The Bifurcation Analysis and Persistence of the Food Chain Ecological Model with Toxicant

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Abstract. In this work, the occurrence conditions of both local Bifurcation and persistence were studied. Saddle-node bifurcation appears near fourth point, near the first point, the second point and the third point a transcritical bifurcation occurred but no pitchfork bifurcation happened near any of the four equilibrium points. In addition to study conditions for Hopf-bifurcation near positive stable point that is the fourth point. Besides discuss persistence occurrence as globally property of the food chain of three species include prey, first predator and top predator with impact of toxin in all species and harvesting effect on the predator's only. Numerical results for the set of hypothetical of parameters come to confirm our analytical results about illustration the occurrence of local bifurcation and persistence of this model.

1. Introduction

The terrestrial ecological system in which we live is truly integrated and interconnected in a striking way, in which the forms of interdependence of all these neighborhoods that live in it are manifested, so the predator and prey model is necessity to study these systems are worthy of attention, including the factors under study, whether they are external or factors related to the structure of the systems Biological. Many researchers have taken an interest in this topic and have made studies on it [1, 2, 3, 4]. The effect of toxins according to their external sources and the pollution they are exposed to that threaten the neighborhoods on the ground or toxins that these creatures use to defend their existence or for the sake of hunting. They have different effects on environmental systems, clearly the extent to which they have had an impact on feeding and growth behaviors in these ecosystems see [5, 6, 7].

Many of factors that affect the system like (shelter, harvesting, toxicity and others). The varying of any parameter within the system, leads to the clarity of a complex behavior that in turn causes the instability of that intended system resulting in what is called bifurcation is a noticeable qualitative change in the behavior of the dynamic system.

Recently the bifurcation theorem had a direct and significant effect on the use new ideas and methods in the theorem of dynamic systems, the local bifurcation and Hopf- bifurcation of proposed systems that include various effects, each according to the model, see [8, 9, 10, 11 ].

Investigating the conditions of persistence in the dynamic system gives biologists a long-term awareness of the persistence of this form of different ecosystem, which contributes in a scientific way to understand the long term survival of all populations and determine precisely whether positive
verify that for any non-hyperbolic property of the equilibrium point is the necessary but not sufficient condition for the bifurcation to happen. Sotomayor's theorem [17] is used here to discuss the local bifurcation analysis of systems; see [12, 13, 14, 15].

2. Model formulation [16]

The following model:
\[
\begin{align*}
\frac{dN_1}{dt} &= rN_1 \left(1 - \frac{N_1}{k}\right) - a_1N_1N_2 - b_1N_1^2 - d_1N_1, \\
\frac{dN_2}{dt} &= e_1a_1N_1N_2 - b_2N_2^2 - a_2N_2N_3 - d_2N_2 - h_1N_2, \\
\frac{dN_3}{dt} &= e_2a_2N_3N_2 - b_3N_3^2 - d_3N_3 - h_2N_3.
\end{align*}
\]

(1)

The variables and parameters illustrated in table (1):

| Variable                  | Description                                                                 |
|---------------------------|------------------------------------------------------------------------------|
| \(N_1(T)\)               | The prey population size at time \(T\)                                      |
| \(N_2(T)\)               | The first predator population size at time \(T\)                            |
| \(N_3(T)\)               | The second predator population size at the time \(T\)                       |
| \(r > 0\)                | The growth rate of prey                                                     |
| \(k > 0\)                | The carrying capacity                                                       |
| \(\alpha_i > 0, i=1,2\). | The predation rates of the first and second predator respectively           |
| \(b_i > 0, i=1,2,3\)     | The toxin rates of all species respectively                                 |
| \(0 < e_i < 1, i=1,2\)   | The rates of conversion of food to first predator and second predator       |
| \(h_i > 0, i=1,2\)       | The rates of harvesting on predators respectively                           |
| \(d_i > 0, i=1,2,3\)     | The death rates of all species in absence of food respectively              |

Table 1. The parameters and positive variables by appearing in the mathematical model.

System (1) dimensionized in the following system:
\[
\begin{align*}
\frac{dx}{dt} &= x [(1 - x) - u_{1y} - u_{2xy} - u_{3z}], \\
\frac{dy}{dt} &= y [u_{4x} - u_{5xy} - u_{6z} - (u_7 + u_8)], \\
\frac{dz}{dt} &= z [u_{9y} - u_{10y} - (u_{11} + u_{12})].
\end{align*}
\]

(2)

Where,
\[
\begin{align*}
u_1 &= \frac{ka_1}{r}, u_2 = \frac{k^2b_1}{r}, u_3 = \frac{d_1}{r}, u_4 = \frac{e_1a_1k}{r}, u_5 = \frac{k^2b_2}{r}, u_6 = \frac{ka_2}{r}, u_7 = \frac{d_2}{r}, \\
u_8 &= \frac{h_1}{r}, u_9 = \frac{e_2k\alpha}{r}, u_{10} = \frac{k^2b_3}{r}, u_{11} = \frac{d_3}{r}, u_{12} = \frac{h_2}{r}.
\end{align*}
\]

3. The Local bifurcation analysis

In this section, since the nonhyperbolic property of the equilibrium point is the necessary but not sufficient condition for the bifurcation to happen. Sotomayor's theorem [17] is used here to discuss the local bifurcation of the system (2). The Jacobian matrix which is given in [16]:
\[
J = [a_{ij}]_{3 \times 3'}
\]

(3)

where, \(a_{11} = 1 - 2x - u_{1y} - 2u_{2xy} - u_{3z}, a_{12} = -u_{1x} - u_{2x}^2, a_{13} = 0, a_{21} = u_{4y} - u_{5xy}^2, a_{22} = u_{4x} - 2u_{5xy} - u_{6z}^2 - (u_7 + u_8), a_{23} = -u_{6y}, a_{31} = 0, a_{32} = u_{8z} - u_{10z}^2, a_{33} = u_{9y} - 2u_{10yz} - (u_{11} + u_{12}).\)

Verify that for any non-zero vector \(V = (v_1, v_2, v_3)^T\) we have:
\[ D^2 f_u(X, \mu)(V,V) = \begin{bmatrix} b_{ij} \end{bmatrix}_{3 \times 1} \]  

where, 
\[ b_{11} = -2[(1 + u_2 y)v_1^2 + (u_1 + 2u_2 x)v_1 v_2], \]
\[ b_{21} = -2[(u_1 x)v_1^2 - (u_4 - 2u_5 y)v_1 v_2 + (u_6)v_2 v_3], \]
\[ b_{31} = -2[(u_1 x y)v_1^2 - (u_9 - 2u_{10} x)v_2 v_3], \]

\[ D^3 f_u(X, \mu)(V,V,V) = \begin{bmatrix} c_{ijl} \end{bmatrix}_{3 \times 3} \]  

where, 
\[ c_{11} = -6u_2 v_1^2 v_2, \]
\[ c_{21} = -6u_5 v_1 v_2^2, \]
\[ c_{31} = -6u_{10} v_1 v_2^2. \]

**Theorem 1:** System (2) near the equilibrium point \( P_0 \) has a transcritical bifurcation at the parameter \( u_3^0 = u_3 = 1 \), but neither saddle-bifurcation nor pitchfork bifurcation can occurred at \( P_0 \).

**Proof:** By substituting \( P_0 = (0,0,0) \) with \( u_3 = u_3^0 = 1 \) in the Jacobian matrix given in equation (3), the characteristic equation of \( J_0 \), has zero eigenvalue \( \lambda_{0x} = 0 \). 

Let, \( \tilde{\Psi}^{[0]} = \left( \tilde{\Psi}_1^{[0]}, \tilde{\Psi}_2^{[0]}, \tilde{\Psi}_3^{[0]} \right)^T \) be the eigenvector of \( J_0 \) related to the eigenvalue \( \lambda_{0x} = 0 \) Thus: \( (J_0 - \lambda_{0x} I)\tilde{\Psi}^{[0]} = 0 \), where: \( J_0 = J(P_0, u_3^0) \).

Then: \( (J_0 - \lambda_{0x} I)^T \tilde{\Psi}^{[0]} = 0 \). Give us \( \tilde{\Psi}^{[0]} = \left( \tilde{\Psi}_1^{[0]}, 0, 0 \right)^T \) and \( \tilde{\Psi}_1^{[0]} \) any nonzero real number.

Let, \( \Psi^{[0]} = \left( \Psi_1^{[0]}, \Psi_2^{[0]}, \Psi_3^{[0]} \right)^T \) be the eigenvector of \( J_0^T \) related to \( \lambda_{0x} = 0 \), of the matrix \( J_0^T \). 

Then by Sotomayor theorem, the occurrence of the saddle-node bifurcation cannot be occurring at \( P_0 \). Moreover,

since, \( D f_{u_3}(X, u_3) = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \),

with the vector \( X = (x, y, z)^T \), \( D f_{u_3}(X, u_3) \) represent derivative of \( f_{u_3}(X, u_3) \).

And then we have:

\[ D f_{u_3}(P_0, u_3^0)\tilde{\Psi}^{[0]} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} \tilde{\Psi}_1^{[0]} \\ \tilde{\Psi}_2^{[0]} \\ \tilde{\Psi}_3^{[0]} \end{bmatrix} = -\begin{bmatrix} \tilde{\Psi}_1^{[0]} \\ \tilde{\Psi}_2^{[0]} \\ \tilde{\Psi}_3^{[0]} \end{bmatrix}, \]

so, we obtain that:

\[ (\tilde{\Psi}^{[0]})^T [D f_{u_3}(P_0, u_3^0)\tilde{\Psi}^{[0]}] = -\tilde{\Psi}_1^{[0]}\tilde{\Psi}_1^{[0]} \neq 0, \]

By using \( \tilde{\Psi}^{[0]} \) in equation (4), we get:

\[ D^2 f_{u_3}(P_0, u_3^0)(\tilde{\Psi}^{[0]}, \tilde{\Psi}^{[0]}) = \begin{bmatrix} -2(\tilde{\Psi}_1^{[0]})^2 \\ 0 \\ 0 \end{bmatrix}, \]

Also \( (\tilde{\Psi}^{[0]})^T D^2 f_{u_3}(P_0, u_3^0)(\tilde{\Psi}^{[0]}, \tilde{\Psi}^{[0]}) = -2(\tilde{\Psi}_1^{[0]})^2 \tilde{\Psi}_1^{[0]} \neq 0. \)

By Sotomayor theorem, the system (2) near \( P_0 \) with \( u_3 = u_3^0 \) possesses a transcritical bifurcation and pitchfork bifurcation cannot occurred at \( P_0 \).

**Theorem 2:** Assume that the stability condition (3a) as in [16] hold. Then system (2) near \( P_1 = (\bar{x}, 0,0) \) possesses a transcritical bifurcation at the parameter value \( \bar{u}_4 = u_4 = \frac{u_4 + \mu u_3}{1 - u_3} \), but neither saddle-node bifurcation nor pitchfork bifurcation can occurred at \( P_1 \).
Proof: By substituting $P_1 = (̄x, 0, 0)$ with $(u_4 = ̄u_4)$ in the Jacobian matrix given in equation (3), the characteristic equation of $f_1$, where $f_1 = f_1(P_1, ̄u_4)$ has zero eigenvalue ($\lambda_{1y} = 0$).

Let, $\bar{v}[1] = (\bar{v}_{1}\[1], \bar{v}_{2}\[1], \bar{v}_{3}\[1] )^T$ be the eigenvector of $\bar{f}_1$ related to the eigenvalue $\lambda_{1y} = 0$ thus, $(\bar{f}_1 - \lambda_{1y} I) V[1] = 0$, which gives:

$\bar{v}_{1}\[1] = \lambda \bar{v}_{2}\[1]$, $\bar{v}_{3}\[1] = 0$, where $\bar{S} = \dfrac{-\bar{u}_{12}}{\bar{u}_{11}} < 0$ and $\bar{v}[1]$ any non zero real number.

Let, $\bar{V}[1] = (\bar{u}[1], \bar{v}[1], \bar{\psi}[1])^T$ be the eigenvector of $\bar{f}_1$ related to $\lambda_{1y} = 0$, of the matrix $\bar{f}_1$ then:

$(\bar{f}_1 - \lambda_{1y} I) \bar{V}[1] = 0$, Give us $\bar{V}[1] = (0, \bar{v}_{2}\[1], 0)^T$ where $\bar{v}_{2}\[1]$ any nonzero real number.

Since, $\dfrac{\partial \bar{f}_1}{\partial u_4} = f_{u_4}(X, u_4) = (\dfrac{\partial f_1}{\partial u_4}, \dfrac{\partial f_2}{\partial u_4}) = (0, xy, 0)^T$.

hence $f_{u_4}(P_1, ̄u_4) = (0,0,0)^T$, therefore, $(\bar{V}[1])^T f_{u_4}(P_1, ̄u_4) = 0$.

Then by Sotomayor theorem, the saddle-node bifurcation cannot be occurring at $P_1$. Moreover, since $Df_{u_4}(X, u_4) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 - u_3 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ with the vector $X = (x, y, z)^T$, $Df_{u_4}(X, u_4)$ represent derivative of $f_{u_4}(X, u_4)$.

And then we have:

$Df_{u_4}(P_1, ̄u_4)V[1] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 - u_3 & 0 \\ 0 & 0 & 0 \end{bmatrix} \bar{S} \bar{v}_{2}\[1] = (1 - u_3) \bar{v}_{2}\[1]$.

Then, by condition (3a) $(\bar{V}[1])^T \begin{bmatrix} Df_{u_4}(P_1, ̄u_4)V[1] \\ 0 \end{bmatrix} = (1 - u_3) \bar{v}_{2}\[1] \bar{v}_{2}\[1] \neq 0$.

By using $V[1]$ in equation (4), we get:

$D^2f_{u_4}(P_1, ̄u_4)(V[1], V[1]) = \begin{bmatrix} \bar{S} \bar{v}_{2}\[1] \bar{S} \bar{v}_{2}\[1] + (u_1 + 2u_2(1 - u_3)) \bar{S} \\ -2(\bar{v}_{2}\[1])^2(u_5(1 - u_3) - ̄u_{4}\bar{S}) \\ 0 \end{bmatrix}$.

So, $(\bar{V}[1])^T D^2f_{u_4}(P_1, ̄u_4)(V[1], V[1]) = -2(\bar{v}_{2}\[1])^2(\bar{v}_{2}\[1])^2(u_5(1 - u_3) - ̄u_{4}\bar{S})$.

Thus, by condition (3a) $(\bar{V}[1])^T D^2f_{u_4}(P_1, ̄u_4)(V[1], V[1]) \neq 0$.

Therefore, by Sotomayor theorem, the system (2) near $P_1$ with $u_4 = ̄u_4$ possesses a transcritical bifurcation, pitchfork bifurcation cannot occurs at $P_1$.

**Theorem 3:** Assume that the local stability conditions (4b), (9b) and (9c) as in [16] and the following condition hold.

$\gamma < \dfrac{u_{12}}{u_{11}}$. \hspace{2cm} (6)

Then system (2) near $P_2 = (̄x, ̄y, 0)$ possesses a transcritical bifurcation at the parameter value $(̄u_0 = u_0 = ̄u_{11} + \dfrac{u_{12}}{γ})$, but neither saddle-node bifurcation nor pitchfork bifurcation occurs at $P_2$.

**Proof:** By substituting $P_2 = (̄x, ̄y, 0)$ with $u_0 = ̄u_0$ in the Jacobian matrix given in equation (3), the characteristic equation of $f_2$, where $f_2 = f_2(P_2, ̄u_0)$ has zero eigenvalue ($\lambda_{2z} = 0$).

Let, $\bar{v}[2] = (\bar{v}_{1}\[2], \bar{v}_{2}\[2], \bar{v}_{3}\[2] )^T$ be the eigenvector of $f_2$ corresponding to the eigenvalue $\lambda_{2z} = 0$ Thus, $(f_2 - \lambda_{2z} I) V[2] = 0$, which gives:

$\bar{v}_{1}\[2] = γ_1\bar{v}_{3}\[2]$, $\bar{v}_{2}\[2] = γ_2\bar{v}_{3}\[2]$, and $\bar{v}_{3}\[2] any non zero real number$.

where $\gamma_1 = \dfrac{a_{12}a_{22} - a_{11}a_{22}}{a_{11}a_{22} - a_{12}a_{21}}$ and $\gamma_2 = \dfrac{a_{11}a_{22} - a_{12}a_{21}}{a_{11}a_{22} - a_{12}a_{21}}$.

So, according to the local stability conditions (4b), (9b), (9c) in [16] and condition (6) we get, $γ_1 > 0$ and $γ_2 < 0$. 


Let, $\hat{\Psi}_0^{[2]} = \begin{pmatrix} \tilde{\psi}_1^{[2]} & \tilde{\psi}_2^{[2]} & \tilde{\psi}_3^{[2]} \end{pmatrix}^T$ be the eigenvector of $\hat{f}_2^T$ related to $\lambda_{2x} = 0$, of the matrix $\hat{f}_2^T$ then:

$$ \left( \hat{f}_2^T - \lambda_{2x} I \right) \hat{\Psi}_0^{[2]} = 0, $$

Give us $\hat{\Psi}_0^{[2]} = \begin{pmatrix} 0,0,0 \end{pmatrix}^T$ where $\lambda_{2x}$ is any nonzero real number.

since, $\frac{\partial f}{\partial u_0} = f_{u_0}(X, u_0) = \begin{pmatrix} \frac{\partial f_1}{\partial u_0} & \frac{\partial f_2}{\partial u_0} & \frac{\partial f_3}{\partial u_0} \end{pmatrix} = \begin{pmatrix} 0,0,yz \end{pmatrix}^T$.

hence, $f_{u_0}(P_2, \tilde{u}_0) = (0,0,0)^T$. Therefore $\left( \hat{\Psi}_0^{[2]} \right)^T f_{u_0}(P_2, \tilde{u}_0) = 0$.

Then by Sotomayor theorem, the occurrence of the saddle-node bifurcation cannot be at $P_2$. Moreover,

since, $Df_{u_0}(X, \tilde{u}_0) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ x & y & z \end{pmatrix}$.

$Df_{u_0}(P_2, \tilde{u}_0)\hat{\Psi}_0^{[2]} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \hat{\Psi}_0^{[2]} \end{pmatrix} = \begin{pmatrix} 0 & 0 & \hat{\Psi}_0^{[2]} \end{pmatrix}$.

so, $\left( \hat{\Psi}_0^{[2]} \right)^T Df_{u_0}(P_2, \tilde{u}_0)\hat{\Psi}_0^{[2]} = \hat{\Psi}_0^{[2]}$.

By using $\hat{\Psi}_0^{[2]}$ in equation (4), we get:

$$ D^2f_{u_0}(P_2, \tilde{u}_0) = \begin{pmatrix} -2(\hat{\Psi}_0^{[2]})^2 [(1 + u_2 \bar{y}) \bar{y}^2 - (u_4 + 2u_2 \bar{x})\bar{y}y_2] \\ -2(\hat{\Psi}_0^{[2]})^2 [u_5 \bar{x} \bar{y}^2 - (u_4 + 2u_5 \bar{y})\bar{y}y_2 + u_6y_2] \\ -2(\hat{\Psi}_0^{[2]})^2 [u_{10}\bar{y} - u_9 \bar{y}_2] \end{pmatrix} $$

So, $\left( \hat{\Psi}_0^{[2]} \right)^T D^2f_{u_0}(P_2, \tilde{u}_0)(\hat{\Psi}_0^{[2]}, \hat{\Psi}_0^{[2]}) = -2(\hat{\Psi}_0^{[2]})^2 [u_{10}\bar{y} - u_9 \bar{y}_2]$. Hence, $\left( \hat{\Psi}_0^{[2]} \right)^T D^2f_{u_0}(P_2, \tilde{u}_0)(\hat{\Psi}_0^{[2]}, \hat{\Psi}_0^{[2]}) \neq 0$.

Therefore, by Sotomayor theorem system (2) near $P_2$ with $u_0 = \tilde{u}_0$ possesses a transcritical bifurcation, but the pitchfork bifurcation cannot occurs at $P_2$.

**Theorem 4:** Assume that the local stability conditions (10a–10c) as in [16] with the following conditions hold:

$\bar{y} < \frac{u_4}{u_6}$,  \hspace{1cm} (7)

$\tilde{u}_0 > \frac{2\bar{u}_0 \bar{z}}{\bar{x}}$.  \hspace{1cm} (8)

$-\bar{\theta}_2[\tilde{u}_0 - 2u_{10} \bar{z}] - (u_1 + 2u_2 \bar{x})\theta_1 \theta_3 - u_6] < (1 + u_2 \bar{y})\bar{y}^2 \theta_2^2 \theta_3 + u_5 \bar{x} \bar{y}^2 \bar{z} \theta_4 - (u_4 + 2u_5 \bar{y})\theta_1 \theta_2 \theta_4 + u_{10} \bar{y}$.  \hspace{1cm} (9)

Where,

$\theta_1 = \frac{\bar{a}_{12}}{\bar{a}_{11} + \bar{a}_{12}} > 0$, \hspace{1cm} $\theta_2 = \frac{-\bar{a}_{11} \bar{a}_{23}}{\bar{a}_{11} \bar{a}_{22} - \bar{a}_{12} \bar{a}_{21}} < 0$, \hspace{1cm} $\theta_3 = \frac{-\bar{a}_{21} \bar{a}_{32}}{\bar{a}_{11} \bar{a}_{22} - \bar{a}_{12} \bar{a}_{21}} > 0$, \hspace{1cm} $\theta_4 = \frac{-\bar{a}_{11} \bar{a}_{32}}{\bar{a}_{11} \bar{a}_{22} - \bar{a}_{12} \bar{a}_{21}} > 0$.

Then the system (2) near $P_3 = (\bar{x}, \bar{y}, \bar{z})$, has only a saddle-node bifurcation at the parameter $\tilde{u}_0 = \frac{\bar{a}_{11} \bar{a}_{22} - \bar{a}_{12} \bar{a}_{21}}{\bar{a}_{11} \bar{a}_{22} + \bar{a}_{12} \bar{a}_{21}}$.

**Proof:** By substituting $P_3 = (\bar{x}, \bar{y}, \bar{z})$ in the equation (3), then the characteristic equation of $\hat{f}_3$, where $\hat{f}_3 = J(P_3, \tilde{u}_0)$ which is given in [16] having zero eigenvalues ($\lambda_{2x} = 0$), if and only if $M_3 = 0$ and then $P_3$ becomes a non-hyperbolic, whenever the parameter takes the value $u_0 = \tilde{u}_0$.

At $P_3$, jacobian matrix of system (2) become: $\hat{f}_3 = J(P_3, \tilde{u}_0) = [\bar{a}_{ij}]_{3 \times 3}$, where, $\bar{a}_{ij} = \bar{a}_{ij}$ for all $i, j = 1, 2, 3$ when $\tilde{u}_0$ in [16] except $\bar{a}_{33} = \frac{\bar{a}_{11} \bar{a}_{22} - \bar{a}_{12} \bar{a}_{21}}{\bar{a}_{11} \bar{a}_{22} - \bar{a}_{12} \bar{a}_{21}}$. Note that; $\tilde{u}_0 > 0$ provided that locally stability conditions (10a–10c) as in [16] and condition (7) hold.
Let, $\vec{\psi}^{[3]} = (\vec{\psi}_1^{[3]}, \vec{\psi}_2^{[3]}, \vec{\psi}_3^{[3]})^T$ be the eigenvector of $\vec{f}_3$ related to the eigenvalue $\lambda_{3x} = 0$. Thus, $(\vec{f}_3 - \lambda_{3x}I)V^{[3]} = 0$, which gives:

$\vec{\psi}_1^{[3]} = \theta_1 \vec{\psi}_3^{[3]}, \vec{\psi}_2^{[3]} = \theta_2 \vec{\psi}_3^{[3]}, \vec{\psi}_3^{[3]} = -\vec{\psi}_3^{[3]},$ where $\vec{v}_3^{[3]}$ any non-zero real number.

And $\theta_1, \theta_2$ are associated to the state of theorem.

Let, $\vec{\psi}^{[3]} = (\vec{\psi}_1^{[3]}, \vec{\psi}_2^{[3]}, \vec{\psi}_3^{[3]})^T$ be the eigenvector related of $\vec{f}_2^T$ to $(\lambda_{3x} = 0)$, of matrix $\vec{f}_2^T$ then:

$(\vec{f}_2^T - \lambda_{3x}I)\vec{\psi}^{[3]} = 0$, Give us: $\vec{\psi}^{[3]}$ = where $\vec{\psi}^{[3]}$ any non-zero real number.

And $\theta_3, \theta_4$ which are associated to the state of theorem. since, $\frac{\partial f}{\partial u_0} = f_{u_0}(X, u_0) = (0, 0, 0, 0, 0)$.

hence $f_{u_0}(P_3, \bar{u}_0) = (0, 0, 0, 0, 0)^T$, therefore $(\vec{\psi}^{[3]})^T f_{u_0}(P_3, \bar{u}_0) = 2\vec{\psi}_3^{[3]} \neq 0$.

Substituting $V^{[3]}$ in equation (4) obtains that:

$$D^2 f_{u_0}(P_3, \bar{u}_0)(V^{[3]}, V^{[3]}) = \begin{bmatrix}
-2(\vec{\psi}_3^{[3]})^2[(1 + u_2 \bar{y})\theta_3^2 - (u_1 + 2u_2 \bar{x})\theta_1 \theta_2] \\
-2(\vec{\psi}_3^{[3]})^2[u_5 \bar{x} \theta_2^2 - (u_4 - 2u_5 \bar{y})\theta_1 \theta_2 - u_6 \theta_2] \\
-2(\vec{\psi}_3^{[3]})^2[u_{10} \bar{y} + (\bar{u}_9 - 2u_{10} \bar{x})\theta_2]
\end{bmatrix}.$$ 

So, $(\vec{\psi}^{[3]})^T D^2 f_{u_0}(P_3, \bar{u}_0)(\vec{\psi}^{[3]}, \vec{\psi}^{[3]}) = -2(\vec{\psi}_3^{[3]})^2 \vec{\psi}_3^{[3]} \vec{W}.$

Where,

$$\vec{W} = (1 + u_2 \bar{y})\theta_3^2 + (u_1 + 2u_2 \bar{x})\theta_1 \theta_2 \theta_3 + u_5 \bar{x} \theta_2^2 \theta_4 - (u_4 - 2u_5 \bar{y})\theta_1 \theta_2 \theta_4 + u_6 \theta_2 \theta_4 + u_{10} \bar{y} + (\bar{u}_9 - 2u_{10} \bar{x}) \theta_2.$$

So, by conditions (7), (8) and (9), we get:

$(\vec{\psi}^{[3]})^T D^2 f_{u_11}(P_3, \bar{u}_11)(\vec{\psi}^{[3]}, \vec{\psi}^{[3]}) \neq 0$.

Therefore, by Sotomayor theorem system (2) near $P_3$ with $u_0 = \bar{u}_0$ system (2) has only saddle-node bifurcation, neither a transcritical bifurcation nor pitchfork bifurcation occurs at $P_3$.

4. The Hopf-bifurcation analysis

In this section, the occurrence of Hopf bifurcation, according to the Haque and Venturino methods [18], near the equilibrium points of system (2) investigated as blow.

**Theorem 5:** Suppose that the locally conditions (10a-10d) as in [16] and the following conditions hold:

$$l > 0. \quad (10)$$

$$lL_1 + M_1 M_2 > L_2 + L_3. \quad (11)$$

$$l(-L_4 + L_6 + L_6) + L_6 > l^2 L_4 + L_4 L_1 - (L_7 + L_8). \quad (12)$$

$$2M_2 L_2 < 2M_2^2 - 2M_1 M_2 \sqrt{M_2} L_4. \quad (13)$$

Where, $\bar{a}_{ij}$ and $M_i$, $(i, j = 1, 2, 3)$ is mentioned in [16]. Then for the parameter value $\bar{u}_0 = \frac{lL_1 + M_1 M_2 - (L_2 + L_3)}{lL_4}$, system (2) has a Hopf- bifurcation at $P_3$. Where,

$$l = u_4 \bar{x} + 2u_2 \bar{x} \bar{y} - u_6 \bar{x} - u_7, L_1 = \bar{a}_{11} \bar{a}_{33} > 0, L_2 = \bar{a}_{11} \bar{a}_{23} \bar{a}_{32} > 0, L_3 = \bar{a}_{12} \bar{a}_{21} \bar{a}_{33} > 0, L_4 = \bar{a}_{11} + \bar{a}_{33} < 0, L_5 = \bar{a}_{23} \bar{a}_{32} < 0, L_6 = \bar{a}_{12} \bar{a}_{21} < 0, L_7 = \bar{a}_{14} \bar{a}_{24} \bar{a}_{33} > 0, L_8 = \bar{a}_{23} \bar{a}_{32} \bar{a}_{33} > 0.$$

**Proof:** The characteristic equation of system (2) at $P_3$ mentioned in local stability in [16].

$$\lambda^3 + M_1 \lambda^2 + M_2 \lambda + M_3 = 0. \quad (14)$$

we need to find the parameter $(\bar{u}_0)$ to verify the necessary and sufficient conditions for Hopf bifurcation to occur that satisfy: $M_i(\bar{u}_0) > 0$, $(i = 1, 2)$ and $\Delta(\bar{u}_0) = M_1 M_2 - M_3 = 0$.

Provided conditions of locally (10a-10d) in [16], $M_i(\bar{u}_0) > 0$, $(i = 1, 2)$
And ($\bar{u}_b > 0$) provided conditions (10) and (11).

It is observed that $\Delta = 0$, gives:

$$B_1\bar{u}_b^2 + B_2\bar{u}_b + B_3 = 0,$$

(15)

where,

$$B_1 = -L_4 > 0,$$

$$B_2 = -2L_4 - (-L_4^2 + L_5 + L_6) > 0,$$

and

$$B_3 = -L_4^2 L_4 + l(-L_4^2 + L_5 + L_6) + (-L_4 L_4 + L_6 + L_5 + L_3).$$

Where: $l, L_4, L_5, L_6, L_7 and L_8$ are mentioned in the state theorem.

By using Descartes rule of sign, equation (15) has a unique positive root($\bar{u}_b$), provided condition (12) hold.

Now, at($u_b = \bar{u}_b$), the characteristic equation (14) can be rewritten as:

$$P_3(\lambda) = (\lambda + M_1)(\lambda^2 + M_2) = 0,$$

(16)

Which; have two roots: $\lambda_1 = -M_1 < 0$ and $\lambda_{2,3} = \pm i\sqrt{M_2}$, observe that at($u_b = \bar{u}_b$), there are two pure imaginary eigenvalues ($\lambda_{2,3}$) and one eigenvalue ($\lambda_1$), which is real and negative.

Now for all values of $u_b$ in the neighborhood of $\bar{u}_b$, the roots in general of the following form: $\lambda_{2,3} = \delta_1(u_b) \pm i\delta_2(u_b)$.

Now, to verify the transversally condition we must prove that:

$$\Psi(\bar{u}_b)\Theta(\bar{u}_b) + \Gamma(\bar{u}_b)\Phi(\bar{u}_b) \neq 0.$$

Note that for $u_b = \bar{u}_b$ we have:

$$\delta_1 = 0 \text{ and } \delta_2 = \sqrt{M_2},$$

substituting the value of $\delta_2$ gives the following simplifications:

$$\Psi(\bar{u}_b) = -2M_2; \quad \Theta(\bar{u}_b) = 2M_1\sqrt{M_2}; \quad \Gamma(\bar{u}_b) = L_4 + M_2 \sqrt{M_2}.$$  

Thus, provided conditions of locally (10a-10d) in [16] with condition (13), give us:

$$\Psi(\bar{u}_b)\Theta(\bar{u}_b) + \Gamma(\bar{u}_b)\Phi(\bar{u}_b) = -2M_2^2 + 2M_1 L_1 + 2M_1 M_2 \sqrt{M_2} L_4 \neq 0.$$  

Therefore system (2) at $P_3$ with the parameter $\bar{u}_b$ has a Hopf-bifurcation.

5. The Persistence analysis

In this section, employing method of average Lyapunov function [15], to study the conditions for the persistence . In general persistence is a global property of a dynamical system, is dependent upon solution behavior near extinction boundaries (boundary planes), but it is not dependence upon interior solution space structure. Biologically, the system is persists means for all future time all populations will survival. However, mathematically it means that strictly positive solutions do not have omega limit set on the boundary of the non-negative cone [14]. But the dynamical system faces extinction, if the dynamical system does not persists then the solution have omega limit set on the boundary of the nonnegative cone. We need to study the global dynamics in the boundary $xy-\text{plane}$ illustrated by theorem.

Theorem6: Suppose that the equilibrium point $P_2 = (\bar{x}, \bar{y}, 0)$ is locally asymptotically stable in the $\text{Int}R^2_+$, then it is a globally asymptotically stable in the $\text{Int}R^2_+$ of $xy-\text{plane}$.  

Proof: System (2) has a subsystem that appears when the top predator absence, take the form:

$$\begin{align*}
\frac{dx}{dt} &= x(1 - x - u_4 y - u_2 xy - u_3) \\
\frac{dy}{dt} &= y(u_4 x - u_5 xy - (u_7 + u_8))
\end{align*}$$

(17)
The subsystem (17), has a unique interior equilibrium point \( P_2 = (\hat{x}, \hat{y}, 0) \), as it is referred in [16] in Int. \( R^3_+ \) of \( xy - \text{plane} \).

Consider \( H(x,y) = \frac{1}{xy} \).

Note that, \( H(x,y) \) is positive for all \((x,y) \in \text{Int.} \ R^2_+ \) and is \( C^1 \) function.

Since, \( \Delta(x,y) = \frac{\partial}{\partial x}(Hf_1) + \frac{\partial}{\partial y}(Hf_2) \)

\[ = \frac{1}{y(\hat{u}_2 + u_0)} \]

hence, \( \Delta(x,y) \) is not identically zero and does not change sign in the \( \text{Int.} \ R^2_+ \) of \( xy - \text{plane} \).

Therefore, by Bendixon-Dulac criterion [17], subsystem (17) has no periodic dynamic in the interior of positive quadrant of \( xy - \text{plane} \). Since, \( P_2 \) is the only positive equilibrium point of the subsystem (17) in the interior of positive quadrant of \( xy - \text{plane} \). So, according to Poincare-Bendixon theorem; \( P_2 \) is globally asymptotically stable in the interior of positive quadrant.

**Theorem 7:** Assume that there are no periodic dynamics of system (2) in the boundary \( xy - \text{plane} \). According to conditions of locally (4b) and (9a) as in [16] and the following condition hold:

\[ \hat{x} > \frac{u_t + u_8}{u_4 - u_9 \hat{y}} \]  

(18)

Then system (2) is uniformly persistent.

**Proof:** Consider the following average Lyapunov function:

\[ \tau(x,y,z) = x^d_1 y^d_2 z^d_3, \]

Such that \( d_1, d_2, d_3 \) are positive constants. Obviously, \( \tau(x,y,z) \) is a nonnegative

\( C^1 \) defined in \( R^3 \). We obtain:

\[ \Omega(x,y,z) = \frac{\tau'(x,y,z)}{\tau(x,y,z)}, \]

\[ \Omega(x,y,z) = d_1[1-x] - y(u_1 + u_2 x) - u_3 + d_2[x(u_4 - u_5 y) - u_6 z - (u_7 + u_9)] + d_3[y(u_9 - u_10 z) - u_{11} + u_{12})]. \]

Now,

- For \( P_0 = (0,0,0) \), we get:

Violate condition (3a) as in [16] therefore we have:

\[ \Omega(P_0) = d_1[1-u_3] - d_2(u_7 + u_9)] - d_3(u_{11} + u_{12})]. \]

By increasing \( d_1 \) to sufficiently large value, imply that \( \Omega(P_0) > 0 \)

- For \( P_1 = (\hat{x},0,0) \), we get:

Since, \( \hat{x} = 1 - u_3 \) hence the first term of \( \Omega(P_1) \) equal zero, therefore

\[ \Omega(P_1) = d_2[u_4 \hat{x} - (u_7 + u_9)] - d_3(u_{11} + u_{12})]. \]

Violate condition of stability (8) as in [16] and by increasing \( d_2 \) to sufficiently large value, imply that \( \Omega(P_1) > 0 \).

- For \( P_2 = (\hat{x}, \hat{y}, 0) \), we get:

Since, \( \hat{y} = -\frac{\hat{x} + u_3}{u_1 + u_2 \hat{x}} \) hence the first term of \( \Omega(P_1) \) equal zero. Therefore, \( \Omega(P_2) = d_2[\hat{x}(u_4 - u_5 \hat{y}) - (u_7 + u_9)] + d_3[u_9 \hat{y} - (u_{11} + u_{12})]. \) Provided that conditions (18) also apply conditions (9a) and Violate condition (9c) as in [16], by increasing \( d_2 \) to sufficiently large value, imply that \( \Omega(P_2) > 0 \). Hence, system (2) is uniformly persistent.

6. The Numerical results

The importance of studying numerical simulations, it gives us a good explain about the effect of varying the parameters values of the system (2), also another benefit that our obtained analytical results is confirmed. For the parameters below in equation (19), system (2) has a globally asymptotically stable to positive equilibrium point \( P_3 = (0.880209, 0.0885339, 0.152867) \).

\[ u_1 = 0.8, u_2 = 0.5, u_3 = 0.01, u_4 = 0.3, u_5 = 0.8, u_6 = 0.6, u_7 = 0.01, u_8 = 0.1, u_9 = 0.4, u_{10} = 0.4, u_{11} = 0.01, u_{12} = 0.02 \]

(19)
Therefore, system (2) persists in the form of a globally asymptotically stable positive point $P_3$, in the interior of $R_3^+$. Varying the parameters values $u_i$, $i = 1, 2, 5, 6, 10$, the solution still approaches to a positive equilibrium point $P_3 = (0.880209, 0.0885339, 0.152867)$.

Thus the system (2) still persists, as shown in Figure 1. (a-b) for typical value $u_4 = 0.9$.

![Figure 1](image1.png)

**Figure 1.** For the data given in equation (19) beginning at $(1.3, 0.2, 1.3)$ the trajectory and time series of system (2). (a) The trajectory of system (2) with $u_4 = 0.9$, approaches asymptotically to the positive point $P_3 = (0.8702, 0.0882, 0.1501)$, (b) Time series of the trajectory in (a).

Change the rate of prey death $u_3$, in the range $0.01 \leq u_3 < 0.43$, it’s seen that trajectory of system (2) approaches asymptotically to the positive equilibrium point $P_3$, see Figure 2. (a-b), for typical value of $u_3 = 0.1$. While increasing $u_3$ in the range, $0.43 \leq u_3 < 0.58$, the system (2) loses the persistence and the solution of system (2) approaches asymptotically to $P_2$, see Figure 2. (c-d), for typical value of $u_3 = 0.45$, where the bifurcation occurred. Moreover, $0.58 \leq u_3 < 1$, loses the persistence and the solution of system (2) approaches asymptotically to $P_1$, as shown Figure 2. (e-f), for idealistic value of $u_3 = 0.77$, where the bifurcation occurred, finally when $u_3 \geq 1$, the system (2) loses the persistence and the solution of system (2) approaches asymptotically to $P_0$, see Figure 2. (g-h), for typical value of $u_3 = 1$, where the bifurcation occurred.

![Figure 2](image2.png)
Figure 2. For the data given in eq. (19) beginning at (1.3, 0.5, 1.3) the trajectory and time series of system (2).

(a) The trajectory of system (2) with $u_3 = 0.1$, approaches asymptotically to the positive point $P_3 = (0.797, 0.085, 0.124)$. (b) Time series of the trajectory in (a). (c) The trajectory of system (2) with $u_3 = 0.45$ approaches asymptotically to the point $P_2 = (0.466, 0.080, 0)$ (d) Time series of the trajectory in (c).

(e) The trajectory of system (2) with $u_3 = 0.77$, which approaches asymptotically to the point $P_1 = (0.23, 0, 0)$. (f) Time series of the trajectory in (e).

Varying harvesting rate of second predator $u_{11}$, in the range $0.01 \leq u_{11} < 0.06$, it's seen that trajectory of system (2) approaches asymptotically to the positive equilibrium point $P_3$, see Figure3 (a-b), for typical value of $u_{11} = 0.01$. While increasing $u_{11}$ in the range, $0.06 \leq u_{11} < 1$, the system (2) loses the persistence and the solution of system (2) approaches asymptotically to $P_2$, see Figure 3 (c-d), for idealistic value of $u_{11} = 0.4$, where the bifurcation occurred.
Figure 3. For the data given in eq. (19) beginning at (1.3, 0.5, 1.3) the trajectory and time series of system (2). (a) The trajectory of system (2) with \( u_{11} = 0.01 \), approaches asymptotically to \( P_3 = (0.88, 0.088, 0.152) \). (b) Time series of the trajectory in (a), (c) the trajectory of system (2) with \( u_{11} = 0.4 \), approaches asymptotically to \( P_2 = (0.76, 0.194, 0) \). (d) Time series of the trajectory in (c).

According to the above, doing the same for the rest parameters of the system (2) The obtaining results can be summarized in the following table:

Table 2: Numerical behavior for bifurcation and persistence of system (2) as varying some parameters keeping the rest fixed as in equation (20).

| Parameters varied in system (2) | Numerical behavior of system (2) | Persistence of system (2) | bifurcation point |
|----------------------------------|----------------------------------|---------------------------|-------------------|
| 0.6 ≤ \( u_1 \) ≤ 1.01         | Approach to \( P_2 \)           | Persists                  |                   |
| 0.5 ≤ \( u_1 \) ≤ 1.99         | Approach to \( P_3 \)           | Persists                  | \( u_3 = 0.43 \)  |
| 0.01 ≤ \( u_1 \) ≤ 0.42        | Approach to \( P_1 \)           | Persists                  | \( u_4 = 0.44 \)  |
| 0.43 ≤ \( u_1 \) ≤ 0.57        | Approach to \( P_2 \)           | Not Persists              | \( u_2 = 0.58 \)  |
| 0.58 ≤ \( u_1 \) ≤ 0.99        | Approach to \( P_2 \)           | Not Persists              | \( u_4 = 1 \)     |
| \( u_2 ≥ 1 \)                   | Approach to \( P_2 \)           | Not Persists              |                   |
| 0.01 ≤ \( u_4 \) ≤ 0.13        | Approach to \( P_3 \)           | Not Persists              | \( u_4 = 0.14 \)  |
| 0.14 ≤ \( u_4 \) ≤ 0.18        | Approach to \( P_2 \)           | Not Persists              |                   |
| 0.19 ≤ \( u_4 \) ≤ 0.3          | Approach to \( P_1 \)           | Persists                  | \( u_4 = 0.19 \)  |
| 0.8 ≤ \( u_4 \) ≤ 1.99         | Approach to \( P_1 \)           | Persists                  |                   |
| 0.6 ≤ \( u_1 \) ≤ 1.99         | Approach to \( P_2 \)           | Persists                  |                   |
| 0.01 ≤ \( u_1 \) ≤ 0.11        | Approach to \( P_3 \)           | Not Persists              | \( u_3 = 0.12 \)  |
| 0.12 ≤ \( u_1 \) ≤ 0.17        | Approach to \( P_2 \)           | Not Persists              | \( u_3 = 0.18 \)  |
| 0.18 ≤ \( u_1 \) ≤ 0.99        | Approach to \( P_1 \)           | Not Persists              |                   |
| 0.1 ≤ \( u_4 \) ≤ 0.2           | Approach to \( P_3 \)           | Persists                  | \( u_4 = 0.21 \)  |
| 0.21 ≤ \( u_4 \) ≤ 0.26        | Approach to \( P_2 \)           | Not Persists              | \( u_4 = 0.27 \)  |
| 0.27 ≤ \( u_4 \) ≤ 0.89        | Approach to \( P_1 \)           | Not Persists              |                   |
| 0.01 ≤ \( u_4 \) ≤ 0.16        | Approach to \( P_3 \)           | Not Persists              | \( u_4 = 0.17 \)  |
| 0.17 ≤ \( u_4 \) ≤ 0.4          | Approach to \( P_2 \)           | Not Persists              |                   |
| 0.4 ≤ \( u_4 \) ≤ 1.99         | Approach to \( P_2 \)           | Persists                  |                   |
| 0.01 ≤ \( u_{11} \) ≤ 0.05     | Approach to \( P_3 \)           | Persists                  | \( u_{11} = 0.06 \)|
| 0.06 ≤ \( u_{11} \) ≤ 0.99     | Approach to \( P_2 \)           | Not Persists              | \( u_{11} = 0.07 \)|
| 0.01 ≤ \( u_{11} \) ≤ 0.06     | Approach to \( P_2 \)           | Persists                  |                   |
| 0.07 ≤ \( u_{11} \) ≤ 0.99     | Approach to \( P_2 \)           | Not Persists              |                   |
7. Conclusion and Discussion

In this work, the occurrence of local bifurcation and persistence are discussed with an appropriate conditions of food chain which contains a prey–first predator–second predator model under effect of toxin on all species and with harvesting on the predators only have been studied, transcritical bifurcation occurrence near $P_0, P_1$ and $P_2$, while a saddle-node bifurcation occurs near $P_3$, on the other hand there is no pitchfork bifurcation near all of these equilibrium points. Further investigations for the Hopf bifurcation near $P_3$ was done. Finally, numerical simulations are used to illustration the occurrence of local bifurcation and persistence of this system. And the following observations are obtained:

1- It is observed that in system (2), whatever varying the parameters $u_i, i = 1, 2, 5, 6$ and $10$, the solution of system (2) still approaches to positive stable point $P_3$ and there parameters don’t have any effect on the dynamical behavior of system (2) for the data given in equation (19) therefore, any type of bifurcation cannot happen but the system still persists.

2- Varying the parameters $u_i, i = 3, 4, 7, 8, 9, 11, 12$, keeping other parameters as in equation (19) which represent the natural death of prey, rate of conversion of food to the first predator, death rate of the first predator, harvesting rate of the first predator, rate of conversion of food to the second predator, death rate of the second predator, harvesting rate of the second predator respectively. It played an important role in dynamics of system (2) in terms of persistence and bifurcation.

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