Perturbations of Spatially Closed Bianchi III Spacetimes

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Abstract
Motivated by the recent interest in dynamical properties of topologically non-trivial spacetimes, we study linear perturbations of spatially closed Bianchi III vacuum spacetimes, whose spatial topology is the direct product of a higher genus surface and the circle. We first develop necessary mode functions, vectors, and tensors, and then perform separations of (perturbation) variables. The perturbation equations decouple in a way that is similar to but a generalization of those of the Regge–Wheeler spherically symmetric case. We further achieve a decoupling of each set of perturbation equations into gauge-dependent and independent parts, by which we obtain wave equations for the gauge-invariant variables. We then discuss choices of gauge and stability properties. Details of the compactification of Bianchi III manifolds and spacetimes are presented in an appendix. In the other appendices we study scalar field and electromagnetic equations on the same background to compare asymptotic properties.

1 Introduction
Astronomical observations strongly support the picture of an isotropic and homogeneous universe which is expanding at such a rate, relative to its energy density, that it will continue to expand forever. Isotropy and homogeneity must be interpreted as valid only when the universe is viewed in a coarse-grained sense — on a sufficiently large scale that fine details such as clusters of galaxies are blended into a uniform background. For this reason most studies of cosmological perturbations have begun with a homogeneous and isotropic background and thus with a Friedman-Robertson-Walker (FRW) metric to perturb.

There are however several logical limitations to this point of view. First of all since one cannot really observe the entire universe it is perfectly conceivable that the actual universe

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is only locally but not globally homogeneous and isotropic. For example the negatively curved $k = -1$ FRW models can be spatially compactified in infinitely many ways to yield spacetimes which, though indistinguishable from the conventional FRW models on sufficiently small scales (which might however exceed the observable universe in spatial extent) are nevertheless inhomogeneous in the global sense. In particular, at a given instant of cosmic time, the length of the shortest closed spacelike geodesic through a given point (roughly the shortest distance around the universe from there) would depend upon the location of that point in a compactified model. This is a clear (though perhaps unobservable) violation of global homogeneity. Furthermore since astronomical observations are limited to (past) lightlike directions for any given observer, any hypothetical breakdown of global homogeneity might be unobservable, even in principle, at a given epoch of cosmic time but became observable at a later epoch.

Both classes of “open” FRW models (the $k = -1$ hyperbolic and the $k = 0$ flat models) are amenable to spatial compactification but the $k = 0$ models are somewhat unstable towards perturbation into locally homogeneous but non-isotropic, Kasner-like behavior in which the effective Hubble constant would be directionally dependent and thus (in principle at least) observable from any vantage point in the universe. By contrast, the compactified $k = -1$ models are known to be stable in the direction of cosmological expansion [6] and thus do not seem to require an excessive “fine-tuning” of the initial conditions (or perhaps an inflationary scenario) to protect them from the development of non-isotropic expansion.

An even more radical departure from the conventional FRW point of view is conceivable however and even suggested by recent results and conjectures on possible 3-manifold topologies and recent insights into the dynamics of expanding cosmological models implied by Einstein’s equations. Aside from the compact hyperbolic manifolds mentioned above which can support $k = -1$ locally (but not globally) homogeneous and isotropic metrics there is a more general class of prime compact 3-manifolds that one can form by “glueing” finite volume hyperbolic manifolds across certain special surfaces (called “incompressible 2-tori”) to so-called graph manifolds of infinite fundamental group. This set of graph manifolds includes for example the trivial and non-trivial circle bundles over compact higher genus surfaces. This glueing construction (which is distinct from the connected sum operation in which the glueing occurs across embedded spheres instead of tori) leads to a class of manifolds which no longer admit even an everywhere locally homogeneous and isotropic metric since the graph manifolds pieces are incompatible with the extension of such a structure from the hyperbolic pieces.

At first sight it would seem that such manifolds are ruled out, as models for the actual universe, by the most basic of astronomical observations. However some recent studies of the so-called reduced Hamiltonian for Einstein’s equations and their implications for cosmological dynamics suggest a different point of view. They support the scenario that, under cosmological expansion, the spatial metric (after rescaling by the cube of the mean curvature to take out the overall effect of expansion) converges to the hyperbolic metric on the hyperbolic components of the manifold but collapses the graph manifold components to zero rescaled volume asymptotically. This picture is suggested by the known behavior of the reduced Hamiltonian (and its quasi-local variant) which monotonically decreases for every non-selfsimilar solution of the field equations and whose value at any instant (in
a constant mean curvature or CMC time slicing) is the rescaled volume at that instant. The infimum of the reduced Hamiltonian is known to be zero for a pure graph manifold of the type we are considering and the asymptotic vanishing of this Hamiltonian corresponds to the Cheeger–Gromov style collapse of the rescaled volume mentioned above. This total “collapse of the graphs” is seen to occur for the explicitly known, spatially compactifiable Bianchi vacuum spacetimes of types I, II, III, VI$_0$, and VIII [5] which are all graph manifolds of the type we want. By contrast the infimum of the reduced Hamiltonian for compact hyperbolic manifolds is conjectured (through its direct connection to the topological invariant known as the $\sigma$-constant) to be achieved by convergence to the hyperbolic metric and it is known that this canonical metric provides at least an isolated local minimum to the reduced Hamiltonian [1, 2, 3, 4].

Thus one is led to consider a model of the universe in which the spatial topology is a composite of hyperbolic and graph manifold pieces but for which the Einsteinien dynamics predicts that the spatial volume is asymptotically dominated by the (locally homogeneous and isotropic) hyperbolic components. Whether a physically plausible cosmological model could be developed along these lines (e.g., whether structure formation could be largely confined to the hyperbolic components) is not known but, from the mathematical point of view, it is worth noting that such manifolds which are composite with respect to the torus decomposition described above can still be prime (i.e., indecomposible into more fundamental prime manifolds through the sphere decomposition). As such they are still among the most basic building blocks of compact 3-manifolds and indeed, taken together with the special case of pure compact hyperbolic manifolds, form perhaps the largest distinguished subset of such prime manifolds. Together with the (pure) graph manifolds of infinite fundamental group they are conjectured to exhaust the family of so-called prime, $K(\pi, 1)$ manifolds. If current conjectures (including the still unproven Poincaré conjecture) on 3-manifold topology are correct the most general compact, connected, orientable 3-manifold is constructable as the connect sum of a finite number of $K(\pi, 1)$ factors, a finite number of $S^2 \times S^1$ handles and a finite number of spherical space forms (i.e., quotients of $S^3$ by finite subgroups of SO(4) which act freely and properly discontinuously on $S^3$).

Of course there is no known exact solution of Einstein’s equations on any of these more exotic torus decomposable or sphere decomposable manifolds. However there are known solutions available for many of the constituents of such composite manifolds. In particular, there are known Bianchi solutions, both vacuum and fluid filled, for most of the graph manifold examples mentioned above. Since these could conceivably play an important role in the structure and evolution of a plausible universe model we are motivated to study their stability properties for much the same reason that people have studied the stability properties of the FRW models.

Going beyond the FRW models and the Bianchi I, V and IX families which include such models the simplest manifolds to consider would seem to be those of the Bianchi III (or Thurston’s $H^2 \times \mathbb{R}$) type which can be compactified to trivial $S^1$-bundles over higher genus surfaces. Starting with a family of explicitly known vacuum solutions we here develop the tensor harmonics needed for the separation of the spatial coordinates on such backgrounds and carry out this separation explicitly for the linearized Einstein equations as well as for the Maxwell and massless scalar wave equations. These harmonics have much
in common with those developed by Regge and Wheeler [18] for the study of spherically symmetric backgrounds (such as Schwarzschild or Nariai-Kantowski-Sachs solutions) but with eigenfunctions of the scalar Laplacian on the higher genus surface playing the role of the ordinary spherical harmonics on $S^2$ in the Regge–Wheeler analysis. For completeness though we develop also the harmonics based on the transverse-traceless tensors admitted by the higher genus surfaces as well as those based on the (Hodge-) harmonic one-forms allowed by such surfaces. Since the first Betti number of $S^2$ is vanishing and since $S^2$ has a trivial Teichmüller space these latter classes of tensor harmonics had no analogue in the Regge–Wheeler analysis. Furthermore even the scalar harmonics vary with the Teichmüller parameters determining the higher genus surface and thus depend upon the particular compactification of $H^2$ chosen in the construction. As we shall see the asymptotic decay properties of the gravitational, electromagnetic or scalar field perturbations can vary significantly with the Teichmüller parameters of the chosen compact surface.

To obtain a further simplification of the linearized field equations, we also carry out a decomposition of the metric perturbations into gauge invariant, gauge dependent and constrained variables, the last two subsets being essentially canonically conjugate to each other. We then derive the decoupled wave equations satisfied by the various gauge invariant perturbation modes and discuss the behavior of their solutions on the chosen backgrounds. By choosing a particular gauge for the remaining, unconstrained variables we show how the full spacetime metrics of the perturbed models can be expressed in terms of the solutions of the gauge invariant wave equations.

Our analysis lays the groundwork for the study of a number of physically interesting questions about these models. For example do the perturbed models decay in some natural, invariant sense towards local homogeneity in the asymptotic regime or do the perturbations get driven away from asymptotic homogeneity by the Cheeger–Gromov “collapse of graph” effect of the background? Does the reduced Hamiltonian for the perturbed solutions asymptotically achieve the infimum attained by the background solutions (corresponding to the vanishing of the $\sigma$-constant for these manifolds) or does it stay bounded away from its infimum for a generic perturbations? Can the methods developed here be generalized to treat the perturbations of Bianchi II (Thurston Nil), Bianchi VI$_0$ (Thurston Sol), and Bianchi VIII (Thurston $\tilde{S}L(2, \mathbb{R})$) backgrounds and, if so, can one answer for those models the same questions raised above for the Bianchi III (Thurston $H^2 \times \mathbb{R}$) models studied here? Finally can one recover the essential features of the asymptotic behavior of these models from energy arguments for the linearized field equations, which avoid the need for an explicit separation of variables and, if so, can one develop these into energy arguments for the fully non-linear equations in the case of sufficiently small perturbations? These are some of the issues we hope to study in the future.

The plan of this paper is as follows. In the next section we briefly describe the background spacetime we consider and set up the background (canonical) variables. The background spacetime is a generalization of the usual Bianchi III model that is spatially compactified. Details of the compactification are discussed in Appendix A. In Sec. 3 we construct mode functions, vectors, and tensors on the spatial manifold. Using these in the subsequent section we perform separations of (perturbation) variables and write down the
It is claimed that we have four kinds of perturbations. The subsequent four sections are devoted to detailed analysis of them, which includes decoupling of each set of perturbation equations into gauge-dependent and independent parts, and deriving the wave equation for the gauge-invariant variable. In Sects. 9 and 10 we discuss choices of gauge. The last section is devoted to a summary with some discussions of the stability properties. In Appendices B and C we study scalar field and electromagnetic equations on the same background. All the appendices deal with subjects that have not appeared elsewhere and are substantial parts of this paper.

We employ the abstract index notation \[23\] and use leading Latin letters \(a, b, \ldots\) to denote abstract indices for vectors and tensors. When useful however we also write a vector or tensor without abstract indices, as long as no confusion occurs. Greek indices \(\mu, \nu, \ldots\) are used to denote spacetime coordinates, running 0 to 3. Middle Latin letters \(i, j, \ldots\) are used for general purposes, e.g. for equations, variables, or generators for infinite groups, as well as for spatial coordinates, running 1 to 3. Capital Latin letters \(I, J, \ldots\) are used to label an invariant basis for a Bianchi group, and run from 1 to 3.

2 The background

Our spatial manifold \(M\) is assumed to be the direct product \(M \simeq \Sigma_g \times S^1\), where \(\Sigma_g\) is the closed surface with genus \(g \geq 2\), and \(S^1\) the circle. The geometry is of Bianchi type III (or \(H^2 \times \mathbf{R}\) in Thurston’s terminology). Let \(\tilde{M} \simeq \mathbf{R}^3\) be the universal cover of \(M\). We can define on \(\tilde{M}\) the invariant 1-forms \(\sigma^I_a\) \((I = 1 \sim 3)\) of Bianchi III. Also, we can define their duals (invariant vectors) \(\chi^I_a\): \(\sigma^I_a \chi^J_a = \delta^I_J\). In local coordinates they are expressed as

\[
\sigma^1 = dx/y, \quad \sigma^2 = dy/y, \quad \sigma^3 = dz, \tag{1}
\]

and

\[
\chi_1 = y\partial_x, \quad \chi_2 = y\partial_y, \quad \chi_3 = \partial_z. \tag{2}
\]

Since \(M\) is compact, the spatial metric \(q_{ab}\) on \(M\) should be locally rotationally symmetric (LRS) to have \(H^2\) symmetry (otherwise we cannot make \(\tilde{M}\) compact by discrete actions of isometry). The background spatial metric is therefore:

\[
q_{ab} = q_1 h_{ab} + q_2 l_{ab}, \quad \text{(the background metric)} \tag{3}
\]

where

\[
h_{ab} = \sigma^1_a \sigma^1_b + \sigma^2_a \sigma^2_b \quad \text{(the standard metric on } H^2\text{)}, \tag{4}
\]

\[
l_{ab} = \sigma^3_a \sigma^3_b \quad \text{(the standard metric on } \mathbf{R}\text{)}, \tag{5}
\]

and the scale factors \(q_i (i = 1, 2)\) are constants on \(M\), and will be functions of time \(t\). We may view the metric \(q_{ab}\) as one on the closed manifold \(M\), as well as one on the universal cover \(\tilde{M}\). We understand that an appropriate set of identifications of points is specified on \(\tilde{M}\) when \(q_{ab}\) is viewed as a metric on the closed manifold \(M\). Similarly, we may view \(h_{ab}\) as a metric on \(\Sigma_g\) as well as one on \(H^2\), and view \(l_{ab}\) as a metric on \(S^1\) as well as one on \(\mathbf{R}\).
on $\mathbb{R}$. (For details of the compactification, see Appendix A, where however we explicitly distinguish metrics on universal covers from ones on compactified manifolds by writing tilde on the metrics of the universal covers.)

We should note that $\sigma^1$ and $\sigma^2$ (or $\chi_1$ and $\chi_2$) cannot be defined on the compact quotient $M$ by themselves, since they are not invariant under the isometries of $H^2$. However, they will always appear in a combined form as in the metric \(\tilde{H}^2\) or in the area 2-form $d\mu_h = \sigma^1 \wedge \sigma^2$, which are both well defined on $M$. On the other hand, $\sigma^3$ and $\chi_3$ are well defined on both $M$ and $\tilde{M}$ by themselves. Note that $\chi_3$ is the natural $S^1$-fiber generator when viewing $M$ as a $S^1$-bundle over $\Sigma_g$. We regard $\chi_3 = \partial_z$ not only as a vector but as an explicit differential operator acting on the functions on $M$ (or $S^1$), i.e.,

$$\chi_3 : C^\infty \to C^\infty.$$  \hspace{1cm} (6)

We also note that in our convention, the scalar curvature for the hyperbolic metric $h_{ab}$ gives $R_h = -2$ (in particular, not normalized to give $-1$).

We think of $q_1$ and $q_2$ as the configuration variables in the background phase space. To find the conjugate momentum $\pi^{ab}$, let $\mu_{q_0}$ be the determinant of the standard (i.e., time-independent) metric $q_{0ab} = h_{ab} + l_{ab}$ with respect to the coordinates $(x, y, z)$. For later convenience we also define three-forms $d\mu_{q_0}$ and $d\omega$ by

$$d\mu_{q_0} \equiv \mu_{q_0} d\omega = \sigma^1 \wedge \sigma^2 \wedge \sigma^3.$$  \hspace{1cm} (7)

The determinant $\mu_q$ of the metric $q_{ab}$ is also defined similarly. The two determinants are related with $\mu_q = q_1 \sqrt{\mu_{q_0}}$. Now, it is easy to see that the conjugate momentum can be written in the form

$$\pi^{ab} = \mu_{q_0}(\frac{\pi_1}{2} h^{ab} + \pi_2 l^{ab}),$$  \hspace{1cm} (the conjugate momentum) \hspace{1cm} (8)

where $h^{ab} = \chi_1 a \chi_1 b + \chi_2 a \chi_2 b$ and $l^{ab} = \chi_3 a \chi_3 b$ are the the inverses of $h_{ab}$ and $l_{ab}$, respectively. In fact, the functions of time $\pi_1$ and $\pi_2$ are chosen so that they are canonically conjugate to $q_1$ and $q_2$, as seen from

$$\Theta_0 \equiv \frac{1}{C_M} \int_M d^3 x \pi^{ab} q_{ab} = \pi_1 \dot{q}_1 + \pi_2 \dot{q}_2,$$  \hspace{1cm} (9)

where $C_M \equiv \int d\mu_{q_0}$, is a constant.

We can calculate the Hamiltonian constraint for the background canonical variables $q_i$ and $\pi_i$ (i = 1, 2), which is

$$\mathcal{H}(q, \pi) \equiv \mu_q^{-1}(\pi^{ab} \pi_{ab} - \frac{1}{2}(\text{tr}\pi)^2) - \mu_q R_q$$

$$= \mu_q(-\pi_1 \pi_2 q_1^{-1} + \frac{1}{2}(\pi_2)^2 q_1^{-2} q_2 + 2 q_1^{-1})$$

$$\approx 0.$$  \hspace{1cm} (10)
On the other hand the momentum constraint is found to be trivial: \( \mathcal{H}^a(q, \pi) \equiv -2D_b \pi^{ab} = 0 \). The background Hamiltonian \( H_0 \) is therefore given by\(^2\)

\[
H_0 = \frac{1}{C_M} \int_M d^3x N \mathcal{H} = N \sqrt{q_2} (-\pi_1 \pi_2 + \frac{1}{2} (\pi_2)^2 q_1^{-1} q_2 + 2), \tag{11}
\]

where \( N \) is the lapse function. With this Hamiltonian we obtain the following Einstein equations:

\[
\dot{q}_1 = \frac{\partial H_0}{\partial \pi_1} = -N \sqrt{q_2} \pi_2,
\]

\[
\dot{\pi}_1 = -\frac{\partial H_0}{\partial q_1} = \frac{1}{2} N \pi_2 q_2^{3/2} q_1^{-2},
\]

\[
\dot{q}_2 = -N \sqrt{q_2} (\pi_1 - \pi_2 q_1^{-1} q_2),
\]

\[
\dot{\pi}_2 = \frac{1}{4} N q_2^{-1/2} (2(\pi_1 \pi_2 - 2) - 3\pi_2 q_1^{-1} q_2),
\tag{12}
\]

where \( \dot{\cdot} \equiv d/dt \). We will refer to \( q_i \) and \( \pi_i \) \((i = 1, 2)\) (and \( N \)) as the background variables, and think of them as given functions of time which satisfy the above evolution equations and the constraint \( \mathcal{H} = 0 \).

The general vacuum solution of Bianchi III is given, up to isometry, by a one parameter family of spacetime metrics (§26 of [17]), for which we write

\[
g_{ab}^{(0)} = -(t_+/t_-)(dt)_a (dt)_b + t_+^2 h_{ab} + (t_-/t_+) l_{ab}, \tag{13}
\]

where

\[
t_+ \equiv t + k,
\]

\[
t_- \equiv t - k,
\tag{14}
\]

and \( k \) is a real parameter. This metric corresponds to the following solution for the background variables:

\[
q_1 = t_+^2, \quad q_2 = t_- t_+^{-1}, \quad \pi_1 = -2t_+ t_-^{-2}, \quad \pi_2 = -2 t_+, \quad N = q_2^{-1/2}. \tag{15}
\]

\(^2\)It is well known that one cannot obtain a correct reduced Hamiltonian by the straightforward procedure from Bianchi class B models including Bianchi III. This is however not the case for the present LRS case, which is spatially compactifiable. (In general, any spatially compactifiable Bianchi models have natural Hamiltonians.)
3 Harmonic expansion of the perturbation variables

Since our manifold \((M, q_{ab})\) is naturally a trivial \(S^1\)-fiber bundle \(\Sigma_g \times S^1\) any function on \(M\) may be expanded by the products of mode functions on \((\Sigma_g, h_{ab})\) and \((S^1, l_{ab})\). This is in fact the case when the natural fiber and base are orthogonal to each other. (See Appendix A for details of our background solution.) We only consider such backgrounds, and call them spatially closed orthogonal Bianchi III backgrounds. In the following therefore we first construct each of the sets of mode quantities, i.e., functions, vectors, and symmetric tensors, for \(S^1\) and for \(\Sigma_g\), and then construct those for \(M \simeq S^1 \times \Sigma_g\) by taking products of them.

**Mode Quantities on \(S^1\)**

The Laplacian on \((S^1, l_{ab})\) is \((\chi_3)^2 = \partial^2/\partial z^2\). The mode functions are therefore, taking the period of \(z\)-coordinate as \(2\pi\), simply given by \(e^{imz}\), where the eigenvalue \(m\) takes values \(m = 0, \pm 1, \cdots\). (16)

We refer to \(m\) as the fiber eigenvalue. It is convenient to redefine the eigenfunctions to be explicitly real so that we can directly calculate perturbation Hamiltonians which are second order functions. This can be done by taking the real or imaginary part of the complex eigenfunction. Let us define pairs of real eigenfunctions \(c_m\) and \(\bar{c}_m\) by the relations

\[
\chi_3 c_m = -mc_m, \quad \chi_3 \bar{c}_m = mc_m.
\]

(17)

We assume that these functions are normalized so that the square integrals on \((S^1, l_{ab})\) give unity:

\[
\int_{S^1} (c_m)^2 dz = \int_{S^1} (\bar{c}_m)^2 dz = 1.
\]

(18)

(For example, we can take \(c_m = (1/\sqrt{\pi}) \cos mz\), \(\bar{c}_m = (1/\sqrt{\pi}) \sin mz\).) Mode vectors and symmetric tensors can be constructed as \(c_m \sigma^3_a\), \(\bar{c}_m \sigma^3_a\), \(c_m l_{ab}\), and \(\bar{c}_m l_{ab}\).

**Mode Quantities on \(\Sigma_g\)**

The mode functions on the hyperbolic surface \((\Sigma_g, h_{ab})\) must satisfy

\[
\triangle_h \hat{S}_\lambda = -\lambda^2 \hat{S}_\lambda,
\]

(19)

where \(\triangle_h\) is the Laplacian with respect to \(h_{ab}\), and \(-\lambda^2\) is its eigenvalue. We will refer to \(\lambda\) as the base eigenvalue. (For simplicity we omit the subscript \(\lambda\) in \(\hat{S}_\lambda\) from now on unless it is necessary.) We assume that \(\hat{S}\) is real and normalized so that the square integral on \((\Sigma_g, h_{ab})\) gives unity, i.e.,

\[
\int_{\Sigma_g} d\mu_h \hat{S}^2 = 1,
\]

(20)

where \(d\mu_h = \sigma^1 \wedge \sigma^2\) is the area two-form associated with the metric \(h_{ab}\).

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3The orthogonal form universal cover metric \(\mathfrak{M}\) does not automatically imply this orthogonality of \((M, q_{ab})\). See Appendix A.
By analogy with the Regge–Wheeler spherically symmetric case [18], we make “even” and “odd” vectors on the $\Sigma_g$ from the scalar $\hat{S}$ when $\lambda > 0$:

\[ \hat{S}_a = \frac{1}{\lambda} \hat{D}_a \hat{S}, \quad \text{(the “even” vector on $\Sigma_g$)} \tag{21} \]

\[ \hat{V}_a = \frac{1}{\lambda} \epsilon_{ab} \hat{D}_b \hat{S}, \quad \text{(the “odd” vector on $\Sigma_g$)} \tag{22} \]

where $\hat{D}_a$ and $\epsilon_{ab} = \sigma^i_{[a} \sigma^j_{b]}$ are, respectively, the covariant derivative operator and volume form associated with the metric $h_{ab}$. We always write hats (\hat{\cdot}) on quantities on $\Sigma_g$ (except $h_{ab}$ and $\epsilon_{ab}$), and raise and lower indices of hatted quantities (and $\epsilon_{ab}$) by $h^{ab}$ and $h_{ab}$, so, e.g., $\epsilon_a^b \equiv \epsilon_{ac} h^{cb}$. The coefficients $1/\lambda$ are determined so that the square integrals give unity, i.e.,

\[ \int_{\Sigma_g} d\mu h h^{ab} \hat{S}_a \hat{S}_b = 1, \tag{23} \]

and the same for $\hat{V}_a$. When $\lambda = 0$, we have $\hat{S}_a = \hat{V}_a = 0$, since $\hat{S}$ is constant.

Other than these even and odd vectors we need to consider the harmonic vectors $\hat{U}_a$:

\[ \triangle_{LB} \hat{U}_a = 0, \tag{24} \]

where $\triangle_{LB} = d \delta + \delta d$ is the Laplace-Beltrami operator. This condition is equivalent, in the notation of standard differential geometry, to $d \hat{U} = \delta \hat{U} = 0$ when the one-form $\hat{U} = \hat{U}_a$ is on a closed surface, and they can be, in our language using the covariant derivatives, expressed as

\[ \hat{D}_a \hat{U}_b = 0, \quad \hat{D}^a \hat{U}_a = 0. \tag{25} \]

(This is not the same as $\triangle_h \hat{U}_a = 0.$) It is a special case of the Hodge decompositions that the space $A^1(M)$ of one-forms on a closed Riemannian surface $M$ can be decomposed into the direct sum as $A^1(M) = H^1(M) \oplus dA^0(M) \oplus \delta A^2(M)$ with respect to the standard $L^2$-norm, where $H^1(M)$ is the space of harmonic forms, $dA^0(M)$ the space of exact forms, and $\delta A^2(M)$ the space of dual exact forms. It can be easily checked that the spaces of even and odd vectors on $\Sigma_g$ can be identified with, respectively, $dA^0(\Sigma_g)$ and $\delta A^2(\Sigma_g)$. Hence the harmonic vectors definitely complete the possible (smooth) vectors on $\Sigma_g$. Recall that harmonic forms are always closed ($d \hat{U} = 0$). From Hodge and de Rham’s theorems we know that $H^1(M)$ is isomorphic to the real-valued cohomology group $H^1(M, \mathbb{R})$. Regge and Wheeler did not have to consider harmonic vectors in the spherically symmetric case, because the cohomology group $H^1(\mathbb{S}^2, \mathbb{R})$ of the sphere is trivial. In the present hyperbolic case, however, $H^1(\Sigma_g, \mathbb{R})$ is nontrivial and given by $\mathbb{R}^{2g}$, so we have $2g$ independent harmonic vectors on $\Sigma_g$. Let us choose these vectors so that they are $L^2$-orthonormal to each other (using the method of Schmidt, if necessary):

\[ \int_{\Sigma_g} d\mu h h^{ab} \hat{U}_a^{(r)} \hat{U}_b^{(r')} = \delta_{rr'}, \tag{26} \]

where $r, r' = 1, \ldots, 2g$. We call parameter $r$ the cohomology label. All the properties we need for the harmonic vectors can be deduced from the definition [25], which does not
depend on $r$. In this sense $r$ can be viewed as a label that distinguishes “degenerated states”. We omit writing these labels when we do not need to specify them explicitly.

Some of the key relations of the three vectors, which we can easily confirm from the definition, are:

$$\triangle_h \hat{Y}_a = - (\text{eig}_{\triangle_h}) \hat{Y}_a, \quad \text{eig}_{\triangle_h} \equiv \begin{cases} 
\lambda^2 + 1, & (\hat{Y}_a = \hat{S}_a) \\
\lambda^2 + 1, & (\hat{Y}_a = \hat{V}_a) \\
1, & (\hat{Y}_a = \hat{U}_a)
\end{cases}$$

(27)

and

$$\hat{D}^b \hat{D}_a \hat{Y}_b = - (\text{eig}_{\hat{D} \hat{D}}) \hat{Y}_a, \quad \text{eig}_{\hat{D} \hat{D}} \equiv \begin{cases} 
\lambda^2 + 1, & (\hat{Y}_a = \hat{S}_a) \\
1, & (\hat{Y}_a = \hat{V}_a) \\
1, & (\hat{Y}_a = \hat{U}_a)
\end{cases}$$

(28)

Let us consider symmetric tensors on $\Sigma_g$. They can be split into scalar, vector, and tensor parts. The scalar part is the part that is constructed from the scalar mode functions $\hat{S}$, which part is subdivided into three parts. One of them is the trace part given by $\hat{S}_{h_{ab}}$, while the others are the (traceless) even and odd parts which are given, when $\lambda > 0$, by

$$\hat{S}_{ab} = \frac{1}{\lambda} \sqrt{\frac{2}{\lambda^2 + 2}} \left( \hat{D}_a \hat{D}_b \hat{S} + \frac{\lambda^2}{2} \hat{S}_{h_{ab}} \right),$$

(29)

$$\hat{V}_{ab} = \sqrt{\frac{2}{\lambda^2 + 2}} \hat{D}_{(a} \hat{V}_{b)}.$$  

(30)

Again, the coefficients have been chosen so that the square integrals give unity, i.e,

$$\int_{\Sigma_g} d\mu_h h^{ac} h^{bd} \hat{S}_{ab} \hat{S}_{cd} = 1,$$

(31)

and the same for $\hat{V}_{ab}$. When $\lambda = 0$, we define that $\hat{S}_{ab} = \hat{V}_{ab} = 0$.

The vector part is the part that cannot be constructed from the mode functions $\hat{S}$ but from the harmonic vectors $\hat{U}_a$. It is given by

$$\hat{U}_{ab} = \hat{D}_a \hat{U}_b.$$  

(32)

The coefficient (= 1) is as always determined so that the square integrals give unity like $\hat{S}_{ab}$ and $\hat{V}_{ab}$. An explicit symmetrization in the right hand side of the above equation is not necessary because of the closedness of $\hat{U}_a$ (See the first equation of Eqs. (25)).

The remaining tensor part is the part that consists of all other possible symmetric tensors on $(\Sigma_g, h_{ab})$. From York’s decomposition \[25\], which is analogous to the Hodge decomposition, we can identify that they are transverse (i.e., divergenceless) and traceless (TT) tensors $\hat{W}_{ab}$:

$$\hat{D}^b \hat{W}_{ab} = 0,$$

(33)

$$h^{ab} \hat{W}_{ab} = 0.$$  

(34)

\[This\ tensor\ is\ not\ a\ zero\ tensor,\ as\ opposed\ to\ the\ flat\ torus\ case.
(\hat{S}h_{ab} \text{ corresponds to York's trace part, while } \hat{S}_{ab}, \hat{V}_{ab}, \text{ and } \hat{U}_{ab} \text{ all correspond to York's vector part. Beware that York's usage of the term “vector” part (or type) is different from ours.) TT tensors appear as variations of the metric \(h_{ab}\) towards the directions that generate Teichmüller deformations \([22]\). The freedom of \(\hat{W}_{ab}\) therefore corresponds to the linearized Teichmüller deformations of \(\Sigma_g\), which means we have \(6g - 6\) independent symmetric TT-tensors on \((\Sigma_g, h_{ab})\). We define \(\hat{W}_{ab}^{(s)}\), \((s = 1, \cdots, 6g - 6)\) so that they are orthonormal to each other, i.e.,

\[
\int_{\Sigma_g} d\mu h \hat{W}_{ab}^{(s)} \hat{W}_{ab}^{(s')*} = \delta_{ss'}.
\]

We call parameter \(s\) the Teichmüller label. Like the cohomology label we can think that this is a label that distinguishes “degenerated states”. We therefore omit writing these labels whenever they are not important.

We can now summarize the symmetric tensors on \((\Sigma_g, h_{ab})\) as in the table below. It is straightforward to check that they are orthogonal to each other with respect to the \(L^2\)-norm.

| \(\hat{S}h_{ab}\) | Even scalar (trace) prt. | Trace part | Generated from | Scalar type | Generated from |
|-----------------|--------------------------|-----------|----------------|-------------|----------------|
| \(\hat{S}_{ab}\) | Even vector prt. | Traceless part | the eigenfunction \(\hat{S} = \hat{S}_\lambda\) | Vector type | Topological |
| \(\hat{V}_{ab}\) | Odd vector prt. | | | Tensor type | origin |
| \(\hat{U}_{ab}\) | Harmonic vec. prt. | | | | |
| \(\hat{W}_{ab}\) | TT part | | | | |

Table 1: The symmetric tensors on \((\Sigma_g, h_{ab})\).

Some of the key relations for these tensors are:

\[
\Delta_h \hat{Y}_{ab} = - (\text{eig}_{\Delta_h}) \hat{Y}_{ab}, \quad \text{eig}_{\Delta_h} \equiv \begin{cases} 
\lambda^2, & (\hat{Y}_{ab} = \hat{S}h_{ab}) \\
\lambda^2 + 4, & (\hat{Y}_{ab} = \hat{S}_{ab}) \\
\lambda^2 + 4, & (\hat{Y}_{ab} = \hat{V}_{ab}) \\
4, & (\hat{Y}_{ab} = \hat{U}_{ab}) \\
2, & (\hat{Y}_{ab} = \hat{W}_{ab})
\end{cases}
\]

and

\[
\hat{D}^c \hat{D}_{(a} \hat{Y}_{b)c} = \begin{cases} 
-\frac{\lambda^2}{2} \hat{Y}_{ab} + \lambda \sqrt{\frac{\lambda^2 + 2}{2}} \hat{S}_{ab}, & (\hat{Y}_{ab} = \hat{S}h_{ab}) \\
-\left(\frac{\lambda^2}{2} + 3\right) \hat{Y}_{ab} + \lambda \sqrt{\frac{\lambda^2 + 2}{2}} \hat{S}_{ab} & (\hat{Y}_{ab} = \hat{S}_{ab}) \\
-\frac{\lambda^2}{2} \hat{Y}_{ab}, & (\hat{Y}_{ab} = \hat{V}_{ab}) \\
3 \hat{Y}_{ab}, & (\hat{Y}_{ab} = \hat{U}_{ab}) \\
2 \hat{Y}_{ab}, & (\hat{Y}_{ab} = \hat{W}_{ab})
\end{cases}
\]

Mode Quantities on \(M \simeq \Sigma_g \times S^1\)
We are now in a position to define the symmetric mode tensors on \( M \). Let \( S, \tilde{S} \in \mathcal{C}^\infty(M) \) be the products
\[
S \equiv c_m \hat{S}, \quad \tilde{S} \equiv c_m \hat{\tilde{S}},
\]
and similarly we make vectors and tensors on \( M \) by letting
\[
\bar{S}_a \equiv c_m \hat{S}_a, \quad \bar{V}_a \equiv c_m \hat{V}_a, \quad \bar{U}_a \equiv c_m \hat{U}_a, \\
S_{ab} \equiv c_m \hat{S}_{ab}, \quad V_{ab} \equiv c_m \hat{V}_{ab}, \quad U_{ab} \equiv c_m \hat{U}_{ab}, \\
W_{ab} \equiv c_m \hat{W}_{ab}.
\]

Then, we define our symmetric mode tensors \( E_1 \sim E_9 \) as
\[
\begin{align*}
(E_1)_{ab} &= S h_{ab}, \quad \text{(the even-trace part)} \\
(E_2)_{ab} &= S_{ab}, \quad \text{(the even-base part)} \\
(E_3)_{ab} &= S l_{ab}, \quad \text{(the even-fiber part)} \\
(E_4)_{ab} &= 2 \tilde{S}_{(a} \sigma^3_{b)} , \quad \text{(the even-cross part)} \\
(E_5)_{ab} &= V_{ab}, \quad \text{(the odd-base part)} \\
(E_6)_{ab} &= 2 \tilde{V}_{(a} \sigma^3_{b)}, \quad \text{(the odd-cross part)} \\
(E_7)_{ab} &= U_{ab}, \quad \text{(the harmonic-base part)} \\
(E_8)_{ab} &= 2 \tilde{U}_{(a} \sigma^3_{b)}, \quad \text{(the harmonic-cross part)} \\
(E_9)_{ab} &= W_{ab}, \quad \text{(the TT part)}
\end{align*}
\]

Here, each of \( E_1 \) to \( E_6 \) are labeled by the eigenvalues \( \lambda \) and \( m \), \( E_7 \) or \( E_8 \) is only by \( m \) with the Laplacian label \( r \), and \( E_9 \) is only by \( m \) with the Teichmüller label \( s \). When the Laplacian \( \Delta_q \) on \( M \) acts on these tensors their response depends in general on the eigenvalues \( \lambda \) and \( m \), but not on the cohomology labels \( r \) and Teichmüller label \( s \), which merely distinguish “degenerated states” for each \( m \), as already remarked.

For convenience we categorize these basis tensors in two ways, “parts” and “types”, which we summarize in Table 2. Note that the types are determined with respect to the associated quantities on the surface \( \Sigma_g \), not to those on the whole \( M \). This usage, which seems natural in the present case, does therefore not directly correspond to the one adopted in the usual perturbation theory of the isotropic cosmologies.

For convenience we define a standard \( L^2 \)-inner product in \( \mathcal{S}_2(M) \) (the space of covariant symmetric tensors on \( M \)) by
\[
F, G \in \mathcal{S}_2(M), \quad (F, G)_0 \equiv \int_M d\mu_0 \, q_{ab}^{ac} q_{bd} F_{ab} G_{cd}.
\]
Table 2: The symmetric tensors on \((M, q_{ab})\).

The subscript 0 in the bracket is a reminder that the integral is taken with respect to
the standard volume element \(\mu_0\). \(^5\)

\[ (E_i; E'_j)_0 = 0, \quad (i \neq j; i, j = 1 \sim 9), \tag{57} \]

where the dash signifies that different values in the labels can be taken. Among the same
kind we have the following relations: \(^6\)

\[ (E_i, E'_i)_0 = \begin{cases} (f_i)^{-1}\delta_{mnm'}\delta_{\lambda\lambda'}, & (i = 1 \sim 6) \\ (f_i)^{-1}\delta_{mnm'}\delta_{rr'}, & (i = 7, 8) \\ (f_i)^{-1}\delta_{mnm'}\delta_{ss'}, & (i = 9) \end{cases} \tag{58} \]

where \(f_i\) are functions of time given explicitly by

\[
\begin{array}{cccccccccc}
 i & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\
 f_i & \frac{q_1}{2} & q_1 & q_2 & \frac{q_1q_2}{2} & q_1' & \frac{q_1q_2}{2} & q_1 & \frac{q_1}{2} & q_2 \\
\end{array}
\tag{59}
\]

4 Perturbation variables and constraints

Let us apply the basic tools of mode expansion presented in the previous section to the
gravitational perturbation problem. \(^7\) Now that we have the complete set of basis tensors

\(^5\)Similarly, we could define another inner product \((E_i, E'_j)\), of which integral is taken with respect to
the time-dependent element \(\mu_0\). The two are simply related by \((E_i, E'_j) = q_1\sqrt{g_1}(E_i, E'_j)_0\). In this note,
however, we only use the standard inner product \((E_i, E'_j)_0\).

\(^6\)For notational simplicity, we assume that \(\lambda\) and \(\lambda'\) are different even if their numerical values are the
same if they correspond to distinct eigenfunctions.

\(^7\)Before proceeding to the perturbation problem however it is also beneficial to investigate simpler field
equations on the same background, especially the massless scalar and source-free electromagnetic fields, to
obtain insights for the properties of the perturbation problem. These materials are given in Appendices B and C.
we can expand perturbation variables in terms of them. Let \( \gamma_{ab} \equiv \delta q_{ab} \) be a perturbation of the spatial metric \( q_{ab} \), and let \( p^{ab} \equiv \delta \pi^{ab} \) be its canonical conjugate. From the relations (58) we find that we should put

\[
\gamma_{ab} = \sum \gamma^i (E_i)_{ab}, \\
p^{ab} = \mu q_0 \sum f_i p_i (E_i)^{ab}
\]

(60)

to keep that the components \( \gamma^i \) and \( p_i \) are canonical. In fact, the above relations show that

\[
\Theta \equiv \int_M d^3x p^{ab} \dot{\gamma}_{ab} = \sum p_i \dot{\gamma}^i.
\]

(61)

Here, \( \dot{\gamma}^i = \gamma^i(t) \) and \( p_i = p_i(t) \) are functions of time \( t \), and specified by the same labels as those of the corresponding basis. The sums are all taken from \( i = 1 \) to \( 9 \), and all possible eigenvalues of and all possible values of other labels (i.e., \( r \) and \( s \)).

**Constraints**

In general, perturbation variables are constrained by the first variations of the Hamiltonian and momentum constraints:

\[
\delta \mathcal{H} \approx 0, \\
\delta \mathcal{H}^a \approx 0.
\]

(62)

(63)

(See Ref. [13] for a general formula for \( \delta \mathcal{H} \equiv D \mathcal{H}(q, \pi) \cdot (\gamma, p) \) and \( \delta \mathcal{H}^a \equiv D^a \mathcal{H}(q, \pi) \cdot (\gamma, p) \).)

For our variables they are all obtained by straightforward computations using Eqs. (60). It is, however, worth noting that since the variation of the Hamiltonian constraint itself is a scalar (density) only the “scalar type” variables (associated with \( E_1 \) to \( E_6 \)) can contribute to it, and it should become a linear combination of \( \gamma^i \) and \( p_i \) \((i = 1 \sim 6)\). However, the odd variables (i.e., \( \gamma^i \) and \( p_i \) for \( i = 5 \) and \( 6 \)) do not actually contribute to this constraint, due to the zeros \( h^{ab}(E_i)_{ab} = l^{ab}(E_i)_{ab} = D^a D^b(E_i)_{ab} = 0 \), for \( i = 5 \) and \( 6 \). Here, \( D_a \) is the covariant derivative operator associated with the spatial (background) metric \( q_{ab} \). (See Eq. (229) for a useful formula.) Therefore this constraint takes contributions only from the even variables \( \gamma^i \) and \( p_i \) \((i = 1 \sim 4)\). We find after an explicit computation that the constraint for each mode is, when \( \lambda > 0 \), given by

\[
\delta \mathcal{H} = \mu_0 (q_1 \sqrt{q_2})^{-1} (D \mathcal{H}) S,
\]

where

\[
D \mathcal{H} \equiv - \left( 2m^2 q_1 + q_2 \left( \frac{2 \lambda^2 q_1 + \pi_2^2 q_2}{2 q_1} \right) \right) \gamma^1 - \lambda \sqrt{\frac{2 + \lambda^2}{2} q_2 \gamma^2} \\
- \left( \lambda^2 - 1 \right) q_1 + \frac{\pi_1 \pi_2 q_1}{2} - \frac{3 \pi_2^2 q_2}{4} \right) \gamma^3 + 2 \lambda m q_1 \gamma^4 \\
- \pi_2 q_1 q_2 p_1 - q_2 (\pi_1 q_1 - \pi_2 q_2) p_3 \approx 0.
\]

(64)

In the case \( \lambda = 0 \), since \((E_2)_{ab}\) and \((E_4)_{ab}\) vanish identically, the above expression is not valid, but still the correct expression can be obtained from the same formula by setting
\(\gamma^2 = \gamma^4 = p_2 = p_4 = 0\), as well as setting \(\lambda = 0\). Thus, it is given by
\[
\mathcal{D}H^{(\lambda=0)} = -\left(2 m^2 q_1 + \frac{\pi_2^2 q_2^2}{2 q_1}\right) \gamma^1 \\
- \left(-q_1 + \frac{\pi_1 \pi_2 q_1}{2} - \frac{3 \pi_2^2 q_2}{4}\right) \gamma^3 \\
- \pi_2 q_1 q_2 p_1 - q_2 (\pi_1 q_1 - \pi_2 q_2) p_3 \approx 0.
\] (65)

In particular, we denote this function when \(m = 0\) as \(\mathcal{D}H^{(\lambda=m=0)}\).

Since the first variation of the momentum constraint is a vector constraint it has four kinds of components with respect to the basis covariant vectors \(S_a, V_a, U_a, \) and \(\tilde{S}_a\). Let us write the corresponding components as, respectively, \(\mathcal{D}H_S, \mathcal{D}H_V, \mathcal{D}H_U, \) and \(\mathcal{D}H_l\), i.e., raising the indices, the variation of the momentum constraint function is of the form
\[
\delta \mathcal{H}^a = -\mu_0 \left[ \sum (\mathcal{D}H_S) S^a + \sum (\mathcal{D}H_V) V^a + \sum (\mathcal{D}H_U) U^a + \sum (\mathcal{D}H_l) q_2^a S^3 \right],
\] (66)
where the sum is taken over all possible eigenvalues and other labels. By straightforward computations the scalar type components, \(\mathcal{D}H_S, \mathcal{D}H_V, \) and \(\mathcal{D}H_l\) are, when \(\lambda > 0\), found to be
\[
\mathcal{D}H_S = \lambda (q_1 p_1 - \pi_2 \gamma^3) - \sqrt{\frac{\lambda^2 + 2}{2}} (\pi_1 \gamma^2 + 2 q_1 p_2) + m(2 \pi_2 \gamma^4 + q_1 p_4) \approx 0,
\] (67)
\[
\mathcal{D}H_l = -\lambda (\pi_1 \gamma^4 + q_2 p_4) + m(\pi_1 \gamma^1 - \pi_2 \gamma^3 - 2 q_2 p_3) \approx 0,
\] (68)
\[
\mathcal{D}H_V = -\sqrt{\frac{\lambda^2 + 2}{2}} (2 q_1 p_5 + \pi_1 \gamma^5) + m(q_1 p_6 + 2 \pi_2 \gamma^6) \approx 0.
\] (69)

Again, when \(\lambda = 0\), since \(E_i\) \((i = 2, 4, 5\) and \(6\)) vanish identically the above expressions are not valid, but still the correct forms are obtained from the same expressions by setting the associated variables zeros, at the same time setting \(\lambda = 0\). As a result, \(\mathcal{D}H_S\) and \(\mathcal{D}H_V\) are found to vanish identically, while \(\mathcal{D}H_l\) gives a nontrivial contribution:
\[
\mathcal{D}H_l^{(\lambda=0)} = m(\pi_1 \gamma^1 - \pi_2 \gamma^3 - 2 q_2 p_3) \approx 0.
\] (70)

When \(m = 0\), however, this also vanishes.

Finally, the vector type component \(\mathcal{D}H_U\) is given by
\[
\mathcal{D}H_U = -(2 q_1 p_7 + \pi_1 \gamma^7) + m(q_1 p_8 + 2 \pi_2 \gamma^8) \approx 0.
\] (71)

Note that this is the same as the one obtained by formally setting \(\lambda = 0\) in \(\mathcal{D}H_V\) (and changing the indices as \(5 \rightarrow 7\) and \(6 \rightarrow 8\)).

The \(\mathfrak{T}\)-basis \(E_9\) does not contribute to any of the constraints, so the associated variables \(\gamma^9\) and \(p_9\) are not constrained.
Decouplings

Before going into further analysis of the evolutions of the perturbations we present in advance how the perturbations decouple. It is direct computations to confirm the following.

Theorem 1 Perturbations of spatially closed orthogonal Bianchi III vacuum spacetimes (with fixed lapse and shift) are excited and evolved independently by either

(i) the set of the even tensors $E_1 \sim E_4$ belonging to the mode for a given pair $(\lambda, m)$,
(ii) the set of the odd tensors $E_5$ and $E_6$ belonging to the mode for a given pair $(\lambda, m)$,
(iii) the set of harmonic tensors $E_7$ and $E_8$ belonging to the mode for given $m$ and $r$, or
(iv) the $TT$ tensor $E_9$ belonging to the mode for given $m$ and $s$.

Here, $\lambda$ is the base eigenvalue, and $m$ is the fiber eigenvalue of the spatial manifold. $r$ is the cohomology label, and $s$ is the Teichmüller label. (When two different eigenstates have the same numerical value(s) of $\lambda$ and/or $m$ we understand that their eigenvalues are distinguished.)

(When we allow the lapse and shift to perturb, the appropriate kind, i.e., even, odd, harmonic, or $TT$ type of mode vectors and scalars are also needed to express the perturbation and those kinds of perturbations still decouple from each other.) We call these kinds of decoupled perturbations, respectively, the even, odd, harmonic, and $TT$ mode perturbations. We denote the even mode phase space for given $\lambda$ and $m$ as $E_{\lambda,m}$. Similarly, $O_{\lambda,m}$ is the odd phase space, $U_m$ the harmonic one, and $W_m$ the $TT$ one.

We summarize the mode phase spaces and their constraints in the table below. The Hamiltonian for each phase space will be shown in subsequent sections.

| Part | Mode phase space | Constraint function(s) |
|------|------------------|------------------------|
| Even | $E_{\lambda,m}$ ($\lambda > 0, m \in \mathbb{Z}$) | $DH, DH_S, DH_T$ |
|      | $E_{0,m}$ ($\lambda = 0, m \neq 0$) | $DH^{(\lambda=0)}, DH_T^{(\lambda=0)}$ |
|      | $E_{0,0}$ ($\lambda = 0, m = 0$) | $DH^{(\lambda=m=0)}$ |
| Odd  | $O_{\lambda,m}$ ($\lambda > 0, m \in \mathbb{Z}$) | $DH_V$ |
| Harmonic | $U_m$ ($m \in \mathbb{Z}$) | $DH_U$ |
| $TT$ | $W_m$ ($m \in \mathbb{Z}$) | $\emptyset$ |

Table 3: Mode phase spaces and their constraints. Each phase space is tacitly assumed to be endowed with an appropriate Hamiltonian function, too. These six kinds of mode phase spaces should be treated separately.

For future convenience let us define some terminology here. We call perturbations corresponding to all zero eigenvalues (i.e., $\lambda = m = 0$ for even or odd, or $m = 0$ for the harmonic or $TT$) zero mode perturbations. We call even and odd perturbations for which
either $\lambda = 0$ or $m = 0$, or all harmonic and $\mathbf{tt}$ perturbations global perturbations. We call the generic perturbations that are not zero mode nor global ones local perturbations. Perturbations with $m = 0$ is also called $U(1)$-symmetric perturbations.

5 Odd Perturbations

In this section we deal with the basic problems of the odd perturbations, mainly the separation of gauge freedom. The odd perturbations are by definition associated with the odd tensors $(E_5)_{ab}$ and $(E_6)_{ab}$. We assume $\lambda > 0$ as these tensors identically vanish for $\lambda = 0$. In the following we work with each $\mathcal{O}_{\lambda,m}$ for a fixed pair of $\lambda > 0$ and $m \in \mathbb{Z}$, and assume that the perturbation canonical tensors are given by (See Eq.(60))

$$\gamma_{ab} = \gamma^5(E_5)_{ab} + \gamma^6(E_6)_{ab},$$

$$p^{ab} = \mu_0(f_5p_5(E_5)^{ab} + f_6p_6(E_6)^{ab}),$$

(72)

where $f_5$ and $f_6$ are functions of time defined in Eq.(59).

The variation of the (covariant) shift vector $\delta N_a = \delta(q_{ab}N^b)$ should be an odd vector of the form

$$\delta N_a = sV_a,$$

(73)

where $s = s(t)$. We call $s$ the odd shift function. The odd perturbations do not contribute to variations of the lapse; $\delta N = 0$.

The canonical variables which span the odd phase space $\mathcal{O}_{\lambda,m}$ are given by $\gamma^5$, $\gamma^6$, $p_5$, and $p_6$. As we have seen in the previous section the odd variables are constrained by a single constraint $\mathcal{H}_V \approx 0$, which defines a three-dimensional subspace in $\mathcal{O}_{\lambda,m}$. Moreover, this constraint function generates gauge transformations in this subspace, so the true (i.e., gauge-invariant) dynamics is lying in the two dimensional space which is obtained by contracting those gauge orbits. We denote this two dimensional subspace as $\tilde{\mathcal{O}}_{\lambda,m}$, and call it the gauge-invariant odd phase space. We call the canonical variables in $\tilde{\mathcal{O}}_{\lambda,m}$ the gauge-invariant (canonical) variables, and denote them as $Q$ and $P$.

One of the main tasks in this section is to find a canonical transformation such that it defines the gauge-invariant canonical variables, and then apply it to the original Hamiltonian in $\mathcal{O}_{\lambda,m}$, which is presented right below. As a result we obtain a gauge-decoupled form of Hamiltonian $H_{\text{odd}}$, for which the gauge-invariant part $H_{\text{GI,odd}}(Q,P)$ determine the dynamics of the gauge-invariant variables. We present the wave equation for the gauge-invariant variable $Q$. The evolution of the other gauge variable will be solved in terms of the gauge-invariant variables in Sec.9.

The evolution equations and Hamiltonian for the odd variables

The evolution equations for the odd variables can be reduced from the general equations of motion by substituting Eqs.(72) and (73). See Ref.[13] for the general perturbation

8 Although a constraint function is also gauge-invariant, we will, unless otherwise stated, only refer to $Q$ or $P$ as a gauge-invariant variable.
Hamilton equations in the case of $N = 1$, $N_a = 0$, and $\delta N = 0 = \delta N_a$. Note that, since the lapse function $N$ for our background is not unity we need to generalize the formula by putting $d/dt \to (1/N)d/dt$. We also add the contributions of the perturbed shift. After a straightforward computation we obtain:

\[
\begin{align*}
\dot{\gamma}^5 &= \frac{N}{q_1 \sqrt{q_2}} \left[ (\pi_1 q_1 - \pi_2 q_2) \gamma^5 + 2q_1^2 p_5 \right] + \sqrt{2(\lambda^2 + 2)} s, \\
\dot{\gamma}^6 &= \frac{N}{q_1 \sqrt{q_2}} \left[ \pi_2 q_2 \gamma^6 + q_1 q_2 p_6 \right] - ms, \\
\dot{p}_5 &= \frac{q_1 \sqrt{q_2}}{N} \left[ -\frac{1}{2} \left( m^2 + \pi_1^2 + \frac{1}{2} \pi_2^{-2} q_2^2 - \pi_1 \pi_2 q_1^{-1} q_2 \right) \gamma^5 \\
&\quad - m \sqrt{\frac{\lambda^2 + 2}{2}} \gamma^6 \\
&\quad - (\pi_1 q_1 - \pi_2 q_2) p_5 \right] - \sqrt{\frac{\lambda^2 + 2}{2}} \pi_1 q_1^{-1} s, \\
\dot{p}_6 &= \frac{N}{q_1 \sqrt{q_2}} \left[ -m \sqrt{\frac{\lambda^2 + 2}{2}} \gamma^5 \\
&\quad - \left( \lambda^2 + \pi_1 \pi_2 + \frac{1}{2} \pi_2^{-1} q_2 \right) \gamma^6 \\
&\quad - \pi_2 q_2 p_6 \right] + 2m \pi_2^{-1} s.
\end{align*}
\]  

We can easily reconstruct the Hamiltonian $\tilde{H}_{\text{odd}}(\gamma^5, \gamma^6, p_5, p_6)$ for the odd variables from the above equations of motion; It is:

\[
\begin{align*}
\tilde{H}_{\text{odd}} &= \frac{N}{q_1 \sqrt{q_2}} \left[ \frac{1}{4} \left( m^2 + \pi_1^2 + \frac{1}{2} \pi_2^{-2} q_2^2 - \pi_1 \pi_2 q_1^{-1} q_2 \right) (\gamma^5)^2 \\
&\quad + \frac{1}{2} \left( \lambda^2 + \pi_1 \pi_2 + \frac{1}{2} \pi_2^{-1} q_2 \right) (\gamma^6)^2 \\
&\quad + m \sqrt{\frac{\lambda^2 + 2}{2}} \gamma^5 \gamma^6 \\
&\quad + q_1^2 p_5^2 + \frac{1}{2} q_1 q_2 p_6^2 \\
&\quad + (\pi_1 q_1 - \pi_2 q_2) \gamma^5 p_5 + \pi_2 q_2 \gamma^6 p_6 \right] - \frac{s}{q_1} \mathcal{D}H_V,
\end{align*}
\]  

where $\mathcal{D}H_V = \mathcal{D}H_V(\gamma^5, \gamma^6, p_5, p_6)$ is the constraint function for the odd perturbation.

We comment that the terms in the shift function $s$ in the Hamiltonian and evolution equations can easily be added after computing them with $s = 0$. The easiest way to do

\footnote{One can in principle get the same Hamiltonian by computing the second variation of the Hamiltonian constraint function. (See, e.g., Ref. \cite{14} for a general formula of the second variation.) It would, however, require tremendous calculations, due to the second order terms.}
this is to note the fact that the perturbation Hamiltonian is obtained by computing (half of) the second variation of the background Hamiltonian:

\[
H = \frac{1}{2} \delta \delta H_0 = \frac{1}{2} \delta \delta \int_M d^3x (N \delta \mathcal{H} + N_a \delta \mathcal{H}^a)
\]

\[
= \int_M d^3x \left( \frac{1}{2} N \delta \delta \mathcal{H} + \delta N \delta \mathcal{H} + \delta N_a \delta \mathcal{H}^a \right).
\]

(We dropped the term \(N_a \delta \delta \mathcal{H}^a\), which is zero for our background with \(N_a = 0\).) The first term in the integration contributes to the Hamiltonian for which \(s = 0\). The second term is zero for the odd perturbations (but not for the even ones). The last term is the one we want to calculate, which is

\[
\int_M d^3x \delta N_a \delta \mathcal{H}^a = - \int_M d\mu_0 s V_a (D \mathcal{H}_V) V^a
\]

\[
= -s q_1^{-1} D \mathcal{H}_V \int_M d\mu_0 (c_m)^2 h^{ab} V_a \dot{V}_b
\]

\[
= -s q_1^{-1} D \mathcal{H}_V.
\]

Thus, we obtain the final form of the Hamiltonian with an arbitrary \(s\). (Note that we already have the explicit form of \(D \mathcal{H}_V\).) The evolution equations with \(s\) may be obtained from that Hamiltonian. This technique will be more helpful for the even perturbation case where much longer computations are required.

**Canonical Transformation**

To decouple the Hamiltonian into gauge-dependent and independent parts and obtain the gauge-invariant phase subspace \(O_{\lambda,m}\) we need to find a canonical transformation such that one of the new momentum variables coincides with the constraint; let \((\gamma^s, p_s)\) be the canonical pair thereof, for which the momentum is constrained \(p_s = 0\). Once such a canonical transformation has been found, the other canonical pair, denoted as \((Q, P)\), automatically becomes gauge-invariant \([12, 14]\). We find such a canonical transformation by the method of generating function \([7]\) (also \([11]\)). Let \(S(\gamma^i, P_i)\) be the generating function we are looking for, which is a function of the original configuration variables \(\gamma^i = (\gamma^5, \gamma^6)\) and new momenta \(P_i = (P, p_s)\). Since both the original and transformed perturbation equations should be linear, it is sufficient for us to consider linear transformations \([11]\).

We can therefore assume that \(S\) is second order:

\[
S = \frac{1}{2} \sum_{i=5}^{6} \sum_{j=5}^{6} \alpha_{ij} \gamma^i \gamma^j + \sum_{i=5}^{6} \beta_i \gamma^i,
\]

\(\alpha_{ij} = \alpha_{(ij)}\). Each \(\beta_i\) should be a linear function of \(P_i\), while \(\alpha_{ij}\) depend only on the background variables.
We remark that we can redefine the constraint function as

\[ C_V \equiv \frac{\theta}{q_1} D H_V, \quad (79) \]

using a (nonzero) arbitrary function of time \( \theta = \theta(t) \). We call this parameter function the *mock shift function*. The factor \( q_1 \) in the above definition is just for convenience. This redefinition is in fact possible, since the original constraint surface specified by \( D H_V = 0 \) is equivalent to that for \( C_V = 0 \). Since we define a new momentum variable \( p_* \) to coincide with \( C_V \), this function is one of those parameters which control the canonical transformation we consider.

We set the following Hamilton-Jacobi-like equation

\[ p_* = C_V(\gamma^i, p_i = \frac{\partial S}{\partial \gamma^i}), \quad (80) \]

Substituting Eq. (78), we obtain the following equations for the generating function coefficients \( \alpha_{ij} \) and \( \beta_i \):

\[ -\theta(\sqrt{2(\lambda^2 + 2)}\beta_5 - m\beta_6) = p_*, \quad (81) \]
\[ -\sqrt{2(\lambda^2 + 2)}\alpha_{56} + m(\alpha_{66} + 2\pi_2 q_1^{-1}) = 0, \quad (82) \]
\[ -\sqrt{\frac{\lambda^2 + 2}{2}}(2\alpha_{55} + \pi_1 q_1^{-1}) + m\alpha_{56} = 0. \quad (83) \]

These equations do not completely determine \( \alpha_{ij} \) and \( \beta_i \); for example, we can regard \( \beta_6 \) and \( \alpha_{66} \) as free functions. This freedom corresponds to the freedom of performing canonical transformations among the gauge-invariant variables like \( (Q, P) \rightarrow (Q', P') \). As we will see below, \( \beta_6 \) solely determines the configuration variable \( Q \), and once \( \beta_6 \) is fixed, then \( \alpha_{66} \) becomes a parameter that controls the canonical transformations of the form \( (Q, P) \rightarrow (Q, P') \). We exploit this freedom to make the final form of the gauge-invariant Hamiltonian neat, especially “diagonal”. For later convenience we “solve” the above equations for \( \alpha_{ij} \) in terms of a single \( \alpha_{66} \):

\[ \alpha_{55} = \frac{\nu^2}{2}(\alpha_{66} + 2\pi_2 q_1^{-1}) - \frac{1}{2}\pi_1 q_1^{-1}, \quad (84) \]
\[ \alpha_{56} = \frac{\nu}{\sqrt{2}}(\alpha_{66} + 2\pi_2 q_1^{-1}), \]

where we have defined a normalized fiber eigenvalue,

\[ \nu \equiv \frac{m}{\sqrt{\lambda^2 + 2}}. \quad (85) \]

We simply set

\[ \beta_6 = P + p_* \quad (86) \]

Then from Eq. (81), we have

\[ \beta_5 = \frac{\nu}{\sqrt{2}} P + \left( \frac{\nu}{\sqrt{2}} - \frac{1}{\sqrt{2(\lambda^2 + 2)} \theta} \right) p_* \quad (87) \]
The gauge-invariant variable $Q$, conjugate to the momentum $P$ is now obtained from the generating function as

$$Q = \frac{\partial S}{\partial P} = \sum_{i=5}^{6} \frac{\partial \beta_i}{\partial P} \gamma^i = \frac{\nu}{\sqrt{2}} \gamma^5 + \gamma^6. \quad (88)$$

On the other hand, the gauge variable $\gamma_*$, conjugate to $p_*$ is given by

$$\gamma_* = \frac{\partial S}{\partial p_*} = \sum_{i=5}^{6} \frac{\partial \beta_i}{\partial p_*} \gamma^i = \left( \frac{\nu}{\sqrt{2}} - \frac{1}{\sqrt{2(\lambda^2 + 2) \theta}} \right) \gamma^5 + \gamma^6. \quad (89)$$

The inverse is therefore

$$\gamma^5 = \sqrt{2(\lambda^2 + 2) \theta} (Q - \gamma_*), \quad \gamma^6 = (1 - m\theta)Q + m\theta \gamma_*.$$  \quad (90)

We also obtain the following transformation between the original and new momentum variables:

$$p_5 = \frac{\partial S}{\partial \gamma^5} = \alpha_{55} \gamma^5 + \alpha_{56} \gamma^6 + \beta_5$$

$$= \left( \alpha_{56} - \sqrt{\frac{\lambda^2 + 2}{2} \pi_1 q_1^{-1} \theta} \right) Q + \frac{\nu}{\sqrt{2}} P$$

$$+ \sqrt{\frac{\lambda^2 + 2}{2} \pi_1 q_1^{-1} \theta} \gamma_* + \left( \frac{\nu}{\sqrt{2}} - \frac{1}{\sqrt{2(\lambda^2 + 2) \theta}} \right) p_*.$$ \quad (91)

$$p_6 = \frac{\partial S}{\partial \gamma^6} = \alpha_{56} \gamma^5 + \alpha_{66} \gamma^6 + \beta_6$$

$$= (\alpha_{66} + 2m \pi_2 q_1^{-1} \theta) Q + P - 2m \pi_2 q_1^{-1} \theta \gamma_* + p_*.$$  

Equations (91) and (90) complete the canonical transformation that takes the Hamiltonian into a desired gauge-decoupled form. The functions $\theta$ and $\alpha_{66}$ are free parameter functions that can be specified at any time one wants.

It is easy to see that the variable $Q = (\nu/\sqrt{2}) \gamma^5 + \gamma^6$ is certainly gauge-invariant. See the induced map (169).

The Gauge-Decoupled Form of Hamiltonian

The total Hamiltonian $H_{\text{odd}}$ for the new set of canonical variables should be of the following (semi-)decoupled form

$$H_{\text{odd}} = H^{GI}_{\text{odd}}(Q, P) + Jp_*,$$  \quad (92)
where $J = J(Q, P, \gamma_s, p_s; s; \theta)$ is a function that possibly depends on all the new variables $Q, P, \gamma_s,$ and $p_s$, and the shift function $s$ (and also parametrically on the mock shift $\theta$). We call $J$ the multiplier function or renormalized shift function for the odd variables. This form of Hamiltonian is a consequence of our system being a first-constrained system (e.g., \[7\]). In fact, the Hamilton equations for the gauge variables

$$\dot{\gamma}_s = J + \frac{\partial J}{\partial p_s} p_s,$$

$$\dot{p}_s = -\frac{\partial J}{\partial \gamma_s} p_s$$

are consistent with the constraint

$$p_s = 0.$$  \hspace{1cm} (94)

Moreover, we can easily see that the evolution of the gauge-invariant variables $Q$ and $P$ is determined, up to constraint, only from the gauge-invariant part Hamiltonian $H_{\text{odd}}^{\text{GI}}(Q, P)$. In fact, while we have

$$\dot{Q} = \frac{\partial H_{\text{odd}}}{\partial P} = \frac{\partial H_{\text{odd}}^{\text{GI}}}{\partial P} + \frac{\partial J}{\partial P} p_s,$$

$$\dot{P} = -\frac{\partial H_{\text{odd}}}{\partial Q} = -\frac{\partial H_{\text{odd}}^{\text{GI}}}{\partial Q} - \frac{\partial J}{\partial Q} p_s$$

from the total Hamiltonian, the last term for each equation vanishes if the constraint $p_s = 0$ is imposed.

We can find the decoupled Hamiltonian $H_{\text{odd}}(Q, P, \gamma_s, p_s)$ from the original Hamiltonian $\tilde{H}_{\text{odd}}(\gamma^5, \gamma^6, p_5, p_6)$ using the definition of the new variables (88), (89) and (91). Since our canonical transformation is time-dependent, however, we must add the time-derivative of the generating function $S$ according to the standard prescription. The correct relation is therefore

$$H_{\text{odd}} = \tilde{H}_{\text{odd}} + \frac{\partial S}{\partial t},$$

where the time derivative operator $(\partial/\partial t)$ acts only on the background variables (i.e., they do not act on the perturbation variables like $Q$ or $P$). From a direct computation we find that the the gauge-invariant part $H_{\text{odd}}^{\text{GI}}(Q, P)$ is given by

$$H_{\text{odd}}^{\text{GI}}(Q, P) = \left(\lambda^2 + \pi_2(\pi_1 + 4\nu^2 \pi_2 + \frac{1}{2} \pi_2 q_1^{-1} q_2) \right) \frac{NQ^2}{2q_1 \sqrt{q_2}}$$

$$+ \frac{1}{2} \alpha_{06} Q^2 + \frac{NuP^2}{2\sqrt{q_2}} + c_{\text{odd}} \frac{NQP}{q_1 \sqrt{q_2}},$$

where we have defined a combined scale function

$$u \equiv \nu^2 q_1 + q_2,$$  \hspace{1cm} (98)
and a cross term function
\[ c_{\text{odd}} \equiv \pi_2 \left( 2\nu^2 q_1 + q_2 \right) + q_1 u \alpha_{66}, \] (99)
for convenience. We have used Eqs. (84) to express \( \alpha_{ij} \) in terms of \( \alpha_{66} \) only, so \( \alpha_{66} \) is now considered as a free function of time. Also, observe that the mock shift \( \theta \) and shift function \( s \) do not appear in \( H_{\text{odd}}^{\text{GI}} \) as they should not.

As for the multiplier function \( J \), we find that
\[
J(Q, P, \gamma_*, p_*, s; \theta) = \left( 2\pi_2 + q_1 \alpha_{66} \right) \left( u - \frac{m}{\lambda^2 + 2} \frac{q_1}{q_1 \sqrt{q_2}} \right) \frac{NP}{\sqrt{q_2}} + \left( \frac{\pi_2 \sqrt{q_2} N}{q_1} + \frac{\dot{\theta}}{\theta} \right) \frac{\dot{\gamma}_*}{\theta} + \left( u + \frac{q_1 (1 - 2m\theta)}{(\lambda^2 + 2)\theta^2} \right) \frac{Np_*}{2\sqrt{q_2}} - \frac{s}{\theta}.
\] (100)
This function determines the evolution of the gauge variable \( \gamma_* \) through the equation of motion
\[
\dot{\gamma}_* = J|_0 \equiv J(Q, P, \gamma_*, p_*, s; \theta).
\] (101)
Note that we can think of the gauge-invariant variables \( Q \) and \( P \) in the above equation as \textit{given functions}, since the evolution of the gauge-invariant variables is determined independently of the gauge variable \( \gamma_* \). We will need the above equation to determine the actual evolution of the metric functions \( \gamma_5 \) and \( \gamma_6 \), especially the evolution expressed in terms of \( Q \) and \( P \) (or \( \dot{Q} \)).

It is interesting to note that even if we leave the shift function unperturbed (i.e., \( s = 0 \)) we can arbitrarily specify the “gauge velocity” \( J \) by suitably specifying \( \theta(t) \), and thereby we can virtually have any desired profile of the gauge variable \( \gamma_*(t) \). In this sense we can say that the mock shift function \( \theta \) mimics the role of the shift function \( s \). We will see however that this feature is superficial in the sense that the profiles of the metric functions \( \gamma_5 \) and \( \gamma_6 \) are not affected by the choice of \( \theta \) as long as \( s = 0 \). In other words, this arbitrariness is an artifact contained in the canonical transformation we consider. If we allow the shift function \( s \) to vary, however, the gauge function \( \theta \) automatically gets related, after solving the gauge equation of motion, to \( s \). (cf. Sec. 9.)

The Wave Equation for the Odd Perturbation
To eliminate unnecessary ambiguities, let us choose
\[
\alpha_{66} = \frac{\pi_2}{q_1} \frac{q_2}{u} - 2
\] (102)
to make \( c_{\text{odd}} = 0 \), with which the gauge-invariant Hamiltonian \( H_{\text{odd}}^\text{GI} \) becomes a diagonal form. In this case, the equations for the gauge-invariant odd variables (up to constraint) become

\[
\dot{Q} = \frac{N u P}{\sqrt{q_2}},
\]

\[
\dot{P} = -\left( (\lambda^2 + 2) - \frac{q_2}{u} + \frac{\pi_2 q_2^2}{4 u q_1} (5 \pi_2 q_2 - 2 \pi_1 q_1) + \frac{\pi_2 q_2^2}{u q_1} (\pi_1 q_1 - 2 \pi_2 q_2) \right) \frac{N Q}{q_1 \sqrt{q_2}}.
\]

Then, eliminating \( P \) we obtain the following wave equation for \( Q \):

\[
\ddot{Q} - \left( \frac{\dot{N}}{N} - N \sqrt{q_2} \left( -\frac{\pi_1 q_1 + 3 \pi_2 q_2}{2 q_1 q_2} + \frac{\pi_1 q_1 - 2 \pi_2 q_2}{u q_1} \right) \right) \dot{Q} + \frac{N^2}{q_1 q_2} \left( (\lambda^2 + 2) u - q_2 + \frac{\pi_2 q_2^2}{4 q_1} (5 \pi_2 q_2 - 2 \pi_1 q_1) + \frac{\pi_2 q_2^2}{u q_1} (\pi_1 q_1 - 2 \pi_2 q_2) \right) Q = 0.
\]

Here, we have used the background Einstein equations \([12]\) to express the time-derivatives of the background variables. This formal equation is valid for any spatially closed orthogonal Bianchi III vacuum background (with vanishing shift). (We also remark that since the definition of \( Q \) is independent of the choice of \( \alpha_{ij} \), the above wave equation for \( Q \) does not depend upon the choice \([12]\)).

If we substitute the exact solution \([15]\) into the above wave equation we obtain an explicit form of wave equation:

\[
\ddot{Q} - \frac{2 \nu^2 (t - 2 k) t_+}{t_- u} \dot{Q} + \left( (\lambda^2 + 2) \frac{u}{t_-^2} + \frac{2 (t - 3 k)}{t_+ t_-} - \frac{4 (t - 2 k)}{t_+^2 u} \right) Q = 0,
\]

where \( u = \nu^2 t_+^2 + t_- t_+^1 \).

### 6 Harmonic Perturbations

We see in this section that the harmonic perturbations can be (formally) viewed as a limit of the odd perturbations and because of that, we are required to perform almost no new calculations on top of those for the odd ones.

Remember that the harmonic perturbations are those associated with the harmonic vectors on \( \Sigma_g \). The corresponding basis tensors are \( E_7 \) and \( E_8 \), which are specified by the fiber eigenvalue \( m \). (More precisely, for each \( m \) there are \( 2g \) independent sets of \( E_7 \) and \( E_8 \), in accordance with the \( 2g \) independent harmonic vectors \( \hat{U}^{(r)} \) \((r = 1, \cdots , 2g)\).) In the following we work with each mode phase space \( U_m = U_m^{(r)} \) for a given \( m \) (and \( r \)). (We hereafter omit writing the cohomology label \( r \) for simplicity). We assume that the canonical perturbation tensors are given by (cf. Eq.(10))

\[
\gamma_{ab} = \gamma^7 (E_7)_{ab} + \gamma^8 (E_8)_{ab},
\]

\[
p^{ab} = \mu_q (f_7 p_7 (E_7)_{ab} + f_8 p_8 (E_8)_{ab}),
\]

(106)

(107)
where the functions $f_7$ and $f_8$ are defined in Eq.(59). Observe that $U_m$ is spanned by $\gamma^7$, $\gamma^8$, $p_7$, and $p_8$, and therefore $\dim U_m = 4$. The variation of the shift vector $\delta N_a$ should be a harmonic type vector of the form

$$\delta N_a = s U_a,$$

(108)

where $s = s(t)$. We call $s$ the \textit{harmonic shift function}. (Using the same notation of the odd shift function $s$ may not cause confusion.) The harmonic perturbations do not contribute to variations of the lapse; $\delta N = 0$.

The variables are constrained by a single constraint $\mathcal{D}H_U$ (See Eq.(71)). A similar count to the odd case gives $4-1-1 = 2$ as the dimension of the gauge invariant phase space $\bar{U}_m$.

What we have to do to obtain a gauge-decoupled Hamiltonian and thereby obtain a set of gauge invariant equations is completely parallel to those for the odd perturbations. Remember that both the odd symmetric tensor and harmonic symmetric tensor are traceless (on $\Sigma_g$); $h^{ab}\hat{V}_{ab} = h^{ab}\hat{U}_{ab} = 0$. Notice also that almost all relations for the harmonic vector $\hat{U}^a$ related to the covariant derivative are the same for the odd vector $\hat{V}^a$ if we assume $\lambda = 0$. For example (see Eq.(27)), $\Box h^{ab}\hat{U}_{ab} = -\hat{U}^a$ is the same as $\Box h^{ab}\hat{V}_{ab} = -(\lambda^2+1)\hat{V}^a$ if we put $\lambda = 0$. (We are talking about “formal” resemblances here, forgetting the fact that the odd vector is not defined for $\lambda = 0$.) The only difference is that the odd vector does not satisfy the closed condition that the harmonic vector does. That is, we have $\hat{D}_a[\hat{U}_b] = 0$ (see Eq.(55)), but $\hat{D}_a[\hat{V}_b] \neq 0$. This difference however does not affect our calculations in any way, since these vectors contribute to the perturbations only through the symmetrized tensors $\hat{V}_{ab} \propto \hat{D}_a[\hat{V}_b]$ or $\hat{U}_{ab} = \hat{D}_a[\hat{U}_b] = \hat{D}_b[\hat{U}_a]$. Due to these properties, all the necessary formulas (e.g., Hamiltonians) for the harmonic perturbations can be obtained from the ones for the odd perturbations by (formally) putting $\lambda = 0$, as well as changing the indices $5 \rightarrow 7$ and $6 \rightarrow 8$, if necessary. So, we are not trying to re-present most of the results.

One of our interests may however be the wave equation for the background solution (15), which is given by

$$\ddot{Q} - \frac{m^2(t - 2k)t_+}{t_-u} \dot{Q} + \left( \frac{m^2 t_+^2}{t_-^2} - \frac{4(t - 2k)}{t_+^2 u} \right) Q = 0,$$

(109)

where $u = (m^2/2)t_+^2 + t_-/t_+$. (This can be obtained from Eq.(105) by putting $\lambda = 0$ and therefore $v = m/\sqrt{2}$.) Here, the gauge-invariant variable $Q$ for the harmonic perturbation is defined by (cf. Eq.(88))

$$Q = \frac{m}{2} \gamma^7 + \gamma^8.$$

(110)

When $m = 0$ (the zero mode), this equation becomes

$$\ddot{Q} + \frac{4k}{t_+^2 t_-} Q = 0,$$

(111)

for which the general solution is given by

$$Q = \frac{t_-}{t_+} \left( C_1 + C_2 \left( t - \frac{4k^2}{t_+} + 4k \log t_- \right) \right),$$

(112)

where $C_1$ and $C_2$ are integration constants.
Transverse-Traceless Perturbations

Remember that the variables for the transverse-traceless (TT) perturbations are not constrained, so we do not need any extra steps to get gauge decoupling.

The TT-perturbation variables can be written

\[ \gamma_{ab} = \gamma^9 (E_9)_{ab}, \]
\[ p^{ab} = \mu_{q_0 f_0 p_0} (E_0)^{ab}. \]

We omit writing the subscripts that distinguish the 6g − 6 independent TT-tensors. We remark again that each mode (for a fixed s) is parameterized by the fiber eigenvalue m, but not by the base one \( \lambda \). The TT perturbations do not contribute to perturbations of the lapse function and shift vector; \( \delta N = 0, \delta N_a = 0 \).

It is not difficult to find by a straightforward calculation the evolution equations for the TT-variables \( \gamma^9 \) and \( p_9 \), which are given by

\[ \dot{\gamma}^9 = N \left( 2 q_1^2 \frac{p_9}{q_1 \sqrt{q_2}} + (\pi_1 q_1 - \pi_2 q_2) \frac{\gamma^9}{q_1 \sqrt{q_2}} \right), \]
\[ \dot{p}_9 = -N \left( (m^2 + \pi_1^2 + \frac{1}{2} \pi_2 q_1 - q_2^2 - \pi_1 \pi_2 q_1^{-1} q_2) \frac{\gamma^9}{2 q_1 \sqrt{q_2}} + (\pi_1 q_1 - \pi_2 q_2) \frac{p_9}{q_1 \sqrt{q_2}} \right). \]

Let us define for coherent notation

\[ Q \equiv \gamma^9, \quad P \equiv p_9. \]

Eliminating \( P \), we find that the wave equation for \( Q \) is given by

\[ \ddot{Q} - \left( \frac{\dot{N}}{N} + \frac{N}{2 q_1 \sqrt{q_2}} (\pi_1 q_1 - 3 \pi_2 q_2) \right) \dot{Q} + N^2 \left( m^2 q_2^{-1} - q_1^{-1} (1 + \frac{1}{2} \pi_1 \pi_2 - \frac{5}{4} \pi_2 q_1^{-1} q_2) \right) Q = 0. \]

Here, we used the background Einstein equation (12) to eliminate time-derivatives of the background variables.

Note however that since \( \gamma^9 \) contributes to the perturbation of metric in the form \( \delta g_{ab} = \gamma^9 (E_0)_{ab} \) and the base \( (E_0)_{ab} \) is tangent to the base \( (E_0)_{ab} \chi^3 = 0 \), we can say that \( \gamma^9 \) belongs to the ‘base’ (not ‘fiber’) part, and the base part of the background metric has unique scale function \( q_1 \). A more natural choice of variable is therefore to take the rescaled variable\(^{10}\)

\[ Q_B \equiv \gamma^9 / q_1 = Q / q_1. \]

(The character B stands for ‘base.’) Let us rewrite the wave equation for \( Q_B \), which is given by

\[ \ddot{Q} - \left( \frac{\dot{N}}{N} + \frac{N}{2 q_1 \sqrt{q_2}} (\pi_1 q_1 + \pi_2 q_2) \right) \dot{Q} + N^2 (m^2 q_2^{-1}) Q_B = 0. \]

\(^{10}\)Similar arguments for the other kinds of perturbations are not trivial, since a gauge-invariant variable is in general a linear combination of variables that belong to different parts ‘base’, ‘fiber’, and ‘cross’ ones.
It is interesting to observe that this wave equation for the TT perturbation is exactly the same as the wave equation for the scalar field in case of $\lambda = 0$. (See Eq. (242) in Appendix B)

On the exact background solution \([15]\), we have

$$\ddot{Q}_B + \frac{2t}{t_+ t_-} \dot{Q}_B + m^2 \frac{t_+^2}{t_-} Q_B = 0.$$  \hspace{1cm} (119)

**Solutions in some special cases**

(i) The flat background ($k = 0$) solution with $m \neq 0$ is

$$Q_B = \frac{1}{t} (C_1 \sin mt + C_2 \cos mt),$$ \hspace{1cm} (120)

where $C_1$ and $C_2$ are integration constants.

(ii) The zero mode ($m = 0$) solution with arbitrary $k$ is

$$Q_B = \begin{cases} 
C_1 + C_2 \log \frac{t}{t_+} & (k \neq 0) \\
C_1 + C_2 \frac{1}{t} & (k = 0)
\end{cases}$$ \hspace{1cm} (121)

8 Even Perturbations

The remaining perturbations are the even perturbations, which are generated by the even mode tensors $(E_i)_{ab}$ ($i = 1 \sim 4$). It is worth bearing in mind that, although any kind of computation becomes much longer and sometimes hideous due to the increased number of basic variables and the complicated couplings among them, the basic procedure to get a decoupled Hamiltonian and wave equation is completely systematic and the same as that for the odd case. (We however do not introduce mock shift and lapse functions similar to the odd mock shift function (cf. Eq. (79)) to simplify the computations and arguments.)

Remember that two of the basis tensors, $(E_2)_{ab}$ and $(E_4)_{ab}$, identically vanish when $\lambda = 0$, so this case should be treated separately. We say that the perturbation is *generic* when $\lambda > 0$, and *nongeneric* when $\lambda = 0$. We first deal with the generic case.

8.1 Generic ($\lambda > 0$) perturbations

In the following we work with the generic even mode phase space $E_{\lambda,m}$ for given $\lambda > 0$ and $m \in \mathbb{Z}$, and suppose that the perturbation canonical tensors are given by (see Eq. (50))

$$\gamma_{ab} = \sum_{i=1}^{4} \gamma^i (E_i)_{ab},$$

$$p_{ab} = \mu q_0 \sum_{i=1}^{4} f_i p_i (E_i)_{ab},$$ \hspace{1cm} (122)
where the functions $f_i$ $(i = 1 \sim 4)$ are defined in Eq. (59). The eight (canonical) variables $\gamma^i, p_i$ $(i = 1 \sim 4)$ span the mode phase space $E_{\lambda,m}$, so $\dim E_{\lambda,m} = 8$. The variation of the lapse function and shift vector should be of the form
\[
\delta N = s_0 S, \\
\delta N_a = s_1 S_a + s_3 \bar{S} \sigma^3_a,
\]

where $s_i$ $(i = 0, 1, 3)$ are functions of time. We call $s_0$ the even lapse function, $s_1$ the even base shift function, $s_3$ the even fiber shift function. The gauge-invariant dynamics is of course not affected by these functions.

As we have seen in Sec. 4 the even variables are constrained by three constraints $\mathcal{D}H \approx 0, \mathcal{D}H_S \approx 0, \mathcal{D}H_l \approx 0$, which define a five $(= 8 - 3)$ dimensional subspace in $E_{\lambda,m}$. Moreover, each function generates independent gauge transformations in this subspace. The gauge-invariant dynamics is therefore in a two $(= 5 - 3)$ dimensional subspace, which we denote $\bar{E}_{\lambda,m}$, and call the gauge-invariant (even) phase space. We call the canonical variables in $\bar{E}_{\lambda,m}$ the gauge-invariant (canonical) variables, and denote them as $Q$ and $P$.

Before performing a canonical transformation to get a decoupled Hamiltonian, we need to find the evolution equations and Hamiltonian for the original variables.

**The evolution equations and Hamiltonian for the even variables**

It is a lengthy but straightforward computation to obtain evolution equations, using Eqs. (122). For simplicity we only present the evolution equations for the vanishing even lapse and shift functions (i.e., synchronous gauge) $s_0 = s_1 = s_3 = 0$. (The general equations with arbitrary even lapse and shift functions may be easily recovered from the general...
Hamiltonian given after the following evolution equations.

\[
\begin{align*}
\frac{q_1 \sqrt{q_2}}{N} \dot{\gamma}^1 &= -\frac{1}{2}\pi_2 q_1 \gamma^3 - q_1 q_2 p_3, \\
\frac{q_1 \sqrt{q_2}}{N} \dot{\gamma}^2 &= (\pi_1 q_1 - \pi_2 q_2) \gamma^2 + 2q_1^2 p_2, \\
\frac{q_1 \sqrt{q_2}}{N} \dot{\gamma}^3 &= -\pi_2 q_1^{-1} q_2^2 \gamma^1 - \frac{1}{2}(\pi_1 q_1 - 3\pi_2 q_2) \gamma^3 \\
&\quad - q_2 (q_1 p_1 - q_2 p_3), \\
\frac{q_1 \sqrt{q_2}}{N} \dot{\gamma}^4 &= \pi_2 q_2 \gamma^4 + q_1 q_2 p_4, \\
\frac{q_1 \sqrt{q_2}}{N} \dot{p}_1 &= -\left(\pi_2 q_1^{-2} q_2^2 - m^2\right) \gamma^1 + \frac{1}{4} \left(2\lambda^2 + 3\pi_2^2 q_1^{-1} q_2\right) \gamma^3 \\
&\quad - m \lambda \gamma^4 + \pi_2 q_1^{-1} q_2^2 p_3, \\
\frac{q_1 \sqrt{q_2}}{N} \dot{p}_2 &= -\frac{1}{2} \left(m^2 + \pi_1^2 - \pi_1 \pi_2 q_1^{-1} q_2 + \frac{1}{2} \pi_2^2 q_1^{-2} q_2^2\right) \gamma^2 \\
&\quad + \sqrt{\frac{\lambda^2 + 2}{2}} \left(\frac{\lambda}{2} \gamma^3 - m \gamma^4\right) \\
&\quad - (\pi_1 q_1 - \pi_2 q_2) p_2, \\
\frac{q_1 \sqrt{q_2}}{N} \dot{p}_3 &= \frac{1}{4} \left(2\lambda^2 + 3\pi_2^2 q_1^{-1} q_2\right) \gamma^1 + \frac{\lambda}{2} \sqrt{\frac{\lambda^2 + 2}{2}} \gamma^2 \\
&\quad - \frac{1}{2} \left(\frac{3}{4} \pi_2^2 + \frac{1}{2} \pi_1 \pi_2 q_1^{-1} q_2^2 - q_1 q_2^{-1}\right) \gamma^3 \\
&\quad + \frac{1}{2} \pi_2 q_1 p_1 + \frac{1}{2} (\pi_1 q_1 - 3\pi_2 q_2) p_3, \\
\frac{q_1 \sqrt{q_2}}{N} \dot{p}_4 &= -m \lambda \gamma^1 - m \sqrt{\frac{\lambda^2 + 2}{2}} \gamma^2 \\
&\quad - \frac{1}{2} \pi_2 q_1^{-1} (2\pi_1 q_1 + \pi_2 q_2) \gamma^4 - \pi_2 q_2 p_4.
\end{align*}
\]

It is at once possible to read off the Hamiltonian which gives rise to these equations, and also easy to generalize it to include the even lapse and shift functions (see the comment below Eq. (124)). The Hamiltonian with arbitrary even lapse and shift functions is given
by

\[
\hat{H}_{\text{even}} = \frac{N}{q_1\sqrt{q_2}} \left[ \frac{1}{2} \left( \pi_2 q_1^{-2} q_2^{-2} - m^2 \right) (\gamma^1)^2 \right.
\]

\[
+ \frac{1}{4} \left( m^2 + \pi_1^2 - \pi_1 \pi_2 q_1^{-1} q_2 + \frac{1}{2} \pi_2 q_1^{-2} q_2^{-2} \right) (\gamma^2)^2
\]

\[
+ \frac{1}{4} \left( \frac{3}{4} \pi_2^2 + \frac{1}{2} \pi_1 \pi_2 q_1^{-1} q_2 - q_1 q_2^{-1} \right) (\gamma^3)^2
\]

\[
+ \frac{1}{4} \pi_2 q_1^{-1} (2 \pi_1 q_1 + \pi_2 q_2) (\gamma^4)^2
\]

\[
- \frac{1}{4} (2\lambda^2 + 3\pi_2^2 q_1^{-1} q_2) \gamma^1 \gamma^3
\]

\[
+ m \lambda \gamma^1 \gamma^4
\]

\[
- \sqrt{\frac{\lambda^2 + 2}{2}} \left( \frac{\lambda}{2} \gamma^3 - m \gamma^1 \right) \gamma^2
\]

\[
+ q_1^2 p_2 + q_2^2 p_3
\]

\[
+ \frac{1}{2} q_1 q_2 \left( p_1^2 - 2p_1 p_3 \right)
\]

\[
- \pi_2 q_1^{-1} \pi_2 q_1^{-1} p_3 + (\pi_1 q_1 - \pi_2 q_2) \gamma^2 p_2
\]

\[
- \frac{1}{2} \pi_2 q_1^{-1} \pi_2 q_1^{-1} p_1 - \frac{1}{2} (\pi_1 q_1 - 3\pi_2 q_2) \gamma^3 p_3 + \pi_2 q_2 \gamma^4 p_4
\]

\[
\left] + \frac{s_0}{q_1\sqrt{q_2}} D H - \frac{s_1}{q_1} D H_S - \frac{s_3}{q_2} D H_t, \right.
\]

where \( D H, D H_S, \) and \( D H_t \) are the constraint functions for the even variables, given in Sec. 4.

**Canonical Transformation**

We need to find a canonical transformation so that three of the new momenta coincide with the three constraint functions. \(^1\)

The remaining new momentum is identified with the gauge-invariant momentum \( P \) that is conjugate to the gauge-invariant configuration variable \( Q \). As in the odd case we write the generating function in the form

\[
S = \frac{1}{2} \sum_{i,j=1}^{4} \alpha_{ij} \gamma^i \gamma^j + \sum_{i=1}^{4} \beta_i \gamma^i,
\]

where \( \alpha_{ij} = \alpha_{(ij)} \). The coefficients \( \alpha_{ij} \) are functions of background variables only, while \( \beta_i \) depend linearly upon \( P \). For convenience, we redefine the constraints in the form

\(^1\)We remark that the constraint functions in linear perturbation theory automatically strongly commute, which is the condition that is necessary (and sufficient) to assure that they can be new momentum variables after a canonical transformation. In fact, since the constraint functions are linear, their brackets do not depend on the perturbation variables, and since the brackets at least weakly (i.e., up to the constraints) commute with each other, this in turn implies that they must strongly commute.
\[ P_i = p_i - K_i \approx 0, \ (i = 1, 2, 4), \] where \( K_i \) is a function of \( \gamma^1, \ \gamma^2, \ \gamma^3, \ \gamma^4, \) and \( p_3. \) (This is actually possible.) Although this redefinition is not necessary, it helps us to make computation more transparent and easier. Let

\[ \{\gamma_*^i\} = (\gamma_*^1, \gamma_*^2, Q, \gamma_*^4) \] (127)

be new configuration variables, and let

\[ \{p_{i*}\} = (p_{1*}, p_{2*}, P, p_{4*}) \] (128)

be their conjugate momenta. The equations coming from the Hamilton-Jacobi-like equations, \( P_1(\gamma^i, p_i = \frac{\partial S}{\partial \gamma^i}) = p_{1*}, \) \( P_2(\gamma^i, p_i = \frac{\partial S}{\partial \gamma^i}) = p_{2*}, \) and \( P_4(\gamma^i, p_i = \frac{\partial S}{\partial \gamma^i}) = p_{4*} \) are the following:

(i) From \( P_1 = c_{10} + c_{11} \gamma^1 + \cdots + c_{14} \gamma^4 = p_{1*}: \)

\[ c_{10} \equiv \beta_1 + \Delta_1 \beta_3 = p_{1*}, \] (129)
\[ c_{11} \equiv \alpha_{11} + \Delta_1 \alpha_{13} + \frac{\lambda^2}{\pi_2 q_1} + \frac{2 m^2}{\pi_2 q_2} + \frac{\pi_2 q_2}{2 q_1^2} = 0, \] (130)
\[ c_{12} \equiv \alpha_{12} + \Delta_1 \alpha_{23} + \frac{\lambda \sqrt{2} + \lambda^2}{\sqrt{2} \pi_2 q_1} = 0, \] (131)
\[ c_{13} \equiv \alpha_{13} + \Delta_1 \alpha_{33} + \frac{2 (2(\lambda^2 - 1) + \pi_1 \pi_2) q_1 - 3 \pi_2^2 q_2}{4 \pi_2 q_1 q_2} = 0, \] (132)
\[ c_{14} \equiv \alpha_{14} + \Delta_1 \alpha_{34} - \frac{2 \lambda m}{\pi_2 q_2} = 0. \] (133)

(ii) From \( P_2 = c_{20} + c_{21} \gamma^1 + \cdots + c_{24} \gamma^4 = p_{2*}: \)

\[ c_{20} \equiv \beta_2 + \Delta_2 \beta_3 = p_{2*}, \] (134)
\[ c_{21} \equiv \alpha_{12} + \Delta_2 \alpha_{13} + \frac{2 m^2 \pi_1 \pi_2 q_1^2 + \lambda^2 (2 \lambda^2 q_1 q_2 + 4 m^2 q_1^2 + \pi_2^2 q_2^2)}{2 \sqrt{2(2 + \lambda^2)} \lambda \pi_2 q_1^2 q_2} = 0, \] (135)
\[ c_{22} \equiv \alpha_{22} + \Delta_2 \alpha_{23} + \frac{\lambda^2 + \pi_1 \pi_2}{2 \pi_2 q_1} = 0, \] (136)
\[ c_{23} \equiv \alpha_{23} + \Delta_2 \alpha_{33} + \frac{4 m^2 \pi_2^2 q_1 + \lambda^2 (2(\lambda^2 - 1) + \pi_1 \pi_2) q_1 + \pi_2^2 q_2}{4 \sqrt{2(2 + \lambda^2)} \lambda \pi_2 q_1 q_2} = 0, \] (137)
\[ c_{24} \equiv \alpha_{24} + \Delta_2 \alpha_{34} + \frac{m (-2 \lambda^2 q_1 + \pi_2 (\pi_1 q_1 - 2 \pi_2 q_2))}{\sqrt{2(2 + \lambda^2)} \pi_2 q_1 q_2} = 0. \] (138)
(iii) From $P_4 = c_{40} + c_{41} \gamma^1 + \cdots + c_{44} \gamma^4 = p_{4*}$:

$$c_{40} \equiv \beta_1 + \frac{2 m \beta_3}{\lambda} = p_{4*}, \quad (139)$$
$$c_{41} \equiv \alpha_{14} + \frac{2 m \alpha_{13}}{\lambda} - \frac{m \pi_1}{\lambda q_2} = 0, \quad (140)$$
$$c_{32} \equiv \alpha_{24} + \frac{2 m \alpha_{23}}{\lambda} = 0, \quad (141)$$
$$c_{43} \equiv \alpha_{34} + \frac{2 m \alpha_{33}}{\lambda} + \frac{m \pi_2}{\lambda q_2} = 0, \quad (142)$$
$$c_{44} \equiv \alpha_{44} + \frac{2 m \alpha_{34}}{\lambda} + \frac{\pi_1}{q_2} = 0. \quad (143)$$

For convenience we have defined

$$\Delta_1 \equiv \frac{\pi_1}{\pi_2} - \frac{q_2}{q_1} = \Sigma - 2 \frac{m^2}{\lambda^2}, \quad (144)$$
$$\Delta_2 \equiv \frac{\lambda}{\sqrt{2(\lambda^2 + 2)}} \Sigma,$$

where

$$\Sigma \equiv \frac{\pi_1}{\pi_2} - \frac{q_2}{q_1} + 2 \frac{m^2}{\lambda^2}. \quad (145)$$

Let us first look at the set of equations $c_{i0} = p_{i*}$ ($i = 1, 2, \text{ and } 4$), which consists of three equations for four unknowns $\beta_i$ ($i = 1 \sim 4$). To determine them we put $\beta_3 = p_{3*} \equiv P$, where $P$ is the gauge-invariant momentum, then we immediately obtain the following solution

$$\beta_1 = p_{1*} - \Delta_1 P,$$
$$\beta_2 = p_{2*} - \Delta_2 P,$$
$$\beta_3 = P,$$
$$\beta_4 = p_{4*} - \frac{2 m P}{\lambda}. \quad (146)$$

The remaining 12 equations, $c_{i1} = c_{2i} = c_{4i} = 0$ ($i = 1 \sim 4$), for 10 unknowns $\alpha_{ij} = \alpha_{(ij)}$ ($i, j = 1 \sim 4$) look overdetermined at first, but this system is underdetermined. In fact, given, say, $\alpha_{33}$, we find we can solve in terms of $\alpha_{33}$ all equations except three equations $c_{21} = c_{24} = c_{41} = 0$, and these three equations are automatically satisfied as long as the background Einstein equations are satisfied. So, the equations are an underdetermined system. In later calculations we will express $\alpha_{ij}$ in terms of $\alpha_{33}$ when it is useful. In such a case, $\alpha_{33}$ is regarded as an arbitrary (prescribed) function of $t$. (“Prescribed” means that $\alpha_{33}$ should be prescribed before solving the perturbation equations.) As pointed out in the section for the odd perturbations, this is a result of the fact that there remains freedom to canonically transform among a given gauge invariant pair of variables.
We can determine the gauge-invariant variable \( Q \) from the generating function \( S \) with the solution (146). We get
\[
Q = \frac{\partial S}{\partial P} = -\Delta_1 \gamma^1 - \Delta_2 \gamma^2 + \gamma^3 - \frac{2m \gamma^4}{\lambda}.
\] (147)

Let us denote the remaining gauge configuration variables as \( \gamma^i_s = \frac{\partial S}{\partial p^i_s} \) \((i = 1, 2, \text{ and } 4)\). They are conjugate to \( p^i_s \). For coherent notation we also define \( \gamma^3_s \equiv Q \). We can easily find
\[
\gamma^i_s = \gamma^i, \quad (i = 1, 2, \text{ and } 4),
\] (148)
and the inverse is
\[
\begin{align*}
\gamma^1 &= \gamma^1_s, \\
\gamma^2 &= \gamma^2_s, \\
\gamma^3 &= \Delta_1 \gamma^1_s + \Delta_2 \gamma^2_s + Q + \frac{2m \gamma^4}{\lambda} \gamma^4_s, \\
\gamma^4 &= \gamma^4_s. 
\end{align*}
\] (149)

As for the momenta, from
\[
p_i = \frac{\partial S}{\partial \gamma^i} = \alpha_{ij} \gamma^j + \beta_i
\] (150)
(and the above relations for \( \gamma^i_s \)) we find
\[
\begin{align*}
p_1 &= (\alpha_{11} + \alpha_{13} \Delta_1) \gamma^1_s + (\alpha_{12} + \alpha_{13} \Delta_2) \gamma^2_s + \alpha_{13} Q + (\alpha_{14} + \alpha_{13} \frac{2m}{\lambda}) \gamma^4_s \\
&+ p_{1s} - \Delta_1 P, \\
p_2 &= (\alpha_{12} + \alpha_{23} \Delta_1) \gamma^1_s + (\alpha_{22} + \alpha_{23} \Delta_2) \gamma^2_s + \alpha_{23} Q + (\alpha_{24} + \alpha_{23} \frac{2m}{\lambda}) \gamma^4_s \\
&+ p_{2s} - \Delta_2 P, \\
p_3 &= (\alpha_{13} + \alpha_{33} \Delta_1) \gamma^1_s + (\alpha_{23} + \alpha_{33} \Delta_2) \gamma^2_s + \alpha_{33} Q + (\alpha_{34} + \alpha_{33} \frac{2m}{\lambda}) \gamma^4_s \\
&+ P, \\
p_4 &= (\alpha_{14} + \alpha_{34} \Delta_1) \gamma^1_s + (\alpha_{24} + \alpha_{34} \Delta_2) \gamma^2_s + \alpha_{34} Q + (\alpha_{44} + \alpha_{34} \frac{2m}{\lambda}) \gamma^4_s \\
&+ p_{4s} - \frac{2m}{\lambda} P.
\end{align*}
\] (151)

Note that the coefficients of \( \gamma^i_s \) \((i = 1, 2, \text{ and } 4)\) in the above relations can be immediately explicitly expressed in terms of the background variables using the equations in (i) to (iii).
The Gauge-Decoupled Form of Hamiltonian

The Hamiltonian $H_{\text{even}}$ for the new variables is obtained from

$$H_{\text{even}} = \tilde{H}_{\text{even}} + \frac{\partial S}{\partial t},$$

(152)

where $(\partial/\partial t)$ acts on the background variables only. The resulting Hamiltonian becomes of the form

$$H_{\text{even}} = H_{\text{even}}^{\text{GI}}(Q, P) + J_1 p_1 + J_2 p_2 + J_4 p_4,$$

(153)

where $H_{\text{even}}^{\text{GI}}(Q, P)$ is the gauge-invariant Hamiltonian, and $J_i = J_i(Q, P, \gamma_1^i, \gamma_2^i, \gamma_4^i, P_i^i)$ ($i = 1, 2, 4$) are the multiplier functions.

From a straightforward computation we find the following gauge-invariant Hamiltonian:

$$H_{\text{even}}^{\text{GI}}(Q, P) = \left\{ \left( \frac{(11\lambda^2 + 4)\pi_2^2 q_2}{16 q_1} + \frac{(\lambda^2 + 2)}{4} \right) \pi_1 \pi_2 + \frac{\lambda^2(\lambda^2 - 1)^2 q_1}{\pi_2^2 q_2} + \frac{5\lambda^2}{2} - \lambda^2 - 3 \right\} \frac{1}{(\lambda^2 + 2)q_2} \right. \left. + \left( \left( \frac{5\lambda^2}{4} + 1 \right) \pi_2^2 + \frac{\lambda^2(\lambda^2 - 1)q_1}{q_2} \right) \frac{\Sigma}{(\lambda^2 + 2)q_2} \right. \left. + \frac{\lambda^2 \pi_2^2 q_1}{4q_2^2} \frac{\Sigma^2}{\lambda^2 + 2} + c_{\text{even}} \frac{\alpha_{33}}{2} \frac{NQ^2}{2q_2^2} \right. \left. + \frac{\alpha_{33}}{2} \frac{Q^2}{2q_1 q_2} + \frac{\lambda^2}{2q_1 q_2} \right. \left. + c_{\text{even}} \frac{NQ^2}{2q_2^2}\right\},$$

(154)

where

$$\Omega \equiv q_2^2 + 2q_1 q_2 \Sigma + \frac{\lambda^2}{\lambda^2 + 2} q_1^2 \Sigma^2,$$

(155)

and

$$c_{\text{even}} \equiv 2(\lambda^2 - 1)\pi_2^{-1} - \pi_1 + \frac{5}{2} \pi_2 q_1^{-1} q_2 \right. \left. + \left( \left( \frac{7\lambda^2}{2} + 4 \right) \pi_2^2 + \frac{2\lambda^2(\lambda^2 - 1)q_1}{\pi_2 q_2} \right) \frac{\Sigma}{(\lambda^2 + 2)} \right. \left. + \frac{\lambda^2 \pi_2 q_1}{(\lambda^2 + 2)q_2} \frac{\Sigma^2}{q_1} \right. \left. + 2 \Omega \frac{\alpha_{33}}{q_1}.\right\}$$

(156)

As we noted earlier we have expressed $\alpha_{ij}$ in terms of $\alpha_{33}$ only, solving the equations $c_{1i} = c_{2i} = c_{4i} = 0$ ($i = 1 \sim 4$) (except the “identities” $c_{21} = c_{24} = c_{41} = 0$). The function $\alpha_{33}$ is therefore now an arbitrary (prescribed) function of $t$. 

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As for the multiplier functions, we find

\[
J_1 = -(\pi_2 + 2q_2\alpha_{33}) \frac{NQ}{2\sqrt{q_2}} - N\sqrt{q_2}P \\
+ \left( (\lambda^2 - 1)\pi_2^{-1}q_1 - \frac{1}{4}\pi_2q_2 \right) \frac{N\gamma_1}{q_1\sqrt{q_2}} \\
+ \frac{\lambda}{\sqrt{2(\lambda^2 + 2)}} \left( (\lambda^2 - 1)\pi_2^{-1}q_1 + \frac{3}{4}\pi_2q_2 \right) \frac{N\gamma_2}{q_1\sqrt{q_2}} \\
- \pi_2\sqrt{q_2}s_0 - \lambda s_1,
\]

\[
J_2 = -\frac{\lambda}{\sqrt{2(\lambda^2 + 2)}} \left( \frac{3}{2}\pi_2 + 2(\lambda^2 - 1)\pi_2^{-1}q_1q_2^{-1} \right) \\
+ (\pi_2q_1q_2^{-1} + 2q_1\alpha_{33}) \Sigma \frac{NQ}{\sqrt{q_2}} \\
- \lambda \frac{\sqrt{2}}{\lambda^2 + 2} N\Sigma q_1P \\
- (\lambda^2\pi_2^{-1}q_1 + \pi_2q_2) \frac{N\gamma_2^2}{q_1\sqrt{q_2}} + \frac{Nq_1p_{2*}}{\sqrt{q_2}} \\
+ \sqrt{2(\lambda^2 + 2)} s_1,
\]

\[
J_4 = -\frac{m}{\lambda} \left( \pi_2 + 2q_2\alpha_{33} \right) \frac{NQ}{\sqrt{q_2}} - 2m\sqrt{q_2}P \\
+ \frac{2\lambda m N\gamma_1}{\pi_2\sqrt{q_2}} \\
+ m \sqrt{\frac{2}{\lambda^2 + 2}} \left( (\lambda^2 - 1)\pi_2^{-1}q_1 + \frac{3}{4}\pi_2q_2 \right) \frac{N\gamma_2}{q_1\sqrt{q_2}} \\
- (\pi_1q_1 - \pi_2q_2) \frac{N\gamma_4}{q_1\sqrt{q_2}} + \frac{N\sqrt{q_2}p_{4*}}{2} \\
- ms_1 + \lambda s_3.
\]

These functions provide the time derivatives of the gauge variables \(\gamma^i_s\) (\(i = 1, 2,\) and \(4\)), i.e.,

\[
\dot{\gamma}^i_s = J_i|_0
\]

for each \(i\), where \(|_0\) stands for imposing the constraints \(p_{is*} = 0\).
The Wave Equation for the Even Perturbation

Let us choose $\alpha_{33}$ so that it makes the Hamiltonian a diagonal form. We therefore choose

$$\alpha_{33} = -\frac{q_1}{2\Omega} \left\{ 2(\lambda^2 - 1)\pi_2^{-1} - \pi_1 + \frac{5}{2} \pi_2 q_1^{-1} q_2 
+ \left( \frac{7\lambda^2}{2} + 4 \right) \pi_2 + \frac{2\lambda^2(\lambda^2 - 1) q_1}{\pi_2 q_2} \right\} \frac{\Sigma}{(\lambda^2 + 2)} \right\}$$

(159)

to make $c_{\text{even}} = 0$. Then, the Hamilton equations are given by

$$\dot{Q} = \frac{N\Omega P}{q_1 \sqrt{q_2}},$$

$$\dot{P} = -\left\{ \left( \frac{(11\lambda^2 + 4)\pi_2 q_2}{16 q_1} + \frac{(\lambda^2 + 2)}{4} \pi_1 \pi_2 
+ \frac{\lambda^2(\lambda^2 - 1)q_1}{\pi_2 q_2} + \frac{5\lambda^4}{2} - \lambda^2 - 3 \right) \frac{1}{(\lambda^2 + 2)q_2} \right\} \frac{\Sigma}{(\lambda^2 + 2)q_2} \right\} \frac{NQ}{\sqrt{q_2}} - \dot{\alpha}_{33} Q.$$  

(160)

It is easy (in principle) to find explicitly the wave equation for $Q$ from the above Hamilton equations, but instead of writing down one that has very complicated form, let us present the wave equation for $Q$ for the exact background solution (15). It is given by

$$\ddot{Q} - 2 \left( \frac{(t - 2k)}{t_{+}t_{-}} - \frac{X}{Z} \right) \dot{Q} + \left( m^2 \frac{t_{+}^2}{t_{-}^2} + \lambda^2 \frac{1}{t_{+}t_{-}} + \frac{Y}{Z} \right) Q = 0,$$

(161)

where

$$X \equiv 8 m^2 (t - 2k)t_{+}^2 + 2\lambda^2 m^2 (2t - k) t_{+}^2 + 2\lambda^2 (2t - 3k) + \frac{\lambda^4 (2t - k)t}{t_{+}},$$

$$Y \equiv 16 m^4 t_{+}^4 - 4 k m^2 t_{+} \left( \lambda^2 \frac{t - 4k}{t_{-}} - 8 \right) + 2 k^2 \lambda^4 \frac{2t + k}{t_{-} t_{+}^3},$$

$$Z \equiv 4 m^4 t_{+}^6 + 8 m^2 t_{-} t_{+}^3 + 4\lambda^2 m^2 t_{+}^3 + 2\lambda^2 t_{-} t_{+} + \lambda^4 t_{+}^2.$$

8.2 Nongeneric ($\lambda = 0$) perturbations

In the following we work with a nongeneric even mode phase space $E_{0,m}$ and suppose that the perturbation canonical tensors are given by (See Eq.(60))

$$\gamma_{ab} = \gamma_1 (E_1)_{ab} + \gamma_3 (E_3)_{ab},$$

$$p_{ab} = \mu_{q_0} (f_1 p_1 (E_1)^{ab} + f_3 p_3 (E_3)^{ab}).$$

(163)
Note that \( \dim \mathcal{E}_{0,m} = 4 \).

As we have seen in Sec. 4, when \( \lambda = 0 \) we have two constraints \( \mathcal{D}H^{(\lambda=0)} \approx 0 \) and \( \mathcal{D}H_t^{(\lambda=0)} \approx 0 \). (See Eqs. (65) and (70)) Note, however, that \( \mathcal{D}H_l^{(\lambda=0)} \approx 0 \) vanishes identically when \( m = 0 \), and thus we are left with a single nontrivial constraint \( \mathcal{D}H^{(\lambda=0,m=0)} \approx 0 \). We must therefore consider the two cases \( m = 0 \) and \( m \neq 0 \) separately.

### 8.2.1 The \( \lambda = 0 \) and \( m \neq 0 \) case: Pure gauge

We have two (nontrivial) constraints \( \mathcal{D}H^{(\lambda=0)} \approx 0 \) and \( \mathcal{D}H_l^{(\lambda=0)} \approx 0 \) when \( \lambda = 0 \) and \( m \neq 0 \). Since the phase space \( \mathcal{E}_{0,m} \) is four dimensional, these two constraints define \( 4 - 2 = 2 \) dimensional constraint subspace and the gauge transformations these constraint functions generate on this subspace contract this subspace to a point: \( \bar{\mathcal{E}}_{0,m \neq 0} = \{ e \} \). In other words, the phase space \( \mathcal{E}_{0,m \neq 0} \) is a space of pure gauge, and therefore we may not be interested in this space.

### 8.2.2 The case \( \lambda = 0 \) and \( m = 0 \): Locally homogeneous perturbations

In this case we have only one constraint \( \mathcal{D}H^{(\lambda=m=0)} \approx 0 \), so the dimension of the gauge-invariant phase space \( \dim \mathcal{E}_{0,0} \) is two(= 4 − 1 − 1). An explicit form of the constraint is obtained from Eq. (65) by putting \( m = 0 \):

\[
\mathcal{D}H^{(\lambda=m=0)} \equiv - p_1 \pi_2 q_1 q_2 - p_3 q_2 (\pi_1 q_1 - \pi_2 q_2) - \gamma^1 \frac{\pi_2^2 q_2^2}{2 q_1} + \gamma^3 \left( q_1 - \frac{\pi_1 \pi_2 q_1}{2} + \frac{3 \pi_2^2 q_2}{4} \right) \approx 0. \tag{164}
\]

We do not have to perform computations from scratch to decouple gauge. As easily seen, our rule to reinterpret the computations presented in Sec. 8.2 for the \( \lambda = m = 0 \) case is (1) first to put \( \gamma^i = p_i = 0 \) for \( i = 2,4 \), (2) then take limit \( m \to 0 \), (3) finally take limit \( \lambda \to 0 \). Keeping the order of steps (2) and (3) is necessary to drop correctly the terms like \( m/\lambda \). This order is particularly important when getting the correct wave equation, e.g., from Eq. (161).

We find that the gauge-invariant configuration variable becomes

\[
Q = - \left( \frac{\pi_1}{\pi_2} - \frac{q_2}{q_1} \right) \gamma^1 + \gamma^3. \tag{165}
\]

The wave equation can be obtained from the generic equation (161) by first putting \( m = 0 \), then taking the limit \( \lambda \to 0 \), as noted. We have

\[
\ddot{Q} + \frac{2}{t_+} \dot{Q} = 0. \tag{166}
\]

The solution is

\[
Q = C_1 t_+^{-1} + C_2, \tag{167}
\]

where \( C_1 \) and \( C_2 \) are integration constants.
9 Gauge Properties of the Odd Perturbations

In this section we discuss gauge issues for the odd perturbation system to obtain the perturbed spacetime metrics in a completed form. Remember that the system contains a gauge-dependent variable \( \gamma^* \) and the odd shift function \( s \). Choosing a particular profile of the shift function \( s(t) \) is called a gauge choice for the odd perturbations. The evolution of the gauge variable \( \gamma^* \) is determined by the evolution equation (the gauge equation) (101), depending on a given gauge choice. The gauge equation also depends (parametrically) on the gauge-invariant variable \( Q \), which is however determined independently of \( \gamma^* \) and \( s \). By solving the gauge equation for a given shift function \( s \) we can express the gauge variable \( \gamma^* \) in terms of \( Q \), and thereby we can express the metric functions \( \gamma^5 \) and \( \gamma^6 \) in terms of \( Q \). We see below how these functions are expressed in some gauge choices.

We first need to obtain the map acting on the odd variables induced by diffeomorphisms for later use.

**Lemma 2 (Induced map)** Let \( v(t) \) be an arbitrary function of time. Then, the one-parameter diffeomorphism generated by the odd vector

\[
Y_a = v(t) V_a
\]

induces the following map \( I_{\text{odd}} \) on the odd perturbation variables \( (s, \gamma^5, \gamma^6) \):

\[
\begin{align*}
I_{\text{odd}}[v] : \quad s &\rightarrow s + \left( \dot{v} - \frac{q_1}{q_1} v \right), \\
\gamma^5 &\rightarrow \gamma^5 + \sqrt{2(\lambda^2 + 2)} v, \\
\gamma^6 &\rightarrow \gamma^6 - mv,
\end{align*}
\]

where \( s = s(t) \) is the odd shift function.

**Proof:** This is a straightforward calculation of the induced map

\[
\delta g_{ab} \rightarrow \delta g_{ab} + 2 \nabla_a Y_b.
\]

Here \( \delta g_{ab} \) is an odd perturbation spacetime metric, and \( \nabla_a \) is the covariant derivative associated with the background spacetime. A straightforward computation shows

\[
2 \nabla_a Y_b = 2 \left( \dot{v} - \frac{q_1}{q_1} v \right) (dt)_a V_b + v(\sqrt{2(\lambda^2 + 2)}(E_5)_{ab} - m(E_6)_{ab}),
\]

from which we can immediately read off the induced map on \( \delta N_a, \gamma^5 \), and \( \gamma^6 \).

This induced map tells us that fixing the odd shift function \( s \) does not imply a complete fixing of the freedom of diffeomorphisms. We obtain the following.

**Proposition 3 (Pure gauge solution)** Consider the odd perturbation equations (74) with a given odd shift function \( s(t) \). Let \( C_1 \) be a constant parameter. Then, the set of functions

\[
\begin{align*}
\gamma^5_{(d)} &= C_1 q_1, \quad \gamma^6_{(d)} &= -C_1 \frac{\nu}{\sqrt{2}} q_1,
\end{align*}
\]

provides the one-parameter solution of the equations which is generated by spatial diffeomorphisms.
Proof: The zero perturbation \( s = \gamma^5 = \gamma^6 = 0 \) is a trivial solution of the perturbation equations. The diffeomorphism generated by the spatial vector (168) induces the map (169), which maps the zero solution to a pure gauge solution. We impose the condition that \( s = 0 \) be retained, then this requires that we choose \( v(\tau) = C_1 q_1 \), where \( C_1 \) is a constant. The pure gauge solution of the claim immediately follows after a redefinition of the constant \( C_1 \). This solution can be superposed with any other solution \((s, \gamma^5, \gamma^6)\) and does not change the odd shift function \( s \), from which the claim holds.

We can express the solution of the odd perturbation equations in terms of the gauge-invariant variable \( Q \) for a given gauge choice. While as we remarked, a gauge choice means a choice of the odd shift function \( s \), it is often useful to express \( s \) with another gauge function.

Proposition 4 (Gauge-variable-free general solution) Let \( Q \) be the gauge-invariant variable for the odd perturbation which evolves according to Eq.(104), and let \( \vartheta = \vartheta(\tau) \) be an arbitrary function of time. Then, the solution of the odd perturbation equations (74), up to diffeomorphisms, is given in terms of \( Q \) and \( \vartheta \) by

\[
\begin{align*}
  s &= O(Q, \dot{Q}; \vartheta), \\
  \gamma^5 &= \sqrt{2(\lambda^2 + 2)} \vartheta Q, \\
  \gamma^6 &= (1 - m\vartheta)Q,
\end{align*}
\]

where the function \( O(Q, \dot{Q}; \vartheta) \) is defined by

\[
O(Q, \dot{Q}; \vartheta) \equiv \left( \dot{\vartheta} - \frac{\dot{q}_1}{q_1} \vartheta - \frac{m}{\lambda^2 + 2} \vartheta \right) Q + \left( \vartheta - \frac{m}{\lambda^2 + 2} \frac{q_1}{u} \right) \dot{Q}.
\]

Proof: Remember that the gauge variable \( \gamma_s \) is governed by the evolution equation \( \dot{\gamma}_s = J\gamma_s \); see Eqs.(100) and (101). For convenience we re-express \( P \) in terms of \( \dot{Q} \) (and \( Q \)), using the Hamilton equations. Similarly, we express \( \pi_1 \) and \( \pi_1 \) (the background momentum variables) in terms of \( q_1, q_2 \) and their time-derivatives, using the background Hamilton equations. The gauge evolution equation \( \dot{\gamma}_s = J\gamma_s \) then becomes

\[
\dot{\gamma}_s = \left( \frac{\dot{q}_1}{q_1} - \frac{\dot{\vartheta}}{\vartheta} \right) \gamma_s = \frac{1}{\vartheta} \left( O(Q, \dot{Q}; \vartheta) - s \right),
\]

where \( O(Q, \dot{Q}; \vartheta) \) is the same as that defined in Eq.(174) (with \( \vartheta \) replaced by \( \theta \)). Here, \( \theta \) is the mock shift function (the parameter function in defining the gauge variable \( \gamma_s \)). From Eq.(171), we can think of \( \gamma_s \) as a function of \( \gamma^5 \) (or similarly of \( \gamma^6 \)),

\[
\gamma_s = Q - \frac{\gamma^5}{\sqrt{2(\lambda^2 + 2)} \vartheta}.
\]

Substituting this into the above equation we obtain the following evolution equation for \( \gamma^5 \),

\[
\dot{\gamma}^5 - \frac{\dot{q}_1}{q_1} \gamma^5 = \sqrt{2} \frac{q_1^2}{u} \left( \frac{Q}{q_1} \right)^{\cdot} + \sqrt{2(\lambda^2 + 2)} s.
\]
The homogeneous solution to this equation is given by $\gamma_{(\text{hom})}^5 = C_1 q_1$, which is however found to be the pure gauge solution (see the previous proposition), so we drop it. To find the special solution we are interested in it is useful to transform like $\gamma^5 = f/q_1$. Using a new function $\vartheta$, we rewrite the shift function $s$ as

$$s = O(Q, \dot{Q}; \vartheta)$$

so that the equation can explicitly be integrated. The solution for $\gamma^5$ is then given by $\gamma^5 = \sqrt{2(\lambda^2 + 2)}q_1$. $\gamma^6$ follows from the definition of $Q$, $\gamma^6 = Q - (\nu/\sqrt{2})\gamma^5$.

Note that there is no $\theta$-dependence in Eq. (178), therefore $\gamma^5$ and similarly $\gamma^6$ do not depend on the mock shift $\theta$.

**Remark** The above solution can be viewed as the “zero-gauge” solution corresponding to the “gauge-fixing” $\gamma_* = 0$. Remember, however, that we have freedom of choosing a function of time, the mock shift $\theta$, to define the gauge variable $\gamma_*$. Because of this arbitrariness the “gauge-fixing” by $\gamma_* = 0$ does not have any invariant meaning. We can obtain the formula led by this observation. In fact, from Eq. (175) the condition $\gamma_* = 0$ corresponds to taking the same relation (178) (with $\vartheta$ replaced by $\theta$), and the same formula for $\gamma^5$ and $\gamma^6$ is easily found from Eq. (90) and $\gamma_* = 0$. Therefore the function $\vartheta$ is the same as the mock shift $\theta$ under the condition $\gamma_* = 0$. Nevertheless the above formula is valid regardless of this condition, and in such a general situation the two functions $\vartheta$ and $\theta$ have nothing to do with each other; $\vartheta$ just parameterizes the odd shift $s$, while $\theta$ parameterizes the freedom of defining $\gamma_*$. In particular, $\vartheta$ can be zero, as opposed to $\theta$.

It is easy to express $\gamma^5$ and $\gamma^6$ directly in terms of the odd shift function $s$. To this we solve the relation $s = O(Q, \dot{Q}; \vartheta)$, which is a simple ODE for $\vartheta$. The solution is

$$\vartheta = q_1 \int \left\{ \frac{m}{\lambda^2 + 2} - \frac{q_1}{u} \left( \frac{Q}{q_1} \right)^{\prime} + \frac{s}{q_1} \right\} dt,$$

where we dropped the homogeneous solution $\vartheta_{(\text{hom})} = C_1 q_1 / Q$, which corresponds to the pure gauge solution. If we substitute this solution into Eqs. (173), we may obtain the explicit solution for a given odd shift function $s$.

One of the particular gauge choices of our interest may be so-called the synchronous gauge, which is characterized by $s = 0$ for the odd perturbations.

**Proposition 5 (Synchronous gauge)** Let $Q$ be the gauge-invariant variable for the odd perturbation which evolves according to Eq. (104). Then, the odd perturbed metric with vanishing shift is up to diffeomorphisms given by

$$s = 0, \quad \gamma^5 = \sqrt{2} \nu q_1 \left( \frac{Q}{u} - \frac{q_1}{u} \left( \frac{Q}{q_1} \right)^{\prime} \right), \quad \gamma^6 = \frac{q_2}{u} Q + \nu^2 q_1 \int \frac{Q}{q_1} \left( \frac{q_1}{u} \right)^{\prime} dt.$$ 

(180)
Proof: This can be immediately obtained by setting \(s = 0\) in Eq. (179):
\[
\vartheta = \vartheta_0 \equiv \frac{m}{\lambda^2 + 2} \frac{q_1}{Q} \int \frac{q_1}{u} \left( \frac{Q}{q_1} \right) dt.
\] (181)

We perform an integration by parts then substitute the resulting \(\vartheta = \vartheta_0\) into the general expression (173).

Formula (173) provides the most general form of the odd metric functions expressed in terms of the gauge invariant variable \(Q\). Let us confirm that they are certainly isometric to each other for any different choices of \(\vartheta\).

**Proposition 6 (Equivalence of the solutions)** The odd perturbed spacetime metrics specified in Proposition 4 with different \(\vartheta(t)\) are all isometric to each other (up to the linear order).

**Proof:** We show that the perturbed metric with an arbitrary \(\vartheta\) in Proposition 4 is isometric to the one in Proposition 5, which in turn implies that all the metrics are isometric to each other. Let \(g^{(0)} + \varepsilon \delta g + O(\varepsilon^2)\) be the perturbed spacetime metric that the perturbation metric in Proposition 4 defines. Let \(Y_a = v(t)V_a\) be an odd vector on this perturbed spacetime, and consider the diffeomorphism generated by this. From the map (169) we find that
\[
\gamma^5 = \sqrt{2} \left( \lambda^2 + 2 \right) \vartheta Q \to \sqrt{2} \left( \lambda^2 + 2 \right) \vartheta Q + \sqrt{2} \left( \lambda^2 + 2 \right) v = \sqrt{2} \left( \lambda^2 + 2 \right) (\vartheta Q + v),
\]
\[
\gamma^6 = Q - m \vartheta Q \to Q - m \vartheta Q - mv = Q - m(\vartheta Q + v),
\] (182)
from which we see that the transformation by the diffeomorphism is equivalent to transforming
\[
\vartheta \to \vartheta' = \vartheta Q + v.
\] (183)

On the other hand to make \(\delta N_a = 0\), it is found from the map (169) that we need to choose \(v(t)\) to satisfy
\[
\dot{v} - \frac{q_1}{q_1} v + O(Q, \dot{Q}; \vartheta) = 0.
\] (184)

We can easily solve this first order ODE, and obtain
\[
v(t) = -q_1 \int \frac{O(Q, \dot{Q}; \vartheta)}{q_1} dt.
\] (185)

With this \(v\), we have
\[
\vartheta' = \vartheta Q + v = \vartheta Q - q_1 \int \frac{O(Q, \dot{Q}; \vartheta)}{q_1} dt.
\] (186)

What we need to confirm is that this \(\vartheta'\) does not depend upon the original choice of \(\vartheta\) and coincides with \(\vartheta_0\) defined in Eq. (181). To this we first differentiate the above equation
(after an appropriate rearrangement) to delete the integral. Then, we have the following ODE for \( \vartheta' \):
\[
\dot{\vartheta}' + \left( \frac{\dot{Q}}{Q} - \frac{\dot{q}_1}{q_1} \right) \vartheta' = \dot{\vartheta} + \left( \frac{\dot{Q}}{Q} - \frac{\dot{q}_1}{q_1} - \frac{O(Q, \dot{Q}; \vartheta)}{\dot{Q}} \right) \vartheta.
\] (187)

Substituting the definition (174) of \( O(Q, \dot{Q}; \vartheta) \), we obtain
\[
\dot{\vartheta}' + \left( \frac{\dot{Q}}{Q} - \frac{\dot{q}_1}{q_1} \right) \vartheta' = \frac{m}{\lambda^2 + 2} \frac{q_1}{u} \left( \frac{\dot{Q}}{Q} - \frac{\dot{q}_1}{q_1} \right).
\] (188)

Now, we can see that the apparent dependence upon the original choice of \( \vartheta \) has disappeared. Moreover this equation is the same as \( O(Q, \dot{Q}; \vartheta') = 0 \), which implies \( \vartheta' = \vartheta_0 \). Hence, all the perturbed metrics with different \( \vartheta \) are isometric to the synchronous perturbed metric given in Corollary 5.

Finally, we introduce a particularly useful gauge choice in the present perturbation system. With this choice the perturbed metric does not contain, in contrast to the synchronous form given in Corollary 5, the time-derivative of \( Q \) and can be written in the simple multiplicative form \( \delta g = Q \times G \), where \( G \) is a symmetric tensor that depends only on the background variables.

**Proposition 7 (Standard gauge)** Let \( Q \) be the gauge-invariant variable for the odd perturbation which evolves according to Eq. (104). When the perturbation of the spacetime metric is written simply in the product form \( \delta g_{ab} = QG_{ab}^{(odd)} \) for a symmetric tensor \( G_{ab}^{(odd)} \) which is independent of \( Q \), the tensor \( G_{ab}^{(odd)} \) is given by
\[
G_{ab}^{(odd)} = \sqrt{2\nu} \frac{q_1}{u} (E_5)_{ab} + \frac{q_2}{u} (E_6)_{ab} + 2 \frac{m}{\lambda^2 + 2} \frac{q_1}{u} V_{(a}dt_{b)}.
\] (189)

**Proof:** Let us choose the function \( \vartheta \) in formula (173) so as to eliminate the \( \dot{Q} \)-dependence of the perturbed metric. When \( m \neq 0 \), we can eliminate the \( \dot{Q} \)-dependence in \( O(Q, \dot{Q}; \vartheta) \) by choosing (see Eq. (174))
\[
\vartheta = \bar{\vartheta} \equiv \frac{m}{\lambda^2 + 2} \frac{q_1}{u},
\] (190)
and in this case we have
\[
\delta N_a = \frac{m}{\lambda^2 + 2} \frac{q_1}{u} Q V_a, \quad \gamma^5 = \sqrt{2\nu} \frac{q_1}{u} Q, \quad \gamma^6 = \frac{q_2}{u} Q.
\] (191)

This completes the proof for the case \( m \neq 0 \). Since it is possible to take limit \( m \to 0 \) in the above expression, the same formula is correct in the case \( m = 0 \), as well. [In fact, when \( m = 0 \) the solution of (177) is given by]
\[
\gamma^5 = \sqrt{2(\lambda^2 + 2)} q_1 \int \frac{s}{q_1} dt
\] (192)
for a given $s$. From the assumption, we must have $s = fQ$ for a function $f$ which is independent of $Q$, but in this case the above $\gamma^5$ is not of the assumed form unless $f = 0$, which reproduces Eqs. (191) for $m = 0$.]

This gauge choice is similar to so-called the longitudinal gauge known in the perturbation theory of isotropic and homogeneous spacetimes. We call this gauge choice the standard gauge for the odd perturbation, due to its simplicity.

Other possible choices of gauge which are of some interest would be those by $\gamma^5 = 0$ ("solid base" gauge), and $\gamma^6 = 0$ ("orthogonality-preserving" or simply "orthogonal" gauge), which, respectively, correspond to $\vartheta = 0$ and $\vartheta = 1/m$. ("Orthogonality" here refers to the one between the base $\Sigma$ and fiber $S^1$.)

10 Gauge Properties of the Even Perturbations

Let us first see how diffeomorphisms act on the even perturbations.

**Lemma 8 (Induced map)** Let $a_i$ ($i = 0, 1, 3$) be arbitrary functions of time. Then, the one-parameter diffeomorphism generated by an even vector

$$Y_a = a_0 S(dt)_a + a_1 S_a + a_3 \bar{S} \sigma^3_a$$

(193)

induces the following map $I_{\text{even}}$ on the even perturbation variables:

$$\begin{align*}
    s_0 &\rightarrow s_0 - \frac{1}{N} \left( \dot{a}_0 + \frac{\dot{N}}{N} a_0 \right), \\
    s_1 &\rightarrow s_1 + \dot{a}_1 - q_1^{-1} q_1 a_1 + \lambda a_0, \\
    s_3 &\rightarrow s_3 + \dot{a}_3 - q_2^{-1} q_2 a_1 - ma_0,
\end{align*}$$

(194)

$I_{\text{even}}[a_0, a_1, a_3] :$

$$\begin{align*}
    \gamma^1 &\rightarrow \gamma^1 - \frac{\dot{a}_1}{N^2} a_0 - \lambda a_1, \\
    \gamma^2 &\rightarrow \gamma^2 + \sqrt{2(\lambda^2 + 2)} a_1, \\
    \gamma^3 &\rightarrow \gamma^3 - \frac{\dot{a}_2}{N^2} a_0 + 2ma_3, \\
    \gamma^4 &\rightarrow \gamma^4 - ma_1 + \lambda a_3,
\end{align*}$$

where $s_0$, $s_1$, and $s_3$ are, respectively, the even lapse function, even base shift function, and even fiber shift function.

**Proof**: This is a straightforward computation of $2 \nabla_a(Y_b)$. For our background metric $g_{ab} = -N^2(dt)_a(dt)_b + q_{ab}$, it holds that

$$\begin{align*}
    \nabla_a(dt)_b &= -\frac{1}{2N^2} \ddot{g}_{ab} \\
    &= \frac{\dot{N}}{N}(dt)_a(dt)_b - \frac{1}{2N^2} \ddot{g}_{ab}.
\end{align*}$$

(195)
Also, for a spatial vector $A_a$ such that $A_a(\partial_t)^a = 0$, we find

$$\nabla_a A_b = D_a A_b - \dot{q}^c d A_c \dot{d}(dt)_b.$$  \hfill (196)

Using these relations we can find

$$2\nabla_{(a} Y_{b)} = 2S \left( \dot{a}_0 + \frac{N}{a_0} (dt)_a \right) (dt)_b$$

$$+ 2 \left( \dot{a}_1 - \dot{q}_1^{-1} \dot{q}_1 a_1 + \lambda a_0 \right) (dt)_a S_b$$

$$+ 2 \left( \dot{a}_3 - \dot{q}_2^{-1} \dot{q}_2 a_1 - ma_0 \right) \bar{S}(dt)_a \sigma^3_b$$

$$- \frac{a_0}{N^2} (\dot{q}_1 (E_1)_{ab} + \dot{q}_2 (E_3)_{ab})$$

$$- a_1 \left( \lambda (E_1)_{ab} - \sqrt{2(\lambda^2 + 2)} (E_2)_{ab} + m (E_4)_{ab} \right)$$

$$+ a_3 \left( 2m (E_3)_{ab} + \lambda (E_4)_{ab} \right),$$  \hfill (197)

from which we can read off the map claimed.

It is straightforward to confirm that the even gauge-invariant perturbation variable $Q = -\Delta_1 \gamma^1 - \Delta_3 \gamma^2 + \gamma^3 - \frac{2m}{\lambda} \gamma^4$ is certainly invariant under this map, $Q \rightarrow \bar{Q}$. (To confirm this we may need the relation $\Delta_1 = \dot{q}_2 \dot{q}_1^{-1}$, which is obtained from the definition (144) using the background Einstein equations (12).)

As in the odd case let us find the pure gauge solution that can be superposed on any solution with given even lapse and shift functions.

**Proposition 9 (Pure gauge solution)** Consider the even perturbation equations with a given even lapse function $s_0$ and even shift functions $s_1$ and $s_3$. Let $C_0$, $C_1$, and $C_2$ be constants and let

$$a_0 = \frac{C_0}{N},$$

$$a_1 = -\lambda q_1 C_0 \int \frac{dt}{Nq_1} + C_1 q_1,$$

$$a_3 = m q_2 C_0 \int \frac{dt}{Nq_2} + C_2 q_2.$$  \hfill (198)

Then, the set of functions

$$\gamma^1_{(d)} = -\frac{\dot{q}_1}{N^2} a_0 - \lambda a_1,$$

$$\gamma^2_{(d)} = \sqrt{2(\lambda^2 + 2)} a_1,$$

$$\gamma^3_{(d)} = -\frac{\dot{q}_2}{N^2} a_0 + 2m a_3,$$

$$\gamma^4_{(d)} = -ma_1 + \lambda a_3,$$  \hfill (199)

represents the three parameter solution of the equations which is generated by diffeomorphisms.
**Proof**: Use the same logic of Proposition 3. Consider the diffeomorphisms generated by the even vector of the form (133). From Eqs. (194) we find that to keep $\delta N_a = 0$

\[
\dot{a}_1 - q_1^{-1}\dot{q}_1 a_1 = -\lambda a_0,
\]

\[
\dot{a}_3 - q_2^{-1}\dot{q}_2 a_1 = m a_0
\]

(200)

should hold. The solution is easily found, provided that $a_0$ is given:

\[
a_1 = -\lambda q_1 \int a_0 dt + C_1 q_1,
\]

\[
a_3 = m q_2 \int a_0 dt + C_2 q_2.
\]

(201)

On the other hand, we also find that to keep $\delta N = 0$

\[
\dot{a}_0 + \frac{\dot{N}}{N} a_0 = 0.
\]

(202)

The solution is

\[
a_0 = C_0 N^{-1}.
\]

(203)

The claim follows by substituting this into the above $a_1$ and $a_3$, and applying the corresponding $I_{\text{even}}$ map to the trivial (i.e., zero) solution. 

As in the odd case, we can find the longitudinal gauge-like solution, where the gauge-invariant variable $Q$ appears in a simple multiplicative form.

**Proposition 10 (Standard gauge)** Let $Q$ be the gauge-invariant (configuration) variable for the even perturbation which evolves according to Eqs. (160). When the perturbation of the spacetime metric is written in the product form $\delta g_{ab} = Q G_{ab}^{(\text{even})}$ for a symmetric tensor $G_{ab}^{(\text{even})}$ which is independent of $Q$ and depend only on background quantities, the tensor $G_{ab}^{(\text{even})}$ is given by

\[
G_{ab}^{(\text{even})} = -\frac{1}{\Omega} \left[ q_1 q_2 (E_1)_{ab} + 2 q_1^2 \Delta_2 (E_2)_{ab} - q_2 (q_1 \Delta_1 + q_2) (E_3)_{ab} + 2 \frac{m}{\lambda} q_1 q_2 (E_4)_{ab} + \frac{N^2}{\lambda^2 + 2} \left( 4 (\lambda^2 - 1) q_1 \pi_2 - 3 q_2 \right) S(dt)_a (dt)_b \right]
\]

\[
+ \frac{2 N^2}{\lambda^2 + 2} \left( \Sigma_1 + \frac{\lambda^2}{\lambda^2 + 2} \right) S(dt)_a (dt)_b
\]

\[
- \frac{2}{\lambda^2 + 2} \left( \Sigma_2 + \Sigma_3 \right) \left( \lambda S(dt)_a (dt)_b + m \bar{S} \sigma^3 (dt)_a (dt)_b \right)
\]

\[
- \frac{4 m}{\lambda^2} \left( \Sigma_1 + \bar{S} \sigma^3 (dt)_a (dt)_b \right)
\]

where

\[
\Sigma_1 \equiv \left( \frac{q_1 q_2}{\Omega} \right), \quad \Sigma_2 \equiv \left( \frac{q_1^2 \Sigma}{\Omega} \right), \quad \Sigma_3 \equiv \frac{q_1^2 \Sigma}{\Omega}
\]

and symbols $\Delta_1$, $\Delta_2$, $\Sigma$, and $\Omega$ are those defined in Eqs. (144), (145), and (155).
Sketch of the derivation of (204): The assumption of the gauge leads us to put $\gamma^i = \vartheta_i Q$ ($i = 1, 2, 4$), where $\vartheta_i$ are functions of the background quantities. ($\gamma^3$ can be determined from the definition of $Q$.) From Eqs. (148), this also means $\gamma^i = \vartheta_i Q$ ($i = 1, 2, 4$). We substitute these relations into the gauge equations (158). We then eliminate $P$ and rewrite the gauge equations in terms of $Q$ and $\dot{Q}$ by using (one of) the Hamilton equations (160), and determine the functions $\vartheta_i$ so that the coefficients of $\dot{Q}$ in those equations vanish. Those equations are simple algebraic equations which can be immediately solved, and the solution is written in terms of the background quantities only. We have

$$
\begin{align*}
\vartheta_1 &= -\frac{q_1 q_2}{\Omega}, \\
\vartheta_2 &= -\frac{2q_1^2 \Delta_2}{\Omega}, \\
\vartheta_4 &= -\frac{2m q_1 q_2}{\lambda}.
\end{align*}
$$

(206)

From these, $\gamma^i$ ($i = 1, 2, 4$), and therefore $\gamma^3$ are determined. The even lapse and shift functions $s_i$ ($i = 0, 1, 3$) are determined from the gauge equations with the above choice of $\vartheta_i$. They are also algebraic (linear) equations for $s_i$, so it is easy to solve, and the solution automatically satisfies the gauge assumption. We have

$$
\begin{align*}
s_0 &= N \left\{ \frac{\Omega^{-1}}{\lambda^2 + 2} \left( 2(\lambda^2 - 1)q_1 \pi_2^{-2} + \frac{3}{2} \pi_1 \right) Q, \\
&\quad - \frac{1}{q_1} \left[ \frac{\lambda^2}{\lambda^2 + 2} \left( \frac{q_1^2 \Sigma}{\Omega} \right) + \left( \frac{q_1 q_2}{\Omega} \right) \right] Q, \right\} Q, \\
s_1 &= -\frac{\lambda}{\lambda^2 + 2} \left( \left( \frac{q_1^2 \Sigma}{\Omega} \right) + \frac{q_2^2 \Sigma}{\Omega} \right) Q, \\
s_3 &= -m \left[ \frac{2}{\lambda^2} \left( \frac{q_1 q_2}{\Omega} \right) + \frac{1}{\lambda^2 + 2} \left( \left( \frac{q_1^2 \Sigma}{\Omega} \right) + \frac{q_2^2 \Sigma}{\Omega} \right) \right] Q.
\end{align*}
$$

(207)

(We have used the background constraint $\mathcal{H} = 0$ to simplify the result, e.g., eliminating one of the background variables $\pi_1$ in $s_0$.) From these we obtain (204).

Due to the complexity (or lengthiness) of the even gauge equations it may be impractical to write down a general solution for an arbitrary gauge. Instead, we can derive any solution with a desired gauge by acting the map $I_{\text{even}}$ on the above standard gauge solution. In this case, the functions which are considered free are the three functions $a_i$ ($i = 0, 1, 3$).

11 Summary and Discussions

A brief summary first. We have seen that mode functions on a closed orthogonal Bianchi III space are specified by two kinds of eigenvalues; the fiber eigenvalue $m$ and the base eigenvalue $\lambda$. Using the metrics and covariant derivatives on the base $\Sigma_g$ and the fiber $S^1$, we have constructed the “even” mode vectors and tensors from the mode functions.
Using the area two-form $\varepsilon_{ab}$ on the base in addition to the same metrics and covariant derivatives, we have constructed the “odd” mode vectors and tensors. The “harmonic” mode vectors and tensors are associated with the (Hodge) harmonic vectors on the base and the mode functions on the fiber. The “TT” mode tensors are associated with the TT tensors on the base and the mode functions on the fiber. This way the mode quantities can naturally be categorized into the even, odd, harmonic, and TT ones. Perturbations of vacuum solutions associated with different kinds (i.e., even, odd, · · ·) and/or different eigenmodes are all independent of each other, in other words, they all decouple, as stated in Theorem 1. This makes it possible to treat them separately.

We have defined mode tensors $(E_i)_{ab}$ ($i = 1 \sim 9$) for a given set of $m$ and $\lambda$ ($i = 1 \sim 6$) or for a given $m$ ($i = 7 \sim 9$), which are eigentensors of the Laplacian and are $L^2$-orthogonal to each other with respect to the standard $L^2$-norm. To separate variables we expand the first variation of the spatial metric in terms of the mode tensors, resulting in getting the coefficients $\gamma_i = \gamma_i(t)$ as the fundamental perturbation configuration variables. The canonical conjugates can also be expanded by the same mode tensors and we obtain $p_i = p_i(t)$ as the conjugates of $\gamma_i$. The Hamiltonians for these perturbation variables are provided in Eq.(75) (odd part) and (125) (even). That the harmonic part can be obtained from the odd one by formally taking limit $\lambda \rightarrow 0$ was pointed out in Sec.6. The Hamilton equations for the TT part are given in Eqs.(114).

A set of Hamilton (i.e., evolution) equations for perturbation variables is generally a system of complicatedly coupled linear ODEs which depend on choice of gauge, i.e., depend on choice of variations of lapse function $\delta N$ and shift vector $\delta N_a$. One of our main tasks in our course of analysis is to decouple the system into gauge-dependent and independent parts for each one of the even and odd perturbation systems. (The harmonic one can be thought of as a limit of the odd system, as already remarked. Since the TT system is not constrained, the fundamental variables for the TT system are themselves gauge-invariant.) One of the advantages of employing Hamiltonian formalism is that this task amounts to finding a certain kind of canonical transformation.

We have used the method of generating function to find such desired canonical transformations. This systematic method was crucial for our achievement of the task. The decoupled-form Hamiltonians are given in Eq.(92) with (97) and (104) (odd) and Eq.(153) with (154) and (157) (even). The wave equations for gauge-invariant variables $Q$ (the definition of $Q$ is different depending on the type of perturbations), or the Hamilton equations for $Q$ and its conjugate $P$ are given in Eq.(104) (odd) and (160) (even). These evolution equations for the gauge-invariant variables are the “heart” of the perturbation dynamics.

To recover an entire perturbed spacetime metric, however, it is unavoidable to choose a particular gauge. As a useful choice, we have defined the standard gauge, in which the gauge-invariant variable $Q$ appears in the perturbed metric in the simplest manner where the metric is expressed in terms of $Q$ only and free from $\dot{Q}$ or $P$. The forms of metrics in this gauge are given in Propositions 7 and 10.

We have thus resolved all the fundamental issues concerning the technical aspects of the perturbations of closed orthogonal Bianchi III vacuum spacetimes, including separation of variables, decoupling into gauge-dependent and independent parts, derivation of wave equations, and choice of gauge.
Our primary interest concerning properties of the perturbations of the Bianchi III spacetime may be their future asymptotic behaviors. For this not only do we need to perform a further careful analysis of the wave equations but we need to clarify how to subtract the effect of the anisotropic expansion of the background and thereby get the “net” effect of the perturbations. This nontrivial task is indispensable to see under what condition the system should be called “stable,” or to say at what rate the perturbations decay or grow. These subjects will be discussed in a separate paper. In the remaining part, let us discuss the scalar and electromagnetic field equations and the $\tau\tau$ perturbation equations, which are simpler than the other perturbation equations.

In Appendix B we have dealt with the wave equation for massless scalar fields. When the background is Minkowski (with a slicing that is Bianchi III), we can obtain the exact solutions for the (mode-decomposed) wave equations for all possible eigenvalues. The solutions tell us that the fields with $m \neq 0$ all decay as $t \to \infty$ at the same rate (See Eq. (250)), while the $U(1)$-symmetric (i.e., $m = 0$) fields decay at several different rates depending on the value of $\lambda$ (See Eq. (252)). It is however true that all modes except the zero mode are decaying. It seems therefore plausible that all the modes are decaying (except the zero mode) for any general orthogonal Bianchi III backgrounds, too.

The (source-free) electromagnetic field equation has been dealt with in the subsequent appendix. Because of the vector-like nature of the electromagnetic fields, the mode-decomposed wave equations belong to either the even type or odd type (or the harmonic type, which can be obtained as the limit $\lambda \to 0$ of the odd type). Interestingly enough, the wave equations for both even and odd fields coincide with those for the scalar field (after a normalization if necessary) when the background is Minkowski. They are therefore decaying again on this background (if we think of the normalizations involved natural).

Among the gravitational perturbations the simplest are the $\tau\tau$ perturbations. This kind of perturbation is, as noted in Sec. 7, the only exceptional case among the four kinds where it is clear how the variable should be normalized to see the stability. We have seen in the same section that the wave equations for the (normalized) $\tau\tau$ perturbation variable, and therefore their asymptotic behaviors, are exactly the same as those for the scalar fields (for $\lambda = 0$). The background is therefore expected to be stable against the $\tau\tau$ perturbations.

As noted, detailed studies of dynamical properties of the perturbations will be reported in a separate paper [24].

A Spatially Closed Bianchi III Spacetimes

In this section we discuss how we can construct the spatially closed Bianchi III spacetimes. The so-called Teichmüller parameters are the key ingredient to understand the compactifications involved. The Teichmüller space of a manifold $M$ modeled on a homogeneous space $X$ is the quotient space $\mathcal{M}/\mathcal{D}_0$, where $\mathcal{M} = \mathcal{M}(M, X)$ is the space of the locally homogeneous metrics on $M$ which are locally isometric to $X$, and $\mathcal{D}_0 = \mathcal{D}_0(M)$ stands for the diffeomorphisms of $M$ that are connected to the identity. Teichmüller parameters are
defined as coordinates in a Teichmüller space \(^{12}\) and they are interpreted as the configuration variables that emerge as a result of compactification. We note that Teichmüller parameters can also be understood as the parameters in the covering group acting on \(X\). In the following we first show how we can compactify Bianchi III homogeneous manifolds by explicitly finding covering groups acting on them, and then generalize the compactification to ones for spacetimes.

Let \(H^2 = (\mathbb{R}^2, \tilde{h}_{ab})\) be the standard hyperbolic plane, and \(E^1 = (\mathbb{R}, \tilde{l}_{ab})\) be the standard Euclidean line. We introduce the standard coordinate systems with which these metrics are expressed\(^{13}\)

\[
\tilde{h}_{ij}dx^i dx^j = \frac{dx^2 + dy^2}{y^2},
\]

and

\[
\tilde{l}_{ij}dx^i dx^j = dz^2.
\]

The coordinates are defined in the region \(x \in (-\infty, \infty), y \in (0, \infty), \) and \(z \in (-\infty, \infty), \) and they span the whole \(H^2\) and \(E^1.\)

As well known \(H^2\) can be compactified to a higher genus surface \(\Sigma_g (g \geq 2)\) by the left-action of a discrete subgroup \(\Gamma_H\) of the orientation-preserving isometry group \(\text{Isom}^+ \mathbb{H}^2.\)

\[
\pi_H : H^2 \rightarrow (\Sigma_g, h_{ab}) = H^2/\Gamma_H.
\]

The topology of the closed surface \(\Sigma_g\) can be specified by its fundamental group \(\pi_1(\Sigma_g)\), which is an infinite group generated by \(2g\) generators \(\alpha_i\) and \(\beta_i (i = 1 \sim g)\) which satisfy a single relation \([\alpha_1, \beta_1] \cdots [\alpha_g, \beta_g] = 1.\) That is, in the standard notation,

\[
\pi_1(\Sigma_g) = \langle \alpha_1, \cdots, \alpha_g, \beta_1, \cdots, \beta_g | [\alpha_1, \beta_1] \cdots [\alpha_g, \beta_g] = 1 \rangle,
\]

where \([\alpha_i, \beta_i] \equiv \alpha_i \beta_i \alpha_i^{-1} \beta_i^{-1}\) is the commutator of group elements. The covering group \(\Gamma_H\) is a representation of this infinite group into \(\text{Isom}^+ \mathbb{H}^2.\)

The group \(\text{Isom}^+ \mathbb{H}^2\) is isomorphic to the projective real special linear group of second rank, \(\text{Isom}^+ \mathbb{H}^2 \simeq \text{PSL}(2, \mathbb{R}).\) Suppose we embed the generators \(\alpha_i\) and \(\beta_i\) into \(\text{PSL}(2, \mathbb{R})\) so that they satisfy the relation for \(\pi_1(\Sigma_g).\) (That is, we identify \(\alpha_i\) and \(\beta_i\) with elements in \(\text{PSL}(2, \mathbb{R})\) so that they satisfy the relation for \(\pi_1(\Sigma_g)\) with respect to the multiplications in \(\text{PSL}(2, \mathbb{R}).\)) However, we do not need an explicit representation of such \(\alpha_i\) and \(\beta_i\) in \(\text{PSL}(2, \mathbb{R}).\) We just note they generate \(\Gamma_H,\) which we denote

\[
\Gamma_H = \{\alpha_1, \cdots, \alpha_g, \beta_1, \cdots, \beta_g\},
\]

where \(\alpha_i, \beta_i \in \text{PSL}(2, \mathbb{R}) (i = 1 \sim g).\)

\(^{12}\)Beware that several different definitions of “Teichmüller space” are used in the literature. In general, they are equivalent for two dimensional surfaces, but not for higher dimensional manifolds, including three-dimensional ones. Ours follow \([10, 16, 9, 20, 21]\).

\(^{13}\)The hyperbolic plane model with this metric defined in the region \(y > 0\) is known as the \(upper-half plane model\), and the plane is often denoted like \(\mathbb{R}_+^2\) to emphasize it is defined on the upper-half plane. In this section however we understand that \(\mathbb{R}_+^2\) is homeomorphic to \(\mathbb{R}^2\) and do not distinguish those two for notational simplicity.
Note that, since $\text{PSL}(2, \mathbb{R})$ is three dimensional, we need (at most) $2g \times 3 = 6g$ parameters to represent $\Gamma^H$. However, since the generators must satisfy the relation $[\alpha_1, \beta_1] \cdots [\alpha_g, \beta_g] = 1$, the number of independent parameters reduces by three, because this relation has three components. Furthermore, remember the fact that different $\Gamma^H$ and $\Gamma^{H'}$ can result in the same closed surface, i.e., $H^2/\Gamma^H$ and $H^2/\Gamma^{H'}$ are isometric to each other, if $\Gamma^H = a^{-1} \circ \Gamma^{H'} \circ a$ for an $a \in \text{Isom} H^2$. (This map is called conjugation. See, e.g., [21].) Since $\text{Isom} H^2$ is three dimensional, these conjugation maps reduce the number at most by another three, and we know they do reduce it by the maximal number, hence resulting in $6g - 6$ independent parameters in $\Gamma^H$. This count corresponds to the well-known fact (e.g. [22]) that a higher genus surface $\Sigma$ has $6g - 6$ Teichmüller parameters.

Let us generalize the covering group $\Gamma^H$ of the closed surface $\Sigma$ to that of the trivial $S^1$-fiber bundle $M \simeq \Sigma_g \times S^1$. The topology of this closed manifold can be uniquely specified by its fundamental group, which can be represented by

\begin{equation}
\pi_1(M) = \langle \alpha_1, \cdots, \alpha_g, \beta_1, \cdots, \beta_g, \varphi \mid [\alpha_1, \beta_1] \cdots [\alpha_g, \beta_g] = [\alpha_1, \varphi] = [\beta_1, \varphi] = 1 \rangle,
\end{equation}

where the new generator $\varphi$ corresponds to the loop along the $S^1$-fiber. (This expression of $\pi_1$ follows from the general theory of $S^1$-bundles over a surface or more general Seifert fiber spaces. See, e.g., [23].)

We consider, as the universal cover, the direct product $H^2 \times E^1$ with the rescaled metric

\begin{equation}
\tilde{g}_{ab} = q_1 \tilde{h}_{ab} + q_2 \tilde{l}_{ab},
\end{equation}

where $q_1$ and $q_2$ are constants. The isometry group of this manifold is the direct product of those of $H^2$ and $E^1$, i.e., $\text{Isom} H^2 \times E^1 \simeq \text{Isom} H^2 \times \text{Isom} E^1$, where the orientation-preserving component of $\text{Isom} E^1, \text{Isom}^+_E E^1$, is the one-dimensional translation group. So, an element $\tilde{\alpha} \in \text{Isom}_0 H^2 \times E^1 \simeq \text{Isom}^+_H H^2 \times \text{Isom}^+_E E^1$ can be\(^{14}\) expressed as

\begin{equation}
\tilde{\alpha} = \begin{pmatrix} \alpha \\ s \end{pmatrix},
\end{equation}

using $\alpha \in \text{Isom}^+_H H^2$ and $s \in \text{Isom}^+_E E^1$. This acts on points $x = (x, y, z) \in M$, viewing $x = (x, y) \times \{z\} \in H^2 \times E^1$. The multiplication for two isometries $\alpha, \alpha' \in \text{Isom}_0 H^2 \times E^1$ is defined by

\begin{equation}
\tilde{\alpha} \cdot \tilde{\alpha}' = \begin{pmatrix} \alpha \\ s \end{pmatrix} \cdot \begin{pmatrix} \alpha' \\ s' \end{pmatrix} = \begin{pmatrix} \alpha \alpha' \\ s + s' \end{pmatrix},
\end{equation}

where $\alpha \alpha'$ is understood as the multiplication in $\text{PSL}(2, \mathbb{R})$.

**Proposition 11** Let

\begin{equation}
\Gamma^H = \{\alpha_1, \cdots, \alpha_g, \beta_1, \cdots, \beta_g\}
\end{equation}

\(^{14}\)The group $\text{Isom}_0 H^2 \times E^1 \simeq \text{Isom}^+_H H^2 \times \text{Isom}^+_E E^1$ is the identity component of $\text{Isom}^+_H H^2 \times E^1$, the orientation-preserving isometry group of $H^2 \times E^1$. We focus on this component to realize a compactification. To generalize the argument to the full $\text{Isom}^+_H H^2 \times E^1$, one may need to consider the orientation-reversing isometries of $H^2$ and $E^1$, as well.
be the covering group for \((\Sigma g, h_{ab})\) acting on \(H^2\). Let \(M\) be the product \(\Sigma g \times S^1\) and let \((M, q_{ab})\) be a locally homogeneous manifold modeled on \(H^2 \times E^1 = (\mathbb{R}^3, \tilde{q}_{ab})\), where \(\tilde{q}_{ab}\) is the rescaled metric \((214)\). Let us represent \((M, q_{ab})\) with the following covering map
\[
\pi : (\mathbb{R}^3, \tilde{q}_{ab}) \to (M, q_{ab}) = (\mathbb{R}^3, \tilde{q}_{ab})/\Gamma.
\] (218)

Then the covering group \(\Gamma \subset \text{Isom}_0 H^2 \times E^1\) can be represented as
\[
\Gamma = \left\{ \left( \begin{array}{c} \alpha_1 \\ v_1 \\ \vdots \\ \alpha_g \\ v_g \\ \beta_1 \\ w_1 \\ \vdots \\ \beta_g \\ w_g \\ 1 \\ \varphi \end{array} \right) \right\},
\] (219)

where \(v_i \in \mathbb{R}, w_i \in \mathbb{R}\) and \(\varphi > 0\) are arbitrary constants. Here, the \(8g - 5\) independent parameters in \(\Gamma\), i.e., \(6g - 6\) parameters in \(\Gamma^H\) plus the \(2g + 1\) parameters \(v_i, w_i\) and \(\varphi\), span the Teichmüller space of \(M\) modeled on \((\mathbb{R}^3, \tilde{q}_{ab})\).

**Proof:** Let us denote the generators of \(\Gamma\) as in
\[
\Gamma = \left\{ \tilde{\alpha}_1, \ldots, \tilde{\alpha}_g, \tilde{\beta}_1, \ldots, \tilde{\beta}_g, \tilde{\varphi} \right\},
\] (220)

where \(\tilde{\alpha}_i, \tilde{\beta}_i, \tilde{\varphi} \in \text{Isom}_0 H^2 \times E^1\). From the relation \([\tilde{\alpha}_1, \tilde{\beta}_1] \cdots [\tilde{\alpha}_g, \tilde{\beta}_g] = 1\), it is obvious that the Isom_+ E^- part of \(\tilde{\alpha}_i\) and \(\tilde{\beta}_i\) should satisfy the unarrowed same relation, from which we have
\[
\tilde{\alpha}_i = \left( \begin{array}{c} \alpha_i \\ v_i \end{array} \right), \quad \tilde{\beta}_i = \left( \begin{array}{c} \beta_i \\ w_i \end{array} \right), \quad (i = 1 \sim g)
\] (221)

where \(v_i\) and \(w_i\) are real constants, and \(\alpha_i\) and \(\beta_i\) are the generators of \(\Gamma^H\). Conversely, if \(\tilde{\alpha}_i\) and \(\tilde{\beta}_i\) are given in this form, from the fact that Isom_+ E^- is abelian, we see
\[
[\tilde{\alpha}_1, \tilde{\beta}_1] \cdots [\tilde{\alpha}_g, \tilde{\beta}_g] = \left( \begin{array}{c} 1 \\ 0 \end{array} \right) = 1,
\] (222)

that is, the \(2g\) generators \((221)\) satisfy the required relation for \(\pi_1(M)\) for arbitrary \(v_i\) and \(w_i\).

For an isometry \(\tilde{a} \in \text{Isom}_0 H^2 \times E^1\) of the form
\[
\tilde{a} = \left( \begin{array}{c} a \\ s \end{array} \right),
\] (223)

the conjugation map acts on \(\tilde{\alpha}_i\) (and similarly on \(\tilde{\beta}_i\)) as
\[
\tilde{a}^{-1} \circ \tilde{\alpha}_i \circ \tilde{a} = \left( \begin{array}{c} a^{-1}\alpha_i a \\ v_i \end{array} \right).
\] (224)

That is, the conjugation is effective only on the Isom_+ H^2 part. From the discussion on \(\Gamma^H\), this fact merely implies that the \(2g\) generators \(\tilde{\alpha}_i\) and \(\tilde{\beta}_i\) contain \(6g - 6\) independent parameters.
It is also easy to check that to meet the remaining commutativity relations \([\vec{\alpha}_i, \vec{\varphi}] = [\vec{\beta}_i, \vec{\varphi}] = 1\) the \(S^1\)-fiber generator \(\vec{\varphi}\) should be of the form

\[
\vec{\varphi} = \begin{pmatrix} 1 \\ \varphi \end{pmatrix},
\]

where \(\varphi > 0\) is an arbitrary positive parameter. (\(\varphi\) cannot be zero for \(M\) not to degenerate. Moreover, \(\vec{\varphi} = (1, \varphi)\) is conjugate to \(\vec{\varphi}_- = (1, -\varphi)\) by the reflection isometry \(r: (x, y, z) \rightarrow (x, y, -z)\). So, it is enough to consider positive \(\varphi\).) The conjugation maps with respect to the isometries connected to the identity do not affect this generator. This completes the proof.

The Teichmüller space of \(M\) was first discussed in [10, 16].

Figure 1 pictures a fundamental domain of a compactified Bianchi III manifold and its projection onto the \(H^2\)-plane of \(z = 0\).

![Figure 1: A fundamental domain of a compactified Bianchi III manifold (the decahedron in the figure) and its projection onto a \(H^2\)-plane (the octagon in the \(x-y\) plane). (For convenience of presentation, the \(x-y\) coordinates in this figure are assumed to be those of the Poincaré model rather than the upper half plane model.)](image)

Note that the universal cover \(H^2 \times E^1\) has two natural structures; the foliation by \(E^1\), and the foliation by \(H^2\). It is easy to see that the foliation by \(E^1\), which is equivalent to the foliation by our \(z\)-axes, descends to a foliation by \(S^1\) after the compactification we discussed (cf. Theorem 4.13 in [19]). In contrast, the above proposition tells us that the \(H^2\)-foliation does not descend to a \(\Sigma_g\)-foliation unless \(v_i = w_i = 0\) for all \(i\). (Note that the \(H^2\)-foliation can be expressed in terms of the coordinates as the foliation by \(z = \text{constant}\) surfaces. So, the condition that the \(H^2\)-foliation descends to a \(\Sigma_g\)-foliation is that each \(z = \text{constant}\) surface is compactified to \(\Sigma_g\). This does not happen unless
\( v_i = w_i = 0 \), as seen from the previous proposition.) We call these \( 2g \) parameters \( v_i \) and \( w_i \) the distortion parameters. \(^{15}\) It may be natural to consider the subclass where the inheritance \( H^2\)-foliation \( \rightarrow \Sigma_g\)-foliation holds:

**Definition 12** When the distortion parameters all vanish, i.e., when the covering group \( \Gamma \) in Proposition 11 is of the form

\[
\Gamma_{\text{orth}} = \left\{ \left( \alpha_1 \right), \ldots, \left( \alpha_g \right), \left( \beta_1 \right), \ldots, \left( \beta_g \right), \left( 1 \right) \right\},
\]

we say that the resulting closed manifold \((M, q_{ab})\) is orthogonal.

Note that the closed hyperbolic surface \((\Sigma_g, q_1 h_{ab})\) does not exist (or, is not embedded) in \((M, q_{ab})\), unless \((M, q_{ab})\) is orthogonal. \(^{16}\)

When \((M, q_{ab})\) is orthogonal, the manifold admits both the foliation by \((\Sigma_g, q_1 h_{ab})\) and foliation (or fibration) by \((S^1, q_2 l_{ab})\). The metric \( q_{ab} \) which is induced from the universal cover one \( \tilde{q}_{ab} \) can be written in the form

\[
q_{ab} = q_1 h_{ab} + q_2 l_{ab}.
\]

A tangent vector \( v^a = v^1 \partial_x + v^2 \partial_y + v^3 \partial_z \) on \( p \in M \) can be uniquely decomposed as

\[
v^a = v^a_H + v^a_S,
\]

where \( v^a_H = v^1 \partial_x + v^2 \partial_y \) is the part tangent to the \((\Sigma_g, q_1 h_{ab})\)-leaf to which the point \( p \) belongs, and \( v^a_S = v^3 \partial_z \) is the part tangent to the \((S^1, q_2 l_{ab})\)-fiber to which \( p \) belongs. \( v^a_H \) and \( v^a_S \) are orthogonal to each other with respect to the metric \( q_{ab} \). We can also check from the above form of the metric that the covariant derivative operator \( D_a \) associated with \( q_{ab} \) can be written in the form

\[
D_a = \hat{D}_a + \sigma^3 a \chi_3,
\]

where \( \hat{D}_a \) is the covariant derivative operator associated with \( h_{ab} \) and \( \chi_3 = \partial_z \) is the derivative operator along the circle fiber. It is easy from this formula to see that the Laplacian \( \triangle_q = q^{ab} D_a D_b \) with respect to \( q_{ab} \) is given by the sum

\[
\triangle_q = q_1^{-1} \triangle_h + q_2^{-1} (\chi_3)^2,
\]

where \( \triangle_h = h^{ab} \hat{D}_a \hat{D}_b \) is the Laplacian with respect to \( h_{ab} \). Therefore the product of eigenfunctions for \((\Sigma_g, h_{ab})\) and for \((S^1, l_{ab})\) becomes an eigenfunction for \((M, q_{ab})\).

Our next step is to generalize our compactification to one for spacetimes. Let \( \hat{Y} \equiv (\mathbb{R}^4, \tilde{g}_{ab}) \) be the universal covering spacetime of a spatially closed Bianchi III spacetime.

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\(^{15}\) We do not use the word “twist” as in \(^{[10]}\) to avoid confusions with those in the so-called pants-decompositions of hyperbolic higher genus surfaces.

\(^{16}\) It is apparent that even if \((M, q_{ab})\) is not orthogonal there exist a foliation of \( M \) by \( \Sigma_g \), since \( M \) is (topologically) the product \( \Sigma_g \times S^1 \). The induced metric \( h_{ab} \) on each leaf of such a foliation is however not hyperbolic, \( h_{ab} \neq q_1 h_{ab} \) (\( q_1 \) is a constant).
Y ≡ (M × R, q_{ab}). Let t be a time function of (R^4, \tilde{g}_{ab}) defined in such a way that each \( t = \text{constant} \) surface coincides with (R^3, \tilde{q}_{ab}), where the scale factors \( q_1 \) and \( q_2 \) in \( \tilde{q}_{ab} \) are regarded as functions of \( t \). Then, we assume that the universal covering spacetime metric is of the synchronous form

\[
\tilde{g}_{ab} = -N^2(\text{d}t)_a(\text{d}t)_b + \tilde{q}_{ab} = -N^2(\text{d}t)_a(\text{d}t)_b + q_1\tilde{h}_{ab} + q_2\tilde{l}_{ab},
\]

where \( N = N(t) \) and \( (\text{d}t)_a \equiv \nabla_a t \). Apparently, Isom\( \tilde{Y} = \text{Isom}H^2 \times \text{Isom}E^1 \). Isom\( \tilde{Y} \) is also equivalent to the isometry group of the spatial manifold, Isom\( \tilde{Y} = \text{Isom}\tilde{M} \equiv \text{Isom}(\tilde{M}, \tilde{q}_{ab}) \). In terminology adopted in [20, 21], this fact can be rephrased as Esom\( \tilde{M} = \text{Isom}\tilde{M} \), where Esom\( \tilde{M} \) is the extendible isometry group of (R^3, \tilde{q}_{ab}).

In general, \( n = \text{dim Isom}\tilde{M} - \text{dim Esom}\tilde{M} \) gives the degrees of freedom of choosing the velocities of the Teichmüller parameters of \( M \) at an initial surface \( t = \text{constant} \) [20, 21]. In the present case where \( n = 0 \) however there is no such freedom, and the compactification of the spatial manifold straightforwardly generalizes to the compactification of the spacetime manifold, i.e., our spatially closed spacetime solution can be represented as

\[
(M \times R, g_{ab}) = (R^4, \tilde{g}_{ab})/\Gamma,
\]

where \( \Gamma \) is of the form [219] and acts on \( R^4 \) as

\[
\Gamma \cdot (t, x) = (t, \Gamma x).
\]

\( \tilde{g}_{ab} \) may be assumed to satisfy Einstein’s equation on the universal cover \( R^4 \). We can write the spacetime metric \( g_{ab} \) as

\[
g_{ab} = -N^2(\text{d}t)_a(\text{d}t)_b + q_{ab}
\]

in both cases where \( (M, q_{ab}) \) is orthogonal and non-orthogonal. It is however only for the orthogonal case that the spatial metric \( q_{ab} \) can be decomposed as Eq.(227).

The vacuum universal cover metrics \( \tilde{g}_{ab} \) form a one parameter family up to isometry. On the other hand, \( \Gamma \) has \( 8g - 5 \) parameters, which are related to the Teichmüller parameters of the spatial section. The general solution is therefore a \( 8g - 4 (= 8g - 5 + 1) \) parameter solution. The orthogonal solution, where \( \Gamma = \Gamma_{\text{orth}} \), has only \( 6g - 4 \) parameters. However, even if the background solution is of orthogonal type, i.e., even if the distortion parameters of the background solution are all zero, it is worth remarking that the distortion parameters can be perturbed off the zeros (through the zero mode harmonic perturbations).

\[17\] The isometry group of a spacetime with time function \( t \) has the subgroup whose action retains the foliation by \( t = \text{constant} \). This subgroup can also be identified with a subgroup of the spatial manifold specified by \( t = \text{constant} \), and called the extendible isometry group of this spatial manifold [20, 21]. Note that in general to compactify a spatially homogeneous spacetime \( \tilde{Y} \) to a spatially locally homogeneous spacetime \( Y \) with spatial manifold \( M \), we need to embed \( \pi_1(M) \) into \( \text{Isom}\tilde{M} \) but into \( \text{Esom}\tilde{M} \). (This is apparent if we think of \( \text{Esom}\tilde{M} \subset \text{Isom}\tilde{Y} \).)
B Wave Equation for the Scalar Field

Consider the scalar wave equation
\[ g^{ab}\nabla_a\nabla_b\Psi = 0. \tag{235} \]

This equation can be derived from the action \( I = (-1/2) \int g^{ab}(\nabla_a\Psi)(\nabla_b\Psi)d\mu_g \), from which we can read the following Lagrangian for our background
\[ L = -\frac{1}{2}Nq_1\sqrt{q_2}\int_M \left(-\frac{1}{N^2}(\dot{\Psi})^2 + q^{ab}(D_a\Psi)(D_b\Psi) \right)d\mu_{q_0}. \tag{236} \]

The momentum \( p_\Psi \) conjugate to \( \Psi \) is
\[ p_\Psi = \frac{\partial L}{\partial \dot{\Psi}} = \frac{q_1\sqrt{q_2}}{N}\dot{\Psi}, \tag{237} \]
so the Hamiltonian is given by
\[ H = \int p_\Psi \dot{\Psi} - L = \frac{N}{2} \int_M \left( \frac{p_\Psi^2}{q_1\sqrt{q_2}} + q_1\sqrt{q_2}q^{ab}(D_a\Psi)(D_b\Psi) \right)d\mu_{q_0}. \tag{238} \]

Let us expand the wave function using the mode function \( S = S_{\lambda,m} \) as
\[ \Psi = \psi S, \quad p_\psi = p_\psi S, \tag{239} \]
where \( \psi = \psi(t) \) and \( p_\psi = p_\psi(t) \) are functions of time, since each mode of this form decouples from the others. Substituting this form into the above Hamiltonian and performing integration by parts (and dropping the constant \( \int S^2 d\mu_{q_0} = 1 \)), we have a reduced Hamiltonian
\[ H = \frac{N}{2} \left( \frac{p_\psi^2}{q_1\sqrt{q_2}} + q_1\sqrt{q_2}\lambda^2q_1^{-1} + m^2q_2^{-1} \right)\psi^2. \tag{240} \]

The Hamilton equations derived from this are
\[ \dot{\psi} = \frac{Np_\psi}{q_1\sqrt{q_2}}, \tag{241} \]
\[ \dot{p}_\psi = -Nq_1\sqrt{q_2}(\lambda^2q_1^{-1} + m^2q_2^{-1})\psi, \]
or eliminating \( p_\psi \) we have a wave equation for \( \psi \),
\[ \ddot{\psi} + \left( \frac{\dot{q}_1}{q_1} + \frac{1}{2}\frac{\dot{q}_2}{q_2} - \frac{N}{N} \right)\dot{\psi} + N^2(\lambda^2q_1^{-1} + m^2q_2^{-1})\psi = 0. \tag{242} \]

Upon substituting the exact solution \( \psi(t) \), we obtain an explicit equation
\[ \ddot{\psi} + \frac{2t}{t_+t_-}\dot{\psi} + \left( \lambda^2\frac{1}{t_+t_-} + m^2\frac{t_2}{t_-^2} \right)\psi = 0. \tag{243} \]

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Another practical form is obtained by transforming $\psi \rightarrow \zeta = \left( t + \frac{t}{t} - t \right)^{1/2} \psi$:

$$\ddot{\zeta} + \frac{1}{t} \dot{\zeta} + Z_{sw} \zeta = 0, \quad (244)$$

where the coefficient $Z_{sw}$ of $\zeta$ is

$$Z_{sw} \equiv \frac{1}{t^2} \left( m^2 t^2 ((t + 2k)^2 + 2k^2) - \left( \frac{1}{4} - \lambda^2 \right) t^2 + k^2 \left( km^2 (4t + k) - \frac{k^2}{4t^2} - \lambda^2 + \frac{3}{2} \right) \right). \quad (245)$$

**Solutions for the Minkowski background ($k = 0$)**

Let us consider the case when $k = 0$ (the Minkowski background). In this case the wave equation (244) becomes Bessel’s equation

$$\ddot{\zeta} + \frac{1}{t} \dot{\zeta} + \left( m^2 - \frac{1 - 4\lambda^2}{4t^2} \right) \zeta = 0. \quad (246)$$

The original field $\psi$ and $\zeta$ are related through $\psi = \zeta / \sqrt{t}$. We can express the general solution to the above equation for $\zeta$ with Bessel functions (when $m \neq 0$):

$$\zeta = C_1 J_{\frac{1}{2} \sqrt{1 - 4\lambda^2}}(mt) + C_2 N_{\frac{1}{2} \sqrt{1 - 4\lambda^2}}(mt), \quad (m \neq 0) \quad (247)$$

where $C_1$ and $C_2$ are integration constants. $J$ and $N$ are, respectively, the Bessel function of the first kind and that of the second kind. It is worth noting that when $\lambda = 0$ this solution can be written only with elementary functions:

$$\zeta = C_1 \sqrt{\frac{2}{\pi mt}} \cos(mt) - C_2 \sqrt{\frac{2}{\pi mt}} \sin(mt). \quad (m \neq 0, \lambda = 0) \quad (248)$$

The asymptotic form of the solution (247) as $t \to \infty$ is of our particular interest, which is, as found in any appropriate formula table, given by

$$\zeta \sim C_1 \sqrt{\frac{2}{\pi mt}} \cos \left( mt - \frac{1 + \sqrt{1 - 4\lambda^2}}{4} \pi \right) + C_2 \sqrt{\frac{2}{\pi mt}} \sin \left( mt - \frac{1 + \sqrt{1 - 4\lambda^2}}{4} \pi \right). \quad (249)$$

Hence, in the range $\lambda^2 < 1/4$ the changes of $\lambda$ result in changes of phase in the asymptotic form, while for $\lambda^2 > 1/4$ they change the amplitude for a fixed $t$ (since, e.g., $\Re(\cos(a+bi)) = \cosh b \cos a$). The decaying rate (of the amplitude) of the field $\psi = \zeta / \sqrt{t}$ is, however, universal, which is given by

$$|\psi| = O\left( \frac{1}{t} \right). \quad (t \to \infty, m \neq 0) \quad (250)$$

The equation for $m = 0$ should be considered separately, but finding the exact solution is not difficult, which is given by

$$\zeta = \begin{cases} 
C_1 t^{\frac{1}{2} \sqrt{1 - 4\lambda^2}} + C_2 t^{-\frac{1}{2} \sqrt{1 - 4\lambda^2}}, & (m = 0, \lambda^2 < 1/4) \\
C_1 + C_2 \log t, & (m = 0, \lambda^2 = 1/4) \\
C_1 \cos \left( \frac{1}{2} \sqrt{4\lambda^2 - 1} \log t \right) + C_2 \sin \left( \frac{1}{2} \sqrt{4\lambda^2 - 1} \log t \right), & (m = 0, \lambda^2 > 1/4)
\end{cases} \quad (251)$$
where $C_1$ and $C_2$ are integration constants. The decaying rate is therefore

$$|\psi| = \begin{cases} 
O(t^{\frac{1}{2}(\sqrt{1-4\lambda^2}-1)}), & (m = 0, \lambda^2 < 1/4) \\
O\left(\frac{\log t}{\sqrt{t}}\right), & (m = 0, \lambda^2 = 1/4) \\
O\left(\frac{1}{\sqrt{t}}\right), & (m = 0, \lambda^2 > 1/4)
\end{cases} \quad (252)$$

A critical point appears at $\lambda^2 = 1/4$ in the present $m=0$ case, as opposed to the $m \neq 0$ generic cases. (We remark that the critical eigenstate for $\lambda^2 = 1/4$ does not necessarily exit. Also $\lambda$ such that $\lambda^2 < 1/4$ (so-called “small eigenvalue”) may not exist. The spectrum varies depending upon the compactification of the background.)

C  Electromagnetic equations

Consider the source-free Maxwell equation

$$g^{ab} \nabla_c F_{ab} = 0. \quad (253)$$

Here, the electromagnetic tensor $F_{ab}$ is obtained from the vector potential $A_a$ through

$$F_{ab} = \partial_a A_b - \partial_b A_a. \quad (254)$$

In local coordinates Eq. (253) reduces to

$$0 = \frac{1}{\sqrt{-g}} \partial_{\nu}(\sqrt{-g}g^{\mu\alpha}g^{\nu\beta}F_{\alpha\beta}) = \frac{1}{Nq_1 \sqrt{q_2} \mu_h} \partial_{\nu}(Nq_1 \sqrt{q_2} \mu_h g^{\mu\alpha}g^{\nu\beta}F_{\alpha\beta}) = \partial_0(Nq_1 \sqrt{q_2}) g^{\mu\nu} g^{00} F_{00} + \partial_0(g^{\mu\nu} g^{00} F_{00}) + \frac{1}{\mu_h} \partial_0(\mu_h g^{\mu\alpha} g^{ij} F_{ij}). \quad (255)$$

**The odd part**

For the odd part we can put

$$A_a = a(t)V_a, \quad (256)$$

where $V_a \equiv c_m \dot{V}_a = c_m \epsilon_a^b \dot{D}_b \dot{S}$. Equation (255) is found to be trivial for $\mu = 0$, and $\mu = 3$, so we are left with $\mu = 1, 2$. In this case, Eq. (255) reduces to

$$0 = \frac{\partial_0(Nq_1 \sqrt{q_2}) g^{AB} g^{00} F_{B0}}{Nq_1 \sqrt{q_2}} + \partial_0(g^{AB} g^{00} F_{B0}) \quad (257)$$

+ \frac{1}{\mu_h} \partial_D(\mu_h g^{AB} g^{DC} F_{BC}) + \partial_3(g^{AB} g^{33} F_{B3}).$$

---

18 Greek indices run 0 to 4, and small Latin ones 1 to 3. $g$ is the determinant of $g_{\mu\nu}$.
19 Capital Latin indices run 1 and 2. Any hatted object is lowered and raised by the standard hyperbolic metric $h^{AB}$ and its inverse $h_{AB}$. 

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Using
\[
F_{B0} = -\dot{a}c_m\dot{V}_B, \quad (258)
\]
\[
F_{B3} = ma\dot{c}_m\dot{V}_B, \quad (259)
\]
\[
F_{BC} = ac_m(\dot{D}_B\dot{V}_C - \dot{D}_C\dot{V}_B), \quad (260)
\]
and
\[
\Delta h \dot{V}_A = -(\lambda^2 + 1)\dot{V}_A, \quad (261)
\]
\[
\hat{D}_B\hat{D}_A\dot{V}_B = -\dot{V}_A, \quad (262)
\]
we obtain the following wave equation:
\[
\ddot{a} + \left(\frac{1}{2}\dot{q}_2 - \frac{\dot{N}}{N}\right)\dot{a} + N^2(\lambda^2q_1^{-1} + m^2q_2^{-1})a = 0. \quad (263)
\]
Upon substituting the exact solution we have
\[
\ddot{a} + \frac{2k}{\dot{t}_+\dot{t}_-}\dot{a} + \left(\lambda^2\frac{1}{\dot{t}_+\dot{t}_-} + m^2\frac{\dot{t}_+^2}{\dot{t}_-^2}\right)a = 0. \quad (264)
\]
To compare with the scalar wave equation we transform the variable as \(a \rightarrow \zeta = a\sqrt{t_-/(t_+t)}\).
We have the following equation for \(\zeta\),
\[
\ddot{\zeta} + \frac{1}{\dot{t}}\dot{\zeta} + Z_{oe}\zeta = 0, \quad (265)
\]
where
\[
Z_{oe} = \frac{1}{\dot{t}_+\dot{t}_-}\left\{m^2\dot{t}_+^2((t + 2k)^2 + 2k^2) - \left(\frac{1}{4} - \lambda^2\right)t^2\right.
\]
\[
+ k\left(2t(1 + 2k^2m^2) - \frac{k^3}{4t^2} - k\left(\frac{1}{2} + \lambda^2\right) + k^3m^2\right)\}. \quad (266)
\]
It is straightforward to see that when \(k = 0\) the above equation for \(\zeta\) becomes exactly the same as the scalar wave equation \((240)\) for \(k = 0\). Therefore in particular the decaying properties for the scalar field and odd electromagnetic field are exactly the same in this case.

**The even part**

The even part of vector potential can be put in the form
\[
A_a = a_0S(dt)_a + a_1S_a + a_3\bar{S}\sigma^3_a, \quad (267)
\]

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where \( a_I = a_I(t), (I = 0, 1, \text{ and } 3) \). It is straightforward to calculate the components of the electromagnetic tensor through Eq. (254). Noting the basic relations for our eigenfunctions \( \partial_A S = \lambda S_A \) and \( \chi_3 S = -mS \), we have \(^{20}\)

\[
F_{0A} = (\dot{a}_1 - \lambda a_0)S_A \\
\quad \equiv \psi_1 S_A, \tag{268}
\]

\[
F_{03} = (\dot{a}_3 + ma_0)\bar{S} \\
\quad \equiv \psi_2 \bar{S}, \tag{269}
\]

\[
F_{3A} = (\lambda a_3 + ma_1)\bar{S}_A \\
\quad \equiv \psi_3 \bar{S}_A, \tag{270}
\]

\[
F_{AB} = 0. \tag{271}
\]

We have defined field strengths of the electric field (\( \psi_1 \) and \( \psi_2 \)) and magnetic field (\( \psi_3 \)) above, which are all functions of time. Note that they are constrained by definition to satisfy

\[
\dot{\psi}_3 = m\psi_1 + \lambda\psi_2. \tag{272}
\]

The strengths \( \psi_1, \psi_2, \text{ and } \psi_3 \) are invariant under the (mode-decomposed) gauge transformation

\[
A_a \rightarrow A_a + \partial_a(fS), \tag{273}
\]

where \( f = f(t) \) is an arbitrary function of time. It is easy to see this invariance using the following induced transformation:

\[
a_0 \rightarrow a_0 + \dot{f}, \\
a_1 \rightarrow a_1 + \lambda f, \tag{274}
\]

\[
a_3 \rightarrow a_3 - mf.
\]

Next, look at the components of the Maxwell equation \(^{25}\), which are given by

(i) \( \mu = 0 \):

\[
-\lambda q_1^{-1}\psi_1 + mq_2^{-1}\psi_2 = 0. \tag{275}
\]

(ii) \( \mu = A \):

\[
\dot{\psi}_1 = \left( \frac{\dot{N}}{N} - \frac{\dot{q}_1}{q_1} - \frac{1}{2} \frac{\dot{q}_2}{q_2} \right) \psi_1 + \frac{\dot{q}_1}{q_1} \psi_1 - N^2 \frac{q_2^{-1}}{q_2} m\psi_3. \tag{276}
\]

(iii) \( \mu = 3 \):

\[
\dot{\psi}_2 = \left( \frac{\dot{N}}{N} - \frac{\dot{q}_1}{q_1} - \frac{1}{2} \frac{\dot{q}_2}{q_2} \right) \psi_2 + \frac{\dot{q}_2}{q_2} \psi_2 - N^2 \frac{q_1^{-1}}{q_1} \lambda\psi_3. \tag{277}
\]

Note that from the two constraints \(^{27}\) and \(^{26}\) functions \( \psi_1 \) and \( \psi_2 \) are solved in terms of \( \dot{\psi}_3 \) as

\[
\psi_1 = (\lambda q_1^{-1} + m^2 q_2^{-1})^{-1} m q_2^{-1} \dot{\psi}_3, \tag{278}
\]

\[
\psi_2 = (\lambda q_1^{-1} + m^2 q_2^{-1})^{-1} \lambda q_1^{-1} \dot{\psi}_3. \tag{279}
\]

\(^{20}\) As in the odd case, capital Latin indices \( A, B \) stand for 1 or 2.
Equations (278) and (276) can virtually be thought of as an unconstrained gauge-invariant system of Hamiltonian equations for the variable $\psi_3$ and its “momentum” $\psi_1$. If we eliminate $\psi_1$ we obtain a wave equation

$$\ddot{\psi}_3 - \left( \frac{\dot{N}}{N} \frac{\dot{q}_1}{q_1} - \frac{1}{2} \frac{\dot{q}_2}{q_2} + \frac{m^2 \dot{q}_1 + \lambda^2 \dot{q}_2}{m^2 q_1 + \lambda^2 q_2} \right) \dot{\psi}_3 + N^2 (\lambda^2 q_1^{-1} + m^2 q_2^{-1}) \psi_3 = 0. \quad (280)$$

If using the exact solution we have

$$\ddot{\psi}_3 + 2 \left( \frac{t}{t_+ t_-} - \frac{m^2 t_+ + \lambda^2 k t_-^2}{m^2 t_+^2 + \lambda^2 t_-^2} \right) \dot{\psi}_3 + \left( \lambda^2 \frac{1}{t_+ t_-} + m^2 \frac{t_+^2}{t_-^2} \right) \psi_3 = 0. \quad (281)$$

Instead of getting an equation for $\psi_3$, we can also get one for $\psi_1$, which is

$$\ddot{\psi}_1 + \frac{6k}{t_+ t_-} \dot{\psi}_1 + \left( \lambda^2 \frac{1}{t_+ t_-} + m^2 \frac{t_+^2}{t_-^2} - \frac{4k(t - 2k)}{t_+^2 t_-^2} \right) \psi_1 = 0. \quad (282)$$

Since $\psi_1$ and the “derivative” of $\psi_3$ are related with Eq.(278), we can reproduce the equation for $\psi_1$ from that of $\psi_3$ by differentiating Eq.(281) (with a further conformal transformation). On the other hand, there is no function $f(t)$ such that only the conformal transformation $\psi_3 \rightarrow \tilde{\psi} = f\psi_3$ reproduces the equation for $\psi_1$. We remark that it is the equation for $\psi_1$ that has a direct resemblance to the odd equation (264), since they coincide when $k = 0$.

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