Comparing strings in $AdS_5 \times S^5$ to planar diagrams: an example

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Abstract

The correlator of a Wilson loop with a local operator in $\mathcal{N} = 4$ SYM theory can be represented by a string amplitude in $AdS_5 \times S^5$. This amplitude describes an overlap of the boundary state, which is associated with the loop, with the string mode, which is dual to the local operator. For chiral primary operators with a large R charge, the amplitude can be calculated by semiclassical techniques. We compare the semiclassical string amplitude to the SYM perturbation theory and find an exact agreement to the first two non-vanishing orders.

1 Introduction

The AdS/CFT correspondence establishes an equivalence of the superstring theory in the $AdS_5 \times S^5$ background and $\mathcal{N} = 4$ supersymmetric Yang-Mills theory (SYM) $[1]$. This equivalence is a true duality: the interaction strength in the string sigma-model
is the inverse of the ’t Hooft coupling in the field theory, which makes any direct comparison of strings with the field theory extremely difficult. On the other hand, planar diagrams of the large-$N$ perturbation theory resemble discretized string world sheets [2], and one may think that a more direct relationship between them exists beyond the fact that they describe one and the same theory in different regions of parameter space. In principle, one has to sum all planar diagrams (solve large-$N$ SYM) in order to reach weakly-coupled stringy regime. As became clear recently, it is still possible to make a comparison to string theory without summing all diagrams. This can be done by considering special states with large quantum numbers. In string theory, such states are semiclassical without any reference to the weakness of interactions in the sigma-model. These states, therefore, have a simple string description all the way down to weak coupling where perturbative SYM theory becomes accurate. Though this logic involves a certain stretch in the arguments, it has worked in a number of examples [3,4]. With the help of the semiclassical string picture, a remarkable progress has been made in understanding operators with large R charge (and also with large spin) in the SYM theory [3, 4]. Correlation functions of these operators have a beautiful world-sheet description, and vice versa, the world-sheet dynamics of the string states with large angular momentum on $S^5$ is encoded in certain set of Feynman diagrams which have relatively simple structure [3, 5, 6, 7, 8].

A local operator in the SYM theory is dual to a closed string state in $AdS_5 \times S^5$. The string theory in AdS also has an open string sector which is associated with Wilson loops [9, 10, 11]. By probing Wilson loops with operators that have parametrically large R charge it is possible to reach the semiclassical regime in the open-string sector too [12]. Whether the semiclassical description in the open string sector captures perturbative regime in the field theory or still requires the 't Hooft coupling to be large is not entirely clear. We will address this question by comparing string-theory calculations with lowest-order Feynman diagrams.

We consider a two-point correlator $\langle W(C)O_J \rangle$ of the Wilson loop with the operator that carries charge $J$ under a $U(1)$ subgroup of the $SO(6)$ R-symmetry group. The operator of interest is chiral primary

$$O_J = \frac{(2\pi)^J}{\sqrt{J\lambda^{J/2}}} \text{tr} Z^J,$$  \hspace{1cm} (1.1)

where $Z = \Phi_1 + i\Phi_2$ is a combination of two of the six adjoint scalars in the $SU(N)$ SYM theory. The correlator $\langle W(C)O_J \rangle$ measures the weight with which operator $O_J$ appears in the local operator expansion of the Wilson loop. We should compare this to an overlap of the closed string boundary state created by the Wilson loop with the supergravity state dual to the operator $O_J$ [13]. There is one special case in which the exact answer is known: It is possible to compute the correlator exactly for the circular Wilson loop [14], which is a chiral operator in a certain sense [15, 16, 17]. Because of the supersymmetry cancellations, only diagrams without internal vertices contribute to its expectation value [18, 17], as well as to its expansion coefficients in chiral primaries [14]. These diagrams can be explicitly resummed which yields the exact results valid at any ’t Hooft coupling. The semiclassical string calculations can also be done to all orders in the sigma-model perturbation theory because of geometric symmetries of the circle [12]. We consider arbitrary contours in this paper, for which the equations of motion in the sigma-model can be solved order by order in $\lambda/J^2$, where $\lambda$ is the
string tension squared*, which according to the AdS/CFT dictionary coincides with the 't Hooft coupling in the SYM theory: \( \lambda = g_{\text{SYM}}^2 N \). We will compare the expansion of the string amplitude in \( \lambda/J^2 \) to the ordinary planar perturbation theory.

The paper is organized as follows. In sec. 2 we set up the notations and review how the correlator of a Wilson loop with the local operator is computed in string theory. In sec. 3 we do a one-loop calculation on the SYM side. The classical solution of the sigma-model, which describes the correlator in string theory, is constructed in sec. 4. We then compare the string amplitude determined by this solution with one-loop SYM perturbation theory. We also show in sec. 5 that diagrams of the SYM perturbation theory exponentiate in accord with predictions of the string theory. We draw the conclusions and discuss the results in sec. 6.

2 Wilson loop correlator in string theory

The supersymmetric Wilson loop operator is defined as \[ W(C) = \text{tr} \text{P} \exp \left[ \int ds \left( iA_\mu x'^\mu + \Phi_1 |x'| \right) \right]. \tag{2.1} \]

This operator is a hybrid of an ordinary non-Abelian phase factor and the unique scalar loop operator which is conformally and Lorentz covariant \[ \dagger \].

The Wilson loop operator is dual to a macroscopic string in \( AdS_5 \times S^5 \). The local operator is dual to a supergravity mode. If we want to compute their correlator, we should find vertex operator associated with the supergravity mode, insert it into the world sheet of the string, combine it with the propagator of the supergravity mode, and then integrate over all string world-sheets and all positions of the vertex operator \[ \dagger \]. In the limit when the distance between the loop and the insertion of the local operator goes to infinity the propagator factorizes, and we are left with the one-point correlation function in the sigma-model:

\[
\langle W(C) O_J(x) \rangle = \frac{1}{|x|^2} \int d^2 \sigma_0 \int D\mathbf{X} e^{-S_{\text{sm}}[X]} V_J(\mathbf{X}(\sigma_0)) + o(1/|x|^{2J}), \tag{2.2} \]

where \( \mathbf{X} \) denotes coordinates, bosonic and fermionic, of the string in the AdS superspace; \( S_{\text{sm}}[X] \) is the Green-Schwarz action of the AdS sigma-model; and \( V_J(\mathbf{X}(\sigma)) \) is the vertex operator which creates string mode dual to the operator \( O_J \). We assume that the conformal gauge is already fixed and regard corresponding Faddeev-Popov factor as a part of the measure in the path integral.

The metric of \( AdS_5 \) in the Poincaré coordinates is

\[
ds^2 = dp^2 + e^{2p} dx^2. \tag{2.3} \]

The boundary is at \( p = \infty \) and the horizon of \( AdS_5 \) is at \( p = -\infty \). We adopt the following parameterization of \( S^5 \): \( \theta^i = (\cos \psi \cos \varphi, \cos \psi \sin \varphi, \sin \psi n) \), where \( n \) is a

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*We express the string tension and all other dimensionful quantities in the units of the AdS radius.

†The most general Wilson loop depends on scalars through an arbitrary linear combination \( \Phi, \theta^i \), where \( \theta^i \) is a unit six-vector, which may depend on \( s \) and thus parameterizes a closed contour in \( S^5 \). We choose a particular \( \theta^i \) for definiteness; results will not be much different for any other constant \( \theta^i \). It would be interesting to consider varying \( \theta^i \) \[ 9, 20, 21 \], but this goes beyond the scope of the present paper.
unit four-vector. The sigma-model action in this parameterization is

\[ S_{\sigma m} = \frac{\sqrt{\lambda}}{4\pi} \int d^2 \sigma \left[ (\partial P)^2 + e^{2P} (\partial X^\mu)^2 + (\partial \Psi)^2 + \cos^2 \Psi (\partial \Phi)^2 + \ldots \right]. \]  

(2.4)

The action for the rest of the angles in \( S^5 \) and for the world-sheet fermions will not be important for us. The operator \( O_J \) is chiral and hence corresponds to the supergravity (ten-dimensional massless) mode. The vertex operator of such string state is a solution of the massless wave equation in \( AdS_5 \times S^5 \). Symmetries and the form of the dual SYM operator \( O_J \) essentially determine the necessary solution: it should have zero momentum along the boundary, since we consider the large-distance asymptotic of the correlator; it should scale as \( V \to e^{-J \omega} V \) under \( P \to P + \omega \), since the operator \( O_J \) has dimension \( J \); and it should transform as \( V \to e^{i\varphi} V \) under an \( SO(6) \) rotation \( \Phi \to \Phi + \varphi \). These simple arguments determine the vertex operator with an exponential accuracy, which will be sufficient for our purposes:

\[ V \propto e^{iJ\Phi - JP}. \]  

(2.5)

Fixing normalization is possible \[13\], but requires much more work.

We will calculate the correlator \[2.2\] in the double-scaling limit of large string tension and large R charge with their ratio \( j = J/\sqrt{\lambda} \) fixed. The string path integral is semiclassical in this limit, so we need just to solve classical equations of motion in the sigma-model, which fortunately leaves behind many hard questions like correct normalization of the measure in the path integral or exact form of the vertex operator. Since the exponent in \[2.5\] is of the same order as the action, we should treat the action and the vertex operator on the same footing \[22\]. This effectively adds a source to the action:

\[ S_{\text{eff}} = S_{\sigma m} + J (P(\sigma_0) - i\Phi(\sigma_0)) \equiv JS. \]

The action with the source term added is the functional that we should minimize in order to compute the string amplitude in the semiclassical approximation. The action evaluated on the classical solution determines the correlator \( \langle W(C)O_J(x) \rangle \) with an exponential accuracy at large \( \lambda \) and large \( J \). Since the rescaled action \( S \) and, consequently, the equations of motion depend on \( \lambda \) and \( J \) only through the ratio \( j = J/\sqrt{\lambda} \), the correlator will have the form:

\[ \langle W(C)O_J(x) \rangle \simeq \frac{1}{|x|^{2j}} e^{-JS(j;C)}. \]  

(2.6)

Solving classical equations of motion of the AdS sigma-model for generic boundary conditions is a hard problem, but it is possible to find an approximate solution when the parameter \( j \) is large. Then the minimal surface is a cylinder to a first approximation \[12\]: the shape of the contour is the same in any slice of AdS parallel to the boundary. It was shown in \[12\] that this solution predicts the following dependence of the correlator on \( J \) and \( \lambda \):

\[ \langle W(C)O_J(x) \rangle \sim \frac{\lambda^{J/2}}{|j|^2}. \]  

(2.7)

Surprisingly, this is the same as one would get by counting combinatorial factors in the lowest-order diagram (fig. 1) of the SYM perturbation theory. We should stress that
the perturbative and the string calculations are valid in different regions of parameters: the semiclassical approximation in string theory requires both $J$ and $\lambda$ to be large, but the ratio $\lambda/J^2$ can be small. The perturbation theory, strictly speaking, is valid only when the coupling is small. Of course, it may turn out that $\lambda/J^2$ is the true parameter of perturbative expansion, then $\lambda$ should not necessarily be small for the perturbation theory to work.

We will develop a systematic method to solve the sigma-model equations of motion order by order in $1/j^2$. This will yield an expansion of the Wilson loop correlator in $\lambda/J^2$, which resembles an ordinary perturbative series. To compare the two, we will first compute the one-loop correction on the SYM side.

## 3 One-loop calculation in the SYM theory

The one-loop planar correction to the correlator of a Wilson loop with the chiral primary operator is described by the single diagram in fig. 2. The diagrams with corrections to the external lines exactly cancel, as shown in appendix B. Combining the tree-level diagram with the one-loop correction gives:

$$\langle W(C)O_J(x) \rangle = \frac{\lambda^{J/2}}{\sqrt{J}(4\pi|X|^2)^J} \int ds_1 \ldots ds_J \theta_c(s_1, \ldots, s_J) \times \left(1 + \lambda \sum_{n=1}^{J} \int_{s_n}^{s_{n+1}} ds \int_{s}^{s_{n+1}} dr G(s, r) + O(\lambda^2)\right) = \frac{\sqrt{J}}{J!} \left(\frac{\sqrt{\lambda}}{4\pi|X|^2}\right)^J \times \left[(2\pi l)^J + \lambda \int_0^{2\pi l} ds \int_0^{2\pi l} dq (2\pi l - q)^J G(s, s + q) + O(\lambda^2)\right]$$

where $\theta_c$ equals one, if its arguments are cyclically ordered along the contour, and equals zero otherwise, $2\pi l = \int ds |x'|$ is the length of the contour $C$, and $G(s, r)$ is the sum of the gauge-boson and the scalar propagators inserted between points $x(s)$ and

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†Our notations and conventions for the SYM perturbation theory are collected in appendix A.
on the contour:

\[ G(s, r) = \frac{1 - x'(s) \cdot x'(r)}{8\pi^2 |x(s) - x(r)|^2}. \]  

(3.9)

From now on, we use the natural parameterization of the contour \( C: x'^2 = 1 \). In going from the second to the third line in (3.8), we used the formula

\[ \int_0^s ds_1 \ldots \int_0^{s_{J-1}} ds_J = \frac{s^J}{J!} \]  

(3.10)

to integrate over the end-points of external legs in the diagrams.

Figure 2: One-loop correction to the correlator \( \langle W(C) O_J(x) \rangle \). The wavy line denotes the sum of the gauge-boson and the scalar propagators.

The answer considerably simplifies in the limit of large \( J \), because then

\[ \int_0^{2\pi l} dq \ (2\pi l - q)^J F(q) = \frac{(2\pi l)^J}{J} F(0) + O(1/J^2) \]  

(3.11)

for any smooth function \( F(q) \), and the propagator (3.9) can be expanded as

\[ G(s, s + q) = \frac{x''(s) \cdot x'(r)}{16\pi^2} + O(q) = \frac{x''(s)}{16\pi^2} + O(1/J^3) \]  

(3.12)

We get:

\[ \langle W(C) O_J(x) \rangle = \sqrt{\frac{\lambda}{J!}} \left( \frac{\sqrt{\lambda} l}{2|x|^2} \right)^J \left[ 1 + \frac{\lambda}{8\pi J} \int_0^{2\pi l} ds \ x''^2 + O(\frac{\lambda}{J^2}) + O(\lambda^2) \right]. \]  

(3.13)

The fact that the large-\( J \) limit is only sensitive to the local limit of the propagator has a simple explanation: since the operator \( O_J \) contains \( J \) scalar fields, there are \( J \) external vertices on the Wilson loop, and the average distance between them is of order \( l/J \). A distance between the end-points of the propagator \( G(s, r) \) is, on average, of the same order: \(|r - s| \sim l/J \). The expansion of the propagator \( G(s, r) \) in \( (r - s) \) therefore generates \( 1/J \) expansion of the Feynman integral.


4 String side

In this section, we will develop a systematic method to solve equations of motion in the sigma-model order by order in $1/j^2$ and to compute the classical string action as a series in this parameter.

We can choose the coordinates $\tau$ and $s$ on the world sheet, such that $s$ coincides with the natural parameter on $C$ when restricted to the boundary at $\tau = 0$. We can also assume that the world sheet has a topology of the cylinder, and that the vertex operator is placed at $\tau = \infty$. The action then takes the following form:

$$S = \frac{1}{4\pi j} \int_{\tau=0}^{\infty} \int_{0}^{2\pi l} dt \, ds \left[ \dot{P}^2 + P'^2 + e^{2P} \left( \dot{X}^2 + X'^2 \right) + \dot{\phi}^2 + \phi'^2 \right] - \frac{l}{j \varepsilon} + (P - i\Phi)|_{\tau=\infty}. \tag{4.14}$$

The cutoff on the integral over $\tau$ is necessary to regularize the divergence of the area at the boundary of AdS space. The area diverges as $2\pi l / \varepsilon$. With our normalization of the action, we should subtract $l/j \varepsilon$ to make the action finite. As was shown in [9, 23], the simple subtraction is the correct way to deal with the divergences in the AdS sigma-model, at least within the semiclassical approximation.

The equations for the minimal surface should be supplemented by boundary conditions. The boundary conditions at $\tau = 0$ are set by the Wilson loop:

$$X^\mu(s,0) = x^\mu(s), \quad P(s,0) = +\infty, \quad \phi(s,0) = 0. \tag{4.15}$$

The source term in the action determines the behavior of the string coordinates at infinity. Let $z$ be a local coordinate near $\sigma_0$, so that the operator is inserted at $z = 0$. Then $X^\mu$ are regular at $z = 0$, but $P$ and $\phi$ have logarithmic singularities determined by the source. With the action normalized as in (2.4), we have:

$$P(z) = -j \ln |z| + \text{regular}, \quad \phi(z) = ij \ln |z| + \text{regular}. \tag{4.16}$$

The exponential parameterization $z = e^{(\tau + is)/l}$ maps the world sheet to a cylinder. The point of operator insertion $z = 0$ is mapped to infinity with the following boundary conditions on the string coordinates:

$$X^\mu(s,\tau) \to x^\mu_\infty, \quad P(s,\tau) \to -\frac{jt}{l}, \quad \phi(s,\tau) \to \frac{ij\tau}{l} \quad (\tau \to \infty). \tag{4.17}$$

We will solve the equations of motion perturbatively in $1/j^2$. In order to do that, it is convenient to introduce new variables in which the solution is non-singular at zeroth order (at infinite $j$). We rescale:

$$t = j\tau, \quad Q = P - \ln j. \tag{4.18}, \tag{4.19}$$

In this way, we get:

$$S = \frac{1}{4\pi} \int_{j\varepsilon}^{\infty} dt \int_{0}^{2\pi l} ds \left[ Q^2 + \frac{1}{j^2} Q'^2 + e^{2Q} \left( j^2 \dot{X}^2 + X'^2 \right) + \phi^2 + \frac{1}{j^2} \phi'^2 \right] - \frac{l}{j \varepsilon} + (Q - i\phi)|_{t=\infty} + \ln j. \tag{4.20}$$

The other $S^5$ angle $\Psi$ in [24] is zero on the classical solution.
The dynamics of the angular coordinate \( \Phi \) is trivial:

\[
\Phi = \frac{it}{l}. \tag{4.21}
\]

Upon substitution of the solution for \( \Phi \), the action combined with the vertex operator (which just imposes the boundary condition for \( Q \) at \( t \to \infty \)) can be written in the following form:

\[
S = \frac{1}{4\pi} \int_{j\varepsilon}^{\infty} dt \int_0^{2\pi l} ds \left[ \left( \dot{Q} + \frac{1}{l} \right)^2 + \frac{1}{l^2} Q'^2 + e^{2Q} \left( j^2 \tilde{X}^2 + X'^2 \right) \right] + Q(j\varepsilon) - \frac{l}{j\varepsilon} + \ln j. \tag{4.22}
\]

Here, we used the equality

\[
(Q(t) - i\Phi(t))|_{t=\infty} = \frac{1}{4\pi} \int_{j\varepsilon}^{\infty} dt \int_0^{2\pi l} ds \frac{2}{l} \left( \dot{Q} + \frac{1}{l} \right) + Q(j\varepsilon). \]

Note that potentially dangerous boundary terms in (4.20) have been completely absorbed into the bulk part of the action. This would have been impossible without cancellations between \( AdS_5 \) and \( S^5 \) contributions which occur because the vertex operator is marginal. Potential divergences at \( t \to \infty \) are of UV nature, since they come from the vicinity of the operator insertion, though in the coordinates we use they might look as an IR effect. Anyway, they cancel and the action is saturated by \( t \sim 1 \). The typical AdS scale \( e^{-P} \) is also small: \( Q \) is finite at large \( j \) and \( P \sim \ln j \) according to (4.19). Hence, the largest contribution to the action comes from the region of AdS space close to the boundary.

The equations of motions are

\[
-\ddot{Q} - \frac{1}{j^2} Q'' + e^{2Q} \left( j^2 \tilde{X}^2 + X'^2 \right) = 0, \tag{4.23}
\]

\[
j^2 \left( e^{2Q} \tilde{X} \right)' + \left( e^{2Q} X' \right)' = 0. \tag{4.24}
\]

They are readily solved at large \( j \). First, we notice that \( X^\mu \) must be \( t \)-independent to the leading order. Hence, the minimal surface is a cylinder with the contour \( C \) as the base:

\[
X^\mu_0(s, t) = x^\mu(s). \tag{4.25}
\]

The equation (4.23) reduces to

\[
-\ddot{Q}_0 + e^{2Q_0} = 0, \tag{4.26}
\]

which is solved by

\[
Q_0 = -\ln \left( \frac{l}{\sinh \frac{t}{l}} \right). \tag{4.27}
\]

To develop a systematic procedure to solve the equations of motion order by order in \( 1/j^2 \), we write:

\[
Q = \sum_{n=0}^{\infty} \frac{1}{j^{2n}} Q_n, \quad X^\mu = \sum_{n=0}^{\infty} \frac{1}{j^{2n}} X^\mu_n. \tag{4.28}
\]
The structure of equations (4.23), (4.24) is such that we can easily solve them recursively. On each step we will need to solve an ordinary linear differential equation for \( Q_n, X_n^\mu \) with an unknown dependence on time only. This is somewhat surprising, since we are dealing with partial differential equations. The reason for the simplification is an enhancement of time derivatives by a factor of \( j^2 \) compared to the derivatives in \( s \). The iterative solution can be constructed as follows: suppose that we know \( Q_m, X_m^\mu \) for \( m < n \). They must solve the first equation (4.23) with an accuracy \( O(1/j^2) \), and the second equation (4.24) with an accuracy \( O(1/j^{2n-2}) \). The next order in the second equation (4.24) allows us to express a linear combination of time derivatives of \( X_n^\mu \) through the known functions. \( Q_n \) drops out at this order because \( \dddot{X}_n^\mu = 0 \). The first equation (4.23) at order \( 1/j^2 \) then reduces to a linear equation for \( Q_n \) and its second time derivative whose coefficients depend on \( X_n^\mu \), which we already know, and on \( Q_m, X_m^\mu \) with \( m < n \).

The iterative procedure is best exemplified by the first three steps:

\[
\begin{align*}
(e^{2Q_0} \dot{X}_1^\mu) \, &= \, - (e^{2Q_0} X_0^\mu')' \implies X_1^\mu, \\
\ddot{Q}_1 - 2 e^{2Q_0} Q_1 \, &= \, e^{2Q_0} \left( \dddot{X}_1^2 + 2 \dot{X}_1' \cdot X_1' \right) \implies Q_1, \\
\left[ e^{2Q_0} \left( 2Q_1 \dot{X}_1^\mu + \ddot{X}_2^\mu \right) \right] \, &= \, - e^{2Q_0} (2Q_1 X_0^\mu' + X_1^\mu) \implies X_2^\mu, \quad (4.29)
\end{align*}
\]

It is straightforward to integrate these equations, though calculations become increasingly difficult with the order of iteration. After rather lengthy algebra we obtain:

\[
\begin{align*}
X_1^\mu &= \frac{j^2}{4} \left( e^{-2\tilde{t}} + 2\tilde{t} - 1 \right) x^{\mu\nu}, \\
Q_1 &= - \frac{j^2}{4} \frac{4 \tilde{t} e^{-2\tilde{t}} + e^{-4\tilde{t}} - 1}{1 - e^{-2\tilde{t}}} x^{\nu2}, \\
X_2^\mu &= \frac{j^4}{16} \left[ \left( 2\tilde{t}^2 + 2\tilde{t} e^{-2\tilde{t}} + e^{-2\tilde{t}} - 1 \right) x^{\mu\nu\nu} \\
&\quad + 4 \left( \tilde{t} e^{-2\tilde{t}} + \tilde{t} + e^{-2\tilde{t}} - 1 \right) (x^{\nu2} x^{\mu\nu})' \\
&\quad - \left( e^{-4\tilde{t}} + 8\tilde{t} e^{-2\tilde{t}} + 4 e^{-2\tilde{t}} + 4\tilde{t} - 5 \right) x^{\nu2} x^{\mu\nu} \right], \quad (4.30)
\end{align*}
\]

where \( \tilde{t} = t/l \).

The classical action can be written as a power series in \( 1/j^2 \):

\[
S = \ln j + \sum_{n=0}^{\infty} \frac{S_n}{j^{2n}} \quad (4.31)
\]
with \footnote{We can forget about regularization everywhere except in $S_0$.}

\[
S_0 = \frac{1}{4\pi} \int_0^{2\pi l} ds \int_0^\infty dt \left[ \left( \dot{Q}_0 + \frac{1}{l} \right)^2 + e^{2Q_0} X'_0 \right] + Q_0(j\varepsilon) - \frac{l}{j\varepsilon},
\]
\[
S_1 = \frac{1}{4\pi} \int_0^{2\pi l} ds \int_0^\infty dt e^{2Q_0} \left( \dot{X}'_1 + 2X'_0 \cdot X'_1 \right),
\]
\[
S_2 = \frac{1}{4\pi} \int_0^{2\pi l} ds \int_0^\infty dt e^{2Q_0} \left(-Q_1 \dot{X}'_1 + \dot{X}_1 \cdot \dot{X}'_2 + 2X'_0 \cdot X'_2 \right).
\]

(4.32)

Calculating the integrals over $t$, we get:

\[
S_0 = \ln \frac{2}{l} - 1,
\]
\[
S_1 = -\frac{l}{8\pi} \int_0^{2\pi l} ds x'^2,
\]
\[
S_2 = \frac{l^3}{64\pi} \int_0^{2\pi l} ds \left[ 2x'^2 - (x'^2)^2 \right].
\]

(4.33)

The appearance of the dimensionful quantity under the logarithm in $S_0$ may seem strange, but in fact is required to reproduce the correct scaling dimension of the correlator, for which we get

\[
\langle W(C)O_J(x) \rangle \simeq \frac{1}{J!} \left( \frac{\sqrt{\lambda} l}{2|x|^2} \right)^J \exp \int_0^{2\pi l} ds \left\{ \frac{\lambda l}{8\pi J} x'^2 - \frac{\lambda^2 l^3}{64\pi J^3} \left[ 2x'^2 - (x'^2)^2 \right] \right\} + O \left( \frac{\lambda^3}{J^5} \right).
\]

(4.34)

Here we used the Stierling formula to approximate $e^{\frac{l}{J}} / J^J \simeq 1 / J!$. Expanding the correlator in $\lambda$, we find the complete agreement with the one-loop perturbation theory \footnote{We can forget about regularization everywhere except in $S_0$.}! We also get a prediction for the two-loop contribution. The two-loop calculation on the SYM side is beyond the scope of the present paper. Instead, we will check that a certain set of diagrams of all orders in $\lambda$ exponentiate, as required by the general structure of the correlator in string theory.

\section{Exponentiation}

If we believe that the string description is valid at weak coupling for large $J$, perturbative series for $\langle W(C)O_J \rangle$ must exponentiate. This is a rather non-trivial statement, since the usual expansion in Feynman diagrams computes the correlator itself, not its logarithm. Perturbative series exponentiate for Abelian Wilson loops, but in the non-Abelian theory, especially at large $N$, exponentiation is not at all obvious. The string-theory prediction for the sum of all planar diagrams is

\[
\langle W(C)O_J(x) \rangle \simeq \frac{1}{|x|^2} e^{-JS(j;C)},
\]

(5.35)
where \( S(j; C) \) has a regular expansion in \( 1/j^2 = \lambda/J^2 \), eq. (4.31), which can be regarded as the weak-coupling expansion. If we reexpand the correlator in \( \lambda \):

\[
\langle W(C)O_J(x) \rangle \simeq \frac{1}{J!} \left( \frac{\sqrt{\lambda} l}{2|x|^2} \right)^J \sum_{n=0}^{\infty} a_n \lambda^n,
\]

we find that the contribution which is least suppressed in \( 1/J \) at any given order of perturbation theory is determined by just one number, \( S_1 \):

\[
a_n = \frac{S_1^n}{J^n n!} + O \left( \frac{1}{J^{n+1}} \right).
\]

This is because the exponent in (5.35) contains an overall factor of \( J \), and if we take the limit of large \( J \) at a fixed order of perturbation theory, only the contribution of the leading term in the expansion of \( S(j) \) survives. Our goal will be to prove this universality by a direct expectation of planar Feynman diagrams in the SYM theory. To do that, we will consider diagrams with largest combinatorial factors at a given order of perturbation theory.

\[\text{Figure 3: The diagrams with largest combinatorial factors at large } J.\]

The leading-order diagram contains \( J \) scalar propagators that connect the local operator to \( J \) points on the Wilson loop. In view of the standard equivalence of planar Feynman graphs and discretized two-dimensional surfaces \[2\], this diagram can be thought of as a prism with \( J \) facets. The base of the prism is the Wilson loop and its apex is the point where operator \( O_J \) is inserted. This point is moved to infinity in our approximation. Higher-order diagrams correspond to decorating the facets with extra propagators and vertices. Until the order of perturbation theory becomes comparable to \( J \), most of the facets will remain empty. It is clear that adding a new element to one of the empty facets produces a combinatorial factor of order \( J \), while adding a propagator or a vertex to an already decorated facet produces a combinatorial factor of order one. Hence, diagrams with the biggest combinatorial factor have \( n \) propagators distributed among \( n \) different facets. Diagrams of this type with internal interaction vertices cancel by a trivial extension of the argument in appendix B. These cancellations leave only the diagrams with \( n \) loop-to-loop propagators sandwiched between \( J \) external legs (fig. 3).
Thus, to the leading order in $1/J$:

\[
 a_n = \frac{(J-1)!}{(2\pi l)^J} \int dq_1 \ldots dq_J \theta_c(q_1, \ldots, q_J)
\times \sum_{1 \leq p_1 \leq \ldots \leq p_n \leq J} \int_{q_{p_1}}^{q_{p_1+1}} ds_1 \int_{s_1}^{s_1+1} dr_1 \ldots \int_{q_{p_n}}^{q_{p_n+1}} ds_n \int_{s_n}^{s_n+1} dr_n
\times G(s_1, r_1) \ldots G(s_n, r_n)
\]  

(5.38)

We can change the order of integration and integrate over $q_i$ using eq. (3.10). Then:

\[
a_n = \frac{(J-1)!}{(2\pi l)^J} \int ds_1 dr_1 \ldots ds_n dr_n \theta_c(s_1, r_1, \ldots, s_n, r_n) G(s_1, r_1) \ldots G(s_n, r_n)
\times \frac{1}{n!} \sum_{k_1 + \ldots + k_n = J} \frac{(s_2 - r_1)^{k_1} \ldots (s_1 - r_n)^{k_n}}{k_1! \ldots k_n!}
\]

\[
= \frac{1}{(2\pi l)^J} \int ds_1 dr_1 \ldots ds_n dr_n \theta_c(s_1, r_1, \ldots, s_n, r_n) G(s_1, r_1) \ldots G(s_n, r_n)
\times [2\pi l - (r_1 - s_1) - \ldots - (r_n - s_n)]^J.
\]  

(5.39)

Applying (3.11) repeatedly $n$ times and keeping only the leading order in $1/J$, we get

\[
a_n \approx \frac{(2\pi l)^n}{n(J+1) \ldots (J+n)} \int ds_1 \ldots ds_n \theta_c(s_1, \ldots, s_n) \frac{x''^2(s_1)}{16\pi^2} \ldots \frac{x''^2(s_n)}{16\pi^2}
\approx \frac{1}{n!} \left( \frac{l}{8\pi J} \int ds x''^2 \right)^n,
\]  

(5.40)

in agreement with the prediction of string theory: the diagrams with largest combinatorial weights indeed exponentiate.

6 Discussion

We made a rather detailed comparison between diagrams of perturbation theory and the semiclassical string amplitude which both compute a two-point correlator of an arbitrary Wilson loop with a local operator in $\mathcal{N} = 4$ SYM theory. The results completely agree to the one-loop accuracy, as well as to all orders in perturbation theory for diagrams with the largest combinatorial weights. We should stress once again that there are no apparent reasons for such an agreement. The string theory and the perturbation theory compute different regimes, which can be most easily seen from the general form of the semiclassical string amplitude (2.6). On the string side, we have to assume that the exponent in (2.6) is large, otherwise the semiclassical approximation breaks down. To compare with SYM perturbation theory, we expand the exponential and thus assume that the exponent is small. Perhaps, one can invoke arguments similar to those of [24] to explain the agreement between the two calculations. The arguments rely on the observation that the 't Hooft coupling appears only in the combination $\lambda/J^2$ on both sides of the correspondence, which for example means that $\lambda/J^2$, and not $\lambda$ itself, is a parameter of the planar perturbation theory. Still, a direct comparison to the string theory requires a non-trivial resummation of perturbative series.
Incidentally, we found that the string calculations are technically simpler than perturbative SYM calculations. It is relatively easy to compute the string amplitude to the two-loop order and it is definitely possible to push string calculation to higher orders in $\lambda/J^2$, while on the SYM side already a two-loop calculation constitutes an enormously hard problem. The simplicity of the string calculation may indicate that various diagrams cancel leaving a simple net result. We indeed observed cancellations at one loop, but those look rather accidental, at least in the way we found them.

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Appendix A $N = 4$ SYM theory

In this appendix we summarize our notations and conventions for the SYM perturbation theory. Feynman rules follow from the SYM action

$$ S = \frac{N}{\lambda} \int d^4x \text{tr} \left\{ \frac{1}{2} F_{\mu\nu}^2 + (D_\mu \Phi_i)^2 - \frac{1}{2} [\Phi_i, \Phi_j]^2 + \text{fermions} \right\}. \quad (A.1) $$

We do all calculations in the Feynman gauge in the coordinate representation, where the propagators are

$$ \langle \Phi_i^{AB}(x_1) \Phi_j^{CD}(x_2) \rangle = \frac{g^2}{8\pi^2} \frac{\delta^{AC} \delta^{BD} \delta_{ij}}{|x_1 - x_2|^2}, \quad (A.2) $$

$$ \langle A_\mu^{AB}(x_1) A_\nu^{CD}(x_2) \rangle = \frac{g^2}{8\pi^2} \frac{\delta^{AC} \delta^{BD} \delta_{\mu\nu}}{|x_1 - x_2|^2}. \quad (A.3) $$

The capital letters denote $U(N)$ indices.

Appendix B Cancellation of diagrams with internal vertices

There are four types of diagrams (fig. 4) that were not taken into account in sections 3 and 5. We show here that they cancel in the large-distance asymptotic of the correlator $\langle W(C) O_J(x) \rangle$. We can amputate those legs of the propagators in diagrams (a), (b), and (c) which couple to the local operator and replace them by $1/(8\pi^2 |x|^2)$. Indeed, the region of integration with intermediate points close to $x$ corresponds to renormalization of the operator $O_J$. Since the operator is chiral and is not renormalized, these contributions mutually cancel. The intermediate region of integration corresponds to
descendants and contributes at higher orders in $1/|x|$. Thus we amputate the external legs and also take into account one half of the diagram (d). Another half participates in cancelling the renormalization of the operator $O_J$.

We start with the diagram (d). The wave function renormalization does not vanish in the Feynman gauge, and was computed in [18]. We can take the wave-function renormalization into account by multiplying each propagator in the tree-level amplitude by a factor $1 + F_d$, where

$$F_d = -\lambda \int d^4x D^2(x),$$

and $D(x)$ is appropriately regularized scalar propagator, which goes to $1/(4\pi^2|x|^2)$ when regularization is removed.

![Figure 4: One-loop diagrams with internal vertices.](image)

The diagram (a) with external legs amputated gives a factor of

$$F_a(x_1, x_2) = \frac{\lambda}{2} \int d^4y d^4z D(y - z) \frac{\partial}{\partial y^\mu} D(y - x_1) \frac{\partial}{\partial z^\mu} D(z - x_2).$$

for each pair of external legs. Here, $x_1$ and $x_2$ are adjacent vertices on the contour $C$. The diagram (b) gives

$$F_b(x_1, x_2) = \frac{\lambda}{2} \int d^4y D(y - x_1) D(y - x_2).$$

Finally, the diagram (c) contributes

$$F_c(x_1, x_2) = \frac{\lambda}{2} \int_{s_1}^{s_2} ds x'^\mu(s) \int d^4y D(y - x) \frac{\partial}{\partial y^\mu} \left( D(y - x_2) - D(y - x_1) \right),$$

where $x(s)$ is a point on the contour between $x_1 \equiv x(s_1)$ and $x_2 \equiv x(s_2)$. Integration by parts yields:

$$F_c = -F_d - 2F_b.$$  

This equation, together with $F_a = F_b$, implies that

$$F_a + F_b + F_c + F_d = 0.$$
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