Controlled Remote Implementation of Operations via Graph States

Xinyu Qiu* and Lin Chen*

Protocols are proposed for controlled remote implementation of operations here. Sharing a \((2N + 1)\)-partite graph state, \(2N\) participants collaborate to prepare the stator and realize the operation \(\bigotimes_{j=1}^{N} \exp[i\alpha_j \sigma_{ij}]\) on \(N\) unknown states for distributed systems \(O_j\), only with the permission of a controller. The control power analysis shows that without the controller’s permission, eight specified operations can be implemented with the success rate of 50%, other operations can be realized with the success rate of 25%, and thus the control power is reliable. All the implementation requirements of this protocol can be satisfied by means of local operations and classical communications, and the experimental feasibility is presented according to current techniques. The entanglement requirement of this protocol is characterized in terms of geometric measure of entanglement. It turns out to be economic to realize the control function from the perspective of entanglement cost.

1. Introduction

As the unique resource, entanglement allows the emergence and development of quantum information processing, such as teleportation,\(^{[1]}\) dense coding,\(^{[2]}\) and cryptography.\(^{[3,4]}\) Being expected to offer substantial speed-ups over classical counterparts, quantum computation has been paid a lot of attention.\(^{[5,6]}\) Several challenges have surfaced in its actual construction, such as decoherence and dissipation, sufficiently manipulating a large number of qubits, and undesirable interactions.\(^{[7]}\) To counter such challenges, distributed quantum computation has been proposed.\(^{[8–10]}\) It requires to transfer states from one place to the other and implementing the operations on a remote state faithfully. The first requirement has been met by quantum teleportation.\(^{[11–13]}\) The second one has been tackled by remote implementation of operation (RIO) and controlled RIO (CRIO). RIO means that the quantum operation performed on the sender’s local system is able to act on an unknown state of a remote system that belongs to the receiver.\(^{[14,15]}\) Then CRIO was proposed by extending RIO to multipartite case with controller.\(^{[16]}\) The idea is to implement remote operations, but only with the permission of controller. It can definitely enhance the security of RIO. Both of RIO and CRIO play an important role not only in distributed quantum computation, but also other tasks in remote quantum information processing such as programming,\(^{[17]}\) operation sharing,\(^{[18]}\) and remote state preparation.\(^{[19]}\) They are realized by local operation and classical communication (LOCC) and consume entanglement resource. Some works concerning RIO have been presented and interesting progress has been made both theoretically\(^{[20–23]}\) and experimentally.\(^{[24–26]}\) As for CRIO, it has been realized in terms of partially unknown quantum operations,\(^{[16,27]}\) various classes of bipartite unitary operations\(^{[28]}\) arbitrary dimensional controlled phase gate\(^{[29]}\) and operators on different remote photon states.\(^{[30]}\) Entangled states including Bell, Greenberger–Horne–Zeilinger (GHZ), and five-qubit cluster states are employed as channels in these protocols. Theoretically, many of these protocols can hardly be scalable, or technically complicated. What is more, the highly entangled states they employed are susceptible to noise, so it is a great challenge to realize them under experimental techniques.

In this paper, we propose CRIO protocols for the operations \(\exp[i\alpha \sigma_{ij}]\) whose control power is reliable, that is, without the controller’s permission, eight specified operations can be implemented with the success rate of 50%, other operations can be realized with the success rate of 25%. The diagram of our protocol via a graph state \(\{|h_i\rangle\}\) is shown in Figure 1. Inspired by the idea of controlled teleportation, the RIO protocol\(^{[30]}\) is generalized into CRIO, which enhances the security of RIO tasks and derives numerous applications. The stators in our protocol are constructed from shared graph states by LOCC, only with the permission of the controller. The protocol via a \((2N + 1)\)-partite graph state can be obtained by generalization, and that is used to implement remote operations on \(N\) distinct quantum systems. Graph states are employed as the channel in our protocol for three reasons. First, they make it possible to realize control function and enable our protocol to be scalable. Second, they are a natural resource for much of quantum information, and known to be most readily available multipartite resource in the laboratory.\(^{[31,32]}\) Third, many of the graph states show advantages for being robust to noise.\(^{[33]}\)
Ithasconnectionswithoptimalentanglementwitnesses, and naturally, they can be characterized by geometric processing.

putting and stimulating more research work on quantum information processing and contributing to improving the ability of distributed quantum communication, extensive applications, and advanced efficiency. It can be shown that our protocol is reliable. Our protocol shows advantage in stronger security, extensive applications, and advanced efficiency. It can contribute to improving the ability of distributed quantum computing and stimulate more research work on quantum information processing.

Graph states have a strong connection with quantum computation. Naturally, they can be characterized by geometric measure of entanglement (GM), a well-known entanglement measure for multipartite systems. GM not only provides a simple geometric picture, but also has significant operational meanings. It has connections with optimal entanglement witnesses,[34] and multipartite state discrimination under LOCC.[35] As one of the most widely used entanglement measures for the multipartite states, GM fulfills all the desired properties of an entanglement monotone.[36] It has been utilized to determine the universality of resource states for one-way quantum computation.[37]

has been employed to show that most entangled states are too entangled to be useful as computational resources.[38]

The rest of this paper is organized as follows. In Section 2, we introduce some basic concepts of graph states, GM, and stator. Then we simply recall the deduction of eigenoperator equation of the stator. In Section 3, we propose our CRO protocol via the tripartite and (2N + 1)-partite graph states, respectively. We also show that our protocol can be flexibly adjusted according to practical requirement. We show GM of the graph states used in our protocols in Section 4. We do the control power analysis in Section 5, and exhibit the experimental feasibility of our protocol in Section 6. We show the application of the proposed protocol in Section 7. Finally, we conclude in Section 8.

2. Preliminaries

In this section, we recall the definitions and some properties of graph states, GM and stator. In Section 2.1, we recall the definition of graph states, and demonstrate a graph state with the help of quantum gates. In Section 2.2, we recall the definition of GM and show a fact we employ to characterize the GM of graph states. In Section 2.3, we introduce the concept of stator and present the eigenoperator equation of the stator used in this paper.

2.1. Graph States

A graph is a pair $G = (V, E)$, where $V$ is the set of vertices and $E \subset [V]^2$ is the set of edges. With each graph, a graph state is associated. An axiomatic framework for mapping graphs to quantum states is proposed in ref. [38]. A graph state is a certain pure state on a Hilbert space $\mathcal{H}^V_2$. Each vertex of the graph labels a qubit. Each vertex $a \in V$ of the graph $G = (V, E)$ is attached to a Hermitian operator

$$K_G^{(a)} = \sigma_x^{(a)} \prod_{b \in N_a} \sigma_z^{(b)} \tag{1}$$

where $\sigma_x^{(a)}$ and $\sigma_z^{(b)}$ are the Pauli matrices and the upper index specifies the Hilbert space on which the operator acts. $K_G^{(a)}$ is an observable of the qubits related to vertex $a$ and all of its neighbors $b \in N_a$.

We demonstrate a graph state with the help of Hadamard gate $H$ and two-qubit controlled-Z gate $CZ_{(i,j)}$, where

$$H = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \tag{2}$$

and the gate

$$CZ_{(i,j)} = \text{diag}\{1, 1, 1, -1\} \tag{3}$$

denotes the gate with control qubit $j_i$ and controlled qubit $j_j$. By the two gates, we can show the preparation of graph states in the quantum circuit conveniently.

A graph state $|H_n\rangle$ is created from a graph $G = \{V, E\}$ of $n$ vertices by assigning a qubit to each vertex and initializing them
by applying the Hadamard gate on each qubit. Let $|+⟩ = (|0⟩ + |1⟩)/\sqrt{2}$. If two vertices $j_i, j_{i'} \in V$ are connected by an edge $e \in E$, then we perform $CZ_{j_i,j_{i'}}$ over the initialized $n$-qubit state $|+⟩^n$.

By implementing all the controlled-Z gates corresponding the edges $e \in E$, we obtain the graph state

$$|H_n⟩ = \prod_{e \in E} CZ_e |+⟩^n$$  \hspace{1cm} (4)

Graph states are useful resources with applications spanning many aspects of quantum information processing, such as computation,[39] cryptography,[40] quantum error correction,[41] and networks.[42] Experimentally, different techniques have been studied to implement graph states including ion traps,[43] superradiant states, isotropic states, generalized Dicke states, mixture of Dicke states, the Smolin state, and Dür’s multipartite entangled states.

Since the graph states we employed in this paper are all locally equivalent to non-negative states, we use Lemma 1 to investigate GM of them. The calculation is presented in Section 4.

2.2. Geometric Measure of Entanglement

Geometric measure of entanglement (GM) is a well-known entanglement measure for multipartite systems.[34] It measures the closest distance in terms of overlap between a given state and the set of separable states, or the set of pure product states. Originally introduced for pure bipartite states, GM was subsequently generalized to multipartite and mixed states. Several inequivalent definitions of GM have been surfaced by now. In this paper, we shall follow the definition as follows. For a pure state $|ψ⟩$, the GM of $|ψ⟩$ is

$$\Lambda^2(|ψ⟩) := \max_{σ ∈ SEP} Tr(|ψ⟩⟨ψ| σ) = \max_{σ ∈ PRO} |⟨ψ|ψ⟩|^2$$  \hspace{1cm} (5)

$$G_{\text{pure}}(|ψ⟩) := 1 − \Lambda^2(|ψ⟩)$$  \hspace{1cm} (6)

Here SEP denotes the separable states and PRO denotes the fully pure product states in the Hilbert space $\mathbb{C}^N \otimes H_i$. The GM of mixed state $ρ$ can be made via the convex hull construction. The essence is a minimization over all decompositions $ρ = \sum_j p_j |ψ_j⟩⟨ψ_j|$ in to pure states,

$$G(ρ) := \min_{\{p_j, |ψ_j⟩\}} \sum_j p_j G_{\text{pure}}(|ψ_j⟩).$$  \hspace{1cm} (7)

By this definition, $G(ρ)$ is an entanglement monotone. GM is known only for a few examples, such as bipartite pure states, GHZ-type states, antisymmetric basis states, pure symmetric three-qubit states and some graph states.[46–48]

We show the following fact given in ref. [49]. It is a useful lemma concerning the closest product states of non-negative states. Here the closest product state denotes any pure product state maximizing (5). The non-negative state means that all its entries in the computational basis are non-negative.

**Lemma 1.** The closest product state to a non-negative state $ρ$ can be chosen to be non-negative.

The proof is presented in Lemma 8 of ref. [49]. This lemma can be used to characterize GM of the states that are non-negative or locally equivalent to non-negative states. In addition, it contributes to prove the strong additivity of GM of the states including Bell diagonal states, maximally correlated generalized Bell diagonal states, isotropic states, generalized Dicke states, mixture of Dicke states, the Smolin state, and Dür’s multipartite entangled states.

2.3. Stator and Eigenoperator Equation

The stator, a hybrid state operator, is an object that expresses quantum correlations between states of one participant and operators of the other participant. It is first proposed in ref. [20] to implement a class of operations $\exp[iασ_z]$ on remote systems. Given a well-prepared stator, the operation on Bob’s system is remotely brought about by Alice’s local operations. The desired operation is determined by Alice and unknown to Bob. Hence it demonstrates advantages in terms of security.

A stator $S_{AB}$ shared by remote observers Alice and Bob is in the space

$$S_{AB} \in \{H_A \otimes O(H_B)\}$$  \hspace{1cm} (8)

where $H_A$ and $H_B$ are the state spaces of Alice and Bob, respectively, and $O(H_B)$ denotes the operators acting on an arbitrary state in $H_B$. A stator has the general form

$$S_{AB} = \sum_{i=1}^{N_A} \sum_{s=1}^{N_B} c_{ij}|s⟩ \otimes B_i$$  \hspace{1cm} (9)

where $N_A = \text{Dim}(H_A), N_B = \text{Dim}(H_B), |s⟩ \in H_A, B_i$ acts on states in $H_B$ and $c_{ij}$ are complex numbers. For each stator, an eigenoperator equation can be constructed. We consider the following stator used in this paper,

$$S = |0⟩_A \otimes I_B + |1⟩_A \otimes σ_{nx}$$  \hspace{1cm} (10)

Here $σ_{nx} = \frac{1}{2} (σ_x \cdot σ) \in O(H_B)$, where $σ_{nx} = [x, y, z]$ is the axis vector and $σ = [σ_x, σ_y, σ_z]$ is the Pauli matrix vector. The states $|0⟩_A, |1⟩_A \in H_A$ are the eigenstates of $σ_{nx}$. One can verify that $σ_{nx}^2 = I$. Obviously, $S$ satisfies the eigenoperator equation

$$σ_{nx} S = σ_{nx} S$$  \hspace{1cm} (11)

Thus for any analytic function $f$, it also satisfies that

$$f(σ_{nx}) S = f(σ_{nx}) S$$  \hspace{1cm} (12)

and particularly,

$$e^{iαS} S = e^{iαS} S$$  \hspace{1cm} (13)

where $α$ is the parameter determined by Alice. Using the stator, a unitary operation on Alice’s qubit gives rise to a similar unitary operation acting on Bob’s system, which is remote to Alice. The construction of stator and the implementation of remote operations are both realized by LOCC.

The target operation $\exp[iασ_z]$ we aim to implement is a useful class of unitary transformations in quantum protocols, where $α$ is a real number, and $σ_z = [x, y, z] \cdot [σ_x, σ_y, σ_z]$. It shows the physical
interpretation of the rotations of the Bloch sphere about an arbitrary axis. To be specific, the operation \( \exp[i\alpha \sigma_n] \) corresponds to a clockwise rotation through an angle \( 2\alpha \) about the axis directed along the unit vector \( n = (x, y, z) \). Since a generic \( 2 \times 2 \) unitary matrix can be seen (up to an overall phase factor) as a rotation about some axis of the Bloch sphere, the operations \( \exp[i\alpha \sigma_n] \) naturally include all the single-qubit unitary operations. Besides, the remote implementation of nonlocal operations can also be realized by the same way. More details are given in section VII of ref. [20].

3. Controlled Remote Implementation of Operations

In this section, we show the implementation of our protocol. A \( (2N + 1) \)-partite graph state is preshared between the participants \( A_1, A_2, \ldots, A_{2N+1} \). The participants cooperate to realize the remote operations \( U_n = \otimes_{i=1}^{N+2} \exp[iu \sigma_{n_i}] \) on \( N \) unknown states for remote systems \( O_i \) by applying LOCC. Here the parameters \( \alpha_{k+i,N} \) and \( \sigma_{n_{k+i,N}} \) is known only by participants \( A_k \) and \( A_{k+i,N} \), respectively, for \( k = 2, 3, \ldots, N + 1 \). Like what we have assumed in controlled quantum teleportation,[60] we consider the situation that participants locate in distributed places. They can neither establish the quantum correlation expect for the preshared graph state nor discard the graph state. Besides, the controller takes the responsibility to decide whether or not and when the implementation of remote operations should be done on each system. Thus our protocol shows advantage in terms of strong security and extensive applications. In Section 3.1, we introduce the implementation of operation \( U_n = \exp[i\sigma_{n_c}] \) on one remote system \( C \) via a tripartite graph state. Then we generalize this protocol into the one via a \( (2N + 1) \)-partite graph state \( (N \geq 2) \), which is presented in Section 3.2. Finally in Section 3.3, we show that our protocol can control all remote operations on subsystems, but also be flexibly adjusted to control some of them. Thus it can be utilized for extensive applications.

3.1. The Protocol via a Tripartite Graph State

We consider the controlled remote implementation of operations via a tripartite graph state, and show the first result of our work.

**Protocol 1.** Let \( \ket{h_i}_{a,b,c} \) be a tripartite graph state shared by three different participants. The controlled remote implementation of operation \( U_n = \exp[i\sigma_{n_c}] \) can be realized via the state \( \ket{h_{3}}_{a,b,c} \), only with the permission of the participant holding qubit \( a \), where

\[
\ket{h_{3}}_{a,b,c} = CZ_{(a,b)}CZ_{(a,c)} \ket{+}^3 = \frac{1}{\sqrt{2}} \left( \ket{000} + \ket{001} + \ket{010} + \ket{011} + \ket{100} \right. \\
\left. - \ket{101} - \ket{110} + \ket{111} \right)_{a,b,c}
\]

\( \ket{+}^3 \) is the superposition of all the basis states of the three qubits. The diagram of this process is shown in Figure 1.

![Figure 1](image1.png)

To show this protocol conveniently, we assume that three participants Alice, Bob, and Charlie share qubit \( a, b, c \) of \( \ket{h_{3}}_{a,b,c} \), respectively. The entangled state \( \ket{h_{3}} \) is prepared by the circuit shown in Figure 2. We will show the process of this protocol, where the target operation \( U_c = \exp[i\sigma_{n_c}] \) can not be realized on an unknown state \( \ket{\Psi_C} \) for system \( C \) without Alice’s permission. Here \( a \) is the parameter determined by Bob, and system \( C \) belongs to Charlie.

![Figure 2](image2.png)

**Figure 2.** The preparation of graph state \( \ket{h_3} \). The expressions of Hadamard and controlled-Z gates used in this circuit is given in Equations (2) and (3), respectively.

First, they construct the stator \( S_t \), this process is performed on the state \( \ket{\Psi_C} \). We do not show \( \ket{\Psi_C} \) in each step for simplicity. Charlie performs the local operation on his qubits \( c \) and \( C \), where

\[
U_{c,C} = \ket{0}_{c} \bra{0} \otimes I_{C} + \ket{1}_{c} \bra{1} \otimes \sigma_{n_c}.
\]

Here the operator \( \sigma_{n_c} = \bar{n}_{c} \cdot \vec{\sigma} \) acting on the system \( C \) satisfies \( \sigma_{n_c}^2 = I \). So the stator is

\[
S'_3 = \frac{1}{2\sqrt{2}} \left( \ket{000}_{a,b,c} \otimes I_{C} + \ket{001}_{a,b,c} \otimes \sigma_{n_c} + \ket{010}_{a,b,c} \otimes \sigma_{n_c} + \ket{011}_{a,b,c} \otimes \sigma_{n_c} \right.
\]

\[
+ \ket{111}_{a,b,c} \otimes \sigma_{n_c} \right) - \ket{100}_{a,b,c} \otimes \sigma_{n_c} - \ket{101}_{a,b,c} \otimes \sigma_{n_c}
\]

\[
= \frac{1}{2} \ket{+}_{a,b,c} \otimes I_{C} + \ket{11}_{a,b,c} \otimes \sigma_{n_c}.
\]

Now Alice’s qubit \( a \) is correlated with qubits \( b \) and \( c \). If Alice does not wish to cooperate with Bob and Charlie, she does nothing or something unknown to them. Then the relation between qubits \( b \) and \( c \) is unknown to Bob and Charlie and hence they can not construct the stator. Otherwise, if Alice wants the operations to be implemented, she performs a measurement of \( \sigma_{n} \) on qubit \( a \), and informs Bob of the measurement result. If the result is \( \ket{+} \), Bob performs the operation \( \sigma_{n} \) on qubit \( b \), otherwise Bob need not perform any operation. They obtain the stator

\[
S''_3 = \ket{00}_{b,c} \otimes I_{C} + \ket{11}_{b,c} \otimes \sigma_{n_c}.
\]

Then Charlie measures qubit \( c \) in the basis \( \ket{+} \) and \( \ket{-} \). If the result is \( \ket{-} \), Bob implements the operation \( \sigma_{n} \) on qubit \( b \), otherwise Bob does nothing. Hence they construct the following stator \( S_t \) successfully,

\[
S_t = \ket{0}_{b} \otimes I_{C} + \ket{1}_{b} \otimes \sigma_{n_c}.
\]

The second step is to implement the operation \( U_c = \exp[i\alpha \sigma_{n_c}] \) on system \( C \), with the help of eigenoperator equation

\[
e^{i\alpha \sigma_{n_c}} S_t = e^{i\alpha \sigma_{n_c}} S_t.
\]

Note that \( \sigma_{n_c}^2 = I \), so we have \( e^{i\alpha \sigma_{n_c}} = \cos \alpha I_{C} + i \sin \alpha \sigma_{n_c} \).

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Employing the Protocol 2.

Bob measures qubit $v_N$ in the Z-basis respectively. Here $V = \{v_a, v_b, v_c, v_d\}$, $E = \{(v_a,v_b), (v_a,v_c), (v_a,v_d), (v_b,v_c), (v_b,v_d), (v_c,v_d)\}$ with $j = 3, 4, \ldots, N+1$. Each vertex is associated to a participant in the protocol. Green vertices $v_{N+2}, v_{N+3}, \ldots, v_{2N+1}$ correspond to participants owning $N$ remote systems, and the blue ones $v_2, v_3, \ldots, v_{N+1}$ correspond to the participants aiming to implement remote operations on these systems.

Now Bob and Charlie implement the remote operation by stator $S_1$. Bob implements the operation $\exp[i\pi a_1\sigma_a]$ on qubit $b$. Based on Equation (19), the resulting state is

$$|s\rangle = e^{i\pi a_1 S_1} |\Psi_C\rangle$$

$$= (|0\rangle_b \otimes I_c + |1\rangle_b \otimes \sigma_c) e^{i\pi a_1 \sigma_a} |\Psi_C\rangle$$

(20)

Bob measures qubit $b$ in the Z-basis $|0\rangle_b$ and $|1\rangle_b$, and informs Charlie of the result. If it is $|1\rangle_b$, Charlie performs the local rotation $\exp[i\pi a_1\sigma_a/2] = i\sigma_c$ on system C. This completes our CRIO protocol via $|h_1\rangle$.

### 3.2. The Protocol via a $(2N+1)$-Partite Graph State

Employing the $(2N+1)$-partite graph state will improve the efficiency of our protocol. That is, $N$ controlled operations can be realized on distinct remote quantum systems simultaneously in a single protocol. From this reason, show the main result of our work as follows.

**Protocol 2.** Let $|h_{2N+1}\rangle_{a_1,a_2,\ldots,a_{2N+1}}$ be a $(2N+1)$-partite graph state shared by $2N+1$ different participants. The controlled remote implementation of operation $\bigotimes_{j=0}^{2N+1} \exp[i\pi x_j\sigma_j]$ can be realized via the state $|h_{2N+1}\rangle_{a_1,a_2,\ldots,a_{2N+1}}$ only with the permission of the participant holding qubit $a_1$, where

$$|h_{2N+1}\rangle_{a_1,a_2,\ldots,a_{2N+1}} = CZ_{(1,2)} CZ_{(1,N+2)} \times \prod_{j=1}^{N+1} \left( CZ_{(2,j)} CZ_{(j,N+2)} \right) \text{ for } j = 3, 4, \ldots, N+1$$

$$= \frac{1}{2^{N-1}} \sum_{i=0}^{N-1} (-1)^{i} |\Psi_i\rangle$$

$$\times |q_1, q_2, \ldots, q_{2N+1}\rangle_{a_1,a_2,\ldots,a_{2N+1}}$$

(21)

The brief description of this protocol is presented here. For a detailed derivation, see Appendix A. The state $|h_{2N+1}\rangle_1$ is prepared by the quantum circuit shown in Figure 3, and its corresponding graph is shown in Figure 4.

Suppose the $2N+1$ participants share the state $|h_{2N+1}\rangle_{a_1,a_2,\ldots,a_{2N+1}}$, where qubit $a_1$ belongs to participant $A_1$, respectively, for $k = 1, 2, \ldots, 2N+1$. Participants $A_{i-N}$ and $A_i$ work as a group to implement the operation $\exp[i\pi x_j\sigma_j]$ on the unknown states $|\Psi_i\rangle$ for system $O_j$, for $j = N+2, N+3, \ldots, 2N+1$. For each group, the parameters $\beta_i$ and $\sigma_{a_i}$ is only available to participants $A_{i-N}$ and $A_i$, respectively. We will show that the target operation can not be implemented without the permission of $A_1$.

Now we show the realization of the remote operations by the cooperation of $2N+1$ participants. First, they prepare the stator $S_{2N+1}$ by LOCC, only with the permission of controller $A_1$, that is, Steps 1–4 in the following statement. Then the operations are implemented on these systems with the help of the stator, that is, Steps 5 and 6. For convenience, we ignore the global factor in the following statement.
Step 1: The participants $A_j$ perform the local operations

$$U_{gj} = |0\rangle_g |0\rangle \otimes I_{O_j} + |1\rangle_g |1\rangle \otimes \sigma_{nO_j}$$ (22)

on the entangled state $|\psi_{N+1}\rangle$ in Equation (21) for qubit $a_i$ and system $O_j$ respectively, where $j = N + 2, N + 3, \ldots, 2N + 1$. The resulting stator is

$$S_{2N+1}' = \sum_{q_0, q_2} (-1)^f |q_2, q_2, \ldots, q_{2N+1}\rangle \otimes \sigma_{nO_{j+2}}^0 \cdots \otimes \sigma_{nO_{j+2}}^{N+1}$$ (23)

where the Boolean function $f(x) = q_0 q_2 \otimes q_2 q_{N+2} \otimes q_{N+1}$ for $x \equiv q_0 q_2 \cdots q_{2N+1}$ and $q_j \in \{0, 1\}, j = 1, 2, \ldots, 2N + 1$.  

Step 2: Participants $A_j, A_{j+1}, \ldots, A_{N+1}$ perform the operation $H$ on their qubits $a_j, a_{j+1}, \ldots, a_{N+1}$, respectively. It yields the following stator $S_{2N+1}'$:

$$S_{2N+1}' = |+\rangle_{a_1} \sum_{q_0, q_2} |q_2, q_2, \ldots, q_{N+1}, q_2, \ldots, q_{N+1}\rangle \otimes \sigma_{nO_{j+2}}^0 \cdots \otimes \sigma_{nO_{j+2}}^{N+1}$$ (24)

where $|+\rangle_{a_1} = \frac{1}{\sqrt{2}} (|0\rangle + |1\rangle)$. The sender does nothing. So they construct the stator

$$S_{2N+1}' = \sum_{q_0, q_2} (-1)^f |q_2, q_2, \ldots, q_{2N+1}\rangle \otimes \sigma_{nO_{j+2}}^0 \cdots \otimes \sigma_{nO_{j+2}}^{N+1}$$ (25)

Step 3: From the expression of $S_{2N+1}'$, one can find that the controller's qubit $a_i$ is correlated with qubits $a_k$ and $a_{k+1}$ for $k = 2, 3, \ldots, N + 1$. If the controller $A_i$ does not wish to cooperate with other groups including $A_k$ and $A_{k+1}$, he does nothing or something unknown to other groups. Then the relation between qubits $a_i$ and $a_{k+1}$ is unknown to participants $A_i$ and $A_{k+1}$ and hence they cannot control the operations at the beginning of this subsection. Otherwise, if the controller $A_i$ permits the realization of these operations, he implements the measurement of $\sigma_{nO_j}$ on qubit $a_i$ and informs $A_j, A_{j+1}, \ldots, A_{N+1}$ of the result. If the result is $|+\rangle$, the receivers $A_j, A_{j+1}, \ldots, A_{N+1}$ perform $\sigma_{nO_j}$ on their qubits, otherwise they do nothing. This yields the stator

$$S_{2N+1}'' = \sum_{q_0, q_2} |q_2, q_2, \ldots, q_{2N+1}\rangle \otimes \sigma_{nO_{j+2}}^0 \cdots \otimes \sigma_{nO_{j+2}}^{N+1}$$ (26)

where $\sigma_{nO_j}^0 = I_{O_j}, j = N + 2, \ldots, 2N + 1$.  

Next the stator $S_{2N+1}$ is employed to implement the remote operations on the unknown state $|\psi_j\rangle$ for remote systems $O_j$.

Step 5: The participant $A_k$ performs the local operation $\exp[i\beta \sigma_{nO_j}]$ on qubit $a_i$, for $k = 2, 3, \ldots, N + 1$. The resulting state is now

$$S_{2N+1}'' = \sum_{j=2N+1}^{2N+1} \exp[i\beta \sigma_{nO_j}] |\psi_j\rangle$$ (27)

Step 6: The participant $A_k$ implements the measurement of $\sigma_{nO_j}$ on qubit $a_i$, and informs $A_{k+N}$ of the measurement result, where $k = 2, 3, \ldots, N + 1$. If the sender $A_k$'s result is $|1\rangle$, the receiver $A_{k+N}$ performs the operation $\exp[i\beta \sigma_{nO_j}] 2 = i\sigma_{nO_j}$, otherwise the receiver need not perform any operation. So they eliminate the stator $S_{2N+1}$. The state is finally

$$\sum_{j=2N+1}^{2N+1} \exp[i\beta \sigma_{nO_j}] |\psi_j\rangle$$ (28)

It indicates that they have implemented the remote operation successfully.

From the Steps 2 and 3 above, one can see that the controller $A_i$ determines whether and when these operations should be implemented, but knows nothing about the parameters of the operations. Participants $A_{k+N}$ and $A_i$ work as a group to implement the operation $\exp[i\beta \sigma_{nO_j}]$ on the unknown states $|\psi_j\rangle$ for system $O_j$, for $j = N + 2, N + 3, \ldots, 2N + 1$. For each group, the parameters $\beta$ and $\sigma_{nO_j}$ are only available to participants $A_{k+N}$ and $A_i$, respectively. That is to say, none of the participants is able to obtain the complete information of the target operations. It guarantees the security of our protocol.

### 3.3. Further Discussions of the Protocol

The above protocol can be flexibly adjusted according to practical requirement. In the former protocol via $|h_{2N+1}\rangle$ in Section 3.2, the controller is able to control $N$ remote operations, in a way that either enables all of them or none of them. If the controller wishes to allow some of $N$ remote operations to be implemented and prohibit the others, a robust protocol using $N$ tripartite graph state between $A_i, A_{i+1}$, and $A_{N+1}$ can be established. The related techniques have been shown in Section 3.1.

On the other hand, the controller can choose to control any $k$ of these $N$ groups, and discard the rest of $N - k$ groups with $k \geq 1$. It can be realized by employing appropriate graph states. Compared with the construction of $|h_{2N+1}\rangle$ in Figures 3 and 4, such states can be constructed by removing corresponding gates $CZ(2, k)$ and $CZ(k, N + 2)$ in pairs, for $k = 3, 4, \ldots, N + 1$. We take the protocol via $|h_i\rangle$ as an example. By setting $N = 2$ in the protocol in Section 3.2, we obtain the controlled protocol via $|h_{3}\rangle$, where

$$|h_{3}\rangle = CZ_{(i\rangle} CZ_{(d\rangle} CZ_{(b\rangle} CZ_{(a\rangle} CZ_{(\mu, b)} |+\rangle^{\otimes 5}$$ (29)
In this protocol, the controller can control both groups. Now we assume the controller $A_i$ only wants to control the group consisting of participants $A_{j}$ and $A_i$. This task can be realized by employing another graph state $|h_j\rangle$. The construction of $|h_j\rangle$ can be done with the help of the graph on the right hand side in Figure 5. That is to say, $|h_j\rangle$ can be constructed by removing the gates $CZ(2, 3)$ and $CZ(2, 4)$ compared with the construction of state $|h_i\rangle$. From this point of view, our protocol can flexibly follow the actual demands and thus it shows advantage in extensive applications.

## 4. Geometric Measure of Entanglement for Graph States

In this section, we investigate the GM of the graph states used in this paper. Then we compare the entanglement requirement between the protocol in ref. [20] and our protocol. We obtain that the two protocols require the same entanglement resource. The controller is added in our protocol, while no more entanglement is required. From this point of view, it is economic to realize the control function from the perspective of entanglement cost.

First we show an example by considering the tripartite graph state $|h_i\rangle$ shown in Equation (14). Since the local unitary operations do not change the entanglement of the states, we perform the local Hadamard gate $H$ on qubit $a$ of $|h_i\rangle$, and obtain the state

$$|g_i\rangle = \frac{1}{2}(|000\rangle + |111\rangle + |110\rangle + |011\rangle)$$

(30)

Now we consider the GM of $\rho_i = |g_i\rangle \langle g_i|$. According to Lemma 1, its closest product state can be chosen to be non-negative. Let $|\varphi_3\rangle = \bigotimes_{j=1}^{N-1} (\cos \theta |0\rangle + \sin \theta |1\rangle)$ be a closest product state, with $0 \leq \theta_j \leq \frac{\pi}{2}$. After some calculations, we obtain that

$$\Lambda^2(\rho_i) = \max_{|\varphi_3\rangle} \langle \varphi_3| \rho_i |\varphi_3\rangle = \max_{|\varphi_3\rangle} \langle \varphi_3| g_i \rangle^2$$

$$= \frac{1}{4} \max_{m, n, \theta_1, \theta_2} \left( \cos \theta_1 \cos (\theta_2 - \theta_3) + \sin \theta_1 \sin (\theta_2 + \theta_3) \right)^2 = \frac{1}{2},$$

(31)

$$G(\rho_i) = 1 - \Lambda^2(\rho_i) = \frac{1}{2}$$

(32)

The maximum in the above equation is obtained at $\theta_1 = \theta_2 = \theta_3 = \pi/4$, that is, $|\varphi_3\rangle = \sum_{\{0,1\}} |0\rangle^\otimes \langle 1|^{\otimes 3}$.

Next we consider the GM of $(2N + 1)$-partite graph state $|h_{2N+1}\rangle$ in Equation (21). We show the following fact:

**Proposition 2.** The geometric measure of entanglement for the state $|h_{2N+1}\rangle$ is equal to $N$, for $N \geq 2$.

**Proof.** We reduce the state $|h_{2N+1}\rangle$ by applying the local Hadamard gates $H$ on qubits $a_1, a_2, \ldots, a_{2N+1}$. Then it can be transformed to the following non-negative state,

$$|g_{2N+1}\rangle = \frac{1}{\sqrt{2^{2N+1}}} \sum_{q_1, q_2, \ldots, q_{2N+1}=0}^1 (|0, q_2, q_3, \ldots, q_{2N+1}, q_1, q_3, \ldots, q_{2N+1}\rangle$$

$$+ |1, q_2, q_3, \ldots, q_{N+1} \oplus 1, 1, q_2, q_3, \ldots, q_{N+1}\rangle)_{q_1, a_2, \ldots, a_{2N+1}}.$$ 

(33)

So the GM of $|h_{2N+1}\rangle$ is equal to that of $|g_{2N+1}\rangle$. Let $\rho_{2N+1} = |g_{2N+1}\rangle \langle g_{2N+1}|$ and $|\varphi_{2N+1}\rangle = \bigotimes_{j=1}^{N+1} (\cos \theta |0\rangle + \sin \theta |1\rangle)$. We denote $q_i \oplus 1$ as $\overline{q}_i$. Obviously, we have

$$(\cos \theta_1 |0\rangle + \sin \theta_1 |1\rangle)_{\overline{q}_i} = \cos \overline{\theta}_i |\sin \overline{\theta}_i \rangle,$$

(34)

where $q_0 = 0, 1$ and $\cos \theta_1 = \sin \overline{\theta}_1 = 1$. Using Equation (34), we obtain that

$$\langle \varphi_{2N+1}| g_{2N+1}\rangle = \frac{1}{\sqrt{2^{2N+1}}} \sum_{q_1, \overline{q}_1, \ldots, \overline{q}_{N+1}} \left[ \cos \theta_1 \left( \prod_{k=2}^{N+1} \cos \overline{\theta}_k \sin \overline{\theta}_k \right) \right.$$

$$\times \left( \prod_{m=2}^{N+1} \cos \overline{\theta}_m \sin \overline{\theta}_m \right)$$

$$+ \sin \theta_1 \left( \prod_{m=2}^{N+1} \cos \overline{\theta}_m \sin \overline{\theta}_m \right)$$

$$\times \left( \prod_{m=2}^{N+1} \cos \overline{\theta}_m \sin \overline{\theta}_m \right) \right]$$

(35)
By some calculations, we have
\[
\langle \varphi_{2N+1} | g_{2N+1} \rangle = \frac{1}{\sqrt{2^{N+1}}} \cos \theta \prod_{j=2}^{N+1} \cos(\theta_j - \theta_{j+N}) \\
+ \sin \theta \prod_{j=2}^{N+1} \sin(\theta_j + \theta_{j+N})
\]

Using Lemma 1, we have \(0 \leq \theta_j \leq \pi/2\) with \(j = 1, 2, \ldots, 2N + 1\). From Equation (36), the value of \(\langle \varphi_{2N+1} | g_{2N+1} \rangle\) reaches the maximum \(1/\sqrt{2^N}\) when \(\theta = \pi/4\). Hence we have
\[
\Lambda^2(\rho_{2N+1}) = \max_{\rho_j} \langle \varphi_{2N+1} | g_{2N+1} \rangle^2 = \frac{1}{2^N}
\]

It holds that
\[
G(\rho_{2N+1}) = 1 - \Lambda^2(\rho_{2N+1}) = 1 - \frac{1}{2^N}
\]

Now we compare the entanglement requirement between the protocol in ref. [20] and our controlled protocol in terms of GM. The protocol for remote operations by stator is first proposed in ref. [20]. In this protocol, \(2N\) participants share the \(2N\)-partite entangled state \(|\phi_{2N}\rangle\) to realize the remote operations on \(N\) systems without the controller. The entangled states used in that protocol are given as follows,
\[
|\phi_j\rangle = \frac{1}{\sqrt{2}} (|00\rangle + |11\rangle),
\]
\[
|\phi_k\rangle = \frac{1}{2} (|0000\rangle + |0101\rangle + |1010\rangle + |1111\rangle),
\]
\[
|\phi_{2N}\rangle = \frac{1}{\sqrt{2^N}} \sum_{q_1, q_2, \ldots, q_N = 0}^{1} |q_1, q_2, \ldots, q_N, q_1, q_2, \ldots, q_N\rangle.
\]

We investigate GM of the states in Equations (39)–(41) and list the entangled states required in the two protocols for a given number of systems. The results are presented in Table 1.

From Table 1, we see that the protocol we proposed in this paper requires the same entanglement resource as the former protocol in ref. [20]. That is to say, although one “controller” who enables control over other groups is added in our protocol, the entanglement cost of our protocol does not increase. Hence it is economic to realize the control function from the perspective of entanglement cost.

5. Control Power Analysis

In this section, we show the control power analysis of our controlled protocol. In analogy to the control power of quantum teleportation\([51]\) the controller’s power here is determined by the situation that the remote operation be accomplished without the controller’s help. If the controller does not wish the remote operation to be executed, he will not perform the measurement and other participants can hardly realize the target operations on remote systems. We show that without the controller’s permission, eight specified operations can be implemented with the success rate of 50%, other operations can be realized with the success rate of 25%. It means that the control power is reliable in our controlled remote implementation protocol. The main result of this section is shown in Proposition 3.

We consider the protocol via \(|h_i\rangle\) in Section 3.1. Other protocols can be analyzed in the same way, as the result we obtained in this section is valid for each group in the protocol via \(|h_{2N+1}\rangle\). To implement the remote operation \(e^{i\alpha_{ac}}\) without the controller Alice’s help, Bob and Charlie collaborate to perform some POVM on qubits \(b\) and \(c\) to eliminate the entanglement between qubits \(a\) and \(b\). In particular, Bob carries out the POVM \(\{M_j = |\beta_j\rangle\langle\beta_j|\}\) and Charlie carries out \(\{N_k = |\gamma_k\rangle\langle\gamma_k|\}\), for \(j, k = 1, 2\). The operators satisfy two basic restrictions,
\[
\sum_{j=1}^{2} M_j M_j^\dagger = \sum_{j=1}^{2} |\beta_j\rangle\langle\beta_j| = I
\]
\[
\sum_{k=1}^{2} N_k N_k^\dagger = \sum_{k=1}^{2} |\gamma_k\rangle\langle\gamma_k| = I
\]

We investigate the probability by which POVM operators \(M_j\) and \(N_k\) occur in the measurement. It is denoted as \(p(j, k)\). We normalize the stator \(S_i\) in Equation (16) and define the normalized stator as \(W_j\), where
\[
W_j = \frac{S_j}{\sqrt{\text{Tr}(S_j^\dagger S_j)}} = \frac{1}{\sqrt{2}} S_j
\]
Note that \( \text{Tr}_c(\sigma_{nc}) = 0 \). So we have

\[
p(j, k) = \frac{1}{8} \text{Tr} \left[ W_{ij}^c (I_c \otimes M^j_b \otimes N^k_c) W_{ij} \right]
\]

\[
= \frac{1}{8} \text{Tr} \left[ I_c + \sin(2\theta) \sin(2\lambda_i) \cos \varphi_j \cos \omega_k \sigma_{nc} \right]
\]

\[
= \frac{1}{8} \text{Tr}_c(I_c) = \frac{1}{4}
\]  

That is to say, the probability of performing \( M^j_b \otimes N^k_c \) on qubits \( b \) and \( c \) is equal to 25% for any \( (j, k) = (1, 1), \ldots (2, 2) \).

Suppose \( |\beta_j\rangle = \cos \theta |0\rangle + e^{i\varphi_j} \sin \theta |1\rangle \) and \( |\gamma_k\rangle = \cos \lambda_k |0\rangle + e^{i\omega_k} \sin \lambda_k |1\rangle \). Without loss of generality, we assume \( \theta_j, \lambda_k \in [0, \frac{\pi}{2}] \) and \( \varphi_j, \omega_k \in [0, 2\pi] \).

Let \( c_{ij} = (\langle \beta_j | s \rangle \langle \gamma_k | t \rangle) \) with \( s, t = 0, 1 \), that is,

\[
c_{0,0} = \cos \theta \cos \lambda_k
\]

\[
c_{0,1} = e^{-i\varphi_j} \cos \theta \sin \lambda_k
\]

\[
c_{1,0} = e^{i\varphi_j} \sin \theta \cos \lambda_k
\]

\[
c_{1,1} = e^{-i\varphi_j} \sin \theta \sin \lambda_k
\]  

We consider the situation that POVM operators \( M_j \) and \( N_k \) occur in the measurement, for \( j, k = 1, 2 \). After the measurement, the stator \( S'_i \) in Equation (16) becomes

\[
T'_s = \frac{1}{2} |+\rangle_1 (M_j |0\rangle_2 N_k |0\rangle_3 \otimes I_c + M_j |1\rangle_2 N_k |1\rangle_3 \otimes \sigma_{nc})
\]

\[
+ \frac{1}{2} |-\rangle_1 (M_j |1\rangle_2 N_k |0\rangle_3 \otimes I_c + M_j |0\rangle_2 N_k |1\rangle_3 \otimes \sigma_{nc})
\]

\[
= \frac{1}{2} |+\rangle_1 (c_{0,0} |\beta_j\rangle \langle \gamma_k | + c_{1,1} |\beta_j\rangle \langle \gamma_k | \otimes \sigma_{nc})
\]

\[
+ \frac{1}{2} |-\rangle_1 (c_{0,1} |\beta_j\rangle \langle \gamma_k | + c_{1,0} |\beta_j\rangle \langle \gamma_k | \otimes \sigma_{nc})
\]

Bob and Charlie aim to implement the operation

\[
e^{i\alpha_{nc}} = \cos \alpha_c + i \sin \alpha \sigma_{nc}
\]  

on system \( C \) without Alice’s help. Their goal is to make qubit \( a \) separate from remaining qubits \( b \) and \( c \) by POVM. We analyze the possible angle \( \alpha \) in the operation \( e^{i\alpha_{nc}} \) that can be realized in this scheme, and obtain the following fact:

**Proposition 3.** For the protocol via \( \{|h_i\}\) in Section 3.1, the probabilistic implementation of remote operation \( e^{i\alpha_{nc}} \) can be realized by POVM without controller’s permission. The success rate to realize the operation \( e^{i\alpha_{nc}} \) is equal to 50% for \( \alpha \in \{0, \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}, \frac{9\pi}{4}\} \), and that is equal to 25% for \( \alpha \in \left\lfloor \frac{\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}, \frac{9\pi}{4}\right\rfloor \).

The proof of Proposition 3 is shown in Appendix B.

Now we show some security analysis of the POVM scheme. In our controlled protocol in Section 3.1, the deterministic implementation of \( e^{i\alpha_{nc}} \) can be realized with the permission of controller Alice. The rotation angle \( \alpha \) can be chosen as any value and Charlie is unaware of any information of the operation that Bob wants to implement. However, in this POVM scheme in Section 5, we claim that Charlie can obtain the value of \( \alpha \) with the probability of 25% or 12.5%. It means that the confidentiality of remote operation \( e^{i\alpha_{nc}} \) may be destroyed. In fact, the rotation angle \( \alpha \) in \( e^{i\alpha_{nc}} \) can only be one of \( \{\frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}\} \) when Charlie prepares the POVM operators with \( \lambda_i = 0 \) and \( \lambda_i = \frac{\pi}{4} \), respectively. So there are four possible rotation angles \( \alpha \) for a group of POVM operators with given parameters. Charlie may guess what the rotation angle \( \alpha \) is, and get to the correct answer with the probability of 25%. When Charlie prepares the operator with \( \lambda_i \in \{0, \frac{\pi}{4}\} \cup \{\frac{\pi}{2}, \frac{3\pi}{4}\} \), the target rotation angle may be any of \( \{\pi - \lambda_i, 2\pi - \lambda_i, \frac{\pi}{2} - \lambda_i, \frac{\pi}{4} - \lambda_i, \lambda_i, \lambda_i + \frac{\pi}{2}, \lambda_i + \frac{3\pi}{4}\} \). So Charlie may be aware of the value of \( \alpha \) with the probability of 12.5%.

To sum up, Bob and Charlie can hardly realize the remote implementation of operation perfectly without the permission of controller Alice. The control power in our controlled protocol is convincing.

### 6. Experimental Feasibility

As the entanglement resource of our protocol, graph states are the most readily available multipartite resource in the laboratory, and they have already been built and used for information processing experimentally. In a recent work, the deterministic protocol is implemented, by which the GHZ states of up to 14 photons and linear cluster states of up to 12 photons have been grown with a fidelity lower bounded by 76(6)% and 56(4)%, respectively.\(^{[52]}\) A scheme to prepare an ultrahigh-fidelity four-photon linear cluster state has been proposed, and it has been experimentally demonstrated with the fidelity of 0.9517 ± 0.0027. This scheme can be directly extended to more photons to generate an N-qubit linear cluster state.\(^{[53]}\) Recently, an efficient scheme is demonstrated to prepare graph states with only a polynomial overhead using long-lived atomic quantum memories.\(^{[54]}\) Such technique can be used to prepare the graph states Equation (21) in our protocol, so the requirement of entanglement resource can be satisfied.

Remote implementation has been demonstrated theoretically and realized experimentally, such as remote state preparation,\(^{[55]}\) nonlocal CNOT operation implementation,\(^{[56]}\) and remote generation of entanglement. In our controlled remote implementation protocol, the participants prepare the stators and implement the remote operation by applying LOCC on shared entangled state. The local operations and measurements required in our protocol are \( U_{\lambda_i,\gamma_j}, H, \sigma_x, \sigma_y, \sigma_z, \) and \( Z \)-basis measurements, respectively. In the work presented in ref.\(^{[26]}\), a kind of memory-enhanced quantum communication is demonstrated experimentally. In this work, some local operations and measurements in the X and Z bases are implemented with a time-delay interferometer (TDI). The device consists of a diamond nanophotonic resonator containing silicon-vacancy quantum memory with an integrated microwave stripline. By reducing the possibility that an additional photon reaches the cavity, high-spin-photon gate fidelities are enabled. Measurements are performed with high-fidelity by keeping track of the timing when the TDI piezo voltage reaches a limiting value, which guarantees that the silicon-vacancy is always resonant with the photonic qubits. The experimental technique in this work can be employed...
The security requirement and efficiency is guaranteed in this computation network. It comes from the following reasons:

1) The client transmits his instructions (whether or not the gates should be implemented) via quantum communication by the shared entanglement. It prevents the information from intercept by outsiders.

2) Any two nodes $B_j$ and $C_j$ of this network transmit the information by quantum communication. The parameter of their target gate is not transmitted via classical communication, and thus it is also protected.

3) The control power of the client is reliable by Protocol 2. Once the client prohibits the gate transfer, other nodes can only implement eight specified gates with the success rate of 50%, other gates can be realized with the success rate of 25%. See more details in Section 5.

4) The quantum gates are transmitted between distant nodes in parallel. The network contributes to realize the parallel computing between several quantum computers.

8. Conclusion

We have proposed the protocol for controlled remote implementation of operations in the form of $U = \exp[i \sigma_j \gamma]$ for each system. A family of graph states is constructed as the entanglement resource in our protocol. Sharing the $(2N + 1)$-partite graph state as channel, $2N$ participants are able to realize the remote operations $U$ on $N$ unknown states for distant systems respectively, only with the permission of a controller. The implementation requirements of our protocol can be satisfied only by means of LOCC. Further we have characterized GM of the graph states in our protocol. Compared with the entanglement requirement of the protocol in ref. [20], the control function of our protocol is realized economically. Based on the result of control power analysis, the control power is reliable in our protocol, that is, others can not realize the implementation without the permission of controller. Further we have exhibited the application and experimental feasibility of our protocol in terms of current techniques.

Many problems arising from this paper can be further explored. As we all know, multilevel systems as qudits feature more advantages than their binary counterpart. Our protocol can be considered in high-dimensional Hilbert spaces, and may be used to implement some other operations. The protocol with more controllers for specific systems can be studied by applying appropriate graph states and local implementations. The interaction between quantum states and environment is unavoidable, leading to the loss of accuracy. Considering the situation that graph states in our protocol are affected by noise, the probabilistic implementation of that can be developed.

Appendix A: Derivation of Protocol 2

We show the derivation of Protocol 2 in detail. We assume that $2N + 1$ participants $A_1, A_2, \ldots, A_{2N + 1}$ are involved. Participant $A_1$ is the controller who supervises $N$ distributed groups. Participants $A_{N+1}$ and $A_j$ work as a group to implement the operation $\exp[i \beta_j \sigma_{n_j}]$ on the unknown states $|\Psi_j\rangle$ for system $O_j$, for $j = N + 2, N + 3, \ldots, 2N + 1$. For each group, the parameters $\beta_j$ and $\sigma_{n_j}$ is only available to participants $A_{N+1}$ and $A_j$, respectively. During this process, the controller $A_1$ determines whether and when these
operations should be implemented, but knows nothing about the parameters of the operations. That is to say, none of the participants is able to obtain the complete information of the target operations. It guarantees the security of our protocol. The $2N + 1$ participants share the following $(2N + 1)$-qubit graph state,

$$|h_{2N + 1}\rangle = CZ_{(1, 2)} CZ_{(1, N + 2)} \prod_{k = 1}^{N + 1} \left[ CZ_{(2, k)} CZ_{(k, N + 2)} CZ_{(k, N + 0)} \right] |\pm\rangle^{\otimes (2N + 1)}$$

$$= \frac{1}{2^{N + 1}} \sum_{q_1, q_2, \ldots, q_{2N + 1} = 0} (-1)^{f(q)} |q_1, q_2, \ldots, q_{2N + 1}\rangle |a_1, a_2, \ldots, a_{2N + 1}\rangle,$$

where qubit $a_k$ belongs to participant $A_k$ respectively, for $k = 1, 2, \ldots, 2N + 1$. The Boolean function $f$ is described as a map $f : \{0, 1\}^{2N + 1} \to \{0, 1\}$, for $x \equiv q_1 q_2 \ldots q_{2N + 1}$ and $q_i \in \{0, 1\}, j = 1, 2, \ldots, 2N + 1$;

$$f(x) = q_1 q_2 \oplus q_1 q_{N + 2} \oplus q_1^2 q_{N + 1} a_1 q_3 a_2 a_3 a_{2N + 1}$$

(A2)

Here $\oplus$ denotes plus modulo two, and

$$q_i = \begin{cases} 
1 & \text{for } q_i = q_i = 1, \\
0 & \text{otherwise}
\end{cases}$$

(A3)

The state $|h_{2N + 1}\rangle$ is prepared by the quantum circuit shown in Figure 3. Now we show the realization of the remote operation by the cooperation of $2N + 1$ participants. First, they prepare the stator $S_{2N + 1}$ by LOCC, only with the permission of controller $A_1$. The states are constructed on the unknown states $|\mathcal{F}_{j}\rangle$ for remote systems $O_j$. Second, the operations are implemented on these systems with the help of the stator. The construction of stator consists of four steps, that is, Steps 1–4 in the following statement. In each step, the participants perform a kind of local operations or measurements. For convenience, we ignore the global factor in the following statement.

Step 1: The participants $A_j$ perform the local operation

$$U_{a_0, O_j} = |0\rangle_{a_0} (|0\rangle \otimes I_{O_j} + |1\rangle) (|1\rangle \otimes \sigma_{O_j}^J)$$

on the entangled state $|h_{2N + 1}\rangle$ in (21) for qubit $a_0$ and system $O_j$ respectively, where $j = N + 2, N + 3, \ldots, 2N + 1$. The operation $\sigma_{O_j}^J$ satisfies that

$$\sigma_{O_j}^J = \sigma_{O_j}^{a_0} I_{O_j}.$$ The resulting stator is

$$S'_{2N + 1} = \sum_{q_1, q_2, \ldots, q_{2N + 1} = 0} (-1)^{f(q)} |q_1, q_2, \ldots, q_{2N + 1}\rangle a_1 a_2 a_3 a_{2N + 1} \otimes \sigma_{O_{N + 3}}^{a_0} \otimes \sigma_{O_{N + 3}}^{a_2} \otimes \cdots \otimes \sigma_{O_{2N + 1}}^{a_2}$$

(A4)

For convenience of the statement, we rearrange the stator $S'_{2N + 1}$ according to qubit $a_j$, in the basis $|\pm\rangle_{a_j}$ and $|\pm\rangle_{a_j}$ as follows,

$$S'_{2N + 1} = |+\rangle_{a_j} \sum_{q_1, q_2, \ldots, q_{2N + 1} = 0} (-1)^{f(q)} (-1)E(q')|q_2, \ldots, q_{2N + 1}\rangle$$

$$\otimes \sigma_{O_{N + 3}}^{a_0} \otimes \cdots \otimes \sigma_{O_{2N + 1}}^{a_2} + |\mp\rangle_{a_j} \sum_{q_2, \ldots, q_{2N + 1} = 0} (-1)^{f(q') - E(q')}|q_2, \ldots, q_{2N + 1}\rangle$$

$$\otimes \sigma_{O_{N + 3}}^{a_0} \otimes \cdots \otimes \sigma_{O_{2N + 1}}^{a_2}$$

where $g_1(x')$ and $g_2(x')$ are Boolean functions of $x' = q_2 q_3 \ldots q_{2N + 1}$:

$$g_1(x') = \oplus_{N = 3}^{N + 1} (q_2 q_3 \oplus q_4 q_{N + 2} \oplus q_5 q_{N + 4})$$

(A7)

$$g_2(x') = q_2 \oplus q_{N + 2} \oplus q_{N + 1} q_3 \oplus q_4 q_{N + 2} \oplus q_5 q_{N + 4}$$

(A8)

From Equations (A7) and (A8), one can obtain that $g_1(x') = g_2(x')$ when $q_2 = q_{N + 2}$, and $g_1(x') = g_2(x') + 1$ when $q_2 = q_{N + 2} + 1$. Hence, ignoring the global factor, the stator in Equation (A8) is equal to

$$S'_{2N + 1} = |+\rangle_{a_j} \otimes M + |\mp\rangle_{a_j} \otimes N$$

where

$$M = \sum_{q_2, q_3, \ldots, q_{2N + 1} = 0} (-1)^{g_1(x')} |q_2, q_3, \ldots, q_{2N + 1}\rangle$$

$$\otimes |q_2, a_0, a_2 \otimes a_3 a_4 \otimes \cdots \otimes a_{2N + 1}\rangle$$

(A10)

$$N = \sum_{q_2, q_3, \ldots, q_{2N + 1} = 0} (-1)^{g_2(x')} |q_2, q_3, \ldots, q_{2N + 1}\rangle$$

$$\otimes |q_2, a_0, a_2 \otimes a_3 a_4 \otimes \cdots \otimes a_{2N + 1}\rangle$$

(A11)

Here $h_1(x')$ and $h_2(x')$ are Boolean functions of $x'' = q_3 q_4 \ldots q_{2N + 1}$:

$$h_1(x'') = \oplus_{N = 3}^{N + 1} q_3 q_{N + 4}$$

(A12)

$$h_2(x'') = \oplus_{N = 3}^{N + 1} (q_3 \oplus q_{N + 4})$$

(A13)

Step 2: Participants $A_2, A_3, \ldots, A_{N + 1}$ perform the operation $H = \mathcal{O}_j$ on their qubits $a_3, a_4, \ldots, a_{N + 1}$, respectively. To show the effect of the operation $\mathcal{O}_j$, we act on the state $S'_{2N + 1}$ in Equation (A4) clearly, we rearrange $M$ in Equation (A10) and $N$ in Equation (A11) as follows

$$M = \sum_{q_2, q_3, \ldots, q_{2N + 1} = 0} |q_2, q_3, \ldots, q_{2N + 1}\rangle$$

$$\otimes |q_2, a_0, a_2 \otimes a_3 a_4 \otimes \cdots \otimes a_{2N + 1}\rangle$$

$$\otimes \mathcal{O}_j |q_2, a_0, a_2 \otimes \cdots \otimes a_{2N + 1}\rangle$$

(A14)

$$N = \sum_{q_2, q_3, \ldots, q_{2N + 1} = 0} |q_2, q_3, \ldots, q_{2N + 1}\rangle$$

$$\otimes |q_2, a_0, a_2 \otimes a_3 a_4 \otimes \cdots \otimes a_{2N + 1}\rangle$$

$$\otimes \mathcal{O}_j |q_2, a_0, a_2 \otimes \cdots \otimes a_{2N + 1}\rangle$$

(A15)

For $k = 3, 4, \ldots, N + 1$, it holds that

$$H_{q_{N + 4}} \left( \sum_{q = 0}^{2N + 1} (-1)^{g_{N + 4} q_{N + 4}} |q_{N + 4}\rangle \langle q_{N + 4}| \right)$$

(A16)

$$H_{q_{N + 4}} \left( \sum_{q = 0}^{2N + 1} (-1)^{g_{N + 4} q_{N + 4}} |q_{N + 4}\rangle \langle q_{N + 4}| \right)$$

(A17)
Applying Equations (A16), (A17) to (A14), (A15) respectively, it yields the following stator $S_{2N+1}'$. The global factor $\sqrt{2}$ is ignored here,

$$
S_{2N+1}' = |+\rangle_{A_k} \sum_{q_2, q_3, \ldots, q_{N+1}} |q_2, q_3, \ldots, q_{N+1}, q_2, q_3, \ldots, q_{N+1}\rangle,
$$

$$
\otimes \sigma^{q_2}_{n_{2N+2}} \otimes \sigma^{q_3}_{n_{2N+3}} \otimes \ldots \otimes \sigma^{q_{N+1}}_{n_{2N+1}}
$$

$$
+ |+\rangle_{A_k} \sum_{q_2, q_3, \ldots, q_{N+1}} |q_2, q_3, \ldots, q_{N+1}, q_2, q_3, \ldots, q_{N+1}\rangle \otimes \otimes \sigma^{q_2}_{n_{2N+2}} \otimes \sigma^{q_3}_{n_{2N+3}} \otimes \ldots \otimes \sigma^{q_{N+1}}_{n_{2N+1}}
$$

(A18)

Step 3: From Equation (A18), one can find that the controller’s qubit $A_k$ is correlated with qubits $a_i$ and $A_{k+N}$, for $k = 2, 3, \ldots, N+1$. If the controller $A_k$ does not wish to cooperate with other groups including $A_k$ and $A_{k+N}$, he does nothing or something unknown to other groups. Then the relation between qubits $a_i$ and $A_{k+N}$ is unknown to participants $A_k$ and $A_{k+N}$, and hence they cannot realize the operations at the beginning of this subsection. Otherwise, if the controller $A_k$ permits the realization of these operations, he implements the measurement of $\sigma_i$ on qubit $a_i$, and informs $A_2, A_3, \ldots, A_{N+1}$ of the result. If the result is $|+\rangle$, the receivers $A_2, A_3, \ldots, A_{N+1}$ perform $\sigma_i$ on their qubits, otherwise they do nothing. This yields the stator

$$
S_{2N+1}' = \sum_{q_2, q_3, \ldots, q_{N+1}} |q_2, q_3, \ldots, q_{N+1}, q_2, q_3, \ldots, q_{N+1}\rangle \otimes \otimes \sigma^{q_2}_{n_{2N+2}} \otimes \sigma^{q_3}_{n_{2N+3}} \otimes \ldots \otimes \sigma^{q_{N+1}}_{n_{2N+1}}
$$

(A19)

Step 4: Participants $A_{k+N}, A_{k+1}, \ldots, A_{2N+1}$ measure their qubits in the X-basis, and inform $A_2, A_3, \ldots, A_{N+1}$ of the results by classical communication, respectively. If the sender’s result is $|+\rangle$, the receiver performs the operation $\sigma_i$ on his qubit, otherwise the receiver does nothing. So they construct the stator

$$
S_{2N+1} = \sum_{q_2, q_3, \ldots, q_{2N+1}} |q_2, q_3, \ldots, q_{2N+1}\rangle \otimes \otimes \sigma^{q_2}_{n_{2N+2}} \otimes \sigma^{q_3}_{n_{2N+3}} \otimes \ldots \otimes \sigma^{q_{2N+1}}_{n_{2N+1}}
$$

(A20)

where $\sigma^0 = I_{O_i}$, $i = N + 2, \ldots, 2N + 1$.

Then the stator $S_{2N+1}$ is employed to implement the remote operations on the unknown state $|\Psi\rangle$ for remote systems $O_i$. Similar to the eigenoperator equation shown in Equation (13), the equation for stator $S_{2N+1}$ still holds,

$$
\left(\prod_{k=2}^{N+1} \exp[\beta_{k+n} \sigma_k^0] \right) S_{2N+1} = \sum_{j=0}^{2N+1} \exp[i\beta_{j+n} \sigma^0_j] S_{2N+1}
$$

(A21)

where $\sigma^0 = I_{O_i}$ and $\beta$ is the parameter determined by the participant $A_{j-N}$. By performing the local operation $\exp[\beta \sigma_j^0]$ on qubit $a_{j-N}$, the participant $A_{j-N}$ realizes the corresponding operation $\exp[\beta \sigma_j^0]$ on remote system $O_j$. The implementation consists of the following two steps.

Step 5: The participant $A_k$ performs the local operation $\exp[\beta \sigma_{k+N}^0]$ on qubit $a_k$, for $k = 2, 3, \ldots, N+1$. Using Equation (A21), the state is now

$$
S_{2N+1} \left(\prod_{k=2}^{N+1} \exp[\beta \sigma_{k+N}^0] |\Psi\rangle \right)
$$

(A22)

Step 6: To eliminate the remaining stator $S_{2N+1}$ on the state $|\Psi\rangle$ and keep only the target operation, the participant $A_k$ performs the measurement of $\sigma_i$ on qubit $a_i$, and informs $A_k$ of the measurement result, where $k = 2, 3, \ldots, N+1$. If the sender $A_k$’s result is $|1\rangle$, the receiver performs the operation $\exp[\pm \sigma^0_{k+N} i/2] = \sigma_{k+N}$, otherwise the receiver does not perform any operation. So they eliminate the stator $S_{2N+1}$. The resulting state is

$$
\exp[i\beta \sigma_{k+N}^0] |\Psi\rangle
$$

(A23)

It indicates that they have implemented the remote operation successfully.

Appendix B: Control Power Analysis in Terms of POVM

In this section, we show the proof of Proposition 3 in Section 5.

Proof: The proof includes two parts: first we consider case I for $c_{0,0} = c_{1,0} = 0$ or $c_{0,1} = c_{1,1} = 0$ in Table B1 in Section B.1, then we consider case II for $c_{0,1} = c_{1,0} \neq 0$ in Table B2 in Section B.2, where $c_{1,0}$ with $s,t=1,2$ are defined in Equations (46)–(49).

Step 1: In case I for $c_{0,0} = c_{1,0} = 0$ or $c_{0,1} = c_{1,1} = 0$.

The first case that $c_{0,0} = c_{1,0} = 0$ or $c_{0,1} = c_{1,1} = 0$ corresponds to $\lambda_2 = \frac{1}{2}$ or $\lambda_2 = \frac{1}{2}$ in Equations (46)–(49), respectively. In this case, the operations $e^{\pm i \pi \sigma_{c,0}}$ with $\alpha = 0$, $\pm \frac{\pi}{2}$, $\pi$ can be realized by Table B1, and the success rate of realization is equal to 50%.

As an example, the operations $e^{\pm i \pi \sigma_{c,0}}$ with $\alpha = 0$ or $\alpha = \frac{1}{2}$ can be realized by different combinations of POVM operators. Bob and Charlie prepare the POVM operators with the parameters $\theta_1 = 0, \theta_2 = \frac{\pi}{2}, \varphi_1 = \varphi_2 = 0$, and $\lambda_1 = \lambda_2 = \frac{1}{2}, \alpha n_0 = \omega n_0 = \frac{1}{2}$. Their POVM operators are

$$
M_1 = N_1 = \begin{pmatrix}
1 & 0 \\
0 & 0
\end{pmatrix}, M_2 = N_2 = \begin{pmatrix}
0 & 1 \\
0 & 1
\end{pmatrix}
$$

(B1)

When the POVM operators $(M_1, N_1)$ occurs in the measurement, the stator $T_3$ in Equation (50) becomes $H^{(k)}_3$, where

$$
H_3^{(11)} = \frac{1}{2} (+\alpha \beta_1 \gamma_1 |b,c\rangle \otimes I_c)
$$

(B2)

$$
H_3^{(12)} = \frac{1}{2} (-\alpha \beta_1 \gamma_1 |b,c\rangle \otimes (-i\sigma_{nc})
$$

(B3)

$$
H_3^{(21)} = \frac{1}{2} (-\alpha \beta_2 \gamma_2 |b,c\rangle \otimes I_c)
$$

(B4)

$$
H_3^{(22)} = \frac{1}{2} (+\alpha \beta_2 \gamma_2 |b,c\rangle \otimes (-i\sigma_{nc})
$$

(B5)

Note that $I_c \propto e^{i \pi \sigma_{nc}} \propto e^{i \pi \sigma_{nc}}$, and $-i\sigma_{nc} \propto e^{i \pi \sigma_{nc}} \propto e^{i \pi \sigma_{nc}}$. Hence, the operation $e^{i \pi \sigma_{nc}}$ with rotation angle $\alpha = 0$ or $\pi$ can be realized when $(M_1, N_1)$ and $(M_2, N_1)$ occur in the measurement; the operation with rotation angle $\alpha = \frac{1}{2}$ or $\frac{1}{2}$ can be realized when $(M_1, N_2)$ and $(M_2, N_2)$ occur.
asthey have been discussed in the former case. To eliminate the entanglement in the measurement. From Equation (45), each of the four combinations of POVM operators occurs with the probability of 25%. So Bob can only implement his desired operation with the success rate of 50%. For example, Bob wishes to implement the operation $e^{i\alpha\pi/4}$ on system C, he can only carry it off when the POVM operators $(M_1, N_1)$ and $(M_2, N_2)$ occur, and the probability of that is 50%.

Table B1. The possible parameters $\theta_j, \phi_j, \lambda_k, \alpha_k$ in Bob and Charlie's POVM operators and its corresponding rotation angle $\alpha$ that can be realized, for $j, k = 1, 2$. Column 9 contains the choice of POVM operators $(M_j, N_j)$, which is associated to different rotation angle $\alpha$. Columns 10–13 contain the coefficients in Equation (50) under specific POVM operators.

| Parameters in Bob's POVM operator | Parameters in Charlie's POVM operator | POVM operators | Coefficients in Equation (50) | Rotation angle in $e^{i\alpha c}$ |
|-----------------------------------|-------------------------------------|----------------|-----------------------------|--------------------------------|
| $\theta_1$ | $\theta_2$ | $\phi_1$ | $\phi_2$ | $\lambda_1$ | $\lambda_2$ | $\omega_1$ | $\omega_2$ | $M_j, N_k$ | $c_{0,0}$ | $c_{1,0}$ | $c_{1,1}$ | $\alpha$ |
| [0, $\frac{\pi}{4}$) | $\frac{\pi}{2} - \theta_1$ | [0, $\frac{\pi}{2}$] | $\phi_1 \pm \frac{\pi}{2}$ | 0 | $\frac{\pi}{2}$ | [0, $\frac{\pi}{2}$] | [0, $\frac{\pi}{2}$] | $M_1, N_1$ | cos $\theta_1$ | 0 | e$^{-i\phi_1}$ sin $\theta_1$ | 0 | $0$ or $\pi$ |
| $M_1, N_1$ | 0 | e$^{-i\phi_1}$ sin $\theta_1$ | 0 | $\frac{\pi}{2}$ or $\frac{3\pi}{2}$ |
| $M_1, N_2$ | e$^{-i\phi_1}$ sin $\theta_1$ | 0 | $\frac{\pi}{2}$ or $\frac{3\pi}{2}$ |
| $M_2, N_1$ | e$^{-i\phi_1}$ sin $\theta_1$ | 0 | $\frac{\pi}{2}$ or $\frac{3\pi}{2}$ |
| $M_2, N_2$ | e$^{-i\phi_1}$ sin $\theta_1$ | 0 | $\frac{\pi}{2}$ or $\frac{3\pi}{2}$ |
| $M_1, N_1$ | 0 | e$^{-i\phi_1}$ sin $\theta_1$ | 0 | $\frac{\pi}{2}$ or $\frac{3\pi}{2}$ |
| $M_1, N_2$ | e$^{-i\phi_1}$ sin $\theta_1$ | 0 | $\frac{\pi}{2}$ or $\frac{3\pi}{2}$ |
| $M_2, N_1$ | e$^{-i\phi_1}$ sin $\theta_1$ | 0 | $\frac{\pi}{2}$ or $\frac{3\pi}{2}$ |
| $M_2, N_2$ | e$^{-i\phi_1}$ sin $\theta_1$ | 0 | $\frac{\pi}{2}$ or $\frac{3\pi}{2}$ |

Table B2. The possible parameters $\theta_j, \phi_j, \lambda_k, \alpha_k$ in Bob and Charlie's POVM operators and its corresponding rotation angle $\alpha$ in operations $e^{i\alpha c}$ that can be realized. The first column contains the choice of POVM operators $(M_k, N_k)$, for $k = 1, 2$.

| Parameters in Bob's POVM operators $M_j, M_j$ | Parameters in Charlie's POVM operators $N_k, N_k$ | Success rate | POVM operators $M_1, N_1$, or $M_2, N_2$ | Coefficients in Equation (B7) | Rotation angle in $e^{i\alpha c}$ |
|----------------------------------------------|-----------------------------------------------|-------------|------------------------------------|-----------------------------|--------------------------------|
| $\theta_1$ | $\theta_2$ | $\phi_1$ | $\phi_2$ | $\lambda_1$ | $\lambda_2$ | $\omega_1$ | $\omega_2$ | $\rho$ | $M_1, N_1$ | $c_{0,1}$ | $c_{1,0}$ | $\alpha$ |
| $\frac{\pi}{4}$ | $\frac{\pi}{4}$ | 0 | $\pi$ | $\frac{\pi}{4}$ | $\frac{\pi}{4}$ | $\frac{\pi}{4}$ | $\frac{\pi}{4}$ | 50% | $M_1, N_1$ | 1 | 0 | $\frac{\pi}{4}$ |
| $M_1, N_1$ | 1 | $\frac{\pi}{4}$ | $\frac{\pi}{4}$ |
| $M_1, N_2$ | 1 | $\frac{\pi}{4}$ | $\frac{\pi}{4}$ |
| $M_2, N_1$ | 1 | $\frac{\pi}{4}$ | $\frac{\pi}{4}$ |
| $M_2, N_2$ | 1 | $\frac{\pi}{4}$ | $\frac{\pi}{4}$ |
| $\frac{\pi}{4}$ | $\frac{\pi}{4}$ | 0 | $\pi$ | $\frac{\pi}{4}$ | $\frac{\pi}{4}$ | $\frac{\pi}{4}$ | $\frac{\pi}{4}$ | 25% | $M_1, N_1$ | $\frac{\pi}{4}$ | 0 | $\frac{\pi}{4}$ |
| $M_1, N_1$ | $\frac{\pi}{4}$ | 0 | $\frac{\pi}{4}$ |
| $M_1, N_2$ | $\frac{\pi}{4}$ | 0 | $\frac{\pi}{4}$ |
| $M_2, N_1$ | $\frac{\pi}{4}$ | 0 | $\frac{\pi}{4}$ |
| $M_2, N_2$ | $\frac{\pi}{4}$ | 0 | $\frac{\pi}{4}$ |

The operations $e^{i\alpha c}$ with $\alpha = 0$ or $\pi$ and $\alpha = \frac{\pi}{4}$ or $\frac{3\pi}{4}$ can also be realized by other POVM operators with different parameters. The process of implementation is the same as that of the former example. Combined with Equation (42) and Equations (46)-(49), we obtain the possible parameters in Bob and Charlie's POVM operators and their corresponding rotation angle $\alpha$ in $e^{i\alpha c}$ that can be realized. They are shown in Table B1. Each of the four combinations of POVM operators $(M_j, N_j)$, occurs with the same probability of 25%, for $j, k = 1, 2$. In this case, Bob can realize his desired operation with the probability of 50%, and his target operation can only be $e^{i\alpha c}$ with the rotation angle $\alpha = \frac{\pi\alpha}{2}$, for $n = 0, 1, 2, 3$.

B.2. Case II: $c_{0,1}c_{1,0} \neq 0$

Next we consider the second case that $c_{0,1}c_{1,0} \neq 0$, that is, $\theta_j, \lambda_k \in \{0, \frac{\pi}{2}\}$ in Equations (46)-(49). We set $\lambda_k \neq 0, \frac{\pi}{2}$ and $\alpha \neq \frac{\pi\alpha}{2}$ with $n = 0, 1, 2, 3$ here, as they have been discussed in the former case. To eliminate the entanglement between qubit $a$ and $b, c$, the coefficients $c_{i,\alpha}$ in Equation (50) should satisfy the following restriction,

$$c_{0,0} = c_{1,1} = K, \quad c_{1,0} = c_{0,1} = 0$$

where $K \in \mathbb{C}$ is a constant. So the stator in Equation (50) becomes

$$T_j = \frac{1}{2} (K \ket{+} + \ket{-}) \otimes \ket{1, 0} \ket{c_{1,0} = c_{0,1} = 0}$$

(B7)

The target operation that Bob and Charlie try to implement on system C is $e^{i\alpha c} = \cos \alpha c_1 + i \sin \alpha c_1$. So the coefficients in Equation (B7) should satisfy

$$c_{1,0} = L \cos \alpha, \quad c_{0,1} = iL \sin \alpha$$

(B8)

where $L = x + iy \in \mathbb{C}$ is a constant with $x \neq 0$ or $y \neq 0, x, y \in \mathbb{R}$. From Equations (46)-(49), (B6), and (B8), we obtain that

$$c_{0,0} = \cos \theta_j \cos \lambda_k = KL \cos \alpha,$$

(B9)

$$c_{1,0} = e^{-i\alpha} \sin \theta_j \sin \lambda_k = iKL \sin \alpha,$$

(B10)

$$c_{1,1} = e^{-i\alpha \sin \theta_j \cos \lambda_k} = iKL \sin \alpha,$$

(B11)

$$c_{1,1} = e^{-i\alpha \sin \theta_j \cos \lambda_k} = iKL \sin \alpha.$$
Now we analyze the possible angle $\alpha$ that satisfies the conditions above. Note that $\theta_2, \lambda_k \in (0, \frac{\pi}{2})$. From (B9)-(B12), we have

$$e^{i\omega_k \cos \theta_2} = \sin^2 \theta_2.$$  

(B13)

So we have $e^{i\omega_k \cos \theta_2} \in \mathbb{R}$ and thus

$$\varphi_1 = 0 \text{ or } \pi.$$  

(B14)

$$\theta_1 = \theta_2 = \frac{\pi}{4}.$$  

(B15)

Then from Equations (B11) and (B14), we have

$$\sin \theta_2 \cos \lambda_k = x \cos \alpha + iy \cos \alpha,$$  

or

$$-\sin \theta_2 \cos \lambda_k = x \cos \alpha + iy \cos \alpha.$$  

(B16)

(B17)

Obviously, $x \cos \alpha + iy \cos \alpha \in \mathbb{R}$. Note that $\theta_1, \lambda_k \in (0, \frac{\pi}{2})$ and thus $\cos \alpha \neq 0$. Hence $y = 0$, that is, $L = x \in \mathbb{R}$. From Equation (B10), we have

$$\omega_k = \frac{\pi}{2} \text{ or } \frac{3\pi}{2}.$$  

(B18)

Using Equations (B10), (B11), and (B15), we have

$$\frac{1}{2}(\cos^2 \lambda_k + \sin^2 \lambda_k) = L^2(\cos^2 \alpha + \sin^2 \alpha) \Rightarrow L = \pm \frac{\sqrt{2}}{2}.$$  

(B19)

From Equations (42), (43), (B14), (B15), and (B18), we obtain that

$$\varphi_1 \neq \varphi_2, \alpha_1 \neq \alpha_2.$$  

(B20)

$$\lambda_1 + \lambda_2 = \frac{\pi}{2}.$$  

(B21)

Considering the restrictions Equations (B14), (B15), (B18)–(B21) on Equations (B10) and (B11), we show that the remote operation $e^{i\omega_k \cos \theta_2}$ with the rotation angle

$$\alpha \in \bigcup_{m=0}^{7} \left( \frac{m\pi}{4}, \frac{(m+1)\pi}{4} \right).$$  

(B22)

can be realized with the success rate of 25%, and the operations corresponding to

$$\alpha = \frac{n\pi}{4} \text{ with } n = 1, 3, 5, 7.$$  

(B23)

can be realized with the success rate of 50%. The implementation of these operations $e^{i\omega_k \cos \theta_2}$ with $\alpha \in \bigcup_{m=0}^{7} \left( \frac{m\pi}{4}, \frac{(m+1)\pi}{4} \right)$ can be realized by Table B2.

As an example, Bob wants to realize the operation $e^{i\omega_k \cos \theta_2}$ with $\alpha = \frac{3\pi}{4}$. For this purpose, Bob and Charlie choose the POVM operators with the parameters $\theta_1 = \theta_2 = \frac{\pi}{4}, \varphi_1 = 0, \varphi_2 = \pi$ and $\lambda_1 = \lambda_2 = \frac{\pi}{4}, \alpha_1 = \frac{\pi}{4}, \alpha_2 = \frac{3\pi}{4}$. Their POVM operators are

$$M_1 = |\beta_1\rangle \langle \beta_1| = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}, \quad M_2 = |\beta_2\rangle \langle \beta_2| = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix},$$  

(M1)

$$N_1 = |\gamma_1\rangle \langle \gamma_1| = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}, \quad N_2 = |\gamma_2\rangle \langle \gamma_2| = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix}.$$  

(N1)

When the POVM operators $(M_1, N_1)$ occurs in the measurement, the stator $T_1$ in Equation (B7) becomes $T_1^{(1)}$, where

$$T_1^{(1)} = \frac{1}{2} \left( |(+\rangle_\alpha + |(-\rangle_\alpha \right) \beta_1, \gamma_1 \rangle \kappa \Omega \left( \frac{1}{2} |c - \frac{1}{2} \sigma_{nc} \right) \right)$$  

(B24)

and

$$T_2^{(1)} = \frac{1}{2} \left( |(+\rangle_\alpha + |(-\rangle_\alpha \right) \beta_1, \gamma_1 \rangle \kappa \Omega \left( \frac{1}{2} |c + \frac{1}{2} \sigma_{nc} \right) \right)$$  

(B25)

Note that $(\frac{1}{2} |c - \frac{1}{2} \sigma_{nc} \right) \alpha e^{i\omega_k \cos \theta_2} \alpha e^{i\omega_k \cos \theta_2} \alpha e^{i\omega_k \cos \theta_2} \alpha e^{i\omega_k \cos \theta_2}$. When the POVM operators $(M_1, N_2), (M_2, N_1), (M_2, N_2), (M_2, N_2)$ occur, the operation $e^{i\omega_k \cos \theta_2}$ can be realized respectively, with $\omega_1 = \frac{3\pi}{4}, \omega_2 = \frac{\pi}{4}$. Bob’s goal is to implement the operation $e^{i\omega_k \cos \theta_2}$ with $\omega_1 = \frac{3\pi}{4}$. It can only be realized when $(M_1, N_1)$ or $(M_2, N_2)$ occur. From Equation (45), each of the four combinations of POVM operators $(M_1, N_2), (M_2, N_1), (M_2, N_2), (M_2, N_2)$ occurs with the same probability of 25%, for $j = 1, 2$. So the success rate of realizing Bob’s target operation is 50%.

Other possible POVM operators with different parameters and its corresponding rotation angle $\alpha$ in the operation $e^{i\omega_k \cos \theta_2}$ are listed in Table B2. For the first set of POVM operators with parameter $\lambda_1 = \frac{\pi}{2}$, Bob can realize his desired operation with the probability of 50% and the target operation that Bob can choose is $e^{i\omega_k \cos \theta_2}$ with $\alpha = \frac{\pi}{4}$, with $n = 1, 3, 5, 7$. By the second set of operators with $\lambda_1 = \frac{\pi}{2}$ or $\frac{3\pi}{2}$, the operation $e^{i\omega_k \cos \theta_2}$ with $\alpha \in \bigcup_{m=0}^{7} \left( \frac{m\pi}{4}, \frac{(m+1)\pi}{4} \right)$ can be realized. If Bob wants to implement that operation, he informs Charlie to prepare the POVM operators with four parameters: $\lambda_1, \lambda_2 = \frac{\pi}{2} - \lambda_1, \alpha_1 = \frac{\pi}{4}, \alpha_2 = \frac{3\pi}{4}$, where

$$\lambda_1 = \begin{cases} \frac{\pi}{2} - \lambda_1 & \text{for } \alpha_1 \in \left( 0, \frac{\pi}{4} \right), \\ \frac{\pi}{2} - \alpha_1 & \text{for } \alpha_1 \in \left( \frac{\pi}{4}, \frac{\pi}{2} \right), \\ \frac{3\pi}{2} - \alpha & \text{for } \alpha_1 \in \left( \frac{\pi}{2}, \frac{3\pi}{2} \right), \\ \frac{3\pi}{2} - \alpha_1 & \text{for } \alpha_1 \in \left( \frac{3\pi}{2}, \frac{5\pi}{2} \right). \end{cases}$$  

(B26)

The operation $e^{i\omega_k \cos \theta_2}$ with $\alpha \in \bigcup_{m=0}^{7} \left( \frac{m\pi}{4}, \frac{(m+1)\pi}{4} \right)$ can be realized only when the operators $(M_1, N_2), (M_1, N_2), (M_2, N_2), (M_2, N_2)$ occur in the measurement, respectively. From Equation (45), each of the combination of POVM operators occurs with the probability of 25%. So the success rate is equal to 25%.

To conclude, we have shown that Bob and Charlie implement the remote operation $e^{i\omega_k \cos \theta_2}$ in the POVM scheme without the permission of controller Alice. They can realize the operation of $\alpha \in \bigcup_{m=0}^{7} \left( \frac{m\pi}{4}, \frac{(m+1)\pi}{4} \right)$ with the success rate of 50%, and the operation corresponding to $\alpha \in \bigcup_{m=0}^{7} \left( \frac{m\pi}{4}, \frac{(m+1)\pi}{4} \right)$ can be realized with the success rate of 25%.

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Conflict of Interest

The authors declare no conflict of interest.

Data Availability Statement

The data that support the findings of this study are available from the corresponding author upon reasonable request.

Keywords

graph states, quantum gates, quantum protocol