Intermutation

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Abstract

This paper proves coherence results for categories with a natural transformation called intermutation made of arrows from \((A \land B) \lor (C \land D)\) to \((A \lor C) \land (B \lor D)\), for \(\land\) and \(\lor\) being two biendofunctors. Intermutation occurs in iterated, or \(n\)-fold, monoidal categories, which were introduced in connection with \(n\)-fold loop spaces, and for which a related, but different, coherence result was obtained previously by Balteanu, Fiedorowicz, Schwänzl and Vogt. The results of the present paper strengthen up to a point this previous result, and show that two-fold loop spaces arise in the manner envisaged by these authors out of categories of a more general kind, which are not two-fold monoidal in their sense. In particular, some categories with finite products and coproducts are such.

Coherence in Mac Lane’s “all diagrams commute” sense is proved here first for categories where for \(\land\) and \(\lor\) one assumes only intermutation, and next for categories where one also assumes natural associativity isomorphisms. Coherence in the sense of coherence for symmetric monoidal categories is proved when one assumes moreover natural commutativity isomorphisms for \(\land\) and \(\lor\). A restricted coherence result, involving a proviso of the kind found in coherence for symmetric monoidal closed categories, is proved in the presence of two nonisomorphic unit objects. The coherence conditions for intermutation and for the unit objects are derived from a unifying principle, which roughly speaking is about preservation of structures involving one endofunctor by another endofunctor, up to a natural transformation that is not an isomorphism. This is related to weakening the notion of monoidal functor. A similar, but less symmetric, justification for intermutation was envisaged in connection with iterated monoidal categories. Unlike the assumptions previously introduced for two-fold monoidal categories, the assumptions for the unit objects of the categories of this paper, which are more general, allow an interpretation in logic.

Mathematics Subject Classification (2000): 18D10, 55P35

Keywords: coherence, associativity, commutativity, monoidal categories, symmetric monoidal categories, iterated monoidal categories, loop spaces
1 Introduction

For $\land$ and $\lor$ being two biendofunctors, we call intermutation the natural transformation $c^k$ whose components are the arrows

$$c^k_{A_1,A_1',A_2,A_2'} : (A_1 \land A_1') \lor (A_2 \land A_2') \to (A_1 \lor A_2) \land (A_1' \lor A_2')$$

(the notation $c^k$ is used in [10]). Intermutation has been investigated in connection with $n$-fold loop spaces in [3], where coherence of categories involving intermutation, called iterated, or $n$-fold, monoidal categories, is of central concern. (The wider context of algebraic topology within which the results of [3] should be placed is described in the introduction of [15].) Here we take over from [3] the main coherence conditions concerning intermutation, and strengthen the previous coherence result, either by considering notions more general than that of two-fold monoidal category with respect to unit objects, or by adding symmetry, i.e. natural commutativity isomorphisms for $\land$ and $\lor$. Instead of having $n$-fold for every $n \ge 2$, we deal only with the case when $n = 2$, and leave open the question to what an extent our approach could be extended to $n > 2$. We suppose however that this can be achieved by relying on the technique of Section 14 of this paper.

Before [3], intermutation was taken in [17] to be an isomorphism, which led to an isomorphism between $\land$ and $\lor$. In [3] it is not assumed that intermutation is an isomorphism, and $\land$ and $\lor$ need not be isomorphic, but the two corresponding unit objects, $\top$ and $\bot$, are assumed to be isomorphic (actually, they coincide), which delivers arrows from $A \lor B$ to $A \land B$. In our approach, $\top$ and $\bot$ are not assumed to be isomorphic; we must only have an arrow from $\bot$ to $\top$, and we need not have arrows from $A \lor B$ to $A \land B$. This generalization of the notion of iterated monoidal category does not sever the connection with loop spaces. Two-fold loop spaces arise in the manner envisaged by [3] out of categories of a more general kind, which are not two-fold monoidal in the sense of [3] (see Sections 12, 13 and 15 for details). In particular some categories with finite products and coproducts are such.

We differ also from [3] in being able to lift from the coherence result in the associative nonsymmetric context, such as the context of [3], a proviso appropriate for symmetric contexts. This proviso can be formulated either with the help of graphs, or by limiting the number of occurrences of letters, as it is done in [3], and as we will do in the symmetric context of Sections 14 and 16, where we have natural commutativity isomorphisms for $\land$ and $\lor$. Our coherence results in the nonsymmetric context are results that say that all arrows with the same source and target are equal. This is coherence in Mac Lane’s “all diagrams commute” sense.

If $\land$ and $\lor$ are interpreted as meet and join respectively, and $\to$ is replaced by $\le$, then intermutation corresponds to an inequality that holds in any lattice. So if $\land$ and $\lor$ are interpreted as conjunction and disjunction respectively, then
intermutation corresponds to an implication that is a logical law, whose converse is not a logical law. This conjunction and this disjunction need not be those of classical or intuitionistic logic, which are tied to distributive lattices. They may be any lattice conjunction and disjunction (see Section 15), such as the additive conjunction and disjunction of linear logic. (Intermutation does not hold for the multiplicative conjunction and disjunction of linear logic.) Our approach is in tune with this logical interpretation also in the presence of the unit objects, which is not the case for [3].

The law generalizing intermutation in logic is quantifier shift:

\[
\exists x \forall y A \rightarrow \forall y \exists x A.
\]

When \( A \) is \( xRy \) and \( R \subseteq \{0, 1\}^2 \), quantifier shift amounts to intermutation:

\[
(0R0 \wedge 0R1) \lor (1R0 \wedge 1R1) \rightarrow (0R0 \lor 1R0) \wedge (0R1 \lor 1R1).
\]

Other generalizations of intermutation in logic are the laws

\[
\forall x A \lor \forall x B \rightarrow \forall x (A \lor B),
\]

\[
\exists x (A \land B) \rightarrow (\exists x A \land \exists x B).
\]

Analogously, the logical laws

\[
\forall x \forall y A \rightarrow \forall y \forall x A,
\]

\[
\exists x \exists y A \rightarrow \exists y \exists x A
\]

generalize respectively the implications underlying the arrows

\[
\hat{c}^m_{A_1, A_1', A_2, A_2'} : (A_1 \land A_1') \land (A_2 \land A_2') \rightarrow (A_1 \land A_2) \land (A_1' \land A_2'),
\]

\[
\hat{c}^m_{A_1, A_1', A_2, A_2'} : (A_1 \lor A_1') \lor (A_2 \lor A_2') \rightarrow (A_1 \lor A_2) \lor (A_1' \lor A_2'),
\]

which we will encounter in Section 14. (In [13], the arrows \( \hat{c}^m \) are investigated in the same spirit as intermutation here.)

In a similar sense, the logical laws

\[
A \land \exists x B \rightarrow \exists x (A \land B),
\]

\[
A \lor \forall x B \rightarrow \forall x (A \lor B),
\]

provided \( x \) is not free in \( A \), together with the converse implications, generalize respectively distribution of conjunction over disjunction and distribution of disjunction over conjunction; the logical law \( \forall x A \rightarrow \exists x A \) generalizes the implication \( A \land B \rightarrow A \lor B \) (which is the type of the mix arrows of [10], Chapter 8).

By restricting the quantifiers in quantifier shift, we obtain the logical laws

\[
(\exists x \in \emptyset) \forall y A \rightarrow \forall y (\exists x \in \emptyset) A,
\]

\[
\exists x (\forall y \in \emptyset) A \rightarrow (\forall y \in \emptyset) \exists x A,
\]

\[
(\exists x \in \emptyset) (\forall y \in \emptyset) A \rightarrow (\forall y \in \emptyset) (\exists x \in \emptyset) A.
\]
These laws correspond respectively to the arrows
\[
\begin{align*}
\hat{w}_-^- : \bot & \to \bot \land \bot, \\
\hat{w}_+^+ : \top & \lor \top \to \top,
\end{align*}
\]
\[
\kappa : \bot \to \top,
\]
which we will encounter for the first time in Section 3, and which will play an important role in Sections 12 and 16 especially. These assumptions for \( \top \) and \( \bot \) flow out of a unifying logical principle, which delivers also intermutation.

As \( \hat{c} \) serves to \textit{atomize} the conjunctive indices of diagonal arrows, so intermutation serves to atomize the disjunctive indices of diagonal arrows, or the conjunctive indices of codiagonal arrows (see the proof of the Proposition in Section 15). Diagonal and codiagonal arrows correspond in proof theory to the structural rules of contraction on the left and on the right respectively. This atomization was exploited in a proof-theoretical, not categorial, context in [7], [5] and [6], where intermutation is called \textit{medial} (an unfortunate denomination, since the different principle underlying \( \hat{c} \) is called so in universal algebra; see [16]). Following these papers, a categorial investigation of intermutation was started in [19] and papers cited therein, where one finds various proposals for axiomatizing structures involving more than what we consider, without concentrating on coherence. The notation of [19] (Section 2.3), analogous to that which may be found in [20] (Session 26), resembles the rectangular notation of Section 8 below. Intermutation plays an important role in the distributive lattice categories of [10] (Chapter 11; see also Sections 9.4 and 13.2).

Here is a summary of our paper. After some preliminary matters in Section 2, in Sections 3 and 4 we justify the introduction of intermutation and of the equations for arrows involving it. We do the same for the arrows \( \hat{w}_-^- \), \( \hat{w}_+^+ \) and \( \kappa \) mentioned above. This justification is governed by a unifying principle, like the principle of [13], and it is related to the justification provided by [3], which is however less symmetrical. Roughly speaking, this principle is about preservation of structures involving one endofunctor by another endofunctor, up to a natural transformation that is not an isomorphism. This has to do with weakening the notion of monoidal functor (see [14], Sections II.1 and III.1, [17], [22], second edition, Section XI.2, and [10], Section 2.8).

Sections 5-7 present auxiliary coherence results involving the unit objects \( \top \) and \( \bot \). In Sections 8-9 we prove coherence for intermutation in the absence of additional assumptions concerning the biendofunctors \( \land \) and \( \lor \). In Sections 10-12 we prove our central coherence result for intermutation in the presence of natural associativity isomorphisms for \( \land \) and \( \lor \). In Section 11, in the absence of the unit objects, we have a full coherence result in Mac Lane’s “all diagrams commute” sense, and in Section 12, in the presence of the unit objects, we have a restricted coherence result. The restriction is of the kind Kelly and Mac Lane had for their coherence result for symmetric monoidal closed categories in [18]. Section 10 is about deciding whether there is an arrow with given source and
target, which is a problem some authors take as being a part of the coherence problem (cf. [18], Theorem 2.1, and [3], Theorem 3.6.2). In Section 13 we compare the coherence results of Sections 11-12 with the related, but different, coherence result of [3]. We show that our restricted coherence result of Section 12 is sufficient for the needs of [3] when \( n \) in \( n \text{-fold} \) is 2. We leave open the question whether this result can be extended to \( n > 2 \). As we said above, we suppose that this can be achieved by relying on the technique of Section 14.

In Sections 14-16 we prove coherence for intermutation in the presence of symmetry, i.e. natural commutativity isomorphisms for \( \land \) and \( \lor \), besides natural associativity isomorphisms. In Section 14 we have a full coherence result in the absence of the unit objects, and in Section 16 a restricted coherence result in the presence of the unit objects, the restriction being analogous to the restriction of Section 12. We formulate coherence in the presence of symmetry without mentioning graphs, but by limiting the number of occurrences of letters. This is however equivalent to coherence with respect to graphs. In Section 15 we show that with natural associativity and commutativity isomorphisms together with intermutation we have caught an interesting fragment of categories with finite nonempty products and coproducts. This fact can serve to obtain loop spaces in the style of [3] out of categories not envisaged by [3].

2 Biassociative and biunital categories

This section is about preliminary matters. In it we fix terminology and state some basic results on which we rely.

For an arrow \( f : A \to B \) in a category, the type of \( f \) is \( A \to B \), which stands for the ordered pair \((A, B)\) made of the source and target of \( f \). We call categorial equations the following usual equations assumed for categories:

\[
\begin{align*}
    f \circ 1_A &= 1_B \circ f = f, & \text{for } f : A \to B, \\
    h \circ (g \circ f) &= (h \circ g) \circ f.
\end{align*}
\]

We call bifunctorial equations for \( \xi \) the equations

\[
\begin{align*}
    1_A \xi 1_B &= 1_{A \times B}, \\
    (f_1 \circ f'_1) \xi (f_2 \circ f'_2) &= (f_1 \xi f_2) \circ (f'_1 \xi f'_2).
\end{align*}
\]

The naturality equation for \( c^k \) is

\[
((f \lor h) \land (g \lor j)) \circ e^k_{A,B,C,D} = (f \land g) \lor (h \land j)).
\]

We have analogous naturality equations for other natural transformations to be encountered in the text.

For \( n \geq 0 \), an \( n \text{-endofunctor} \) of a category \( \mathcal{A} \) is a functor from the product category \( \mathcal{A}^n \) to \( \mathcal{A} \). If \( n = 0 \), then \( \mathcal{A}^0 \) is the trivial category \( \mathcal{T} \) with a unique object \( * \), and a unique arrow \( 1_\ast \), and 0-endofunctors of \( \mathcal{A} \) amount to special objects of \( \mathcal{A} \). Endofunctors are 1-endofunctors, and biendofunctors are 2-endofunctors.
Next we introduce some classes of categories for which coherence results are already known. These results are ultimately based on Mac Lane’s monoidal coherence results of [21] (see also [22], Section VII.2).

We say that \(\langle A, \land, \lor \rangle\) is a biassociative category when \(A\) is a category, \(\land\) and \(\lor\) are biendofunctors of \(A\), and there are two natural isomorphisms \(\xi\), for \(\xi \in \{\land, \lor\}\), with the following components in \(A\):

\[
\xi_{b, A, B, C} : A \xi (B \xi C) \rightarrow (A \xi B) \xi C,
\]

which satisfy Mac Lane’s pentagonal equations:

\[
\xi_{A, B, C, D} \circ \xi_{A, B, C, D} = (\xi_{A, B, C} \xi 1_D) \circ \xi_{A, B, C, D} \circ (1_A \xi \xi_{B, C, D}).
\]

The natural isomorphism inverse to \(\xi\) is \(\xi\). We call the \(\xi\) and \(\xi\) arrows collectively \(\xi\) arrows.

Let \(A\) be the free biassociative category generated by a set of objects. (This set may be conceived as a discrete category.) We take that the objects of \(A\) are the formulae of the propositional language generated by a set of letters (nonempty if the category is to be interesting) with \(\land\) and \(\lor\) as binary connectives. We will later use for letters \(p, q, r, \ldots\), sometimes with indices. Formally, we have the inductive definition:

- every letter is a formula;
- if \(A\) and \(B\) are formulae, then \((A \xi B)\) is a formula, for \(\xi \in \{\land, \lor\}\).

As usual, we take the outermost parentheses of formulae for granted, and omit them. We do the same for other expressions of the same kind later on. The formulae \(A\) and \(B\) are the conjuncts of \(A \land B\), and the disjuncts of \(A \lor B\). For the free biunital category below, and other categories that have the special objects \(\zeta \in \{\top, \bot\}\), we enlarge the inductive definition of formula by the clauses: “\(\zeta\) is a formula.”

The arrow terms of \(A\) are defined by assuming first that \(1_A, \xi_{A, B, C}\) and \(\xi_{A, B, C}\) are arrow terms for all formulae \(A, B\) and \(C\). These primitive arrow terms are then closed under composition \(\circ\) and the operations \(\xi\), provided for composition that the types of the arrow terms composed make them composable.

These arrow terms are then subject to the equations assumed for biassociative categories; namely, the categorial equations, the bifunctorial equations for \(\land\) and \(\lor\), the naturality and isomorphism equations for \(\xi\) and \(\xi\), and Mac Lane’s pentagonal equations. This means that to obtain \(A\) we factor the arrow terms through an equivalence relation engendered by the equations, which is congruent with respect to \(\circ\) and \(\xi\), and we take the equivalence classes as arrows. (A detailed formal definition of such syntactically constructed categories...
may be found in [10], Chapter 2.) We proceed analogously for other freely generated categories we deal with later in the text.

An arrow term of \( A \) in which \( \circ \) does not occur, and in which \( \xi \) occurs exactly once is called a \( \xi \)-term. For example, \( b_{A,B,C}^\xi \), \( 1_D \land b_{A,B,C}^\xi \) and \( (1_D \land b_{A,B,C}^\xi) \lor 1_E \) are all \( \xi \)-terms. We define analogously \( \xi \)-terms for other natural transformations \( \beta \), which will be introduced later in the text. The subterm of a \( \beta \)-term that is a component of the natural transformation \( \beta \) is called its head. For example, the head of the \( \xi \)-term \( (1_D \land \xi_{b \rightarrow A,B,C}) \lor 1_E \) is \( \xi_{b \rightarrow A,B,C} \).

An arrow term of the form \( f_n \circ \ldots \circ f_1 \), where \( n \geq 1 \), such that for every \( i \in \{1, \ldots, n\} \) we have that \( f_i \) is composition-free is called factorized, and \( f_i \) in such a factorized arrow term is called a factor. A factorized arrow term \( f_n \circ \ldots \circ f_1 \circ 1_A \) is developed when for every \( i \in \{1, \ldots, n\} \) we have that \( f_i \) is a \( \beta \)-term for some \( \beta \). Then by using the categorial and bifunctorial equations we prove easily by induction on the length of \( f \) the following lemma for the category \( A \).

**Development Lemma.** For every arrow term \( f \) there is a developed arrow term \( f' \) such that \( f = f' \).

The same lemma will hold for other freely generated categories analogous to \( A \) which we will introduce later.

The category \( A \) is a preorder, which means that there is in \( A \) at most one arrow of a given type, i.e. with a given source and target (for a proof see [10], Section 6.1). We call this fact Biassociative Coherence.

We say that \( \langle A, \land, \lor, \top, \bot \rangle \) is a biunital category when \( A \) is a category, \( \land \) and \( \lor \) are biendofunctors of \( A \), and \( \top \) and \( \bot \) are special objects of \( A \) such that there are four natural isomorphisms \( \delta \) and \( \sigma \), for \( \xi \in \{\land, \lor\} \), with the following components in \( A \):

\[
\delta_{A}^{\xi}: A \xi A \rightarrow A, \quad \sigma_{A}^{\xi}: A \xi A \rightarrow A,
\]

for \( (\xi, \zeta) \in \{\langle \land, \top \rangle, \langle \lor, \bot \rangle\} \), which satisfy the equations

\[
\delta_{\zeta}^{\xi} = \delta_{\zeta}^{\xi}.
\]

The natural isomorphisms inverse to \( \delta^{\xi} \) and \( \sigma^{\xi} \) are \( \delta^{-\xi} \) and \( \sigma^{-\xi} \) respectively. We call all these arrows collectively \( \delta-\sigma \)-arrows.

A coherence result analogous to Biassociative Coherence can be proved for biunital categories (in the style of Normal Biunital Coherence of Section 5 below). We will not dwell on that proof, which we do not need for the rest of this work.
A bimonoidal category is a biunital category \( \langle A, \land, \lor, \top, \bot \rangle \) such that \( \langle A, \land, \lor \rangle \) is a biassociative category, and, moreover, the following equations are satisfied:

\[
\xi \rightarrow b_{A,\xi,C} = \delta_{A} \xi \rightarrow \sigma_{C}
\]

for \( (\xi, \zeta) \in \{ (\land, \top), (\lor, \bot) \} \). A coherence result analogous to Biassociative Coherence can be proved for bimonoidal categories (see [10], Section 6.1).

3 Upward and downward functors and intermutation

In this section we justify the introduction of the arrows \( c^k, \hat{w}_{\bot}, \hat{w}_{\top}, \kappa \) (see Section 1), and of the \( \delta,\sigma \)-arrows of bimonoidal categories (see the preceding section), in terms of notions of functors that preserve the structure induced by an \( m \)-endofunctor up to a natural transformation, which need not be an isomorphism. This justification proceeds out of a unifying principle.

Let \( \vec{A}_m \), where \( m \geq 0 \), be an abbreviation for \( A_1, \ldots, A_m \). If \( m = 0 \), then \( \vec{A}_m \) is the unique object \( * \) of the trivial category \( T \) (see the preceding section).

Let \( M \) and \( M' \) be \( m \)-endofunctors of the categories \( A \) and \( A' \) respectively, and let \( F \) be a functor from \( A' \) to \( A \). We say that \( (F, \psi) \) is an upward functor from \( (A', M') \) to \( (A, M) \) when \( \psi \) is a natural transformation whose components are the following arrows of \( A \):

\[
\psi_{\vec{A}_m} : M(\overline{F\vec{A}_m}) \rightarrow FM'(\vec{A}_m).
\]

We say that \( (F, \overline{\psi}) \) is a downward functor from \( (A', M') \) to \( (A, M) \) when \( \overline{\psi} \) is a natural transformation whose components are the following arrows of \( A \):

\[
\overline{\psi}_{\vec{A}_m} : FM'(\vec{A}_m) \rightarrow M(\overline{F\vec{A}_m}),
\]

with type converse to that of \( \psi_{\vec{A}_m} \).

When \( (F, \psi) \) is an upward functor and \( (F, \overline{\psi}) \) a downward functor from \( (A', M') \) to \( (A, M) \), and, moreover, \( \psi \) is an isomorphism whose inverse is \( \overline{\psi} \), we say that \( (F, \psi, \overline{\psi}) \) is a loyal functor from \( (A', M') \) to \( (A, M) \).

For an \( m \)-endofunctor \( M \) of a category \( A \), we define as usual, in a coordinatewise manner, the \( m \)-endofunctor \( M^n \) of the product category \( A^n \); this means that on objects we have

\[
M^n((A_1, \ldots, A^n_1), \ldots, (A_m, \ldots, A^n_m)) =_{df} (M(A_1, \ldots, A^n_1), \ldots, M(A_m, \ldots, A^n_m)),
\]

and analogously on arrows. Suppose now that \( A' \) is the product category \( A^n \), and that \( M' \) is \( M^n \).

We say that \( M \) intermutes with \( F \) when there is an upward functor \( (F, \psi) \) from \( (A^n, M^n) \) to \( (A, M) \). From an upward functor \( (F, \psi) \) from \( (A^n, M^n) \) to...
(A, M) we obtain a downward functor (M, F) from (A^m, F^m) to (A, F) such that
\( F \hat{\psi} (A_1^1, \ldots, A_m^1) = M \hat{\psi} (A_1^1, \ldots, A_m^1) \),
and vice versa. So M intermutes with F iff there is a downward functor (M, F) from (A^m, F^m) to (A, F). (Note that the relation “intermutes with” is not symmetric.)

We have that (F, M) is a loyal functor iff (M, F) is a loyal functor. If (F, M) is a loyal functor, then M and F intermute with each other.

For \( \wedge \) and \( \vee \) being biendofunctors of the category A, if \( \vee \) intermutes with \( \wedge \), then we write
\[ c_k^{A_1, A_2, A_1', A_2'} : (A_1 \wedge A_1') \vee (A_2 \wedge A_2') \rightarrow (A_1 \vee A_2) \wedge (A_1' \vee A_2') \]
for \( \hat{\psi}_{(A_1, A_1'), (A_2, A_2')} \), which is equal to \( \hat{\psi}_{(A_1, A_2), (A_1', A_2')} \). We call the natural transformation \( c_k \) intermutation.

For \( \xi \) being a biendofunctor of A and \( \zeta \) a special object of A, if \( (\xi, \hat{\psi}) \) is an upward functor from (A^2, (\zeta, \zeta)) to (A, \zeta), then we write
\[ \hat{\psi}_{\xi}^{\zeta} : \zeta \rightarrow \zeta \xi \xi \]
for \( \hat{\psi}_{\xi}^{\zeta} ; \) here \( \zeta \) intermutes with \( \xi \). If \( (\xi, \hat{\psi}) \) is a downward functor from (A^2, (\zeta, \zeta)) to (A, \zeta), then we write
\[ \hat{\psi}_{\xi}^{\zeta} : \zeta \xi \xi \rightarrow \zeta \]
for \( \hat{\psi}_{\xi}^{\zeta} ; \) here \( \xi \) intermutes with \( \zeta \). If \( (\xi, \hat{\psi}, \hat{\psi}) \) is a loyal functor from (A^2, (\zeta, \zeta)) to (A, \zeta), then \( \xi \) and \( \zeta \) intermutes with each other.

For \( \top \) and \( \bot \) being special objects of the category A, if \( \bot \) intermutes with \( \top \), then we write
\[ \kappa : \bot \rightarrow \top \]
for \( \hat{\psi}_{\zeta}^{\top} \). Intuitively, we conceive of \( \top \) as nullary \( \wedge \), and of \( \bot \) as nullary \( \vee \).

In terms of the biendofunctor \( \xi \) and the special object \( \zeta \) of A we define the endofunctors \( \xi \zeta \) and \( \zeta \xi \) of A by
\[ (\xi \zeta) a = a \xi \zeta, \quad (\zeta \xi) a = a \xi a \]
here \( a \) stands either for an object or for an arrow of A, and for arrows we read \( \zeta \) on the right-hand sides as \( 1_{\zeta} \). Let I be the identity functor of A. If \( (I, \psi, \bar{\psi}) \)
is a loyal functor from \((\mathcal{A}, \land \top)\) to \((\mathcal{A}, I)\), then we write
\[
\hat{\delta}^\to_A : A \land \top \to A \quad \text{and} \quad \hat{\delta}^\leftarrow_A : A \to A \land \top
\]
for \(\psi_A\) and \(\overline{\psi}_A\), respectively. A loyal functor from \((\mathcal{A}, \top \land)\) to \((\mathcal{A}, I)\) yields analogously the isomorphisms
\[
\hat{\sigma}^\to : \top \land A \to A \quad \text{and} \quad \hat{\sigma}^\leftarrow : A \to \top \land A.
\]
By replacing \(\land\) and \(\top\) by \(\lor\) and \(\bot\) respectively, we obtain analogously the remaining \(\delta-\sigma\)-arrows. The arrows \(\hat{\delta}_\zeta^\to\) and \(\hat{\delta}_\zeta^\leftarrow\), for \((\xi, \zeta) \in \{(\land, \top), (\lor, \bot)\}\), guarantee that \(\xi\) and \(\zeta\) intermute with each other.

The introduction of the \(b\)-arrows of biassociative categories (see the preceding section) is justified in a similar manner in [13].

4 Preservation

In the preceding section we justified the introduction of the arrows \(e^\zeta, \hat{w}_\top, \hat{w}_\bot\) and \(\kappa\), and of the \(\delta-\sigma\)-arrows. In this section we will justify the equations we will assume later for these arrows. This justification (like that of [13]) proceeds out of a unifying principle of preservation of a natural transformation by an \(m\)-endofunctor, which resembles something that may be found in [3], but is more symmetrical.

First, we define inductively the notion of shape and of its arity:

1. \(\zeta\) is a shape of arity 0;
2. \(\Box\) is a shape of arity 1;
3. if \(M\) and \(N\) are shapes of arities \(m\) and \(n\) respectively, then \((M \xi N)\) is a shape of arity \(m + n\).

As we did for formulae, we take the outermost parentheses of shapes for granted, and omit them. The shapes we have just defined will be called \((\xi, \zeta)\)-shapes. A \(\zeta\)-shape has the clause (0) omitted.

Let \(a_i\), where \(1 \leq i \leq m\), stand either for an object or for an arrow of \(\mathcal{A}\). We shall use the following abbreviations, like the abbreviation \(\overline{A}_m\) of the preceding section:

\[
\begin{align*}
\overline{a}_m & \quad \text{for} \quad a_1, \ldots, a_m, \\
\overline{a}_{\pi(m)} & \quad \text{for} \quad a_{\pi(1)}, \ldots, a_{\pi(m)}, \\
\overline{a}_m, a'_m & \quad \text{for} \quad a_1, a'_1, \ldots, a_m, a'_m, \\
\overline{a}_m \land a'_m & \quad \text{for} \quad a_1 \land a'_1, \ldots, a_m \land a'_m,
\end{align*}
\]

and other analogous abbreviations, made on the same pattern. If \(m = 0\) and \(a_i\) stands for an arrow, then \(\overline{a}_m\) is the unique arrow \(\mathbf{1}_m\) of the trivial category \(\overline{T}\).
A shape $M$ of arity $m$ defines an $m$-endofunctor of $A$, such that $M(\vec{a}_m)$ is obtained by putting $a_i$ for the $i$-th $\Box$, counting from the left, in the shape $M$.

For arrows, we read $\zeta$ as $1_\zeta$. An $m$-endofunctor defined by a shape $M$ and a permutation $\pi$ of $\{1, \ldots, m\}$ define an $m$-endofunctor $M^\pi$ such that

$$M^\pi(\vec{a}_m) = df M(\vec{a}_{\pi(m)}).$$

For the definitions below we make the following assumptions:

- $(\lor \land)$ $\lor$ intermutes with $\land$,
- $(\bot \land)$ $\bot$ intermutes with $\land$,
- $(\lor \top)$ $\lor$ intermutes with $\top$.

The assumption $(\lor \land)$ delivers the natural transformation $c^k$ with components in $A$, the assumption $(\bot \land)$ delivers the arrow $\hat{w}_\bot : \bot \to \bot \lor \bot$ of $A$, and the assumption $(\lor \top)$ delivers the arrow $\check{w}_\top : \top \lor \top \to \top$ of $A$ (see the preceding section).

We define now by induction on the complexity of the $(\lor, \bot)$-shape $M$ of arity $m$ the natural transformation $\psi^M$ whose components are the following arrows of $A$:

$$\psi^M_{A_m, A'_m} : M(\vec{A}_m \land \vec{A}'_m) \to M(\vec{A}_m) \land M(\vec{A}'_m).$$

Here is the definition:

$$\psi^\bot_{A_m, A'_m} = \hat{w}_\bot : \bot \to \bot \land \bot,$n
$$\psi^\lor_{A_m, A'_m} = 1_{A_m \land A'_m},$$
$$\psi^\lor_{A_m, A'_m} = c^k_{M(\vec{A}_m), M(\vec{A}'_m), N(\vec{B}_n), N(\vec{B}'_n)} \circ \psi^\land_{A_m, A'_m} \lor \psi^\land_{A_m, A'_m} \lor \psi^\land_{A_m, A'_m} \lor \psi^\land_{A_m, A'_m}.$$

In the last clause, $M$ is a shape of arity $m$ and $N$ a shape of arity $n$.

We define next by induction on the complexity of the $(\land, \top)$-shape $M$ of arity $m$ the natural transformation $\overline{\psi}^M$ whose components are the following arrows of $A$:

$$\overline{\psi}^M_{A_m, A'_m} : M(\vec{A}_m \lor \vec{A}'_m) \to M(\vec{A}_m \lor \vec{A}'_m).$$

Here is the definition:

$$\overline{\psi}^\top_{A_m, A'_m} = \check{w}_\top : \top \lor \top \to \top,$n
$$\overline{\psi}^\land_{A_m, A'_m} = 1_{A_m \lor A'_m},$$
$$\overline{\psi}^\land_{A_m, A'_m} = c^k_{M(\vec{A}_m), M(\vec{A}'_m), N(\vec{B}_n), N(\vec{B}'_n)} \circ \overline{\psi}^\land_{A_m, A'_m} \lor \overline{\psi}^\land_{A_m, A'_m} \lor \overline{\psi}^\land_{A_m, A'_m} \lor \overline{\psi}^\land_{A_m, A'_m}.$$

Note that

$$\overline{\psi}^\lor_{A, A', B, B'} = \psi_{A, A', B, B'} = c^k_{A, A', B, B'}.$$
If \( \bot \) and \( \top \) do not occur in the shape \( M \), we do not need the arrows \( \hat{w}^{-} \) and \( \hat{w}^{+} \) to define \( \psi^M \) and \( \psi^M \), and we may do without the assumptions \((\bot \land)\) and \((\lor \top)\).

Let now \( \alpha \) be a natural transformation from the \( m \)-endofunctor of \( A \) defined by the \((\lor, \bot)\)-shape \( M_1 \) to the \( m \)-endofunctor of \( A \) defined by the \((\lor, \bot)\)-shape \( M_2 \) and a permutation \( \pi \) of \( \{1, \ldots, m\} \). We say that \( \alpha \) is upward preserved by \( \land \) when diagrams of the following form commute in \( A \):

\[
\begin{array}{c}
M_1(\bar{A}_m) \land M_1(\bar{A}_m') \\
\downarrow \psi_1^M \\
\downarrow \alpha_{\bar{A}_m} \land \alpha_{\bar{A}_m'} \\
M_2(\bar{A}_{\pi(m)}) \land M_2(\bar{A}_{\pi(m)}') \\
\end{array}
\]

\[
\begin{array}{c}
M_1(\bar{A}_m) \land M_1(\bar{A}_m') \\
\downarrow \psi_2^M \\
\downarrow \alpha_{\bar{A}_m} \land \alpha_{\bar{A}_m'} \\
M_2(\bar{A}_{\pi(m)}) \land M_2(\bar{A}_{\pi(m)}') \\
\end{array}
\]

Let now \( \beta \) be a natural transformation as \( \alpha \) above save that the shapes \( M_1 \) and \( M_2 \) are not \((\lor, \bot)\)-shapes but \((\land, \top)\)-shapes. We say that \( \beta \) is downward preserved by \( \lor \) when diagrams of the following form commute in \( A \):

\[
\begin{array}{c}
M_1(\bar{A}_m) \lor M_1(\bar{A}_m') \\
\downarrow \psi_1^M \\
\downarrow \beta_{\bar{A}_m} \lor \beta_{\bar{A}_m'} \\
M_2(\bar{A}_{\pi(m)}) \lor M_2(\bar{A}_{\pi(m)}') \\
\end{array}
\]

\[
\begin{array}{c}
M_1(\bar{A}_m) \lor M_1(\bar{A}_m') \\
\downarrow \psi_2^M \\
\downarrow \beta_{\bar{A}_m} \lor \beta_{\bar{A}_m'} \\
M_2(\bar{A}_{\pi(m)}) \lor M_2(\bar{A}_{\pi(m)}') \\
\end{array}
\]

i.e., we have in \( A \) the equation

\[
(\psi \beta) \quad \beta_{\bar{A}_m} \lor \beta_{\bar{A}_m'} \circ \psi_1^M = \psi_2^M \circ (\beta_{\bar{A}_m} \lor \beta_{\bar{A}_m'}). 
\]

Let \( \langle A, \land, \lor \rangle \) be a biassociative category, and let us make the assumption \((\lor \land)\) (i.e., \( \lor \) intermates with \( \land \)) together with the assumptions
(ψb) \( \delta \to \) is upward preserved by \( \land \),
(ψb) \( \delta \to \) is downward preserved by \( \lor \).

(The biassociative intermuting categories, introduced in Section 10 below, satisfy these assumptions.) It is easy to see that \( \delta \to \) is upward preserved by \( \land \) iff \( \delta \to \) is, and analogously with \( \delta \) and downward preservation by \( \lor \). We take \( \rightarrow \) in (ψb) and \( \leftarrow \) in (ψb) to make duality apparent.

The assumption (ψb) amounts to the equation
\[
\psi_{A_1 \lor A_2, A' \lor A'_2, A_3, A'_3} = \psi_{A_1 \lor A_2, A' \lor A'_2, A_3, A'_3},
\]
and the assumption (ψb) amounts to the equation
\[
\psi_{A_1 \lor A_2, A' \lor A'_2, A_3, A'_3} = \psi_{A_1 \lor A_2, A' \lor A'_2, A_3, A'_3}.
\]

We call collectively these two equations (c b). They stem from [3] (Section 1), where, however, the b-arrows are identity arrows.

Let \( (A, \land, \lor, \top, \bot) \) be a biunital category, and let us make the assumptions (\( \lor \land \)), (\( \ll \land \)) and (\( \lor \top \)) together with the assumptions
\[
(\psi \delta \sigma) \quad \delta \to \text{ and } \sigma \to \text{ are upward preserved by } \land,
(\psi \delta \sigma) \quad \delta \to \text{ and } \sigma \to \text{ are downward preserved by } \lor.
\]

The assumption (ψδσ) amounts to the equations
\[
\psi_{A \lor A'}(\delta_{A \lor A'} \land \sigma_{A \lor A'}) = \psi_{A \lor A'}(\delta_{A \lor A'} \land \sigma_{A \lor A'}),
\]
and the assumption (ψδσ) amounts to the equations
\[
\psi_{A \lor A'}(\delta_{A \lor A'} \land \sigma_{A \lor A'}) = \psi_{A \lor A'}(\delta_{A \lor A'} \land \sigma_{A \lor A'}).
\]

We call collectively these four equations (c δσ).

For the definitions below we make the assumptions (\( \top \land \)) and (\( \lor \top \)) together with the assumption
\[
(\top \top) \quad \top \text{ intermutes with } \top,
\]
which delivers the arrow \( \kappa : \top \to \top \) of \( A \). We define by induction on the complexity of the (\( \lor, \top \))-shape \( M \) the arrow \( j^M : M(\top, \ldots, \top) \to \top \) of \( A \):
\[
\begin{align*}
\emptyset & = \kappa, \\
\top & = \top, \\
j^M \lor N & = (j^M \lor j^N).
\end{align*}
\]
Next we define by induction on the complexity of the \((\land, \top)\)-shape \(M\) the arrow \(\overset{\top}{\psi}^M : \bot \to M(\bot, \ldots, \bot)\) of \(\mathcal{A}\):

\[
\begin{align*}
\overset{\top}{\psi}^\top &= \kappa, \\
\overset{\top}{\psi}^\bot &= \mathbf{1}_\bot \\
\overset{\top}{\psi}^M \land N &= (\overset{\top}{\psi}^M \land \overset{\top}{\psi}^N) \circ \overset{\top}{w}^\bot.
\end{align*}
\]

When \(\top\) and \(\bot\) are conceived as nullary \(\land\) and \(\lor\) respectively, the definitions of \(\overset{\top}{\psi}^M\) and \(\overset{\top}{\psi}^\bot\) are analogous to those of \(\psi^M\) and \(\psi^\bot\) respectively.

Let now \(\alpha\) and \(\beta\) be natural transformations as for the equations \((\psi^\alpha)\) and \((\psi^\beta)\). We say that \(\alpha\) is \textit{upward preserved} by \(\top\) when in \(\mathcal{A}\) we have the equation

\[
(\top (\psi^\alpha)) \overset{\top}{\psi}^M_2 \circ \alpha \top, \ldots, \top = \top \circ \overset{\top}{\psi}^M_1.
\]

We say that \(\beta\) is \textit{downward preserved} by \(\bot\) when in \(\mathcal{A}\) we have the equation

\[
(\bot (\psi^\beta)) \overset{\bot}{\psi}^M_1 = \overset{\bot}{\psi}^M_2 \circ \mathbf{1}_\bot.
\]

We left \(1_\top\) and \(1_\bot\) in these two equations to make apparent the analogy with the equations \((\psi^\top)\) and \((\psi^\bot)\).

Let \(\langle \mathcal{A}, \land, \lor, \top, \bot \rangle\) be a bimonoidal category, and let us make the assumptions \((\bot \land)\), \((\top \lor)\) and \((\bot \lor)\) together with the assumptions

\[
\begin{align*}
(\top (\psi^\delta))^\top \overset{\top}{\delta} \to \top & \circ \overset{\top}{w}^\top \overset{\top}{\delta} \to (1_\top \lor \kappa), \\
(\bot (\psi^\delta))^\bot \overset{\bot}{\delta} \leftarrow \bot & \circ \overset{\bot}{w}^\bot \overset{\bot}{\sigma} \leftarrow (\kappa \land 1_\bot),
\end{align*}
\]

The assumptions \((\top \psi \delta)^\top\) and \((\top \psi \delta)^\bot\) amount respectively to the equations

\[
\begin{align*}
\overset{\top}{w}^\top \circ (\overset{\top}{w}^\top \lor 1_\top) \circ \overset{\top}{\delta}^\top, \top, \top &= \overset{\top}{w}^\top \circ (1_\top \lor \overset{\top}{w}^\top), \\
\overset{\bot}{\delta}^\bot, \bot, \bot \circ (\overset{\bot}{w}^\bot \land 1_\bot) \circ \overset{\bot}{\sigma}^\bot &= (1_\bot \land \overset{\bot}{w}^\bot) \circ \overset{\bot}{w}^\bot,
\end{align*}
\]

which are analogous to the equations \((c^k \psi)^\top\). We call collectively these two equations \((wb)\).

The assumptions \((\top \psi \delta \sigma)^\top\) and \((\top \psi \delta \sigma)^\bot\) amount respectively to the equations

\[
\begin{align*}
\overset{\top}{\delta}^\top &= \overset{\top}{w}^\top \circ (1_\top \lor \kappa), \\
\overset{\bot}{\delta}^\bot &= (1_\bot \land \kappa) \circ \overset{\bot}{w}^\bot, \\
\overset{\top}{\sigma}^\top &= (\kappa \lor 1_\top) \circ \overset{\top}{w}^\top, \\
\overset{\bot}{\sigma}^\bot &= (\kappa \land 1_\bot) \circ \overset{\bot}{w}^\bot.
\end{align*}
\]

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which are analogous to the equations \((c^k \delta \sigma)\). We call collectively these four equations \((\kappa \delta \sigma)\).

To define the bimonoidal intermuting categories in Section 12 we need also the following equations, which we call collectively \((c^k \kappa)\):

\[
\hat{\delta}^\top = (\hat{\delta}^\top \land \hat{\delta}^\top) = c^k_{\top, \bot, \bot, \top} = \kappa = \hat{\delta}^\bot = (\hat{\delta}^\bot \lor \hat{\delta}^\bot),
\]

\[
\check{\delta}^\top = (\check{\delta}^\top \land \check{\delta}^\top) = c^k_{\bot, \top, \bot, \bot} = \kappa = \check{\delta}^\bot = (\delta^\bot \lor \check{\delta}^\bot).
\]

From these two equations we obtain either two alternative definitions of \(\kappa\) in terms of \(c^k_{\top, \bot, \bot, \top}\) or \(c^k_{\bot, \top, \bot, \bot}\), or the definitions of these two arrows in terms of \(\kappa\). These equations bear an analogy to the equations \((\omega^k \delta \sigma)\) (see the equation \((\iota \alpha \iota)\) in Section 7). As in the equation \((\hat{\psi} \check{b})\) the arrow \(\check{b}^\top_{\bot, \bot, \bot}\) is “shifted” to \(1_\top\) by simplifying arrows made of \(\hat{w}^\top_{\bot}\), so in the equations \((c^k \kappa)\) the arrows \(c^k_{\top, \bot, \bot, \top}\) and \(c^k_{\bot, \top, \bot, \bot}\) are shifted to \(\kappa\) by analogous simplifying arrows.

A justification of the assumptions for biassociative categories in the spirit of this section and of the preceding one may be found in [13]. This involves in particular a justification of Mac Lane’s pentagonal equation. In Sections 14 and 16 we will mention further justifications in the same spirit of the assumptions made for categories with symmetry, i.e. natural commutativity isomorphisms.

5 Normal biunital categories

In this and in the next two sections we present auxiliary coherence results involving the unit objects \(\top\) and \(\bot\). These results will serve for the coherence results of Sections 12 and 16.

A normal biunital category is a biunital category \(\langle A, \land, \lor, \bot, \top \rangle\) (see Section 2) such that in \(A\) we have the isomorphisms \(\hat{w}^\top_{\bot}\): \(\top \Rightarrow \bot \land \bot\) and \(\check{w}^\top_{\bot}\): \(\top \lor \top \Rightarrow \bot\). According to the terminology of Section 3, in a normal biunital category for every \(\xi\) in \(\{\land, \lor\}\) and every \(\zeta\) in \(\{\top, \bot\}\) we have that \(\xi\) and \(\zeta\) intermute with each other. The inverses of \(\hat{w}^\top_{\bot}\) and \(\check{w}^\top_{\bot}\) are \(\hat{w}^\bot_{\bot}\) and \(\check{w}^\bot_{\bot}\) respectively. We call these four arrows collectively \(w\text{-arrows}\. The normal biunital category freely generated by a set of objects is called \(\textbf{N}_{\top, \bot}\).

For every formula \(A\) we define the formula \(\nu(A)\), which is the normal form of \(A\), in the following way:

\[
\nu(p) = p, \text{ for } p \text{ a letter}, \quad \nu(\zeta) = \zeta, \text{ for } \zeta \in \{\top, \bot\},
\]

for \(\zeta \in \{\land, \lor\}\) and \(i \in \{1, 2\}\),

\[
\nu(A_1 \xi A_2) = \begin{cases} 
\zeta & \text{if } \nu(A_1) = \nu(A_2) = \zeta, \\
\nu(A_i) & \text{if } \nu(A_{3-i}) = \zeta \neq \nu(A_i) \text{ and } (\xi, \zeta) \in \{(\land, \top), (\lor, \bot)\}, \\
\nu(A_1) \xi \nu(A_2) & \text{otherwise}.
\end{cases}
\]
If no letter occurs in \( A \), then \( \nu(A) \) is either \( \top \) or \( \bot \). It is clear that for every formula \( A \) we have an isomorphism of \( N_{\top,\bot} \) of the type \( A \to \nu(A) \).

An arrow term of \( N_{\top,\bot} \) is called directed when \( \top \) does not occur in it as a superscript. We can prove the following.

**Directedness Lemma.** If \( f, g : A \to \nu(B) \) are directed arrow terms of \( N_{\top,\bot} \), then \( f = g \) in \( N_{\top,\bot} \).

The proof of this lemma is analogous to Mac Lane’s proof of a lemma that delivered monoidal coherence (see [21], [22], Section VII.2, or [10], Sections 4.3 and 4.6). Whenever for \( i \in \{1, 2\} \) we have that the arrow terms \( f_i : A \to A_i \) are factors of two directed developed arrow terms, we establish that we have two arrow terms \( f'_i : A_i \to C \) such that \( f'_i = f''_i \), and we have a directed arrow term of the type \( C \to \nu(B) \), because \( \nu(B) \) is in normal form. For that we use bifunctorial and naturality equations, except for the case where the heads of \( f_1 \) and \( f_2 \) are \( \delta\top \) and \( \delta\bot \) for \( (\xi, \zeta) \in \{ (\wedge, \top), (\vee, \bot) \} \). Then we use the equation \( \delta\top = \delta\bot \) of biunital categories (which we have also in monoidal categories). We can then prove the following.

**Normal Biunital Coherence.** The category \( N_{\top,\bot} \) is a preorder.

This is obtained from the Directedness Lemma, as Mac Lane obtained monoidal coherence (see the proof of Associative Coherence in [10], Section 4.3).

Normal Biunital Coherence guarantees that there is a unique arrow of \( N_{\top,\bot} \) of the type \( A \to \nu(A) \), and of the converse type. These arrows are isomorphisms.

### 6 \( \kappa \)-Normal biunital categories

A **\( \kappa \)-normal biunital category** is a normal biunital category \( (\mathcal{A}, \wedge, \vee, \top, \bot) \) (see the preceding section) such that in \( \mathcal{A} \) we have an arrow \( \kappa : \bot \to \top \) that satisfies the equations \( (\kappa\delta\sigma) \) (see Section 4). We call \( K^\theta_{\top,\bot} \) the \( \kappa \)-normal biunital category freely generated by the empty set of objects. So there are no objects of \( K^\theta_{\top,\bot} \) in which letters occur. We prove the following.

**\( K^\theta_{\top,\bot} \) Coherence.** The category \( K^\theta_{\top,\bot} \) is a preorder.

**Proof.** For every arrow term \( f \) of \( K^\theta_{\top,\bot} \) either there is an arrow term \( f' \) of \( N_{\top,\bot} \) such that \( f = f' \) in \( K^\theta_{\top,\bot} \), or there are two arrow terms \( f' \) and \( f'' \) of \( N_{\top,\bot} \) such that \( f = f'' \circ \kappa \circ f' \) in \( K^\theta_{\top,\bot} \). This is established by using the equations \( (\kappa\delta\sigma) \) and the following consequences of the naturality equations for the \( \delta\sigma \)-arrows:

\[
\begin{align*}
\kappa \wedge \mathbf{1} \top &= \delta\top \circ \kappa \circ \delta\top, \\
\kappa \vee \mathbf{1} \bot &= \delta\bot \circ \kappa \circ \delta\bot,
\end{align*}
\]

\[
\mathbf{1} \wedge \kappa = \delta\top \circ \kappa \circ \delta\top,
\]

\[
\mathbf{1} \vee \kappa = \delta\bot \circ \kappa \circ \delta\bot.
\]

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Note that there are no arrow terms of $K^0_{\top,\bot}$ of the form $\kappa \circ g \circ \kappa$.

Suppose now for $i \in \{1, 2\}$ that $f_i : A \to B$ is an arrow term of $K^0_{\top,\bot}$. We
have either $f_1 = f_1'$ and $f_2 = f_2'$, or $f_1 = f_1'' \circ \kappa \circ f_1'$ and $f_2 = f_2'' \circ \kappa \circ f_2'$, for $f_1'$ and
$f_1''$ arrow terms of $N^\top_{\bot,\bot}$. It is impossible that $f_i = f_i'$ and
$f_{3-i} = f_{3-i}'' \circ \kappa \circ f_{3-i}'$, because $f_i = f_i'$ requires that $A$ be isomorphic to $B$, while $f_{3-i} = f_{3-i}'' \circ \kappa \circ f_{3-i}'$
prevents that. Then we just apply Normal Biunital Coherence.

We suppose that if to $\kappa$-normal biunital categories we add the equations

$$\hat{\delta}^{-}_{\bot \land A} \circ (\kappa \land 1_{\bot \land A}) = 1_{\bot} \land (\hat{\delta}^{-}_{A} \circ (\kappa \land 1_{A})), \quad (\kappa \lor 1_{\top \lor A}) \circ \hat{\delta}^{-}_{\top \lor A} = 1_{\top} \lor ((\kappa \lor 1_{A}) \circ \hat{\delta}^{-}_{A}),$$

then we could prove coherence in the sense of preordering for the resulting
categories without assuming that the set of generators for the freely generated
category is empty. (For categories where these equations hold see the end of the
next section.)

7 Normal bimonoidal categories

A normal bimonoidal category is a bimonoidal category (see Section 2) that is
also normal biunital (see Section 5), in which, moreover, the equations (wb) (see
Section 4) are satisfied. We call $NA_{\top,\bot}$ the normal bimonoidal category freely
generated by a set of objects. We can prove the following.

Normal Bimonoidal Coherence. The category $NA_{\top,\bot}$ is a preorder.

Proof. By Biassociative Coherence (see Section 2) and the results of [10]
(Chapter 3), we can replace the category $NA_{\top,\bot}$ by an equivalent strictified
category $NA^\text{st}_{\top,\bot}$, where the $b$-arrows are identity arrows. Then to show that
$NA^\text{st}_{\top,\bot}$ is a preorder, which implies that $NA_{\top,\bot}$ is a preorder, we proceed as
for the proof of Normal Biunital Coherence in Section 5. Here we apply the
equations (wb).

A $\kappa$-normal bimonoidal category is a normal bimonoidal category that is
also $\kappa$-normal biunital (see the preceding section). We call $KA^\emptyset_{\top,\bot}$ the $\kappa$-normal
bimonoidal category freely generated by the empty set of objects. We can prove
the following.

$KA^\emptyset_{\top,\bot}$ Coherence. The category $KA^\emptyset_{\top,\bot}$ is a preorder.

Proof. We enlarge the proof of $K^\emptyset_{\top,\bot}$ Coherence of the preceding section by
using the following equations of monoidal categories:
for \((\xi, \zeta) \in \{(\wedge, \top), (\vee, \bot)\}\). We use also analogous equations derived by isomorphism, where the superscript \(\rightarrow\) of \(b\) is replaced by \(\leftarrow\). With these equations and the equations \((wb)\) we can eliminate every occurrence of \(b\). To achieve that we use also equations derived from naturality equations, like the following equation:

\[
\overrightarrow{b}_{A,\xi,\zeta,\nu} = \delta_{A,\xi,\zeta} \circ \overrightarrow{b}_{A,\xi,\zeta,\nu} = \delta_{A,\xi,\zeta} \circ (\overrightarrow{b}_{A,\xi,\zeta,\nu}) \circ (\overrightarrow{b}_{A,\xi,\zeta,\nu}),
\]

where \(i_B : B \to \nu(B)\) is an arrow term of \(N_{\top, \bot}\) standing for an isomorphism, and \(i_B^{-1}\) stands for its inverse.

The equations \((wb)\) can be justified in a manner somewhat different from that of Section 4 by appealing to the isomorphisms \(i_B : B \to \nu(B)\) mentioned at the end of the proof above. For \(\alpha_{\xi,...,\zeta} : M(\zeta,...,\zeta) \to N(\zeta,...,\zeta)\) a component of a natural transformation, we can take that the equations \((wb)\) are obtained from the following equation:

\[
(i\alpha) \quad i_{N(\zeta,...,\zeta)} \circ \alpha_{\xi,...,\zeta} = \mathbf{1}_{\xi} \circ i_M(\zeta,...,\zeta).
\]

We read this equation by saying that \(\alpha_{\xi,...,\zeta}\) is shifted by two isomorphisms to \(\mathbf{1}_{\xi}\) (see the end of Section 4 for shifting to \(\kappa\)).

In \(\kappa\)-normal bimonoidal categories we have the equations mentioned after the proof of \(K_{\top, \bot}\) Coherence at the end of the preceding section, and we can prove coherence in the sense of preordering for these categories without assuming that the set of generators for the freely generated \(\kappa\)-normal bimonoidal category is empty. We will however not dwell on this proof, for which we have no application in the rest of the paper.

The categories treated in this and in the preceding two sections make the following chart:
8 Rectangular notation

To prove coherence results for categories with intermutation we will find very helpful a planar notation for propositional formulae involving $\land$ and $\lor$, i.e. for the objects of our freely generated categories. We first introduce a representation of formula trees obtained by subdividing rectangles. To every formula there will correspond a rectangle subdivided into further rectangles corresponding to the subformulae; if the formula is a letter, there is just one rectangle.

Out of this representation we obtain our planar notation, which we call the rectangular notation for formulae. We deal first with binary trees involving two binary connectives, and next with trees with finite branching, possibly bigger than binary, involving two strictly associative binary connectives. Something similar to our rectangular notation was used to explain the Eckmann-Hilton argument in [1] (Section VI) and [2] (Section 1.2). We do not know whether our rectangular notation, and the use we make of it, may be connected significantly with the little $n$-cubes studied in [3] (Section 6).

To obtain our rectangular notation we go through the following construction. At the beginning, we index every point in the plane by the empty set. At the end of the construction the points that are vertices of rectangles will receive as indices one of the following sets:

$$\emptyset, \{\downarrow\}, \{\uparrow\}, \{\to\}, \{\leftarrow\}, \{\rightarrow, \leftarrow\},$$

while all the other points will be indexed by $\emptyset$.

The construction then proceeds as follows. For a formula $A$ we draw a rectangle enclosing $A$. If during the construction we have a rectangle enclosing a subformula $B \land C$ of $A$:

\[
\begin{array}{c}
\alpha \\
B \land C \\
\beta
\end{array}
\]

then we delete $B \land C$ and subdivide the rectangle in the middle by a vertical line segment so as to obtain two rectangles enclosing $B$ and $C$ respectively:

\[
\begin{array}{c}
\alpha \cup \{\downarrow\} \\
B \\
\beta \cup \{\uparrow\} \\
C
\end{array}
\]

Here $\alpha$ and $\beta$ are the index sets of the middle points of the horizontal sides in the former figure, and these index sets are changed after the subdivision as...
in the latter figure. All the other points have index sets unchanged in passing from the former to the latter figure. The vertical line segment introduced by the subdivision corresponds to the main $\land$ of $B \land C$, and we say that the big rectangle that has previously enclosed $B \land C$ corresponds to $B \land C$.

If during the construction we have a rectangle enclosing a subformula $B \lor C$ of $A$ as in the figure on the left, then by subdividing it horizontally in the middle we pass to the figure on the right:

\[
\begin{array}{c}
\alpha & B \lor C & \beta \\
\hline
\end{array}
\quad \begin{array}{c}
\alpha \cup \{\to\} & B & \beta \cup \{\leftarrow\} \\
\hline
\end{array}
\]

with the new horizontal line segment corresponding to the main $\lor$ of $B \lor C$, and the big rectangle on the right corresponding to $B \lor C$. The construction is over when rectangles enclose only letters, and the end result of the construction is the binary rectangular grid of $A$, which we denote by $\rho(A)$. It is easy to see that $\rho$ is a one-one map.

For example, for $A$ being $((p \land q) \land r) \lor (((s \land t) \lor (u \land q)) \land ((v \land p) \lor w))$ we obtain the following binary rectangular grid $\rho(A)$:

\[
\begin{array}{ccc}
\{\uparrow\} & \{\downarrow\} \\
\hline
p & q & r \\
\hline
\{\rightarrow\} & s & t & \{\rightarrow, \leftarrow\} \\
\hline
\{\rightarrow\} & u & \{\rightarrow, \leftarrow\} & \{\rightarrow\} \\
\hline
\{\downarrow\} & \{\uparrow\} & w
\end{array}
\]

All the points in the plane except those whose index set is mentioned in this picture are indexed by $\emptyset$. By not making the sides of the rectangles meet at crossing points we indicate how these points are indexed, and we may omit mentioning in the picture even the index sets we have mentioned.

If in the construction we have presented above we do not require that the subdividing line segments, vertical and horizontal, be in the middle, though they may be there, then we obtain a rectangular grid $\gamma(A)$ where $\gamma$ differs from $\rho$ in not being one-one any more. To $\gamma(A)$ there will correspond a set of formulae obtained from $A$ by associating parentheses in different manners; more precisely, we will obtain the set of all objects isomorphic to $A$ in the free biassociative category $A$ of Section 2.
9 Intermuting categories

We call \( \langle A, \wedge, \vee \rangle \) an intermuting category when \( \wedge \) and \( \vee \) are biendofunctors of \( A \) and there is a natural transformation \( c^k \) whose components are the following arrows of \( A \):

\[
c_{A,B,C,D}^k: (A \wedge B) \vee (C \wedge D) \rightarrow (A \vee C) \wedge (B \vee D).
\]

In the terminology of Section 3, we assume that \( \vee \) intermutes with \( \wedge \). Let \( C^k \) be the free intermuting category generated by a set of objects.

For every arrow term \( f: A \rightarrow B \) of \( C^k \) the binary rectangular grids \( \rho(A) \) and \( \rho(B) \) differ only with respect to the index sets; these grids are otherwise the same. This is because \( c_{A,B,C,D}^k \) corresponds to passing from the figure on the left to the figure on the right:

\[
\begin{array}{c|c}
\rho(A) & \rho(B) \\
\hline
\rho(C) & \rho(D)
\end{array}
\quad \begin{array}{c|c}
\rho(A) & \rho(B) \\
\hline
\rho(C) & \rho(D)
\end{array}
\]

with all the index sets being unchanged except that indexing the crossing in the middle of the figure. This crossing corresponds in a unique way to \( c_{A,B,C,D}^k \). So to every occurrence of \( c^k \) in \( f \) there corresponds a unique crossing in \( \rho(A) \) and \( \rho(B) \), which is only indexed differently in \( \rho(A) \) and \( \rho(B) \). Hence, for two arrow terms \( f \) and \( g \) of the same type, there is a bijection from the set of occurrences of \( c^k \) in \( f \) to the set of occurrences of \( c^k \) in \( g \) such that this bijection maps an occurrence of \( c^k \) in \( f \) to the occurrence of \( c^k \) in \( g \) with the same corresponding crossing.

We say that a \( c^k \)-term corresponds to a crossing when its head corresponds to this crossing. (The notions of \( c^k \)-term and of its head are defined in Section 2.)

Note that not every crossing indexed with \( \{\downarrow, \uparrow\} \) is such that it can be reindexed with \( \{\rightarrow, \leftarrow\} \) through \( c^k \), i.e. by intermuting. In the example in the preceding section, the crossing involving \( p, q, s \) and \( t \) is not ready for intermuting. It will become such after intermuting is performed at the two other crossings indexed with \( \{\downarrow, \uparrow\} \). But a crossing indexed with \( \{\downarrow, \uparrow\} \) may be such that it can never become ready for intermuting, because it is not in the centre of a rectangle.

Although in a composition \( f_2 \circ f_1: A \rightarrow B \) of two \( c^k \)-terms the two intermutings corresponding to \( f_1 \) and \( f_2 \) cannot always be “permuted”, they can be “permuted” if in \( \rho(A) \) the crossings corresponding to \( f_1 \) and \( f_2 \) are ready for intermuting. We can first easily establish the following lemma.
Lemma 1. If $f: A \to B$ and $g: A \to C$ are two $c^k$-terms such that $B$ differs from $C$, then there are two $c^k$-terms $g': B \to D$ and $f': C \to D$ such that $g' \circ f = f' \circ g$, and for $h \in \{f, g\}$ the $c^k$-terms $h$ and $h'$ correspond to the same crossing in $\rho(A)$.

Next we have the following lemma about “permutation”.

Lemma 2. Let $f_n \circ \ldots \circ f_1: A \to B$, for $n \geq 2$ be a composition of $c^k$-terms such that for $i \in \{1, \ldots, n\}$ the factor $f_i$ corresponds to the crossing $x_i$ in $\rho(A)$, and let $g_1: A \to C$ be a $c^k$-term that corresponds to $x_j$ for some $j \in \{1, \ldots, n\}$. Then there is a composition of $c^k$-terms $g_n \circ \ldots \circ g_2: C \to B$ such that $f_n \circ \ldots \circ f_1 = g_n \circ \ldots \circ g_1$ in $C^k$.

Proof. Let the type of $f_1$ be $A \to A_1$. If $A_1$ is $C$, then $f_1 = g_1$ and $g_n \circ \ldots \circ g_2$ is $f_n \circ \ldots \circ f_2$. If $A_1$ is not $C$, then by Lemma 1, there are two $c^k$-terms $g_1': A_1 \to D$ and $f_1': C \to D$ such that $g_1' \circ f_1 = f_1' \circ g_1$.

If $D$ is $B$, then $n = 2$, and $g_2$ is $f_1'$. Since $f_2$ and $g_1'$ must be the same $c^k$-term, we have $f_2 \circ f_1 = g_2 \circ g_1$.

If $D$ is not $B$, then $n > 2$, and, because the crossing corresponding to $g_1'$ is $x_j$ for $j \neq 1$, we may apply the induction hypothesis to $f_n \circ \ldots \circ f_2: A_1 \to B$ and $g_1': A_1 \to D$ to obtain $g_n \circ \ldots \circ g_3: D \to B$ such that $f_n \circ \ldots \circ f_2 = g_n \circ \ldots \circ g_3 \circ g_1'$. We take that $g_2$ is $f_1'$, and we obtain

$$f_n \circ \ldots \circ f_2 \circ f_1 = g_n \circ \ldots \circ g_3 \circ g_1' \circ f_1$$

$$= g_n \circ \ldots \circ g_3 \circ g_2 \circ g_1$$

By applying Lemma 2 we can easily prove the following proposition by induction on the length of a developed arrow term (see Section 2).

Intermuting Coherence. The category $C^k$ is a preorder.

10 Biassociative intermuting categories

A biassociative intermuting category is a biassociative category $\langle A, \wedge, \vee \rangle$ (see Section 2) that is an intermuting category (see the preceding section), and, moreover, the equations $(c^k b)$ (see Section 4) are satisfied. Let $A C^k$ be the free biassociative intermuting category generated by a set of objects.

One can show that every natural transformation defined by an arrow term of $A$ such that $\wedge$ does not occur in it is upward preserved by $\wedge$ in $A C^k$, and analogously when $\wedge$ is replaced by $\vee$ and “upward” by “downward”. For that we rely essentially on the following. Suppose that $\beta'$ and $\beta''$ are upward preserved by $\wedge$. Then to show that $\beta' \lor \beta''$ is also upward preserved by $\wedge$ we rely on bifunctorial equations and the naturality of $c^k$. To show that the natural transformation $\beta'$ obtained from $\beta$ by substituting a $\lor$-shape (see Section 4) in one of the indices
of $\beta$ is upward preserved by $\land$ if $\beta$ is, we rely on the naturality of $\beta$. In showing that, we have equations like the following instance of the naturality of $b^{-1}$:

$$(1_{A_1} \land A'_1 \land 1_{A_2} \land A'_2) \circ b^{-1}_1 = b^{-1}_{A_1} \land A'_1, A_2 \land A'_2, (A_3 \land A'_3) \lor (A_4 \land A'_4) = b^{-1}_{A_1} \land A'_1, A_2 \land A'_2, (A_3 \lor A_4) \land (A'_3 \lor A'_4) \circ (1_{A_1} \land A'_1 \land 1_{A_2} \land A'_2) \circ (c_{A_3, A'_3, A_4, A'_4} B).$$

Something analogous can be shown for categories more complex than $\mathbf{AC}^k$, which we will encounter later in this paper (cf. Section 14).

As we did in the proof of normal Bimonoidal Coherence in Section 7, we can apply Biassociative Coherence to obtain a strictified category $\mathbf{AC}^{k\text{-}st}$ equivalent to $\mathbf{AC}^k$ where the $b$-arrows are identity arrows. Our ultimate goal is to show that $\mathbf{AC}^{k\text{-}st}$ is a preorder, which implies that $\mathbf{AC}^k$ is a preorder too.

Before working towards that goal, we will consider in this section the problem whether there is an arrow of a given type in $\mathbf{AC}^{k\text{-}st}$. This sort of problem (which is called the theoremhood problem in [10], Section 1.1) is sometimes taken to be a part of a coherence result (cf. [18], Theorem 2.1, and [3], Theorem 3.6.2). We do not need to solve this problem to show that $\mathbf{AC}^{k\text{-}st}$ is a preorder, which is properly coherence, but the techniques used in this section will be analogous to those used in the next section to demonstrate coherence.

The objects of $\mathbf{AC}^{k\text{-}st}$ are equivalence classes of formulae $[A]$ such that $[A]$ is the set of all formulae isomorphic to $A$ in the free biassociative category $\mathbf{A}$ of Section 2. Such an equivalence class $[A]$ corresponds in a one-to-one manner to the rectangular grid $\gamma(A)$ mentioned at the end of Section 8. We may denote the equivalence class $[A]$ by deleting from the formula $A$ parentheses tied to an occurrence of $\xi$ within the immediate scope of another occurrence of $\xi$, for $\xi \in \{\land, \lor\}$. We call the result of this deleting procedure a form sequence (which is short for form sequence in natural notation, according to the terminology of [10], Section 6.2).

We call a form sequence diversified when every letter occurs in it at most once. To simplify matters, we speak from now on only about diversified form sequences. It is easier if we speak about letters, rather than their occurrences, and with diversified form sequences we may do so. We denote by $\text{let}(X)$ the set of letters occurring in the form sequence $X$.

If $B$ is in the equivalence class $[A]$, then the rectangular grids $\gamma(A)$ and $\gamma(B)$ may be taken to be the same, and instead of $\gamma(A)$ we write $\gamma(X)$, where $X$ is the form sequence obtained from $A$ by the deleting procedure above. Formally, we take $\gamma(X)$ to be an equivalence class.

For a form sequence $X$ we define inductively four sequences of letters taken from $X$ which we call $T(X)$, $B(X)$, $L(X)$ and $R(X)$ ($T$ stands for top, $B$ for bottom, $L$ for left and $R$ for right). The sequences $T(p)$, $B(p)$, $L(p)$ and $R(p)$ are all the one-member sequence $p$. For

$$(S, \xi) \in \{(T, \land), (B, \land), (L, \lor), (R, \lor)\}$$
we have that \( S(X_1 \xi X_2) \) is the sequence obtained by concatenating the sequences \( S(X_1) \) and \( S(X_2) \). In the remaining cases we have

\[
T(X_1 \lor X_2) = T(X_1), \quad B(X_1 \lor X_2) = B(X_2),
\]
\[
L(X_1 \land X_2) = L(X_1), \quad R(X_1 \land X_2) = R(X_2).
\]

If, for example, \( X \) is the form sequence

\[
(p_1 \land q_1 \land r) \lor ((s \land t) \lor (u \land q_2)) \land ((v \land p_2) \lor w),
\]

which is obtained from the formula in the example in Section 8, then we have the following:

\[
T(X) = p_1 q_1 r, \quad B(X) = u q_2 w,
\]
\[
L(X) = p_1 s u, \quad R(X) = r p_2 w.
\]

Note that these sequences can easily be read from \( \gamma(X) \), which is obtained from the grid in Section 8 just by adding subscripts to \( p \) and \( q \).

For \( x \) an occurrence of \( \land \) in a form sequence \( X \) such that \( X_1 x X_2 \) is a \textit{subsequence} of \( X \), i.e. a form sequence that is subword of \( X \), we define the sequences \( L_x \) and \( R_x \) as \( L(X_2) \) and \( R(X_1) \) respectively. Intuitively, we may read \( L_x \) as “the vertical sequence of letters that have \( x \) immediately on the left in the rectangular grid \( \gamma(X) \)”, and analogously for \( R_x \) (and also for \( T_y \) and \( B_y \) below). Note that the same \( x \) may occur in two different subsequences \( X_1 x X_2 \) and \( X'_1 x X'_2 \) of \( X \). For example, \( p \land q \) and \( p \land q \land r \) are both subsequences of \( p \land q \land r \). It is however easy to check that the definition above is correct, since if \( X'_1 x X'_2 \) is also a subsequence of \( X \), then \( L(X_2) = L(X'_2) \) and \( R(X_1) = R(X'_1) \).

For \( y \) an occurrence of \( \lor \) in a form sequence \( X \) such that \( X_1 y X_2 \) is a \textit{subsequence} of \( X \), we define the sequence \( T_y \) as \( T(X_2) \) and \( B_y \) as \( B(X_1) \).

Note that if \( x_1 \) and \( x_2 \) are two different occurrences of \( \land \) in \( X \), then no letter is both in \( L_{x_1} \) and \( L_{x_2} \), and the same holds for \( R_{x_1} \) and \( R_{x_2} \). We have an analogous situation with \( y \), \( T \) and \( B \).

For \( X \) and \( Y \) being form sequences, let \( f : X \rightarrow Y \) be an arrow of \( \text{AC}^{\mathbb{k}^{\text{st}}}. \) (It is easy to see that \( X \) is diversified if \( Y \) is, and our assumption is that both are diversified.) By considering \( f \) in a developed form (see Section 2), and what is intermuted by each of its factors, it can be checked easily that the following conditions hold for \( X \) and \( Y \).

**Condition \( \land \).** There is a function \( c \) from the set of occurrences of \( \land \) in \( X \) onto the set of occurrences of \( \land \) in \( Y \) such that for every occurrence \( x \) of \( \land \) in \( Y \) there are occurrences \( x_1, \ldots, x_k \) of \( \land \) in \( X \) for which

\[
L_x = L_{x_1} \cdots L_{x_k}, \quad R_x = R_{x_1} \cdots R_{x_k} \quad \text{and} \quad c^{-1}(\{x\}) = \{x_1, \ldots, x_k\}.
\]

We may say that \( x_1, \ldots, x_k \) merge into \( x \).
CONDITION ∨. There is a function $d$ from the set of occurrences of $∨$ in $Y$ onto the set of occurrences of $∨$ in $X$ such that for every occurrence $y$ of $∨$ in $X$ there are occurrences $y_1, \ldots, y_l$ of $∨$ in $Y$ for which

$$T_y = T_{y_1} \cdots T_{y_l}, \quad B_y = B_{y_1} \cdots B_{y_l} \quad \text{and} \quad d^{-1}(\{y\}) = \{y_1, \ldots, y_l\}.$$  

We may say that $y$ is split into $y_1, \ldots, y_l$.

For $X$ and $Y$ being form sequences, we say that $(X, Y)$ is a legitimate pair when $\text{let}(X) = \text{let}(Y)$ and Conditions $∧$ and $∨$ are satisfied.

So we know that if $X$ and $Y$ are respectively the source and the target of an arrow of $\text{AC}^{k*}$, then $(X, Y)$ is a legitimate pair. Our purpose in this section is to show that the converse holds too, which will give us a criterion for the existence of arrows in $\text{AC}^{k*}$. It is a decidable question whether $(X, Y)$ is a legitimate pair.

We introduce first the following notions. Let $X$ be a form sequence, and let $p$ and $q$ be letters in $X$. Let $q \prec p$ in $X$ mean that there exists an occurrence $y$ of $∨$ in $X$ such that $q$ belongs to $B_y$ and $p$ belongs to $T_y$. If $q \prec p$, then in $\gamma(X)$ we have a horizontal dividing line segment to which the top side of the rectangle enclosing $p$ and the bottom side of the rectangle enclosing $q$ both belong. Let $\prec \ast$ be the transitive closure of the relation $\prec$ in $X$. Then we have the following.

**Lemma for $\prec \ast$.** Let $(X, Y)$ be a legitimate pair. If $q \prec \ast p$ in $Y$, then $q \prec \ast p$ in $X$.

This is an easy consequence of Condition $∨$, and becomes clear when we consider $\gamma(X)$ and $\gamma(Y)$. Note that the converse of this lemma does not hold, but we will use this lemma to establish a related equivalence in the Position-Preservation Lemma below.

We define $q \mid p$ and $q \mid \ast p$ analogously, and there is an analogous lemma to the one above for $\mid \ast$, which is a consequence of Condition $∧$.

When there is no occurrence $x$ of $∧$ in the form sequence $X$ such that $p$ belongs to $L_x$ we say that $p$ is a left-border letter of $X$. In the grid $\gamma(X)$, the letter $p$ is at the left border of the rectangle corresponding to $X$. We define analogously the right-border, top-border and bottom-border letters of $X$; we just replace $(L, ∧)$ by $(R, ∧)$, $(T, ∨)$ and $(B, ∧)$.

Let $y_1, \ldots, y_i$ for $i \geq 1$ be occurrences of $∨$ in a form sequence $X$, and let $T_{y_i}$, for $i \in \{1, \ldots, l\}$, be the sequence $p_1^i \cdots p_k^i$, for $k_i \geq 1$. We say that the sequence $y_1 \cdots y_i$ is a horizontal transversal of $X$ when $p_1^i$ is a left-border letter of $X$ and $p_k^i$ is a right-border letter of $X$, while for every $i \in \{1, \ldots, l-1\}$ we have $p_{k_i}^i \mid p_{k_i}^{i+1}$. (Horizontal transversals could equivalently be defined by using $B_{y_i}$ instead of $T_{y_i}$.) In $\gamma(X)$ a horizontal transversal appears as
where each \( y_i \) is a horizontal line segment, and the rectangle drawn corresponds to \( X \). For example, if \( y_1, y_2 \) and \( y_3 \) are the three occurrences of \( \lor \) (counting from the left) in the example of Section 8, then the sequences \( y_1 \) and \( y_2y_3 \) are horizontal transversals. Contrary to what we have in this example, horizontal transversals can be of length greater than 2, and not every occurrence of \( \lor \) in \( X \) need be included in a horizontal transversal. We can define analogously vertical transversals, and rely in the remainder of the exposition on this other notion, rather than on the notion of horizontal transversal.

It is easy to ascertain the following lemma.

**Transversal-Preservation Lemma.** Let \((X,Y)\) be a legitimate pair. If \( y_1 \ldots y_l \) is a horizontal transversal of \( X \), then the members of \( \bigcup_{i \in \{1,\ldots,l\}} d^{-1}\{y_i\} \) make a horizontal transversal of \( Y \).

We denote by \( d^{-1}(y_1 \ldots y_l) \) the horizontal transversal of \( Y \) whose existence is claimed by this lemma.

There are analogous definitions of being above a horizontal transversal for letters and occurrences of \( \lor \). With vertical transversals we would define being on the left-hand side and on the right-hand side for letters and occurrences of \( \land \). Any of these notions could be taken as central for the exposition, as below is in ours.

We can prove the following.

**Position-Preservation Lemma.** Let \((X_1yX_2,Y)\) be a legitimate pair for \( y \) an occurrence of \( \lor \). Then \( p \) is below the horizontal transversal \( y \) of \( X_1yX_2 \) iff \( p \) is below the horizontal transversal \( d^{-1}(y) \) of \( Y \).

**Proof.** From left to right we proceed by induction. If \( p \) is in \( T_y \) in \( X_1yX_2 \), then we are done. If \( p \), which is below \( y \), is not in \( T_y \) in \( X_1yX_2 \), then there must be an occurrence \( y' \) of \( \lor \) in \( X_1yX_2 \) below \( y \), different from \( y \), such that \( p \) is in \( T_{y'} \), and by the induction hypothesis for every \( q \) in \( B_{y'} \) we have that \( q \) is below
By Condition $\lor$ there is a sequence $y'_1, \ldots, y'_n$ of occurrences of $\lor$ in $Y$ such that for some $i \in \{1, \ldots, n\}$ we have that $p$ is in $T_{y'_i}$ in $Y$, and for every $q$ in $B_{y'_i}$ we have that $q$ is in $B_{y'_i}$. So, by the induction hypothesis, for every $q$ in $B_{y'_i}$ we have that $q$ is below $d^{-1}(y)$, and hence $p$ is below $d^{-1}(y)$. From right to left we just apply the Lemma for $\neg\neg\ast$. \hfill \dashv

An analogous Position-Preservation Lemma holds for above, on the left-hand side and on the right-hand side, defined as indicated above.

For $X$ a form sequence, let $p$ be a letter such that $X$ is not $p$. We define inductively the form sequence $X^{-p}$:

- if $p$ is not in $X$, then $X^{-p}$ is $X$;
- for $\xi \in \{\&, \lor\}$, if $X$ is of the form $Y \xi p$ or $p \xi Y$, then $X^{-p}$ is $Y$, and if $X$ is of the form $Y_1 \xi Y_2$ for $p$ occurring in $Y_i$, for some $i \in \{1, 2\}$, but different from $Y_i$, then $X^{-p}$ is $Y_1^{-p} \xi Y_2^{-p}$.

Since the same $X$ can be of the form $Y_1 x Y_2$ and $Y'_1 x' Y'_2$ for $x$ and $x'$ different occurrences of $\xi$, and $Y_i$ different from $Y'_i$, one can raise the question whether the definition above is unambiguous. That it is indeed such can easily be checked with the formal notation for form sequences of [10] (Section 6.2). For example, $(q \& p \& r)^{-p}$ is $q \& r$, irrespectively of whether we interpret $q \& p \& r$ as $(q \& p) \& r$ or as $q \& (p \& r)$.

Since it is easy to see that $(X^{-p})^{-q} = (X^{-q})^{-p}$, we can define in the following manner $X^{-P}$ for $X$ a form sequence and $P = \{p_1, \ldots, p_n\}$ a set of letters such that $\text{let}(X) - P \neq \emptyset$:

$$X^{-P} = \text{df} \ (\ldots (X^{-p_1})^{-p_n}).$$

For form sequences $X$ and $Y$, we abbreviate $X^{-\text{let}(Y)}$ by $X^{-Y}$. We can then prove the following.

**INTERPOLATION LEMMA $\lor$.** *If $(X_1 \lor X_2, Y)$ is a legitimate pair, then $(X_1, Y^{-X_2})$, $(X_2, Y^{-X_1})$ and $(Y^{-X_2} \lor Y^{-X_1}, Y)$ are legitimate pairs.*

**PROOF.** The proof of this lemma is pretty clear from $\gamma(X)$, $\gamma(Y)$ and the Position-Preservation Lemma. We have the following picture:

\[
\begin{array}{ccccc}
X_1 & y^{-X_2} & 1 & 1 & Y \\
X_2 & y^{-X_1} & 1 & 1 \\
Y & \mid & \mid & \mid \\
\end{array}
\]

Formally, for the legitimacy of $(X_1, Y^{-X_2})$, we demonstrate what is the result $Z^{-X_2}$ of deleting the letters in $\text{let}(X_2)$ from a subformsequence $Z$ of $Y$. We distinguish three cases depending on the form of $Z$. The first case is when $Z$
is of the form $Z_1y'Z_2$ for $y'$ an occurrence of $\lor$ in $Y$ such that $d(y') = y$ where $X$ is $X_1yX_2$; then $Z^{-X_2}$ is $Z_1$. In the second case, when $Z$ is of the form $Z_1y'Z_2$ such that $d(y') \neq y$, we have two subcases: if $d(y')$ is below $y$, then $Z^{-X_2}$ is $Z_1^{-X_2}$, and if $d(y')$ is above $y$, then $Z^{-X_2}$ is $Z_1 \lor Z_2^{-X_2}$. Finally, if $Z$ is of the form $Z_1 \land Z_2$, then $Z^{-X_2}$ is $Z_1^{-X_2} \land Z_2^{-X_2}$.

We define the functions $c'$ and $d'$ that make legitimate $(X_1,Y^{-X_1})$ in the following manner. First we define a partial map $\kappa_{Z,X_2}$ from the set of occurrences of $\land$ in a subformsequence $Z$ of $Y$ to the occurrences of $\land$ in $Z^{-X_2}$ by distinguishing the same three cases as above. When from the domain of $\kappa_{Z,X_2}$ we omit the elements for which $\kappa_{Z,X_2}$ is undefined, we obtain a bijection. The function $c'$ is defined by composing the restriction of $c$ with $\kappa_{Y,X_2}$. Next we define a map $\delta_{Z,X_2}$ from the set of occurrences of $\lor$ in $Z^{-X_2}$ to the set of occurrences of $\lor$ in $Z$ by distinguishing the same three cases. There is also an obvious partial map $\varepsilon$ from the set of occurrences of $\lor$ in $X_1 \lor X_2$ to the set of occurrences of $\lor$ in $X_1$. When from the domain of $\varepsilon$ we omit the elements for which $\varepsilon$ is undefined, we obtain a bijection. The function $d'$ is defined by composing $\delta_{Y,X_2}$ with $d$ and $\varepsilon$.

We proceed analogously to show that $(X_2, Y^{-X_1})$ is legitimate. To define the functions $e''$ and $d''$ that make legitimate $(Y^{-X_2} \lor Y^{-X_1}, Y)$ we use the inverses of the maps $\kappa_{Y,X_2}$ and $\kappa_{Y,X_1}$ for $e''$, and the maps $\delta_{Y,X_2}$ and $\delta_{Y,X_1}$ for $d''$.

We can prove analogously the following dual lemma, where $\lor$ is replaced by $\land$.

**Interpolation Lemma $\land$.** If $(X, Y_1 \land Y_2)$ is a legitimate pair, then $(X^{-Y_2}, Y_1)$, $(X^{-Y_1}, Y_2)$ and $(X, X^{-Y_2} \land X^{-Y_1})$ are legitimate pairs.

As a consequence of the foregoing results (cf. in particular the first case in the proof of the Interpolation Lemma $\lor$) we have the following lemma, and its analogue immediately below.

**Auxiliary Lemma $\lor$.** If $(X, Y_1yY_2)$ is a legitimate pair for $y$ an occurrence of $\lor$, then $X$ is of the form $X_1d(y)X_2$ and $(X_i, Y_i)$ is a legitimate pair for every $i \in \{1, 2\}$.

**Auxiliary Lemma $\land$.** If $(X_1xX_2, Y)$ is a legitimate pair for $x$ an occurrence of $\land$, then $Y$ is of the form $Y_1c(x)Y_2$ and $(X_i, Y_i)$ is a legitimate pair for every $i \in \{1, 2\}$.

Then we can prove the following.

**Theorem.** There is an arrow $f : X \to Y$ of $\text{AC}^{k \ast t}$ iff $(X, Y)$ is a legitimate pair.

**Proof.** We have already established this theorem from left to right. From right to left we proceed by induction on the sum $n$ of the number of letters in
X and the number of occurrences of $\wedge$ in $X$. If $n = 1$, then $X$ and $Y$ are the same letter $p$.

If $n > 1$ and $Y$ is $Y_1 \lor Y_2$ or $X$ is $X_1 \land X_2$, then we apply the Auxiliary Lemmata $\lor$ and $\land$ above.

If $n > 1$ and $X$ is $X_1 \lor X_2$, while $Y$ is $Y' \lor Y''$, for $x$ an occurrence of $\wedge$, then we proceed as follows. By applying the Interpolation Lemma $\lor$ we obtain that $(X_i, Y_i - X_3^{i-1})$, for every $i \in \{1, 2\}$, and $(Y_i - X_2 \lor Y_1, Y)$ are legitimate pairs.

By the induction hypothesis, we obtain the arrows $f_i : X_i \to Y_i - X_3^{i-1}$, and hence the arrow $f_1 \lor f_2 : X_1 \lor X_2 \to Y_i - X_2 \lor Y_1$.

If either $f_1$ or $f_2$ is not an identity arrow, then $Y_i - X_2 \lor Y_1$ has at least one occurrence of $\wedge$ less than $X$, and we may apply the induction hypothesis to obtain an arrow $g : Y_i - X_2 \lor Y_1 \to Y$. Hence we have $g \circ (f_1 \lor f_2) : X \to Y$.

If both $f_1$ and $f_2$ are identity arrows, then $Y_i - X_3^{i-1}$ is of the form $Y_i' x_i Y_i''$ for $x_i$ an occurrence of $\wedge$ such that $\kappa_{Y_i X_3^{i-1}}(x) = x_i$, where $\kappa_{Y_i X_3^{i-1}}$ is defined as in the proof of the Interpolation Lemma $\lor$. We have the arrow

$$c_{Y_1', Y_1'', Y_2', Y_2''} : (Y'_1 \lor Y''_1) \lor (Y'_2 \lor Y''_2) \to (Y'_1 \lor Y'_2) \lor (Y''_1 \lor Y''_2),$$

and we have to ascertain that $(Y'_1 \lor Y'_2, Y')$ is a legitimate pair for every $j \in \{', ''\}$. This follows from the Interpolation Lemma $\lor$.

In terms of rectangular grids we have

$$
\begin{array}{c|c|c}
Y'_1 & Y''_1 & Y' \\
\hline
Y'_2 & Y''_2 & Y'' \\
\end{array}
$$

By applying the induction hypothesis we obtain the arrows $f_j : Y'_j \lor Y''_j \to Y'$ for $j \in \{', ''\}$. Hence we have $(f' \lor f'') \circ c_{Y_1', Y_1'', Y_2', Y_2''} : X \to Y$.

11 Biassocciative intermuting coherence

Suppose $(X, Y)$ is a legitimate pair, $x$ is an occurrence of $\wedge$ in $Y$, and $y$ is an occurrence of $\lor$ in $X$. We say that $(x, y)$ is a crossing of $(X, Y)$ when the sets $R_x \cap B_y, L_x \cap B_y, R_x \cap T_y$ and $L_x \cap T_y$ are all nonempty.

If these four sets are nonempty, then they are singletons. This is because for every pair of letters $p$ and $q$ in a form sequence there is a minimal subformsequence in which both $p$ and $q$ occur, and this subformsequence is either a conjunction or a disjunction, but not both.
The arrow terms of $\text{AC}^k$ where indices are replaced by form sequences are the arrow terms of $\text{AC}^{k\text{st}}$ (see [10], Section 3.2). In these arrow terms of $\text{AC}^{k\text{st}}$ we may replace all the $b$-arrows by identity arrows.

For $f : X \to Y$ an arrow term of $\text{AC}^{k\text{st}}$ there is a natural one-to-one correspondence between the occurrences of $c^k$ in $f$ and the crossings of $(X, Y)$. If the representatives for the rectangular grids $\gamma(X)$ and $\gamma(Y)$ are well chosen, then $c^k_{S,T,U,V}$ in $f$ corresponds to the crossing in the following two fragments of these grids:

$$
\begin{array}{c|c}
S & T \\
\hline
y & U \\
\hline & V
\end{array}
\quad
\begin{array}{c|c}
S & T \\
\hline x & U \\
\hline & V
\end{array}
$$

These two pictures explain why we call $(x, y)$ a crossing. If these representatives are not well chosen, we may have, for example,

$$
\begin{array}{c|c}
S & T \\
\hline
y & U \\
\hline & V
\end{array}
\quad
\begin{array}{c|c}
S & T \\
\hline x & U \\
\hline & V
\end{array}
$$

We say that $(x, y)$ is the pair of coordinates of $c^k_{S,T,U,V}$ in $(X, Y)$. For a $c^k$-term $f$, we say that the coordinates of $f$ are the coordinates of the head of $f$. (The notions of $c^k$-term and of its head are defined in Section 2.)

If $f, g : X \to Y$ are two arrow terms of $\text{AC}^{k\text{st}}$, then there is one-to-one correspondence between the occurrences of $c^k$ in $f$ and the occurrences of $c^k$ in $g$, such that the occurrences with the same coordinates correspond to each other.

Let $f_2 \circ f_1$ be a composition of two $c^k$-terms that is not equal to an arrow term of the form $f'_i \circ f'_i$ for $f'_i$, where $i \in \{1, 2\}$, a $c^k$-term with the same coordinates as $f_1$. Then we say that $f_2$ is checked by $f_1$.

It is straightforward to verify the following lemma by appealing to bifunctorial and naturality equations.

**Checking Lemma.** Let $f_2 \circ f_1$ be a composition of two $c^k$-terms such that $f_i$, where $i \in \{1, 2\}$, has the coordinates $(x_i, y_i)$ with $x_1 \neq x_2$ and $y_1 \neq y_2$. Then $f_2$ is not checked by $f_1$.

As a corollary of this lemma we have that if $f_2$ is checked by $f_1$ and $(x_i, y_i)$ are the coordinates of $f_i$, then either $x_1 = x_2$, and we say that $f_2$ is vertically checked by $f_1$, or $y_1 = y_2$ and we say that $f_2$ is horizontally checked by $f_1.$
For $f_n \circ \ldots \circ f_1$, with $n \geq 1$, a composition of $c^k$-terms, we say that $f_i$ is **horizontally blocked** in this composition when either $i = 1$ or $f_i$ is horizontally checked by $f_{i-1}$.

We say that a $\beta$-term $f$ is **conjunctively headed** when its head $\beta$ occurs in a subterm of $f$ of the form $1_X \land \beta$ or $\beta \land 1_X$. We are interested in this notion when $\beta$ is $c^k$. We can prove the following series of lemmata, leading to the theorem below.

**Horizontal Checking Lemma.** If $f_2 \circ f_1$ is a composition of two $c^k$-terms such that $f_2$ is conjunctively headed, then $f_2$ is not horizontally checked by $f_1$.

**Proof.** Suppose $f_2$ is conjunctively headed, and the coordinates of $f_i$, for $i \in \{1, 2\}$, are $(x_i, y_i)$. If $y_1 \neq y_2$, then $f_2$ is not horizontally checked by $f_1$. If $y_1 = y_2 = y$, then we have to consider many cases, for which the following three fragments of rectangular grids corresponding to the source of $f_1$ are typical:

\[
\begin{array}{c|c|c}
\hline
y & x_2 & x_1 \\
\hline
\end{array}
\quad
\begin{array}{c|c|c}
\hline
y & x_2 & x_1 \\
\hline
\end{array}
\quad
\begin{array}{c|c|c}
\hline
y & x_2 & x_1 \\
\hline
\end{array}
\]

We can always apply the equation (ψb) of Section 4. Note that in the second and third case we could not have

\[
\begin{array}{c|c|c}
\hline
y & x_1 & x_2 \\
\hline
\end{array}
\quad
\begin{array}{c|c|c}
\hline
y & x_1 & x_2 \\
\hline
\end{array}
\]

because then the source of $f_1$ would not be a form sequence.

**Horizontal Blocking Corollary.** Let $f_n \circ \ldots \circ f_1 : X_1 y X_2 \rightarrow Y$, for $n \geq 1$ and $y$ an occurrence of $\forall$, be a composition of $c^k$-terms, and let $i \in \{1, \ldots, n\}$ be such that the second coordinate of $f_i$ is $y$. If $f_i$ is conjunctively headed, then $f_i$ is not horizontally blocked in $f_n \circ \ldots \circ f_1$.

**Vertical Checking Lemma.** If $f_2 \circ f_1$ is a composition of two $c^k$-terms such that $f_1$ is not conjunctively headed, then $f_2$ is not vertically checked by $f_1$.

**Proof.** Suppose $f_1$ is not conjunctively headed, and the coordinates of $f_i$, for $i \in \{1, 2\}$, are $(x_i, y_i)$. If $x_1 \neq x_2$, then $f_2$ is not vertically checked by $f_1$. If $x_1 = x_2 = x$, then we have a situation as in the following fragment of a rectangular grid corresponding to the source of $f_1$: 

\[
\begin{array}{c|c|c}
\hline
x_1 & x_2 \\
\hline
\end{array}
\quad
\begin{array}{c|c|c}
\hline
x_1 & x_2 \\
\hline
\end{array}
\]
We can always apply the equation ($\psi \beta$) of Section 4. \hfill \dagger

**Permuting Lemma.** Let $g = f : X_1 y X_2 \to Y$, for $y$ an occurrence of $\lor$, be such that $g$ is a $c^k$-term whose second coordinate is different from $y$, and every occurrence of $c^k$ in $f$ has the second coordinate $y$. Then $g = f = f' \circ g'$ where $g'$ has the same coordinates as $g$.

**Proof.** Let $f_n \circ \ldots \circ f_1 \circ 1_{X_1 y X_2}$ be a developed arrow term equal to $f$ (see Section 2 for the notion of developed arrow term). We proceed by induction on $n$. If $n = 0$, then $g \circ 1_{X_1 y X_2} = 1_{X_1 y X_2} \circ g$ and $g' = g$.

Let $n > 0$. Suppose the coordinates of $g$ are $(x, y_1)$. If there is an $i \in \{1, \ldots, n\}$ such that $f_i$ has the coordinates $(x, y)$, then we can assume that $f_n \circ \ldots \circ f_1$ is such that $f_i$ is horizontally blocked in this composition. Suppose $g$ is checked by $f_n$. Then, since $y_1 \neq y$, the $c^k$-term $g$ must be vertically checked by $f_n$. Hence the coordinates of $f_n$ are $(x, y)$. By the Vertical Checking Lemma, $f_n$ is conjunctively headed. By the Horizontal Blocking Corollary, $f_n$ is not horizontally blocked in the composition $f_n \circ \ldots \circ f_1$, which contradicts the assumption. So, $g = f_n = f_n' \circ g'$ where $g'$ has the same coordinates as $g$, and we may apply the induction hypothesis. \hfill \dagger

**Theorem.** If $f', f : X \to Y$ are arrows of $AC^{kst}$, then $f = f'$.

**Proof.** As in the proof of the Theorem of the preceding section, we proceed by induction on the sum $n$ of the number of letters in $X$ and the number of occurrences of $\land$ in $X$. If $n = 1$, then $X = Y = p$, and $1_p : p \to p$ is the unique arrow from $p$ to $p$.

If $X$ is $X_1 \land X_2$, then $Y$ must be of the form $Y_1 \land Y_2$, and it is easy to prove that every $f : X \to Y$ is equal to an arrow $f_1 \land f_2$ for $f_i : X_i \to Y_i$, for $i \in \{1, 2\}$. Then we may apply the induction hypothesis to $f_i$. We proceed analogously when $Y$ is $Y_1 \lor Y_2$.

Suppose now that $X$ is $X_1 y X_2$ for $y$ an occurrence of $\lor$, and $Y$ is $Y_1 x Y_2$ for $x$ an occurrence of $\land$. By the Permuting Lemma, we may conclude that $f = h \circ g$ where the second coordinate of every occurrence of $c^k$ in $h$ is $y$, and there is no occurrence of $c^k$ in $g$ with the second coordinate $y$. Then it is easy to see that $g$ must be equal to an arrow $g_1 \lor g_2 : X_1 y X_2 \to Y_1^x X_2 \lor Y_1^x Y_1$ (cf. the Interpolation Lemma in the preceding section). If $g_1 \lor g_2$ is not an identity arrow, then we may apply the induction hypothesis to $g_i$, for $i \in \{1, 2\}$, and $h$. This is because the “interpolated” $Y_1^x X_2 \lor Y_1^x Y_1$ is uniquely determined by $X$.
and $Y$, and is hence the same for $f$ and $f'$, and because in the source of $h$ there is at least one occurrence of $\land$ less than in $X$. If, on the other hand, $g_1 \lor g_2$ is the identity arrow $1_{X_1 y X_2}$, then $h = h_m \circ \ldots \circ h_1$, for $m \geq 1$, where $h_m \circ \ldots \circ h_1$ is a composition of $c^k$-terms, and for some $j \in \{1, \ldots, m\}$ we have that $h_j$ has the coordinates $(x, y)$. Then, since $x$ is the main connective of $Y$, we have that $h_j$ is either $h_1$ or it is conjunctively headed. In the second case, by the Horizontal Checking Lemma we may again assume that $h_j$ is $h_1$. Since $h_1$ is of the type $X_1 y X_2 \rightarrow Y'_1 \land Y'_2$ (it is a $c^k$-term), we may apply the induction hypothesis to $h_m \circ \ldots \circ h_2: Y'_1 \land Y'_2 \rightarrow Y_1 x Y_2$, in whose source there is one occurrence of $\land$ less than in $X$. 

The proof of the foregoing theorem suggests how to construct a unique normal form for arrow terms.

The Theorem we have proved above does not amount to showing that $\text{AC}^{k \text{st}}$ is a preorder because it is formulated with the assumption that $X$ and $Y$ are diversified. We can however lift this assumption. To achieve that we need the following.

We say that the sequence of letters $p_1 \ldots p_n$ for $n \geq 1$ is the left border of a form sequence $X$ when \{p_, \ldots, p_n\} is the set of all the left-border letters of $X$ and for every $i \in \{1, \ldots, n-1\}$ we have $\frac{p_i}{p_{i+1}}$ (see the preceding section for the definition of left-border letter and of the relation $\leftarrow$). The left border of $X$ appears indeed as the left border of the rectangular grid $\gamma(X)$. We could define analogously the right border, the top border and the bottom border. We can then prove the following.

**Border Lemma.** If $(X, Y)$ is a legitimate pair, then $p_1 \ldots p_n$ is the left border of $X$ iff $p_1 \ldots p_n$ is the left border of $Y$.

**Proof.** We establish first that for a legitimate pair $(X, Y)$ we have that $p$ is a left-border letter of $X$ iff $p$ is a left-border letter of $Y$. From left to right we appeal to the fact that the function $c$ of Condition $\land$ is onto. For the other direction we appeal to the fact that $c$ is a function. The matter becomes clear by considering the rectangular grids $\gamma(X)$ and $\gamma(Y)$. So \{p_, \ldots, p_n\} is the set of left-border letters of $X$ iff $\{p_1, \ldots, p_n\}$ is the set of left-border letters of $Y$.

Suppose $p_1 \ldots p_n$ is the left border of $Y$, and suppose $n > 1$. Then by Condition $\lor$ from $\frac{p_i}{p_{i+1}}$ in $Y$ we may infer $\frac{p_i}{p_{i+1}}$ in $X$. So $p_1 \ldots p_n$ is the left border of $X$.

The same lemma holds when we replace “left” by “right”, “top” and “bottom”. For “top” and “bottom” we appeal to the function $d$ of Condition $\lor$ in the first part of the proof, and to Condition $\land$ in the second part. We can prove the following.

**Uniqueness Lemma.** If $f_1: X \rightarrow Y_1$ and $f_2: X \rightarrow Y_2$ are arrow terms of $\text{AC}^{k \text{st}}$
such that $Y_1$ and $Y_2$ are substitution instances of each other, then $Y_1$ and $Y_2$ are the same form sequence.

Proof. The form sequences $Y_1$ and $Y_2$ are substitution instances of each other iff their rectangular grids $\gamma(Y_1)$ and $\gamma(Y_2)$ are the same when letters are omitted. We will show that $X$ imposes the letters to be put in this grid. So $Y_1$ and $Y_2$ will be the same form sequence.

Since $(X, Y_1)$ and $(X, Y_2)$ are legitimate pairs, by the Border Lemma, the left border $p_1 \ldots p_n$ of $X$ is also the left border of $Y_1$ and $Y_2$. If this left border is also the right border, then we are done. Otherwise, there must be an occurrence $x$ of $\land$ in $Y_1$ and $Y_2$ such that the sequence $p_1 \ldots p_{j-1}R_xp_{j+m} \ldots p_n$ for some $m \geq 1$. Then, by Condition $\land$, for some $k \geq 1$ there are occurrences $x_1, \ldots, x_k$ of $\land$ in $X$ such that $R_x = R_{x_1} \ldots R_{x_k}$. So $Y_1$ and $Y_2$ must be such that $L_x = L_{x_1} \ldots L_{x_k}$.

If the sequence $p_1 \ldots p_{j-1}L_xp_{j+m} \ldots p_n$ is the right border of $Y_1$ and $Y_2$, then we are done. Otherwise, there is an occurrence $x'$ of $\land$ in $Y_1$ and $Y_2$ such that this sequence is of the form

$$p'_{1} \ldots p'_{j'-1}R_xp'_{j'+m'} \ldots p'_{n'},$$

and we proceed as before until we are done. $\dashv$

We are now ready to prove the following.

**AC$^{kst}$ Coherence.** The category $\text{AC}^{kst}$ is a preorder.

Proof. For $f_1, f_2: X \rightarrow Y$ arrows of $\text{AC}^{kst}$ with $X$ and $Y$ not diversified, we find $f'_1: X' \rightarrow Y_1$ and $f'_2: X' \rightarrow Y_2$ such that $X'$, and hence also $Y_1$ and $Y_2$, are diversified, while $f_1$ and $f_2$ are substitution instances of $f'_1$ and $f'_2$ respectively. Then by the Uniqueness Lemma we obtain that $Y_1 = Y_2$, and by the Theorem established in this section we have that $f'_1 = f'_2$. So $f_1 = f_2$. $\dashv$

As a consequence of this and of the equivalence of the categories $\text{AC}^k$ and $\text{AC}^{kst}$ we obtain the following.

**Biassociative Intermuting Coherence.** The category $\text{AC}^k$ is a preorder.

12 Bimonoidal intermuting categories

In this section we add unit objects to the biassociative intermuting categories of the preceding two sections, and prove a restricted coherence result for the ensuing notion (more general than the notion of two-fold monoidal category of
A bimonoidal intermuting category is a $\kappa$-normal bimonoidal category $\langle A, \wedge, \vee, T, \bot \rangle$ (see Section 6) such that $\langle A, \wedge, \vee \rangle$ is a biaffine intermuting category (see Section 10), and, moreover, the equations $(c^k\delta\sigma)$ and $(c^k\kappa)$ (see Section 4) are satisfied. This means that besides the assumptions for bimonoidal categories (see Section 2) we have the natural transformation $c^k$, the isomorphisms $\tilde{w}_-^r : \bot \to \bot \wedge \bot$ and $\tilde{w}_+^r : T \vee T \to T$, and the arrow $\kappa : \bot \to T$, for which we assume the equations $(c^k\beta), (c^k\delta\sigma), (\xi\beta), (\kappa\delta\sigma)$ and $(c^k\kappa)$ of Section 4.

Let $\mathbf{AC}^{k\emptyset}_{\top, \bot}$ and $\mathbf{AC}^{k\emptyset}_{\top, \bot}$ be the free bimonoidal intermuting categories generated respectively by a nonempty set of objects and the empty set of objects. In $\mathbf{AC}^{k\emptyset}_{\top, \bot}$ there are no arrows of the type $A \to B$ such that in one of $A$ and $B$ there are no letters and in the other there are. The coherence result we will prove below for $\mathbf{AC}^{k\emptyset}_{\top, \bot}$ will enable us to ascertain that $\mathbf{AC}^{k\emptyset}_{\top, \bot}$ is a full subcategory of $\mathbf{AC}^{k}_{\top, \bot}$.

According to what we had for the category $\mathbf{N}_{\top, \bot}$ in Section 5, there is an isomorphism of the type $A \to \nu(A)$ in $\mathbf{AC}^{k}_{\top, \bot}$. In $\mathbf{AC}^{k\emptyset}_{\top, \bot}$ the formula $\nu(A)$ is either $T$ or $\bot$. We can prove the following.

**AC**$^{k\emptyset}_{\top, \bot}$ Coherence. The category $\mathbf{AC}^{k\emptyset}_{\top, \bot}$ is a preorder.

**Proof.** We enlarge the proof of $\mathbf{KA}^{0}_{\top, \bot}$ Coherence of Section 7. We now have additional equations obtained from the equations $(c^k\delta\sigma)$ and $(c^k\kappa)$, which enable us to eliminate every occurrence of $c^k$, together with equations derived from the naturality equations for $c^k$.

We say that an object $A$ of $\mathbf{AC}^{k}_{\top, \bot}$ is $\zeta$-pure for $\zeta \in \{T, \bot\}$ when there is no occurrence of $\zeta$ in $\nu(A)$. For $(\zeta, \xi) \in \{(\bot, \wedge), (T, \vee)\}$, it is easy to see that $A$ is not $\zeta$-pure iff either $\nu(A)$ is $\zeta$ or there is a subformula of $A$ of the form $B_1 \xi B_2$ such that for some $i \in \{1, 2\}$ we have that $\nu(B_i)$ is $\zeta$ and a letter occurs in $B_3-i$. A formula is called pure when it is both $\bot$-pure and $T$-pure. We can prove the following.

**Purity Lemma.** Let $f : A \to B$ be an arrow of $\mathbf{AC}^{k}_{\top, \bot}$. If $A$ is $\bot$-pure, then $B$ is $\bot$-pure, and if $B$ is $T$-pure, then $A$ is $T$-pure.

**Proof.** By the Development Lemma (see Section 2), it is enough to verify the lemma for $f : A \to B$ a $\beta$-term of $\mathbf{AC}^{k}_{\top, \bot}$, where $\beta$ is $\xi \rightarrow, \xi \leftarrow, \xi \rightarrow, \xi \leftarrow, \zeta \rightarrow, \zeta \leftarrow, \zeta \rightarrow, \zeta \leftarrow, \kappa$ or $c^k$. The only interesting case arises when $f : A \to B$ is a $c^k$-term.

Suppose $B$ is not $\bot$-pure. If $\nu(B)$ is $\bot$, then we can easily conclude that $\nu(A)$ is $\bot$ too. Suppose, on the other hand, that there is a subformula of $B$ of the form $K \times D$ or $D \times K$ for $x$ an occurrence of $\wedge$ such that $\nu(K)$ is $\bot$ and a letter occurs in $D$. Suppose the head of $f$ is

$$c^k_{E,F,G,H} : (E \wedge F) \vee (G \wedge H) \to (E \vee G) \wedge (F \vee H).$$
The only interesting case is when $x$ is the main $\land$ of $(E \lor G) \land (F \lor H)$. Suppose $K$ is $E \lor G$. Then $\nu(E) = \nu(G) = \bot$, and there is a letter in either $F$ or $H$. So $A$ is not $\bot$-pure. We reason analogously in other cases.

As an immediate corollary of this lemma we have the following.

**Purity Corollary.** If $f : A \to B$ and $g : B \to C$ are two arrows of $\mathbf{AC}^k_{\top, \bot}$ such that both $A$ and $C$ are pure, then $B$ is pure.

We say that a formula is constant-free when there is no occurrence of $\top$ or $\bot$ in it. For $\xi \in \{\land, \lor\}$, the arrow terms $\hat{\xi}^\rightarrow_{A,B,C}$, $\hat{\xi}^\leftarrow_{A,B,C}$ and $\hat{\xi}^k_{A,B,C,D}$ are constant-free when their indices $A$, $B$, $C$ and $D$ are constant-free. We can then prove the following.

**Constant-Free Indices Lemma.** If $f : A \to B$ is an arrow of $\mathbf{AC}^k_{\top, \bot}$ such that $A$ and $B$ are pure, then there is an arrow term $f' : A \to B$ of $\mathbf{AC}^k_{\top, \bot}$ in which every occurrence of $\hat{\xi}^\rightarrow$, $\hat{\xi}^\leftarrow$ and $\hat{\xi}^k$ is in a constant-free arrow term, $\kappa$ does not occur in $f'$ and $f = f'$ in $\mathbf{AC}^k_{\top, \bot}$.

**Proof.** We call the assumption that $A$ and $B$ are pure the purity assumption. By the Development Lemma (see Section 2) and the Purity Corollary, it is enough to verify the lemma for $f : A \to B$ a $\beta$-term of $\mathbf{AC}^k_{\top, \bot}$. By reasoning as in the proof of $\mathbf{KA}^0_{\top, \bot}$ Coherence in Section 7, we may assume that every index of every $\beta$-term, for $\beta$ being $\hat{\xi}^\rightarrow$, $\hat{\xi}^\leftarrow$ or $\hat{\xi}^k$, is $\nu(D)$ for some $D$; so each of these indices is either $\top$ or $\bot$, or it is constant-free.

Suppose $\beta$ is $\hat{\xi}^\rightarrow$. If any of its indices is $\top$, then we eliminate $\hat{\xi}^\rightarrow$ by relying on the equations mentioned in the proof of $\mathbf{KA}^0_{\top, \bot}$ Coherence. If all its indices are $\bot$, then we eliminate $\hat{\xi}^\rightarrow$ by relying on the $(w;b)$ equation $(\hat{\xi}^\rightarrow; \hat{\xi}^\leftarrow)$ of Section 4. If any of these indices were $\bot$ without all of them being such, then this would contradict the purity assumption. We proceed in such a manner for all $\hat{\xi}^\rightarrow$-terms and $\hat{\xi}^\leftarrow$-terms.

Suppose $\beta$ is $\hat{\xi}^k$. Then we have the following cases. If only one of its indices is $\top$ or $\bot$, while the remaining three indices are constant-free, then this contradicts the purity assumption. If two of its indices are $\top$ or $\bot$, while the remaining two indices are constant-free, then we may either apply the equations $(\hat{\xi}^k \delta \sigma)$ to eliminate $\hat{\xi}^k$, or we would contradict again the purity assumption. We have an analogous situation when three indices of $\hat{\xi}^k$ are $\top$ or $\bot$, while the remaining one is constant-free. The last case is when all the indices of $\hat{\xi}^k$ are $\bot$. Then we apply either the equations $(\hat{\xi}^k \delta \sigma)$ to eliminate $\hat{\xi}^k$, or the equations $(\hat{\xi}^k \kappa)$ to reduce $\hat{\xi}^k$ to $\kappa$. If we apply $(\hat{\xi}^k \kappa)$, then we obtain a new occurrence of $\kappa$, with which we deal as in the last part of the proof, into which we go now.

Suppose we have a $\kappa$-term. By the purity assumption, this term cannot be
just $\kappa$. So it has a subterm of the form $1_E \kappa$ or $\kappa 1_E$. We may assume as above that $E$ is $\nu(D)$ for some $D$. No letter can occur in $E$, because this would contradict the purity assumption. So $E$ is $\top$ or $\bot$. If $(\xi, E) \in \{(\land, \bot), (\lor, \top)\}$, then we apply the equations $(\kappa \delta \sigma)$ to eliminate $\kappa$. If $(\xi, E) \in \{(\land, \top), (\lor, \bot)\}$, then we apply the equations mentioned in the proof of $K_{\top, \bot}^k$ Coherence in Section 6, which are consequences of naturality equations. These equations do not eliminate $\kappa$, but they replace a $\kappa$-term of greater complexity by a $\kappa$-term of lesser complexity. This enables us to proceed by induction. 

We prove now the main result of this section.

**Restricted Bimonoidal Intermuting Coherence.** If $f, g: A \to B$ are arrows of $\mathbf{AC}_{\top, \bot}^k$ such that both $A$ and $B$ are pure, or no letter occurs in them, then $f = g$ in $\mathbf{AC}_{\top, \bot}^k$.

**Proof.** When no letter occurs in $A$ and $B$, we apply $\mathbf{AC}_{\top, \bot}^k$ Coherence. When $A$ and $B$ are pure, we proceed as follows. We replace first $f$ and $g$ by $f'$ and $g'$ respectively, which satisfy the conditions specified in the Constant-Free Indices Lemma above; we have $f = f'$ and $g = g'$ in $\mathbf{AC}_{\top, \bot}^k$.

As we applied Bissociative Coherence to obtain $\mathbf{AC}_{\top, \bot}^{k_{st}}$ in Section 10 (and as we did previously in the proof of Normal Bimonoidal Coherence in Section 7), we can apply Normal Biunital Coherence of Section 5 to obtain a strictified category $\mathbf{AC}_{\top, \bot}^{k_{st}}$ equivalent to $\mathbf{AC}_{\top, \bot}^k$, where the $\delta$-$\sigma$-arrows (see Section 2) and the $w$-arrows (see Section 5) are identity arrows. So the arrows $f'$ and $g'$ are mapped by the functor underlying this equivalence of categories into the arrows $f''$, $g'': \nu(A) \to \nu(B)$ of $\mathbf{AC}_{\top, \bot}^{k_{st}}$.

The arrows $f''$ and $g''$ may be represented by two arrow terms of $\mathbf{AC}^k$, and, by Bissociative Intermuting Coherence of the preceding section, we may conclude that $f'' = g''$ in $\mathbf{AC}^k$, and hence also in $\mathbf{AC}_{\top, \bot}^{k_{st}}$. So, by equivalence of categories, $f' = g'$, and hence also $f = g$ in $\mathbf{AC}_{\top, \bot}^k$. 

The restriction imposed by this restricted coherence result is of the same kind as the restriction Kelly and Mac Lane had for their restricted coherence result for symmetric monoidal closed categories (see [18] and [11], end of Section 3.1, or [12], Section 8).

**13 Bimonoidal intermuting categories and two-fold loop spaces**

In this section we consider the relationship of our results to the paper [3], mentioned in the introduction. We will just summarize matters and will not go into details, known either from [3] or other references.
Restricted Bimonoidal Intermuting Coherence, which we have proved in the preceding section, enables us to strengthen up to a point Theorem 2.1 of [3], one of the main results of that paper. This theorem is about the notion of \( n \)-fold monoidal category, which for \( n = 2 \) is a bimonoidal intermuting category in which the \( b \)-arrows, the \( \delta \)-\( \sigma \)-arrows, the \( w \)-arrows and the arrow \( \kappa \) are identity arrows. In two-fold monoidal categories \( \top \) and \( \bot \) coincide. The theorem also relies on the notion of lax functor from a category to the category \( \text{Cat} \) of categories (see [23]; the notion originates in [4]) and in the topologists’ notion of simplicial category, called \( \Delta \) (which in [22], Section VII.5, is called \( \Delta^+ \)). This theorem is formulated as follows.

**Theorem 2.1** [3]. An \( n \)-fold monoidal category \( C \) determines a lax functor \( C_{*,\ldots,*} : \Delta^{op} \times \Delta^{op} \times \cdots \times \Delta^{op} \to \text{Cat} \) such that \( C_{p_1,p_2,\ldots,p_n} = C^{p_1,p_2,\ldots,p_n} \).

From this theorem one obtains immediately the main result of [3], which says that the group completion of the nerve of an \( n \)-fold monoidal category is an \( n \)-fold loop space. We can however prove the following theorem for strict bimonoidal intermuting categories, in which the \( b \)-arrows and the \( \delta \)-\( \sigma \)-arrows are identity arrows.

**Theorem.** A strict bimonoidal intermuting category \( C \) determines a lax functor \( C_{*,\ldots,*} : \Delta^{op} \times \Delta^{op} \to \text{Cat} \) such that \( C_{p_1,p_2} = C^{p_1,p_2} \).

**Proof.** We proceed as in the proof of Theorem 2.1 in [3], and rely on the notions introduced in that paper. For \( \gamma = (\alpha, \beta) \) a pair of arrows of \( \Delta^{op} \) such that \( \alpha : m \to m' \) and \( \beta : n \to n' \), we denote by \( \gamma^* \) the functor \( C_{\alpha, \beta} : C^m \to C^{m'} \) defined as in [3]. For \( * \) one can easily prove the following.

For \( k \geq 1 \) and \( i \in \{1, \ldots, k\} \), let \( \gamma_i = (\alpha_i, \beta_i) \) be a pair of arrows of \( \Delta^{op} \) such that \( \alpha_i : m_{i-1} \to m_i \) and \( \beta_i : n_{i-1} \to n_i \). Then \( \gamma_1 \circ \cdots \circ \gamma_k \) is a functor from \( C^{m_0 \cdot m_1} \) to \( C^{m_k \cdot n_k} \).

For \( \vec{p} \) being \( p_1^1, \ldots, p_{m_0}^1, \ldots, p_1^{m_0}, \ldots, p_{m_0}^{m_0} \) and

\[
\gamma_1 \circ \cdots \circ \gamma_k (\vec{p}) = (A_1^1, \ldots, A_{m_k}^1, \ldots, A_1^{m_0}, \ldots, A_{m_k}^{m_0}),
\]

where we treat every object \( p_j^i \) of \( C \) as a letter, we have that if \( \nu(A_j^i) = \top \), then for every \( l \in \{1, \ldots, m_k\} \) no letter occurs in \( A_j^l \), and for every \( l \in \{1, \ldots, m_k\} \) the elements of \( \{\nu(A_1^l), \ldots, \nu(A_{m_k}^l)\} \) different from \( \top \) are all \( \bot \) or all constant-free. As a corollary we have that every \( A_j^l \) is either pure or no letter occurs in it.

Hence, by our Restricted Bimonoidal Intermuting Coherence of the preceding section, we conclude that any two \( m_3 \cdot n_3 \)-tuples of “canonical” arrows of \( C \) from \( \gamma_3 \circ \gamma_2 \circ \gamma_1 (\vec{p}) \) to \( (\gamma_3 \circ \gamma_2 \circ \gamma_1)^*(\vec{p}) \) are equal, which is sufficient to show that \( C_{*,\ldots,*} \) is a lax functor.

\( \diamond \)

We leave open the question whether this theorem could be generalized to cover a notion of \( n \)-fold intermuting category for \( n \geq 2 \), which generalizes the notion
of \( n \)-fold monoidal category as our notion of bimonoidal intermuting category generalizes the notion of two-fold monoidal category. As we said in the introduction, we suppose that this can be achieved by relying on the technique of the next section.

Our strengthening of Theorem 2.1 of [3] should be considered in the light of the statement of the main result of [3], made in the first sentence of the abstract and in the last paragraph of the Introduction. This is the statement that for all \( n \) the notion of \( n \)-fold monoidal category corresponds precisely to the notion of \( n \)-fold loop space. There are interesting categories that are not two-fold monoidal, but which, according to our strengthening, give rise to a two-fold loop space. The dicartesian categories of [10] (Section 9.6), which are categories with all finite products and coproducts (including the empty ones) and with \( w \)-isomorphisms added, are such. A concrete dicartesian category obtained by extending the category of pointed sets with the empty set is described in [10] (Section 9.7).

Bimonoidal intermuting categories do not make like two-fold monoidal categories assumptions that preclude a logical interpretation. Such is the assumption that the two unit objects, \( \top \) and \( \bot \), are isomorphic, which delivers the arrows of the type \( A \lor B \to A \land B \). The assumptions of dicartesian categories can be naturally interpreted in conjunctive-disjunctive logic, including the constants \( \top \) and \( \bot \).

Dicartesian categories are lattice categories, in the sense of [10] (Section 9.4). We will examine lattice categories in Section 15 below, and we will show that every such category has as its fragment a symmetric biassociative intermuting structure, which we define in the next section.

Our approach differs from [3] also in the following respect. Our Biassociative Intermuting Coherence of Section 11 states that \( AC^k \) is a preorder. For the category of [3] closest to \( AC^k \) one cannot obtain this result, but only that it is what we call in the next section a diversified preorder, which is a result like the Theorem of Section 11, where a diversification proviso is involved (see Section 10).

## 14 Symmetric biassociative intermuting categories

In this section we add symmetry, i.e. natural commutativity isomorphisms for \( \land \) and \( \lor \), to the biassociative intermuting categories of Section 10, and prove an appropriate coherence result for the ensuing notion.

A symmetric biassociative category is a biassociative category \( \langle A, \land, \lor \rangle \) (see Section 2) in which there are two natural self-inverse isomorphisms \( \xi \), for \( \xi \in \{\land, \lor\} \), with the following components in \( A \):

\[
\xi_{A,B}: A \xi B \to B \xi A,
\]
which satisfy Mac Lane’s hexagonal equations:

\[
\xi_{A,B,C}^{-1} = b_{B,C,A}^{-1} \circ (1_B \xi \xi_{A,C}) \circ b_{B,A,C}^{-1} \circ (\xi_{A,B} \xi 1_C) \circ b_{A,B,C}^{-1}.
\]

We call the arrows \(\hat{c}\) and \(\check{c}\) collectively \(c\)-arrows. Let \(S\) be the free symmetric biassociative category generated by a set of objects.

We say that a formula is \textit{diversified} when every letter occurs in it at most once. (We already had diversified form sequences in Section 10). We say that a category whose objects are formulae is a \textit{diversified preorder} when for every pair of arrows \(f, g: A \rightarrow B\) such that \(A\) and \(B\) are diversified we have \(f = g\) in this category. In [10] (Section 6.3) one can find a proof, based on Mac Lane’s symmetric monoidal coherence, that \(S\) is a diversified preorder. We call this fact Symmetric Biassociative Coherence. In [10] we call by the same name an equivalent result about the existence of a faithful functor from the category \(S\) to the category whose objects are finite ordinals and whose arrows are bijections.

A \textit{symmetric biassociative intermuting} category is a symmetric biassociative and biassociative intermuting category (see Section 10) that satisfies moreover the equations

\[
(\psi \check{c}) \quad c_{A_2,A'_2,A_1,A'_1}^k \circ \check{c}_{A_1 \land A'_1,A_2 \land A'_2} = (\check{c}_{A_1,A_2} \land \check{c}_{A'_1,A'_2}) \circ c_{A_1,A'_1,A_2,A'_2}^k,
\]

\[
(\overline{\psi} \check{c}) \quad \check{c}_{A_1 \lor A'_1,A_2 \lor A'_2} \circ c_{A_2,A'_2,A_1,A'_1}^k = c_{A_2,A_1,A'_2,A'_1}^k \circ (\check{c}_{A_1,A_2} \lor \check{c}_{A'_1,A'_2}),
\]

which we call collectively \((c^k c)\) (these equations may be found in [19], Section 2.3). The equation \((\psi \check{c})\) says that \(\check{c}\) is upward preserved by \(\land\), while \((\overline{\psi} \check{c})\) says that \(\check{c}\) is downward preserved by \(\lor\) (see Section 4).

In every symmetric biassociative intermuting category the following equations hold:

for \(\xi \in \{\land, \lor\}\) and

\[
\check{c}_{A,B,C,D}^m \Rightarrow \hat{b}_{A,B,C,D}^m \circ (1_A \xi (\hat{b}_{B,C,D}^m \circ (\check{c}_{B,C} \xi 1_D) \circ \hat{b}_{B,A,C,D}^m) \circ \hat{b}_{A,B,C,D}^m): (A \xi B) \xi (C \xi D) \rightarrow (A \xi C) \xi (B \xi D),
\]

\[
(\psi \check{c}^m) \quad c_{A_1 \lor A_2,A'_1 \lor A'_2,A_3,A'_3}^k \circ c_{A_3 \lor A_4,A'_3 \lor A'_4} = c_{A_1 \lor A_2 \lor A_3 \lor A_4,A'_1 \lor A'_2 \lor A'_3 \lor A'_4},
\]

\[
(\overline{\psi} \check{c}^m) \quad \check{c}_{A_1 \lor A_2,A'_1 \lor A'_2,A_3 \lor A_4,A'_3 \lor A'_4} \circ c_{A_3 \lor A_4,A'_3 \lor A'_4} = \check{c}_{A_1 \lor A_2 \lor A_3 \lor A_4,A'_1 \lor A'_2 \lor A'_3 \lor A'_4},
\]

\[
(\check{c}^m) \quad \check{c}_{A_1 \lor A_2,A'_1 \lor A'_2,A_3 \lor A_4,A'_3 \lor A'_4} \circ c_{A_3 \lor A_4,A'_3 \lor A'_4} = \check{c}_{A_1 \lor A_2 \lor A_3 \lor A_4,A'_1 \lor A'_2 \lor A'_3 \lor A'_4},
\]

\[
(\check{c}^m) \quad \check{c}_{A_1 \lor A_2,A'_1 \lor A'_2,A_3 \lor A_4,A'_3 \lor A'_4} \circ c_{A_3 \lor A_4,A'_3 \lor A'_4} = \check{c}_{A_1 \lor A_2 \lor A_3 \lor A_4,A'_1 \lor A'_2 \lor A'_3 \lor A'_4}.
\]
These two equations, which we call collectively \((c^k c^m)\), are both obtained from the equation \((\psi c^m)\) of [13] (Section 2) by replacing some occurrences of \(\land\) by \(\lor\). (The equation \((\psi c^m)\) is the equation \((\tilde{\psi} c^m)\) with \(c^k\) and \(\lor\) replaced everywhere by \(\tilde{c}^m\) and \(\land\) respectively.) The equation \((\psi c^m)\) says that \(\tilde{c}^m\) is upward preserved by \(\land\), and \((\tilde{\psi} c^m)\) says that \(\tilde{c}\) is downward preserved by \(\lor\). The equations \((c^k c^m)\) are derived from the equations assumed for symmetric biassociative intermuting categories by relying on the fact that the natural transformations used to define \(\tilde{c}^m\) are upward preserved by \(\land\), and those used to define \(\tilde{c}^m\) are downward preserved by \(\lor\) (cf. the beginning of Section 10).

The equations \((c^k c^m)\) are related to the equation corresponding to the hexagonal interchange diagram used for defining \(n\)-fold monoidal categories for \(n > 2\) in [3] (end of Definition 1.7). Let us call this equation \(HI\). Taking that \(\tilde{c}^m\) and \(\tilde{\psi} c^m\) are \(\eta^{11}\) and \(\eta^{22}\) respectively, the equation \((\tilde{\psi} c^m)\) is obtained by putting \(i = j = 1\) and \(k = 2\) in \(HI\), and the equation \((\tilde{\psi} c^m)\) is obtained by putting \(i = 1\) and \(j = k = 2\) in \(HI\). The equation \((\psi c^m)\) mentioned above is obtained by putting \(i = j = k\) in \(HI\), which amounts to omitting the superscripts of \(\eta\).

Let \(\mathbf{SC}^k\) be the free symmetric biassociative intermuting category generated by a set of objects. We can proceed as in [10] (Section 7.6) to obtain a strictified category \(\mathbf{SC}^k\) where the \(b\)-arrows and the \(c\)-arrows are identity arrows. The category \(\mathbf{SC}^k\) is not equivalent to \(\mathbf{SC}^k\), but we have that if \(\mathbf{SC}^k\) is a diversified preorder in a sense to be defined below, then \(\mathbf{SC}^k\) is a diversified preorder. The remainder of this section is devoted to proving that \(\mathbf{AC}^{k\text{st}}\) is indeed such a diversified preorder. This proof could be adapted to prove the Theorem of Section 11, which says that \(\mathbf{AC}^{k\text{st}}\) is a diversified preorder. (We have however preferred to rely on rectangular grids in Section 11, because they give a clearer picture.)

The objects of \(\mathbf{SC}^{k\text{st}}\) are equivalence classes of formulae \([A]\) such that \([A]\) is the set of all formulae isomorphic to \(A\) in the free symmetric biassociative category \(\mathbf{S}\). We may represent unequivocally the equivalence class \([A]\) by an equivalence class of form sequences, such that form sequences in the class differ from each other in the order of conjuncts and disjuncts. We call such equivalence classes of form sequences \emph{form multisets} (see [10], Section 7.7). For example, \(p \land q \land (p \lor r \lor p)\) and \(q \land (r \lor p \lor p) \land p\) stand for the same form multiset.

As we did in Section 10 we assume that we deal with \emph{diversified} form multisets, i.e. with form multisets in which every letter occurs at most once. Such form multisets are called \emph{form sets}. That \(\mathbf{SC}^{k\text{st}}\) is a diversified preorder means that if \(f', f'': X \to Y\) are arrows of \(\mathbf{SC}^{k\text{st}}\) where \(X\) and \(Y\) are form sets, i.e. diversified form multisets, then \(f' = f''\).

We denote by \(\text{let}(X)\) the set of letters occurring in the form set \(X\). Let \(p\) be a letter such that \(X\) is not \(p\). We define inductively the form set \(X^{-p}\) as we did in Section 10 when \(X\) was a form sequence, and we use for \(X\) and \(Y\) form sets and \(P\) a set of letters the notation \(X^{-P}\) and \(X^{-Y}\) defined analogously, with the same provisos, as in Section 10.
For \( X \) a form set, let \( X' \) be a subformset of \( X \) when \( X' \), which stands for a form set, is a subword of a representative of \( X \). We can then state the following.

**Lemma 1.** If \( f : X \to Y \) is an arrow of \( \textbf{SC}^{kst} \), and \( P \) is a set of letters such that for every subformset \( U \land V \) of \( X \)

\[
\text{let}(U) \subseteq P \quad \text{iff} \quad \text{let}(V) \subseteq P,
\]

then this equivalence holds for every subformset \( U \land V \) of \( Y \).

**Proof.** It is sufficient to prove this lemma for \( f \) being a \( c^k \)-term. We proceed by induction on the complexity of this \( c^k \)-term. For the basis, suppose that \( f \) is

\[
c_{x_1,x_2,x_3,x_4}^s : (X_1 \land X_2) \lor (X_3 \land X_4) \to (X_1 \lor X_3) \land (X_2 \lor X_4).
\]

We have \( \text{let}(X_1 \lor X_3) \subseteq P \) iff \( \text{let}(X_1) \subseteq P \) and \( \text{let}(X_3) \subseteq P \),

\[
\quad \quad \text{iff} \quad \text{let}(X_2) \subseteq P \) and \( \text{let}(X_4) \subseteq P \), by the assumption concerning the source \( X \) of \( f \),

\[
\quad \quad \text{iff} \quad \text{let}(X_2 \lor X_4) \subseteq P,
\]

and the main \( \land \) of \( (X_1 \lor X_3) \land (X_2 \lor X_4) \) is the only “new” \( \land \) in the target of \( f \).

For the induction step, suppose \( f \) is of the form \( f' \land 1_{X_2} \) for \( f' : X_1 \to Z \). If \( Z = Z_1 \lor Z_2 \), then we just apply the induction hypothesis to \( f' \).

Suppose \( Z = Z_1 \land Z_2 \). If \( \text{let}(X_2) \subseteq P \), then \( \text{let}(X_1) \subseteq P \), which implies that \( \text{let}(Z_1 \land Z_2) \subseteq P \), and hence \( \text{let}(Z_i) \subseteq P \) for every \( i \in \{1,2\} \). If \( \text{let}(Z_i) \subseteq P \) for some \( i \in \{1,2\} \), then by the induction hypothesis \( \text{let}(Z_{3-i}) \subseteq P \), and so \( \text{let}(X_1) \subseteq P \). Hence \( \text{let}(X_2) \subseteq P \).

If \( f \) is of the form \( f' \lor 1_{X_2} \), then we just apply the induction hypothesis to \( f' \).

As a corollary we have the following lemma.

**Lemma 2.** If \( f : X_1 \lor X_2 \to Y \) is an arrow of \( \textbf{SC}^{kst} \), then for every subformset \( \text{let}(Y_1) \subseteq \text{let}(X_i) \) for every \( i \in \{1,2\} \)

\[
\text{let}(Y_1) \subseteq \text{let}(X_i) \quad \text{iff} \quad \text{let}(Y_2) \subseteq \text{let}(X_i).
\]

The arrow terms of \( \textbf{SC}^k \) where indices are replaced by form sets are the arrow terms of \( \textbf{SC}^{kst} \). In these arrow terms of \( \textbf{SC}^{kst} \) we may replace all the \( b \)-arrows and \( c \)-arrows by identity arrows. Since in developed \( \textbf{SC}^{kst} \) arrow terms we need to consider only \( c^k \)-terms, we easily establish the following.

**Lemma 3.** Every arrow term \( f : X_1 \land X_2 \to Y \) of \( \textbf{SC}^{kst} \) is equal to \( f_1 \land f_2 \) for some \( f_i : X_i \to Y_i \) where \( i \in \{1,2\} \).
Since the equations assumed for $\mathbf{SC}^k$ are such that the number of occurrences of $c^k$ is equal on the two sides of the equations, we have in general the following.

**Lemma 4.** If $f = g$ in $\mathbf{SC}^k$ or $\mathbf{SC}^{k^*}$, then the number of occurrences of $c^k$ is equal in the arrow terms $f$ and $g$.

Let $f : X \to Y$ be an arrow term of $\mathbf{SC}^{k^*}$, and let $P$ be a set of letters such that $\text{let}(X) - P \neq \emptyset$ (which implies that $\text{let}(Y) - P \neq \emptyset$, since $\text{let}(X) = \text{let}(Y)$), and such that, as in Lemma 1, for every subformset $U \wedge V$ of $X$ we have $\text{let}(U) \subseteq P$ iff $\text{let}(V) \subseteq P$. Then we define inductively the arrow $f^{-P} : X^{-P} \to Y^{-P}$ of $\mathbf{SC}^{k^*}$ in the following manner:

- If $\text{let}(X) \cap P = \emptyset$, then $f^{-P} = f$.
- If $f$ is $1_X$, then $f^{-P} = 1_{X^{-P}}$.
- If $f$ is $c^k_{X_1 \wedge X_2, X_3 \wedge X_4}$, then $f^{-P} = 1_{X^{-P}}$ in case either $\text{let}(X_1 \wedge X_2) \subseteq P$ or $\text{let}(X_3 \wedge X_4) \subseteq P$, and $f^{-P} = c^k_{X_1^{-P}, X_2^{-P}, X_3^{-P}, X_4^{-P}}$ otherwise.
- If $f$ is $f_1 \xi f_2$, for $\xi \in \{\land, \lor\}$ and $f_i : X_i \to Y_i$, where $i \in \{1, 2\}$, then $f^{-P} = f_3^{-P}$ in case $\text{let}(X_i) \subseteq P$, and $f^{-P} = f_1^{-P} \xi f_2^{-P}$ in case for no $i \in \{1, 2\}$ the inclusion $\text{let}(X_i) \subseteq P$ holds.
- If $f$ is $f_2 \circ f_1$, then by Lemma 1, both $f_1^{-P}$ and $f_2^{-P}$ are defined, and $f^{-P} = f_2^{-P} \circ f_1^{-P}$.

For $X_1$ and $X_2$ form sets, we say that $c^k_{S,T,U,V}$ is $(X_1, X_2)$-splitting when $\text{let}(S \wedge T) \subseteq \text{let}(X_i)$ and $\text{let}(U \vee V) \subseteq \text{let}(X_{3-i})$ for some $i \in \{1, 2\}$. We say that an arrow term of $\mathbf{SC}^{k^*}$ is $(X_1, X_2)$-splitting when every occurrence of $c^k$ in it is $(X_1, X_2)$-splitting, and we say that it is $(X_1, X_2)$-nonsplitting when every occurrence of $c^k$ in it is not $(X_1, X_2)$-splitting. One can easily check that if $f = g$ and $f$ is an $(X_1, X_2)$-splitting arrow term, then $g$ is an $(X_1, X_2)$-splitting arrow term too. (This is not the case when we replace “splitting” by “nonsplitting”.) It is clear that every $(X_1, X_2)$-splitting arrow term is equal to a developed $(X_1, X_2)$-splitting arrow term, and analogously with “splitting” replaced by “nonsplitting”. We can then prove the following.

**Lemma 5.** If $f : X_1 \vee X_2 \to Y$ is $(X_1, X_2)$-nonsplitting, then $f$ is equal to $f_1 \vee f_2$ for $f_i : X_i \to Y_i$ and $i \in \{1, 2\}$.

**Proof.** It is enough to establish this lemma for $f$ being a $c^k$-term, which is trivial. We then proceed by induction on the number of factors in a developed arrow term equal to $f$.

Lemmata 6-14 below will yield a normal form for $(X_1, X_2)$-splitting arrow terms of $\mathbf{SC}^{k^*}$ with source $X_1 \vee X_2$. After these lemmata we establish
with the help of this normal form and of Lemma 5 that every arrow term \( f : X_1 \lor X_2 \to Y \) of \( \text{SC}^{k^{\text{st}}} \) is equal to \( h \circ (g_1 \lor g_2) \) for \( h \) being \((X_1, X_2)\)-splitting and for \( g_i : X_i \to Y^{-X_{3-i}} \). After that we will be able to proceed by induction to show that \( \text{SC}^{k^{\text{st}}} \) is a diversified preorder. First we prove the following.

**Lemma 6.** If \( f : X_1 \lor X_2 \to Y \) is \((X_1, X_2)\)-splitting, then \( Y^{-X_i} \) is \( X_{3-i} \) for every \( i \in \{1, 2\} \).

**Proof.** Since \( f^{\lor X_i} : (X_1 \lor X_2)^{-X_i} \to Y^{-X_i} \) is \( 1_{Y^{-X_i}} \), and since \((X_1 \lor X_2)^{-X_i} \) is \( X_{3-i} \), the lemma follows.

**Lemma 7.** If \( f : X_1 \lor X_2 \to Y_1 \land Y_2 \) is \((X_1, X_2)\)-splitting, then \( X_i \) is of the form \( X_1' \land X_1'' \) for every \( i \in \{1, 2\} \).

**Proof.** We have that

\[
f^{-X_{3-i}} = 1_{X_i} : X_i \to (Y_1 \land Y_2)^{-X_{3-i}}.
\]

By Lemma 2, we have that \((Y_1 \land Y_2)^{-X_{3-i}} \) is \( Y_1^{-X_{3-i}} \land Y_2^{-X_{3-i}} \), which is \( X_i \).

For \( X_1 \) being \( X_1' \land X_1'' \) and \( X_2 \) being \( X_2' \land X_2'' \), consider an arrow term

\[
f \circ c_{X_1', X_1'', X_2', X_2''}^k : X_1 \lor X_2 \to Y_1 \land Y_2
\]

of \( \text{SC}^{k^{\text{st}}} \). For \( i, j \in \{1, 2\} \) and \( l \in \{', ''\} \) we say that \( X_j^l \) is exactly in \( Y_i \) when \( X_j^l \) is \( Y_i^{-X_{3-j}} \). We say that \( X_j^l \) is properly in \( Y_i \) when \( X_j^l \) is \( P_j \) for \( P_1 \land P_2 \) being \( Y_i^{-X_{3-j}} \). We say that \( X_j^l \) extends \( Y_i \) when \( X_j^l \) is \( Y_i^{-X_{3-j}} \land Q_j \) for \( Q_1 \land Q_2 \) being \( Y_i^{-X_{3-j}} \). Finally, we say that \( X_j^l \) partakes in \( Y_i \) and \( Y_2 \) when \( X_j^l \) is \( P_j \land Q_j \) for \( P_1 \land P_2 \) being \( Y_1^{-X_{3-j}} \) and \( Q_1 \land Q_2 \) being \( Y_2^{-X_{3-j}} \). We say that \( X_1' \) and \( X_2' \) are related in the same way to \( Y_i \) when either they are both exactly in \( Y_i \), or they are both properly in \( Y_i \), or they both extend \( Y_i \), or they both partake in \( Y_i \) and \( Y_2 \).

For Lemmata 8-10 below we assume that \( f \circ c_{X_1', X_1'', X_2', X_2''}^k \), of the type displayed above, is \((X_1, X_2)\)-splitting.

**Lemma 8.** For every \( i, j \) and \( l \), we have that \( X_j^l \) is either exactly in \( Y_1 \) or \( Y_2 \), or \( X_j^l \) is properly in \( Y_i \), or \( X_j^l \) extends \( Y_i \), or \( X_j^l \) partakes in \( Y_1 \) and \( Y_2 \).

**Proof.** By Lemma 6 and Lemma 2 we have that

\[
X_j^l \land X_j^l'' \land Y_1^{-X_{3-j}} \land Y_2^{-X_{3-j}}.
\]

From this the lemma follows.
**Lemma 9.** For every $i$, $j$ and $l$, we have that $X^i_j$ is properly in $Y_i$ iff $X^i_{3-j}$ extends $Y_{3-i}$, and $X^i_j$ partakes in $Y_1$ and $Y_2$ iff $X^i_{3-j}$ partakes in $Y_1$ and $Y_2$.

The proof of Lemma 8 yields this lemma too.

**Lemma 10.** For every $i$, $j$, and $l$, we have that $X^i_1$ and $X^i_2$ are related in the same way to $Y_i$.

**Proof.** Suppose $X^i_1$ and $X^i_2$ are not related in the same way to $Y_i$. Then, by Lemma 8, we have to consider a number of cases, and show that we have a contradiction in each of these cases. We will consider just one, typical, case. In all the other cases we proceed more or less in the same way.

Suppose $X^i_1$ is exactly in $Y_1$ and $X^i_2$ is properly in $Y_1$. Then, by Lemma 9, we have that $X''^i_1$ is exactly in $Y_2$ and $X''^i_2$ extends $Y_2$, which means that $X''^i_1$ is $Y_2^{-X_2}$ and $X''^i_2$ is $Y_2^{-X_1} \land Q_2$ for $Q_1 \land Q_2$ being $Y_1^{-X_1}$. By Lemma 3, we have that

$$f = f' \land f'' : (X''^i_1 \lor X''^i_2) \land (X''^i_1 \lor X''^i_2) \rightarrow Z' \land Z''$$

for $f': X''^i_1 \lor X''^i_2 \rightarrow Z'$ where $l \in \{', ''\}$, and $Z' \land Z''$ the same form set as $Y_1 \land Y_2$. So $f''$ is of the type $Y_2^{-X_2} \lor (Y_2^{-X_1} \land Q_2) \rightarrow Z''$ and $\text{let}(Z'') = \text{let}(Y_2) \cup \text{let}(Q_2)$.

Since $\text{let}(Q_2) \subseteq \text{let}(Y_1)$ and $Z' \land Z''$ is the same form set as $Y_1 \land Y_2$, the form set $Z''$ must be $Z''_1 \land Z''_2$ for $\text{let}(Z''_2) = \text{let}(Q_2)$. Hence $f = c^{k}_{X''^i_1, X''^i_2, X''^i_2}$ is of the type $X_1 \lor X_2 \rightarrow (Z' \land Z''_1) \land Z''_2$. We have $\text{let}(Z''_2) \subseteq \text{let}(X_2)$, because $\text{let}(Q_2) \subseteq \text{let}(Y_1^{-X_1})$. So, by Lemma 2, $\text{let}(Z' \land Z''_1) \subseteq \text{let}(X_2)$. But this would mean that $\text{let}(X_1) = \emptyset$, which is a contradiction.

We can also prove the following.

**Lemma 11.** If $f : X_1 \lor X_2 \rightarrow Y_1 \land Y_2$ is $(X_1, X_2)$-splitting, then for some $(X_1, X_2)$-splitting arrow term $g$ we have $f = g \circ c^{k}_{Y_1^{-X_2}, Y_2^{-X_2}, Y_1^{-X_1}, Y_2^{-X_1}}$.

**Proof.** Suppose $f : X_1 \lor X_2 \rightarrow Y_1 \land Y_2$ is $(X_1, X_2)$-splitting. By the remark made after the definition of $(X_1, X_2)$-splitting arrow terms, and by Lemma 7, we have that $f = g \circ c^{k}_{X_1', X_2', X_2'_{\prime\prime}}$ for $g$ an $(X_1, X_2)$-splitting developed arrow term, where $X_j$ is $X_j' \land X_j''$ for $j \in \{1, 2\}$. We proceed by induction on the number $n$ of $c^{k}$-terms in $g$. If $n = 0$, then $g = 1_{(X_1' \lor X_2') \land (X_1'' \lor X_2'')}$ for $Y_1$ being $X_1' \lor X_2'$ and $Y_2$ being $X_2'' \lor X_2''$. So $f = c^{k}_{Y_1^{-X_2}, Y_2^{-X_2}, Y_1^{-X_1}, Y_2^{-X_1}}$.

If $n > 0$, and $c^{k}_{X_1', X_1', X_2', X_2'} = c^{k}_{Y_1^{-X_2}, Y_2^{-X_2}, Y_1^{-X_1}, Y_2^{-X_1}}$, then we are done.

If this equation does not hold, then, by Lemmata 10 and 9, we have two cases to consider:

(I) $X_1^i$ and $X_2^i$ both extend $Y_i$ for some $i \in \{1, 2\}$.

(II) $X_1^i$ and $X_2^i$ both partake in $Y_1$ and $Y_2$. 

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We conclude similarly that \( P \) by the equation (induction hypothesis, for an \((X_1', X_2')\)-splitting arrow term). So, by the induction hypothesis, for an \((X_1', X_2')\)-splitting arrow term \( h \) we have that
\[
g' = h \circ c^k_{y_1^{-x_2}, y_2^{-x_2}, y_1^{-x_1}, y_2^{-x_1}}.
\]

By Lemma 6 applied to \( g' \), we have that \((Y_1 \land Y_2')^{-X_i'}\) is \( X_{l-1}' \). So \( Y_1^{-X_2} \land Y_2^{-X_2} \) is \( Y_1^{-X_2} \land Q \) and \( Y_1^{-X_1} \land Y_2'^{-X_1'} \) is \( Y_1^{-X_1} \land R \). Since \( let(Y_1^{-X_1'}) \subseteq let(Y_1) \) and \( let(Q_1) \cup let(R_1) \subseteq let(Y_2) \), we conclude that
\[
Y_1^{-X_2'} = Y_1^{-X_2}, \quad Y_2' = Y_2, \quad Y_1^{-X_1'} = Y_1^{-X_1} \quad \text{and} \quad Y_2' = Y_2
\]

So
\[
f = (h \land g^\prime) = (c^k_{y_1^{-x_2}, y_2^{-x_2}, y_1^{-x_1}, y_2^{-x_1}} \circ \mathbf{1}_{X_1' \lor X_2'}) = c^k_{y_1^{-x_2}, y_2^{-x_2}, y_1^{-x_1}, y_2^{-x_1}}
\]

by the equation (\( \psi(h) \)) of Section 4. Here \( Q \land X_2'' \) is \( Y_2^{-X_2} \) and \( R \land X_2'' \) is \( Y_2^{-X_1} \), as desired. We proceed analogously when \( l \) is \( ' \) and \( i \) is 2, and when \( l \) is \( ' \) and \( i \) is 1 or 2.

Next we consider case (II). So \( X_1' \) is \( P' \land Q' \) for \( P' \land P'' \) being \( Y_1^{-X_2} \) and \( Q' \land Q'' \) being \( Y_2^{-X_2} \), while \( X_2' \) is \( O' \land R' \) for \( O' \land O'' \) being \( Y_1^{-X_1} \) and \( R' \land R'' \) being \( Y_2^{-X_1} \). We have that \( X_2'' \) is \( P'' \land Q'' \) and \( X_2'' \) is \( O'' \land R'' \).

We conclude by Lemma 3 that \( g = g' \land g'' \) for \( g' : X_1' \lor X_2' \rightarrow Y_1' \land Y_2' \), where \( l \in \{', '' \} \) and \( Y_1' \lor Y_2' \) is \( Y_1 \) for \( i \in \{1, 2\} \). Since \( g \) is \((X_1, X_2')\)-splitting, \( g' \) for \( l \in \{', '' \} \) is \((X_1, X_2')\)-splitting, and hence also \((X_1', X_2')\)-splitting. So, by the induction hypothesis, for an \((X_1', X_2')\)-splitting arrow terms \( h' \) we have that
\[
g' = h' \circ c^k_{y_1^{-x_2}, y_2^{-x_2}, y_1^{-x_1}, y_2^{-x_1}}.
\]

By Lemma 6 applied to \( g' \) we have that \((Y_1' \land Y_2')^{-X_i'}\) is \( X_{l-1}' \). So \( Y_1'^{-X_2} \land Y_2'^{-X_2} \) is \( P' \land Q' \). Since \( let(P') \subseteq let(Y_1) \) and \( let(Q') \subseteq let(Y_2) \), we conclude that
\[
Y_1'^{-X_2} = P' \quad \text{and} \quad Y_2'^{-X_2} = Q'.
\]

We conclude similarly that
\[
Y_1'^{-X_1'} = O' \quad \text{and} \quad Y_2'^{-X_1'} = R',
\]
\[
Y_1''^{-X_2''} = P'' \quad \text{and} \quad Y_2''^{-X_2''} = Q'',
\]
\[
Y_1''^{-X_1''} = O'' \quad \text{and} \quad Y_2''^{-X_1''} = R''.
\]

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Lemma 14. By replacing $f$ for $i$, then either prime conjunct is a Lemma 13. If $c$ the target of
Lemma 12. If $f_1, f_2 : (p_1 \land \ldots \land p_n) \lor (q_1 \land \ldots \land q_n) \rightarrow (p_1 \lor q_1) \land \ldots \land (p_n \lor q_n)$, for $n \geq 1$ and $p_1, \ldots, p_n, q_1, \ldots, q_n$ distinct letters, are arrow terms of $\mathbf{SC}^{k^{mt}}$, then $f_1 = f_2$.

Proof. If $n = 1$, then it is clear that $f_1 = f_2 = 1_{p_1 \lor q_1}$. If $n > 1$, then it is easy to see that $f_1$ and $f_2$ are $(p_1 \land \ldots \land p_n, q_1 \land \ldots \land q_n)$-splitting. Then by Lemmata 11 and 3, it follows that for every $i \in \{1, 2\}$

$$f_i = (1_{p_1 \lor q_1} \land f'_i) \ast c^i_{p_1,p_2,\ldots,p_n,q_1,q_2,\ldots,q_n}$$

for $f'_i : (p_2 \land \ldots \land p_n) \lor (q_2 \land \ldots \land q_n) \rightarrow (p_2 \lor q_2) \land \ldots \land (p_n \lor q_n)$. By the induction hypothesis, $f'_1 = f'_2$.

Let $c^i_{p_1,\ldots,p_n,q_1,\ldots,q_n}$ stand for any arrow term of $\mathbf{SC}^{k^{mt}}$ of the type

$$(p_1 \land \ldots \land p_n) \lor (q_1 \land \ldots \land q_n) \rightarrow (p_1 \lor q_1) \land \ldots \land (p_n \lor q_n).$$

In every such arrow term there are $n-1$ occurrences of $c^k$. If $n = 1$, then $c^k_{p_1,q_1}$ stands for $1_{p_1 \lor q_1}$. We write $c^k_{p_1,\ldots,p_n,q_1,\ldots,q_n}$ for the arrow term obtained from $c^k_{p_1,\ldots,p_n,q_1,\ldots,q_n}$ by substituting the form sets $P_i$ and $Q_i$ for $p_i$ and $q_i$, respectively. We can then easily prove the following.

Lemma 13. If $f : X_1 \lor X_2 \rightarrow Y_1 \land \ldots \land Y_n$, for $n \geq 1$, is $(X_1, X_2)$-splitting, then

$$f = (f_1 \land \ldots \land f_n) \ast c^i_{X_{i-2},\ldots,X_{i-3},X_{i-1},Y_{i-2},\ldots,Y_{i-1},Y_{i-1}}$$

for $f_i : Y_{i-2} \land Y_{i-1} \rightarrow Y_i$ an $(Y_{i-2}, Y_{i-1})$-splitting arrow term, where $1 \leq i \leq n$.

This lemma is of particular interest when $Y_1, \ldots, Y_n$ are all the conjuncts of the target of $f$; i.e., $Y_i$ is not of the form $Y_i' \land Y_i''$. We say that in this case $Y_i$ is a prime conjunct. We define analogously a prime disjunct of a form set, just by replacing $\land$ by $\lor$.

By the dual of Lemma 3 and by Lemma 7 we have the following lemma.

Lemma 14. If $f : X_1 \lor X_2 \rightarrow Y' \lor Y''$ is an $(X_1, X_2)$-splitting arrow term such that $Y'$ is a prime disjunct of $Y' \lor Y''$, then $f = f' \lor f''$ for $f' : X' \rightarrow Y'$ where either
$X'$ is a prime disjunct of $X_i$ for some $i \in \{1, 2\}$ and $f' = 1_{X'}$, or

$X' = X'_1 \vee X'_2$ for $X'_i$ being a prime disjunct of $X_i$ for $i \in \{1, 2\}$ and $f'$ being $(X'_1, X'_2)$-splitting.

We define as follows a class $N$ of arrow terms, for which we say that they are in $(X_1, X_2)$-splitting normal form. For every $X$ we have that $1_X$ is in $N$. If for $n \geq 2$ we have that $c_{P_1, \ldots, P_n, Q_1, \ldots, Q_n}$ is $(X_1, X_2)$-splitting and that $f_1, \ldots, f_n$, which are not all identity arrows, are in $N$, then

- $c_{P_1, \ldots, P_n, Q_1, \ldots, Q_n}$ is in $N$,
- $f_1 \vee \ldots \vee f_n$ is in $N$,
- $(f_1 \land \ldots \land f_n) \circ c_{P_1, \ldots, P_n, Q_1, \ldots, Q_n}$ is in $N$ (provided the composition is defined).

It is easy to see that an arrow term in $(X_1, X_2)$-splitting normal form is $(X_1, X_2)$-splitting. Lemmata 13 and 14 guarantee that every $(X_1, X_2)$-splitting arrow term with the source $X_1 \vee X_2$ is equal to an arrow term in $(X_1, X_2)$-splitting normal form.

For the proof of next lemma we define the notion of $(\land, \lor)$-arrow-shape, analogous up to a point to the notion of $\xi$-shape of Section 4. We have:

- $\square$ is a $(\land, \lor)$-arrow-shape;
- if $M$ is a $(\land, \lor)$-arrow-shape, then $M \land 1_X$ and $M \lor 1_X$ are $(\land, \lor)$-arrow-shapes.

For $M$ a $(\land, \lor)$-arrow-shape and $f$ an arrow term, the arrow term $M(f)$ is obtained from $M$ by replacing $\square$ by $f$.

We will also use in the proof of the next lemma the following abbreviations for $m \geq 1$:

\[
X_m = df X_1, \ldots, X_m, \quad \land X_m = df X_1 \land \ldots \land X_m.
\]

**Lemma 15.** Let $g \circ h$ be such that $g$ is a $c^k$-term that is not $(X_1, X_2)$-splitting, and $h$ is an $(X_1, X_2)$-splitting arrow term whose source is $X_1 \lor X_2$. Then there exists a $c^k$-term $g'$ that is not $(X_1, X_2)$-splitting such that $g \circ h = f \circ g'$ for some arrow term $f$.

**Proof.** As remarked above, by Lemmata 13 and 14, we may assume that $h$ is in $(X_1, X_2)$-splitting normal form. We proceed by induction on the number $n$ of occurrences of $c^k$ in $h$. (We find $m - 1$ occurrences of $c^k$ in each $c_{P_1, \ldots, P_m, Q_1, \ldots, Q_m}$.)

By the assumption that $g$ is not $(X_1, X_2)$-splitting and that $h$ is in $(X_1, X_2)$-splitting normal form, we have only two interesting cases, for which the following two cases are typical:
(1) \( g \circ h = g \circ h'' \circ h' = M(\langle P_m^k \cup P_n^k, \wedge (Q_m^k \cup Q_n^k), R^k, S^k \rangle \circ (\langle P_m^k, Q_m^k, P_n^k, Q_n^k \rangle \cup 1_{R^k \cup S^2}) \circ h') \)
(2) \( g \circ h = g \circ h'' \circ h' = M(\langle P_m^k, Q_m^k, P_n^k, Q_n^k \rangle \cup (\langle P_m^k \cup P_n^k, \wedge (Q_m^k \cup Q_n^k), R^k, S^k \rangle \circ (\langle P_m^k, Q_m^k, P_n^k, Q_n^k \rangle \cup 1_{R^k \cup S^2}) \circ h') \)

where \( M \) is a \((\wedge, \vee)\)-arrow shape, and for every set form \( Y^i \) in whose name there is a superscript \( i \in \{1, 2\} \), we have that \( let(Y^i) \subseteq let(X_i) \).

In case (1) we have that

\[
\psi^k_{P_m^k, Q_m^k, P_n^k, Q_n^k} \circ (\langle P_m^k \cup P_n^k, \wedge (Q_m^k \cup Q_n^k), R^k, S^k \rangle \cup 1_{R^k \cup S^2}) =
((\langle P_m^k, Q_m^k, P_n^k, Q_n^k \rangle \cup 1_{R^k}) \wedge \left( (c^k_{m}, P_m^k, Q_m^k, P_n^k, Q_n^k \rangle \cup 1_{S^k}) \wedge \left( \langle P_m^k, \wedge (Q_m^k \cup Q_n^k), R^k, S^k \rangle \cup 1_{R^k \cup S^2} \right) \right) \wedge \left( (\langle P_m^k, Q_m^k, P_n^k, Q_n^k \rangle \cup 1_{R^k \cup S^2}) \wedge (c^k_{m}, Q_m^k, P_n^k, Q_n^k \rangle \cup 1_{R^k \cup S^2}) \right)
\]

by taking that \( c^k_{m}, Q_m^k, P_n^k, Q_n^k \rangle \cup 1_{R^k \cup S^2} \) is \( (c^k_{m}, P_m^k, \wedge (Q_m^k \cup Q_n^k), R^k, S^k \rangle \cup 1_{R^k \cup S^2}) \), and by applying the naturality equation for \( c^k \) and the equation \((\psi^k)\) of Section 4.

Then we apply the induction hypothesis to the composition of

\[
M(\langle P_m^k, Q_m^k, P_n^k, Q_n^k \rangle \cup 1_{R^k \cup S^2})
\]

which is not \((X_1, X_2)\)-splitting, with \( h' \), which is in \((X_1, X_2)\)-splitting normal form, and with at least one occurrence of \( c^k \) less than in \( h \).

In case (2) we have that

\[
\psi^k_{P_m^k, Q_m^k, P_n^k, Q_n^k} \circ (\langle P_m^k \cup P_n^k, \wedge (Q_m^k \cup Q_n^k), R^k, S^k \rangle \cup 1_{R^k \cup S^2}) =
((\langle P_m^k, Q_m^k, P_n^k, Q_n^k \rangle \cup 1_{R^k}) \wedge \left( (c^k_{m}, Q_m^k, P_n^k, Q_n^k \rangle \cup 1_{S^k}) \wedge \left( \langle P_m^k, \wedge (Q_m^k \cup Q_n^k), R^k, S^k \rangle \cup 1_{R^k \cup S^2} \right) \right) \wedge \left( (\langle P_m^k, Q_m^k, P_n^k, Q_n^k \rangle \cup 1_{R^k \cup S^2}) \wedge (c^k_{m}, Q_m^k, P_n^k, Q_n^k \rangle \cup 1_{R^k \cup S^2}) \right)
\]

by taking that \( c^k_{m}, Q_m^k, P_n^k, Q_n^k \rangle \cup 1_{R^k \cup S^2} \) is as above, and analogously for \( c^k_{R^k \cup S^2} \), and by applying the naturality equation for \( c^k \) and the equation \((\psi^k)\) from the beginning of this section.

Then we apply the induction hypothesis to the composition of

\[
M(\langle P_m^k, Q_m^k, P_n^k, Q_n^k \rangle \cup 1_{R^k \cup S^2})
\]

which is not \((X_1, X_2)\)-splitting, with \( h' \), which is in \((X_1, X_2)\)-splitting normal form, and with at least two occurrence of \( c^k \) less than in \( h \).

Then we can prove the following key lemma.

**Lemma 16.** For every arrow \( f : X_1 \cup X_2 \to Y \) of \( SC^{k^{st}} \) there is an \((X_1, X_2)\)-nonsplitting arrow term \( f' \) and an \((X_1, X_2)\)-splitting arrow term \( f'' \) such that \( f = f'' \circ f' \).
PROOF. By the Development Lemma \( f \) is equal to a developed arrow term. Every developed arrow term of \( \text{SC}^{k*} \) such that some of its factors are \( (X_1, X_2) \)-splitting \( c^k \)-terms is of the form

\[
f'' \circ f'' \circ \cdots \circ f'' \circ g_{n+k} \circ \cdots \circ g_{n+1} \circ f'' \circ \cdots \circ f'' \circ f' \circ f',
\]

for \( m, n \geq 1 \) and \( k, l \geq 0 \), where \( f'_1, \ldots, f'_m, g_{n+1} \) and \( g_{n+k} \) are \((X_1, X_2)\)-non-splitting, while \( f''_1, \ldots, f''_{n+k+1}, \ldots, f''_{n+k+1} \) are \((X_1, X_2)\)-splitting. (Note that identity arrow terms are \((X_1, X_2)\)-splitting, as well as \((X_1, X_2)\)-splitting.)

By Lemma 5, the target of \( f''_m \circ \cdots \circ f''_1 \) is of the form \( Y_1 \lor Y_2 \) such that \( \text{let}(X_i) \subseteq \text{let}(Y_i) \) for every \( i \in \{1, 2\} \). Hence being \((X_1, X_2)\)-splitting is the same as being \((Y_1, Y_2)\)-splitting.

If \( k = 0 \), then we are done. If \( k > 0 \), then we call \( n + k \) the *tail length*. By applying Lemma 15, the Development Lemma and Lemma 4, we obtain that

\[
g_{n+1} \circ f''_m \circ \cdots \circ f''_1 = h_n \circ \cdots \circ h_1 \circ g'
\]

for \( g' \) a \( c^k \)-term that is not \((X_1, X_2)\)-splitting. Then after replacing above the left-hand side by the right-hand side we have either obtained \( f = f'' \circ f' \) as desired, or we have obtained an arrow term with a strictly smaller tail length.

Formally, we make an induction on a multiset ordering; see [8].

We can now establish the following.

\textbf{SC}^{k*} \textbf{COHERENCE.} If \( f', f'' : X \rightarrow Y \) are arrows of \( \text{SC}^{k*} \), then \( f' = f'' \); i.e., the category \( \text{SC}^{k*} \) is a diversified preorder.

PROOF. As in the proofs of the Theorems of Sections 10 and 11, we proceed by induction on the sum \( n \) of the number of letters in the form set \( X \) and the number of occurrences of \( \land \) in \( X \). If \( n = 1 \), then \( X = Y = p \), and \( 1_p : p \rightarrow p \) is the unique arrow from \( p \) to \( p \).

If \( X = X_1 \land X_2 \), then, by Lemma 3, we have that \( Y \) must be of the form \( Y_1 \land Y_2 \) and \( f^l = f_1^l \land f_2^l \) for every \( l \in \{\prime, \prime'\} \), where both \( f_1^l \) and \( f_2^l \) are of the type \( X_i \rightarrow Y_i \) for \( i \in \{1, 2\} \). Then we may apply the induction hypothesis to \( f_1^l \) and \( f_2^l \). We proceed analogously when \( Y \) is \( Y_1 \lor Y_2 \) just by relying on the dual of Lemma 3.

Suppose now that \( X = X_1 \lor X_2 \) and \( Y = Y_1 \land Y_2 \). Then, by Lemma 16 and Lemmata 5-6, we have that \( f^l = h^l \circ (g'_i \lor g''_i) \) for every \( l \in \{\prime, \prime'\} \), where \( h^l \) is \((X_1, X_2)\)-splitting (which is the same as being \((Y_1, Y_2)\)-splitting) and \( g'_i \) and \( g''_i \) are of the type \( X_i \rightarrow Y_i \). If \( i \in \{1, 2\} \). If there is at least one occurrence of \( c^k \) in \( g'_i \) for some \( i \in \{1, 2\} \) and some \( l \in \{\prime, \prime'\} \), then we may apply the induction hypothesis to \( g'_i \) and \( g''_i \), since at least the number of letters has decreased in their source, and we may apply the induction hypothesis to \( h' \) and \( h'' \), since in their source \((Y_1, Y_2) \lor (Y_1, Y_2)\) there is at least one occurrence of \( \land \) less than in \( X_1 \lor X_2 \).
If \( g'_i = g''_i = 1_X \), for \( i \in \{1, 2\} \), then, by Lemma 11, for every \( l \in \{', ''\} \)
\[
 f^l = u^l \circ c^k_{Y_1 \rightarrow x_2, Y_2 \rightarrow x_1, Y_1 \rightarrow x_1, Y_2 \rightarrow x_1}
\]
and we may apply the induction hypothesis to \( u' \) and \( u'' \).

\[\dashv\]

From this result we obtain as a corollary the main result of this section.

**Symmetric Biassociative Intermuting Coherence.** The category \( \text{SC}^k \) is a diversified preorder.

### 15 Lattice categories and symmetric biassociative intermuting categories

In this section we consider the relationship between the symmetric biassociative intermuting categories of the preceding section and an important type of categories for which coherence was previously established. We will just summarize matters, and will not go into all the details, known either from [10], or other, earlier, references.

A **lattice** category is a category with all finite nonempty products and coproducts. We call \( \text{L} \) the free lattice category generated by a set of objects. A detailed equational presentation of \( \text{L} \), in several possible languages, may be found in [10] (Chapter 9). The language on which we rely in this section has as primitive arrow terms those of \( \text{SC}^k \) extended with the arrow terms corresponding to the diagonal maps, the projections, and their duals:

\[
\hat{w}_A : A \to A \land A, \quad \hat{k}^i_{A_1, A_2} : A_1 \land A_2 \to A_i, \\
\hat{w}_A : A \lor A \to A, \quad \hat{k}^i_{A_1, A_2} : A_i \to A_1 \lor A_2,
\]

for \( i \in \{1, 2\} \); the arrow terms are closed under composition \( \circ \) and the operations \( \land \) and \( \lor \). The equations assumed for \( \text{L} \) are first categorial, bifunctorial and naturality equations. Next we have equations that guarantee that \( \land \) is binary product, with \( \hat{w} \) and \( (k^1, k^2) \) being respectively the unit and counit of the underlying adjunction (see the equations \( (\hat{w} \hat{k}) \) and \( (\hat{w} \hat{k} \hat{k}) \) in [10], Section 9.1). We also have dual equations that guarantee that \( \lor \) is binary coproduct. Next we have definitional equations for \( \hat{b}, \hat{c} \) and \( \hat{c}^k \):

\[
\hat{b}_{A,B,C} = ((1_A \land \hat{k}^1_{B,C}) \land (\hat{k}^2_{B,C} \circ \hat{k}^2_{A,B,C})) \circ \hat{w}_{A \land (B \land C)}, \\
\hat{b}_{C,B,A} = ((\hat{k}^1_{C,B} \circ \hat{k}^1_{C \land B, A}) \land (\hat{k}^2_{C,B} \land 1_A)) \circ \hat{w}_{(C \land B) \land A}, \\
\hat{c}_{A,B} = (\hat{k}^2_{A,B} \land \hat{k}^1_{A,B}) \circ \hat{w}_{A \land B}, \\
\hat{c}^k_{A,B,C,D} = ((\hat{k}^1_{A,B} \lor \hat{k}^1_{C,D}) \land (\hat{k}^2_{A,B} \lor \hat{k}^2_{C,D})) \circ \hat{w}_{(A \land B) \land (C \land D)},
\]

or, alternatively,
and dual equations for \( \hat{b} \) and \( \hat{c} \).

There exists a faithful functor \( G \) from the category \( \mathbf{L} \) to the category \( \mathbf{Rel} \) whose arrows are relations between finite ordinals. The existence of this faithful functor is called Lattice Coherence (see [10], Section 9.4). Symmetric Biassociative Intermuting Coherence could also be expressed by stating that a functor whose arrows are relations between finite ordinals. The existence of this faithful functor into \( \mathbf{Rel} \), the image under this functor being a discrete subcategory of \( \mathbf{Rel} \), is a faithful functor. (Coherence in the sense of preordering that we had previously seen in this paper can also be expressed as the existence of a faithful functor into \( \mathbf{Rel} \); see [10], Section 2.9.) From Symmetric Biassociative Intermuting Coherence it follows that \( \mathbf{SC}^k \) is isomorphic to a subcategory of \( \mathbf{L} \).

We can then prove that \( \mathbf{SC}^k \) catches an interesting fragment of \( \mathbf{L} \). (This result may be understood as extending the result of [9].)

**Proposition.** If for the arrow \( f : A \to B \) of \( \mathbf{L} \) we have that \( G(f) \) is a bijection, then there is an arrow term \( f' : A \to B \) of \( \mathbf{SC}^k \) such that \( f = f' \) in \( \mathbf{L} \).

**Proof.** By relying on the following equations of \( \mathbf{L} \):

\[
\begin{align*}
\hat{w}_{A \land B} &= \hat{c}_{A,B,B} \circ (\hat{w}_A \land \hat{w}_B), \\
\hat{w}_{A \lor B} &= \hat{c}_{A,B,B} \circ (\hat{w}_A \lor \hat{w}_B), \\
\hat{w}_A &= (\hat{w}_A \land \hat{w}_B) \circ \hat{c}_{A,B,A,B}, \\
\hat{w}_{A \lor B} &= (\hat{w}_A \land \hat{w}_B) \circ \hat{c}_{A,B,A,B},
\end{align*}
\]

we may assume that every \( \hat{\xi} \), for \( \xi \in \{\land, \lor\} \), has as its index a letter; i.e. we have only \( \hat{w}_p \) for \( p \) a letter (see the preceding section for the definition of \( \hat{c} \)).

By relying on the following equation of \( \mathbf{L} \):

\[
\hat{k}_{1,C,A \land B} = \hat{k}_{1,C,A} \circ (1 \land \hat{c}_{A,B}).
\]

which follows from the naturality of \( \hat{k}_1 \), we may assume that for every occurrence of \( \hat{k}_{1,C,B} \) the second index \( D \) is either a letter or of the form \( D_1 \lor D_2 \). With the analogous equations for \( \hat{k}_2 \), \( \hat{k}_1 \) and \( \hat{k}_2 \), we are allowed to make analogous assumptions concerning \( \hat{k}_{1,D,C}, \hat{k}_{1,D} \) and \( \hat{k}_{2,D} \).

Next we rely on bifunctorial and naturality equations, and the following equations of \( \mathbf{L} \):

\[
\begin{align*}
(\hat{k}_{A,D} \land 1) \circ \hat{b}_{A,D,C} &= 1_A \land \hat{k}_{D,C}, \\
\hat{k}_{A,B,D} \circ \hat{b}_{A,B,D} &= 1_A \land \hat{k}_{B,D}, \\
(\hat{k}_{2,D,B} \land 1) \circ \hat{b}_{D,B,C} &= \hat{k}_{D,B \land C}, \\
\hat{k}_{A,D} \circ \hat{c}_{D,A} &= \hat{k}_{2,D,A}, \\
\hat{k}_{A,D,B} \circ \hat{c}_{D,B} &= \hat{k}_{2,D_B}, \\
\hat{k}_{A,B,D_1 \lor D_2} \circ \hat{c}_{A,D_1,B,D_2} &= \hat{k}_{A,D_1} \lor \hat{k}_{B,D_2}. \\
\end{align*}
\]

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\[
\hat{k}_{p,p} \cdot \hat{w}_p = 1_p,
\]

together with the analogous equations involving \(\hat{b}^\leftarrow\) and the dual equations involving \(\hat{k}^i, \hat{b}^\rightarrow, \hat{c}\) and \(\hat{w}\), in order to eliminate every occurrence of \(\hat{k}^i\).

(The first three equations displayed above are related to the three equations displayed in the proof of \(\text{KA}^\emptyset\) Coherence in Section 7.)

Since \(G(f)\) is a bijection, no occurrence of \(\hat{k}^i\) can remain. That no occurrence of \(\hat{w}\) can remain after having eliminated all the occurrences of \(\hat{k}^i\) follows from the Composition Elimination result of [10] (Section 9.4) combined with the existence of a functor \(G'\) from \(\textbf{L}\) to the category \(\text{Mat}\), which is isomorphic to the skeleton of the category of finite-dimensional vector spaces over a fixed number field with linear transformations as arrows (see [10], Section 12.5). For example, \(c_{p,q,r,s}^k\) is mapped by \(G\) to the relation whose diagram is

\[
\begin{array}{c}
(p \land q) \lor (r \land s) \\
\downarrow \\
(p \lor r) \land (q \lor s)
\end{array}
\]

while by \(G'\) it is mapped to the matrix

\[
\begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
\]

The category \(\text{Mat}\) takes into account whether in the diagram corresponding to \(G(f)\) two occurrences of the same letter may be joined by more than one line. \(\dashv\)

### 16 Symmetric bimonoidal intermuting categories

In this section, which is parallel to Section 12, we establish as our final result a symmetric variant of Restricted Bimonoidal Intermuting Coherence. This result is based essentially on Symmetric Biaassociative Intermuting Coherence of Section 14. Before defining symmetric bimonoidal intermuting categories, we introduce some preliminary notions of bimonoidal categories with natural commutativity isomorphisms, and prove auxiliary coherence results for them.

A symmetric bimonoidal category is a bimonoidal category \(\langle \mathcal{A}, \land, \lor, \top, \bot \rangle\) (see Section 2) such that \(\langle \mathcal{A}, \land, \lor \rangle\) is a symmetric biaassociative category (see Section 14). Symmetric bimonoidal categories are coherent in the sense that the symmetric bimonoidal category freely generated by a set of objects is a diversified preorder (see [10], Section 6.4).
In [13] one can find a justification in the spirit of Sections 3 and 4 of all the assumptions made for symmetric bimonoidal categories. In these categories $\xi$, for $\xi \in \{\land, \lor\}$, intermutes with itself, and $\xi \land \to, \xi \lor \to$ are upward and downward preserved by $\xi$, where this preservation is understood quite analogously to what we had in Section 4. Mac Lane’s pentagonal and hexagonal equations follow from this preservation. The role of $\xi^k$ in that is played by the natural isomorphisms $\xi^m$ of Section 14. In terms of these isomorphisms and of the unit objects one defines the $b$-arrows and the $c$-arrows.

A symmetric normal bimonoidal category is a symmetric bimonoidal category that is also normal bimonoidal (see Section 7), and that satisfies moreover the two equations

\[
\begin{align*}
(\psi \hat{c}) & \qquad \hat{w}_{\land} \land \hat{c}_{\land, \land} = \hat{w}_{\land}, \\
(\psi \check{c}) & \qquad \check{c}_{\land, \land} \land \hat{w}_{\land} = \check{w}_{\land},
\end{align*}
\]

called collectively ($wc$). The equation $(\psi \check{c})$ says that $\check{c}$ is upward preserved by $\land$, and $(\psi \hat{c})$ says that $\hat{c}$ is downward preserved by $\bot$ (see Section 4). From these two equations we obtain immediately the equations

\[
\begin{align*}
(c1) & \qquad \check{c}_{\land, \land} = 1_{\land \land}, \\
& \qquad \hat{c}_{\land, \land} = 1_{\land \land}.
\end{align*}
\]

We call $\text{NS}_{\land} \land$ the symmetric normal bimonoidal category freely generated by a set of objects, and we can prove the following.

**Symmetric Normal Bimonoidal Coherence.** The category $\text{NS}_{\land} \land$ is a diversified preorder.

**Proof.** By Symmetric Biassociative Coherence (see the beginning of Section 14) and the results of [10] (Chapter 3, in particular Section 3.3), we can replace the category $\text{NS}_{\land} \land$ by a strictified category $\text{NS}_{\land}^{st} \land$ where the $b$-arrows and the $c$-arrows are identity arrows. We can show that $\text{NS}_{\land}^{st} \land$ is a diversified preorder, in the sense in which $\text{SC}^{st}$ is a diversified preorder (see the beginning of Section 14). For that we proceed as for the proof of Normal Biunital Coherence in Section 6. This will imply that $\text{NS}_{\land} \land$ is a diversified preorder.

A symmetric $\kappa$-normal bimonoidal category is a symmetric normal bimonoidal category that is also $\kappa$-normal bimonoidal (see Section 7). We call $\text{KS}_{\land}^{\emptyset} \land$ the symmetric $\kappa$-normal bimonoidal category freely generated by the empty set of objects. We can prove the following.

**KS^{\emptyset}_{\land} \land Coherence.** The category $\text{KS}_{\land}^{\emptyset} \land$ is a preorder.

**Proof.** We enlarge the proofs of $\text{K}^{\emptyset}_{\land} \land$ Coherence and $\text{K}^{\emptyset}_{\land} \land$ Coherence of Sections 6 and 7 by using the following equations of symmetric monoidal
categories:
\[
\xi c_A = \delta_A^{-1} = \delta_A, \\
\xi A,\zeta = \delta_A^{-1} = \delta_A.
\]
for \((\xi, \zeta) \in \{(\land, \top), (\lor, \bot)\}\). With these equations and the equations \((c1)\) we can eliminate every occurrence of \(c\).

A *symmetric bimonoidal intermuting* category is a symmetric \(\kappa\)-normal bimonoidal category \(\langle A, \land, \lor, \top, \bot \rangle\) that is also a bimonoidal intermuting category (see Section 12) such that \(\langle A, \land, \lor \rangle\) is a symmetric biassociative intermuting category (see Section 14). This means that we assume all the equations we have assumed for various notions of categories considered up to now, excluding the lattice categories of the preceding section. In addition to the equations listed for bimonoidal intermuting categories at the beginning of Section 12 we assume the equations \((c_k)\) of Section 14 and the equations \((wc)\) from the beginning of this section.

The \(b\)-arrows and the \(c\)-arrows are definable in terms of the isomorphisms \(\xi m\) and of the unit objects, as we mentioned above. We could moreover replace the equations \((c_k b)\) and \((c_k c)\) by the equations \((c_k c_m)\) of Section 14. According to [13], the equation \((\psi c_m)\) (see Section 14), and an analogous equation with \(\land\) replaced by \(\lor\), deliver Mac Lane’s pentagonal and hexagonal equations. We have mentioned in Section 14 that the equations \((c_k c_m)\) and \((\psi c_m)\) may be viewed as instances of the hexagonal interchange equation of [3] (end of Definition 1.7), which we have called \(HI\), and which serves to define \(n\)-fold monoidal categories for \(n > 2\). In an \(n\)-fold version of our notion of symmetric bimonoidal intermuting category, for \(n \geq 2\), the scheme of the equation \(HI\) would deliver all the coherence conditions, save those involving the unit objects.

Let \(SC_k\) and \(SC_k^{\emptyset}\) be the free symmetric bimonoidal intermuting categories generated respectively by a nonempty set of objects and the empty set of objects. By combining the proof of \(AC_k^{\emptyset}\) Coherence of Section 12 and the proof of \(KS_k^{\emptyset}\) Coherence above we obtain a proof of \(SC_k^{\emptyset}\) Coherence, which says that \(SC_k^{\emptyset}\) is a preorder.

By proceeding as in Section 12, and by relying on Symmetric Biassociative Intermuting Coherence of Section 14, we can then establish the following.

**Restricted Symmetric Bimonoidal Intermuting Coherence.**

*If \(f,g : A \to B\) are arrows of \(SC_k\) such that both \(A\) and \(B\) are either pure and diversified, or no letter occurs in them, then \(f = g\) in \(SC_k\).*

**Acknowledgement.** Work on this paper was supported by the Ministry of Science of Serbia (Grants 144013 and 144029).
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