On a separation criterion for symmetric elliptic bluff body flows

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Abstract

A new analytical criterion that captures the onset of separation of flow past elliptic cylinders is derived by considering the variation of the wall normal velocity in Reynolds number parameter space. It is shown that this criterion can be used to calculate the separation Reynolds number ($Re_s$) for the classical problem of flow past a circular cylinder, a contentious and unresolved issue till date. The two dimensional Navier-Stokes equations are solved computationally and an exact value of $Re_s$ is obtained by applying the aforementioned criterion.

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A great deal of research in the past century has been focused on bluff body wakes and considerable progress has been made towards understanding them. A body is considered “bluff” if the spatial extent of the body along the flow direction is of comparable or lesser order to that normal to it. The bluff bodies that are of concern here are a family of symmetric elliptical cylinders, with their minor axes aligned with the flow direction. The flow past such a bluff body is steady for very low Reynolds numbers \( Re = Ua/\nu \) where \( a \) is the length of the body normal to the flow and \( \nu \) is the coefficient of viscosity), which in case of a circular cylinder happens for \( Re < 47 \). In two dimensions, as \( Re \) is increased, a few well defined features of the flow are observed. Up to a particular value of \( Re \) (say \( Re_s \)), the streamlines, while being asymmetric about the axis normal to that of the flow, are attached to the body \([1]\). But for \( Re > Re_s \), the flow separates and two well defined separation “bubbles” or eddies are observed \([1]\). These “bubbles” are regions of closed streamlines where the flow direction just next to the cylinder is in the opposite direction to the mean flow \((i.e.\) a region of backflow). The characteristic features of these bubbles have been documented comprehensively both in computational and experimental studies, mostly for the case of flow past a circular cylinder. Investigations of flow past elliptic cylinders are few \([2]\) and none of them seem to focus on aspects of flow separation. Computational results \([3, 4]\) indicate that for the case of a circular cylinder, \( 4 < Re_s < 7 \), but even recent investigations, for example \([5]\), have been unable to capture the bubble for \( Re < 6 \) inspite of having a much higher numerical resolution in the wake region than \([3, 4]\). Experimental studies \([6, 7, 8]\) suffer from similar ambiguities and an effort to summarize and compile various such experiments was undertaken by \([9]\) who concluded that \( Re_s \sim 5 \). The criterion derived here, as we shall see later, can be used to obtain the exact value of \( Re_s \) and resolve this ambiguity.

The complexity of the problem means that only a few comprehensive theories have been put forth in the past and their scope and success have been limited. One theory of note by \([10, 11, 12]\) uses the triple-deck model to show that the length of the bubbles (in the direction of the flow) increases linearly with Reynolds number. This theoretical result has been confirmed by many experiments and numerical simulations \([13]\). However, this theory is unable to predict the Reynolds number at which the bubbles appear, \( Re_s \). Further, it does not attempt to explicitly understand and characterize the flow when bubbles form. Neither does it attempt to generalize the same for other bluff bodies like symmetric elliptic cylinders in which the bubble formation is of a similar nature.
The aim of this Rapid Communication is an attempt, a possible first step, in this direction. Here we derive a simplistic analytical criterion that captures the Reynolds number at which the bubbles start forming, \( \text{i.e. } Re_s \). This criterion is general and valid for a whole class of elliptic cylinders. It seems similar to the Prandtl shear stress criterion which states that, at the point where the streamline separates the wall shear stress, \( \tau_{\text{wall}} = 0 \). The distinction between the Prandtl criterion and the present one though is quite fundamental and should pose no source of confusion. The criterion derived here is valid only at \( Re_s \), i.e. it captures the Reynolds number at which the bubble starts forming. Whereas once the bubble forms, the upper and lower streamlines ending at the wall (the “separatrices” since they separate the “outer” flow region from the flow “inside” the bubble) always satisfy the Prandtl shear stress criterion at the point that they separate, irrespective of the Reynolds number (of course, until the onset of unsteadiness). The Prandtl criterion, it must be noted, has relevance only after the bubble is formed and the point that the separatrices separate from the cylinder surface has \( \tau_{\text{wall}} = 0 \), but this has absolutely no bearing on \( Re_s \).

We study uniform flow past a 2D elliptic cylinder such that the flow direction is along the minor axis, as shown in Fig. 1. A body-fitting orthogonal coordinate system given by \( \zeta - \eta \) (specifically, an elliptic cylindrical coordinate system) coincides with the \( x - y \) axis at the point P, the intersection of the symmetry plane with the cylinder. One expects that, as the \( Re \) is increased from zero through \( Re_s \), it is at P that the bubbles start forming. Examining the flow in the vicinity of P in \( Re \) parameter space would give us the required criterion. This is done by considering the flow around P for two Reynolds numbers \( Re_1 \) and \( Re_2 \) such that

\[
0 < Re_1 < Re_s < Re_2 < Re_{\text{uns}}
\]

Here \( Re_{\text{uns}} \) is the Reynolds number at which the steady wake of the cylinder becomes
unstable and vortex shedding sets in, e.g. $Re_{uns} = 47$ for a circular cylinder. Now for $Re = Re_1$, the bubble has not started forming and there is no region of backflow; whereas for $Re = Re_2$, the bubble has formed and is of finite size and has a region of backflow. Then one can find a distance $\delta \zeta$, which is less than the bubble length, so that,

$$u(\zeta_P + \delta \zeta, \eta_P, Re_2) < 0 \quad (2)$$

$$u(\zeta_P + \delta \zeta, \eta_P, Re_1) > 0 \quad (3)$$

Here $P \equiv (\zeta_P, \eta_P)$, $P' \equiv (\zeta_P + \delta \zeta, \eta_P)$ and the flow velocity in $\zeta - \eta$ coordinates is written as $u \equiv u_\zeta$ and $v \equiv u_\eta$ along the respective coordinates.

Note that the no slip condition at the wall implies that $u|_P = v|_P = 0$. Since $v = 0$ along the cylinder, $\frac{\partial u}{\partial \eta}|_P = 0$ (so are derivatives to higher orders along that direction). Using the continuity equation at $P$, $\frac{1}{h_\eta} \frac{\partial u}{\partial \eta} + \frac{1}{h_\zeta} \frac{\partial u}{\partial \zeta} = 0$ (here $h_\eta$ and $h_\zeta$ are the scale factors for the coordinate transformation $(x, y) \rightarrow (\zeta, \eta)$) and the fact that $\frac{\partial u}{\partial \eta}|_P = 0$, we get

$$\left. \frac{\partial u}{\partial \zeta} \right|_P = 0 \quad (4)$$

Now we write the $u$ velocity at point $P'$ as a Taylor expansion about the point $P$, keeping terms till second order. Since $\delta \zeta$ is along the $\zeta$ direction, we get

$$u|_{P'} = u|_P + \frac{\partial u}{\partial \zeta}|_P \delta \zeta + \frac{\partial^2 u}{\partial \zeta^2}|_P \frac{\delta \zeta^2}{2!} + O(\delta \zeta^3) \quad (5)$$

From no slip and Eq. (4) we get

$$u|_{P'} = \frac{\partial^2 u}{\partial \zeta^2}|_P \frac{\delta \zeta^2}{2!} + O(\delta \zeta^3) \quad (6)$$

Clearly, the relations (2) and (3) imply that $u|_{P'}(Re) = 0$ at some $Re_1 < Re < Re_2$. By letting the limits $Re_1 \rightarrow Re^-_s$ and $Re_2 \rightarrow Re^+_s$ alongwith $\delta \zeta \rightarrow 0$, we see that this Reynolds number is precisely $Re_s$, i.e. $u|_{P'}(Re_s) = 0$. And from (6) we finally get,

$$\frac{\partial^2 u}{\partial \zeta^2}(\zeta_P, \eta_P, Re_s) = 0 \quad (7)$$

which is precisely the criterion required for separation. The above derivation is almost trivial and the result is fairly intuitive. Further, it is seen that the condition (7) is necessary and sufficient. This is because $u$ near P is always positive for $Re < Re_s$ and so is the second derivative. Similarly for $Re_{uns} > Re > Re_s$ it is always negative, again implying a negative
value of the second derivative. An equivalent pressure condition can also be derived by using (7) and considering the Navier-Stokes (NS from here on) equation at P. Setting all derivatives of velocity components along the wall to be zero and using Eq. (4), we get,

\[
\frac{1}{h_\zeta \zeta^2} \frac{\partial^2 u}{\partial \zeta^2} (\zeta_P, \eta_P, Re) = Re \frac{\partial p}{\partial \zeta} (\zeta_P, \eta_P, Re)
\] (8)

for \( Re = Re_s \), we have from (7),

\[
\frac{\partial p}{\partial \zeta} (\zeta_P, \eta_P, Re_s) = 0
\] (9)

Of the two equivalent conditions, (7) and (9), it is not clear which would be more useful in computational efforts in determining \( Re_s \). This can be resolved by examining the behavior of \( \frac{\partial^2 u}{\partial \zeta^2} (\zeta_P, \eta_P, Re) \) and \( \frac{\partial p}{\partial \zeta} (\zeta_P, \eta_P, Re) \) in \( Re \) space around \( Re_s \).

It is instructive to note that relations similar to (7) and (9) would also be valid in three dimensions for flow past a family of symmetric ellipsoids. This family, to which the sphere also belongs, has rotation symmetry about an axis parallel to the flow direction. The derivation of the separation criteria for this family mirrors the one above. While flow past an ellipsoid has not been investigated in any detail, for a sphere, experiments imply that \( Re_s = 25 \) [14, 15]. However, unlike the case of a flow past a cylinder, there is excellent agreement between various experimental and computational results [15] about this value. In the light of this fact, the criterion (7) has little application here. Our present study is therefore restricted to the study of two dimensional flows.

We consider flow past a circular cylinder, which is a special case of the family of ellipses and relatively easy to compute. In this case, \((\zeta, \eta) \equiv (r, \theta)\) but we continue to use \((\zeta, \eta)\) for the sake of uniformity of notation. We solve the 2D NS equations computationally in the streamfunction-vorticity formulation. This approach has an advantage over primitive variable formulations since an explicit operator splitting is not required. The unsteady NS equations are solved in the range \( 2.5 < Re < 10 \) by impulsively starting the flow, unlike the authors [4] who solve the steady NS equation. Impulsively started flow past a cylinder has been studied extensively using a variety of numerical formulations - finite difference, finite volume and vortex methods [16]. The present work uses a finite difference formulation for its simplicity as done by [3]. It is formulated using an explicit time stepping scheme, the time stepping being varied from a fourth Runge Kutta scheme to an Euler scheme with identical results. The discretization of the non-linear terms was done using a 3rd
order upwind scheme [17] while all other derivatives were effected using central differencing. The streamfunction-vorticity Poisson equation was solved using a stabilized Biconjugate Gradient Method [18]. The grid used was a body fitting grid, with grid clustering in the radial direction. Computations were performed on three different grids with the highest resolution being $300 \times 150$ and the largest ratio of the outer boundary to the cylinder diameter being 40. Each computation was performed till steady state was reached. About a hundred hours of computation time were required for all the cases on the most refined grid, the results of which are presented below.

Once the flow is started impulsively from rest, a pair of separation bubbles can be seen for $Re > 6$, below which it was hard to resolve the bubbles. The computations were continued up to a time $t=10$ (where $U = \frac{1}{2}D = 1$, $U$ and $D$ being the free stream velocity and cylinder diameter respectively) when steady state was approximately reached. The difficulty in resolving the bubbles close to separation is not a new problem and is common to computational investigations of this nature [5]. In order to do this, we first look at the $u$ velocity variation along the symmetry plane close to the cylinder. This has been done and the results are plotted in Fig. 2 for a range of Reynolds numbers as indicated. Clearly $u < 0$ near P indicates the presence of a bubble. As seen from Fig. 1, $4 < Re_s < 6$, which is in
good agreement with existing computational results.

In Fig. 3, $\frac{\partial^2 u}{\partial \zeta^2}(\zeta_P, \eta_P, Re)$, which is obtained from the $u$ velocity data by central differencing, is plotted against Reynolds number. As seen in Fig. 3 we obtain the rather surprising result that $\frac{\partial^2 u}{\partial \zeta^2}(\zeta_P, \eta_P, Re)$ varies linearly with $Re$ in the neighborhood of $Re_s$ and so, by virtue of (8), $\frac{\partial p}{\partial \zeta}(\zeta_P, \eta_P, Re)$ does not. Therefore, numerical simulations attempting to compute $Re_s$ can do so by calculating $\frac{\partial^2 u}{\partial \zeta^2}(\zeta_P, \eta_P, Re)$ for a few Reynolds numbers around $Re_s$ and then extrapolating the curve linearly to zero to obtain $Re_s$. $\frac{\partial p}{\partial \zeta}(\zeta_P, \eta_P, Re)$ on the other hand is highly non-linear around $Re_s$ and cannot be used in similar manner, even though (9) itself remains valid. So from Fig. 3 and applying (7) one gets $Re_s=4.8$. The aforementioned approach underlines the advantage of using (7) in order to get $Re_s$ as opposed to attempts in resolving the bubble by solving the NS equation for a large number of values of $Re$. One must realize that, though (7) is exact, the linearity of $\frac{\partial^2 u}{\partial \zeta^2}(\zeta_P, \eta_P, Re)$ around $Re_s$ is observed only through computations and that too for the special case of a circular cylinder. But from the seemingly general nature of the problem of separation for the family of ellipses, as seen in deriving (7), one might expect that the linearity property holds well across the family. But an ad hoc argument of generality can hardly suffice and therefore the author is currently investigating this aspect of the problem, both analytically and numerically.

Also, a note of caution regarding the value of $Re_s$ obtained here, which it must be re-
marked is still contingent on the computational approach used. And the present computation while reasonable, can be improved with respect to the grid refinement and also the outer domain size, which might have some bearing on the value of $Re_s$. In fact there was some dependence of the exact value of $Re_s$ on the outer domain size (about 7% between 30 and 40 times the cylinder diameter for the most refined grid). But the linearity property remained unchanged and the variation in the slope of the $\frac{\partial^2 u}{\partial \zeta^2}(\zeta_P, \eta_P, Re)$ curve was also negligible.

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