STRICT FEASIBILITY OF VARIATIONAL INCLUSION PROBLEMS IN REFLEXIVE BANACH SPACES

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Abstract. In this paper, we are denoted to introducing the strict feasibility of a variational inclusion problem as a novel notion. After proving a new equivalent characterization for the nonemptiness and boundedness of the solution set for the variational inclusion problem under consideration, it is proved that the nonemptiness and boundedness of the solution set for the variational inclusion problem with a maximal monotone mapping is equivalent to its strict feasibility in reflexive Banach spaces.

1. Introduction. Throughout this paper, let $X$ be a reflexive Banach space with its dual space $X^*$. Denote by $\|\cdot\|$ and $\langle \cdot, \cdot \rangle$ the norm on the Banach space $X$ and the dual pair between $X^*$ and $X$, respectively. In this paper, we focus on the study of the following variational inclusion problem:

Find $x \in C$ such that

$$0 \in T(x),$$

where $C \subset X$ is a nonempty, closed, and convex subset of $X$, $T : C \to 2^{X^*}$ be a set-valued mapping with nonempty values in $X^*$, and 0 denotes the zero element in the space $X^*$. It is obvious that the solution set of the variational inclusion problem (1) can be defined as $\{ x \in C | 0 \in T(x) \}$.

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As an important generalization of the optimization problems, variational inequality problems, equilibrium problems and their related problems have been studied intensively. A large number of research results have been obtained by many scholars due to their wide applications in many disciplines such as mechanics, engineering, economics and transportation etc. (see, for example, [9, 10, 15, 16, 18, 19, 20, 21, 22, 25, 26, 27, 29]). In particular, the variational inclusion problem has been studied intensively by many authors in recent years (see, for example, [11, 12, 13, 30]). Among the main research issues on the variational inclusion problem, one of the most interesting problems in the theoretical aspect is the development of an efficient and implementable algorithm. Various kinds of iterative algorithms have been designed and studied to find solutions for various kinds of variational inclusions (see, for example, [11, 12, 13]).

It is well known that strict feasibility has played a basic and significant role in the development of algorithms for optimization problems, especially interior-point algorithms and continuation methods. It has been shown in many literature that strict feasibility is an important and useful condition guaranteeing the nonemptiness and boundedness of the solution sets for the optimization problems, variation inequality problems, and other related problems. In [1, 2, 17, 31], the authors of these papers concerns the variational inequality problems or complementarity problems with monotone-type operators. It has been shown by different approaches that these problems have nonempty and bounded solution sets if and only if they are strictly feasible. For more references on the study of strict feasibility for variational inequalities and other related problems, we refer the readers to [5, 6, 7, 14]. However, to the best of the authors’ knowledge, few research works study the strict feasibility for the variational inclusion problems in the literature. Therefore, we mainly focus on the study of strict feasibility for the variational inclusion problem in reflexive Banach spaces.

The main purpose of this paper is to investigate the strict feasibility of the variational inclusion problem in reflexive Banach spaces. Firstly, we generalize the results of [30] from a Hilbert space to a reflexive Banach space, and present a different equivalent characterization of the nonemptiness and boundedness of the solution set for the variational inclusion problem. Based on such the equivalent characterization, we prove that the strict feasibility of the variational inclusion problem with the maximal monotone mapping is equivalent to the nonemptiness and boundedness of its solution set. Our research method is quite different from Theorem 3.1 of [3] and Theorem 4.3 of [4] where the strict feasibility of generalized variational inequalities are discussed.

The rest of this paper is organized as follows. In Section 2, we recall some basic notations and present some preliminary results. We present in Section 3 an equivalent characterization of the nonemptiness and boundedness of the solution set for the variational inclusion problem in reflexive Banach spaces. Then, in Section 4, we consider the strict feasibility of the variational inclusion problem with the maximal monotone mapping in reflexive Banach spaces. Finally, we conclude this paper in Section 5.

2. Notations and preliminaries. In this section, we recall some basic notations and present some preliminary results, which can be found in [8, 23, 24, 28] and will be used in the following sections.
Let $X$, $X^*$ and $C$ be as those in Sect. 1. The symbol "→" and "⇒" are used to denote the strong and weak convergence, respectively. Let
\[
\text{barr}(C) := \{x^* \in X^* : \sup_{x \in C} (x^*, x) < \infty\}
\]
denoting the barrier cone of $C$. The recession cone of $C$ is the closed and convex cone defined by
\[
C_{\infty} := \{d \in X : \exists t_n \downarrow 0, \exists x_n \in C, t_n x_n \rightharpoonup d\}.
\]
It is known that, given $x_0 \in C$,
\[
C_{\infty} = \{d \in X : x_0 + \lambda d \in C \text{ for all } \lambda > 0\}.
\]
The normalized duality mapping $J : X \to 2^{X^*}$ is defined by
\[
J(x) = \{f \in X^* : \langle f, x \rangle = \|f\| \|x\|, \|f\| = \|x\|\}, \quad \forall x \in X.
\]
For a nonempty set $D$ in $X$, $D^- := \{x^* \in X^* : \langle x^*, x \rangle \leq 0, \forall x \in D\}$.

**Definition 2.1.** Let $C$ be a nonempty, closed, and convex subset of $X$ with $int(barr(C)) \neq \emptyset$. The variational inclusion problem (1) is called strictly feasible iff,
\[
T(C) \cap int(-barrC) \neq \emptyset.
\]
**Definition 2.2.** Let $T : C \to 2^{X^*}$ be a set-valued mapping with nonempty values. $T$ is said to be

(i) **monotone on** $C$ if for each pair of points $x, y \in C$ and for all $x^* \in T(x)$ and $y^* \in T(y)$,
\[
\langle y^* - x^*, y - x \rangle \geq 0;
\]

(ii) **maximal monotone on** $C$ if $T$ is monotone, and for any $y \in C$, $\langle \xi - x^*, y - x \rangle \geq 0$
for all $x \in C$ and $x^* \in T(x)$ implies that $\xi \in T(y)$.

It is well known that if $T$ is maximal monotone on $C$, then $T$ is monotone and
\[
(J + \lambda T)(X) = X^*, \quad \forall \lambda > 0,
\]
where $J$ is the normalized duality mapping.

Some preliminary results are quoted below.

**Lemma 2.3** ([4]). Let $C$ be a nonempty, closed, and convex subset in $X$ with $int(barr(C)) \neq \emptyset$, then there does not exist\{ $x_n$ $\subset$ $C$ with $\|x_n\|$ $\to$ $\infty$ such that $\frac{x_n}{\|x_n\|}$ $\rightharpoonup$ $0$.\}
If additionally $C$ is a cone, then there does not exist\{ $d_n$ $\subset$ $C$ with $\|d_n\|$ $= 1$ such that $d_n$ $\to$ $0$.\}

3. **Nonemptiness and boundedness of the solution set.** In this section, we give a sufficient and necessary condition for the nonemptiness and boundedness of the solution set for the variational inclusion problem (1).

**Theorem 3.1.** Let $C$ be a nonempty, closed, and convex subset in $X$. Suppose that $int(barr(C)) \neq \emptyset$ and that $T : C \to 2^{X^*}$ be a maximal monotone set-valued mapping. Then $C_{\infty} \cap T(C)^- = \{0\}$ if and only if the set $\{x \in C | 0 \in T(x)\}$ is nonempty and bounded.

**Proof.** The “only if” part: Firstly, we need to prove that there exists $x_0 \in C$ such that $0 \in T(x_0)$. Suppose that $C_{\infty} \cap T(C)^- = \{0\}$. Since $T$ is maximal monotone, for $n \in \mathbb{N}$, let $x_n := (J + nT)^{-1}(0)$, i.e., $0 \in J(x_n) + nT(x_n)$. This means that for some $u_n \in J(x_n)$ such that $-\frac{u_n}{n} \in T(x_n)$. 

Now we claim that \( \{x_n\} \) is bounded. If not, then \( \|x_n\| \geq n \) for each \( n \). Without loss of generality, we may assume that \( \frac{x_n}{\|x_n\|} \rightarrow d \in C_\infty \) by the definition of the recession cone. It follows from \( \text{int}(\text{barr}C) \neq \emptyset \) and Lemma 2.3 that \( d \neq 0 \).

Let \( y \in C \) and \( y^* \in T(y) \). Then for all \( \|y\| < n \), the monotonicity of \( T \) implies that

\[
\langle -\frac{u_n}{n} - y^*, x_n - y \rangle \geq 0,
\]

which yields that

\[
0 \leq \langle -\frac{u_n}{n} - y^*, \frac{x_n}{\|x_n\|} - \frac{y}{\|x_n\|} \rangle = -\frac{1}{n\|x_n\|} \langle u_n, x_n \rangle + \frac{1}{n\|x_n\|} \langle u_n, y \rangle - \langle y^*, \frac{x_n}{\|x_n\|} \rangle + \langle y^*, \frac{y}{\|x_n\|} \rangle \leq -\frac{\|x_n\|}{n} + \frac{\|y\|}{n} - \langle y^*, \frac{x_n}{\|x_n\|} \rangle + \langle y^*, \frac{y}{\|x_n\|} \rangle \leq \frac{\|y\|}{n} - \langle y^*, \frac{x_n}{\|x_n\|} \rangle + \langle y^*, \frac{y}{\|x_n\|} \rangle = \langle y^*, \frac{y}{\|x_n\|} \rangle. \tag{2}
\]

Passing to the limit as \( n \rightarrow \infty \) in (2), we obtain \( \langle y^*, d \rangle \leq 0 \) for all \( y^* \in T(C) \), and hence \( d \in T(C)^- \). Thus, \( 0 \neq d \in C_\infty \cap T(C)^- \), which is a contradiction. So the claim is verified.

Without loss of generality, we can assume that \( x_n \rightharpoonup x_0 \). For any \( z \in C \) and any \( z^* \in T(z) \), we have

\[
0 \leq \langle -\frac{u_n}{n} - z^*, x_n - z \rangle = \langle \frac{u_n}{n}, z - x_n \rangle + \langle -z^*, x_n - z \rangle \leq \frac{\|u_n\|}{n} \|z - x_n\| + \langle -z^*, x_n - z \rangle. \tag{3}
\]

By the definition of the normal duality mapping, \( \|u_n\| = \|x_n\| \) and so \( \{u_n\} \) is bounded. Letting \( n \rightarrow \infty \) in (3), we get

\[
\langle -z^*, x_0 - z \rangle \geq 0.
\]

Hence, the maximal monotonicity of \( T \) implies that \( 0 \in T(x_0) \).

Next we need to prove that \( \{x \in C : 0 \in T(x)\} \) is bounded. If not, then for every \( m > 0 \), there exists \( x_m \in C \) with \( \|x_m\| \geq m \) such that \( 0 \in T(x_m) \). Without loss of generality, \( \frac{x_m}{\|x_m\|} \rightarrow \tilde{d} \in C_\infty \). It follows from \( \text{int}(\text{barr}C) \neq \emptyset \) and Lemma 2.3 that \( \tilde{d} \neq 0 \).

Let \( \tilde{y} \in C \) and \( \tilde{y}^* \in T(\tilde{y}) \). Then for all \( \|\tilde{y}\| < m \), by the monotonicity of \( T \), we have \( \langle -\tilde{y}^*, x_m - \tilde{y} \rangle \geq 0 \). It turns out that

\[
\langle -\tilde{y}^*, \frac{x_m}{\|x_m\|} - \frac{\tilde{y}}{\|x_m\|} \rangle \geq 0. \tag{4}
\]

Similarly, letting \( m \rightarrow \infty \) in (4), we obtain \( \langle \tilde{y}^*, \tilde{d} \rangle \leq 0 \) for all \( \tilde{y}^* \in T(C) \) and so \( 0 \neq \tilde{d} \in C_\infty \cap T(C)^- \). Thus, we obtain a contradiction. Therefore, the set \( \{x \in C : 0 \in T(x)\} \) is bounded.

The “if” part: Let \( x_0 \in \{x \in C : 0 \in T(x)\} \). For any \( \rho > 0 \) with \( \|x_0\| < \rho \). Suppose that the conclusion does not hold, then \( C_\infty \cap T(C)^- \neq \{0\} \). Since \( C_\infty \cap T(C)^- \) is a closed and convex cone, we can select \( d_0 \in C_\infty \cap T(C)^- \) such that \( \|d_0\| = 1 \). Moreover, \( d_0 \in C_\infty \) implies that \( x_0 + 2\rho d_0 \in C \).
For any $y \in C$ and any $y^* \in T(y)$, by the monotonicity of $T$, we have
\[
\langle -y^*, x_0 - y \rangle \geq 0.
\] (5)
Since $d_0 \in T(C)^-$, $\langle y^*, d_0 \rangle \leq 0$. Thus
\[
\langle y^*, 2\rho d_0 \rangle \leq 0.
\] (6)
It follows from (5) and (6) that $\langle -y^*, x_0 + 2\rho d_0 - y \rangle \geq 0$. The maximal monotonicity of $T$ yields that
\[
0 \in T(x_0 + 2\rho d_0).
\]
However,
\[
\|x_0 + 2\rho d_0\| \geq 2\rho\|d_0\| - \|x_0\| > 2\rho - \rho = \rho.
\] (7)
Since $\|x_0 + 2\rho d_0\| > \rho$ and $\rho$ is arbitrary, (7) contradicts the assumption that the set $\{x \in C | 0 \in T(x)\}$ is bounded. This completes the proof. \qed

Remark 1. The method of proof in Theorem 3.1 is quite different from the one used in Theorem 3.1 and Theorem 4.1 of [30]. In fact, we give another equivalent characterization for the solution set of the variational inclusion problem to be nonempty and bounded. Moreover, Theorem 3.1 generalize the results of [30] from a Hilbert space to a reflexive Banach space.

4. Strict feasibility. In this section, we shall establish our main result which states that the variational inclusion problem (1) has a nonempty and bounded solution set if and only if it is strictly feasible in reflexive Banach spaces.

Theorem 4.1. Let $C$ be a nonempty, closed and convex subset in $X$. Suppose that $\text{int}(\text{barr}C) \neq \emptyset$ and that $T : C \to 2^{X^*}$ be a maximal monotone set-valued mapping with nonempty values. Then the set $\{x \in C | 0 \in T(x)\}$ is nonempty and bounded if and only if the variational inclusion problem (1) is strictly feasible, i.e.,
\[
T(C) \cap \text{int}(-\text{barr}C) \neq \emptyset.
\]
Proof. The “only if” part: Suppose that the set $\{x \in C | 0 \in T(x)\}$ is nonempty and bounded. By Theorem 3.1, we have
\[
C_\infty \cap T(C)^- = \{0\}. \quad (8)
\]
Hence, for all $d \in C_\infty \setminus \{0\}$, there exists $y^*_0 \in T(C)$, $\langle y^*_0, d \rangle > 0$.

Now we assume that the conclusion does not hold, then $T(C) \cap \text{int}(-\text{barr}C) = \emptyset$, that is, for any $y^* \in X^*$ such that $y^* \in T(C)$, but $y^* \notin \text{int}(-\text{barr}C)$.

For any $t > 0$, $td \in C$ because $d \in C_\infty$. It follows that
\[
\langle -y^*_0, td \rangle = t\langle -y^*_0, d \rangle < 0.
\] (9)
By $y^*_0 \in T(C)$, we have $y^*_0 \notin \text{int}(-\text{barr}C)$. Thus, there exists $x_0 \in C$ such that
\[
\langle -y^*_0, x_0 \rangle = +\infty,
\]
which is a contradiction with (9) as $d \in C_\infty$ is arbitrary.

The “if” part: Suppose that the variational inclusion problem is strictly feasible. Now we claim that
\[
C_\infty \cap T(C)^- = \{0\}. \quad (10)
\]
If not, then $C_\infty \cap T(C)^- \neq \{0\}$. We can select a sequence $\{d_n\} \subset C_\infty \cap T(C)^-$ such that $\|d_n\| = 1$ for each $n$. Without loss of generality, $d_n \rightharpoonup d_0$ as $n \to \infty$. Since $C_\infty$
is a closed and convex cone, it is weakly closed and so \( d_0 \in C_\infty \). By Lemma 2.3, we have \( d_0 \neq 0 \).

For any \( y^* \in T(C) \), we have
\[
\langle y^*, d_n \rangle \leq 0.
\]
Combining with \( d_n \rightharpoonup d_0 \), it follows that
\[
\langle y^*, d_0 \rangle \leq 0. \tag{11}
\]
Since \( \text{int}(T(C) \cap (-\text{barr}C)) \neq \emptyset \). Let \( \xi \in \text{int}(T(C) \cap (-\text{barr}C)) \). We claim that
\[
\langle -\xi, d_0 \rangle > 0. \tag{12}
\]
In fact, if not, then \( \langle -\xi, d_0 \rangle = 0 \) as (11) holds.

Since \( \xi \in \text{int}(T(C) \cap (-\text{barr}C)) \), for any \( x^* \in X^* \), there exists \( \lambda \in (0, 1) \) such that \( \lambda \xi + (1 - \lambda)x^* \in T(C) \cap (-\text{barr}C) \). It implies that
\[
\langle -\lambda \xi + (1 - \lambda)x^*, d_0 \rangle \geq 0
\]
and so \( \langle -x^*, d_0 \rangle \geq 0 \). Similarly, we can obtain \( \langle x^*, d_0 \rangle \geq 0 \). Thus, \( \langle x^*, d_0 \rangle = 0 \), which contradicts with \( d_0 \neq 0 \). Therefore, the claim (12) is proved.

By \( \xi \in \text{int}(-\text{barr}C) \), we have
\[
\sup_{x \in C} \langle -\xi, x \rangle < +\infty. \tag{13}
\]
Let \( x_0 \in C \). \( d_0 \in C_\infty \) implies that \( x_0 + td_0 \in C \) for all \( t > 0 \). It follows that
\[
\langle -\xi, x_0 + td_0 \rangle < +\infty
\]
and so
\[
t \langle -\xi, d_0 \rangle < +\infty. \tag{14}
\]
However, letting \( t \to +\infty \), (12) implies that \( t \langle -\xi, d_0 \rangle \) has no upper bound, which is a contradiction with (14). Thus, the claim (10) is verified. By Theorem 3.1, we know that the solution set of the variational inclusion problem is nonempty and bounded.

**Remark 2.** Based on Theorem 3.1 of Sect. 3, Theorem 4.1 give the strict feasibility of a variational inclusion problem is a sufficient and necessary condition for guaranteeing the nonemptiness and boundedness of its solution set in reflexive Banach spaces. The proof method is quite different from the one used in [3, 4] which discuss the strict feasibility of generalized variational inequalities. We would like to mention that the approach in this paper is simpler and easier than the one used in [3, 4].

Now we shall give an example in an infinite-dimensional Hilbert space \( \ell^2 \). The following example indicates that strict feasibility can ensure the nonemptiness and boundedness of the solution set for the variational inclusion problem.

**Example 4.1.** Let \( X = \ell^2 \) and so \( X^* = \ell^2 \). The inner product in \( \ell^2 \) is defined by
\[
\langle x, y \rangle = \sum_{n=1}^{\infty} x_n y_n, \quad \forall x, y \in \ell^2.
\]
Let \( C = \{ x = (x_1, x_2, \ldots, x_n, \ldots) \in \ell^2 : x_n \geq 0, \forall n \in \mathbb{N} \} \). Define \( T : C \to \ell^2 \) by
\[
T(x) := x, \quad \forall x \in C.
\]
Then for all \( x, y \in C \),
\[
(T(x) - T(y), x - y) = \langle x - y, x - y \rangle = \sum_{i=1}^{\infty} (x_i - y_i)^2 = \| x - y \|_2^2 \geq 0
\]
and
\[
(\lambda T)(\ell^2) = \ell^2,
\]
which means that \( T \) is maximal monotone. Now we shall prove that the strict feasibility of the variational inclusion problem (1). In fact, we have
\[
\sup_{x \in C} \langle 0, x \rangle = 0 < +\infty,
\]
which yields that \( 0 \in T(C) \cap \text{int}(\text{bar} C) \). Moreover, by a trivial computation, we know that the solution set is a bounded single-point set \( \{0\} \).

The following example shows that the maximal monotonicity assumption in Theorem 4.1 is essential.

**Example 4.2.** Let \( X = \mathbb{R} = [\infty, +\infty) \) and \( C = \mathbb{R}_+ = [0, +\infty) \). Then \( X^* = \mathbb{R} \).
Define \( T : C \to 2^\mathbb{R} \) by
\[
T(x) := [-1, 1], \quad \forall x \in C.
\]
It is clear that \( T \) is not monotone and so not maximal monotone. Now we shall prove that the strict feasibility of the variational inclusion problem (1). In fact, we have
\[
\sup_{x \in [0, +\infty]} \langle -1, x \rangle = \sup_{x \in [0, +\infty]} \{-x\} = 0 < +\infty.
\]
which implies that \( 1 \in T(C) \cap \text{int}(\text{bar} C) \). However, by a simple calculation, it yields that the solution set \([0, +\infty]\) is unbounded.

5. **Conclusions.** This paper is concerning with the strict feasibility of variational inclusion problems in reflexive Banach spaces. Firstly, we proved an equivalent characterization of the nonemptiness and boundedness of the solution set for a variational inclusion problem. Based on the equivalent characterization, we gave the strict feasibility of the variational inclusion problem is equivalent to the solution set being nonempty and bounded. While in the proof of \([3, 4]\), the authors employed some complicated analytical skills to study the strict feasibility of variational inequalities. Therefore, the method used in this paper is simpler and easier. Further research works should be carried out to study the strict feasibility for some other kinds of optimization problems.

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