Counterfactual restrictions and Bell’s theorem

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We show that the ability to consider counterfactual situations is a necessary assumption of Bell’s theorem, and that, to allow Bell inequality violations while maintaining all other assumptions, we just require certain measurement choices be counterfactually restricted, rather than the full removal of counterfactual definiteness. We illustrate how the counterfactual definiteness assumption formally arises from the statistical independence assumption. Counterfactual restriction therefore provides a way to interpret statistical independence violation different to what is typically assumed (i.e. that statistical independence violation means either retrocausality or superdeterminism). We tie counterfactual restriction to contextuality, and show the similarities to that approach.

I. INTRODUCTION

Counterfactual definiteness is the intuitive idea that it is meaningful to consider what the values of any currently-defined observable, or the result of a given measurement, would instead be, if the world was different in some specific way. This forms part of our day-to-day experience; we expect a small change to affect the universe in a measurable way. Quantum mechanics casts doubt on this. Standard quantum mechanics has been described as acting as though unperformed experiments have no results\textsuperscript{1}, making it meaningless to discuss the impact of a small change in circumstances on a result. In other words, quantum mechanics has been said to require counterfactual indefiniteness.

By looking in more detail at each of the assumptions required to make a Bell inequality, we investigate the argument that it is necessary to exclude counterfactual definiteness in order to allow us to violate that inequality while still preserving all other assumptions which go into that inequality. We find that it is not necessary to remove counterfactual definiteness entirely to allow Bell inequality violation; the violation just requires a restriction on those counterfactual scenarios which are nomologically possible (a counterfactual restriction).

This paper is laid out as follows. In Section II, we discuss Bell’s Theorem, starting in Section II A by going over the EPR paradox, laying out how to generate (and violate) a Bell inequality (specifically the CHSH inequality) in this scenario in Section II B, then finally going through experimental demonstrations of Bell inequality violations, and experimental loopholes in these demonstrations, in Section II C. Section III looks at the assumptions required to generate a Bell inequality, starting in Section III A with hidden variables, then factorisability in Section III B which is formed of the intersection of No Superluminal Interaction (Section III C) and Statistical Independence (Section III D), then the ability to compare measurement results from different trials in Section III E. Finally, we look in more detail at this comparison assumption in Section IV, showing in Section IV A that Bell inequality violation only requires counterfactual restriction, rather than a full removal of counterfactual definiteness; in Section IV B that this counterfactual restriction links to, and provides another interpretation of, the violation of Statistical Independence; and in Section IV C that this interpretation links naturally to contextuality. We summarise our argument in Section V.

II. EPR AND BELL’S THEOREM

In this section, we go over Bell’s Theorem. This states that, given certain (reasonable-sounding) classical assumptions, one can generate an inequality from the expected results of measurements in an experimental scenario, which can be violated by expected results of measurements in this scenario when represented using quantum mechanics\textsuperscript{2}. The idea of specifically considering cases where quantum mechanics gives differing predictions to those expected from reasonable assumptions about the nature of the world, originated with Einstein, along with Podolsky and Rosen (EPR)\textsuperscript{3}. They first came up with a scenario in which interpreting quantum mechanics as fully describing a given situation seemingly leads to a paradox. In this section, we first go through this EPR paradox, and Bohm’s simplification of it, before showing how to use this EPR-Bohm set-up to derive a Bell inequality: the CHSH inequality. We then describe experimental attempts to violate this inequality (e.g. Freedman and Clauser’s\textsuperscript{4}, and Aspect et al’s\textsuperscript{5, 6}), loopholes in those experiments, and how those loopholes were resolved. This serves as the groundwork necessary for the discussion in the next Section of the assumptions necessary to form a Bell inequality.
FIG. 1. The EPR-Bohm experiment [7], where a source emits pairs of spin-entangled particles in opposite directions. The joint spin state of the two particles is $|\Psi\rangle = (|\uparrow\rangle_A \otimes |\downarrow\rangle_B + |\downarrow\rangle_A \otimes |\uparrow\rangle_B)/\sqrt{2}$. The left-hand particle (particle $A$) is subjected to either test $a_0$ or $a_1$, and the right-hand particle (particle $B$), is subjected to either test $b_0$ or $b_1$. All tests have result either $+1$ or $-1$. This set-up (alongside various assumptions, discussed below) was used to derive the Clauser-Horne-Shimony-Holt (CHSH) inequality [8], which was experimentally violated by Aspect et al [6].

A. The EPR Paradox

Einstein, Podolsky and Rosen came to a paradox by combining two key concepts in quantum mechanics: conjugacy (one variable’s uncertainty increasing as the other’s decreases) and entanglement (two particles’ states being so correlated they cannot be written independently) [3].

When we measure one of a set of conjugate variables for one of a pair of entangled particles, quantum mechanics says the conjugate variable for the other particle becomes uncertain. This is despite the two particles potentially being spatially separated, and standard quantum mechanics providing no physical mechanism for a signal to propagate between them.

Bohm simplified this scenario, doing away with non-commuting observables, and focusing on joint states—those which quantum mechanics says cannot be written as the (tensor) product of single-particle states [7]. He gave the example of the joint spin state of two entangled electrons, where the total spin of the two particles is 0 (see Fig 1).

Using bra-ket notation [9], we write the spin-states of the two electrons as the Bell state

$$|\Psi\rangle = \frac{1}{\sqrt{2}} (|\uparrow\rangle_A \otimes |\downarrow\rangle_B + |\downarrow\rangle_A \otimes |\uparrow\rangle_B)$$

where indices $A$ and $B$ label the spin states of particles $A$ and $B$ respectively. Note, while normally we need to define a basis for the result of a spin measurement (e.g., for a spin-1/2 particle, spin-up or spin-down in the $X$, $Y$ or $Z$ direction), the Bell state in Eq. [1] takes the same form regardless of the basis in which we represent it, so long as we represent both particles in the state in the same basis—i.e.,

$$|\Psi\rangle = \frac{1}{\sqrt{2}} (|\uparrow\rangle_A \otimes |\downarrow\rangle_B + |\downarrow\rangle_A \otimes |\uparrow\rangle_B)$$

(2)

and similarly for any other choice of representation direction).

As this state cannot be written as the tensor product of the two individual electrons’ states, quantum mechanics says we cannot describe these two electrons’ spins independently of one another. Given the collapse postulate, if the description in Eq. [1] completely describes the physical reality of the scenario, then when one of the electrons’ spins is measured, it somehow collapses the overall state, pushing the other electron into its corresponding spin (and making its spin in any of the other two mutually unbiased bases uncertain). If we are being fully realistic about both the wavefunction (i.e. saying there is nothing in this system not described by the wavefunction as given/no hidden variables) and this collapse, this happens instantaneously, and so information must travel superluminally. To reinforce this, the state into which the first electron collapses is completely random (by this account) that the second electron can be said to be ‘pre-prepared’ into the corresponding state in advance, to avoid the need for superluminal information transfer.

Einstein dismissed this effect as “spooky action at a distance” (spukhafte Fernwirkung) [10]—information seemingly passing from one place to another, instantly and without mechanism. To him, this meant there must be a deeper description of what was happening in the system than standard quantum mechanics provided—that there was some “hidden variable”, not described by the formalism of quantum mechanics (and so missing from Eq. [1]) which governed the underlying situation in order to avoid this superluminal update. Models which describe this possibility are referred to as “hidden-variables” models [11].

B. Bell’s Theorem

For nearly thirty years after the EPR paper, whether you believed quantum mechanics was incomplete, failing to account for some ‘hidden variables’ which further explain its peculiarities, or instead violates some fundamental assumption about the nature of the universe, was just considered a matter of interpretation. However, in 1964 Bell proposed an experiment to identify testable differences between these two options [12]. For the EPR-Bohm experiment, Bell derived the upper limit for measurable correlations between the two particles, assuming
they obeyed a local hidden-variable model (among other assumptions).

These local hidden variable models posit a variable $\lambda$, which, from the moment an entangled state is generated, holds information as to the final state its constituents will end up in, above and beyond that given by the wavefunction. For instance, for Eq. 1 this information would be whether the spins of the two electrons will collapse to $|↑\rangle_1 \otimes |↓\rangle_2$ or $|↓\rangle_1 \otimes |↑\rangle_2$. This variable is hidden (i.e., not necessarily accessible to the experimenter), and local (i.e., cannot be used as a channel to superluminally transmit information between the two particles). The space $\Lambda$ spans all possible values for this local hidden variable (i.e. $\lambda \in \Lambda, \forall \lambda$). (We go into more detail about these assumptions in Section III A.)

By using properties of these local hidden variable models alongside certain assumptions, Bell (and others) derived inequalities for sums of correlation functions between measurement results. The most famous of these Bell inequalities is the Clauser-Horne-Shimony-Holt (CHSH) inequality [8] (of which we use the Wigner form [13-14]). As in the EPR-Bohm set-up, a source emits pairs of particles in opposite directions. The left-hand particle is subjected to test $a$ (either $a_0$ or $a_1$), and the right-hand one to test $b$ (either $b_0$ or $b_1$). All tests have result either +1 or −1 (see Fig. 1). For example, for tests of single-photon polarisation, where $a_0$ and $a_1$ ($b_0$ and $b_1$) are differently oriented polarisers, +1 corresponds to the photon being transmitted, and -1 to it being absorbed.

For a given trial $i$, we call the result $A_i$ of the experiment of the left either $A_{0,i}$ or $A_{1,i}$ (depending on whether we choose test $a$ to be $a_0$ or $a_1$ for that trial), and similarly $B_i$ as either $B_{0,i}$ or $B_{1,i}$. Therefore, we can write the four pairs of products of the results as $A_{0,i}B_{0,i}, A_{1,i}B_{0,i}, A_{0,i}B_{1,i}$, and $A_{1,i}B_{1,i}$.

We use these 4 result combinations to give

$$\gamma_i(a_0, a_1, b_0, b_1) = A_{0,i}B_{0,i} + A_{1,i}B_{0,i} + A_{0,i}B_{1,i} - A_{1,i}B_{1,i}$$

As $A_{0,i}, A_{1,i}, B_{0,i}, B_{1,i} \in \{-1, 1\}$,

$$\gamma_i(a_0, a_1, b_0, b_1) = A_{0,i}(B_{0,i} + B_{1,i}) + A_{1,i}(B_{0,i} - B_{1,i})$$

$$= \pm 2$$

(4)

For $N$ events, given $\gamma_i = \pm 2$,

$$\left| \frac{1}{N} \sum_{i=1}^{N} \gamma_i \right| = \left| \frac{1}{N} \sum_{i=1}^{N} A_{0,i}B_{0,i} + \frac{1}{N} \sum_{i=1}^{N} A_{1,i}B_{0,i} + \frac{1}{N} \sum_{i=1}^{N} A_{0,i}B_{1,i} - \frac{1}{N} \sum_{i=1}^{N} A_{1,i}B_{1,i} \right| \leq 2$$

(5)

As $A_i \in \{A_{0,i}, A_{1,i}\}$ and $B_i \in \{B_{0,i}, B_{1,i}\}$, the expectation value for the product of the results of a given pair of measurements $a, b$ is

$$E(a, b) = \lim_{N \to \infty} \frac{1}{N} \sum_{i=1}^{N} A_{i}B_{i}$$

(6)

so Eq. 5 becomes

$$S = |E(a_0, b_0) + E(a_1, b_0) + E(a_0, b_1) - E(a_1, b_1)| \leq 2$$

(7)

This is the CHSH inequality. If, for any set of measurement settings a quantum version of the system can get a value of $S$ greater than 2, then this shows that quantum mechanics must violate one of the assumptions we used to create our Bell inequality, and gives us a way of experimentally testing whether or not the universe follows all of these assumptions. Note, the CHSH inequality is typically used as, unlike Bell’s 1964 inequality, it doesn’t require perfect anticorrelation between the measurement outcomes of the two particles when measured using the same setting. This allows it to better account for experimental errors; with perfect anticorrelation, it simplifies to Bell’s initial inequality [2].

Quantum mechanics can violate this limit. Returning to the state defined in Eq. 1, the quantum mechanical expectation value for the product of the results of two tests $a, b$ on the two parts of the entangled state (i.e., test $a$ on the part of the state labelled subscript 1, and test $b$ on the part labelled subscript 2) is

$$E(a, b) = \langle \Psi | (\sigma_{1,a} \otimes \sigma_{2,b}) | \Psi \rangle$$

(8)

where $\sigma_{1,a}$ is the spin operator on particle 1 in direction $a$, and $\sigma_{2,b}$ the spin operator on particle 2 in direction $b$, such that for instance

$$\sigma_{1,z} |↑z\rangle_1 = |↑z\rangle_1; \sigma_{1,z} |↓z\rangle_1 = - |↓z\rangle_1$$

(9)

As spin lives on the Bloch sphere, we can simplify Eq. 8 by decomposing spin measurements in the $b$ direction, into the component of the measurement parallel to $a$, and the component perpendicular to $a$ (denoted $a \perp$):

$$E(a, b) = \langle \Psi | (\sigma_{1,a} \otimes (\cos \theta_{ab} \sigma_{2,a} + \sin \theta_{ab} \sigma_{2,a\perp})) | \Psi \rangle$$

(10)

where $\theta_{ab}$ is the angular distance along the Bloch sphere between $a$ and $b$.

As $|\Psi\rangle$ is the state given in Eq. 1 which keeps the same form regardless of the basis we represent it in, Eq. 10 simplifies further to

$$E(a, b) = - \cos \theta_{ab}$$

(11)

This means, given

$$S = |E(a_0, b_0) + E(a_1, b_0) + E(a_0, b_1) - E(a_1, b_1)|$$

(12)

in the quantum case,

$$S = | - \cos \theta_{a_0b_0} - \cos \theta_{a_1b_0} - \cos \theta_{a_0b_1} + \cos \theta_{a_1b_1}|$$

(13)
This is maximal when \( \theta_{a_0b_0} = \theta_{a_1b_0} = \theta_{a_0b_1} = \pi/4 \) and \( \theta_{a_1b_1} = 3\pi/4 \), such as when

\[
a_0 = \sigma_X; \ a_1 = \sigma_Z; \ b_0 = \frac{\sigma_Z + \sigma_X}{\sqrt{2}} \ b_1 = \frac{\sigma_Z - \sigma_X}{\sqrt{2}}
\]

and gives

\[
S_{\text{max}} = \left| -\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} \right| = 2\sqrt{2}
\]

This is a proof of the Tsirelson bound [15]: the maximal value of \( S \) achievable using quantum mechanics. (Note, for the EPR-Bohm scenario, Popescu and Rohrlich showed \( S \) can go above this bound, up to \( S \leq 4 \), for general nonlocal theories [16], though Carmi and Cohen showed models where \( S > 2\sqrt{2} \) must at the least violate a subtle form of relativistic causality [17]). While there are more formal proofs, generalised for all quantum systems (rather than just dimension-2, like the spin-1/2 qubits above), this proof serves our purpose. It shows the CHSH inequality can be violated using the quantum formalism: local-hidden-variable models of the sort used to generate the CHSH inequality above fail to account for predicted quantum correlations between the two particles. Therefore, an experiment giving \( S > 2 \) would show the Universe violates one of the assumptions used to form this inequality.

C. Experimental Tests and Loopholes

The first experimental attempt to violate a Bell inequality was performed by Freedman and Clauser in 1972 [4]. This used polarisation-entangled photon pairs, generated through the atomic cascade of Calcium, to try and violate the CHSH inequality that Clauser et al had proposed two years earlier. However, this experiment left a number of loopholes—ways a local hidden-variable theory could still explain the correlations observed, given practical details of the experiment. In this Subsection, we discuss these loopholes, and how later Bell tests managed to avoid them, such that loophole-free violations of Bell inequalities have now been performed. We follow Larsson et al’s categorisation of these loopholes into the Locality loophole, the Memory loophole, the Detection loophole, and the Coincidence loophole [18].

Locality: If local (i.e., luminal or subluminal) communication between the two measurement sites is possible in the time between a measurement choice being decided at one measurement site, and the measurement being performed at the other measurement site, then it is perfectly consistent that we could observe correlations which violate our Bell inequality even if the Universe obeys a local hidden-variable theory. This was the Locality loophole found in Freedman and Clauser’s experiment—we could imagine some mechanism by which the measurement settings at the first site (the polariser direction being set) locally affect the state of the particle at the second site, in such a way as to generate these correlations. Note this loophole is also behind supposed macroscopic violations of Bell inequalities, such as those presented by Aerts et al [19], which serve to show very little except an inequality based on an assumption of separation between two subsystems can easily be violated if the two subsystems are not in fact separate. This loophole was first closed in Aspect et al’s tests, proposed in 1976 [20], and performed in the early 1980s [5, 6]. These experiments also used polarisation-entangled single photons, but used time-varying analysers to perform measurement. These analysers are effectively variable polarisers, which jump between two polarisation orientations in a time far shorter than the time it takes each photon to travel from the source to its respective polariser.

Memory: While Aspect et al’s experiment closed the locality loophole, it raised a new loophole—the memory loophole. This loophole comes from the analysers in Aspect et al’s experiment shifting between the two polarisations deterministically, at a fixed frequency—the state of the polariser at some given time in the future was predictable. While unlikely, one could imagine a way the system could extrapolate what the state would be at a given time to allow the generation of Bell inequality-violating coincidences. Aspect himself noted this loophole in his proposal [20], saying the set-up required the supplemental assumption that the polarisers have no “memory” as to what state they had been in previously, so the system couldn’t extrapolate from any regularity in the settings what their state would be in the future. This assumption however is not needed (and the memory loophole closed) if the measurement settings for each experimental run are chosen in some unpredictable way, so there is no way to infer what the measurement setting will be for a given run. The first Bell tests to close this loophole were those of Weihs et al [21]. Here, random number generators at each measurement site determined which of the two possible measurement settings to use for a given run, after the photons were emitted from the source. (This also avoids the locality loophole.)

Detection: Next, we look at the detection, or efficiency loophole, whereby the possibility of losing or not detecting particles during the experiment can affect the correlations observed. This can occur in two ways. Firstly, when using polarisers (like Freedman and Clauser), one of the measurement results (-1) is recorded when there is no detection, as one assumes the photon, having had an orthogonal polarisation to the polariser’s measurement setting, was absorbed by the polariser. However, there are many other reasons the photon might be lost during the experiment. For instance, the detector might not have perfect efficiency, the lenses and transmissive optics between the source and the polariser might not be perfectly transmissive, or polarisers might not be perfectly transmissive even at their transmission polarisation. Therefore, -1 would be recorded far more often in practice than we would expect with a theoretically perfect experiment. This could overshadow
any correlation from entanglement, and lead to the data not violating the inequality. To avoid this (and to compensate for not receiving a click when observing a -1 detection), Freedman and Clauser adapted the CHSH inequality to give an inequality in terms of rates of coincidence detection as a function of angle between the two polarisers, divided by rate of coincidences when both polarisers were removed. This however relies on an assumption of fair sampling—that the detection efficiency is the same for different polarisations. This could be violated if, for instance, the detectors had a bias towards detecting certain polarisations more efficiently than others, or the lenses used subtly act as polarisers.

The detection loophole also appears in Aspect et al’s experiments. These use polarising beam splitters of variable basis rather than variable polarisers, so there is a detection associated with both a +1 and a -1 click. However, they still neglects runs (photon pairs) where one or both photons are absorbed. Therefore, forming a Bell inequality for these experiments still requires some assumption of fair sampling—that observed runs are a fair sample of all runs. This loophole is only closed by tests where this loss can be bounded and accounted for. Initially, this was only possible in tests using solid-state platforms rather than optics, such as Rowe et al’s [22], which then suffered from the locality loophole. However recent experiments have been performed which close both detection and locality loopholes simultaneously—e.g. Hensen et al’s [23], which use a combination of optics and nitrogen-vacancy centre qubits, and Giustina et al’s [24] and Shalm et al’s, which both use entangled photon pairs.

Coincidence: A related loophole is the coincidence loophole: given we rely on simultaneous detection to tell us which particles where initially generated in the same pair, we could imagine a local hidden variable model whereby state/setting-dependent delays cause correlations which violate the Bell inequality [23]. This loophole is most problematic in photon-pair experiments, such as Freedman and Clauser’s, and Aspect et al’s, but can again be mitigated in solid-state experiments. In these, it is easier to assign measurement results to a given run, and so they don’t require coincidence detection. Larsson showed that later photonic Bell tests (e.g., Giustina et al’s and Shalm et al’s) also avoid this loophole [26].

Bell tests have been performed which close all these loopholes. These experiments show one of the assumptions which go into generating a theoretical Bell inequality must be false, rather than there potentially being some convoluted experimental reason why we observe such a violation.

III. FORMAL ASSUMPTIONS AND THE CONSEQUENCES OF THEIR VIOLATION

Above, we showed that observed Bell inequality violations are not due to experimental loopholes, but must instead necessarily be due to one of the assumptions used to generate such a Bell inequality being false. We now look in more detail at these assumptions. In this paper, we divide the assumptions into Hidden Variables, the Matching Condition, and Factorisability (which we subdivide into No Superluminal Interaction and Statistical Independence). Since Bell Inequalities are violated by quantum particles, it follows that at least one of these assumptions must be violated. Note there are many other categorisations (e.g. Wiseman and Cavalcanti’s [27]), which subdivide many of these assumptions further (especially the No Superluminal Interaction assumption). However this simple categorisation is sufficient for our purposes, given we are mainly focussing on the statistical independence assumption (which is universal to these categorisations), and counterfactual definiteness/ the Matching Condition, which we demonstrate to be a necessary assumption.

A. Hidden variables

The first assumption needed for the Bell inequality derivation in Section 11B is the existence of hidden variables, and the result of a given measurement being a function both of the measurement setting, and some hidden variables. These hidden variables are typically viewed as localised at each particle in some way (i.e. as hidden system properties of each particle). However, this neglects that the hidden variables can also be taken to describe the detectors, and the environment, or may not even be localisable in any meaningful way. Consideration of the “location” of hidden variables becomes important when combined with the No Superluminal Signalling assumption mentioned later. Note, these variables need not completely determine the measurement result (i.e., there can still be some classical randomness to the process of measurement result generation). However, given these variables are hidden, we can just add an additional hidden variable to the model which differentiates between the different random options. This allows us to return to a deterministic relationship between the combination of hidden variables and measurement setting, and measurement results, without affecting the conclusions of the model (see Section 4.2 of [25]).

We can represent this dependence on hidden variables by writing our product result AB as

\[ AB \equiv A(a, b, \lambda)B(a, b, \lambda) \] (16)

where a and b are our two measurement setting choices \((a \in \{a_0, a_1\}, b \in \{b_0, b_1\})\), and \(\lambda\) represents our hidden variables. For greatest generality, we write each measurement result A, B as a function of both the measurement choice applied on its side, and the measurement choice applied to the other side. This is restricted further by the Factorisability assumption discussed below.

The hidden-variables assumption is often claimed to be rebutted by the violation of Bell inequalities. Those
who hold to this interpretation claim that what Bell’s Theorem shows is that quantum mechanics provides a complete description of the world, and hidden variable models are unnecessary classical baggage we bring with us. However, given it requires the intersection of the hidden variables assumption with a number of other assumptions to generate a Bell inequality, it is obviously not true that hidden variables are necessarily ruled out by Bell inequality violations. Further, taking the quantum formalism at face value still requires the violation of another key assumption necessary to generate a Bell inequality: factorisability.

B. Factorisability

The next assumption, which is necessary to form a Bell inequality, is factorisability. This is the assumption we can write the product of the measurement results in the form

\[ AB \equiv A(a, \lambda_A)B(b, \lambda_B) \]  

(17)

where there are two implicit assumptions. The first is that the hidden variables \( \lambda_A \) are only able to be influenced by events in the past or future light cone of the measurement event giving \( A \), and the same for \( \lambda_B \) and \( B \). The second is that the measurement settings \( a \) and \( b \) can be treated as completely free variables—they are not themselves influenced by each other, or the hidden variables, nor can they influence the hidden variables.

This notion of factorisability therefore can be decomposed more fully into these two assumptions—no superluminal interaction, and statistical independence.

Given the quantum formalism represents entangled states as the superposition of different tensored states, which cannot be written in such a way that the state of each subsystem is separate (i.e. as the tensor product of some state/superposition of states of particle \( A \), and some state/superposition of states of particle \( B \)), the quantum formalism in some way looks to violate some notion of factorisability. However, this tensor-product-factorisability is formally different to the notion of measurement-result factorisability we give above.

C. No superluminal interaction

The No Superluminal Interaction assumption is that there can be no way for spacelike-separated events (those outside each others’ past or future light-cones) to affect each other. By having the two measurement events spacelike-separated (as is required to avoid the locality loophole), this assumption prevents the measurement setting on one side of the set-up directly affecting the measurement result on the other side of the set-up (as in, affecting the result beyond such correlation as would be allowed by the shared part of the hidden variables). Therefore, it allows us to partially-factorise the product of the measurement results into

\[ AB \equiv A(a(\lambda), \lambda)B(b(\lambda), \lambda) \]  

(18)

However, this does not necessarily mean this assumption requires the measurement settings on one side cannot affect the measurement results from the other, as this correlation between the two could be carried through a correlation between the measurement settings and the hidden variables. (This possibility is only blocked by combining the No Superluminal Interaction assumption with the Statistical Independence assumption below.)

A concern with models which break this assumption is that we haven’t observed any other systems where interaction occurs superluminally. Shimony therefore proposed the idea of quantum interactions exhibiting superluminal ‘passion’, rather than action, at a distance [29]. This is as, even if factorisability was violated by some superluminal effect for entangled particles, no observer would be able to extract or transmit information superluminally through this mechanism, due to the No-Signalling Theorem [11, 30]. Therefore, the violation of this assumption doesn’t necessarily allow observations which would contradict Special Relativity.

That this assumption is the one violated is one of the most common interpretations of Bell’s Theorem, to the extent that Bell’s Theorem is often claimed to show that nature is nonlocal [31], and EPR-correlation (or at least the quantum formalism’s violation of tensor-product-factorisability) is often referred to in quantum information theory as nonlocality [10, 32, 33]. However, while such an interpretation respects No-Signalling, it still seems to violate our intuitions about Special Relativity, given this prima facie prohibits any signal, not just observable information, travelling superluminally.

D. Statistical Independence

Statistical independence is the assumption that there are no correlations between our measurement choices and our hidden variables. This can be represented in multiple equivalent ways.

Statistical Independence generally refers to the assumption that hidden variables \( \lambda \) are uncorrelated with the choices of measurement settings for a given measurement. For the Bell Inequality we derive above, we can write this as

\[ a \neq a(\lambda); b \neq b(\lambda); \lambda \neq \lambda(a); \lambda \neq \lambda(b) \]  

(19)

Alternatively, if we imagine some probability distribution over our hidden variables describing their likelihood, \( \rho \), then we can write this assumption as

\[ \rho(\lambda, a, b) = \rho(\lambda), \forall a \in \{a_0, a_1\}, \forall b \in \{b_0, b_1\} \]  

(20)
where \( \rho \) is the probability of the hidden variables having a certain value \( \lambda \).

It is a common misconception that this assumption is not needed, and so Bell’s Theorem rules out all local hidden-variable models [31] [34]. However, the assumption is mathematically necessary to formulate a Bell inequality [32] [37]—otherwise, the measurement result on one side of the experiment could trivially depend on the measurement setting on the other side of the experiment through the hidden variables, which would violate factorisability without requiring superluminal interaction. Hall showed that remarkably little violation of statistical independence is necessary to allow a local hidden variable model to reproduce quantum mechanics [38], and Kimura et al showed this holds even for a remarkably small space of hidden variables [39]. Therefore it makes sense to consider models which violate this statistical independence assumption.

We can imagine three cases (ignoring just repeated coincidence) where this assumption is not true (i.e., where there is some correlation between the hidden variables and measurement settings). First, this correlation could be due to the hidden variables in some way influencing the choice of measurement settings; secondly, the measurement settings could in some way influence the value of the hidden variables; or thirdly, the choice of measurement settings and value of the hidden variables could have some common cause. Models following the first or third option are commonly referred to as superdeterministic [40] [41] (although in [42] and below we discuss another set of models which fall under the third option: supermeasured models). Models following the second option are termed retrocausal: we do not discuss these further in this paper, but recommend [43] for a thorough review of these models.

While violation of the statistical independence assumption explains the apparently instantaneous, mechanism-free collapse of one particle based on the other (due to measurement choices themselves being correlated with the hidden variables), it has been argued that it presents both epistemic issues (making it impossible to empirically derive physical laws), and physical issues (such as being fine-tuned to the point of conspiracy) [44] [45]. While such bold claims have been refuted elsewhere [40] [41], such a discussion is not the subject of our paper.

Examples of superdeterministic models include Brans’s model [46], ’t Hoof’s cellular automaton model [47], Ciepielewski, Okon and Sudarsky’s model [48], Donadi and Hossenfelder’s toy model [49], and Palmer’s Invariant Set Theory [50]. Invariant Set Theory is an example of a model which fits more neatly in a new subcategory of statistical independence-violating model we proposed in [12]—supermeasured models.

While authors often talk about classical superdeterministic models [51], all the models mentioned above use entanglement, superpositions, density matrices and wavefunctions just like standard quantum mechanics [52]. This illustrates why it is worth investigating local hidden-variable models which avoid Bell’s Theorem: not just to return to something which looks like classical mechanics, but to resolve issues which the standard formalism of quantum mechanics cannot. An example of such an issue is the Measurement Problem [28].

1. Supermeasured

An alternative understanding of how statistical independence may be violated, is the application of a supermeasure (a non-trivial measure) to the relation between probability space and state space [42]. First we must acknowledge the existence of a measure \( \mu \) to move from probability space to state space - i.e.

\[
\rho_{\text{Bell}}(\lambda|a, b) := \rho(\lambda|a, b)\mu(\lambda|a, b) \tag{21}
\]

where \( \rho_{\text{Bell}} \) is the response function on state space for a given hidden variable \( \lambda \), used to formulate Bell inequalities.

Therefore, the actual assumption used in Bell’s theorem, rather than that given in Eq. 20 is

\[
\rho_{\text{Bell}}(\lambda|a, b) = \rho_{\text{Bell}}(\lambda) \tag{22}
\]

This means, instead of treating the Statistical Independence assumption as a primitive (often referred to as the “Free Choice” assumption, as in Wiseman and Cavalcanti [27]), we should instead view it as the intersection of Eq. 20 and

\[
\mu(\lambda|a, b) = \mu(\lambda), \forall a, b, \lambda \tag{23}
\]

A violation of this implies a non-trivial measure taking us to probability space from state space. Palmer [50] [53] constructed a model which violates Bell inequalities by proposing non-trivial measures (and also is suggested to be extendable to reproduce General Relativity). However, ‘supermeasured’ models may also imply limits on counterfactual definiteness, as we will discuss below.

E. Ability to compare cases with different measurement settings

1. Statistical vs Counterfactual

The final assumption necessary to form a Bell inequality is that there is some way to compare scenarios with different measurement settings. This comparison can be done in one of two ways statistically, or counterfactually. We here first look at the statistical method, which is the method typically used experimentally, where results from multiple different Bell tests (different is, to use the terminology above) are combined together. However, an issue with this method is it assumes that the
hidden variable $\lambda$ will remain the same across different tests (different is). This is a fundamentally unverifiable assumption, given the hidden-ness of the hidden variables $i$. While arguments are often given in analogy to classical experiments, that we should assume that we can hold $\lambda$ fixed, as we can screen off causal influences from the experiment (e.g. isolate labs from the environment, keep apparatus as similar as possible between tests, etc), this often assumes only classical influences, which given we are investigating non-classical correlations, is dubious at best. Bell himself admitted this, saying "In this matter of causality it is a great inconvenience that the real world is given to us once only... We cannot repeat an experiment changing just one variable; the hands of the clock will have moved, and the moons of Jupiter." This indicates the issue with such a statistical approach, and the reason for instead comparing results counterfactually.

2. Counterfactual Definiteness

Informally, counterfactual definiteness, or determinateness, is the idea that there is a definite outcome for what an observable would be, were something done differently (e.g. for Bell tests, a different measurement basis chosen for one of the two spin measurements). While this is commonly confused with determinism - that the universe evolves deterministically, from a set of initial conditions - counterfactual definiteness is not necessarily deterministic. The universe may be probabilistic and counterfactually definite; so long as there are always definite values for all observables in counterfactual scenarios. This is shown by probabilistic Bell inequalities (e.g. the CHSH inequality [8]), which, despite being probabilistic, still gives an inequality which quantum phenomena can violate.

Therefore, being able to violate Bell inequalities could mean, for both the Wigner- and probabilistic forms of these inequalities, that the universe isn’t counterfactually definite.

This makes us ask just how reduced counterfactual definiteness needs to be to allow a Bell inequality to be violated while keeping all other assumptions. The answer to this is that Bell inequality violation does not require the removal of all counterfactual definiteness—just the violation of the matching condition.

3. The Matching Condition

The final assumption required to generate a Bell inequality is the matching condition [11]. This is a strong version of the assumption of counterfactual definiteness—that there is a matter of fact about the results of unperformed measurements. In Section III-B, we define $\gamma_i$ for a given trial $i$ by summing the four different pairs of measurement results. However, this requires there to be a matter of fact about what result we would get, had we performed each of the four possible pairs of measurements on the particle pair. For instance, imagine for a given trial that we had measured the two particles with measurement settings $a_0$ and $b_0$, and so got measurement result pair $A_{0,0}$. To generate a value of $\gamma$ for that trial, we are now required to also imagine what the results of a measurement would be, had that trial, had we instead measured $a_1$ instead of $a_0$ (to get a value of $A_{1,0}$).

Therefore, moving to probabilistic models does not affect the measurement results on the other, which is instead part of the locality assumption above. Nor is it the same as saying we can simultaneously measure $A_{0,i}$ and $A_{1,i}$ (or $B_{0,i}$ and $B_{1,i}$)—just that they have definite values which can be simultaneously considered in those pairs. All the matching condition assumption requires is that the measurement results of unperformed measurements are counterfactually definite for all four pairs of measurement choices:

$$\forall (a \in \{a_0, a_1\}, b \in \{b_0, b_1\}, \lambda \in \Lambda), \exists A_i B_i$$ (24)

There are ways to generate Bell inequalities without making explicit use of the Matching Condition, as in stochastic hidden variables models. However, this involves considering probability distributions over possible measurement result pairs, which requires assuming Kolmogorov’s axioms [59]. When all of these probability distributions are nonzero, as is necessary to represent the EPR-Bohm experiment, these axioms lead to one of two conclusions. The first option is, the probability distributions could be an incorrect representation of the situation (representing instead the experimenter’s incomplete knowledge). Alternatively, if the distributions provide accurate representations of the situation, these probability axioms place counterfactual definiteness conditions on the four result pairs at least as strong as the Matching Condition. This is as it requires the situation where a certain result pair has a nonzero probability distributions to be meaningful enough to assign nonzero probability to. Therefore, moving to probabilistic models does not remove the dependency on counterfactual definiteness, or show the Matching Condition to be unnecessary.

From our discussion above, one can see that we either need a statistic comparison assumption, or a counterfactual definiteness assumption (e.g. the Matching Condition). While authors such as Maudlin challenge the necessity of a counterfactual definiteness assumption, as they claim it is “not in the maths” [57], we see from above that such an assumption appears obviously when we try to construct a Bell inequality. This shows that Bell’s Theorem has an assumption that isn’t yet suffi-
IV. COUNTERFAC TUAL RESTRICTION

A. Counterfactual restriction rather than no
counterfactual definiteness

Given the full specification of a scenario, counterfactual definiteness implies there is a fact of the matter on what would happen in any counterfactual situation. Given the full specification of a counterfactual situation, there is a matter of fact about what you’d get. (Note this is distinct from “and that definite value is invariant under a given change”.) Think of a world with certain observables, indexed under a given change. This is distinct from “and that definite value is invariant under a given change.”

To consider this, let us look again at the matching condition. As we say above, it requires the value of all four result products in our CHSH inequality are simultaneously defined for any λ, and so that any triple of a, b, and λ is nomologically possible. If we take a specific state as being defined if it has a nonzero measure on state space, the matching condition is equivalent to saying

\[ \mu(\lambda|a, b) \neq 0, \forall (a \in \{a_0, a_1\}, b \in \{b_0, b_1\}, \lambda) \]  

Therefore, a violation of the matching condition is just a case where, for at least one triple of a, b, and λ, μ is equal to 0.

As \( \rho_{Bell} \) is what goes into Bell’s theorem, rather than ρ, this still allows us to maintain physical statistical independence (or to say, for any given triple of a, b, and λ, calculate the expectation value for each result product)—it just means we cannot combine the four different expectation values together to form a Bell inequality.

Despite being a violation of what formally enters Bell’s theorem as the Statistical Independence assumption, this isn’t what most people think of when they think of statistical independence being violated—these cases can still obey

\[ \rho(\lambda|a, b) = \rho(\lambda), \forall (a \in \{a_0, a_1\}, b \in \{b_0, b_1\}, \lambda) \]  

and can even still obey Bell-Statistical Independence for all allowed counterfactual situations—that is

\[ \rho_{Bell}(\lambda|a, b) = \rho_{Bell}(\lambda), \forall (a \in \{a_0, a_1\}, b \in \{b_0, b_1\}, \lambda) \text{ where } \mu \neq 0 \]  

The fact that violations of the matching condition can be represented as violations of Bell-Statistical Independence (but not physical statistical independence), or to use other terminology, as a form of supermeasured theory, makes us ask whether we should reassess how we typically consider statistical independence-violating models.

Let us look at the archetypal supermeasured theory—Palmer’s invariant set theory.

In Palmer’s Invariant Set Theory, there is a clear (but uncomputable) split between those states which are on the invariant set, and so allowed by the theory, and those which are not on the set, and so are prohibited by the theory. This split occurs in such a way that, if a pair
of states exist such that each state has a definite value for one of a pair of conjugate variables (e.g. one state has a definite value for spin in the \(x\)-direction, and the other a definite value for spin in the \(z\)-direction), the two states will never both be on the invariant set, and so at least one of the two states will never be a (nomologically) possible state. This restriction on states with definite values of conjugate variables both being on the invariant set looks in some sense like a counterfactual restriction on the pair of states for a given value of \(\lambda\)—if one of the pair is counterfactually definite, the other by definition is counterfactually restricted.

Another set of models which look to be cases of a counterfactual restriction are those described as having an epistemic restriction, such as Spekkens’ Toy Model \([59]\), and those models derived from it \([60, 64]\). While these models often don’t fully reproduce quantum mechanics, they reproduce certain properties of quantum mechanics despite being nominally classical, through the application of an epistemic restriction. Given the epistemic restriction serves to change the behaviour of the system in ways far stronger than one would normally attribute to limiting knowledge about a system, the “knowledge balance principle” underlying these models looks more like a counterfactual restriction than a real restriction on knowledge. Interpreting the principle this way, these models also look to show that quantum features can be regained by a counterfactually-restricted classical model.

C. Links to Contextuality

The assumption going into Bell’s Theorem that we are free to choose all variables we view as “free” classically (i.e. statistical independence, or that there is no counterfactual restriction) ties into the burgeoning research area of contextuality theory \([65]\). This investigates into the idea that the key difference between classical and quantum phenomena is that classical phenomena are always the same regardless of measurement context, whereas quantum phenomena are context-dependent. While the theory of contextuality has been tied into statistical independence violation previously \([66, 67]\), this is (surprisingly) a fringe position. This is despite statistical independence being the assumption that the choice of measurement (i.e. measurement context) plays no effect on a quantum system, and so models being contextual (where quantum systems are dependent on measurement contexts) and models violating statistical independence (where quantum systems are dependent on measurement settings used) being obviously equivalent. This cognitive dissonance between contextuality being accepted but statistical independence violation being rejected may be due to the typical scenarios these two model behaviours are demonstrated in: a system depending on the measurement context (the totality of measurement settings for the scenario) is possibly more palatable in Kochen-Specker-style scenarios (where spatial separation between elements of the measurement context isn’t emphasised) than in Bell scenarios (where spatial separation between elements of the measurement context is emphasised). The former seems more obviously some form of restriction of possibilities given the context, whereas the latter seems to push for some realist physical mechanism whereby the separate parts of the measurement context (in the EPR-Bohm scenario, the measurement choices at Alice and Bob) coordinate with one another. This links to typical claims that statistical independence violating models are in some sense fine-tuned, or conspiratorial, even when those terms are not well-defined. However, considering statistical independence violation instead as a restriction on counterfactuals, rather than requiring a conspiratorial physical mechanism, both helps resolve this worry and reiterates the link between SI-violation and contextuality (which makes sense, given the EPR-Bohm scenario, and any other Bell Inequality-violating scenario, is also inherently contextual).

V. CONCLUSION

We introduced Bell’s Theorem, and looked at both the assumptions necessary to formulate a Bell inequality, and the experimental loopholes which can allow models which still meet all these assumptions to violate such an inequality.

By looking at the assumptions underpinning Bell inequalities, we showed that one way of interpreting the theorem is that some form of counterfactual restriction allows a model where, despite all other assumptions necessary to create a Bell inequality being allowed, quantum correlations between measurement results can occur. We linked this counterfactual restriction to statistical independence-violating models (especially those coming under the recently-introduced subcategorisation of supermeasured models), and then linked both counterfactual restriction and statistical independence-violation to contextuality. This serves not only to undermine claims like Maudlin’s that counterfactual definiteness/the matching condition are not necessary to generate Bell inequalities, as they don’t appear in the maths (they do, albeit through the statistical independence assumption), but also provides a way of interpreting statistical independence-violating models at odds with their typical conception as either superdeterministic or retrocausal. We hope this helps motivate investigations of links between contextuality, statistical independence violation, and counterfactuality in quantum foundations.

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[1] Asher Peres and Daniel R. Terno. Quantum information and relativity theory. Rev. Mod. Phys., 76:93–123, 1 2004. doi:10.1103/RevModPhys.76.93
[2] John S Bell. Introduction to the hidden-variable question. Technical report, CM-P00058691, 1971.
[3] A. Einstein, B. Podolsky, and N. Rosen. Can quantum-mechanical description of physical reality be considered complete? Phys. Rev., 47:770–780, May 1935. doi:10.1103/PhysRev.47.777
[4] Stuart J. Freedman and John F. Clauser. Experimental test of local hidden-variable theories. Phys. Rev. Lett., 28:938–941, Apr 1972. doi:10.1103/PhysRevLett.28.938
[5] Alain Aspect, Philippe Grangier, and Gérard Roger. Experimental tests of realistic local theories via bell’s theorem. Phys. Rev. Lett., 47:460–463, 8 1981. doi:10.1103/PhysRevLett.47.460
[6] Alain Aspect, Philippe Grangier, and Gérard Roger. Experimental realization of einstein-podolsky-rosen-bohm gedankenexperiment: A new violation of bell’s inequalities. Phys. Rev. Lett., 49:91–94, Jul 1982. doi:10.1103/PhysRevLett.49.91
[7] D. Bohm. Quantum Theory. Dover Books on Physics Series. Dover Publications, 1951. URL: google.co.uk/books?id=9DWim3RhymsC
[8] John F. Clauser, Michael A. Horne, Abner Shimony, and Richard A. Holt. Proposed experiment to test local hidden-variable theories. Phys. Rev. Lett., 23:880–884, Oct 1969. doi:10.1103/PhysRevLett.23.880
[9] Paul Adrien Maurice Dirac. A new notation for quantum mechanics. In Mathematical Proceedings of the Cambridge Philosophical Society, volume 35, pages 416–418. Cambridge University Press, 1939. doi:10.1017/S0305004100021162
[10] Albert Einstein. Letter from einstein to max born, 3 march 1947. In Max Born and Irene Born, editors, The Born-Einstein Letters; Correspondence between Albert Einstein and Max and Hedwig Born from 1916 to 1955, pages 157–158. Macmillan, London, 1971.
[11] Emily Adlam, Jonte R Hance, Sabine Hossenfelder, and Tim N Palmer. Taxonomy for physics beyond quantum mechanics. arXiv preprint arXiv:2309.12293, 2023. doi:10.48550/arXiv.2309.12293
[12] J. S. Bell. On the einstein podolsky rosen paradox. Physics Physique Fizika, 1:195–200, Nov 1964. doi:10.1103/PhysicsPhysiqueFizika.1.195
[13] Eugene P. Wigner. On hidden variables and quantum mechanical probabilities. American Journal of Physics, 38(8):1005–1009, 1970. doi:10.1119/1.1976526
[14] Michael Redhead. Incompleteness, nonlocality, and realism: a prolegomenon to the philosophy of quantum mechanics. Oxford University Press, 1987.
[15] Boris S Cirbel’son. Quantum generalizations of bell’s inequality. Letters in Mathematical Physics, 4(2):93–100, 1980. doi:10.1007/BF00417500
[16] Sandu Popescu and Daniel Rohrlich. Quantum nonlocality as an axion. Foundations of Physics, 24(3):379–385, 3 1994. doi:10.1007/BF02058098
[17] Avishy Carmi and Eliahu Cohen. Relativistic independence bounds nonlocality. Science Advances, 5(4):eaav8370, 2019. doi:10.1126/sciadv.aav8370
[18] Jan-Åke Larsson. loopholes in bell inequality tests of local realism. Journal of Physics A: Mathematical and Theoretical, 47(42):424003, 2014. doi:10.1088/1751-8113/47/42/424003
[19] Diederik Aerts, Sven Aerts, Jan Broekaert, and Liane Gabora. The violation of bell inequalities in the macroworld. Foundations of Physics, 30(9):1387–1414, Sep 2000. doi:10.1023/A:1026449716544
[20] Alain Aspect. Proposed experiment to test the nonseparability of quantum mechanics. Phys. Rev. D, 14:1944–1951, 10 1976. doi:10.1103/PhysRevD.14.1944
[21] Gregor Weihs, Thomas Jennewein, Christoph Simon, Harald Weinfurter, and Anton Zeilinger. Violation of bell’s inequality under strict einstein locality conditions. Physical Review Letters, 81(23):5039, 1998. doi:10.1103/PhysRevLett.81.5039
[22] M. A. Rowe, D. Kielpinski, V. Meyer, C. A. Sackett, W. M. Itano, C. Monroe, and D. J. Wineland. Experimental violation of a bell’s inequality with efficient detection. Nature, 409(6822):791–794, 2 2001. doi:10.1038/35057215
[23] B. Hensen, H. Bernien, A. E. Dréau, A. Reiserer, N. Kalb, M. S. Blok, J. Ruitenberg, R. F. L. Vermeulen, R. N. Schouten, C. Abellán, W. Amaya, V. Pruneri, M. W. Mitchell, M. Markham, D. J. Twitchen, D. Elkouss, S. Wehner, T. H. Taminiau, and R. Hanson. Loophole-free bell inequality violation using electron spins separated by 1.3 kilometres. Nature, 526(7575):682–686, 10 2015. doi:10.1038/nature15759
[24] Marissa Giustina, Marijn A. M. Versteegh, Sören Wengerowsky, Johannes Handsteiner, Armin Hochrainer, Kevin Phelan, Fabian Steinlechner, Johannes Kofler, Jan-Åke Larsson, Carlos Abellán, Waldimar Amaya, Valerio Pruneri, Morgan W. Mitchell, Jörn Beyer, Thomas Gerrits, Adriana E. Lita, Lynden K. Shalm, Sae Woo Nam, Thomas Scheidl, Rupert Ursin, Bernhard Wittmann, and Anton Zeilinger. Significant-loophole-free test of bell’s theorem with entangled photons. Phys. Rev. Lett., 115:250401, Dec 2015. doi:10.1103/PhysRevLett.115.250401
[25] J-A Larsson and Richard David Gill. Bell’s inequality and the coincidence-time loophole. EPL (Europhysics Letters), 67(5):707, 2004. doi:10.1209/epl/i2004-10124-7
[26] Jan-Åke Larsson, Marissa Giustina, Johannes Kofler, Bernhard Wittmann, Rupert Ursin, and Sven Ramelow. Bell-inequality violation with entangled photons, free of the coincidence-time loophole. Phys. Rev. A, 90:032107, 9 2014. doi:10.1103/PhysRevA.90.032107
[27] Howard M Wiseman and Eric G Cavalcanti. Causuriam investigatio and the two bell’s theorems of john bell. In Quantum [Un] Speakables II, pages 119–142. Springer, 2017. doi:10.1007/978-3-319-38987-5_6
[28] Jonte R Hance and Sabine Hossenfelder. What does it take to solve the measurement problem? Journal of Physics Communications, 6(10):102001, 2022. doi:10.1088/2399-6528/ac96cf
[29] Abner Shimony. Controllable and uncontrollable nonlocality. volume 2, page 130–139. Cambridge University Press, 1993. doi:10.1017/CBO9781139172196.010
[30] Gian-Carlo Ghirardi, Alberto Rimini, and Tullio Weber.
A general argument against superluminal transmission through the quantum mechanical measurement process. *Lettere al Nuovo Cimento*, 27(10):293–298, 1980. doi:10.1007/BF02817189

[31] Nature Physics Editorial Team. Survey the foundations. *Nature Physics*, 18(9):961–961, 9 2022. doi:10.1038/s41567-022-01766-x

[32] Sandu Popescu. Nonlocality beyond quantum mechanics. *Nature Physics*, 10(4):264–270, 4 2014. doi:10.1038/nphys2916

[33] Nicolas Brunner, Daniel Cavalcanti, Stefano Pironio, Valerio Scarani, and Stephanie Wehner. Bell nonlocality. *Reviews of Modern Physics*, 86(2):419, 2014. doi:10.1103/RevModPhys.86.419

[34] Maximilian Schlosshauer, Johannes Koffer, and Anton Zeilinger. A snapshot of foundational attitudes toward quantum mechanics. *Studies in History and Philosophy of Science Part B: Studies in History and Philosophy of Modern Physics*, 44(3):222–230, 2013. doi:10.1016/j.shpsb.2013.04.004

[35] Sujeevan Sivasundaram and Kristian Hvidtfelt Nielsen. Surveying the attitudes of physicists concerning foundational issues of surveym theory. arXiv preprint arXiv:1612.00676, 2016. URL: https://arxiv.org/abs/1612.00676

[36] Eddy Keming Chen. Bell’s theorem, quantum probabilities, and superdeterminism. In *The Routledge Companion to Philosophy of Physics*, pages 184–199. Routledge, 2021.

[37] Jonte R. Hance and Sabine Hossenfelder. Bell’s theorem allows local theories of quantum mechanics. *Nature Physics*, 18(12):1382–1382, 12 2022. doi:10.1038/s41567-022-01831-5

[38] Michael J. W. Hall. Relaxed bell inequalities and kochen-specker theorems. *Phys. Rev. A*, 84:022102, Aug 2011. doi:10.1103/PhysRevA.84.022102

[39] Gen Kimura, Yugo Susuki, and Kei Morisue. Relaxed bell inequality as a trade-off relation between measurement dependence and hiddenness. *Physical Review A*, 108(2):022214, 2023. doi:10.1103/PhysRevA.108.022214

[40] Sabine Hossenfelder and Tim Palmer. Rethinking superdeterminism. *Frontiers in Physics*, 8:139, 2020. doi:10.3389/fphy.2020.00139

[41] Sabine Hossenfelder. Superdeterminism: A guide for the perplexed. arXiv preprint arXiv:2010.01324, 2020. URL: https://arxiv.org/abs/2010.01324

[42] Jonte R. Hance, Sabine Hossenfelder, and Tim N. Palmer. Supermeasured: Violating bell-statistical independence without violating physical statistical independence. *Foundations of Physics*, 52(4):81, Jul 2022. doi:10.1007/s10701-022-00602-9

[43] K. B. Wharton and N. Argaman. Colloquium: Bell’s theorem and locally mediated reformulations of quantum mechanics. *Rev. Mod. Phys.*, 92:021002, 2020. doi:10.1103/RevModPhys.92.021002

[44] Indrajit Sen and Antony Valentini. Superdeterministic hidden-variables models i: non-equilibrium and signalling. *Proceedings of the Royal Society A*, 476(2243):20200212, 2020. doi:10.1098/rspa.2020.0212

[45] Indrajit Sen and Antony Valentini. Superdeterministic hidden-variables models ii: conspiracy. *Proceedings of the Royal Society A*, 476(2243):20200214, 2020. doi:10.1098/rspa.2020.0214

[46] Carl H. Brans. Bell’s theorem does not eliminate fully causal hidden variables. *International Journal of Theoretical Physics*, 27(2):219–226, 2 1988. doi:10.1007/BF00670750

[47] Gerard t Hooft. *The cellular automaton interpretation of quantum mechanics*. Springer Nature, 2016. doi:10.1007/978-3-319-41285-6

[48] Gerardo Sanjuán Ciepielewski, Elias Okon, and Daniel Sudarsky. On superdeterministic restrictions of settings independence. *British Journal for the Philosophy of Science*, 2020. doi:10.1086/714819

[49] Sandro Donadi and Sabine Hossenfelder. Toy model for local and deterministic wave-function collapse. *Phys. Rev. A*, 106:022212, Aug 2022. doi:10.1103/PhysRevA.106.022212

[50] TN Palmer. Discretization of the bloch sphere, fractal invariant sets and bell’s theorem. *Proceedings of the Royal Society A*, 476(2230):20190212, 2020. doi:10.1098/rspa.2019.0350

[51] Patrick J. Daley, Kevin J. Resch, and Robert W. Spekkens. Experimentally adjudicating between different causal accounts of Bell-inequality violations via statistical model selection. *Phys. Rev. A*, 105:042220, 4 2022. doi:10.1103/PhysRevA.105.042220

[52] Jonte R Hance and Sabine Hossenfelder. Comment on “experimentally adjudicating between different causal accounts of bell-inequality violations via statistical model selection”. arXiv preprint arXiv:2206.10619, 2022. doi:10.48550/arXiv.2206.10619

[53] TN Palmer. A local deterministic model of quantum spin measurement. *Proceedings of the Royal Society of London, Series A: Mathematical and Physical Sciences*, 451(1943):585-608, 1995. doi:10.1098/rspa.1995.0145

[54] TN Palmer. The invariant set postulate and bell’s theorem. *Proceedings of the Royal Society A: Mathematical, Physical and Engineering Sciences*, 465(2110):3165-3185, 2009. doi:10.1098/rspa.2009.0080

[55] TN Palmer. Invariant set theory. arXiv preprint arXiv:1605.01051, 2016. URL: https://arxiv.org/abs/1605.01051

[56] Andrei Nikolaevich Kolmogorov and Albert T Bharucha-Reid. *Foundations of the theory of probability*: Second edition, 2021.

[57] Tim Maudlin. What bell proved: A reply to belaylock. *American Journal of Physics*, 78(1):121–125, 2010. doi:10.1119/1.3243280

[58] Boris Kment. Varieties of Modality. In Edward N. Zalta, editor, *The Stanford Encyclopedia of Philosophy*. Metaphysics Research Lab, Stanford University, Spring 2021 edition, 2021.

[59] Robert W. Spekkens. Evidence for the epistemic view of quantum states: A toy theory. *Phys. Rev. A*, 75:032110, Mar 2007. doi:10.1103/PhysRevA.75.032110

[60] S. J. van Enk. A toy model for quantum mechanics. *Foundations of Physics*, 37(10):1447–1460, Oct 2007. doi:10.1007/s10701-007-9171-3

[61] T. Paterek, B Dakić, and Č Brukner. Theories of systems with limited information content. *New Journal of Physics*, 12(5):053037, may 2010. doi:10.1088/1367-2630/12/5/053037
[62] Bob Coecke and Bill Edwards. Toy quantum categories (extended abstract). *Electronic Notes in Theoretical Computer Science*, 270(1):29–40, 2011. Proceedings of the Joint 5th International Workshop on Quantum Physics and Logic and 4th Workshop on Developments in Computational Models (QPL/DCM 2008). doi:10.1016/j.entcs.2011.01.004

[63] Stephen D. Bartlett, Terry Rudolph, and Robert W. Spekkens. Reconstruction of gaussian quantum mechanics from liouville mechanics with an epistemic restriction. *Phys. Rev. A*, 86:012103, Jul 2012. doi:10.1103/PhysRevA.86.012103

[64] Robert W. Spekkens. *Quasi-Quantization: Classical Statistical Theories with an Epistemic Restriction*, pages 83–135. Springer Netherlands, Dordrecht, 2016. doi:10.1007/978-94-017-7303-4_4

[65] Costantino Budroni, Adán Cabello, Otfried Gühne, Matthias Kleinmann, and Jan-Åke Larsson. Kochen-specker contextuality. *Rev. Mod. Phys.*, 94:045007, Dec 2022. doi:10.1103/RevModPhys.94.045007

[66] Ehtibar N. Dzhafarov, Janne V. Kujala, and Jan-Åke Larsson. Contextuality in three types of quantum-mechanical systems. *Foundations of Physics*, 45(7):762–782, Jul 2015. doi:10.1007/s10701-015-9882-9

[67] Ehtibar N. Dzhafarov, Janne V. Kujala, and Victor H. Cervantes. Contextuality-by-default: A brief overview of ideas, concepts, and terminology. In Harald Atmanspacher, Thomas Filk, and Emmanuel Pothos, editors, *Quantum Interaction*, pages 12–23, Cham, 2016. Springer International Publishing. doi:10.1007/978-3-319-28675-4_2