CONSTRUCTION OF FLOWS OF FINITE-DIMENSIONAL ALGEBRAS

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Abstract. Recently, we introduced the notion of flow (depending on time) of finite-dimensional algebras. A flow of algebras (FA) is a particular case of a continuous-time dynamical system whose states are finite-dimensional algebras with (cubic) matrices of structural constants satisfying an analogue of the Kolmogorov-Chapman equation (KCE). Since there are several kinds of multiplications between cubic matrices one has fix a multiplication first and then consider the KCE with respect to the fixed multiplication. The existence of a solution for the KCE provides the existence of an FA. In this paper our aim is to find sufficient conditions on the multiplications under which the corresponding KCE has a solution. Mainly our conditions are given on the algebra of cubic matrices (ACM) considered with respect to a fixed multiplication of cubic matrices. Under some assumptions on the ACM (e.g. power associative, unital, associative, commutative) we describe a wide class of FAs, which contain algebras of arbitrary finite dimension. In particular, adapting the theory of continuous-time Markov processes, we construct a class of FAs given by the matrix exponent of cubic matrices. Moreover, we remarkably extend the set of FAs given with respect to the Maksimov’s multiplications of our paper [8]. For several FAs we study the time-dependent behavior (dynamics) of the algebras. We derive a system of differential equations for FAs.

1. Introduction

It is known that (see e.g. [16]) if each element of a family (depending on time) of matrices satisfying the Kolmogorov-Chapman equation (KCE) is stochastic, then it generates a Markov process. But what kind of process or dynamical systems can be generated by a family of non-stochastic matrices satisfying KCE? Depending on the matrices, it can be a non-Markov process [4], a deformation [12], etc. Other motivations of consideration of non-stochastic solutions of KCE are given in recent papers [2, 11, 13, 14, 15]. These papers devoted to study some chains of evolution algebras. In each of these papers the matrices of structural constants (time-dependent on the pair \((s, t)\)) are square or rectangular and satisfy the KCE. In other words, a chain of evolution algebras is a continuous-time dynamical system which in any fixed time is an evolution algebra.

In [3], [7] some cubic stochastic matrices are used (as matrices of structural constants) to investigate algebras and dynamical systems of bisexual populations.

In [8] we generalized the notion of chain of evolution algebras (given for algebras with rectangular matrices) to a notion of flow of arbitrary finite-dimensional algebras (i.e. their matrices of structural constants are cubic matrices). In this paper we continue our investigations of flows of algebras (FAs).

The paper is organized as follows. In Section 2 we give the main definitions related to algebras of cubic matrices, several kinds of multiplications of cubic matrices and FAs. Note that an FA is defined by a family (depending on time) of cubic matrices of structural constants, which satisfy an analogue of KCE. Since there are several types of multiplication of cubic matrices, one has to fix a multiplication, say \(\mu\), first and then consider the KCE with respect to this multiplication. In Section 3 we find some conditions on \(\mu\) under which the KCE has at least one solution. For several multiplications we give a wide class of solution of KCE, i.e. a wide class of FAs. For the multiplications of Maksimov [9] we extend the class of FAs given in [8]. Section 4 contains

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some differential equations for FAs. For several FAs we study the time-dependent behavior of the flows.

2. Definitions

2.1. Algebras of cubic matrices. Given a field $F$, any finite-dimensional algebra $A$ can be specified up to isomorphism by giving its dimension (say $m$), and specifying $m^3$ structural constants $c_{ijk}$, which are scalars in $F$. These structure constants determine the multiplication in $A$ via the following rule:

$$e_i e_j = \sum_{k=1}^{m} c_{ijk} e_k,$$

where $e_1, \ldots, e_m$ form a basis of $A$. Thus the multiplication of a finite-dimensional algebra is given by a cubic matrix $(c_{ijk})$.

A cubic matrix $Q = (q_{ijk})_{i,j,k=1}^{m}$ is a $m^3$-dimensional vector which can be uniquely written as

$$Q = \sum_{i,j,k=1}^{m} q_{ijk} E_{ijk},$$

where $E_{ijk}$ denotes the cubic unit (basis) matrix, i.e. $E_{ijk}$ is a $m^3$-cubic matrix whose $(i, j, k)$th entry is equal to 1 and all other entries are equal to 0.

Denote by $\mathcal{C}$ the set of all cubic matrices over a field $F$. Then $\mathcal{C}$ is an $m^3$-dimensional vector space over $F$, i.e. for any matrices $A = (a_{ijk}), B = (b_{ijk}) \in \mathcal{C}$, $\lambda \in F$, we have

$$A + B := (a_{ijk} + b_{ijk}) \in \mathcal{C}, \quad \lambda A := (\lambda a_{ijk}) \in \mathcal{C}.$$

In general, one can fix an $m^3 \times m^3 \times m^3$- cubic matrix $\mu = \left( C_{uvw}^{ijkl,lnr} \right)$ as a matrix of structural constants and give a multiplication of basis cubic matrices as

$$E_{ijk} \ast_{\mu} E_{lnr} = \sum_{uvw} C_{uvw}^{ijkl,lnr} E_{uvw}. \quad (2.1)$$

Then the extension of this multiplication by bilinearity to arbitrary cubic matrices gives a general multiplication on the set $\mathcal{C}$ and it becomes an algebra of cubic matrices (ACM), denoted by $\mathcal{C}_{\mu}$. Under known conditions (see [5]) on structural constants one can make this general ACM as a commutative or/and associative algebra, etc.

2.2. Maksimov’s multiplications. Here we introduce some simple versions of multiplications (2.1). Denote $I = \{1, 2, \ldots, m\}$. Following [9] define the following multiplications for basis matrices $E_{ijk}$:

$$E_{ijk} \ast_{a} E_{lnr} = \delta_{kl} E_{i(a(j,n))r}, \quad (2.2)$$

where $a: I \times I \to I$, $(j, n) \mapsto a(j, n) \in I$, is an arbitrary associative binary operation and $\delta_{kl}$ is the Kronecker symbol.

Denote by $\mathcal{O}_m$ the set of all associative binary operations on $I$.

The general formula for the multiplication is the extension of (2.2) by bilinearity, i.e. for any two cubic matrices $A = (a_{ijk}), B = (b_{ijk}) \in \mathcal{C}$ the matrix $A *_{a} B = (c_{ijk})$ is defined by

$$c_{ijr} = \sum_{l,n: a(l,n)=j} \sum_{k} a_{ilk} b_{knr}. \quad (2.3)$$

Denote by $\mathcal{C}_{a} \equiv \mathcal{C}^{(a)}_{a} = (\mathcal{C}, *_{a}), a \in \mathcal{O}_m$, the ACM given by the multiplication $*_{a}$.
2.3. Flow of algebras. Following [8] we define a notion of flow of algebras (FA). Consider a family \( \{ A[s,t] : s, t \in \mathbb{R}, 0 \leq s \leq t \} \) of arbitrary \( m \)-dimensional algebras over the field \( F \), with basis \( e_1, \ldots, e_m \) and multiplication table
\[
e_i e_j = \sum_{k=1}^{m} c_{ijk}^{[s,t]} e_k, \quad i, j = 1, \ldots, m.
\]

Here parameters \( s, t \) are considered as time.

Denote by \( M[s,t] = (c_{ijk}^{[s,t]})_{i,j,k=1,\ldots,m} \) the matrix of structural constants of \( A[s,t] \).

**Definition 1.** Fix an arbitrary multiplication (not necessarily the Maksimov’s multiplication) of cubic matrices, say \( *_{\mu} \).

A family \( \{ A[s,t] : s, t \in \mathbb{R}, 0 \leq s \leq t \} \) of \( m \)-dimensional algebras over the field \( F \) is called an FA of type \( \mu \) if the matrices \( M[s,t] \) of structural constants satisfy the Kolmogorov-Chapman equation (for cubic matrices):
\[
M[s,t] \star_{\mu} M[t,s] = M[s,\tau] \star_{\mu} M[\tau,t], \quad \text{for all } 0 \leq s < \tau < t.
\]

(2.4)

See [8] for useful remarks and several examples of FAs.

**Definition 2.** An FA is called a (time) homogenous FA if the matrix \( M[s,t] \) depends only on \( t - s \). In this case we write \( M[t-s] \).

**Definition 3.** An FA is called a discrete-time FA if the matrix \( M[s,t] \) depends only on \( t, s \in \mathbb{N} \). In this case we write \( M[n,m] \), \( n, m \in \mathbb{N} \).

To construct an FA of type \( \mu \) one has to solve (2.4). In this paper our aim is to construct FAs, i.e. find solutions to the equation (2.4).

### 3. Constructions and time dynamics of FAs

#### 3.1. Power-associative multiplications.
Recall that an algebra is called power-associative if for each element \( a \) the subalgebra generated by \( a \) is associative, that is \( a^n a^m = a^{m+n} \), for all nonnegative integers \( n, m \).

Fix an arbitrary multiplication rule between cubic matrices, say \( *_{\mu} \).

For a cubic matrix \( Q \) denote
\[
Q^{*_{\mu} n} = Q *_{\mu} Q *_{\mu} \cdots *_{\mu} Q, \quad n = 1, 2, \ldots.
\]

**Condition 1.** Assume that the algebra \( C_{\mu} \) of cubic matrices is power-associative.

**Proposition 1.**

1. If Condition 1 is satisfied for the multiplication \( *_{\mu} \) then \( M[n,m] = Q^{*_{\mu} (m-n)} \) generates a discrete time FA of type \( \mu \).

2. If ACM \( C_{\mu} \) has an idempotent, i.e. there exists \( I \in C_{\mu} \) such that \( I *_{\mu} I = I \) then \( M[s,t] = I \), for all \( 0 \leq s < t \), generates an FA of type \( \mu \).

**Proof.**

1. Using power-associativity of the ACM for any natural numbers \( n < k < m \) we get
\[
Q^{*_{\mu} (m-n)} = Q^{*_{\mu} (k-n)} *_{\mu} Q^{*_{\mu} (m-k)}.
\]

Thus \( M[n,m] = Q^{*_{\mu} (m-n)} \) is a solution to (2.4) and therefore it generates a discrete-time FA of type \( \mu \), denote it by \( A_1^{[n,m]} \).

2. Straightforward. Denote the corresponding FA by \( A_2^{[s,t]} \). □
Remarks on time dynamics. The time dynamics of $A_1^{[n,m]}$ depends on the fixed matrix $Q$. This FA is a time homogenous; it can be periodic iff the powers (with respect to multiplication $\mu$) of the cubic matrix $Q$ have periods, i.e. if there is $p \in \mathbb{N}$ such that $Q^{*n} = Q$; if $Q$ is such that $\lim_{n \to \infty} Q^{*n} = Q$ exists then the FA has a limit algebra:

$$\tilde{A}_1 = \lim_{m,n \to \infty} A_1^{[n,m]}.$$  

The time dynamics of $A_2^{[s,d]}$ is trivial: it does not depend on time.

3.2. Exponential solutions. In the theory of continuous-time Markov chains, the matrix exponent of square matrices plays a crucial role (see e.g. [11, Chapter 3], [16, Chapter 2]). Here we adapt this notion of matrix exponent for cubic matrices.

Recall that an algebra is unitial if it has an element $u$ with $ux = x = xu$ for all $x$ in the algebra. The element $u$ is called unit element.

Condition 2. Assume $*_{\mu}$ is such that the corresponding ACM, $C_{\mu} = (C,*_{\mu})$ is unitial, i.e. it has a unit matrix denoted by $I$.

In the unital ACM $C_{\mu}$ for each cubic matrix $Q$ we define $Q^{*0} = I$.

A matrix exponent $\exp_{\mu}(tQ)$ is defined by

$$\exp_{\mu}(tQ) = 1 + \sum_{n \geq 1} \frac{(tQ)^{*n}}{n!} = \sum_{n \geq 0} \frac{(tQ)^{*n}}{n!}, \quad \text{i.e.} \quad \left( \exp_{\mu}(tQ) \right)_{ijk} = \sum_{n \geq 0} \frac{t^n(Q^{*n})_{ijk}}{n!}. \quad (3.1)$$

For a cubic matrix $Q$, the parameter $t$ in (3.1) can be any real number. In our paper we consider the case $t \geq 0$.

Condition 3. Assume that $C_{\mu}$ is a normed space with norm $\|\cdot\|$ such that for any two cubic matrices $A, B$, we have

$$\|A *_{\mu} B\| \leq \|A\| \|B\|. \quad (3.2)$$

Condition 4. Assume $C_{\mu}$ is an associative algebra.

Proposition 2. Assume Conditions 1–4 are satisfied. Let $Q$ be a cubic matrix. Then

(a) The series in (3.1) converges.

(b) The function $\exp_{\mu}(tQ)$ is differentiable with respect to $t$ with

$$\frac{d}{dt}\exp_{\mu}(tQ) = Q *_{\mu} \exp_{\mu}(tQ) = \exp_{\mu}(tQ) *_{\mu} Q, \quad t \in \mathbb{R}. \quad (c)$$

(c) The semigroup property:

$$\exp_{\mu}((s + t)Q) = \exp_{\mu}(sQ) *_{\mu} \exp_{\mu}(tQ), \quad \text{for all } s, t \in \mathbb{R}. \quad (3.3)$$

(d) The function $\exp_{\mu}(tQ)$ is the only solution to

$$\frac{d}{dt}Y(t) = Y(t) *_{\mu} Q = Q *_{\mu} Y(t), \quad \text{with } Y(0) = I. \quad (3.4)$$

These are called the Kolmogorov’s forward and backward equations.

Proof. (a) Using properties of the norm and (3.2) we get

$$\|\exp_{\mu}(tQ)\| = \left\| \sum_{n \geq 0} \frac{(tQ)^{*n}}{n!} \right\| \leq \sum_{n \geq 0} \frac{|t|^n\|Q\|^n}{n!} = \exp(|t|\|Q\|) < +\infty.$$

(b) Write

$$\frac{d}{dt}\exp_{\mu}(tQ) = \frac{d}{dt} \sum_{n \geq 0} \frac{(tQ)^{*n}}{n!} = \sum_{n \geq 1} \frac{t^{n-1}Q^{*n}}{(n-1)!} = Q *_{\mu} \sum_{n \geq 1} \frac{(tQ)^{*n-1}}{(n-1)!} = Q *_{\mu} \exp_{\mu}(tQ).$$
It is obvious that
\[ Q \ast \mu \exp_{\mu}(tQ) = Q \ast \mu \sum_{n \geq 1} \frac{(tQ)^{n\mu}}{(n-1)!} = \sum_{n \geq 1} \frac{(tQ)^{n\mu}}{(n-1)!} \ast \mu Q = \exp_{\mu}(tQ) \ast \mu Q. \]

(c) From (3.1) we have
\[ \exp_{\mu}(sQ) \ast \mu \exp_{\mu}(tQ) = \left( \sum_{n \geq 0} \frac{(sQ)^{n\mu}}{n!} \right) \ast \mu \left( \sum_{k \geq 0} \frac{(tQ)^{k\mu}}{k!} \right) \]
\[ = \sum_{n \geq 0} \sum_{k \geq 0} \frac{Q^{n\mu}(n+k)s^k t^k}{n!k!}. \]

Denoting \( j = n + k \) from the last equality, we get
\[ \exp_{\mu}(sQ) \ast \mu \exp_{\mu}(tQ) = \sum_{j \geq 0} \sum_{k \geq 0} Q^{\mu j}(s+j) \frac{j!}{(j-k)!k!} s^k t^{j-k} \]
\[ = \sum_{j \geq 0} \frac{Q^{\mu j}(s+t)^j}{j!} = \exp_{\mu}((s+t)Q). \]

(d) From part (b) it follows that \( Y(t) = \exp_{\mu}(tQ) \) is a solution to (3.3). We show its uniqueness. Assume \( Z(t) \) is another solution. Then take
\[ \frac{d}{dt} (Z(t) \ast \mu \exp_{\mu}(-tQ)) = \frac{d}{dt} Z(t) \ast \mu \exp_{\mu}(-tQ) + Z(t) \ast \mu \frac{d}{dt} \exp_{\mu}(-tQ) \]
\[ = Q \ast \mu Z(t) \ast \mu \exp_{\mu}(-tQ) - Z(t) \ast \mu Q \ast \mu \exp_{\mu}(-tQ) = 0. \]

Thus \( Z(t) \ast \mu \exp_{\mu}(-tQ) \) is independent on \( t \). Consequently, since at \( t = 0 \) we have this function equal to \( I \) it follows that \( Z(t) \ast \mu \exp_{\mu}(-tQ) = I \), consequently by part (c) we get \( Z(t) = \exp_{\mu}(tQ) \).

Using (3.3) one easily checks that \( M_{[s,t]} = \exp_{\mu}((t-s)Q) \) satisfies (2.4).

Summarizing, we get the following,

**Theorem 1.** Let \( \mu \) be a multiplication such that Conditions 1–4 are satisfied. Let \( Q \) be a cubic matrix. Then the family of matrices \( \{ M_{[s,t]} = \exp_{\mu}((t-s)Q) \} \), \( 0 \leq s < t \), generates a time-homogeneous FA of type \( \mu \), denoted by \( A_{3}^{[s,t]} \).

Remarks on time dynamics. To study time dynamics of FA \( A_{3}^{[s,t]} \) one need to compute the matrix exponent \( \exp_{\mu}(tQ) \). We note that even in case of square matrices finding methods to compute the matrix exponent is difficult, and this is still a topic of considerable current research. But for some simple \( Q \) one can compute the matrix exponent. For example, if \( Q \) is nilpotent, i.e. there exists \( q \) such that \( Q^{\mu q} = 0 \). Then we have
\[ \exp_{\mu}(tQ) = I + tQ + \frac{t^2}{2} Q^{\mu 2} + \cdots + \frac{t^{q-1}}{(q-1)!} Q^{\mu (q-1)}. \]
For the corresponding \( A_{3}^{[s,t]} \) we see that if \( Q \) is nilpotent then does not exist a limit algebra at \( t \to \infty \).

3.3. **Time non-homogenous FAs.** The following theorem gives an example of \( m \)-dimensional time non-homogenous FA.

Let \( I \in \mathcal{C}_{\mu} \) be a unit matrix (under Condition 2). Matrix \( A^{-1} \in \mathcal{C}_{\mu} \) is called inverse of an \( A \in \mathcal{C}_{\mu} \) if
\[ A \ast \mu A^{-1} = A^{-1} \ast \mu A = I. \]
Theorem 2. Assume \( *_\mu \) such that Condition 2 and 4 are satisfied. Let \( \{ A^t, t \geq 0 \} \subset \mathcal{C}_\mu \) be a family of invertible (for all \( t \)) \( m^3 \)-matrices. Then the matrix

\[ \mathcal{M}^{[s,t]} = A^s *_\mu (A^t)^{-1}, \]

generates an FA of type \( \mu \), where \((A^t)^{-1}\) is the inverse of \( A^t \).

Proof. By associativity of the multiplication \( *_\mu \) of matrices we get

\[ \mathcal{M}^{[s,r]} *_\mu \mathcal{M}^{[r,t]} = A^s *_\mu \left((A^r)^{-1} *_\mu A^r\right) *_\mu (A^t)^{-1} = A^s *_\mu (A^t)^{-1} = \mathcal{M}^{[s,t]}. \]

Thus \( \mathcal{M}^{[s,t]} \) satisfies (2.4), consequently each family (with one parameter) of invertible cubic matrices defines an FA, denoted by \( \mathcal{A}_4^{[s,t]} \), which is time non-homogenous, in general. But will be a time homogenous FA, for example, if \( A^t \) is equal to \( t \)th power of an invertible matrix \( A \). □

Remarks on time dynamics. Depending on the given family \( A^t \) one can study time dynamics of the FA \( \mathcal{A}_4^{[s,t]} \) (see [2] Example 4] for a simple case).

3.4. Constructions for Maksimov’s multiplications. Denote by \( \mathcal{O}_m \) the set of all associative binary operations on \( I = \{1,2,\ldots,m\} \).

Definition 4. We say that an operation \( a \in \mathcal{O}_m \) is uniformly distributed if

\[ \sum_{l,n:a(l,n)=j} 1 = m, \]

independently on \( j \in I \).

For example, an operation \( a \) with \( a(i,j) = j \), for all \( i,j \in I \), is uniformly distributed. But \( a(i,j) = 1 \), for all \( i,j \in I \), is not uniformly distributed.

Theorem 3. Consider the Maksimov’s multiplication (2.3) with respect to a uniformly distributed \( a \). Take arbitrary functions \( f_i(t) \) and \( g_i(t) \), \( i = 1,\ldots,m \), of \( t \in \mathbb{R} \) such that

\[ \sum_{k=1}^m f_k(t)g_k(t) = \frac{1}{m}, \text{ for any } t \in \mathbb{R}. \quad (3.5) \]

Then the family of cubic matrices \( \mathcal{M}^{[s,t]} = (f_i(s)g_k(t))_{i,j,k=1}^m \) generates an FA of type \( a \), denoted by \( \mathcal{A}_5^{[s,t]} \).

Notice that the entries of these cubic matrices do not depend on the middle index \( j \).

Proof. Let \( \mathcal{M}^{[s,t]} = (f_i(s)g_k(t))_{i,j,k=1}^m \) then the equation (2.4) has the form

\[ f_i(s)g_r(t) = \sum_{l,n:a(l,n)=j} \sum_k f_i(s)g_k(\tau)f_k(\tau)g_r(t). \quad (3.6) \]

Simplify the system (3.6) to get

\[ f_i(s)g_r(t) \left[ \sum_{l,n:a(l,n)=j} \sum_k f_k(\tau)g_k(\tau) - 1 \right] = 0. \]

By conditions of the theorem we have

\[ \sum_{l,n:a(l,n)=j} \sum_k f_k(\tau)g_k(\tau) - 1 = 0, \text{ for all } \tau \in \mathbb{R}. \]

This completes the proof. □
Now we construct a very rich class of FAs of type $a_0$, where $a_0$ is an operation in $O_m$ such that $a_0(j,n) = j$ for any $j,n \in I$. Note that this operation is uniformly distributed. Then the corresponding Maksimov’s multiplication for arbitrary cubic matrices

$$A = (a_{ijk})_{i,j,k=1}^m, \quad B = (b_{ijk})_{i,j,k=1}^m, \quad C = (c_{ijk})_{i,j,k=1}^m,$$

is $C = A \ast a_0 B$ where

$$c_{ijr} = \sum_{k,n=1}^m a_{ijk}b_{knr}. \quad (3.7)$$

**Theorem 4.** Let $g_i(t)$ and $\gamma_{ij}(t)$, $i,j \in I$, be arbitrary functions of $t \in \mathbb{R}$ such that $g_i(t) \neq 0$ and

$$m \sum_{j=1}^m \gamma_{ij}(s) = g_i(s), \quad \text{for all } i \in I, s \in \mathbb{R}. \quad (3.8)$$

Then the cubic matrix

$$M[s,t] = \left( \gamma_{ij}(s) \right)_{i,j,r=1}^m / (g_r(t))$$

generates an FA of type $a_0$ denoted by $A_0^{[s,t]}$.

**Proof.** For $\ast a_0$, using (3.7), the equation (2.4) can be written as

$$M[s,t]_{ijr} = \sum_{k,n=1}^m M[s,\tau]_{ijk}M[\tau,t]_{knr}, \quad \text{for all } i,j,r \in I. \quad (3.9)$$

Denote

$$f_{ir}(s,t) = \sum_{j=1}^m M[s,t]_{ijr}. \quad (3.10)$$

Then from (3.9) we get

$$f_{ir}(s,t) = \sum_{k=1}^m f_{ik}(s,\tau)f_{kr}(\tau,t), \quad \text{for all } i,r \in I. \quad (3.11)$$

Consider arbitrary functions $g_i(t)$, $i = 1, \ldots, m$, such that $g_i(t) \neq 0$. Then it is easy to see that the system of functional equations (3.11) has the following solution

$$f_{ij}(s,t) = g_i(s) / m g_j(t).$$

Using this equality, by (3.10) and (3.9), we get

$$M[s,t]_{ijr} g_r(t) = \frac{1}{m} \sum_{k=1}^m M[s,\tau]_{ijk} g_k(\tau), \quad \text{for all } i,j,r \in I.$$

From this equality it is clear that the quantity $M[s,t]_{ijr} g_r(t)$ should not depend on $t$ and $r$, i.e. $M[s,t]_{ijr} g_r(t) = \gamma_{ij}(s)$, for some function $\gamma_{ij}(s)$. Thus

$$M[s,t]_{ijr} = \frac{\gamma_{ij}(s)}{g_r(t)}, \quad \text{for all } i,j,r \in I.$$

Now the equality (3.10) gives the condition (3.8) on $\gamma_{ij}$. \hfill \Box

**Remarks on time dynamics.** In general the behavior of $A_0^{[s,t]}$ depends on given functions $f_i(t)$ and $g_i(t)$. One can choose these function suitable to an expected property of the dynamics. Here we consider one example to make $A_0^{[s,t]}$ a periodic FA. Consider

$$f_k(t) = 2 + \sin(kt), \quad g_k(t) = \frac{1}{m^2(2 + \sin(kt))}, \quad k = 1, \ldots, m.$$
Then the condition (3.5) is satisfied. Consequently, the matrix

$$M^{[s,t]} = \left( \frac{2 + \sin(is)}{m^2(2 + \sin(kt))} \right)_{i,j,k=1}^m$$

generates a periodic FA $A_b^{[s,t]}$.

The time behavior of $A_b^{[s,t]}$ (is similar to the behavior of $A_b^{[s,t]}$) depends on its parameter-functions $g_i(t)$ and $\gamma_{ij}(t)$, $i, j \in I$.

Let $S_m$ be the group of permutations on $I$.

Take $a \in O_m$ and define an action of $\pi \in S_m$ on $a$ (denoted by $\pi a$) as

$$\pi a(i, j) = \pi a(\pi^{-1}(i), \pi^{-1}(j)), \quad \text{for all } i, j \in I.$$

**Theorem 5.** Let $a \in O_m$ and $*_a$ be a Maksimov’s multiplication (see (2.3)). If there exists a permutation $\pi \in S_m$ such that $\pi b = a$, that is

$$a(j, n) = \pi^{-1}(b(\pi(j), \pi(n))), \quad \text{for all } j, n \in I,$$

then there is an FA of type $a$ iff there is an FA of type $b$.

**Proof.** Assume there is an FA of type $a$, i.e. the equation (2.4) has a solution, say $M^{[s,t]} = (M^{[s,t]}_{ijr})$, with respect to the multiplication $*_a$. That is

$$M^{[s,t]}_{ijr} = \sum_{l;n:a(l,n)=j} \sum_k M^{[s,\tau]}_{ilk} M^{[\tau,t]}_{knr}.$$ (3.13)

Denote $\pi(i) = i'$, then using (3.12) from (3.13) we get

$$M^{[s,t]}_{\pi^{-1}(i')\pi^{-1}(j')\pi^{-1}(r')} = \sum_{l;n:b(l,n)=j'} \sum_k M^{[s,\tau]}_{\pi^{-1}(i')\pi^{-1}(l')\pi^{-1}(k')\pi^{-1}(n')\pi^{-1}(r')}.$$ (3.12)

Thus $N^{[s,t]} = (N^{[s,t]}_{ijr})$ with $N^{[s,t]}_{ijr} = M^{[s,t]}_{\pi^{-1}(i)\pi^{-1}(j)\pi^{-1}(r)}$ is a solution of the equation (2.4) with respect to the multiplication $*_b$. This completes the proof. 

3.5. **Multiplication of solutions of the equation (2.4).** In this subsection we assume the following

Condition 5. Assume $C_\mu$ is a commutative algebra.

**Theorem 6.** Assume $*_\mu$ such that Condition 4 and 5 are satisfied. Let $\{M^{[s,t]}\}, \{N^{[s,t]}\} \subset C_\mu$ be two families of $m^3$-matrices which generate two FAs of type $\mu$. Then the matrix

$$B^{[s,t]} = M^{[s,t]} *_\mu N^{[s,t]},$$

generates an FA of type $\mu$.

**Proof.** By associativity and commutativity of the multiplication $*_\mu$ of matrices we get

$$B^{[s,\tau]} *_\mu B^{[\tau,t]} = \left( M^{[s,\tau]} *_\mu N^{[s,\tau]} \right) *_\mu \left( M^{[\tau,t]} *_\mu N^{[\tau,t]} \right) = \left( M^{[s,\tau]} *_\mu M^{[\tau,t]} \right) = M^{[s,t]} *_\mu N^{[s,t]} = B^{[s,t]},$$

**Remark 1.** We note that Theorem 6 can be generalized, i.e. under conditions 4 and 5, let $\{M^{[s,t]}\} \subset C_\mu$, $i = 1, \ldots, k$, be $k$ families of $m^3$-matrices which generate $k$ FAs of type $\mu$. Then the matrix

$$B^{[s,t]} = M_1^{[s,t]} *_\mu M_2^{[s,t]} *_\mu \cdots *_\mu M_k^{[s,t]},$$

generates an FA of type $\mu$. 

This formula is useful to construct new FAs by known ones (using above mentioned examples of FAs and other examples given in [8]).

**Remark 2.** To check our Condition 1–5 for an algebra $\mathcal{C}_\mu$ one has to check known conditions (5) on $\mu = (c_{ijk}^{uvw})$ of the matrix of structural constants. Indeed, one can numerate the elements of the set \{ijk : i, j, k = 1, ..., m\} to write it in the form $\mathcal{J} = \{1, 2, ..., m^3\}$ then the matrix $\mu$ can be written as usual form: $\mu = (c_{ijk}^k)$ where $i, j, k \in \mathcal{J}$. Then the following conditions are known for an algebra with matrix $\mu$:

- **Commutative iff:**
  
  \[ c_{ij}^k = c_{ji}^k, \quad \text{for all } i, j, k \in \mathcal{J}. \]

- **Associative iff:**

  \[
  \sum_{r=1}^n c_{ij}^r c_{rk}^l = \sum_{r=1}^n c_{ir}^r c_{jk}^l, \quad \text{for all } i, j, k, l \in \mathcal{J}.
  \]

- **Existence of an idempotent element:** this problem is equivalent to the existence of a fixed point of the map $V: \mathbb{R}^{m^3} \to \mathbb{R}^{m^3}$, $x \mapsto V(x) = x'$, given by

  \[
  V : x'_k = \sum_{i, j = 1}^{m^3} c_{ij}^k x_i x_j, \quad k = 1, \ldots, m^3.
  \]

  For example, if $\mu$ is a stochastic cubic matrix in the sense that

  \[
  c_{ij}^k \geq 0, \quad \sum_{k=1}^{m^3} c_{ij}^k = 1, \quad \text{for all } i, j,
  \]

  then from known theorems about fixed points it follows that the corresponding operator $V$ has at least one fixed point (i.e. the algebra $\mathcal{C}_\mu$ has at least one idempotent element).

Let us give an example of ACM for which all above mentioned conditions can be easily checked.

**Example.** Let $\alpha : \mathcal{J} \times \mathcal{J} \to \mathcal{J}$ be a binary operation on $\mathcal{J} = \{1, 2, \ldots, m^3\} \equiv \{ijk : i, j, k = 1, \ldots, m\}$ and assume that $(\mathcal{J}, \alpha)$ is a group with the operation $\alpha$, i.e., the operation satisfies axioms: associativity, has identity element (denoted by $i_{0j0k0}$) and each element has an inverse (for each $ijk$ its inverse is denoted by $ijk$).

Define the following multiplication for basis matrices $E_{ijk}$:

\[
E_{ijk} *_{\alpha} E_{inr} = E_{\alpha(ijk, inr)}. \tag{3.14}
\]

Since $(\mathcal{J}, \alpha)$ is a group it is easy to see that the ACM $\mathcal{C}_\alpha$ is associative, with unit matrix $\mathbb{I} = E_{i0j0k0}$ and each basis element $E_{ijk}$ has its inverse denoted by $E_{ij0k0}^{-1}$. Indeed, since $\alpha(ijk, ijk) = i_{0j0k0}$ we have

\[
E_{ijk} *_{\alpha} E_{ij0k0}^{-1} = E_{\alpha(ijk, inr)} = E_{i0j0k0}.
\]

Moreover, if the group is commutative then the algebra $\mathcal{C}_\alpha$ is also commutative.

4. **Differential equations for flows of algebras**

For a continuous-time Markov process, Kolmogorov derived forward equations and backward equations, which are a pair of systems of differential equations that describe the time-evolution of the probability transition probabilities $P_{ij}^{[s,t]}$ giving the process [9]. For quadratic stochastic processes derived partial differential equations with delaying argument were derived. These equations then were used to describe some processes (see [10]).

In this section we shall derive partial differential equations for the matrices $\mathcal{M}_{ij}^{[s,t]}$. 
Let $\mathcal{M}^{[s,t]} = \left( M_{i,j,k}^{[s,t]} \right)_{i,j,k=1}^m$ generate an FA of type $\mu$. Take a small $h > 0$ such that $s + h < \tau < t$ and using the equation (2.4), we write

$$\mathcal{M}^{[s+h,t]} - \mathcal{M}^{[s,t]} = \mathcal{M}^{[s+h,\tau]} \ast \mu \mathcal{M}^{[\tau,t]} - \mathcal{M}^{[s,\tau]} \ast \mu \mathcal{M}^{[\tau,t]} = \left( \mathcal{M}^{[s+h,\tau]} - \mathcal{M}^{[s,\tau]} \right) \ast \mu \mathcal{M}^{[\tau,t]}.$$ 

Dividing this expression by $h$ and assuming the existence of the limits we obtain

$$\lim_{h \to 0} \frac{\mathcal{M}^{[s+h,t]} - \mathcal{M}^{[s,t]}}{h} = \lim_{h \to 0} \frac{\mathcal{M}^{[s+h,\tau]} - \mathcal{M}^{[s,\tau]}}{h} \ast \mu \mathcal{M}^{[\tau,t]},$$

i.e.

$$\frac{\partial}{\partial s} \mathcal{M}^{[s,t]} = \left( \frac{\partial}{\partial s} \mathcal{M}^{[s,t]} \right)^m i,j,k=1 \ast \mu \mathcal{M}^{[\tau,t]}, \quad (4.1)$$

here $\frac{\partial}{\partial s} \mathcal{M}^{[s,t]} = \left( \frac{\partial}{\partial s} M_{i,j,k}^{[s,t]} \right)_{i,j,k=1}^m$.

Similarly with respect $t$ we get

$$\frac{\partial}{\partial t} \mathcal{M}^{[s,t]} = \mathcal{M}^{[s,\tau]} \ast \mu \left( \frac{\partial}{\partial t} \mathcal{M}^{[\tau,t]} \right). \quad (4.2)$$

Summarize this in the following theorem.

**Theorem 7.** If $\mathcal{M}^{[s,t]}$ generates an FA of type $\mu$ then it satisfies the partial differential equations (4.1) and (4.2).

**Remark 3.** The equation (4.1) is called forward equation and (4.2) is called backward equation. We note that the equations (3.4) are particular cases of these equations. As it was shown above the solution of the equation (3.4) gives only time-homogenous FAs. Each matrix of the FAs mentioned in the previous section satisfy equations (1.1), (1.2).

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