Mirror symmetry of 3d $\mathcal{N} = 4$ gauge theories and supersymmetric indices

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Abstract

We compute supersymmetric indices to test mirror symmetry of three-dimensional $\mathcal{N} = 4$ gauge theories and dualities of half-BPS enriched boundary conditions and interfaces in four-dimensional $\mathcal{N} = 4$ Super-Yang Mills theory. We find the matching of indices as strong evidences for various dualities of the 3d interfaces conjectured by Gaiotto and Witten under the action of $S$-duality in Type IIB string theory.

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1 Introduction and conclusions

Three-dimensional $\mathcal{N} = 4$ supersymmetric gauge theories have a moduli space of supersymmetric vacua consisting of a Higgs branch $\mathcal{M}_H$ and a Coulomb branch $\mathcal{M}_C$ which are hyperkähler manifolds.
The theories have an intriguing duality, known as mirror symmetry, which relates theories with completely different UV descriptions where the Higgs branch and the Coulomb branch are exchanged and FI parameters and mass parameters are exchanged \[1\]. This mapping is very non-trivial. The Higgs branch is a hyperkähler quotient realized as a zero locus of the $D$-term constraints divided by the gauge group. On the Higgs branch the gauge symmetry is broken completely and the Higgs branch is not affected by the quantum correction. On the Coulomb branch the gauge group is generically broken to its maximal torus. Unlike the Higgs branch, the Coulomb branch receives perturbative and non-perturbative quantum corrections.

The quantum Coulomb branch is studied in terms of Hilbert series in \[2\]. The Hilbert series is a generating function that counts chiral operators on the branches $\mathcal{M}_{H/C}$ of vacua, graded by their dimensions and quantum numbers under global symmetry. It encodes the quantum numbers of generators and relations of the chiral ring $\mathbb{C}[\mathcal{M}_{H/C}]$ of the corresponding branches $\mathcal{M}_{H/C}$ of vacua.

Three-dimensional $\mathcal{N} = 4$ gauge theories can be realized in Type IIB string theory using the brane configuration \[7\]. Mirror symmetry can be viewed as arising from S-duality of Type IIB string theory. Type IIB brane construction is extended to further study of half-BPS boundary conditions and interfaces for four-dimensional $\mathcal{N} = 4$ Super Yang-Mills (SYM) theory \[8\ \[9\ \[10\], quarter-BPS corner configuration for four-dimensional $\mathcal{N} = 4$ SYM theory \[11\ \[12\], half-BPS boundary conditions for three-dimensional $\mathcal{N} = 4$ gauge theory \[13\] and two-dimensional $\mathcal{N} = (0, 4)$ supersymmetric gauge theories \[14\]. S-duality turns out to give a physical underpinning to Geometric Langlands Program \[15\ \[16\ \[17\] and Symplectic Duality \[18\ \[19\].

In this paper we compute supersymmetric full- and half-indices to test mirror symmetry of 3d $\mathcal{N} = 4$ gauge theories and to extend the analysis in \[12\] for the dualities of half-BPS boundary conditions and interfaces in 4d $\mathcal{N} = 4$ SYM theory. These dualities were originally conjectured by Gaiotto and Witten \[10\]. There have been plenty of works on the subject of 3d $\mathcal{N} = 4$ full superconformal indices \[20\ \[21\ \[22\ \[23\ \[24\ \[25\], however, general tests of mirror symmetry for 3d $\mathcal{N} = 4$ gauge theories have not appeared in the literature except for the simplest Abelian mirror symmetry \[26\], in contrast to 3d $\mathcal{N} = 2$ gauge theories \[27\ \[28\ \[29\]. Additionally, one drawback of the 3d full-indices is that they are insensitive to the boundary conditions for 4d $\mathcal{N} = 4$ gauge theory, which involve singular boundary conditions specified by the Nahm poles \[8\]. In order to address these open issues, we evaluate supersymmetric 4d full-indices and 3d half-indices which count local operators both in four and three dimensions and present vast identities of indices by checking several terms of series expansion. As discussed in \[26\], after the special limits, the full-indices of 3d $\mathcal{N} = 4$ gauge theories reduce to the Hilbert series. As a result, the identities of indices provide promotions of identities of Hilbert series discussed in \[21\ \[26\]. Furthermore, the full-indices of 3d $\mathcal{N} = 4$ gauge theories can be used to generalize the analysis in \[12\] of the half-BPS boundary conditions and interfaces for 4d $\mathcal{N} = 4$ SYM theory. We present general half-indices for enriched half-BPS boundary conditions and interfaces in 4d $\mathcal{N} = 4$ gauge theory and check precise matching of the indices for dual pairs.

The organization of this paper is straightforward. In section \[2\] we briefly review the supersymmetric indices introduced in \[12\] and present formulae and some properties of half-indices for 4d $\mathcal{N} = 4$ gauge theories and full-indices for 3d $\mathcal{N} = 4$ gauge theories. In section \[3\] we evaluate full-indices for 3d $\mathcal{N} = 4$ Abelian gauge theories and check mirror symmetry. In section \[4\] we further examine mirror
symmetry by computing full-indices for 3d $\mathcal{N} = 4$ non-Abelian gauge theories. We also briefly check in section 5 Seiberg-like duality for 3d $\mathcal{N} = 4$ gauge theories proposed in [10] between ugly theory and good theory in terms of full-indices. In section 6 we discuss the half-BPS enriched boundary conditions for 4d $\mathcal{N} = 4$ SYM theory which involve 3d $\mathcal{N} = 4$ gauge theories. We present strong evidences for dualities between them conjectured from string theory by calculating half-indices, which contain non-regular Nahm pole b.c. Finally in section 7 we study the half-BPS interfaces in 4d $\mathcal{N} = 4$ U(1) gauge theory including 3d $\mathcal{N} = 4$ Abelian gauge theories. We test dualities between the interfaces by computing half-indices.

2 Indices

We begin with a definition of the quarter-index introduced in [12]. It is a generalization of superconformal index in that it can count local operators living in different dimensions, i.e. in 4d bulk, 3d boundary and 2d junction. When the configuration has a trivial junction, it becomes the half-index that counts boundary local operators, while for the trivial interface, it becomes the full-index that counts bulk local operators. The quarter-index can be defined as the trace over the cohomology of the preserved supercharges

$$\text{IV}(t, x; q) := \text{Tr}_{\text{Op}}(-1)^F q^{J + H + C} t^H C^f.$$  (2.1)

Here $F$ is the Fermion number, $J$ is the generator of the $U(1)J$ rotational symmetry in the space-time on which local operators are supported. $H$ and $C$ stands for the Cartan generators of the $SU(2)_H$ and $SU(2)_C$ R-symmetry groups respectively. $f$ is the Cartan generator of the global symmetry. The choice of fugacity in the index (2.1) is fixed in such a way that the power of $q$ is always strictly positive for a non-trivial local operator, by a unitarity bound. This ensures the convergence of the index. Consequently, the index can be a formal power series in $q$ whose coefficients are Laurent polynomials in the other fugacities.

In this paper we focus on the configurations of 3d $\mathcal{N} = 4$ gauge theories which may couple to 4d $\mathcal{N} = 4$ gauge theories so that the indices (2.1) reduce to the full-indices I of 3d $\mathcal{N} = 4$ gauge theories and/or the half-indices II of 4d $\mathcal{N} = 4$ gauge theories. One may compute the indices for appropriate configurations by a localization procedure. However, we will not pursue that direction in this paper, instead we will count local operators seriously from physical consideration.

In the description of indices we use the following notation by defining $q$-shifted factorial

$$(a; q)_0 := 1, \quad (a; q)_n := \prod_{k=0}^{n-1} (1 - aq^k), \quad (q)_n := \prod_{k=1}^{n} (1 - q^k), \quad n \geq 1,$$

$$(a; q)_\infty := \prod_{k=0}^{\infty} (1 - aq^k), \quad (q)_\infty := \prod_{k=1}^{\infty} (1 - q^k),$$

$$(a^{\pm}; q)_\infty := (a; q)_\infty (a^{-1}; q)_\infty.$$  (2.2)

where $a$ and $q$ are complex variables with $|q| < 1$.

We compute the indices to test the 3d dualities and dualities of the half-BPS boundary conditions/interfaces which are conjectured from string theory [10]. We consider five types of branes in Type IIB string theory whose world-volumes span the following directions:

- D3-branes extended along $x^0 x^1 x^2 x^6$, 


• NS5-branes extended along $x^0 x^1 x^2 x^3 x^4 x^5$,
• D5-branes extended along $x^0 x^1 x^2 x^7 x^8 x^9$,
• NS5$'$-branes extended along $x^0 x^1 x^6 x^7 x^8 x^9$,
• D5$'$-branes extended along $x^0 x^1 x^3 x^4 x^5 x^6$.

In other words, the brane configuration is summarized as

| 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|---|---|
| D3 | o | o | o | - | - | - | - | - | - |
| NS5 | o | o | o | o | o | o | - | - | - |
| D5 | o | o | o | - | - | - | - | - | o |
| NS5$'$ | o | o | - | - | - | o | o | o | o |
| D5$'$ | o | o | - | - | - | o | o | o | - |

(2.3)

The 3d $\mathcal{N} = 4$ gauge theories are realized by considering D3-branes which are finite segments in the $x^6$ direction between NS5-branes and may intersect with D5-branes [7]. The half-BPS boundaries and interfaces in 4d $\mathcal{N} = 4$ gauge theories are realized by considering D3-branes which are (semi-)infinite D3-branes which end on or pass through a sequence of NS5- and D5-branes [8].

Such brane setup is a nice tool for finding mirror pairs of 3d $\mathcal{N} = 4$ gauge theories and dual pairs of half-BPS boundary conditions and interfaces in 4d $\mathcal{N} = 4$ SYM theory by studying the action of S-duality [10].

### 2.1 Indices of 4d $\mathcal{N} = 4$ SYM theory

#### 2.1.1 Full-indices

Four-dimensional $\mathcal{N} = 4$ SYM theory has $SU(4)_R$ R-symmetry. It contains the adjoint scalar fields transforming as 6 under the $SU(4)_R$. Let $X$ and $Y$ be the scalar fields transforming as $(1, 3)$ and $(3, 1)$ under the $SU(2)_C \times SU(2)_H \subset SU(4)_R$. In the brane construction (2.3), the scalar fields $X$ and $Y$ describe the positions of D3-branes along the $(x^7, x^8, x^9)$ directions and $(x^3, x^4, x^5)$ directions respectively. The theory also has the 4d gauginos $\lambda$ transforming as $(2, 2)$ under the $SU(2)_C \times SU(2)_H$.

The local operators in 4d $\mathcal{N} = 4$ gauge theory of gauge group $G$ which contribute to index have charges

\[
\begin{array}{c|cccc}
G & \partial^n X & \partial^n Y & \partial^n \lambda & \partial^n \bar{\lambda} \\
U(1)_j & \text{adj} & \text{adj} & \text{adj} & \text{adj} \\
U(1)_1 & n & n & n + \frac{1}{2} & n + \frac{1}{2} \\
U(1)_C & 0 & 2 & + & + \\
U(1)_H & 2 & 0 & + & + \\
\text{fugacity} & q^{n+\frac{1}{2}t^2} & q^{n+\frac{1}{2}t^2} & -q^{n+1} & -q^{n+1} \\
\end{array}
\]

(2.4)

From (2.4) one can express the index for 4d $\mathcal{N} = 4$ $U(N)$ gauge theory as

\[
\text{I}^{4d \ U(N)}(t; q) = \frac{1}{N!} \frac{(q)^{2N}}{(q^{2t^2}; q)_{\infty}^N (q^{2t^{-2}}; q)_{\infty}^N} \int \prod_{i=1}^N ds_i \prod_{i \neq j} \frac{(q^{s_i}; q)_{\infty} (q^{s_i + s_j}; q)_{\infty}}{(q^{\frac{1}{2}t^2 s_i}; q)_{\infty} (q^{\frac{1}{2}t^{-2} s_i}; q)_{\infty}}.
\]

(2.5)

Here the denominator comes from the scalar fields $X$ and $Y$ while the numerator captures the 4d gauginos. The integration contour for gauge fugacities $s_i$ is taken as a unit torus $T^N$. 

6
2.1.2 Half-indices

Four-dimensional $\mathcal{N} = 4$ SYM theory admits half-BPS boundary conditions which preserve three-dimensional $\mathcal{N} = 4$ supersymmetry with the R-symmetry group $SU(4)_R$ broken down to $SU(2)_C \times SU(2)_H$. In the brane setup $[2,3]$, they arise when parallel D3-branes end on a single fivebrane. There are two types of three-dimensional boundaries/interfaces at $x^2 = 0$ realized by NS5'- and D5'-branes and those at $x^6 = 0$ realized by NS5- and D5-branes.

Let us consider the half-BPS boundary conditions for 4d $\mathcal{N} = 4$ $U(1)$ gauge theory. When the NS5-brane and D5-brane end on a single D3-brane, one finds the Neumann b.c. $\mathcal{N}$ and Dirichlet b.c. $\mathcal{D}$ at $x^6 = 0$ for $U(1)$ gauge theory respectively:

$$
\mathcal{N} : \quad F_{6\mu}\partial = 0, \quad \partial_\mu X|_\partial = 0, \quad \partial_\mu Y|_\partial = 0 \quad \mu, \nu = 0, 1, 2
$$

On the other hand, when the NS5'-brane and D5'-brane end on a single D3-brane one obtains the Neumann b.c. $\mathcal{N}'$ and Dirichlet b.c. $\mathcal{D}'$ at $x^2 = 0$ for $U(1)$ gauge theory respectively:

$$
\mathcal{N}' : \quad F_{2\mu}\partial = 0, \quad \partial_2 X|_\partial = 0, \quad \partial_2 Y|_\partial = 0
$$

$$
\mathcal{D}' : \quad F_{\mu\nu}\partial = 0, \quad \partial_\mu X|_\partial = 0, \quad \partial_2 Y|_\partial = 0 \quad \mu, \nu = 0, 1, 6
$$

The half-indices of the Neumann b.c. $\mathcal{N}$ and Dirichlet b.c. $\mathcal{D}$ for 4d $\mathcal{N} = 4$ $U(1)$ gauge theory takes the form

$$
\Pi_{\mathcal{N}}^{4d} U(1)(t; q) = \Pi_{\mathcal{D}}^{4d} U(1)(t; q) = \frac{(q)_{\infty}}{(q^{\frac{1}{2}}t^{-2}; q)_{\infty}}.
$$

The denominator is associated to the scalar fields $Y$ charged under $U(1)_C$ while the numerator correspond to a half of the 4d gauginos. Likewise, the half-indices of Neumann b.c. $\mathcal{N}'$ and Dirichlet b.c. $\mathcal{D}'$ is

$$
\Pi_{\mathcal{D}}^{4d} U(1)(t; q) = \Pi_{\mathcal{D}}^{4d} U(1)(t; q) = \frac{(q)_{\infty}}{(q^{\frac{1}{2}}t^{2}; q)_{\infty}}.
$$

The denominator captures the scalar fields $X$ charged under $U(1)_H$ whereas the numerator is associated to a half of the 4d gauginos.

The half-BPS boundary conditions corresponding to $N$ D3-branes ending on a single NS5-brane (or NS5') are also Neumann b.c. for the $U(N)$ gauge theory. We can denote them as $\mathcal{N}$ and $\mathcal{N}'$ as in the Abelian case. By contrast, when $N$ multiple D3-branes end on a single D5-brane, one finds a singular boundary condition associated to a regular Nahm pole [30, 8]. A single D5-brane or D5'-brane on which $N$ D3-branes end give rise to the Nahm or Nahm' pole boundary conditions:

$$
\text{Nahm :} \quad F_{\mu\nu}\partial = 0, \quad D_6 \vec{X} + \vec{X} \times \vec{X}|_\partial = 0, \quad D_\mu \vec{Y}|_\partial = 0 \quad \mu, \nu = 0, 1, 6
$$

$$
\text{Nahm' :} \quad F_{\mu\nu}\partial = 0, \quad D_\mu \vec{X}|_\partial = 0, \quad D_2 \vec{Y} + \vec{Y} \times \vec{Y}|_\partial = 0
$$

where we denote the scalar fields by $\vec{X}$ and $\vec{Y}$ as they are the $SU(2)_H$ triplet and the $SU(2)_C$ triplet respectively. The Nahm’s equations for the scalar fields $\vec{X}$ and $\vec{Y}$ have singular solutions

$$
\vec{X}(x^6) = \frac{\vec{t}}{x^6}, \quad \vec{Y}(x^2) = \frac{\vec{t}}{x^2}
$$

where $\vec{t} = (t_1, t_2, t_3)$ is a triplet of elements of the Lie algebra $\mathfrak{g} = \mathfrak{u}(N)$ obeying the commutation relation $[t_1, t_2] = t_3$ and cyclic permutation thereof. The choice of $\vec{t}$ specifies a homomorphism of
Lie algebras $\rho: \mathfrak{su}(2) \to \mathfrak{g}$ which maps the fundamental representation of $U(N)$ to the dimension $N$ irreducible representation of $\mathfrak{su}(2)$. When $N$ multiple D3-branes end on a multiple D5-branes, one finds other Nahm poles, including the Dirichlet b.c. as the trivial Nahm pole corresponding to the case with $N$ D5-branes.

The half-index of Neumann b.c. $N'$ for 4d $\mathcal{N} = 4$ $U(N)$ gauge theory takes the form

$$\Pi_{N'}^{4d} U(N)(t; q) = \frac{1}{N!} (q)^N \left( q^{2t^2}; q \right) \prod_{i=1}^N \frac{d s_i}{2 \pi i s_i} \prod_{i \neq j} \left( \frac{q^{s_i}; q}{q^{s_j}; q} \right)_{\infty}. \tag{2.12}$$

Again the integration contour for gauge fugacities $s_i$ is a unit torus $\mathbb{T}^N$. The half-index of Dirichlet b.c. $D$ for 4d $\mathcal{N} = 4$ $U(N)$ gauge theory is given by

$$\Pi_D^{4d} U(N)(t, z_i; q) = \frac{(q)^N}{(q^{2t^2}; q)_{\infty}} \prod_{i \neq j} \left( \frac{q^{z_i}; q}{q^{z_j}; q} \right)_{\infty}. \tag{2.13}$$

where $z_i$ is the fugacities associated to the boundary $U(N)$ global symmetry. The half-index for Nahm pole boundary conditions in 4d $\mathcal{N} = 4$ $U(N)$ gauge theory is

$$\Pi_{\text{Nahm}}^{4d} U(N)(t; q) = \prod_{k=1}^N \left( \frac{q^{k+1} t^{2(k-1)}; q}{(q^{2t^2}; q)_{\infty}} \right). \tag{2.14}$$

As discussed in [12], the duality between the Neumann b.c. and the regular Nahm pole b.c. implies equality between the half-indices (2.12) and (2.14). In section 6 we discuss more general dual descriptions of the half-indices, including half-index (2.13) for Dirichlet b.c.

We can get similar expressions for the mirror boundary conditions, i.e. $N'$, $D'$ and Nahm' by setting $t \to t^{-1}$.

### 2.2 Indices of 3d $\mathcal{N} = 4$ gauge theory

The 3d $\mathcal{N} = 4$ hypermultiplet consists of a pair of complex scalars $\mathbb{H}, \tilde{\mathbb{H}}$ forming a doublet of $SU(2)_H$ and a pair of complex fermions $\psi_+^H, \psi_-^H$ forming a doublet of $SU(2)_C$. The charges of 3d $\mathcal{N} = 4$ hypermultiplet is

\[
\begin{array}{c|ccccc}
\text{U(1)}_C & \mathbb{H} & \tilde{\mathbb{H}} & \psi_+^H & \psi_-^H & \psi_+^H & \psi_-^H \\
\hline
\text{U(1)}_H & 0 & 0 & - & - & + & + \\
\end{array}
\]

The 3d $\mathcal{N} = 4$ Abelian vector multiplet consists of a 3d $U(1)$ gauge field $A^{3d}_\mu$, three scalars, which we denote by real and complex scalars $\sigma, \phi$ forming the $SU(2)_C$ triplet, and two complex fermions $(\chi^{3d}_\pm, \eta^{3d}_\pm)$. The charges of 3d $\mathcal{N} = 4$ vector multiplet is

\[
\begin{array}{c|cccccccc}
\text{U(1)}_C & A^{3d}_\mu & \sigma & \chi^{3d}_+ & \chi^{3d}_- & \eta^{3d}_+ & \eta^{3d}_- \\
\hline
\text{U(1)}_H & 0 & 0 & 2 & - & - & - & - \\
\end{array}
\]

The 3d $\mathcal{N} = 4$ superalgebra has an outer automorphism that interchanges $SU(2)_C$ and $SU(2)_H$. This automorphism makes the ordinary supermultiplets into twisted supermultiplets. The twisted hyper and vector multiplets can be obtained by exchanging the $U(1)_H$ and $U(1)_C$ charges of the hypermultiplet and vector multiplet respectively.
In three dimensions, a photon is electric magnetic dual to a scalar field, which we call dual photon. The dual photon is periodic when the gauge group is compact and the shift symmetry of the dual photon is a classical topological symmetry whose conserved current is $\ast F$ where $\ast$ is the Hodge star. The conservation of $\ast F$ follows from the Bianchi identity $dF = 0$.

2.2.1 Matter multiplets

The index of a 3d $\mathcal{N} = 2$ chiral multiplet of charge +1 under a $U(1)_f$ flavor symmetry with fugacity $x$ is

$$I_{3d\; CM}(x; q) = \frac{(qx^{-1}; q)_{\infty}}{(x; q)_{\infty}}. \quad (2.17)$$

Its denominator counts complex scalar and its $\partial$ derivatives while its numerator counts fermions and its $\partial$ derivatives included in the 3d $\mathcal{N} = 2$ chiral multiplet.

3d $\mathcal{N} = 4$ hypermultiplet has the following operators which contribute to index:

| $U(1)_f$ | $U(1)_f$ | $U(1)_f$ | $U(1)_f$ |
|-----------|-----------|-----------|-----------|
| $n$       | $n$       | $n + \frac{1}{2}$ | $n + \frac{1}{2}$ |

The index for 3d $\mathcal{N} = 4$ hypermultiplet is

$$I_{3d\; HM}(t, x; q) = \prod_{n=0}^{\infty} \left( \prod_{k=0}^{n} \frac{(q\frac{3}{4}t^{-1}x; q)_{\infty}(q\frac{3}{4}t^{-1}x; q)_{\infty}}{(q^{\frac{3}{4}}tx; q)_{\infty}(q^{\frac{3}{4}}tx^{-1}; q)_{\infty}} \right) \prod_{n=0}^{\infty} \left( \prod_{k=0}^{n} \frac{(q\frac{3}{4}t^{-1}x; q)_{\infty}(q\frac{3}{4}t^{-1}x^{-1}; q)_{\infty}}{(q^{\frac{3}{4}}tx; q)_{\infty}(q^{\frac{3}{4}}tx^{-1}; q)_{\infty}} \right). \quad (2.19)$$

It can be expanded as

$$I_{3d\; HM}(t, x; q) = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{(q\frac{3}{4}t^{-1}x; q)_{\infty}(q\frac{3}{4}t^{-1}x; q)_{\infty}}{(q^{\frac{3}{4}}tx; q)_{\infty}(q^{\frac{3}{4}}tx^{-1}; q)_{\infty}} \frac{\prod_{k=0}^{n} \frac{(q\frac{3}{4}t^{-1}x; q)_{\infty}(q\frac{3}{4}t^{-1}x^{-1}; q)_{\infty}}{(q^{\frac{3}{4}}tx; q)_{\infty}(q^{\frac{3}{4}}tx^{-1}; q)_{\infty}}}{(q^{\frac{3}{4}}tx; q)_{\infty}(q^{\frac{3}{4}}tx^{-1}; q)_{\infty}}. \quad (2.20)$$

The free 3d $\mathcal{N} = 4$ hypermultiplet has no Coulomb branch local operators surviving in the H-twist. Therefore in the H-twist limit $t \to q^{\frac{3}{4}}$, the indices (2.19) and (2.20) become trivial

$$I_{3d\; HM}(t = q^{\frac{3}{4}}, x; q) = 1. \quad (2.21)$$

On the other hand, in the C-twist limit $t \to q^{-\frac{3}{4}}$, the indices reduce to

$$I_{3d\; HM}(t = q^{-\frac{3}{4}}, x; q) = \frac{1}{(1 - x)(1 - x^{-1})}. \quad (2.22)$$

This counts two bosonic generators in the algebra of Higgs branch local operators.

The operators in 3d $\mathcal{N} = 4$ twisted hypermultiplet which contribute to index are

| $U(1)_f$ | $U(1)_f$ | $U(1)_f$ | $U(1)_f$ |
|-----------|-----------|-----------|-----------|
| $n + \frac{1}{2}$ | $n + \frac{1}{2}$ | $0$ | $0$ |

The operators in 3d $\mathcal{N} = 4$ twisted hypermultiplet which contribute to index are
The index for 3d $\mathcal{N} = 4$ twisted hypermultiplet can be obtained from the index (2.19) by setting $t \rightarrow t^{-1}$

$$\mathbb{I}^{3d \text{ tHM}}(t, x; q) = \frac{(q^{\frac{1}{2}} t x; q)_{\infty} (q^{\frac{1}{2}} t x^{-1}; q)_{\infty}}{(q^{\frac{1}{2}} t^{-1} x; q)_{\infty} (q^{\frac{1}{2}} t^{-1} x^{-1}; q)_{\infty}} = \mathbb{I}^{3d \text{ CM}}(q^{\frac{1}{2}} t^{-1} x) \cdot \mathbb{I}^{3d \text{ CM}}(q^{\frac{1}{2}} t^{-1} x^{-1}).$$

Again it has an expansion

$$\mathbb{I}^{3d \text{ tHM}}(t, x; q) = \sum_{n=0}^{\infty} \sum_{k=0}^{n} (q^{1/2} t^2 q_{\infty}) (q^{1/2} t^2 q_{\infty}) \cdot \frac{q^{k} (q)_{n-k}}{(q)_{k}(q)_{n-k}} x^{n-2k} q^{\frac{3}{4}} t^{-n}. \quad (2.25)$$

### 2.2.2 Gauge multiplets

While in four-dimensional case the index only involves integration over the gauge group [31, 32], in three-dimensional case the index would have non-perturbative contributions of monopole operators and contain the sum over the magnetic fluxes of monopole operators for all backgrounds [27, 20].

Let us firstly consider the perturbative contributions to the index. The charges of operators in 3d $\mathcal{N} = 4$ vector multiplet contributing to the index are

$$\begin{array}{c|cc|cc}
 & D^{n} (\sigma + iq) & D^{n} \varphi & D^{n} \lambda_{-}^{3d} & D^{n} \eta_{-}^{3d} \\
\hline
G & \text{adj} & \text{adj} & \text{adj} & \text{adj} \\
U(1)_{J} & n & n & n + \frac{1}{2} & n + \frac{1}{2} \\
U(1)_{C} & 0 & 2 & - & - \\
U(1)_{H} & 0 & 0 & - & + \\
fugacity & q^{n} s^{\alpha} & q^{n+1/2} t^{-2} s^{\alpha} & -q^{n} s^{\alpha} & -q^{n+1/2} t^{2} s^{\alpha} \\
\end{array}$$

The perturbative index contributed from the local operators in (2.26) of 3d $\mathcal{N} = 4$ $U(1)$ vector multiplet takes the form

$$\mathbb{I}^{3d \text{ pert}} U(1)(t; q) = \frac{(q^{1/2} t^2 q_{\infty})}{(q^{1/2} t^{-2} q_{\infty})} \int ds \frac{d}{2\pi i s} \quad (2.27)$$

where the integration contour of gauge fugacity $s$ is a unit circle. Similarly, the perturbative index for 3d $\mathcal{N} = 4$ $U(N)$ vector multiplet takes the form

$$\mathbb{I}^{3d \text{ pert}} U(N)(t; q) = \frac{1}{N!} \frac{(q^{1/2} t^2 q_{\infty})^{N}}{(q^{1/2} t^{-2} q_{\infty})^{N}} \int ds_{1} \frac{d}{2\pi i s_{1}} \prod_{i=1}^{N} 1 - s_{i} \prod_{i \neq j} \left( 1 - \frac{s_{i}}{s_{j}} \right) \frac{(q^{1/2} t^{2} \bar{s}_{i})}{(q^{1/2} t^{-2} \bar{s}_{i})} \quad (2.28)$$

where the integration contour of gauge fugacities $s_{i}$ is a unit torus $\mathbb{T}^{N}$.

Likewise, charges of operators in 3d $\mathcal{N} = 4$ twisted vector multiplet are

$$\begin{array}{c|cc|cc}
 & D^{n} (\bar{\sigma} + i\bar{\rho}) & D^{n} \bar{\varphi} & D^{n} \lambda_{-}^{3d} & D^{n} \eta_{-}^{3d} \\
\hline
G & \text{adj} & \text{adj} & \text{adj} & \text{adj} \\
U(1)_{J} & n & n & n + \frac{1}{2} & n + \frac{1}{2} \\
U(1)_{C} & 0 & 0 & + & + \\
U(1)_{H} & 0 & 2 & + & - \\
fugacity & q^{n} s_{\alpha} & q^{n+1/2} t^{2} s_{\alpha} & -q^{n} s_{\alpha} & -q^{n+1/2} t^{-2} s_{\alpha} \\
\end{array}$$

We can obtain the index for 3d $\mathcal{N} = 4$ twisted vector multiplet by setting $t \rightarrow t^{-1}$ for the index of 3d $\mathcal{N} = 4$ vector multiplet.
As 3d $\mathcal{N} = 4$ $U(1)$ gauge theory appears from 4d $\mathcal{N} = 4$ $U(1)$ gauge theory on a segment with Neumann b.c. $\mathcal{N}$ at each end, we have a schematic relation
\[
\mathbb{F}_{3d \ \text{pert}}^U(1) = \frac{\Pi_{\mathcal{N}}^{4d \ U(1)} \times \Pi_{\mathcal{N}'}^{4d \ U(1)}}{\Pi_{\mathcal{D}}^{4d \ U(1)}}.
\]
When a 4d $\mathcal{N} = 4$ $U(1)$ gauge theory is put on a slab with Neumann b.c. $\mathcal{N}$ and Dirichlet b.c. $\mathcal{D}$ at each end, we have
\[
1 = \frac{\Pi_{\mathcal{N}}^{4d \ U(1)} \times \Pi_{\mathcal{D}}^{4d \ U(1)}}{\Pi_{\mathcal{U}}^{4d \ U(1)}}.
\]
This indicates that the resulting 3d theory is a trivial theory. For a 4d $\mathcal{N} = 4$ $U(1)$ gauge theory on a segment obeying Dirichlet b.c. at both ends, we find that
\[
\mathbb{F}_{3d \ \text{pert}}^U(1) = \frac{\Pi_{\mathcal{N}}^{4d \ U(1)} \times \Pi_{\mathcal{N}'}^{4d \ U(1)}}{\Pi_{\mathcal{U}}^{4d \ U(1)}}.
\]
This reflects the fact that the resulting 3d theory is mirror to 3d $\mathcal{N} = 4$ twisted $\widetilde{U}(1)$ gauge theory appearing from a 4d theory on a segment with Neumann b.c. $\mathcal{N}'$ at both ends.

Now consider the non-perturbative contributions to indices from monopole operators. Classically monopole operators are charged under the topological symmetry. In addition, quantum mechanically they can acquire non-trivial quantum numbers. Let us consider the canonical $U(1)$ R-charge as $\frac{1}{2}(C - H)$ so that the complex scalar $\mathbb{H}$ in the hypermultiplet carries charge $-1/2$ and the adjoint complex scalar $\varphi$ in the vector multiplet have charge $+1$. Then the R-charge of a BPS bare monopole operator of magnetic charge $m$ in the IR CFT is given by
\[
\Delta(m) = \frac{1}{2} \sum_{i=1}^{N_f} \sum_{\lambda_i \in \mathbb{R}_i} |\lambda_i(m)| - \sum_{\alpha \in \Delta_+} |\alpha(m)|.
\]
This formula was firstly proposed in \cite{10} and later verified in \cite{33, 34}. The first term in (2.33) is the contribution from $N_f$ hypermultiplets labeled by $i = 1, \ldots, N_f$ transforming as representations $\{R_i\}_{i=1, \ldots, N_f}$ under the gauge group. The sum is taken over the weights $\lambda_i$ of $R_i$. The second term is the contribution from vector multiplet. The sum is taken over the positive roots $\alpha \in \Delta_+$.

The R-charge (2.33) of bare monopole can specify its energy, or equivalently conformal dimension since the bare monopole is a BPS state. If all BPS monopole operators carry $\Delta(m) > \frac{1}{2}$, the theory is called good. If all BPS monopole operators have $\Delta(m) \geq \frac{1}{2}$ and some saturate the unitarity bound $\Delta(m) = \frac{1}{2}$, the theory is called ugly. Otherwise, the theory is called bad. In this paper, we focus on the good or ugly theories in which $\Delta(m)$ is identified with the conformal dimension and the quantum numbers $J + \frac{H + C}{4}$ of a bare monopole operator with magnetic charge $m$ is fixed to $\Delta(m) = \frac{1}{2}$ in the IR SCFT.

Taking into account the above, we are led to propose the full-index of 3d $\mathcal{N} = 4$ gauge theory with gauge group $G$ and $N_f$ hypermultiplets which takes the form
\[
\mathbb{I}^{3d \ G}(t, x_1, z_i; q) = \frac{1}{|\text{Weyl}(G)|} \left( \frac{q^{\frac{1}{2}} t^2; q \infty}{|q^{\frac{1}{2}} t^{-2}; q \infty|} \sum_{m \in \text{cochar}(G)} \prod_{\alpha \in \text{roots}(G)} ds 2\pi i s \left( 1 - q^{-1 - m \cdot \alpha} s^\alpha \right) \left( q^{\frac{1}{2} - m \cdot \alpha} t^{2s^\alpha}; q \infty \right) \right.
\]
\[
\times \prod_{i=1}^{N_f} \prod_{\lambda_i \in \mathbb{R}_i} \left( q^{\frac{1}{2} + \frac{m \cdot \lambda_i}{2}} t^{-1 - s^\lambda_i x_i^\pm}; q \infty \right)^{\Delta(m)} \left( q^{\frac{1}{2} + \frac{m \cdot \lambda_i}{2}} t s^\lambda_i x_i^\pm; q \infty \right)^{\Delta(m)} \times q^{\frac{\Delta(m)}{2}} t^{-2\Delta(m)} z^m.
\]
Here the second line is the contribution from vector multiplet of gauge group $G$. The third line is the contribution from $N_f$ hypermultiplets transforming as representation $\{R_i\}_{i=1,\ldots,N_f}$ of gauge group $G$. The fugacities $x$ are associated to the flavor symmetry that rotates $N_f$ hypermultiplets. For non-zero magnetic flux $m$, the expression is shifted from the index (2.19) for 3d hypermultiplet. This reflects the fact that in the presence of magnetic flux, the electrically charged states get an effective quantum numbers. The third line is the contribution from bare monopole operators. The fugacities $z$ are associated to the topological symmetry.

In order to check mirror symmetry and dual boundary conditions, one also needs to evaluate the mirror version of full-index for 3d $\mathcal{N} = 4$ gauge theory. The mirror index of 3d $\mathcal{N} = 4$ gauge theory consisting of twisted vector multiplet of gauge group $G$ and $\tilde{N}_f$ twisted hypermultiplets transforming as representation $\{\tilde{R}_i\}_{i=1,\ldots,\tilde{N}_f}$ is given by

$$\prod_{i=1}^{\tilde{N}_f} G(t, x_i, z_i; q) = \frac{1}{|\text{Weyl}(G)|} \frac{(q^{\frac{1}{2}} t^{-2}; q)_{\infty}^{\text{rank}(G)}}{(q^{\frac{1}{2}} t; q)_{\infty}^{\text{rank}(G)}} \sum_{m \in \text{cochar}(\tilde{G})} \oint \prod_{s \in \text{roots}(G)} ds \frac{1 - q^{\frac{1}{2} m s \alpha}}{1 - q^{\frac{1}{2} m s \alpha} \bigl(1 - q^{\frac{1}{2} m s \alpha} t^{-2} s^\alpha; q\bigr)_{\infty}} \times \prod_{i=1}^{\tilde{N}_f} \frac{1}{\prod_{i=1}^{\tilde{N}_f} t^{-1} s^{\pm \lambda_i} z_i^{\pm}; q_{\infty}} \times q^{\frac{1}{2} (m \cdot s)} t^{2 \Delta(m)} x^m,$$

(3.5)

where the fugacities $z_i$ and $x_i$ are associated to the flavor symmetry and the topological symmetry in the twisted 3d $\mathcal{N} = 4$ theories as they are exchanged under mirror symmetry. In the following sections, we show various identities between indices (2.34) and (2.35) for mirror pairs.

### 3 Abelian mirror symmetry

In this section, we consider the 3d $\mathcal{N} = 4$ Abelian gauge theory and its mirror. The mirror of 3d $\mathcal{N} = 4$ $U(1)$ gauge theory with $N_f$ hypermultiplets is a twisted $U(1)^{N_f-1}$ quiver gauge theory with $N_f$ twisted hypermultiplets [35]. The quiver diagram and the corresponding brane configuration are depicted in Figure 1. We check that two indices for mirror pairs coincide with each other.

#### 3.1 SQED$_1$

We begin with a 3d $\mathcal{N} = 4$ Abelian gauge theory with a single charged hypermultiplet, which we call SQED$_1$. The 3d $\mathcal{N} = 4$ SQED$_1$ is mirror to 3d $\mathcal{N} = 4$ twisted hypermultiplet [35]. The flavor symmetry of the twisted hypermultiplet is mapped to the topological symmetry in SQED$_1$. This is the simplest 3d $\mathcal{N} = 4$ Abelian mirror symmetry and index computation has been already computed in [26]. We further extract a Higgsing interpretation of the indices by picking up residues at poles in the indices.

The full index of SQED$_1$ is given by

$$\prod_{i=1}^{\tilde{N}_f} G(t, x; q) = \frac{(q^{\frac{1}{2}} t^2; q)_{\infty}}{(q^{\frac{1}{2}} t^{-2}; q)_{\infty}} \sum_{m \in \mathbb{Z}} \oint \frac{ds}{2 \pi i s} \frac{(q^{\frac{1}{2}} + \frac{1}{2} m t^{-1} s; q)_{\infty} (q^{\frac{1}{2}} + \frac{1}{2} m t^{-1} s^{-1}; q)_{\infty} q^{\frac{1}{2} m} t^{-|m|} x^m}{(q^{\frac{1}{2}} + \frac{1}{2} m t s; q)_{\infty} (q^{\frac{1}{2}} + \frac{1}{2} m t s^{-1}; q)_{\infty}} q^{\frac{1}{2} m}$$

(3.1)
Figure 1: (a) The quiver diagrams of SQED$_{N_f}$ and its mirror $U(1)^{N_f-1}$ quiver gauge theory. (b) The brane configurations of SQED$_{N_f}$ and its mirror $U(1)^{N_f-1}$ quiver gauge theory.

where $m$ is the magnetic monopole charge and the fugacity $x$ is associated to a $U(1)_t$ topological symmetry. The factors $q^{|m|} t^{-|m|} x^m$ are associated to contributions from bare monopole operators with the R-charge $\Delta(m) = \frac{|m|}{2}$. The remaining factors in the integrand involve shifts of quantum numbers due to the background magnetic flux. The index for SQED$_1$ already appeared in [28, 26] and the index (3.1) coincides with the index (2.24) of 3d $\mathcal{N} = 4$ twisted hypermultiplet.

As discussed in [12], poles in the integrand and their residues may have a physical interpretation. When FI-like parameter turned on, elementary fields with gauge charges get non-trivial vevs, which leads to a Higgsing the gauge group. Consequently, the index is written as a sum over residues which are associated to the index of the Higgsed theory. In the brane setup, this corresponds to a certain deformation of the brane configuration. In order to extract a Higgsing interpretation by expanding the index (3.1), it is enough to consider the perturbative SQED$_1$ index corresponding to $m = 0$ that takes the form

$$
\mathcal{I}^{\text{3d pert SQED}_1}(t; q) = \left( \frac{q^{\frac{1}{2}} t^2; q}{q^\frac{1}{2} t^{-2}; q} \right)_\infty \frac{d s}{2 \pi i s} \left( \frac{q^{\frac{3}{2}} t^{-s}; q}{q^\frac{3}{2} t^{-1-s}; q} \right)_\infty \left( \frac{q^s t s; q}{q^s t s^{-1}; q} \right)_\infty .
$$

(3.2)

This can be evaluated by considering the residues at poles of charged hypermultiplet $s = q^{\frac{1}{2} + m} t$ as

$$
\mathcal{I}^{\text{3d pert SQED}_1}(t; q) = \left( \frac{q^{\frac{1}{2}} t^2; q}{q^\frac{1}{2}} \right)_\infty \sum_{m=0}^{\infty} \left( \frac{q^{1+m}; q}{q^2 m t^2; q} \right)^2 q^{-2 m} .
$$

(3.3)

As the residue sum begins with 1, the Higgsed theory is a trivial theory. This would imply that the expansion (3.2) is associated to a Higgsing process which splits a D3-brane along the D5-brane and separates one of the NS5-brane in the $x^{7,8,9}$ directions (see Figure 2).
Correspondingly, we get the mirror index by selecting out the zero charged sector from the index (2.24) of 3d $\mathcal{N} = 4$ twisted hypermultiplet

$$\int \frac{dx}{2\pi ix} \mathbb{I}^{3d \ \text{thHM}}(t, x; q) = \int \frac{dx}{2\pi ix} \left( \frac{(q^{\frac{1}{2}}tx; q)^\infty}{(q^{\frac{1}{2}}tx^{-1}; q)^\infty} \right). \quad (3.4)$$

The indices (3.3) and (3.4) can be shown to be equal. To see the equivalence, we can firstly calculate the index (3.4) by taking the sum over residues at poles of bi-fundamental twisted hypermultiplet $x = q^{\frac{1}{2}+m}t^{-1}$ as

$$\int \frac{dx}{2\pi ix} \mathbb{I}^{3d \ \text{thHM}}(t, x; q) = \frac{(q^{\frac{1}{2}}t^2; q)^\infty}{(q^{\frac{1}{2}}t^{-2}; q)^\infty} \sum_{m=0}^{\infty} \frac{(q^{\frac{1}{2}}t^{-2}; q)^\infty(q^{\frac{1}{2}}t^2; q)^\infty}{(q^{\frac{1}{2}+m}t^{-2}; q)^2} q^{m} t^{2m}. \quad (3.5)$$

On the other hand, we can expand the index (3.2) of SQED$_1$ in terms of the expansion (2.20) of the hypermultiplet index as

$$\mathbb{I}^{3d \ \text{pert SQED}_1}(t; q) = \frac{(q^{\frac{1}{4}}t^2; q)^\infty}{(q^{\frac{1}{4}}t^{-2}; q)^\infty} \sum_{n=0}^{\infty} \sum_{k=0}^{n} \frac{(q^{\frac{1}{4}}t^{-2}; q)^\infty(q^{\frac{1}{4}}t^2; q)^\infty}{(q)^k(q^k)^{n-k}} s^{n-2k} q^{2} t^{n}. \quad (3.6)$$

This agrees with the expression (3.5). Also we observe that the expansion in the sum (3.6) starts from the perturbative index $\mathbb{I}^{3d \ \text{pert U}(1)}$ of 3d $\mathcal{N} = 4$ U(1) vector multiplet. This would be associated to the Higgsing process of separating the D5-brane from the stretched D3-brane between the NS5-branes (see Figure 2).

Introducing a Wilson line operator $W_n$ of charge $n$, we obtain the index

$$\mathbb{I}^{3d \ \text{SOED}_1}_{W_n}(t, x; q) = \frac{(q^{\frac{1}{4}}t^2; q)^\infty}{(q^{\frac{1}{4}}t^{-2}; q)^\infty} \sum_{m\in\mathbb{Z}} \int \frac{ds}{2\pi is} \frac{(q^{\frac{1}{4}+\frac{|m|}{2}}t^{-1}s; q)^\infty(q^{\frac{1}{4}+\frac{|m|}{2}}t^{-1}s^{-1}; q)^\infty}{(q^{\frac{1}{4}+\frac{|m|}{2}}ts; q)^\infty(q^{\frac{1}{4}+\frac{|m|}{2}}ts^{-1}; q)^\infty} s^{n} q^{\frac{|m|}{2}} t^{-|m|} x^{m}. \quad (3.7)$$

This agrees with

$$\mathbb{I}^{3d \ \text{thHM}}_{W_n}(t, x; q) = \frac{(q^{\frac{1}{4}+\frac{|m|}{2}}tx; q)^\infty(q^{\frac{1}{4}+\frac{|m|}{2}}tx^{-1}; q)^\infty}{(q^{\frac{1}{4}+\frac{|m|}{2}}tx; q)^\infty(q^{\frac{1}{4}+\frac{|m|}{2}}t^{-1}x^{-1}; q)^\infty} q^{\frac{|m|}{2}} t^{n}. \quad (3.8)$$

We see that the index (3.8) has the contributions from bare monopole operator associated with flavor symmetry and the quantum numbers of twisted hypermultiplets are affected by the flux.

### 3.2 $T[SU(2)]$

Next example is a 3d $\mathcal{N} = 4$ Abelian gauge theory with two charged hypermultiplets, which we call $T[SU(2)]$. The dimension of the Higgs branch is $\text{dim}_C \mathcal{M}_H^{T[SU(2)]} = 2 \cdot (1 \times 2 - 1^2) = 2$ and the dimension of the Coulomb branch is $\text{dim}_C \mathcal{M}_C^{T[SU(2)]} = 2 \cdot 1 = 2$. This is self-mirror with two global symmetries, a $SU(2)$ flavor symmetry on $\mathcal{M}_H$ and enhanced $SU(2)_t$ topological symmetry on $\mathcal{M}_C$ which are exchanged under mirror symmetry.
The Higgsing procedure of $3d \, \mathcal{N} = 4$ SQED$_1$ splitting a D3-brane along the D5-brane and separating one of the NS5-brane in the $x^{7,8,9}$ directions. (b) The Higgsing procedure of $3d \, \mathcal{N} = 4$ SQED$_1$ separating the D5-brane from the stretched D3-brane between the NS5-branes.

The index of $T[SU(2)]$ reads

\[
\begin{align*}
\Pi^{T[SU(2)]}(t, x_\alpha, z_\alpha; q) &= \frac{(q^{1/2} t^2; q)_\infty}{(q^{1/2} t^{-2}; q)_\infty} \sum_{m \in \mathbb{Z}} \oint ds \frac{2\pi i s}{(q^{1/2} + im t^{-1/2} s^{1/2} x_1^+; q)_\infty} \cdot \frac{(q^{3/2} + im t^{-1} s^{1/2} x_2^+; q)_\infty}{(q^{1/2} + im t^{1/2} s^{1/2} x_2^+; q)_\infty} \\
&\times q^{\frac{4m}{2} t^{-2} |m|} x_1^m x_2^{-m}
\end{align*}
\]

where the fugacities $x_\alpha$ are associated to the $SU(2)$ flavor symmetry with $x_1 x_2 = 1$ and $z_\alpha$ are the fugacities for the topological symmetry.

The index (3.9) of $T[SU(2)]$ coincides with the index of $\widetilde{T[SU(2)]}$ which is obtained from $T[SU(2)]$ by replacing their supermultiplets with the twisted counterparts:

\[
\begin{align*}
\Pi^{T[SU(2)]}(t, x_\alpha, z_\alpha; q) &= \frac{(q^{1/2} t^2; q)_\infty}{(q^{1/2} t^{-2}; q)_\infty} \sum_{m \in \mathbb{Z}} \oint ds \frac{2\pi i s}{(q^{1/2} + im t^{1/2} s^{1/2} z_1^+; q)_\infty} \cdot \frac{(q^{3/2} + im t^{-1} s^{1/2} z_2^+; q)_\infty}{(q^{3/2} + im t^{-1} s^{1/2} z_2^+; q)_\infty} \\
&\times q^{\frac{4m}{2} t^{-2} |m|} x_1^m x_2^{-m}
\end{align*}
\]

Here the role of fugacities $x_\alpha$ and $z_\alpha$ are exchanged.
We can draw a lesson from the expansion of the index. To get a Higgsing procedure, we simply look at the sector with zero flavor charge of the perturbative index of \( T[SU(2)] \)

\[
\oint \frac{dx_1}{2\pi i x_1} \frac{dx_2}{2\pi i x_2} \mathbb{P}^{3d \text{ pert}} T[SU(2)](t;q) = \frac{(q^{\frac{3}{2} t^2}; q)_\infty}{(q^{t^{-2}}; q)_\infty} \int \frac{ds_1}{2\pi i s_1} \int \frac{dx_1}{2\pi i x_1} \int \frac{dx_2}{2\pi i x_2} \nonumber
\]

\[
\times \frac{(q^{\frac{3}{2} t^{-1} s^+ x^+_1}; q)_\infty}{(q^{\frac{3}{2} ts^+ x^+_1}; q)_\infty} \frac{(q^{\frac{3}{2} t^{-1} s^+ x^+_2}; q)_\infty}{(q^{\frac{3}{2} ts^+ x^+_2}; q)_\infty}. \tag{3.11}
\]

This can be evaluated as

\[
\oint \frac{dx_1}{2\pi i x_1} \frac{dx_2}{2\pi i x_2} \mathbb{P}^{3d \text{ pert}} T[SU(2)](t;q) = \frac{(q^{\frac{3}{2} t^2}; q)_\infty}{(q^{t^{-2}}; q)_\infty} \sum_{n,m=0}^{\infty} \frac{(q^{1+n}; q)_\infty^2 (q^{1+m}; q)_\infty^2}{(q^{\frac{3}{2} + iT^{-2}}; q)_\infty^2 (q^{\frac{3}{2} + MT^{-2}}; q)_\infty^2} q^{n+m} t^{2(n+m)}. \tag{3.12}
\]

Again the expansion \([3.12]\) has a Higgsing interpretation. As its first term is identified with the index \( \mathbb{P}^{3d \text{ pert}} U(1) \) of 3d \( N = 4 \) \( U(1) \) vector multiplet. It is associated to a Higgsing manipulation of separation of two D5-branes from a suspended D3-brane between two NS5-branes.

### 3.3 SQED\(_{3}\)

Consider the Abelian gauge theory with three charged hypermultiplets, which we denote by SQED\(_{3}\). It is a good but unbalanced quiver. The dimension of the Higgs branch is \( \dim_c \mathcal{M}^{\text{SQED}_3}_H = 2 \cdot (1 \times 3 - 1^2) = 4 \) while the dimension of the Coulomb branch is \( \dim_c \mathcal{M}^{\text{SQED}_3}_C = 2 \cdot 1 = 2 \).

The index of SQED\(_{3}\) takes the form

\[
\mathbb{P}^{3d \text{ SQED}_3}(t,x_\alpha,z_\alpha;q) = \frac{(q^{\frac{3}{2} t^2}; q)_\infty}{(q^{t^{-2}}; q)_\infty} \sum_{m \in \mathbb{Z}} \int \frac{ds}{2\pi i s} \prod_{n=1}^{3} \frac{(q^{\frac{1}{2} + \frac{1}{3} m}; t^{-3} s^+ x^+_\alpha}; q)_\infty}{(q^{\frac{1}{2} + \frac{1}{3} m}; t^{-3} s^+ x^+_\alpha}; q)_\infty} q^{\frac{3|m|}{2} t^{-3} |m| z^m_{1,2} 2} \tag{3.13}
\]

where the fugacities \( x_\alpha \) are associated to the \( SU(3) \) flavor symmetry with \( \prod_{i=1}^{3} x_\alpha = 1 \) and \( z_\alpha \) are the fugacities for the topological symmetry.

The mirror of SQED\(_{3}\) is a \( U(1)^2 \) quiver gauge theory consisting of two twisted vector multiplets of \( U(1) \) gauge groups and three twisted hypermultiplets which carry charges \((+,0),(+,+)\) under the \( U(1) \times U(1) \) gauge group. We denote this theory by \([1] - (1)^2 - [1]\). Here and in the following the round brackets \((N)\) represent \( U(N) \) gauge symmetry and the square brackets \([M]\) denote \( U(M) \) global symmetry. The dash lines are bi-fundamental (twisted)hypermultiplets. The tildes indicate that the corresponding gauge theories are constructed by twisted supermultiplets. The dimension of the Higgs branch is \( \dim_c \mathcal{M}^{[1] - (1)^2 - [1]}_H = 2 \cdot (1 + 1 - 1^2) = 2 \) while the dimension of the Coulomb branch is \( \dim_c \mathcal{M}^{[1] - (1)^2 - [1]}_C = 2 \cdot (1 + 1) = 4 \).
The index $[3.13]$ of SQED$_3$ agrees with the index of the mirror $\hat{U}(1)^2$ quiver gauge theory
\[
\Gamma^{3d} [1] - (1)^2 - [1](t, x_\alpha, z_\alpha; q)
\]
\[
= \frac{(q^{\frac{1}{2}}t^{\frac{1}{2}}; q)_\infty^2}{(q^{\frac{1}{2}}t^{\frac{1}{2}}; q)_\infty^2} \sum_{m_1, m_2 \in \mathbb{Z}} \oint_{2\pi i s_1} \oint_{2\pi i s_2} \oint_{2\pi i s_3} \prod_{\alpha=1}^4 \frac{\left(q^{\frac{3}{4} + \frac{|m_\alpha|}{2}} t^{\frac{1}{4}} s_{\alpha}^{\mp} x_{\alpha}^{\mp}; q \right)_\infty}{\left(q^{\frac{3}{4} + \frac{|m_\alpha|}{2}} t^{\frac{1}{4}} s_{\alpha}^{\mp} x_{\alpha}^{\mp}; q \right)_\infty} q^{4|m| - |m_\alpha|} z_1 z_2 m \tag{3.14}
\]
Here the fugacities $x_\alpha$ are associated to the topological symmetry while the fugacities $z_\alpha$ are associated to the flavor symmetry.

### 3.4 SQED$_4$

The last example is the Abelian gauge theory with four charged hypers, SQED$_4$.

The index of SQED$_4$ is
\[
\Gamma^{3d} \text{SQED}_4(t, x_\alpha, z_\alpha; q)
\]
\[
= \frac{(q^{\frac{1}{2}}t^{\frac{1}{2}}; q)_\infty^4}{(q^{\frac{1}{2}}t^{\frac{1}{2}}; q)_\infty^4} \sum_{m_1, m_2, m_3, m_4 \in \mathbb{Z}} \oint_{2\pi i s_1} \oint_{2\pi i s_2} \oint_{2\pi i s_3} \oint_{2\pi i s_4} \prod_{\alpha=1}^4 \frac{\left(q^{\frac{3}{4} + \frac{|m_\alpha|}{2}} t^{\frac{1}{4}} s_{\alpha}^{\mp} x_{\alpha}^{\mp}; q \right)_\infty}{\left(q^{\frac{3}{4} + \frac{|m_\alpha|}{2}} t^{\frac{1}{4}} s_{\alpha}^{\mp} x_{\alpha}^{\mp}; q \right)_\infty} q^{4|m| - 4|m_\alpha|} z_1 z_2 m \tag{3.15}
\]
where $x_\alpha$ are the fugacities for the $SU(4)$ flavor symmetry with $\prod_{\alpha=1}^4 x_\alpha = 1$ and $z_\alpha$ are the fugacities for the topological symmetry.

The index $[3.15]$ of SQED$_4$ coincides with the index of its mirror $\hat{U}(1)^3$ quiver gauge theory
\[
\Gamma^{3d} [1] - (1)^3 - [1](t, x_\alpha, z_\alpha; q)
\]
\[
= \frac{(q^{\frac{1}{2}}t^{\frac{1}{2}}; q)_\infty^3}{(q^{\frac{1}{2}}t^{\frac{1}{2}}; q)_\infty^3} \sum_{m_1, m_2, m_3, m_4 \in \mathbb{Z}} \oint_{2\pi i s_1} \oint_{2\pi i s_2} \oint_{2\pi i s_3} \oint_{2\pi i s_4} \prod_{\alpha=1}^4 \frac{\left(q^{\frac{3}{4} + \frac{|m_\alpha|}{2}} t^{\frac{1}{4}} s_{\alpha}^{\mp} x_{\alpha}^{\mp}; q \right)_\infty}{\left(q^{\frac{3}{4} + \frac{|m_\alpha|}{2}} t^{\frac{1}{4}} s_{\alpha}^{\mp} x_{\alpha}^{\mp}; q \right)_\infty} q^{4|m| - 4|m_\alpha|} z_1 z_2 m \tag{3.16}
\]
where $x_\alpha$ are identified with the fugacities for the topological symmetry while the fugacities $z_\alpha$ are associated to the flavor symmetry.

### 3.5 SQED$_{N_f}$

Now we would like to discuss the generalization of full-indices for 3d $\mathcal{N} = 4$ Abelian gauge theories. Consider a 3d $\mathcal{N} = 4$ Abelian gauge theory with $N_f$ charged hypermultiplets, which we call SQED$_{N_f}$. 

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For SQED$_{N_f}$, the R-charge of a bare monopole is $\Delta(m) = \frac{N_f|m|}{2}$ and the index takes the form

$$\text{I}^{3d \text{ SQED}_{N_f}}(t, x, \alpha; q) = \frac{(q \frac{1}{2} t^2; q)_{\infty}}{(q \frac{1}{2} t^{-2}; q)_{\infty}} \sum_{m \in \mathbb{Z}} \frac{ds}{2\pi i s} \prod_{\alpha=1}^{N_f} (q^{\frac{1}{2} + \frac{1}{2} \alpha} t^{-1} s^\alpha x_\alpha; q)_{\infty} q^{\frac{N_f|m|}{2} t - N_f|m| z_1 z_2 - m}$$  \hspace{1cm} (3.17)

where $x_\alpha$ are the fugacities for the SU($N_f$) flavor symmetry with $\prod_{\alpha=1}^{N_f} x_\alpha = 1$ and $z_\alpha$ are the fugacities for the topological symmetry.

As discussed in [26], one can obtain the Hilbert series for branches of vacua in the special fugacity limit of the indices. We briefly check this in our notation as follows. The Coulomb branch of SQED$_{N_f}$ for SQED$_{N_f}$ is the $A_{N_f-1}$ singularity $\mathbb{C}^2/\mathbb{Z}_{N_f}$ [1]. Taking the Coulomb limit $q \rightarrow 0$ while keeping $\overline{t} z = q^{\frac{1}{2}} t^{-1}$ constant, and setting $z = z_1 z_2^{-1}$ the index (3.17) reduces to

$$\text{I}^{3d \text{ SQED}_{N_f}}(\overline{t}, z) = \frac{1}{(1 - \overline{t})} \sum_{m \in \mathbb{Z}} \frac{\overline{t}^{N_f|m|} z^m}{1 - \overline{t}^{N_f|f|} z^{2}}.$$

This is identified with the refined Hilbert series of the Coulomb branch of SQED$_{N_f}$ [2]. The factors $1/(1 - \overline{t})$, $1/(1 - \overline{t}^{N_f} z)$ and $1/(1 - \overline{t}^{N_f} z^{-1})$ would correspond to the scalar field $\varphi$, the monopole operator $V_+$ of magnetic flux +1 and the monopole operator $V_-$ of magnetic flux -1 respectively. They obey the relation $V_+ V_- = \varphi^{N_f}$ at dimension $N_f$ and topological charge 0 [36] which is encoded by the numerator.

In fact, by setting $z = 1$ in (3.18), we get the Hilbert series of the $A_{N_f-1}$ singularity $\mathbb{C}^2/\mathbb{Z}_{N_f}$ [37]

$$\text{I}^{3d \text{ SQED}_{N_f}}(\overline{t}, z = 1) = \frac{1 - \overline{t}^{N_f}}{(1 - \overline{t})(1 - \overline{t}^{N_f})} = \text{Hilb}[\mathbb{C}^2/\mathbb{Z}_{N_f}].$$  \hspace{1cm} (3.19)

The mirror of SQED$_{N_f}$, which we denote by $[1] - (1)^{\overline{N}_f-1} [1]$ has a gauge group $U(1)^{N_f}/U(1)$ and the twisted hypermultiplets that are associated to the links of the extended Dynkin diagram of $A_{N_f-1}$. The twisted hypers carry charges $\{+, -, 0, \cdots, 0\}, \{0, +, -, \cdots, 0\}, \cdots, \{0, \cdots, 0, +\}$ under the $U(1)^{N_f}$. For $[1] - (1)^{\overline{N}_f-1} [1]$ theory, the magnetic fluxes are labeled by $N_f - 1$ integers $(m_1, \cdots, m_{N_f-1})$ corresponding to $(N_f - 1) U(1)$ topological symmetries.

The conformal dimension of bare monopole is $\Delta(m) = \frac{1}{2}(|m_1| + |m_{N_f-1}| + \sum_{i=1}^{N_f-2} |m_i - m_{i+1}|)$. 

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Then the index of the mirror quiver gauge theory \([1] - (1)^{N_f-1} - [1]\) reads

\[
\mathbb{P}^3 [1] - (1)^{N_f-1} - [1](t, x_\alpha, z_\alpha; q)
\]

\[
\times = \frac{(q^{\frac{1}{2}} t^{-\frac{1}{2}}; q)_\infty}{(q^{\frac{1}{2}} t^2; q)_\infty} \sum_{m_1, \cdots, m_{N_f-1} \in \mathbb{Z}} \prod_{i=1}^{N_f-1} \frac{ds_i}{2\pi is_i} \times \frac{(q^{\frac{1}{2}} + \frac{|m_1|}{2} ts_1 t^{-\frac{1}{2}}; q)_\infty}{(q^{\frac{1}{2}} - \frac{|m_1|}{2} ts_1 t^{-\frac{1}{2}}; q)_\infty} \prod_{i=1}^{N_f-2} \frac{(q^{\frac{1}{2}} + \frac{|m_{i+1}|}{2} ts_i t^{-\frac{1}{2}}; q)_\infty}{(q^{\frac{1}{2}} - \frac{|m_{i+1}|}{2} ts_i t^{-\frac{1}{2}}; q)_\infty} \times q^{\frac{1}{2} |m_1| + \frac{|m_{N_f-1}|}{2} + \sum_{i=1}^{N_f-2} \frac{|m_i|}{2} ts_i t^{-\frac{1}{2}} \sum_{i=1}^{N_f-1} |m_i - m_{i+1}|} \prod_{\alpha=1}^{N_f-1} \left( \frac{x_\alpha}{x_{\alpha+1}} \right)^{m_\alpha} \left( \frac{x_{N_f}}{x_1} \right)^{m_1 + m_{N_f-1}}.
\]

(3.20)

Here the fugacities \(z_\alpha\) are used for the flavor symmetry while the fugacities \(x_\alpha\) are associated to the topological symmetry. The index (3.20) would be equal to the index (3.17) of \(\text{SQED}_{N_f}\).

The Coulomb branch of \([1] - (1)^{N_f-1} - [1]\) is the reduced moduli space of one instanton of \(SU(n)\).

Taking the Higgs limit \(q \to 0\) while keeping \(t^2 = q^2 t\) constant, setting \(z_\alpha = x_\alpha x_{\alpha+1}^{-1}\) and \(m_0 = m_{N_f}\) \(\equiv 0\), the index (3.20) becomes

\[
\mathbb{P}^3 \text{Higgs} [1] - (1)^{N_f-1} - [1](t, z_i) = \frac{1}{(1-t)^{N_f-1}} \sum_{m_1, \cdots, m_{N_f-1}} \left( \frac{t^2 \sum_{i=0}^{N_f-1} |m_i - m_{i+1}|}{1 - t} \right) \prod_{i=1}^{N_f-1} z_i^{m_i}.
\]

(3.21)

where we have removed decoupled \(U(1)\) by gauge fixing to set \(m_1 + m_{N_f-1} = 0\). This is the refined Hilbert series of the Coulomb branch of \([1] - (1)^{N_f-1} - [1]\).

4 Non-Abelian mirror symmetry

In this section we test 3d \(\mathcal{N} = 4\) mirror symmetry for non-Abelian gauge theories by computing 3d full-indices. We confirm that two indices nicely agree to each other.

4.1 \((2) - [4]\)

Consider 3d \(\mathcal{N} = 4\) \(U(2)\) gauge theory with four fundamental hypermultiplets, which we denote by \((2) - [4]\). This is the simplest balanced non-Abelian gauge theory. The quiver diagram and the brane construction are shown in Figure 3.

The dimension of the Coulomb branch is

\[
\dim_{\mathbb{C}} \mathcal{M}_{C}^{(2) - [4]} = 2 \cdot 2 = 4
\]

(4.1)

and the dimension of the Higgs branch is

\[
\dim_{\mathbb{C}} \mathcal{M}_{H}^{(2) - [4]} = 2 \cdot (2 \cdot 4 - 2^2) = 8.
\]

(4.2)

The magnetic fluxes for the \(U(2)\) gauge theory are labeled by two integers \(m_1, m_2\). The R-charge of bare monopole is given by

\[
\Delta(m) = 2 \sum_{i=1}^{2} |m_i| - |m_1 - m_2|
\]

(4.3)
where the first terms are the contributions from the four fundamental hypers while the second term is the contribution from the $U(2)$ vector multiplets.

The index of $(2) - [4]$ is

$$
\Pi^{3d} (2) - [4] (t, x_\alpha, z_\alpha; q)
= \frac{1}{2} \left( \frac{q^2 t^2; q^2}{q^2 t^{-2}; q^2} \right)_{\infty} \sum_{m_1, m_2 \in \mathbb{Z}} \frac{ds_1}{2\pi i s_1} \frac{ds_2}{2\pi i s_2} \times \left( \frac{1 - q^{\pm |m_1 - m_2|/2}}{q^{\pm |m_1 - m_2|/2}} \right) \left( \frac{q^{1+|m_1 - m_2|} t^{2 z_1}; q}{q^{1+|m_1 - m_2|} t^{-2 z_1}; q} \right)_{\infty} \times \prod_{i=1}^{4} \left( \frac{q^{1+|m_i|/2} t^{-1 s_i^\pm}}{q^{1+|m_i|/2} t s_i^\pm} \right)_{\infty}
$$

$$
\times q^{m_1 + m_2 - |m_1 - m_2|/2} t^{-4|m_1| - 4|m_2| + 2|m_1 - m_2|} z_1^{m_1 + m_2} z_2^{-m_1 - m_2}
$$

(4.4)

where $x_\alpha$ are the fugacities for the $SU(4)$ flavor symmetry with $\prod_{\alpha=1}^{4} x_\alpha = 1$ and $z_\alpha$ are the fugacities for the topological symmetry.

In the Coulomb limit where one keeps $t_0^2 = q^4 t^{-1}$ constant and sends $q \to 0$ and $t \to \infty$, the index (4.4) reduces to

$$
\Pi^{3d}_{\text{Coulomb}} (2) - [4] (t, z)
= \frac{1}{2} \left( \frac{1}{1 - t^2} \right) \sum_{m_1, m_2 \in \mathbb{Z}} \frac{ds_1}{2\pi i s_1} \frac{ds_2}{2\pi i s_2} \delta_{m_1, m_2} \left( \frac{1 - s_1^\pm s_2^\mp}{1 - t s_1^\pm s_2^\mp} \right) \Delta(m) z^{m_1 + m_2}
$$

(4.5)

where we have defined $z := z_1 z_2^{-1}$. Let $\lambda(m)$ be a partition of 2 which is associated to the magnetic
flux $m_i$ obeying $\sum_{i=1}^2 \lambda_j(m_i) = 2$ and $\lambda_j(m_i) \geq \lambda_{j+1}(m_i)$. Making use of the formula

$$
\frac{1}{N!} \frac{1}{(1-x)^N} \int \prod_{i=1}^N d\bar{s}_i \prod_{i \neq j} \frac{1-\frac{z_i}{s_j}}{1-x \frac{z_i}{s_j}} = \prod_{k=1}^N \frac{1}{1-x^{s_k}},
$$

we get

$$
\tilde{r}^{3d \ (2) - [4]}_{\text{Coulomb}}(\bar{t}, z) = \sum_{m_1, m_2 \in \mathbb{Z}} \tilde{t}^{2m_1+2|m_2|-|m_1-m_2|} \cdot z^{m_1+m_2} \cdot P_{U(2)}(\bar{t}, m_i).
$$

(4.7)

Here

$$
P_{U(2)}(\bar{t}, m_i) = \prod_{k=1}^2 \frac{1}{(1 - \bar{t}^k)^{\lambda^T_k(m_i)}}
$$

(4.8)

is the factor which counts the number of Casimir where $\lambda^T_k(m_i)$ is the length of the $k$-th row of the transposed Young tableau $\lambda^T(m_i)$. The expression (4.7) is identified with the refined Hilbert series for the Coulomb branch of $(2) - [4]$ which can be further written as

$$
\tilde{r}^{3d \ (2) - [4]}_{\text{Coulomb}}(\bar{t}, z) = \text{Hilb}^{3d \ (2) - [4]}(\bar{t}, z)
$$

$$
= \sum_{i=1}^2 \frac{1 - \bar{t}^i}{(1 - \bar{t}^i) (1 - z^{3-i}) (1 - z^{-1\bar{t}^i-1})}.
$$

(4.9)

The factors $1/(1 - \bar{t}^i)$ with $i = 1, 2$ correspond to the 2 generators $\text{Tr} \varphi^i$ where $\varphi$ is the adjoint scalar field. The factors $1/(1 - z^{3-i})$ and $1/(1 - z^{-1\bar{t}^i-1})$ with $i = 1, 2$ describe the monopole operators $V_+$ with magnetic flux $(+,0)$ and $V_-$ with magnetic flux $(-,0)$ dressed by the adjoint complex scalar field.

The mirror of $(2) - [4]$ is the quiver gauge theory $^{(1)-(2)-(1)}_{[2]}$. The dimension of the Coulomb branch is

$$
\dim \mathcal{M}_C^{(1)-(2)-(1)}_{[2]} = 2 \cdot (1 + 2 + 1) = 8
$$

(4.10)

and the dimension of the Higgs branch is

$$
\dim \mathcal{M}_H^{(1)-(2)-(1)}_{[2]} = 2 \cdot (1 \times 2 + 2 \times 2 + 2 \times 1 - 1^2 - 2^2 - 1^2) = 4.
$$

(4.11)

As predicted by mirror symmetry, the dimensions (4.10) and (4.11) agree with (4.2) and (4.1) respectively.

The magnetic fluxes for $^{(1)-(2)-(1)}_{[2]}$ are labeled by four integers $m_1, m_2, m_3, m_4$. The dimension of bare monopole is given by

$$
\Delta(m) = \frac{1}{2} (|m_1 - m_2| + |m_1 - m_3| + |m_2| + |m_3| + \frac{1}{2} (|m_2 - m_4| + |m_3 - m_4|) - |m_2 - m_3|)
$$

(4.12)

where the terms $\frac{1}{2} (|m_1 - m_2| + |m_1 - m_3|)$ are the contributions from bi-fundamental twisted hypers transforming as $(+, \bar{2}, 0) \oplus (-, 2, 0)$, the terms $|m_2| + |m_3|$ are the contributions from two fundamental
twisted hypers transforming as \((0, 2, 0)\), the terms \(\frac{1}{2}(|m_2 - m_4| + |m_3 - m_4|)\) are the contributions from bi-fundamental twisted hypers transforming as \((0, 2, -) \oplus (0, 2, +)\), and the terms \(|m_2 - m_3|\) are the contributions from the \(U(2)\) twisted vector multiplet.

We can compute the index for the mirror quiver theory \(\text{(1)} - \text{(2)} - \text{(1)}\) as

\[
\prod_{i=2}^{3d} \frac{(q^2 t^{-2}; q)_\infty}{(q^2 t^2; q)_\infty} \sum_{m_1 \in \mathbb{Z}} \oint \frac{ds_1}{2\pi i s_1} \cdot \frac{1}{2} \left( \frac{q^2 t^{-2}; q}_\infty \right)^2 \sum_{m_2, m_3 \in \mathbb{Z}} \oint \frac{ds_2}{2\pi i s_2} \oint \frac{ds_3}{2\pi i s_3} \\
	imes \left( 1 - q^{\frac{1}{2} m_2 - m_3 \mp \frac{s_2}{2}} \right) \left( 1 - q^{\frac{1}{2} m_2 - m_3 \mp \frac{s_3}{2}} \right) \left( q^{1 + |m_2 - m_3|} t^{-2} s_2; q \right)_\infty \left( q^{1 + |m_2 - m_3|} t^{-2} s_3; q \right)_\infty \\
\times \frac{\left( q^{\frac{1}{2} m_2 - m_3 \mp \frac{s_2}{2}} \right) \left( q^{1 + |m_2 - m_3|} t^{-2} s_2; q \right)_\infty \left( q^{1 + |m_2 - m_3|} t^{-2} s_3; q \right)_\infty \\
\times \prod_{i=2}^{3d} \frac{(q^2 t^{-2}; q)_\infty}{(q^2 t^2; q)_\infty} \sum_{m_4 \in \mathbb{Z}} \oint \frac{ds_4}{2\pi i s_4} \\
\times \prod_{i=2}^{3d} \left( q^{\frac{1}{2} \frac{m_1 - m_i}{2} + \frac{m_2 + m_4}{4} + \frac{m_3 + m_4}{4} + \frac{1}{2} - \alpha i} t^{-1} s_i^\pm t s_i^\mp \alpha \right)_\infty \left( q^{\frac{1}{2} \frac{m_1 - m_i}{2} + \frac{m_2 + m_4}{4} + \frac{m_3 + m_4}{4} + \frac{1}{2} + \alpha i} t^{-1} s_i^\pm t s_i^\mp \alpha \right)_\infty \\
\times q^{m_1 - m_2\pm |m_1 - m_3\pm |} q^{m_2\pm |m_2 - m_4\pm |} q^{m_3\pm |m_2 - m_3\pm |} q^{m_4\pm |m_2 - m_4\pm |} q^{m_4\pm |m_2 - m_3\pm |} \\
\times \left( x_1 \frac{m_1}{x_2} \right)^{m_1} \left( x_2 \frac{m_2 + m_3}{x_3} \right)^{m_2 + m_3} \left( x_3 \frac{m_4}{x_4} \right)^{m_4} \left( x_4 \frac{m_1 + m_4}{x_1} \right)^{m_1 + m_4} (4.13)
\]

where \(x_\alpha\) are the fugacities for the topological symmetry while \(z_\alpha\) are the fugacities for the flavor symmetry. As expected, we find that the indices (4.4) and (4.13) coincide with each other.

### 4.2 \((N) - [2N]\)

Let us argue for the general balanced \(U(N)\) gauge theory with \(2N\) hypers which we describe as \((N) - [2N]\). The magnetic charges for \(U(N)\) gauge theory are given by \(N\)-tuples of integers \((m_1, \cdots, m_N)\). The dimension of Coulomb branch is

\[
dim \mathcal{M}_C^{(N) - [2N]} = 2N (4.14)
\]

while the dimension of Higgs branch is

\[
dim \mathcal{M}_H^{(N) - [2N]} = 2 \cdot (N \times 2N - N^2) = 2N^2. (4.15)
\]

The R-charge of bare monopole is

\[
\Delta(m) = N \sum_{i=1}^{N} |m_i| - \sum_{i < j} |m_i - m_j|. (4.16)
\]

Here the first terms are contributed from the \(2N\) fundamental hypers while the second terms are the contributions from the \(U(N)\) vector multiplet.
The index of $(N - [2N])$ takes the form

\[
I^{(N-[2N])}(t, x_i, z; q) = \frac{1}{N!} \left( \frac{q^{3} t^2}{t-2} : q \right)^{N} \sum_{m_1, \ldots, m_N \in \mathbb{Z}} \prod_{i=1}^{N} ds_i \prod_{i<j} \left( 1 - q^{\frac{|m_i - m_j|}{2}} s_i^{-} s_j^{+} \right) \left( q^{\frac{1+|m_i - m_j|}{2}} t^2 s_i^{+} s_j^{-} ; q \right)^{\infty} \right.

\[
\left. \times \prod_{i=1}^{N} \prod_{\alpha=1}^{2} \left( q^{\frac{3}{2} + \frac{|m_i|}{2} t^{-1} s_i^{+} x_{\alpha}^{-} ; q \right)^{\infty} \right]
\]

\[
\times q^{\frac{N}{2} \sum_{i=1}^{N} |m_i| - \sum_{i<j} \frac{|m_i - m_j|}{2}} t^{-2 N \sum_{i=1}^{N} m_i + 2 \sum_{i<j} |m_i - m_j|} \left( \frac{z_1}{z_2} \right)^{\sum_{i=1}^{N} m_i} \tag{4.17}
\]

where $x_\alpha$ are the fugacities for the $SU(2N)$ flavor symmetry with $\prod_{i=1}^{2N} x_\alpha = 1$ and $z_\alpha$ are the fugacities for the topological symmetry.

The mirror of $(N - [2N])$ is the quiver gauge theory

\[
\begin{array}{c}
(1)-(2) - \cdots - (N) - \cdots - (2)-(1) \\
\text{dim}_{\mathbb{C}} M_C \\
= 2 \cdot \left( 2 \sum_{k=1}^{N-1} k + N \right) = 2 N^2
\end{array}
\tag{4.18}
\]

and the dimension of Higgs branch is

\[
\begin{array}{c}
(1)-(2) - \cdots - (N) - \cdots - (2)-(1) \\
\text{dim}_{\mathbb{C}} M_H \\
= 2 \cdot \left( 2 \times \sum_{k=1}^{N-1} k(k + 1) + 2 \times N - 2 \sum_{k=1}^{N-1} k^2 - N^2 \right) = 2 N.
\end{array}
\tag{4.19}
\]

As predicted by mirror symmetry, the dimensions \eqref{4.18} and \eqref{4.19} agree with the dimensions \eqref{4.15} and \eqref{4.14} respectively.

Classically there is a $U(1)^{N^2}$ topological symmetry and we label the magnetic fluxes by $N$ sets of $k$-tuple of integers $(m^{(k)}_{i})_{i=1, \ldots, k}$ with $k = 1, \ldots, N$ and $(N - 1)$ sets of $(2N - k)$-tuple of integers $(m^{(k)}_{i})_{i=1, \ldots, 2N-k}$ with $k = N+1, \ldots, 2N-1$, which are in total $N^2$ integers. The conformal dimension of bare monopole is

\[
\Delta(m) = \frac{1}{2} \sum_{k=1}^{2N-2} \sum_{i} \sum_{j} |m^{(k)}_{i} - m^{(k+1)}_{j}| + \sum_{i=1}^{N} |m^{(N)}_{i}| - \sum_{k=1}^{2N-1} \sum_{i<j} |m^{(k)}_{i} - m^{(k)}_{j}|. \tag{4.20}
\]

The first terms are the contributions from the bi-fundamental twisted hypermultiplets. The sum over $i$ runs from 1 to $k$ for $k \leq N$ and from 1 to $2N - k$ for $N < k$ while the sum over $j$ runs from 1 to $k+1$ for $k \leq N - 1$ and from 1 to $2N - k - 1$ for $N - 1 < k$. The second terms are the contributions from the two twisted hypermultiplets transforming as fundamental representation under the $U(N)$ gauge symmetry, and the third terms are the contributions from the twisted vector multiplets.
Then the index of the quiver gauge theory \((1)-(2)\ldots \overline{(N)} \ldots -(2)-(1)\) reads

\[
(1)-(2)\ldots \overline{(N)} \ldots -(2)-(1)
\]

\[
\prod_{k=1}^{N} \frac{1}{k!} \frac{(q^{1/2}t^{-2}; q)^{k}}{(q^{1/2}t^{2}; q)^{k}} \sum_{m_{1}^{(k)}, \ldots, m_{N}^{(k)} \in \mathbb{Z}} \prod_{i=1}^{k} \frac{ds_{i}^{(k)}}{2\pi i s_{i}^{(k)}} \left(1 - q^{\frac{|m_{i}^{(k)} - m_{j}^{(k)}|}{2}} s_{i}^{(k)} s_{j}^{(k)} \mp \sum_{N_{i}^{(k)}} \sum_{N_{j}^{(k)}} \left(1 + i |m_{i}^{(k)} - m_{j}^{(k)}| \right) t^{2} s_{i}^{(k)} s_{j}^{(k)} ; q \right) \infty
\]

\[
\prod_{i<j} \left(1 - q^{\frac{|m_{i}^{(N+k) - m_{j}^{(N+k)}|}{2}} s_{i}^{(N+k)} s_{j}^{(N+k)} \mp \sum_{N_{i}^{(N+k)}} \sum_{N_{j}^{(N+k)}} \left(1 + i |m_{i}^{(N+k) - m_{j}^{(N+k)}|} ; q \right) t^{2} s_{i}^{(N+k)} s_{j}^{(N+k)} ; q \right) \infty
\]

\[
\prod_{i<j} \left(1 - q^{\frac{|m_{i}^{(N+k) - m_{j}^{(N+k)}|}{2}} s_{i}^{(N+k)} s_{j}^{(N+k)} \mp \sum_{N_{i}^{(N+k)}} \sum_{N_{j}^{(N+k)}} \left(1 + i |m_{i}^{(N+k) - m_{j}^{(N+k)}|} ; q \right) t^{2} s_{i}^{(N+k)} s_{j}^{(N+k)} ; q \right) \infty
\]

where \(z_{\alpha}\) are the fugacities for the flavor symmetry and \(x_{\alpha}\) are the fugacities for the topological symmetry. We expect that the index \(4.17\) coincides with the index \(4.21\).

### 4.3 \(T[SU(3)]\)

Let us consider \(T[SU(3)]\) theory. This is a self-mirror theory and the quiver diagram and the brane construction are illustrated in Figure.\[\]

The dimension of the Coulomb branch of \(T[SU(3)]\) is

\[
\dim_{C} \mathcal{M}^{T[SU(3)]} = 2 \cdot (1 + 2) = 6
\]

and the dimension of the Higgs branch of \(T[SU(3)]\) is

\[
\dim_{C} \mathcal{M}^{T[SU(3)]} = 2 \cdot (1 + 2 \times 3 - 1^2 - 2^2) = 6.
\]

The dimensions \(4.22\) and \(4.23\) are equal as \(T[SU(3)]\) is a self-mirror theory.
where the topological symmetry.

\[ U \text{ are the contributions from the bi-fundamental hypers, the next two terms are the contributions from three fundamental hypers for } U(2) \text{ gauge symmetry, and the last term is the contribution from the } U(2) \text{ vector multiplet.} \]

The index of \( T[SU(3)] \) reads

\[
\prod_{i=2}^{3} \prod_{\alpha=1}^{3} \prod_{\beta=1}^{3} \prod_{i=2}^{3} \prod_{\alpha=1}^{3} \prod_{\beta=1}^{3} \left( q^{\frac{1}{2}+\frac{|m_1-m_3|}{2}} t s_1 \frac{s_2}{s_i} ; q^{\infty} \right) \left( q^{\frac{1}{2}+\frac{|m_2-m_3|}{2}} t s_1 \frac{s_2}{s_i} ; q^{\infty} \right) \left( q^{\frac{1}{2}+\frac{|m_1-2|}{2}} t s_1 \frac{s_2}{s_i} ; q^{\infty} \right) \left( q^{\frac{1}{2}+\frac{|m_2-2|}{2}} t s_1 \frac{s_2}{s_i} ; q^{\infty} \right) \\
\times q^{-|m_1-m_2|+|m_1-m_3|+3|m_2|+3|m_3|+3|m_2-m_3|} t^{-|m_1-m_2|-|m_1-m_3|-3|m_2|-3|m_3|+2|m_2-m_3|} \times \left( \frac{z_1}{z_2} \right)^{m_1} \left( \frac{z_2}{z_3} \right)^{m_2+m_3} \left( \frac{z_3}{z_1} \right)^{m_1+m_2+m_3}
\]

(4.25)

where \( x_\alpha \) are the fugacities for the \( SU(3) \) flavor symmetry with \( x_1 x_2 x_3 = 1 \) and \( z_\alpha \) are the fugacities for the topological symmetry.

Figure 4: (a) The quiver diagrams of \( T[SU(3)] \). (b) The brane configurations of \( T[SU(3)] \).
The index (4.25) agrees with the index of $T[SU(3)]$ which consists of twisted supermultiplets

$$\tilde{\text{I}}^{T[SU(3)]}(t, x_\alpha, z_\gamma; q)$$

\[= \frac{(q^\frac{1}{2}t^{-2}; q)_\infty}{(q^\frac{1}{2}t^2; q)_\infty} \sum_{m_1 \in \mathbb{Z}} \int \frac{ds_1}{2\pi i s_1} \frac{1}{2} \frac{(q^\frac{1}{2}t^{-2}; q)_\infty^2}{(q^\frac{1}{2}t^2; q)_\infty^2} \sum_{m_2, m_3 \in \mathbb{Z}} \int \frac{ds_2}{2\pi i s_2} \frac{ds_3}{2\pi i s_3} \left(1 - q^{\frac{m_1 - m_2 - m_3}{s_3}} \right) \left(1 - q^{\frac{s_3}{s_2}} \right) \left(q^{\frac{4m_1}{s_2}} t^{-2} \right) \left(q^{\frac{4m_2}{s_2}} t^{-2} \right) \left(q^{\frac{4m_3}{s_2}} t^{-2} \right) \left(q^{\frac{4s_3}{s_2}} \right) \times \prod_{i=1}^3 \left(\frac{q^{\frac{1}{2} + \frac{m_1 - m_i}{s_i}} t s_i^{\frac{1}{2}} s_i^{\frac{1}{2}}; q)_\infty}{(q^{\frac{1}{2} + \frac{m_1 - m_i}{s_i}} t s_i^{\frac{1}{2}} s_i^{\frac{1}{2}}; q)_\infty} \prod_{i=1}^3 \prod_{\alpha=1}^3 \left(q^{\frac{1}{2} + \frac{m_1 - m_i}{s_i}} t^{-1} s_i^{\frac{1}{2}} s_i^{\frac{1}{2}}; q)_\infty}{(q^{\frac{1}{2} + \frac{m_1 - m_i}{s_i}} t^{-1} s_i^{\frac{1}{2}} s_i^{\frac{1}{2}}; q)_\infty} \right) \times \frac{x_1^{m_1}}{x_2^{m_1 + m_2 + m_3}} \frac{x_2^{m_2}}{x_3^{m_1 + m_2 + m_3}} \frac{x_3^{m_3}}{x_1^{m_1 + m_2 + m_3}} \right) (4.26)$$

where $z_\alpha$ are the fugacities for the flavor symmetry with $z_1 z_2 z_3 = 1$ and $x_\alpha$ are the fugacities for the topological symmetry.

### 4.4 $T[SU(N)]$

Consider the quiver gauge theory (1) - (2) - · · · - (N-1) - [N], which we call $T[SU(N)]$. The dimension of the Coulomb branch of $T[SU(N)]$ is

$$\dim \mathcal{M}_C^{T[SU(N)]} = 2 \sum_{k=1}^{N-1} k = N(N-1) \quad (4.27)$$

and the dimension of the Higgs branch of $T[SU(N)]$ is

$$\dim \mathcal{M}_H^{T[SU(N)]} = 2 \left[ \sum_{k=1}^{N-1} k(k+1) - \sum_{k=1}^{N-1} k^2 \right] = N(N-1). \quad (4.28)$$

The quiver gauge theory $T[SU(N)]$ is a self-mirror theory whose Coulomb branch and Higgs branch are identical as the dimensions (4.27) and (4.28) are equal.

From brane construction, a D3-brane ending on the NS5-brane can be viewed as a magnetic monopole. Hence the Coulomb branch or Higgs branch of $T[SU(N)]$ is interpreted as the moduli space of $SU(N)$ monopoles which are formed by one with magnetic charge (+, −, 0, · · · , 0), two with magnetic charge (0, +, −, · · · , 0), · · · , (N − 1) with magnetic charge (0, 0, · · · , +, −) together with N fixed Dirac monopoles with magnetic charge (0, 0, · · · , 0, +).

The magnetic fluxes for $T[SU(N)]$ is labeled by $N-1$ sets of $k$-tuple of integers, that is $\frac{N(N-1)}{2}$ integers $\{m_i^{(k)}\}_{i=1, \ldots, k}$ with $k = 1, \ldots, N-1$. The canonical R-charge of bare monopole is

$$\Delta(m) = \frac{1}{2} \sum_{k=1}^{N-1} \sum_{i=1}^{k+1} |m_i^{(k)} - m_j^{(k+1)}| + \frac{N}{2} \sum_{i=1}^{N-1} |m_i| - \sum_{k=1}^{N-1} |m_i^{(k)} - m_j^{(k)}| \quad (4.29)$$

where the first terms are the contributions from the bi-fundamental hyper, the second terms are those from the N fundamental hyper, and the third terms are those from the vector multiplets.
The index of $T[SU(N)]$ takes the form

\[
P^{SU(N)}(t, x, z; q) = \prod_{k=1}^{N-1} \left[ \frac{1}{k!} \left( \frac{q^{3/2} t^2}{z} \right)^k \sum_{m_1, \ldots, m_k \in \mathbb{Z}} \int \prod_{i=1}^{k} \frac{ds_i^{(k)}}{2\pi i s_i^{(k)}} \left( 1 - q^{-\frac{1}{2} |m_i - m_j|} s_i^{(k)} s_j^{(k)\mp} \right) \left( q^{1 + \frac{1}{2} |m_i - m_j|} t^2 s_i^{(k)} s_j^{(k)\mp} ; q \right)_\infty \right]
\times \prod_{i < j} \left( \frac{1 + q^{-\frac{1}{2} |m_i - m_j|}}{z^{-1} t^2 s_i^{(k)} s_j^{(k)\mp} ; q \infty} \right)
\times \prod_{k=1}^{N-2} \prod_{i=1}^{k+1} \left( q^{\frac{1}{2} |m_i - m_{i+1}|} t^{-1} s_i^{(k)} s_{i+1}^{(k)\mp} ; q \infty \right)
\times \prod_{i=1}^{N-1} \prod_{\alpha=1}^{N} \left( q^{\frac{1}{2} |m_i - m_{\alpha}|} t^{-1} s_i^{(k)} s_{\alpha}^{(k)\mp} ; q \infty \right)
\times \prod_{k=1}^{N-1} \left( x_k \right)_{z_k} \left( \frac{x_N x_1}{z_1} \right)_{x_1}^{N-1} m_i^{(k)} m_i^{(k)} (4.30)
\]

where $x_\alpha$ are the fugacities for the $SU(N)$ flavor symmetry with $\prod_{\alpha=1}^{N} x_\alpha = 1$ and $z_\alpha$ are the fugacities for the topological symmetry.

We expect that the index $[4.30]$ is equal to the index of $T[SU(N)]$

\[
P^{SU(N)}(t, x, z; q) = \prod_{k=1}^{N-1} \left[ \frac{1}{k!} \left( \frac{q^{3/2} t^2}{z} \right)^k \sum_{m_1, \ldots, m_k \in \mathbb{Z}} \int \prod_{i=1}^{k} \frac{ds_i^{(k)}}{2\pi i s_i^{(k)}} \left( 1 - q^{-\frac{1}{2} |m_i - m_j|} s_i^{(k)} s_j^{(k)\mp} \right) \left( q^{1 + \frac{1}{2} |m_i - m_j|} t^2 s_i^{(k)} s_j^{(k)\mp} ; q \right)_\infty \right]
\times \prod_{i < j} \left( \frac{1 + q^{-\frac{1}{2} |m_i - m_j|}}{z^{-1} t^2 s_i^{(k)} s_j^{(k)\mp} ; q \infty} \right)
\times \prod_{k=1}^{N-2} \prod_{i=1}^{k+1} \left( q^{\frac{1}{2} |m_i - m_{i+1}|} t^{-1} s_i^{(k)} s_{i+1}^{(k)\mp} ; q \infty \right)
\times \prod_{i=1}^{N-1} \prod_{\alpha=1}^{N} \left( q^{\frac{1}{2} |m_i - m_{\alpha}|} t^{-1} s_i^{(k)} s_{\alpha}^{(k)\mp} ; q \infty \right)
\times \prod_{k=1}^{N-1} \left( x_k \right)_{z_k} \left( \frac{x_N x_1}{z_1} \right)_{x_1}^{N-1} m_i^{(k)} m_i^{(k)} (4.31)
\]

with $z_\alpha$ being the fugacities for the flavor symmetry satisfying $\prod_{\alpha=1}^{N} z_\alpha = 1$ and $x_\alpha$ being the fugacities for the topological symmetry.
Consider a quiver gauge theory\( (1) - (2) - (1) \) whose quiver diagram and the brane construction are illustrated in Figure 5. This is a self-mirror quiver theory.

The dimension of the Coulomb branch is
\[
\dim \mathcal{M}_C^{(1) - (2) - (1)} = 2 \cdot (1 + 2 + 1) = 8 \tag{4.32}
\]
and the dimension of the Higgs branch is
\[
\dim \mathcal{M}_H^{(1) - (2) - (1)} = 2 \cdot (1 \times 1 + 1 \times 2 + 2 \times 2 + 2 \times 1 + 1 \times 1 - 1^2 - 2^2 - 1^2) = 8. \tag{4.33}
\]

Let us label the magnetic fluxes for \( (1) - (2) - (1) \) by four integers \( m_1, m_2, m_3, m_4 \). The R-charge of bare monopole operator is given by
\[
\Delta(m) = \frac{|m_1|}{2} + \frac{|m_1 - m_2| + |m_1 - m_3|}{2} + |m_2| + |m_3| + \frac{|m_2 - m_4| + |m_3 - m_4|}{2} + \frac{|m_4|}{2} - |m_2 - m_3|. \tag{4.34}
\]

The terms \( \frac{|m_1|}{2} + \frac{|m_4|}{2} \) are contributed from the charged hypers under the \( U(1) \) gauge factors while the terms \( |m_2| + |m_3| \) are the contributions from the fundamental hypers of \( U(2) \) gauge symmetry. The terms \( \prod_{i<j} |m_i - m_j| \) are the contributions from the bi-fundamental hypers. The term \( -|m_2 - m_3| \) are the contribution from the \( U(2) \) vector multiplet.

Figure 5: (a) The quiver diagrams of the self-mirror \( U(1) \times U(2) \times U(1) \) quiver gauge theory. (b) The brane configurations of the self-mirror \( U(1) \times U(2) \times U(1) \) quiver gauge theory.
The index of \((1) - (2) - (1)\) is evaluated as

\[
\frac{(q^2 t^2; q)_\infty}{(q^{2t-2}; q)_\infty} \sum_{m_1 \in \mathbb{Z}} \int \frac{ds_1}{2\pi i s_1} \times \frac{1}{2} \frac{(q^{\frac{1}{2} t^2}; q)_\infty}{(q^{2t-2}; q)_\infty} \sum_{m_2, m_3 \in \mathbb{Z}} \int \frac{ds_2}{2\pi i s_2} \frac{ds_3}{2\pi i s_3} \left(1 - q^{\frac{m_2 - m_3}{2}} s_2^\pm s_3^\mp \right) \left(q^{\frac{1 + |m_2 - m_3|}{2}} t^2 s_2^\pm s_3^\mp ; q\right)_\infty
\]

where \(x_\alpha\) are the fugacities for the flavor symmetry and \(z_\alpha\) are the fugacities for the topological symmetry.

As expected, we have confirmed that the index \((4.35)\) coincides with the index of \((1) - (2) - (1)\).
which consists of twisted supermultiplets

\[
\frac{3d}{\prod} \frac{1^{(1)} \cdot 2^{(2)} \cdot 1^{(1)}}{|1| \ |2| \ |1|} (t, x_\alpha, z_\alpha; q) = (q^{\frac{t}{2}} t^{-2}; q)^{\infty} \sum_{m_1 \in \mathbb{Z}} \frac{d t_1}{2 \pi i s_1} \times \frac{(q^{\frac{t}{2}} t^{-2}; q)^{\infty}}{2 \sum_{m_2, m_3 \in \mathbb{Z}} \frac{d s_2}{2 \pi i s_2} \sum_{m_4 \in \mathbb{Z}} \frac{d s_4}{2 \pi i s_4}} \times \left( q^{\frac{3}{4} + \frac{m_1}{2}} t \frac{3}{2} s_1^3 + \frac{2}{3} z_1^3; q \frac{3}{2} s_1^3 + \frac{2}{3} z_1^3; q \right)^{\infty} \prod_{i=2}^{3} \left( q^{\frac{3}{4} + \frac{m_i}{2}} t \frac{3}{2} s_i^3 + \frac{2}{3} z_i^3; q \frac{3}{4} s_i^3 + \frac{2}{3} z_i^3; q \right)^{\infty} \prod_{i=2}^{3} \left( q^{\frac{3}{4} + \frac{m_i}{2}} t \frac{3}{2} s_i^3 + \frac{2}{3} z_i^3; q \frac{3}{4} s_i^3 + \frac{2}{3} z_i^3; q \right)^{\infty}
\]

where \(z_\alpha\) are the fugacities for the flavor symmetry and \(x_\alpha\) are the fugacities for the topological symmetry.

As a next example, let us consider a quiver gauge theory \(1^{(1)} 2^{(2)} 1^{(1)}\). This is not self-mirror and the quiver diagram and the brane construction are drawn in Figure 6. The dimension of the
Coulomb branch is

\[
\text{dim}_{\mathbb{C}} \mathcal{M}_C = \frac{(1) - (2) - (1)}{2} \cdot (1 + 2 + 1) = 8
\]  

(4.37)

while the dimension of the Higgs branch is

\[
\text{dim}_{\mathbb{C}} \mathcal{M}_H = 2 \cdot (1 \times 2 + 2 \times 2 + 2 \times 1 \times 1 - 1^2 - 2^2 - 1^2) = 6.
\]  

(4.38)

Again the magnetic fluxes are labeled by integers \(m_1, m_2, m_3, m_4\). The canonical R-charge of bare monopole is given by

\[
\Delta(m) = \frac{|m_1 - m_2| + |m_1 - m_3| + |m_2| + |m_3|}{2} + \frac{|m_2 - m_4| + |m_3 - m_4|}{2} + \frac{|m_4|}{2} - |m_2 - m_3|,
\]  

(4.39)

which is obtained by eliminating the contribution \(\frac{|m|}{2}\) of the charged hyper from the R-charge for self-mirror quiver gauge theory

\[
(1) - (2) - (1)
\]  

\[
\begin{array}{c|c|c}
1 & 2 & 3 \\
\hline
1 & 2 & 1 \\
\end{array}
\]

(4.34)

We can evaluate the index of

\[
\begin{array}{c|c|c}
1 & 2 & 3 \\
\hline
1 & 2 & 3 \\
\end{array}
\]

as

\[
\frac{3!}{3} \frac{(q^{\frac{3}{2}} t^2; q)_\infty}{(q^{\frac{1}{2}} t^{-2}; q)_\infty} \sum_{m_1, \ldots, m_4} \int \frac{ds_1}{2\pi i s_1} \int \frac{ds_2}{2\pi i s_2} \int \frac{ds_3}{2\pi i s_3} \int \frac{ds_4}{2\pi i s_4}
\]

\[
\times \frac{(q^{\frac{3}{2}} t^2; q)_\infty}{(q^{\frac{1}{2}} t^{-2}; q)_\infty} \sum_{m_1, m_2, m_3} \int \frac{ds_2}{2\pi i s_2} \int \frac{ds_3}{2\pi i s_3} \int \frac{ds_4}{2\pi i s_4}
\]

\[
\times \prod_{i=2}^3 \left( q^{\frac{3}{2} + \frac{|m_i - m_1|}{2} - \frac{1}{2} - |s_i|}; q \right)_\infty \prod_{i=2}^3 \left( q^{\frac{3}{2} + \frac{|m_i - m_3|}{2} - \frac{1}{2} - |s_i|}; q \right)_\infty
\]

\[
\times \prod_{i=1}^2 \left( q^{\frac{3}{2} + \frac{|m_i - m_1|}{2} + \frac{1}{2} + |s_i|}; q \right)_\infty \prod_{i=1}^2 \left( q^{\frac{3}{2} + \frac{|m_i - m_3|}{2} + \frac{1}{2} + |s_i|}; q \right)_\infty
\]

\[
\times t^{-|m_1 - m_2| - |m_1 - m_3| - 2|m_2| - 2|m_3| - |m_2 - m_4| - |m_3 - m_4| - |m_4| + 2|m_2 - m_3|}
\]

\[
\times \left( \frac{z_1}{z_2} \right)^{2} \left( \frac{z_2}{z_3} \right)^{m_1} \left( \frac{z_3}{z_1} \right)^{m_2+3} \left( \frac{z_4}{z_1} \right)^{m_4} \left( \frac{z_1}{z_4} \right)^{m_1+m_3} \right)
\]  

(4.40)

where the fugacities \(x_\alpha\) are associated to the flavor symmetry while the fugacities \(z_\alpha\) are associated to the topological symmetry.

The mirror of

\[
\begin{array}{c|c|c}
1 & 2 & 3 \\
\hline
1 & 2 & 1 \\
\end{array}
\]

(2) - (1)

\[
\begin{array}{c|c|c}
1 & 2 & 3 \\
\hline
2 & 1 & 1 \\
\end{array}
\]

(4.35)

whose quiver diagram and brane construction are illustrated in Figure 6.
The dimension of the Coulomb branch is

\[
\dim_{\mathcal{C}}\mathcal{M}_C^{(2)-(1)} = 2 \cdot (2 + 1) = 6
\]

(4.41)

and the dimension of the Higgs branch is

\[
\dim_{\mathcal{H}}\mathcal{M}_H^{(2)-(1)} = 2 \cdot (2 \times 3 + 2 \times 1 + 1 \times 1 - 2^2 - 1^2) = 8.
\]

(4.42)

The magnetic fluxes are labeled by three integers \(m_1, m_2, m_3\). We have the bare monopole operator with dimension

\[
\Delta(m) = \frac{3}{2} |m_1| + \frac{3}{2} |m_2| + \frac{|m_1 - m_3|}{2} + \frac{|m_2 - m_3|}{2} + \frac{|m_3|}{2} - |m_1 - m_2|.
\]

(4.43)

Here the first two terms are contributed from three fundamental twisted hypers for \(U(2)\) gauge symmetry, the next two terms are the contributions from bi-fundamental twisted hypers, the second from the last is contributed from charged twisted hyper for \(U(1)\) gauge symmetry, and the last comes from the \(U(2)\) twisted vector multiplet.

The index of \([3] \quad [1]\) is given by

\[
\begin{align*}
\left(\frac{2}{3d} \left| \begin{array}{c}
|t, x, z| \quad [q]
\end{array} \right| \right) = \\
\prod_{i=1}^{3} (q^2 t_i^2; q)_{\infty} \prod_{i=1}^{3} (q^{1/2} t_i; q)_{\infty}
\end{align*}
\]

(4.44)

As predicted from mirror symmetry, the associated global symmetry of fugacities \(x, z\) are swapped and the indices (4.40) and (4.44) coincide with each other.

4.6 \quad (1) - (2) - [4]

Consider the 3d \(\mathcal{N} = 4\) quiver gauge theory \((1) - (2) - [4]\). The quiver diagram and the brane construction are illustrated in Figure 7.

The dimension of the Coulomb branch is

\[
\dim_{\mathcal{C}}\mathcal{M}_C^{(1)-(2)-[4]} = 2 \cdot (1 + 2) = 6
\]

(4.45)
and the dimension of the Higgs branch is

\[ \dim M_{\mathcal{H}}^{(1)-(2)-[4]} = 2 \cdot (1 \times 2 + 2 \times 4 - 1^2 - 2^2) = 10. \] (4.46)

For (1) – (2) – [4] we can label the magnetic fluxes in terms of three integers \( m_1, m_2, m_3 \). The R-charge of bare monopole operator is expressed as

\[ \Delta(m) = \frac{|m_1 - m_2|}{2} + \frac{|m_1 - m_3|}{2} + 2|m_2| + 2|m_3| - |m_2 - m_3| \] (4.47)

where the first two terms are the contributions from bi-fundamental hypers, the next two terms are those from four fundamental hypers for the \( U(2) \) gauge node and the last is contributed from the \( U(2) \) vector multiplet.

The index of (1) – (2) – [4] is

\[
\begin{align*}
&\prod_{m_1 \in \mathbb{Z}} \frac{(q^2 t^2; q)^\infty}{(q^2 t^{-2}; q)^\infty} \sum_{m_2, m_3 \in \mathbb{Z}} \int \frac{ds_1}{2\pi i s_1} \int \frac{ds_2}{2\pi i s_2} \int \frac{ds_3}{2\pi i s_3} \left(1 - q^{-\frac{|m_2 - m_3|}{2}} s_2^\pm s_3^\mp \right) \left(q^{\frac{1 + |m_2 - m_3|}{2}} t s_2^\pm s_3^\mp; q\right)^\infty \\
&\times \frac{1}{2} \left(\frac{q^2 t^2; q)^\infty}{(q^2 t^{-2}; q)^\infty} \sum_{m_2, m_3 \in \mathbb{Z}} \int \frac{ds_2}{2\pi i s_2} \int \frac{ds_3}{2\pi i s_3} \left(1 - q^{-\frac{|m_2 - m_3|}{2}} s_2^\pm s_3^\mp \right) \left(q^{\frac{1 + |m_2 - m_3|}{2}} t s_2^\pm s_3^\mp; q\right)^\infty \\
&\times \prod_{i=2}^3 \left(\frac{q^2 t^2; q)^\infty}{(q^2 t^{-2}; q)^\infty} \sum_{m_2, m_3 \in \mathbb{Z}} \int \frac{ds_2}{2\pi i s_2} \int \frac{ds_3}{2\pi i s_3} \left(1 - q^{-\frac{|m_2 - m_3|}{2}} s_2^\pm s_3^\mp \right) \left(q^{\frac{1 + |m_2 - m_3|}{2}} t s_2^\pm s_3^\mp; q\right)^\infty \\
&\times q^{\frac{|m_1 - m_2|^2}{2} + \frac{|m_1 - m_3|^2}{2} + \frac{|m_2 - m_3|^2}{2}} \times t^{-|m_1 - m_2| - |m_1 - m_3| - |m_2 - m_3| + 2|m_2 - m_3|} \\
&\times \left(\frac{z_3}{z_1}\right)^{m_1} \left(\frac{z_1}{z_2}\right)^{m_2 + m_3} \left(\frac{z_2}{z_3}\right)^{m_1 + m_2 + m_3}
\end{align*}
\] (4.48)

where \( x_\alpha \) and \( z_\alpha \) are the fugacities for flavor symmetry and topological symmetry respectively.
The mirror of (1) – (2) – [4] is the quiver gauge theory \( \text{(2)} \rightarrow \text{(2)} \rightarrow \text{(1)} \). The corresponding quiver diagram and brane configuration are shown in Figure 7.

The dimension of the Coulomb branch is

\[
\dim_{\text{C}} \mathcal{M}^{[2]}_{\text{C}} = 2 \cdot (2 + 2 + 1) = 10
\]

and the dimension of the Higgs branch is

\[
\dim_{\text{H}} \mathcal{M}^{[2]}_{\text{H}} = 2 \cdot (2 \times 2 + 2 \times 2 + 2 \times 1 + 2 \times 1 - 2^2 - 2^2 - 1^2) = 6.
\]

The dimension of monopole operator is

\[
\Delta(m) = |m_1| + |m_2| + \sum_{i=1}^{2} \sum_{j=3}^{4} \frac{|m_i - m_j|}{2} + \sum_{i=3}^{4} \frac{|m_i - m_4|}{2} - \frac{|m_3 - m_4| + |m_1 - m_2|}{2}
\]

where the first line includes the contributions from the twisted hyper multiplets while the second line describes the contributions from two \( U(2) \) twisted vector multiplets.

We have checked that the index \((4.48)\) beautifully coincides with the index of the mirror quiver gauge theory

\[
I_{3\text{d}}^{\text{(2)} \rightarrow \text{(2)} \rightarrow \text{(1)}}(t, x_\alpha, z_\alpha; q) = \frac{1}{2} \frac{(q^2 t^{-2}; q)_{2\infty}}{(q^2 t^2; q)_{2\infty}} \sum_{m_1, m_2 \in \mathbb{Z}} \int \frac{ds_1}{2\pi i s_1} \frac{ds_2}{2\pi i s_2} (1 - q^{1 + |m_1 - m_2|} s_1^\pm s_2^\pm) (q^{1 + |m_1 - m_2|} t^{-2} s_1^\pm s_2^\pm; q)_{\infty}
\]

\[
\times \frac{1}{2} \frac{(q^2 t^{-2}; q)_{2\infty}}{(q^2 t^2; q)_{2\infty}} \sum_{m_3, m_4 \in \mathbb{Z}} \int \frac{ds_3}{2\pi i s_3} \frac{ds_4}{2\pi i s_4} (1 - q^{1 + |m_3 - m_4|} s_3^\pm s_4^\pm) (q^{1 + |m_3 - m_4|} t^{-2} s_3^\pm s_4^\pm; q)_{\infty}
\]

\[
\times \frac{(q^2 t^{-2}; q)_{\infty}}{(q^2 t^2; q)_{\infty}} \sum_{m_5 \in \mathbb{Z}} \int \frac{ds_5}{2\pi i s_5} \cdot \prod_{i=1}^{2} \prod_{\alpha=1}^{2} \left( q^{1 + |m_i|} t s_i^\pm s_i^\pm; q \right)\left( q^{1 + |m_j|} t^{-1} s_i^\pm s_i^\pm; q \right) \cdot \prod_{i=1}^{4} \prod_{j=3}^{4} \left( q^{1 + |m_i - m_j|} t s_i^\pm s_j^\pm; q \right)
\]

\[
\times \prod_{i=3}^{4} \frac{q^{1 + |m_i|} t s_i^\pm s_i^\pm; q}{q^{1 + |m_i - m_3|} t^{-1} s_i^\pm s_i^\pm; q} \cdot \prod_{i=3}^{4} \frac{q^{1 + |m_i - m_4|} t s_i^\pm s_i^\pm; q}{q^{1 + |m_i - m_4|} t^{-1} s_i^\pm s_i^\pm; q}
\]

\[
\times q^{\sum_{i=1}^{2} |m_i| + \sum_{i=1}^{4} |m_i - m_3| + \sum_{i=1}^{4} |m_i - m_4|} \cdot \prod_{i=1}^{4} \frac{m_i + m_{i+2} + m_{i+3} + m_{i+4}}{m_i + m_{i+2} + m_{i+3} + m_{i+4}} \cdot \prod_{i=1}^{4} \frac{m_i + m_{i+2} + m_{i+3} + m_{i+4}}{m_i + m_{i+2} + m_{i+3} + m_{i+4}}
\]

\[
\times \frac{m_1 + m_2}{m_1 + m_2 + m_3 + m_4 + m_5 + m_6 + m_7 + m_8}
\]

where the fugacities \( z_\alpha \) and \( x_\alpha \) are associated to the flavor and topological symmetries of the mirror theory respectively.
5 Seiberg-like duality

5.1 (2) – [3] and SQED₃

The 3d \( \mathcal{N} = 4 \) \( U(N) \) gauge theory with \( (2N-1) \) hypermultiplets is expected to be equivalent in the IR to a \( \mathcal{N} = 1 \) gauge theory with \( (2N-1) \) hypermultiplets times a free twisted hypermultiplet \[10\]. This is interpreted as Seiberg-like duality between the ugly and good theories. The equality of the Hilbert series for the Coulomb branch is shown in \[2\] and that for the Hilbert series for the Higgs branch is shown in \[20\]. Here we briefly check the equalities of 3d full-indices.

The simplest example is \( N = 2 \), i.e. \( U(2) \) gauge theory with three flavors, which we denote by \((2) – [3]\).

The index of \((2) – [3]\) is

\[
\begin{align*}
\mathbb{I}^{3d} (2) – [3] (t, x_{\alpha}, z_{\alpha}; q) &= \frac{1}{2} \left( \frac{q^{\frac{1}{2}} t^2; q)_\infty}{(q^2 t^{-2}; q)_\infty} \right)^3 \sum_{m_1, m_2 \in \mathbb{Z}} \oint \frac{ds_1}{2\pi is_1} \frac{ds_1}{2\pi is_2} \left( \frac{1 - q^{\alpha - m_1 - m_2} s_1^{\pm} s_2^{\mp}}{q^{\alpha + m_1 - m_2} t^{2 s_1^{\pm} s_2^{\mp}; q}_\infty} \right) \\
&\times 2 \prod_{i=1}^{3} \prod_{\alpha=1}^{3} \left( \frac{q^{\frac{1}{2}} + \frac{|m_1|}{2} t^{-1} s_1^{\pm} x_{\alpha}^{\pm}; q}_\infty \right) \\
&\times q^{\frac{3}{4} |m_1| + \frac{3}{4} |m_2| - \frac{|m_1 - m_2|}{4} + 2|m_2|} t^{-3|m_1| - 3|m_2| + 2|m_1 - m_2|} \left( \frac{z_1}{z_2} \right)^{m_1 + m_2} (5.1)
\end{align*}
\]

where the fugacities \( x_{\alpha} \) are associated to the \( SU(3) \) flavor symmetry with \( x_1 x_2 x_3 = 1 \) and the fugacities \( z_{\alpha} \) are associated to the topological symmetry.

It can be checked that the index (5.1) coincides with

\[
\mathbb{I}^{3d} \text{SQED}_3 (t, x_{\alpha}, z_{\alpha}; q) \times \mathbb{I}^{3d} \text{tHM} \left( t, \frac{z_1}{z_2}; q \right) (5.2)
\]

where \( \mathbb{I}^{3d} \text{SQED}_3 \) is the index (3.13) of the SQED₃ and \( \mathbb{I}^{3d} \text{tHM} \) is the index (2.24) for the 3d twisted hypermultiplet.

5.2 (3) – [5] and (2) – [5]

Next consider the case with \( N = 3 \), i.e. \( U(3) \) gauge theory with five fundamental hypermultiplets.

We can compute the index as

\[
\begin{align*}
\mathbb{I}^{3d} (3) – [5] (t, x_{\alpha}, z_{\alpha}; q) &= \frac{1}{3!} \left( \frac{q^{\frac{1}{2}} t^2; q)_\infty}{(q^2 t^{-2}; q)_\infty} \right)^3 \sum_{m_1, m_2, m_3 \in \mathbb{Z}} \oint \frac{ds_1}{2\pi is_1} \frac{ds_1}{2\pi is_2} \frac{ds_1}{2\pi is_3} \left( 1 - q^{\alpha - m_1 - m_2 - m_3} s_1^{\pm} s_2^{\mp} s_3^{\mp} \right) \\
&\times 3 \prod_{i=1}^{5} \prod_{\alpha=1}^{3} \left( \frac{q^{\frac{1}{2}} + \frac{|m_1|}{2} t^{-1} s_1^{\pm} x_{\alpha}^{\pm}; q}_\infty \right) \\
&\times q^{\frac{5}{4} |m_1| + \frac{5}{4} |m_2| + \frac{5}{4} |m_3| - \frac{|m_1 - m_2| + |m_1 - m_3| + |m_2 - m_3|}{2}} \\
&\times t^{-5|m_1| - 5|m_2| - 5|m_3| + 2|m_1 - m_2| + 2|m_1 - m_3| + 2|m_2 - m_3|} \left( \frac{z_1}{z_2} \right) (5.3)
\end{align*}
\]
where the fugacities \( x_\alpha \) are associated to the \( SU(5) \) flavor symmetry with \( \prod_{\alpha=1}^5 x_\alpha = 1 \) and the fugacities \( z_\alpha \) are associated to the topological symmetry.

We have checked that the index \((5.3)\) of \((3) - [5]\) agrees with

\[
\mathbb{I}^{3d} (2) - [5] (t, x_\alpha, z_\alpha; q) \times \mathbb{I}^{3d \text{ tHM}} \left( t, \frac{z_1}{z_2}; q \right) \tag{5.4}
\]

where

\[
\mathbb{I}^{3d} (2) - [5] (t, x_\alpha, z_\alpha; q) = \frac{1}{2} \left(\frac{q^{1/2} t^2; q}{q^{1/2} t^{-2}; q}\right)_\infty \sum_{m_1, m_2} \int \prod_{i=1}^2 \frac{ds_i}{2\pi s_i} \left( 1 - q^{\frac{|m_1 - m_2|}{2}} s_1^\pm s_2^\mp \right) \left( q^{1 + |m_1 - m_2|} t^2 s_1^\pm s_2^\mp; q \right)_\infty \times \prod_{i=1}^2 \prod_{\alpha=1}^5 \left( q^{\frac{3 + |m_1|}{2}} t^{-1} s_i^\pm x_\alpha^\pm; q \right)_\infty \times q^{\frac{5m_1 + 5m_2 - |m_1 - m_2|}{2}} t^{-5|m_1| - 5|m_2| + 2|m_1 - m_2|} \cdot \left( \frac{z_1}{z_2} \right) \tag{5.5}
\]
is the index of \((2) - [5]\).

5.3 \((N) - [2N - 1]\) and \((N - 1) - [2N - 1]\)

Now we are led to propose the generalization from the previous examples. For general \( N \) we expect the identity of indices

\[
\mathbb{I}^{3d} (N) - [2N - 1] (t, x_\alpha, z_\alpha; q) = \mathbb{I}^{3d} (N - 1) - [2N - 1] (t, x_\alpha, z_\alpha; q) \times \mathbb{I}^{3d \text{ tHM}} \left( t, \frac{z_1}{z_2}; q \right) \tag{5.6}
\]

where

\[
\mathbb{I}^{3d} (N) - [N_f] (t, x_\alpha, z_\alpha; q) = \frac{1}{N!} \left(\frac{q^{1/2} t^2; q}{q^{1/2} t^{-2}; q}\right)_\infty \sum_{m_1, \ldots, m_N} \int \prod_{i=1}^N \frac{ds_i}{2\pi s_i} \prod_{i<j} \left( 1 - q^{\frac{|m_i - m_j|}{2}} s_i^\pm s_j^\mp \right) \left( q^{1 + |m_i - m_j|} t^2 s_i^\pm s_j^\mp; q \right)_\infty \times \prod_{i=1}^N \prod_{\alpha=1}^{N_f} \left( q^{\frac{3 + |m_1|}{2}} t^{-1} s_i^\pm x_\alpha^\pm; q \right)_\infty \times q^{\sum_{i=1}^{N_f} |m_i| - \sum_{j<i} |m_i - m_j|} \cdot t^{-N_f} \sum_{i=1}^{N_f} |m_i| + 2 \sum_{i<j} |m_i - m_j| \cdot \left( \frac{z_1}{z_2} \right)^{\sum_{i=1}^{N_f} |m_i|} \tag{5.7}
\]
is the index for 3d \( \mathcal{N} = 4 \) \( U(N) \) gauge theory with \( N_f \) fundamental hypermultiplets.

6 Dualities of boundary conditions

Making use of the results in section 3 and 4 we construct the dual half-BPS boundary conditions for 4d \( \mathcal{N} = 4 \) gauge theories by including 3d \( \mathcal{N} = 4 \) gauge theories. In order to check the dualities conjectured from the action of S-duality in string theory, we compute the half-indices which encode the half-BPS boundary conditions for 4d \( \mathcal{N} = 4 \) gauge theories as well as the full-indices for 3d \( \mathcal{N} = 4 \) gauge theories. The indices for some half-BPS interfaces in 4d \( \mathcal{N} = 4 \) gauge theory were studied in 35 [72]. The analysis in this section and in section 7 provides more general examples which realize 4d-3d dualities.
Figure 8: (a) The quiver diagrams of the enriched Neumann b.c. 4d $U(N)\parallel 3d \ U(M)$ and its mirror. (b) The corresponding brane constructions. While the enriched Neumann b.c. 4d $U(N)\parallel 3d \ U(M)$ involves 4d gauge and 3d gauge symmetries, its mirror b.c. breaks 4d gauge symmetry.

6.1 4d $U(N)\parallel 3d \ U(M)$

Now let us consider the enriched Neumann b.c. $\mathcal{N}$ for 4d $\mathcal{N} = 4 \ U(N)$ gauge theory which is coupled to 3d $\mathcal{N} = 4 \ U(M)$ vector multiplet by the 3d bi-fundamental hypermultiplet. We denote this boundary condition by 4d $U(N)\parallel 3d \ U(M)$. It can be realized in the brane construction as in Figure 8. There are finite $M$ D3-branes stretched between the NS5-branes and semi-infinite $N$ D3-branes extending along the one side of the NS5-brane in the $x^6$ direction. The computation of the half-indices for the enriched b.c. 4d $U(N)\parallel 3d \ U(M)$ involves two sets of contour integral corresponding to the 4d gauge and 3d gauge symmetries. On the other hand, the mirror boundary condition is rather simple as it has no gauge symmetry.

6.1.1 4d $U(1)\parallel 3d \ U(1)$

We start from the boundary condition 4d $U(1)\parallel 3d \ U(1)$, that is Neumann b.c. $\mathcal{N}$ for 4d $\mathcal{N} = 4 \ U(1)$ gauge theory which couples to 3d $U(1)$ vector multiplet through the 3d bi-fundamental hypermultiplet. We label the magnetic fluxes for 3d $U(1)$ gauge theory by an integer $m$.

The half-index for the boundary condition 4d $U(1)\parallel 3d \ U(1)$ takes the form

$$
\II_{\mathcal{N}}^{4d \ U(1)\parallel 1}(t, z_\alpha; q) = \frac{(q)_\infty}{(q^2 t^{-2}; q)_\infty} \oint ds_1 \frac{(q^2 t^2; q)_\infty}{(q^2 t^{-2}; q)_\infty} \sum_{m \in \mathbb{Z}} \oint ds_2 \frac{2\pi is_2}{2\pi is_1} \cdot \frac{\left(q^{2|m|} t^{-|m|} q^{\frac{|m|}{2}} \cdot t^{-|m|} \left(z_1\right)^m \right)}{(q^2 + \frac{|m|}{t^{-1} s_1 s_2}; q)_\infty}. 
$$

(6.1)

where $z_\alpha$ are the fugacities for the topological symmetry. The contributions in the second line are 4d $U(1)$ gauge theory with Neumann b.c. $\mathcal{N}$ and 3d $U(1)$ vector multiplet. The third line is the contributions of the 3d bi-fundamental hyper and monopole operators.
The S-dual boundary condition, which we denote by 4d $U(1)[[1]]$ includes two D5'-branes. For the D5'-brane on which a single D3-brane end, the 4d $U(1)$ gauge theory should obey Dirichlet b.c. $D'$. In addition, the other D5'-brane interface would couple to a twisted hypermultiplet.

The half-index for the boundary condition 4d $U(1)[[1]]$ is

$$
\Pi_{D'}^{4d \ U(1)[[1]]} (t, z_\alpha; q) = \frac{(q^2 t^{-2}; q)_\infty}{(q^2 t^{-2}; q)_\infty} \cdot \frac{(q^2 t^{2}; q)_\infty}{(q^2 t^{2}; q)_\infty} \cdot \frac{(q^2 t^{2}; q)_\infty}{(q^2 t^{2}; q)_\infty}
$$

where $z_\alpha$ are now associated with the boundary global symmetry, that is the broken gauge symmetry. The half-index (6.2) coincides with the half-index (6.1).

### 6.1.2 4d $U(2)[3d \ U(1)]$

Next consider the boundary condition 4d $U(2)[3d \ U(1)]$, that is Neumann b.c. $N$ for 4d $N = 4$ $U(2)$ gauge theory which couples to 3d $U(1)$ vector multiplet through the 3d bi-fundamental hypermultiplet.

The half-index reads

$$
\Pi_{N}^{4d \ U(2)[3d \ U(1)]} (t, z_\alpha; q) = \frac{1}{2} (q^2 t^{-2}; q)_\infty \oint \frac{ds_1}{2\pi i s_1} \oint \frac{ds_2}{2\pi i s_2} \oint \frac{ds_3}{2\pi i s_3} \frac{(z_1^{2}; q)_\infty}{(z_1^{2}; q)_\infty} \frac{(z_2^{2}; q)_\infty}{(z_2^{2}; q)_\infty} \frac{(z_3^{2}; q)_\infty}{(z_3^{2}; q)_\infty}
$$

$$
\times \frac{(q^2 t^{2}; q)_\infty}{(q^2 t^{2}; q)_\infty} \sum_{m \in \mathbb{Z}} \cdot \frac{(q^{4} t^{-1} s_1^{\pm} s_3^{\pm}; q)_\infty}{(q^{4} t^{-1} s_1^{\pm} s_3^{\pm}; q)_\infty} \cdot \left( t^{-2|\bar{m}|} \right)^m.
$$

In the second line we have the contributions from 4d $U(2)$ gauge theory with Neumann b.c. $N$. In the third line we have the contributions from 3d $U(1)$ vector multiplet. The last line includes the contributions from the 3d bi-fundamental hyper and monopole operator.

The S-dual boundary condition for the boundary condition 4d $U(2)[3d \ U(1)]$ is identified with Dirichlet b.c. $D'$ for 4d $U(2)$ gauge theory [8] (see Figure 9).

As discussed in section 2.1.2, the half-index of Dirichlet b.c. $D'$ for 4d $U(2)$ gauge theory is given by

$$
\Pi_{D'}^{4d \ U(2)[3d \ U(1)]} (t, z_\alpha; q) = \frac{(q^2 t^{2}; q)_\infty}{(q^2 t^{2}; q)_\infty} \cdot \frac{(q^{2} t^{2}; q)_\infty}{(q^{2} t^{2}; q)_\infty} \cdot \frac{(q^{2} t^{2}; q)_\infty}{(q^{2} t^{2}; q)_\infty}
$$

We have confirmed that the half-indices (6.3) and (6.4) coincide with each other.
6.1.3 4d $U(3)|3d$ $U(1)$

Let us consider the boundary condition $4d$ $U(3)|3d$ $U(1)$, which is the enriched Neumann b.c. $\mathcal{N}$ for $4d$ $\mathcal{N} = 4$ $U(3)$ gauge theory with a coupling to $3d$ $U(1)$ vector multiplet through the $3d$ bi-fundamental hypermultiplet.

The half-index is

$$\Pi_{\mathcal{N}}(t, z; q) = \frac{1}{3!} \frac{(q^2 t^{-2}; q^3)^{3}}{(q^2 t^{-2}; q^3)^{3}} \left[ \prod_{i=1}^{3} \frac{ds_{i}}{2\pi i s_{i}} \prod_{i \neq j}^{3} \left( \frac{s_{i}}{s_{j}} \right)^{\infty} \prod_{m \in \mathbb{Z}^+} \int ds_{4} \frac{(q^{4} t^{2}; q^{3})^{\infty}}{(q^{4} t^{2}; q^{3})^{\infty}} \sum_{m \in \mathbb{Z}^+} \frac{\prod_{i=1}^{3} (q^{4} t^{-1} s_{i}^{\pm} s_{4}^{\pm}; q)^{\infty} q^{3|m|} t^{-3|m|} (z_{1} z_{2})^{m}}{(q^{4} t^{2} s_{i}^{\pm} s_{4}^{\pm}; q)^{\infty}} \right].$$

Under the action of S-duality, we find the boundary condition $4d$ $U(3)|[1]|$ which is realized by two D5'-branes. The first D5'-brane breaks the $U(3)$ gauge symmetry down to $U(1)$ and the second D5'-brane further breaks the $U(1)$. Hence the dual boundary condition involves the Nahm pole of rank 2 and the Dirichlet boundary condition $\mathcal{D}'$ for $U(1)$. In addition, as discussed in [12], we also expect that there are contributions to half-index corresponding to $4d$ $U(3)$ gauginos and scalars.
Then we have the half-index for the boundary condition 4d $U(3)[1]$
\[
\Pi_{\mathcal{D}'}^{4d \ U(3)[1]}(t, z_\alpha; q) = \frac{(q)_\infty (q^{\frac{3}{2}} t^{-2}; q)_\infty}{(q^{\frac{3}{2}} t^{-2}; q)_\infty (q t^{-4}; q)_\infty} \frac{(q)_\infty (q^{\frac{3}{2}} t^{-1} z_1^{\pm}; q)_\infty}{(q^{\frac{3}{2}} t^{-1} z_1^{\pm}; q)_\infty} \frac{(q^{\frac{3}{2}} t^{-3} z_2^{\pm}; q)_\infty}{(q^{\frac{3}{2}} t^{-3} z_2^{\pm}; q)_\infty}.
\]
(6.6)

The last terms are the contributions from 4d $U(3)$ gauginos and scalars which are not contained in the $U(1)$ gauge theory [12]. It follows that the half-index (6.5) agrees with (6.6).

6.1.4 4d $U(N)$|3d $U(1)$

For the boundary condition 4d $U(N)|3d \ U(1)$, we have the Neumann b.c. $N$ for 4d $\mathcal{N} = 4 \ U(N)$ gauge theory which couples to 3d $U(1)$ vector multiplet through the 3d bi-fundamental hypermultiplet.

The half-index is expressed as
\[
\Pi_N^{4d \ U(N)[1]}(t, z_\alpha; q) = \frac{1}{N!} \frac{\langle q \rangle^N_\infty}{\langle q z_1^{\pm} \rangle^N_\infty} \prod_{i=1}^{N} \frac{d s_i}{2 \pi i s_i} \prod_{i \neq j} \left(\frac{z_i^2}{z_j^2}; q\right)_\infty \prod_{i=1}^{N} \frac{d s_{N+1}}{2 \pi i s_{N+1}} \prod_{i=1}^{N} \left(\frac{q^{\frac{3}{2}} t^{2}; q^\infty}{(q^{\frac{3}{2}} t^{-2}; q^\infty) \sum_{m \in \mathbb{Z}} \frac{d s_{N+1}}{2 \pi i s_{N+1}}} \prod_{i=1}^{N} \left(\frac{q^{3 + m} s_{N+1}; q^\infty}{(q^{3 + m} t s_{N+1}; q^\infty)} \right) q^{\frac{m}{2} |m|} t^{-N |m|} \left(\frac{z_i^{\pm}}{z_j^{\pm}}\right)^{m} \right) \left(\frac{z_1^{\pm}}{z_2^{\pm}}\right)^{m}.
\]
(6.7)

The contributions in the second and third lines are 4d $U(N)$ SYM theory with Neumann b.c. $\mathcal{N}$ and 3d $U(1)$ gauge theory. The last line includes the contributions from the 3d bi-fundamental hypermultiplet and monopole operator with dimension $\Delta(m) = \frac{N |m|}{2}$.

The S-dual boundary condition, which we denote by 4d $U(N)[1]$, would involve the singular boundary condition specified by the Nahm pole of rank $(N - 1)$ corresponding to the D5'-brane interface and the Dirichlet b.c. $\mathcal{D}'$ corresponding to the other D5'-brane on which the D3-brane terminate. Besides, there would be the contributions to the half-index from the 4d $U(N)$ gauginos and scalars [12].

The half-index for the boundary condition 4d $U(N)[1]$ takes the form
\[
\Pi_{\mathcal{D}'}^{4d \ U(N)[1]}(t, z_\alpha; q) = \prod_{k=1}^{N-1} \frac{\left(q^{\frac{k+1}{2}} t^{-2(k-1)}; q\right)_\infty}{\left(q^{\frac{k+1}{2}} t^{-2k}; q\right)_\infty} \frac{\langle q \rangle^N_\infty}{\langle q z_1^{\pm} \rangle^N_\infty} \prod_{i=1}^{N} \frac{d s_{N+1}}{2 \pi i s_{N+1}} \prod_{i=1}^{N} \left(\frac{q^{3 + m} s_{N+1}; q^\infty}{(q^{3 + m} t s_{N+1}; q^\infty)} \right) q^{\frac{m}{2} |m|} t^{-N |m|} \left(\frac{z_i^{\pm}}{z_j^{\pm}}\right)^{m}.
\]
(6.8)

It is expected that the half-index (6.7) gives the same answer as the half-index (6.8).

6.1.5 4d $U(3)$|3d $U(2)$

Now consider the case where 3d gauge symmetry is non-Abelian. For 3d $U(2)$ gauge theory, the canonical R-charge of bare monopole may not be positive when the rank of 4d bulk gauge symmetry
is less than three, which does not ensure the convergence of the half-indices. Thus we start with the boundary condition 4d $U(3)|3d U(2)$, where we have the Neumann b.c. $N$ for 4d $\mathcal{N} = 4$ $U(3)$ gauge theory coupled to 3d $U(2)$ vector multiplet via 3d hypermultiplet transforming as $(\mathbf{3}, \overline{2}) \oplus (\overline{3}, 2)$ under the $U(3) \times U(2)$ gauge symmetry.

The half-index is

$$
\mathbb{I}^{4\mathcal{N}}_{U(3)|[2]}(t, z_{\alpha}; q) = \frac{1}{3!} \left[ \sum_{m_1, m_2} \frac{1}{2 \pi i s_1} \prod_{i=1}^{3} ds_i \prod_{i \neq j} \left( \frac{q^{\frac{2}{3}} t^{2}; q}{q^{\frac{2}{3}} t^{2}; q} \right)^{\infty} \left( \frac{q^8 t^{-2} z^{\frac{3}{2}}; q}{q^8 t^{-2} z^{\frac{3}{2}}; q} \right)_{\infty}^{1} \right]
$$

Here the second and third lines include half-index for 4d $U(3)$ gauge theory with Neumann b.c. $N$ and full-index for 3d $U(2)$ gauge theory. The last line has the contributions from the 3d bi-fundamental hypermultiplet and monopole operator of dimension $\Delta(m) = \frac{3|m_1|}{2} + \frac{3|m_2|}{2} - |m_1 - m_2|$.

The S-dual boundary condition, which we represent as 4d $U(3)|[2]|$, contains the Dirichlet b.c. $D'$ which is realized by the D5$'$ brane interface on which a single D3-brane end and the singular boundary condition of rank 2 corresponding to the other D5$'$-brane on which two D3-branes terminate. The additional contributions would appear from the broken 4d gauge theory.

We find that the half-index (6.9) agrees with

$$
\mathbb{I}^{4\mathcal{D}'}_{U(3)|[2]}(t, z_{\alpha}; q) = \frac{1}{3!} \left[ \sum_{m_1, m_2} \frac{1}{2 \pi i s_1} \prod_{i=1}^{3} ds_i \prod_{i \neq j} \left( \frac{q^{\frac{2}{3}} t^{2}; q}{q^{\frac{2}{3}} t^{2}; q} \right)^{\infty} \left( \frac{q^8 t^{-2} z^{\frac{3}{2}}; q}{q^8 t^{-2} z^{\frac{3}{2}}; q} \right)_{\infty}^{1} \right]
$$

The second line has the expected contributions, which are the half-index of Dirichlet b.c. $D'$ for 4d $U(1)$ gauge theory and half-index of Nahm' b.c. for 4d $U(2)$ gauge theory. The next line would correspond to the two kinds of contributions from the broken 4d gauge theory. The result indicates that there remain two D3-branes between the D5$'$-branes unlike the boundary condition 4d $U(N)|[1]$.

6.1.6 4d $U(4)|3d U(2)$

For the boundary condition 4d $U(4)|3d U(2)$, we find the Neumann b.c. $N$ for 4d $\mathcal{N} = 4$ $U(4)$ gauge theory with a coupling to 3d $U(2)$ vector multiplet via 3d hypermultiplet transforming as $(\mathbf{4}, \overline{2}) \oplus (\overline{4}, 2)$ under the $U(4) \times U(2)$ gauge symmetry.
We can evaluate the half-index as

\[ \Pi_{\mathcal{N}}^{4d} U(4)|^{(2)}(t, z_\alpha; q) \]

\[ = \frac{1}{4!} (q^{\frac{3}{2}} t^{-2}; q)_\infty \prod_{i=1}^{4} ds_i \prod_{i \neq j} (q^{\frac{3}{2} t^{-2} s_i}; q)_\infty \]

\[ \times \frac{1}{2} (q^{\frac{3}{2} t^{-2}}; q)_\infty \sum_{m_1, m_2 \in \mathbb{Z}} \oint ds_5 \oint ds_6 \left( \frac{1 - q^{\frac{|m_1 - m_2|}{2}} s_5^\pm s_6^\mp}{\left( q^{\frac{1+|m_1 - m_2|}{2}} t^2 s_5^\pm s_6^\mp; q \right)_\infty} \right) \]

\[ \times \prod_{i=1}^{4} \prod_{j=5}^{6} \frac{(q^{\frac{3}{2} t^{-2} - 2}; q^4; q)_\infty}{(q^{\frac{3}{2} t^{-2}}; q^4; q^4; q^4)_\infty} \]

\[ \times \frac{(q^{\frac{3}{2} t^{-2} s_1^\pm s_2^\pm}; q)_\infty}{(q^{\frac{3}{2} t^{-2} s_1^\pm s_2^\pm}; q)_\infty} \]

\[ \times \frac{(q^{\frac{3}{2} t^{-2} z_1^\pm z_2^\pm}; q)_\infty}{(q^{\frac{3}{2} t^{-2} z_1^\pm z_2^\pm}; q)_\infty} \]

(6.11)

The contributions in the second and third lines correspond to the 4d $U(4)$ SYM theory with Neumann b.c. $\mathcal{N}$ and 3d $U(2)$ gauge theory. The contributions from the last line are the 3d bi-fundamental hyper and monopole operator of R-charge $\Delta(m) = 2|m_1| + 2|m_2| - |m_1 - m_2|$.

The S-dual boundary condition 4d $U(4)|^{[2]}$ should involve two singular boundary conditions characterized by Nahm poles of rank 2 as the number of D3-branes jumps from 4 to 2 across the D5′-brane interface and from 2 to 0 across the another D5′-brane.

In fact, the half-index (6.11) coincides with

\[ \Pi_{\mathcal{D}'}^{4d} U(4)|^{[2]}(t, z_\alpha; q) \]

\[ = \frac{(q^{\frac{3}{2} t^{-2}}; q)_\infty}{(q^{\frac{3}{2} t^{-2}}; q)_\infty} \]

\[ \times \frac{(q^{\frac{3}{2} t^{-2} s_1^\pm s_2^\pm}; q)_\infty}{(q^{\frac{3}{2} t^{-2} s_1^\pm s_2^\pm}; q)_\infty} \]

(6.12)

This contains a square of half-index of Nahm’ boundary condition for $U(2)$ gauge theory in the second line and additional contributions in the next line. As in the half-index (6.10) for the boundary condition 4d $U(3)|^{[2]}$, there are two types of contributions from 4d local operators.

### 6.1.7 4d $U(5)|3d$ $U(2)$

Let us consider the boundary condition 4d $U(5)|3d$ $U(2)$ which includes Neumann b.c. $\mathcal{N}$ for 4d $U(5)$ SYM theory coupled to 3d $U(2)$ gauge theory through the 3d bi-fundamental hypermultiplet.
Similarly, we can compute the half-index for the boundary condition 4d \( U(5)|3d \ U(2) \) as

\[
\prod_{N}^{4d} U(5) | 2I(2) | (t, z_{\alpha}; q) = \frac{(q)^{5}}{5} \prod_{i=1}^{5} ds_{i} \prod_{i \neq j} \left( \frac{q_{j} t^{-2} s_{j}}{s_{j}} \right)_{\infty} \sum_{k_{1}, k_{2}, \ldots} \frac{1}{2} \left( \frac{q_{j} t^{-2}; q_{j}}{q_{j} t^{-2}; q_{j}} \right)_{\infty} \int_{0}^{5} ds_{6} \int_{0}^{5} ds_{7} \left( 1 - q^{\frac{1}{2}} \frac{q_{j} t^{-2}}{s_{j} + s_{j} \pm 2} \right) \left( q^{\frac{1}{2}} \frac{q_{j} t^{-2}}{s_{j} + s_{j} \pm 2} ; q \right)_{\infty} \left( q^{\frac{1}{2}} \frac{q_{j} t^{-2}}{s_{j} + s_{j} \pm 2} ; q \right)_{\infty}
\]

\[
\times \prod_{i=1}^{7} \left( q^{\frac{1}{2}} \frac{q_{j} t^{-2}}{s_{j} + s_{j} \pm 2} \right)_{\infty} \left( q^{\frac{1}{2}} \frac{q_{j} t^{-2}}{s_{j} + s_{j} \pm 2} ; q \right)_{\infty} \left( q^{\frac{1}{2}} \frac{q_{j} t^{-2}}{s_{j} + s_{j} \pm 2} ; q \right)_{\infty} \left( q^{\frac{1}{2}} \frac{q_{j} t^{-2}}{s_{j} + s_{j} \pm 2} ; q \right)_{\infty}
\]

(6.13)

The S-dual boundary condition would involve two singular boundary conditions specified by the Nahm pole of rank 3 and rank 2 corresponding to the two D5'-branes.

One can check that the half-index (6.13) agrees with

\[
\prod_{N}^{4d} U(5) | 2I(2) | (t, z_{\alpha}; q) = \frac{(q)^{5}}{5} \prod_{i=1}^{5} ds_{i} \prod_{i \neq j} \left( \frac{q_{j} t^{-2} s_{j}}{s_{j}} \right)_{\infty} \sum_{k_{1}, k_{2}, \ldots} \frac{1}{2} \left( \frac{q_{j} t^{-2}; q_{j}}{q_{j} t^{-2}; q_{j}} \right)_{\infty} \int_{0}^{5} ds_{6} \int_{0}^{5} ds_{7} \left( 1 - q^{\frac{1}{2}} \frac{q_{j} t^{-2}}{s_{j} + s_{j} \pm 2} \right) \left( q^{\frac{1}{2}} \frac{q_{j} t^{-2}}{s_{j} + s_{j} \pm 2} ; q \right)_{\infty} \left( q^{\frac{1}{2}} \frac{q_{j} t^{-2}}{s_{j} + s_{j} \pm 2} ; q \right)_{\infty}
\]

\[
\times \prod_{i=1}^{7} \left( q^{\frac{1}{2}} \frac{q_{j} t^{-2}}{s_{j} + s_{j} \pm 2} \right)_{\infty} \left( q^{\frac{1}{2}} \frac{q_{j} t^{-2}}{s_{j} + s_{j} \pm 2} ; q \right)_{\infty} \left( q^{\frac{1}{2}} \frac{q_{j} t^{-2}}{s_{j} + s_{j} \pm 2} ; q \right)_{\infty} \left( q^{\frac{1}{2}} \frac{q_{j} t^{-2}}{s_{j} + s_{j} \pm 2} ; q \right)_{\infty}
\]

(6.14)

As we expect, the half-index (6.14) contains the product of half-index of Nahm' boundary condition for \( U(3) \) gauge theory and half-index of Nahm' boundary condition for \( U(2) \) gauge theory. Again it also involves two types of contributions corresponding to the broken \( U(2) \) gauge symmetry.

**6.1.8 4d \( U(N) \mid 3d \ U(M) \)**

Taking into account the analysis above, we would like to present the general half-index of the boundary condition 4d \( U(N) \mid 3d \ U(M) \), i.e. Neumann b.c. \( N \) for 4d \( N = 4 \) \( U(N) \) SYM theory coupled to 3d \( \mathcal{N} = 4 \) \( U(M) \) vector multiplet through the 3d hypermultiplet transforming as \( (N, \bar{M}) \oplus (\overline{N}, M) \) under \( U(N) \) and \( U(M) \) gauge symmetries.
We can compute the half-index as
\[
\begin{align*}
\Pi^4_{U(N)(M)}(t, z; q) &= \frac{1}{N!} \left( \frac{q^N}{q^{\frac{1}{2} t^{-2}}; q^N} \right) \sum_{i=1}^{N} \frac{d s_i}{2 \pi i s_i} \prod_{i \neq j} \left( q^{rac{1}{2} t^{-2} s_i}; q^{s_j} \right) \\
\times \prod_{i=1}^{N} \prod_{j=1}^{N} \left( \frac{1 - q^{s_{i<j}}}{1 - q^{s_{i>j}}} \right) \left( q^{\frac{1}{2} t^{-2} s_i}; q^{s_j} \right)^{\infty} \\
\times q^{\sum_{i=1}^{N} m_i} \prod_{i<j} \left( 1 - q^{s_i s_j} \right) \left( q^{\frac{1}{2} t^{-2} s_i}; q^{s_j} \right)^{\infty} \\
\times \prod_{i=1}^{N} \prod_{j=1}^{M} \left( q^{\frac{1}{2} t^{-2} s_i}; q^{s_j} \right)^{\infty}
\end{align*}
\]

The contributions in the second line and third line are the half-index for 4d $U(N)$ gauge theory satisfying Neumann b.c. $\mathcal{N}$ and full-index for 3d $U(M)$ gauge theory respectively. The contributions in the fourth line count the 3d bi-fundamental hypermultiplet. The last line involves the contribution from the monopole operator whose canonical R-charge is
\[
\Delta(m) = \sum_{i=1}^{M} \frac{N m_i}{2} - \sum_{i<j} |m_i - m_j|.
\]

We assume the condition $N \geq 2M - 1$, which guarantees that all monopole operators are above the unitarity bound so that the half-indices are convergent.

The S-dual boundary condition 4d $U(N)[|M|]$ is associated with two D5'-branes. The D5'-brane defect would break the 4d gauge group $U(N)$ down to $U(M)$ block-diagonal subgroup. This leads to the singular boundary condition specified by the Nahm pole of rank $(N - M)$. The another D5'-brane on which $M$ D3-branes terminate requires the singular boundary specified by the Nahm pole of rank $M$. There would be $M$ different types of contributions to half-index from broken gauge theory characterized by the Nahm pole of rank $(N - M)$, as discussed in [12].

The half-index for the boundary condition 4d $U(N)[|M|]$ takes the form
\[
\Pi^4_{U(N)[|M|]}(t, z; q) = \prod_{k=1}^{N-M} \left( q^{\frac{1}{2} t^{-(k-1)}}; q^{\infty} \right)^{\infty} \prod_{l=1}^{M} \left( q^{\frac{1}{2} t^{-2(l-1)}}; q^{\infty} \right)^{\infty}
\]

We expect that the half-index [6.15] for the enriched Neumann boundary condition 4d $U(N)[3d U(M)$ is equal to the half-index [6.17] for its dual boundary condition 4d $U(N)[|M|]$.

6.1.9 4d $U(3)|3d U(2) \times U(1)$

Making use of the basic identity of half-indices [6.15] and [6.17], one can engineer a variety of Nahm poles associated with multiple D5-branes.
Let us consider the enriched Neumann b.c. $4d\ U(3)|(2) - (1)|$ which is Neumann b.c. $\mathcal{N}$ for $4d\ U(3)$ gauge theory with a coupling to $3d$ quiver gauge theory $(2) - (1)$. It is expected to be dual to the Dirichlet b.c. $\mathcal{D}'$, i.e. a trivial Nahm pole boundary condition, for $4d\ \mathcal{N} = 4\ U(3)$ gauge theory.

The half-index of the enriched Neumann b.c. $4d\ U(3)|(2) - (1)|$ is evaluated as

$$\Pi_{\mathcal{N}}^{4d\ U(3)|(2) - (1)|}(t, z_\alpha; q)$$

\begin{equation}
= \frac{1}{3!} \frac{(q)^3}{(q^2 t^{-2}; q)_\infty} \prod_{i=1}^3 \frac{ds_i}{2\pi is_i} \prod_{i\neq j} \frac{\left(\frac{q}{s_i}; q\right)_{\infty}}{\left(\frac{q}{s_i}; q\right)_{\infty}} \prod_{m_1, m_2 \in \mathbb{Z}} \left(\frac{1}{2\pi is_4} \frac{ds_5}{2\pi is_5} \left(1 - q^{m_1-m_2} s_4^\pm s_5^\mp q^{m_1-m_2} t^\pm s_4^\pm s_5^\mp q^{i+|m_1-m_2| t^\pm s_4^\pm s_5^\mp q_{\infty}}\right) \right)
\end{equation}

\begin{equation}
\times \left(\frac{(q^2 t^{-2}; q)_{\infty}^2}{(q^2 t^{-2}; q)_{\infty}} \sum_{m_3 \in \mathbb{Z}} \int ds_6 \prod_{i=1}^5 \frac{1}{2\pi is_6} \frac{(q^{2 + \frac{|m_1-3|}{2}} t^{-1} s_i^\pm s_j^\mp q_{\infty})}{(q^{2 + \frac{|m_1-3|}{2}} t^{-1} s_i^\pm s_j^\mp q_{\infty})} \prod_{i=4}^5 \frac{(q^{2 + \frac{|m_1-3|}{2}} t^{-1} s_i^\pm s_j^\mp q_{\infty})}{(q^{2 + \frac{|m_1-3|}{2}} t^{-1} s_i^\pm s_j^\mp q_{\infty})} \right)
\end{equation}

\begin{equation}
\times \left(\frac{\left(z_1 \right)^{m_1+m_2}}{\left(z_2 \right)^{m_3}} \frac{\left(z_2 \right)^{m_3}}{\left(z_1 \right)^{m_1+m_2}} \right)
\end{equation}

In fact, the half-index (6.18) agrees with the half-index of Dirichlet b.c. $\mathcal{D}'$ for $4d\ U(3)$ gauge theory

$$\Pi_{\mathcal{D}'}^{4d\ U(3)}(t, z_\alpha; q)$$

\begin{equation}
= \frac{(q)^3}{(q^2 t^{-2}; q)_{\infty}} \prod_{i=1}^3 \frac{\left(q^{z_1^\pm z_2^\mp q_{\infty}}(q^{z_2^\pm z_3^\mp q_{\infty}}(q^{z_3^\pm z_1^\mp q_{\infty}}(q^{z_2^\pm z_3^\mp q_{\infty}}(q^{2 t^{-2} z_1^\pm z_2^\mp z_3^\mp q_{\infty}}q^{2 t^{-2} z_2^\pm z_3^\mp z_1^\mp q_{\infty}})q^{2 t^{-2} z_3^\pm z_1^\mp z_2^\mp q_{\infty}})q^{2 t^{-2} z_1^\pm z_2^\mp z_3^\mp q_{\infty}})q^{2 t^{-2} z_2^\pm z_3^\mp z_1^\mp q_{\infty}})q^{2 t^{-2} z_3^\pm z_1^\mp z_2^\mp q_{\infty}})q^{2 t^{-2} z_1^\pm z_2^\mp z_3^\mp q_{\infty}}\right)}{(q^2 t^{-2}; q)_{\infty}}}
\end{equation}

\textbf{6.1.10 $4d\ U(3)|3d\ U(N - 1) \times \cdots \times U(1)$}

Now we present the half-index of the enriched Neumann boundary condition $4d\ U(3)|3d\ U(N - 1) \times \cdots \times U(1)$, which is the Neumann b.c. $\mathcal{N}$ for $4d\ \mathcal{N} = 4\ U(N)$ gauge theory coupled to $3d\ \mathcal{N} = 4\ U(N - 1)$ quiver gauge theory through the 3d hypermultiplet transforming as $(\mathbf{N}, \mathbf{N-1}) \oplus (\mathbf{N}, \mathbf{N-1})$ under the $U(N) \times U(N - 1)$ gauge group. This is expected to be dual to the Dirichlet boundary condition $\mathcal{D}'$ for $4d\ \mathcal{N} = 4\ U(N)$ SYM theory.

One can compute the half-index of the enriched Neumann boundary condition $4d\ U(3)|3d\ U(N - 1) \times \cdots \times U(1)$.
1) \times \cdots \times U(1) \text{ as}

\begin{align*}
\Pi_{N}^{4d} U(N)(N-1) - (N-2) - \cdots - (1) & (t, z_\alpha; q)

= \frac{1}{N!} \left( \frac{(q)}{\sqrt{q}} \right)_{\infty}^{N} \int \prod_{i=1}^{N} \frac{ds_i}{2 \pi i s_i} \prod_{i \neq j} \left( \frac{q^{\frac{1}{2}} t^{-2} z_i}{s_j} \right)_{\infty}

\times \prod_{k=1}^{N-1} \left[ \frac{1}{k!} \left( \frac{q^{\frac{1}{2}} t^2; q}{} \right)^{k} \sum_{m_1^{(k)}, \ldots, m_k^{(k)}} \left( \frac{q^{\frac{1}{2}} t^2}{s_j} \right)^{k} \prod_{j=1}^{k} \left( \frac{q^{\frac{1}{2}} t^2 z_i}{s_j} \right)_{\infty} \right]

\times \prod_{i=1}^{N} \prod_{j=1}^{N-1} \left( \frac{q^{\frac{1}{2}} + \frac{|m_1^{(k)}|}{2}}{t^{\frac{1}{2}} \sqrt{s_j (N-1)^{\frac{1}{2}}} ; q} \right)_{\infty} \prod_{k=1}^{N-2} \prod_{i=1}^{k} \prod_{j=1}^{k+1} \left( \frac{q^{\frac{1}{2}} + \frac{|m_1^{(k)} - m_{(k+1)}^{(k+1)}|}{2}}{t^{\frac{1}{2}} \sqrt{s_j (k+1)^{\frac{1}{2}}} ; q} \right)_{\infty}

\times \sum_{k=1}^{N-1} \sum_{j=1}^{N-2} |m_j^{(k)}| - |m_i^{(k)}| + \sum_{k=1}^{N-1} \sum_{j=1}^{k+1} \frac{|m_i^{(k)} - m_j^{(k+1)}|}{2} - \sum_{k=1}^{N-1} \sum_{j=1}^{k+1} |m_i^{(k)} - m_j^{(k+1)}| + \sum_{k=1}^{N-1} \sum_{j=1}^{k+1} |m_i^{(k)} - m_j^{(k+1)}| + 2 \sum_{k=1}^{N-1} \sum_{j=1}^{k+1} |m_i^{(k)} - m_j^{(k+1)}|

\times \prod_{k=1}^{N-1} \left( \frac{z_k}{z_{k+1}} \right)^{m_1^{(k)}} \left( \frac{z_N}{z_1} \right)^{m_1^{(k)} + \sum_{i=1}^{N-1} m_i^{(k)}}.
\end{align*}

The half-index \red{[6.20]} would agree with the half-index of Dirichlet boundary $\mathcal{D}'$ for 4d $\mathcal{N} = 4$ $U(N)$ gauge theory:

\begin{align*}
\Pi_{\mathcal{D}'}^{4d} U(N)(t, z_\alpha; q) &= \left( \frac{q}{\sqrt{q}} \right)_{\infty}^{N} \prod_{i \neq j} \left( \frac{q^{\frac{1}{2}} t^{-2} z_i}{s_j} \right)_{\infty}.
\end{align*}

### 6.2 4d $U(N)|3d$ $(M) - [2M]$

Let us consider the enriched Neumann b.c. $\mathcal{N}$ for 4d $\mathcal{N} = 4$ $U(N)$ gauge theory coupled to balanced 3d $\mathcal{N} = 4$ $U(M)$ gauge theory with $2M$ hypermultiplets. We represent this boundary condition by 4d $U(N)|3d$ $(M) - [2M]$. The corresponding quiver diagram and brane construction are shown in Figure \red{[10]}. In contrast to the enriched boundary conditions in section \red{6.1}, the dual boundary conditions can admit both 4d and 3d gauge symmetries. The dualities of boundary conditions produce a generalization of mirror symmetry for $(N) - [2N]$ discussed in section \red{4.2}.

### 6.2.1 4d $U(1)|3d$ $(1) - [2]$

Consider the simplest case where 4d $U(1)$ gauge theory with Neumann b.c. $\mathcal{N}$ couples to $(1) - [2]$ or equivalently $T\text{[SU}(2)]$ through the 3d $\mathcal{N} = 4$ bi-fundamental hypermultiplet. We denote this by 4d $U(1)|3d$ $(1) - [2]$.
Figure 10: (a) The quiver diagrams of the enriched Neumann b.c. 4d $U(N)|3d (M) − [2M]$ and its mirror. (b) The corresponding brane constructions.

We can evaluate the half-index as

$$\mathcal{I}_{4d}^{U(1)/(1)−[2]}(t, x_\alpha, z_\alpha; q) = (q)_{\infty} \int \frac{ds_1}{2\pi is_1} \frac{(q^{\frac{1}{2}}t^2; q)_{\infty}}{(q^{\frac{1}{2}}t^{-2}; q)_{\infty}} \sum_{m \in \mathbb{Z}} \int \frac{ds_2}{2\pi is_2} \frac{(q^{\frac{1}{2}}t^{-2}; q)_{\infty}}{(q^{\frac{1}{2}}t^2; q)_{\infty}} \prod_{\alpha=1}^2 \frac{(q^{\frac{1}{2}}t^{-1}s_1^\pm s_2^\pm; q)_{\infty}}{(q^{\frac{1}{2}}t^{1}s_1^\pm s_2^\pm; q)_{\infty}} \cdot \frac{(q^{\frac{1}{2}}t^{-1}s_1^\pm s_2^\pm; q)_{\infty}}{(q^{\frac{1}{2}}t^{1}s_1^\pm s_2^\pm; q)_{\infty}} \cdot \frac{1}{q^{\frac{1}{2}}t^{-1}s_1^\pm s_2^\pm; q)_{\infty}}.$$  

This is a generalization of full-index (3.9) for $T[SU(2)]$. The canonical R-charge of bare monopole is $\Delta(m) = \frac{3|\alpha|}{2}$, which has contributions from two fundamental hypermultiplets and additional bi-fundamental hypermultiplet.

One can read the dual boundary condition from the brane setup in Figure 10. Although $T[SU(2)]$ is self-mirror, the boundary condition 4d $U(1)|3d (1) − [2]$ is not self-mirror. The dual boundary condition is enriched Neumann b.c. $\mathcal{N}'$ for 4d $U(1)$ gauge theory, which has a co-dimension one defect on which 3d charged twisted hypermultiplet lives and a coupling through bi-fundamental twisted hyper to twisted SQED$_1$, i.e. 3d $\mathcal{N} = 4 \tilde{U}(1)$ gauge theory with a charged twisted hypermultiplet.
Then the half-index for the dual boundary condition is given by

\[
\sum_{\mathcal{N}'}^{4d \ U(1)+ \ \text{hyper} \ |_{\text{b,c.}}^{(3)}}(t, x_\alpha, z_\alpha; q) = \frac{(q)_{\infty}}{(q^2 t^2; q)_{\infty}} \int \frac{ds_1}{2\pi i s_1} \frac{ds_2}{2\pi i s_2} \sum_{m \in \mathbb{Z}} \prod_{i=1}^{4d \ U(1)} \left(\frac{q^2 + i|t| s_i^+ s_i^-; q)_{\infty}}{q^2 t^{-2} s_i^\pm; q)_{\infty}} \right)^m.
\]

There are three kinds of twisted hypermultiplet. One lives on a defect in 4d \(\mathcal{N} = 4\ U(1)\) gauge theory which has no contribution to the canonical R-charge of bare monopole. The other two are the bi-fundamental twisted hyper which couples to 4d \(U(1)\) gauge theory and 3d \(U(1)\) twisted vector multiplet and the fundamental twisted hyper in 3d SQED\(_1\). They have the contributions to the canonical R-charge of bare monopole.

We have confirmed that the half-index \(6.22\) agrees with \(6.23\).

### 6.2.2 4d \(U(2)|3d\ (1) - [2]\)

Next consider the enriched Neumann boundary condition 4d \(U(2)|3d\ (1) - [2]\), that is Neumann b.c. \(\mathcal{N}\) for \(U(2)\) gauge theory with a coupling to \(T[SU(2)]\) through the 3d boundary bi-fundamental hypermultiplet.

The half-index is given by

\[
\sum_{\mathcal{N}'}^{4d \ U(2)|1) - [2]}(t, x_\alpha, z_\alpha; q) = \frac{1}{2} \frac{(q^2 t^2; q)_{\infty}}{(q^2 t^{-2}; q)_{\infty}} \int \frac{ds_1}{2\pi i s_1} \frac{ds_2}{2\pi i s_2} \frac{(s_2^\pm; q)_{\infty}}{(s_1^\pm; q)_{\infty}} \prod_{m \in \mathbb{Z}} \frac{(q^2 + i|t| s_i^+ s_i^-; q)_{\infty}}{(q^2 t^{-2} s_i^\pm; q)_{\infty}} \left(\frac{q^2 t^{-2} s_i^\pm; q)_{\infty}}{q^2 t^2; q)_{\infty}} \right)^m.
\]

For the S-dual boundary condition, we see from Figure 10 that the D5’-brane defect breaks 4d gauge group down to \(U(1)\). In contrast to the case with equal numbers of D3-branes, there is no fundamental twisted hypermultiplet at the defect. However, there are contributions to the half-index from the broken \(U(2)\) gauge theory \([12]\). The surviving 4d \(U(1)\) gauge theory should obey the enriched Neumann b.c. \(\mathcal{N}'\) corresponding to the NS5’-brane with a coupling to twisted SQED\(_1\) through the 3d bi-fundamental twisted hyper.

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In fact, we can check that the half-index \([6.24]\) coincides with the following half-index

\[
\Pi_{N'}^{4d} U(2) \rightarrow U(1) \left| \begin{array}{l}
(1) \\
(t, x_\alpha, z_\alpha; q)
\end{array} \right.
\]

\[
= \frac{(q)_\infty}{(q^2 t^2; q)_\infty} \oint \frac{ds_1}{2\pi is_1} \frac{(q^2 t^{-2}; q)_\infty}{(q^2 t^2; q)_\infty} \sum_{m\in\mathbb{Z}} \oint \frac{ds_2}{2\pi is_2} \frac{(q^2 t^{-2}; q)_\infty}{(q^2 t^2; q)_\infty} \times \frac{(qs_1^2 z_1; q)_\infty}{(qs_1^2 z_1; q)_\infty} \cdot \left( q^{\frac{1}{2} + \frac{|m|}{2}} ts_2^\pm z_2; q \right)_\infty \cdot \left( q^{\frac{1}{2} + \frac{|m|}{2}} t^{-1}s_2^\pm z_2; q \right)_\infty \cdot \left( q^{\frac{1}{2} + \frac{|m|}{2}} t^{-1}s_2^\pm z_2; q \right)_\infty \cdot \left( x_1 x_2 \right)^m . \quad (6.25)
\]

The contributions in the second line include the half-index of Dirichlet b.c. \(D'\) for 4d \(U(1)\) gauge theory corresponding to the reduction \(U(2) \rightarrow U(1)\) of 4d gauge symmetry, the half-index of Neumann b.c. \(N'\) for surviving 4d \(U(1)\) gauge theory and full-index of 3d \(U(1)\) vector multiplet. The first factors in the last line correspond to the contributions from 4d gauginos and scalars in the broken \(U(2)\) gauge theory [12]. The other terms are the 3d bi-fundamental and fundamental twisted hypers as well as monopole operators with \(\Delta(m) = |m|\).

### 6.2.3 4d \(U(2)|3d\) (2) \(-[4]\)

Let us consider the enriched Neumann b.c. \(N\) for 4d \(U(2)\) gauge theory with a coupling to (2) \(-[4]\) via 3d bi-fundamental hyper, which we describe as 4d \(U(2)|3d\) (2) \(-[4]\).

The half-index of the enriched Neumann b.c. 4d \(U(2)|3d\) (2) \(-[4]\) is

\[
\Pi_{N'}^{4d} U(2)|3d|^{(2)} \left| \begin{array}{l}
(2) \left| \begin{array}{l}
(1) \\
(t, x_\alpha, z_\alpha; q)
\end{array} \right.
\end{array} \right.
\]

\[
= \frac{1}{2} \left( q^2 z_\alpha; q \right)_\infty \oint \frac{ds_1}{2\pi is_1} \frac{ds_2}{2\pi is_2} \frac{\left( \frac{z_1}{s_1}; q \right)_\infty \left( \frac{z_2}{s_1}; q \right)_\infty}{\left( q^2 t^{-2}; q \right)_\infty} \times \frac{1}{2} \left( q^2 t^2; q \right)_\infty \sum_{m_1, m_2 \in \mathbb{Z}} \oint \frac{ds_3}{2\pi is_3} \frac{ds_4}{2\pi is_4} \frac{\left( 1 - q^{|m_1 - m_2|} s_3^{\pm} s_4^{\mp}; q \right)_\infty \left( \frac{1 + |m_1 - m_2|}{2} t^2 s_3^{\pm} s_4^{\mp}; q \right)_\infty}{\left( q^\frac{1}{2} + \frac{|m_1 - m_2|}{2} t^{-1}s_3^{\pm} s_4^{\mp}; q \right)_\infty} \times \prod_{i=1}^{2} \prod_{j=3}^{4} \frac{q^\frac{1}{2} + \frac{|m_1 - m_2|}{2} t^{-1}s_3^{\pm} s_4^{\mp}; q \infty}{(q^\frac{1}{2} + \frac{|m_1 - m_2|}{2} t^{-1}s_3^{\pm} s_4^{\mp}; q \infty)} \times q^{\frac{1}{2} + \frac{|m_1 + m_2|}{2} t^{-1}s_3^{\pm} s_4^{\mp}; q \infty} \cdot \left( \frac{x_1}{x_2} \right)^{m_1 + m_2} \cdot \left( q^\frac{1}{2} + \frac{|m_1 - m_2|}{2} t^{-1}s_3^{\pm} s_4^{\mp}; q \infty \right) . \quad (6.26)
\]

This is a generalization of full-index \([4.4]\) for (2) \(-[4]\). The second line is the half-index of Neumann b.c. \(N\) for 4d \(U(2)\) gauge theory. The fourth line includes the contributions from the boundary 3d bi-fundamental hypermultiplet. The bare monopole now has dimension \(\Delta(m) = 3|m_1| + 3|m_2| - |m_1 - m_2|\).

The dual boundary condition is the enriched Neumann b.c. \(N'\) for 4d \(U(2)\) gauge theory. The 4d \(U(2)\) gauge theory has a defect 3d fundamental twisted hypermultiplet corresponding to D5'-brane and additionally couples to 3d twisted quiver gauge theory \((2) \rightarrow (1)\) through the 3d boundary twisted hypermultiplet transforming as \((2, \overline{2}) \oplus (\overline{2}, 2)\) under the 4d \(U(2)\) gauge symmetry and 3d \(U(2)\) gauge symmetry factor at the end of the quiver.
The half-index for the dual enriched boundary condition, which we denote by 4d $U(2)+\text{thyper}$, is given by

$$
\begin{align*}
\Pi_{\mathcal{N}'}^{4d \ U(2) + \text{thyper}} & \quad (t, x, z; q) \\
= & \frac{1}{2} \left( \frac{q_1}{q_2} \right)^2 \int \frac{ds_1}{\pi is_1} \frac{ds_2}{\pi is_2} \left( \frac{s_1}{s_2} \right)_\infty \left( \frac{s_2}{s_1} \right)_\infty \\
& \times \frac{1}{2q_2 \ t^2} \sum_{m_1, m_2 \in \mathbb{Z}} ds_3 \ ds_4 \left( 1 - q^{m_1-m_2} s_3^+ s_4^- \right) \left( q^{1+|m_1-m_2|} t^{-2} s_3^+ s_4^- ; q \right)_\infty \\
& \times \frac{1}{2q_1 \ t^2} \sum_{m_3, m_4 \in \mathbb{Z}} ds_5 \ ds_6 \left( 1 - q^{m_3-m_4} s_5^+ s_6^- \right) \left( q^{1+|m_3-m_4|} t^{-2} s_5^+ s_6^- ; q \right)_\infty \\
& \times \frac{1}{2} \left( \frac{q_1}{q_2} \right)^2 \sum_{m_5 \in \mathbb{Z}} ds_7 \left( 1 - q^{m_5} s_7^- \right) \left( q^{1+|m_5|} t^{-2} s_7^- ; q \right)_\infty \\
& \times \prod_{i=1}^2 \left( \frac{q^{\frac{3}{2}} t^{-\frac{1}{2}} s_i^+ z_i^-; q \infty}{\pi is_1 s_1^- s_2^-} \right) \prod_{i=1}^2 \left( q^{\frac{3}{2}} t^{\frac{1}{2}} s_i^+ s_j^-; q \infty \right) \prod_{j=3}^4 \left( q^{\frac{3}{2}} t^{\frac{1}{2}} s_i^+ s_j^-; q \infty \right) \prod_{j=5}^6 \left( q^{\frac{3}{2}} t^{\frac{1}{2}} s_i^+ s_j^-; q \infty \right) \\
& \times \prod_{i=5}^6 \left( q^{\frac{3}{2}} t^{\frac{1}{2}} s_i^+ z_i^-; q \infty \right) \prod_{j=5}^6 \left( q^{\frac{3}{2}} t^{\frac{1}{2}} s_i^+ s_j^-; q \infty \right) \\
& \times q^{\frac{|m_1|+m_2}{2} + \sum_{i=1}^2 \sum_{j=3}^4 \sum_{k=5}^6 \frac{|m_1-m_2|}{4} + \frac{|m_3+m_4|}{4} + \frac{|m_3-m_4|}{4} + \frac{|m_4-m_5|}{4} - \frac{|m_1-m_2|}{2} - \frac{|m_3-m_4|}{2}} \\
& \times t^{2|m_1+2|m_2| + \sum_{i=1}^2 \sum_{j=3}^4 \sum_{k=5}^6 |m_1-m_2| + |m_3+m_4| + |m_3-m_4| + |m_4-m_5| - 2|m_1-m_2| - 2|m_3-m_4|} \\
& \times \left( \frac{x_1}{x_2} \right)^{m_1+m_2} \left( \frac{x_2}{x_3} \right)^{m_3+m_4} \left( \frac{x_3}{x_4} \right)^{m_5} \left( \frac{x_4}{x_1} \right)^{m_1+m_2+m_5}.
\end{align*}
\tag{6.27}
$$

In fact, the half-index $[6.26]$ beautifully coincides with the half-index $[6.27]$.

### 6.2.4 4d $U(3)|3d$ (2) $-$ [4]

To gain more insight, we consider the enriched Neumann b.c. 4d $U(3)|3d$ (2) $-$ [4], that is the enriched Neumann b.c. $\mathcal{N}$ for 4d $\mathcal{N} = 4 U(3)$ gauge theory with a coupling to (2) $-$ [4] via 3d boundary bi-fundamental hyper.
The half-index of the enriched Neumann b.c. 4d $U(3)|3d (2) - [4]$ is

$$\Pi_{N}^{4d} U(3)|[2] - [4] (t, x_{a}, z_{a}; q)$$

$$= \frac{1}{3!} \left( \frac{q^{3}}{t^{2}} ; q \right)^{3} \prod_{i=1}^{3} \int \frac{ds_{i}}{2\pi i s_{i}} \prod_{i \neq j} \left( \frac{q^{1/2} - 2 s_{i}}{s_{j}} ; q \right)^{\infty}$$

$$\times \prod_{m_{1}, m_{2} \in \mathbb{Z}} \left( \frac{q^{1/2} t^{2}; q^{2}}{q^{3} t^{2} ; q^{2}} \right)^{\infty} \sum \prod_{i=1}^{3} \prod_{j=4}^{5} \left( q^{1/2} + \frac{|m_{j}|}{2} l^{-1} s_{i}^{\pm} s_{j}^{\mp} ; q \right)^{\infty} \prod_{i=4}^{5} \prod_{\alpha=1}^{4} \left( q^{1/2} + \frac{|m_{\alpha}|}{2} l s_{i}^{\pm} x_{\alpha}^{\mp} ; q \right)^{\infty}$$

$$\times q^{7|m_{1}| + 7|m_{2}| - |m_{1} - m_{2}|} \cdot l^{-7|m_{1}| - 7|m_{2}| + 2|m_{1} - m_{2}|} \cdot \left( \frac{2}{z} \right)^{m_{1} + m_{2}} . \tag{6.28}$$

Again this is a generalization of full-index (4.4) for $(2) - [4]$. The second line is the half-index of Neumann b.c. $N'$ for 4d $U(3)$ gauge theory. The fourth line includes the contributions from boundary 3d bi-fundamental hypermultiplet. The bare monopole carries the R-charge $\Delta(m) = \frac{2}{7}|m_{1}| + \frac{2}{7}|m_{2}| - |m_{1} - m_{2}|$.

From the brane configuration in Figure 10 we get the dual boundary condition which breaks 4d $U(3)$ gauge symmetry down to $U(2)$ according to the D5'-brane interface. As the number of D3-branes jumps across the D5'-brane, there is no defect hypermultiplet on D5' interface. However, the half-index would receive the contributions from 4d gauginos and scalars in the broken $U(3)$ gauge theory [12].

The surviving 4d $U(2)$ gauge theory should obey Neumann b.c. $N'$ corresponding to the NS5'-brane on which two semi-infinitie D3-branes end. It further couples to 3d twisted quiver gauge theory $(2) - (2) - (1)$ through the 3d boundary twisted hypermultiplet transforming as $(2, \overline{2}) \oplus (\overline{2}, 2)$ under the 4d $U(2)$ gauge symmetry and 3d $U(2)$ gauge symmetry factor at the end of the quiver.
In fact, we can check that the half-index \(6.28\) coincides with the half-index

\[
\Pi_{N'} \rightarrow U(2) + \text{hyper} \quad \frac{\Pi}{|1|} (t, x_\alpha, z_\alpha, q)
\]

\[
= \frac{1}{2} \left( \frac{q^{2t^2}; q^2}{q^{2t^2}; q^2} \right)^2 \int_{\Pi_{N'}^{4d} U(2)} ds_1 ds_2 \left( \frac{q^{2x_\alpha}; q}{q^{2x_\alpha}; q} \right)_\infty \left( \frac{q^{2z_\alpha}; q}{q^{2z_\alpha}; q} \right)_\infty \frac{(q^{2t-2}; q^2)_\infty}{(q^{2t-2}; q^2)_\infty} \frac{(q^{2t^2}; q^2)_\infty}{(q^{2t^2}; q^2)_\infty}
\]

\[
\times \frac{1}{2} \left( \frac{q^{2t^2}; q^2}{q^{2t^2}; q^2} \right)^2 \sum_{m_1, m_2 \in \mathbb{Z}} \int_{\Pi_{N'}^{4d} U(2)} ds_3 ds_4 \left( 1 - \frac{q^{m_1-m_2}}{s_3^+ s_3^-} \right) \left( \frac{q^{t^{i_1}}}{t^{i_2}} \right)^\infty \frac{(q^{t^{i_1}}; q^2)_\infty}{(q^{t^{i_1}}; q^2)_\infty}
\]

\[
\times \frac{1}{2} \left( \frac{q^{2t^2}; q^2}{q^{2t^2}; q^2} \right)^2 \sum_{m_3, m_4 \in \mathbb{Z}} \int_{\Pi_{N'}^{4d} U(2)} ds_5 ds_6 \left( 1 - \frac{q^{m_3-m_4}}{s_5^+ s_5^-} \right) \left( \frac{q^{t^{i_1}}}{t^{i_2}} \right)^\infty \frac{(q^{t^{i_1}}; q^2)_\infty}{(q^{t^{i_1}}; q^2)_\infty}
\]

\[
\times \frac{1}{2} \left( \frac{q^{2t^2}; q^2}{q^{2t^2}; q^2} \right)^2 \sum_{m_5 \in \mathbb{Z}} \int_{\Pi_{N'}^{4d} U(2)} ds_7 \left( 1 - \frac{q^{m_5}}{s_7^+ s_7^-} \right) \left( \frac{q^{t^{i_1}}}{t^{i_2}} \right)^\infty \frac{(q^{t^{i_1}}; q^2)_\infty}{(q^{t^{i_1}}; q^2)_\infty}
\]

\[
\times \frac{2}{(q^{t^{i_1}}; q^2)_\infty} \prod_{i=1}^{2} \frac{q^{t^{i_1}}}{(q^{t^{i_1}}; q^2)_\infty} \prod_{i<j}^{4} \frac{q^{t^{i_1}}}{(q^{t^{i_1}}; q^2)_\infty} \prod_{i=3}^{4} \frac{q^{t^{i_1}}}{(q^{t^{i_1}}; q^2)_\infty} \prod_{i=5}^{6} \frac{q^{t^{i_1}}}{(q^{t^{i_1}}; q^2)_\infty} \prod_{j=5}^{6} \frac{q^{t^{i_1}}}{(q^{t^{i_1}}; q^2)_\infty}
\]

\[
\times q^{\frac{1}{2} t^{m_1+2} m_2 + \sum_{j=3}^{3} h_{j=3}^{i=4} \frac{m_1+2 m_3+|m_3-m_4|+|m_4-m_5|+|m_4-m_5|+|m_4-m_5|-2|m_4-m_5|-2|m_4-m_5|}{4} \times \frac{x_1}{x_2} \frac{x_2}{x_3} \frac{x_3}{x_4} \frac{x_4}{x_1} ^{m_1+2+m_5}
\]

This includes the half-index of Dirichlet b.c. \(D'\) for 4d \(U(1)\) gauge theory corresponding to the reduction \(U(3) \rightarrow U(2)\) of gauge symmetry, the half-index of Neumann b.c. \(N'\) for 4d \(U(2)\) gauge theory, and the full-indices for boundary 3d local operators. In particular, the first factors in the sixth line describe the 4d gauginos and scalars in the broken \(U(3)\) gauge theory [12].

### 6.2.5 4d \(U(N)\) 3d \((M) = [2M]\)

Let us propose the generalization of enriched Neumann b.c. 4d \(U(N)\) 3d \((M) = [2M]\), which is the Neumann b.c. \(N'\) for 4d \(N = 4\) \(U(N)\) gauge theory coupled to 3d \(N = 4\) \(U(M)\) balanced gauge theory with \(2M\) fundamental hypermultiplets via 3d boundary hypermultiplets transforming as \((N, \overline{M}) \oplus (\overline{N}, M)\) under the \(U(N) \times U(M)\) gauge symmetry. Here we assume that \(N \geq M\)
The half-index for the enriched Neumann b.c. 4d $U(N)|3d \ (M) - [2M]$ reads

$$
\Pi_N^{4d} U(N)|[M] - [2M] (t, x_a, z_\alpha; q) \\
= \frac{1}{N!} \frac{(q)^N}{(q^2 t^{-2}; q)^N} \prod_{i=1}^{N} \left( \frac{d s_i}{2 \pi i s_i} \right) \prod_{i \neq j} \left( \frac{q^{1/2} t^{-2 s_i}}{s_i} ; q \right) \infty \\
\times \left( q^2 t^{-2} ; q \right)^{M} \sum_{m_1, \cdots, m_M \in \mathbb{Z}} \prod_{i=1}^{N+M} \left( 1 - q^{\frac{1}{2}} t^{-2 s_i} s_i \pm \pi i s_i \right) \left( q^{1/2} t^{-2 s_i} s_i \mp \pi i s_i \right) \\
\times q^{\frac{N+M}{2} \sum_{i=1}^{M} |m_i| - \sum_{i<j} \frac{|m_i - m_j|}{2}} t^{-(N+2M) \sum_{i=1}^{M} |m_i| + 2 \sum_{i<j} |m_i - m_j|} \left( \frac{z_1}{z_2} \right) \sum_{i=1}^{M} m_i. 
$$

(6.30)

The second and third line describe the half-index of Neumann b.c. $N'$ for 4d $U(N)$ gauge theory and the full-index of 3d $U(M)$ vector multiplet respectively. The fourth line counts the boundary bi-fundamental hypermultiplet and the fundamental hypermultiplet in $(M) - [2M]$. The last line counts the monopole operator with the canonical R-charge

$$
\Delta(m) = \frac{N + 2M}{2} \sum_{i=1}^{M} |m_i| - \sum_{i<j} |m_i - m_j|. 
$$

(6.31)

This is generalized from the formula (4.16) by including the additional contributions from the boundary bi-fundamental hypermultiplet coupled to 4d $U(N)$ SYM theory.

The S-dual boundary condition can be read off from the brane configuration in Figure 10. The 4d gauge symmetry is required to break down to $U(M)$ due to the presence of the D5'-brane interface. When $N = M$, there is a defect 3d twisted hypermultiplet at the D5'-brane interface transforming under fundamental representation under the $U(N)$ gauge group. For $N > M$, there is no fundamental twisted hypermultiplet at the defect, however, there are contributions to the half-index from the broken $U(N)$ gauge theory associated with the Nahm pole of rank $(N - M)$ [12].

The surviving 4d $U(M)$ gauge theory should obey the Neumann b.c. $N'$ due to the NS5'-brane. It further couples to 3d twisted quiver gauge theory through the 3d bi-fundamental twisted hypermultiplet. The 3d twisted quiver gauge theory has $U(M)^M \times U(M - 1) \times U(M - 2) \times U(1)$ gauge symmetry and a single $U(1)$ flavor node for the $M$-th $U(M)$ gauge node, which we denote by $\frac{(M)^{M-1} - (M) - (M-1) - (M-2) - \cdots - (1)}{[1]}$.
We expect that the half-index (6.30) agrees with

\[
4d \ U(N) \to U(M) \quad (M)_{M-1}^{M-1} = (M) - (M-1) - (M-2) - \ldots - (1)
\]

\[
\mathbb{P}_{\mathcal{N}'}(M)_{M-1}^{M-1} = \prod_{i=1}^{M} \frac{\sum_{m} s_i(m)}{2\pi i s_i(k)} \prod_{i < j} \frac{(q^2 - t^2 s_i m)}{2\pi i s_i(k)} \times \prod_{k=1}^{N-M} \frac{(q^2 - t^2 s_i m)}{2\pi i s_i(k)}
\]

\[
\times \left( \prod_{i=1}^{M} \frac{(q^2 - t^2 s_i m)}{2\pi i s_i(k)} \prod_{i > j} \frac{(q^2 - t^2 s_i m)}{2\pi i s_i(k)} \times \prod_{k=1}^{N-M} (q^2 - t^2 s_i m) \prod_{i=1}^{M} \frac{(q^2 - t^2 s_i m)}{2\pi i s_i(k)}
\]

The second line includes the half-index of Nahm' boundary condition for 4d \( U(N-M) \) gauge theory corresponding to the defect of D5' brane and the half-index of Neumann b.c. \( \mathcal{N}' \) for 4d \( U(N) \) gauge theory corresponding to the NS5'-brane on which semi-infinite \( N \) D3-branes terminate. The contributions from third to sixth line are the full-index of 3d twisted vector multiplets. The first factors in the seventh line describe the defect twisted hypermultiplet for \( N = M \) and the local operators appearing from the broken \( U(N) \) gauge theory for \( N > M \). The remaining contributions are the 3d twisted hypers and monopole operator with conformal dimension

\[
\Delta(m) = \frac{M}{2} \sum_{i=1}^{M} |m_i| + \frac{2M-2}{2} \sum_{k=1}^{M} \sum_{i<j} \frac{|m_{i+k} - m_{j+k}|}{2} + \frac{1}{2} \sum_{i=1}^{M} |m_i| - \sum_{k=1}^{M-1} \sum_{i<j} |m_{i+k} - m_{j+k}|
\]

(6.33)

where the first terms are the contributions from the bi-fundamental twisted hyper coupled to 4d \( U(M) \) and 3d \( U(M) \) gauge symmetries, the second and third are the contributions from the bi-fundamental
Figure 11: (a) The quiver diagrams of the enriched Neumann b.c. 4d $U(N)|T[SU(M)]$ and its mirror. (b) The corresponding brane constructions.

twisted hyper in the 3d quiver gauge theory and the last terms are the contributions from the 3d twisted vector multiplets.

### 6.3 4d $U(N)|T[SU(M)]$

Let us study the enriched Neumann b.c. 4d $U(N)|T[SU(M)]$ for 4d $U(N)$ SYM theory. The corresponding quiver diagram and brane configuration are depicted in Figure 11. We have already examined the case with $M = 2$ in section 6.2.1 and 6.2.2. Although $T[SU(N)]$ is self-mirror, the enriched boundary condition 4d $U(N)|T[SU(M)]$ is not self-mirror. We further check the dualities for $M = 3$ and propose the generalization.

#### 6.3.1 4d $U(1)|T[SU(3)]$

Consider the enriched Neumann b.c. 4d $U(1)|T[SU(3)]$, which is Neumann b.c. $\mathcal{N}$ for 4d $U(1)$ gauge theory coupled to $T[SU(3)]$ through the 3d bi-fundamental hypermultiplet.
One can compute the half-index as

\[
\begin{align*}
\mathbb{I}^{4d U(1)}_{N'}(t, x, z, \alpha; q) &= \frac{(q^2 t^2; q) \infty}{(q^2 t^{-2}; q) \infty} \int \frac{ds_1}{2\pi is_1} \frac{(q^2 t^2; q) \infty}{(q^2 t^{-2}; q) \infty} \sum_{m_1 \in \mathbb{Z}} \int \frac{ds_2}{2\pi is_2} \\
&\times \frac{1}{2} \frac{(q^{\frac{3}{2}} t^2; q) \infty}{(q^{\frac{3}{2}} t^{-2}; q) \infty} \sum_{m_2, m_3 \in \mathbb{Z}} \int \frac{ds_2}{2\pi is_2} \frac{ds_3}{2\pi is_3} \left( 1 - q^{m_2 - m_3 \pm s_3 \pm s_4} \right) \left( q^{1 + m_2 - m_3 / 2} t^2 s_3 \pm s_4 / q \right) \infty \\
&\times \left( q^{\frac{3}{2} + \frac{|m_1|}{2}} - \frac{1}{2} q_{1 \pm} s_{1 \pm} q \right) \infty \cdot \prod_{i=3}^4 \left( q^{\frac{3}{2} + \frac{|m_1 - m_{i-1}|}{2}} s_{2 \pm} s_{i \pm} q \right) \infty \cdot \prod_{i=3}^4 \prod_{\alpha=1}^3 \left( q^{\frac{3}{2} + \frac{|m_1 - m_{i-1}|}{2}} l_{1 \pm} s_{1 \pm} x_{\alpha \pm} q \right) \infty \\
&\times t^{-|m_1| - |m_2| - |m_1 - m_3| - |m_2 - m_3| + 2|m_2 - m_3|} \\
&\times \left( \frac{z_1}{z_2} \right)^m \left( \frac{z_2}{z_3} \right)^{m_2 + m_3} \left( \frac{z_3}{z_1} \right)^{m_1 + m_2 + m_3}.
\end{align*}
\]

(6.34)

The second and third lines describes the Neumann half-index for 4d $U(1)$ gauge theory and the full-index for 3d $U(1) \times U(2)$ vector multiplet. The remaining contributions are 3d hypermultiplets and monopole operator of dimension

\[
\Delta(m) = \frac{|m_1|}{2} + \frac{|m_1 - m_2|}{2} + \frac{|m_1 - m_3|}{2} + \frac{3}{2} l |m_2| + \frac{3}{2} m_3 - |m_2 - m_3|.
\]

(6.35)

This is a generalization of the formula (4.24). The first term in (6.35) describes the contribution from the boundary bi-fundamental hyper coupled to 4d and 3d gauge theories.

Under the action of S-duality, we find the brane configuration shown in Figure [11]. The dual boundary condition is the enriched Neumann b.c. $N'$ for $U(1)$ gauge theory. It involves a defect 3d twisted hypermultiplet arising from D3-D5 string which are charged under the 4d $U(1)$ gauge symmetry. The 4d $U(1)$ gauge theory obeys Neumann b.c. $N'$ corresponding to the NS5'-brane and couples to 3d twisted quiver gauge theory \((2) - (1)\) through the 3d bi-fundamental twisted hypermultiplet.
We can compute the half-index for the dual boundary condition as

\[
\begin{aligned}
\mathbb{I}_{N'}^{4d U(1) + U(1) \rightarrow N'} & = \left( t, x_\alpha, z_\alpha ; q \right) \\
& = \frac{(q)_\infty}{(q^2 t^2 ; q)_\infty} \oint \frac{ds_1}{2\pi i s_1} \\
& \times \frac{1}{2} \frac{(q^2 t^{-2}; q)_\infty}{(q^2 t^2 ; q)_\infty} \sum_{m_1, m_2 \in \mathbb{Z}} \oint \frac{ds_2}{2\pi i s_2} \oint \frac{ds_3}{2\pi i s_3} \\
& \times \frac{(q^2 t^{-2}; q)_\infty}{(q^2 t^2 ; q)_\infty} \sum_{m_3 \in \mathbb{Z}} \oint \frac{ds_4}{2\pi i s_4} \\
& \times \frac{(q^3 t s_1^\pm z_\alpha^\pm ; q)_\infty}{(q^2 t^{-1} s_1^\pm ; q)_\infty} \prod_{i=2}^{3} \left( q^{\frac{1}{2} + \frac{|m_i - m_{i-1}|}{2}} t s_i^\pm s_{i-1}^\mp ; q \right)_\infty \\
& \times \frac{2}{3} \prod_{i=2}^{3} \prod_{\alpha = 1}^{2} \left( q^{\frac{1}{2} + \frac{|m_i - m_\alpha|}{2}} t^{-1} s_i^\pm z_i^\mp \right)_\infty \\
& \times q^{3|m_1 + |2|m_2| + |m_1 - m_3| + |m_2 - m_3| - 2|m_1 - m_3|} \\
& \times \frac{x_1}{x_2}^{m_1 + m_2} \cdot \frac{x_2}{x_3}^{m_3} \cdot \frac{x_3}{x_1}^{m_1 + m_2 + m_3}. \\
\end{aligned}
\]

(6.36)

The contributions in the second to fourth are the half-index of Neumann b.c. \( N \) for 4d \( U(1) \) gauge theory and the full-index of 3d \( U(1) \times U(2) \) vector multiplet. The remaining terms are the contributions from 3d twisted hypermultiplets and monopole operator with the R-charge

\[
\Delta(m) = \frac{3}{2} |m_1| + \frac{3}{2} |m_2| + \frac{|m_1 - m_3|}{2} + \frac{|m_2 - m_3|}{2} - |m_1 - m_3|. 
\]

(6.37)

We have checked that the half-indices (6.34) and (6.36) coincide with each other.

\subsection*{6.3.2 4d \( U(2) \)|\( T[SU(3)] \)}

Let us study the enriched Neumann b.c. 4d \( U(2) \)|\( T[SU(3)] \), i.e. Neumann b.c. \( N \) for 4d \( U(2) \) gauge theory coupled to \( T[SU(3)] \) via 3d bi-fundamental hypermultiplet.
The half-index reads

\[
\mathbb{I}^{4d}_{U(2)}(1) - (2) - [3](t, x_α, z_α; q) \\
= \frac{1}{2} \left( \frac{q^{\frac{1}{2}}}{q^{\frac{1}{2}} t^{-2}; q^2} \right) \int \frac{ds_1}{2\pi is_1} \frac{ds_2}{2\pi is_2} \left( q^{\frac{2}{2}} t^{-2 s_1}; q \right) \left( q^{\frac{2}{2}} t^{-2 s_2}; q \right) \frac{\left( \frac{s_1}{s_2}; q \right) \left( \frac{s_2}{s_1}; q \right)}{\prod_{m_{1,2,3} \in \mathbb{Z}} s_1^{m_1} s_2^{m_2} s_3^{m_3} + q^{\frac{1}{2}} t^{s_1 s_2 s_3}; q^\infty} \\
\times \left( \frac{q^{\frac{1}{2}} t^2; q^2}{q^{\frac{1}{2}} t^{-2}; q^2} \right) \sum_{m_1, m_2, m_3 \in \mathbb{Z}} \int \frac{ds_4}{2\pi is_4} \frac{ds_5}{2\pi is_5} \left( 1 - q^{\frac{1}{2}} m_2 - m_3 \right) \left( q^{\frac{1}{2}} t^{s_4 s_5}; q \right) \frac{\left( q^{\frac{1}{2}} t^{s_4 s_5}; q \right)}{\prod_{i=1}^5 \left( q^{\frac{3}{2} + \frac{|m_{i-1}|}{2}} t^{-1 s_i^3 s_i^3}; q \right) \prod_{i=4}^5 \left( q^{\frac{3}{2} + \frac{|m_{i-1}|}{2}} t^{-1 s_i^3 s_i^3}; q \right) \prod_{i=1}^5 \left( q^{\frac{3}{2} + \frac{|m_{i-1}|}{2}} t^{-1 s_i^3 s_i^3}; q \right)} \\
\times \prod_{i=1}^{m_1} \left( q^{\frac{3}{2} + \frac{|m_{i-1}|}{2}} t^{-1 s_i^3 s_i^3}; q \right) \prod_{i=4}^{m_3} \left( q^{\frac{3}{2} + \frac{|m_{i-1}|}{2}} t^{-1 s_i^3 s_i^3}; q \right) \prod_{i=1}^{m_3 - m_2 - m_3 + 1} t^{-2|m_1| - |m_1 - m_2| - |m_1 - m_3| - 3|m_2| - 3|m_3| + 2|m_2 - m_3|} \\
\times \left( \frac{z_1}{z_2} \right)^{m_1} \left( \frac{z_2}{z_3} \right)^{m_2 + m_3} \left( \frac{z_3}{z_1} \right)^{m_1 + m_2 + m_3} \\
\right)
\]

The second line is the half-index of Neumann b.c. \( \mathcal{N} \) for \( U(2) \) SYM theory while the third and fourth lines are the full-index for 3d \( U(1) \times U(2) \) vector multiplet. The other terms count 3d hypermultiplets and monopole operator with R-charge

\[
\Delta(m) = |m_1| + \frac{|m_1 - m_2|}{2} + \frac{|m_1 - m_3|}{2} + \frac{3|m_2|}{2} + \frac{3|m_3|}{2} - |m_2 - m_3| 
\]

where the first term is contributed from the bi-fundamental hypermultiplet coupled to 4d and 3d gauge symmetries while the other terms are the contributions from the 3d multiplets in \( T[SU(3)] \).

We can identify the dual boundary condition from the brane configuration in Figure 11. It requires the 4d \( U(2) \) gauge symmetry to break down to \( U(1) \) due to the D5' defect. Unequal numbers of D3-branes across the D5' defect do not admit 3d twisted hypermultiplet at the defect. Instead, there would be contributions to the half-index from the broken \( U(2) \) gauge theory [12]. Then the surviving \( U(1) \) gauge theory obeys the enriched Neumann b.c. 4d \( U(1)[T[SU(3)] \) studied in section 6.3.1.
We find that the half-index \((6.38)\) coincides with
\[
\Pi_{\mathcal{N}'}^{4d \ U(2) \rightarrow U(1)} \left( t, \alpha, x; q \right)
\]
\[
= \frac{(q)_\infty}{(q^2 t^2; q)_\infty} \int_{\Pi_{\mathcal{N}'}^{4d \ U(1)}} \frac{ds_1}{2\pi is_1} \frac{(q)_\infty}{(q^2 t^2; q)_\infty} \int_{\Pi_{\mathcal{N}'}^{4d \ U(1)}} \frac{ds_2}{2\pi is_2} \frac{ds_3}{2\pi is_3} \frac{1 - q^{\frac{m_1 - m_2}{2}} s_2^2 s_3^2}{q^{\frac{1 + m_1 - m_2}{2}} t^2 s_2^2 s_3^2} \frac{(q^{1 + m_1 - m_2} t^{-2} s_2^2 s_3^2; q)_\infty}{(q^{\frac{1 + m_1 - m_2}{2}} t^2 s_2^2 s_3^2; q)_\infty}
\]
\[
\times \frac{(q^2 t^2; q)_\infty}{(q^2 t^2; q)_\infty} \sum_{m_1, m_2 \in \mathbb{Z}} \int_{\Pi_{\mathcal{N}'}^{4d \ U(1)}} \frac{ds_4}{2\pi is_4}
\]
\[
\times \left( \frac{qs_1 s_3^2; q}_\infty \right) \frac{1}{(q^2 t^2; q)_\infty} \prod_{i=2}^3 \left( q^{\frac{m_{i-1}}{2}} t^{-1} s_i^2 s_i^\perp; q)_\infty \right. \!
\]
\[
\times \prod_{i=2}^2 \left( \frac{q^{\frac{m_{i-1}}{2}} t^{-1} s_i^2 s_i^\perp; q)_\infty \right. \!
\]
\[
\times \prod_{i=2}^3 \left( q^{\frac{m_{i-1}}{2}} t^{-1} s_i^2 s_i^\perp; q)_\infty \right. \!
\]
\[
\times \prod_{i=2}^3 \left( q^{\frac{m_{i-1}}{2}} t^{-1} s_i^2 s_i^\perp; q)_\infty \right. \!
\]
\[
\times \left( \frac{q^{\frac{1}{2}} + \frac{m_1 - m_2}{2} t s_1^2 s_1^\perp; q)_\infty \right. \!
\]
\[
\times \left( \frac{q^{\frac{1}{2}} + \frac{m_1 - m_2}{2} t s_1^2 s_1^\perp; q)_\infty \right. \!
\]
\[
\times \left( x_1 \right)^{m_1 + m_2} \left( x_2 \right)^{m_3} \left( x_3 \right)^{m_1 + m_2 + m_3}
\]
\[
= \left( x_1 \right)^{m_1 + m_2} \left( x_2 \right)^{m_3} \left( x_3 \right)^{m_1 + m_2 + m_3}
\]
\[
(6.40)
\]

The contributions in the second line are the half-index of Dirichlet b.c. \(\mathcal{D}'\) for 4d \(U(1)\) gauge theory and the half-index of Neumann b.c. \(\mathcal{N}'\) for 4d \(U(1)\) gauge theory. The fourth and fifth lines describe the full-index for 3d \(U(2) \times U(1)\) twisted vector multiplet. The first terms in the fifth line would correspond to the local operators arising from the broken gauge theory \[12\]. The other terms count 3d twisted hypermultiplets and monopole operator of dimension \(6.37\).

### 6.3.3 4d \(U(N)\) \([T[SU(M)]]\)

We want to propose the generalization of the results so far. The enriched Neumann boundary condition 4d \(U(N)\) \([T[SU(M)]]\) is constructed from the brane setup in Figure [11] which is the Neumann b.c. \(\mathcal{N}'\) for 4d \(U(N)\) SYM theory coupled to \([T[SU(M)]]\) via the boundary 3d hypermultiplet transforming as \((N, -) \oplus (N, +)\) under the 4d \(U(N)\) gauge and 3d \(U(1)\) gauge symmetries.
We can write the half-index as

\[
\begin{align*}
\mathbb{I}_{\mathcal{N}}^{U(N)\mathcal{T}[SU(M)]}(t, x, z; q) &= \frac{1}{N!} \frac{(q)^N}{(q^2 t^{-2}; q)^N} \oint \prod_{i=1}^{N} ds_i \prod_{i<j} \frac{\left(\frac{z_i}{s_i}; q\right)_{\infty}}{\left(\frac{z_j}{s_j}; q\right)_{\infty}} \\
&\times \prod_{k=1}^{M-1} \left[ \frac{1}{k!} \frac{(q^{2} t^{2}; q)^k}{(q^{2} t^{-2}; q)^k} \sum_{m_{k}^{(1)}, \ldots, m_{k}^{(k)} \in \mathbb{Z}} \oint \prod_{i=1}^{k} ds_{i}^{(k)} \\
&\quad \times \prod_{i<j} \left( 1 - q^{\frac{|m_{k}^{(1)}|}{2} + \frac{|m_{k}^{(k)}|}{2}} s_{i}^{(k)} s_{j}^{(k)}; q \right)_{\infty} \left( q^{\frac{1+|m_{k}^{(1)}|-|m_{k}^{(k)}|}{2}} t^{2} s_{i}^{(k)} s_{j}^{(k)}; q \right)_{\infty} \right] \\
&\times \prod_{i=1}^{N} \frac{(q^{\frac{3}{2} + \frac{|m_{i}^{(1)}|}{2}} t^{-1} s_{i}^{(1)} s_{j}^{(2)}; q)_{\infty}}{(q^{\frac{3}{2} + \frac{|m_{i}^{(1)}|}{2}} t^{1} s_{i}^{(1)} s_{j}^{(2)}; q)_{\infty}} \prod_{k=1}^{M-2} \prod_{i=1}^{k} \prod_{j=1}^{k+1} \left( q^{\frac{3}{2} + \frac{|m_{i}^{(1)}|}{2}} t^{-1} s_{i}^{(k)} s_{j}^{(k)}; q \right)_{\infty} \left( q^{\frac{1+|m_{i}^{(1)}|-|m_{i}^{(k)}|}{2}} t s_{i}^{(k)} s_{j}^{(k)}; q \right)_{\infty} \\
&\times \prod_{i=1}^{M-1} \prod_{\alpha=1}^{M} \frac{(q^{\frac{3}{2} + \frac{|M-1|}{2}} t^{-1} s_{i}^{(M)}; q)_{\infty}}{(q^{\frac{3}{2} + \frac{|M-1|}{2}} t s_{i}^{(M)}; q)_{\infty}} \\
&\times q^{N \frac{|m_{i}^{(1)}|}{2} + \frac{1}{2} \sum_{k=2}^{M-2} \sum_{i=1}^{k} \sum_{j=1}^{k+1} |m_{i}^{(k)}| - |m_{j}^{(k+1)}| + M \sum_{k=1}^{M-1} |m_{i}^{(M-1)}| - \sum_{k=1}^{M-1} \sum_{i<j} |m_{i}^{(k)} - m_{j}^{(k)}|} \\
&\times t^{-N |m_{i}^{(1)}| - \sum_{k=2}^{M-2} \sum_{i=1}^{k} \sum_{j=1}^{k+1} |m_{i}^{(k)}| - |m_{j}^{(k+1)}| - M \sum_{k=1}^{M-1} |m_{i}^{(M-1)}| + 2 \sum_{k=1}^{M-1} |m_{i}^{(k)} - m_{j}^{(k)}|} \\
&\times \prod_{k=1}^{M-1} \left( \frac{z_{k}}{z_{k+1}} \right)^{\frac{1}{2} \sum_{i=1}^{k} m_{i}^{(k)}} \left( \frac{z_{M}}{z_{1}} \right)^{\frac{1}{2} \sum_{i=1}^{M-1} m_{i}^{(M-1)}}. \tag{6.41}
\end{align*}
\]

The terms appearing from the second to fourth line describe the half-index of Neumann b.c. \( \mathcal{N} \) for 4d \( \mathcal{N} = 4 \) \( U(N) \) SYM theory and the full-index for 3d \( U(1) \times U(2) \times \cdots \times U(M-1) \) vector multiplet. The associated magnetic fluxes are labeled by \( \frac{M(M-1)}{2} \) integers \( \{m_{i}^{(k)}\}_{i=1, \ldots, k} \) where \( k = 1, \ldots, M-1 \). The contributions in the fifth and sixth lines describe 3d \( \mathcal{N} = 4 \) hypermultiplets. The other terms count the bare monopole with the R-charge

\[
\Delta(m) = \frac{N}{2} |m_{i}^{(1)}| + \frac{1}{2} \sum_{k=1}^{M-2} \sum_{i=1}^{k} \sum_{j=1}^{k+1} |m_{i}^{(k)}| - |m_{j}^{(k+1)}| + \frac{M}{2} \sum_{i=1}^{M-1} |m_{i}^{(M-1)}| - \sum_{k=1}^{M-1} \sum_{i<j} |m_{i}^{(k)} - m_{j}^{(k)}|. \tag{6.42}
\]

This formula is generalized from \([4.29]\) by the additional contributions appearing in the first terms. They come from the boundary hypermultiplet transforming as \( (\mathbb{N}, -) \oplus (\bar{\mathbb{N}}, +) \) under the 4d \( U(N) \) gauge and 3d \( U(1) \) gauge symmetries.

The dual quiver diagram and the corresponding brane setup are illustrated in Figure \([11]\). It is identifies with the boundary condition for 4d \( U(N) \) gauge theory including the Nahm' pole of rank \( (N-1) \) that breaks the 4d gauge symmetry down to \( U(1) \). While for \( N = 1 \) one finds a defect twisted hypermultiplet arising from D3-D5' string, for \( N > 1 \), the half-index receives contributions from broken gauge theory as discussed in [12].

In addition, the surviving \( U(1) \) gauge theory should satisfy the enriched Neumann b.c. \( \mathcal{N}' \) corre-
sponding to NS5'-brane with a coupling to 3d twisted quiver gauge theory through the 3d boundary twisted hypermultiplet.

We find the half-index

\[
\begin{align*}
4d \ U(N) & \rightarrow U(1) \bigg|_{M-1} \bigg|_{M-2} \cdots \bigg|_{M-1} \\
\Pi_{\mathcal{N}'} & = \frac{(q)^\infty}{(q^2t^2; q^2)^\infty} \oint_{\partial^{4d}/U(1)} ds_1 \prod_{i=1}^{N-1} \frac{(q^{1+i} t^{-2(l-1)}; q)^\infty}{(q^2t^{2l}; q)^\infty} \\
& \times \prod_{k=1}^{M-1} \left[ \frac{1}{k!} \frac{(q^2t^{-2}; q)^k}{(q^2t^2; q)^k} \sum_{m_1^{(k)} \cdots m_{\infty}^{(k)} \in \mathbb{Z}} \oint_{\partial^{4d}/U(1)} ds_1^{(k)} \prod_{i=1}^{k} \frac{ds_i^{(k)}}{2\pi is_i^{(k)}} \right] \\
& \times \prod_{i<j} \left( 1 - q^\frac{|m_1^{(k)} - m_2^{(k)}|}{2} s_i^{(k)} s_j^{(k)}; q \right)^\infty \left( \frac{q}{q^2 t^2 s_i^{(k)} s_j^{(k)}; q} \right)^\infty \\
& \times \frac{(q^2 t^2 s_i^{(k)}; q)^\infty}{(q^2 t^2; q)^\infty} \prod_{i=1}^{M-1} \frac{(q^{\frac{|M-1|}{2}} t s_i^{(k)}; q)^\infty}{(q^{\frac{|M-1|}{2}} t^{-1} s_i^{(k)}; q)^\infty} \\
& \times \prod_{k=1}^{M-1} \prod_{\alpha=1}^{M-1} \frac{(q^{\frac{|M-1|}{2}} t s_i^{(k)} s_j^{(k)}; q)^\infty}{(q^{\frac{|M-1|}{2}} t^{-1} s_i^{(k)} s_j^{(k)}; q)^\infty} \\
& \times q^{M \sum_{i=1}^{M-1} m_i^{(M-1)} + \sum_{i=1}^{M-1} k_{i-1} \sum_{k=0}^{k_{i-1}} \frac{|m_i^{(k)} - m_{i+1}^{(k)}|}{2} - \sum_{k=1}^{M-1} \sum_{i<j} \frac{|m_i^{(k)} - m_j^{(k)}|}{2}} \\
& \times \prod_{k=1}^{M-1} \left( \frac{x_k}{x_{k+1}} \right)^{m_i^{(M-1)}} \left( \frac{x_{M}}{x_1} \right)^{m_i^{(M-1)}}. 
\end{align*}
\]

The contributions in the second line are the half-index of Nahm' b.c. for 4d $U(N-1)$ gauge theory and the half-index of Neumann b.c. for 4d $U(1)$ gauge theory. The next two lines are the full-index of 3d $U(M-1) \times U(M-2) \times \cdots \times U(1)$ twisted vector multiplet. The first terms in the fifth line correspond to the defect hypermultiplet for $N = 1$ or the extra fields appearing from the broken gauge theory [12]. The remaining terms count 3d $\cal{N} = 4$ twisted hypers and monopole operator of dimension

\[
\Delta(m) = \frac{M}{2} \sum_{i=1}^{M} m_i^{(M-1)} + \sum_{k=1}^{M-2} k_{i-1} \sum_{k=0}^{k_{i-1}} \frac{|m_i^{(k)} - m_{i+1}^{(k)}|}{2} - \sum_{k=1}^{M-1} \sum_{i<j} |m_i^{(k)} - m_j^{(k)}|.
\]

We expect that the half-indices (6.41) and (6.43) lead to the same result.

\section{Dualities of interfaces}

In this section we would like to study the dualities of interfaces for a pair of 4d $\cal{N} = 4$ gauge theories including additional 3d $\cal{N} = 4$ gauge theories.

\section{Dualities of interfaces}

In this section we would like to study the dualities of interfaces for a pair of 4d $\cal{N} = 4$ gauge theories including additional 3d $\cal{N} = 4$ gauge theories.
7.1 \(4d\ U(N)\mid3d\ U(N)^{k-1}\mid4d\ U(N)\)

We study the interface for a pair of two \(4d\ \mathcal{N} = 4\ U(N)\) gauge theories which involves \(3d\ \mathcal{N} = 4\ U(N)^{k-1}\) quiver gauge theory. We denote this by \(4d\ U(N)\mid3d\ U(N)^{k-1}\mid4d\ U(N)\). The corresponding quiver diagram and brane configuration are given in Figure 12. It is realized by \(N\) D3-branes intersecting with \(k\) NS5-branes. The dual configuration is realized by \(N\) D3-branes intersecting with \(k\) D5'-branes. According to the presence of D5' interface, the \(4d\ U(N)^{k-1}\) gauge symmetry is broken down to a diagonal \(U(N)\) so that four-dimensional \(U(N)\) gauge fields couple to \(3d\ U(N)^{k-1}\) twisted hypermultiplets. The analysis for \(k = 1\) where the interface has no \(3d\) gauge theory is already given in [12]. We present more general results and check that the half-indices for the dual interfaces agree with each other.

7.1.1 \(4d\ U(1)\mid3d\ U(1)\mid4d\ U(1)\)

Let us consider the interface \(4d\ U(1)\mid3d\ U(1)\mid4d\ U(1)\). The \(4d\ U(1)\) gauge theories satisfy Neumann b.c. \(\mathcal{N}\) corresponding to the NS5-branes. Each of \(4d\) theories is coupled to \(3d\ U(1)\) vector multiplet through the \(3d\) bi-fundamental hypermultiplet. Consequently, the interface has \(U(1)\times U(1)\times U(1)\) gauge symmetry. We label the magnetic flux associated to \(3d\ U(1)\) gauge theory by an integer \(m\).

The half-index is evaluated as

\[
\begin{align*}
\Pi_{X}^{4d\ U(1)/(1)\mid4d\ U(1)}(t, z; q) &= \frac{(q)_{\infty}}{(q^{2}t^{-2}; q)_{\infty}} \sum_{m \in \mathbb{Z}} \frac{ds_{2}}{2\pi is_{2}} \frac{(q^{2}t^{2}; q)_{\infty}}{(q^{2}t^{-2}; q)_{\infty}} \frac{ds_{3}}{2\pi is_{3}} \frac{(q)_{\infty}}{(q^{2}t^{-2}; q)_{\infty}} \\
&\times \frac{(q^{2}t^{-1}s_{1}^{-1}s_{2}^{-1}; q)_{\infty}}{(q^{2}t^{1}ts_{1}^{1}s_{2}^{1}; q)_{\infty}} \frac{(q^{2}t^{-1}s_{1}^{1}s_{2}^{-1}; q)_{\infty}}{(q^{2}t^{1}ts_{1}^{1}s_{2}^{1}; q)_{\infty}} q^{im} t^{-2|m|} \left( \frac{z_{1}}{z_{2}} \right)^{m}.
\end{align*}
\]

(7.1)

The second line contains the two half-indices of Neumann b.c. \(\mathcal{N}\) for \(4d\ U(1)\) gauge theory and the full-index for \(3d\ U(1)\) vector multiplet. The last line describes the \(3d\) bi-fundamental hypermultiplets.
and monopole operator of dimension $\Delta(m) = |m|$. The S-dual interface is realized by a single D3-brane intersecting with two D5′-branes. The two D5′ defects would break the $U(1) \times U(1) \times U(1)$ gauge symmetry down to a diagonal $U(1)$ and couple it to two fundamental twisted hypermultiplets. Hence it is described by 4d $\mathcal{N} = 4$ $U(1)$ gauge theory on the full space-time coupled to two 3d twisted hypermultiplets.

We can compute the corresponding half-index as

$$
\prod_{D^4}^{4d} U(1|1) + 2 {\text{thypers}}(t, z_\alpha; q) = \frac{(q)_{\infty}^2}{(q_{\frac{1}{2}} t^2; q)_{\infty}(q_{\frac{1}{2}} t^{-2}; q)_{\infty}} \int_{4d \ U(1)} \frac{ds_{1}}{2\pi is_{1}} \sum_{m_{1},m_{2}\in\mathbb{Z}} \int_{4d \ U(1)} \frac{ds_{2}}{2\pi is_{2}} \int_{4d \ U(1)} \frac{ds_{3}}{2\pi is_{3}} \frac{(q)_{\infty}}{(q_{\frac{1}{2}} t^{-2}; q)_{\infty}} \prod_{\alpha=1}^{2} \frac{(q_{\frac{1}{2}} t s_{\alpha}^{\pm} z_{\alpha}; q)_{\infty}}{(q_{\frac{1}{2}} t^{-1} s_{\alpha}^{\pm} z_{\alpha}; q)_{\infty}}.
$$

We have checked that this coincides with the half-index (7.1).

### 7.1.2 4d $U(1)|3d \ U(1)^2|4d \ U(1)$

Let us proceed the analysis for the interface 4d $U(1)|3d \ U(1)^2|4d \ U(1)$. This is realized by a single D3-brane interacting with three NS5-branes. It contains a pair of 4d $U(1)$ gauge theories with Neumann b.c. $\mathcal{N}$ and 3d $U(1)^2$ gauge theory at the interface. The coupling between 4d and 3d gauge theories are realized by the bi-fundamental 3d hypermultiplets. We label the magnetic fluxes in terms of two integers $m_1, m_2$.

The half-index is

$$
\prod_{\mathcal{N}}^{4d} U(1|1) - (1)\prod_{\mathcal{N}}^{4d} U(1)(t, z_\alpha; q) = \frac{(q)_{\infty}}{(q_{\frac{1}{2}} t^{-2}; q)_{\infty}} \int_{4d \ U(1)} \frac{ds_{1}}{2\pi is_{1}} \sum_{m_{1},m_{2}\in\mathbb{Z}} \int_{4d \ U(1)} \frac{ds_{2}}{2\pi is_{2}} \int_{4d \ U(1)} \frac{ds_{3}}{2\pi is_{3}} \frac{(q)_{\infty}}{(q_{\frac{1}{2}} t^{-2}; q)_{\infty}} \prod_{\alpha=1}^{2} \frac{(q_{\frac{1}{2}} t s_{\alpha}^{\pm} z_{\alpha}; q)_{\infty}}{(q_{\frac{1}{2}} t^{-1} s_{\alpha}^{\pm} z_{\alpha}; q)_{\infty}}
\times \frac{(q_{\frac{1}{2}} t^{2}; q)_{\infty}}{(q_{\frac{1}{2}} t^{-2}; q)_{\infty}} \left(1 + \frac{1}{4} \sum_{m_{1},m_{2}\in\mathbb{Z}} t^{-|m_{1}|-|m_{1}-m_{2}|}-|m_{2}| \right) \left(\frac{z_{1}}{z_{2}}\right)^{m_{1}} \left(\frac{z_{2}}{z_{3}}\right)^{m_{2}} \left(\frac{z_{3}}{z_{1}}\right)^{m_{1}+m_{2}}.
$$

Again the second line describes two half-indices of Neumann b.c. $\mathcal{N}$ for 4d $U(1)$ gauge theory and the full index of 3d $U(1)^2$ vector multiplet. The other terms correspond to 3d hypermultiplets and monopole operator with the R-charge $\Delta(m) = \frac{1}{2}|m_{1}| + \frac{1}{2}|m_{1} - m_{2}| + \frac{1}{2}|m_{2}|$.

The S-dual interface is constructed by a single D3-brane interacting with three D5′-branes. The three D5′-brane interfaces break down the 4d gauge symmetry down to $U(1)$ and couple 4d $U(1)$ gauge theory to three fundamental twisted hyper multiplets. In other words, the dual interface is described by 4d $U(1)$ gauge theory that includes three twisted fundamental hypers at the defects.

The half-index reads

$$
\prod_{D^4}^{4d} U(1|1) + 3 {\text{thypers}}(t, z_\alpha; q) = \frac{(q)_{\infty}^2}{(q_{\frac{1}{2}} t^2; q)_{\infty}(q_{\frac{1}{2}} t^{-2}; q)_{\infty}} \int_{4d \ U(1)} \frac{ds}{2\pi is} \prod_{\alpha=1}^{3} \frac{(q_{\frac{1}{2}} t s_{\alpha}^{\pm} z_{\alpha}; q)_{\infty}}{(q_{\frac{1}{2}} t^{-1} s_{\alpha}^{\pm} z_{\alpha}; q)_{\infty}}.
$$

As we expect, this agrees with the half-index (7.3).
7.1.3 4d U(2)|3d U(2)|4d U(2)

Now consider the non-Abelian examples. For the interface 4d U(2)|3d U(2)|4d U(2), we have a pair of 4d U(2) SYM theories with Neumann b.c. \( \mathcal{N} \). Each of them is coupled to 3d U(2) vector multiplet via 3d bi-fundamental hyper at the interface. The magnetic fluxes can be labeled by two integers \( m_1, m_2 \).

We can write the half-index as

\[
\mathbb{I}^{4d \ U(2)|3d \ U(2)|4d \ U(2)}_{\mathcal{N}} (t, z_\alpha; q) \\
= \frac{1}{2} \left( \frac{q}{q^2 t^{-2}; q^2} \right)_\infty \oint \frac{ds_1}{2 \pi i s_1} \frac{ds_2}{2 \pi i s_2} \left( \frac{\alpha}{s_1}; q \right)_\infty \left( \frac{\alpha}{s_1}; q \right)_\infty \\
\left( \frac{q^4 t^2; q^4}{q^4 t^{-2}; q^4} \right)_\infty \sum_{m_1, m_2 \in \mathbb{Z}} \oint \frac{ds_3}{2 \pi i s_3} \frac{ds_4}{2 \pi i s_4} \left( \frac{1 - q^{m_1 - m_2}}{q^{1 + |m_1 - m_2|} t s_3^\pm s_4^\pm; q} \right)_\infty \\
\left( \frac{q^4 t^2; q^4}{q^4 t^{-2}; q^4} \right)_\infty \left( \frac{q}{q^2 t^{-2}; q^2} \right)_\infty \sum_{m_1, m_2 \in \mathbb{Z}} \oint \frac{ds_5}{2 \pi i s_5} \frac{ds_6}{2 \pi i s_6} \left( \frac{\alpha}{s_5}; q \right)_\infty \left( \frac{\alpha}{s_6}; q \right)_\infty \\
\left( \frac{q^4 t^2; q^4}{q^4 t^{-2}; q^4} \right)_\infty \prod_{i=3}^{6} \prod_{j=5}^{6} \left( \frac{q^4 t^2; q^4}{q^4 t^{-2}; q^4} \right)_\infty \\
q^{m_1 + m_2} t^{-4|m_1| - 4|m_2| + 2|m_1 - m_2|} . (\frac{z_1}{z_2})^{m_1 + m_2} . (7.5)
\]

The contributions appearing from the second to fourth line are the two half-indices of Neumann b.c. for 4d U(2) SYM theory and the full-index of 3d U(2) vector multiplet. The other contributions are 3d hypers and monopole operator of the R-charge \( \Delta(m) = 2|m_1| + 2|m_2| - |m_1 - m_2| \).

Similarly, the dual interface is a whole 4d U(2) gauge theory with two twisted hypers at the defects. In fact, we find that the half-index \([7.5]\) matches with

\[
\mathbb{I}^{4d \ U(2)+2 \ \text{thypers}}_{\mathcal{D} \times \mathcal{D'}} (t, z_\alpha; q) \\
= \frac{1}{2} \left( \frac{q}{q^2 t^2; q^2} \right)_\infty \oint \frac{ds_1}{2 \pi i s_1} \frac{ds_2}{2 \pi i s_2} \left( \frac{s_1^\pm s_2^\pm; q}{q^{1/2} t^{1/2} s_1^\pm s_2^\pm; q} \right)_\infty \\
\left( \frac{q^4 t^2; q^4}{q^4 t^{-2}; q^4} \right)_\infty \left( \frac{q^4 t^2; q^4}{q^4 t^{-2}; q^4} \right)_\infty \left( \frac{q^4 t^2; q^4}{q^4 t^{-2}; q^4} \right)_\infty \\
\prod_{i=1}^{2} \prod_{\alpha=1}^{2} \left( \frac{q^4 t^2 s_i^\pm z_{i\alpha}^\pm; q}{q^{1/2} t^{1/2} s_i^\pm z_{i\alpha}^\pm; q} \right)_\infty . (7.6)
\]

7.1.4 4d U(2)|3d U(2)²|4d U(2)

As a further check, we examine the interface 4d U(2)|3d U(2)²|4d U(2).
The half-index reads

$$ \Pi_{N'N}^{4d} U(2)(2) - (2)4d U(2) (t, z_{\alpha}; q) $$

$$ = \frac{1}{2} \left( q; z_{\alpha} \right)_{\infty} \int \frac{ds_1}{2\pi is_1} \frac{ds_2}{2\pi is_2} \left( q^{2} t^{-2} s_1; q \right)_{\infty} \left( q^{2} t^{-2} s_2; q \right)_{\infty} $$

$$ \prod_{i=1}^{4} U(2) $$

$$ \times \frac{1}{2} \left( q^{2} t^{2}; q \right)_{\infty} \sum_{m_1, m_2 \in \mathbb{Z}} \int \frac{ds_3}{2\pi is_3} \frac{ds_4}{2\pi is_4} \left( 1 - q^{m_1 - m_2} s_3^{+} s_4^{+} \right) \left( q^{1 + m_1 - m_2} t^{2} s_3^{+} s_4^{+}; q \right)_{\infty} $$

$$ \times \frac{1}{2} \left( q^{2} t^{2}; q \right)_{\infty} \sum_{m_3, m_4 \in \mathbb{Z}} \int \frac{ds_5}{2\pi is_5} \frac{ds_6}{2\pi is_6} \left( 1 - q^{m_3 - m_4} s_5^{+} s_6^{+} \right) \left( q^{1 + m_3 - m_4} t^{2} s_5^{+} s_6^{+}; q \right)_{\infty} $$

$$ \times \frac{1}{2} \left( q^{2} t^{2}; q \right)_{\infty} \int \frac{ds_7}{2\pi is_7} \frac{ds_8}{2\pi is_8} \left( q^{2} t^{-2} s_7^{+}; q \right)_{\infty} \left( q^{2} t^{-2} s_8^{+}; q \right)_{\infty} $$

$$ \prod_{i=1}^{4} U(2) $$

$$ \times \frac{2}{2} \prod_{i=1}^{4} \left( q^{2} t^{2}; q \right)_{\infty} \prod_{i=3}^{6} \left( q^{2} t^{2}; q \right)_{\infty} \prod_{i=5}^{8} \left( q^{2} t^{2}; q \right)_{\infty} \prod_{i=7}^{10} \left( q^{2} t^{2}; q \right)_{\infty} $$

$$ \times q^{-|m_1| - |m_2| - |m_3| - |m_4| - |m_5| - |m_6| - |m_7| - |m_8|} $$

$$ \times \left( z_1 \right)_{m_1 + m_2} \left( z_2 \right)_{m_3 + m_4} \left( z_3 \right)_{m_1 + m_3 + m_4} \left( z_4 \right)_{m_1 + m_3 + m_4} $$

(7.7)

The half-index [7.7] agrees with

$$ \Pi_{D'D'}^{4d} U(2)^{+3} (t, z_{\alpha}; q) $$

$$ = \frac{1}{2} \left( q; z_{\alpha} \right)_{\infty} \int \frac{ds_1}{2\pi is_1} \frac{ds_2}{2\pi is_2} \left( s_1^{+} s_2^{+}; q \right)_{\infty} \left( q^{2} t^{2}; q \right)_{\infty} $$

$$ \times \prod_{i=1}^{3} \frac{1}{2} \left( q^{3} t^{2} s_1^{+}; q \right)_{\infty} \left( q^{3} t^{2} s_2^{+}; q \right)_{\infty} $$

(7.8)

### 7.1.5 4d $U(N)$|3d $U(N)^{k-1}$|4d $U(N)$

Let us discuss the generalization of the interface $4d U(N)$|3d $U(N)^{k-1}$|4d $U(N)$ and its dual. The interface $4d U(N)$|3d $U(N)^{k-1}$|4d $U(N)$ has a pair of 4d $U(N)$ SYM theories obeying Neumann b.c. $\mathcal{N}$ corresponding to the NS5-branes. Both of them are coupled to 3d $U(N)^{k-1}$ vector multiplet at the interface through the 3d bi-fundamental hypermultiplets. Let us label the magnetic fluxes by $k - 1$ sets of $N$-tuple of integers, i.e. $N(k-1)$ integers $\{m^{(l)}_i\}_{i=1,\ldots,N}$ with $l = 1, \cdots, k-1$. 

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The half-index takes the form

\[
\mathbb{I}_{\mathcal{N}^4 \mathbb{U}(N)}^4 \cdot \mathbb{U}(N)^k \cdot \mathbb{U}(N)^k \left( t, z_\alpha; q \right)
\]

\[
= \frac{1}{N!} \left( \frac{q^2 t^{-2}}{q} \right)^N \int \prod_{i=1}^{N} d s_i \prod_{i \neq j}^{N} \left( \frac{s_i t^{-2} z_j}{s_j} q \right)^{\infty}
\]

\[
\times \left[ \frac{1}{N!} \left( \frac{q^2 t^2}{q} \right)^N \prod_{i=1}^{N} \left( \frac{m_i^{(l)}}{s_i} \right) \prod_{i \neq j}^{N} \left( \frac{m_i^{(l)} - m_j^{(l)}}{s_j} q \right)^{\infty} \right]
\]

\[
\times \left( 1 - q^{\frac{|m_i^{(l)} - m_j^{(l)}|}{s_i}} \left( \frac{1 + |m_i^{(l)} - m_j^{(l)}|}{s_j} t^2 s_j^{(l)} \right) \right)^{\infty}
\]

\[
\times \frac{1}{N!} \left( \frac{q^2 t^{-2}}{q} \right)^N \int \prod_{i=N+1}^{N} d s_i \prod_{i \neq j}^{N} \left( \frac{s_i t^{-2} z_j}{s_j} q \right)^{\infty}
\]

\[
\times \prod_{i=1}^{N} \prod_{j=1}^{N} \left( \frac{q^2 + |m_i^{(l)} - m_j^{(l)}|}{s_i t^2 s_j^{(l)} \pm q} \right)^{\infty} \prod_{i=1}^{N} \prod_{j=1}^{N} \left( \frac{q^2 + |m_i^{(l+1)} - m_j^{(l+1)}|}{t^2 s_i^{(l+1)} \pm s_j^{(l+1)} \mp q} \right)^{\infty}
\]

\[
\times \prod_{i=1}^{N} \prod_{j=N+1}^{N} \left( \frac{q^2 + |m_i^{(k-1)}|}{s_i t^{-2} s_j^{(l)} \pm q} \right)^{\infty} \prod_{i=1}^{N} \prod_{j=1}^{N} \left( \frac{q^2 + |m_i^{(k-1)} - m_j^{(k-1)}|}{s_i t^{-2} s_j^{(k-1)} \pm q} \right)^{\infty}
\]

\[
\times q^{\sum_{i=1}^{N} N|m_i^{(1)}|} + \sum_{i=1}^{k-2} \sum_{i=1}^{N} \sum_{j=1}^{N} N|m_i^{(l)} - m_j^{(l+1)}| - \sum_{i=1}^{N} \sum_{j=1}^{N} N|m_i^{(k-1)}| + 2 \sum_{i=1}^{k-1} \sum_{j=1}^{N} |m_i^{(l)} - m_j^{(l)}|
\]

\[
\times \prod_{\alpha=1}^{k-1} \left( \frac{z_\alpha}{z_\alpha + 1} \right) \prod_{i=1}^{N} m_i^{(1)} + \prod_{j=1}^{N} m_j^{(k-1)} \right). \quad (7.9)
\]

The terms appearing from the second to fifth lines are the two half-indices of Neumann b.c. $\mathcal{N}$ for 4d $\mathcal{N} = 4$ $\mathbb{U}(N)$ SYM theory and the full-index of 3d $\mathcal{N} = 4$ $\mathbb{U}(N)^k$ vector multiplet. The terms in the sixth and seventh lines are the contributions from 3d $\mathcal{N} = 4$ hypermultiplets. The remaining terms describe the monopole contributions whose canonical R-charge is

\[
\Delta(m) = \sum_{i=1}^{N} \frac{N|m_i^{(1)}|}{2} + \sum_{l=1}^{k-2} \sum_{i=1}^{N} \sum_{j=1}^{N} \frac{|m_i^{(l)} - m_j^{(l+1)}|}{2} + \sum_{i=1}^{N} \frac{N|m_i^{(k-1)}|}{2} - \sum_{l=1}^{k-1} \sum_{i<j}^{N} |m_i^{(l)} - m_j^{(l)}| \quad (7.10)
\]

where the first and third terms are contributed from bi-fundamental hypers coupled to 4d and 3d gauge theories, the second terms are the contributions from 3d bi-fundamental hypers between gauge nodes in 3d quiver gauge theory and the last terms are contributed from 3d $\mathbb{U}(N)^k$ vector multiplet.

Under S-duality we find the dual interface which is identified with a 4d $\mathcal{N} = 4$ $\mathbb{U}(N)$ SYM theory with $k$ defects corresponding to $k$ D5'-branes, which couple the 4d $\mathbb{U}(N)$ gauge theory to $k$ 3d $\mathcal{N} = 4$ fundamental twisted hypermultiplets.
The half-index for the dual interface should be computed as

\[
\mathbb{I}_{4d}^{U(N) + k \text{ hypers}}(t, z_\alpha; q) = \frac{1}{N!} \left( \frac{q^{2N}}{(q; q^2)_\infty} \right)^N \left( \frac{q^{2N}}{(q; q^2)_\infty} \right) \prod_{i=1}^{N} \frac{ds_i}{2\pi i s_i} \prod_{i \neq j} \left( \frac{q^{\frac{z_i}{s_j}}; q}{q^{\frac{z_j}{s_i}}; q} \right)_{\infty} \left( \frac{q^{\frac{2z_i}{s_j}}; q}{q^{\frac{2z_j}{s_i}}; q} \right)_{\infty} \prod_{i=1}^{N} \prod_{\alpha=1}^{k} \left( \frac{q^{\frac{2z_i}{s_j}}; q}{q^{\frac{2z_j}{s_i}}; q} \right)_{\infty}.
\]  

(7.11)

We expect that the half-index (7.9) is equal to (7.11).

### 7.2 4d $U(L)|3d U(M)|4d U(N)$

We consider the interface 4d $U(L)|3d U(M)|4d U(N)$, which involves a pair of 4d $\mathcal{N} = 4$ $U(L)$ and $U(N)$ gauge theories with Neumann b.c. $\mathcal{N}$ and 3d $U(M)$ vector multiplet where $L$, $M$ and $N$ are not all equal. The corresponding quiver diagram and brane setup are depicted in Figure 13. According to unequal numbers of D3-branes, the dual interface can involve singular boundary conditions specified by Nahm poles corresponding to the D5'-branes. We compute the half-indices for a pair of dual interfaces and check the matching. We find that the half-indices have additional contributions when one of 4d gauge symmetries is smaller than 3d gauge symmetry.

#### 7.2.1 4d $U(1)|3d U(1)|4d U(2)$

For the interface 4d $U(1)|3d U(1)|4d U(2)$, we have 4d $U(1)$ and $U(2)$ gauge theories obeying Neumann b.c. $\mathcal{N}$, corresponding to the NS5-branes. They couple to 3d $U(1)$ vector multiplet via 3d bifundamental hypers. We label the magnetic fluxes by an integer $m$. 

![Figure 13: (a) The quiver diagrams of the 4d $U(L)|3d U(M)|4d U(N)$ interface and its mirror. (b) The corresponding brane configurations.](image-url)
One finds the half-index
\[
\Pi^{4d}_{-N,N} U^{(1) | (1) | 4d U^{(2)} } (t, z_\alpha; q) \\
= \frac{ (q)_{\infty} }{ (q^{\frac{1}{2}} t^{2}; q)_{\infty} } \frac{ ds_1 }{ 2\pi i s_1 } \left( \frac{ (q^{\frac{1}{2}} t^{2}; q)_{\infty} }{ 2\pi i s_2 } \right)^m \\
\times \frac{ (q)_{\infty}^2 }{ (q^{\frac{1}{2}} t^{2}; q)_{\infty}^2 } \frac{ ds_3 }{ 2\pi i s_3 } \frac{ ds_4 }{ 2\pi i s_4 } \left( \frac{ (q^{\frac{1}{2}} t^{2}; q)_{\infty} }{ 2\pi i s_5 } \right)^m \\
\times \prod_{i=1,3,4} \frac{ (q^{\frac{1}{2}} q^{\frac{1}{2}} ; q)_{\infty} }{ (q^{\frac{1}{2}} q^{\frac{1}{2}} ; q)_{\infty} } \frac{ (z_1)_{m} }{ (z_2)_{m} } . \tag{7.12}
\]
Here the contributions in the second and third lines describe the half-indices of Neumann b.c. \(N\) for 4d \(U(1)\) and \(U(2)\) gauge theories and the full-index of 3d \(U(1)\) vector multiplet. The terms in the last line count 3d hypermultiplet and monopole operator of dimension \(\Delta(m) = \frac{3|m|}{2}\).

The dual interface has two defects corresponding to two D5'-brane defects. They breaks the \(U(1)\times U(1)\times U(2)\) gauge symmetry down to \(U(1)\). The number of D3-brane does not change across one of the D5'-brane defect, while it jumps from two to one across the other D5'-brane defect. The former couple the surviving 4d \(U(1)\) gauge theory to 3d fundamental twisted hyper at the defect. The latter defect does not admit 3d fundamental twisted hyper. Instead, it would have the contributions to the half-index from broken \(U(2)\) gauge theory [12].

We find that the half-index (7.12) agrees with
\[
\Pi^{4d}_{-D'} U^{(1) | (1) | 4d U^{(2)} } (t, z_\alpha; q) \\
= \frac{ (q)_{\infty}^2 }{ (q^{\frac{1}{2}} t^{2}; q)_{\infty} (q^{\frac{1}{2}} t^{2}; q)_{\infty} } \frac{ ds }{ 2\pi i s } \left( \frac{ (q^{\frac{1}{2}} t^{2}; q)_{\infty} }{ 2\pi i s } \right)^m \\
\times \frac{ (q^{\frac{1}{2}} t^{2}; q)_{\infty}^2 }{ (q^{\frac{1}{2}} t^{2}; q)_{\infty}^2 } \frac{ ds_1 }{ 2\pi i s_1 } \frac{ ds_4 }{ 2\pi i s_4 } \left( \frac{ (q^{\frac{1}{2}} t^{2}; q)_{\infty} }{ 2\pi i s_5 } \right)^m \\
\times \prod_{i=1,3,4} \frac{ (q^{\frac{1}{2}} q^{\frac{1}{2}} ; q)_{\infty} }{ (q^{\frac{1}{2}} q^{\frac{1}{2}} ; q)_{\infty} } \frac{ (z_1)_{m} }{ (z_2)_{m} } . \tag{7.13}
\]
The terms in the second line are the full-index for 4d \(U(1)\) gauge theory and the half-index of Dirichlet b.c. \(D'\) for 4d \(U(1)\) gauge theory. The terms in the third line includes full-index for 3d fundamental twisted hyper and the contributions from the broken \(U(2)\) gauge theory [12].

### 7.2.2 4d \(U(2) | 3d U(1) | 4d U(2)\)

Next consider the interface 4d \(U(2) | 3d U(1) | 4d U(2)\) in which a pair of two 4d \(U(2)\) gauge theories obeying Neumann b.c. \(N\) and 3d \(U(1)\) gauge theory are included. In this case the both 4d gauge symmetries are non-Abelian and not equal to 3d gauge symmetry.
The half-index is expressed as
\[
\prod_{i=1,2,4,5} \left( \frac{q^{t+\frac{1}{4}|\pm t^{-1}s_{1}^\pm s_{3}^\pm|;q})_{\infty} (q^{t-2s_{1}s_{3};q}_{\infty})}{(q^{s_{1}s_{3};q}_{\infty})\prod_{\alpha=1}^{2} (q^{s_{a}z_{\alpha};q}_{\infty})} \right).
\] (7.14)

In the dual interface the two defects corresponding to D5'-branes break down the \( U(2)\times U(1)\times U(2) \) gauge symmetry down to \( U(1) \) so that there is a 4d \( U(1) \) gauge theory in the whole space. Unequal numbers of D3-branes on both sides of two D5'-branes would give rise to contributions to the half-index from the broken part of gauge theory instead of 3d hypermultiplets.

We find that the half-index (7.14) coincides with
\[
\prod_{i=1,2,4,5} \left( \frac{q^{t+\frac{1}{4}|\pm t^{-1}s_{1}^\pm s_{3}^\pm|;q})_{\infty} (q^{t-2s_{1}s_{3};q}_{\infty})}{(q^{s_{1}s_{3};q}_{\infty})\prod_{\alpha=1}^{2} (q^{s_{a}z_{\alpha};q}_{\infty})} \right).
\] (7.15)

The contributions in the second line include full-index for 4d \( U(1) \) gauge theory as well as the square of half-index of Dirichlet b.c. \( D' \) for 4d \( U(1) \) gauge theory. The last terms describe the contributions to the half-index from the broken gauge theory.

### 7.2.3 4d \( U(2)|3d \ U(1)|4d \ U(3) \)

Let us turn to the interface 4d \( U(2)|3d \ U(1)|4d \ U(3) \). In this case all the gauge symmetries are different and each of 4d gauge symmetries is larger than 3d gauge symmetry.
The half-index is computed as

\[
\Pi_{\mathcal{N}'}^{4d\ U(2)|1|4d\ U(3)}(t, z_\alpha; q) = \frac{1}{2} \left( \frac{q^2}{(q^{\frac{3}{2}}t^{-2}; q)_\infty} \right) \int_\mathbb{R} ds_1 ds_2 \left( \frac{e^{s_1}}{2\pi i s_1} \right)_\infty \left( \frac{e^{s_2}}{2\pi i s_2} \right)_\infty \frac{d^{2} \theta}{(q^{\frac{3}{2}}t^{-2} e^{s_1}; q)^{\infty} (q^{\frac{3}{2}}t^{-2} e^{s_2}; q)^{\infty}} \\
\times \left( \frac{q^{\frac{3}{2}}t^{2}; q)^{\infty}}{(q^{\frac{3}{2}}t^{-2}; q)^{\infty}} \right) \sum_{m \in \mathbb{Z}} \int_\mathbb{R} ds_3 \left( \frac{e^{s_3}}{2\pi i s_3} \right)_\infty \frac{d^{3} \theta}{(q^{\frac{3}{2}}t^{-2} e^{s_3}; q)^{\infty}} \\
\times \left( \frac{1}{3!} \left( \frac{q^{3}}{(q^{\frac{3}{2}}t^{-2}; q)^{\infty}} \right)^3 \int_\mathbb{R} d^{6} \theta \left( \frac{e^{s_1}}{2\pi i s_1} \right)_\infty ... \\
\times \prod_{i=1,2,4,5,6} \left( \frac{q^{\frac{3}{2}+\frac{1}{2}}t^{-1} e^{s_i}; q)^{\infty}}{(q^{\frac{3}{2}}t^{-2}; q)^{\infty}} \right) q^{\frac{5}{2}[m]}. \left( \frac{z_1}{z_2} \right)^{m}.
\] (7.16)

For the dual interface the 4d $U(2) \times U(1) \times U(3)$ gauge symmetry should break down to $U(1)$ according to the D5'-brane interface. The reduction of gauge symmetry has two processes for the two D5'-branes, $U(2) \rightarrow U(1)$ and $U(3) \rightarrow U(1)$. The former is associated to a trivial Nahm pole, i.e. Dirichlet b.c. and the latter is to a Nahm pole of rank 2.

It is confirmed that the half-index (7.16) coincides with

\[
\Pi_{\mathcal{N}'}^{4d\ U(2)|1|4d\ U(3)}(t, z_\alpha; q) = \left( \frac{q^2}{(q^{\frac{3}{2}}t^{2}; q)_\infty} \right)^{\infty} \int_\mathbb{R} ds_1 ds_2 \left( \frac{e^{s_1}}{2\pi i s_1} \right)_\infty \left( \frac{e^{s_2}}{2\pi i s_2} \right)_\infty \frac{d^{2} \theta}{(q^{\frac{3}{2}}t^{2} e^{s_1}; q)^{\infty} (q^{\frac{3}{2}}t^{2} e^{s_2}; q)^{\infty}} \\
\times \left( \frac{q^{\frac{3}{2}}t^{2}; q)^{\infty}}{(q^{\frac{3}{2}}t^{2}; q)^{\infty}} \right) \sum_{m \in \mathbb{Z}} \int_\mathbb{R} ds_3 \left( \frac{e^{s_3}}{2\pi i s_3} \right)_\infty \frac{d^{3} \theta}{(q^{\frac{3}{2}}t^{2} e^{s_3}; q)^{\infty}} \\
\times \left( \frac{1}{3!} \left( \frac{q^{3}}{(q^{\frac{3}{2}}t^{2}; q)^{\infty}} \right)^3 \int_\mathbb{R} d^{6} \theta \left( \frac{e^{s_1}}{2\pi i s_1} \right)_\infty ... \\
\times \prod_{i=1,2,4,5,6} \left( \frac{q^{\frac{3}{2}+\frac{1}{2}}t^{-1} e^{s_i}; q)^{\infty}}{(q^{\frac{3}{2}}t^{2}; q)^{\infty}} \right) q^{\frac{5}{2}[m]}. \left( \frac{z_1}{z_2} \right)^{m}.
\] (7.17)

The terms in the second line are the full-index for 4d $U(1)$ gauge theory, the half-index of Dirichlet b.c. $\cal{D}'$ for 4d $U(1)$ gauge theory and the half-index of Nahm pole b.c. for 4d $U(2)$ gauge theory. The contributions in the last line correspond to the local operators arising from the broken $U(2)$ and $U(3)$ gauge theories [12].

### 7.2.4 4d $U(1)|3d\ U(2)|4d\ U(2)$

Now we consider the interface 4d $U(1)|3d\ U(2)|4d\ U(2)$ in which 3d gauge symmetry is non-Abelian and one of 4d gauge symmetries is smaller than 3d gauge symmetry. Again the 4d and 3d gauge theories are coupled by 3d bi-fundamental hypermultiplet. We label the magnetic fluxes for 3d $U(2)$ gauge theory by two integers $m_1, m_2$. 

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We can compute the half-index for the interface 4d $U(1)|3d$ $U(2)|4d$ $U(2)$ as

$$
\Pi_{\mathcal{N}\mathcal{N}'}^{4d} U(1)|2|4d U(2)\left(t, z_\alpha; q \right) \\
= \frac{(q)_\infty}{(q^{\frac{1}{2}} t^{-2}; q)_\infty} \frac{1}{2\pi is} \int_{\Pi_{\mathcal{N}\mathcal{N}'}^{4d} U(1)} ds \left( \frac{1 - q^{m_1 - m_2} s^2_2 s^2_3}{q^{m_1 + m_2} t^2 s^2_2 s^2_3; q}_\infty \frac{q^{-1 + m_1 - m_2} t^{-2} s^2_2 s^2_3; q}_\infty \right)
$$

$$
\times \frac{1}{2} \frac{(q^{\frac{1}{2}} t^{-2}; q)^2}_\infty \sum_{m_1, m_2 \in \mathbb{Z}} \int_{2\pi is \mathbb{I}^{2}} ds_2 \frac{ds_3}{2\pi is \mathbb{I}^{3}} \frac{(q^{\frac{1}{2}}; q)_\infty (q^{\frac{1}{2}}; q)_\infty}{(q^{\frac{1}{2}} t^{-2} s_2 s_3; q)_\infty (q^{\frac{1}{2}} t^{-2} s_2 s_3; q)_\infty}
$$

$$
\times \prod_{i=2}^{3} \frac{(q^{\frac{1}{2}} t^{-1}; q)_\infty (q^{\frac{1}{2}} t^{-1}; q)_\infty}{(q^{\frac{1}{2}} t^{-1} s^2_1; q)_\infty (q^{\frac{1}{2}} t^{-1} s^2_1; q)_\infty} \cdot \prod_{i=2}^{5} \frac{(q^{\frac{1}{2}} t^{-1}; q)_\infty (q^{\frac{1}{2}} t^{-1}; q)_\infty}{(q^{\frac{1}{2}} t^{-1} s^2_1; q)_\infty (q^{\frac{1}{2}} t^{-1} s^2_1; q)_\infty}
$$

$$
\times q^{\frac{3m_1}{4} + \frac{3m_2}{4} - \frac{m_1 - m_2}{2}} \cdot 3^{-3|m_1|-3|m_2| + 2|m_1 - m_2|} \left( \frac{z_1}{2} \right)^{m_1 + m_2}.
$$

The dual interface has the two D5′ defects in 4d gauge theory which break $U(1) \times U(2) \times U(2)$ gauge symmetry down to $U(1)$. From the brane picture of Figure 13 the number of D3-branes jumps from one to two across one of the D5′-brane and it does not change across the other D5′-brane. For the former defect D5′-brane, we should get Dirichlet b.c. $D'$ for 4d $U(1)$ gauge theory together with the associated extra local operators from the broken gauge theory. On the other hand, the latter defect D5′-brane would couple the $U(2)$ broken gauge theory between two D5′-branes to fundamental twisted hypermultiplet. As a result, one finds the twisted hypermultiplet which transforms as the fundamental representation under the surviving $U(1)$ gauge theory and that which does not couple to the surviving $U(1)$ gauge theory.

In fact, the half-index (7.18) agrees with

$$
\Pi_{\mathcal{N}\mathcal{N}'}^{4d} U(1)|2|4d U(2)\left(t, z_\alpha; q \right) \\
= \frac{(q)_\infty}{(q^{\frac{1}{2}} t^{-2}; q)_\infty} \frac{1}{2\pi is} \int_{\Pi_{\mathcal{N}\mathcal{N}'}^{4d} U(1)} ds \left( \frac{q^{\frac{1}{2}} t^{-2}; q}_\infty \frac{(q^{\frac{1}{2}}; q)_\infty}{(q^{\frac{1}{2}} t^{-2}; q)_\infty} \right)
$$

$$
\times \frac{(g s^{\pm} z^2_1; q)_\infty}{(q^{\frac{1}{2}} t^{-2} s^2 \pm z^2_1; q)_\infty} \cdot \frac{(q^{\frac{1}{2}} t s^{\pm} z^2_2; q)_\infty}{(q^{\frac{1}{2}} t^{-1} s^2 \pm z^2_2; q)_\infty} \cdot \frac{(q^{\frac{1}{2}} t^{-1} s^{\pm} z^2_2; q)_\infty}{(q^{\frac{1}{2}} t^{-1} s^2 \pm z^2_2; q)_\infty}.
$$

The terms in the second line are the full-index for 4d $U(1)$ gauge theory and the half-index of Dirichlet b.c. $D'$ for 4d $U(1)$ gauge theory. The next terms are the extra contributions from the broken $U(2)$ gauge theory, the fundamental twisted hypermultiplet coupled to the surviving $U(1)$ gauge theory and the another twisted hypermultiplet which has no coupling to the surviving $U(1)$ gauge theory.

7.2.5 4d $U(2)|3d$ $U(2)|4d$ $U(3)$

Consider the interface 4d $U(2)|3d$ $U(2)|4d$ $U(3)$. In this case 3d non-Abelian gauge symmetry is smaller than each of 4d gauge symmetries.
The half-index is given by
\[ \Pi_{N_N}^{4d} U(2) U(2) U(3) (t, z_0; q) \]
\[ = \frac{1}{2} (q^{2} t^{-2}; q)^{\infty} \int \frac{ds_1}{2 \pi i s_1} \frac{ds_2}{2 \pi i s_2} \left( \frac{s_1}{s_2}; q \right)^{\infty} \left( \frac{s_2}{s_1}; q \right)^{\infty} \]
\[ \times \left( q^{2} t^{-2}; q \right)^{\infty} \sum_{m_1, m_2 \in \mathbb{Z}} \int \frac{ds_3}{2 \pi i s_3} \frac{ds_4}{2 \pi i s_4} \left( \frac{s_4}{s_3}; q \right)^{\infty} \left( \frac{s_3}{s_4}; q \right)^{\infty} \]
\[ \times \frac{1}{3!} (q^{3} t^{-2}; q)^{3} \int \prod_{i=5}^{7} \frac{ds_i}{2 \pi i s_i} \prod_{i \neq j} \left( \frac{s_i}{s_j}; q \right)^{\infty} \]
\[ \times \prod_{i=1}^{2} \prod_{j=3}^{4} \left( q^{2} t^{-2}; q \right)^{\infty} \sum_{m_1, m_2 \in \mathbb{Z}} \int \prod_{i=5}^{7} \frac{ds_i}{2 \pi i s_i} \prod_{i \neq j} \left( \frac{s_i}{s_j}; q \right)^{\infty} \]
We have the half-index

\[
\mathbb{II}_{N'N}^{4d \, U(1)[2] | 4d \, U(3)} (t, z_\alpha ; q) = \frac{(q)_\infty}{(q^2 t^2 ; q)_\infty} \int \frac{ds}{2 \pi i s} \frac{(q^2 t^2 ; q)_\infty}{(q^2 t^{-2} ; q)_\infty} \frac{(q^2 t^2 ; q)_\infty}{(q^2 t^{-2} ; q)_\infty} \prod_{i=1}^{2} \frac{((q^2 t^{-2} s^{\pm 1} \mp z_\alpha ; q)_\infty \cdot (q^2 t^{-2} s^{\pm 1} \mp z_\alpha ; q)_\infty)}{((q^2 t^{-2} s^\pm \mp z_\alpha ; q)_\infty \cdot (q^2 t^{-2} s^\pm \mp z_\alpha ; q)_\infty)}.
\]

(7.22)

For the dual interface, the 4d \( U(1) \times U(2) \times U(3) \) gauge symmetry should be broken to \( U(1) \) so that there is a 4d \( U(1) \) gauge theory in a whole space. As the numbers of D3-branes are all different, the interface has no fundamental twisted hypermultiplet. For the two defect D5'-branes we should find a pair of Dirichlet boundary conditions \( D' \) for 4d \( U(1) \) gauge theory and the associated additional local operators from broken gauge theory. For the defect D5'-brane corresponding to the reduction \( U(3) \to U(2) \), we would find extra local operators which are neutral under the gauge symmetry as the \( U(2) \) is further broken to \( U(1) \).

We find that the half-index (7.22) coincides with

\[
\mathbb{II}_{N'N}^{4d \, U(1)[2] | 4d \, U(3)} (t, z_\alpha ; q) = \frac{(q)_\infty}{(q^2 t^2 ; q)_\infty} \int \frac{ds}{2 \pi i s} \frac{(q)_\infty}{(q^2 t^2 ; q)_\infty} \frac{(q)_\infty}{(q^2 t^{-2} ; q)_\infty} \prod_{i=1}^{2} \frac{((q^2 t^{-2} s^{\pm 1} \mp z_\alpha ; q)_\infty \cdot (q^2 t^{-2} s^{\pm 1} \mp z_\alpha ; q)_\infty)}{((q^2 t^{-2} s^\pm \mp z_\alpha ; q)_\infty \cdot (q^2 t^{-2} s^\pm \mp z_\alpha ; q)_\infty)}.
\]

(7.23)

As expected, the contributions in the second line include the full-index for 4d \( U(1) \) gauge theory and the square of half-index of Dirichlet b.c. \( D' \) for 4d \( U(1) \) gauge theory. The contributions in the last line describe the extra operators from broken gauge theory. The last terms are the extra local operators which are neutral under the gauge symmetry.

### 7.2.7 4d \( U(1) \times 3d \ U(2) \times 4d \ U(4) \)

Let us examine the interface 4d \( U(1) \times 3d \ U(2) \times 4d \ U(4) \). This is the another example where all the gauge symmetries are different and 3d gauge symmetry is non-Abelian which is larger than one of the 4d gauge symmetry.
One obtains the half-index
\[
\Pi^{4d \ U(1)[(2)]4d \ U(4)}_{\mathcal{N}'} (t, z; q) = \left( \frac{q}{q^t t^{-2}; q} \right)_{\infty} \int_{\mathcal{N}'} ds_i \left( 1 - q^{1+|m_1-m_2|} s_2^+ s_3^+ \right) \left( q^{1+|m_1-m_2|} t^2 s_2^+ s_3^+ ; q \right)_{\infty} \\
\times \frac{1}{2} \left( \frac{q}{q^t t^{-2}; q} \right)^2_{\infty} \sum_{m_1, m_2 \in \mathbb{Z}} \int_{\mathcal{N}'} ds_i \left( 1 - q^{1+|m_1-m_2|} s_2^+ s_3^+ \right) \left( q^{1+|m_1-m_2|} t^2 s_2^+ s_3^+ ; q \right)_{\infty} \\
\times \frac{1}{4!} \left( \frac{q}{q^t t^{-2}; q} \right)^3_{\infty} \prod_{i=4}^6 ds_i \prod_{i \neq j} \frac{(q^t s_i^+ s_j^+)}{q^t s_i^+ s_j^+ ; q} \left( q^t s_i^+ s_j^+ ; q \right)_{\infty} \\
\times \prod_{i=2}^3 \left( \frac{q^{1+|m_1-i|} t^{-1}s_i^+ s_j^+ ; q} q^{1+|m_1-i|} t^{-1}s_i^+ s_j^+ ; q \right)_{\infty} \prod_{i=2}^7 \left( q^{1+|m_1-i|} t^{-1}s_i^+ s_j^+ ; q \right)_{\infty} \\
\times q^{\frac{|m_1-1|+|m_2-1|}{2} - \frac{|m_1-m_2|}{2}} t^{-5|m_1| - |m_2| + 2|m_1 - m_2|} \left( \frac{21}{22} \right)^{m_1 + m_2}.
\] (7.24)

The S-dual interface has the two defects which break 4d $U(1) \times U(2) \times U(4)$ gauge symmetry down to $U(1)$. It includes Dirichlet b.c. $D'$ for the reduction $U(2) \to U(1)$ and rank 2 Nahm' boundary condition for the reduction $U(4) \to U(2)$. There is no 3d twisted hypermultiplet. Instead, there are extra fields from broken gauge theories which are specified by the reduced gauge symmetries. In particular, for the Nahm pole of rank 2, we would find the extra fields which are neutral under the surviving $U(1)$ gauge symmetry as the relevant $U(2)$ symmetry is further broken.

The half-index \(7.24\) coincides with
\[
\Pi^{4d \ U(1)[(2)]4d \ U(4)}_{D'D'} (t, z; q) = \left( \frac{q}{q^t t^{-2}; q} \right)_{\infty} \int_{\mathcal{N}'} ds_i \left( 1 - q^{1+|m_1-m_2|} s_2^+ s_3^+ \right) \left( q^{1+|m_1-m_2|} t^2 s_2^+ s_3^+ ; q \right)_{\infty} \\
\times \frac{1}{2} \left( \frac{q}{q^t t^{-2}; q} \right)^2_{\infty} \sum_{m_1, m_2 \in \mathbb{Z}} \int_{\mathcal{N}'} ds_i \left( 1 - q^{1+|m_1-m_2|} s_2^+ s_3^+ \right) \left( q^{1+|m_1-m_2|} t^2 s_2^+ s_3^+ ; q \right)_{\infty} \\
\times \frac{1}{4!} \left( \frac{q}{q^t t^{-2}; q} \right)^3_{\infty} \prod_{i=4}^6 ds_i \prod_{i \neq j} \frac{(q^t s_i^+ s_j^+)}{q^t s_i^+ s_j^+ ; q} \left( q^t s_i^+ s_j^+ ; q \right)_{\infty} \\
\times \prod_{i=2}^3 \left( \frac{q^{1+|m_1-i|} t^{-1}s_i^+ s_j^+ ; q} q^{1+|m_1-i|} t^{-1}s_i^+ s_j^+ ; q \right)_{\infty} \prod_{i=2}^7 \left( q^{1+|m_1-i|} t^{-1}s_i^+ s_j^+ ; q \right)_{\infty} \\
\times q^{\frac{|m_1-1|+|m_2-1|}{2} - \frac{|m_1-m_2|}{2}} t^{-5|m_1| - |m_2| + 2|m_1 - m_2|} \left( \frac{21}{22} \right)^{m_1 + m_2}.
\] (7.25)

We see that the contributions include the full-index for 4d $U(1)$ gauge theory, the half-index of Nahm' b.c. for $U(2)$ gauge theory, the half-index of Dirichlet b.c. $D'$ for $U(1)$ gauge theory. The other terms are the expected contributions from the broken gauge theories.

### 7.2.8 4d $U(2)|3d \ U(3)|4d \ U(3)$

As a last check we examine the interface 4d $U(2)|3d \ U(3)|4d \ U(3)$ where all the gauge symmetries are non-Abelian. It is described by a pair of 4d $U(2)$ and $U(3)$ gauge theories with Neumann b.c. $N$ coupled to 3d $U(3)$ vector multiplet through the 3d bi-fundamental hypermultiplets. The magnetic fluxes for 3d $U(3)$ gauge theory is labeled by three integers $m_1, m_2, m_3$. 

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One obtains the half-index for the interface 4d $U(2)|3d U(3)|4d U(3)$
\[ \mathcal{I}_{N^c_N}^{4d U(2)/3|4d U(3)}(t, z_0; q) \]
\[ = \frac{1}{2} \frac{(q^2)_\infty^2}{(q^{\frac{t}{2}} t^{-2}; q^2)_\infty^2} \int \frac{ds_1}{2\pi is_1} \frac{ds_2}{2\pi is_2} \left( \frac{\ell_{s_1}; q}{\ell_{s_2}; q} \right)_\infty \left( \frac{\ell_{s_2}; q}{\ell_{s_1}; q} \right)_\infty \]
\[ \times \frac{1}{3!} \frac{(q^2 t^2; q^3)_\infty^3}{(q^{\frac{t}{2}} t^{-2}; q^3)_\infty^3} \sum_{m_1, m_2, m_3 \in \mathbb{Z}} \prod_{i=3}^5 \frac{ds_i}{2\pi is_i} \prod_{i<j} \left( 1 - \frac{q^{m_1 - m_2 - m_3}}{s_i \ell_{s_j}^\mp} \right) \left( \frac{q^{1+m_1 - m_2 - m_3}}{t^2 \ell_{s_j}^\mp ; q} \right)_\infty \]
\[ \times \frac{1}{3!} \frac{(q^2 t^2; q^3)_\infty^3}{(q^{\frac{t}{2}} t^{-2}; q^3)_\infty^3} \prod_{i=6}^8 \frac{ds_i}{2\pi is_i} \prod_{i \neq j} \left( \frac{\ell_{s_i}; q}{\ell_{s_j}; q} \right)_\infty \]
\[ \times \frac{2}{5!} \prod_{i=1}^2 \left( \frac{q^{\frac{t}{2}} t^{-2}; q^3}_\infty \prod_{j=3}^5 \left( q^{\frac{t}{2}} t^{-2}; q^3 \right)_\infty \prod_{j=6}^8 \left( q^{\frac{t}{2}} t^{-2}; q^3 \right)_\infty \right)^{m_1 + m_2 + m_3} \]
\[ \times \frac{t^{-5}[m_1 - 2m_2 - 5m_3 + 2|m_1 - m_2| + 2m_1 - m_3] - 2|m_2 - m_3]}{(7.26)} \]

The contributions from the second to fourth line are the half-indices of Neumann b.c. $N$ for 4d $U(2)$ and $U(3)$ gauge theories and the full-index for 3d $U(3)$ vector multiplet. The terms in the fifth line count 3d hypermultiplets. The remaining terms count monopole operator of dimension $\Delta(m) = \sum_{i=1}^3 \frac{5|m_i|}{2} - \sum_{i<j} |m_i - m_j|$. 

The dual interface includes the defects which break 4d $U(2) \times U(3) \times U(3)$ gauge symmetry down to $U(2)$ in such a way that we have the surviving 4d $U(2)$ SYM theory and Dirichlet b.c. $D'$ for 4d $U(1)$ gauge theory and the associated extra local operators corresponding to the reduction $U(3) \rightarrow U(2)$. The defect $D'\text{brane}$ which has three D3-branes in either sides couple 3d twisted hypers to the surviving 4d $U(2)$ gauge theory and to the broken $U(3)$ theory.

We have confirmed that the half-index \[(7.26)\] coincides with
\[ \mathcal{I}_{N^c_N}^{4d U(2)/3|4d U(3)}(t, z_0; q) \]
\[ = \frac{1}{2} \frac{(q^2)_\infty^2}{(q^{\frac{t}{2}} t^{-2}; q^2)_\infty^2} \int \frac{ds_1}{2\pi is_1} \frac{ds_2}{2\pi is_2} \left( \frac{\ell_{s_1}; q}{\ell_{s_2}; q} \right)_\infty \left( \frac{\ell_{s_2}; q}{\ell_{s_1}; q} \right)_\infty \left( q^{\frac{t}{2}} t^{-2}; q \right)_\infty \]
\[ \times \frac{(q^2)_\infty}{(q^{\frac{t}{2}} t^{-2}; q)_\infty^2} \prod_{i=1}^2 \left( \frac{q^{\frac{t}{2}} t^{-2}; q^2}_\infty \prod_{j=1}^2 \left( q^{\frac{t}{2}} t^{-2}; q^2 \right)_\infty \right)^{q^{\frac{t}{2}} t^{-2}; q^2}_\infty \right)^{q^{\frac{t}{2}} t^{-2}; q^2}_\infty \]
\[ \times \frac{t^{-5}[m_1 - 2m_2 - 5m_3 + 2|m_1 - m_2| + 2m_1 - m_3] - 2|m_2 - m_3]}{(7.27)} \]

As we expect, the second line describes the full-index for 4d $U(2)$ gauge theory and the next line contains the half-index of Dirichlet b.c. $D'$ for 4d $U(1)$ gauge theory, the relevant extra contributions and the full-indices for 3d twisted hypermultiplets.
7.2.9 4d \textit{U}(L)|3d \textit{U}(M)|4d \textit{U}(N)

Let us discuss the general 4d \textit{U}(L)|3d \textit{U}(M)|4d \textit{U}(N) interface whose quiver diagram and brane setup are shown in Figure 13. It has a pair of 4d \( \mathcal{N} = 4 \) \textit{U}(L) and \textit{U}(N) gauge theories which obey Neumann b.c. \( \mathcal{N} \) at the 3d boundary. Each of them couples to 3d \( \mathcal{N} = 4 \) \textit{U}(M) vector multiplet through the 3d boundary bi-fundamental hypermultiplets. We label magnetic fluxes for 3d \textit{U}(M) gauge theory by \( M \) integers \( m_1, \cdots, m_M \). The interface includes bare monopole whose canonical R-charge is given by

\[
\Delta(m) = \sum_{i=1}^{M} \frac{(L+N)|m_i|}{2} - \sum_{i<j} |m_i - m_j|.
\]

(7.28)

To ensure that all monopole operators are above the unitarity bound, we assume that \( L + N \geq 2M - 1 \).

In fact, for \( L + N < 2M - 1 \) the half-indices may not be convergent. Without loss of generality we further assume that \( L \leq N \).

The half-index takes the form

\[
\text{III}^{4d \textit{U}(L)||\text{3d \textit{U}(M)||4d \textit{U}(N)}}(t, z_\alpha; q)
\]

\[
= \frac{1}{L!} \frac{(q^{L/2}; q^L)_{\infty}^L}{(q^{\frac{L}{2}} t^{-2}; q^L)_{\infty}^L} \frac{1}{2\pi i} \prod_{i=1}^{L} ds_i \prod_{i<j} \left( q^{\frac{L}{2}} t^{-\frac{2}{L}}; q^\frac{L}{2} \right)_{\infty}^i \left( q^{\frac{L}{2}} t^{-\frac{2}{L}}; q^\frac{L}{2} \right)_{\infty}^j \\
\times \frac{1}{M!} \frac{(q^2 t^{2}; q^M)_{\infty}^M}{(q^2 t^{-2}; q^M)_{\infty}^M} \sum_{m_1, \cdots, m_M \in \mathbb{Z}} \prod_{i=L+1}^{L+M} ds_i \prod_{i<j} \left( q^{\frac{L}{2}} t^{\frac{2}{L}}; q^\frac{L}{2} \right)_{\infty}^i \left( q^{\frac{L}{2}} t^{\frac{2}{L}}; q^\frac{L}{2} \right)_{\infty}^j \\
\times \frac{1}{N!} \frac{(q^N; q^N)_{\infty}^N}{(q^N t^{-2}; q^N)_{\infty}^N} \prod_{i=L+M+1}^{L+M+N} ds_i \prod_{i<j} \left( q^{\frac{L}{2}} t^{-\frac{2}{L}}; q^\frac{L}{2} \right)_{\infty}^i \left( q^{\frac{L}{2}} t^{-\frac{2}{L}}; q^\frac{L}{2} \right)_{\infty}^j \\
\times L \sum_{i=1}^{L} \sum_{j=L+1}^{L+M} \frac{(q^{\frac{L}{2}} + m_i - m_j + L)}{2} \frac{1}{t^{L}} \frac{1}{s_i - s_j} \frac{1}{q^{\frac{L}{2}} s_i - s_j} \frac{1}{q^{\frac{L}{2}} s_i + s_j} \frac{1}{q^{\frac{L}{2}} s_i - s_j} \frac{1}{q^{\frac{L}{2}} s_i + s_j} \\
\times q^{\sum_{i=1}^{M} (L+N)|m_i|} \frac{1}{4} \sum_{i<j} \frac{|m_i - m_j|}{2} \frac{1}{t^{L}} \cdot t^{-\sum_{i=1}^{L} (L+N)|m_i|+2 \sum_{i<j} |m_i - m_j|} \cdot \left( \frac{z_1}{z_2} \right)^{\sum_{i=1}^{M} m_i}.
\]

(7.29)

The terms from the second to fourth line describe the half-indices of Neumann b.c. \( \mathcal{N} \) for 4d \( \mathcal{N} = 4 \) \textit{U}(L) and \textit{U}(N) gauge theories and full-index for 3d \( \mathcal{N} = 4 \) \textit{U}(M) vector multiplet. The first and second terms in the fifth line count 3d \( \mathcal{N} = 4 \) bi-fundamental hypermultiplets which couple to 4d \textit{U}(L) and 3d \textit{U}(M) gauge theories and those which couple to 4d \textit{U}(N) and 3d \textit{U}(M) gauge theories respectively. The remaining terms count bare monopole operator of dimension (7.28).

Now consider the dual interface. The corresponding quiver and brane configuration are illustrated in Figure 13. When \( M < L \), the 4d \textit{U}(L) \times \textit{U}(M) \times \textit{U}(N) \) gauge symmetry is broken down to \textit{U}(M) and the whole 4d \textit{U}(M) gauge theory remains. For the two defects corresponding to two D5' branes, we would have two Nahm' boundary conditions of rank \( L - M \) and rank \( N - M \). In addition, there are extra contributions to the half-index which come from the broken \textit{U}(L) and \textit{U}(N) gauge theories.
The half-index (7.29) for the dual interface with $M < L$ will be computed as

$$\llbracket_{D^0}^{4d} U(L)|[M]|^{4d} U(N)(t, z; q) = \frac{1}{M!} \left( \frac{q^{2M}}{(q^{\frac{1}{M}} t^2; q)_\infty} \right) \oint \frac{d s_i}{2\pi i s_i} \prod_{i \neq j} \left( \frac{q^n; q}{q^{\frac{1}{M}} t^{2s_i}; q} \right)_\infty \left( q^{\frac{2n}{s_i}}; q \right)_\infty \left( q^{\frac{2n}{s_j}}; q \right)_\infty$$



$$\times \prod_{k=1}^{M-L} \left( \frac{q^{\frac{k-1}{M}} t^{-2(k-1)}; q}{q^{\frac{k}{M}} t^{-2k}; q} \right)_\infty \prod_{j=1}^{M-N} \left( \frac{q^{\frac{1}{M}} t^{-2(j-1)}; q}{q^{\frac{1}{M}} t^{-2j}; q} \right)_\infty \prod_{i=1}^{M-N} \left( \frac{q^{\frac{4-N-M}{2}} t^{1-(N-M)} s_i^{\pm \pm \pm}; q}{q^{\frac{4-N-M}{2}} t^{1-(N-M)} s_i^{\pm \pm \pm}; q} \right)_\infty. \quad (7.30)$$

The terms in the second and third line are the full-index for 4d $U(M)$ gauge theory and half-indices for Nahm’ boundary conditions of rank $(L - M)$ and rank $(N - M)$. The terms in the last line describe the extra contributions from broken gauge theories.

It is expected that the half-indices (7.29) and (7.30) give the same answer for $M < L$.

For $L \leq M$, the gauge symmetry $U(L) \times U(M) \times U(N)$ is broken to $U(L)$ and the $U(L)$ gauge symmetry is kept in the whole 4d theory. For $L < M$ the defect of D5’-brane which has $L$ and $M$ D3-branes in their sides may give rise to the Nahm’ b.c. of rank $L - M$ and the associated extra degrees of freedom at the interface which couple to the surviving 4d $U(L)$ gauge theory. For $L = M$, it couples 3d fundamental twisted hypermultiplet to the 4d $U(L)$ gauge theory. On the other defect of D5’-brane which has $M$ and $N$ D3-branes in their sides would lead to the Nahm’ b.c. of rank $N - M$ and the associated extra degrees of freedom at the interface. These extra operators would couple to the surviving $U(L)$ gauge theory however, further degrees of freedom which do not couple to the surviving $U(L)$ gauge theory would appear in contrast to the case with $M < L$.

Thus the half-index will be evaluated as

$$\llbracket_{D^0}^{4d} U(L)|[M]|^{4d} U(N)(t, z; q) = \frac{1}{L!} \left( \frac{q^{2L}}{(q^{\frac{1}{L}} t^2; q)_\infty} \right) \oint \frac{d s_i}{2\pi i s_i} \prod_{i \neq j} \left( \frac{q^n; q}{q^{\frac{1}{L}} t^{2s_i}; q} \right)_\infty \left( q^{\frac{2n}{s_i}}; q \right)_\infty \left( q^{\frac{2n}{s_j}}; q \right)_\infty$$



$$\times \prod_{k=1}^{M-L} \left( \frac{q^{\frac{k-1}{L}} t^{-2(k-1)}; q}{q^{\frac{k}{L}} t^{-2k}; q} \right)_\infty \prod_{j=1}^{M-N} \left( \frac{q^{\frac{1}{L}} t^{-2(j-1)}; q}{q^{\frac{1}{L}} t^{-2j}; q} \right)_\infty \prod_{i=1}^{M-N} \left( \frac{q^{\frac{4-N-M}{2}} t^{1-(N-M)} s_i^{\pm \pm \pm}; q}{q^{\frac{4-N-M}{2}} t^{1-(N-M)} s_i^{\pm \pm \pm}; q} \right)_\infty. \quad (7.31)$$

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The terms in the second and third line are the full-index for 4d $U(L)$ gauge theory and half-indices for Nahm’ boundary conditions of rank $(M - L)$ and rank $(N - M)$. The terms in the fourth line are the extra contributions from broken gauge theories which couple to the surviving gauge theory. The terms in the last line is those which are neutral under the surviving gauge symmetry.

For $L \leq M$, the half-index (7.29) will be equal to the half-indices (7.31).

### 7.3 4d $U(1) \mid 3d \text{SQED}_{N_f} \mid 4d \ U(1)$

Let us turn to the interfaces which include 3d $\mathcal{N} = 4$ gauge theories with flavors. We consider the interface which has a pair of 4d $\mathcal{N} = 4$ $U(1)$ gauge theories with Neumann b.c. $\mathcal{N}$ coupled to 3d $\mathcal{N}' = 4$ SQED$_{N_f}$ through the 3d bi-fundamental hypermultiplets. We denote this interface by 4d $U(1)\mid 3d \text{SQED}_{N_f} \mid 4d \ U(1)$. The corresponding quiver diagram and brane configuration are illustrated in Figure 14. It is realized by a single D3-brane intersecting with two NS5-branes and $N_f$ D5-branes. On the other hand, the S-dual configuration contains a single D3-brane interacting with $N_f$ NS5'-branes and two D5'-branes. For the corresponding dual interface involves 3d Abelian quiver gauge theories as for the Abelian mirror symmetry, however, the topological symmetry in the original interface is mapped to the flavor symmetry for the twisted hypers living at defects in the bulk 4d theory which are decoupled from the 3d quiver gauge theories. We compute the half-indices to check the dualities of the interfaces.

#### 7.3.1 4d $U(1) \mid 3d \text{SQED}_1 \mid 4d \ U(1)$

We start with the simplest case, i.e. the interface 4d $U(1)\mid 3d \text{SQED}_1 \mid 4d \ U(1)$. A pair of 4d $U(1)$ gauge theories obeying Neumann b.c. $\mathcal{N}$ is coupled to 3d $U(1)$ vector multiplet via 3d bi-fundamental hyper. In addition, 3d $U(1)$ vector multiplet is coupled to a fundamental hyper. We label the magnetic fluxes for the 3d $U(1)$ gauge theory by an integer $m$. 

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Figure 14: (a) The quiver diagrams of the interface 4d $U(1)\mid 3d \text{SQED}_{N_f} \mid 4d \ U(1)$ and its mirror. (b) The corresponding brane configurations.
The half-index of the 4d $U(1)|$SQED$_1|4d U(1)$ interface reads

\[
\Pi_{\mathcal{N}',\mathcal{N}}^{4d U(1)|\text{SQED}_1|4d U(1)}(t, z_\alpha; q)
= \frac{(q)_{\infty}}{(q^z t^2; q)_{\infty}} \int \frac{ds_1}{2\pi i s_1} \frac{(q^z t^2; q)_{\infty}}{(q^z t^{-2}; q)_{\infty}} \sum_{m \in \mathbb{Z}} \int \frac{ds_2}{2\pi i s_2} \frac{(q)_{\infty}}{(q^z t^{-2}; q)_{\infty}} \int \frac{ds_3}{2\pi i s_3}
\times \frac{(q^z t^2; q)_{\infty}}{(q^z t^2; q)_{\infty}} \frac{(q^z t^2; q)_{\infty}}{(q^z t^{-2}; q)_{\infty}} \frac{(q^z t^2; q)_{\infty}}{(q^z t^{-2}; q)_{\infty}} \frac{(q^z t^2; q)_{\infty}}{(q^z t^{-2}; q)_{\infty}}
\times q^{\frac{3|m|}{2} t^{-3|m|}} z_{\alpha}^{-m}.
\]

(7.32)

The contributions in the second line are the square of half-index of Neumann b.c. $\mathcal{N}$ for 4d $U(1)$ gauge theory and the full-index for 3d $U(1)$ vector multiplet. The terms in the next line are the contributions from 3d hypermultiplets. The terms in the last line count bare monopole operator of dimension $\Delta(m) = \frac{3|m|}{2}$.

From the brane picture in Figure 14 the dual interface is realized by a single D3-brane which passes through two D5'-branes and a single NS5'-brane. It does not have 3d gauge theory, instead it has a pair of 4d $\mathcal{N} = 4$ $U(1)$ gauge theories with Neumann b.c. $\mathcal{N}'$ corresponding the NS5'-brane. Each of 4d theories also includes a defect of D5'-brane that couples a 3d fundamental twisted hypermultiplet to 4d $U(1)$ gauge theory. In addition, there would be the 3d bi-fundamental twisted hypermultiplet charged under two 4d $U(1)$ gauge symmetries.

In fact, we find that the half-index (7.32) coincides with

\[
\Pi_{\mathcal{N}',\mathcal{N}}^{4d U(1)+\text{hyper}|4d U(1)+\text{hyper}}(t, z_\alpha; q)
= \frac{(q)_{\infty}}{(q^z t^2; q)_{\infty}} \int \frac{ds_1}{2\pi i s_1} \frac{(q)_{\infty}}{(q^z t^2; q)_{\infty}} \int \frac{ds_2}{2\pi i s_2} \frac{(q)_{\infty}}{(q^z t^2; q)_{\infty}} \int \frac{ds_3}{2\pi i s_3}
\times \frac{(q^z t^2; q)_{\infty}}{(q^z t^2; q)_{\infty}} \frac{(q^z t^2; q)_{\infty}}{(q^z t^{-2}; q)_{\infty}} \frac{(q^z t^2; q)_{\infty}}{(q^z t^{-2}; q)_{\infty}} \frac{(q^z t^2; q)_{\infty}}{(q^z t^{-2}; q)_{\infty}}
\times q^{\frac{3|m|}{2} t^{-3|m|}} z_{\alpha}^{-m}.
\]

(7.33)

We see that the topological symmetry associated to the fugacities $z_\alpha$ is mapped to the flavor symmetry for the defect twisted hypermultiplets.

### 7.3.2 4d $U(1)|T[SU(2)]|4d U(1)$

Next consider the interface 4d $U(1)|T[SU(2)]|4d U(1)$. The interface has a pair of 4d $U(1)$ gauge theories with Neumann b.c. $\mathcal{N}$ coupled to $T[SU(2)]$ via 3d bi-fundamental hypermultiplets.
The half-index of the 4d \( U(1) | T[SU(2)] \) \( | 4d \ U(1) \) interface is

\[
\frac{1}{\mathcal{N}^\ast} \int \frac{d4d U(1)}{4d U(1)} (t, x_\alpha, z_\alpha; q) = \frac{(q)_\infty}{(q^2 t^2; q)_\infty} \frac{ds_1}{2 \pi i s_1} \frac{(q^2 t^2 q)_\infty}{(q^2 t^2; q)_\infty} \sum_{m \in \mathbb{Z}} (q^2 t^2; q) \int \frac{ds_3}{2 \pi i s_3} \left( \prod_{i=1}^2 \frac{(q^{1/2 + |m|} t^{-1} s_2^2 x_1^i; q)_\infty}{(q^{1/2 + |m|} t s_2^2 x_1^i; q)_\infty} \right) \frac{(q^{1/2} + |m|) t^{-1} s_2^2 x_1^i; q)_\infty}{(q^{1/2} + |m|) t s_2^2 x_1^i; q)_\infty} \times q^{|m| t^{-4|m|} z_2^i},
\]

\((7.34)\)

where \( x_\alpha \) are the fugacities for the \( SU(2) \) flavor symmetry of \( T[SU(2)] \) and \( z_\alpha \) are the fugacities for the topological symmetry of \( T[SU(2)] \) respectively.

Under the action of S-duality we find the dual interface which has a pair of 4d \( U(1) \) gauge theories with Neumann b.c. \( \mathcal{N}^\ast \) corresponding to the NS5'-branes. Each of 4d theories has a defect of D5'-brane that couple 3d fundamental twisted hyper to a 4d \( U(1) \) gauge theory. Besides, the interface includes 3d \( U(1) \) twisted vector multiplet coupled to both 4d \( U(1) \) gauge theories via 3d twisted hypermultiplets. We let \( m \) be the label of magnetic fluxes for the 3d \( \tilde{U}(1) \) gauge theory.

Then the half-index takes the form

\[
\frac{1}{\mathcal{N}^\ast} \int \frac{d4d U(1) + \text{thyper} (\tilde{1})}{4d U(1) + \text{thyper} (\tilde{1})} (t, x_\alpha, z_\alpha; q) = \frac{(q)_\infty}{(q^2 t^2; q)_\infty} \frac{ds_1}{2 \pi i s_1} \frac{(q^2 t^2 q)_\infty}{(q^2 t^2; q)_\infty} \sum_{m \in \mathbb{Z}} \frac{ds_2}{2 \pi i s_2} \frac{(q)_\infty}{(q^2 t^2; q)_\infty} \frac{ds_3}{2 \pi i s_3} \left( \prod_{i=1}^2 \frac{(q^{1/2 + |m|} t^{-1} s_2^2 x_1^i; q)_\infty}{(q^{1/2 + |m|} t s_2^2 x_1^i; q)_\infty} \right) \frac{(q^{1/2} + |m|) t^{-1} s_2^2 x_1^i; q)_\infty}{(q^{1/2} + |m|) t s_2^2 x_1^i; q)_\infty} \times q^{|m| t^{-4|m|} z_2^i}.
\]

\((7.35)\)

Here the fugacities \( z_\alpha \) are associated with the flavor symmetry for the defect hypermultiplets which are decoupled from the 3d \( \tilde{U}(1) \) gauge theory and the fugacities \( x_\alpha \) are associated with the topological symmetry of the 3d \( \tilde{U}(1) \) gauge theory.

We have checked that the half-indices \((7.34)\) and \((7.35)\) agree with each other.

### 7.3.3 4d \( U(1) | 3d \text{ SQED}_3 \) \( | 4d \ U(1) \)

To check the dualities of interfaces, let us further examine the interface 4d \( U(1) | 3d \text{ SQED}_3 \) \( | 4d \ U(1) \).
The half-index of the 4d $U(1)|3d$ SQED$_3|4d$ $U(1)$ interface takes the form

$$
\Pi_{NN'}^{4d U(1)|3d SQED_3|4d U(1)}(t, x_\alpha, z_\alpha; q) = \frac{(q)_\infty}{(q^{1/2} t^2; q)_\infty} \oint_{\Pi_{NN'}^{4d U(1)}} \frac{ds_1}{2\pi i s_1} \left( \frac{(q^{1/2} t^{2}; q)_\infty}{(q^{1/2} t^{2}; q)_\infty} \sum_{m \in \mathbb{Z}} \oint_{\Pi_{NN'}^{4d U(1)}} \frac{ds_2}{2\pi i s_2} \frac{(q)_\infty}{(q^{1/2} t^{2}; q)_\infty} \oint_{\Pi_{NN'}^{4d U(1)}} \frac{ds_3}{2\pi i s_3} \right)
$$

where $x_\alpha$ and $z_\alpha$ are the fugacities for the flavor and topological symmetries of 3d SQED$_3$.

For the dual interface we have a pair of 4d $U(1)$ gauge theories obeying Neumann b.c. $N'$. Each of 4d theories has a defect which couples the 3d fundamental twisted hyper to a 4d $U(1)$ gauge theory. In addition, they are coupled to 3d $U(1) \times U(1)$ twisted vector multiplet through the 3d bi-fundamental twisted hyper.

As expected from the above, we check that the half-index (7.36) coincides with

$$
\Pi_{NN'}^{4d U(1) + th\text{y}(1) - (1)} 4d U(1) + th\text{y}(t, x_\alpha, z_\alpha; q) = \frac{(q)_\infty}{(q^{1/2} t^{2}; q)_\infty} \oint_{\Pi_{NN'}^{4d U(1)}} \frac{ds_1}{2\pi i s_1} \left( \frac{(q^{1/2} t^{2}; q)_\infty}{(q^{1/2} t^{2}; q)_\infty} \sum_{m_1, m_2 \in \mathbb{Z}} \oint_{\Pi_{NN'}^{4d U(1)}} \frac{ds_2}{2\pi i s_2} \oint_{\Pi_{NN'}^{4d U(1)}} \frac{ds_3}{2\pi i s_3} \frac{(q)_\infty}{(q^{1/2} t^{2}; q)_\infty} \oint_{\Pi_{NN'}^{4d U(1)}} \frac{ds_4}{2\pi i s_4} \right)
$$

Here $z_\alpha$ are the fugacities for the flavor symmetries for the defect twisted hypermultiplets and $x_\alpha$ are the fugacities for the topological symmetry for the boundary 3d $U(1) \times U(1)$ twisted vector multiplet.

### 7.3.4 4d $U(1)|3d$ SQED$_N$, 4d $U(1)$

Now we would like to propose the generalization. For the interface 4d $U(1)|3d$ SQED$_N$, 4d $U(1)$, we have a pair of 4d $N = 4$ $U(1)$ gauge theories living in half-spaces obeying Neumann b.c. $N$. The interface has 3d $N = 4$ SQED$_N$ coupled to the 4d theories in either sides.
The half-index of the 4d $U(1)|3d$ SQED$_{N_f}$|4d $U(1)$ interface is

$$\prod_{N'N}^{4d \ U(1)|\text{SQED}_{N_f}|4d \ U(1)} (t, x_\alpha, z_\alpha; q) = \frac{(q)_\infty}{(q^2 t^-; q)_\infty} \oint \frac{ds_1}{2\pi i s_1} \frac{(q^2 t^-; q)_\infty}{(q^2 t^2; q)_\infty} \sum_{m_1 \in \mathbb{Z}} \oint \frac{ds_2}{2\pi i s_2} \frac{(q^2 t^-; q)_\infty}{(q^2 t^2; q)_\infty} \oint \frac{ds_3}{2\pi i s_3}$$

$$\times \frac{(q^{\frac{N_f}{2}} + t^2 s_1^\pm; q)_\infty}{(q^{\frac{N_f}{2}} + t s_1^\pm; q)_\infty} \prod_{\alpha=1}^{N_f} \frac{N_f}{(q^{\frac{N_f}{2}} + t^{-1} s_2^\pm x_{\alpha}; q)_\infty}{(q^{\frac{N_f}{2}} + t^{-1} s_2^\pm; q)_\infty} \times \left( q^\frac{1}{2} \prod_{i=1}^{N_f-1} (q^{2} + \frac{m_i}{t} - 1 s_{N_f-1}^\pm; q)_\infty \right) \times \left( q^\frac{1}{2} \prod_{i=1}^{N_f-1} (q^{2} + \frac{m_i}{t} - 1 s_{N_f-1}^\pm; q)_\infty \right) \times q^{-\sum_{i=1}^{N_f-1} (m_i + m_{N_f-1})} \cdot \frac{N_f}{m_1 + m_{N_f-1} + \sum_{i=1}^{N_f-2} (m_i - m_{i+1})} \cdot \prod_{i=1}^{N_f-1} \left( \frac{x_i}{x_{i+1}} \right)^{m_i} \left( x_{N_f} \right)^{m_1 + m_{N_f-1}}$$

(7.38)

where $x_\alpha$ are the fugacities for the $SU(N_f)$ flavor symmetry of 3d SQED$_{N_f}$ with $\prod_{\alpha=1}^{N_f} x_\alpha = 1$ and $z_\alpha$ are the fugacities for the topological symmetries of the 3d SQED$_{N_f}$. The terms in the second line are the two half-indices of Neumann b.c. $N'$ for 4d $N = 4$ U(1) gauge theory and the full-index for 3d $N = 4$ U(1) vector multiplet. The terms in the next line describe the 3d $N = 4$ hypermultiplets. The terms in the last line count monopole operator of dimension $\Delta(m) = (N_f+2)|m|$. The dual interface involves a pair of 4d $N = 4$ U(1) gauge theories living in a half-space obeying Neumann b.c. $N'$. Unlike the original interface, each of 4d theories has a defect that couple a 3d $N = 4$ fundamental twisted hypermultiplet to the associated 4d U(1) gauge theories. In addition, the interface has a 3d $N = 4$ U(1) twisted vector multiplet coupled to both 4d gauge theories through the 3d $N = 4$ bi-fundamental twisted hypermultiplets.

The half-index for the dual interface should take the form

$$\prod_{N'N}^{4d \ U(1)} \text{thyper}(t, x_\alpha, z_\alpha; q)$$

$$= \frac{(q)_\infty}{(q^2 t^2; q)_\infty} \oint \frac{ds_1}{2\pi i s_1} \frac{(q^2 t^2; q)_\infty}{(q^2 t^-; q)_\infty} \sum_{m_1 \in \mathbb{Z}} \oint \frac{ds_2}{2\pi i s_2} \frac{(q^2 t^-; q)_\infty}{(q^2 t^2; q)_\infty} \oint \frac{ds_3}{2\pi i s_3}$$

$$\times \frac{(q^{\frac{N_f}{2}} + t^2 s_1^\pm; q)_\infty}{(q^{\frac{N_f}{2}} + t^{-1} s_1^\pm; q)_\infty} \prod_{\alpha=1}^{N_f} \frac{N_f}{(q^{\frac{N_f}{2}} + t s_2^\pm x_{\alpha}; q)_\infty}{(q^{\frac{N_f}{2}} + t s_2^\pm; q)_\infty} \times \left( q^\frac{1}{2} \prod_{i=1}^{N_f-1} (q^{2} + \frac{m_i}{t} - 1 s_{N_f-1}^\pm; q)_\infty \right) \times \left( q^\frac{1}{2} \prod_{i=1}^{N_f-1} (q^{2} + \frac{m_i}{t} - 1 s_{N_f-1}^\pm; q)_\infty \right) \times q^{-\sum_{i=1}^{N_f-1} (m_i + m_{N_f-1})} \cdot \frac{N_f}{m_1 + m_{N_f-1} + \sum_{i=1}^{N_f-2} (m_i - m_{i+1})} \cdot \prod_{i=1}^{N_f-1} \left( \frac{x_i}{x_{i+1}} \right)^{m_i} \left( x_{N_f} \right)^{m_1 + m_{N_f-1}}$$

(7.39)

where $z_\alpha$ is the fugacities for the flavor symmetry of defect 3d twisted hypermultiplets and $x_\alpha$ is the fugacities for the topological symmetry of the 3d $U(1)N_f^{-1}$ twisted vector multiplet. The contributions in the second line are the square of half-index of Neumann b.c. $N'$ for 4d U(1) gauge theory and the full-index for 3d $U(1)N_f^{-1}$ twisted vector multiplet. The terms in the third and fourth lines correspond to the contributions of 3d $N = 4$ twisted hypermultiplets. The terms in the last line count monopole operator of dimension $\Delta(m) = \frac{1}{2}(m_1 + m_{N_f-1} + \sum_{i=1}^{N_f-2} (m_i - m_{i+1}))$.

It is expected that the half-indices (7.38) and (7.39) agree with each other.
Figure 15: (a) The quiver diagrams of the interface $4d\ U\!(N)|3d\text{ SQED}_{N_f}|4d\ U\!(M)$ and its mirror.  
(b) The corresponding brane configurations.

7.4 $4d\ U\!(N)|3d\text{ SQED}_{N_f}|4d\ U\!(M)$

We next turn to the interface $4d\ U\!(N)|3d\text{ SQED}_{N_f}|4d\ U\!(M)$ where $N, M > 1$. The corresponding quiver diagram and brane construction are drawn in Figure 15. Unlike the interface discussed in section 7.3, the dual interface may include singular boundary conditions for $4d\ U\!(N)$ and $U\!(M)$ gauge theories which are characterized by the Nahm poles of rank $(N - 1)$ and rank $(M - 1)$. We calculate the half-indices for a pair of dual interfaces and find the matching.

7.4.1 $4d\ U\!(2)|3d\text{ SQED}_1|4d\ U\!(3)$

Let us consider the interface $4d\ U\!(2)|3d\text{ SQED}_1|4d\ U\!(3)$. This contains a pair of $4d\ \mathcal{N} = 4\ U\!(2)$ and $U\!(3)$ gauge theories obeying Neumann b.c. $\mathcal{N}$ and $3d\ \mathcal{N} = 4\text{ SQED}_1$ which is coupled to the both $4d$ theories through the $3d\ \mathcal{N} = 4$ hypermultiplets.
The half-index of the 4d \( U(2) | 3d \text{ SQED}_1 | 4d \ U(3) \) interface is

\[
\mathcal{I}_{N_{\mathcal{N}}}^{4d} U(2)|3d \text{ SQED}_1|4d U(3) (t, z_\alpha; q) \\
= \frac{1}{2} \left( \frac{q^2}{z_\alpha^2} \right) \int \frac{ds_1}{2\pi is_1} \frac{ds_1}{2\pi is_2} \left( q^{2} t^{-2} \frac{z_1}{z_2}; q \right)_\infty \left( q^{2} t^{-2} \frac{z_2}{z_1}; q \right)_\infty \\
\times (q^{2} t^2; q)_\infty \sum_{m \in Z} \int \frac{ds_3}{2\pi is_3} \frac{1}{3!} \left( q^{2} t^{-2}; q \right)^3 \int \prod_{i=1}^{6} \frac{ds_i}{2\pi is_i} \prod_{j \neq i} \left( q^{2} t^{-2} \frac{z_i}{z_j}; q \right)_\infty \\
\times \prod_{i=1}^{2} \left( q^{\frac{1}{2} + \frac{1}{2}l} t^{-1} s_i^+ s_i^- ; q \right)_\infty \left( q^{\frac{1}{2} + \frac{1}{2}l} t^{-1} s_i^+ s_i^- ; q \right)_\infty \left( q^{\frac{1}{2} + \frac{1}{2}l} t s_i^+ s_i^- ; q \right)_\infty \\
\times q^{\frac{1}{2} + \frac{1}{2}l} t^{-6} l \prod_{m} \left( z_m \right)^{-m} (7.40)
\]

where \( z_\alpha \) are the fugacities associated to the topological symmetry for SQED$_1$.

For the S-dual interface, according to the presence of D5'-branes, the 4d bulk \( U(2) \) and \( U(3) \) gauge symmetries should break down to \( U(1) \)'s. Correspondingly, one finds Dirichlet b.c. \( \mathcal{D}' \) for 4d \( U(1) \) gauge theory for the reduction \( U(2) \rightarrow U(1) \) and Nahm' b.c. for 4d \( U(2) \) gauge theory for the reduction \( U(3) \rightarrow U(1) \). There is no defect twisted hypermultiplet, however, from the broken gauge theories one finds extra degrees of freedom as we have discussed. In addition, surviving two 4d \( U(1) \) gauge theories should further satisfy Nahm b.c. \( \mathcal{N}' \) due to the NS5'-branes. The interface has no 3d gauge multiplet, however it has 3d bi-fundamental twisted hyper which couples to the surviving two 4d \( U(1) \) gauge theories.

In fact, we find that the half-index \([7.40]\) coincides with

\[
\mathcal{I}_{N_{\mathcal{N}'}}^{4d} U(2) \rightarrow U(1) | 4d U(3) \rightarrow U(1) (t, z_\alpha; q) \\
= \left( \frac{q}{z_\alpha^2} \right) \int \frac{ds_1}{2\pi is_1} \frac{ds_1}{2\pi is_2} \left( q^{2} t^{-2} ; q \right)_\infty \left( q^{2} t^{-2} ; q \right)_\infty \left( q^{2} t^{-2} ; q \right)_\infty \left( q^{2} t^2 ; q \right)_\infty \\
\times (q^{2} t^2; q)_\infty \sum_{m \in Z} \int \frac{ds_3}{2\pi is_3} \frac{1}{3!} \left( q^{2} t^{-2}; q \right)^3 \int \prod_{i=1}^{6} \frac{ds_i}{2\pi is_i} \prod_{j \neq i} \left( q^{2} t^{-2} \frac{z_i}{z_j}; q \right)_\infty \\
\times \prod_{i=1}^{2} \left( q^{\frac{1}{2} + \frac{1}{2}l} t^{-1} s_i^+ s_i^- ; q \right)_\infty \left( q^{\frac{1}{2} + \frac{1}{2}l} t^{-1} s_i^+ s_i^- ; q \right)_\infty \left( q^{\frac{1}{2} + \frac{1}{2}l} t s_i^+ s_i^- ; q \right)_\infty \\
\times q^{\frac{1}{2} + \frac{1}{2}l} t^{-6} l \prod_{m} \left( z_m \right)^{-m} (7.41)
\]

The contributions in the second line include the expected four half-indices for 4d gauge theories. The first terms in the last line correspond to the extra local operators from the broken gauge theory for the reduction \( U(2) \rightarrow U(1) \), the second terms there describe the 3d bi-fundamental twisted hypermultiplets, the last ones are the extra contributions from the broken gauge theory for the reduction \( U(3) \rightarrow U(1) \).

\textbf{7.4.2 4d} \( U(2) | T[SU(2)] | 4d \ U(3) \)

Consider the interface 4d \( U(2) | T[SU(2)] | 4d \ U(3) \) whose dual interface admits 3d gauge theory.
Similarly, we can write the half-index of the interface 4d $U(2)|T[SU(2)]|U(3)$ as

$$
\Pi_{N'N}^{4d} U(2)\rightarrow U(1)|D(1)\rightarrow U(1)\rightarrow U(1)(t, x_\alpha, z_\alpha; q)
$$

$$
= \frac{1}{2} \frac{(q^2 t^2; q)_\infty}{(q^2 t^{-2}; q)_\infty} \int \frac{ds_1}{2\pi i s_1} \frac{ds_2}{2\pi i s_2} \frac{(z_{s_1}; q)_\infty}{(z_{s_2}; q)_\infty} \left( \frac{q^2 t^{-2} s_1; q}{q^2 t^{-2} s_2; q} \right)_\infty \left( \frac{q^2 t^{-2} s_2; q}{q^2 t^{-2} s_1; q} \right)_\infty
$$

$$
\times \frac{(q^2 t^2; q)_\infty}{(q^2 t^{-2}; q)_\infty} \sum_{m \in \mathbb{Z}} \int \frac{ds_3}{2\pi i s_3} \frac{1}{3!} \frac{(q^3; q)_\infty}{(q^3 t; q)_\infty} \int \frac{ds_4}{2\pi i s_4} \prod_{i \neq j} \frac{z_{s_i}; q)_\infty}{(z_{s_j}; q)_\infty} \left( \frac{q^2 t^{-2} s_i; q}{q^2 t^{-2} s_j; q} \right)_\infty
$$

$$
\times \prod_{i=1}^{2} \frac{(q^4 + |i| t^{-1} s_i t^{-1} s_i^2; q)_\infty}{(q^4 t^2; q)_\infty} \prod_{\alpha=1}^{2} \frac{(q^4 + |i| t^{-1} s_i^2 t^{-1} s_i^3; q)_\infty}{(q^4 t^2; q)_\infty} \prod_{i=1}^{6} \frac{(q^4 + |i| t^{-1} s_i t^{-1} s_i^3; q)_\infty}{(q^4 t^2; q)_\infty} \prod_{i=1}^{3} \frac{(q^4 + |i| t^{-1} s_i t^{-1} s_i^3; q)_\infty}{(q^4 t^2; q)_\infty} \prod_{i=1}^{4} \frac{(q^4 + |i| t^{-1} s_i t^{-1} s_i^3; q)_\infty}{(q^4 t^2; q)_\infty}
$$

$$
\times q^{7|m| t^{-7|m|} m \cdot -m}
$$

(7.42)

where $x_\alpha$ and $z_\alpha$ are the fugacities for the flavor and topological symmetries of $T[SU(2)]$.

For the dual interface one has a pair of 4d gauge theories. Each of them has a defect corresponding to the D5'-brane that breaks the gauge symmetry down to $U(1)$. One is specified by a trivial Nahm pole while the other is by a Nahm pole of rank 2. These defects give rise to extra contributions from the broken gauge theories. In addition, the surviving $U(1)$ gauge theories should obey Neumann b.c. $N'$ corresponding to the NS5'-branes and couple to 3d $U(1)$ twisted vector multiplet through the 3d twisted hypermultiplets.

Then the half-index is given by

$$
\Pi_{N'N}^{4d} U(2)\rightarrow U(1)|D(1)\rightarrow U(1)\rightarrow U(1)(t, x_\alpha, z_\alpha; q)
$$

$$
= \frac{(q^2; q)_\infty}{(q^2 t^2; q)_\infty} \int \frac{ds_1}{2\pi i s_1} \frac{(q^2 t^{-2}; q)_\infty}{(q^2 t^2; q)_\infty} \sum_{m \in \mathbb{Z}} \int \frac{ds_2}{2\pi i s_2} \frac{(q^2 t^2; q)_\infty}{(q^2 t^{-2}; q)_\infty} \int \frac{ds_3}{2\pi i s_3}
$$

$$
\times \frac{(q^2 t^2; q)_\infty}{(q^2 t^{-2}; q)_\infty} \frac{(q^2 t^2; q)_\infty}{(q^2 t^{-2}; q)_\infty} \frac{(q^2 t^2; q)_\infty}{(q^2 t^{-2}; q)_\infty} \frac{(q^2 t^2; q)_\infty}{(q^2 t^{-2}; q)_\infty} \frac{(q^2 t^2; q)_\infty}{(q^2 t^{-2}; q)_\infty} \frac{(q^2 t^2; q)_\infty}{(q^2 t^{-2}; q)_\infty}
$$

$$
\times q^{7|m| t^{-7|m|} m \cdot -m}
$$

(7.43)

where $z_\alpha$ are associated to the flavor symmetry for the extra local operators on the defects from the broken gauge theories and $x_\alpha$ are the fugacities for the topological symmetry for the 3d $U(1)$ twisted vector multiplet.

We have confirmed that the half-index [7.42] coincides with the half-index [7.43].

### 7.4.3 4d $U(2)|3d$ SQED$_3$|4d $U(3)$

Our next example is the interface 4d $U(2)|3d$ SQED$_3$|4d $U(3)$.
Similarly, we can evaluate the half-index of the 4d $U(2)|3d$ SQED$_3|4d$ $U(3)$ interface as

$$
\Pi_{\mathcal{N}'}^{4d \ U(2)} (t, x_\alpha, z_\alpha; q)
= \frac{1}{2} \frac{(q^2 t^{-2}; q)^2_\infty}{(q^2 t^2; q)^2_\infty} \int \frac{ds_1}{2\pi i s_1} \frac{ds_2}{2\pi i s_2} \frac{(\frac{s_1}{s_2}; q)_\infty}{(\frac{s_2}{s_1}; q)_\infty} \frac{(q^2 t^{-2} s_1^2; q)_\infty}{(q^2 t^{-2} s_2^2; q)_\infty}
$$

$$
\times \frac{(q^2 t^2; q)_\infty}{(q^4 t^2; q)_\infty} \sum_{m \in \mathbb{Z}} \int \frac{ds_3}{2\pi i s_3} \frac{(q^4 t^2; q)_\infty}{(q^4 t^2; q)_\infty} \frac{1}{3!} \frac{(q^2 t^{-2}; q)^3_\infty}{(q^2 t^2; q)^3_\infty} \int \prod_{i \neq j} \frac{ds_i}{2\pi i s_i} \frac{(\frac{z_i}{s_j}; q)_\infty}{(\frac{s_j}{z_i}; q)_\infty}
$$

$$
\times \prod_{i=1}^{2} (\frac{q^{\frac{1}{4}} t^{\frac{1}{2}}}{s_i^2} - q^{-1} s_i^{\pm \frac{1}{2}}; q)_\infty \prod_{i=1}^{3} (\frac{q^{\frac{1}{4}} t^{\frac{1}{2}}}{s_i^2} - q^{-1} s_i^{\pm \frac{1}{2}}; q)_\infty \prod_{i=4}^{6} (\frac{q^{\frac{1}{4}} t^{\frac{1}{2}}}{s_i^2} - q^{-1} s_i^{\pm \frac{1}{2}}; q)_\infty \prod_{m=1}^{\infty} \frac{q^{2|m| - 8|m|} z_{1} z_{2}^{-m}}{z_{1} z_{2}^{m}}
$$

(7.44)

where $x_\alpha$ and $z_\alpha$ are the fugacities for the flavor and topological symmetries of SQED$_3$ respectively.

Again we can obtain the dual interface from the brane setup in Figure 15. Each of D5'-brane interfaces breaks the gauge symmetry down to $U(1)$. One is associated to a trivial Nahm pole while the other is to a Nahm pole of rank 2. These defects yield additional contributions from the broken gauge theories. In addition, the surviving $U(1)$ gauge theories satisfy Neumann b.c. $\mathcal{N}'$ corresponding to the NS5'-branes and couple to 3d $U(1) \times U(1)$ twisted vector multiplet through the 3d twisted hypermultiplets.

As we expect, the half-index (7.44) coincides with

$$
\Pi_{\mathcal{N}'}^{4d \ U(2)} \rightarrow U(1)(\mathbb{1}^2) \Pi_{\mathcal{N}'}^{4d \ U(3)} \rightarrow U(1)(t, x_\alpha, z_\alpha; q)
$$

$$
= \frac{(q^2 t; q)^2_\infty}{(q^2 t^2; q)^2_\infty} \int \frac{ds_1}{2\pi i s_1} \frac{(q^2 t^2; q)_\infty}{(q^2 t^2; q)_\infty} \sum_{m_1, m_2 \in \mathbb{Z}} \int \frac{ds_2}{2\pi i s_2} \frac{ds_3}{2\pi i s_3}
$$

$$
\times \frac{(q^2 t^{-2}; q)_\infty}{(q^4 t^{-2}; q)_\infty} \frac{(q^4 t^2; q)_\infty}{(q^4 t^2; q)_\infty} \int \frac{ds_3}{2\pi i s_3}
$$

$$
\times \frac{(q^{\frac{1}{4}} t^{\frac{1}{2}} s_1^2; q)_\infty}{(q^{\frac{1}{4}} t^{\frac{1}{2}} s_1^2; q)_\infty} \frac{(q^{\frac{1}{4}} t^{\frac{1}{2}} s_2^2; q)_\infty}{(q^{\frac{1}{4}} t^{\frac{1}{2}} s_2^2; q)_\infty} \frac{(q^{\frac{1}{4}} t^{-\frac{1}{2}} s_3^2; q)_\infty}{(q^{\frac{1}{4}} t^{-\frac{1}{2}} s_3^2; q)_\infty}
$$

(7.45)

Here the fugacities $z_1, z_2$ are associated with the degrees of freedom appearing at the two defects while the fugacities $x_\alpha$ are associated with the topological symmetry for $U(1) \times U(1)$ twisted vector multiplet.
7.4.4 4d $U(N)|3d$ SQED$_{N_f}|4d$ $U(M)$

We would like to propose the generalization of dualities for the interface 4d $U(N)|3d$ SQED$_{N_f}|4d$ $U(M)$. The interface involves a pair of 4d $\mathcal{N} = 4 U(N)$ and $U(M)$ SYM theories satisfying Neumann b.c. $\mathcal{N}$ corresponding to the NS5-branes. Both of 4d theories couple to 3d $\mathcal{N} = 4$ SQED$_{N_f}$ through 3d bi-fundamental hypermultiplets. The magnetic fluxes for 3d SQED$_{N_f}$ can be labeled by an integer $m$. H

The half-index of the 4d $U(N)|3d$ SQED$_{N_f}|4d$ $U(M)$ interface is

\[
\prod_{\mathcal{N}}^{4d} \frac{U(N)|3d \text{ SQED}_{N_f}|4d \text{ } U(M)}{t,x_\alpha,z_\alpha;q} = \frac{1}{N!} \left(\frac{q^{\frac{t}{2}}}{q^{\frac{t}{2}}}; q^\infty\right) \prod_{i=1}^{N} \frac{ds_i}{2\pi i s_i} \prod_{i \neq j} \left(\frac{q^{\frac{s_i}{2}}}{q^{\frac{s_j}{2}}}; q^\infty\right)
\]

\[
\times \frac{(q^{\frac{t}{2}}, q^t)^{M}}{(q^{\frac{t}{2}}, q^t)^{M}} \prod_{i=N+1}^{N+M+1} \frac{ds_i}{2\pi i s_i} \prod_{i \neq j} \left(\frac{q^{\frac{s_i}{2}}}{q^{\frac{s_j}{2}}}; q^\infty\right)
\]

\[
\times \prod_{i=1}^{N} \frac{(q^{\frac{1}{2} + i m t}t^{-1}s_i^{-1}s_{N+1}^+; q^\infty)}{(q^{\frac{1}{2} + i m t}ts_i^{-1}s_{N+1}^+; q^\infty)} \prod_{\alpha=1}^{N_f} \frac{(q^{\frac{1}{2} + i m t}t^{-1}s_{N+1}^{-1}s_i^+; q^\infty)}{(q^{\frac{1}{2} + i m t}ts_{N+1}^{-1}s_i^+; q^\infty)} \prod_{i=N+2}^{N+M+1} \frac{(q^{\frac{1}{2} + i m t}t^{-1}s_i^{+}; q^\infty)}{(q^{\frac{1}{2} + i m t}ts_i^{+}; q^\infty)}
\]

\[
\times q^{\frac{(N+M+N_f)}{4}(m)} t^{-(N+M+N_f)|m|} \frac{z_1^{m} z_2^{-m}}{z_1^{-m} z_2^{m}} \quad (7.46)
\]

where the fugacities $x_\alpha$ with $\prod_{\alpha=1}^{N_f} x_\alpha = 1$ are associated with the $SU(N_f)$ flavor symmetry of SQED$_{N_f}$ while the fugacities $z_\alpha$ are associated with the topological symmetry of SQED$_{N_f}$. The terms from the second to fourth line describe gauge multiplets in the interface, including the half-indices of Neumann b.c. $\mathcal{N}$ for 4d $\mathcal{N} = 4 U(N)$ and $U(M)$ gauge theories and the full-index for 3d $\mathcal{N} = 4 U(1)$ vector multiplet. The terms in the fifth line correspond to the 3d $\mathcal{N} = 4$ hypermultiplets. The terms in the last line describe monopole operator with the R-charge $\Delta(m) = (N+M+N_f)|m|$

Under the action of S-duality one finds the dual interface which also has a pair of 4d $\mathcal{N} = 4$ gauge theories. The D5-brane intersecting with $N$ D3-brane breaks $U(N) \times U(1)$ gauge symmetry down to $U(1)$ while the other D5'-brane intersecting with $M$ D3-branes breaks $U(M) \times U(1)$ gauge symmetry down to $U(1)$. The reductions of 4d gauge symmetries are described by the boundary conditions specified by the two Nahm poles of rank $(N - 1)$ and rank $(M - 1)$. Correspondingly, extra contributions would appear at the defects from the broken gauge theories [12].

According to the presence of NS5'-branes, the surviving 4d $\mathcal{N} = 4 U(1)$ gauge theories further satisfy Neumann b.c. $\mathcal{N}'$. They are couple to 3d $\mathcal{N} = 4 U(1)\mathcal{N}_f^{-1}$ quiver gauge theory via 3d $\mathcal{N} = 4$ bi-fundamental twisted hypermultiplets.
The half-index should take the form

\[
\mathbb{I}^4d \text{ U}(N) \rightarrow \text{U}(1)^{[N-1]} \rightarrow \text{U}(M) \rightarrow \text{U}(1) \left( t, x_\alpha, x_\alpha'; q \right) \]

\[
= \frac{(q)_\infty}{(q^2 t^2; q)_\infty} \prod_{k=1}^{N-1} \frac{1}{2 \pi i s_k} \frac{d s_k}{(q^{2} t^{2(k-1)}; q)_\infty} (q^{2} t^{2-2(k-1)}; q)_\infty \frac{1}{(q^{2} t^{2k}; q)_\infty} \sum_{m_1, \ldots, m_{N-1} \in \mathbb{Z}^+} \prod_{i=2}^{N_i} \frac{d s_{N_i}}{2 \pi i s_{N_i}} 
\]

\[
\times \prod_{i=1}^{M-1} \frac{(q^2 t^{2l}; q)_\infty}{(q^2 t^{2l-2}; q)_\infty} (q^2 t^{-2l}; q)_\infty \prod_{i=2}^{N_i} \frac{d s_{N_i}}{2 \pi i s_{N_i}} 
\]

\[
\times \frac{\left( q^{2} t^{2-l} \prod_{i=1}^{N_i} (q^{2} t^{2l-2}; q)_\infty \prod_{i=2}^{N_i} \frac{d s_{N_i}}{2 \pi i s_{N_i}} \right)}{\prod_{i=2}^{N_i} \frac{d s_{N_i}}{2 \pi i s_{N_i}} \prod_{i=2}^{N_i} \frac{d s_{N_i}}{2 \pi i s_{N_i}}} 
\]

\[
\left( q^{2} t^{2-l} \prod_{i=1}^{N_i} (q^{2} t^{2l-2}; q)_\infty \prod_{i=2}^{N_i} \frac{d s_{N_i}}{2 \pi i s_{N_i}} \right) \prod_{i=2}^{N_i} \frac{d s_{N_i}}{2 \pi i s_{N_i}} \prod_{i=2}^{N_i} \frac{d s_{N_i}}{2 \pi i s_{N_i}} 
\]

\[
\left( q^{2} t^{2-l} \prod_{i=1}^{N_i} (q^{2} t^{2l-2}; q)_\infty \prod_{i=2}^{N_i} \frac{d s_{N_i}}{2 \pi i s_{N_i}} \right) \prod_{i=2}^{N_i} \frac{d s_{N_i}}{2 \pi i s_{N_i}} \prod_{i=2}^{N_i} \frac{d s_{N_i}}{2 \pi i s_{N_i}} 
\]

\[
\times q^{\frac{1}{2} (\sum_{i=1}^{N_i} m_{N_i-1} + \sum_{i=1}^{N_i-1} m_{N_i-1} - 1) + \sum_{i=1}^{N_i-1} m_{i-1} - m_{i+1} + 1} \prod_{\alpha=1}^{N_f} \frac{x_{\alpha}}{x_{\alpha+1}} \frac{m_{1} + m_{N_f-1} - 1}{1} 
\]

(7.47)

where $z_1, z_2$ are associated with the extra local operators at the two defects and $x_{\alpha}$ are associated with the topological symmetry for 3d $\text{U}(1)$ quiver gauge theory. The terms from the second to third line includes the square of half-indices of Neumann b.c. $N'$ for 4d $\mathcal{N} = 4$ $\text{U}(1)$ gauge theory, the two half-indices of Nahm b.c. of rank $(N - 1)$ and rank $(M - 1)$, and the full-index for 3d $\mathcal{N} = 4$ $\text{U}(1)$ vector multiplet. The terms from the fourth to fifth line describe 3d $\mathcal{N} = 4$ twisted hypermultiplets. The contributions in the last line count bare monopole of dimension $\Delta(m) = \frac{1}{2} (|m_1| + |m_{N_f-1}| + \sum_{i=1}^{N_f-2} |m_i - m_{i+1}|)$.

We expect that the half-indices (7.46) and (7.47) agree with each other.

### 7.5 4d $\text{U}(1) + N$ hypers | 3d SQED $N_f$ | 4d $\text{U}(1) + M$ hypers

The local operators on the D5-brane defect in 4d $\mathcal{N} = 4$ gauge theories carry flavor symmetry charges which is identified with the topological symmetry charges in the dual side. We investigate the interface in which one of 4d $\text{U}(1)$ gauge theories has multiple $N$ D5-brane defects and the other has $M$ D5-brane defects and both 4d theories obey Neumann b.c. $\mathcal{N}'$ with a coupling to 3d $\mathcal{N} = 4$ SQED $N_f$. We denote this interface by 4d $\text{U}(1) + N$ hypers $| 3d$ $\text{SQED} N_f | 4d$ $\text{U}(1) + M$ hypers. The corresponding quiver diagram and brane construction are shown in Figure 16. The $N$ and $M$ D5-branes introduce $N$ and $M$ fundamental hypermultiplets living at the defects.

#### 7.5.1 4d $\text{U}(1) + 2$ hypers | 3d $\text{U}(1)$ | 4d $\text{U}(1) + 2$ hypers

Let us begin with the interface 4d $\text{U}(1) + 2$ hypers $| 3d$ $\text{U}(1) | 4d$ $\text{U}(1) + 2$ hypers. A pair of 4d $\text{U}(1)$ gauge theories with Neumann b.c. $\mathcal{N}'$ is coupled to 3d $\text{U}(1)$ vector multiplet through the 3d bi-fundamental hypermultiplets. Each of the 4d gauge theories has two defects that couple two
Figure 16: (a) The quiver diagrams of the interface $4d \ U(1) + N$ hypers $| 3d \ SQED_{N_f} | 4d \ U(1) + M$ hypers and its mirror. (b) The corresponding brane configurations.
fundamental hypermultiplets to the 4d gauge theories. The magnetic fluxes for 3d $U(1)$ gauge theory can be labeled by an integer $m$.

The half-index reads

$$
\Pi_{\mathcal{N}}^{4d \text{ } U(1)+2 \text{ hypers}}(t, x_\alpha, z_\alpha; q)
= \frac{(q^\frac{1}{2} t^2; q)_\infty}{(q^\frac{1}{2} t^2; q)_\infty} \prod_{\alpha=1}^{2} \left( \frac{q^\frac{1}{2} t^2; (q^\frac{1}{2} t^2; q)_\infty; q^\frac{1}{2} t^2; q)_\infty}{(q^\frac{1}{2} t^2; q)_\infty} \right) \prod_{i=1}^{4} \left( \frac{(q^\frac{1}{2} t^2; q)_\infty}{(q^\frac{1}{2} t^2; q)_\infty} \right) \left( \frac{q^\frac{1}{2} t^2; q)_\infty}{(q^\frac{1}{2} t^2; q)_\infty} \right) \prod_{\alpha=1}^{4} \frac{d s_\alpha}{2 \pi i s_\alpha} \frac{d s_\alpha}{2 \pi i s_\alpha} \frac{d s_\alpha}{2 \pi i s_\alpha} \frac{d s_\alpha}{2 \pi i s_\alpha} \frac{d s_\alpha}{2 \pi i s_\alpha} \frac{d s_\alpha}{2 \pi i s_\alpha} \frac{d s_\alpha}{2 \pi i s_\alpha}
$$

where $x_\alpha$ are the fugacities for flavor symmetry of the defect hypermultiplets while $z_\alpha$ are those for topological symmetry of 3d $U(1)$ gauge theory. The terms in the second line include the square of half-index of Neumann b.c. $N$ for 4d $U(1)$ gauge theory and the full-index for 3d $U(1)$ vector multiplet. The first and third terms in the next line are the contributions from 3d fundamental hypermultiplets living at the defects and the second terms correspond to 3d bi-fundamental hypermultiplets. The terms in the last line count monopole operator of dimension $\Delta(m) = |m|$.

One can identify the S-dual interface from the brane picture in Figure 16. It contains a pair of 4d $\mathcal{N} = 4$ $U(1)$ gauge theories satisfying Neumann b.c. $\mathcal{N}'$. The 4d gauge theories are coupled to 3d $\mathcal{N} = 4$ quiver gauge theory through the 3d bi-fundamental twisted hypermultiplets. For the 3d $U(1) \times U(1) \times U(1)$ twisted vector multiplet, we label the magnetic fluxes by three integers $m_1, m_2, m_3$.

In fact, the half-index (7.48) agrees with the half-index for the dual interface

$$
\Pi_{\mathcal{N}}^{4d \text{ } U(1)}(t, x_\alpha, z_\alpha; q)
= \frac{(q^\frac{1}{2} t^2; q)_\infty}{(q^\frac{1}{2} t^2; q)_\infty} \prod_{\alpha=1}^{2} \left( \frac{q^\frac{1}{2} t^2; (q^\frac{1}{2} t^2; q)_\infty; q^\frac{1}{2} t^2; q)_\infty}{(q^\frac{1}{2} t^2; q)_\infty} \right) \prod_{i=1}^{4} \left( \frac{(q^\frac{1}{2} t^2; q)_\infty}{(q^\frac{1}{2} t^2; q)_\infty} \right) \left( \frac{q^\frac{1}{2} t^2; q)_\infty}{(q^\frac{1}{2} t^2; q)_\infty} \right) \prod_{\alpha=1}^{4} \frac{d s_\alpha}{2 \pi i s_\alpha} \frac{d s_\alpha}{2 \pi i s_\alpha} \frac{d s_\alpha}{2 \pi i s_\alpha} \frac{d s_\alpha}{2 \pi i s_\alpha} \frac{d s_\alpha}{2 \pi i s_\alpha} \frac{d s_\alpha}{2 \pi i s_\alpha} \frac{d s_\alpha}{2 \pi i s_\alpha}
$$

where the fugacities $z_\alpha$ and $x_\alpha$ are associated with the flavor and topological symmetry for respectively. The contributions in the second line include the half-indices of Neumann b.c. $\mathcal{N}'$ for 4d
$U(1)$ gauge theory and full-index for 3d $U(1) \times U(1) \times U(1)$ twisted vector multiplet. The terms in the next two lines are contributed from 3d twisted hypermultiplets. The remaining terms count bare monopole operators of dimension $\Delta(m) = \frac{|m_1|}{2} + \frac{|m_1 - m_2|}{2} + \frac{|m_2 - m_3|}{2} + \frac{|m_3|}{2} + |m_2|$. 

### 7.5.2 4d $U(1) + 2$ hypers | 3d SQED$_1$ | 4d $U(1) + 2$ hypers

Next consider the interface 4d $U(1) + 2$ hypers | 3d SQED$_1$ | 4d $U(1) + 2$ hypers. The interface can be obtained by additionally introducing a 3d fundamental hypermultiplet coupled to 3d vector multiplet from the interface 4d $U(1) + 2$ hypers | 3d $U(1)$ | 4d $U(1) + 2$ hypers.

The half-index is

$$
\mathbb{I}_{N'N}^{4d \ U(1) + 2 \ hypers | 3d \ SQED_1 | 4d \ U(1) + 2 \ hypers} (t, x_\alpha, z_\alpha; q) = \left( \frac{q^2 t^2; q}{q^2 t^{-2}; q} \right)^{\infty} \sum_{m \in \mathbb{Z}} \frac{ds_1}{2\pi is_1} \frac{ds_2}{2\pi is_2} \frac{ds_3}{2\pi is_3} \left( \frac{q^2 t^{-2}; q}{q^2 t^2; q} \right)^{\infty} \prod_{\alpha = 1}^2 \left( \frac{q^2 t^{-1} s_{\alpha}^{\pm} s_{\alpha}^{\mp}; q}{q^2 t^{1} s_{\alpha}^{\pm} s_{\alpha}^{\mp}; q} \right)_{\infty} \prod_{i = 1, 3} \left( \frac{(q^2 t^{-1} s_{i}^{\pm} x_{i}^{\mp}; q)}{(q^2 t^{1} s_{i}^{\pm} x_{i}^{\mp}; q)} \right)_{\infty} \prod_{\alpha = 3}^4 \left( \frac{(q^2 t^{-1} s_{\alpha}^{\pm} x_{\alpha}^{\mp}; q)}{(q^2 t^{1} s_{\alpha}^{\pm} x_{\alpha}^{\mp}; q)} \right)_{\infty} \left( \frac{q^2 |m| \cdot t^{-3|m|} \cdot \left( z_{2i} \right)^m}{ds_{\infty}} \right)_{\infty} (7.50)
$$

where $x_\alpha$ are associated with the flavor symmetry of the defect hypermultiplets and $z_\alpha$ are the fugacities for topological symmetry of SQED$_1$.

One can find the dual interface from the dual of the interface 4d $U(1) + 2$ hypers | 3d $U(1)$ | 4d $U(1) + 2$ hypers by replacing the 3d quiver gauge theory with

$$
(1) - (1) - (1) - (1) \quad | \quad (1) - (1) - (1) \quad \left| \right| [1] \quad [1] .
$$

In fact, the half-index (7.50) agrees with

$$
\mathbb{I}_{N'N'}^{4d \ U(1)} (t, x_\alpha, z_\alpha; q) = \left( \frac{q^2 t^2; q}{q^2 t^{-2}; q} \right)^{\infty} \sum_{m_1, \ldots, m_4 \in \mathbb{Z}} \frac{ds_1}{2\pi is_1} \frac{ds_2}{2\pi is_2} \frac{ds_3}{2\pi is_3} \frac{ds_4}{2\pi is_4} \frac{ds_6}{2\pi is_6} \left( \frac{q^2 t^{-2}; q}{q^2 t^2; q} \right)^{\infty} \prod_{i = 2}^5 \left( \frac{(q^2 t^{-1} s_{i}^{\pm} x_{i}^{\mp}; q)}{(q^2 t^{1} s_{i}^{\pm} x_{i}^{\mp}; q)} \right)_{\infty} \left( \frac{q^2 |m| \cdot t^{-3|m|} \cdot \left( z_{2i} \right)^m}{ds_{\infty}} \right)_{\infty} (7.51)
$$
where the fugacities \( x_\alpha \) and \( z_\alpha \) are associated to the topological and flavor symmetries for 3d quiver gauge theory.

\[
\frac{1}{(1) - (1) - (1) - (1)}
\]

### 7.5.3 4d \( U(1)+3 \) hypers\( |3d \ U(1)|4d \ U(1)+3 \) hypers

To gain more insight about mirror map between flavor and topological symmetries, we consider the interface 4d \( U(1)+3 \) hypers\( |3d \ U(1)|4d \ U(1)+3 \) hypers. In this case each of 4d \( U(1) \) gauge theories has three defects of D5-branes that couple three fundamental hypermultiplets to the 4d bulk gauge theory.

The half-index reads

\[
\Pi_{N^\prime N^\prime}^{4d \ U(1)+3 \ hypers}(1) \frac{1}{(1) - (1) - (1) - (1)} 4d \ U(1)
\]

\[
= \frac{(q)_\infty}{(q^{\frac{1}{2}} t^2 q^2)_\infty} \int \frac{ds_1}{2\pi is_1} \frac{(q^{\frac{1}{2}} t^2 q^2)_\infty}{(q^{\frac{1}{2}} t^{-2} q^2)_\infty} \sum_{m \in \mathbb{Z}} \int \frac{ds_2}{2\pi is_2} \frac{(q^{\frac{1}{2}} t^{2} q^2)_\infty}{(q^{\frac{1}{2}} t^{-2} q^2)_\infty} \int \frac{ds_3}{2\pi is_3}
\]

\[
\times \prod_{\alpha=1}^{3} \frac{(q^{\frac{1}{2}} t^{-1} s_1^{\pm} s_2^{\mp}; q)_\infty}{(q^{\frac{1}{2}} t^{2} s_1^{\pm} s_2^{\mp}; q)_\infty} \prod_{i=1}^{6} \frac{(q^{\frac{1}{2}} t^{-1} s_1^{\pm} s_2^{\mp}; q)_\infty}{(q^{\frac{1}{2}} t^{2} s_1^{\pm} s_2^{\mp}; q)_\infty}
\]

\[
\times q^{|m_i|} t^{-2|m_i|} \left( \frac{z_1}{s_2} \right)^m
\]

where \( x_\alpha \) are associated with the flavor symmetry of the defect hypermultiplets and \( z_\alpha \) are the fugacities for topological symmetry of 3d \( U(1) \) gauge theory.

Due to the increase in the number of defects of D5-branes in the original interface, the 3d quiver gauge theory in the dual interface gets more gauge nodes. The dual interface can be obtained from the previous examples by taking the 3d quiver gauge theory

\[
(1) - (1) - (1) - (1)
\]

We have checked that the half-index (7.52) coincides with

\[
\Pi_{N^\prime N^\prime}^{4d \ U(1)+3 \ hypers}(1) \frac{1}{(1) - (1) - (1) - (1)} 4d \ U(1)
\]

\[
= \frac{(q)_\infty}{(q^{\frac{1}{2}} t^2 q^2)_\infty} \int \frac{ds_1}{2\pi is_1} \frac{(q^{\frac{1}{2}} t^{-2} q^2)_\infty}{(q^{\frac{1}{2}} t^2 q^2)_\infty} \sum_{m_1, \ldots, m_3 \in \mathbb{Z}} \int \frac{ds_i}{2\pi is_i} \frac{(q^{\frac{1}{2}} t^{-2} q^2)_\infty}{(q^{\frac{1}{2}} t^2 q^2)_\infty} \int \frac{ds_7}{2\pi is_7}
\]

\[
\times \frac{(q^{\frac{1}{2}} + \frac{[m_1]}{2} t s_1^{\pm} s_2^{\mp}; q)_\infty}{(q^{\frac{1}{2}} + \frac{[m_1]}{2} t^{-1} s_1^{\pm} s_2^{\mp}; q)_\infty} \prod_{i=2}^{5} \frac{(q^{\frac{1}{2}} + \frac{[m_{i-1} - m_i]}{2} t^{-1} s_1^{\pm} s_2^{\mp}; q)_\infty}{(q^{\frac{1}{2}} + \frac{[m_{i-1} - m_i]}{2} t s_1^{\pm} s_2^{\mp}; q)_\infty}
\]

\[
\times \frac{(q^{\frac{1}{2}} + \frac{[m_i]}{2} t s_1^{\pm} s_2^{\mp}; q)_\infty}{(q^{\frac{1}{2}} + \frac{[m_i]}{2} t^{-1} s_1^{\pm} s_2^{\mp}; q)_\infty} \prod_{\alpha=1}^{2} \frac{(q^{\frac{1}{2}} + \frac{[m_{i-1} - m_i]}{2} t s_1^{\pm} s_2^{\mp}; q)_\infty}{(q^{\frac{1}{2}} + \frac{[m_{i-1} - m_i]}{2} t^{-1} s_1^{\pm} s_2^{\mp}; q)_\infty}
\]

\[
\times q^{|m_i|} t^{2|m_i|} \left( \frac{z_1}{s_2} \right)^m
\]

where the fugacities \( x_\alpha \) and \( z_\alpha \) are associated to the topological and flavor symmetries for 3d quiver gauge theory

\[
(1) - (1) - (1) - (1)
\]
7.5.4 4d $U(1) + N$ hypers$|3d$ SQED$_{N_f}|$4d $U(1) + M$ hypers

Finally, we discuss the generalization of the dualities for the interface 4d $U(1) + N$ hypers$|3d$ SQED$_{N_f}|$4d $U(1) + M$ hypers whose quiver diagram and brane construction are illustrated in Figure 16. Although we have restricted to the Abelian interface in this section, the generalization to the non-Abelian interface is straightforward by using the results so far.

For the interface there are a 4d $\mathcal{N} = 4$ $U(1)$ gauge theory with $N$ defects which couple the 4d theory to the $N$ fundamental 3d hypermultiplets and another 4d $\mathcal{N} = 4$ $U(1)$ gauge theory with $M$ defects which couple the 4d theory to the $M$ fundamental 3d hypermultiplets. A pair of 4d gauge theories obeys Neumann b.c. $\mathcal{N}$ and interacts with 3d $\mathcal{N} = 4$ SQED$_{N_f}$ through a coupling given by 3d $\mathcal{N} = 4$ bi-fundamental hypermultiplets. The interface has 4d $U(1) \times U(1)$ gauge symmetry and 3d $U(1)$ gauge symmetry. We can label the magnetic fluxes for 3d $U(1)$ gauge theory by an integer $m$.

Then the half-index for the interface 4d $U(1) + N$ hypers$|3d$ SQED$_{N_f}|$4d $U(1) + M$ hypers takes the form

$$
\begin{align*}
\mathbb{I}^{4d U(1) + N \text{ hypers}}_\mathcal{N}^{4d U(1) + M \text{ hypers}} & = \frac{(q)_\infty}{(q^2 t^2 - z_1^2 q)_\infty} \frac{ds_1}{2\pi i s_1} \frac{(q^2 t^2 - z_2^2 q)_\infty}{(q^2 t^2 - z_2^2 q)_\infty} \sum_{m \in \mathbb{Z}} \frac{ds_2}{2\pi i s_2} \frac{(q^2 t^2 - z_3^2 q)_\infty}{(q^2 t^2 - z_3^2 q)_\infty} \prod_{\alpha=1}^{N+N_f}\frac{(q^2 t^2 - \frac{1}{s_1} x_{\alpha}^\pm)^\infty}{(q^2 t^2 - \frac{1}{s_2} x_{\alpha}^\pm)^\infty} \prod_{\alpha=N+1}^{N+N_f+M}\frac{(q^2 t^2 - \frac{1}{s_3} x_{\alpha}^\pm)^\infty}{(q^2 t^2 - \frac{1}{s_3} x_{\alpha}^\pm)^\infty} \\
& \times \prod_{\alpha=1}^{N+N_f+M}\frac{(q^2 t^2 - \frac{1}{s_1} x_{\alpha}^\pm)^\infty}{(q^2 t^2 - \frac{1}{s_1} x_{\alpha}^\pm)^\infty} \prod_{\alpha=1}^{N+M+1}\frac{(q^2 t^2 - \frac{1}{z_1} x_{\alpha}^\pm)^\infty}{(q^2 t^2 - \frac{1}{z_1} x_{\alpha}^\pm)^\infty} \prod_{\alpha=1}^{N+M+1}\frac{(q^2 t^2 - \frac{1}{z_2} x_{\alpha}^\pm)^\infty}{(q^2 t^2 - \frac{1}{z_2} x_{\alpha}^\pm)^\infty} \prod_{\alpha=1}^{N+M+1}\frac{(q^2 t^2 - \frac{1}{z_3} x_{\alpha}^\pm)^\infty}{(q^2 t^2 - \frac{1}{z_3} x_{\alpha}^\pm)^\infty} \\
& \times q^{(N+N_f+M)|m|} t^{-(N+N_f+M)|m|} \left( \frac{z_1}{z_2} \right)^m
\end{align*}
$$

(7.54)

where we have used $\{x_{\alpha}\}_{\alpha=1,\ldots,N}$ for the fugacities of the $SU(N)$ flavor symmetries of the defect hypermultiplets, $\{x_{\alpha}\}_{\alpha=N+1,\ldots,N+N_f}$ for the fugacities of the $SU(N_f)$ flavor symmetry of 3d SQED$_{N_f}$, $\{x_{\alpha}\}_{\alpha=N+N_f+1,\ldots,N+N_f+M}$ for the fugacities of the $SU(M)$ flavor symmetries of the defect hypermultiplets. The fugacities $z_{\alpha}$ are associated to the topological symmetry of 3d SQED$_{N_f}$. The contributions in the second line are the two half-indices of Neumann b.c. $\mathcal{N}$ for 4d $U(1)$ gauge theory and the full-index for 3d $U(1)$ vector multiplet. The terms appearing in the next two lines are the contributions from the 3d hypermultiplets. The terms in the last line count bare monopole of dimension $\Delta(m) = (N+N_f+M)|m|$. 

From the S-dual configuration of the brane setup in Figure 16 we can read off the dual interface. It has a pair of 4d $\mathcal{N} = 4$ $U(1)$ gauge theories which satisfy Neumann b.c. $\mathcal{N}'$ and couple to 3d quiver gauge theory $^{(1)^{N-1} - (1) - (1)^{N_f} - (1)}$ through the 3d bi-fundamental twisted hypermultiplets. We can label the magnetic fluxes for the 3d quiver gauge theory by $N+N_f+M-1$ integers $m_1, \ldots, m_{N+N_f+M-1}$.
The half-index for the dual configuration is given by

\[
4d \ U(1) \ |(1)^{N-1} - (1)^{N-1} \cdot \prod_{\mathcal{N} \in \mathcal{N}'} U(1) = \frac{(q)_\infty}{(q^2 t^2; q)_\infty} \int_{\mathbb{R}^d U(1)} ds_1 \frac{2\pi is_1}{2\pi i s_1} \times \frac{(q^\frac{t}{2} t^{-2} \cdot q)_\infty}{(q^\frac{t}{2} t^2; q)_\infty} \sum_{m_1, \ldots, m_{N+N_f+M-1} \in \mathbb{Z}} \int_{\mathbb{R}^d U(1)} ds_{N+N_f+M-1} \prod_{i=2} \frac{2\pi i s_i}{2\pi i s_i} \times \frac{(q^\frac{t}{2} t^2; q)_\infty}{(q^\frac{t}{2} t^2; q)_\infty} \int_{\mathbb{R}^d U(1)} ds_{N+N_f+M-1} \prod_{i=2} \frac{2\pi i s_i}{2\pi i s_i}
\]

where \( m_0 = m_{N+N_f+M} = 0 \). The fugacities \( x_\alpha \) and \( z_\alpha \) are now associated to the topological and flavor symmetries for 3d quiver gauge theory \( (1)^{N-1} - (1)^{N-1} \cdot \prod_{\mathcal{N} \in \mathcal{N}'} U(1) \). The terms from the second to fourth line are the two half-indices of Neumann b.c. \( \mathcal{N} \) for 4d \( U(1) \) gauge theory and the full-index for 3d \( U(1)^{N+N_f+M-1} \) twisted vector multiplet. The terms appearing in the next two lines describe the contributions from 3d twisted hypermultiplets. The remaining terms count bare monopole of dimension

\[
\Delta(m) = \frac{|m|}{2} + \frac{m_N}{2} + \frac{|m_{N+N_f}|}{2} + \frac{|m_{N+N_f+M-1}|}{2} + \sum_{i=1}^{N+N_f+M-2} \frac{|m_i - m_{i+1}|}{2}.
\]

We expect that the half-index \((7.54)\) is equal to \((7.55)\).

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