A note on “On the classification of Landsberg spherically symmetric Finsler metrics”

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Abstract

In this paper, we prove that all spherically symmetric Landsberg surfaces are Berwaldian. We modify the classification of spherically symmetric Finsler metrics, done by the author in [S. G. Elgendi, On the classification of Landsberg spherically symmetric Finsler metrics, Int. J. Geom. Methods Mod. Phys. 18 (2021)], of Berwald type of dimension \(n \geq 3\). Precisely, we show that all Berwald spherically symmetric metrics of dimension \(n \geq 3\) are Riemannian or given by a certain formula. As a simple class of Berwaldian metrics, we prove that all spherically symmetric metrics in which the function \(\phi\) is homogeneous of degree \(-1\) in \(r\) and \(s\) are Berwaldian.

Keywords: spherically symmetric metrics; Berwald metrics; Landsberg metrics

MSC 2020: 53B40; 53C60

1 Introduction

In Finsler geometry, the existence of a regular non-Berwaldian Landsberg Finsler metric is still an open problem. In the two-dimensional (2D) case, that problem seems more complicated. Some non-regular Landsberg Finsler metrics which are not Berwaldian are known in higher dimensions (cf. [1, 3, 6]). But in dimension two, to the best of our knowledge, no concrete examples are given. There is a class of examples of non-Berwaldian Landsberg spherically symmetric surfaces obtained by L. Zhou [8]. But in a joint paper of the author (cf. [4]), it was proven that this class is, in fact, Berwaldian.

In [2], we have classified all Landsberg spherically symmetric Finsler metrics of dimension \(n \geq 3\). Precisely, we prove that all Landsberg spherically symmetric metrics of dimension \(n \geq 3\) are Riemannian or its geodesic spray is given by a certain formula. In this paper, we complete this classification by showing that all 2D Landsberg spherically symmetric Finsler metrics (regular or non-regular) are Berwaldian.
Also, in [2], we have proven that all Berwaldian spherically symmetric metrics of dimension \( n \geq 3 \) are Riemannian. But we discovered a missing case in the proof. In this paper, we modify this result and prove that all Berwaldian spherically symmetric metrics of dimension \( n \geq 3 \) are Riemannian or the function \( \phi \) is given by

\[
\phi = s \psi \left( \frac{s^2}{g(r) + s^2 \int 4rc_0(r)g(r)dr} \right) e^{-\int \left( \frac{2}{r} - 2r^3c_0(r) \right) dr}
\]

where \( c_0(r) \) is a smooth function and \( g(r) = e^{\int \left( \frac{2}{r} - 4r^{3c_0(r)} \right) dr} \).

At the end of this paper we provide a table classifying all spherically symmetric Finsler metrics of Landsberg and Berwald types.

2 Spherically symmetric metrics

Throughout, we use the notations and terminology of [2]. A spherically symmetric Finsler metric \( F \) on \( B^n(r_0) \subset \mathbb{R}^n \) is defined by

\[
F(x, y) = |y| \phi \left( |x|, \frac{\langle x, y \rangle}{|y|} \right),
\]

where \( \phi : [0, r_0) \times \mathbb{R}^n \to \mathbb{R}, (x, y) \in T\mathbb{B}^n(r_0) \setminus \{0\} \), and \( |\cdot| \) and \( \langle \cdot, \cdot \rangle \) are the standard Euclidean norm and inner product on \( \mathbb{R}^n \). Or simply, \( F = u \phi (r, s) \) where \( r = |x|, u = |y| \) and \( s = \frac{\langle x, y \rangle}{|y|} \).

The spherically symmetric metrics are a special general \( (\alpha, \beta) \)-metrics. Therefore, a spherically symmetric metric \( F = u\phi (r, s) \) on \( B^n(r_0) \) is regular if and only if \( \phi \) is positive, \( C^\infty \) function such that

\[
\phi - s\phi_s > 0, \quad \phi - s\phi_s + (r^2 - s^2)\phi_{ss} > 0.
\]

Moreover, in case of \( n = 2 \), the regularity condition is

\[
\phi - s\phi_s + (r^2 - s^2)\phi_{ss} > 0
\]

for all \( |s| \leq r < r_0 \). The subscript \( s \) (resp. \( r \)) refers to the derivative with respect to \( s \) (resp. \( r \)). The spherically symmetric Finsler metrics are studied in many papers for more details, we refer to, [5] [7] [8].

Since the components of the metric tensor associated with the Euclidean norm are just the Kronecker delta \( \delta_{ij} \), then we lower the indices of \( y^i \) and \( x^i \) as follows

\[
y_i := \delta_{ih}y^h, \quad x_i := \delta_{ih}x^h.
\]

It should be noted that \( y_i \) and \( x_i \) are the same as \( y^i \) and \( x^i \) respectively. So we confirm that \( y_i \neq F \frac{\partial F}{\partial y^i} \) rather \( y_i = u \frac{\partial u}{\partial y^i} \). Moreover, we have the following properties

\[
y^iy_i = u^2, \quad x^ix_i = r^2, \quad y^iy_i = \langle x, y \rangle.
\]

By making use of the above notations and properties, we are able to keep the indices consistent. Many articles in the literature study the geometric objects associated to a spherically
symmetric Finsler metric with some kind of inconsistency with the indices. For example, in \[8\], a tensorial equation has an up index in one side and in the other side the same index is lower index. We tried to solve this problem in \[2\] but with a bit long formulae. In this paper, we fix it with a simple and natural way.

The components \(g_{ij}\) of the metric tensor of the spherically symmetric metric \(F = u\phi(r, s)\) are given by
\[
g_{ij} = \sigma_0 \delta_{ij} + \sigma_1 x_i x_j + \frac{\sigma_2}{u} (x_i y_j + x_j y_i) + \frac{\sigma_3}{u^2} y_i y_j, \tag{2.1}
\]
where
\[
\sigma_0 = \phi(\phi - s\phi_s), \quad \sigma_1 = \phi_s^2 + \phi\phi_{ss}, \quad \sigma_2 = (\phi - s\phi_s)\phi_s - s\phi\phi_{ss}, \quad \sigma_3 = s^2\phi\phi_{ss} - s(\phi - s\phi_s)\phi_s.
\]

The following geometric objects can be found in \[5, 7, 8\]. The components \(G_{jk\ell}\) of the Berwald curvature are defined by
\[
G_{jk\ell} = \frac{\partial^2}{\partial y^k \partial y^\ell} G^i - \frac{2}{u^2} y^k y^\ell G^i,
\]
where
\[
G^i = u P y^i + u^2 Q x^i, \tag{2.3}
\]
where the functions \(P\) and \(Q\) are defined by
\[
Q := \frac{1}{2r} \frac{-\phi_r + s\phi_{rs} + r\phi_{ss}}{\phi - s\phi_s + (r^2 - s^2)\phi_{ss}}, \quad P := -\frac{Q}{\phi} (s\phi + (r^2 - s^2)\phi_s) + \frac{1}{2r\phi} (s\phi_r + r\phi_s). \tag{2.4}
\]

The components \(G_{ijk\ell}^i\) of the Berwald curvature are defined by
\[
G_{ijk\ell}^i = \frac{\partial^2}{\partial y^k \partial y^\ell} G^i.
\]

For a spherically symmetric Finsler metric \(F = u\phi(r, s)\), the components \(G_{ijk\ell}^i\) are calculated as follows
\[
G_{ijk\ell}^i = \frac{P_{ss}}{u^2} (\delta_j^i x_k x_\ell + \delta_k^i x_j x_\ell + \delta_\ell^i x_j x_k) + \frac{1}{u} (P - sP_s) (\delta_j^i \delta_k^\ell + \delta_k^i \delta_j^\ell + \delta_\ell^i \delta_{jk})
- \frac{sP_{ss}}{u^2} (\delta_j^i (x_k y_\ell + x_\ell y_k) + \delta_k^i (x_j y_\ell + x_\ell y_j) + \delta_\ell^i (x_j y_k + x_k y_j))
- \frac{sP_{ss}}{u^2} (\delta_{jk} x_\ell + \delta_{j\ell} x_k + \delta_{k\ell} x_j) y^i + \frac{1}{u} (Q_s - sQ_{ss}) (\delta_{jk} x_\ell + \delta_{j\ell} x_k + \delta_{k\ell} x_j) x^i
+ \frac{1}{u^3} (s^2 P_{ss} + sP_s - P) (\delta_j^i y_k y_\ell + \delta_k^i y_j y_\ell + \delta_\ell^i y_j y_k) + (\delta_{jk} y_\ell + \delta_{j\ell} y_k + \delta_{k\ell} y_j) y^i
+ \frac{1}{u^3} (3P - s^3 P_{sss} - 6s^2 P_{ss} - 3sP_s) y_j y_k y_\ell y^i + \frac{P_{ss}}{u^2} x_j x_k x_\ell y^i \tag{2.5}
\]
Theorem 2.3.

where conditions are satisfied.

a

Theorem 2.2.

Proposition 2.1.

\[ (2.2) \]

\[ A \text{ Landsberg spherically symmetric Finsler metric of dimension } n \geq 3 \]

\[ (2.9) \]

where \( c_0, c_1, c_2, c_3 \) are arbitrary functions of \( r \).

Theorem 2.3.

A spherical symmetric Finsler surface is Berwaldian if and only if

\[ P = b_1 s + \frac{b_2}{\sqrt{r^2 - s^2}} + \frac{b_3 (r^2 - 2s^2)}{\sqrt{r^2 - s^2}}, \]

\[ Q = b_0 s^2 + \frac{1}{2 b_1} + \frac{b_2 s (r^2 - 2s^2)}{r^4 \sqrt{r^2 - s^2}} - \frac{b_3 s (3r^2 - 2s^2)}{r^2 \sqrt{r^2 - s^2}} - \frac{a}{r^2 s} \sqrt{r^2 - s^2}, \]

where \( a, b_0, b_1, b_2, b_3 \) are arbitrary functions of \( r \) and to be chosen such that the compatibility conditions are satisfied.
Proposition 2.4. [2] A spherically symmetric surface is Landsbergian if and only if \((r^2 - s^2)L_1 + 3L_2 = 0\), that is,
\[
(r^2 - s^2)L_1 + 3L_2 = ((r^2 - s^2)Q_{sss} + 3(Q_s - sQ_{ss}) + 3P_{ss}) + 3(P - sP_s)\phi_s
\]
\[
+ ((r^2 - s^2)(sQ_{ss} + P_{ss}) + 3s(Q_s - sQ_{ss}) - 3sP_{ss})\phi
\]
\[
= \frac{1}{s}(sH - (r^2 - s^2)H_s)\phi_s + ((r^2 - s^2)K_s - 3sK)\phi = 0,
\]
where \(K := P_{ss} - Q_s + sQ_{ss}\).

We end this section by proving the following result.

Proposition 2.5. A spherically symmetric Finsler surface \(F = u\phi\) is Berwaldian if and only if \(sH - (r^2 - s^2)H_s = 0\).

Proof. Let \(F = u\phi\) be a Berwaldian spherically symmetric Finsler surface, then the Berwald curvature vanishes. Then, we conclude that the mean curvature \(E_{ij}\) vanishes as well. Now, by [2, Proposition 6.1], we obtain that \(sH - (r^2 - s^2)H_s = 0\).

Conversely, assume that \(sH - (r^2 - s^2)H_s = 0\). Then, by the proof of Theorem 6.3 in [2], the functions \(P\) and \(Q\) are given by Theorem 2.3. Consequently, \(F\) is Berwaldian.

3 Berwald case

The following theorem is a modified version of [2, Theorem 5.4].

Theorem 3.1. All Berwaldian spherically symmetric metrics of dimension \(n \geq 3\) are Riemannian or the function \(\phi\) is given by
\[
\phi = s \psi \left( \frac{s^2}{g(r) + s^2} \int \frac{dr}{4rc_0(r)g(r)} \right) e^{-\int \left( \frac{r^2}{\sqrt{r^2 - s^2}} \right) dr}
\]
where \(c_0(r)\) is a smooth function and \(g(r) = e^{\int \left( \frac{r^2}{\sqrt{r^2 - s^2}} \right) dr}\).

Proof. Let \(F\) be a Berwald spherically symmetric of dimension \(n \geq 3\). Since every Berwald metric is Landsbergian, then the geodesic spray of \(F\) is determined by (2.9);
\[
P = c_1 s + \frac{c_2}{r^2} \sqrt{r^2 - s^2}, \quad Q = \frac{1}{2} c_0 s^2 - \frac{c_2 s}{r^4} \sqrt{r^2 - s^2} + c_3.
\]
Calculating the quantities
\[
P - sP_s = \frac{c_2}{\sqrt{r^2 - s^2}}, \quad Q_s - sQ_{ss} = -\frac{c_2}{(r^2 - s^2)^{3/2}};
\]
\[
P_{ss} = -\frac{c_2}{(r^2 - s^2)^{3/2}}, \quad Q_{sss} = \frac{3c_2}{(r^2 - s^2)^{5/2}}.
\]
Since \(F\) is Berwaldian, then the mean curvature \(E_{ij} = 0\). Now, plugging the above quantities into the equation \(E_{ij} = 0\) implies
\[
\frac{nc_2}{u \sqrt{r^2 - s^2}} \left( \delta_{ij} - \frac{r^2}{u^2 (r^2 - s^2)} y_i y_j - \frac{1}{u^2 (r^2 - s^2)} x_i x_j + \frac{s}{u (r^2 - s^2)} (x_i y_j + x_j y_i) \right) = 0.
\]
Contracting the above equation by $\delta^{ij}$ and using the properties $\delta^{ij}y_iy_j = u^2$, $\delta^{ij}x_ix_j = r^2$ and $\delta^{ij}x_iy_j = (x, y)$, we have

$$n(n - 2)c_2\sqrt{r^2 - s^2} = 0.$$  

Since $n \geq 3$ and the above equation holds for all $r$ and $s$, we must have $c_2 = 0$. Thus, we have

$$P = c_1s, \quad Q = \frac{1}{2}c_0s^2 + c_3. \quad (3.1)$$

Plugging the above formulae of $P$ and $Q$ into the compatibility conditions \((2.3)\), we get

$$C_1 = (1 - 2c_3r^2 + (c_1 + 2c_3)s^2)\phi_s - s(c_1 + 2c_3)\phi = 0,$$

$$C_2 = \frac{1}{r}\phi_r - (c_1s + c_0s(r^2 - s^2))\phi_s - (c_1 + c_0s^2)\phi = 0.$$

Now, we have two cases; the first case is $1 - 2c_3r^2 + (c_1 + 2c_3)s^2 \neq 0$, then solving the above two equations algebraically implies

$$\frac{\phi_s}{\phi} = \frac{(c_1 + 2c_3)s}{1 + (c_1 + 2c_3)s^2 - 2c_3r^2},$$

$$\frac{\phi_r}{\phi} = \frac{r(c_0(1 + c_1r^2)s^2 + 2c_1(c_1 + 2c_3)s^2 + c_1(1 - 2c_3r^2))}{1 + (c_1 + 2c_3)s^2 - 2c_3r^2}. \quad (3.2)$$

Integrating $\frac{\phi_r}{\phi}$ with respect to $s$ yields

$$\phi = a(r)\sqrt{(c_1 + 2c_3)s^2 - 2c_3r^2 + 1},$$

where $a(r)$ is to be chosen such that both formulae of \((3.2)\) are satisfied, that is, calculating $\frac{\phi_r}{\phi}$ and equaling it with the second formula of \((3.2)\) we obtain $\phi$. Consequently, the metric

$$F = u\phi = a(r)\sqrt{(c_1 + 2c_3)(x, y)^2 + (-2c_3|x|^2 + 1)|y|^2}$$

is Riemannian.

The second case is $1 - 2c_3r^2 + (c_1 + 2c_3)s^2 = 0$. This implies $c_1 = -\frac{1}{r^2}$ and $c_3 = \frac{1}{2r^2}$. In this case, the functions $P$ and $Q$ are given by

$$P = -\frac{s}{r^2}, \quad Q = \frac{1}{2}c_0s^2 + \frac{1}{2r^2}.$$  

Hence, the compatibility conditions \((2.3)\) reduced to

$$C_2 = \frac{1}{r}\phi_r - \left(-\frac{s}{r^2} + c_0s(r^2 - s^2)\right)\phi_s + \left(\frac{1}{r^2} - c_0s^2\right)\phi = 0.$$

According to \([9]\) Lemma 3.3, the general solution of the above equation is given by

$$\phi = s\, \psi\left(\frac{s^2}{g(r) + s^2\int 4rc_0(r)g(r)dr}\right) e^{-\int \left(\frac{2}{r} - 2r^3c_0(r)\right)dr}$$

where $g(r) = e^{\int \left(\frac{2}{r} - 2r^3c_0(r)\right)dr}$.  \(\Box\)
Corollary 3.2. A spherically symmetric Finsler metric of dimension \( n \geq 3 \) is Berwaldian if and only if

\[
P = f_1 s, \quad Q = f_2 s^2 + f_3,
\]

where \( f_1, f_2 \) and \( f_3 \) are arbitrary functions of \( r \).

Proof. Assume that \( F \) is a spherically symmetric metric with the geodesic spray given by the functions

\[
P = f_1 s, \quad Q = f_2 s^2 + f_3.
\]

Then, we have

\[
P - sP_s = 0, \quad P_{ss} = 0, \quad Q_s - sQ_{ss} = 0, \quad Q_{sss} = 0.
\]

That is, the Berwald curvature (2.5) vanishes and hence the metric is Berwaldian. Conversely, assume that \( F \) is Berwaldian, then by (3.1), the functions \( P \) and \( Q \) can be written in the form

\[
P = f_1 s, \quad Q = f_2 s^2 + f_3.
\]

This completes the proof. \( \square \)

As a simple class of spherically symmetric Finsler metrics of Berwald type, we have the following class.

Theorem 3.3. All spherically symmetric metrics in which the function \( \phi \) is homogeneous of degree \( -1 \) in \( r \) and \( s \) are Berwaldian.

Proof. Let \( F = u\phi \) be a spherically symmetric metric such that \( \phi \) is homogeneous of degree \( -1 \) in \( r \) and \( s \). Then by Euler’s Theorem of homogeneous functions, we have

\[
r\phi_r + s\phi_s = -\phi.
\]

Differentiating the above equation with respect to \( s \), we get

\[
r\phi_{rs} + s\phi_{ss} = -2\phi_s.
\]

Then, by substituting from the above two equations into (2.4) to get the functions \( P \) and \( Q \) as follows: to get \( Q \), we substitute by \( \phi \) and \( \phi_{rs} \). To obtain \( P \), we substitute by \( \phi \). Therefore, we find that

\[
Q = \frac{1}{2r^2}, \quad P = -\frac{s}{r^2}.
\]

By Corollary 3.2, \( F \) is Berwaldian. Moreover, for the above formulae of \( P \) and \( Q \), one can see that the compatibility conditions are satisfied. \( \square \)

4 Landsberg surfaces

The following theorem completes the classification, done in [2], of Landsberg spherically symmetric Finsler metrics.

Theorem 4.1. All Landsberg spherically symmetric Finsler surfaces are Berwaldian.
Proof. Let $F = u\phi$ be a Landsberg spherically symmetric Finsler surface. Then the condition (2.10) together with the compatibility conditions (2.8) are satisfied.

Now, at each point $(x, y) \in T\mathbb{B}^n(r_0) \setminus \{0\}$, we can consider the compatibility condition $C_1 = 0$ and the Landsberg condition (2.10) are algebraic homogeneous equations in $\phi$ and $\phi_s$. Therefore, at each point the two equations must be dependent so that $\phi$ has non-zero values, since in the regular case, $\phi$ is positive. If we allow the surface to be singular in certain direction, then it may happen that $\phi$ is zero at that direction, in this case, the two equations should be dependent on a subset of $T\mathbb{B}^n(r_0)$. Otherwise, the functions $\phi$ and $\phi_s$ vanish.

We claim that the two equations are not compatible in the sense that will be explained below and hence the coefficients of $\phi_s$ and $\phi$ in (2.10) must vanish. Therefore, $sH - (r^2 - s^2)H_s = 0$ and by Proposition (2.3), the surface is Berwaldian.

Now, let’s show that the conditions (2.10) and (2.8) are not compatible. We consider a Berwald surface, then by Theorem (2.3) the functions $P$ and $Q$ are given by

\begin{align*}
P &= b_1 s + \frac{b_2}{\sqrt{r^2 - s^2}} + \frac{b_3(r^2 - 2s^2)}{\sqrt{r^2 - s^2}}, \\
Q &= b_0 s^2 + \frac{1}{2} b_1 + \frac{b_2 s(r^2 - 2s^2)}{r^4 \sqrt{r^2 - s^2}} - \frac{b_3 s(3r^2 - 2s^2)}{r^2 \sqrt{r^2 - s^2}} - \frac{a}{r^2} s \sqrt{r^2 - s^2}.
\end{align*}

(4.1)

Substituting by the above formulae of $P$ and $Q$ into the compatibility condition $C_1$, we get

\[
\phi_s = \frac{2b_1 s \sqrt{r^2 - s^2} - a s^2 + 2b_3 s^2 - 4b_3 s^2 + 2b_2}{\sqrt{r^2 - s^2}(1 - b_1 r^2 + 2b_1 s^2 + s \sqrt{r^2 - s^2}(a + 4b_3))}.
\]

(4.2)

It should be noted that the denominator $1 - b_1 r^2 + 2b_1 s^2 + s \sqrt{r^2 - s^2}(a + 4b_3) \neq 0$. Indeed, assume that $1 - b_1 r^2 + 2b_1 s^2 + s \sqrt{r^2 - s^2}(a + 4b_3) = 0$. Removing the square root and combining the like terms imply

\[(a^2 + 8ab_3 + 4b_1^2 + 16b_3^2) s^4 - (a^2 r^2 + 8ab_3 r^2 + 4b_1^2 r^2 + 16b_3^2 r^2 - 4b_1) s^2 + b_1^2 r^4 - 2b_1 r^2 + 1 = 0.
\]

Since the above equation is valid for all $s$, we have $b_1 = \frac{1}{r^2}, a = -4b_3$. Substituting these formulae in to the coefficient of $s^4$, we have

\[a^2 + 8ab_3 + 4b_1^2 + 16b_3^2 = 4 \neq 0.
\]

Which is a contradiction.

Now, we can rewrite the Landsberg condition (2.10) as follows

\[
(s\phi + (r^2 - s^2)\phi_s)((r^2 - s^2)Q_{sss} + 3(Q_s - sQ_{ss}))
+ ((r^2 - s^2)P_{sss} - 3sP_{ss})\phi + (3(r^2 - s^2)P_{ss} + 3(P - sP_s))\phi_s = 0.
\]

(4.3)

By the formula of $\phi_s$ given in (4.2), one can see that the compatibility condition $C_1$ is satisfied.

For a choice of the functions $b_2(r)$ and $b_3(r)$ such that $b_2 - b_3 r^2 \neq 0$, we have

\[\frac{(r^2 - s^2)Q_{sss} + 3(Q_s - sQ_{ss})}{(r^2 - s^2)^{5/2}} = \frac{-6r^2(b_2 - b_3 r^2)}{(r^2 - s^2)^{5/2}} \neq 0.
\]

Then, we have

\[
(s\phi + (r^2 - s^2)\phi_s) = -\frac{(r^2 - s^2)P_{sss} - 3sP_{ss})\phi + (3(r^2 - s^2)P_{ss} + 3(P - sP_s))\phi_s}{(r^2 - s^2)Q_{sss} + 3(Q_s - sQ_{ss})}
\]

(4.4)
Moreover, the condition $C_1$ can be rewritten in the form

$$C_1 = -(s\phi + (r^2 - s^2)\phi_s)(2Q - sQ_s) + (1 + sP)\phi_s - (sP_s - 2P)\phi = 0 \quad (4.5)$$

Now, substituting from (4.4) into (4.5), we get

$$\frac{(2Q - sQ_s)[((r^2 - s^2)P_{sss} - 3sP_{ss})\phi + (3(r^2 - s^2)P_{ss} + 3(P - sP_s))\phi_s]}{(r^2 - s^2)Q_{sss} + 3(Q_s - sQ_{ss})} + (1 + sP)\phi_s + (sP_s - 2P)\phi = 0 \quad (4.6)$$

Dividing both sides of (4.6) by $\phi$ we have

$$\frac{(2Q - sQ_s)[((r^2 - s^2)P_{sss} - 3sP_{ss}) + (3(r^2 - s^2)P_{ss} + 3(P - sP_s))\frac{\phi_s}{\phi}]}{(r^2 - s^2)Q_{sss} + 3(Q_s - sQ_{ss})} + (1 + sP)\frac{\phi_s}{\phi} + (sP_s - 2P) = 0 \quad (4.7)$$

The substitution from (4.4) and (4.2) into (4.7) implies that the left hand side of (4.7) is non zero. For example, consider the choice

$$a = 0, \quad b_1 = -\frac{1}{r^2}, \quad b_2 = r^2, \quad b_3 = 0.$$  

Then, the left hand side of (4.7) is given by

$$s\left((r^2 - s^2)^2(1 + r^2) + r^8\right) - r^6\sqrt{r^2 - s^2(r^2 - s^2 + 1)} \over r^2(r^2 - s^2)^2.$$  

And this is a contradiction and the proof is completed. \hfill \Box

**Remark 4.2.** It should be noted that the above proof of the above theorem does not depend on whether the surface is regular or not.
Table 1: Classification of Landsberg and Berwald spherically symmetric metrics

| Type          | Landsberg                      | Berwald                      |
|---------------|-------------------------------|------------------------------|
| Dimension     | \( n = 2 \) \( \quad \) \( n \geq 3 \) | \( n = 2 \) \( \quad \) \( n \geq 3 \) |
| P             | as the Berwaldian surface \( P = c_1 s + \frac{c_2 s}{r^2} \sqrt{r^2 - s^2} \) | \( P = b_1 s + \frac{b_2 s}{\sqrt{r^2 - s^2}} + \frac{b_3 (r^2 - 2s^2)}{\sqrt{r^2 - s^2}} \) \( \quad \) \( P = c_1 s \) |
| Q             | as the Berwaldian surface \( Q = \frac{1}{2} c_0 s^2 - \frac{c_3 s}{r^2} \sqrt{r^2 - s^2} + c_3 \) | \( Q = b_0 s^2 + \frac{1}{2} b_1 + \frac{b_2 s (r^2 - 2s^2)}{r^2 \sqrt{r^2 - s^2}} - \frac{b_3 s (3r^2 - 2s^2)}{r^2 \sqrt{r^2 - s^2}} - \frac{a r^2 s}{r^2 \sqrt{r^2 - s^2}} \) | \( Q = \frac{1}{2} c_0 s^2 + c_3 \) |
| Regular examples | only Berwaldian examples exist | \[2] Examples 1 and 2 | \[2] Examples 1 and 2 |
| Non-regular examples | Berwaldian surfaces exist | Some examples exist | Some examples exist | Some examples exist |
| Concrete examples | \( F = \frac{u}{r} \phi \left( \frac{s}{r} \right) \) | see \[2\] Examples 1 and 2 | \( F = \frac{u}{r} \phi \left( \frac{s}{r} \right) \) | \( F = \frac{u}{r} \phi \left( \frac{s}{r} \right) \) |

Where, \( c_0, c_1, c_2, c_3, b_1, b_2, b_3, \) and \( a \) are arbitrary functions of \( r \).
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