LOW REGULARITY OF SOLUTIONS TO THE
ROTATION-CAMASSA-HOLM TYPE EQUATION WITH THE
CORIOLIS EFFECT

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Abstract. Studied herein is the local well-posedness of solutions with the
low regularity in the periodic setting for a class of one-dimensional general-
ized Rotation-Camassa-Holm equation, which is considered as an asymptotic
model to describe the propagation of shallow-water waves in the equatorial re-
region with the weak Coriolis effect due to the Earth’s rotation. With the aid of
the semigroup approach and a refined viscosity technique, the local existence,
uniqueness and continuity of periodic solutions in the spatial space $C^1$ is estab-
lished based on the local structure of the dynamics along the characteristics.

1. Introduction. This paper is concerned with the following one-dimensional gen-
eralized Rotation-Camassa-Holm (g-RCH) equation

$$u_t - u_{txx} + cu_x + 3uu_x - \frac{\beta_0}{\beta} u_{xxx} + \frac{\omega_1}{\alpha^2} u^2 u_x + \frac{\omega_2}{\alpha^3} u^3 u_x = \sigma(2u_x u_{xx} + uu_{xxx}),$$

which can be seen as an application of the transformation

$$x \rightarrow \sqrt{\beta} \mu x, \quad t \rightarrow \sqrt{\beta} \mu t, \quad u \rightarrow \alpha \epsilon u$$
on the original g-RCH equation

$$u_t - \beta \mu u_{txx} + cu_x + 3\alpha \epsilon u u_x - \beta_0 \mu u_{xxx} + \omega_1 \epsilon^2 u^2 u_x + \omega_2 \epsilon^3 u^3 u_x = \sigma \alpha \epsilon \mu (2u_x u_{xx} + uu_{xxx}),$$

wherein the solution $u(t, x)$ represents the horizontal velocity field at height $0 \leq z_0 \leq 1$ defined by

$$z_0^2 = \frac{(3c^4 + 5c^2 + 3)\sigma - (6c^4 + 7c^2 + 10)}{3(\sigma - 3)(c^2 + 1)^2}, \quad \sigma \neq 3,$$

c = \sqrt{1 + \Omega^2} - \Omega is the wave speed depending on the constant rotational frequency \( \Omega \) describing the Coriolis effect due to the Earth rotation, $\alpha = \frac{c^2}{1 + c^2}, \omega_1$ and $\omega_2$ are

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two coefficients of nonlinear high-order terms concerning the rotational frequency $\Omega$ defined by

$$\omega_1 = \frac{-3c(c^2 - 1)(c^2 - 2)}{2(1 + c^2)^3} \quad \text{and} \quad \omega_2 = \frac{(c^2 - 2)(c^2 - 1)^2(8c^2 - 1)}{2(1 + c^2)^5}.$$ 

The other coefficient $\beta$ of the term $u_{txx}$ and $\beta_0$ of the term $u_{xxx}$ in g-RCH Equation (2) respectively rely on both the rotational frequency $\Omega$ and the real parameter $|\sigma| < +\infty$ satisfying

$$\beta = \begin{cases} 3c^4 + 8c^2 - 1, & \sigma \neq 3, \\ 0, & \sigma = 3 \end{cases}, \quad \text{and} \quad \beta_0 = \begin{cases} c\left((\sigma\epsilon^{\sigma+1}c^{\sigma-1})/2\right), & \sigma \neq 3, \\ -c^3(1 + c^2), & \sigma = 3. \end{cases}$$

Observe that Equation (2) was derived by the formal asymptotic procedures in the equatorial zone along with the constant Coriolis frequency in the rotating fluid caused by the Earth’s rotation in [7], where the authors succeed to use the idea of the formal derivation of the Camassa-Holm (CH) equation with the Coriolis effect in the equatorial region by applying the CH scaling of the form as

$$\mu \ll 1 \quad \text{and} \quad \epsilon = O(\sqrt{\mu})$$

with $O(\epsilon^4, \mu^7)$ under the situation that the interface between the air and the water is a free surface, the water flows are incompressible and inviscid with a constant density $\rho$, here $\epsilon = \frac{a}{h_0} \ll 1$ is the amplitude parameter, $\mu = \frac{h_0^2}{\lambda^2} \ll 1$ is the shallowness parameter, $a$ is small amplitude, $h_0$ is mean level of water surface and $\lambda$ is the long wavelength.

It was also found that Equation (2) can be rewritten in the following two compatible Hamiltonian forms [7]

$$m_t = -B_1 \frac{\delta F}{\delta m} = -B_2 \frac{\delta E}{\delta m},$$

if one takes

$$m = u - \beta \mu \partial_x^3 u,$$

where the two skew-symmetric differential operators $B_1$ and $B_2$ are defined by

$$B_1 = \partial_x - \beta \mu \partial_x^3$$

and

$$B_2 = \partial_x \left((\sigma\alpha m + \frac{c}{2})\partial_x \right) + (\sigma\alpha m + \frac{c}{2}) \partial_x + (1 - \sigma) \alpha \epsilon \partial_x \partial_x + \beta \mu \partial_x^3 + \frac{2}{3} \omega_1 \epsilon^2 \partial_x (u \partial_x^{-1}(u \partial_x^{-1})),$$

respectively, associated with the following conserved quantities

$$E(u) = \frac{1}{2} \int \left(u^2 + \beta \mu u_x^2\right)dx,$$

$$F(u) = \frac{1}{2} \int \left(cu^2 + \alpha \epsilon u^3 + \sigma \alpha \epsilon \beta \mu u_x^2 + \beta \mu u_x^2 + \frac{1}{6} \omega_1 \epsilon^2 u^4 + \frac{1}{10} \omega_2 \epsilon^3 u^5\right)dx.$$ 

In fact, the g-RCH Equation (2) with $\sigma = 1$ was first proposed in [6] where the authors pointed out that its solution corresponding to physically relevant initial perturbations is more accurate on a much longer time scale and the deviation of the free surface can be determined by the horizontal velocity at a certain depth in the second-order approximation. Meanwhile, a blow-up criteria and wave-breaking
phenomena associated with the effects of the Coriolis force caused by the Earth rotation and nonlocal higher nonlinearities were drawn by the authors via the method of characteristics and conserved quantities to the Riccati-type differential inequality. More recently, some peaked solutions for Equation (2) with $\sigma = 1$ were established in [17]. It is found that the parameter $\Omega$ introduced in [6, 17, 7, 32] was viewed as a fixed $O(1)$ in terms of the nonlinear parameter $\epsilon$ and the shallowness parameter $\mu$, which makes some clues to see the formation of singularity under the influence of those high-order degree nonlinearities arising from the parameter $\Omega$. Also, such a model equation is analogous to the CH approximation of the two-dimensional incompressible and irrotational Euler equations and has a formal bi-Hamiltonian structure with the following three conserved quantities

$$I(u) = \int_{\mathbb{R}} u \ dx, \quad E(u) = \frac{1}{2} \int_{\mathbb{R}} \left( u^2 + u_x^2 \right) dx,$$

$$F(u) = \frac{1}{2} \int_{\mathbb{R}} \left( cu^2 + u^3 + uu_x^2 + \frac{\beta_0}{\beta} u_x^2 + \frac{\omega_1}{6\alpha^2} u^4 + \frac{\omega_2}{10\alpha^3} u^5 \right) dx.$$

Recalling the limit case of the Coriolis effect, i.e., $\Omega \to 0$ with $\sigma \neq 3$, it is easily seen

$$c \to 1, \quad \beta \to \frac{5}{6(3-\sigma)}, \quad \beta_0 \to \frac{\sigma + 2}{6(3-\sigma)}, \quad \omega_1 \to 0, \quad \omega_2 \to 0, \quad \alpha \to \frac{1}{2}.$$

In this regard, the scaling

$$x \to x - \frac{2}{5} t, \quad u \to u - \frac{1}{5}, \quad t \to t$$

allows Equation (2) with $\sigma = 1$ to be the classical CH equation

$$m_t + um_x + 2u_x m = 0, \quad m = u - u_{xx}, \quad x \in \mathbb{R}, \quad t > 0$$

that proposed in [3] as a model for describing the motion of shallow water waves over a flatbed, which was also obtained in [16] as a bi-Hamiltonian generalized of KdV. It is well-known that if we take $m = (1 - \alpha^2 \partial_x^2) u$ into Equation (3), then this famous CH equation can be rewritten as the following form

$$m_t = -B_1 \frac{\partial E}{\partial m} = -B_2 \frac{\partial F}{\partial m},$$

where both

$$B_1 = \partial_x - \partial_x^3 \quad \text{and} \quad B_2 = \partial_x m + m \partial_x$$

are Hamiltonian operators along with the following corresponding Hamiltonians

$$E(u) = \frac{1}{2} \int_{\mathbb{R}} \left( u^2 + u_x^2 \right) dx$$

and

$$F(u) = \frac{1}{2} \int_{\mathbb{R}} \left( u^3 + uu_x^2 \right) dx.$$
These details mentioned above reveal that the Coriolis forcing in the propagation of shallow-water waves cannot be neglected. Especially, the one-dimensional direction of propagation of the equatorial waves was due to the Coriolis effect (see [11, 13] for more information) and the data provided in [12, 18] claimed that the nonlinearity plays an important role for the equatorial ocean dynamics, so we herein continue to see the parameter Ω describing the Coriolis effect as a fixed constant and evaluate how these high-order degree nonlinearities affect the propagation of regularity of solutions for the g-RCH Equation (1). Recently, we observe that the well-posedness of local solution for the g-RCH Equation (1) in the Sobolev spaces of whole line was recently shown in [7], that is, for the initial data $u_0 \in H^s(\mathbb{R})$ with $s > \frac{3}{2}$, there exists some $T = T(u_0) > 0$ such that Equation (1) involving the non-periodic initial data admits a unique solution $u \in C([0,T]; H^s(\mathbb{R})) \cap C^1([0,T]; H^{s-1}(\mathbb{R}))$ with the continuity on the initial data $u_0$ [7]. So, the purpose of our work is to reveal the effect of high-order degree nonlinearities caused by the parameter Ω on the propagation of regularity of local solutions by lowering the regularity requirement on the initial data from $u_0 \in H^s(\mathbb{R})$ with $s > \frac{3}{2}$ to the class $C^1$ for the g-RCH Equation (1), or an equivalent form

$$u_t + \sigma uu_x + \frac{\beta_0}{\beta} u_x + \partial_x \left(1 - \partial_x^2\right)^{-1} g(u) = 0$$

involving the following periodic initial condition

$$u(t, x + \sigma) = u(t, x), \quad u(x, 0) = u_0(x), \quad x \in \mathbb{R},$$

hereafter

$$g(u) = \frac{3 - \sigma}{2} u^2 + \frac{\sigma}{2} u_x^2 + \lambda_1 u + \lambda_2 u^3 + \lambda_3 u^4$$

with

$$\lambda_1 = c - \frac{\beta_0}{\beta}, \quad \lambda_2 = \frac{\omega_1}{3\alpha^2} \quad \text{and} \quad \lambda_3 = \frac{\omega^2}{4\alpha^3}.$$

The following theorem provides the main result of our work.

**Theorem 1.1.** Let $r \in (0, 1)$ be given and $K > 0$ be fixed. Assume that $\sigma \neq 0$, $\sigma \neq 3$ and $v(t, x)$ is the local solution to the initial value problem

$$\begin{cases}
\partial_t v = P(\varphi_v, v), \\
v(0) = v_0 = u_0 \in C^1_1(\mathbb{R}),
\end{cases}$$

hereafter

$$P(\varphi, v) = -\partial_x \left(1 - \partial_x^2\right)^{-1} V \circ \varphi,$$

$$V = \frac{3 - \sigma}{2} (v \circ \varphi^{-1})^2 + \frac{\sigma}{2} \left(\partial_x (v \circ \varphi^{-1})\right)^2$$

$$+ \lambda_1 (v \circ \varphi^{-1}) + \lambda_2 (v \circ \varphi^{-1})^3 + \lambda_3 (v \circ \varphi^{-1})^4$$

and

$$\varphi_v(t, x) = x + \int_0^t v(s, x) ds.$$  

Then, problem (4), (5) admits a unique solution $u(t, x) : [0, T] \times \mathbb{R} \to \mathbb{R}$ given by

$$u \left(t, \sigma x + \frac{\beta_0}{\beta} t\right) = v \left(t, \varphi_v^{-1}(t, x)\right),$$

where
which is periodic of period \( \sigma \) with respect to \( x \) such that

\[
u \in C^1([0,T], C_1^1(\mathbb{R})) \cap C([0,T], C_1^1(\mathbb{R}))
\]

for

\[
0 < T < \min \left\{ \frac{r}{\|v_0\|_{C_1^1(\mathbb{R})} + K}, \frac{K}{c_0(r + K) + \|P(Id, v_0)\|_{C_1^1(\mathbb{R})}} \right\}.
\]

Furthermore, the map \( u_0 \mapsto u(\cdot; u_0) \) is continuous from \( C_1^1(\mathbb{R}) \) to

\[
C([0,T], C_1^1(\mathbb{R})) \cap C([0,T], C_1^1(\mathbb{R})).
\]

Although we do not discuss the case that \( \sigma = 3 \), i.e. \( \beta = 0 \), it is worthwhile to mention that when \( \beta = 0 \), the so-called g-RCH Equation (2) arrives at

\[
u_t + \frac{c}{3} \delta \mu u_{xxx} + cu_x + 3\delta \epsilon uu_x + \tilde{\omega}_1 \frac{c^2}{3}(u^3)_x + \tilde{\omega}_2 c^3 u^3 u_x = 0,
\]

where the solution \( \nu \) is horizontal velocity at the height

\[
\tilde{z}_0 = \frac{3c^4 + 5c^2 + 3}{3(c^2 + 1)^2} < 1, \quad \tilde{\alpha} = \frac{c^2}{1 + c^2},
\]

\[
\tilde{\omega}_1 = \frac{-3c(c^2 - 1)(c^2 - 2)}{2(1 + c^2)^3}, \quad \tilde{\omega}_2 = \frac{(c^2 - 2)(c^2 - 1)^2(8c^2 - 1)}{2(1 + c^2)^5}
\]

with

\[
c = \sqrt{\frac{1}{3} \left( \sqrt{19} - 4 \right)} \quad \text{or} \quad \Omega = \sqrt{\frac{1}{6} (1 + 2\sqrt{19})}.
\]

In this case, it is resemble that Equation (12) can be viewed as a class of generalized KdV equations with higher-order nonlinearities. For the background and dynamical behavior of this type generalized KdV equation, we refer the reader to [20, 21, 25, 26] and the references therein.

Our approach is inspired by the idea performed on the CH equation in [28] and on the Dullin-Gottwald-Holm (DGH) equation in [29] where the regularity requirements on the initial data for the corresponding equation were lowered to some differentiable periodic functions, but the difficulty in our case is that the g-RCH Equation (1) of the present paper contains some higher-order nonlinear terms, which need some more refined estimates and detailed analysis not only to switch between Lagrangian coordinates and Eulerian coordinates, but also to construct a suitable Lipschitzian operator. Based on the semigroup approach of quasilinear hyperbolic evolution equations via the Lagrangian coordinates, it is shown that the local existence, uniqueness and continuity of solutions with continuously differentiable, periodic initial data for the g-RCH Equation (1) are established, which covers some previous results in view of the variability of the parameter \( \sigma \). For more recent applications of the semigroup associated with relevant linear operator on various partial differential equations, we refer to [5, 19, 24, 30].

The rest of the present paper is organized as follows. Section 2 introduces some notations and preliminaries. Section 3 is devoted to the local well-posedness of solutions to the g-RCH Equation (1) involving the periodic initial data (5) with low regularity in \( C^1 \) space along the characteristics by an application of Banach contraction mapping principle.
2. Some notations and preliminaries. Some notations and propositions used to establish the main result of the present paper are prepared in this section.

(1) $C_1(\mathbb{R}) \subset C(\mathbb{R})$ is the usual linear space over $\mathbb{R}$ of all real-valued continuously periodic functions $f : \mathbb{R} \to \mathbb{R}$ with period 1, which equipped with the obvious norm

$$
\|f\|_{C_1(\mathbb{R})} = \sup_{x \in \mathbb{R}} |f(x)|
$$

is a Banach space;

(2) $C_i(\mathbb{R}) = C_1(\mathbb{R}) \cap C^i(\mathbb{R}) (i = 1, 2)$ connote the linear space over $\mathbb{R}$ of all real-valued continuously $i$-order differentiable functions $f : \mathbb{R} \to \mathbb{R}$ that are periodic with period 1, which inherited the linear operations and endowed with the subsequent norm

$$
\|f\|_{C_i(\mathbb{R})} = \sup_{x \in \mathbb{R}} |f(x)| + \sum_{j=1}^{i} \sup_{x \in \mathbb{R}} |f^{(j)}(x)|
$$

reaches to a Banach space, respectively;

(3) $X(\mathbb{R}) \subset C^1(\mathbb{R})$ denotes the collection of all functions $g \in C^1(\mathbb{R})$ with $g(1) = g(0) + 1$ and the first derivative $g'$ periodic of period 1, which supplied with the metric

$$
d(g_1, g_2) = |g_1(0) - g_2(0)| + \sup_{x \in \mathbb{R}} |g'_1(x) - g'_2(x)|
$$

is a complete metric space;

(4) $v \circ \varphi$ stands for the function composition of $v$ and $\varphi$;

(5) $\varphi^{-1}$ means the inverse function of $\varphi$;

(6) $(X(\mathbb{R}) \times C_1^i(\mathbb{R}), D)$ denotes a complete metric space endowed with the metric

$$
D((\varphi_1, v_1), (\varphi_2, v_2)) = d(\varphi_1, \varphi_2) + \|v_1 - v_2\|_{C^1(\mathbb{R})};
$$

(7) $\bar{B}(v_0, K)$ is defined as a closed ball of radius $K$ and center $v_0$ in

$$
C \left([0, T], C^1(\mathbb{R})\right).
$$

The following provides some useful propositions.

**Proposition 1** ([28]). For given $r \in (0, 1)$, the set

$$
U_r = \{g \in X(\mathbb{R}) \mid d(Id, g) < r, \; g'(x) > 0 \; \text{for} \; x \in \mathbb{R}\} \subset X(\mathbb{R})
$$

is an open set in $(X(\mathbb{R}), d)$, wherein

$$
d(Id, g) = |g(0)| + \sup_{x \in \mathbb{R}} |1 - g'(x)|.
$$

**Proposition 2** ([28]). Let $r \in (0, 1)$ be fixed, then for $\varphi \in U_r$ and $v \in C^1(\mathbb{R})$, there holds a 1-order Lipschitz function $\varphi^{-1} \in X(\mathbb{R})$ with the (uniform) Lipschitz coefficient $L = \frac{1}{r}$ and $v \circ \varphi^{-1} \in C^1(\mathbb{R})$.

**Proposition 3** ([28]). Let $V \in C^1(\mathbb{R})$, then there exists a unique $y \in C^2(\mathbb{R})$ such that

$$
y - y'' = V \; \text{in} \; \mathbb{R},
$$

which formula is given by

$$
\begin{align*}
y(x) &= \int_0^1 G(x - [x] - s)V(s)ds, \\
y'(x) &= \int_0^1 G'(x - [x] - s)V(s)ds,
\end{align*}
$$
wherein

\[ G(q) = \begin{cases} \frac{\cosh(q - \frac{1}{2})}{2 \sinh \frac{1}{2}}, & q \geq 0, \\ \frac{\cosh(q + \frac{1}{2})}{2 \sinh \frac{1}{2}}, & q < 0. \end{cases} \]  

Here, \([x]\) represents the integer part of \(x\).

3. **The local well-posedness of solutions with low regularity in** \(C^1\). This section pays attention to the proof of Theorem 1.1. We first construct a local solution in Lagrangian coordinates and then transfer it in Eulerian coordinates.

3.1. **Local solution in Lagrangian coordinates.** The goal of this subsection is to construct a solution in Lagrangian coordinates. The first step herein is to introduce the following corresponding differential system

\[
\begin{aligned}
\partial_t \varphi &= v, \\
\partial_t v &= P(\varphi, v)
\end{aligned}
\]  

involving the initial data

\[
\varphi(0) = Id, \quad v(0) = v_0 \in C^1_1(\mathbb{R})
\]  

with \(\varphi(t) \in X(\mathbb{R})\) and \(v(t) \in C^1_1(\mathbb{R})\), where \(P(\varphi, v)\) is defined by (7) with (8).

From Proposition 2 and Proposition 3, it is shown that the system (16),(17) is well-defined.

The proof of Theorem 1.1 (the local well-posedness of low regularity solution to problem (4),(5) for \(\sigma \neq 0\) and \(\sigma \neq 3\)) is approached based on the following Lemma 3.1 and Lemma 3.2. A Lipschitz operator is first constructed in Lemma 3.1.

**Lemma 3.1.** Let \(r \in (0,1)\) be given and \(K > 0\) be fixed. Then, the operator \(P : X(\mathbb{R}) \times C^1_1(\mathbb{R}) \to C^1_1(\mathbb{R})\) given by (7) is Lipschitzian in \(U_r \times \bar{B}(v_0, K)\).

**Proof.** Observer that (13) allows

\[ y' = \partial_x (1 - \partial_x^2)^{-1}V, \]

which together with (7) transpires that

\[ P(\varphi, v) = -y' \circ \varphi \]

and

\[ \partial_x P(\varphi, v) = -(y'' \circ \varphi) \varphi' = -((y - V) \circ \varphi) \varphi'. \]

Then the norm of \(P(\varphi, v)\) in \(C^1_1(\mathbb{R})\) is equivalent to

\[ \sup_{x \in \mathbb{R}} |y' \circ \varphi| + \sup_{x \in \mathbb{R}} |((y - V) \circ \varphi) \varphi'|. \]

In order to get the target of the proof, some estimates on \(y'(\varphi(x))\) and \((y - V) \circ \varphi\) \(\varphi'\)

are performed as follows. From (14) and

\[ G'(q) = \begin{cases} \frac{\sinh(q - \frac{1}{2})}{2 \sinh \frac{1}{2}}, & q \geq 0, \\ \frac{\sinh(q + \frac{1}{2})}{2 \sinh \frac{1}{2}}, & q < 0. \end{cases} \]
it is inferred that
\[ y'(\varphi(x)) \]
\[ = \int_{\varphi(0)}^{\varphi(x)} G'(\varphi(x) - s)V(s)ds \]
\[ = \int_{\varphi(0)}^{\varphi(x)} \frac{\text{sh}(\varphi(x) - s - \frac{1}{2})}{2 \text{ sh} \frac{1}{2}} V(s)ds + \int_{\varphi(0)}^{\varphi(1)} \frac{\text{sh}(\varphi(x) - s + \frac{1}{2})}{2 \text{ sh} \frac{1}{2}} V(s)ds \]
\[ = \int_{0}^{x} \frac{\text{sh}(\varphi(x) - \varphi(q) - \frac{1}{2})}{2 \text{ sh} \frac{1}{2}} \left( \lambda_1 v(q) + \lambda_2 v^3(q) + \lambda_3 v^4(q) \right) \frac{\sigma}{2} \left( \frac{v'(q)}{\varphi'(q)} \right)^2 + 3 - \frac{\sigma}{2} v^2(q) \varphi'(q) dq \]
\[ + \int_{x}^{1} \frac{\text{sh}(\varphi(x) - \varphi(q) + \frac{1}{2})}{2 \text{ sh} \frac{1}{2}} \left( \lambda_1 v(q) + \lambda_2 v^3(q) + \lambda_3 v^4(q) \right) \frac{\sigma}{2} \left( \frac{v'(q)}{\varphi'(q)} \right)^2 + 3 - \frac{\sigma}{2} v^2(q) \varphi'(q) dq \]
\[ = P_1(\varphi, v)(x) + P_2(\varphi, v)(x), \quad x \in [0, 1]. \]

Where the change of variables \( s = \varphi(q) \) and the formula \( \partial_s (v \circ \varphi^{-1}) = \frac{(\partial_s v) \circ \varphi^{-1}}{(\partial_s \varphi) \circ \varphi^{-1}} \) have been used.

The following task is to control the two terms \( P_1(\varphi, v)(x) \) and \( P_2(\varphi, v)(x) \). In order to deal with the term \( P_1(\varphi, v)(x) \), an auxiliary function \( F_1 : \mathbb{R}^5 \rightarrow \mathbb{R} \) is defined by
\[ F_1(a, b, c, d, e) = \frac{\text{sh} (a - b - \frac{1}{2})}{2 \text{ sh} \frac{1}{2}} \left( \lambda_1 c + \frac{3 - \sigma}{2} c^2 + \lambda_2 c^3 + \lambda_3 c^4 + \frac{\sigma}{2} d^2 \right) e \]  
(21)

with \( a, b, c, d, e \in \mathbb{R} \). Then, from \( \varphi_1, \varphi_2 \in U_r \) and \( v_1, v_2 \in \bar{B}(v_0, K) \) it reveals
\[ \left| P_1(\varphi_1, v_1)(x) - P_1(\varphi_2, v_2)(x) \right| \]
\[ \leq \int_{0}^{1} \left| F_1(a_1, b_1, c_1, d_1, e_1)(q) - F_1(a_2, b_2, c_2, d_2, e_2)(q) \right| dq, \]  
(22)

where
\[ a_i(q) = \varphi_i(x), \quad b_i(q) = \varphi_i(q), \quad c_i(q) = v_i(q), \quad d_i(q) = \frac{v'_i(q)}{\varphi'_i(q)}, \quad e_i(q) = \varphi'_i(q), \quad i = 1, 2. \]

First, we claim that the terms \( a_i(q), b_i(q), c_i(q), d_i(q), e_i(q) \) are uniformly bounded for \( q \in [0, 1] \). Indeed, from
\[ \varphi(q) = \varphi(0) + q - \int_{0}^{q} (1 - \varphi'(s)) ds, \]  
(23)

\[ \varphi \in U_r \subset X(\mathbb{R}) \] and Proposition 1 it follows
\[ |a(q)| \]
\[ = |\varphi(0) + x - \int_{0}^{x} (1 - \varphi'(s)) ds| \]
Similarly, $|b(q)| < 1 + r$ and

$$0 < e(q) = \varphi'(q) \leq 1 + \sup_{s \in [0,1]} |1 - \varphi'(s)| \leq 1 + d(Id, \varphi) < 1 + r.$$  

From $v \in \bar{B}(v_0, K) \subset C^1_c(\mathbb{R})$ and Proposition 2, it thus transpires that

$$|c(q)| \leq \|v\|_{C^1_c(\mathbb{R})} \leq \|v_0\|_{C^1_c(\mathbb{R})} + K$$

and

$$|d(q)| = \left| \frac{v'(q)}{\varphi'(q)} \right| = \left| (\varphi^{-1}(q))' \right| \left| v'(q) \right| \leq \frac{1}{1 - r} \|v\|_{C^1_c(\mathbb{R})} \leq \frac{1}{1 - r} \left( \|v_0\|_{C^1_c(\mathbb{R})} + K \right).$$

Thus, the desired claim as indicated above is complete.

Second, we prove that the auxiliary function $F_i(a, b, c, d, e)$ is Lipschitzian. An application of the mean value theorem on (22) gives

$$|F_1(a_1, b_1, c_1, d_1, e_1)(q) - F_1(a_2, b_2, c_2, d_2, e_2)(q)|$$

$$\leq \sup_{w \in [w_1, w_2]} \|\nabla F(w)\| \left( |a_1(q) - a_2(q)| + |b_1(q) - b_2(q)| + |c_1(q) - c_2(q)| + |d_1(q) - d_2(q)| + |e_1(q) - e_2(q)| \right)$$

$$\leq \sup_{w \in \mathbb{R}^5, \|w\| \leq 5C} \|\nabla F(w)\| \left( |a_1(q) - a_2(q)| + |b_1(q) - b_2(q)| + |c_1(q) - c_2(q)| + |d_1(q) - d_2(q)| + |e_1(q) - e_2(q)| \right)$$

$$= C(F) \left( |a_1(q) - a_2(q)| + |b_1(q) - b_2(q)| + |c_1(q) - c_2(q)| + |d_1(q) - d_2(q)| + |e_1(q) - e_2(q)| \right),$$

within

$$C(F) = \sup_{w \in \mathbb{R}^5, \|w\| \leq 5C} \|\nabla F(w)\|,$$  

$w_i = (a_i, b_i, c_i, d_i, e_i)(q) \in \mathbb{R}^5, \ i = 1, 2$

and

$$C = \max \left\{1 + r, \frac{1}{1 - r} \left( \|v_0\|_{C^1_c(\mathbb{R})} + K \right) \right\}.$$  

Again, by (23) and $\varphi_1, \varphi_2 \in U_r$, it is found that

$$|a_1(q) - a_2(q)|$$

$$\leq |\varphi_1(0) - \varphi_2(0)| + \int_0^x |\varphi'_1(s) - \varphi'_2(s)| ds$$

$$\leq |\varphi_1(0) - \varphi_2(0)| + \sup_{s \in [0,1]} |\varphi'_1(s) - \varphi'_2(s)|$$

$$\leq d(\varphi_1, \varphi_2).$$
Similarly,
\[ |b_1(q) - b_2(q)| \leq |\varphi_1(0) - \varphi_2(0)| + \int_0^q |\varphi'_1(s) - \varphi'_2(s)| ds \leq d(\varphi_1, \varphi_2) \]  
(27)
and
\[ |e_1(q) - e_2(q)| = |\varphi'_1(q) - \varphi'_2(q)| \leq d(\varphi_1, \varphi_2). \]  
(28)
As \( v_1, v_2 \in \bar{B}(v_0, K) \subset C^1_1(\mathbb{R}) \) it is given that
\[ |c_1(q) - c_2(q)| = |v_1(q) - v_2(q)| \leq \|v_1 - v_2\|_{C^1_1(\mathbb{R})} \]  
(29)
and
\[
|d_1(q) - d_2(q)| = \frac{|v_1'(q) - v_2'(q)|}{\varphi'_1(q) \varphi'_2(q)} \leq \frac{|v_1'(q) - v_2'(q)|}{\varphi'_1(q)} + \frac{|\varphi_1'(q) - \varphi_2'(q)|}{\varphi'_2(q)}
\]
\[ = (\varphi^{-1}_1(q))' |\varphi'_1(q) - \varphi'_2(q)| - (\varphi^{-1}_2(q))' |\varphi'_1(q) - \varphi'_2(q)| \leq \frac{1}{1 - r} \|v_1 - v_2\|_{C^1_1(\mathbb{R})} + \frac{C}{(1 - r)^2} d(\varphi_1, \varphi_2) \]
\[ \leq \frac{C}{1 - r} D((\varphi_1, v_1), (\varphi_2, v_2)). \]  
(30)
Then, a substitution of (24)-(30) into (22) yields
\[
\left| P_1(\varphi_1, v_1)(x) - P_1(\varphi_2, v_2)(x) \right| \leq C(F) \left( 3d(\varphi_1, \varphi_2) + \|v_1 - v_2\|_{C^1_1(\mathbb{R})} + \frac{C}{1 - r} D((\varphi_1, v_1), (\varphi_2, v_2)) \right)
\]
\[ \leq \left( 3 + \frac{C}{1 - r} \right) C(F) D((\varphi_1, v_1), (\varphi_2, v_2)). \]  
(31)
Similar to the several treatment above performed on (31), it can be concluded that
\[
\left| P_2(\varphi_1, v_1)(x) - P_2(\varphi_2, v_2)(x) \right| \leq \bar{C} D((\varphi_1, v_1), (\varphi_2, v_2)).
\]  
(32)
Further, for
\[- \partial_x P(\varphi, v) = g(\varphi(x)) \varphi' - V(\varphi(x)) \varphi' \]
\[ = \varphi'(x) \int_{\varphi(0)}^{\varphi(x)} G(\varphi(x) - s)V(s) ds - \varphi'(x) V(\varphi(x)) \]
\[ = \varphi'(x) \left( \int_{\varphi(0)}^{\varphi(x)} \frac{\text{ch}(\varphi(x) - s) + \frac{1}{2}}{2 \text{sh} \frac{1}{2}} V(s) ds + \int_{\varphi(x)}^{\varphi(1)} \frac{\text{ch}(\varphi(x) - s + \frac{1}{2})}{2 \text{sh} \frac{1}{2}} V(s) ds \right) \]
\[ - \varphi'(x) \left( \lambda_1 v(x) + \lambda_2 v^3(x) + \lambda_3 v^4(x) + \frac{\sigma}{2} \left( \frac{v'(x)}{\varphi'(x)} \right)^2 + \frac{3 - \sigma}{2} v^2(x) \right) \]
\[ = \varphi'(x) \left( \int_0^x \frac{\text{ch}(\varphi(x) - \varphi(q) - \frac{1}{2})}{2 \text{sh} \frac{1}{2}} \left( \lambda_1 v(q) + \lambda_2 v^3(q) + \lambda_3 v^4(q) \right) \right) \]
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\[ + \frac{\sigma}{2} \left( \frac{v'(q)}{\varphi'(q)} \right)^2 + \frac{3 - \sigma}{2} v^2(q) \varphi'(q) dq \]

\[ \varphi'(x) \left( \int_x^1 \frac{\text{ch}(\varphi(x) - \varphi(q) + \frac{1}{2})}{2 \text{sh} \frac{1}{2}} \left( \lambda_1 v(q) + \lambda_2 v^3(q) + \lambda_3 v^4(q) \right. \right. \]

\[ + \frac{\sigma}{2} \left( \frac{v'(q)}{\varphi'(q)} \right)^2 + \frac{3 - \sigma}{2} v^2(q) \varphi'(q) dq \right) \right) \]

\[ - \varphi'(x) \left( \lambda_1 v(x) + \lambda_2 v^3(x) + \lambda_3 v^4(x) + \frac{\sigma}{2} \left( \frac{v'(x)}{\varphi'(x)} \right)^2 + \frac{3 - \sigma}{2} v^2(x) \right) . \]

by utilizing the similar arguments performed in (31) and (32), it can be evaluated

\[ \left| \partial_x P(\varphi_1, v_1) - \partial_x P(\varphi_2, v_2) \right| \leq C_2 D((\varphi_1, v_1), (\varphi_2, v_2)). \] (34)

Consequently, in view of (31), (32) and (34), this completes the proof of Lemma 3.1.

Lemma 3.2 focuses on the well-posedness of problem (6) based on Lemma 3.1.

**Lemma 3.2.** Assume (11). Then problem (6) admits a unique solution in

\[ C^1 \left( [0, T], C^1_1(\mathbb{R}) \right) \]

and also the map \( v_0 \mapsto v \) is continuous in the \( C^1 \) norm.

**Proof.** Define an operator

\[ \mathcal{T} : C \left( [0, T], C^1_1(\mathbb{R}) \right) \to C \left( [0, T], C^1_1(\mathbb{R}) \right) \]

as

\[ (\mathcal{T} v)(t) = v_0 + \int_0^t P(\varphi_v, v)(s) ds, \quad t \in [0, T]. \] (35)

First, we claim that \( \mathcal{T}(B(v_0, K)) \subseteq \bar{B}(v_0, K) \) for some small enough \( T \) defined by (11). From \( v(t, x) \in \bar{B}(v_0, K) \), it implies that \( v(t, x) \in C^1_1(\mathbb{R}) \). Then by (9), it can be derived that

\[ \varphi_v(t, 1) = 1 + \int_0^t v(s, 1) ds = 1 + \int_0^t v(s, 0) ds = 1 + \varphi_v(t, 0) \]

and

\[ \varphi_v'(t, x) = 1 + \int_0^t \frac{\partial v(s, x)}{\partial x} ds = 1 + \int_0^t \frac{\partial v(s, x + 1)}{\partial x} ds = \varphi_v'(t, x + 1), \]

which shows

\[ \varphi_v(t, x) \in X(\mathbb{R}). \]
Further, from (11) it follows
\[
d(\text{Id}, \varphi_v(t, x)) = \left| \varphi_v(t, 0) \right| + \sup_{x \in \mathbb{R}} \left| 1 - \frac{\partial}{\partial x} \varphi_v(t, x) \right|
\]
\[
= \left| \int_0^t v(s, 0) ds \right| + \sup_{x \in \mathbb{R}} \left| \int_0^t \frac{\partial v(s, x)}{\partial x} ds \right|
\]
\[
\leq t \sup_{s \in [0,t]} \|v(s, x)\|_{C_1^1(\mathbb{R})}
\]
\[
= T \left( \|v_0\|_{C_1^1(\mathbb{R})} + K \right)
\]
\[
< r,
\]
and hence
\[
\varphi_v(t, x) \in U_r \quad \text{for} \quad t \in [0, T].
\]
A simple calculation with (35), Lemma 3.1, (36), \(v(t, x) \in \bar{B}(v_0, K)\) and (11) reveals that
\[
\| (Tv)(t) - v_0 \|_{C_1^1(\mathbb{R})}
\]
\[
= \left\| \int_0^t P(\varphi_v, v)(s) ds \right\|_{C_1^1(\mathbb{R})}
\]
\[
\leq \int_0^T \|P(\varphi_v, v)(s)\|_{C_1^1(\mathbb{R})} ds
\]
\[
\leq \int_0^T \left( c_0 D\left( (\varphi_v(s), v(s)), (\text{Id}, v_0) \right) + \|P(\text{Id}, v_0)(s)\|_{C_1^1(\mathbb{R})} \right) ds
\]
\[
= \int_0^T \left( c_0 \left( d(\text{Id}, \varphi_v) + \|v_0 - v(s)\|_{C_1^1(\mathbb{R})} \right) + \|P(\text{Id}, v_0)(s)\|_{C_1^1(\mathbb{R})} \right) ds
\]
\[
\leq T \left( c_0(r + K) + \|P(\text{Id}, v_0)(s)\|_{C_1^1(\mathbb{R})} \right)
\]
\[
< K,
\]
and thereby
\[
T(\bar{B}(v_0, K)) \subseteq \bar{B}(v_0, K).
\]

Second, we prove that the operator \(T\) defined by (35) is a contraction. Combining (35), Lemma 3.1 and (9) yields
\[
\| (Tv_1)(t) - (Tv_2)(t) \|_{C_1^1(\mathbb{R})}
\]
\[
= \left\| \int_0^t \left( P(\varphi_{v_1}, v_1)(s) - P(\varphi_{v_2}, v_2)(s) \right) ds \right\|_{C_1^1(\mathbb{R})}
\]
\[
\leq \int_0^t \left\| P(\varphi_{v_1}, v_1)(s) - P(\varphi_{v_2}, v_2)(s) \right\|_{C_1^1(\mathbb{R})} ds
\]
\[
\leq \int_0^t c_0 D\left( (\varphi_{v_1}(s), v_1(s)), (\varphi_{v_2}(s), v_2(s)) \right) ds
\]
\[
= c_0 \int_0^t \left( d(\varphi_{v_1}(s), \varphi_{v_2}(s)) + \|v_1(s) - v_2(s)\|_{C_1^1(\mathbb{R})} \right) ds
\]
Proposition 2, Lemma 3.2, Lemma 3.1 and (25), it transpires that

\[
\varphi = \varphi_x
\]

Throughout the proof of this theorem, we let

This theorem is proved by considering the following six steps.

**Proof of Theorem 1.1.**

1. The detailed proof of Theorem 1.1 is presented as follows via Lemma 3.1 and Lemma 3.2.

2. Then, the conclusion of this lemma is directly followed from the Banach contraction principle along with the Picard iteration technique for ODE [23].

3. **Local well-posedness of solution in the corresponding Eulerian coordinates.** The detailed proof of Theorem 1.1 is presented as follows via Lemma 3.1 and Lemma 3.2.

4. **Proof of Theorem 1.1.** This theorem is proved by considering the following six steps. Throughout the proof of this theorem, we let \( \varphi = \varphi_x \) for simplicity.

5. **Step 1.** We show \( \varphi^{-1}(-,x) \in C^1([0,T],\mathbb{R}) \). Consider \( t_1, t_2 \in [0,T] \). Then, from Proposition 2, Lemma 3.2, Lemma 3.1 and (25), it transpires that

\[
= c_0 \int_0^t \left( |\varphi_{v_1}(s,0) - \varphi_{v_2}(s,0)| + \sup_{x \in \mathbb{R}} |\varphi'_{v_1}(s,x) - \varphi'_{v_2}(s,x)| \right) ds
\]

\[
+ c_0 \int_0^t \|v_1(s) - v_2(s)\|_{C^1_c(\mathbb{R})} ds
\]

\[
= c_0 \int_0^t \int_0^s \left( |v_1(q,0) - v_2(q,0)| + \sup_{x \in \mathbb{R}} |\partial_x v_1(q,x) - \partial_x v_2(q,x)| \right) dq ds
\]

\[
+ c_0 \int_0^t \|v_1(s) - v_2(s)\|_{C^1_c(\mathbb{R})} ds
\]

\[
\leq c_0 (1 + T) \int_0^t \sup_{q \in [0,s]} \|v_1(q) - v_2(q)\|_{C^1_c(\mathbb{R})} ds
\]

\[
\leq c_0 (1 + T) d_k(v_1, v_2) \int_0^t e^{ks} ds
\]

\[
= c_k (1 + T) d_k(v_1, v_2)(e^{kt} - 1)
\]

and, consequently

\[
d_k \left( (Tv_1)(t) - (Tv_2)(t) \right) \leq \frac{1}{k} c_0 (1 + T) d_k(v_1, v_2),
\]

wherein

\[
d_k(\mu_1, \mu_2) = \sup_{s \in [0, T]} \left( e^{-k s} \sup_{q \in [0, s]} \|\mu_1(q) - \mu_2(q)\|_{C^1_c(\mathbb{R})} \right), \quad \mu_1, \mu_2 \in C^1_c(\mathbb{R})
\]

and

\[
k > c_0(1 + T)
\]

is fixed. Therefore, the advertised claim is confirmed.

Then, the conclusion of this lemma is directly followed from the Banach contraction principle along with the Picard iteration technique for ODE [23].

3. **Local well-posedness of solution in the corresponding Eulerian coordinates.** The detailed proof of Theorem 1.1 is presented as follows via Lemma 3.1 and Lemma 3.2.

**Proof of Theorem 1.1.** This theorem is proved by considering the following six steps. Throughout the proof of this theorem, we let \( \varphi = \varphi_x \) for simplicity.

1. **Step 1.** We show \( \varphi^{-1}(-,x) \in C^1([0,T],\mathbb{R}) \). Consider \( t_1, t_2 \in [0, T] \). Then, from Proposition 2, Lemma 3.2, Lemma 3.1 and (25), it transpires that

\[
\left| \varphi^{-1}(t_1, \varphi(t_2,x)) - \varphi^{-1}(t_2, \varphi(t_2,x)) \right|
\]

\[
= \left| \varphi^{-1}(t_1, \varphi(t_2,x)) - \varphi^{-1}(t_1, \varphi(t_1,x)) \right|
\]

\[
\leq \left| \int_{\varphi(t_1,x)}^{\varphi(t_2,x)} |\partial_z \varphi^{-1}(t_1, z)| dz \right|
\]

\[
\leq \frac{1}{1 - r} \left| \varphi(t_2,x) - \varphi(t_1,x) \right|
\]

\[
\leq \frac{1}{1 - r} \left| \int_{t_1}^{t_2} |\partial_x \varphi(\tau, x)| d\tau \right|
\]
We claim that which means \( \varphi \) shows that from (9) it is clear that also observe that the term on the right side of (42) is a continuous function, then and consequently \( \varphi(t_2, x) \) in (37) is replaced by \( x \), then

\[
|\varphi^{-1}(t_1, x) - \varphi^{-1}(t_2, x)| \leq C|t_1 - t_2|, \tag{38}
\]

since the estimate in (37) is independent of \( x \). At this point, from (38) it is adduced that \( \varphi^{-1}(\cdot, x) \) is a.e. differentiable. Differentiating the identity \( \varphi^{-1}(t, \varphi(t, x)) = x \) (39) with respect to \( t \) yields

\[
\partial_t \varphi^{-1} + \partial_x \varphi^{-1} \partial_t \varphi(t, x) = 0. \tag{40}
\]

If (39) is differentiated along \( x \), there appears the relation \( \partial_x \varphi^{-1} \partial_x \varphi(t, x) = 1. \) (41)

A simple inspection of (41) into (40) shows

\[
\partial_t \varphi^{-1}(t, \varphi(t, x)) = -\frac{\partial_t \varphi(t, x)}{\partial_x \varphi(t, x)},
\]

and consequently

\[
\partial_t \varphi^{-1}(t, x) = -\frac{\partial_t \varphi(t, \varphi^{-1}(t, x))}{\partial_x \varphi(t, \varphi^{-1}(t, x))}, \quad \text{a.e. in } (0, T). \tag{42}
\]

Also observe that the term on the right side of (42) is a continuous function, then from (9) it is clear that

\[
\varphi^{-1}(t, x) = x - \int_0^t \frac{\partial_t \varphi(s, \varphi^{-1}(s, x))}{\partial_x \varphi(s, \varphi^{-1}(s, x))} ds, \quad t \in [0, T], \tag{43}
\]

which means \( \varphi^{-1}(t, x) \in C^1([0, T], \mathbb{R}) \). Hence we complete this claim.

**Step 2.** We claim that \( u \in C([0, T], C_1(\mathbb{R})) \) is Lipschitzian. A direct computation shows that

\[
|u(t_1, x) - u(t_2, x)|
\]

\[
\leq \left| v\left(t_1, \varphi^{-1}\left(t_1, \frac{1}{\sigma}(x - \frac{\alpha_0}{\beta}t_1)\right)\right) - v\left(t_2, \varphi^{-1}\left(t_2, \frac{1}{\sigma}(x - \frac{\alpha_0}{\beta}t_1)\right)\right) \right|
\]

\[
+ \left| v\left(t_1, \varphi^{-1}\left(t_1, \frac{1}{\sigma}(x - \frac{\alpha_0}{\beta}t_1)\right)\right) - v\left(t_1, \varphi^{-1}\left(t_2, \frac{1}{\sigma}(x - \frac{\alpha_0}{\beta}t_2)\right)\right) \right|
\]

\[
+ \left| v\left(t_2, \varphi^{-1}\left(t_2, \frac{1}{\sigma}(x - \frac{\alpha_0}{\beta}t_1)\right)\right) - v\left(t_2, \varphi^{-1}\left(t_2, \frac{1}{\sigma}(x - \frac{\alpha_0}{\beta}t_2)\right)\right) \right|
\]

\[
= L_1(t_1, t_2) + L_2(t_1, t_2) + L_3(t_1, t_2).
\]
The crucial step herein is to control the terms $L_1(t_1, t_2)$, $L_2(t_1, t_2)$ and $L_3(t_1, t_2)$. First, from the conclusion (38) of Step 1, Lemma 3.2 and (25) it implies

$$L_1(t_1, t_2)$$

$$\leq \sup_{q \in \mathbb{R}} |\partial_x v(t_1, q)| \left| \varphi^{-1} \left( t_1, \frac{1}{\sigma} \left( x - \frac{\beta_0}{\beta} t_1 \right) \right) - \varphi^{-1} \left( t_2, \frac{1}{\sigma} \left( x - \frac{\beta_0}{\beta} t_2 \right) \right) \right|$$

$$\leq \|v(t_1)\|_{C^1(\mathbb{R})} |t_1 - t_2|$$

(45)

$$\leq \left( \|v_0\|_{C^1(\mathbb{R})} + K \right) C|t_1 - t_2|$$

$$\leq C^2(1 - r)|t_1 - t_2|.$$

Second, it is not hard to see that

$$L_2(t_1, t_2)$$

$$\leq \sup_{q \in \mathbb{R}} |\partial_x v(t_1, q)| \left| \varphi^{-1} \left( t_2, \frac{1}{\sigma} \left( x - \frac{\beta_0}{\beta} t_1 \right) \right) - \varphi^{-1} \left( t_2, \frac{1}{\sigma} \left( x - \frac{\beta_0}{\beta} t_2 \right) \right) \right|$$

$$\leq \|v(t_1)\|_{C^1(\mathbb{R})} \frac{1}{1 - r} \frac{|\beta_0|}{|\beta| |\sigma|} |t_1 - t_2|$$

(46)

$$\leq C|\beta_0| \frac{1}{|\beta| |\sigma|} |t_1 - t_2|.$$

where use has been made of Lemma 3.2, Proposition 2 and (25). To the end, from Lemma 3.2, it is inferred that $v(t, x) \in C^1 \left( [0, T], C^1(\mathbb{R}) \right)$ and thus

$$\partial_t v(t, x) \in C \left( [0, T], C^1(\mathbb{R}) \right),$$

which gives

$$L_3(t_1, t_2)$$

$$\leq \sup_{t \in [0, T]} \left| \partial_t v \left( t, \varphi^{-1} \left( t_2, \frac{1}{\sigma} \left( x - \frac{\beta_0}{\beta} t_2 \right) \right) \right) \right| |t_1 - t_2|$$

$$\leq \sup_{q \in \mathbb{R}} \sup_{t \in [0, T]} |\partial_t v(t, q)| |t_1 - t_2|$$

(47)

$$\leq c_1 |t_1 - t_2|.$$
Further, from Lemma 3.2, (50) and (25), it transpires that
\begin{equation}
\frac{v(t_1, \varphi^{-1}(t_1, x)) - v(t_2, \varphi^{-1}(t_2, x))}{\partial_x \varphi(t_1, \varphi^{-1}(t_1, x))} + \frac{v(t_2, \varphi^{-1}(t_2, x))}{\partial_x \varphi(t_1, \varphi^{-1}(t_1, x))} - \frac{v(t_2, \varphi^{-1}(t_2, x))}{\partial_x \varphi(t_2, \varphi^{-1}(t_2, x))}
\end{equation}
\begin{equation}
= L_4(t_1, t_2) + L_5(t_1, t_2).
\end{equation}

In the following, we estimate \(L_4(t_1, t_2)\) and \(L_5(t_1, t_2)\), respectively. Differentiating the identify
\begin{equation}
\varphi(t, \varphi^{-1}(t, x)) = x
\end{equation}
with respect to \(x\), there appears the relation
\begin{equation}
\partial_x \varphi(t, \varphi^{-1}(t, x)) \partial_x \varphi^{-1}(t, x) = 1,
\end{equation}
which from (10) and the proof of Step 2, follows that
\begin{equation}
L_4(t_1, t_2)
= \partial_x \varphi^{-1}(t_1, x) \left| u\left( t_1, \sigma x + \frac{\beta_0}{\beta} t_1 \right) - u\left( t_2, \sigma x + \frac{\beta_0}{\beta} t_2 \right) \right| \leq \frac{1}{1 - r} \tilde{C}\| t_1 - t_2 \|.
\end{equation}

Further, from Lemma 3.2, (50) and (25), it transpires that
\begin{equation}
L_5(t_1, t_2)
= |v(t_2, \varphi^{-1}(t_2, x))| \left| \frac{\partial_x \varphi(t_1, \varphi^{-1}(t_1, x)) - \partial_x \varphi(t_2, \varphi^{-1}(t_2, x))}{\partial_x \varphi(t_1, \varphi^{-1}(t_1, x))} \right|
\end{equation}
\begin{equation}
\leq \| v(t_2) \|_{C^1_{1}(\mathbb{R})} \frac{1}{(1 - r)^2} \left| \partial_x \varphi(t_1, \varphi^{-1}(t_1, x)) - \partial_x \varphi(t_2, \varphi^{-1}(t_2, x)) \right|
\end{equation}
\begin{equation}
= \left( \| v_0 \|_{C^1_{1}(\mathbb{R})} + K \right) \frac{1}{(1 - r)^2} \left| \partial_x \varphi(t_1, \varphi^{-1}(t_1, x)) - \partial_x \varphi(t_2, \varphi^{-1}(t_2, x)) \right|
\end{equation}
\begin{equation}
= C \frac{1}{1 - r} \left| \partial_x \varphi(t_1, \varphi^{-1}(t_1, x)) - \partial_x \varphi(t_2, \varphi^{-1}(t_2, x)) \right|
\end{equation}
\begin{equation}
= \frac{C}{1 - r} L_5(t_1, t_2).
\end{equation}

Next, we give some analysis on the term \(\tilde{L}_5(t_1, t_2)\). As already discussed in the proof of Lemma 3.2, \(\varphi(t, x)\) lies in \(X(\mathbb{R})\), which implies that \(\partial_x \varphi(t, x)\) is periodic with period 1, its restriction to \([0, T] \times \mathbb{R}\) is uniformly continuous, namely, for every \(\varepsilon > 0\), there exists \(\delta = \delta(\varepsilon) > 0\) such that
\begin{equation}
|\partial_x \varphi(t_1, x_1) - \partial_x \varphi(t_2, x_2)| < \varepsilon
\end{equation}
for all \(t_1, t_2 \in [0, T]\) with \(|t_1 - t_2| < \delta\) and \(x_1, x_2 \in \mathbb{R}\) with \(|x_1 - x_2| < \delta\). At this point, if we recall (38) and take
\begin{equation}
\delta_1 = \delta_1(\varepsilon) = \min \left( \delta, \frac{\delta}{C} \right) > 0,
\end{equation}
then for \(t_1, t_2 \in [0, T]\) with \(|t_1 - t_2| < \delta_1\), it is thus inferred that
\begin{equation}
\tilde{L}_5(t_1, t_2) < \varepsilon.
\end{equation}
Combining (49), (52), (53) and (54) confirms our claim.

**Step 4.** We claim that \( \varphi^{-1}(t, x) \in C^1 \left([0, T], C([R, R])\right) \). Consider \( t_1, t_2 \in [0, T] \) with \( h = t_2 - t_1 > 0 \). Then, inspired by (49), (52), (53), (54) and (43), taking

\[
    h < \delta_2(\varepsilon) = \min \left( \delta_1 \left( \frac{\varepsilon(1-r)}{2C} \right), \frac{\varepsilon(1-r)}{2C} \right),
\]

it is ascertained that

\[
    \frac{\varepsilon^{-1}(t_2, x) - \varepsilon^{-1}(t_1, x)}{h} - \partial_t \varepsilon^{-1}(t_1, x) = \frac{\varepsilon^{-1}(t_2, x) - \varepsilon^{-1}(t_1, x)}{h} - \frac{\partial_t \varepsilon^{-1}(t_1, x)}{\partial_t \varepsilon^{-1}(t_1, x)} \\
    = \frac{1}{h} \int_{t_2}^{t_1} \partial_t \varphi(s, \varepsilon^{-1}(s, x)) \, ds - \frac{1}{h} \int_{t_1}^{t_2} \partial_t \varphi(t, \varepsilon^{-1}(t, x)) \, ds \\
    \leq \frac{1}{h} \int_{t_1}^{t_2} \left| \partial_t \varphi(s, \varepsilon^{-1}(s, x)) - \partial_t \varphi(t, \varepsilon^{-1}(t, x)) \right| \, ds \\
    \leq \frac{\varepsilon}{2} + \frac{1}{1-r} \tilde{C}h \\
    < \varepsilon, \quad x \in R.
\]

**Step 5.** We claim that \( u(t, x) \in C^1 \left([0, T], C_1([R])\right) \). Consider \( t_1, t_2 \in [0, T] \) with \( h = t_2 - t_1 > 0 \). In order to make the proof clear, attention is turned to the particular case \( \sigma = 1 \) and \( \beta_0 = 0 \). For the general case \( \sigma \neq 1 \) and \( \beta_0 \neq 0 \), some estimates introduced in Step 3 and Step 4 can be utilized. In fact, by the rescaling \( u \left(t, t, \frac{\sigma x + \beta_0 t}{\beta_0} \right) \to \tilde{v}(t, x) \), the conclusion can also be derived.

Recalling Lemma 3.2, we see that \( v(t, x) \in C^1 \left([0, T], C_1([R])\right) \), which together with (10) and (55) follows

\[
    \frac{u(t_2, x) - u(t_1, x)}{h} - \partial_t u(t_1, x) = \frac{v(t_2, \varepsilon^{-1}(t_2, x)) - v(t_1, \varepsilon^{-1}(t_1, x))}{h} \\
    - \partial_t v(t, \varepsilon^{-1}(t, x)) \partial_t \varepsilon^{-1}(t_1, x) - \partial_t v(t, \varepsilon^{-1}(t_1, x)) \\
    \leq \frac{v(t_2, \varepsilon^{-1}(t_2, x)) - v(t_1, \varepsilon^{-1}(t_1, x))}{h} \\
    - \partial_t v(t_1, \varepsilon^{-1}(t_1, x)) \partial_t \varepsilon^{-1}(t_1, x) - \partial_t v(t_1, \varepsilon^{-1}(t_1, x)) \\
    + \frac{v(t_2, \varepsilon^{-1}(t_2, x)) - v(t_1, \varepsilon^{-1}(t_1, x))}{h} \\
    - \partial_t v(t_1, \varepsilon^{-1}(t_1, x)) \partial_t \varepsilon^{-1}(t_1, x) - \partial_t v(t_1, \varepsilon^{-1}(t_1, x)) \\
    \leq \frac{1}{h} \int_{\varepsilon^{-1}(t_2, x)}^{\varepsilon^{-1}(t_1, x)} \left| \partial_x v(t_2, s) - \partial_x v(t_1, \varepsilon^{-1}(t_1, x)) \right| \, ds,
\]
The similar treatment on $\tilde{L}\_0(t_1, t_2)$ of Step 3 is applied as follows to estimate the  
\[ L_6(t_1, t_2) \leq C\varepsilon. \]  
(58) 

Substituting (58) into (56) along with (55) yields our claim.

**Step 6.** We claim that $u(t, x) \in C([0, T], C^1(\mathbb{R}))$. Recalling Step 2, it suffices to prove $\partial_\sigma u(t, x) \in C([0, T], C^1(\mathbb{R}))$. Consider $t_1, t_2 \in [0, T]$. From (10), (51), Proposition 2, Lemma 3.2 and (25), it follows

\[
\begin{aligned}
&\left| \partial_\sigma u(t_1, x) - \partial_\sigma u(t_2, x) \right| \\
= &\left| \partial_\sigma v\left(t_1, \varphi^{-1}\left(t_1, \frac{1}{\sigma}\left(x - \frac{\beta_0}{\beta} t_1\right)\right)\right) - \partial_\sigma v\left(t_2, \varphi^{-1}\left(t_2, \frac{1}{\sigma}\left(x - \frac{\beta_0}{\beta} t_2\right)\right)\right) \right| \\
&\leq \left| \partial_\sigma v\left(t_1, \varphi^{-1}\left(t_1, \frac{1}{\sigma}\left(x - \frac{\beta_0}{\beta} t_1\right)\right)\right) - \partial_\sigma v\left(t_2, \varphi^{-1}\left(t_2, \frac{1}{\sigma}\left(x - \frac{\beta_0}{\beta} t_2\right)\right)\right) \right| \\
&\quad + \left| \partial_\sigma v\left(t_2, \varphi^{-1}\left(t_2, \frac{1}{\sigma}\left(x - \frac{\beta_0}{\beta} t_2\right)\right)\right) - \partial_\sigma v\left(t_1, \varphi^{-1}\left(t_1, \frac{1}{\sigma}\left(x - \frac{\beta_0}{\beta} t_1\right)\right)\right) \right| \\
&\leq \frac{1}{1 - r} L_8(t_1, t_2) + \frac{1}{(1 - r)^2} \left| \partial_\sigma v\left(t_2, \varphi^{-1}\left(t_2, \frac{1}{\sigma}\left(x - \frac{\beta_0}{\beta} t_2\right)\right)\right) \right| L_7(t_1, t_2) \\
&\leq \frac{C}{1 - r} L_7(t_1, t_2) + \frac{1}{1 - r} L_8(t_1, t_2),
\end{aligned}
\]
where
\[ L_7(t_1, t_2) = \left| \partial_x \varphi(t_2, \varphi^{-1}(t_2, 1/\sigma (x - \beta_0 / \beta t_2))) - \partial_x \varphi(t_1, \varphi^{-1}(t_1, 1/\sigma (x - \beta_0 / \beta t_1))) \right| \]
and
\[ L_8(t_1, t_2) = \left| \partial_x v(t_1, \varphi^{-1}(t_1, 1/\sigma (x - \beta_0 / \beta t_1))) - \partial_x v(t_2, \varphi^{-1}(t_2, 1/\sigma (x - \beta_0 / \beta t_2))) \right| \]

Hence, to show that
\[ C_1 - r L_7(t_1, t_2) + \frac{1}{1-r} L_8(t_1, t_2) < \tilde{\varepsilon}, \]

it is sufficient to control \( L_7(t_1, t_2) \) and \( L_8(t_1, t_2) \). Combining (38) and Proposition 2 yield
\[
\begin{align*}
\left| \varphi^{-1}(t_1, 1/\sigma (x - \beta_0 / \beta t_1)) - \varphi^{-1}(t_2, 1/\sigma (x - \beta_0 / \beta t_2)) \right| \\
\leq \left| \varphi^{-1}(t_1, 1/\sigma (x - \beta_0 / \beta t_1)) - \varphi^{-1}(t_2, 1/\sigma (x - \beta_0 / \beta t_2)) \right| \\
+ \left| \varphi^{-1}(t_1, 1/\sigma (x - \beta_0 / \beta t_1)) - \varphi^{-1}(t_2, 1/\sigma (x - \beta_0 / \beta t_2)) \right| \\
\leq \left( C + \frac{1}{1-r} \frac{||\beta_0||}{||\beta||} \right) |t_1 - t_2| \\
= C_1 |t_1 - t_2|.
\end{align*}
\]

In this regard, it is seen to be as advertised by taking the choose of \(|t_1 - t_2| < \delta_2 = \delta(\varepsilon) = \min \left( \delta, \frac{\delta}{\sigma} \right) \) and iterating the similar control on \( L_5(t_1, t_2) \) and \( L_6(t_1, t_2) \). Hence, the result in the claim is established. This completes the proof of Theorem 1.1.

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