CONNECTIONS ON METRIPLECTIC MANIFOLDS

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Abstract. In this note we discuss conditions under which a linear connection on a manifold equipped with both a symmetric (Riemannian) and a skew-symmetric (almost-symplectic or Poisson) tensor field will preserve both structures.

If \((M, g)\) is a (pseudo-)Riemannian manifold, then classical results due to T. Levi-Civita, H. Weyl and E. Cartan [7] show that for any \((1, 2)\) tensor field \(T_{i}^{jk}\) which is skew-symmetric by lower indices, there exists a unique linear connection \(\Gamma\) preserving the metric \(\nabla_{\Gamma} g = 0\), with \(T\) as its torsion tensor: \(T_{i}^{jk} = \frac{1}{2}(\Gamma_{i}^{jk} - \Gamma_{j}^{ik})\).

It has also been shown [4] that given any symmetric (by lower indices) \((1, 2)\) tensor \(S_{i}^{jk}\) on a symplectic manifold \((M, \omega)\), there exists a unique linear connection preserving \(\omega\) which has \(S\) as its symmetric part, i.e., \(S_{i}^{jk} = \frac{1}{2}(\Gamma_{i}^{jk} + \Gamma_{j}^{ik})\). Moreover, it is known [9] that if \(\omega\) is a regular Poisson tensor on \(M\), then there always exists a linear connection on \(M\) with respect to which \(\omega\) is covariantly constant. Such connections are called Poisson connections, and can be chosen to coincide with the Levi-Civita connection of the metric \(g\) (if \(g\) is Riemannian) in certain cases.

Considering these results, one is naturally led to the question: Given a skew-symmetric \((0, 2)\) tensor \(\omega\), and a (pseudo-)Riemannian metric \(g\) on a manifold \(M\), when do there exist linear connections preserving \(\omega\) and \(g\) simultaneously:

\[\nabla_{\Gamma} \omega + \nabla_{\Gamma} g = 0?\] (1)

Motivated by the terminology of P.J. Morrison [6], we call the a manifold equipped with both a (pseudo-)Riemannian metric \(g\) and a skew-symmetric \((2, 0)\) tensor \(P\) a metriplectic manifold, and a connection which preserves both tensors will be called a metriplectic connection. In the first section we restrict ourselves to the case in which both \(\omega = P^{-1}\) and \(g\) are nondegenerate, that is \(\omega\) is almost-symplectic and \(g\) is Riemannian. We combine the results from [7] and [4] to derive a necessary condition for a connection \(\Gamma\) to be a metriplectic connection. We also discuss the form of \(\Gamma\) in the almost-Hermitian and symplectic cases. The main result of this section is the following

**Proposition** Any connection \(\Gamma\) with symmetric part \(\Pi\) and torsion \(T\) that preserves both a Riemannian metric \(g\) and an (almost-)symplectic form \(\omega\) has the form \(\Gamma = \Pi + T\)

\[\Pi = L(g) + \tilde{g}(T), \quad T = \omega^{-1}\nabla g \omega - (1/2)\omega^{-1}d\omega - \tilde{\omega}\tilde{g}(T),\]

where \(L(g)\) is the Levi-Civita connection defined by the metric \(g\), and the “bar” operators \(\tilde{g}(T)\) and \(\tilde{\omega}\tilde{g}(T)\) are related to the symmetries of the torsion \(T\).

In the second section we give the proof of a theorem due to Shubin [8] which states that if \(M\) admits a metriplectic connection, and \(P = \omega^{-1}\) is nondegenerate, then \(M\) is a Kähler manifold. We also formulate an observation made by Vaisman [9] as a generalization of Shubin’s theorem in the case that \(P\) is degenerate.
1. Necessary Conditions on the Metriplectic Connection $\Gamma$

We consider here the case where $\omega = P^{-1}$ is skew-symmetric and nondegenerate (not necessarily closed), and $g$ is Riemannian. Suppose that $\Gamma = \Pi + T$ is a connection on the Poisson-Riemannian manifold $(M, g, \omega)$ with $\Pi$ and $T$ symmetric and skew-symmetric (torsion) tensors respectively. Assume that $\Gamma$ satisfies (1). Since $\nabla^\Pi g = 0$, we know from [4] that the symmetric tensor $\Pi$ must have local components

$$\Pi^i_{jk} = L(g)_{ij} + g^{is}(T^q_{sj}g_{kq} + T^q_{sk}g_{jq}),$$

where $L(g)_{ij} = (1/2)g^{is}(g_{js,k} + g_{ks,j} - g_{jk,s})$ is the Levi-Civita connection for $g$. On the other hand, since $\nabla^\Pi \omega = 0$, we have (2)

$$T^i_{jk} = L(\omega)_{jk} - \omega^{is}(\Pi^q_{sj}\omega_{kq} + \Pi^q_{sk}\omega_{jq}),$$

where $L(\omega)_{jk} = (1/2)\omega^{is}(\omega_{js,k} + \omega_{ks,j} + \omega_{jk,s})$. We introduce the following operator: for any nondegenerate $(0, 2)$ tensor $h$ define the linear operator $\bar{h}$ on $(1, 2)$ tensors by

$$\bar{h}(B)_{ijk} = h^{is}(B^q_{sj}h_{kq} + B^q_{sk}h_{jq}).$$

With this definition, we can write (2) and (3) as

$$\Pi = L(g) + \bar{g}(T) \quad \text{and} \quad T = L(\omega) - \bar{\omega}(\Pi).$$

Thus, the original connection $\Gamma$ has the form

$$\Gamma = L(g) + L(\omega) - \bar{\omega}(\Pi) + \bar{g}(T),$$

$$= L(g) + L(\omega) - \bar{\omega}(L(g)) - \bar{\omega}(\bar{g}(T)) + \bar{g}(T).$$

Notice that when $\bar{h}$ operates on a connection form $A$, the result is related to the covariant derivative of $h$ with respect to $A$ as follows:

$$\nabla^A h = \partial h - h(\bar{h}(A)) \quad \text{or} \quad \bar{h}(A) = h^{-1}\partial h - h^{-1}\nabla^A h.$$ (5)

In particular, $\bar{\omega}(L(g)) = \omega^{-1}(\partial \omega - \nabla^g \omega)$ where $\nabla^g$ is the covariant derivative with respect to $g$. So we have

$$\Gamma = L(g) + L(\omega) - \omega^{-1}(\partial \omega - \nabla^g \omega) - \bar{\omega}(\bar{g}(T)) + \bar{g}(T).$$

Rewriting $L(\omega)$ as $\omega^{-1}\partial \omega - (1/2)\omega^{-1}\omega$, we have the following

**Proposition.** Any connection $\Gamma$ the preserves both a Riemannian metric $g$ and an (almost-)symplectic form $\omega$ has the form $\Gamma = \Pi + T$ with

$$\Pi = L(g) + \bar{g}(T), \quad T = \omega^{-1}\nabla^g \omega - (1/2)\omega^{-1}\omega - \bar{\omega}(\bar{g}(T)).$$

If $\omega$ is closed (i.e. $(M, \omega)$ is symplectic), then $\Gamma = L(g) + \bar{g}(T) + \omega^{-1}\nabla^g \omega - \bar{\omega}(\bar{g}(T)).$

1.1. Almost-Hermitian Connections with Totally Skew Torsion. Suppose now that $g$ and $\omega$ are related by an almost-complex structure $J = g^{-1}\omega$ satisfying $J^2 = -I$. Observe that $\bar{g}(T) = 0$ if and only if $T$ is totally skew-symmetric with respect to the metric $g$ (that is, $T_{ijk} = T^q_{ij}g_{kq}$ is an exterior 3-form). In this case,

$$T = \omega^{-1}\nabla^g \omega - (1/2)\omega^{-1}\omega.$$ (6)

Thus $\Gamma$ reduces to the canonical almost-hermitian connection with totally skew torsion (see e.g. [3] [4]). Indeed, the torsion 3-form $T(X, Y, Z) = < T(X, Y), Z >_g$ can be expressed in terms of the Nijenhuis tensor $N(X, Y, Z) = < N(X, Y), Z >_g$ of the almost-complex structure $J$ (see [2]) as follows.
Proposition. If the torsion of an almost-Hermitian connection $\Gamma$ is totally skew-symmetric, then

$$T(X, Y, Z) = (1/2)N(X, Y, Z) - (1/2)d\omega(JX, JY, JZ)$$

for all $X, Y, Z$.

Proof.

$$2T_{ijk} = (2\omega^{ri} \nabla^g_{\omega ij} - \omega^{ri} d\omega_{ijk})q_{ik}$$

$$= 2J^k_i \nabla^g_{\omega ij} - J^k_i (\nabla^g_{\omega ij} + \nabla^g_{j \omega ni} + \nabla^g_{i \omega jn})$$

$$= J^k_i \nabla^g_{\omega ij} - J^k_i \nabla^g_{j \omega ni} - J^k_i \nabla^g_{i \omega jn}$$

$$= J^k_i \nabla^g_{\omega ij} - J^k_i \nabla^g_{j \omega ni} + J^k_i \nabla^g_{i \omega jn}$$

$$= -J^k_i \nabla^g_{\omega ji} - J^k_i \nabla^g_{j \omega ki} - J^k_i \nabla^g_{i \omega kj} + J^k_i \nabla^g_{\omega jk} - J^k_i \nabla^g_{\omega k} - J^k_i \nabla^g_{j \omega kn}$$

Using the fact that $\omega^{\alpha n} g_{\alpha} = -J^k_i \nabla^g_{\omega ij} - J^k_i \nabla^g_{j \omega ki} - J^k_i \nabla^g_{i \omega kj} + J^k_i \nabla^g_{\omega jk} - J^k_i \nabla^g_{\omega k} - J^k_i \nabla^g_{j \omega kn}$, we have

$$J^k_i \nabla^g_{\omega ij} = J^k_i \nabla^g_{\omega ji} + J^k_i \nabla^g_{j \omega ki} + J^k_i \nabla^g_{i \omega kj} - J^k_i \nabla^g_{\omega jk} + J^k_i \nabla^g_{\omega k} + J^k_i \nabla^g_{j \omega kn}.$$ 

Permuting the indices $i, j, k$ gives us

$$2T_{ijk} = (\nabla^g_{\omega sr} + \nabla^g_{\omega ts} + \nabla^g_{\omega st})J^r_i J^s_j J^k_k + N_{ijk}$$

$$= -d\omega_{rst}J^r_i J^s_j J^k_k + N_{ijk}. \square$$

Using this expression for $T$ together with (6), it is easy to see that $M$ is:

- Hermitian ($J$ is integrable) $\leftrightarrow T(X, Y, Z) = -(1/2)d\omega(JX, JY, JZ)$,
- Symplectic ($\omega$ is closed) $\leftrightarrow T = (1/2)N$, and
- Kähler ($\omega$ is closed and $g$-parallel) $\leftrightarrow \Gamma$ is the Levi-Civita connection determined by $g$.

1.2. The Symplectic Case. As mentioned above, if $M, \omega$ is symplectic, then the connection $\Gamma$ takes the form

$$\Gamma = L(g) + \bar{g}(T) + \omega^{-1} \nabla^g \omega - \bar{\omega}g(T).$$

Applying (7) to $\omega$ and $\Gamma$, we see that the condition $\nabla^\Gamma \omega = 0$ is equivalent to

$$0 = \partial \omega - \omega(\bar{\omega}(\Gamma))$$

$$= \partial \omega - \omega(\bar{\omega}(L(g))) - \omega(\bar{\omega}(g(T))) - \omega(\bar{\omega}(T))$$

$$= \nabla^g \omega - \omega(\bar{\omega}(\bar{g}(T))) - \omega(\bar{\omega}(T)).$$

However, $0 = \nabla^g \omega - \omega(\bar{\omega}(\bar{g}(T))) - \omega(T)$. Thus we obtain the following condition on $T$:

$$\bar{\omega}(T) = T \text{ or } (\bar{\omega} - I)T = 0.$$

Remark. This result can be derived directly from the condition $\nabla^\Gamma \omega = 0$ together with the Jacobi condition for $\omega$, and is easily seen to be equivalent to the following cyclic condition on the indices of the tensor $T \omega$,

$$T_{ik}^s \omega_{kj} + T_{kj}^s \omega_{sj} + T_{ij}^s \omega_{si} = 0.$$ 

Thus, we may substitute $\bar{g}(\bar{\omega}(T))$ for $\bar{g}(T)$ in (7). Writing the difference $\bar{g}(\bar{\omega}(T)) - \bar{\omega}(\bar{g}(T))$ as $[\bar{g}, \bar{\omega}](T)$, we arrive at the following...
Proposition. A linear connection $\Gamma$ on a symplectic-Riemannian manifold $(M, g, \omega)$ which preserves both $g$ and $\omega$ has the form

$$\Gamma = L(g) + \omega^{-1} \nabla^g \omega + [\bar{g}, \bar{\omega}](T).$$

2. Necessary Conditions on the structure of $(M, g, P)$

It is known [5] that if $M$ is a Kähler manifold, then the Kähler form is parallel with respect to the Levi-Civita connection $L$ on $M$ defined by the Kähler metric, in which case [11] clearly holds (with $\Gamma = L$). In this section we will discuss a partial converse to this fact due to M. Shubin.

2.1. Shubin’s Theorem. On a manifold $M$, let $g_0$ be a Riemannian metric and let $\omega = P^{-1}$ be an almost-symplectic form (non-degenerate and skew-symmetric). Let $L(g_0)$ denote the Levi-Civita connection associated with $g_0$, and let $K = g_0 + \omega$. We denote the covariant derivatives with respect and $L(g_0)$ by $\nabla^0$. The following theorem is a reformulation of a result by Shubin [8], and its proof follows Shubin’s proof, with some variations.

Theorem. If $(M, g_0, \omega)$ is an almost-symplectic Riemannian manifold, and $\nabla^0 K = 0$, then there exists a complex structure $J$ on $M$ such that the metric $g$ defined by $g(X, Y) = \omega(X, JY)$ is parallel with respect to $g_0$ (thus $L(g) = L(g_0)$), and defines a Kähler structure on $M$.

Proof. First note that if $\nabla^0 K = 0$, then $\nabla^0 \omega = 0$. Since $\nabla^0$ is symmetric, it follows (see remark 1.4 in [11]) that $d\omega = 0$, and so $\omega$ is symplectic. Now, any symplectic manifold [8] admits an almost-complex structure $J$ defined by $J = A(-A^2)^{-1/2}$, where $A$ is the linear operator defined by

$$g_0(AX, Y) = \omega(X, Y).$$

Since both $g_0$ and $\omega$ are parallel with respect to $\nabla^0$, it is clear that the operator $A$ will also be parallel, thus $\nabla^0 J = 0$. The integrability of $J$ then follows from the expression

$$N_J(X, Y) = (\nabla^0_{XJ} Y - (\nabla^0_{YJ} X + J(\nabla^0_{YX}) X - J(\nabla^0_{X}) Y$$

for the Nijenhuis torsion of $J$ (see [11]).

The metric $g(X, Y) = \omega(X, JY)$ is Hermitian with respect to $J$. Therefore, it defines a Kähler structure on $M$. Furthermore, the equality $g_{ij} = \omega_{ikj}^k$ shows that $\nabla^0 g = 0$. The connection $L(g_0)$ is symmetric and compatible with $g$, so it must coincide with the Levi-Civita connection $L(g)$. \qed

2.2. Generalization to a degenerate $P$. If the tensor $P$ is degenerate, then we cannot construct the covector $\omega = P^{-1}$ on $M$. In order to deal with this possibility, we change setting from the cotangent to the tangent bundle.

With $g_0$ and $\nabla^0$ as above, suppose that $M$ is equipped with a (possibly degenerate) Poisson tensor $P$, and let $K = g_0^{-1} + P$. If $\nabla^0 K = 0$, then $\nabla^0 P = 0$ and $M$ is a regular Poisson manifold with symplectic foliation $\mathcal{S}(M)$ defined by the kernel of $P$ (see [9]). The restriction $P_S$ of $P$ to a symplectic leaf $S$ is nondegenerate, and $S$ is endowed with a symplectic form $\omega = P_S^{-1}$.

A classical result of Lichnerowicz [5] states that there exist local coordinates $x^i$ along $\mathcal{S}(M)$ and $y^i$ along $\mathcal{N}$ (the transverse foliation orthogonal to $\mathcal{S}(M)$) in which
$g_0$ and $\omega$ have the form $g_0 = g' + g''$ where

$$
g' = (g_0)_{ij}(y)dy^i dy^j, \quad g'' = (g_0)_{ij}(x)dx^i dx^j, \quad \omega = \omega_{ij}(x)dx^i \wedge dx^j.
$$

By restricting $\omega$ and $g''$ to a symplectic leaf $S$, we are in the situation described by Shubin’s Theorem above. Thus, we have a complex structure $J$ which defines a Hermitian metric $g_s = (g_0)_{ij}(x)dx^i dx^j$ on $S$ which is parallel with respect to $g''$ (and, therefore, with respect to $g_0$). We can extend $J$ by 0 to all of $M$, and define a new metric $\hat{g}$ on $M$ by the formula $\hat{g} = g' + g_s$.

This metric is called a partially Kähler metric. It is parallel with respect to $g_0$ and, when restricted to the symplectic leaf $S$, is a Hermitian metric on $S$. In his book [9], Vaisman concludes from these remarks that “the parallel Poisson structures of a Riemannian manifold $(M, g_0)$ are exactly those defined by the Kähler foliation of the $g_0$-parallel partially-Kähler metrics of $M$ (if any)”.

One can view this statement as the following generalization of Shubin’s Theorem.

**Theorem.** If $\nabla^0 K = 0$, then $M$ is a regular Poisson manifold (with the Poisson tensor $P$), and there exists a complex structure $J$ on the symplectic leaves of $M$ such that the metric $\hat{g}$ defined above is parallel with respect to $g_0$ (thus $\nabla^g = \nabla^0$), and the restriction of $\hat{g}$ to the symplectic leaf $S$ defines a Kähler foliation on $M$.

### 3. Related Questions

We have shown that the preservation of both Riemannian and Poisson structures on $M$ by a linear connection imposes certain conditions on the connection itself, as well as on the structure of the manifold $(M, g, P)$. In future work we will address some related questions, including: When can one guarantee the existence of a metriplectic connection on a manifold $M$, and is there an optimal or canonical choice of such a connection, similar to the canonical connection given in [3]?

### References

[1] M. DeLeon and P. Rodrigues. *Methods of Differential Geometry in Analytical Mechanics*. Elsevier Science Pub., 1989.

[2] T. Friedrich and S. Ivanov. *Parallel Spinors and Connections with Skew-Symmetric Torsion in String Theory*. Asian J. Math. 6 303-336, math.dg/0102142, 2002.

[3] P. Gauduchon. *Hermitian Connections and Dirac Operators*. Bolletino U.M.I. (7) 11-B, Suppl. fasc. 2, 257-288, 1997.

[4] I. Gelfand, V. Retakh, and M. Shubin. *Fedosov Manifolds*. Adv.Math. 136, 104-140, 1998.

[5] A. Lichnerowicz. *Global Theory of Connections and Holonomy Groups*. Noordhoff Int’l Pub., 1976.

[6] P. J. Morrison. *A Paradigm for Joined Hamiltonian and Dissipative Systems*. Physica 18D,410-419, 1986.

[7] J. A. Schouten. *Ricci-Calculus*. Springer-Verlag, Berlin, 1954.

[8] M. Shubin. *A sequence of connections and a Characterization of Kähler Manifolds*. Contemporary Mathematics, AMS vol231, 265-270, 1999.

[9] I. Vaisman. *Lectures on the Geometry of Poisson Manifolds*. BirkhauserVerlag, 1994.

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