Noise cross-correlation and Cooper pair splitting efficiency in multi-terminal superconductor junctions.

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Abstract

We analyze the non-local shot noise in a multi-terminal junction formed by two Normal metal leads connected to one superconductor. Using the cross Fano factor and the shot noise, we calculate the efficiency of the Cooper pair splitting. The method is applied to $d$-wave and iron based superconductors. We determine that the contributions to the noise cross-correlation are due to crossed Andreev reflections (CAR), elastic cotunneling, quasiparticles transmission and local Andreev reflections. In the tunneling limit, the CAR contribute positively to the noise cross-correlation whereas the other processes contribute negatively. Depending on the pair potential symmetry, the CAR are the dominant processes, giving as a result a high efficiency for Cooper pair split. We propose the use of the Fano factor to test the efficiency of a Cooper pair splitter device.

Keywords: Superconductivity, Pairing symmetries, Iron Pnictides, Andreev reflection, SN junctions, Nanostructures, Nanocontacts.

1. Introduction

Entanglement states between photons have been very well developed \cite{1}. However, entanglement between electrons in solid system is difficult to create because the electrons are immersed in a macroscopic ground state, which prevents the straightforward generation of entangled pairs of electrons.

The use of a superconductor to produce entangled electrons has been proposed \cite{2,3,4,5,6,7,8} because its ground state is composed of Cooper pairs. A Cooper pair in the superconductor can be break up into two nonlocal entangled electrons that enter into different normal metal leads via the Cooper pair splitting (CPS) \cite{9,10,11,12,13}. The CPS has been studied theoretically and experimentally \cite{14,15,16,17,18}, opening a door to test Bell inequalities in the solid state \cite{19,20}.

The Cooper pair splitting process is analog to CAR, where an incoming electron from one of the leads is reflected as a hole in the other one, inducing a Cooper pair in the superconductor. Typically, to analyze the CAR, the electrical current has been used to test the Bell inequalities; it is important to analyze not only the currents through every lead, but also the correlations between them, which can be determined through the noise.

According to its origin, the noise is classified into two types: one due to thermal fluctuations, which is known as Nyquist-Johnson noise \cite{21}, and the other due to the discrete behavior of the electric charge, known as shot noise \cite{22}. It is possible to obtain with shot noise measurements information not commonly obtained with conductance measurements.

In high $T_C$ superconductors, the shot noise has been studied for plain junctions where shot noise is affect by the pair potential symmetry \cite{23,24}.

The nonlocal shot noise or noise cross-correlation between two electrodes connected to a superconductor reveals a change of the sign in the correlation between currents \cite{25,26}. For example, for two electrodes connected to a normal metal, the nonlocal shot noise shows a negative crossed correlation, which indicates that when the electrical current in one lead increases, the current in the other lead
decreases \([28, 30]\). However the nonlocal shot noise for two leads connected to a superconductor can exhibits positive values \([31, 32]\) that is, the electrical current in the two leads could increase or decrease at the same time.

The noise cross-correlation has been studied for two quantum point leads connected to a superconductor \([33–38]\); nevertheless, the nonlocal shot noise has not been studied for two electrodes connected to \(d\)-wave or iron based superconductors, where the pair potential symmetry can affect the transport properties of the system. In the present paper we show an analytic approach that allows us to find the current-current correlation and separate the contributions due to the different processes. In particular we analyze the Cooper pair split contribution. For this, we use the Hamiltonian approach and the non-equilibrium Green functions in Keldysh formalism. We consider typical pair potential symmetries for cuprates and iron based superconductors which means \(d\), \(s_+\) and \(s_-\) symmetries \([39–46]\). In addition, we consider a \(s\)-wave compound added to the \(d\)-wave superconductors, thus we study how the magnitude and phase of the pair potential affect the noise cross-correlation. We analyze the symmetric case when the two leads are connected to the same voltage \((V_a = V_b = V)\) and the non-symmetric case when the two leads are connected to a voltage difference \(V\) \((V_a = 0, V_b = V)\). We find that the positive noise cross-correlation is favored by symmetric applied voltages.

2. Shot noise cross-correlation

The system considered is formed by two one-dimensional normal metal leads connected to a semi-infinite superconducting region Fig. (1). The lead \(a(b)\) is connected to voltage \(V_{a(b)}\), while the superconductor is grounded. It has been demonstrated that there are two processes that contribute to the cross conductance; the CAR and the elastic cotunneling (EC) \([47]\), see Fig. 2 When the \(V_{a} = 0\) and \(V_{b} = V\) the nonlocal differential conductance is given by

\[
\sigma_{ab} = \frac{dI_a}{dV_b} = \frac{2e^2}{h} (T_{\text{CAR}} - T_{\text{EC}}),
\]

(1)

where \(T_{\text{CAR}}\) and \(T_{\text{EC}}\) are the transmission coefficients of the CAR and EC respectively and that can be written in terms of the Green function as

\[
T_{\text{CAR}} = 4t^4\left|G_{ab, ch}(E)\right|^2,
\]

(2)

\[
T_{\text{EC}} = 4t^4\left|G_{ab, ee}(E)\right|^2,
\]

(3)

where \(t\) is the hopping parameter coupling the leads and the superconductor, \(G_{ab}(E)\) is the nonlocal green function in the superconducting region between \(a\) and \(b\), the subindex \(ee(h)\) denotes the electron-electron (hole) component. The contributions of \(T_{\text{CAR}}\) and \(T_{\text{EC}}\) processes decrease when the distance between the leads increases and depend on the symmetry of the pair potential \([17]\).

![Diagram of two electrodes](image)

**Fig. 1.** Diagram of two electrodes \(a(b)\) separated by a distance \(d\) and connected to a superconductor. The superconductor is grounded and the leads are at voltages \(V_a\) and \(V_b\) respectively.

Our aim is to obtain the noise cross-correlations and the Fano factor. For \(s\)-wave superconductors, the nonlocal shot noise has been calculated \([32]\), showing that, whereas the CAR contribute positively to the crossed correlation between the currents in the two leads \([48–50]\), the EC contributes negatively \([51]\). Local processes like the Andreev reflections (AR) and quasiparticles transmission (Q) contribute negatively to the nonlocal shot noise (see Fig. 2). Whereas the AR are equivalent to Cooper pair tunneling in one lead, the CAR are equivalent to CPS \([52, 53]\); hence, we are interested in analyzing the positive contributions to the nonlocal shot noise (see Fig. 3).

We use the Hamiltonian approach to study this system and the Green functions, and Keldysh formalism \([54]\) in order to find the nonlocal shot noise between the leads \(\beta\) and \(\beta'\) at frequency \(\omega\) (for details see appendix A),

\[
S_{\beta\beta'}(\omega) = \frac{2e^2t^4}{h} \int dE |K_{\beta\beta'}(E, \epsilon)|
\]

(4)
\[ K_{\beta\beta'}(E, \varepsilon) = 4\pi^2(1 - f(\varepsilon))\text{Tr} \left[ \tilde{N}_\beta \tilde{A}_L(E) + \tilde{f}_L(E)\tilde{\rho}_L(E)\tilde{q}_{\beta'}(E, \varepsilon)\tilde{\rho}_R(\varepsilon)\tilde{A}_R(\varepsilon) \right] + \tilde{K}_{\beta\beta'}(E, \varepsilon), \]

where
\[ \varepsilon = E + \hbar\omega. \]

The kernel \( K_{\beta\beta'}(E, \varepsilon) \) can be rewritten as

\[ K_{\beta\beta'}(E, \varepsilon) = K_{\beta\beta'}^1(E, \varepsilon) + K_{\beta\beta'}^2(E, \varepsilon) + K_{\beta\beta'}^3(E, \varepsilon) + K_{\beta\beta'}^4(E, \varepsilon), \]

where

\[ K_{\beta\beta'}^1(E, \varepsilon) = 4\pi^2(1 - f(\varepsilon))\text{Tr} \left[ \tilde{N}_\beta \tilde{A}_L(E) + \tilde{f}_L(E)\tilde{\rho}_L(E)\tilde{q}_{\beta'}(E, \varepsilon)\tilde{\rho}_R(\varepsilon)\tilde{A}_R(\varepsilon) \right], \]
\[ K_{\beta\beta'}^2(E, \varepsilon) = 4\pi^2 f(E)(1 - f(\varepsilon))\text{Tr} \left[ \tilde{N}_\beta \tilde{G}_{LL}(E)\tilde{\rho}_R(E)\tilde{q}_{\beta'}(E, \varepsilon) \right], \]
\[ K_{\beta\beta'}^3(E, \varepsilon) = 4\pi^2 f(E)^2\text{Tr} \left[ \tilde{N}_\beta \tilde{G}_{LL}(E)\tilde{\rho}_R(\varepsilon)\tilde{q}_{\beta'}(E, \varepsilon)(1 - \tilde{f}_L(E)) \right], \]
\[ K_{\beta\beta'}^4(E, \varepsilon) = 4\pi^2 f(E)^2\text{Tr} \left[ \tilde{N}_\beta \tilde{G}_{LL}(E)\tilde{\rho}_R(E)\tilde{q}_{\beta'}(E, \varepsilon)(1 - \tilde{f}_L(E)) \right]. \]

with

\[ \tilde{q}_{\beta'}(E, \varepsilon) = \tilde{A}_L(\varepsilon)\tilde{N}_\beta\tilde{A}_R(\varepsilon) - \tilde{\rho}, \]
\[ \tilde{G}_{RR}(E)\tilde{N}_\beta\tilde{G}_{LL}(E)\tilde{\rho}, \]
\[ \tilde{q}_{\beta'}^2(E, \varepsilon) = \tilde{p}\tilde{G}_{LL}(E)\tilde{N}_\beta\tilde{A}_R(\varepsilon), \]
\[ -\tilde{A}_L^\dagger(\varepsilon)\tilde{N}_\beta^\dagger\tilde{G}_{LL}(E)\tilde{\rho}, \]
\[ \tilde{q}_{\beta'}^3(E, \varepsilon) = \tilde{A}_L^\dagger(\varepsilon)\tilde{N}_\beta^\dagger\tilde{G}_{RR}(E)\tilde{\rho}, \]
\[ -\tilde{p}\tilde{G}_{RR}(E)\tilde{N}_\beta^\dagger\tilde{A}_L(\varepsilon), \]
\[ \tilde{q}_{\beta'}^4(E, \varepsilon) = \tilde{p}\tilde{G}_{LL}(E)\tilde{N}_\beta^\dagger\tilde{G}_{RR}(E)\tilde{\rho}, \]
\[ -\tilde{A}_L^\dagger(\varepsilon)\tilde{N}_\beta\tilde{A}_R(\varepsilon), \]
\[ \tilde{A}_L(E) = \tilde{I} + \tilde{G}_{RR}(E)\tilde{\rho}, \]
\[ \tilde{A}_R(E) = \tilde{I} + \tilde{G}_{LL}(E)\tilde{\rho}. \]

In this equations, \( \tilde{G}_{ij}^r \) is the retarded Green function between the region \( i \) and \( j \) in Nambu \( \otimes \) electrodes space, the subindex \( L \) denotes the "left" region of leads and the subindex \( R \) the right superconducting region, \( \tilde{\rho}_i \) is the local density of states,
\( \hat{\rho}_s = -\text{Im}(g_b^\dagger) / \pi \), with \( g_b \) the unperturbed Green function, \( \tilde{\rho} = t \ast \bar{\sigma}_z \) is the hopping matrix, and

\[
\hat{N}_a = \left( \begin{array}{cc} \hat{I} & 0 \\ 0 & 0 \end{array} \right), \quad \hat{N}_b = \left( \begin{array}{cc} 0 & 0 \\ 0 & \hat{I} \end{array} \right),
\]

\[
\hat{\sigma}_z = \left( \begin{array}{cc} \hat{\sigma}_z & 0 \\ 0 & \hat{\sigma}_z \end{array} \right),
\]

\[
\hat{f}_L = \left( \begin{array}{cc} \hat{f}_{La} & 0 \\ 0 & \hat{f}_{Lb} \end{array} \right),
\]

\[
\hat{f}_{L\beta} = \left( \begin{array}{cc} \hat{f}_{\beta} & 0 \\ 0 & \hat{f}_{h\beta} \end{array} \right),
\]

with \( f_{\beta} = f(E - eV_b) \), \( f_{h\beta} = f(E + eV_b) \) the Fermi distributions for electron and hole like quasiparticles at the lead \( \beta \). Each kernel \( K_{\beta\beta}'(E, \epsilon) \) describes a process that contributes to the electrical current fluctuations and can be classified according to its origin \[4\]. The kernel \( K_{\beta\beta}'(E, \epsilon) \) can be interpreted as the dispersion of particles injected from the lead to the superconductor, where the product \( (1 - f(\epsilon)) f_L(E) \) relates the electron transmission from the left side to the superconductor. Similarly in \( K_{3\beta\beta}'(E, \epsilon) \) the product \( f(E)(1 - f(\epsilon)) \) can be interpreted as the dispersion of particles from the superconductor to the lead. The \( K_{3\beta\beta}'(E, \epsilon) \) term corresponds to the quasiparticles dispersion and the product \( f(E)(1 - f(\epsilon)) \) is related with thermal fluctuations; and finally in \( K_{3\beta\beta}'(E, \epsilon) \) the product \( \hat{f}_L(E)\hat{\rho}_L(E)\hat{\sigma}_z(E, \epsilon)(1 - \hat{f}_L(E))\hat{\rho}_L(E) \) corresponds to hole-hole, electron-hole and electron-electron reflections (further details can be found in Appendix A).

At zero temperature and zero frequency all the products \( f(E)(1 - f(E)) \) vanish \[53\], therefore the thermal fluctuations become zero and we obtain the shot noise. For \( \beta = \beta' \) we obtain the local shot noise at the same lead, while for \( \beta \neq \beta' \) we obtain the nonlocal shot noise, which we analyze in two cases with respect to the applied voltage: the non-symmetric (n-sy) case, where the lead \( a \) is grounded and the lead \( b \) is at voltage \( V \); and the symmetric case (sy), where the two leads are at the same voltage \( V \).

For non-symmetric voltages and in the tunneling limit, the noise cross-correlation expression becomes to

\[
S_{ba}^{n-sy}(V) = S_{CAR}(V) + S_{EC}(V),
\]

\[
S_{CAR}(V) = \frac{4e^2}{h} \int_0^{eV} dE[T_{CAR}(E)],
\]

\[
S_{EC}(V) = -\frac{4e^2}{h} \int_0^{eV} dE[T_{EC}(E)],
\]

in this case \( T_{CAR} \) and \( T_{EC} \) are calculated from the unperturbed Green function of the superconducting region \( G_{RR}^\gamma(E) = \tilde{g}^\gamma(E) \)

\[
T_{CAR}(E) = 4t^4|\tilde{g}_{ab,ea}^\gamma(E)|^2,
\]

\[
T_{EC}(E) = 4t^4|\tilde{g}_{ab,ec}^\gamma(E)|^2.
\]

The noises \( S_{CAR} \) and \( S_{EC} \) correspond to noise due to crossed electron-hole and electron-electron reflection probabilities respectively. While \( CAR \) contributes positively to the shot noise cross correlations, \( EC \) contributes negatively. The differential nonlocal shot noise is defined as

\[
\frac{dS_{ba}^{n-sy}(V)}{dV} = \frac{4e^2}{h} (T_{CAR}(eV) - T_{EC}(eV)),
\]

and is proportional to \( \sigma_{ab} \), Eq. 1. \( dS_{ba}^{n-sy}/dV = 2e\sigma_{ab} \). Therefore for low transparencies and non-symmetric voltages, we get only \( CAR \) and \( EC \) contributions to the differential crossed shot noise. However, for symmetric voltages in the tunneling limit, the contributions due to \( EC \) for voltages lower than \( |\Delta(k)| \) are zero, then the shot noise can be written as

\[
S_{ba}^{sy}(V) = \frac{8e^2}{h} \int_0^{eV} dE[T_{CAR}(E)],
\]

and therefore the differential nonlocal shot noise is written as

\[
\frac{dS_{ba}^{sy}(V)}{dV} = 8e^3T_{CAR}(eV).
\]

For higher transparencies, other processes contribute to the noise cross-correlations \[57\]. Then, for the symmetric case with voltages smaller than \( |\Delta(k)| \), the main contribution to the shot noise is due to \( CAR \). For higher voltages, there are other processes like quasiparticles transmission and Andreev reflections which can contribute positively or negatively to shot noise. \( EC \) contributes negatively and appear by means of intermediate propagators. For any voltage the nonlocal shot noise in tunnelling limit can be written as
\[ S_{ba}^{\alpha \beta} (V) = \frac{4e^2}{\hbar} \int_{V_0}^{eV} dE \left[ 2T_{CAR} (E) - 2T_{EC} (E) + \delta T_{EC} (E) \right], \]  

with

\[ \delta T_{EC} (E) = 4t^4 \left( \delta \tilde{g}_{ba,ee} (E) \tilde{g}_{ba,ee}^* (E) + \delta \tilde{g}_{ab,ee} (E) \tilde{g}_{ab,ee}^* (E) \right). \]

From equation 22, we can see that for voltages lower than \(|\Delta (k)|\), EC contribution cancels itself out because the nonlocal Green functions are always real, \( \delta T_{EC} = 2T_{EC} \) recovering the equation 20.

### 2.1. The Fano factor and the efficiency

The Fano factor \( (F) \) is defined as the ratio between the shot noise and the 2\( e \) multiplied by the electrical current \[ F = \frac{S}{2eI}. \]  

In the tunneling limit the Fano factor provides information about the effective electric charge. We analyze the cross correlations using the Fano factor in terms of the noise cross-correlation and cross current considering the equation used by Samuelsson et. al

\[ F_{\beta \beta'} = \frac{S_{\beta \beta'}}{2e \sqrt{T_{\beta \beta'}}}, \]  

where \( I_\beta \) is the current at the lead \( \beta \). According to equation 13 in the tunneling limit, with non-symmetric voltages, shot noise and current contributions are due to CAR and EC

\[ F_{ab}^{n - sy} = \frac{S_{\text{CAR}} + S_{\text{EC}}}{2e \sqrt{T_a T_b}}, \]

and in the case of symmetric voltages and the shot noise contributions get reduced to only CAR, for \( V < |\Delta (k)| \), therefore the cross Fano is

\[ F_{ab}^{sy} = \frac{S_{\text{CAR}}}{2e \sqrt{T_a T_b}}. \]

It is expected that the nonlocal Fano factor takes values between \(-1\) and \(1\). The explanation arises from the effective electric charge. If we examine the electrical current in every normal metal lead, we find that the electrical current in every lead is due to electron or hole transmission; thus the maximum and the minimum values that the effective electric charge can have are \( e \) and \(-e\) respectively. This Fano factor sign also gives information about sign of the noise cross-correlation. A Fano factor bigger than zero means positive cross correlations dominance, whereas a Fano factor smaller than zero means negative cross correlations dominance.

Considering all the processes that contribute to the nonlocal shot noise in the tunneling limit, we define the efficiency of the device \( (\eta) \) as the ratio between the differential nonlocal shot noise due to CAR and the sum of the absolute values of the contributions due to CAR and EC 21.

\[ \eta = \frac{S_{\text{CAR}}}{|S_{\text{CAR}}| + |S_{\text{EC}}|}. \]  

For the symmetric case, in the tunneling limit and voltages lower than \(|\Delta (k)|\) the contributions to the nonlocal shot noise are only due to CAR; then the efficiency and the Fano factor are equal to one. If we set higher voltages, the other processes contribute to the noise cross-correlation, which indicates that the efficiency and the Fano factor decrease. This result implies that the Fano factor provides information concerning the efficiency of the device. In the next section, we show the efficiency and the Fano factor using different pair potential symmetries and voltages at the leads.

### 3. Results

We calculate the nonlocal shot noise as a function of the distance between the leads and the transmission coefficient which is related to the hopping parameter \( t \) by

\[ T_N = \frac{4t^2}{(1 + t^2)^2}. \]

We analyze the shot noise and the Fano factor using typical symmetries for HTc superconductors (HTcS), with \( d, d + i s \) and, \( s_{++} \) and \( s_{+-} \) symmetries. In this work we have fixed \( \Delta_0 = 20 \text{ meV} \) and the ratio \( \Delta_0 / E_F \approx 10^{-1} \), which are typical values for a HTcS. We consider non-isotropic symmetries for the superconductor pair potential which depends on the wave vector. For \( d \)-wave superconductors the pair potential is \( \Delta (\theta) = \Delta_0 \cos 2(\theta - \alpha) \), with \( \alpha \) the angle between the crystallographic axes of the superconductor and the normal direction to
the two leads, see Fig. 5 b).

Finally, to describe the shot noise in iron based superconductors, we consider two multiband models the $s_{\pm}$ and the $s_{\pm\pm}$. In the first one the phase difference between the two gaps is 0, whereas in the second one it is $\pi$ so that $\Delta_1/\Delta_2 = \pm|\Delta_1|/|\Delta_2|$. $\Delta_1(2)$ is the pair potential in the band 1(2) [38] [40].

3.1. $s$-wave superconductors

If we consider an isotropic symmetry for the superconductor pair potential $\Delta(k) = \Delta_0$. For non-symmetric case and in the tunneling limit, the differential nonlocal shot noise exhibits oscillations as a function dependent on the distance between the two leads. At this limit the contribution to the noise cross-correlations due to Q and AR is much smaller than CAR and EC, it meaning that CAR and EC compete between them. For higher values of the transmission $T_N$, the noise cross-correlation becomes negative due to the increase in processes that contribute negatively to the cross-correlations such as quasiparticles transmission (see Fig. 3). Finally for high transparencies $T_N \to 1$, the differential nonlocal shot noise is completely positive because for a transparent lead the CAR have additional contributions of intermediate propagators in the superconducting region [32]. In addition, the noise cross-correlation exhibits an exponential decay with respect to the distance between the leads (see Fig. 5 a)). These results are in agreement with those obtained by R. Melin et al. [32].

When $V_a = V_b$ at the tunneling limit, the cross correlation between the currents is positive because EC contribution is zero and no local shot noise is due only to CAR. Similarly to the non-symmetric case, when $T_N$ increases the shot noise takes negative values. This behavior occurs because of the higher AR and Q contributions, see Fig. 4. Finally, for high transparencies, the shot noise becomes positive due to contributions from various processes with intermediate propagators between the two leads, see Fig. 4 b).

For symmetric voltages smaller than $|\Delta(k)|$ and at the tunneling limit, due to the cancellation of the EC and the negligible contributions from AR and Q, CAR become the dominant processes giving as a result an efficiency equal to one. For higher voltages, Q and AR contributions increase causing a decrease in the efficiency and the Fano factor (Fig. 2).

3.2. $d$-wave superconductors

Unlike $s$-wave superconductors, for $d_{x^2-y^2}$ symmetry, when $V_a = 0$ and $V_b = V$, the differential shot noise is positive for every transmission value, indicating that this symmetry favors positive cross correlations, in particular for low transparencies.

When we set $V_a = V_b$, the contribution to the shot noise due to CAR is twice the obtained for non-symmetric voltages whereas the EC contribution cancel. This is reflected in the increase of the differential cross correlation shot noise in the tunneling limit. Similarly to $s$-wave superconductors, the shot noise is positive for low transparencies; however, as $T_N$ increases the differential shot noise becomes negative due to AR and Q. For high transparencies, the differential shot noise becomes positive.

For this pair potential symmetry, the cross correlation oscillations do not disappear for high transparencies. This is due to the relative phase of the pair potential, which allows constructive and destructive interferences in the leads. We appreciate an algebraic decay of the shot noise with respect to the distance between the leads, proportional to $1/d^2$ (Fig. 4 a) and b)), in contrast to the exponential decay of the $s$ wave superconductors.

We add an isotropic component to the $d_{x^2-y^2}$ pair potential $\Delta_\pm = \Delta_0 \cos (2(\theta \mp \alpha)) + i\Delta_\parallel$ to get the $d_{x^2-y^2} + is$ symmetry, with $\Delta_s = 0.05\Delta_0$. 

![Fig. 4. The spatial-averaged differential shot noise cross-correlation as a function of the transmission coefficient for a $s$ wave superconductor, at $120$ and $eV = 0$, for non-symmetric voltages $V_a=0$ and symmetric voltages $V_a = V_b$.](image-url)
For non-symmetric voltages, in the tunneling limit, we get positive differential shot noise similarly to those observed with the $d_{x^2-y^2}$ symmetry, see Fig. 7 c). For intermediate transparencies, the EC, Q and AR contributions can be more relevant than CAR, producing an oscillating behavior around zero. For symmetric voltages, the CAR domains over Q and AR for small transparencies and similarly to the non-symmetric voltages, see Fig. 7 d), we appreciate that the differential noise cross-correlation has been displayed for positive values.

Due to the $\Delta_s$ component an exponential decay is added to the algebraic decay that we observed with the $d_{x^2-y^2}$ wave superconductors, such that the behavior is proportional to $e^{-d/\pi \xi_0 / d^2}$.

When we consider a $d_{xy}$ symmetry the AR are suppressed by the diffraction in the interphase between the lead and superconductor [61]; thus, the processes that contribute to the shot noise are CAR, EC and Q. In NIS (N: normal-metal, I: Insulator, S: Superconductor) plane junctions, when the symmetry of the pair potential is $d_{xy}$, a zero bias conductance peak (ZBCP) appears in the differential conductance due to the induction of states at the interface. The Andreev reflection coefficient is 1 and rapidly decays as $V$ increases. In the differential shot noise the peak is split and at zero bias the shot noise is equal to zero [23, 24, 62, 63]. These two results lead to a Fano factor equal to zero at zero bias for this kind of symmetry. However, in NIS quantum point contact ZBCP does not appear because the Andreev reflections are zero. The wave functions in the channel are a superposition of two plane waves with wave numbers $k_y = \pm p/W$. Each wave experiences a pair potential phase 0 and $\pi$, respectively, and therefore the Andreev reflection coefficient for each wave is out of phase by $\pi$, such that the waves of the reflected holes interfere destructively and the Andreev reflections vanish.

When the leads are connected to non-symmetric voltages and at the tunneling limit, the EC dominates over CAR, so that the noise cross-correlation displays negative values. When $T_N$ increases, the Q contributions increase also, causing that the differential noise cross-correlation becomes more negative. Unlike the $d_{x^2-y^2}$ symmetry, for $d_{xy}$ symme-
tive values for the shot noise cross correlations. As the EC contributions to the cross correlations cancel and the AR are suppressed due to the pair potential symmetry, we only have the CAR and Q contributions. For low transparencies we appreciate a dominance of CAR that decreases as $T_N$ increases.

Whereas for $d_{xy}$ and non-symmetric voltages the dominance is due to EC, with an isotropic component the AR occur and contributions due to CAR increase. This phenomenon can be appreciated in the negative values obtained in the differential shot noise with an isotropic component, see Fig. 8a).

For the $d_x$ symmetry we obtain an algebraic decay with respect to the distance of the leads proportional to $1/d^4$ (Fig. 8a) and b)), whereas for the $d_y$ symmetry, the decay is proportional to $e^{-d/(\pi \xi)}/d^4$ (Fig. 8c) and d)).

3.3. Multiband superconductors, $s_{++}$ and $s_{+-}$ symmetries

The main feature of iron based superconductors is their multiple band structure near to the Fermi level. In a simplified scheme, the structure is reduced to two band models, where the symmetry of the pair potential in each band could be different, and experimental evidence has been favorable to the $s_{++}$ and $s_{+-}$ symmetries.

In order to calculate the transport properties for this kind of symmetries, we find the equilibrium Green functions for two $s$-wave superconductors with different pair potentials, $\Delta_1 = 0.5 \Delta_0$ and $\Delta_2 = \Delta_0$. We use two phase differences between the two gaps, which for the $s_{++}$ is 0 and for the $s_{+-}$ is $\pi$. Then by means of a weight factor ($\alpha$), defined as the ratio of the probability amplitudes for an incoming electron from one of the two leads to tunnel into the first or second band, we write the total Green function of the system as

\[
\hat{g}^R_{RR}(E) = \sum_{k_y} |t(k_y)|^2 \left[ \hat{g}^R_{\Delta_1}(E, k_y) + \alpha \hat{g}^R_{\Delta_2}(E, k_y) \right] .
\]

For the $s_{+-}$ symmetry and non-symmetric voltages, we appreciate destructive effects over AR due to the phase difference in the pair potential (Fig. 8a) and b)). However, the contributions due to AR do not disappear completely because the magnitude of the two gaps are not equal. At the tunneling limit, the cross-correlations are negative due to the larger EC contribution. For high transparencies the positive contributions to the noise cross correlation increase because of non-local processes of higher order.
We have determined analytical equations for shot noise cross correlation in multi-terminal superconductors from the Green’s functions and the Keldysh formalism. We have considered two leads (conductors from the Green’s functions and the Keldysh noise cross correlation in multi-terminal superconductors).

4. Conclusions

When we set symmetric voltages, the EC contributions cancel and the CAR are reduced, the CAR contributes positively to the cross-correlation; thus, the shot noise is positive.

The results for the \( s_{++} \) symmetry in Fig. 10 are quite similar to those obtained for the \( s \) symmetry. We also appreciate that the nonlocal shot noise for symmetric voltages becomes negative for intermediate values of the transparency according to the behavior observed in the Fig. 4. This behavior occurs because for this pair potential there is no phase difference; thus, the contributions of the two bands to the nonlocal shot noise are added, yielding similar magnitudes and signs as those for \( s \) symmetry.

4. Conclusions

We have determined analytical equations for shot noise cross correlation in multi-terminal superconductors from the Green’s functions and the Keldysh formalism. We have considered two leads (a and b) and the cases where the applied voltages are \( V_a = 0 \) and \( V_b = V \) (non-symmetric case), and \( V_a = V_b = V \) (symmetric case). We have considered pair potential symmetries, \( s, d, d + i s \) and multiband \( s_{++} \) and \( s_{++} \). We found that when we apply symmetric voltages lower than \( |\Delta(k)| \) in the tunneling limit, positive cross correlations are favored. This result is obtained due to the fact that the EC, which are the main source of negative cross correlations, is canceled allowing the CAR to be the dominant processes.

We calculate the crossed Fano factor of the system for symmetric voltages. The sign of the Fano factor reveals the sign of the cross correlation dominance and we find that this factor in the tunneling limit is analogous to the CPS efficiency of the device.

In particular, we observe that when symmetric voltages are applied to a \( d_{xy} \)-wave superconductor, and in the tunneling limit, the CAR become the dominant processes making the device to exhibits good efficiency even for voltages higher than \( \Delta_0 \). By on the other side, for non symmetric voltages EC dominates over CAR. For \( d_{x^2−y^2} \) and \( d_{xy} \) symmetries, the isotropic component causes negative contributions reflected in negative values for the cross correlation.

In general the shot noise cross correlation between two leads for \( d \)-wave superconductors exhibits algebraic decay with the increase of the distance between the leads, in contrast to the exponential behavior typically observed for isotropic superconductors. These properties would allow the development of devices that are capable of detecting positive cross correlation at distances several times larger than the characteristic coherence length with good efficiency.

To prove the entanglement, it is still needed to show Bells inequality by means of the coherence
and spin correlation that could be accomplished by means of ferromagnetic leads.

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Appendix A. Modeling the system

From the superconductor surface Green functions in momentum representation $(\hat{g}_{S}(E, ky))$, we calculate the local $\hat{g}_{S,aa(\bar{b}b)}(E)$ and the non-local $\hat{g}_{S,ab(ab)}(E)$ equilibrium Green functions [47]

$$
\hat{g}_{S,aa(\bar{b}b)}(E) = \sum_{k_y} |f(k_y)|^2 \hat{g}_{S}(E, k_y),
$$

$$
\hat{g}_{S,ab(ab)}(E) = \sum_{k_y} |f(k_y)|^2 \hat{g}_{S}(E, k_y),
$$

(A.1)

where $f(k_y)$ is the weighting factor and is proportional to the perpendicular wave vector $k_{yF}$ [61], $\xi$ is the superconductor coherence longitudinal defined as $\xi(E, \theta) = \xi_0 / Re[\sqrt{1 - E^2/\Delta^2(\theta)}]$ and $\xi_0 = \hbar v_F / (\pi \Delta_0)$ the BCS coherence longitudinal.

We write the retarded Green function of the un-coupled superconductor $\hat{g}_{RR}(E)$ as

$$
\hat{g}_{S}(E) = \begin{pmatrix}
\hat{g}_{S,aa}(E) & \hat{g}_{S,ab}(E) \\
\hat{g}_{S,ba}(E) & \hat{g}_{S,bb}(E)
\end{pmatrix},
$$

(A.2)

where the symbol $'$ denotes a $2 \times 2$ matrix in Nambu space, whereas the symbol $\ast$ denotes a $4 \times 4$ matrix in the Nambu electrodes space.

We obtain the non-equilibrium Green functions of the coupled system $\hat{G}_{ij,\beta\beta'}^{++(\mp)(\pm)}(E)$ and the perturbed Green Function $\hat{G}_{ij,\beta\beta'}^{(a)}(E)$ solving the Dyson equation for two leads, where $\beta$ and $\beta'$ denote the $a$ or $b$ lead respectively, and $i$ and $f$ denote the $L$ or $R$ region respectively.

$$
\hat{G}_{ij,\beta\beta'}^{++(\mp)(\pm)}(E) = [\hat{I} + \hat{G}_{ij}^{(r)}(E)\hat{p}]\hat{g}_{ij,\beta\beta'}^{++(\mp)(\pm)}(E),
$$

$$
\hat{G}_{ij,\beta\beta'}^{(a)}(E) = \hat{g}_{ij,\beta\beta'}^{(a)}(E) + \hat{g}_{ij,\beta\beta'}^{(a)}(E)\hat{p}
$$

(A.3)

(A.4)

where $t$ is the hopping parameter related to the transmission of particles from the left side to the right side [61], $(+)\text{-}(\mp)$ are the two branches of the Keldysh space, $\hat{g}_{ij,\beta\beta'}^{++(\mp)(\pm)}$ is the non-equilibrium Green function (leads or superconductor) in Keldysh space without coupling [47].

The electrical current in the $\beta$ lead is

$$
I_{\beta} = \frac{e t^2}{\hbar} \int_{-\infty}^{\infty} dE \text{Tr} \{ \hat{N}_{\beta} (\hat{g}_{LL}^{\dagger}(E)\hat{\sigma}_z) \}
$$

(A.5)

$$
\hat{G}_{RR}^{\dagger}(E) - \hat{g}_{LL}^{\dagger}(E)\hat{\sigma}_z\hat{G}_{RR}^{\dagger}(E)
$$

with, $\hat{N}_a, \hat{N}_b, \hat{\sigma}_z$ given by the Eq. 13. To calculate the noise cross-correlation we define the spectral density of the electrical current fluctuations between the electrodes $\beta$ and $\beta'$ as

$$
S_{\beta\beta'}(\omega) = h \int d(\tau') e^{i\omega(\tau')} \begin{bmatrix}
\langle \delta I_{\beta}(\tau')\delta I_{\beta'}(\tau) \rangle \\
\langle \delta I_{\beta'}(\tau')\delta I_{\beta}(\tau) \rangle
\end{bmatrix},
$$

(A.6)

where $\delta I_{\beta}(\tau)$ is the deviation of the electrical current regarding to its mean value, $\delta I_{\beta}(\tau) = I_{\beta}(\tau) - \langle I_{\beta}(\tau) \rangle$.

$$
\langle \delta I_{\beta}(\tau)\delta I_{\beta'}(\tau) \rangle = 2 \epsilon^2 \text{Tr} \{ \hat{N}_{\beta}^{\dagger} \hat{\sigma}_z \hat{p} \}
$$

(A.7)

$$
\hat{G}_{LL}^{\dagger}(\tau, \tau')\hat{\sigma}_z \hat{p}
$$

(A.8)

$$
\hat{G}_{RR}^{\dagger}(\tau, \tau') - \hat{N}_{\beta}^{\dagger} \hat{\sigma}_z \hat{p}
$$

(A.9)

with $\hat{p} = t \ast \hat{\sigma}_z$. We write the noise cross-correlation as

$$
S_{\beta\beta'}(\omega) = \frac{2 \epsilon^2 t^4}{\hbar} \int dE [K_{\beta\beta'}(E, \epsilon) + K_{\beta'\beta}(\epsilon, E)],
$$

(A.10)

$$
\epsilon = E + \hbar \omega
$$

where the kernel $K_{\beta\beta'}(E, \epsilon)$ is given by

$$
K_{\beta\beta'}(E, \epsilon) = \text{Tr} \{ \hat{N}_{\beta}^{\dagger} \hat{G}_{LL}^{\dagger}(E) \hat{N}_{\beta'} \}
$$

(A.11)

$$
\hat{G}_{RR}^{\dagger}(\epsilon) - \hat{N}_{\beta} \hat{G}_{LR}^{\dagger}(E) \hat{N}_{\beta'}
$$

(A.12)

$$
\hat{N}_{\beta}^{\dagger} \hat{G}_{LR}^{\dagger}(\epsilon)
$$

(A.13)

From the non-equilibrium Green function obtained by solving the Dyson equation Eq. (A.1) we calculate the kernels $K_{\beta\beta'}$ Eq. (A.10) and the noise cross-correlations, Eq. (A.11).
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