INSTABILITY OF EVAPORATION FRONTS IN THE INTERSTELLAR MEDIUM

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ABSTRACT

The neutral component of the interstellar medium is segregated into the cold neutral medium (CNM) and warm neutral medium (WNM) as a result of thermal instability. It was found that a plane-parallel CNM–WNM evaporation interface, across which the CNM undergoes thermal expansion, is linearly unstable to corrugational disturbances, in complete analogy with the Darrieus–Landau instability (DLI) of terrestrial flames. We perform a full linear stability analysis as well as nonlinear hydrodynamic simulations of the DLI of such evaporation fronts in the presence of thermal conduction. We find that the DLI is suppressed at short length scales by conduction. The length and time scales of the fastest growing mode are inversely proportional to the evaporation flow speed of the CNM and its square, respectively. In the nonlinear stage, the DLI saturates to a steady state where the front deforms to a finger-like shape protruding toward the WNM, without generating turbulence. The evaporation rate at nonlinear saturation is larger than the initial plane-parallel value by a factor of \( \sim 2.4 \) when the equilibrium thermal pressure is \( 1800 \, \text{Kg cm}^{-3} \). The degrees of front deformation and evaporation-rate enhancement at nonlinear saturation are determined primarily by the density ratio between the CNM and WNM. We demonstrate that the Field length in the thermally unstable medium should be resolved by at least four grid points to obtain reliable numerical outcomes involving thermal instability.

Key words: conduction – hydrodynamics – instabilities – ISM: kinematics and dynamics – ISM: structure – methods: analytical

Online-only material: color figures

1. INTRODUCTION

The interstellar medium (ISM) is inhomogeneous, consisting of multiple components with a wide range of densities and temperatures. In a simple description of the two-phase model (Spitzer 1958; Field et al. 1969), a diffuse gas suffers from thermal instability and segregates into a cold neutral medium (CNM) with temperature \( T \sim 10^2 \, \text{K} \) and a warm neutral medium (WNM) with \( T \sim 10^3 \, \text{K} \) (Field 1965; see also Meerson 1996 and Cox 2005 for reviews). Strong radiative and mechanical heating by supernova explosions produces a hot third phase that fills most of the volume in galaxies (Cox & Smith 1974; McKee & Ostriker 1977). The ISM is highly responsive and thus changes its phase readily depending on environmental conditions. For example, local compression and radiative cooling turn a hot gas to a WNM and then to a CNM, while the reverse phase transitions can occur due to expansion and heating (e.g., McKee & Ostriker 2007). Despite pervasive presence of supersonic turbulence in the ISM (e.g., Mac Low & Klessen 2004; Heiles 2004), pressure equilibrium among different phases roughly holds as long as the characteristic time between shocks is longer than the cooling time (Wolfire et al. 2003).

Phase transitions of the ISM usually involve interfaces or thermal fronts between different phases (e.g., Stone 2011). The thickness of thermal fronts is of the order of the “Field length” (Field 1965), across which conductive heat flux balances the radiative heating and cooling (see also Begelman & McKee 1990). In the case of diffuse ISM, thermal fronts are occupied by gas in the TI-unstable temperature range whose mass fraction is non-negligible compared to the CNM and WNM (e.g., Piontek & Ostriker 2005, 2007; Hennebelle & Audit 2007; Kim et al. 2008, 2010). Thermal fronts are further termed evaporation fronts when a colder component becomes hotter as it moves across them, or condensation fronts in the opposite situations.

In pioneering studies, Zel’dovich & Pikel’Ner (1969) and Penston & Brown (1970) independently examined the steady-state structure of planar thermal fronts. They found that there exists saturation pressure, \( P_{\text{sat}} \), at which a front experiences no net cooling, and that the equilibrium thermal pressure, \( P_{\text{eq}} \), determines the type of thermal fronts such that the fronts are static (i.e., no gas motion across them) when \( P_{\text{eq}} = P_{\text{sat}} \), while \( P_{\text{eq}} > P_{\text{sat}} \) for condensation fronts, and \( P_{\text{eq}} < P_{\text{sat}} \) for evaporation fronts due to excessive cooling or heating (see also Inoue et al. 2006; Iwasaki & Inutsuka 2012). Stone & Zweibel (2010) considered magnetized thermal fronts and showed that magnetic fields are distributed almost uniformly due to efficient ambipolar diffusion, making the temperature profile almost the same as in the unmagnetized cases.

As has often been noted, the mathematical problem of determining the structure of thermal fronts in the ISM is identical to that of terrestrial flames in combustion theory. In the case of evaporation fronts, for example, an upstream CNM changing to a downstream WNM due to radiative heating is analogous exactly to upstream unburnt gas transforming to downstream burnt ash with chemical reactions as a heating source. It has long been well recognized in combustion theory that planar flame fronts are unconditionally unstable to front distortions owing to thermal expansion across them (e.g., Williams 1985; Zel’dovich et al. 1985; Liberman et al. 1994; Law 2006; Searby 2009; see Bychkov & Liberman 2000 for an in-depth review). This corrugational instability is usually referred to as the Darrieus–Landau instability (DLI) after the original studies of Darrieus (unpublished) and Landau (1944). When
the flow is assumed incompressible and the flame front is taken infinitesimally thin (i.e., ignoring the effect of thermal conduction), the growth rate $\Omega_0$ of the DLI is given by

$$\Omega_0 = k v_x - \frac{\mu}{1+\mu} (\sqrt{1+\mu} - \mu^{-1} - 1),$$

where $k$ is the wavenumber of perturbations transverse to the flow direction, $v_x$ is the velocity of the upstream unburnt gas with respect to the front, and $\mu$ ($> 1$) is the expansion factor defined as the ratio of unburnt to burnt gas densities (e.g., Williams 1985; Landau & Lifshitz 1987). Liberman et al. (1994) studied the linear stability of a flame front with finite thickness, showing that thermal conduction stabilizes short-wavelength perturbations. Travnikov et al. (1997) further considered the effect of flow compressibility and found that the maximum growth rate increases with the flow Mach number.

The DLI has been of special interest in the study of explosive nucleosynthesis occurring in Type Ia supernova flames as it is considered as one of the candidate mechanisms that may trigger the deflagration-to-detonation transition in thermonuclear burning (Niemeyer & Hillebrandt 1995; Röpke et al. 2003; Dursi et al. 2003; Bell et al. 2004a). In particular, Bell et al. (2004a) carried out high-resolution numerical simulations of the DLI of C/O thermonuclear flames with $\mu \sim 1.4-1.7$ by including the effect of finite flame thickness, corresponding to the late stages of a Type Ia supernova event. They found that the DLI in the linear stage accelerates the flames by increasing their surface area, and saturates in the nonlinear stage by forming round cusps in the flames. In their models, the maximum enhancement in the flame speeds is only a few percent, about an order of magnitude smaller than the results of Röpke et al. (2003) that treated the flames. In their models, the maximum enhancement in the flame speeds is only a few percent, about an order of magnitude smaller than those in the chemical or thermonuclear flames, so that it is interesting to study the effect of $\mu$ on the changes in the evaporation rate and front shapes in the nonlinear stage. In addition, a full linear stability analysis of the DLI, applicable to a CNM–WNM interface in the diffuse ISM, that properly takes allowance for finite front thickness is still lacking. Therefore, in this paper, we investigate both linear stability and nonlinear evolution of evaporating fronts in the ISM by including the effect of thermal conduction. For the linear stability analysis, we follow the eigenvalue approach of Liberman et al. (1994) and find numerical dispersion relations for general $k$. We run hydrodynamic simulations to study nonlinear development of the DLI. We also study the effect of $\mu$ on the nonlinear state of the DLI by employing a modified form of the heating function, and demonstrate that sufficient resolution is required to resolve interfaces between the CNM and WNM.

The rest of this paper is organized as follows. In Section 2, we introduce basic equations of hydrodynamics and calculate the structure of thermal fronts in a steady state. In Section 3, we present the result of full linear stability analyses and the scaling relations for the most unstable modes. In Section 4.1, we present the requirement of numerical resolution to resolve the thermal interfaces between CNM and WNM. The results of two-dimensional simulations of the DLI using both single-mode and multi-mode perturbations as well as the effects of varying expansion factor are presented in Section 4.2. We summarize and discuss our main results in Section 5.

2. STEADY FRONTS

2.1. Basic Equations

In this paper we consider gas flows across an evaporating front between the CNM and WNM of the ISM, and study stability of the front against distortional perturbations in the presence of thermal conduction. We do not consider the effect of magnetic fields and gaseous self-gravity in the present work. The governing equations of ideal hydrodynamics read

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho v) = 0, \quad (2)$$

$$\frac{\partial \rho v}{\partial t} + \nabla \cdot (\rho vv + P) = 0, \quad (3)$$

$$\frac{\partial E}{\partial t} + \nabla \cdot ((E + P) v - \kappa \nabla T) = -\rho \mathcal{L}, \quad (4)$$

$$P = \frac{\rho k_B T}{m}, \quad (5)$$

where $\rho$, $v$, and $P$ are the gas density, velocity, and pressure, respectively, $E = P/(\gamma - 1) + \rho v^2/2$ is the total energy density with the adiabatic index $\gamma = 5/3$, $\kappa$ is the thermal conductivity, and $\rho \mathcal{L}$ is the net radiative cooling rate per unit volume. For the thermal pressure, we take an ideal gas law

$$P = \frac{\rho k_B T}{m}, \quad (5)$$

where $k_B$ is the Boltzmann constant and $m = 1.37 m_H$ denotes the mean mass per hydrogen atom, corresponding to the solar abundances. In this work, we ignore the effect of viscosity since it has little influence on the dynamics of thermal fronts (Pelce & Clavin 1982).

The net volumetric heat-loss rate is given by

$$\rho \mathcal{L} = n^2 \Lambda(T) - n \Gamma, \quad (6)$$

with $n = \rho / \overline{m}$ being the number density of hydrogen. In the diffuse ISM, the heating rate $\Gamma$ is primarily by the photoelectric...
Figure 1. Thermal equilibrium curve with $\mathcal{L} = 0$ (solid line) in the density–pressure plane overlaid shaded contours of our adopted heat-loss function. The regions above and below the curve correspond to gas with net cooling and heating, respectively. Two-phase equilibria for the coexistence of CNM and WNM can coexist only when the pressure is in the range $P_{\text{min}}/k_B < P < 5005$ cm$^{-3}$ K and $P_{\text{max}}/k_B = 5005$ cm$^{-3}$ K, and a static equilibrium is attained at the saturation pressure $P_{\text{sat}}/k_B = 2282$ cm$^{-3}$ K, marked as the horizontal dashed line. (A color version of this figure is available in the online journal.)

**In the neutral ISM, thermal conduction is mostly due to collisions of hydrogen atoms. The corresponding conductivity depends on the temperature as $\kappa = 2.5 \times 10^7 \frac{\sqrt{T}}{T} \text{erg cm}^{-1} \text{s}^{-1} \text{K}^{-1}$ (Parker 1953; Spitzer 1962). As we shall show below, it is important to resolve the Field length in numerical simulations in order to obtain reliable results. While it is desirable to use the above form of thermal conductivity, we found that this requires extremely high resolution to resolve a transition layer close to the CNM. Therefore, throughout this work, we take a constant value $\kappa_0 = 10^3 \text{erg cm}^{-1} \text{s}^{-1} \text{K}^{-1}$, corresponding to $T = 1600$ K. While using a constant conductivity slightly changes the thickness of a transition layer between the CNM and WNM, it does not alter the essential physics involved in the DLI.**

### 2.2. Front Structure

We seek one-dimensional, steady-state solutions of Equations (2)–(4) that allow a phase transition between the CNM and WNM. We work in a frame in which the transition front is stationary. Following Inoue et al. (2006) and Iwasaki & Inutsuka (2012), we take a convention that the CNM and WNM are located at the left- and right-hand sides of the $x$-axis, respectively. Equations (2)–(4) are then simplified to

$$j_{x0} = G_0 v_x = \text{constant},$$

$$M_{x0} = P + \rho v_x^2 = \text{constant},$$

$$\kappa \frac{d^2 T}{dx^2} = j_{x0} c_p \frac{dT}{dx} + \rho \mathcal{L}(T).$$

where $j_{x0}$ and $M_{x0}$ denote the mass flux and the momentum flux, respectively, and $c_p = \gamma (\gamma - 1) k_B/m$ is the specific heat at constant pressure. Note that in deriving Equation (11), we make the isobaric approximation $P \approx \text{constant}$, the validity of which will be justified a posteriori.

Equation (11) suggests that there are two characteristic length scales related to a transition layer:

$$\ell_D = \frac{c_p}{v_x}, \quad \text{and} \quad \ell_F = \sqrt{\frac{k T}{n \Lambda}}.$$

The first one is the heat diffusion length occurring over the advection time scale ($\ell_D/v_x$), while the second one is the Field length over which the conductive heat transport balances the cooling (Begelman & McKee 1990). The latter also corresponds to the maximum wavelength of TI in the presence of thermal conduction (Field 1965). For thermal fronts in the ISM we consider here, the advection term in Equation (11) is much smaller than the heating and conduction terms (i.e., $\ell_F \gg \ell_D$), so that $\ell_F$ naturally corresponds to the thickness of transition layers in the ISM.\footnote{This is unlike in terrestrial flame fronts where $\ell_D \ll \ell_F$, so that the front thickness is determined primarily by $\ell_D$ (e.g., Zel’dovich et al. 1985).}

Note that $\ell_F/\ell_D = t_{\text{cool}}/t_{\text{flow}}$, where $t_{\text{cool}} = \gamma (\gamma - 1) P/n^2 \Lambda$ is the cooling time and $t_{\text{flow}} = \ell_F/v_x$ is the time it takes a fluid element to pass through the front.

Equation (11) can be integrated numerically subject to the conditions

$$T \bigg|_{-\infty} = T_1, \quad T \bigg|_{+\infty} = T_2, \quad \frac{dT}{dx} \bigg|_{\pm \infty} = 0.$$  

Here and hereafter, the subscripts “1” and “2” indicate the physical quantities of the CNM and WNM very far away from...
the front, respectively, at an equilibrium pressure, $P_{\text{eq}}$. Thus, finding $T(x)$ constitutes an eigenvalue problem with eigenvalue $j_{x0}$. For given $P_{\text{eq}}$, we take a trial value of $j_{x0}$ and integrate Equation (11) from $x = \pm \infty$ toward a midpoint where $T$'s from both sides match with each other. We check if $dT/dx$ from each side is the same at the midpoint as well, and vary $j_{x0}$ iteratively until the smoothly connecting solutions are obtained. By repeating the procedures, one can find $j_{x0}$ as a function of $P_{\text{eq}}$ for steady equilibria.

Figure 2 plots (a) $j_{x0}/\bar{m}$ and (b) the equilibrium CNM velocity $v_{x1} = j_{x0}/\rho_{1}$ at $x = -\infty$ as functions of $P_{\text{eq}}$. The front is static (i.e., $j_{x0} = v_{x1} = 0$) at the saturation pressure $P_{\text{sat}}/k_B = 2282$ cm$^{-3}$ K marked by the dashed line in Figures 1 and 2. Both $|j_{x0}|$ and $|v_{x1}|$ increase as $P_{\text{eq}}$ departs from $P_{\text{sat}}$. Since $T_2 \gg T_1$, Equation (11) is further integrated to

$$j_{x0} \approx Q/(c_p T_2),$$

where $Q \equiv -\int_{-\infty}^{\infty} \rho L d\tau$, indicating that $Q = 0$ for static fronts (Zel’dovich & Pikel’ner 1969). When $P_{\text{eq}} < P_{\text{sat}}$, the equilibrium densities are smaller than the static cases and the gas is thus dominated by heating with $Q > 0$. In this case, the transition layer corresponds to an evaporation front since the CNM moves in the positive $x$-direction and undergoes thermal expansion to turn to the WNM downstream. When $P_{\text{eq}} > P_{\text{sat}}$, on the other hand, the radiative cooling dominates to have $Q < 0$ and $v_{x1} < 0$, so that the WNM moves in the negative $x$-direction to change to the CNM after passing through a condensation front. The eigenvalue $j_{x0}$ is identical to the evaporation or condensation rate of the gas per unit area across the front. Figure 2 shows that for evaporation fronts, the inflowing CNM velocity at far upstream is larger for smaller $P_{\text{eq}}$, but is limited to below 2.5 m s$^{-1}$. The corresponding WNM velocity at far downstream amounts to $v_{x2} = (\rho_1/\rho_2)v_{x1} < 137$ m s$^{-1}$. Since the associated Mach number is less than 0.016, one can ignore the $\rho u_1^2$ term in Equation (10) to obtain $P \approx M_{x0} = \text{constant}$.

This proves the validity of the isobaric approximation for steady fronts.

Figure 3 displays the exemplary distributions of (top) gas density, (middle) temperature, and (bottom) net cooling rate for an evaporation front with $P_{\text{eq}}/k_B = 1800$ cm$^{-3}$ K (red), a static front with $P_{\text{eq}}/k_B = 2282$ cm$^{-3}$ K (black), and a condensation front with $P_{\text{eq}}/k_B = 3000$ cm$^{-3}$ K (blue). Cooling regions are highly localized near to the CNM side of the transition layer where $\rho$ and $T$ vary steeply.

(A color version of this figure is available in the online journal.)

3. LINEAR DISPERSION RELATION

We explore the DLI of a steady evaporation front in the linear regime. We begin by summarizing the physics behind the DLI in the long-wavelength limit: the reader is referred to Williams (1985) and Zel’dovich et al. (1985) for a more
is displaced sinusoidally (thick solid line) along the detailed explanation. Figure 4 sketches a situation where a front near a distorted evaporation front (thick curve) that was originally parallel to the y-direction. The CNM and WNM are located at the upstream and downstream sides, respectively. The dashed arrows indicate the directions of heat flows via conduction. See text for details.

![Figure 4](image)

**Figure 4.** Schematic diagram showing directions of gas flows (solid arrows) near a distorted evaporation front (thick curve) that was originally parallel to the y-direction. The CNM and WNM are located at the upstream and downstream sides, respectively. The dashed arrows indicate the directions of heat flows via conduction. See text for details.

Table 1

| P_{eq}/k_B | μ   | v_{i,1} | \ell_{D,1} | \ell_{f,1} | L_f | \lambda_{max} | \ell_{gr} |
|------------|-----|--------|-------------|------------|-----|---------------|--------|
| 1700       | 59.1| 231.7  | 0.028       | 0.0020     | 0.12 | 0.43          | 0.798  |
| 1800       | 68.9| 113.6  | 0.045       | 0.0017     | 0.12 | 0.29          | 1.05   |
| 1900       | 76.5| 63.1   | 0.069       | 0.0015     | 0.11 | 0.22          | 1.43   |
| 2000       | 83.0| 35.8   | 0.106       | 0.0013     | 0.11 | 0.19          | 2.10   |
| 2100       | 88.7| 18.8   | 0.179       | 0.0011     | 0.11 | 0.16          | 3.58   |

**Notes.** Column 1: equilibrium pressure (cm\(^{-3}\) K). Column 2: \(\mu = \rho_1/\rho_2\) is the expansion factor. Column 3: the inflow velocity of the CNM at \(x = -\infty\) (cm s\(^{-1}\)). Columns 4–7: the diffusion length in the CNM, field length in the CNM, field length in the WNM, and front thickness, respectively (pc). Columns 8–9: the wavelength (pc) and the growth time (Myr) of the most unstable mode of the DLI, respectively.

![Figure 5](image)

**Figure 5.** Dimensionless growth rate \(\Omega_{\ell_{D,1}}/v_{i,1}\) of the DLI for \(P_{eq}/k_B = 1700, 1800,\) and \(1900\) cm\(^{-3}\) K as functions of the dimensionless wavenumber \(k\ell_{D,1}\). Solid curves are the results of the full linear stability analysis, while dotted lines draw Equation (1) with the corresponding \(\mu\).

(A color version of this figure is available in the online journal.)

should advance further downstream (upstream). This causes the front to bend in a runaway fashion, indicating an instability.

The above argument is valid as long as the wavelengths of perturbations are much longer than the front width. The growth rate \(\Omega_0\) given in Equation (1) implies that modes with smaller wavelengths grow faster. However, small-scale modes would be stabilized due to heat conduction by the following manner. As indicated by the dashed arrows in Figure 4, excess heat in the WNM can be easily transferred via conduction to the CNM ahead of the convex parts of the front. The enhanced heating would speed up the evaporation rate, compensating at least partly for the increased mass flux there and thereby reducing a need for the front to advance further. If this conduction-mediated evaporation rate exceeds the increased mass flux, the front would be drawn back to the original position and the DLI would be completely suppressed. The effect of conduction is important when the perturbation wavelength is comparable to or less than the front thickness.

In the Appendix, we present the detailed procedure for finding the linear dispersion relations of the DLI of an evaporation front in the presence of conduction. Following the method of Liberman et al. (1994), we describe the perturbations as a linear combination of incompressible, vortex, and thermal modes in the far upstream and downstream sides separately, and make them connect smoothly to each other at the front, which allows us to obtain the growth rate, \(\Omega\), uniquely for given wavenumber \(k\) and \(\mu\). We also show analytically that \(\Omega \rightarrow \Omega_0\) as \(k \rightarrow 0\). While our method requires cumbersome iterative integrations of the linearized equations, it does not require to make long- or short-wavelength approximations.

Figure 5 plots \(\Omega\) against \(k\) as solid lines for \(P_{eq}/k_B = 1700, 1800,\) and \(1900\) cm\(^{-3}\) K. The growth rate and wavenumber are normalized using \(v_{i,1}\) and \(\ell_{D,1}\). The dotted lines draw the corresponding \(\Omega_0\), which are in good agreement with \(\Omega\) in the limit of \(k\ell_{D,1} \ll 1\). The growth rate achieves its peak value...
at $k_{\text{max}}\ell_{\text{D}1} \sim 0.23–0.29$, is slightly asymmetric with respect to $k_{\text{max}}$, and becomes zero at $k_{\text{crit}}\ell_{\text{D}1} \sim 0.52–0.54$. Columns 8 and 9 of Table 1 give $\lambda_{\text{max}} = 2\tau/k_{\text{max}}$ and the growth time $t_{\text{gr}} = 1/\Omega_{\text{max}}$ of the fastest growing mode in physical units. These are fitted approximately as

$$\lambda_{\text{max}} = 1.75F_{\lambda}(\frac{v_{x1}}{1 \text{ m s}^{-1}})^{-1} (\frac{n_1}{10^3 \text{ cm}^{-3}})^{-1} (\frac{k}{k_0}),$$

and

$$t_{\text{gr}} = 55.4F_{\lambda} \text{ Myr} (\frac{v_{x1}}{1 \text{ m s}^{-1}})^{-2} (\frac{n_1}{10^3 \text{ cm}^{-3}})^{-1} (\frac{k}{k_0}),$$

where $F_{\lambda}(P_{\text{eq}}) = 1 + 30.3(1 - P_{\text{eq}}/P_{\text{sat}})^3$ and $F_{\lambda}(P_{\text{eq}}) = 1 + 111(1 - P_{\text{eq}}/P_{\text{sat}})^3$ are the correction factors for $P_{\text{eq}}$ in the range of $0.74 \leq P_{\text{eq}}/P_{\text{sat}} \leq 0.92$. The dependence of $\lambda_{\text{max}}$ and $t_{\text{gr}}$ on $v_{x1}$, $n_1$ and $k$ follows simply from $\lambda_{\text{max}} \propto \ell_{\text{D}1}$ and $t_{\text{gr}} \propto \lambda_{\text{max}}/v_{x1}$. These fitting formulae are accurate within 7%. The DLI of evaporation fronts at higher equilibrium pressure takes longer time to grow, owing to a smaller $v_{x1}$ in the background state.

### 4. NUMERICAL SIMULATIONS

To study nonlinear development of the DLI of an equilibrium configuration found in the preceding section, we evolve the set of Equations (2)–(4) by using the Athena code (Stone et al. 2008). *Athena* is a general-purpose Eulerian code for magnetohydrodynamics based on high-order Godunov methods. Among the various algorithms implemented in it, we use the constrained transport scheme for directionally unsplit integration, the HLLC Riemann solver for flux computation, and the piecewise linear method for spatial reconstruction. The thermal conduction and heat-loss terms are solved explicitly.

In this section, we first address the issue of proper resolution required to resolve an interface between the CNM and WNM, and then present the numerical results for the DLI in the nonlinear regime.

#### 4.1. Constraint on Spatial Resolution

To check our implementation of the heating, cooling, and conduction terms in the *Athena* code, we have tested the code to the growth of TI by running one-dimensional simulations. For this purpose, we initially consider a static, thermally unstable medium with $n = 2.80 \text{ cm}^{-3}$ and $T = 814 \text{ K}$ in the domain with size $L_x = 20 \text{ pc}$, and impose random perturbations to the pressure with amplitudes of 0.1%. We employ the periodic boundary conditions at both ends of the domain. We run various models with differing numbers of grid points from $N_x = 2^7$ to $2^{14}$. Table 2 gives the results of these one-dimensional simulations. As Column 3 of Table 2 shows, all of our runs successfully reproduce, within $\sim 4\%$, the analytic growth rate $\tau_{\text{gr}} = 0.85 \text{ Myr}^{-1}$ of the most unstable mode in the linear regime, consistent with the results of previous studies (e.g., Piontek & Ostriker 2004; Kim et al. 2008; Choi & Stone 2012).

We, however, find that the density and temperature profiles, at the saturated state of TI, of the interfaces between the CNM and WNM and the corresponding equilibrium pressure are dependent upon numerical resolution. To illustrate this, Figure 6 compares the results for (a) temperature distribution and (b) scatter plots in the $n-P$ plane at $t = 500 \text{ Myr}$ from the runs with $N_x = 2^{10}$ (with the grid spacing of $\Delta x = 1.95 \times 10^{-2} \text{ pc}$; black triangles) and $2^{11}$ (with $\Delta x = 2.4 \times 10^{-3} \text{ pc}$; red circles) zones.

#### Table 2

| $N_x$ | $\Delta x$ | $\tau_{\text{gr}}$ | $P_{\text{sat}}/k_B$ | $\delta v_{x1}$ |
|-------|-------------|---------------------|------------------------|-----------------|
| (1)   | (2)         | (3)                 | (4)                    | (5)             |
| $2^7$ | $1.6 \times 10^{-1}$ | 0.87 | 1818 | 3.6 $\times 10^1$ |
| $2^8$ | $7.8 \times 10^{-2}$ | 0.84 | 1813 | 2.7 $\times 10^1$ |
| $2^9$ | $3.9 \times 10^{-2}$ | 0.84 | 1865 | 1.9 $\times 10^1$ |
| $2^{10}$ | $2.0 \times 10^{-2}$ | 0.85 | 1989 | 1.4 $\times 10^1$ |
| $2^{11}$ | $9.8 \times 10^{-3}$ | 0.84 | 2158 | 6.6 $\times 10^0$ |
| $2^{12}$ | $4.9 \times 10^{-3}$ | 0.86 | 2267 | 2.4 $\times 10^0$ |
| $2^{13}$ | $2.4 \times 10^{-3}$ | 0.85 | 2282 | 4.7 $\times 10^{-2}$ |
| $2^{14}$ | $1.2 \times 10^{-3}$ | 0.85 | 2282 | 1.2 $\times 10^{-2}$ |

Notes. Columns 1–3: number of zones, the zone spacing (pc), and the numerical growth rate of TI (Myr$^{-1}$). Columns 4–5: numerically found equilibrium pressure (cm$^{-3}$ K) and the velocity dispersion (m s$^{-1}$) averaged over $t = 200$–500 Myr.

Due to TI, the perturbations grow into a highly nonlinear state where cold clumps are surrounded by a warm intercloud gas. Some clumps merge together into larger ones at late times, and the system reaches a quasi-steady state at around 50 Myr. Note that the high-resolution model recovers the saturation pressure $P_{\text{sat}} = 2282 k_B \text{ cm}^{-3} \text{ K}$ discussed in Section 2.2 almost exactly with a root-mean-square velocity of $\delta v_{x1} \lesssim 0.01 \text{ m s}^{-1}$. In the low-resolution model, on the other hand, $P \approx 1870 k_B \text{ cm}^{-3} \text{ K}$ with small fluctuations; the corresponding velocity field has $\delta v_{x1} = 14.0 \text{ m s}^{-1}$, showing that the velocity dispersion induced by TI also depends on numerical resolution. Figures 6(c) and (d) directly compares the profiles of (c) temperature and (d) heat-loss function across a CNM–WNM interface between the low- and high-resolution models. The parts indicated by the black and red arrows in Figure 6(a) are enlarged and shifted so as to make the front position $x_1$ coincide. The solid lines representing the solution of Equations (6) and (11) at $P = P_{\text{sat}}$ are almost identical to the results of the high-resolution run, while they deviate considerably from those of the low-resolution model. The equilibrium pressure and velocity dispersion averaged over $t = 200$–500 Myr are listed in Columns 4 and 5 of Table 2.

The discrepancies of the saturation pressure and interface profiles in our low-resolution runs from the analytic predictions are a numerical artifact caused by overcooling in the cooling-dominated region. Figure 6(d) shows that strong radiative cooling is highly localized to a narrow layer where temperature changes steeply. Its thickness is $\sim 10^{-2} \text{ pc}$, comparable to the Field length ($\ell_{F,2}$), in the thermally unstable medium. On the other hand, the heating zone is relatively widely distributed over $\sim 0.1 \text{ pc}$, comparable to $\ell_{F,2}$. The $N_x = 2^{10}$ model with $\Delta x = 0.02 \text{ pc}$ resolves the heating zone quite well, but is unable to resolve the cooling zone. This results in net overcooling across the interface, and thus reduction in the equilibrium pressure (e.g., Piontek & Ostriker 2004). This in turn leads to larger temperatures and lower densities of the CNM than the values at the saturation pressure as seen in Choi & Stone (2012).

In Figure 7 we plot the dependence on numerical resolution of the equilibrium pressure obtained from our one-dimensional test runs: circles and errorbars represent the mean values and standard deviations over $t = 200$–500 Myr. Note that the grid spacing is shown as the physical length in the top $x$-axis, while it is in terms of the field number $n_F \equiv \Delta x/\ell_{F,2}$ in the bottom $x$-axis. It is apparent that the equilibrium pressure converges to $P_{\text{sat}}$ (the dashed line) as $n_F$ decreases. Note that...
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Figure 6. Comparisons between the low-resolution run with \(N_x = 2^{10}\) (\(\Delta x = 1.95 \times 10^{-2}\) pc; black triangles) and the high-resolution run with \(N_x = 2^{13}\) (\(\Delta x = 2.4 \times 10^{-3}\) pc; red circles) of the nonlinear static equilibrium at \(t = 500\) Myr obtained from TI for (a) overall temperature distributions, (b) scatter plots in the \(n-P\) plane, with \(P_{\text{sat}}\) equal to the analytic saturation pressure, and the profiles of (c) temperature and (d) heat-loss function near the CNM–WNM interface marked by the black or red arrow in (a). In (c) and (d), the solid lines draw the analytic predictions at \(P = P_{\text{sat}}\).

(A color version of this figure is available in the online journal.)

4.2. Two-dimensional Simulations

We now turn to our central problem: the nonlinear evolution of the DLI. We first restrict ourselves to the most unstable mode, and study its linear and nonlinear growth as well as various physical properties at saturation in detail. We then explore the case with multi-mode perturbations.

### 4.2.1. Single-mode Case

As a background state, we select an evaporation front attained at \(P_{\text{eq}}/k_B = 1800\) cm\(^{-3}\) K as our fiducial model, and study its response to the most unstable mode of the DLI. The results of other equilibrium states at different \(P_{\text{eq}}\) are qualitatively similar.

The initial state is generated using the method described in Section 2.2 for a given pressure, placing the evaporating front at \(x = 0\). As our computation domain, we take a rectangular box that spans \(-(3/5)L_x \leq x \leq (2/5)L_x\) and \(0 \leq y \leq L_y\).

\[
\frac{P}{P_{\text{sat}}} \approx 1 \quad \text{only when} \quad n_F \lesssim 0.25, \tag{17}
\]

suggesting that it is necessary to resolve the Field length of the transition layer by at least four zones in order to obtain accurate solutions for CNM–WNM interfaces. In what follows, we present the results of two-dimensional simulations for the DLI that satisfy the condition (17).

4.2. Two-dimensional Simulations
The box size is chosen as $L_x \times L_y = 5.64 \text{ pc} \times 1.06 \text{ pc}$, which is large enough in the $x$-direction to encompass the asymptotic regions of the flows and equals $\lambda_{\text{max}}$ in the $y$-direction. We set up a $N_x \times N_y = 2048 \times 384$ Cartesian grid with the cell size of $\Delta x = \Delta y = 2.8 \times 10^{-3} \text{ pc}$, which has the field number $n_F = 0.20$, fulfilling the resolution requirement of Equation (17). For the boundary conditions, we implement the inflow boundary condition at the left $x$-boundary in which the density and velocity are set equal to the unperturbed values every time step. This is not only to reduce the effects of reflection of outgoing waves at the boundary but also to make the upstream region at far field retain its unperturbed state. We impose the outflow boundary condition at the right $x$-boundary, and the periodic conditions at the $y$-boundaries. The model parameters and simulation outcomes of the fiducial model (Model MU69) are listed in the top row of Table 3.

We have first checked that the initial front structure remains stationary over a long period of time in the absence of any perturbation. This confirms that our realization of the equilibrium configuration is in a steady state. Next, we add small perturbations to the initial configuration by shifting the front position slightly as $x_t = -D_s(0) \cos(2\pi y/L_y)$ with the initial displacement amplitude of $D_s(0)/L_y = 10^{-2}$, which seeds the most unstable DLI mode. We run the simulation until 700 Myr, corresponding to $\sim 17 t_{\text{gr}}$, well beyond the nonlinear saturation of the DLI.

Figure 8 displays temperature snapshots in logarithmic scales at $t/t_{\text{gr}} = 0, 3, 6, 9, 12$ of Model MU69. The characteristic distortion of the front becomes noticeable at $t/t_{\text{gr}} = 1$, which is growing exponentially with time. Figure 9 compares the perturbed temperature $T'(x, y = 0.5 L_y)$ and $y$-velocity $v_y(x, y = 0.25 L_y)$ at $t/t_{\text{gr}} = 2$ with the analytic eigenfunctions (solid lines) obtained from the linear stability analysis. At this time, the system is still in the linear regime. The agreement between the numerical results and the predictions of the linear theory is excellent. We note that the profile of the perturbed temperature closely resembles that of the initial temperature gradient, i.e., $T' \propto -dT/dx$ in the linear regime, as proven by Liberman et al. (1994).

To describe the front shape at arbitrary $t$, we introduce a curve $C_t = x - s(y, t) = 0$, where $s(y, t)$ denotes the $x$-position of the isotherm with $T_I = \sqrt{T_t T_{\text{gr}}}$ at given $x$ and $t$. Then, the evaporation rate of the CNM per unit area is given by

$$\epsilon = \rho_f \left( \hat{s} \cdot \mathbf{v}_I - \frac{1}{|\nabla C_t|} \frac{\partial s}{\partial t} \right),$$  

(18)

where $\hat{s} = \nabla C_t/|\nabla C_t|$ is the unit vector normal to the front directed toward the WNM, $\rho_f$ and $\mathbf{v}_I$ denote the gas density and velocity at the front, respectively. The total evaporation rate in the computational domain is then $\mathcal{E} = \int_{C_t} \epsilon \, dl$, where the integration is carried along the front. Note that $s$ is constant for a vertically straight, steady front, yielding $\epsilon_0 = j_{s0} = 2066 \, m \, \text{cm}^{-2} \, \text{s}^{-1}$ and $\mathcal{E}_0 = j_{s0} L_y$ in the unperturbed state.

Figure 10 plots the temporal changes of (a) the distortion amplitude $D_s \equiv (\max(s) - \min(s))/2$ and (b) the total evaporation rate in Model MU69. The DLI grows exponentially at early time, whose rate is consistent with the linear-theory prediction plotted as a short dotted line. As Figure 8 shows, the front becomes increasingly more distorted as the DLI grows, and has $D_s$ comparable to $\lambda_{\text{max}}$ at $t/t_{\text{gr}} \sim 6$. The front eventually develops into a finger-like structure that protrudes toward the downstream direction. The distorted front does not grow further after $t/t_{\text{gr}} \sim 7$, indicating that the DLI saturates nonlinearly. The front distortion increases the length of the front where the inflowing CNM turns into the WNM. Thus, the growth of the DLI inevitably results in an increase in the evaporation rate, which in turn causes the distorted front to move toward the upstream direction in our simulation, as evidenced in Figure 8.

According to Zel’Dovich (1966), the nonlinear saturation of the flame instability occurs due to the Huygens principle which states that every point of the front can be regarded as a source of a secondary spherical wave (see also Zel’dovich et al. 1985). Suppose a curved front that is moving relative to the CNM. Since waves launched from the concave (convex) parts of the front to the WNM diverge (converge), the convex parts eventually develop cusps in the limit of infinitesimally thin front. The propagation velocity of the convex parts is larger than that of the concave part, which balances the growing tendency of the distortion amplitude of the front, resulting in a steady configuration. Smoothed by thermal conduction, the distorted front in our model does not display a sharp cusp.

Figure 8(e) plots gas streamlines (white lines) around the front at $t/t_{\text{gr}} = 12$ in Model MU69. Although the refracted flow field indicates a production of some vorticity at the distorted front ($|\nabla \times \mathbf{v}| \lesssim 1.79 \times 10^{-13} \text{ s}^{-1}$), the DLI of an initially laminar flow does not lead to turbulence at nonlinear saturation, consistent with the results of Bell et al. (2004a). The local evaporation rate $\epsilon$ varies along the front in such a way that it is largest ($4.68 j_{s0}$) at the tip of the finger due to the largest curvature and hence the efficient conductive heating from the surrounding WNM, and becomes smallest ($\sim 0.77 j_{s0}$) at the wing sides. When integrated over the front length, the total evaporation is $\mathcal{E} = 2.4 j_{s0} L_y$ at saturation, 2.4 times larger than $\mathcal{E}_0$. This increase of $\mathcal{E}$ is in complete accordance with the larger inflow velocity of the CNM relative to the front, which is $\sim 2.4$ times larger than the initial plane-parallel value. Because the fractional increase in the front length is also a factor of 2.4 in the saturated state, the increase

| Model    | $\mu$ | $v_{t1}$ | $N_x \times N_y$ | $L_x \times L_y$ | $\Delta x$ | $D_s$ | $\mathcal{E}/(j_{s0} L_y)$ |
|----------|-------|----------|-------------------|------------------|----------|------|---------------------|
| MU69     | 68.9  | 114      | $2048 \times 384$ | $5.64 \times 1.06$ | $2.8 \times 10^{-3}$ | 0.54 | 2.44                |
| MU69mul  | 68.9  | 114      | $3072 \times 1536$ | $8.46 \times 4.23$ | $2.8 \times 10^{-3}$ | 0.59 | 2.49                |
| MU38     | 37.5  | 256      | $2048 \times 384$ | $6.33 \times 1.18$ | $3.5 \times 10^{-3}$ | 0.45 | 2.00                |
| MU11     | 11.2  | 404      | $2048 \times 384$ | $10.5 \times 1.97$ | $9.6 \times 10^{-3}$ | 0.43 | 1.44                |
| MU03     | 3.24  | 834      | $1024 \times 384$ | $9.22 \times 3.46$ | $2.6 \times 10^{-3}$ | 0.33 | 1.10                |

Notes. Column 1: model name. Columns 2–3: the expansion factor and the CNM velocity at far upstream (cm s$^{-1}$). Columns 4–6: number of zones, domain size (pc $\times$ pc), and zone spacing (pc) of the simulation. Columns 7–8: the distortion amplitude (pc) and the evaporation rate relative to the value at nonlinear saturation.
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Figure 8. Snapshots of temperature distribution in logarithmic scale at \( t/t_{fr} = 0 \), 3, 6, and 12 of Model MU69 that starts from single-mode perturbations with \( L_y = \lambda_{\text{max}} \). The DLI grows exponentially at \( t/t_{fr} \approx 6 \) to distort the front, leading to a finger-like shape pointing downstream in the nonlinear stage. There is no change in the front shape other than a translational shift between (d) and (e), suggesting a saturation of the DLI with an enhanced evaporation rate. In (e), gas streamlines are overlaid to show steady flow structures across the distorted front.

(A color version of this figure is available in the online journal.)

Figure 9. Distributions of (a) the perturbed temperature \( T' \) along the \( y = 0.5L_y \) cut relative to the initial temperature \( T \) and (b) perturbed velocity \( v_y \) along the \( y = 0.25L_y \) cut relative to \( v_{y1} \) at \( t/t_{fr} = 2 \) of Model MU69 shown as dots. The results of the linear stability analysis are compared as solid lines.

(A color version of this figure is available in the online journal.)

This model is to explore whether the system readily picks up the most unstable mode of the DLI. We take the same background state as in Model MU69, and displace the front positions in \( x \) randomly with amplitude of \( 10^{-3}L_y \) from the equilibrium location. Figure 11 displays temperature snapshots in logarithmic scale at \( t/t_{fr} = 0, 3, \) and 6. Perturbations grow at rates depending on their wavelengths. At early time, perturbations with large initial amplitudes emerge first, which happen to be the mode with \( m = L_y/\lambda = 5 \) at \( t/t_{fr} = 3 \). But, it is eventually the most unstable \( m = 4 \) mode that dominates to form finger-like nonlinear structures at late time. Although the growth of other unstable modes makes the spacing between the fingers irregular to some extent, the overall morphology and the increase in the evaporation rate at nonlinear saturation are consistent with the results of Model MU69. We again note that the ratio of the kinetic energy to the thermal energy in Model MU69mul at \( t/t_{fr} = 6 \) is \( 4 \times 10^{-4} \) and the system remains laminar, without evolving into a turbulent state.

4.2.3. Effects of Density Ratio

As shown in the preceding sections, the DLI of an evaporation front in the ISM leads to front deformation and an increase in the evaporation rate, without driving turbulence. This is qualitatively consistent with the results of numerical studies of the DLI in the context of C/O thermonuclear flames in Type Ia supernovae (e.g., Bell et al. 2004a) and combustion in heat engines (e.g., Bychkov et al. 1996; Travnikov et al. 1999). However, our results differ quantitatively in that while the front deformation and the associated increase of the flame propagation speed are only a few percent in the other studies, they are more than 100% in our simulations. These differences are most likely caused by differences in the expansion factor \( \mu \) between the models. Note that \( \mu \lesssim 10 \) in terrestrial flames and \( \mu \lesssim 2 \) in supernova thermonuclear flames, which is about an order of magnitude smaller than that of evaporation fronts in the ISM.

4.2.2. Multi-mode Perturbation

We also run Model MU69mul that has a larger simulation domain with \( L_x \times L_y = 8.46 \text{ pc} \times 4.23 \text{ pc} \) than in the single-mode case, which can accommodate perturbations with wavelength up to \( 4\lambda_{\text{max}} \) (see the second row of Table 3).
Figure 10. Temporal changes of (a) the distortion amplitude $D_s$ of the front and (b) the total evaporation rate $\dot{E}$ in Model MU69. The predicted linear growth rate is indicated as a dotted line segment in (a). The DLI saturates at $t/t_{gr} \sim 6$–7, with the evaporation rate enhanced by a factor of 2.4.

Figure 11. Temperature snapshots at $t/t_{gr} = 0,$ 3, and 6 in Model MU69mul with $L_y = 4\lambda_{\text{max}}$ that starts from random perturbations. While various modes grow in the linear stage, the system at late time is dominated by the most unstable mode that produces four finger-like structures. (A color version of this figure is available in the online journal.)

To directly assess the impact of varying $\mu$ on the nonlinear saturation of DLI, we conduct simulations of heuristically modeled fronts with smaller $\mu$. For this purpose, we modify the density-independent heating rate to

$$\Gamma = \Gamma_0 \times \frac{\exp[(n/n_0)^2]}{1 + n/n_0} \quad \text{(19)}$$

where $n_0$ is a free parameter. Note that $\Gamma \to \Gamma_0$ as $n_0 \to \infty$. For finite $n_0$, $\Gamma$ increases rapidly with $n \gtrsim n_0$, lowering the equilibrium CNM density without much effect on the WNM density. Thus, a smaller value of $n_0$ results in smaller $\mu$ in an equilibrium configuration.

Figure 12(a) plots a few equilibrium density profiles for $n_0 = 10, 4,$ and 2 $\text{cm}^{-3}$; the corresponding density contrasts, pressures, and the far upstream inflow speeds are $\mu = 38,$ 11, 3.2, $P_{eq}/k_B = 1500,$ 2000, 2800 $\text{cm}^{-3}\text{K}$, and $v_{x1} = 256,$ 404, 834 $\text{cm s}^{-1}$, respectively. The case with $n_0 = \infty$ is also plotted for comparison. For these steady fronts, we perform the linear stability analysis and plot the resulting growth rates in Figure 12(b). Clearly, the DLI grows slower with decreasing $\mu$, which is expected from Equation (1). The critical wavenumber becomes smaller with decreasing $\mu$.

For these evaporation fronts with the modified heating rate, we run numerical simulations of the DLI by taking $L_y$ equal
to the wavelength of the fastest growing mode. The parameters for these runs, named MU38, MU11, and MU03, are given in Table 3. Note that the models with small \( \mu \) easily meet the resolution requirement because of the higher CNM temperature (leading to the increase of \( \ell_F \)). In all models, we displace the fronts sinusoidally with an amplitude of \( 10^{-3} \ell_F \). As in Model MU69, the evaporation fronts in these lower-\( \mu \) models are increasingly more distorted with time as a result of the DLI. In Figure 12(b), we mark as open circles the growth rates measured from numerical simulations in the linear stage, in good agreement with the linear-theory results. The DLI soon enters from the simulations in the linear stage, in good agreement with the linear-theory results. The overall front width is comparable to the Field length in the WNM.

Figure 13 compares the front shapes from models with different \( \mu \) at \( t/t_F = 12 \). Apparently, the distortion amplitude and the evaporation rate at saturation become smaller with decreasing \( \mu \). For example, the ratio of the distortion amplitude to the wavelength of the most unstable mode is \( \sim 0.095 \) and 0.38 in Models MU03 and MU38, respectively. The total evaporation rate at saturation can be fitted by

\[ \mathcal{E}/\mathcal{E}_0 \approx 1 + 0.41 (\log \mu)^2. \]

For \( 3 \leq \mu \leq 70 \). Extrapolating this result to \( \mu = 1.52 \) corresponding to C/O thermonuclear flames, we obtain \( (\mathcal{E} - \mathcal{E}_0)/\mathcal{E}_0 = 1.4\% \), roughly consistent with the result of Bell et al. (2004a). Again, the evaporation fronts in all models do not develop a cusp at the location of the maximum distortion due to the smoothing effect of thermal conduction.

5. SUMMARY AND DISCUSSION

While the behavior of a thermally bistable fluid consisting of the CNM and WNM is becoming increasingly more important for numerical studies of the ISM, relatively little attention has been directed to dynamics of their interfaces. In this paper, we have presented the results of the full linear stability analysis and numerical simulations for the corrugational instability, or the DLI, of evaporation fronts in the ISM. As an unperturbed state, we consider an evaporation front in plane-parallel geometry and take a constant value for thermal conductivity. Our key findings are summarized as follows.

1. The type and structure of a thermal front between the CNM and WNM in steady equilibrium are determined by the equilibrium thermal pressure \( P_{\text{eq}} \) such that the front becomes a condensation front when \( P_{\text{sat}} < P_{\text{eq}} < P_{\text{max}} \) across which the WNM changes to the CNM, and an evaporating front when \( P_{\text{min}} < P_{\text{eq}} < P_{\text{sat}} \), where \( P_{\text{sat}} \) is the saturation pressure for static front and \( P_{\text{max}} \) and \( P_{\text{min}} \) refer to the maximum and minimum pressures for two-phase equilibrium, respectively. For our adopted heat-loss function and thermal conductivity, \( P_{\text{sat}}/k_B = 2282 \text{ cm}^{-3} \text{ K}, P_{\text{max}}/k_B = 5005 \text{ cm}^{-3} \text{ K}, \) and \( P_{\text{min}}/k_B = 1597 \text{ cm}^{-3} \text{ K}. \) The incident velocity \( v_{\text{in}} \) of the CNM at far upstream relative to the evaporating front is limited to below 250 cm s\(^{-1}\), much smaller than the sound speed, making the isobaric approximation valid. The overall front width is comparable to the Field length in the WNM.

2. We perform the full linear stability analysis of the DLI in the presence of thermal conduction following the eigenvalue approach of Liberman et al. (1994). While the front thickness is determined by the Field length, the length and time scales of the instability are well characterized by the diffusion length \( \ell_D = \sqrt{D/k_B} \) and the corresponding crossing time \( t_D/v_{\text{in}} \) (see Figure 5 and Equations (15) and (16)). The linear dispersion relations show that perturbations with \( \lambda/\ell_D < 12 \) are completely stabilized by conduction, while they are well approximated by Equation (1) for very long-wavelength
perturbations. The growth rate also depends on the expansion factor \( \mu \) defined by the density ratio of the CNM to WNM.

3. Using one-dimensional simulations of TI, we demonstrate that it is important to resolve the field length of a transition layer between the CNM and WNM by at least four grid points in order to obtain accurate density and temperature distributions as well as the correct saturation pressure (see Equation (17)). Otherwise, the region of strong radiative cooling near the CNM would be unresolved, giving rise to overcooling and reduction in the equilibrium pressure.

4. Two-dimensional simulations of the DLI of an evaporating front show that small perturbations grow exponentially in the linear regime to bend the front, and saturate nonlinearly typically at \( t/t_{gr} \sim 7 \). The numerical growth rates in the linear stage are in good agreement with the predictions of the linear theory. In the nonlinear regime, the front is in a steady state and has a finger-like shape pointing toward the WNM, without developing turbulent flows. The presence of thermal conduction smooths out the front that would otherwise be cuspy with infinitesimal front thickness. The increase in the front length at saturation directly translates into an increase in the evaporation rate. For our fiducial model with \( P_{eq}/k_B = 1800 \text{ cm}^{-3} \text{ K} \) and \( \mu = 68.9 \), the saturated evaporation rate is increased by a factor of \( \sim 2.4 \) relative to the initial plane-parallel value. By running control models with the modified heating rate, we find that the evaporation rate at saturation relative to the initial value depends on \( \mu \) and is given by Equation (20).

The importance of resolving interfaces accurately has been emphasized by a number of authors in various contexts. For example, Koyama & Inutsuka (2004) ran various simulations of TI with differing resolutions, and found that numerical convergence for the density distribution is achieved only when the grid size is less than one third of the local field length, similar to our results. Krumholz et al. (2007) discussed numerically induced cooling in an ionization front advancing into a surrounding molecular medium. By comparing their numerical results with the analytic solutions, they showed that overcooling arises if the size of a computational cell is larger than the true thickness of the front, slowing down the front expansion. Overcooling is due to numerical mixing, leading to the overestimation of the amount of molecular gas around the front, a far more efficient coolant than ions and atoms. An analogous situation takes place in cosmological simulations that often lack sufficient resolution to resolve interfaces between gases of different temperatures, giving rise to the classical overcooling problem (e.g., Katz 1992; Mac Low 2013). In our models, unresolved cooling in the interfaces leads to non-vanishing \( Q \) even for a static front, which results in non-zero gas motions whose speed is roughly \( \sim c_{ Tamb}/t_{cool} \) from Equation (14) (see also Iwasaki & Inutsuka 2012).

As represented by Equations (15) and (16), the time and length scales of the DLI depend on the inflow speed \( v_{i} \), and the density \( n_1 \) of the CNM, which in turn depends rather sensitively on the adopted heat-loss function. In this work, we considered the ISM parameters representing the solar neighborhood conditions and found that the most unstable mode has a typical wavelength of \( \sim 1 \text{ pc} \) and a growth time of \( \sim 50 \text{ Myr} \) for \( n_1 = 10 \text{ cm}^{-3} \), \( v_{i1} = 1 \text{ m s}^{-1} \), and \( \kappa = 10^5 \text{ erg cm}^{-2} \text{ s}^{-1} \text{ K}^{-1} \). However, the radiative cooling and heating may vary considerably in space and time, depending on local conditions such as star-forming activity, gas column density, abundances of the main coolants, etc., all of which can affect the density and temperature profiles of an evaporating front in equilibrium. In the inner regions of a galactic disk, for example, elevated star formation rates together with stronger ionizing background radiation lead to a significantly enhanced heating rate, which would make the thermal equilibrium curve in the \( n-P \) plane shifted upward and rightward (e.g., Parravano et al. 2003; Wolfire et al. 2003; Cox 2005). When the heating rate is five times larger than the one we adopt in the present work, for instance, an evaporation front has \( v_{i1} = 254 \text{ cm s}^{-1} \) and \( n_1 = 91 \text{ cm}^{-3} \) at \( P_{eq}/k_B = 9000 \text{ cm}^{-3} \text{ K} \). The growth time and wavelength of the fastest growing mode is then \( t_{gr} = 1.7 \text{ Myr} \) and \( \lambda_{max} = 0.094 \text{ pc} \), suggesting that the growth of the DLI is highly subject to environmental conditions.

Inoue et al. (2006) examined the linear stability of evaporation fronts in the ISM accounting for the effect of temperature-dependent conductivity. They obtained the growth rate of the DLI by considering only thermal modes, while neglecting incompressible and vortex modes (see the Appendix). The cooling function they adopted is different from ours in that they took the first terms (due to Lyα emissions) and second terms (due to C II lines) in Equation (8) about 30 times larger and smaller than those in our paper, respectively. The resulting pressure range for two-phase equilibrium is \( 637 \text{ cm}^{-3} \text{ K} < P/k_B < 12,600 \text{ cm}^{-3} \text{ K} \), much wider than ours. The typical inflow speed and density of the CNM in their models are \( v_{i1} = 5 \text{ m s}^{-1} \) and \( n_1 = 30 \text{ cm}^{-3} \), larger by about a factor of five and two than our values. They found that the most unstable mode has a growth time of \( \sim 0.3 \text{ Myr} \) and a wavelength of \( \sim 0.1 \text{ pc} \). Although it is difficult to make a direct comparison due to the differences in the cooling function and thermal conductivity, our results are overall consistent with their results if \( v_{i1} \) and \( n_1 \) are taken appropriately in Equations (15) and (16).

In this work, we have investigated the DLI under initially laminar conditions, showing that the DLI itself does not lead to turbulence in the ISM, consistent with the results of the DLI in terrestrial flames (e.g., Bychkov et al. 1996; Travnikov et al. 1999) and thermonuclear flames (e.g., Röpke et al. 2003; Bell et al. 2004a, 2004b). The real ISM, however, is shaped by turbulence on a wide range of length scales. Since the DLI involves deflection of gas streamlines at the front, the presence of non-uniform distribution of pressure and velocity in the background flows may obstruct the development of the instability. Heyer & Brunt (2004) reported that the velocity dispersions of clouds behave as \( v(\ell) = v(\ell_0)(\ell/\ell_0)^{3/2} \), with \( v(\ell_0 = 1 \text{ pc}) = 0.9 \text{ km s}^{-1} \) and \( q = 0.56 \), from 30 pc down to 0.03 pc scales (see also McKee & Ostriker 2007). Assuming that the evaporating flow decouples from turbulence, the DLI grows only if its growth time is larger than the eddy turnover time at \( \ell = \lambda_{max} \) or if \( v_{i1} \gtrsim 15 \text{ m s}^{-1}(n_1/10 \text{ cm}^{-3})^{-1/4}(\kappa/k_0)(q/0.56)^{1/4} \) from Equations (15) and (16). For our choice of the heat-loss function, \( v_{i1} \) is less than \( \sim 3 \text{ m s}^{-1} \) for steady evaporation fronts, suggesting that the DLI of CNM–WNM evaporation fronts in the neutral ISM is unlikely to grow into the nonlinear regime unless the interfaces are strongly protected from the ISM turbulence.
In addition, the growth time of the DLI based on our results is 0.35 Myr, much smaller than the expected evaporation time scale $\sim$18 Myr, suggesting that the evaporation front of a spherical cloud in a hot medium may suffer from the DLI. Of course, the real assessment of the DLI in this situation requires consideration of the curvature effect as well as realistic thermal conductivity and heating/cooling rates applicable for the hot phase that can substantially alter the background states and evaporation processes.

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APPENDIX
LINEAR STABILITY ANALYSIS

Here, we present the method to obtain linear dispersion relations of the DLI of evaporating fronts in the presence of thermal conduction. Our approach essentially follows Liberman et al. (1994) who studied instability of terrestrial flames (see also Liberman 2008).

A.1. Perturbation Equations

We initially consider a one-dimensional steady evaporation front located at $x = 0$, like the one shown in Figure 3, in which the density, velocity, and temperature vary with $x$. We apply two-dimensional perturbations to the steady configuration, and seek exponentially growing modes. Assuming that the perturbation amplitudes are small, Equations (2)–(5) are linearized to

$$\frac{\partial \rho'}{\partial t} + \frac{\partial}{\partial x} \left( \rho v' + \rho' v_s \right) + \frac{\partial}{\partial y} \left( \rho v'_y \right) = 0,$$

(A1)

$$\frac{\partial}{\partial t} \left( \rho v'_x + \rho' v_s \right) + \frac{\partial}{\partial x} \left( P' + 2 \rho v'_x + \rho' v^2_s \right) + \frac{\partial}{\partial y} \left( \rho v'_y \right) = 0,$$

(A2)

$$\frac{\partial}{\partial t} \left( \rho v'_y \right) + \frac{\partial}{\partial x} \left( \rho v'_x v_s + \rho' v^2_s \right) + \frac{\partial P'}{\partial y} = 0,$$

(A3)

$$\kappa \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) T' = \rho c_p \frac{\partial T'}{\partial t} + (\rho v'_x + \rho v'_y) c_p \frac{dT'}{dx} + \rho v'_x c_p \frac{dT'}{dx} + \frac{d(\rho L)}{dT'} T',$$

(A4)

and

$$\frac{P'}{P} = \frac{\rho'}{\rho} + \frac{T'}{T},$$

(A5)

where the primes indicate the perturbed quantities. In deriving Equation (A4), we have made the isobaric approximation under which $\rho L$ is a univariate function of $T$. The isobaric assumption is valid since the fractional change of thermal pressure is proportional to the square of the Mach number that is much less than unity even for fastest evaporating flows (e.g., Liberman 2008).

It is convenient to take $j' = \rho v'_x + \rho' v_s$ and $M' = P' + 2 \rho v'_x + \rho' v^2_s$, instead of $\rho'$ and $v'_x$, as independent perturbed variables. We decompose the perturbations as $\alpha \exp(\Omega t + iky)$, where $k$ and $\Omega$ denote the wavenumber and growth rate, respectively. We introduce the dimensionless perturbed variables as

$$\left( \frac{j'/j_{0}}{M'/(\rho v^2_{c1})} \right) = \text{Re} \left[ \left( \frac{\mathcal{J}'(\xi)}{\mathcal{M}'(\xi)} \right) \frac{J(\xi)}{T(\xi)} e^{i\Omega t + iky} \right],$$

(A6)

where the quantities with subscript “1” are evaluated at $x = -\infty$. Then, Equations (A1)–(A5) can be written as

$$\frac{dJ'}{d\xi} = -\frac{\nu T'}{T'} + \nu \sigma T,$$

(A7)

$$\frac{d\mathcal{M}'}{d\xi} = -\nu \sigma J' - \nu \nu'',$$

(A8)

$$\frac{d\nu'}{d\xi} = -2\nu T J' + \nu \mathcal{M}' - \nu \sigma \frac{\nu'}{T'} - \nu T',$$

(A9)

$$\frac{d^2 T'}{d\xi^2} = \frac{dT'}{d\xi} + \frac{dH}{dT} \frac{dT}{d\xi} + \frac{dT}{d\xi} J' + \nu \sigma \frac{T'}{T' + v^2 T'},$$

(A10)

where $\xi \equiv x/\ell_{D,1}$, $T \equiv T/T_1$, $\sigma \equiv \Omega/(kv^2_{c1})$, $\nu \equiv k\ell_{D,1}$, and $H \equiv -\left(\ell_{D,1}/\ell_{F,1}\right)^2 \rho L/(\eta_1 T_1)$. Let $U(\xi)$ and $D(\xi)$ denote the vectors, $(J', \nu', \mathcal{M}', T', dT'/d\xi)$, that describe the perturbations in the upstream and downstream sides of the front, respectively. Our strategy is to first obtain $U$ and $D$ by integrating Equations (A7)–(A10) from $\xi = \pm \infty$ to zero and then find $\sigma$ by the requirement $U = D$ at $\xi = 0$. To do this, we need appropriate boundary conditions at far-field zones as described below.

A.2. State Vectors

While $T$ in our problem varies with $\xi$, there are regions far away from the front where $T$ can be treated constant, thereby allowing algebraic solutions for perturbations. More specifically, let $\xi_- (< 0)$ and $\xi_+ (> 0)$ be the positions in the upstream and downstream flow, respectively, such that

$$\left| \frac{d \ln T}{d \xi} \right| \ll \min(1, \nu),$$

(A11)

at the far-field zones with $\xi < \xi_-$ or $\xi > \xi_+$. The boundary conditions for the perturbed variables in these regions are that they should be regular as $\xi \rightarrow \pm \infty$, that is, the perturbations should behave as $\propto \exp(\xi)$ for $|\xi| \gg 1$, with $\text{Re}(\beta) > 0$ at $\xi < \xi_-$ and $\text{Re}(\beta) < 0$ at $\xi > \xi_+$. Substituting the perturbations of this form into the perturbed continuity and momentum equations (Equations (A7)–(A9)), one obtains

$$\left( \beta^2 - \nu^2 \right) (T \beta + \nu \sigma) J' = \frac{V}{T^2} (T \beta + \nu \sigma \beta + \nu T) T',$$

(A12)

while Equation (A10) leads to

$$\left( \beta^2 - \beta + dH/dT - \nu^2 - \nu \sigma /T \right) T' = 0,$$

(A13)

in the far-field zones.
Clearly, there are five distinct values that $\beta$ can take. The first three values can be obtained from Equation (A12) by imposing $T' = 0$, corresponding to hydrodynamic waves propagating from the front. These are $\beta = \pm \nu$ representing incompressible modes, and $\beta = -\nu \sigma / T$ representing a vortex mode carried by the background flow (e.g., Landau & Lifshitz 1987). The remaining two solutions are

$$\beta_i = \frac{1}{2} \pm \sqrt{\frac{1}{4} + \nu^2 + \frac{\nu \sigma}{T} - \frac{dH}{dT}}, \quad (A14)$$

obtained from Equation (A13), corresponding to thermal modes. The associated state vectors can be obtained by substituting $\beta$’s back to Equations (A7)–(A10).

In general, perturbations that grow at far fields are a superposition of these five basic modes, but the boundary conditions mentioned above limit the number of the basic modes by requiring $\beta > 0$ in the upstream flow and $\beta < 0$ in the downstream flow. In addition, the vortex mode is related to the advection of vorticity which is generated when the front is curved. Since there is no source of vorticity generation other than the front itself (Zel’dovich et al. 1985), the vortex mode can exist only in the downstream flow if the flow at $\xi = -\infty$ is irrotational. Therefore, we are left with the following five state vectors.

1. Upstream incompressible mode:
$$U_i = e^{i\xi} (1, -1, -(\sigma - 1), 0, 0). \quad (A15)$$

2. Downstream incompressible mode:
$$D_i = e^{-i\xi} (1, \mu, \sigma + \mu, 0, 0). \quad (A16)$$

3. Downstream vortex mode:
$$D_v = e^{-i\nu \xi / \mu} (1, 2 \mu, 0, 0). \quad (A17)$$

4. Upstream thermal mode:
$$U_t = e^{\beta_+-i\xi} \left( \frac{\nu \beta_+ - v \sigma}{\beta_+^2 - v^2}, -\frac{\nu \beta_+ + v \sigma}{\beta_+^2 - v^2}, \frac{v^2 \beta_+^2 - \sigma^2}{\beta_+^2 - v^2}, 1, \beta_+ \right), \quad (A18)$$

5. Downstream thermal mode:
$$D_t = e^{\beta_+ i\xi} \left( \frac{\nu \beta_+ - v \sigma + \mu v}{\mu^2 \beta_+^2 - v^2}, \frac{-\nu \beta_+ + v \sigma + \mu v}{\mu^2 \beta_+^2 - v^2}, \frac{v^2 \beta_+^2 - \sigma^2 + \mu^2 v^2}{\mu^2 \beta_+^2 - v^2}, 1, \beta_+ \right), \quad (A19)$$

where $\beta_+$ and $\beta_-$ denote the positive and negative values of $\beta$, respectively, from Equation (A14). The perturbations at far fields can then be written as

$$D(\xi) = C_1 \mathbf{D}_1 + C_2 \mathbf{D}_2 + C_3 \mathbf{D}_3, \quad \text{for} \quad \xi \geq \xi_+, \quad (A20)$$

where $C_i$’s are constants to be determined.

Unlike hydrodynamic waves for which $\beta \propto \nu$, thermal waves always have $|\beta| \sim \max(1, \nu)$, decaying on a length scale shorter than the front thickness $\ell_D$. The role of the thermal modes thus becomes important when $\nu \lesssim 1$, while it can be ignored in the long-wavelength limit. We note that the analysis presented by Inoue et al. (2006) for short-wavelength perturbations considered only thermal modes as the basic states and ignored hydrodynamic modes.

A.3. Dispersion Relations

Since Equations (A7)–(A10) are linear in the perturbed variables, we may take $C_1 = 1$ without loss of generality. Therefore, the problem is reduced to finding the eigenvalue $\sigma$ and four proportionality constants $C_2, \ldots, C_5$ subject to five constraints, $\mathbf{D} = \mathbf{U}$ at $\xi = 0$.

A.3.1. Long-wavelength Limit

Before explaining the computation method for obtaining dispersion relations for general $\nu$, we revisit the case of long-wavelength perturbations with $\nu \ll 1$, for which the thermal front can be treated as a discontinuous surface at $\xi = 0$. In this case, the temperature distribution of the background flow can be taken as $T = 1$ for $\xi < 0$ and $T = \mu$ for $\xi > 0$, and the upstream and downstream far-field zones extend to $\xi_+ = 0$ and $\xi_- = 0^+$, respectively.

It can be shown that the terms in the left-hand side of Equation (A10) is of zeroth order in $\nu$, while the terms in the right-hand side are of higher order (e.g., Liberman 2008). Using the equilibrium condition (Equation (11)), one can show that the solution of the zeroth-order terms in Equation (A10) is given by

$$T' = \xi T \frac{dT}{d\xi}, \quad (A22)$$

where $\xi_T$ is a small constant representing a shift of the front in $\xi$ (Liberman 2008). This indicates that the thermal modes are absent except near the discontinuous front (i.e., $C_2 = C_3 = 0$).

By integrating Equations (A7)–(A9) across the front and by keeping the first-order terms in $\nu$, one obtains

$$\mathcal{J}_t = \frac{1}{\mu} \mathcal{J}_v', \quad \mathcal{M}_t = \mathcal{M}_v', \quad \nu_\mathcal{J}' = \frac{\mu - 1}{\sigma} \nu_\mathcal{J}', \quad (A23)$$

where the subscripts “-$\nu$” and “+$\nu$” indicate the values evaluated at $\xi = \xi_-$ and $\xi = \xi_+$, respectively. Inserting Equation (A23) into Equations (A20) and (A21) and using the hydrodynamic state vectors given in Equations (A15)–(A17), one obtains the quadratic equation

$$\sigma^2 + \frac{2\mu}{\mu + 1} \sigma - \frac{\mu(\mu - 1)}{\mu + 1} = 0, \quad (A24)$$

for $\sigma$, the positive (unstable) solution of which is identical to Equation (1).

A.3.2. General Cases

To obtain $\sigma$ for arbitrary $\nu$, we proceed by taking $\xi_-$ and $\xi_+$ sufficiently large enough to satisfy Equation (A11) for a background configuration at given $P_{eq}$. We then choose five trial values for $\sigma$ as well as $C_2, \ldots, C_5$, and integrate Equations (A7)–(A10) from $\xi = \xi_-$ to $\xi = 0$ to find $\mathbf{D}(0)$ and
U(0), respectively. We then check if the two vectors connect smoothly at \( \xi = 0 \). If the relative difference \( |D(0)/U(0) - 1| \) is larger than the tolerance limit (say, \( \sim 10^{-3} \)), we change \( \sigma \) and \( C_2, \cdots, C_5 \) iteratively based on the Newton–Raphson technique until the smoothly connecting solutions are obtained. We repeat the calculations by varying \( \nu \) to find a dispersion relation for given \( \rho_{eq} \). Figure 5 plots as the solid lines the growth rate \( \sigma \) for \( \rho_{eq}/k_B = 1700, 1800, \) and \( 1900 \) cm\(^{-3}\) K. Equation (1) is compared as the dotted lines, which agree very well with the true dispersion relations at \( \nu \ll 1 \). Note that the DLI is stabilized by thermal conduction at \( \nu > \nu_{crit} \sim 0.52-0.54 \).

REFERENCES

Bakes, E. L. O., & Tielens, A. G. G. M. 1994, ApJ, 427, 822
Begelman, M. C., & McKee, C. F. 1990, ApJ, 358, 375
Bell, J. B., Day, M. S., Rendleman, C. A., Woosley, S. E., & Zingale, M. 2004a, ApJ, 606, 1029
Bell, J. B., Day, M. S., Rendleman, C. A., Woosley, S. E., & Zingale, M. 2004b, JCoPh, 195, 677
Bychkov, V. V., Golberg, S. M., Liberman, M. A., & Eriksson, L. E. 1996, PhRvE, 54, 3713
Bychkov, V. V., & Liberman, M. A. 2000, PhR, 325, 115
Choi, E., & Stone, J. M. 2012, ApJ, 747, 86
Cox, D. P. 2005, ARA&A, 43, 337
Cox, D. P., & Smith, B. W. 1974, ApJL, 189, L105
Field, G. B. 1965, ApJ, 142, 531
Field, G. B., Goldsmith, D. W., & Habing, H. J. 1969, ApJL, 155, L149
Heiles, C. 2004, in ASP Conf. Ser. 323, Star Formation in the Interstellar Medium: In Honor of David Hollenbach, ed. D. Johnstone, F. C. Adams, D. N. C. Lin, D. A. Neufeeld, & D. A. Neufeeld (San Francisco, CA: ASP), 79
Heeneman, P., & Audit, E. 2007, A&A, 465, 431
Heyer, M. H., & Brunt, C. M. 2004, ApJL, 615, L45
Inoue, T., Inutsuka, S.-I., & Koyama, H. 2006, ApJ, 652, 1331
Iwasaki, K., & Inutsuka, S.-I. 2012, MNRAS, 423, 3638
Katz, N. 1992, ApJ, 391, 502
Kim, C.-G., Kim, W.-T., & Ostriker, E. C. 2008, ApJ, 681, 1148
Kim, C.-G., Kim, W.-T., & Ostriker, E. C. 2010, ApJ, 720, 1454
Koyama, H., & Inutsuka, S.-I. 2002, ApJL, 564, L97
Koyama, H., & Inutsuka, S.-I. 2004, ApJL, 602, L25
Krumholz, M. R., Stone, J. M., & Gardiner, T. A. 2007, ApJ, 671, 518
Landau, L. D., & Lifshitz, E. M. 1987, Fluid Mechanics (2nd ed.; New York: Pergamon)
Law, C. 2006, Combustion Physics (Cambridge: Cambridge Univ. Press)
Liberman, M. A. 2008, Introduction to Physics and Chemistry of Combustion: Explosion, Flame, Detonation (New York: Springer)
Liberman, M. A., Bychkov, V. V., Golberg, S. M., & Book, D. L. 1994, PhRvE, 49, 445
Mac Low, M.-M. 2013, Sci, 340, 1541
Mac Low, M.-M., & Klessen, R. S. 2004, RVMP, 76, 125
McKee, C. F., & Cowie, L. L. 1977, ApJ, 215, 213
McKee, C. F., & Ostriker, E. C. 2007, ARA&A, 45, 565
McKee, C. F., & Ostriker, J. P. 1977, ApJ, 218, 148
Meerson, B. 1996, RVMP, 68, 215
Niemeyer, J. C., & Hillebrandt, W. 1995, ApJ, 452, 779
Parker, E. N. 1953, ApJ, 117, 431
Parravano, A., Hollenbach, D. J., & McKee, C. F. 2003, ApJ, 584, 797
Pelce, P., & Clavin, P. 1982, IPM, 124, 219
Penston, M. V., & Brown, F. E. 1970, MNRAS, 150, 373
Piontek, R. A., & Ostriker, E. C. 2004, ApJ, 601, 905
Piontek, R. A., & Ostriker, E. C. 2005, ApJ, 629, 849
Piontek, R. A., & Ostriker, E. C. 2007, ApJ, 663, 183
Röpke, F. K., Niemeyer, J. C., & Hillebrandt, W. 2003, ApJ, 588, 952
Sharp, G. 2009, in Combustion Phenomena: Selected Mechanisms of Flame Formation, Propagation, and Extinction, ed. J. Jarosinski & B. Veyssiere (Boca Raton, FL: CRC Press), 67
Spitzer, L. 1958, RVMP, 30, 1108
Spitzer, L. 1962, Physics of Fully Ionized Gases (2nd ed.; New York: Interscience)
Stone, J. M. 2011, PhD thesis, Univ. Wisconsin-Madison
Stone, J. M., Gardiner, T. A., Teuben, P., Hawley, J. F., & Simon, J. B. 2008, ApJS, 178, 137
Stone, J. M., & Zweibel, E. G. 2010, ApJ, 724, 131
Travnikov, O. Y., Bychkov, V. V., & Liberman, M. A. 1999, PhFl, 11, 2657
Travnikov, O. Y., Liberman, M. A., & Bychkov, V. V. 1997, PhFl, 9, 3935
Vázquez-Semadeni, E., Gómez, G. C., Jappsen, A. K., et al. 2007, ApJ, 657, 870
Williams, F. A. 1985, Combustion Theory: The Fundamental Theory of Chemically Reacting Flow Systems (2nd ed.; Menlo Park, CA: Benjamin/Cummings)
Wolfire, M. G., McKee, C. F., Hollenbach, D., & Tielens, A. G. G. M. 2003, ApJ, 587, 278
Zel’dovich, Y. B. 1966, JAMTP, 7, 68
Zel’dovich, Y. B., Barenblatt, G. I., Librovich, V. B., & Makhviladze, G. M. 1985, The Mathematical Theory of Combustion and Explosions (New York: Consultants Bureau)
Zel’dovich, Y. B., & Pikel’ner, S. B. 1969, JETP, 29, 170