Determination of vertical well flow rate at elliptical approximation of external boundary

P A Larin

Ufa State Petroleum Technological University, Branch of the University in the City of Oktyabrsky, 54a, Devonskaya St., Oktyabrsky, Republic of Bashkortostan, 452607, Russian Federation

E-mail: kafedra.itmen@mail.ru, info@of.ugntu.ru

Abstract. The external boundary typically has an arbitrary shape. The Dupuis formula obtained by circular approximation of a closed external boundary is usually used to determine the flow rate of a hydrodynamically perfect well. The paper proposes an elliptical approximation in which the wellhead may be located anywhere inside the bounding ellipse. The purpose of the study is to obtain a formula for calculating the well flow rate during elliptical approximation of the external boundary. Objects of study: a vertical well surrounded by an arbitrary external boundary. Research methods: differential and integral calculus. Results: formula for the flow rate of hydrodynamically perfect well is obtained based on known characteristics of formation, flowing fluid and parameters of an ellipse approximating the external boundary; an example is given showing the difference between the flow rates counted according to this formula and the Dupuis formula, which is its special case.

1. Introduction
When solving the problem of fluid influx into a vertical well, the closed external boundary, which usually has an arbitrary shape, is usually averaged with a circle centered on the well axis [1]. In this case, the well flow rate is calculated using the Dupuis formula. This formula and its modifications are used in the study of the fluid flow under various conditions [5-8]. The next step would be a more accurate consideration of the shape of the external boundary and the location of the wellhead within that boundary. Let us try to put this idea into practice and average the external boundary with an ellipse.

2. Problem statement
We have a vertical well of radius $r_c$ penetrating the feed bed with thickness $h$ (Fig. 1, 3). The external boundary is an average ellipse. We believe that the external boundary and the ellipse are guide lines of vertical cylinders with height $h$, and the borehole passes inside these cylinders. Ellipse dimensions: major semi-axis $a$, minor semi-axis $b$. Known values: $k$ – permeability of the formation, $\mu$ – dynamic viscosity of the formation fluid, $p_{km}$ – formation pressure at the external boundary, $p_w$ – formation pressure near the well wall. We will consider the fluid incompressible and there are no sources or drains inside the external
boundary. In this case, the flow rate $Q$ through any closed surface surrounding the well is the same. Then we have to determine the well flow rate.

3. Generalized ellipse equation

Let us draw an imaginary horizontal plane in the feed bed. The plane intersects the axis of the well at point $O$. On this plane let us place the Cartesian rectangular coordinate system $Oxy$, thus directing the coordinate axis $x$ parallel to the large axis of the ellipse.

![Diagram of a horizontal plane and ellipse](image)

**Fig. 1.** Horizontal plane on which the coordinate system $Oxy$ is placed intersects the borehole 1, the external boundary 2 and the ellipse 3 averaging the external boundary.

In the $Oxy$ coordinate system, the circle representing the well wall will be given by equation [2, 3]

$$x'^2 + y'^2 = r'^2.$$  

(1)

the center of ellipse $A$ will get the coordinates $(x_A, y_A)$, and the ellipse will be represented by the equation

$$\frac{(x-x_A)^2}{a^2} + \frac{(y-y_A)^2}{b^2} = 1.$$  

(2)

A circle (1) may be converted into an ellipse (2) through the equation

$$\frac{(x-x_A)^2}{(a-r_f)^2} + \frac{(y-y_A)^2}{(b-r_f)^2} = 1,$$  

(3)

with variable parameter $\tau \in [0,1]$. If we take an arbitrary point $M$ on the $OA$ segment, then $\tau = OM/OA$. The series of ellipses given by equation (3) is shown in Figure 2.

![Diagram of series of ellipses](image)

**Fig. 2.** Elliptical external boundary 2 and ellipses 3 arranged around the well 1. Fluid flow lines 4 to the well are orthogonal to ellipses.
4. Well flow rate formula

The extreme lines described in equations (1) and (2) are isobars. Therefore, it will be assumed that the intermediate ellipses (3) are also isobars, the pressure on which depends only on the parameter \( \tau: p = p(\tau) \).

We will also assume that the pressures are normal to these ellipses. Each ellipse defines a vertical elliptical cylinder with height \( h \). Among the cylinder family, let us choose an arbitrary value corresponding to the fixed value \( \tau \). On this cylinder let us highlight an infinitely narrow vertical platform \( dS = h dL_{\text{ell}} \), where \( dL_{\text{ell}} \) – element of the ellipse length (Fig. 3). Let us denote \( n \) from normal to \( dS \). The site vector will be equal to \( dS = n dS \).

![Fig. 3. A section (dS), through which the fluid flow is drawn, is marked on the elliptical cylinder 2 corresponding to the fixed value of the parameter \( \tau \) and located around the well 1.](image)

The volume of the liquid proceeding through \( dS \) per unit of time is defined by a scalar product \( dQ = \mu \cdot \nabla p dS \), where \( \mu \) – fluid velocity. Let us assume that the fluid velocity is proportional to the pressure gradient

\[
v = -\frac{k}{\mu} \nabla p.
\]

The minus sign is explained by the fact that the vectors \( v \) and \( \nabla p \) are directed opposite. Therefore

\[
dQ = -\frac{k}{\mu} \nabla p dS.
\]

Since vectors \( \nabla p = (p_x, p_y) \) and \( n \) are parallel, then \( \nabla p n = |\nabla p| |n| = |\nabla p| \),

\[
dQ = -\frac{k}{\mu} \nabla p h dL_{\text{ell}}.
\]

For the convenience of subsequent transformations, let us write equation (3) in the form of a system

\[
\begin{align*}
  x &= x, \tau + [(a-r) \tau + r_j] \cos \varphi, \\
  y &= y, \tau + [(b-r) \tau + r_j] \sin \varphi.
\end{align*}
\]

where \( \tau \) and \( \varphi \) – independent variables, \( \varphi \in [0, 2\pi] \). When moving along the ellipse, only the variable \( \varphi \) changes, so

\[
dL_{\text{ell}} = \sqrt{dx^2 + dy^2} = \sqrt{[(a-r) \tau + r_j] \sin \varphi + [(b-r) \tau + r_j] \cos \varphi} \, d\varphi.
\]

Let us consider the definition of the value \( |\nabla p| = \sqrt{(p_x')^2 + (p_y')^2} \). Since

\[
\begin{align*}
  p_x' &= p_x x_\tau + p_y y_\tau, \\
  p_y' &= p_x x_\varphi + p_y y_\varphi,
\end{align*}
\]

then, by substituting the values (6) and \( p_x' = 0 \) (because the pressure depends only on \( \tau \)) we will have
\[
\left\{ \begin{array}{l}
p'\{x_1 + (a-r)\cos \varphi\} + p'\{y_1 + (b-r)\sin \varphi\} = p', \\
-p'\{(a-r)\tau + r\} \sin \varphi + p'\{(b-r)\tau + r\} \cos \varphi = 0.
\end{array} \right.
\]

Hence

\[p'_{\varphi} = \frac{\{(b-r)\tau + r\} \cos \varphi}{u(\varphi)} p', \quad p'_{\varphi} = \frac{\{(a-r)\tau + r\} \sin \varphi}{u(\varphi)} p',\]

where

\[u(\varphi) = [x_1 + (a-r)\cos \varphi][\{(b-r)\tau + r\} \cos \varphi + [y_1 + (b-r)\sin \varphi][\{(a-r)\tau + r\} \sin \varphi].\]

Therefore,

\[\left[\nabla p\right] = \sqrt{(p'_{\varphi})^2 + (p'_{\varphi})^2} = \sqrt{\frac{\{(a-r)\tau + r\}^2 \sin^2 \varphi + \{(b-r)\tau + r\}^2 \cos^2 \varphi}{u(\varphi)} p'}, \quad \text{(8)}\]

The flow direction does not matter, so in (5) let us omit the minus sign. The substitution of (7) and (8) in (5) gives

\[dQ = \frac{hk(\tau \varphi)}{\mu} \frac{\{(a-r)\tau + r\}^2 \sin^2 \varphi + \{(b-r)\tau + r\}^2 \cos^2 \varphi}{u(\varphi)} p'_{\varphi} d\varphi. \quad \text{(9)}\]

It is taken into account that the permeability coefficient in different places of the feed formation may be different: \(k = k(\tau, \varphi).\) Having integrated the expression according to \(\varphi\) from 0 to \(2\pi\), we get the flow rate – the influx of fluid into the well from all sides per unit time:

\[Q = \frac{h}{\mu} F(\tau) \frac{dp}{d\tau}, \quad \text{(9)}\]

where

\[F(\tau) = \int_0^{2\pi} \frac{k(\tau \varphi)}{u(\varphi)} \frac{\{(a-r)\tau + r\}^2 \sin^2 \varphi + \{(b-r)\tau + r\}^2 \cos^2 \varphi}{u(\varphi)} p'_{\varphi} d\varphi. \quad \text{(10)}\]

In (9), \(Q\) does not depend on \(\tau\). If \(\tau\) changes from 0 to 1, the pressure varies from \(p_c\) to \(p_{pr}\), so from (9) we get

\[Q = \frac{h (p_{pr} - p_c)}{\mu} \left[ \int_0^1 \frac{dp}{F(\tau)} \right]. \quad \text{(11)}\]

We obtained a formula for calculating the flow rate with an elliptical approximation of the external boundary.

When the permeability coefficient is constant, \(k = k_0\), and the circle of the radius \(r_{in}\) serves as the external boundary, then \(x_1 = y_1 = 0, a = b = r_{in}\), and from (10) we get

\[F(\tau) = \frac{2\pi k_0 [(r_{pr} - r) \tau + r]}{r_{in} - r}. \]

By substituting this value in (11) we receive the Dupuis formula

\[Q = \frac{2\pi hk_0 (p_{pr} - p_c)}{\mu ln \frac{r_{in} - r}{r_c}}. \quad \text{(12)}\]

5. Example of Formula 11

It is known that the radius of the well is \(r_c = 0.11\) m, the thickness of the feed formation is \(h = 10\) m, the viscosity of the leaking fluid is \(\mu = 8\) mPa·s, the pressure on the external boundary is \(p_{pr} = 22\) MPa, the pressure at the wall of the well is \(p_c = 2\) MPa. After averaging the external boundary with an ellipse, its semi-axes \(a = 2500\) m, \(b = 1600\) m were found. In the \(Oxy\) coordinate system with the origin on the well axis and the \(x\) axis of the ellipse parallel to the large axis, the center \(A\) of the ellipse received the following
coordinates: \( x_A = -300 \) m and \( y_A = 200 \) m. We need to determine the well flow rate \( Q \) for the following cases:

a) permeability coefficient of the feed formation is constant and equal to \( k = 1.3 \times 10^{-15} \) m\(^2\);

b) permeability coefficient decreases according to the linear law as it approaches the well:
\[
k(\tau) = k_c + (k_m - k_c) \tau, \quad \text{where} \quad k_c = 1 \times 10^{-15} \text{m}^2, \quad k_m = 1.6 \times 10^{-15} \text{m}^2 \quad \text{(Fig. 4 a)};
\]

c) permeability coefficient decreases according to the quadratic law
\[
k(\tau) = k_m - (k_m - k_c)(1 - \tau^2), \quad \text{where} \quad k_c = 0.7 \times 10^{-15} \text{m}^2, \quad k_m = 1.6 \times 10^{-15} \text{m}^2 \quad \text{(Fig. 4, b)};
\]
d) external boundary is replaced by a circle covering the same area as this ellipse, centered on the axis of the well.

Solution. By substituting these tasks into the calculation formulas (10), (11), we get:

a) \( Q = 2.30 \times 10^{-5} \text{m}^3/\text{s}, \) or \( 2.00 \text{ m}^3/\text{day} \);

b) \( Q = 2.35 \times 10^{-5} \text{m}^3/\text{s}, \) or \( 2.03 \text{ m}^3/\text{day} \);

c) \( Q = 2.38 \times 10^{-5} \text{m}^3/\text{s}, \) or \( 2.05 \text{ m}^3/\text{day} \);

d) Area bounded by ellipse \( S = \pi ab = 1.26 \times 10^7 \text{ m}^2 \). The radius of round external boundary enclosing the same area with center on well axis \( r_o = \sqrt{S/\pi} = 200 \text{ m} \). At \( k = 1.3 \times 10^{-15} \text{m}^2 \) by formula (12) we get \( Q = 2.08 \times 10^{-5} \text{m}^3/\text{s}, \) or \( 1.80 \text{ m}^3/\text{day} \). This value is 10% less than the result obtained in a). Thus, the calculations of the flow rate with elliptical and circular approximation may differ significantly.

\[\text{Fig. 4. Permeability of feed beds around the well depending on the parameter } \tau = \frac{OM}{OA}; \quad \text{a) linear law of dependence; b) quadratic law of dependence}\]

6. Conclusion

Using mathematical transformations (differential and integral calculus) in this paper, we obtained a formula for calculating the flow rate of a well with an elliptical approximation of the power supply circuit. To calculate the flow rate, the reservoir characteristics are required as initial data. A comparative analysis of the determination of the depts calculated using this formula and the Dupuy formula, which is a special case of it, shows a higher accuracy of the obtained formula.

The formula for calculating the well flow rate in an elliptical approximation of the external boundary may be useful if it is required to accurately determine the well flow rate. The work [4] shows how to deliver the specified volumes of fluid to remote tanks with the least energy consumption.

Acknowledgments

The author is grateful to Robert Shakurovich Mufazalov, who readily provided advice on the soil parameters around the well.

References

[1] Moufazalov R Sh 2013 Skin factor. Historical errors and misconceptions made in the theory of hydrodynamics of the oil layer Georesources 2013 5 34-48

[2] Korn G, Korn T 1974 Handbook of mathematics. For scientists and engineers (Moscow: Nauka)
[3] Myshkis A D 2007 *Lectures in Higher Mathematics: Textbook* (SPb: Lan)

[4] Bikbulatova G I, Galeev A S, Boltmeva Yu A, Larin P A, Suleymanov R N, Filimonov O V 2019 Optimization of the process of pumping fixed volumes of fluid in two *directions Bulletin of the Tomsk Polytechnic University. Geo-Resource Engineering* **330** (1) 134-144

[5] Brito A, Cabello R, Guzman N, Marcano L, Trujillo J 2015 Hydrodynamic study of multiphase flow transport of highly viscous foamy fluids *Journal of Petroleum Science and Engineering* **135** 367–374.

[6] Dong T, Cao S, Xu G 2015 Highly porous oil sorbent based on hollow fibers as the interceptor for oil on static and running water *Journal of Hazardous Materials* **305** 1–7

[7] He J G, Song K P, Yang J 2014 Study on experiment of advance water injection an example from low permeability oil reservoir of Fuyu oil reservoir in an oil field *Science Technology and Engineering* **14** (11) 181–183

[8] Wang W, Peng H, Li S 2015 Opening mechanism of dynamic fractures caused by water injection and effective adjustment in low permeability reservoirs, Daqing oilfield in Songliao basin. *Oil & gas geology* **36** (5) 842–847