Triaxial Deformation and Asynchronous Rotation of Rocky Planets in the Habitable Zone of Low-Mass Stars

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ABSTRACT
Rocky planets orbiting M-dwarf stars in the habitable zone tend to be driven to synchronous rotation by tidal dissipation, potentially causing difficulties for maintaining a habitable climate on the planet. However, the planet may be captured into asynchronous spin-orbit resonances, and this capture may be more likely if the planet has a sufficiently large intrinsic triaxial deformation. We derive the analytic expression for the maximum triaxiality of a rocky planet, with and without a liquid envelope, as a function of the planet’s radius, density, rigidity and critical strain of fracture. The derived maximum triaxiality is consistent with the observed triaxialities for terrestrial planets in the solar system, and indicates that rocky planets in the habitable zone of M-dwarfs can in principle be in a state of asynchronous spin-orbit resonances.

Key words: planets and satellites: dynamical evolution and stability - planets and satellites: fundamental parameters - planets and satellites: general - planets and satellites: terrestrial planets

1 INTRODUCTION

With current technology, we may detect rocky exoplanets in the habitable zone (HZ) of M-dwarf stars (Charbonneau & Deming 2007; Shields et al. 2016). Indeed, the TRansiting Planets and Planetesimals Small Telescope (TRAPPIST) survey has already discovered several potentially habitable planets around the low-mass (0.08 M⊙) star TRAPPIST-1 (Gillon et al. 2016, 2017), and radial velocity measurements have revealed the earth-massed planet Proxima Centauri b in the HZ around the closest star to our sun (Anglada-Escudé et al. 2016). Statistics of planets discovered by the Kepler mission suggests that ∼ 50% of stars with effective temperatures cooler than 4000° K have earth-sized planets (Dressing & Charbonneau 2013, 2015; Morton & Swift 2014), and ∼20% of these cool stars have rocky planets in the HZ (Morton & Swift 2014; Dressing & Charbonneau 2015).

Because the HZ of M-dwarfs is located at a small orbital semi-major axis (a ≲ 0.1 AU), planets in this region are often expected to be in a state of tidally synchronized rotation. This could potentially create difficulties for maintaining a habitable climate over the lifetime of the planet, and even lead to atmosphere collapse (e.g., Kasting et al. 1993; Joshi et al. 1997; Kite et al. 2011; Heng & Kopparapu 2012; Yang et al. 2013; Kopparapu et al. 2016; Turbet et al. 2016). Thermal tide associated with a sufficiently massive atmosphere can in principle drive the planet’s rotation away from synchronicity. This is the case for Venus (Gold & Soter 1969; Ingersoll & Dobrovolskis 1978), and may also operate for planets in the HZ around stars more massive than 0.5 M⊙ (Leconte et al. 2015). Another possibility to avoid tidal locking is the planet retains a small orbital eccentricity, while spin is captured into a non-synchronous resonance (such as 3 : 2) with the orbit during spindown, as in the case of Mercury (Goldreich & Peale 1966, 1968).

A critical parameter for determining if a planet is susceptible to be captured into a spin-orbit resonance is its intrinsic triaxiality [see Eq. (22)]. This triaxiality is sustained by the rigidity the rocky planet, and determines the strength of the torque keeping the planet in resonance. For the simplest frequency-independent rheologies, this resonant triaxial torque must overcome the dissipative tidal torque working to drive the planet toward synchronization (Goldreich & Peale 1966, 1968; Murray & Dermott 2000). With frequency dependent rheologies (Makarov 2012; Efroimsky 2012), spin-orbit resonant capture may occur without the resonant triaxial torque due to the behavior of the tidal torque near spin-orbit resonances (Ribas et al. 2016; Bartuccelli, Deane, & Gentile 2017). However, the resonant triaxial torque is often necessary for spin-orbit resonant capture, even with frequency-dependent rheologies (see Fig. 4 of Ribas et al. 2016). The main goal of this paper is to calculate the maximum triaxial deformation a rocky planet (with and without a fluid envelope) can sustain, as a function of its physical and material properties (density, size, elastic rigidity and...
the critical strain for fracture), and to evaluate the possibility of resonant spin-orbit capture of planets in the HZ of M-dwarfs.

In Section 2 we provide an order-of-magnitude estimate of the maximum traxial deformation $\epsilon_{\text{max}}$ of a bare rocky planet. Section 3 contains detailed calculations of $\epsilon_{\text{max}}$ for rocky planets with and without a fluid envelope or atmosphere. In Section 4 we summarize our result and discuss its implications for resonant spin-orbit capture and asynchronous planet rotation.

2 ORDER OF MAGNITUDE ESTIMATE OF THE MAXIMUM TRIAXIALITY OF ROCKY PLANETS

For a rocky planet of density $\rho_c$ and radius $R_c$, the anisotropic stress associated with the weight of its traxial deformation is or order

$$|T_{\text{grav}}| \sim \rho_c g_c R_c \epsilon,$$  \hspace{1cm} (1)

where $g_c = GM_c/R_c^2$ is the gravitational acceleration, and $\epsilon$ is the dimensionless triaxiality [defined in Eq. (22) below]. The stress $T_{\text{grav}}$ must be balanced by internal elastic stress. A rough magnitude of the elastic stresses is

$$|T_{\text{elast}}| \sim \mu u,$$  \hspace{1cm} (2)

where $\mu$ is the shear modulus and $u$ is the strain. This gives

$$u \sim \rho_c g_c R_c \epsilon / \mu.$$  \hspace{1cm} (3)

The planet can yield plastically or fracture when $u$ exceeds a critical value $u_{\text{crit}}$ (of order $10^{-5} - 10^{-3}$). Thus the maximum triaxiality is

$$\epsilon_{\text{max}} \sim \left( \frac{\mu}{\rho_c g_c R_c} \right) u_{\text{crit}}.$$  \hspace{1cm} (4)

Detailed calculation in Section 3 reproduces the same scaling relation except $\epsilon_{\text{max}}$ is a factor of 7.9 larger. Thus

$$\epsilon_{\text{max}} \sim 7.9 \left( \frac{\mu}{\rho_c g_c R_c} \right) u_{\text{crit}}.$$  \hspace{1cm} (5)

3 QUANTITATIVE CALCULATION

We model the planet to as a constant density core (density = $\rho_c$) with radius $R_c$, with a fluid envelope extending to radius $R$. We consider two types of envelopes:

(i) An isothermal atmosphere, with equation of state $\rho = p/c_s^2$.  

(ii) A constant density ocean, with $\rho = \rho_o$.

Here, $p$ is the pressure and $c_s$ is the (constant) sound speed. We assume the atmosphere is thin, with scale height $c_s^2 / g_c \ll R_c$, where $g_c = (4\pi/3)\rho_c R_c$ is the gravitational acceleration at $r = R_c$.

1 A real planet may consist of a solid/liquid core, a mantle, a crust and an liquid envelope/atmosphere. In our simple planet model, the region inside $R_c$ (with constant density and rigidity) is termed “rocky core” or “core”, while the region outside $R_c$ (with zero rigidity) termed envelope.

We take the equilibrium state to be spherically symmetric with no shear stress. The equations of hydrostatic equilibrium are

$$\frac{dp}{dr} - \frac{\rho}{r^2} \frac{d\phi}{dr} = 0$$  \hspace{1cm} (6)

and

$$\frac{1}{r^2} \frac{d}{dr} \left( r^2 \frac{d\phi}{dr} \right) = 4\pi G \rho,$$  \hspace{1cm} (7)

where $\phi$ is the gravitational potential. We require $d\phi/dr|_{r_o} = 0$, $p(R) = 0$, and continuity of $p$, $\phi$, and $d\phi/dr$ at $r = R_c$. For a bare rocky planet, the solutions for $\phi$ and $p$ are (for $r < R_c$)

$$\phi(r) = -\pi G \rho_c \left( 2R_c^2 - \frac{2}{3} r^2 \right),$$  \hspace{1cm} (8)

$$p(r) = \frac{2\pi}{3} G \rho_c^2 (R_c^2 - r^2).$$  \hspace{1cm} (9)

For a planet with an isothermal atmosphere, the solutions are (to leading order in $c_s^2 / R_c g_c$)

$$\phi(r) = \left\{ \begin{array}{ll}
-\pi G \rho_o (2R_c^2 - \frac{2}{3} r^2) & r \leq R_c \\
-\frac{4\pi}{5} \frac{G}{R_c} R_c^2 / r & r > R_c
\end{array} \right.$$  \hspace{1cm} (10)

$$p(r) = \left\{ \begin{array}{ll}
\frac{2\pi}{5} G \rho_o (R_c^2 - r^2) + c_s^2 \rho_o & r \leq R_c \\
\rho_o c_s^2 \exp[-(r - R_c)c_s / c_s^2] & r > R_c
\end{array} \right.$$  \hspace{1cm} (11)

where $\rho_o$ is the density at the base of the atmosphere, and is a free parameter. For a planet with a constant density ocean, the solutions are (for $r \leq R$)

$$\phi(r) = \left\{ \begin{array}{ll}
-\pi G \rho_o (2R_c^2 - \frac{2}{3} r^2) & r \leq R_c \\
-\frac{4\pi}{5} R_c^2 (\rho_o - \rho_c) / r - \pi G \rho_o (2R_c^2 - \frac{2}{3} r^2) & r > R_c
\end{array} \right.$$  \hspace{1cm} (12)

$$p(r) = -\rho_o \phi(r) + \rho_o \phi(R).$$  \hspace{1cm} (13)

We then perturb the surface of the core, so that the core’s radius is given by

$$r_c = R_c + \delta R_c \mathrm{Re}[Y_{22}(\theta, \varphi)],$$  \hspace{1cm} (14)

where $\delta R_c \ll R_c$, $Y_{22}$ are spherical harmonics, and $\mathrm{Re}$ denotes the real part. The perturbed surface of the fluid envelope is

$$r = R + \delta R \mathrm{Re}[Y_{22}(\theta, \varphi)].$$  \hspace{1cm} (15)

The associated perturbation to the gravitational potential is

$$\delta \phi = \delta \phi_{22}(r) \mathrm{Re}[Y_{22}(\theta, \varphi)].$$  \hspace{1cm} (16)

Solving the perturbed Poisson’s equation, we find that for a bare rocky planet and a thin-atmosphere planet, $\delta \phi_{22}$ is given by (for $r \leq R$)

$$\delta \phi_{22}(r) = -\frac{4\pi}{5} \frac{G \rho_c R_c^2 \alpha_s^2 \delta R_c}{\mathrm{max}(r, R_c)},$$  \hspace{1cm} (17)

and for a constant density ocean:

$$\delta \phi_{22}(r) = \delta \phi_o(r) + \delta \phi_c(r),$$  \hspace{1cm} (18)

where

$$\delta \phi_c(r) = -\frac{4\pi}{5} \frac{G \rho_c R_c^2 \alpha_s^2 \delta R_c}{\mathrm{max}(r, R_c)},$$  \hspace{1cm} (19)

$$\delta \phi_o(r) = -\frac{4\pi}{5} \frac{G \rho_o \left[ R^2 \alpha_s^2 \delta R - R_o^2 \alpha_s^2 \delta R_c \right]}{\mathrm{max}(r, R_c)}.$$  \hspace{1cm} (20)
are the perturbations in the gravitational potential from the perturbed core and ocean, respectively, and
\[ \alpha = \frac{\operatorname{min}(r, R)}{\operatorname{max}(r, R)}, \quad \alpha_c = \frac{\operatorname{min}(r, R_c)}{\operatorname{max}(r, R_c)}. \] (21)

Let \( I = I_{zz} \geq I_{yy} \geq I_{xx} \) be the principal components of the planet’s moment of inertia tensor. To linear order in all perturbed quantities, the triaxiality \( \epsilon \) is
\[ \epsilon \equiv \frac{I_{yy} - I_{xx}}{I_{yy}} = \frac{32\pi Q_{22}}{15} \frac{R^3}{\epsilon^2}, \] (22)
where \( Q_{22} \) is the second gravitational moment of the planet. Because \( Q_{22} \) is related to the perturbed gravitational potential at the surface of the planet through
\[ \delta\phi_{22}(R) = -\left(\frac{4\pi}{3}\right) GQ_{22} \frac{R^3}{\epsilon^2}, \] (23)
we may write
\[ \epsilon = -\sqrt{\frac{10}{3\pi G}} \delta\phi_{22}(R). \] (24)
Thus, the triaxiality may be obtained by evaluating the perturbed potential at the surface of the planet. For a planet with an isothermal atmosphere (which formally extends to infinity), we evaluate Eq. (24) at \( r = R_c \). Corrections to Eq. (24) from the gravitational potential of the atmosphere are of order \( c_s^2/(gR_c) \ll 1 \).

Elastic stresses are required to resist the non-isotropic weight on the core from the planet’s ellipticity. Assuming the core to be homogeneous (shear modulus \( \mu = \) constant) and incompressible, the perturbed equations of elastostatic equilibrium in the core are (Landau & Lifshitz 1959)
\[ -\nabla \delta p + \rho \nabla^2 \delta \phi = 0, \] (25)
\[ \nabla \cdot \xi = 0, \] (26)
where \( \xi \) is the Lagrangian displacement, \( \delta \rho, \delta p \), and \( \delta \phi \) are the Eulerian perturbations. In the fluid envelope, the equations of hydrostatic equilibrium are
\[ -\nabla \delta p - \delta \rho \nabla \phi - \rho \nabla \delta \phi = 0, \] (27)
coupled with the perturbed equation of state
\[ \delta \rho = \frac{dp}{d\delta p}. \] (28)

Equations (25)–(28) are solved with the boundary conditions \( \xi(0) = 0 \), \( [-\delta p - \xi \cdot \nabla p]_{r=R_c} = 0 \), and at the core-envelope boundary \( (r = R_c) \), we require continuity of \( \nabla \cdot \xi \) and the Lagrangian perturbed radial traction
\[ \Delta T = -\delta p \hat{r} + \mu \hat{r} \cdot \left((\nabla \xi) + (\nabla \xi)^T \right) - (\xi, \nabla p) \hat{r}. \] (29)

The strain tensor for an incompressible material is
\[ \mathbf{u} = \frac{1}{2} \left( (\nabla \xi) + (\nabla \xi)^T \right). \] (30)

We define the strain amplitude \( u \) via
\[ u^2 \equiv \frac{1}{2} \text{Tr}(\mathbf{u}^2), \] (31)
where \( \text{Tr}(\mathbf{U}) \) denotes the trace of the tensor \( \mathbf{U} \). When \( u \) exceeds a critical value \( u_{\text{crit}} \), the rocky planet no longer behaves elastically, and begins to either plastically deform or fracture (the von Mises yield criterion, see Turcotte & Schubert 2002). The critical strain \( u_{\text{crit}} \) is a material property of the rocky planet (more specifically, the planet’s mantle), and is related to the yield stress \( Y \) via \( u_{\text{crit}} = Y/\mu \). Laboratory studies of the strength of rocks which make up the Earth’s crust (Kohlstedt et al. 1999) and theoretical arguments on the initiation of subduction by plastic yielding in the earth’s lithosphere (Powell 1993; Trompert & Hansen 1998; Wong & Solomatov 2015) give estimates of \( Y = 10^9 \text{–} 10^{10} \text{dyn/cm}^2 \) for the earth’s lithosphere. The characteristic shear modulus value for the Earth is \( \mu = 10^2 \text{dyn/cm}^2 \) (Turcotte & Schubert 2002), thus the critical strain is in the range \( u_{\text{crit}} = 10^{-5} \text{–} 10^{-4} \).

The strain required to resist the anisotropic weight of a triaxial planet is non-uniform, and assumes a maximum (peak) value \( u_{\text{peak}} \) at a certain location in the planet. When \( u_{\text{peak}} \) exceeds \( u_{\text{crit}} \), the core either plastically deforms or fractures, reducing the strain in the surface and core, and hence reducing \( \epsilon \). Therefore, the maximal triaxiality \( \epsilon_{\text{max}} \) of the rocky planet is set by \( u_{\text{peak}} = u_{\text{crit}} \).

### 3.1 Bare Rocky Planet

We show in the Appendix that the solution \( \xi \) of Eqs. (24)–(26) may be written in the form
\[ \xi = \xi_r(r) \hat{r} \text{Re}[Y_{22}(\theta, \varphi)] + \xi_\perp(r) \text{Re}[/ \text{V}_{22}(\theta, \varphi)], \] (32)
where
\[ \xi_r(r) = \xi_1 \left(\frac{r}{R_c}\right)^3 + 2\xi_3 \left(\frac{r}{R_c}\right), \] (33)
\[ \xi_\perp(r) = \xi_2 \left(\frac{r}{R_c}\right)^3 + 3\xi_3 \left(\frac{r}{R_c}\right), \] (34)
and \( \xi_1, \xi_2, \) and \( \xi_3 \) are constants. Applying the boundary conditions at \( r = 0 \) and \( r = R_c \), we find
\[ \xi_1 = \frac{6}{95} \left(\frac{\rho g R_c}{\mu}\right) \delta R_c, \] (35)
\[ \xi_2 = \frac{1}{19} \left(\frac{\rho g R_c}{\mu}\right) \delta R_c, \] (36)
\[ \xi_3 = -\frac{8}{95} \left(\frac{\rho g R_c}{\mu}\right) \delta R_c. \] (37)

From Eq. (31), the corresponding strain amplitude \( u \) is given by
\[ u^2 = \frac{1}{2} u_r^2 + \frac{1}{2} u_\theta^2 + \frac{1}{2} u_\varphi^2 + u_r^2 + u_\theta^2 + u_\varphi^2, \] (38)
where (Landau & Lifshitz 1959)
\[ u_r = A_{222} \frac{d \xi_r}{dr} \sin \theta \cos 2\varphi, \] (39)
\[ u_\theta = A_{222} \left[ \xi_r \sin^2 \theta + \frac{2 \xi_\perp}{r} \cos^2 \theta - \sin^2 \theta \right] \cos 2\varphi, \] (40)
\[ u_\varphi = A_{222} \left[ \xi_r \sin^2 \theta - \frac{2 \xi_\perp}{r} \sin^2 \theta + 1 \right] \cos 2\varphi, \] (41)
\[ u_\theta = -6A_{222} \frac{\xi_\perp}{r} \cos \sin 2\varphi, \] (42)
\[ u_\varphi = A_{222} \left[ \frac{\xi_r}{r} + \frac{d \xi_\perp}{dr} - \frac{\xi_\perp}{r} \right] \sin \theta \cos \theta \cos 2\varphi, \] (43)
\[ u_\varphi = -A_{222} \left[ \frac{\xi_r}{r} + \frac{d \xi_\perp}{dr} - \frac{\xi_\perp}{r} \right] \sin \theta \sin 2\varphi, \] (44)
Figure 1. Rescaled strain magnitude \( \bar{u}^2 \) [Eq. (45)] for a bare rocky planet as a function of the coordinates \((r, \theta)\). The values of azimuth \( \varphi \) are as indicated.

and \( A_{22} = \sqrt{15/32\pi} \). As expected, the strain amplitude \( u \) scales as \( \rho_g \delta R_c / \mu \). Therefore, we define the rescaled strain magnitude

\[
\bar{u} \equiv \left( \frac{\mu}{\rho_g \delta R_c} \right) u. \tag{45}
\]

In Figure 1 we plot the rescaled \( \bar{u}^2 \) over coordinates \((r, \theta)\), for \( \varphi = 0 \) and \( \varphi = \pi/4 \). Deep in the planetary interior \((r \sim 0.0 - 0.2R_c)\) is where the planetary strain is highest, with a maximal value of

\[
\bar{u}_\text{peak} \equiv \max_{r \in [0,R_c]} [\bar{u}(r,\theta,\varphi)] = 0.195. \tag{46}
\]

The source of this strain is not the direct response to the weight of the planet’s triaxiality, which scales with radius as \( \bar{u}^2 \propto r^4 \) [\( \xi_1 \) and \( \xi_2 \) terms in Eqs. (53)-(54)]. Instead, the strain deep in the planetary interior comes from additional stresses to make the radial traction [Eq. (29)] vanish on the planet’s surface, which scales with radius as \( \bar{u}^2 \propto \text{constant} \) [\( \xi_3 \) terms in Eqs. (55)-(56)]. In comparison, we find the maximal rescaled strain on the planetary surface \( \bar{u}_\text{surf} \) to be

\[
\bar{u}_\text{surf} \equiv \max_{r=R_c} [\bar{u}(r,\theta,\varphi)] = 7.32 \times 10^{-2}. \tag{47}
\]

The value \( \bar{u}_\text{peak} \) differs from \( \bar{u}_\text{surf} \) by a factor of \( \sim 3 \). There is some uncertainty as to what is the correct location one should equate \( u \) with \( u_{\text{crit}} \) to obtain the planet’s maximal triaxiality. For instance, a substantial portion of the Earth’s core is fluid (Turcotte & Schubert 2002), so it is unable to sustain any anisotropic strain \( u \). Due to the crudeness of our model, we still equate \( u_{\text{crit}} \) with \( u_{\text{peak}} \) to calculate \( \epsilon_{\text{max}} \), and note that realistic equations of state for terrestrial planets may change this result by factors of order unity.

From Eqs. (17) and (24), we have

\[
e^c = \sqrt{15/2\pi} (\delta R_c/R_c),
\]

thus

\[
u_{\text{peak}} \approx \frac{1}{7.9} \frac{\rho_g R_c}{\mu} \epsilon.
\tag{48}
\]

Equating \( u_{\text{peak}} \) with \( u_{\text{crit}} \) gives the maximum triaxiality of the bare rocky planet:

\[
\epsilon_{\text{max}} \approx 7.9 \frac{\mu}{\rho_g R_c} u_{\text{crit}}
\]

\[
\approx 1.9 \times 10^{-5} \left( \frac{\mu}{10^{12}\text{dyn/cm}^2} \right)
\times \left( \frac{\rho}{6\text{g/cm}^3} \right)^{-2} \left( \frac{R_c}{R_\oplus} \right)^{-2} \left( \frac{\epsilon_{\text{crit}}}{10^{-8}} \right).
\tag{49}
\]

3.2 Planet with a Thin Isothermal Atmosphere

Applying the boundary conditions at the core-envelope boundary, we find

\[
\xi_1 = \frac{6}{95} \left( 1 - \frac{\rho_a}{\rho_c} \right) \frac{\rho_g R_c}{\mu} \delta R_c,
\tag{50}
\]

\[
\xi_2 = \frac{1}{19} \left( 1 - \frac{\rho_a}{\rho_c} \right) \frac{\rho_g R_c}{\mu} \delta R_c,
\tag{51}
\]

\[
\xi_3 = -\frac{8}{95} \left( 1 - \frac{\rho_a}{\rho_c} \right) \frac{\rho_g R_c}{\mu} \delta R_c,
\tag{52}
\]

where \( \rho_a \) is the density at the base of the atmosphere [see Eq. (11)]. Clearly, unless \( \rho_a \sim \rho_c \), the reduced strain in the core from the addition of a thin isothermal atmosphere is negligible. The peak strain amplitude in the core is

\[
u_{\text{peak}} \approx \frac{1}{7.9} \frac{\rho_g R_c}{\mu} \epsilon \left( 1 - \frac{\rho_a}{\rho_c} \right).
\tag{53}
\]

The maximum triaxiality is larger than Eq. (49) by the factor \( (1 - \rho_a/\rho_c)^{-1} \).

3.3 Planet with a Constant Density Ocean

When the core is surrounded by a constant density ocean, we apply the boundary conditions at the core-envelope interface and find

\[
\xi_1 = \frac{6}{95} F_1 \left( \frac{R_c}{R} \frac{\rho_o}{\rho_c} \right) \frac{\rho_g R_c}{\mu} \delta R_c,
\tag{54}
\]

\[
\xi_2 = \frac{1}{19} F_1 \left( \frac{R_c}{R} \frac{\rho_o}{\rho_c} \right) \frac{\rho_g R_c}{\mu} \delta R_c,
\tag{55}
\]

\[
\xi_3 = -\frac{8}{95} F_1 \left( \frac{R_c}{R} \frac{\rho_o}{\rho_c} \right) \frac{\rho_g R_c}{\mu} \delta R_c,
\tag{56}
\]

where

\[
F_1(x, y) = \left( 1 + \frac{3y}{2} \right) (1 - y) - \frac{9yx^2(1-y)}{10x^2(1-y) + 2y/3}.
\tag{57}
\]

Notice that \( F_1(R_c/R, 0) = 1 \), reproducing the results of Section 3.1, and also \( F_1(R_c/R, 1) = 0 \), showing that when
\[ u_{\text{peak}} = \frac{\mu}{\rho_{\text{o}} g_\text{e} \delta R_c} u_{\text{peak}} = 0.195 F \left( \frac{R_c}{R}, \frac{\rho_o}{\rho_c} \right). \] (58)

Plotted in Figure 2 is the square of Eq. (58) as a function of \( \rho_o/\rho_c \), with values of \( R_c/R \) as indicated. We see that when the core is small (\( R_c \lesssim 0.5R \)) with a low density ocean (\( \rho_o \lesssim 0.2 \rho_c \)), the presence of an ocean increases the strain in the core. When the core is large (\( R_c \gtrsim 0.5R \)) or the ocean is dense (\( \rho_o \gtrsim 0.2 \rho_c \)), the weight of the ocean works to cancel the weight on the core from the planet’s triaxiality \( \epsilon \).

With a constant density ocean, \( \epsilon \) and \( \delta R_c/R_c \) are related by

\[ \epsilon = \sqrt{\frac{15}{2\pi}} F_2 \left( \frac{R_c}{R}, \frac{\rho_o}{\rho_c} \right) \delta R_c/R_c. \] (59)

where

\[ F_2(x, y) = \frac{1 - y}{1 - y + y/x^5} \left[ 1 + \frac{3y}{5x^3(1 - y) + 2y} \right]. \] (60)

Notice that \( F_2(R_c/R, 0) = 1 \), recovering the bare rocky planet result. Also note that \( F_2(R_c/R, 1) = 0 \), showing when \( \rho_o = \rho_c \) the planet has no triaxiality. Using Eqs. (58) and (59), and setting \( u_{\text{peak}} = u_{\text{crit}} \), we obtain the maximum triaxiality

\[ \epsilon_{\text{max}} = 7.9 \frac{\mu}{\rho_{\text{e}} g_\text{e} u_{\text{crit}}} F \left( \frac{R_c}{R}, \frac{\rho_o}{\rho_c} \right), \] (61)

where

\[ F(x, y) = F_2(x, y)/F_1(x, y). \] (62)

Figure 3 shows \( \epsilon_{\text{max}} \) [Eq. (61)] as a function of \( R_c/R \), with values of \( \rho_o/\rho_c \) as indicated. We see that the maximal triaxiality of the planet may be significantly decreased by the presence of an ocean. This is because as \( R_c/R \to 0 \), the bulge at the planetary surface induced by the planet’s triaxiality becomes increasingly negligible, even though the strain in the core may be reduced by the presence of an ocean (see Fig. 2).

4 SUMMARY AND DISCUSSION

We have derived the analytic expression for the maximum triaxial deformation \( \epsilon_{\text{max}} \) of a rocky planet as a function of its density and radius [see Eq. (19)]. This maximum triaxiality depends on the rigidity (shear modulus) and critical strain of the rocky material. A thin atmosphere surrounding the rocky core has a negligible impact on \( \epsilon_{\text{max}} \) [see Eq. (59)], while a liquid ocean envelope may lower the maximal triaxiality by a factor of a few or more than an order of magnitude, depending on the thickness and density of the ocean [see Eq. (61) and Figs. 2, 3].
The rigidity $\mu$ and critical strain $u_{\text{crit}}$ for rocky planets are unknown. The value of $u_{\text{crit}}$ is particularly uncertain, and probably depends on the assembly history of the planet. Applying our result to terrestrial bodies in the solar system, we find that the observed values of $\epsilon$ are consistent with our predicted $\epsilon_{\text{max}}$ (see Table 1) for reasonable $\epsilon$ (about $10^{12}$ dynes/cm²) and $u_{\text{crit}} = 10^{-5}$. Interestingly, for Mercury, Earth, Mars, and the Moon, these observed $\epsilon$ values are close to $\epsilon_{\text{max}}$ with $u_{\text{crit}} = 10^{-5}$, the lower range of the critical strain for the Earth.

### 4.1 Implication for Spin-Orbit Resonance Capture

As noted in Section 1, tidal dissipation tends to drive a close-in planet toward synchronous rotation. The magnitude of the tidal torque on the planet reads

$$T_{\text{tide}} \simeq \frac{3GM_{\star}R_{\star}^5}{2a^6} \frac{k_2}{Q}, \quad (63)$$

where $k_2$ and $Q$ are the Love number and tidal quality factor of the planet, respectively. This gives the tidal synchronization time

$$t_{\text{sync}} = \frac{\Omega}{T_{\text{tide}}} = 3.5 \times 10^5 \left(\frac{Q/k_2}{10^3}\right) \left(\frac{\rho_c}{6 \text{ g/cm}^3}\right)$$

$$\times \left(\frac{M_{\star}}{0.3 M_{\odot}}\right)^{-3/2} \left(\frac{a}{0.1 \text{ AU}}\right)^{9/2} \text{ years}, \quad (64)$$

where $M_{\star}$ is the host star mass, $a$ is the planetary semimajor axis, and $\Omega \simeq \sqrt{GM_{\star}/a^3}$ is the planetary orbital angular frequency. We have scaled $a$ to 0.1 AU, the characteristic HZ distance for $0.3M_{\odot}$ M-dwarfs (e.g. Shields et al. 2016). On the other hand, the tidal circularization time of the orbit is

$$t_{\text{circ}} = \frac{M_{\star}a^2\Omega}{T_{\text{tide}}} = 4.8 \times 10^{12} \left(\frac{M_{\star}}{0.3 M_{\odot}}\right)^{-3/2} \left(\frac{a}{0.1 \text{ AU}}\right)^{13/2}$$

$$\times \left(\frac{\rho_c}{6 \text{ g/cm}^3}\right) \left(\frac{R_{\star}}{R_{\odot}}\right)^{-2} \left(\frac{Q/k_2}{10^3}\right) \text{ years}, \quad (65)$$

where $M_{\odot}$ is the mass of the planet. The planet can retain its initial (“primordial”) eccentricity at formation if $t_\text{circ}$ is longer than the age of the system. With a finite eccentricity, the planet may be captured into the 3:2 spin-orbit resonance during its tidal spin-down (Goldreich & Peale 1966). The resonance torque on the planet due to its intrinsic triaxial deformation has a magnitude (for the 3:2 resonance)

$$T_{\text{tri}} \simeq \frac{21}{4} \epsilon I_c \left(\frac{GM_{\star}}{a^3}\right). \quad (66)$$

If the planet’s rheology is a frequency-independent constant-$Q$ tidal model, a necessary condition for resonance capture is $T_{\text{tri}} > T_{\text{tide}}$, giving

$$\epsilon > 1.0 \times 10^{-6} \left(\frac{M_{\star}}{0.3 M_{\odot}}\right) \left(\frac{a}{0.1 \text{ AU}}\right)^{-3}$$

$$\times \left(\frac{\rho_c}{6 \text{ g/cm}^3}\right) ^{-1} \left(\frac{k_2/Q}{10^{-3}}\right) \left(\frac{\epsilon}{0.01}\right)^{-1}, \quad (67)$$

where we have scaled the planetary eccentricity $\epsilon$ to 0.01, characteristic of super-Earth systems discovered by the Kepler mission (Wu & Lithwick 2013). Although we have assumed a simple constant-$Q$ model for the planet’s rheology, one may obtain a similar lower bound for $\epsilon$ using an Andrade model, as long as the planet’s eccentricity is low enough (Ribas et al. 2016). We also note that planets with eccentricities of order 0.01 have low probabilities for capture into 3:2 spin-orbit resonances, regardless of the rheology (Murray & Dermott 2000).

Makarov, Berghea, & Efroimsky 2012.

To avoid chaotic spin behavior associated with the overlap of the synchronous and 3:2 resonances, the planet’s triaxiality must satisfy (Wisdom et al. 1984)

$$\epsilon < \frac{1}{3} \left(\frac{1}{2 + \sqrt{14\epsilon}}\right)^2 \simeq \frac{1}{12}. \quad (68)$$

Comparing (67) and (68) to our derived $\epsilon_{\text{max}}$, we see that capture into stable asynchronous spin-orbit resonance is a distinct possibility for planets in the HZ of M-dwarfs.

If we take $M_{\star} = 0.08 M_{\odot}$ and $a = 0.03$ AU, appropriate for the “habitable” planets around TRAPPIST-1 (Gillon et al. 2017), then the numerical factor in front of Eq. (65) becomes $1.4 \times 10^{10}$ years, and that of Eq. (67) becomes $10^{-5}$. This suggests that tidal dissipation cannot damp the planet’s eccentricity, and the planet can be sufficiently triaxial to allow for capture into the 3:2 resonance.

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Eqs. (71)-(73) are satisfied if
\[ 0 = -V + 2\mu \left( \frac{\xi_1}{R_c} + 3\frac{\xi_2}{R_c} \right), \]  
\[ 0 = 5\xi_1 - 6\xi_2. \]  

In addition, we may add to Eq. (74) any solutions to the equations
\[ \nabla \cdot \xi = 0, \quad \nabla^2 \xi = 0. \]  

It is sufficient to take
\[ \xi = R_c \nabla \left[ \xi_3 \left( \frac{r}{R_c} \right)^2 \text{Re}(Y_{22}) \right]. \]  

Thus the general solutions to Eqs. (71)-(73) take the forms
\[ \xi_r(r) = \xi_1 \left( \frac{r}{R_c} \right)^3 + 2\xi_3 \left( \frac{r}{R_c} \right). \]  
\[ \xi_{\perp}(r) = \xi_2 \left( \frac{r}{R_c} \right)^3 + \xi_3 \left( \frac{r}{R_c} \right). \]

These equations are completed by requiring continuity of \( \xi_r \) and the radial traction \( \Delta T \) [Eq. (29)] at the planet-envelope boundary \( (r = R_c) \). Specifically, we require
\[ \xi_r(R_c) = \xi_r(R_c^+), \]  
\[ \left[ -\delta p_{22} - \xi_r \frac{dp}{dr} + 2\mu \frac{d\xi_r}{dr} \right]_{r=R_c^-} = \left[ -\delta p_{22} - \xi_r \frac{dp}{dr} \right]_{r=R_c^+}, \]  
\[ \left[ \frac{\xi_r}{r} + \frac{d\xi_r}{dr} - \frac{\xi_{\perp}}{r} \right]_{r=R_c^-} = 0. \]

To solve equations (25)–(26) for an incompressible planet, we note that \( \delta p \) satisfies \( \nabla^2 \delta p = 0 \). This, with the form of \( \delta \phi \) [Eq. (16)] and the requirement that \( \delta p \) be finite at \( r = 0 \), implies that \( \delta p = \delta p_{22}(r) \text{Re}[Y_{22}(\theta, \varphi)] \), with \( \delta p_{22} \propto r^2 \).

Define
\[ \delta v \equiv \delta p + \rho G \delta \phi = V \left( \frac{r}{R_c} \right)^2 \text{Re}[Y_{22}(\theta, \varphi)], \]  
where \( V \) is an undetermined constant. We decompose \( \xi \) into radial and tangential components:
\[ \xi = \xi_r(r) \hat{r} \text{Re}[Y_{22}(\theta, \varphi)] + \xi_{\perp}(r) \hat{\theta} \text{Re}[r \nabla Y_{22}(\theta, \varphi)]. \]  

Equations (25)–(26) then become
\[ 0 = -\frac{2V}{R_c} \left( \frac{r}{R_c} \right)^2 + \mu \left[ \frac{d^2 \xi_r}{dr^2} + \frac{2d\xi_r}{r dr} - 8\frac{\xi_r}{r^2} + 12\frac{\xi_1}{r^2} \right], \]  
\[ 0 = -\frac{V}{R_c} \left( \frac{r}{R_c} \right)^2 + \mu \left[ \frac{2\xi_r}{r^2} + \frac{d^2 \xi_r}{dr^2} + 2\frac{d\xi_r}{dr} - 6\frac{\xi_1}{r^2} \right], \]  
\[ 0 = \frac{d\xi_r}{dr} + 2\frac{\xi_r}{r} - 6\frac{\xi_1}{r}. \]  

Solutions to the inhomogeneous Eqs. (71)–(72) require \( \xi_r, \xi_{\perp} \propto r^3 \). Specifically, taking
\[ \xi_r = \xi_1 \left( \frac{r}{R_c} \right)^3, \quad \xi_{\perp} = \xi_2 \left( \frac{r}{R_c} \right)^3, \]