An improved Haar wavelet quasilinearization technique for a class of
generalized Burger’s equation

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Abstract
Solving Burgers’ equation always poses challenge to researchers as for small values of viscosity the analytical
solution breaks down. Here we propose to compute numerical solution for a class of generalised Burgers’
equation described as
\[ \frac{\partial w}{\partial t} + w^\mu \frac{\partial w}{\partial x^*} = \nu w^\delta \frac{\partial^2 w}{\partial x^*^2}, \quad a \leq x^* \leq b, \quad t \geq 0, \]
based on the Haar wavelet (HW) coupled with quasilinearization approach. In the process of numerical
solution, finite forward difference is applied to discretize the time derivative, Haar wavelet to spatial derivative
and non-linear term is linearized by quasilinearization technique. To discuss the accuracy and efficiency of
the method \( L_\infty \) and \( L_2 \)-error norm are computed and they are compared with some existing results. We have
proved the convergence of the proposed method. Computer simulations show that the present method gives
accurate and better result even for small number of grid points for small values of viscosity.

Keywords: Quasilinearization, Haar wavelets, Finite difference, Generalized Burgers’ equation.

1 Introduction

The study of Burger’s equation \cite{1, 2, 3} is popular among the scientific community. Different form of Burgers’
equation appears in various areas such as chemical kinetics, optical fibers, fluid dynamics, biology, solid state
physics, plasma physics etc. The main aim of this article is to find numerical solution of a class of generalized
Burger’s equation of the following form
\[ \frac{\partial w}{\partial t} + w^\mu \frac{\partial w}{\partial x^*} = \nu w^\delta \frac{\partial^2 w}{\partial x^*^2}, \quad a \leq x^* \leq b, \quad t \geq 0, \]  \hspace{1cm} (1)
with the boundary conditions (BCs)
\[ w(a, t) = f_1(t), \quad w(b, t) = f_2(t), \quad t \in [0, T], \]  \hspace{1cm} (2)
and the initial condition
\[ w(x^*, 0) = f(x^*), \quad x^* \in [a, b], \]  \hspace{1cm} (3)
where \( w(x^*, t) \) represents the velocity for the spacial dimension \( x^* \), time \( t \), and \( \nu > 0 \) is kinematic coefficient and
\( \mu, \delta \) are non negative integer such that \( \mu + \delta \geq 1 \). When \( \mu = 1, \delta = 0 \), equation (1) is called Burgers’ equation.
In 1915, it was first proposed by Bateman \cite{1}. Later in 1948, it was introduced by Burger \cite{2, 3} as a class of
equation which delineate the mathematical model of turbulence. Due to his immense works on the model it is
termed as Burgers’ equation. For arbitrary initial condition, it was solved analytically by both Hopf \cite{4} and Cole
\cite{5} independently. Since these analytical solution are in infinite series and converge very slowly for small value
of viscosity coefficient \( \nu \). In many cases these solution fails for \( \nu < 0.01 \) and not easy to capture the solution.

For \( \mu + \delta \geq 2 \) Eq. (1) is called generalized Burger’s equation. The generalized Burgers’ equation has
been solved numerically and analytically by several researcher. For \( \mu \geq 2, \delta = 0 \), Ramadan EL-Danaf \cite{6} used
collocation method coupled with quintic splines, Ramadan et al. \cite{7} discussed collocation of septic over finite
element to find numerical solution. Saka and Dag \cite{8} used time and space splitting techniques and then applied
quintic B-spline collocation method. Petrov-Galerkin technique in [9] applied by Roshan and Bhamra. Brastos [10]-[11] used various explicit finite difference scheme to find the numerical solution. Zhang et al. [12] have used local discontinuous Galerkin method. Several others methods have been developed to solve the equation discussed in [13, 14, 15] etc.

Wavelets method has been used to solve PDEs (Partial differential equations) numerically since 1990s. The best feature of the wavelet approach is the capability to detect the irregular structure, singularities and transient phenomena revealed by the analyzed equation. Most of the algorithm for the numerical solution of PDEs by the wavelet method are based on collocation [16]-[17] or the Galerkin technique method [16, 18, 19]. Based on the Haar wavelets method, Chen and Hsiao in [20] proposed a method to find the numerical solution of ordinary differential equations. They replaced highest order derivative function by Haar series. Recently, many ordinary and partial differential equation have been solved by several authors by Haar wavelet method. Lepik used Haar wavelet method to solve nonlinear ordinary differential equation and diffusion equation in [21], Poisson equation in [22], Sin-Gordon and Burgers’ equation in [23]. Verma et al. solved Lane-Emden Equations in [24]. Celik [25] discussed Haar wavelet method for the numerical solutions of Burger-Huxley equation and later applied to magneto hydrodynamic flow equation. Jiwari [26] used Haar wavelet quasilinearization approach to solve Burgers’ equation. Lane-Emden equation arising in astrophysics has been solved numerically using Haar wavelet method in [27]. Haar wavelet method is used in [27] for the numerical solution of biharmonic and 2D and 3D Poisson equation.

In this paper, we propose a technique to solve a class of generalized Burgers’ equation with the combination of finite forward difference and Haar wavelet. We discretize the time derivative $\eta_j$ by forward finite difference. Also the other terms $w^\mu w_x, w^\delta w_{xx}$ are approximated by their average at $j$ and $j + 1$th level. Numerical simulations suggest that this averaging improves the results. The spatial discretization is taken care by Haar wavelet and quasilinearization technique is used to deal with the non-linear term. Convergence analysis is also presented by computing $L^2$ error estimate analytically. Test problems are considered at the end of the section to validate the robustness of the present method.

This article is organised as follows. In section 2, introduction about the Haar wavelets and approximation of the function by using Haar wavelet is discussed. In section 3 discretization of the problem has been done. Then nonlinear function is linearized by the help of quasilinearization. Further we discuss how to implement this technique involving Haar wavelet to the so called generalized Burgers’ equation. In section 4, $L^2$ error estimate is analysed. In section 5, numerical solution of the generalized Burgers’ equation by proposed method are tabulated and illustrated graphically for test problems. Lastly in section 6, we conclude the paper.

## 2 Preliminary

For the numerical solutions of differential equations, integro differential equations and integral equations, one of the simplest mathematical tool that is being used, is Haar wavelet method. Haar wavelets are one of the simplest wavelet among the various type of wavelets. It have been used from 1910 when these were established by the Hungarian mathematicin Alfred Haar. Haar functions are unit step functions which takes three values 0, 1 and −1. Haar function is the oldest and simplest orthonormal wavelet with compact support. For $x \in [0, 1)$, the family of Haar wavelet is defined as:

$$h_i(x) = \begin{cases} 1, & x \in [\eta_1, \eta_2] \\ -1, & x \in [\eta_2, \eta_3] \\ 0, & \text{otherwise} \end{cases}$$

where

$$\eta_1 = \frac{k}{m}, \quad \eta_2 = \frac{k + 0.5}{m} \quad \text{and} \quad \eta_3 = \frac{k + 1}{m}.$$  

In equation (5), we have the integer $m = 2^j$, $j = 0, 1, 2, \ldots, J$ defines the level of wavelets and $k$ is the translation parameter given as $k = 0, 1, 2, \ldots, m - 1$. The indices in $h_i$ in equation (4) can be evaluated from the formula $i = m + k + 1$. For the minimal values of $k = 0, m = 1$, we have the minimal value of $i = 2$. For the maximal value of $k = m - 1$ we have the maximum value of $i$ given by $i = 2M = 2^{j+1}$, $J$ is the maximum resolutions. $h_1(x)$ is the scaling function which is the member of the Haar function defined as

$$h_1(x) = \begin{cases} 1, & x \in [0, 1) \\ 0, & \text{otherwise}. \end{cases}$$
For the solution of any differential equation, we need to integrate the Haar function which is given in the following integrals:

\[
p_{\sigma,l}(x) = \left( \int_0^x \int_0^{\xi_{\sigma-1}} \cdots \int_0^{\xi_1} h_i(\xi) \, d\xi \, d\xi_1 \cdots d\xi_{\sigma-1} \right)
\]

\[
= \frac{1}{(\sigma-1)!} \int_0^x (x-\xi)^{\sigma-1} h_i(\xi) \, d\xi,
\]

(7)

where \( \sigma = 1, 2, \ldots, n; \quad l = 1, 2, \ldots, 2M. \) By considering the equation (4) the integrals (7) can be determined analytically. Thus we have

\[
p_{\sigma,i}(x) = \begin{cases} 
0, & x < \eta_1 \\
\frac{(x-\eta_1)^{\sigma}}{\sigma!}, & x \in [\eta_1, \eta_2] \\
\frac{(x-\eta_1)^{\sigma}}{\sigma!} - \frac{(x-\eta_2)^{\sigma}}{\sigma!}, & x \in [\eta_2, \eta_3] \\
\frac{(x-\eta_1)^{\sigma}}{\sigma!} - \frac{(x-\eta_2)^{\sigma}}{\sigma!} + \frac{(x-\eta_3)^{\sigma}}{\sigma!}, & x > \eta_3.
\end{cases}
\]

(8)

The above formula is applicable for \( i > 1. \) For the case \( \sigma = 1 \) and \( \sigma = 2, \) we have the following

\[
p_{1,i}(x) = \begin{cases} 
x - \eta_1, & x \in [\eta_1, \eta_2] \\
\eta_3 - x, & x \in [\eta_2, \eta_3] \\
0, & \text{otherwise}
\end{cases}
\]

(9)

\[
p_{2,i}(x) = \begin{cases} 
\frac{(x-\eta_2)^{2}}{2}, & x \in [\eta_1, \eta_2] \\
\frac{1}{4M^2} - \frac{(x-\eta_3)^2}{2}, & x \in [\eta_2, \eta_3] \\
\frac{1}{4M^2}, & x \in [\eta_3, 1] \\
0, & \text{otherwise}
\end{cases}
\]

(10)

Since all the Haar wavelets are orthogonal to each other i.e.

\[
\int_0^1 h_i(x)h_l(x)dx = 2^{-j}\delta_{i,l} = \begin{cases} 
2^{-j}, & i = l = 2^j + k \\
0, & i \neq l,
\end{cases}
\]

(11)

therefore any function \( w(x) \in L^2[0,1) \) can be expressed as the sum of infinite series form

\[
w(x) = \sum_{i=1}^{\infty} c_i h_i(x)
\]

(12)

where the coefficient \( c_i \) can be calculated by

\[
c_i = 2^j \int_0^1 w(x)h_i(x),
\]

(13)

where \( i = 2^j + k, \quad j \geq 0, \quad 0 \leq k < 2^j. \) The above series contains infinite number of terms. In case, the function \( w(x) \) is piecewise constant by itself or approximated as piecewise constant then the series terminates at finite terms and can be expressed as

\[
w(x) = \sum_{i=1}^{2M} c_i h_i(x) = C^T_{(2M)}h_{(2M)}(x),
\]

(14)

Here \( T \) represent transpose and \( 2M = 2^{j+1} \) is the length of the vector given by

\[
C^T_{(2M)} = [c_1, c_2, \ldots, c_{(2M)}]
\]

\[
h_{(2M)}(x) = [h_{(1)}(x), h_{(2)}(x), \ldots, h_{(2M)}(x)]^T.
\]

(15)
3 Derivation of the scheme

Since the Haar wavelets described only for $x \in [0, 1]$ so first we transfer the interval $x_\ast \in [a, b]$ in to unit interval $x \in [0, 1]$. Let us use the expression $x = \frac{x_\ast - a}{b - a}$, where $L = b - a$. Hence equation (1) becomes

$$\frac{\partial w}{\partial t} + \frac{1}{L} w^{\mu} w_x = \frac{\nu}{L^2} w^{\delta} w_{xx}, \quad 0 \leq x \leq 1, \quad t \geq 0, \quad (16)$$

with the boundary conditions (BCs)

$$w(0, t) = f_1(t), \quad w(1, t) = f_2(t), \quad t \in [0, T], \quad (17)$$

and the initial conditions

$$w(x, 0) = f(x), \quad x \in [0, 1]. \quad (18)$$

Now, we apply finite forward difference for the time and average in time for $w^{\mu} w_x$ and $w^{\delta} w_{xx}$ in equation (16), we have

$$\frac{w_{j+1} - w_j}{\Delta t} + \frac{1}{2L} \left[ w^{\mu}_{j+1}(w_{j+1})_x + w^{\mu}_{j}(w_{j})_x \right] = \frac{\nu}{2L^2} \left[ w^{\delta}_{j+1}(w_{j+1})_{xx} + w^{\delta}_{j}(w_{j})_{xx} \right] \quad 0 \leq j \leq N - 1, \quad (19)$$

with the BCs

$$w_{j+1}(0) = f_1(t_{j+1}), \quad w_{j+1}(1) = f_2(t_{j+1}) \quad 0 \leq j \leq N - 1, \quad (20)$$

and initial condition

$$w_0 = f(x), \quad (21)$$

where $t_{j+1} = \Delta t(j + 1)$, $N\Delta t = T$ ($\Delta t$ is time step size and $T$ is final time) and $w_{j+1}$ is the approximated solution at $(j + 1)$th time level. Equation (19) can be expressed as follows

$$\frac{\nu \Delta t}{2L^2} w_{j+1}^{\delta}_{j+1}(w_{j+1})_{xx} - \frac{\Delta t}{2L} w_{j+1}^{\mu}_{j+1}(w_{j+1})_x - w_{j+1} = -w_j + \frac{\Delta t}{2L} w_{j}^{\mu}(w_{j})_x - \frac{\nu \Delta t}{2L^2} w_{j}^{\delta}(w_{j})_{xx} \quad 0 \leq j \leq N - 1, \quad (22)$$

with the BCs

$$w_{j+1}(0) = f_1(t_{j+1}), \quad w_{j+1}(1) = f_2(t_{j+1}) \quad 0 \leq j \leq N - 1, \quad (23)$$

and initial condition

$$w_0 = f(x). \quad (24)$$

Equation (22) is the non-linear ordinary differential equations at $(j + 1)$th time level.

Nonlinearity in equation (22) can be handled by several methods. One of the possible method is quasi-linearization process [28]. For $\mu \geq 1, \delta \geq 1$ the nonlinear term in equation (22) is replaced by the following equations

$$w_{j+1}^{\mu}(w_{j+1})_x = \mu w_{j}^{\mu-1}(w_{j})_x + w_{j}^{\mu}(w_{j+1})_x - \mu w_{j}^{\mu}(w_{j})_x \quad (25)$$
$$w_{j+1}^{\delta}(w_{j+1})_{xx} = \delta w_{j}^{\delta-1}(w_{j})_{xx} + w_{j}^{\delta}(w_{j+1})_{xx} - \delta w_{j}^{\delta}(w_{j})_{xx} \quad (26)$$

Thus, we have the following equations

$$\frac{\nu \Delta t}{2L^2} w_{j+1}^{\delta}(w_{j+1})_{xx} - \frac{\Delta t}{2L} w_{j}^{\mu}(w_{j})_x + \frac{\nu \Delta t}{2L^2} \delta w_{j}^{\delta-1}(w_{j})_{xx} + w_{j+1} - \frac{\Delta t}{2L} \mu w_{j}^{\mu}(w_{j})_x w_{j+1} - w_{j+1}$$
$$= -w_j + \frac{\Delta t}{2L} (1 - \mu) w_{j}^{\mu}(w_{j})_x - \frac{\nu \Delta t}{2L^2} (1 - \delta) w_{j}^{\delta}(w_{j})_{xx}, \quad (27)$$

with the boundary conditions

$$w_{j+1}(0) = f_1(t_{j+1}), \quad w_{j+1}(1) = f_2(t_{j+1}), \quad 0 \leq j \leq N - 1, \quad (28)$$

and initial condition

$$w_0 = f(x). \quad (29)$$
Now, we discretize the second order spatial derivative present in equation (27) using the Haar wavelets as follows

\[(w_{j+1})_{xx}(x) = \sum_{i=1}^{2M} c_{i} h_{i}(x) = C_{2M}^{T} h_{2M}(x)\] (30)

Integrating the equation (30) from 0 to \(x\), we have

\[(w_{j+1}(x) = \sum_{i=1}^{2M} c_{i} p_{i,1}(x) + (w_{j+1})_{x}(0),\] (31)

\(w_{j+1}(0)\) is unknown in the equation (31). To find this we integrate (31) from 0 to 1 and using BCs, we have

\[(w_{j+1})_{x}(x) = \sum_{i=1}^{2M} c_{i} \left[p_{i,1}(x) - p_{i,2}(1)\right] + f_{2}(t_{j+1}) - f_{1}(t_{j+1}),\] (32)

again integrating equation (32) from 0 to \(x\), we get

\[w_{j+1}(x) = \sum_{i=1}^{2M} c_{i} \left[p_{i,2}(x) - p_{i,2}(1)\right] + x(f_{2}(t_{j+1}) - f_{1}(t_{j+1})) + f_{1}(t_{j+1}).\] (33)

Now, putting equations (30)-(33) in (27), we have

\[\sum_{i=1}^{2M} c_{i} \left[\frac{\nu}{2L} \mu w_{j}^{2} h_{i}(x) - \frac{\Delta t}{2L} w_{j}^{\mu}(p_{i,1}(x) - p_{i,2}(1))\right]
+ \left[\frac{\nu}{2L} \delta w_{j}^{(\delta-1)}(w_{j})_{xx} - \frac{\Delta t}{2L} \mu w_{j}^{(\mu-1)}(w_{j})_{x} - 1\right] \left(p_{i,2}(x) - p_{i,2}(1)\right)\]
\[= -w_{j} + \frac{\Delta t}{2L} (1 - \mu) w_{j}^{\mu}(w_{j})_{x} - \frac{\nu}{2L} \delta w_{j}^{(\delta-1)} w_{j}^{\mu}(w_{j})_{xx} + \frac{\Delta t}{2L} w_{j}^{\mu}\left(f_{2}(t_{j+1}) - f_{1}(t_{j+1})\right)\]
\[= \left(-w_{j} + \frac{\Delta t}{2L} (1 - \mu) w_{j}^{\mu}(w_{j})_{x} - \frac{\nu}{2L} \delta w_{j}^{(\delta-1)} w_{j}^{\mu}(w_{j})_{xx} + \frac{\Delta t}{2L} w_{j}^{\mu}\left(f_{2}(t_{j+1}) - f_{1}(t_{j+1})\right)\right)\] (34)

where \(p_{2,i}(1)\) can easily calculated from equation (10) and are given by

\[p_{2,i}(1) = \begin{cases} 0.5, & i = 1 \\ \frac{i}{2M}, & i > 1. \end{cases}\] (35)

Now, let us take the collocation points \(x_{k} = \frac{k-0.5}{2M}, k = 1, 2, ..., 2M\) and applying discretization on equation (34), we get the following linear system

\[\sum_{i=1}^{2M} c_{i} \left[\frac{\nu}{2L} \mu w_{j}^{2} h_{i}(x_{k}) - \frac{\Delta t}{2L} w_{j}^{\mu}(p_{i,1}(x_{k}) - p_{i,2}(1))\right]
+ \left[\frac{\nu}{2L} \delta w_{j}^{(\delta-1)}(w_{j})_{xx} - \frac{\Delta t}{2L} \mu w_{j}^{(\mu-1)}(w_{j})_{x} - 1\right] \left(p_{i,2}(x_{k}) - p_{i,2}(1)\right)\]
\[= -w_{j} + \frac{\Delta t}{2L} (1 - \mu) w_{j}^{\mu}(w_{j})_{x} - \frac{\nu}{2L} \delta w_{j}^{(\delta-1)} w_{j}^{\mu}(w_{j})_{xx} + \frac{\Delta t}{2L} w_{j}^{\mu}\left(f_{2}(t_{j+1}) - f_{1}(t_{j+1})\right)\]
\[= \left(-w_{j} + \frac{\Delta t}{2L} (1 - \mu) w_{j}^{\mu}(w_{j})_{x} - \frac{\nu}{2L} \delta w_{j}^{(\delta-1)} w_{j}^{\mu}(w_{j})_{xx} + \frac{\Delta t}{2L} w_{j}^{\mu}\left(f_{2}(t_{j+1}) - f_{1}(t_{j+1})\right)\right)\] (36)

By solving the above linear system, we can obtain the wavelets coefficient \(C_{2M}^{T}\). For the first time step the value of \((w_{j}), (w_{j})_{x}, (w_{j})_{xx}\) can be taken from initial conditions. For the next time step the value of \((w_{j}), (w_{j})_{x}, (w_{j})_{xx}\) are calculated by solving the above equation for the wavelet coefficient \(C_{2M}^{T}\) and putting in equation (33),(32) and (30). The value of \((w_{j}), (w_{j})_{x}, (w_{j})_{xx}\) for each time step can be obtained in the same way. To start the iterations, we use \(w_{0}(x_{k}) = f(x_{k}), (w_{j})_{0}(x_{k}) = f'(x_{k}), (w_{j})_{0}(x_{k}) = f''(x_{k})\).
4 \textbf{L}^2 \textbf{Error}

For the convergence of the projected method, we analyze the asymptotic expression of the equation (33) and the corresponding equation is below

$$ w(x) = \sum_{i=1}^{\infty} c_i \left[ p_{i,2}(x) - p_{i,2}(1) \right] + x(f_2(t_{j+1}) - f_1(t_{j+1})) + f_1(t_{j+1}). \quad (37) $$

\textbf{Lemma 4.1.} Let us assume that $w(x) \in L^2(R)$ with $|w_x| \leq K$, $\forall \ x \in (0,1)$; $K > 0$ and $w(x) = \sum_{i=0}^{\infty} c_i h_i(x)$. Then $|c_i| \leq K 2^{-(3j-2)/2}$.

\textbf{Proof.} See [29].

\textbf{Lemma 4.2.} Let $w(x) \in L^2(R)$ be a continuous function in the interval $(0,1)$. Then at J th level, the error norm is bounded by

$$ ||E_J||_2^2 \leq \frac{K^2}{12} 2^{-2J}, \quad (38) $$

where $|w_x| \leq K$, $\forall \ x \in (0,1)$; $K > 0$, $M$ is the positive given by $M = 2^J$.

\textbf{Proof.} See [29].

\textbf{Theorem 4.3.} Let $w(x)$ and $w_{2M}$ are the exact and approximated solution of the equation (33), then

$$ ||E_J||_2 = ||w(x) - w_{2M}(x)||_2 \leq 2K \left( 2^{-(3j+1)/2} \right). \quad (39) $$

\textbf{Proof.}

$$ ||E_J||_2^2 = \int_0^1 \left[ \sum_{j=J+1}^{\infty} \sum_{i=1}^{2^j-1} \sum_{k=0}^{2^j-1} \sum_{l=0}^{2^j-1} \sum_{s=0}^{2^j-1} c_{2^j+k+1} \left( p_{2^j+k+1}(x) - x p_{2^j+k+1}(1) \right) \right]^2 dx $$

$$ = \sum_{j=J+1}^{\infty} \sum_{k=0}^{2^j-1} \sum_{l=0}^{2^j-1} \sum_{s=0}^{2^j-1} \sum_{i=1}^{2^j-1} \int_0^1 \left( p_{2^j+k+1}(x) - x p_{2^j+k+1}(1) \right) $$

$$ \left( p_{2^j+k+1}(x) - x p_{2^j+k+1}(1) \right) dx \quad (40) $$

First we will evaluate the upper bound for the function $p_{2,i}(x)$ in all the subinterval $[0,1]$. We have $p_{2,i} = 0$ for $x \in [0, \eta_1]$. In the interval $[\eta_1, \eta_2]$, function $p_{2,i}(x)$ is monotonic increasing and its maximum value can be obtained by putting $x = \eta_2$ and therefor

$$ p_{2,i}(x) = p_{2,2^j+k+1}(x) \leq \frac{(\eta_2 - \eta_1)^2}{2} = \frac{1}{2 \left( \frac{1}{2^j+1} \right)^2}, \ \forall x \in [\eta_1, \eta_2]. \quad (42) $$

In the interval $[\eta_2, \eta_3]$ it can be easily prove that function $p_{2,i}$ is monotonically increasing by using the equation (5),(8) and the condition $\frac{\partial p_{2,i}(x)}{\partial x} > 0$ if $x < \eta_3$ which is true. Hence $p_{2,i}(x)$ attains its maximum value at the end point $x = \eta_3$. Hence

$$ p_{2,i}(x) = p_{2,2^j+k+1}(x) \leq \left( \frac{1}{2^j+1} \right)^2, \ \forall x \in [\eta_2, \eta_3]. \quad (43) $$

For the subinterval $[\eta_3, 1]$, $p_{2,i}(x)$ can be express as (Equation (22) [30])

$$ p_{2,i}(x) = p_{2,2^j+k+1}(x) = \left( \frac{1}{2^j+1} \right)^2. \quad (44) $$

Thus the function $p_{2,i}(x)$ has an upper bound in $[0,1]$ given by

$$ p_{2,i}(x) = p_{2,2^j+k+1}(x) \leq \left( \frac{1}{2^j+1} \right)^2, \ \forall x \in [0,1]. \quad (45) $$
Now, we have
\[
(p_{2,i}(x) - xp_{2,i}(1)) \leq |p_{2,i}(x)| + |x||p_{2,i}(1)| \\
\leq |p_{2,i}(x)| + |p_{2,i}(1)| \\
\leq 2\left(\frac{1}{2^{j+1}}\right)^2 .
\] (46)

Similarly, we have
\[
(p_{2,i}(x) - xp_{2,i}(1)) \leq \left|p_{2,i}(x)\right| + \left|xp_{2,i}(1)\right| \\
\leq 2\left(\frac{1}{2^{j+1}}\right)^2 .
\] (47)

Putting (46) and (47) in equation (41), we get
\[
\left|E_J\right|^2 \leq 4 \sum_{j=J+1}^{2^j-1} \sum_{k=0}^{2^j-k+1} c_{2^j+k+1} c_{2^j+k+1} \left(\frac{1}{2^{j+1}}\right)^2 \left(\frac{1}{2^{j+1}}\right)^2 \text{ by lemma 4.1}
\leq 4K^2 \sum_{j=J+1}^{\infty} \sum_{l=j+1}^{\infty} 2^{-2\left(\frac{j}{2}\right) - l} \left(1 - 2^{-5/2}\right)^2 \left(\frac{1}{2^{j+1}}\right)^2
\leq 4K^2 \left(\frac{2^{-2\left(\frac{j}{2}\right) - l} \left(1 - 2^{-5/2}\right)^2}{1 - 2^{-5/2}}\right)^2
\]

Hence
\[
\left|E_J\right|^2 \leq 2K \left(\frac{2^{-2\left(\frac{j}{2}\right) - l} \left(1 - 2^{-5/2}\right)^2}{1 - 2^{-5/2}}\right). \quad (48)
\]

It is clear that equation (47) indicates that the error bound is proportional to the level of resolutions $J$ of the Haar wavelet. Also $\left|E_J\right|^2 \to 0$ as $J \to \infty$. Thus the proposed scheme converges to the solutions as $J$ approaches to infinity.

5 Numerical Illustration

In this part, we measure the efficiency of the proposed method by taking some example with the help of mean square root error norm $L_2$ and maximum error norm $L_\infty$ defined as
\[
L_2 = \left(\sum_{j=0}^{2^M} \Delta x |w_j^{exact} - (w_{2M})_j|^2\right)^{1/2}, \quad L_\infty = \|w^{exact} - w_{2M}\|_\infty = \max_j |w_j^{exact} - (w_{2M})_j|,
\]
where $w_{2M}$ is the approximated result by Haar Wavelet method.

5.1 Test Problem 1

Let us take $\mu = 2, \delta = 0$, in equation (1) and the BCs
\[
w(0, t) = 0 = w(1, t) \quad t > 1,
\] (49)

with the initial condition
\[
w(1, t) = \frac{x_*}{1 + e^{\frac{x_*}{\sqrt{t/c_0}}}}, \quad x_* \in (0, 1)
\] (50)

which is obtained from the exact solution [10]
\[
w(x_*, t) = \frac{x_*^t}{1 + \sqrt{t/c_0} e^{\frac{x_*^t}{\sqrt{t/c_0}}}}, \quad t \geq 1, x_* \in (0, 1)
\] (51)
where $0 < c_0 < 1$.

For the comparison purpose, in table 1, we take $\Delta t = 0.01, 2M = 16, \nu = 0.01$ and $C_0 = 0.5$ and we compute the $L_\infty$-error and $L_2$-error at $T = 2$ and $T = 4$. We observe that the result by present scheme is better than the existing result published in [31]-[8] even though $\Delta t$ and $\Delta x$ taken in present scheme is very large compare to the value of $\Delta t$ and $\Delta x$ taken in [31]-[8] for $\nu = 0.01$. In table 2, the result are computed for small value of $\nu$ and $\Delta x$ taken in present scheme which is very large compare to the value of $\Delta t$ and $\Delta x$ taken in [31]-[8].

In fig. 1, we plot the numerical result (Left) for $\nu = 0.005, 2M = 16$ and $C_0 = 0.5$ and find that the computed result follows the physical behaviour of the solution at different time for the time step $\Delta t = 0.01$. In fig 1 (Right) the absolute error is plotted and observe that the absolute error at different discrete point are less than 0.0005 which are acceptable.

Table 1: Comparison of numerical result with the existing result by the help of $L_\infty$ and $L_2$ error of the problem (5.1) at $T = 2, 4$ for $\nu = 0.01, \Delta t = 0.01$ and $C_0 = 0.5, J = 3$.

| $\nu = 0.01$ | $\Delta x$ | $\Delta t$ | $L_\infty$-error | $L_2$-error | $L_\infty$-error | $L_2$-error |
|----------------|-------------|-------------|------------------|-------------|------------------|-------------|
| Present        | 1/16        | 0.01        | 0.76E-03         | 0.347E-03   | 0.582E-03         | 0.311E-03   |
| [31]           | 1/100       | 0.00001     | 0.81387E-03      | 0.38291E-03 | 0.60474E-03       | 0.31718E-03 |
| [6]            | 1/100       | 0.00001     | 1.21698E-03      | 0.52308E-03 | 0.93136E-03       | 0.51625E-03 |
| [7]            | 1/100       | 0.00001     | 1.70309E-03      | 0.79043E-03 | 0.99645E-03       | 0.55767E-03 |
| [8]            | 1/100       | 0.00001     | 0.81680E-03      | 0.37932E-03 | 0.60537E-03       | 0.31724E-03 |

Table 2: Comparison of numerical result with the existing result by the help of $L_\infty$ and $L_2$ error of the problem (5.1) at $T = 2, 4$ for $\nu = 0.01, \Delta t = 0.01$ and $C_0 = 0.5, J = 3$.

| $\nu = 0.001$ | $\Delta x$ | $\Delta t$ | $L_\infty$-error | $L_2$-error | $L_\infty$-error | $L_2$-error |
|----------------|-------------|-------------|------------------|-------------|------------------|-------------|
| Present        | 1/32        | 0.01        | 0.2236E-03       | 0.054998E-03| 0.1823E-03       | 0.0575806E-03|
| [31]           | 1/100       | 0.00001     | 0.26595E-03      | 0.07173E-03 | 0.19549E-03      | 0.05727E-03 |
| [6]            | 1/100       | 0.00001     | 0.27967E-03      | 0.06703E-03 | 0.21856E-03      | 0.06670E-03 |
| [7]            | 1/100       | 0.00001     | 0.81852E-03      | 0.18355E-03 | 0.35635E-03      | 0.11441E-03 |
| [8]            | 1/100       | 0.00001     | 0.26094E-03      | 0.06811E-03 | 0.19288E-03      | 0.05652E-03 |

Figure 1: Numerical solution (left) and absolute error(right) for the problem (5.1) for the time step $\Delta t = 0.01, \nu = 0.005, J = 3$ at different time $T$. 
5.2 Test Problem 2

Let us take \( \mu = 1, \delta = 1 \) with the BCs

\[ w(0, t) = 0 = w(1, t) \quad t > 0 \]  \hspace{1cm} (52)

and initial condition

\[ w(x, 0) = \frac{1 - e^{x/\nu} + (e^{1/\nu} - 1)x}{\sigma}, \quad x \in (0, 1), \]  \hspace{1cm} (53)

which is obtained from the exact solution [32]

\[ w(x, t) = \frac{1 - e^{x/\nu} + (e^{1/\nu} - 1)x}{(e^{1/\nu} - 1)t + \sigma}, t \geq 0, \]  \hspace{1cm} (54)

where \( \sigma > 0 \) is a parameter. We compute the \( L_\infty \) and \( L_2 \)-norm error. In table 3, we summarize the results computed by the present method for \( \nu = 1, \sigma = 2 \) with the time step \( \Delta t = 0.001, 0.01 \) and \( 2M = 8 \) and 32 at different times \( T \). We can see in table 3 that as the value of \( 2M \) is increased, \( L_\infty \) and \( L_2 \) error decrease and hence we can say that when \( J \to \infty \) the error will tend to zero.

In fig.2 (Left), the exact solution and the numerical solution are plotted for the time step \( \Delta t = 0.001, \nu = 1, \sigma = 1 \) at different times \( T \) and we see that the numerical result are very close to the exact solution. In fig. 2 (Right) absolute error is plotted at discrete points for the different times with the time step \( \Delta t = 0.001, \nu = 1 \) and \( \sigma = 1 \). We see that the absolute error are very small and less than 0.0000012 and is maximum near the point 0.7 for all different times. In fig 3 (Left) we have graphed the numerical solution and exact solution of the problem (5.2) for small values of \( \nu = 0.01, \Delta t = 0.01 \) at different \( T \) for \( 2M = 32 \). We can see that the numerical solution is almost same as the exact solution except near the boundary point 1. In fig 3 (Right), we have graphed the absolute error of the problem at different times \( T \) for \( \nu = 0.01, \Delta t = 0.01 \) and \( 2M = 32 \). It is observed that the absolute error throughout the domain is almost zero except near the boundary point 1. This error can be reduced by taking the more number of grid points as seen in the table 3.

Table 3: \( L_\infty \) and \( L_2 \) error of the problem (5.2) at different \( T \) for \( J = 2, 4, \nu = 1 \) and \( \sigma = 2 \).

| \( J \) | \( T = 0.01 \) | \( T = 0.1 \) | \( T = 0.2 \) |
|-------|-------------|-------------|-------------|
|       | \( L_\infty \) error | \( L_\infty \) error | \( L_\infty \) error |
|       | \( \Delta t = 0.001 \) | \( \Delta t = 0.01 \) | \( \Delta t = 0.01 \) |
| 2     | 1.1533E-06 | 9.9506E-06 | 7.3036E-05 |
| 4     | 8.18486E-07 | 7.00077E-06 | 1.22587E-05 |

| \( J \) | \( L_2 \) error | \( L_2 \) error | \( L_2 \) error |
|-------|-------------|-------------|-------------|
|       | \( \Delta t = 0.001 \) | \( \Delta t = 0.01 \) | \( \Delta t = 0.01 \) |
| 2     | 7.31654E-08 | 6.26645E-07 | 1.09634E-06 |
| 4     | 5.12615E-08 | 4.40074E-07 | 7.72171E-07 |

Figure 2: Numerical solution (left) and absolute error (right) for the problem (5.2) with \( \Delta t = 0.001, \nu = 1, \sigma = 1 \) at different time \( T \) for \( J = 4 \).
5.3 Test Problem 3

Here we take $\mu = 1, \delta = 0$, then equation (1) becomes one dimensional Burgers’ equation. We take BCs

$$w(0, t) = 0 = w(1, t), \quad t > 0,$$

and the initial conditions

$$w(x_*, 0) = \frac{2\pi \nu \sin(\pi x_*)}{\sigma + \cos(\pi x_*)}, \quad x_* \in (0, 1),$$

which is obtained from the exact solution [33]

$$w(x_*, t) = \frac{2\pi \nu e^{-\pi^2\nu t} \sin(\pi x_*)}{\sigma + e^{-\pi^2\nu t} \cos(\pi x_*)}, \quad x_* \in (0, 1),$$

where $\sigma > 1$ is a parameter.

The numerical results of the example for $\nu = 0.01$ are presented in table 4 with the time step $\Delta t = 0.01$ at $T = 1$ for the parameter $\sigma = 100$. Table summarise $L_\infty$ and $L_2$-error norm. The result by the present method is compared with the result published in [34] and [35] and is found that result is much better than the result in [34] and [35]. In the present method the spacial step size $\Delta x$ is greater than the spacial step size taken in [34] and [35] and found that the error is comparatively small. In fig. 4 (Left) numerical solution and exact solution is depicted for small value of $\nu = 0.005$ at different $T$ with the time step $\Delta t = 0.01$ and $\sigma = 4$. It is observed that numerical result are very closed to the exact solution. In fig.4 (Right) the absolute error is plotted for different $T$. It can been seen that the absolute error are very small and less than 0.0000010 which is acceptable.

### Table 4: Comparision of numerical result with the existing result by the help of $L_\infty$ and $L_2$ error of the problem (5.3) at $T = 1$ for $\nu = 0.01, \Delta t = 0.01$ and $\sigma = 100$.

| $\Delta x$ | $L_2$ - error | $L_\infty$ - error | Mittal and Jain [35] | Present |
|-----------|---------------|---------------------|----------------------|--------|
| 1/10      | 3.4545E-07  | 4.8808E-07  | 3.2840E-07  | 4.6280E-07 |
| 1/20      | 1.0124E-07  | 1.4305E-07  | 8.1921E-08  | 1.1640E-07 |
| 1/40      | 4.0028E-08  | 5.6677E-08  | 2.0470E-08  | 2.9068E-08 |
| 1/80      | 1.59079E-08 | 2.26455E-08 | 5.66586E-09 | 7.2706E-09 |

5.4 Test Problem 4

Here we take $\mu = 2, \delta = 1$, then the equation (1) becomes

$$w_t + w^2 w_{x_*} = \nu w w_{x_*}^2.$$

(58)
Let us take boundary condition
\[ w(0, t) = 0 = w(5, t) \quad t > 0, \tag{59} \]
with initial condition
\[ w(x_*, 0) = \sin(\pi x_*) \tag{60} \]
To the best of our knowledge the analytical solution of the problem (5.4) does not exist in the literature but the existence of solution is discussed in [36]. We do not have exact solution so we only compute numerical result and plot in figure. In fig.5 (Left) We plot the numerical result of the problem 5.4 for different values of \( \nu \) at \( T = 0.1 \) and time step \( \Delta t = 0.01 \) and \( 2M = 32 \). It is observed that the numerical solution of the problem follows the physical behaviour of the solution for all values of \( \nu \). In fig.5 (Right) numerical solution are plotted at different time \( T \) for small values of \( \nu = 0.005 \) and observed that it also follows physical behaviour of the solution.

6 Conclusion
In this work, Haar wavelet with the combination of quasilinearization and finite forward difference which involves averaging is discussed. The performance of the present method is shown by testing the method over several
examples and accuracy of the method is measured by $L_2$ and $L_{\infty}$-error norm. It is observed that the present method gives better accuracy than the result published in the literature even for small number of grid points. Based on the performance of the present method, it is observed that the our method is competitive with the existing method such as finite difference, finite element etc., and this method can be also used for different type of PDEs that models real life problems in different field of engineering and science.

References

[1] Harry Bateman. Some recent researches on the motion of fluids. *Monthly Weather Review*, 43(4):163–170, 1915.

[2] JM Burgers. Mathematical examples illustrating relations occurring in the theory of turbulent fluid motion, *verh. Nederl. Akad. Wetensch. Afd. Wetensch. Afd. Natuurk. Sect.*, 1:17, 1939.

[3] Johannes Martinus Burgers. A mathematical model illustrating the theory of turbulence. In *Advances in applied mechanics*, volume 1, pages 171–199. Elsevier, 1948.

[4] Eberhard Hopf. The partial differential equation $u_t + uu_x = \mu u_{xx}$. *Communications on Pure and Applied mathematics*, 3(3):201–230, 1950.

[5] Julian D Cole. On a quasi-linear parabolic equation occurring in aerodynamics. *Quarterly of applied mathematics*, 9(3):225–236, 1951.

[6] Mohamed A Ramadan and Talaat S El-Danaf. Numerical treatment for the modified burgers equation. *Mathematics and Computers in Simulation*, 70(2):90–98, 2005.

[7] Mohamed A Ramadan, Talaat S El-Danaf, and Faisal EI Abd Alal. A numerical solution of the burgers equation using septic b-splines. *Chaos, Solitons & Fractals*, 26(4):1249–1258, 2005.

[8] Bülent Saka and İdris Dağ. A numerical study of the burgers equation. *Journal of the Franklin Institute*, 345(4):328–348, 2008.

[9] Thoudam Roshan and KS Bhamra. Numerical solutions of the modified burgers equation by petrov–galerkin method. *Applied Mathematics and Computation*, 218(7):3673–3679, 2011.

[10] Athanassios G Bratsos. A fourth-order numerical scheme for solving the modified burgers equation. *Computers & Mathematics with Applications*, 60(5):1393–1400, 2010.

[11] AG Bratsos and LA Petrakis. An explicit numerical scheme for the modified burgers’ equation. *International Journal for Numerical Methods in Biomedical Engineering*, 27(2):232–237, 2011.

[12] Zhang Rong-Pei, Yu Xi-Jun, and Zhao Guo-Zhong. Modified burgers’ equation by the local discontinuous galerkin method. *Chinese Physics B*, 22(3):030210, 2013.

[13] RS Temsah. Numerical solutions for convection–diffusion equation using el-gendi method. *Communications in Nonlinear Science and Numerical Simulation*, 14(3):760–769, 2009.

[14] A Griewank and Talaat S El-Danaf. Efficient accurate numerical treatment of the modified burgers equation. *Applicable Analysis*, 88(1):75–87, 2009.

[15] Yali Duan, Ruxun Liu, and Yanqun Jiang. Lattice boltzmann model for the modified burgers equation. *Applied Mathematics and Computation*, 202(2):489–497, 2008.

[16] Silvia Bertoluzza, Giovanni Naldi, and Jean Christophe Ravel. Wavelet methods for the numerical solution of boundary value problems on the interval. In *Wavelet Analysis and Its Applications*, volume 5, pages 425–448. Elsevier, 1994.

[17] Valeriano Comincioli, Giovanni Naldi, and Terenzio Scapolla. A wavelet-based method for numerical solution of nonlinear evolution equations. *Applied Numerical Mathematics*, 33(1-4):291–297, 2000.

[18] M-Q CHEN, Chyi Hwang, and Y-P SHIH. The computation of wavelet-galerkin approximation on a bounded interval. *International journal for numerical methods in engineering*, 39(17):2921–2944, 1996.
[19] A Avudainayagam and C Vani. Wavelet-galerkin solutions of quasilinear hyperbolic conservation equations. *Communications in numerical methods in engineering*, 15(8):589–601, 1999.

[20] CF Chen and CH Hsiao. Haar wavelet method for solving lumped and distributed-parameter systems. *IEE Proceedings-Control Theory and Applications*, 144(1):87–94, 1997.

[21] Úlo Lepik. Numerical solution of differential equations using haar wavelets. *Mathematics and computers in simulation*, 68(2):127–143, 2005.

[22] Úlo Lepik. Solving pdes with the aid of two-dimensional haar wavelets. *Computers & Mathematics with Applications*, 61(7):1873–1879, 2011.

[23] Úlo Lepik. Numerical solution of evolution equations by the haar wavelet method. *Applied Mathematics and Computation*, 185(1):695–704, 2007.

[24] Amit K Verma and Diksha Tiwari. Higher resolution methods based on quasilinearization and haar wavelets on lane–emden equations. *International Journal of Wavelets, Multiresolution and Information Processing*, 17(03):1950005, 2019.

[25] İbrahim Çelik. Haar wavelet method for solving generalized burgers–huxley equation. *Arab Journal of Mathematical Sciences*, 18(1):25–37, 2012.

[26] Ram Jiwari. A haar wavelet quasilinearization approach for numerical simulation of burgers equation. *Computer Physics Communications*, 183(11):2413–2423, 2012.

[27] Harpreet Kaur, RC Mittal, and Vinod Mishra. Haar wavelet approximate solutions for the generalized lane–emden equations arising in astrophysics. *Computer Physics Communications*, 184(9):2169–2177, 2013.

[28] RE Bellman and RE Kalaba Quasilinearization. Nonlinear boundary value problems, 1965.

[29] S Saha Ray. On haar wavelet operational matrix of general order and its application for the numerical solution of fractional bagley torvik equation. *Applied Mathematics and Computation*, 218(9):5239–5248, 2012.

[30] J Majak, BS Shvartsman, M Kirs, M Pohlak, and H Herranen. Convergence theorem for the haar wavelet based discretization method. *Composite Structures*, 126:227–232, 2015.

[31] Athanasios G Bratsos and Abdul QM Khaliq. An exponential time differencing method of lines for the burgers and the modified burgers equations. *Numerical Methods for Partial Differential Equations*, 34(6):2024–2039, 2018.

[32] Ronald E Mickens. *Nonstandard finite difference models of differential equations*. World Scientific, 1994.

[33] Asai Asaithambi. Numerical solution of the burgers equation by automatic differentiation. *Applied Mathematics and Computation*, 216(9):2700–2708, 2010.

[34] Kaysar Rahman, Nurnantat Helil, and Rahmatjan Yimin. Some new semi-implicit finite difference schemes for numerical solution of burgers equation. In *2010 International Conference on Computer Application and System Modeling (ICCASM 2010)*, volume 14, pages V14–451. IEEE, 2010.

[35] RC Mittal and RK Jain. Numerical solutions of nonlinear burgers equation with modified cubic b-splines collocation method. *Applied Mathematics and Computation*, 218(15):7839–7855, 2012.

[36] Ar S Tersenov. On solvability of some boundary value problems for a class of quasilinear parabolic equations. *Siberian Mathematical Journal*, 40(5):972–980, 1999.