1 Introduction

$L_\infty$-algebras (also called strongly homotopy Lie algebras) were first introduced in [1] and [2] and are a generalisation of graded Lie algebras in which a system of antisymmetric $n$-ary bracket satisfies a generalised Jacobi identity. The first part of this article serves as a self-contained introduction to $L_\infty$-algebras, in which we discuss different characterisations of $L_\infty$-algebras and their representations (up to homotopy), closely following [3].

The $L_\infty$-algebra cohomology with values in the adjoint representation was introduced in [4] using a Lie bracket on the space of cochains. We extend this approach to arbitrary representations, which leads to a characterisation of certain $L_\infty$-algebras as abelian extensions of $L_\infty$-algebras by 2-cocycles. This generalises a theorem from [5] that characterises certain $L_\infty$-algebras in terms of Lie algebra cohomology.

This article is largely based on my same-titled Bachelor’s thesis, which I wrote under the supervision of Chenchang Zhu at the University of Göttingen in 2018.

2 Mathematical background

In this section, we discuss exterior and symmetric powers, algebras and coalgebras in the graded framework. In particular, we show that antisymmetric and symmetric maps are related by a shift in degree and that coderivations of the symmetric coalgebra are in one-to-one correspondence with their weight one components. These results are later key to the characterisations of $L_\infty$-structures in terms of symmetric brackets and codifferentials. The main references for this section are [3,4,6,7].

2.1 Graded vector spaces

A graded vector space is a vector space $V$ together with a decomposition $V \cong \bigoplus_{p \in \mathbb{Z}} V_p$ for a family of vector spaces $\{V_p\}_{p \in \mathbb{Z}}$. An element $v \in V_p$ is then called homogeneous of degree $p$ and we write $|v| = p$.

Here and subsequently, we assume all vector spaces to be over a fixed ground field $\mathbb{k}$ of characteristic zero. We always denote by $V$ and $W$ graded vector spaces and by $v_1, \ldots, v_n \in V$ arbitrary homogeneous elements.

A linear map $f : V \to W$ is called homogeneous (of degree $p$) if there is $p \in \mathbb{Z}$ such that $f(V_n) \subset W_{n+p}$ for all $n \in \mathbb{Z}$. We denote by $\text{Hom}_p(V, W)$ the vector space of all homogeneous linear maps $V \to W$ of degree $p$ and by $\text{Hom}(V, W)$ the graded vector space $\bigoplus_{p \in \mathbb{Z}} \text{Hom}_p(V, W)$. Elements in $\text{Hom}_0(V, W)$ are also called degree preserving.

Note that we can identify ungraded vector spaces with graded ones that are concentrated in degree zero, that is $V_k = 0$ for $k \neq 0$.

There is a canonical grading on the direct sum of $V$ and $W$ given by $(V \oplus W)_p = V_p \oplus W_p$. The isomorphism $V \otimes W \cong \bigoplus_{p \in \mathbb{Z}} \bigoplus_{i+j=p} (V_i \otimes W_j)$ allows us to define a grading on $V \otimes W$ by $$(V \otimes W)_p = \bigoplus_{i+j=p} (V_i \otimes W_j).$$
This extends to a grading on $V^\otimes n := \bigotimes_{i=1}^{n} V$ given by
\[(V^\otimes n)_p = \bigoplus_{i_1 + \ldots + i_n = p} V_{i_1} \otimes \ldots \otimes V_{i_n}.\]

We denote by $\tau_{V,W}$ the linear degree preserving map
\[\tau_{V,W} : V \otimes W \to W \otimes V, \quad v \otimes w \mapsto (-1)^{|v||w|} w \otimes v.\]

If $f \in \text{Hom}(V, W)$ and $g \in \text{Hom}(V', W')$ are homogeneous for graded vector spaces $V'$ and $W'$, we define the linear map $f \otimes g : V \otimes V' \to W \otimes W'$ by
\[(f \otimes g)(v \otimes v') = (-1)^{|v||g|} f(v) \otimes g(v')\]
for $v \in V$, $v' \in V'$ homogeneous. Note that $|f \otimes g| = |f| + |g|$. This generalises to tensor products of three or more vector spaces in the obvious way and we abbreviate $f \otimes \ldots \otimes f : V^\otimes n \to W^\otimes m$ to $f^\otimes n$.

For the composition of such functions, (1) implies
\[(f' \otimes g') \circ (f \otimes g) = (-1)^{|g'|} f' \circ f \otimes (g' \circ g),\]
when $f'$ and $g'$ are homogeneous linear maps with domains $W$ and $W'$, respectively.

When working in the framework of graded vector spaces, the general rule of thumb for the signs is that whenever two “graded symbols” of degree $p$ and $q$, respectively, change their order in an equation, there should be the sign $(-1)^{|p||q|}$. This is called the Koszul sign convention.

We denote by $\mathfrak{S}_n$ the symmetric group, the group of all permutations of the set $\{1, \ldots, n\}$, and by $s_i \in \mathfrak{S}_n$ for $1 \leq i \leq n - 1$ the transposition with $s_i(i) = i + 1$ and $s_i(i + 1) = i$. There are two natural linear right actions of $\mathfrak{S}_n$ on $V^\otimes n$. These are given on the generating subset $\{s_1, \ldots, s_{n-1}\} \subset \mathfrak{S}_n$ by
\[
\varepsilon(s_i)(v_1 \otimes \ldots \otimes v_n) = (-1)^{|v_i||v_{i+1}|} v_1 \otimes \ldots \otimes v_{i-1} \otimes v_{i+1} \otimes v_i \otimes v_{i+2} \otimes \ldots \otimes v_n,
\]
\[
\chi(s_i)(v_1 \otimes \ldots \otimes v_n) = (-1)^{|v_i||v_{i+1}|} v_1 \otimes \ldots \otimes v_{i-1} \otimes v_{i+1} \otimes v_i \otimes v_{i+2} \otimes \ldots \otimes v_n.
\]
We call $\varepsilon$ and $\chi$ the (graded) symmetric and (graded) antisymmetric action of $\mathfrak{S}_n$ on $V^\otimes n$, respectively. Note that $\varepsilon(\sigma)$ is degree preserving for all $\sigma \in \mathfrak{S}_n$ as
\[
\varepsilon(\sigma)(v_1 \otimes \ldots \otimes v_n) = \pm v_{\sigma(1)} \otimes \ldots \otimes v_{\sigma(n)}.
\]
We then denote the sign in (3) by $\varepsilon(\sigma; v_1, \ldots, v_n)$ and similarly by $\chi(\sigma; v_1, \ldots, v_n)$ the sign such that
\[
\chi(\sigma)(v_1 \otimes \ldots \otimes v_n) = \chi(\sigma; v_1, \ldots, v_n)v_{\sigma(1)} \otimes \ldots \otimes v_{\sigma(n)}.
\]
We abbreviate $\varepsilon(\sigma; v_1, \ldots, v_n)$ and $\chi(\sigma; v_1, \ldots, v_n)$ to $\varepsilon(\sigma)$ and $\chi(\sigma)$, when no confusion arises.

Let $U_S \subset V^\otimes n$ be the graded subspace spanned by all elements of the form
\[
v_1 \otimes \ldots \otimes v_n - \varepsilon(\sigma)(v_1 \otimes \ldots \otimes v_n)
\]
for $\sigma \in \mathfrak{S}_n$. The space $S^n(V) := V^\otimes n / U_S$ is called the $n$th symmetric power of $V$. Similarly, the $n$th exterior power of $V$ is defined as the quotient of $V^\otimes n$ by the graded subspace spanned by all elements of the form
\[
v_1 \otimes \ldots \otimes v_n - \chi(\sigma)(v_1 \otimes \ldots \otimes v_n)
\]
for $\sigma \in \mathfrak{S}_n$ and is denoted by $\bigwedge^n V$.

An $n$-linear map $f : V^n \to W$ is called (graded) symmetric if for all $\sigma \in \mathfrak{S}_n$
\[f(v_1, \ldots, v_n) = \varepsilon(\sigma)f(\sigma v_1, \ldots, v_n)\]
holds. We can write this conveniently as $f \circ \varepsilon(\sigma) = f$. Similarly, $f$ is called (graded) antisymmetric if $f \circ \chi(\sigma) = f$ for all $\sigma \in \mathfrak{S}_n$.

**Proposition 1.** Let $f : V^\otimes n \to W$ be a symmetric linear map. There is a unique linear map $\varphi : S^n(V) \to W$ such that the following diagram commutes:
\[
\begin{array}{ccc}
V^\otimes n & \overset{\pi_S}{\longrightarrow} & S^n(V) \\
\downarrow f & & \downarrow \varphi \\
W & \overset{\uparrow}{\longleftarrow} &
\end{array}
\]
where $\pi_S : V^\otimes n \to S^n(V)$ is the canonical projection.

**Proof.** As $f$ is symmetric, it vanishes on the generators of $U_S$ and factors through $\pi_S$ to a linear map $\varphi : S^n(V) \to W$ such that the diagram above commutes. This map is unique as $\pi_S$ is surjective.

**Remark 2.** As the symmetric $\mathfrak{S}_n$-action on $V^\otimes n$ is degree preserving, $S^n(V)$ inherits a canonical grading from $V^\otimes n$ such that $\pi_S$ is degree preserving. It is then immediate that if $f$ is homogeneous in Proposition 1, so is the map $\varphi$ and $|\varphi| = |f|$. As $\pi_S$ is symmetric by construction of $S^n(V)$, Proposition 1 yields an isomorphism between the subspace of $\text{Hom}(V^\otimes n, W)$ consisting of all symmetric maps and $\text{Hom}(S^n(V), W)$. An analogue of Proposition 1 holds for $\bigwedge^n V$ and induces an isomorphism between the subspace $\text{Hom}(V^\otimes n, W)$ of all anti-symmetric maps and $\text{Hom}(\bigwedge^n V, W)$.

An element in $V^\otimes n$ is called symmetric if it is invariant under the symmetric $\mathfrak{S}_n$-action on $V^\otimes n$. We claim that $S^n(V)$ is isomorphic to the subspace of $V^\otimes n$ of all symmetric elements. Indeed, letting $v_1 \lor \ldots \lor v_n$ denote the image of $v_1 \otimes \ldots \otimes v_n$ under $\pi_S$, the linear map
\[\varphi : S^n(V) \to \frac{1}{n!} \sum_{\sigma \in \mathfrak{S}_n} \varepsilon(\sigma)_x \sigma(1) \otimes \ldots \otimes \sigma(n)\]
is well-defined and satisfies $\pi_S \circ \varphi = \text{id}_{S^n(V)}$ and $\varphi \circ \pi_S = \frac{1}{n!} \sum_{\sigma \in \mathfrak{S}_n} \varepsilon(\sigma)$. As the latter is a projection of $V^\otimes n$ onto said subspace, the claim follows. A similar statement clearly holds for $\bigwedge^n V$.

For $n \in \mathbb{Z}$, we define the graded vector space $V^n$ to be the vector space $V$ with the grading defined by $V|_p = V^{p+n}$. We denote by $\downarrow^n : V \to V^n$ the identity map on $V$, which becomes a linear isomorphism of degree $-n$, and by $\uparrow^n$ its inverse. We abbreviate $\downarrow^1$ and $\uparrow^1$.
to $\downarrow$ and $\uparrow$, respectively. Note that $\left(\frac{1}{k}\right)^{-1} = (1)^{-1} = 1$ as a consequence of (2).

Proposition 3 (The décalage isomorphism). For $\sigma \in S_n$, $\sigma \in S_n$, $\varepsilon(\sigma) = \varepsilon(\sigma)$. There is then a degree preserving isomorphism

$$S^n(V[1]) \cong \left(\bigwedge^n V\right)[n].$$

Proof. Note that for the first part, we only have to check (4) on the generating subset $\{s_1, \ldots, s_{n-1}\} \subset S_n$. This is an easy computation left to the reader. Let $\pi_A : V \to \bigwedge^n V$ be the canonical projection. The linear maps

$$\pi_S \circ \downarrow \circ \varepsilon = \downarrow \circ \varepsilon \circ \pi_A,$$

are then antisymmetric and symmetric, respectively. The induced linear maps between $\bigwedge^n V$ are then easily seen to be inverse to each other. As these maps are of degree $-n$ and $n$, respectively, we obtain a degree preserving isomorphism $\bigwedge^n V \cong \left(\bigwedge^n V\right)[n]$.

Corollary 4. There is for each $p \in \mathbb{Z}$ a one-to-one correspondence between symmetric linear maps $\lambda : \bigwedge^n V \to \bigwedge^n V[1]$ of degree $p$ and antisymmetric linear maps $l : V \to V$ of degree $p + 1 - n$ given by

$$l = \uparrow \circ \lambda \circ \downarrow \circ \varepsilon,$$

$$\lambda = (1)^{-1} \downarrow \circ l \circ \uparrow \circ \varepsilon.$$

A differential on the graded vector space $V$ is a linear map $d : V \to V$ of degree one such that $d^2 = 0$. We then call the pair $(V, d)$ a differential graded vector space (DG vector space for short). A homomorphism between DG vector spaces $(V, d)$ and $(W, d')$ is a degree preserving linear map $f : V \to W$ such that $d' \circ f = f \circ d$.

DG vector spaces are sometimes called cochain complexes. Given a cochain complex $(V, d)$, one then calls an element $v \in V_n$ an $n$-coboundary if $d(v) = 0$ and an $n$-coboundary if $v = d(w)$ for some $w \in V_{n-1}$. The graded vector space $H(V) = \ker(d)/\text{im}(d)$ measures the non-exactness of the sequence

$$\cdots \to V_{n-1} \overset{d}{\to} V_n \overset{d}{\to} V_{n+1} \overset{d}{\to} \cdots$$

and is called the cohomology of $(V, d)$. We then call $H_n(V) = \{n\text{-coboundaries} \mathbb{n}\text{coboundaries}\}$ the $n$th cohomology group.

2.2 Graded algebras

By an algebra we mean a vector space $A$ together with a linear map $\mu : A \otimes A \to A$; the multiplication $\mu$ is in general not assumed to be associative.

A graded algebra $A$ is an algebra that is also a graded vector space in which the multiplication is degree preserving. If also $ab = (-1)^{|a||b|}ba$ for all $a, b \in A$, we call $A$ (graded) commutative. A homomorphism of graded algebras is a degree preserving algebra homomorphism.

A (two-sided) ideal $I$ in $A$ is called homogeneous if $I \subset A$ is a graded subspace. Note that an ideal is homogeneous if and only if it is spanned by homogeneous elements.

Remark 5. If $I \subset A$ is a homogeneous ideal, the canonical isomorphism $A/I \cong \bigoplus_{\mathbb{Z} \geq 0} A/I$ makes $A/I$ into a graded algebra such that the canonical projection $A \to A/I$ is a homomorphism of graded algebras.

Remark 6. Let $A$ and $B$ be two graded associative algebras. The multiplication defined by

$$(a \otimes b)(a' \otimes b') = (-1)^{|a||a'|}aa' \otimes bb'.$$

for $a, a' \in A$, $b, b' \in B$ homogeneous makes $A \otimes B$ into a graded associative algebra. If $A$ and $B$ are both unital/commutative, then so is $A \otimes B$.

Example 7 (The tensor algebra). We denote by $T(V) := \bigoplus_{n \geq 0} V^\otimes n$ the tensor algebra of $V$. It carries the multiplication induced by the canonical isomorphism $V^\otimes r \otimes V^\otimes s \cong V^\otimes (r+s)$, making it into a unital associative algebra. The grading on $T(V)$ induced by the grading on $V^\otimes n$ is given by

$$T(V)_p = \bigoplus_{i_1 + \cdots + i_n = p} V_{i_1} \otimes \cdots \otimes V_{i_n}$$

and is called the interior grading. On the other hand, $T(V)$ carries the grading given by $T(V) = \bigoplus_{n \geq 0} V^\otimes n$, which is called the exterior grading or grading by weight. If not specified otherwise, we always understand $T(V)$ to carry its interior grading. Note that both gradings make $T(V)$ into a graded algebra.

Example 8 (The symmetric and exterior algebra). Let $I_S \subset T(V)$ be the two-sided homogeneous ideal generated by elements of the form $v_1 \otimes v_2 - (-1)^{|v_1||v_2|}v_2 \otimes v_1$. We call $S(V) := T(V)/I_S$ the symmetric algebra of $V$. Similarly, the exterior algebra of $V$, denoted by $\bigwedge V$, is defined as the quotient of $T(V)$ by the two-sided homogeneous ideal generated by elements of the form $v_1 \otimes v_2 + (-1)^{|v_1||v_2|}v_2 \otimes v_1$. We denote the multiplication in $S(V)$ and $\bigwedge V$ by $\otimes$ and $\wedge$, respectively. Note that $S(V)$ and $\bigwedge V$ also admit an exterior grading or grading by weight as $V^\otimes n \cap I_S = U_n$, we have $S(V) = \bigoplus_{n \geq 0} S^n(V)$ and similarly $\bigwedge V = \bigoplus_{n \geq 0} \bigwedge^n V$.

It is easy to see that if $A$ is a graded unital associative algebra, there is for each linear degree preserving map $f : V \to A$ a unique homomorphism of unital graded algebras $\varphi : T(V) \to A$ that agrees on $V$ with $f$ (see for example [6], Proposition 1.1.1). It is then immediate that if $A$ is commutative, $\varphi$ factors to a unique homomorphism of unital graded algebras $S(V) \to A$. Applying this to the linear map

$$V \otimes W \to S(V) \otimes S(W), \quad (v, w) \mapsto v \otimes 1 + 1 \otimes w$$

yields a homomorphism of graded unital algebras $S(V \otimes W) \to S(V) \otimes S(W)$ that is easily seen to be an isomorphism with inverse $S(V) \otimes S(W) \to S(V \otimes W)$, $v \otimes w \mapsto v \wedge w$. With a slight modification of the sign in
(6), similar arguments show that \( \wedge(V \oplus W) \cong \wedge V \otimes \wedge W. \) In particular, we have
\[
S^n(V \oplus W) \cong \bigoplus_{p+q=n} S^p(V) \otimes S^q(W),
\]
where \( n \geq 0 \).

For a graded algebra \( A \), a derivation of \( A \) of degree \( p \) is a linear map \( d : A \to A \) of degree \( p \) satisfying
\[
d(ab) = d(a)b + (-1)^{p[a]} ad(b)
\]
for all \( a, b \in A \) homogeneous. We denote by \( \text{Der}_p(A) \) the vector space of all derivations of degree \( p \) and \( \text{Der}(A) \) the graded vector space \( \bigoplus_{p \in \mathbb{Z}} \text{Der}_p(A) \). A differential on the graded algebra \( A \) is an element \( d \in \text{Der}(A) \) of degree one such that \( d^2 = 0 \). The pair \( (A, d) \) is then called a differential graded algebra (DGLA for short). A homomorphism of DGLAs is a homomorphism of graded algebras that is also a homomorphism of DG vector spaces.

### 2.3 Graded Lie algebras and unshuffle permutations

**Definition 9.** A graded Lie algebra is a graded vector space \( L \) together with a (graded) antisymmetric degree preserving linear map \([\cdot, \cdot] : L \otimes L \to L \) called the Lie bracket satisfying the (graded) Jacobi identity
\[
[x, [y, z]] = [[x, y], z] + (-1)^{|x||y|}[y, [x, z]] = 0
\]
for all \( x, y, z \in L \) homogeneous.

If \( L \) is ungraded, we recover the usual definition of a Lie algebra. Note that (9) is nothing else than \([x, \cdot] \) being a derivation of the graded algebra \((L, [\cdot, \cdot])\).

**Example 10.** For a graded associative algebra \( A \), we can define the graded commutator \([\cdot, \cdot] : A \otimes A \to A \) by \([a, b] = ab - (-1)^{|a||b|}ba\) for \( a, b \in A \) homogeneous. This makes \( A \) into a graded Lie algebra. In particular, \( \text{gl}(V) := \text{Hom}(V, V) \) becomes a graded Lie algebra. If \( V \) is itself a graded algebra, one can check that \( \text{Der}(V) \subset \text{gl}(V) \) is a Lie subalgebra.

**Definition 11.** A differential graded Lie algebra (DGLA for short) is a DGLA in which the underlying algebra is a graded Lie algebra.

**Example 12.** Let \( L \) be a graded Lie algebra and \( x \in L \) a degree one element such that \( \frac{1}{2} [x, x] = 0 \). Then \( d := [x, \cdot] \) satisfies \( d^2 = 0 \) by the Jacobi identity (9) and \((L, [\cdot, \cdot], d)\) is a DGLA. In particular, for \((V, \partial)\) a DG vector space, this makes \((\text{gl}(V), [\cdot, \cdot], [\partial, \cdot])\) canonically into a DGLA as \( \frac{1}{2} [\partial, \partial] = d^2 = 0 \).

**Definition 13.** For a DGLA \((L, [\cdot, \cdot], d)\), a Maurer–Cartan element is an element \( x \in L \) of degree one such that
\[
d(x) + \frac{1}{2} [x, x] = 0.
\]

The equation (10) is called the Maurer–Cartan equation.

**Example 14.** Let \((L, [\cdot, \cdot], d = [x, \cdot])\) be as in Example 12. For \( y \in L \) of degree one, we have \( \frac{1}{2} [x + y, x + y] = 0 \) if and only if \( y \) satisfies the Maurer–Cartan equation.

For \( 0 \leq i \leq n \), an \((i, n - i)\)-unshuffle is a permutation \( \sigma \in \mathfrak{S}_n \) satisfying \( \sigma(1) < \ldots < \sigma(i) \) and \( \sigma(i + 1) < \ldots < \sigma(n) \). Following the notation in [3], we denote the set of all \((i, n - i)\)-unshuffles by \( \mathfrak{S}_n^{i, n-i} \). Using the antisymmetry of the Lie bracket, one can rewrite (9) as
\[
\sum_{\sigma \in \mathfrak{S}_n^{i, n-i}} \chi(\sigma)[[x_{\sigma(1)}, x_{\sigma(2)}], x_{\sigma(3)}] = 0
\]
for all \( x_1, x_2, x_3 \in L \) homogeneous.

**Lemma 15.** Each element \( \sigma \in \mathfrak{S}_n \) has for each \( i \in \{0, \ldots, n\} \) a unique decomposition \( \sigma = \tau(\alpha, \beta) \), where \( \tau \in \mathfrak{S}_{i, n-i} \) and \((\alpha, \beta) \in \mathfrak{S}_i \times \mathfrak{S}_{n-i} \). Here, \( \mathfrak{S}_i \times \mathfrak{S}_{n-i} \) is considered as a subgroup of \( \mathfrak{S}_n \) in the obvious way.

**Proof.** Clearly, \( \tau \) has to be the unique \((i, n - i)\)-unshuffle such that \( \{\tau(1), \ldots, \tau(i)\} = \{\sigma(1), \ldots, \sigma(i)\} \) and \( \{\tau(i + 1), \ldots, \tau(n)\} = \{\sigma(i + 1), \ldots, \sigma(n)\} \). We then have \( \tau^{-1} \sigma \in \mathfrak{S}_i \times \mathfrak{S}_{n-i} \).

### 2.4 Graded coalgebras

A (graded) coalgebra \((C, \Delta)\) is a graded vector space \( C \) together with a degree preserving linear map \( \Delta : C \to C \otimes C \) called the coproduct. If the diagram
\[
\begin{array}{ccc}
C & \xrightarrow{\Delta} & C \otimes C \\
\downarrow{\Delta} & & \downarrow{\Delta \otimes \text{id}_C} \\
C \otimes C & \xrightarrow{\text{id}_C \otimes \Delta} & C \otimes C \otimes C
\end{array}
\]
commutes, \( C \) is called coassociative. We call \( C \) counital if there is a degree preserving linear map \( \varepsilon : C \to \mathbb{k} \) such that the diagram
\[
\begin{array}{ccc}
C & \xrightarrow{\Delta} & C \otimes C \\
\downarrow{\varepsilon \otimes \text{id}_C} & & \downarrow{\varepsilon \otimes \varepsilon} \\
\mathbb{k} \otimes C & \xrightarrow{\varepsilon \otimes \text{id}_C} & \mathbb{k} \otimes C \otimes C
\end{array}
\]
commutes. The map \( \varepsilon \) is then called the counit of \( C \). If \( \tau_C \circ \Delta = \Delta \), then \( C \) is called cocommutative. A linear degree preserving map \( f : C \to D \) between coalgebras \((C, \Delta_C)\) and \((D, \Delta_D)\) is called a homomorphism of coalgebras if
\[
(f \otimes f) \Delta_C = \Delta_D \circ f.
\]
If \( C \) and \( D \) are counital with counits \( \varepsilon \) and \( \eta \), respectively, and if \( f \) also satisfies \( \eta \circ f = \varepsilon \), we call \( f \) a homomorphism of counital coalgebras.
For a coassociative coalgebra \((C, \Delta)\) and \(n \in \mathbb{N}\), we define the \emph{iterated coproduct} \(\Delta^n : C \to C \otimes (n+1)\) by \(\Delta^0 = \text{id}_C\) and \(\Delta^n = (\Delta \otimes \text{id}_C \otimes \ldots \otimes \text{id}_C)\Delta^{n-1}\) for \(n \geq 1\). It is convenient to then use \emph{Sweedler notation} and to write \(\Delta^n(x) \in C \otimes (n+1)\) for \(x \in C\) as

\[
\Delta^n(x) = \sum x_{(1)} \otimes \ldots \otimes x_{(n+1)}.
\]

In this notation, for example, the condition for \(C\) to be cocommutative becomes \(\sum x_{(1)} \otimes x_{(2)} = \sum(-1)^{x_{(1)}||x_{(2)}} x_{(2)} \otimes x_{(1)}\) for all \(x \in C\).

**Lemma 16.** Let \((C, \Delta_C)\) be a coassociative coalgebra. Then for all \(p, q \in \mathbb{N}\),

\[
(\Delta_C^p \otimes \Delta_C^q)\Delta_C = \Delta_C^{p+q+1}.
\]

If \((D, \Delta_D)\) is another coassociative coalgebra and if \(f : C \to D\) is a coalgebra homomorphism,

\[
f^{\otimes (n+1)} \circ \Delta_C^n = \Delta_D^n \circ f
\]

holds for all \(n \in \mathbb{N}\).

**Proof.** One obtains (15) and (16) by iterating (12) and (14); the details are left to the reader or can be found in [7], Lemma-Definition VIII.10).

### 2.4.1 Coaugmented coalgebras

A \emph{coaugmented coalgebra} \((C, \Delta, \varepsilon, u)\) is a comonoidal coassociative coalgebra \((C, \Delta, \varepsilon)\) together with a homomorphism of counital coalgebras \(u : \mathbb{k} \to C\). The coproduct on \(\mathbb{k}\) is given by \(1_{\mathbb{k}} \mapsto 1_{\mathbb{k}} \otimes 1_{\mathbb{k}}\) and its counit is the identity on \(\mathbb{k}\). Denoting \(u(1_{\mathbb{k}}) \in C\) as 1, the conditions for \(u\) to be a homomorphism of counital coalgebras become \(\Delta(1) = 1 \otimes 1\) and \(\varepsilon \circ u = \text{id}_\mathbb{k}\). A homomorphism between coaugmented coalgebras \(C, D\) and \(f : C \to D\) such that \(f(1) = 1\).

Given a coaugmented coalgebra \((C, \Delta, \varepsilon, u)\), set \(\overline{C} := \ker(\varepsilon)\). We claim that \(C \cong \overline{C} \oplus \mathbb{k}\). Indeed, as \(\varepsilon \circ u = \text{id}_\mathbb{k}\), the short exact sequence of graded vector spaces

\[
0 \to \ker(\varepsilon) \to C \xrightarrow{\Delta} \mathbb{k} \to 0
\]

splits. For \(x \in \overline{C}\), we then define \(\overline{\Delta}(x) := \Delta(x) - x \otimes 1 - 1 \otimes x\). Using (13) and \(\varepsilon(x) = 0\), one easily sees that

\[
\overline{\Delta}(x) \in \ker(\varepsilon \circ \text{id}_C) \cap \ker(\text{id}_C \otimes \varepsilon) = \overline{C} \otimes \overline{C};
\]

for the last equality, note that \(C \cong \overline{C} \otimes \overline{C} \cong (\overline{C} \otimes \mathbb{k}) \oplus (\mathbb{k} \otimes \overline{C}) \oplus (\mathbb{k} \otimes \mathbb{k})\). We call \(\overline{\Delta} : \overline{C} \to \overline{C} \otimes \overline{C}\) the \emph{reduced coproduct} on \(\overline{C}\). It is straightforward to check that \((\overline{C}, \overline{\Delta})\) is a coassociative coalgebra.

Conversely, given a coassociative coalgebra \((\overline{C}, \overline{\Delta})\), we define a coproduct on \(C := \overline{C} \oplus \mathbb{k}\) by \(\Delta(1) = 1 \otimes 1\) and \(\Delta(x) = \overline{\Delta}(x) + x \otimes 1 + 1 \otimes x\) for \(x \in \overline{C}\). This makes \(C\) into a coaugmented coalgebra; the counit and coaugmentation map are given by the projection \(C \to \mathbb{k}\) and the inclusion \(\mathbb{k} \hookrightarrow C\), respectively. These constructions are clearly inverse to each other (up to isomorphism).

Let \(C\) and \(D\) be coaugmented coalgebras with counits \(\varepsilon\) and \(\eta\), respectively. A linear degree preserving map \(f : C \to D\) satisfying \(\eta \circ f = \varepsilon\) and \(f(1) = 1\) decomposes as

\[
f = \widetilde{f} \otimes \text{id}_\mathbb{k} : \overline{C} \oplus \mathbb{k} \to \overline{D} \oplus \mathbb{k}
\]

for a unique degree preserving linear map \(\widetilde{f} : \overline{C} \to \overline{D}\). It is then easy to see that \(f\) is a homomorphism of coaugmented coalgebras if and only if \(\widetilde{f}\) is a homomorphism of coalgebras. This yields a one-to-one correspondence between coalgebra homomorphisms \(\overline{C} \to \overline{D}\) and homomorphisms of coaugmented coalgebras \(C \to D\).

Loosely speaking, this let us choose if we want to work with coaugmented coalgebras or non-coaugmented ones. In more technical terms, we have an equivalence between the category of coaugmented coalgebras and the category of coassociative coalgebras.

We call a coaugmented coalgebra \((C, \Delta)\) \emph{conilpotent} if for all \(x \in \overline{C}\) there is an \(n \in \mathbb{N}\) such that \(\overline{\Delta}(x) = 0\).

### 2.4.2 Examples of coalgebras

There is a coproduct \(\overline{\Delta}_A\) on \(\overline{T}(V) := \bigoplus_{n \geq 1} V \otimes_n\) given by

\[
\overline{\Delta}_A(v_1 \ldots v_n) = \sum_{i=1}^{n-1} (v_1 \ldots v_i) \otimes (v_{i+1} \ldots v_n),
\]

where we now denote the multiplication in \(T(V)\) by concatenation to avoid ambiguities. This makes \(\overline{T}(V)\) into a coassociative graded coalgebra. The induced coaugmented coalgebra \((\overline{T}(V), \overline{\Delta}_A)\) is called the \emph{tensor coalgebra}. Inductively, one finds

\[
\overline{\Delta}(v_1 \ldots v_n) = \sum_{1 \leq i_1 < \ldots < i_m < n} (v_1 \ldots v_{i_1}) \otimes \ldots \otimes (v_{i_{m+1}} \ldots v_n),
\]

which shows that \(T(V)\) is conilpotent.

**Proposition 17.** Let \((C, \Delta)\) be a conilpotent coalgebra and \(f : C \to V\) a linear degree preserving map. There is a unique homomorphism of coaugmented coalgebras \(\tilde{f} : C \to T(V)\) such that \(f = \text{pr}_V \circ \tilde{f}\), where here and subsequently, \(\text{pr}_{(-)}\) denotes the projection onto a subspace under a given decomposition.

**Proof.** It clearly suffices to show that there is a unique homomorphism of coalgebras \(\tilde{f} : C \to T(V)\) satisfying \(\text{pr}_V \circ \tilde{f} = f\). For the uniqueness, assume that there is such \(\tilde{f}\). By Lemma 16, we have

\[
\tilde{f}^{\otimes (n+1)} \circ \overline{\Delta}^n = \overline{\Delta}^n \circ \tilde{f}
\]

for all \(n \in \mathbb{N}\). Composing both sides with \(\text{pr}_{V^{\otimes (n+1)}}\) and noting that \(\text{pr}_{V^{\otimes (n+1)}} \circ \overline{\Delta}^n = \text{pr}_{V^{\otimes (n+1)}} \circ \tilde{f}^{\otimes (n+1)}\) then yields

\[
\text{pr}_{V^{\otimes (n+1)}} \circ \tilde{f} = \text{pr}_{V^{\otimes (n+1)}} \circ \tilde{f}^{\otimes (n+1)} \circ \overline{\Delta}^n = f^{\otimes (n+1)} \circ \overline{\Delta}^n.
\]
This shows that \( \tilde{f} \) is completely determined by \( f \) and therefore unique. For the existence, consider the linear map 
\[
\sum_{n=0}^{\infty} \Delta^n : C \to T(C).
\]
This is well-defined as \( C \) is cocommutative. As \( T(f) = \bigoplus_{n \geq 1} f^\otimes n : T(C) \to T(V) \) is easily seen to be also a coalgebra homomorphism, \( \tilde{f} := T(f) \circ \sum_{n=0}^{\infty} \Delta^n : C \to T(V) \) is a homomorphism of coalgebras with \( \text{pr}_V \circ \tilde{f} = f \).

Let \( \mathcal{S}(V) := \bigoplus_{n \geq 1} S^n(V) \). Consider the linear maps 
\[
\pi : T(V) \to \mathcal{S}(V), \quad v_1 \otimes \ldots \otimes v_n \mapsto \frac{1}{n!} v_1 \otimes \ldots \otimes v_n,
\]
\[
N : \mathcal{S}(V) \to T(V), \quad v_1 \otimes \ldots \otimes v_n \mapsto \sum_{\sigma \in S_n} \varepsilon(\sigma) v_{\sigma(1)} \otimes \ldots \otimes v_{\sigma(n)}.
\]
It is immediate that \( \pi \circ N = \text{id}_S \), where here and subsequently, we abbreviate \( \mathcal{S}(V) \) and \( S(V) \) to \( S \) in subscripts. Using Lemma 15, we compute
\[
\Delta_S(v_1 \otimes \ldots \otimes v_n) = \sum_{i=1}^{n-1} \sum_{\tau \in S_n} \varepsilon(\tau) (v_{\tau(1)} \otimes \ldots \otimes v_{\tau(i)}) 
\]
\[
\otimes (v_{\tau(i+1)} \otimes \ldots \otimes v_{\tau(n)}), \quad (18)
\]
It is immediate that \( S(V) \) is conilpotent, as it is a subcoalgebra of \( T(V) \). We claim that \( S(V) \) is even cocommutative. Indeed, let \( \sigma_i \in S_n \) be for \( 0 \leq i \leq n \) the permutation given by \( (\sigma_1(1), \ldots, \sigma_1(n)) = (i + 1, \ldots, n, 1, \ldots, i) \). We then have \( \text{Sh}_{n-i,i} \circ \sigma_i = \text{Sh}_{n-i,i} \) and therefore
\[
\tau_{S,S} \circ \Delta_S(v_1 \otimes \ldots \otimes v_n)
\]
\[
= \sum_{i=0}^{n} \sum_{\tau \in S_{n-i}} \varepsilon(\tau) (v_{\tau(1)} \otimes \ldots \otimes v_{\tau(n-i)}) 
\]
\[
\otimes (v_{\tau(n-i+1)} \otimes \ldots \otimes v_{\tau(n)})
\]
\[
= \sum_{i=0}^{n} \sum_{\tau \in S_{n-i}} \varepsilon(\tau) (v_{\tau(1)} \otimes \ldots \otimes v_{\tau(i)}) 
\]
\[
\otimes (v_{\tau(i+1)} \otimes \ldots \otimes v_{\tau(n)})
\]
\[
= \Delta_S(v_1 \otimes \ldots \otimes v_n).
\]

**Proposition 18.** Let \( (C, \Delta) \) be a cocommutative conilpotent coalgebra and \( f : C \to V \) a degree preserving linear map. There is a unique homomorphism of coaugmented coalgebras \( f : C \to S(V) \) such that \( f = \text{pr}_V \circ \tilde{f} \).

**Proof.** As in the proof of Proposition 17, it suffices to show that there is a unique homomorphism of coalgebras \( \tilde{f} : C \to S(V) \) satisfying \( \text{pr}_V \circ \tilde{f} = f \). Recall that \( T(f) \circ \sum_n \Delta^n \) is the unique coalgebra homomorphism \( C \to T(V) \) extending \( f \). For \( 0 \leq i \leq n-1 \), we have
\[
(id_V^{\otimes i} \otimes \tau_{C,C} \otimes id_V^{\otimes (n-i-1)})T(f) \circ \Delta^n
\]
\[
= T(f)(id_V^{\otimes i} \otimes \tau_{C,C} \otimes id_V^{\otimes (n-i-1)}) \Delta^n
\]
\[
= T(f)(id_C^{\otimes i} \otimes \tau_{C,C} \otimes id_C^{\otimes (n-i-1)}) \Delta^{n-1}
\]
\[
= T(f)(id_C^{\otimes i} \otimes \Delta \otimes id_C^{\otimes (n-i-1)}) \Delta^{n-1}
\]
\[
= T(f) \circ \Delta^n
\]
since \( C \) is cocommutative. As \( (id_V^{\otimes i} \otimes \tau_{C,C} \otimes id_V^{\otimes (n-i-1)}) = \varepsilon(\sigma_i) \), the image of \( T(f) \circ \sum_n \Delta^n \) is contained in the subspace of \( T(V) \) of all symmetric elements, which is \( \text{im}(N) \).
We obtain an induced homomorphism of coalgebras
\[
\tilde{f} = \pi \circ T(f) \circ \sum_{n=0}^{\infty} \Delta^n : \mathcal{C} \to \mathcal{S}(V),
\]
\[
x \mapsto \frac{1}{n!} \sum_{n=1}^{\infty} f(x(1)) \vee \ldots \vee f(x(n))
\]
with \( \text{pr}_V \circ \tilde{f} = f \). Similarly, a coalgebra homomorphism \( \tilde{f} : \mathcal{C} \to \mathcal{S}(V) \) gives rise to a coalgebra homomorphism \( N \circ \tilde{f} : \mathcal{C} \to \mathcal{T}(V) \) that is uniquely determined by \( \text{pr}_V \circ N \circ \tilde{f} = f \) by Proposition 17. As \( N \) is injective, this shows uniqueness of \( \tilde{f} \).

**Example 19.** A linear degree preserving map \( f : V \to W \) can be extended by zero to a linear map \( \tilde{S}(V) \to W \). The induced homomorphism of coaugmented coalgebras \( \tilde{S}(V) \to \tilde{S}(W) \) is denoted by \( S(f) \) and is given by \( S(f)(v_1 \vee \ldots \vee v_n) = f(v_1) \vee \ldots \vee f(v_n) \).

### 2.4.3 Comodules and coderivations

Let \( (C, \Delta) \) be a coalgebra. A left comodule over \( C \) is a graded vector space \( M \) together with a degree preserving linear map \( \Delta_l : M \to C \otimes M \) satisfying
\[
(\Delta \otimes \text{id}_M) \Delta_l = (\text{id}_C \otimes \Delta_l) \Delta_l.
\]
(20)

Similarly, a right comodule over \( C \) is a graded vector space \( M \) together with a degree preserving linear map \( \Delta_r : M \to M \otimes C \) such that
\[
(\text{id}_M \otimes \Delta) \Delta_r = (\Delta_r \otimes \text{id}_C) \Delta_r.
\]
(21)

If \( M \) is both a left and a right comodule over \( C \) and if the compatibility relation
\[
(\Delta_l \otimes \text{id}_C) \Delta_r = (\text{id}_C \otimes \Delta_r) \Delta_l
\]
(22)
is satisfied, \( M \) is called a (bi)comodule over \( C \). Given such \( M \), we define a coderivation of degree \( p \) to be a homogeneous linear map \( d : M \to C \) of degree \( p \) such that
\[
\Delta \circ d = (d \otimes \text{id}_C) \Delta_r + (\text{id}_C \otimes d) \Delta_l.
\]
(23)

We denote the vector space of all these maps by \( \text{Coder}_p(M,C) \) and by \( \text{Coder}(M,C) \) the graded vector space \( \bigoplus_{p \in \mathbb{Z}} \text{Coder}_p(M,C) \).

Let \( (C, \Delta_C) \) and \( (D, \Delta_D) \) be coassociative coalgebras and \( f : D \to C \) a coalgebra homomorphism. Then \( \Delta_r := (\text{id}_D \otimes f) \Delta_D \) and \( \Delta_l := (f \otimes \text{id}_D) \Delta_D \) make \( D \) into a comodule over \( C \). In particular, \( C \) is a comodule over itself and we abbreviate \( \text{Coder}(C, C) \) to \( \text{Coder}(C) \). If \( C \) and \( D \) are coaugmented and if \( f \) is a homomorphism of coaugmented coalgebras, the comodule structure is compatible with the counit in the sense that the diagram
\[
\begin{array}{ccc}
C \otimes D & \xrightarrow{\Delta_l} & D \otimes C \\
\varepsilon \otimes \text{id}_D & \parallel & \Delta_r \\
\text{id}_C \otimes \varepsilon & \text{commutes, where } \varepsilon \text{ is the counit on } C. \text{ Observe that } f : D \to C \text{ then makes } D \text{ into a comodule over } C. \text{ The following proposition relates elements in } \text{Coder}(D, C) \text{ to coderivations } d : D \to C \text{ that satisfy } d(1) = 0; \text{ the latter is called a coderivation of coaugmented coalgebras.}
\end{array}
\]

**Proposition 20.** Let \( f : D \to C \) be a homomorphism of coaugmented coalgebras. There is a one-to-one correspondence between coderivations \( d : D \to C \) satisfying \( d(1) = 0 \) and coderivations \( \tilde{d} : D \to \tilde{C} \) given by \( d = \tilde{d} \oplus 0 : D \oplus k \to C \oplus k \).

**Proof.** Given a linear map \( \tilde{d} \in \text{Hom}(D, C) \), one easily checks that \( \tilde{d} \) is a coderivation if and only if \( \tilde{d} \oplus 0 : D \oplus k \to C \oplus k \) is. It then suffices to show that each coderivation \( d : D \to C \) with \( d(1) = 0 \) is of this form. Let \( \varepsilon \) be the counit of \( C \) and \( \mu_k : k \otimes k \to k \) the multiplication on \( k \).

From (13), (23) and the compatibility with the counit (24) it then follows that
\[
\varepsilon \circ d = \mu_k((\varepsilon \otimes \varepsilon) \Delta_C \circ d)
\]
\[
= \mu_k((\varepsilon \otimes (\varepsilon \circ d)) \Delta_l + (\varepsilon \circ (\varepsilon \circ d)) \Delta_r)
\]
\[
= 2(\varepsilon \circ d),
\]
which shows that \( d(D) \subset \tilde{C} \). Hence, \( d \) decomposes as \( d \oplus 0 : D \oplus k \to C \oplus k \).

For a coassociative coalgebra \( C \), we call an element \( d \in \text{Coder}(C) \) of degree one with \( d^2 = 0 \) a coderivation or \( C \). We then call the pair \((C, d)\) a differential graded coassociative coalgebra (DGC for short). If \( C \) is coaugmented and \( d(1) = 0 \), we call \((C, d)\) a coaugmented DGC. A homomorphism of DGCs is then a coalgebra homomorphism that is also a homomorphism of DG vector spaces; homomorphisms of coaugmented DGCs are defined accordingly. From Proposition 20 and Section 2.4.1, we then obtain an equivalence between the categories of DGCs and coaugmented DGCs.

**Proposition 21.** Let \( C \) be a coassociative coalgebra. Then \( \text{Coder}(C) \subset \mathfrak{gl}(C) \) is closed under the graded commutator. Also, if \( f : D \to C \) is a homomorphism of coassociative coalgebras, \( d \in \text{Coder}(C) \) and \( d' \in \text{Coder}(D) \), then \( f \circ d' \) or \( d' \circ f \in \text{Coder}(D, C) \).

**Proof.** Both parts of the proposition are straightforward computations which are left to the reader.

**Theorem 22.** Let \( D \) be a cocommutative coaugmented coalgebra and \( f : D \to \tilde{S}(V) \) a homomorphism of coaugmented coalgebras. The linear map
\[
\text{Coder}(D, \tilde{S}(V)) \to \text{Hom}(D, V), \quad d \mapsto \text{pr}_V \circ d
\]
is then an isomorphism. Its inverse is given by
\[
\text{Hom}(D, V) \to \text{Coder}(D, \tilde{S}(V)), \quad \lambda \mapsto \mu_S(\lambda \otimes f) \Delta_D,
\]
where \( \mu_S : \tilde{S}(V) \otimes \tilde{S}(V) \to \tilde{S}(V) \) denotes the multiplication on \( \tilde{S}(V) \) and \( \Delta_D \) the coproduct on \( D \).

It is immediate that \( d(1) = 0 \) if and only if \( \lambda = \text{pr}_V \circ d \) vanishes on \( k \). Together with Proposition 20, this shows \( \text{Coder}(D, \tilde{S}(V)) \cong \text{Hom}(D, V) \).
Let $\Delta_S (v_1) \ldots \Delta_S (v_{n+1})$
\begin{align*}
&= \Delta_S (v_1 \vee \ldots \vee v_n) \Delta_S (v_{n+1}) \\
&= \sum_{i=0}^{n} \sum_{\tau \in Sh_{n,i-1}} (\varepsilon (\tau) (v_{\tau (1)} \vee \ldots \vee v_{\tau (i)}) \otimes (v_{\tau (i+1)} \vee \ldots \vee v_{\tau (n)})) (v_{n+1} \otimes 1 + 1 \otimes v_{n+1}) \\
&= \sum_{i=0}^{n} \sum_{\sigma \in Sh_{n,i-1}} (-1)^{i+1} \sum_{\sigma (1) = i, \sigma(n) = n} \varepsilon (\sigma) (v_{\sigma (1)} \vee \ldots \vee v_{\sigma (n)} \vee v_{n+1}) \otimes (v_{\sigma (i+1)} \vee \ldots \vee v_{\sigma (n)}) \\
&+ \sum_{i=0}^{n+1} \sum_{\sigma \in Sh_{n,i-1}} \varepsilon (\sigma) (v_{\sigma (1)} \vee \ldots \vee v_{\sigma (i)}) \otimes (v_{\sigma (i+1)} \vee \ldots \vee v_{\sigma (n+1)}) \\
&= \Delta_S (v_1 \vee \ldots \vee v_{n+1})).
\end{align*}

The first part of Theorem 22 actually holds for a broader class of comodules over $S(V)$ (see for example [8], Lemma 2.4); the inverse formula $d = \mu_S (\Delta \otimes id_M) \Delta_r$ then continues to hold for comodules $M$ in which $\tau_{M,S} \circ \Delta_r = \Delta_l$.

**Remark 23.** For $1 \leq i \leq n-1$ and $\tau \in Sh_{n,i-1}$ either $\tau (i) = n$ or $\tau (n) = n$. In the first case, there is a unique $\sigma \in Sh_{n-1,i-1}$ such that $(\tau (1), \ldots, \tau (n)) = (\sigma (1), \ldots, \sigma (i-1), \sigma (i), \ldots, \sigma (n-1))$, while in the second case $(\tau (1), \ldots, \tau (n)) = (\sigma (1), \ldots, \sigma (n-1), \sigma (n))$ for a unique $\sigma \in Sh_{n-1,i-1}$. This yields a bijection $Sh_{n-1,i-1} \approx Sh_{n-1,n-1} \cup Sh_{n-1,n-i-1}$. By setting $Sh_{n,0,1} = \emptyset$, this also holds for $i = 0, n$.

**Lemma 24.** The map $\Delta_S : S(V) \to S(V) \otimes S(V)$ is a homomorphism of graded algebras.

**Proof.** We show by induction over $n \in \mathbb{N}$ that
\begin{equation}
\Delta_S (v_1 \vee \ldots \vee v_n) = \Delta_S (v_1) \ldots \Delta_S (v_n). \quad (25)
\end{equation}

For $n = 1$ there is nothing to do. Assume that (25) holds for $n \geq 1$. We compute
\begin{equation*}
\text{See this equation above.}
\end{equation*}

In the fourth equality, we shifted the summation index of the first sum and used Remark 23.

**Proof of Theorem 22.** Let $d : D \to S(V)$ be a coderivation, that is
\begin{equation*}
\Delta_S \circ d = (d \otimes f + f \otimes d) \Delta_D.
\end{equation*}
Inductively, we then get
\begin{equation*}
\Delta^n_S \circ d = \sum_{k=0}^{n} (f^k \otimes d \otimes f^{\otimes (n-k)}) \Delta^n_D.
\end{equation*}

For $n \in \mathbb{N}$, let $\pi_n : T^n(V) \to S^n(V)$ be the linear map defined by
\begin{equation*}
\pi_n (v_1 \otimes \ldots \otimes v_n) = \frac{1}{n!} v_1 \vee \ldots \vee v_n.
\end{equation*}

From $S(V)$ being a subcoalgebra of $T(V)$ and (17), it follows that $\pi_{n+1} \circ pr_V^{\otimes (n+1)} \circ \Delta^n_S = pr_{S^{n+1}}(V)$. We then have
\begin{equation*}
pr_{S^{n+1}}(V) \circ d = \pi_{n+1} \circ pr_V^{\otimes (n+1)} \circ \sum_{k=0}^{n} (f^k \otimes d \otimes f^{\otimes (n-k)}) \Delta^n_D = \pi_{n+1} \circ \sum_{k=0}^{n} ((pr_V \circ f)^k \otimes (pr_V \circ d) \otimes (pr_V \circ f)^{\otimes (n-k)}) \Delta^n_D.
\end{equation*}

As this holds for all $n \in \mathbb{N}$ and as $pr_k \circ d = 0$ by the same computation as in the proof of Proposition 20, $d$ is completely determined by $pr_V \circ d$.

What is left is to show that given $\lambda \in \text{Hom}(D,V)$ homogeneous, $d := \mu_S (\lambda \otimes f) \Delta_D$ is a coderivation with $pr_V \circ d = \lambda$. While the latter holds by construction, we compute for $x \in D$ homogeneous
\begin{equation*}
\text{See this equation next page}
\end{equation*}

where we used Lemma 24 in the first and cocommutativity of $D$ in the fifth equality.

### 2.5 Dual spaces

The graded vector space $V^* := \text{Hom}(V, k)$ is called the dual space of $V$. By degree reasons, $(V^*)^k = \text{Hom}_k(V, k) = \text{Hom}_p(V_k, k) = (V_{-k})^*$. For $f \in \text{Hom}_p(V, W)$, the linear map $f^* \in \text{Hom}_p(W^*, V^*)$ is defined by $f^*(\varphi) = (-1)^{|f||\varphi|} f \circ \varphi$ for $\varphi \in W^*$ homogeneous. Note that
\[(\Delta_S \circ d)(x) = \sum \Delta_S(\lambda(x_1)) \triangledown \Delta_S(f(x_2)) = \sum (\lambda(x_1) \otimes 1 + 1 \otimes \lambda(x_1)) \triangledown (f \otimes f)(\Delta_D(x_2)) = \sum (\lambda(x_1) \otimes 1 + 1 \otimes \lambda(x_1))(f(x_2) \otimes f(x_3)) = \sum (\lambda(x_1) \triangledown f(x_2)) \otimes f(x_3) + (\lambda(x_1) \triangledown f(x_3)) \triangledown f(x_2) = \sum (\mu_S(\lambda \otimes f) \triangledown f + f \otimes \mu_S(\lambda \otimes f))(x_1 \otimes x_2 \otimes x_3) = (\mu_S(\lambda \otimes f)\Delta_D \otimes f + f \otimes \mu_S(\lambda \otimes f)\Delta_D)(\Delta_D(x)).\]

We say that \(V\) is of finite type if \(V_k\) is finite-dimensional for all \(k \in \mathbb{Z}\). Note that if \(V\) is of finite type, the canonical inclusion \(V \hookrightarrow V^{**}\) is an isomorphism.

If \(V_k = 0\) for \(k > 0\), then \(V\) is called \(\mathbb{Z}_{<0}\)-graded. Notions as \(\mathbb{Z}_{<0}\)-graded or \(\mathbb{Z}_{\geq0}\)-graded are defined accordingly. In the following, we denote \(V^{**}\) as \(T(V)\) for better readability.

**Proposition 25.** If \(V\) is of finite type and if for all \(k \in \mathbb{Z}\) the decomposition

\[T^n(V)_k = \bigoplus_{i_1 + \ldots + i_n = k} (V_{i_1} \otimes \ldots \otimes V_{i_n})\]

has only finitely many non-trivial summands, then the canonical inclusion \(T^n(V^*) \hookrightarrow T^n(V)^*\) is an isomorphism.

**Proof.** It is well-known that for finite-dimensional (ungraded) vector spaces \(V_1, \ldots, V_n\), the canonical inclusion \(V_1^* \otimes \ldots \otimes V_n^* \hookrightarrow (V_1 \otimes \ldots \otimes V_n)^*\) is an isomorphism. We then have

\[(T^n(V^*)) = \bigoplus_{i_1 + \ldots + i_n = k} (V_{i_1} \otimes \ldots \otimes V_{i_n})^* = \bigoplus_{i_1 + \ldots + i_n = k} (V^*_{i_1} \otimes \ldots \otimes V^*_{i_n}) = T^n(V^*)_k.\]

**Remark 26.** If \(V\) is of finite type and \(\mathbb{Z}_{<0}\)-graded, \(V^*\) is also of finite type and \(\mathbb{Z}_{\geq0}\)-graded and they both satisfy the hypothesis of Proposition 25. It is then easy to see that \(\Delta: V \to V \otimes V\) makes \(V\) into a graded coassociative/cocommutative coalgebra if and only if \(\Delta^*: V^* \otimes V^* \cong (V \otimes V)^* \to V^*\) makes \(V^*\) into an associative/cummutative algebra. A linear map \(d: V \to V\) is then a coderivation of \((V, \Delta)\) if and only if \(-d^*\) is a derivation of \((V^*, \Delta^*)\). The map \(gl(V) \to gl(V^*)\), \(f \mapsto -f^*\) preserves the graded commutator and therefore restricts to an isomorphism of graded Lie algebras \(\text{Coder}(V) \cong \text{Der}(V^*)\).

**Corollary 27.** If \(V\) is of finite type and \(\mathbb{Z}_{<0}\)-graded, the canonical inclusion \(T(V^*) \hookrightarrow T(V)^*\) is an isomorphism.

**Proof.** Note that for all \(k \in \mathbb{Z}\) and \(n > k\), we have \(T^n(V^*)_{-k} = 0\). Then

\[(T(V^*)) = \bigoplus_{n \geq 0} (T^n(V^*)) = \bigoplus_{n \geq 0} (T^n(V^*)) = \bigoplus_{n \geq 0} (T^n(V)\kappa) = T^n(V)\kappa.\]

**Lemma 28.** Let \(\xi \in T^n(V^*) \subset T^n(V)^*\) and \(\sigma \in \mathfrak{g}_n\). Then

\[\xi(\sigma) = \xi \circ \xi(\sigma^{-1}).\]

**Proof.** It suffices to show this for \(\sigma = s_1, \ldots, s_n\), in which case it is an easy computation.

**Proposition 29.** If \(V\) is of finite type and \(\mathbb{Z}_{<0}\)-graded, \(S(V)^* \cong S(V^*)\). Under this identification, \(\Delta_S^*\) is the usual multiplication on \(S(V^*)\).

**Proof.** Fix \(n \geq 0\). By Lemma 28, the isomorphism \(T^n(V^*) \cong T^n(V^*)\) maps the subspace of symmetric elements in \(T^n(V^*)\) onto the space of symmetric linear maps \(V^\otimes n \to k\). While the latter is isomorphic to \(\text{Hom}(S^n(V), k) \cong S^n(V)^*\) by Remark 2, the former is isomorphic to \(S^n(V)^*\) via the linear map

\[S^n(V^*) \to T^n(V^*), v_1 \ldots v_n \mapsto \sum_{\sigma \in \mathfrak{g}_n} \xi(\sigma)v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(n)}.\]

This yields an isomorphism \(S^n(V^*) \cong S^n(V)^*\). With a similar reasoning as in the proof of Corollary 27, one obtains \(S(V)^* \cong S(V^*)\). It is then a straightforward computation to show that \(\Delta_S^*\) is indeed the usual multiplication on \(S(V^*)\).

### 3 \(L_\infty\)-algebras

We start this section with a theorem from [5] that characterizes certain \(L_\infty\)-algebras using Lie algebra cohomology; later, we seek to generalise it in the context of \(L_\infty\)-algebra cohomology. After that, we discuss different characterisations of \(L_\infty\)-structures using the key results of Section 2. Different points of view naturally lead to different notions of homomorphisms between \(L_\infty\)-algebras; we will finish the section with a comparison of those. For this, we will mostly
\[ l_1(l_1(x_1)) = 0, \]
\[ l_1(l_2(x_1, x_2)) = l_2(l_1(x_1), x_2) + (-1)^{|x_1|}l_2(x_1, l_1(x_2)), \]
\[ 0 = \sum_{\sigma \in Sh_{0,1}} \chi(\sigma) l_2(x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)}) + l_1(l_3(x_1, x_2, x_3)) + l_3(l_1(x_1, x_2, x_3)) + (-1)^{|x_1|}l_3(x_1, l_1(x_2), x_3) + (-1)^{|x_1|+|x_2|}l_3(x_1, x_2, l_1(x_3)) \]

follow [3], although the original references for Section 3.2 are [1] and [8].

**Definition 30.** An \( L_{\infty}-\text{algebra} \) is a graded vector space \( L \) together with antisymmetric linear maps \( l_k: L^\otimes k \to L \) called (higher) brackets of degree \( |l_k| = 2 - k \) for \( 1 \leq k < \infty \) such that the generalized Jacobi identity
\[
\sum_{i+j,n+1} \sum_{\sigma \in Sh_{i,j,n}} (-1)^{i(j-1)} \chi(\sigma) l_j(l_i(x_1, \ldots, x_i), x_{i+1}, \ldots, x_n) = 0 \tag{27}
\]
holds for all \( n \geq 1 \) and \( x_1, \ldots, x_n \in L \) homogeneous. We then call the set \( \{ l_k \mid 1 \leq k < \infty \} \) an \( L_{\infty} \)-structure on \( L \).

Writing out (27) for \( n = 1, 2, 3 \) yields
\[
\text{See this equation above}
\]

for all \( x_1, x_2, x_3 \in L \) homogeneous. While the first two equations may be summarized by saying that \( l_1 \) is a differential on the (non-associative) graded algebra \( (L, l_2) \), a comparison with (11) shows that the third one describes the defect of the Jacobi identity in \( (L, l_2) \). In particular, an \( L_{\infty} \)-algebra with \( l_k = 0 \) for \( k \geq 3 \) is nothing else than a DGLA.

If \( L \) is concentrated in degree zero, \( l_k = 0 \) for \( k \neq 2 \) by degree reasons and \( (L, l_2) \) is an (ungraded) Lie algebra.

**Definition 31.** Let \( L \) and \( L' \) be \( L_{\infty} \)-algebras with \( L_{\infty} \)-structures \( \{ l_k \}_{k \in \mathbb{N}} \) and \( \{ l'_k \}_{k \in \mathbb{N}} \), respectively. A strict \( L_{\infty} \)-algebra homomorphism is a degree preserving linear map \( f: L \to L' \) satisfying
\[
f \circ l_k = l'_k \circ f^\otimes k \tag{28}
\]
for all \( 1 \leq k < \infty \).

These homomorphisms are strict in the sense that they strictly preserve all brackets. A different characterisation of \( L_{\infty} \)-algebras will later lead to a more general notion of \( L_{\infty} \)-algebra homomorphisms.

### 3.1 Characterisation via Lie algebra cohomology

For an (ungraded) Lie algebra \((g, [\cdot, \cdot])\), a representation of \( g \) on an (ungraded) vector space \( V \) is a homomorphism of Lie algebras \( \rho: g \to \mathfrak{gl}(V) \). Given such a \( \rho \), the \emph{Lie algebra cohomology with values in} \( V \) is the cohomology of the Chevalley–Eilenberg (cochain) complex \( \bigoplus_{n \geq 0} \text{Hom}(\Lambda^n g, V, \delta) \), where for an antisymmetric linear map \( \omega: g^\otimes n \to V \), we define \( \delta \omega \) by
\[
\delta \omega(x_1, \ldots, x_{n+1}) = \sum_{i=1}^{n+1} (-1)^{i+1} \rho(x_i)(\omega(x_1, \ldots, \hat{x}_i, \ldots, x_{n+1})) + \sum_{1 \leq j < k \leq n+1} (-1)^{j+k} \omega([x_j, x_k], x_1, \ldots, \hat{x}_j, \ldots, \hat{x}_k, \ldots, x_{n+1}),
\]
for \( x_1, \ldots, x_{n+1} \in g \). In the sums above, elements with \( \hat{\cdot} \) are to be omitted.

**Theorem 32** ([5], Theorem 55). There is for each \( n \geq 1 \) a one-to-one correspondence between \( L_{\infty} \)-algebras \( L \) such that \( l_k = 0 \) for \( k \neq -n, 0 \) and \( l_1 = 0 \) and quadruples \((g, V, \rho, l_{n+2})\) consisting of a Lie algebra \( g \), a representation \( \rho \) of \( g \) on a vector space \( V \) and an \((n+2)-\)cocycle \( l_{n+2} \).

**Sketch of proof.** For an \( L_{\infty} \)-algebra \( L = L_0 \oplus L_n \) with \( l_1 = 0 \), all brackets except for \( l_2 \) and \( l_{n+2} \) have to vanish by degree reasons. Also, \( l_2 \) has to vanish on \( \Lambda^2 L_n \) and \( l_{n+2} \) can only be non-trivial on \( \Lambda^{n+2} L_0 \) with image in \( L_n \). Using (8), we can decompose \( l_2 \) into linear maps \([\cdot, \cdot]: \Lambda^2 L_0 \to L_0 \) and \( \rho: L_0 \otimes L_n \to L_n \). It is then a matter of computation to show that \( l_2 \) and \( l_{n+2} \) satisfying (27) amounts to \((L_0, [\cdot, \cdot])\) being a Lie algebra, \( \rho \) being a representation of \( L_0 \) on \( L_n \) and \( l_{n+2} \) being a cocycle.

### 3.2 Symmetric brackets and codifferentials

Recall that by Corollary 4, an antisymmetric map \( l_k: L^\otimes k \to L \) of degree \( 2 - k \) is equivalent to a symmetric degree one map \( -\lambda_k: S^k(L[1]) \to L[1] \) such that
\[
l_k = \uparrow \circ \lambda_k \circ \downarrow^\otimes k.
\]
If we now rewrite (27) in terms of the maps \( \lambda_k \), we obtain a characterisation of \( L_{\infty} \)-structures on \( L \) in terms of symmetric brackets. Note that for fixed \( n \), we can write (27) as
\[
\sum_{i+j,n+1} \sum_{\sigma \in Sh_{i,j,n}} (-1)^{i(j-1)} l_j(l_i \otimes \text{id}_L^\otimes (j-1)) \hat{\chi}(\sigma) = 0
\]
As \( \downarrow^\otimes n \) and \( \uparrow^\otimes n \) are isomorphisms, this is equivalent to
\[
\text{See this equation next page}
\]
where we used (2) and (4). We have proved the following.
0 = (-1)^{n(n-1)\over 2} \sum_{i+j=n+1} \sum_{\sigma \in S_n} \varepsilon(\sigma) 
\lambda_j(\lambda_i(x_{\sigma(1)}, \ldots, x_{\sigma(i)}), x_{\sigma(i+1)}, \ldots, x_{\sigma(n)}) = 0
\tag{29}
holds for all for all \ n \geq 1 \ and \ x_1, \ldots, x_n \in L[1]
homogeneous.

**Corollary 34.** An \( L_\infty \)-structure on the graded vector space \( L \) is equivalent to a linear degree one map \( \lambda : S(L[1]) \to L[1] \) such that
\[ \lambda \circ \mu_S(\lambda \otimes id_S)\Delta_S = 0. \tag{30} \]

**Proof.** Combine the brackets in Proposition 33 to a single element \( \lambda = \sum_k \lambda_k \in \text{Hom}(S(L[1]), L[1]) \) of degree one and compare with (18).

By abuse of notation, we then also refer to the pair \( (L[1], \lambda) \) as an \( L_\infty \)-algebra. Strict homomorphisms of \( L_\infty \)-algebras can then be described as degree preserving linear maps that preserve the symmetric brackets.

**Proposition 35.** Let \( (L[1], \lambda) \) and \( (L'[1], \lambda') \) be \( L_\infty \)-algebras. There is a one-to-one correspondence between strict \( L_\infty \)-algebra homomorphisms \( f : L \to L' \) and linear degree preserving maps \( g = \downarrow \circ f \uparrow : L[1] \to L'[1] \) satisfying
\[ \lambda \circ \lambda_k = \lambda'_k \circ g^\otimes k \tag{31} \]
for all \( k \geq 1 \).

The equation (31) can be written more conveniently as
\[ g \circ \lambda = \lambda' \circ S(g). \tag{32} \]
We then also refer to \( g \) as a strict \( L_\infty \)-algebra homomorphism.

**Theorem 36.** An \( L_\infty \)-structure on the graded vector space \( L \) is equivalent to a codifferential \( d \in \text{Coder}(S(L[1])) \) with \( d(1) = 0 \).

In this case, we also refer to the pair \( (S(L[1]), d) \) as an \( L_\infty \)-algebra.

**Proof.** Let \( \lambda : S(L[1]) \to L[1] \) be of degree one and \( d = \mu_S(\lambda \otimes id_S)\Delta_S \) be the unique coderivation extending \( \lambda \) in the sense of Theorem 22. As \( d^2 = 1 \over 2 [d, d] \) is a coderivation of \( S(L[1]) \) by Proposition 21, we have by Theorem 22 that \( d^2 = 0 \) if and only if
\[ 0 = \text{pr}_{L[1]} \circ d^2 = \lambda \circ \mu_S(\lambda \otimes id_S)\Delta_S. \]

**Corollary 37.** If \( L \) is of finite type and \( \mathbb{Z}_{\leq 0} \)-graded, an \( L_\infty \)-structure on \( L \) is equivalent to a differential on the graded algebra \( S(L[1])^* \). Explicitly, consider \( d_{CE} = -d^* \) for \( d \) as in Theorem 36.

**Proof.** See Remark 26 and note that as \( L \) is \( \mathbb{Z}_{\leq 0} \)-graded, each \( d \in \text{Coder}(S(L[1])) \) of degree one vanishes on \( \k \) by degree reasons.

### 3.3 Weak homomorphisms

Let \( (S(L[1]), d) \) and \( (S(L'[1]), d') \) be \( L_\infty \)-algebras, \( \lambda = \text{pr}_{L[1]} \circ d \) and \( \lambda' = \text{pr}_{L'[1]} \circ d' \). The characterisation of \( L_\infty \)-structures as codifferentials on the symmetric coalgebra leads to another notion of homomorphisms of \( L_\infty \)-algebras, namely as homomorphisms of (coaugmented) DGCs.

**Definition 38.** A (weak) homomorphism of \( L_\infty \)-algebras between \( L \) and \( L' \) is a homomorphism of coaugmented DGCs \( f : S(L[1]) \to S(L'[1]) \).

**Remark 39.** By Proposition 21 and Theorem 22, a homomorphism of coaugmented coalgebras \( f : S(L[1]) \to S(L'[1]) \) is a homomorphism of \( L_\infty \)-algebras if and only if
\[ (\text{pr}_{L[1]} \circ f) \circ d = \lambda' \circ f. \]

Note that by Proposition 20, it makes sense to also refer to DGC homomorphisms \( \overline{S}(L[1]) \to \overline{S}(L'[1]) \) as homomorphisms of \( L_\infty \)-algebras.

From the dualised standpoint, we immediately get the following.

**Proposition 40.** Assume that \( L \) and \( L' \) are \( \mathbb{Z}_{\leq 0} \)-graded and of finite type. Then \( f : S(L[1]) \to S(L'[1]) \) is a homomorphism of \( L_\infty \)-algebras if and only if \( f^* : S(L'[1])^* \to S(L[1])^* \) is a homomorphism of unital DG algebras.
With now two different notions of $L_\infty$-algebra homomorphisms at hand, it is reasonable to ask if there is a connection between them. As commented in ([8], Remark 5.3), strict homomorphisms are essentially the weak homomorphisms that preserve the exterior degree.

**Lemma 41.** Let $g: L[1] \to L'[1]$ be a linear degree preserving map. Then $g$ is a strict $L_\infty$-algebra homomorphism if and only if $S(g)$ is a weak one.

**Proof.** Observe that $\operatorname{pr}_{L'[1]} \circ S(g) \circ d = g \circ \operatorname{pr}_{L[1]} \circ d = g \circ \lambda$.

**Lemma 42.** A homomorphism of coalgebras $f: S(L[1]) \to S(L'[1])$ preserves the exterior degree if and only if $f = S(g)$ for a linear degree preserving map $g: L[1] \to L'[1]$.

**Proof.** Assume that $f: S(L[1]) \to S(L'[1])$ is a homomorphism of coalgebras such that $f(S^n(L[1])) \subseteq S^n(L'[1])$ for all $n$ and let $g := \operatorname{pr}_{L'[1]} \circ f|_{L[1]}$. Then $\operatorname{pr}_{L'[1]} \circ f = g \circ \operatorname{pr}_{L[1]} = \operatorname{pr}_{L'[1]} \circ S(g)$. Hence, $f = S(g)$ by Proposition 18.

**Proposition 43.** Let $f: S(L[1]) \to S(L'[1])$ be a (weak) $L_\infty$-algebra homomorphism. Then $f$ preserves the exterior degree if and only if $f = S(g)$ for a strict $L_\infty$-algebra homomorphism $g$.

**Proof.** Combine Lemma 41 and Lemma 42.

From this it follows for example that all (weak) $L_\infty$-algebra homomorphisms between Lie algebras are induced by Lie algebra homomorphisms.

### 4 Representations (up to homotopy)

While representations (up to homotopy) of $L_\infty$-algebras are often defined in terms of antisymmetric maps, we start with a definition that keeps the symmetric point of view of the last section. While it is a straightforward computation to show equivalence between these definitions, it is convenient to save this for Section 5.1. We then show that representations (up to homotopy) are nothing else than weak $L_\infty$-algebra homomorphisms into $\mathfrak{gl}(V)$ for a DG vector space $V$, a characterisation due to Lada and Markl [8]. In [3], representations (up to homotopy) were described (under some finiteness assumptions) as differentials on $S(L[1])^* \otimes V$. We discuss this point of view in the second half of this section, which also leads us to $L_\infty$-algebra cohomology.

From now on, $L$ denotes an $L_\infty$-algebra with $L_\infty$-structure $\{l_k \mid 1 \leq k < \infty\}$ and $\lambda$ and $d$ are as in Corollary 34 and Theorem 36, respectively.

**Definition 44.** A representation (up to homotopy) of $L$ on $V$ is a linear map $\rho: S(L[1]) \otimes V \to V$ of degree one that satisfies

\[
\rho(d \otimes \text{id}_V) + \rho(\text{id}_S \otimes \rho)(\Delta_S \otimes \text{id}_V) = 0. \quad (33)
\]

**4.1 Representations as (weak) homomorphisms**

We prove the following version of ([8], Theorem 5.2).

**Theorem 45.** There is a one-to-one correspondence between representations of $L$ on $V$ and pairs $(\vartheta, f)$, where $\vartheta$ is a differential on $V$ and $f: S(L[1]) \to S(\mathfrak{gl}(V)[1])$ a homomorphism of $L_\infty$-algebras. Here, $\mathfrak{gl}(V)$ carries the DGLA structure induced by $\vartheta$, see Example 12.

One should therefore really think of $L$ being represented on a DG vector space. The following lemma characterises $L_\infty$-algebra homomorphisms into DGLAs and is a symmetric version of ([8], Definition 5.2).

**Lemma 46.** Let $(L', l'_2, l'_3)$ be a DGLA and $\lambda'_1 = \lambda_1^1 + \lambda_2^2$ be the corresponding linear degree one map $\mathfrak{S}(L'[1]) \to L'[1]$. For $f: S(L[1]) \to L'[1]$ a linear degree preserving map, the induced homomorphism of coalgebras $\tilde{f}: S(L[1]) \to \mathfrak{S}(L'[1])$ is a homomorphism of $L_\infty$-algebras if and only if

\[
f \circ \Delta = \lambda'_1 \circ f + \frac{1}{2} \lambda'_2(f \otimes f) \Delta_S. \quad (34)
\]

This is the case if and only if the linear degree one map $\rho: S(L[1]) \to L'$ defined by $f(x) = (-1)^{|x|+1} \rho(x)$ satisfies

\[
\rho \circ \Delta + l'_1 \circ \rho + \frac{1}{2} l'_2(\rho \otimes \rho) \Delta_S = 0. \quad (35)
\]

**Proof.** The first part follows immediately from Remark 39 and the explicit construction of $f$ (see Proposition 18). It is straightforward to check that for $x \in S(L[1])$ homogeneous,

\[
f(d(x)) = (-1)^{|x|+1} \rho(d(x)),
\]

\[
\lambda'_1(f(x)) = (-1)^{|x|+1} l'_1(\rho(x)),
\]

\[
(\lambda'_2(f \otimes f) \Delta_S)(x) = (-1)^{|x|+1} l'_2(\rho \otimes \rho) \Delta_S(x),
\]

from which the second part then follows.

**Proof of Theorem 45:** As $\text{Hom}(S(L[1]) \otimes V, V) \cong \text{Hom}(S(L[1]), \mathfrak{gl}(V))$, a linear degree one map $\rho: S(L[1]) \otimes V \to V$ can be decomposed into linear degree one maps $\tilde{\rho}: S(L[1]) \to \mathfrak{gl}(V)$ and $\rho_0: k \to \mathfrak{gl}(V)$; the latter being equivalent to the choice of a degree one element $\partial = \rho_0(1_k) \in \mathfrak{gl}(V)$. If we show that under this identification $\rho$ satisfying (33) is equivalent to $\partial^2 = 0$ and $\tilde{\rho}$ satisfying (35), the assertion follows by Lemma 46. For $x \in S(L[1])$ homogeneous,

\[
\frac{1}{2} \left(\begin{array}{c} \rho \otimes \rho \end{array}\right) \Delta_S(x) = \frac{1}{2} \sum (-1)^{|x(1)|} [\tilde{\rho}(x(1)), \tilde{\rho}(x(2))] = \frac{1}{2} \sum (-1)^{|x(1)|} \rho(x(1)) \circ \tilde{\rho}(x(2)) + (-1)^{|x(1)|} \tilde{\rho}(x(1)) \circ x(2) + (-1)^{|x(1)|} x(2) \otimes x(1), \cdot
\]

\[
= \rho(\text{id}_S \otimes \rho)(\Delta_S(x), \cdot)
\]
by cocommutativity of $\mathcal{S}(L[1])$ and
\[
[\partial, \hat{p}(x)] = \partial \circ \hat{p}(x) - (\partial \hat{p}(x)) \circ \partial \\
= \rho_0(1) \circ \hat{p}(x) + (-1)^{|x|} \hat{p}(x) \circ \rho_0(1) \\
= \rho(\text{id}_S \otimes \rho)(1 \otimes x + x \otimes 1, \cdot).
\]
As $(\rho \circ d)(x) = \rho(d \otimes \text{id}_V)(x, \cdot)$, $\hat{p}$ satisfying (35) is equivalent to (33) holding on $\mathcal{S}(L[1]) \otimes V$. We also have
\[
(\rho(d \otimes \text{id}_V) + \rho(\text{id}_S \otimes \rho)(\Delta_S \otimes \text{id}_V))(1, \cdot) \\
= \rho(1, \rho(1, \cdot)) = \partial^2,
\]
which completes the proof as $\mathcal{S}(L[1]) = \mathbb{k} \oplus \mathcal{S}(L[1])$.

Example 47 (The trivial representation on a DG vector space). Let $(V, \partial)$ be a DG vector space. There is a strict homomorphism of $L_\infty$-algebras $0: L \to \mathfrak{gl}(V)$. The induced representation $\mathcal{S}(L[1]) \otimes V \to V$ is on $\mathbb{k} \otimes V \cong V$ given by $\partial$ and zero elsewhere and is called the trivial representation of $L$ on $V$. In particular, there is a trivial representation of $L$ on $\mathbb{k}$.

Remark 48. Let $\rho$ be a representation of $L$ on $V$ and $(\partial, f)$ as in Theorem 45.

(1) Then $-\partial^*$ is a differential on $V^*$ and the map $\mathfrak{gl}(V) \to \mathfrak{gl}(V^*)$, $g \mapsto -g^*$ is a homomorphism of DGLAs. By composing the corresponding weak homomorphism with $f$, we obtain an $L_\infty$-algebra homomorphism $\mathcal{S}(L[1]) \to \mathcal{S}(\mathfrak{gl}(V^*))[1]$. The induced representation is given by
\[
\rho^\vee: \mathcal{S}(L[1]) \otimes V^* \to V^*, \quad x \otimes \xi \mapsto -\rho(x, \cdot)^* \xi
\]
and is called the representation dual to $\rho$.

(2) Fix $n \in \mathbb{Z}$. Then $(-1)^n \partial^* \circ \partial \circ \uparrow^n$ is a differential on $V[n]$ and
\[
\mathfrak{gl}(V) \to \mathfrak{gl}(V[n]), \quad g \mapsto (-1)^n[g] \circ \rho \circ \uparrow^n
\]
is a DGLA homomorphism. The induced representation of $L$ on $V[n]$ is given by
\[
\mathcal{S}(L[1]) \otimes V[n] \to V[n], \quad x \otimes \uparrow^n v \mapsto (-1)^{n+n|x|} \rho(x, v).
\]

4.2 Representations as coderivations

Observe that the map $\Delta_V := \Delta_S \otimes \text{id}_V: \mathcal{S}(L[1]) \otimes V \to \mathcal{S}(L[1]) \otimes (\mathcal{S}(L[1]) \otimes V)$ satisfies
\[
(\Delta_S \otimes \text{id}_S \otimes \text{id}_V) \Delta_V = (\text{id}_S \otimes \Delta_V) \Delta_V,
\]
which makes $\mathcal{S}(L[1]) \otimes V$ into a left $\mathcal{S}(L[1])$-comodule.

Definition 49. Let $d' \in \text{Coder}(\mathcal{S}(L[1]))$ be of degree $p$. A coderivation of $\mathcal{S}(L[1]) \otimes V$ extending $d'$ is a linear map $D: \mathcal{S}(L[1]) \otimes V \to \mathcal{S}(L[1]) \otimes V$ of degree $p$ such that
\[
\Delta_V \circ D = (d' \otimes \text{id}_S \otimes V + \text{id}_S \otimes D) \Delta_V.
\]

Proposition 50 ([6], Proposition 1.5.3, p. 31). Let $d' \in \text{Coder}(\mathcal{S}(L[1]))$ be of degree $p$. There is a one-to-one correspondence between coderivations $D$ of $\mathcal{S}(L[1]) \otimes V$ extending $d'$ and linear maps $\rho: \mathcal{S}(L[1]) \otimes V \to V$ of degree $p$ given by
\[
D = d' \otimes \text{id}_V + (\text{id}_S \otimes \rho) \Delta_V, \\
\rho = \text{pr}_V \circ D,
\]
where $\text{pr}_V: \mathcal{S}(L[1]) \otimes V \to V$ is the projection of $\mathcal{S}(L[1]) \otimes V$ onto $\mathbb{k} \otimes V \cong V$.

Proof. Let $D$ be a coderivation of $\mathcal{S}(L[1]) \otimes V$ extending $d'$. As $(\text{id}_S \otimes \text{pr}_V)(\Delta_S \otimes \text{id}_V) = \text{id}_S \otimes \text{id}_V$, we obtain from (37) that
\[
D = (\text{id}_S \otimes \text{pr}_V) \Delta_V \circ D \\
= (\text{id}_S \otimes \text{pr}_V)(d' \otimes \text{id}_S \otimes \text{id}_V + \text{id}_S \otimes D) \Delta_V \\
= (d' \otimes \text{id}_V)(\text{id}_S \otimes \text{pr}_V) \Delta_V + (\text{id}_S \otimes (\text{pr}_V \circ D)) \Delta_V \\
= d' \otimes \text{id}_V + (\text{id}_S \otimes (\text{pr}_V \circ D) \Delta_V).
\]
This shows that $D$ is completely determined by $\text{pr}_V \circ D$.

Let conversely $\rho \in \text{Hom}(\mathcal{S}(L[1]) \otimes V, V)$ be of degree $p$. Using (36) and that $d'$ is a coderivation, we compute
\[
\Delta_V \circ (\text{id}_S \otimes \rho) \Delta_V = (\Delta_S \otimes \rho) \Delta_V \\
= (\text{id}_S \otimes \text{id}_S \otimes \rho) \Delta_V + (\text{id}_S \otimes D) \Delta_V, \\
\Delta_V \circ (d' \otimes \text{id}_V) = ((d' \otimes \text{id}_S + \text{id}_S \otimes d') \Delta_S) \otimes \text{id}_V \\
= (d' \otimes \text{id}_S \otimes \text{id}_V + \text{id}_S \otimes d' \otimes \text{id}_V) \Delta_V,
\]
which, combined, show that $D := d \otimes \text{id}_V + (\text{id}_S \otimes \rho) \Delta_V$ is a coderivation of $\mathcal{S}(L[1]) \otimes V$ extending $d$. It is easy to see that then $\text{pr}_V \circ D = \rho$, which completes the proof.

Corollary 51. There is a one-to-one correspondence between representations (up to homotopy) of $L$ on $V$ and coderivations $D: \mathcal{S}(L[1]) \otimes V \to \mathcal{S}(L[1]) \otimes V$ extending $d$ such that $D^2 = 0$.

Proof. It is a straightforward computation to check that $D^2 = \frac{1}{2}[D, D]$ is a coderivation of $\mathcal{S}(L[1]) \otimes V$ extending $\frac{1}{2}[d, d] = d^2 = 0$. By Proposition 50, $D^2 = 0$ if and only if $\rho = \text{pr}_V \circ D$ satisfies
\[
0 = \text{pr}_V \circ D^2 = \rho(d \otimes \text{id}_V) + \rho(\text{id}_S \otimes \rho)(\Delta_S \otimes \text{id}_V).
\]

4.3 A first approach to $L_\infty$-algebra cohomology

Assume now that the $L_\infty$-algebra $L$ is $\mathbb{Z}_{\geq 0}$-graded and of finite type and that $V$ is either finite-dimensional or of finite type and trivial in the negative degrees. We then have $\mathcal{S}(L[1])^* \cong \mathcal{S}(L[1]^*)^*$, $V \cong V^*$ and $(\mathcal{S}(L[1]) \otimes V^*)^* \cong \mathcal{S}(L[1]^*) \otimes V$. Let $d_{CE} = -d^*$ denote the differential on $\mathcal{S}(L[1]^*)$. The map
\[
\mathcal{S}(L[1]^*) \otimes (\mathcal{S}(L[1]^*) \otimes V) \to \mathcal{S}(L[1]^*) \otimes V, \\
(\xi \otimes (\eta \otimes v)) \mapsto (\xi \otimes \eta \otimes v)
\]
makes $\mathcal{S}(L[1]^*) \otimes V$ into a left $\mathcal{S}(L[1]^*)$-module. Similarly to Definition 49, we call a linear map $D_{CE}: \mathcal{S}(L[1]^*) \otimes$
V → S(L[1]*) ⊗ V of degree one a derivation of S(L[1]*) ⊗ V extending dCE if

\[ D_{CE} (ξ ⊗ (ν ⊗ ν)) = d_{CE} ξ ⊗ (ν ⊗ ν) + (-1)^{[ξ]} ξ ⊗ D_{CE} (ν ⊗ ν) \]

holds for all ξ, ν ∈ S(L[1]*), ν ∈ V homogeneous.

Note that a representation of L on V is equivalent to a representation on V^* by Remark 48 and V ∼= V^*. As the notion of a derivation extending dCE is dual to the one of a coderivation extending d, we get the following dualized version of Corollary 51.

**Proposition 52.** A representation ρ of L on V is equivalent to a derivation DCE: S(L[1]*) ⊗ V → S(L[1]*) ⊗ V extending dCE with DCE^2 = 0. Explicitly, we have DCE = −D^*, where D is the codereivation extending d induced by the dual representation ρ^*.

For a fixed representation ρ of L on V, we can then see S(L[1]*) ⊗ V as our generalized Chevalley–Eilenberg complex with coboundary operator DCE.

### 4.4 A dead-end

This not only provides us with an explicit construction of the coboundary operator from a given representation, but also gives it the additional structure of a derivation extending dCE. Unfortunately, this came at the cost of the finiteness assumptions we imposed on L on V at the beginning of Section 4.3. As our goal is to establish a generalisation of Theorem 32 – which does not need such assumptions – in terms of L∞-algebra cohomology, this is not the appropriate framework for our purposes. We can, however, make the following observation.

**Remark 53.** With our finiteness assumptions on L and V, we have S(L[1]*) ⊗ V ∼= Hom(S(L[1]), V), where ξ ⊗ v ∈ S(L[1]*) ⊗ V is identified with the linear map S(L[1]) → V, x → (−1)^{|ξ||v|} ξ(ν) · v. For f ∈ Hom(S(L[1]), V) homogeneous, one finds that DCEf is then given by

\[ D_{CE} f = ρ(id_S ⊗ f) Δ_S - (-1)^{|f|} f ⊙ d. \]  

(38)

One could then simply define DCE: Hom(S(L[1]), V) → Hom(S(L[1]), V) by (38), even if L and V do not meet our finiteness assumptions. Although there is a priori no reason for DCE^2 = 0 to hold in the general case, a straightforward computation shows that it actually does. While this leaves us with nothing but the formula (38) to work with, it also suggests that there should be another approach to L∞-algebra cohomology that gets by without the need of finiteness assumptions.

In [4], the L∞-algebra cohomology with values in the adjoint representation was introduced in terms of the commutator bracket of coderivations and the isomorphism Coder(S(L[1]), S(L[1])) ∼= Hom(S(L[1]), L[1]). In the next section, we extend this approach to arbitrary representations, which leads to a generalisation of Theorem 32 in a rather natural way.

### 5 L∞-algebra cohomology

#### 5.1 The Lie bracket on Hom(S(L[1] ⊕ V), L[1] ⊕ V)

Recall from Proposition 21 that Coder(S(L[1])) is closed under the graded commutator. Together with Theorem 22, this induces a Lie bracket on Hom(S(L[1]), L[1]). Its explicit formula is

\[ [f, g] = f ◦ μ_S (g ◦ id_S) Δ_S - (-1)^{|f||g|} g ◦ μ_S (f ◦ id_S) Δ_S \]  

(39)

for f, g ∈ Hom(S(L[1]), L[1]) homogeneous.

As L∞-structures correspond to coderivations with d(1) = 0 and elements in Hom(S(L[1]), L[1]), it is only natural to restrict ourselves to the Lie subalgebra Hom(S(L[1]), L[1]). Keeping the Hom(k, L[1]) part corresponds to the framework of curved L∞-algebras, which are L∞-algebras that also allow for a 0-ary bracket k → L[1].

**Remark 54.** The same construction also makes Hom(S(L[1] ⊕ V), L[1] ⊕ V) into a graded Lie algebra. The decomposition S(L[1] ⊕ V) ∼= S(L[1]) ⊕ S(V) implies that

\[ \overline{S}(L[1] ⊕ V) ≃ \overline{S}(L[1]) ⊕ \overline{S}(V) ⊕ \overline{S}(L[1]) ⊕ \overline{S}(V). \]  

(40)

We can then consider spaces like Hom(\overline{S}(L[1] ⊕ V), L[1] ⊕ V) and Hom(\overline{S}(L[1]) ⊕ V, L[1] ⊕ V) as subspaces of Hom(\overline{S}(L[1] ⊕ V), L[1] ⊕ V) in the obvious way. The inclusion of Hom(\overline{S}(L[1]), L[1]) into Hom(\overline{S}(L[1] ⊕ V), L[1] ⊕ V) is then easily seen to preserve the Lie bracket.

**Remark 55.** In terms of the Lie bracket on Hom(S(L[1]), L[1]), the condition (30) for a linear map λ: \overline{S}(L[1]) → L[1] of degree one to define an L∞-algebra structure on L[1] becomes

\[ \frac{1}{2} [λ, λ] = 0. \]  

(41)

By Example 12 and Remark 54, this makes Hom(\overline{S}(L[1] ⊕ V), L[1] ⊕ V) into a DGLA. Solutions of the Maurer–Cartan equation then induce new L∞-structures on L[1] by Example 14.

By abuse of notation, we now denote the (co)products on S(L[1]) and S(L[1] ⊕ V) both by μ_S and Δ_S. This is justified, as they coincide on S(L[1]) ⊂ S(L[1] ⊕ V).

In (38), d = μ_S(λ ⊗ id_S) Δ_S and μ_S(id_S ⊗ f) Δ_S = μ_S(f ⊗ id_S) Δ_S due to S(L[1]) being (co)commutative. The similarity between (38) and (39) suggests to approach L∞-algebra cohomology using the Lie bracket on Hom(\overline{S}(L[1] ⊕ V), L[1] ⊕ V).

**Proposition 56.** Let ρ ∈ Hom(S(L[1]) ⊕ V, V) be of degree one. Then ρ is a representation of L on V if and only if

\[ ρ ◦ μ_S (ρ ⊗ id_S) Δ_S + ρ ◦ μ_S (λ ⊗ id_S) Δ_S = 0, \]  

(42)

where λ and ρ are considered as elements of Hom(\overline{S}(L[1] ⊕ V), L[1] ⊕ V).
Proof. Note that \( \rho \circ \mu_S(\rho \otimes \text{id}_S)\Delta_S \text{ and } \rho \circ \mu_S(\rho \otimes \text{id}_S)\Delta_S \) are only possibly nonzero on \( S(L[1]) \otimes V \). For \( x_1, \ldots, x_{n-1} \in L[1] \) and \( x_n \in V \), a routine computation using Lemma 24 shows that
\[
(\rho \circ \mu_S(\lambda \otimes \text{id}_S)\Delta_S)(x_1 \otimes \ldots \otimes x_n) = \rho(d(x_1 \otimes \ldots \otimes x_{n-1}), x_n),
(\rho \circ \mu_S(\lambda \otimes \text{id}_S)\Delta_S)(x_1 \otimes \ldots \otimes x_n) = \rho(\text{id}_S \otimes \rho)(\Delta_S(x_1 \otimes \ldots \otimes x_{n-1}), x_n).
\]
As again \( \mu_S(\rho \otimes \text{id}_S)\Delta_S = \mu_S(\text{id}_S \otimes \rho)\Delta_S \) by (co)commutativity of \( \mathcal{S}(L[1] \oplus V) \), \( \rho \) satisfies (33) if and only if it satisfies (42).

Corollary 57. An element \( \rho \in \text{Hom}(S(L[1]) \otimes V) \) of degree one is representation of \( L \) on \( V \) if and only if \( (L[1] \oplus V, \lambda + \rho) \) is an \( L_\infty \)-algebra.

Proof. We have
\[
\frac{1}{2}[\lambda + \rho, \lambda + \rho] = \frac{1}{2}[\lambda, \lambda] + [\lambda, \rho] + \frac{1}{2}[\rho, \rho] = \rho \circ \mu_S(\lambda \otimes \text{id}_S)\Delta_S + \rho \circ \mu_S(\rho \otimes \text{id}_S)\Delta_S.
\]

Corollary 58. The subspace \( \text{Hom}(S(L[1]) \otimes V) \) is invariant under the Lie bracket \([\cdot, \cdot]\) and the differential \([\cdot, \cdot]\). Representations (up to homotopy) of \( L \) on \( V \) are then exactly the Maurer–Cartan elements in \( \text{Hom}(S(L[1]) \otimes V) \).

By applying Proposition 33 to Corollary 57 and using that a representation on \( V \) is equivalent to one on \( V[1] \) by Remark 48, we obtain the following.

Proposition 59. A representation of \( L \) on \( V \) is equivalent to a system of linear maps \( \rho_k : \bigwedge^{k-1} L \otimes V \to V \) of degree \( 2-k \) for \( k \geq 1 \) such that \( \{k + \rho_k : \bigwedge^{k-1} L \otimes V \to L \otimes V | 1 \leq k < \infty \} \) is an \( L_\infty \)-structure on \( L \otimes V \).

Remark 60. It is easy to see that the generalized Jacobi identity (27) for \( \{k + \rho_k | 1 \leq k < \infty \} \) has only to be checked on \( \bigwedge^{n-1} L \otimes V \) for each \( n \geq 1 \). Representations of \( L_\infty \)-algebras are often defined in terms of these equations, see for example (8), Definition 5.1 and (3), Definition 18. Similarly, equation (42) on \( S(L[1]) \otimes V \) is easily seen to be the condition imposed on \( \rho \) in (3), Definition 19.

For a fixed representation \( \rho \) of \( L \) on \( V \), \( [\lambda + \rho, \cdot] \) makes \( \text{Hom}(\mathcal{S}(L[1] \otimes V), L[1] \otimes V) \) into a DGLA. The space \( \text{Hom}(\mathcal{S}(L[1]), V) \) is then an abelian Lie subalgebra that is invariant under \([\lambda + \rho, \cdot]\). Explicitly, we have for \( f \in \text{Hom}(\mathcal{S}(L[1]), V) \) homogeneous
\[
[\lambda + \rho, f] = \rho(\text{id}_S \otimes f)\Delta_S - (-1)^{|f|}f \circ d. \tag{43}
\]

Definition 61. The map \( \delta := [\lambda + \rho, \cdot] : \text{Hom}(\mathcal{S}(L[1]), V) \to \text{Hom}(\mathcal{S}(L[1]), V) \) is called the \( L_\infty \)-coboundary operator. The cohomology of the cochain complex \( (\text{Hom}(\mathcal{S}(L[1]), V), \delta) \) is called the \( L_\infty \)-algebra cohomology with values in \( V \).

Remark 62. For \( L \) and \( V \) as in Section 4.3, we clearly have \( \delta = D_{CE} \). If \( L = \mathfrak{g} \) and \( V \) are concentrated in degree zero, the décalage isomorphism (5) implies that
\[
\text{Hom}_p(S(\mathfrak{g}[1]), V) \cong \prod_{n \geq 1} \text{Hom}_{p-n}(\bigwedge^n S(\mathfrak{g}), V)
\]
for all \( p \geq 1 \). This way, we recover the usual Lie algebra cohomology.

Example 63 (The adjoint representation). The adjoint representation of \( L \) on \( L[1] \) is given by \( S(L[1]) \otimes L[1] \to L[1], x \otimes y \mapsto \lambda(x \otimes y) \). While there are now two distinct copies of \( L[1] \) involved, it is evident by (43) that \( \delta = [\lambda, \cdot] \), the bracket being the one on \( \text{Hom}(\mathcal{S}(L[1]), L[1]) \). This is the case discussed in [4].

5.2 \( L_\infty \)-structures induced by 2-cocycles

The description of \( L_\infty \)-structures, representations (up to homotopy) and the \( L_\infty \)-coboundary operator all by the same Lie bracket yields the following generalisation of Theorem 32.

Theorem 64. Let \( L \) and \( V \) be graded vector spaces and \( \lambda \in \text{Hom}(\mathcal{S}(L[1]), L[1]), \rho \in \text{Hom}(\mathcal{S}(L[1]) \otimes V, V) \) and \( \omega \in \text{Hom}(\mathcal{S}(L[1]), V) \) be all of degree one. Then \( (L[1] \otimes V, \lambda + \rho + \omega) \) is an \( L_\infty \)-algebra if and only if \( (L[1], \lambda) \) is an \( L_\infty \)-algebra, \( \rho \) is a representation of \( L \) on \( V \) and \( \omega \) is a \( V \)-valued cocycle.

Proof. The map \( \frac{1}{2}[\lambda + \rho + \omega, \lambda + \rho + \omega] = \frac{1}{2}[\lambda, \lambda] + [\lambda, \rho] + \frac{1}{2}[\rho, \rho] + [\lambda + \rho, \omega] \) decomposes itself into linear maps
\[
\frac{1}{2}[\lambda, \rho] : S(L[1]) \to L[1],
[\lambda, \rho] + \frac{1}{2}[\rho, \rho] : S(L[1]) \otimes V \to V,
[\lambda + \rho, \omega] : S(L[1]) \to V.
\]

The assertion then follows from Remark 55, Corollary 58 and the definition of \( \delta \).

In terms of antisymmetric brackets, Theorem 64 characterises \( L_\infty \)-structures on \( L \oplus V \) in which for each \( n \in \mathbb{N} \), the \( n \)-ary bracket decomposes into linear maps
\[
\bigwedge^n L \to L,
\bigwedge^{n-1} L \otimes V \to V,
\bigwedge^n L \to V.
\]

These then correspond to cocycles in \( \text{Hom}_1(S(L[1]), V[1]) \cong \text{Hom}_2(S(L[1]), V) \). So, it is the 2-cocycles that characterise these \( L_\infty \)-structures, as in the Lie algebra case (cf. [9], Proposition 7.5.18, p. 202).
5.3 Extensions of $L_\infty$-algebras

We conclude with a brief discussion of extensions of $L_\infty$-algebras. This puts some constructions we discussed in context. The notions are completely analogous to the Lie algebra case, see for example ([9], Sections 5.1.3 and 7.5.2).

A graded subspace $I \subset L$ of an $L_\infty$-algebra $(L[1],\lambda)$ is called an ideal if $\lambda(x \vee y) \in I[1]$ for all $x \in I[1]$ and $y \in S(L[1])$. Then $L/I$ carries a canonical $L_\infty$-structure such that the projection $L \to L/I$ is a strict homomorphism of $L_\infty$-algebras. An ideal $I \subset L$ is always an $L_\infty$-subalgebra as in particular $\lambda(x) \in I[1]$ for all $x \in S(I[1])$.

Definition 65. An extension of an $L_\infty$-algebra $(L[1],\lambda_1)$ by another $L_\infty$-algebra $(L_2[1],\lambda_2)$ is an exact sequence of $L_\infty$-algebras and strict homomorphisms

$$0 \to L_2 \to L \xrightarrow{\lambda} L_1 \to 0. \quad (44)$$

Given such an exact sequence (44), the graded subspace $L_2 \cong \ker(\lambda) \subset L$ is an ideal and $\lambda$ induces a strict isomorphism $L/L_2 \cong L_1$ of $L_\infty$-algebras.

We then always have $L \cong L_1 \oplus L_2$ (non-canonically) as graded vector spaces, so we are essentially concerned with $L_\infty$-structures on $L_1 \oplus L_2$ such that the canonical maps $L_2 \to L_1 \oplus L_2$ and $L_1 \oplus L_2 \to L_1$ are strict $L_\infty$-algebra homomorphisms. With the decomposition (40), we can decompose such an $L_\infty$-structure $\lambda: S((L_1 \oplus L_2)[1]) \to (L_1 \oplus L_2)[1]$ into linear degree one maps

$$\lambda_1: S(L_1[1]) \to L_1[1], \quad \omega: S(L_1[1]) \to L_2[1], \quad 0: S(L_2[1]) \to L_1[1], \quad \lambda_2: S(L_2[1]) \to L_2[1],$$

$$0: S(L_1[1]) \otimes S(L_2[1]) \to L_1[1], \quad \lambda_m: S(L_1[1]) \otimes S(L_2[1]) \to L_2[1].$$

5.3.1 Abelian and central extensions

An $L_\infty$-algebra $L$ is called abelian if only its 1-ary bracket is nontrivial. An abelian $L_\infty$-algebra is then nothing else than a DG vector space.

An $L_\infty$-algebra extension $L_2 \to L \to L_1$ is called abelian if $L_2$ is abelian. The $L_\infty$-structures constructed in Theorem 64 are examples of abelian extensions of $L$ by $V$.

Similarly, an extension $L_2 \to L \to L_1$ is called central if $\lambda(x \vee y) = 0$ for $x \in L_2[1], y \in S(L[1])$. It is immediate that this is the case if and only if $L_2$ is abelian and $\lambda_m = 0$.

For abelian $L_2$, the central extensions $L_2 \to L_1 \oplus L_2 \to L_1$ are by Theorem 64 characterised by 2-cocycles of $L_1$ with values in the trivial representation of $L_1$ on $L_2$.

5.3.2 Semidirect sums

An $L_\infty$-algebra $((L_1 \oplus L_2)[1],\lambda)$ is said to be a semidirect sum of the $L_\infty$-algebras $(L_1[1],\lambda_1)$ and $(L_2[1],\lambda_2)$ if the canonical sequence $L_2 \to L_1 \oplus L_2 \to L_1$ is an $L_\infty$-algebra extension and if the canonical map $L_1 \to L_1 \oplus L_2$ is a strict homomorphism of $L_\infty$-algebras. This is clearly the case if and only if $\omega = 0$ in the decomposition above. A semidirect sum of $L_1$ and $L_2$ is therefore characterised by $\lambda_m$. Note that $L_1 \subset L_1 \oplus L_2$ is an ideal if and only if $\lambda_m = 0$. In this case, $L_1 \oplus L_2$ carries the $L_\infty$-structure $\lambda_1 + \lambda_2$ and is called the direct sum of $L_1$ and $L_2$.

For an arbitrary $\lambda_m \in \text{Hom}_2(S(L_1[1]) \otimes S(L_2[1]), L_2[1])$, the condition for $\lambda_1 + \lambda_2 + \lambda_m$ to define an $L_\infty$-structure on $L_1 \oplus L_2$ becomes

$$[\lambda_1 + \lambda_2, \lambda_m] + \frac{1}{2} [\lambda_m, \lambda_m] = 0. \quad (45)$$

The isomorphism $\text{Hom}_2(S(L_1[1]) \otimes S(L_2[1]), L_2[1]) \cong \text{Hom}_2(S(L_1[1]), \text{Hom}_2(S(L_2[1]), L_2[1]))$ allows for the following characterisation of semidirect sums.

Theorem 66. Let $\lambda_m \in \text{Hom}_2(S(L_1[1]) \otimes S(L_2[1]), L_2[1])$ be of degree one. Then $\lambda_m$ satisfies (45) if and only if the corresponding linear degree one map $\delta: S(L_1[1]) \to \text{Hom}_2(S(L_2[1]), L_2[1])$ is a weak homomorphism of $L_\infty$-algebras in the sense that it satisfies (35).

Proof. Note that $\text{Hom}_2(S(L_1[1]) \otimes S(L_2[1]), L_2[1])$ is closed under $[\cdot, \cdot]$ and $[\lambda_1 + \lambda_2, \cdot]$. Therefore, (45) has only to be checked on $S(L_1[1]) \otimes S(L_2[1])$. Let $d_1$ and $d_2$ denote the codifferentials on $S(L_1[1])$ and $S(L_2[1])$, respectively. For $x \in S(L_1[1])$ and $y \in S(L_2[1])$, we then compute

See this equation above

and $[\lambda_1, \lambda_m](x \vee y) = \lambda_m(d_1(x) \vee y) = (\delta \circ d_1)(x)(y)$.

Example 67. The $L_\infty$-structure on $L \oplus V$ induced by a representation of $L$ on $V$ is a semidirect sum. For
compliance with Theorem 66, note that $gl(L_2[1]) \subset \Hom(S(L_2[1]), L_2[1])$ is a Lie subalgebra.

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