Locally covariant quantum field theory *

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Abstract

The principle of local covariance which was recently introduced admits a generally covariant formulation of quantum field theory. It allows a discussion of structural properties of quantum field theory as well as the perturbative construction of renormalized interacting models on generic curved backgrounds and opens in principle the way towards a background independent perturbative quantization of gravity.

1 Introduction

Quantum field theory may be understood as the incorporation of the principle of locality ("Nahwirkungsprinzip"), which is at the basis of classical field theory, into quantum theory, formally encoded in the association of points $x$ of spacetime to observables $\varphi(x)$ of quantum theory \cite{1}. The physical picture behind is that experiments in a laboratory are analyzed in terms of spacetime positions of measuring devices, as for instance in scattering theory.

This picture assumes an a priori notion of spacetime. Such a notion is in conflict with principles of general relativity where a point should be characterized by intrinsic properties of geometry, in the spirit of Leibniz, and in contrast to Newton’s idea of an absolute spacetime. In order to incorporate gravity into quantum theory one therefore should aim at a formulation of the theory which uses only intrinsic and local properties of spacetime.

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The traditional formalism of quantum field theory, however, is full of nonlocal concepts which have no obvious local geometric interpretation. Most remarkably, the concepts of a vacuum and of particles are meaningful only for spacetimes which have special structures. As a consequence, an S-matrix cannot be introduced, in the generic case. On the more technical side, the euclidean formulation of quantum field theory and the corresponding path integral admit a covariant formalism on compact Riemannian spaces, but, in general, there is no analogue of the Osterwalder-Schrader-Theorem [2] which allows the transition to a spacetime with a Lorentzian metric.

There seem to be two reasons for the use of nonlocal concepts in quantum field theory. One is the occurrence of divergences which prohibit a direct translation of the intrinsically local classical field theory into quantum theory; the other are the nonlocal features of quantum physics, as they become visible in the violation of Bell’s inequalities.

The problem of singularities can be solved by replacing momentum space techniques by methods of microlocal analysis. This program was successfully carried through by Brunetti and Fredenhagen [3] on the basis of a crucial observation of Radzikowski [4]. The nonlocal aspects of quantum physics can be circumvented by shifting the emphasis from the states (with the essentially nonlocal phenomenon of entanglement) to the algebras of observables where locality is encoded in algebraic relations. We therefore developed a generalization of the framework of algebraic quantum field theory [5, 6] which is

- intrinsically local
- generally covariant
- based on Haag’s concept of algebras of local observables

We show that the concept can be realized in renormalized perturbation theory, and that structural theorems of quantum field theory as the spin statistics connection [7] and the existence of PCT symmetry [8] can be formulated and, in a somewhat weaker form, proven. Moreover, we indicate how the theory can be physically interpreted. Some tentative remarks on a consistent framework for perturbative quantization of gravity conclude the paper.
2 Quantum field theory as a functor

Let me first recall the formalism of algebraic quantum field theory ("local quantum physics"). There the basic structure consists in an association of spacetime regions \( \mathcal{O} \subset \mathcal{M} \) to algebras of observables \( \mathfrak{A}(\mathcal{O}) \subset \mathfrak{A}(\mathcal{M}) \) satisfying the Haag-Kastler axioms \([9,10]\):

\[
\mathcal{O}_1 \subset \mathcal{O}_2 \implies \mathfrak{A}(\mathcal{O}_1) \subset \mathfrak{A}(\mathcal{O}_2) \quad \text{(isotony)}
\]

\[
\mathcal{O}_1 \text{ spacelike to } \mathcal{O}_2 \implies [\mathfrak{A}(\mathcal{O}_1), \mathfrak{A}(\mathcal{O}_2)] = \{0\} \quad \text{(locality)}
\]

The identity component of the group \( G \) of isometries \( g \) of \( \mathcal{M} \) is represented by automorphisms \( \alpha_g \) of \( \mathfrak{A}(\mathcal{M}) \) such that

\[
\alpha_g(\mathfrak{A}(\mathcal{O})) \subset \mathfrak{A}(g\mathcal{O})
\]

If \( \mathcal{O} \) is a neighbourhood of a Cauchy surface of \( \mathcal{M} \) then

\[
\mathfrak{A}(\mathcal{O}) = \mathfrak{A}(\mathcal{M}) \quad \text{(time slice axiom)}
\]

A map \( \mathfrak{A} \) with the properties above is called a local net. It is the principle of algebraic quantum field theory that the local net characterizes the theory uniquely. This principle has been tested mainly on Minkowski space and was found to be satisfied in all cases considered so far (see e.g. the contribution of Buchholz to the proceedings of ICMP2000 in London [11]).

In the application of the framework of algebraic quantum field theory to curved spacetime, there is, however, the problem that the group \( G \) of isometries is trivial in the generic case, and that its replacement by the group of diffeomorphisms is in conflict with the axiom of locality.

We therefore proposed in [5] the following generalization of the Araki-Haag-Kastler framework of algebraic quantum field theory (see also [7] and [6]): Instead of defining the theory on a specific spacetime, we define it on all spacetimes (of a suitable class) in a coherent way. Let \( \mathfrak{Man} \) denote a category whose objects are globally hyperbolic spacetimes which are oriented and time oriented and whose arrows are isometric embeddings of one spacetime into another one, such that orientations and causal relations are preserved.

A locally covariant quantum field theory is now defined as a covariant functor from \( \mathfrak{Man} \) into a category \( \mathfrak{Alg} \) of operator algebras, where the arrows are faithful homomorphisms. It depends on the
problem under consideration whether one assumes the objects to be C*-algebras or topological algebras.

This definition contains the original Haag-Kastler framework. Namely, if one restricts the functor to the full subcategory whose objects are globally hyperbolic, causally convex subsets of a fixed spacetime $\mathcal{M}$, one obtains a local net which satisfies the conditions of isotony and covariance.

The analogue of the remaining Haag-Kastler axioms can easily be formulated. The axiom of locality (in the sense of commutativity of spacelike separated observables) means that, provided $\psi_i : \mathcal{M}_i \rightarrow \mathcal{N}$, $i = 1, 2$ are arrows of $\text{Man}$ such that the images of $\mathcal{M}_1$ and $\mathcal{M}_2$ are spacelike separated in $\mathcal{N}$, the corresponding subalgebras of $\mathfrak{A}(\mathcal{N})$ commute. The time slice axiom, on the other hand, says that if the image of $\psi$ is a neighbourhood of a Cauchy surface of the target spacetime, then the corresponding subalgebra already coincides with the full algebra.

The latter property allows a comparison of time evolution on different spacetimes. Namely, let $\mathcal{M}_1$ and $\mathcal{M}_2$ have Cauchy surfaces with mutually isometric neighbourhoods $\mathcal{N}_\pm$, and let $\psi_{i,\pm} : \mathcal{N}_\pm \rightarrow \mathcal{M}_i$ denote the corresponding embeddings. Then, using the fact that the corresponding embeddings $\mathfrak{A}\psi_{i,\pm} =: \alpha_{i,\pm}$ of operator algebras are surjective, we can define an automorphism of $\mathfrak{A}(\mathcal{M}_1)$ by

$$\beta = \alpha_{1,+}^{-1} \alpha_{2,+}^{-1} \alpha_{2,-} \alpha_{1,-}^{-1}.$$ 

$\beta$ describes the effect of the change of the metric from $\mathcal{M}_1$ to $\mathcal{M}_2$ between the two Cauchy surfaces and is called the relative Cauchy evolution. It may be interpreted as a time evolution in the interaction picture. In particular, if $\beta_h$ arises from the addition of a symmetric tensor $h$ with compact support to the original metric, then its functional derivative at $h = 0$ can be interpreted as a commutator with the energy momentum tensor (up to a factor of $\frac{1}{2\pi}$). In fact, it is always covariantly conserved, and for the free Klein-Gordon field it coincides with the adjoint action of the canonical energy momentum tensor ($\frac{1}{2\pi}$) [5].

Based on the locally covariant system of operator algebras described by the functor $\mathfrak{A}$, one can introduce other locally covariant structures. They are, by definition, natural transformations between functors defined on $\text{Man}$.

First of all, two locally covariant theories are equivalent, if the functors are naturally equivalent. Namely, let $\mathfrak{A}_1$ and $\mathfrak{A}_2$ be two func-
 tors from Man to Alg. They are naturally equivalent, if there exists a family $\gamma_M$ of isomorphisms from $A_1(M)$ to $A_2(M)$ such that

$$\gamma_M \circ A_1 \psi = A_2 \psi \circ \gamma_M.$$ 

One may also define a locally covariant state to be a family $\omega_M$ of states on $A(M)$ with the covariance property

$$\omega_M = \omega_N \circ A_\psi.$$ 

But it turns out, that states with this property do not exist in typical cases. This may be interpreted as nonexistence of a vacuum state in a generally covariant framework, and is the root of many problems of quantum field theories on curved spacetimes; in particular, a covariant path integral, even in a formal perturbative sense, does not exist.

One can however show, for the free Klein Gordon field, that a locally covariant minimal folium of states does exist. A folium of states on a C*-algebra $\mathcal{B}$ is a norm closed set of states which is invariant under the operations

$$\omega \rightarrow \omega_B = \frac{\omega(B^* \cdot B)}{\omega(B^*B)} , \ B \in \mathcal{B}.$$ 

and under convex combinations. A folium is minimal, if it does not contain proper subfolia. We have the following theorem \[5\], which might be considered as a version of the principle of local definiteness \[12\].

**Theorem 1** For the Klein-Gordon theory, there exists a family of folia $\mathcal{F}_M$ of $A(M)$ such that

$$(A_\psi)^* \mathcal{F}_N \subset \mathcal{F}_M$$

and is minimal if $\psi(M)$ is relatively compact.

The most important natural transformations are the local quantum fields. A locally covariant scalar field $A$ is a family of linear continuous maps $A_M : D(M) \rightarrow A(M)$ (here we prefer to work with topological operator algebras), such that

$$A_\psi \circ A_M = A_N \circ D_\psi$$

where $D_\psi \equiv \psi_*$ denotes the push forward on the test function spaces,

$$(\psi_* f)(x) = \begin{cases} f(\psi^{-1}(x)) & , \ x \in \psi(M) \\ 0 & , \ else \end{cases}.$$
Vector, tensor and spinor fields are analogously defined as natural transformations with the functor of spaces of compactly supported sections of suitable bundles. One may also look at nonlinear functionals, an example being the relative Cauchy evolution $\beta$. There one introduces, for every spacetime $\mathcal{M}$, the set $\mathcal{L}_{\text{or}}(\mathcal{M})$ of Lorentz metrics which differ from the given metric only on a compact region. Let $\beta_{\mathcal{M},g}$ be the relative Cauchy evolution between $\mathcal{M}$ and the spacetime obtained by replacing the metric by $g \in \mathcal{L}_{\text{or}}(\mathcal{M})$, and let $\beta_\mathcal{M}: \mathcal{L}_{\text{or}}(\mathcal{M}) \to \text{Aut}(\mathfrak{A}(\mathcal{M}))$ be defined by

$$\beta_\mathcal{M}(g) = \beta_{\mathcal{M},g}.$$ 

Then $\beta$ is a natural transformation in the sense that

$$\mathfrak{A}_\psi \circ \beta_{\mathcal{M},g} = \beta_{\mathcal{N},\psi^*g} \circ \mathfrak{A}_\psi.$$ 

### 3 Free fields and Wick polynomials

As a first example we construct the theory of the free Klein Gordon field. There the algebra $\mathfrak{A}(\mathcal{M})$ is the unital *-algebra generated by elements $\varphi_\mathcal{M}(f)$, $f \in \mathcal{D}(\mathcal{M})$ with the following relations: First we require that the map

$$f \to \varphi_\mathcal{M}(f)$$

is linear and the involution is given by

$$\varphi_\mathcal{M}(f)^* = \varphi_\mathcal{M}(\overline{f}).$$

Then we assume that the field equation is satisfied in the weak sense,

$$\varphi_\mathcal{M}(Kf) = 0$$

with the Klein Gordon operator $K$ on $\mathcal{M}$. Finally, the commutation relations are given by

$$[\varphi_\mathcal{M}(f), \varphi_\mathcal{M}(g)] = i(f, E_\mathcal{M}g)$$

where $E_\mathcal{M}$ is the fundamental solution of the Klein-Gordon equation on $\mathcal{M}$.

The action $\mathfrak{A}_\psi \equiv \alpha_\psi$ of $\mathfrak{A}$ on arrows $\psi: \mathcal{M} \to \mathcal{N}$ is given by

$$\alpha_\psi(\varphi_\mathcal{M}(f)) = \varphi_\mathcal{N}(\psi^*f).$$
The crucial fact which guarantees that $\alpha_\psi$ is an homomorphism of algebras, is that the restriction of the fundamental solution $E_\mathcal{N}$ to $\psi(\mathcal{M})$ coincides with the push forward of $E_\mathcal{M}$.

By this construction, we also obtained a locally covariant field, namely $\varphi = (\varphi_\mathcal{M})_\mathcal{M}$.

One may now ask whether there exists a locally covariant state which then could be chosen as the analogue of the vacuum state. In order to exclude pathologies we restrict ourselves to states $\omega_\mathcal{M}$ whose $n$-point functions $\omega_\mathcal{M}^{(n)}$ are distributions on $\mathcal{M}^n$,

$$\omega_\mathcal{M}(\varphi_\mathcal{M}(f_1) \cdots \varphi_\mathcal{M}(f_n)) = \langle \omega_\mathcal{M}^{(n)}, f_1 \otimes \cdots \otimes f_n \rangle .$$

The 2-point function on Minkowski space $\mathbb{M}$ is then, due to the required covariance property, Poincaré invariant and must therefore be the usual vacuum 2-point function $\omega_\mathbb{M}^2(x, y) = \Delta_+(x - y)$. On the cylinder spacetime $\text{Cyl} = \mathbb{R} \times T^3$, on the other hand, covariance requires that the 2-point function is invariant under space and time translations,

$$\omega_\text{Cyl}^2(x, y) = G(x - y) ,$$

where $G$ is a distribution on Minkowski space which is periodic with period $2\pi$ in all spatial coordinates. If $G$ coincided on subregions which are isometrical to subregions of Minkowski space with $\Delta_+$ it had to be in the time variable the boundary value of an analytic function on the upper halfplane, hence it must be a ground state 2 point function of the cylinder spacetime, which in the massive case is unique and is given by

$$\Delta_\text{Cyl}^+ (x) = \sum_{n \in \mathbb{Z}^3} \Delta_+ (x + 2\pi (0, n))$$

and thus does not coincide with $\Delta_+$ in an open region. In the massless case a ground state on the cylinder spacetime does not exist because of zero modes.

We conclude that no locally covariant Hadamard state of the free Klein-Gordon field exists. This proves the nonexistence of a vacuum state on generic curved spacetimes mentioned in the previous section.

Also the familiar concept of a particle has no obvious locally covariant formulation. Therefore the standard interpretation of field theory in terms of scattering of particles does not apply to generic curved spacetimes. We may use, however, the existence of other locally covariant fields.
In the case of the Klein-Gordon field, one may try to define in addition to the free field itself other members of the Borchers class. In [13] Wick polynomials of the free field were defined as operator valued distributions. Their definition depended on the choice of a specific Hadamard state. In the case of the Wick square the definition was

\[ \varphi^2_\omega(x) = \lim_{y \to x} (\varphi(x) \varphi(y) - \omega_2(x, y)) . \]

Two such Wick squares with respect to different Hadamard states \( \omega, \omega' \) differ by a smooth function \( H_{\omega, \omega'} \) which satisfies the covariance condition

\[ H_{\omega \circ \alpha \psi, \omega' \circ \alpha \psi}(x) = H_{\omega, \omega'}(\psi(x)) \]

and the cocycle condition

\[ H_{\omega, \omega'} + H_{\omega', \omega''} + H_{\omega'', \omega} = 0 . \]

The problem is now to find functions \( h_\omega \) which transform covariantly

\[ h_{\omega \circ \alpha \psi}(x) = h_\omega(\psi(x)) \]

and trivialize the cocycle condition,

\[ h_\omega - h_{\omega'} = H_{\omega, \omega'} . \]

Then the definition

\[ \varphi^2 = \varphi^2_\omega - h_\omega \]

gives a locally covariant field.

A solution was given by Hollands and Wald [6] in terms of the construction of Hadamard states. A Hadamard state has a 2-point function of the form

\[ \omega_2 = \frac{u}{\sigma} + v \ln \sigma + w \]

with smooth functions \( u, v, w \) where \( u \) and \( v \) are determined by local geometry and \( \sigma \) is the square of the geodesic distance between the two arguments. One then can choose

\[ h_\omega(x) = w(x, x) . \]
4 Renormalized perturbation theory

Renormalized perturbation theory can be formulated as the construction of the local S-matrix as a formal power series of time ordered products

\[ S(gL) = \sum \frac{i^n}{n!} \int dx_1 \cdots dx_n g(x_1) \cdots g(x_n) T L(x_1) \cdots L(x_n) \]

where \( L \) is the Lagrangian and \( g \) is a test function with compact support. The time ordered products are operator valued distributions which are well defined on noncoinciding points. In this approach which is due to Stückelberg [14], Bogoliubov [15], Epstein and Glaser [16], the ultraviolet divergences show up as ambiguities in the extension to coinciding points. The local S-matrices can then be used for a construction of local algebras of observables, see e.g. [17].

The construction can be generalized to curved spacetimes by methods of microlocal analysis [3]. In the case of Minkowski space the ambiguities are restricted by the requirement of Poincaré invariance; as shown by Hollands and Wald [6] a corresponding restriction in the general covariant situation can be obtained by imposing the condition of local covariance.

Namely, if \( \psi \) is a morphism from \( \mathcal{M} \) to \( \mathcal{N} \), then one requires

\[ \alpha_\psi(S(gL_M)) = S(\psi^*gL_N) . \]

This condition fixes the ambiguities up to terms which depend locally on the metric. By additional assumptions concerning the scaling behaviour, continuity and analyticity, Hollands and Wald were able to show that the renormalization freedom on generic curved spacetimes is up to a possible coupling to curvature terms the same as in Minkowski space.

5 Spin-statistics, PCT, and all that

The principle of local covariance also admits a formulation of important structural theorems of quantum field theory [18]. An example is the connection between spin and statistics. The proof on Minkowski space heavily relies on the structure of the Poincaré group, so at first sight a generalization to curved spacetimes seems to be hopeless. But actually Verch [7] was able to use the covariance principle to relate the
statistics on a generic spacetime to that on Minkowski spacetime and thus obtained an elegant proof of a general spin statistics theorem.

Another famous theorem of general quantum field theory is the PCT theorem. Since it involves elements of the Poincaré group even the formulation of the theorem on generic curved spacetimes is unclear. But in the locally covariant framework one may associate to every time oriented spacetime $\mathcal{M} \in \text{Obj}(\text{Man})$ the manifold $\overline{\mathcal{M}}$ by reversing the time orientation. One then defines the PCT transformed functor $\overline{\mathfrak{A}}$ by

$$
\overline{\mathfrak{A}}(\mathcal{M}) = \overline{\mathfrak{A}(\overline{\mathcal{M}})}
$$

where for an algebra over the complex numbers the overlining means complex conjugation.

The statement of PCT invariance would then be that the functors $\mathfrak{A}$ and $\overline{\mathfrak{A}}$ are naturally equivalent.

A first result in this direction was obtained by Hollands [8]. He could show that at least the operator product expansions (provided they exist) in both theories are equivalent.

6 Outlook

We have seen that the standard interpretation of quantum field theory in terms of scattering of particles is not meaningful on generic spacetimes. Instead one should interpret the theory in terms of locally covariant fields. As an example one may use the Buchholz-Ojima-Roos theory of local equilibrium states [19]. One selects a subspace $V$ of locally covariant fields and compares a given state $\omega$ on a given spacetime with convex combinations of homogeneous KMS states on Minkowski space. If at a point $x$ the expectation values of fields $A \in V$ can be described by a probability measure on the set of homogeneous KMS states $\sigma$,

$$
\omega(A_{\mathcal{M}}(x)) = \int d\mu(\sigma) \sigma(A_{\mathcal{M}}(0)) , A \in V ,
$$

this measure can be interpreted in terms of local thermodynamic properties of $\omega$ at the point $x$, as measured by $V$. In particular this provides a definition of temperature in nonstationary spacetimes.

The proposed framework also allows the formulation of the action of compactly supported diffeomorphisms. Let $\beta_{\mathcal{M},g}$ be the relative Cauchy evolution induced by a change of the metric of $\mathcal{M}$ as defined
in Sect.3. If $g$ arises from the application of a compactly supported
diffeomorphism $\phi$, $g = \phi_\ast g_M$, then by the general covariance of the
formalism we find

$$\beta_{M,\phi_\ast g_M} = \text{id}$$

One may also introduce quantities which transform covariantly un-
der diffeomorphisms. An example are the retarded fields which were
introduced by Hollands and Wald \[6\]. They are given by

$$A_{M,g}^{\text{ret}}(f) = \alpha_{\psi} \alpha_{\psi_g}^{-1}(A_{M,g}(f)) .$$

where $\psi, \psi_g$ denote appropriate embeddings of another spacetime onto
a neighbourhood of a Cauchy surface which lies sufficiently far in the
past such that $g$ differs from $g_M$ only in its future. If now the met-
ic is changed by a compactly supported diffeomorphism $\phi$, then the
retarded field transforms as

$$A_{M,\phi_\ast g_M}^{\text{ret}}(f) = A_{M}(\phi_\ast f) .$$

One may ask whether the covariant framework which was pre-
sented here also allows a generally covariant perturbative approach to
the quantization of gravity. This in fact possible. Namely, one has to
quantize the theory on all spacetimes simultaneously. The fact that
the metric is dynamical means that the split into background metric
and quantized fluctuations is arbitrary. This can be expressed by the
postulate that the relative Cauchy evolution $\beta_g$ is the identity for all
metrics $g \in \mathcal{L}(\mathcal{M})$. A corresponding consideration applies to other
background fields, see e.g. \[20\].

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