On a two particles system associated to the one spatial Galilei group

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Abstract

In [8] ,M.Daumens and M. Perroud studied a two Galilean free particles system by realizing the three spatial dimensional Galilei group on its maximal coadjoint orbit. In this paper we realize a similar study for the one spatial dimensional Galilei group . As its maximal coadjoint orbit describes a non free massive particle [6], this gives us a two non free galilean particles system. Use of the barycenter coordinates gives rise to the notions of a total force and a relative force similar to those of a total linear momentum and a relative linear momentum ([8], [9],[10]). We will distinguish the case of a non isolated system from that of an isolated one . We will the show that the barycenter is accelerated in the first case while it moves with a constant velocity in the second one.

1 Introduction

Let $G$ be a Lie group. As we know [1], all generic $G$–hamiltonian spaces are coadjoint orbits of the $G$-action on the dual of the Lie algebra of the central extension of $G$.

We also know [2] that the projective unitary representations in quantum mechanics are, in classical mechanics, the symplectic realizations. In this paper we revisit the one spatial Galilei group [6]. If $e$, $p$ and $k$ are respectively the dual of time translations, space translations and boost, we know ([6],[7]) that there are three kinds of galilean systems corresponding each to a coadjoint orbit (a $G$– homogeneous symplectic manifold)on which the Galilei group is represented faithfully. These orbits are:
• One $O_{(m,f,U)}$ endowed with the symplectic form $\sigma = dp \wedge dq$ and characterized by a mass $m$, a force $f$ and an internal energy $U = e - \frac{p^2}{2m} + fq$ with $q = \frac{k}{m}$.

• One $O_{(m,U)}$ endowed with the symplectic form $\sigma = dp \wedge dq$, $q = \frac{k}{m}$ and characterized by a mass $m$ and an internal energy $U = e - \frac{p^2}{2m}$.

• One $O_{(f,K)}$ endowed with the symplectic form $\sigma = dp \wedge dq$, $q = -\frac{\lambda}{f}$ and characterized by a force $f$ and an invariant $K = k - \frac{p^2}{2f}$. This is the result of [6]. In the next section we identify $O_{(f,K)}$ with a space-time curved by the force $f$.

The second section in this paper recall the galilean momenta on the three coadjoint orbits. In the third section, similarly to the work [3], we study a two particles systems corresponding to the orbits $O_{(m_1, f_1, U_1)}$, $a = 1, 2$ (non free massive particles). The study of a two particles system corresponding to the orbits $O_{(m_a, U_a)}$, $a = 1, 2$ (free massive particles) does not give any new result.

2 Galilean momenta

Let us recall some useful ingredients on the momentum map and the apply then them to the Galilean coadjoint orbits. Let $G$ be a Lie group and let $G^*$ be the dual of the Lie algebra $G$ of $G$. If $(V, \sigma)$ is a given $G$–symplectic manifold, there exist a momentum map ([3], [4]) $J : V \to G^*$ defined by

$$< J(x), X > = [\lambda(X)](x)$$

where $\lambda : G \to C^\infty(V, \mathbb{R})$ is the comomentum defined by

$$X_\ast \sigma = d\lambda(X)$$

The components of the $J(x)$, $x \in V$ in a given basis of $G^*$ are the classical fundamental observables associated to $G$. If $(X_i)$ is a basis of the Lie algebra $G$ whose the dual basis is $X_i^\ast$, then $J(x) = J_i(x)X_i^\ast \in G^*$ gives rise to

$$J_i(x) = [\lambda(X_i)](x)$$

which defines the $i^{th}$ component of the momentum. We will then denote by $J_X(x)$ the momentum corresponding to the generator $X$ of $G$.

Let us denote the galilean generators by $K$ for the boosts, $P$ for the space translations and $E$ for the time translations. From ([6], [7]) we verify that
• the symplectic realization of the Galilei group on $O_{(m,f,U)}$ is

$$L_{(x,t,v)}(p,q) = (p - mv + ft, q + \frac{p}{m} t + \frac{f}{m} t^2 + x - vt)$$

(4)

The Galilei Lie algebra is then realized by the hamiltonian vector fields

$$\rho(K) = m \frac{\partial}{\partial p}, \quad \rho(P) = -\frac{\partial}{\partial q}, \quad \rho(E) = -f \frac{\partial}{\partial p} - \frac{p}{m} \frac{\partial}{\partial q}$$

(5)

and the Galilean momentum components are

$$J_K(p,q) = mq, \quad J_P(p,q) = p, \quad J_E(p,q) = \frac{p^2}{2m} - f q$$

(6)

• the symplectic realization of the Galilei group on $O_{(m,U)}$ is

$$L_{(x,t,v)}(p,q) = (p - mv, q + \frac{p}{m} t + x - vt)$$

(7)

The Galilei Lie algebra is then realized by the hamiltonian vector fields

$$\rho(K) = m \frac{\partial}{\partial p}, \quad \rho(P) = -\frac{\partial}{\partial q}, \quad \rho(E) = -\frac{p}{m} \frac{\partial}{\partial q}$$

(8)

and the Galilean momentum components are

$$J_K(p,q) = mq, \quad J_P(p,q) = p, \quad J_E(p,q) = \frac{p^2}{2m}$$

(9)

• the symplectic realization of the Galilei group on $O_{(f,K)}$ is

$$L_{(x,t,v)}(\tau,q) = (\tau + t, q + v \tau + x)$$

(10)

where we have used the non canonical space-time coordinates with $\tau = \frac{f}{j}$. The Galilei Lie algebra is then realized by the hamiltonian vector fields

$$\rho(K) = -\tau \frac{\partial}{\partial q}, \quad \rho(P) = -\frac{\partial}{\partial q}, \quad \rho(E) = -\frac{\partial}{\partial \tau}$$

(11)

and the Galilean momentum components are

$$J_K(q,\tau) = f \frac{\tau^2}{2}, \quad J_P(q,\tau) = f \tau, \quad J_E(p,q) = -fq$$

(12)

Looking to (6) and (9) permit us to interpret $J_P$ and $J_E$ as respectively a linear momentum and an energy while the relation (12) tell us that $J_P$ and $J_E$ are respectively an impulse and a work. Nothing is clear for the component $J_K$. This is the object of this article. We will do it by considering the two particles systems. The phase space of this system will be a cartesian product of two orbits of the same kind. We will endow it with a symplectic form which is the sum of the corresponding symplectic forms.
3 Two galilean particles system

Following [8], let us consider a symplectic manifold \((V, \sigma)\) where \(V\) is the cartesian product of two orbits \(O(m_1, f_1, U_1)\) and \(O(m_2, f_2, U_2)\) while \(\sigma\) is the sum of the symplectic forms \(\sigma_a\) on \(O(m_a, f_a, U_a)\):

\[
\sigma = dp_1 \wedge dq_1 + dp_2 \wedge dq_2 \quad (13)
\]

The active action of the Galilei group on \(V\) is given by

\[
L_{(x_a, t_a, v_a)}(p_a, q_a)
= (p_a - m_a v_a + f_a t_a, q_a + \frac{p_a}{m_a} t_a + \frac{f_a t_a^2}{m_a} + x_a - v_a t_a), \ a = 1, 2 \quad (14)
\]

where \(v_a\) is the velocity of the particle \(a\) with respect an observer, \((x_a, t_a)\) is the space-time "position" of the particle \(a\) with respect the observer. This means that \(V\) describes a two particles system whose the masses are \(m_1\) and \(m_2\), while the force acting on \(O(m_1, f_1, U_1)\) is \(f_1\), that on \(O(m_2, f_2, U_2)\) is \(f_2\). The force \(f_1\) (\(f_2\)) is produced by \(O(m_2, f_2, U_2)\) (\(O(m_1, f_1, U_1)\)) for an isolated system or the force \(f_1\) (\(f_2\)) is produced by \(O(m_2, f_2, U_2)\) (\(O(m_1, f_1, U_1)\)) and other objects for an non isolated system.

3.1 Barycenter decomposition

Let us introduce the barycenter decomposition \(B : V \to V_{CM} \times V_{INT}\) where \(V_{CM} = \{(p, q)\}\) and \(V_{INT} = \{\pi, \rho\}\) defined by [8]

\[
p = p_1 + p_2, \ \pi = \frac{m_2 p_1 - m_1 p_2}{m_1 + m_2}, \ q = \frac{m_1 q_1 + m_2 q_2}{m_1 + m_2}, \ \rho = q_1 - q_2 \quad (15)
\]

where \(m = m_1 + m_2\) is the total mass, \(p = p_1 + p_2\) is the total linear momentum, \(\pi = \frac{m_2 p_1 - m_1 p_2}{m_1 + m_2}\) is the relative linear momentum while \(\rho = q_1 - q_2\) is the relative position. Note that the Galilei group has to transform \((p, q)\) and \((\pi, \rho)\) according [4]. But we verify that [4] does not preserve the barycenter decomposition. This will be done by the subgroup \(B = \{(x_1, t, v_1), (x_2, t, v_2)\} \subset G \times G\) which act on \((p, q, \pi, \rho)\) according

\[
L_{((x, t, v), (r, t, u))}(p, q, \pi, \rho)
= (p - mv + ft, q + \frac{p}{m} t + \frac{f}{m} t^2 + x - vt, \pi - mu + \varphi t, \rho + \frac{\pi}{\mu} t + \frac{\varphi}{m} t^2 + r - ut) \quad (16)
\]
where
\[ f = f_1 + f_2, \mu = \frac{m_1 m_2}{m_1 + m_2}, \varphi = \frac{m_2 f_1 - m_1 f_2}{m_1 + m_2} \]  \hspace{1cm} (17)

and
\[ r = x_1 - x_2, x = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2}, v = \frac{m_1 v_1 + m_2 v_2}{m_1 + m_2}, u = v_1 - v_2 \]  \hspace{1cm} (18)

We recognize in $\mu$ the reduced mass ([8], [9], [10]). Let us call $f$ and $\varphi$ the total force and the relative force respectively. The non isolated system is characterized by $f \neq 0$ while the isolated one is characterized by $f = 0$.

Note that the relations defining $\pi$ and $\varphi$ can be rewritten as
\[ \frac{\pi}{\mu} = \frac{p_1}{m_1} - \frac{p_2}{m_2}, \frac{\varphi}{\mu} = \frac{f_1}{m_1} - \frac{f_2}{m_2} \]  \hspace{1cm} (19)

They define the relative velocity $\frac{\pi}{\mu}$ and the relative acceleration $\frac{\varphi}{\mu}$.

### 3.1.1 Non isolated system

It is possible to choose a Galilei frame (the center of mass frame) in which $p = 0, q = 0$. The internal Galilei group $G_{INT}$ for the two-particles system is then the subgroup of $B$ which stabilizes $(p = 0, q = 0)$ [8]. We then verify that
\[ G_{INT} = \{ (\frac{f t^2}{m} + \frac{m_2}{m} r, t, \frac{f t}{m} + \frac{m_2}{m} u), (\frac{f t^2}{m} - \frac{m_1}{m} r, t, \frac{f t}{m} - \frac{m_1}{m} u) \} \]  \hspace{1cm} (20)

In the barycenter coordinates (15) the symplectic form (13) is
\[ \sigma = dp \wedge dq + d\pi \wedge d\rho \]  \hspace{1cm} (21)

From (16) we verify that the Lie algebra of $B$ is realized on $V$ by the hamiltonian vector fields
\[ P_{CM} = -\frac{\partial}{\partial q}, K_{CM} = m \frac{\partial}{\partial p} \]  \hspace{1cm} (22)

for the mass center,
\[ P_{INT} = -\frac{\partial}{\partial \rho}, K_{INT} = \mu \frac{\partial}{\partial \pi} \]  \hspace{1cm} (23)
for the fictitious particle and the time translation is realized by

\[
E = -\frac{p}{m} \frac{\partial}{\partial q} - f \frac{\partial}{\partial p} - \frac{\pi}{\mu} \frac{\partial}{\partial \rho} - \frac{\varphi}{\mu} \frac{\partial}{\partial \pi} \tag{24}
\]

We then verify that the corresponding momenta components are

\[
[J(P_{CM})](p, q; \pi, \rho) = p , \quad [J(K_{CM})](p, q; \pi, \rho) = mq \tag{25}
\]

for the mass center,

\[
[J(P_{INT})](p, q; \pi, \rho) = \pi , \quad [J(K_{INT})](p, q; \pi, \rho) = \mu \rho \tag{26}
\]

for the fictitious particle. Moreover the momentum component conjugated to time (energy) is

\[
[J(E)](p, q; \pi, \rho) = \frac{p^2}{2m} - f q + \frac{\pi^2}{2\mu} - \varphi r \tag{27}
\]

We rewrite it as

\[
J_E(p, q; \pi, r) = T(p, \pi) + V(q, r) \tag{28}
\]

where \( T(p, \pi) = \frac{p^2}{2m} + \frac{\pi^2}{2\mu} \) is the sum of the kinetic energies while the second term of the right hand side, \( V(q, r) = -f q - \varphi r \), is the sum of the potential energies.

We then see that only the energy component depend on the center of mass and on the fictitious particle.

From the fact that (use of (16))

\[
L_{(0,t,0)}(p, q, \pi, \rho) = (p + ft, q + \frac{p}{m} t, \pi + \varphi t, \rho + \frac{\pi}{\mu} t + \frac{\varphi}{\mu} t^2) \tag{29}
\]

we verify that the motion equations are

\[
p = m \dot{q} , \quad \dot{q} = \frac{f}{m} \tag{30}
\]

for the particle in the center of mass and

\[
\pi = \mu \dot{\rho} , \quad \dot{\rho} = \frac{\varphi}{\mu} \tag{31}
\]

for the fictitious particle. Note also that from (15) and (26), we write

\[
\frac{[J(K_{CM})](p, q, \pi, \rho)}{m} = \frac{m_1 q_1 + m_2 q_2}{m_1 + m_2} \tag{32}
\]

which relates the momentum component \([J(K_{CM})](p, q, \pi, \rho)\) to the center of mass position \( q \). We then see that the center of mass is accelerated by the total force \( f \), the acceleration being \( a = \frac{f}{m} \) while the fictitious particle is also accelerated by the relative force \( \varphi \), the acceleration being \( \gamma = \frac{\varphi}{\mu} \).
3.1.2 Isolated system

In this case, \( f_1 = -f_2 = \varphi \). The internal Galilei group is

\[
G_{\text{INT}} = \{(\frac{m_2}{m} r, t, \frac{m_2}{m} w), (-\frac{m_1}{m} r, t, -\frac{m_1}{m} w)\}
\]

which is similar to that of \([8]\). Moreover the the action of the group \( \mathcal{B} \) is

\[
L_{(x,t,v),(r,t,u)}(p, q, \pi, \rho) = (p - mv, q + \frac{p}{m} t + x - vt, \pi - \mu u + \varphi t, \rho + \frac{\pi}{\mu} t + \frac{\varphi t^2}{\mu} + r - ut)
\]

(34)

The only changed momentum component is the energy which becomes

\[
[J(E)](p, q; \pi, \rho) = \frac{p^2}{2m} + \frac{\pi^2}{2\mu} - \varphi \rho
\]

(35)

Moreover the motion equations are now

\[
p = m\dot{q}, \quad \ddot{q} = 0
\]

(36)

for the particle in the center of mass,

\[
\pi = \mu \ddot{r}, \quad \ddot{\rho} = \frac{\varphi}{\mu}
\]

(37)

for the fictitious particle. They tell us that the barycenter moves on a straight line with a constant velocity \( \frac{p}{m} \) as it is usual (\([9],[10]\)) while the fictitious particle is still accelerated by the relative force \( \varphi = f_1 = -f_2 \) like in the non isolated case.

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