POSITIVE SOLUTIONS TO \( p \)-KIRCHHOFF-TYPE ELLIPTIC EQUATION WITH GENERAL SUBCRITICAL GROWTH

HUIXING ZHANG AND RAN ZHANG

Abstract. In this paper, we study the existence of positive solutions to the \( p \)-Kirchhoff elliptic equation involving general subcritical growth

\[
(a + \lambda \int_{\mathbb{R}^N} |\nabla u|^p \, dx + \lambda b \int_{\mathbb{R}^N} |u|^p \, dx)(-\Delta_p u + b|u|^{p-2}u) = h(u), \quad \text{in } \mathbb{R}^N,
\]

where \( a, b > 0 \), \( \lambda \) is a parameter and the nonlinearity \( h(s) \) satisfies the weaker conditions than the ones in our known literature. We also consider the asymptotics of solutions with respect to the parameter \( \lambda \).

1. Introduction

In this paper, we study the existence of positive solutions to \( p \)-Kirchhoff-type problem

\[
\begin{cases}
(a + \lambda \int_{\mathbb{R}^N} |\nabla u|^p \, dx + \lambda b \int_{\mathbb{R}^N} |u|^p \, dx)(-\Delta_p u + b|u|^{p-2}u) = h(u), & \text{in } \mathbb{R}^N, \\
u(x) > 0, & x \in \mathbb{R}^N, \quad u(x) \in W^{1,p}(\mathbb{R}^N),
\end{cases}
\]

where \( a \) and \( b \) are positive constants, \( \Delta_p u = \text{div}(|\nabla u|^{p-2}\nabla u) \), \( 1 < p < N \), \( \lambda > 0 \) is a parameter and the general nonlinearity \( h(s) \) satisfies the following conditions:

\( h_1 \) \( \lim_{s \to +\infty} \frac{h(s)}{s^p} = 0 \), with \( p^* = \frac{pN}{N-p} \);

\( h_2 \) \( h \in C([\mathbb{R}_+, \mathbb{R}_+]) \) with \( \mathbb{R}_+ = [0, +\infty) \) and \( \lim_{s \to 0} \frac{h(s)}{s^{p^*}} = 0 \);

\( h_3 \) there exists \( \xi > 0 \) such that \( G(\xi) = \int_0^\xi g(s) \, ds > 0 \), where \( g(s) = h(s) - ab|s|^{p-2} \).

When \( p = 2 \) in the problem (1.1), the equation reduces to

\[
(a + \lambda \int_{\mathbb{R}^N} |\nabla u|^2 \, dx + \lambda b \int_{\mathbb{R}^N} |u|^2 \, dx)(-\Delta u + bu) = h(u), \quad x \in \mathbb{R}^N.
\]

The problem (1.2) on a bounded domain \( \Omega \subset \mathbb{R}^N \) is viewed as the Kirchhoff-type problem which was proposed by Kirchhoff [15]. Kirchhoff-type problem

Received June 3, 2016; Revised September 5, 2016.

2010 Mathematics Subject Classification. 35J20, 35J15, 35J60.

Key words and phrases. \( p \)-Kirchhoff-type equation, subcritical growth, asymptotics.
is linked with a generalization of well-known D’Alembert’s wave equations for free vibration of elastic strings, especially, considering the changes in length of string produced by transverse oscillations. In addition, the problem (1.2) also models several biological systems, where $u$ describes a process which depends on the average of itself (see [1]).

In recent years, Kirchhoff-type problems in $\mathbb{R}^N$ have been studied by many authors, for example, see [13, 16, 17, 18] and references therein. In [16], Li et al. utilized a cut-off functional to obtain the bounded Palais-Smale sequences and proved the existence of a positive solution to the Kirchhoff-type problem (1.2). Subsequently, in [17], Liu, Liao, and Tang also considered problem (1.2) under the following conditions:

- $(g_1)$ $h \in C([0, +\infty))$ with $R_+ = [0, +\infty)$ and $\lim_{s \to 0} \frac{h(s)}{s} = 0$;
- $(g_2)$ $\lim_{s \to +\infty} \frac{h(s)}{s^{2^*}} = 0$, with $2^* = \frac{2N}{N-2}$;
- $(g_3)$ there exists $\eta > 0$, such that $H(\eta) = \int_0^\eta h(t)dt > \frac{a^N}{2}$.

The conditions $(g_1)$-$(g_3)$ are weaker than the ones in [16]. The result in [17] covered the asymptotically linear case and superlinear case at infinity.

For the $p$-Kirchhoff-type problem (1.1), there have been some results. In [9], Cheng and Dai proved the existence of positive solutions for $p$-Kirchhoff type problem under the following assumptions:

- $(f_1)$ there exists a $C > 0$ such that $|h(t)| \leq C(|t|^{p-1} + |t|^{q-1})$ for all $t \geq 0$ and some $q \in (p, p^*)$, here $p^* = \frac{Np}{N-p}$;
- $(f_2)$ $\lim_{t \to 0^+} \frac{h(t)}{t} = 0$;
- $(f_3)$ $\lim_{t \to +\infty} \frac{h(t)}{t^p} = +\infty$. For more results, we refer the reader to [2, 7, 6] and the references therein.

In this paper, we are motivated by [9, 16, 17] and study the existence of positive solutions to the problem (1.1). We will adopt the totally different approaches with the ones (namely, cut-off functional techniques and monotonicity methods) in [9, 16, 17] to obtain bounded (PS) sequence. We believe that the conditions $(h_1)$-$(h_3)$ on the general nonlinearity $h$ are almost optimal.

In order to state the results clearly, we introduce some Sobolev spaces. Denote $W^{1,p}(\mathbb{R}^N)$ be the usual Sobolev space equipped with the norm

$$||u|| = (\int_{\mathbb{R}^N} (|\nabla u|^p + b|u|^p)dx)^{\frac{1}{p}}$$

and $D^{1,p}(\mathbb{R}^N) := \{u \in L^p(\mathbb{R}^N); \nabla u \in L^p(\mathbb{R}^N)\}$ endowed with the norm $||u||_{D^{1,p}} = (\int_{\mathbb{R}^N} |\nabla u|^pdx)^{\frac{1}{p}}$. Let $W_{rad}^{1,p}(\mathbb{R}^N)$ be the subspace of $W^{1,p}(\mathbb{R}^N)$ of radially symmetric functions. $||u||_q = (\int_{\mathbb{R}^N} |u|^qdx)^{\frac{1}{q}}$ for $q \geq 1$ with $u \in L^q(\mathbb{R}^N)$. $C_i$ denote positive constants, $i = 1, 2, \ldots, S$ and $C_q$ denote the best constants of Sobolev embeddings $D^{1,p}(\mathbb{R}^N) \hookrightarrow L^p(\mathbb{R}^N)$ and
\[ W^{1,p}(\mathbb{R}^N) \hookrightarrow L^q(\mathbb{R}^N), \]

\[ S(\int_{\mathbb{R}^N} |u|^p \, dx)^{p/p^*} \leq \int_{\mathbb{R}^N} |\nabla u|^p \, dx \quad \text{for all } u \in D^{1,p}(\mathbb{R}^N), \]

\[ C_q(\int_{\mathbb{R}^N} |u|^p \, dx)^{p/q} \leq \int_{\mathbb{R}^N} (|\nabla u|^p + b|u|^p) \, dx \quad \text{for all } u \in W^{1,p}(\mathbb{R}^N). \]

The following theorem is the first main result in the paper.

**Theorem 1.1.** Assume that \((h_1)-(h_3)\) hold. There exists a constant \(\lambda_0 > 0\) such that, for any \(\lambda \in (0, \lambda_0)\), the problem (1.1) admits at least one positive solution \(u_\lambda\).

**Remark 1.1.** Theorem 1.1 covers the result in [17]. Indeed, when \(p = 2\), Theorem 1.1 is the main result in [17].

**Remark 1.2.** We easily prove that the conditions \((f_1)\) and \((f_3)\) in [9] are stronger than the ones \((h_1)\) and \((h_3)\) respectively. In this sense, we improve the main result in [9].

When \(\lambda = 0\) in the equation (1.1), the problem reduces to

\[ -a\Delta_p u + ab|u|^{p-2}u = h(u), \quad x \in \mathbb{R}^N. \]

The problem (1.5) is viewed as the limit problem of (1.1) when \(\lambda \to 0\). We can now state the second main result in this paper.

**Theorem 1.2.** If the general nonlinearity \(h\) satisfies \((h_1)-(h_3)\), then, as \(\lambda \to 0\), \(u_\lambda\) converges to \(u\) in \(W^{1,p}(\mathbb{R}^N)\), where \(u\) is a ground state solution to the problem (1.5).

**Remark 1.3.** In order to prove the existence of a ground state solution for the problem (1.5), the assumptions \((h_1)\), \((h_3)\) and the additional condition \((h_2)\) there exists some \(q \in (p - 1, p^* - 1)\) such that

\[ \lim_{t \to \infty} \sup_{t \neq 0} \frac{h(t)}{t^q} < \infty \]

were already used by Berestycki and Lions [3], for \(p = 2\), and by J. M. do \'O and E. Medeiros [12], for the \(1 < p \leq N\) case. Obviously, the condition \((h_2)\) in this paper is weaker than the one \((h_2')\). In this sense, we improve the results in [3, 12].

The rest of the paper is organised as follows. In Section 2, we prove that the limit problem (1.5) has at least a ground state solution. In Section 3, we will find a solution in some neighborhood of the solutions to the limit problem (1.5). Indeed, we view the problem (1.1) as the perturbed problem of (1.5) if \(\lambda\) is sufficiently small. Because of the lack of Ambrosetti-Rabinowitz condition, we use a local deformation approach from Byeon and Jeanjean [4] to obtain a bounded (PS) sequence. In addition, due to the appearance of nonlocal terms \(\int_{\mathbb{R}^N} |\nabla u|^p \, dx\) and \(\int_{\mathbb{R}^N} |u|^p \, dx\), we make a crucial modification on the min-max value which is defined by \(C_\lambda\), where all paths are requested to be uniformly bounded with respect to \(\lambda\). Finally, we give the proofs of the main results.
2. Existence of ground state solutions to limit problem

In this section, we prove that the problem (1.5) has at least one ground state solution. Since we consider the positive solutions, we can assume that \( h(s) = 0 \) for \( s \leq 0 \). Meanwhile, as the problems (1.1) and (1.5) are autonomous, we can work in \( W^{1,p}_r(\mathbb{R}^N) \) (see Theorem 1.28 in [20]). Define the energy functionals of problems (1.1) and (1.5) respectively by

\[
I_\lambda(u) = \frac{a}{p} \|u\|^p + \frac{\lambda}{2p} \|u\|^{2p} - \int_{\mathbb{R}^N} H(u)dx
\]

and

\[
I(u) = \frac{a}{p} \|u\|^p - \int_{\mathbb{R}^N} H(u)dx,
\]

where \( u \in W^{1,p}_r(\mathbb{R}^N) \) and \( H(t) = \int_0^t h(s)ds \).

By the conditions (h1)-(h3), we can prove that \( I_\lambda, I \in C^1(W^{1,p}_r(\mathbb{R}^N), \mathbb{R}) \).

Indeed, the weak solutions of the problem are the critical points of the corresponding energy functional.

**Proposition 2.1.** Suppose that (h1)-(h3) hold. Then the limit problem (1.5) has at least one ground state solution \( u \in W^{1,p}_r(\mathbb{R}^N) \).

In order to prove the main results, we need the following lemmas.

**Lemma 2.2** (Pohožăev equality). If \( u \) is a nontrivial solution of the equation

\[
a(-\Delta_p u + b|u|^{p-2}u) = h(u), \quad x \in \mathbb{R}^N,
\]

then \( u \) satisfies the following Pohožăev equality

\[
\frac{a(N-p)}{p} \int_{\mathbb{R}^N} |\nabla u|^p dx = N \int_{\mathbb{R}^N} G(u)dx, \text{ where } G(u) = H(u) - \frac{ab}{p} |u|^p.
\]

**Proof.** The proof is similar to the one of Lemma 2.6 in [16]. We omit the details. \( \square \)

For convenience, we give the following notations.

\( \mathcal{L} := \{ u \in W^{1,p}_r(\mathbb{R}^N) \setminus \{0\} : \int_{\mathbb{R}^N} G(u)dx = 1 \} \)

and

\( \mathcal{P} := \{ u \in W^{1,p}_r(\mathbb{R}^N) \setminus \{0\} : \frac{a(N-p)}{p} \int_{\mathbb{R}^N} |\nabla u|^p dx = N \int_{\mathbb{R}^N} G(u)dx \} \).

From (h3), we have \( \mathcal{L} \neq \emptyset \) and \( \mathcal{P} \neq \emptyset \). Set \( L = \frac{1}{p} \inf_{u \in \mathcal{L}} \|\nabla u\|^p_p \), \( \beta_0 = \inf_{u \in \mathcal{P}} I(u) \)
and the mountain pass value

\[
k = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} I(\gamma(t)),
\]

where \( \Gamma = \{ \gamma \in C([0,1], W^{1,p}_r(\mathbb{R}^N)) : \gamma(0) = 0, I(\gamma(1)) < 0 \} \).
Lemma 2.3. Assume that \((h_1)-(h_3)\) hold. Then \(\beta_0 \leq k\) and
\[
\beta_0 = \frac{p}{N-p} \left( \frac{a(N-p)}{N} \right)^\frac{N}{2} L^\frac{N}{2}.
\]

Proof. In order to prove \(\beta_0 \leq k\), it suffices to prove that \(\gamma([0,1]) \cap P \neq \emptyset\) for all \(\gamma \in \Gamma\).

Set
\[
P(u) = \frac{N-p}{p} \int_{\mathbb{R}^N} |\nabla u|^p dx - \frac{N}{a} \int_{\mathbb{R}^N} G(u) dx.
\]
By \((h_1)\) and \((h_2)\), we easily obtain that there exists \(\rho > 0\) such that \(P(u) > 0\), \(0 < \|u\| \leq \rho\). For any \(\gamma \in \Gamma\), we get \(P(\gamma(0)) = 0\) and \(P(\gamma(1)) \leq \max\{\frac{N-p}{p}, \frac{N}{a}\} I(\gamma(1)) < 0\). Thus, there exists a \(t_0 \in (0,1)\) such that \(P(\gamma(t_0)) = 0\) with \(\|\gamma(t_0)\| > \rho\). This implies that \(\gamma([0,1]) \cap P \neq \emptyset\) for all \(\gamma \in \Gamma\).

In the following, we prove that \(\beta_0 = \frac{p}{N-p} \left( \frac{a(N-p)}{N} \right)^\frac{N}{2} L^\frac{N}{2}\). Firstly, we claim that \(L > 0\). In fact, if \(L = 0\), there is \(\{u_n\} \subset L\) with \(\int_{\mathbb{R}^N} G(u_n) dx = 1\) such that \(\|\nabla u_n\|_p \to 0\) as \(n \to \infty\). From the Sobolev’s embedding \(D^{1,p}(\mathbb{R}^N) \hookrightarrow L^p(\mathbb{R}^N)\), we have \(\|u_n\|_{p^*} \to 0\) as \(n \to \infty\). Together with the assumptions \((h_1)\) and \((h_2)\), we get
\[
\lim_{n \to \infty} \sup_{R} \int_{\mathbb{R}^N} G(u_n) dx \leq \lim_{n \to \infty} \sup_{R} C_1 \int_{\mathbb{R}^N} |u_n|^p dx = 0.
\]
This is a contradiction with \(\int_{\mathbb{R}^N} G(u_n) dx = 1\). Thus, \(L > 0\).

For any \(u \in \mathcal{L}\), define \((\Phi_t(u))(x) = u(\frac{x}{t})\), \(T(u) = \frac{1}{p} \int_{\mathbb{R}^N} |\nabla u|^p dx\) and \(V(u) = \int_{\mathbb{R}^N} G(u) dx\). We have
\[
T(u(\frac{x}{t})) = t^{N-p} T(u)
\]
and
\[
V(u(\frac{x}{t})) = t^N \int_{\mathbb{R}^N} G(u) dx.
\]
Thus, choosing \(t_u = (\frac{a(N-p)}{Np})^\frac{1}{N} \|\nabla u\|_p\), we get that \(\Phi_{t_u}\) is a bijection from \(\mathcal{L}\) to \(\mathcal{P}\). For any \(u \in \mathcal{L}\),
\[
I(\Phi_{t_u}(u)) = at_u^{N-p}T(u) - t_u^N V(u)
\]
\[
= \frac{p}{N-p} \left( \frac{a(N-p)}{Np} \right)^\frac{N}{2} \|\nabla u\|^N_p.
\]
Furthermore,
\[
\inf_{u \in \mathcal{P}} I(u) = \inf_{u \in \mathcal{L}} I(\Phi_{t_u}(u)),
\]
which implies that
\[
\beta_0 = \frac{p}{N-p} \left( \frac{a(N-p)}{N} \right)^\frac{N}{2} L^\frac{N}{2}.
\]

\[\square\]

Lemma 2.4. If \(h \in C(\mathbb{R}^N \times \mathbb{R})\) and assume that
\[
\lim_{t \to 0} \frac{h(x,t)}{t^{p-1}} = 0
\]
and

$$\lim_{t \to \infty} \sup_{t \in [0, 1]} \frac{|h(x, t)|}{|t|^{p-1}} < \infty$$

hold uniformly in $x \in \mathbb{R}^N$. For any $\{u_n\}$ with $u_n \to u_0$ weakly in $W^{1, p}(\mathbb{R}^N)$ and $u_n \to u_0$ a.e. in $\mathbb{R}^N$, we have

$$\int_{\mathbb{R}^N} H(x, u_n) dx = \int_{\mathbb{R}^N} (H(x, u_n - u_0) + H(x, u_0)) dx + o(1),$$

where $H(x, t) = \int_0^t h(x, s) ds$.

**Proof.** For the subcritical case, we refer to the reference [11]. We omit the details. □

**Proof of Proposition 2.1.** Assume that there exists $\{u_n\} \subset W^{1, p}(\mathbb{R}^N)$ such that $\int_{\mathbb{R}^N} G(u_n) dx = 1$ and $\int_{\mathbb{R}^N} |\nabla u_n|^p dx \to pL$ as $n \to \infty$. By the conditions (h1) and (h2), we get that $\|u_n\|_p$ is bounded. So, $\{u_n\}$ is bounded in $W^{1, p}_r(\mathbb{R}^N)$. We may assume that $u_n \to u^*$ weakly in $W^{1, p}_r(\mathbb{R}^N)$. By Lemma 2.4, we have

$$\int_{\mathbb{R}^N} H(u_n) dx = \int_{\mathbb{R}^N} (H(u_n - u^*) dx + \int_{\mathbb{R}^N} H(u^*) dx + o(1).$$

From the conditions (h1) – (h2), for any $\xi > 0$, there exists $C_\xi > 0$ such that

$$H(s) \leq \xi |s|^p + \xi |s|^{p^*} + C_\xi |s|^{k_0}, \quad k_0 \in (p, p^*).$$

Thus

$$\left| \int_{\mathbb{R}^N} H(u_n - u^*) dx \right| \leq \xi \int_{\mathbb{R}^N} |u_n - u^*|^p dx + \xi \int_{\mathbb{R}^N} |u_n - u^*|^{p^*} dx + C_\xi \int_{\mathbb{R}^N} |u_n - u^*|^{k_0} dx$$

$$\leq \xi J_1 + \xi J_2 + C_\xi J_3,$$

where

$$J_1 = \int_{\mathbb{R}^N} |u_n - u^*|^p dx,$$

$$J_2 = \int_{\mathbb{R}^N} |u_n - u^*|^{p^*} dx,$$

and

$$J_3 = \int_{\mathbb{R}^N} |u_n - u^*|^{k_0} dx.$$

From the Sobolev’s imbedding $W^{1, p}_r(\mathbb{R}^N) \hookrightarrow L^{k_0}(\mathbb{R}^N), k_0 \in [p, p^*]$, we obtain $\|u_n\|_{k_0}$ is bounded. In connection with Minkowski inequality, one has

$$|J_1|, \quad |J_2| \leq C_1,$$

where $C_1 > 0$.

In addition, since the imbedding $W^{1, p}_r(\mathbb{R}^N) \hookrightarrow L^{k_0}(\mathbb{R}^N), k_0 \in (p, p^*)$ is compact, we have $J_3 \to 0$ as $n \to \infty$. So, $\int_{\mathbb{R}^N} H(u_n - u^*) dx \to 0$ as $n \to \infty$. Furthermore, it follows from (2.1) that

$$\int_{\mathbb{R}^N} H(u_n) dx = \int_{\mathbb{R}^N} H(u^*) dx + o(1).$$
Next, since \( u_n \to u^* \) weakly in \( W^{1,p}_r(\mathbb{R}^N) \), we get
\[
\lim_{n \to \infty} \int_{\mathbb{R}^N} |u_n|^p dx \geq \int_{\mathbb{R}^N} |u^*|^p dx.
\]
Then
\[
1 = \lim_{n \to \infty} \int_{\mathbb{R}^N} G(u_n) dx = \lim_{n \to \infty} \int_{\mathbb{R}^N} \left( H(u_n) - \frac{ab}{p} |u_n|^p \right) dx \\
\leq \int_{\mathbb{R}^N} \left( H(u^*) - \frac{ab}{p} |u^*|^p \right) dx \\
= \int_{\mathbb{R}^N} G(u^*) dx.
\]

**Case 1.** \( V(u^*) = \int_{\mathbb{R}^N} G(u^*) dx = 1 \). Combining with \( \int_{\mathbb{R}^N} G(u_n) dx = 1 \) and (2.2), then we have
\[
\|u_n\|_p^p \to \|u^*\|_p^p \quad \text{as} \quad n \to \infty.
\]
Since \( T(u_n) \to L \) as \( n \to \infty \) and \( T(u^*) = L \), we obtain
\[
\|\nabla u_n\|_p^p \to \|\nabla u^*\|_p^p \quad \text{as} \quad n \to \infty.
\]
It follows from (2.3) and (2.4) that \( \|u_n\| \to \|u^*\| \) as \( n \to \infty \). Therefore,
\[
u_n \to u^* \quad \text{strongly in} \quad W^{1,p}_r(\mathbb{R}^N) \quad \text{as} \quad n \to \infty.
\]

**Case 2.** \( V(u^*) = \int_{\mathbb{R}^N} G(u^*) dx > 1 \). There exists \( t_0 > 0 \) such that
\[
\int_{\mathbb{R}^N} G(u(\frac{x}{t_0})) dx = 1.
\]
Together with \( V(u(\frac{x}{t_0})) = t_0^N V(u) = 1 \), we get that \( t_0 = (V(u))^{-\frac{1}{N}} \). Then we have
\[
T(u(\frac{x}{t_0})) = t_0^{N-p} T(u) \geq L.
\]
Namely,
\[
T(u) \geq t_0^{-(N-p)} L \\
\geq (V(u))^{\frac{N-p}{N}} L \\
> L.
\]
This is a contradiction with \( T(u) \leq L \). So we obtain that \( V(u^*) = 1 \) and \( T(u^*) = L \). Setting \( t_{u^*} = (\frac{a(N-p)}{2Np})^{\frac{1}{2}} \|\nabla u^*\|_p \), it follows from Coleman, Glazer and Martin [10] that \( w = u^*(\frac{x}{t_{u^*}}) \in \mathcal{P} \) is a ground state solution to the limit problem (1.5).

Let \( A_r \) be the set of the radial ground state solution \( U \) of the problem (1.5). From Proposition 2.1, we know that \( A_r \neq \emptyset \).

**Lemma 2.5.** \( A_r \) is compact in \( W^{1,p}_r(\mathbb{R}^N) \).
Proof. For any sequence \( \{u_n\} \subset A_r \), it follows from similar arguments [4] that \( u_n \) is a minimizer of \( T(u) \) on the set

\[
\{ u \in W^{1,p}_r(\mathbb{R}^N) : \int_{\mathbb{R}^N} G(u) = \beta_1 \},
\]

where \( \beta_1 = \left( \frac{a(N-p)L}{N} \right)^{\frac{N}{p}} \).

Set \( v_n(x) = u_n(\beta_1^{-\frac{1}{p}} x) \), then \( v_n \) is a minimizer of \( T(u) \) on \( L \). Namely,

\[
\|\nabla v_n\|_p^p \to pL \text{ with } \int_{\mathbb{R}^N} G(v_n)dx = 1.
\]

From the conditions \((h_1)-(h_2)\) and the Sobolev’s imbedding theorem, we can prove that \( \{v_n\} \) is bounded in \( W^{1,p}_r(\mathbb{R}^N) \). Similar arguments in Proposition 2.1 show that there exists \( v_0 \in L \) such that \( v_n \to v_0 \) strongly in \( W^{1,p}_r(\mathbb{R}^N) \). Furthermore, we can obtain that \( u_n \to u_0 \) in \( A_r \), where \( u_0 = v_0(\beta_1^{-\frac{1}{p}} x) \). The proof is completed. □

**Lemma 2.6.** The mountain pass value corresponds with the least energy level, namely, \( k = \beta_0 = I(u_0) \), where \( u_0 \in A_r \).

**Proof.** By the assumptions \((h_1)-(h_3)\), we know that the mountain pass value \( k \) is well defined. On the one hand, we get that \( \beta_0 \leq k \). On the other hand, since \( u_0 \) is a ground state solution to the limit problem (1.5), we adopt the similar idea in [5] and can prove that there exists a path \( \gamma \in \Gamma \) satisfying \( \gamma(0) = 0, I(\gamma(1)) < 0 \) and \( \max_{t \in [0,1]} I(\gamma(t)) = I(u_0) \). This implies that \( k \leq \beta_0 \). The proof is completed. □

### 3. Proofs of main results

Set \( U_t(x) = U_{\left(\frac{t}{t_1}\right)}, U \in A_r \). By Lemma 2.2, we have

\[
I(U_t) = \frac{a}{p} \int_{\mathbb{R}^N} |\nabla U_t|^p dx - \int_{\mathbb{R}^N} G(U_t)dx
\]

\[
= \frac{a}{p} t^{N-p} \int_{\mathbb{R}^N} |\nabla U|^p dx - t^N \int_{\mathbb{R}^N} G(U)dx
\]

\[
= \left( \frac{a}{p} t^{N-p} - \frac{a(N-p)}{Np} t^N \right) \int_{\mathbb{R}^N} |\nabla U|^p dx.
\]

This shows that \( I(U_t) \to -\infty \) as \( t \to \infty \). Thus, there exists \( t_1 > 1 \) such that \( I(U_t) < -3 \) for \( t \in [t_1, +\infty) \).

Define \( D_\lambda = \max_{t \in [0,t_1]} I_\lambda(U_t) \). By Lemma 2.5 and Lemma 2.6, we can get that

\[
\lim_{\lambda \to 0} D_\lambda = k.
\]

In order to get the uniformly bounded set of the mountain pathes, we give the following result.

**Lemma 3.1.** There exist \( \lambda_0 > 0 \) and \( C_2 > 0 \), such that for any \( \lambda \in (0, \lambda_0) \), \( I_\lambda(U_t) < -3, \|U_t\| \leq C_2, \forall t \in (0, t_1) \) and \( \|U\| \leq C_2, U \in A_r \).
Proof. By Lemma 2.5, there is a constant $C_3 > 0$ such that $\|U\| \leq C_3$ for any $U \in A_r$. Meanwhile,

$$
\|U\|^p = t^{N-p}\|\nabla U\|^p + t^N \|U\|^p \\
\leq (t^{N-p} + t^N)\|U\|^p \\
\leq ((t_1)^{N-p} + (t_1)^N)C_3^p.
$$

We choose $C_2 = \max\{C_3, ((t_1)^{N-p} + (t_1)^N)\frac{1}{2} C_3\}$ and obtain that

$$
\|U\|, \|U\| \leq C_2 \text{ for any } U \in A_r.
$$

Furthermore

$$
I_\lambda(U_1) = I(U_1) + \frac{\lambda}{2p}\|U_1\|^{2p} \\
\leq I(U_1) + \frac{\lambda}{2p}C_2^{2p}.
$$

It follows from $I(U_{t_1}) < -3$ that there exists $\lambda_0 > 0$ such that

$$
I_\lambda(U_{t_1}) < -3 \text{ for any } \lambda \in (0, \lambda_0).
$$

The proof is completed. \hfill \Box

By Lemma 3.1, we will define a min-max value

$$
C_\lambda = \inf_{\gamma \in \Gamma_\lambda} \max_{s \in [0, t_1]} I_\lambda(\gamma(s)),
$$

where $\Gamma_\lambda = \{\gamma \in C([0, t_1], W^{1,p}_r(\mathbb{R}^N)) : \gamma(0) = 0, \gamma(t_1) = U_1, \|\gamma(t)\| \leq C_2 + 2\}$. Obviously, $\Gamma_\lambda \neq \emptyset$ and $C_\lambda \leq D_\lambda$ for $\lambda \in (0, \lambda_0)$.

**Lemma 3.2.** One has $\lim_{\lambda \to 0} C_\lambda = k$.

**Proof.** It is clear that $C_\lambda \leq D_\lambda \to k$ as $\lambda \to 0$. On the other hand, for any $\gamma \in \Gamma_\lambda$, we have $\tilde{\gamma}(\cdot) = \gamma(t_1 \cdot) \in \Gamma$. Together with $I_\lambda(u) \geq I(u)$, we obtain that $C_\lambda \geq k$. So, $\lim_{\lambda \to 0} C_\lambda = k$. \hfill \Box

For $\alpha, d > 0$, set

$$
I_\lambda^\alpha = \{u \in W^{1,p}_r(\mathbb{R}^N) : I_\lambda(U) \leq \alpha\}
$$

and

$$
A^d = \{u \in W^{1,p}_r(\mathbb{R}^N) : \inf_{v \in A_r} \|u - v\| \leq d\}.
$$

Obviously, for all $d > 0$, $A^d \neq \emptyset$. In the following, we will find a solution to the problem (1.1) in the neighborhood of $A_r$ for $\lambda > 0$ small enough.

**Lemma 3.3.** For any $\{u_{\lambda_i}\} \subset A^d$ satisfying $\lim_{i \to \infty} I_{\lambda_i}(u_{\lambda_i}) \leq k$ and $\lim_{i \to \infty} I'_{\lambda_i}(u_{\lambda_i}) = 0$, there exists $u_0 \in A^d$ such that $u_{\lambda_i} \to u_0$ strongly in $W^{1,p}_r(\mathbb{R}^N)$ as $i \to \infty$, where $\lim_{i \to \infty} \lambda_i = 0$, provided that

$$
0 < d < \min\{1, (\frac{Nk}{a})^{\frac{1}{p-1}}\}. \tag{3.1}
$$
Proof. For convenience, we replace $\lambda_i$ by $\lambda$. Since $u_\lambda \in A^d$, we have $u_\lambda = U_\lambda + v_\lambda$, where $U_\lambda \in A_r$ and $v_\lambda \in W^{1,p}_r(\mathbb{R}^N)$ with $\|v_\lambda\| \leq d$. Because $A_r$ is compact, there exist $U_0 \in A_r$ and $v_0 \in W^{1,p}_r(\mathbb{R}^N)$ such that $U_\lambda \to U_0$ strongly in $W^{1,p}_r(\mathbb{R}^N)$, $v_\lambda \to v_0$ weakly in $W^{1,p}_r(\mathbb{R}^N)$ and $v_\lambda \to v_0$ a.e. in $\mathbb{R}^N$. Let $u_0 = U_0 + v_0$, then $u_0 \in A_d$ and $u_\lambda \to u_0$ weakly in $W^{1,p}_r(\mathbb{R}^N)$.

Firstly, we claim that $u_0 \not\equiv 0$. It follows from $\lim_{i \to \infty} I'_X(u_{\lambda_i}) = 0$ that $I'(u_0) = 0$. Otherwise, if $u_0 \equiv 0$, then $\|U_0\| = \|v_0\| \leq d$. By Lemma 2.2, we obtain that $\|\nabla U_0\|_p = \left(\frac{Nk}{a}\right)^{\frac{1}{p}}$. On the other hand, by (3.1), we have

$$\|\nabla U_0\|_p \leq \|U_0\| \leq d < \left(\frac{Nk}{a}\right)^{\frac{1}{p}}.$$  

This is a contradiction. So $u_0 \not\equiv 0$ and $I(u_0) \geq k$.

Secondly, we prove that $u_{\lambda_i} \to u_0$ strongly in $W^{1,p}_r(\mathbb{R}^N)$. Indeed, $\{u_{\lambda_i}\}$ is a (PS) sequence of $I_X$, that is, $\{u_{\lambda_i}\}$ and $\{I_X(u_{\lambda_i})\}$ are bounded, $I_X(u_{\lambda_i}) \to 0$ as $i \to \infty$. We obtain

$$u_{\lambda_i} \to u_0 \text{ in } L^q(\mathbb{R}^N), \quad q \in (p,p^*)$$

and

$$u_{\lambda_i} \to u_0 \text{ a.e. in } \mathbb{R}^N.$$  

For convenience, we write $u_i$ for $u_{\lambda_i}$. By (H1) and (H2), for any $\xi > 0$, there exists $C_\xi > 0$ such that

$$|h(t)| \leq \xi |t|^{p-1} + |t|^{p^*-1} + C_\xi |t|^{q-1}, \quad t \in \mathbb{R}, \quad q \in (p,p^*).$$

Thus, by Hölder inequality, we have

$$\left| \int_{\mathbb{R}^N} h(u_i)(u_i - u)dx \right|$$

$$\leq \int_{\mathbb{R}^N} |h(u_i)||u_i - u|dx$$

$$\leq \int_{\mathbb{R}^N} (\xi|u_i|^{p-1}|u_i - u| + \xi|u_i|^{p^*-1}|u_i - u| + C_\xi|u_i|^{q-1}|u_i - u|)dx$$

$$\leq \xi\|u_i\|_p^{p-1}\|u_i - u\|_p + \xi\|u_i\|_p^{p^*-1}\|u_i - u\|_p + C_\xi\|u_i\|_q^{q-1}\|u_i - u\|_q$$

$$= \xi\delta_1 + \xi\delta_2 + C_\xi\delta_3,$$  

where

$$\delta_1 = \|u_i\|_p^{p-1}\|u_i - u\|_p,$$

$$\delta_2 = \|u_i\|_p^{p^*-1}\|u_i - u\|_p,$$

and

$$\delta_3 = \|u_i\|_q^{q-1}\|u_i - u\|_q.$$  

By the Sobolev’s imbedding $W^{1,p}_r(\mathbb{R}^N) \hookrightarrow L^q(\mathbb{R}^N)$ and Minkowski inequality, we get that $\delta_1$ and $\delta_2$ are bounded. It follows from $q \in (p,p^*)$ that $\|u_i - u\|_q \to 0$. 

namely $\delta_3 \to 0$. Thus, we have
\[
\int_{\mathbb{R}^N} h(u_i)(u_i - u)dx \to 0.
\]
So
\[
(a + \lambda\|u_i\|^p)(u_i, u_i - u) = \langle I_\lambda'(u_i), u_i - u \rangle + \int_{\mathbb{R}^N} h(u_i)(u_i - u)dx \to 0,
\]
where $(u_i, u_i - u) = \int_{\mathbb{R}^N}(|\nabla u_i|^{p-2}\nabla u_i \cdot \nabla (u_i - u) + b|u_i|^{p-2}u_i(u_i - u))dx$.
Noticing that max
\[
\text{we have}
\]
(3.3)
\[
(u_i, u_i - u) = 0.
\]
In addition, together with $u_i \to u$ weakly in $W_r^1, p(\mathbb{R}^N)$, we have
(3.4)
\[
(u, u_i - u) = 0.
\]
It follows from (3.3) and (3.4) that
\[
\int_{\mathbb{R}^N}((|\nabla u_i|^{p-2}\nabla u_i - |\nabla u|^{p-2}\nabla u) \cdot \nabla (u_i - u) + b|u_i|^{p-2}u_i(u_i - u)dx \to 0.
\]
Combining with the following standard inequality in $\mathbb{R}^N$ given by
\[
\langle |\alpha|^{p-2}\alpha - |\beta|^{p-2}\beta, \alpha - \beta \rangle \geq \begin{cases} 
C_p|\alpha - \beta|^p, & p \in [2, +\infty), \\
C_p|\alpha - \beta|^2(|\alpha| + |\beta|)^{p-2}, & p \in (1, 2), 
\end{cases}
\]
we can prove that $u_n \to u$ strongly in $W_r^1, p(\mathbb{R}^N)$.

By Lemma 3.3, there exists a constant $d$ satisfying (3.1) and $C_4 > 0, \lambda_0 > 0$ such that $\|I_\lambda(u)\| \geq C_4$ for $u \in I_\lambda^{\lambda_0} \cap (A^{\frac{d}{N}}A^{\frac{d}{2}})$ and $\lambda \in (0, \lambda_0)$.

**Lemma 3.4.** There exists $C_4 > 0$ such that for small $\lambda > 0$, $I_\lambda(\gamma(s)) \geq C_\lambda - C_4$, this shows that $\gamma(s) \in A^{\frac{d}{2}}$, where $\gamma(s) = U(\frac{1}{s})$, $s \in (0, t_1]$.

**Proof.** By Pohožaev equality,
\[
I_\lambda(\gamma(s)) = I(\gamma(s)) + \frac{\lambda}{2p}\|\gamma(s)\|^{2p} = \left(\frac{a}{p}t^{N-p} - \frac{(N-p)a}{Np}t^N\right)\int_{\mathbb{R}^N} |\nabla U|^pdx + \frac{\lambda}{2p}\|U(\frac{1}{s})\|^{2p}.
\]
From Lemma 3.1, we have
\[
I_\lambda(\gamma(s)) = \left(\frac{a}{p}t^{N-p} + \frac{(N-p)a}{Np}t^N\right)\int_{\mathbb{R}^N} |\nabla U|^pdx + O(\lambda).
\]
Noticing that $\max_{s \in (0, t_1]} I(\gamma(s)) = k$ can be achieved at $s = 1$, there exists $C_5 > 0$ so small that $\gamma(s) = U(\frac{1}{s}) \in A^{\frac{d}{2}}$ for $|s - 1| \leq C_5$. Combining with $C_\lambda \to k$ as $\lambda \to 0$, there is $C_4 > 0$ such that
\[
I(\gamma(s)) \geq C_\lambda - C_4
\]
for $\lambda > 0$ small enough. This implies that $|s - 1| \leq C_5$ and $\gamma(s) \in A^d$. \hfill \Box

**Lemma 3.5.** For any $\lambda > 0$ small enough, there exists a sequence $\{u_n\} \subset I^D_\lambda \cap A^d$ such that $I^D_\lambda(u_n) \to 0$ as $n \to \infty$.

**Proof.** Assume by contradiction, there exists $\beta(\lambda) > 0$ such that $|I^D_\lambda(u)| \geq \beta(\lambda)$, $u \in I^D_\lambda \cap A^d$ for some $\lambda > 0$. Then there exists a pseudo-gradient vector field [19] $\Phi_1$ in $W^{1,p}(\mathbb{R}^N)$ on a neighborhood $Y_\lambda$ of $I^D_\lambda \cap A^d$ such that

$$\|I^D_\lambda(u)\| \leq 2 \min\{1, |I^D_\lambda(u)|\}$$

and

$$\langle I^D_\lambda(u), \Phi_\lambda(u) \rangle \geq \min\{1, |I^D_\lambda(u)|\}|I^D_\lambda(u)|.$$ 

Denote $\zeta_\lambda$ be a Lipschitz continuous function on $W^{1,p}(\mathbb{R}^N)$ such that $\zeta_\lambda \in [0, 1]$ and

$$\zeta_\lambda(u) = \begin{cases} 1, & u \in I^D_\lambda \cap A^d, \\ 0, & u \in W^{1,p}(\mathbb{R}^N) \setminus Y_\lambda. \end{cases}$$

Define $\mu_\lambda$ be a Lipschitz continuous function on $\mathbb{R}$ such that $\mu_\lambda \in [0, 1]$ and

$$\mu_\lambda(t) = \begin{cases} 1, & |t - C_\lambda| \leq \frac{C_4}{4}, \\ 0, & |t - C_\lambda| \geq \frac{C_4}{4}, \end{cases}$$

where $C_4$ is given in Lemma 3.4. Set

$$\eta_\lambda(u) = \begin{cases} -\zeta_\lambda(u)\mu_\lambda(I^D_\lambda(u))\Phi_\lambda(u), & u \notin Y_\lambda, \\ 0, & u \in W^{1,p}(\mathbb{R}^N) \setminus Y_\lambda. \end{cases}$$

Then, the following initial value problem

$$\begin{cases} \frac{d}{dt}Z_\lambda(u, t) = \eta_\lambda(Z_\lambda(u, t)), \\ Z_\lambda(u, 0) = u, \end{cases}$$

admits a unique global solution $Z_\lambda : W^{1,p}(\mathbb{R}^N) \times \mathbb{R}_+ \to W^{1,p}(\mathbb{R}^N)$ which satisfies

(i) $Z_\lambda(u, t) = u$, if $t = 0$ or $u \notin Y_\lambda$ or $|I^D_\lambda(u) - C_\lambda| \geq C_4$;

(ii) $\|\frac{d}{dt}Z_\lambda(u, t)\| \leq 2$ for $(u, t) \in W^{1,p}(\mathbb{R}^N) \times \mathbb{R}_+$;

(iii) $\frac{d}{dt}I^D_\lambda(Z_\lambda(u, t)) \leq 0$.

We adopt similar idea in [8] and obtain that for any $s \in (0, t_4]$, there is $t_s > 0$ such that

$$Z_\lambda(\gamma(s), t_s) \in I^D_\lambda - \frac{C_4}{4},$$

where $\gamma(s) = U(s\gamma(s), u)$, $s \in (0, t_4]$. Let $\gamma_0(s) = Z_\lambda(\gamma(s), t_s(s))$, where $t_s(s) = \inf\{t \geq 0, Z_\lambda(\gamma(s), t) \in I^D_\lambda - \frac{C_4}{4}\}$. By similar ideas in [8, 21], we can prove that $\gamma_0(s)$ is continuous in $[0, t_4]$ and $\|\gamma_0(s)\| \leq \|\gamma(s)\| \leq C_2 + 2$. Therefore, with $\max_{t \in [0, t_4]} I_\lambda(\gamma_0(t)) \leq C_4 - \frac{C_4}{4}\lambda$. This is a contradiction with $C_\lambda = \inf_{\gamma \in \Gamma_\lambda, s \in [0, t_4]} I_\lambda(\gamma(s))$. The proof is completed. \hfill \Box
Now, we give the proofs of the main results.

Proof of Theorem 1.1. By Lemma 3.5, there exists a bounded (PS) sequence \( \{ u_n \} \subset I^\lambda_D \cap A^d \). Without loss of generality, we may assume that \( u_n \rightharpoonup u_\lambda \) weakly in \( W^{1,p}(\mathbb{R}^N) \). In connection with Lemma 3.3 and Lemma 3.4, we can obtain that \( I'_\lambda(u_\lambda) = 0 \) and \( u_\lambda \in I^\lambda_D \cap A^d \). Furthermore, it follows from similar arguments in Lemma 3.3 that \( u_\lambda \not\equiv 0 \) under the proper choice of \( d \) satisfying (3.1). By the strong maximum principle, we adopt similar idea in [18] and can prove that \( u_\lambda \) is a positive solution of the problem (1.1). □

Proof of Theorem 1.2. For any \( \phi \in C^\infty_0(\mathbb{R}^N) \), we have
\[
I'_\lambda(u_\lambda)\phi = I'(u_\lambda)\phi + \lambda\|u_\lambda\|^p \int_{\mathbb{R}^N} |u_\lambda|^{p-2}u_\lambda \phi dx = 0.
\]
Then
\[
I'(u_\lambda)\phi = -\lambda\|u_\lambda\|^p \int_{\mathbb{R}^N} |u_\lambda|^{p-2}u_\lambda \phi dx \to 0 \quad \text{as} \quad \lambda \to 0.
\]
Combining with that \( I_\lambda(u_\lambda) = I(u_\lambda) + \frac{\lambda}{2p}\|u_\lambda\|^{2p} \), we have
\[
I(u_\lambda) \leq C_\lambda \quad \text{and} \quad I'(u_\lambda) \to 0 \quad \text{as} \quad \lambda \to 0.
\]
Namely, \( \{ u_n \} \) is a bounded (PS) sequence for the energy functional \( I \). We may assume that \( u_\lambda \rightharpoonup u^* \) weakly in \( W^{1,p}(\mathbb{R}^N) \), then \( I'(u^*) = 0 \). Similar proof as the one in Lemma 3.3 demonstrates that \( u_\lambda \to u^* \) strongly in \( W^{1,p}(\mathbb{R}^N) \).

By the proper choice of \( d > 0 \), we can prove that \( u^* \not\equiv 0 \). Hence \( I(u^*) \geq k \).

Meanwhile, we have \( I(u^*) \leq k \) since \( I(u_\lambda) \leq D_\lambda \to k \) as \( \lambda \to 0 \). So \( I(u^*) = k \).

By Lemma 2.6, \( u^* \) is a ground state solution to the limit problem (1.5). The proof is completed. □

Acknowledgements. The authors would like to appreciate the referees for their precious comments and suggestions about the original manuscript. This research was supported by the Fundamental Research Funds for the Central Universities (2015XKMS072).

References

[1] C. Alves, F. Corrêa, and T. Ma, Positive solutions for a quasilinear elliptic equation of Kirchhoff type, Comput. Math. Appl. 49 (2005), no. 1, 85–93.
[2] G. Autuori, F. Colasuonno, and P. Pucci, On the existence of stationary solutions for higher-order \( p \)-Kirchhoff problems, Commun. Contemp. Math. 16 (2014), no. 5, 1–43.
[3] H. Berestycki and P. Lions, Nonlinear scalar field equations I. Existence of a ground state, Arch. Rational Mech. Anal. 82 (1983), no. 4, 313–345.
[4] J. Byeon and L. Jeanjean, Standing waves for nonlinear Schrödinger equations with a general nonlinearity, Arch. Rational Mech. Anal. 185 (2007), no. 2, 185–200.
[5] J. Byeon, J. Zhang, and W. Zou, Singularly perturbed nonlinear Dirichlet problems involving critical growth, Calc. Var. 47 (2013), no. 1-2, 65–85.
[6] C. Chen, J. Huang, and L. Liu, *Multiple solutions to the nonhomogeneous p-Kirchhoff elliptic equation with concave-convex nonlinearities*, Appl. Math. Lett. **36** (2013), no. 7, 754–759.

[7] C. Chen and Q. Zhu, *Existence of positive solutions to p-Kirchhoff type-problem without compactness conditions*, Appl. Math. Lett. **28** (2014), 82–87.

[8] Z. Chen and W. Zou, *Standing waves for a coupled system of nonlinear Schrödinger equations*, Ann. Mat. Pura. Appl. **194** (2015), no. 1, 183–220.

[9] X. Cheng and G. Dai, *Positive solutions for p-Kirchhoff type problems on $\mathbb{R}^N$*, Math. Meth. Appl. Sci. **38** (2015), no. 12, 2650–2662.

[10] S. Coleman, V. Glaser, and A. Martin, *Action minima among solutions to a class of Euclidean scalar field equations*, Comm. Math. Phys. **58** (1978), no. 2, 211–221.

[11] V. Coti Zelati and P. Rabinowitz, *Homoclinic type solutions for a semilinear elliptic PDE on $\mathbb{R}^N$*, Comm. Pure Appl. Math. **45** (1992), no. 10, 1217–1269.

[12] J. do O and E. Medeiros, *Remarks on the least energy solutions for quasilinear elliptic problems in $\mathbb{R}^N$*, Electron. J. Differential Equations **2003** (2003), 1–14.

[13] Y. Huang and Z. Liu, *On a class of Kirchhoff type problems*, Arch. Math. **102** (2014), 127–139.

[14] L. Jeanjean and K. Tanaka, *A remark on least energy solution in $\mathbb{R}^N$*, Proc. Amer. Math. Soc. **13** (2002), 2399–2408.

[15] G. Kirchhoff, *Mechanik*, Teubner, Leipzig, 1983.

[16] Y. Li, F. Li, and J. Shi, *Existence of a positive solution to Kirchhoff type problems without compactness conditions*, J. Differential Equations **253** (2012), no. 7, 2285–2294.

[17] J. Liu, J. Liao, and C. Tang, *Positive solution for the Kirchhoff-type equations involving general subcritical growth*, Commun. Pure Appl. Math. **15** (2016), no. 2, 445–455.

[18] Z. Liu and S. Guo, *On ground states for the Kirchhoff type problems with a general critical nonlinearity*, J. Math. Anal. Appl. **426** (2015), no. 1, 267–287.

[19] M. Struwe, *Variational Methods*, Application to nonlinear partial differential equations and Hamiltonian systems, Springer-Verlag, 1990.

[20] M. Willem, *Minimax Theorems*, Birkhäuser, 1996.

[21] J. Zhang, J. Marcos Do Ó, and M. Squassina, *Schrödinger-Poisson systems with a general critical nonlinearity*, Commun. Contemp. Math. (published online), 2016.

HUIXING ZHANG
**DEPARTMENT OF MATHEMATICS**
**CHINA UNIVERSITY OF MINING AND TECHNOLOGY**
**XUZHOU, JIANGSU 221116, P. R. CHINA**
**E-mail address:** zhx20110906@cumt.edu.cn

RAN ZHANG
**DEPARTMENT OF MATHEMATICS**
**CHINA UNIVERSITY OF MINING AND TECHNOLOGY**
**XUZHOU, JIANGSU 221116, P. R. CHINA**
**E-mail address:** 152620191020163.com