Recent progress in the theory of two-proton radioactivity and three-body decay

L. V. Grigorenko\textsuperscript{1,2} and M. V. Zhukov\textsuperscript{3}

\textsuperscript{1} Flerov Laboratory of Nuclear Reactions, JINR, RU-141980 Dubna, Russia
\textsuperscript{2} Gesellschaft f"{u}r Schwerionenforschung mbH, Planckstr. 1, D-64291 Darmstadt, Germany
\textsuperscript{3} Fundamental Physics, Chalmers University of Technology, S-41296 G"{o}teborg, Sweden

Abstract. The current situation in the theory of three-body decay and experimental advances are briefly overviewed. The integral formula for decay widths of three-body states is introduced. Theoretical approach with a simplified Hamiltonian allows a semianalytical treatment of the process. The problem of the width of the first excited $3/2^-$ state of $^{17}$Ne is resolved. The qualitative consequences of the model for simplified approaches are discussed. These include a better understanding of the diproton decay approximation and an introduction of a corrected “R-matrix” formula for three-body width.

1. Brief overview

The idea of the “true” two-proton radioactivity was proposed about 50 years ago in a classical paper of Goldansky [1]. The word “true” denotes here that we are dealing not with a relatively simple emission of two protons, which becomes possible in every nucleus above two-proton decay threshold, but with a specific situation where one-proton emission is energetically (due to the proton separation energy in the daughter system) or dynamically (due to various reasons) prohibited. Only simultaneous emission of two protons is possible in that case. The dynamics of such decays can not be reduced to a sequence of two-body decays and from theoretical point of view we have to deal with a three-body Coulomb problem in the continuum, which is known to be very complicated.

Theoretic progress in this field was quite slow. Only recently a consistent quantum mechanical theory of the process was developed [2, 3, 4], which allows to study the two-proton (three-body) decay phenomenon in a three-body cluster model. It has been applied to a range of a light nuclear systems ($^{12}$O, $^{16}$Ne [5], $^{6}$Be, $^{8}$Li\textsuperscript{*}, $^{9}$Be\textsuperscript{*} [6], $^{17}$Ne\textsuperscript{*}, $^{19}$Mg [7]). Systematic exploratory studies of heavier prospective $2p$ emitters $^{30}$Ar, $^{34}$Ca, $^{45}$Fe, $^{48}$Ni, $^{54}$Zn, $^{58}$Ge, $^{62}$Se, and $^{66}$Kr [4, 8] have also been performed providing predictions of lifetime ranges and possible correlations among fragments.

Experimental studies of the two-proton radioactivity is presently an extremely actively developing field. Since the first experimental identification of $2p$ radioactivity in $^{45}$Fe [9, 10] it was also found in $^{54}$Zn [11]. Some fingerprints of the $^{48}$Ni $2p$ decay were observed and the $^{45}$Fe lifetime and the decay energy were measured with improved accuracy [12]. There was an intriguing discovery of the extreme enhancement of the $2p$ decay mode for the high-spin $21^+$ isomer of $^{94}$Ag, interpreted so far only in terms of the hyperdeformation of this state [13]. New experiments, aimed at more detailed $2p$ decay studies (e.g. the observation of correlations), are
under way at GSI, MSU, GANIL, and Jyväskylä. The first results of some of these experiments
have become recently available. This include the discovery of the $^{19}$Mg isotope as $2p$ emitter and
the observation of the transversal $2p$ distributions [14]. The lifetime and distributions observed
in this decay are in a reasonable agreement with the theoretical predictions [7]. The direct proof
of the $2p$ character of the $^{45}$Fe decay was obtained in [15] using the time projection chamber
(TPC). Practically simultaneously the angular and energy correlations in $^{45}$Fe were measured
using the novel optical TPC instrumentation [16]. The more precise $2p$ lifetime value obtained
in the latter experiment and the correlation patterns were found to be consistent within the
predicted in Ref. [4] connection between these observables.

Another, possibly very important, field of application of the two-proton decay studies was
shown in Refs. [17, 18]. It was demonstrated in [17] that the importance of direct resonant two-
proton radiative capture processes was underestimated in earlier treatment of the rp-process
waiting points [19]. The scale of modification of the astrophysical $2p$ resonant radiative capture
rates can be as large as several orders of magnitude in certain temperature ranges. In paper
[18] it has been found that nonresonant E1 contributions to three-body (two-proton) capture
rates can also be much larger than was expected before. The updated $2p$ astrophysical capture rate
for the $^{15}$O($2p$,γ)$^{17}$Ne reaction appears to be competing with the standard $^{15}$O($\alpha$,γ)$^{19}$Ne
breakout reaction for the hot CNO cycle. The improvements of the $2p$ capture rates obtained in
[17, 18] are connected to consistent quantum mechanical treatment of the three-body Coulomb
continuum in contrast to the essentially quasiclassical approach typically used in astrophysical
calculations of three-body capture reactions (e.g. [19, 20]).

The growing quality of the experimental studies of the $2p$ decays and the high precision
required for for certain astrophysical calculations inspired us to revisit the issues connected with
different uncertainties and technical difficulties of our studies. In paper [21] we extended the well
known two-body formalism of the integral formulae for decay width [22, 23] to the three-body case
and formulated a simplified three-body model which has many dynamical features similar
to the realistic case, but allows the exact semianalytical treatment and thus makes possible a
precise calibration of three-body calculations.

Several other theoretical approaches were applied to the problem in the recent years. We
should mention the “diproton” model [24, 25], “R-matrix” approach [26, 27, 28, 29], continuum
shell model [30, 31], and adiabatic hyperspherical approach of [32, 33]. Some issues of a
compatibility between different approaches are addressed in the recent papers [21, 34] and we
are going to discuss some qualitative results of these studies below.

2. Integral formula for $2p$ width for simplified Hamiltonian

From formal point of view the main problem in construction of the $2p$ decay theory is absence
of the Green’s function for the systems with three-body Coulomb interaction. However, a
semianalytical model can be formulated for certain simplified three-body Hamiltonians. One
of three interactions should be neglected in such a Hamiltonian and the others are assumed to
depend only on the Jacobi coordinates $X$ and $Y$:

$$
H = T + V = T + V_3(\rho) + V_x^{\text{coul}}(X) + V_y^{\text{nuc}}(X) + V_y^{\text{coul}}(Y) + V_y^{\text{nuc}}(Y). 
$$ (1)

In this expression the three-body potential $V_3(\rho)$ controls the position of the three-body
resonance. The auxiliary Hamiltonian of the form

$$
\tilde{H} = T + \tilde{V} = T + V_x^{\text{coul}}(X) + V_y^{\text{nuc}}(X) + V_y^{\text{coul}}(Y) + V_y^{\text{nuc}}(Y) 
$$ (2)

allows separate treatment of $X$ and $Y$ variables and has the analytical Green’s function

$$
G_E^{(+)}(XY, X'Y') = \frac{1}{2\pi i} \int_{-\infty}^{\infty} d\varepsilon \ G_E^{(+)}(X, X') \ G_E^{(+)}(Y, Y') 
$$ (3)
The two-body Green’s functions in the expressions above are defined via the eigenfunctions of the subhamiltonians
\[
\begin{align*}
H_x - \varepsilon &= \tilde{T}_x + V_c^{\text{coul}}(X) + V_{nuc}^{\text{nuc}}(X) - \varepsilon \\
H_y - (E - \varepsilon) &= \tilde{T}_y + V_y^{\text{coul}}(Y) + V_{y}^{\text{nuc}}(Y) - (E - \varepsilon)
\end{align*}
\]

Using the auxiliary Green’s function (3) the WF with correct outgoing asymptotic \(\bar{\Psi}^{(+)}(XY)\) can be found in a very good approximation using the eigenfunction of (1) \(\Psi^{(+)}(XY)\) with approximate asymptotic
\[
\bar{\Psi}^{(+)}(XY) = \frac{1}{2\pi i} \int dX'dY' \int_0^\infty d\varepsilon G^{(+)}(\varepsilon) G^{(+)}_{E-\varepsilon}(Y,Y')(\bar{V} - V) \Psi^{(+)}(XY),
\]
and the width is defined via the flux of one of the protons (here that on the \(X\) Jacobi variable) through the sphere of a large radius
\[
\Gamma(E) = \Im \left[ X^2 \int d\Omega_x \int dX \left( \bar{\Psi}^{(+)} \nabla_X \Psi^{(+)} \right) \right] \bigg|_{X \to \infty} = \frac{8}{\pi} \int_0^E d\varepsilon \frac{1}{v_x(\varepsilon)v_y(\varepsilon)} |A(\varepsilon)|^2, \tag{4}
\]
where \(v_x(\varepsilon) = \sqrt{2\varepsilon/M_x}, \ v_y(\varepsilon) = \sqrt{2(E - \varepsilon)/M_y}, \) and
\[
A(\varepsilon) = \int_0^\infty dX' \int_0^\infty dY' \varphi_x(k_x(\varepsilon)X') \varphi_y(k_y(\varepsilon)Y')(\bar{V} - V) \varphi_{\text{Ll},Lp}(X',Y').
\]

The semianalytical model for three-body decay width introduced for simplified Hamiltonian has important features: (i) it explicitly isolates different degrees of freedom in a complex three-body system and thus simplifies the understanding of certain dynamical aspects and (ii) it does not suffer from numerical and basis size convergence issues, which could be important in certain full three-body calculations.

3. Interpretation of the “diproton” model
So far the diproton model has been treated by us as a reliable upper limit for three-body width [8]. With some technical improvements (namely the momentum distribution inside the diproton is assumed) this model was used for the two-proton width calculations in Refs. [26, 27, 28, 29]. Essentially it defines the width by expression
\[
\Gamma_{dp}(E) = \frac{S_{2p}}{M_{\text{red}}r_{ch}^2(dp)} P_{l=0}(E - \varepsilon, r_{ch}(dp), 2Z_{\text{core}}), \tag{5}
\]
where the diproton channel radius \(r_{ch}^2(dp)\) for nucleus with the mass number \(A\) is defined as a “touching radius” for the “diproton particle” and the \((A - 2)\) remnant
\[
r_{ch}^2(dp) = \tau_0 \left[ (A - 2)^{1/3} + 2^{1/3} \right] = 1.45 \left[ (A - 2)^{1/3} + 1.26 \right]. \tag{6}
\]

With the semianalytical model discussed above we can build an appropriately formulated diproton model. The assumed nuclear structure is very simple, but the diproton penetration process is treated exactly — without assumptions about the emission of diproton from some nuclear surface, which should be made in “R-matrix” approach. The widths obtained in such a model appear to be unexpectedly small which lead us to understanding that the interpretation and the usage of the diproton model should be revised.

In Fig. 1 we compare the results of the semianalytical model (4) calculations for \(^{45}\text{Fe}\) with the diproton width estimated by Eq. (5). The energy for the relative \(^{43}\text{Cr-pp}\) motion is taken
Figure 1. Effective equivalent channel radius \( r_{ch}(dp) \) for “diproton emission”: (a) as a function of radius \( \rho_0 \) of the three-body potential (7), the value \( \rho_0/\sqrt{2} \) should be comparable with typical nuclear sizes, (b) as a function of the position of the peak \( Y_{peak} \) in the three-body WF \( \bar{\Psi}^{(+)} \) in \( Y \) coordinate. The dashed lines are given to guide the eye.

\[ E - \varepsilon = 0.95E \text{ basing on the energy distribution in the } p-p \text{ channel, calculates in Ref. [21].} \]

In Fig. 1a we show the effective equivalent channel radii for diproton emission obtained by fulfilling condition \( \Gamma_{dp} \equiv \Gamma \) for semianalytical model calculations with different radii \( \rho_0 \) of the three-body potential

\[ V_3(\rho) = V_3^0 \exp\left[-(\rho/\rho_0)^2\right], \tag{7} \]

in the simplified Hamiltonian (1). It is easy to see that for realistic values of these radii (\( \rho_0 \approx 6 \text{ fm for } ^{45}\text{Fe} \)) the equivalent diproton model radii should be very small (\( \sim 1.5 \text{ fm} \)). This happens because in reality the “diproton” is too large to be considered as emitted from nuclear surface of a small \( \rho_0 \) radius. Technically it can be seen as the nonlinearity of the \( r_{ch}(dp) - \rho_0 \) dependence, with linear region achieved at \( \rho_0 \sim 15 - 20 \text{ fm} \). Only at such unrealistically large \( \rho_0 \) values the typical nuclear radius (when it becomes comparable with the “size” of the diproton) can be reasonably interpreted as the surface, off which the “diproton” is emitted. It is interesting to note that in the nonlinear region for Fig. 1a there exists practically an exact correspondence between the \( Y \) coordinate of the WF peak in the internal region and the channel radius for the diproton emission (Fig. 1b). This fact is reasonable to interpret in such a way that the diproton is actually emitted not from the nuclear surface [as it is presumed by the existing systematics of diproton calculations (6)] but from the interior region, where the WF is mostly concentrated. Thus the diproton model is applicable to estimate the contribution of the corresponding decay mechanism, but the systematics (6) of the channel radii used for these studies should be revised.

4. Comparison with “R-matrix” model

The quasiclassical expression for \( 2p \) width was obtained in the paper [34] neglecting \( p-p \) interaction and assuming existence of narrow resonances with energies \( E_{xa} \) and \( E_{ya} \) in the subsystems

\[ \Gamma(E) = \frac{D_3(E - E_{xa} - E_{ya})^2}{2\pi} \int_0^E dE_x \frac{D_x \Gamma_{xa}(E_x)}{(E_x - E_{xa})^2 + \Gamma_{xa}(E_x)^2/4} \frac{D_y \Gamma_{ya}(E_y)}{(E_y - E_{ya})^2 + \Gamma_{ya}(E_y)^2/4}. \tag{8} \]

Coefficients \( D_3, D_x, D_y \) are all close to unity and \( E_y = E - E_x \). In a reasonable approximation only energies \( E_x = E_y = E/2 \) contribute the expression under integral and slow varying
denominators can be pulled out of integral:

\[ \Gamma(E) = \frac{D_3 D_x D_y}{2\pi} \frac{(E - E_{xa} - E_{ya})^2}{(E/2 - E_{xa})^2(E/2 - E_{ya})^2} \int_0^E dE_x \Gamma_{xa}(E_x) \Gamma_{ya}(E_y). \] (9)

Let us consider some special cases. In the case \( E_{xa} \equiv E_{ya} \) we obtain

\[ \Gamma(E) = \frac{D_3 D_x D_y}{2\pi} \frac{4}{(E/2 - E_{xa})^2} \int_0^E dE_x \Gamma_{xa}(E_x) \Gamma_{ya}(E_y). \] (10)

In the case \( E_{ya} \gg E, E_{xa} \) we can approximate the \((E_{3\gamma} - E_{xa} - E_{ya})^2\) by \(E_{ya}^2\) in the numerator and \((E_{ya} - E_{3\gamma}/2)^2\) by \(E_{ya}^2\) in the denominator and obtain

\[ \Gamma(E) = \frac{D_3 D_x D_y}{2\pi} \frac{1}{(E/2 - E_{xa})^2} \int_0^E dE_x \Gamma_{xa}(E_x) \Gamma_{ya}(E_y). \] (11)

The formulae (10) and (11) look formally identical except the coefficient 4. The “R-matrix formulae” for three-body decays utilized in the series of works [26, 27, 28, 29] has “asymmetric” form, which can be derived from Eq. (8) in the limit \( E_{ya} \gg E, E_{xa} \) (see Ref. [34]). However, in the work [32], providing very similar to [7] properties of the \( 2^p \) emitters evidently has the condition \( E_x \equiv E_y \) fulfilled. Thus the approach of [26, 27, 28, 29] applied to this class of decays could lead to 4 times underestimation of the decay width.

5. Problem of the \( ^{17}\)Ne \( 3/2^- \) state width

It was demonstrated in papers [17, 18] that the astrophysical radiative capture rate for the \( ^{15}\)O(2\(p\),\(\gamma\))\(^{17}\)Ne reaction depends strongly on the two-proton width of the first excited \( 3/2^- \) state in \(^{17}\)Ne. This width was calculated in Ref. [7] as \(4.1 \times 10^{-16} \) MeV (some confusion can be connected with a misprint in Table III of Ref. [7], see erratum). However, in the work [32], providing very similar to [7] properties of the \(^{17}\)Ne WFs for the ground and the lowest excited states, the width of the \( 3/2^- \) state was found to be \(3.6 \times 10^{-12} \) MeV. It was supposed in [32] that such a strong disagreement is connected with poor subbarrier convergence of the HH method in [7] compared to Adiabatic Faddeev HH method of [32]. This point was further reiterated in Ref. [33].

In paper [21] the semianalytical model was used to analyse this problem. The results of calculations Ref. [7] were revised, but the obtained increase of the width was found to be far insufficient to close the gap with the results of [32, 33]. The most conservative upper limit \( \Gamma \sim 5 \times 10^{-14} \) MeV was obtained in a model with simplified Hamiltonian (neglecting \( p-p \) Coulomb interaction). The other models systematically produce smaller values, with realistic calculations confined to the narrow range \( \Gamma \sim (5 - 8) \times 10^{-15} \) MeV. Thus the value \( \Gamma \sim 4 \times 10^{-12} \) MeV obtained in paper [32] is very likely to be erroneous. That result is possibly connected to a simplistic quasiclassical procedure for width calculations employed in this work.

6. Conclusion

Here we introduce the integral formula for the widths of the resonances decaying into the three-body channel for simplified Hamiltonians and discuss various aspects of its practical application. Having in mind the origin and scale of the uncertainties introduced by reducing the realistic three-body model to a simplified three-body model (discussed in detail in Ref. [21]) and by reducing the simplified three-body model to the quasiclassical three-body model (discussed in Ref. [34]), we now have a basis for the appropriate (within the limits of its reliability) application of this model for estimates of the two-proton widths.
References

[1] V. I. Goldansky, Nucl. Phys. 19, 482 (1960).
[2] L. V. Grigorenko, R. C. Johnson, I. G. Mukha, I. J. Thompson, and M. V. Zhukov, Phys. Rev. C 64 054002 (2001).
[3] L. V. Grigorenko, R. C. Johnson, I. G. Mukha, I. J. Thompson, and M. V. Zhukov, Phys. Rev. Lett. 85, 22 (2000).
[4] L. V. Grigorenko, and M. V. Zhukov, Phys. Rev. 68 C, 054005 (2003).
[5] L. V. Grigorenko, I. G. Mukha, I. J. Thompson, and M. V. Zhukov, Phys. Rev. Lett. 88, 042502 (2002).
[6] L. V. Grigorenko, R. C. Johnson, I. G. Mukha, I. J. Thompson, and M. V. Zhukov, Eur. Phys. J. A 15 125 (2002).
[7] L. V. Grigorenko, I. G. Mukha, and M. V. Zhukov, Nucl. Phys. A713, 372 (2003); A740, 401(E) (2004).
[8] L. V. Grigorenko, I. G. Mukha, and M. V. Zhukov, Nucl. Phys. A714, 425 (2003).
[9] M. Pfutzner et al., Eur. Phys. J. A 14, 279 (2002).
[10] J. Giovinazzo et al., Phys. Rev. Lett. 89, 102501 (2002).
[11] B. Blank et al., Phys. Rev. Lett. 94, 232501 (2005).
[12] C. Dossat et al., Phys. Rev. C 72, 054315 (2005).
[13] Ivan Mukha et al., Nature 439, 298 (2006).
[14] I. Mukha et al., Phys. Rev. Lett. 99, 182501 (2007).
[15] J. Giovinazzo et al., Phys. Rev. Lett. 99, 102501 (2007).
[16] K. Miernik et al., Phys. Rev. Lett., in print.
[17] L. V. Grigorenko and M. V. Zhukov, Phys. Rev. C 72 015803 (2005).
[18] L. V. Grigorenko, K. Langanke, N. B. Shul’gina, and M. V. Zhukov, Phys. Lett. B641, 254 (2006).
[19] J. Görres, M. Wiescher, and F.-K. Thielemann, Phys. Rev. C 51, 392 (1995).
[20] K. Nomoto, F. Thielemann, and S. Miyaji, Astron. Astrophys. 149, 239 (1985).
[21] L. V. Grigorenko and M. V. Zhukov, Phys. Rev. C 76, 014008 (2007).
[22] K. Harada and E. A. Rauscher, Phys. Rev. 169, 818 (1968).
[23] S. G. Kadmensky and V. E. Kalechits, Yad. Fiz. 12, 70 (1970) [Sov. J. Nucl. Phys. 12, 37 (1971)].
[24] B. A. Brown, Phys. Rev. C 43, R1513 (1991); 44, 924(E) (1991).
[25] W. Nazarewicz et al., Phys. Rev. C 53, 740 (1996).
[26] F. C. Barker, Phys. Rev. C 63, 047303 (2001).
[27] F. C. Barker, Phys. Rev. C 66, 047603 (2002).
[28] F. C. Barker, Phys. Rev. C 68, 054602 (2003).
[29] B. A. Brown and F. C. Barker, Phys. Rev. C 67, 041304(R) (2003).
[30] J. Rotureau, J. Okołowicz, and M. Ploszajczak, Phys. Rev. Lett. 95, 042503 (2005).
[31] J. Rotureau, J. Okołowicz, and M. Ploszajczak, Nucl. Phys. A767, 13 (2006).
[32] E. Garrido, D. V. Fedorov, and A. S. Jensen, Nucl. Phys. A733, 85 (2004).
[33] E. Garrido, D. V. Fedorov, A. S. Jensen, H.O.U. Fynbo, Nucl. Phys. A748, 39 (2005).
[34] L. V. Grigorenko and M. V. Zhukov, Phys. Rev. C 76, 014009 (2007).