THE TRACE OF THE LOCAL $A^1$-DEGREE

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Abstract. We prove that the local $A^1$-degree of a polynomial function at an isolated zero with finite separable residue field is given by the trace of the local $A^1$-degree over the residue field. This fact was originally suggested by Morel’s work on motivic transfers and Kass and Wickelgren’s work on the Scheja-Storch bilinear form.

1. Introduction

The $A^1$-degree, first defined by Morel [Mor04, Mor12], provides a foundational tool for solving problems in $A^1$-enumerative geometry, a field which seeks to resolve enumerative questions over arbitrary fields by wielding the machinery of $A^1$-homotopy theory [MV99]. One often encounters problems for which an $A^1$-degree is needed at a point which is not rational. At points whose residue field is a finite extension of the ground field, it was suggested by Morel’s work on cohomological transfer maps [Mor12] that one may first compute the $A^1$-local degree over the residue field, and then trace down to the ground field. Kass and Wickelgren show that the Scheja-Storch bilinear form [SS75] is equal to the local $A^1$-degree at rational points [KW19]. They also prove that at points with finite separable residue field, the Scheja-Storch form is given by taking the trace of the Scheja-Storch form over the residue field [KW17, Proposition 32]. We show that an analogous statement is true for the local $A^1$-degree.

Theorem 1.1. Let $f : A^n_k \to A^n_k$ be an endomorphism of affine space, and let $p \in A^n_k$ be an isolated root of $f$ such that $k(p)$ is a separable extension of finite degree over $k$, and let $\tilde{p}$ denote a canonical point above $p$. Then

$$\deg^A_\tilde{p} f = \text{Tr}_{k(p)/k} \deg^A_p (f \otimes_k k(p))$$

is an equality in $GW(k)$.

As a corollary, we strengthen the result that the local $A^1$-degree is the Scheja-Storch form at rational points [KW19] by weakening the requirement that the point be rational.

Corollary 1.2. At points whose residue fields are finite separable extensions of the ground field, the local $A^1$-degree coincides with the Scheja-Storch form.

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1For the sake of brevity, we neglect to include exposition of the basic notions used in $A^1$-enumerative geometry; instead we refer the reader to the expository paper [WW19], as well as the exposition found throughout the following papers: [KW17, Lev17, BKW18, SW18, KW19, LV19, KT19].
In this paper we utilize the machinery of stable $\mathbb{A}^1$-homotopy theory, initially developed by Morel and Voevodsky [MV99], to prove this theorem. In particular, we rely heavily on work of Hoyois [Hoy15].

**Terminology 1.3.** If $p$ is a point with $k(p) \mid k$ a separable extension of finite degree, we call $p$ a *finite separable point*. We may also say that $p$ has a *finite separable residue field* in this context.

**Remark 1.4.** Throughout this paper, we will often discuss bilinear forms and their isomorphism classes. If $\beta$ is a bilinear form, then we may, by abuse of notation, denote the isomorphism class of $\beta$ by $\beta$.

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2. **Proof of Theorem 1.1**

For any $k$-scheme $X$ and any point $p \in X$, we have a canonical $k(p)$-rational point $\tilde{p} \in X_{k(p)}$ sitting over $p$, defined via the following pullback diagram:

$$
\begin{array}{ccc}
\text{Spec } k(p) & \xrightarrow{\text{id}} & \text{Spec } k(p) \\
\downarrow & & \downarrow \\
X_{k(p)} & \xrightarrow{\text{id}} & \text{Spec } k(p) \\
\downarrow & & \downarrow \\
X & \xrightarrow{\text{id}} & \text{Spec } k
\end{array}
$$

To simplify notation, we write $L = k(p)$, and write the base change of $f$ as $f_L = f \otimes_k L$. Let $\pi : \mathbb{A}^n_L \to \mathbb{A}^n_k$ denote the canonical morphism of affine space given by the inclusion of fields $k \hookrightarrow L$. We may consider the following diagram

$$
\begin{array}{c}
\mathbb{A}^n_L \xrightarrow{f_L} \mathbb{A}^n_L \\
\pi \downarrow \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \pi \downarrow \\
\mathbb{A}^n_k \xrightarrow{f} \mathbb{A}^n_k \\
\end{array}
\quad \text{which maps} \quad
\begin{array}{c}
\tilde{p} \longrightarrow 0 \\
\downarrow \\
p \longrightarrow 0
\end{array}
$$

We note that the point $\tilde{p}$ in the top left corner is a root of $f$ when we base change to the residue field $L = k(p)$. Thus we have that $f_L$ has an isolated rational zero at $\tilde{p}$. 
We now have a diagram of pointed schemes, so by passing to sufficiently small local neighborhoods around each of these points, we obtain an induced diagram

\[
\begin{array}{ccc}
\mathbb{P}^n_L/(\mathbb{P}^n_L \setminus \{\tilde{p}\}) & \xrightarrow{\gamma} & \mathbb{P}^n_L/(\mathbb{P}^n_L \setminus \{0\}) \\
\pi_p & \downarrow & \pi_0 \\
\mathbb{P}^n_k/(\mathbb{P}^n_k \setminus \{p\}) & \xrightarrow{\gamma} & \mathbb{P}^n_k/(\mathbb{P}^n_k \setminus \{0\})
\end{array}
\]

(1)

For any point \( q \in \mathbb{P}^n_k \), there is a canonical \( \mathbb{A}^1 \)-homotopy equivalence

\[
\mathbb{P}^n_k/(\mathbb{P}^n_k \setminus \{q\}) \simeq \mathbb{P}^n_{k(q)}/\mathbb{P}^{n-1}_{k(q)}.
\]

This is a standard result of purity \([MV99, \text{Proposition 2.17}]\), which allows us to identify \( \mathbb{P}^n_k/(\mathbb{P}^n_k \setminus \{y\}) \) with the Thom space \( \text{Th}(\mathcal{O}_{k(q)}) \). We then apply the canonical \( \mathbb{A}^1 \)-weak equivalence \( \text{Th}(\mathcal{O}_{k(q)}) \simeq \mathbb{P}^n_{k(q)}/\mathbb{P}^{n-1}_{k(q)} \) which may be found in \([MV99, \text{Proposition 2.17(3)}]\).

As an application of this argument, we obtain the following \( \mathbb{A}^1 \)-equivalences:

\[
\begin{align*}
\mathbb{P}^n_L/(\mathbb{P}^n_L \setminus \{\tilde{p}\}) & \simeq \mathbb{P}^n_L/\mathbb{P}^{n-1}_L, \\
\mathbb{P}^n_L/(\mathbb{P}^n_L \setminus \{0\}) & \simeq \mathbb{P}^n_L/\mathbb{P}^{n-1}_L, \\
\mathbb{P}^n_k/(\mathbb{P}^n_k \setminus \{0\}) & \simeq \mathbb{P}^n_k/\mathbb{P}^{n-1}_k, \\
\mathbb{P}^n_k/(\mathbb{P}^n_k \setminus \{p\}) & \simeq \mathbb{P}^n_k/\mathbb{P}^{n-1}_k.
\end{align*}
\]

**Lemma 2.1.** In the stable homotopy category \( \mathcal{SH}(k) \), the collapse map

\[
\mathbb{P}^n_k/\mathbb{P}^{n-1}_k \to \mathbb{P}^n_k/(\mathbb{P}^n_k \setminus \{p\}) \simeq \mathbb{P}^n_L/\mathbb{P}^{n-1}_L
\]

is \( \mathbb{P}^n_k/\mathbb{P}^{n-1}_k \wedge (-) \) applied to the transfer map

\[
\eta : \mathbf{1}_k \to \rho_*\rho^*\mathbf{1}_k \simeq \rho_*\mathbf{1}_k,
\]

where \( \rho : \text{Spec}L \to \text{Spec}k \) is the canonical structure map.

**Proof.** The case \( n = 1 \) may be found in \([Hoy15, \text{Lemma 5.5}]\), and the proof generalizes to higher \( n \) as in \([KW19, \text{Lemma 13}]\). \( \square \)

**Definition 2.2.** Let \( f : \mathbb{A}^n_k \to \mathbb{A}^n_k \) be an endomorphism of affine space, with \( q \) a point such that \( f(q) \) is \( k \)-rational. Let \( U \ni q \) be an open neighborhood, chosen sufficiently small so that \( U \cap f^{-1}(f(q)) = \{q\} \). Then the local \( \mathbb{A}^1 \)-degree \( \deg_{q}^{\mathbb{A}^1} f \) is defined to be Morel’s \( \mathbb{A}^1 \)-degree homomorphism applied to the element in the stable homotopy category

\[
\mathbb{P}^n_k/\mathbb{P}^{n-1}_k \to \mathbb{P}^n_k/(\mathbb{P}^n_k \setminus \{q\}) \xrightarrow{\simeq} U/(U \setminus \{q\}) \xrightarrow{f|_U} \mathbb{P}^n_k/(\mathbb{P}^n_k \setminus \{f(q)\}) \xrightarrow{\simeq} \mathbb{P}^n_k/\mathbb{P}^{n-1}_k,
\]

where this first map is the collapse map defined in Lemma 2.1.
We may rewrite Diagram 1 as:

\[
\begin{array}{ccc}
P_L/P_{L}^{n-1} & \xrightarrow{\deg_{L}^{A}} & P_{L}^{n}/P_{L}^{n-1} \\
\downarrow{r} & & \downarrow{g} \\
P_{L}^{n}/P_{L}^{n-1} & \xrightarrow{\tilde{f}} & P_{k}^{n}/P_{k}^{n-1}.
\end{array}
\]

Via the definition of Morel’s $A^1$-degree $\deg_{A}^{A^1} : [P_{L}^{n}/P_{L}^{n-1}, P_{L}^{n}/P_{L}^{n-1}]_{A^1} \cong GW(L)$, the homotopy class of the top map can be identified with the local degree of $f_L$ at $\tilde{p}$. The bottom map, which we have denoted $\tilde{f}$, is not exactly the local degree of $f$, since the domain is a sphere over $L$. To rectify this, we will precompose with a collapse map from a sphere over $k$.

**Proposition 2.3.** The degree of the composite

\[
P_k^{n}/P_k^{n-1} \xrightarrow{\pi^*T\mathbb{A}_k^{n}/\eta} P_L^{n}/P_L^{n-1} \xrightarrow{\tilde{f}} P_k^{n}/P_k^{n-1}
\]

is the local degree $\deg_{\tilde{p}}^{A^1}(f) \in GW(k)$.

*Proof.* The first map in Definition 2.2 is the collapse map, which is equal to $P_k^{n}/P_k^{n-1} \wedge \eta$ by Lemma 2.1. Applying Definition 2.2 to Diagram 2 gives the desired result. □

**Proposition 2.4.** [Hoy15, p.15] The map $g$ in Diagram 2 is $\pi^*T\mathbb{A}_k^{n}/(-)$ applied to the map

\[
\epsilon : \rho_*\rho^*1_k \cong \rho_*1_L \to 1_k
\]

in the stable homotopy category $SH(k)$.

**Proposition 2.5.** The stable homotopy class of the map $r$ in Diagram 2 is $\langle 1 \rangle \in GW(L)$.

*Proof.* Consider the map $\pi : \mathbb{A}_L^{n} \to \mathbb{A}_k^{n}$ sending the canonical point $\tilde{p}$ to $p$. This induces a map on the cofiber of tangent spaces

\[
\begin{array}{ccc}
T_{\tilde{p}}\mathbb{A}_L^{n} \setminus \{0\} & \xrightarrow{(\pi^*T\mathbb{A}_k^{n})_{\tilde{p}}} & (\pi^*T\mathbb{A}_k^{n})_{\tilde{p}} \setminus \{0\} \\
T_{\tilde{p}}\mathbb{A}_k^{n} \setminus \{0\} & \xrightarrow{\sim} & \mathbb{A}_L^{n} \setminus \{0\}.
\end{array}
\]

Via the standard trivialization of the tangent space of affine space, we obtain a canonical isomorphism

\[
\begin{array}{ccc}
T_{\tilde{p}}\mathbb{A}_L^{n} \setminus \{0\} & \sim & \mathbb{A}_L^{n} \setminus \{0\}.
\end{array}
\]

Together with our choice of isomorphism $k(p) \cong L$, we get canonical weak equivalences

\[
\begin{array}{ccc}
(\pi^*T\mathbb{A}_k^{n})_{\tilde{p}} \setminus \{0\} & \sim & L \otimes_k \mathbb{A}_k^{n} \setminus \{p\} = \mathbb{A}_L^{n} \setminus \{p\} \sim \mathbb{A}_L^{n} \setminus \{0\},
\end{array}
\]

where this last equivalence is given by translation. We can see that, on the tangent spaces, the base change map $\pi$ sends $\frac{d}{dx_i}$ to $\frac{d}{dx_i}$, so Equation 3 can be rewritten as
the identity map on $\frac{A^n}{\mathbb{A}^n \setminus \{0\}}$. The result then follows from applying the $A^1$-equivalence $P^n_L/P^{n-1}_L \simeq A^n_L/(\mathbb{A}^n_L \setminus \{0\})$.

Since the stable homotopy class of $r$ is $\langle 1 \rangle$, it is in particular a weak equivalence, and its inverse $r^{-1}$ is given by $\langle 1 \rangle$ as well. Putting the previous propositions together, we obtain a commutative diagram, where we can associate the homotopy classes of the top and bottom maps with their associated degrees in $GW(L)$ and $GW(k)$, respectively:

(4)

\[
\begin{array}{ccc}
\mathbb{P}^n_L/P^{n-1}_L & \overset{\deg_{A^1} f_L}{\longrightarrow} & \mathbb{P}^n_L/P^{n-1}_L \\
\downarrow \langle 1 \rangle & & \downarrow \mathbb{P}^n_k/P^{n-1}_k \wedge \epsilon \\
\mathbb{P}^n_k/P^{n-1}_k & \overset{\deg_{A^1} f}{\longrightarrow} & \mathbb{P}^n_k/P^{n-1}_k \\
\end{array}
\]

**Lemma 2.6.** Let $L$ be a finite separable extension of $k$, and let $\rho : \text{Spec} L \to \text{Spec} k$ be the corresponding morphism of schemes. For any element $\omega \in GW(L)$, we have that $Tr_{L/k}\omega$ is the composite

\[
1_k \xrightarrow{\eta} \rho_*1_L \simeq \rho_#1_L \xrightarrow{\rho_#\omega} \rho_#1_L \xrightarrow{\epsilon} 1_k.
\]

**Proof.** This is due to [Hoy15, Proposition 5.2, Lemma 5.3].

**Proof of Theorem 1.1.** By applying Lemma 2.6 to Diagram 4, we can factor $\deg_{A^1} f \in GW(k)$ as the maps along the top to obtain an equality in $GW(k)$. Therefore

\[
\deg_{A^1} f = (\mathbb{P}^n_k/P^{n-1}_k \wedge \eta) \circ \langle 1 \rangle \circ \deg_{A^1} f_L \circ (\mathbb{P}^n_k/P^{n-1}_k \wedge \epsilon) \\
= (\mathbb{P}^n_k/P^{n-1}_k \wedge \eta) \circ \deg_{A^1} f_L \circ (\mathbb{P}^n_k/P^{n-1}_k \wedge \epsilon) \\
= Tr_{L/k} \deg_{A^1} f_L. \quad \square
\]

### 3. A Brief Proof of Corollary 1.2

In [KW17, Proposition 32], the authors prove that the Scheja-Storch bilinear form, denoted $\text{ind}_p f$, is computed by the trace

\[
\text{ind}_p f = Tr_{k(p)/k}\text{ind}_{k(p)} f_{k(p)}.
\]

Moreover in [KW19], the authors prove that at any rational point, the Scheja-Storch form agrees with the local $A^1$-degree. Combining these two results with Theorem 1.1 for any isolated zero $p$ with finite separable residue field we have that

\[
\text{ind}_p f = Tr_{k(p)/k}\text{ind}_{k(p)} f_{k(p)} = Tr_{k(p)/k} \deg_{A^1} f_{k(p)} = \deg_{A^1} f.
\]
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