A link between two elliptic quantum groups

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March 22, 2018

Abstract

We consider the category $\mathcal{C}_B$ of meromorphic finite-dimensional representations of the quantum elliptic algebra $B$ constructed via Belavin’s $R$-matrix, and the category $\mathcal{C}_F$ of meromorphic finite-dimensional representations of Felder’s elliptic quantum group $E_{\tau,\gamma}(gl_n)$. For any fixed $c \in \mathbb{C}$, we use a version of the Vertex-IRF correspondence to construct two families of (generically) fully faithful functors $H^c \times : \mathcal{C}_B \to \mathcal{D}_B$ and $F^c \times : \mathcal{C}_F \to \mathcal{D}_B$ where $\mathcal{D}_B$ is a certain category of infinite-dimensional representations of $B$ by difference operators. We use this to construct an equivalence between the abelian subcategory of $\mathcal{C}_B$ generated by tensor products of vector representations and the abelian subcategory of $\mathcal{C}_F$ generated by tensor products of vector representations.

1 Categories of meromorphic representations

In this section, we recall the definitions of various categories of representations of quantum elliptic algebras.

Notations: let us fix $\tau \in \mathbb{C}$, $\text{Im}(\tau) > 0$, $\gamma \in \mathbb{R} \setminus \mathbb{Q}$ and $n \geq 2$. Denote by $(v_i)_{i=1}^n$ the canonical basis of $\mathbb{C}^n$ and by $(E_{ij})_{i,j=1}^n$ the canonical basis of $\text{End}(\mathbb{C}^n)$, i.e $E_{ij}v_k = \delta_{jk}v_i$. Let $\mathfrak{h} = \{ \sum \lambda_i E_{ii} \mid \sum \lambda_i = 0 \}$ be the space of diagonal traceless matrices. We have a natural identification $\mathfrak{h}^* = \{ \sum \lambda_i E_{ii}^* \mid \sum \lambda_i = 0 \}$. In particular, the weight of $v_i$ is $\omega_i = E_{ii}^* - \frac{1}{n} \sum_k E_{kk}^*$.

Classical theta functions: the theta function $\theta_{\kappa,\kappa'}(t; \tau)$ with characteristics $\kappa, \kappa' \in \mathbb{R}$ is defined by the formula

$$\theta_{\kappa,\kappa'}(t; \tau) = \sum_{m \in \mathbb{Z}} e^{i\pi(m+\kappa)(m+\kappa)\tau + 2i(m+\kappa')t}.$$ 

It is an entire function whose zeros are simple and form the (shifted) lattice $\{ \frac{1}{2} - \kappa + (\frac{1}{2} - \kappa')\tau \} + \mathbb{Z} + \tau\mathbb{Z}$.

Theta functions satisfy (and are characterized up to renormalization by) the following fundamental monodromy relations

$$\theta_{\kappa,\kappa'}(t+1; \tau) = e^{2i\pi\kappa} \theta_{\kappa,\kappa'}(t; \tau), \quad (1)$$
$$\theta_{\kappa,\kappa'}(t+\tau; \tau) = e^{-i\pi\tau - 2i\pi(t+\kappa')} \theta_{\kappa,\kappa'}(t; \tau). \quad (2)$$

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Theta functions with different characteristics are related to each other by shifts of \( t \):

\[
\theta_{\kappa_1+\kappa_2,\kappa_1'}(t; \tau) = e^{i\pi \kappa_2^2 \tau + 2i\pi \kappa_2 (t + \kappa_1' + \kappa_1') \theta_{\kappa_1,\kappa_1'}(t + \kappa_2 \tau + \kappa_1') \tau}. \tag{3}
\]

In particular, we set \( \theta(t) = \theta_{\frac{2}{3}, \frac{2}{3}}(t; \tau) \).

### 1.1 Meromorphic representations of the Belavin quantum elliptic algebra

Consider the two \( n \times n \) matrices

\[
A = \begin{pmatrix}
1 & 0 & \ldots & 0 \\
0 & \xi & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & \xi^{n-1}
\end{pmatrix}, \quad B = \begin{pmatrix}
0 & 1 & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & 1 \\
1 & 0 & \ldots & 0
\end{pmatrix}
\]

where \( \xi = e^{2i\pi/n} \). We have \( A^n = B^n = \text{Id}, \ BA = \xi AB \); i.e. \( A, B \) generate the Heisenberg group. Belavin ([2]) introduced the matrix \( R^B(z) \in \text{End}(\mathbb{C}^n) \otimes \text{End}(\mathbb{C}^n) \), uniquely determined by the following properties:

1. Unitarity: \( R^B(z)R^B_{21}(-z) = 1 \),
2. \( R^B(z) \) is meromorphic, with simple poles at \( z = \gamma + \mathbb{Z} + \tau \mathbb{Z} \),
3. \( R^B(0) = P : x \otimes y \mapsto y \otimes x \) for \( x, y \in \mathbb{C}^n \) (permutation),
4. Lattice translation properties:

\[
R^B(z + 1) = A_1 R^B(z) A_1^{-1} = A_2^{-1} R^B(z) A_2,
R^B(z + \tau) = e^{-2i\pi \frac{\gamma}{n}} B_1 R^B(z) B_1^{-1} = e^{-2i\pi \frac{\gamma}{n}} B_2^{-1} R^B(z) B_2.
\]

In particular, \( R^B(z) \) commutes with \( A \otimes A \) and \( B \otimes B \). The matrix \( R^B(z) \) satisfies the quantum Yang-Baxter equation with spectral parameters:

\[
R^B_{12}(z-w)R^B_{13}(z)R^B_{23}(w) = R^B_{23}(w)R^B_{13}(z)R^B_{12}(z-w).
\]

### The category \( \mathcal{C}_B \)

following Faddeev, Reshetikhin, Takhtajan and Semenov-Tian-Shansky, one can define an algebra \( \mathcal{B} \) from \( R^B(z) \), using the RLL formalism—see [8], [10]. However, we will only need to consider a certain category of modules over this algebra, defined as follows.

Let \( \mathcal{C}_B \) be the category whose objects are pairs \((V, L(z))\) where \( V \) is a finite dimensional vector space and \( L(z) \in \text{End}(\mathbb{C}^n) \otimes \text{End}(V) \) is an invertible meromorphic function (the L-operator) such that \( L(z+n) = L(z) \) and \( L(z+n\tau) = L(z) \), satisfying the following relation in the space \( \text{End}(\mathbb{C}^n) \otimes \text{End}(V) \otimes \text{End}(V) \):

\[
R^B_{12}(z-w)L_{13}(z)L_{23}(w) = L_{23}(w)L_{13}(z)R^B_{12}(z-w) \tag{4}
\]

(as meromorphic functions of \( z \) and \( w \)); morphisms \((V, L(z)) \to (V', L'(z))\) are linear maps \( \varphi : V \to V' \) such that \((1 \otimes \varphi)L(z) = L'(z)(1 \otimes \varphi)\) in the space \( \text{Hom}(\mathbb{C}^n \otimes V, \mathbb{C}^n \otimes V') \). The quantum Yang-Baxter relation for \( R^B \) implies that
(\mathbb{C}^n, \chi(z) R^B(z - w)) \in \text{Ob}(\mathcal{C}_B)$ for all $w \in \mathbb{C}$, where we set $\chi(z) = \frac{\theta(z - (1 - \frac{1}{g}))}{\theta(z)}$.

This object is called the vector representation and will be denoted simply by $V_B(w)$.

The category $\mathcal{C}_B$ is naturally a tensor category with tensor product

$$(V, L(z)) \otimes (V', L'(z)) = (V \otimes V', L_{12}(z)L_{13}(z))$$

(5)

at the level of objects and with the usual tensor product at the level of morphisms.

There is a notion of a dual representation in the category $\mathcal{C}_B$: the (right) dual of $(V, L(z))$ is $(V^*, L^*(z))$ where $L^*(z) = L^{-1}(z)^t$ (first apply inversion, then apply the transposition in the second component $t_2$). If $V, W \in \text{Ob}(\mathcal{C}_B)$ and $\varphi \in \text{Hom}_{\mathcal{C}_B}(V, W)$ then $\varphi^\dagger \in \text{Hom}_{\mathcal{C}_B}(W^*, V^*)$.

We will also need an extended category $\mathcal{C}_B^\ast$ defined as follows: objects of $\mathcal{C}_B^\ast$ are objects of $\mathcal{C}_B$ but we set

$$\text{Hom}_{\mathcal{C}_B^\ast}(V, V') = \text{Hom}_{\mathcal{C}_B}(V, V') \otimes M_\mathbb{C}$$

where $M_\mathbb{C}$ is the field of meromorphic functions of a complex variable $x$. In other words, morphisms in $\mathcal{C}_B^\ast$ are meromorphic 1-parameter families of morphisms in $\mathcal{C}_B$.

The category $\mathcal{D}_B$: We now define a difference-operator variant of the categories $\mathcal{C}_B, \mathcal{C}_B^\ast$. Let us denote by $M_{h^\ast}$ the field of $(n\omega_i)$-periodic meromorphic functions $h^\ast \rightarrow \mathbb{C}$ and by $D_{h^\ast}$ the $\mathbb{C}$-algebra generated by $M_{h^\ast}$ and shift operators $T_\mu : M_{h^\ast} \rightarrow M_{h^\ast}, f(\lambda) \mapsto f(\lambda + \mu)$ for $\mu \in h^\ast$. If $V$ is a finite-dimensional vector space, we set $V_{h^\ast} = M_{h^\ast} \otimes V$, and $D(V) = D_{h^\ast} \otimes \text{End}(V)$. Let $\mathcal{D}_B$ be the category whose objects are pairs $(V, L(z))$ where $V$ is a finite-dimensional $\mathbb{C}$-vector space and $L(z) \in \text{End}(\mathbb{C}^n) \otimes D(V)$ is an invertible operator with meromorphic coefficients satisfying $\frac{\partial}{\partial z}$ in $\text{End}(\mathbb{C}^n) \otimes D(V) \otimes D(V)$; morphisms $(V, L(z)) \rightarrow (V', L'(z))$ are $(n\omega_i)$-periodic meromorphic functions $\varphi : h^\ast \rightarrow \text{Hom}(V, V')$ such that $(1 \otimes \varphi)L(z) = L(z)(1 \otimes \varphi)$ in $\text{Hom}_C(\mathbb{C}^n \otimes V_{h^\ast}, \mathbb{C}^n \otimes V_{h^\ast})$ (i.e morphisms are $M_{h^\ast}$-linear).

The category $\mathcal{D}_B$ is a right-module category over $\mathcal{C}_B$, i.e we have a (bi)functor $\otimes : \mathcal{D}_B \times \mathcal{C}_B \rightarrow \mathcal{D}_B$ defined by $[\otimes]$.

The category $\mathcal{D}_B^\ast$ is defined in an analogous way: objects are pairs $(V, L(z, x))$ as in $\mathcal{D}_B$ but the $L$-operator is now a meromorphic function of $z$ and $x$, and morphisms $(V, L(z, x)) \rightarrow (V', L'(z, x))$ are meromorphic maps $\varphi(\lambda, x) : h^\ast \times \mathbb{C} \rightarrow \text{Hom}_C(V, V')$ satisfying $(1 \otimes \varphi)L(z, x) = L(z, x)(1 \otimes \varphi)$.

1.2 Meromorphic representations of the elliptic quantum group $\mathcal{E}_{\tau, \gamma/2}(\mathfrak{g}_n)$

Felder’s dynamical $R$-matrix: Let us consider the functions of two complex variables

$$\alpha(z, l) = \frac{\theta(l + \gamma)\theta(z)}{\theta(l)\theta(z - \gamma)}, \quad \beta(z, l) = \frac{\theta(z - l)\theta(\gamma)}{\theta(l)\theta(z - \gamma)}.$$
As functions of $z$, $\alpha$ and $\beta$ have simple poles at $z = \gamma + \mathbb{Z} + \tau \mathbb{Z}$ and satisfy

\[
\alpha(z+1,l) = \alpha(z,l), \quad \alpha(z+\tau,l) = e^{-2i\pi\gamma} \alpha(z,l), \quad \beta(z+1,l) = \beta(z,l), \quad \beta(z+\tau,l) = e^{-2i\pi(\gamma-1)} \beta(z,l).
\]

Felder introduced in [1] the matrix $R^F(z,\lambda) : \mathbb{C} \times \mathfrak{h}^* \to \text{End}(\mathbb{C}^n) \otimes \text{End}(\mathbb{C}^n)$:

\[
R^F(z,\lambda) = \sum_i E_{ii} \otimes E_{ii} + \sum_{i \neq j} \alpha(z,\lambda_i - \lambda_j) E_{ii} \otimes E_{jj} + \sum_{i \neq j} \beta(z,\lambda_i - \lambda_j) E_{ji} \otimes E_{ij}
\]

where $\lambda = \sum_i \lambda_i E_{ii} \in \mathfrak{h}^*$.

This matrix is a solution of the quantum dynamical Yang-Baxter equation with spectral parameters

\[
R^F_{12}(z - w,\lambda - \gamma h_3) R^F_{13}(z,\lambda) R^F_{23}(w,\lambda - \gamma h_1)
= R^F_{23}(w,\lambda) R^F_{13}(z,\lambda - \gamma h_2) R^F_{12}(z - w,\lambda)
\]

where we have used the following convention: if $V_i$ are diagonalizable $\mathfrak{h}$-modules with weight decomposition $V_i = \bigoplus \mu V_i^\mu$ and $a(\lambda) \in \text{End}(\bigotimes V_i)$ then

\[
a(\lambda - \gamma h) \bigotimes_{i} V_i^\mu_i = a(\lambda - \gamma \mu)
\]

As usual, indices indicate the components of the tensor product on which the operators act.

In addition, $R(z,\lambda)$ satisfies the following two conditions:

1. Unitarity: $R_{12}(z,\lambda) R_{21}(-z,\lambda) = Id$,
2. Weight zero: $\forall h \in \mathfrak{h}, [h^{(1)} + h^{(2)}, R(z,\lambda)] = 0$.

The category $\mathcal{C}_F$: It is possible to use $R(z,\lambda)$ to define an algebra by the RLL-formalism (see [2]): the elliptic quantum group $E_{\tau,\gamma/2}(\mathfrak{gl}_n(\mathbb{C}))$. However, we will only need the following category of its representations $\mathcal{C}_F$, introduced by Felder in [3] and studied by Felder and Varchenko in [4]: objects are pairs $(V, L(z,\lambda))$ where $V$ is a finite-dimensional diagonalizable $\mathfrak{h}$-module and $L(z,\lambda) : \mathbb{C} \times \mathfrak{h}^* \to \text{End}(\mathbb{C}^n) \otimes \text{End}(V)$ is an invertible meromorphic function which is $(n\omega_i)$-periodic in $\lambda$ and which satisfies the following two conditions:

\[
[h_1 + h_2, L(z,\lambda)] = 0,
\]

\[
R_{12}(z - w,\lambda - \gamma h_3) L_{13}(z,\lambda) L_{23}(w,\lambda - \gamma h_1)
= L_{23}(w,\lambda) L_{13}(z,\lambda - \gamma h_2) R_{12}(z - w,\lambda)
\]

(6)

Morphisms $(V, L(z,\lambda)) \to (V', L'(z,\lambda))$ are $(n\omega_i)$-periodic meromorphic weight zero maps $\varphi(\lambda) : V \to V'$ such that $L'(z,\lambda)(1 \otimes \varphi(\lambda - \gamma h_1)) = (1 \otimes \varphi(\lambda)) L(z,\lambda)$. The dynamical quantum Yang-Baxter relation for $R^F(z,\lambda)$ implies that $(\mathbb{C}^n, R^F(z - w,\lambda)) \in \text{Ob}(\mathcal{C}_F)$ for all $w \in \mathbb{C}$. This is the vector representation and it will be denoted by $V_F(w)$.

The category $\mathcal{C}_F$ is naturally equipped with a tensor structure: it is defined on objects by

\[
(V, L(z,\lambda)) \otimes (V', L'(z,\lambda)) = (V \otimes V', L_{12}(z,\lambda - \gamma h_3) L'_{13}(z,\lambda)),
\]
and if \( \varphi \in \text{Hom}_{C_F}(V, W), \varphi' \in \text{Hom}_{C_F}(V', W') \) then
\[
(\varphi \otimes \varphi')(\lambda) = \varphi(\lambda - \gamma h_2) \otimes \varphi'(\lambda) \in \text{Hom}_{C_F}(V \otimes V', W \otimes W').
\]

There is a notion of a dual representation in the category \( C_F \): the (right) dual of \((V, L(z, \lambda))\) is \((V^*, L^*(z, \lambda))\) where \( L^*(z, \lambda) = L^{-1}(z, \lambda + \gamma h_2)^t \) (apply inversion, shifting and then apply the transposition in the second component \( t_2 \)). If \( V, W \in \text{Ob}(C_B) \) and \( \varphi(\lambda) \in \text{Hom}_{C_B}(V, W) \) then \( \varphi^*(\lambda) := \varphi(\lambda + \gamma h_1)^t \in \text{Hom}_{C_B}(W^*, V^*) \).

The extended category \( C_F^e \) is defined by \( \text{Ob}(C_F^e) = \text{Ob}(C_F) \) and
\[
\text{Hom}_{C_F^e}(V, V') = \text{Hom}_{C_F}(V, V') \otimes M_{C}
\]
i.e morphisms in \( C_F^e \) are meromorphic 1-parameter families of morphisms in \( C_F \).

# 2 The functor \( F^C_\lambda : C_F \to D_B \)

In this section, we define a family of functors from meromorphic (finite-dimensional) representations of \( E_\tau \tilde{\mathfrak{g}}_n(\mathbb{C}) \) to infinite-dimensional representations of the quantum elliptic algebra \( B \).

## 2.1 Twists by difference operators:

For any finite-dimensional diagonalizable \( \mathfrak{h} \)-module \( V \), let \( e^{\gamma D} \in \text{End}(V) \) denote the shift operator: \( e^{\gamma D} \sum \mu f_\mu(\lambda) v_\mu = \sum \mu f(\lambda + \gamma \mu) v_\mu, v_\mu \in V_\mu \). Now let \((V, L(z, \lambda)) \in C_F\), and let \( S(z, \lambda), S'(z, \lambda) : \mathbb{C} \times \mathfrak{h}^* \to \text{End}(\mathbb{C}^n) \) be meromorphic and nondegenerate. Define the difference-twist of \((V, L(z, \lambda))\) to be the pair \((V, L^{S, S'}(z))\) where
\[
L^{S, S'}(z) = S_1(z, \lambda - \gamma h_2)L(z, \lambda)e^{-\gamma D_1}S_1'(z, \lambda)^{-1} \in \text{End}(\mathbb{C}^n) \otimes D(V).
\]

This is a difference operator acting on \( \mathbb{C}^n \otimes V_{\mathfrak{h}} \).

**Lemma 1** The difference operator \( L^S(z, \lambda) \) satisfies the following relation in \( \text{End}(\mathbb{C}^n) \otimes D(V) \otimes D(V) \):
\[
T_{12}(z, w, \lambda - \gamma h_3)L_{13}^{S, S'}(z)L_{23}^{S, S'}(w) = L_{23}^{S, S'}(w)L_{13}^{S, S'}(z)T_{12}'(z, w, \lambda)
\]
where
\[
T(z, w, \lambda) = S_2(w, \lambda)S_1(z, \lambda - \gamma h_2)R_{12}^F(z - w, \lambda)S_2(w, \lambda - \gamma h_1)^{-1}S_1(z, \lambda)^{-1}
\]
\[
T'(z, w, \lambda) = S_1'(z, \lambda)S_2'(w, \lambda + \gamma h_1)R_{12}^F(z - w, \lambda)S_1'(z, \lambda + \gamma h_1)^{-1}S_2'(w, \lambda)^{-1}
\]

**Proof:** the proof is straightforward, using relation (6) for \( L(z, \lambda) \) and the weight zero property of \( R^F(u, \lambda) \) and \( L(u, \lambda) \). \( \square \)
2.2 The Vertex-IRF transform

Let \( \phi_1(u) = e^{2i\pi(x_1 + i\pi)\theta_{0,0}(u + l\tau; n\tau)} \) for \( l = 1, \ldots, n \). Then the vector \( \Phi(u) = (\phi_1(u), \ldots, \phi_n(u)) \) is, up to renormalization, the unique holomorphic vector in \( \mathbb{C}^n \) satisfying the following monodromy relations:

\[
\Phi(u + 1) = A\Phi(u), \quad \Phi(u + \tau) = e^{-i\pi \frac{1}{2} - 2i\pi B} B\Phi(u)
\]

(10) (11)

Now let \( S(z, \lambda) : \mathbb{C} \times \mathfrak{h}^* \to \text{End}(\mathbb{C}^n) \) be the matrix whose columns are \( (\Phi_1(z, \lambda), \ldots, \Phi_n(z, \lambda)) \) where \( \Phi_j(z, \lambda) = \Phi(z - n\lambda) \). Using \([9],[10]\), it is easy to see that we have \( \det(S(z, \lambda)) = \text{Const}(\lambda)\theta(z) \) and hence that \( S(z, \lambda) \) is invertible for \( z \neq 0 \) and generic \( \lambda \).

Lemma 2 We have

\[
R^B(z - w)S_1(z, \lambda)S_2(w, \lambda - \gamma h_1) = S_2(w, \lambda)S_1(z, \lambda - \gamma h_2)R^F(z - w, \lambda)
\]

\[
R^B(z - w)S_2(w, \lambda)S_1(z, \lambda + \gamma h_2) = S_1(z, \lambda)S_2(w, \lambda + \gamma h_1)R^F(z - w, \lambda)
\]

Proof: the first relation is equivalent to the following identities for \( i, j = 1, \ldots, n \):

\[
R^B(z - w)\Phi_i(z, \lambda) \otimes \Phi_j(w, \lambda - \gamma \omega_i) = \Phi_i(z, \lambda - \gamma \omega_i) \otimes \Phi_j(w, \lambda)
\]

\[
R^B(z - w)\Phi_i(z, \lambda) \otimes \Phi_j(w, \lambda - \gamma \omega_i) = \alpha(z - w, \lambda_i - \lambda_j)\Phi_i(z, \lambda - \gamma \omega_j) \otimes \Phi_j(w, \lambda)
\]

\[
+ \beta(z - w, \lambda_i - \lambda_j)\Phi_j(z, \lambda - \gamma \omega_i) \otimes \Phi_i(w, \lambda)
\]

These identities are proved by comparing poles and transformation properties under lattice translations as functions of \( z \) and \( w \), and using the uniqueness of \( \Phi \). The second relation of the lemma is proved in the same way. These identities are essentially the Vertex/Interaction-Round-a-Face transform of statistical mechanics (see \[9],[10\] and \[11\] for the case \( n = 2 \)). \( \square \)

2.3 Construction of the functor \( F^c_x : \mathcal{C}_F \to \mathcal{D}_B \)

Let us fix some \( c \in \mathbb{C} \). We can now define the family of functors \( F^c_x : \mathcal{C}_F \to \mathcal{C}_B \) indexed by \( x \in \mathbb{C} \): for \((V, L(z, \lambda)) \in \mathcal{C}_F\), set \( F^c_x((V, L(z, \lambda))) = (V, L^{S_x,S_{\gamma c}}(z)) \) with \( S_u(z, \lambda) = S(z - u, \lambda) \) as above and let \( F^c_x \) be trivial at the level of morphisms.

Proposition 1 \( F^c_x : \mathcal{C}_F \to \mathcal{D}_B \) is a functor.

Proof: it follows from Lemma 2 that \((V, L^{S_x,S_{\gamma c}}(z)) \in \text{Ob}(\mathcal{D}_B)\). Furthermore, if \( \varphi(\lambda) \in \text{Hom}_{\mathcal{C}_F}((V, L(z, \lambda)), (V', L'(z, \lambda))) \) then by definition we have \( L'(z, \lambda) \) \( (1 \otimes \varphi(\lambda - \gamma h_1))(1 \otimes \varphi(\lambda))(L(z, \lambda)), \) so that

\[
S_1(z - x, \lambda - \gamma h_2)L'(z, \lambda)e^{-\gamma D_1}S_1(z - x - c, \lambda)^{-1}(1 \otimes \varphi(\lambda))
\]

\[
= S_1(z - x, \lambda - \gamma h_2)L'(z, \lambda)(1 \otimes \varphi(\lambda - \gamma h_1))e^{-\gamma D_1}S_1(z - x - c, \lambda)^{-1}
\]

\[
= S_1(z - x, \lambda - \gamma h_2)(1 \otimes \varphi(\lambda))L'(z, \lambda)e^{-\gamma D_1}S_1(z - x - c, \lambda)^{-1}
\]

\[
= (1 \otimes \varphi(\lambda))S_1(z - x, \lambda - \gamma h_2)L'(z, \lambda)e^{-\gamma D_1}S_1(z - x - c, \lambda)^{-1}
\]

since \( \varphi(\lambda) \) is of weight zero. Thus \( F^c_x(\varphi(\lambda)) \) is an intertwiner in the category \( \mathcal{D}_B \). \( \square \)
We can also think of the family of functors $F^c_x$ as a single functor $F^c: C^c_x \rightarrow D^c_x$.

**Remark:** we can think of the difference-twist and the relations in Lemma 2 as a dynamical analogue of the notion of equivalence of R-matrices due to Drinfeld and Belavin-see [1].

3 The image of the trivial representation and the functor $H^c_x: C_B \rightarrow D_B$

Applying the functor $F^c_x$ to the trivial representation $(\mathbb{C}, \text{Id}) \in \text{Ob}(C_F)$ yields

$$F^c_x((\mathbb{C}, \text{Id})) = (\mathbb{C}, S(z - x, \lambda)e^{-\gamma D_1}S(z - x - c, \lambda)^{-1}).$$

We will denote this object by $I^c_x$. For instance, when $n = 2$, we obtain a representation of the Belavin quantum elliptic algebra as difference operators acting on the space of periodic meromorphic functions in one variable $\lambda$, i.e.

$$L(z) = \begin{pmatrix} a(z) & b(z) \\ c(z) & d(z) \end{pmatrix}$$

where $a(z), b(z), c(z), d(z)$ are operators of the form $f(z)T_{-\gamma} + g(z)$ where $T_{-\gamma}$ is the shift by $-\gamma$.

Such representations of $B$ by difference operators already appeared in the work of Krichever, Zabrodin ([9]) (for $n = 2$) and Hasegawa ([7],[8]) (for the general case), where they were also derived by some Vertex-IRF correspondence.

**Definition:** Let $c \in \mathbb{C}$ and let $H^c_x: C_B \rightarrow D_B$ be the functor defined by the assignment $V \rightarrow I^c_x \otimes V$ and which is trivial at the level of morphisms. The family of functors $H^c_x$ gives rise to a functor $H^c: C^c_B \rightarrow D^c_B$.

4 Full Faithfulness of the functor $H^c_x: C_B \rightarrow D_B$

In this section, we prove the following result

**Proposition 2** Let $V, V' \in \text{Ob}(C_B)$. Then for all but finitely many values of $x \mod \mathbb{Z} + \mathbb{Z} \tau$, the map

$$H^c_x: \text{Hom}_{C_B}(V, V') \rightarrow \text{Hom}_{D_B}(H^c_x(V), H^c_x(V'))$$

is an isomorphism.

**Proof:** since $\text{Hom}_{C_B}(V, V') \simeq \text{Hom}_{C^c}(\mathbb{C}, V' \otimes V^*)$, $\text{Hom}_{D_B}(I^c_x \otimes V, I^c_x \otimes V') \simeq \text{Hom}_{D_B}(I^c_x, I^c_x \otimes V' \otimes V^*)$, it is enough to show that the map $H^c_x: \text{Hom}_{C_B}(\mathbb{C}, W) \rightarrow \text{Hom}_{D_B}(I^c_x, I^c_x \otimes W)$ is an isomorphism for all $W \in \text{Ob}(C_B)$. Since $H^c_x$ is trivial at the level of morphisms, this map is injective. Now let $W \in \text{Ob}(C_B)$ and
let $\varphi(\lambda) \in \text{Hom}_B(I^*_x, I^*_x \otimes W)$, that is, $\varphi(\lambda)$ is a $(n\omega_i)$-periodic meromorphic function $b^* \to W$ satisfying the equation

$$\varphi_2(\lambda)S_1(z - x, \lambda)e^{-\gamma D_1}S_1(z - x - c, \lambda)^{-1} = S_1(z - x, \lambda)e^{-\gamma D_1}S_1(z - x - c, \lambda)^{-1}L_{12}(z)\varphi_2(\lambda)$$

where $L(z)$ is the L-operator of $W$. This is equivalent to

$$L_{12}(z)\varphi_2(\lambda) = S_1(z - x - c, \lambda)\varphi_2(\lambda + \gamma h_1)S_1(z - x - c, \lambda)^{-1} \quad (12)$$

Now $L(z)$ is an elliptic function (of periods $n$ and $n\tau$) so it is either constant or it has a pole. Restricting $W$ to the subrepresentation $\text{Span}(\varphi(\lambda), \lambda \in b^*)$, we see that the latter case is impossible for generic $x$ as the RHS of (12) has a pole at $z = x + c$ only; hence $L(z)$ is constant. Furthermore, from (12) we see that the matrix

$$M(\lambda) = S_1(z - x - c, \lambda)^{-1}L_{12}S_1(z - x - c, \lambda)$$

is independent of $z$. In particular, setting $z \mapsto z + 1$ and using the transformation properties (11) of $S(z, \lambda)$, we obtain $[A_1, L_{12}] = 0$. This implies that $L = \sum_i E_{ii} \otimes D_i$ for some $D_i \in \text{End}(W)$.

**Lemma 3** Let $U$ be a finite dimensional vector space, let $T \in \text{End}(\mathbb{C}^n) \otimes \text{End}(U)$ be an invertible solution of the equation

$$R_{12}(z)T_{12} + T_{23} = T_{23}R_{12}(z)$$

such that $T = \sum_i E_{ii} \otimes D_i$ for some $D_i \in \text{End}(U)$. Then $[D_i, D_j] = 0$ for all $i, j$ and there exists $X \in \text{End}(U)$ such that $X^n = 1$ and $D_{i+1} = XD_i$ for all $i = 1, \ldots, n$.

**Proof:** let us write $R_{12}(z) = \sum_{p,q,r,s} R_{p,q,r,s}(z)E_{pq} \otimes E_{rs}$. Then equation (13) is equivalent to $R_{p,q,r,s}(z)D_pD_q = R_{p,q,r,s}(z)D_rD_r$ for all $p, q, r, s$. But it follows from the general formula for $R_{12}(z)$ that $R_{p,q,r,s}(z) \neq 0$ if and only if $p + q \equiv r + s \pmod{n}$. Thus we have $[D_i, D_j] = 0$ for all $i, j$ and $X := D_1D_{i+1}$ independent of $i$, and satisfies $X^n = 1$. \[\square\]

By the above lemma, there exists $X \in \text{End}(W)$ such that $X^n = 1$ and $D_{k+1} = XD_k$. Suppose that $X \neq 1$ and choose $c \in W$ such that $X(e) = \xi^k e$ with $\xi^k \neq 1$. Now we apply the transformation $z \mapsto z + \tau$ to the matrix $M(\lambda)$. Noting that, by (11), $S(z - x - c + \tau, \lambda) = e^{-i\pi\tau/2 - 2i\pi(z-x-c)/n}BS(z - x - c, \lambda)F(\lambda)$ where $F(\lambda) = \text{diag}(e^{-2i\pi\lambda}, \ldots, e^{-2i\pi\lambda_n})$, we obtain the equality

$$F(\lambda)^{-1}S_1(z - x - c, \lambda)^{-1}B_1^{-1}L_{12}B_1S_1(z - x - c, \lambda)F(\lambda) = S_1(z - x - c, \lambda)^{-1}L_{12}S_1(z - x - c, \lambda)$$

Applying this to the vector $e$ yields $\text{Ad} F(\lambda)(M(\lambda))(e) = \xi^{-k}M(\lambda)(e)$. This is possible for all $\lambda$ only if $k \equiv 0 \pmod{n}$. Hence $X = 1$ and (12) reduces to the equation $D\varphi_2(\lambda) = \varphi_2(\lambda + \gamma h_1)$. In particular $\varphi(\lambda)$ is $\gamma(\omega_i - \omega_j)$-periodic. But by our assumption, $\varphi(\lambda)$ is $(n\omega_i)$-periodic and $\gamma$ is real and irrational. Therefore $\varphi(\lambda)$ is constant and it is a morphism in the category $C_B$. \[\square\]

**Corollary 1** The functor $\mathcal{H}^e : C_B^r \to D_B^r$ is fully faithful.
Remark: equation (12) shows that $\text{Hom}_{\mathcal{D}_B}(I^*_x, I^*_x \otimes V) = \text{Hom}_{\mathcal{C}_B}(V^*, I^0_{x+\tau})$. Thus the above proposition states that for any finite-dimensional representation $V \in \mathcal{O}b(\mathcal{C}_F)$ and for all but finitely many $x \mod \mathbb{Z} + \tau \mathbb{Z}$, we have $\text{Hom}_{\mathcal{D}_B}(V^*, I^0_\nu) = \text{Hom}_{\mathcal{C}_B}(V^*, \mathbb{C})$, where the isomorphism is induced by the embedding $\mathbb{C} \subset I^0_\nu$ (constant functions). However, for finitely many values of $x \mod \mathbb{Z} + \tau \mathbb{Z}$, this may not be true: see [9] and [8] where some finite-dimensional subrepresentations of $I^0_\nu$ are considered.

5 Full faithfulness of the functor $F^c_x : \mathcal{C}_F \rightarrow \mathcal{D}_B$

In this section, we prove the following result:

Proposition 3 The functor $F^c_x : \mathcal{C}_F \rightarrow \mathcal{D}_B$ is fully faithful.

Proof: we have to show that for any two objects $V, V'$ in $\mathcal{C}_F$ there is an isomorphism $F^c_x : \text{Hom}_{\mathcal{C}_F}(V, V') \rightarrow \text{Hom}_{\mathcal{D}_B}(F^c_x(V), F^c_x(V'))$. Since $F^c_x$ is trivial at the level of morphisms, this map is injective. Now let $V, W \in \mathcal{O}b(\mathcal{C}_F)$ and let $\phi(\lambda) \in \text{Hom}_{\mathcal{D}_B}(F^c_x(V), F^c_x(W))$. By definition, $\phi(\lambda) : V \rightarrow W$ satisfies the relation

$$
\phi_2(\lambda)S_1(z-x, \lambda - \gamma h_2)L^V_{12}(z, \lambda)e^{-\gamma D^1}S_1(z-x-c, \lambda)^{-1} = S_1(z-x, \lambda - \gamma h_2)L^W_{12}(z, \lambda)e^{-\gamma D^1}S_1(z-x-c, \lambda)^{-1}\phi_2(\lambda)
$$

where $L^V(z, \lambda)$ (resp. $L^W(z, \lambda)$) is the L-operator of $V$ (resp. $W$). This is equivalent to

$$
\phi_2(\lambda)S_1(z-x, \lambda - \gamma h_2)L^V_{12}(z, \lambda) = S_1(z-x, \lambda - \gamma h_2)L^W_{12}(z, \lambda)\phi_2(\lambda - \gamma h_1)
$$

(14)

Introduce the following notations: write $W = \bigoplus_\xi W_\xi$, $V = \bigoplus_\mu V_\mu$, $\phi(\lambda) = \sum_\nu \phi_\nu(\lambda)$ for the weight decompositions (so that $\phi_\nu : V_\xi \rightarrow W_{\xi + \nu}$). Also let $S(z-x, \lambda) = \sum_{i,j} S^{ij}(z-x, \lambda)e_{ij}$, $L^V_{12}(z, \lambda) = \sum_{i,j} E_{ij} \otimes L^V_{ij}(z, \lambda)$ and use the same notation for $L^W(z, \lambda)$. Applying (13) to $v_i \otimes \zeta_\mu$ for some $i$ and $\zeta_\mu \in V_\mu$ yields

$$
\sum_{j,k,\nu} S^{kj}(z-x, \lambda - \gamma (\mu + \omega_i - \omega_j))v_k \otimes \phi_\nu(\lambda)(L^H_V(z, \lambda))\zeta_\mu
$$

$$
= \sum_{l,k,\sigma} S^{kl}(z-x, \lambda - \gamma (\mu + \omega_i - \omega_l + \sigma))v_k \otimes L^H_W(z, \lambda)\phi_\sigma(\lambda - \gamma \omega_i)\zeta_\mu
$$

(15)

where we used the weight-zero property of $L^V(z, \lambda)$ and $L^W(z, \lambda)$. Applying $v^*_k$ to (15) and projecting on the weight space $W_{\mu + \omega_i + \xi}$ gives the relation

$$
\sum_{\nu - \omega_j = \xi} S^{kj}(z-x, \lambda - \gamma (\mu + \omega_i - \omega_j))\phi_\nu(\lambda)(L^H_V(z, \lambda))\zeta_\mu
$$

$$
= \sum_{\sigma - \omega_j = \xi} S^{kl}(z-x, \lambda - \gamma (\mu + \omega_i - \omega_j + \sigma))L^H_W(z, \lambda)\phi_\sigma(\lambda - \gamma \omega_i)\zeta_\mu
$$

(16)
for any $i, k, \xi$ and $\zeta \in V_\mu$. Now let $A = \{\chi \mid \varphi_\chi(\lambda) \neq 0\}$.
Fix some $j$ and let $\beta \in A$ be an extremal weight in the direction $-\omega_j$ (i.e. $\beta - \omega_j + \omega_k \notin A$ for $k \neq j$). Then $[14]$ for $\xi = \beta - \omega_j$ reduces to
\[
S_k^j(z - x, \lambda - \gamma(\mu + \omega_i - \omega_j)) \varphi_\beta(\lambda)(L^j(z, \lambda)\zeta) = S_k^j(z - x, \lambda - \gamma(\mu + \omega_i - \omega_j + \beta))L^j(z, \lambda)\varphi_\beta(\lambda - \gamma\omega_i)\zeta \mu
\]
(17)

**Claim:** there exists $i \in \{1, \ldots, n\}, \mu$ and $\zeta \in V_\mu$ such that $\varphi_\beta(\lambda)(L^i(z, \lambda)\zeta) \neq 0$ for generic $z$ and $\lambda$.

**Proof:** recall the central element $Q\text{det}(z, \lambda) \in E_{\tau, z}^\varnothing(gl_n)$. By definition, its action on $\Omega$ is invertible. Expanding $Q\text{det}(z, \lambda)$ along the $j$th-line, we have $Q\text{det}(z, \lambda) = \sum j_i^j L^j(z, \lambda)P_i(z, \lambda)$ for some operators $P_i(z, \lambda) \in \text{End}(V)$. In particular, $\sum j_i^j \text{Im} L^j(z, \lambda) = V$, and the claim follows.

Thus, the ratio $S_k^j(z - x, \lambda - \gamma(\mu + \omega_i - \omega_j + \beta))/S_k^j(z - x, \lambda - \gamma(\mu + \omega_i - \omega_j))$ is independent of $k$. This is possible only if $\beta \in \sum j_i^j \mathbb{C}E^\varnothing_r$. Applying this to $j = 1, \ldots, n$, we see that $A = \{0\}$. Hence $\varphi_\lambda(\lambda) = 0$ is an $h$-module map. But then relation (14) reduces to $\varphi_\lambda(\lambda) L^j(z, \lambda) = L^j(z, \lambda)\varphi_\lambda(\lambda - \gamma h_1)$, and $\varphi_\lambda(\lambda)$ is an intertwiner in the category $\mathcal{C}F$.

**Corollary 2** The functor $\mathcal{F}^c : \mathcal{C}_F \to \mathcal{D}_B^c$ is fully faithful.

### 6 The image of the vector representation

Let us denote $\tilde{V}_F(w) = (\mathbb{C}^n, \chi(w)R^F(w, \lambda))$. It is an object of $\mathcal{C}_F$ which equals the tensor product of the vector representation $V_F(w)$ by the one-dimensional representation $(\mathbb{C}, \chi(z))$.

**Proposition 4** For any $x, w, x + c \neq w$ (mod $\mathbb{Z} + r\mathbb{Z}$), we have $\mathcal{F}^c_\lambda(V_F(w)) \simeq \mathcal{H}^c_\lambda(V_B(w))$.

**Proof:** by definition, we have
\[
\mathcal{F}^c_\lambda(\tilde{V}_F(w)) = (\mathbb{C}^n, \chi(z)S_1(z - x, \lambda - \gamma h_2)R^F(z - w, \lambda)e^{-\gamma D_1} \times S_1(z - x - c, \lambda)^{-1},
\]
\[
I^c_\lambda \otimes V_B(w) = (\mathbb{C}^n, \chi(z)S_1(z - x, \lambda)e^{-\gamma D_1}S_1(z - x - c, \lambda)R^B(z - w))
\]

We claim that the map $\varphi(\lambda) = e^{\gamma D}(S(w - x - c, \lambda)^{-1})e^{\gamma D} \in \text{End}(\mathbb{C}^n)$ is an intertwiner $\mathcal{H}^c_\lambda(V_B(w)) \simeq I^c_\lambda \otimes V_B(w) \rightarrow \mathcal{F}^c_\lambda(\tilde{V}_F(w))$. Indeed, we have
\[
S_1(z - x, \lambda - \gamma h_2)R^F(z - w, \lambda)e^{-\gamma D_1}S_1(z - x - c, \lambda)^{-1}(1 \otimes \varphi(\lambda))
\]
\[
= e^{-\gamma D_2}S_1(z - x, \lambda)e^{\gamma D_1}R^F(z - w, \lambda)e^{-\gamma D_1 + D_2}S_1(z - x - c, \lambda + \gamma h_2)^{-1}S_2(w - x - c, \lambda)^{-1}e^{\gamma D_2}
\]
\[
= e^{-\gamma D_2}S_1(z - x, \lambda)e^{-\gamma D_1}R^F(z - w, \lambda)S_1(z - x - c, \lambda + \gamma h_2)^{-1}S_2(w - x - c, \lambda)^{-1}e^{\gamma D_2}
\]
\[
= e^{-\gamma D_2}S_1(z - x, \lambda)e^{-\gamma D_1}S_2(w - x - c, \lambda + \gamma h_1)^{-1}S_1(z - x - c, \lambda)^{-1}R^B(z - w)e^{\gamma D_2}
\]
\[
= e^{-\gamma D_2}S_2(w - x - c, \lambda)e^{\gamma D_1}S_1(z - x - c, \lambda)^{-1}R^B(z - w)e^{\gamma D_2}
\]
\[
= (1 \otimes \varphi(\lambda))S_1(z - x, \lambda)e^{-\gamma D_1}S_1(z - x - c, \lambda)^{-1}R^B(z - w)
\]
where we used Lemma 2 and the zero-weight property of $R^F(u, \lambda)$.
Lemma 4 Let $V, V' \in \text{Ob}(C_F)$, $W, W' \in \text{Ob}(C_B)$ and suppose that $F^c_x(V) \simeq H^c_x(W)$ and $F^c_x(V') \simeq H^c_x(W')$. Then $F^c_x(V \otimes V') \simeq H^c_x(W \otimes W')$.

Proof: If $\varphi(\lambda) : V \to W$ and $\varphi'(\lambda) : V' \to W'$ are intertwiners then it is easy to check using the methods above that $\varphi_2(\lambda - \gamma h_1)\varphi_1(\lambda) : V \otimes V' \to W \otimes W'$ is an intertwiner. □

Applying this to tensor products of the vector representations, we obtain

Corollary 3 For any $x \in \mathbb{C}$ and $w_1, \ldots, w_r \in \mathbb{C}\setminus\{x + c + Z + \tau Z\}$, we have

$$F^c_x(\tilde{V}_F(w_1) \otimes \ldots \tilde{V}_F(w_r)) \simeq H^c_x(V_B(w_1) \otimes \ldots V_B(w_r)).$$

Corollary 4 For any $w_1, \ldots, w_r \in \mathbb{C}$, we have

$$F^c_x(\tilde{V}_F(w_1) \otimes \ldots \tilde{V}_F(w_r)) \simeq H^c_x(V_B(w_1) \otimes \ldots V_B(w_r)).$$

Notice that in this case, we have a canonical intertwiner, given by the formula

$$\varphi_{1\ldots r}(\lambda, w_1, \ldots, w_r) = \tilde{S}_r^{-1}(w_r - x - c, \lambda - \gamma \sum_{i=1}^{r-1} h_i) \ldots \tilde{S}_1^{-1}(w_1 - x - c, \lambda),$$

where we set $\tilde{S}(z, \lambda) = e^{-\gamma D} S(z, \lambda)e^{\gamma D}$.

7 Equivalence of subcategories

Let us summarize the results of sections 4-8. By proposition 2, we can identify $C^v_B$ with a full subcategory $D^v_1$ of $D^v_B$. By proposition 3, we can identify $C^c_F$ with a full subcategory $D^c_2$ of $D^c_B$. Moreover, $D^c_2$ and $D^v_1$ intersect (at least if we replace $D^v_B$ by the equivalent category $\tilde{D}^v_B$ whose objects are isomorphism classes of objects of $D^v_B$), and the intersection contains objects of the form $F^c_x(\bigotimes_i V_F(w_i)) \simeq H^c_x(\bigotimes_i V_B(w_i))$, where $i = 1, \ldots, r$ and $w_i \in \mathbb{C}$. Hence,

Theorem 1 The abelian subcategory $V^*_B$ of $C^v_B$ generated by objects $\bigotimes_i V_B(w_i)$ for $i = 1, \ldots, r$, $r \in \mathbb{N}$ and $w_i \in \mathbb{C}$ and the abelian subcategory $V^*_F$ of $C^v_F$ generated by objects $\bigotimes_j V_F(w_j)$ for $j = 1, \ldots, s$, $s \in \mathbb{N}$ and $w_j \in \mathbb{C}$ are equivalent.

Note that for numerical values of $x$, $F^c_x : C_F \to D_B$ is always fully faithful, and $F^c_x(C_F)$ a full subcategory of $D_B$, but this is not true of $H^c_x$, because of the existence of nontrivial finite-dimensional subrepresentations of $I^0_x$.

Acknowledgments: The authors were supported by the NSF grant DMS-9700477. O.S would like to thank Harvard University Mathematics Department for its hospitality without which this work would not have been possible.
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