NEW RESULT ON CHERN CONJECTURE FOR MINIMAL HYPERSURFACES AND ITS APPLICATION

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Dedicated to Professor Shing-shen Chern on the occasion of his 105th birthday

Abstract. We verify that if $M$ is a compact minimal hypersurface in $S^{n+1}$ whose squared length of the second fundamental form satisfying $0 \leq |A|^2 - n \leq \frac{n}{22}$, then $|A|^2 \equiv n$ and $M$ is a Clifford torus. Moreover, we prove that if $M$ is a complete self-shrinker with polynomial volume growth in $\mathbb{R}^{n+1}$ whose equation is given by (1.1), and if the squared length of the second fundamental form of $M$ satisfies $0 \leq |A|^2 - 1 \leq \frac{1}{21}$, then $|A|^2 \equiv 1$ and $M$ is a round sphere or a cylinder. Our results improve the rigidity theorems due to Q. Ding and Y. L. Xin [14, 16].

1. Introduction

The famous Chern Conjecture for minimal hypersurfaces in a sphere was proposed by S. S. Chern [10, 11] in 1968 and 1970, and was listed in the Problem Section by S. T. Yau [46] in 1982, as stated

Chern Conjecture. (A) (Standard version) Let $M$ be a compact minimal hypersurface with constant scalar curvature in the unit sphere $S^{n+1}$. Then the possible values of the scalar curvature of $M$ form a discrete set.

(B) (Refined version) Let $M$ be a compact minimal hypersurface with constant scalar curvature in the unit sphere $S^{n+1}$. Then $M$ is isoparametric.

(C) (Stronger version) Let $M$ be a compact minimal hypersurface in the unit sphere $S^{n+1}$. Denote by $A$ the second fundamental form of $M$. Set $a_k = (k - \text{sgn}(5 - k))n$, for $k \in \{m \in \mathbb{Z}^+; 1 \leq m \leq 5\}$. Then we have

(i) For any fixed $k \in \{m \in \mathbb{Z}^+; 1 \leq m \leq 4\}$, if $a_k \leq |A|^2 \leq a_{k+1}$, then
M is isoparametric, and $|A|^2 \equiv a_k$ or $|A|^2 \equiv a_{k+1}$.

(ii) If $|A|^2 \geq a_5$, then $M$ is isoparametric, and $|A|^2 \equiv a_5$.

It is well-known that the Chern Conjecture consists of several pinching problems. In the late 1960s, Simons, Lawson, and Chern-do Carmo-Kobayashi [11, 21, 32] solved the first pinching problem for compact minimal hypersurfaces in the unit sphere $S^{n+1}$ and verified that if $|A|^2 \leq n$, then $|A|^2 \equiv 0$ and $M$ is the great sphere $S^n$, or $|A|^2 \equiv n$ and $M$ is one of the Clifford torus $S^k(\sqrt{k_n}) \times S^{n-k}(\sqrt{n-k_n})$, $1 \leq k \leq n-1$. More generally, they proved a rigidity theorem for compact minimal submanifolds in a sphere under a sharp pinching condition. Further developments on the first pinching problem have been made by many other authors [7, 13, 18, 23, 25, 31, 36, 37, 45], etc.

As a special part of the Chern Conjecture, the following problem has been open for more than 40 years.

**The Second Pinching Problem.** Let $M$ be a compact minimal hypersurface in the unit sphere $S^{n+1}$.

(i) If $|A|^2$ is constant, and if $n \leq |A|^2 \leq 2n$, then $|A|^2 = n$, or $|A|^2 = 2n$.

(ii) If $n \leq |A|^2 \leq 2n$, then $|A|^2 \equiv n$, or $|A|^2 \equiv 2n$.

In 1983, Peng and Terng [28, 29] initiated the study of the second pinching problem for minimal hypersurfaces in a unit sphere, and made the following breakthrough on the Chern Conjecture.

**Theorem A.** Let $M$ be a compact minimal hypersurface in the unit sphere $S^{n+1}$.

(i) If $|A|^2$ is constant, and if $n \leq |A|^2 \leq n + \frac{1}{12n}$, then $|A|^2 = n$.

(ii) If $n \leq 5$, and if $n \leq |A|^2 \leq n + \tau_1(n)$, where $\tau_1(n)$ is a positive constant depending only on $n$, then $|A|^2 \equiv n$.

During the past three decades, there have been some important progresses on the Chern Conjecture [11, 14, 15, 17, 27, 30, 33, 34, 35, 38, 39, 40, 42, 43, 44, 47], etc. In 1993, Chang [14] proved Chern Conjecture (A) in dimension three. Yang-Cheng [42, 43, 44] improved the pinching constant $\frac{1}{12n}$ in Theorem A(i) to $\frac{\tau_1}{4}$. Later, Suh-Yang [33] improved this pinching constant to $\frac{\tau_1}{2n}$.

In 2007, Wei and Xu [35] proved that if $M$ is a compact minimal hypersurface in $S^{n+1}$, $n = 6, 7$, and if $n \leq |A|^2 \leq n + \tau_2(n)$, where $\tau_2(n)$ is a positive constant depending only on $n$, then $|A|^2 \equiv n$. Later, Zhang [47] extended the second pinching theorem due to Peng-Terng [29] and Wei-Xu [35] to the case of $n = 8$. In 2011, Ding and Xin [14] verified the following important rigidity theorem, as stated
Theorem B. Let $M$ be an $n$-dimensional compact minimal hypersurface in the unit sphere $S^{n+1}$. If $n \geq 6$, the squared norm of the second fundamental form satisfies $0 \leq |A|^2 - n \leq \frac{n}{23}$, then $|A|^2 \equiv n$, i.e., $M$ is a Clifford torus.

In the present paper, we first give a refined version of Ding-Xin’s rigidity theorem for minimal hypersurfaces in a sphere.

Theorem 1.1. Let $M$ be an $n$-dimensional compact minimal hypersurface in the unit sphere $S^{n+1}$. If the squared norm of the second fundamental form satisfies $0 \leq |A|^2 - n \leq \frac{n}{22}$, then $|A|^2 \equiv n$ and $M$ is a Clifford torus.

Moreover, we consider the isometric immersion $X : M \to \mathbb{R}^{n+1}$. If the position vector $X$ evolves in the direction of the mean curvature vector $\vec{H}$, then it gives rise to a solution to the mean curvature flow
\[
\begin{aligned}
\frac{\partial}{\partial t} X(x, t) &= \vec{H}(x, t), \quad x \in M, \\
X(x, 0) &= X(x).
\end{aligned}
\]

An important class of solutions to the above mean curvature flow equations are self-shrinkers, which satisfy
\[
(1.1) \quad H = -X^{\langle N},
\]
where $X^N$ is the projection of $X$ on the unit inner normal vector $\xi$, i.e., $X^N = \langle X, \xi \rangle$.

Rigidity problems of self-shrinkers have been studied extensively [2, 8, 12, 16, 20, 22, 24]. In 2011, Le and Sesum [22] proved that any $n$-dimensional complete self-shrinker with polynomial volume growth in $\mathbb{R}^{n+1}$ whose squared norm of the second fundamental form satisfies $|A|^2 < 1$ must be a hyperplane. Afterwards, Cao and Li [2] generalized this rigidity result to arbitrary codimensional cases and proved that if $M$ is an $n$-dimensional complete self-shrinker with polynomial volume growth in $\mathbb{R}^{n+q}$, and if $|A|^2 \leq 1$, then $M$ must be one of the generalized cylinders. Under the assumption that $|A|$ is constant, Cheng and Wei [8] proved that if $M$ is an $n$-dimensional complete self-shrinker with polynomial volume growth in $\mathbb{R}^{n+1}$, and if $0 \leq |A|^2 - 1 \leq \frac{3}{7}$, then $|A|^2 = 1$.

In 2014, Ding and Xin [16] proved the second pinching theorem for self-shrinkers in the Euclidean space.

Theorem C. Let $M$ be an $n$-dimensional complete self-shrinker with polynomial volume growth in $\mathbb{R}^{n+1}$. If the squared norm of the second fundamental form satisfies $0 \leq |A|^2 - 1 \leq 0.022$, then $|A|^2 \equiv 1$ and $M$ is a round sphere or a cylinder.
In Section 3, we improve Ding-Xin’s pinching constant in Theorem 1 and prove the following rigidity theorem for self-shrinkers in the Euclidean space.

**Theorem 1.2.** Let \( M \) be an \( n \)-dimensional complete self-shrinker with polynomial volume growth in \( \mathbb{R}^{n+1} \). If the squared norm of the second fundamental form satisfies \( 0 \leq |A|^2 - 1 \leq \frac{1}{|A|^2} \), then \( |A|^2 = 1 \) and \( M \) is a round sphere or a cylinder.

2. Minimal hypersurfaces in the unit sphere

Let \( M \) be an \( n \)(\( \geq 2 \))-dimensional compact minimal hypersurface in the unit sphere \( S^{n+1} \). We shall make use of the following convention on the range of indices:

\[ 1 \leq i, j, k, \ldots \leq n. \]

We choose a local orthonormal frame \( \{e_1, e_2, \ldots, e_{n+1}\} \) near a fixed point \( x \in M \) over \( S^{n+1} \) such that \( \{e_1, e_2, \ldots, e_n\} \) are tangent to \( M \).

Let \( \{\omega_1, \omega_2, \ldots, \omega_{n+1}\} \) be the dual frame fields of \( \{e_1, e_2, \ldots, e_{n+1}\} \). Denote by \( R_{ijkl} \) and \( A = \sum_{i,j} h_{ij} \omega_i \otimes \omega_j \) the Riemannian curvature tensor and the second fundamental form of \( M \), respectively. Then we have

\[ \text{Trace } A = \sum_i h_{ii} = 0. \]

Put \( S = |A|^2 = \sum_{i,j} h_{ij}^2 \). We denote the first, the second and the third covariant derivatives of the second fundamental form of \( M \) by

\[ \nabla A = \sum_{i,j,k} h_{ijk} \omega_i \otimes \omega_j \otimes \omega_k, \]

\[ \nabla^2 A = \sum_{i,j,k,l} h_{ijkl} \omega_i \otimes \omega_j \otimes \omega_k \otimes \omega_l, \]

\[ \nabla^3 A = \sum_{i,j,k,l,m} h_{ijklm} \omega_i \otimes \omega_j \otimes \omega_k \otimes \omega_l \otimes \omega_m. \]

We have the Gauss and Codazzi equations.

\[ R_{ijkl} = \delta_{ik} \delta_{jl} - \delta_{il} \delta_{jk} + h_{ik} h_{jl} - h_{il} h_{jk}, \]

and

\[ h_{ijk} = h_{ikj}. \]

By the Gauss equations, the scalar curvature of \( M \) is given by

\[ R = n(n-1) - S. \]

We also have the Ricci identities.

\[ h_{ijkl} = h_{ijlk} + \sum_m h_{mj} R_{mikl} + \sum_m h_{im} R_{mjkl}, \]

\[ h_{ijk} = h_{ikj}. \]
(2.4) \[ h_{ijklm} = h_{ijkm} + \sum_r h_{rjk}R_{rilm} + \sum_r h_{irk}R_{rjlm} + \sum_r h_{ijr}R_{rklm}. \]

Choose a suitable orthonormal frame \( \{e_1, e_2, \ldots, e_n\} \) at \( x \) such that \( h_{ij} = \lambda_i \delta_{ij} \) for all \( i, j \). Then we have

\[ \frac{1}{2} \Delta S = S(n - S) + |\nabla A|^2. \]

Hence

\[ (2.5) \quad \frac{1}{2} \Delta S = S(n - S) + |\nabla A|^2. \]

We get

\[ \frac{1}{2} \Delta |\nabla A|^2 = (2n + 3 - S)|\nabla A|^2 - \frac{3}{2} |\nabla S|^2 + |\nabla^2 A|^2 \]

\[ + 3(2B_2 - B_1), \]

where \( B_1 = \sum_{i,j,k,l,m} h_{ijk}h_{ijl}h_{km}h_{ml} \) and \( B_2 = \sum_{i,j,k,l,m} h_{ij}h_{klm}h_{im}h_{jl} \).

Following [29] (also see [14, 40]), we have

\[ (2.6) \quad |\nabla S|^2 = \frac{1}{2} \Delta S^2 - S\Delta S = \frac{1}{2} \Delta S^2 + 2S^2(S - n) - 2S|\nabla A|^2. \]

(2.7) \[ \frac{1}{2} \Delta |\nabla A|^2 = (2n + 3 - S)|\nabla A|^2 - \frac{3}{2} |\nabla S|^2 + |\nabla^2 A|^2 \]

\[ + 3(2B_2 - B_1), \]

where \( B_1 = \sum_{i,j,k,l,m} h_{ijk}h_{ijl}h_{km}h_{ml} \) and \( B_2 = \sum_{i,j,k,l,m} h_{ij}h_{klm}h_{im}h_{jl} \).

Following [29] (also see [14, 40]), we have

\[ (2.8) \quad |\nabla^2 A|^2 - \frac{3S(S - n)^2}{2(n + 4)} \geq \frac{3}{4} \sum_{i \neq j} t_{ij}^2 = \frac{3}{4} \sum_{i,j} t_{ij}^2, \]

(2.9) \[ 3(B_1 - 2B_2) \leq \sigma S|\nabla A|^2, \]

where \( t_{ij} = h_{ij} - h_{ji} \) and \( \sigma = \sqrt{17 + 1} \).

Moreover, we have

\[ (2.10) \quad \int_M (B_1 - 2B_2)dM = \int_M (Sf_4 - f_3^2 - S^2 - \frac{|\nabla S|^2}{4})dM, \]

where \( f_k = \text{Trace} A^k = \sum_i \lambda_i^k \).

Set \( G = \sum_{i,j} t_{ij}^2 = \sum_{i,j} (\lambda_i - \lambda_j)^2 (1 + \lambda_i \lambda_j)^2 \). Thus, we have

\[ (2.11) \quad G = 2Sf_4 - 2f_3^2 - 2S^2 - 2S(S - n). \]

This implies that

\[ (2.12) \quad \frac{1}{2} \int_M GdM = \int_M [(B_1 - 2B_2) - |\nabla A|^2 + \frac{1}{4} |\nabla S|^2]dM. \]

Now we are in a position to prove our rigidity theorem for minimal hypersurfaces in a sphere.
Proof of Theorem 7.1. It follows from (2.7), (2.8) and (2.12) that

\[(2.13) \quad \int_M \left[ \left( S - 2n - \frac{3}{2} \right)|\nabla A|^2 + \frac{3}{2}(B_1 - 2B_2) + \frac{9}{8}|\nabla S|^2 \right] dM \geq 0.\]

Under the pinching condition \( n \leq S \leq n + \frac{n}{k} \), (2.13) together with (2.6) implies

\[(2.14) \quad 0 \leq \int_M \left[ \left( S - 2n - \frac{3}{2} \right)|\nabla A|^2 + \frac{3}{2}(B_1 - 2B_2) + \frac{9}{4}(S^3 - nS^2 - S)|\nabla A|^2 \right] dM.\]

When \( n = 2, 3 \), combining (2.9) and (2.14), we get

\[(2.15) \quad 0 \leq \int_M \left( \frac{k + 9}{4k}n - \frac{5}{4}S - \frac{3}{2} \right)|\nabla A|^2 dM.\]

When \( n = 4, 5 \), by Lemma 3.4 in [47], we have

\[(2.16) \quad 3(B_1 - 2B_2) \leq \eta_n S|\nabla A|^2,\]

where \( \eta_4 = 2.16 \) and \( \eta_5 = 2.23 \). It follows from (2.14) and (2.16) that

\[(2.17) \quad 0 \leq \int_M \left( \frac{1 + \frac{9}{4}n}{4} - \frac{5 - 2\eta_n}{4}S - \frac{3}{2} \right)|\nabla A|^2 dM.\]

Taking \( k = 22 \), we see that the coefficients of the integrals in (2.15) and (2.17) are both negative. Therefore, \( \nabla A = 0 \) and \( |A|^2 = n \), for \( n \leq 5 \).

When \( n \geq 6 \), the following lemma can be found in [14].

**Lemma 2.1.** If \( n \geq 6 \), \( n \leq S \leq \frac{16}{13}n \), then \( 3(B_1 - 2B_2) \leq (S + 4 + C_1(n)G^{1/3})|\nabla A|^2 \), where \( C_1(n) = \left( \frac{3 \sqrt{6} - 4p}{\sqrt{6} - 1 + 3p} (6 - \sqrt{6} - 13p)\right)^{1/3} \) and \( p = \frac{1}{13(n-2)} \).
For \( \theta \in (0, 1) \), from (2.7), (2.8) and (2.12), using Lemma 2.1 and Young’s inequality, we drive the following inequality.

\[
\frac{3(1-\theta)}{4} \int_M G dM + \int_M \frac{3S(S-n)^2}{2(n+4)} dM \\
\leq \int_M \left[ (S-2n-3)|\nabla A|^2 + \frac{3}{2} |\nabla S|^2 + 3(B_1 - 2B_2) - \frac{3\theta}{4} G \right] dM \\
= \int_M (S-2n-3 + \frac{3\theta}{2})|\nabla A|^2 dM + \left( \frac{3}{2} - \frac{3\theta}{8} \right) \int_M |\nabla S|^2 dM \\
+ (3 - \frac{3\theta}{2}) \int_M (B_1 - 2B_2) dM \\
\leq \int_M (S-2n-3 + \frac{3\theta}{2})|\nabla A|^2 dM + \left( \frac{3}{2} - \frac{3\theta}{8} \right) \int_M |\nabla S|^2 dM \\
+ (1 - \frac{\theta}{2}) \int_M \left( S + 4 + C_1(n)G^{1/3} \right) |\nabla A|^2 dM \\
\leq \int_M \left[ (2-\frac{\theta}{2})S-2n - 1 - \frac{\theta}{2} \right]|\nabla A|^2 dM + \left( \frac{3}{2} - \frac{3\theta}{8} \right) \int_M |\nabla S|^2 dM \\
(2.18) + \frac{3(1-\theta)}{4} \int_M G dM + C_3 \int_M |\nabla A|^3 dM,
\]

where \( C_3 = C_3(n, \theta) = \frac{1}{3} C_1(n)^{3/2} (1 - \frac{\theta}{2})^{3/2} (1 - \theta)^{-1/2} \).

For \( \epsilon > 0 \), (2.5) together with the divergence theorem and Cauchy-Schwarz’s inequality implies

\[
\int_M |\nabla A|^3 dM \leq \int_M S(S-n)|\nabla A| dM + \epsilon \int_M |\nabla^2 A|^2 dM \\
+ \frac{1}{16 \epsilon} \int_M |\nabla S|^2 dM.
\]

(2.19)

Under the pinching condition \( n \leq S \leq n + \delta(n)(\leq \frac{n}{1-\epsilon}) \), for \( \epsilon_1 = \theta_1 \epsilon > 0 \), we estimate \( \int_M S(S-n)|\nabla A| dM \) as follows.

\[
\int_M S(S-n)|\nabla A| dM \leq 2(n+\delta) \epsilon_1 \int_M S(S-n) dM \\
+ \frac{1}{8(n+\delta) \epsilon_1} \int_M S(S-n)|\nabla A|^2 dM \\
(2.20) \leq \left[ 2(n+\delta) \epsilon_1 + \frac{S-n}{8 \epsilon_1} \right] \int_M |\nabla A|^2 dM.
\]
From (2.7) and (2.9), we have

\[
\int_{M} \left| \nabla^2 A \right|^2 dM \leq \frac{3}{2} \int_{M} \left| \nabla S \right|^2 dM \]

(2.21) \quad + \int_{M} \left[ (\sigma + 1)S - 2n - 3 \right] \left| \nabla A \right|^2 dM.

Substituting (2.19), (2.20) and (2.21) into (2.18), we obtain

\[
0 \leq \int_{M} \left\{ \left[ (2 - \frac{\theta}{2}) S - 2n + 1 - \frac{\theta}{2} \right] \left| \nabla A \right|^2 dM \\
+ \left( \frac{3}{2} - \frac{3\theta}{8} \right) \int_{M} \left| \nabla S \right|^2 dM - \int_{M} \frac{3S(S-n)^2}{2(n+4)} dM \\
+C_3 \left\{ 2(n+\delta)\epsilon_1 + \frac{S-n}{8\epsilon_1} \right\} \int_{M} \left| \nabla A \right|^2 dM \\
+ \frac{3\epsilon}{2} \int_{M} \left| \nabla S \right|^2 dM + \frac{1}{16\epsilon} \int_{M} \left| \nabla S \right|^2 dM \\
+ \int_{M} \epsilon[(\sigma + 1)S - 2n - 3] \left| \nabla A \right|^2 dM \right\} \\
= \int_{M} \left\{ \left[ (2 - \frac{\theta}{2}) S - 2n + 1 - \frac{\theta}{2} + \epsilon C_3[(\sigma + 1)S - 2n - 3] \\
+ C_3 \left[ 2(n+\delta)\epsilon_1 + \frac{S-n}{8\epsilon_1} \right] \right\} \left| \nabla A \right|^2 dM - \int_{M} \frac{3S(S-n)^2}{2(n+4)} dM \\
+ \left( 3 - \frac{3\theta}{4} + 3\epsilon C_3 + \frac{C_3}{8\epsilon} \right) \int_{M} [S(S-n)^2 - (S-n) \left| \nabla A \right|^2] dM \\
\leq \int_{M} \left\{ \left[ 1 - \frac{\theta}{2}(n+1) + \epsilon C_3(n\sigma - n - 3) + 2\epsilon_1 C_3(n+\delta) \\
- \left[ 1 - \frac{\theta}{4} + \frac{C_3}{8\epsilon} - \frac{C_3}{8\epsilon_1} + \epsilon C_3(2-\sigma) \right](S-n) \right\} \left| \nabla A \right|^2 dM \\
+ \delta \left( 3 - \frac{3\theta}{4} + 3\epsilon C_3 + \frac{C_3}{8\epsilon} - \frac{3}{2(n+4)} \right) \int_{M} \left| \nabla A \right|^2 dM \\
= \int_{M} \left\{ \left[ 1 - \frac{\theta}{2}(n+1) + \epsilon C_3(n\sigma + n - 3 + 5\delta) + \frac{C_3\delta}{8\epsilon} \right. \\
+ \delta \left( 3 - \frac{3\theta}{4} - \frac{3}{2(n+4)} \right) + 2\epsilon(\theta_1 - 1) C_3(n+\delta) \\
\left. - \left[ 1 - \frac{\theta}{4} + \frac{C_3}{8\epsilon} - \frac{C_3}{8\epsilon_1} + \epsilon C_3(2-\sigma) \right](S-n) \right\} \left| \nabla A \right|^2 dM. \right. \]

(2.22)
Let $\epsilon = \sqrt{\frac{\delta}{8(\sigma n + n - 3 + 5\delta + 2(n + \delta)(\theta_1 - 1))}}$, $\theta = 0.866$ and $\theta_1 = 0.83$. Then

$$C_3(n) = \frac{4}{9} \times 0.567^{3/2} \times 0.134^{-1/2} \times \sqrt{3 - \sqrt{6} - 4p} \frac{6 - \sqrt{6} - 13p}{\sqrt{6} - 1 + 13p},$$

where $p = \frac{1}{13(n-2)}$.

Thus, (2.22) is reduced to

$$0 \leq \left[-0.433n + C_3\sqrt{\frac{\delta}{2}}(\sigma n + n - 3 + 5\delta - 0.34(n + \delta)) + (2.3505 - \frac{3}{2(n + 4)})\delta + 0.567\right] \int_M |\nabla A|^2 dM$$

$$+ \left[-0.7835 - C_3(2\epsilon - \sigma \epsilon - \frac{0.17}{6.64\epsilon})\right] \int_M (S - n)|\nabla A|^2 dM. \quad (2.23)$$

We put

$$\delta = \frac{n}{22}$$

$$g_1(x) = -0.433x + \sqrt{\frac{16 \times 0.567^3}{81 \times 0.134}} \times \sqrt{\frac{3 - \sqrt{6} - 4p}{\sqrt{6} - 1 + \frac{1}{x-2}}} \times (6 - \sqrt{6} - \frac{1}{x-2}) \sqrt{\frac{x}{44} \left(\frac{\sqrt{17} + 3}{2} \times 3 - \frac{5x}{22} - \frac{0.34 \times 23x}{22}\right)}$$

$$+ \frac{2.3505x}{22} - \frac{3x}{44(x + 4)} + 0.567,$$

$$g_2(x) = -0.7835 - \sqrt{\frac{16 \times 0.567^3}{81 \times 0.134}} \times \sqrt{\frac{3 - \sqrt{6} - 4p}{\sqrt{6} - 1 + \frac{1}{x-2}}} \times (6 - \sqrt{6} - \frac{1}{x-2}) \left[\sqrt{\frac{(1 - \sqrt{17})^2x}{4 \times 176}} \left(\frac{\sqrt{17} + 3}{2} \times 3 - \frac{5x}{22} - \frac{0.34 \times 23x}{22}\right)\right] \frac{17}{664} \times \sqrt{\frac{176}{x} \left(\frac{\sqrt{17} + 3}{2} \times 3 - \frac{5x}{22} - \frac{0.34 \times 23x}{22}\right)}.$$
Case I. Noting that
\[ 0.433x - \frac{2.3505x}{22} + \frac{3x}{44(x + 4)} - 0.567 > 0, \text{ for } x \geq 6, \]
we take
\[
g_1(x) = \left[ \sqrt{\frac{16 \times 0.567^3}{81 \times 0.134}} \times \left[ \frac{3 - \sqrt{6} - \frac{4}{13(x - 2)}}{1 \sqrt{6 - 1 + \frac{1}{x - 2}}} \right] (6 - \sqrt{6} - \frac{1}{x - 2}) \right.
\]
\[\times \sqrt{\frac{x}{44}} \left( \frac{\sqrt{17} + 3}{2} x - 3 + \frac{5x}{22} - \frac{0.34 \times 23x}{22} \right)\]
\[+ 0.433x - \frac{2.3505x}{22} + \frac{3x}{44(x + 4)} - 0.567 ]^{-1}\]
\[\times \left( \frac{1}{(x - 2)(x + 4)} \right)^2 \frac{Z(x) - W(x)}{Z(x) - W(x)} + 1, \]
where
\[Z(x) = \frac{0.567^3 \times 16}{0.134 \times 81 \times 44} \left[ (6 - \sqrt{6})(x - 2) - 1 \right] \left[ (3 - \sqrt{6})(x - 2) - \frac{4}{13} \right] \]
\[\times \left[ (11\sqrt{17} + 38 - 0.34 \times 23)x - 3 \right] (x + 4)^2 x, \]
and
\[W(x) = \left[ 0.433x(x + 4) - \frac{2.3505x(x + 4)}{22} + \frac{3x}{44} - 0.567(x + 4) \right]^2 \]
\[\times [(\sqrt{6} - 1)(x - 2) + 1](x - 2)^2. \]

We get
\[(2.24) \quad Z(x) - W(x) \leq Q_1(x), \text{ for } x \geq 0, \]
where
\[Q_1(x) = -0.00868x^7 + 0.0575x^6 + 0.207x^5 - 3.126x^4 + 2.331x^3 + 30.434x^2 - 69.56x + 40. \]

By calculating the determinant of the Sylvester matrix associated to \(Q_1\) and \(Q'_1\), we get
\[(2.25) \quad R(Q_1, Q'_1) \approx -32.12. \]
It implies that the discriminant of \(Q_1(x)\) is negative, which equals to
\[\frac{(-1)^{21}}{0.00868} R(Q_1, Q'_1). \]

By a classical result for discriminant of polynomials, there exists at least one pair of complex conjugate roots for \(Q_1(x)\). By a direct computation, we have
\[Q_1(-6) = 501.124, \quad Q_1(-5) = -166.787, \quad Q_1(0) = 40, \]
\[Q_1(1.5) = -1.74319, \quad Q_1(2) = 0.44096, \quad Q_1(3) = -11.8077. \]
Thus, $Q_1(x)$ possesses five real roots in the interval $(-6, 3)$. Hence

$$Q_1(x) < 0, \quad \text{for } x \geq 3.$$ 

Therefore, we have

$$g_1(x) < 0, \quad \text{for } x \geq 6.$$ 

Case II. Put

$$U_1(x) = \sqrt{\frac{3 - \sqrt{6} - \frac{4}{13(x-2)}}{\sqrt{6}-1+\frac{1}{x-2}}}, \quad U_2(x) = (6 - \sqrt{6} - \frac{1}{x-2})$$

and

$$U_3(x) = \sqrt{\frac{22x}{(11\sqrt{17} + 38 - 0.34 \times 23)x - 66}}.$$ 

By a direct computation, we have

$$\frac{\partial}{\partial x} U_1(x) = \frac{35 - 9\sqrt{6}}{26U_1[(\sqrt{6} - 1)(x-2) + 1]^2},$$

$$\frac{\partial}{\partial x} U_2(x) = \frac{1}{(x-2)^2}, \quad \frac{\partial}{\partial x} U_3(x) = -\frac{3U_3^3(x)}{2x^2}.$$ 

Thus, we have

$$\frac{\partial}{\partial x} g_2(x) = \sqrt{\frac{16 \times 0.567^3}{81 \times 0.134}} \frac{\partial}{\partial x} \left\{ U_1(x)U_2(x) \right\} \times \left[ \frac{17\sqrt{11}}{166U_3(x)} + \frac{(\sqrt{17} - 1)\sqrt{11}}{88}U_3(x) \right]$$

$$= \sqrt{\frac{16 \times 0.567^3}{81 \times 0.134}} \left\{ \frac{(35 - 9\sqrt{6})U_2}{26U_1[(\sqrt{6} - 1)(x-2) + 1]^2} + \frac{U_1}{(x-2)^2} \right\}$$

$$\times \left[ \frac{17\sqrt{11}}{166U_3} + \frac{(\sqrt{17} - 1)\sqrt{11}}{88}U_3 \right]$$

$$- \left[ \frac{(\sqrt{17} - 1)\sqrt{11}}{88} - \frac{17\sqrt{11}}{166U_3^2} \right] \left\{ \frac{9U_1U_2U_3^2}{2x^2} \right\}.$$
\[
\sqrt{\frac{16 \times 0.5673}{81 \times 0.134}} ((11 \sqrt{17} + 38 - 0.34 \times 23)x - 66)^{-2}
\]
\[
U_1 U_3 (x - 2)^2 x [\sqrt{6} - 1)(x - 2) + 1]^2
\]
\[
\times \left\{ \left[ \frac{35 - 9\sqrt{6}}{26} U_2 (x - 2)^2 x + U_1^2 [\sqrt{6} - 1)(x - 2) + 1]^2 \right] \right. 
\]
\[
\times \left[ \frac{17 \sqrt{11}}{166} + \frac{(\sqrt{17} - 1) \sqrt{11}}{88} U_3^2 \right] [(11 \sqrt{17} + 30.18)x - 66]^2 x 
\]
\[- \frac{3U_2^2}{x} [11 \sqrt{17} + 30.18]x - 66]^2 [\sqrt{6} - 1)(x - 2) + 1]^2 
\]
\[\times U_1^2 \left[ \frac{(\sqrt{17} - 1) \sqrt{11} U_3^2}{176} - \frac{17 \sqrt{11}}{332} U_2 (x - 2)^2 \right] \}
\]
\[
\sqrt{\frac{16 \times 0.5673}{81 \times 0.134}} ((11 \sqrt{17} + 38 - 0.34 \times 23)x - 66)^{-2}
\]
\[
U_1 U_3 (x - 2)^2 x [\sqrt{6} - 1)(x - 2) + 1]^2 R(x),
\]

where
\[
R(x) = \left\{ \frac{(35 - 9\sqrt{6})[(6 - \sqrt{6})(x - 2) - 1](x - 2)}{26} 
\]
\[+ \left[ (3 - \sqrt{6})(x - 2) - \frac{4}{13} \right] [(\sqrt{6} - 1)(x - 2) + 1]\right. 
\]
\[
\times \left\{ \frac{17 \sqrt{11}}{166} [(11 \sqrt{17} + 30.18)x - 66]^2 x 
\]
\[- \frac{(\sqrt{17} - 1) \sqrt{11}}{4} [(11 \sqrt{17} + 30.18)x - 66]^2 x \}
\]
\[-66 \left[ (3 - \sqrt{6})(x - 2) - \frac{4}{13} \right] [(\sqrt{6} - 1)(x - 2) + 1] 
\]
\[
\times \left\{ \frac{(\sqrt{17} - 1) \sqrt{11} x}{8} - \frac{17 \sqrt{11}}{332} [(11 \sqrt{17} + 30.18)x - 66] \right\} 
\]
\[- \frac{(6 - \sqrt{6})(x - 2) - 1](x - 2).\]

We have
\[
(2.26) \quad R(x) \geq 10000Q_2(x), \quad \text{for } x \geq 0,
\]
where \(Q_2(x) = 0.7633x^5 - 5.1552x^4 + 13.4534x^3 - 17.435x^2 + 11.598x - 3.2064.\)

By calculating the determinant of the Sylvester matrix associated to \(Q_2\) and \(Q_2'\), we get
\[
(2.27) \quad R(Q_2, Q_2') \approx -0.145.
\]

Hence, the discriminant of \(Q_2(x)\) is negative. So, there exists at least one pair of complex conjugate roots for \(Q_2(x)\).
On the other hand, we get
\[ Q_2(0) = -3.2064, \quad Q_2(1) = 0.0181, \]
\[ Q_2(2) = -0.1808, \quad Q_2(3) = 5.8251. \]
Hence
\[ Q_2(x) > 0, \quad \text{for } x \geq 3. \]
It implies that \( g_2(x) \) is increasing for \( x \geq 3 \). Notice that
\[ \lim_{x \to \infty} g_2(x) \approx -0.044. \]
Therefore, we have
\[ g_2(x) < 0, \quad \text{for } x \geq 6. \]
From (2.23), we have \( \nabla A = 0 \) and \( S = n \), i.e., \( M \) is a Clifford torus.

\[ \square \]

**Remark.** In fact, we can enlarge the second pinching interval in Theorem 1.1 to \([n, n + \frac{n}{21.6}]\) by modifying the parameters \( \theta \) and \( \theta_1 \).

### 3. Self-shrinkers in the Euclidean space

Let \( M \) be an \( n \)-dimensional complete self-shrinker with polynomial volume growth in \( \mathbb{R}^{n+1} \). We shall make use of the following convention on the range of indices:
\[ 1 \leq i, j, k, \ldots \leq n. \]
We choose a local orthonormal frame \( \{e_1, e_2, \ldots, e_{n+1}\} \) near a fixed point \( x \in M \) over \( \mathbb{R}^{n+1} \) such that \( \{e_i\} \) are tangent to \( M \) and \( e_{n+1} \) equals to the unit inner normal vector \( \xi \). Let \( \{\omega_1, \omega_2, \ldots, \omega_{n+1}\} \) be the dual frame fields of \( \{e_1, e_2, \ldots, e_{n+1}\} \). Denote by \( A, H, S \) and \( R_{ijkl} \) the second fundamental form, the mean curvature, the squared length of the second fundamental form and the Riemannian curvature tensor of \( M \), respectively. Then we have
\begin{align*}
(3.1) \quad A &= \sum_{i,j} h_{ij} \omega_i \otimes \omega_j, \quad h_{ij} = h_{ji}, \\
(3.2) \quad S &= |A|^2 = \sum_{i,j} h_{ij}^2, \\
(3.3) \quad H &= \text{Trace } A = \sum_i h_{ii}.
\end{align*}
The Gauss and Codazzi equations are given by
\begin{align*}
(3.4) \quad R_{ijkl} &= h_{ik}h_{jl} - h_{il}h_{jk},
\end{align*}
and
\( h_{ijk} = h_{ikj}. \)

We have the Ricci identities.
\[
(3.6) \quad h_{ijkl} - h_{ijlk} = \sum_m h_{im} R_{mkl} + \sum_m h_{mj} R_{mikl}.
\]

(3.7) \( h_{ijklm} - h_{ijkml} = \sum_r h_{rjk} R_{rilm} + \sum_r h_{irk} R_{rijlm} + \sum_r h_{ijr} R_{rklm}. \)

To simplify computation, we choose the local orthonormal frame such that its tangential covariant derivatives vanish at \( x \in M, i.e., \nabla_{e_i} e_j = h_{ij} \xi. \)

Then we have
\[
(3.8) \quad \nabla_{e_i} H = -\nabla_{e_i} < X, \xi > = h_{ik} < X, e_k >, \quad (3.9) \quad \text{Hess} H(e_i, e_j) = -\nabla_{e_i} \nabla_{e_j} < X, \xi > = 2 \sum_{i,j} h_{ij} \Delta h_{ij} + 2 \sum_{i,j,k} h_{ij} h_{ik} h_{kj}.
\]

In [12], Colding and Minicozzi introduced the linear operator
\[
\mathcal{L} = \Delta - < X, \nabla(\cdot) > = e^{-|X|^2} \text{Div}(e^{-|X|^2} \nabla(\cdot)).
\]

They showed that \( \mathcal{L} \) is self-adjoint respect to the measure \( \rho dM \), where \( \rho = e^{-|X|^2/2}. \)

Thus, we get
\[
(3.10) \quad \mathcal{L}|A|^2 = \Delta |A|^2 - < X, \nabla |A|^2 > = 2 \sum_{i,j} h_{ij} \Delta h_{ij} + 2|A|^2 - 2 \sum_{i,j,k} h_{ij} h_{ijk} < X, e_k > = 2(|A|^2 - 2|A|^4) + 3(B_1 - 2B_2) + \frac{3}{2} |\nabla S|^2.
\]

We have the following equalities for \( |\nabla S|^2 \) and \( |\nabla^2 A|^2 \).
\[
(3.11) \quad |\nabla S|^2 = \frac{1}{2} \mathcal{L} S^2 + 2S^2(S - 1) - 2S|\nabla A|^2,
\]
\[
(3.12) \quad |\nabla^2 A|^2 = \frac{1}{2} \mathcal{L}|\nabla A|^2 + (|A|^2 - 2)|\nabla A|^2 + 3(B_1 - 2B_2) + \frac{3}{2} |\nabla S|^2.
\]
where \( B_1 = \sum_{i,j,k,l,m} h_{ijk} h_{ijm} h_{ml} \) and \( B_2 = \sum_{i,j,k,l,m} h_{ijklm} h_{im} h_{jl} \)

Following [16, 29], we have

\[
3(B_1 - 2B_2) \leq \sigma |\nabla A|^2,
\]

where \( \sigma = \sqrt{17 + 1} \). We choose a local orthonormal frame \( \{ e_i \} \) such that \( h_{ij} = \mu_i \delta_{ij} \) at \( x \).

By Ricci identity (3.6), we have

\[
t_{ij} := h_{ijij} - h_{jiji} = \mu_i \mu_j (\mu_i - \mu_j).
\]

By a computation, we obtain

\[
\int_M (B_1 - 2B_2) \rho dM = \int_M \left( \frac{1}{2} G - \frac{1}{4} |\nabla S|^2 \right) \rho dM,
\]

where \( G = \sum_{i,j} t_{ij}^2 = 2(S f_4 - f_3^2) \) and \( f_k = \text{Trace} A^k = \sum_i \mu_k^i \).

By Lemma 4.2 in [16], we have

\[
3(B_1 - 2B_2) \leq (S + C_2 G^{1/3}) |\nabla A|^2,
\]

where \( C_2 = \frac{2\sqrt{6} + 3}{\sqrt{21\sqrt{6} + 103/2}} \).

Set \( u_{ijkl} = \frac{1}{4}(h_{ijkl} + h_{lijk} + h_{klij} + h_{jkl}) \). Notice that \( u_{ijkl} \) is symmetric in \( i, j, k, l \). Then we have

\[
\sum_{i,j,k,l} (h_{ijkl}^2 - u_{ijkl}^2) = \frac{1}{16} \sum_{i,j,k,l} [(h_{ijkl} - h_{iij})^2 + (h_{ijkl} - h_{klj})^2 + (h_{ijkl} - h_{jkl})^2 + (h_{ijkl} - h_{jli})^2 + (h_{ijkl} - h_{jkl})^2 + (h_{ijkl} - h_{jli})^2 + (h_{ijkl} - h_{jkl})^2 + (h_{ijkl} - h_{jli})^2]
\]

\[
\geq \frac{6}{16} \sum_{i\neq j} [(h_{ijij} - h_{ijij})^2 + (h_{ijji} - h_{ijij})^2]
\]

i.e.,

\[
|\nabla^2 A|^2 \geq \frac{3}{4} G.
\]

Now we are in a position to prove our rigidity theorem for self-shrinkers in the Euclidean space.

**Proof of Theorem 1.2.** Combining (3.12), (3.15), (3.16) and (3.18), using Young’s inequality, we drive the following inequality for arbitrary
\[ \theta \in (0, 1). \]

\[
\frac{3}{4}(1 - \theta) \int_M G \rho dM + \frac{3\theta}{8} \int_M |\nabla S|^2 \rho dM \\
\leq \int M |\nabla^2 A|^2 \rho dM - \frac{3\theta}{2} \int_M B \rho dM \\
= \int M [(S - 2)|\nabla A|^2 + \frac{3}{2}|\nabla S|^2] \rho dM \\
+ 3\left(1 - \frac{\theta}{2}\right) \int_M (B_1 - 2B_2) \rho dM \\
\leq \int M [(S - 2)|\nabla A|^2 + \frac{3}{2}|\nabla S|^2] \rho dM \\
+ \left(1 - \frac{\theta}{2}\right) \int M (S + C_2 G^{1/3}) |\nabla A|^2 \rho dM \\
\leq \int M \left\{ \left[\left(2 - \frac{\theta}{2}\right) S - 2\right]|\nabla A|^2 + \frac{3}{2}|\nabla S|^2 \right\} \rho dM \\
(3.19) + C_4 \int_M |\nabla A|^3 \rho dM + \frac{3}{4}(1 - \theta) \int_M G \rho dM,
\]

where \( C_4(\theta) = \frac{4}{9} C_2^{3/2} \left(1 - \frac{\theta}{2}\right)^{3/2} (1 - \theta)^{-1/2} \). This implies that

\[
0 \leq \int M \left[\left(2 - \frac{\theta}{2}\right) S - 2\right]|\nabla A|^2 \rho dM \\
(3.20) + \left(\frac{3}{2} - \frac{3\theta}{8}\right) \int_M |\nabla S|^2 \rho dM + C_4 \int_M |\nabla A|^3 \rho dM.
\]

Applying the divergence theorem, we get

\[
\begin{align*}
- \int_M \nabla |\nabla A| \cdot \nabla S \rho dM \\
= \int_M |\nabla A| \Delta S \rho dM + \int_M |\nabla A| \nabla S \cdot \nabla \rho dM \\
= \int_M |\nabla A| \Delta S \rho dM + \int_M |\nabla A| (- < X, \nabla S >) \rho dM \\
= \int_M |\nabla A| \mathcal{L} S \rho dM.
\end{align*}
(3.21)
\]
From (3.10), (3.12), (3.13) and (3.21), for arbitrary $\epsilon > 0$, we have

$$\int_M |\nabla A|^3 \rho dM = \int_M (S^2 - S + \frac{1}{2} \mathcal{L}|A|^2)|\nabla A|\rho dM$$

$$= \int_M (S^2 - S)|\nabla A|\rho dM - \frac{1}{2} \int_M \nabla|\nabla A| \cdot \nabla A \rho dM$$

$$\leq \int_M (S^2 - S)|\nabla A|\rho dM + \epsilon \int_M |\nabla^2 A|^2 \rho dM$$

$$+ \frac{1}{16\epsilon} \int_M |\nabla S|^2 \rho dM$$

$$\leq \int_M (S^2 - S)|\nabla A|\rho dM + \left(\frac{3\epsilon}{2} + \frac{1}{16\epsilon}\right) \int_M |\nabla S|^2 \rho dM$$

$$+ \epsilon \int_M (\sigma S + S - 2)|\nabla A|^2 \rho dM.$$  

(3.22)

From (3.11), we have

$$\frac{1}{2} \int_M |\nabla S|^2 \rho dM = \int_M S(S - 1)^2 \rho dM - \int_M (S - 1)|\nabla A|^2 \rho dM.$$  

(3.23)

When $1 \leq S \leq 1 + \delta$, for $\epsilon_1 = \theta_1 \epsilon > 0$, we obtain

$$\int_M S(S - 1)|\nabla A|\rho dM \leq (2 + 2\delta)\epsilon_1 \int_M S(S - 1)\rho dM$$

$$+ \frac{1}{(8 + 8\delta)\epsilon_1} \int_M S(S - 1)|\nabla A|^2 \rho dM$$

$$\leq (2 + 2\delta)\epsilon_1 \int_M S(S - 1)\rho dM$$

$$+ \frac{1}{8\epsilon_1} \int_M (S - 1)|\nabla A|^2 \rho dM.$$  

(3.24)

Substituting (3.22), (3.23) and (3.24) into (3.20), we have

$$0 \leq \int_M \left[\left(2 - \frac{\theta}{2}\right) S - 2\right]|\nabla A|^2 \rho dM + \left(\frac{3}{2} - \frac{3\theta}{8}\right) \int_M |\nabla S|^2 \rho dM$$

$$+ C_4 (2 + 2\delta)\epsilon_1 \int_M S(S - 1)\rho dM + \frac{1}{8\epsilon_1} \int_M (S - 1)|\nabla A|^2 \rho dM$$

$$+ \epsilon \int_M (\sigma S + S - 2)|\nabla A|^2 \rho dM + \left(\frac{3\epsilon}{2} + \frac{1}{16\epsilon}\right) \int_M |\nabla S|^2 \rho dM.$$
\[
\leq \int_M \left[ \left( 2 - \frac{\theta}{2} \right) S - 2 + \frac{C_4}{8\epsilon_1} (S - 1) + 2C_4\epsilon_1 (1 + \delta) \\
+ C_4\epsilon (\sigma S + S - 2) \right] |\nabla A|^2 \rho dM + \left[ 3 - \frac{3\theta}{4} + C_4 (3\epsilon + \frac{1}{8\epsilon}) \right] \times \left[ \int_M \delta S (S - 1) \rho dM - \int_M (S - 1) |\nabla A|^2 \rho dM \right] \]
\[
= \int_M \left[ - \frac{\theta}{2} + \frac{\sqrt{17} + 3}{2} C_4\epsilon + \left( 5\epsilon + \frac{1}{8\epsilon} \right) \frac{\delta C_4}{4} \\
+ \left( 3 - \frac{3\theta}{4} \right) \delta + 2C_4 (1 + \delta) (\epsilon_1 - \epsilon) \right] |\nabla A|^2 \rho dM \\
(3.25) \quad - \int_M \left[ 1 - \frac{\theta}{4} - \frac{\sqrt{17} - 3}{2} C_4\epsilon + \frac{C_4}{8\epsilon} - \frac{C_1}{8\epsilon_1} \right] (S - 1) |\nabla A|^2 \rho dM.
\]

Let \( \epsilon_1 = \theta_1 \epsilon, \theta_1 = 0.81, \epsilon = \sqrt{\frac{\delta}{4(\sqrt{17} + 3) + 16(1 + \delta)(\theta_1 - 1) + 40\delta}}, \theta = 0.836. \)
Then we have \( C_4 \leq 1.066218. \) Thus, (3.25) is reduced to
\[
0 \leq \left[ \frac{C_4}{4} \sqrt{40\delta^2 - 3.04(1 + \delta)\delta + 4(\sqrt{17} + 3)\delta} \\
- 0.418 + 2.373\delta \right] \int_M |\nabla A|^2 \rho dM \\
+ \left[ \frac{C_4}{8} \left( 4\sqrt{17}\epsilon - 12\epsilon - \frac{1}{\epsilon} + \frac{1}{0.81\epsilon} \right) - 0.791 \right] \times \int_M (|A|^2 - 1) |\nabla A|^2 \rho dM.
(3.26)
\]

We take \( \delta = 1/21. \) Then the coefficients of the integrals in (3.26) are both negative. Therefore, we have \( |\nabla A| \equiv 0 \) and \( |A| \equiv 1, \) i.e., \( M \) is a round sphere or a cylinder. \( \Box \)

To attack the Chern conjecture, we would like to suggest a program that divides the second optimal pinching problem for minimal hypersurfaces in a sphere into several steps. Precisely, we propose the following unsolved problem.

**Problem 3.1.** Let \( M \) be a compact minimal hypersurface in \( \mathbb{S}^{n+1}. \) Denote by \( A \) the second fundamental form of \( M. \) Is it possible to prove that there exists a positive integer \( k \) with \( 1 \leq k \leq 21, \) such that if \( 0 \leq |A|^2 - n \leq \frac{n}{k}, \) then \( |A|^2 \equiv n \) and \( M \) is a Clifford torus, or \( |A|^2 \equiv 2n \) and \( k = 1? \)

For self-shrinkers in the Euclidean space, we would like to propose the following problem.
Problem 3.2. Let $M$ be an $n$-dimensional complete self-shrinker with polynomial volume growth in $\mathbb{R}^{n+1}$. Denote by $A$ the second fundamental form of $M$. Is it possible to prove that there exists a positive integer $k$ with $1 \leq k \leq 20$, such that if $0 \leq |A|^2 - 1 \leq \frac{1}{k}$, then $|A|^2 \equiv 1$ and $M$ is a round sphere or a cylinder, or $|A|^2 \equiv 2$ and $k = 1$?

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