An efficient Fisher-scoring algorithm for fitting latent class models with individual covariates

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Abstract

For latent class models where the class weights depend on individual covariates, we derive a simple expression for computing the score vector and a convenient hybrid between the observed and the expected information matrices which is always positive definite. These ingredients, combined with a maximization algorithm based on line search, provides an efficient tool for maximum likelihood estimation. In particular, the proposed algorithm is such that the log-likelihood never decreases from one step to the next and the choice of starting values is not crucial for reaching a local maximum. We show how the same algorithm may be used for numerical investigation of the effect of model mispecifications. An application to education transmission is used as an illustration.

Keywords: Latent class models, individual covariates, Fisher-scoring algorithm, line search.

1. Introduction

The latent class models considered in this paper are those where subjects belong to one among a finite set of disjoint latent classes with probabilities which may depend on individual covariates. Observations are based on a collection of discrete responses whose distribution depends on the latent type but not on covariates. The literature on latent class models is very extensive, see Vermunt (2010) for a convenient selection of some of the most relevant contributions; a slightly more extended framework, dealing with missing data and known groups of distinct respondents is presented by Chung (2003).

The EM (expectation-maximization) algorithm is generally used to compute the maximum likelihood estimates, though, for instance, the Latent GOLD software combines EM and Newton Raphson. As regards the EM algorithm, its numerical stability and the fact that the likelihood always increases from one step to the next, are mentioned as its main advantages. The Newton-Raphson algorithm, though faster, is known for being likely to diverge, unless the starting values are close to a local maximum; in particular, in the context of latent class models, the algorithm cannot be used safely on their own. The performance of the Newton-Raphson or Fisher-scoring algorithms may be greatly improved by performing a line search to optimize the step length (see for example...
Potra and Shi (1995; Turner, 2008) and adopt suitable strategies that prevents the likelihood from decreasing. Bolck et al. (2004) have proposed a three step algorithm which, in the first step estimates a latent class model without covariates, assigns subjects to latent classes and estimates the latent regression model with weights derived from the estimated matrix of classification errors. An extension of this approach is proposed by Vermunt (2010).

In this paper we propose a convenient matrix formulation that allow to derive simple expressions for computing the score and the observed or the expected information matrix. The expected information matrix has the advantage of being always positive definite whenever the model is identifiable; on the other hand, it has been argued (see Efron and Hinkley, 1978) that the observed information matrix is preferable for the asymptotic distribution of the maximum likelihood estimates because it is data dependent. We show that there is a component of the observed information which is easier to compute, is always positive definite and such that its expectation is still equal to the expected information; we suggest using this hybrid information matrix in the maximization algorithm. In addition we describe the main features of an intelligent software which combines line search and strategies to prevent the likelihood from decreasing. With a minor modification, the same algorithm may be used to maximizes the expected log-likelihood; this could be used as a numerical tool to assess consistency of estimates under possible misspecifications of the model, when theoretical results are not easily available.

In section 2, after introducing the notations, we derive an expression for the score and an approximation of the information matrix which, we show, is positive definite, under suitable conditions. In section 3 we discuss the computation of the previous quantities and describe a suitable line search algorithm; in addition we show how the same algorithm may be used for numerical assessment of the effect of model misspecifications. In section 4 we present an application from the field of education transmission.

2. Notations and main results

Suppose there are $c$ disjoint latent classes and let $\pi_i$, $i = 1, \ldots, n$, be the vector of prior probabilities for the $i$th subject to belong to one of the $c$ latent classes; let $X_i$ be a $c \times k$ matrix depending on individual covariates and assume that

$$\pi_i = \frac{\exp(X_i\beta)}{\sum_l \exp(X_l\beta)}.$$

Let $r$ be the number of possible configurations of the response variables; their joint distribution conditional on $U = j$, $j = 0, \ldots, c - 1$, may be represented by the $r \times 1$ vector of probabilities

$$q_j = \frac{\exp(G\theta_j)}{\sum_l \exp(G\theta_l)},$$

where $G$ is a $r \times g$ full rank design matrix and $\theta_j$ a suitable vector of log-linear parameters. The dimension $r$ of $q_j$ is equal to the product of the number of
categories of the response variables with entries in a given lexicographic order. This formulation does not necessarily assume conditional independence among the responses; the conditional dependence structure is determined by the $G$ matrix and we assume that this is such that the model is identifiable. Finally, let $Q$ be the matrix whose $j$th column is $q_j$, so that $p_i = Q \pi_i$ is the marginal distribution of the responses. If we stack the vectors $\theta_j$ one below the other into the vector $\theta$, the contribution of the $i$th subject to the log-likelihood may be written as $\ell(\beta; \theta; y_i, X_i) = y_i' \log(p_i)$.

2.1. Score vector and information matrix

Under the assumption that observations from different subjects are independent, the score and the information may be written as sums across subjects. By application of the chain rule, the score relative to $\beta$ is

$$s_\beta = \sum_i X_i' \Omega_{\pi_i} Q' \text{diag}(p_i)^{-1} y_i$$

where $\Omega_{\pi_i} = \text{diag}(\pi_i) - \pi_i \pi_i'$. Noting that $p_i = \sum_j \pi_{ij} q_j$, by the chain rule, the score relative to $\theta_j$ is

$$s_j = \sum_i \pi_{ij} G_j' \Omega_{j} \text{diag}(p_i)^{-1} y_i,$$

where $\Omega_j = \text{diag}(q_j) - q_j q_j'$. It is convenient to think of the observed information matrix as made of two components: call $F$ the matrix which we obtain by treating the score vector as a function of $\text{diag}(p_i)$ while all the rest is held constant and call $D$ the matrix which we obtain by differentiating the score vector while $\text{diag}(p_i)$ is held constant. Let

$$A_i = (Q \Omega_{\pi_i} X_i \quad \pi_{i1} G \quad \ldots \quad \pi_{ic} G)$$

and $\text{diag}(d_i) = \text{diag}(p_i)^{-2} \text{diag}(y_i)$.

Lemma 1. The matrix $F$ is equal to $\sum_i A_i' \text{diag}(d_i) A_i$ and $E(D) = 0$.

Proof. See the Appendix.

When individual observations are available, $y_i = e_u(i)$, a vector of 0’s except for the $u(i)$th entry which is 1; let $t_i = A_i' e_u(i)$, let $T$ the $n \times (k + cg)$ matrix with rows $t_i'$; let $\bar{p}_i$ be the $u(i)$th element of $(p_i)$ and $\bar{p}$ the vector with elements $\bar{p}_i$.

Lemma 2. The hybrid information matrix $F$ is positive definite if and only if $T$ is of full rank $k + cg$.

Proof. The result follows because, by simple algebra, $F = \sum_i t_i t_i' / \bar{p}_i^2 = T' \text{diag}(\bar{p})^{-2} T$.
In practice, $F$ is positive definite whenever, within the $n$ observations, there are at least $n \geq k + cg$ distinct patterns of covariate configurations; in that case the model is identifiable.

The result of Lemma 2 seems to suggest that $F$ may be used as an approximation for both the observed and the expected information matrix. Relative to the observed information, it has the advantage that it is positive definite like the expected information. However, as we show below, it is more easily computed than the expected information and, in addition, it is partly data dependent.

3. Computational aspects

First we note that the whole score vector may be computed as

$$s = \begin{pmatrix} s_\beta \\ s_1 \\ \vdots \\ s_c \end{pmatrix} = \sum_i A_i' \text{diag}(p_i)^{-1} y_i = \sum_i t_i / \tilde{p}_i.$$ 

Though the $A_i$ matrices involve, apparently, several matrix multiplication, as we show below, they need not be computed explicitly. Each $t_i$ vector may be constructed by stacking one below the other the following components, where $\tilde{q}_i = Q' e_{u(i)}$ is the $u(i)$th column of $Q'$,

$$X_i' \Omega_{\pi_i} Q' e_{u(i)} = X_i' \text{diag}(\pi_i) \tilde{q}_i = X_i' \pi_i \pi_i' \tilde{q}_i,$$

and, for $j = 0, \ldots, c - 1$,

$$\pi_{ij} (g_i' \tilde{q}_{ij} - G' q_j \tilde{q}_{ij}),$$

where $g_i$ is the $u(i)$th column of $G'$.

3.1. Line search

Let $\psi$ be the vector obtained stacking $\beta$ and $\theta$ one below the other; after $h - 1$ steps, the basic updating equation takes the form

$$\hat{\psi}^{(h)} = \hat{\psi}^{(h-1)} + a(h-1) \left( F^{(h-1)} \right)^{-1} s^{(h-1)},$$

where $a_{h-1}$ is the step length. When the log-likelihood is not concave and may have two or more local maxima, an algorithm with $a_h = 1$ is almost certain to diverge, unless the starting value is very close to a local maximum; one possibility would be to set $a_0$ very small and let it increase with $h$. In a related context, Turner (2008) suggest using the Levenberg-Marquardt algorithm which combines Newton-Raphson and steepest ascent steps; this would be less efficient in our context where the information matrix is positive definite. Our algorithm uses a proper line search where the log-likelihood is never allowed to decrease. Its main features are given below:

1. set $a_0$ to some value possibly smaller than 1, say, 0.5;
2. at the \((h-1)\)th step, first use the updating equation to compute the first
guess, say, \(\hat{\psi}^{(h,a)}\);
3. compute the log-likelihood and the score at the first guess;
4. with these elements find the step length that maximizes a cubic approxi-
mation to the log-likelihood, let \(\hat{\psi}^{(h,b)}\) be the second guess;
5. compute the log-likelihood at \(\hat{\psi}^{(h,b)}\) and select the best guess;
6. in case of no improvement, first shorten the step and, if even this does not
work, perform a steepest ascent step.

Few other adjustments are made to check whether the log-likelihood is locally
concave or that the second derivative is negative along the given direction in
order to perform some conditional adjustments to the step length. In any case,
a starting point is never updated unless a better one has been found.

In order to increase the probability of reaching a global maximum, after con-
vergence, a random perturbation is applied to the estimates and the algorithm
restarted for a few times.

3.2. Numerical assessment of the effect of mispecifications

We now show how the expressions for the score and the information matrix
may be used to assess the effect of model misspecifications. Suppose that \(\mathcal{M}\)
is the true model and \(\tilde{\mathcal{M}}\) is a mispecified model; misspecifications may concern
the number of latent classes, the regression model determined by \(X_i\) or the
dependence structure of responses encoded into the \(G\) matrix.

Suppose, for simplicity, that the \(n\) covariate configurations are kept fixed
while the number \(m\) of the replicates \(y_{it}\), corresponding to each configuration,
increases. Then, the law of large numbers may be used to show that the average
log-likelihood function converges to its expected value at the true model. Thus,
if we want to assess the effect that a given misspecifications of the model has
on the estimates of the parameters of the mispecified model when no theory is
easily available, we simply maximize the appropriate expected log-likelihood.

This may be easily performed by the same adjusted Fisher scoring algorithm
described above: we may use the expressions for the score vector and information
matrix and simply replace the observations \(y_i\) with their expected value under
the true model. The only difference is that the simplified expressions described
above can no longer be used. On the other hand, based on our experience,
the expected log-likelihood seems to very well behaved so that convergence is
usually reached in very few steps.

4. Application

4.1. The data

We use data from the National Child Development Survey (NCDS), a UK
cohort study targeting all the population born in the UK between the 3rd to
the 9th of March 1958. Information on family background and on schooling
and social achievements for the subjects in the sample were collected at different stages of their lives. In the application below we use, as covariates, the number of years of education and the amount of concern for the child education (as graded by the teachers), separately for father and mother. As response variables we consider the performance in mathematics and reading test scores taken when the child was 7, 11 and 16 years old, an overall measure of non cognitive attitudes (as reported by teachers) and the academic qualification achieved (none, O-level, A-level, university degree). Overall we use 8 responses, all, except for academic qualification, were coded into three categories based on quantiles. A complete description of the original data is available at http://www.esds.ac.uk/longitudinal/access/ncds.

4.2. The model

The vector of prior weights \( \pi \) was assumed to depend on the four covariates (education and interest for each parent) as in a multinomial logistic regression, this requires \( 4(c-1) \) regression parameters and \( c-1 \) logit intercepts. The response variables were assumed conditionally independent, except for a first order autoregressive model within Math and Read test scores taken at adjacent dates; because each of these variables has three ordered categories, to use a parsimonious model, in place of the 4 interactions, we used a vector of scores with values 1, 0.5 and 0 according to whether the categories of the two response variables (say Math at 16 and Math at 11) were equal, differed of 1 or of 2.

For simplicity, in this application we restrict attention only to the 2568 females with no missing data for the selected variables. Because the relative size of the selected sub-sample is slightly less than 30%, results should be interpreted with care. Similar models with a number of latent classes ranging from 2 to 5 were fitted and the Bayesian Information criteria was used to determine that the model with four latent classes was the most adequate.

| \( c \) | 2   | 3   | 4   | 5   |
|--------|-----|-----|-----|-----|
| Bic    | 37165 | 36577 | 36507 | 36592 |

4.3. Main results

The estimated regression coefficients and \( z \) ratios for the logits of belonging to the the different latent classes relative to the first are displayed in Table 2. All regression coefficients are positive and most are also significant; this seems to suggest that the first latent class contains subjects with the lowest cognitive abilities and that the parents concern, probably associated to their pressure, is important in pushing up. The education of the father seems to have a positive significant effect most of the times, not so the education of the mother; however father education might be a proxy for family income.
Table 2: Estimated regression coefficients for the latent weights

| Model | $\hat{\beta}$ | z | $\hat{\beta}$ | z | $\hat{\beta}$ | z |
|-------|----------------|---|----------------|---|----------------|---|
| Int.  | 0.962          | 6.40 | 0.518          | 5.16 | 1.317         | 8.34 |
| F.Ed. | 1.239          | 8.87 | 0.135          | 1.34 | 2.001         | 10.02 |
| M.Ed. | 0.028          | 0.16 | 0.145          | 1.53 | 0.388         | 2.44 |
| F.In. | 0.296          | 2.85 | 0.385          | 3.9  | 0.831         | 5.70 |
| M.In. | 0.334          | 3.52 | 0.607          | 3.72 | 1.239         | 6.53 |

To characterize the nature of the latent classes better, we display the conditional distributions of the academic qualification and that of the non cognitive score tests in Table 3. It emerges that academic qualifications and latent classes are stochastically ordered, thus, relative to this response, classes are ordered from worst to best. Instead, relative to non cognitive tests, latent classes are in reverse order and this seems to indicate that non cognitive scores are probably a measure of problematic behaviour. A similar picture emerges from Table 4: essentially subjects in latent class 3 are the most talented both in Math and Read.

Table 3: Conditional distributions of academic qualification and non cognitive scores

| Academic qual. | Non cognitive |
|----------------|---------------|
| U              | None | O-lev | A-lev | Univ | 0 | 1 | 2 |
|----------------|----------------|-------|-------|------|---|---|---|
| 0              | 0.9584 | 0.0328 | 0.0079 | 0.0009 | 0.0534 | 0.2027 | 0.7439 |
| 1              | 0.6198 | 0.3037 | 0.0038 | 0.0736 | 0.2443 | 0.3814 | 0.3743 |
| 2              | 0.1908 | 0.5579 | 0.0830 | 0.1683 | 0.4344 | 0.3845 | 0.1811 |
| 3              | 0.0611 | 0.2243 | 0.2225 | 0.4920 | 0.5644 | 0.3131 | 0.1225 |

Table 4: Estimated conditional distribution of Math and Read scores at the age of 16

| U | Math | Reading |
|---|------|---------|
|   | 0    | 1       | 2      | 0   | 1   | 2   |
| 0 | 0.8720 | 0.1280 | 0.0000 | 0.8982 | 0.1018 | 0.0000 |
| 1 | 0.5703 | 0.4029 | 0.0268 | 0.5023 | 0.4519 | 0.0459 |
| 2 | 0.1285 | 0.5475 | 0.3239 | 0.0736 | 0.5547 | 0.3717 |
| 3 | 0.0040 | 0.0653 | 0.9307 | 0.0014 | 0.1403 | 0.8583 |
Appendix

Proof of Lemma 1

Information relative to $\beta$: to differentiate $s_\beta$ with $\Omega_\pi$, held constant, use the properties of the inverse and diagonal operators to differentiate with respect to the $j$th element of $\pi_i$ which gives

$$-X'_i\Omega_\pi, Q'\text{diag}(p_i)^{-2}\text{diag}(y_i)q_j;$$

the result follows by stacking these row vectors one to the side of the other and then apply the chain rule. When $\text{diag}(p_i)^{-1}$ is held fixed, let $v_i = Q'\text{diag}(p_i)^{-1}y_i$, then compute the derivative with respect to $\pi'_i$ and apply the chain rule to obtain

$$X'_i(\text{diag}(v_i) - (\pi'_i v_i)I - \pi'_i v'_i)\Omega_\pi_iX_i.$$  

To show that the expected value of this expression is 0 note that $E(v_i) = Q'1 = 1_r$, $\text{diag}[E(v_i)] = I$, the identity matrix and that $v'_i\Omega_\pi_i = 0'$.

Information relative to $\theta$: The derivative of $s_h$ with $\Omega_j$ held constant may be computed as above giving terms of the form

$$-\pi_{ij}\pi_{ih}G'\Omega_j\text{diag}(p_i)^{-2}\text{diag}(y_i)\Omega_jG.$$  

Let $v_i = \text{diag}(p_i)^{-1}y_i$ and $g_h$ the $h$th column of $G'$, to compute the derivative with $v_i$ held fixed, first differentiate with respect to the elements of $q_j$ and then use the chain rule to get

$$\pi_{ij}G'(\text{diag}(v_i) - (q'_j v_i)I - q'_j v'_i)\Omega_jG.$$  

Because $E(v_i) = 1_r$, this expression has 0 expectation.

The mixed information: In practice it is convenient to differentiate each $s_j$ with respect to $\beta'$. With techniques similar to those used above, the component where the initial $\pi_j$ is held fixed is

$$-\pi_{ij}G'\Omega_j\text{diag}(p_i)^{-2}\text{diag}(y)Q\Omega_\pi X_i$$  

and the other component is simply

$$s_j(x'_j - \pi'X_i),$$  

this has 0 expectation because $E(s_j) = 0$.

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