SCHUR-WEYL DUALITY FOR $U_{v,t}(sl_n)$

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Abstract. In [8], the authors get a new presentation of two-parameter quantum algebra $U_{v,t}(\mathfrak{g})$. Their presentation can cover all Kac-Moody cases. In this paper, we construct a suitable Hopf pairing such that $U_{v,t}(sl_n)$ can be realized as Drinfeld double of certain Hopf subalgebras with respect to the Hopf pairing. Using Hopf pairing, we construct a $R$-matrix for $U_{v,t}(sl_n)$ which will be used to give the Schur-Weyl dual between $U_{v,t}(sl_n)$ and Hecke algebra $H_k(v,t)$. Furthermore, using the Fusion procedure we construct the primitive orthogonal idempotents of $H_k(v,t)$. As a corollary, we give the explicit construction of irreducible $U_{v,t}(sl_n)$-representations of $V^\otimes k$.

1. Introduction

Classical Schur-Weyl duality related irreducible finite-dimensional representations of the general linear and symmetric groups [18]. The quantum version for the quantum enveloping algebra $U_q(sl_n)$ and the Hecke algebra $H_q(S_m)$ has been one of the pioneering examples [13] in the fervent development of quantum groups. Two-parameter general linear quantum groups were introduced by Takeuchi in 1990 [17]. The related references are [1, 4, 6, 7, 12, 17]. In 2001, Benkart and Witherspoon obtained the structure of two-parameter quantum groups corresponding to the general linear Lie algebra $gl_n$ and the special linear Lie algebra $sl_n$ with a different motivation [3]. They showed that the quantum groups can be realized as Drinfeld doubles of certain Hopf subalgebras with respect to Hopf pairings. Using Hopf pairing, Benkarat and Witherspoon constructed $R$-matrix which is used to establish an analogue of Schur-Weyl duality [2].

In [8], using geometric construction, the authors got a new presentation of generators and relations for a two-parameter quantum algebra $U_{v,t}$ determined by a certain matrix which may served as a generalized Cartan matrix. The two parameters $v$ and $t$ they used are different from the one $(r, s)$. Furthermore, their presentation covered all Kac-Moody cases, unlike the one in literature which mainly studies finite type and some affine types. A two-parameter quantum algebra $U_{v,t}$ is a two-cocycle deformation, depending only on the second parameter $t$, of its one-parameter analogue. And the algebra $U_{r,s}$ in [3], is a two-cocycle deformation depending both parameters $(r = s^{-1})$.

We focus on the two-parameter quantum group $U_{v,t}(sl_n)$ for the purpose of giving the Shur-Weyl dual between $U_{v,t}(sl_n)$ and $H_k(v,t)$. We will show that $U_{v,t}(sl_n)$ also has a Drinfeld double realization by two certain Hopf subalgebras, but the $R$-matrix constructed by similar way in [2] can not afford a representation of $H_k(v,t)$, we need to take a suitable modification.

This paper is organized as follows. In section 2 we give a Hopf pairing between two certain Hopf subalgebras of $U_{v,t}(sl_n)$. Then we prove that $U_{v,t}(sl_n)$ can be realized as the
Drinfeld double of certain Hopf subalgebras with respect to the Hopf pairing. In section 3 we construct the tensor power representation \( V \otimes k \) of \( U_{v,t}(sl_n) \) and \( R \)-matrix \( \tilde{R} \). In section 4 we prove that the \( U_{v,t}(sl_n) \)-module \( V \otimes k \) affords a representation of Hecke algebra \( H_k(v,t) \). This leads to a Schur-Weyl duality between \( U_{v,t}(sl_n) \) and Hecke algebra \( H_k(v,t) \). In section 5 we give a family of primitive orthogonal idempotents of \( H_k(v,t) \). In section 6 irreducible representations of \( U_{v,t}(sl_n) \) are constructed by using the fusion procedure.

2. Two-Parameter Quantum Group \( U_{v,t}(sl_n) \) and Its Drinfeld Double

In this section, we revisit the definitions of two-parameter quantum algebra \( U_{v,t}(sl_n) \) introduced by Fan and Li in \cite{FanLi} and its Hopf algebra structure. In particular, \( U_{v,t}(sl_n) \) also can be realized as a Drinfeld double of its certain subalgebras.

2.1. Two-Parameter Quantum Group \( U_{v,t}(sl_n) \).

In this paper, we fix the Cartan datum \((\Omega_{n-1}, \cdot)\) of type \( A_{n-1} \), where

\[
\Omega_{n-1} = (\Omega_{ij}) = \begin{pmatrix}
1 & 0 & 0 & \ldots & 0 & 0 \\
-1 & 1 & 0 & \ldots & 0 & 0 \\
0 & -1 & 1 & \ldots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \ldots & 1 & 0 \\
0 & 0 & 0 & \ldots & 1 & 0
\end{pmatrix}_{(n-1) \times (n-1)},
\]

and for any \( 1 \leq i, j \leq n - 1 \), denote \( \langle i, j \rangle = \Omega_{ij} \), then \( i \cdot j \) can be defined as \( \langle i, j \rangle + \langle j, i \rangle \).

**Definition 1** (\cite{FanLi}). The two-parameter quantum algebra \( U_{v,t}(sl_n) \) associated to \( \Omega_{n-1} \) is an associative \( \mathbb{Q}(v,t) \)-algebra with 1 generated by symbols \( E_i, F_i, K_i^{\pm 1}, K_i'^{\pm 1} \), \( \forall 1 \leq i < n \) and subject to the following relations.

\[
\begin{align*}
(K1) \quad & K_i^{\pm 1}K_j^{\pm 1} = K_j^{\pm 1}K_i^{\pm 1}, \quad K_i'^{\pm 1}K_j'^{\pm 1} = K_j'^{\pm 1}K_i'^{\pm 1}, \\
& K_i^{\pm 1}K_j'^{\pm 1} = K_j'^{\pm 1}K_i^{\pm 1}, \quad K_i'^{\pm 1}K_j^{\pm 1} = 1 = K_i^{\pm 1}K_i'^{\pm 1}, \\
(R2) \quad & K_iE_jK_i'^{-1} = v^{ij}t^{(i,j)-(j,i)}E_j, \quad K_i'iE_jK_i'^{-1} = v^{-ij}t^{(i,j)-(j,i)}E_j, \\
& K_iF_jK_i'^{-1} = v^{-ij}t^{(j,i)-(i,j)}F_j, \quad K_i'iF_jK_i'^{-1} = v^{ij}t^{(j,i)-(i,j)}F_j, \\
(R3) \quad & E_iF_j - F_jE_i = \delta_{ij}K_i - K_i', \\
(R4) \quad & [E_i, E_j] = [F_i, F_j] = 0 \text{ if } |i - j| > 1, \\
(R5) \quad & E_i^2E_{i+1} - t(v + v^{-1})E_iE_{i+1}E_i + t^2E_{i+1}E_i^2 = 0, \\
& E_iE_{i+1}^2 - t(v + v^{-1})E_iE_{i+1}E_i + t^2E_{i+1}^2E_i = 0, \\
(R6) \quad & F_i^2F_{i+1} - t^{-1}(v + v^{-1})F_iF_{i+1}F_i + t^{-2}F_{i+1}F_i^2 = 0, \\
& F_iF_{i+1}^2 - t^{-1}(v + v^{-1})F_iF_{i+1}F_i + t^{-2}F_{i+1}^2F_i = 0.
\end{align*}
\]

There is a Hopf algebra structure on the algebra \( U_{v,t}(sl_n) \) with the comultiplication \( \Delta \) by

\[
\Delta(K_i^{\pm 1}) = K_i^{\pm 1} \otimes K_i^{\pm 1}, \quad \Delta(K_i'^{\pm 1}) = K_i'^{\pm 1} \otimes K_i', \\
\Delta(E_i) = E_i \otimes 1 + K_i \otimes E_i, \quad \Delta(F_i) = 1 \otimes F_i + F_i \otimes K_i',
\]
the counit \( \varepsilon \) by

\[
\varepsilon(K_i^{\pm 1}) = \varepsilon(K_i'^{\pm 1}) = 1, \quad \varepsilon(E_i) = \varepsilon(F_i) = 0, \quad S(K_i^{\pm 1}) = K_i'^{\pm 1},
\]

and the antipode \( S \) by

\[
S(K_i'^{\pm 1}) = K_i'^{\mp 1}, \quad S(E_i) = -K_i'^{-1}E_i, \quad S(F_i) = -F_iK_i'^{-1}.
\]

2.2. Drinfeld Double Realization Of \( U_{v,t}(sl_n) \).

**Definition 2.** (See [14], 3.2.1) A Hopf paring of two Hopf algebras \( H \) and \( H' \) is a bilinear form \((-,-): H' \times H \to K \) (a field) such that

1. \( (1, h) = \epsilon_H(h), \quad (h', 1) = \epsilon_{H'}(h'); \)
2. \( (h', hk) = (\Delta_H(h'), h \otimes k) = \sum(h'_1, h)(h'_2, k); \)
3. \( (h'k', h) = (h' \otimes k', \Delta_H(h)) = \sum(h', h_{(1)})(k', h_{(2)}); \)

for all \( h, k \in H, h', k' \in H' \), where \( \epsilon_H \) and \( \epsilon_{H'} \) are the counits of \( H \) and \( H' \) respectively, and \( \Delta_H \) and \( \Delta_{H'} \) are their comultiplications. For \( h \in H, \Delta(h) = \sum h_{(1)} \otimes h_{(2)}. \)

A direct consequence is that

\[
(S_H(h'), h) = (h', S_H(h))
\]

for all \( h \in H \) and \( h' \in H' \), where \( S_{H'} \) and \( S_H \) are the antipodes of \( H' \) and \( H \) respectively.

Let \( B \) (resp. \( B' \)) be the Hopf subalgebra of \( U_{v,t}(sl_n) \) generated by \( E_i, K_i^{\pm 1} \) (resp. \( F_i, K_i'^{\pm 1} \)) for \( 1 \leq i < n \). \( B^{\text{coop}} \) is the Hopf algebra having the opposite comultiplication to the Hopf algebra \( B' \) and \( S_{B^{\text{coop}}} = S_{B'}^{-1}, \Delta_{B^{\text{coop}}} = \Delta_{B'}^{\text{op}}. \)

**Proposition 1.** There exists a unique Hopf paring \((-,-): B^{\text{coop}} \times B \to \mathbb{Q}(v,t)\) such that

\[
(F_i, E_j) = \frac{\delta_{ij}}{v^{i-1} - v},
\]

\[
(K_i', K_j) = v^{ij}i^{(j,i)-(i,j)},
\]

for any \( 1 \leq i, j < n \), and all other pairs of generators are 0. Moreover, we have

\[
(S(a), S(b)) = (a, b)
\]

for any \( a \in B^{\text{coop}}, b \in B. \)

**Proof.** Any Hopf paring of bialgebras is determined by the values on the generators, so the uniqueness is clear. The process of proof reduces to the existence.

The pairings defined by (2) and (3) in the proposition can be extended to a bilinear form on \( B^{\text{coop}} \times B \) by requiring that the conditions (1), (2) and (3) in definition 2 hold. We only need to verify that the relations (2) and (3) in \( B' \) and \( B \) are preserved.

It is straightforward to check that the bilinear form preserves all the relations among the \( K_i^{\pm 1} \) in \( B \) and the \( K_i'^{\pm 1} \) in \( B' \). Next, for any \( 1 \leq i, j < n \), we check

\[
(X, K_j E_i) = (X, v^{ij}i^{(j,i)-(i,j)} E_i K_j),
\]

where \( X \) is any word in the \( F_i \) and \( K_i'^{\pm 1}, 1 \leq i < n. \) If \( X = K_k^l F_i \), the left hand side
\((X, K_j E_i) = (\Delta^{op}(K'_k) \Delta^{op}(F_i), K_j \otimes E_i)\)
\[= (K'_k F_i \otimes K_k + K'_k K'_i \otimes K'_k F_i, K_j \otimes E_i)\]
\[= (K'_k F_i, K_j) (K'_k, E_i) + (K'_k K'_i, K_j) (K'_k F_i, E_i)\]
\[= \frac{1}{v^{-1} - v} v^{j \cdot t(j, i) - (i, j)} (K'_k, K_j) (K'_k, K_i) \quad (l = i)\]

the right hand side
\((X, v^{j \cdot t(j, i) - (i, j)} E_i K_j) = v^{j \cdot t(j, i) - (i, j)} (\Delta^{op}(K'_k F_i), E_i \otimes K_j)\)
\[= v^{j \cdot t(j, i) - (i, j)} (K'_k F_i \otimes K_k + K'_k K'_i \otimes K'_k F_i, E_i \otimes K_j)\]
\[= \frac{1}{v^{-1} - v} v^{j \cdot t(j, i) - (i, j)} (K'_k, K_j) (K'_k, K_i) \quad (l = i)\]

Hence, \((X, K_j E_i) = (X, v^{j \cdot t(j, i) - (i, j)} E_i K_j)\).

In particular, it can be similarly checked that the bilinear form preserves all the other relations in \(B\) and \(B'\).

\[\square\]

**Definition 3.** (See \[14\], 3.2) If there is a Hopf pairing between Hopf algebras \(H\) and \(H'\), then we may form the Drinfeld double \(D(H, H'^{\text{coop}})\), where \(H'^{\text{coop}}\) is the Hopf algebra having the opposite coproduct to \(H\). \(D(H, H'^{\text{coop}})\) is a Hopf algebra whose underlying vector space is \(H \otimes H'\) with the tensor product coalgebra structure. The algebra structure is given by as follows:
\[(a \otimes f)(a' \otimes f') = \sum (S_{H'^{\text{coop}}}(f(1)), a'_1(1)) (f(3), a'_3(3)) a a' \otimes f(2) f'\]
for \(a, a' \in H\) and \(f, f' \in H'\). And the antipode \(S\) is given by
\[S(a \otimes f) = (1 \otimes S_{H'^{\text{coop}}}(f))(S_H(a) \otimes 1)\]

Clearly, the algebras \(H\) and \(H'^{\text{coop}}\) are identified with \(H \otimes 1\) and \(1 \otimes H'^{\text{coop}}\) respectively in \(D(H, H'^{\text{coop}})\).

**Proposition 2.** \(D(\cal B, \cal B'^{\text{coop}})\) is isomorphic to \(U_{v, t}(\mathfrak{sl}_n)\).

**Proof.** Define the embedding maps
\[\iota : \cal B \rightarrow D(\cal B, \cal B'^{\text{coop}})\]
\[E_i \mapsto \hat{E}_i := \iota(E_i) = E_i \otimes 1\]
\[K_i^{\pm 1} \mapsto \hat{K}_i^{\pm 1} := \iota(K_i^{\pm 1}) = K_i^{\pm 1} \otimes 1,\]
and
\[\iota' : \cal B'^{\text{coop}} \rightarrow D(\cal B, \cal B'^{\text{coop}})\]
\[F_i \mapsto \hat{F}_i := \iota(F_i) = 1 \otimes F_i\]
\[K_i^{\prime \pm 1} \mapsto \hat{K}_i^{\prime \pm 1} := \iota(K_i^{\prime \pm 1}) = 1 \otimes K_i^{\prime \pm 1}.\]
Then $\mathcal{B}$ and $\mathcal{B}'_{coop}$ can be viewed as subalgebras in $D(\mathcal{B}, \mathcal{B}'_{coop})$. A map $\varphi$ between $D(\mathcal{B}, \mathcal{B}'_{coop})$ and $U_{v,t}(sl_n)$ is defined as follows:

$$\varphi : D(\mathcal{B}, \mathcal{B}'_{coop}) \rightarrow U_{v,t}(sl_n)$$

$$\hat{E}_i \mapsto \varphi(\hat{E}_i) = E_i,$$

$$\hat{F}_i \mapsto \varphi(\hat{F}_i) = F_i,$$

$$\hat{K}^{\pm 1}_i \mapsto \varphi(\hat{K}^{\pm 1}_i) = K^{\pm 1}_i,$$

$$\hat{K}^{\pm 1}_i \mapsto \varphi(\hat{K}^{\pm 1}_i) = K'^{\pm 1}_i.$$

Note that, $\varphi$ preserves the coalgebra structures. Next we will check that $\varphi$ preserves the relations (R1-R6) in $U_{v,t}(sl_n)$. Consider $\varphi(\hat{K}_j \hat{K}_i' - \hat{K}_i' \hat{K}_j)$. By definition,

$$\hat{K}_j \hat{K}_i' = (K_j \otimes 1)(1 \otimes K_i') = K_j \otimes K_i'.$$

To calculate $\hat{K}_i' \hat{K}_j$, we have

$$\Delta^2(K_j) = K_j \otimes K_j \otimes K_j,$$

so that

$$\hat{K}_i' \hat{K}_j = (1 \otimes K'_i)(K_j \otimes 1)$$

$$= (S_{\mathcal{B}'_{coop}}(K'_i), K_j)(K'_i, K_j)K_j \otimes K'_i$$

$$= K_j \otimes K'_i.$$
3. Finite Dimensional Representations Of $U_{v,t}(sl_n)$

3.1. The Natural Representation Of $U_{v,t}(sl_n)$. Set $\Lambda = \mathbb{Z}\epsilon_1 \oplus \mathbb{Z}\epsilon_2 \oplus \cdots \oplus \mathbb{Z}\epsilon_n$. For any $\lambda = \sum_{j=1}^{n} \lambda_j \epsilon_j \in \Lambda$, one defines the algebra homomorphism $\hat{\lambda} : U^0 \to \mathbb{Q}(v,t)$ by

$$
\hat{\lambda}(K_i) = v^{\sum_{i=1}^{n} \lambda_j} \sum_{t=1}^{n} \lambda_j (i-j)^{-1}, \quad \hat{\lambda}(K'_i) = v^{-\sum_{j=1}^{n} \lambda_j} \sum_{t=1}^{n} \lambda_j (i-j)^{-1},
$$

where $\langle i, n \rangle = \delta_{in}$, and $\langle n, i \rangle = \begin{cases} 0, & 1 \leq i < n - 1; \\ -1, & i = n - 1; \\ 1, & i = n; \end{cases}$ for $1 \leq i \leq n$.

Let $V_n$ be the $n$-dimensional $\mathbb{Q}(v,t)$ vector space with basis $\{v_1, v_2, \cdots, v_n\}$. For any $1 \leq i, j \leq n$, set $E_{ij}$ be the $n \times n$ matrices with entry 1 in row $i$ and column $j$ and other entries 0.

We define an $U_{v,t}(sl_n)$ representation $\rho'_n : U_{v,t}(sl_n) \times V_n \to V_n$ by the following way:

$$
\rho'_n(E_i) = E_{i+1}, \quad \rho'_n(F_i) = E_{i+1,i},
$$

$$
\rho'_n(K_i) = t^{-1}I_n + (v - t^{-1})E_{i,i} + (v^{-1} - t^{-1})E_{i+1,i},
$$

$$
\rho'_n(K'_i) = t^{-1}I_n + (v^{-1} - t^{-1})E_{i,i} + (v - t^{-1})E_{i+1,i}.
$$

This follows from the fact that $K_i v_j = v^{k_{i,j}} \sum_{k=1}^{n} \epsilon_k v_j$ for all $1 \leq i \leq n - 1$, $1 \leq j \leq n$ that $v_j$ corresponds to the weight $\sum_{k=1}^{n} \epsilon_k$. Thus, $V_n = \bigoplus_{j=1}^{n} V_{\epsilon_j}$ is the natural analogue of the $n$-dimensional representation of $sl_n$, and $(\rho'_n, V_n)$ is an irreducible representation of $U_{v,t}(sl_n)$.

3.2. Tensor Power representations Of $U_{v,t}(sl_n)$.

Definition 4. Let $M, N$ be $U_{v,t}(sl_n)$-modules. For $a \in U_{v,t}(sl_n)$, $u \in M$ and $v \in N$, we define $a(u \otimes v) = \Delta(a) (u \otimes v)$, then under such action, $M \otimes N$ is a $U_{v,t}(sl_n)$-module. We call it the tensor power of modules $M$ and $N$.

Definition 5. More generally, suppose that $M_1, M_2, \cdots M_n$ are $U_{v,t}(sl_n)$-modules. For $a \in U_{v,t}(sl_n)$, $m_i \in M_i$, we define $a(m_1 \otimes m_2 \otimes \cdots \otimes m_n) = \Delta^{-1}(a)(m_1 \otimes m_2 \otimes \cdots \otimes m_n)$, where $\Delta^k = (\Delta \otimes id^{k-1}) \Delta^{k-1}$ for any $k \in \mathbb{N}$. Then $M_1 \otimes M_2 \otimes \cdots \otimes M_n$ is a $U_{v,t}(sl_n)$-module. We call it the tensor product of modules $M_1, M_2, \cdots, M_n$.

Remark 1. It is easy to know that for any $k \in \mathbb{N}$, $1 \leq j < n$,

$$
\Delta^{-1}(E_j) = \sum_{i=1}^{k} K_j \otimes \cdots \otimes K_j \otimes E_j \otimes 1 \otimes \cdots \otimes 1.
$$

$$
\Delta^{-1}(F_j) = \sum_{i=1}^{k} 1 \otimes \cdots \otimes 1 \otimes F_j \otimes K'_j \otimes \cdots \otimes K'_j.
$$

$$
\Delta^{-1}(K_i) = K_i \otimes K_i \otimes \cdots \otimes K_i.
$$

$$
\Delta^{-1}(K'_i) = K'_i \otimes K'_i \otimes \cdots \otimes K'_i.
$$
Generally, the $k$-fold tensor power $(\rho_n, V_n^\otimes k)$ of $(\rho_n', V_n)$ is also a $U_{v,t}(sl_n)$-module, where $V_n^\otimes k = V_n \otimes V_n \otimes \cdots \otimes V_n$ ($k$ factors).

**Proposition 3.** For $n \geq k$, if $(\rho'_n, V_n)$ is the natural representation of $U_{v,t}(sl_n)$, then $(\rho_n, V_n^\otimes k)$ is a cyclic $U_{v,t}(sl_n)$-module generated by $\{v_1, v_2, \cdots, v_n\}$.

**Proof.** The proof is similarly to lemma 6.2 in [2]. We omit it. \qed

### 3.3. The $R$-Matrix

To obtain a $H_k(v, t)$-representation from $U_{v,t}(sl_n)$-representation $(\rho_n, V_n^\otimes k)$, we shall construct a $R$-matrix.

As above notation, denote by $U^+$ (resp. $U^-$) the subalgebra of $U_{v,t}(sl_n)$ generated by 1 and $E_i$ (resp. 1 and $F_i$) for all $1 \leq i \leq n-1$. Then $U^+$ has a decomposition $U^+ = \bigoplus_{\xi \in \Lambda^+} U_{\xi}^+$, where

$$U_{\xi}^+ = \{ x \in U^+ \mid K_i x = \sum_{j=1}^{n} \xi_{i,j} x K_i, 1 \leq i \leq n-1 \}$$

for $\xi = \sum_{j=1}^{n} \xi_j \in \Lambda^+$ and $\xi_j \geq 0$.

It can be checked that $U_{\xi}^+$ is spanned by all the monomials $E_{i_1}E_{i_2}\cdots E_{i_m}$ such that $\xi = \epsilon_{i_1} + \epsilon_{i_2} + \cdots + \epsilon_{i_m}$. For $U^-$ we have similar decomposition $U^- = \bigoplus_{\xi \in \Lambda^-} U_{\xi}^-$. Let $d_{\xi}$ be the $\mathbb{Q}(v, t)$ dimension of $U_{\xi}^+$. Assume $\{u_{k,j}\}_{k=1}^{d_{\xi}}$ is a basis for $U_{\xi}^+$, and $\{v_{k,j}\}_{k=1}^{d_{\xi}}$ is the dual basis for $U_{-\xi}$ with respect to to the Hopf pairing defined by (2) and (3). That is to say, if $\{E_1\}$ is a basis for $U_{e_1}^+$, then $\{(v - v)F_1\}$ is the dual basis for $U_{-e_1}$.

If $\lambda = \sum_{j=1}^{n} \lambda_j \epsilon_j \in \Lambda$, set

$$K_{\lambda} = K_1^{\lambda_1} \cdots K_{n-1}^{\lambda_{n-1}} A_n^{\lambda_n}, \quad K'_{\lambda} = (K_1')^{\lambda_1} \cdots (K_{n-1}')^{\lambda_{n-1}} B_n^{\lambda_n},$$

and

$$\Theta = \sum_{\xi \in \Lambda^+} \sum_{k=1}^{d_{\xi}} u_k^\xi \otimes u_k^\xi.$$

Let $\widetilde{R} = \widetilde{R}_{V_n, V_n} : V_n \otimes V_n \longrightarrow V_n \otimes V_n$ be the $R$-matrix defined by

$$\widetilde{R}_{V_n, V_n}(v_i \otimes v_j) = \Theta \circ f(v_j \otimes v_i)$$

where $f(v_j \otimes v_i) = (K_1', K_\mu)^{-1}$ when $v_i \in V_\mu$ and $v_j \in V_\lambda$, the Hopf pairing $(-, -)$ is defined by (2), (3) and

$$(B_n, A_n) = 1, \quad (B_n, K_j) = v^{(n,j)} t^{-(n,j)}, \quad (K'_i, A_n) = (vt)^{\langle n,i \rangle}.$$

More precisely, $\widetilde{R}$ acts on $V_n \otimes V_n$ by

$$\widetilde{R} = \sum_{i<j} vt E_{j,i} \otimes E_{i,j} + \sum_{i<j} v^{-1} E_{i,j} \otimes E_{j,i} + t^{-1} (1 - v^2) \sum_{i<j} E_{j,i} \otimes E_{i,j} + \sum_{i} E_i \otimes E_i.$$
Proposition 4. Let $\tilde{R}_i$ be the $U_{v,t}(sl_n)$-module isomorphism on $V_n^{\otimes k}$ defined by

$$\tilde{R}_i(z_1 \otimes z_2 \otimes \cdots \otimes z_k) = z_1 \otimes z_2 \otimes \cdots \otimes z_{i-1} \otimes \tilde{R}(z_i \otimes z_{i+1}) \otimes z_{i+2} \otimes \cdots \otimes z_k,$$

where $z_1, z_2, \ldots, z_k \in V_n$. Then $\tilde{R}_i$ satisfy the Yang-Baxter equations. That is to say, the following braid relations hold:

$$\tilde{R}_i \tilde{R}_j = \tilde{R}_j \tilde{R}_i, \quad |i - j| \neq 1;$$

$$\tilde{R}_i \tilde{R}_{i+1} \tilde{R}_i = \tilde{R}_{i+1} \tilde{R}_i \tilde{R}_{i+1}, \quad 1 \leq i < k.$$

4. HECKE ALGEBRA AND THE SCHUR-WEYL DUALITY FOR $U_{v,t}(sl_n)$

Let $v, t$ be any formal variables. We introduce the Hecke algebra $H_k(v, t)$ as follows.

Definition 6. The Hecke algebra $H_k(v, t)$ be the unital associate algebra over $\mathbb{Q}(v, t)$ with generators $T_i, 1 \leq i < k$, subject to the relations:

(H1) $T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1}, \quad 1 \leq i < k$;

(H2) $T_i T_j = T_j T_i$, if $|i - j| \geq 2$;

(H3) $(T_i - v^{-1} t)(T_i + vt) = 0, \quad 1 \leq i < k$.

The $R$-matrix $\tilde{R}$ defined in section 3 only satisfies the braid relations. In order to construct an action for two-parameter Hecke algebra $H_k(v, t)$ on $V_n^{\otimes k}$, we must modify the $R$-matrix $\tilde{R}$.

Set

$$R = \sum_{i<j} t^{2}E_{j,i} \otimes E_{i,j} + \sum_{i<j} E_{i,j} \otimes E_{j,i} + (v^{-1} - v) t \sum_{i<j} E_{j,i} \otimes E_{i,j} + v^{-1} t \sum_{i} E_{i,i} \otimes E_{i,i}.$$
For the last case \( l_i < l_{i+1} \), we have
\[
(R_i)^2(v_l \otimes \cdots \otimes v_l) = t^2 R_i(v_l \otimes \cdots \otimes v_{l-1} \otimes v_{l+i} \otimes v_l \otimes \cdots \otimes v_l)
\]
\[
= t^2 (v_l \otimes \cdots \otimes v_l) + (v^{-1} - v)t^2 (v_l \otimes \cdots \otimes v_{l-1} \otimes v_{l+i} \otimes v_l \otimes \cdots \otimes v_l)
\]
\[
= (t^2 + (v^{-1} - v)tR_i)(v_l \otimes \cdots \otimes v_l).
\]

This leads to the Schur-Weyl duality between \( U_{v,t}(sl_n) \) and \( H_k(v,t) \).

**Theorem 1.** Assume \( v^2 \) is not a root of unity. Then

1. \( \delta_n(H_k(v,t)) = \text{End}_{U_{v,t}(sl_n)}(V_n^\otimes)^k \);
2. for \( n \geq k \), we have \( \text{End}_{U_{v,t}(sl_n)}(V_n^\otimes)^k \cong H_k(v,t) \).

**Proof.** For conclusion (1), the proof is similarly to \([3]\). We consider the conclusion (2).
Assume \( f \in \text{End}_{U_{v,t}(sl_n)}(V_n^\otimes)^k \), \( \underline{v} = v_1 \otimes v_2 \otimes \cdots \otimes v_k \in V_n^\otimes \), then \( f(\underline{v}) \) must be the linear combinations of \( v_{\sigma(1)} \otimes v_{\sigma(2)} \otimes \cdots \otimes v_{\sigma(k)} \) for some \( \sigma \in \mathfrak{S}_k \). We will show that there is an element \( T^\sigma \in H_k(v,t) \) such that \( R^\sigma \cdot v = \delta_n(T^\sigma) \otimes v_{\sigma(1)} \otimes v_{\sigma(2)} \otimes \cdots \otimes v_{\sigma(k)} \). For any element \( \sigma \in \mathfrak{S}_k \), \( \sigma \) can be written as a product of transpositions, denoted by \( \sigma = \tau_{i_1} \cdots \tau_{i_m} \), where \( \tau_{i_i} = (i_i i_{i+1} + 1) \). For distinct index \( j_1, \cdots, j_k \), we set
\[
R^\tau_{j_1}(v_{j_1} \otimes \cdots \otimes v_{j_k}) = \begin{cases} \ ((v^{-1} - v)Id + R_i)(v_{j_1} \otimes \cdots \otimes v_{j_k}) & \text{if } j_{i_i} > j_{i_{i+1}}; \\ t^{-2}R_i(v_{j_1} \otimes \cdots \otimes v_{j_k}) & \text{if } j_{i_i} < j_{i_{i+1}}. \end{cases}
\]

Then defining \( R^\sigma = R^\tau_{j_1} \circ \cdots \circ R^\tau_{j_k} \) in \( \text{End}_{U_{v,t}(sl_n)}(V_n^\otimes)^k \), it can be checked that \( R^\sigma \cdot (v_{\sigma(1)} \otimes v_{\sigma(2)} \otimes \cdots \otimes v_{\sigma(k)}) \). Therefore, the map \( \delta_n : H_k(v,t) \rightarrow \text{End}_{U_{v,t}(sl_n)}(V_n^\otimes)^k \) is a surjective, and
\[
\text{End}_{U_{v,t}(sl_n)}(V_n^\otimes)^k = \text{span}_{\mathbb{Q}(v,t)} \{ R^\sigma | \sigma \in \mathfrak{S}_k \}.
\]

Consequently, \( \dim_{\mathbb{Q}(v,t)} \text{End}_{U_{v,t}(sl_n)}(V_n^\otimes)^k = k! = \dim_{\mathbb{Q}(v,t)} H_k(v,t) \). It follows that (2) holds.

**Corollary 2.** Assume \( v^2 \) is not a root of unity. The space \( V_n^\otimes \) as an \( U_{v,t}(sl_n) \otimes H_k(v,t) \)-module has the decomposition
\[
V_n^\otimes = \bigoplus \lambda \ V_\lambda \otimes V^\lambda,
\]
where the partition \( \lambda \) of \( k \) runs over the set of partitions such that \( l(\lambda) \leq n \), \( V_\lambda \) is the \( U_{v,t}(sl_n) \)-module associated to \( \lambda \), \( V^\lambda \) is the \( H_k(v,t) \)-module corresponding to \( \lambda \).

**5. The Primitive Orthogonal Idempotents Of \( H_k(v,t) \)**

For any \( i = 1, 2, k-1 \), let \( s_i = (i, i+1) \) be the transposition in the symmetric group \( S_k \). Choose a reduced decomposition \( w = s_i_1 \cdots s_i_l \) for \( w \in S_k \), denote \( T_w = T_i_1 \cdots T_i_l \). Then \( T_w \) does not depend on the reduced decomposition, and the set \( \{ T_w \mid w \in S_k \} \) is a basis of \( H_k(v,t) \) over \( \mathbb{Q}(v,t) \).

The Jucys-Murphy elements \( y_1, \cdots, y_k \) of \( H_k(v,t) \) are defined inductively by
\[
y_1 = 1, \quad y_{i+1} = t^{-2}T_i y_i T_i \quad \text{for } i = 1, \cdots, k-1.
\]
These elements satisfy
\[ y_i T_l = T_l y_i \quad \text{for} \ l \neq i, \ i - 1. \]

Furthermore, the elements \( y_i \) can be written as follows:
\[ y_i = 1 + (v^{-1} - v) t^{-1} (T_{(1, i)} + T_{(2, i)} + \cdots + T_{(i-1, i)}), \]
where \( T_{(m, n)} \) belong to \( H_k(v, t) \) associated to the transposition \( (m, n) \in S_k \). In particular, \( y_1, \cdots, y_k \) generate a commutative subalgebra of \( H_k(v, t) \).

For any \( i = 1, \cdots, k \), we let \( w_i \) denote the unique longest element of the symmetric group \( S_i \) which is regarded as the natural subgroup of \( S_k \). The corresponding elements \( T_{w_i} \in H_k(v, t) \) are then given by \( T_{w_i} = 1 \) and
\[
T_{w_i} = T_1(T_2T_1) \cdots (T_{i-2}T_{i-1} \cdots T_1)(T_{i-1}T_{i-2} \cdots T_1)
= (T_1 \cdots T_{i-2}T_{i-1})(T_1 \cdots T_{i-3}T_{i-2}) \cdots (T_1T_2)T_1, \ i = 2, \cdots, k.
\]

It is easily check that
\[
T_{w_i}T_j = T_{i-j}T_{w_i}, \ 1 \leq j < i \leq k,
T_{w_i}^2 = t^{2(k-1)} y_1y_2 \cdots y_i, \ i = 1, \cdots, k.
\]

Following [16], for any \( i = 1, \cdots, k \), we define the elements:
\[
T_i(x, y) = t^{-1} T_i + \frac{(v^{-1} - v)x}{y - x},
\]
where \( x \) and \( y \) are complex variables. We will regard the \( T_i(x, y) \) as rational functions in \( x \) and \( y \) with values in \( H_k(v, t) \). These functions satisfy the braid relations:
\[
T_i(x, y)T_{i+1}(x, z)T_i(y, z) = T_{i+1}(y, z)T_i(x, z)T_{i+1}(x, y),
\]
and
\[
T_i(x, y)T_i(y, x) = \frac{(x - v^{-2}y)(x - v^2y)}{(x - y)^2}.
\]

Following [5], we will identify a partition \( \lambda = (\lambda_1, \cdots, \lambda_l) \) of \( k \) with its Young diagram which is a left-justified array of rows of cells such that the first row contains \( \lambda_1 \) cells, the second row contains \( \lambda_2 \) cells, etc. A cell \( \tau \) outside \( \lambda \) is called addable to \( \lambda \) if the union of the cell \( \tau \) and \( \lambda \) is a Young diagram. A tableau \( \mathcal{T} \) of shape \( \lambda \) is obtained by filling in the cells of the diagram bijectively with the numbers \( 1, \cdots, k \). A tableau \( \mathcal{T} \) is called standard if its entries increase along the rows and down the columns. If a cell occupied by \( i \) occurs in row \( m \) and column \( n \), its \( (v, v^{-1}) \)-content \( \sigma_i \) will be defined as \( v^{-2(m-n)} \).

In accordance to [5], a set of primitive orthogonal idempotents \( \{E^\lambda_T\} \) of \( H_k(v, t) \), parameterized by partitions \( \lambda \) of \( k \) and standard tableaux \( \mathcal{T} \) of shape \( \lambda \) can be constructed inductively as follows. If \( k = 1 \), set \( E^\lambda_T = 1 \). For \( k \geq 2 \), one defines inductively that
\[
E^\lambda_T = E_\mathcal{U}^\mu \frac{(y_k - \rho_1) \cdots (y_k - \rho_l)}{\prod (\sigma - \rho_i)},
\]
where \( \mathcal{U} \) is the tableau of shape \( \mu \) obtained form \( \mathcal{T} \) by removing the cell \( \alpha \) occupied by \( k \), and \( \rho_1, \cdots, \rho_l \) are the \( (v, v^{-1}) \)-contents of all the addable cells of \( \mu \) except for \( \alpha \), while \( \sigma \) is the \( (v, v^{-1}) \)-content of \( \alpha \).
These elements become a family of primitive orthogonal idempotents of $H_k(v,t)$. Indeed, if $\lambda$ and $\lambda'$ are distinct partitions of $k$, and $T$ (respectively $T'$) is any standard tableau of shape $\lambda$ (respectively $\lambda'$), then we have

$$E^\lambda_T E^\lambda'_{T'} = \delta_{\lambda,\lambda'} \delta_{T,T'} E^\lambda_T.$$

Moreover,

$$\sum_\lambda \sum_T E^\lambda_T = 1,$$

summed over all partitions $\lambda$ of $k$ and all the standard tableaux $T$ of shape $\lambda$.

**Example 1.** For $k = 2$, then $\lambda_1 = (1,1)$ and $\lambda_2 = (2)$ are all possible partitions of $k$. Set $T_1 = \begin{array}{c} 1 \\ 2 \end{array}$ (resp. $T_2 = \begin{array}{c} 1 \\ 2 \\ 1 \end{array}$) be the only standard tableau of $\lambda_1$ (resp. $\lambda_2$).

Consider $E^{\lambda_1}_{T_1}$. Obviously, $U = \begin{array}{c} 1 \\ 1 \end{array}$ is the tableau of shape $\mu = (1)$ obtained from $T_1$ by removing the cell occupied by 2, and $\rho_1 = v^{-2}$, $\sigma = v^2$. Then

$$E^{\lambda_1}_{T_1} = \frac{\sigma - \rho_1}{\sigma + \rho_1} = \frac{v^2 - v^{-2}}{v^2 + v^{-2}} = -\frac{v^{1} - v^{-1}}{1 + v^{4}} T_1 + \frac{1}{1 + v^2}.$$

Similarly, we can calculate $E^{\lambda_2}_{T_2} = \frac{v^{-1}}{1 + v^2} T_1 + \frac{v^2}{1 + v^2}$. As we see, the set of $E^{\lambda_1}_{T_1}$ and $E^{\lambda_2}_{T_2}$ is a complete set of primitive orthogonal idempotents of $H_2(v,t)$.

## 6. Fusion Formulas for $E^\lambda_T$ of $H_k(v,t)$

We now apply the fusion formulas [11] for the primitive orthogonal idempotents of two-parameter Hecke algebra $H_k(v,t)$.

Let $\lambda = (\lambda_1, \cdots, \lambda_l)$ be a partition of $k$, $\lambda' = (\lambda'_1, \cdots, \lambda'_l)$ be the conjugate partition of $\lambda$ obtained by turning the rows into columns. If a cell $\alpha$ occurs in the $(i,j)$-th position of $\lambda$, denoted by $\alpha = (i,j)$, then the corresponding hook is defined as $h_\alpha = \lambda_i + \lambda'_j - i - j + 1$. Set

$$f(\lambda) = v^{-b(\lambda)} t^{k(k-1)} (1 - v^{-2})^k \prod_{\alpha \in \lambda} (1 - v^{-2h_\alpha})^{-1},$$

where $b(\lambda) = \sum_{i \geq 1} \lambda_i (\lambda_i - 1)$, the sum is carried out all cells $\alpha$ of $\lambda$.

Now we introduce the rational function $\Psi(u_1, \cdots, u_k)$ in complex variables $u_1, \cdots, u_k$ with values in $H_k(v,t)$ by the following way:

$$\Psi(u_1, \cdots, u_k) = \prod_{i=1, \cdots, k-1} (T_i(u_1, u_{i+1}) T_{i-1}(u_2, u_{i+1}) \cdots T_1(u_i, u_{i+1})) \cdot T^{-1}_{w_k},$$

where the product is carried out in the order of $i = 1, \cdots, k - 1$.

**Proposition 6.** For the partition $\lambda$ of $k$ and a standard tableau $T$ of shape $\lambda$, the primitive orthogonal idempotents $E^\lambda_T$ can be obtained by the consecutive evaluations

$$E^\lambda_T = f(\lambda) \Psi(u_1, \cdots, u_k)|_{u_1 = \sigma_1, u_2 = \sigma_2, \cdots, u_k = \sigma_k}.$$
Example 2. As example [1] we take \( k = 2, \lambda_2 = (2) \). Then
\[
f(\lambda_2) = v^{-2}t(1-v^{-2})^2(1-v^{-2})^{-1}(1-v^{-4})^{-1}
\]
where \( b(\lambda_2) = 2, h_{(1,1)} = 2, h_{(1,2)} = 1 \). Since \( \sigma_1 = 1, \sigma_2 = v^{-2} \), so
\[
\Psi(u_1, u_2)\big|_{u_1=\sigma_1, u_2=\sigma_2} = (t^{-1}T_1 + \frac{v^{-1}-v}{v^2-1})T_1^{-1}
\]
\[
= vt^{-2}T_1 + v^2t^{-1}.
\]
Thus, the idempotent
\[
E_{T_2}^{\lambda_2} = f(\lambda_2)\Psi(u_1, u_2)\big|_{u_1=\sigma_1, u_2=\sigma_2}
\]
\[
= \frac{vt^{-1}}{1+v^2}T_1 + \frac{v^2}{1+v^2}T_1.
\]
The result coincide with example [1]

Since \( E_T^\lambda \) is a primitive idempotent of \( H_k(v,t) \), \( E_T^\lambda \) acts on the simple module \( V^\lambda \) of \( H_k(v,t) \) as a projector on a 1-dimensional subspace and when \( \lambda \neq \lambda' \), \( E_T^\lambda \) annihilates the irreducible \( H_k(v,t) \)-module \( V^{\lambda'} \). Furthermore, using Corollary [2] we can get the following explicit description of the irreducible modules of \( U_{v,t}(sl_n) \).

Theorem 2. For a partition \( \lambda = (\lambda_1, \ldots, \lambda_l) \) of \( k \) with length \( l \leq n \) and \( T \) a standard tableau of type \( \lambda \), then
\[
V(\lambda) = E_T^\lambda(V^{\otimes k})
\]
is the finite dimensional irreducible representation of \( U_{v,t}(sl_n) \).

Example 3. There are two partitions \( \lambda_1 = (1,1) \) and \( \lambda_2 = (2) \) for \( k = 2 \), and both of their length are not bigger than \( n \geq 2 \). Combine with the results in example [1] a computation shows that
\[
E_{T_1}^{\lambda_1}(V^{\otimes 2}) = \text{span}_{Q(v,t)}\{v_i \otimes v_j - vt v_j \otimes v_i \, | \, 1 \leq i < j \leq n\},
\]
\[
E_{T_2}^{\lambda_2}(V^{\otimes 2}) = \text{span}_{Q(v,t)}\{v_i \otimes v_i \, | \, 1 \leq i \leq n\} \cup \{v_i \otimes v_j + v^{-1}t v_j \otimes v_i \, | \, 1 \leq i < j \leq n\},
\]
they are precisely the irreducible \( U_{v,t}(sl_n) \)-submodules of \( V \otimes V \).

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Acknowledgements: This work is supported by NSFC 11571119 and NSFC 11475178.

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