A Short Proof of a Grothendieck-Lefschetz Theorem for Equivariant Picard Groups

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Abstract

We give a short proof of a Grothendieck-Lefschetz Theorem for equivariant Picard groups of nonsingular varieties with the action of an affine algebraic group.

1 Introduction

The Grothendieck-Lefschetz Theorem for Picard groups, see Cor XII.3.6 and Cor XII.3.7 in [Gro68] or Cor IV.3.3 in [Har70], states that if $X$ is a nonsingular projective variety over a field of characteristic zero and $Y$ is a subvariety of $X$, such that $\dim(Y) \geq 3$ and $Y$ is a complete intersection in $X$, then the natural map $\text{Pic}(X) \to \text{Pic}(Y)$ is an isomorphism.

If $G$ is an algebraic group acting on $X$, we let $\text{Pic}^G(X)$ denote the equivariant Picard group of $X$, as defined in Chapter 1.3 of [MFK94]; in other words, the elements of $\text{Pic}^G(X)$ are line bundles over $X$ with a $G$-linearisation, and this is a group via the tensor product. It has been shown recently in [Rav18] that a version of Grothendieck-Lefschetz holds for the equivariant Picard group, provided that the group $G$ is assumed to be finite. This was done by adapting the proof of Grothendieck-Lefschetz in [Har70] to deal with the group action.

In this paper, we drop the finiteness hypothesis and prove the result for any affine algebraic group. More specifically, we prove the following:

**Theorem 1.** Let $k$ be a field. Let $Y \subset X$ be normal proper $k$-varieties. Let $G$ be an affine algebraic group over $k$ acting on $X$, and suppose $Y$ is $G$-invariant. If the natural map $\text{Pic}(X) \to \text{Pic}(Y)$ is injective (resp. an isomorphism), then the natural map on equivariant Picard groups $\text{Pic}^G(X) \to \text{Pic}^G(Y)$ is also injective (resp. an isomorphism).

As a corollary, we obtain the equivariant version of Grothendieck-Lefschetz:

**Corollary 2.** Let $k$ be a field of characteristic zero. Let $X$ be a nonsingular projective variety over $k$ and let $G$ be an affine algebraic group over $k$ acting on $X$. Let $Y$ be a $G$-invariant normal subvariety of $X$, such that $\dim(Y) \geq 3$ and $Y$ is a complete intersection in $X$. Then the natural map $\text{Pic}^G(X) \to \text{Pic}^G(Y)$ is an isomorphism. If instead we assume that $\dim(Y) = 2$, then the natural map $\text{Pic}^G(X) \to \text{Pic}^G(Y)$ is injective.

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2 Equivariant Grothendieck-Lefschetz

Before proving Theorem 1, we need a technical lemma that will give us an exact sequence involving $\text{Pic}^G(X)$. Throughout, we assume that varieties are irreducible.

**Lemma 3.** Let $X$ be a normal proper variety over $k$ and $G$ be an affine algebraic group over $k$ acting on $X$. Then there exists an exact sequence of abelian groups

$$0 \to \text{Hom}_{\text{GpSch}}(G, k^\times) \to \text{Pic}^G(X) \to \text{Pic}(X)^G \to \text{Ext}^1_{\text{GpSch}}(G, k^\times).$$

**Proof.** Let $\text{Ext}^1_{\text{Gp}}(G, k^\times)$ denote the set of (not necessarily central) extensions of abstract groups, where $k^\times = \mathcal{O}_X(X)^\times$ is viewed as a $G$-module via the morphism $G \to \text{Aut}(X)$. We first remark that, since $k^\times$ is an abelian group, then the set $\text{Ext}^1_{\text{Gp}}(G, k^\times)$ is itself endowed with the structure of an abelian group, with the Baer sum as the group operation; see Ex. 7 on page 114 of [Mac95]. We can then consider the subset $\text{Ext}^1_{\text{GpSch}}(G, k^\times)$ of extensions of group schemes. In fact, from the definition of the Baer sum, we can see that $\text{Ext}^1_{\text{GpSch}}(G, k^\times)$ is closed under the group operation, so that it is a subgroup of $\text{Ext}^1_{\text{Gp}}(G, k^\times)$.

We define the map $\epsilon: \text{Pic}(X)^G \to \text{Ext}^1_{\text{GpSch}}(G, k^\times)$ as in Section 3.4 of [Bri]. We introduce some notation, so that we can spell out this definition: For a line bundle $L$ over $X$, we define $\text{Aut}(X)_L = \{ \phi \in \text{Aut}(X): \phi^* L \cong L \}$ to be the subgroup of $\text{Aut}(X)$ that stabilises $L$, and we let $\text{Aut}^G_m(L)$ denote the group of automorphisms of $L$ that commute with the action of $\mathbb{G}_m$ by multiplication on the fibres of $L$.

By Lemma 3.4.1 in [Bri], we have an extension $1 \to k^\times \to \text{Aut}^G_m(L) \to \text{Aut}(X)_L \to 1$. If, furthermore, $L \in \text{Pic}(X)^G$, then the image of the homomorphism $G \to \text{Aut}(X)$ is contained in $\text{Aut}(X)_L$. Thus, we can pull back the given group extension by the homomorphism $G \to \text{Aut}(X)_L$, and we obtain another short exact sequence $1 \to k^\times \to \mathcal{G}(L) \to G \to 1$, where $\mathcal{G}(L)$ is called the lifting group associated to $L$. We then define $\epsilon$ by $L \mapsto (1 \to k^\times \to \mathcal{G}(L) \to G \to 1)$. We observe that since the morphism $G \to \text{Aut}(X)$ is algebraic, then $\epsilon(L)$ is an extension of group schemes.

We let $\alpha: \text{Pic}^G(X) \to \text{Pic}(X)^G$ be the homomorphism that forgets the linearisation. By Cor 7.1 of [Dol03], we have that $\ker(\alpha) = \text{Hom}_{\text{GpSch}}(G, k^\times)$.

As in Remark 3.4.3(i) in [Bri], $L \in \text{Pic}(X)^G$ is linearisable if and only if the extension $\epsilon(L)$ splits as an extension of abstract groups and the resulting action of $G$ on $L$ is algebraic. Note that this latter condition is automatic when $G$ is finite; in general, what we then require is that the extension $\epsilon(L)$ splits when viewed as an extension of algebraic groups, rather than just abstract groups. Indeed, in this situation, the induced group homomorphism $G \to \mathcal{G}(L) \to \text{Aut}^G_m(L)$ will be algebraic, and hence we obtain a linearisation of $L$. Thus, we have $\im(\alpha) = \ker(\epsilon)$, which gives our assertion.

We are now ready to prove our main theorem.

**Proof of Theorem 1.** By Lemma 3, we have the following commutative diagram with exact rows:

\[
\begin{array}{cccccc}
0 & \longrightarrow & \text{Hom}(G, k^\times) & \longrightarrow & \text{Pic}^G(X) & \longrightarrow & \text{Pic}(X)^G & \longrightarrow & \text{Ext}^1_{\text{GpSch}}(G, k^\times) \\
& & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & \text{Hom}(G, k^\times) & \longrightarrow & \text{Pic}^G(Y) & \longrightarrow & \text{Pic}(Y)^G & \longrightarrow & \text{Ext}^1_{\text{GpSch}}(G, k^\times)
\end{array}
\]


The two leftmost and the rightmost vertical arrows are obviously isomorphisms. By hypothesis, the natural map $\text{Pic}(X) \to \text{Pic}(Y)$ is injective (resp. an isomorphism), and because it is $G$-equivariant, it induces an injection (resp. an isomorphism) of the $G$-invariant Picard groups $\text{Pic}(X)^G \to \text{Pic}(Y)^G$. In other words, the second vertical arrow (counting from the right) is injective (resp. an isomorphism). By the Five Lemma, the middle arrow is injective (resp. an isomorphism) as well.

\[\square\]

**Proof of Corollary 2.** In view of Theorem 1, we just need to show that $\text{Pic}(X) \to \text{Pic}(Y)$ is injective when $\dim(Y) = 2$ and an isomorphism when $\dim(Y) \geq 3$. This follows immediately from the Grothendieck-Lefschetz Theorem for Picard groups.

\[\square\]

**Remark.** We observe that if $X = \mathbb{P}^n_k$, then we can drop the hypothesis that $k$ has characteristic zero in Corollary 2, due to Cor XII.3.7 of [Gro68].

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