ON THE HOMOGENEOUS IDEAL OF A
PROJECTIVE NONSINGULAR TORIC VARIETY

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Abstract. Using techniques connected with the idea of Frobenius splitting, due to Mehta and Ramanathan, we show that a projectively embedded nonsingular and proper toric variety is projectively normal, and that its ideal is generated by quadrics.

1.1. Introduction. Assume, using the terminology of [O1], that $T$ is an algebraic torus of dimension $n$ over an algebraically closed field $k$, that $\Delta = \{\sigma_i, \; i \in I\}$ is a fan and that $X = T\text{emb}(\Delta)$ is the corresponding toric variety. Suppose that $X$ is proper and that $\mathcal{L}$ is a very ample invertible sheaf defining an embedding of $X$ in a projective space. Define the graded $k$-algebra

$$R(\mathcal{L}) := \bigoplus_{i \geq 0} H^0(X, \mathcal{L}^\otimes i).$$

If $X$ is proper, this algebra is finite dimensional in each degree as a $k$-vector space, and it is the integral closure of the homogeneous coordinate ring of the image of $X$ (cf., e.g., [H2, Exc. II 5.14]). Use the notation $\Gamma\mathcal{L} := H^0(X, \mathcal{L})$. The main result of this paper is the following.

Theorem 1. If $X$ is a nonsingular and proper toric variety and $\mathcal{L}$ is a very ample invertible sheaf, then $R(\mathcal{L})$ is generated by elements in degree one. Hence $R(\mathcal{L})$ is a quotient $k[\Gamma\mathcal{L}]/I$ of the polynomial algebra $k[\Gamma\mathcal{L}]$, and furthermore $I$ is generated, as a homogeneous ideal, by elements of degree 2.

The first statement implies that the homogeneous coordinate ring of the image of the embedding equals $R(\mathcal{L})$, and hence is normal (since all toric varieties are by construction assumed to be normal), i.e., $X$ is projectively normal with respect to this embedding (cf. [loc. cit.]).

The second statement says, in more geometric language, that the projective cone in $k^{s+1} = \text{Spec } k[\Gamma\mathcal{L}]$ of the image of $X$ in $\mathbb{P}^s$ is cut out by quadrics.

For the case of toric surfaces Koelman [K] has, by completely different methods, given a complete description of when the conclusion of the theorem holds, without assuming nonsingularity. In particular, the theorem is not true for all toric varieties, though the exceptions in the case of surfaces are few and their existence does not seem to be connected with nonsingularity. There is at least a bound on the degrees. A result by Batyrev is quoted in [ES], saying that the degree of the relations of the homogeneous coordinate ring of an arbitrary (and so possibly non-smooth) toric variety is less than $n+1$. We have not investigated to what extent the methods of this paper can be used to study singular varieties.
Ewald and Schmeinck [ES] prove Theorem 1 in the case of smooth toric varieties such that the number of facets is at most \( n + 2 \). They also ask whether this result may be extended to all smooth toric varieties.

The methods of proof are mostly taken from the proof of a similar result for Schubert varieties by Mehta and Ramanathan, using the concept of Frobenius splitting (cf. [MR] and [R]). Direct application of the proof in [R], given in Section 2, however, only suffices to prove the result for a certain subclass of toric varieties. A trick, described in Section 3, then gives a reduction to this case.

Generalizing the properties of the theorem to higher syzygies of the ring \( R(\mathcal{L}) \) over itself (see Lemma 1.2), the concept of a Koszul algebra is obtained, as described, e.g. in [BF]. It is in fact true that homogeneous coordinate rings of smooth toric varieties and Schubert varieties are Koszul algebras. This again follows from the Frobenius splitting of higher diagonals proved in this paper. The details will appear elsewhere. It would also be interesting to see whether it is possible to use Frobenius splitting to study the higher syzygies of \( R(\mathcal{L}) \) as a module over \( k[\Gamma \mathcal{L}] \) (cf. [EL]).

1.2. More notation. Generators and relations. Let \( X = \text{Emb}(\Delta) \) as in the introduction. Denote by \( N \cong \mathbb{Z}^n \) the lattice in the real vector space \( N_{\mathbb{R}} \), which contains the cones \( \sigma_i \in \Delta \) and by \( M \cong \text{Hom}_{\mathbb{Z}}(N, \mathbb{Z}) \) the dual of \( N \). In the sequel, all toric varieties are assumed to be proper. A toric variety \( X \) is proper if and only if \( \cup_{\sigma \in \Delta} \sigma = N_{\mathbb{R}} \). Each cone is of the form

\[
\sigma = \sum_{i=1}^{s} R_0 n_i,
\]

where \( R_0 \) denotes the positive real numbers, and all \( n_i \in N \). If \( X \) is nonsingular, and \( \sigma \) is a cone of maximal rank, then \( s = n \) and \( \{n_i : 1 \leq i \leq n\} \) is chosen to be a basis of \( N \).

Each invertible sheaf \( \mathcal{L} \) on \( X \) is determined by a real-valued function \( h_{\mathcal{L}} : N_{\mathbb{R}} \to \mathbb{R} \), called the support function of \( \mathcal{L} \) and satisfying some properties (\( \Delta \)-linearity) defined purely in terms of the fan.

It is useful to give a homological description of generators and relations. Suppose that \( A = \bigoplus_{i \geq 0} A_i \) is a graded commutative and connected (i.e. \( A_0 = k \)) Noetherian \( k \)-algebra. Then, as is perhaps seen most easily from the bar complex, all torsion groups have induced gradings

\[
\text{Tor}_i^A(k, k) = \bigoplus_{j \geq i} \text{Tor}_{ij}^A(k, k).
\]

If \( A = k[V]/(R) \) is a minimal graded presentation (i.e. the graded vector spaces \( V \) and \( R \in k[V] \) have minimal vector space dimension), then \( V \cong \bigoplus_{j \geq 1} \text{Tor}_{1,j}^A(k, k) \). If \( V \) only has elements in degree 1 or equivalently \( \text{Tor}_{1,j}^A(k, k) = 0 \), if \( j > 1 \), then \( \bigoplus_{j \geq 3} R_j \cong \bigoplus_{j \geq 3} \text{Tor}_{2,j}^A(k, k) \). (These results are well-known even though it is difficult to find a reference. They are, e.g., proved in Lemaire’s book [L, Chap.1], in the category of graded non-commutative algebras. The changes needed in his argument to adapt them to the case of commutative rings are easy and obvious. Note that \( \text{Tor}_{2}^A(k, k) \) really gives the relations of \( A \) considered as a non-commutative algebra. These relations differ from \( R \) precisely by the presence of all commutators on \( V \) in degree 2.) Thus we get:
**Lemma 1.2.** The graded connected $k$-algebra $A$ is generated by the elements in degree 1 if and only if $\text{Tor}_{1,j}(k,k) = 0$, for all $j > 1$. $A$ has a generating set of relations in degree 2 if and only if $\text{Tor}_{2,j}(k,k) = 0$, for all $j > 2$.

1.3. The characteristic. Reduction of the theorem to positive characteristics is fairly standard. We give an argument for completeness. Both a toric variety $X = T \text{emb}(\Delta)$ and an invertible sheaf $\mathcal{L}$ on $X$ are given by combinatorial data, independent of the characteristic of the ground field $k$. In particular, they are really objects defined over $\mathbb{Z}$. Realized as such, call them $\tilde{X}$ and $\tilde{\mathcal{L}}$. Let $k_p$ be a field of arbitrary characteristic $p$, possibly 0, and let $i_p : \text{Spec } k_p \to \text{Spec } \mathbb{Z}$, be the canonical map. Suppose that $\mathcal{L} = i_0^*\tilde{\mathcal{L}}$ is very ample. Then it is clear from the criterion in [O1, Cor. 2.15], formulated solely in terms of the support function of $\mathcal{L}$, that $i_p^*\tilde{\mathcal{L}}$ is also very ample for all $p$. Furthermore, it is clear from the combinatorial description of coherent cohomology groups, given in [O1, Lemma 2.3], that $R(i_p^*\mathcal{L}) = i_p^*R(\mathcal{L})$, and that $R(\mathcal{L})$ is a graded ring flat over $\mathbb{Z}$. It follows, by the universal coefficient theorem, that

$$k_p \otimes \mathbb{Z} \text{Tor}^R_{i,*}(\mathcal{L}, \mathcal{Z}) \subset \text{Tor}^R_{i,*}(k_p, k_p)$$

Thus it is clear that knowledge of the vanishing for a fixed pair $(i, j)$ of $\text{Tor}^R_{i,j}(k_p, k_p)$ for all algebraically closed fields in positive characteristics suffices to prove the vanishing of $\text{Tor}^R_{i,j}(\mathcal{Z}, \mathcal{Z})$ and hence of $\text{Tor}^R_{i,j}(k_0, k_0)$. In view of Lemma 1.2 this means that it suffices to prove the theorem in all positive characteristics. Hence we may, in the rest of the paper, assume that the characteristic is positive.

1.4. Frobenius splitting. In this subsection $X$ is an arbitrary variety over a field $k$ of positive characteristic $p$. The (absolute) Frobenius map $F : X \to X$ is given as the identity on the underlying topological space and as the $p$-th power map on the structure sheaves $\mathcal{O}_X \to F_*\mathcal{O}_X$. The last map is an injection, and if it makes $\mathcal{O}_X$ a direct summand in $F_*\mathcal{O}_X$ (as an $\mathcal{O}_X$-module), then the variety is said to be Frobenius split (cf. [MR]).

Frobenius split varieties have nice properties. For example, if $\mathcal{L}$ is an ample line bundle, there is a split map $\mathcal{L} \to \mathcal{L} \otimes F_*\mathcal{O}_X \cong F_\ast F^\ast \mathcal{L} \cong F_*\mathcal{L} \otimes p$, and hence an injection in cohomology $H^i(X, \mathcal{L}) \to H^i(X, F_*\mathcal{L} \otimes p) = H^i(X, \mathcal{L} \otimes p)$. By iterating and using the ampleness of $\mathcal{L}$ we then see that $H^i(X, L) = 0$, for $i > 0$. (cf. [loc.cit. Prop. 1]).

Let $a : F_*\mathcal{O}_X \to \mathcal{O}_X$ be a splitting map. A closed subvariety $Z \subset X$, defined by an ideal $I$, is said to be compatibly split if $a(F_\ast I) = I$. If this is the case and $\mathcal{L}$ is an ample line bundle on $X$, then the restriction map $H^0(X, \mathcal{L}) \to H^0(Z, \mathcal{L} \otimes \mathcal{O}_Z \mathcal{O}_Z)$ is surjective (cf. [loc.cit. Prop. 3]). In particular if the diagonal $\Delta_X \subset X \times X$ is compatibly split, then the map $H^0(X \times X, \mathcal{L} \otimes \mathcal{L}) \to H^0(\Delta_X, (\mathcal{L} \otimes \mathcal{L}) \otimes \mathcal{O}_{X \times X} \mathcal{O}_{\Delta_X})$ is surjective. But this map is isomorphic to the multiplication map $H^0(X, \mathcal{L}) \otimes H^0(X, \mathcal{L}) \to H^0(X, \mathcal{L} \otimes 2)$ and it follows that $R(\mathcal{L})$ is generated in degree one. There is a similar criterion implying that $R(\mathcal{L})$ has relations only in degree 2; both criteria are due to Ramanathan. We formulate them in a lemma.

**Lemma 1.4.** (cf. [R, Cor. 2.3 and Prop. 2.7]). Assume that the diagonal $\Delta_X$ is compatibly split in $X \times X$. Then $R(\mathcal{L})$ is generated by degree one elements. If furthermore the partial diagonals $\Delta_{12}$ and $\Delta_{23}$ (defined in Section 2) are compatibly split in $X \times X \times X$, then $R(\mathcal{L})$ has relations only in degree 2.
Note that a Frobenius splitting induces splittings for all open subsets, and that conversely if a globally defined map $F_*\mathcal{O}_X \to \mathcal{O}_X$ induces a splitting on an open subset, then it is globally a splitting (cf. [MR]).

1.5. Toric varieties are Frobenius split.

Let the group algebra of $M \cong \mathbb{Z}^n$ be $k[M] := \oplus_{m \in M} ke(m)$ with multiplication defined by $e(m_1)e(m_2) = e(m_1 + m_2)$. The Frobenius map on $k[M]$ is determined by $F(e(m)) = e(pm)$ and a splitting $a$ is given by

$$a(e(m)) = \begin{cases} e(m_1), & \text{if } pm_1 = m \\ 0, & \text{otherwise}. \end{cases}$$

Suppose that $k[S] \subset k[M]$ is the monoid algebra of a finitely generated sub-monoid $S \subset M$. Then the splitting $a$ clearly induces a splitting of $k[S]$ precisely when $pM \cap S = pS$. This is a property weaker than the property of $S$ to be saturated (which is the same property when $p$ is an arbitrary integer). A toric variety is constructed by gluing a set of $U_i = \text{Spec } k[S_i]$ for some finitely generated saturated submonoids $S_i$ of $M$, along open sets of the same type (cf. [O1, Prop. 1.1-2]). Obviously $a$ as above defines compatible maps on each of the $k[S_i]$ and hence the first part of the following proposition is clear.

**Proposition 1.5.1.** Toric varieties are Frobenius split by a above. All T-invariant subvarieties are compatibly split.

**Proof.** Let $Z$ be a $T$-invariant closed subvariety. On each $T$-invariant affine open set, the ideal $\mathcal{I}_Z$ of $Z$ is completely reducible as a $T$-representation. Assume that $e(m) \in \mathcal{I}_Z$ and that $a(e(m)) = e(m_1) \neq 0$. Then $(e(m_1))^p = e(m) \in \mathcal{I}_Z$ implies $e(m_1) \in \mathcal{I}_Z$ since $\mathcal{O}_X/\mathcal{I}_Z$ is a reduced ring.

Thus one sees that splitness is already exploited in the more powerful property that the coordinate ring is completely reducible as a $T$-representation. Note, however, that it gives a quick proof of the vanishing of the higher cohomology of ample line bundles (see 1.4). This is well-known and proved in, e.g., [O1, Thm. 2.7] for invertible sheaves which satisfy the weaker property of being generated by global sections.

**Corollary 1.5.2.** (cf.[O1, Thm. 2.7]) If $\mathcal{L}$ is an ample line bundle on the toric variety $X$, then $H^i(X, \mathcal{L}) = 0$, for $i > 0$.

In spite of the triviality of this, the technique of Ramanathan [R] of using Frobenius splitness of, for example, the diagonal $\Delta_X \subset X \times X$, to obtain information on the homogeneous coordinate ring of $X$, seems to give new results. In order to avail ourselves of these techniques, it is convenient to use a different description of Frobenius splitting, given in the lemma below. Note that, if $X$ is nonsingular, by duality for a finite morphism,

$$\text{Hom}_{\mathcal{O}_X}(F_*\mathcal{O}_X, \mathcal{O}_X) \cong H^0(X, \mathcal{K}^{\otimes(1-p)}),$$

where $\mathcal{K}$ denotes the canonical bundle. (cf. [MR, Prop. 5]). If $X = T \text{emb}(\Delta)$ is a nonsingular $n$-dimensional toric variety, the canonical bundle is, by [O1, p. 71], $\mathcal{K} = \mathcal{L}(-\Sigma D_i)$, where $D_i$ is the closure of orb$(R_0n_i)$ and the sum is taken over all 1-dimensional faces $R_0n_i$, $1 \leq i \leq r$, of $\Delta$. 


Lemma 1.5.3. Assume that $X$ is nonsingular. Considered as a global section of $K^\otimes(1-p)$, $a$ has the divisor $(p-1)\sum D_i$, where the sum is over all $i$, $1 \leq i \leq r$.

Proof. Let $U_\sigma = \text{Spec } k[\hat{\sigma} \cap M]$ be the affine $T$-invariant open subset corresponding to a maximal dimensional cone

$$\sigma = \sum_{i=1}^n R_0 n_i.$$ 

Let $\{m_i, 1 \leq i \leq n\}$ be a basis dual to the $n_i$. Then $\{x_1 := e(m_1), \ldots, x_n := e(m_n)\}$ forms a system of local coordinates. Use multi-index $\alpha = (\alpha_i) \in \mathbb{Z}^n$ to denote the monomial $\prod x_\alpha$. Let $dx = dx_1 \wedge \ldots \wedge dx_n$ and $dx_1^{-p}$ the corresponding local section of $K^\otimes(1-p)$. The isomorphism $F^* K^\otimes(1-p) \cong \text{Hom}_{O_X}(F^* O_X, O_X)$ is then given (cf. [MR]) in local coordinates by

$$x_\alpha dx_1^{-p} \mapsto x_\beta \mapsto \begin{cases} x^{(\alpha+\beta+1)/p-1}, & \text{if } (\alpha + \beta + 1)/p \text{ is an integer} \\ 0 & \text{otherwise} \end{cases}.$$ 

In view of the definition, it is then clear that $a$ corresponds to

$$x_1^{p-1} \ldots x_n^{p-1} dx_1^{-p},$$

and hence

$$\text{div } a|_{U_\sigma} = (p-1)V(x_1) + \ldots + (p-1)V(x_n) = (p-1)(D_1 \cap U_\sigma + \ldots + D_n \cap U_\sigma).$$

Repeating this argument for all other cones gives the lemma.

In the notation of the above proof, define $\tilde{D}_\sigma := \sum_{i>n} D_i$, $D_\sigma := \sum_{i\leq n} D_i$, so

$$K^{-1} = L(D_\sigma + \tilde{D}_\sigma).$$

2. Application of the technique of Ramanathan. Let $X = T_{\text{emb}}(\Delta)$ be a nonsingular $n$-dimensional toric variety, and use the notation of 1.4-5. The main result of the present section is the following proposition.

Proposition 2.2.1. Define the partial diagonal $D_{i,i+1}^s \subset X \times \ldots \times X = X(s)$ (s factors), as the set

$$\{(x_1, \ldots, x_s), x_k \in X, 1 \leq k \leq s, x_i = x_{i+1}\}.$$ 

Then there is a Frobenius splitting of $X(s)$ which simultaneously splits all $D_{i,j}^s$, $1 \leq i \leq j \leq s$, if a certain numerical parameter $e(X)$, defined below in the course of the proof (Definition 2.2.3), satisfies $e(X) \geq -1$.

Corollary 2.2.2. If $X$ is a nonsingular toric variety with $e(X) \geq -1$ and $L$ is a very ample invertible sheaf, then $R(L)$ is generated in degree one and has relations in degree 2.
case of Borel varieties, and it will occupy the rest of the section. It should perhaps be noted that the numerical condition of the proposition is certainly far from being necessary. It is only a technical condition under which a certain construction of a splitting works. For example \(c(P^n) = -n\), but the requisite splitting of the diagonals is proven by Ramanathan (using a splitting of the flag variety associated to \(P^n\) as a starting point). We do not know whether all toric varieties satisfy the conclusion of the proposition.

Let \(X = \text{emb}(\Delta)\) be a nonsingular \(n\)-dimensional toric variety, and fix a cone \(\sigma \in \Delta\) of maximal dimension. There is a unique closed \(T\)-invariant point \(p(\sigma) : = \text{orb}(\sigma) \in U_\sigma\) (cf. [O1,1.6]). The product space \(X(s)\) is Frobenius split by the map \(a(s) = a \times \ldots \times a\) (using the splitting defined in 1.5), and this map in fact compatibly splits all sets

\[
X_{\sigma,i_1, \ldots, i_t}(s) = \{(x_1, \ldots, x_s) \in X : x_k \in X, 1 \leq k \leq s, x_{i_j} = p(\sigma), 1 \leq j \leq t\}
\]

\((t \leq s\) arbitrary positive integers), but it does not split the partial diagonals. The idea of Ramanathan’s proof is to use a rational, i.e. partially defined, automorphism \(\alpha\) of \(X(s)\) to exchange the role of these two kinds of subsets. Assume \(s = 3\) (the general case is similar), and denote \(a(3)\) by \(a\), for readability, and furthermore \(U_\sigma \times U_\sigma \times U_\sigma\) by \(U_\sigma(3)\). There is an isomorphism \(U_\sigma \cong \mathbb{A}^n\), the affine \(n\)-space, which takes \(p(\sigma)\) to 0. Hence we can define an isomorphism \(\alpha : U_\sigma(3) \to U_\sigma(3)\), which, identifying \(U_\sigma\) with \(\mathbb{A}^n\), is given by

\[
\alpha(x_1, x_2, x_3) = (x_1, x_1 - x_2, x_2 - x_3).
\]

The inverse image by \(\alpha\) of \(X_{\sigma,i+1} \cap U_\sigma(3)\) is \(D_{i,i+1}^3 \cap U_\sigma(3)\), if \(i = 1, 2\). Hence \(\alpha^*(a)\) compatibly splits \(D_{i,i+1}^3 \cap U_\sigma(3)\), and if it could be proven that this, a priori only rational, section of \(K_{X(3)}^{\infty(1-p)}\), actually is regular, then the theorem would follow. For simplicity we will mostly use \(D\) to denote both a divisor \(D\) on \(X\) and its restriction \(D \cap U_\sigma\) to \(U_\sigma\). The divisor of zeroes in \(U_\sigma(3)\) of \(a\) is, by Lemma 1.5.3, precisely \(p - 1\) times

\[
(D_\sigma \times U_\sigma \times U_\sigma) + (U_\sigma \times D_\sigma \times U_\sigma) + (U_\sigma \times U_\sigma \times D_\sigma).
\]

Hence the part of \(\text{div} \alpha^*(a)\) with support not entirely outside \(U_\sigma(3)\) is

\[
(p - 1)D_\sigma \times X \times X + (p - 1)E_2 + (p - 1)E_3,
\]

where

\[
E_i = \{(x_1, x_2, x_3) \in U_\sigma(3) : x_{i-1} - x_i \in D_\sigma \cap U_\sigma\}.
\]

To determine the linear equivalence class of \(E_2\) and \(E_3\) we proceed as follows. Every line bundle \(L\) on \(X(3)\) is the product \(L_1 \times L_2 \times L_3\) of line bundles \(L_j, j = 1, 2, 3\), on \(X\), and \(L_j\) is determined by pulling back \(L\) to the subvariety \(F_j = \{(x_k) \in X(3) : x_k = p(\sigma), k \neq j\}\), for \(j = 1, 2, 3\). Doing this with the line bundle corresponding to \(E_2\) implies that this line bundle is

\[
\mathcal{O}_X(D_\sigma) \times \mathcal{O}_X(D_\sigma^-) \times \mathcal{O}_X,
\]
and similarly $\overline{E}_3$ corresponds to
\[ \mathcal{O}_X \times \mathcal{O}_X(D_\sigma) \times \mathcal{O}_X(D^-_\sigma). \]
Here $D^-_\sigma := \{-d : d \in D_\sigma \cap U_\sigma\}$ is linearly equivalent to $D_\sigma$, since $D_\sigma$ corresponds to the union of the coordinate axes in $A^n$. Hence the part of the divisor of $\alpha^*(a)$ with support not entirely outside $U_\sigma(3)$ is precisely
\begin{equation}
2(p-1)D_\sigma \times X \times X + 2(p-1)X \times D_\sigma \times X + (p-1)X \times X \times D_\sigma.
\end{equation}
The part $H$ of $\text{div} \alpha^*(a)$ which we have not obtained with the above argument has support on $X(3) - U_\sigma(3) = \text{supp}(\tilde{D}_\sigma) \times X \times X \cup \ldots \cup X \times X \times \text{supp}(\tilde{D}_\sigma)$, and it may be written
\begin{equation}
H = G_1 + G_2 + G_3
\end{equation}
where $G_1$ is the part of the divisor $\text{div} \alpha^*(a)$ with support on $\text{supp}(\tilde{D}_\sigma) \times X \times X$, and so on. Now, on the other hand, the anticanonical bundle $K^{-1}_X(3)$ corresponds to
\begin{equation}
(D_\sigma + \tilde{D}_\sigma) \times X \times X + \ldots + X \times X \times (D_\sigma + \tilde{D}_\sigma).
\end{equation}
Hence $p-1$ times this divisor is linearly equivalent to the divisor of the rational section $\text{div} \alpha^*(a)$, so that adding (1) and (2) and equating the result with (3) gives the following three linear equivalence identities, one in each variable.
\[ (p-1)D_\sigma \times X \times X + G_1 \sim (p-1)\tilde{D}_\sigma \times X \times X \]
\[ X \times (p-1)D_\sigma \times X + G_2 \sim X \times (p-1)\tilde{D}_\sigma \times X \]
\[ G_3 \sim (p-1)X \times X \times \tilde{D}_\sigma. \]
Noting that the support of $G_3$ is contained in $X \times X \times \text{supp}(\tilde{D}_\sigma)$ and since the group of divisors with support in $\tilde{D}_\sigma$ is mapped injectively to $\text{Pic} X$, by taking linear equivalence classes (see the description of Pic $X$ below), the only way the last equation can be satisfied is, if there are no poles of $\alpha^*(a)$ on $X \times X \times \tilde{D}_\sigma$, and the zeroes are precisely $(p-1)X \times X \times \tilde{D}_\sigma$. The two remaining identities coincide except for the coordinate in which they take place. So, since the goal is to prove that $\alpha^*(a)$ is a regular section or, equivalently, that the divisor of $\alpha^*(a)$ is effective, it suffices to study the equation
\begin{equation}
(p-1)(D_1 + \ldots + D_n) + \sum_{j=n+1}^{r} a_j D_j = (p-1)\tilde{D}_\sigma = \sum_{j=n+1}^{r} D_j,
\end{equation}
in $\text{Pic} X$ and to prove that this equation has a uniquely determined solution in positive numbers $a_j$, $j = n+1, \ldots, r$. The Picard group of $X$ is the quotient
\[ \oplus_{i=1}^{r} \mathbb{Z} D_i / \{ \text{div} m : m \in M \}, \]
and \( \text{div} m = \sum_{i=1}^{r} \langle m, n_i \rangle D_i \). Let \( \{ m_i : 1 \leq i \leq n \} \) be the dual basis of \( \{ n_i : 1 \leq i \leq n \} \). In \( \text{Pic} X \) the relations

\[
0 \sim D_i + \sum_{j=n+1}^{r} \langle m_i, n_j \rangle D_j
\]

are then true and so

\[
\sum_{j=n+1}^{r} (a_j - (p - 1)(\sum_{i=1}^{n} \langle m_i, n_j \rangle - 1))D_j \sim 0,
\]

and hence (since the linear equivalence classes of \( \{ D_i : i > n \} \) obviously form a basis of \( \text{Pic} X \)) it follows that

\[
a_j = (p - 1)(1 + \sum_{i=1}^{n} \langle m_i, n_j \rangle)
\]

for all \( j > n \). Define a linear functional

\[
\xi_{\sigma} : \mathbb{N}_R \to \mathbb{Z}, \ n \mapsto \sum_{i=1}^{n} \langle m_i, n \rangle.
\]

It then follows that all \( a_j \) are positive, if and only if

\[
\min\{\xi_{\sigma}(n_j); \ n_j \text{ generates a 1-dimensional face of } \Delta.\}
\]

is larger than \(-1\).

Taking into account that \( \sigma \) was an arbitrary cone, the theorem has been proved using the following definition of \( e(X) \).

**Definition 2.2.3.** Let

\[
e(X) := \max\{\min\{\xi_{\sigma}(n_j); \ n_j \text{ generates a 1-dimensional face of } \Delta.\}; \ \sigma \in \Delta\}.
\]

**3.1. The use of \( P(L \oplus M) \).** Let \( L \) and \( M \) be invertible sheaves on \( X \), and consider the projective line bundle \( P(L \oplus M) \) over \( X \). The idea of the present section is to show that, for \( L \) very ample and \( M \) a high enough power of a very ample sheaf, the criterion for Frobenius splitting of diagonals, deduced in the preceding section, is applicable to \( P(L \oplus M) \), and that this suffices to derive the desired properties of \( R(L) \) for \( L \).

There is a canonical invertible sheaf \( O(1) \) on \( P(L \oplus M) \) and a projection morphism

\[
\pi : P(L \oplus M) \to X.
\]
Lemma 3.1.1. \( \mathcal{O}(1) \) is a very ample invertible sheaf, if both \( \mathcal{L} \) and \( \mathcal{M} \) are very ample.

Proof. This follows e. g. from [H1] together with the fact that ample equals very ample on a nonsingular toric variety (cf. [O1, Cor 2.15]).

We have

\[
R(\mathcal{O}(1)) = \sum_{r \geq 0} H^0(\mathbb{P}(\mathcal{L} \oplus \mathcal{M}), \mathcal{O}(r)) \cong \sum_{r \geq 0} H^0(X, \pi_* \mathcal{O}(r)),
\]

\[
= H^0(X, S(\mathcal{L} \oplus \mathcal{M})) \cong H^0(X, (S(\mathcal{L}) \otimes_{\mathcal{O}_X} S(\mathcal{M}))).
\]

(cf. [H2, II.7.11]). Here \( S(\mathcal{M}) \) denotes the symmetric \( \mathcal{O}_X \)-algebra of the locally free \( \mathcal{O}_X \)-module \( \mathcal{M} \).

We want to transfer nice properties from \( R(\mathcal{O}(1)) \) to \( R(\mathcal{L}) \). Obviously \( S(\mathcal{L}) \) is a quotient ring of \( S(\mathcal{L}) \otimes_{\mathcal{O}_X} S(\mathcal{M}) \) by an ideal \( \mathcal{I} \), which is the direct sum of the modules \( \mathcal{L}^{\otimes a} \otimes \mathcal{M}^{\otimes b} \), for all \( a \geq 0 \), \( b > 0 \). If \( \mathcal{L} \) and \( \mathcal{M} \) are generated by global sections then \( H^q(X, \mathcal{L}^{\otimes a} \otimes \mathcal{M}^{\otimes b}) = 0 \), if \( q > 0 \), and \( a \geq 0 \), \( b > 0 \) (cf. [O1, Thm. 2.7] or our Prop. 1.5). Hence \( H^q(X, \mathcal{I}) = 0 \), if \( q > 0 \), and by taking global sections, it is seen that the subring \( R(\mathcal{L}) \) of \( R(\mathcal{O}(1)) \) is a quotient of \( R(\mathcal{O}(1)) \) by the ideal \( H^0(X, \mathcal{I}) \). We have thus shown that \( R(\mathcal{L}) \) is a ring-retract of \( R(\mathcal{O}(1)) \).

Lemma 3.1.2. Assume that \( \mathcal{L} \) is generated by global sections, so that \( R(\mathcal{L}) \) is a retract of \( R(\mathcal{O}(1)) \). Then \( R(\mathcal{L}) \) has generators in degree 1 and relations in degree 2, if this is true for \( R(\mathcal{O}(1)) \).

Proof. The inclusion of graded rings \( i : R = R(\mathcal{L}) \subset S = R(\mathcal{O}(1)) \) induces a degree-preserving map

\[
\text{Tor}(i) : \text{Tor}^{R}_{*,*}(k, k) \to \text{Tor}^{S}_{*,*}(k, k).
\]

The retraction \( s : S \to R \) also preserves the degree and satisfies \( s \circ i = Id \), and hence it induces a left inverse \( \text{Tor}(s) \), which satisfies \( Id = \text{Tor}(s \circ i) = \text{Tor}(s) \circ \text{Tor}(i) \). Thus \( \text{Tor}(i) \) has to be an injection and the vanishing of \( \text{Tor}^S_{i,j}(k, k) \), for certain pair of indices \( i, j \), implies the vanishing of \( \text{Tor}^R_{i,j}(k, k) \) for the same indices. An application of Lemma 1.2 now finishes the proof.

3.2. \( \mathbb{P}(\mathcal{L} \oplus \mathcal{M}) \) as a toric variety. Suppose that \( \mathcal{L} = \mathcal{O}(D_1) \) and \( \mathcal{M} = \mathcal{O}(D_2) \) are \( T \)-linearized invertible sheaves on \( X \). Then the projective space bundle \( \mathbb{P}(\mathcal{L} \oplus \mathcal{M}) \) is again a toric variety \( \tilde{T} \text{emb}(\Delta_P) \), belonging to an algebraic torus \( \tilde{T} \) of dimension \( n + 1 \), if \( T \) has dimension \( n \). The construction of \( \Delta_P \) below is taken from [O2, p. 40].

Let \( N^1 = \mathbb{Z} \) with \( \mathbb{Z} \)-basis \( l_1 \), and let \( N_P = N \times N^1 \). There is, for all cones \( \sigma \) in \( \Delta \), an \( m_L(\sigma) \in M \) such that the restriction \( \mathcal{L}|U_\sigma \) is generated by \( m_L(\sigma) \) as an \( \mathcal{O}_{U_\sigma} \)-module, where \( U_\sigma = \text{Spec} k[\bar{\sigma} \cap M] \) is the affine open subset of \( X \) corresponding to \( \sigma \). Similarly for \( \mathcal{M} \).

Let \( l_0 = -l_1 \), and let \( \bar{\sigma} \) be the image of \( \sigma \) under the injective linear map \( N_\mathbb{R} \to (N_P)_\mathbb{R} \) sending \( y \) to...
\[ y - \langle m_L(\sigma), y \rangle l_0 - \langle m_M(\sigma), y \rangle l_1 = y + (\langle m_L(\sigma), y \rangle - \langle m_M(\sigma), y \rangle) l_1. \]

The set \( \tilde{\Delta} = \{ \tilde{\sigma}; \ \sigma \in \Delta \} \) forms a lifting of \( \Delta \) to \( N(N_P)_R \). Then
\[
P(L \oplus M) = \tilde{T} \text{emb}(\Delta_P)
\]
where \( \Delta_P \) is the set of all \( \sigma_P = \tilde{\sigma} + R_0 l_i, \ i = 0, 1 \), where \( \tilde{\sigma} \) runs through \( \tilde{\Delta} \).

Reformulated in terms of generators of maximal cones, the description says that if \( \{n_1, \ldots, n_n\} \) generate \( \sigma \), then \( \{\tilde{n}_1, \ldots, \tilde{n}_n\} \), where
\[
\tilde{n}_j = n_j + (\langle m_L(\sigma), n_j \rangle - \langle m_M(\sigma), n_j \rangle) l_1
\]
together with either \( R_0 l_0 \) or \( R_0 l_1 \) generate a cone of maximal rank in \( (N_P)_R \), and all such cones are obtained in this way. Note that, using the support function of the invertible sheaves, this may be expressed as
\[
\tilde{n}_j = n_j + (h_L(n_j) - h_M(n_j)) l_1.
\]

3.3. Proof of Theorem 1. We now show that, under suitable conditions on \( X, M \) and \( L \) the space \( P(L \oplus M) \) satisfies the condition given in Section 2 for Frobenius splitting. Assume that \( X \) is nonsingular and that \( \sigma \) is generated by the basis \( \{n_1, \ldots, n_n\} \), and, furthermore, that \( M \) and \( L \) are very ample. Let \( \sigma_P = \tilde{\sigma} + R_0 l_1 \). Suppose that \( y \) is another fundamental generator of some cone \( \tau \), and that \( y = \Sigma a^i n_i \). Then
\[
\tilde{y} = \Sigma a^i n_i + (h_L(y) - h_M(y)) l_1
\]
\[
= \Sigma a^i (\tilde{n}_i - (h_L(n_i) - h_M(n_i)) l_1) + (h_L(y) - h_M(y)) l_1
\]
\[
= \Sigma a^i \tilde{n}_i + ((h_L(y) - \Sigma a^i h_L(n_i)) - (h_M(y) - \Sigma a^i h_M(n_i))) l_1
\]
\[
= \Sigma a^i \tilde{n}_i + ((h_L(y) - \langle y, m_L(\sigma) \rangle) - (h_M(y) - \langle y, m_M(\sigma) \rangle)) l_1.
\]

From this it immediately follows that
\[
\xi_{\sigma_P}(\tilde{y}) = \Sigma a^i + h_L(y) - \langle y, m_L(\sigma) \rangle + \langle y, m_M(\sigma) \rangle - h_M(y).
\]

Note that, if the invertible sheaf \( M \) is generated by global sections belonging to \( M \), then all \( m_M(\sigma), \ \sigma \in \Delta \), belong to \( H^0(X, M) \) and the inequality \( h[y] \leq \langle m, y \rangle \) is true for all \( y \) and \( m \in \Gamma M \) (cf. [O1, Thm. 2.7]). If, furthermore, \( M \) is ample, the inequality \( h[y] < \langle m_M(\sigma), y \rangle \) is true for \( y \) not in \( \sigma \) (cf. [O1, Cor. 2.14, Lemma 2.12]). Note also that
\[
\langle y, m_M \otimes \sigma (\sigma) \rangle - h_M \otimes \sigma (\sigma) = b(\langle y, m_M(\sigma) \rangle - h_M(y)),
\]
and hence, by substituting the tensor power \( M \otimes \sigma \) for \( M \), with \( b \) sufficiently large, it is possible to make \( \xi_{\sigma_P}(\tilde{y}) \) arbitrarily large. Assume thus that \( \xi_{\sigma_P}(\tilde{y}) \geq -1 \), for all \( y \) not in \( \sigma \). Since \( \xi_{\sigma_P}(l_0) = -1 \) and \( \xi_{\sigma_P}(\tilde{y}) = 1 \) if \( y \in \sigma \), this means that there is to any \( L \) an \( M \) such that the condition for compatible splitting of diagonals is fulfilled for \( P(L \oplus M) \). Hence by the corollary to Proposition 2 and Lemma 3.1.2, \( R(L) \) is generated in degree 1 and has relations in degree 2. Thus Theorem 1 is proved.

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