ON DEFECT OF COMPACTNESS FOR SOBOLEV SPACES ON MANIFOLDS

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Abstract. Defect of compactness, relative to an embedding of two Banach spaces $E \hookrightarrow F$, is a difference between a weakly convergent sequence $u_k \rightharpoonup u$ in $E$ and $u$, taken up to a remainder that vanishes in the norm of $F$. For Sobolev embeddings in particular, defect of compactness is expressed as a profile decomposition - a sum of terms, called elementary concentrations, with asymptotically disjoint supports. We discuss a profile decomposition for the Sobolev space $H^{1,2}(M)$ of a Riemannian manifold with bounded geometry is a sum of elementary concentrations associated with concentration profiles defined on manifolds different from $M$, that are induced by a limiting procedure. The profiles satisfy an inequality of Plancherel type, and a similar relation, related to the Brezis-Lieb Lemma, holds for $L^p$-norms of profiles on the respective manifolds.

1. Introduction

Defect of compactness of an embedding $E \hookrightarrow F$ of two Banach spaces (the difference between a weakly convergent sequence and its weak limit up to a remainder vanishing in $F$), takes, under general conditions the form of profile decomposition - a sum of, in some sense, decoupled terms, called elementary concentrations, which reflect certain asymptotic behavior of the sequence. Profile decomposition for the Sobolev embedding $\dot{H}^{1,p}(\mathbb{R}^N) \hookrightarrow L^\frac{\mathbb{R}^N}{N}$, $N > p > 1$, was found by Solimini [13], and later, independently, by Gérard [6] and Jaffard [7]. It is a sum of decoupled terms of the form $g_k w \overset{\text{def}}{=} t_k^{\frac{N}{N-p}} w(t_k(x-y_k))$ with $y_k \in \mathbb{R}^N$ and $t_k > 0$. Decoupling of two rescaling operator sequences, $g_k$ and $g_k'$, of this form refers to $g_k^{-1} g_k' \rightharpoonup 0$, and the asymptotic profile is defined
by the inverse rescaling of the original sequence:

\[ g_k^{-1}u_k = t_k^{N-p} u_k(t_k^{-1}x + y_k) \rightarrow w. \]

If the sequence \( u_k \) is furthermore bounded in the homogeneous space \( H^{1,p}(\mathbb{R}^N) \), it has a profile decomposition with no scaling factors \( (t_k = 1) \) and with remainder vanishing in \( L^q \), \( p < q < \frac{pN}{N-p} \). This profile decomposition extends to the Sobolev spaces \( H^{1,2}(M) \) of Riemannian manifolds \( M \) possessing a rich isometry group, with concentrations taking the form \( w \circ \eta_k, \eta_k \in \text{Iso}(M) \) and the profiles \( w \) defined by \( u_k \circ \eta_k^{-1} \rightarrow w \) in \( H^{1,2}(M) \), see [3]. Profile decomposition for a general embedding \( E \hookrightarrow F \) of two Banach spaces, which is cocompact relative to a general group of linear isometries on \( E \), is provided in [14]. In relation to Sobolev spaces of Riemannian manifolds, a profile decomposition similar to Solimini’s was obtained by Struwe [15], but only for a specific class of sequences and only on compact manifolds. Elementary concentrations in Struwe’s profile decomposition are based on asymptotic profiles defined on the tangent spaces to the manifold at the points of concentration. Applications of Struwe’s profile decomposition are elaborated in the monograph [2]. Struwe’s result was extended to the case of general sequences in a recent paper [1].

The result we describe in this announcement generalizes the profile decomposition of [3] to manifolds that may have no nontrivial isometry group. Complete proofs of all statements here are given in [12]. The problem was proposed to one of the authors several years ago by Richard Schoen [9].

### 2. A “SPOTLIGHT” LEMMA

Let \( M \) be a smooth, complete \( N \)-dimensional Riemannian manifold with metric \( g \) and a positive injectivity radius \( r(M) \). We assume that \( M \) is a connected non-compact manifold of bounded geometry. The latter is defined as follows, e.g. cf. [10].

**Definition 2.1.** A smooth Riemannian manifold \( M \) is of bounded geometry if the following two conditions are satisfied:

(i) The injectivity radius \( r(M) \) of \( M \) is positive.

(ii) Every covariant derivative of the Riemann curvature tensor \( R^M \) of \( M \) is bounded, i.e., \( \nabla^k R^M \in L^\infty(M) \) for every \( k = 0, 1, \ldots \)

In what follows \( B(x, r) \) will denote a geodesic ball in \( M \) and \( \Omega_r \) will denote the ball in \( \mathbb{R}^N \) of radius \( r \) centered at the origin. Let \( r \in (0, r(M)) \) be fixed. Then the Riemannian exponential map \( \exp_x \) is a diffeomorphism of \( \{v \in T_x M : g_x(v, v) < r\} \) onto \( B(x, r) \). For
each $x \in M$ we choose an orthonormal basis for $T_x M$ which yields an identification $i_x : \mathbb{R}^N \to T_x M$. Then $e_x : \Omega_r \to B(x, r)$ will denote a geodesic normal coordinates at $x$ given by $e_x = \exp_x \circ i_x$. We do not require smoothness of the map $i_x$ with respect to $x$. We will consider the maps $e_x$ as defined on the balls $\Omega_a$ with $a = \frac{3}{4} r(M)$.

**Definition 2.2.** A subset $Y$ of Riemannian manifold $M$ is called $\varepsilon$-discretization of $M$, $\varepsilon > 0$, if the distance between any two distinct points of $Y$ is greater than or equal to $\varepsilon$ and

$$M = \bigcup_{y \in Y} B(y, \varepsilon).$$

Any connected Riemannian manifold $M$ has a $\varepsilon$-discretization for any $\varepsilon > 0$, and if $M$ is of bounded geometry then for any $t \geq 1$ the covering $\{B(y, t \varepsilon)\}_{y \in Y}$ is uniformly locally finite.

The following statement is a counterpart of the "cocompactness" lemma proved in the Euclidean case by Lieb [8].

**Lemma 2.3** ("Spotlight lemma"). Let $M$ be an $N$-dimensional Riemannian manifold of bounded geometry and let $Y \subset M$ be a $r$-discretization of $M$, $r < r(M)$. Let $(u_k)$ be a bounded sequence in $H^{1, 2}(M)$. Then, $u_k \to 0$ in $L^p(M)$ for any $p \in (2, \frac{2N}{N-2})$ if and only if $u_k \circ e_{y_k} \to 0$ in $H^{1, 2}(\Omega_r)$ for any sequence $(y_k)$, $y_k \in Y$.

3. MANIFOLD-AT-INFINITY

In what follows we consider the radius $\rho < \frac{r(M)}{8}$ and a $\hat{\rho}$-discretization $Y$ of $M$, $\frac{\rho}{2} < \hat{\rho} < \rho$. We will write $\mathbb{N}_0 \overset{\text{def}}{=} \mathbb{N} \cup \{0\}$.

**Definition 3.1.** Let $(y_k)_{k \in \mathbb{N}}$ be a sequence in $Y$ that is an enumeration of the infinite subset of $Y$. A countable family $\{(y_{k;i})_{i \in \mathbb{N}_0}\}_{i \in \mathbb{N}_0}$ of sequences on $Y$ is called a trailing system for $(y_k)_{k \in \mathbb{N}}$ if for every $k \in \mathbb{N}$ $(y_{k;i})_{i \in \mathbb{N}_0}$ is an ordering of $Y$ by the distance from $y_k$, that is, an enumeration of $Y$ such that $d(y_{k;i}, y_k) \leq d(y_{k;i+1}, y_k)$ for all $i \in \mathbb{N}_0$. In particular, $y_{k;0} = y_k$.

It is easy to see that any enumeration of the infinite subset of $Y$ admits a trailing system. With a given trailing system $\{(y_{k;i})_{k \in \mathbb{N}}\}_{i \in \mathbb{N}_0}$ we associate a manifold $M_{\infty}^{(y_{k;i})}$ defined by a gluing data described below.

We give only a rough sketch of construction that involves many technical details. For each such pair $(i, j) \in \mathbb{N}_0$ we consider maps on $\Omega_{2\rho}$

$$\psi_{ij,k} \overset{\text{def}}{=} e_{y_{k;i}}^{-1} \circ e_{y_{k;j}},$$
The range of $e_{y_{k;i}}$, which is $B(y_{k;j}, 2\rho)$, may not necessarily fall into the domain of $e^{-1}_{y_{k;i}}$, which is $B(y_{k;i}, \alpha)$, so the maps $\psi_{ij,k}$ are defined only for a subset of $i, j$ and $k$. There is, however, a certain non-empty set $\mathcal{K} \subset \mathbb{N}_0 \times \mathbb{N}_0$ such that $\psi_{ij,k} \equiv e^{-1}_{y_{k;i}} \circ e_{y_{k;j}}$ is a map $\Omega_{2\rho} \rightarrow \Omega_{\alpha}$ for all $k$ sufficiently large whenever $(i, j) \in \mathcal{K}$.

From boundedness of the geometry and the Ascoli-Arzelà theorem, it follows that there is a renamed subsequence of $(\psi_{ij,k})_{k \in \mathbb{N}}$, $(i, j) \in \mathcal{K}$, that converges in $C^\infty(\bar{\Omega}_{2\rho})$ to some smooth function $\psi_{ij} : \Omega_{2\rho} \rightarrow \Omega_{\alpha}$, and, moreover, we may assume that the same extraction of $(\psi_{ji,k})_{k \in \mathbb{N}}$ converges in $C^\infty(\bar{\Omega}_{2\rho})$ as well. We define $\Omega_{ij} \equiv \psi_{ij}(\Omega_{\rho}) \cap \Omega_{\rho}$. This set may generally be empty. Let us define

$$\mathcal{K} \equiv \{(i, j) \in \mathcal{K} : \Omega_{ij} \neq \emptyset\}. \quad (3.1)$$

Basing on the gluing theorem in [5] we can associate the family of sets $\Omega_{ij}$ and the maps $\psi_{ij}$ with a differentiable manifold as follows.

**Proposition 3.2.** Let $M$ be a Riemannian manifold with bounded geometry and let $Y$ be its discretization.

For any trailing system $\{(y_{k;i})_{k \in \mathbb{N}}\}_{i \in \mathbb{N}_0}$ related to the sequence $(y_k)$ in $Y$ there exists a smooth manifold $M_{\infty}^{(y_{k;i})}$ with an atlas $\{(U_i, \tau_i)\}_{i \in \mathbb{N}_0}$ such that:

1) $\tau_i(U_i) = \Omega_{\rho}$,

2) there exists a renamed subsequence of $k$ such that for any two charts $(U_i, \tau_i)$ and $(U_j, \tau_j)$ with $U_i \cap U_j \neq \emptyset$ the corresponding transition map $\psi_{ij} : \tau_j(U_j \cap U_i) \rightarrow \tau_i(U_j \cap U_i)$ is the following $C^\infty$-limit:

$$\psi_{ij} = \lim_{k \rightarrow \infty} e^{-1}_{y_{k;i}} \circ e_{y_{k;j}}.$$

For convenience we introduce "inverse" charts $\varphi_i = \tau^{-1}_i$ so that $\varphi_j^{-1} \circ \varphi_i = \psi_{ji} : \Omega_{ij} \rightarrow \Omega_{ji}$.

We can endow the manifold $M_{\infty}^{(y_{k;i})}$ with a metric which is also related to the asymptotic properties of $M$. For any $i \in \mathbb{N}_0$ we define a metric tensor $g^{(i)}$ on $\Omega_{\rho}$, by the limiting procedure on a suitable renamed subsequence:

$$g^{(i)}_\xi(v, w) \equiv \lim_{k \rightarrow \infty} g_{e_{y_{k;i}}(\xi)}(de_{y_{k;i}}(v), de_{y_{k;i}}(w)), \xi \in \Omega_{\rho} \text{ and } v, w \in \mathbb{R}^N,$$

$$(3.2)$$

Existence of the limit follows from the boundedness of the geometry of $M$. Afterwards we pull the metric tensor back onto $U_i = \varphi_i(\Omega_{\rho}) \subset$
Definition 3.3. A manifold at infinity $M_{\infty}^{(y_{k,i})}$ of a manifold $M$ with bounded geometry, generated by a trailing system $\{(y_{k,i})_{k \in \mathbb{N}}\}_{i \in \mathbb{N}_0}$ of a sequence $(y_k)$ in $Y$, is a differentiable manifold supplied with a Riemannian metric tensor $\tilde{g}$ defined by (3.3).

Since all the limits in construction are uniform $C^\infty$-limits, manifolds at infinity of $M$ are also of bounded geometry.

Proposition 3.4. Let $M$ be a Riemannian manifold with bounded geometry and let $Y$ be its $\rho$-discretization, $\rho/2 < \hat{\rho} < \rho < \frac{r(M)}{2}$. Then for every discrete sequence $(y_k)$ in $Y$ and its trailing system $\{(y_{k,i})_{k \in \mathbb{N}}\}_{i \in \mathbb{N}_0}$ there exists a renamed subsequence $(y_k)$ that generates a Riemannian manifold at infinity $M_{\infty}^{(y_{k,i})}$ of the manifold $M$. The manifold $M_{\infty}^{(y_{k,i})}$ has bounded geometry with an injectivity radius not less than $\rho$.

Remark 3.5. If $M'$ is another manifold such that $M$ and $M'$ have respective compact subset $M_0$ and $M'_0$ such that $M \setminus M_0$ is isometric to $M' \setminus M'_0$, i. e. if $M'$ is $M$ up to a compact perturbation, then their manifolds at infinity for corresponding trailing systems coincide. From this follows that manifold at infinity of the manifold $M$ is not necessarily diffeomorphic to $M$.

4. Concentration profiles. The main result

Definition 4.1. Let $M$ be a manifold of bounded geometry and $Y$ be its discretization. Let $(u_k)$ be a bounded sequence in $H^{1,2}(M)$. Let $(y_k)$ be a sequence of points in $Y$ and let $\{(y_{k,i})_{k \in \mathbb{N}}\}_{i \in \mathbb{N}_0}$ be its trailing system. One says that $w_i \in H^{1,2}(\Omega_\rho)$ is a local profile of $(u_k)$ relative to a trailing sequence $(y_{k,i})_{k \in \mathbb{N}}$, if, on a renamed subsequence, $u_k \circ e_{y_{k,i}} \rightharpoonup w_i$ in $H^{1,2}(\Omega_\rho)$ as $k \to \infty$. If $(y_k)$ is a renamed (diagonal) subsequence such that $u_k \circ e_{y_{k,i}} \rightharpoonup w_i$ in $H^{1,2}(\Omega_\rho)$ as $k \to \infty$ for all $i \in \mathbb{N}_0$, then the family $\{w_i\}_{i \in \mathbb{N}_0}$ is called an array of local profiles of $(u_k)$ relative to the trailing system $\{(y_{k,i})_{k \in \mathbb{N}}\}_{i \in \mathbb{N}_0}$ of the sequence $(y_k)$.

Proposition 4.2. Let $M$ be a manifold of bounded geometry and let $Y$ its discretization. Let $(u_k)$ be a bounded sequence in $H^{1,2}(M)$. Let $\{w_i\}_{i \in \mathbb{N}_0}$ be an array of local profiles of $(u_k)$ associated with a trailing system $\{(y_{k,i})_{k \in \mathbb{N}}\}_{i \in \mathbb{N}_0}$ related to the sequence $(y_k)$ in $Y$. Then there
exists a function \( w : M_\infty^{(y_{k,i})} \rightarrow \mathbb{R} \) such that \( w \circ \varphi_i = w_i, \; i \in \mathbb{N}_0 \), where \( \varphi_i : \Omega_\rho \rightarrow M_\infty^{(y_{k,i})} \) are local coordinate maps of \( M_\infty^{(y_{k,i})} \).

**Definition 4.3.** Let \( \{w_i\}_{i \in \mathbb{N}_0} \) be a local profile array of a bounded sequence \( \langle u_k \rangle \) in \( H^{1,2}(M) \) relative to a trailing system \( \{(y_{k,i})_{k \in \mathbb{N}}\}_{i \in \mathbb{N}_0} \). The function \( w : M_\infty^{(y_{k,i})} \rightarrow \mathbb{R} \) given by Proposition 4.2 is called the global profile of the sequence \( \langle u_k \rangle \) relative to \( (y_{k,i}) \).

Since \( M \) has bounded geometry, we may fix a uniformly smooth partition of unity \( \{\chi_y\}_{y \in Y} \) subordinated to the uniformly finite covering of \( M \) by geodesic balls \( \{B(y, \rho)\}_{y \in Y} \).

**Definition 4.4.** Let \( M \) be a manifold of bounded geometry and let \( Y \) be its discretization. Let \( M_\infty^{(y_{k,i})} \) be a manifold at infinity of \( M \) generated by a trailing system \( \{(y_{k,i})_{k \in \mathbb{N}}\}_{i \in \mathbb{N}_0} \). An elementary concentration associated with a function \( w : M_\infty^{(y_{k,i})} \rightarrow \mathbb{R} \) is a sequence \( \langle W_k \rangle_{k \in \mathbb{N}} \) of functions \( M \rightarrow \mathbb{R} \) given by

\[
W_k = \sum_{i \in \mathbb{N}_0} \chi_{y_{k,i}} w \circ \varphi_i \circ e^{-1}_{y_{k,i}}, \quad k \in \mathbb{N}.
\]

where \( \varphi_i \) are the local coordinate maps of manifold \( M_\infty^{(y_{k,i})} \).

In heuristic terms, after we find limits \( w_i, \; i \in \mathbb{N}_0 \), of the sequence \( \langle u_k \rangle \) under the “trailing spotlights” \( \langle e_{y_{k,i}} \rangle_{k \in \mathbb{N}_0} \) that follow different trailing sequences \( \langle y_{k,i} \rangle_{k \in \mathbb{N}} \) of \( (y_k) \), we give an approximate reconstruction \( W_k \) of \( u_k \) “centered” on the moving center \( y_k \) of the “core spotlight”. We do that by first splitting \( w \) into local profiles \( w \circ \varphi_i, \; i \in \mathbb{N}_0 \), on the set \( \Omega_\rho \), casting them onto the manifold \( M \) in the vicinity of \( y_{k,i} \) by composition with \( e^{-1}_{y_{k,i}} \), and patching all such compositions together by the partition of unity on \( M \). Such reconstruction approximates \( u_k \) on geodesic balls \( B(y_k, R) \) with any \( R > 0 \), but it ignores the values of \( u_k \) for \( k \) large on the balls \( B(y'_k, R) \), with \( d(y_k, y'_k) \rightarrow \infty \), where \( u_k \) is approximated by a different local concentration. It has been shown in [3] for the case of manifold \( M \) with cocompact action of a group of isometries (in particular, for homogeneous spaces) that a global reconstruction of \( u_k \), up to a remainder vanishing in \( L^p(M) \), is a sum elementary concentrations associated with all such mutually decoupled sequences.

Similarly, the profile decomposition theorem below, which is the main result of this paper, says that any bounded sequence \( \langle u_k \rangle \) in \( H^{1,2}(M) \) has a subsequence that, up to a remainder vanishing in \( L^p(M) \), \( p \in (2, 2^*) \), equals a sum of decoupled elementary concentrations. To simplify the notation we will index the sequences, the related trailing
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systems and the corresponding manifold by \( n \), i.e. below we write \( y^{(n)} \), \( y_{k;i}^{(n)} \), and \( M_{\infty}^{(n)} \).

**Theorem 4.5.** Let \( M \) be a manifold of bounded geometry and let \( Y \) be its discretization. Let \((u_k)\) be a sequence in \( H^{1,2}(M) \) weakly convergent to some function \( w^{(0)} \) in \( H^{1,2}(M) \). Then there exists a renamened subsequence of \((u_k)\), sequences \((y^{(n)}_k)_{k \in \mathbb{N}} \) in \( Y \), and associated with them global profiles \( w^{(n)} \) on the respective manifolds at infinity \( M_{\infty}^{(n)} \), \( n \in \mathbb{N} \), such that \( d(y^{(n)}_k, y^{(m)}_k) \to \infty \) when \( n \neq m \), and

\[
    u_k - w^{(0)} - \sum_{n \in \mathbb{N}} W_k^{(n)} \to 0 \quad \text{in} \quad L^p(M), \quad p \in (2, 2^*), \tag{4.2}
\]

where \( W_k^{(n)} = \sum_{i \in \mathbb{N}_0} \chi_i^{(n)} w^{(n)} \circ \varphi_i^{(n)} \circ e^{-1}_{y_{k;i}} \) are elementary concentrations, \( \varphi_i^{(n)} \) are the local coordinates of the manifolds \( M_{\infty}^{(n)} \) and \( \{\chi_i^{(n)}\}_{i \in \mathbb{N}_0} \) the corresponding partitions of unity. The series \( \sum_{n \in \mathbb{N}} W_k^{(n)} \) converges in \( H^{1,2}(M) \) unconditionally and uniformly in \( k \in \mathbb{N} \). Moreover,

\[
    \|w^{(0)}\|_{H^{1,2}(M)}^2 + \sum_{n=1}^{\infty} \|w^{(n)}\|_{H^{1,2}(M_{\infty}^{(n)})}^2 \leq \limsup_{k \to \infty} \|u_k\|_{H^{1,2}(M)}^2, \tag{4.3}
\]

and

\[
    \int_M |u_k|^p dv_g \to \int_M |w^{(0)}|^p dv_g + \sum_{n=1}^{\infty} \int_{M_{\infty}^{(n)}} |w^{(n)}|^p dv_g. \tag{4.4}
\]

**Remark 4.1.** In [12] it is shown that Theorem 4.5 implies the profile decomposition of [3] in the case when \( M \) is cocompact relative to a discrete isometry group.

It is interesting to compare the objects at infinity in the profile decomposition (4.2) and in the profile decompositions in [1, 15]. In the latter, loss of compactness occurs due to blowup concentrations, and concentration profiles, defined by behavior of the sequence near a given point, are functions on the tangent space, which can be seen as the manifold-at-infinity created by the concentration mechanism at work - zooming into the manifold \( M \) at a given point. In (4.2) concentration profiles are generated by localized shifts to infinity, followed by a reassembly on a new manifold. In both profile decompositions, sum of the energies of profiles on respective manifolds at infinity is dominated by the energy of the sequence, and analogous relations hold for the \( L^p \)-norms.
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