COHOMOLOGY OF LIE ALGEBROIDS OVER ALGEBRAIC SPACES

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Abstract. We consider Lie algebroids over an algebraic space (or topological ringed space) as quasicoherent sheaves of Lie-Rinehart algebras. We express hypercohomology for a locally free Lie algebroid (not necessarily of finite rank) as a derived functor, and simplify it via Čech cohomology. Furthermore, we define the Hochschild hypercohomology of a sheaf of generalized bialgebras and study the cases of the universal enveloping algebroid and the jet algebroid of a Lie algebroid. In the sequel, we present a version of Hochschild-Kostant-Rosenberg theorem for a locally free Lie algebroid, as well as its dual version.

1. Introduction

The notion of Lie algebroids plays a prominent role in differential geometry and mathematical physics as they represent generalized infinitesimal symmetries of spaces, which are related to the corresponding global symmetries of spaces described by Lie groupoids [21]. Lie algebroids over a $C^\infty$-manifold are joint generalization of tangent vector bundle over the manifold and Lie algebras [26]. An algebraic analogue of Lie algebroids, known as Lie-Rinehart algebras, is used to study general situations [3,16,18,22]. In the context of complex geometry and algebraic geometry, Lie algebroids over analytic spaces [32] and over algebraic varieties (or schemes) [3,7,18] respectively have been studied in sheaf theoretic language, which are joint generalization of the tangent sheaf and sheaves of Lie algebras [18,32,33]. We study Lie algebroids over an algebraic space (shortly, $a$-space) or over a topological ringed space $(X,\mathcal{O}_X)$ as certain quasicoherent sheaves of Lie-Rinehart algebras [18,29,38], where $\mathcal{O}_X$ is a sub-sheaf of algebras of the sheaf of continuous functions $\mathcal{C}_X^0$ on $X$ [30,33]. This framework unifies the concept of Lie algebroids across these three types of base spaces as special cases. To study calculus on these (smooth and singular) geometric objects in a consistent manner, we need the notion of Lie algebroids within algebro-geometric settings. It allows one to treat several geometric structures, such as Poisson analytic spaces [32], singular foliations or generalized involutive distributions [10,20,32], as well as free Lie algebroids [18]. In particular, the sheaf of logarithmic derivations for some (principal or free) divisor of a complex manifold (or a smooth algebraic variety) [4,29,32], and the path algebroid of a smooth manifold [18], are key object of study in this context.

Associated with a Lie algebroid over an $a$-space, there are two canonical sheaves of generalized bialgebras: the universal enveloping algebroid and the jet algebroid [1,6,7,18,29,32]. Both play crucial roles in studying the homological algebra of a Lie algebroid. Here, the sheaves of generalized bialgebras serve as a sheaf-theoretic analogue (or global version) of the notion of $R/k$-bialgebras [26] or left $R$-bialgebroids [19], referred to as $\mathcal{O}_X/k_X$-bialgebroids [29]. The universal enveloping algebroid $\mathcal{U}(\mathcal{O}_X,\mathcal{L})$ of a Lie algebroid $\mathcal{L}$ generalizes the notion of sheaf of differential operators on a manifold [29,35]. It is sheafification of the presheaf of universal enveloping algebras of Lie-Rinehart algebras associated with each space of sections of $\mathcal{L}$ [3,29]. It has a canonical $\mathcal{O}_X/k_X$-bialgebra structure [29], similar structures are present in [7,18] by different names. Moreover the dual of $\mathcal{U}(\mathcal{O}_X,\mathcal{L})$, $\mathcal{J}(\mathcal{O}_X,\mathcal{L}) := \text{Hom}_{\mathcal{O}_X}(\mathcal{U}(\mathcal{O}_X,\mathcal{L}),\mathcal{O}_X)$ is the jet algebroid of $\mathcal{L}$, generalizes the notion of sheaf of jets on a manifold (see[1,5,7,18,32]). It is sheafification of the presheaf of jet algebras associated with the sheaf of Lie-Rinehart algebras [1,29]. It also has a canonical $\mathcal{O}_X/k_X$-bialgebra structure induced from the structure of $\mathcal{U}(\mathcal{O}_X,\mathcal{L})$ [29]. Moreover, an $\mathcal{O}_X/k_X$-bialgebra gives rise to a Lie algebroid over the $a$-space $(X,\mathcal{O}_X)$, consisting of its sheaf of primitive elements [29].

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Cohomology of a Lie algebroid generalizes both Lie algebra cohomology (Chevalley-Eilenberg cohomology) and de Rham cohomology, and is commonly referred to as Chevalley-Eilenberg-de Rham cohomology [1–3,21,32]. The cohomology of a Lie algebroid \( \mathcal{L} \), which is a locally free \( \mathcal{O}_X \)-module of finite rank, over a Noetherian separated scheme [3] or over a complex manifold \((X, \mathcal{O}_X)\) [32], is described using the derived functor \( Ext \). In non-commutative geometry [13,17,35], the dual pair consisting of the Hochschild cohomology and homology \((HH^\bullet(A), HH_\bullet(A))\) of an associative \( \mathbb{K} \)-algebra \( A \) serves as a non-commutative analogue to the classical dual pair \( (\wedge^\bullet \mathcal{O}_X \mathcal{T}_X, \wedge^\bullet \mathcal{O}_X \Omega^1_X) \), which consists of the sheaf of multivector fields and the sheaf of differential forms over a smooth \( a \)-space \((X, \mathcal{O}_X)\). This correspondence is given by the standard Hochschild-Kostant-Rosenberg (shortly, HKR) theorem. In [5–7], the authors establish a version of the HKR theorem, using the dual pair \( \mathfrak{U}(\mathcal{O}_X, \mathcal{L}) \) and \( \mathfrak{J}(\mathcal{O}_X, \mathcal{L}) \), to explore precalculus up to homotopy for locally free Lie algebroids of finite rank. For that, they introduce an isomorphism of Gerstenhaber algebras, given by a twisted version of the HKR morphism. In this paper, we deduce analogous cohomological results using a different approach to include more general situations, such as relaxing the finite rank condition. To derive such results, we present a global version of cohomological results on Lie-Rinehart algebras [16,19,25,34].

In Section 2, we recall some of the preliminary notions associated with a Lie algebroid that are required for the article. In Section 3, we consider Lie algebroid hypercohomology over certain special \( a \)-spaces. Here, we show that one can define Lie algebroid cohomology, where the underlying \( \mathcal{O}_X \)-module, while locally free, is not necessarily of finite rank, thus encompassing all three geometric setups. The cohomology of a locally free Lie algebroid \( \mathcal{L} \) over an \( a \)-space \((X, \mathcal{O}_X)\) with coefficient in a \( \mathcal{L} \)-module \( \mathcal{E} \), is given by \( \mathbb{H}^\bullet(\mathcal{L}, \mathcal{E}) \cong Ext_{\mathcal{U}(\mathcal{O}_X, \mathcal{L})}^\bullet(\mathcal{O}_X, \mathcal{E}) \). Here, we use the result of Lie-Rinehart cohomology as derived functor for a projective Lie-Rinehart algebra [16,34] as a local description. We apply this result for the sheaf of logarithmic derivations and for \( \mathcal{O}_X/\mathbb{K}_\mathbb{X} \)-bialgebras. Furthermore, we express the Lie algebroid cohomology in terms of Čech cohomology by considering a good open cover [2,33,36]. As an application of the cohomology groups, we consider abelian Lie algebroid extensions and (first) Chern classes [24,25] in this general set up. In Section 4, we consider algebraic (analytic) de Rham cohomologies for some free divisors associated with principal ideal sheaves [29] and compute the corresponding logarithmic de Rham cohomologies [4,27]. We view these cohomologies as Lie algebroid (hyper)cohomology as described in [2,3,32].

In the first part of Section 5, we define Hochschild hypercohomology of an \( \mathcal{O}_X/\mathbb{K}_\mathbb{X} \)-bialgebra. As special cases of it, we study Hochschild hypercohomology of \( \mathfrak{U}(\mathcal{O}_X, \mathcal{L}) \) and \( \mathfrak{J}(\mathcal{O}_X, \mathcal{L}) \) associated with a Lie algebroid \( \mathcal{L} \). After that, we present a version of Hochschild-Kostant-Rosenberg (HKR) theorem for locally free Lie algebroids (locally free \( \mathcal{O}_X \)-modules but not necessarily of finite rank) over any of the special \( a \)-spaces. It provides an isomorphism of graded vector spaces between Hochschild hypercohomology of \( \mathfrak{U}(\mathcal{O}_X, \mathcal{L}) \) and hypercohomology of the sheaf of \( \mathcal{L} \)-poly vector fields, that is \( \mathbb{H}^\bullet(\mathfrak{U}(\mathcal{O}_X, \mathcal{L})) \cong \mathbb{H}^\bullet(\mathcal{L}) \). This result we get by locally using the algebraic counterpart for projective Lie-Rinehart algebras described in [19]. We derive a result in sheaf cohomology context to simplify the hypercohomology \( \mathbb{H}^\bullet(X, \wedge^\bullet \mathcal{O}_X \mathcal{T}_X) \). Moreover, we present the HKR theorem in some of the special cases. Next we discuss about the dual version of HKR theorem in the generalize setup. Thus we show that for a locally free Lie algebroids \( \mathcal{L} \) of finite rank over any of the special \( a \)-space \((X, \mathcal{O}_X)\), we get a canonical isomorphism of vector spaces between Hochschild hypercohomology of \( \mathfrak{J}(\mathcal{O}_X, \mathcal{L}) \) and the Lie algebroid (hyper)cohomology of \( \mathcal{L} \), that is \( \mathbb{H}^\bullet(\mathfrak{J}(\mathcal{O}_X, \mathcal{L})) \cong \mathbb{H}^\bullet(\mathcal{L}, \mathcal{O}_X) \). This result we get by locally using the algebraic counterpart as described in [19]. Both of these hypercohomologies are obtained by the derived functor \( Cotor \). At last, we apply the dual HKR theorem for the tangent sheaf \( \mathcal{L} := \mathcal{T}_X \) over non-singular \( a \)-spaces and obtain some interesting results using the (smooth, analytic or algebraic) de Rham cohomology [8,36] of \( X \).

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2. Preliminaries

In this section, we recall some standard notions, namely algebraic spaces (or topological ringed spaces), Lie algebroids, universal enveloping algebroids, generalized bialgebras etc in algebro-geometric language. Then, we recall some important relationships among them. In the later sections, we use these notions to consider cohomology theoretic results for such Lie algebroids.

The notation $K$ is used for $\mathbb{R}, \mathbb{C}$ (the real or complex number fields respectively) or a general algebraically closed field of characteristic zero. We denote the constant sheaf by $\mathbb{K}_X$ on a topological space $X$ with stalks being isomorphic to $\mathbb{K}$. The sheaf of $K$-valued continuous functions on $X$ is denoted by $\mathcal{C}^0_X$.

2.1. Lie algebroids over algebraic spaces. Here we consider some special locally ringed spaces [39], and then define Lie algebroids over these spaces [29].

Definition 2.1. Let $X$ be a topological space and $\mathcal{O}_X$ a $\mathbb{K}_X$-subalgebra of the sheaf of continuous functions $\mathcal{C}^0_X$ on $X$. The pair $(X, \mathcal{O}_X)$ is said to be an algebraic space, or simply, an a-space.

For an a-space $(X, \mathcal{O}_X)$, the sheaf of $\mathbb{K}_X$-linear derivations of $\mathcal{O}_X$ is denoted by $\mathfrak{Der}_{\mathbb{K}_X}(\mathcal{O}_X)$.

Note 2.2. Consider smooth manifolds, complex manifolds, analytic spaces and algebraic varieties, all with their associated structure sheaf, as special a-spaces. The tangent sheaf over such an a-space $(X, \mathcal{O}_X)$ is $\mathfrak{Der}_{\mathbb{K}_X}(\mathcal{O}_X)$, has a standard compatible $\mathbb{K}_X$-Lie algebra and a (quasi)coherent $\mathcal{O}_X$-module structure [18, 32]. A special a-space $(X, \mathcal{O}_X)$ is said to be non-singular or smooth if its tangent sheaf is a locally free $\mathcal{O}_X$-module.

The standard notion of Lie algebroids is defined on a smooth (real) vector bundle over a smooth manifold. The space of global sections of a Lie algebroid forms a Lie-Rinehart algebra, capturing the entire structure. In contrast, for holomorphic (or algebraic) vector bundles over complex manifolds (or algebraic varieties), the space of global sections does not capture the whole information. A sheaf-theoretic approach is necessary for analytic and algebraic geometry, applicable to both singular and non-singular cases.

Definition 2.3. A Lie algebroid $\mathcal{L}$ over an a-space $(X, \mathcal{O}_X)$ is a quasicoherent sheaf of $(\mathbb{K}_X, \mathcal{O}_X)$-Lie-Rinehart algebras. That is, $\mathcal{L}$ is a $\mathbb{K}_X$-Lie algebra and a quasicoherent $\mathcal{O}_X$-module equipped with a homomorphism $\alpha : (\mathcal{L}, [\cdot, \cdot]) \to (\mathfrak{Der}_{\mathbb{K}_X}(\mathcal{O}_X), [\cdot, \cdot]_e)$ of $\mathcal{O}_X$-modules and $\mathbb{K}_X$-Lie algebras, called the anchor map. The map $\alpha$ satisfies the Leibniz rule: $[D, fD'] = f[D, D'] + \alpha(D)(f)D'$ for $f \in \mathcal{O}_X$ and $D, D' \in \mathcal{L}$.

We denote this Lie algebroid as $(\mathcal{L}, [\cdot, \cdot], \alpha)$ or simply by $\mathcal{L}$.

Remark 2.4. Lie algebroids over a smooth manifold $X$ is equivalent to locally free Lie algebroids of finite rank over the a-space $(X, \mathcal{C}^\infty_X)$, where $\mathcal{C}^\infty_X$ is the sheaf of $C^\infty$-functions on $X$.

A homomorphism of Lie algebroids over an a-space $(X, \mathcal{O}_X)$

$$\phi : (\mathcal{L}_1, [\cdot, \cdot]_1, a_1) \rightarrow (\mathcal{L}_2, [\cdot, \cdot]_2, a_2)$$

is a sheaf homomorphism of Lie-Rinehart algebras, i.e.

- $\phi : \mathcal{L}_1 \rightarrow \mathcal{L}_2$ is an $\mathcal{O}_X$-module homomorphism,
- $\phi : (\mathcal{L}_1, [\cdot, \cdot]_1) \rightarrow (\mathcal{L}_2, [\cdot, \cdot]_2)$ is a $\mathbb{K}_X$-Lie algebra homomorphism,
- compatibility condition: $a_2 \circ \phi = a_1$.

Example 2.5. The standard Lie algebroid structure on the tangent sheaf of an a-space $(X, \mathcal{O}_X)$ is described by the pair $(\mathfrak{Der}_{\mathbb{K}_X}(\mathcal{O}_X), [\cdot, \cdot]_e, id) =: \mathcal{T}_X$. In particular, when $X$ is a real smooth manifold (complex manifold) with the structure sheaf $\mathcal{O}_X$, the sheaf of smooth (holomorphic) vector fields $\mathfrak{X}_X$ on $X$ is isomorphic to $\mathfrak{Der}_{\mathbb{K}_X}(\mathcal{O}_X)$, forming a locally free Lie algebroid of rank equals to dimension of the manifold $X$.

The sheaf of differential 1-forms $\Omega^1_X$ over a Poisson manifold $X$ has a canonical Lie algebroid structure. For smooth and analytic cases the details can be found in [10] and [32] respectively.

Example 2.6. A singular foliation $F$ on a real smooth manifold or a complex manifold $(X, \mathcal{O}_X)$ is an $\mathcal{O}_X$-submodule of the Lie algebroid $\mathcal{T}_X$ (or $\mathfrak{X}_X$), which is (i) stable under the Lie bracket and (ii) locally finitely generated. It provides a Stefan-Sussmann distribution on $X$, forms a Lie algebroid over $(X, \mathcal{O}_X)$. 

In particular, regular foliations arise from involutive sub-bundles of the tangent bundle (providing Frobenius distributions). See [10,20,32] for details.

**Example 2.7.** Let \( X \) be a complex manifold and \( \mathcal{O}_X \) be the sheaf of holomorphic functions. The vanishing set (or the zero locus) \( Y := V(\mathcal{I}) \) of a ideal-sheaf \( \mathcal{I} \subset \mathcal{O}_X \) is a subspace of \( X \) which is not necessarily a submanifold (it may have singularity as well). The sheaf of functions \( \mathcal{O}_Y := \mathcal{O}_X/\mathcal{I} \) on \( Y \) is its structure sheaf and the pair \((Y, \mathcal{O}_Y)\) is an analytic space (see in [32]).

Here, the tangent sheaf \( \mathcal{T}_Y := (\mathcal{D}er_{\mathcal{I}_Y} \mathcal{O}_Y) = (\cdot, \cdot)_c, d\mathcal{I}_Y \) is not necessarily a locally free Lie algebroid over \((Y, \mathcal{O}_Y)\). Consider the sheaf of logarithmic derivations (forms a generalized involutive distribution or singular foliation) \([29,32]\) as

\[
\mathcal{T}_X(-\log Y) := \{ D \in \mathcal{T}_X : D(\mathcal{I}) \subset \mathcal{I} \} \hookrightarrow \mathcal{T}_X
\]

(geometrically, it represents the sheaf of vector fields on \( X \) that are tangent to \( Y \) for a smooth divisor \( Y \)) with the canonical Lie algebroid structure. It is associated with \( \mathcal{T}_Y \) via the Lie algebroid epimorphism

\[
\rho : \mathcal{T}_X(-\log Y) \to \mathcal{T}_Y
\]
defined by \( \rho(D) = \bar{D} \), \( \bar{D}(f) = [D(f)] \) for any sections \( f \in \mathcal{O}_X, D \in \mathcal{T}_X(-\log Y) \).

For example, let \( X = \mathbb{C}^2 \) with global coordinate functions \( x, y \) and consider the analytic space as \((Y, \mathcal{O}_Y)\) associated to the vanishing set of the ideal sheaf \( \mathcal{I} \) generated by the function \( xy \). Here, the subspace \( Y \) is the union of points on the axes, known as a normal crossing divisor, has a singularity at the origin \( 0 := (0,0) \). Thus, the Lie algebroid \( \mathcal{T}_X(-\log Y) \) is a locally free \( \mathcal{O}_X \)-module generated by the derivations \( x\partial_x \) and \( y\partial_y \), but the Lie algebroid \( \mathcal{T}_Y \) is not locally free. See [29,32] for details.

### 2.2. Universal enveloping algebroid of a Lie algebroid

Let \((\mathcal{L}, [\cdot, \cdot], a)\) be a Lie algebroid over any of the special a-space \((X, \mathcal{O}_X)\). For each open set \( U \) of \( X \), we find the universal enveloping algebra \( \mathcal{U}(\mathcal{O}_X(U), \mathcal{L}(U)) \) of the \((\mathbb{K}, \mathcal{O}_X(U))-\text{Lie-Rinehart algebra} \mathcal{L}(U)\) or of the Lie-Rinehart pair \((\mathcal{O}_X(U), \mathcal{L}(U))\) (see [16,26,34]). The sheafification of the canonical presheaf:

\[
U \mapsto \mathcal{U}(\mathcal{O}_X(U), \mathcal{L}(U)),
\]

is known as the universal enveloping algebroid of the Lie algebroid \( \mathcal{L} \) and denoted by \( \mathcal{U}(\mathcal{O}_X, \mathcal{L}) \) (see [3,5,29,32,35] for details).

From the construction of \( \mathcal{U}(\mathcal{O}_X, \mathcal{L}) \), it is an associative \( \mathbb{K}_X \)-algebra and \( \mathcal{O}_X \)-bimodule. Moreover, there is a canonical \( \mathbb{K}_X \)-algebra monomorphism \( \iota_X : \mathcal{O}_X \hookrightarrow \mathcal{U}(\mathcal{O}_X, \mathcal{L}) \) and an \( \mathcal{O}_X \)-linear map \( \iota_L : \mathcal{L} \to \mathcal{U}(\mathcal{O}_X, \mathcal{L}) \). The associative \( \mathbb{K}_X \)-algebra \( \mathcal{U}(\mathcal{O}_X, \mathcal{L}) \) is generated by \( \mathcal{O}_X \) and \( \iota_L(\mathcal{L}) \) satisfy the following identities:

\[
D \cdot D' = D' \cdot D, \quad D f - f D = a(D)(f),
\]

where \( D, D' \in \mathcal{L}, f \in \mathcal{O}_X, \) and \( D = \iota_L(D) \) for all \( D \in \mathcal{L} \).

Hence, the map \( \iota_L \) can also be viewed as a \( \mathbb{K}_X \)-Lie algebra homomorphism.

**Remark 2.8.** The universal enveloping algebroid \( \mathcal{U}(\mathcal{O}_X, \mathcal{L}) \) of \( \mathcal{L} \) is characterized by the following universal property:

Let \( \mathcal{A} \) be a sheaf of unital associative \( \mathbb{K}_X \)-algebras with sheaf homomorphisms \( \phi : \mathcal{O}_X \to \mathcal{A} \) of \( \mathbb{K}_X \)-unital algebras and \( \psi : (\mathcal{L}, [\cdot, \cdot]) \to (\mathcal{A}, [\cdot, \cdot]_\mathbb{K}) \) of \( \mathbb{K}_X \)-Lie algebras such that \( \phi(D) = \psi(D) \) and \( (\psi(D), \phi(f))_{\mathbb{K}} = \phi(a(D)(f))) \) holds for \( f \in \mathcal{O}_X \) and \( D \in \mathcal{L} \). Then, there exists a unique \( \mathbb{K}_X \)-linear homomorphism \( \iota : \mathcal{U}(\mathcal{O}_X, \mathcal{L}) \to \mathcal{A} \) such that \( \psi \circ \iota_X = \phi \) and \( \psi \circ \iota_L = \psi \).

**Note 2.9.** For a locally free Lie algebroid \( \mathcal{L} \) over an a-space \((X, \mathcal{O}_X)\), the homomorphism \( \iota_L : \mathcal{L} \to \mathcal{U}(\mathcal{O}_X, \mathcal{L}) \) becomes an embedding (as \( \mathbb{K}_X \)-algebras and \( \mathcal{O}_X \)-modules).

**Note 2.10.** For a non-singular special a-space \((X, \mathcal{O}_X)\), the universal enveloping algebroid is isomorphic to the sheaf of differential operators \( \mathcal{D}_X \) on \( X \) (i.e. the sheaf of differential operators over \( \mathcal{O}_X \), sometimes denoted as \( \mathcal{D}iff(\mathcal{O}_X) \)), i.e.

\[
\mathcal{U}(\mathcal{O}_X, \mathcal{T}_X) \cong \mathcal{D}iff(\mathcal{O}_X) =: \mathcal{D}_X.
\]
Note 2.11. The sheaf of logarithmic derivations and sheaf of logarithmic differential operators are denoted as $T_X(-\log Y)$ and $D_X(-\log Y)$ respectively for a principal divisor $Y$ in some complex manifold or smooth algebraic variety $X$ [29].

In the case of a free divisor $Y$ in $X$ (i.e. $T_X(-\log Y)$ is locally free $O_X$-module [4,29,32]) we have (sheafifying the local description given for the module of logarithmic derivations in [22]) the isomorphism
\[ U(O_X, T_X(-\log Y)) \cong D_X(-\log Y). \]

Remark 2.12. In [18], the notion of path algebroid $P_X$ of a smooth manifold or smooth algebraic variety $X$ is constructed as the free Lie algebroid generated by tangent sheaf $T_X$. It forms a locally free sheaf of Lie-Rinehart algebras over $(X, O_X)$ of infinite rank. The universal enveloping algebroid $U(O_X, P_X) =: \mathfrak{U} X$ is described as sheaf of non commutative differential operators on $X$.

2.3. $O_X/\mathbb{K}_X$-bialgebras. The notation $O_X/\mathbb{K}_X$-bialgebras [29] is considered as a sheaf theoretic analogue of the notion $R/\mathbb{K}_X$-bialgebras [26] or left bialgebroids [19]. As an example, we consider the universal enveloping algebroid $U(O_X, L)$ (and its dual) of a Lie algebroid $L$ over $(X, O_X)$. Similar structure is appeared in [1,5,7,18], in complex and algebraic geometry context.

Definition 2.13. A tuple $(A, \Delta, \epsilon)$ is called $O_X/\mathbb{K}_X$-bialgebra if the following conditions hold.

- $A$ is a sheaf of unital associative $\mathbb{K}_X$-algebra extending the sheaf $O_X$;
- $A$ is equipped with the morphism of $O_X$-modules, the comultiplication $\Delta : A \to A \otimes_{O_X} A$ and counit $\epsilon : A \to O_X$;
- The image $\Delta(A)$ lies in a certain $\mathbb{K}_X$-subalgebra of $A \otimes_{O_X} A$;
- $\Delta(1) = 1 \otimes 1$ and $\epsilon(1) = 1$, where $1$ is the unit of $O_X$;
- $\Delta(ab) = \Delta(a)\Delta(b)$, $\epsilon(ab) = \epsilon(a)\epsilon(b))$ for any two sections $a, b$ in $A$.

We recall the canonical $O_X/\mathbb{K}_X$-bialgebra structures on $U(O_X, L)$ and its dual in the following.

The universal enveloping algebroid $U(O_X, L)$ as an $O_X/\mathbb{K}_X$-bialgebra:

Let $(L, [\cdot, \cdot], a)$ be a Lie algebroid over one of the special $\alpha$-space $(X, O_X)$. The sheaf $U(O_X, L)$ has a canonical associative unital $\mathbb{K}_X$-algebras and an $O_X$-bimodule structure as discussed above.

Observe that $U(O_X, L)$ is a cocommutative counital $O_X$-coalgebra in the sense that there is a natural cocommutative coassociative comultiplication $\Delta : U(O_X, L) \to U(O_X, L) \otimes_{O_X} U(O_X, L)$, and counit $\epsilon : U(O_X, L) \to O_X$ respectively, which are locally given by the following formulas [5]
\[
\Delta(f) = f \otimes 1 + 1 \otimes f,
\Delta(D) = D \otimes 1 + 1 \otimes D,
\Delta(D'D'') = \sum D'_{(1)}D''_{(1)} \otimes D'_{(2)}D''_{(2)},
\epsilon(D) = D(1),
\]
for a section $f$ of $O_X$, for a section $D$ of $L$ with $D = \iota_L(D)$, and for sections $D', D''$ of $U(O_X, L)$. Consequently, $(U(O_X, L), \Delta, \epsilon)$ is a sheaf of $O_X/\mathbb{K}_X$-bialgebras.

The Jet enveloping algebroid $\mathfrak{j}(O_X, L)$ as an $O_X/\mathbb{K}_X$-bialgebra: The notion of jet algebroid of a Lie algebroid $L$ over $(X, O_X)$, is defined as the dual of the universal enveloping algebroid:
\[
\mathfrak{j}(O_X, L) := \text{Hom}_{O_X}(U(O_X, L), O_X).
\]
For each open set $U$ of $X$, we find the jet algebras $\mathcal{J}(O_X(U), L(U))$ of the $(\mathbb{K}, O_X(U))$-Lie-Rinehart algebra $L(U)$ (see [1,7,19]). The jet algebroid $\mathfrak{j}(O_X, L)$ is the sheafification of the canonical presheaf:
\[
U \mapsto \mathcal{J}(O_X(U), L(U)).
\]

One can dualize the structures on $U(O_X, L)$. The product on $\mathfrak{j}(O_X, L)$ is induced from the coproduct of $U(O_X, L)$ on each space of sections, which is locally defined by
\[
(\phi_1\phi_2)(D) := \phi_1(D_{(1)})\phi_2(D_{(2)})
\]
for sections $\phi_1, \phi_2 \in \mathfrak{j}(O_X, L)$ and a section $D \in U(O_X, L)$. By cocommutativity of $U(O_X, L)$, this defines a commutative algebra structure on $\mathfrak{j}(O_X, L)$. The unit for this multiplication is locally given by the left
count \( \epsilon : \mathcal{U}(O_X, \mathcal{L}) \to O_X \), since
\[
(\epsilon \phi)(D) = \epsilon(D_{(1)}) \phi(D_{(2)}) = \phi(\epsilon(D_{(1)})D_{(2)}) = \phi(D)
\]
for a section \( \phi \) in \( \mathcal{J}(O_X, \mathcal{L}) \) and a section \( D \) in \( \mathcal{U}(O_X, \mathcal{L}) \). The left and right \( O_X \)-module structure on \( \mathcal{U}(O_X, \mathcal{L}) \) provides \( O_X - O_X \)-bimodule structure on \( \mathcal{J}(O_X, \mathcal{L}) \).

The product on \( \mathcal{U}(O_X, \mathcal{L}) \) descends to a sheaf homomorphism \( m : \mathcal{U}(O_X, \mathcal{L}) \otimes_{K_X} \mathcal{U}(O_X, \mathcal{L}) \to \mathcal{U}(O_X, \mathcal{L}) \). We can therefore dualize the product to obtain a coproduct \( \Delta^*: \mathcal{J}(O_X, \mathcal{L}) \to \mathcal{J}(O_X, \mathcal{L}) \otimes_{O_X} \mathcal{J}(O_X, \mathcal{L}) \) locally defined as
\[
\phi(DD') =: \Delta^*(\phi)(D \otimes D') = \phi(1)(D\phi(2)(D'))
\]
Associativity of the multiplication implies that \( \Delta^* \) is coassociative. The counit for this coproduct is given by \( \epsilon^* : \phi \mapsto \phi(1_{\mathcal{U}(O_X, \mathcal{L})}) \). It follows that \( (\mathcal{J}(O_X, \mathcal{L}), \Delta^*, \epsilon^*) \) is an \( O_X/k_X \)-bialgebra.

**Note 2.14.** The jet algebroid is a formal groupoid that serves as the formal exponentiation of the Lie algebroid [5]. In [18], the author studied this structure for the path algebroid \( \mathcal{P}_X \) of a smooth variety \( X \), referring to it as the formal path groupoid.

**Note 2.15.** For an \( O_X/k_X \)-bialgebra \( (A, \Delta, \epsilon) \), the sheaf of premitive elements is determined by the following assignment (see [29])
\[
U \mapsto \{ a \in A(U) \mid \Delta(U)(a) = a \otimes 1 + 1 \otimes a \},
\]
denoted by \( \mathcal{P}(A) \), provides a sheaf of \( (k_X, O_X) \)-Lie-Rinehart algebras as follows:
\[
a : (\mathcal{P}(A), [\cdot, \cdot]_c) \to (\text{Der}_{k_X}(O_X), [\cdot, \cdot]_c)
\]
is defined by \( a(D)(f) = \epsilon(D \ f) \) for sections \( f \in O_X \) and \( D \in \mathcal{P}(A) \).

**Note 2.16.** For a smooth (holomorphic) Lie algebroid [2] i.e. for a locally free sheaf of Lie-Rinehart algebras of finite rank over a smooth (complex) manifold, the universal enveloping algebroid is a cocomplete locally graded free \( O_X/k_X \)-bialgebra (of finite type) [29]. On the other hand, the tangent sheaf over the affine scheme \( \text{Spec}(A^N) \) (where \( A^N := k[x_i]_{i \in \mathbb{N}} \) [9,29] and the free Lie algebroid \( \mathcal{P}_X \) over a smooth manifold \( X \) [18] are locally free sheaves of Lie-Rinehart algebras (of infinite rank). Thus, their universal enveloping algebroids are cocomplete locally graded free \( O_X/k_X \)-bialgebras (of infinite type) [29].

3. **Lie algebroid Cohomology over \( a \)-spaces**

Lie algebroids over an algebraic space (shortly, an \( a \)-space) and homomorphisms among them are discussed in Section 2.1. These form a category, where kernel and image of a morphism provides new objects in that category. We recall the notion of representations or module of a Lie algebroid in this context. The category of representations of a Lie algebroid has enough injectives (since the category of quasi-coherent sheaves of left \( O_X \)-modules is enough injectives [14]), useful for doing homological algebra.

The Chevalley-Eilenberg-de Rham cohomology (or, Lie algebroid cohomology) of a coherent Lie algebroid over a complex manifold or over a Noetherian separated scheme is expressed as derived functor \( \text{E} \text{xt} \) [3,32]. In [32], the result is stated for locally free Lie algebroid of finite rank over a complex manifold and in [3], the result is proved for locally free Lie algebroid of finite rank over a Noetherian separated scheme. Here, we construct a more general cochain complex of sheaves, which coincides with the cochain complexes of sheaves used in the classical contexts [2,3,21,32]. For that we do not require the coherent (or locally finitely presented) condition on the underlying \( O_X \)-module of a Lie algebroid. Notably, for analytic and algebraic geometric setups, the base space \( X \) does not need to be non-singular. However, in that case, the tangent sheaf, which becomes a coherent Lie algebroid, does not retain the property of being a locally free Lie algebroid. For more details, see [29,32]. Moreover, we present the result in a simplified form by using Čech cohomology. As an application of Lie algebroid cohomology we consider Chern classe of a module (locally free \( O_X \)-module of finite rank) over the Lie algebroid.

To consider Lie algebroid cohomology for \( \mathcal{L} \) over \( \mathcal{(X, O_X)} \) with coefficient in some \( O_X \)-module \( \mathcal{E} \), we need to consider the followings.
3.1. Atiyah algebroid. For an (quasicoherent) $\mathcal{O}_X$-module $\mathcal{E}$, we form a Lie algebroid consisting of the sheaf of differential operators on $\mathcal{E}$ of order $\leq 1$ [3,18,32], defined by

$$\mathcal{A}(\mathcal{E}) = \{ D \in \text{End}_{\mathcal{O}_X}(\mathcal{E}) | D(f s) = f D(s) + \sigma_D(f) s \text{ for a unique } \sigma_D \in T_X, $$

where $f \in \mathcal{O}_X$ and $s \in \mathcal{E}$ }, with the anchor map defined by

$$\sigma : \mathcal{A}(\mathcal{E}) \to T_X $$

where $D \mapsto \sigma_D$

and the Lie bracket is commutator bracket. This Lie algebroid structure is so-called Atiyah algebroid of the $\mathcal{O}_X$-module $\mathcal{E}$. It provides a short exact sequence (s.e.s.) of Lie algebroids over $(X, \mathcal{O}_X)$ (an abelian Lie algebroid extension) as

$$0 \to \text{End}_{\mathcal{O}_X}(\mathcal{E}) \to \mathcal{A}(\mathcal{E}) \to T_X \to 0. $$

In particular, when $\mathcal{E} = \mathcal{O}_X$ for a non singular $a$-space $X$, we have $\mathcal{A}(\mathcal{O}_X) \cong \mathcal{O}_X \oplus T_X$, equals to the sheaf of differential operators on $X$ of order $\leq 1$ with scalar symbol. The universal enveloping algebroid of the Atiyah algebroid $\mathcal{A}(\mathcal{O}_X)$ is the sheaf of twisted differential operators over $X$ [1].

A connection $\nabla : T_X \to \mathcal{A}(\mathcal{E})$ that satisfies $\sigma \circ \nabla = 1d_{T_X}$ provides a splitting of the s.e.s. (1) as $\mathcal{O}_X$-modules. If its curvature is zero then the s.e.s. splits as Lie algebroids. More generally, for a Lie algebroid $(\mathcal{L}, [-, -], a)$ on an $a$-space $(X, \mathcal{O}_X)$, a $\mathcal{L}$-connection on $\mathcal{E}$ is defined by an $\mathcal{O}_X$-linear map (see [1,3,32])

$$\nabla : \mathcal{L} \to \mathcal{A}(\mathcal{E})$$

$$D \mapsto \nabla_D$$

satisfying the Leibniz rule $\nabla(f s) = f \nabla_D(s) + a(D)(f) s$ for sections $f \in \mathcal{O}_X$, $D \in \mathcal{L}$ and $s \in \mathcal{E}$ (to ensure compatibility, we choose $\sigma \circ \nabla = a$, i.e. $\sigma_{\nabla_D} = a(D)$ for $D \in \mathcal{L}$). Equivalently, it is described by a $\mathbb{K}_X$-linear map

$$\nabla : \mathcal{E} \to \Omega^1_{\mathcal{L}} \otimes_{\mathcal{O}_X} \mathcal{E}$$

satisfying the Leibniz rule $\nabla(f s) = f \nabla s + \tilde{a}(df) \otimes s$, where $\Omega^1_{\mathcal{L}} := \mathcal{H}om_{\mathcal{O}_X}(\mathcal{L}, \mathcal{O}_X)$ for a Lie algebroid $\mathcal{L}$ and $\Omega^1_{T_X} := \Omega^1_{T_X}$ together with $\tilde{a} : \Omega^1_{\mathcal{L}} \to \Omega^1_{T_X}$ is the dual of the anchor map.

The $\mathcal{L}$-connection on $\mathcal{E}$ is said to be flat if the map (2) is a Lie algebroid homomorphism (i.e. the $\mathcal{L}$-curvature is zero), i.e. the map (2) additionally satisfies

$$\nabla_{[D,D]} = [\nabla_D, \nabla_{D'}]_c$$

In that case, $(\mathcal{E}, \nabla)$ is said to be a representation of $\mathcal{L}$ or call it a $\mathcal{L}$-module.

Next, we consider some special cases where $X$ is non-singular $a$-spaces:

(i) For $\mathcal{L} = T_X$ and $\mathcal{E}$ is a locally free $\mathcal{O}_X$-module of finite rank (or vector bundle over $X$), we have the standard covariant derivative as a $T_X$-connection on $\mathcal{E}$.

(ii) For $\mathcal{L} = T_X(-\text{log } Y)$, if a connection exists on an $\mathcal{O}_X$-module $\mathcal{E}$, it is called a logarithmic connection on $\mathcal{E}$. It forms a meromorphic connection with poles along the divisor $Y$ [4,32].

If $(\mathcal{E}, \nabla)$ is a $\mathcal{L}$-module, then the Lie algebroid morphism

$$\nabla : \mathcal{L} \to \mathcal{A}(\mathcal{E})$$

extends to an $\mathcal{O}_X$-linear homomorphism of $\mathbb{K}_X$-algebras (using Remark 2.8)

$$\nabla : \mathcal{U}(\mathcal{O}_X, \mathcal{L}) \to \text{End}_{\mathbb{K}_X}(\mathcal{E}),$$

making $\mathcal{E}$ into a left $\mathcal{U}(\mathcal{O}_X, \mathcal{L})$-module. If $\mathcal{L}$ is locally free $\mathcal{O}_X$-module and $\mathcal{E}$ be a $\mathcal{U}(\mathcal{O}_X, \mathcal{L})$-module, then the restriction of the action of $\mathcal{U}(\mathcal{O}_X, \mathcal{L})$ to $\mathcal{L}$ (by using the canonical embedding of $\mathcal{L}$ in $\mathcal{U}(\mathcal{O}_X, \mathcal{L})$ as described in Note 2.9), provides a $\mathcal{L}$-module structure on $\mathcal{E}$. In this case, the category of $\mathcal{L}$-modules and the category of left $\mathcal{U}(\mathcal{O}_X, \mathcal{L})$-modules are equivalent. It helps in studying homological algebra with $\mathcal{L}$-modules ($\mathcal{L}$ is a sheaf of non-associative $\mathbb{K}$-algebras) by treating them as modules over the associative $\mathbb{K}_X$-algebra $\mathcal{U}(\mathcal{O}_X, \mathcal{L})$.

**Notation:** Let $\mathcal{O}$ be a sheaf of associative $\mathbb{K}$-algebras over a topological space $X$ with $\mathcal{O}$-modules $\mathcal{C}^i$, for $i \geq 0$. Consider a cochain complex $(\mathcal{C}^\bullet, d)$ where $\mathcal{C}^\bullet := \oplus \mathcal{C}^i$ is a graded $\mathcal{O}$-module and $d : \mathcal{C}^\bullet \to \mathcal{C}^{\bullet+1}$ is the co-boundary map, i.e. $d$ is a $\mathbb{K}_X$-linear map of degree 1 satisfying $d^2 = 0$. Similarly, for a chain
complex, instead of a co-boundary map we have a boundary map \( \partial : C^\bullet \to C^{\bullet-1} \), a \( \mathbb{K}_X \)-linear map of degree \(-1\) satisfying \( \partial^2 = 0 \). To address the following topics of discussion consistently, we consider such (co)chain complexes where \( \mathcal{O} \)-linearity of the (co)boundary maps is not necessarily required.

For a \( \mathcal{O} \)-module \( \mathcal{C} \), we can canonically form the following sheaves of graded vector spaces \( \wedge_{\mathcal{O}} \mathcal{C} := \oplus_i \wedge_{\mathcal{O}}^i \mathcal{C} \) and \( \mathcal{C}^{\otimes i} := \oplus_i \mathcal{C}^{\otimes i} \), where \( i \geq 0 \) or \( i \in \mathbb{N} \cup \{0\} \). These form exterior algebra and tensor algebra of \( \mathcal{C} \) with respect to the products \( \wedge \) and \( \otimes \) over \( \mathcal{O} \) respectively.

**Note 3.1.** For a \( \mathcal{O} \)-module \( \mathcal{F} \) on \( X \), there exists an injective resolution \( \mathcal{F} \rightarrowto \mathcal{C}^\bullet(\mathcal{F}) \) (a quasi-isomorphism) of \( \mathcal{O} \)-modules, known as the flabby Godement resolution of \( \mathcal{F} \). Thus, for a complex of \( \mathcal{O} \)-modules \( \mathcal{F}^\bullet \), we consider the associated bicomplex of sheaves \( \mathcal{C}^\bullet(\mathcal{F}^\bullet) = (\mathcal{C}^p(\mathcal{F}^q)) \) (\( p, q \in \mathbb{N} \cup \{0\} \)) of injective resolutions. The original complex is embedded in the total complex \( K^\bullet = \operatorname{tot}(\mathcal{C}^\bullet(\mathcal{F}^\bullet)) \), and this embedding is a quasi-isomorphism. The cohomology of the associated complex \( K^\bullet(X) = \operatorname{tot}(\mathcal{C}^\bullet(\mathcal{F}^\bullet))(X) \) of global sections is called the hypercohomology of \( \mathcal{F}^\bullet \), and denoted by \( H^\bullet(X, \mathcal{F}^\bullet) \) (see [8]).

**Note 3.2.** Denote the category of cochain complexes of sheaves of \( \mathcal{O}_X \)-modules on an \( \alpha \)-space \((X, \mathcal{O}_X)\) by \( \operatorname{Shv}(X)^\bullet \) and the category of \( \mathbb{K} \)-vector spaces by \( \operatorname{Vect} \).

For \( \mathcal{F}^\bullet \in \operatorname{Shv}(X)^\bullet \), the \( k \)-th cohomology sheaf of the complex \( \mathcal{F}^\bullet \) is

\[
\mathcal{H}^k(\mathcal{F}^\bullet) := \operatorname{Ker}(\mathcal{F}^k \to \mathcal{F}^{k+1})/\operatorname{Im}(\mathcal{F}^{k-1} \to \mathcal{F}^k)
\]

(where \( \mathcal{F}^{-1} := 0 \)) for \( k \in \mathbb{N} \cup \{0\} \), considers as quotient of \( \mathbb{K}_X \)-vector spaces.

Moreover, a map of complexes \( \mathcal{F}^\bullet \to \mathcal{G}^\bullet \) is a quasi-isomorphism if the induced map on cohomology sheaves \( \mathcal{H}^k(\mathcal{F}^\bullet) \to \mathcal{H}^k(\mathcal{G}^\bullet) \) is an isomorphism for all \( k \).

The \( k \)-th hypercohomology is a functor \( H^k(\mathcal{F}(X, -)) : \operatorname{Shv}(X)^\bullet \to \operatorname{Vect} \) for \( k \in \mathbb{N} \cup \{0\} \) (see [8, 36] for details) satisfying the following two conditions:

A quasi-isomorphism of complexes \( f^\bullet : \mathcal{F}^\bullet \to \mathcal{G}^\bullet \) induces an isomorphism \( H^k(X, f^\bullet) \), and if \( \mathcal{I}^\bullet \) is a complex of injective sheaves then \( H^k(X, \mathcal{I}^\bullet) = H^k(\Gamma(X, \mathcal{I}^\bullet)) \), where \( \Gamma(X, \mathcal{I}^\bullet) \) is the space of global sections of \( \mathcal{I}^\bullet \).

Denote the graded vector space \( \bigoplus_{n=0}^\infty H^n(X, \mathcal{F}^\bullet) \) as \( H^\bullet(X, \mathcal{F}^\bullet) \).

**Remark 3.3.** We use sheaf-theoretic generalizations of the well-known derived functors \( \operatorname{Ext} \) and \( \operatorname{Cotor} \) (see [34, 19]), by sheafifying the standard cochain complexes, and subsequently considering the associated hypercohomologies.

### 3.2. Chevalley-Eilenberg-de Rham complex and Koszul-Rinehart resolution

For a Lie algebroid \( \mathcal{L} \) over \((X, \mathcal{O}_X)\) with a representation \((\mathcal{E}, \nabla)\), consider the cochain complex (consists with \( \mathcal{O}_X \)-modules with a \( \mathbb{K}_X \)-linear degree 1 map), a generalization of the well known Chevalley-Eilenberg-de Rham complex [21]

\[
(4) \quad \Omega^\bullet_{\mathcal{E}}(\mathcal{X}) := (\operatorname{Hom}_{\mathcal{O}_X}(\wedge_{\mathcal{O}_X} \mathcal{L}, \mathcal{E}), d).
\]

It is the sheafification of the presheaf of cochain complexes [34, 40]

\[
U \mapsto (\operatorname{Hom}_{\mathcal{O}_X(U)}(\wedge_{\mathcal{O}_X(U)} \mathcal{L}(U), \mathcal{E}(U)), d_U) := \Omega^\bullet_{\mathcal{E}(U)}(\mathcal{E}(U)),
\]

where the differential \( d_U \) associated with an open subset \( U \) of \( X \) is given by

\[
d_U(\omega)(D_1 \wedge \cdots \wedge D_{k+1}) = \sum_{i=1}^{k+1} (-1)^{i+1} \nabla_{D_i}^U(\omega(D_1 \wedge \cdots \wedge \hat{D_i} \wedge \cdots \wedge D_{k+1})) + \sum_{i<j} (-1)^{i+j} \omega([D_i, D_j] \wedge D_1 \wedge \cdots \wedge \hat{D_i} \wedge \cdots \wedge \hat{D_j} \wedge \cdots \wedge D_{k+1}),
\]

where \( D_1, \ldots, D_{k+1} \in \mathcal{L}(U) \) and \( \omega \in \Omega^k_{\mathcal{E}(U)}(\mathcal{E}(U)) \), and \( \nabla : U \to \nabla^U \) (see the notion in (2)) is the sheafification. Notice that, the differential \( d \) (i.e. \( d^2 = 0 \)) is a \( \mathbb{K}_X \)-linear map, but not an \( \mathcal{O}_X \)-module homomorphism (satisfies the Leibniz rule: \( df \omega = df \wedge \omega + f d \omega \), for \( f \in \mathcal{O}_X(U), \omega \in \Omega^k_{\mathcal{E}(U)}(\mathcal{E}(U)) \)).

If both \( \mathcal{L} \) and \( \mathcal{E} \) are locally free \( \mathcal{O}_X \)-modules of finite rank, then we get

\[
\operatorname{Hom}_{\mathcal{O}_X}(\wedge_{\mathcal{O}_X} \mathcal{L}, \mathcal{E}) \cong \wedge_{\mathcal{O}_X} \mathcal{L}^* \otimes_{\mathcal{O}_X} \mathcal{E},
\]
which implies \( \Omega^* (\mathcal{E}) \cong \Omega^* \otimes_{\mathcal{O}_X} \mathcal{E} \) where \( \Omega^*_X \) is the sheafification of the presheaf of cochain complexes (see (6)). This complex is equivalent to the standard complex associated with a smooth Lie algebroid [21] or a holomorphic Lie algebroid [32]. In particular, we get the Chevalley-Eilenberg complex when \( \mathcal{L} \) is a Lie algebroid over a point; and the de Rham complex \( \Omega^*_X \) when \( \mathcal{L} = T_X \) and \( \mathcal{E} = \mathcal{O}_X \) over a non-singular (or smooth) \( \alpha \)-space \( X \) (see Note 2.2).

The associated hypercohomology of the cochain complex (4) of \( \mathcal{O}_X \)-modules is called the Lie algebroid cohomology of \( \mathcal{L} \) with coefficient in \( \mathcal{E} \) and denoted by \( \mathbb{H}^*(\mathcal{L}, \mathcal{E}) \).

Consider the chain complex of left \( \mathcal{U}(\mathcal{O}_X, \mathcal{L}) \)-modules, called the Koszul-Rinehart complex:

\[
\mathcal{K}^*_{\mathcal{O}_X} \mathcal{L} := \mathcal{U}(\mathcal{O}_X, \mathcal{L}) \otimes_{\mathcal{O}_X} \mathcal{L},
\]

it is the sheafification of the presheaf of cochain complexes (see [34, 40])

\[
U \mapsto (\mathcal{U}(\mathcal{O}_X(U), \mathcal{L}(U))) \otimes_{\mathcal{O}_X(U)} \mathcal{L}(U), \partial_U := \mathcal{K}^*_{\mathcal{O}_X(U)} \mathcal{L}(U)
\]

where the differential \( \partial_U \) associated with an open subset \( U \) of \( X \) is given by

\[
\partial_U (\tilde{D} \otimes D_1 \wedge \cdots \wedge D_k) = \sum_{i=1}^{k} (-1)^{i-1} \tilde{D} \tilde{D}_i \otimes D_1 \wedge \cdots \wedge \hat{D}_i \wedge \cdots \wedge D_k
\]

\[+ \sum_{i<j} (-1)^{i+j} \tilde{D} \otimes [D_i, D_j] \wedge D_1 \wedge \cdots \wedge \hat{D}_i \wedge \cdots \wedge \hat{D}_j \wedge \cdots \wedge D_k,
\]

where \( \tilde{D}, \hat{D}_i \) define by the canonical map \( \mathcal{U} : \mathcal{L}(U) \rightarrow \mathcal{U}(\mathcal{O}_X(U), \mathcal{L}(U)) \), for all \( \tilde{D} \in \mathcal{U}(\mathcal{O}_X(U), \mathcal{L}(U)) \) and \( D_1, \ldots, D_k \in \mathcal{L}(U) \). Notice that the boundary (or chain) map \( \partial \) is \( \mathcal{U}(\mathcal{O}_X, \mathcal{L}) \)-linear.

When \( \mathcal{L} \) is locally free as \( \mathcal{O}_X \)-module, it provides a locally free resolution of \( \mathcal{O}_X \) as a left \( \mathcal{U}(\mathcal{O}_X, \mathcal{L}) \)-module (implies the stalk-wise exactness), known as the Koszul-Rinehart resolution. That is, it is given by the (augmented) chain complex of left \( \mathcal{U}(\mathcal{O}_X, \mathcal{L}) \)-modules

\[
\mathcal{K}^*_X \mathcal{L} \rightarrow \mathcal{O}_X,
\]

defined by the counit map (or augmentation map) \( \varepsilon : \mathcal{U}(\mathcal{O}_X, \mathcal{L}) \rightarrow \mathcal{O}_X \) (is given as \( \varepsilon(\tilde{D}) = \tilde{D}(1) \) for \( \tilde{D} \in \mathcal{U}(\mathcal{O}_X, \mathcal{L}) \)), provides the quasi-isomorphism.

**Note 3.4.** In the last section we consider analogous resolutions for \( \mathcal{O}_X \) in the category of left \( \mathcal{S}_{\mathcal{O}_X} \)-comodules, where we replace the complex \( \mathcal{K}^*_{\mathcal{O}_X} \mathcal{L} \) with its dual complex, the dual Koszul-Rinehart complex \( \mathcal{K}^*_{\mathcal{O}_X} \mathcal{L} \) (sheafifying the local counterpart as described in [19] accordingly).

**Remark 3.5.** In [2, 3, 32], the Lie algebroid cohomology have considered for a special kind of Lie algebroid \( \mathcal{L} \) (consists by coherent sheaf or locally free sheaf of finite rank) over a Noetherian separated scheme or a complex manifold \( (X, \mathcal{O}_X) \). The Lie algebroid cohomology of \( \mathcal{L} \) with values in a \( \mathcal{L} \)-module \( \mathcal{E} \) (a coherent \( \mathcal{O}_X \)-module) is the hypercohomology of the complex (\( \wedge_{\mathcal{O}_X} \mathcal{L}^* \otimes_{\mathcal{O}_X} \mathcal{E}, d \)). This appears as a special case (5) of the complex described in (4).

**Notations:** From now on we use a special type of open cover \( \{ U_x \mid x \in X \} \) of \( X \) for a locally free Lie algebroid \( \mathcal{L} \) where the restrictions \( \mathcal{L}|_{U_x} \) are free \( \mathcal{O}_X|_{U_x} \)-modules for every \( U_x \). These open sets \( U_x \)'s are called special open sets.

We now describe Lie algebroid cohomology in terms of a derived functor.

**Theorem 3.6.** Let \( \mathcal{L} \) be a locally free Lie algebroid over \( (X, \mathcal{O}_X) \) and \( \mathcal{E} \) a representation of \( \mathcal{L} \). Then we get an isomorphism of graded vector spaces

\[
\mathbb{H}^*(\mathcal{L}, \mathcal{E}) \cong Ext^*_\mathcal{L}(\mathcal{O}_X, \mathcal{E}).
\]

**Proof.** For every \( x \in X \), the space of sections \( \mathcal{L}(U_x) \) is a \( (\mathbb{K}, \mathcal{O}_X(U_x)) \)-Lie-Rinehart algebra and a free module over \( \mathcal{O}_X(U_x) \). Thus, the Koszul-Rinehart resolution (7) is a locally free resolution of the sheaf of local rings \( \mathcal{O}_X \). Note that, here the \( \mathcal{L} \)-module \( \mathcal{E} \) can be considered as a \( \mathcal{U}(\mathcal{O}_X, \mathcal{L}) \)-module.

Here, we consider the two naturally associated presheaf of cochain complexes (using notions 3.2 and 3.2)
We have a canonical morphism between these two presheaves of cochain complexes as \(k_X\)-modules, which is a local isomorphism or stalk-wise isomorphism, using results from [34] as local descriptions (applying on each special open neighborhoods \(U_x\)’s of \(x \in X\)). Thus, the associated sheaves of cochain complexes are isomorphic as \(k_X\)-modules (using Lemma 3.2. and Lemma 3.3. from [29]), i.e.

\[
\Omega^*_x(\mathcal{E}) \cong Hom_{U(\mathcal{O}_X(\mathcal{L}(U)), \mathcal{E}(U))}.
\]

Hence, the Chevalley-Eilenberg-de Rham complex of \(\mathcal{L}\) with coefficient in \(\mathcal{E}\) is isomorphic to dual of the Koszul-Rinehart resolution of \(\mathcal{O}_X\) by \(\mathcal{E}\) in the category of left \(\mathcal{U}(\mathcal{O}_X, \mathcal{L})\)-modules.

Therefore, the associated cohomology sheaves (see Note 3.2) are isomorphic, i.e.

\[
H^k(\Omega_x^*, \mathcal{E}) \cong Ext^k_{U(\mathcal{O}_X, \mathcal{L})}(\mathcal{O}_X, \mathcal{E}), \text{ for every } k.
\]

Now, applying the hypercohomology functor \(H^*\) (see Note 3.2) we get the required result (since \(H^*(\mathcal{L}, \mathcal{E}) = H^*(X, \Omega^*_x(\mathcal{E}))\) and \(\text{Ext}^k_{U(\mathcal{O}_X, \mathcal{L})}(\mathcal{O}_X, \mathcal{E}) = H^*(X, \text{Hom}_{U(\mathcal{O}_X, \mathcal{L})}(\mathcal{K}^*_x, \mathcal{L}, \mathcal{E}))\)).

**Note 3.7.** One can find the proof for a locally free Lie algebroid \(\mathcal{L}\) of finite rank over a Noetherian separated scheme \((X, \mathcal{O}_X)\) discussed in [3], by using the ideas of [34] on the level of stalks \(\mathcal{L}_x\) (since \(\mathcal{L}_x\) is a \((\mathbb{K}, \mathcal{O}_X)\)-Lie-Rinehart algebra, projective (in fact, free) module over \(\mathcal{O}_X, x\) for all \(x \in X\)).

**Corollary 3.8.** Let \((X, \mathcal{O}_X)\) be a complex manifold or smooth algebraic variety and \(Y\) a free divisor in \(X\). Therefore, the Lie algebroid \(T_X(−\log Y)\) of logarithmic derivation is locally free \(\mathcal{O}_X\)-module (of finite rank). Using Note 2.11, the logarithmic de Rham cohomology (hypercohomology of the complex (9) described in Remark 4.1 and Remark 4.2 for some special cases, as given in (10)) can be expressed as

\[
H^*(X, \Omega^*_X(\log Y)) \cong \text{Ext}^*_{T_X(−\log Y)}(\mathcal{O}_X, \mathcal{O}_X).
\]

**Corollary 3.9.** Consider the path algebroid \(\mathcal{P}_X\) for a non-singular special a-space \((X, \mathcal{O}_X)\) (see Note 2.2) and its universal enveloping algebroid \(\mathbb{D}_X\) (see Remark 2.12). Note that, a module over the free Lie algebroid \(\mathcal{P}_X\) can be viewed as a vector bundle with \(T_X\)-connection (not necessarily flat), and a \(\mathbb{D}_X\)-module describes some system of linear PDE on the space of paths (see [18] for details).

The Theorem 3.6 for the Lie algebroid \(\mathcal{P}_X\) provides the isomorphism

\[
H^*(\mathcal{P}_X, \mathcal{E}) \cong \text{Ext}^*_{\mathcal{P}(\mathcal{A})}(\mathcal{O}_X, \mathcal{E}).
\]

**Corollary 3.10.** Let \(\mathcal{A}\) be a cocomplete locally graded free \(\mathcal{O}_X/\mathbb{K}_X\)-bialgebra [29] of finite or infinite type (for examples consider Note 2.16). Then the Lie algebroid \(\mathcal{P}(\mathcal{A})\) (sheaf of primitive elements, see Note 2.15) is a locally free \(\mathcal{O}_X\)-module (of finite or infinite rank accordingly). Thus, we have \(\mathcal{A} \cong U(\mathcal{O}_X, \mathcal{P}(\mathcal{A}))\) [29]. Therefore, applying Theorem 3.6 for \(\mathcal{L} = \mathcal{P}(\mathcal{A})\) provides the isomorphism

\[
H^*(\mathcal{P}(\mathcal{A}), \mathcal{E}) \cong \text{Ext}^*_{\mathcal{P}(\mathcal{A})}(\mathcal{O}_X, \mathcal{E}).
\]

**Remark 3.11.** In complex geometry (algebraic geometry), for a cochain complex of coherent sheaves \(\mathcal{F}^*\) over an analytic space (algebraic variety), we compute the hypercohomology \(H^*(X, \mathcal{F}^*)\) via Chech cohomology \(H^*(U, \mathcal{F}^*)\) associated with some good open cover \(U = \{U_i\}_{i}\) of \(X\) (since \(H^*(X, \mathcal{F}^*) \cong H^*(U, \mathcal{F}^*)\), given by a canonical isomorphism). Specifically, we provide a good open cover by connected Stein spaces (affine varieties) and we use Leray’s theorem [8,33] and Cartan-Serre’s vanishing theorem [11] for sheaf cohomology.

When we consider a locally free Lie algebroid \(\mathcal{L}\) of finite rank over an analytic space (algebraic variety) \((X, \mathcal{O}_X)\), we can compute the Lie algebroid cohomology as derived functor using Chech cohomology.

In both cases we have \(H^*(\mathcal{L}, \mathcal{E}) \cong H^*(U, \Omega^*_x(\mathcal{E}))\). Since over Stein spaces (affine varieties) \(U_i, \mathcal{L}(U_i)\) is a projective \(\mathcal{O}_X(U_i)\)-module [28,39], thus we get an isomorphism

\[
H^*(\mathcal{L}(U_i), \mathcal{E}(U_i)) \cong \text{Ext}^*_{U(\mathcal{O}_X(U_i), \mathcal{L}(U_i))}(\mathcal{O}_X(U_i), \mathcal{E}(U_i)).
\]
Hence,
\[ \tilde{H}^* (\mathcal{U}, \Omega^*_2 (\mathcal{E})) \cong \tilde{H}^* (\mathcal{U}, \mathcal{K}om_{\mathcal{U}, \mathcal{O}_X, \mathcal{L}} (\mathcal{K}^*_2 (\mathcal{L}, \mathcal{E}))). \]

Also, note that the whole idea works for the classical case of Lie algebroids over smooth manifolds (i.e. for smooth Lie algebroids). In [2], the Lie algebroid (hyper)cohomology of holomorphic (algebraic) Lie algebroids over a complex manifold (smooth scheme) has been expressed by Čech cohomology using a good open cover consisting with connected Stein manifolds (affine schemes). However, the case associated with analytic spaces is not considered. Additionally, this result can be extended to the quasicoherent \( \mathcal{O}_X \)-module cases.

More generally, for a cochain complex of quasicoherent sheaves \( \mathcal{F}^* \) over a Noetherian separated scheme, by considering an affine open cover \( \mathcal{U} = \{ U_i \} \), of \( X \), we get a canonical isomorphism \( \mathbb{H}^* (\mathcal{X}, \mathcal{F}^*) \cong \tilde{H}^* (\mathcal{U}, \mathcal{F}^*) \) [14]. Thus, we get a similar result for a locally free Lie algebroid over a Noetherian separated scheme.

3.3. Chern classes: Lie algebroid (or Lie-Rinehart algebra) connections, curvatures and the associated Characteristic classes (e.g. Picard group, Chern class) has been studied in the context of differential geometry for smooth Lie algebroids. In considering an affine open cover \( \mathcal{U} \), we get an abelian Lie algebroid extension (as an equivalence class)

\[ \text{Proof.} \]

analogous result for a locally free Lie algebroid over a Noetherian separated scheme.

This result can be viewed in the algebraic geometry set up (over the affine scheme \( \text{Spec} R \)). We derive an analogous result for a locally free Lie algebroid \( \mathcal{L} \) over an \( a \)-space \((X, \mathcal{O}_X)\) with a \( \mathcal{L} \)-module \((\mathcal{E}, \nabla)\) as follows.

We denote the set of equivalence classes of abelian extensions of a Lie algebroid \( \mathcal{L} \) by a flat \( \mathcal{L} \)-connection \((\mathcal{E}, \nabla)\) as \( \text{Ext}^1 (\mathcal{L}, \mathcal{E}, \nabla) \). An element \( [\mathcal{E}'] \in \text{Ext}^1 (\mathcal{L}, \mathcal{E}, \nabla) \) represent a s.e.s. of Lie algebroids

\[ \mathcal{E} \to \mathcal{E}' \to \mathcal{L} \]

upto isomorphism in the usual sense, where \( \mathcal{E} \) is the trivial Lie algebroid. Thus, as a local description, on the level of stalks, for each point \( x \in X \), we have the s.e.s. of \((\mathbb{K}, \mathcal{O}_{X,x})\)-Lie-Rinehart algebras

\[ \mathcal{E}_x \to \mathcal{E}'_x \to \mathcal{L}_x. \]

**Theorem 3.12.** Let \((\mathcal{L}, [-], \mathfrak{a})\) be a locally free Lie algebroid over an \( a \)-space \((X, \mathcal{O}_X)\) with a \( \mathcal{L} \)-module \((\mathcal{E}, \nabla)\). Then, we get a canonical isomorphism of \( \mathbb{K} \)-vector spaces

\[ \mathbb{H}^2 (\mathcal{L}, \mathcal{E}) \cong \text{Ext}^1 (\mathcal{L}, \mathcal{E}, \nabla). \]

**Proof.** Given a Lie algebroid \((\mathcal{L}, [-], \mathfrak{a})\) over \((X, \mathcal{O}_X)\) and an element (an isomorphism class) \([\omega] \in \mathbb{H}^2 (\mathcal{L}, \mathcal{E})\) where \( \omega \in \Omega^2_1 (\mathcal{E}) \), we get a Lie algebroid extension of \( \mathcal{L} \) by \( \mathcal{E} \) (upto isomorphism) as follows.

Consider the \( \mathcal{O}_X \)-module \( \mathcal{E} \oplus \mathcal{L} \) with the \( \mathbb{K}_X \)-Lie algebra structure given by

\[ [ (s, D), (s', D') ]_\omega := (\nabla_D (s') - \nabla_{D'} (s) + \omega (D, D')) , [D, D']) , \]

for \( s, s' \in \mathcal{E} \) and \( D, D' \in \mathcal{L} \), together with the map

\[ \mathfrak{a}_\omega : \mathcal{L}_\omega \to \mathcal{T}_X, \ (s, D) \mapsto \mathfrak{a}(D). \]

Thus, \( \mathcal{L}_\omega := (\mathcal{E} \oplus \mathcal{L}, [-], \mathfrak{a}_\omega, \mathfrak{a}_\omega) \) forms a Lie algebroid over \((X, \mathcal{O}_X)\). Let \([\omega] = [\omega']\) for some \( \omega' \in \Omega^2_1 (\mathcal{E}) \), i.e. \( \omega - \omega' = \theta \mathfrak{d} \theta \) for some \( \theta \in \Omega^1_1 (\mathcal{E}) \). Then the map \( \tilde{\theta} : \mathcal{L}_\omega \to \mathcal{L}_\omega' \) defined by \( \tilde{\theta} (s + D) = (s + \theta (D)) + D \) forms an isomorphism of Lie algebroids. Thus, we get an abelian Lie algebroid extension (as an equivalence class) of \( \mathcal{L} \) by the module \((\mathcal{E}, \nabla)\) i.e. a s.e.s. of Lie algebroids where \( \mathcal{E} \) is the trivial Lie algebroid

\[ \mathcal{E} \hookrightarrow \mathcal{L}_\omega \to \mathcal{L}, \]
and the resultant isomorphism class \([\mathcal{L}_c] \in \text{Ext}^1(\mathcal{L}, \mathcal{E}, \nabla)\).

Consider an abelian Lie algebroid extension \(\mathcal{L}'\) of \(\mathcal{L}\) by \(\mathcal{E}\) up to isomorphism, i.e. \([\mathcal{L}'] \in \text{Ext}^1(\mathcal{L}, \mathcal{E}, \nabla)\) as
\[
\mathcal{E} \rightarrow \mathcal{L}' \rightarrow \mathcal{L},
\]
where \((\mathcal{E}, \nabla)\) is a \(\mathcal{L}\)-module. Since \(\mathcal{L}\) is locally free \(\mathcal{O}_X\)-module, the s.e.s. (8) splits as an \(\mathcal{O}_X\)-module, i.e. there is an isomorphism \(\phi : \mathcal{L}' \rightarrow \mathcal{E} \oplus \mathcal{L}\) for the split (or section map) \(\phi : \mathcal{L} \rightarrow \mathcal{L}'\). Thus, the possible choice for a \(\mathbb{K}_X\)-Lie bracket on \(\mathcal{E} \oplus \mathcal{L}\) is given as follows:
\[
[(s, D), (s', D')] := [s, s']' + [D, s']' - [D', s]' + [D, D]',
\]
where \([s, s]' = 0\), \([D, s]' = \nabla_D(s')\), \([D', s]' = \nabla_{D'}(s)\), \([D, D]' = \omega(D, D') + [D, D']\) for some \(\omega \in \Omega^2_\mathcal{E}(\mathcal{E})\) with \(d(\omega) = 0\), for \(s, s' \in \mathcal{E}, D, D' \in \mathcal{L}\). Let \(\phi' : \mathcal{L} \rightarrow \mathcal{L}'\) be another such splitting map and \(\omega' \in \Omega^2_\mathcal{E}(\mathcal{E})\) associated to the isomorphism \(\phi' : \mathcal{L}' \rightarrow \mathcal{E} \oplus \mathcal{L}\). Thus, using the automorphism \(\tilde{\phi}' \circ \tilde{\phi}^{-1}\) on \(\mathcal{E} \oplus \mathcal{L}\) we get \(\omega - \omega' = d(\phi - \phi')\). Thus, we get the required element \([\omega] \in \mathbb{H}^2(\mathcal{L}, \mathcal{E})\) associated to the equivalence class \([\mathcal{L}]\).

This one-to-one correspondence between the \(\mathbb{K}\)-vector spaces \(\mathbb{H}^2(\mathcal{L}, \mathcal{E})\) and \(\text{Ext}^1(\mathcal{L}, \mathcal{E}, \nabla)\) (with the standard vector space structure) can be extended to an isomorphism. \(\square\)

**Remark 3.13.** In particular, for \(\mathcal{E} = \mathcal{O}_X\) with the standard \(\mathcal{L}\)-module structure given by the anchor map, this kind of result is proved in the holomorphic context using Čech cohomology [37].

Note that the Chern class of a finitely generated projective \(R\)-module \(E\) with a \(L\)-connection \(\nabla\) on \(E\), defined by the trace of the curvature \(R\) associated with the connection \(\nabla\) [24,25]. If \(R\) is a regular \(\mathbb{K}\)-algebra (implies \(L := \text{Der}_X(R)\) is a finitely generated projective \(R\)-module) then by considering the covariant connection on \(E\) we get the first Chern class given by trace of the curvature
\[
c_1(E) = \text{Trace}(R) \in H^2(\text{Der}_X(R), R).
\]

Analogous result holds for a locally free \(\mathcal{O}_X\)-module \(\mathcal{E}\) of finite rank over the affine scheme \(X = \text{Spec}R\). We compute analogous result in the general set up by sheafifying the local descriptions given in [23,24].

For a complex manifold \(X\), a locally free \(\mathcal{O}_X\)-module \(\mathcal{E}\) of finite rank (equivalently a holomorphic vector bundle over \(X\)) always have covariant derivative as a flat \(\mathcal{T}_X\)-connection on \(\mathcal{E}\). An arbitrary \(\mathcal{T}_X\)-connection on \(\mathcal{E}\) is given by an \(\mathcal{O}_X\)-linear map
\[
\nabla : \mathcal{T}_X \rightarrow \text{End}_{\mathcal{O}_X}(\mathcal{E})
\]
satisfying the Leibniz rule, and its curvature is the \(\mathcal{O}_X\)-linear map
\[
R_{\nabla} : \bigwedge^2_\mathcal{O}_X \mathcal{T}_X \rightarrow \text{End}_{\mathcal{O}_X}(\mathcal{E})
\]
defined as
\[
R_{\nabla}(D \wedge D') = [\nabla_D, \nabla_{D'}]_c - \nabla_{[D, D']_c},
\]
for any two sections \(D, D'\) of \(\mathcal{T}_X\). More generally, for a \(\mathcal{L}\)-connection \(\nabla\) on \(\mathcal{E}\), one verifies that the curvature \(R_{\nabla} \in \text{Hom}_{\mathcal{O}_X}(\bigwedge^2_\mathcal{O}_X \mathcal{L}, \text{End}_{\mathcal{O}_X}(\mathcal{E}))\), where \(\mathcal{L}\) is a Lie algebroid.

Consider the induced \(\mathcal{T}_X\)-connection \(ad \nabla = [\nabla, \cdot]_c\) on \(\text{End}_{\mathcal{O}_X}(\mathcal{E})\), a flat connection. Thus by considering the Chevalley-Eilenberg-de Rham complex with coefficient in \(\text{End}_{\mathcal{O}_X}(\mathcal{E})\), we get the cohomology class \([R_{\nabla}] \in H^2(\mathcal{X}, \text{End}_{\mathcal{O}_X}(\mathcal{E}))\) (since \(d^2(R_{\nabla}) = 0\)). Hence, the Chern class of the \(\mathcal{T}_X\)-module \((\mathcal{E}, \nabla)\) is
\[
c_1(\mathcal{E}) := \text{Trace}(R_{\nabla}) \in H^2(\mathcal{X}, \mathcal{O}_X)
\]
where \(\text{Trace} : \text{End}_{\mathcal{O}_X}(\mathcal{E}) \rightarrow \mathcal{O}_X\) is the trace map.

The restriction of a \(\mathcal{T}_X\)-connection to the Lie algebroid \(\mathcal{L} := \mathcal{T}_X(-\log Y)\) (i.e. \(\Omega^1_\mathcal{L} := \Omega^1_X(\log Y)\)) induces a logarithmic connection which is a meromorphic connection with poles along the divisor \(Y\). Moreover, if \(Y\) is a free divisor, then for a locally free \(\mathcal{O}_X\)-module \(\mathcal{E}\) of finite rank with a \(\mathcal{L}\)-connection we get the first Chern class \(c_1(\mathcal{E}) \in H^2(\mathcal{L}, \mathcal{O}_X)\). Note that a \(\mathcal{L}\)-connection might not exists in general. We can extend the notion of Chern classes for \(\mathcal{T}_X\) or \(\mathcal{T}_X(-\log Y)\) to arbitrary Lie algebroids.
4. Logarithmic de Rham cohomology

We compute algebraic (analytic) de Rham cohomology groups of a family of non-singular varieties $Y_t$ for $t \in \mathbb{C} \setminus \{0\}$ and their associated singular variety $Y_0$ (which appears as principal sheaf $\mathcal{I}$ of $\mathcal{O}_X$). Thus, we derive cohomology of the Lie algebroid $\mathcal{T}_X$ for $t \in \mathbb{C} \setminus \{0\}$ and study the singular case analogously. Then, we compute hypercohomology of the logarithmic de Rham complexes [27], which simplifies to the Lie algebroid cohomology for $\mathcal{T}_X(-\log Y_t)$ [32], as we work over some free divisors $Y_t \subset X$ for $t \in \mathbb{C}$ [4,32].

We use the notion $\{(s_1, \ldots, s_n)\}$ for $R$-module generated by $s_1, \ldots, s_n \in E$, where $R$ is a $k$-algebra and $E$ is an $R$-module (also extend to sheaf theoretic settings).

(1) Rectangular Hyperbolae: We consider the affine space $X := \mathbb{A}^2$ (with the Zariski topology) and its coordinate ring $R = \mathcal{O}_X(X) = \mathbb{C}[x,y]$. The principal ideal $I = \langle xy - t \rangle \subset R$ provides rectangular hyperbolae $Y_t := V(I)$ (vanishing set of $I$) for every $t \in \mathbb{C} \setminus \{0\}$. For $t \in \mathbb{C} \setminus \{0\}$, $Y_t$ is a non-singular affine variety with its coordinate ring $R_t := \mathcal{O}_{Y_t}(Y_t) = \mathbb{C}[x,y]/(xy - t)$.

Here, $Y := Y_1$ is homeomorphic to $\mathbb{C} \setminus \{0\}$ and $\mathbb{C} \setminus \{0\}$ is same homotopy type of $S^1$. Thus, on considering the singular cohomology we get that

$$H^i(Y, \mathbb{C}) = H^i(S^1, \mathbb{C}) = \mathbb{C} \text{ for } i = 0, 1$$

and

$$H^i(Y, \mathbb{C}) = H^i(S^1, \mathbb{C}) = \{0\} \text{ for } i \geq 2.$$ 

We can use algebraic de Rham theorem [8,36] to get algebraic de Rham cohomology of $Y_t$ by the singular cohomology of $S^1$. Now we recall the computation which helps us to follow the associated singular case when $t = 0$ (which is a normal crossing divisor).

On $Y$, the differential $df = xdy + ydx = 0$ (where $f = xy - 1$) and $\frac{1}{y} \in R_1 = \mathbb{C}[x, x^{-1}] =: R$, hence $dy = -\frac{dx}{xy} \in \langle dx \rangle$. Thus the space of differential 1-forms on $Y$ (similar to Kähler differentials for $\text{Spec} \, R$) is

$$\Omega^1_R := \langle \langle dx, dy \rangle \rangle = \langle dx \rangle.$$ 

Note that $\frac{dx}{xy}$ is an algebraic differential 1-form on $Y$ (here $\Omega^1_R$ is a free $R$-module of rank 1). The space of algebraic differential 2-forms on $Y$ is $\Omega^2_R := \wedge^2_R \Omega^1_R$. Now, $dx \wedge dy = dx \wedge -\frac{dy}{xy} = 0$ on $Y$ (since $y = \frac{1}{x}$ in $Y$) and thus we get $\Omega^2_R = \{0\}$.

Thus, the algebraic de Rham complex of $Y$ (or $\text{Spec} \, R$) is

$$R \xrightarrow{d^0} \Omega^1_R \xrightarrow{d^1} \cdots,$$

where the first differential is given by $x \mapsto dx$ and $\frac{1}{y} \mapsto -\frac{dx}{xy}$. Here, $\frac{dx}{xy}$ is in the kernel of $d^1$ but $\frac{dx}{x}$ is not in the image of $d^0$ (the only possible choice of preimage is $\log x$, but it is not a polynomial or regular function). It follows that $H^1_{\text{dR}}(Y)$ is a $\mathbb{C}$-vector space of dimension 1. Now, $H^n_{\text{dR}}(Y) \cong \mathbb{C}$ and $H_{\text{dR}}^n(Y) = \{0\}$ for $n \geq 2$.

This is studied in [15], as a topological invariant for non singular spaces, but the analogous computation for the associated singular case is not derived there.

Note that the affine algebraic set $Y$ can be considered as an analytic space. Moreover, this space can be viewed as an analytic divisor [29,32].

We consider the associated logarithmic de Rham cohomology [4,27] in the context of complex geometry.

**Remark 4.1.** Here, $\mathcal{O}_X$ is the sheaf of holomorphic functions on $X := \mathbb{C}^2$ and $(Y := V(I), \mathcal{O}_Y := \mathcal{O}_X/I)$ is the analytic space associated with the principal ideal sheaf $\mathcal{I} = \langle xy - 1 \rangle$ of $\mathcal{O}_X$. The sheaf of logarithmic derivations for $Y$ in $X$ is $\mathcal{T}_X(-\log Y) := \langle \{x \partial_x - y \partial_y, f \nabla f\} \rangle$ where $\nabla f = y \partial_x + x \partial_y$ (since $f = xy - 1$). The sheaf of logarithmic 1-forms for $Y$ in $X$ (i.e. meromorphic 1-forms $\omega$ of $X$ with poles along the divisor $Y$ such that $f \omega \in \mathcal{O}_X$ and $df \wedge \omega \in \Omega^1_X$ [27]) is

$$\Omega^1_X(\log Y) = \mathcal{H}om_{\mathcal{O}_X}(\mathcal{T}_X(-\log Y), \mathcal{O}_X) = \langle \langle dx, dy, \frac{df}{f} \rangle \rangle,$$

where $df = ydx + xdy$. Note that, the relation $y \, dx + x \, dy - (xy - 1) \, \frac{df}{f} = 0$ holds.
The sheaf of logarithmic differential 2-forms on $X$ associated with $Y$ is

$$\Omega^2_X(\log Y) := \bigwedge^2 \Omega^1_X(\log Y) = \{dx \wedge \frac{df}{f}, dy \wedge \frac{df}{f}, dx \wedge dy\}.$$ 

To determine hypercohomology [8,36] of the logarithmic de Rham complex

$$\Omega^\bullet_X(\log Y) : \mathcal{O}_X \xrightarrow{d_0} \Omega^1_X(\log Y) \xrightarrow{d_1} \bigwedge^2 \Omega^1_X(\log Y) \xrightarrow{d_2} \cdots,$$

we have $\ker d_0 = \mathbb{C}X$; $\ker d_1$ is the $\mathbb{C}X$-vector space generated by

$$\{dy \mid g \in \mathcal{O}_X\} \cup \{\frac{df}{f} \}$$

i.e. $\ker d_1 = \mathfrak{m} \cdot d \mathbb{C}X \{\frac{df}{f}\}$ (here $d_0 = d$); and $\ker d^2$ is the $\mathbb{C}X$-vector space generated by $\{dy \wedge \frac{df}{f} \mid g \in \mathcal{O}_X\} \cup \{dx \wedge dy\} = \Omega^2_X(\log Y)$. Therefore,

$$\ker d^2 = \mathbb{C}X \{d^1(g \frac{df}{f}) \mid g \in \mathcal{O}_X\} \oplus \mathbb{C}X \{d^1(x \ dy)\} = \mathfrak{m} \cdot d^2.$$

Therefore, the hypercohomology groups (taking the space of global sections of the complex, since $X$ is a Stein manifold) are given by the following

$$H^0(X, \Omega^2_X(\log Y)) = \mathbb{C}, \quad H^1(X, \Omega^2_X(\log Y)) = \mathbb{C}, \quad H^n(X, \Omega^2_X(\log Y)) = \{0\} \quad (n \geq 2).$$

On $Y$ we have $df = ydx + xdy = 0$ implies $\frac{dy}{y} + \frac{dx}{x} = 0 \quad (x \neq 0, y \neq 0)$ and by integrating it we get $xy = t$ are the solutions which parametrize $t \in \mathbb{C} \{0\}$.

Thus, we can view the normal crossing divisor $Y_0 := \{(x, y) \in \mathbb{C}^2 \mid xy = 0\}$ (a singular analytic space and a free divisor) as deformation of family of rectangular hyperbolas $Y_i$ (appears in [31] as deformation of a scheme).

(2) Normal Crossing Divisor: Here we compute the algebraic de Rham cohomology of the normal crossing divisor $Y_0$ which is a singular affine algebraic set in $X$ with the space of global section of its structure sheaf

$$R_0 := \mathcal{O}_{Y_0}(Y_0) = \mathbb{C}[x, y] \langle xy \rangle.$$

The algebraic de Rham complex for $Y_0$ (or Spec $R_0$) is

$$R_0 \xrightarrow{d} \Omega^1_{R_0} \xrightarrow{d} \Omega^2_{R_0} \xrightarrow{d} \cdots.$$

For $Y_0$ we have $d(xy) = 0$. Thus, $xdy = -ydx$, which implies $xdx \wedge_R dy = 0$ or $ydx \wedge_R dy = 0$. But we cannot conclude that $dx \wedge_R dy = 0$ on $Y_0$.

The space of algebraic differential (Kähler differential) 1-forms for $R_0$ is

$$\Omega^1_{R_0} = \frac{\langle dx, dy \rangle}{(ydx + xdy)}.$$

as a $R_0$-module. Note that for all $n, m \geq 2$, $d_1^n(x^ndy) = 0 = d_1(y^ndx)$ on $Y_0$ and $x^ndy, y^ndx$ both are not in $\text{Im} \ d$, but both these elements vanish on $Y_0$. Also, for all $n, m \geq 0$, $x^ndx$ and $y^ndx$ are in $\text{Im} \ d$, provides zero cohomology class. The Kähler differential 1-form $xdy$ (or $ydx$) on $Y_0$ is not in $\text{Im} \ d$, but since it is also not in $\ker d^1$ (since $d_1(xdy) = dx \wedge_R dy \neq 0$), does not provides a cohomology class. Thus, $H^1_{dR}(Y_0) = \ker d \cong \mathbb{C}$ and $H^1_{dR}(Y_0) \cong \{0\}$. Next, we compute its second algebraic de Rham cohomology class, for that first consider the $R$-module of algebraic (Kähler) differential 2-forms

$$\Omega^2_{R_0} = \frac{\langle dx \wedge dy \rangle}{(ydx + xdy)}.$$

Since the only possible choice for a cohomology class in $H^2_{dR}(Y_0)$ is the algebraic 2-form $dx \wedge dy = d_1(xdy)$ is cohomologous with zero element, thus $H^2_{dR}(Y_0) = \{0\}$.

Note that $Y_0$ is of same homotopy type to a point, thus singular cohomologies of $Y_0$ vanishes in all higher dimensions ($\geq 1$). Fortunately, for this normal crossing variety the algebraic de Rham cohomology and the singular cohomology are same (see [12]).

This approach does not provide a topological invariant for all singular algebraic varieties, to resolve this problem a different approach is considered in [15].
We consider the associated cohomology in complex geometry context (see [29]).

Remark 4.2. Here $\mathcal{O}_X$ is the sheaf of holomorphic functions on $X := \mathbb{C}^2$ and $(Y_0 := V(\mathcal{I}), \mathcal{O}_Y := \mathcal{O}_X/\mathcal{I})$ is the analytic space associated with the ideal sheaf $\mathcal{I} = (xy)$. Here, the sheaf of logarithmic derivation is $\mathcal{T}_X(-\log Y_0) = \{[x\partial_x, y\partial_y]\}$ and the sheaf of logarithmic 1-forms $\Omega^1_X(\log Y_0) := \{[\frac{dx}{x}, \frac{dy}{y}]\}$. Both are locally free $\mathcal{O}_X$-modules of rank 2. The sheaf of logarithmic 2-forms for the divisor $Y_0$ is $\Omega^2_X(\log Y_0) := \{[\frac{dx}{x} \wedge \frac{dy}{y}]\}$, a locally free $\mathcal{O}_X$-module of rank 1.

Note that, the sheaf $\mathcal{T}_X(-\log Y_0)$ has a canonical Lie algebroid structure, and $Y_0$ is a free divisor [29,32]. Since, $\log x$ and $\log y$ both are not in $\mathcal{O}_X$, thus $\frac{dx}{x}$ and $\frac{dy}{y}$ are not in $\Im d^0$ but both are in $\Ker d^1$. Similarly, $\frac{dx}{x} \wedge \frac{dy}{y}$ is in $\Ker d^2$ but not in $\Im d^1$. Other choices for cocycles are appears as coboundaries.

Thus, the hypercohomology of the logarithmic de Rham complex

\begin{equation}
\Omega^\bullet_X(\log Y) : \mathcal{O}_X \xrightarrow{d^0} \Omega^1_X(\log Y_0) \xrightarrow{d^1} \wedge^2 \Omega^1_X(\log Y_0) \xrightarrow{d^2} \cdots
\end{equation}

is $H^0(X, \Omega_X^\bullet(\log Y_0)) = \mathbb{C}$, $H^1(X, \Omega_X^\bullet(\log Y_0)) = \mathbb{C}^2$ and $H^2(X, \Omega_X^\bullet(\log Y_0)) = \mathbb{C}$, all higher cohomologies are zero.

Now, consider the holomorphic Lie algebroid cohomology $H^*_\text{hol}(\mathcal{T}_X(-\log Y_0)) := H^*_\text{hol}(X, \Omega_X^\bullet(\log Y_0)) \cong H^*_\text{dR}(X \setminus Y_0, \mathbb{C})$ (using the standard Lemma of Atiyah-Hodge [36]), similar results appear in [4,37].

Therefore, the standard Lie algebroid cohomology of $\mathcal{T}_X(-\log Y_0)$ followed by the isomorphism (10) provides the analytic (algebraic) de Rham cohomology of the torus $\mathbb{T}^2$ in $\mathbb{R}^4 \cong \mathbb{C}^2$ (since $X \setminus Y_0 = \{x, y \in \mathbb{C}^2 \mid x \neq 0$ and $y \neq 0\} = (\mathbb{C} \setminus \{0\}) \times (\mathbb{C} \setminus \{0\}$) is a complex manifold and homotopic to $S^1 \times S^1 = \mathbb{T}^2$). This agrees with the logarithmic de Rham cohomology for $\Omega_X^\bullet(\log Y_0)$ (described in (9)) as computed.

Note 4.3. In general for smooth algebraic variety and complex manifold, to compute its cohomology we need to consider sheaf of de Rham complex because the space of global sections not necessarily captures the whole information. In these cases we can use Čech cohomology [8,33] by considering an affine open cover or Stein open cover (known as good open cover) [2,36] accordingly.

5. Hochschild Cohomology for a Lie algebroid over a-space

In this section, we define Hochschild cohomology of an $\mathcal{O}_X/\mathcal{K}_X$-bialgebra (see Section 2.3), using (Hopf-)Hochschild cohomology of a left bialgebroid [19] as local descriptions. In particular, we describe the cases associated with universal enveloping algebroid and jet algebroid of a locally free Lie algebroid over an a-space. The cohomology groups can be computed using suitable standard complexes. We prove a version of Hochschild-Kostant-Rosenberg (HKR) theorem and dual HKR theorem. It is done by following the local counterpart as can be found in [19], where the authors proved (dual) HKR theorem for (finitely generated) projective Lie-Rinehart algebras. In particular, we present the HKR theorem for the sheaf of logarithmic differential operators and the sheaf of noncommutative differential operators.

5.1. Hochschild hypercohomology of an $\mathcal{O}_X/\mathcal{K}_X$-bialgebra. Let $(\mathcal{A}, \Delta, \epsilon)$ be an $\mathcal{O}_X/\mathcal{K}_X$-bialgebra. Consider the presheaf of cochain complexes of $\mathcal{O}_X$-modules

\[ U \mapsto C^\bullet(\mathcal{A}(U)), \]

where $C^\bullet(\mathcal{A}(U)) := ((\mathcal{A}(U))^{\otimes \mathcal{K}_X^{(U)}}, b)$ is the Hochschild cochain complex of the $\mathcal{O}_X(U)/\mathcal{K}_X$-bialgebra $\mathcal{A}(U)$ with the differential $b : (\mathcal{A}(U))^{\otimes \mathcal{K}_X^{(U)}} \to (\mathcal{A}(U))^{\otimes \mathcal{K}_X^{(U)}}$ defined for any $a_1, \ldots, a_n \in \mathcal{A}(U)$ as follows:

\[ b(a_1 \otimes \cdots \otimes a_n) = 1 \otimes a_1 \otimes \cdots \otimes a_n + \sum_{i=1}^n (-1)^i a_1 \otimes \cdots \otimes \Delta(a_i) \otimes \cdots \otimes a_n + (-1)^{n+1} a_1 \otimes \cdots \otimes a_n \otimes 1. \]
The sheafification of the presheaf provides a cochain complexes of \( \mathcal{O}_X \)-modules, we call it as Hochschild cochain complex for \( \mathcal{A} \) and denote it by \( \mathcal{C}^\bullet(\mathcal{A}) \). Its associated hypercohomology is called Hochschild hypercohomology of \( \mathcal{A} \), and denoted by \( \mathbb{H}H^\bullet(\mathcal{A}) \). To compute it, it is often useful to consider an appropriate resolution. The canonical choice is given as follows.

Consider the presheaf of cobar complex of left \( \mathcal{A} \)-comodules

\[
U \mapsto \mathcal{C}ob^\bullet(\mathcal{A}(U))
\]

where \( \mathcal{C}ob^\bullet(\mathcal{A}(U)) := ((\mathcal{A}(U))^\otimes n_{+1}, b') \) is the cobar complex for \( \mathcal{O}_X(U) \) in the category of left \( \mathcal{A}(U) \)-comodules with the differential \( b' : (\mathcal{A}(U))^\otimes n_{+1} \to (\mathcal{A}(U))^\otimes n_{+2} \) defined for any \( a_0, \ldots, a_n \in \mathcal{A}(U) \) as follows:

\[
b'(a_0 \otimes a_1 \otimes \cdots \otimes a_n) = \sum_{i=0}^{n} (-1)^i a_0 \otimes \cdots \otimes \Delta(a_i) \otimes \cdots \otimes a_n + (-1)^{n+1} a_0 \otimes \cdots \otimes a_n \otimes 1.
\]

The sheafification of the presheaf \( U \mapsto \mathcal{C}ob^\bullet(\mathcal{A}(U)) \), denotes as \( \mathcal{C}ob^\bullet(\mathcal{A}) \). It provides a resolution of \( \mathcal{O}_X \) by cofree left \( \mathcal{A} \)-comodules (a quasi-isomorphism) given by the left \( \mathcal{O}_X \)-module structure on \( \mathcal{A} \) as

\[
\mathcal{O}_X \to \mathcal{C}ob^\bullet(\mathcal{A}),
\]

call it cobar resolution of \( \mathcal{A} \). Then by applying cotensor product functor \( \mathcal{O}_X \boxtimes \mathcal{A} \) (sheafifying the cotensor functor described in [19]) on it and considering hypercohomology (i.e. applying hypercohomology \( \mathbb{H}^\bullet(\mathcal{X}, -) \) functor on the induced cochain complex \( \mathcal{O}_X \boxtimes \mathcal{A} \mathcal{C}ob^\bullet(\mathcal{A}) \) of \( \mathbb{K}_X \)-vector spaces), we get the following isomorphism (sheafifying the local descriptions of relationship among Hochschild cochain complex and cobar resolution [19] and considering their associated hypercohomologies)

\[
\mathbb{H}H^\bullet(\mathcal{A}) \cong \mathbb{H}^\bullet(\mathcal{X}, \mathcal{O}_X \boxtimes \mathcal{A} \mathcal{C}ob^\bullet(\mathcal{A})) = \mathfrak{C}otor^\mathcal{A}_\mathcal{O}(\mathcal{O}_X, \mathcal{O}_X).
\]

Note 5.1. The standard notion of the Hochschild chain complex is defined for an associative \( \mathbb{K} \)-algebra using its multiplication, and its dual complex is then considered as the Hochschild cochain complex. In our case, the algebra structure over \( \mathcal{O}_X \) is important to consider. For an \( \mathcal{O}_X \)-bimodule algebra, we do not have an associative algebra structure over \( \mathcal{O}_X \), but rather a co-associative algebra structure. Therefore, for an \( \mathcal{O}_X \)-bimodule algebra the co-associative algebra structure over \( \mathcal{O}_X \) defines Hochschild cohomology using co-multiplication.

Now, we describe Hochschild (hyper)cohomology for the \( \mathcal{O}_X \)-bimodule algebras \( \mathfrak{U}(\mathcal{O}_X, \mathcal{L}) \) and \( \mathfrak{J}(\mathcal{O}_X, \mathcal{L}) \) of a (locally free) Lie algebroid \( \mathcal{L} \) over an \( n \)-space \( (\mathcal{X}, \mathcal{O}_X) \) (see Section 2 for these notions).

5.2. Hochschild hypercohomology of universal enveloping algebroid. First we recall some of the essential ideas related for our next topic of discussion.

The sheaf of \( \mathcal{L} \)-poly-vector fields on \( \mathcal{X} \) is defined as \( \mathcal{T}^{poly}_\mathcal{L}(\mathcal{O}_X) := \bigoplus_n \Lambda^n \mathcal{L}_\mathcal{X} \). This generalizes the space of multisections of the tangent bundle (i.e. multi-vector fields) described for different \( n \)-spaces occurs in geometries. It has a canonical sheaf of Gerstenhaber algebra structure on \( \mathcal{X} \). For the case of a smooth manifold \( \mathcal{X} \), Kontsevich introduced the (sheaf of) poly-differential operators on \( \mathcal{X} \). It induces a subcomplex of the Hochschild cochain complex for \( \mathcal{O}_X \). Its analogue in the context of Lie algebroid is the sheaf of \( \mathcal{L} \)-poly differential operators \( \mathcal{D}^{poly}_\mathcal{L}(\mathcal{O}_X) := \bigoplus_n \mathcal{U}(\mathcal{O}_X, \mathcal{L})^{\otimes n_{+1}} \). It has a canonical sheaf of Gerstenhaber algebra structure on \( \mathcal{X} \). In particular, when \( \mathcal{L} = \mathcal{T}_\mathcal{X} \) we denote these sheaves by the standard notions \( \mathcal{T}^{poly}_\mathcal{L} \) and \( \mathcal{D}^{poly}_\mathcal{L} \) respectively. The so-called HKR map is a quasi-isomorphism between the complexes associated with \( \mathcal{T}^{poly}_\mathcal{L} \) and \( \mathcal{D}^{poly}_\mathcal{L} \) (of differential graded \( \mathbb{K}_X \)-vector spaces). See [7] for more generalities.

Using ideas from proof of HKR theorem (and canonical PBW coalgebra isomorphism) for Lie-Rinehart algebras [19] we get the following results. To state these results first we need to recall some notations associated with a \( (\mathbb{K}, \mathcal{R}) \)-Lie-Rinehart algebra \( \mathcal{L} \) and an \( \mathcal{R}/\mathbb{K} \)-bialgebra \( \mathcal{A} \). The symmetric algebra of \( \mathcal{L} \) over \( \mathcal{R} \) (using underlying \( \mathcal{R} \)-module structure) is denoted by \( S_{\mathcal{R}} \mathcal{L} \) and \( S_{\mathcal{R}} \mathcal{L}^* \) is the symmetric algebra for the \( \mathcal{R} \)-module \( \mathcal{R}^* := \text{Hom}_{\mathbb{K}}(\mathcal{R}, \mathcal{R}) \), the universal enveloping algebra of the \( (\mathbb{K}, \mathcal{R}) \)-Lie-Rinehart algebra \( \mathcal{L} \) is denoted by \( \mathcal{U}(\mathcal{R}, \mathcal{L}) \) and its dual is the jet algebra \( \mathcal{J}(\mathcal{R}, \mathcal{L}) \), the cobar resolution of \( \mathcal{A} \) (using underlying \( \mathcal{R} \)-coalgebra structure) is denoted by \( \mathcal{C}ob^\bullet(\mathcal{A}) \) and the associated cohomology (applying the functor \( \mathfrak{U}(\mathcal{R}, \mathcal{L}) \)
Lemma 5.3. If a Lie algebroid $\mathcal{L}$ is locally free $\mathcal{O}_X$-module, then the generalized PBW map (or symmetrization map)

$$S_{\mathcal{O}_X} \mathcal{L} \to \mathcal{U}(\mathcal{O}_X, \mathcal{L})$$

is an isomorphism of $\mathcal{O}_X$-coalgebras.

Proof. First we take the PBW map associated with space of sections of $\mathcal{L}$ over each special open subset $U_x$ in $X$. It provides an $\mathcal{O}_X(U_x)$-coalgebra isomorphism (between the cocommutative $\mathcal{O}_X(U_x)$-coalgebras)

$$S_{\mathcal{O}_X(U_x)} \mathcal{L}(U_x) \cong \mathcal{U}(\mathcal{O}_X(U_x), \mathcal{L}(U_x)).$$

It induces stalkwise isomorphisms and by using the sheafification functor over the underlying presheaf homomorphism we get the generalized PBW map

$$S_{\mathcal{O}_X} \mathcal{L} \to \mathcal{U}(\mathcal{O}_X, \mathcal{L})$$

as an isomorphism of $\mathcal{O}_X$-coalgebras. \qed

Lemma 5.3. If a Lie algebroid $\mathcal{L}$ is locally free $\mathcal{O}_X$-module, then there is an isomorphism of graded $\mathcal{K}$-vector spaces associated with each special open set $U_x$.

$$HH^\bullet(\mathcal{U}(\mathcal{O}_X(U_x), \mathcal{L}(U_x))) \cong \wedge^\bullet_{\mathcal{O}_X(U_x)} \mathcal{L}(U_x).$$

Proof. The $(\mathcal{K}, \mathcal{O}_X(U_x))$-Lie-Rinehart algebra $\mathcal{L}(U_x)$ is a free $\mathcal{O}_X(U_x)$-module for each special open set $U_x$ (3.2). As in [19], we consider the isomorphism of cochain complexes associated with each special open set $U_x$ given as $\text{Coh}^\bullet(\mathcal{U}(\mathcal{O}_X(U_x), \mathcal{L}(U_x))) \cong \text{Coh}^\bullet(S_{\mathcal{O}_X(U_x)} \mathcal{L}(U_x))$. The isomorphism induces an isomorphism of graded vector spaces

$$HH^\bullet(S_{\mathcal{O}_X(U_x)} \mathcal{L}(U_x)) \cong HH^\bullet(\mathcal{U}(\mathcal{O}_X(U_x), \mathcal{L}(U_x))).$$

associated with every special open set $U_x$ for $x \in X$. Now for an open set $U \subset X$ we have the anti-symmetrization (or skew-symmetrization) map

$$\Lambda_{\mathcal{U}U} : \Lambda_{\mathcal{O}_X(U)} \mathcal{L}(U) \to (S_{\mathcal{O}_X(U)} \mathcal{L}(U))^\otimes \text{ given by }$$

$$D_1 \wedge \cdots \wedge D_n \mapsto \frac{1}{n!} \sum_{\sigma \in S_n} (-1)^{\sigma} D_{\sigma(1)} \otimes \cdots \otimes D_{\sigma(n)},$$

as well as the map

$$P_{\mathcal{U}U} : (S_{\mathcal{O}_X(U)} \mathcal{L}(U))^\otimes \to \Lambda_{\mathcal{O}_X(U)} \mathcal{L}(U) \text{ defined as }$$

$$\bar{D}_1 \otimes \cdots \otimes \bar{D}_n \mapsto P_{\mathcal{U}U}(\bar{D}_1) \wedge \cdots \wedge P_{\mathcal{U}U}(\bar{D}_n),$$

Here $P_{\mathcal{U}U} : S_{\mathcal{O}_X(U)} \mathcal{L}(U) \to S^{1}_{\mathcal{O}_X(U)} \mathcal{L}(U) = \mathcal{L}(U)$ is the projection map on the direct summand $S^{1}_{\mathcal{O}_X(U)} \mathcal{L}(U) = \mathcal{L}(U) = \Lambda^{1}_{\mathcal{O}_X(U)} \mathcal{L}(U)$ and $D_i \in \mathcal{L}(U), \bar{D}_i \in S_{\mathcal{O}_X(U)} \mathcal{L}(U)$ ($i = 1, \ldots, n$).

These morphisms defines cochain equivalence (or quasi isomorphism) $\text{Coh}^\bullet(S_{\mathcal{O}_X(U_x)} \mathcal{L}(U_x)) \cong (\Lambda_{\mathcal{O}_X(U_x)} \mathcal{L}(U_x), 0)$ on each special open set $U_x$ in $X$ and using that we get the isomorphism of graded vector spaces

$$HH^\bullet(\mathcal{U}(\mathcal{O}_X(U_x), \mathcal{L}(U_x))) \cong \Lambda_{\mathcal{O}_X(U_x)} \mathcal{L}(U_x).$$
On each open set $U$ of $X$, we can define the anti-symmetrization map
\[ \tilde{\Lambda}(U) : \Lambda_{O_X(U)}^i(U) \to (U(O_X(U), L(U)))^{\otimes i} \]
for every $k \geq 0$ (or $k$ is any non negative integer).

**Lemma 5.4.** If $\mathcal{F}$ is a sheaf of $O_X$-modules, then by considering the exterior algebra as a chain complex $\Lambda_{O_X}^\cdot \mathcal{F} := (\oplus_i \Lambda_{O_X}^i \mathcal{F}, 0)$ we get a canonical isomorphism of $\mathbb{K}$-vector spaces
\[ H^k(X, \Lambda_{O_X}^\cdot \mathcal{F}) \cong \bigoplus_{i+j=k} H^j(X, \Lambda_{O_X}^i \mathcal{F}) \]
for every $k \geq 0$.

**Proof.** Here we apply the notion of hypercohomology (Note 3.1, Note 3.2) for the cochain complex of sheaves $\Lambda_{O_X}^\cdot \mathcal{F}$. For each sheaf of $O_X$-modules (or $O_X$-module) $\Lambda_{O_X}^i \mathcal{F}$, we have an injective resolution (given by the flabby Godement resolution) $C^\cdot(\Lambda_{O_X}^\cdot \mathcal{F})$, provides a quasi-isomorphism
\[ \Lambda_{O_X}^i \mathcal{F} \cong C^i(\Lambda_{O_X}^\cdot \mathcal{F}). \]
Thus, the sheaf cohomology $H^\cdot(X, \Lambda_{O_X}^i \mathcal{F})$ of $\Lambda_{O_X}^i \mathcal{F}$ is isomorphic to the cohomology (in usual sense) of the complex of $O_X(X)$-modules $C^\cdot(\Lambda_{O_X}^\cdot \mathcal{F})(X)$.

The corresponding bicomplex for the complex of sheaves $\Lambda_{O_X}^\cdot \mathcal{F}$ is $C^\cdot(\mathcal{F}) := (C^i(\Lambda_{O_X}^\cdot \mathcal{F}))_{i,j \geq 0}$. The original complex can be embedded in the total complex $\mathcal{K}^\cdot := \text{tot}(C^\cdot(\mathcal{F}))$. Moreover, this embedding is a quasi-isomorphism. The cohomology of the associated complex of global sections $\mathcal{K}^\cdot(X) = \text{tot}(C^\cdot(\mathcal{F}))(X)$ is the hypercohomology of the complex $\Lambda_{O_X}^\cdot \mathcal{F}$. Since here differential of the complex is zero, thus the computation of cohomology of the total complex is become a simpler one. It is directly expressed through the sheaf cohomologies $H^j(X, \Lambda_{O_X}^i \mathcal{F})$ as stated in the Lemma.

**Notations:** For an open set $U$ of $X$, the cotensor product with $O_X(U)$ in the category of left $S_{O_X(U)}L(U)$-comodules is $O_X(U) \otimes_{S_{O_X(U)}L(U)} -$ and the associated cohomology groups is given by the Cotor groups (functor) as $\text{Cotor}_{S_{O_X(U)}L(U)}(O_X(U), -)$. Instead of $S_{O_X(U)}L(U)$, we use $\mathcal{F}(O_X(U), L(U))$ when it requires.

**Theorem 5.5.** (Generalized HKR theorem) Let $\mathcal{L}$ be a locally free Lie algebroid over $(X, O_X)$. Then the anti-symmetrization map (known as HKR morphism)
\[ \Lambda_{O_X}^\cdot \mathcal{L} \to \mathcal{U}(O_X, \mathcal{L})^{\otimes \cdot} \]
provides a quasi-isomorphism of the associated cochain complexes of sheaves. It induces an isomorphism of graded vector spaces
\[ \mathbb{H}H^\cdot(\mathcal{U}(O_X, \mathcal{L})) \cong \mathbb{H}^\cdot(X, \Lambda_{O_X}^\cdot \mathcal{L}) \]

**Proof.** Assume first $\mathcal{L}$ to be a locally free Lie algebroid of finite rank. Since for each $x \in X$ we have an open set $U_x$ containing $x$ with $L|_{U_x}$ is a free $O_X|_{U_x}$-module, thus on each $U_x$ we use results from the proof of the HKR Theorem [19] for the $(\mathbb{K}, O_X(U_x))$-Lie-Rinehart algebra $L(U_x)$. Then, for each open special sets $U_x$ we have isomorphisms of cochain complexes of left $S_{O_X(U_x)}L(U_x)$-comodules
\[ (O_X(U_x) \otimes_{S_{O_X(U_x)}L(U_x)} \mathcal{K}_{U_x}^\cdot \mathcal{L}, \text{id}_{O_X(U_x)} \otimes_{O_X(U_x)} \partial_{U_x}) \xrightarrow{\phi_{U_x}} (\Lambda_{O_X(U_x)}^\cdot \mathcal{L}(U_x), 0) \]
compatible with restrictions, where $\mathcal{K}_{U_x}^\cdot \mathcal{L} := S_{O_X(U_x)}L(U_x) \otimes_{S_{O_X(U_x)}L(U_x)} \Lambda_{O_X(U_x)}^\cdot L(U_x)$ is with the canonical $S_{O_X(U_x)}L(U_x)$-comodule structure and the differential $\partial_{U_x} : \mathcal{K}_{U_x}^\cdot \mathcal{L} \to \mathcal{K}_{U_x}^{\cdot+1} \mathcal{L}$ is given by the dual Koszul resolution of $O(U_x)$ as $S_{O_X(U_x)}L(U_x)$-comodules. Indeed, the dual Koszul-Rinehart complex for $L(U)$ is given as $\mathcal{K}_{U_x}^\cdot \mathcal{L}(U) := (\mathcal{K}_{U_x}^\cdot \mathcal{L}, \partial_{U_x})$, for any open set $U$ in $X$. 

\[ \square \]
Now, an open set $U \subset X$ is expressed by taking a partition as $\bigcup_{x \in X} (U \cap U_x)$ and thus we have cochain homomorphisms of left $S_{O_X(U)} L(U)$-comodules

$$\left( \mathcal{O}_X(U) \square_{S_{O_X(U)} L(U)} \mathcal{K}^\bullet_{O_X} L , \text{id}_{\mathcal{O}_X(U)} \otimes \mathcal{O}_X(U) \partial_U \right) \overset{\partial_U}{\to} \left( \mathcal{K}^\bullet_{O_X} L(U), 0 \right)$$

compatible with restrictions. Then it induces a homomorphism on the associated presheaves of cochain complexes, provides stalkwise isomorphism. Thus, applying sheafification functor we get a canonical isomorphism of cochain complexes of left $S_{O_X} L$-comodules

$$(\mathcal{O}_X \square_{S_{O_X} L} \mathcal{K}^\bullet_{O_X} L = (\mathcal{O}_X \square_{S_{O_X} L} (S_{O_X} L \otimes \wedge_{O_X}^\bullet L), \text{id}_{\mathcal{O}_X} \otimes \mathcal{O}_X \partial) \overset{\partial}{\to} (\wedge_{O_X}^\bullet L, 0) .$$

To compute Hochschild hypercohomology $\mathbb{H}^\bullet(S_{O_X} L)$ via a derived functor, we consider the standard cofibrant complex associated with each open set $U$, denoted as $\text{Co}b^\bullet(S_{O_X(U)} L(U))$ (considering $A(U) = S_{O_X(U)} L(U)$ as described in Section 5.1).

We have two presheaves of vector spaces, one is presheaf of cofibrant complexes of the sheaf $S_{O_X} L$ of $O_X/\mathcal{K}_{X,b}$-bialgebra, defined as

$$U \mapsto \text{Co}b^\bullet(S_{O_X(U)} L(U))$$

with its restriction morphism induced from the restrictions of the sheaf $S_{O_X} L$, and another one is the presheaf of Koszul-Rinehart complex of $O_X$ in the category of left $S_{O_X} L$-comodules, defined as

$$U \mapsto \mathcal{K}^\bullet_{O_X(U)} L(U) .$$

These two presheaves are quasi isomorphic on special open sets $U_x$ for every $x \in X$ (for the $C^\infty$ case these holds for every open set $U$). In stalkwise, it induces quasi-isomorphism and by applying sheafification functor we get quasi-isomorphism (chain homotopy)

$$\text{Co}b^\bullet(S_{O_X} L) \simeq \mathcal{K}^\bullet_{O_X} L$$

(13) using the alternating map and projection map (5.2), we have quasi isomorphism

$$S_{O_X(U_x)} L(U_x) \otimes_{O_X(U_x)} \wedge_{O_X(U_x)}^\bullet (L(U_x)) \overset{\text{id} \otimes \text{Alt}_{U_x}}{\to} S(O_X(U_x), L(U_x)) \otimes (\cdot \cdot \cdot) .$$

Then on taking cotensor product by $O_X$ from left (as standard left $S_{O_X} L$-comodule) to the cochain complexes (13) of sheaves of left $S_{O_X} L$-comodules we get from (12)

$$\mathcal{O}_X \square_{S_{O_X} L} \text{Co}b^\bullet(S_{O_X} L) \simeq (\wedge_{O_X}^\bullet L, 0) .$$

Applying hypercohomology functor $\mathbb{H}^\bullet(X, -)$, we get Hochschild hypercohomology of $\mathbb{L}(O_X, L)$ (using Lemma 5.2) through derived functor Cotor as

$$\mathbb{H}^\bullet(\mathbb{L}(O_X, L)) \cong \text{Cotor}^\bullet_{S_{O_X} L}(O_X, O_X) \overset{\mathbb{L}(\cdot \cdot \cdot)}{\cong} \mathbb{H}^\bullet(X, \wedge_{O_X}^\bullet L) .$$

An alternative approach for proof of the theorem for finite rank case is given as follows.

Consider the presheaves of graded vector spaces

$$U \mapsto \mathbb{H}^\bullet(S_{O_X(U)} L(U)) ,$$

$$U \mapsto \wedge_{O_X(U)}^\bullet L(U)$$

are isomorphic on each special open sets $U_x$'s (see Lemma 5.3). The associated presheaf homomorphism is induced from the homomorphisms $\text{Alt}_U$ and $\partial_U$ (described in the proof of the Lemma 5.3). Thus, by considering sheafifications (of these two presheaf of cohomology spaces which are isomorphic stalkwise) we get an isomorphism in the associated cohomology sheaves (described in Definition 3.2) as

$$\mathcal{H}^\bullet(\mathcal{L}(O_X, L)) \equiv \mathcal{H}^\bullet(S_{O_X} L) \equiv \wedge_{O_X}^\bullet L .$$

Next on the induced hypercohomology groups (using Note 3.2) we have

$$\mathbb{H}^\bullet(\mathcal{L}(O_X, L)) \equiv \mathbb{H}^\bullet(S_{O_X} L) \equiv \mathbb{H}^\bullet(X, \wedge_{O_X}^\bullet L) .$$
In general case, where $\mathcal{L}$ is locally free $\mathcal{O}_X$-module of infinite rank (quasicoherent sheaf), there exists a filtered ordered set $J$ as well as an inductive system of free $\mathcal{O}_X(U_x)$-modules $\{\mathcal{L}(U_x)_j \mid j \in J\}$ such that

$$\mathcal{L}(U_x) \cong \lim_{j \in J} \mathcal{L}(U_x)_j,$$

for each special open set $U_x$ (3.2) (using results from [19]). Since both $\mathbb{H}^\bullet$ (which is isomorphic to the derived functor $\text{Cotor}^\bullet$) and the functor $S$ commute with inductive limits over a filtered ordered set, we get the result using the result follows from the locally free $\mathcal{O}_X$-module of finite rank case (local counterpart is described in [19]).

\begin{remark}
The HKR morphism is not a ring isomorphism, but composing it together with the Todd genus provides a ring isomorphism (moreover it provides a canonical Gerstenhaber algebra isomorphism). It is required in the study of formality (or quantization) for Lie algebroids over a ringed site [7].
\end{remark}

Corollary 5.7. By applying the Lemma 5.4, the above HKR theorem reduces to

$$\mathbb{H}^\bullet(\mathcal{U}(\mathcal{O}_X, \mathcal{L})) \cong \bigoplus_{i,j} H^i(X, \wedge^j_{\mathcal{O}_X} \mathcal{L}).$$

Corollary 5.8. In the special case of $\mathcal{L} = T_X$ when $X$ is non-singular or smooth (see Note 2.2), we get isomorphism of graded vector spaces

$$\mathbb{H}^\bullet(D_X) = \mathbb{H}^\bullet(X, D^\text{poly}_X) \cong \mathbb{H}^\bullet(X, T^\text{poly}_X).$$

Corollary 5.9. Using Note 2.11 for a free divisor $Y$ in $X$, we get isomorphism of graded vector spaces

$$\mathbb{H}^\bullet(D_X(-\log Y)) \cong \mathbb{H}^\bullet(X, \wedge^\bullet_{\mathcal{O}_X} T_X(-\log Y)).$$

Corollary 5.10. (Noncommutative analogue of the HKR Theorem) The free Lie algebroid $\mathcal{P}_X$ can be identified with sheaf of regular noncommutative vector fields on $X$ and we have seen already that the sections of its universal enveloping algebra $D_X$ are noncommutative differential operators on $X$ (see Remark 2.12). Thus, in this case from the HKR theorem we get a canonical isomorphism

$$\mathbb{H}^\bullet(D_X) \cong \mathbb{H}^\bullet(X, \wedge^\bullet_{\mathcal{O}_X} \mathcal{P}_X)$$

Corollary 5.11. For a cocomplete graded free $\mathcal{O}_X/K_X$-bialgebra $\mathcal{A}$ (of finite or infinite type), the isomorphism $\mathcal{A} \cong \mathcal{U}(\mathcal{O}_X, \mathcal{P}(\mathcal{A}))$ holds [29] and thus we get the graded vector space isomorphism

$$\mathbb{H}^\bullet(\mathcal{A}) \cong \mathbb{H}^\bullet(X, \wedge^\bullet_{\mathcal{O}_X} \mathcal{P}(\mathcal{A})).$$

5.3. Hochschild hypercohomology of jet algebroid. Here, we present a dual version of the HKR theorem by considering Hochschild cohomology of a jet algebroid $\mathcal{J}(\mathcal{O}_X, \mathcal{L})$ for a Lie algebroid $\mathcal{L}$ over $(X, \mathcal{O}_X)$ (see the part (2.3) mentioned in Section 2).

Note 5.12. There is a canonical left $\mathcal{U}(\mathcal{O}_X, \mathcal{L})$-module structure on $\mathcal{J}(\mathcal{O}_X, \mathcal{L})$ constructed as follows (see [6,7,19] for the local descriptions):

A canonical flat $\mathcal{L}$-connection on $\mathcal{J}(\mathcal{O}_X, \mathcal{L})$, called Grothendieck connection is given by

$$\nabla_D(\phi)(D') := \tilde{D}(\phi(D')) - \phi(\tilde{D} D')$$

for all sections $D \in \mathcal{L}$, $\phi \in \mathcal{J}(\mathcal{O}_X, \mathcal{L})$ and $D' \in \mathcal{U}(\mathcal{O}_X, \mathcal{L})$, where $\tilde{D} := \mathcal{A}(D)$ and $\tilde{D} := \mathcal{I}_L(D)$. Thus, we get a $\mathcal{L}$-module structure on $\mathcal{J}(\mathcal{O}_X, \mathcal{L})$. We can extend it to the induced $\mathcal{U}(\mathcal{O}_X, \mathcal{L})$-module structure on $\mathcal{J}(\mathcal{R}, \mathcal{L})$.

Theorem 5.13. (Generalized dual HKR theorem) Let $\mathcal{L}$ be a locally free Lie algebroid over $(X, \mathcal{O}_X)$ which is of finite rank. Then the anti-symmetrization map (dual HKR morphism)

$$\wedge^\bullet_{\mathcal{O}_X} \mathcal{L} \to \mathcal{J}(\mathcal{O}_X, \mathcal{L})^\otimes$$

provides a quasi-isomorphism between the associated cochain complexes of sheaves. It induces canonical isomorphism of graded vector spaces

$$\mathbb{H}^\bullet(\mathcal{J}(\mathcal{O}_X, \mathcal{L})) \cong \mathbb{H}^\bullet(X, \Omega^\bullet_\mathcal{L}).$$
Proof. For a locally free Lie algebroid $\mathcal{L}$ over $(X, \mathcal{O}_X)$ of finite rank (say $r$, for some $r \in \mathbb{N}$), the dual of $\mathcal{L}$, denoted by $\mathcal{L}^*$, is a locally free $\mathcal{O}_X$-module of the same rank. Thus, $\mathcal{L}^*(U_x)$ has a basis $\{w_1, \ldots, w_r\}$ (say) for each special open set $U_x$ around $x \in X$. Since $\mathcal{J}(\mathcal{O}_X, \mathcal{L})$ is a commutative $\mathcal{O}_X$-algebra, thus $\mathcal{J}(\mathcal{O}_X(U_x), \mathcal{L}(U_x)) \cong \mathcal{O}_X(U_x)[[w_1, \ldots, w_r]]$ as $\mathcal{O}_X(U_x)$-algebra. Hence, by applying sheafification functor we get the isomorphism of $\mathcal{O}_X$-algebras $\mathcal{J}(\mathcal{O}_X, \mathcal{L}) \cong \mathcal{S}_{\mathcal{O}_X}^{\mathcal{L}^*}$ (this sheaf of symmetric algebras is formally completed with respect to the degree) [1,5]. Now, consider the dual Koszul-Rinehart resolution of $\mathcal{O}_X$ in the category of left $\mathcal{J}(\mathcal{O}_X, \mathcal{L})$-comodules (by using local counterpart from [19]) is

$$
\mathcal{O}_X \hookrightarrow (\mathcal{J}(\mathcal{O}_X, \mathcal{L}) \otimes_{\mathcal{O}_X} \wedge_{\mathcal{O}_X} \mathcal{L}^*, \nabla) :=: \mathcal{K}_{\mathcal{O}_X}^{\mathcal{L}^*},
$$

where $\nabla$ is the Grothendieck connection (Note 5.12), sheafification of the presheaf of canonical left $\mathcal{L}(U)$-connection $U \mapsto \nabla^U$ on $\mathcal{J}(\mathcal{O}_X(U), \mathcal{L}(U))$. It is basically the sheafification of the presheaves of cochain complex

$$
U \mapsto \{\mathcal{O}_X(U) \rightarrow \mathcal{K}_{\mathcal{O}_X(U)}^{\mathcal{L}(U)^*}\},
$$

where $\mathcal{K}_{\mathcal{O}_X(U)}^{\mathcal{L}(U)^*} = (\mathcal{J}(\mathcal{O}_X(U), \mathcal{L}(U)) \otimes_{\mathcal{O}_X(U)} \wedge_{\mathcal{O}_X(U)} \mathcal{L}(U)^*, \nabla^U)$. The unit of $\mathcal{J}(\mathcal{O}_X, \mathcal{L})$ as an $\mathcal{O}_X$-algebra provides the morphism from $\mathcal{O}_X$ to $\mathcal{K}_{\mathcal{O}_X}^{\mathcal{L}^*}$.

Applying cotensor product by $\mathcal{O}_X$ (with canonical $\mathcal{J}(\mathcal{O}_X, \mathcal{L})$-comodule structure, which induces from the $\mathsf{U}(\mathcal{O}_X, \mathcal{L})$-module structure on $\mathcal{O}_X$) to the dual Koszul-Rinehart complex $\mathcal{K}_{\mathcal{O}_X}^{\mathcal{L}^*}$, we get the canonical isomorphism (sheafifying local descriptions from [19]) of cochain complexes of sheaves of graded vector spaces as

$$
\mathcal{O}_X \otimes_{\mathcal{J}(\mathcal{O}_X, \mathcal{L})} \mathcal{K}_{\mathcal{O}_X}^{\mathcal{L}^*} \cong (\wedge_{\mathcal{O}_X} \mathcal{L}^*, d).
$$

To describe the isomorphism (15) in an explicit way, we use the following steps.

$$
\mathcal{O}_X(U) \sqcup_{\mathcal{J}(\mathcal{O}_X(U), \mathcal{L}(U))} (\mathcal{J}(\mathcal{O}_X(U), \mathcal{L}(U)) \otimes_{\mathcal{O}_X(U)} \wedge_{\mathcal{O}_X(U)} \mathcal{L}(U)^*) \cong \wedge_{\mathcal{O}_X(U)} \mathcal{L}(U)^*.
$$

Since the unit $1_U \in \mathcal{J}(\mathcal{O}_X(U), \mathcal{L}(U))$ is given by the counit $\epsilon_U$ of $\mathsf{U}(\mathcal{O}_X(U), \mathcal{L}(U))$, the induced differential is exactly the Lie-Rinehart coboundary $d_U$ (4), on an open set $U \subset X$. These isomorphisms are compatible with restrictions. Now consider the canonical presheaves from the above complexes associated with each open sets, which are isomorphic on each special open sets (3.2) and compatible with the natural restrictions. Then considering these sheafifications and using the above isomorphisms, we get isomorphism between the complexes of sheaves, described in (15).

Since $\mathcal{L}$ is locally free $\mathcal{O}_X$-module of finite rank, there exists open sets $U_x$ around each point $x \in X$, where $\mathcal{L}|_{U_x}$ is free $\mathcal{O}_X|_{U_x}$-module of finite rank. Thus, for each $U_x$ we can use results from the proof of the dual HKR Theorem [19] for the $(\mathbb{K}, \mathcal{O}_X(U_x))$-Lie-Rinehart algebra $\mathcal{L}(U_x)$. Here, we get

$$
\mathcal{K}_{\mathcal{O}_X(U_x)}^{\mathcal{L}(U_x)^*} \cong (\wedge_{\mathcal{O}_X(U_x)} \mathcal{L}(U_x)^*).
$$

Hence, $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{L}(U_x), \mathcal{L}(X)) \cong \wedge_{\mathcal{O}_X} \mathcal{L}^*$ and the Chevalley-Eilenberg-de Rham complex of $\mathcal{L}$ is $\Omega^*_\mathcal{L} \cong (\wedge_{\mathcal{O}_X} \mathcal{L}^*, d)$. Therefore, we get (using isomorphism (15))

$$
\mathcal{O}_X \otimes_{\mathcal{J}(\mathcal{O}_X, \mathcal{L})} \mathcal{K}_{\mathcal{O}_X}^{\mathcal{L}^*} \cong \Omega^*_\mathcal{L}.
$$

To express Hochschild hypercohomology of $\mathcal{J}(\mathcal{O}_X, \mathcal{L})$ as derived functor, we need to consider standard cobar resolution of $\mathcal{O}_X$ as $\mathcal{J}(\mathcal{O}_X, \mathcal{L})$-comodules and cotensor it with $\mathcal{O}_X$ (putting $\mathcal{A} = \mathcal{J}(\mathcal{O}_X, \mathcal{L})$ in the relation appears in (11)). Thus,

$$
\mathbb{H}H^\bullet(\mathcal{J}(\mathcal{O}_X, \mathcal{L}) \cong \mathcal{Cotor}_{\mathcal{J}(\mathcal{O}_X, \mathcal{L})}^\bullet(\mathcal{O}_X, \mathcal{O}_X).
$$

Since for each open set $U_x$ of $X$, the $(\mathbb{K}, \mathcal{O}_X(U_x))$-Lie-Rinehart algebras $\mathcal{L}(U_x)$ is finitely generated (in fact free), thus we get quasi-isomorphisms

$$
\mathcal{J}(\mathcal{O}_X(U_x), \mathcal{L}(U_x)) \otimes_{\mathcal{O}_X(U_x)} \mathcal{O}_X(U_x) \cong \mathcal{J}(\mathcal{O}_X(U_x), \mathcal{L}(U_x)) \otimes_{\mathcal{O}_X(U_x)} \wedge_{\mathcal{O}_X(U_x)} \mathcal{L}(U_x)^*,
$$

$$
\mathcal{J}(\mathcal{O}_X(U_x), \mathcal{L}(U_x)) \otimes_{\mathcal{O}_X(U_x)} \wedge_{\mathcal{O}_X(U_x)} \mathcal{L}(U_x)^+ \cong \mathcal{J}(\mathcal{O}_X(U_x), \mathcal{L}(U_x)) \otimes_{\mathcal{O}_X(U_x)} \wedge_{\mathcal{O}_X(U_x)} \mathcal{L}(U_x)^{\bullet+1},
$$

$$
\mathcal{J}(\mathcal{O}_X(U_x), \mathcal{L}(U_x)) \otimes_{\mathcal{O}_X(U_x)} \wedge_{\mathcal{O}_X(U_x)} \mathcal{L}(U_x)^{\bullet+1} \cong \mathcal{J}(\mathcal{O}_X(U_x), \mathcal{L}(U_x)) \otimes_{\mathcal{O}_X(U_x)} \wedge_{\mathcal{O}_X(U_x)} \mathcal{L}(U_x)^{\bullet+1}.
$$

Cotor(\mathcal{O}_X, \mathcal{O}_X) is finitely generated over \mathcal{O}_X, thus the complex $\mathcal{O}_X \otimes_{\mathcal{J}(\mathcal{O}_X, \mathcal{L})} \mathcal{K}_{\mathcal{O}_X}^{\mathcal{L}^*}$ is a finite projective resolution of $\mathcal{O}_X$ as a $\mathcal{O}_X$-module, thus $\mathcal{O}_X \otimes_{\mathcal{J}(\mathcal{O}_X, \mathcal{L})} \mathcal{K}_{\mathcal{O}_X}^{\mathcal{L}^*}$ is a finite projective resolution of $\mathcal{O}_X$ as a $\mathcal{O}_X$-module. Therefore, the Hochschild hypercohomology of $\mathcal{J}(\mathcal{O}_X, \mathcal{L})$ is the cohomology of the complex $\mathbb{H}H^\bullet(\mathcal{J}(\mathcal{O}_X, \mathcal{L}) \cong \mathcal{Cotor}_{\mathcal{J}(\mathcal{O}_X, \mathcal{L})}^\bullet(\mathcal{O}_X, \mathcal{O}_X)$.
for each $U_x$ associated with $x \in X$, where $p_{U_x}$ is given by the canonical projections $pr_1 : \mathcal{J}(\mathcal{O}_X(U_x), \mathcal{L}(U_x)) \cong S_{\mathcal{O}_X(U_x)} \mathcal{L}(U_x)^* \to \mathcal{L}(U_x)^*$ and the anti-symmetrization map $Alt_{U_x} : S^*_{\mathcal{O}_X(U_x)} \mathcal{L}(U_x)^* \to \mathcal{J}(\mathcal{O}_X(U_x), \mathcal{L}(U_x))^\circ$ for every special open set $U_x$ of $X$ (for smooth manifold we get these quasi-isomorphisms on each open sets). Sheafification of the associated presheaves provides the quasi-isomorphism between the cobar complex $\mathcal{C}^0(\mathcal{J}(\mathcal{O}_X, \mathcal{L}))$ and the dual Koszul-Rinehart complex $\mathcal{K}^0_{\mathcal{O}_X} \mathcal{L}^*$. Applying de Rham theorems in different settings of a manifold, smooth algebraic variety or smooth scheme over the field, sheafification of the associated presheaves provides the quasi-isomorphism between the cobar complex $\mathcal{C}^0(\mathcal{J}(\mathcal{O}_X, \mathcal{L}))$ and the dual Koszul-Rinehart complex $\mathcal{K}^0_{\mathcal{O}_X} \mathcal{L}^*$. Thus, using the isomorphisms (17) and (18), the Hochschild hypercohomology groups of jet algebroid $\mathcal{J}(\mathcal{O}_X, \mathcal{L})$ is expressed by the Chevally-Eilenberg-de Rham hypercohomology of $\mathcal{L}$.

\[ \text{Corollary 5.16.} \] Applying the Theorem 5.15 to $\mathcal{L} = \mathcal{T}_X$ over some non-singular a-space $X$, we get isomorphism of graded vector spaces

\[ HH^*(\mathcal{J}(\mathcal{O}_X, \mathcal{T}_X)) \cong H^*(X, \mathcal{O}_X^*) \cong H^*(X, \mathbb{K}) \text{ or } H^*(X^\text{an}, \mathbb{K}) \]

by applying de Rham theorems in different settings (smooth, analytic, algebraic) [9,36], where $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ accordingly and $X^\text{an}$ is the analytification of the algebraic variety (scheme) $X$.

\[ \text{Corollary 5.16.} \] Applying the Theorem 3.6 for $\mathcal{L} = \mathcal{T}_X$ over some non-singular a-space $X$ and using the Corollary 5.15 we get the isomorphism

\[ HH^*(\mathcal{J}(\mathcal{O}_X, \mathcal{T}_X)) \cong \text{Ext}^*_{\mathcal{D}_X}(\mathcal{O}_X, \mathcal{O}_X), \]

where $\mathcal{D}_X := \text{Diff}(\mathcal{O}_X) \cong \mathcal{U}(\mathcal{O}_X, \mathcal{T}_X)$ is the sheaf of differential operators of $X$. Thus, in this case we get a canonical isomorphism

\[ \text{Cotor}^*_{\mathcal{J}_X}(\mathcal{O}_X, \mathcal{O}_X) \cong \text{Ext}^*_{\mathcal{D}_X}(\mathcal{O}_X, \mathcal{O}_X), \]

where $\mathcal{J}_X := \mathcal{J}(\mathcal{O}_X, \mathcal{T}_X)$ is the usual sheaf of jets on $X$ [5,32].

### 5.4. Remarks on Lie algebroids over Schemes

All the previous discussions are applicable to Lie algebroids over Noetherian separated schemes (or schemes of finite type) over a field of characteristic zero (see [3,18]). Moreover, these results can be extended to the more general notion of Lie algebroids over ringed sites ([5–7]). It requires more of an algebraic viewpoint rather than a geometric one.

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