QUANTUM FERMION HAIR

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Abstract

It is shown that the Dirac operator in the background of a magnetic Reissner-Nordström black hole and a Euclidean vortex possesses normalizable zero modes in theories containing superconducting cosmic strings. One consequence of these zero modes is the presence of a fermion condensate around magnetically charged black holes which violates global quantum numbers.
1. Introduction

Classically black holes are among the most perfect objects in the universe, being traditionally characterized according to the classical no hair theorems by only their mass, angular momentum, and electric and magnetic charges. It has been realized recently that the structure of black holes is possibly much richer. Quantum mechanically black holes can carry a variety of hair which can be detected by means of Aharonov-Bohm interference effects \([1]\). In \([1]\) axion hair which can be detected through scattering of axion strings or superstrings off black holes was discussed. In \([2]\) it was argued that discrete gauge hair could also be carried by black holes and a number of consequences of such hair were explored in \([3]\), \([4]\), \([5]\).

One of the striking conclusions of \([3]\) was the fact that Euclidean vortex solutions in a black hole background give rise to a non-perturbative, exponentially decaying electric field outside the horizon of a Schwarzschild black hole. Thus in principle the effects of quantum hair can be detected not only by interference effects, but directly through measurement of the electric field. The vortex solutions which give rise to these effects are mathematically very similar to cosmic string solutions which arise in spontaneously broken \(U(1)\) gauge theories. It has been known for some time that fermion fields in the presence of such cosmic strings can have normalizable zero modes \([3]\). In some models these zero modes lead to superconducting cosmic strings \([4]\). It is natural to wonder whether these zero modes can also play a role in the structure of black holes. In this paper we answer this in the affirmative for magnetically charged Reissner-Nordström black holes by showing that operators constructed out of fermion fields acquire non-zero expectation values outside the horizon.

The outline of the paper is as follows. The second section contains a review of the relevant results regarding cosmic string superconductivity. In the third section we discuss euclidean vortices in black hole backgrounds and extend previous results to the case of Reissner-Nordström solutions. In the fourth section we combine and extend these results to demonstrate the existence of fermion zero modes in the combined black hole-vortex configuration. In the fifth section we discuss some physical effects of these zero modes, in particular we show that magnetic black holes develop a fermion condensate outside the horizon which violates global quantum numbers.

2. Cosmic Strings and Fermion Zero Modes

The model of fermion zero modes of interest to us was introduced by Jackiw and Rossi in \([3]\) and generalized by Witten in \([4]\). This theory (in flat Minkowski space) has a \(U(1)_R\) gauge symmetry with gauge field strength \(G_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu\), a fermion field \(\psi\) with charge \(r_\psi = -1/2\), and a charge one scalar field \(\phi\). The lagrangian is given by

\[
\mathcal{L} = -\frac{1}{4} G^{\mu\nu} G_{\mu\nu} + i \bar{\psi} \gamma^\mu D_\mu \psi + \frac{1}{2} i \lambda \phi \psi^T C \psi - \frac{1}{2} i \lambda \phi^* \bar{\psi} C \psi^T + D_\mu \phi^* D^\mu \phi - V(\phi),
\]

where \(V(\phi)\) is a potential for \(\phi\) with a minimum at \(|\langle \phi \rangle| = v\), the covariant derivative is given by \(D_\mu = (\partial_\mu + i e R B_\mu)\) with \(R\) the \(U(1)_R\) charge, \(C\) is the charge conjugation.
matrix, we take \( \lambda \) to be real and positive, and we may for simplicity choose \( \psi \) to have definite chirality.

It is then convenient to use a chiral basis for the gamma matrices with

\[
\gamma^5 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix}, \quad C = \begin{pmatrix} -i\sigma^2 & 0 \\ 0 & i\sigma^2 \end{pmatrix}
\]  

(2.2)

and to set

\[
\psi = \begin{pmatrix} \psi_L \\ 0 \end{pmatrix}
\]  

(2.3)

where \( \sigma^\mu = (1, \sigma^i), \bar{\sigma}^\mu = (1, -\sigma^i) \) in terms of the usual Pauli matrices.

A static vortex configuration running along the \( z \)-axis with winding number \( k = 1 \) is given in polar coordinates \( (\rho, \theta) \) by

\[
\phi(\rho) = vX_0(\rho)e^{i\theta} \\
B_\theta = \frac{1}{eR}(P_0(\rho) - 1)
\]

(2.4)

with \( X_0(\rho) \to 0 \) and \( P_0(\rho) \to 1 \) as \( \rho \to 0 \), while as \( \rho \to \infty \) \( P_0 \) approaches zero and \( X_0 \) approaches unity.

The \( \psi \) equation of motion following from (2.1) is

\[
i\bar{\sigma}^\mu D_\mu \psi_L + \lambda \phi^* \sigma^2 \psi^*_L = 0.
\]

(2.5)

A static zero-energy solution to the \( \psi \) equation of motion in the background (2.4) will thus obey the equation

\[
\bar{\sigma} \cdot (i\nabla - r_\psi e_R \vec{B})\psi_L - \lambda \phi^* \sigma^2 \psi^*_L = 0.
\]

(2.6)

Going to polar coordinates and writing the two-component spinor \( \psi_L \) as

\[
\psi_L = \begin{pmatrix} e^{r_\psi \int_0^\rho d\rho' (P_0(\rho') - 1)/\rho'} \psi_U \\ e^{-r_\psi \int_0^\rho d\rho' (P_0(\rho') - 1)/\rho'} \psi_D \end{pmatrix}
\]

(2.7)

yields the decoupled equations

\[
e^{-i\theta}(i\partial_\rho \psi_D + \frac{1}{\rho} \partial_\theta \psi_D) + i\lambda \phi^* \psi^*_D = 0 \\
e^{i\theta}(i\partial_\rho \psi_U - \frac{1}{\rho} \partial_\theta \psi_U) - i\lambda \phi^* \psi^*_U = 0.
\]

(2.8)

For a single vortex configuration given by (2.4), (2.8) has a single normalizable solution

\( \psi^0_L \) given by \( \psi^0_U = 0 \),

\[
\psi^0_D = e^{-\lambda v \int_0^\rho X_0(\rho') d\rho'}
\]

(2.9)
which obeys $i\gamma_1\gamma_2\psi_L^0 = -\psi_L^0$. In [3] Jackiw and Rossi analyze the spectrum of the Dirac operator in a more general $k$-vortex configuration and show that it includes precisely $k$ normalizable zero modes in accordance with the general index theorem of [3].

Returning to the full equation of motion (2.5) we can look for solutions of the form

$$\psi_L = \eta(z, t)\psi_L^0(\rho, \theta).$$

(2.10)

Since $\gamma_5\psi_L = i\gamma_0\gamma_1\gamma_2\gamma_3\psi_L = -\psi_L$ and $i\gamma_1\gamma_2\psi_L^0 = -\psi_L^0$, (2.5) implies that $\eta(z + t) = \eta(z, t)$ is a two-dimensional Majorana-Weyl spinor which represents fermions localized on the vortex which propagate down the vortex at the speed of light.

In order to obtain charged zero-modes on the vortex it is necessary to double the degrees of freedom so as to obtain complex zero modes. Thus following [7] we can generalize the lagrangian (2.3) to include an additional left-handed fermion field $\chi$ and an additional unbroken $U(1)_Q$ gauge symmetry with field strength $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. The lagrangian is

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{4}G_{\mu\nu}G^{\mu\nu} + i\bar{\psi}\gamma^\mu D_\mu\psi + i\bar{\chi}\gamma^\mu D_\mu\chi + D_\mu\phi^*D^\mu\phi - V(\phi)$$

$$+ i\lambda\phi\bar{\psi}^T C\chi - i\lambda\phi^*\bar{\psi}C\chi^T.$$  

(2.11)

The $U(1)_R$ symmetry is spontaneously broken by the expectation value of $\phi$ with $Q = 0, R = +1$ and again gives rise to vortex solutions as given by (2.4). The Yukawa couplings are consistent with the choice of quantum number assignments $Q = q, R = r$ for $\psi$ and $Q = -q, R = -r - 1$ for $\chi$, although note that $\text{Tr}QR = 2q(r + 1/2)$ so that $Q$ is orthogonal to $R$ only for $r = -1/2$.

The $\psi$ and $\chi$ equations of motion are

$$i\bar{\sigma}^\mu D_\mu\psi_L + \lambda\phi^*\sigma^2\chi_L^* = 0,$$

$$i\bar{\sigma}^\mu D_\mu\chi_L + \lambda\phi^*\sigma^2\psi_L^* = 0.$$  

(2.12)

Repeating the previous analysis we now find a normalizable zero mode in a $k = 1$ vortex background given by

$$\chi_L = e^{-i\alpha}\psi_L^0 x_-(\rho) , \quad \psi_L = e^{i\alpha}\psi_L^0 x_+/(\rho),$$

(2.13)

with $\alpha$ an arbitrary phase, and where $x_\pm \in \mathbb{R}$, and satisfy

$$x'_\pm \pm r_-(\rho - 1) x_\pm \pm \lambda v X_0(\rho)(x_- - x_+) = 0.$$  

(2.14)

$r_- = (r_\psi - r_\chi)/2 = r + 1/2$ represents the overlap of $U(1)_Q$ with $U(1)_R$. We have not been able to solve (2.14) analytically, other than as a perturbation series in $r_-$, but it is easy to see that there exists a solution with asymptotic behavior

$$x_+ \approx 1 - \frac{r_-(1 - r_-)}{2\lambda v \rho} , \quad x_- \approx 1 + \frac{r_-(1 + r_-)}{2\lambda v \rho}.$$  

(2.15)

1 We could of course work with orthogonal $U(1)$ factors to start with in which case the unbroken $U(1)$ would in general be a linear combination of the original $U(1)$ factors.
as $\rho \to \infty$, which makes the zero modes (2.13) normalizable.

Thus we see that the previous real Majorana-Weyl zero mode has been extended to a complex Weyl zero mode which carries charge under the unbroken $U(1)_Q$.

As emphasized in [7], as it stands both the original four-dimensional theory and the two-dimensional low-energy effective theory of the zero modes is anomalous. This may be addressed by adding an additional multiplet of left-handed spinors $\hat{\psi}, \hat{\chi}$ with quantum numbers $Q = \hat{q}, R = \hat{r}$ and $Q = -\hat{q}, R = -\hat{r} + 1$ respectively which acquire mass from coupling to $\phi^*$ rather than $\phi$:

$$\hat{\mathcal{L}} = i\gamma \phi^* \hat{\psi}^T C\hat{\chi} + \text{h.c.}$$

In the general situation with $M$ pairs $(\psi_i, \chi_i)$ and $N$ pairs $(\hat{\psi}_j, \hat{\chi}_j)$ the conditions for anomaly cancellation are

$$\text{Tr} R^3 = 0 = \sum_{i=1}^{M} (-3r_i^2 - 3r_i - 1) + \sum_{j=1}^{N} (3\hat{r}_j^2 - 3\hat{r}_j + 1);$$

$$\text{Tr} Q^2 R = 0 = -\sum_{i=1}^{M} q_i^2 + \sum_{j=1}^{N} \hat{q}_j^2;$$  \hspace{1cm} (2.17)

$$\text{Tr} Q R^2 = 0 = -\sum_{i=1}^{M} q_i (2r_i + 1) + \sum_{j=1}^{N} \hat{q}_j (2\hat{r}_j - 1).$$

The simplest anomaly free model has $M = N = 1$ with $\hat{q} = q$ and $\hat{r} = r + 1$. An embedding of this structure in a grand unified theory is given by the $O(10)$ vortex model discussed in [7]. In this model a vortex is formed in the breaking of $O(10)$ to the standard model and the dynamics require that the ordinary electroweak Higgs doublet have a phase change of $2\pi$ in going around the vortex. The fields and their quantum numbers are then (for a single generation) $\psi_1 = e_L, \chi_1 = e_c^L$ with $q_1 = -1, r_1 = 1; \psi_2 = d_L, \chi_2 = d_c^L$ with $q_2 = -1/3, r_2 = -1/3$ and $\hat{\psi}_1 = u_L, \hat{\chi}_1 = u_c^L$ with $\hat{q}_1 = 2/3, \hat{r}_1 = -1/3$. The zero modes in this model consist of left-moving down quarks and anti-quarks, left-moving electrons and positrons, and right-moving up quarks and anti-quarks.

3. Euclidean Black Hole Vortices

In the following section we will discuss fermion zero modes in a background consisting of a Euclidean vortex in a Reissner-Nordström background. In this section we discuss the basic features and existence of such a vortex solution. In order to search for a vortex-Reissner-Nordström solution we would ideally couple the Euclidean matter action to gravity, employing suitable boundary conditions which indicate the presence of a vortex. In particular the vortex fields must have an asymptotic winding number

$$\phi \to v e^{2\pi i k \tau / \beta}$$

$$B \to -\frac{2\pi k}{e_R \beta} d\tau$$

$$\text{(3.1)}$$
as $r \to \infty$, and be regular as $r \to r_+$, the horizon. The formalism for dealing with such a system was developed in [5], although there it was used for a Schwarzschild black hole. The Reissner-Nordström black hole represents an increase in calculational complexity, although many of the qualitative features of the vortex solution remain. Let us first review the salient points of [5] before proceeding to the Reissner-Nordström vortex.

In [5] it was noted that existence of solutions to the non-linear coupled vortex-gravity equations is in general a very difficult problem. What enabled a tractable solution to be found was the observation that the situation was very similar to that of a self-gravitating cosmic string, and [5] proved that the black hole vortex satisfied similar criteria to the cosmic string. Specifically, the metric was shown to be asymptotically Schwarzschild, provided the following conditions were satisfied:

(i) the system was weakly gravitating, and
(ii) the energy momentum decayed at least as fast as $r^{-5}$ outside the core, where

$$\hat{r} = \int_{r_+}^{r} \frac{dr'}{A}(3.2)$$

is the proper distance from the horizon. This argument, coupled with a rewriting of the equations as a perturbative expansion in $\epsilon$, the gravitational strength of the vortex, and a numerical background vortex solution allowed a strong plausibility argument for existence of a fully coupled Einstein-Abelian Higgs vortex solution. The back-reaction of the vortex on the geometry was easily extracted, and was shown to take the form of cutting a slice out of the euclidean black hole cigar geometry - a gravitational effect analogous to the self-gravitating cosmic string.

For a Reissner Nordström vortex, we expect that if we find a background solution of a similar form to that in Schwarzschild, then the rest of the argument will go through in an analogous fashion. Since we are concerned with the physics of the vortex itself here, rather than its gravitational effect, we restrict ourselves to proving existence of a background solution. Questions of gravitational backreaction and thermodynamics will be discussed elsewhere.

In order to search for a vortex solution we make the usual decomposition (with $B = \beta/2\pi$)

$$\phi = vX(r)e^{ikr/B}$$

$$B_\mu = \frac{1}{Be}(P(r) - k)\partial_\mu \tau = \frac{1}{Be}(P_\mu - k\partial_\mu \tau), \quad (3.3)$$

which yields the equations of motion

$$\frac{1}{r^2}(r^2P,_r),_r = \frac{\lambda v^2 X^2 P}{A^2}$$

$$\frac{1}{r^2}(r^2A^2X,_r),_r = \frac{P^2 X}{A^2 B^2} + \frac{\lambda v^2}{2} (X^2 - 1), \quad (3.4)$$

where $A^2 = g_{00}$, and $\nu = \lambda/2e^2$ is the Bogomoln’yi parameter representing the relative strengths of the Higgs and gauge interactions. $\nu = 1$ is often called the supersymmetric limit.
The asymptotic behaviour of the background solutions to (3.4) is

\[ X \propto (r - r_+)^{k/2} \quad P = k - O(r - r_+) \quad \text{as } r \to r_+ \]

\[ 1 - X \propto r^{-1-a} e^{-\sqrt{\lambda} vr} \quad P \propto r^{-1-a/\sqrt{\nu}} e^{-\sqrt{\lambda} vr/\sqrt{\nu}} \quad \text{as } r \to \infty \]

(3.5)

where \( a = \sqrt{\lambda} vr_+(1 + r_+/B) \). In order to verify that these correspond to the asymptotic behaviours of a full solution, we must integrate (3.4) with \( A^2 = (r - r_+)(r - r_-)/r^2 \). Setting \( \tilde{r} = \sqrt{\lambda} v(r - r_+) \), \( R = \sqrt{\lambda} v \) and \( \Delta = 2Rr_+/B \) gives

\[ \frac{1}{(\tilde{r} + R)^2} [(\tilde{r} + R)^2 P'] = \frac{X^2 P (\tilde{r} + R)^2}{\nu \tilde{r}(\tilde{r} + \Delta)} \]

\[ \frac{1}{(\tilde{r} + R)^2} [\tilde{r}(\tilde{r} + \Delta) X'] = \frac{P^2 X (\tilde{r} + R)^2 \Delta^2}{4R^4 \tilde{r}(\tilde{r} + \Delta)} + \frac{1}{2} X(X^2 - 1). \]

(3.6)

Note that \( \Delta \to 0 \) gives extremal Reissner-Nordström, and \( R \to \infty \) gives the thin string limit. We have integrated the equations numerically using a relaxation technique, and the results show that the bounded solutions at the horizon do indeed integrate out to the exponentially decaying solutions at infinity. Figure 1a shows a plot of \( P \) with \( k, \nu = 1, \) and \( R = 1 \) for varying \( \Delta \), and Figure 1b a plot of \( X \) for the same values of the parameters. Note the bunching up of the solutions near to the horizon as we lower \( \Delta \). This represents the divergence of the proper distance to the horizon, which from (3.2) is

\[ \hat{r} = \{ \sqrt{r - r_+} + \sqrt{r - r_-} + (r_+ + r_-) \log \left[ \frac{\sqrt{r - r_+} + \sqrt{r - r_-}}{\sqrt{r_+ - r_-}} \right] \} \]

\[ \sim r_+ \log(\hat{r}/\Delta) \quad \text{as} \quad \Delta \to 0. \]

(3.7)

As a final remark, note that if we replace \( r \) by \( \hat{\rho} = \sqrt{\lambda} v \hat{r} \), (3.4) becomes

\[ \frac{d}{d\hat{\rho}} \left( \frac{1}{\hat{\rho}} \frac{dP}{d\hat{\rho}} \right) = \frac{X^2 P}{\nu \hat{\rho}} + O(\Delta/R^3) \]

\[ \frac{d}{d\hat{\rho}} \left( \frac{dX}{d\hat{\rho}} \right) = \frac{P^2 X}{\hat{\rho}} + \frac{1}{2} \hat{\rho} X(X^2 - 1) + O(\Delta/R^3). \]

(3.8)

Thus either in the thin string or near extremal limit, the vortex becomes a Nielsen-Olesen vortex, and we can write

\[ X = X_0(\hat{r}) \]

\[ P = P_0(\hat{r}) \]

(3.9)

where \( X_0, P_0 \) are the functions appearing in (2.4).

4. Euclidean Fermion Zero Modes

In this section we find fermion zero modes in a combined vortex Reissner-Nordström background of the type discussed in the previous section. In order to do so we must first
discuss the euclidean version of the action (2.11) in this background. In the following section we will discuss the physical consequences of these zero modes.

First let us discuss (2.11). In order to euclideanise we must set $t \rightarrow i\tau$, and $g_{\mu\nu} \rightarrow -g_{\mu\nu}$ (since we were working with signature $+\quad-\quad-\quad-\quad$). We must then make an overall sign change to the Lagrangian in order to comply with the convention that the action for matter fields

$$S_M = \int \sqrt{g}L_M d^4x > 0.$$  

(4.1)

In this case the Einstein equations take the form

$$G_{\mu\nu} = 8\pi GT_{\mu\nu} = \frac{16\pi G \delta(L_M \sqrt{g})}{\sqrt{g} \delta g^{\mu\nu}}.$$  

(4.2)

Clearly the bosonic part of (2.11) becomes

$$L_B = \frac{1}{4}F_{\mu\nu}^2 + \frac{1}{4}G_{\mu\nu}^2 + |D_\mu \phi|^2 + V(\phi).$$  

(4.3)

For the fermionic part, note that it is only the spatial part of the metric that reverses sign, hence

$$\hat{\gamma}^\tau_E = \gamma^t_L, \quad \hat{\gamma}^i_E = i\gamma^i_L$$  

(4.4)

and therefore

$$L_F = -\bar{\psi}\gamma^\mu D_\mu \psi - \bar{\chi}\gamma^\mu D_\mu \chi - i\lambda(\phi \psi^T C\chi - \phi^* \bar{\psi} C\bar{\chi}^T).$$  

(4.5)

As is well known, in euclidean space $\psi$ and $\bar{\psi}$ are no longer related by complex conjugation and must be treated as independent fields. ‘Lorentz’ invariance in fact requires that $\bar{\psi}$ have opposite chirality to $\psi$. Therefore in what follows we are implicitly taking $\psi, \chi$ to be left-handed, and $\bar{\psi}, \bar{\chi}$ right-handed.

We thus wish to look for left-handed $\psi$ and right-handed $\bar{\chi}$ solutions of the equations

$$\gamma^\mu D_\mu \psi = i\lambda\phi^* C\bar{\chi}^T$$  

$$\gamma^\mu D_\mu (\bar{\chi}^T)^* = i\lambda\phi^* C\psi^*$$  

(4.6)

plus the analogous equations involving (right-handed) $\bar{\psi}$ and (left-handed) $\chi$.

Our conventions are as follows. We will use coordinates $x^1 = r$, $x^2 = \theta$, $x^3 = \phi$ and $x^4 = \tau$ to describe the Reissner-Nordström metric with $(\theta, \phi)$ being angular coordinates on $S^2$, $r$ the radial coordinate, and $\tau$ the euclidean time coordinate. We will use hatted indices for tangent space indices. We start with a set of euclidean gamma matrices $\gamma^\mu$, $\hat{\mu} = 1 \cdots 4$ obeying $\{\gamma^\mu, \gamma^{\hat{\mu}}\} = 2\delta^{\mu\hat{\mu}}$. An explicit chiral basis from (1.4) is given by

$$\gamma^5 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \gamma^{\hat{\mu}} = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^{\hat{\mu}} & 0 \end{pmatrix}, \quad C = \begin{pmatrix} -i\sigma^2 & 0 \\ 0 & i\sigma^2 \end{pmatrix}$$  

(4.7)
with $\tilde{\sigma} = (i\sigma^i, 1)$ and $\tilde{\sigma}^\mu = (-i\sigma^i, 1)$. The curved space gamma matrices are related to these by $\gamma^\mu = e^{\mu \hat{\mu}} \gamma^{\hat{\mu}}$ with $e^{\mu \hat{\mu}}$ the vierbein. The vierbein is then given by $e^{\mu \hat{\mu}} = \text{diag}(\frac{A(r)}{r}, 1/r, 1/r \sin \theta, 1/A(r))$ with

$$A^2(r) = \left(1 - \frac{2M}{r} + \frac{4\pi g_Q^2}{r^2}\right) \quad (4.8)$$

and $g_Q$ the magnetic charge.

The covariant derivative appearing in (4.6) is given by

$$D_\mu = \partial_\mu - \Gamma_\mu + iRe_RB_\mu + iQe_QA_\mu \quad (4.9)$$

with $\Gamma_\mu = -\frac{1}{2} \omega_\mu^{\hat{\mu} \hat{\nu}} \Sigma^{\hat{\mu} \hat{\nu}}$ in terms of the spin connection $\omega_\mu^{\hat{\mu} \hat{\nu}}$ and the euclidean Lorentz generators $\Sigma^{\hat{\mu} \hat{\nu}} = \frac{1}{4} [\gamma^{\hat{\mu}}, \gamma^{\hat{\nu}}]$. $B_\mu$ is the broken $U(1)_R$ gauge field with coupling constant $e_R$ and $A_\mu$ is the unbroken $U(1)_Q$ gauge field with coupling constant $e_Q$.

The background fields are given by the euclidean vortex configuration

$$\phi = vX(r)e^{2\pi i r/\beta}$$

$$B_4 = \frac{2\pi}{e_R\beta} (P(r) - 1) \quad (4.10)$$

and the monopole field of the unbroken $U(1)_Q$,

$$A_3 = g_Q(1 - \cos \theta), \quad (4.11)$$

with $g_Q$ the magnetic charge. The Dirac quantization condition implies that $g_Qe_Q = n_M/2$ with $n_M$ integer (we assume that $e_Q$ is normalized so that the charges $q, \hat{q}$ are integers). The non-vanishing components of the connection $\Gamma_\mu$ are

$$\Gamma_1 = 0$$

$$\Gamma_2 = \frac{1}{2} A(r)\gamma^1 \gamma^2$$

$$\Gamma_3 = \frac{1}{2} (\sin \theta A(r)\gamma^1 \gamma^3 + \cos \theta \gamma^2 \gamma^3)$$

$$\Gamma_4 = \frac{1}{2} A' A \gamma^1 \gamma^4 \quad (4.12)$$

Corresponding to the $S^2 \times \mathbb{R}^2$ structure of the Reissner-Nordström black hole manifold, we would like to decompose the Dirac operator $\gamma^\mu D_\mu$ into an angular part depending on coordinates $(\theta, \phi)$ and a remaining part depending on coordinates $(r, \tau)$. To do this we first write

$$\psi(x^\mu) = \frac{W_-(x^\mu)}{\sqrt{Ar^2 \sin \theta}} \quad (4.13)$$

We then find that

$$\gamma^\mu D_\mu \psi = \frac{1}{\sqrt{Ar^2 \sin \theta}} [A\gamma^1 \partial_r W_- + A^{-1} \gamma^3 (\partial_\tau + i r \psi e_RB_\tau) W_- + \frac{1}{r} KW_-] \quad (4.14)$$
with \( K \) given by
\[
K = \left( \gamma^2 \partial_2 + \frac{\gamma^3}{\sin \theta} (\partial_3 + iQeQA_3) \right).
\] (4.15)

In what follows we take the \( Q \) charge of \( \psi(\chi) \) to be \(-1(+1)\).

The operator \( K \) is essentially the Dirac operator for a charged fermion interacting with a \( U(1) \) magnetic monopole field on the two-sphere and according to the index theorem has \( n_M \) normalizable chiral zero modes. For \( n_M = 1 \) the zero mode is given explicitly by
\[
W^0_-(\theta, \phi) = \sqrt{\frac{\sin \theta}{\tan \theta/2}} \xi_-
\] (4.16)

with \( \xi_- \) a constant left-handed spinor obeying \( i\gamma^2\gamma^3\xi_- = -\xi_- \).

We can also decompose \( \chi^T \) as
\[
[\chi^T(x^\mu)]^* = \frac{W_+(x^\mu)}{\sqrt{Ar^2 \sin \theta}}.
\] (4.17)

Remembering that \( \chi \) has a \( U(1)_Q \) charge which is the negative of \( \psi \) we again find a zero mode of the angular operator of the form (4.18) but with \( \xi_- \) replaced by \( \xi_+ \), a right handed spinor which obeys \( i\gamma^2\gamma^3\xi_+ = \xi_+ \). Similarly, if we are solving the \((\chi, \bar{\psi})\) system, we find that \( \chi \propto \xi_+^\prime, \bar{\psi} \propto \xi_-^\prime \) where \( i\gamma^2\gamma^3\xi_+^\prime = \pm \xi_-^\prime \), but \( \xi_\pm^\prime \) are required to have the opposite four-dimensional chiralities to \( \xi_\pm \).

We now look for solutions of (4.6) of the form
\[
\psi = \frac{W^0_0}{\sqrt{Ar^2 \sin \theta}} f_-(r, \tau), \quad (\chi^T)^* = \frac{W^0_+}{\sqrt{Ar^2 \sin \theta}} g_+(r, \tau)
\]
\[
\chi = \frac{W^0_+}{\sqrt{Ar^2 \sin \theta}} f_+(r, \tau), \quad (\bar{\psi}^T)^* = \frac{W^0_-}{\sqrt{Ar^2 \sin \theta}} g_-(r, \tau)
\] (4.18)

Now, in our basis, we may choose \( C\xi_\pm = \hat{\gamma}^1 \xi_\pm \) (in general there will be some arbitrary but constant phase), and the equality
\[
\hat{\gamma}^1 \xi_\pm = \mp i\gamma^3 \gamma^5 \xi_\pm
\] (4.19)

implies that inputting these Ansätze into (4.4) yields:
\[
Af_+^\prime - \frac{2\pi r_\psi}{A\beta} (P-1) f_- + iA f_+ e^{-2\pi i\tau/\beta} g_+^* = 0
\]
\[
Ag_+^\prime - \frac{2\pi r_\chi}{A\beta} (P-1) g_+ + iA g_+ e^{-2\pi i\tau/\beta} f_+^* = 0
\] (4.20)

with the corresponding equation for \( f_- \) and \( g_- \) having a minus sign in front of the first term. Thus, using the \( \hat{r} \) coordinate from (3.2), and noting that over the range of interest \( A \sim 2\pi \hat{r}/\beta \) and (3.9) applies, we see
\[
\pm f_+^\prime - \frac{r_\psi}{\hat{r}} (P_0 - 1) f_+ + \frac{i}{\hat{r}} \partial f_+^\prime - \frac{r_\chi}{\hat{r}} (P_0 - 1) g_+ + \frac{i}{\hat{r}} \partial g_+^\prime + \lambda v X_0 e^{-2\pi i\tau/\beta} f_+^* = 0
\]
\[
\pm g_+^\prime - \frac{r_\psi}{\hat{r}} (P_0 - 1) g_+ + \frac{i}{\hat{r}} \partial g_+^\prime - \frac{r_\chi}{\hat{r}} (P_0 - 1) f_+ + \lambda v X_0 e^{-2\pi i\tau/\beta} f_+^* = 0
\] (4.21)
Clearly this requires \( f_- = g_+ = 0 \) for a non-singular solution, and for \( f_+, g_- \) to take the form

\[
\begin{align*}
    f_+ &= \sqrt{r} e^{i\alpha} x_+ (\hat{r}) \exp \left\{ \int \left[ \frac{1}{2} \frac{2\pi (P-1)}{A \beta} - \lambda v X \right] d\hat{r} \right\} e^{-i\pi \tau / \beta} \\
    g_- &= \sqrt{r} e^{-i\alpha} x_- (\hat{r}) \exp \left\{ \int \left[ \frac{1}{2} \frac{2\pi (P-1)}{A \beta} - \lambda v X \right] d\hat{r} \right\} e^{-i\pi \tau / \beta}
\end{align*}
\]

(4.22)

where \( x_\pm \) are, for small \( \hat{r} \), the same functions as appear in (2.14). Note that these solutions are anti-periodic in the angular coordinate \( \tau \) as compared to the solutions (2.13) which are periodic in \( \theta \). This is due to the fact that we have used an orthonormal cylindrical basis which is naturally adapted to the euclidean Schwarzschild solution rather than the cartesian basis used in (2.13) and [6]. For a detailed discussion of the relation between these bases see [9].

Clearly for an anti-vortex/anti-monopole background this solution will be unchanged, whereas for a vortex/anti-monopole or anti-vortex/monopole background the situation will be reversed; it will be \( \psi, \bar{\chi} \) that has the zero-mode and \( \bar{\psi}, \chi \) which does not.

Notice that the presence of the \( \sqrt{A} \) term in the denominator of (4.18) implies that \( \chi, \bar{\psi} \) are of order unity as \( \hat{r} \to 0 \), i.e., at the horizon. Thus \( (\bar{\psi}, \chi) \) can be regarded as hair on the black hole (a non-singular non-trivial field configuration supported by an event horizon) albeit in its euclidean section.

5. Fermion Hair

Having established the existence of vortex solutions in a magnetic Reissner-Nordström background and the presence of fermion zero modes when fermions are coupled as in the model of [7] we now wish to discuss some physical consequences of these zero modes.

The basic formalism for carrying out calculations of the effects of vortices on black holes carrying discrete charges has been discussed in some depth in [4] so we shall be rather brief here. The Euclidean partition function

\[
Z(\beta) \equiv e^{-\beta F} = \int e^{-S_E}
\]

for a black hole with both discrete charge and magnetic charge is to be evaluated by inserting a projection operator onto states of definite discrete charge and restricting the path integral to configurations of definite magnetic flux on the \( S^2 \) at radial infinity. The Euclidean path integral is then saturated by solutions with topology \( \mathbb{R}^2 \times S^2 \) which in addition satisfy the constraints imposed by the projection operator. In addition the bosonic fields in the path-integral are required to be periodic in Euclidean time with period \( \beta \) while fermion fields are required to be anti-periodic.

In our model this corresponds to saturating the functional integral (ignoring backreaction) with the euclidean magnetic Reissner-Nordström black hole and a vortex of the broken \( U(1)_R \) of vorticity \( k \) with \( k \) integer. The dominant effects come from \( k = -1, 0, +1 \) and the effects of discrete charge appear only in the sectors with \( k \neq 0 \).
The first immediate consequence of the zero modes we have found is the vanishing of the \( k \neq 0 \) contributions to the partition function and to all other correlation functions which do not involve the requisite number of fermion fields. It follows in this model that the temperature of the black hole is not modified by the presence of discrete charge, nor does the screened electric field acquire a exponentially small probability outside the horizon of the black hole - it is completely extinguished.

The only operators which will acquire non-zero expectation values in the vorticity \( k = \pm 1 \) sectors will be those with the correct number of fermion operators to soak up the zero modes. With a single \((\psi, \chi)\) multiplet and a single \((\hat{\psi}, \hat{\chi})\) multiplet the simplest anomaly free theory (as mentioned in sec. 2) has \( \hat{q} = q \) and \( \hat{r} = r + 1 \). If we choose \( r = 1/N \) then the \( U(1)_R \) symmetry is broken down to \( \mathbb{Z}_N \) and we can in the usual way assign a \( \mathbb{Z}_N \) charge to black holes. In this theory we find that the vortex sector gives rise to zero modes for \((\bar{\chi}, \psi)\) and \((\hat{\bar{\psi}}, \hat{\chi})\) leading to an expectation value for the operator \( O = \psi^T C \hat{\chi} \hat{\psi} C \bar{\chi} \) while the anti-vortex sector gives an expectation value to the hermitian conjugate operator. Note that these operators are neutral under the unbroken \( U(1)_Q \) as would be expected for a non-anomalous symmetry.

The situation described above is reminiscent of the the instanton calculation of the \( \theta \)-dependence of correlation functions in massless QCD with the discrete charge of the black hole playing the role of the the \( \theta \)-parameter of QCD. In massless QCD the proper interpretation is that the \( \theta \)-vacua are no longer physically distinguishable but rather label the globally degenerate vacua resulting from spontaneous breaking of the \( U(1)_A \) axial symmetry by the QCD vacuum. The key to this identification is the chiral anomaly

\[
\partial_\mu j^{\mu 5} = \frac{g^2}{16\pi^2} \text{Tr} F F
\]

(5.2)

and the resulting chiral Ward identities \([10]\).

We should inquire whether a similar mechanism is operative here. The answer in the models we have examined is yes. To see how this arises in the above example first note that the theory also possesses a global symmetry \( U(1)_X \) which is orthogonal to both \( U(1)_R \) and \( U(1)_Q \) with \( \psi \) and \( \hat{\chi} \) having \( X = 1 \) and \( \chi \) and \( \hat{\psi} \) having \( X = -1 \). This global symmetry is however anomalous as can be seen by noting that e.g. \( \text{Tr} XQR = -2q \). It is also clear that the operator \( O \) which acquires a non-zero expectation value transforms non-trivially under \( X \). We thus find that the anomalous \( U(1)_X \) symmetry is spontaneously broken in the black hole background and hence any putative discrete charge can be absorbed by performing a \( U(1)_X \) transformation.

We thus come to a rather surprising conclusion in this class of theories. In the background of a magnetically charged black hole there is no physical effect of discrete charge, rather the space outside the horizon of the black hole is filled with a fermion condensate which violates a global, anomalous symmetry. In contrast, if we were to consider an electrically charged black hole we would find no fermion zero modes, there would be no fermion condensate, and one could measure the effects of discrete charge as usual. It is striking that electric and magnetic black holes which are so similar classically can have such distinct quantum structure.
In semi-realistic models such as the $O(10)$ vortex model of [4] we would generically find a fermion condensate outside the black hole which violates baryon plus lepton number. This condensate would presumably lead to baryon number violating scattering processes off the black hole which are in some ways analogous to the Callan-Rubakov effect except that here they would be driven by the anomaly in the baryon number current rather than by superheavy $X$-boson exchange in the core of the monopole.

It is worth emphasizing two points regarding the fermion hair we have discovered. The first is that it is secondary hair in the language of [3]. That is, it does not enlarge the space of quantum states of the black hole but is rather determined by the quantum numbers (in this case magnetic charge) carried by the black hole. The second point is that this hair, unlike that discussed in [3], is not dependent on the existence of discrete gauge charge for its existence. In fact, as argued above, the condensate removes the final wisps of discrete gauge hair and replaces them with a rich mane of fermionic hair.

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