Charged Dilatonic Black Holes: String frame vs Einstein frame

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The descriptions of Reissner-Nordström and Kerr-Newman dilatonic black holes in the Einstein frame are compared to those in the string frame. We describe various physical measurements in the two frames and show which experiments can distinguish between the two frames. In particular we discuss the gyromagnetic ratios of black holes, the decay law via Hawking radiation and the propagation of light on black hole backgrounds.

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I. INTRODUCTION

Superstring theory [1] compactified down to physical four space-time dimensions is generally accepted as the description of curved backgrounds as non-vanishing expectation values of massless string excitations in a language which reproduces Einstein’s general relativity. However it is not a priori clear in which frame such a description should be better formulated, since there are several frameworks possible.

Specifically, upon compactification of extra dimensions and nearly independently of both the content of the starting theory and the compactification scheme, one expects to obtain a low-energy, tree-level, effective 4-dimensional action $S_\sigma$ for the evolution of the uncompactified degrees of freedom of the string, which is in the form of a conformally invariant non-linear $\sigma$-model. A spin 2 gravitational field $G_{ij}$ (Latin indices run from 0 to 3) and an antisymmetric field $B_{ij}$ which couple respectively to the string vertex operators for their own emission (see [2,3] and Refs. therein), emerge in a natural way in $S_\sigma$

$$S_\sigma = \frac{1}{2\lambda_s^2} \int d^4x \left[ h^{\alpha\beta} G_{ij} \partial_\alpha x^i \partial_\beta x^j + \epsilon^{\alpha\beta} B_{ij} \partial_\alpha x^i \partial_\beta x^j + \ldots \right],$$

where $\lambda_s$ is the string length, $h_{\alpha\beta}$ the world-sheet metric tensor, $\epsilon^{\alpha\beta}$ the Levi-Civita symbol in two dimensions and ellipses stand for fermionic terms as well as gauge fields and other moduli.

Of course were one able to perform the derivation completely, the background expectation values of the various fields could be fully determined from the action of the fundamental theory from which one has started. This purely descriptive picture is hard to perform in general, if possible at all. The most serious obstacle to obtaining such a picture is that there are at present no models which describe compactification as a dynamical process taking place in the higher dimensional space of (super)string theory and leading to an evolution towards the present state of the Universe as the preferred vacuum (for a review of early attempts see e.g. [4]). Thus one is naturally led to seek other ways of solving for the fields. A possible guide is the requirement of conformal invariance on the world-sheet for a generalized $D$-dimensional $\sigma$-model which furnishes us a complete set of renormalization equations. Further, it turns out that in the latter set one has also to include the scalar excitation $\phi$ (the dilaton) which couples to the world-sheet scalar curvature.

The aforementioned equations, supplemented with suitable boundary conditions, are sufficient to determine the background fields without making any further reference to string theory. This is a result of the fact that the same equations can also be obtained by varying an effective action which, for the particular case of $B_{ij} \equiv 0$ and non zero Maxwell strength tensor of electrodynamics $F_{ij}$, can be written (see [5], but we work in four space-time dimensions)

$$S_{SF} = \frac{1}{2} \int d^4x \sqrt{-G} e^{-\phi} \left[ \frac{1}{\lambda_s^2} \left( R_{(G)} + G^{ij} \nabla_i \phi \nabla_j \phi \right) - \frac{1}{\alpha^2} e^{(1-a)\phi} F^2 \right],$$

where $R_{(G)}$ is the scalar curvature of the metric $G_{ij}$, $\alpha$ is the electromagnetic coupling constant and $a$ is the dilaton coupling constant with the dilaton assumed to remain massless. This is the effective action which describes the
The uncompactified degrees of freedom of the string move along a geodesic of the metric $g_{ij}$ as can be inferred from (1.3). The equations of motion following from (1.2) can be written as (we omit the subscript ($G$) and set $\lambda_s = \alpha = 1$)

\[
R_{ij} - \frac{1}{2} G_{ij} R + \frac{1}{2} G_{ij} (\nabla \phi)^2 - G_{ij} \nabla^2 \phi + \nabla_i \nabla_j \phi - 2 e^{(1-a)\phi} T_{ij}^{EM} = 0
\]

\[
\nabla^2 \phi - (\nabla \phi)^2 + a e^{(1-a)\phi} F^2 = 0
\]

\[
\nabla_i (e^{-a\phi} F^{ij}) = 0
\]

(1.3)

where $\nabla$ denotes the covariant derivative with respect to the metric $G_{ij}$ and the electromagnetic energy-momentum tensor is given as

\[
T_{ij}^{EM} = F_{ik} F^k_j - \frac{1}{4} G_{ij} F^2.
\]

(1.4)

The effective action and equations of motion can be further modified by applying the following conformal transformation

\[
G_{ij} = e^{\phi_0 - \phi} g_{ij},
\]

(1.5)

where the constant $\phi_0$ can be taken to be the average value of the dilaton in the present Universe. Since in these notes we consider only asymptotically flat cases, we will assume $\phi_0 \equiv 0$ in the forthcoming sections. We further define the Planck length as

\[
\ell_p^2 = e^{\phi_0} \lambda_s^2,
\]

(1.6)

thus obtaining the action in the Einstein frame (EF),

\[
S_{EF} = \frac{1}{2} \int d^4 x \sqrt{-g} \left[ \frac{1}{\ell_p^2} \left( R_{(g)} - \frac{1}{2} g^{ij} \nabla_i \phi \nabla_j \phi \right) - \frac{1}{\alpha^2} e^{-a\phi} F^2 \right],
\]

(1.7)

in which we point out that space-time coordinates have been left unchanged and $R_{(g)}$ is the curvature related to $g_{ij}$. The new equations of motion are obtained by applying the same mapping to the previous ones given in (1.3) (again we omit the subscript ($g$) and set $\ell_p = \alpha = 1$),

\[
R_{ij} = \frac{1}{2} \nabla_i \phi \nabla_j \phi + 2 e^{-a\phi} T_{ij}^{EM}
\]

\[
\nabla^2 \phi + a e^{-a\phi} F^2 = 0
\]

\[
\nabla_i (e^{-a\phi} F^{ij}) = 0
\]

(1.8)

where $\nabla$ is now the covariant derivative with respect to the metric $g_{ij}$. We observe that the dilaton is obviously left unchanged and that the equation for the electromagnetic field is (formally) the same as in SF. Indeed, since $\nabla_i (e^{-a\phi} F^{ij}) \equiv \partial_i (\sqrt{-g} e^{-a\phi} [F^{ij}]_{SF}) = \partial_i (\sqrt{-G} e^{-a\phi} [F^{ij}]_{EF})$ from (1.3) and (1.8), solutions of Maxwell’s equations in the two frames must be related by

\[
e^{2\phi} [F^{ij}]_{SF} = [F^{ij}]_{EF}.
\]

(1.9)

Therefore the physical (covariant) components of the electromagnetic field are the same in both frames:

\[
[F_{ij}]_{SF} = [F_{ij}]_{EF}.
\]

(1.10)

Furthermore, the uncompactified degrees of freedom of the string do not move along a geodesic of the metric $g_{ij}$, and the scalar curvatures are in general different in the two frames because of the dilaton,

\[
R_{(G)} = 2 (\nabla \phi)^2 - 3 \nabla^2 \phi
\]

\[
R_{(g)} = \frac{1}{2} (\nabla \phi)^2.
\]

(1.11)

Another general observation is that $S_{EF}$ is invariant under the following transformation
\[ T : \begin{cases} a \rightarrow -a \\ \phi \rightarrow -\phi \end{cases}, \quad (1.12) \]

which is not an invariance of \( S_{SF} \). Therefore, although the mapping (1.3) looks almost trivial at a first sight, it is clear that the physics in the two frames can be significantly different.

Which frame is more suitable as a description of the present state of our Universe is an open question which will eventually be settled by experiment [8]. The issue raised in the present notes has already been extensively discussed in the framework of scalar-tensor theories of gravity and observable consequences have been deduced mainly in cosmology. Because of the direct coupling between the dilaton and matter (in our case the electromagnetic field), both actions in Eqs. (1.2) and (1.7) fail to be of the Brans-Dicke type, thus the equivalence principle does not hold in general. Specifically, one expects the equivalence principle to be violated whenever the gradient of the dilaton field is not negligible. However, such violations might be allowable provided they occurred far in the past, e.g. in the early stages of the Universe (see [4], [9] and Refs. therein, [10]) or take place in regions of space which have not been tested at present.

In the following we will analyze some of the physical implications of the differences between the two frames for two black hole geometries, in which the gradient of the static dilaton field is appreciably strong only in a relatively small region of space outside the event horizon. We consider a charged black hole (RND) [11,5] in section II and a rotating black hole with small charge-to-mass ratio (KND) [12] in section III and suggest possible experiments. Finally, in section IV we compare the propagation of light on these black hole backgrounds in the two frames.

II. RND BLACK HOLES

The line element representing a 4-dimensional charged dilatonic black hole in \( EF \) is [11,6]

\[ ds^2_{EF} = -e^{2\Phi} dt^2 + e^{2\Lambda} dr^2 + R^2 d\Omega^2, \quad (2.1) \]

with \( d\Omega^2 = d\theta^2 + \sin^2 \theta d\phi^2 \) and

\[
e^{2\Phi} = e^{-2\Lambda} = \left(1 - \frac{r_+}{r}\right) \left(1 - \frac{r_-}{r}\right)^\frac{1-a^2}{1+a^2}, \]

\[
R^2 = r^2 \left(1 - \frac{r_-}{r}\right)^\frac{2a^2}{1-a^2}. \quad (2.2)\]

One encounters an outer horizon at

\[ r_+ = M + \sqrt{M^2 - (1 - a^2)Q^2}, \quad (2.3) \]

while

\[ r_- = (1 + a^2) \frac{Q^2}{r_+}, \quad (2.4) \]

is a real singularity for \( a \neq 0 \). The corresponding electromagnetic field has only one non-vanishing component (the static electric field),

\[ F_{tr} = \frac{Q}{r^2}, \quad (2.5) \]

and the dilaton field is given by

\[ e^{-\phi} = \left(1 - \frac{r_-}{r}\right)^{-\frac{2a^2}{1-a^2}} = 1 + O(r^{-1}) \quad (2.6) \]

We observe that for \( a = 0 \) the above expressions reduce to the pure Reissner-Nordström (RN) solution (with a constant \( \phi_0 = 0 \) dilaton field) and that \( M \) and \( Q \) represent the physical (ADM) mass and charge of the black hole.

As mentioned in the Introduction, from (2.6) one sees that the dilaton gradient falls off and becomes negligible sufficiently far away from \( r_- \). To be more specific, since \( e^{\phi} \sim G_N \), one can write the total force acting on a test mass \( m \) as the sum of the force \( F_{\phi} \) due to the spatial dependence of \( G_N \) and the Newtonian contribution \( F_N \).
\[
F_{\text{tot}} \sim -\partial_r \left( G_N \frac{M m}{r} \right) = F_\phi + G_N \frac{M m}{r^2},
\]

and finds that \( F_\phi \) becomes of the same order as \( F_N \) at

\[
r \sim \frac{1 + 3 a^2}{1 + a^2} r_-. \tag{2.8}
\]

This implies that, if one can perform a measurement with the precision of one part over \(10^N\), one has to go closer than \( r_c \sim 10^N r_-\) to the black hole centre in order to test any violation of the equivalence principle. Further, \( r_c \) must lie outside \( r_+\) and this gives an estimate for the smallest charge-to-mass ratio that the black hole must possess in order that any deviation can be tested, namely

\[
\frac{Q}{M} > 10^{-N/2}. \tag{2.9}
\]

For a solar mass black hole and \( N \sim 10 \) this means a charge of about \(10^{14}\) electron charges or \(10^{15}\) C. On the other hand, for a Planck mass black hole with one electron charge the ratio \( Q/M \sim 0.1 \) and one only needs a precision of \( N = 2\).

Upon transforming to SF one obtains

\[
ds^2_{SF} = -\left(1 - \frac{r_+}{r}\right) \left(1 - \frac{r_-}{r}\right) \frac{1 + 2 a^2}{1 + a^2} dt^2 + \left(1 - \frac{r_+}{r}\right)^{-1} \left(1 - \frac{r_-}{r}\right)^{\frac{a^2 + 2 a - 1}{1 + a^2}} dr^2 + r^2 \left(1 - \frac{r_-}{r}\right)^{\frac{2 a (1 + a)}{1 + a^2}} d\Omega^2. \tag{2.10}
\]

An interesting feature of the latter metric is that \( G_{\theta\theta} \) (as well as \( G_{\phi\phi} \)) is regular at \( r = r_-\) for \( a = 0 \) (RN), as expected, and also for \( a = -1 \) \([13]\). Further, the physical (ADM) mass of the black hole is shifted according to

\[
M_{\text{phys}}|_{SF} = M + \frac{a Q^2}{r_+} = M \left(1 + \frac{a Q^2}{2 M^2}\right) \mathcal{O}\left(\frac{Q^4}{M^4}\right). \tag{2.11}
\]

The latter result is however not particularly interesting from the experimental point of view unless a way can be found to measure \( M \) and \( M_{\text{phys}} \) separately.

As models of astrophysical black holes, both the line elements \((2.1)\) and \((2.10)\) are unsatisfactory. They describe non-rotating spherical objects, but real black holes are expected to be spinning fast because of angular momentum conservation during the collapse of the original star (for further evidence see \( e.g. \) \([14]\)). If this limitation is ignored, the fact that \( M \) also appears in the expression for \( r_+ \) (and \( r_-\) as well, but the latter singularity is hidden) allows us to propose an experimental test consisting of the following steps:

\(i\) by comparing the acceleration of a charged test particle to the acceleration of a neutral particle of equal mass the electric field \( F_e \) can be measured;

\(ii\) the value of \( M_{\text{phys}} \) is obtained directly from the acceleration of the neutral particle at large distance;

\(iii\) the radius \( r_+ \) can be estimated by inferring the largest distance from which light can escape or by localizing the inner edge of the accreting disk.

\(F_e\) is still given by Eq. \((2.3)\), as can be inferred from the general relation \((1.10)\). Thus step \(i\) allows the computation of \( Q \) and the insertion of \( Q \) into the definition of \( r_+ \) in \((2.3)\) which, together with the measured value of \( r_+ \) from \(iii\), gives \( M \). If \( M \) is equal to \( M_{\text{phys}} \) from \(ii\), then EF is the physical frame and one might question the stringy origin of the action \( S_g\); in case they are not equal, SF is the physical frame and \((2.11)\) can be used to estimate \( a \) \([9]\).

As a further consequence of the metric being different in the two frames, we notice that the expression of the area of the surface of the outer horizon is given respectively by

\[
A|_{EF} = 4 \pi r_+^{2 (1+a)/a} \left(r_+ - r_-\right)^{\frac{2 a^2}{1 + a^2}} \tag{2.12}
\]

\[
A|_{SF} = 4 \pi r_+^{2 (1+a)/a} \left(r_+ - r_-\right)^{\frac{2 a (1+a)}{1 + a^2}},
\]

so that

\[
\frac{A|_{EF}}{A|_{SF}} = \left(\frac{r_+ - r_-}{r_+}\right)^{\frac{1}{1+a^2}} < 1, \tag{2.13}
\]

for \( a > 0 \) (the same ratio is \( > 1 \) for \( a < 0 \)). As we shall show, this affects the evaporation of the black hole.
We assume that the area law holds in both frames, so that the internal degeneracy of the black hole is given by (see [15] and Refs. therein)

$$\Omega \sim e^{A/4},$$

(2.14)

where $A$ is the area of the horizon. Then we study the evolution of the mass of the black hole in time assuming it can only decrease by emitting Hawking quanta (matter possibly falling into the black hole and increasing its mass will not be considered). The occupation number density of the Hawking radiation, properly computed in the microcanonical ensemble where total energy of the system is conserved and equal to $M_{phys}$, is [15]

$$n(\omega) = \frac{M_{phys}/\omega}{\sum_{l=1}^{M_{phys}/\omega} \Omega\left(M_{phys} - l\omega\right) \Omega(M_{phys})}.$$  

(2.15)

For the sake of simplicity we specialize to the case $\alpha = 1$ and also assume that the ratio $x \equiv Q/M$ is small and remains constant along the evaporation. Thus we find for the area of the horizon

$$A \simeq 4\pi M^2 (1 - f x^2),$$

(2.16)

where $M$ is now the physical mass as measured in each frame and $f = 1/2$ in EF ($f = 3/2$ in SF). The occupation number (2.15) becomes

$$n_f(\omega) \sim \sum_{l=1}^{M/\omega} \left[ e^{4\pi l^2 \omega^2 - 8\pi M l \omega} (1 - f x^2) \right].$$

(2.17)

from which the energy emitted per unit time can be computed according to

$$\frac{dM}{d\tau} \sim -A \int d\omega \omega^3 \Gamma(\omega) n_f(\omega).$$

(2.18)

We have numerically integrated the above expression and the results for the two frames are shown in Fig. 1 for $\Gamma = 1$ and $x = 1/2$ (a relatively large ratio chosen for the purpose of stressing the difference between the two frames). From that figure one sees that for large values of $M$ the emission is higher in SF, thus leading to a faster decay, but then the curve in EF overcomes the curve in SF for values around the Planck mass $M_p \equiv \sqrt{\hbar c G_N}$ and smaller. Both curves reach a maximum and then vanish for zero mass, a feature which is a direct consequence of the use of the microcanonical approach (that is energy conservation).

![FIG. 1. Energy emitted by RND black holes per unit time and ratio $Q/M$ fixed in the two frames. The mass is in units of the Planck mass. The vertical scale is arbitrary.](image)
The time evolutions which result from Eq. (2.18) are shown in Fig. 2. When the mass $M$ is of the order of the Planck mass (or bigger), Eq. (2.18) can be approximated by the thermal distribution with inverse temperature $\beta = 8\pi M (1 - f x^2)$. This behaviour would lead to a complete evaporation in a finite time. However, as can be seen from Fig. 2 after the maximum slope is reached the curves switch to a power law decay and fall to zero in infinite time (see Ref. [15] for the details). The difference between the two frames is that $M$ is smaller in SF than in EF during the thermal phase but falls off less rapidly in SF than in EF during the late stages.

**FIG. 2.** Time evolution of the mass of RND black holes with $M(0) = M_p$ in the two frames. The time scale is arbitrary.

### III. KND BLACK HOLES

In Ref. [12] a new solution of the Einstein-Maxwell dilatonic equations (1.8) was found in EF which represents a rotating, charged black hole with static dilaton field in the small charge-to-mass $(Q/M \ll 1)$ approximation. In a subsequent paper [16] we showed that the corresponding metric can be simplified by shifting the radial coordinate $r$ and suitably redefining the parameters $M$ (mass), $J$ (angular momentum) and $Q$ (charge) to make them equal the corresponding physical (ADM) quantities $M_{\text{phys}}$, $J_{\text{phys}}$ and $Q_{\text{phys}}$ which determine the Newtonian motion in the asymptotically flat region. One then has

$$g_{ij} = g_{ij}^{KN} + O\left(\frac{Q^4}{M^4}\right), \quad (3.1)$$

where $g_{ij}^{KN}$ is the usual Kerr-Newman (KN) metric which we write in Boyer-Lindquist coordinates [17],

$$ds_{\text{KN}}^2 = -\sqrt{\Delta} \sin \theta \left[ \frac{\chi}{\lambda} dt^2 - \frac{1}{\chi} (dt - \omega d\varphi)^2 \right] + \rho^2 \left[ \frac{(dr)^2}{\Delta} + (d\theta)^2 \right], \quad (3.2)$$

in which ($\alpha \equiv J/M$)

$$\chi = \frac{\sqrt{\Delta} \sin \theta}{\Psi}, \quad \Delta = r^2 - 2M r + \alpha^2 + Q^2,$$

$$\rho^2 = r^2 + \alpha^2 \cos^2 \theta,$$

$$\Psi = -\frac{\Delta - \alpha^2 \sin^2 \theta}{\rho^2}$$

$$\omega = -\alpha \sin^2 \theta [1 + \Psi^{-1}] . \quad (3.3)$$

This also implies that the causal structure is not affected by the presence of the dilaton at that order in EF. In fact one still has two horizons for $\Delta = 0$, that is
\[ r_\pm = M \pm \sqrt{M^2 - \alpha^2 - Q^2} \]. 

(3.4)

However the presence of a non-zero dilaton field,

\[ \phi = -a \frac{Q^2}{\rho^2} \frac{M}{r} \],

(3.5)
affects the electric and magnetic field potentials \( A \) and \( B \) (see Ref. [12] for the definitions),

\[
A = Q \frac{r}{\rho^2} \left[ 1 - \left( \frac{1}{2} \frac{r}{\rho^2} \right) \frac{a^2 Q^2}{3M} \right] \\
B = -Q \alpha \frac{\cos \theta}{\rho^2} \left[ 1 - \left( \frac{1}{2} \frac{M}{r} - \frac{r}{\rho^2} \right) \frac{a^2 Q^2}{3M} \right],
\]

(3.6)

where the terms proportional to \( Q^2 \) inside the brackets correspond to the corrections with respect to the KN potentials [17]. These corrections also affect the electric and magnetic fields in the limit \( r \to \infty \),

\[
\mathcal{E}_r \approx \frac{Q}{r^2} \\
\mathcal{E}_\theta \approx -2 \frac{\alpha^2 Q}{r^4} \sin \theta \cos \theta \\
\mathcal{B}_r \approx \frac{\alpha Q}{r^3} \cos \theta \left[ 1 - \frac{a^2 Q^2}{6M^2} \right] \\
\mathcal{B}_\theta \approx \frac{\alpha Q}{r^3} \sin \theta \left[ 1 - \frac{a^2 Q^2}{6M^2} \right] = \frac{\mu_{phys}}{r^3} \sin \theta,
\]

(3.7)

where \( \mathcal{E}_\hat{a} \) and \( \mathcal{B}_\hat{a} \) (\( \hat{a} = \hat{r}, \hat{\theta} \)) are respectively the electric and the magnetic field components with respect to the usual spherical coordinate tetrad basis [18]. Eq. (3.7) shows a relative shift in the intensity of the field \( \mathcal{B} \) with respect to \( \mathcal{E} \). This is also the source for the anomalous gyromagnetic ratio

\[ [g]_{EF} = 2 \frac{\mu_{phys}}{Q_{phys}} \frac{M_{phys}}{J_{phys}} \approx 2 \left[ 1 - \frac{a^2 Q^2}{6M^2} \right], \]

(3.8)

which for the KND black hole cannot be greater than 2 and is equal to 2 for the KN black hole [19].

An obvious consequence of the above form of the solution is that the geodesic motion of neutral particles, which do not couple directly to the dilaton, are unaffected by the presence of static dilaton field (up to order \( Q^3/M^3 \)). As we have mentioned in the Introduction, this should not be the case for fundamental strings, from whose action \( S_{SF} \) has been derived by compactifying extra dimensions. To see that this is indeed the case we compute the SF metric corresponding to \( g_{ij}^{KN} \) with the dilaton field given in (3.5), thus obtaining

\[ G_{ij} = g_{ij}^{KN} \left( 1 - a \frac{r}{\rho^2} \frac{Q^2}{M} \right) + O \left( \frac{Q^4}{M^4} \right). \]

(3.9)

Then we observe that (3.9) cannot be remapped into \( g_{ij}^{KN} \) by a change of coordinates since the curvature scalars are different in the two frames. In fact, from (1.11) one obtains

\[
R_{(G)} \approx 12 a \frac{Q^2}{M} \alpha^2 \cos 2\theta \frac{1}{r^3} \\
R_{(g)} \approx \frac{a^2}{2} \frac{Q^4}{M^2} \frac{1}{r^4},
\]

(3.10)

where, for the sake of simplicity, we have displayed only the behaviors for large \( r \).

Now we analyze further physical differences between the two frames. To start with, since

\[ G_{tt} \approx -\left[ 1 - \frac{2M}{r} \left( 1 + \frac{aQ^2}{2M^2} \right) \right], \]

(3.11)

for \( r \to +\infty \), the ADM mass in SF is shifted to
\[ [M_{\text{phys}}]_{SF} = [M_{\text{phys}}]_{EF} \left( 1 + \frac{a Q^2}{2 M^2} \right), \]

which is the same expression that was obtained on expanding the physical mass in \( Q/M \) for RND. Thus, the same experimental test described in Section II after Eq. (2.11) can be repeated for the present case. We remind the reader here that the KND metric is a more realistic candidate for the description of astrophysical black holes, since it describes black holes which possess angular momentum.

We notice that
\[ G_t \varphi \approx -2 \frac{\alpha M}{r} \sin^2 \theta \equiv -2 J_{\text{phys}} r \sin^2 \theta, \]

therefore \( J_{\text{phys}} \) is left unchanged by the conformal mapping as well as \( \mu_{\text{phys}} \), see (1.10) and (3.7). The gyromagnetic ratio in SF is then given by
\[ [g]_{SF} \approx 2 \left[ 1 + \frac{a Q^2}{2 M^2} \left( 1 - \frac{a}{3} \right) \right]. \]

The above expression for \([g]_{SF}\) shows a remarkable difference with respect to that of EF, that is \([g]_{SF}\) can be either smaller than 2 (for \( a < 0 \) or \( a > 3 \)), equal to 2 for \( (a = 0, 3) \) or greater than 2 (for \( 0 < a < 3 \)). The existence of the latter case provides another way of testing which frame is the physical one. In fact, since \([g]_{EF}\) can be at most equal to 2, the measurement of a value greater than 2 for the gyromagnetic ratio of a black hole would prove that Physics has to be described in SF (this should indeed be the case [6]). On the other hand, the measurement of any value smaller than 2, although crucial for proving the existence of static dilaton field, would not suffice for discriminating between EF and SF, unless an independent way of measuring \( a \) along with the mass and charge of the black hole can be found.

IV. LIGHT PROPAGATION

In this section we study the differences which emerge in the propagation of electromagnetic signals in the two frames.

We start from the approximation of geometric optics and look at the deflection angle of a null ray impinging upon the black hole from far away and then escaping to \( r = +\infty \). Since the relation between SF and EF is given by a conformal transformation of the metric, the naive expectation is that no changes occur. In fact it is easy to prove that the picture is exactly the same in both frames, and the eikonal path followed by any null ray is unaffected by the choice of the frame.

We consider for the metric in EF a generic form \( g_{ij} = g_{ij}(r, \theta) \), with only one possible off-diagonal term \( (g_{t\varphi}) \), which can be easily specialized to both RND and KND. The Lagrangian for a null particle moving on such a metric and in the equatorial plane \( \theta = \phi/2 \) is given by [17]
\[ 2 \mathcal{L} = g_{tt} \dot{t}^2 + 2 g_{t\varphi} \dot{t} \dot{\varphi} + g_{\varphi\varphi} \dot{\varphi}^2 + g_{rr} \dot{r}^2 = 0, \]

where a dot denotes the derivative with respect to an affine parameter \( \lambda \). The conserved momenta are
\[ p_t = g_{tt} \dot{t} + g_{t\varphi} \dot{\varphi} \equiv E \]
\[ p_{\varphi} = g_{t\varphi} \dot{\varphi} + g_{t \varphi} \dot{t} \equiv L, \]

corresponding respectively to the energy and angular momentum of the null particle. In particular, from the conservation of \( p_{\varphi} \) one obtains
\[ [\dot{\varphi}]_{EF} = D^{-1} (g_{tt} L - g_{t\varphi} E), \]

with \( D = g_{tt} g_{\varphi\varphi} - g_{t\varphi}^2 \), and, after substituting for \( \dot{t} \) and \( \dot{\varphi} \) in the Lagrangian,
\[ [\dot{r}^2]_{EF} = \frac{D^{-1}}{g_{rr}} \left( g_{tt} L^2 + g_{\varphi\varphi} E^2 - 2 g_{t\varphi} E L \right). \]

From the above expressions for \( \dot{\varphi} \) and \( \dot{r} \) one finally obtains

8
\[
\left[ \frac{d\varphi}{dr} \right]_{EF} = \left[ \frac{d\varphi}{d\lambda} \right]_{EF} \left[ \frac{d\lambda}{dr} \right]_{EF} = \sqrt{\frac{g_{rr}}{D}} \frac{g_{tt} L - g_{t\varphi} E}{\sqrt{g_{tt} L^2 + 2 g_{t\varphi} E L}}. \tag{4.5}
\]

The deflection angle is then given by
\[
\Delta \varphi = 2 \left( \varphi(\bar{r}) - \varphi(\infty) \right) - \pi, \tag{4.6}
\]

where
\[
\varphi(r) = \int_{\bar{r}}^{r} \left[ \frac{d\varphi}{dr} \right]_{EF} dr, \tag{4.7}
\]

and \( \bar{r} \) is the minimum radial coordinate reached by the particle, that is
\[
[\bar{r}]_{EF} = 0. \tag{4.8}
\]

Explicit expressions can then be obtained for the two kinds of black holes by simply substituting in the corresponding metric elements \([20]\).

Switching to SF, one obtains
\[
[\dot{r}]_{SF} = e^{-\phi} [\dot{r}]_{EF} \quad \text{and} \quad [\dot{\varphi}]_{SF} = e^{-\phi} [\dot{\varphi}]_{EF}. \tag{4.9}
\]

Thus, since \( e^{-\phi} \) is a positive function, one finds that the turning point at which the particle stops approaching the black hole has radial coordinate \( \bar{r} \) which does not depend on the frame. Further
\[
\left[ \frac{d\varphi}{dr} \right]_{SF} = \left[ \frac{d\varphi}{dr} \right]_{EF}. \tag{4.10}
\]

One can then conclude that \( \Delta \varphi \) is frame-independent and that eikonal trajectories of null waves are exactly the same in both EF and SF. In particular, for KND this means that to lowest order in \( (Q/M)^2 \) eikonals do not sense the dilaton at all \([20]\).

Of course the above conclusion does not preclude the possibility of detecting other differences in the propagation of light waves in SF with respect to EF. To begin with, we notice that, although the radial coordinate of the turning points defined by Eq. (4.8) is frame-independent, the actual proper distance to an observer placed at \( r = r_o \) depends on the frame. If we consider the RND case and \( a = 1 \), we see that the difference between proper distances to \( r_o \) is given by
\[
[l]_{EF} = [l]_{SF} + r - \ln \left( \frac{r_o - r_+}{r_b - r_+} \right), \tag{4.11}
\]

which would result in a delay for the time of flight from \( \bar{r} \) to \( r_o \) in EF with respect to SF.

A possible way of detecting such a time delay is displayed in Fig. 3 where one considers a source of light at \( r = r_s \) which emits both towards the observer placed at \( r_o \) (ray 1) and towards the black hole (ray 2). The latter ray then bounces back at \( r_b \) and reaches the observer with a delay with respect to ray 1 given by twice the time it takes to go from the source to \( r_b \) (this is a simple model for the so called \textit{reverberation} in the accreting disk, see \([21]\)). In EF this delay is given by (again assuming RND with \( a = 1 \))
\[
\tau|_{EF} \sim 2 \left[ r_s - r_b + r_+ \ln \left( \frac{r_s - r_+}{r_b - r_+} \right) \right], \tag{4.12}
\]

while in SF one has
\[
\tau|_{SF} \sim \tau|_{EF} - 2 r_+ \ln \left( \frac{r_s - r_+}{r_b - r_+} \right). \tag{4.13}
\]
The difference between the metrics in the two frames also affects the red-shift \( z \) of waves emitted at \( r_s \). For instance, in RND with \( a = 1 \) one has

\[
\begin{align*}
\left. z \right|_{EF} &= -\frac{r_+}{r_s} - \frac{2M}{r_s} \\
\left. z \right|_{SF} &= \left. z \right|_{EF} - \frac{r_+ - r_+}{r_s^2} = \left. z \right|_{EF} - \frac{Q^2}{r_s} \left( \frac{1}{M} - \frac{2}{r_s} \right).
\end{align*}
\]

(4.14)

Finally, there are effects that cannot be accounted for without fully considering the wave nature of electromagnetic radiation. In fact, the electromagnetic waves couple directly to the dilaton (as can be seen in the action, see also Refs. [13,20] for explicit expressions in EF) and, although at leading orders the eikonal trajectories look the same in both frames, the intensity of the waves is different because of the different metric backgrounds. From (1.10) one can estimate the intensity of electromagnetic radiation produced in scattering events involving other fields in the model according to

\[
I \sim \left| \vec{E} \right|^2 + \left| \vec{B} \right|^2 \sim F^2.
\]

Therefore from (1.10) one gets

\[
[I]_{SF} \sim e^{-2\phi} \left[ F^2 \right]_{EF} \sim e^{-2\phi} [I]_{EF}.
\]

(4.15)

This implies \([I]_{SF} \sim [I]_{EF} \left( 1 + O(r^{-1}) \right)\) and the difference is not appreciable when the scatterings occur far away from the hole.

V. CONCLUSION

The resolution of the issue of which frame is more suitable for the description of physical processes will have to wait at least until the next generation of astrophysical instruments. The measurement of the gyromagnetic ratio of a black hole will be a sensitive test of SF vs. EF, but precise measurements of the magnetic moment, the mass, the charge and the angular momentum of the black hole are necessary for the determination of this ratio. Existing telescopes are incapable of making such measurements, but future instruments will no doubt be able to ‘see’ close enough to the horizon of the black hole to determine these quantities with sufficient precision. Measurement of the intensity of electromagnetic radiation passing near the horizon of a black hole would also, in principle, be a very sensitive way of distinguishing between the two frames. This would require measuring the intensity of light from the stellar component of a black hole binary system for two different locations of the star – once when the star is occulting and once when it is eclipsing the black hole. The mass and charge of the black hole would also have to be known, as well as the angular momentum if frame dragging effects are to be taken into account. Although such binary systems are known to exist, precise measurement of the appropriate parameters is beyond present capabilities and will have to wait until the next generation of instruments become operative.

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