ON CRITICALITY THEORY FOR ELLIPTIC MIXED BOUNDARY
VALUE PROBLEMS IN DIVERGENCE FORM

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ABSTRACT. The paper is devoted to the study of positive solutions of a second-order
linear elliptic equation in divergence form in a domain $\Omega \subseteq \mathbb{R}^n$ that satisfy an oblique
boundary condition on a portion of $\partial \Omega$. First, we study the degenerate mixed boundary
value problem

$$\begin{cases}
Pu = f & \text{in } \Omega, \\
Bu = 0 & \text{on } \partial \Omega_{\text{Rob}}, \\
u = 0 & \text{on } \partial \Omega_{\text{Dir}},
\end{cases} \tag{P,B}$$

where $\Omega$ is a bounded Lipschitz domain, $\partial \Omega_{\text{Rob}}$ is a relatively open portion of $\partial \Omega$, $\partial \Omega_{\text{Dir}}$ is a closed set of $\partial \Omega$, and $B$ is an oblique (Robin) boundary operator defined on $\partial \Omega_{\text{Rob}}$. In particular, we discuss the unique solvability of the above problem, the existence of a principal eigenvalue, and the existence of a positive minimal Green function. Then we establish a criticality theory for positive weak solutions of the operator $(P,B)$ in a general domain $\Omega$ with no boundary condition on $\partial \Omega_{\text{Dir}}$ and no growth condition at infinity. The paper generalizes and extends results obtained by Pinchover and Saadon (2002) for classical solutions of such a problem, where stronger regularity assumptions on the coefficients of $(P,B)$, and the boundary $\partial \Omega_{\text{Rob}}$ are assumed.

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1. Introduction

Let $P$ be a second-order, linear, elliptic operator, defined on a domain $\Omega \subseteq \mathbb{R}^n$, of the divergence form

$$Pu := -\text{div} \left[ A(x)\nabla u + u\tilde{b}(x) \right] + \tilde{b}(x) \cdot \nabla u + c(x)u \quad x \in \Omega, \tag{1.1}$$

with real measurable coefficients. Let $\partial \Omega_{\text{Rob}}$ be a relatively open Lipschitz-portion of $\partial \Omega$, and consider the oblique boundary operator

$$Bu := \beta(x) \left( A(x)\nabla u + u\tilde{b}(x) \right) \cdot \bar{n}(x) + \gamma(x)u \quad x \in \partial \Omega_{\text{Rob}}, \tag{1.2}$$

where $\xi \cdot \eta$ denotes the Euclidean inner product of the vectors $\xi, \eta \in \mathbb{R}^n$, $\bar{n}(x)$ is the outward unit normal vector to $\partial \Omega$ at $x \in \partial \Omega_{\text{Rob}}$, and $\beta, \gamma$ are real measurable functions defined on $\partial \Omega_{\text{Rob}}$. The boundary of $\Omega$ is then naturally decomposed to a disjoint union of its
Robin part $\partial\Omega_{\text{Rob}}$, and its Dirichlet part $\partial\Omega_{\text{Dir}}$. That is, $\partial\Omega = \partial\Omega_{\text{Rob}} \cup \partial\Omega_{\text{Dir}}$, where $\partial\Omega_{\text{Rob}} \cap \partial\Omega_{\text{Dir}} = \emptyset$.

The aim of the paper is to study properties of positive weak solutions of the equation $Pu = 0$ in $\Omega$ satisfying weakly the oblique boundary conditions $Bu = 0$ on $\partial\Omega_{\text{Rob}}$, under minimal regularity assumptions on $(P, B)$ and $\partial\Omega$ (see Assumptions 2.3). In particular, we study the principal generalized eigenvalue, Green function, and the general properties of the cone of positive solutions. Such a study appears under the name criticality theory.

The case $\partial\Omega_{\text{Rob}} = \emptyset$ has been studied extensively in the past four decades, see for example [2, 22, 26]. Moreover, criticality theory for classical solutions of the degenerate mixed boundary value problem

$$
\begin{aligned}
- \sum_{i,j=1}^{n} a^{ij}(x) D_{ij} u + \sum_{i=1}^{n} b^{i}(x) D_{i} u + c(x) u &= 0 \quad \text{in } \Omega, \\
\gamma(x) u + \beta(x) \frac{\partial u}{\partial \vec{n}} &= 0 \quad \text{on } \partial\Omega_{\text{Rob}},
\end{aligned}
$$

was established in [27] under the stronger regularity assumptions

$$a^{ij}, b^{i}, c \in C^{\alpha}(\Omega \setminus \partial\Omega_{\text{Dir}}), \quad \partial\Omega_{\text{Rob}} \subset C^{2,\alpha}, \quad \gamma, \beta \in C^{1,\alpha}(\partial\Omega_{\text{Rob}}), \quad \beta > 0, \gamma \geq 0. \quad (1.3)$$

We note that the results in [27] relies heavily on the Hopf boundary point lemma which holds on $\partial\Omega_{\text{Rob}}$ once the regularity assumptions (1.3) are assumed. Furthermore, criticality theory for the adjoint operator $(P^{*}, B^{*})$ was not discussed in [27], and in particular, properties of the positive Green function $G_{P, B}(x, y)$ as a function of $x$ and $y$ were not established. We mention also the related paper [10], where D. Daners studied the case of a mixed nondegenerate boundary value problem in divergence form under the assumptions that $\Omega$ is a Lipschitz bounded domain, the coefficients of $P$ and $B$ are bounded, and both $\partial\Omega_{\text{Rob}}$ and $\partial\Omega_{\text{Dir}}$ are relatively open and closed subsets of $\partial\Omega$.

In the present paper we generalize the results obtained in [27] (and also in [10]) by passing from the realm of classical solutions to the realm of weak solutions assuming significantly weaker regularity assumptions (see Assumptions 2.4 for bounded Lipschitz domains, and Assumptions 2.3 for the general case). We note that under these assumptions, the boundary point lemma does not necessarily hold. Furthermore, our regularity assumptions on $\partial\Omega_{\text{Rob}}$ force us to overcome a non-trivial geometric difficulty, namely, the existence of a bounded Lipschitz exhaustion of $\Omega \setminus \partial\Omega_{\text{Dir}}$ (see Definition 3.25). Existence of such an exhaustion is known for the case where $\partial\Omega = \partial\Omega_{\text{Rob}} \subset C^{3} [12]$.

The paper is organized as follows. In Section 2 we introduce some necessary notation and assumptions, and define the notion of weak solutions to the problem

$$
\begin{aligned}
Pu &= 0 \quad \text{in } \Omega, \\
Bu &= 0 \quad \text{on } \partial\Omega_{\text{Rob}}.
\end{aligned} \quad (1.4)
$$
Section 3 is devoted to the local theory. In particular, we study the coercivity of the bilinear form associated to the mixed boundary value problem (1.4) in the appropriate functional space $H^1_{\partial \Omega_{\text{Dir}}}(\Omega)$, where $\Omega$ is a bounded domain satisfying Assumptions 2.4. In addition, we recall some known local regularity results needed for the rest of the paper, and discuss the compactness of the resolvent operator, the generalized maximum principle, the existence of a principal eigenvalue, and the Harnack convergence principle. In Section 4, we develop a criticality theory when $(P, B)$ and $\Omega$ satisfy Assumptions 2.3. More precisely, we define criticality/subcriticality of the operator $(P, B)$ in $\Omega$, obtain characterizations of subcritical and critical operators, and prove the existence of a ground state for critical operators. Section 5 is devoted to the construction of the positive (minimal) Green function for a subcritical operator $(P, B)$ in $\Omega$. Finally, in Section 6, we discuss the symmetric case, where $\tilde{b} = \bar{b}$, and in particular, prove the appropriate Allegretto-Piepenbrink-type theorem (cf. [2, 26, 30]).

2. Preliminaries and notations

Let $\Omega$ be a domain in $\mathbb{R}^n$, $n \geq 2$, and let $\partial \Omega = \partial \Omega_{\text{Rob}} \cup \partial \Omega_{\text{Dir}}$, where $\partial \Omega_{\text{Rob}} \cap \partial \Omega_{\text{Dir}} = \emptyset$. We assume that $\partial \Omega_{\text{Rob}}$, the Robin-portion of $\partial \Omega$, is a relatively open subset of $\partial \Omega$, and $\partial \Omega_{\text{Dir}}$, the Dirichlet part of $\partial \Omega$, is a closed set of $\partial \Omega$. Moreover, if $\Omega$ is a bounded domain, we further assume that in the relative topology of $\partial \Omega$, we have $\text{int}(\partial \Omega_{\text{Dir}}) \neq \emptyset$. Throughout the paper we use the following notation and conventions:

- For any $\xi \in \mathbb{R}^n$ and a positive definite symmetric matrix $A \in \mathbb{R}^{n \times n}$, let $|\xi|_A := \sqrt{A\xi \cdot \xi}$, where $\xi \cdot \eta$ denotes the Euclidean inner product of $\xi, \eta \in \mathbb{R}^n$.
- From time to time we use the Einstein summation convention.
- The gradient of a function $f$ will be denoted either by $\nabla f$ or $Df$.
- For $x \in \mathbb{R}^n$, we denote $x = (x', x_n)$, where $x_n \in \mathbb{R}$.
- For $R > 0$ and $x \in \mathbb{R}^n$, we denote by $B_R(x)$ the open ball of radius $R$ centered at $x$.
- $\chi_B$ denotes the characteristic function of a set $B \subset \mathbb{R}^n$.
- We write $A_1 \Subset A_2$ if $A_1$ is a compact set, and $A_1 \subset A_2$.
- Let $\Omega$ be a domain and let $\Omega'$ be a subdomain. We write $\Omega' \Subset_R \Omega$ if $\Omega' \subset \overline{\Omega}$, $\partial \Omega' \cap \partial \Omega_{\text{Dir}} = \emptyset$, and $\partial \Omega' \cap \partial \Omega_{\text{Rob}} \subset \partial \Omega_{\text{Rob}}$ with respect to the relative topology on $\partial \Omega_{\text{Rob}}$.
- For a subdomain $\omega \Subset_R \Omega$, we define $\partial \omega_{\text{Rob}} := \text{int}(\partial \omega \cap \partial \Omega_{\text{Rob}})$, and $\partial \omega_{\text{Dir}} := \partial \omega \Delta \partial \omega_{\text{Rob}}$.
- For any $1 \leq p \leq \infty$, $p'$ is the Hölder conjugate exponent of $p$ satisfying $p' = p/(p-1)$.
- For $1 \leq p < n$, $p^* := np/(n-p)$ is the corresponding Sobolev critical exponent.
- For a Banach space $V$ over $\mathbb{R}$, we denote by $V^*$ the space of continuous linear maps from $V$ into $\mathbb{R}$.
- $C$ refers to a positive constant which may vary from line to line.
• Let \(g_1, g_2\) be two positive functions defined in \(\Omega\). We use the notation \(g_1 \asymp g_2\) in \(\Omega\) if there exists a positive constant \(C\) such that
\[
C^{-1} g_2(x) \leq g_1(x) \leq C g_2(x) \quad \text{for all } x \in \Omega.
\]

• Let \(g_1, g_2\) be two positive functions defined in \(\Omega\), and let \(x_0 \in \Omega\). We use the notation \(g_1 \sim g_2\) near \(x_0\) if there exists a positive constant \(C\) such that
\[
\lim_{x \to x_0} \frac{g_1(x)}{g_2(x)} = C.
\]

• For any real measurable function \(u\) and \(\omega \subset \mathbb{R}^n\), we denote
\[
\inf_{\omega} u := \text{ess inf}_{\omega} u, \quad \sup_{\omega} u := \text{ess sup}_{\omega} u, \quad u^+ := \max(0, u), \quad u^- := \max(0, -u).
\]

• \(\text{sign } u(x) = u(x)/|u(x)|\) if \(u(x) \neq 0\), and \(\text{sign } u(x) = 0\) if \(u(x) = 0\).

Let \(R, K > 0\), and let \(w\) be a real-valued Lipschitz continuous function defined on \(B'_R := \{x' \in \mathbb{R}^{n-1} : |x'| < R\}\) with
\[
|w(x') - w(y')| \leq K|x' - y'| \quad \forall x', y' \in B'_R(0),
\]
and \(w(0) \in (0, R)\). We denote
\[
\Omega[R] = \{x \in \mathbb{R}^n : x_n > w(x'), \ |x| < R\},
\]
\[
\sigma[R] = \{x \in \mathbb{R}^n : x_n \geq w(x'), \ |x| = R\},
\]
\[
\Sigma[R] = \{x \in \mathbb{R}^n : |x| < R, \ x_n = w(x')\}.
\]

**Definition 2.1** (Lipschitz and \(C^1\)-portions). Let \(x_0 \in \partial \Omega\) and \(R > 0\) such that \(\Omega[x_0, R] := \Omega \cap B_R(x_0)\) is a Lipschitz (resp., \(C^1\)) domain. The set \(\Sigma[x_0, R] = \partial \Omega \cap B_R(x_0)\) is called a Lipschitz (resp., \(C^1\))-portion of \(\partial \Omega\).

Further, we introduce some functional spaces. Denote \(\mathcal{D}(\Omega, \partial \Omega_{\text{Dir}}) := C^\infty_c(\overline{\Omega} \setminus \partial \Omega_{\text{Dir}})\). So, \(u \in \mathcal{D}(\Omega, \partial \Omega_{\text{Dir}})\) if \(u\) has compact support and
\[
\text{supp } u := \{x \in \Omega \mid u(x) \neq 0\} \subset \overline{\Omega} \setminus \partial \Omega_{\text{Dir}}.
\]

For \(q \geq 1\), we define \(W^{1,q}_{\partial \Omega_{\text{Dir}}}(\Omega)\) to be the closure of \(\mathcal{D}(\Omega, \partial \Omega_{\text{Dir}})\) with respect to the Sobolev norm of \(W^{1,q}(\Omega)\). We also consider the following spaces:
\[
L^q_{\text{loc}}(\overline{\Omega} \setminus \partial \Omega_{\text{Dir}}) := \{u \mid \forall x \in \Omega \cup \partial \Omega_{\text{Rob}}. \ \exists r_x > 0 \text{ s.t. } u \in L^q(\Omega \cap B_{r_x}(x))\},
\]
\[
W^{1,q}_{\text{loc}}(\overline{\Omega} \setminus \partial \Omega_{\text{Dir}}) := \{u \mid \forall x \in \Omega \cup \partial \Omega_{\text{Rob}}. \ \exists r_x > 0 \text{ s.t. } u \in W^{1,q}(\Omega \cap B_{r_x}(x))\}.
\]

In the case \(q = 2\) we omit the index \(q\) and write \(H^{1}_{\text{loc}}(\overline{\Omega} \setminus \partial \Omega_{\text{Dir}}) := W^{1,2}_{\text{loc}}(\overline{\Omega} \setminus \partial \Omega_{\text{Dir}})\).

**Remark 2.2.** For every Lipschitz subdomain \(\Omega' \Subset_R \Omega\) and \(1 < q < \infty\) the space \(W^{1,q}(\Omega')\) is a reflexive Banach space and therefore, \(W^{1,q}_{\partial \Omega_{\text{Dir}}}(\Omega')\) is reflexive as well.
Consider an elliptic operator $P$ of the form (1.1) and a Robin boundary operator $B$ of the form (1.2). Throughout the paper we assume the following regularity assumptions on $P$, $B$ and $\partial \Omega$:

**Assumptions 2.3.**
- $\partial \Omega = \partial \Omega_{\text{Rob}} \cup \partial \Omega_{\text{Dir}}$, where $\partial \Omega_{\text{Rob}} \cap \partial \Omega_{\text{Dir}} = \emptyset$, and $\partial \Omega_{\text{Rob}}$ is a relatively open subset of $\partial \Omega$.
- For each $x_0 \in \partial \Omega_{\text{Rob}}$ there exists $R > 0$ such that $\Sigma[x_0, R]$ is a $C^1$-portion.
- $A = (a^{ij})_{i,j=1}^n \in L^\infty(\overline{\Omega} \setminus \partial \Omega_{\text{Dir}}; \mathbb{R}^{n \times n})$ is a symmetric positive definite matrix valued function which is locally uniformly elliptic in $\overline{\Omega} \setminus \partial \Omega_{\text{Dir}}$, that is, for any compact $K \subset \overline{\Omega} \setminus \partial \Omega_{\text{Dir}}$ there exists $\Theta_K > 0$ such that
  \[
  \Theta_K^{-1} \sum_{i=1}^n \xi_i^2 \leq \sum_{i,j=1}^n a^{ij}(x)\xi_i\xi_j \leq \Theta_K \sum_{i=1}^n \xi_i^2 \quad \forall \xi \in \mathbb{R}^n \text{ and } \forall x \in K.
  \]
- $\tilde{\mathbf{b}}, \bar{\mathbf{b}} \in L^p(\overline{\Omega} \setminus \partial \Omega_{\text{Dir}}; \mathbb{R}^n)$, and $c \in L^{p/2}(\overline{\Omega} \setminus \partial \Omega_{\text{Dir}})$ for some $p > n$.
- $\beta > 0$, and $\gamma/\beta \in L^\infty(\partial \Omega_{\text{Rob}})$.

Moreover, sometimes we need to assume for bounded domains the following regularity requirements:

**Assumptions 2.4.**
- $\Omega$ is a bounded Lipschitz domain, and $\text{int}(\partial \Omega_{\text{Dir}}) \neq \emptyset$ in the relative topology of $\partial \Omega$.
- $\partial \Omega_{\text{Rob}} \subset \partial \Omega$ is a relatively open and locally Lipschitz subset of $\partial \Omega$.
- $\partial \Omega_{\text{Dir}} \subset \partial \Omega$ is a finite disjoint union of closures of Lipschitz-portions.
- $A = (a^{ij})_{i,j=1}^n \in L^\infty(\overline{\Omega}; \mathbb{R}^{n \times n})$ is a symmetric positive definite matrix valued function which is uniformly elliptic in $\overline{\Omega}$, that is, there exists $\Theta > 0$ such that
  \[
  \Theta^{-1} \sum_{i=1}^n \xi_i^2 \leq \sum_{i,j=1}^n a^{ij}(x)\xi_i\xi_j \leq \Theta \sum_{i=1}^n \xi_i^2 \quad \forall \xi \in \mathbb{R}^n \text{ and } \forall x \in \overline{\Omega}.
  \]
- $\tilde{\mathbf{b}}, \bar{\mathbf{b}} \in L^p(\Omega; \mathbb{R}^n)$, and $c \in L^{p/2}(\Omega)$ for some $p > n$.
- $\beta > 0$, and $\gamma/\beta \in L^\infty(\partial \Omega_{\text{Rob}})$.

**Remark 2.5.** The $C^1$-smoothness of $\partial \Omega_{\text{Rob}}$ in Assumptions 2.3 is needed only in two parts of the paper: in the construction of the Green function (see Section 5), and in the construction of a Lipschitz-exhaustion of $\overline{\Omega} \setminus \partial \Omega_{\text{Dir}}$ (See Appendix A). In the rest of the paper it is enough to assume that $\partial \Omega_{\text{Rob}}$ is locally Lipschitz.
Remark 2.6. If Assumptions 2,3 hold in $\Omega$, then Assumptions 2,4 hold in any Lipschitz subdomain $\omega \Subset R \Omega$ with $\partial \omega_{\text{Dir}} = \partial \omega \cap \Omega$.

Next, we define (weak) solutions and supersolutions of the boundary value problem

$$
\begin{cases}
Pu = 0 & \text{in } \Omega, \\
Bu = 0 & \text{on } \partial \Omega_{\text{Rob}}.
\end{cases}
$$

(P, B)

Definition 2.7. We say that $u \in H^1_{\text{loc}}(\Omega \setminus \partial \Omega_{\text{Dir}})$ is a weak solution (resp., supersolution) of the problem $(P, B)$ in $\Omega$, if for any (resp., nonnegative) $\phi \in D(\Omega, \partial \Omega_{\text{Dir}})$ we have

$$
B_{P,B}(u, \phi) := \int_{\Omega} \left[ (a^{ij}D_j u + u \tilde{b}^i)D_i \phi + (\tilde{b}^iD_i u + cu)\phi \right] dx + \int_{\partial \Omega_{\text{Rob}}} \frac{\gamma}{\beta} u \phi d \sigma = 0 \text{ (resp., } \geq 0),
$$

where $d \sigma$ is the $(n - 1)$-dimensional surface measure. In this case we write $(P, B)u = 0$ (resp., $(P, B)u \geq 0$). Furthermore, $u$ is a weak subsolution of $(P, B)$ in $\Omega$ if $-u$ is a supersolution of $(P, B)$ in $\Omega$.

The above definition should be compared with the following standard definition of weak (super)solutions in a domain $\Omega$.

Definition 2.8. We say that $u \in H^1_{\text{loc}}(\Omega)$ is a weak solution (resp., supersolution) of the equation $Pu = 0$ in $\Omega$ if for any (resp., nonnegative) $\phi \in C^\infty_0(\Omega)$

$$
\int_{\Omega} \left[ (a^{ij}D_j u + u \tilde{b}^i)D_i \phi + (\tilde{b}^iD_i u + cu)\phi \right] dx = 0 \text{ (resp., } \geq 0).
$$

Hence, any weak solution (resp., supersolution) of the equation $(P, B)u = 0$ in $\Omega$ is a weak solution (resp., supersolution) of $Pu = 0$ in $\Omega$. In the sequel, by a (super)solution of $(P, B)$ we always mean a weak (super)solution.

The formal $L^2$-adjoint of the operator $(P, B)$ is given by the operator $(P^*, B^*)$

$$
\begin{cases}
P^* u := -\text{div} \left[ A\nabla u + \tilde{b} u \right] + \tilde{b} \cdot \nabla u + cu, \\
B^* u := \beta (A\nabla u + u \tilde{b}) \cdot \bar{n} + \gamma u.
\end{cases}
$$

(2.1)

Indeed, if $A, \tilde{b}, \tilde{b}$ and $\partial \Omega$ are sufficiently smooth, then for any $\phi, \psi \in D(\Omega, \partial \Omega_{\text{Dir}})$ satisfying $B\psi = B^* \phi = 0$ on $\partial \Omega_{\text{Rob}}$ in the classical sense, we have

$$
\int_{\Omega} P(\psi) \phi dx = \int_{\Omega} \left( - \text{div} (A\nabla \psi + \tilde{b} \psi) + \tilde{b} \cdot \nabla \psi + c \psi \right) \phi dx =
$$

$$
\int_{\partial \Omega_{\text{Rob}}} \psi \phi \frac{\gamma}{\beta} \bar{n} d \sigma + \int_{\Omega} \left( (A\nabla \psi + \tilde{b} \psi) \cdot \nabla \phi - \psi \nabla \cdot (\tilde{b} \phi) + c \psi \phi \right) dx =
$$

$$
\int_{\Omega} \psi \left( - \text{div} (A\nabla \phi + \tilde{b} \phi) + \tilde{b} \cdot \nabla \phi + c \phi \right) dx = \int_{\Omega} \psi P^*(\phi) dx.
$$

Finally, we define the notion of nonnegativity of the operator $(P, B)$.
Definition 2.9. We denote the cones of all positive solutions and positive supersolutions of the equation \((P,B)u = 0 \text{ in } \Omega\) by \(\mathcal{H}_{P,B}^0(\Omega)\) and \(\mathcal{S}H_{P,B}(\Omega)\), respectively. The operator \((P,B)\) is said to be nonnegative in \(\Omega\) (in short, \((P,B) \geq 0\)) if \(\mathcal{H}_{P,B}^0(\Omega) \neq \emptyset\).

3. Local theory

In the present section we study the mixed value problem \([P,B]\) in a bounded Lipschitz domain \(\Omega \subset \mathbb{R}^n\). In particular, we discuss the coercivity in \(H^1_{\partial \Omega_{\text{Dir}}}(\Omega)\) of the bilinear form \(B_{P,B}\) associated with the operator \((P,B)\), the validity of a weak maximum principle, and the existence of a principal eigenfunction. Unless otherwise stated:

We assume throughout this section that Assumptions 2.4 are satisfied in \(\Omega\).

3.1. Coerciveness, solvability and compactness. In this subsection we discuss the coercivity of \(B_{P,B}\) in a bounded Lipschitz domain. First, we recall the trace inequality (see for example [21, Corollary 5.23]).

Lemma 3.1 (Trace inequality). Assume that \(\Omega\) is a bounded Lipschitz domain. Then there exists \(C(\Omega) > 0\) such that for any \(u \in H^1(\Omega)\) and \(\varepsilon > 0\)

\[
\int_{\partial \Omega} u^2 \, d\sigma \leq C(\Omega) \left( \varepsilon \| \nabla u \|_{L^2(\Omega)}^2 + \frac{1}{\varepsilon} \| u \|_{L^2(\Omega)}^2 \right).
\] (3.1)

The trace inequality implies the following Poincaré inequality:

Lemma 3.2 (Poincaré inequality). Assume that \(\Omega\) is a bounded Lipschitz domain. Then there exists \(C(\Omega) > 0\) such that

\[
\| u \|_{H^1(\Omega)} \leq C(\Omega) \left( \| \nabla u \|_{L^2(\Omega)}^2 + \int_{\partial \Omega_{\text{Rob}}} |u|^2 \, d\sigma \right) \quad \forall u \in H^1_{\partial \Omega_{\text{Dir}}}(\Omega). \tag{3.2}
\]

Proof. The Poincaré inequality [21, Lemma 5.22] implies that there exists \(C(\Omega) > 0\) such that

\[
\| \phi \|_{L^2(\Omega)}^2 \leq C(\Omega) \left( \| \nabla \phi \|_{L^2(\Omega)}^2 + \| \phi \|_{L^2(\Omega_{\text{Rob}})}^2 \right) \quad \forall \phi \in H^1(\Omega).
\]

Therefore, for any \(\phi \in \mathcal{D}(\Omega, \partial \Omega_{\text{Dir}})\) (which in particular, vanishes on a neighborhood of \(\partial \Omega_{\text{Dir}}\)), we have

\[
\| \phi \|_{H^1(\Omega)}^2 = \| \nabla \phi \|_{L^2(\Omega)}^2 + \| \phi \|_{L^2(\Omega_{\text{Rob}})}^2 \leq (1 + C) \left( \| \nabla \phi \|_{L^2(\Omega)}^2 + \| \phi \|_{L^2(\Omega_{\text{Rob}})}^2 \right) = C \left( \| \nabla \phi \|_{L^2(\Omega)}^2 + \| \phi \|_{L^2(\Omega_{\text{Rob}})}^2 \right).
\]

For a general \(u \in H^1_{\partial \Omega_{\text{Dir}}}(\Omega)\), let \(\{\phi_k\}_{k \in \mathbb{N}} \subset \mathcal{D}(\Omega, \partial \Omega_{\text{Dir}})\) be an approximating sequence which converges to \(u\) in \(H^1(\Omega)\). Then,

\[
\| u \|_{H^1(\Omega)} \leq \| u - \phi_k \|_{H^1(\Omega)} + \| \phi_k \|_{H^1(\Omega)} \leq \| u - \phi_k \|_{H^1(\Omega)} + C \left( \| \nabla \phi_k \|_{L^2(\Omega)} + \| \phi_k \|_{L^2(\Omega_{\text{Rob}})} \right).
\]
The trace inequality in \( H^1(\Omega) \) then implies
\[
\lim_{k \to \infty} \int_{\partial \Omega_{\text{Rob}}} |\phi_k|^2 \, d\sigma = \int_{\partial \Omega_{\text{Rob}}} |u|^2 \, d\sigma,
\]
and therefore, we are done. \( \square \)

Lemma 3.1 and Lemma 3.2 imply the coercivity of the bilinear form associated with the operator \((P + \mu, B)\) for a large enough constant \( \mu \).

**Theorem 3.3** (Coercivity and compactness). Assume that \( \Omega \) is a bounded Lipschitz domain, and that Assumptions 2.4 are satisfied in \( \Omega \). Consider the quadratic form
\[
J_{P,B}[u] := B_{P,B}(u, u) = \int_{\Omega} [(A \nabla u + u \tilde{b}) \cdot \nabla u + (\tilde{b} \cdot \nabla u + cu) u] \, dx + \int_{\partial \Omega_{\text{Rob}}} \frac{\gamma}{\beta} | \nabla u |^2 \, d\sigma, \quad u \in H^1_{\partial \Omega_{\text{Dir}}}(\Omega).
\]

Then
(i) The bilinear form \( B_{P,B}(\cdot, \cdot) \) is bounded on \( H^1_{\partial \Omega_{\text{Dir}}}(\Omega) \times H^1_{\partial \Omega_{\text{Dir}}}(\Omega) \), and there exists \( \mu_0 \in \mathbb{R} \) such that for any \( \mu > \mu_0 \) the quadratic form \( J_{P,B}[u] + \mu \int_{\Omega} |u|^2 \, dx \) is coercive on \( H^1_{\partial \Omega_{\text{Dir}}}(\Omega) \).
(ii) For any \( f \in L^2(\Omega) \) and \( \mu > \mu_0 \) there exists a unique function \( u_f = L^{-1}_\mu f \in H^1_{\partial \Omega_{\text{Dir}}}(\Omega) \) satisfying
\[
B_{P,B}(u_f, v) + \mu \int_{\Omega} u_f v \, dx = \int_{\Omega} f v \, dx \quad \forall v \in H^1_{\partial \Omega_{\text{Dir}}}(\Omega).
\]
(iii) For any \( \mu > \mu_0 \), \( L^{-1}_\mu : L^2(\Omega) \to L^2(\Omega) \) is a compact linear operator.

**Proof.** We follow the method in [31].
(i) Using the Sobolev embedding theorem, the trace inequality, together with Assumptions 2.4 and Young’s inequality, we obtain the following estimates for any \((u, v) \in H^1_{\partial \Omega_{\text{Dir}}}(\Omega) \times H^1_{\partial \Omega_{\text{Dir}}}(\Omega)\) and \( \varepsilon > 0 \):

1. \( \left| \int_{\Omega} a^{ij} D_i u D_j v \, dx \right| \leq \|a^{ij}\|_{L^\infty(\Omega)} \|\nabla u\|_{L^2(\Omega)} \|\nabla v\|_{L^2(\Omega)}, \)
2. \( \Theta^{-1} \|\nabla u\|_{L^2(\Omega)}^2 \leq \int_{\Omega} a^{ij} D_i u D_j u \, dx \leq \Theta \|\nabla u\|_{L^2(\Omega)}^2, \)
3. \( \left| \int_{\Omega} \tilde{b}^i u D_j v \, dx \right| \leq \|\tilde{b}\|_{L^\infty(\Omega)} \|u\|_{L^2(\Omega)} \|\nabla v\|_{L^2(\Omega)} \leq C \left( \|\nabla u\|_{L^2(\Omega)} \|\nabla v\|_{L^2(\Omega)} \right), \)
4. \( \left| \int_{\Omega} \tilde{b}^i u D_j u \, dx \right| \leq \|\tilde{b}\|_{L^\infty(\Omega)} \|u\|_{L^2(\Omega)} \|\nabla u\|_{L^2(\Omega)} \leq C \left( \varepsilon \|\nabla u\|_{L^2(\Omega)}^2 + \frac{1}{\varepsilon} \|u\|_{L^2(\Omega)}^2 \right), \)
5. \( \left| \int_{\Omega} \tilde{b}^i u D_j v \, dx \right| \leq \|\tilde{b}\|_{L^\infty(\Omega)} \|u\|_{L^2(\Omega)} \|\nabla v\|_{L^2(\Omega)} \leq C \left( \|\nabla u\|_{L^2(\Omega)} \|\nabla v\|_{L^2(\Omega)} \right), \)
6. \( \left| \int_{\Omega} \tilde{b}^i u D_j u \, dx \right| \leq \|\tilde{b}\|_{L^\infty(\Omega)} \|u\|_{L^2(\Omega)} \|\nabla u\|_{L^2(\Omega)} \leq C \left( \varepsilon \|\nabla u\|_{L^2(\Omega)}^2 + \frac{1}{\varepsilon} \|u\|_{L^2(\Omega)}^2 \right). \)
Hence, there exists $C > 0$ such that
\[
|\mathcal{B}_{P,B}(u,v)| \leq C \|u\|_{H^1(\Omega)} \|v\|_{H^1(\Omega)} \quad \forall u, v \in H^1_{\partial \Omega_{\text{dir}}}(\Omega).
\]

Moreover, the trace inequality (3.1), and the Poincaré type inequality (Lemma 3.2) imply
\[
\|\nabla u\|_{L^2(\Omega)}^2 \geq \frac{1}{C} \|u\|_{H^1(\Omega)}^2 - \int_{\partial \Omega_{\text{Rob}}} |u|^2 d\sigma.
\]

Combining this with the obtained estimates (1)-(9) for the terms of $J_{P,B}$, it follows that there exists $C = C(n, P, B, \Omega) > 0$ such that
\[
J_{P,B}[u] \geq (\Theta^{-1} - \varepsilon C) \|u\|_{H^1(\Omega)}^2 - \frac{C}{\varepsilon} \|u\|_{L^2(\Omega)}^2 \quad \forall u \in H^1_{\partial \Omega_{\text{dir}}}(\Omega).
\]

Hence, there exists $\mu$ and $\delta > 0$ such that
\[
J_{P,B}[u] + \mu \int_{\Omega} |u|^2 dx \geq \delta \|u\|_{H^1(\Omega)}^2 \quad \forall u \in H^1_{\partial \Omega_{\text{dir}}}(\Omega). \tag{3.3}
\]

(ii) For a given $f \in L^2(\Omega)$ the functional $f(v) := \int_{\Omega} f v dx$ is a bounded linear functional on $H^1_{\partial \Omega_{\text{dir}}}(\Omega)$. By part (i), $J_{P+\mu,B}$ is coercive on $H^1_{\partial \Omega_{\text{dir}}}$. Therefore, the Lax-Milgram theorem implies the existence and uniqueness of the required function $u = u_f$. Moreover, the mapping $f \mapsto u_f$ defines a bounded linear operator $L^{-1}_\mu : L^2(\Omega) \to H^1_{\partial \Omega_{\text{dir}}}(\Omega)$ by $L^{-1}_\mu f = u_f$.

(iii) For any $f \in L^2(\Omega)$ and $u = L^{-1}_\mu f$ we have by (3.3)

\[
\delta \|u\|_{H^1(\Omega)}^2 \leq \mathcal{B}_{P,B}(u,u) + \mu \int_{\Omega} |u|^2 dx = (f,u) \leq \|f\|_{L^2(\Omega)} \|u\|_{H^1(\Omega)}.
\]

Hence,
\[
\|L^{-1}_\mu f\|_{H^1(\Omega)} \leq C \|f\|_{L^2(\Omega)}.
\]

The compact embedding $H^1(\Omega) \hookrightarrow L^2(\Omega)$ implies the compactness of the embedding $H^1_{\partial \Omega_{\text{dir}}}(\Omega) \hookrightarrow L^2(\Omega)$. Hence, the operator $L^{-1}_\mu : L^2(\Omega) \to L^2(\Omega)$ is compact. \qed

3.2. Regularity near a Robin-portion. Next, we discuss regularity properties of solutions near a Lipschitz-portion of $\partial \Omega_{\text{Rob}}$.

**Lemma 3.4** ([21] Theorems 5.36 and 5.42). Fix $x_0 \in \partial \Omega_{\text{Rob}}$ and $R > 0$ such that $\Omega[x_0, 4R] \subset \Omega$ and $\Sigma[x_0, 4R] \subset \partial \Omega_{\text{Rob}}$. 


(1) Let $u$ be a nonnegative weak subsolution of
\[
\begin{align*}
Pu &= 0 \quad \text{in } \Omega[x_0, 4R], \\
Bu &= 0 \quad \text{on } \Sigma[x_0, 4R].
\end{align*}
\] (3.4)

Then for any $l > 1$ there exists $C = C(P, B, R, l, n) > 0$ such that
\[
\sup_{\Omega[x_0, R]} u \leq C(R^{-n/l}\|u\|_{L^l(\Omega[x_0, 2R])}).
\]

(2) Let $u$ be a nonnegative weak supersolution of (3.4). Then there exists $l > 1$ and $C = C(P, B, R, n) > 0$ such that the following weak Harnack inequality holds
\[
C(R^{-n/l}\|u\|_{L^l(\Omega[x_0, 2R])}) \leq \inf_{\Omega[x_0, R]} u.
\]

By combining both inequalities in Lemma 3.4 we obtain

**Corollary 3.5 (Up to the boundary Harnack inequality).** Fix $x_0 \in \partial\Omega_{Rob}$ and $R > 0$ such that $\Omega[x_0, 4R] \subset \Omega$ and $\Omega[x_0, 4R] \subset \partial\Omega_{Rob}$. Let $u$ be a nonnegative weak solution of (3.4). Then there exists a positive constant $C = C(P, B, R, n)$ such that
\[
\sup_{\Omega[x_0, R]} u \leq C \inf_{\Omega[x_0, R]} u.
\]

**Lemma 3.6 (Up to the boundary H"older continuity [21, Theorem 5.45]).** Fix $x_0 \in \partial\Omega_{Rob}$ and $R > 0$ such that $\Omega[x_0, 4R] \subset \Omega$ and $\Omega[x_0, 4R] \subset \partial\Omega_{Rob}$. Let $u$ be a weak solution of (3.4), and $0 < r < R$. Then there exist $C = C(P, B, R, n) > 0$ and $0 < \alpha < 1$ such that
\[
\operatorname{osc}_{\Omega[x_0, r]} u \leq C \left(\frac{r^\alpha}{R^\alpha} \sup_{\Omega[x_0, R]} u\right).
\]

In particular, if $u$ is bounded in $\Omega[x_0, R]$, then $u \in C^\alpha(\Omega[x_0, R])$.

As a corollary of Lemma 3.5 and Lemma 3.6 we obtain the following result.

**Lemma 3.7.** Fix $x_0 \in \partial\Omega_{Rob}$ and $R > 0$ such that $\Omega[x_0, 4R] \subset \Omega$ and $\Omega[x_0, 4R] \subset \partial\Omega_{Rob}$.

(i) Let $u$ be a nonzero nonnegative weak solution of (3.4). Then $u \in C^\alpha(\Omega[x_0, R])$ and $u > 0$ in $\Omega[x_0, R]$.

(ii) If $u$ is a positive weak supersolution of (3.4), then $\inf_{\Omega[x_0, R]} u > 0$.

**Proof.** (i) By part (1) of Lemma 3.4, $u$ is bounded in $\Omega[x_0, R]$, and consequently, Corollary 3.6 implies that $u \in C^\alpha(\Omega[x_0, R])$. Hence, Lemma 3.5 implies $\inf_{\Omega[x_0, R]} u \geq \frac{1}{C} \sup_{\Omega[x_0, R]} u > 0$.

(ii) Follows immediately from part (2) of Lemma 3.4.

In the sequel, we shall use the following terminology.

**Definition 3.8.** Let $u \in H^1_{\text{loc}}(\Omega \setminus \partial\Omega_{\text{Dir}})$, where $\Omega$ is a Lipschitz bounded domain. We say that $u \geq 0$ on $\partial\Omega_{\text{Dir}}$ if $u^- \in H^1_{\partial\Omega_{\text{Dir}}}(\Omega)$. 

3.3. **Maximum principles.** The proof of the following weak maximum principle is hinged on the method in [21, Theorem 5.15], though under a different setting and regularity assumptions:

**Lemma 3.9 (Weak maximum principle).** Consider the problem \((P,B)\) and suppose that Assumptions [2, 4] are satisfied in \(\Omega\). Assume further that
\[
\int_{\Omega} (\tilde{b}^i D_i \phi + c \phi) \, dx + \int_{\partial \Omega_{Rob}} \frac{\gamma}{\beta} \phi \, d\sigma \geq 0 \quad \forall \phi \in H^1_{\partial \Omega_{Dir}}(\Omega), \, \phi \geq 0. \tag{3.5}
\]
Let \(u \in H^1(\Omega)\) be a weak supersolution of \((P,B)\) in \(\Omega\) satisfying \(u^- \in H^1_{\partial \Omega_{Dir}}(\Omega)\). Then \(u \geq 0\) in \(\Omega\).

**Proof.** Assume by contradiction that \(\sup_{\Omega} (u^-) > 0\). For any nonnegative \(\phi \in H^1_{\partial \Omega_{Dir}}(\Omega)\)
\[
\int_{\Omega} \left[ (a^{ij} D_j u + u \tilde{b}^i) D_i \phi + (\tilde{b}^i D_i u) \phi \right] \, dx \geq - \int_{\Omega} c u \phi \, dx - \int_{\partial \Omega_{R}} \frac{\gamma}{\beta} u \phi \, d\sigma.
\]
Therefore,
\[
\int_{\Omega} \left[ a^{ij} D_j u D_i \phi + (\tilde{b}^i - \hat{b}^i) D_i u \phi \right] \, dx \geq - \int_{\Omega} (\tilde{b}^i D_i (u \phi) + c u \phi) \, dx - \int_{\partial \Omega_{R}} \frac{\gamma}{\beta} u \phi \, d\sigma.
\]
Consequently, by (3.5), for any nonnegative \(\phi \in H^1_{\partial \Omega_{Dir}}(\Omega)\) satisfying \(\phi u \leq 0\) in \(\Omega\), we have
\[
- \int_{\Omega} a^{ij} D_j u D_i \phi \, dx \leq \int_{\Omega} (\tilde{b}^i - \hat{b}^i) D_i u \phi \, dx. \tag{3.6}
\]
Let \(k\) satisfy \(- \sup_{\Omega} (u^-) < k < 0\) and consider the function \(\phi = - \min(u - k, 0)\), and let \(F_k = \{x \in \Omega : u(x) < k\}\) be the support of \(\phi\). Since \(u^- \in H^1_{\partial \Omega_{Dir}}(\Omega)\), it follows that \(\phi \in H^1_{\partial \Omega_{Dir}}(\Omega)\). Moreover,
\[
\phi u \leq 0 \text{ in } \Omega, \quad Du = - D\phi \text{ in } F_k, \text{ and } D\phi = 0 \text{ in } \Omega \setminus F_k.
\]
Consequently, the uniform ellipticity, Hölder’s inequality, and (3.6) imply
\[
\Theta^{-1} \| \nabla \phi \|_{L^2(F_k)}^2 \leq \int_{F_k} a^{ij} D_j \phi D_i \phi \, dx = - \int_{\Omega} a^{ij} D_j u D_i \phi \, dx \leq \int_{F_k} (\tilde{b}^i - \hat{b}^i) D_i u \phi \, dx \leq \| \tilde{b}^i - \hat{b}^i \|_{L^p(F_k)} \| \nabla u \|_{L^2(F_k)} \| \phi \|_{L^q(F_k)},
\]
where \(q = \frac{2p}{p-2} < \frac{2n}{n-2}\). The Gagliardo-Nirenberg-Sobolev inequality [21, Theorem 5.8] implies that for any \(N > 2 \geq n\)
\[
\left( \int_{\Omega} |\phi|^{\frac{2N}{N-2}} \, dx \right)^{\frac{N-2}{N}} \leq C \left( \int_{\Omega} |\phi|^2 \, dx \right)^{\frac{N-n}{N}} \left( \int_{\Omega} |D\phi|^2 \, dx \right)^{\frac{n}{N}}.
\]
Let $N$ be such that $\frac{2N}{N-2} > q$. Then,

$$\left(\int_{\Omega} |\phi|^{\frac{2N}{N-2}} \, dx\right)^{\frac{N-2}{N}} \leq C \left(\int_{\Omega} |\phi|^2 \, dx\right)^{\frac{N-2}{N}} \left(\int_{\Omega} |D\phi|^2 \, dx\right)^{\frac{N}{N}} \leq \tilde{C}\|\tilde{b} - \tilde{b}^i\|_{L^p(F_k)} \left(\int_{\Omega} |\phi|^2 \, dx\right)^{\frac{N-2}{2N}} \left(\int_{\Omega} |\phi|^q \, dx\right)^{\frac{N}{Nq}}. $$

By Hölder inequality we obtain for $s = 2N/(N-2)$

$$\left(\int_{\Omega} |\phi|^2 \, dx\right)^{\frac{N-2}{2N}} \left(\int_{F_k} |\phi|^q \, dx\right)^{\frac{N}{Nq}} \leq |\Omega|^{\frac{2N-2s}{N-2}} \left(\int_{\Omega} |\phi|^s \, dx\right)^{\frac{2N-2s}{N-2s}} |F_k|^{\frac{s-q}{Nq}} \left(\int_{\Omega} |\phi|^s \, dx\right)^{\frac{N}{Nq}} =$$

$$|\Omega|^{\frac{2N-2s}{N-2}} |F_k|^{\frac{s-q}{Nq}} \left(\int_{\Omega} |\phi|^s \, dx\right)^{\frac{N}{Nq}}.$$

If $\|\tilde{b} - \tilde{b}\|_{L^p(\Omega,\mathbb{R}^n)} \neq 0$, then $|F_k|$ is bounded from below by a positive constant independent of $k$, and therefore, the set $\cap_{F_k}$ has a positive measure. Consequently, letting $k \to -\sup_{\Omega}(u^-)$, it follows that $u$ attains a finite infimum on a set $F^*$ of positive measure. On $F^*$ we have $Du = 0$, however, $F^*$ contains $\cap_{F_k}$ and $Du \neq 0$ on $\cap_{F_k}$, a contradiction.

On the other hand, if $\|\tilde{b} - \tilde{b}\|_{L^p(\Omega,\mathbb{R}^n)} = 0$, it follows that $\phi = 0$, and then $u \geq k$. Letting $k \to 0$ implies $u \geq 0$. \hfill $\square$

**Remark 3.10.** The assumption in Lemma 3.9 that $\partial \Omega_{\text{Dir}}$ contains a nonempty Lipschitz-portion of $\partial \Omega$ is essential. Indeed, if $\partial \Omega_{\text{Dir}} = \emptyset$, then the lemma’s assumptions imply that either $u = \text{constant}$ or else $u \geq 0$, see [21, Theorem 5.15].

We recall the following (interior) weak Harnack inequality.

**Lemma 3.11 (Weak Harnack inequality [23 Theorem 3.13]).** Let Assumptions 2.3 hold in a domain $\Omega \subset \mathbb{R}^n$, and let $u$ be a weak supersolution of the equation $Pu = 0$ in $\Omega$. Assume that $u \geq 0$ in some open ball $B(r) \subset \Omega$ of radius $r > 0$. Then for any $\varrho, \tau \in (0,1)$ and $\gamma \in (0, n/(n-2))$, there is a positive constant $C > 0$ depending on $P, n, \gamma, \varrho, \tau$ and $r$, such that

$$\left(\frac{1}{|B(\varrho r)|} \int_{B(\varrho r)} u^\gamma \, dx\right)^{1/\gamma} \leq C \inf_{B(\tau r)} u. \quad (3.7)$$

Lemma 3.4 and Lemma 3.11 imply:

**Lemma 3.12 (Strong maximum principle).** Let Assumptions 2.3 hold in a domain $\Omega \subset \mathbb{R}^n$, and let $u$ be a nonnegative supersolution of $[P,B]$ in $\Omega$. Then either $u$ is strictly positive in $\Omega \cup \partial \Omega_{\text{Rob}}$, or else, $u = 0$ in $\Omega$. 


Definition 3.13. We say that the generalized maximum principle holds in a bounded domain Ω if for any \( u \in H^1(\Omega) \) satisfying \((P, B)u \geq 0 \) in Ω and \( u^- \in H_{\partial\Omega_{\text{Dir}}}^1(\Omega) \), we have \( u \geq 0 \) in Ω.

We recall the notion of the ground state transform which implies a generalized maximum principle that holds when assumption (3.5) is replaced by

\[
\mathcal{RSH}_{P,B}(\Omega) := \left\{ u \in \mathcal{SH}_{P,B}(\Omega) \mid u, u^{-1}, u P u \in L^\infty_{\text{loc}}(\bar{\Omega} \setminus \partial\Omega_{\text{Dir}}) \text{ and } \frac{u B u}{\beta} \in L^\infty_{\text{loc}}(\partial\Omega_{\text{Rob}}) \right\} \neq \emptyset.
\]

Definition 3.14. \( u \in \mathcal{RSH}_{P,B}(\Omega) \) is called a regular positive supersolution. Note that any \( u \in H_{P,B}^0(\Omega) \) is a regular positive supersolution of \((P, B)\) in Ω.

Definition 3.15 (Ground state transform). Assume that \( u \in \mathcal{RSH}_{P,B}(\Omega) \), and consider the bilinear form

\[
B_{P^u, B^u}(\phi, \psi) := B_{P,B}(u \phi, u \psi),
\]

where \( \phi, \psi \in \mathcal{D}(\Omega, \partial\Omega_{\text{Dir}}) \). We note that the form \( B_{P^u, B^u} \) corresponds to the elliptic operator \((P^u, B^u)\), where

\[
P^u(w) := -\frac{1}{u^2} \text{div}(u^2 A(x) \nabla w) + \left[ b - \tilde{b} \right] \nabla w + w \frac{P u}{u}, \quad \text{and} \quad B^u(w) = \beta A \nabla w \cdot \vec{n} + w \frac{B u}{u}. \tag{3.8}
\]

We say that \( w \) is a weak (resp., regular super) solution of \((P^u, B^u)\) in Ω, if \( w \in H^1_{\text{loc}}(\Omega \setminus \partial\Omega_{\text{Dir}}) \) and for any (resp., nonnegative) \( \phi \in \mathcal{D}(\Omega, \partial\Omega_{\text{Dir}}) \) we have

\[
B_{P^u, B^u}(w, \phi) = \int_{\Omega} \left[ a^{ij} D_i w D_j \phi + (\tilde{b}^i - \tilde{b}^i) D_i w \phi \right] u^2 \, dx + \int_{\Omega} w \phi (P u) u \, dx + \int_{\partial\Omega_{\text{Rob}}} w \phi \frac{u B u}{\beta} \, d\sigma = 0 \text{ (resp., } \geq 0) \tag{3.9}
\]

Note that \( B_{P^u, B^u} \) is defined on \( L^2(\Omega, u^2 dx) \).

Remark 3.16. (1) Let \( u, v \in \mathcal{H}_{P,B}^0(\Omega) \) (resp., \( u \in \mathcal{H}_{P,B}^0(\Omega), v \in \mathcal{SH}_{P,B}(\Omega) \)). Assume that \( v/u \in H^1_{\text{loc}}(\Omega \setminus \partial\Omega_{\text{Dir}}, u^2 dx) \). Then \( v/u \) is a weak positive (resp., super) solution of the equation \((P^u, B^u)w = 0 \) in Ω.

(2) If \( u \in \mathcal{H}_{P,B}^0(\Omega) \), then for any \( w \in H^1_{\text{loc}}(\Omega \setminus \partial\Omega_{\text{Dir}}) \) and \( \phi \in \mathcal{D}(\Omega, \partial\Omega_{\text{Dir}}) \) we have

\[
B_{P^u, B^u}(w, \phi) = \int_{\Omega} \left[ a^{ij} D_i w D_j \phi + (\tilde{b}^i - \tilde{b}^i) D_i w \phi \right] u^2 \, dx = B_{P,B}(uw, u\phi).
\]

Consequently, we have:
Corollary 3.17. Let Assumptions 2.4 hold in a bounded Lipschitz domain \( \Omega \), and let \( u \in \mathcal{RSH}_{P,B}(\Omega) \). Then \( u > 0 \) on \( \bar{\Omega} \setminus \partial \Omega_{\text{Dir}} \). Moreover, the weak maximum principle holds for the operator \((P^u, B^u)\) in any Lipschitz domain \( \Omega' \subseteq_R \Omega \).

Furthermore, if \( u \in \mathcal{H}^0_{P,B}(\Omega) \) and \( u, u^{-1} \in L^\infty(\Omega) \), then \( u \in C^\alpha(\bar{\Omega} \setminus \partial \Omega_{\text{Dir}}) \), and the weak maximum principle holds for the operator \((P^u, B^u)\) in \( \Omega \).

Remark 3.18. For the relationship between the validity of the generalized maximum principle and the existence of a positive solution such that \( u \) and \( u^{-1} \) are bounded, see [7] for the case \( \partial \Omega_{\text{Rob}} = \emptyset \), and [3] for the case \( \partial \Omega_{\text{Rob}} = \emptyset \).

Corollary 3.17 implies the following generalized maximum principle.

Lemma 3.19 (Generalized maximum principle). Let Assumptions 2.3 hold in a domain \( \Omega \), and let \( u \in \mathcal{RSH}_{P,B}(\Omega) \). If \( \Omega' \subseteq_R \Omega \) is a Lipschitz subdomain, then the generalized maximum principle holds for \((P, B)\) in \( \Omega' \). Moreover, if in addition, Assumptions 2.4 hold in a bounded Lipschitz domain \( \Omega \), \( u \in \mathcal{H}^0_{P,B}(\Omega) \), and \( u, u^{-1} \in L^\infty(\bar{\Omega} \setminus \partial \Omega_{\text{Dir}}) \). Then, the generalized maximum principle holds for the operator \((P^u, B^u)\) in \( \Omega \).

Proof. Apply the ground state transform \((P^u, B^u)\) (see Definition 3.15). By Corollary 3.17 the weak maximum principle holds for \((P^u, B^u)\) in \( \Omega' \). Consequently, if \( v \) is a supersolution of \((P, B)\) in \( \Omega' \) with \( v^- \in H^1_{\partial \Omega_{\text{Dir}'}(\Omega')} \), then \( v/u \) is a supersolution of \((P^u, B^u)\) in \( \Omega' \) with \( (v/u)^- \in H^1_{\partial \Omega_{\text{Dir}'}(\Omega')} \), and by Corollary 3.17, \( v/u \geq 0 \) in \( \Omega' \). Thus, \( v \geq 0 \) in \( \Omega' \).

The second statement of the lemma follows from the second part of Corollary 3.17.

Next, we present a priori interior estimates for positive supersolution.

Lemma 3.20. Let Assumptions 2.3 hold in a Lipschitz domain \( \Omega \), and \( f \in L^2_{\text{loc}}(\bar{\Omega} \setminus \partial \Omega_{\text{Dir}}) \). Assume that \( v \) is a (continuous) solution of the equation \( Pu = f \) in a domain \( \Omega \subseteq \mathbb{R}^n \) and satisfies \( Bu = 0 \) on \( \partial \Omega_{\text{Rob}} \) (in short, \( (P, B)v = f \) in \( \Omega \)). Then for any Lipschitz subdomains \( \omega \subseteq_R \omega' \subseteq_R \Omega \), there exists a constant \( C > 0 \) independent of \( v \), such that

\[
\|v\|_{H^1(\omega)} \leq C \left( \|v\|_{L^\infty(\omega')} + \|f\|_{L^2(\omega')} \right).
\]

Proof. The definition of \( v \) being a solution of \((P, B)u = f\) in \( \Omega \) reads as

\[
\int_{\Omega} [(a^{ij}D_i v + \nabla \tilde{b}^i)D_i \phi + (\tilde{b}'D_i v)\phi + cv\phi] d\sigma = \int_{\Omega} f\phi d\sigma \quad \forall \phi \in \mathcal{D}(\Omega, \partial \Omega_{\text{Dir}}).
\]

(3.10)

For any such \( \phi \), we may pick \( v\phi^2 \) as a test function in (3.11) to get

\[
\int_{\Omega} a^{ij}D_j vD_i v\phi^2 d\sigma = -2 \int_{\Omega} [a^{ij}D_i vD_j \phi v + v^2 \phi \tilde{b}'D_i \phi] d\sigma - \int_{\Omega} \phi^2 v\tilde{b}'D_i v d\sigma - \int_{\Omega} \phi^2 v \tilde{b}'D_i v d\sigma - \int_{\Omega} (\gamma/\beta) \phi^2 d\sigma + \int_{\partial \Omega_{\text{Rob}}} f v\phi^2 d\sigma.
\]
Let \( \omega \in \Omega \) and \( \omega' \in \Omega \) be Lipschitz subdomains of \( \Omega \), and let \( \phi \in D(\omega', \partial \omega'_{\text{Dir}}) \) satisfy
\[
0 \leq \phi \leq 1, \quad \phi = 1 \quad \text{in } \omega, \quad |\nabla \phi| \leq \frac{1}{\text{dist}(\partial \omega_{\text{Dir}}, \partial \omega'_{\text{Dir}})} \quad \text{in } \omega'.
\]
Then one obtains as in the proof of Theorem 3.3 that
\[
\Theta_{\omega}^{-1}\|\nabla v\|_{L^2(\omega)}^2 \leq \int_{\omega'} a^{ij} D_i v D_j v \phi^2 \, dx = -2 \int_{\omega'} [a^{ij} D_i v D_j \phi \phi v + \phi^2 \phi \tilde{b}^i D_i \phi] \, dx - \int_{\omega'} \phi^2 v \tilde{b}^i D_i v \, dx - \int_{\omega'} (\tilde{b}^i D_i v + cv) v \phi^2 \, dx - \int_{\partial \omega_{\text{Rob}}} (\gamma/\beta)v^2 \phi^2 d\sigma + \int_{\omega'} f v \phi^2 \, dx \leq C_1(P, B, \omega, \omega') \left( \varepsilon \|\nabla v\|_{L^2(\omega')}^2 + \frac{1}{\varepsilon} \|v\|_{L^2(\omega')}^2 \right) + \varepsilon \|v\|_{L^2(\omega')} + \frac{1}{\varepsilon} \|f\|_{L^2(\omega')}.
\]
Therefore, for \( \varepsilon > 0 \) sufficiently small there exists \( C = C(P, B, \omega, \omega', \varepsilon) \) such that
\[
\|v\|_{H^1(\omega)} \leq C\|v\|_{L^2(\omega')} + \frac{1}{\varepsilon} \|f\|_{L^2(\omega')} \leq C\|\omega'\|\|v\|_{L^\infty(\omega')} + \frac{1}{\varepsilon} \|f\|_{L^2(\omega')}.
\]

3.4. The spectrum of \((P, B)\). Next, we discuss spectral properties of the operator \((P, B)\) in a bounded Lipschitz domain \( \Omega \) satisfying Assumptions 2.4.

**Definition 3.21.** Let \((\tilde{P}, \tilde{B})\) be the realization of the operator \((P, B)\) in \( \Omega \) with the domain
\[
D(\tilde{P}, \tilde{B}) := \{ u \in H^1_{\partial \Omega_{\text{Dir}}}(\Omega) \subset L^2(\Omega) \mid (P, B)u = f \in L^2(\Omega) \},
\]
and let \(\sigma(\tilde{P}, \tilde{B})\) and \(\rho(\tilde{P}, \tilde{B})\) be the spectrum and the resolvent set of \((\tilde{P}, \tilde{B})\) in \(L^2(\Omega)\), respectively. We define the ‘bottom’ of the spectrum of \((\tilde{P}, \tilde{B})\) in \( \Omega \) by
\[
\Gamma = \Gamma(P, B, \Omega) := \inf \{ \text{Re}(\lambda) \mid \lambda \in \sigma(\tilde{P}, \tilde{B}) \}.
\]
For \( \lambda \in \rho(\tilde{P}, \tilde{B}), \) we denote the resolvent operator of \((\tilde{P} - \lambda, \tilde{B})\) in \(L^2(\Omega)\) by \(R_\Omega(\lambda)\). Set
\[
\Lambda = \Lambda(P, B, \Omega) := \sup \{ \lambda : \mathcal{B}_{P - \lambda, B}(\phi, \phi) \geq 0, \forall \phi \in \mathcal{D}(\Omega, \partial \Omega_{\text{Dir}}) \}.
\]
In the sequel we might omit the dependence on \(P, B, \Omega\) when there is no danger of ambiguity.

**Lemma 3.22.** \((\tilde{P}, \tilde{B}) : D(\tilde{P}, \tilde{B}) \to L^2(\Omega)\) is a closed operator.

**Proof.** Let \(\{u_k\}_{k \in \mathbb{N}} \subset D(\tilde{P}, \tilde{B})\) satisfying
\[
\lim_{k \to \infty} \|u_k - u\|_{L^2(\Omega)} = 0, \quad (\tilde{P}, \tilde{B})u_k = f_k \in L^2(\Omega), \quad \text{and} \quad \lim_{k \to \infty} \|f_k - f\|_{L^2(\Omega)} = 0.
\]
Then \((P + \mu, B)u_k = \mu u_k + f_k\) and thus \(u_k = (P + \mu, B)^{-1}(\mu u_k + f_k)\). By Theorem 3.3 the operator \((P + \mu, B)^{-1} : L^2(\Omega) \to H^1_{\partial \Omega_{\text{Dir}}}(\Omega)\) is linear and bounded. By letting \(k \to \infty\) we
obtain that \( u = (P + \mu, B)^{-1}(\mu u + f) \in H^1_{\partial\Omega_{\text{Dir}}}(\Omega). \)
Hence, \( u \in D(\tilde{P}, \tilde{B}) \) and \( (P, B)u = f. \)

**Remark 3.23.** Let Assumptions 2.4 hold in a domain \( \Omega. \) Recall that by Theorem 3.3, the resolvent operator \( R_\Omega(\lambda) \) is compact on \( L^2(\Omega) \) for large enough \( \lambda \in \mathbb{R}. \) Therefore, \( R_\Omega(\lambda) \) is compact for any \( \lambda \in \rho(\tilde{P}, \tilde{B}) \) [18 Proposition 2.7.6]. In particular, for any \( \lambda \in \rho(\tilde{P}, \tilde{B}), \) and \( f \in C_0^\infty(\Omega), \) the function \( R_\Omega(\lambda)f \) is a solution to the problem

\[
(P - \lambda)v = f \quad \text{in } \Omega, \\
Bu = 0 \quad \text{on } \partial\Omega_{\text{Rob}}, \quad v \in H^1_{\partial\Omega_{\text{Dir}}}(\Omega).
\]

Moreover, Lemma 3.22 and the compactness of \( R_\Omega(\lambda) \) implies that \( \sigma(\tilde{P}, \tilde{B}) = \sigma_{\text{point}}(\tilde{P}, \tilde{B}) \) (see for example, [18 Theorem 2.7.8]). In Theorem 3.40 we show that in fact, \( \Gamma \in \sigma_{\text{point}}(\tilde{P}, \tilde{B}). \)

**Corollary 3.24.** Fix \( \lambda \in \rho(\tilde{P}, \tilde{B}). \) Then \( \lambda' \) in an eigenvalue of \( (\tilde{P}, \tilde{B}) \) if and only if \( \lambda' = \mu^{-1} + \lambda, \) where \( \mu \neq 0 \) is an eigenvalue of \( R_\Omega(\lambda). \) In particular, the spectrum of \( (\tilde{P}, \tilde{B}) \) consists of only isolated eigenvalues of finite multiplicity.

**Proof.** By Remark 3.23, \( \sigma(\tilde{P}, \tilde{B}) = \sigma_{\text{point}}(\tilde{P}, \tilde{B}). \) Hence, by the definition of \( R_\Omega(\lambda), \) we see that \( \mu \in \sigma(R_\Omega(\lambda)) \) and \( \mu \neq 0 \) if and only if \( \lambda' := \mu^{-1} + \lambda \in \sigma_{\text{point}}(\tilde{P}, \tilde{B}). \)

### 3.5. Exhaustion and Harnack convergence principle

We proceed with Harnack convergence principle for a sequence of positive solutions in a general domain \( \Omega. \) First, we define a (Lipschitz) exhaustion of \( \overline{\Omega} \setminus \partial\Omega_{\text{Dir}}. \)

**Definition 3.25.** A sequence \( \{\Omega_k\}_{k \in \mathbb{N}} \) is called an exhaustion of \( \overline{\Omega} \setminus \partial\Omega_{\text{Dir}} \) if it is an increasing sequence of Lipschitz bounded domains such that \( \Omega_k \subseteq R \) \( \Omega_{k+1} \subseteq R \) \( \Omega, \) and \( \bigcup_{k \in \mathbb{N}} \overline{\Omega}_k = \overline{\Omega} \setminus \partial\Omega_{\text{Dir}}. \) For \( k \geq 1 \) we denote:

\[
\partial\Omega_{k,\text{Rob}} := \text{int}(\partial\Omega_k \cap \partial\Omega_{\text{Rob}}), \quad \partial\Omega_{k,\text{Dir}} := \partial\Omega_k \setminus \partial\Omega_{k,\text{Rob}}, \quad \Omega^*_k := (\overline{\Omega} \setminus \partial\Omega_{\text{Dir}}) \setminus \overline{\Omega}_k.
\]

**Remark 3.26.** In Appendix A we show that such an exhaustion exists once we impose the following stronger regularity assumption on \( \partial\Omega_{\text{Rob}}; \) for each \( x_0 \in \partial\Omega_{\text{Rob}} \) there exists \( R > 0 \) such that \( \Sigma[x_0, R] \) is a \( C^1 \)-portion of \( \partial\Omega_{\text{Rob}}. \) This extra regularity assumption is needed to ensure that \( \partial\Omega_k \) meets \( \partial\Omega_{\text{Rob}} \) in ‘good directions’ (see Definition A.1) which implies that \( \Omega_k \) is indeed a Lipschitz domain.

**Lemma 3.27** (Harnack convergence principle). Suppose that Assumptions 2.5 hold in \( \Omega, \) and let \( \{\Omega_k\}_{k \in \mathbb{N}} \) be an exhaustion of \( \overline{\Omega} \setminus \partial\Omega_{\text{Dir}}. \) Let \( x_0 \in \Omega \) be a fixed reference point. For each \( k \geq 1, \) let \( u_k \in H^1_{\text{loc}}(\Omega_k \cup \partial\Omega_{k,\text{Rob}}) \) be a positive solution of the problem

\[
\begin{align*}
Pu &= 0 \quad \text{in } \Omega_k, \\
Bu &= 0 \quad \text{on } \partial\Omega_{k,\text{Rob}},
\end{align*}
\]
satisfying \( u_k(x_0) = 1 \). Then the sequence \( \{u_k\}_{k \in \mathbb{N}} \) admits a subsequence converging locally uniformly in \( \Omega \setminus \partial \Omega_{\text{Dir}} \) to a positive solution \( u \in H^0_{P,B}(\Omega) \).

Moreover, the same conclusion holds if in (3.11) one replaces \( P \) by \( P + V_k \), with \( V_k \to 0 \) in \( L^{p/2}(\Omega) \), where \( p > n \).

**Proof.** Fix \( k \in \mathbb{N} \). By the local Harnack inequality \cite[Theorem 8.20.]{13}, and the up to the boundary Harnack inequality (Corollary 3.29), the sequence \( \{u_j\}_{j \geq k} \) is locally uniformly bounded in \( \Omega_k \). Moreover, Lemma 3.26 implies that this sequence is bounded in \( C^{\alpha}(\Omega_{k-1}) \). Hence, by Arzelá-Ascoli theorem \( \{u_j\}_{j \in \mathbb{N}} \) admits a subsequence \( \{u_{j_i}\}_{i \in \mathbb{N}} \) converging uniformly to \( \tilde{u}_k \in C^{\alpha}(\Omega_{k-1}) \). Moreover, by Lemma 3.20, \( \|\nabla u_{j_i}\|_{L^2(\Omega_{k-1})} \) is uniformly bounded. In fact, in light of Lemma 3.20 by extracting a subsequence, we may assume that \( \{u_{j_i}\}_{j \in \mathbb{N}} \) also converges weakly in \( H^1(\Omega_k) \) and almost everywhere to the positive function \( \tilde{u}_k \). In particular, for each \( 1 \leq i \leq n \), we clearly have that \( D_i \tilde{u}_k \) is a linear functional on \( L^2(\Omega_k) \) satisfying for all \( \phi \in C^\infty_0(\Omega_k) \)

\[
\int_{\Omega_k} D_i \tilde{u}_k \phi \, dx = \lim_{l \to \infty} \int_{\Omega_k} D_i u_{j_i} \phi \, dx = - \lim_{l \to \infty} \int_{\Omega_k} u_{j_i} D_i \phi \, dx = - \int_{\Omega_k} \tilde{u}_k D_i \phi \, dx. \tag{3.12}
\]

By Riesz representation theorem, \( D_i \tilde{u}_k \in L^2(\Omega_k) \) and by (3.12) \( \nabla \tilde{u}_k \) is indeed the weak gradient of \( \tilde{u}_k \). Therefore, we may apply the estimates in Theorem 3.3 to deduce that for any \( \phi \in \mathcal{D}(\Omega_k, \partial \Omega_{k,\text{Dir}}) \), \( \lim_{i \to \infty} B_{P,B}(u_{j_i}, \phi) = B_{P,B}(\tilde{u}_k, \phi) \). Hence, \( \tilde{u}_k > 0 \) is a positive solution of (3.11).

Since \( k \) was arbitrary, we may use the Cantor diagonal argument to extract a subsequence \( \{u_{j_{i_k}}\}_{j \in \mathbb{N}} \) of \( \{u_j\}_{j \in \mathbb{N}} \) converging locally uniformly in \( \overline{\Omega} \setminus \partial \Omega_{\text{Dir}} \) to a function \( u \in C^{\alpha}(\overline{\Omega} \setminus \partial \Omega_{\text{Dir}}) \cap H^1_{\text{loc}}(\Omega \setminus \partial \Omega_{\text{Dir}}) \) which is a positive solution of the problem \( (P,B) \) in \( \Omega \).

**3.6. The generalized principal eigenvalue.** Next, we introduce the notion of the generalized principal eigenvalue of \((P,B)\), and study its relation to \( \Gamma \), the ‘bottom’ of \( \sigma(\hat{P}, \hat{B}) \).

**Definition 3.28.** Let \( \Omega \) be a domain in \( \mathbb{R}^n \), and let \( 0 \leq V \in L^{p/2}(\Omega \setminus \partial \Omega_{\text{Dir}}) \). The **generalized principal eigenvalue** of \((P,B)\) in \( \Omega \) with respect to \( V \) is defined by

\[
\lambda_0 = \lambda_0(P, B, V, \Omega) := \sup\{\lambda \in \mathbb{R} \mid \mathcal{H}^0_{P-\lambda V,B}(\Omega) \neq \emptyset\}.
\]

Unless otherwise stated, we always assume that \( V = 1 \) in the definition of \( \lambda_0 \), and we usually omit the dependence on \( P, B, V \) and \( \Omega \).

The Harnack convergence principle (Lemma 3.27) implies:

**Corollary 3.29.**

\[
\lambda_0(P, B, V, \Omega) := \max\{\lambda \in \mathbb{R} \mid \mathcal{H}^0_{P-\lambda V,B}(\Omega) \neq \emptyset\}, \quad \text{i.e.} \quad \mathcal{H}^0_{P-\lambda_0 V,B}(\Omega) \neq \emptyset.
\]

**Lemma 3.30** (Generalized maximum principle, cf. \cite[Theorem 2.6]{2}). Let \( \Omega \) be a domain in \( \mathbb{R}^n \). For any \( \lambda \leq \lambda_0(P, B, 1, \Omega) \), the operator \((P - \lambda, B)\) satisfies the generalized maximum principle in any Lipschitz subdomain \( \Omega' \Subset_R \Omega \). Hence, \((\infty, \lambda_0(P, B, 1, \Omega)) \subset \rho(\hat{P}, \hat{B}, \Omega') \).
Lemma 3.31. Let \( \lambda \) be a Lipschitz domain, and suppose that \( u \in H^1_{\text{loc}}(\Omega \setminus \partial \Omega_{\text{Dir}}) \) satisfies \((P - \lambda, B)u \geq 0\) in \( \Omega \) for some \( \lambda \in \mathbb{R} \). Then \((P - \lambda, B)u \leq 0\) in \( \Omega \).

Proof. Without loss of generality, we may assume that \( \lambda = 0 \). Clearly \( u^- \) and \(|u|\) belong to \( H^1_{\text{loc}}(\Omega \setminus \partial \Omega_{\text{Dir}}) \). For any \( \varepsilon > 0 \) define \( u_\varepsilon = \sqrt{u^2 + \varepsilon^2} \). Then \( u_\varepsilon, u/u_\varepsilon \) belong to \( H^1_{\text{loc}}(\Omega \setminus \partial \Omega_{\text{Dir}}) \) and \( u_\varepsilon \to |u| \) as \( \varepsilon \to 0 \). Next, let \( 0 \leq \phi \in \mathcal{D}(\Omega, \partial \Omega_{\text{Dir}}) \). Then

\[
A(x)\nabla u_\varepsilon \cdot \nabla \phi = A(x) \frac{u}{u_\varepsilon} \nabla u \cdot \nabla \phi \\
\leq A(x) \nabla u \cdot \frac{u}{u_\varepsilon} \nabla \phi + A(x) \nabla u \cdot \left( 1 - \frac{u^2}{u_\varepsilon^2} \right) \nabla u = A(x) \nabla u \cdot \nabla \left( \frac{u}{u_\varepsilon} \phi \right) \quad \text{a.e in } \Omega.
\]

Consider the function \( \phi_\varepsilon := \frac{1}{2}(1 - \frac{u}{u_\varepsilon}) \), then

\[
A(x) \frac{\nabla (u_\varepsilon - u)}{2} \cdot \nabla \phi \leq -A(x) \nabla u \cdot \nabla \phi_\varepsilon.
\]

Note that \( 0 \leq \phi_\varepsilon \in H^1_{\text{loc}}(\Omega) \cap L^\infty(\Omega) \). Since \((P - \lambda, B)u \geq 0\) it follows that

\[-\int \Omega A(x) \nabla u \cdot \nabla \phi_\varepsilon \, dx \leq \int \Omega (u \tilde{b} \cdot \nabla \phi_\varepsilon + \tilde{b} \cdot \nabla u \phi_\varepsilon + cu \phi_\varepsilon) \, dx + \int_{\partial \Omega_{\text{Rob}}} \frac{\gamma}{\beta} u \phi_\varepsilon \, d\sigma.
\]

Therefore,

\[
\int \Omega A(x) \nabla (u_\varepsilon - u) \cdot \nabla \phi \, dx \leq \int \Omega (u \tilde{b} \cdot \nabla \phi_\varepsilon + \tilde{b} \cdot \nabla u \phi_\varepsilon + cu \phi_\varepsilon) \, dx + \int_{\partial \Omega_{\text{Rob}}} \frac{\gamma}{\beta} u \phi_\varepsilon \, d\sigma. \tag{3.13}
\]

Notice that \( \phi_\varepsilon \to (\text{sign } u^-) \phi \) as \( \varepsilon \to 0 \) and \( 0 \leq \phi_\varepsilon \leq \phi \). Moreover,

\[
\lim_{\varepsilon \to 0} \nabla \phi_\varepsilon = \lim_{\varepsilon \to 0} \frac{1}{2} \left( 1 - \frac{u}{u_\varepsilon} \right) \nabla \phi - \frac{u_\varepsilon \nabla u - u^2 \nabla u}{u_\varepsilon^2} = (\text{sign } u^-) \nabla \phi.
\]

Letting \( \varepsilon \to 0 \) in (3.13), we see that

\[
\int \Omega A(x) \nabla u^- \cdot \nabla \phi \, dx \leq -\int \Omega (u \tilde{b} \cdot \nabla \phi + \tilde{b} \cdot \nabla u^- \phi + cu^- \phi) \, dx - \int_{\partial \Omega_{\text{Rob}}} \frac{\gamma}{\beta} u^- \phi \, d\sigma. \tag{\*)
\]
We need the following Kato-type inequality for the operator \((P, B)\).

**Lemma 3.32** (cf. [2, Lemma 2.8]). Let \(u \in H^1_{\text{loc}}(\overline{\Omega} \setminus \partial \Omega_{\text{Dir}}; \mathbb{C})\) satisfy \((P, B)u = f\) in \(\Omega\), where \(f \in L^1_{\text{loc}}(\overline{\Omega} \setminus \partial \Omega_{\text{Dir}})\). Then

\[
(P, B)|u| \leq \Re \left( \frac{\overline{u}}{|u|} f \right).
\] (3.14)

**Proof.** Assume first that \(\overline{b} = \tilde{b} = c = 0\). Let \(\varepsilon > 0\) and let \(0 \leq \phi \in D(\Omega, \partial \Omega_{\text{Dir}})\). Then \(\nabla u = \frac{\nabla u \overline{u} + u \nabla \overline{u}}{2u} \). Moreover, (cf. [16, (5.6)]), \(|\nabla u|_A^2 - |\nabla u|_{\overline{A}}^2 \geq 0\) a.e in \(\Omega\). Thus,

\[
\int_{\partial \Omega} A \nabla u \cdot \nabla \phi \, d\sigma + \int_{\partial \Omega} \frac{\gamma}{\beta} \overline{u} \frac{u}{2u} \, d\sigma = \int_{\partial \Omega} \left(\gamma \frac{\overline{u}}{2u} \phi \right) A(x) \nabla u \cdot \nabla \left(\frac{\overline{u}}{2u} \phi \right) \nabla u \cdot \nabla \left(\frac{\overline{u}}{2u} \phi \right) + \int_{\partial \Omega} \frac{\gamma}{\beta} \overline{u} \frac{u}{2u} \, d\sigma = \int_{\partial \Omega} \left(\frac{\overline{u}}{u} f \phi - \frac{2}{u} (|\nabla u|_A^2 - |\nabla u|_{\overline{A}}^2) \phi \right) \, d\sigma \leq \int_{\Omega} \Re \left( \frac{\overline{u}}{u} f \right) \phi \, d\sigma.
\]

Letting \(\varepsilon \to 0\) we obtain (3.14).

The general case follows as in to the proof of Lemma 2.8 in [2]. \(\square\)

An immediate corollary is the following lemma.

**Lemma 3.33** (cf. [2, Lemma 2.8]). Let \(u \in H^1_{\text{loc}}(\overline{\Omega} \setminus \partial \Omega_{\text{Dir}})\) satisfy \((P - \lambda, B)u = 0\) in \(\Omega\). Then

\[
(P, B)|u| \leq \Re(\lambda)|u|.
\]

We use Lemma 3.31 to obtain a seemingly stronger generalized maximum principle which holds for \(\lambda < \Gamma(P, B, \Omega)\), where \(\Omega\) is a Lipschitz bounded domain. Since \((P - \lambda, B)\) is coercive for any \(\lambda < \Lambda\), it follows that \(\Lambda \leq \Gamma(P, B, \Omega)\).

**Lemma 3.34.** Let Assumptions 2.4 hold in a domain \(\Omega\), and assume that \(\lambda < \Gamma\). Then \((P - \lambda, B)\) satisfies the generalized maximum principle in \(\Omega\).

**Proof.** We apply Agmon’s method in [2, Lemma 2.6], where the Dirichlet problem is considered. Fix \(u \in H^1_{\text{loc}}(\overline{\Omega} \setminus \partial \Omega_{\text{Dir}})\) with \(u^- \in H^1_{\partial \Omega_{\text{Dir}}}(\Omega)\) such that \((P - \lambda, B)u \geq 0\) in \(\Omega\). We need to show that \(u^- = 0\) in \(\Omega\).

Assume first that \(\lambda < \Lambda\). Lemma 3.31 implies that \((P - \lambda, B)u^- \leq 0\) in \(\Omega\), and therefore,

\[
\mathcal{B}_{P - \lambda, B}(u^-, u^-) \leq 0.
\]
On the other hand, the definition of $\Lambda$ implies
\[ B_{P-\lambda,B}(u^-, u^-) \geq (\Lambda - \lambda)\|u^-\|^2_{L^2}, \]
and therefore $u^- = 0$. So, the generalized maximum principle holds true for $(P - \lambda, B)$ in $\Omega$ for $\lambda < \Lambda$.

Next, specify $\lambda' < \lambda < \lambda < \Gamma$, and let $u$ as above. Then by Lemma 3.31 we have,
\[ (P - \lambda', B)u^- \leq (\lambda - \lambda')u^- \quad \text{in } \Omega. \tag{3.15} \]
By the generalized maximum principle for $(P - \lambda', B)$ in $\Omega$ (proved above), we have that
\[ u^- \leq (\lambda - \lambda')R_\Omega(\lambda')u^- \quad \text{in } \Omega. \]
By (3.15) it follows that
\[ (P - \lambda')u^- \leq (\lambda - \lambda')^2R_\Omega(\lambda')u^- \quad \forall k \in \mathbb{N}. \tag{3.16} \]
The inequality $\|R_\Omega(\lambda')\| \leq \frac{1}{\text{dist}(\lambda', \sigma(\tilde{P}, \tilde{B}))} \leq \frac{1}{|\Gamma - \lambda'|}$ implies
\[ |\lambda - \lambda'|\|R_\Omega(\lambda')\| \leq \frac{|\lambda - \lambda'|}{|\Gamma - \lambda'|} < q < 1. \]
Letting $k \to 0$ in (3.16), we obtain $u^- = 0$. \hfill \Box

**Remark 3.35.** Suppose that $\Gamma \notin \sigma(\tilde{P}, \tilde{B})$, then for $\lambda' < \Gamma$ we have
\[ |\lambda' - \Gamma| < \text{dist}(\lambda', \sigma(\tilde{P}, \tilde{B})). \]
Therefore, the discreteness of $\sigma(\tilde{P}, \tilde{B})$ implies that there exists $\varepsilon > 0$ such that for any $\lambda < \Gamma + \varepsilon$ we have $|\lambda' - \lambda||R_\Omega(\lambda)| < \tilde{q} < 1$. Repeating the proof of Lemma 3.34, it follows that $(P - \Gamma - \varepsilon/2, B)$ satisfies the generalized maximum principle in $\Omega$. We note that in fact, $\Gamma \in \sigma(\tilde{P}, \tilde{B})$, see Theorem 3.40 below.

3.7. **Principal eigenvalue.** In the present subsection we prove the existence of a principal eigenvalue in a Lipschitz bounded domain (cf. [24, 5]).

**Definition 3.36.** We say that $\lambda_c$ is a principal eigenvalue of the operator $(\tilde{P}, \tilde{B})$ in $\Omega$ and $u_c$ is its associated principal eigenfunction if $u_c \in H^0_{P-\lambda_c,B}(\Omega) \cap H^1_{\text{Dir}}(\Omega)$.

To establish the existence of $\lambda_c$ we use the following version of Krein-Rutman theorem.

**Theorem 3.37** (Krein-Rutman-type theorem [8, Corollary 2.2.3]). Let
\[ L^2_+(\Omega) = \{f \in L^2(\Omega) \mid f \geq 0\}. \]
Suppose that $T : L^2(\Omega) \to L^2(\Omega)$ is a compact linear operator mapping $L^2_+(\Omega)$ into itself, and there exists $0 \neq e \in L^2_+(\Omega)$ and a constant $\varrho > 0$ such that
\[ (T - \varrho)e \in L^2_+(\Omega). \]
Then the spectral radius of $T$, denoted by $r(T)$, is positive and there exists a nontrivial $v \in L^2_+(\Omega)$ satisfying $Tv = r(T)v$.

**Theorem 3.38.** Let Assumptions 2.4 hold in a bounded domain $\Omega$. Then the operator $(\hat{P}, \hat{B})$ admits a principal eigenvalue $\lambda_c$ with a positive principal eigenfunction $u_c$. Hence, $H^0_{\hat{P}-\lambda, B}(\Omega) \cap H^{1}_{\partial \Omega \text{Dir}}(\Omega) \neq \emptyset$. In particular, $\lambda_c \leq \lambda_0$.

**Proof.** Consider the operator $T_\lambda := R_{\Omega}(\lambda)$, where $\lambda \in \mathbb{R}$. By Corollary 3.24 $T_\lambda : L^2(\Omega) \to L^2(\Omega)$ is a compact operator. Moreover, the generalized maximum principle (which holds by Lemma 3.34) implies that for all $f \in L^2(\Omega)$, we have $T_\lambda f \in L^2(\Omega)$. Next, we let $0 \leq \phi \in C_0^\infty(\Omega)$. By the generalized maximum principle and the strong maximum principle, $T_\lambda \phi > 0$ in $\Omega$. Let $\frac{1}{T_\lambda \phi} = 1 - \frac{1}{T_\lambda \phi}$. By definition, $T_\lambda \phi - \phi \phi = T_\lambda \phi \geq 0$ in $\Omega \setminus \text{supp}(\phi)$. Moreover, $\frac{\phi}{2\|\phi\|_{L^\infty(\Omega)}} \left( \inf_{\text{supp}(\phi)} T_\lambda \phi \right) \leq T_\lambda \phi \leq T_\lambda \phi$. Hence, $T_\lambda \phi - \phi \phi \in L^2_+(\Omega)$. By Theorem 3.37 $r(T_\lambda) > 0$ and there exists $0 \leq u_c \in L^2(\Omega) \cap D(\hat{P}, \hat{B})$ satisfying $T_\lambda u_c = r(T_\lambda)u_c$. in $\Omega$. Therefore, $u_c \in H^1_{\partial \Omega \text{Dir}}(\Omega)$ and $Pu_c = \frac{1}{r(T_\lambda) + \lambda} u_c$, $Bu = 0$ on $\partial \Omega \text{Rob}$. By the strong maximum principle, $u_c > 0$ in $\Omega$. Thus, $\lambda_c = \lambda_c(\lambda) = r(T_\lambda)^{-1} + \lambda$ is a principal eigenvalue of $(\hat{P}, \hat{B})$. □

**Remark 3.39.** The simplicity of the principal eigenvalue of $(P, B)$ in $\Omega$, when $(P, B)$ is nonselfadjoint and nonsmooth, remains open even in the case $\partial \Omega \text{Rob} = \emptyset$.

We have

**Theorem 3.40.** Let Assumptions 2.4 hold in a bounded domain $\Omega$. Then $\Gamma = \lambda_c \in \sigma(\hat{P}, \hat{B})$. In particular, $\lambda_c$ does not depend on $\lambda$.

Moreover, for any $\lambda \in \mathbb{R}$, the operator $(P - \lambda, B)$ satisfies the generalized maximum principle in $\Omega$ if and only if $\lambda < \Gamma = \lambda_c$.

For the proof of Theorem 3.40 we need the following auxiliary lemma.

**Lemma 3.41.** Let Assumptions 2.4 hold in a bounded domain $\Omega$, and assume that the generalized maximum principle for $(P - \lambda, B)$ holds in $\Omega$ for some $\lambda \in \mathbb{R}$. Then $\lambda \leq \Gamma$.

**Proof of Lemma 3.41.** It is enough to show that if $(P - \lambda, B)$ satisfies the generalized maximum principle in $\Omega$, then for any eigenvalue $z \in \sigma(\hat{P}, \hat{B})$ we have $\lambda < \text{Re}(z)$. Indeed,
let $u \in H^1_{\partial \Omega_{\text{Dir}}} (\Omega), u \neq 0$, satisfy $(P - \lambda, B)u = 0$ with $\text{Re}(z) \leq \lambda$. By Lemma 3.33, $(P - \lambda, B)|u| \leq 0$, and therefore, $(P - \lambda, B)(-|u|) \geq 0$. By the generalized maximum principle, $u \equiv 0$, which is a contradiction. □

Proof of Theorem 3.40. First, we claim that $\Gamma \in \sigma (\tilde{P}, \tilde{B})$. Otherwise, Remark 3.35 implies now that there exists $\varepsilon > 0$ such that $(P - \Gamma - \varepsilon, B)$ satisfies the generalized maximum principle in $\Omega$, a contradiction to Lemma 3.41.

Corollary 3.24 implies that $\Gamma = \mu - 1 + \lambda$, where $\lambda$ is sufficiently small such that $(P - \lambda, B)$ is coercive in $\Omega$ for any $\lambda' \leq \lambda$, and $\mu \neq 0$ is a real eigenvalue of $R_{\Omega}(\lambda)$. Since $\lambda$ belongs to the resolvent set, $\mu > 0$, and we have

$$\Gamma = \mu - 1 + \lambda \geq 1 + r(R_{\Omega}(\lambda)) + \lambda = \lambda_c.$$  

By the definition of $\Gamma$ we also have $\Gamma \leq \lambda_c$. Hence, $\Gamma = \lambda_c$, and $\lambda_c$ does not depend on $\lambda$.

Since $\Gamma = \lambda_c$, it follows that the generalized maximum principle does not hold for $\lambda = \Gamma$. In addition, Lemma 3.41 implies that the generalized maximum principle does not hold for $\lambda > \Gamma$. On the other hand, by Lemma 3.34, the generalized maximum principle holds for $\lambda < \Gamma$, and the second assertion of the theorem follows. □

Corollary 3.29 and Lemma 3.41 imply the following.

Lemma 3.42. Let Assumptions 2.3 hold in a domain $\Omega$, and let $u \in H^0_{P,B}(\Omega)$, or more generally, assume that $u$ is a regular positive supersolution of $(P,B)$ in $\Omega$. Then:

(1) for any Lipschitz bounded domain $\Omega_0 \subset \subset \Omega$, we have $0 < \lambda_c(P,B,\Omega_0) \leq \lambda_0(P,B,\Omega_0)$;

(2) if $\{\Omega_k\}_{k \in \mathbb{N}}$ is an exhaustion of $\overline{\Omega} \setminus \partial \Omega_{\text{Dir}}$, then the sequences $\{\lambda_c(P,B,\Omega_k)\}_{k \in \mathbb{N}}$ and $\{\lambda_0(P,B,\Omega_k)\}_{k \in \mathbb{N}}$ are strictly decreasing.

Proof. (1) We claim that $\inf_{\overline{\Omega}} u > 0$. Indeed, let $\Omega_0 \subset \subset \Omega \subset \subset \Omega$ be a Lipschitz subdomain. For any $x \in \overline{\Omega} \cap \overline{\Omega_0}$ there exists an open neighborhood $x \in B_x \subset \tilde{\Omega}$ such that $\inf_{B_x} u \geq C_x > 0$. Since $\Omega_0$ is precompact we may subtract a finite subcover $\{B_{x_j}\}_{j=1}^m$ from which we obtain $\inf_{\Omega_0} u \geq \min_{1 \leq j \leq m} C_{x_j} > 0$. By Lemma 3.19, $(P,B)$ satisfies the generalized maximum principle in $\Omega_0$, which in light of Theorem 3.40 implies that $\lambda_c(P,B,\Omega_0) > 0$. Part (2) follows from Part (1) and Lemma 3.19. □

We conclude this section with the following lemma.

Lemma 3.43. Let Assumptions 2.3 hold in $\Omega$, and let $\{\Omega_k\}_{k \in \mathbb{N}} \subset \Omega$ be an exhaustion of $\overline{\Omega} \setminus \partial \Omega_{\text{Dir}}$. Denote $\tilde{\lambda}_0(\Omega) := \lim_{k \to \infty} \lambda_0(\Omega_k)$, and $\tilde{\lambda}_c(\Omega) := \lim_{k \to \infty} \lambda_c(\Omega_k)$. Then

$$\lambda_0(\Omega) = \tilde{\lambda}_0(\Omega) = \tilde{\lambda}_c(\Omega).$$
Assume further that Assumptions 2.4 hold in $\Omega$, and that $\lim_{k \to \infty} u_{c,k} = u_c$, where $u_{c,k}$ and $u_c$ are principal eigenfunctions in $\Omega_k$ and $\Omega$ respectively. Then

$$\lambda_0(\Omega) = \lambda_c(\Omega) = \Gamma(P, B, \Omega).$$

**Proof.** By Lemma 3.42, the sequences $\{\lambda_c(\Omega_k)\}_{k \in \mathbb{N}}$ and $\{\lambda_0(\Omega_k)\}_{k \in \mathbb{N}}$ are monotone decreasing and satisfy

$$\lambda_0(\Omega) < \lambda_c(\Omega_k) \leq \lambda_0(\Omega_k).$$

Let $u_{c,k} \in \mathcal{H}^0_{P-\lambda_c(\Omega_k)}$ be a principal eigenfunction satisfying $u_{c,k}(x_0) = 1$, and let $u_k \in \mathcal{H}^0_{P-\lambda_0(\Omega_k)}$ satisfy $u_0(x_0) = 1$. Then the Harnack convergence principle implies that $\lambda_0(\Omega) \leq \bar{\lambda}_c(\Omega) \leq \lambda_0(\Omega)$. On the other hand, by the definition of $\lambda_0(\Omega)$ we have $\lambda_0(\Omega) \leq \bar{\lambda}_c(\Omega) \leq \lambda_0(\Omega)$.

Under the further assumptions we clearly have $\bar{\lambda}_c(\Omega) = \lambda_c(\Omega)$, and by the first part, we have, $\lambda_0(\Omega) = \lambda_c(\Omega) = \Gamma(P, B, \Omega)$. \qed

4. Criticality theory

In the present section we discuss a criticality theory for $(P, B)$ in a domain $\Omega$. More precisely, we study the relation between the validity of the generalized maximum principle of $(P, B)$ in bounded Lipschitz subdomains of $\Omega$, the existence of positive (super)solutions of $(P, B)u = 0$ in $\Omega$, and the nonnegativity of the generalized principal eigenvalue. Moreover, we define the notion of positive solutions of minimal growth at infinity in $\Omega$, criticality and subcriticality of $(P, B)$ in $\Omega$ and discuss some related properties. Unless otherwise stated, we assume throughout the section that Assumptions 2.3 are satisfied in $\Omega$.

4.1. Characterization of $\lambda_0$. Let $\lambda_0$ be the generalized principal eigenvalue (see Definition 3.28). As a consequence of the results in the previous section, we obtain the following characterization of $\lambda_0$.

**Theorem 4.1.** The following assertions are equivalent:

1. $\mathcal{H}^0_{P,B}(\Omega) \neq \emptyset$. In other words, $\lambda_0(P, B, 1, \Omega) \geq 0$.
2. $(P, B)$ admits a regular positive supersolution in $\Omega$.
3. $\lambda_0(P, B, 1, \Omega') > 0$ for any Lipschitz subdomain $\Omega' \Subset_R \Omega$.
4. $(P, B)$ satisfies the generalized maximum principle in any Lipschitz subdomain $\Omega' \Subset_R \Omega$.

**Proof.** (1) $\Longrightarrow$ (2) : Obvious.
(2) $\Longrightarrow$ (3) : Follows from Lemma 3.42
(3) $\Longrightarrow$ (4) : Follows from Corollary 3.29 and Lemma 3.30
(4) $\Longrightarrow$ (1) : By Lemma 3.41, $\Gamma(P, B, 1, \Omega') \geq 0$ for any Lipschitz subdomain $\Omega' \Subset_R \Omega$.

Since the generalized maximum holds for any $\lambda < \Gamma$, it follows that for any $\lambda < 0$ the resolvent operator is positive. Let $\{\Omega_k\}_{k \in \mathbb{N}}$ be an exhaustion of $\overline{\Omega} \setminus \partial \Omega_{Dir}$ and fix $x_0 \in \Omega_1$. 


For $k \geq 1$, denote by $R_k$ the resolvent operator of $(\tilde{P} + 1/k, \tilde{B})$ in $\Omega_k$, and let $f_k \in C_0^\infty(\Omega_k \setminus \Omega_{k-1})$ be a nonzero nonnegative function. Using the Harnack convergence principle (Lemma 3.27), it follows that the sequence
\[
\left\{ \frac{R_k(f_k)}{R_k(f_k)(x_0)} \right\}_{k \in \mathbb{N}}
\]
admits a subsequence converging to $u \in H^0_{P,B}(\Omega)$. □

Theorem 4.1 clearly implies the strict monotonicity of the generalized principal eigenvalues in bounded subdomains (cf. Lemma 3.43).

**Corollary 4.2.** Let $\Omega_1$ and $\Omega_2$ be two Lipschitz nonempty subdomains of $\Omega$ satisfying $\Omega_1 \Subset R \Omega_2 \Subset R \Omega$. Then
\[
\lambda_0(P, B, \Omega) < \lambda_0(P, B, \Omega_2) < \lambda_0(P, B, \Omega_1).
\]
In particular, if $\{\Omega_k\}_{k \in \mathbb{N}}$ is an exhaustion of $\overline{\Omega} \setminus \partial \Omega_{\text{Dir}}$, then $\lambda_0(\Omega_k) \to \lambda_0(\Omega)$.

**Proof.** By the monotonicity of $\lambda_0$ with respect to increasing subdomains domain, it follows that $\lambda_0(\Omega) \leq \lambda_1$, where $\lambda_1 := \lim_{k \to \infty} \lambda_0(\Omega_k)$. We need to prove that $\lambda_0(\Omega) = \lambda_1$. Suppose that $\lambda_0(\Omega) < \lambda_1$, then by the Harnack convergence principle (Lemma 3.27), it follows that $H^0_{P-\lambda_1,B}(\Omega) \neq \emptyset$, which is a contradiction to the definition of $\lambda_0(\Omega)$. □

As a consequence of Theorem 4.1 we obtain the existence of positive (super)solutions to the following mixed boundary value problem.

**Lemma 4.3.** Assume that $(P, B) \geq 0$ in $\Omega$, and let $\Omega' \Subset R \overline{\Omega} \setminus \partial \Omega_{\text{Dir}}$ be a Lipschitz subdomain of $\Omega$. Let $K \Subset \Omega'$ be a Lipschitz subdomain.

Then for any nonzero nonnegative function $f \in C_0^\infty(\Omega' \setminus K)$ there exists a unique positive weak solution $u$ to the problem
\[
\begin{align*}
Pw &= f \quad \Omega' \setminus K, \\
Bw &= 0 \quad \partial \Omega'_{\text{Rob}}, \\
\text{Trace}(w) &= 0 \quad \partial \Omega'_{\text{Dir}} \cup \partial K.
\end{align*}
\] (4.1)

**Proof.** By Theorem 3.40
\[
\lambda_\epsilon(P, B, 1, \Omega' \setminus K) = \Gamma(P, B, \Omega' \setminus K) > 0.
\]
Therefore, $(\tilde{P}, \tilde{B})$ is invertible in $\Omega' \setminus K$, and the corresponding resolvent operator is positive. Hence, there exists a unique nonnegative solution $u \in H^1_{(\partial \Omega' \cup \partial K) \setminus \partial \Omega_{\text{Rob}}}(\Omega' \setminus K)$ to problem (4.1). The strict positivity of $u$ follows from the strong maximum principle. □

4.2. Minimal Growth. Next, we introduce the notion of positive solution of minimal growth for $(P, B)$ (cf. [2, 27]). In the sequel $\{\Omega_k\}_{k \in \mathbb{N}}$ is an exhaustion of $\overline{\Omega} \setminus \partial \Omega_{\text{Dir}}$. 
Definition 4.4. A function $u$ is said to be a positive solution of $(P,B)$ of minimal growth in a neighborhood of infinity in $\Omega$ if $u \in \mathcal{H}^{0}_{P,B}(\Omega^*_j)$ for some $j \geq 1$ and for any $l > j$ and $v \in C(\Omega^*_l \cup \partial \Omega_{l,\text{Dir}}) \cap \mathcal{S} \mathcal{H}_{P,B}(\Omega^*_l)$, $u \leq v$ on $\partial \Omega_{l,\text{Dir}}$ $\implies$ $u \leq v$ on $\Omega^*_l$.

Lemma 4.5. Assume that $(P,B) \geq 0$ in $\Omega$. Then for any $x_0 \in \Omega$ the equation $(P,B)u = 0$ admits (up to a multiplicative constant) a unique positive solution $v$ in $\Omega \setminus \{x_0\}$ of minimal growth in a neighborhood of infinity in $\Omega$.

Proof. Existence: Fix an exhaustion $\{\Omega_k\}_{k \in \mathbb{N}}$ of $\overline{\Omega} \setminus \partial \Omega_{\text{Dir}}$. We may assume that $x_0 \in \Omega_1$. For $k \in \mathbb{N}$, let $B_k = B(x_0, \delta/k)$, where $\delta > 0$ is sufficiently small such that $B_1 \Subset \Omega_1$. Let $f_k \in C^0_0(B_{k-1} \setminus B_k)$ be a nonzero nonnegative function. By Lemma 4.4, there exists a unique positive solution $v_k$ to the problem

$$\begin{cases}
Pw = f_k & \Omega_k \setminus B_k, \\
Bw = 0 & \partial \Omega_{k,\text{Rob}}, \\
\text{Trace}(w) = 0 & \partial \Omega_{k,\text{Dir}} \cup \partial B_k.
\end{cases}$$

Fix $x_1 \in \Omega \setminus B_1$, and consider the sequence $\{u_k := v_k/v_k(x_1)\}_{k \in \mathbb{N}}$. By the Harnack convergence principle (Lemma 3.34), $\{u_k\}_{k \in \mathbb{N}}$ converges (up to a subsequence) to a positive solution $u \in H^1_{\text{loc}}((\overline{\Omega} \setminus \partial \Omega_{\text{Dir}}) \setminus \{x_0\})$ of $(P,B)u = 0$ in $\Omega \setminus \{x_0\}$.

We claim that $u$ is a positive solution of minimal growth in a neighborhood of infinity in $\Omega$. Indeed, fix $l > 1$ and let $s \in \mathcal{S} \mathcal{H}_{P,B}(\Omega^*_l) \cap C(\Omega^*_l \cup \partial \Omega_{l,\text{Dir}})$ such that $u \leq s$ on $\partial \Omega_{l,\text{Dir}}$. For each $k > l$ the boundary condition $\text{Trace}(u_k) = 0$ on $\partial \Omega_{k,\text{Dir}}$, and the generalized maximum principle (Lemma 3.34) imply that $u_k \leq s$ in $\Omega_k \setminus \Omega_l$. By letting $k \to \infty$, we obtain $u \leq s$ in $\Omega^*_l$.

Uniqueness: Let $u$ and $v$ be positive solutions of $(P,B)$ in $\Omega \setminus \{x_0\}$ having minimal growth at infinity in $\Omega$. Clearly, $u$ has a removable singularity at $x_0$ if and only if $v$ has. If the singularity at $x_0$ is removable, we may assume that $u(x_0) = v(x_0)$, and a simple comparison argument implies that $u = v$. Otherwise, by [29, Theorems 1 and 5], $v \sim u \sim G^B_P(x,x_0)$ near $x_0$, where $G^B_P(x,x_0)$ is the positive (Dirichlet) minimal Green function of $P$ in $B_1$, and again, a comparison argument implies that $u = Cv$ with $C > 0$. \qed

4.3. Criticality vs. subcriticality. For a nonnegative operator $(P,B)$ we introduce the notions of criticality and subcriticality.

Definition 4.6. Assume that $(P,B) \geq 0$ in $\Omega$. The operator $(P,B)$ is subcritical in $\Omega$ if there exists $0 \leq W \in L^p_{\text{loc}}(\overline{\Omega} \setminus \partial \Omega_{\text{Dir}})$, $p > n/2$, such that $\mathcal{H}^0_{P,-W,B}(\Omega) \neq \emptyset$; such a $W$ is called a Hardy-weight for $(P,B)$ in $\Omega$. Otherwise, $(P,B)$ is said to be critical in $\Omega$.

If $\mathcal{H}^0_{P,B}(\Omega) = \emptyset$, then $(P,B)$ is called supercritical in $\Omega$.

Lemma 4.7. Let $(P,B) \geq 0$ in $\Omega$. Then $(P,B)$ is critical in $\Omega$ if and only if $(P,B)$ admits (up to a multiplicative constant) a unique regular positive supersolution in $\Omega$. 
Proof. Assume that \((P, B)\) is subcritical in \(\Omega\) and let \(W \geq 0, W \in L^{p/2}_{\text{loc}}(\overline{\Omega} \setminus \partial \Omega_{\text{Dir}})\) be a Hardy-weight. Then any \(u \in \mathcal{H}^0_{P,B}(\Omega)\) and \(v \in \mathcal{H}^0_{P-B}(\Omega)\) are regular positive supersolutions of \((P, B)\) in \(\Omega\) which are linearly independent.

If \((P, B)\) has two linearly independent regular positive supersolutions, \(v\) and \(u\) in \(\Omega\), then a direct calculation (see [14, Lemma 5.1]) shows that \(\sqrt{uv} \in \mathcal{RSH}_{P-B}(\Omega)\), where \(W = \frac{1}{4} \left| \nabla (\log(v/u)) \right|^2 \geq 0\) is the corresponding Hardy-weight. Clearly, \(W \in L^1_{\text{loc}}(\Omega)\), and therefore there exists \(0 \leq W' \leq W\) such that \(W' \in C_0^\infty(\Omega)\). For such \(W'\) we have that \(\sqrt{uv} \in \mathcal{RSH}_{P-W',B}(\Omega)\), and in light of Theorem [1.1] \((P, B)\) is subcritical in \(\Omega\). \(\square\)

Remark 4.8. Lemma [4.7] and the proof of Lemma [3.19] clearly imply that \((P, B)\) is critical (resp., subcritical) in \(\Omega\) if and only if \((P^u, B^u)\) is critical (resp., subcritical) in \(\Omega\), where \(u\) is a regular positive supersolution of \((P, B)\) in \(\Omega\).

Definition 4.9. A function \(u \in \mathcal{H}^0_{P,B}(\Omega)\) is called a ground state if \(u\) has minimal growth in a neighborhood of infinity in \(\Omega\).

Lemma 4.10. Assume that \((P, B) \geq 0\) in \(\Omega\). Then \((P, B)\) admits a ground state in \(\Omega\) if and only if \((P, B)\) is critical in \(\Omega\).

Proof. Suppose that \((P, B)\) admits a ground state \(\varphi\) in \(\Omega\). Let \(\{\Omega_k\}_{k \in \mathbb{N}}\) be an exhaustion of \(\overline{\Omega} \setminus \partial \Omega_{\text{Dir}}\), and let \(x_1 \in \Omega_1\). Let \(v\) be a regular positive supersolution of \((P, B)\) in \(\Omega\). In light of Lemma [4.7] it suffices to show that \(v = C\varphi\) for some constant \(C > 0\). Since \(\varphi\) has minimal growth, it follows that there exists \(C > 0\) satisfying \(C\varphi \leq v\). Let

\[
C_0 := \sup \{ C > 0 \mid C\varphi \leq v \text{ in } \Omega \}.
\]

Consider the function \(v_0 := v - C_0\varphi \geq 0\) and note that \((P, B)v_0 \geq 0\) in \(\Omega\). Assume by contradiction that \(v_0 \not\geq 0\). By the strong maximum principle (Lemma [3.12] \(v_0 > 0\) in \(\Omega\), and therefore, \(v_0 \in \mathcal{RSH}_{P,B}(\Omega)\). By repeating the above argument with \(\varphi\) and \(v_0\), it follows that there exists \(\mu > 0\) such that \(\mu\varphi \leq v_0\). Hence, \(v \geq (C_0 + \mu)\varphi\), a contradiction to the maximality of \(C_0\). Hence, \(v = C_0\varphi\).

Assume now that \((P, B)\) does not admit a ground state in \(\Omega\). Let \(u_{x_0}\) be the positive solution of minimal growth having a nonremovable singularity at \(x_0\), and \(\{\Omega_k\}_{k \in \mathbb{N}}\) be an exhaustion of \(\overline{\Omega} \setminus \partial \Omega_{\text{Dir}}\) (see Lemma [4.5]). Let \(0 \leq f \in C_0^\infty(\overline{\Omega})\), where \(\Omega\) is a small neighborhood of \(x_0\), and let \(w_k\) be the solution of the problem:

\[
\begin{cases}
Pw = f & \text{in } \Omega_k, \\
Bw = 0 & \text{on } \partial \Omega_{k,\text{Rob}}, \\
\text{Trace}(w) = 0 & \text{on } \partial \Omega_{k,\text{Dir}}.
\end{cases}
\]

By the generalized maximum principle \(\{w_k\}_{k \in \mathbb{N}}\) is monotone increasing. If there exists \(x_1 \neq x_0\) such that the sequence \(\{w_k(x_1)\}_{k \in \mathbb{N}}\) is bounded, then by Harnack convergence principle (Lemma [3.27]) and elliptic regularity, \(\{w_k\}_{k \in \mathbb{N}}\) converges locally uniformly in \(\Omega\) to
a positive solution \(w\) of \((P, B)u = f \geq 0\) in \(\Omega\). Clearly \(w\) is a regular positive supersolution of \((P, B)\) in \(\Omega\). In light of Lemma 4.7, \((P, B)\) is subcritical in \(\Omega\).

Otherwise, fix \(x_1 \neq x_0\). Then once more, by the Harnack convergence principle and interior elliptic regularity, the sequence \(\{z_k := w_k/w_k(x_1)\}_{k \in \mathbb{N}}\) converges to a positive solution \(z \in \mathcal{H}_{P, B}(\Omega)\). Note that \(z < u_{x_0}\) near \(x_0\) (since \(u_{x_0}\) has a singularity at \(x_0\)), and therefore, by a standard comparison argument, \(z\) is smaller than \(u_{x_0}\) in \(\Omega \setminus \{x_0\}\). Hence, \(z\) is a positive solution of \((P, B)u = 0\) in \(\Omega\) of minimal growth at infinity. Namely, \(z\) is a ground state of \((P, B)\) in \(\Omega\), a contradiction to our assumption.

\[\square\]

### 5. Positive minimal Green function

#### 5.1. Green function in a Lipschitz bounded subdomain

Throughout the present subsection, we assume that \((P, B)\) is nonnegative in a domain \(\Omega\) and that Assumptions 2.3 hold true. We construct in the subcritical case the corresponding positive minimal Green function \(G_{P_{-\lambda B}}(x, y)\) of \((P, B)\) in \(\Omega\). We follow Stampacchia’s Green function construction for the Dirichlet boundary value problem in a bounded domain [31, Section 9]. We mention the following result for subsequent applications (cf. [\[1\] Theorem 3.9]).

**Proposition 5.1.** Let \(\Omega' \Subset \Omega\) and \(q > 1\). Then \(\Upsilon \in (W^{1,q}_{\partial\Omega'}(\Omega'))^*\) if and only if there exist \(g_0 \in L^q(\Omega')\) and \(g \in L^q(\Omega', \mathbb{R}^n)\) such that for any \(u \in W^{1,q}_{\partial\Omega'}(\Omega')\),

\[
\Upsilon(u) = \int_{\Omega'} (ug_0 - \nabla u \cdot g) \, dx,
\]

and in this case we write \(\Upsilon = g_0 + \text{div } g\).

Moreover, there exist \(g_0 \in L^q(\Omega')\) and \(g \in L^q(\Omega', \mathbb{R}^n)\) satisfying (5.1) such that

\[
\|\Upsilon\|_{(W^{1,q}_{\partial\Omega'}(\Omega'))^*} \approx \left(\|g_0\|_{L^q(\Omega')} + \|g\|_{L^q(\Omega', \mathbb{R}^n)}\right).
\]

Let \(\Omega' \Subset \Omega\) be a Lipschitz subdomain of \(\Omega\). Then for \(\lambda < \lambda_c(\Omega')\) the resolvent operator \(R_{\Omega'}(\lambda)\) exists, and \(R_{\Omega'}(\lambda) : L^2(\Omega') \to H^1_{\partial\Omega'}(\Omega')\) is bounded. By the Fredholm alternative, for any \(\Upsilon = g_0 + \text{div } g \in (H^1_{\partial\Omega'}(\Omega'))^*\) there exists a unique solution \(u_\Upsilon \in H^1_{\partial\Omega'}(\Omega')\) to the problem

\[
\begin{cases}
(P - \lambda)w = g_0 + \text{div } g & \text{in } \Omega', \\
Bw = -g \cdot \vec{n} & \text{on } \partial\Omega'_{\text{Rob}},
\end{cases}
\]

(in short, \((P - \lambda, B)w = g_0 + \text{div } g\) in \(\Omega'\)) in the following sense: For all \(\phi \in \mathcal{D}(\Omega', \partial\Omega'_{\text{Rob}})\),

\[
\int_{\Omega'} [A\nabla u_\Upsilon \cdot \nabla \phi + u_\Upsilon \vec{b} \cdot \nabla \phi + \vec{b} \cdot \nabla u_\Upsilon \phi + (c - \lambda)u_\Upsilon \phi] \, dx + \int_{\partial\Omega'_{\text{Rob}}} \vec{n} \cdot u_\Upsilon \phi \, d\sigma = \int_{\Omega'} (g_0 \phi - g \cdot \nabla \phi) \, dx.
\]

Note that in Theorem 3.3 we proved the unique solvability of (5.2) only for the case \(g = 0\), but the proof applies also for the general case \(g \neq 0\) (see for example, [15, Theorem 8.3]).
Lemma 5.2. Assume that \((P, B) \geq 0\) in \(\Omega\), and let \(\Omega' \Subset \Omega\) be a Lipschitz subdomain. Let 
\[
F : (H^{1,p}_{\partial \Omega'_{\text{Dir}}}(\Omega'))^* \to H^1_{\partial \Omega'_{\text{Dir}}}(\Omega')
\]
be the linear operator mapping \(\Upsilon \mapsto u_\Upsilon\), where \((P, B)u_\Upsilon = \Upsilon\) in \(\Omega'\). Then \(F\) is continuous.

Proof. It is enough to show that the graph of \(F\) is closed. Let \((\Upsilon_k, u_k) \to (\Upsilon, u)\) in the 
graph norm, where \(F(\Upsilon_k) = u_k\). We need to show that \(F(\Upsilon) = u\), that is \((P, B)u = \Upsilon\).

By the definition of \(F\), for any \(\phi \in \mathcal{D}(\Omega', \partial \Omega'_{\text{Dir}})\) we have \(\mathcal{B}_{P, B}(u_k, \phi) = \Upsilon_k(\phi)\). Clearly, 
\[
\Upsilon_k(\phi) \to \Upsilon(\phi),
\]
and by Theorem 3.3
\[
\mathcal{B}_{P, B}(u_k, \phi) \to \mathcal{B}_{P, B}(u, \phi) \quad \text{as } k \to \infty.
\]

Hence, \(F(\Upsilon) = u\). \(\square\)

The following lemma utilizes standard tools of functional analysis to obtain the Green function of \((P, B)\) in a Lipschitz subdomain \(\Omega' \Subset \Omega\). Recall that according to Assumptions 23 the exponent \(p\) satisfies \(p > n\).

Lemma 5.3. Suppose that \((P, B) \geq 0\) in \(\Omega\), and let \(\Omega' \Subset \Omega \setminus \partial \Omega_{\text{Dir}}\) be a Lipschitz subdomain. Let \(\lambda < \lambda_c(\Omega')\), and let \(u_\Upsilon \in H^1_{\partial \Omega'_{\text{Dir}}}(\Omega')\) be the unique weak solution of \((5.2)\), where \(g_0 \in L^p(\Omega'), g \in L^p(\Omega', \mathbb{R}^n)\). Then for any \(\Omega'' \Subset \Omega'\)
\[
\sup_{\Omega''} |u_\Upsilon| \leq C(\|g_0\|_{L^p(\Omega')} + \|g\|_{L^p(\Omega', \mathbb{R}^n)}),
\]
where \(C\) depends only on \(\Omega', \Omega'', \lambda\) and the coefficients of the operator \((P, B)\) in \(\Omega\).

In particular, for any \(x \in \Omega''\), the functional
\[
J_x : (W^{1,p'}_{\partial \Omega'_{\text{Dir}}}(\Omega'))^* \to \mathbb{R}, \quad \Upsilon \mapsto u_\Upsilon(x)
\]
is bounded, and there exists \(0 < G_{P-\lambda, B}^{\Omega'}(x, \cdot) \in (W^{1,p'}_{\partial \Omega'_{\text{Dir}}}(\Omega'))^* \subset W^{1,p'}_{\partial \Omega'_{\text{Dir}}}(\Omega')\) satisfying
\[
J_x(\Upsilon) = G_{P-\lambda, B}^{\Omega'}(x, \cdot)(\Upsilon) \quad \forall \Upsilon \in (W^{1,p'}_{\partial \Omega'_{\text{Dir}}}(\Omega'))^*,
\]
and the following Green representation formula holds:
\[
uu^u \Upsilon \fff^u x = \int_{\Omega'} [G_{P-\lambda, B}^{\Omega'}(x, y)g_0(y) - \nabla_y G_{P-\lambda, B}^{\Omega'}(x, y) \cdot g(y)] \, dy.
\]

Proof. Denote \(u = u_\Upsilon\). It follows from [21, Theorem 5.36] that
\[
\sup_{\Omega''} |u| \leq C(\|g_0\|_{L^p(\Omega')} + \|g\|_{L^p(\Omega', \mathbb{R}^n)}) + \|u\|_{L^2(\Omega')}.
\]

On the other hand, Lemma 5.2 implies that
\[
\|u\|_{L^2(\Omega')} \leq C\|\Upsilon\|_{\left(W^{1,2}_{\partial \Omega'_{\text{Dir}}}(\Omega')\right)^*} \leq C\|\Upsilon\|_{\left(W^{1,p'}_{\partial \Omega'_{\text{Dir}}}(\Omega')\right)^*} \approx \|g_0\|_{L^p(\Omega')} + \|g\|_{L^p(\Omega', \mathbb{R}^n)}.
\]

Moreover, in light of the generalized and the strong maximum principle, if \((P, B)u \geq 0\) in \(\Omega'\), then \(u > 0\) in \(\Omega'\). Since the space \(W^{1,p'}_{\partial \Omega'_{\text{Dir}}}(\Omega')\) is a closed subspace of a reflexive Sobolev space, it is reflexive as well. In particular, there exists \(0 < G_{P-\lambda, B}^{\Omega'}(x, \cdot) \in W^{1,p'}_{\partial \Omega'_{\text{Dir}}}(\Omega')\) such
that for any \( \Upsilon = g_0 + \nabla g \),

\[
J_x(\Upsilon) = \Upsilon(G_{P-\lambda,B}^{\Omega''}(x, \cdot)) = \int_{\Omega'} (G_{P-\lambda,B}^{\Omega''}(x, y)g_0(y) - \nabla_y G_{P-\lambda,B}^{\Omega''}(x, y) \cdot g(y))\,dy. \tag{5.5}
\]

The reflexivity also implies that

\[
G_{P-\lambda,B}^{\Omega''}(x, \cdot)(\Upsilon) = \Upsilon(G_{P-\lambda,B}^{\Omega''}(x, \cdot)) = J_x(\Upsilon).
\]

Moreover, it follows that for each fixed \( x \in \Omega' \), \( G_{P-\lambda,B}^{\Omega''}(x, \cdot) \) does not depend on subdomains \( \Omega'' \) containing \( x \). Therefore, we obtain (5.4). \( \square \)

**Corollary 5.4.** Fix \( x_0 \in \Omega'' \subseteq R \). \( \Omega' \subseteq R \), \( \Omega'' \) are Lipschitz subdomains of \( \Omega \). Then the operator \( H : (W_1^{1,p'}(\Omega'))^* \rightarrow L^\infty(\Omega'') \), which maps \( \Upsilon \) to \( u_{\Omega'} \), is bounded. Moreover, \( H^*: L^1(\Omega'') \rightarrow (W_1^{1,p'}(\Omega'))^* \) is bounded as well.

Next, we are interested in the regularity properties of the function \( G_{P-\lambda,B}^{\Omega''}(x, y) \). It turns out that \( G_{P-\lambda,B}^{\Omega''}(x, y) \) satisfies \( (P - \lambda, B)G_{P-\lambda,B}^{\Omega''}(\cdot, y) = \delta_y \) in a generalized sense, where \( \delta_y \) is the Dirac measure centered at \( y \in \Omega' \). Therefore, it is natural to extend the meaning of solutions of (5.2) by introducing the notion of distributional solutions to our boundary value problem.

**Definition 5.5.** Let Assumptions 2.3 hold in a bounded Lipschitz domain \( \Omega \). Let \( \eta \) be a compactly supported measure on \( \Omega \) with bounded variation. We say that \( u \in W_1^{1,p'}(\Omega) \) is a distributional solution of the boundary value problem \( (P, B)u = \eta \) in \( \Omega \) if for any (continuous) \( \phi \in H_{\partial\Omega_{\text{Rob}}}^1(\Omega) \) satisfying

\[
\begin{cases}
P^*\phi = g_0 + \nabla g & \text{in } \Omega, \\B^*\phi = -g \cdot \vec{n} & \text{on } \partial\Omega_{\text{Rob}},
\end{cases}
\tag{5.6}
\]

with \( g_0, g^i \in \mathcal{D}(\Omega, \partial\Omega_{\text{Rob}}) \) for all \( 1 \leq i \leq n \), we have

\[
\int_{\Omega} (ug_0 - \nabla u \cdot g)\,dx = \int_{\Omega} \phi \,d\eta. \tag{5.7}
\]

**Remark 5.6.** Note that (5.7) reads as \( \Upsilon(u) = \int_{\Omega} \phi \,d\eta \), where \( \Upsilon = g_0 + \nabla g \in (W_1^{1,p'}(\Omega'))^* \).

**Lemma 5.7.** Let \( x_0 \in \Omega' \subseteq R \). For \( \lambda < \lambda_c(P, B, \Omega') \), the function \( G_{P-\lambda,B}^{\Omega''}(x_0, \cdot) \) is a positive distributional solution of the problem \( (P^* - \lambda, B^*)u = \delta_{x_0} \) in \( \Omega' \).

**Proof.** By Lemma 5.3 \( G_{P-\lambda,B}^{\Omega''}(x_0, \cdot) \in W_1^{1,p'}(\Omega') \). Let \( \phi \in H_{\partial\Omega_{\text{Dir}}}^1(\Omega') \) be a weak solution of (5.2). We need to show that for any \( x_0 \in \Omega' \) we have

\[
\int_{\Omega'} \left[ G_{P-\lambda,B}^{\Omega''}(x_0, y)g_0(y) - \nabla G_{P-\lambda,B}^{\Omega''}(x_0, y)g(y) \right]dy = \phi(x_0).
\]
Recall that by (5.4)
\[ \int_{\Omega'} [G_{P-\lambda,B}^{\Omega'}(x_0,y)g_0(y) - \nabla G_{P-\lambda,B}^{\Omega'}(x_0,y)g(y)] \, dy = u(x_0), \]
where \( u \in H_{\partial \Omega'}^1(\Omega') \) is a weak solution to the problem (5.2).

Since \( \lambda < \lambda_0(\bar{P},B,\Omega') \), the generalized maximum principle for \((P - \lambda, B)\) in \( \Omega' \) implies that the weak solution \( \phi \) to problem (5.2) is unique. Hence, \( \phi(x_0) = u(x_0) \). Moreover, if \( \Upsilon = g_0 \geq 0 \), and \( x \in \Omega' \), then
\[ \int_{\Omega'} G_{P-\lambda,B}^{\Omega'}(x,y)g_0(y) \, dy = \phi(x) = u(x) > 0. \]
Since, \( x \) and \( g_0 \) are arbitrary, we deduce that \( G_{P-\lambda,B}^{\Omega'}(x, \cdot) \) is positive a.e. in \( \Omega' \). \( \square \)

**Lemma 5.8.** Assume that \( G_1 \) and \( G_2 \) are distributional solutions of the problem \((P^* - \lambda, B^*) = \delta_\infty \) in a bounded Lipschitz domain \( \Omega' \subset \Omega \setminus \partial \Omega_{\text{Dir}} \), where \( x_0 \in \Omega' \) and \( \lambda < \lambda_0(\Omega') \). Then \( G_1 = G_2 \) in \( \Omega' \).

**Proof.** According to our assumptions the function \( G := G_1 - G_2 \) is a distributional solutions to the problem \((P^* - \lambda, B^*) = 0 \) in \( \Omega' \). Recall that for any \( \psi \in C_0^\infty(\Omega') \), there exists a unique solution to the problem \((P - \lambda, B)\phi = \psi \) in \( \Omega' \), and therefore,
\[ \int_{\Omega'} G(x,y)\psi(y) \, dx = 0. \]
Since \( \psi \) is arbitrary \( G = 0 \). \( \square \)

We proceed with regularity properties of distributional solutions near \( \partial \Omega_{\text{Rob}} \). Recall that for \( x \in \mathbb{R}^n \), we use also the notation \( x = (x', x_n) \).

**Lemma 5.9.** Let Assumptions [2,3] hold in \( \Omega \), and let \( \Omega' \subset \subset \Omega \) be a Lipschitz subdomain. Assume that \( (P, B) \geq 0 \) in \( \Omega \), and let \( \eta \) be a compactly supported measure on \( \Omega \) with bounded variation such that \( \emptyset \neq \text{supp}(\eta) \subset \Omega \). Finally, let \( v \) be a positive distributional solution of the problem \((P^*, B^*)v = \eta \) in \( \Omega' \). Then \( v \in H^1_{P^*,B^*}(\Omega' \setminus \text{supp}(\eta)) \).

**Proof.** By the ground state transform, we may assume that \((P, B)1 = 0 \). In other words, we may assume that the coefficients of \((P, B)\) satisfy \( \gamma = c = \tilde{b} = 0 \) (see (3.8)).

Assume first that
\[ \Omega' = B_1(0)^+ = \{ x \in \mathbb{R}^n \mid x_n > 0 \text{ and } |x| < 1 \}, \quad \partial \Omega'_{\text{Rob}} = \{ x \in \mathbb{R}^n \mid x_n = 0 \text{ and } |x| < 1 \}. \]

Denote \( B_1(0)^- = \{ x \in \mathbb{R}^n \mid x_n < 0 \text{ and } |x| < 1 \} \), and \( \text{supp}(\eta) \cap \Omega' = \emptyset \). Consider the operator \((\tilde{P}^*, \tilde{B}^*)\), the ‘even’ extension of \((P^*, B^*)\) into \( B_1(0) \), namely,
\[ \tilde{a}^{ij}(x',x_n) := \begin{cases} a^{ij}(x',x_n) & x_n \geq 0, \\ a^{ij}(x',-x_n) & x_n < 0, \end{cases} \quad \tilde{b}(x',x_n)^j := \begin{cases} b^j(x',x_n) & x_n \geq 0, 1 \leq j \leq n, \\ b^j(x',-x_n) & x_n < 0, 1 \leq j \leq n. \end{cases} \]
Similarly, define the ‘even’ extension of the given solution $v$ into $B_1(0)$ by

$$
\tilde{v}(x', x_n) := \begin{cases} 
v(x', x_n) & x_n \geq 0, \\
v(x', -x_n) & x_n < 0.
\end{cases}
$$

We claim that $\tilde{v} \in L^1(B_1(0))$ is a distributional solution to the Dirichlet problem $\tilde{P}^\ast(u) = 0$ in $B_1(0)$ in the following sense (see, [31, Definition 9.1]):

$$
\int_{B_1(0)} \tilde{v}\psi \, dx = \int_{B_1(0)} \tilde{v}\phi \, dx = 0,
$$

for all admissible $\phi$, i.e., $\phi \in H^1_0(B_1(0)) \cap C(B_1(0))$ and $\tilde{P}(\phi) = \psi \in C(B_1(0))$.

Since $v \in W^{1,p'}(B_1(0)^+)$, it follows that $\tilde{v} \in L^1(B_1(0))$. Let us fix an admissible function $\phi$, and consider the functions

$$
\phi_1 = \phi \chi_{B_1(0)^+}, \quad \psi_1 = \psi \chi_{B_1(0)^+}, \quad \phi_2 = \phi \chi_{B_1(0)^-}, \quad \psi_2 = \psi \chi_{B_1(0)^-}.
$$

Then

$$
\int_{B_1(0)} \tilde{v}\psi \, dx = \int_{B_1(0)^+} v\psi_1 \, dx + \int_{B_1(0)^-} \tilde{v}\psi_2 \, dx = \int_{B_1(0)^+} v\psi_1 \, dx + \int_{B_1(0)^+} v(x', x_n)\psi_2(x', -x_n) \, dx.
$$

It remains to show that

$$
(P,B)(\phi_1 + \phi_2(x', -x_n)) = \psi_1 + \psi_2(x', -x_n) \quad \text{in } B_1(0)^+.
$$

Indeed, let $\varphi \in \mathcal{D}(B_1(0)^+, \partial B_1(0)^+_{Du})$ and let

$$
\tilde{\vartheta}(x) := \begin{cases} 
\varphi(x', x_n) & \text{in } B_1(0)^+,
\\
\varphi(x', -x_n) & \text{in } B_1(0)^-.
\end{cases}
$$

Then $\tilde{\vartheta} \in H^1_0(B_1(0))$. Note that for any $\psi \in C^{\infty}_0(B_1(0))$,

$$
\int_{B_1(0)} D_n \tilde{\vartheta} \psi \, dx + \int_{B_1(0)} \tilde{\vartheta} D_n \psi \, dx = \int_{\partial B_1(0)} \phi(x', 0) \tilde{\vartheta} - \phi(x', 0) \vartheta \, d\sigma = 0.
$$

Next, we compute

$$
\int_{B_1(0)^+} \left[A \nabla (\phi_1 + \phi_2(x', -x_n)) \cdot \nabla \varphi + \bar{b} \cdot \nabla (\phi_1 + \phi_2(x', -x_n)) \varphi\right] \, dx =
$$

$$
\int_{B_1(0)^+} \left[\tilde{A} \nabla \phi \cdot \nabla \vartheta + \tilde{b} \cdot \nabla \phi \vartheta \right] \, dx + \int_{B_1(0)^-} \left[\tilde{A} \nabla \phi(x', x_n) \cdot \nabla \vartheta + \tilde{b} \cdot \nabla \phi(x', x_n) \vartheta \right] \, dx = \int_{B_1(0)} \psi \vartheta \, dx.
$$

On the other hand,

$$
\int_{B_1(0)^+} [\varphi \psi_1 + \varphi \psi_2(x', -x_n)] \, dx = \int_{B_1(0)^+} \varphi \psi_1 \, dx + \int_{B_1(0)^-} \varphi(x', -x_n) \psi_2(x', x_n) \, dx = \int_{B_1(0)} \vartheta \psi \, dx.
$$

By a standard argument of local ‘flattening’ $\partial\Omega_{Rob}$ (see for example, [13, Appendix C.5]), we deduce that if $v$ is a distributional solution to the problem $(P^*, B^*)v = \eta$ in $\Omega'$, then
for any $\Omega'' \Subset \Omega'$ and any $y_0 \in \partial\Omega''_{\text{Rob}}$, $\tilde{P}\tilde{v} = 0$ (in the distributional sense) in some ball $B_\varepsilon(y_0)$. Extend $\tilde{v}$ to $\Omega'$ by letting $\tilde{v} = v$ in $\Omega'$. As a consequence, $\tilde{P}\tilde{v} = \eta$ in $\int (\Omega'' \cup (B_\varepsilon(y_0)))$ in the distributional sense. By [31 Theorem 9.3] (see also [17 Section 5]), $\tilde{v} \in H^1_{\text{loc}}\left(\int (\Omega' \cup (B_\varepsilon(y_0))) \setminus \text{supp}(\mu)\right)$.

Let $\{v_l\}_{l \in \mathbb{N}} \subset \mathcal{D}(\partial\Omega'_{\text{Dir}}, \Omega')$ be a sequence which converges in $W^{1,p'}(\Omega' \setminus \partial\Omega'_{\text{Dir}})$ to $v$. By the definition of weak solution, for any $\phi \in \mathcal{D}(\partial\Omega'_{\text{Dir}}, \Omega')$ supported outside $\text{supp}(\eta)$, $(P, B)\phi = \Upsilon$ in $\Omega'$ implying

$$\Upsilon(v_l) = \int_{\Omega'} A\nabla\phi \nabla v_l + \bar{b}\nabla\phi v_l \, dx.$$  \hspace{1cm} (5.8)

Letting $l \to \infty$ and the definition of distributional solution imply

$$0 = \Upsilon(v) = \int_{\Omega'} A\nabla\phi \nabla v + \bar{b}\nabla\phi v \, dx,$$

i.e., $(P^*, B^*)v = 0$ in $\Omega' \setminus \text{supp}(\eta)$ in the weak sense. \hspace{1cm} \Box

As a result of Lemma 5.9, we obtain the following interior regularity of $G_{P-\lambda,B}^{\Omega'}$.

**Lemma 5.10.** Let $\Omega' \Subset \Omega \setminus \partial\Omega_{\text{Dir}}$ be a Lipschitz subdomain of $\Omega$, and let $\lambda < \lambda_0(P, B, 1, \Omega)$.

Then for any $x_0 \in \Omega'' \Subset \Omega'$, and $\varepsilon > 0$ such that $B_\varepsilon(x_0) \Subset \Omega'$, $G_{P-\lambda,B}^{\Omega'}(x_0, \cdot) \in H^1(\Omega'' \setminus B_\varepsilon(x_0)) \cap W^{1,p'}(\Omega')$ is positive, and satisfies $(P^*, B^*)G_{P-\lambda,B}^{\Omega'}(x_0, \cdot) = 0$ in $\Omega' \setminus B_\varepsilon(x_0)$ in the weak sense.

**Lemma 5.11.** Assume that $(P, B) \geq 0$ in $\Omega$ and let Assumptions 2.3 hold in $\Omega$. Then $(P^*, B^*) \geq 0$ in $\Omega$.

**Proof.** Let $\{\Omega_k\}_{k \in \mathbb{N}}$ be an exhaustion of $\Omega \setminus \partial\Omega_{\text{Dir}}$. Choose a sequence $\{x_k\}_{k \in \mathbb{N}} \subset \Omega$ converging either to a point $\xi$ that belongs to a Dirichlet (Lipschitz)-portion of $\partial\Omega$ if $\Omega$ is bounded, or otherwise, to infinity. Without loss of generality, we may assume that $x_k \in \Omega_k$. By lemmas 5.10 and 5.9, the operator $(P, B)$ admits a positive Green function $G_{P,B}^{\Omega_k}(x_k, \cdot)$ with singularity at $x_k$ which solves weakly the equation $(P^*, B^*)u = 0$ in $\Omega_k \setminus \{x_k\}$. Consider the sequence

$$u_k(y) := \frac{G_{P,B}^{\Omega_k}(x_k, y)}{G_{P,B}^{\Omega_k}(x_k, x_1)}.$$

By the Harnack convergence theorem (Lemma 3.27), $u_k$ admits a subsequence converging to a positive solution of $(P^*, B^*)u = 0$ in $\Omega$. \hspace{1cm} \Box

**Lemma 5.12.** Let $\Omega' \Subset \Omega$ be a Lipschitz subdomain and let $\eta$ be a Radon measure compactly supported on $B_\varepsilon(x_0) \Subset \Omega'$. Let $u \in W^{1,p'}(\Omega')$ be a positive distributional solution to the problem $(P, B)w = \eta$ in $\Omega'$. Then

$$\|u\|_{W^{1,p'}(\Omega')} \leq C(\varepsilon) \int_{\Omega'} d\eta,$$  \hspace{1cm} (5.9)
where \( C(\varepsilon) \) does not depend on \( u \).

If in addition, \( u \in H^1(\Omega' \setminus \text{supp}(\eta)) \) is a weak solution to the problem \((P, B)w = 0 \) in \( \Omega' \setminus \text{supp}(\eta) \), then for any Lipschitz subdomain \( \Omega'' \Subset \Omega' \setminus \text{supp}(\eta) \) we have

\[
\sup_{\Omega''} |u| \leq C(\Omega'', \varepsilon) \int_{\Omega''} \eta \, \text{d} \eta. \tag{5.10}
\]

**Proof.** Let \( g_0 \) and \( g \) satisfying \( g_0, g^i \in \mathcal{D}(\Omega', \partial \Omega'_{\text{Dir}}), \forall 1 \leq i \leq n \). Writing \( \Upsilon = g_0 + \text{div} \, g \in (W^{1,p'}(\Omega'))^* \) we obtain

\[
u(\Upsilon) = \int_{\Omega'} (u g_0 - \nabla u \cdot g) \, \text{d}x = \int_{B_{\varepsilon}(x_0)} \phi \, \text{d} \eta,
\]

where \( \phi \) is a weak solution of (5.6) with \( \Omega \) replaced by \( \Omega' \). By Lemma 5.3

\[
\left| \int_{B_{\varepsilon}(x_0)} \phi \, \text{d} \eta \right| \leq \| \Upsilon \|_{(W^{1,p'}(\Omega'))^*} C(\varepsilon) \int_{B_{\varepsilon}(x_0)} \text{d} \eta.
\]

By the denseness of smooth functions in \( L^p \)-spaces, we obtain that for any \( \Upsilon \in (W^{1,p'}(\Omega'))^* \),

\[
|u(\Upsilon)| \leq \| \Upsilon \|_{(W^{1,p'}(\Omega'))^*} C(\varepsilon) \int_{B_{\varepsilon}(x_0)} \text{d} \eta,
\]

implying that \( \|u\|_{W^{1,p'}(\Omega')} \leq C(\varepsilon) \int_{B_{\varepsilon}(x_0)} \eta \text{d} \eta \).

Let \( q = p' \), and note that \( 1 < q < 2 \). The Sobolev embedding theorem [21, Theorem 5.8] states that

\[
\|u\|_{L^{q^*}(\Omega')} \leq C\|u\|_{W^{1,q}(\Omega')} \quad \forall u \in W^{1,q}(\Omega'),
\]

where \( q^* = nq/(n - q) \) is the critical Sobolev exponent.

Assume in addition that \( u \in H^1(\Omega' \setminus \text{supp}(\eta)) \) is a weak solution to the problem \((P, B)u = 0 \) in \( \Omega' \setminus \text{supp}(\eta) \), then [21, Theorem 5.36] implies that for any \( \Omega'' \Subset \Omega' \setminus \text{supp}(\eta) \),

\[
\sup_{\Omega''} |u| \leq C\|u\|_{L^{q^*}(\Omega')},
\]

and by the first part of the proof we are done. \( \square \)

The following well known proposition is a consequence of the Lebesgue differentiation theorem.

**Proposition 5.13.** Let \( \Omega' \Subset \Omega \) be a Lipschitz subdomain containing \( x \). For all \( k \in \mathbb{N} \), consider the functions \( f_{k,x} := |B_{1/k}(x)|^{-1} \chi_{B_{1/k}(x)} \). Then, for any \( \phi \in H^1(\Omega') \)

\[
\lim_{k \to \infty} \int_{\Omega'} \phi(y) f_{k,x}(y) \, \text{d}y = \phi(x) \quad \text{for a.e. } x \in \Omega'. \tag{5.11}
\]

If \( \phi \) is continuous in \( \Omega' \), then (5.11) holds for all \( x \in \Omega' \).

As a corollary of Proposition 5.13, Lemma 5.11 and estimates (5.9), and (5.10), we obtain the following approximation result.
Corollary 5.14. Let \( q = p' \). For all \( k \in \mathbb{N} \), consider the functions \( f_k := |B_{1/k}(x_0)|^{-1} \chi_{B_{1/k}(x_0)} \), where \( x_0 \in \Omega' \subset \Omega \) and \( \Omega' \) is a Lipschitz bounded subdomain. Let \( u_k \in H^1_{\partial \Omega'_{\text{Dir}}} (\Omega') \) be a sequence of positive (Hölder continuous) solutions to the problem \((P^*, B^*) u_k = f_k \) in \( \Omega' \). Then, \( u_k \) converges weakly (up to a subsequence) in \( W^{1,q}(\Omega') \) and locally uniformly in \( \overline{\Omega'} \setminus (\partial \Omega'_{\text{Dir}} \cup \{x_0\}) \) to a positive distributional solution of the problem \((P^*, B^*) u = \delta_{x_0} \) in \( \Omega' \), namely, to \( G^w_{P,B}(x_0, \cdot) \).

Remark 5.15. By Corollary 5.14 \( G^w_{P-\lambda,B}(x_0, \cdot) \) is Hölder continuous in \( \Omega' \setminus \{x_0\} \) and satisfies weakly \((P^*, B^*) u = 0 \) in \( \Omega' \setminus \{x_0\} \). Moreover, Corollary 5.14 and [15, Theorem 8.29] imply that if \( \mathcal{T} \) is a Lipschitz-portion of \( \partial \Omega'_{\text{Dir}} \), then \( G^w_{P,B}(x_0, \cdot) \) vanishes continuously on \( \mathcal{T} \). We remark that this conclusion can also be deduced from the approximation method in [31, Theorem 9.2] in which one obtains that \( G^w_{P-\lambda,B}(x_0, \cdot) \) is a (locally uniformly) limit of positive solutions \( \{u_k\}_{k \in \mathbb{N}} \) of \((P^k)^*, (B^*) u_k = 0 \) in \( \Omega' \setminus B_{1/k}(x_0) \). Here \((P^k)^*\) is the mollification of the operator \( P^* \) whose coefficients are smooth in \( \mathbb{R}^n \).

Remark 5.16. Suppose that \((P, B)\) admits a positive Green function \( G = G^w_{P,B}(x_0, \cdot) \) in a bounded Lipschitz domain \( \Omega \). Let \( v \) be a positive solution of the Dirichlet problem \( P^* v = 0 \) in \( B_{\varepsilon}(x_0) \subset \Omega \) satisfying \( v = G \) on \( \partial B_{\varepsilon}(x_0) \). Then \( G - v \) is a distributional solution (in the Dirichlet sense [31, Definition 9.1]) of the Dirichlet problem \( P^* u = \delta_{x_0} \) in \( B_{\varepsilon}(x_0) \), \( u = 0 \) on \( \partial B_{\varepsilon}(x_0) \). Therefore, by the uniqueness of the Dirichlet Green function, \( G^w_{P,B}(x_0, x) = v(x) + G^w_{P}(x_0, x) \) in \( B_{\varepsilon}(x_0) \), where \( G^w_{P}(x_0, x) \) is the Dirichlet Green function in \( B_{\varepsilon}(x_0) \) with a pole at \( x_0 \) (see also [29, Theorems 1 and 5]).

As a consequence of the previous subsection we obtain the following important result.

Lemma 5.17. Let \( \Omega' \subset \Omega \) be a Lipschitz subdomain. Then \( G^w_{P,B}(x, y) = G^w_{P^*,B^*}(y, x) \) for all \( x, y \in \Omega' \), \( x \neq y \).

Proof. Fix \( x_0 \neq y_0 \) in \( \Omega' \), and let \( u_k \) (resp., \( v_k \)) be an approximating sequence of \( G^w_{P,B}(x_0, \cdot) \) (resp., \( G^w_{P^*,B^*}(y_0, \cdot) \)) with \( g_k \) (resp., \( f_k \)) as in Corollary 5.14 where \( g_k \) is an approximating sequence of \( \delta_{x_0} \) (resp., \( f_k \) is an approximating sequence of \( \delta_{x_0} \)). For all \( k \),

\[
B_{P^*,B^*}(u_k, v_k) = \int_{\Omega'} f_k v_k \, dx \rightarrow G^w_{P,B}(y_0, x_0) \quad \text{as} \quad k \rightarrow \infty.
\]

On the other hand,

\[
B_{P^*,B^*}(u_k, v_k) = B_{P,B}(v_k, u_k) = \int_{\Omega'} g_k u_k \, dx \rightarrow G^w_{P^*,B^*}(x_0, y_0) \quad \text{as} \quad k \rightarrow \infty. \quad \square
\]

5.2. Green function for \((P, B)\) satisfying Assumptions 2.3. We proceed with the construction of the positive minimal Green function of \((P, B)\) in a domain \( \Omega \).

Theorem 5.18. Let \((P, B) \geq 0\) in \( \Omega \) and satisfies Assumptions 2.3. Then either \((P, B)\) admits a positive minimal Green function in \( \Omega \), or else, \((P, B)\) admits a ground state.
Proof. Let \( \{\Omega_k\}_{k \in \mathbb{N}} \) be an exhaustion of \( \overline{\Omega} \setminus \partial \Omega_{\text{Dir}} \). Recall that for any \( \Omega' \in R \overline{\Omega} \setminus \partial \Omega_{\text{Dir}} \), the generalized maximum principle holds in \( \Omega' \), hence, \((P,B)\) admits a positive Green function \( G_{P,B}^{\Omega'} \) in \( \Omega' \). Let \( x_0, x_1 \in \Omega \) with \( x_0 \neq x_1 \). For each \( k \in \mathbb{N} \), consider the function \( G_k(x,y) = G_{P,B}^{\Omega_k}(x,y) \).

We claim that the sequence \( \{G_k(x,y)\}_{k \in \mathbb{N}} \) is monotone nondecreasing. Indeed, for any \( 0 \leq \phi \in \mathcal{D}(\Omega_k, \partial \Omega_{k,\text{Dir}}) \), and \( x \in \Omega_k \),

\[
    u_{k+1}(x) - u_k(x) = \int_{\Omega_{k+1}} G_{k+1}(x,y)(x,y) \phi \, dy - \int_{\Omega_k} G_k(x,y) \phi \, dy = \int_{\Omega_k} (G_{k+1}(x,y) - G_k(x,y)) \phi \, dy,
\]

where \( u_{k+1} \) and \( u_k \) are positive solutions to the equation \((P,B)u = \phi \) in \( \Omega_k \) and \( \Omega_{k+1} \) respectively. Moreover, \( u_k < u_{k+1} \) on \( \partial \Omega_{k,\text{Dir}} \) implying that \( u_k < u_{k+1} \) in \( \Omega_k \). Since \( \phi \) is arbitrary, the monotonicity of \( G_k(x,y) \) follows.

Next we show the following dichotomy:

**Case 1:** Assume first that for some \( y_0 \neq x_0 \in \Omega \), the sequence \( \{G_k(x_0,y_0)\}_{k \in \mathbb{N}} \) is bounded, then by the monotonicity and Harnack convergence principle, the sequence \( \{G_k(x,y)\}_{k \in \mathbb{N}} \) converges locally uniformly in \( \Omega \setminus \{y\} \) to a positive solution of \( (P,B)u = 0 \) in \( \Omega \setminus \{y\} \), denoted by \( G_{P,B}(x,y) \). Consider again the function \( u_k(x) = \int_{\Omega_k} G_k(x,y) \phi(y) \, dy \) which satisfies \((P,B)u_k = \phi \) in \( \Omega_k \). Then, for any \( 0 \leq \phi \in C_0^\infty(\Omega) \), the monotone convergence theorem implies that

\[
    \lim_{k \to \infty} u_k(x) = \lim_{k \to \infty} \int_{\Omega_k} G_k(x,y) \phi(y) \, dy = \int_{\Omega} G_{P,B}(x,y) \phi(y) \, dy.
\]

Further, for \( x_1 \notin \text{supp}(\phi) \) the sequence \( \{u_k(x_1)\}_{k \in \mathbb{N}} \) is bounded by \( \int_{\Omega} G_{P,B}(x_1,y) \phi(y) \, dy \). Therefore, \( u_k \to u \) locally uniformly in \( \Omega \) and \((P,B)u = \phi \) in \( \Omega \). In particular, \( u \) is a regular positive supersolution of \((P,B)\), and therefore, \((P,B)\) is subcritical in \( \Omega \). Moreover, \((P,B)G_{P,B}(\cdot,y) = \delta_y \) in \( \Omega \) in the distributional sense, and \((P,B)G_{P,B}(\cdot,y) = 0 \) in \( \Omega \setminus \{y\} \) in the weak sense. Clearly, the uniqueness and minimality of such a Green function follows from a standard comparison argument. We call \( G_{P,B}^{\Omega} \) the positive minimal Green function of the operator \((P,B)\) in \( \Omega \).

**Case 2:** Assume that the sequence \( \{G_k(x_0,y_0)\}_{k \in \mathbb{N}} \) is unbounded, and again by the monotonicity and Harnack convergence principle, the sequence \( \{G_k(x,y)\}_{k \in \mathbb{N}} \) converges locally uniformly in \( \Omega \setminus \{y\} \) to \( \infty \). Let

\[
    u_k(x) = \int_{\Omega_k} G_k(x,y) \phi(y) \, dy \in H_{\partial \Omega_{k,\text{Dir}}}^1(\Omega_k).
\]

Fix \( x_1 \in \Omega \). By the monotone convergence theorem,

\[
    \lim_{k \to \infty} u_k(x_1) = \lim_{k \to \infty} \int_{\Omega_k} G_k(x_1,y) \phi(y) \, dy = \infty.
\]
Therefore, the sequence \( \{ \varphi_k := u_k/u_k(x_1) \}_{k \in \mathbb{N}} \) converges locally uniformly to a positive solution \( \varphi \) of the equation \((P, B)v = 0\) in \( \Omega \).

We claim that the function \( \varphi \) is a ground state. Indeed, let \( K \Subset \Omega \) be a Lipschitz subdomain such that \( \supp(\phi) \Subset K \). Let \( v \) be a positive continuous supersolution of the equation \((P, B)v = 0\) in \( \Omega \setminus K \) such that \( \varphi \leq Cv \) in \( \partial K_{\text{Dir}} \). Recall that \( \varphi_k \to \varphi \) uniformly on \( K \) and therefore for any \( \varepsilon > 0 \) there exists \( \varphi_k \) satisfying \( \varphi_k \leq (C + \varepsilon)v \) on \( \partial K_{\text{Dir}} \). By the generalized maximum principle in \( \Omega_k \setminus K \), we have \( \varphi_k \leq (C + \varepsilon)v \) in \( \Omega_k \setminus K \). Letting \( k \to \infty \) we deduce that \( \varphi \leq (C_1 + \varepsilon)v \) in \( \Omega \setminus K \). Since \( \varepsilon > 0 \) is arbitrarily small we have, \( \varphi \leq Cv \) in \( \Omega \setminus K \), implying that \( \varphi \) has minimal growth. By the uniqueness of the ground state, it follows that \( \varphi \) does not depend on \( \phi \). \( \square \)

As a corollary of Lemma 5.17 and Theorem 5.18 we have:

**Corollary 5.19.** Assume that \((P, B) \geq 0\) in \( \Omega \). Then \((P, B)\) is subcritical in \( \Omega \) if and only if \((P^*, B^*)\) is subcritical in \( \Omega \).

The approximation argument in Corollary 5.14 and Harnack convergence principle readily imply the following result.

**Lemma 5.20.** Assume that \((P, B)\) is subcritical in \( \Omega \). Let \( \{ \Omega_k \}_{k \in \mathbb{N}} \) be an exhaustion of \( \overline{\Omega} \setminus \partial \Omega_{\text{Dir}} \), and let \( q = p' \). Consider the functions \( f_k := |B_{1/k}(x_0)|^{-1} \chi_{B_{1/k}(x_0)} \) where \( x_0 \in \Omega_1 \). Let \( u_k \in H^1_{\partial \Omega_{\text{Dir}}} (\Omega_k) \) be a sequence of positive (Hölder continuous) solutions to the problem \((P, B)u_k = f_k\) in \( \Omega_k \). Then, \( u_k \) converges locally uniformly in \( \overline{\Omega} \setminus \partial \Omega_{\text{Dir}} \) to \( G_{P, B}^\Omega(\cdot, x_0) \). In particular, \( G_{P, B}^\Omega(\cdot, x_0) \in \mathcal{H}^0_{P, B}(\Omega \setminus \{x_0\}) \) is a positive solution in \( \Omega \setminus \{x_0\} \) of minimal growth in a neighborhood of infinity in \( \Omega \).

**Remark 5.21.** Assume that \((P, B)\) is subcritical in \( \Omega \), and let \( u \in \mathcal{H}^0_{P, B}(\Omega) \). Consider the corresponding ground state transform \((P^u, B^u)\) in \( \Omega \). Then in \( L^p_{\text{loc}}(\Omega) \)

\[
G_{P^u, B^u}^\Omega(x, y) = \frac{G_{P, B}^\Omega(x, y)u(y)}{u(x)}.
\]

We conclude the present section with some basic properties of critical and subcritical operators \((P, B)\).

**Remark 5.22.** Using the same proofs as in [27] and references therein, one deduce the following assertions for \((P, B) \geq 0\) in \( \Omega \):

1. Let \( V \in L^{p/2}_{\text{loc}}(\overline{\Omega} \setminus \partial \Omega_{\text{Dir}}), \ p > n \). Then

\[
S := \{ \lambda \in \mathbb{R} : \mathcal{H}^0_{P-\lambda V, B} \neq \emptyset \}
\]

is a closed interval, and if \( \lambda \in \text{int}(S) \), then \((P - \lambda, B)\) is subcritical in \( \Omega \). Furthermore, \( S \) is unbounded if and only if \( V \) does not change its sign in \( \Omega \). Moreover, if \( V \) has a compact support in \( \Omega \), and \((P, B)\) is subcritical in \( \Omega \), then \( \text{int}(S) \neq \emptyset \) and \( \lambda \in \partial S \) if and only if \((P - \lambda, B)\) is critical in \( \Omega \).
(2) Protter-Weinberger formula [27, Theorem 7.12]: Let $V \in L^{p'/2}_{\text{loc}}(\overline{\Omega} \setminus \partial \Omega_{\text{Dir}})$, $p > n$, be a positive potential, and consider the sets:

$$K := \{0 < u \in H^1_{\text{loc}}(\overline{\Omega} \setminus \partial \Omega_{\text{Dir}})\}, \quad M := \{\phi \geq 0 \mid \phi \in \mathcal{D}(\Omega, \partial \Omega_{\text{Dir}})\}.$$ 

Then

$$\lambda_0(P, B, V, \Omega) = \sup_{u \in K} \inf_{\phi \in M} \frac{\mathcal{B}_{P,B}(u, \phi)}{\int_{\Omega} V u \phi \, dx}.$$ 

(3) Let $V_1, V_2 \in L^{p'/2}_{\text{loc}}(\overline{\Omega} \setminus \partial \Omega_{\text{Dir}})$, $V_1 \leq V_2$. Then $\lambda_0(P, B, V_2, \Omega) \leq \lambda_0(P, B, V_1, \Omega)$, and if $\mathcal{S} \mathcal{H}_{P,B}(\Omega) = \emptyset$ and $V_1 \geq 0$, then $\lambda_0(P, B, V_2, \Omega) \leq \lambda_0(P, B, V_1, \Omega)$ [27, Lemma 7.10].

(4) If $W, V_1, V_2 \in L^{p'/2}_{\text{loc}}(\overline{\Omega} \setminus \partial \Omega_{\text{Dir}})$ such that $V_1 \leq V_2$ then $\lambda_0(P + V_2, B, W, \Omega) \geq \lambda_0(P + V_1, B, W, \Omega)$. Moreover, $\lambda_0(P, B, V, \Omega)$ is a concave function of $V$, [27, Lemma 7.9].

(5) Assume that $\Omega_1 \Subset \Omega_2$ and assume that $(P, B)$ is subcritical in $\Omega_2$. Let $V_1, V_2 \in L^{p'/2}(\Omega_2)$ satisfying $0 \leq V_1 \leq V_2$. Then $G_{P+V_2,B}^{\Omega_2}(\cdot, y) \leq G_{P+V_1,B}^{\Omega_2}(\cdot, y)$ in $\Omega_1$ [27, Corollary 8.22].

6. Symmetric operators

Throughout this section we assume that $(P, B)$ satisfies Assumptions 2.3 in $\Omega$, and that $(P, B)$ is symmetric, in other words, we assume that $\tilde{b} = \tilde{b}$. We note that if $P$ is symmetric and $\tilde{b}$ is smooth enough, then $P$ is in fact a Schrödinger-type operator of the form

$$Pu := -\text{div} \left( A(x) \nabla u \right) + (c(x) - \text{div} \tilde{b}(x)) u \quad x \in \Omega.$$ 

We prove the appropriate Allegretto-Piepenbrink-type theorem for $[P,B]$ and characterize criticality via the existence of a null-sequence.

6.1. Allegretto-Piepenbrink theorem. This theorem is well known if $\partial \Omega_{\text{Rob}} = \emptyset$, namely, when $\hat{P}$ is the Friedrichs extension of $P$ (see, [2] [20] [30] and references therein). We prove:

**Theorem 6.1** (Allegretto-Piepenbrink-type theorem). Let Assumptions 2.3 hold in $\Omega$. Then a symmetric operator $(P, B)$ is nonnegative in $\Omega$ if and only if $\mathcal{B}_{P,B}(\phi, \phi) \geq 0$ for all $\phi \in \mathcal{D}(\Omega, \partial \Omega_{\text{Dir}})$. In other words, in the symmetric case we have $\lambda_0(P, B, 1, \Omega) = \Lambda(P, B, \Omega)$.

**Proof.** Fix an exhaustion $\{\Omega_k\}_{k \in \mathbb{N}}$ of $\overline{\Omega} \setminus \partial \Omega_{\text{Dir}}$. If $\mathcal{B}_{P,B}(\phi, \phi) \geq 0$ for all $\phi \in \mathcal{D}(\Omega, \partial \Omega_{\text{Dir}})$, then for any $k > 0$ the form $\mathcal{B}_{P+1/k,B}$ is coercive in $H^1_{\partial \Omega_{k,\text{Dir}}}(\Omega_k)$, and the corresponding resolvent is a positive operator.

For $k \in \mathbb{N}$, let $f_k \in C^\infty_0(\Omega_k \setminus \Omega_{k-1})$ be a nonzero nonnegative function. By Lemma 4.3 there exists a unique positive solution $v_k$ to the problem

$$\begin{cases}
(P + \frac{1}{k}) w = f_k & \Omega_k, \\
Bw = 0 & \partial \Omega_{k,\text{Rob}}, \\
\text{Trace}(w) = 0 & \partial \Omega_{k,\text{Dir}}.
\end{cases}$$
Fix \( x_1 \in \Omega \setminus \Omega_1 \), and consider the sequence \( \{ u_k := v_k / v_k(x_1) \}_{k \in \mathbb{N}} \). By the Harnack convergence principle (Lemma 3.27), there exists a subsequence of \( \{ u_k \} \) converging to a positive solution \( u \in H^1_{\text{loc}}(\Omega) \) of \((P, B)u = 0 \) in \( \Omega \). Hence, \( \lambda_0(P, B, 1, \Omega) \geq \Lambda(P, B, \Omega) \).

Assume now that \((P, B) \geq 0 \) in \( \Omega \), and let \( u \in \mathcal{H}^0_{P, B}(\Omega) \). By the ground-state transform (Definition 3.15) in subdomains \( \Omega' \Subset R_{\Omega} \), we obtain

\[
\mathcal{B}_{P, B}(u\phi, u\phi) = \mathcal{B}_{P^\ast, B^\ast}(\phi, \phi) = \int_{\Omega} (a^{ij} D_j \phi D_i \phi) u^2 \, dx \geq 0 \quad \forall u\phi \in \mathcal{D}(\Omega, \partial \Omega_{\text{Dir}}).
\]

Hence, \( \mathcal{B}_{P, B} \) is nonnegative on \( \mathcal{D}(\Omega, \partial \Omega_{\text{Dir}}) \). Hence, \( \lambda_0(P, B, 1, \Omega) \leq \Lambda(P, B, \Omega) \). \( \square \)

**Remark 6.2.** If \((P, B)\) is symmetric in \( \Omega \) and Assumptions 2.4 hold in \( \Omega \), then the above proof and Theorem 3.40 imply that

\[
\lambda_0(P, B, 1, \Omega) = \Lambda(P, B, \Omega) = \Gamma(P, B, \Omega) = \lambda_c.
\]

### 6.2. Null-sequence and criticality.

**Definition 6.3.** A sequence \( \{ u_k \}_{k \in \mathbb{N}} \subset \mathcal{D}(\Omega, \partial \Omega_{\text{Dir}}) \) is called a **null-sequence** with respect to \((P, B)\) in \( \Omega \) if the sequence satisfies the following three properties:

1. \( u_k \geq 0 \) for all \( k \in \mathbb{N} \),
2. there exists a fixed open set \( O \Subset \Omega \) such that \( \| u_k \|_{L^2(O)} = 1 \) for all \( k \in \mathbb{N} \),
3. \( \lim_{k \to \infty} \mathcal{B}_{P, B}(u_k, u_k) = 0 \).

For the characterization of criticality by the existence of a null-sequence in the particular case \( \partial \Omega_{\text{Rob}} = \emptyset \), see, [22, Theorem 2.7] and [28, Theorem 1.4].

**Theorem 6.4.** Let Assumptions 2.3 hold in \( \Omega \). Assume that \((P, B)\) is symmetric and \((P, B) \geq 0 \) in \( \Omega \). Then \((P, B)\) admits a null-sequence in \( \Omega \) if and only if \((P, B)\) is critical in \( \Omega \).

**Proof.** Let \( u \in \mathcal{H}^0_{P, B}(\Omega) \), then by the ground-state transform we have

\[
\mathcal{B}_{P, B}(u\phi, u\phi) = \mathcal{B}_{P^\ast, B^\ast}(\phi, \phi) = \int_{\Omega} (a^{ij} D_j \phi D_i \phi) u^2 \, dx,
\]

where \( u\phi \in \mathcal{D}(\Omega, \partial \Omega_{\text{Dir}}) \).

We claim that if \((P, B)\) is critical in \( \Omega \), then for any nonempty open set \( O \Subset \Omega \), we have

\[
C_O := \inf_{\| \phi \|_{L^2(O)} = 1} \mathcal{B}_{P, B}(\phi, \phi) = 0. \tag{6.1}
\]

Indeed, pick \( 0 \leq W \in C^\infty_0(\Omega) \) such that \( 0 \leq W \leq 1 \). Then for all \( 0 \leq \phi \in \mathcal{D}(\Omega, \partial \Omega_{\text{Dir}}) \) with \( \| \phi \|_{L^2(O)} = 1 \) we have

\[
C_O \int_{\Omega} W \phi^2 \leq C_O \leq \mathcal{B}_{P, B}(\phi, \phi).
\]
Hence, the criticality of \((P, B)\) clearly implies that \(C_O = 0\), and therefore, there exists a minimizing sequence for \((6.1)\) which is obviously a null-sequence.

Now, let \(\{\phi_k\}_{k \in \mathbb{N}}\) be a null-sequence of \((P, B)\) in \(\Omega\), let \(v \in \mathcal{SH}_{P,B}(\Omega)\), and let \(w_k := \phi_k/v\). Recall that by the weak Harnack inequality (Lemma 3.11), \(v > 0\), and therefore, the sequence \(\{w_k\}_{k \in \mathbb{N}}\) is well defined. Since \(\phi_k\) is a null-sequence, it follows that for any \(K \in R \Omega \setminus \partial\Omega_{Dir}\) there exists \(C_K > 0\) such that

\[
C_K \int_K |\nabla w_k|^2 v^2 dx \leq \int_K |\nabla w_k|^2_A v^2 dx \leq \int_{\Omega} |\nabla w_k|^2_A v^2 dx \leq B_{P,B}(w_k, w_k) \leq B_{P,B}(\phi_k, \phi_k) \to 0
\]

as \(k \to \infty\), where the above inequality is a consequence of \((3.9)\) and the weak Harnack inequality (Lemma 3.11).

Consequently, \(\nabla w_k \to 0\) in \(L^2_{loc}(\Omega \setminus \partial\Omega_{Dir})\). Rellich-Kondrachov theorem (see for example [20, Theorem 8.11]) implies that up to a subsequence \(w_k \to C \geq 0\) in \(H^1_{loc}(\Omega \setminus \partial\Omega_{Dir})\). Hence, up to a subsequence \(\phi_k \to Cv\) pointwise in \(\Omega\) and also in \(L^2_{loc}(\Omega \setminus \partial\Omega_{Dir})\). Therefore, any two positive supersolutions in \(\mathcal{SH}_{P,B}(\Omega)\) are equal up to a multiplicative constant. In view of Lemma 4.7 \((P, B)\) is critical in \(\Omega\).

\[\square\]

**Appendix A.**

The appendix is devoted to a construction (under Assumptions 2.3) of a Lipschitz exhaustion of \(\overline{\Omega} \setminus \partial\Omega_{Dir}\) (see, Definition 3.25). It is well known (see for example, [9, Proposition 8.2.1]), that for any domain \(\Omega \subset \mathbb{R}^n\) there is an exhaustion \(\{\Omega_k\}_{k \in \mathbb{N}}\) of bounded smooth domains satisfying \(\Omega_k \Subset \Omega_{k+1} \Subset \Omega\), and \(\bigcup_{k \in \mathbb{N}} \Omega_k = \Omega\).

In order to obtain an exhaustion of \(\overline{\Omega} \setminus \partial\Omega_{Dir}\), we construct an exhaustion of \(\partial\Omega_{Rob}\) and glue it through appropriate ‘cylinders’ to a smooth compact exhaustion of \(\Omega\). The difficulty that arises in such a process comes from the fact that \(\partial\Omega_{Rob}\) might be unconnected and only \(C^1\)-smooth. Moreover, the ‘cylinders’ should touch \(\partial\Omega_{Rob}\) in ‘good directions’, i.e., directions with respect to which \(\partial\Omega_{Rob}\) can be locally represented as the graph of a continuous function.

Existence of such an exhaustion is known for the case, where \(\partial\Omega = \partial\Omega_{Rob} \subset C^3\) [12]. In our paper we consider the case \(\partial\Omega_{Rob} \subset \partial\Omega\), and \(\partial\Omega_{Rob} \subset C^1_{loc}\) which forces us to take a different approach. We remark that our construction is inspired by [4].

We begin with the following geometric preliminaries.

**Definition A.1.** Let \(\Omega \subset \mathbb{R}^n\) be a domain of class \(C^0\). For a point \(x_0 \in \mathbb{R}^n\), we define a **good direction at** \(x_0\), **with respect to a ball** \(B(x_0, \delta)\), \(\delta > 0\), with \(B(x_0, \delta) \cap \partial\Omega \neq \emptyset\), to be a vector \(N = N(x_0) \in S^{n-1}\) such that there is an orthonormal coordinate system \(Y = (y', y_n) = (y_1, \ldots, y_n)\) with origin at the point \(x_0\), so that \(N = e_n\) is the unit vector in the \(y_n\) direction, together with a continuous function \(f : \mathbb{R}^{n-1} \to \mathbb{R}\) (depending on \(x_0, \delta\)
and $N$), such that
\[ \Omega \cap B_{x_0}(\delta) = \{ y \in \mathbb{R}^n : f(y') < y_n, |y| < \delta \}. \]

We say that $N$ is a good direction at $x_0$ if it is a good direction with respect to some ball $B(x_0, \delta)$ with $B(x_0, \delta) \cap \partial \Omega \neq \emptyset$.

If $x_0 \in \partial \Omega$, then a good direction $N$ at $x_0$ is called a pseudonormal at $x_0$ (see [4]).

**Remark A.2.** If $\Omega \in C^1$, then any good direction at $x_0 \in \partial \Omega$ is never tangent to $\partial \Omega$ at $x_0$. Indeed, the normal direction at $x_0$ is given by the vector $(\nabla f, -1)$ which is not orthogonal to the vector $e_n = (0, \ldots, 0, 1)$.

**Proposition A.3.** Let $\Omega \subset \mathbb{R}^n$ be a $C^1$-domain. Let $x_0 \in \partial \Omega$ and let $v \in S^{n-1} \setminus T_{x_0} \partial \Omega$, where $T_{x_0} \partial \Omega$ is the tangent space to $\partial \Omega$ at $x_0$. Then $v$ is a good direction at $x_0$.

**Proof.** Let us assume without loss of generality that $x_0 = 0$. There exists a local coordinate system, $\delta > 0$, and a $C^1$ function $\phi : \mathbb{R}^{n-1} \to \mathbb{R}$ such that $\phi(0) = 0, B_{x_0}(\delta) \cap \Omega = \{ x \in \mathbb{R}^n : \phi(x') < x_n, |x| < \delta \}$.

Assume first that $v \in \mathcal{H}_i$ where $1 \leq i \leq n$ is fixed and
\[ \mathcal{H}_i := \{ z \in \mathbb{R}^n \mid z = (0, \ldots, 0, z_i, 0, \ldots, 0, z_n), z_i, z_n \neq 0 \}. \]

Denote by $0 < \theta < \pi$ the angle between $v$ and $e_n$. Let $\mathcal{O}$ be the rotation which maps $e_n$ to $v$ and fixes $e_j$ for all $j \neq n$ and $j \neq i$. Namely,
\[
\mathcal{O}_{n \times n} = \begin{pmatrix}
1 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\
0 & \cdots & \cos \theta & 0 & \cdots & \cdots & \sin \theta & \\
0 & \cdots & 1 & 0 & \cdots & \cdots & 0 & \\
0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \\
0 & \cdots & -\sin \theta & 0 & \cdots & \cdots & \cos \theta & \\
0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \\
0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \\
0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 
\end{pmatrix}.
\]

By the implicit function theorem, $\mathcal{O} \left( \begin{array}{c} x' \\ \phi(x') \end{array} \right)$ is a graph of a $C^1$ function and $e_n$ is a good direction as long as
\[ \cos \theta + \frac{\partial \phi}{\partial x_i}(0) \sin \theta \neq 0. \]

The latter condition is satisfied once $v \notin T_{x_0} \partial \Omega$.

For a general $v \in S^{n-1} \setminus T_{x_0} \partial \Omega$, we can write $v = \sum_{i=1}^{n+1} \alpha_i u_i$ where $u_i \in \mathcal{H}_j$ for some $j$, and $u_i$ is a good direction at $x_0$, $0 \leq \alpha_i \leq 1$, $u_{n+1} = e_n$ and $\sum_{i=1}^{n+1} \alpha_i = 1$. By [4, Lemma 2.2], $v/|v|$ is a good direction.

**Proposition A.4.** [4, Proposition 2.1] Let $\Omega \subset \mathbb{R}^n$ be a bounded, open set with boundary of class $C^0$. Then there exists a neighborhood $U$ of $\partial \Omega$ and a smooth function $\vec{N} : U \to S^{n-1}$ so that for each $x_0 \in U$ the unit vector $\vec{N}(x_0)$ is a good direction at $x_0$. 
We conclude our preliminaries with the following lemma.

**Lemma A.5.** Let $T, S \subset \mathbb{R}^n$ be $C^1$ domains and let $H := T \cap S$. For $x \in \partial H$, denote by $\vec{N}_T(x)$ and $\vec{N}_S(x)$ the corresponding normal vector fields to $\partial T$ and $\partial S$, respectively. Assume that for all $x \in \partial H$, $\vec{N}_T(x)$ and $\vec{N}_S(x)$ are linearly independent. Then each connected component of $H$ is a Lipschitz domain.

**Proof.** By [4, Lemma 7.1], it is enough to find $n$ linearly independent good directions at each $x_0 \in \partial H$. By Proposition A.3 at each $x_0 \in \partial H$ we can find an exterior cone of good directions with respect for both $S$ and $T$. This cone contains $n$ linearly independent good directions. □

Let Assumptions 2.3 hold in a domain $\Omega \subset \mathbb{R}^n$ with non-empty boundary. The signed distance $d_\Omega$ to $\partial \Omega$ is given by

$$d_\Omega(x) := \begin{cases} \inf_{y \in \partial \Omega} |x - y| & \text{if } x \in \Omega, \\ -\inf_{y \in \partial \Omega} |x - y| & \text{if } x \notin \Omega. \end{cases}$$

Let $\rho(x) : \mathbb{R}^n \to \mathbb{R}$ be a regularized signed distance to $\partial \Omega$ [4, Proposition 3.1], namely, $\rho(x) \in C^\infty(\mathbb{R}^n \setminus \partial \Omega) \cap C^{0,1}(\mathbb{R}^n)$ satisfies the following properties:

1. For all $x \in \mathbb{R}^n \setminus \partial \Omega$, $\frac{1}{2} \leq \frac{\rho(x)}{d_\Omega(x)} \leq 2$.
2. If $\Omega$ is bounded with continuous boundary, then there exists a relative neighborhood $U$ of $\partial \Omega$ in $\Omega$ such that $|\nabla \rho| \neq 0$ in $U \setminus \partial \Omega$.

For $\varepsilon > 0$ small enough, let

$$D_\varepsilon := \{x \in \Omega : \rho(x) > \varepsilon\} \neq \emptyset.$$  

By Sard’s theorem [19, Theorem 6.8], for almost every $\varepsilon > 0$, the open sets $D_\varepsilon$ are smooth. For each $\delta > 0$ let

$$D_{\varepsilon,\delta} := D_\varepsilon \cap B_0(1/\delta).$$

By [12, Lemma 1] and Lemma A.5 there exists $c_0$ such that for a.e. $0 < \varepsilon, \delta < c_0$, we have that $D_{\varepsilon,\delta}$ is a Lipschitz set having a finite number of connected components.

**Lemma A.6.** For each $k > 0$ there exist $\varepsilon, \delta > 0$ such that such that $D_\varepsilon$ is smooth, and if $x \in E$, where $E$ is a connected component of $\partial D_{\varepsilon,\delta}$, and $|x| < k$, then $|\nabla \rho(x)| \neq 0$ for all $x \in E \setminus \partial \Omega$.

**Proof.** As mentioned before, Sard’s theorem implies that there exists a sequence $\{\varepsilon_l\}_{l \in \mathbb{N}}$ satisfying $0 < \varepsilon_l \to 0$ as $l \to \infty$, and such that $D_{\varepsilon_l}$ is smooth. By [4, (3.4)], a regularized signed distance function $\rho(x)$ is given by the implicit equation

$$G(x, \tau) = \int_{|z|<1} d_\Omega\left(x - \frac{\tau}{2}z\right) \varphi(z) \, dz,$$
where \( \varphi \) is a smooth nonnegative function on \( \mathbb{R}^n \) supported on the unit ball such that 
\[
\int_{|z|<1} \varphi(z) \, dz = 1, 
\]
and
\[
\rho(x) = G(x, \rho(x)). \tag{A.1}
\]

If \(|x| \leq k\) then for \(\tau, \delta\) sufficiently small we have
\[
d_{\Omega} \left( x - \frac{\tau}{2}z \right) = d_{D_{\epsilon,\delta}} \left( x - \frac{\tau}{2}z \right).
\]
Hence, by (A.1) and the properties of \(\rho(x)\),
\[
\nabla \rho(x) \neq 0.
\]

Next, we modify the exhaustion \(\{\Omega_k\}_{k \in \mathbb{N}}\) of \(\Omega\) to obtain an exhaustion of \(\overline{\Omega} \setminus \partial \Omega_{\text{Dir}}\). Since \(\Omega\) is connected we may construct \(\Omega_k\) such that for each \(k\), \(\Omega_k\) is a connected component of some \(D_{\epsilon,\delta}\) (see the last paragraph in the proof of [9, Proposition 8.2.1]). We proceed by the following steps:

**Step 1:** Let \(V\) be any noncompact (connected) component of \(\partial \Omega_{\text{Rob}}\), and consider \(V\) as a \(C^1\)-differentiable manifold (without boundary). Then \(V\) admits a \(C^1\)-compact exhaustion \(\{U_m^V\}_{m \in \mathbb{N}}\) (see for example, [19, Proposition 2.28] and use [9, Proposition 8.2.1]).

**Step 2:** Assume that \(V \in C^1\) is any noncompact (connected) component of \(\partial \Omega_{\text{Rob}}\), and let \(\{U_m^V\}_{m \in \mathbb{N}}\) be a \(C^1\)-compact exhaustion of \(V\). Fix some \(U_m^V\) and consider the integral hypersurface \(\partial Z_m^V\), where \(Z_m^V\) is the union of all integral curves defined by the flow
\[
\dot{\gamma}(t) = \vec{N}(t) \in S^{n-1}
\]
with initial condition \(\gamma(0) \in U_m^V\). Here \(\vec{N}\) is smooth vector field of good directions which is defined in some \(n\)-dimensional relative neighborhood of \(U_m^V\), see Proposition A.4. In other words, \(\partial Z_m^V\) is the boundary of the union of all integral curves of the vector field \(\vec{N}\) which start from \(U_m^V\). By Remark A.2, the vector field \(\vec{N}\) is never tangent to \(U_m^V\). Consequently, \(\partial Z_m^V\) is a well defined integral hypersurface ‘starting’ from \(U_m^V\).

**Step 3:** If the component \(V\) is compact, then we take \(V\) as a trivial exhaustion of itself, and we define \(Z_1^V \subset \mathbb{R}^n\) to be a smooth bounded relative neighborhood of \(V\) in \(\Omega\).

**Step 4:** For each \(k, m\) we define the following set:
\[
C_{k,m}^V := (Z_m^V \cap (\overline{\Omega} \setminus \Omega_k)) \cup \Omega_k.
\]
By Lemma A.6 for each \(m \in \mathbb{N}\) there exist \(k_V(m)\) such that for all \(k \geq k_V(m)\),
\[
\nabla \rho(x) \neq 0 \quad \text{for all} \quad x \in Z_m^V \cap \partial \Omega_{k_V(m)}. \tag{A.2}
\]

**Remark A.7.** One can visualize the set \(C_{k,m}^V\) as follows: The integral surface \(Z_m^V\) is a perturbed cylinder attaching \(U_m^V \subset V \subset \partial \Omega_{\text{Rob}}\) to its image under some projection on \(\partial \Omega_k\). \(C_{k,m}^V\) is then the union of the cylinder and \(\Omega_k\) as depicted by the gray area in Figure 1.

**Step 5:** The last step of the construction is given by the following lemma.
Lemma A.8. Let \( \{ V^l \}_{l \in \mathbb{N}} \) be the connected components of \( \partial \Omega_{\text{Rob}} \) and let \( \{ U^V_m \}_{m \in \mathbb{N}} \) be a \( C^1 \) exhaustion of \( V^l \). For each \( m \) let
\[
q^l_m := \max \{ k_{V^l}(m) : 1 \leq l \leq m \}.
\]
Then the following assertions hold.
1. \[
\bigcup_{m \in \mathbb{N}} \left( \bigcup_{l=1}^m C^{V^l}_{q^l_m,m} \right) = \overline{\Omega} \setminus \partial \Omega_{\text{Dir}}.
\]
2. For each \( m \in \mathbb{N} \), \( \tilde{\Omega}_m := \bigcup_{l=1}^m C^{V^l}_{q^l_m,m} \) is a Lipschitz connected set.

Hence, \( \{ \tilde{\Omega}_m \}_{m \in \mathbb{N}} \) is a Lipschitz exhaustion of \( \overline{\Omega} \setminus \partial \Omega_{\text{Dir}} \).

**Proof.** (1) Follows directly from our construction.

(2) For each \( l, m \in \mathbb{N} \), \( Z^V_m \) is \( C^1 \) and \( \Omega_{q^l_m} \) is Lipschitz. Moreover, \( x \in \partial Z^V_m \cap \partial \Omega_{q^l_m} \) implies that \( \rho(x) = \varepsilon_{q^l_m} \). By [4, Remark 3.1], the latter equality and (A.2) imply that the intersection \( Z^V_m \cap \Omega_{q^l_m} \) is transversal and therefore Lipschitz. Moreover, the intersection \( Z^V_m \cap (\overline{\Omega} \setminus \Omega_{q^l_m}) \) is transversal by the definition of \( Z^V_m \), and therefore Lipschitz.

Finally, since \( Z^V_m \) is path connected we deduce that \( \tilde{\Omega}_m \) is connected as well. \( \square \)

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References

[1] R.A. Adams, “Sobolev Spaces”, Pure and Applied Mathematics, Vol. 65., Academic Press, New York-London, 1975.

[2] S. Agmon, On positivity and decay of solutions of second order elliptic equations on Riemannian manifolds, in “Methods of Functional Analysis and Theory of Elliptic Equations” (Naples, 1982), pp. 19–52, Liguori, Naples, 1983.

[3] W. Arendt, A.F.M ter Elst, and J. Glick, Strict positivity for the principal eigenfunction of elliptic operators with various boundary conditions, Adv. Nonlinear Stud. 20 (2020), 633–650.

[4] J. Ball, and A. Zarnescu, Partial regularity and smooth topology-preserving approximations of rough domains, Calc. Var. Partial Differential Equations 56 (2017), Paper No. 13, 32 pp.

[5] H. Berestycki, L. Nirenberg, S.R.S. Varadhan, The principal eigenvalue and maximum principle for second-order elliptic operators in general domains, Comm. Pure Appl. Math. 47 (1994), 47–92.

[6] D. Bucur, A. Giacomini, and P. Trebeschi, Best constant in Poincaré inequalities with traces: A free discontinuity approach, Ann. Inst. H. Poincaré Anal. Non Linéaire 36 (2019), 1959–1986.

[7] H. Berestycki, and L. Rossi, Maximum principle and generalized principal eigenvalue for degenerate elliptic operators, J. Math. Pures Appl. (9) 103 (2015), 1276–1293.

[8] I. Birindelli, Second-order elliptic equations in general domains: Hopf’s lemma and anti-maximum principle, Thesis (Ph. D.), New York University, 1992.

[9] D. Daners, Domain perturbation for linear and semi-linear boundary value problems, in “Handbook of Differential Equations: Stationary Partial Differential Equations”, Vol. 6, Handb. Differ. Equ., Elsevier/North-Holland, Amsterdam, 2008, 1–81.

[10] D. Daners, Inverse positivity for general Robin problems on Lipschitz domains, Arch. Math. (Basel) 92 (2009), 57–69.

[11] B. Devyver, M. Fraas, and Y. Pinchover, Optimal Hardy weight for second-order elliptic operator: an answer to a problem of Agmon, J. Funct. Anal. 266 (2014), 4422–4489.

[12] R. Ducasse, and L. Rossi, Blocking and invasion for reaction-diffusion equations in periodic media, Calc. Var. Partial Differential Equations 57 (2018), Paper No. 142, 39 pp.

[13] L.C. Evans, “Partial Differential Equations”, Second edition, Graduate Studies in Mathematics, 19, American Mathematical Society, Providence, 2010.

[14] L.C. Evans, and R.F. Gariepy, “Measure Theory and Fine Properties of Functions”, Studies in Advanced Mathematics, CRC Press, Boca Raton, FL, 1992.

[15] D. Gilbarg, and N. S. Trudinger, “Elliptic Partial Differential Equations of Second Order”, Reprint of the 1998 edition, Classics in Mathematics, Springer-Verlag, Berlin, 2001.

[16] T. Kato, Schrödinger operators with singular potentials, Israel J. Math. 13 (1972), 135–148.

[17] S. Kim, and G. Sakellaris, Green’s function for second order elliptic equations with singular lower order coefficients, Comm. Partial Differential Equations 44 (2019), 228–270.

[18] O. Lablée, “Spectral Theory in Riemannian Geometry” EMS Textbooks in Mathematics, European Mathematical Society (EMS), Zürich, 2015.

[19] J.M. Lee, “Introduction to Smooth Manifolds”, Second edition, Graduate Texts in Mathematics, 218, Springer, New York, 2013.

[20] E. Lieb, and M. Loss, “Analysis”, Graduate Studies in Mathematics 14. American Mathematical Society, Providence, RI, 2001.

[21] G.M. Lieberman, “Oblique Derivative Problems for Elliptic Equations”, World Scientific Publishing Co, Pte. Ltd., Hackensack, NJ, 2013.

[22] M. Murata, Structure of positive solutions to $-(\Delta + V)u$ in $\mathbb{R}^n$, Duke Math. J. 53 (1986), 869–943.
[23] J. Maly, and W. P. Ziemer, “Fine Regularity of Solutions of Elliptic Partial Differential Equations”, Mathematical Surveys and Monographs 51, American Mathematical Society, Providence, RI, 1997.

[24] R.D. Nussbaum, and Y. Pinchover, On variational principles for the generalized principal eigenvalue of second order elliptic operators and some applications, Festschrift on the occasion of the 70th birthday of Shmuel Agmon, J. Anal. Math. 59 (1992), 161–177.

[25] Y. Pinchover, On positive solutions of second-order elliptic equations, stability results, and classification. Duke Math. J. 57 (1988), 955–980.

[26] Y. Pinchover, Topics in the theory of positive solutions of second-order elliptic and parabolic partial differential equations. in: “Spectral Theory and Mathematical Physics: a Festschrift in Honor of Barry Simon’s 60th Birthday”, eds. F. Gesztesy, et al., Proceedings of Symposia in Pure Mathematics 76 Part 1, American Mathematical Society, Providence, RI, 2007, 329–356.

[27] Y. Pinchover, and T. Saadon, On positivity of solutions of degenerate boundary value problems for second-order elliptic equations, Israel J. Math. 132 (2002), 125–168.

[28] Y. Pinchover, and K. Tintarev, A ground state alternative for singular Schrödinger operators, J. Functional Analysis 230 (2006), 65–77.

[29] J. Serrin, Isolated singularities of solutions of quasi-linear equations., Acta Math. 113 (1965), 219–240.

[30] B. Simon, Schrödinger semigroups, Bull. Amer. Math. Soc. (N.S.) 7 (1982), 447–526.

[31] G. Stampacchia, Le problème de Dirichlet pour les équations elliptiques du second ordre à coefficients discontinus, Ann. Inst. Fourier (Grenoble) 15 (1965), 189–258.

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