Toward a relativistic von Neumann no hidden variables theorem

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(Dated: April 8, 2014)

We prove a version of Bell’s Theorem that does not assume locality. Beside quantum mechanics and weak forms of relativistic causality and of realism, we only assume one correlation to be twice differentiable at zero. Since we prove that said correlation is once differentiable at zero, this brings us one degree of differentiability away from a relativistic von Neumann no hidden variables theorem.

Bell’s Theorem [1, 2] (as it is now reformulated under the influence of many, including Bell himself) establishes that quantum mechanics (QM) is incompatible with the conjunction of locality (the absence of any causal relation between events that are space-like separated in the usual sense in the kinematics of special relativity (SR)) and realism (according to which values of observables pre-exist before measurements). This well known fact hides a question: Is there a realistic theory compatible with what we know of microscopic physics? In a form that was important at the time of [6], the question reads: Is there a predictive hidden variables (PHVs) theory with averages predicted by QM (and fully consistent with all that is known of microphysics [3])? A negative answer would close an old wound: the proof of von Neumann’ 1932 theorem [6] on the non-existence of (dispersion-free) HVs is faulty [3], [4] and has so far not been fully repaired (although several proofs assuming a general form of non-contextuality have been found [3], [7], [8]). Here we take on the question of whether there might be a realist theory behind QM while deliberately not assuming any form of locality: we assume realism and look for extra hypotheses as weak as we can find to get a violation of Bell’s inequality (or more generally a contradiction). It is known that any four sequences of elements in \{-1,1\} must obey some inequalities discovered by Boole. It follows that Bell’s theorem, to the effect that some inequalities must be obeyed by some spin correlations for pairs in a singlet state when assuming realism and locality, can be reinterpreted as follows: the corresponding sequences of spins that could coexist if one assumes realism, cannot coexist if one also assumes locality. Since assuming realism besides QM reveals insufficient, we invoke a simple condition weaker than locality (the REACP, see below) that we can deduce from the kinematics of SR (which anyway is part of the traditional Bell’s theorem discussion in the hypothesis of locality): see Theorem 3 and its corollary. Thus, assuming QM, SR and realism, we prove (see Theorem 1) that some correlations that are central in Bell’s theory are once differentiable at the origin. This result seems incompatible with Bohm-de Broglie theory, the main realistic hidden variable theory. It then appears as reasonable to assume that said correlations are in fact twice differentiable; we are in a context where all correlations appearing in any theory are either real-analytic or fail to be differentiable. Under this mild further smoothness assumption, we get (our Theorem 2) a contradiction as in usual Bell’s Theorem. It follows that one at least of QM, SR, realism (in the precise form we postulate below), and the extra differentiability assumption that we make has to be false. Of course, realism is also ruled out by the Bell-Kochen-Specker theorems, now even proven in forms [8] that permit experimental verification [9], but this is under an assumption, non-contextuality, that is stronger than locality as already discussed by Bell [1] (cf. [2]).

We consider sequences of EPRB pairs, i.e., pairs of spin-\frac{1}{2} particles \( (p, \overrightarrow{p}) \) with the singlet state as wave function’s spin part: \( \Psi = \frac{1}{\sqrt{2}}( |+\rangle_p \otimes |-\rangle_{\overrightarrow{p}} - |-\rangle_p \otimes |+\rangle_{\overrightarrow{p}} ) \), and in particular a sequence of EPRB pairs \( (p_i, \overrightarrow{p}_i) \). The “EPRB” name evokes a reformulation by Bohm [10] of the “EPR paper” [11] using spin-\frac{1}{2} particles pairs. In what follows, corresponding measurements made by Alice and Bob are measurements respectively made by these two agents on the members \( p_i \) and \( \overrightarrow{p}_i \) of any given singlet state pair that they get. By definition \( p_i \) is the member of the \( i \)th pair that flies to Alice while \( \overrightarrow{p}_i \) is the member of that pair that flies to Bob. Thus Alice measures the normalized projections \( a = \{ a_i \} \) of the spins of the \( (p_i)’s \) along the vector \( \overrightarrow{v}_{\theta_a} \) at angle \( \theta_a \) while Bob measures the normalized projections \( b = \{ b_i \} \) of the spins of the \( (\overrightarrow{p}_i)’s \) along \( \overrightarrow{v}_{\theta_b} \). All our vectors belong to some fixed plane, and the angle \( \theta_{a,b} \) is measured relative to a reference vector \( \overrightarrow{v}_0 \).
at angle 0 chosen once and for all in that plane. We shall set \( \theta_{cd} \equiv \theta_c - \theta_d \). Quantum mechanics (QM) teaches us that the correlation \( \langle a, b \rangle \) is given by the twisted Malus law (TML):

\[
\langle a, b \rangle \equiv \lim_{n \to \infty} \frac{1}{n} \sum_{i=1}^{n} a_i b_i = -\cos(\theta_{ab}).
\]

A Bell type experiment (i.e., an experiment of the type considered in Bell’s Theory: see [1] and subsequent papers on the same theme) only provides two sequences of observed normalized spin projections: \( a = \{a_i\} \) and \( b = \{b_i\} \). PHVs theories have often been proposed as a mean to obtain a deterministic theory for micro-physics. The characteristic property of PHVs of relevance for us here is the fact that they are realistic so that observable values have a meaning before (and in fact independently of) being measured. Instead of using PHVs as Bell in [1] one tends now (see e.g., [12] and [13]) to rather use one of the weaker realism assumptions that go collectively under the name of classical microscopic realism (CMR). CMR tells us that all the observables that could be measured have well defined values. The weakest form of CMR needed to discuss Bell’s Theory is used in [14] and [15]. We will stick to CMR to which we have adapted the following two conventions that go back at least to [1].

Convention 1. We assume that any \( n \) quantities that are not measured, but that exist according to CMR, have the values that would have been obtained if they had been jointly measured quantities. We assume here that said sets of \( n \) quantities can indeed be co-measured.

Convention 2. We assume that the statistical predictions (e.g., average values and in particular correlations) are the ones given by QM.

We call physical angles the angles used for measurements. To mark the corresponding previously defined sequences \( a \) and \( b \) so as to also be prepared to denote other possibilities, we use \( (a, b) \equiv (a^{o}, b^{o}) \) where the left (resp. right) hollow circle in the superscript of \( c \) in \( \{a, b\} \) marks that Alice (resp. Bob) has used a physical angle. We use a filled circle instead to represent the fact that so marked sequence is a CRM-dependent sequence of values (inferred to exist from CMR along a non-physical angle).

For instance, besides measurements on both elements of \((a, b)\), one may consider the following alternate situations: [AS1a] Alice could have chosen another angle of possible measurement \( \theta_a \) with Bob keeping the angle \( \theta_b \), in which case they would have respectively obtained the sequences \( a^{o} \equiv \{a_i^{o}\} \) and \( b^{o} \equiv \{b_i^{o}\} \) (or together the pair of sequences \( (a^{o}, b^{o}) \)). [AS1b] Bob could have chosen another angle of possible measurement \( \theta_b' \), while Alice kept the angle \( \theta_a \), in which case they would have obtained the pair of sequences \( (a^{o}, b^{o'}) \). [AS2] Alice and Bob could both have chosen the alternate angles \( \theta_a \) and \( \theta_b' \), and thus would have obtained the pair of sequences of counterfactual values, \( (a^{o}, b^{o'}) \) that is used in [16]. The word “measured” when applied to one sequence does not tell us that the corresponding sequence is known: only the co-measured sequences \( (a^{o}, b^{o}) \) are assumed to be known a priori. While the two sequences that make sense together must carry the same super-scripts, we will also consider mixed pairs such as, e.g., \( (a^{o}, b^{o'}) \) (cf. [2], [3]).

Remark 1. We stress that the sequences which are not measured have a physical meaning only when assuming the CMR hypothesis. We shall continue assuming CMR but our goal is to prove ab absurdo that CMR makes no physical sense.

Recall now that locality means that the outputs (either measured or inferred to exist by assuming CMR) do not depend on the choice of vector made by the other observer when the measurements by Alice and Bob are space-like separated (i.e., \( ||\Delta s|| > c \cdot ||\Delta t|| \) for the space and time distances where \( c \) is the speed of light in vacuum).

Remark 2. With the set of hypotheses made so far, there is no reason to expect (as holds true when assuming locality) that for \( c \in \{a, b\} \), \( c^{\diamond_1} \diamond_1 = c^{\diamond_2} \diamond_2 \) beyond the case when \( \diamond_1 = \diamond_2 \) and \( \diamond_1 = \diamond_2 \). In particular, eight different sequences could have been generated .

We do not assume locality here as it is known since [1] that assuming QM, PHVs, and locality implies a contradiction (while CMR instead of PHVs works as well).

Without further assumption than QM, the TML lets us compute \((a^{o}, b^{o})\), but when we assume both QM and CMR, more observables values make sense, and it follows from Conventions 1 and 2 that the TML applies to more pairs, so that we have (see also [10]):

\[
\langle a^{o}, b^{o} \rangle = -\cos(\theta_{ab}), \quad \langle a^{o}, b^{o} \rangle = -\cos(\theta_{ab}), \quad \langle a^{o}, b^{o} \rangle = -\cos(\theta_{ab}), \quad \langle a^{o}, b^{o} \rangle = -\cos(\theta_{ab}).
\]

Consider now the inequalities \( |x_1 + x_2| + |x_3 - x_4| \leq 2 \), \( |x_1 - x_2| + |x_3 + x_4| \leq 2 \). The points in \( \mathbb{R}^4 \) that satisfy both inequalities define a polytope \( P \subset I^4 \subset \mathbb{R}^4 \), where \( I = [-1, +1] \). Let us recall [17], [18]:

Boole’s Lemma:

\[
|\langle u, x \rangle + \langle v, x \rangle| + |\langle u, y \rangle - \langle v, y \rangle| \leq 2 \quad (1)
\]

where, \( u, x, v, \) and \( y \) are sequences of numbers in the doubleton \( \{-1, 1\} \) such that the limits defining the 4 correlations involved in [1] exist.

Notice that we could apply Boole’s lemma to all correlations associated to the \( n \)-tuples \( a^{o}, b^{o}, a^{o}, b^{o}, a^{o}, b^{o}, a^{o}, b^{o}, a^{o}, b^{o} \) obtained before. Recall however that in order to apply that lemma, only four of these correlations can be used simultaneously. Also, following a tradition that goes back to [19], we only
consider correlations of the form \( \langle a^{\circ}, b^{\circ} \rangle \). Since all the sequences that we can consider in these correlations have values in \([-1, +1]\), it follows from Boole’s Lemma under standard convergence hypotheses that one has the **CHSH inequalities** \(19\):

\[
\begin{align*}
| \langle a^{\circ}, b^{\circ} \rangle + \langle a^{\circ}, b^{\circ} \rangle | + | \langle a^{\circ}, b^{\circ} \rangle - \langle a^{\circ}, b^{\circ} \rangle | & \leq 2 \quad (2) \\
| \langle a^{\circ}, b^{\circ} \rangle - \langle a^{\circ}, b^{\circ} \rangle | + | \langle a^{\circ}, b^{\circ} \rangle + \langle a^{\circ}, b^{\circ} \rangle | & \leq 2 \quad (3)
\end{align*}
\]

Recall that with the hypotheses that we have made so far, the correlations \( \langle a^{\circ}, b^{\circ} \rangle \), \( \langle a^{\circ}, b^{\circ} \rangle \), and \( \langle a^{\circ}, b^{\circ} \rangle \) are all unknown to us.

The **Restricted effect after cause principle (REACP)** states that for any Lorentz observer (LO), the registered observable value after a physical choice of angle cannot change because of further choices that are made after that. We prove below the REACP to be a direct consequence of SR, something that we could not prove (so far) for the effect after cause principle of \(14\) and \(15\).

When assuming the REACP we do not ask from CMR-dependent values the same that is asked from actual values that are generated by measurements because of the following simple observation (in the spirit of the title of a famous short paper \(21\)): - What happens may have consequences but what could have happened (at the atomic scale at least) has no consequence (on the subsequent actual world). - In particular, “Only an actual measurement by Alice may have an effect on the sequence measured by Bob, and vice versa.”

Recall now that in order to use inequalities \(2\) we need to compute \( \langle a^{\circ}, b^{\circ} \rangle \), \( \langle a^{\circ}, b^{\circ} \rangle \), and \( \langle a^{\circ}, b^{\circ} \rangle \), or at least we need to know enough about these correlations.

**Claim 1.** \(\text{[REACP]} \Rightarrow [b^{\circ} = b^{\circ} \text{ and } a^{\circ} = a^{\circ}].\)

This claim follows from choosing different LO’s: (i) Considering one LO for which the measurements done by Bob happen before the corresponding ones done by Alice, the REACP tells us directly that no change performed by Alice can affect the measurement done by Bob which implies that \(b^{\circ} = b^{\circ}\); (ii) Considering instead another LO for whom Alice measures before Bob on corresponding particles, we get \(a^{\circ} = a^{\circ}\) mutatis mutandis.

**Corollary 1.** Assuming QM and CMR it follows from the REACP that \( \langle a^{\circ}, b^{\circ} \rangle = \langle a^{\circ}, b^{\circ} \rangle = -\cos(\theta_{ab}) \) and that \( \langle a^{\circ}, b^{\circ} \rangle = \langle a^{\circ}, b^{\circ} \rangle = -\cos(\theta_{ab}) \).

**Remark 3.** The REACP is the EACP of \(14\), \(15\) restricted to actual measurements.

Let us denote \(T \equiv [-\pi, \pi]/(-\pi \sim \pi)\) and \(I \equiv [-1, 1] \). There is a function \(C : T^4 \rightarrow \Sigma\) which represents how the following correlations depend on choices of quadruplets of angles \(\theta = (\theta_a, \theta_{ab}, \theta_{b}, \theta_{ab})\) in \(T^4\). We have:

\[
C(\theta) = \langle a^{\circ}, b^{\circ} \rangle = \langle a^{\circ}, b^{\circ} \rangle = \langle a^{\circ}, b^{\circ} \rangle = \langle a^{\circ}, b^{\circ} \rangle (\theta),
\]

and it follows from the CHSH inequalities \(2\) and \(3\) that:

\[
\forall \theta \in T^4 \ , \ C(\theta) \in P.
\]

We shall prove some weak regularity properties of the correlation \(\langle a^{\circ}, b^{\circ} \rangle\) as a function of \(\theta\), and then also prove that we get a contradiction if we further make the mild assumption of a bit more of regularity of that function. In view of the computed and measured correlation functions in the context that we are in, assuming one degree more of differentiability for a correlation that we prove differentiable would appear *a priori* much weaker than assuming the first degree of differentiability.

Observing that \(\theta_{ab} + \theta_{ab} - \theta_{ab} + \theta_{ab} = 0\) we can consider variables \(\theta_1, \theta_2, \theta_3, \) and \(\theta_4\) such that:

\[
\theta_1 = \theta_{ab}, \theta_2 = \theta_{ab}, \theta_3 = \theta_{ab}, \theta_4 = \theta_1 - (\theta_2 + \theta_3).
\]

Let us denote by \(\theta\) the triplet of angles differences \(\theta_{ab}, \theta_{ab}, \theta_{ab}\), so that \(\theta = (\theta_1, \theta_2, \theta_3)\). Also, let:

\[
\mathcal{Q}_+ = \{ \theta = (\theta_1, \theta_2, \theta_3) \in R^3 : 0 \leq \theta_i \leq \pi, i = 1, 2, 3 \},
\]

and consider the function \(F_4 : \mathcal{Q}_+ \rightarrow R\) defined by \(F_4(\theta) = \langle a^{\circ}, b^{\circ} \rangle\).

**Theorem 1.** Suppose \([\text{QM+CMR+REACP}]\) and suppose also that \(C\) satisfies \(21\). Then, \(F_4\) is differentiable at \(\theta = (0, 0, 0)\) and \(DF_4(O) = (0, 0, 0)\).

From \(2\) and the definition of \(F_4\) it follows that \(F_4(O) = -1\). Moreover, for \(\theta\) close to \(O\), the CHSH inequalities imply:

\[
|F_4(\theta) + 1| \leq 3 - \cos \theta_1 - \cos \theta_2 - \cos \theta_3.
\]

Dividing by the norm of \(\theta\), we get a quotient that clearly goes to zero as \(\theta \rightarrow O\).

We thus have established that \(F_4\) is differentiable at \(O \in \mathcal{Q}_+\). The typical expectation is that if such a correlation function is once differentiable, it is real analytic, but we shall not assume that much on top of the regularity that we have just proved.

**Theorem 2.** Suppose \([\text{QM+CMR+REACP}]\), and also both that \(C\) satisfies \(4\) and that \(F_4\), hence \(\langle a^{\circ}, b^{\circ} \rangle\), is twice differentiable at \(\theta = O\) as a function restricted to \(\mathcal{Q}_+\). Then, it would follow that \(3 = 1\).

First, it follows from general considerations that under the \([\text{QM+CMR}]\) assumptions \(F_4(\theta, \theta, \theta) = -\cos(\theta)\) for any \(\theta\). Now, we illustrate the main point of the proof by assuming that \(F_4\) is a degree 2 polynomial near \(\theta = O\), whence \(F_4(\theta) = -1 + \sum_i c_i \theta_i + \sum_{i<j} c_{ij} \theta_i \theta_j\), (the general case follows readily by using the elementary theory of Taylor polynomials.) We accordingly replace the function \(-\cos(x)\) by its order-two approximating polynomial \(-1 + \frac{1}{2} x^2\). When assuming \([\text{QM+CMR+REACP}]\), by Claim \(11\) all linear coefficients must vanish: \(c_i = 0\) for \(i = 1, 2, 3\). Next, it is not hard to check that if \(|w| \leq 1\) and \(|1 + z| + |1 + w| \leq 2\) then \(z = w\) (cf. \(10\)).
In this way, we thus obtain that \( c_{11} = \frac{1}{2} \) by considering \( \theta = (x, 0, 0) \). Analogously, we obtain \( c_{22} = c_{33} = \frac{1}{2} \).

Now, taking \( \theta = (x, x, x) \) and noting that \( F_3 \) coincides with the cosine on the diagonal, we have:

\[
\frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \sum_{i < j} c_{ij} = \frac{1}{2}.
\] (5)

To finish the proof, we show that \( c_{ij} = 0 \) when \( i < j \). For that purpose, consider the second order terms in the CHSH inequalities, taking \( \theta = (0, x, kx) \) (with \( x \) small and \( kx \) also small but fixed); we get that for all \( k \):

\[
2 - \frac{1}{2}x^2 + \frac{1}{2}x^2 + c_{23}kx^2 \leq 2 + o(x^2).
\]

Canceling 2, dividing by \( x^2 \) and taking \( k \) arbitrarily large, we see that the inequality holds only when \( c_{23} = 0 \). The equalities \( c_{13} = 0 = c_{23} \) are shown in the same way. Therefore, equation (5) implies that \( 3 = 1 \).

**Remark 4.** Since Theorem 3 only requires functional regularity when restricted to \( Q_+ \), it also holds when the correlation \( \langle a^{*i}, b^{*j} \rangle \) or its derivatives are allowed to jump as some of the \( \theta_i \)’s \( (i = 1, 2, 3) \) changes sign; in particular, Theorem 4 can be applied to correlations that depends on absolute value.

**Theorem 3.** The REACP is implied by SR, and more specifically by the impossibility of sending superluminal messages.

To prove this statement we show that the negation of the REACP allows superluminal signaling. To fix the ideas, assume that two slots \(-1_i \) and \(+1_i \) are associated to each event in a sequence of events labeled by \( i \). The result of an experiment is then marked by putting a slot marker in one of these two slots. Thus, if the REACP fails, the marker of some slot would move because of later manipulations in the time frame of some LO. Even if only some sufficient proportion of slots would get removed or displaced, trivial error correcting techniques by repetition of symbols and intertwining of words (to better correct bunched errors) would allow to send YES or NO messages with high probability backward in time, which is the basic form of superluminal transmission. This argument also shows that the negation of the REACP is stronger than the negation of SR, so that the REACP follows from SR. This proves Theorem 3.

We know (see, e.g., [21]) that the negation of locality does not imply the possibility of transmitting superluminal messages (symbolic sequences with information content). From that one can deduce the following result:

**Corollary 2.** The conjunction of the REACP with the regularity hypothesis of Theorem 4 is strictly weaker than locality when assuming \( [QM + CRM] \).

In conclusion, our three theorems and Corollary 2 taken together express a non-realism statement that is real progress on the traditional form of Bell’s Theorem, thus also on the quite interesting Bell-Kochen-Specker type results that have flourished recently, beyond what we could describe here. Indeed, for those who believe in both QM and (the kinematics part of) SR, the only small gap that remains to achieve a full replacement of von Neumann’s faulty theorem is one degree of differentiability for one of the correlations that comes up in Bell’s theory, a correlation that we do prove to be differentiable at 0, in contradiction with what one expects from a realism-compatible correlation [22].

**Acknowledgments:** This work was made possible by support from FAPESP through “Proyecto Temático Dinámica em Baixas Dimensões”, Proc. FAPESP 2011/16265-2, and also Proc. FAPESP 2012/19995-0. The support of Palis-Balzan’s Prize is also gratefully acknowledged. We thank A. Fine for his comments on a former version.