A Proof of Solomon’s Rule

Stephanie J. van Willigenburg
Department of Mathematics and Statistics,
York University, 4700 Keele St,
North York, ON, M3J 1P3, CANADA.

February 1, 2008

Abstract

We put forward a proof of Solomon’s rule, in terms of matrices, for multiplication in the descent algebra of the symmetric group. Our proof exploits the graphs that we can obtain from all the subsets of the set of transpositions, \((i, i + 1)_{i=1}^{n-1}\).

Let \(W\) be a Coxeter group with generating set, \(S\), of fundamental reflections. If \(J\) is any subset of \(S\), let \(W_J\) be the subgroup generated by \(J\). Let \(X_J\) be the unique set of minimal length left coset representatives of \(W_J\). Note that \(X_J^{-1} = \{x^{-1} | x \in X_J\}\) is the unique set of minimal length representatives for the right cosets of \(W_J\). Let \(l(y)\) denote the length of \(y\) in \(W\).

Solomon [9] then gives us the following theorem:

**Theorem 1** For every subset \(K\) of \(S\), let

\[
\mathcal{X}_K = \sum_{\sigma \in X_K} \sigma.
\]

Then for subsets \(J\) and \(K\) in \(S\)

\[
\mathcal{X}_J \mathcal{X}_K = \sum_{x \in X_J^{-1} \cap X_K} \mathcal{X}_{x^{-1} J x \cap K}
\]

From this it follows that the set of all \(\mathcal{X}_K\) form a basis for an algebra, the descent algebra of \(W\).

Independently, interpretations of this theorem involving certain matrices have been developed for the descent algebras of the Coxeter groups of types \(A\) and \(B\) as a means of obtaining further results about these algebras, [6], [5], [1], [2], [3]. However, no such matrix interpretation was known for the Coxeter groups
of type \(D\). In this paper we shall develop some tools and a lemma from which we can easily deduce the matrix interpretation of Theorem 1 for the descent algebra of the Coxeter groups of type \(A\). The purpose of this paper, however, is not just to give another proof of a well known result, instead it is to act as a precursor to a subsequent paper in which we formulate the missing matrix interpretation for the descent algebras of the Coxeter groups of type \(D\), [3].

To develop our tools, let us take our Coxeter group \(W\) to be the Coxeter group of type \(A\) with \(n - 1\) fundamental reflections, that is the symmetric group \(S_n\). More specifically, let us take \(S_n\) to be the group of permutations acting on the set \(N = \{1, \ldots, n\}\), with generating set \(S\), where \(S\) is the set of \(n - 1\) transpositions \(s_1, s_2, \ldots, s_{n-1}\), such that \(s_i = (i, i + 1)\).

If \(J\) is a subset of \(S\), then we define the graph \(J = (N, E)\) of \(J\) to be the graph with vertex set \(\{1, \ldots, n\}\), and edge set \(E = \{(i, i + 1) \in J\}\). In general we shall use roman capitals \(J, K, \ldots\) for subsets of \(S\), and their calligraphic counterparts \(J, K, \ldots\) for their associated graphs. Suppose now that \(J\) has \(r\) connected components. A set of vertices is associated with each component, and we can order these sets by their least elements in a natural way. Once ordered, we can label them \(J_1, \ldots, J_r\) such that \(1 \in J_1\), and define the ordered presentation of \(J\) to be the ordered list

\[(J_1, \ldots, J_r).\]

Note that this is the canonical ordered set partition associated to \(J\), and that if \(u \in J_i\) and \(v \in J_j\), and \(i < j\), then \(u < v\).

**Example 1** In \(S_9\). If \(J = \{(2, 3), (3, 4), (7, 8)\}\), then \(J\) is

\[
\begin{array}{cccccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9
\end{array}
\]

with

\[
J_1 = \{1\}, J_2 = \{2, 3, 4\}, J_3 = \{5\}, J_4 = \{6\}, J_5 = \{7, 8\}, J_6 = \{9\}.
\]

The ordered presentation of \(J\) is

\[\{(\{1\}, \{2, 3, 4\}, \{5\}, \{6\}, \{7, 8\}, \{9\}\)}\].

Let us define \(W_J\) to be the subgroup of \(S_n\) that consists of all permutations of \(S_n\) that fix all points outside of \(J_i\), and let

\[W_J = W_{J_1} \times \cdots \times W_{J_r}.
\]

Observe that \(W_J = W_J\). If we let \(\kappa\) be a composition of \(n\), with components \(\kappa_1, \kappa_2, \ldots, \kappa_r\), and define

\[S_\kappa = S_{\kappa_1} \times \cdots \times S_{\kappa_r},
\]
then \( S_\kappa \cong W_J \), where the sets \( J_i \) of \( J \) satisfy
\[
|J_i| = \kappa_i. \tag{1}
\]

Let us now take \( J \) and \( K \) to be any subsets of \( S \), and \( x \in W \), and let \( xJ \) denote the image of the graph \( J \) under \( x \); that is \((x(i), x(j))\) is an edge in \( xJ \) if and only if \((i, j)\) is an edge in \( J \). Let \( J \cap K \) be the graph with vertex set \( N \) whose edges are those present in both \( J \) and \( K \). The ordered presentation of \( x^{-1}J \cap K \), where \( x \in X^{-1}_J \cap X_K \), is given by the following lemma.

**Lemma 1** Let \( J \) and \( K \) be subsets of \( S \), and let \( x \in X^{-1}_J \cap X_K \). Let the ordered presentation of \( x^{-1}J \cap K \) be

\[
(x^{-1}J_1 \cap K_1, x^{-1}J_2 \cap K_1, \ldots, x^{-1}J_r \cap K_1, x^{-1}J_1 \cap K_2, x^{-1}J_2 \cap K_2, \ldots, x^{-1}J_r \cap K_2, \ldots, x^{-1}J_1 \cap K_s, x^{-1}J_2 \cap K_s, \ldots, x^{-1}J_r \cap K_s)
\]

with empty sets removed.

**Proof** To prove that \( (2) \) is the ordered presentation of \( x^{-1}J \cap K \), it is sufficient to prove the following two statements

1. The elements of each set are less than those that appear in any set later in the list.
2. Each non-empty set \( x^{-1}J_q \cap K_m \) is indeed the vertex set of a connected component of \( x^{-1}J \cap K \).

To prove statement 1 we must show that

1.1 The elements in \( x^{-1}J_r \cap K_m \) are less than those in \( x^{-1}J_1 \cap K_{m+1} \)
1.2 The elements in \( x^{-1}J_q \cap K_m \) are less than those in \( x^{-1}J_{q+1} \cap K_m \).

Case 1.1 follows immediately, since all vertices in \( K_m \) are less than those in \( K_{m+1} \) by definition.

To prove case 1.2 we need only show that if the vertex \( i \in x^{-1}J_q \cap K_m \) and \( j \in x^{-1}J_{q+1} \cap K_m \), then \( i < j \). To do this we shall first prove that all vertices in \( K_m \) appear from left to right in increasing order in the list \( x^{-1}(1), x^{-1}(2), \ldots, x^{-1}(n) \).

From the definition of \( X_K \) as a set of minimal length coset representatives, it follows that if \( x \in X_K \) then \( l(xk) > l(x) \) for all \( k \in K \). In \( S_n \), \( l(x) \) is the number
of inversions in $x$, that is the number of $h < l$ for which $x(l) < x(h)$, \[7\]. Hence it follows that for all $k = (h, h + 1) \in K$ we have $x(h) < x(h + 1)$, since $xk$, $x$ differ only in the reversing of $h$ and $h + 1$. From this we can deduce that $h$ is to the left of $h + 1$ in the list

$$x^{-1}(1), x^{-1}(2), \ldots, x^{-1}(n).$$

Now suppose that $i \in x^{-1}J_q \cap K_m$, $j \in x^{-1}J_{q+1} \cap K_m$. Then $x(i) = u \in J_q$, and $x(j) = v \in J_{q+1}$. It follows that $u < v$, and so $x^{-1}(u)$ appears before $x^{-1}(v)$ in $x^{-1}(1), x^{-1}(2), \ldots, x^{-1}(n)$. However, $x^{-1}(u) = i$ and $x^{-1}(v) = j$, and since we know that the vertices of $K_m$ appear in increasing order from left to right in $x^{-1}(1), x^{-1}(2), \ldots, x^{-1}(n)$, it follows that $i < j$.

Statement 2 will follow if we can prove the following assertions.

2.1 The sets $x^{-1}J_q \cap K_m$ are all disjoint.

2.2 No edge in $x^{-1}J \cap K$ connects vertices in different subsets $x^{-1}J_q \cap K_m$ and $x^{-1}J_{q'} \cap K_{m'}$.

2.3 For every $i, i + 1 \in xJ_q \cap K_m$, an edge exists in $x^{-1}J \cap K$ between $i, i + 1$.

Again, assertion 2.1 follows since all $J_q$ and $K_m$ are disjoint and $x$ is a bijection from $N$ to itself.

To prove assertion 2.2, let $(u, v)$ be an edge in $x^{-1}J \cap K$, such that $u \in x^{-1}J_q \cap K_m$ and $v \in x^{-1}J_{q'} \cap K_{m'}$. We know that $J_q$ and $J_{q'}$ are vertex sets of connected components of $J$, so $x^{-1}J_q$ and $x^{-1}J_{q'}$ must be vertex sets of connected components of $x^{-1}J$. Hence, $q = q'$. Similarly, $K_m$ and $K_{m'}$ are vertex sets of connected components of $K$, and so $m = m'$.

For assertion 2.3, let $i, i + 1 \in x^{-1}J_q \cap K_m$ and let $x(i) = u$, and $x(i + 1) = u + l$. Since we know from the proof of case 1.2 that all $i \in K_m$ appear in increasing order from left to right in the list $x^{-1}(1), x^{-1}(2), \ldots, x^{-1}(n)$, we can deduce that $l \geq 1$. We can also deduce that because $X^{-1}_q$ is defined as a set of minimal length right coset representatives, we have that $x^{-1}(v) < x^{-1}(v + 1)$ for all $(v, v + 1) \in J$.

Therefore, since $u, u + l \in J_q$, we have that $u + k \in J_q$ for all $k = 0, \ldots, l$ such that

$$x^{-1}(u) < x^{-1}(u + 1) < \ldots < x^{-1}(u + l - 1) < x^{-1}(u + l).$$

However, $x^{-1}(u) = i$, $x^{-1}(u + l) = i + 1$, so it follows that $l = 1$, and so, by definition $(u, u + l)$ is an edge in $J$. Therefore, since $x^{-1}(u) = i$ and $x^{-1}(u + l) = i + 1$, it follows that $(i, i + 1)$ is an edge in $x^{-1}J \cap K$, and we are done. \[\blacksquare\]
As a consequence of Lemma 2 \cite{9}, and our Lemma 1

\[
x^{-1}W_J x \cap W_K = \{x^{-1}J x \cap K \}
= W_{x^{-1}J x \cap K}
= W_{x^{-1}J x \cap K_1} \times \ldots \times W_{x^{-1}J x \cap K_s}
= (x^{-1}W_J x \cap W_{K_1}) \times \ldots \times (x^{-1}W_J x \cap W_{K_s})
\cong (x^{-1}S_{\kappa_1} x \cap S_{\nu_1}) \times \ldots \times (x^{-1}S_{\kappa_r} x \cap S_{\nu_s})
\]

where \(\kappa\) and \(\nu\) are suitable compositions of \(n\) determined by \(J, K\) respectively, according to condition (1). Note that the final isomorphism symbol is an equality if \(x^{-1}S_{\kappa_i} x \cap S_{\nu_j}\) is regarded as the group of permutations on \(x^{-1}J_i \cap K_j\).

Let

\[z_{ij} = |x^{-1}J_i \cap K_j|
\]

then, by Theorem 1.3.10 \cite{8}, we have a bijective mapping

\[\zeta : x \mapsto (z_{ij})\]

from \(X^{-1}_J \cap X_K\) into the set of \(s \times r\) matrices with non-negative integer entries, \(z = (z_{ij})\), which satisfy

\[\sum_i z_{ij} = \kappa, \quad \sum_j z_{ij} = \nu.\]

Observe that reading the non-zero entries of the matrix \(z\) by row give a composition, \(\eta\), of \(n\). We say that \(\eta\) is the reading word of \(z\), and note that \(S_\eta\) is isomorphic to \(W_{x^{-1}J x \cap K}\). We also observe that each matrix corresponds to one \(x \in X^{-1}_J \cap X_K\), given in Solomon’s Theorem. Therefore, if we now rename the basis elements such that \(X_J\) becomes \(B_\kappa\), where the components of \(\kappa\) in order are the sizes of the vertex sets of \(J\) taken in the natural order, we can recast Solomon’s Theorem in terms of compositions and matrices as follows.

**Theorem 2** For every composition \(\nu\) of \(n\), let \(X_\nu\) be the unique set of minimal length left coset representatives of \(S_n/S_\nu\). Let

\[B_\nu = \sum_{\sigma \in X_\nu} \sigma.\]

If \(\kappa, \nu\) are compositions of \(n\), then

\[B_\kappa B_\nu = \sum_z B_\eta\]

where the sum is over all matrices \(z = (z_{ij})\) with non-negative integer entries that satisfy
1. $\sum_i z_{ij} = \kappa_j,$

2. $\sum_j z_{ij} = \nu_i.$

For each matrix, $z$, $\eta$ is the reading word of $z$.

This is precisely the classical matrix interpretation of Solomon’s Theorem for the symmetric groups, for instance Proposition 1.1, [6].

Acknowledgements The author is indebted to Michael Atkinson for many useful discussions, and grateful to Nantel Bergeron for critical comments.

References

[1] MD Atkinson. Solomon’s descent algebra revisited. *Bulletin of the London Mathematical Society*, 24: 545-551, 1992.

[2] F Bergeron and N Bergeron. A decomposition of the descent algebra of the hyperoctahedral group 1. *Journal of Algebra*, 148:86-97, 1992.

[3] N Bergeron. A decomposition of the descent algebra of the hyperoctahedral group 2. *Journal of Algebra*, 148:98-122, 1992.

[4] N Bergeron and SJ van Willigenburg. On the descent algebra of type $D$. In preparation.

[5] AM Garsia and J Remmel. Shuffles of permutations and the Kronecker product. *Graphs and Combinatorics*, 1:217-263, 1985.

[6] AM Garsia and C Reutenauer. A decomposition of Solomon’s descent algebra. *Advances in Mathematics*, 77:189-262, 1989.

[7] JE Humphrey. *Reflection groups and Coxeter groups*, Cambridge University Press, 1990.

[8] G James and A Kerber. *Encyclopedia of mathematics and its applications: The representation theory of the symmetric group*, volume 16. Addison-Wesley, Massachusetts, 1981.

[9] L Solomon. A formula in the group ring of a Coxeter group. *Journal of Algebra* 41:255-268, 1976.