Generic coverings of plane with A-D-E-singularities.

V.S. Kulikov and Vic.S. Kulikov *

Abstract

We investigate a presentation of an algebraic surface $X$ with A-D-E-singularities as a generic covering $f : X \to \mathbb{P}^2$, i.e. a finite morphism, having at most folds and pleats apart from singular points, isomorphic to a projection of a surface $z^2 = h(x, y)$ onto the plane $x, y$ in neighbourhoods of singular points, and the branch curve $B \subset \mathbb{P}^2$ of which has only nodes and ordinary cusps except singularities originated from the singularities of $X$. It is deemed that classics proved that a generic projection of a non-singular surface $X \subset \mathbb{P}^r$ is of such form. In this paper this result is proved for an embedding of a surface $X$ with A-D-E-singularities, which is a composition of the given one and a Veronese embedding. We generalize results of the paper [K], in which Chisini’s conjecture on the unique reconstruction of $f$ by the curve $B$ is investigated. For this fibre products of generic coverings are studied. The main inequality bounding the degree of a covering in the case of existence of two nonequivalent coverings with the branch curve $B$ is obtained. This inequality is used for the proof of the Chisini conjecture for $m$-canonical coverings of surfaces of general type for $m \geq 5$.

Introduction

Let $S \subset \mathbb{P}^r$ be a non-singular projective surface, $f : S \to \mathbb{P}^2$ be its generic projection to the plane, $B \subset \mathbb{P}^2$ be the branch curve, which we call the discriminant curve. It is deemed that classics proved (see [Z], p.104) that (i) the map $f$ is a finite covering, which has as singularities at most double points (folds), or singular points of cuspidal type (pleats); (ii) with this $f^*(B) = 2R + C$, where the double curve $R$ is non-singular and irreducible, and the curve $C$ is reduced; (iii) the curve $B$ is cuspidal, i.e. has at most nodes and ordinary cusps; over a node there lie two double points, and over a cusp – one point of cuspidal type; (iv) the restriction of $f$ to $R$ is of degree one. Any finite morphism $f : S \to \mathbb{P}^2$ is called a generic (or simple) covering, if it possesses the same properties as a generic projection. Two coverings of plane $(S_1, f_1)$ and $(S_2, f_2)$ are called equivalent, if there is a morphism $\varphi : S_1 \to S_2$ such that $f_1 = f_2 \circ \varphi$.

In this paper we consider a generalization of the notion of a generic covering to the case of surfaces with A-D-E-singularities. First of all we want to explain why we need such a generalization. A presentation of an algebraic variety as a finite covering of the projective space is one of the affective ways of studying projective varieties as well as their moduli. To

*Partly supported by RFFI (No. 99-01–01133) and INTAS-OPEN-97-2072.
Theorem 0.1 compare we recall what such an approach gives in the case of curves. For a curve $C$ of genus $g$ a generic covering $f : C \to \mathbb{P}^1$ is such a covering that in every fibre there is at most one ramification point which is a double point (or a singular point of $f$ of type $A_1$). Let $B \subset \mathbb{P}^1$ be the set of branch points, and $d = \deg B$, i.e. $d = \sharp(B)$. Then according to the Hurwitz formula $d = 2N + 2g - 2$, where $N = \deg f$. If $N \geq g + 1$, then any curve of genus $g$ can be presented as a simple covering of $\mathbb{P}^1$ of degree $N$. The set of all simple coverings (up to equivalence) $f : C \to \mathbb{P}^1$ of degree $N$ with $d$ branch points is parametrized by a Hurwitz variety $H = H^{N,d}$. Let $\mathbb{P}^d \setminus \Delta$ ($\Delta$ – discriminant) be the projective space parametrizing the sets of $d$ different points of $\mathbb{P}^1$, and let $M_g$ be the moduli space of curves of genus $g$. There are two maps: a map $h : H \to \mathbb{P}^d \setminus \Delta$, sending $f$ to the set of branch points $B \subset \mathbb{P}^1$, and a map $\mu : H \to M_g$, sending $f$ to the class of curves isomorphic to $C$. Hurwitz introduced and investigated the variety $H$ in 1891. He proved that the variety $H$ is connected, and $h$ is a finite unramified covering. In modern functorial language $H$ was studied also by W. Fulton in 1969. The map $\mu$ is surjective (and has fibres of dimension $N + (N - g + 1)$). This gives one of the proofs of irreducibility of the moduli space $M_g$.

In the case of surfaces we also can consider an analog of Hurwitz variety $H$ of all generic coverings (up to equivalence) $f : S \to \mathbb{P}^2$ of degree $N$ and with discriminant curve $B$ of degree $d$ with given number $n$ of nodes and given number $c$ of cusps. Let $\mathbb{P}^\nu$, $\nu = \frac{(d(N+3))}{2}$, be a projective space parametrizing curves of degree $d$, and $h : H \to \mathbb{P}^\nu$ be a map sending a covering $f$ to its discriminant curve $B$. In [K] a Chisini conjecture is studied. It claims that if $B$ is the discriminant curve of a generic covering $f$ of degree $N \geq 5$, then $f$ is uniquely up to equivalence defined by the curve $B$. In other words, it means that the map $h$ is injective (and, besides, $N = \deg f$ is determined by $B$). In [K] it is proved that the Chisini conjecture is true for almost all generic coverings. In particular, it is true for generic coverings defined by a multiple canonical class. A construction of the moduli space of surfaces of general type uses pluricanonical maps. As is known [BPV], if $S$ is a minimal surface of general type, then for $m \geq 5$ the linear system $|mK_S|$ blows down only (-2)-curves and gives a birational map of $S$ to a surface $X \subset \mathbb{P}^r$ (the canonical model) with at most A-D-E-singularities (in other terms, rational double points, Du Val singularities, simple singularities of Arnol’d and etc.). This requires a generalization of the notion of a generic covering to the case of surfaces with A-D-E-singularities.

In this paper we, firstly, generalize a classical result on singularities of generic projections of non-singular surfaces to the case of surfaces with A-D-E-singularities. We prove that if a surface $X \subset \mathbb{P}^r$ has at most A-D-E-singularities, then (may be after a ”twist”) for a generic projection $f : X \to \mathbb{P}^2$ the discriminant curve $B$ also has at most A-D-E-singularities. It follows from a slightly more general theorem.

**Theorem 0.1** Let $X \subset \mathbb{P}^r$ be a surface with at most isolated singularities of the form $z^2 = h(x, y)$ (= ”double planes”), $X \to \mathbb{P}^2$ be the restriction to $X$ of a generic projection $\mathbb{P}^r \setminus L \to \mathbb{P}^2$ from a generic linear subspace $L$ of dimension $r - 3$. Then

(i) $f$ is a finite covering;

(ii) at non-singular points of $X$ the covering $f$ has as singularities at most either double points (folds), or singular points of cuspidal type (pleats); in a neighbourhood of these points $f$
is equivalent to a projection of a surface \( x = z^2 \), respectively \( y = z^3 + xz \), to the plane \( x, y \);

(iii) in a neighbourhood of a point \( s \in \text{Sing } X \) the covering \( f \) is analytically equivalent to a projection of a surface \( z^2 = h(x, y) \) to the plane \( x, y \); from (ii) and (iii) it follows that the ramification divisor is reduced, i.e. \( f^*(B) = 2R + C \), where \( B = f(R) \), and \( R \) and \( C \) are reduced curves;

(iv) except singular points \( f(\text{Sing } X) \) the discriminant curve \( B \) is cuspidal;

(v) the restriction of \( f \) to \( R \) is of degree 1.

Actually, the main difficulty in the proof of this theorem lies in the classical case, when the surface \( X \) is non-singular. Unfortunately, authors do not know a complete (and modern) proof of this theorem, and it seems that such a proof does not exist. Thus, the proof, even in the case of a non-singular surface, take interest. In this paper we prove a weakened version of Theorem 0.1, in which the initial embedding is ‘twisted’ by a Veronese embedding. This is quite enough for the purposes described above.

Thus, the curve \( B \) has, firstly, ‘the same’ singularities as the surface \( X \) (and as the curve \( R \)), which are locally defined by the equation \( h(x, y) = 0 \). These singularities on \( B \) we call \( s \)-singularities, in particular, \( s \)-nodes and \( s \)-cusps. Besides, there are nodes and cusps on \( B \) originated from singularities of the map \( f \), which we call \( p \)-nodes and \( p \)-cusps. There are two double points of \( f \) over a \( p \)-node, at which \( f \) is defined locally as a projection of surfaces \( z_1 = x^2 \) and \( z_2 = y^2 \) to the plane \( x, y \).

If \( S \) is a surface with A-D-E-singularities, then a covering \( f : S \to \mathbb{P}^2 \) is called generic, if it satisfies the properties of Theorem 0.1.

Secondly, we generalize the central result of [K] to the case of surfaces with A-D-E-singularities. It is proved there that if a generic covering \( f : S \to \mathbb{P}^2 \) of a non-singular surface \( S \) with discriminant curve \( B \) is of sufficiently big degree \( \deg f = N \), namely under condition

\[
N > \frac{4(3\bar{d} + g - 1)}{2(3\bar{d} + g - 1) - c},
\]

where \( 2\bar{d} = \deg B \), \( g \) be the geometric genus of \( B \), and \( c \) be the number of cusps, then \( B \) is the discriminant curve of a unique generic covering (the Chisini conjecture holds for \( B \)).

We can’t expect an analogous result in the case of singular surfaces, because for a curve \( B \) of even degree with at most A-D-E-singularities there always exists a double covering, which is generic. But if two generic coverings with given discriminant curve \( B \) are coverings of sufficiently big degree, then they are equivalent. More exactly, we prove the following theorem. Let there are two generic coverings \( f_1 : X_1 \to \mathbb{P}^2 \) and \( f_2 : X_2 \to \mathbb{P}^2 \) of surfaces with A-D-E-singularities and with the same discriminant curve \( B \subset \mathbb{P}^2 \). Let \( f_i^*(B) = 2R_i + C_i \), \( i = 1, 2 \). With respect to a pair of coverings \( f_1 \) and \( f_2 \) nodes and cusps of \( B \) are partitioned into four types: ss-, sp-, ps- and pp-nodes and cusps. For example, a sp-node \( b \in B \) is a node, which is a \( s \)-node for \( f_1 \) and a \( p \)-node for \( f_2 \). The number of sp-nodes is denoted by \( n_{sp} \). Then \( n = n_{ss} + n_{sp} + n_{ps} + n_{pp} \).

The analogous terminology is used for cusps.

**Theorem 0.2** If \( f_1 \) and \( f_2 \) are nonequivalent generic coverings, then

\[
\deg f_2 \leq \frac{4(3\bar{d} + g_1 - 1)}{2(3\bar{d} + g_1 - 1) - \iota_1},
\]

where \( 2\bar{d} = \deg B \), \( g_1 \) be the geometric genus of \( B \), and \( \iota_1 \) be the number of cusps on \( B \).
where \( g_1 = p_a(R_1) \) is the arithmetic genus of the curve \( R_1 \), and \( t_1 = 2n_{sp} + 2c_{sp} + c_{pp} \).

We apply the main inequality (2) to the proof of the Chisini conjecture in the case of generic pluricanonical coverings. Let \( S \) be a minimal model of a surface of general type. According to a theorem of Bombieri \( [BPV] \), if \( m \geq 5 \), then the \( m \)-canonical map \( \varphi_m : S \to \mathbb{P}^{p_m - 1} \), defined by the complete linear system numerically equivalent to \( |mK_S| \), is a birational morphism, which blows down \((-2)\)-curves on \( S \). Then the canonical model \( X = \varphi_m(S) \) has at most A-D-E-singularities. A generic projection \( f : X \to \mathbb{P}^2 \) is called a generic \( m \)-canonical covering for \( S \). We prove the following theorem.

**Theorem 0.3** Let \( S_1 \) and \( S_2 \) be minimal models of surfaces of general type with the same \( (K_S^2) \) and \( \chi(S) \), and let \( f_1 : X_1 \to \mathbb{P}^2 \), \( f_2 : X_2 \to \mathbb{P}^2 \) be generic \( m \)-canonical coverings with the same discriminant curve. Then for \( m \geq 5 \) the coverings \( f_1 \) and \( f_2 \) are equivalent.

Consider a subvariety \( \mathcal{H} \subset \text{Hilb} \times \text{Gr} \), parametrizing \( m \)-canonical coverings. Here \( \text{Hilb} \) is a subscheme of the Hilbert scheme, parametrizing numerically \( m \)-canonical embeddings \( X \subset \mathbb{P}^M \) of surfaces with A-D-E-singularities and fixed \( (K^2_S) \) and \( \chi(S) \), \( \text{Gr} \) is the Grassmann variety of projection centres from \( \mathbb{P}^M \) to \( \mathbb{P}^2 \), and \( \mathcal{H} \) consists of pairs \( (X \subset \mathbb{P}^M, L) \) such that a restriction to \( X \) of a projection with centre \( L \) is a generic covering. By theorem \( [7] \) there is a one-to-one correspondence between the set of irreducible (respectively, connected) components of \( \text{Hilb} \) and \( \mathcal{H} \). Let \( h : \mathcal{H} \to \mathbb{P}^\nu \) be a map, taking a covering to its discriminant curve. Denote by \( \mathcal{D} \) a variety of plane curves of degree \( d \) with A-D-E-singularities, among which the number of nodes \( \geq n_p \), and the number of cusps \( \geq p \), where \( d, n_p, c_p \) are defined by invariants of \( S \) (see §6). By theorem \( [3] \) it follows (cf. \( [3] \), §5)

**Corollary.** The map, induced by \( h \), from the set of irreducible (respectively, connected) components of the variety \( \mathcal{H} \) to the set of irreducible (respectively, connected) components of the variety \( \mathcal{D} \) is injective.

The proof of the main inequality (2) in \( [3] \) in the case of non-singular surfaces runs as follows. To compare two coverings \( f_1 \) and \( f_2 \), a normalization \( X \) of the fibre product \( X_1 \times_{\mathbb{P}^2} X_2 \) is considered. Let \( g_i : X \to X_i, \ i = 1, 2 \), be the corresponding mappings to the factors. The preimage \( g_i^{-1}(R_i) = R + C \) falls into two parts, where \( R \) is the curve mapped by \( g_2 \) to \( R_2 \), and \( C \) is the curve mapped by \( g_2 \) to \( C_2 \). If \( f_1 \) and \( f_2 \) are nonequivalent, then the surface \( X \) is irreducible, and if \( X_i \) are non-singular, then \( X \) is non-singular too. The main inequality is obtained by applying the Hodge index theorem to the pair of divisors \( R \) and \( C \) on \( X \). We use the same idea also in the case of surfaces with A-D-E-singularities. For this we carry out the local analysis of the normalization of the fibre product \( X \) in the case of generic coverings of surfaces with A-D-E-singularities.

In §1 we generalize to the case of surfaces with A-D-E-singularities the theorem on generic projections. In §2 a local analysis of a normalization of the fibre product \( X \) is carried out. In §3 we investigate the canonical cycle of an A-D-E-singularity, with the help of which we compute numerical invariants of a generic covering in §4. In §5 the main inequality (2) is proved. Finally, in §6 the Chisini conjecture for generic \( m \)-canonical coverings of surfaces of general type is proved.
1 Singularities of a generic projection of a surface with A-D-E-singularities.

In this section we prove Theorem 1.1.

1.1. A generic projection to $\mathbb{P}^3$. Let $X \subset \mathbb{P}^r$ be a surface of degree $\deg X = N$ with at most isolated hypersurface singularities $x_1, \ldots, x_k$, i.e. such that the dimension of the tangent spaces $\dim T_{X,x_i} = 3$. Denote by $\pi_L : \mathbb{P}^r \setminus L \to \mathbb{P}^{r-1}$ a projection from a linear subspace $L$ of codimension $e$. It can be obtained as a composition of projections with centers at points. The Theorem 1.1 on projections of $X$ to the plane ($e = 3$) is one of a series of theorems on generic projections for different $e$, beginning with projections from points ($e = r$) and finishing by projections to the line ($e = 2$), i.e. Lefschetz pencils.

A classical result is that, if $r > 5$ ($= 2 \dim X + 1$), then the projection from a generic point gives an isomorphic embedding of $X$ into $\mathbb{P}^{r-1}$. It follows that, if $e \geq 6$, then the projection from a generic subspace $L$ gives an isomorphic embedding of $X$ into $\mathbb{P}^{e-1}$. In particular, by a generic projection the surface $X$ is embedded into $\mathbb{P}^5$. When projecting to $\mathbb{P}^3$, $e = 5$, there appears isolated singularities on $\pi_L(X)$, which is not difficult to describe. To prove Theorem 1.1 we are going to consider a generic projection of $X$ into $\mathbb{P}^3$, $e = 4$, and to take advantage of the following theorem.

**Theorem 1.1** If $X \subset \mathbb{P}^r$ is a surface with at most isolated hypersurface singularities $x_i$, then the restriction of a projection $\pi_L : \mathbb{P}^r \setminus L \to \mathbb{P}^3$ with the centre in a generic subspace $L \subset \mathbb{P}^r$ of codimension 4 gives a birational map of $X$ onto a surface $Y \subset \mathbb{P}^3$, which is an isomorphism outside the double curve $D \subset X$ not passing through the points $x_i$, and $Y$ has, except the points $\pi_L(x_i)$, at most ordinary singularities – the double curve $\Delta = \pi_L(D)$, on which there lie a finite number of ordinary triple points and a finite number of pinches. In neighbourhoods of these points in appropriate local analytic coordinates $Y$ has normal forms as follows: $uv = 0$ for ordinary double points, $uvw = 0$ for ordinary triple points, $u^2 - vw^2 = 0$ for pinches (or "Whitney umbrellas").

The contemporary proof of this theorem one can find in the textbook [G-H]. The presence of singular points $x_i$ do not add extra troubles: we need only to see to thecentre of the projection $L$ not to intersect the tangent spaces $T_{X,x_i}$, $\dim T_{X,x_i} = 3$. A proof of this theorem one can find also in [M].

We want to prove that for a generic point $\xi \in \mathbb{P}^3$ the composition of projections $\pi_L$ and $\pi_\xi : \mathbb{P}^3 \setminus \xi \to \mathbb{P}^2$, i.e. the projection $\mathbb{P}^r \setminus \pi^{-1}_L(\xi) \to \mathbb{P}^2$ with the centre $\pi^{-1}_L(\xi)$, restricted to $X$, $\hat{f} = \pi_\xi \circ \pi_{L|x} : X \to \mathbb{P}^2$, gives a covering satisfying the properties stated in Theorem 1.1.

1.2. The dispostion of lines with respect to a surface $\mathbb{P}^3$. To describe a projection $\pi_\xi$ we need to investigate the disposition of lines $l \subset \mathbb{P}^3$ with respect to the surface $Y$. A line $l$ is called **transversal** to $Y$ at a point $y$, if it is transversal to the tangent cone to $Y$ at this point. It means that $(l \cdot Y)_y = 1$, if $y \notin \text{Sing } Y$; $(l \cdot Y)_y = 2$, if $y \in \Delta \setminus \Delta_t$ and $(l \cdot Y)_y = 3$, if $y \in \Delta_t$. We denote by $\Delta_t$ and $\Delta_p$ the set of triple points and the set of pinches. If $l$ is not transversal to $Y$ at a point $y$, we say that it is tangent to $Y$ at this point. A line $l$ is called a **simple tangent** to $Y$ at $y$, if $y \notin \text{Sing } Y$ and $(l \cdot Y)_y = 2$, or if $y \in \Delta \setminus (\Delta_t \cup \Delta_p)$ and $(l \cdot Y)_y = 3$, i.e. $(l \cdot Y)_y = 2$.
for one of two branches $Y_i$ at the point $y$. A line $l$ is called stationary tangent, respectively simple stationary tangent to $Y$ at $y$, if $y \notin \text{Sing } Y$ and $(l \cdot Y)_y \geq 3$, respectively $= 3$. A line $l$ is called stationary tangent, respectively simple stationary tangent to $Y$, if $l$ is transversal to $Y$ at all points, except one, at which $l$ is stationary tangent, respectively simple stationary tangent, and, besides the other points of intersection $l \cap Y$ are non-singular on $Y$. Finally, $l$ is called simple bitangent, if $l$ is transversal to $Y$ at all points, except two of them, at which the contact is simple, the tangent planes at them are distinct, and, besides, $l \cap \text{Sing } Y = \emptyset$. We want to prove that for a generic point $\xi \in \mathbb{P}^3$ all lines $l \ni \xi$ are at most simple bitangents and simple stationary tangents with respect to $Y$.

To study the disposition of lines $l \subset \mathbb{P}^3$ with respect to $Y$, we consider the Grassmann variety $G = G(1, 3)$ and the flag variety $\mathbb{F} = \{ (\xi, l) \in \mathbb{P}^3 \times G \mid \xi \in l \}$. There are two projections $\text{pr}_1 : \mathbb{F} \to \mathbb{P}^3$ and $\text{pr}_2 : \mathbb{F} \to G$, which are $\mathbb{P}^2$- and $\mathbb{P}^1$-bundles respectively; $\dim \mathbb{F} = 5$, and $\dim G = 4$. In the sequel we consider points $\xi \in \mathbb{P}^3$ as centres of projection $\pi_\xi : \mathbb{P}^3 \setminus \xi \to \mathbb{P}$. The fibre $\text{pr}_1^{-1}(\xi) \simeq \mathbb{P}^2$ is mapped by the projection $\text{pr}_2$ isomorphically onto $\mathbb{P}^2$. For $\xi \in \mathbb{P}^3$ there is a section $s_\xi : \mathbb{P}^3 \setminus \xi \to \mathbb{F}$ of the projection $\text{pr}_1$, $y \mapsto (y, \overline{\xi y})$. Then $\pi_\xi$ coincides with the restriction of the projection $\text{pr}_2$ to $s_\xi(\mathbb{P}^3 \setminus \xi)$.

Firstly, we consider the case, when a surface $Y$ is non-singular, and then we describe the necessary modifications and supplements in the case, when there is a double curve $\Delta$ and isolated singularities $s_i$ on $Y$.

Consider a filtration of the variety $\mathbb{F}$ by subvarieties

$$Z_k = \{ (\xi, l) \in \mathbb{F} \mid (l \cdot Y)_\xi \geq k \}.$$

Then $Z_1 = \text{pr}_1^{-1}(Y), \dim Z_1 = 4$. Over a generic point $l \in G$ the map $\varphi = \text{pr}_2|_{Z_1} : Z_1 \to G$ is an unramified covering of degree $N$. If there are no lines on $Y$, then $\varphi$ is a finite covering, and $Z_2$ is the ramification divisor of the covering.

Now consider restrictions of the projection $\text{pr}_1$. The variety $Z_2$ is isomorphic to a projectivized tangent bundle, $Z_2 \simeq \mathbb{P}(\Theta_Y)$, and $Z_2 \to Y$ is a $\mathbb{P}^1$-fibre bundle, $\dim Z_2 = 3$. At a generic point $y \in Y$ there are two asymptotic directions $l_1$ and $l_2$ in $T_{Y,y}$, for which $(l_1 \cdot Y)_y$ and $(l_2 \cdot Y)_y \geq 3$. Therefore, over a generic point the restriction of $\text{pr}_1$ onto $Z_3, \psi : Z_3 \to Y$, is a two-sheeted covering, the branch curve of which $P \subset Y$ is the parabolic curve consisting of points with coinciding asymptotic directions. Some fibres of the projection $\text{pr}_1$ are exceptional curves of the map $\psi$. Their images on $Y$ are points $y$, at which the restriction of the second differential of the local equation of $Y$ onto the tangent plane $T_{Y,y}$ vanishes. Such points $y$ are called the planar points of the surface $Y$. The curve $H = \psi(Z_4) \subset Y$ consists of points $y$, at which at least one of the numbers $(l_i \cdot Y)_y \geq 4$ ($H$ is a curve, if the surface $Y$ is not a quadric).

### 1.3. Absence of non simple stationary tangents

Consider a product $Y \times \mathbb{F} \subset Y \times \mathbb{P}^3 \times G$ and projections $\text{pr}_1'$ and $\text{pr}_2'$ onto $Y \times \mathbb{F}$ and $Y \times G$. We can consider the varieties $Z_k$ as subvarieties in $Y \times G \subset \mathbb{P}^3 \times G$. Consider a variety

$$I_4 = \{ (y; \xi, l) \in Y \times \mathbb{F} \mid (l \cdot Y)_y \geq 4 \} = (\text{pr}_2')^{-1}(Z_4).$$

The projection $\text{pr}_2' = \text{id}_Y \times \text{pr}_2$, as well as $\text{pr}_2 : \mathbb{F} \to G$, is a $\mathbb{P}^1$-bundle. Therefore, $\dim I_4 = 2$ and $\dim \Sigma_4 \leq 2$, where $\Sigma_4 = p_2(I_4)$, and $p_2$ is a projection of $Y \times \mathbb{P}^3 \times G$ to $\mathbb{P}^3$. Then, if $\xi \in \mathbb{P}^3 \setminus \Sigma_4$, we have that $(l \cdot Y)_y \leq 3$ for any line $l \ni \xi$ at any point $y \in Y$. 

6
1.4. Absence of non simple bitangents. Consider a variety $\Sigma_{2,3} \subset \mathbb{P}^3$, made up of non simple bitangents, and show that $\Sigma_{2,3} \leq 2$. Consider a product $Y \times Y \times \mathbb{P} \subset Y \times Y \times \mathbb{P}^3 \times G$ and subvarieties $I_{i,j}$, which are closures of

$$I_{i,j}^0 = \{(y_1, y_2; \xi, l) \in Y \times Y \times \mathbb{P} : (Y \cdot l)_{y_1} \geq i, (Y \cdot l)_{y_2} \geq j, y_1 \neq y_2\}.$$ 

Denote a projection of $Y \times Y \times \mathbb{P}$ to $Y \times Y \times G$ by $pr''_2$, and let $pr''_2(I_{i,j}) = \tilde{I}_{i,j}$. The projection $pr''_2$ and its restriction to $I_{i,j}, I_{i,j} \rightarrow \tilde{I}_{i,j}$ are $\mathbb{P}^1$-bundles.

Lemma 1.1

$$\dim I_{2,3} \leq 2.$$ 

Proof. Consider subvarieties

$$Y \times Y \times G \supset \tilde{I}_{1,1} \supset \tilde{I}_{2,1} \supset \tilde{I}_{2,2} \supset \tilde{I}_{2,3},$$

and let $q_1$ be a projection onto the first factor. Obviously, $\tilde{I}_{1,1}$ is an irreducible variety of dimension $\dim \tilde{I}_{1,1} = 4$, birationally isomorphic to $Y \times Y$. The projection $q_1 : \tilde{I}_{2,1} \rightarrow Y$ is a fibration, fibers of which are curves $q_1^{-1}(y) \simeq C_y$, where

$$C_y = Y \cap T_{Y,y}.$$ 

The curve $C_y$ has a singularity at the point $y$, which is a node for a generic point $y$.

Furthermore, the restriction of the projection to $\tilde{I}_{2,2}, q_1 : \tilde{I}_{2,2} \rightarrow Y$, is surjective, and its fibre over a point $y \in Y$ is

$$q_1^{-1}(y) = \{(y, y', l) \mid l \subset T_{Y,y} \text{ and } l \text{ is tangent to } y \text{ at } y'\}.$$ 

We want to prove that $q_1(\tilde{I}_{2,3})$ doesn’t coincide with $Y$, i.e. the embedding $Y \subset \mathbb{P}^3$ possesses the following property $(L_1)$: there exists a point $y \in Y$ such that all lines $l \subset T_{Y,y}$, passing through $y$, have at most simple contact with $C_y \setminus \{y\}$. We prove this below in 1.6 (Proposition 1.2) under the assumption that the embedding $Y \subset \mathbb{P}^3$ is obtained by a projection of an embedding "improved" by a Veronese embedding $v_k$, $k \geq 2$.

Thus, $\dim q_1(\tilde{I}_{2,3}) \leq 1$. A generic fibre of the map $q_1 : \tilde{I}_{2,3} \rightarrow Y$ is of dimension zero (it being one, $Y$ is a ruled surface and we obtain a contradiction to the property $(L_1)$), therefore, $\dim \tilde{I}_{2,3} \leq 1$ and, consequently, $\dim I_{2,3} \leq 2$.

Set $\Sigma_{2,3} = p_3(I_{2,3})$, where $p_3$ is a projection of $Y \times Y \times \mathbb{P}^3 \times G$ to $\mathbb{P}^3$. It follows from Lemma 1.1 that $\dim \Sigma_{2,3} \leq 2$. If $\xi \notin \Sigma_{2,3}$, then any line $l \ni \xi$, touching $Y$ at two points $y_1$ and $y_2$, has a simple contact at these points.

1.5. Absence of 3-tangents. Consider a product $Y \times Y \times Y \times \mathbb{P} \subset Y \times Y \times Y \times \mathbb{P}^3 \times G$ and subvarieties $I_{i,j,k}$, which are closures of

$$I_{i,j,k}^0 = \{(y_1, y_2, y_3; \xi, l) \in Y \times Y \times Y \times \mathbb{P} : (Y \cdot l)_{y_1} \geq i, (Y \cdot l)_{y_2} \geq j, (Y \cdot l)_{y_3} \geq k\},$$

where $y_1 \neq y_2 \neq y_3 \neq y_1$. Denote a projection of $Y \times Y \times Y \times \mathbb{P}$ onto $Y \times Y \times Y \times G$ by $pr''_2$, and let $I_{i,j,k} = pr''_2(I_{i,j,k})$. As above, it is clear that $\dim I_{1,1,1} = 4$, and $pr''_2$ being a $\mathbb{P}^1$-bundle, we have $\dim I_{1,1,1} = 5$. 

7
Lemma 1.2

\[ \dim I_{2,2,2} \leq 2. \]

Proof. Again consider a projection of \( X \times Y \times Y \times G \) and of its subvarieties

\[ Y \times Y \times Y \times G \supset \bar{I}_{1,1,1} \supset \bar{I}_{2,1,1} \supset \bar{I}_{2,2,1} \supset \bar{I}_{2,2,2}, \]

to the first factor. Consider \( q_1 : \bar{I}_{2,2,2} \to Y \). For a point \( y \in Y \) we have \( q_1^{-1}(y) = \{(y, y_2, y_3; l) \mid l \subset T_{Y,y}, l \text{ is tangent to } y \text{ at points } y_2 \text{ and } y_3 \in l\} \). Just as in Lemma 1.1 it is sufficient to prove that \( q_1(\bar{I}_{2,2,2}) \) doesn’t coincide with \( Y \). It means that there exists a point \( y \in Y \), possessing the following property \((L_2)\): none of the lines \( l \subset T_{Y,y} \), passing through \( y \), is not a bitangent, i.e. can’t touch \( C_y \setminus \{y\} \) at two different points. We prove this below in the following 1.6 (Proposition 1.2) under the assumption that the embedding \( Y \subset \mathbb{P}^3 \) is obtained by a projection of an embedding “improved” by a Veronese embedding \( v_k \).

Set \( \Sigma_{2,2,2} = p_4(I_{2,2,2}) \), where \( p_4 \) is a projection of \( X \times Y \times Y \times \mathbb{P}^3 \times G \) to \( \mathbb{P}^3 \). Then \( \dim \Sigma_{2,2,2} \leq 2 \) and if \( \xi \notin \Sigma_{2,2,2} \), then any line \( l \ni \xi \) touches \( Y \) at most at two points.

1.6. Embeddings with a Lefschetz property. The properties \((L_1)\) and \((L_2)\) in the two previous subsections mean that there exists a point \( y \in Y \), for which the projection \( \pi_y \) of the curve \( C_y \setminus \{y\} \subset T_{Y,y} \simeq \mathbb{P}^2 \) from the point \( y \) is a Lefschetz pencil. Thus, to prove Lemmas 1.1 and 1.2 it is necessary to prove the existence of a point \( y \in Y \) possessing the following “Lefschetz property” \((L)\) with respect to the embedding into \( \mathbb{P}^3 \). We formulate it for a surface \( X \) embedded into a projective space of any dimension.

Let \( X \subset \mathbb{P}^r \) be an embedding into the projective space. We say, that a hyperplane \( L_1 \subset \mathbb{P}^r \) possesses a property \((L)\), if the curve \( X \cap L_1 \) has at most one node, i.e. \( L_1 \) touches \( X \) at a unique point \( x \), at which the curve \( X \cap L_1 \) has an ordinary quadratic singularity. In other words, the point \([L_1] \in \mathbb{P}^r\), corresponding to \( L_1 \), is a non-singular point of the dual variety \( X^\vee \).

We say that a pair \((L_1, L_3)\), where \( L_3 \subset L_1 \) is a linear subspace of dimension \( r - 3 \), possesses a property \((L)\), if \( L_1 \) possesses the property \((L)\), \( x \in L_3 \), and a projection of the curve \( X \cap L_1 \rightarrow \mathbb{P}^3 \) from the centre \( L_3 \) is a Lefschetz pencil, i.e. any fibre of this (rational) mapping contains one singular point, and this point is at most quadratic (is of multiplicity 2). We say that an embedding \( X \subset \mathbb{P}^r \) possesses a property \((L)\), if \( \exists x \in X \), for which \( L_1 = T_{X,x} \) possesses the property \((L)\), and \( L_1 \) can be added to a pair \((L_1, L_3)\) with the property \((L)\).

It is clear that, if a pair \((L_1, L_3)\) possesses the property \((L)\) and \( Y \subset \mathbb{P}^3 \) is obtained from \( X \) by projection from a centre \( L_4 \subset L_3, \dim L_4 = r - 4 \), then the embedding \( Y \subset \mathbb{P}^3 \) possesses the property \((L)\).

Proposition 1.1 If \( S \subset \mathbb{P}^d \) is an embedding of a non-singular surface, and \( X \subset \mathbb{P}^r \) is its composition with the Veronese embedding \( v_k \) defined by polynomials of degree \( k \), then the embedding \( X \subset \mathbb{P}^r \) possesses the property \((L)\).

Proof. Consider the hyperplane \( L_1 \) corresponding to a point \([L_1] \in X^\vee \setminus \text{Sing } X^\vee \). Then the curve \( C = X \cap L_1 \) contains a unique singular point – a node \( x \in C \). Let \( i : C \to X \) be the embedding. Consider a projection \( \pi_{k,x} : \mathbb{P}^r \setminus x \to \mathbb{P}^{r-1} \) from the point \( x \). To prove Proposition 1.1 it is enough to show that the image \( \pi_{k,x}(C) \) is a non-singular curve in \( \mathbb{P}^{r-1} \). For then,
Let $I_x$ be the ideal sheaf of the point $x$ on $S$, and $\mathcal{O}_S(1)$ be the sheaf of hyperplane sections. Under the identification $v_k : S \simeq X$, the map $\pi_{k,x}$ is given by sections of $H^0(S, \mathcal{O}_S(k) \otimes I_x)$. Let $k = 2$ and let $\sigma : \overline{S} \to S$ be a $\sigma$-process with centre at the point $x$. We can assume that $\overline{S}$ is embedded into $\mathbb{P}^{r-1}$, where $r_2 - 1 = q(q + 3)/2$, and the rational map $\sigma^{-1} : S \to \overline{S}$ is given by sections of $H^0(S, \mathcal{O}_S(2) \otimes I_x)$, i.e. it coincides with $\pi_{2,x}$. Since the proper transform $\overline{C} = \sigma^{-1}(C) \subset \overline{S}$ is a non-singular curve, we obtain Proposition [1] in the case $k = 2$. Besides, note that sections of $i^*(H^0(S, \mathcal{O}_S(2) \otimes I_x))$ give an embedding of $\overline{C}$ into $\mathbb{P}^{r-1}$. Consequently, for $k > 2$ sections of $i^*(H^0(S, \mathcal{O}_S(k) \otimes I_x))$ also give an embedding of $\overline{C}$ into $\mathbb{P}^{r-1}$, since there is a natural injection $H^0(S, \mathcal{O}_S(k - 2)) \otimes H^0(S, \mathcal{O}_S(2) \otimes I_x) \subset H^0(S, \mathcal{O}_S(k) \otimes I_x)$, and sections of $H^0(S, \mathcal{O}_S(k - 2))$ have no base points and fixed components. Therefore, sections of $i^*(H^0(S, \mathcal{O}_S(k) \otimes I_x))$ separate points and tangent directions on $\overline{C}$.

We say that a linear subspace $L_4$ of dimension $r - 4$ possesses a property (L) with respect to an embedding $X \subset \mathbb{P}^r$, if the projection $\pi_{L_4}$ to $\mathbb{P}^3$ from the centre $L_4$ maps $X$ onto a surface $Y = \pi_{L_4}(X)$ with ordinary singularities.

As is known (see [G-H]), there is an open subset $U$ in the Grassmannian $G_4 = Gr(r - 4, r)$, points of which correspond to linear subspaces with the property (L).

**Proposition 1.2** If an embedding $X \subset \mathbb{P}^r$ possesses the property (L), then there exists a linear subspace $L_4$ with the property (L), which can be added to a flag $L_1 \supset L_3 \supset L_4$ such that the pair $(L_1, L_3)$ possesses the property (L). In other words, there exists a projection to $\mathbb{P}^3$, for which the embedding $Y \subset \mathbb{P}^3$, where $Y$ is the image of $X$, possesses the property (L).

**Proof.** Let $G_1 = \mathbb{G}r$ be the dual space to $\mathbb{P}^r$, $G_3 = G(r - 3, r)$ be the Grassmann variety of linear subspaces $L_3$ of dimension $r - 3$, and $F = F_1, 3, 4 \subset \mathbb{G}r \times G_3 \times G_4$ be the variety of flags $L_1 \supset L_3 \supset L_4$. Let $X^\vee$ be the dual variety, $W \subset X \times X^\vee \subset \mathbb{G}r \times \mathbb{G}r$ be a closed subvariety $W = \{(x, L_1) : L_1 \supset T_{X,x}\}$. Then the projection of $W \to X^\vee$ is an isomorphism over $X_0^\vee = X^\vee \setminus \text{Sing } X^\vee$, $W_0 \simeq X_0^\vee$.

Denote by $Z \subset X \times F$ a closed subvariety

$$Z = \{(x; L_1 \supset L_3 \supset L_4) \mid (x, L_1) \in W, L_3 \supset x\},$$
and by $Z_0 \subset Z$ an open subset: $(x, L_1) \in W_0$. Then $Z$ is an irreducible variety. Consider a projection $Z \to W_0$. The fibres are not empty by the previous proposition, and each of the fibres contains an open set of points $z$, for which the pair $(L_1, L_3)$ possesses the property (L) (because the centres of projections for Lefschetz pencils form an open set). Therefore, $Z$ contains an open set $Z_L$, for points of which the pair $(L_1, L_3)$ possesses the Lefschetz property.

Obviously, the map $Z \to G_4$ is surjective. Therefore, $p_4^{-1}(U)$, where $p_4$ is a projection of $Z$ to $G_4$, is a non empty Zariski open set. Then $Z_L \cap p_4^{-1}(U)$ is not empty, and if $(x; L_1 \supset L_3 \supset L_4)$ is a point of this set, then $L_4$ possesses the desired property.

1.7. Projecting in a neighbourhood of a generic point of the double curve $\Delta$. Now let $Y \subset \mathbb{P}^3$ has ordinary singularities along the double curve $\Delta$ and isolated singularities $s_i \in$
there are two asymptotic directions for each of two branches of $Y$, including triple points and pinches, and also at singular points.

Consider $Y \times \mathbb{F}$. In addition to notations in 1.3, let $q_1$ and $q_2$ the projections of $Y \times G$ to $Y$ and $G$. Consider the intersection $\tilde{A} = (\Delta \times G) \cap Z_3$. Then the restriction of the projection $Y \times G \to Y$ to $\tilde{A}$, $\tilde{A} \to \Delta$, is a covering of degree 4 over a generic point: at a point $y \in \Delta$ there are two asymptotic directions for each of two branches of $Y$ at $y$. Therefore, $\tilde{A}$ is a curve.

Denote by $\Sigma_0$ the union of planes in $\mathbb{P}^3$ composing the tangent cones at the rest points of $\Delta$, including triple points and pinches, and also at singular points $s_i \in Y$.

1.8. Projecting in a neighbourhood of a triple point. If $\xi \notin \Sigma_0$, then in a neighbourhood of a point $y \in \Sigma_0$ all lines $l \ni \xi$ are transversal to each of the three branches of $Y$ at $y$, and therefore, locally these branches are mapped isomorphically onto $\mathbb{P}^2$.

1.9. Projecting in a neighbourhood of a pinch. In a neighbourhood of a pinch $y \in Y$ there are coordinates, by which $Y$ is locally defined by an equation $u^2 = vw^2$. The double curve $\Delta \subset Y$ is a line $u = w = 0$, and the tangent cone $C_{Y,y}$ to $Y$ at $y$ has an equation $u = 0$. In a neighbourhood of a pinch a normalization $\mathbb{C}^2 \to Y$ is defined by formulae $u = tw, v = t^2, w = w$.

Since $X$ is non-singular and $\pi_L$ is a finite map, we can assume that the projection $\pi_L$ is the normalization. If a point $\xi$ does not belong to the tangent cone $C_{Y,y}$, then the projection $\pi_\xi$ locally is a map of gedrr 2. A projection $f : X \to \mathbb{P}^2$ a neighbourhood of the preimage of a pinch is a two-sheeted covering of non-singular varieties, and, hence, locally is defined by equations $v = t^2, w = w$. Thus, the curve $\tilde{R} \subset Y$ goes through the pinch, and pinches are projected to non-singular points of the discriminant curve $B$.

1.10. Normal forms of a generic projection at points of the ramification curve.

Lemma 1.3 ([A]) Let $(X,0) \subset (\mathbb{C}^3,0)$ be a non-singular surface, and $(\mathbb{C}^3,0) \to (\mathbb{C}^2,0)$ be a smooth morphism, the restriction of which $f : X \to \mathbb{C}^2$ is a finite covering of degree $\mu$. Then one can choose local coordinates $x,y$ in $\mathbb{C}^2$ and $x,y,z$ in $\mathbb{C}^3$ such, that $X$ is defined by an equation

$$y = z^\mu + \lambda_1(x)z^{\mu-2} + \ldots + \lambda_{\mu-2}(x)z,$$

and $f$ is a projection along $z$ axis.

Proof. This is Lemma 1 in Arnol’d paper [A]. It is obtained, if we consider the covering $f$ as a 2-parameter family of 0-dimensional hypersurface singularities of multiplicity $\mu$, and, consequently, $f$ is induced by the miniversal deformation of the singularity of type $A_{\mu-1}$.

We proved that at a generic point $P$ of the ramification curve a projection $f : X \to \mathbb{P}^2$ is of degree $\mu = 2$, and at isolated points it is of degree $\mu = 3$. By Lemma 1.3 for $\mu = 2$ we obtain that at a generic point of the ramification curve a generic projection is equivalent to a projection of the surface $X : x = z^2$ to the $x,y$-plane, i.e. it is a fold. For $\mu = 3$ we obtain
Corollary 1.1 For a non-singular surface $X$ a finite covering $X \to \mathbb{C}^2$ of degree 3 locally is a projection to the $x, y$-plane of one of the surfaces

$$y = z^3 + x^k z, \; k = 1, 2, \ldots, \text{ or } y = z^3 (k = \infty).$$

In the case $k \neq \infty$ the ramification curve $C$ is reduced and has an equation $3z^2 + x^k = 0$ in local coordinates $x, z$ on $X$. The curve $C$ is non-singular only for $k = 1$. The discriminant curve $B$ has an equation $4x^{3k} + 27y^2 = 0$, i.e. $B$ is a cusp. It is an ordinary cusp only for $k = 1$.

Proof. By lemma [1.3] we have that $X$ is defined by an equation $y = z^3 + \lambda_1(x)z$. We obtain the stated normal form of the covering $f$, where $k$ is the order of vanishing of $\lambda_1(x)$ at the point $x = 0$. The ramification curve $C$ is defined by equation $J = 0$, where $J = 3z^2 + x^k$ is the Jacobian of the covering $f$. The discriminant curve $B$ is defined by 0th Fitting ideal $F_0(f_*\mathcal{O}_C)$ of the sheaf $f_*\mathcal{O}_C$. To obtain an equation of $B$ — the generator of the Fitting ideal, we need to take a finite presentation $f_*\mathcal{O}_X \to f_*\mathcal{O}_X \to f_*\mathcal{O}_C \to 0$ of the sheaf $f_*\mathcal{O}_C$, where $(f_*\mathcal{O}_X)_0 = \mathbb{C}\{x, z\} = \mathbb{C}\{x, y\} \oplus \mathbb{C}\{x, z\}z \oplus \mathbb{C}\{x, y\}z^2$, and to compute a determinant of the $\mathbb{C}\{x, y\}$-linear map $J$, which is a multiplication by the Jacobian $J$. ■

Now we show that for a generic projection the discriminant curve $B$ has at most ordinary nodes and cusps. Let $b \in B$ be a point corresponding to a bitangent $l$ under projecting $\pi_\xi : \mathbb{P}^3 \setminus \xi \to \mathbb{P}^2$ from a point $\xi$. Let $l$ touches $Y$ at points $P_1$ and $P_2$, to which correspond branches $B_1$ and $B_2$ of the discriminant curve $B$ at a point $b$. We have to show that for a generic projection the point $b$ is a node, i.e. the branches $B_1$ and $B_2$ have different tangents. Determine where does the centres $\xi$ of ”bad” projections lie. Let a line $\lambda \subset \mathbb{P}^2$ is a common tangent to branches $B_1$ and $B_2$ at a point $b$. Then the plane $\pi_\xi^{-1}(\lambda)$ is bitangent — it touches the surface $Y$ at points $P_1$ and $P_2$. Consider the dual surface $Y^\vee \subset \mathbb{P}^3$. Then the point $[\pi_\xi^{-1}(\lambda)] \in \text{Sing } Y^\vee = \gamma^\vee$. Set $\gamma = \tau^{-1}(\gamma^\vee)$, where $\tau : Y \to Y^\vee$ is the Gauss map. Let $\Sigma_\mu \subset \mathbb{P}^3$ be a ruled surface composed by lines $P_1P_2$, where $P_1, P_2 \in \gamma$, $\tau(P_1) = \tau(P_2) = [\pi_\xi^{-1}(\lambda)]$. Then, if $\xi \notin \Sigma_\mu$, then at points $b$, corresponding to bitangents $l$, the curve $B$ has at most nodes.

Now let $b \in B$ be a point corresponding to a stationary tangent $l$ at a point $P \in Y$. As was noted above, in a neighbourhood of $P$ the projection $\pi_\xi$ is equivalent to a projection of a surface $y = z^3 + x^k z$ to the $x, y$-plane. We have to show that for a generic projection the exponent $k = 1$. The fact is that, if $k > 1$, then the point $P$ is a planar point of the surface $Y$. Excepting the centres of projection lying in tangent planes to $Y$ at planar points, we obtain that in a neighbourhood of a point with $\mu = 3$ the projection $f$ is equivalent to a projection of a surface $y = z^3 + xz$ to the $x, y$-plane, i.e. it is a pleat.

1.11. Projecting in a neighbourhood of an isolated double plane singularity.

Lemma 1.4 If $(X, 0) \subset (\mathbb{C}^3, 0)$ is an (isolated) double plane singularity $z^2 = h(x, y)$, $\pi : X \to \mathbb{C}^2$ be a projection from any point $p \in \mathbb{C}^3$, not lying in the tangent cone $z = 0$, then the ramification curve of $\pi$ is reduced, and the discriminant curve $B \subset \mathbb{C}^2$ is locally analytically isomorphic to the singularity $h(x, y) = 0$.

Proof. The singularity $(X, 0)$ is of multiplicity 2. Therefore, $\pi$ is a covering of degree 2, and, consequently, is locally a projection of a double plane $w^2 = g(u, v)$ to the $(u, v)$-plane. 11
Thus, the germs of singularities $h(x, y)$ and $g(u, v)$ are stably isomorphic, and hence isomorphic ([AGN], vol.1).

\section{Local structure of fibre products.}

\subsection{Local structure of a generic covering.}

Let $f : X \to \mathbb{P}^2$ be a generic covering of the plane by a surface $X$ with A-D-E-singularities, and let $B \subset \mathbb{P}^2$ be the discriminant curve, $f^*(B) = 2R + C$. Singular points $o \in \text{Sing} X$ will be called $s$-points of the surface $X$ (from the word singularity). In a neighbourhood of a $s$-point $o$ the covering $f$ is isomorphic to the projection to $x, y$-plane of a surface $z^2 = h(x, y)$, where $h(x, y)$ has one of the A-D-E-singularities. Singular points $o$ on $X$ correspond to singular points of the same type as $o$ on $B$.

With respect to $f$ non-singular points of $X$ are partitioned into $r$-points (from the word regular), at which the morphism $f$ is étale, and $p$-points (from the words singularity of projection) – they are points of the ramification curve $R$. A $p$-point is either a fold (or a singular $p$-point of type $A_1$), in a neighbourhood of which $f : (x, z) \mapsto (x, y), y = z^2$, or a pleat $o \in R \cap C$ (or a singular $p$-point of type $A_2$), in a neighbourhood of which $f : (x, z) \mapsto (x, y), y = z^3 - 3xz$ (more details about this see in section 2.4 below).

The singular points of $B$ ‘originated’ from singular points $\text{Sing} X$ we call $s$-points. There are additional singular points of type $A_1$ (nodes) and of type $A_2$ (cusps), which we call $p$-nodes and $p$-cusps. Over a generic point $b \in B$ there lie one fold and $N - 2$ $r$-points; over $p$-node there lie two folds and $N - 4$ $r$-points; over a $p$-cusp there lie one pleat and $N - 3$ $r$-points; over a $s$-node or a $s$-cusp, as over any $s$-point $b \in B$, there lie one singular point of $X$ and $N - 2$ $r$-points.

\subsection{Types of points on the fibre product.}

With respect to a pair of generic coverings $f_1 : X_1 \to \mathbb{P}^2$ and $f_2 : X_2 \to \mathbb{P}^2$ with the same discriminant curve $B \subset \mathbb{P}^2$ nodes and cusps on $B$ are partitioned by this time into 4 types: $ss$-, $sp$-, $ps$- and $pp$-nodes and cusps. For example, a $ps$-node it is a node $b \in B$, such that there are two folds on $X_1$ over $b$, and on $X_2$ over $b$ there is a singular point of type $A_1$. The analogous terminology is used for the classification of points $\bar{x} = (x_1, x_2)$ on the fibre product $X^x = X_1 \times_{\mathbb{P}^2} X_2$ : we say about $rs$-points, $ss$-points, $sp$-points, etc. For example, we say that $\bar{x}$ is a $ps$-point of type $A_2$, if $x_1 \in X_1$ is a $p$-point of type $A_2$, and $x_2 \in \text{Sing} X_2$ is a singular $s$-point of type $A_2$.

In this section we describe the structure of a normalization $\nu : X = (X^x)^{(\nu)} \to X^x$ of the fibre product $X^x$. Denote by $g_1, g_2$ and $f$ the morphisms of $X$ to $X_1, X_2$ and $\mathbb{P}^2$. Since the normalization is defined locally, we can replace $\mathbb{P}^2$ by a neighbourhood of the point $0 \in \mathbb{C}^2$ and to assume that $X_1$ and $X_2$ are neighbourhoods of points $x_1 \in X_1$ and $x_2 \in X_2$. We pass on to an item-by-item examination of all possible types of points $\bar{x} = (x_1, x_2) \in X^x$. We do it up to the permutation of factors $X_1$ and $X_2$.

At first we consider quite trivial cases.

\subsubsection{If $\bar{x}$ is a $r*$-point (where $* = r, s, p$), then $X^x$ at the point $\bar{x}$ is locally the same as $X_2$ at the point $x_2$, and $f^* : X^x \to \mathbb{C}^2$ is locally the same as $f_2 : X_2 \to \mathbb{C}^2$.}

\subsubsection{If $\bar{x}$ is a $2 \times 2$-point, i.e. $x_1$ and $x_2$ are points of ‘double planes’, $z_1^2 = h(x, y), z_2^2 =
$h(x,y)$, then $X^* = X_1 \times_{C^2} X_2$ in a neighbourhood of the point $\bar{x} = (x_1, x_2)$ is a surface in $\mathbb{C}^4 \ni (x,y,z_1,z_2)$, defined by equations $z_1^2 = h(x,y)$, $z_2^2 = h(x,y)$. We obtain that $z_1^2 = z_2^2$ and hence $X^* = X_1^* \cup X_2^*$, where $X_1^* : z_1^2 = h(x,y), z_2 = z_1$, $X_2^* : z_2^2 = h(x,y), z_1 = -z_2$. The surfaces $X_1^*$ and $X_2^*$ meet along a curve $z_1 = z_2 = 0, h(x,y) = 0$. We obtain that a normalization $X = \tilde{X}^* = X_1^* \sqcup X_2^*$ locally consists of two disjoint components $X_1^*$ and $X_2^*$ isomorphic to $X_1$ and $X_2$.

In particular, we obtain a description of the normalization in a neighbourhood of a pp-point $(x_1, x_2) \in X^*$ lying over a non-singular point of $B$, $B : x = 0$.

Every ss-point is a $2 \times 2$-point. Thus, in a neighbourhood of a ss-point the normalization has the same local structure as in the case of a pp-point above: $X$ locally consists of two disjoint components isomorphic to $X_1$ and $X_2$.

It remains to examine less trivial cases when $\bar{x}$ is a pp- or sp-point of type $A_1$ or $A_2$. This is done in the following two subsections.

2.3. On fibre product of double planes.

2.3.1. The ordinary quadratic singularity – the singularity of type $A_1$ on a surface $X_0 : z^2 = xy$ can be considered as a 2-sheeted covering of the plane $f_0 : X_0 \to \mathbb{C}^2$ branched along a node $B : xy = 0$. As is known, the singularity $X_0$ itself can be considered as a quotient singularity under the action of cyclic group $\mathbb{Z}_2 = \{\pm 1\}, X_0 = X/\mathbb{Z}_2$, where $X = \mathbb{C}^2 \ni (z_1, z_2)$, and a generator of $\mathbb{Z}_2$ acts by the rule: $z_1 \mapsto -z_1, z_2 \mapsto -z_2$. The factorization morphism $g_0 : X \to X_0$ is defined by formulae

$x = z_1^2, y = z_2^2, z = z_1z_2$.

We obtain a 4-sheeted covering $f = f_0 \circ g_0 : X \to \mathbb{C}^2$, which can be considered as a factorization under the action of the group $G = \mathbb{Z}_2 \times \mathbb{Z}_2$ on $X$. Then the factorization $g_0$ corresponds
to a subgroup of order two $\mathbb{Z}_2 = G_0 = \{(1, 1), (−1, −1)\}$, imbedded diagonally into $G$. In $G$ there are two more subgroups of order two: $G_1 = \{1\} \times \mathbb{Z}_2$ and $G_2 = \mathbb{Z}_2 \times \{1\}$. Considering $X_1 = \mathbb{C}_2/G_1 \simeq \mathbb{C}_2$ and $X_2 = \mathbb{C}_2/G_2 \simeq \mathbb{C}_2$, we obtain two more decompositions of $f$ and a commutative diagram

\[
X = \mathbb{C}^2 \ni (z_1, z_2)
\]

\[
\begin{array}{ccc}
g_1 & & g_2 \\
\uparrow & & \uparrow \\
(z_1, y) & \in & \mathbb{C}^2 = X_1 \\
\downarrow & & \downarrow \\
X_0 & \rightarrow & X_2 = \mathbb{C}^2 \ni (x, z_2) \\
\downarrow & & \downarrow \\
(x, y) & \in & \mathbb{C}^2
\end{array}
\]

\text{(2)}

where $g_1 : y = z_2^2$, $f_1 : x = z_1^2$, $g_2 : x = z_1^2$, $f_2 : y = z_2^2$.

Denote by $B_1 : x = 0$, $B_2 : y = 0$ the branches of $B : xy = 0$, and by $R' : z_1 = 0$, $R'' : z_2 = 0$ the branches of their proper transform $z_1 z_2 = 0$ on $X$.

The diagram (2) shows that we can consider $X$ as a normalization in three cases:

2.3.2. $X$ is a normalization in a neighbourhood of a ps-point of type $A_1$, $\bar{x} \in X_1^x = X_1 \times_{\mathbb{C}_2} X_0$, $f_1^*(B_1) = 2R_1$, $f_1^*(B_2) = C_1$; $g_1^*(R_1) = R'$, $g_1^*(C_1) = 2R''$; $f_0^*(B) = 2R_2 = 2(R'_2 + R''_2)$; $g_0^*(R_2') = R'$; $g_0^*(R''_2) = R''$.

($g_0$ is unramified outside the point $0 \in X_0$).
2.3.3. $X$ is a normalization in a neighbourhood of a sp-point of type $A_1$, $\bar{x} \in X_2^x = X_0 \times \mathbb{C}^2 X_2$ (the case symmetric to 2.3.2.)

2.3.4. $X$ is a normalization in a neighbourhood of a pp-point of $A_1$, $\bar{x} \in X_2^x = X_1 \times \mathbb{C}^2 X_2$, which is not a $2 \times 2$-point, $f_1 : x = z_1^2$, $f_2 : y = z_2^2$.

Fig. 3

$f_1^*(B) = 2R_1 + C_1$, $g_1^*(R_1) = R'$, $g_1^*(C_1) = 2R''$,

$f_2^*(B) = 2R_2 + C_2$, $g_2^*(R_2) = R''$, $g_2^*(C_2) = 2R'$.

Using 2.3.2-2.3.4, now we can describe a normalization $X$ over a neighbourhood of a node $b \in B$.

2.3.5 Over a neighbourhood of a ps-node $b \in B$ (as well as a sp-node) a normalization of $X^x$ in a neighbourhood of a ps-point looks like as

Fig. 4
\[ g_1^*(R'_1) = R'' + R'' + C'' , \quad g_1^*(R''_1) = R'' + R'' + C'' \]
\[ g_2^*(R'_2) = R' + R' + C' , \quad g_2^*(R''_2) = R'' + R'' \]

On Fig. 4 the normalization in neighbourhoods of pr-, rs- and rr-points of \( X^\times \) is not pictured.

2.3.6. Over a neighbourhood of a pp-node \( b \in B \) a normalization of \( X^\times \) in a neighbourhood of a pp-point looks like as:

2.4. On coverings of \( \mathbb{C}^2 \) unbranched outside a cusp \( B : y^2 = x^3 \). To describe a normalization of the fibre product in a neighbourhood of a sp- and pp-point of type \( A_2 \) in a natural context, we begin with reminding of a small topic from singularity theory.

2.4.1 The singularity of cuspidal type of a map (pleat) and the miniversal deformation of a singularity of type \( A_2 \). A cusp \( (B, 0) \subset (\mathbb{C}^2, 0) \) is defined by a germ of function \( x^3 - y^2 \) stable equivalent to a germ of function \( x^3 \). It is a simple singularity of type \( A_2 \). It is interesting that a cusp (a singularity of type \( A_2 \) ) appears also on the discriminant in the base of the miniversal deformation of the same singularity of type \( A_2 \).

As is known, the miniversal unfolding of the function \( t = z^3 \) is

\[ \mathbb{C} \times \mathbb{C}^2 \rightarrow \mathbb{C} \times \mathbb{C}^2 , \quad (z, a_2, a_3) \mapsto (z^3 + a_2 z + a_3, a_2, a_3) . \]

The restriction of this map over \( \{0\} \times \mathbb{C}^2 \) gives a miniversal deformation \( F \) of a zero-dimensional singularity \( z^3 = 0, \mathbb{C}^3 \supset X \xrightarrow{F} \mathbb{C}^2 \). Here \( X \) is a surface \( z^3 + a_2 z + a_3 = 0 \), and \( F \) is induced by projection onto \( (a_2, a_3) \). The surface \( X \) is a graph of function \( -a_3 = z^3 + a_2 z ; z \) and \( a_2 \) are local coordinates on \( X \),
\[(a_2, z) \in \mathbb{C}^2 \xrightarrow{\sim} X \subset \mathbb{C}^3\]

\[G \downarrow F \quad , \quad G: \begin{cases} a_2 &= a_2 \\ -a_3 &= z^3 + a_2z. \end{cases}\]

We obtain a 3-sheeted covering \(G : \mathbb{C}^2 \to \mathbb{C}^2\), the ramification curve of which \(R\) is defined by the equation \(3z^2 + a_2 = 0\), and the discriminant (branch) curve \(B = G(R)\) is defined by equation

\[4a_2^3 + 27a_3^2 = 0.\]

To bring the equation of \(B\) to the form \(y^2 = x^3\), we make a substitution

\[a_2 = -3x, \ a_3 = 2y,\]

and denote \(\mathbb{C}^2 \simeq X\) by \(X_3\), and \(G\) by \(f_3\).

**Fig. 6**

We obtain a 3-sheeted covering \(f_3 : X_3 \to \mathbb{C}^2\),

\[f_3 : \ x = x, \ y = -\frac{1}{2}(z^3 - 3xz).\]

Then \(x^3 - y^2 = x^3 - \frac{1}{4}(z^3 - 3xz)^2 = (x - z^2)^2(x - \frac{1}{4}z^2)\) and, consequently,

\[f_3^*(B) = 2R + C,\]

where \(R : x = z^2\) is the ramification curve, and \(C : x = \frac{1}{4}z^2\). Note that \(C\) and \(R\) are tangent of order two, \((C \cdot R) = 2\).

By Lemma 1.3 the singular point of the covering \(f_3\) is uniquely characterized as a singular point of a 3-sheeted covering \(f : X \to \mathbb{C}^2\) by a non-singular surface \(X\), the discriminant curve of which is an ordinary cusp.
2.4.2 The Viète map $f_6$. We produce a well known regular covering of $\mathbb{C}^2$ with group $S_3$ branched along a cusp $B : y^2 = x^3$, which appears to be a normalization of the fibre product in a neighbourhood of a sp-point of type $A_2$. This covering naturally appears in singularity theory.

Consider a quotient of the space $\mathbb{C}^3$ under the action of permutation group $S_3$. We get the Viète map

$$v : \mathbb{C}^3 \rightarrow \mathbb{C}^3, \ (z_1, z_2, z_3) \mapsto (a_1, a_2, a_3),$$

where $(z - z_1)(z - z_2)(z - z_3) = z^3 + a_1 z^2 + a_2 z + a_3$, i.e.

$$a_1 = -(z_1 + z_2 + z_3), \ a_2 = z_1 z_2 + z_2 z_3 + z_3 z_1, \ a_3 = -z_1 z_2 z_3.$$ 

The map $v$ is a map of degree 6 unramified outside $\Delta = \bigcup_{i \neq j} \{z_i = z_j\}$, and $v(\Delta) = D$ is defined by the discriminant of a polynomial of degree three.

The action of $S_3$ on $\mathbb{C}^3$ is reducible: $\mathbb{C}^3$ is a direct sum $\mathbb{C}^3 = \mathbb{C} \oplus \mathbb{C}^2$ of invariant subspaces of the line $\mathbb{C} = \{z_1 = z_2 = z_3\}$ and of the plane $\mathbb{C}^2 = \{z_1 + z_2 + z_3 = 0\}$. Consider the restriction of $v$ to this plane $\mathbb{C}^2$,

$$(z_1, z_2, z_3) \in \mathbb{C}^3 \supset \{z_1 + z_2 + z_3 = 0\} = \mathbb{C}^2 \rightarrow \mathbb{C}^2 = \{a_1 = 0\} \subset \mathbb{C}^3 \ni (a_1, a_2, a_3).$$

Set $\mathbb{C}^2 \cap \Delta = L \ , \ \mathbb{C}^2 \cap D = B$. Then $L$ consists of three lines

$$L = L_1 + L_2 + L_3, \text{ where } L_i : z_j = z_k, \ z_1 + z_2 + z_3 = 0 \ , \ \{i, j, k\} = \{1, 2, 3\},$$

and the curve $B : 4a_2^3 + 27a_3^2 = 0$ is defined by the discriminant of the polynomial $z^3 + a_2 z + a_3$. Since $\pi_1(\mathbb{C}^2 \setminus L) = \pi_1(\mathbb{C}^3 \setminus \Delta), \ \pi_1(\mathbb{C}^2 \setminus B) = \pi_1(\mathbb{C}^3 \setminus D) = B_{13}$, we obtain a covering $v : \mathbb{C}^2 \rightarrow \mathbb{C}^2$ of degree 6 unbranched apart from $B$. Denote this map by $f_6$. In coordinates $x, y$, where $a_2 = -3x, a_3 = 2y$, this map

$$\mathbb{C}^2 = \{z_1 + z_2 + z_3 = 0\} = X_6 \xrightarrow{f_6} \mathbb{C}^2 \ni (x, y)$$

is defined by formulae

$$f_6 : x = -\frac{1}{3}(z_1 z_2 + z_2 z_3 + z_3 z_1), \ y = -\frac{1}{2}z_1 z_2 z_3,$$

the discriminant $B$ has equation $y^2 = x^3$, and $f^* (B) = 2L = 2L_1 + 2L_2 + 2L_3$ (it is easy to see that $x^3 - y^2 = \frac{1}{12}(z_2 - z_1)^2(z_3 - z_2)^2(z_1 - z_3)^2$ under condition $z_1 + z_2 + z_3 = 0$).

Consider a two-sheeted covering unbranched outside $B$

$$(x, y, w) \in \mathbb{C}^3 \supset X_2 \xrightarrow{f_2} \mathbb{C}^2 \ni (x, y),$$

where $X_2$ is defined by equation $w^2 = x^3 - y^2$, and $f_2$ is induced by projection. Such a structure has a generic covering $f : X \rightarrow \mathbb{P}^2$ in a neighbourhood of a sp-point of type $A_2$.

**Lemma 2.1** ([C]) If $f : (X, 0) \rightarrow (\mathbb{C}^2, 0)$ is a finite covering by a normal irreducible surface $X$, unbranched outside an ordinary cusp $B \subset \mathbb{C}^2$, and the ramification curve of which is reduced, i.e. $f^* (B) = 2R + C$ ($R$ and $C$ reduced curves), then $f$ is equivalent to one of the coverings $f_2, f_3$ and $f_6$. ■

18
The proof is obtained by means of studying the possible monodromy homomorphisms $\rho : \pi_1 \to S_N$, where $\pi_1 = \pi_1(C^2 \setminus B) = Br$ is the fundamental group of a cusp, and $N = \deg f$.

We obtain one more characterization of the covering $f_3$ as a finite covering $f : (X, 0) \to (C^2, 0)$ by a normal irreducible surface, unbranched outside a cusp $B$, and with a reduced and non-singular ramification curve $R$.

2.4.3 *Description of a normalization of the fibre product in a neighbourhood of a sp-point of type $A_2$.* The map $f_6$ factors through the maps $f_2$ and $f_3$, and we have a commutative diagram

$$
\begin{array}{ccc}
X_6 &=& \{z_1 + z_2 + z_3 = 0\} \\
&\xrightarrow{g_3} & \xrightarrow{g_2} & \{w^2 = x^3 - y^2\} = X_2 \\
&\xrightarrow{f_6} & \xrightarrow{f_3} & X_3 = \{z^3 - 3xz^2 + 2y = 0\}, \\
&\xrightarrow{f_2} & & (x, y) \in C^2 \ni B : y^2 = x^3
\end{array}
$$

where $g_2$ and $g_3$ are defined by formulae: $x$ and $y$ are defined by the same formulae as $f_6$, and $z = z_1$ for $g_2$, and $w = \frac{1}{6\sqrt{3}}(z_2 - z_1)(z_3 - z_2)(z_1 - z_3)$ for $g_3$. It is easy to see that $g_3$ is a factorization under the action of a cyclic group $Z_3 = A_3 \subset S_3$, $X_2 = X_6/A_3$, and $g_2$ is a factorization under the action of a cyclic group of order two $Z_2 \simeq S_2 = \{(1), (2, 3)\} \subset S_3$.

By the property of universallity of fibre products we have a morphism $X_6 \to X_2 \times_{C^2} X_3$. The fibre product $X_2 \times_{C^2} X_3$ is irreducible, since each its component $Z$ is mapped onto $X_2$ and $X_3$, and, therefore, the degree of $Z \to C^2$ have to be divided by 2 and 3, i.e. have to be equal to 6. Thus, $X_6$ is a normalization of $X_2 \times_{C^2} X_3$, and the diagram $(*)_3$ describes a normalization of the fibre product in a neighbourhood of a sp-point of type $A_2$.  

19
The diagram (\(*_3\)) can be visually-schematic presented as follows

![Diagram](image)

Fig. 7

Direct computations show that \(x - z^2 = \frac{1}{3}(z_2 - z_1)(z_1 - z_3)\), and \(x - \frac{1}{4}z^2 = \frac{1}{12}(z_3 - z_2)^2\), i.e.

\[g_2^*(R) = L_2 + L_3, \quad g_2^*(C) = 2L_1.\]

And, besides, \(g_3^*(R_1) = L_1 + L_2 + L_3\).

2.4.4 Description of a normalization of the fibre product in a neighbourhood of a pp-point of type \(A_2\). Let \(x_1 \in X_1\) and \(x_2 \in X_2\) be p-points of type \(A_2\) for \(f_1\) and \(f_2\); \(f_1^*(B) = 2R_1 + C_1, f_2^*(B) = 2R_2 + C_2\). In this case the 3-sheeted coverings \(f_1\) and \(f_2\) are the same (equivalent), and the monodromy homomorphisms \(\varphi_1, \varphi_2 : \pi_1 = \pi_1(\mathbb{C}^2 \setminus B, y_0) \to S_3\) are epimorphic. The fibre \((f^*)^{-1}(y_0)\) of the 9-sheeted covering \(f^* : X^* = X_1 \times \mathbb{C}^2 X_2 \to \mathbb{C}^2\) consists of pairs \(f_1^{-1}(y_0) \times f_2^{-1}(y_0) = \{(i, j), 1 \leq i, j \leq 3\}\), and the monodromy homomorphism is (equivalent to) a diagonal homomorphism \(\varphi : \pi_1 \to S_3 \times S_3 \subset S_9\). Since \(\varphi_i\) are epimorphic, the fibre \((f^*)^{-1}(y_0)\) consists of two orbits w.r.t. the action of \(\pi_1\) — the orbit of the point \((1, 1)\), which consists of 3 elements, and the orbit of the point \((1, 2)\), which consists of 6 elements. From this and from Lemma [2.1] it follows that in a neighbourhood of the \(\bar{x} = (x_1, x_2)\) a normalization \(X\) of the product \(X^*\) consists of two disjoint components \(X = X_3 \coprod X_6\), and on \(X_3\) the morphism \(f : X \to \mathbb{C}^2\) coincides with \(f_3\), the morphisms \(g_1\) and \(g_2\) are isomorphisms, and on \(X_6\) the morphism \(f = f_6\), the morphisms \(g_1\) and \(g_2\) are the same as \(g_2\) in the diagram \((*_3)\)
There are 4 curves on $X^\times$: $C_1 \times_B C_2$, $R_1 \times_B C_2$, $C_1 \times_B R_2$, $R_1 \times_B R_2$, preimages of which on the normalization $X$ are $C_3$, $L_2$, $L_1$, and $L_3$, $R_3$. Under such a numeration of the lines $L_i$ we have

$$g_1^*(R_1) = R_3 + L_2 + L_3$$
$$g_1^*(C_1) = C_3 + 2L_1$$
$$g_2^*(R_2) = R_3 + L_1 + L_3$$
$$g_2^*(C_2) = C_3 + 2L_2$$

2.4.5 A lift of the diagram (*3). Consider the diagram (*3). For computation of intersection numbers in §5 we need to resolve the singular point of type $A_2$ on the surface $X_2$, and to ‘disjoint’ the curves $L_2$ and $L_3$ on $X$. A resolution of the singular point of type $A_2$, as of any ‘double plane’, can be obtained, if we firstly take an imbedded resolution $\sigma : \mathbb{C}^2 \to \mathbb{C}^2$ of the branch curve $B \subset \mathbb{C}^2$, and then take a normalization of $X_2 \times_{\mathbb{C}^2} \mathbb{C}^2$.

Actually we’ll make more – we lift the whole of the diagram (*3) on $\mathbb{C}^2$.

1) The singular point of $B$ is resolved by one $\sigma$-process $\sigma_1$. It is enough for the resolving of the singular point on $X_2$, but to resolve the total transform of $B$ up to a divisor with normal crossings, one need two more $\sigma$-processes. We picture the resolution process schematically by ‘drawing’ the total transform of the curve $B$. Denote by $E_i$ the curve glued in under the $i$-th $\sigma$-process, and also its proper transform under subsequent $\sigma$-processes.

Along each curve we indicate two numbers: the negative is the self-intersection number, the positive is its multiplicity in the total transform of the curve $B$. 

21
2) Denote by $\sigma : \bar{\mathbb{C}}^2 \to \mathbb{C}^2$ the composition $\sigma_3 \circ \sigma_2 \circ \sigma_1$. We add on the diagram $(\ast_3)$ over $\bar{\mathbb{C}}^2$ and obtain a diagram as follows, on which all morphisms on the right face are finite coverings.

The right square of the diagram $(\ast)$ is obtained as a fibre product $(\ast_3) \times_{\mathbb{C}^2} \bar{\mathbb{C}}^2$, i.e. $\bar{X}_i$ are normalizations of $X_i \times_{\mathbb{C}^2} \bar{\mathbb{C}}^2$, and morphisms are induced by morphisms of the diagram $(\ast_3)$ and projections. We describe how one can construct the diagram $(\ast)$ not uniformly as a normalization of the lift, but step-by-step. To facilitate the following of the description we begin with the final picture. We draw the right square of the diagram $(\ast)$ by replacing the varieties at its vertices by the total transforms of the curve $B$.

Fig. 10
The rule of notation is as follows. The exceptional curves $E_1, E_2, E_3$ on $\mathbb{C}^2$ are already denoted. Under double indexing $E_{i,j}$ the first index indicates the variety $X_i$, where $E_{i,j}$ lies, and the second index indicates to what curve $E_j$ the curve $E_{i,j}$ is mapped on $\mathbb{C}^2$.

3) We begin a description of the diagram $(\ast)$ with $X_6$ (‘from the top’). To disjoint the lines $L_i$, we make $\sigma$-process with centre at the point $0 \in X_6 = \mathbb{C}^2 = \{z_1 + z_2 + z_3 = 0\} \subset \mathbb{C}^3$. By this the curve $E_{6,3} = \mathbb{P}^1 = \{t_1 + t_2 + t_3 = 0\} \subset \mathbb{P}^2 \ni (t_1 : t_2 : t_3)$ is glued, and we obtain a variety $X'_6$. The action of $S_3$ on $X_6$ is extended to $X'_6$ and, in particular, to $\mathbb{P}^1$. On $\mathbb{P}^1$ there are 8 exceptional points forming exceptional orbits:

$$p_1 = E_{6,3} \cap L_1 = (-2 : 1 : 1), \quad p_2 = E_{6,3} \cap L_2 = (1 : -2 : 1), \quad p_3 = E_{6,3} \cap L_3 = (1 : 1 : -2);$$

$$P_1 = (0 : 1 : -1), \quad P_2 = (1 : 0 : -1), \quad P_3 = (1 : -1 : 0);$$

$$Q_1 = (1 : \zeta : \zeta^2), \quad Q_2 = (1 : \bar{\zeta} : \bar{\zeta}^2),$$

where $\zeta = \sqrt[3]{1}$ is a primitive root, and $\bar{\zeta} = \zeta^2$. Denote by $\xi = (123)$ a generator of the cyclic group of order three $\mathbb{Z}_3 = A_3 = \{(1), (123), (132)\} \subset S_3$, and by $\varepsilon = (23)$ a generator of the cyclic group of order two $\mathbb{Z}_2 = S_2 = \{(1), (23)\} \subset S_3$. Then

$$\xi(p_1) = p_2, \quad \xi(p_2) = p_3, \quad \xi(p_3) = p_1; \quad \xi(P_1) = P_2, \quad \xi(P_2) = P_3, \quad \xi(P_3) = P_1;$$

$$\varepsilon(p_1) = p_2, \quad \varepsilon(p_2) = p_3, \quad \varepsilon(p_3) = p_1; \quad \varepsilon(P_1) = P_2, \quad \varepsilon(P_2) = P_3, \quad \varepsilon(P_3) = P_2; \quad \varepsilon(Q_1) = Q_2.$$

If we take a quotient $X'_6$ under the action of $\mathbb{Z}_3 = A_3$, then the stationary points $Q_1$ and $Q_2$ give two quotient singularities on $X'_2 = X'_6/A_3$, resolving of which $X''_2 \to X'_2$ glues the curves $E''_{2,1}$ and $E''_{2,1}$ with $(E''_{2,1}) = -3, \ (E''_{2,1}) = -3$. To lift $X'_6 \to X'_2$ onto $X''_2$, we have to blow up the points $Q_1$ and $Q_2$, $X''_6 \to X'_6$, and by this we obtain $X''_6 \to X''_2$.

![Diagram 11](image)

**Fig. 11**

4) The map $f_2$ is a factorization under the cyclic group $\mathbb{Z}_2 = S_2$. The action extends to $X''_2$. The stationary point on $E_{6,3}$ – the image of the point $P_1$ on $E_{6,3}$ gives a singular point of type $A_2$ on $X''_2/\mathbb{Z}_2 \ (= \bar{\mathbb{C}}^2)$. A resolution of this point glues a (-2)-curve $E_2$, and we obtain $\bar{\mathbb{C}}^2$. To lift $X''_2 \to X''_2/\mathbb{Z}_2$ onto the resolution $\bar{\mathbb{C}}^2$, we have to blow up a point on $X''_2$. By this a (-1)-curve $E_{2,2}$ is glued, and we obtain $\bar{X}_2$. To obtain $\bar{g}_3 : \bar{X}_6 \to \bar{X}_2$, we have to perform 3
σ-processes with centres at points $P_1, P_2, P_3$ on $X''$, by which three lines $E''_{6,2}; E''_{6,2}$ and $E''_{6,2}$ are glued. We obtain the left side $\bar{g}_3$ and $\bar{f}_2$ of the right square of diagram (*), pictured on Fig. 10. Note that the map $\bar{g}_3$ is ramified along the curves $E''_{6,1}$ and $E''_{6,1}$, and the map $\bar{f}_2$ is ramified along the curves $E_{2,2}$ and $R_1$.

We can blow down the (-1)-curve $E_{2,2}$ on $\bar{X}_2$, and then to blow down the (-1)-curve $E_{2,3}$. By this we obtain a minimal resolution of the singular point of type $A_2$ on $X_2$,

![Diagram](image1)

Fig. 12

5) The map $\bar{g}_2$ is a factorization under the group $\mathbb{Z}_2 = S_2 = \{(1), (23)\}$. We obtain the surface $\bar{X}_3 = \bar{X}_6/S_2$, $\bar{g}_2 : \bar{X}_6 \to \bar{X}_3$. The map $\bar{g}_2$ is ramified along the curves $E''_{6,2}$ and $L_2$, which are mapped onto $E''_{3,2}$ and $C$ correspondingly. The diagram is completed by the map $\bar{f}_3 : \bar{X}_3 \to \mathbb{C}^2$. The surface $\bar{X}_3$ is obtained from $X_3 = \mathbb{C}^2$, if we at first blow up the point of tangency of curves $C$ and $R$ gluing $E''_{3,2}$; then we blow up the point of intersection of $C$ and $R$ gluing $E_{3,3}$; finally, we blow up two more points on $E_{3,3}$.

![Diagram](image2)

Fig. 13

3 The canonical cycle of a Du Val singularity

We intend to apply Hodge index theorem to obtain the basic inequality for generic coverings of $\mathbb{P}^2$ by surfaces with A-D-E-singularities. For this we need intersection theory and, therefore, a resolution of singularities of $X$. In this section we examine the local situation and find out how the resolution affects the canonical class and the ramification curve.

3.1. Definition of canonical cycle. Let $(X, x)$ be a 2-dimensional A-D-E-singularity. Let $\pi : \bar{X} \to X$ be a minimal resolution, $L = \pi^{-1}(x)$ be the exceptional curve. As is known, the
canonical class $K_X$ is trivial in a neighbourhood of $L$, that is we can choose a divisor in $K_X$ with a support not intersecting $L$. In other words, there is a differential form $\omega$ on $X$, which has neither poles nor zeroes in a neighbourhood of $L$. Such a form can be obtained, for example, as follows. As is known, $(X, x)$ is a quotient singularity, $X = \mathbb{C}^2/G$, where $G \subset SL(2, \mathbb{C})$. The form $du \wedge dv$ on $\mathbb{C}^2 \ni (u, v)$ is invariant w.r.t. $G$ and it defines a form on $X$ $(\varphi^*(\omega) = du \wedge dv$, where $\varphi : \mathbb{C}^2 \to X$). Hence, the divisor $(\omega) = \sum k_i L_i$. Since $L_i$ are $(-2)$-curves, $(L_i \cdot (\omega)) = 0$, and we obtain $(\omega) = 0$.

On the other hand, $(X, x)$ can be considered as a double plane, that is as a 2-sheeted covering $X \xrightarrow{\bar{f}} Y$ of the plane $Y = \mathbb{C}^2$ (locally). Let $z^2 = h(x, y)$ be an equation of $(X, x)$, $B : h(x, y) = 0$ be the discriminant curve, $f^{-1}(B) = R$, defined by the equation $z = 0$, be the ramification curve. We can consider the differential form $\omega = f^*(dx \wedge dy)$ lifted from $Y$. Then on $\bar{X}$ the divisor $(\omega)(z) = \bar{R} + Z$, where $\bar{R} \subset \bar{X}$ is the proper transform of $R$, $Z = \sum \gamma_i L_i$ is a cycle on $L = \pi^{-1}(x)$. We shall say that $Z$ is the canonical cycle of a 2-dimensional A-D-E-singularity. Thus, $-Z$ is a cycle on the exceptional curve $L$, which is equivalent to the ramification curve $\bar{R}$ in a neighbourhood of $L$. Let us calculate the canonical cycle for all A-D-E-singularities.

3.2. On resolution of double planes. As for any double plane, a resolution of an A-D-E-singularity can be obtained by means of a resolution of the discriminant curve $B \subset Y = \mathbb{C}^2$, $B : h(x, y) = 0$. Let $\sigma : \bar{Y} \to Y$ be a composition of $\sigma$-processes, such that the total transform of $B$ is a divisor with normal crossings. Let $\sigma^*(B) = \bar{B} + \sum_{i=1}^r \alpha_i l_i$, where $\bar{B}$ is the proper transform of $B$, $l_i \simeq \mathbb{P}^1$, $i = 1, \ldots, r$, are the exceptional curves, as well as their proper transforms, glued by $\sigma$-processes. Let $\bar{X}$ be the normalization of $\bar{Y} \times_Y X$, and $\bar{f}$ and $\pi$ be induced by projections,

$$\pi^{-1}(x) = L = L_1 \cup \ldots \cup L_r \subset \bar{X} \xrightarrow{\pi} X \supset R \ni x, R : z = 0$$

$$\begin{align*}
\bar{f} &\quad & \pi \\
\sigma^*(B) = \bar{B} + \sum_{i=1}^r \alpha_i l_i \subset \bar{Y} &\quad & \sigma \\
f &\quad & \bar{f}
\end{align*}$$

(2)

Set $\bar{f}^{-1}(l_i) = L_i$. The curve $L_i$ is either irreducible or consists of two components $L_i = L_i' + L_i''$, where $L_i' \simeq \mathbb{P}^1$, $L_i'' \simeq \mathbb{P}^1$. The mapping $\bar{f}$ is a 2-sheeted covering branched along the curve $\bar{B} + \sum l_i$. To be more graphic we denote the curves $l_i$, for which $\alpha_i$ are odd, also by $\bar{l}_i$, and $L_i$ respectively by $\bar{L}_i$. The surface $\bar{X}$ has singularities of type $A_1$ over nodes of the branch curve $\bar{B} + \sum \bar{l}_i$. If this curve is non-singular, that is, a disconnected union of components (one can reach this by performing one additional $\sigma$-processes for each node), then $X$ is non-singular and is a resolution of the singularity $(X, x)$. Let $\bar{R}$ be the proper transform of $R$ w.r.t. $\pi$ (= the proper transform of $\bar{B}$ w.r.t. $\bar{f}$). We have $\bar{f}^*(\bar{l}_i) = 2\bar{L}_i$, if $\alpha_i$ is odd, and $\bar{f}^*(\bar{l}_i) = L_i$, if $\alpha_i$ is even. We have

$$((\sigma \circ \bar{f})^*h(x, y))(z^2) = 2\bar{R} + \sum_{\alpha_i \text{ odd}} 2\alpha_i \bar{L}_i + \sum_{\alpha_i \text{ even}} \alpha_i L_i$$
and, consequently, \((z) = \bar{R} + Z\), where

\[
Z = \sum_{\alpha_i \text{-odd}} \alpha_i \bar{L}_i + \sum_{\alpha_i \text{-even}} \frac{1}{2} \alpha_i L_i.
\]

Let us compute the cycle \(Z\) for each type of A-D-E-singularities (despite of abundance of papers concerning Du Val singularities, the authors do not know any of them, where the cycle \(Z\) is written out; so we have to perform these computations).

3.3. **Computation of the canonical cycle.** Consider the minimal resolution of each type of A-D-E-singularities described above. The following lemma contains the results of computations of \(\sigma^* (B)\), of the exceptional curve \(\pi^{-1}(x) = L\) and of the canonical cycle \(Z\).

**Lemma 3.1** Below we picture schematically the total transform \(\sigma^* (B) = \bar{B} + \sum_{i=1}^r \alpha_i l_i\) (near each curve \(l_i\) a positive number \(\alpha_i\), and a negative number \((l_i^2)\) are shown), and over it we picture the curve \(\pi^{-1}(\bar{R})\), consisting of \(\bar{R}\) and \((-2)\)-curves, and besides we write down the canonical cycle \(Z\):

1) The singularity \(A_{2k-1}: y^2 = x^{2k},\ k \geq 1,\)

\[
Z = L_1 + 2L_2 + \ldots + kL_k;
\]
2) The singularity $A_{2k} : y^2 = x^{2k+1}$, $k \geq 1$,

$$Z = L_1 + 2L_2 + \ldots + kL_k;$$

3) The singularity $D_{2k+2} : x(y^2 + x^{2k})$, $k \geq 1$,

$$Z = 3L_1 + 5L_2 + \ldots + (2k+1)L_k + 2L_{k+1} + 4L_{k+2} + \ldots + 2kL_{2k} + (k+1)L_{2k+1} + (k+1)L_{2k+2};$$
4) The singularity $D_{2k+3} : x(y^2 + x^{2k+1})$, $k \geq 1$,

\[
4) \text{ The singularity } D_{2k+3} : x(y^2 + x^{2k+1}), \quad k \geq 1,
\]

\[
Z = 3L_1 + 5L_2 + \ldots + (2k+1)L_k + 2L_{k+1} + 4L_{k+2} + \ldots + 2kL_{2k} + (2k+2)L_{2k+2} + (k+1)L_{2k+1};
\]

\[
L_{2k+1} = L'_{2k+1} + L''_{2k+1};
\]

5) The singularity $E_6 : x^3 + y^4$, 

\[
5) \text{ The singularity } E_6 : x^3 + y^4,
\]

\[
Z = 3L_1 + 2L_2 + 4L_3 + 6L_4;
\]
6) The singularity $E_7 : x(x^2 + y^3)$,

\[
Z = 3L_1 + 5L_2 + 9L_3 + 6L_4 + 5L_5 + 8L_6 + 3L_7;
\]

7) The singularity $E_8 : x^3 + y^5$,

\[
Z = 3L_1 + 5L_2 + 9L_3 + 15L_4 + 10L_5 + 8L_6 + 12L_7 + 6L_8. \quad \blacksquare
\]

3.4. Defect of a singularity. Define a defect $\delta$ of a A-D-E-singularity by the formula

\[
\delta = \frac{1}{2} (\bar{R} \cdot Z).
\]
Corollary 3.1 For different types of A-D-E-singularities the defect equals
\[ \delta = \begin{cases} 
\frac{n+1}{2} & \text{for type } A_n; \\
\frac{n}{2} + 1 & \text{for type } D_n; \\
\frac{n+1}{2} & \text{for types } E_n, \ n = 6, 7, 8.
\end{cases} \]

In particular, for the type $A_1$ (nodes) and $A_2$ (cusps) the defect $\delta = 1$.

Actually one can show that defect $\delta$ is the $\delta$-invariant (genus) of the one-dimensional A-D-E-singularity.

4 Numerical invariants of a generic covering

Now we consider a global situation. Let $X$ be a surface with A-D-E-singularities,
\[ \text{Sing } X = \sum_{k \geq 1} a_k A_k + \sum_{k \geq 4} d_k D_k + \sum_{k = 6, 7, 8} e_k E_k, \]
that means that $X$ has $a_k$ singularities of type $A_k$, $d_k$ of type $D_k$ and $e_k$ of type $E_k$.

Let $f : X \to \mathbb{P}^2$ be a generic covering of degree $N$, and $B \subset \mathbb{P}^2$ be the discriminant curve. Let $\deg B = d$ and let $B$ has $n$ nodes and $c$ cusps, $n_s = a_1$ and $c_s = a_2$ of which originates from $\text{Sing } X$, and $n_p$ and $c_p$ are p-nodes and p-cusps. Let $R \subset X$ be the ramification curve, $f^*(B) = 2R + C$, and $L \subset X$ be the preimage of a generic line $l \subset \mathbb{P}^2$. Denote by $\bar{R}$ and $\bar{L}$ the proper transforms of $R$ and $L$ on $S$. Then $\bar{R}$ is a normalization of the curve $R \simeq B$, and $\bar{L} \simeq L$.

4.1. The canonical class $K_S$ and the canonical cycle $Z$. Let
\[ Z = \sum_{x \in \text{Sing } X} Z_x \]
be the canonical cycle of $S$, where $Z_x$ are the canonical cycles of singularities $x \in \text{Sing } X$. It follows from 3.2 that
\[ K_S = (f \circ \pi)^* K_{\mathbb{P}^2} + \bar{R} + Z = -3\bar{L} + \bar{R} + Z. \]
Besides, the singularities of $X$ being Gorenstein, the divisor $R$ is locally principal, and
\[ \pi^*(R) = \bar{R} + Z. \]

4.2. The intersection numbers.

Lemma 4.1 The intersection numbers of $L$, $\bar{R}$ and $Z$ on $S$ are equal
\[ (L^2) = N, \ L \cdot R = d, \ L \cdot Z = 0, \ \bar{R} \cdot Z = 2\delta_X, \ (Z^2) = -2\delta_X, \]
where
\[ \delta_X = \sum_{x \in \text{Sing } X} \delta_x = \sum a_k \left[ \frac{k+1}{2} \right] + \sum d_k \left( \frac{k}{2} + 1 \right) + \sum e_k \left( \frac{k+1}{2} \right) \]
is the defect of the surface $X$.  

30
Proof. Obviously, we have \( (L^2) = \deg f = N \), and \( \bar{L} \cdot \bar{R} = \deg B = d \). By 3.4 we have \( \bar{R} \cdot Z = 2\delta_X \). The divisor \( Z \) being exceptional, we have \( \pi(Z) = \text{Sing} \ X \), \( \dim \pi(Z) = 0 \), and \( \bar{L} = \pi^*(\bar{L}) \), \( \bar{R} + Z = \pi^*(R) \), and therefore, \( \bar{L} \cdot Z = 0 \), and \( (\bar{R} + Z) \cdot Z = 0 \), and, consequently, \( (Z^2) = -(\bar{R} \cdot Z) \).\hfill \blacksquare

It remains to compute \( (\bar{R}^2) \).

4.3. The evenness of degree \( \deg B = d = 2\bar{d} \). The restriction of \( \bar{f} \) to \( \bar{L} \), \( \bar{L} \to l \sim \mathbb{P}^1 \), is a covering of degree \( N \), with ramification indices 2 at the points of intersection of \( \bar{L} \) and \( \bar{R} \). We have \( \bar{L} \cdot \bar{R} = d \), and from Hurwitz formula we obtain \( 2g(\bar{L}) - 2 = -2N + d \). It follows that \( \deg B = d \) is even. Let \( d = 2\bar{d} \). Besides, since

\[
g(\bar{L}) = \frac{1}{2}d + 1 - N \geq 0,
\]
we obtain a bound for the degree of covering,

\[
N \leq \bar{d} + 1.
\]

4.4. The self-intersection number \((\bar{R}^2)\) and the arithmetical genus of the curve \(R\). Denote by \( \delta \) the defect of the curve \( B \),

\[
\delta = \delta_B = \sum_{s \in \text{Sing} B} \delta_s = n + c + \delta_0,
\]

where

\[
\delta_0 = \sum_{x \in \text{Sing} B, \ x \ not \ A_1 \ and \ A_2} \delta_x.
\]

The numbers \( \delta \) and \( \delta_0 \) are the extremal values of defects \( \delta_X \) of surfaces \( X \) with given discriminant curve \( B : \delta_0 \) corresponds to a surface \( X \), all nodes and cusps of which are p-nodes and p-cusps, \( n = n_p \), \( c = c_p \), and \( \delta \) corresponds to a surface \( X \) (for example, a 2-sheeted covering of \( \mathbb{P}^2 \)), all nodes and cusps of which are s-nodes and s-cusps, \( n = n_s \), \( c = c_s \).

At first we express the geometric genus of \( B \), \( g = g(B) = g(\bar{R}) \), in terms of the defect \( \delta \). For this we consider a surface \( X \), which is a 2-sheeted covering of \( \mathbb{P}^2 \) with the discriminant curve \( B \). In this case \( (Z^2) = -(\bar{R} \cdot Z) = -2\delta \), and \( f^*(B) = 2R \) and, consequently, \( d \cdot \bar{L} \sim 2\bar{R} + 2Z \), because \( B \sim d \cdot l \). From (4.1) and the adjunction formula \( g(\bar{R}) = \frac{(\bar{R}, \bar{R} + K_X)}{2} + 1 \) we obtain

\[
g = \frac{(d - 1)(d - 2)}{2} - \delta.
\]

If it is known that the defect \( \delta \) coincides with the \( \delta \)-invariant of a one-dimensional singularity, then this formula coincides with the well known formula for the geometric genus \( g(R) \) of a singular curve \( R \), \( g(R) = p_a(R) - \sum_{x \in \text{Sing} R} \delta_x \).

We return to a generic covering \( X \) of degree \( N \), \( n = n_s + n_p \), \( c = c_s + c_p \). Then

\[
\delta_X = n_s + c_s + \delta_0 = \delta - n_p - c_p.
\]
Lemma 4.2 The self-intersection number of the proper transform of the ramification curve $\bar{R} \subset S$ is equal

$$(\bar{R}^2) = 3\bar{d} + g - 1 - \delta_X,$$

(4.9)

and

$$(\bar{R} + Z)^2 = 3\bar{d} + g - 1 + \delta_X = 3\bar{d} + p_a(R) - 1,$$

(4.10)

where

$$p_a(R) = g + \delta_X = \frac{(d-1)(d-2)}{2} - n_p - c_p$$

(4.11)

is the arithmetical genus of $R$.

Proof. From (4.1) and the adjunction formula $2g(R) - 2 = (\bar{R}, R + K_S) = (\bar{R}, -3\bar{L} + 2\bar{R} + Z)$ we obtain $$(\bar{R}^2) = \frac{3}{2}(\bar{R} \cdot \bar{L}) + g - 1 - \frac{1}{2}(\bar{R} \cdot Z).$$ Applying formulae (4.3), we obtain the proof. ■

From formulae (4.1), (4.3) and (4.9) we obtain a corollary.

Corollary 4.1

$$(K_S^2) = 9N - 9\bar{d} + p_a(R) - 1,$$

or, substituting $p_a(R)$ from (4.11),

$$(K_S^2) = 9N + \frac{1}{2}d(d - 12) - n_p - c_p.$$  

(4.12')

4.5. A bound for the covering degree.

Lemma 4.3 For a generic covering of degree $N$ with discriminant curve of degree $d = 2\bar{d}$ and genus $g$, we have

$$N \leq 4\bar{d}^2 \frac{d}{3d + g - 1 + \delta_X},$$

(4.13)

where $\delta_X$ is the defect of singularities of $X$, and moreover, the equality holds if and only if $\bar{L} \equiv mK_S$ for some $m \in \mathbb{Q}^*$, or $mK_S \equiv 0$.

Proof. Applying Hodge index theorem to divisors $\bar{L}$ and $\pi^*(R) = \bar{R} + Z$ on $S$, we obtain

$$\left| \begin{array}{cc} \bar{L}^2 & (\bar{L}, \bar{R} + Z) \\ (\bar{L}, \bar{R} + Z) & (\bar{R} + Z)^2 \end{array} \right| = \left| \begin{array}{cc} N & d \\ d & 3\bar{d} + g - 1 + \delta_X \end{array} \right| \leq 0,$$

and it is the desired inequality. The equality holds only if $\bar{L}$ and $\bar{R} + Z$ are linear dependent in the Néron-Severi group $NS(\bar{X}) \otimes \mathbb{Q}$. Since $K_S = -3\bar{L} + \bar{R} + Z$, we obtain the assertion about possible equality. ■

4.6. The topological Euler characteristic $e(S)$.

Lemma 4.4 The topological Euler characteristic of a surface $S$, obtained by the minimal resolution of singularities of $X$, is connected with the defect $\delta_X$ and invariants of a generic covering $f$ by a formula

$$e(S) = 3N + 2g - 2 + 2\delta_X - c_p,$$

(4.14)

where $N = \deg f$, and $c_p$ is the number of $p$-cusps on $B$ (or the number of pleats of $f$).
Proof is obtained in the same way as in the case of a non-singular surface $X$ ([K], §1 Lemma 7), considering a generic pencil of lines on $\mathbb{P}^2$ and the corresponding hyperplane sections on $S$, and lifting the morphism $\bar{f}: S \to \mathbb{P}^2$ to a morphism of fiberings of curves over $\mathbb{P}^1$. One can obtain a proof by direct computations. At first we find $e(X) = 3N - e(B) - n_p - c_p$ by considering the finite covering $f: X \to \mathbb{P}^2$, the stratification $\mathbb{P}^2 = (\mathbb{P}^2 \setminus B) \cup (B \setminus \text{Sing } B) \cup \text{Sing } B$, and applying the additivity property of Euler characteristic, and then we find $e(S)$. ■

From Noether’s formula $(K_S^2) + e(S) = 12p_a$ and formulae (4.12) and (4.14) we have $12p_a = 12N - 9\bar{d} + 3p_a(R) - 3 - c_p$. Substituting $p_a(R)$ from (4.11), we obtain a corollary.

**Corollary 4.2** The Euler characteristic of the structure sheaf $\mathcal{O}_S$ equals

$$p_a = 1 - q - p_g = N + \frac{\bar{d}(\bar{d} - 3)}{2} - \frac{n_p}{4} - \frac{c_p}{3}. \quad (4.15)$$

Thus, as in the case of a non-singular surface $X$, we obtain

**Corollary 4.3**

$n_p \equiv 0 \pmod{4}$, $c_p \equiv 0 \pmod{3}$.

5 Proof of the main inequality.

5.1. A fiber product of two generic coverings. Let a curve $B$ be a common discriminant curve for two generic coverings $f_1: X_1 \to \mathbb{P}^2$ and $f_2: X_2 \to \mathbb{P}^2$ of degrees $\deg f_1 = N_1$ and $\deg f_2 = N_2$. Let

$$\text{Sing } B = nA_1 + cA_2 + \sum_{k>2} a_k A_k + \sum_{k \geq 4} d_k D_k + \sum_{k=6,7,8} e_k E_k.$$ 

With respect to a pair of coverings $f_1$ and $f_2$ nodes and cusps of $B$ are subdivided into four types,

$$n = n_{ss} + n_{sp} + n_{ps} + n_{pp}, \quad c = c_{ss} + c_{sp} + c_{ps} + c_{pp}, \quad (5.1)$$

where $n_{\sharp\sharp}$ and $c_{\sharp\sharp}$ are numbers of $\sharp\sharp$-nodes and $\sharp\sharp$-cusps of $B$. In particular, $n_{ss} + n_{sp} = a_1$ is the number of singularities of type $A_1$, and $ss + sp = a_2$ is the number of singularities of type $A_2$ on the surface $X_1$.

Consider a normalization $X$ of the fiber product $X^\times = X_1 \times_{\mathbb{P}^2} X_2$ and the corresponding commutative diagram

$$X = \mathbb{C}^2 \ni (z_1, z_2)$$

$$(z_1, y) \in \mathbb{C}^2 = X_1 \ni (x_1, y) \ni X_0$$

$$(x, y) \in \mathbb{C}^2$$

$$(x, y) \ni X_2 = \mathbb{C}^2 \ni (x, z_2)$$

$$(x, y) \ni X_2 = \mathbb{C}^2 \ni (x, z_2)$$
The surface $X$ is a $N_1N_2$-sheeted covering of $\mathbb{P}^2$ and it has at most A-D-E-singularities, which lie over $\text{Sing} \ B$.

**Lemma.** If coverings $f_1$ and $f_2$ are non equivalent, then the surface $X$ is irreducible.

*Proof* is word for word the same as in the case of generic coverings of non-singular surfaces ([K] Proposition 2). □

We set

$$g_1^{-1}(R) = R + C,$$

where $R$ is a part, which is mapped by $g_2$ onto $R_2$, and $C$ is a part, which is mapped $g_2$ onto $C_2$. We are interested in the intersection number of $R$ and $C$ after a resolution of singularities of $X$ in a neighbourhood of the curve $R + C$.

Consider a restriction $R + C \to R_1$ of the covering $g_1$ over the curve $R_1$. As follows from 2.2.1 and 2.2.2, it is an étale covering of degree $N_2$ over a generic point $x_1 \in R_1$, where $R \to R_1$ is a 2-sheeted, and $C \to R_1$ is a $(N_2 - 2)$-sheeted covering. The same picture is over a point $x_1 \in R_1$, which is a $s$-point of $X_1$, lying over a ss-point of $B$.

Denote by $\tilde{\pi} : S \to X$ a minimal resolution of singularities of $X$, and denote by $\tilde{R}$ and $\tilde{C}$ the proper transforms of $R$ and $C$ on $S$. Our goal is to calculate the intersection numbers $(\tilde{R}^2)$, $(\tilde{R} \cdot \tilde{C})$ and $(\tilde{C}^2)$, and also the analogous intersection numbers for divisors $\tilde{\pi}^{-1}(R) = \tilde{R} + Z_R$ and $\tilde{\pi}^{-1}(C) = Z_C$, where $Z_R$ and $Z_C$ are the sums of canonical cycles corresponding to singular points $x \in \text{Sing} \ X$ and lying on $R$ and $C$ respectively.

**5.2. The structure of a fibre product over a neighbourhood of a singular point of the discriminant curve.** Let $U \subset \mathbb{P}^2$ be a sufficiently small neighbourhood (in complex topology) of a point $b \in \text{Sing} \ B$. The preimage $f_1^{-1}(U)$ is a disjoint union of two parts, $f_1^{-1}(U) = V_1 \sqcup V'_1$, where $V_1$ is a part containing the ramification curve $R_1$, and $V'_1$ is a part not containing $R_1$ and étale mapped to $U$. Analogously $f_2^{-1}(U) = V_2 \sqcup V'_2$. Then $f^{-1}(U)$ is a disjoint union of four open sets – of normalizations of fibre products $W = V_1 \times_U V_2$, $W' = V_1 \times_U V'_2$, $V'_1 \times_U V'_2$. And only $W$ and $W'$ meet the curve $g_1^{-1}(R_1)$. The open sets $W \subset X$ were studied in detail in §2. The surface $X$ in the neighbourhood $W$ is non-singular except the case of ss-points $b$. The open set $W'$ consists of $N_2 - k$ components ($k = 2, 3$ or $4$ depending on the type of the singular point $b$), which are mapped isomorphically onto $V_1$. And $W'$ does not meet $R$, and $W' \cap C$ consists of $N_2 - k$ components isomorphic to $V_1 \cap R_1$.

It follows from the investigation of the local structure of $X$ in §2 that $X$ and the curves $R$ and $C$ are of the following form over neighbourhoods of singular points $b \in \text{Sing} \ B$ of different types.

1) Over a ss-point $b$ the neighbourhood $W$ has 2 , and $W'$ has $N_2 - 2$ components, which are mapped isomorphically onto $V_1$ by the map $g_1$. Correspondingly $R \cap W$ consists of two, and $C \cap W'$ consists of $(N_2 - 2)$ components isomorphic to $R_1 \cap V_1$.

2) Over a sp-point $b \in B$ of type $A_1$ the neighbourhood $W'$ consists of $(N_2 - 4)$ components isomorphic to $V_1$ and having a singular point of type $A_1$. Correspondingly $C$ consists of $N_2 - 4$ nodal curves. The neighbourhood $W$ consists of two components: see Fig. 4, where

$$R = R' + R'' \ , \quad C = C' + C'' \ ,$$

(it ought to change places of the left and right parts of Fig. 4, $g_1$ stands for $g_2$, and $g_2$ – for
3) Over a ps-point \( b \in B \) of type \( A_1 \) the neighbourhood \( V_1 \subset X_1 \) consists of two components and on each of them the map \( f_1 \) has a fold. The neighbourhood \( W' \) consists of disjoint union of \( (N_2 - 2) \) pieces isomorphic to \( V_1 \). The neighbourhood \( W \) consists of two components: see Fig. 4, on which 
\[ R = R'' + R''', \text{ and } C = \emptyset. \]
We see that on \( W \) the curve \( R \) is non-singular and does not meet \( C \).

4) Over a pp-point \( b \in B \) of type \( A_1 \) the neighbourhood \( V_1 \subset X_1 \) consists of two irreducible components and on each of them \( f_1 \) has a fold. The neighbourhood \( W' \) is non-singular and consists of \( N_2 - 4 \) components isomorphic to \( V_1 \). The neighbourhood \( W \) is represented on Fig. 5, on which 
\[ R = R' + R'' + R'''', \text{ and } C = C'' + C''''. \]
We see that the curves \( R \) and \( C \) are non-singular and do not meet.

5) Over a sp-point \( b \in B \) of type \( A_2 \) the neighbourhood \( V_1 \) has a singular point of type \( A_2 \), and \( W' \) consists of \( (N_2 - 3) \) components isomorphic to \( V_1 \). The neighbourhood \( W \) is pictured on Fig. 7, on which 
\[ R = L_2 + L_3, C = L_1. \]
We see that \( R \) has a double point, \( C \) is non-singular and intersect transversally each of the branches of \( R \) at the intersection point, and, consequently, \((R \cdot C) = 2\).

6) Over a ps-point \( b \in B \) of type \( A_2 \) the neighbourhood \( V_1 \) is non-singular, and \( W' \) consists of \( (N_2 - 2) \) components isomorphic to \( V_1 \). The neighbourhood \( W \) is pictured on Fig. 7 (on which it ought to change places of the left and right parts, \( g_1 \) stands for \( g_2 \), and \( g_2 \) for \( g_3 \) ), where 
\[ R = L_2 + L_3, C = \emptyset. \]
We see that \( R \) has a double point and does not meet \( C \).

7) Over a pp-point \( b \in B \) of type \( A_2 \) the neighbourhood \( W' \) consists of \( N_2 - 3 \) components isomorphic to \( V_1 \). The neighbourhood \( W \) is pictured on Fig. 8, on which 
\[ R = R_3 + L_3, C = L_2. \]
We see that \( R \) is non-singular and meets with \( C \) transversally at one point.

From the obtained local description it follows that the surface \( X \) is non-singular at the points of intersection of \( R \) and \( C \), and the intersection is not void only over the points \( b \in B \) of types: over sp-points of type \( A_1 \), where \((R \cdot C) = 2\), over sp-points of type \( A_2 \), where \((R \cdot C) = 2\), and over pp-pointe of type \( A_2 \), where \((R \cdot C) = 1\). Therefore, 
\[ (\tilde{R} \cdot \tilde{C}) = 2n_{sp} + 2c_{sp} + c_{pp}. \] 
(5.3)

5.3. A lift of the fibre product to a resolution of the discriminant curve. To compute intersection numbers on \( S \) we consider firstly an auxiliary surface \( \bar{X} \), which is not a minimal
resolution of \(X\), and then we ‘descend’ to \(S\). Let \(\sigma : \mathbb{P}^2 \to \mathbb{P}^2\) be a composition of \(\sigma\)-processes resolving the curve \(B\) and needed to obtain a minimal resolution of a double plane singularities, lying over \(B\) (see \(\S3\)), and, besides, let \(\sigma\) includes two additional \(\sigma\)-processes as in 2.4.5 for each cusp, which is not a \(ss\)-cusp. Consider a lift of the diagram \((\ast_1)\) to \(\mathbb{P}^2\), namely consider the diagram

\[
(5.4)
\]

in which \(\bar{X}_i\) and \(\bar{X}\) are normalizations of \(X_i \times_{\mathbb{P}^2} \mathbb{P}^2\) and \(X \times_{\mathbb{P}^2} \mathbb{P}^2\). Then morphisms ‘on the right wall’ of diagram (5.4) are finite coverings. The surface \(\bar{X}\) is non-singular, and \(\bar{\pi} : \bar{X} \to S\) blows down the ‘superfluous’ exceptional curves of the first kind. Let \(\bar{R}_1\) be the proper transform of \(R_1\) on \(\bar{X}_1\), and \(\bar{R}\) and \(\bar{C}\) (respectively \(R_2\) and \(C_2\)) be the proper transforms of \(R\) and \(C\) on \(\bar{X}\) (respectively on \(S\)). Then \(\bar{g}_1(\bar{R}_1) = \bar{R} + \bar{C}\), and \(\bar{R} \to \bar{R}_1\) and \(\bar{C} \to \bar{R}_1\) are finite coverings of degree 2 and \(N_2 - 2\) respectively, and \(\bar{R}\) and \(\bar{C}\) are disjoint. Therefore,

\[
(\bar{R}^2) = 2 (\bar{R}_1^2), \quad (\bar{C}^2) = (N_2 - 2) (\bar{R}_1^2), \quad \bar{R} \cdot \bar{C} = 0. \tag{5.5}
\]

Actually from 3) and 4) one can see that over \(ps\)- and \(pp\)-nodes \(b\) in a neighbourhood of \(R + C\) the surface \(X\) is non-singular, and \(\bar{\pi} : \bar{X} \to S\) blows down the ‘superfluous’ exceptional curves of the first kind. Let \(\bar{R}_1\) be the proper transform of \(R_1\) on \(\bar{X}_1\), and \(\bar{R}\) and \(\bar{C}\) (respectively \(R\) and \(C\)) be the proper transforms of \(R\) and \(C\) on \(\bar{X}\) (respectively on \(S\)). Then \(\bar{g}_1(\bar{R}_1) = \bar{R} + \bar{C}\), and \(\bar{R} \to \bar{R}_1\) and \(\bar{C} \to \bar{R}_1\) are finite coverings of degree 2 and \(N_2 - 2\) respectively, and \(\bar{R}\) and \(\bar{C}\) are disjoint. Therefore,

\[
(\bar{R}^2) = 2 (\bar{R}_1^2), \quad (\bar{C}^2) = (N_2 - 2) (\bar{R}_1^2), \quad \bar{R} \cdot \bar{C} = 0. \tag{5.5}
\]

5.4. Computation of intersection numbers. First we find \((\bar{R}_1^2)\). Recall that by (4.9) we have on the minimal resolution \(\bar{X}_1\) of the surface \(X_1\)

\[
(\bar{R}_1^2) = 3\bar{d} + g - 1 - \delta_1, \tag{5.6}
\]

where \(\delta_1 = \delta_{X_1} = n_s + c_s + \delta_0\), and \(n_s = n_{ss} + n_{sp}\) and \(s = ss + sp\) are the numbers of singular points of type \(A_1\) and \(A_2\) on the surface \(X_1\).

Let \(\pi_1 = \bar{\pi}_1 \circ \pi_1\), where \(\bar{\pi}_1 : \bar{X}_1 \to X_1\) is a minimal resolution, and \(\bar{\pi}_1 : \bar{X}_1 \to \bar{X}_1\) is the blowing down of the "superfluous" exceptional curves. The surface \(\bar{X}_1\) differs from the surface \(\bar{X}_1\) only over the cusps of \(B\), which are not \(ss\)-cusps. Let \(U = \sigma^{-1}(U)\), and \(\bar{V}_1 = \pi_1^{-1}(V_1), \bar{V}_1 = \bar{\pi}_1^{-1}(V_1)\) be neighbourhoods of \(X_1\) and \(\bar{X}_1\) lying over \(U\) and containing the proper transform of \(R_1\). Analogously \(\bar{V}_1'\) and \(\bar{V}_1'\).
For a sp-cusp \( b \in B \) the blowing down \( \tilde{V}_1 \rightarrow \tilde{V}_1 \) is represented on Fig. 12. We see that \( \tilde{\pi} \) includes one \( \sigma \)-process with a centre \( \tilde{R}_1 \). For ps- and pp-cusps \( b \in B \) the blowing down \( \tilde{V}_1 \rightarrow \tilde{V}_1 \) is represented on Fig. 13 (where \( R \) stands for \( R_1 \), and \( C \) stands for \( C_1 \)). We see that it needs two \( \sigma \)-process with centres on \( R_1 \) to disjoint \( C_1 \) and \( R_1 \). Therefore,

\[
(R_1^2) = (\tilde{R}_1^2) - c_{sp} - 2c_{ps} - 2c_{pp},
\]

(5.7)

Now we examine how the intersection numbers \((R^2)\) and \((C^2)\) change under the blowing down \( \tilde{\pi} \). For a neighbourhood \( U \subset \mathbb{P}^2 \) of a point \( b \in \text{Sing } B \) set \( \tilde{W} = \pi^{-1}(W) \), \( \tilde{W}' = \pi^{-1}(W') \), \( \tilde{W} = \tilde{\pi}^{-1}(W) \), \( \tilde{W}' = \tilde{\pi}^{-1}(W') \). Then \( \tilde{g}_1^{-1}(\tilde{V}_1) = \tilde{W} \cup \tilde{W}' \). We examine one after another the blowing down \( \tilde{\pi} : \tilde{X} \to \tilde{S} \) in neighbourhoods \( \tilde{W} \cup \tilde{W}' \subset \tilde{X} \) separately for different types of singular points \( b \in \text{Sing } B \) (the numbering of cases corresponds to the numbering of cases in 5.2).

2) For a sp-point \( b \) of type \( A_1 \) the neighbourhood \( \tilde{W}' \) is a disjoint union of \((N_2 - 4)\) open sets isomorphic to \( \tilde{V}_1 \) to the minimal resolution of singular points of type \( A_1 \). The neighbourhood \( \tilde{W} \) is represented on Fig. 4, and \( \tilde{\pi} : \tilde{W} \to \tilde{W} \) is a blowing up of two points \( R' \cap R'' \) and \( R''' \cap R'''' \). Therefore, the blowing down \( \tilde{\pi} : \tilde{W} \to \tilde{W} \simeq W \) increases \((\tilde{R}^2)\) and \((\tilde{C}^2)\) on \( 2 \) for one point \( b \) and, consequently, on \( 2n_{sp} \) for all points of this type.

5) For a sp-point of type \( A_2 \) the neighbourhood \( \tilde{W} \) is represented on the upper part of Fig. 10. It is obtained from the neighbourhood \( \tilde{W} \), pictured on Fig. 7, by blowing up the point of intersection of lines \( L_1, L_2 \) and \( L_3 \), and then by blowing up 5 points on the glued line \( E_{0,3} \) and not lying on the proper transform of these lines. The blowing down \( \tilde{\pi} : \tilde{W} \to \tilde{W} \simeq W \) is the converse procedure, i.e. the blowing down of five exceptional curves of the first kind, and then blowing down the curve \( E_{0,3} \). In this case \( R = L_2 + L_3 \), and \( C = L_1 \). Since \((\tilde{R}^2) = (L_2^2) + 2(L_2 \cdot L_3) + (L_3^2)\) and \((L_2^2), (L_3^2)\) are diminished on 1, and \( L_2 \) and \( L_3 \) are no longer intersected after the \( \sigma \)-process with the centre at the point \( L_2 \cap L_3 \), the blowing down \( \tilde{\pi} \) increases \((\tilde{R}^2)\) on 4 for one point \( b \) and on \( 4c_{sp} \) for all points of this type.

6) For a ps-point \( b \) of type \( A_2 \) the neighbourhood \( \tilde{W} \) and the blowing down \( \tilde{\pi} : \tilde{W} \to \tilde{W} \simeq W \) are the same as in 5), but in this case \( R = L_2 + L_3 \), and \( C \cap W = \emptyset \). Therefore, as in 5) we obtain that the blowing down \( \tilde{\pi} \) increases \((\tilde{R}^2)\) on \( 4c_{ps} \).

The neighbourhood \( \tilde{W}' \) consists of \((N_2 - 2)\) components isomorphic to \( \tilde{V}_1 \), for each of which \( \tilde{\pi} \) is represented on Fig. 12. As above for \((\tilde{R}_1^2)\), we see that the blowing down \( \tilde{\pi} \) increases \((\tilde{C}^2)\) on \((N_2 - 3) + 1\) (taking account of the neighbourhood \( \tilde{W} \) ) for one point \( b \) and on \((N_2 - 2)c_{sp} \) for all points of this type.

7) For a pp-point \( b \) of type \( A_2 \) the neighbourhood \( \tilde{W} \) consists of two components: one is the same as in 5) and the other is the same as \( \tilde{V}_1 \) and represented on the left side of Fig. 13. Since in the neighbourhood, represented on Fig. 8, \( R = R_3 + L_3 \), and \( C = L_2 \), we obtain that the blowing down \( \tilde{\pi} : \tilde{W} \to \tilde{W} \simeq W \) increases \((\tilde{R}^2)\) on \( 2(N_2 - 2) \) for one point \( b \) and on \( 2(N_2 - 2)c_{ps} \) for all points of this type. Besides, \((\tilde{C}^2)\) is increased on \( c_{pp} \).

The neighbourhood \( \tilde{W}' \) consists of \((N_2 - 3)\) components isomorphic to \( \tilde{V}_1 \), and is represented
on Fig. 13 (on which C stands for R ). Therefore, taking account of the neighbourhood W, the blowing down π increases \((\tilde{C}^2)\) on \(2(N_2 - 3)c_{pp} + c_{pp} = (2N_2 - 5)c_{pp}\).

Summing all modifications of \((\tilde{R}^2)\) and \((\tilde{C}^2)\), we obtain

\[
(\tilde{R}^2) = (\tilde{R}^2) + 2n_{sp} + 4c_{sp} + 4c_{ps} + 3c_{pp},
\]

\[
(\tilde{C}^2) = (\tilde{C}^2) + 2n_{sp} + (N_2 - 2)c_{sp} + 2(N_2 - 2)c_{ps} + (2N_2 - 5)c_{pp}.
\]

Applying (5.5) and substituting \((\tilde{R}^2)\) from (5.7), we obtain

\[
(\tilde{R}^2) = 2((\tilde{R}^2) - c_{sp} - 2c_{ps} - 2c_{pp}) + 2n_{sp} + 4c_{sp} + 4c_{ps} + 3c_{pp} =
\]

\[
= 2(\tilde{R}^2_1) + 2n_{sp} + 2c_{sp} - c_{pp},
\]

\[
(\tilde{C}^2) = (N_2 - 2)((\tilde{R}^2) - c_{sp} - 2c_{ps} - 2c_{pp}) + 2n_{sp} + (N_2 - 2)c_{sp} + 2(N_2 - 2)c_{ps} + (2N_2 - 5)c_{pp} =
\]

\[
= (N_2 - 2)(\tilde{R}^2_1) + 2n_{sp} - c_{pp}.
\]

5.5. Computation of intersection numbers (continuation). Now we find \((\tilde{R} + Z_R)^2\), \((\tilde{C} + Z_C)^2\) and \((\tilde{R} + Z_R) \cdot (\tilde{C} + Z_C)\), where the divisor \(Z_R\), respectively \(Z_C\), equals to \(\sum Z_x\), where \(Z_x\) is the canonical cycle of a point \(x \in Sing X\), and the summation runs over \(x \in \tilde{R}\), respectively \(x \in C\). The analogous sums \(\sum \delta_x\) we denote by \(\delta_R\) and \(\delta_C\) respectively. By (4.2) we have

\[
(\tilde{R} \cdot Z_R) = -(Z_R^2) = 2\delta_R, (\tilde{C} \cdot Z_C) = -(Z_C^2) = 2\delta_C.
\]

Obviously,

\[
(\tilde{R} + Z_R) \cdot (\tilde{C} + Z_C) = \tilde{R} \cdot \tilde{C},
\]

and

\[
(\tilde{R} + Z_R)^2 = (\tilde{R}^2) + 2(\tilde{R} \cdot Z_R) + (Z_R^2) = (\tilde{R}^2) + 2\delta_R.
\]

Analogously, \((\tilde{R}_C + Z_C)^2 = (\tilde{C}^2) + 2\delta_C\).

It remains to determine how many singular points \(x \in Sing X\) lie on \(R\), respectively, on \(C\). From 5.2 it follows that over each ss-point on \(R\) there lie 2, and on \(C\) there lie \((N_2 - 2)\) singular points. There are no other singular points on \(R\). There are singular points on \(C\) of the following type: over a sp-point of type \(A_1\) there are \((N_2 - 4)\) singular points of type \(A_1\), over a sp-point of type \(A_2\) there are \((N_2 - 3)\) singular points of type \(A_2\). We obtain

\[
\delta_R = 2(\delta_0 + n_{ss} + c_{ss}) = 2(\delta_1 - n_{sp} - c_{sp}),
\]

\[
\delta_C = (N_2 - 2)(\delta_0 + n_{ss} + c_{ss}) + (N_2 - 4)n_{sp} + (N_2 - 3)c_{sp} =
\]

\[
= (N_2 - 2)(\delta_1 - n_{sp} - c_{sp}).
\]

Substituting \((\tilde{R}^2)\) from (5.10) and \(\delta_R\) from (5.14) to (5.13), we obtain

\[
(\tilde{R} + Z_R)^2 = 2(\tilde{R}_1^2) + 2n_{sp} + 2c_{sp} - c_{pp} + 4(\delta_1 - n_{sp} - c_{sp}) =
\]

\[
= 2((\tilde{R}_1^2) + 2\delta_1) - 2n_{sp} - 2c_{sp} - c_{pp}.
\]

38
Analously we find
\[(\tilde{R}_C + Z_C)^2 = (N_2 - 2)(\tilde{R}_1^2) + 2n_{sp} - c_{pp} + 2(N_2 - 2)\delta_1 - 4n_{sp} - 2c_{sp} =
\]
\[= (N_2 - 2)((\tilde{R}_1^2) + 2\delta_1) - 2n_{sp} - 2c_{sp} - c_{pp}.
\]
Set
\[2n_{sp} + 2c_{sp} + c_{pp} = \iota_1,
\]
and let
\[g_1 = p_a(R_1) = g + \delta_1
\]
be the arithmetic genus of the curve $R_1$. Since by (5.6) $(\tilde{R}_1^2) + 2\delta_1 = 3\bar{d} + g - 1 + \delta_1 = 3\bar{d} + g_1 - 1$, finally we obtain:
\[(\tilde{R} + Z_R)^2 = 2(3\bar{d} + g_1 - 1) - \iota_1,
\]
\[(\tilde{C} + Z_C)^2 = (N_2 - 2)(3\bar{d} + g_1 - 1) - \iota_1,
\]
\[(\tilde{R} + Z_R) \cdot (\tilde{C} + Z_C) = \iota_1.
\]

**5.7. The self-intersection number of the divisor $\tilde{R} + Z_R$ is positive.**

**Lemma 5.1**
\[(\tilde{R} + Z_R)^2 > 0.
\]

**Proof.** Recall that $2\bar{d} = d = \deg B$, and $\delta_1 = \delta_0 + n_{sp} + c_{sp}$. Therefore,
\[(\tilde{R} + Z_R)^2 = 2(3\bar{d} + g - 1 + \delta_1) - 2n_{sp} - 2c_{sp} - c_{pp} =
\]
\[d + (2d + 2g - 2) + 2\delta_0 - c_{pp}.
\]

Now we apply the Hurwitz formula for a generic projection $\varphi : B \to \mathbb{P}^1$ of the curve $B$ from a point $P \in \mathbb{P}^2$ onto the line $\mathbb{P}^1$, more precisely for the covering $\tilde{\varphi} : \tilde{B} \to \mathbb{P}^1$, where $\tilde{\varphi} = \varphi \circ n$, and $n : B \to B$ is a normalization of the curve $B$. Obviously, the covering $\tilde{\varphi}$ is ramified at the following points. Firstly, $\tilde{\varphi}$ has a ramification of the second order at points $\tilde{b} \in \tilde{B}$, which correspond to non-singular points $b \in B$, for which the line $\overline{Pb}$ is tangent to $B$. The number of such points is $\bar{d} = \deg \tilde{B}$, where $\tilde{B}$ is a curve dual to $B$. Secondly, $\tilde{\varphi}$ has a ramification of order $m_k$ at points $\tilde{b}$, which correspond to the branches $B_k$ of the curve $B$ at the singular points $b$. Here $m_k$ is the multiplicity (order) of the corresponding branch. Denote by
\[\nu = \sum_k (m_k - 1),
\]
where the summation runs over all branches of the curve (at singular points). The covering $\tilde{\varphi}$ is of degree $d = \deg B$. By the Hurwitz formula we obtain
\[2g - 2 = -2d + \hat{d} + \nu.
\]

**Remark 5.1** *Actually we derived one of the Plücker formulae*
\[\hat{d} = 2d + (2g - 2) - \nu
\]
*for a plane curve with singularities.*
Obviously, the number \( \nu \) for A-D-E-singularities is equal:
\[
\nu(A_{2k-1}) = \nu(D_{2k+2}) = 0, \quad \nu(A_{2k}) = \nu(D_{2k+3}) = \nu(E_7) = 1, \quad \nu(E_6) = \nu(E_8) = 2.
\]
Therefore, for the curve \( B \) the number \( \nu = \nu(B) \) is equal:
\[
\nu = c + \nu', \quad \text{where} \quad \nu' = \sum_{k>1} a_{2k} + \sum d_{2k+3} + 2e_6 + e_7 + 2e_8.
\] (5.22)

Returning to the proof of the inequality, we obtain from (5.19), (5.21) and (5.22)
\[
(\bar{R} + Z_R)^2 = d + (\bar{d} + \nu') + 2\delta_0 - c_{pp} = d + \bar{d} + 2\delta_0 + \nu' + (c - c_{pp}) > 0. \quad \blacksquare \quad (5.23)
\]

5.8. Conclusion of the main inequality. Applying the Hodge index theorem to divisors \( \bar{R} + Z_R \) and \( \bar{C} + Z_C \) on the surface \( S \), we obtain
\[
\begin{vmatrix}
2(3\bar{d} + g_1 - 1) - \nu_1 \\
\nu_1
\end{vmatrix}
\begin{vmatrix}
\nu_1 \\
(N_2 - 2)(3\bar{d} + g_1 - 1) - \nu_1
\end{vmatrix} \leq 0.
\]
Therefore,
\[
2(N_2 - 2)(3\bar{d} + g_1 - 1)^2 - N_2(3\bar{d} + g_1 - 1)\nu_1 \leq 0
\]
or
\[
N_2[2(3\bar{d} + g_1 - 1) - \nu_1] \leq 4(3\bar{d} + g_1 - 1). \quad (5.24)
\]
Thus, if there are two nonequivalent generic coverings \( f_1 \) and \( f_2 \), then
\[
N_2 \leq \frac{4(3\bar{d} + g_1 - 1)}{2(3\bar{d} + g_1 - 1) - \nu_1}. \quad (5.25)
\]

6 Proof of the Chisini conjecture for pluricanonical embeddings of surfaces of general type.

6.1. The numerical invariants in the case of a \( m \)-canonical embedding. Let \( S \) be a minimal model of a surface of general type with numerical invariants \((K_S^2) = k \) and \( e(S) = e \). Let \( X \) be a canonical model of the surface \( S \), and \( \pi : S \rightarrow X \) be the blowing down of \((-2)\)-curves. Let \( f : X \rightarrow \mathbb{P}^2 \) be a generic \( m \)-canonical covering, i.e. a generic projection onto \( \mathbb{P}^2 \) of \( X = \varphi_m(S) \), where \( \varphi_m \) is a \( m \)-canonical map, \( \varphi_m : S \rightarrow \mathbb{P}^{p_m-1} \), defined by the complete linear system \( |mK_S| \), \( p_m = \frac{1}{2}m(m-1)k + \chi(S) \). As is well known [BPV], by a theorem of Bombieri \( \varphi_m(S) \simeq X \) for \( m \geq 5 \), and \( \varphi_m \) gives the blowing down \( \pi \).

Let \( B \subset \mathbb{P}^2 \) be the discriminant curve. We conserve the notations of \S 4. Then
\[
L = mK_S, \quad K_S \cdot Z = 0, \quad R = (3m + 1)K_S - Z. \quad (6.1)
\]
By formulae (4.3), we obtain
\[
N = m^2 k, \quad d = m(3m + 1)k. \quad (6.2)
\]
By formulae (4.10), we find
\[ 3d + p_a(R) - 1 = (3m + 1)^2k, \] (6.3)
and
\[ p_a(R) - 1 = \frac{1}{2}(3m + 1)(3m + 2)k. \] (6.4)

6.2. Invariants of a surface and of the discriminant curve define invariants of the covering. Now let \( S_1 \) and \( S_2 \) be two surfaces of general type with numerical invariants \( k \) and \( e \). Let \( f_i : X_i \to \mathbb{P}^2, i = 1, 2, \) be their \( m_i \)-canonical coverings having the same discriminant curve \( B \subset \mathbb{P}^2 \). Then by the second formula of (6.2) it follows that \( m_1 = m_2 = m \). Then also \( \deg f_1 = \deg f_2 = N \). We show that the other numerical invariants of \( f_1 \) and \( f_2 \) are the same.

By formula (6.4) it follows that \( p_a(R_1) = p_a(R_2) \), and since \( p_a(R) = g + \delta_X \), we have \( \delta_X_1 = \delta_X_2 \).

By formulae (4.14) and (4.11) it follows that the number of \( p \)-cusps \( c_p \) and the number of \( p \)-nodes \( n_p \) for both coverings are the same. Then \( n_p = n_{pp} + n_{sp} = n_{pp} + n_{sp}, c_p = c_{pp} + c_{ps} = c_{pp} + c_{sp} \) and, consequently, \( n_{ps} = n_{sp} \) and \( c_{ps} = c_{sp} \).

6.3. The main inequality in the case of surfaces of general type. To prove that \( m \)-canonical projections \( f_1 \) and \( f_2 \) are equivalent, by (5.24) it is sufficient to show that an inequality
\[ N \left( 2(3d + p_a(R) - 1) - \iota \right) > 4(3d + p_a(R) - 1) \]
holds (here \( R \) stands for \( R_1 \)), or
\[ (N - 2)(3d + 2p_a(R) - 2) - N \cdot \iota > 0, \] (6.6)
where
\[ \iota = 2n_{sp} + 2c_{sp} + c_{pp} = 2n_{sp} + c_{sp} + c_p. \] (6.7)

Let us obtain an estimate for the number \( \iota \). We can express \( c_p \) by formulae (4.14)
\[ c_p = 3N + 2p_a(R) - 2 - e. \] (6.8)

To estimate \( 2n_{sp} + c_{sp} \) we use the Hirzebruch-Miyaoka inequality ([BPV], p.215): if the minimal surface of general type \( S \) contains \( s \) disjoint \((-2)\)-curves, then
\[ s \leq \frac{2}{9} (3e(S) - (K_S^2)) . \] (6.9)

Since we can take one \((-2)\)-curve for each of the singular points of types \( A_1 \) and \( A_2 \) on \( X \), we have
\[ n_s + c_s \leq \frac{2}{9} (3e - k) . \] (6.10)

Remark 6.1 Instead of the Hirzebruch-Miyaoka inequality we can use the estimate \( 2n_{sp} + c_{sp} \leq 2(h^{1,1} - 1) = 2(e - 2 + 4q - 2p_g - 1) \) and the inequalities \( p_g \geq q, p_g \leq \frac{1}{2}(K_S^2) + 2 \) (the Noether’s inequality).
By (6.7), (6.8) and (6.10), we obtain an estimate

\[ \iota \leq \frac{4}{9}(3e - k) + 3N + 2p_a(R) - 2 - e = \frac{1}{9}e - \frac{4}{9}k + 3N + 2p_a(R) - 2. \]

Applying the Noether’s inequality ([BPV], p. 211),

\[ e \leq 5k + 36, \quad (6.11) \]

we obtain

\[ \iota \leq \frac{11}{9}k + 12 + 3N + 2p_a(R) - 2. \quad (6.12) \]

Combining (6.12) and (6.6), we obtain a corollary.

**Lemma 6.1** If the inequality

\[ 3N(d - N) - 6d - 4(p_a(R) - 1) - \left( \frac{11}{9}k + 12 \right)N > 0 \quad (6.13) \]

holds, then a generic m-canonical projection of a surface of general type $S$ with given $k$ and $e$ is unique. ■

**6.4. Proof of Theorem 0.3.** Express the inequality (6.13) in terms of $m$. Substitute $N$ and $d$ from (6.2) and $p_a(R) - 1$ from (6.4) to (6.13). We obtain

\[ 3m^3(2m + 1)k^2 - 6m(3m + 1)k - 2(3m + 1)(3m + 2)k - \left( \frac{11}{9}k + 12 \right)km^2 > 0, \]

i.e.

\[ 3m^3(2m + 1)k - 4(3m + 1)^2 - \left( \frac{11}{9}k + 12 \right)m^2 > 0. \]

Dividing by $m^2$, we obtain

\[ 3m(2m + 1)k - \left( \frac{11}{9}k + 12 \right) - 4 \left( 3 + \frac{1}{m} \right)^2 > 0, \]

or, dividing by $k$,

\[ 3m(2m + 1) > \frac{11}{9} + \frac{1}{k} \left( 12 + 4(3 + \frac{1}{m})^2 \right). \quad (6.14) \]

The right side of inequality decreases, when $k$ and $m$ increase. This inequality holds for all $k \in \mathbb{N}$, if it holds for $k = 1$. For $k = 1$ and $m = 3$ the right side equals $\frac{11}{9} + 12 + 4 \cdot \left( \frac{4}{3} \right)^2 = \frac{173}{9} < 9 \cdot 7 = 63$. Thus, the inequality (6.14), and, consequently, the inequality (6.6), holds for $m \geq 3$ and for all $k$. This completes the proof of Theorem 0.3.

We can mention in addition that for $m = 2$ the inequality (6.14) holds, if $k > 2$, and for $m = 1$ it holds, if $k > 9$. 

42
References

[A] Arnol’d, V.I.: Indices of singular points of 1-forms on a manifold with a boundary, convolution of invariants of groups generated by reflections, and singular projections of smooth surfaces. – Usp. Math. Nauk., 34, No. 2 (1979), 3–38. (Engl. translation in Russ. Math. Surv., 34, No. 2 (1979), 1–42.)

[AGV] Arnol’d, V.I., Gusein-Zade, S.M., Varchenko, A.N.: Singularities of differentiable Maps, Vol. I, II. – Birkhäuser, (1985).

[BPV] Barth W., Peters C., Van de Ven A.: Compact complex surfaces.– Springer (1984).

[C] Catanese F.: On a Problem of Chisini. – Duke Math. J., 53, No. 1 (1986), 33-42.

[G-H] Griffiths, Ph., Harris, J.: Principles of algebraic geometry.– John Wiley & Sons, New York (1978).

[K] Kulikov, Vic.S. On a Chisini Conjecture. – Izvestiya: Mathematics, V.63, No.6 (1999).

[M] Moishezon B.: Complex Surfaces and Connected Sums of Complex Projective Planes. – LNM 603, Springer (1977).

[Z] Zariski, O.: Algebraic surfaces.– Berlin,Verlag von Julius Springer (1935) (Springer-Verlag (1971)).

V.S. Kulikov
Chair of Mathematics,
Moscow State University of Printing,
E-mail: valentin@masha.ips.ras.ru

Vic. S. Kulikov
Department of algebra,
Steklov Mathematical Institute,
E-mail: victor@olya.ips.ras.ru

43