The energy identity of Sacks-Uhlenbeck operator and infinitely many solutions for Brezis-Nirenberg problem

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Abstract: Let $\Omega$ be a bounded smooth domain in $\mathbb{R}^N$ with $N \geq 3$, $1 < \alpha$, $2^* = \frac{2N}{N-2}$ and $\{u_\alpha\} \subset H^{1,2\alpha}_0(\Omega)$ be a critical point of the functional

$$I_{\alpha,\lambda}(u) = \frac{1}{2\alpha} \int_\Omega [(1 + |\nabla u|^2)^\alpha - 1] dx - \frac{\lambda}{2} \int_\Omega u^2 dx - \frac{1}{2^{*\alpha}} \int_\Omega |u|^{2^*} dx.$$

In this paper, we obtain the limit behaviour of $u_\alpha$ ($\alpha \to 1$), energy identity, Pohozaev identity, some integral estimates, etc. And using these results, we prove infinitely many solutions for the following Brezis-Nirenberg problem for $N \geq 7$:

$$\begin{cases} -\Delta u = |u|^{2^*-2}u + \lambda u \quad \text{in } \Omega, \\ u = 0, \quad \text{on } \partial \Omega. \end{cases}$$

Keywords: Critical growth, Energy identity, Infinitely many solutions, Brezis-Nirenberg problem.

1 Introduction

We know that the energy of harmonic maps does not satisfy the Palais-Smale condition (see [16, 17, 18, 25]). So, from the viewpoint of calculus of variation, it is difficult to show the existence of harmonic maps from a surface. In order to obtain harmonic maps, Sacks and Uhlenbeck in [25] introduced the so called $\alpha$-energy $E_\alpha$ instead of $L^2$ energy $E$ as the following

$$E_\alpha = \frac{1}{2\alpha} \sum [(1 + |\nabla u|^2)^\alpha - 1] dV_g,$$

where $\alpha > 1$, $(\Sigma, g)$ is a Riemann surface, $(N, h)$ is an $n$-dimensional smooth compact Riemannian manifold which is embedded in $\mathbb{R}^k$ and $u$ is a map between $\Sigma$ and $N$. Using $\alpha$-energy $E_\alpha$ Sacks and Uhlenbeck proved that there is a sequence such that $u_\alpha$ converges to a harmonic map $u_1$ outside a finite set of points $X$, as $\alpha \to 1$. And the energy identity of a sequence of $u_\alpha$ was considered in [16, 17, 18].

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Motivated by the ideas of Sacks and Uhlenbeck, we consider following boundary value problem
\[\begin{align*}
-\text{div}((1 + |\nabla u|^2)^{\alpha-1}\nabla u) &= |u|^{2^*-2}u + \lambda u & \text{in } \Omega \\
u &= 0 & \text{on } \partial \Omega,
\end{align*}\]
where \(\alpha > 1\), \(2^* = \frac{2N}{N-2}\) and \(\Omega\) is a bounded smooth domain in \(\mathbb{R}^N\). We call the nondegenerate operator \(-\text{div}((1 + |\nabla \cdot |^2)^{\alpha-1}\nabla \cdot)\) Sacks-Uhlenbeck operator. The energy functional of problem \((1.1)\) is
\[I_{\alpha,\lambda}(u) = \frac{1}{2\alpha} \int_{\Omega} [(1 + |\nabla u|^2)^\alpha - 1] dx - \frac{\lambda}{2} \int_{\Omega} u^2 dx - \frac{1}{2^*} \int_{\Omega} |u|^{2^*} dx, \quad u \in H_0^{1,2\alpha}(\Omega)\]
and limit problem of \((1.1)\) is the well-known Brezis-Nirenberg problem
\[\begin{align*}
-\Delta u &= |u|^{2^*-2}u + \lambda u & \text{in } \Omega \\
u &= 0 & \text{on } \partial \Omega.
\end{align*}\]
Since the embedding \(H_0^{1,2\alpha}(\Omega) \hookrightarrow L^{2^*}(\Omega)\) is compact, we easily show that the functional \(I_{\alpha,\lambda}(u)\) satisfies the Palais-Smale condition, then by symmetric Mountain Pass theorem (see [22]), the functional \(I_{\alpha,\lambda}(u)\) has infinitely many critical points \(u_{\alpha,k}, k = 1, 2, \ldots\).

Our first result is the following energy identity:

**Theorem 1.1.** Let \(\{u_\alpha\}\) be a critical point of the functional \(I_\alpha\), and \(u_\alpha \rightharpoonup u\) in \(H_0^{1}(\Omega)\) as \(\alpha \to 1\). Then there exists a nonnegative integer \(k\), \(x_{\alpha,j} \in \Omega\), \(R_{\alpha,j} \in \mathbb{R}_+\), \(U_j \in D^{1,2}(\mathbb{R}^N)\) such that
\[\left\| u_\alpha - u - \sum_{j=1}^{k} (R_{\alpha,j})^{\frac{N-2}{2}} U_j (R_{\alpha,j}(x - x_{\alpha,j})) \right\| = o(1), \quad (1.3)\]
\[I_{\alpha,\lambda}(u_\alpha) = I_{\alpha,\lambda}(u) + \sum_{j=1}^{k} I_{1,0}(U_j, \mathbb{R}^N) + o(1), \quad (1.4)\]
\[R_{\alpha,j}\text{dist}(x_{\alpha,j}, \partial \Omega) \to +\infty, \quad (1.5)\]
where \(U_j\) is a positive solution of
\[\frac{N-2}{2} \int_{\mathbb{R}^N} |\nabla u|^2 dx - \frac{1}{2^*} \int_{\mathbb{R}^N} |u|^{2^*} dx. \quad (1.7)\]

As an application, we use Theorem \((1.1)\) to prove the existence of infinitely many solutions for the well-known Brezis-Nirenberg problem:

**Theorem 1.2.** Suppose that \(\lambda > 0\), \(N \geq 7\). Then \((1.2)\) has infinitely many solutions.
directly to obtain the existence of solutions for (1.2). The pioneering paper on problem (1.7) was by Brézis-Nirenberg [1] in 1983 where the authors showed that for $N \geq 4$ and $\lambda \in (0, \lambda_1)$ problem (1.7) has at least one positive solution, where $\lambda_1$ denotes the principal eigenvalue of $-\Delta$ on $\Omega$. The same conclusion was proved in [1] for $N = 3$ when $\Omega$ is a ball and $\lambda \in (\frac{N}{4}, \lambda_1)$. In this case, by using the Pohozaev identity, equation (1.7) has no radial solution when $\lambda \in (0, \frac{N}{4})$. Note that, using the Pohozaev identity, (1.7) has no nontrivial solution when $\lambda \leq 0$ and $\Omega$ is star-shaped.

Since 1983, there has been a considerable number of papers on problem (1.2). Let us now briefly enumerate the multiplicity results obtained to date as follows:

1. Cerami et al. in [5] proved that the number of solutions of (1.2) is bounded below by the number of eigenvalues of $(-\Delta, \Omega)$ lying in the open interval $(\lambda, \lambda + S|\Omega|^{-\frac{2}{N}})$, where $S$ is the best constant for the Sobolev embedding $D^{1,2}(\mathbb{R}^N) \to L^2(\mathbb{R}^N)$ and $|\Omega|$ is the Lebesgue measure of $\Omega$.

2. If $N \geq 4$ and $\Omega$ is a ball, then for any $\lambda > 0$, infinitely many sign-changing solutions which were built using particular symmetries of the domain $\Omega$ were obtained by Fortunato and Jannelli (see [10]).

3. In Cerami et al. [6] it was proved for $N \geq 6$, that (1.2) has two pairs of solutions on any smooth bounded domain.

4. Using Pohozaev identity and the global compact result Devillanova and Solimini [8] showed that, if $N \geq 7$, problem (1.2) has infinitely many solutions for each $\lambda > 0$. For low dimensions, that is, $N = 4, 5, 6$, in [9], Devillanova and Solimini proved the existence of at least $N + 1$ pairs of solutions provided $\lambda$ is small enough. In [7], Clapp and Weth extended this last result to all $\lambda > 0$.

5. Schecter and Zou [23] showed that in any bounded and smooth domain, for $N \geq 7$ and for each fixed $\lambda > 0$, problem (1.2) has infinitely many sign changing solutions

Using the methods of [8], Cao et al. in [3] obtained infinitely many solutions for the semilinear elliptic equations involving Hardy potentials and critical Sobolev exponents for $N \geq 7$. The result of the existence of the infinitely many solutions was also extended to $p$-Laplacian equation ($1 < p < \infty$) with critical growth for $N > p^2 + p$ in [4].

In [8], the well known global compactness result which gives a complete description of the noncompact (P.S.)c sequence for all energy levels $c$ of the functional was used to obtain the proof of the existence of the infinitely many solutions for (1.2). The global compactness result was firstly obtained for Brezis-Nirenberg problem by M. Struwe [24]. For $p$-Laplacian case, C. Mercuri and M. Willem [20] obtained the global compactness result for all $1 < p < N$. And the result was proved in [2] for singular elliptic problem. When $\alpha \to 1$, the solution $u_\alpha$ of problem (1.1) converges weakly to $u$, and $u$ is a solution of (1.2). Our Theorem 1.1 describes the limit behaviour of $u_\alpha$, and is similar to the global compactness result.

Note that Theorem 1.2 is similar to the result which was obtained by Devillanova and Solimini [8]. In [8], they firstly considered the following approximation problem

\[
\begin{cases}
-\Delta u = |u|^{2^*-2-\varepsilon} u + \lambda u, & \text{in } \Omega, \\
u = 0, & \text{on } \partial \Omega,
\end{cases}
\]

where $\varepsilon > 0$, and then, they set up the global compactness result, Pohozaev identity, some integral estimates for the approximation problem. So they used these results to prove infinitely many solutions for Brezis-Nirenberg problem for $N \geq 7$. In this work,
our problem (1.1) can be regarded as the approximation problem of Brezis-Nirenberg problem. So using the methods of [8], we can also obtain infinitely many solutions for Brezis-Nirenberg problem for $N \geq 7$. Since the two approximation problems are different, the two sets of infinitely many solutions may also be different. In a forthcoming paper, we will prove the conclusion.

Let $P(t) = \frac{1}{2^{\alpha}}((1 + t^2)^{\alpha} - 1)$, then $p(t) = P'(t) = (1 + t^2)^{\alpha-1}t$ and $P(t)$ and $p(t)$ satisfy the following inequalities

\begin{align}
(1) & : 2 \leq \frac{tp(t)}{P(t)} \leq 2\alpha, \text{ for } t > 0, \\
(2) & : t^{2\alpha}P(u) < P(tu) \leq t^2P(u), \text{ if } 0 \leq t \leq 1, \\
(3) & : t^2P(u) \leq P(tu) \leq t^{2\alpha}P(u), \text{ if } t \geq 1.
\end{align}

Since the operator $-\text{div}((1 + |\nabla \cdot |^2)^{\alpha-1}\nabla \cdot)$ is inhomogeneous, (1.9) is used to overcome the nonhomogeneous difficulty.

## 2 Proof of Theorem 1.1

The Hilbert space $D^{1,2}(\mathbb{R}^N)$ is the completion of the space $C^\infty_0(\mathbb{R}^N)$ with respect to the norm

$$
\|u\|_{D^{1,2}(\mathbb{R}^N)} = \left\{ \int_{\mathbb{R}^N} |\nabla u|^2 dx \right\}^{\frac{1}{2}}.
$$

The well-known Sobolev inequalities state that for all $u \in D^{1,2}(\mathbb{R}^N)$,

$$
\left( \int_{\mathbb{R}^N} |u(x)|^{\frac{2N}{N-2}} dx \right)^{\frac{N-2}{N}} \leq C \left( \int_{\mathbb{R}^N} |\nabla u(x)|^2 dx \right)^{\frac{1}{2}},
$$

where $C$ depends only on $N$.

Set

$$
S_0 = \inf \left\{ \frac{\int_{\mathbb{R}^N} |\nabla u(x)|^2 dx}{(\int_{\mathbb{R}^N} |u(x)|^{2^*})^{\frac{N-2}{2}}} : w \in D^{1,2}(\mathbb{R}^N) \right\}. \tag{2.3}
$$

It is well-known that $S_0$ is attained by the extremal functions

$$
U_\varepsilon(x) = \left( \frac{\varepsilon}{x^2 + \varepsilon^2} \right)^{\frac{N-2}{2}}, \tag{2.4}
$$

where $\varepsilon > 0$ is arbitrary.

Now using the proof of the concentration-compactness principle in Orlicz space in [12], we can prove the following lemma:
Lemma 2.1. Let \( \{u_\alpha\} \) be a bounded sequence in \( \mathcal{D}^{1,2}(\mathbb{R}^N) \). We may assume \( u_\alpha \) converges a.e. \( u \in \mathcal{D}^{1,2}(\mathbb{R}^N) \), \( (1 + |\nabla u_\alpha|^2)^{\alpha-1} |\nabla u_\alpha|^2 \), \( u_\alpha^2 \) converges weakly to some bounded, nonnegative measures \( \mu, \nu \) on \( \mathbb{R}^N \), as \( \alpha \to 1 \).

(1): Then we have for some at most countable family \( J \), for some families \( \{x_j\}_{j \in J} \) for distinct points in \( \mathbb{R}^N \), \( \{\nu_j\}_{j \in J} \) in \( (0, \infty) \)

\[
\nu = |u|^2^* + \sum_{j \in J} \nu_j \delta_{x_j}, \tag{2.5}
\]

\[
\mu \geq |\nabla u|^2 + \sum_{j \in J} S_0 \nu_j^2 \delta_{x_j}. \tag{2.6}
\]

(2): If \( \mu \equiv 0 \) and \( \mu(\mathbb{R}^N) \leq S_0 \nu(\mathbb{R}^N)^{2^*/2} \), then \( J = \{x_0\} \) for some \( x_0 \in \mathbb{R}^N \) and \( \nu = c_0 \delta_{x_0} \), \( \mu = S_0 c_0^{-2/2} \delta_{x_0} \) for some \( c_0 > 0 \).

Lemma 2.2. Let \( \{u_\alpha\} \) be a critical point of the functional \( I_\alpha \), and \( u_\alpha \rightharpoonup u \) in \( H_0^1(\Omega) \). Then there exist at most finitely many points \( x_1, x_2, \ldots, x_l \in \Omega \) such that

\[
u_j > 0 \text{ such that } |u_\alpha|^2^* \rightharpoonup \nu = |u|^2^*(x, 0) + \sum_{j \in J} \nu_j \delta_{x_j}, \tag{2.8}
\]

\[
(1 + |\nabla u_\alpha|^2)^{\alpha-1} |\nabla u_\alpha|^2 \rightharpoonup \mu \geq |\nabla u|^2 + \sum_{j \in J} S_0 \nu_j^2 \delta_{x_j}. \tag{2.9}
\]

Now we show that \( X := \{x_1, x_2, \ldots\} \) is a finite set. We choose \( \psi = \phi u_\alpha \) in \( \langle I_\alpha'(u_\alpha), \psi \rangle = o(1) \|\psi\| \), and let \( m \to +\infty \). Then by (2.7), we have

\[
\int \phi d\nu = \int \phi d\mu - \int T_1 D_\psi u dx + \int |u|^2^* \phi dx.
\]

Let \( \phi \) concentrate on \( x_i \), then \( \mu(\{x_i\}) = \nu(\{x_i\}) \). Moreover, according to the relation \( \mu(\{x_j\}) \geq S_0 (\nu(\{x_j\}))^{2/2}, \) if \( \nu_j = \nu(\{x_j\}) > 0 \), then \( \nu_j \geq S_0^{-N/2} \). Note that \( \nu(\Omega) < \infty \).

This implies that \( X \) is a finite set.

Choose a function \( \varphi \in C_0^\infty(\Omega) \), \( \varphi \geq 0, \varphi \equiv 0 \) on \( \partial \Omega \), and \( \varphi(x_j) = 0, \forall x_j \in X \). We get

\[
\int |\varphi u_\alpha|^2^* dx \to \int |\varphi u|^2^* dx + \sum_j \nu_j \varphi^2^*(x_j) = \int |\varphi u|^2^* dx.
\]
So, \( \varphi_{u_\alpha} \to \varphi u \) in \( L^2(\Omega) \). Furthermore, we easily obtain

\[
\int _\Omega \varphi (1 + |\nabla u_\alpha|^2)^{\alpha - 1} \nabla u_\alpha \nabla (u_\alpha - u) dx - \int _\Omega \varphi \nabla u \nabla (u_\alpha - u) dx = \int _\Omega (|u_\alpha|^{2^* - 1} - |u|^{2^* - 1})(u_\alpha - u) \varphi dx + \lambda \int _\Omega \varphi (u_\alpha - u)^2 dx
\]

\[
- \int _\Omega (1 + |\nabla u_\alpha|^2)^{\alpha - 1} \nabla u_\alpha \nabla \varphi (u_\alpha - u) dx - \int _\Omega \nabla u \nabla \varphi (u_\alpha - u) dx + o(1) \to 0 \text{ as } n \to \infty.
\]

We easily know that

\[
\int _\Omega \varphi (1 + |\nabla u_\alpha|^2)^{\alpha - 1} \nabla u_\alpha \nabla (u_\alpha - u) dx - \int _\Omega \varphi \nabla u \nabla (u_\alpha - u) dx \sim \int _\Omega \varphi (|\nabla u_\alpha|^2 - |\nabla u|^2) dx
\]

So, we have

\[
\int _\Omega \varphi (|\nabla u_\alpha|^2 - |\nabla u|^2) dx \sim \int _\Omega (|u_\alpha|^{2^* - 1} - |u|^{2^* - 1})(u_\alpha - u) \varphi dx + \lambda \int _\Omega \varphi (u_\alpha - u)^2 dx
\]

\[
- \int _\Omega (1 + |\nabla u_\alpha|^2)^{\alpha - 1} \nabla u \nabla \varphi (u_\alpha - u) dx - \int _\Omega \nabla u \nabla \varphi (u_\alpha - u) dx + o(1) \to 0 \text{ as } n \to \infty.
\]

This implies that

\[
u_\alpha \to u \text{ in } H^1_{0,\text{loc}}(\Omega \setminus \{x_1, x_2, \cdots, x_l\}).\]

\[
\square
\]

**Lemma 2.3.** Suppose that \( u_\alpha \in H^1_0(\Omega) \) satisfy \( I_{\alpha,0}'(u_\alpha) \to 0 \) and \( u_\alpha \rightharpoonup 0 \) in \( H^1_0(\Omega) \). Then there exist three sequences \( \{x_\alpha\} \subset \Omega, R_\alpha \to \infty \) and \( w_\alpha \in H^1_0(\Omega) \) such that

\[
w_\alpha(x) = u_\alpha - R^{(N-2)/2}_\alpha U_0(R_\alpha(x - x_\alpha)) + o(1),
\]

\[
|I_{\alpha,0}(u_\alpha) - I_{\alpha,0}(w_\alpha) - I_{\alpha,0}(U_0, \mathbb{R}^N)| \to 0,
\]

\[
I_{\alpha,0}'(w_\alpha) \to 0,
\]

\[
R_\alpha \text{dist}(x_\alpha, \partial \Omega) \to +\infty,
\]

where \( o(1) \to 0 \) in \( \mathcal{D}^{1,2}(\mathbb{R}^{N+1}) \) and \( U_0 \) is a solution of

\[
- \Delta u = u^{2^* - 1}, \text{ in } \mathbb{R}^N.
\]

**Proof.** We assume that

\[
|u_\alpha|^{2^*} \to \sum \nu_j \delta_{x_j}, x_j \in \bar{\Omega}.
\]

Then there exists at least one \( \nu_j \neq 0 \). Otherwise, \( u_\alpha \to 0 \) in \( L^{2^*}(\Omega) \). And by \( I_{\alpha,0}'(u_\alpha) \to 0 \), we have \( u_\alpha \to 0 \) in \( H^1_0(\Omega) \).

Set
\[ Q_\alpha(r) = \sup_{x \in \Omega} \left\{ \int_{B_r(x)} |\nabla u_\alpha|^2 \, dx + \int_{B_r(x)} |u_\alpha|^{2^*} \, dx \right\}. \] (2.14)

For sufficiently small \( \tau \in (0, S^{N/2}_0) \), choose \( R_\alpha = R_\alpha(\tau) > 0, x_\alpha \in \bar{\Omega} \) such that
\[
\left\{ \int_{B_{1/R_\alpha}(x_\alpha)} |\nabla u_\alpha|^2 \, dx + \int_{B_{1/R_\alpha}(x_\alpha)} |u_\alpha|^{2^*} \, dx \right\} = Q_\alpha \left( \frac{1}{R_\alpha} \right) = \tau. \] (2.15)

Set
\[
\tilde{u}_\alpha(x) = R^{(2-N)/2}_\alpha u_\alpha \left( \frac{x}{R_\alpha} + x_\alpha \right), x \in \Omega_\alpha = \left\{ x : \frac{x}{R_\alpha} + x_\alpha \in \Omega \right\}. \] (2.16)

Then, we obtain
\[
\tilde{Q}_\alpha(r) = \sup_{x \in \mathbb{R}^N} \left\{ \int_{B_r(x)} |\nabla \tilde{u}_\alpha|^2 \, dx + \int_{B_r(x)} |\tilde{u}_\alpha|^{2^*} \, dx \right\}
= \sup_{x \in \mathbb{R}^N} \left\{ \int_{B_{1/R_\alpha}(x)} |\nabla u_\alpha|^2 \, dx + \int_{B_{1/R_\alpha}(x)} |u_\alpha|^{2^*} \, dx \right\}
= Q_\alpha \left( \frac{r}{R_\alpha} \right). \] (2.17)

Hence,
\[
\left\{ \int_{B_{1/R_\alpha}(x_\alpha)} |\nabla u_\alpha|^2 \, dx + \int_{B_{1/R_\alpha}(x_\alpha)} |u_\alpha|^{2^*} \, dx \right\} = \tilde{Q}_\alpha(1) = \tau. \] (2.18)

Now we will show that there exists a sufficiently small \( \tau \in (0, S^{N/2}_0) \) such that \( R_\alpha(\tau) \to +\infty \), as \( m \to +\infty \), if not, for each \( \varepsilon > 0 \), there is a constant \( M_\varepsilon > 0 \) such that \( R_\alpha(\varepsilon) \leq M_\varepsilon \).

So,
\[
\int_{B_{1/M_\varepsilon}(x)} |u_\alpha|^{2^*} \, dx \leq \sup_{x \in \Omega} \left\{ \int_{B_{1/R_\alpha}(x_\alpha)} |\nabla u_\alpha|^2 \, dx + \int_{B_{1/R_\alpha}(x_\alpha)} |u_\alpha|^{2^*} \, dx \right\}
= Q_\alpha \left( \frac{1}{R_\alpha} \right) = \varepsilon, \forall x \in \bar{\Omega}. \] (2.19)

Futhermore,
\[
\nu_j \leq \int_{B_{1/M_\varepsilon}(x)} |u_\alpha|^{2^*} \, dx + o(1) \leq \varepsilon + o(1), \forall \varepsilon > 0.
\]

Contradiction!

Now we distinguish two case:

(i) \( R_\alpha \text{dist}(x_\alpha, \partial \Omega) \to +\infty \), in this case \( \Omega_\alpha \to \Omega^\infty = \mathbb{R}^N \).
(ii) $R_a \operatorname{dist}(x_a, \partial \Omega) \to M < +\infty$, uniformly. Then after an orthogonal transformation,

$$\Omega_\alpha \to \Omega^\infty = \mathbb{R}^N_+ = \{ x = (x_1, \cdots, x_N), x_1 > 0 \}.$$ 

If $(x, y) \notin \Omega_\alpha$, we define $\tilde{u}_\alpha = 0$. Since

$$\int_{\mathbb{R}^N} |\nabla \tilde{u}_\alpha|^2 dx = \int_{\Omega_\alpha} |\nabla \tilde{u}_\alpha|^2 dx = \int_{\Omega} |\nabla u_\alpha|^2 dx,$$

we can assume that $\tilde{u}_\alpha \rightharpoonup U_0$ in $\mathcal{D}^{1,2}(\mathbb{R}^N_+)$. 

Since in each case for any $\varphi \in C^\infty(\Omega^\infty)$, we have that $\varphi \in C^\infty(\tilde{\Omega}_\alpha)$, for large $m$, there holds

$$\int_{\mathbb{R}^N} \nabla \tilde{u}_\alpha \nabla \varphi dx - \int_{\mathbb{R}^N} |\tilde{u}_\alpha|^{2^* - 1} \varphi dx$$

$$= \int_{\mathbb{R}^N} \nabla u_\alpha \nabla \varphi^*_\alpha dx - \int_{\mathbb{R}^N} |u_\alpha|^{2^* - 1} \varphi^*_\alpha dx$$

$$= \int_{\mathbb{R}^N} (1 + |\nabla \tilde{u}_\alpha|^2)^{\alpha - 1} \nabla u_\alpha \nabla \varphi^*_\alpha dx - \int_{\mathbb{R}^N} |u_\alpha|^{2^* - 1} \varphi^*_\alpha dx + o(1)$$

$$= o(1) \| \varphi^*_\alpha \|_{H^1(\Omega)} = o(1) \| \varphi \|_{\mathcal{D}^{1,2}(\mathbb{R}^N)}, \quad (2.20)$$

where $\varphi^*_m(x) = R_a^{(2-p)/2}(\varphi(R_a(x - x_a))) \in H^1_0(\Omega)$. We are going to prove that

$$\tilde{u}_\alpha \to U_0 \quad \text{in} \quad \mathcal{D}^{1,2}_{\text{loc}}(\mathbb{R}^N). \quad (2.21)$$

To do this, by (2.20), we next only need to show that

$$\tilde{u}_\alpha \to U_0 \quad \text{in} \quad L^{2^*}_{\text{loc}}(\mathbb{R}^N).$$

Indeed, choose a smooth cut-off function $\eta \in H^1_0(\Omega)$ such that $0 \leq \eta \leq 1$, $\eta = 1$ in $B_1$, $\eta = 0$ outside $B_2$, where, $B_\rho = \{(x) : |x| \leq \rho \}$. Assume that

$$|\eta \tilde{u}_\alpha|^2 \to \nu = |\eta U_0|^2 \chi(x, 0) + \sum_{j \in J} \nu_j \delta_{x_j}, \quad (2.22)$$

$$|\nabla(\eta \tilde{u}_\alpha)|^2 \to \mu \geq |\nabla(\eta U_0)|^2 + \sum_{j \in J} S_0 \nu_j^2 \delta_{x_j}. \quad (2.23)$$

Similar to Lemma 2.1, it is easy to show that if $\nu_j = \nu(\{x_j\}) > 0$, then $\nu_j \geq S_0^{N/2}$, where $x_j \in B_k$. Moreover

$$S_0^{N/2} > \tau = \tilde{Q}_a(1) \geq \int_{B_1(x_j)} |\nabla \tilde{u}_\alpha|^2 dx \geq \mu(\{x_j\}) \geq S_0^{N/2}.$$ 

Contradiction. So, $\nu_j = \nu(\{x_j\}) = 0$, and (2.21) holds.

Next we will show that case (ii) doesn’t occur. In fact, using (2.20) and (2.21), one
has
\[ - \Delta U_0 = U_0^{2^* - 1}, \text{ in } \mathbb{R}^N, \]  
(2.24)

Similar to the proof in Theorem 1.1 in [24], we can use Pohozaev identity and Strong maximum principle (see [21]) to show that \( U_0 = 0 \). This contradicts the following relation

\[
\int_{B_1(0)} |\nabla U_0|^2 dx + \int_{B_1(0)} |U_0|^{2^*} dx = \lim_{\alpha \to 1} \int_{B_1(0)} (1 + |\nabla \tilde{u}_\alpha|^2)^{\alpha - 1}|\nabla \tilde{u}_\alpha|^2 dx + \int_{B_1(0)} |\tilde{u}_\alpha|^2 dx = \tau > 0. \tag{2.25}
\]

Hence, \( R_\alpha \text{dist}(x_\alpha, \partial \Omega) \to +\infty \).

Let \( \alpha \in C(B_2) \) satisfying \( 0 \leq \alpha \leq 1, \alpha = 1 \) in \( B(0, 1) \), and \( \alpha = 0 \) outside \( B(0, 2) \). Set

\[ w_\alpha(x) = u_\alpha(x) - R_\alpha^{(N-2)/2} U_0(R_\alpha(x - x_\alpha)) \in H^1_{0,L}(C), \]

where the sequence \( R_\alpha \) is chosen such that \( R_\alpha \text{dist}(0, \partial \Omega) \to +\infty \) and \( \tilde{R}_\alpha := \frac{R_\alpha}{R_\alpha} \to +\infty \), then we have

\[ \tilde{w}_\alpha(x) = \tilde{u}_\alpha(x) - U_0(x) \alpha \left( \frac{x}{\tilde{R}_\alpha} \right). \]

Similar to [24], it is easy to show that

\[ w_\alpha(x) = u_\alpha - R_\alpha^{(N-2)/2} U_0(R_\alpha(x - x_\alpha)) + o(1), \]  
(2.26)

where \( o(1) \to 0 \) in \( D^{1,2}(\mathbb{R}^{N+1}) \).

Therefore, by (2.21), we get

\[
\int_{\Omega} [(1 + |\nabla u_\alpha|^2)^\alpha - 1] dx - \int_{\Omega} [(1 + |\nabla w_\alpha|^2)^\alpha - 1] dx \\
= \int_{\Omega} |\nabla u_\alpha|^2 dx - \int_{\Omega} |\nabla u_\alpha - R_\alpha^{(N-2)/2} U_0(R_\alpha(x - x_\alpha))|^2 dx + o(1) \\
= \int_{\mathbb{R}^N} |\nabla \tilde{u}_\alpha|^2 dx - \int_{\mathbb{R}^N} |\nabla \tilde{u}_\alpha - U_0|^2 dx + o(1) \\
= \int_{\mathbb{R}^N} \int_{0}^{1} 2(t \nabla \tilde{u}_\alpha + (1-t) \nabla (\tilde{u}_\alpha - U_0)) \nabla U_0 dt dx + o(1) \\
\to \int_{\mathbb{R}^N} \int_{0}^{1} 2|\nabla U_0|^2 dt dx = \int_{\mathbb{R}^N} |\nabla U_0|^2 dx. \tag{2.28}
\]
To proceed, observe that like (2.28), we have
\[
\int_{\Omega} |u_\alpha|^2 dx - \int_{\Omega} |w_\alpha|^2 dx \to \int_{\mathbb{R}^N} |u|^2 dx.
\]
So
\[
|I_{\alpha,0}(u_\alpha) - I_{\alpha,0}(w_\alpha) - I_{\alpha,0}(u, \mathbb{R}^N)| \to 0.
\]
On the other hand, according to (2.26) and (2.27), we easily infer
\[
|\langle I'_{\alpha,0}(u_\alpha), \varphi \rangle - \langle I'_{\alpha,0}(w_\alpha), \varphi \rangle| \leq C \left[ \left( \int_{\mathbb{R}^N} |\nabla \tilde{u}_\alpha - \nabla \tilde{w}_\alpha - \nabla U_0|^2 dx \right)^{\frac{1}{2}} + \left( \int_{\mathbb{R}^N} (|\tilde{u}_\alpha|^{2^* - 1} - |\tilde{w}_\alpha|^{2^* - 1} - |U_0|^{2^* - 1})^{2'} dx \right)^{\frac{1}{2'}} \right] \|\varphi\|_{D^{1,2}(\mathbb{R}^{N+1})} \to 0. \tag{2.29}
\]

Proof of Theorem 1.1. Applying Lemma 2.3 to the sequences,
\[
v_1^\alpha = u_\alpha - u,
\]
\[
v_j^\alpha = u_\alpha - u - \sum_{i=1}^{j-1} U_i^\alpha = v_{j-1}^\alpha - U_{j-1}^\alpha, j > 1,
\]
where \(U_i^\alpha(x) = (R_i^\alpha)_{\frac{N-2}{2}} U_i(x - x_n^i)\).

By induction
\[
I_{\alpha,0}(v_j^\alpha) = I_{\alpha,\lambda}(u_\alpha) - I_{\alpha,\lambda}(u) - \sum_{i=1}^{j-1} I_{\alpha,0}(U^j, \mathbb{R}^N),
\]
\[
\leq I_{\alpha,\lambda}(u_\alpha) - (j - 1)\beta^*,
\]
where \(\beta^* = \frac{1}{2N} S_N^N\).

Since the latter will be negative for large \(j\), the iteration must stop after finite steps; moreover, for this index we have
\[
v_{m+1}^k = u_\alpha - u - \sum_{j=1}^{k} U_j^\alpha \to 0 \ \text{in}
\]
and
\[
I_{\alpha,\lambda}(u_\alpha) - I_{\alpha,\lambda}(u) - \sum_{j=1}^{k} I_{\alpha,0}(U_j, \mathbb{R}^N) \to 0.
\]
The proof is complete. \[\square\]
3 Proof of Theorem 1.2

In this section, we assume that $1 < \alpha < \frac{N}{N-2}$. This section will be divided into three subsections. In subsection 3.1, we will give some integral estimates. In subsection, we will prove some estimates on safe regions. Subsection 3.3 is devoted to proving Theorem 1.2.

3.1 Some integral estimates

For any $p_2 < 2^{*} < p_1$, $\beta > 0$ and $R \geq 1$, let us consider the following relation:

$$
\begin{align*}
\|u_1\|_{p_1} &\leq \beta, \\
\|u_2\|_{p_2} &\leq \beta R^{\frac{N}{2p_2} - \frac{N}{2p_1}}.
\end{align*}
$$

(3.1)

Based the idea of [8], we define

$$
\|u\|_{p_1,p_2,R} = \inf \{ \beta : \text{there are } u_1 \text{ and } u_2, \text{such that(3.1) holds and } |u| \leq u_1 + u_2 \}.
$$

In this subsection, our main result is the following proposition.

**Proposition 3.1.** Let $u_\alpha$ be a weak solution of (1.1) with $\alpha \to 1$. For any $p_1, p_2$ satisfying $(1 - \frac{1}{2\alpha}) < p_2 < 2^{*} < p_1$, there exists a constant $C$, depending on $p_1$ and $p_2$, such that

$$
\|u_\alpha\|_{p_1,p_2,R,\alpha} \leq C.
$$

**Lemma 3.2.** Assume that $\Omega_1$ is a bounded domain satisfying $\Omega \subset \Omega_1$. For any functions $f_1(x) \geq 0$ and $f_2(x) \geq 0$, let $w \geq 0$ be the solution of

$$
\begin{align*}
- \text{div}((1 + |\nabla w|^2)^{\alpha-1}\nabla w) = f_1(x) + f_2(x) &\text{ in } \Omega_1, \\
w = 0 &\text{ on } \partial \Omega_1.
\end{align*}
$$

(3.2)

Let $w_i, i = 1, 2$, be the solution of

$$
\begin{align*}
- \text{div}((1 + |\nabla w_i|^2)^{\alpha-1}\nabla w_i) = f_i(x) &\text{ in } \Omega_1, \\
w_i = 0 &\text{ on } \partial \Omega_1.
\end{align*}
$$

(3.3)

Then, there is a constant $C > 0$, depending only on $r = \frac{1}{3}\text{dist}(\Omega, \partial \Omega_1)$, such that

$$
w(x) \leq C \inf_{y \in B_r(x_0)} w(y) + C w_1(x) + C w_1(x), \text{ for all } x \in \Omega.
$$

(3.4)

**Proof.** Let $r = \frac{1}{3}\text{dist}(\Omega, \partial \Omega_1)$. Then it follows from [15, Theorem 2] that for any $x_0 \in \Omega$,

$$
w(x_0) \leq C_2 \inf_{x \in B_r(x_0)} w(x) + C_3 W_{1,2\alpha}(x_0, 2r, f_1 + f_2),
$$

where $W_{1,2\alpha}(x_0, r, f)$ is the Wolff potential for the function $f$:

$$
W_{1,2\alpha}(x_0, r, f) = \int_0^r \left( \int_{B_t(x_0)} |f| \right)^\frac{1}{2\alpha-1} \frac{dt}{t^{\frac{N}{2\alpha-1}+1}}.
$$
On the other hand, it is easy to show that
\[ W_{1,2\alpha}(x_0, 2r, f_1 + f_2) \leq 2^{\frac{1}{2\alpha - 1}} W_{1,2\alpha}(x_0, 2r, f_1) + 2^{\frac{1}{2\alpha - 1}} W_{1,2\alpha}(x_0, 2r, f_2). \]
So, using [15, Theorem 2] again, we obtain
\[
w(x_0) \leq C_2 \inf_{x \in B_r(x_0)} w(x) + C_3 W_{1,2\alpha}(x_0, 2r, f_1) + C_3 W_{1,2\alpha}(x_0, 2r, f_2) \\
\leq C_2 \inf_{x \in B_r(x_0)} w(x) + C_4 w_1(x_0) + C_5 w_1(x_0).
\] (3.5)

Lemma 3.3. Let \( w \) be the solution of
\[
\begin{cases}
- \text{div}((1 + |\nabla w|^2)^{\alpha - 1} \nabla w) = a(x) v^{2\alpha - 1} & \text{in } \Omega, \\
w = 0 & \text{on } \partial \Omega,
\end{cases}
\] (3.6)
where \( a(x) \geq 0 \) and \( v \geq 0 \) are functions satisfying \( a, v \in L^\infty(\Omega) \). Then, for any \( p_1 > 2^* > p_2 > (1 - \frac{1}{2\alpha}) 2^* \), there is a constant \( C = C(p_1, p_2, |\Omega|) \), such that for any \( R \geq 1 \),
\[
\| w \|_{p_1, p_2, R} \leq C \| a \|_{\frac{2\alpha - 1}{N}} \| v \|_{p_1, p_2, R}.
\]

Proof. For any small \( \theta > 0 \), let \( v_1 \geq 0 \) and \( v_2 \geq 0 \) be the functions such that \( v \leq v_1 + v_2 \), and (3.1) holds with \( \alpha = \| v \|_{p_1, p_2, R} + \theta \). Choose a domain \( \Omega_1 \) with \( \Omega \subset \Omega_1 \). We let \( a(x) = 0 \) and let \( v_1 = 0 \) in \( \Omega_1 \setminus \Omega \). Consider
\[
\begin{cases}
- \text{div}((1 + |\nabla w_i|^2)^{\alpha - 1} \nabla w_i) = a(x) v_i^{2\alpha - 1} & \text{in } \Omega_1, \\
w_i = 0 & \text{on } \partial \Omega_1.
\end{cases}
\] (3.7)
Then, it follows from Corollary A that
\[
\| w_i \|_{p_i} \leq C \| a \|_{\frac{2\alpha - 1}{N}} \| v \|_{p_i}, \quad i = 1, 2.
\]
On the other hand, by the comparison principle, we can deduce \( w \leq \tilde{w} \) in \( \Omega \), where \( \tilde{w} \) is the solution of
\[
\begin{cases}
- \text{div}((1 + |\nabla \tilde{w}|^2)^{\alpha - 1} \nabla \tilde{w}) = 2^{2\alpha - 1} a(x) (v_1^{2\alpha - 1} + v_2^{2\alpha - 1}), & \text{in } \Omega, \\
\tilde{w} = 0, & \text{on } \partial \Omega.
\end{cases}
\] (3.8)
It follows from Corollary A again that
\[
\| \tilde{w} \|_{p_2} \leq C \| a \|_{\frac{2\alpha - 1}{N}} \| (v_1^{2\alpha - 1} + v_2^{2\alpha - 1}) \|_{p_2}^{\frac{1}{2\alpha - 1}} \\
\leq C \| a \|_{\frac{2\alpha - 1}{N}} \| (\| v_1 \|_{p_2} + \| v_2 \|_{p_2}) \\
\leq C \| a \|_{\frac{2\alpha - 1}{N}} \| (\| v_1 \|_{p_1} + \| v_2 \|_{p_2}).
\] (3.9)
As before, let \( r = \frac{1}{3} \operatorname{dist}(\Omega, \partial \Omega_1) \). So, we obtain
\[
\sup_{x \in B_r(x_0)} \tilde{w}(x) \leq Cr^{-N} \int_{B_r(x_0)} \tilde{w} dx \leq Cr^{-\frac{N}{p_2}} \| \tilde{w} \|_{p_2}
\]
\[
= C\| a \|_{\frac{N}{2N-\alpha}}^{-1} (\| v_1 \|_{p_1} + \| v_2 \|_{p_2}) \quad \text{for all } x_0 \in \Omega. \tag{3.10}
\]

On the other hand, it follows from Lemma 3.2 that there is \( C_1 > 0 \), such that
\[
\tilde{w}(x_0) \leq C \inf_{x \in B_r(x_0)} \tilde{w}(x) + Cw_1(x_0) + Cw_2(x_0)
\]
\[
\leq C\| a \|_{\frac{N}{2N-\alpha}}^{-1} (\| v_1 \|_{p_1} + \| v_2 \|_{p_2}) + Cw_1(x_0) + Cw_2(x_0). \tag{3.11}
\]
Combining \( w(x) \leq \tilde{w}(x) \), for all \( x \in \Omega \), and (3.11), we obtain
\[
w(x) \leq \tilde{w}_1(x) + \tilde{w}_2(x) \quad \text{for all } x \in \Omega, \tag{3.12}
\]
where
\[
\tilde{w}_1(x) = C\| a \|_{\frac{N}{2N-\alpha}}^{-1} (\| v_1 \|_{p_1} + \| v_2 \|_{p_2}) + C_1 w_1(x),
\]
and \( \tilde{w}_2(x) = C_1 w_2(x) \).

Noting that \( R^{\frac{N}{2N-\alpha}} \leq 1 \) as \( R \geq 1 \), by (3.9), we deduce
\[
\| \tilde{w}_1 \|_{p_1} \leq C\| a \|_{\frac{N}{2N-\alpha}}^{-1} (\| v_1 \|_{p_1} + \| v_2 \|_{p_2}) + C\| w_1 \|_{p_1}
\]
\[
\leq C'\| a \|_{\frac{N}{2N-\alpha}}^{-1} (\| v_1 \|_{p_1} + \| v_2 \|_{p_2})
\]
\[
\leq C\| a \|_{\frac{N}{2N-\alpha}}^{-1} (\| v_1 \|_{p_1, p_2, R + \theta}), \tag{3.13}
\]
and
\[
\| \tilde{w}_2 \|_{p_2} \leq C\| a \|_{\frac{N}{2N-\alpha}}^{-1} \| v_2 \|_{p_2} \leq C'\lambda^{\frac{N}{2N-\alpha}} \| a \|_{\frac{N}{2N-\alpha}}^{-1} (\| v_1 \|_{p_1, p_2, R + \theta}). \tag{3.14}
\]

So, the result follows. \( \square \)

**Lemma 3.4.** Let \( w \) be a weak solution of
\[
\left\{ \begin{array}{ll}
- \operatorname{div}(1 + |\nabla v|^{2})^{\alpha-1} \nabla v = 2v^{2r-1} + A & \text{in } \Omega, \\
 w = 0, & \text{on } \partial \Omega, \tag{3.15}
\end{array} \right.
\]
where \( v \geq 0 \) and \( v \in L^\infty(\Omega) \). For any \( p_1, p_2 \) with \( 2^* - 1 < p_2 < 2^* < p_1 < \frac{N(2^*-1)}{2\alpha} \), let
\[
\frac{1}{q_i} = \frac{2^* - 1}{(2\alpha - 1)p_i} - \frac{2\alpha}{N(2\alpha - 1)}, i = 1, 2. \tag{3.16}
\]
Then there is a constant \( C = C(p_1, p_2) \), such that for any \( R > 0 \),
\[
\| w \|_{q_1, q_2, R} \leq C\| v \|^{\frac{2^*-1}{p_1, p_2, R}} + C.
\]

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Proof. For any small $\theta > 0$, let $v_1 \geq 0$ and $v_2 \geq 0$ be the functions such that $v \leq v_1 + v_2$, and (3.1) holds with $\alpha = \|v\|_{p_1, p_2, R} + \theta$. Let $\hat{w}$ be the solution of

$$
\begin{align*}
- \text{div}((1 + |\nabla \hat{w}|^2)^{\alpha - 1} \nabla \hat{w}) &= 2^{2^*} v_1^{2^*-1} + 2^{2^*} v_2^{2^*-1} + A \quad \text{in } \Omega_1, \\
\hat{w} &= 0, \quad \text{on } \partial \Omega_1.
\end{align*}
$$

(3.17)

Then $w \leq \hat{w}$, It follows from Proposition A that there is a $\tilde{p} > 0$, such that $\|\hat{w}\|_{\tilde{p}} \leq C$. Thus,

$$
\inf_{x \in B_r(x_0)} \hat{w}(x) \leq C, \quad \text{for all } x_0 \in \Omega
$$

Now we consider

$$
\begin{align*}
- \text{div}((1 + |\nabla w_1|^2)^{\alpha - 1} \nabla w_1) &= 2^{2^*} v_1^{2^*-1} + A \quad \text{in } \Omega_1, \\
w_1 &= 0, \quad \text{on } \partial \Omega_1,
\end{align*}
$$

(3.18)

and

$$
\begin{align*}
- \text{div}((1 + |\nabla w_2|^2)^{\alpha - 1} \nabla w_2) &= 2^{2^*} v_2^{2^*-1} \quad \text{in } \Omega_1, \\
w_2 &= 0, \quad \text{on } \partial \Omega_1.
\end{align*}
$$

(3.19)

Then, by Lemma 3.2,

$$
\hat{w}(x_0) \leq C + C w_1(x_0) + C w_2(x_0), \quad \text{for all } x_0 \in \Omega.
$$

Let $\hat{p}_i = \frac{p_i}{2^*-1}$, then $q_i = \frac{N(2^*-1)\hat{p}_i}{N-2^*\hat{p}_i}$, $i = 1, 2$. Besides, for $p_i \in \left(2^* - 1, \frac{N(2^*-1)}{2^*}\right)$, we have $\hat{p}_i \in \left(p_i, \frac{N}{2^*}\right)$.

By Proposition A, we have

$$
\|C + C w_1\|_{q_1} \leq C' + C'\|w_1\|_{q_1} \leq C' + C'\|v_1^{2^*-1} + A\|_{\frac{\alpha}{\hat{p}_1}} \leq C \left(1 + \|v_1\|_{\frac{2^*-1}{\hat{p}_1}}^{\frac{2^*-1}{2^*-1}}\right),
$$

(3.20)

and

$$
\|w_2\|_{q_2} \leq C\|v_2\|_{\frac{2^*-1}{p_2}}^{\frac{2^*-1}{p_2}} \leq C \left(R_\Omega^{\frac{N}{2^*}} \frac{\alpha}{\frac{N}{2^*}} (\theta + \|v_2\|_{p_1, p_2, R})\right)^{\frac{2^*-1}{2^*-1}},
$$

(3.21)

since

$$
\left(\frac{N}{2^*} \frac{2^* - 1}{p_2} - \frac{N}{q_2}\right) = N \left(\frac{2\alpha}{N(2\alpha - 1)} - \frac{2^* - 1}{p_2(2\alpha - 1) + \frac{1}{q_2}}\right) = 0.
$$

So, the result follows. \qed

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Let $u_\alpha$ be a solution of (1.1), and $w_\alpha$ be the solution of
\[
\begin{cases}
- \text{div}((1 + |\nabla w|^2)^{\alpha-1}\nabla w) = 2u_\alpha^{2\alpha-1} + A & \text{in } \Omega, \\
w = 0, & \text{on } \partial \Omega,
\end{cases}
\tag{3.22}
\]
where $A > 0$ is a large constant. By the comparison principle, we have
\[
|u_\alpha(x)| \leq w_\alpha(x), \quad x \in \Omega.
\tag{3.23}
\]

**Lemma 3.5.** Let $w_\alpha(x)$ be a solution of (3.22). Then there exist constants $C > 0$, and $p_1, p_2 \in ((1 - \frac{1}{2\alpha})2^*, +\infty)$ with $p_2 < 2^* < p_1$, such that
\[
\|w_\alpha\|_{p_1, p_2, \alpha} \leq C.
\]

**Proof.** From Theorem 1.1, we have
\[
u_\alpha = u + k \sum_{j=1}^{k} \rho_{x_{\alpha,j}, \alpha}(U_j) + \omega_\alpha,
\tag{3.24}
\]
where $\rho_{x_{\alpha,j}, \alpha}(U_j) = (R_{\alpha}^\frac{N-2}{2}) U_j (R_{\alpha}^\frac{N-2}{2}(x-x_{\alpha,j}))$, $u_0$ is a $C^{1,\alpha}$ function, and $||\omega_\alpha|| \to 0$ as $\alpha \to 1$. Let $\tilde{w}_\alpha$ be a solution of
\[
\begin{cases}
- \text{div}((1 + |\nabla \tilde{w}_\alpha|^2)^{\alpha-1}\nabla \tilde{w}_\alpha) = C \left(|u|^{2^*-2\alpha} + \sum_{j=1}^{k} |\rho_{x_{\alpha,j}, \alpha}(U_j)|^{2^*-2\alpha} + |\omega_\alpha|^{2^*-2\alpha}\right) w_\alpha^{2\alpha-1} + A & \text{in } \Omega_1, \\
\tilde{w}_\alpha = 0, & \text{on } \partial \Omega_1,
\end{cases}
\tag{3.25}
\]
where $\Omega_1$ is a bounded domain in $\mathbb{R}^N$ satisfying $\Omega \subset \Omega_1$, $C > 0$ is a fixed large constant. Then, by the comparison principle,
\[
\tilde{w}_\alpha(x) \leq w_\alpha(x) \quad x \in \Omega.
\]
Moreover, it is easy to check that
\[
\int_{\Omega_1} |\tilde{w}_\alpha(x)|^{2^*} \, dx \leq C.
\tag{3.26}
\]

Let $w = Gv$ be the solution of
\[
\begin{cases}
- \text{div}((1 + |\nabla w|^2)^{\alpha-1}\nabla w) = v & \text{in } \Omega_1, \\
w = 0, & \text{on } \partial \Omega_1.
\end{cases}
\tag{3.27}
\]

Then, it follows from Lemma 3.2 and (3.26) that
\[
\tilde{w}_\alpha \leq C + G \left(|u|^{2^*-2\alpha} w_\alpha^{2\alpha-1} + A\right) + \sum_{j=1}^{k} G \left(|\rho_{x_{\alpha,j}, \alpha}(U_j)|^{2^*-2\alpha} w_\alpha^{2\alpha-1}\right) + G \left(|\omega_\alpha|^{2^*-2\alpha} w_\alpha^{2\alpha-1}\right).
\]

Firstly, we treat the term $G \left(|u|^{2^*-2\alpha} w_\alpha^{2\alpha-1} + A\right)$. Take any $q \in \left(\frac{2\alpha N}{2\alpha N - N + 2\alpha}, \frac{2^*}{2\alpha-1}\right)$.
Then

\[ p_1 = \frac{Nq(2\alpha - 1)}{N - 2\alpha q} > (2\alpha)^* > 2^*. \]

It follows from Proposition A that

\[ \|G\left( |u|^2 - 2\alpha w_\alpha^{2\alpha - 1} + A \right) \|_{p_1} \leq C \| |u|^2 - 2\alpha w_\alpha^{2\alpha - 1} + A \|_{\frac{1}{q(2\alpha - 1)}}^{\frac{1}{q}} \]

\[ \leq C + C \left( \int_\Omega |u|^2 - 2\alpha w_\alpha^{2\alpha - 1} |^q dx \right)^{\frac{1}{q}} \]

\[ \leq C + C \left( \int_\Omega |w_\alpha|^{(2\alpha - 1)q} dx \right)^{\frac{1}{q}} \leq C. \quad (3.28) \]

Next, we treat the term \( G \left( |\rho_{x,j,R_{a,j}}(U_j)|^{2 - 2\alpha w_\alpha^{2\alpha - 1}} \right) \). Let \( p_2 \in \left( \frac{(2\alpha - 1)N}{N - 2\alpha}, 2^* \right) \) be a constant. By Corollary B, we obtain

\[ \|G \left( |\rho_{x,j,R_{a,j}}(U_j)|^{2 - 2\alpha w_\alpha^{2\alpha - 1}} \right) \|_{p_2} \leq C \| |\rho_{x,j,R_{a,j}}(U_j)|^{2 - 2\alpha} \|_{\frac{1}{p_2}}^{\frac{1}{p_2 - 1}} \| w_\alpha \|_{2^*} \]

\[ \leq C \| |\rho_{x,j,R_{a,j}}(U_j)|^{2 - 2\alpha} \|_{\frac{1}{p_2}}^{\frac{1}{p_2 - 1}}, \quad (3.29) \]

where \( r \) is determined by

\[ \frac{1}{r} = \frac{2\alpha - 1}{p_2} + \frac{2\alpha}{N} - \frac{2\alpha - 1}{2^*}. \]

But

\[ \int_\Omega |\rho_{x,j,R_{a,j}}(U_j)|^{(2 - 2\alpha)r} dx = (R_{a,j})^{-N + (N - \alpha N + 2\alpha)r} \int_{\Omega_{x,j,R_{a,j}}} |U_j|^{2N - 2\alpha N + 4\alpha} dx, \]

where \( \Omega_{x,\lambda} = \{ y : x + \lambda^{-1} y \in \Omega \} \).

For \( j = 1, \cdots, k \), using the Kelvin transformation \( v(x) = |x|^{2-N} U \left( \frac{x}{|x|^2} \right) \), we have

\[ |U_j(x)| \leq \frac{C}{|x|^{N-2}}, \quad (3.30) \]

So for any \( r > \frac{N}{4\alpha - (2\alpha - 2)N} \), we obtain

\[ \int_{\Omega_{x,j,R_{a,j}}} |U_j|^{\frac{2N - 2\alpha N + 4\alpha}{N - 2}} dx \leq C, \; j = 1, \cdots, k. \]

Note that \( \frac{N}{2\alpha} > \frac{N}{4\alpha - (2\alpha - 2)N} \) and \( r \rightarrow \frac{N}{2\alpha} \) as \( p_2 \rightarrow 2^* \). So we such that the corresponding \( r > \frac{N}{4\alpha - (2\alpha - 2)N} \). Thus, we have proved that there is a \( p_2 < 2^* \), such that

\[ \|G \left( |\rho_{x,j,R_{a,j}}(U_j)|^{2 - 2\alpha w_\alpha^{2\alpha - 1}} \right) \|_{p_2} \leq C(R_{a,j})^{\left( \frac{N}{2\alpha} + (N - \alpha N + 2\alpha) \right)\frac{1}{p_2 - 1}} = C(R_{a,j})^{\frac{N}{2\alpha} + \frac{N - \alpha N}{p_2 - 1}}, \quad (3.31) \]
Finally, we treat the term $G \left( |\omega_\alpha|^{2^* - 2\alpha} w_\alpha^{2\alpha - 1} \right)$. It follows from Lemma 3.3 that

$$
\|G \left( |\omega_\alpha|^{2^* - 2\alpha} w_\alpha^{2\alpha - 1} \right)\|_{p_1, p_2, R_\alpha} \leq C \|\omega_\alpha|^{2^* - 2\alpha} \|_{p_2, R_\alpha}^{1 - \frac{2\alpha}{2^*}} \|w_\alpha\|_{p_1, p_2, R_\alpha} \\
\leq \frac{1}{2} \|w_n\|_{p_1, p_2, R_\alpha} + C. \quad (3.32)
$$

Since $\|\omega_\alpha|^{2^* - 2\alpha}\|_{\frac{2^*}{2\alpha}} \to 0$ as $\alpha \to 1$.

Combining (3.29), (3.31) and (3.32), we obtain

$$
\|w_\alpha\|_{p_1, p_2, R_\alpha} \\
\leq C + \|G \left( |u|^{2^* - 2\alpha} w_\alpha^{2\alpha - 1} + A \right)\|_{p_1, p_2, R_\alpha} + \sum_{j=1}^{k} \|G \left( |\rho_{x_{\alpha,j}, R_{\alpha,j}}(U_j)|^{2^* - 2\alpha} w_\alpha^{2\alpha - 1} \right)\|_{p_1, p_2, R_\alpha} \\
+ \|G \left( |\omega_\alpha|^{2^* - 2\alpha} w_\alpha^{2\alpha - 1} \right)\|_{p_1, p_2, R_\alpha}. \\
\leq C + \|G \left( |u|^{2^* - 2\alpha} w_\alpha^{2\alpha - 1} + A \right)\|_{p_1} + C \sum_{j=1}^{k} \|G \left( |\rho_{x_{\alpha,j}, R_{\alpha,j}}(U_j)|^{2^* - 2\alpha} w_\alpha^{2\alpha - 1} \right)\|_{p_2} (R_{\alpha,j})^{\frac{N}{2^*} - \frac{2\alpha}{2^*} + \frac{N/2 - \alpha}{2\alpha - 1}} \\
+ C\|G \left( |\omega_\alpha|^{2^* - 2\alpha} w_\alpha^{2\alpha - 1} \right)\|_{p_1, p_2, R_\alpha} \\
\leq \frac{1}{2} \|w_\alpha\|_{p_1, p_2, R_\alpha} + C. \quad (3.33)
$$

So the result follows.

\[\square\]

**Proof of Proposition 3.1.** The constants $q_1$ and $q_2$ defined in Lemma 3.4 satisfy $q_1 > 2^*$ and $q_1 \to +\infty$ as $p_1 \to \frac{N}{2\alpha} (2^* - 1)$, while $q_2 < 2^*$ and $q_2 \to (1 - \frac{1}{2\alpha}) 2^*$ as $p_2 \to 2^* - 1$.

Using (3.23), we just need to show the result for $w_n$.

Since $w_n$ satisfies (3.22), we can use Lemmas 3.4 and 3.5 to prove that

$$
\|w_\alpha\|_{p_1, p_2, R_\alpha} \leq C
$$

holds for any $p_1, p_2$ with $p_1 > 2^* > p_2 > \left(1 - \frac{1}{2\alpha}\right) 2^*$.

\[\square\]

### 3.2 Estimates on safe regions

Noting that the number of the bubbles of $u_n$ is finite and using Theorem 1.1, we can always find a constant $C > 0$, independent of $n$, such that the region

$$
\mathcal{A}_\alpha^1 = \left( B_{(C+5)R_\alpha^\frac{1}{2}}(x_\alpha) \setminus B_{CR_\alpha^\frac{1}{2}}(x_\alpha) \right) \cap \Omega
$$

does not contain any concentration point of $u_\alpha$ for any $\alpha$. We call this region a safe region for $u_\alpha$.

Let

$$
\mathcal{A}_\alpha^2 = \left( B_{(C+4)R_\alpha^\frac{1}{2}}(x_\alpha) \setminus B_{(C+1)R_\alpha^\frac{1}{2}}(x_\alpha) \right) \cap \Omega.
$$

In this section, we will prove the following result.
Proposition 3.6. Let $u_\alpha$ be a weak solution of (1.1) with $\alpha \to 1$. Then there exists a constant $C > 0$, independent of $\alpha$, such that

$$\int_{A^2} |u_\alpha(x)| dx \leq C R_\alpha^{\frac{2N}{N+2}}$$

for all $x \in A^2_\alpha$.

To prove Proposition 3.6, we need the following lemma.

Lemma 3.7. There is a constant $C > 0$, independent of $\alpha$, such that

$$\frac{1}{r^N} \int_{\partial B_r(y) \cap \Omega} |u_\alpha| dx \leq C$$

for all $r \geq CR_\alpha^{-\frac{1}{2}}$.

Proof. Let $\tilde{w}_\alpha$ be the solution of

$$\begin{cases}
-\Delta \tilde{w}_\alpha = 2|u_\alpha|^{2^* - 1} + A & \text{in } \Omega_1, \\
\tilde{w}_\alpha = 0 & \text{on } \partial \Omega_1,
\end{cases} \quad (3.34)$$

and we know that $\tilde{w}_\alpha > 0$. Then we have

$$0 \leq \text{div}((1 + |\nabla u_\alpha|^2)^{\alpha-1}\nabla u_\alpha) - \Delta \tilde{w}_\alpha$$

$$= \text{div}((1 + |\nabla u_\alpha|^2)^{\alpha-1}\nabla u_\alpha) - \text{div}((1 + |\nabla \tilde{w}_\alpha|^2)^{\alpha-1}\nabla \tilde{w}_\alpha)$$

$$+ \text{div}((1 + |\nabla \tilde{w}_\alpha|^2)^{\alpha-1}\nabla \tilde{w}_\alpha) - \Delta \tilde{w}_\alpha,$$

$$= \text{div}((1 + |\nabla u_\alpha|^2)^{\alpha-1}\nabla u_\alpha) - \text{div}((1 + |\nabla \tilde{w}_\alpha|^2)^{\alpha-1}\nabla \tilde{w}_\alpha)$$

$$+ \text{div}((1 + |\nabla \tilde{w}_\alpha|^2)^{\alpha-1} - 1)\nabla \tilde{w}_\alpha). \quad (3.35)$$

If $\text{div}((1 + |\nabla \tilde{w}_\alpha|^2)^{\alpha-1} - 1)\nabla \tilde{w}_\alpha) \leq 0$, using (3.35) and the comparison principle of uniform elliptic operator, we have $|u_\alpha| \leq \tilde{w}_\alpha$ in $\Omega$.

If $\text{div}((1 + |\nabla \tilde{w}_\alpha|^2)^{\alpha-1} - 1)\nabla \tilde{w}_\alpha) > 0$, using the by the comparison principle in [11], we have $\tilde{w}_\alpha < 0$, which is a contradiction.

Hence, we have $|u_\alpha| \leq \tilde{w}_\alpha$ in $\Omega$.

Now we have the following formula

$$\frac{1}{r^{N-1}} \int_{\partial B_r(y)} \tilde{w}_\alpha dS = \frac{1}{r_0^{N-1}} \int_{\partial B_{r_0}(y)} \tilde{w}_\alpha dS + \int_r^{r_0} \frac{1}{t^{N-1}} \int_{B_t(y)} (2|u_\alpha|^{2^* - 1} + A) dx dt. \quad (3.36)$$

By continuity, since $\{u_\alpha\}$ is bounded in $L^{2^*} \subset L^1$, we can suppose

$$\int_{B_1(y)} u_\alpha dx \leq C$$

with a constat $C$ independent of $\alpha$. So, there is $r_0 \in [\frac{1}{2}, 1]$, such that

$$\frac{1}{r_0^{N-1}} \int_{\partial B_{r_0}(y)} u_\alpha dS \leq C.$$
We now estimate
\[ \int_{r}^{r_0} \frac{1}{t^{N-1}} \int_{B_t(y)} |u_\alpha|^{2^*-1}dx \, dt, \]
for all \( r \geq \tilde{C}R_\alpha^{-\frac{1}{2}} \).

By Proposition 3.1, we know that \( \|u_\alpha\|_{p_1,p_2,R_\alpha} \leq C \) for any \( p_1, p_2 \) satisfying \( p_1 > 2^* > p_2 > (1 - \frac{1}{2\alpha})2^* \).

Let \( p_2 = 2^* - 1 \), and let \( p_1 > 2^* \). Then we can choose \( v_{1,\alpha} \) and \( v_{2,\alpha} \), such that
\[ |u_\alpha| \leq v_{1,\alpha} + v_{2,\alpha}, \]
and
\[ \|v_{1,\alpha}\|_{p_1} \leq C, \quad \|v_{2,\alpha}\|_{p_2} \leq CR_\alpha^{\frac{N}{p_2}}, \]

If we choose \( p_1 > 2^* \) large enough, then
\[ \int_{r}^{r_0} \frac{1}{t^{N-1}} \int_{B_t(y)} v_{1,\alpha}^{2^*-1}dx \, dt \leq \int_{r}^{r_0} \frac{1}{t^{N-1}} \left( \int_{B_t(y)} v_{1,\alpha}^{p_1}dx \right)^{\frac{2^*}{p_1}} t^{N\left(1 - \frac{2^*}{p_1}\right)}dt, \]
\[ \leq C \int_{r}^{r_0} \frac{1}{t^{N-1}} t^{N\left(1 - \frac{2^*}{p_1}\right)}dt \leq C, \quad (3.37) \]
and
\[ \int_{r}^{r_0} \frac{1}{t^{N-1}} \int_{B_t(y)} v_{2,\alpha}^{2^*-1}dx \, dt \leq C \int_{r}^{r_0} \frac{1}{t^{N-1}} R_\alpha^{(2^*-1)(\frac{N}{2^*} - \frac{N}{2})}dt, \]
\[ \leq C \frac{1}{r^{N-2}} R_\alpha^{\frac{N-2}{2}} \leq C, \quad (3.38) \]
since \( r \geq \tilde{C}R_\alpha^{-\frac{1}{2}} \).

Combining (3.37) and (3.38), we obtain
\[ \int_{r}^{r_0} \left( \frac{1}{t^{N-2}} \int_{B_t(y)} |u_\alpha|^{2^*-1}dx \right) \frac{dt}{t} \leq C \int_{r}^{r_0} \frac{1}{t^{N-1}} \int_{B_t(y)} v_{1,\alpha}^{2^*-1}dx \, dt \]
\[ + C \int_{r}^{r_0} \frac{1}{t^{N-1}} \int_{B_t(y)} v_{2,\alpha}^{2^*-1}dx \, dt \leq C. \quad (3.39) \]

The conclusion for \( r \geq r_0 \) is obvious and thus we complete our proof.

Now we are ready to prove Proposition 3.6.
Proof of Proposition 3.6. It follows from Lemma 3.7 that for any \( y \in A_\alpha^2 \), we have
\[
\frac{1}{r^N} \int_{\partial B_r(y) \cap \Omega} |u_\alpha| dx \leq C, \quad \text{for all } y \in \Omega,
\]
for all \( r \geq \tilde{C} R_\alpha^{-\frac{1}{2}} \). Let
\[
v_\alpha(x) = u_\alpha \left( R_\alpha^{-\frac{1}{2}} x \right), \quad x \in \Omega_\alpha,
\]
where \( \Omega_\alpha = \{ x : R_\alpha^{-\frac{1}{2}} x \in \Omega \} \).

Then from (1.9), \( v_\alpha \) satisfies
\[
\int_{\Omega_\alpha} |\nabla v_\alpha|^2 dx \leq \int_{\Omega_\alpha} \left( (1 + |\nabla v_\alpha|^2)^{\alpha-1} |\nabla v_\alpha|^2 \right) dx
\leq R_\alpha^{\frac{N}{2}} \int_{\Omega} \left( (1 + |R_\alpha^{-\frac{1}{2}} \nabla u_\alpha|^2)^{\alpha-1} |R_\alpha^{-\frac{1}{2}} \nabla u_\alpha|^2 \right) dx
\leq R_\alpha^{\frac{N}{2}} R_\alpha^{-\frac{1}{2}} \int_{\Omega} \left( (1 + |\nabla u_\alpha|^2)^{\alpha-1} |\nabla u_\alpha|^2 \right) dx
\leq CR_\alpha^{\frac{N}{2}} R_\alpha^{-1} \int_{\Omega} \left( 2 |v_\alpha|^{2^{*}-1} + Au_\alpha^2 \right) dx
\leq CR_\alpha^{-1} \int_{\Omega_\alpha} \left( 2 |v_\alpha|^{2^{*}-1} + A |v_\alpha| \right) |v_\alpha| dx. \tag{3.40}
\]

Let \( z = R_\alpha^{\frac{1}{2}} y \). Since \( B_{R_\alpha^{-\frac{1}{2}}}(y), \ y \in A_\alpha^2 \) does not contain any concentration point of \( u_\alpha \), we can deduce that
\[
R_\alpha^{-1} \int_{B_1(z)} \left( 2 |v_\alpha|^{2^{*}-1} + A |v_\alpha| \right)^{\frac{N}{2}} dx \leq \int_{B_{R_\alpha^{-\frac{1}{2}}}(y)} |u_\alpha(x)|^{2^{*}} dx + CR_\alpha^{-\frac{N}{2}} \to 0, \quad \alpha \to 1. \tag{3.41}
\]

Since \( v_\alpha \) satisfies (3.40), by Moser iteration, we obtain
\[
\|v_\alpha\|_{L^{\infty}(B_{R_\alpha^{-\frac{1}{2}}}(z))} \leq C \int_{B_1(z)} |v_\alpha| dx + C
= CR_\alpha^{\frac{N}{2}} \int_{B_{R_\alpha^{-\frac{1}{2}}}(y)} |u_\alpha| dx + C \leq C. \tag{3.42}
\]

As a result,
\[
\int_{B_{R_\alpha^{-\frac{1}{2}}}(y)} |u_\alpha| dx \leq CR_\alpha^{-\frac{N}{2}}, \quad \text{for all } y \in A_\alpha^2. \tag{3.43}
\]
Let
\[ A_3^\alpha = \left( B_{(C+3)R_\alpha^{-\frac{1}{2}}(x_\alpha)} \setminus B_{(C+2)R_\alpha^{-\frac{1}{2}}(x_\alpha)} \right) \cap \Omega. \]

**Proposition 3.8.**

\[ \int_{A_3^\alpha} |\nabla u_\alpha|^2 dx \leq C \int_{A_2^\alpha} (|u_\alpha|^{2^*} + 1) dx + CR_\alpha \int_{A_2^\alpha} |u_\alpha|^2 dx. \] \hspace{1cm} (3.44)

In particular,
\[ \int_{A_3^\alpha} |\nabla u_\alpha|^{2\alpha} dx \leq CR_\alpha^{\frac{2-N}{2}}. \] \hspace{1cm} (3.45)

**Proof.** Let \( \phi_\alpha \in C_0^\infty(A_3^\alpha) \) be a function with \( \phi_\alpha = 1 \) in \( A_3^\alpha \), \( 0 \leq \phi_\alpha \leq 1 \) and \( |\nabla \phi_\alpha| \leq CR_\alpha^{\frac{1}{2}} \).

From
\[ \int_\Omega (1 + |\nabla u_\alpha|^{2\alpha}) (\nabla u_\alpha \nabla (\phi_\alpha^2 u_\alpha)) dx \leq C \int_\Omega (2|u_\alpha|^{2^*} + A) \phi_\alpha^2 |u_\alpha|^2 dx, \] \hspace{1cm} (3.46)

we can prove (3.46).

From (3.46) and Proposition (3.6), we get
\[ \int_{A_3^\alpha} |\nabla u_\alpha|^{2\alpha} dx \leq CR_\alpha^{\frac{2-N}{2}} + CR_\alpha^{\frac{2-N}{2}} \leq CR_\alpha^{\frac{2-N}{2}}. \] \hspace{1cm} (3.47)

\[ \square \]

### 3.3 Proof of Theorem 1.2

Take a \( t_n \in [\tilde{C} + 2, \tilde{C} + 3] \), satisfying

\[ \int_{\partial B_{t_nR_\alpha^{-\frac{1}{2}}}^\alpha(y)} (R_{\alpha}^{-1}|u_\alpha|^{2^*} + |u_\alpha|^2 + R_{\alpha}^{-1}|\nabla u_\alpha|^{2\alpha}) dS \leq CR_\alpha^{\frac{1}{2}} \int_{A_3^\alpha} (R_{\alpha}^{-1}|u_\alpha|^{2^*} + |u_\alpha|^2 + R_{\alpha}^{-1}|\nabla u_\alpha|^{2\alpha}) dS. \] \hspace{1cm} (3.48)

Using Proposition 3.6, (3.45) and (3.48), we obtain
\[ \int_{\partial B_{t_nR_\alpha^{-\frac{1}{2}}}^\alpha(y)} (R_{\alpha}^{-1}|u_\alpha|^{2^*} + |u_\alpha|^2 + R_{\alpha}^{-1}|\nabla u_\alpha|^{2\alpha}) dS \leq CR_\alpha^{\frac{1}{2} - \frac{N}{2}}. \] \hspace{1cm} (3.49)

**Proof of Theorem 1.2.** We have two different cases:

(i) \( B_{t_nR_\alpha^{-\frac{1}{2}}(x_\alpha)} \cap (\mathbb{R}^N \setminus \Omega) \neq \emptyset; \)

(ii) \( B_{t_nR_\alpha^{-\frac{1}{2}}(x_\alpha)} \subset \Omega. \)
We have the following local Pohozaev identity for $u_\alpha$ on $B_\alpha = B_{t_\alpha R_\alpha^{-\frac{1}{2}}}(x_\alpha) \cap \Omega$:

$$\lambda \int_{B_\alpha} u_\alpha^2 dx + \frac{N}{2} \int_{B_\alpha} |\nabla u_\alpha|^2 (1 + |\nabla u_\alpha|^{a-1}) dx - \frac{N}{2\alpha} \int_{B_\alpha} [(1 + |\nabla u_\alpha|^2)^{\alpha/2} - 1] dx$$

$$= \frac{1}{2^*} \int_{\partial B_\alpha} |u_\alpha|^2 \nu \cdot dS + \frac{\lambda}{2} \int_{\partial B_\alpha} |u_\alpha|^2 (x - x_0) \cdot \nu dS$$

$$- \frac{1}{2\alpha} \int_{\partial B_\alpha} [(1 + |\nabla u_\alpha|^2)^{\alpha/2} - 1] (x - x_0) \cdot \nu dS + \frac{N}{2^*} \int_{\partial B_\alpha} (1 + |\nabla u_\alpha|^2)^{\alpha-1} (\nabla u_\alpha \cdot \nu) u_\alpha dS$$

$$+ \int_{\partial B_\alpha} (1 + |\nabla u_\alpha|^2)^{\alpha-1} (\nabla u_\alpha \cdot (x - x_0)) \cdot (\nabla u_\alpha \cdot \nu) dS, \quad (3.50)$$

where $\nu$ is the outward normal to $\partial B_\alpha$. The point $x_0$ in (3.50) is chosen as follows. In case (i), we take $x_0 \in \mathbb{R}^N \setminus \Omega$ with $|x_0 - x_\alpha| \leq 2t_\alpha R_\alpha^{-\frac{1}{2}}$ and $\nu \cdot (x - x_0) \leq 0$ in $\partial \Omega \cap B_\alpha$. In case (ii), we take a point $x_0 = x_\alpha$.

By (1.9), we know that the first term in the left-hand side of (3.50) is non-negative. We thus obtain from (3.50) that

$$\lambda \int_{B_\alpha} u_\alpha^2 dx \leq \frac{1}{2^*} \int_{\partial B_\alpha} |u_\alpha|^2 \nu \cdot dS + \frac{\lambda}{2} \int_{\partial B_\alpha} |u_\alpha|^2 (x - x_0) \cdot \nu dS$$

$$- \frac{1}{2\alpha} \int_{\partial B_\alpha} [(1 + |\nabla u_\alpha|^2)^{\alpha/2} - 1] (x - x_0) \cdot \nu dS + \frac{N}{2^*} \int_{\partial B_\alpha} (1 + |\nabla u_\alpha|^2)^{\alpha-1} (\nabla u_\alpha \cdot \nu) u_\alpha dS$$

$$+ \int_{\partial B_\alpha} (1 + |\nabla u_\alpha|^2)^{\alpha-1} (\nabla u_\alpha \cdot (x - x_0)) \cdot (\nabla u_\alpha \cdot \nu) dS. \quad (3.51)$$

Now we decompose $\partial B_\alpha$ into

$$\partial B_\alpha = \partial_1 B_\alpha \cup \partial_\nu B_\alpha,$$

where $\partial_1 B_\alpha = \partial B_\alpha \cap \partial \Omega$ and $\partial_\nu B_\alpha = \partial B_\alpha \cap \partial \Omega$.

Noting $u_\alpha = 0$ on $\partial \Omega$, we find

$$\frac{1}{2^*} \int_{\partial_1 B_\alpha} |u_\alpha|^2 \nu \cdot dS + \frac{\lambda}{2} \int_{\partial_1 B_\alpha} |u_\alpha|^2 (x - x_0) \cdot \nu dS$$

$$- \frac{1}{2\alpha} \int_{\partial_1 B_\alpha} [(1 + |\nabla u_\alpha|^2)^{\alpha/2} - 1] (x - x_0) \cdot \nu dS + \frac{N}{2^*} \int_{\partial_1 B_\alpha} (1 + |\nabla u_\alpha|^2)^{\alpha-1} (\nabla u_\alpha \cdot \nu) u_\alpha dS$$

$$+ \int_{\partial_1 B_\alpha} (1 + |\nabla u_\alpha|^2)^{\alpha-1} (\nabla u_\alpha \cdot (x - x_0)) \cdot (\nabla u_\alpha \cdot \nu) dS$$

$$= -\frac{1}{2\alpha} \int_{\partial_\nu B_\alpha} [(1 + |\nabla u_\alpha|^2)^{\alpha/2} - 1] (x - x_0) \cdot \nu dS$$

$$+ \int_{\partial_\nu B_\alpha} (1 + |\nabla u_\alpha|^2)^{\alpha-1} (\nabla u_\alpha \cdot (x - x_0)) \cdot (\nabla u_\alpha \cdot \nu) dS \leq 0. \quad (3.52)$$
So, we can rewrite (3.51) as
\[
\lambda \int_{B_\alpha} u_\alpha^2 \, dx \leq \frac{1}{2} \int_{\partial B_\alpha} |u_\alpha|^2 \langle x - x_0 \rangle \cdot \nu \, dS + \lambda \frac{1}{2} \int_{\partial B_\alpha} |u_\alpha|^2 \langle x - x_0 \rangle \cdot \nu \, dS
\]
\[
- \frac{1}{2\alpha} \int_{\partial B_\alpha} [(1 + |\nabla u_\alpha|^2)^{\alpha - 1} - 1] \langle x - x_0 \rangle \cdot \nu \, dS + \frac{N}{2\alpha} \int_{\partial B_\alpha} (1 + |\nabla u_\alpha|^2)^{\alpha - 1} (\nabla u_\alpha \cdot \nu) u_\alpha \, dS
\]
\[
+ \int_{\partial B_\alpha} (1 + |\nabla u_\alpha|^2)^{\alpha - 1} (\nabla u_\alpha \cdot (x - x_0)) \cdot (\nabla u_\alpha \cdot \nu) \, dS, \quad \text{(by (1.9)).}
\]
\[
(3.53)
\]
Using (3.49), noting that \(|x_0 - x_\alpha| \leq CR_\alpha^{-3/2}\) for \(\partial_1 B_\alpha\), we see
\[
\text{RHS of (3.53)} \leq CR_\alpha^{-3} + \int_{\partial B_\alpha} (|u_\alpha|^2 + |u_\alpha|^2 + |\nabla u_\alpha|^2) \, dS + C \int_{\partial B_\alpha} |\nabla u_\alpha||u_\alpha| \, dS
\]
\[
\leq CR_\alpha^{-3/2}.
\]
\[
(3.54)
\]
Recall that in the proof of Lemma 3.5, we have the decomposition
\[
u_\alpha = u + \sum_{j=1}^k \rho_{x_\alpha,j, R_\alpha,j} (U_j) + \omega_\alpha := u_0 + u_{\alpha,1} + u_{\alpha,2},
\]
\[
(3.55)
\]
with \(\|u_{\alpha,2}\| \to 0\) as \(\alpha \to 1\). We easily find that if \(N > 4\),
\[
\int_{\mathbb{R}^N} |U_j|^2 \, dx < +\infty, \quad j = 1, \ldots, k.
\]
\[
(3.56)
\]
On the other hand, \(B_\alpha' = B_{LR_\alpha^{-1}}(x_\alpha)\), where \(L > 0\) is so large that
\[
\int_{B_{LR_\alpha^{-1}}(0)} |U_j|^2 \, dx > 0, \quad j = 1, \ldots, k.
\]
\[
(3.57)
\]
Since \(u_\alpha = 0\) in \(\mathbb{R}^N \setminus \Omega\), we have
\[
\int_{B_\alpha} |u_\alpha|^2 \, dx = \int_{B_{LR_\alpha^{-1}}(x_\alpha)} |u_\alpha|^2 \, dx \geq \int_{B_{LR_\alpha^{-1}}(x_\alpha)} |u_\alpha|^2 \, dx
\]
\[
\geq \frac{1}{2} \int_{B_{R_\alpha^{-1}}(x_\alpha)} |u_{\alpha,1}|^2 \, dx - C \int_{B_{R_\alpha^{-1}}(x_\alpha)} |u_0|^2 \, dx - C \int_{B_{R_\alpha^{-1}}(x_\alpha)} |u_{\alpha,2}|^2 \, dx.
\]
\[
(3.58)
\]
But
\[
C \int_{B_{R_\alpha^{-1}}(x_\alpha)} |u|^2 \, dx \leq CR_\alpha^{-N} = o(1) R_\alpha^{-2},
\]
\[
(3.59)
\]
and
\[
\int_{B'_\alpha} |u_{\alpha,2}|^2 \, dx \leq C \left( \int_{B'_\alpha} |u_{\alpha,2}|^{2^*} \, dx \right)^{\frac{2}{2^*}} R^{-2}_\alpha = o(1) R^{-2}_\alpha. \tag{3.60}
\]

Since \(\|u_{\alpha,2}\| \to 0\) as \(\alpha \to 1\).

On the other hand, let us assume that \(\rho_{x_{\alpha,1}, R_{\alpha,1}}(U_1)\) is the bubble with slowest concentration rate. Then
\[
\int_{B'_\alpha} |u_{\alpha,1}|^2 \, dx \geq \frac{1}{2} \int_{B'_\alpha} |\rho_{x_{\alpha,1}, R_{\alpha,1}}(U_1)|^2 \, dx + O \left( \sum_{j=2}^k \int_{B'_\alpha} |\rho_{x_{\alpha,j}, R_{\alpha,j}}(U_j)|^2 \, dx \right). \tag{3.61}
\]

Direct calculations implies that
\[
\int_{B'_\alpha} |\rho_{x_{\alpha,1}, R_{\alpha,1}}(U_1)|^2 \, dx = R^{-2}_{\alpha,1} \int_{B(0)} |U_1|^2 \, dx \geq C_1 R^{-2}_{\alpha,1},
\]
for some constant \(C_1 > 0\). Similarly,
\[
\int_{B'_\alpha} |\rho_{x_{\alpha,j}, R_{\alpha,j}}(U_j)|^2 \, dx = R^{-2}_{\alpha,j} \int_{(B'_\alpha)_{x_{\alpha,j}, R_{\alpha,j}}} |U_j|^2 \, dx. \tag{3.62}
\]

Here we use the notation \(S_{x,R} = \{y : R^{-1}y + x \in S\}\) for any set \(S\).

If \(\frac{R_{\alpha,j}}{R_{\alpha,1}} \to +\infty\), then we obtain from (3.62).
\[
\int_{B'_\alpha} |\rho_{x_{\alpha,j}, R_{\alpha,j}}(U_j)|^2 \, dx = o \left( R^{-2}_{\alpha,1} \right). \tag{3.63}
\]

If \(\frac{R_{\alpha,j}}{R_{\alpha,1}} \leq C < +\infty\), then
\[
(B'_{\alpha})_{x_{\alpha,j}, R_{\alpha,j}} = \{y : R^{-1}_{\alpha,j}y + x_{\alpha,j} \in B'_\alpha\} = \{y : |R^{-1}_{\alpha,j}y + x_{\alpha,j} - x_{\alpha,1}| \leq LR^{-1}_{\alpha,1}\} \subset \{y : |y + R_{\alpha,j}(x_{\alpha,j} - x_{\alpha,1})| \leq C\}. \tag{3.64}
\]

Since \(|R_{\alpha,j}(x_{\alpha,j} - x_{\alpha,1})|\) as \(\alpha \to 1\), we find that \((B'_{\alpha})_{x_{\alpha,j}, R_{\alpha,j}}\) moves to infinity. So we obtain from (3.56 and (3.62) that
\[
\int_{B'_\alpha} |\rho_{x_{\alpha,j}, R_{\alpha,j}}(U_j)|^2 \, dx = o \left( R^{-2}_{\alpha,1} \right). \tag{3.65}
\]

So, we have proved that there is a constant \(C_1 > 0\), such that
\[
\int_{B'_\alpha} |u_{\alpha,1}|^2 \, dx \geq C_1 R^{-2}_{\alpha}. \tag{3.66}
\]
Therefore, by (3.58), (3.59), (3.60) and (3.66), we have proved

\[ LHS (3.53) \geq \frac{C_1}{4} R_\alpha^{-2}. \] (3.67)

From (3.54) and (3.67), we find

\[ R_\alpha^{-2} \leq CR_\alpha^{-\frac{N-2}{2}}. \] (3.68)

which is a contradiction if \( N > 6. \)

**Proof of Theorem 1.2.** For any \( k \in \mathbb{N}, \) define the \( Z_2 \)-homotopy class \( \mathcal{F}_k \) by

\[ \mathcal{F}_k = \{ A : A \in H_1^{1,2\alpha}(\Omega) \text{ is compact, } Z_2 \text{ - invariant, and } \gamma(A) \geq k \}, \]

where the genus \( \gamma(A) \) is smallest integer \( m, \) such that there exists an odd map \( \phi \in C(A, \mathbb{R}^m \setminus \{0\}). \)

For \( k = 1, 2, \cdots, \) we can define the minimax value (see [13, p.134])

\[ c_{k,\alpha} = \inf_{A \in \mathcal{F}_k} \max_{u \in A} I_{\alpha,\lambda}(u). \]

From Corollary 7.12 in [13], for each small \( \alpha > 0, \) \( c_{k,\alpha} \) is a critical value of \( I_{\alpha,\lambda}(u), \) since \( I_{\alpha,\lambda}(u) \) satisfies the Palais-Smale condition. Thus (1.3) has a solution \( u_{k,\alpha} \) such that \( I_{\alpha,\lambda}(u) = c_{k,\alpha}. \)

We fix a \( \alpha_0 < \frac{N}{2} \) and \( \alpha < \frac{\alpha_0}{2}. \) Since \( c_{k,\alpha} \) is is increasing in \( \alpha > 1, \) we obtain \( c_{k,\alpha} \leq c_{k,\alpha_0}. \) So \( c_{k,\alpha} \) is uniformly bounded for fixed \( k. \)

From \( I_{\alpha,\lambda}(u_{k,\alpha}) = c_{k,\alpha} \) and the equation satisfied by \( c_{k,\alpha_0}, \) we get

\[ \int_\Omega u_{k,\alpha}^2 dx = \int_\Omega (1 + |\nabla u_{k,\alpha}|^2)^{\alpha-1} |\nabla u_{k,\alpha}|^2 dx - \lambda \int_\Omega u_{k,\alpha}^2 dx \] (3.69)

and

\[ \frac{1}{2^*} \int_\Omega |u_{k,\alpha}|^{2^*} dx + c_{k,\alpha} = \frac{1}{2\alpha} \int_\Omega [(1 + |\nabla u_{k,\alpha}|^2)^\alpha - 1] dx - \frac{\lambda}{2} \int_\Omega u_{k,\alpha}^2 dx. \] (3.70)

So from (3.69) and (3.70) we have

\[ c_{k,0} > c_{k,\alpha} = \left( \frac{1}{2\alpha} - \frac{1}{2^*} \right) \int_\Omega |u_{k,\alpha}|^{2^*} dx - \left( \frac{\lambda}{2} - \frac{\lambda}{2\alpha} \right) \int_\Omega |u_{k,\alpha}|^2 dx \]

\[ \geq \left( \frac{1}{2\alpha} - \frac{1}{2^*} \right) \int_\Omega |u_{k,\alpha}|^{2^*} dx - o(|\alpha - 1|) C \lambda \left( \int_\Omega |u_{k,\alpha}|^2 dx \right)^{\frac{2}{2^*}}, \] (3.71)

where \( C \) depends on \( |\Omega|, 2\alpha \) and \( N \) only.
Therefore there is a positive constant $C$ independent of $N$ such that

$$
\int_{\Omega} |\nabla u_{k,\alpha}|^2 dx \leq C \int_{\Omega} |\nabla u_{k,\alpha}|^{2\alpha} dx \leq C.
$$

Thus $u_{k,\alpha}$ is uniformly bounded with respect to $\alpha$. And the bubble $\rho_{x_{\alpha,j}, r_{\alpha,j}}(U_j)$ does not appear in (3.55). So we have a subsequence of $\{u_{k,\alpha}\}$, such that, $u_{k,\alpha} \rightarrow u_k$ in $H_0^1(\Omega)$, and $c_{k,\alpha} \rightarrow c_k$ as $\alpha \rightarrow 1$. Then $u_k$ is a critical point of $I_{1,\lambda}(u)$ and $I_{1,\lambda}(u_k) = c_k$.

We are now ready to show that $I_{1,\lambda}(u)$ has infinitely many critical points. Noting that $c_k$ is non-decreasing in $k$, we have the following two cases:

**Case I.** There are $1 < k_1 < \cdots < k_i < \cdots$, satisfying $c_{k_1} < \cdots < c_{k_i} < \cdots$.

In this case, $I_{1,\lambda}(u)$ has infinitely many critical points $u_i$ such that $I(u_i) = c_{k_i}$.

**Case II.** There is a positive integer $m$ such that $c_k = c$ for all $k \geq m$.

If for any $\delta > 0$, $I_{1,\lambda}(u)$ has a critical point $u$ with $I_{1,\lambda}(u) \in (c - \delta, c + \delta)$ and $I_{1,\lambda}(u) \neq c$, then we are done. So from now on we assume that there exists a $\delta > 0$, such that $I_{1,\lambda}(u)$ has no critical point $u$ with $I_{1,\lambda}(u) \in (c - \delta, c) \cup (c, c + \delta)$.

Let

$$K_c = \{u \in H_0^{1,2}(\Omega) : I'_{1,\lambda}(u) = 0, I_{1,\lambda}(u) = c\}.$$

If we can prove that

$$\gamma(K_c) \geq 2,$$

then $I_{1,\lambda}(u)$ has infinitely many critical points.

Suppose, on the contrary, that $\gamma(K_c) = 1$. Take a small $\delta_1 > 0$, such that $\gamma(K) = 1$, where $K = \{u \in H_0^{1,2}(\Omega) : \|u - K_c\| \leq \delta_1\}$. Then $\gamma(K \cap H_0^{1,2}(\Omega)) \leq 1$.

Define

$$D_\alpha = \left(K_\alpha^{c+\delta} \setminus K_\alpha^{c-\delta}\right) \setminus (K \cap H_0^{1,2\alpha}(\Omega)),$$

where

$$K_\alpha^t = \{u \in H_0^{1,2\alpha}(\Omega) : I_{\alpha,\lambda}(u) \leq t\}.$$

We now claim that if $\alpha - 1 > 0$ is small, $I_{\alpha,\lambda}(u)$ has no critical point $u \in D_\alpha$. Otherwise, suppose that there are $\alpha \rightarrow 1$ and $u_n \in D_\alpha$ satisfying

$$I'_{\alpha,\lambda}(u_n) = 0, u_n \notin K \cap H_0^{1,2\alpha}(\Omega).$$

And the bubble $\rho_{x_{\alpha,j}, r_{\alpha,j}}(U_j)$ does not appear in (3.55). So $u_\alpha$ (up to a subsequence) converges strongly to $u$ in $H_0^1(\Omega)$ as $\alpha \rightarrow 1$. Then

$$I'_{1,\lambda}(u) = 0, I_{1,\lambda}(u) \in (c - \delta, c + \delta), u \notin K.$$

This contradicts to the assumption.

So, for any $\varepsilon > 0$ small, there exists a constant $c_\alpha^* > 0$, such that

$$\|I'_{1,\lambda}(u)\| \geq c_\alpha^* > 0, \text{ for all } u \in D_\alpha.$$

Standard techniques show that we can find an odd homeomorphism $\eta : H_0^{1,2\alpha}(\Omega) \rightarrow H_0^{1,2\alpha}(\Omega)$ such that

$$\eta((K_\alpha^{c+\delta} \setminus (K \cap H_0^{1,2\alpha}(\Omega)))) \subset K_\alpha^{c-\delta}. \quad (3.72)$$

See for example the proof of Theorem 1.9 in [22].
Fix $k > m$, Since $c_{k, \alpha}, c_{k+1, \alpha} \to c$ as $\alpha \to 1$, we can find an $\alpha - 1 > 0$ small, such that
\[ c_{k, \alpha}, c_{k+1, \alpha} \in \left( c - \frac{1}{4} \delta, c + \frac{1}{4} \delta \right). \]
By the definition of $c_{k+1, \alpha}$, we can find a set $A \in F_{k+1}$, such that
\[ I_{\alpha, \lambda}(u) < c_{k+1, \alpha} + \frac{1}{4} \delta < c + \delta, \quad u \in A. \]
Hence, $A \subset K_{\alpha}^{c-\delta}$. From (3.72), \( \tilde{A} =: \eta(A \setminus (K \cap H_{0}^{1, 2\alpha}(\Omega))) \subset K_{\alpha}^{c-\delta} \). That is
\[ I_{\alpha, \lambda}(u) < c - \delta, \quad u \in \tilde{A}. \]
On the other hand, if $\gamma((K \cap H_{0}^{1, 2\alpha}(\Omega))) = 0$, then $\eta(K_{\alpha}^{c+\delta}) \subset K_{\alpha}^{c-\delta}$, which is contradiction with the definition of $c_{k, \alpha}$.
If $\gamma((K \cap H_{0}^{1, 2\alpha}(\Omega))) = 1$, by Lemma 3.32 of [22], we find that $A \setminus (K \cap H_{0}^{1, 2\alpha}(\Omega)) \subset F_{k}$.
Using Theorem 1.9 in [22], we conclude $\tilde{A} \subset F_{k}$. As a result,
\[ c_{k, \alpha} \leq \sup_{u \in \tilde{A}} I_{\alpha, \lambda}(u) < c - \delta. \]
This is a contradiction to $c_{k, \alpha} > c - \frac{1}{4} \delta$.

Appendix. Some estimates for solutions

In this section, we assume that $\Omega_{1}$ is a bounded domain in $\mathbb{R}^{N}$. We give some estimates for solutions of the equation $-\text{div}((1 + |\nabla \cdot |^{2})^{\alpha-1} \nabla \cdot) = f$. These estimates are very similar to the estimates of $p$-Laplacian equation which were obtained in (see[4]). But for the readers convenience, we give the details of these estimates.

**Proposition A**. Let $w \in H_{0}^{1, 2\alpha}(\Omega_{1})$ be the solution of
\[ -\text{div}((1 + |\nabla w|^{2})^{\alpha-1} \nabla w) = f(x) \quad \text{in} \ \Omega_{1}. \quad (A.1) \]
Suppose that $f \geq 0, f \in L^{\infty}(\Omega_{1})$. Then, for any $\frac{N}{2\alpha} > q \geq 1$, there is a constant $C = C(q)$, such that
\[ \|w\|_{L^{q}(\Omega_{1})}^{\frac{1}{1-q}} \leq C \|f\|_{L^{q}(\Omega_{1})}^{\frac{1}{1-q}}. \]

**Proof**. We now prove that for $r > 1 - \frac{1}{2\alpha}$,
\[ \int_{\Omega_{1}} (1 + |\nabla w|^{2})^{\alpha-1} \nabla w \nabla (w^{1+2\alpha(r-1)}) dx \leq \int_{\Omega_{1}} f(x) w^{1+2\alpha(r-1)} dx. \quad (A.2) \]
Firstly, we assume $r \geq 1$ and $\eta = w^{1+2\alpha(r-1)}$. Then from
\[ \nabla \eta = (1 + 2\alpha(r-1)) w^{2\alpha(r-1)} \nabla w, \]

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and \( r \geq 1 \), it is easy to show that \( \eta \in H^{1,2\alpha}(\Omega_1) \) since \( w \in L^\infty(\Omega_1) \). So, we obtain

\[
\int_{\Omega_1} (1 + |\nabla w|^2)^{\alpha-1} \nabla w \nabla (w^{1+2\alpha(r-1)}) \, dx = \int_{\Omega_1} f(x) w^{1+2\alpha(r-1)} \, dx.
\]

Hence Proposition A holds.

Next we consider the case \( r \in (1 - \frac{1}{2\alpha}, 1) \). In this case, \( w^{1+2\alpha(r-1)} \) may not be in \( H^{1,2\alpha}(\Omega_1) \). So we need to proceed differently. By the comparison principle, we know that \( w \geq 0 \) in \( \Omega_1 \). For any \( \theta > 0 \) being a small number, let \( \eta = w + \theta \). Then \( \eta \in H^{1,2\alpha}(\Omega_1) \), and

\[
\nabla \eta = (w + \theta)^{2\alpha(r-1)} \nabla w + 2\alpha r(w - 1)(w + \theta)^{2\alpha(r-1)-1} \nabla w.
\]

So we deduce

\[
\int_{\Omega_1} (1 + |\nabla w|^2)^{\alpha-1} \nabla w \nabla \eta \, dx \\
= \int_{\Omega_1} (1 + |\nabla w|^2)^{\alpha-1} |\nabla w|^2 ((w + \theta)^{2\alpha(r-1)} + 2\alpha r(w - 1)(w + \theta)^{2\alpha(r-1)-1}) \, dx \\
= \int_{\Omega_1} f(x) w^{1+2\alpha(r-1)} \, dx. \quad (A.3)
\]

On the other hand, by Fatou’s lemma,

\[
(1 + 2\alpha(r - 1)) \int_{\Omega_1} w^{2\alpha(r-1)}(1 + |\nabla w|^2)^{\alpha-1} |\nabla w|^2 \, dx \\
= \int_{\Omega_1} \liminf_{\theta \to 0} (1 + |\nabla w|^2)^{\alpha-1} \nabla w \nabla \eta \, dx \\
= \int_{\Omega_1} \liminf_{\theta \to 0} (1 + |\nabla w|^2)^{\alpha-1} |\nabla w|^2 ((w + \theta)^{2\alpha(r-1)} + 2\alpha r(w - 1)(w + \theta)^{2\alpha(r-1)-1}) \, dx \\
\leq \liminf_{\theta \to 0} \int_{\Omega_1} (1 + |\nabla w|^2)^{\alpha-1} |\nabla w|^2 ((w + \theta)^{2\alpha(r-1)} + 2\alpha r(w - 1)(w + \theta)^{2\alpha(r-1)-1}) \, dx \\
= \liminf_{\theta \to 0} \int_{\Omega_1} f(x) w^{1+2\alpha(r-1)} \, dx \\
\leq \int_{\Omega_1} f(x) w^{1+2\alpha(r-1)} \, dx. \quad (A.4)
\]

Hence, (A.2) holds.

From (A.2), by Sobolev embedding, Hölder inequality and \( \nabla w^r = r w^{r-1} \nabla w \), we have

\[
C \| w \|_{L^{2\alpha}(\Omega_1)}^{2\alpha r} \leq \int_{\Omega_1} f(x) w^{1+2\alpha(r-1)} \, dx \leq \| f \|_q \left( w^{1+2\alpha(r-1)} \right)^{\frac{q}{q+r}} \left( w^{1+2\alpha(r-1)} \right)^{\frac{1}{q+r}}. \quad (A.5)
\]
where \((2\alpha)^* = \frac{2\alpha}{N-2\alpha}\). Choose \(r\), such that

\[(1 + 2\alpha(r - 1)) \frac{q}{q - 1} = (2\alpha)^r.\]

Then,

\[r = \frac{q(2\alpha - 1)}{2q\alpha - (2\alpha)^*(q - 1)}. \tag{A.6}\]

Note that \(2q\alpha - (2\alpha)^*(q - 1) > 0\) since \(q < \frac{N}{2\alpha}\). So, \(r > 0\) and

\[r > \frac{q(2\alpha - 1)}{2q\alpha} = 1 - \frac{1}{2\alpha}.\]

For such \(r\),

\[(2\alpha)^r = \frac{(2\alpha)^q(2\alpha - 1)}{2q\alpha - (2\alpha)^*(q - 1)} = \frac{Nq(2\alpha - 1)}{N - 2\alpha q}.\]

Thus, (A.5) implies that

\[C\|w\|^{2\alpha - (2\alpha)^r(1 - \frac{1}{q})}_{(2\alpha)^r} \leq \|f\|_q. \tag{A.7}\]

Moreover, by (A.6), we know that

\[2\alpha r - (2\alpha)^r \left(1 - \frac{1}{q}\right) = 2\alpha - 1.\]

So, from the above discussion, Proposition A holds.

**Corollary A.** Let \(w \in H^{1,2\alpha}_0(\Omega_1)\) be the solution of

\[-\text{div}((1 + |\nabla w|^{2\alpha - 1})\nabla w) = a(x)v^{2\alpha - 1} \quad \text{in} \ \Omega_1, \tag{A.1}\]

where \(a(x) \geq 0, v \geq 0\) are functions satisfying \(a, v \in L^\infty(\Omega_1)\). Then, for any \(q > (1 - \frac{1}{2\alpha})\frac{2\alpha}{N - 2\alpha q}\), there is a constant \(C = C(q)\), such that

\[\|w\|_q \leq C\|a\|^{\frac{1}{N-2\alpha}} \|v\|_q. \tag{A.2}\]

**Proof.** For any \(q_1 \in (1, \frac{N}{2\alpha})\), set \(q = \frac{Nq_1(2\alpha - 1)}{N - 2\alpha q_1}\). We easily prove that \(\frac{2N}{2\alpha} > q_1 > 1\) is equivalent to \((1 - \frac{1}{2\alpha})\frac{2\alpha}{N - 2\alpha q} < q < +\infty\). By Proposition A, there exists a constant \(C(q_1) > 0\) such that

\[\|w\|_q \leq C(q_1)\|av^{2\alpha - 1}\|^{\frac{1}{2\alpha - 1}}. \tag{A.2}\]

On the other hand, from H"{o}lder inequality we deduce

\[\|av^{2\alpha - 1}\|^{\frac{1}{q_1 - 1}} \leq \|a\|^{\frac{1}{N-2\alpha}} \|v\|_q. \tag{A.3}\]

which, together with (A.2), completes our proof of Corollary A.
**Corollary B.** Let \( w \in H^1(\Omega_1) \) be the solution of
\[
- \text{div}((1 + |\nabla w|^2)^{\alpha - 1} \nabla w) = a(x)v^{2\alpha - 1} \quad \text{in } \Omega_1,
\]
where \( a(x) \geq 0, v \geq 0 \) are functions satisfying \( a, v \in L^\infty(\Omega_1) \). Then, for any \( p_2 \in \left(\frac{2\alpha - 1}{N - 2\alpha}, (2\alpha)^*\right) \), there is a constant \( C = C(p_2) \), such that
\[
\|w\|_{p_2} \leq C\|a\|^{\frac{1}{r-1}}\|v\|_{(2\alpha)^*},
\]
where \( r \) is determined by \( \frac{1}{r} = \frac{2\alpha - 1}{p_2} + \frac{2\alpha}{N} - \frac{2\alpha - 1}{(2\alpha)^*} \).

**Proof.** By the assumption we know that \( w \geq 0 \). Set \( p_2 = \frac{Nq_2(2\alpha - 1)}{N - 2\alpha q_2} \) and choose \( r > 0 \) such that
\[
\frac{1}{r} = \frac{2\alpha - 1}{p_2} + \frac{2\alpha}{N} - \frac{2\alpha - 1}{(2\alpha)^*},
\]
then \( \frac{(2\alpha - 1)q_2}{r - q_2} = (2\alpha)^* \). It is easy to show that \( \frac{(2\alpha - 1)q_2}{N - 2\alpha} < p_2 < (2\alpha)^* \) is equivalent to
\[
1 < q_2 < \frac{N}{2\alpha + (N - 2\alpha)\left(1 - \frac{1}{2\alpha}\right)},
\]
which implies that \( 1 < q_2 < \frac{N}{2\alpha} \). For such \( p_2 \) and \( q_2 \) we can apply Proposition A to obtain a constant \( C(q_2) > 0 \) such that
\[
\|w\|_{p_2} \leq C(q_2)\|av^{2\alpha - 1}\|_{q_2}^{\frac{1}{q_2}}.
\]
On the other hand, from Hölder inequality we get
\[
\|av^{2\alpha - 1}\|_{q_2}^{\frac{1}{q_2}} \leq \|a\|_r^{\frac{1}{r-1}}\|v\|_{(2\alpha)^*}. \tag{B.3}
\]
Hence, Corollary B holds. \( \square \)

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