Magnetic microswimmers exhibit Bose-Einstein-like condensation

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We study an active matter system comprised of magnetic microswimmers confined in a microfluidic channel and show that it exhibits a new type of self-organized behavior. Combining analytical techniques and Brownian dynamics simulations, we demonstrate how the interplay of non-equilibrium activity, external driving, and magnetic interactions leads to the condensation of swimmers at the center of the channel via a non-equilibrium phase transition that is formally akin to Bose-Einstein condensation. We find that the effective dynamics of the microswimmers can be mapped onto a diffusivity-edge problem, and use the mapping to build a generalized thermodynamic framework, which is verified by a parameter-free comparison with our simulations. Our work reveals how driven active matter has the potential to generate exotic classical non-equilibrium phases of matter with traits that are analogous to those observed in quantum systems.

The interplay between non-equilibrium collective dynamics of active matter systems [1, 2] and external control provides a wide range of possibilities for new classes of self-organization [3–14]. Due to the versatility it offers [15], magnetic actuation [16–18] and magnetic steering [19–20] has received increasing attention in the recent years, with experimental realizations consisting of both biological [21, 22] and synthetic [23–27] microswimmers.

The coupling between non-equilibrium activity and long-range magnetic dipole-dipole interaction can lead to new emergent properties for magnetic microswimmers [8, 10, 28–32]. Similarly rich phenomenology is known to emerge from long-range interactions in phoretic active matter [33], and in particular, due to the interplay between translational and orientational degrees of freedom [34–37]. In determining the potential for such emergent effects, a key difference between biological and artificial magnetic swimmers is in the strength of their respective interactions. While magnetotactic bacteria carry a typical magnetization of the order of ~10 A m−1 [21, 22], the magnetization can reach values of up to ~104 A m−1 for swimmers with magnetite [38]. The effect of such strong magnetic dipole-dipole interaction on the collective response of magnetic swimmers is still largely unexplored, despite the growing interest in their potential applications for cargo and drug delivery in microscopic environments [39]. Here, we illustrate how strong dipole-dipole interactions affect the collective behavior of magnetic microswimmers confined in a microfluidic channel.

Using Brownian dynamics simulations and a coarse-grained analytical framework, we show that the radial dynamics of microswimmers across the channel is equivalent to that of particles diffusing in an effective potential and presenting a diffusivity-edge [13]. Consequently, the system is found to exhibit a transition leading to the formation of a condensate at the channel center, which coexists with a surrounding gas. By means of a generalized thermodynamic framework, we characterize the singular behavior of the system and find it to be analogous to the characteristics of Bose-Einstein condensation (BEC) transition. These concrete predictions are moreover quantitatively verified by simulations, without the need of tuning parameters. Finally, our extensive simulations across the entire parameter space allow us to construct a phase diagram, with phase boundaries that show agreement with the simple criteria obtained from our analytical framework.

We start by introducing the microscopic model of magnetic microswimmers which we simulated and used as a starting point for the derivation of a field theory. We consider swimmers that carry a magnetic dipole moment m0n, along which they self-propel with a constant speed v0, under the influence of a uniform magnetic field Bext = −Bextez; see Fig. 1(a). They swim in a 3-dimensional cylindrical channel oriented along ez, and experience a Poiseuille flow described as \( V_f = v_f (1 - r^2/R_0^2) e_z \), with r denoting the radial distance from the center and R0 being the channel radius. The Langevin equations governing the dynamics of their position r and orientation n thus read

\[ \dot{r} = v_0 n + V_f + \frac{m_0}{\zeta} \nabla (n \cdot B_{int}) + \xi, \quad (1) \]

\[ \dot{n} = \left[ \frac{m_0}{\zeta} n \times (B_{ext} + B_{int}) + \frac{1}{2} \nabla \times V_f + \xi_r \right] \times n, \quad (2) \]

where \( \zeta \) and \( \zeta_r \) denote the translational and rotational friction coefficients that are taken to be scalar for simplicity, \( \xi \) and \( \xi_r \) are thermal noises of respective variances \( 2D \) and \( 2D_r \), with \( D = k_B T/\zeta \) and \( D_r = k_B T/\zeta_r \), and \( T \) being the medium temperature. \( B_{int} \) in Eqs. (1) and (2) is the effective magnetic field induced by other swimmers, and is obtained from Ampère’s law. In what follows, \( N \) denotes the number of swimmers in the channel, which is set to 1000 (Simulation details can be found in [10]).
Their mean density $\rho_0 = N / (\pi R_0^2 L)$ is adjusted by varying the channel length $L$. Moreover, we fix $v_0$, $R_0$, $m_0$, $\zeta$, $\zeta_r$, and $T$ to realistic values [8,10] (see Table I in [10]), such that only $B_{\text{ext}}$, $v_J$ and $\rho_0$ are varied. As we shall see later, the dimensionless number

$$J \equiv \frac{m_0 B_{\text{ext}} DD_r}{k_B T v_0 v_J}$$

which combines the relative strength of the magnetic energy versus thermal energy and the strength of propulsion and shear velocities versus diffusion, plays a key role in determining the behavior of the system.

When magnetic interactions are negligible compared to the effect of external driving, the radial dynamics of particles relaxes over a finite timescale $\tau \equiv D_r R_0^2 m_0 B_{\text{ext}} / (v_0 v_J k_B T) = J R_0^2 / D$. The dynamics along the channel direction $e_z$ is then determined by the value of the dimensionless parameter

$$B \equiv \frac{\mu_0 \rho_0 m_0^2 B_{\text{ext}} v_J}{4 k_B T} \frac{v_J}{v_0}.$$  

(4)

When $B < 1$, the distribution of swimmers along $e_z$ is uniform on average, whereas for $B \geq 1$ the system undergoes an instability leading to a dynamical steady state made of a periodic arrangement of traveling clusters, characterized by strong inhomogeneities in the particle distribution along the channel axis (see Fig. 1(d)).

In this study, we focus on the stationary radial distribution of swimmers, $\phi_r(r) \equiv \langle \rho(r, t) / \rho_0 \rangle _{\text{st}}$, where $\rho(r, t)$ denotes particle density. The Poiseuille flow $V_J$ generates a vorticity that orients the swimmers—that are already aligned by the external magnetic field—towards the center of the channel, essentially acting as a confining potential in the radial direction [8,30]. Figure 1(b) shows how for $B \ll 1$ $\phi_r(r)$ is well approximated by a Gaussian, which corresponds to the case where the effective potential is quadratic. Deep in the clustering phase, $\phi_r(r)$ is not Gaussian and dramatically shoots up in the vicinity of $r = 0$ (see Fig. 1(c)). The scaling of the maximum of $\phi_r$ at $r = 0$, denoted $\phi_0$, with the mean particle density $\rho_0$ is shown in Fig. 1(e). When $B$ is sufficiently small, $\phi_0$ barely varies with $\rho_0$ as expected from a radial focusing by an effective potential. On the contrary, for large values of $B$ the system exhibits anomalous accumulation of particles at $r = 0$, as indicated by the abrupt increase of $\phi_0$ with $\rho_0$.

The parameter $B$ essentially measures how dipole-dipole interactions and alignment with the external magnetic field dominate over thermal fluctuations, as well as how self-propulsion competes with the external flow. The condensation phenomenon described above occurs when $B \gg 1$, and thus relies on the key role of magnetic dipolar interactions between swimmers. As a first simplification, we consider the case where the alignment with $B_{\text{ext}}$ dominates over thermal fluctuations: $m_0 B_{\text{ext}} \gg k_B T$. We moreover assume that the induced magnetic field $B_{\text{int}}$ is negligible compared to $B_{\text{ext}}$ in (2) for the orientational dynamics.

Within the above two assumptions, which are met in most experimentally relevant cases [8,10], the rotational dynamics becomes a fast process and can be decoupled from the translational dynamics. The resulting orientational equilibrium is essentially determined by the balance between magnetic torque ($m_0 n \times B_{\text{ext}}$) and the vorticity ($\frac{1}{2} \nabla \times V_J$). Denoting $\theta$ the angle between $n$ and $-e_z$ (see Fig. 1(a)), its stationary value averaged over thermal fluctuations obeys $\langle \sin \theta \rangle \approx r / (\tau v_0) \ll 1$. Projecting (4) along the radial direction of the channel $e_r$, the equation governing the dynamics of $r$ reads

$$\dot{r} = -\frac{r}{\tau} - \frac{m_0}{\zeta} \partial_r (e_z \cdot B_{\text{int}}) + \vartheta_r,$$

(5)
where the effective Gaussian noise $\tilde{\eta}$ is delta-correlated with variance $2D_{\text{eff}}$, where $D_{\text{eff}} \equiv D + \frac{\nu_0^2}{D} \left[ k_B T / (m_0 B_{\text{ext}}) \right]^2$. The dynamics of the system in the radial direction is therefore equivalent to that of interacting dipoles that experience an effective temperature

$$k_B T_{\text{eff}} \equiv k_B T \left[ 1 + \frac{\nu_0^2}{DD_\tau} \left( \frac{k_B T}{m_0 B_{\text{ext}}} \right)^2 \right], \quad (6)$$

and a confining effective harmonic potential $U(r) \equiv \frac{1}{2} kr^2$ with stiffness $k \equiv \zeta / \tau = k_B T_{\text{eff}} / (D_{\text{eff}} \tau)$ which focuses the particles at the center of the channel. Numerical simulations of Eqs. (1) and (2) reveal that the clustering instability leads to a highly dynamic regime where clusters assemble and disassemble continuously due to thermal fluctuations (see SM movie [40], clustering panel). Therefore, longitudinal density inhomogeneities are expected to have little influence on $B_{\text{int}}$ in the steady state, suggesting that the dynamics will be dominated by the leading contribution of $B_{\text{int}}(r) \simeq -\mu_0 m_0 \rho_0 \phi(r) \varepsilon_z$ in this limit [10].

Inserting this expression into (6), the radial dynamics completely decouples from the longitudinal one. Using the following ansatz for the particle density inside the channel $\rho(r, t) = \rho_0 \phi_{\text{r}}(r, t)$, the radial equilibrium condition thus follows

$$k_B T_{\text{eff}} \left( 1 - \frac{\phi_{\text{r}}}{\phi_c} \right) \partial_r \phi_{\text{r}}(r) + \phi_{\text{r}}(r) \partial_r U(r) = 0, \quad (7)$$

where

$$\phi_c \equiv \frac{1}{4B} \frac{\nu_0^2}{DD_\tau} = \frac{k_B T_{\text{eff}}}{\mu_0 \rho_0 m_0^2}. \quad (8)$$

While $T_{\text{eff}}$ plays the role of an effective temperature for $\phi_r$ in the dilute limit (corresponding to $\phi_r \to 0$) as mentioned above, the strength of the collective effects leading to density-dependent effective diffusion in (7) is set by $\phi_c$. Importantly, we observe that the associated density-dependent effective diffusion coefficient vanishes at $\phi_r = \phi_c$, which will place the system of magnetic bacteria in a shear flow in the class of systems that can exhibit a classical analogue of Bose-Einstein condensation of particles in the ground state $U = 0$ [12, 14].

It follows from (7) that for $\phi_r < \phi_c$, $\phi_r$ is a monotonously decreasing function of $U$ [14]. We denote $\phi_0$ as the maximum of $\phi_r$ that corresponds to $U = 0$, and define $\beta \equiv (k_B T_{\text{eff}})^{-1}$. Using these definitions, the solution of (7) reads

$$\phi_r(U) = -\phi_c W_0 \left[ -\frac{\phi_0}{\phi_c} e^{-\beta U} - \frac{\phi_0}{\phi_c} \right] \quad (\phi_0 < \phi_c), \quad (9)$$

where $W_0(x)$ is the principal branch of the Lambert W function that satisfies $W_0(x e^x) = x$. When $\phi_0 < \phi_c$,

![FIG. 2. Quantitative characterization of the condensate. (a) Maximum of the radial distribution $\phi_0$, (b) the condensate fraction $N_c / N$, and (c) the mean potential energy per particle $U$, as functions of $T_{\text{eff}} / T_c$ and fixed mean density $\rho_0$. For (b) and (c) $\rho_0 = 2 \cdot 10^{16} \text{ m}^{-3}$. (d) Pressure exerted on the condensate $\Delta p$ as a function of $\rho_0^{-1}$ for $v_f = 80 \text{ km} \cdot \text{s}^{-1}$, $B_{\text{ext}} = 4.4 \text{ mT}$ (red) and $v_f = 60 \text{ km} \cdot \text{s}^{-1}$, $B_{\text{ext}} = 3.3 \text{ mT}$ (blue). In all panels the simulation data (points) are compared to the theoretical predictions (dashed lines) with no free parameters.](image-url)

$\phi_r$ is a smooth function of $r$ and its normalization

$$\frac{\Delta p}{kR_0^2} \int_0^\infty r \, d\phi_r(U(r)) = \frac{kR_0^2}{2} \int_0^\infty \sqrt{1 - \frac{1}{T_c / T_{\text{eff}}}} \, dU \phi_r(U) = \frac{1}{2} \Delta p \equiv \Delta p_{\text{eff}} \equiv \frac{1}{2} \frac{kR_0^2}{\phi_0^2} \exp(-\beta U),$$

in agreement with our numerical simulations (see Fig. 1b). As the systematic derivation of (7) from the particle-level stochastic dynamics gives the expressions of $T_{\text{eff}}$, $\phi_c$, and $U$ as functions of the microscopic parameters, a quantitative and parameter-free comparison between the theory and the simulations is possible. This is shown in Fig. 2 whose panel (a) verifies that (10) is in excellent agreement with the simulation results for $T_{\text{eff}} > T_c$. As $\phi_0$ approaches $\phi_c$ (or equivalently as $T_{\text{eff}} \to T_c$), $\phi_r$ becomes non-analytical at $U = 0$, which reflects the formation of a condensate (see Fig. 1c). Consequently, the distribution cannot be normalized and an additional contribution from the ground state needs to be added by hand. Denoting the number of particles in the condensate as $N_c$ and $N_c / N$ as the corresponding ground state
Following previous works \cite{14,41}, the analogy with BEC can be further extended by defining and calculating thermodynamic quantities for the system. The mean potential energy per particle defined as \( \bar{U} \equiv \frac{2}{k R_0} \int_0^\infty d\bar{U} U \phi_r(\bar{U}) \), can be explicitly calculated to give

\[
\bar{U} \equiv \frac{k_B T_{\text{eff}}}{8} \left\{ \begin{array}{ll}
\frac{5}{3} + \frac{1}{3} \phi_c \left( \frac{T_{\text{eff}}}{T_c} - 1 \right) & (T_{\text{eff}} > T_c) \\
\frac{5}{6} T_{\text{eff}} \frac{\phi_c}{T_c} & (T_{\text{eff}} \leq T_c) \end{array} \right.
\]  

(13)

As in the case of BEC \cite{41,14}, (13) predicts a change of slope of \( \bar{U} \) at \( T_{\text{eff}} = T_c \). In particular, for \( T_{\text{eff}} > T_c \) it gives \( \bar{U} \sim k_B T_{\text{eff}} \), which corresponds to a 2-dimensional ideal gas law. Figure 2(c) shows that the theoretical prediction (13) is well reproduced by the simulations. We note that \( \bar{U} \) is generally overestimated in the condensed phase as a consequence of the finite radial extension of the condensate.

With the parameters used in the simulations, the radial focusing of particles occurs on scales much smaller than the channel radius (see Figs. 1(b) and (c)). Therefore, the radial confinement of particles is essentially due to the effective potential \( U(r) \) and the effect of the channel boundary is negligible. Within the generalized thermodynamics of the BEC in the steady state, a pressure \( p \) can be defined for the active fluid via \( \rho \phi \tilde{U} \). We find that the longitudinal clustering instability occurs for \( \rho \ll \rho_0 \), because the solution given in (11) admits values of \( \phi \) larger than \( \phi_c \) at a single point only.

From the normalization of \( \phi_r \), we find that the fraction of particles in the condensate satisfies the following relation

\[
\frac{N_c}{N} = 1 - \frac{T_{\text{eff}}}{T_c} \quad (T_{\text{eff}} \leq T_c),
\]  

(12)

which is consistent with the BEC law in two-dimensional free space \cite{43,44}. Equation (11) predicts the formation of a point-wise condensate at \( U = 0 \). We have measured \( N_c/N \) in our Brownian dynamics simulations by defining it as \( \frac{2}{k R_0^2} \int_{\phi_r(U) \geq \phi_c} d\bar{U} \phi_r(\bar{U}) \), and found a good agreement with the theoretical prediction of (12) as shown (by the red dots) in Fig. 2(b). We have found that measuring the average number of particles \( N_0 \) in a cylinder of radius 0.005\( R_0 \) around the channel center leads to an underestimation of the number of condensed particles (see the hollow squares in Fig. 2(b)). We thus conclude from these observations that the condensate emerging in our simulations occupies a finite volume. This feature can moreover be read directly from the distribution \( \phi_r \) and is linked to the regularization of the near-field magnetic interactions, whose details are discussed in \cite{40}. The addition of short-range repulsion between the swimmers is expected to have similar consequences.
In the limit of small $\phi_0/\phi_c$ and with parameter values considered in Ref. 30, (15) reduces to $B \geq 1$, as expected. Below the condensation threshold where $T_{\text{eff}} < T_c$, the lhs of (15) diverges due to the singularity of $\phi$, at $U = 0$, and the inequality is always satisfied.

Our theoretical investigations thus predict that magnetic microswimmers in a quasi-one dimensional channel exhibit three types of dynamical behavior. When collective effects are negligible, the swimmers are radially focused due to an effective quadratic potential created by the interplay between the external flow and the magnetic field, while being uniformly distributed along the channel axis. When $T_{\text{eff}} > T_c$ and the inequality (15) is satisfied, the system undergoes an instability that gives rise to the formation of clusters that travel along the channel. This longitudinal structure formation persists when $T_{\text{eff}} \leq T_c$, while in that case a macroscopic number of swimmers form a condensate at the center of the channel in a BEC-like fashion. A phase diagram in the $(T_{\text{eff}}/T_c, J/\phi_c)$ parameter space summarizing this phase behavior is provided in Fig. 3(a). Our Brownian dynamics simulations at fixed values of $J = 2 \times 10^{-4}$ and $5 \times 10^{-4}$ verify our theoretically predicted phase behavior of the system, as shown in Figs. 3(b) and (c) (see 40 for simulations details).

To conclude, we have fully characterized the collective behavior of magnetic microswimmers suspended in a microfluidic channel. We have the system to exhibit a novel type of nonequilibrium condensation transition, which shows striking similarities with Bose-Einstein condensation. These findings not only enrich the broad set of many-body dynamics exhibited by active matter systems, but also provide guidelines for future designs of controllable functional micro-robotic active matter systems with desired emergent properties.

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