RESONANCES FOR QUASI-ONE-DIMENSIONAL DISCRETE
SCHRÖDINGER OPERATORS

M. ASSAL, O. BOURGET, P. MIRANDA AND D. SAMBOU

Abstract. Consider a multichannel Laplace type operator $H_0$ on $\ell^2(\mathbb{Z}) \otimes \mathcal{G}$, where $\mathcal{G}$ is an auxiliary separable Hilbert space. For suitable compact perturbations $V$, we study the distribution of resonances of the operator $H_0 + \omega V$, for small values of $|\omega|$, near the thresholds of the spectrum of $H_0$. We distinguish two cases. If $\mathcal{G}$ is of finite dimension, resonances do not accumulate at the thresholds. We give a geometrical description of their distribution and we control exactly their number at least in the non-degenerate case. If $\mathcal{G}$ is of infinite dimension, an accumulation phenomenon occurs at some thresholds. We describe it by means of an asymptotical analysis of the counting function of resonances.

Mathematics subject classification 2020: 47A10, 81Q10, 81U24.

1. Introduction

The resonance phenomena in quantum mechanics has been mathematically tackled from various perspectives. For instance, resonances of a stationary quantum system can be defined as the poles of a suitable meromorphic extension of either the Green function, the resolvent of the Hamiltonian or the scattering matrix. The imaginary part of such poles is sometimes interpreted as the inverse of the lifetime of some associated quasi-eigenstate. A related idea is to identify the resonances of a quantum Hamiltonian with the discrete eigenvalues of some non-selfadjoint operator obtained from the original one by the methods of spectral deformations. Another point of view consists in defining the resonances dynamically, i.e., in terms of quasi-exponential decay for the time evolution of the system. This property is somewhat encoded in the concept of sojourn time. The equivalence between these different perspectives and formalisms is also an issue.

The study of the existence and the asymptotic behavior of resonances in different asymptotic regimes has witnessed a lot of progress during the last thirty years mainly in the context of continuous configuration spaces. This has been achieved thanks to the development of many mathematical approaches such as scattering methods, spectral and variational techniques, semiclassical and microlocal analysis (many references to this vast literature can be found in the monographs [20, 21, 11]).

On the other side, the qualitative spectral properties of the discrete Laplace operator and some selfadjoint generalizations exhibiting dispersive properties, have been extensively investigated. We primarily refer to [10, 34] for the multidimensional lattice case $\mathbb{Z}^d$, [14] and references therein for trees, [27, 28] and [2, 31] for periodic graphs and perturbed graphs, respectively. Analyses of the continuum limit are performed in [29, 22]. The role of the thresholds of the discrete Laplace operator are specifically studied in [23, 24].
However, there are only few works dealing with resonances of quantum Hamiltonians on discrete structures (see for instance [23, 7]). This suggests that a more systematic analysis of resonances in the spirit of [6, 5] should be performed in this context. The present paper is the first of a sequence of studies about resonances for operators on various graph structures. In what follows, we focus on some generalizations of the 1D discrete Laplace operator and study the asymptotic distribution of the resonances which appear in the neighborhoods of the thresholds, in perturbative regimes.

The asymptotic behavior of resonances near thresholds have been studied in an abstract setting in [18] (when there is no accumulation). This result does not include the models of the present work due to the type of the singularity of the resolvents in our case. On the other side, in some continuous waveguides models, the singularities at the thresholds are similar in structure to the ones that appear in our case. Thus, related results to ours are obtained in [9, 8]. However, using a somewhat different approach, the conclusions that we obtain here are sharper, since we provide a precise control on the number of resonances and on their location. Moreover, we treat also the accumulation of resonances, which do not appear in the above works.

We go on with a presentation of our framework and the main issues addressed in this paper. The resonances are defined in Section 2. The main results, namely Theorems 3.1 and 3.3, are stated in Section 3. In Section 4 we briefly describe some models where our results can be applied. The proofs of the main theorems are postponed to Sections 5-7. Finally, in the appendix we prove a result on the multiplicity of resonances needed in our study. We present this result in an abstract form since it may be of independent interest.

**General setting.** Let $\Delta$ be the positive one-dimensional discrete Laplacian defined on the Hilbert space $\ell^2(\mathbb{Z}) = \{u : \mathbb{Z} \to \mathbb{C}; \|u\|^2 = \sum_{n \in \mathbb{Z}} |u(n)|^2 < \infty\}$ by $(\Delta u)(n) := 2u(n) - u(n + 1) - u(n - 1), \ u \in \ell^2(\mathbb{Z}).$ Consider a complex separable Hilbert space $\mathcal{G}$ and let $M$ be a compact operator on $\mathcal{G}$. On the Hilbert space $\ell^2(\mathbb{Z}) \otimes \mathcal{G} \cong \ell^2(\mathbb{Z}; \mathcal{G})$, we introduce the operator $H_0$ defined by

$$H_0 := \Delta \otimes I_{\mathcal{G}} + I_{\ell^2(\mathbb{Z})} \otimes M,$$

where $I_{\mathcal{G}}$ and $I_{\ell^2(\mathbb{Z})}$ denote the identity operators on $\mathcal{G}$ and $\ell^2(\mathbb{Z})$ respectively.

It is well known that the operator $\Delta$ is bounded and selfadjoint in $\ell^2(\mathbb{Z})$ and its spectrum is absolutely continuous given by $\sigma(\Delta) = \sigma_{ac}(\Delta) = [0, 4]$. We refer to the beginning of section 5 for a short background on the basic properties of the operator $\Delta$. It follows that the spectrum of $H_0$ has the following band structure in the complex plane

$$\sigma(H_0) = \bigcup_{\lambda \in \sigma(M)} \Lambda_{\lambda}, \quad \Lambda_{\lambda} := [\lambda, \lambda + 4],$$

where the edge points $\{\lambda, \lambda + 4\}_{\lambda \in \sigma(M)}$ play the role of spectral thresholds. These bands are parallel to the real axis, and may be pairwise disjoint, overlapping or intersecting (see Fig. 1).

Our main purpose in this paper is to study the distribution of resonances of operators of the form $H_\omega = H_0 + \omega V, \ \omega \in \mathbb{C}$, near the spectral thresholds of $\sigma(H_0)$. The specific class of
perturbations $V : \ell^2(\mathbb{Z}) \otimes \mathcal{G} \to \ell^2(\mathbb{Z}) \otimes \mathcal{G}$ to be considered will be introduced later on (see Assumption 2.1). The resonances will be defined as the poles of the meromorphic extension of the resolvent of $H_\omega$ in some weighted spaces (see section 2).

We will study two different cases: the first one is when $\mathcal{G}$ is finite-dimensional, and the second one is when $\mathcal{G}$ is infinite-dimensional and $M$ is a finite-rank operator on $\mathcal{G}$. Although both cases may look similar, they present an important difference in terms of the distribution of resonances. More specifically, in the first case the spectrum of $M$ consists only on eigenvalues with finite multiplicities, while in the second case, 0 is an eigenvalue of $M$ with infinite multiplicity. This splits the set of spectral thresholds into two classes. The first class consists on the thresholds \{\lambda, \lambda + 4\}_{\lambda \in \sigma(M)} such that $\lambda$ is an eigenvalue of $M$ with finite multiplicity. Near such points, we prove the existence of resonances of the operator $H_\omega$ and precise their location depending on the perturbation parameter $\omega$, for $|\omega|$ small. More precisely, we obtain the exact number of resonances near non-degenerate thresholds and an upper bound near degenerate ones (see Definition 2.1 and Theorem 3.1). The second class consists of the thresholds \{0, 4\} in the infinite dimensional case. In contrast to the previous case, we show that an accumulation of resonances of the operator $H_\omega$ occurs near these points, for $|\omega|$ sufficiently small (see Theorem 3.3).

Notations. Let us fix some notations used throughout the paper. Let $\mathcal{K}$ be a separable Hilbert space. We denote by $\mathcal{B}(\mathcal{K})$ the algebra of bounded linear operators acting on $\mathcal{K}$. $\mathfrak{S}_\infty(\mathcal{K})$ and $\mathfrak{S}_n(\mathcal{K})$, $n \geq 1$, stand for the ideal of compact operators and the Schatten classes respectively. In particular, $\mathfrak{S}_1(\mathcal{K})$ and $\mathfrak{S}_2(\mathcal{K})$ are the ideals of trace class operators and Hilbert-Schmidt operators on $\mathcal{K}$, endowed with the norms $\|\cdot\|_1$ and $\|\cdot\|_2$ respectively. If $U \subset \mathbb{C}$ is an open set, we denote $\text{Hol}(U; \mathcal{K})$ the set of holomorphic functions on $U$ with values in $\mathcal{K}$.

We denote by $(\delta_n)_{n \in \mathbb{Z}}$ the canonical basis of $\ell^2(\mathbb{Z})$. Throughout the paper, $\mathcal{H}$ stands for the Hilbert space $\ell^2(\mathbb{Z}) \otimes \mathcal{G}$ endowed with its orthonormal basis $(\delta_n \otimes e_j)_{(n,j) \in \mathbb{Z} \times I}$, where $(e_j)_{j \in I}$ stands for an orthonormal basis of $\mathcal{G}$, for some index set $I \subseteq \mathbb{Z}_+$.

For $s > 0$, let $W_s$ be the multiplication operator by the function $Z \ni n \mapsto \|e^{-\frac{s}{2}|n|}\|_{\ell^2(\mathbb{Z})}e^{\frac{s}{2}|n|}$ acting on $e^{-\frac{s}{2}|\cdot|}\ell^2(\mathbb{Z})$ with values in $\ell^2(\mathbb{Z})$. $W_{-s}$ stands for the multiplication operator by the function $Z \ni n \mapsto \|e^{-\frac{s}{2}|n|}\|_{\ell^2(\mathbb{Z})}^{-1}e^{-\frac{s}{2}|n|}$ acting on $\ell^2(\mathbb{Z})$ with values in $\ell^2(\mathbb{Z})$. We set $W_{\pm s} := W_{\pm s} \otimes I_\mathcal{G}$. The first quadrant of the complex plane will be denoted $\mathbb{C}_1$, i.e., $\mathbb{C}_1 :=$
\{z \in \mathbb{C}; \Re z > 0, \Im z > 0\}. For \(\varepsilon > 0\) and \(z_0 \in \mathbb{C}\), we set \(D_\varepsilon(z_0) := \{z \in \mathbb{C}; |z - z_0| < \varepsilon\}\) and \(D_\varepsilon^*(0) := D_\varepsilon(z_0) \setminus \{z_0\}\).

### 2. Resonances

In this section, we define the resonances of the operator \(H_\omega = H_0 + \omega V\) near the spectral thresholds of \(\sigma(H_0)\) for some class of perturbations \(V\) satisfying Assumption 2.1 below. We consider the following two cases:

**Case (A).** \(\mathcal{G}\) is finite-dimensional.

**Case (B).** \(\mathcal{G}\) is infinite-dimensional and \(M\) is finite-rank.

In both cases, we assume that \(M\) is diagonalizable. From now on, we denote by \(H_{0,A}\) (respectively \(H_{0,B}\)) the operator \(H_0\) defined by (1.1) in case (A) (respectively in case (B)). The same notation is used for the perturbed operators, i.e.,

\[ H_{\omega,A} := H_{0,A} + \omega V, \quad H_{\omega,B} := H_{0,B} + \omega V, \quad \omega \in \mathbb{C}. \]

In the following \(N\) stands for the dimension of \(\mathcal{G}\) in case (A) and for the rank of \(M\) in case (B). Let us denote by \(\{\lambda_q\}_{q=1}^N\) the set of eigenvalues of \(M\) in case (A). In case (B) we still denote by \(\{\lambda_q\}_{q=1}^N\) the non-zeros eigenvalues of \(M\) and set \(\lambda_0 = 0\).

The spectra of the operators \(H_{0,A}\) and \(H_{0,B}\) are given by

\[ \sigma(H_{0,A}) = \bigcup_{j=1}^N \Lambda_j, \quad \sigma(H_{0,B}) = \bigcup_{j=1}^N \Lambda_j \cup [0, 4], \quad \Lambda_j := [\lambda_j, \lambda_j + 4]. \]

Let \(\{\lambda_q\}_{q=1}^d\) be the subset of \(\{\lambda_q\}_{q=1}^N\) consisting of its distinct elements, \(1 \leq d \leq N\). The sets of the spectral thresholds of \(\sigma(H_{0,A})\) and \(\sigma(H_{0,B})\) are denoted by

\[ \mathcal{T}_A := \{\lambda_q, \lambda_q + 4\}_{q=1}^d, \quad \mathcal{T}_B := \{\lambda_q, \lambda_q + 4\}_{q=0}^d, \]

respectively. In the sequel, we shall use the notation \(\bullet\) to refer either to \(A\) or \(B\).

**Definition 2.1.** A threshold \(\zeta \in \mathcal{T}_\bullet\) is degenerate if there exist \(p \neq q \in \{1, \ldots, d\}\) such that \(\zeta = \lambda_q = \lambda_p + 4 \in \mathcal{T}_\bullet\). Otherwise, \(\zeta\) is non-degenerate.

For instance, in the example shown in Figure 1, \(\lambda_3\) is a degenerate threshold.

As noticed in [23, Appendix A] in the case of the free discrete Laplacian, there is a simple relation between the right thresholds and the left ones that makes possible to reduce the study near the threshold \(\lambda_q + 4\) to that near \(\lambda_q\). In order to keep the paper at a reasonable length, in the following we will state our results only for the left thresholds \(\lambda_q\). Analogous results for the right thresholds \(\lambda_q + 4\) hold with natural modifications.

For \(q \in \{0, 1, \ldots, d\}\), denote by \(\nu_q\) the dimension of \(\ker(M - \lambda_q)\). Of course \(\nu_0 = \infty\) in case (B). Let us denote by \(\pi_q\) the projection onto \(\ker(M - \lambda_q)\) defined by

\[ \pi_q := \frac{1}{2\pi i} \oint_{|z-M|=\varepsilon} (z-M)^{-1} dz, \quad 0 < \varepsilon \ll 1. \]

Given a threshold \(\lambda_q \in \mathcal{T}_\bullet\) one introduces the parametrization

\[ k \mapsto z_q(k) = \lambda_q + k^2, \]

where \(k\) is a complex variable in a small neighborhood of 0.
Throughout this paper, we assume the following condition on the perturbation $V$.

**Assumption 2.1.** The operator $V$ acting on $\mathcal{H}$ satisfies:

**Case (A).** $V \in \mathcal{B}(\mathcal{H})$ and there exist constants $\rho, C > 0$ such that
\[ \| V(n, m) \|_{\mathfrak{D}} \leq C e^{-\rho |n| + |m|}, \quad \forall (n, m) \in \mathbb{Z}^2, \]
where $(V(n, m))_{(n,m)\in\mathbb{Z}^2}$ is the matrix of $V$ in the basis $(\delta_n \otimes e_j)_{(n,j)\in\mathbb{Z} \times \{1, \ldots, N\}}$.

**Case (B).** There exist bounded operators $\mathcal{V}$ and $K$ acting on $\mathcal{H}$ and $\mathfrak{D}$, respectively, such that
\[ |V|^{\frac{1}{2}} = \mathcal{V}(W_{-\rho} \otimes K), \]
for some $\rho > 0$, with either $\mathcal{V} \in \mathfrak{D}_{p}(\mathcal{H})$ or $K\pi_0 \in \mathfrak{D}_p(\mathfrak{D})$ for some $p \in [1, +\infty)$.

Notice that if $V$ satisfies Assumption 2.1 (A) or (B), then $W_{\rho}VW_{\rho} \in \mathcal{B}(\mathcal{H})$.

**Proposition 2.2.** Let $\lambda_q \in \mathcal{T}_\omega$. Under Assumption 2.1 (•), there exists $\varepsilon_0 > 0$ such that for all $|\omega|$ sufficiently small, the operator-valued function
\[ D^*_\varepsilon_0(0) \cap \Sigma_1 \ni k \mapsto W_{-\rho}(H_{\omega,\bullet} - z_q(k))^{-1}W_{-\rho} \]
admits a meromorphic extension to $D_{\varepsilon_0}(0)$, with values in $\mathcal{B}(\mathcal{H})$. We denote by $\mathcal{R}^{(q)}_{\omega,\bullet}(k)$ this extension.

The proof of Proposition 2.2 is postponed to Section 5. Taking into account the above result one defines the resonances of the operator $H_{\omega,\bullet}$ near a threshold $\lambda_q \in \mathcal{T}_\omega$ as follows:

**Definition 2.3.** The resonances of the operator $H_{\omega,\bullet}$ near a threshold $\lambda_q \in \mathcal{T}_\omega$ are defined as the points $z_q(k)$ such that $k \in D_{\varepsilon_0}(0)$ is a pole of the meromorphic extension $\mathcal{R}^{(q)}_{\omega,\bullet}$ given by Proposition 2.2. The multiplicity of a resonance $z_0 = z_q(k_0)$ is defined by
\[ \text{mult}(z_0) := \text{rank} \oint_{\gamma} \mathcal{R}^{(q)}_{\omega,\bullet}(k) dk, \]
where $\gamma$ is a positively oriented circle centered on $k_0$, that not contain any other pole of $\mathcal{R}^{(q)}_{\omega,\bullet}$.

The set of resonances of $H_{\omega,\bullet}$ will be denoted $\text{Res}(H_{\omega,\bullet})$.

Notice that by (2.1), the resonances near a threshold $\lambda_q \in \mathcal{T}_\omega$ are defined in a Riemann surface $\mathcal{S}_q$, which is locally two sheeted.

### 3. Main results

In this section one formulates our main results on the existence and the asymptotic properties of the resonances of the operators $H_{\omega,A}$ and $H_{\omega,B}$ near the spectral thresholds.

Let $a_{-1}$ and $b_{-1}$ be the operators in $\ell^2(\mathbb{Z})$ defined by: for $u = (u(n))_{n \in \mathbb{Z}} \in \ell^2(\mathbb{Z})$,
\[ (a_{-1}u)(n) := \sum_{m \in \mathbb{Z}} \frac{i}{2} W_{-\rho}(n)W_{-\rho}(m) u(m), \]
\[ (b_{-1}u)(n) := \sum_{m \in \mathbb{Z}} \frac{(-1)^{n+m+1}}{2} W_{-\rho}(n)W_{-\rho}(m) u(m). \]
In case (A) or (B), for \( q \neq p \in \{0, 1, \ldots, d\} \), define the projections in \( \mathcal{H} \)
\[
\Pi_q := \frac{2}{i} a_{-1} \otimes \pi_q, \quad \Pi_{q,p} := \frac{2}{i} a_{-1} \otimes \pi_q - 2b_{-1} \otimes \pi_p.
\]
Notice that \( \text{rank } \Pi_q = \nu_q \) and \( \text{rank } \Pi_{q,p} = \nu_q + \nu_p \). Introduce the operators \( E_q \) and \( E_{q,p} \) defined in \( \mathcal{H} \) by
\[
E_q := \Pi_q W_p V W_p \Pi_q \quad \text{and} \quad E_{q,p} := \Pi_{q,p} W_p V W_p (a_{-1} \otimes \pi_q + b_{-1} \otimes \pi_p).
\]

### 3.1. Asymptotic distribution of the resonances: non-accumulation case.

Our first results consist on the existence, non-accumulation and asymptotic dependence on \( \omega \) of the resonances of the operator \( H_{\omega,A} \) near the thresholds \( T_A \), and the ones of \( H_{\omega,B} \) near the thresholds \( T_B \setminus \{0, 4\} \).

For \( q \neq 0 \), let \( \{\alpha_1^{(q)}, \alpha_2^{(q)}, \ldots, \alpha_r^{(q)}\}, 1 \leq r \leq \nu_q \) be the set of distinct eigenvalues of \( E_q|_{\text{Ran } \Pi_q} \), each \( \alpha_j^{(q)} \) of multiplicity \( m_{q,j} \). Analogously let \( \{\beta_1^{(q)}, \beta_2^{(q)}, \ldots, \beta_r^{(q)}\}, 1 \leq r \leq \nu_q + \nu_p \) be the set of distinct eigenvalues of \( E_{q,p}|_{\text{Ran } \Pi_q \Pi_p} \), each \( \beta_j^{(q)} \) of multiplicity \( m_{q,p,j} \). Of course \( \sum_{j=1}^{r} m_{q,j} = \nu_q \) and \( \sum_{j=1}^{r} m_{q,p,j} = \nu_q + \nu_p \).

**Theorem 3.1.** Assume that \( \mathcal{G} \) is finite-dimensional and let Assumption 2.1 (A) holds. Let \( \lambda_q \in T_A \) be a spectral threshold.

1) Suppose that \( \lambda_q \) is non-degenerate. Then, there exist \( \varepsilon_0, \delta_0 > 0 \) such that for all \( |\omega| < \delta_0 \) we have the following:
   (i) For any \( z_q(k) \in \text{Res}(H_{\omega,A}) \) with \( k = k_q(\omega) \in D_{\varepsilon_0}|\omega|(0) \), there exists a unique \( \alpha_j^{(q)} \in \sigma(E_q|_{\text{Ran } \Pi_q}) \) such that
   \[
   k_q(\omega) = -\frac{i}{2} \alpha_j^{(q)}(\omega) + O(|\omega|^{1+1/m_{q,j}}),
   \]
   where the \( O \) is uniform with respect to \( \omega \). Conversely for any \( \alpha_j^{(q)} \in \sigma(E_q|_{\text{Ran } \Pi_q}) \) there exists at least one and at most \( m_{q,j} \) resonances \( z_q(k) = \lambda_q + k(\omega)^2 \) of \( H_{\omega,A} \) with \( k(\omega) \) satisfying (3.5).
   (ii) Moreover if \( V \) is selfadjoint, then for any \( \alpha_j^{(q)} \in \sigma(E_q|_{\text{Ran } \Pi_q}) \) there exist \( m_{q,j} \) resonances \( z_q(k) \) (counting multiplicities) with \( k = k_q(\omega) \) satisfying (3.5). In particular
   \[
   \sum_{z_q(k) \in \text{Res}(H_{\omega,A}), k \in D_{\varepsilon_0}|\omega|(0)} \text{mult}(z_q(k)) = \nu_q.
   \]
2) Suppose that \( \lambda_q \) is degenerate and let \( p \neq q \in \{1, \ldots, d\} \) such that \( \lambda_q = \lambda_p + 4 \). Then, the assertion (i) holds true with \( \frac{i}{2} \alpha_j^{(q)} \) replaced by \( \beta_j^{(q)} \) and \( m_{q,j} \) replaced by \( m_{q,p,j} \).

The above result states that for any fixed threshold \( \lambda_q \in T_A \) and \( |\omega| \) small enough, the operator \( H_{\omega,A} \) has resonances \( z_q(k) = \lambda_q + k(\omega)^2 \) near \( \lambda_q \). All of them are distributed (in variable \( k \)) in clusters around the points \( -\frac{i}{2} \alpha_j^{(q)}(\omega) \) (resp. \( -\beta_j^{(q)}(\omega) \)), \( j \in \{1, \ldots, r\} \). In each cluster, there is at least one and at most \( m_{q,j} \) (resp. \( m_{q,p,j} \)) resonances. If the threshold \( \lambda_q \) is non-degenerate and \( V \) is selfadjoint, one has exactly \( m_{q,j} \) resonance(s) in each cluster and the total number of resonances near \( \lambda_q \) is equal to \( \nu_q \) as shown by Figure 2.

**Remark 3.2.** The same result occurs in case (B) near the thresholds \( \lambda_q \in T_B \setminus \{0, 4\} \).
3.2. Asymptotic distribution of the resonances: accumulation case. For a compact self-adjoint operator $T$ and $I$ a real interval let us introduce

$$n_I(T) := \text{Tr} \chi_I(T),$$

which is the function that counts the number of eigenvalues of $T$ in $I$, including multiplicities.

**Theorem 3.3.** Let Assumption 2.1 (B) holds with $V \geq 0$ and $\omega$ small. Then, there exists $0 < \epsilon_0 \ll 1$ such that the resonances $z_0(k) = k^2$ of $H_{\omega,B}$ with $k \in D^{*}_{\epsilon_0}(0)$ satisfy:

(i) $\text{Im}(k/\omega) \leq 0$, $|\text{Re}(k/\omega)| = o(|k/\omega|)$.

(ii) Suppose that $E_0$ is of infinite rank. Then, the number of resonances of $H_{\omega,B}$ near 0 is infinite. More precisely, there exists a sequence of positive numbers $(\epsilon_j)_j$ tending to zero such that, counting multiplicities, we have

$$\lim_{j \to +\infty} \frac{\# \{z_0(k) \in \text{Res}(H_{\omega,B}) : |\omega| \epsilon_j < |k| < \epsilon_0 |\omega| \}}{n_{[\epsilon_j,1]}(E_0)} = 1.$$  

**Remark 3.4.** For $\alpha, \beta, \theta > 0$ one considers the sectorial domain

$$C_\theta(a,b) := \{x + iy \in \mathbb{C} : a \leq x \leq b, |y| \leq \theta |x| \}.$$  

**Theorem 3.3** entails the following consequences:
a) The resonances of $H_{\omega,B}$ satisfy for $\epsilon |\omega| < |k| < \epsilon_0 |\omega|$, $k \in -i\omega \mathcal{C}_{\theta_0}(\epsilon, \epsilon_0)$, for some $\theta_0 > 0$.

b) If $\text{Arg}(\omega) \neq \pm \frac{\pi}{2}$, then there exists an interval $I$ centered at 0 such that $H_{\omega,B}$ don’t have eigenvalues in $I$ except maybe at 0.

4. Illustrative examples

In this part, we give some models for which Theorem 3.1 can be applied.

4.1. Perturbations of the discrete Laplace operator on the strip (with DBC). Let $\mathcal{G} = \mathbb{C}^m$ for some $m \in \mathbb{N}$ and $(e_k)_{k \in \{1, \ldots, m\}}$ its canonical orthonormal basis. The operator $M$ is defined on $\mathcal{G}$ by $M e_1 = 2 e_1 - e_2$, $M e_m = -e_{m-1} + 2 e_m$ and $M e_k = -e_{k-1} + 2 e_k - e_{k+1}$ for $k \in \{2, \ldots, m-1\}$.

Then the operator $H_0$ may be considered as the Hamiltonian of the system describing the behavior of a free particle moving in the discrete strip $\mathbb{Z} \times \{1, \ldots, m\}$.

The eigenvalues and a corresponding basis of (normalized) eigenvectors of $M$ are respectively given by

$$\lambda_j = 4 \sin^2 \left( \frac{\pi j}{2(m+1)} \right), \quad v_j = \sum_{k=1}^{m} \sqrt{\frac{2}{m+1}} \sin \left( \frac{j k \pi}{m+1} \right) e_k, \quad j \in \{1, \ldots, m\}.$$

4.2. Coupling with a non-Hermitian Anderson model. The non-Hermitian Anderson model was first proposed for the analysis of vortex pinning in type-II superconductors [19] and has also been applied to the study of population dynamics [30].

Let $\mathcal{G} = \ell^2(\mathbb{Z}_m)$ for some $m \geq 2$ and $(e_k)_{k \in \mathbb{Z}_m}$ its canonical orthonormal basis. Let $g \in \mathbb{R}$ and $(\omega_k)_{k \in \mathbb{Z}_m} \subset \mathbb{R}$. Consider the operator $M$ defined by:

$$M e_k = e^g e_{k-1} + \omega_k e_k + e^{-g} e_{k+1}, \quad k \in \mathbb{Z}_m.$$

Assume first that $\omega_k = 0$ for any $k \in \mathbb{Z}_m$. The operator $M$ is diagonalizable, with eigenvalues and a corresponding basis of eigenvectors given respectively by:

$$\lambda_j = e^{g} e^{j \theta_m} + e^{-g} e^{j \theta_m} = 2 \cosh g \cos j \theta_m + 2i \sinh g \sin j \theta_m, \quad \theta_m := \frac{2\pi}{m},$$

$$v_j = \sum_{k \in \mathbb{Z}_m} e^{j k \theta_m} e_k, \quad j \in \mathbb{Z}_m.$$

Similar to the previous example, if $g = 0$, the operator $H_0$ is nothing but the discrete Laplace operator on the tube $\mathbb{Z} \times \mathbb{Z}_m$. If $g \neq 0$ and $(\omega_k)$ are random variables drawn from some fixed appropriate probability distribution, the eigenvalues of $M$ form a ”bubble with wings” in the complex plane. We refer to [12, 17] for more details.

4.3. Coupling with a $\mathcal{PT}$-symmetric Hamiltonian. The pertinence of $\mathcal{PT}$-symmetric Hamiltonians in physics is discussed in e.g. [3, 26]. For illustrative purpose, we consider in the sequel a minimal example of a non-Hermitian, $\mathcal{PT}$-symmetric system (see [26] for details).
Let $\mathfrak{G} = \mathbb{C}^2$ and $(e_k)_{k \in \{1,2\}}$, its canonical orthonormal basis. Identifying the operators with their matrix representations in this basis, let $M$ be defined by

$$M = \kappa \sigma_1 + i\gamma \sigma_3$$

with

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

where $\kappa$ and $\gamma$ are non negative.

If $\kappa \neq \gamma$, direct calculations show that the eigenvalues and a corresponding basis of eigenvectors of $M$ are respectively given by:

$$\lambda_j = (-1)^{j+1} \sqrt{\kappa^2 - \gamma^2}$$

$$v_j = (i\gamma + (-1)^{j+1} \sqrt{\kappa^2 - \gamma^2}) e_1 + \kappa e_2.$$

where $j \in \{1,2\}$. In this case, $M$ is diagonalizable but the eigenprojectors are not orthogonal unless $\gamma = 0$.

The linear operator $M$ commutes with the antilinear operator $PT$, where $P$ stands for the linear (and unitary) operator $\sigma_1$ while $T$ stands for the (antilinear) complex conjugation operator. If $\gamma < \kappa$, $\sigma(M) \subset \mathbb{R}$ while if $\gamma > \kappa$, $\sigma(M) \subset i\mathbb{R}$.

If $\kappa = \gamma > 0$, the $PT$ symmetry is spontaneously broken in the sense that $M$ is not diagonalizable anymore.

5. Preliminary results and proof of Proposition 2.2

5.1. Study of the resolvent of the free Hamiltonian near the spectral thresholds.

The first step in our analysis is the study of the behavior of the resolvent of the free Hamiltonian $H_{0,\bullet}$ near the spectral thresholds. In order to unify the analysis for both cases (A) and (B), we introduce the operator $\tilde{\pi}_{0,\bullet}$ on $\mathfrak{G}$ defined by

$$\tilde{\pi}_{0,\bullet} := \begin{cases} 0 & \text{if } \bullet = A \\ \pi_0 & \text{if } \bullet = B, \end{cases}$$

where we recall that $\pi_0$ denotes the projection onto the infinite-dimensional subspace $\text{Ker}(M)$ in case (B). Using the fact that $M$ is diagonalizable, for any $z \in \mathbb{C} \setminus \sigma(H_{0,\bullet})$, one has

$$(H_{0,\bullet} - z)^{-1} = \sum_{j=1}^{d} (\Delta + \lambda_j - z)^{-1} \otimes \pi_j + (\Delta - z)^{-1} \otimes \tilde{\pi}_{0,\bullet}.$$ (5.1)

Let us recall the following basic properties of the one-dimensional discrete Laplacian on $\ell^2(\mathbb{Z})$. Let $\mathcal{F} : \ell^2(\mathbb{Z}) \to L^2(\mathbb{T})$ be the unitary discrete Fourier transform defined by

$$(\mathcal{F}u)(\theta) = \frac{1}{2\pi} \sum_{n \in \mathbb{Z}} e^{-in\theta} u(n), \quad u \in \ell^2(\mathbb{Z}), \theta \in \mathbb{T} := \mathbb{R}/2\pi\mathbb{Z}.$$  

The operator $\Delta$ is unitarily equivalent to the multiplication operator on $L^2(\mathbb{T})$ by the function $f(\theta) = 2 - 2 \cos(\theta)$, more precisely, one has

$$(\mathcal{F}(\Delta u))(\theta) = f(\theta)(\mathcal{F}u)(\theta), \quad u \in \ell^2(\mathbb{Z}), \theta \in \mathbb{T}. $$ (5.2)

Hence, the operator $\Delta$ is selfadjoint in $\ell^2(\mathbb{Z})$ and its spectrum is absolutely continuous and coincides with the range of the function $f$, that is $\sigma(\Delta) = \sigma_{ac}(\Delta) = [0,4]$.
For any \( z \in \mathbb{C} \setminus [0, 4] \), the kernel of \((\Delta - z)^{-1}\) is given by (see for instance [25])
\[
R_0(z; n, m) = \frac{e^{-i\theta(z)|n-m|}}{2i\sin(\theta(z))}, \quad (n, m) \in \mathbb{Z}^2,
\]
where \( \theta(z) \) is the unique solution to the equation \( 2 - 2\cos(\theta) = z \) lying in the region \{ \theta \in \mathbb{C}; -\pi \leq \text{Re}\theta \leq \pi, \text{Im}\theta < 0 \}.

Let us start with following preliminary result.

**Lemma 5.1.** Let \( z_0 \in [0, 4) \) and \( \rho > 0 \). There exists \( 0 < \varepsilon_0 \ll 1 \) such that the operator-valued function
\[
D^*_{\varepsilon_0}(0) \cap C_1 \ni k \mapsto W_{-\rho}(\Delta - (z_0 + k^2))^{-1}W_{-\rho}
\]
admits an analytic extension to \( D^*_{\varepsilon_0}(0) \) if \( z_0 = 0 \) and to \( D^*_{\varepsilon_0}(0) \) if \( z_0 \in (0, 4) \), with values in the trace class operators \( \mathfrak{S}_1(\ell^2(\mathbb{Z})) \). This extension will be denoted \( R_0(z_0 + k^2) \).

**Proof.** The proof follows the ideas of [13, Lemma 3.1]. Let \( z_0 \in [0, 4) \). Then \( \frac{2-z_0}{2} \in (-1, 1) \) if \( z_0 \in (0, 4) \) and \( \frac{2-z_0}{2} = 1 \) if \( z_0 = 0 \). It follows that for \( \varepsilon > 0 \) small enough,
\[
\frac{2-z_0}{2} - \frac{k^2}{2} \in \mathbb{C} \setminus \left((\infty, -1] \cup [1, +\infty)\right)
\]
for \( 0 \leq |k| < \varepsilon_0 \) if \( z_0 \in (0, 4) \) and for \( 0 < |k| < \varepsilon_0 \) if \( z_0 = 0 \). Therefore, for such \( k \), the unique solution to the equation \( 2 - 2\cos(\theta) = z_0 + k^2 \) is given by
\[
\theta_{z_0}(k) = \arccos\left(\frac{2-z_0}{2} - \frac{k^2}{2}\right),
\]
where \( \arccos(z) \) denotes the principal branch of \( \cos^{-1}(z) \), defined for \( z \in \mathbb{C} \setminus \left((\infty, -1] \cup [1, +\infty)\right) \). The function \( k \mapsto \theta_{z_0}(k) \) is analytic in \( D^*_{\varepsilon_0}(0) \) if \( z_0 \in (0, 4) \) and in \( D^*_{\varepsilon_0}(0) \) if \( z_0 = 0 \). Moreover, \( \sin(\theta_{z_0}(k)) \neq 0 \) since \( \theta_{z_0}(k) \neq 0 \). Hence the kernel of \( W_{-\rho}(\Delta - (z_0 + k^2))^{-1}W_{-\rho} \)
given by
\[
\frac{1}{\|e^{-\frac{\theta_{z_0}(k)}{2}m}\|_{\ell^2(\mathbb{Z})}} e^{-\frac{\theta_{z_0}(k)}{2}|m|} e^{-i\theta_{z_0}(k)|n-m|} = \frac{e^{-\frac{\theta_{z_0}(k)}{2}|n|}}{2i\sin(\theta_{z_0}(k))} e^{-\frac{\theta_{z_0}(k)}{2}|n|}
\]
extends analytically with respect to \( k \) in \( D^*_{\varepsilon_0}(0) \) if \( z_0 \in (0, 4) \) and in \( D^*_{\varepsilon_0}(0) \) if \( z_0 = 0 \). Consequently, the operator-valued function \( k \mapsto W_{-\rho}(\Delta - (z_0 + k^2))^{-1}W_{-\rho} \) can be extended via the kernel \( (5.4) \) from \( D^*_{\varepsilon_0}(0) \cap C_1 \to D^*_{\varepsilon_0}(0) \) if \( z_0 \in (0, 4) \) and to \( D^*_{\varepsilon_0}(0) \) if \( z_0 = 0 \). We shall denote this extension \( R_0(z_0 + k^2) \).

Let us show that the extension \( R_0(z_0 + k^2) \) is trace class. Without loss of generality we assume that \( z_0 = 0 \). The proof for \( z_0 \in (0, 4) \) follows the same ideas.

For simplicity, in the following we use the notation \( \theta_k := \theta_0(k) \). Suppose first that \( \text{Im}\theta_k < 0 \). Then combining the fact that \( (\Delta - k^2)^{-1} \in \mathcal{B}(\ell^2(\mathbb{Z})) \) and \( W_{-\rho} \in \mathfrak{S}_1(\ell^2(\mathbb{Z})) \) one gets that \( W_{-\rho}(\Delta - k^2)^{-1}W_{-\rho} \) is trace class.

Suppose now that \( \text{Im}\theta_k \geq 0 \). Let \( L \) be a positive integer, and introduce \( \chi_L \) the characteristic function of \([-L, L] \) in \( \mathbb{Z} \). Define the operator \( R_L(k) := \chi_L(\Delta - k^2)^{-1} \chi_L \). Then, from \( (5.4) \) we can compute the kernel of \( R_L(k)R_L(\tilde{k}) \) to obtain the following relation, for any \( k, \tilde{k} \neq 0 \) sufficiently close to 0,
\[
R_L(k) = R_L(\tilde{k}) + g(k, \tilde{k})R_L(k)R_L(\tilde{k}) + F(k, \tilde{k}),
\]
where \( g(k, \tilde{k}) := (e^{-i\theta_k} - e^{-i\theta_{\tilde{k}}})(e^{i\theta_k} - e^{-i\theta_{\tilde{k}}}) \) and \( F(k, \tilde{k}) \) is the rank two operator with kernel
\[
(n, m) \mapsto \frac{e^{-i\theta_k} - e^{-i\theta_{\tilde{k}}}}{4\sin \theta_k \sin \theta_{\tilde{k}}} \chi_L(n) \left( e^{-i\theta_k(n+L)} e^{-i\theta_{\tilde{k}}(m+L)} - e^{-i\theta_k(n-L)} e^{-i\theta_{\tilde{k}}(m-L)} \right) \chi_L(m).
\]
Let \( \Im \theta_k > 0 \) and take \( \tilde{k} \) such that \( \theta_{\tilde{k}} = -\theta_k \). Then, using (5.5) and the above case \( \Im \theta_k < 0 \) we obtain that \( R_L(k) \) is the sum of a trace class operator and a rank 2 operator whose norm can be estimated by
\[
\frac{1}{2|\sin \theta_k|} \left( \| \chi_L e^{-i\theta_k(\cdot+L)} \|_{\ell^2} \| \chi_L e^{i\theta_k(\cdot+L)} \|_{\ell^2} + \| \chi_L e^{-i\theta_k(\cdot-L)} \|_{\ell^2} \| \chi_L e^{i\theta_k(\cdot-L)} \|_{\ell^2} \right) \leq \frac{C L e^{4|\Im \theta_k|}}{|\sin \theta_k|}.
\]
For \( \Im \theta_k = 0 \), one gets using the kernel (5.4) that \( \| R_L(k) \|_2 \leq C/|\sin \theta_k| \). Therefore, taking \( \theta_k = -i\theta_k \) we obtain from (5.4) and (5.5)
\[
\| R_L(k) \|_1 \leq \frac{C}{|\sin \theta_k|} \| R_L(k) \|_2 \| R_L(\tilde{k}) \|_1 + C \frac{L e^{2L|\theta_k|}}{|\sin \theta_k|^2}.
\]
Now, let us show that \( W_{-\rho} R_L(k) W_{-\rho} \) converges to \( R_0(k^2) \) in the trace class norm as \( L \to \infty \). Let \( L \geq L_0 \). One has, setting \( R_L := R_L(k) \), \( R_{L_0} := R_{L_0}(k) \) and \( R_0 := R_0(k^2) \),
\[
\| W_{-\rho} R_L W_{-\rho} - W_{-\rho} R_{L_0} W_{-\rho} \|_1 \leq \| W_{-\rho} \chi_L R_0(\chi_L - \chi_{L_0}) W_{-\rho} \|_1 + \| W_{-\rho} (\chi_L - \chi_{L_0}) R_0 \chi_{L_0} W_{-\rho} \|_1,
\]
and
\[
\| W_{-\rho} \chi_L R_0(k)(\chi_L - \chi_{L_0}) W_{-\rho} \|_1 = C \left\| \sum_{|l|, |L, L_0| < |j| \leq L} e^{-\frac{\theta}{2}(|l|+|j|)} \delta_l \chi_l R_0(k) \chi_j \delta_j \right\|_1
\]
\[
\leq C \sum_{|l|, |L, L_0| < |j| \leq L} e^{-\frac{\theta}{2}(|l|+|j|)} \| \chi_j R_0(k) \chi_l \|_1
\]
\[
+ C \sum_{|l|, |L, L_0| < |j| \leq L} e^{-\frac{\theta}{2}(|l|+|j|)} \| \chi_l R_0(k) \chi_j \|_1
\]
\[
\leq C \sum_{|l|, |L, L_0| < |j| \leq L} e^{-\frac{\theta}{2}(|l|+|j|)} \| l \| e^{4|\Im \theta_k|}
\]
\[
+ C \sum_{|l|, |L, L_0| < |j| \leq L} e^{-\frac{\theta}{2}(|l|+|j|)} \| l \| e^{4|\Im \theta_k|}
\]
which tends to 0 as \( L_0 \) tends to \( +\infty \) for \( k \) small enough, where in the last inequality we used (5.6). Finally, taking \( L \to +\infty \) in (5.5) one can get the analyticity in \( \mathcal{S}_1 \). This ends the proof of the Lemma.

The next result states that the weighted resolvent of the free Hamiltonian \( H_{0,\bullet} \) extends meromorphically near any \( \lambda_q \in \mathcal{T}_* \). More precisely, one has:

**Lemma 5.2.** Let \( \lambda_q \in \mathcal{T}_* \). There exists \( 0 < \varepsilon_0 \ll 1 \) such that the operator-valued function
\[
D^*_{\varepsilon_0}(0) \cap C_1 \ni k \mapsto W_{-\rho}(H_{0,\bullet} - z_q(k))^{-1} W_{-\rho}
\]
admits an analytic extension to \( D^*_{\varepsilon_0}(0) \), with values in the space \( \mathcal{H}_\bullet \), where
\[
\mathcal{H}_\bullet := \begin{cases} \mathcal{S}_\infty(\mathcal{H}) & \text{if } \bullet = A \\ B(\mathcal{H}) & \text{if } \bullet = B. \end{cases}
\]
We denote this extension $R_{0,\bullet}^{(q)}(k)$.

**Proof.** Let us first consider the case (A). Setting $z_j^{(q)} := \lambda_q - \lambda_j$ it follows from (5.1) that

$$
W_{-\rho}(H_{0,A} - z_q(k))^{-1}W_{-\rho} = \sum_{j=1}^{d} W_{-\rho}(\Delta - (z_j^{(q)} + k^2))^{-1}W_{-\rho} \otimes \pi_j.
$$

The above sum splits into the following two terms

$$
W_{-\rho}(H_{0,A} - z_q(k))^{-1}W_{-\rho} = \sum_{j \in \{0, A\}} W_{-\rho}(\Delta - (z_j^{(q)} + k^2))^{-1}W_{-\rho} \otimes \pi_j
$$

The second term in the RHS is clearly analytic with respect to $k$ in a small neighborhood of 0. On the other hand, by Lemma 5.1, the first term in the RHS extends to an analytic function of $k$ in $D_{\varepsilon_0}^\ast(0)$ for $\varepsilon_0 > 0$ small enough. The extension is clearly in $\mathcal{G}_\infty(\mathcal{H})$.

Consider now the case (B). The only difference with respect to the above case come from $q = 0$. From (5.1) again one has

$$
W_{-\rho}(H_{0,B} - z_0(k))^{-1}W_{-\rho} = \sum_{j=1}^{d} W_{-\rho}(\Delta - (\lambda_j + k^2))^{-1}W_{-\rho} \otimes \pi_j + W_{-\rho}(\Delta - k^2)^{-1}W_{-\rho} \otimes \pi_0.
$$

In this case since the operator $\pi_0$ is not compact, the extension belongs to $\mathcal{B}(\mathcal{H})$. 

**Remark 5.3.** It is easy to see that in case (B), replacing one of the weight operators $W_{-\rho}$ in (5.7) by $W_{-\rho} \otimes K$ with $K$ a compact operator in $\mathcal{G}$ ensures that the analytic extension holds in $\mathcal{G}_\infty(\mathcal{H})$. For instance,

$$
D_{\varepsilon_0}^\ast(0) \cap \mathbb{C} \ni k \mapsto (W_{-\rho} \otimes K)(H_{0,B} - z_q(k))^{-1}W_{-\rho}
$$

admits an analytic extension to $D_{\varepsilon_0}^\ast(0)$, with values in $\mathcal{G}_\infty(\mathcal{H})$. This will be important for the proof of Proposition 2.2.

Now, we precise the form of the singularity of the weighted resolvent at the thresholds.

**Proposition 5.4.** Let $\lambda_q \in \mathcal{T}_\bullet$. There exists $0 < \varepsilon_0 < 1$ such that for $k \in D_{\varepsilon_0}^\ast(0)$ the following statements hold:

(i) Assume that $\lambda_q$ is non-degenerate, then

$$
R_{0,\bullet}^{(q)}(k) - \frac{a^{-1} \otimes \pi_q}{k} \in \text{Hol}(D_{\varepsilon_0}(0); \mathcal{H}_\bullet).
$$

(ii) Assume that $\lambda_q$ is degenerate such that $\lambda_q = \lambda_p + 4$ for some $p \in \{1, ..., d\}$, then

$$
R_{0,\bullet}^{(q)}(k) - \frac{a^{-1} \otimes \pi_q + b^{-1} \otimes \pi_p}{k} \in \text{Hol}(D_{\varepsilon_0}(0); \mathcal{H}_\bullet).
$$

**Proof.** We only prove (i) in case (A). The others assertions work similarly.
Recall from (5.8) that we have
\[ W_\rho(H_{0,A} - z_q(k))^{-1} W_\rho = \sum_{j \neq q} W_\rho(\Delta - (z_j^{(q)} + k^2))^{-1} W_\rho \otimes \pi_j + W_\rho(\Delta - k^2)^{-1} W_\rho \otimes \pi_q. \]

Since \( \lambda_q \neq \lambda_j + 4 \) for all \( j \neq q \) it follows that \( z_j^{(q)} \notin \{0, 4\} \) for any \( j \neq q \in \{1, \ldots, d\} \). Consequently, by Lemma 5.1, the first term in the RHS of the above equation extends analytically in a small neighborhood of 0 with values in \( \mathcal{S}_\infty(H) \). On the other hand, the kernel of the operator \( W_\rho(\Delta - k^2)^{-1} W_\rho \) is given by
\[ R_0(k^2; n, m) = \frac{e^{-\frac{2|m|}{2k}} e^{-\frac{2|n|}{2k}}}{\|2|n|\|_{\ell^2(\mathbb{Z})}} R_0(k; n, m), \]
where \( R_0(k^2; n, m) \) is defined by (5.3).

One can write
\[ R_0(k^2; n, m) = \frac{i}{k \sqrt{4 - k^2}} + \frac{i}{k \sqrt{4 - k^2}} \left( e^{\frac{|n-m|}{2} \text{arcsin} \frac{k}{2} - 1} \right) = \frac{i}{2k} + r(k; n, m), \]
with
\[ r(k; n, m) := i \left( \frac{1}{k \sqrt{4 - k^2}} - \frac{1}{2k} \right) + \frac{i}{k \sqrt{4 - k^2}} \left( e^{\frac{|n-m|}{2} \text{arcsin} \frac{k}{2} - 1} \right). \]

One easily verifies that the function \( r \) extends to a holomorphic function in a small neighborhood of 0. Therefore, putting together (5.12) and (5.13), one obtains
\[ W_\rho(\Delta - k^2)^{-1} W_\rho \otimes \pi_q = \frac{a_1 \otimes \pi_q}{k} + \mathcal{A}(k) \otimes \pi_q, \]
where \( \mathcal{A}(k) \) is the operator on \( \ell^2(\mathbb{Z}) \) with kernel \( W_\rho(n)r(k; n, m)W_\rho(m) \). This ends the proof. \( \square \)

5.2. **Proof of Proposition 2.2.** The proof is a consequence of Lemma 5.2 and the analytic Fredholm extension Theorem. From the resolvent identity
\[ (H_\omega, \bullet - z)\big((I + \omega V(H_0, \bullet - z))^{-1} = (H_0, \bullet - z)^{-1}, \]

it follows that
\[ W_{-\rho}(H_\omega, \bullet - z_q(k))^{-1} W_{-\rho} = W_{-\rho}(H_0, \bullet - z_q(k))^{-1} W_{-\rho}(I + \mathcal{P}_{\omega, \bullet}(z_q(k)))^{-1}, \]
where
\[ \mathcal{P}_{\omega, \bullet}(z) := \omega W_\rho V(H_0, \bullet - z)^{-1} W_{-\rho}. \]

Lemma 5.2 implies that there exists \( \varepsilon_0 > 0 \) such that the operator-valued function \( k \mapsto \mathcal{P}_{\omega, \bullet}(z_q(k)) \) defined by (5.16) extends to an analytic function in \( D_{\varepsilon_0}(0) \). In case (A), Lemma 5.2 again ensures that this extension is with values in \( \mathcal{S}_\infty(H) \). On the other hand, in case (B), using the polar decomposition \( V = J|V|^\frac{1}{2} |V|^\frac{1}{2} \) and Assumption 2.1 (B), one has
\[ \mathcal{P}_{\omega, B}(z_q(k)) = \omega W_\rho V(H_{0,B} - z_q(k))^{-1} W_{-\rho} = \omega W_\rho |V|^\frac{1}{2} J|V|^\frac{1}{2}(H_{0,B} - z_q(k))^{-1} W_{-\rho} = \omega W_\rho |V|^\frac{1}{2} J\nu W_{-\rho} \otimes K(H_{0,B} - z_q(k))^{-1} W_{-\rho}. \]
According to Remark 5.3, \((W_{-\rho} \otimes K)(H_{0,B} - z_q(k))^{-1}W_{-\rho}\) extends to an analytic extension in a small neighborhood of 0 with values in \(S_{\infty}(\mathcal{H})\). Since the operator
\[
W_{\rho}|V|^{1/2} = W_{\rho}(|V|^{1/2})^{*} = W_{\rho}(W_{-\rho} \otimes K^{*})V^{*} = (I_{\ell^2(Z)} \otimes K^{*}) \otimes V^{*}
\]
is bounded, it follows that the analytic extension of \(\mathcal{P}_{\omega,B}(z_q(k))\) is also with values in \(S_{\infty}(\mathcal{H})\). Therefore the analytic Fredholm theorem ensures that
\[
D_{\epsilon_0}^{*}(0) \cap \mathbb{C} \ni k \leftrightarrow (I + \mathcal{P}_{\omega,•}(z_q(k)))^{-1}
\]
adopts a meromorphic extension to \(D_{\epsilon_0}^{*}(0)\). We use the same notation for the extended operator. Hence, the operator-valued function \(k \mapsto W_{-\rho}(H_{\omega,•} - z_q(k))^{-1}W_{-\rho}\) extends to a meromorphic function of \(k \in D_{\epsilon_0}^{*}(0)\). This ends the proof of Propositions 2.2.

6. Proof of Theorem 3.1

We split the proof into two parts, namely, part 1) (i) and part 1) (ii). In the end of this section we will point out the modifications needed in order to prove assertion 2).

6.1. Proof of part 1) (i) of Theorem 3.1. Let \(\lambda_q \in \mathcal{T}_A\) be a fixed threshold and assume that \(\lambda_q\) is non-degenerate in the sense of Definition 2.1.

According to Proposition 5.4, there exists \(\epsilon_0 > 0\) and an analytic function \(G\) in \(D_{\epsilon_0}(0)\) with values in \(S_{\infty}(\mathcal{H})\) such that for all \(k \in D_{\epsilon_0}^{*}(0)\) we have
\[
(6.1) \quad \mathcal{R}_{0,A}^{(q)}(k) = \frac{a_1 \otimes \pi_q}{k} + G(k).
\]

It follows from equation (5.15) that for all \(k \in D_{\epsilon_0}^{*}(0)\),
\[
(6.2) \quad \mathcal{R}_{\omega,A}^{(q)}(k) = \left(\frac{a_1 \otimes \pi_q}{k} + G(k)\right) [I + \mathcal{P}_{\omega,A}(z_q(k))]^{-1},
\]
where \(\mathcal{P}_{\omega,A}(z_q(k))\) is defined by (5.16). More precisely, setting \(V_\rho := W_{\rho}VW_{\rho}\), one has
\[
(6.3) \quad [I + \mathcal{P}_{\omega,A}(z_q(k))]^{-1} = \left(1 + \omega V_\rho \mathcal{R}_{0,A}^{(q)}(k)\right)^{-1}.
\]

Since \(G\) is analytic near 0 it follows that for \(|\omega|\) small enough, the operator-valued function \(I + \omega V_\rho G(k)\) is invertible. Using (6.1), one writes
\[
(6.4) \quad [I + \mathcal{P}_{\omega,A}(z_q(k))]^{-1} = \left(1 + \frac{\omega}{k} \mathcal{L}_\omega(k)\right)^{-1} (I + \omega V_\rho G(k))^{-1},
\]
where \(\mathcal{L}_\omega(k)\) is the operator in \(\mathcal{H}\) defined by
\[
\mathcal{L}_\omega(k) := (I + \omega V_\rho G(k))^{-1} V_\rho (a_1 \otimes \pi_q).
\]

Putting together (6.2) and (6.4), we obtain, for all \(k \in D_{\epsilon_0}^{*}(0)\),
\[
(6.5) \quad \mathcal{R}_{\omega,A}^{(q)}(k) = \left(\frac{a_1 \otimes \pi_q}{k} + G(k)\right) \left[1 + \frac{\omega}{k} \mathcal{L}_\omega(k)\right]^{-1} (I + \omega V_\rho G(k))^{-1}.
\]

Consequently, the poles of \(k \mapsto \mathcal{R}_{\omega,A}^{(q)}(k)\) near 0 coincide with those of the operator-valued function
\[
J_\omega(k) := \left(\frac{a_1 \otimes \pi_q}{k} + G(k)\right) \left[1 + \frac{\omega}{k} \mathcal{L}_\omega(k)\right]^{-1}.
\]
We shall make use of the following elementary result whose proof is omitted.
Lemma 6.1. Let $\mathcal{K}$ be a Hilbert space and consider two linear operators $A, \Pi : \mathcal{K} \to \mathcal{K}$ such that $\Pi^2 = \Pi$ and $A \Pi = \Pi$. Then, $I + A$ is invertible if and only if $\Pi(I + A) : \text{Ran} \Pi \to \text{Ran} \Pi$ is invertible, and in this case one has

$$(I + A)^{-1} = (I - \tilde{\Pi} A \Pi) B^{-1} + \tilde{\Pi},$$

where $\tilde{\Pi} := I - \Pi$ and $B^{-1} := (\Pi(I + A) \Pi)^{-1} \oplus 0$ with respect to the decomposition $\mathcal{K} = \text{Ran} \Pi \oplus \text{Ran} \tilde{\Pi}$.

Let $\Pi_q$ be the projection on $\mathcal{H}$ defined by (3.3). Applying the above result with $A = \frac{\omega}{k} L_\omega(k)$ and $\Pi = \Pi_q$, we get

$$(I + \frac{\omega}{k} L_\omega(k))^{-1} = \left(I - \frac{\omega}{k} \tilde{\Pi}_q L_\omega(k) \Pi_q\right) \left(\left(\Pi_q \left(I + \frac{\omega}{k} L_\omega(k)\right) \Pi_q\right)^{-1} \oplus 0\right) + \tilde{\Pi}_q.$$

Here $\tilde{\Pi}_q := I - \Pi_q$. Therefore, a straightforward computation yields

$$J_\omega(k) = \left(\frac{i}{2} \Pi_q - \omega \mathcal{G}(k) \left(\frac{k}{\omega} - \tilde{\Pi}_q L_\omega(k) \Pi_q\right)\right) \left([\Pi_q (k + \omega L_\omega(k)) \Pi_q]^{-1} \oplus 0\right) + \mathcal{G}(k) \tilde{\Pi}_q.$$

Since $[\Pi_q (k + \omega L_\omega(k)) \Pi_q]^{-1} : \text{Ran} \Pi_q \to \text{Ran} \Pi_q$ it follows that $[\Pi_q (k + \omega L_\omega(k)) \Pi_q]^{-1} \oplus 0$ is stable by $\Pi_q : \mathcal{H} \to \mathcal{H}$. Consequently,

$$(6.6) \quad J_\omega(k) = \left(\frac{i}{2} - \omega \mathcal{G}(k) \left(\frac{k}{\omega} - \tilde{\Pi}_q L_\omega(k) \Pi_q\right)\right) \left([\Pi_q (k + \omega L_\omega(k)) \Pi_q]^{-1} \oplus 0\right) + \mathcal{G}(k) \Pi_q.$$

Using the analyticity of $\mathcal{G}$ and $L_\omega$ near 0, one sees that $\left(\frac{i}{2} - \omega \mathcal{G}(k) \left(\frac{k}{\omega} - \tilde{\Pi}_q L_\omega(k) \Pi_q\right)\right)$ is invertible for $|k|$ and $|\omega|$ small enough. Therefore, we conclude that the poles near 0 of $J_\omega$ are the same as those of the operator-valued function

$$k \mapsto (\Pi_q (k + \omega L_\omega(k)) \Pi_q)^{-1} : \text{Ran} \Pi_q \to \text{Ran} \Pi_q.$$

Let $M_\omega(k)$ be the matrix of the operator $\Pi_q (k + \omega L_\omega(k)) \Pi_q : \text{Ran} \Pi_q \to \text{Ran} \Pi_q$. We have

$$\Pi_q (k + \omega L_\omega(k)) \Pi_q = k \Pi_q + \omega \Pi_q (I + \omega V_\rho \mathcal{G}(k))^{-1} \rho(a_{-1} \otimes \pi_q) \Pi_q$$

$$(6.7) \quad = k \Pi_q + \frac{i}{2} \omega \Pi_q V_\rho \Pi_q + \omega^2 \Pi_q S_\omega(k) \Pi_q,$$

where $S_\omega(k) := \frac{i}{2} \sum_{n \geq 1} (-1)^n \omega^{n-1} (V_\rho \mathcal{G}(k))^n V_\rho$ is an operator-valued function which is analytic near $k = 0$ for $|\omega| > 0$ small enough and $\|S_\omega(k)\| = O(1)$ uniformly w.r.t. $k$.

The usual expansion formula for the determinant allows to write

$$\det(M_\omega(k)) = \omega^{\nu_q} \left(\prod_{j=1}^r \left(\frac{k}{\omega} + \alpha_j^{(q)}\right)^{m_{q,j}} + \omega s_\omega(k)\right),$$

with $\alpha_j^{(q)} := \frac{i}{2} \alpha_j^{(q)}$, where $\{\alpha_j^{(q)}\}_{j=1}^r$ are the distinct eigenvalues of $E_q := \Pi_q V_\rho \Pi_q : \text{Ran} \Pi_q \to \text{Ran} \Pi_q$ and $s_\omega$ is an analytic scalar-valued function satisfying

$$(6.8) \quad |s_\omega(k)| \leq C_0,$$
for some constant $C_0 > 0$ independent of $k$ and $\omega$. We are therefore led to study the roots of the equation

$$
(6.9) \quad \prod_{j=1}^{r} \left( \frac{k}{\omega} + \alpha_j(q) \right)^{m_{q,j}} + \omega s_\omega(k) = 0.
$$

On one hand, by a simple contradiction argument one shows that all the roots of the above equation satisfy (3.5). On the other hand, let $j_0 \in \{1, \ldots, r\}$ and let $C > 0$ be a constant independent of $k$ and $\omega$. We set $\delta_{j_0, C} := C|\omega|^{1+\frac{1}{m_{q,j_0}}}$. There exists a constant $C' > 0$ such that for any $k \in \partial D_{\delta_{j_0, C}}(-\alpha_{j_0}, \omega)$, one has

$$
\prod_{j=1, j \neq j_0}^{r} \left| \frac{k}{\omega} + \alpha_j(q) \right|^{m_{q,j}} \geq C'.
$$

Consequently,

$$
\prod_{j=1}^{r} \left| \frac{k}{\omega} + \alpha_j(q) \right|^{m_{q,j}} \geq CC'|\omega| > |\omega s_\omega(k)|, \quad \forall k \in \partial D_{\delta_{j_0, C}}(-\alpha_{j_0}, \omega),
$$

where $C > 0$ is chosen such that $CC' > C_0$, with $C_0$ given by (6.8).

Since both terms in (6.9) are analytic functions of $k$ near $k = 0$, it follows by Rouché Theorem that for $|\omega|$ small, det($M_\omega(k)$) admits exactly $m_{q,j_0}$ zeros in $D_{\delta_{j_0, C}}(-\alpha_{j_0}, \omega)$, counting multiplicities. This ends the proof of statement (i).

### 6.2. Proof of part 1) (ii) of Theorem 3.1.

Fix $q \in \{1, \ldots, d\}$. Equation (3.5) implies that in variable $k$, the resonances of $H_{\omega,A}$ are distributed in “clusters” around the points $-\frac{i}{2} \omega \alpha_j(q)$, $j \in \{1, \ldots, r\}$. Fix $j \in \{1, \ldots, r\}$ and let $C > 0$ and $1 < \delta < 1 + 1/m_{q,j}$ so that the disk $D_{\frac{1}{2} \omega \alpha_j(q)}$ contains all the resonances of the $j$-th cluster and only them. Set $\Gamma_j := \partial D_{\frac{1}{2} \omega \alpha_j(q)}$. We will show that

$$
(6.10) \quad \text{rank} \int_{\Gamma_j} \mathcal{R}^{(q)}_{\omega,A}(k) dk = \text{rank} \int_{\Gamma_j} (k \Pi_q + \frac{i}{2} \omega \Pi_q \mathbf{V} \Pi_q)^{-1} dk.
$$

Since the RHS is equal to the multiplicity of $-\frac{i}{2} \omega \alpha_j(q)$, part ii) of Theorem 3.1 follows.

Equality (6.10) is a consequence of the following Lemma and standard arguments (see for instance [32, Page 14]).

**Lemma 6.1.** As $|\omega| \to 0$, one has

a) $\left\| \int_{\Gamma_j} \mathcal{R}^{(q)}_{\omega,A}(k) - (k \Pi_q + \frac{i}{2} \omega \Pi_q \mathbf{V} \Pi_q)^{-1} dk \right\| = o(1)$.

b) $\left\| \left( \int_{\Gamma_j} \mathcal{R}^{(q)}_{\omega,A}(k) dk \right)^2 - \int_{\Gamma_j} \mathcal{R}^{(q)}_{\omega,A}(k) dk \right\| = o(1)$.

**Proof.** We simplify the notations by putting

$$
\mathcal{A}_0(k) := k \Pi_q + \frac{i}{2} \omega \Pi_q \mathbf{V} \Pi_q \quad \text{and} \quad \mathcal{A}(k) := \Pi_q (k + \omega \mathcal{L}_\omega(k)) \Pi_q.
$$

...
Using (6.5) and (6.7), one has

\[ \oint_{\Gamma_j} R_{\omega}\mathcal{A}(k)dk = \oint_{\Gamma_j} T_\omega(k)\mathcal{A}(k)^{-1}U_\omega(k)dk, \]

where \( T_\omega(k) = I + \omega \tilde{T}_\omega(k) \) and \( U_\omega(k) = I + \omega \tilde{U}_\omega(k) \) are holomorphic operator-valued functions in \( k \) and uniformly bounded inside \( \Gamma_j \), for \( |k| \leq |\omega| \ll 1 \). One writes

\[ T_\omega(k)\mathcal{A}(k)^{-1}U_\omega(k) = \mathcal{A}(k)^{-1} + \omega \tilde{T}_\omega(k)\mathcal{A}(k)^{-1} + \omega \mathcal{A}(k)^{-1}\tilde{U}_\omega(k) + \omega^2 \tilde{T}_\omega(k)\mathcal{A}(k)^{-1}\tilde{U}_\omega(k). \]

On the other hand, from (6.7), one has for any \( k \in \Gamma_j \),

\[ \mathcal{A}(k) = \mathcal{A}_0(k)(I + \omega^2 \mathcal{A}_0(k)^{-1}\tilde{S}_\omega(k)), \]

with \( \tilde{S}_\omega(k) := \Pi_q S_\omega(k)\Pi_q \). Using the self-adjointness of \( V \), for any \( k \in \Gamma_j \), we have

\[ \| \mathcal{A}_0(k)^{-1} \| = \| (k\Pi_q + \frac{i}{2}\omega \Pi_q V_q \Pi_q)^{-1} \| = \frac{1}{\text{dist}(k, \sigma(-\frac{1}{2}\omega \Pi_q V_q \Pi_q))} = \mathcal{O}(|\omega|^{-\delta}). \]

Therefore, \( \| I + \omega^2 \mathcal{A}_0(k)^{-1}\tilde{S}_\omega(k) \| \leq 2 \) and it follows from (6.13) that we have

\[ \left\| \oint_{\Gamma_j} \omega \tilde{T}_\omega(k)\mathcal{A}(k)^{-1}U_\omega(k)dk \right\| \leq C|\omega| \oint_{\Gamma_j} \| \mathcal{A}_0(k)^{-1} \| d|k| \leq C|\omega|. \]

The integral over \( \Gamma_j \) of each one of the two last terms in the RHS of (6.12) can be estimated in the same way and we obtain

\[ \oint_{\Gamma_j} \omega \mathcal{A}(k)^{-1}\tilde{U}_\omega(k)dk = \mathcal{O}(|\omega|) \quad \text{and} \quad \oint_{\Gamma_j} \omega^2 \tilde{T}_\omega(k)\mathcal{A}(k)^{-1}\tilde{U}_\omega(k)dk = \mathcal{O}(|\omega|). \]

Now, using (6.14) again, we get \( \| (I + \omega^2 \mathcal{A}_0(k)^{-1}\tilde{S}_\omega(k))^{-1} - I \| = \mathcal{O}(|\omega|^{2-\delta}) \), which yields

\[ \left\| \oint_{\Gamma_j} \mathcal{A}(k)^{-1} - \mathcal{A}_0(k)^{-1} \| d|k| \leq \oint_{\Gamma_j} \left\| (I + \omega^2 \mathcal{A}_0(k)^{-1}\tilde{S}_\omega(k)^{-1}) - I \right\| \mathcal{A}_0(k)^{-1} d|k| = \mathcal{O}(|\omega|^{2-\delta}). \]

Putting together (6.11), (6.12), (6.15), (6.16) and (6.17) we get statement a).

Let us now prove statement b). We set \( \tilde{\Gamma}_j(t) = -\frac{i\omega}{2}\alpha_j^{(q)} + e^{-it}|\omega|^\eta \), \( t \in [0, 2\pi] \) and \( 0 < \eta < 1 \). For \( k \in \Gamma_j \) and \( \tilde{k} \in \tilde{\Gamma}_j \), using (6.12) and proceeding as above, one obtains

\[ \left\| \oint_{\Gamma_j} \oint_{\tilde{\Gamma}_j} R_{\omega}\mathcal{A}(k)R_{\omega}\mathcal{A}(\tilde{k}) - \mathcal{A}(k)^{-1}\mathcal{A}(\tilde{k})^{-1}d\tilde{k}dk \right\| = o(|\omega|). \]

Next, from the resolvent identity one writes

\[ \mathcal{A}(k)^{-1}\mathcal{A}(\tilde{k})^{-1} = \frac{\mathcal{A}(k)^{-1}}{k - \tilde{k}} - \frac{\mathcal{A}(\tilde{k})^{-1}}{\tilde{k} - k} - \omega^2 \mathcal{A}(k)^{-1}(\tilde{S}_\omega(\tilde{k}) - \tilde{S}_\omega(k))\mathcal{A}(\tilde{k})^{-1}. \]

Further,

\[ \oint_{\Gamma_j} \oint_{\tilde{\Gamma}_j} \frac{\mathcal{A}(k)^{-1}}{k - \tilde{k}} d\tilde{k}dk - \oint_{\Gamma_j} \oint_{\tilde{\Gamma}_j} \frac{\mathcal{A}(\tilde{k})^{-1}}{\tilde{k} - k} d\tilde{k}dk = \oint_{\Gamma_j} \mathcal{A}(k)^{-1} dk. \]
Using the fact that \( \|I + \omega^2A_0(k)^{-1}\tilde{S}_\omega(k)\| \leq 2 \) together with (6.13), we obtain
\[
|\omega|^2 \left| \oint_{\Gamma_j} \oint_{\tilde{\Gamma}_j} A(k)^{-1} \left( \frac{\tilde{S}_\omega(\tilde{k}) - (S_\omega(k))}{\tilde{k} - k} \right) \right| \leq |\omega|^2 \oint_{\Gamma_j} \oint_{\tilde{\Gamma}_j} \left| \frac{\|A_0(k)^{-1}\| \|\tilde{A}_0(\tilde{k})^{-1}\|}{|\tilde{k} - k|} \right| d|\tilde{k}|d|k|
\]
(6.21)

Putting together (6.18)-(6.21) and using (6.12) and (6.15) we get statement b). \( \square \)

6.3. Proof of part 2) of Theorem 3.1. The proof of 2) can be performed in a similar manner as that of 1) (i). The only difference is to define the operator \( \mathcal{L}_\omega(k) \) by
\[
\mathcal{L}_\omega(k) = (I + \omega V_\rho \mathcal{G}(k))^{-1} V_\rho (a_{-1} \otimes \pi_q + b_{-1} \otimes \pi_p)
\]
to get the analogous equation of (6.5) in this case. Then Lemma 6.1 can be applied with \( \Pi = \Pi_{q,p} \) and (6.6) is also obtained. The rest of the proof is the same.

7. Proof of Theorem 3.3

7.1. Resonances as characteristic values. We start this section by giving an alternative definition of the multiplicity of a resonance. Our first task will be to show that this new definition coincides with the one given in (3.6).

To begin with, let us recall some definitions and results on characteristic values. For more details on the subject, one refers to [15] and the book [16, Section 4]. Let \( \mathcal{U} \) be a neighborhood of a fixed point \( w \in \mathbb{C} \), and \( F: \mathcal{U} \setminus \{w\} \rightarrow \mathcal{B}(\mathcal{K}) \) be a holomorphic operator-valued function. The function \( F \) is said to be finite meromorphic at \( w \) if its Laurent expansion at \( w \) has the form
\[
F(z) = \sum_{n=m}^{+\infty} (z - w)^n A_n, \quad m > -\infty,
\]
where (if \( m < 0 \)) the operators \( A_m, \ldots, A_{-1} \) are of finite rank.

Assume that the set \( \mathcal{U} \) is open connected, \( F \) is finite meromorphic and Fredholm at each point of \( \mathcal{U} \), and there exists \( w_0 \in \mathcal{U} \) such that \( F(w_0) \) is invertible. Then, there exists a closed and discrete subset \( Z' \) of \( \mathcal{U} \) such that \( F(z) \) is invertible for each \( z \in \mathcal{D} \setminus Z' \) and
\[
F^{-1}: \mathcal{U} \setminus Z' \rightarrow \mathrm{GL}(\mathcal{K})
\]
is finite meromorphic and Fredholm at each point of \( \mathcal{U} \) [16, Proposition 4.1.4].

**Definition 7.1.** The points of \( Z' \) where the function \( F \) or \( F^{-1} \) is not holomorphic are called the characteristic values of \( F \). The index of \( F \) with respect to the contour \( \partial \Omega \) is defined by
\[
\text{Ind}_{\partial \Omega} F := \frac{1}{2\pi i} \text{Tr} \oint_{\partial \Omega} F'(z)F(z)^{-1} dz,
\]
where \( \partial \Omega \) is the boundary of a connected domain \( \Omega \subseteq \mathcal{D} \) not intersecting \( Z' \). This number is actually an integer (see section [16, Section 4]).
Lemma 7.4. The multiplicity of \( I \) of \( z \) implies that
\[
\text{which is analytic in } D \text{ and Lemma 5.2 that } z \in D^+_\omega(0) \text{ is a characteristic value of the operator-valued function } I + P_{\omega,\dagger}(z(\cdot)), \text{ where } P_{\omega,\dagger}(\cdot) \text{ is defined by (5.16)}.
\]

**Definition 7.2.** For \( z_0 \in \text{Res}(H_{\omega,\dagger}) \), we define
\[
(7.2) \quad \text{mult}_T(z_0) := \text{Ind}_{\gamma_z}(I + P_{\omega,\dagger}(z(\cdot))).
\]
Here \( \gamma_z \) is a positively oriented circle chosen sufficiently small so that \( k_0 \) is the only characteristic value enclosed by \( \gamma_z \).

The following result states that both definitions (7.2) and (3.6) of the multiplicity of a resonance coincide.

**Lemma 7.3.** Let Assumption 2.1 \((\dagger)\) holds and let \( z_0 \in \text{Res}(H_{\omega,\dagger}) \). Then one has
\[
\text{mult}_T(z_0) = \text{mult}(z_0).
\]

**Proof.** This result is a consequence of Proposition 8.1 applied to \( P = H_{\omega,\dagger}, Q = H_0, \) and \( S = \omega V \). Set \( B := \mathcal{H} = \ell^2(\mathbb{Z}) \otimes \mathfrak{G} \), \( B_0 := \ell^2_p(\mathbb{Z}) \otimes \mathfrak{G} \) and \( B_1 := \ell^2_{-p}(\mathbb{Z}) \otimes \mathfrak{G} \), where \( \ell^2_p(\mathbb{Z}) := \{ u : \mathbb{Z} \to \mathbb{C} : \forall n \in \mathbb{Z} \ |u(n)|^2 e^{|\omega|n} < \infty \} \). It is standard to see that \( B_0 \) is dense in \( B \), \( B \) is dense in \( B_1 \) and that the corresponding inclusions are continuous. From Assumption 2.1 \((\bullet)\), the operator \( V \) is bounded. Then Conditions (I) and (II) are satisfied since \( H_0 \) is \( \omega V \) is bounded. Thus all the requirements of Proposition 8.1 are met and the result follows. \( \Box \)

Under Assumption 2.1 \((\bullet)\), define the operator-valued function
\[
(7.3) \quad k \mapsto \mathcal{T}_\omega(z_0(k)) := \omega V^{1/2}(H_{0,B} - k^2)^{-1}V^{1/2},
\]
which is analytic in \( D^s_{\omega}(0) \) with values in compact operators, as follows immediately from Remark 5.3. Hence, identity
\[
(I + \omega V^{1/2}(H_{0,B} - z_0(k))^{-1}V^{1/2})(I - \omega V^{1/2}(H_{\omega,B} - z_0(k))^{-1}V^{1/2}) = I
\]
implies that \( z_0(k) = k^2 \in S_0 \) is a resonance of \( H_{\omega,B} \) if and only if \( k_0 \) is a characteristic value of \( I + \mathcal{T}_\omega(z_0(\cdot)) \). Moreover, the following holds:

**Lemma 7.4.** The multiplicity of \( z_0 = k_0^2 \) given by (7.2) coincides with the multiplicity of \( k_0 \) as a characteristic value of \( I + \mathcal{T}_\omega(z_0(\cdot)) \). That is
\[
\text{mult}_T(z_0) = \text{Ind}_{\gamma_z}(I + \mathcal{T}_\omega(z_0(\cdot))).
\]

**Proof.** Under Assumption 2.1 \((\bullet)\), as in the proof of Lemma 5.2, one can show that for \( k \in D^s_{\omega}(0) \) the operators \( P_{\omega,B}(z_0(k)) \) and \( \mathcal{T}_\omega(z_0(k)) \) are in the Schatten class \( \mathfrak{S}_p(\mathcal{H}) \). Therefore, one can define the \( p \)-regularized determinant of \( I + F_j(k) \), \( j = 1, 2 \) where \( F_1(k) = P_{\omega,B}(z_0(k)) \) and \( F_2(k) = \mathcal{T}_\omega(z_0(k)) \), by
\[
f_j(k) := \text{det}_p(I + F_j(k)) := \text{det}\left[(I + F_j(k))e^{\sum_{i=0}^{p-1} \frac{(-F_j(k))^i}{i!}}\right].
\]
Moreover \( D^s_{\omega}(0) \ni k \mapsto f_j(k) \) is a holomorphic function and \( k_0 \) is a zero of \( f_j \). The Residue theorem implies that the multiplicity of \( k_0 \) as zero of \( f_j \) is equal to
\[
\frac{1}{2i\pi} \int_{\gamma_z} \frac{f_j'(k)}{f_j(k)} dk = \frac{1}{2i\pi} \int_{\gamma_z} \partial_k \ln f_j(k) dk
\]
for \( \epsilon \) which is analytic near 0. Then, (7.5) yields
\[
I = \frac{1}{2\pi i} \int_{z_0} \text{Tr} \left( (I + F_j(k))' (I + F_j(k))^{-1} - \sum_{s=0}^{m-1} (I + F_j(k))' (-F_j(k))^{s+1} \right) dk = \text{Ind}_{\gamma} (I + F_j(k)).
\]
Now, it suffices to note that the functions \( f_1 \) and \( f_2 \) satisfy for \( k \in D_{\epsilon_0}^* (0) \cap C_1 \)
\[
f_1(k) = \det_p (I + \omega V (H_{0,B} - z_0(k))^{-1}) = f_2(k).
\]
Thus \( f_1 \) and \( f_2 \) coincide in \( D_{\epsilon_0}^* (0) \) and the result follows. \( \square \)

Introduce the operator
\[
Q_0 := -i V^{\frac{1}{2}} W_\rho (a_{-1} \otimes \pi_0) W_\rho V^{\frac{1}{2}},
\]
and let \( P_0 \) be the orthogonal projection onto Ker\( (Q_0) \). Then, according to Proposition 5.4
\[
T_\omega(z_0(k)) = \frac{i \omega}{k} Q_0 + \omega T_V(k)
\]
for \( T_V(k) = V^{\frac{1}{2}} W_\rho G(k) W_\rho V^{\frac{1}{2}} (G(k) \text{ being defined in (6.1))}.\)

7.2. **Proof of Theorem 3.3.** From the above discussion, the poles different from zero of \( W_\rho (H_{0,B} - z_0(k))^{-1} W_\rho \) coincide with the characteristic values \( k \) of \( I + T_\omega(z_0(\cdot)) \).

Set
\[
\mathcal{A}_\omega(\zeta) := Q_0 - \zeta T_V(-i \omega \zeta),
\]
which is analytic near 0. Then, (7.5) yields \( I + T_\omega(z_0(k)) = I - \frac{\mathcal{A}_\omega(\zeta)}{\zeta} \), with \( \zeta = i \omega^{-1} k \).

Now notice the following:
- \( \mathcal{A}_\omega(0) = Q_0 \) is self adjoint.
- Assumption 2.1 (B) implies that \( \mathcal{A}_\omega(\zeta) \) is compact-valued.
- \( I - \mathcal{A}_\omega(0) P_0 = I - \omega T_V(0) P_0 \), so for \( \omega \ll 1 \) the operator \( I - \mathcal{A}_\omega(0) P_0 \) is invertible.

Thus the conditions of [5, Corollary 3.4. (i) and (ii)] are met. This result states that in our situation, for \( \epsilon_0 > 0 \) small enough and for any \( 0 < \epsilon < \epsilon_0 \), the characteristic values \( \zeta = i \omega^{-1} k \) such that \( |\omega| < |k| < \epsilon_0 |\omega| \), satisfy
\[
\text{Re}(ik/\omega) \geq 0, \quad |\text{Im}(ik/\omega)| = o(|k/\omega|),
\]
which in turn imply (i) of Theorem 3.3.

Let us denote the set of characteristic values of \( I + T_\omega \) by \( \text{Char}(\bullet) \). Equation (7.6) shows that for \( \epsilon |\omega| < |k| < \epsilon_0 |\omega| \), these characteristic values \( \zeta = i \omega^{-1} k \) are concentrated in a sector \( C_\theta(\epsilon, \epsilon_0) \), \( \theta > 0 \). In particular, using Lemmas 7.3 and 7.4, one has that (counting multiplicities)
\[
\# \{ z_0(k) \in \text{Res}(H_{\omega,B}) : |\epsilon| |\omega| < |k| < \epsilon_0 |\omega| \} = \# \{ z = i \omega^{-1} k \in \text{Char}(\bullet) \cap C_\theta(\epsilon, \epsilon_0) \} + O(1),
\]
for \( \epsilon \searrow 0 \).

Now, by Assumption 2.1 (B), \( \mathcal{A}_\omega(0) = Q_0 \in \mathcal{G}_p(\mathcal{H}) \). If it also has infinite rank, then [5, Corollary 3.9] implies that there exists a sequence \( (\epsilon_j)_j \) of positive numbers tending to zero such that
\[
\# \{ z = i \omega^{-1} k \in \text{Char}(\bullet) \cap C_\theta(\epsilon_j, \epsilon_0) \} = \text{Tr} \chi_{[\epsilon_j, 1]}(Q_0)(1 + o(1)), \quad j \to \infty.
\]
To conclude the proof it suffices to notice that one has
\[
Q_0 = \frac{1}{2} (V^{\frac{1}{2}} W_\rho P_0)(V^{\frac{1}{2}} W_\rho P_0)^* \quad \text{and} \quad E_0 = (V^{\frac{1}{2}} W_\rho P_0)^*(V^{\frac{1}{2}} W_\rho P_0),
\]
so that both operators has the same rank and $n_{\ell,j,1}(Q_0) = n_{\ell,j,1}(\frac{1}{2}E_0) = n_{\ell,j,2}(E_0) = n_{\ell,j,1}(E_0) + O(1)$.

8. Appendix

In this appendix we prove an abstract result concerning the multiplicity of resonances. We use the abstract setting for the theory of resonances as it appears in [1]. The presentation in this section is given in a more general framework than what is required for our study. It can be applied in other settings as well [33, 7].

Let $B$ be a Banach space and $B_0, B_1$ two reflexive Banach spaces such that $B_0 \subset B \subset B_1$, $B_0$ is dense in $B$ and $B$ is dense in $B_1$. Further, the natural injections $I_0 : B_0 \hookrightarrow B$ and $I : B \surj B_1$ are continuous.

Let $Q : Dom(Q) \subset B \to B$ be a closed linear operator and $S : B \to B_0$ linear, such that $S : B \to B$ is bounded and extends as a bounded operator from $B_1$ to $B_0$. Define

$$P := Q + S \quad \text{with} \quad Dom(P) = Dom(Q).$$

Let $D \subset \mathbb{C}$ be an open subset of the resolvent set of $Q$ and the resolvent set of $P$. We assume that $D \ni z \to I(P - z)^{-1}I_0 \in B(B_0, B_1)$ has a finite meromorphic extension to $D^+ \supset D$. In the same way we assume that $D \ni z \to I(Q - z)^{-1}I_0 \in B(B_0, B_1)$ has an analytic extension to $D^+$. Denote these extensions by $R(z), R_0(z)$ respectively.

Therefore, if $z_0 \in D^+$ is a pole of $R(z)$, we have the expansions

$$R_0(z) = \sum_{j=0}^{\infty} M_j(z - z_0)^j \quad ; \quad R(z) = A_{-L}(z - z_0)^{-L} + \cdots + A_{-1}(z - z_0)^{-1} + \operatorname{Hol}(z),$$

with $\operatorname{rank}(A_{-j}) < \infty$ for $0 < j \leq L$.

We assume the following conditions:

**Condition I:** The operator $\mathcal{P}$ such that $Dom(\mathcal{P}) := IDom(P)$ and $\mathcal{P}u := IPu$, is closable in $B_1$.

**Condition II:** The set $\mathcal{F}_0 := \{u \in B_0 \cap Dom(\mathcal{P}) : Pu \in B_0\}$ is dense in $B_0$.

Let $P_1$ be the closure of $\mathcal{P}$. From [1, Proposition 5.2] the image of $R(z)$ is contained in $Dom(P_1)$, and using (8.1) it follows from [1, Theorem 5.5] that

$$A_{-j-1} = (P_1 - z_0)^j A_{-1},$$

$$\operatorname{Ran}(A_{-j-1}) \subset \operatorname{Ran}(A_{-j}), \quad j \leq 1.$$  

**Proposition 8.1.** Let $z_0 \in D^+$ be a pole of $R(z)$ and let $\gamma_{z_0}$ be a positively oriented curve containing $z_0$ and no other pole of $R(z)$. Then

$$\operatorname{rank}(A_{-1}) = \operatorname{Ind}_{\gamma_{z_0}}(I + SR_0(z)).$$

**Proof.** The proof follows the ideas of the particular case of [4, Proposition 3]. Using the resolvent identities $R(z) - R_0(z) = R(z)SR_0(z) = -R_0(z)SR(z)$, valid for $z \in D$, algebraic computations show that

$$A_{-1} = -\sum_{j \geq 0} \sum_{k < 0} A_k SM_{j-k} S A_{j-1},$$

with $\operatorname{rank}(A_{-j}) < \infty$ for $0 < j \leq L$.
where \(A_{-j} = 0\) if \(j > L\) and \(M_{j-k} = 0\) if \(j - k < 0\). This sum is then finite (see the proof of Lemma 2 of [4] for details).

Using (8.2) and (8.3) one can define the operator \(\Pi_{-1} : \text{Ran}(A_{-1}) \to \text{Ran}(A_{-1})\)

\[
\Pi_{-1} := -\sum_{j \geq 0} \sum_{k < 0} A_k S M_{j-k} S (P_1 - z_0)^j.
\]

It follows from (8.5) and (8.2) that one has

\[
A_{-1} = \Pi_{-1} A_{-1}.
\]

Moreover, \(\Pi_{-1}^2 = \Pi_{-1}\). Indeed, let \(f \in \text{Ran}(A_{-1})\) and by (8.3), take \(g_{k,l}\) such that \(A_{-1} g_{k,l} = A_k S M_{j-k} S (P_1 - z_0)^j f\). Then one has

\[
\Pi_{-1} f = -\sum_{j \geq 0} \sum_{k < 0} A_k S M_{j-k} S (P_1 - z_0)^j f = -A_{-1} \sum_{j \geq 0} \sum_{k < 0} g_{k,l},
\]

and by (8.6) this implies \(\Pi_{-1}^2 f = -\Pi_{-1} A_{-1} \sum_{j \geq 0} \sum_{k < 0} g_{k,l} = -A_{-1} \sum_{j \geq 0} \sum_{k < 0} g_{k,l} = \Pi_{-1} f\).

Using (8.3) and (8.6) one can show that \(\text{Ran}(A_{-1}) = \text{Ran}(\Pi_{-1})\). Hence \(\text{rank}(A_{-1}) = \text{Tr}(\Pi_{-1})\). Now, using (8.2) and the cyclicity of the trace, we have

\[
\text{Tr}(\Pi_{-1}) = -\text{Tr} \sum_{j \geq 0} \sum_{-L \leq k < 0} S M_{j-k} S (P_1 - z_0)^j A_k
\]

\[
= -\text{Tr} \sum_{j \geq 0} \sum_{j-L \leq k < 0} S M_{j-k} S A_{k-j}.
\]

On the other hand, we can see that \((I + SR_0(z))(I - SR(z)) = I\). Then, we have

\[
\text{Ind}_{z_0} (I + SR_0(z)) = -\text{Tr} \frac{1}{2\pi i} \oint_{\gamma_{z_0}} SR(z) S \partial_z R_0(z) dz
\]

\[
= -\text{Tr} \sum_{L \geq l \geq 1} l S A_{-l} S M_l
\]

\[
= -\text{Tr} \sum_{L \geq k \geq 1} k S M_k S A_{-k}.
\]

Noticing that \(\sum_{L \geq l \geq 1} l S M_l S A_{-l} = \sum_{j \geq 0} \sum_{j-L \leq k < 0} S M_{j-k} S A_{k-j}\), this ends the proof. \(\square\)

References

[1] Shmuel Agmon. A perturbation theory of resonances. *Communications on Pure and Applied Mathematics*, 51(11-12):1255–1309, 1998.

[2] Kazunori Ando, Hiroshi Isozaki, and Hisashi Morioka. Spectral properties of Schrödinger operators on perturbed lattices. In *Annales Henri Poincaré*, volume 17, pages 2103–2171. Springer, 2016.

[3] Carl M Bender. Making sense of non-hermitian hamiltonians. *Reports on Progress in Physics*, 70(6):947, 2007.

[4] Jean-François Bony, Vincent Bruneau, and Georgi Raikov. Resonances and spectral shift function near the Landau levels. *Annales de l’institut Fourier*, 57(2):629–672, 2007.

[5] Jean-François Bony, Vincent Bruneau, and Georgi Raikov. Counting function of characteristic values and magnetic resonances. *Comm. Partial Differential Equations*, 39(2):274–305, 2014.

[6] Jean-François Bony, Vincent Bruneau, and Georgi Raikov. Resonances and spectral shift function for magnetic quantum Hamiltonians. *RIMS Hokyuroku Bessatsu*, B45:77–100, 2014.
[7] Olivier Bourget, Diomba Sambou, and Amal Taarabt. On the spectral properties of non-selfadjoint discrete Schrödinger operators. *Journal de Mathématiques Pures et Appliquées*, 141:1–49, 2020.

[8] Vincent Bruneau, Pablo Miranda, Daniel Parra, and Nicolas Popoff. Eigenvalue and resonance asymptotics in perturbed periodically twisted tubes: twisting versus bending. *Ann. Henri Poincaré*, 21(2):377–403, 2020.

[9] Vincent Bruneau, Pablo Miranda, and Nicolas Popoff. Resonances near thresholds in slightly twisted waveguides. *Proc. Amer. Math. Soc.*, 146(11):4801–4812, 2018.

[10] Anne Boutet de Monvel and Jaouad Sahbani. On the spectral properties of discrete Schrödinger operators: the multi-dimensional case. *Reviews in Mathematical Physics*, 11(09):1061–1078, 1999.

[11] Semyon Dyatlov and Maciej Zworski. *Mathematical theory of scattering resonances*, volume 200 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2019.

[12] Joshua Feinberg and A Zee. Non-hermitian localization and delocalization. *Physical Review E*, 59(6):6433, 1999.

[13] Richard Froese. Asymptotic distribution of resonances in one dimension. *Journal of differential equations*, 137(2):251–272, 1997.

[14] Vladimir Georgescu and Sylvain Golénia. Isometries, Fock spaces, and spectral analysis of Schrödinger operators on trees. *J. Funct. Anal.*, 227(2):389–429, 2005.

[15] I. Gohberg and E. Sigal. An operator generalization of the logarithmic residue theorem and the theorem of Rouché. *Mathematics of the USSR-Sbornik*, 13(4):603, 1971.

[16] Israel Gohberg, Seymour Goldberg, and Marius A Kaashoek. *Classes of linear operators*, volume 63. Birkhäuser, 2013.

[17] Ilya Ya Goldsheid and Boris A Khoruzhenko. Distribution of eigenvalues in non-Hermitian Anderson models. *Physical review letters*, 80(13):2897, 1998.

[18] Alain Grigis and Frédéric Klopp. Valeurs propres et résonances au voisinage d’un seuil. *Bulletin de la Société Mathématique de France*, 124(3):477–502, 1996.

[19] Naomichi Hatano and David R Nelson. Localization transitions in non-Hermitian quantum mechanics. *Physical review letters*, 77(3):570, 1996.

[20] Bernard Helffer and Johannes Sjöstrand. Résonances en limite semi-classique. *Mémoires de la Société Mathématique de France*, 24-25:1–228, 1986.

[21] David Nelson and Nadav Shnerb. Non-Hermitian localization and population biology. *Physical Review E*, 58(2):1383, 1998.

[22] Hiroshi Isozaki and Arne Jensen. Continuum limit for lattice Schrödinger operators. *Reviews in Mathematical Physics*, page 2250001, 2021.

[23] Vladimir V Konotop, Jianke Yang, and Dmitry A Zezyulin. Nonlinear waves in pt-symmetric systems. *Reviews of Modern Physics*, 88(3):035002, 2016.

[24] Evgeny Korotyaev and Natalia Saburova. Schrödinger operators on periodic discrete graphs. *Journal of Mathematical Analysis and Applications*, 420(1):576–611, 2014.

[25] Evgeny Korotyaev and Natalia Saburova. Spectral band localization for Schrödinger operators on discrete periodic graphs. *Proceedings of the American Mathematical Society*, 143(9):3951–3967, 2015.

[26] Shu Nakamura and Yukihide Tadano. On a continuum limit of discrete Schrödinger operators on square lattice. *arXiv preprint arXiv:1903.10656*, 2019.

[27] Michael Reed and Barry Simon. *IV: Analysis of Operators*, volume 4. Elsevier, 1978.

[28] Diomba Sambou. Résonances près de seuils d’opérateurs magnétiques de Pauli et de Dirac. *Canadian Journal of Mathematics*, 65(5):1095–1124, 2013.
[34] Yukihide Tadano. Long-range scattering for discrete Schrödinger operators. *Annales Henri Poincaré*, 20(5):1439–1469, 2019.

Marouane Assal, Pablo Miranda, Departamento de Matemática y Ciencia de la Computación, Universidad de Santiago de Chile, Las Sophoras 173, Santiago, Chile.

Email address: marouane.assal@usach.cl
Email address: pablo.miranda.r@usach.cl

Olivier Bourget, Facultad de Matemáticas, Pontificia Universidad Católica de Chile, Vicuna Mackenna 4860, Santiago, Chile.

Email address: bourget@mat.uc.cl

Diomba Sambou, Institut Denis Poisson, Université d’Orléans, CNRS-UMR 7013, 45067 Orléans Cedex 2, France.

Email address: diomba.sambou@univ-orleans.fr