RANDOM SAMPLING AND RECONSTRUCTION OF CONCENTRATED SIGNALS IN A REPRODUCING KERNEL SPACE

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Abstract

In this talk, I will discuss random sampling of signals concentrated on a bounded Corkscrew domain $\Omega$ of a metric measure space, and reconstructing concentrated signals approximately from their (un)corrupted sampling data taken on a sampling set contained in the domain $\Omega$. This talk is based on a joint paper with Yaxu Li and Jun Xian, ACHA 2021.
1. Introduction: Sampling and reconstruction
2. Concentrated signals in reproducing kernel spaces
3. Random sampling and stability
4. Finite algorithm to approximate concentrated signals from their random samples
5. Finite algorithm to approximate concentrated signals from their corrupted random samples
6. Numerical simulations
1. Introduction: Sampling and Reconstruction

- **Signals** (Analog signals, spatial signals, discrete signals), data on networks, *functions on a domain* $\Omega$

**Figure:** A Capture Of A Digital Audio Signal In The DAW Logic Pro X

- Signals in real life have certain properties (bandlimited, regularity, finite rate of innovation, sparsity) or reside in some linear spaces (PW, SIS, RKS)

- *Signals in RKS with energy mainly concentrated on a domain*  

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1. https://mynewmicrophone.com/microphone-audio-signal/
**Sampling**: from signals to sampling data

\[ f \mapsto (\psi(f))_{\psi \in \Psi}, \]

where \( \Psi \) (for instance \( \{\delta(x_i), x_i \in X\} \)) is a family of (non)linear sampling functionals.

**Two categories**: Non-Probability Sampling (*deterministic sampling*) and Probability Sampling (*random sampling*). The difference is whether the sample selection is based on randomization or not. ²

²https://towardsdatascience.com/sampling-techniques-a4e34111d808
Reconstruction in signal processing, (also known as interpolation and data fitting), attempts to produce a continuous time signal coinciding with the points of the discrete time signal. ³

Figure: Cardinal spline interpolations of the discrete time signal of degree \( d = 0, 1, 2, \infty \). The interpolations become increasingly smooth and approach the sinc interpolation as the order increases.

³https://cnx.org/contents/d2CEAGW515.4:gWKrY9L410/Signal-Reconstruction
Sampling from signals to sampling data $f \mapsto (\psi(f))_{\psi \in \Psi}$.

Reconstruction from sampling data to signal approximation $(\psi(f))_{\psi \in \Psi} \mapsto \tilde{f}$ is an inverse problem. The inverse problem is well-posedness, i.e., $\tilde{f} \approx f$,

- when the original signal has additional information, such as bandlimited, regularity, finite rate of innovation, sparsity, or residing in some linear spaces (PW, SIS, RKS)
- when the sampling procedure is appropriate (the sampling family $\Psi$, such as random sampling with enough samples, deterministic sampling with large density)

Reconstruction algorithms: Iterative (infinite steps) and non-iterative (finite steps)

In this talk, we consider random sampling and finite reconstruction of concentrated signals in a reproducing kernel space.
2. Concentrated signals in reproducing kernel spaces

- **Paley-Wiener space** $PW$:

$$PW = \left\{ f \in L^2(\mathbb{R}) : \text{supp} \hat{f} \subset [-\pi, \pi] \right\}$$

$$= \left\{ \sum_{k \in \mathbb{Z}} c(k) \text{sinc}(x - k), \sum_{k \in \mathbb{Z}} |c(k)|^2 < \infty \right\}$$

where $\text{sinc}(t) = \frac{\sin \pi t}{\pi t}$ is the cardinal sine function and $
\{\text{sinc}(\cdot - k), k \in \mathbb{Z}\}$ is ON.

- **Shift-invariant spaces**

$$V(\phi) = \left\{ \sum_{k \in \mathbb{Z}} c(k) \phi(x - k), \sum_{k \in \mathbb{Z}} |c(k)|^2 < \infty \right\},$$

where $\{\phi(\cdot - k), k \in \mathbb{Z}\}$ forms a Riesz basis.

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4A. Aldroubi and K. Grochenig, Nonuniform sampling and reconstruction in shift-invariant spaces. SIAM Review, 2001
Let \( L^p := L^p(X, \rho, \mu) \), \( 1 \leq p \leq \infty \) be the linear space of all \( p \)-integrable functions on the metric measure space \( (X, \rho, \mu) \). Consider the range space

\[
V_p = \{ Tf, f \in L^p \} = \{ f \in L^p, Tf = f \}
\]

of an idempotent integral operator

\[
Tf(x) = \int_X K(x, y)f(y)d\mu(y)
\]

The range space \( V_p \) is a **Reproducing Kernel Subspace** of \( L^p \) \(^5\) if the integral kernel \( K \) has certain off-diagonal decay and Hölder continuity,

\[
\|K\|_{S, \theta} = \|K\|_S + \sup_{0 < \delta < 1} \delta^{-\theta} \|\omega_\delta(K)\|_S < \infty.
\]

For instance \( |K(x, y)| \leq (1 + \rho(x, y))^{-N} \) and

\[
|K(x, y) - K(x', y')| \leq (\rho(x, x') + \rho(y, y'))^\theta (1 + \rho(x, y) + \rho(x', y'))^{-N}.
\]

\(^5\)Z. Nashed and Q. Sun, Sampling and reconstruction of signals in a reproducing kernel subspace of \( L^p(R^d) \), JFA, 2010
The range space $V_p = \{ Tf, f \in L^p \} = \{ f \in L^p, Tf = f \}$ is a RKS, where

$$Tf(x) = \int_X K(x, y)f(y)d\mu(y)$$

$PW = \left\{ f \in L^2, f = f * \text{sinc} \right\}$ with kernel $K(x, y) = \text{sinc}(x - y)$.

$SIS = \{ f \in L^2, f(x) = \int f(y)(\sum_{k \in \mathbb{Z}} \tilde{\phi}(y - k)\phi(x - k))dy \}$ where $\{\tilde{\phi}(x - k)\}$ is a dual Riesz basis.

In the Euclidean space setting, $V_p = \{ \sum_{\lambda \in \Lambda} c_\lambda \phi_\lambda, \sum_\lambda |c_\lambda|^p < \infty \}$, where $\phi_\lambda, \lambda \in \Lambda$ is a $p$-frame.

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6 A Aldroubi, Q Sun, WS Tang, $p$-Frames and Shift Invariant Subspaces of $L_p$, JFAA 2021; PG Casazza, D Han, DR Larson, Frames for Banach spaces, Contemporary Mathematics 247, 1999.
A Corkscrew domain $\Omega$ if for any $x \in \partial \Omega$ and radius $0 < r \leq \text{diam} \partial \Omega$, there exists $y \in \Omega$ such that $B(y, cr) \subset \Omega \cap B(x, r)$, where $c \in (0, 1)$ is an absolute constant.  

Concentrated signals $V_{p,\Omega,\epsilon} = \{f \in V_p, \|f\|_{p,\Omega^c} \leq \epsilon \|f\|_p\}$

Figure: $\sum_{i=-50}^{50} r_i \exp(-(x - i - \theta_i)^2)$, where $1/2 \leq |r_i| \leq 1$

Matthew Badger, Lipschitz Approximation to Corkscrew Domains https://badger.math.uconn.edu/talk/corkrain.pdf

Used in time-frequency analysis, phase retrieval, and (random) sampling of bandlimited and wavelet signals by Alaifari, Bass, Cheng, Daubechies, Fuhr, Grochenig, Grohs etc
sampling in linear spaces (PW, SIS, Reproducing kernel), via inverse problem

sampling of signal with properties, such as smoothness and sparsity, via optimization approach.

Random sampling in Paley-Wiener space and shift-invariant space has been discussed by Bass, Grochenig etc. ⁹

Bass and Grochenig use the spectral approach of the composition projection onto the domain and the space. We use completely different approach.

The challenge: The set $V_{p,\Omega,\epsilon}$ is not a convex set (also not a linear space)

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⁹R.F. Bass, K. Gröchenig, Relevant sampling of band-limited functions, Ill. J. Math. 57 (2013) 43–58; R.F. Bass, K. Gröchenig, Random sampling of bandlimited functions, Isr. J. Math. 177 (2010) 1–28
3. Random Sampling and Stability

Concentrated signals $f$ in a reproducing kernel space

\[ V_{p,\Omega,\epsilon} = \{ f \in V_p, \|f\|_{p,\Omega} \leq \epsilon\|f\|_p \} \]

Stability of bi-Lipschitz type vs. uniqueness:

**Theorem**

(Li, S. Xian ¹⁰)  If $\Gamma_{\Omega} \subset \Omega$ is a discrete sampling set with sufficiently small Hausdorff distance $d_H(\Gamma_{\Omega}, \Omega)$ between $\Gamma_{\Omega}$ and $\Omega$, then for all $f, g \in V_{p,\Omega,\epsilon}$,

\[
(1 - \epsilon - \|K\|_{S,\theta}(d_H(\Gamma_{\Omega}, \Omega))^\theta) \|f - g\|_p - 2\epsilon \min(\|f\|_p, \|g\|_p) \\
\leq \| (f(\gamma) - g(\gamma))_{\gamma \in \Gamma_{\Omega}} \|_{p,\mu(\Gamma_{\Omega})} \\
\leq (1 + \|K\|_{S,\theta}(d_H(\Gamma_{\Omega}, \Omega))^\theta) \|f - g\|_p,
\]

where $I_{\gamma}, \gamma \in \Gamma_{\Omega}$ is a Voronoi partition of the domain $\Omega$.

¹⁰Y Li, Q Sun, J Xian, Random sampling and reconstruction of concentrated signals in a reproducing kernel space, ACHA, 54(2021), 273-302
With high probability, an original $\varepsilon$-concentrated signal can be “reconstructed” approximately from its random samples corrupted by i.i.d. random noises, when the random sampling size is large enough.

**Theorem**

If $\{\gamma, \gamma \in \Gamma_{\Omega}\}$ are i.i.d. random positions drawn on $\Omega$ w.r.t. the probability measure $(\mu(\Omega))^{-1}d\mu$, then for any $\tilde{\varepsilon} \in (0, 1 - \varepsilon)$, then

$$(1 - \varepsilon - \tilde{\varepsilon})\|f - g\|_p - 2\varepsilon \min(\|f\|_p, \|g\|_p) \leq \|(f(\gamma) - g(\gamma))_{\gamma \in \Gamma_{\Omega}}\|_{p, \mu(\Gamma_{\Omega})} \leq (1 + \tilde{\varepsilon})\|f - g\|_p, f, g \in V_{p, \Omega, \varepsilon}$$

hold with probability at least

$$1 - \frac{10^d \mu(\Omega)}{c^d D_1(\mu)(\tilde{\varepsilon}/\|K\|_{S, \theta})^{d/\theta}} \left(1 - \frac{c^d D_1(\mu)(\tilde{\varepsilon}/\|K\|_{S, \theta})^{d/\theta}}{10^d \mu(\Omega)}\right)^N,$$

where $N$ is the size of the sampling set $\Gamma_{\Omega}$.
4. RECONSTRUCTION OF CONCENTRATED SIGNALS FROM RANDOM SAMPLES

- Non-uniqueness from i.i.d. random samples $f(\lambda), \lambda \in \Omega$, of concentrated signals $f$.
- For any $f \in V_p$ we define

$$g_0 = \sum_{\gamma \in \Gamma_\Omega} \mu(I_{\gamma})f(\gamma)K(\cdot, \gamma) \in V_p,$$

(3a)

and $g_n \in V_p, n \geq 1$, inductively by

$$g_n = g_0 + g_{n-1} - S_\Gamma g_{n-1}, n \geq 1,$$

(3b)

where the preconstruction operator $S_\Gamma$ on $L^p$ is given in $S_\Gamma g(x) = \sum_{\gamma \in \Gamma} \mu(T_{\gamma})(Tg)(\gamma)K(x, \gamma)$ with $\Gamma = \Gamma_\Omega \cup \Gamma_{\Omega^c}$, and $\Gamma_{\Omega^c}$ being a discrete sampling set of the complement $\Omega^c$ of the domain $\Omega$ satisfying

$$d_H(\Gamma_{\Omega^c}, \Omega^c) \leq \min \left( \varepsilon^{1/\theta} \| K \|_{S,\theta}^{-1/\theta}, (2 \| K \|_{S,\theta}^2)^{-1/\theta} \right).$$
Finite iterative algorithm $g_n, n \geq 0$, to construct a concentrated signal $\tilde{f}$ to approximate the original signal $f$:

$$g_n = g_0 + g_{n-1} - S_{\Gamma} g_{n-1}, \ n \geq 1,$$

with $g_0 = \sum_{\gamma \in \Gamma_{\Omega}} \mu(I_{\gamma}) f(\gamma) K(\cdot, \gamma) \in V_p$

**Theorem**

(Li, S. Xian, ACHA 2021) Suppose that $\{\gamma, \gamma \in \Gamma_{\Omega}\}$ are i.i.d. random positions drawn on $\Omega$ with respect to probability measure $(\mu(\Omega))^{-1} d\mu$, and denote the size of $\Gamma_{\Omega}$ by $N$. Then for $n + 1 \geq \frac{\ln(1/\varepsilon) - \ln \|K\|_{S,\theta}}{\ln 2}$, the following reconstruction error estimates

$$\|g_n - f\|_p \leq 8 \|K\|_{S,\theta} \varepsilon \|f\|_p, \ f \in V_{p,\Omega,\varepsilon},$$

hold with probability at least

$$1 - \frac{10^d (2 \|K\|_{S,\theta}^2)^{d/\theta} \mu(\Omega)}{c^d D_1(\mu)} \left( 1 - \frac{c^d D_1(\mu)}{10^d (2 \|K\|_{S,\theta}^2)^{d/\theta} \mu(\Omega)} \right)^N.$$
Consider i.i.d. random samples $f(\gamma), \gamma \in \Gamma \subset \Omega$, of concentrated signals $f$ corrupted by i.i.d. random noises, $\xi(\gamma)$, and define the iterative algorithm

$$\tilde{g}_0 = \sum_{\gamma \in \Gamma \Omega} \mu(I_\gamma)(f(\gamma) + \xi(\gamma))K(\cdot, \gamma) \in V_p,$$

and $g_n \in V_p, n \geq 1$, inductively by

$$\tilde{g}_n = \tilde{g}_0 + \tilde{g}_{n-1} - S_\Gamma \tilde{g}_{n-1}, \ n \geq 1,$$

where the preconstruction operator $S_\Gamma$ on $L^p$ is given by

$$S_\Gamma g(x) = \sum_{\gamma \in \Gamma} \mu(I_\gamma)(Tg)(\gamma)K(x, \gamma)$$

with $\Gamma = \Gamma \Omega \cup \Gamma \Omega^c$, and $\Gamma \Omega^c$ being a discrete sampling set of the complement $\Omega^c$ of the domain $\Omega$ satisfying

$$d_H(\Gamma \Omega^c, \Omega^c) \leq \min \left( \varepsilon^{1/\theta} \|K\|_{S,\theta}^{-1/\theta}, (2\|K\|^2_{S,\theta})^{-1/\theta} \right).$$
Theorem

Suppose that $\tau \in (0, 1/2)$ and $\xi(\gamma), \gamma \in \Gamma_{\Omega}$, are i.i.d. random variables with mean zero and variance $\sigma^2$, i.e., $E(\xi(\gamma)) = 0$ and $\text{Var}(\xi(\gamma)) = \sigma^2$, $\gamma \in \Gamma_{\Omega}$. Let $f \in V_{p, \Omega, \varepsilon}$ and set

$$\tilde{\delta}_1 = \min \left( (2\|K\|^2_{S, \theta})^{-1/\theta}, \left( \frac{\tau \varepsilon^2 \sigma^{-2} \|f\|_p^2}{D_2(\mu)(D_1(\mu))^{2/p-1}} \right)^{1/d} \right).$$

If the size $N$ of the random sampling set $\Gamma_{\Omega}$ satisfies

$$N \geq \frac{10^d \mu(\Omega)}{c^d D_1(\mu) \tilde{\delta}_1^d} \ln \frac{10^d \mu(\Omega)}{c^d D_1(\mu) \tau \tilde{\delta}_1^d},$$

then for any integer $n + 1 \geq \frac{\ln(1/\varepsilon) - \ln \|K\|_{S, \theta}}{\ln 2}$,

$$\|\tilde{g}_n - f\|_\infty \leq 10(D_1(\mu))^{-1/p} \|K\|^2_{S, \theta \varepsilon} \|f\|_p$$

hold with probability at least $1 - 2\tau$. 
6. Numerical Simulations

- In our simulations, we consider the following family of signals

\[ f_{L,\alpha} = \sum_{i=-L}^{L} r_i (1 + |i|)^{-\alpha} \exp(- (x - i - \theta_i)^2) \]  

concentrated on the interval \( \Omega_L = [-L, L] \), where \( L \geq 1, \alpha \geq 0 \), and random variables \( r_i, -L \leq i \leq L \), are independently selected in \([-1, 1] \setminus (-1/2, 1/2)\) with uniform distribution, and \( \theta_i \in [-1/10, 1/10], i \in \mathbb{Z} \), are randomly selected.

- The sampling data of a concentrated signal \( f \) on the sampling set \( \Gamma_{N,L} = \{ \gamma_k \subset [-L, L], 1 \leq k \leq N \} \) being corrupted by i.i.d random noises \( \xi(\gamma_k) \in [-\delta, \delta], \gamma_k \in \Gamma_{N,L} \), with uniform distribution,

\[ \tilde{f}_{\gamma_k} = f(\gamma_k) + \xi(\gamma_k), \quad \gamma_k \in \Gamma_{N,L}. \]
Figure: Plotted on the left is noisy sampling data with $N = 2L^2 = 5000$ and $\alpha = 0$, and the right is the difference $\tilde{g}_{N,L,\alpha}^{(n)} - f_{L,\alpha}$ between the reconstructed signal $\tilde{g}_{N,L,\alpha}^{(n)}$ at the sixth iteration ($n = 6$) from noisy sampling data and the original signal $f_{L,\alpha}$. The relative approximation error $\left\| g_{N,L,\alpha}^{(n)} - f_{L,\alpha} \right\|_2 / \left\| f_{L,\alpha} \right\|_2$ and the concentration ratio $\left\| f_{L,\alpha} \right\|_2, \Omega_L^c / \left\| f_{L,\alpha} \right\|_2$ are 0.1036 and 0.0886.
Figure: Plotted on the left is the noisy sampling data with $N = 2L^2 = 5000$ and $\alpha = 0.8$. Shown on the right is the difference $\tilde{g}_{N,L,\alpha}^{(n)} - f_{L,\alpha}$ between the reconstructed signal $\tilde{g}_{N,L,\alpha}^{(n)}$ at the sixth iteration ($n = 6$) from noisy sampling data in the middle figures and the original signal $f_{L,\alpha}$. The relative approximation error $\|g_{N,L,\alpha}^{(n)} - f_{L,\alpha}\|_2/\|f_{L,\alpha}\|_2$ and the concentration ratio $\|f_{L,\alpha}\|_{2,\Omega^c}/\|f_{L,\alpha}\|_2$ are 0.0948 and 0.0145.
7. SUMMARY

- concentrated signals (nonconvex) vs signals with regularity, sparsity or residing in linear space
- random vs deterministic sampling
- Exact reconstruction vs approximation
- random sampling on networks and feature learning from sampling data?
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