Topological Bounds from Label Translation Symmetry of Non-Barotropic MHD

Asher Yahalom
Ariel University, Ariel 40700, Israel
E-mail: asya@ariel.ac.il

Abstract. The theorem of Noether dictates that for every continuous symmetry group of an Action the system must possess a conservation law. In this paper we discuss some subgroups of Arnold’s labelling symmetry diffeomorphism related to non-barotropic magnetohydrodynamics (MHD) and the conservations laws associated with them. Furthermore, we will study the dynamical bounds resulting from the topological Noether currents associated with label translation symmetry groups.

1. Introduction
The theorem of Noether dictates that for every continuous symmetry group of an Action the system must possess a conservation law. For example time translation symmetry results in the conservation of energy, while spatial translation symmetry results in the conservation of linear momentum and rotation symmetry in the conservation of angular momentum to list some well known examples. But sometimes the conservation law is discovered without reference to the Noether theorem by using the equations of the system. In that case one is tempted to inquire what is the hidden symmetry associated with this conservation law and what is the simplest way to represent it.

The concept of metage as a label for fluid elements along a vortex line in ideal fluids was first introduced by Lynden-Bell & Katz [1]. A translation group of this label was found to be connected to the conservation of Moffat’s [2] helicity by Yahalom [3]. The concept of metage was later generalized by Yahalom & Lynden-Bell [4] for barotropic MHD, but now as a label for fluid elements along magnetic field lines which are comoving with the flow in the case of ideal MHD. Yahalom & Lynden-Bell [4] has also shown that the translation group of the magnetic metage is connected to Woltjer [5, 6] conservation of cross helicity for barotropic MHD. Recently the concept of metage was generalized also for non barotropic MHD in which magnetic field lines lie on entropy surfaces [7]. This was later generalized by dropping the entropy condition on magnetic field lines [8].

Cross Helicity was first described by Woltjer [5, 6] and is give by:

\[ H_C \equiv \int \vec{B} \cdot \vec{v} d^3x, \]  
(1)
in which \( \vec{B} \) is the magnetic field, \( \vec{v} \) is the velocity field and the integral is taken over the entire flow domain. \( H_C \) is conserved for barotropic or incompressible MHD and is given a topological
interpretation in terms of the knottiness of magnetic and flow field lines. A generalization of barotropic fluid dynamics conserved quantities including helicity to non barotropic flows including topological constants of motion is given by Mobbs [9]. However, Mobbs did not discuss the MHD case.

Both conservation laws for the helicity in the fluid dynamics case and the barotropic MHD case were shown to originate from a relabelling symmetry through the Noether theorem [3, 10, 11, 4]. Webb et al. [13] have generalized the idea of relabelling symmetry to non-barotropic MHD and derived their generalized cross helicity conservation law by using Noether’s theorem but without using the simple representation which is connected with the metage variable. The conservation law deduction involves a divergence symmetry of the action. These conservation laws were written as Eulerian conservation laws of the form $D_t + \nabla \cdot \vec{F} = 0$ where $D$ is the conserved density and $F$ is the conserved flux. Webb et al. [15] discuss the cross helicity conservation law for non-barotropic MHD in a multi-symplectic formulation of MHD. Webb et al. [12, 13] emphasize that the generalized cross helicity conservation law, in MHD and the generalized helicity conservation law in non-barotropic fluids are non-local in the sense that they depend on the auxiliary nonlocal variable $\sigma$, which depends on the Lagrangian time integral of the temperature $T(x,t)$. Notice that a potential vorticity conservation equation for non-barotropic MHD is derived by Webb, G. M. and Mace, R.L. [16] by using Noether’s second theorem.

It should be mentioned that Mobbs [9] derived a helicity conservation law for ideal, non-barotropic fluid dynamics, which is of the same form as the cross helicity conservation law for non-barotropic MHD, except that the magnetic field induction is replaced by the generalized fluid helicity $\Omega = \nabla \times (\vec{v} - \sigma \vec{\nabla}s)$. Webb et al. [12, 13] also derive the Eulerian, differential form of Mobbs [9] conservation law (although they did not reference Mobbs [9]). Webb and Anco [14] show how Mobbs conservation law arises in multi-symplectic, Lagrangian fluid mechanics.

Variational principles for magnetohydrodynamics were introduced by previous authors both in Lagrangian and Eulerian form. Sturrock [17] has discussed in his book a Lagrangian variational formalism for magnetohydrodynamics. Vladimirov and Moffatt [22] in a series of papers have discussed an Eulerian variational principle for incompressible magnetohydrodynamics. However, their variational principle contained three more functions in addition to the seven variables which appear in the standard equations of incompressible magnetohydrodynamics which are the magnetic field $\vec{B}$, the velocity field $\vec{v}$ and the pressure $P$. Kats [23] has generalized Moffatt’s work for compressible non barotropic flows but without reducing the number of functions and the computational load. Sakurai [21] has introduced a two function Eulerian variational principle for force-free magnetohydrodynamics and used it as a basis of a numerical scheme, his method is discussed in a book by Sturrock [17]. Yahalom & Lynden-Bell [4] combined the Lagrangian of Sturrock [17] with the Lagrangian of Sakurai [21] to obtain an Eulerian Lagrangian principle for barotropic magnetohydrodynamics which will depend on only six functions. The variational derivative of this Lagrangian produced all the equations needed to describe barotropic magnetohydrodynamics without any additional constraints. The equations obtained resembled the equations of Frenkel, Levich & Stilman [24] (see also [25]). Yahalom [29] have shown that for the barotropic case four functions will suffice. Moreover, it was shown that the cuts of some of those functions [30] are topological local conserved quantities.

Previous work was concerned only with barotropic magnetohydrodynamics. Variational principles of non barotropic magnetohydrodynamics can be found in the work of Bekenstein & Oron [26] in terms of 15 functions and V.A. Kats [23] in terms of 20 functions. Morrison [27] has suggested a Hamiltonian approach but this also depends on 8 canonical variables (see table 2 [27]). The variational principle introduced in [18, 19] show that only five functions will suffice to describe non barotropic MHD in the case that we enforce a Sakurai [21] representation for the magnetic field.
The variational principle was found as an efficient tool for deducing a generalized cross helicity using the translation symmetry groups of labels to both the global non-barotropic cross helicity conservation law and the conservation law of circulations of topological velocity along magnetic field lines. The latter were shown to be equivalent to the amount of non-barotropic cross helicity per unit of magnetic flux [31, 32]. Furthermore, we have shown that two additional cross helicity conservation laws exist: the $\chi$ and $\eta$ cross helicities [33]. Possible applications for MHD constraints of the generalized cross helicity and magnetic helicity are described in [32].

The plan of this paper is as follows: First we introduce the basic quantities and equations of non-barotropic MHD. Then we discuss some translation subgroups of Arnold’s labelling symmetry diffeomorphism related to non-barotropic MHD and the conserved laws associated with them. Finally we introduce dynamical bounds associated with these Noether currents.

2. Basic Equations
Consider the equations of non-barotropic MHD [17, 18]:

$$\frac{\partial \vec{B}}{\partial t} = \vec{\nabla} \times (\vec{v} \times \vec{B}),$$

(2)

$$\vec{\nabla} \cdot \vec{B} = 0,$$

(3)

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot (\rho \vec{v}) = 0,$$

(4)

$$\rho \frac{d\vec{v}}{dt} = \rho \left( \frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \vec{\nabla}) \vec{v} \right) = -\vec{\nabla} p(\rho, s) + \frac{(\vec{\nabla} \times \vec{B}) \times \vec{B}}{4\pi},$$

(5)

$$\frac{ds}{dt} = \frac{\partial s}{\partial t} + (\vec{v} \cdot \vec{\nabla}) s = 0.$$  

(6)

In the above the following notations are utilized: $\frac{\partial}{\partial t}$ is the temporal derivative, $\frac{d}{dt}$ is the temporal material derivative and $\vec{\nabla}$ has its standard meaning in vector calculus. $\rho$ is the fluid density and $s$ is the specific entropy. Finally $p(\rho, s)$ is the pressure which depends on the density and entropy (the non-barotropic case). Equation (2) describes the fact that the magnetic field lines are moving with the fluid elements (“frozen” magnetic field lines), equation (3) describes the fact that the magnetic field is solenoidal, equation (4) describes the conservation of mass and equation (5) is the Euler equation for a fluid in which both pressure and Lorentz magnetic forces apply. Equation (6) describes the fact that heat is not created (zero viscosity, zero resistivity) in ideal non-barotropic MHD and is not conducted, thus only convection occurs. The number of independent variables for which one needs to solve is eight ($\vec{v}, \vec{B}, \rho, s$) and the number of equations (2,4,5,6) is also eight. Notice that equation (3) is a condition on the initial $\vec{B}$ field and is satisfied automatically for any other time due to equation (2).

3. The labelling symmetry group and its subgroups
It is obvious that the choice of fluid labels is quite arbitrary. However, when enforcing the $\chi, \eta, \mu$ coordinate system (see [33] for details) such that:

$$\rho = \frac{\partial (\chi, \eta, \mu)}{\partial (x, y, z)},$$

(7)

The choice is restricted to $\tilde{\chi}, \tilde{\eta}, \tilde{\mu}$ such that:

$$\frac{\partial (\tilde{\chi}, \tilde{\eta}, \tilde{\mu})}{\partial (\chi, \eta, \mu)} = 1.$$  

(8)
Moreover the Euler potential magnetic field representation:
\[
\vec{B} = \vec{\nabla} \chi \times \vec{\nabla} \eta,
\]
reduces the choice further to:
\[
\frac{\partial (\tilde{\chi}, \tilde{\eta})}{\partial (\chi, \eta)} = 1.
\]

3.1. Metage translations

In what follows we consider the transformation:
\[
\tilde{\chi} = \chi, \tilde{\eta} = \eta, \tilde{\mu} = \mu + a(\chi, \eta)
\]

Hence \(a\) is a label displacement which may be different for each magnetic field line, as the field line is closed one need not worry about edge difficulties. This transformation satisfies trivially the conditions (8,10). If \(a = \delta \mu\) is small we can calculate the associated fluid element displacement with this relabelling (see [33] for details).
\[
\vec{\zeta} = -\frac{\partial \vec{r}}{\partial \mu} \delta \mu = -\delta \mu \frac{\vec{B}}{\rho}.
\]
Inserting this expression into the boundary term in equation (32) of [33] will result in:
\[
\delta A_B = \int dt \oint d\vec{S} \cdot [\vec{B}(\frac{1}{2}v^2 - w(\rho, s)) - \vec{v}((\vec{v} - \sigma \vec{\nabla} s) \cdot \vec{B})] \delta \mu = 0,
\]
which is the condition for magnetic cross helicity conservation. Inserting equation (12) into equation (34) of [33] we obtain the conservation law:
\[
\delta J = \int d^3x \rho (\vec{v} - \sigma \vec{\nabla} s) \cdot \vec{\zeta} = -\int d^3x \delta \mu (\vec{v} - \sigma \vec{\nabla} s) \cdot \vec{B}
\]
In the simplest case we may take \(\delta \mu\) to be a small constant, hence:
\[
\delta J = -\delta \mu \int d^3x (\vec{v} - \sigma \vec{\nabla} s) \cdot \vec{B} = -\delta \mu H_{CNB}
\]
Where \(H_{CNB}\) is the non barotropic global cross helicity [12, 31, 32] defined as:
\[
H_{CNB} \equiv \int d^3x (\vec{v} - \sigma \vec{\nabla} s) \cdot \vec{B} = \int d^3x \hat{v}_t \cdot \vec{B}
\]
in which \(\hat{v}_t \equiv \vec{v} - \sigma \vec{\nabla} s\) is the topological velocity field. We thus obtain the conservation of non-barotropic cross helicity using the Noether theorem and the symmetry group of metage translations. Of course one can perform a different translation on each magnetic field line, in this case one obtains:
\[
\delta J = -\int d^3x \delta \mu \hat{v}_t \cdot \vec{B} = -\int d\chi d\eta \delta \mu \int_{\chi, \eta} d\mu \rho^{-1} \hat{v}_t \cdot \vec{B}
\]
Now since \(\delta \mu\) is an arbitrary (small) function of \(\chi, \eta\) it follows that:
\[
I = \int d\mu \rho^{-1} \hat{v}_t \cdot \vec{B}
\]
is a conserved quantity for each magnetic field line. Along a magnetic field line the following equations hold:

\[ d\mu = \nabla \mu \cdot d\vec{r} = \nabla \mu \cdot \hat{B} dr = \frac{\rho}{B} dr \]  

(19)
in the above \( \hat{B} \) is an unit vector in the magnetic field direction and equation (15) of [33] is used. Inserting equation (19) into equation (18) we obtain:

\[ I = \oint_{\chi,\eta} d\vec{r} \cdot \vec{v}_t = \oint_{\chi,\eta} d\vec{r} \cdot \nabla \nu = [\nu]. \]  

(20)

which is just the circulation of the topological velocity along the magnetic field lines. This quantity can be written in terms of the generalized Clebsch representation of the velocity [18]:

\[ \vec{v} = \nabla \nu + \alpha \nabla \chi + \beta \nabla \eta + \sigma \nabla s, \quad \vec{v}_t = \nabla \nu + \alpha \nabla \chi + \beta \nabla \eta \]  

(21)
as:

\[ I = \oint_{\chi,\eta} d\vec{r} \cdot \vec{v}_t = \oint_{\chi,\eta} d\vec{r} \cdot \nabla \nu = [\nu]. \]  

(22)

\([\nu]\) is the discontinuity of \( \nu \). This was shown to be equal to the amount of non barotropic cross helicity per unit of magnetic flux [31, 32].

\[ I = [\nu] = \frac{dH_{CNB}}{d\Phi}. \]  

(23)

3.2. Transformations of magnetic surfaces

Consider the following transformations:

\[ \tilde{\eta} = \eta + \delta \eta(\chi,\eta), \quad \tilde{\chi} = \chi + \delta \chi(\chi,\eta), \quad \tilde{\mu} = \mu \]  

(24)
in which \( \delta \eta, \delta \chi \) are considered small in some sense. Inserting the above quantities into equation (10) and keeping only first order terms we arrive at:

\[ \partial_\eta \delta \eta + \partial_\chi \delta \chi = 0. \]  

(25)

This equation can be solved as follows:

\[ \delta \eta = \partial_\chi \delta f, \quad \delta \chi = -\partial_\eta \delta f, \]  

(26)
in which \( \delta f = \delta f(\chi,\eta) \) is an arbitrary small function. In this case we obtain a particle displacements of the form:

\[ \vec{\xi} = -\frac{\partial \vec{r}}{\partial \chi} \delta \chi - \frac{\partial \vec{r}}{\partial \eta} \delta \eta = -\frac{1}{\rho} \left( \nabla \eta \times \nabla \mu \delta \chi + \nabla \mu \times \nabla \chi \delta \eta \right) = \frac{\nabla \mu}{\rho} \times (\nabla \eta \delta \chi - \nabla \chi \delta \eta) \]  

(27)

A special case that satisfies equation (25) is the case of a constant \( \delta \chi \) and \( \delta \eta \), those two independent displacements lead to two new topological conservation laws:

\[ \delta J_\chi = \delta \chi \int d^3x \vec{v}_t \cdot \nabla \mu \times \nabla \eta = \delta \chi H_{CNB\chi}, \quad \delta J_\eta = \delta \eta \int d^3x \vec{v}_t \cdot \nabla \chi \times \nabla \mu = \delta \eta H_{CNB\eta}. \]  

(28)

Where the new non barotropic global cross helicities are defined as:

\[ H_{CNB\chi} \equiv \int d^3x \vec{v}_t \cdot \nabla \mu \times \nabla \eta, \quad H_{CNB\eta} \equiv \int d^3x \vec{v}_t \cdot \nabla \chi \times \nabla \mu \]  

(29)
We will find it useful to introduce the abstract "magnetic fields" as follows:

$$\vec{B}_\chi \equiv \vec{\nabla}_\mu \times \vec{\nabla}_\eta, \quad \vec{B}_\eta \equiv \vec{\nabla}_\chi \times \vec{\nabla}_\mu$$  \hspace{1cm} (30)

In terms of which we obtain the new helicities in a more conventional form:

$$H_{CNB\chi} = \int d^3x \vec{v}_t \cdot \vec{B}_\chi, \quad H_{CNB\eta} = \int d^3x \vec{v}_t \cdot \vec{B}_\eta$$  \hspace{1cm} (31)

We remark that those topological constants will only be conserved under special boundary conditions satisfying:

$$\oint d\vec{S} \cdot \left[ -\rho \vec{\xi} \left( \frac{1}{2} \vec{v}^2 - w(\rho, s) \right) + \rho \vec{v}(\vec{v}_t \cdot \vec{\xi}) + \frac{1}{4\pi} \vec{B} \times (\vec{\xi} \times \vec{B}) \right] = 0$$  \hspace{1cm} (32)

for:

$$\vec{\xi}_\chi = \frac{\vec{B}_\chi}{\rho} \delta \chi, \quad \vec{\xi}_\eta = \frac{\vec{B}_\eta}{\rho} \delta \eta$$  \hspace{1cm} (33)

This is more plausible for magnetic field lines which lie on topological torii. In this case \( \eta \) is non single valued [4] and thus the translation in this direction resembles moving fluid elements along closed loops. Both those helicities suffer a topological interpretation in terms of the knottiness of the abstract magnetic field lines and the flow lines. Finally we remark that for barotropic MHD \( \vec{v}_t \) can be replaced with \( \vec{v} \).

4. Possible Application

In his important review paper "Physics of magnetically confined plasmas" A. H. Boozer [34] states that: "A spiky current profile causes a rapid dissipation of energy relative to magnetic helicity. If the evolution of a magnetic field is rapid, then it must be at constant helicity." Usually topological conservation laws are used in order to deduce lower bounds on the "energy" of the flow. Those bounds are only approximate in non ideal flows but due to their topological nature simulations show that they are approximately conserved even when the "energy" is not. For example it is easy to show that the "energy" is bounded from below by the non-barotropic cross helicity as follows (see [33]):

$$H_{CNB} = \int \vec{B} \cdot \vec{v}_t d^3x \leq \frac{1}{2} \int \left( \vec{B}^2 + \vec{v}_t^2 \right) d^3x, \quad (34)$$

$$H_{CNB} = \int \vec{B} \cdot \vec{v}_t d^3x \leq \sqrt{\int \vec{v}_t^2 d^3x} \sqrt{\int \vec{B}^2 d^3x}, \quad (35)$$

the second equation is a result of the Cauchy-Schwartz inequality. In this sense a configuration with a highly complicated topology is more stable since its energy is bounded from below. It is a simple thing to show that similar bounds occur also for the \( \chi \) and \( \eta \) helicities:

$$H_{CNB\chi} = \int \vec{B}_\chi \cdot \vec{v}_t d^3x \leq \frac{1}{2} \int \left( \vec{B}_\chi^2 + \vec{v}_t^2 \right) d^3x, \quad (36)$$

$$H_{CNB\chi} = \int \vec{B}_\chi \cdot \vec{v}_t d^3x \leq \sqrt{\int \vec{v}_t^2 d^3x} \sqrt{\int \vec{B}_\chi^2 d^3x}, \quad (37)$$

$$H_{CNB\eta} = \int \vec{B}_\eta \cdot \vec{v}_t d^3x \leq \frac{1}{2} \int \left( \vec{B}_\eta^2 + \vec{v}_t^2 \right) d^3x, \quad (38)$$

$$H_{CNB\eta} = \int \vec{B}_\eta \cdot \vec{v}_t d^3x \leq \sqrt{\int \vec{v}_t^2 d^3x} \sqrt{\int \vec{B}_\eta^2 d^3x}, \quad (39)$$

Hence the kinetic energy is bounded by three differen bounds and so it the "total" energy. The importance of each of those bounds is dependent on the flow.
5. Conclusion
We have shown the connection of the translation symmetry groups of labels to both the
global non barotropic cross helicity conservation law and the conservation law of circulations
of topological velocity along magnetic field lines. The latter were shown to be equivalent to
the amount of non barotropic cross helicity per unit of magnetic flux [31, 32, 33]. Further
more we have shown that two additional cross helicity conservation laws exist the $\chi$ and $\eta$ cross helicities. Those lead to new bounds on MHD flows in addition to the bounds of the standard
non-barotropic cross helicity discussed in [32] for ideal non-barotropic MHD. The importance
of constants of motion for stability analysis is also discussed in [35]. The significance of those
constraints for non-ideal MHD and for plasma physics in general remains to be studied in future
works.

References
[1] Lynden-Bell D and Katz J 1981 Proceedings of the Royal Society of London. Series A, Mathematical and
 Physical Sciences, 378, No. 1773, 179
[2] Moffatt H K 1969 J. Fluid Mech. 35 117
[3] Yahalom A 1995 J. Math. Phys. 36 1324
[4] Yahalom A and Lynden-Bell D 2008 Journal of Fluid Mechanics 607 235
[5] Woltjer L 1958a Proc. Nat. Acad. Sci. U.S.A. 44 489
[6] Woltjer L 1958b Proc. Nat. Acad. Sci. U.S.A. 44 833
[7] Yahalom A 2017 Proceedings of the Chaotic Modeling and Simulation International Conference CHAOS ed
Christos H. Skiadas , 859
[8] Yahalom A 2018 Quantum Theory and Symmetries with Lie Theory and Its Applications in Physics Volume
2 (Springer Proceedings in Mathematics & Statistics) 255 387
[9] Mobbs S D 1981 Journal of Fluid Mechanics 108 475
[10] Padiyhe N and Morrison P J 1996 Phys. Lett. A 219 287
[11] Padiyhe N and Morrison P J 1996 Plasma Phys. Rep. 22 869
[12] Webb et al 2014 J. Phys. A, Math. and Theoret 47 095501
[13] Webb et al 2014 J. Phys A, Math. and Theoret 47 095502
[14] Webb G M and Anco S. C. 2016 J. Phys A, Math. and Theoret. 49 075501
[15] Webb G M, McKenzie J F and Zank G P 2015 J. Plasma Phys. 81 905810610
[16] Webb G M and Mac R L 2015 J. Plasma Phys.81, Issue 1, 905810115
[17] Sturrock P A 1994 Plasma Physics (Cambridge University Press, Cambridge)
[18] Yahalom A 2016 J. Plasma Phys. 82 905820204
[19] Yahalom A 2016 International Journal of Geometric Methods in Modern Physics 13 1650130
[20] Yahalom A 2016 International Journal of Mechanics 10 336
[21] Sakurai T 1979 Pub. Ast. Soc. Japan 31 209
[22] Vladimirrov V A and Moffatt H K 1995 J. Fluid. Mech 283 125
[23] Kats A V 2003 JETP Lett. 77 657
[24] Frenkel A , Levich E and Stilman L 1982 Phys. Lett. A 88 461
[25] Zakharov V E and Kuznetsov E A 1997 Usp. Fiz. Nauk 40 1087
[26] Bekenstein J D and Oron A 2000 Physical Review E 62 5594
[27] Morrison P J 1982 AIP Conference proceedings 88, Table 2, 13
[28] Yahalom A and Lynden-Bell D 2014 Geophysical & Astrophysical Fluid Dynamics 108 6
[29] Yahalom A 2010 EPL 89 34005
[30] Yahalom A 2013 Physics Letters A 377 1898
[31] Yahalom A 2017 Journal of Geophysical & Astrophysical Fluid Dynamics 111 131
[32] Yahalom A 2017 Fluid Dynamics Research 50 011406
[33] Yahalom A 2019 Journal of Physics: Conf. Series 1194 012113
[34] Boozer A. H. 2004 Rev. Mod. Phys. 76 1071
[35] Katz J, Inagaki S and Yahalom A 1993 Pub. Astro. Soc. Japan 45 421