The \(^{12}\text{C}\) nucleus with \(N=6\) and \(Z=6\) is a doubly-magic closed-shell nucleus in a toroidal potential and Wheeler’s triangular resonating group model of \(^{12}\text{C}\) as three clusters of alpha particles will naturally generate a toroidal density if the nucleons interchanging between the clusters are allowed to circulate continuously from one cluster to another. Experimentally, many excited states of \(^{12}\text{C}\) decay predominantly into three alpha particles, and the triangular clusters of three alpha particles have a high degree of overlap with a torus. We explore a toroidal description for some excited states of \(^{12}\text{C}\) and search for the signature that will reveal a possible toroidal configuration of the nucleus. We find that the \(^{12}\text{C}\) nucleus in a toroidal configuration distinguishes itself by toroidal multiplets of particle-hole excitations between one toroidal shell to another. Subject to further confirmation, the Hoyle state and many of its higher excited states may be tentatively attributed to those of a \(^{12}\text{C}\) nucleus in a toroidal configuration.

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I. INTRODUCTION

The study of the intrinsic structure of the \(^{12}\text{C}\) nucleus has a long history. Wheeler proposed an \(\alpha\) particle model for \(\alpha\)-conjugate nuclei and suggested in 1937 that the \(^{12}\text{C}\) nucleus may be described as a triangular resonating grouping of three alpha clusters obeying Bose-Einstein statistics and exchanging nucleons between them [1, 2]. Later in 1953 Hoyle postulated an excited state of \(^{12}\text{C}\) as the doorway for the triplet alpha reaction in nucleosynthesis, in which two alpha particles fuse into beryllium-8 and then capture a third alpha particle to form the \(^{12}\text{C}\) nucleus. Related theoretical investigations have been carried out to understand the properties of the \(^{12}\text{C}\) nucleus. Recent experimental results and related reviews have been presented in [7–17]. In addition to Wheeler’s \(\alpha\) particle model, related theoretical models include the cluster model of Brink [18], an interacting cluster model of three alpha particles [19–21], a Nilsson oblate ellipsoidal model [22] with a commensurate axis ratio [23], an algebraic U(7) model with a D\(_{3h}\) symmetry [24, 25], an antisymmetric molecular dynamics (AMD) model [26–28], a microscopic fermionic molecular dynamics (FMD) model [29, 30], a Bose-Einstein condensate-type cluster model [31], an \(ab\ initio\) no-core shell model [32, 33], a no-core simplectic model (NCSpM) [34], \(ab\ initio\) lattice effective field theory (L-EFT) [35], microscopic cluster models [36], energy-density functional mean-field models [37–44], and a rod model of excited states [45–47].

We explore here an additional toroidal model of the \(^{12}\text{C}\) nucleus for many reasons. First and foremost is the reason that the nature of many excited states of \(^{12}\text{C}\) has not been fully understood [7–17]. On the other hand, in the single-particle energy level diagram for a toroidal nucleus [48, 49], the \(^{12}\text{C}\) nucleus with \(N=6\) and \(Z=6\) is a doubly-magic closed-shell nucleus. Toroidal excited states have been predicted in the mass region of \(40 \leq A \leq 70\) [48, 49] by employing the shell-correction method [50]. The extrapolation from these results points to a possible toroidal local equilibrium in \(^{12}\text{C}\) in the low excitation energy region. Furthermore, recent experimental observation of narrow resonances in excited \(^{28}\text{Si}\) [51] suggests possible population of toroidal high-spin isomers as predicted previously in \(^{28}\text{Si}\) [52]. Similar toroidal high-spin isomers have also been predicted in the light-mass region from \(^{24}\text{Mg}\) to \(^{56}\text{Ni}\) in relativistic and non-relativistic mean-field theories [52–58]. It is also interesting to note that Wheeler’s triangular resonating group model of the \(^{12}\text{C}\) nucleus with the \(D_{3h}\) symmetry [24, 25] represents a model of three quasi-static clusters with the interchange of nucleons between them. In a more general dynamical description, if the nucleons interchanging between the triangular clusters are allowed to circulate continuously and self-consistently from one cluster to another, they will generate naturally a toroidal density distribution. Experimentally, many excited states of \(^{12}\text{C}\) decay predominantly into three alpha particles [13–17], and a triangular cluster of three alpha particles has a high degree of overlap with a torus. For all these reasons, it is of interest to explore whether some excited states of the \(^{12}\text{C}\) nucleus may be attributed to those of the nucleus in a toroidal configuration.

The band of \(0^+(\text{ground})\) and \(2^+(4.33\text{ MeV})\) states of...
the $^{12}$C nucleus have been identified as collective states of an oblate spheroid with $\beta = -0.6$ [59]. Toroidal states of $^{12}$C, if exist, can only be associated with higher excited states of $^{12}$C above the $2^+(4.33 \text{ MeV})$ state. The next higher excited state is the $0^+$ Hoyle state at 7.654 MeV, which decays predominantly into three alpha particles. Since the triangular clusters of three alpha particles have a large overlap with a toroidal distribution, the Hoyle state presents itself as a good candidate to be a member of the toroidal states of $^{12}$C. Because the toroidal configuration and the oblate spheroid configuration have distinct topologies, the toroidal states and the oblate spheroid states are likely to be independent without substantial mixing, if they co-exist in $^{12}$C.

The intrinsic shape of a $^{12}$C nucleus in the toroidal configuration shows up as a distinct spectrum of a set of states that are intimately tied to each other through its toroidal geometry. Such a distinct spectrum of states constitute the toroidal signature that will reveals a possible toroidal configuration of the nucleus. We shall look for such a signature among the excited states of $^{12}$C and explore whether the Hoyle state at 7.654 MeV may be the ground state of the toroidal band of the $^{12}$C nucleus. By comparison with experimental data, we find that subject to further confirmation, the Hoyle state and many of its higher excited states may be tentatively attributed to those of a $^{12}$C nucleus in a toroidal configuration.

The paper is organized as follows. In Section II, we write down the single-particle state energies of a toroidal nucleus in the approximation of a large major radius with the neglect of the small spin orbit interaction. They depend only on the toroidal major radius $R$ and the orbital angular momentum component $\Lambda_z$ along the symmetry z-axis. The method to calculate the quantum numbers and the excitation energies of the particle-hole excitations from these single-particle states are examined in Section III. The spectrum of $^{12}$C in the toroidal configuration is investigated in Section IV with the details of the calculations for higher excited states included in the Appendix. In Section V, we confront the theoretical predicted spectrum of toroidal states with observed $^{12}$C spectrum. We find tentatively that the Hoyle state and many of its higher states may be attributed to those of a $^{12}$C nucleus in a toroidal configuration. In Section VI, we propose a toroidal constraint in mean-field dynamics to study the energy surface in the toroidal degree of freedom and to locate local energy minima in the toroidal configuration. In Section VII, we present our summary and discussions.

## II. SINGLE-PARTICLE STATES IN A TOROIDAL POTENTIAL

Nucleons in a toroidal nucleus generate a mean-field potential that has the same shape as the toroidal density distribution. To get the spectrum of a toroidal nucleus, we place the nucleons in the lowest states in the toroidal potential, make the single-particle particle-hole excitations, and record their excitation energies and quantum numbers. The single-particle potential can be represented by a toroidal harmonic oscillator potential with minima at $\rho=R$ and $z=0$ in cylindrical coordinates $(\rho, z, \phi)$ [48, 49]

$$V_0(\rho, z) = \frac{1}{2} m \omega^2 \{ (\rho - R)^2 + z^2 \},$$  \hspace{1cm} (1)

where $\rho = \sqrt{x^2 + y^2}$, the symmetry axis lies along the $z$-direction, $\omega$ is the harmonic oscillator frequency, and $m$ is the rest mass of a nucleon. By introducing the radial difference $q = \rho - R$, we re-write the above potential as

$$V_0(\rho, z) = \frac{1}{2} m \omega^2 (q^2 + z^2).$$  \hspace{1cm} (2)

Approximate analytical solutions of the single-particle eigenenergies have been obtained by making the large major radius approximation, in which the major radius $R$ is much greater than the minor radius $d$ [60]. In that approximation, the variable $q$ can be extended to $-\infty$ without serious errors so that the above potential (2) is just a two-dimensional simple harmonic oscillator potential in $q$ and $z$. The eigenstates are labeled by $|n_\rho n_z \Lambda_z \Omega_z \rangle$, where $n_\rho$ and $n_z$ are the harmonic oscillator quantum numbers of the potential (2), $\Omega_z$ is $\Lambda_z + s_z$, $\Lambda_z$ is the orbital angular momentum component $\Lambda_z$, and $s_z$ is the intrinsic spin component. The single-particle energies are given approximately by [60]

$$\epsilon(n_\rho n_z \Lambda_z \Omega_z) = (n_\rho + \frac{1}{2}) \hbar \omega' + (n_z + \frac{1}{2}) \hbar \omega_z$$
$$+ \frac{\hbar^2 \Lambda_z^2}{2m R^2} + \frac{1}{2} a_0,$$  \hspace{1cm} (3)

where

$$\omega_z^2 = \omega_z^2 (1 + a_2),$$  \hspace{1cm} (4)

$$a_2 = \frac{1}{m \omega_z^2} \left\{ \frac{\hbar^2 \Lambda_z^2}{2m R^2} + \frac{2n_\rho (\hbar \omega_z)^2}{\hbar \omega_0} s_z \Lambda_z \frac{2}{R^2} \right\},$$  \hspace{1cm} (5)

$$a_0 = \frac{1}{2 m \omega_z^2 q_0^2} + \frac{\hbar^2 \Lambda_z^2}{2m R^2} \left( \frac{2n_\rho}{R} + \frac{3q_0^2}{R^2} \right)$$
$$- \frac{2n_\rho (\hbar \omega_z)^2}{\hbar \omega_0} s_z \Lambda_z \left( \frac{q_0}{R} + \frac{q_0^2}{R^2} \right),$$  \hspace{1cm} (6)

$$q_0 = \frac{1}{m \omega_z^2 (1 + a_2)} \left\{ \frac{\hbar^2 \Lambda_z^2}{2m R^2} + \frac{2n_\rho (\hbar \omega_z)^2}{\hbar \omega_0} s_z \Lambda_z \right\},$$  \hspace{1cm} (7)

where $\kappa$ is the strength of the spin-orbit interaction [22] and $\hbar \omega_0$ is the harmonic oscillator constant of an equivalent spherical nucleus with the same volume [61]

$$\hbar \omega_0 = 41 \text{ MeV}/A^{1/3}.$$  \hspace{1cm} (8)

We shall limit our attention only to low-lying excited states involving nucleons with $n_\rho=0$ and $n_z=0$ and the lowest few $\Lambda_z$. For convenience of notation, we shall omit
$n_p$ and $n_z$ from the state labels. The nuclear spin unit $\hbar$ will be implicitly understood. Upon neglecting further the small spin-orbit splitting for these single-particle states with $n_p=0$ and $n_z=0$, the single-particle energies are given approximately by

$$\epsilon(\Lambda_z\Omega_z) \equiv \epsilon(n_p=0, n_z=0, \Lambda_z\Omega_z) \approx \frac{\hbar^2\Lambda_z^2}{2m} - \frac{\bar{\epsilon}}{R^2} + \hbar\omega_{\perp} \langle 9 \rangle$$

III. PARTICLE-HOLE EXCITATIONS OF A CLOSED SHELL TOROIDAL NUCLEUS

The density of single-particle states in the energy space is far from being uniform in a toroidal nucleus. With the single-particle state energies given by Eq. (3) or (9) for a toroidal nucleus, the single-particle state energies cluster together into “shells” in the single-particle energy space, with a rather large energy gap between one shell and the next, the small spin-orbit splitting for these single-particle states as basis states as in [62–65].

The doubly-magic nature of the $^{12}$C nucleus in the toroidal configuration means that the energy gap between the occupied states in the $\Lambda_z=1$ shell and the unoccupied states in the $\Lambda_z=2$ shell is expected to be quite large, presumably much larger than the spin-orbit and residual interactions. Thus, the single-particle particle-hole excitations will yield the gross structure, while the spin-orbit and residual interactions will provide the fine structure of the spectrum. In the present first survey we shall confine ourselves only with the gross structure by studying particle-hole excitations built on toroidal single-particle shells without spin-orbit and residual interactions. Refinement of the energy spectrum can be carried out in the projected shell model calculations using the toroidal states as basis states as in [62–65].

We use the single-particle energies Eq. (9) to evaluate the energy spectrum of a double-closed shell toroidal nucleus by calculating the particle-hole excitation energy $E_I = E_0$ and its associated angular momentum component $I_z$, at the equilibrium toroidal major radius $R$. We call the angular momentum component $I_z$ along the symmetry axis with $I_z=I$ the spin $I$ (or $I_z$) of the toroidal state.

The multiplet of $(n \text{ particle})-(n \text{ hole})$ excitations from the ground state constructed by promoting $n$ nucleons from the occupied $\Lambda_{zi}$ shell to an unoccupied $\Lambda_{zf}$ shell will be labeled as $(n\text{pnh})^{\pi}_{\Lambda_{zi},\Lambda_{zf}}$, where $\pi$, the parity of the multiplet, is equal to $(-1)^{\Lambda_{zf}-\Lambda_{zi}}$. In particular, the $(1p1h)^{\pi}_{\Lambda_{zi},\Lambda_{zf}}$ multiplet of particle-hole excitations can be constructed by promoting a nucleon from an occupied initial state $|\Lambda_{zi}\Omega_{zi}\rangle$ in the $\Lambda_{zi}$ shell to an unoccupied final state $|\Lambda_{zf}\Omega_{zf}\rangle$ in the $\Lambda_{zf}$ shell. From Eq. (9), such a particle-hole excitation leads to a spin increment $\Delta_I I_z$ for such an $i \rightarrow f$ particle-hole excitation given by

$$\Delta_I I_z |[\Lambda_{zi}\Omega_{zi}] \rightarrow [\Lambda_{zf}\Omega_{zf}] \rangle = \Omega_{zf} - \Omega_{zi}, \langle 10 \rangle$$

and an excitation energy increment $\Delta_I E_{\Lambda_i\Lambda_f}$ given by

$$\Delta_I E_{\Lambda_i\Lambda_f} |[\Lambda_{zi}\Omega_{zi}] \rightarrow [\Lambda_{zf}\Omega_{zf}] \rangle = \epsilon([\Lambda_{zf}\Omega_{zf}]_i) - \epsilon([\Lambda_{zi}\Omega_{zi}]_i) = \frac{\hbar^2}{2mR^2} (\Lambda_{zf}^2 - \Lambda_{zi}^2). \langle 11 \rangle$$

Here the energy unit $\hbar^2/2mR^2$ appears so frequently in the excitation energy expressions that it deserves a symbol of its own. We call it $\epsilon_0$, 

$$\epsilon_0 = \frac{\hbar^2}{2mR^2}. \langle 12 \rangle$$

The spin $I_z$ and excitation energy $E_x$ of a state in the $(n\text{pnh})^{\pi}_{\Lambda_{zi},\Lambda_{zf}}$ multiplet, in the simplest approximation with the neglect of spin-orbit and residual interactions, are just additive sums of the spin and energy increments from independent $i \rightarrow f$ particle-hole excitations, subject to the Pauli exclusion principle,

$$I_z = \sum_{if} \Delta_{Iz} |[\Lambda_{zi}\Omega_{zi}] \rightarrow [\Lambda_{zf}\Omega_{zf}] \rangle. \langle 13 \rangle$$

The parity $\pi$ of the state is

$$\pi = \prod_{if} \{(-1)^{\Lambda_{zf}-\Lambda_{zi}}\}. \langle 14 \rangle$$
Because there are many different \( \Omega_z \) states in a single-particle \( \Lambda_z \) shell, there are many different spin increments in an \((nph)_\Lambda \) multiplet. Consequently there are many different \( I_z \) spins in the multiplet states of the toroidal nucleus. All these states with different spins \( I = I_z \) within the multiplet have the same parity \( \pi \) and are degenerate with the excitation energy

\[
E_x = E_I - E_0 = \sum_{ij} \Delta_{ij} E_{\Lambda_i\Lambda_j} ([\Lambda_z \Omega_z], \rightarrow [\Lambda_z \Omega_z]), \tag{15}
\]

where \( E_0 \) is the energy of the toroidal ground state relative to the \( ^{12}\text{C} \) ground state. The inclusion of spin-orbit and residual interactions in a more refined calculation will split the degeneracy of the different \( I_z \) states in the multiplet.

IV. SPECTRUM OF THE \( ^{12}\text{C} \) NUCLEUS IN THE TOROIDAL CONFIGURATION

In this section, we shall show explicitly how the spectrum of the low-lying toroidal \( ^{12}\text{C} \) can be determined from the toroidal single-particle energies. We shall include the determination of higher \( ^{12}\text{C} \) toroidal states in the Appendix.

A. \((1p1h)_{\Lambda_i}^{\pi} \) Toroidal States

As one can see from Fig. 1, different \((1p1h)_{\Lambda_i}^{\pi} \) multiplets of exited \( ^{12}\text{C} \) toroidal states arise by promoting nucleons from \([0(\pm 1/2)], [1(\pm 1/2)], [1(\pm 3/2)]\) toroidal states in the \( \Lambda_z = 0 \) and 1 shells to occupy empty \([2(\pm 5/2)], [2(\pm 3/2)], [3(\pm 7/2)], [3(\pm 5/2)]\) toroidal states in the \( \Lambda_z = 2 \) and 3 shells. The knowledge of the particle and hole state quantum numbers gives the spin and parity of the excitation, and the excitation energy can be determined from the single-particle energy (9) or the energy increment (15). In particular, for a member \( I_z \) with \( I = I_z \) in the \((1p1h)_{\Lambda_i}^{\pi} \) multiplet, the excitation energy \( E_I \) of the member relative to the toroidal ground state of energy \( E_0 \) is given by

\[
E_I - E_0 = \Delta E_{\Lambda_i \Lambda_j}. \tag{16}
\]

All members of a multiplet have the same excitation energy in the present idealized approximation with the neglect of the spin-orbit and residual interactions.

The (one particle)-(one hole) \( \Delta E_{\Lambda_i \Lambda_j} \) for different shell-to-shell excitations are

\[
\begin{align*}
\Delta E_{02} &= \epsilon(|\Lambda_z| = 1) - \epsilon(|\Lambda_z| = 2) = \frac{3h^2}{2mR^2} = 3\epsilon_0, \\
\Delta E_{03} &= \epsilon(|\Lambda_z| = 0) - \epsilon(|\Lambda_z| = 2) = \frac{4h^2}{2mR^2} = 4\epsilon_0, \\
\Delta E_{13} &= \epsilon(|\Lambda_z| = 1) - \epsilon(|\Lambda_z| = 3) = \frac{8h^2}{2mR^2} = 8\epsilon_0, \\
\Delta E_{03} &= \epsilon(|\Lambda_z| = 0) - \epsilon(|\Lambda_z| = 3) = \frac{9h^2}{2mR^2} = 9\epsilon_0. \tag{17}
\end{align*}
\]

These excitation energies depend only on a single-parameter, the major radius \( R \), which can be determined by confronting the predicted theoretical spectrum with the experimental data in the next Section.

The parity of a member \( I_z \) member of the \((1p1h)_{\Lambda_i}^{\pi} \) multiplet, is

\[
\pi = (-1)^{\Lambda_i - \Lambda_j}. \tag{18}
\]

| particle-hole excitation | particle-hole configuration | \( I_z^\pi \) | \( E_I - E_0 \) |
|--------------------------|----------------------------|-------------|-------------|
| \((1p1h)_{12}^0\) \(\Lambda_z = 0 \) shell \( \rightarrow \Lambda_z = 2 \) shell | [1(-3/2)] \(-2(5/2)) \) | 1 | 3 |
| \((1p1h)_{13}^+\) \(\Lambda_z = 1 \) shell \( \rightarrow \Lambda_z = 3 \) shell | [3(5/2)] \(-2(5/2)) \) | 4 | 8 |
| \((1p1h)_{13}^-\) \(\Lambda_z = 1 \) shell \( \rightarrow \Lambda_z = 3 \) shell | [3(-3/2)] \(-2(5/2)) \) | 9 |

TABLE I. The \((1p1h)_{\Lambda_i}^{\pi} \) excitation in toroidal \( ^{12}\text{C} \) where \([\Lambda_i \Omega_z]^i \) represents a particle state and \([\Lambda_i \Omega_z]^{i-1} \) the hole state.

We show the spin quantum number \( I_z \), parity \( \pi \), and the excitation energy \( E_I - E_0 \) of different \((1p1h)_{\Lambda_i}^{\pi} \) multiplets of excited toroidal states in Table I. We have kept the sign of \( \Omega_z \) of the particle state positive. If we study the remaining case by reversing the sign of \( \Omega_z \) of the particle state, we obtain the same set of states as in Table I, except that the signs of \( I_z \) is reversed if it is non-zero, and it has a different particle-hole combination if \( I_z \) is zero. Thus, each of the total set of states in Table I is doubly degenerate. The double degeneracy occurs repeatedly in all toroidal states, and we shall make its double degeneracy implicit and shall consider only non-negative values of \( I_z \) with double degeneracy in what follows, unless explicitly specified otherwise. Furthermore, it should be kept in mind that Table I is applicable to neutron as well as to proton \((1p1h) \) excitations.

Table I and Fig. 2 show that the \((1p1h)^{12}_{12} \) for the excitation of a nucleon from the \( \Lambda_z = 1 \) shell
to the $\Lambda_z=2$ shell consists of a set of eight doubly-degenerate states, \{4$^-$, 2(3$^-$), 2(2$^-$), 2(1$^-$), 0$^-$\}, lying at $E_x=E_1-E_0=3\omega_0$. The (1p1h)$_{12}^-$ multiplet for the excitation of a nucleon from the $\Lambda_z=0$ shell to the $\Lambda_z=2$ shell consists of a set of four states, \{(3$^+$), 2(2$^+$), 1$^+$\}, lying at an excitation energy $E_x=4\omega_0$. The (1p1h)$_{03}^+$ multiplet for the excitation of a nucleon from the $\Lambda_z=1$ shell to the $\Lambda_z=3$ shell consists of a set of eight states of \{(5$^+$), 2(4$^+$), 2(3$^+$), 2(2$^+$), 1$^+$\} lying at $E_x=8\omega_0$. Finally, the (1p1h)$_{03}^-$ multiplet for the excitation of a nucleon from the $\Lambda_z=0$ shell to the $\Lambda_z=3$ shell consists of a set of four doubly-degenerate states, \{(4$^-$), 2(3$^-$), 2$^-$\}, lying at $E_x=9\omega_0$. The spectrum of these states are shown in Fig. 2.

The lowest lying toroidal states are the (1p1h)$_{12}^-$ and (1p1h)$_{03}^-$ multiplets lying above the ground toroidal $^{12}$C state. They should be prominently excited by stripping ($^3$He,$d$) reactions that add a proton to excite and combine with a $^{11}$B nucleus to a toroidal configuration.

B. \[(2p2h)_{12}^+\] at $E_x=6\omega_0$ for exciting two identical nucleons from $\Lambda_z=1$ shell to $\Lambda_z=2$ shell

We consider next the \[(2p2h)_{12}^+\] excitations of two identical particles (neutrons or protons) from the $\Lambda_z=1$ shell to the $\Lambda_z=2$ shell. Because of the Pauli exclusion principle, the two identical particle or holes cannot occupy the same $|\Lambda_z, \Omega_z\rangle$ state. Consequently, to get the (2 particle)-(2 hole) excitations involving identical particles, it is simplest to combine the angular momentum components of the two particles and two holes separately first, under the restriction of the Pauli principle, before combining them together. For this purpose, we list all combinations of states of two holes in Table II and two particles in Table III, under the restriction of the Pauli principle.

![FIG. 2. (Colour online.) (a) Experimental excitation energy $E$ of $^{12}$C excited states relative to the $^{12}$C ground state [13], with the axis of the excitation energy $E$ on the left. (b) The theoretical spectrum of the toroidal states in different multiplets, with the axis of the excitation energy $E_x = E - E_0$, relative to the energy $E_0$ of the toroidal ground state, given on the right. Comparison between the experimental and theoretical spectra is made by identifying the Hoyle state as the toroidal ground state and the lowest lying 3$^-$, 1$^-$, 2$^-$, 4$^-$ as members of the toroidal (1p1h)$_{12}^-$ multiplet, leading to $\omega_0 = 1.25$ MeV (see text). The spins and parities of members of the multiplets are presented in the figure and given in Table VI.](image)

| Table II. Combination of two holes states in $|1(\pm3/2)\rangle$ and $|1(\pm1/2)\rangle$ states in the $\Lambda_z=1$ shell under the restriction of the Pauli principle |
|---|
| Two hole configuration | $I_x$ |
| $|\{[1(3/2)][1(1/2)]\}$ | 2$^+$ |
| $|\{[1(3/2)][1(-1/2)]\}$ | 2$^-$ |
| $|\{[1(1/2)][1(-1/2)]\}$ | 0$^+$ |
| $|\{[1(-3/2)][1(3/2)]\}$ | 0$^-$ |
| $|\{[1(-3/2)][1(1/2)]\}$ | 2$^-$ |
| $|\{[1(-3/2)][1(-1/2)]\}$ | 2$^+$ |

| Table III. Combination of two particles states in $|2 \pm 5/2\rangle$ and $|2 \pm 3/2\rangle$ orbitals in the $\Lambda_z=2$ shell under the restriction of the Pauli principle |
|---|
| Two particle configuration | $I_x$ |
| $|\{[2(5/2)][2(3/2)]\}$ | 4$^+$ |
| $|\{[2(5/2)][2(3/2)]\}$ | 1$^+$ |
| $|\{[2(3/2)][2(3/2)]\}$ | 0$^+$ |
| $|\{[2(5/2)][2(5/2)]\}$ | (-1)$^+$ |
| $|\{[2(-5/2)][2(-3/2)]\}$ | (-4)$^+$ |

Now combine the two hole states with all two particle states, we get the angular momentum component $I_x$ and parity $\pi$ as listed in Table IV. As one observes in Table IV, the \[(2p2h)_{12}^+\] multiplet consists of a set of 18 doubly-degenerate positive parity states, \{6$^+$, 5$^+$, 2(4$^+$), 2(3$^+$), 4(2$^+$), 5(1$^+$), 3(0$^+$)\}, lying at $E_x=6\omega_0$ as shown in Fig. 2(b). It is applicable to a pair of neutrons or protons.
C. [(1p1h)_{12p}-(1p-1h)_{12p}]^{\pi} toroidal multiplet involving one neutron and one proton

In the last subsection, we have considered the (2p2h) excitations involving two identical nucleons. The case of the (2p2h) excitations involving two different types of nucleons from the \( \Lambda_z = 1 \) shell to the \( \Lambda_z = 2 \) shell differ from the previous case, because the pairs of particles or holes do not need to be restricted by the Pauli principle. We can consider such (2p-2h) excitations by combining a (1p1h) neutron excitation with an independent (1p1h) proton excitation. Such a \([(1p1h)_{12p}-(1p-1h)_{12p}]^{\pi}\) multiplet has the excitation energy \( E_x = 2\Delta E_{12} = 6\epsilon_0 \) and positive parity. Their spin quantum numbers \( I_z \) and parities are given in Table V.

There are altogether \( \{8^+,4(7^+),8(6^+),12(5^+),15(4^+),16(3^+),16(2^+),15(1^+),15(0^+),1(-4^+),4(-3^+)\},9(-2^+),12(-1)^+\} \) for a total of 128 states. If we reverse the sign of the \( (1p1h)_{12p} \) state, we get the same set of states but with the signs of \( I_z \) is reversed if it is non-zero, and it has a different particle-hole combination if \( I_z \) is zero. The \( (I_z)^+ \) and the \( (-I_z)^+ \) states in these two sets can be grouped together. In this new grouping, the combined set contains 128 doubly-degenerate states of the set \( \{8^+,4(7^+),8(6^+),12(5^+),16(4^+),20(3^+),25(2^+),27(1^+),15(0^+)\} \) at the excitation energy \( E_x = 2\Delta E_{12} = 6\epsilon_0 \).

We have thus obtained the spectrum of the lowest-lying multiplets of states. The spectrum of the higher states are included in the Appendix. We can summarize the theoretical energy spectrum of the toroidal \(^{12}\text{C} \) nucleus in Table VI and in Fig. 2.
TABLE VI. The theoretical spectrum of $^{12}$C in a toroidal configuration. Here, $E_e = E_f - E_0$, $E_f$ is the energy of a state in the multiplet, $E_0$ is the toroidal ground state energy, $e_0=1.25$ MeV by matching with the experimental spectrum, and $N$ is the number of doubly-degenerate states in the multiplet.

| Ground State & Multiplets | $E_e$ (MeV) | $I_{1}=I$ states | $N$ |
|-------------------------|------------|-----------------|-----|
| Toroidal Ground State   |            |                 |     |
| [(1p1h)$_{12}$]$^{-}$   | 0          | 0$^+$           | 1   |
| [(1p1h)$_{02}$]$^{+}$   | 3          | 10$^+$, 2(1$^-$), 2(2$^-$), 2(3$^-$), 4$^+$ | 8   |
| [(2p2h)$_{12}$]$^{+}$   | 4          | 11$^+$, 2(2$^-$), 3$^+$ | 4   |
| [(1p1h)$_{12}$,(1p1h)$_{22}$]$^{+}$ | 6      | 15.16, 3(0$^+$), 5(1$^+$), 4(2$^+$), 2(3$^+$), 2(4$^+$), 5$^+$, 6$^+$ | 128 |
| [(1p1h)$_{12}$,(1p1h)$_{02}$]$^{-}$ | 7      | 15.16, 15(0$^-$), 27(1$^-$), 25(2$^-$), 20(3$^-$), 16(4$^-$), 12(5$^-$), 8(6$^-$), 4(7$^-$), 8$^+$ | 24 |
| [(1p1h)$_{12}$,(1p1h)$_{02}$]$^{-}$ | 7      | 16.42, 15(0$^-$), 15(1$^-$), 12(2$^-$), 9(3$^-$), 4(8$^-$), 7(5$^-$), 4(6$^-$), 7$^+$ | 64 |
| [(1p1h)$_{13}$]$^{+}$   | 8          | 17.67, 1$^+$, 2(2$^+$), 2(3$^+$), 4(4$^+$), 5$^+$ | 8   |
| [(2p2h)$_{02}$]$^{+}$   | 8          | 17.67, 1$^+$, 1$^+$, 4$^+$ | 3   |
| [(1p1h)$_{02}$,(1p1h)$_{22}$]$^{+}$ | 8      | 18.92, 1$^+$, 6(0$^+$), 8(1$^+$), 3(2$^+$), 4(3$^+$), 6(4$^+$), 4(5$^+$), 6$^+$ | 32 |
| [(1p1h)$_{03}$]$^{-}$   | 9          | 18.92, 2$^+$, 2(2$^+$), 4$^+$ | 8   |
| [(3p3h)$_{12}$]$^{-}$   | 9          | 18.92, 30(0$^-$), 59(1$^-$), 55(2$^-$), 47(3$^-$), 36(4$^-$), 25(5$^-$), 16(6$^-$), 10(7$^-$), 6(8$^-$), 3(9$^-$), 10$^+$ | 288 |
| [(2p2h)$_{22}$,(1p1h)$_{12}$]$^{-}$ | 9      | (same as above) | 288 |
| [(2p2h)$_{12}$,(1p1h)$_{02}$]$^{+}$ | 10      | 3(0$^-$), 6(1$^+$), 6(2$^+$), 5(3$^-$), 3(4$^-$), 5$^+$ | 24 |
| [(2p2h)$_{12}$,(1p1h)$_{02}$]$^{+}$ | 10      | 15(0$^-$), 30(1$^-$), 29(2$^-$), 25(3$^-$), 18(4$^-$), 11(5$^-$), 7(6$^-$), 5(7$^-$), 3(8$^-$), 9$^+$ | 144 |
| [(2p2h)$_{12}$,(1p1h)$_{02}$]$^{+}$ | 10      | 15(0$^-$), 30(1$^-$), 29(2$^-$), 25(3$^-$), 18(4$^-$), 11(5$^-$), 7(6$^-$), 5(7$^-$), 3(8$^-$), 9$^+$ | 144 |
| [(2p2h)$_{02}$,(1p1h)$_{12}$]$^{-}$ | 11      | 4$^+$, 2(2$^+$), 2(2$^-$), 2(1$^-$), 0$^+$ | 8   |
| [(4p4h)$_{12}$]$^{+}$   | 12         | 22.65$^+$       | 1   |
| [(2p2h)$_{12}$,(2p2h)$_{12}$]$^{+}$ | 12      | 69(0$^-$), 130(1$^+$), 117(2$^+$), 96(3$^-$), 78(4$^-$), 58(5$^+$), 42(6$^-$), 26(7$^-$), 16(8$^-$), 8(9$^-$), 5(10$^-$), 2(11$^-$), 12$^+$ | 628 |
| [(3p3h)$_{12}$,(1p1h)$_{02}$]$^{-}$ | 13      | 23.90, 0$^-$, 2(1$^-$), 0$^+$ | 4   |
| [(3p3h)$_{12}$,(1p1h)$_{02}$]$^{-}$ | 13      | 23.90, 0$^-$, 2(1$^-$), 4(5$^-$), 8(6$^-$), 15(7$^-$), 9(8$^-$), 3(9$^-$), 10(10$^-$), 2(11$^-$), 12$^+$ | 64 |
| [(3p3h)$_{12}$,(1p1h)$_{02}$]$^{-}$ | 13      | (same as above) | 64 |
| [(2p2h)$_{12}$,(2p2h)$_{12}$]$^{+}$ | 14      | 25.15, 0$^+$, 1$^+$, 2$^+$ | 3   |
| [(2p2h)$_{12}$,(2p2h)$_{12}$]$^{+}$ | 14      | 25.15, 13(0$^+$), 23(1$^+$), 20(2$^+$), 15(3$^-$), 13(4$^-$), 10(5$^-$), 7(6$^-$), 3(7$^-$), 2(8$^-$), 9$^+$, 10$^+$ | 108 |

For simplicity, we have represented the single-particle potential of a toroidal nucleus by a simple harmonic oscillator potential, Eq. (1). Such a representation is adequate for the lowest few toroidal shells. However, in a more realistic case, the single-particle potential should follow the toroidal density and should be a diffused potential with a finite depth. The higher toroidal shells are expected to be unbound. As a consequence, there will be a termination of the (npnh) toroidal excitations, indicated by the absence of toroidal particle-hole excitations at high energies and a rapid decrease of the density of toroidal particle-hole excitation states. It will be of interest to search for the excitation energy maximum in $^{12}$C at which the toroidal particle-hole excitation states cease to appear.

V. EXPLORATORY COMPARISON OF THEORETICAL TOROIDAL $^{12}$C STATES WITH EXPERIMENTAL SPECTRUM

In the last few sections, we show that the $^{12}$C nucleus in a toroidal configuration possesses a distinct spectrum of states with a well-defined pattern of spins, parities, and excitation energies. They arise from particle-hole excitations from one parent toroidal shell to another toroidal shell and can be conveniently called “toroidal states”. The spectrum constitutes the signature of the $^{12}$C nucleus in the toroidal configuration. Using such a signature, we would like to explore whether there may be toroidal states in $^{12}$C with the $0^+$ Hoyle state at 7.654 MeV as the ground state of the toroidal band.

In our exploratory survey, we expect that if it coexists with the oblate spheroidal ground state, the toroidal ground state of $^{12}$C and the low-lying toroidal multiplets will show up as $^{12}$C resonances with a large probability to breakup into 3 alpha particles, because the triangular clusters of three alpha particles has a large overlap with the configuration of a toroidal nucleus. Accordingly, we search for good candidate states in $^{10}$B$(^{3}$He,$p)^{12}$C$^*$→3α and $^{11}$B$(^{3}$He,d)$^{12}$C$^*$→3α reactions [14–16] as shown in Figs. 3, 4, and 5. We shall discuss the results of our search in the following subsections.

A. Search for toroidal states with the $^{11}$B$(^{3}$He,d)$^{12}$C$^*$→3α reaction

In the experiments of Kirsebom et al. [14, 15], excited intermediate $^{12}$C$^*$ states were produced by the bombardment of the $^{3}$He projectile at 8.5 MeV onto a $^{11}$B target in the reaction $^{3}$He + $^{11}$B → $d + ^{12}$C$, with the subsequent breakup of the $^{12}$C$^*$ intermediate state into three alpha particles, $^{12}$C$^*$ → 3α. The complete kinematic data of the final deuteron $d$ and the 3 alpha particles have been recorded with detectors of fine resolutions and segmentation. The complete kinematics data allow the determination of the energy of the intermediate $^{12}$C$^*$ state, the history of its subsequent decay, whether through the $0^+$ or the $2^+$ state of $^8$Be, and the energy distribution Dalitz plot of the 3 alpha particles. From these pieces of information, the spins and parities of the prominent $^{12}$C resonances can be inferred. The complete kinematic data allows the removal of most of the random coincidences and decay channels that do not involve the production
of intermediate excited states of $^{12}$C*. The measurement is essentially free of background [66]. The spectrum as shown in Figs. 3(a) and 4(a) from Kirsebom et al. [14, 15] can be considered to be the spectrum for the produced intermediate excited $^{12}$C* states, which include both the identified $^{12}$C resonances as well as unresolved and unidentified excited $^{12}$C* states, as we shall discuss below.

It is instructive to review how (i) the $^{11}$B($^3$He,d)$^{12}$C* reaction mechanism, (ii) the decay of $^{12}$C* → 3α, and (iii) the knowledge of the toroidal nuclear structure information help guide us in the search for toroidal $^{12}$C particle-hole excitation states. The ($^3$He,d) process of [14, 15] strips a proton from the incident projectile nucleus, $^3$He, turns it into a deuterium, and deposits the stripped proton onto the target $^{11}$B nucleus, which has 5 protons and and 6 neutrons. With this addition of the stripped proton, the system with 6 protons and 6 neutrons completes a doubly-closed shell for a toroidal shape. Previous theoretical work indicates that the extrapolation from heavier toroidal α-conjugate nuclei points to a possible toroidal state with doubly-closed toroidal shells in $^{12}$C in the low excitation energy region [48, 49]. If such a state exists, then by the selection of the breakup into three alpha particles, we search for the energy location for the $^{12}$C toroidal ground state for processes in which the stripped proton can excite and polarize the nucleons in the $^{11}$B system to re-configure themselves into a doubly closed shell toroidal nucleus with only a single unoccupied state in the proton $\Lambda_z = 1$ shell, and the stripped proton can fill up the unoccupied proton single-particle state to lead to the (0p0h) toroidal $^{12}$C ground state.

With the selection of decaying into 3 alpha particles, we can look for events at another energy for the toroidal (1p1h) $^{12}$C state, in which the stripped proton excite nucleons in the $^{11}$B system to re-configure themselves into a toroidal configuration with a hole in the $\Lambda_z=1$ shell, while the stripped proton goes on to an unoccupied proton state in the $\Lambda_z=2$ shell, leading to the (1p1h) state of toroidal $^{12}$C. We expect that at the appropriate energies, the $^{11}$B($^3$He,d)$^{12}$C* → 3α reaction should favorably populate (0p0h) and (1p1h) states of the toroidal $^{12}$C nucleus. As a consequence, the toroidal (0p0h) 0+ ground state and the lowest lying [(1p1h)$_{12}^-$] multiplet of \{0+, 2(1+), 2(2+), 2(3+), 4+\}, and the [(1p1h)$_{02}^+$] multiplets of \{1+, 2(2+), 3+\} in Table VI should be favorably excited in the $^{11}$B($^3$He,d)$^{12}$C* → 3α reaction.

Figure 3(a) gives the experimental excitation function for the $^{11}$B($^3$He,d)$^{12}$C* → 3α reaction in the range $7 \leq E \leq 13$ MeV in logarithmic scale, and Fig. 4(a) gives the same excitation function in the range $9 \leq E \leq 15$ MeV.

![Figure 3](image_url)

**FIG. 3.** (Colour online.) (a) Experimental excitation function for the reaction $^{11}$B($^3$He,d)$^{12}$C* → 3α in logarithmic scale as a function of the $^{12}$C* excitation energy $E$ in the range of $7 \leq E \leq 13$ MeV obtained by Kirsebom et al. [14], with the axis of the excitation energy on the left. (b) Experimental level scheme from the compilation of [13]. (c) Theoretical level scheme of toroidal states with the axis of $E_x = (E - E_0)/\epsilon_0$ on the right. Subsequent matching between theoretical and experimental excitation energies leads to $\epsilon_0=1.25$ MeV.

![Figure 4](image_url)

**FIG. 4.** (Colour online.) Experimental excitation function the reaction $^{11}$B($^3$He,d)$^{12}$C* → 3α in linear scale as a function of the $^{12}$C* excitation energy $E$ in the range of $9 \leq E \leq 15$ MeV obtained by Kirsebom et al. [14], with the axis of the excitation energy on the left. (b) Experimental level scheme from the compilation of [13]. (c) Theoretical toroidal states with the axis of $E_x/\epsilon_0 = (E - E_0)/\epsilon_0$ on the right.
in linear scale. Figs. 3(b) and 4(b) give the energy levels from the experimentally of identified $^{12}$C excited states from the compilation of [13]. The excitation energy $E$ to the energy of $^{12}$C ground state is given on the left axis.

The experimental excitation functions in Figs. 3(a) and 4(a) indicate that the $0^+$ Hoyle state at $E=7.654$ MeV is prominently exited, as are the $3^-$ state at 9.654 MeV, the $1^-$ state at 10.847 MeV, the $2^-$ state at 11.837 MeV, the $4^-$ state at 13.314 MeV, and the $1^+$ state at 12.71 MeV.

The theoretical toroidal states energy levels are presented in Figs. 3(c) and 4(c). In the large radius approximation with the neglect of the small spin-orbit and residual interactions, the gross pattern of the toroidal states depends only on a single parameter, the major radius $R$. The simplified theoretical spectrum indicates that the lowest excited states are the $[(1p1h)_{12}]^-$ multiplet of $\{4^-,2(3^-),2(2^-),2(1^-),0^-\}$ at $E_x = 3\epsilon_0 = 3h^2/2mR^2$, and another $[(1p1h)_{02}]^+$ multiplet of $\{1^+,2(2^+),3^+\}$ at $E_x = 4\epsilon_0$, as given in Table VI and Fig. 2, 3(c), and 4(c). The degeneracy within the multiplet arises from the neglect of spin-orbit splitting and residual interactions.

By comparison of the experimental and the theoretical spectrum, we find that the lowest theoretical multiplet in toroidal $^{12}$C contains states with spins and parities that coincide with those of the experimental states. It is therefore reasonable to identify the $0^+$ Hoyle state at 7.654 MeV to be the $(0p0h)$ ground state of the toroidal configuration and the set of lowest $3^- (9.654$ MeV), $1^- (10.847$ MeV), $2^- (11.837$ MeV), $4^- (13.314$ MeV)) states to be members of the $(1p1h)_{12}^-$ multiplet.

We get additional support to identify these four states as members of a multiplet from the strengths of the excitation function. Upon treating the stripping reaction $^{11}$B($^3$He, d)$^{12}$C$^*$ leading to toroidal particle-hole states in $^{12}$C$^*$ as a step process, the probability amplitude is the product of the amplitude (i) for the stripped proton to excite and to combine with the $^{11}$B system to come to the toroidal doubly-close shell of $^{12}$C$^*$, and the amplitude (ii) for the particle-hole excitation from the close shell to the toroidal (one particle)-(one hole) excitation. The amplitude (i) is independent of the members of the multiplet, and the latter amplitude (ii) depends only on the degree of fullness of the closed occupied $\Lambda_z = 1$ (or 0) shell and the degree of emptiness of the unoccupied $\Lambda_z = 2$ shell, which are the same for all members of the shell-to-shell particle-hole multiplet. Consequently, all members of the $(1p1h)_{12}^-$ multiplet should be approximately equally populated. An examination of the widths and heights of the identified lowest $\{3^-,1^-,2^-,4^\}$ states indicate that these four are approximately equally populated, supporting their identification as members of the $(1p1h)_{12}^-$ multiplet.

Following such an identification, we set the average excitation energy of the four states $\{3^-,1^-,2^-,4^\}$ at 11.41 MeV to be the excitation energy of the $(1p1h)_{12}^-$ multiplet. Such a matching leads to the theoretical energy scale,

$$\epsilon_0 = 1.25 \text{ MeV.} \quad (19)$$

By the definition of $\epsilon_0$ as $\hbar^2/2mR^2$, we obtain the corresponding equilibrium major radius,

$$R = 4.06 \text{ fm.} \quad (20)$$

The knowledge of $\epsilon_0$ allows the determination of the theoretical toroidal spectrum of $^{12}$C as tabulated in Table VI and shown in Figs. 2-5. In particular, we find that the $1^+$ state at $E_z = 12.710$ MeV approximately matches the $1^+$ state in the multiplet of $(1p1h)_{12}^+$. The $1^+$ state at $E_z = 15.111$ MeV approximately matches the $1^+$ state in the multiplet of $(2p2h)_{12}^+$. There also appears to be a hint of an alternating change of parity of the states, similar to the alternating parity of the theoretical levels. However, whether such coincidental comparison can remain in the presence of residual interactions remains to be investigated.

The excited states of $^{12}$C contains a $4^+$ state at 13.3 MeV [70] and another $4^+$ state at 14.079 MeV [13]. The sequence of the Hoyle state $0^+ (7.654$ MeV), $2^+ (9.870$ MeV), and either one of the two $4^+$ states follow approximately the relation between the angular momentum and the rotational energy of a rotor. From the analysis of the interacting cluster model, it is found that the sequence of the $0^+_1$ (ground state), $2^+_1 (4.33$ MeV), and $4^+_1 (14.079$ MeV) form the rotational states of the ground band, whereas the sequence of the Hoyle state $0^+_2 (7.654$ MeV), $2^+_2 (9.870$ MeV), and $4^+_2 (13.3$ MeV) state [70] form a rotational band with a different moment of inertia [71]. The presence of a rotational band built on the Hoyle state is consistent with the identification of the Hoyle state as the head of the toroidal multiplet suggested by particle-hole excitation data from the stripping reaction we have presented here. In contrast to the Hoyle state and the $(1p1h)$ states, the stripping reaction $(^3$He,d) does not prominently excite the $2^+ (9.870$ MeV), and $4^+ (13.3$ MeV) states of the rotational band of the Hoyle state because they are probably collective states in nature.

While the above comparison indicates that the Hoyle state and many of its higher excited states may be tentatively attributed to the $^{12}$C nucleus in a toroidal configuration, further confirmation and careful examination are necessary. There appear to be many more theoretical states compared to the experimentally identified states. The large number of theoretical states means that further experimental works are needed. The experimental search for these states may be difficult as they may be so broad that they may be hidden under the resonances of the other observed and identified states. Theoretically, refinement of the theoretical work using toroidal basis to include spin-orbit and residual interactions will determine more precisely the locations of these states so to ascertain better whether such a toroidal description has approximate validity.

With regard to the question of uncovering the remaining members of the multiplets, it is interesting to note
the remarkable recent advances in employing complete kinematics to study the breakup of $^{12}$C resonances into three $\alpha$ particles in Ref. [14]. The measurement of [14] is essentially free of background [66]. Interesting enough, as noted in [14], the strength of the excitation function away from the clean peaks of these states in Fig. 3(a) indicate that additional overlapping resonances of $^{12}$C may have been produced in the process. They constitute non-vanishing strengths in the Dalitz plot and may represent the un-identified members of the toroidal multiplet. Thus, the search and identify these missing members of the [(1p1h)$_{12}$ and (1p1h)$_{02}$ multiplets from the underlying strengths of the excitation function in Fig. 3 may provide a test of the toroidal description of the $^{12}$C states. In this regard, the technique of identifying resonances by the patterns in the Dalitz plot [69] may find its useful application in the determination of the spin and parity of the states in the underlying structure.

We can seek additional support for the toroidal multiplet description of the $^{12}$C states. We focus our attention on the excitation function in the excitation energy range $9 \leq E \leq 14.5$ MeV in Fig. 4(a), where the $1^-$, $2^-$, and $4^-$ resonances are well separated and identified. However, in the same excitation energy range in Fig. 4(a), there appears an underlying broad structure beneath the peaks of the identified resonances. We take the view that by nature of the background-free measurement with full kinematics in the determination of the spectrum of Fig. 4(a), the board structure likely originates from the remaining members of the toroidal multiplets that are also produced with each state approximately in equal strength in the stripping $^{11}$B($^3$He,d)$^{12}$C$^*$→3$\alpha$ reaction.

It is illuminating to estimate the number of the produced but un-identified $^{12}$C$^*$ resonances in the underlying broad structure in the range of excitation energies in Fig. 4(a) by using the excitation function strengths of the resolved resonances in the range as a yard stick. In the data of Fig. 4(a), the $1^-(10.847$ MeV), $2^-(11.837$ MeV), and $4^-(13.314$ MeV) resonances are members of the multiplet arising from single-particle excitations populating a one-particle-one-hole state in the $\Lambda_z=2$ shell. The strengths of each of the three peaks defines roughly a definite measure which we call “single-particle unit”, with approximately the same area for each of the three peaks, and each single-particle unit leads to the production of one $^{12}$C$^*$ state. On that basis of using that (average) area as a yardstick, we find that in the energy range of Fig. 4(a) up to the instrumental cut-off excitation energy of $\sim 14.5$ MeV, there are approximately 3 to 5 units of single-particle strength for natural parity states and 2 to 4 units for the un-natural parity states in the underlying overlapping resonances, with a considerable degrees of uncertainty in these numbers due to the uncertainty in separating out the resolved peaks from the underlying broad structure. It is gratifying that the numbers of remaining un-identified members in the multiplet fall within the range of numbers of states estimated to be present in the underlying broad structure in Fig. 4(a).

By this comparison, it appears that the total number of identified and unidentified resonances produced in the $^{11}$B($^3$He,d)$^{12}$C$^*$→3$\alpha$ reaction matches approximately the total number of states of the (1p1h)$_{12}$ and (1p1h)$_{02}$ multiplets, providing an additional support for the tentative toroidal multiplet description of these states in $^{12}$C.

B. Toroidal states production in the $^{10}$B($^3$He,p)$^{12}$C$^*$ reaction

![FIG. 5. (Colour online.) Experimental excitation functions of the reaction $^{10}$B($^3$He,p)$^{12}$C$^*$→3$\alpha$ as a function of the proton energy $E_p$ from Alcorta et al.\[16\] The decay of $^{12}$C$^*$ goes through the $0^+$ state of $^8$Be in (a) and through the $2^+$ state of $^8$Be in (b). The excitation energies, spins, and parities of identified resonances of $^{12}$C$^*$ are indicated. Please note the different count scales on the y-axes in Figs. (a) and (b).]

In the experiment to use the $^{10}$B($^3$He,p)$^{12}$C→3$\alpha$ reaction to study the excited states of $^{12}$C by Alcorta et al.\[16\], the $^3$He projectile collides with a $^{10}$B target nucleus at a beam energy of 4.9 MeV to produce a proton and 3 alpha particles. The complete kinematics data of all four final particles were collected using detectors of fine resolution and segmentation. Again, the full knowledge of the complete kinematics provides information on the energy of the intermediate $^{12}$C$^*$ state, the history of its decay through $^8$B, and the three alpha particle Dalitz plots to facilitate the assignment of spins and parities. Contributions from direct reactions leading to the production of intermediate states other than $^{12}$C$^*$ have been eliminated as much as possible. The excitation function of the 3$\alpha$ spectrum can be considered to be essentially free of background, pending future removal of the small $^3$He+$^{10}$B→$^8$Be+$^5$Li→$^8$p+$^5$Li and $^3$He+$^{10}$B→$^8$p+$^9$B→$^8$α+$^5$Li contributions [66].

The ($^3$He,p) reaction strips a neutron and a proton from the $^3$He projectile and deposits them onto the $^{10}$B target. The selection of the final state of three alpha particles and a scattered proton allow one to judiciously examine those events in which the stripped pair of nucleons interact with the $^{10}$B system to re-arrange themselves.
into a toroidal configuration with a subsequent decay into three alpha particles. At the energy appropriate for the production of the toroidal ground state, the stripped neutron and proton can excite the nucleons in $^{10}$B to re-arranged themselves into a toroidal configuration leaving two holes at the top of the fermi surface. Filling the two holes with the stripped nucleons in the closed toroidal shell will lead to the production of the $(0p0h)$ toroidal ground state. Filling one hole in the $\Lambda_z=1$ shell and another unoccupied state by the stripped nucleons in the $\Lambda_z=2$ shell at another appropriate excitation energy will lead to a $(1p1h)$ toroidal state in $^{12}$C$^\ast$, and filling up two empty states in the $\Lambda_z=2$ shell will lead to a $(2p2h)$ toroidal state at a still higher excitation energy. The $^{10}$B($^3$He,$p$)3$\alpha$ reaction should favorably populate the $(0p0h)$, $(1p1h)$, and $(2p2h)$ toroidal excitations of $^{12}$C$^\ast$ at different excitation energies.

Experiment data $^{10}$B($^3$He,$p$)$^{12}$C→3$\alpha$ reaction [16] as shown in Fig. 5 in the three alpha breakup indicate that the Hoyle state, the lowest $\{1^-,2^-,3^-,4^-\}$ and $1^+$ states are prominently produced, leading additional support to the description that the Hoyle state may be tentatively identified as the toroidal ground state, the $\{1^-,2^-,3^-,4^-\}$ as members of the toroidal $(1p1h)_{12}$ multiplet, and the $1^+$ $(12.71$ MeV) state to be a member of the $(1p1h)_{02}$ multiplet. The possibility of the $(2p2h)$ excitations in the $^{10}$B($^3$He,$p$)$^{12}$C$^\ast$ reaction lead to an enhanced excitation function at higher excitation energies. It is interesting to note that the experimental $4^+$ $(14.08$ MeV) state and the $2^+$ $(16.11$ MeV) falls in the vicinity of the theoretical $(2p2h)_{12}$ multiplet whose members include a $2^+$ and a $4^+$ state. Whether these two states can be identified as two members of the $(2p2h)_{12}$ multiplet will require further theoretical and experimental investigations.

In Fig. 5 the produced resolved states appear as sharp peaks on top of an underlying broad structure. If we employ the earlier method using the areas of the excitation function covered by known resonances as a ‘single-particle’ yard stick to estimate the number of states involved in the underlying broad structure, we would come up with the result that the number of states comprising the underlying broad structure in Fig. 5 is an order of magnitude greater than number of resolved and identified resonances. Thus in addition to the production of the resolved and identified states that have been mentioned as possible members of the toroidal multiplets, the underlying structure represents also possible produced members of the toroidal multiplets in the $^{10}$B($^3$He,$p$)$^{12}$C$^\ast$→3$\alpha$ reaction of Ref. [16]. They have the same property of decaying into three alpha particles, they are in the same energy region as the resolved members of the multiplet, and all these state in the multiplet may share similar shell-to-shell excitation probability amplitudes.

The experimental excitation function in Fig. 5 provides valuable information on the variation of the the number of excited $^{12}$C$^\ast$ states as a function of excitation energy. The excitation functions rise up as a function of increasing excitation energy $E$, starting at around $E \sim 12-14$ MeV, reaching a maximum at around $E \sim 18.5$ MeV, and decrease rapidly at around $E \sim 22$ MeV. Theoretically in the toroidal model of $^{12}$C$^\ast$, there is an increase in numbers of toroidal states with increasing excitation energy as shown in Table VI and plotted in Fig. 6. It rises up around $E \sim 14.5$ MeV and has two peaks structure with fine details at 19 and 23 MeV. It decreases rapidly for $E_x$ greater than $\sim 22$ MeV. The inclusion of the spin-orbit and residual interactions as well as the large width of these toroidal states will smear the theoretical peaks, but the general structure of a peak around the excitation region of $E \sim 18$ MeV mimicks qualitatively the shape of the experimental excitation function in Fig. 5. It will be of interest to investigate whether such a correlation with the shape of the experimental excitation function may indeed corresponds to the theoretical variation of the density of toroidal states. It will also be of interest to find out whether there is also the termination of the toroidal particle-hole excitation multiplets at high excitation energies, when the unoccupied single-particle states at the higher shells become unbound.

In addition to the experimental results discussed in this section, many recent experiments have been carried out to study the Hoyle state and the higher excited states. Whelen et al. use the $^{12}$C($^3$He,$^3$He)$^{12}$C$^\ast$→3$\alpha$ reaction [7] at a beam energy of $E(^3$He$)=46$ MeV. Barbui et al.[17] studied 3 alpha particles in coincidence in forward angles using the inverse kinematic reaction $^{20}$Ne + α → 3α + X with $^{20}$Ne on a gaseous $^4$He target at a maximum beam energy of 12 MeV/nucleon. These measurements uncover many relatively narrow resonances in $^{12}$C$^\ast$ at high excitation energies in conjunction with the Hoyle state at
7.654 MeV. Many of the resonances at very high energies have not been completely identified and analyzed to gain information on their spins and the parities. Further future analysis of the quantum numbers and the nature of these many resonances will enhance our knowledge of the excited states of the $^{12}$C nucleus and will provide a test of different models.

VI. TOROIDAL $^{12}$C IN TOROIDAL CONSTRAINT DYNAMICS

Our exploratory investigation points to the need to devise tools within the mean-field theory that can constrain the $^{12}$C nucleus so to possess local toroidal energy equilibrium configurations. As the ground state of $^{12}$C has the topology of an oblate spheroid, we should be prepared to examine the toroidal configuration as excited diabatic states of the system. Success for formulating such tools will be useful not only for the $^{12}$C nucleus but also for the many diabatic states that may be associated with the large region of toroidal high-spin isomers in $\alpha$-conjugate nuclei up to $A \sim 70$ [56–58]. It will also be useful for the investigation of diabatic toroidal states in the intermediate and superheavy mass region for which some recent progress has been made [60, 67, 68].

We consider a Hamiltonian $H_0$ which contains already many constraints that have been imposed on the system. We wish to impose an additional toroidal constraint to hold the nucleus in a toroidal shape. For this purpose, we introduce a radial moment $d_0^2$ to characterize the radial property of the density distribution that can constrain the density into a toroidal shape,

$$d_0^2 = \left\{ \int \rho(r) \rho(r - \langle \hat{\rho} \rangle)^2 dr \right\},$$

where $\langle \hat{\rho} \rangle$ is the expectation value of the coordinate $\rho = \sqrt{x^2 + y^2}$,

$$\langle \hat{\rho} \rangle = \int \rho(r) \rho \psi_i(r),$$

and $D(\rho, z)$ is the nuclear density which can be determined self-consistently from the set of the wave functions $\{\psi_i\}$ of occupied states

$$D(r) = \sum_{i=1}^{A} \psi_i^*(r) \psi_i(r).$$

The constraint of a fixed toroidal radial moment $d_0^2$, can be imposed with the additional Lagrange multiplier $\lambda$,

$$H = H_0 + \lambda \left\{ \int \rho(r) \rho(r - \langle \hat{\rho} \rangle)^2 - d_0^2 \right\}.$$  (24)

Upon a minimization of the Hamiltonian $H$ with respect to a variation in $\psi_i^*$, we have

$$\frac{\delta H}{\delta \psi_i^*} = \frac{\delta H_0}{\delta \psi_i^*} + \lambda \int \rho(r - \langle \hat{\rho} \rangle)^2 \psi_i$$

+(terms involving $\frac{\delta (\hat{\rho})}{\delta \psi_i^*}$)  (25)

To the extend that the change of the bulk $\langle \hat{\rho} \rangle$ with respect to the change of individual single-particle states $\psi_i^*$ is small when the occupation numbers of the states have settled down and do not change, the last term of the above variation can be neglected. Writing $H_0$ in terms of the single-particle Hamiltonian $h_0$ as

$$H_0 = \int dr \sum_{i=1}^{A} \psi_i^*(r) h_0 \psi(r),$$

and we have

$$\frac{\delta H_0}{\delta \psi_i^*} = \int dr \ h_0 \psi_i.$$  (27)

Under the toroidal constraint of Eq. (24), the minimization of $H$ with respect to $\psi_i^*$ leads to the single-particle Hamiltonian under the toroidal constraint

$$h' = h_0 + \lambda (\rho - \langle \hat{\rho} \rangle)^2.$$  (28)

In the case with a large $R/d$ ratio, $\langle \hat{\rho} \rangle \sim R$. Thus, we observe that the last term of Eq. (28) is approximately in the same form as the toroidal potential of Eq. (1), with the Lagrange multiplier $\lambda$ playing the role of the harmonic oscillator frequency $\omega_\perp$ multiplied by the nucleon mass $m$. We obtain the result that with the addition of the toroidal constraint Eq. (21), the variation principle lead to a single-particle toroidal potential that is similar to the potential of Eq. (1) with the $\omega_\perp$ appearing as a variational parameter. For a given quadrupole moment that leads to the proper $R$, the variation $\lambda$ will lead to the proper radial width $d$ of the transverse degree of freedom.

Another approach to examine toroidal nuclei can be carried out in diabatic mean-field calculations [53]. This is achieved by constraining the occupation of the single-particle states so that at the locations of the crossing of two single-particle states at the top of the fermi surface, one does not choose to occupy the state of the lowest energy. Instead, one maintains the diabatic configuration with the occupation of the state with the highest overlap with the earlier state before the level crossing, leading to a diabatic energy equilibrium as in [53]. Diabatic calculations have been performed successfully for $^{24}$Mg [53] and $^{28}$Si [51]. It will be of interest to see whether diabatic calculations for $^{12}$C will reveal the toroidal $^{12}$C states as discussed here.

VII. CONCLUSIONS AND DISCUSSIONS

In spite of many investigations on the excited states of $^{12}$C, the intrinsic structure of many excited states of $^{12}$C nucleus remains an unresolved problem [32]. We explore a novel description for some excited states of $^{12}$C for the reason that the $^{12}$C nucleus, with 6 neutrons 6 protons, is a doubly-magic closed-shell nucleus in a toroidal potential. The strong shell effects associated with a toroidal
shape may allow a toroidal $^{12}$C nucleus to co-exist with the ground state oblate spheroid. A dynamical generalization of the motion of the exchange nucleons between the clusters in Wheeler’s classical model of $^{12}$C as three α clusters will also generate a toroidal density if the nucleons exchanged between the clusters are allowed to circulate continuously from cluster to cluster. The toroidal description of $^{12}$C is therefore a dynamical derivative of Wheeler’s triangular cluster model. Furthermore, the importance of the toroidal shell structure has recently been highlighted by possible populations of toroidal high-spin isomers as was predicted by a number of theoretical investigations [51]. Many excited states of $^{12}$C also decays predominantly into three alpha particles which have a large overlap with a toroidal configuration.

Nucleons in a toroidal potential will generate single-particle states obeying the proper shell structure appropriate for such a peculiar density. Consequently the toroidal shell structure gives rise to a set of excited states bearing the imprint of the intrinsic properties of the toroidal nucleus. Specifically, the clustering of the single-particle state energy levels into shells lead to the classification of toroidal $\Lambda_z$ shells, which show up as different particle-hole excitations from one toroidal shell to another. The spectrum of a toroidal nucleus is characterized by these multiple shell-to-shell step-wise excitation multiplets of various spins, parities, and excitation energies.

We have obtained the theoretical signature of the $^{12}$C nucleus in the toroidal configurations and have confronted the data with theory. We find that the Hoyle state may be tentatively attributed to be the ground state of a $^{12}$C nucleus in the toroidal configuration and the lowest excited states of $3^-, 1^+, 2^-$ and $4^-$ may be members of the $(1p1h)_{12}$ multiplet and the $1^+ (12.71$ MeV) state may be members of the $(1p1h)_{02}$ multiplet. The sequence of the Hoyle state, the $2^+$ state, and $4^+$ state may be the rotational band of the Hoyle state. There is a hint that parity of the states appear to be approximately alternating, in a way similar to the prediction of the theory.

In spite of the tentatively encouraging results, the evidence is not yet completely conclusive as the complete set of multiplets have not been found. There are many more states in the theoretical toroidal description, as compared to the identified resonances. Experimentally, the search for the remainder states in the multiplet will be needed. The presence of prominent underlying broad structures in the excitation functions of the $^{11}$B$(^{4}$He,$d)^{12}$C→3α reaction of Ref. [14] indicates that the remainder members of the produced multiplet may be contained in the underlying broad structures. Experimental data from the $^{10}$B$(^{4}$He,$p)^{12}$C→3α reaction in [16] provide additional support for the possible toroidal description of the Hoyle state and the low-lying excited states. Furthermore, the $^{10}$B$(^{4}$He,$p)^{12}$C→3α data at higher energies suggests possible copious production of toroidal states which appear as a large underlying broad structure underneath the resolved resonances. Theoretical predictions and their comparisons with the experimental data therefore suggest that there may be a large number of toroidal $^{12}$C* states over a large energy region that readily breakup into three alpha particles, which may have implications in energy-producing mechanisms.

In future theoretical work, it is necessary to include spin-orbit and residual interactions so as to define better the splitting of the multiplets and to guide where these states may be more precisely located. The knowledge of a better wave function will allow the evaluation of the toroidal moment of inertia of the Hoyle state, for comparison with the observed moment of inertia. Theoretical mean-field calculations with a toroidal constraint will need to be carried out to show whether there are adiabatic toroidal $^{12}$C states that can co-exist with the ground oblate spheroid state.

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**Appendix A: (nphl) multiplets of states of toroidal $^{12}$C at higher excitation energies**

1. $[(1p1h)_{12}(1p-1h)_{02}]^-$ multiplet at $E_x=7\epsilon_0$ involving two identical nucleons

We like to examine the $I_z$ states of $[(1p1h)_{12}(1p1h)_{02}]^-$ at $E_x=\Delta E_{12}+\Delta E_{02}+7\epsilon_0$ involving identical particles with $(|\Lambda_z|=1)\rightarrow(|\Lambda_z|=2)$ and $(|\Lambda_z|=0)\rightarrow(|\Lambda_z|=2)$ excitations. The particles in the $\Lambda_z=2$ shell of identical particles must obey the Pauli Principle, we need to combine the particle pair first before we combine them with the hole pairs.

The pair of particles in the $\Lambda_z=2$ shell combine as in the following Table VII below

| two particle configuration | $I_z$ |
|---------------------------|-------|
| $\{2(5/2)\}2(3/2)$ | $4^+$ |
| $\{2(5/2)\}2(3/2)$ | $1^+$ |
| $\{2(3/2)\}2(-3/2)$ | $0^+$ |
| $\{2(5/2)\}2(5/2)$ | $0^+$ |
| $\{-5/2\}2(3/2)$ | $(1-)^+$ |
| $\{-5/2\}2(3/2)$ | $(4-)^+$ |

So the pair of $2p$ in the $\Lambda_z=2$ shell combine to be $\{4^-, 1^+, 0^+, 0^+, -(1-)^+, (4-)^+\}$. It suffices to consider only $4^+$, $1^+$ and $0^+$ and get the other possibility by reversing the sign. Now combine with the hole states $|0\pm 1/2\rangle$ with $|1\pm 3/2\rangle$ and $|1\pm 1/2\rangle$, we get
We find that the \([(1p1h)_{12}(1p1h)_{02}]^-\) multiplet at \(E_x = 7\epsilon_0\) contains 24 doubly-degenerate negative-parity states: \(\{6^-, 2(5^-), 2(4^-), 3(3^-), 5(2^-), 7(1^-), 4(0^-)\}\).
As one observes in Table XI, there are all together 3 doubly-degenerate \((2\text{ particle})-(2\text{ hole})\) states, with \(I_x^Z = (±4)^+, (±1)^+\), and \((0^+)^2\) at an excitation energy \(E_x = 2\Delta E_{02} \approx 8\varepsilon_0\). So, the \((2p2h)^+\) multiplet consists of 3 doubly-degenerate states of \([4^+, 1^+, 0^+]\).

4. \([1p1h)_{02}, (1p-1h)_{02p}]^+\) multiplet at \(E_x = 8\varepsilon_0\) involving a neutron and a proton

We have considered the \((2p2h)\) excitations involving only one type of nucleon. We can consider \((2p-2h)\) excitations involving neutrons and protons by combining the neutron and proton \((1p-1h)\) excitations independently.

Table XII contains the states \([6^+, 4(5^+), 6(4^+), 4(3^+), 2(2^+), 4(1^+), 6(0^+), 4(-1)^+, 1(-2)^+\]. Grouping the plus with the minus together, we find that the \([(1p1h)_{02}, (1p1h)_{02p}]^+\) multiplet at \(E_x = 8\varepsilon_0\) contains the set of 32 doubly-degenerate states: \([6^+, 4(5^+), 6(4^+), 4(3^+), 2(2^+), 4(1^+), 6(0^+), 4(-1)^+, 1(-2)^+\].

5. \((3p3h)_{12}\) at \(E_x = 9\varepsilon_0\) involving 3 identical nucleons

While the number of states in the \((2\text{ particle})-(2\text{ hole})\) excitation in the \(\Lambda_x = 1\) state to the \(\Lambda_x = 2\) states are numerous in number, the number of state become fewer in number for the \((3\text{ particle})-(3\text{ hole})\) excitation because they approach the completion of a closed shell configuration with the excited single-particle \(\Lambda_x = 2\) states.

Thus, the 3p-3h states have parity \(\pi = -1\) and are \((±4)^-, (±3)^-, (±2)^-, (±1)^-, \) for a single population of the \([2(±5/2)]\) state, and \((±3)^-, (±2)^-, (±1)^-, 0^+\) for a single population of the \([2(±3/2)]\) state. So, for the \((3p3h)_{12}\) multiplet, there are all together 8 doubly-degenerate negative-parity states at an excitation energy \(E_x = 9\varepsilon_0\): \([4^-, 2(3^-), 2(2^-), 2(1^-), 0^-]\).

6. \([(2p2h)_{12}, (1p1h)_{12p}]^-\) multiplet at \(E_x = 9\varepsilon_0\) involving three identical nucleons

As one observes in Table IV, the \((2p2h)_{12p}\) states consists of a set of 18 doubly-degenerate states with spin and parity: \([6^+, 5^+, 2(4^+), 2(3^+), 4(2^+), 5(1^+), 3(0^+)\]). The \((1p1h)_{12p}\) states consists of a set of 8 doubly-degenerate states with spin and parity: \([0, 2(1), 2(2), 2(3), 4]\). They can be combined independently as in table XIV.

The table consists of the set of negative parity states:
The Λ

The Λ

We combine all three particles and all three holes separately following the Pauli principle. The three particles behave as a single particle in that shell, with spins ±3/2 and ±5/2: \{5/2, 3/2, -3/2, -5/2\}. Next, we combine the three holes: one in the Λ = 0 shell, and the other two in the Λ = 1 shell. The two holes in Λ = 1 shell combine together to form spin \{2, 1, 0, 0, -1, -2\} while the hole in the Λ = 0 shell has spin \{1/2, -1/2\}. We combine the spin of the three holes in the following table:

TABLE XV. The two holes in Λ = 2 combined with 1 hole in Λ = 0 shell gives spins in this table

| two holes in Λ = 1 shell | one hole in Λ = 0 shell |
|--------------------------|-------------------------|
| 2                        | 5/2                     |
| 1                        | 3/2                     |
| 0                        | 1/2                     |
| -1                       | -1/2                    |
| -2                       | -3/2                    |

For the \([(1p1h)_{12}(2p2h)_{02}]^+\) multiplet at \(E_x = 10\varepsilon_0\), the three hole states combine to form states with spin \{5/2, 2(3/2), 3(1/2), 3(-1/2), 2(-3/2), -5/2\}. Now combine the three hole states with the three particle states, we get the spin configurations of the \([(1p1h)_{12}(2p2h)_{02}]^+\) multiplet in Table XVI.

### 7. \([(2p2h)_{12}(1p1h)_{02}]^+\) multiplet at \(E_x = 10\varepsilon_0\) involving identical nucleons

The final state has three nucleons in the Λ = 0 shell gives spins in this table:

\[
\begin{align*}
E_x &= 14\varepsilon_0 \text{ comprising of } 288 \text{ doubly-degenerate states of } \\
&\{(10^{-}, 3(9^{-}), 6(8^{-}), 10(7^{-}), 16(6^{-}), 25(5^{-}), 33(4^{-}), 36(3^{-}), 35(2^{-}), 33(1^{-}), 30(0^{-}), 26(-1^{-}), 20(-2^{-}), 11(-3^{-}), 3(-4^{-})\}.
\end{align*}
\]

### 8. \([(2p2h)_{12p}(1p1h)_{02p}]^-\) multiplet at \(E_x = 10\varepsilon_0\) involving different nucleons

As one observes in Table IV, the \((2p2h)_{12p}\) states consist of a set of 18 doubly-degenerate states with spin and parity: \{6^+, 5^+, 2(4^+), 2(3^+), 4(2^+), 5(1^+)\}. The \((1p1h)_{02p}\) states consists of a set of 4 doubly-degenerate states with spin and parity: \{1^+, 2(2^+), 3^+\}. They can be combined independently as in table XVII.

TABLE XVII. The spin states of the \([(1p1h)_{02}(2p2h)_{12}]^+\) multiplet at \(E_x = 10\varepsilon_0\). The parities of these states are all negative.

| \([(1p1h)_{02}(2p2h)_{12}]^+\) State |
|-------------------------------------|
| \begin{tabular}{c|c|c|c}
| two holes in Λ = 1 shell & one hole in Λ = 0 shell & 5/2 & -3/2 & -5/2 \\
| \hline
| 2                        & 5/2                     & 3/2   \\
| 1                        & 3/2                     & 1/2   \\
| 0                        & 1/2                     & -1/2  \\
| -1                       & -1/2                    & -3/2  \\
| -2                       & -3/2                    & -5/2  \\
| \end{tabular} |
|-------------------------------------|

From Table XVI, we find that the \([(1p1h)_{02}(2p2h)_{12}]^+\) multiplet at \(E_x = 10\varepsilon_0\) comprises of 24 doubly-degenerate positive parity states: \{5^+, 3(4^+), 5(3^+), 6(2^+), 6(1^+), 3(0^+)\}.

The table consists of the set of 144 negative parity states: \{9^-, 3(8^-), 5(7^-), 7(6^-), 11(5^-), 18(4^-), 22(3^-), 18(2^-), 13(1^-), 15(0^-), 17(-1^-), 11(-2^-), 3(3^-)\}. The other complimentary case gives the same set except with the sign of \(I_z\) reversed. Combining the two sets gives the the \([(1p1h)_{02p}(2p2h)_{12p}]^-\) multiplet at \(E_x = 10\varepsilon_0\) comprising of 144 doubly-degenerate positive parity states.
of \{9^+, 3(8^+), 5(7^+), 7(6^+), 11(5^+), 18(4^+), 25(3^+), 29(2^+), 30(1^+), 15(0^+)\}.

9. \([2p2h]_{12}^{(1p1h)_{12}}\) multiplet at \(E_x = 11\epsilon_0\) involving identical nucleons

The final state has three nucleons in the \(\Lambda = 2\) shell. We combine all three particles and all three holes separately following the Pauli principle. The three particles behave as a single particle in that shell, with spin \(+3/2\) and \(-5/2\): \{5/2, 3/2, -3/2, -5/2\}. Next, we combine the three holes: one in the \(\Lambda_z = 1\) shell, and the other two in the \(\Lambda_z = 0\) shell. The two holes in the \(\Lambda_z = 0\) shell combine together to form spin \{0\} while the hole in the \(\Lambda_z = 1\) shell has spin \{3/2, 1/2, -1/2, -3/2\}.

| \(\Lambda_z = 0\) shell | \(\Lambda_z = 1\) shell |
|--------------------------|------------------------|
| 5/2                      | 3/2, -3/2, -5/2        |
| 3/2                      | 4, 3, 0, -1            |
| 1/2                      | 3, 2, -1, -2           |
| -1/2                     | 2, 1, -2, -3           |
| -3/2                     | 1, 0, -3, -4           |

From Table XVIII, we find that the \([1p1h]_{12}(2p2h)_{12}\) multiplet at \(E_x = 10\epsilon_0\) comprises of 8 doubly-degenerate states: \(\{4^-, 2(3^-), 2(2^+), 2(1^-), 0^-\}\).

10. \((4p4h)_{12}^+\) multiplet at \(E_x = 12\epsilon_0\) involving four identical nucleons

In making the (4 particle)-(4 hole) excitation from the \(\Lambda = 1\) shell to the \(\Lambda = 2\) shell the four identical nucleons complete a closed shell configuration with the full occupation of the \(\Lambda = 2\) shell, at an excitation energy \(E_x = 12\epsilon_0\).

11. \([2p2h]_{12}^{(2p2h)_{12}}\) multiplet at \(E_x = 12\epsilon_0\)

As one observes in Table IV, the \((2p2h)_{12}\) states consists of a set of 18 doubly-degenerate states with spin and parity \(\{6^+, 5^+, 2(4^+), 2(3^+), 4(2^+), 5(1^+), 3(0^+)\}\). The multiplet \([2p2h]_{12}^{(2p2h)_{12}}\) involving two neutrons and two protons can combine their multiplet states independently as in Table XX.

| \([2p2h]_{12}\) states | \([2p2h]_{12}\) multiplet |
|--------------------------|---------------------------|
| \(6\)                     | 12, 11, 10(2), 9(4), 8(4) |
| \(5\)                     | 11, 10(2), 9(4), 8(4), 7(4) |
| \(4\)                     | 10(2), 9(4), 8(4), 7(4), 6(4) |
| \(3\)                     | 9(4), 8(4), 7(4), 6(4), 5(4) |
| \(2\)                     | 8(4), 7(4), 6(4), 5(4), 4(4) |
| \(1\)                     | 7(4), 6(4), 5(4), 4(4), 3(4) |
| \(0\)                     | 6(4), 5(4), 4(4), 3(4), 2(4) |

We collect all spin states, and get the set: \(\{12^+, 2(11^+), 5(10^+), 8(9^+), 16(8^+), 26(7^+), 39(6^+), 50(5^+), 63(4^+), 74(3^+), 83(2^+), 80(1^+), 69(0^+), 50(-1^+), 34(-2^+), 22(-3^+), 15(-4^+), 8(-5^+), 3(-6^+)\}\). The other complimentary case case with negative \(I_z\) of the \((2p2h)_{12}\) multiplet gives the same set with the sign of \(I_z\) reversed. So, the \([2p2h]_{12}^{(2p2h)_{12}}\) multiplet at \(E_x = 12\epsilon_0\) comprises of 648 doubly-degenerate states of the set:

\(\{12^+, 2(11^+), 5(10^+), 8(9^+), 16(8^+), 26(7^+), 42(6^+), 58(5^+), 78(4^+), 96(3^+), 117(2^+), 130(1^+), 169(0^+)\}\).

12. \([3p3h]_{12}^{(1p1h)_{12}}\) multiplet at \(E_x = 13\epsilon_0\) involving identical nucleons

The four particles in the \(\Lambda_z = 2\) complete a closed shell and have \(I_z = 0\). The three holes in the \(\Lambda_z = 1\) shell have spins \{3/2, 1/2, -1/2, -3/2\} and the hole in the \(\Lambda_z = 0\) shell has spins \{1/2, -1/2\}. As in Table VIII, the \([3p3h]_{12}^{(1p1h)_{12}}\) multiplet at \(E_x = 13\epsilon_0\) involving identical nucleons form the set of 4 doubly-degenerate states \(\{2^-, 2(1^-), 0^-\}\)
13. \([3p3h]_{12\nu}(1p1h)_{02\nu}\] multiplet at \(E_x = 13\epsilon_0\) involving three neutrons and one proton

The \([3p3h]_{12\nu}\) multiplet contains 8 doubly-degenerate states \(\{4^-, 2(3^-), 2(2^-), 2(1^-), 0^-\}\) and the \((1p1h)_{02\nu}\) multiplet contains 4 doubly-degenerate states \(\{3^+, 2(2^+), 1^+\}\). The \([3p3h]_{12\nu}(1p1h)_{02\nu}\] at \(E_x = 13\epsilon_0\) involving three neutrons and one proton contains 64 doubly-degenerate states: \{7^-, 4(6^-), 7(5^-), 8(4^-), 9(3^-), 12(2^-), 15(1^-), 8(0^-)\}.

14. \([2p2h]_{02}(2p2h)_{12\nu}\] multiplet at \(E_x = 14\epsilon_0\) involving identical nucleons

The four particles in the \(\Lambda_z = 2\) shell completes a closed shell and a total spin \(I_z = 0\). It is only necessary to find the \(I_z\) of the two holes in the \(\Lambda_z = 0\) and two holes in the \(\Lambda_z = 2\) shells. The two holes in the \(\Lambda_z = 0\) complete a closed shell, and have a total spin of \(I_z = 0\). The two holes in the \(\Lambda_z = 1\) shell have spins \(\{2, 1, 0, 0, -1, -2\}\), so the \([2p2h]_{02}(2p2h)_{12\nu}\] at \(E_x = 14\epsilon_0\) contains 3 doubly-degenerate states: \(\{2^+, 1^+, 0^+\}\).

15. \([2p2h]_{12\nu}(2p2h)_{02\nu}\] multiplet at \(E_x = 14\epsilon_0\)

The \((2p2h)_{12\nu}\) multiplet contains a set of 18 doubly-degenerate states with spin and parity \(\{6^+, 5^+, 2(4^+), 2(3^+), 4(2^+), 5(1^+), 3(0^+)\}\). The \((2p2h)_{02\nu}\) states contains a set of 3 doubly-degenerate states with spin and parity \(\{0^+, 1^+, 4^+\}\). Combining these two sets of states, we get the spins of the multiplet in table XXI.

| \([(2p2h)_{12\nu}]\) State | \([((2p2h)_{02\nu}]\) | \(\epsilon_0\) |
|--------------------------|----------------|---------|
| \(4\)                     | 10             | 1        |
| \(-1\)                   | 0              | 1        |
| \(-4\)                   | 0              | 1        |

This table contains of states: \(\{10^+, 9^+, 2(8^+), 3(7^+), 7(6^+), 10(5^+), 10(4^+), 16(2^+), 18(1^+), 13(0^+), 5(-1^+), 4(-2^+), 5(-3^+), 3(-4^+)\}\). The other complimentary case gives the same set except with the sign of \(I_z\) reversed. Combine the complementary set, we find that the \([(2p2h)_{12\nu}(2p2h)_{02\nu}]\) multiplet at \(E_x = 14\epsilon_0\) contains 18 doubly-degenerate positive parity states of the set: \(\{10^+, 9^+, 2(8^+), 3(7^+), 7(6^+), 13(5^+), 10(4^+), 15(3^+), 20(2^+), 23(1^+), 13(0^+)\}\).

\[\text{TABLE XXI. The spin states of the } [(2p2h)_{12\nu}(2p2h)_{02\nu}]^+ \text{ multiplet at } E_x = 14\epsilon_0. \text{ All these states have a positive parity.}\]

\[\text{TABLE XXI. The spin states of the } [(2p2h)_{12\nu}(2p2h)_{02\nu}]^+ \text{ multiplet at } E_x = 14\epsilon_0. \text{ All these states have a positive parity.}\]

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