Stability of a Vortex in a Rotating Trapped Bose-Einstein Condensate

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1 Time-Dependent Gross-Pitaevskii Equation

The remarkable achievement of Bose-Einstein condensation in dilute trapped alkali-metal atomic gases has stimulated the (now successful) search for quantized vortices that are usually associated with external rotation. An essential feature of these condensates is the order parameter, characterized by a complex macroscopic wave function \( \Psi(\mathbf{r}, t) \). Theoretical descriptions of vortices in trapped low-temperature condensates have relied on the time-dependent Gross-Pitaevskii (GP) equation, which omits dissipation. For a trap rotating at angular velocity \( \Omega \) about \( \hat{z} \), it is a nonlinear Schrödinger equation

\[
\hat{i} \hbar \frac{\partial \Psi}{\partial t} = (T + V_{\text{tr}} - \Omega L_z + g|\Psi|^2) \Psi, \tag{1}
\]

where \( T = -\hbar^2 \nabla^2 / 2M \) is the kinetic-energy operator, \( V_{\text{tr}} = \frac{1}{2} M \sum_j \omega_j^2 x_j^2 \) is the harmonic trap potential, \( L_z = xp_y - yp_x \) is the \( z \) component of angular momentum, and \( g \equiv 4\pi\hbar^2 a / M \) characterizes the strength of the short-range interparticle potential (here, \( a \) is the positive \( s \)-wave scattering length for repulsive two-body interactions; typically \( a \) is a few nm). Current experiments involve “dilute” systems, so that the zero-temperature condensate contains nearly all \( N \) particles, with \( \int dV |\Psi|^2 \approx N \). For a steady solution, \( \Psi(\mathbf{r}, t) = \Psi(\mathbf{r}) e^{-i\mu t/\hbar} \), where \( \mu \) is the chemical potential.

1.1 Equivalent Hydrodynamics of Compressible Isentropic Fluid

If the condensate wave function is written as \( \Psi = e^{iS} |\Psi| \), the real and imaginary parts of Eq. (1) precisely reproduce the time-dependent irrotational hydrodynamics of a compressible isentropic fluid, written in terms of the number density \( n = |\Psi|^2 \) and the velocity potential \( \Phi = \hbar S / M \), where \( \mathbf{v} = \nabla \Phi \). In principle, the dynamics of a curved vortex line in a rotating trap follows directly from general hydrodynamics, but the trap potential and resulting nonuniform density complicate the problem considerably. Thus it is preferable to start with simple situations.

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1.2 Thomas-Fermi Limit for Large Condensates

For a noninteracting condensate, the macroscopic wave function is the ground state of the harmonic oscillator, with spatial extent \( d_j = (\hbar/M\omega_j)^{1/2} \) and \( j = x, y, z \); typically \( d_j \) is a few \( \mu \)m. The repulsive interactions act to expand the condensate, and the relevant dimensionless interaction parameter is \( Na/d_0 \), where \( d_0 = (d_xd_yd_z)^{1/3} \) is a suitable geometric mean. Recent experiments focus on the regime \( Na/d_0 \gg 1 \), when the quantum-mechanical energy associated with the density gradients is negligible. In this Thomas-Fermi (TF) limit \cite{13}, the nonrotating GP equation has the solution
\[
|\Psi_{TF}|^2 + V_{tr} \approx \mu ,
\]
with a parabolic density profile
\[
n_{TF}(r) \approx n_{TF}(0) \left( 1 - \frac{x^2}{R_x^2} - \frac{y^2}{R_y^2} - \frac{z^2}{R_z^2} \right) .
\]
(2)

Here \( R_j = (2\mu/M\omega_j^2)^{1/2} \gg d_j \) fixes the condensate’s dimensions and \( n_{TF}(0) = \mu/g \) is the central density. The (large) dimensionless ratios \( R_0^2/d_0^2 = 15Na/d_0 \) and \( \mu/\hbar\omega_0 = 1/2R_0^2/d_0^2 \) characterize the effect of the repulsive interactions, where \( \omega_0 \) and \( R_0 \) are appropriate geometric means.

2 Energy of a Vortex in a Large Rotating Trap

The time-dependent GP equation \cite{8} follows from a variational energy functional
\[
E(\Omega) = \int dV \Psi^* \left( T + V_{tr} + \frac{1}{2}g|\Psi|^2 - \Omega L_z \right) \Psi .
\]
(3)

If \( \Psi \) represents a straight singly quantized vortex in a rotating disk-shape trap \( (R_\perp \gg R_z) \), the TF density yields the increase in energy \( \Delta E(r_0, \Omega) \) associated with the vortex at the transverse position \( r_0 \) \cite{14}. For a nonrotating trap with \( \Omega = 0 \), this energy \( \Delta E(r_0, 0) \) decreases monotonically with lateral displacement. Energy conservation requires that the allowed motion is a precession at fixed trap potential (the trajectory is elliptical for an anisotropic trap). In the presence of weak dissipation, the vortex slowly spirals outward, lowering its energy.

For arbitrary \( \Omega \) and small lateral displacements, \( \Delta E(r_0, \Omega) \) has the form
\[
\Delta E(r_0, \Omega) \approx \frac{4\pi}{3} \frac{R_\perp n(0) \hbar^2}{M} \left\{ \ln \left( \frac{R_\perp}{\xi} \right) - \frac{2M\Omega R_\perp^2}{5\hbar} \right\}
- \frac{1}{2} \left( \frac{x_0^2}{R_x^2} + \frac{y_0^2}{R_y^2} \right) \left[ 3 \ln \left( \frac{R_\perp}{\xi} \right) - \frac{2M\Omega R_\perp^2}{\hbar} \right] ,
\]
(4)

where \( \xi = \hbar/(2M\mu)^{1/2} \) is the vortex-core radius and \( 2/R_\perp^2 = 1/R_x^2 + 1/R_y^2 \) defines the mean transverse radius of the condensate. In the TF limit, \( \xi \) is small, with \( \xi R_0 = d_0^2 \), ensuring a clear separation of length scales \( \xi \ll d_0 \ll R_0 \).

- For small positive \( \Omega \), \( \Delta E(r_0, \Omega) \) decreases with increasing lateral displacements so that a vortex will spiral outward in the presence of weak dissipation.
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- The curvature at \( r_0 = 0 \) vanishes at \( \Omega_m = \frac{3}{2}(\hbar/MR^2)\ln(R/\xi) \), signaling the onset of metastability. Since \( \Delta E(0, \Omega_m) > 0 \), this state is not truly stable.
- When \( \Omega > \Omega_m \), the trap center becomes a local minimum. For weak dissipation, a vortex that is slightly displaced from the center will spiral inward.
- When \( \Omega \) reaches \( \Omega_c = \frac{5}{2}(\hbar/MR^2)\ln(R/\xi) = \frac{5}{2}\Omega_m \), a central vortex becomes stable because \( \Delta E(0, \Omega_c) = 0 \). Hence \( \Omega_c \) is the thermodynamic critical angular velocity for vortex creation.
- For \( \Omega > \Omega_c \), the form of \( \Delta E(r_0, \Omega) \) shows that a vortex at the outer edge remains metastable, but the barrier for entry into the condensate decreases. Spontaneous nucleation of a vortex may eventually occur through a surface instability. The observed critical angular velocity for vortex creation is \( \approx 70\% \) higher than the predicted TF value \( \Omega_c \), in qualitative agreement with such a surface mechanism (but see Ref. 18 and Sec. 4.2 for an alternative explanation).

3 Small-Amplitude Excitation of a Vortex in a Rotating Trap

A macroscopic Bose condensate acts like an external particle source, in the sense that the same physical excited state (here labeled by \( j \)) can be achieved either by adding or by subtracting one particle. The actual eigenstates involve “quasiparticle” operators \( \alpha_j^\dagger \) and \( \alpha_j \) that are linear combinations of the two quantum-mechanical states. For a given normal mode, the resulting pair of coupled complex amplitudes \( u_j(r) \) and \( v_j(r) \) obey the Bogoliubov equations that determine the corresponding eigenfrequency \( \omega_j \). Imposing Bose-Einstein commutation relations \( [\alpha_j, \alpha_k^\dagger] = \delta_{jk} \) requires that these amplitudes obey the particular normalization \( \int dV (|u_j|^2 - |v_j|^2) = 1 \), and the resulting quasiparticle Hamiltonian reduces to a set of uncoupled harmonic oscillators \( \sum_j \hbar \omega_j \alpha_j^\dagger \alpha_j \), summed over all modes with positive normalization. In the simplest case of a uniform condensate, the eigenfrequencies are all positive, which ensures that the system is stable because the energy then has a lower bound. For a uniform condensate moving with velocity \( v \), however, some of the eigenfrequencies can become negative as soon as \( |v| \) exceeds the speed of sound, which corresponds to the instability associated with the Landau critical velocity. Physically, the system can spontaneously generate quasiparticles because the Hamiltonian is no longer bounded from below.

These general ideas have direct relevance to the stability of a singly quantized vortex in a rotating trapped condensate. For simplicity, it is convenient to consider an axisymmetric condensate in equilibrium with a trap rotating at an angular velocity \( \Omega \). In this case, states can be characterized by their azimuthal angular quantum numbers \( m_j \), and the transformation to rotating coordinates \( H \rightarrow H - \Omega L_z \) ensures that the eigenfrequencies \( \omega_j(\Omega) \) in the rotating frame are simply \( \omega_j(\Omega) = \omega_j - m_j \Omega \), where \( \omega_j \) is the corresponding eigenfrequency in the nonrotating frame.
3.1 Stability of a Vortex

The first numerical study of the Bogoliubov equations for a singly quantized vortex in a nonrotating trap found a single negative-frequency mode (the “anomalous mode”) \[21,22\], implying that the vortex in the condensate is unstable. Physically, this instability arises because the condensate with a vortex has a higher energy than the vortex-free condensate.

This situation is especially clear for a noninteracting Bose gas in an axisymmetric harmonic trap, when the first excited (vortex) state has an excitation energy \(\hbar \omega_\perp\) and an angular momentum \(\hbar\). The transition back to the true ground state involves the (negative) frequency \(-\omega_\perp\) and the (negative) change in angular quantum number \(-1\). More generally, the numerical analysis for small and medium interaction strength \(Na/d_\perp\)\[21\] found that the anomalous frequency \(\omega_a\) remained negative (with \(m_a = -1\) unchanged). In a frame rotating with angular velocity \(\Omega\), the anomalous frequency becomes \(\omega_a(\Omega) = \omega_a + \Omega\). Since \(\omega_a < 0\), the eigenfrequency \(\omega_a(\Omega)\) in the rotating frame rises toward 0 with increasing \(\Omega\) and vanishes at a critical rotation speed \(\Omega^* = -\omega_a = |\omega_a|\). The Bogoliubov description of small oscillations implies that a condensate with a singly quantized vortex is unstable for \(\Omega < \Omega^*\) but becomes stable for \(\Omega > \Omega^*\).

In contrast to the numerical study \[21\] for small and medium values of the dimensionless coupling parameter \(Na/d_0\), a direct perturbation analysis is feasible for a large condensate (\(Na/d_0 \gg 1\)) containing an axisymmetric singly quantized vortex (the TF limit). Detailed study of the Bogoliubov equations \[14\] yields the explicit expression for the anomalous frequency in the rotating frame \(\omega_a(\Omega) = -\frac{3}{2} \left(\frac{\hbar}{MR_\perp^2}\right) \ln \left(\frac{R_\perp}{\xi}\right) + \Omega\), which has the expected form. The resulting critical angular velocity \(\Omega^* = \frac{3}{2} \left(\frac{\hbar}{MR_\perp^2}\right) \ln \left(\frac{R_\perp}{\xi}\right)\) for the onset of local stability agrees with \(\Omega_m\) inferred from Eq. (4) for the onset of metastability.

3.2 Splitting of Normal-Mode Frequencies Caused by a Vortex

The ground-state condensate can sustain dynamical oscillations driven by the mean-field repulsive interaction (analogous to plasma oscillations in a charged medium). These normal modes become particularly simple in the TF limit of a large condensate \[23\], and experiments have confirmed the predictions for the lowest few modes in considerable detail \[11\]. For an axisymmetric condensate, the normal modes can be classified by their azimuthal quantum number \(m\), and modes with ±\(m\) are degenerate.

When the condensate contains a vortex, the asymmetric circulating flow affects the preceding normal modes. In particular, the originally degenerate modes are split by the Doppler shift of the local frequency (analogous to the splitting of magnetic sublevels in the Zeeman effect). In the TF limit, this small fractional splitting of the degenerate modes is proportional to \(|m|d_\perp^2/R_0^2\)\[24,25\], it has served to detect the presence of a vortex and to infer the circulation and angular momentum \[6,26\].
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4 Vortex Dynamics

At zero temperature, the time-dependent GP equation (1) determines the dynamics of the condensate in a rotating trap. A vortex line will move in response to the nonuniform trap potential, the external rotation, and its own local curvature. This problem is especially tractable in the TF limit, because the small vortex core radius \( \xi \) permits a clear separation of length scales; the method of matched asymptotic expansions yields an explicit expression for the local velocity of a vortex line \([27,28]\).

4.1 Dynamics of Straight Vortex

It is simplest to consider a straight vortex \([14]\), which applies to a disk-shape condensate with \( R_\perp \gg R_z \). Assume that the vortex is located near the trap center at a transverse position \( r_0(t) \). In this region, the trap potential does not change significantly on a scale of order \( \xi \). The solution proceeds in two steps.

(a) First, consider the region near the vortex core that is assumed to move with a transverse velocity \( V_\perp \hat{z} \). Transform to a co-moving frame centered on the vortex core, where the trap potential exerts a force proportional to \( \nabla_\perp V_{\text{tr}} \) evaluated at \( r_0(t) \). The resulting solution includes both the detailed core structure and the “asymptotic” region \( |r_\perp - r_{\perp 0}| \gg \xi \).

(b) Second, consider the region far from the vortex, where the core can be treated as a distant singularity. The short-distance behavior of this solution includes the region \( \xi \ll |r_\perp - r_{\perp 0}| \). The two solutions must agree in the common region, which determines the translational velocity \( V \) of the vortex line.

The details become intricate, but the final answer is elegant and physical:

\[
V = \frac{3\hbar}{4M\mu} \left[ \ln \left( \frac{R_\perp}{\xi} \right) - \frac{2M\Omega R_\perp^2}{3\hbar} \right] \hat{z} \times \nabla_\perp V_{\text{tr}},
\]

(5)

where \( R_\perp \) for an asymmetric trap is defined below Eq. (4). This expression has several important aspects

- The motion follows an equipotential line along the direction \( \hat{z} \times \nabla_\perp V_{\text{tr}} \), conserving energy, as appropriate for the GP equation at zero temperature. For an asymmetric harmonic trap, the trajectory is elliptical.
- For a nonrotating trap, the motion is counterclockwise, in the positive sense.
- With increasing external rotation, the translational velocity \( V \) decreases and vanishes at the special value \( \Omega_m = \frac{3}{4}(\hbar/MR_\perp^2) \ln(R_\perp/\xi) \) discussed below Eq. (4).
- For \( \Omega > \Omega_m \), the motion as seen in the rotating frame is clockwise.
- A detailed analysis of the normalization of the Bogoliubov amplitudes shows that the positive-norm state has the frequency

\[
\omega = \frac{2\omega_x \omega_y}{\omega_x^2 + \omega_y^2} (\Omega - \Omega_m),
\]

(6)
This normal-mode frequency is negative (and hence locally unstable) for \( \Omega < \Omega_m \), but it becomes locally stable for \( \Omega > \Omega_m \), in agreement with the discussion below Eq. (4).

4.2 Inclusion of Curvature

Equation (5) for the local velocity of a straight vortex oriented along \( \hat{z} \) can be generalized to include the possibility of a different orientation of the vector \( \hat{t} \) locally tangent to the vortex core. In addition, local curvature \( k \) of the vortex line defines a plane that includes both \( \hat{t} \) and the local normal \( \hat{n} \), inducing an additional translational velocity along the binormal vector \( \hat{b} \equiv \hat{t} \times \hat{n} \). A detailed analysis \[29\] yields the translational velocity of the element located at \( r_0 \)

\[
V(r_0) = -\frac{\hbar}{2M} \left( \hat{t} \times \nabla V_{tr}(r_0) + kb \right) \ln \left( \xi \sqrt{\frac{1}{R_x^2} + \frac{k^2}{8}} \right) + \frac{2\nabla V_{tr}(r_0) \times \Omega}{\nabla^2 V_{tr}(r_0)},
\]

(7)

where \( \nabla^2 \) is the Laplacian in the plane perpendicular to \( \Omega \). In the first term, \( |\Psi_T|^2 \) vanishes near the condensate boundary; hence \( \hat{t} \times \nabla V_{tr}(r_0) \) must also vanish there, implying that the vortex is locally perpendicular to the surface.

Equation (7) allows a study of the dynamics of small-amplitude displacements of the vortex from the \( z \) axis, when \( x(z,t) \) and \( y(z,t) \) obey coupled equations. In the limit \( \omega_z = 0 \), there is no confinement in the \( z \) direction, and the density is independent of \( z \). The resulting two-dimensional dynamics exhibits helical solutions that are linear combinations of two plane standing waves.

More generally, for \( \omega_z \neq 0 \), the density near the \( z \) axis has the TF parabolic form, and the solutions become more complicated. It is convenient to define the asymmetry parameters \( \alpha = R_x^2/R_z^2 \) and \( \beta = R_y^2/R_z^2 \), where \( \alpha > 1, \beta > 1 \) indicate a disk shape and \( \alpha < 1, \beta < 1 \) indicate a cigar shape.

- For a nonrotating trap with the special asymmetry values \( \alpha = 2/[n(n+1)] \) (here, \( n \) a positive integer), the effects of the nonuniform trap potential and the curvature just balance, and the condensate has stationary solutions with the vortex at rest in the \( xz \) plane. A disk-shape trap has no such states, and the first one occurs for the spherical trap with \( \alpha = 1 \). The next such state occurs for \( \alpha = \frac{1}{3} \), when the condensate is significantly elongated. Similar considerations for \( \beta \) apply to stationary states in the \( yz \) plane.
- For other values of \( \alpha \) and \( \beta \), solutions necessarily involve motion of the vortex line relative to the stationary condensate.
- Analytical solutions can be found for an extremely flat disk with \( \alpha \gg 1 \) and \( \beta \gg 1 \), reproducing the frequency \( \Omega_m \) found in Eq. (4).
- For small deformations of a vortex line in an axisymmetric trap with \( \alpha = \beta \), a disk-shape or spherical condensate (\( \alpha \geq 1 \)) has only a single (unstable) precessing normal mode with a negative frequency \( \omega_a < 0 \). In this case, an external rotation \( \Omega \geq \Omega_m = |\omega_a| \) stabilizes the vortex. For these geometries, \( \Omega_m \) is less than the thermodynamic critical value \( \Omega_c \). In a spherical
condensate, the one anomalous mode $|\omega_a|$ agrees with the observed vortex precession frequency seen in the JILA experiments [8,18].

- In contrast, an axisymmetric cigar-shape condensate has additional negative-frequency precessing modes, and $\Omega_m$ can exceed $\Omega_c$ for sufficiently elongated condensates; such behavior seems relevant for the ENS experiments [5,7,18], where $\Omega_m \approx 1.7\Omega_c$ is close to the observed rotation speed for creating the first vortex.

- For small axisymmetric deviations from a spherical trap, a straight vortex line can execute large-amplitude periodic trajectories. In this case, the vortex line becomes invisible when it tips away from the line of sight, and it then periodically returns to full visibility. Such revivals agree with preliminary observations at JILA [26].

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