Scaling for the Percolation Backbone

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We study the backbone connecting two given sites of a two-dimensional lattice separated by an arbitrary distance \( r \) in a system of size \( L \). We find a scaling form for the average backbone mass:

\[
\langle M_B \rangle \sim L^{d_B} G(r/L),
\]

where \( G \) can be well approximated by a power law for \( 0 < x < 1 \):

\[
G(x) \sim x^\psi
\]

with \( \psi = 0.37 \pm 0.02 \). This result implies that \( \langle M_B \rangle \sim L^{d_B} r^\psi \) for the entire range \( 0 < r < L \).

We also propose a scaling form for the probability distribution \( P(M_B) \) of backbone mass for a given \( r \). For \( r \approx L \), \( P(M_B) \) is peaked around \( L^{d_B} \), whereas for \( r \ll L \), \( P(M_B) \) decreases as a power law, \( M_B^{-r_B} \), with \( r_B \approx 1.20 \pm 0.03 \). The exponents \( \psi \) and \( \tau_B \) satisfy the relation \( \psi = d_B(\tau_B - 1) \), and \( \psi \) is the codimension of the backbone, \( \psi = d - d_B \).

The percolation problem is a classical model of phase transitions, as well as a useful model for describing connectivity phenomena, and in particular for describing porous media. At the percolation threshold \( p_c \), the mass of the largest cluster scales with the system size \( L \) as \( M \sim L^{d_f} \). The fractal dimension \( d_f \) is related to the space dimension \( d \) and to the order parameter and correlation length exponents \( \beta \) and \( \nu \) by \( d_f = d - \beta/\nu \). In two dimensions, \( d_f = 91/48 \) is known exactly.

An interesting subset of the percolation cluster is the backbone which is obtained by removing the non-current carrying bonds from the percolation cluster. The backbone has an interesting history and its exact value is not known. A current state of the art is the numerical estimate [11] is exponent and its exact value is not known. A current current carried \( \psi \) is the codimension of the backbone, \( \psi = d - d_B \).

We study here the backbone connecting these two points for \( 0 < r < L \). We consider the backbone mass probability distribution \( P(M_B) \). We show that \( P(M_B) \) obeys a simple scaling form in the entire range of \( r/L \):

\[
P(M_B) \sim \frac{1}{r^{d_B}} F \left( \frac{M_B}{r^{d_B}} \right),
\]

where \( F(x) \) is a scaling function, whose shape depends on the ratio \( r/L \).

For \( r \approx L \), it seems reasonable to assume that \( P(M_B) \) will be peaked around its average value \( < M_B > \sim L^{d_B} \). The data collapse predicted by Eq. [1] is represented in Fig. 1(a). In this case, the scaling function \( F \) is peaked at approximately \( L^{d_B} \).

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However, the case \( r \ll L \) is far less clear. In fact, we expect for \( r \ll L \) that the backbone mass fluctuates greatly from one realization to another, since its minimum value can be \( r \) and its maximum can be of order \( L^d \). Fig. 1(b) shows a log-log plot of \( P(M_B) \). It has a lower cut-off of order \( r \) (since the backbone must connect points \( A \) and \( B \)) and a upper cut-off of order \( L^d \). We find good data collapse (Fig. 1(c)), which indicates that the scaling function \( F \) is a power law in the range from \( r^d \) to \( L^d \), with exponent approximately \( \tau_B \approx 1.20 \pm 0.03 \) (there is a cut-off at \( M_B \approx L^d \) not shown here). The exponent \( \tau_B \) is connected to the blob size distribution \( \Phi(x) \) since typically, the two sites belong to the same blob, and the sampling of backbones is equivalent to sampling of the blobs. From \[5\],

\[
\frac{d}{d_B} = \tau_B.
\] (2)

This relation gives the estimate \( \tau_B \approx 1.22 \) in good agreement with our numerical simulation.

We note that for larger values of \( M_B \), a "bump" (indicated by an arrow on Fig. 1(b)) located at approximately \( L^d \) appears and assumes increasing importance when \( r \) approaches \( L \).

We now study the average backbone mass \( \langle M_B \rangle \). From dimensional considerations, the \( r \) dependence can only be a function of \( r/L \). We thus propose the following Ansatz:

\[
\langle M_B(r, L) \rangle = L^d G \left( \frac{r}{L} \right).
\] (3)

In Fig. 2(a), we show a double logarithmic scale \( M_B \) versus \( r \) for different values of \( L \). In order to test the Eq. \[3\], we scale the data of Fig. 2(a). The data collapse is obtained using \( d_B = 1.65 \) and is shown on Fig. 2(b). This (log-log) plot supports the scaling Ansatz \[3\]. Moreover, one can see that the scaling function \( G \) is, surprisingly, a pure power law on the entire range \([0, 1]\), with exponent \( \psi = 0.37 \pm 0.02 \).

The results \[1\] and \[3\] are consistent, since if \[1\] holds with a power law behavior for the scaling function \( F(x) \sim x^{-\psi} \) for \( x > 1 \), and \( F(x) = 0 \) for \( x < 1 \), then the average mass is given by

\[
\langle M_B(r, L) \rangle = \int_{r}^{L^d} F \left( \frac{M}{r^d} \right) \frac{dM}{r^d} M.
\] (4)

Assuming that \( L/r \) is large enough, the integral in \[4\] can be approximated as \( L^d \psi r^{-\psi} \), where

\[
\psi = d_B (\tau_B - 1)
\] (5)

In our simulation \( \tau_B \approx 1.20 \pm 0.03 \), which leads to the value \( \psi \approx 0.33 \pm 0.05 \) in reasonable agreement with the value measured directly on the average mass.

Moreover, using Eq. \[2\] together with Eq. \[5\], we obtain

\[
\psi = d - d_B
\] (6)

which means that \( \psi \) is the codimension of the fractal backbone.

To summarize, we find that for any value of \( r/L \), the scaling form, Eq. \(1\), for the probability distribution is valid. The shape of the scaling function \( F \) depends on \( r/L \), being a peaked distribution for \( r \approx L \), and a power law for \( r \ll L \). The average backbone mass varies with \( r \) and \( L \) according to Eq. \(1\). For fixed system size, it varies as \( \langle M_B \rangle \sim r^\psi \) (for \( 0 < r < L \)). The value of \( \psi \) is small (\( \psi \approx 0.37 \)) indicating that the backbone mass does not change drastically as \( r \) changes. On the other hand, the exponent governing the variation of \( \langle M_B \rangle \) with \( L \) for fixed \( r \) is expected to be larger, with \( \langle M_B \rangle \sim r^{d_B - \psi} \). This exponent \( d_B - \psi \) is not equal to the fractal dimension \( d_B \) of the backbone, but is smaller by an amount equal to \( \psi \).

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FIG. 1. (a) Data collapse of $P(M_B)$ using Eq. (1) for three different values of $r \approx L$. (b) Probability distribution of the backbone mass for $L = 1000$ and $r = 2$ (computed with $10^5$ configurations). The exponent $\tau_B$ is obtained by a linear fit over the range $30 < M_B < 3 \times 10^4$ and the error bar on $\tau_B$ is around 0.03. The arrow denotes the fact that $M_B$ peaks as $L^{d_B}$. (c) Data collapse of $P(M_B)$ for $L = 400$ using Eq. (1) for three different values of $r$.

FIG. 2. (a) Log-log plot of the average backbone mass $\langle M_B \rangle$ versus $r$ for four different values of $L$. (b) Data from Fig. 2(a) collapsed with the use of the scaling form proposed in Eq. (2). The error on $\psi$ is typically 0.02.