_0 - s_t \geq \frac{1}{4}, [s]_t \leq 10t) + \mathbb{P} ([s]_t \geq 10t) \quad \text{(By (2))}
$$

$$
\leq 4 \exp\left( -\frac{1}{4} \min(\eta, t) r^2 \right) + \exp\left( -\frac{1}{320t} \right) + \exp\left( -\frac{c_1}{t} \right) \quad \text{(By Lemmas 1.4 and 3.6, respectively)}
$$

$$
\leq 4 \left( \exp\left( -\frac{1}{4} \min(\eta, t) r^2 \right) + \exp\left( -\frac{c_1}{t} \right) \right) \quad \text{(with } c_4 = \min(c_1, \frac{1}{320}) \text{)}
$$

Define $\beta = \min(\eta, T)$ and choose $t(r) = \min(\eta, T, \frac{1}{r}) = \min(\beta, \frac{1}{r})$ we get that :

- If $r \geq \frac{1}{\beta}$,
  $$\mu(S^c_r) \leq 8 \exp(-c_5 r) \quad \text{(19)}$$
  where $c_5 = \min(1/4, c_4)$

- If $r \leq \frac{1}{\beta}$,
  $$\mu(S^c_r) \leq 4 \left( \exp\left( -\frac{1}{4} \min(\eta, T) r^2 \right) + \exp\left( -\frac{c_4}{\min(\eta, T)} \right) \right) \leq 8 \exp\left( -c_5 \beta r^2 \right) \quad \text{(20)}$$

Overall this implies that for all $r > 0$,

$$\mu(S^c_r) \lesssim \exp(-\min(c_0 r, c_1 \beta r^2))$$

which is the desired result.

References

[1] Christer Borell. Convex measures on locally convex spaces. *Arkiv för Matematik*, 12(1-2):239 – 252, 1974.

[2] Yuansi Chen. An almost constant lower bound of the isoperimetric coefficient in the KLS Conjecture. *Geometric and Functional Analysis*, 31(1):34–61, Feb 2021.

[3] Ronen Eldan. *Distribution of Mass in Convex Bodies*. Tel Aviv University, 2012.

[4] Ronen Eldan. Thin shell implies spectral gap up to polylog via a stochastic localization scheme. *Geometric and Functional Analysis*, 23(2):532–569, Apr 2013.

[5] David A Freedman. On tail probabilities for martingales. *the Annals of Probability*, pages 100–118, 1975.

[6] M. Gromov and V. D. Milman. A topological application of the isoperimetric inequality. *American Journal of Mathematics*, 105(4):843–854, 1983.
[7] Ravi Kannan, László Lovász, and Miklós Simonovits. Isoperimetric problems for convex bodies and a localization lemma. *Discrete & Computational Geometry*, 13(3):541–559, 1995.

[8] Bo'az Klartag. Eldan’s stochastic localization and tubular neighborhoods of complex-analytic sets. *The Journal of Geometric Analysis*, 28(3):2008–2027, Jul 2018.

[9] Bo'az Klartag and Joseph Lehec. Bourgain’s slicing problem and kls isoperimetry up to polylog. *arXiv preprint arXiv:2203.15551*, 2022.

[10] M. Ledoux. *The Concentration of Measure Phenomenon*. Mathematical surveys and monographs. American Mathematical Society, 2001.

[11] Yin Tat Lee and Santosh Srinivas Vempala. Eldan’s stochastic localization and the KLS Hyperplane conjecture: An improved lower bound for expansion. In *2017 IEEE 58th Annual Symposium on Foundations of Computer Science (FOCS)*, pages 998–1007, 2017.

[12] Emanuel Milman. On the role of convexity in isoperimetry, spectral gap and concentration. *Inventiones mathematicae*, 177(1):1–43, Jul 2009.

[13] Bernt Øksendal. Stochastic differential equations. In *Stochastic differential equations*, pages 65–84. Springer, 2003.

[14] G. Paouris. Concentration of mass on convex bodies. *Geometric & Functional Analysis GAFA*, 16(5):1021–1049, Dec 2006.