Thermodynamics of black holes in Einstein-Gauss-Bonnet gravity with dark matter

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Abstract

In this paper, we study Einstein-Gauss-Bonnet (EGB) static black holes surrounded by three phenomenological density profiles of dark matter halos. The main result is the obtention of analytical solutions for the metric and all thermodynamic quantities, such as Hawking temperature, entropy, heat capacity at constant volume, and Gibbs free energy. With these, we could find a non-null horizon radius in which the black hole halts its evaporation by vanishing the temperature. Finally, by studying the behavior of the heat capacity and Gibbs free energy, we find the occurrence of local and global phase transitions.

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1 Introduction

The actual cosmological model for the universe, namely, ΛCDM, is well-supported by observational data, which have unveiled that the stuff of our universe consists of about 4% baryonic matter, 29.6% dark matter, and 67.4% dark energy [1]. Concerning the galaxies and their agglomerates, dark matter contributes to the formation, evolution, and coalescence of such structures [2] through the gravitational interaction, although it still be an open problem what is the true nature of that stuff (see [3] for a review).

Dark matter halos are present in practically all galaxies, and thus one has considered the connection between the galactic central black holes and these halos (see, e.g., [4–6]). Also one has cogitated the possibility of dark matter to be source for the black holes [7–10] and, vice-versa, black holes as source for dark matter, specially the primordial ones [11–14]. Wormholes sourced by dark matter also has been considered in General Relativity and modified gravity [15–18].

Modified Einstein–Gauss–Bonnet gravity has also drawn much attention in the last few years. It is based on a generalization of Einstein’s field equations by adding to the usual gravitational action higher derivatives terms in the form $S_{GB} = \int d^Dx \sqrt{-g} \mathcal{G}$, where $\mathcal{G}$ is the so-called Gauss–Bonnet invariant term, given by [19]

$$\mathcal{G} = R_{\mu\nu} R^{\mu\nu} - 4 R_{\mu} R^{\mu} + R^2,$$

(1)

which represents a topological invariant term, i.e. a total derivative, in 3+1 dimensions, not yielding an effective gravitational dynamics, therefore.

According to Lovelock’s theorem [19–21], Einstein’s general relativity with the cosmological constant is the unique theory of gravity if we assume: (i) the space-time is 3 + 1 dimensional, (ii) diffeomorphism invariance, (iii) metricity, and (iv) second order equations of motion. However, Glavan and Lin discover a general covariant modified theory of gravity in $D = 4$ space-time dimensions which propagates only the massless graviton and bypasses Lovelock’s theorem. Their theory is formulated in $D > 4$ dimensions. Its action consists of the Einstein-Hilbert term with a cosmological constant, and the Gauss-Bonnet term multiplied by a factor $1/(D-4)$. The four-dimensional theory is defined as the limit $D \to 4$.

In this singular limit, the Gauss-Bonnet invariant gives rise to non-trivial contributions to gravitational dynamics, while preserving the number of graviton degrees of freedom and being free from Ostrogradsky instability [22,23]. A non-minimum coupling of the Einstein–Gauss–Bonnet Lagrangian with the scalar field is a consistent extension for such a theory, and has been explored in some scenarios, including cosmological ones [24,25]. In this context, novel gravitational solutions were obtained and studied in several recent papers, as wormholes [26–29], black strings [30], and black holes [31–37], including black holes in connection with dark matter [38].

In this work, we study the thermodynamic properties of EGB gravitational theory considering dark matter profiles. We find a static and spherically symmetric solution that depends, generically, on a dark matter density profile. We find analytically the thermodynamic quantities, temperature, entropy, heat capacity, and free energy for our solution considering three dark matter profiles: (i) NFW profile, (ii) Burkert profile, and (iii) pseudo-isothermal profile. Throughout this work, we consider $G = c = h = k = 1$ and we will work in the convention ($-,+,+,+$).

The paper is organized as follows. In section 2, we revisit the EGB theory and find the static and spherically symmetric solution of EGB black hole with dark matter. In section 3, we present the dark matter density profiles that we apply in our solution. Next, we find the thermodynamic quantities and investigate their consequences in section 4. Finally, in section 5, we summarize the results.
2 Black hole solution in the EGB gravitational theory with a generic spherical distribution of matter

The EGB action can be written as

\[ S = \frac{1}{16\pi} \int d^Dx \sqrt{-g} [R + \alpha \mathcal{G}], \]

where \( R \) is the Ricci scalar, \( \mathcal{G} \) is given by Eq.(1) and \( \alpha \) is the Gauss-Bonnet constant. Henceforth, we consider a \textit{ansatz} static and spherically symmetric spacetime in 4D,

\[ ds^2 = -f(r)dt^2 + \frac{1}{f(r)}dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2). \]

Although in \( D = 4 \), the GB term is a total derivative and hence does not contribute to the gravitational dynamics, an extra scalar field can be coupled with the GB term, known as Einstein-dilaton Gauss-Bonnet theory \[39,40\], influencing that dynamics. However, recently Glavan and Lin \[22\] studied the implications of GB term considering a rescaling of the coupling constant,

\[ \alpha \rightarrow \frac{\alpha}{D-4}. \]

In this case, the field equation of component 00, taking into account a material source, becomes

\[ \frac{1}{r} \frac{df}{dr} + \frac{f}{r^2} - \frac{1}{r^2} - \alpha \left[ \frac{2(f - 1)}{r^3} \frac{df}{dr} - \frac{(f - 1)^2}{r^4} \right] = -\rho(r), \]

where \( \rho(r) = -T^{0}_{0} \). The exact solution of this equation is

\[ f_{\pm}(r) = 1 + \frac{r^2}{2\alpha} \left[ 1 \pm \sqrt{1 - \frac{4\alpha}{r^3} \left[ \text{const} - \frac{E(r)}{4\pi} \right]} \right], \]

where \( E(r) = 4\pi \int dr r^2 \rho(r) \). We want a solution which becomes the Schwarzschild solution when \( \alpha \rightarrow 0 \) and \( \rho(r) = 0 \). So the solution which does that is \( f_{-}(r) \) with \( \text{const} = -2M \), explicitly

\[ f_{-}(r) = 1 + \frac{r^2}{2\alpha} \left[ 1 - \sqrt{1 + \frac{8M\alpha}{r^3} + \frac{\alpha E(r)}{4\pi r^3}} \right], \]

with \( \rho(r) \) i.e. \( E(r) = 0 \), equation (7) agrees with reference \[41\]. The Einstein-Gauss-Bonnet field equations generate \( T^{0}_{0} = T^{1}_{1} \) and \( T^{2}_{2} = T^{3}_{3} \). The field equation for the component 22 is

\[ \frac{1}{2} \frac{d^2 f}{dr^2} + \frac{1}{r} \frac{df}{dr} - \alpha \left[ \frac{2f}{r^2} \frac{d^2 f}{dr^2} - \frac{2f}{r^2} \frac{df}{dr} + \frac{2}{r^2} \left( \frac{df}{dr} \right)^2 - \frac{4(f - 1)}{r^3} \frac{df}{dr} + \frac{2(f - 1)^2}{r^4} \right] = p_l, \]

where \( p_l = T^{2}_{2} = T^{3}_{3} \). After some manipulations and the using of equation (5), we find

\[ r \frac{d\rho}{dr} + 2\rho + 2p_l = 0. \]

Equation (9) is the same of \( \nabla_{\mu} T^{\mu}_{r} = 0 \) and does not bring new physics. This way, the solution of equation (7) is correct.
2.1 Thermodynamics of the EGB black hole solution surrounded by a generic spherical distribution of matter

Now, we will investigate the thermodynamic properties of the solution of eq. (7), from horizons given by the expression \( f(r_h) = 0 \). Then, the relationship between mass parameter \( M \) and \( r_h \) is

\[
M = \frac{r_h}{2} + \frac{\alpha}{2r_h} - \frac{E(r_h)}{32\pi}. \tag{10}
\]

The Hawking temperature for the metric given by eq. (3) is \( T_H = \frac{1}{4\pi} \left( \frac{df}{dr} \right)_{r=r_h} \). \( \tag{11} \)

Thus, we explicitly find the Hawking temperature as

\[
T_H = \frac{1}{4\pi} \left[ \frac{1}{r_h} - \frac{\alpha}{r_h^2} - \frac{r_h^2}{4} \rho(r_h) \right] \left( 1 + \frac{2\alpha}{r_h^2} \right)^{-1}, \tag{12}
\]

equation (12) agrees with reference [41] in limit of \( \rho = 0 \) and becomes the hawking temperature for Schwarzschild blackhole, \( T_H = 1/4\pi r_h \), when \( \alpha \to 0 \) and \( \rho = 0 \).

Another quantity of interest is the entropy, which is obtained from \( dS = dM/T_H \). We can compute it by using equations (10) and (12), finding

\[
dS = 2\pi r_h \left( 1 + \frac{2\alpha}{r_h^2} \right) dr_h, \tag{13}
\]

and then we can integrate it, arriving at

\[
S = \frac{A}{4} + 2\pi \alpha \ln \left( \frac{A}{A_0} \right), \tag{14}
\]

where \( A = 4\pi r_h^2 \) is the horizon area and \( A_0 \) is a constant with dimension of area. An interesting inference is that the matter distribution surrounding the black hole does not influence the entropy, which is therefore the same as the EGB black hole vacuum solution [41]. For more details on the discussions of the logarithmic behavior of the entropy, one can refer to [51].

Another quantity that brings physical insights is the heat capacity. We can compute this quantity as \( C_v = dM/dT_H \), which results in

\[
C_v = 2\pi r_h^2 \left( 1 + \frac{2\alpha}{r_h^2} \right)^2 \left[ 1 - \frac{\alpha}{r_h^2} - \frac{r_h^2}{4} \rho(r_h) \right] \times \left\{ \frac{5\alpha}{r_h^2} - 1 - \frac{2\alpha^2}{r_h^4} - \frac{1}{4} \rho(r_h) \left[ 1 + \frac{6\alpha}{r_h^2} \right] - \frac{r_h^3}{4} \frac{d\rho}{dr_h} \left[ 1 + \frac{2\alpha}{r_h^2} \right] \right\}^{-1} \tag{15}
\]

The last quantity that we are going to compute is the Gibbs free energy \( F = M - TS \). We find directly this quantity by using equations (10), (12) and (14), finding

\[
F = \frac{r_h}{2} + \frac{\alpha}{2r_h} - \frac{E(r_h)}{32\pi} - \frac{1}{4\pi r_h} \left[ 1 - \frac{\alpha}{r_h^2} - \frac{r_h^2}{4} \rho(r_h) \right] \left( 1 + \frac{2\alpha}{r_h^2} \right)^{-1} \left[ \pi r_h^2 + 2\pi \alpha \ln \left( \frac{r_h^2}{r_0^2} \right) \right], \tag{16}
\]

where \( r_0 \) is the radius in \( A_0 = 4\pi r_0^2 \). The equations (12), (14), (15), (16) cover, in an analytical way, all thermodynamic properties of an EGB black hole with a generic spherical distribution of matter surrounding it. In next subsection we are going to analyze these thermodynamic properties to three dark matter density profiles.
3 Thermodynamic properties of EGB black holes surrounded by dark matter

We are going to investigate the solution of equation (7) and the thermodynamic properties discussed in subsection 2.1 concerning three dark matter profiles obtained from observations combined with numerical simulations.

3.1 Dark matter density profiles

The first one is the Burkert density profile [42]. This is a cored profile of dark matter halos, which is popular due to its agreement with phenomenology [43–46]. Its density is

\[ \rho_B(r) = \frac{\rho_c}{\left(1 + \frac{r}{r_c}\right)^2 \left(1 + \frac{r^2}{r_c^2}\right)} \]  

(17)

The Burkert density profile is approximately constant for small radial distances from the center and is proportional to \( r^{-3} \) to the large ones. The \( E_B = 4\pi \int dr r^2 \rho_B(r) \), explicitly

\[ E_B(r) = \pi \rho_c r_c^3 \left[ 2 \ln \left(1 + \frac{r}{r_c}\right) + \ln \left(1 + \frac{r^2}{r_c^2}\right) - 2 \arctan \left(\frac{r}{r_c}\right) \right] . \]  

(18)

The second dark matter density profile is due to Navarro, Frenk and White (NFW); this profile was found by using high-resolution N-body simulations in studying the equilibrium density profiles of dark matter halos. They found that all profiles should have the same shape regardless of the halo mass, initial density fluctuation spectrum and the values of the cosmological parameters [47, 48]. The dark matter profile density is

\[ \rho_{NFW}(r) = \frac{\rho_s}{r_s \left(1 + \frac{r}{r_s}\right)^2} \]  

(19)

where \( \rho_s \) and \( r_s \) are two characteristic parameters of the model. Note that for large radial distances from the center, the NFW density profile is proportional to \( r^{-3} \) like the Burkert profile. When this distance is small the NFW profile increases without limit, but the mass \( E_{NFW}(r) \) is finite for all \( r \). It is given by

\[ E_{NFW}(r) = \pi \rho_s r_s^3 \left[ \ln \left(1 + \frac{r}{r_s}\right) + \frac{1}{1 + \frac{r}{r_s}} \right] . \]  

(20)

The last dark matter profile which we will analyze is the pseudo-isothermal profile [49]. The corresponding dark matter profile density is

\[ \rho_{iso}(r) = \rho_c \left[1 + \frac{r^2}{r_c^2}\right]^{-1} . \]  

(21)

This profile has not the same behavior as the two others to large radial distances, namely, its behavior is proportional to \( r^{-1} \) in this limit. On the other hand, when \( r \) is small this profile approximates to \( \rho_c \), i.e., the core density. The integrated mass is

\[ E_{iso}(r) = 4\pi \rho_c r_c^3 \left[ \frac{r}{r_c} - \arctan \left(\frac{r}{r_c}\right) \right] . \]  

(22)

It is worth pointing out that, in the limit for large distances, such a mass will cause Keplerian circular trajectories with approximately constant orbital velocities, like the ones observed in the curves of galaxy rotations.
3.2 Thermodynamic properties of EGB black holes with dark matter

Now, we will investigate the thermodynamic properties of the EGB black hole to different dark matter profiles presented in section 3.1. In figure 1, we can see the behavior of $f(r)$ for each dark matter profile. To suit values of the parameters there are horizons given by $f(r_h) = 0$ and, according to these parameters, it is possible to obtain two, one, or no horizon. We can also see the relationship between the mass parameter $M$ and the horizon radius through equation (10) with $E(r) = 4\pi \int dr r^2 \rho_{DM}(r)$, where $\rho_{DM}(r)$ is one of the dark matter density profiles, see figure 3. Solving equation (10) to find all horizons’ radii, in case of their existence, can be very hard due to the dependence of dark matter manifested in $E(r)$. Then we made figure 2 to exemplify the possibility of no one, one, or two horizons’ radii. In figure 2, we plot the set of metric functions with the Burkert profile, and without dark matter, each set has a different value of the mass parameter: $M = 0.5, M = 0.7, M = 1$. The behavior of the other dark matter profiles is similar.

Let us investigate the Hawking temperature for the dark matter profiles. As already discussed, the Hawking temperature for a generic spherical distribution of matter is given by equation (12). We can see the behavior of the temperature to three dark matter profiles in figure 4. The temperature is equal to zero for the horizon radius which satisfies

$$4r_h^2 - 4\alpha - r_h^4 \rho(r_h) = 0,$$

in a specific way, the temperature vanishes for each dark matter profile for all $r_h$ which satisfies the equation (23) where $\rho(r)$ is the dark matter profile density. When the black hole reaches this critical $r_h$, the evaporation ceases and a remnant survives. For instance, on considering the pseudo-isothermal dark matter profile (21), this critical horizon radius is given by

$$r_h^c = \sqrt{2} \sqrt{\frac{\sqrt{\alpha^2 + \alpha r_c^4 \rho_c} + 2\alpha r_c^2 + r_c^4 + \alpha - r_c^2}{r_c^2 \rho_c + 4}}.$$

Figure 1: $f(r)$ for different DM profiles. $\rho_c = \rho_s = 0.5, r_c = r_s = 1.5, M = 1$ and $\alpha = 0.5$
Figure 2: $f(r)$ for different DM profiles. $\rho_c = \rho_s = 0.5$, $r_c = r_s = 1.5$ and $\alpha = 0.5$. The solid line is the metric function with Burkert dark matter and the dotter line is the metric function without dark matter. The blue ones has $M = 0.5$, the green ones has $M = 0.7$, and the red ones has $M = 1$.

Notice that, in this case, for $\alpha = 0$ (i.e., in general relativity), there is no such remnant, since $r_c^e = 0$, even in presence of dark matter.

As already stated in subsection 2.1, the entropy is not changed by a generic spherical distribution of matter surrounding the EGB black hole. Thus, the entropy for the three dark matter profiles is the same of equation (14).

Let us present now the results of the heat capacity at constant volume for the three dark matter
profiles. The behavior of this quantity can be seen in figure 5. The black hole presents local phase transitions when the heat capacity vanishes, i.e., at the horizon radius which obeys

\[
\frac{5\alpha}{r_h^3} - 1 - \frac{2\alpha^2}{r_h} - \frac{r_h^2}{4} \rho(r_h) \left[ 1 + \frac{6\alpha}{r_h^2} \right] - \frac{r_h^3}{4} \frac{\rho'(r_h)}{dr_h} \left[ 1 + \frac{2\alpha}{r_h^2} \right] = 0, \tag{25}
\]

where \( \rho \) is the dark matter profile density. Notice that the black holes present smooth (abrupt) phase transitions, from unstable (stable) regions to the stable (unstable) ones (\( C_v < 0 \), unstable; \( C_v > 0 \), stable).

Figure 4: Behavior of \( T_H \) for different DM profiles, \( \rho_c = \rho_s = 0.5 \), \( r_c = r_s = 1.5 \), \( M = 1 \) and \( \alpha = 0.5 \).

Figure 5: Behavior of \( C_v \) for different DM profiles, \( \rho_c = \rho_s = 0.5 \), \( r_c = r_s = 1.5 \), \( M = 1 \) and \( \alpha = 0.5 \).
We also considering the Gibbs free energy, equation (16), for each dark matter profile. The behavior of this quantity can be seen in figure 6. We can see by means of the plot that the Gibbs free energy encompasses only smooth (global) phase transitions ($F < 0$, stable; $F > 0$, unstable).

Figure 6: Behavior of $F$ for different DM profiles, $\rho_c = \rho_s = 0.5$, $r_c = r_s = 1.5$, $M = 1$, $r_0 = 0.1$ and $\alpha = 0.5$.
4 Conclusion

In this paper, we have obtained exact black hole solutions surrounded by dark matter halos, in the context of Einstein–Gauss–Bonnet gravity, and studied their thermodynamic properties. Thus, in section 2, we found the EGB black hole solution with a generic spherical distribution (density) of matter and in section 2, we have analytically found the corresponding thermodynamic properties: temperature, entropy, heat capacity at constant volume, and Gibbs free energy (see equations (12), (14), (15), (16)).

In section 3, we analyzed the EGB black hole solution particularizing to the following phenomenological density profiles of dark matter: (i) Burkert, (ii) NFW profile, and (iii) pseudo-isothermal. The behavior of the solution summarized in equation (7) depends on the EGB coupling constant $\alpha$, as well as on the dark matter characteristic parameters of density and radius. This solution can present zero, one, or two horizons according to the setting of parameters. Figure 1 exhibits the solution behavior with two horizons for each dark matter profile. It is worth point out that these horizons are smaller than the one associated to the Schwarzschild black hole solution.

Using the same set of parameters, we studied the thermodynamic properties. Figure 4 presents the temperature for each dark matter profile according to equation (12). An interesting point is that the temperature goes to zero for every horizon radius which satisfies equation (23), indicating the emergence of a remnant, when the black hole halts its evaporation. Figure 4 presents graphically these points for the dark matter cases. We saw also that a generic spherically symmetric distribution of matter as a gravitational source does not contribute to the entropy in equation (14), then the entropy is the same as a pure EGB black hole.

We also analyzed the heat capacity at constant volume and the Gibbs free energy. We presented the behavior of the former for each dark matter profile in figure 5, according to equation (15), where we have identified continuous and discontinuous (local) phase transitions for critical horizon radii, which satisfy equation (25). We also analyzed the behavior of Gibbs free energy for each dark matter profile in figure 6, according to equation (16). In terms of this quantity, the EGB black hole surrounded by dark matter exhibits continuous (global) phase transitions.

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