Finite volume mass gap and free energy of the SU(\(N\)) × SU(\(N\)) chiral sigma model

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ABSTRACT: We compute the free energy in the presence of a chemical potential coupled to a conserved charge in the effective SU(\(N\)) × SU(\(N\)) scalar field theory to third order for asymmetric volumes in general \(d\)-dimensions, using dimensional regularization (DR). We also compute the mass gap in a finite box with periodic boundary conditions.
1 Introduction

Chiral perturbation theory ($\chi$PT) [1, 2] is the effective theory describing the low energy dynamics of the lowest lying pseudoscalar mesons. The parameters of the theory are couplings appearing in the effective chiral Lagrangian, the pion decay constant $F_\pi$ ($= F$ in the chiral limit) and other low energy constants (LEC’s). These parameters can be determined by phenomenology, or by lattice simulations of QCD. For a detailed summary of various determinations of the LEC’s the reader is referred to the FLAG review [3].

For $N_f = 2$ the relevant $\chi$PT has $\text{SU}(2) \times \text{SU}(2) \simeq \text{O}(4)$ symmetry. As a consequence in the past many theoretical $\chi$PT computations, in particular those pertaining to finite volume, have been performed for the slightly simpler model with $\text{O}(n)$ symmetry. One special environment is the so called $\delta$–regime first discussed by Leutwyler [4] where the system
is in a periodic spatial box of sides $L_s$ and $m_\pi L_s$ is small (i.e. small or zero quark mass) whereas $F_\pi L_s$ is large. In 2009 Hasenfratz [5] computed the mass gap in the delta-regime to third order $\chi$PT with the hope that a comparison with a precise lattice measurement of the low-lying stable masses in this regime may be used to determine some combination of the LEC’s.

In a previous paper [6] we computed the change in the free energy due to a chemical potential coupled to a conserved charge in the non-linear $O(n)$ sigma model with two regularizations, lattice regularization (with standard action) and DR in a general $d$-dimensional asymmetric volume with periodic boundary conditions (pbc) in all directions. This freedom allowed us for $d = 4$ to establish two independent relations among the 4-derivative couplings appearing in the effective Langrangians and in turn this allows conversion of results for physical quantities computed by the lattice regularization to those involving scales introduced in DR.

In particular we could convert the computation of the mass gap in a periodic box, by Niedermayer and Weiermann [7] using lattice regularization to a result involving parameters of the dimensionally regularized effective theory, and we verified this result by a direct computation [6] (which disagrees slightly with the previous computation [5]).

Although $N_f = 2$ is the phenomenologically most relevant case due to the low mass of the physical pions, $\chi$PT with $N_f > 2$ can also have useful applications [3]. With this in mind in this paper we extend the computations to the case of $SU(N) \times SU(N)$. After recollecting the structure of the effective Lagrangian in the next section we compute the free energy is in sect. 3 and the mass gap in a finite periodic box in sect. 4.

In this paper we do not analyze explicit chiral symmetry breaking. In QCD the effect of including a small quark mass on the finite volume spectrum has been computed for $N_f = 2$ to leading order in [4], and to next-to-leading order by Weingart [8, 9]. Furthermore Matzelle and Tiburzi [10] have studied the effect of small symmetry breaking in the quantum mechanical (QM) rotator picture ($N_f = 2$), and extended the results to small non-zero temperatures. In a related recent paper [11] we have computed the isospin susceptibility in the effective $O(n)$ scalar field theory, to third order $\chi$PT in the delta-regime using the QM rotator picture including an explicit symmetry breaking term, and showed consistency with standard $\chi$PT computations.

2 The effective Lagrangian

The dynamical fields are matrices $U(x) \in SU(N)$. In the chiral limit the action is invariant under global $SU(N)_L \times SU(N)_R$ transformations of the fields

$$U(x) \to g_L U(x) g_R^\dagger.$$  \hspace{1cm} (2.1)

In this limit the leading order effective Lagrangian is given by [1]:

$$\mathcal{L}_1 = \frac{1}{4g_0^2} \text{tr} \left( \partial_{\mu} U^\dagger \partial_{\mu} U \right).$$  \hspace{1cm} (2.2)
For $N \geq 4$ there are four linearly independent\footnote{up to higher derivatives} four-derivative terms in the effective Lagrangian [1]

$$\mathcal{L}_2 = \sum_{i=0}^{3} \frac{G_4^{(i)}}{4} \mathcal{L}_2^{(i)},$$  

\text{(2.3)}

with

$$\mathcal{L}_2^{(0)} = \text{tr} \left( \partial_\mu U^\dagger \partial_\nu U \partial_\rho U^\dagger \partial_\sigma U \right),$$  

\text{(2.4)}

$$\mathcal{L}_2^{(1)} = \text{tr}^2 \left( \partial_\mu U^\dagger \partial_\mu U \right),$$  

\text{(2.5)}

$$\mathcal{L}_2^{(2)} = \text{tr} \left( \partial_\mu U^\dagger \partial_\nu U \right) \text{tr} \left( \partial_\rho U^\dagger \partial_\sigma U \right),$$  

\text{(2.6)}

$$\mathcal{L}_2^{(3)} = \text{tr} \left( \partial_\mu U^\dagger \partial_\nu U \partial_\rho U^\dagger \partial_\sigma U \right).$$  

\text{(2.7)}

The 4-derivative couplings in (2.3) are related to the standard ones [1] as $G_4^{(i)} = -4L_i$. Since we work here in Euclidean space-time, our couplings differ in sign.\footnote{To avoid confusion with the box size $L_\mu$ we shall use the renormalized couplings $L_i^r$ only in the final results.} Note also the absence of the 4-derivative term $\text{tr} \left( \Box U^\dagger \Box U \right)$ in the above list; As explained in [12], this term can be eliminated by redefinition of the field $U$. The argument is reproduced for completeness in Appendix C.

For $N < 4$ these four operators are not all independent. One has [12]

$$\mathcal{L}_2^{(0)} = -\frac{1}{2} \mathcal{L}_2^{(1)} + \mathcal{L}_2^{(2)}, \quad \mathcal{L}_2^{(3)} = \frac{1}{2} \mathcal{L}_2^{(1)}, \quad (N = 2),$$  

\text{(2.8)}

$$\mathcal{L}_2^{(0)} = \frac{1}{2} \mathcal{L}_2^{(1)} + \mathcal{L}_2^{(2)} - 2\mathcal{L}_2^{(3)}, \quad (N = 3).$$  

\text{(2.9)}

A proof of (2.9) is given in Appendix B. Accordingly, in (2.3) one can restrict the summation to $i = 1, 2$ for $N = 2$ and to $i = 1, 2, 3$ for $N = 3$.

From these relations it follows that the results obtained for general $N$ should at $N = 2$ be invariant under the transformation

$$G_4^{(0)} \rightarrow G_4^{(0)} + \alpha_1, \quad G_4^{(1)} \rightarrow G_4^{(1)} + \alpha_2,$$

$$G_4^{(2)} \rightarrow G_4^{(2)} - \alpha_1, \quad G_4^{(3)} \rightarrow G_4^{(3)} + \alpha_1 - 2\alpha_2, \quad (N = 2),$$  

\text{(2.10)}

while at $N = 3$ under

$$G_4^{(0)} \rightarrow G_4^{(0)} + 2\alpha, \quad G_4^{(1)} \rightarrow G_4^{(1)} - \alpha,$$

$$G_4^{(2)} \rightarrow G_4^{(2)} - 2\alpha, \quad G_4^{(3)} \rightarrow G_4^{(3)} + 4\alpha, \quad (N = 3).$$  

\text{(2.11)}

The SU($N$) $\times$ SU($N$) model for $N = 2$ flavors is equivalent to the O(4) non-linear sigma model [13] (with fields $S_i$, $i = 0, \ldots, 3$ and $S^2 = 1$) where

$$\mathcal{L}_1 = \frac{1}{2g_0^2} \left( \partial_\mu S \cdot \partial_\mu S \right),$$  

\text{(2.12)}
and

\[ \mathcal{L}_2 = \sum_{i=2,3} \frac{g_4^{(i)}}{4} \mathcal{L}_2^{(i)} \]  

(2.13)

with

\[ \mathcal{L}_2^{(2)} = (\partial_\mu S \cdot \partial_\nu S)^2, \]  

(2.14)

\[ \mathcal{L}_2^{(3)} = (\partial_\mu S \cdot \partial_\nu S)(\partial_\rho S \cdot \partial_\sigma S). \]  

(2.15)

Writing

\[ U(x) = S_0(x) + i \sum_{a=1}^3 \sigma^a S_a(x), \quad S^2(x) = 1, \]  

(2.16)

where \( \sigma^a \) are the Pauli matrices, one obtains

\[ \mathcal{L}_2^{(2)} = \frac{1}{4} \mathcal{L}_2^{(1)}, \quad \mathcal{L}_2^{(3)} = \frac{1}{4} \mathcal{L}_2^{(2)}. \]  

(2.17)

This leads to the identification [1]

\[ g_4^{(2)} = 4G_4^{(1)}, \quad g_4^{(3)} = 4G_4^{(2)}. \]  

(2.18)

These and the relations (2.10), (2.11) can serve as checks on the final results.

2.1 Perturbative expansion

Here we work in a continuum volume \( V = L_t \times L_s^{d_s}, \ d_s = d - 1 \). In this section we impose periodic boundary conditions (pbc) on the dynamical variables in all directions. We dimensionally regularize by adding \( q \) extra compact dimensions of size \( \hat{L} \) (also with pbc) and analytically continue the resulting loop formulae to \( q = -2\epsilon \). We define \( D = d + q, V_D = V \hat{L}^q \), and the aspect ratios \( \ell = L_t/L_s, \hat{\ell} \equiv \hat{L}/L_s^3 \).

For the perturbative expansion we parameterize \( U \) with scalar fields \( \xi_a(x) \)\footnote{It is advantageous to treat these extra dimensions with a different size, since an extra check of the calculation is provided by the requirement that physical quantities are independent of this choice.}

\[ U(x) = u \bar{U}(x), \quad \bar{U}(x) = \exp(i g_0 \xi(x)), \]  

(2.19)

where \( u \) is a constant matrix and

\[ \xi = \sum_{a=1}^{N_1} \lambda^a \xi_a, \]  

(2.20)

where the hermitian \( \lambda \)-matrices are defined and some of their properties noted in Appendix A. Further

\[ N_1 \equiv N^2 - 1, \]  

(2.21)

and the fields \( \xi \) satisfy the constraints

\[ \int_x \xi_a(x) = 0, \quad \forall a. \]  

(2.22)
$A_{2,e\text{ff}}$ has a perturbative expansion

$$A_{2,e\text{ff}} = A_{2,0} + g_0^2 A_{2,1} + g_0^4 A_{2,2} + \mathcal{O}(g_0^6),$$  \quad (2.23)

where

$$A_{2,0} = \frac{1}{2} \int_x \partial_\mu \xi_a(x) \partial_\mu \xi_a(x),$$  \quad (2.24)

$$A_{2,1} = A_{2,1}^{(a)} + A_{2,1}^{(b)},$$  \quad (2.25)

$$A_{2,1}^{(a)} = \frac{N}{3 V_D} \int_x \sum_a \xi_a(x) \xi_a(x),$$  \quad (2.26)

$$A_{2,1}^{(b)} = \frac{1}{48} \int_x \text{tr} \left( \left[ \xi(x), \partial_\mu \xi(x) \right]^2 \right),$$  \quad (2.27)

and

$$A_{2,2} = A_{2,2}^{(a)} + A_{2,2}^{(b)} + A_{2,2}^{(c)},$$  \quad (2.28)

$$A_{2,2}^{(a)} = \frac{1}{1440 V_D} \int_x \sum_a \text{tr} \left( \lambda^a \left[ \left[ \xi(x), \left[ \xi(x), \left[ \xi(x), \lambda^a] \right] \right] \right] \right),$$  \quad (2.29)

$$A_{2,2}^{(b)} = \frac{1}{1152 V_D^2} \int_{xy} \sum_{a,b} \text{tr} \left( \lambda^b \left[ \left[ \xi(x), \left[ \xi(x), \lambda^b] \right] \right] \right) \text{tr} \left( \lambda^a \left[ \left[ \xi(y), \left[ \xi(y), \lambda^b] \right] \right] \right),$$  \quad (2.30)

$$A_{2,2}^{(c)} = \frac{1}{1440} \int_x \text{tr} \left( \left[ \xi(x), \left[ \xi(x), \partial_\mu \xi(x) \right] \right]^2 \right),$$  \quad (2.31)

where the terms $A_{2,1}^{(a)}, A_{2,2}^{(a)}, A_{2,2}^{(b)}$ come from the zero mode action derived in Appendix D.

The total effective action has a perturbative expansion of the form

$$A = \sum_{r=0} A_r g_0^{2r},$$  \quad (2.32)

with

$$A_r = A_{2,r} + \sum_{i=0}^3 \frac{3}{4} G_4^{(i)} A_{4,r}^{(i)}.$$

Note

$$A_{4,0}^{(i)} = 0 = A_{4,1}^{(i)}, \quad \forall i,$$

and

$$A_{4,2}^{(0)} = \int_x \text{tr} \left( \partial_\mu \xi(x) \partial_\nu \xi(x) \partial_\mu \xi(x) \partial_\nu \xi(x) \right),$$  \quad (2.35)

$$A_{4,2}^{(1)} = \int_x \text{tr}^2 \left( \partial_\mu \xi(x) \partial_\mu \xi(x) \right),$$  \quad (2.36)

$$A_{4,2}^{(2)} = \int_x \text{tr} \left( \partial_\mu \xi(x) \partial_\nu \xi(x) \right) \text{tr} \left( \partial_\mu \xi(x) \partial_\nu \xi(x) \right),$$  \quad (2.37)

$$A_{4,2}^{(3)} = \int_x \text{tr} \left( \partial_\mu \xi(x) \partial_\nu \xi(x) \partial_\nu \xi(x) \partial_\nu \xi(x) \right).$$  \quad (2.38)
The free 2-point function is given by
\[
\langle \xi_a(x)\xi_b(y) \rangle_0 = \delta_{ab}G(x-y),
\] (2.39)
with propagator
\[
G(x) = \frac{1}{V_D} \sum_p e^{ipx} p^2, \tag{2.40}
\]
where the sum is over momenta \( p_\mu = 2\pi n_\mu/L_\mu \), \( n_\mu \in \mathbb{Z} \) and the prime on the sum means that \( p = 0 \) is omitted.

3 The chemical potential

The chemical potential \( h \) is introduced by the substitution:
\[
\partial_0 \to \partial_0 + h \left[ \frac{\lambda^3}{2}, \cdot \right]. \tag{3.1}
\]
This gives an additional \( h \)-dependent part \( A_h \) to the total action of the form
\[
A_h = A_{2h} + \sum_{i=0}^{3} \frac{G_4^{(i)}}{4} A_{4h}^{(i)}. \tag{3.2}
\]
Further writing
\[
A_{2h} = i h B_2 + h^2 C_2 + \ldots, \tag{3.3}
\]
\[
A_{4h}^{(i)} = i h B_4^{(i)} + h^2 C_4^{(i)} + \ldots, \tag{3.4}
\]
we have
\[
B_2 = -\frac{i}{4g_0^2} \int_x \text{tr} \left( \lambda^3 \left[ U(x), \partial_0 U^\dagger(x) \right] \right), \tag{3.5}
\]
\[
C_2 = \frac{1}{16 g_0} \int_x \text{tr} \left( \left[ \lambda^3, U(x) \right] \left[ \lambda^3, U^\dagger(x) \right] \right). \tag{3.6}
\]
The 4-derivative operators \( B_4^{(i)}, C_4^{(i)} \) are given in Appendix F.

The \( h \)-dependent part of the free energy \( f_h \) is defined as
\[
e^{-V f_h} = \langle e^{-A_h} \rangle_A = 1 - \langle A_h \rangle_A + \frac{1}{2} \langle A_h^2 \rangle_A + \ldots \tag{3.7}
\]
giving up to the order \( h^2 \):
\[
V f_h = \langle A_h \rangle_A - \frac{1}{2} \langle A_h^2 \rangle_A + \frac{1}{2} \langle A_h^4 \rangle_A + \ldots \tag{3.8}
\]
Note for an observable \( X \):
\[
\langle X \rangle_A = \langle X \rangle_0 - g_0^2 \langle X A_1 \rangle_0^c - g_0^4 \langle X A_2 \rangle_0^c + \frac{1}{2} g_0^4 \langle X A_4 \rangle_0^c + \ldots \tag{3.9}
\]
Now
\[ \langle B_2 \rangle_A = 0 = \langle B_4^{(i)} \rangle_A \quad \forall i, \] (3.10)
so we have
\[ \chi = -2 \sum_{s=1}^{5} F_s, \] (3.11)
with
\[ F_1 = \frac{1}{V_D} \langle C_2 \rangle_A, \] (3.12)
\[ F_2 = \frac{1}{2 V_D} \langle B_2 \rangle_A, \] (3.13)
\[ F_3 = \sum_{i=0}^{3} \frac{G_4^{(i)}}{4} \frac{1}{V_D} \langle C_4^{(i)} \rangle_A, \] (3.14)
\[ F_4 = \sum_{i=0}^{3} \frac{G_4^{(i)}}{4} \frac{1}{V_D} \langle B_2 B_4^{(i)} \rangle_A, \] (3.15)
\[ F_5 = \frac{1}{2} \sum_{j=0}^{3} \frac{G_4^{(i)}}{4} \frac{1}{V_D} \langle B_4^{(i)} B_4^{(j)} \rangle_A. \] (3.16)

Averaging over the zero modes, denoting \( \overline{U}(x) = e^{ig_0 \lambda \xi(x)} \),
\[ \frac{1}{V_D} \int du \, C_2 = \frac{1}{8g_0^2} \int_x \, du \, \text{tr} \left( \lambda^3 u \overline{U}(x) \lambda^3 \overline{U}^\dagger(x) u^\dagger - (\lambda^3)^2 \right) \]
\[ = -\frac{1}{4g_0^2}, \] (3.17)
where we used (E.3). So
\[ F_1 = -\frac{1}{4g_0^2}. \] (3.18)

Next
\[ \frac{1}{V_D} \int du \, B_2 = \frac{1}{g_0^2} W, \] (3.19)
with \( W \) given by
\[ W = -\frac{1}{16V_D} \int_{xy} \int \, du \, \text{tr} \left( \lambda^3 \left[ u \overline{U}(x), \partial_0 \overline{U}^\dagger(y) u^\dagger \right] \right) \text{tr} \left( \lambda^3 \left[ u \overline{U}(y), \partial_0 \overline{U}^\dagger(x) u^\dagger \right] \right). \] (3.20)

For the averages we have
\[ \langle W \rangle_A = \frac{1}{N_1} \langle [W] \rangle_A, \] (3.21)
where \( [W] \) is obtained from \( W \) by replacing \( \lambda_{ij}^3 \lambda_{kl}^3 \) by \( \sum_a \lambda_{ij}^a \lambda_{kl}^a \). Using completeness in the form (A.12) we get
\[ [W] = -\frac{1}{8V_D} \int_{xy} \int \, du \, \text{tr} \left( \left[ u \overline{U}(x), \partial_0 \overline{U}^\dagger(x) u^\dagger \right] \left[ u \overline{U}(y), \partial_0 \overline{U}^\dagger(y) u^\dagger \right] \right) \]
\[ = \frac{1}{8V_D} \int_{xy} \int \, du \, \text{tr} \left( J_-(x) J_-(y) + J_+(x) J_+(y) \right) \]
\[ + u J_-(x) u^\dagger J_+(y) + J_+(x) u J_-(y) u^\dagger \right), \] (3.22)
where
\[ J_+(x) \equiv i \partial_0 \mathcal{U}^\dagger(x) \mathcal{U}(x), \quad J_-(x) \equiv i \partial_0 \mathcal{U}(x) \mathcal{U}^\dagger(x). \] (3.23)

Note \( J_\pm \) are hermitian \( J_\pm^\dagger(x) = J_\pm(x) \) and traceless
\[ \text{tr}(J_\pm(x)) = 0, \] (3.24)
and have a perturbative expansion\(^5\)
\[ J_\pm(x) = \pm g_0 \left( \frac{\exp(\text{Ad}[\pm ig_0 \xi(x)]) - 1}{\text{Ad}[\pm ig_0 \xi(x)]} \right) \partial_0 \xi(x) \] (3.25)
\[ = \pm g_0 \sum_{r=1}^{\infty} \frac{1}{r!} \left( \text{Ad}[\pm ig_0 \xi(x)] \right)^{r-1} \partial_0 \xi(x) \]
\[ = \pm g_0 \partial_0 \xi(x) + ig_0^2 \frac{1}{2} [\xi(x), \partial_0 \xi(x)] + g_0^3 \frac{1}{6} [\xi(x), [\xi(x), \partial_0 \xi(x)]] \]
\[ - i g_0^4 \frac{1}{24} [\xi(x), [\xi(x), [\xi(x), \partial_0 \xi(x)]]] + O(g_0^5). \] (3.26)

Note that for pbc
\[ \int_x J_\pm(x) = 0. \] (3.27)

Using (E.3) we get simply
\[ [W] = \frac{1}{8V_D} \int_{xy} \text{tr} \left( (J_- (x) J_-(y) + J_+ (x) J_+(y)) \right). \] (3.28)

This has a perturbative expansion
\[ [W] = g_0^4 W_2 + g_0^6 W_3 + \ldots \] (3.29)
with
\[ W_2 = -\frac{1}{16V_D} \int_{xy} \text{tr} \left( [[\xi(x), \partial_0 \xi(x)] \xi(y), \partial_0 \xi(y)] \right) \]
\[ = \frac{1}{2V_D} f_{abc} f_{cde} \int_{xy} \xi_a(x) \partial_0 \xi_b(x) \xi_c(y) \partial_0 \xi_d(y), \] (3.30)
and
\[ W_3 = W_3^{(1)} + W_3^{(2)}, \] (3.31)
with
\[ W_3^{(1)} = \frac{1}{144V_D} \int_{xy} \text{tr} \left( [[\xi(x), \xi(y), \partial_0 \xi(x)] \xi(y), [\xi(y), \partial_0 \xi(y)] \right) \]
\[ W_3^{(2)} = \frac{1}{96V_D} \int_{xy} \text{tr} \left( [[\xi(x), \partial_0 \xi(x)] \xi(y), [\xi(y), [\xi(y), \partial_0 \xi(y)] \right) \]. \] (3.32) (3.33)

Expanding (3.13) in a perturbative series
\[ F_2 = \sum_{r=0}^{\infty} F_{2r} g_0^{2r}, \] (3.34)

\(^5\)For \( a, b \) in the Lie algebra \( \text{Ad}(a)b = [a, b] \)
we have at leading order

\[ F_{2,0} = \frac{1}{2N_1} \langle W_2 \rangle_0. \]  

(3.35)

Now

\[ \langle W_2 \rangle_0 = \frac{1}{8} Z_1 \int_x [\partial_0 G(x)]^2 \]
\[ = \frac{1}{8} Z_1 I_{21}. \]  

(3.36)

Here \(Z_1\) is a group factor defined in \((A.16)\) in Appendix A where also other such factors \(Z_i, i = 2, \ldots, 8\) appearing below are defined and evaluated. Further the dimensionally regularized sums \(I_{nm}\) are formally defined by

\[ I_{nm} = \frac{1}{V_D} \sum'_p \frac{(p_0^2)^m}{(p^2)^n}. \]  

(3.37)

So we have at leading order

\[ F_{2,0} = \frac{N}{2} I_{21}. \]  

(3.38)

At next order

\[ F_{2,1} = \frac{1}{2N_1} \left[ \langle W_3 \rangle_0 - \langle W_2 A_{2,1} \rangle_0 \right]. \]  

(3.39)

First

\[ \langle W_3^{(1)} \rangle_0 = \frac{1}{96} (Z_2 + Z_3) \mathcal{W} = \frac{1}{2} N^2 N_1 \mathcal{W}, \]  

(3.40)

where

\[ \mathcal{W} = - \int_x G(x)^2 \partial_0^2 G(x). \]  

(3.41)

This 2-loop function, the “massless sunset diagram”, is calculated in detail in [14].

Secondly

\[ \langle W_3^{(2)} \rangle_0 = \frac{1}{48} Z_5 G(0) \int_x [\partial_0 G(x)]^2 \]
\[ = -\frac{5}{3} N^2 N_1 I_{10} I_{21}. \]  

(3.42)

Next

\[ \langle W_2 A_{2,1}^{(a)} \rangle_0 = -\frac{N}{48 V_D^2} \int_{xyu} \langle \text{tr} \left[ [\xi(x), \partial_0 \xi(x)] [\xi(y), \partial_0 \xi(y)] \right] \xi_a(u) \xi_a(u) \rangle_0 \]
\[ = \frac{4N^2 N_1}{3V_D^2} \int_{xyu} G(x - u)G(y - u)\partial_0^2 \partial_0^2 G(x - y) \]
\[ = \frac{4}{3} N^2 N_1 \frac{1}{V_D} I_{31}. \]  

(3.44)

Furthermore

\[ \langle W_2 A_{2,1}^{(b)} \rangle_0 = -\frac{1}{768 V_D} \int_{xyu} \langle \text{tr} \left[ [\xi(x), \partial_0 \xi(x)] [\xi(y), \partial_0 \xi(y)] \right] \text{tr} \left( [\xi(u), \partial_0 \xi(u)]^2 \right) \rangle_0 \]
\[ = w_2^{(a)} + w_2^{(b)} + w_2^{(c)}, \]  

(3.45)

\[^{\text{6}}\text{we used} \int_y \partial_0^2 [G(x - y)^2 \partial_0^2 G(x - y)] = 0\]
with
\[ w^{(a)}_2 = \frac{N}{96V_D} G(0) \int_{xyu} \langle \text{tr} \left( \left[ \xi(x), \partial_0 \xi(x) \right] \left[ \xi(y), \partial_0 \xi(y) \right] \partial_\mu \xi_a(u) \partial_\mu \xi_a(u) \right) \rangle^c_0 \]
\[ = -\frac{N}{12V_D} G(0) Z_1 \int_{xyu} \partial_\mu G(x - u) \partial_\mu G(y - u) \partial_\rho \partial_0 G(x - y) \] (3.46)
\[ = -\frac{2}{3} N^2 N_1 \overline{t}_{10} \overline{t}_{21} . \]

\[ w^{(b)}_2 = \frac{N}{96V_D} \square G(0) \int_{xyu} \langle \text{tr} \left( \left[ \xi(x), \partial_0 \xi(x) \right] \left[ \xi(y), \partial_0 \xi(y) \right] \xi_a(u) \xi_a(u) \right) \rangle^c_0 \]
\[ = \frac{2}{3} N^2 N_1 \frac{1}{V_D} \overline{t}_{31} . \] (3.47)

Note that \( \overline{t}_{00} = -\square G(0) = -1/V_D \) since with dimensional regularization one sets \( \delta(0) = 0 \).

Finally
\[ w^{(c)}_2 = -\frac{1}{96V_D} (Z_6 + Z_7) \int_{xyu} G(x - u) \partial_0^2 G(x - u) G(y - u) \partial_0^2 G(y - u) \]
\[ = -N^2 N_1 \overline{t}_{21}^2 . \] (3.48)

### 3.1 Contribution from the 4-derivative terms

For the averages we have
\[ \langle C^{(i)}_4 \rangle_A = \frac{1}{N_1} \langle \left[ C^{(i)}_4 \right] \rangle_A , \] (3.49)
where \( \left[ C^{(i)}_4 \right] \) is obtained in Appendix F from \( C^{(i)}_4 \) by replacing \( \lambda^3_{ij} \lambda^3_{kl} \) by \( \sum_a \lambda^a_{ij} \lambda^a_{kl} \) and averaging over the constant modes. From these expressions we obtain
\[ F_{3,1} = -\frac{G^{(0)}_4}{N} \left\{ \frac{1}{V_D} + 2N_1 \overline{t}_{11} \right\} + G^{(4)}_4 \left\{ \frac{N_1}{V_D} - 2\overline{t}_{11} \right\} \]
\[ + G^{(2)}_4 \left\{ \frac{1}{V_D} - N^2 \overline{t}_{11} \right\} + G^{(3)}_4 \left\{ \frac{N_1}{V_D} - (N^2 - 2)\overline{t}_{11} \right\} . \] (3.50)

One can check that \( F_{3,1} = 0 \) for \( N = 3 \) when one sets \( G^{(0)}_4 = 2, G^{(1)}_4 = -1, G^{(2)}_4 = -2, G^{(3)}_4 = 4 \), as required by (2.9).

Finally
\[ F_{4,1} = F_{5,1} = 0 . \] (3.51)

### 3.2 Summary

Collecting the results together, the expansion of the susceptibility with DR is given by
\[ \chi = \frac{1}{2g_0^2} \left( 1 + g_0^2 R_1 + g_0^4 R_2 + \ldots \right) , \] (3.52)
with
\[ R_1 = -2N \overline{t}_{21} . \] (3.53)
\[ R_2 = R_2^{(a)} + R_2^{(b)}, \quad (3.54) \]

with

\[
R_2^{(a)} = N^2 \left\{ -\mathcal{W} + 2T_{21} [T_{10} - T_{21}] + \frac{4}{V_D} T_{31} \right\}, \quad (3.55)
\]

\[
R_2^{(b)} = -4 \left[ -\frac{G_4^{(0)}}{N} \left\{ \frac{1}{V_D} + 2N_1 T_{11} \right\} \right. + \left. \frac{G_4^{(1)}}{N} \left\{ \frac{N_1}{V_D} - 2T_{11} \right\} \right.
\]
\[
\left. + \frac{1}{V_D} - N^2 T_{11} \right\} + \frac{G_4^{(3)}}{N} \left\{ \frac{N_1}{V_D} - (N^2 - 2)T_{11} \right\} \right]\]
\[
= -\frac{4}{N} \left[ -G_4^{(0)} + NN_1 G_4^{(1)} + NG_4^{(2)} + N_1 G_4^{(3)} \right] \frac{1}{V_D}
\]
\[
+ \frac{4}{N} \left[ 2N_1 G_4^{(0)} + 2NG_4^{(1)} + N^3 G_4^{(2)} + (N^2 - 2)G_4^{(3)} \right] T_{11}. \quad (3.56)
\]

For \( N = 2, 3 \) the relations (2.10), (2.11) are satisfied.

### 3.3 Renormalization of the free energy in \( d = 4 \)

We first recall some results obtained in [14] for the behavior of the functions as \( q \to 0 \):

\[
T_{10} = -\beta_1(\ell) L_s^{-2} + \mathcal{O}(q), \quad (3.58)
\]

\[
T_{11} = \frac{1}{L_s^4} \left\{ \frac{1}{2} (1 - q \ln L_s) \left[ \gamma_1(\ell) - \frac{1}{2} \right] + q W_1(\ell, \hat{\ell}) \right\} + \mathcal{O}(q^2), \quad (3.59)
\]

\[
T_{21} = \frac{1}{8\pi L_s^2} (\gamma_2(\ell) - 1) + \mathcal{O}(q), \quad (3.60)
\]

\[
T_{31} = -\frac{1}{32\pi^2} \left[ \frac{1}{q} - \ln L_s - \frac{1}{2} \gamma_3(\ell) \right] + \mathcal{O}(q), \quad (3.61)
\]

where the shape functions \( \beta_1(\ell), \gamma_i(\ell) \) and \( \mathcal{W}(\ell) \) are given in [6], and for the 2-loop function

\[
\mathcal{W} = \frac{1}{16\pi^2 L_s^4} \left\{ \left[ \frac{1}{q} - 2 \ln L_s \right] W_0(\ell) + \frac{1}{3\ell} \ln(\hat{\ell}) - \frac{10}{3} W_1(\ell, \hat{\ell}) + \mathcal{W}(\ell) \right\} + \mathcal{O}(q), \quad (3.62)
\]

with the non-singular shape function [6]:

\[
W_0(\ell) = \frac{5}{3} \left( \frac{1}{2} - \gamma_1(\ell) \right) - \frac{1}{3\ell}. \quad (3.63)
\]

The shape function \( W_1(\ell, \hat{\ell}) \) occurring in (3.59) and (3.62) is not needed here (see below).

Below we switch to the conventional couplings \( L_i = -G_4^{(i)}/4 \) and express the bare couplings through the renormalized ones by

\[
L_i = L_i^r + \frac{v_i}{16\pi^2} \mu^{D-4} \left( \frac{1}{D-4} + \bar{C} \right). \quad (3.64)
\]

where

\[
\bar{C} = \log \bar{c} = -\frac{1}{2} (\ln(4\pi) - \gamma_E + 1) = -1.476904292. \quad (3.65)
\]
By convention [1] the renormalized couplings are taken at the scale $\mu = M_\pi$, where $M_\pi$ is the mass of the charged pion.

Requiring cancellation of the $\propto 1/(D - 4)$ terms in $R_2$ one obtains two relations,

$$
Nv_2 - v_0 + (N^2 - 1)(Nv_1 + v_3) = \frac{5}{48} N^3,
$$

$$
2Nv_0 - Nv_3 + (N^2 - 2)(v_2 - 2v_1) = 0.
$$

(3.66)

Due to these relations the terms $\ln(\hat{\ell})$ and $W_1(\ell, \hat{\ell})$ depending on auxiliary, unphysical box size, also cancel. The relations (3.66) are satisfied by the coefficients $v_i$ which were calculated in an elegant way by Gasser and Leutwyler [1]7:

$$
v_0 = N/48, \quad v_1 = 1/16, \quad v_2 = 1/8, \quad v_3 = N/24.
$$

(3.67)

Finally one has

$$
L_s^2 \chi = \frac{1}{2} F_2^2 L_s^2 \left(1 + \frac{1}{F_2^2 L_s^2} (L_s^2 R_1) + \frac{1}{F_4^2 L_s^4} (L_s^4 R_2) + \mathcal{O} \left( (FL_s)^{-6} \right) \right)
$$

(3.68)

where

$$
L_s^2 R_1 = -\frac{N}{4\pi} (\gamma_2 - 1),
$$

$$
L_s^4 R_2 = -\frac{N^2}{32\pi^2} \left[ (\gamma_2 - 1)^2 + 8\pi (\gamma_2 - 1) \beta_1 + 2\mathcal{W} - \frac{2}{\ell} \gamma_3 \right]
$$

$$
+ \frac{5N^2}{48\pi^2} \left[ \frac{1}{\ell} - \gamma_1 + \frac{1}{2} \right] \log (\tau L_s M_\pi)
$$

$$
- \frac{8}{N} \left[ 2(N^2 - 1)L_0^r + 2NL_1^r + N^3 L_2^r + (N^2 - 2)L_3^r \right] \left( \gamma_1 - \frac{1}{2} \right)
$$

$$
+ \frac{16}{N} \left[ -L_0^r + N(N^2 - 1)L_1^r + NL_2^r + (N^2 - 1)L_3^r \right] \frac{1}{\ell}, \quad (N \geq 4).
$$

(3.69)

For $N = 3$ one should here omit the term proportional to $L_0^r$. Similarly, for $N = 2$ one should omit $L_0^r$ and $L_3^r$. In addition, to use the conventional notation (stemming from the O(4) formulation), one should make the replacement $L_1^r \rightarrow l_1^r/4$, $L_2^r \rightarrow l_2^r/4$. This result is also invariant under the transformations corresponding to (2.10) and (2.11).

For the O($n$) case one has [6]

$$
L_s^4 R_2^{O(n)} = -\frac{n - 2}{16\pi^2} \left[ (\gamma_2 - 1)^2 + 8\pi (\gamma_2 - 1) \beta_1 + 2\mathcal{W} - \frac{(n - 2)}{\ell} \gamma_3 \right]
$$

$$
+ \frac{n - 2}{24\pi^2} \left[ \frac{3n - 7}{\ell} - 5 \left( \gamma_1 - \frac{1}{2} \right) \right] \log (\tau L M_\pi)
$$

$$
- 2 \left( 2l_1^r + nl_2^r \right) \left( \gamma_1 - \frac{1}{2} \right) + 4 \left( (n - 1)l_1^r + l_2^r \right) \frac{1}{\ell}.
$$

(3.70)

Our result (3.69) for $N = 2$ flavors agrees with this taken at $n = 4$.

---

7The coefficients in [1] are written out explicitly only for $N = 3$ but the previous steps are done for general $N$. The $N = 3$ coefficients $\Gamma_i$ in [1] are given by $\Gamma_1 = v_1 + v_0/2 = 3/32$, $\Gamma_2 = v_2 + v_0 = 3/16$ and $\Gamma_3 = v_3 - 2v_0 = 0$. 

4 Computation of mass gap on a periodic strip

In this section we will compute the mass gap of the 4d chiral SU($N \times SU(N$) model on a periodic strip. We will follow the method first used in [15] and later in [7]. In the latter references the computation was done using lattice regularization. Here we will employ dimensional regularization as we did in [6]. The dynamical fields $U(x)$ are now defined in a volume

$$\Lambda = \left\{ x; x_0 \in [-T, T], x_\mu \in [0, L], \text{for } \mu = 1, \ldots, d - 1, x_\mu \in [0, \hat{L}], \text{for } \mu = d, \ldots, D - 1 \right\},$$

(4.1)

with periodic boundary conditions in the $D - 1$ “spatial” directions, and free boundary conditions in the time direction.

Here we will only give a brief description of the computation since it follows closely that for the O($n$) model [6]. We first compute the 2-point function

$$C_0(x) = \lim_{T \to \infty} \frac{1}{N} \langle \text{tr} \left( U^\dagger(x) U(0) \right) \rangle \propto e^{- (E_1 - E_0) |x_0|}, \quad (|x_0| \to \infty).$$

(4.2)

It follows that the mass gap

$$E_1 - E_0 = - \lim_{x_0 \to \infty} \frac{\partial}{\partial x_0} \ln C(x_0).$$

(4.3)

Since $C_0(x)$ has a perturbative expansion of the form

$$C_0(x) = 1 + \sum_{\nu=1}^{\infty} g_0^{2\nu} C_0^{(\nu)}(x),$$

(4.4)

equation (4.3) yields the power series

$$E_1 - E_0 = \frac{1}{2V_D} \sum_{\nu=1}^{\infty} g_0^{2\nu} \Delta^{(\nu)}.$$ 

(4.5)

If for $x_0 \to \infty$:

$$C_0^{(\nu)}(x) \sim \sum_{r=0}^{\nu} \hat{c}_r^{(\nu)} \left( \frac{x_0}{2V_D} \right)^r + \text{exponentially damped}$$

(4.6)

then

$$\Delta^{(1)} = -\hat{c}_1^{(1)},$$

$$\Delta^{(2)} = -\hat{c}_1^{(2)} + \hat{c}_1^{(1)} \hat{c}_0^{(1)} = -\hat{c}_1^{(2)} - \Delta^{(1)} \hat{c}_0^{(1)},$$

$$\Delta^{(3)} = -\hat{c}_1^{(3)} + \hat{c}_1^{(2)} \hat{c}_0^{(1)} + \hat{c}_1^{(1)} \hat{c}_0^{(2)} - \hat{c}_1^{(1)} \hat{c}_0^{(1)} \hat{c}_0^{(2)}$$

$$= -\hat{c}_1^{(3)} - \Delta^{(2)} \hat{c}_0^{(1)} - \Delta^{(1)} \hat{c}_0^{(2)}.$$ 

(4.7)

(4.8)

(4.9)
It thus suffices to compute the coefficients $c_i^{(r)}$ with $i = 0, 1^8$.

The fields $U(x)$ are parameterized as in (2.19) but now the $\xi(x)$-field satisfies Neumann boundary conditions [15]

$$\partial_0 \xi(x) = 0 \quad \text{for} \quad x_0 = \pm T,$$

and periodic boundary conditions in the spatial directions.

The corresponding free 2-point function is given by

$$\langle \xi_a(x) \xi_b(y) \rangle_0 = \delta_{ab} G(x, y),$$

with

$$G(x, y) = \frac{1}{V_D} \left( \frac{x_0^2 + y_0^2}{4T} - \frac{1}{2} |x_0 - y_0| + \frac{T}{6} \right)$$

$$+ \sum_{m=-\infty}^{\infty} \left\{ R(x_0 - y_0 + 4mT, x - y)$$

$$+ R(x_0 + y_0 + 2(2m + 1)T, x - y) \right\},$$

where

$$R(z) = \frac{1}{2V_D} \sum_{\mathbf{p} \neq 0} \frac{1}{\omega_{\mathbf{p}}} e^{-\omega_{\mathbf{p}} |z_0|} e^{i \mathbf{pz}},$$

where the sum goes over $p_\mu = \frac{2\pi \nu_\mu}{L_\mu}$, $\mu = 1, \ldots, D - 1$ with $\nu_\mu \in \mathbb{Z}$, and

$$\omega_{\mathbf{p}} = |\mathbf{p}|.$$

Expanding

$$\frac{1}{N} \text{tr} \left( U^\dagger(x)U(0) \right) = 1 + \sum_{\nu=1}^\infty g_0^{2\nu} \theta_\nu + \sum_{\nu=1}^\infty g_0^{2\nu+1} \rho_\nu,$$

the operators $\rho_\nu$ are not of interest to us here since their expectation values with operators even in $\xi$ are zero, and the 2-point correlation function has a perturbative expansion of the form

$$\langle \frac{1}{N} \text{tr} \left( U^\dagger(x)U(0) \right) \rangle = 1 + \sum_{\nu=1}^\infty g_0^{2\nu} \omega_\nu,$$

with

$$\omega_1 = \langle \theta_1 \rangle_0,$$

$$\omega_2 = \langle \theta_2 \rangle_0 - \langle \theta_1 A_1 \rangle_0^6,$$

$$\omega_3 = \langle \theta_3 \rangle_0 - \langle \theta_2 A_1 \rangle_0^6 - \langle \theta_1 A_2 \rangle_0^6 + \frac{1}{2} \langle \theta_1 A_1^2 \rangle_0^6,$$

where $\langle \ldots \rangle^c$ denote connected parts.

The interaction terms in the total action have the same form as in the previous section apart from the integration range which is now $\Lambda$, and the volume factors $V_D$ in the expressions for $A_{2,1}^{(a)}, A_{2,2}^{(a)}, A_{2,2}^{(b)}$ should be replaced by $|\Lambda| = 2T V_D$, $V_D = L^{D-1} L^{D-d}$.

---

8Computation of higher coefficients $c_i^{(r)}$ $i > 1$ can serve as useful checks since these are fixed by the requirement of exponentiation.
The computation now proceeds as in [6], and here we only give the final results. In lowest order
\[
\bar{c}^{(1)}_1 = -\frac{2N_1}{N}, \quad \bar{c}^{(1)}_0 = -\left(\frac{2N_1}{N}\right) R(0),
\]
where \(R(0)\) is dimensionally regularized [6]. So for the energy shift, first computed by Leutwyler [4],
\[
\triangle^{(1)} = \frac{2N_1}{N}. \quad (4.22)
\]
In the next order
\[
\bar{c}^{(2)}_1 = \frac{2N_1(N_1 - 1)}{N^2} R(0), \quad \bar{c}^{(2)}_0 = \frac{N_1(N_1 - 1)}{N^2} R(0)^2,
\]
yielding
\[
\triangle^{(2)} = 2N_1 R(0). \quad (4.25)
\]
Finally at third order we obtain
\[
\bar{c}^{(3)}_1 = \frac{N_1}{N^3} \left[ N^4 + 2N^2 - 4 \right] R(0)^2 - 2N_1 N \left( W + \frac{3}{8} Y \right) + \bar{c}^{(3;13)}_1,
\]
where
\[
W = -\int_{-\infty}^{\infty} dz_0 \int |z| R(z)^2 \partial_R R(z),
\]
and
\[
Y = \frac{1}{V_D} \sum_{p \neq 0} \frac{1}{P^2}. \quad (4.28)
\]
The term \(\bar{c}^{(3;13)}_1\) appearing in (4.26) is the contribution to the correlator from the 4-derivative terms:
\[
\bar{c}^{(3;13)}_1 = -\frac{8N_1}{N^2} \left[ 2(N^2 - 1)G_4^{(0)} + 2NG_4^{(1)} + N^3G_4^{(2)} + (N^2 - 2)G_4^{(3)} \right] \bar{R}(0). \quad (4.29)
\]
The 3rd order energy shift is given by:
\[
\triangle^{(3)} = N_1 N \left[ 2W + \frac{3}{4} Y + R(0)^2 \right] + \frac{8N_1}{N^2} \left[ 2(N^2 - 1)G_4^{(0)} + 2NG_4^{(1)} + N^3G_4^{(2)} + (N^2 - 2)G_4^{(3)} \right] \bar{R}(0). \quad (4.30)
\]
Defining the moment of inertia \(\Theta\) through\(^9\)
\[
m_1 = \frac{N_1}{N\Theta}, \quad (4.31)
\]
\(^9\)For \(N = 2\) this is consistent with the standard definition of \(\Theta\) for O(4).
then
\[ \frac{\Theta}{V_D} = \frac{1}{g_0^2} \left[ 1 + \Theta_1 g_0^2 + \Theta_2 g_0^4 + \ldots \right], \]
(4.32)

with
\[ \Theta_1 = -NR(0), \]
(4.33)
\[ \Theta_2 = -\frac{N}{2N_1} \triangle^{(3)} + N^2 R(0)^2 \]
(4.34)
\[ = -N^2 \left( W + \frac{3}{8} Y - \frac{1}{2} R(0)^2 \right) \]
\[ - \frac{4}{N} \left[ 2(N^2 - 1) G_4^{(0)} + 2NG_4^{(1)} + N^3 G_4^{(2)} + (N^2 - 2) G_4^{(3)} \right] \ddot{R}(0). \]
(4.35)

For \( O(n) \) we had [6] for \( d = 4 \):
\[ m_1 = \frac{(n - 1)}{2\Theta}, \]
(4.36)

with
\[ \frac{\Theta}{F^2 L^3} = 1 + \Theta_1 (FL)^{-2} + \Theta_2 (FL)^{-4} + \ldots \]
(4.37)

and
\[ \Theta_1 = -(n - 2)L^2 R(0), \]
(4.38)
\[ \Theta_2 = (n - 2)L^4 \left[ -2W + R(0)^2 - \frac{3}{4} Y \right] + 4 (2l_1 + nl_2) \ddot{R}(0). \]
(4.39)

We can check (for \( d = 4 \)) using (2.18) (and recalling \( l_1 = -g_4^{(2)}/4, l_2 = -g_4^{(3)}/4 \)) and setting \( F^2 = 1/g_0^2 \) that
\[ L^2 [\Theta_1]_{N=2} = [\Theta_1]_{n=4}, \]
(4.40)
\[ L^4 [\Theta_2]_{N=2} = [\Theta_2]_{n=4}. \]
(4.41)

4.1 Renormalization of the mass gap in \( d = 4 \)

The mass gap does not lead to a new renormalization condition beyond (3.66) required by the free energy considered in this paper. As discussed in [11], the reason is that they are closely related: knowing \( \Theta \) determines the mass spectrum of Hamiltonian states and these determine the free energy.

In [14] we find
\[ W = \frac{5}{24\pi^2} \ddot{R}(0) \left[ \frac{1}{D - 4} - \ln L \right] + \frac{c_w}{L^4}, \]
(4.42)

with
\[ c_w = 0.0986829798. \]
(4.43)

Further
\[ -L^2 R(0) = -\beta_1^{(3)} = 0.2257849594, \]
(4.44)
and \( L^4 \tilde{R}(0) \) can be expressed through the known shape coefficients as\(^{10}\)

\[
-\frac{L^4 \tilde{R}(0)}{4!} \equiv \rho = 8\pi^2 \beta_2^{(3)}(1) - \frac{1}{2} \alpha_2^{(3)}(1) + \frac{3}{4} = \lim_{\ell \to \infty} \left[ \frac{1}{2} \left( \frac{\gamma_1^{(4)}(\ell)}{4} - \frac{1}{2} \right) + \frac{1}{\ell} \right]
\]

\[
= 0.8375369107 .
\]

Also

\[
Y = L^{-3} T_{10}^{(3)} = L^{-3} \left[ \frac{C_1}{L_{\mathrm{HL}}} \right]_A^{(d=3)} = -\beta_1^{(3)} L^{-4} .
\]

After introducing the renormalized couplings (3.64) one obtains

\[
L^2 \Theta_1 = \mathcal{N} \beta_1^{(3)} ,
\]

\[
L^4 \Theta_2 = \frac{1}{2} \mathcal{N}^2 \left[ \beta_1^{(3)} \left( \beta_1^{(3)} + 3 \right) \right] - 2c_w - \frac{5\mathcal{N}^2 \rho}{24\pi^2} \log(c LM)
\]

\[
- \frac{16\rho}{\mathcal{N}} \left[ 2(N^2 - 1)L_0^4 + 2NL_1^4 + N^3 L_2^4 + (N^2 - 2)L_3^4 \right] , \quad (N \geq 4) .
\]

Note that the combination of the renormalized couplings is the same as one of the combinations appearing in (3.57). Again for \( N = 3 \) one should omit \( L_0^4 \), while for \( N = 2 \) the couplings \( L_0^4 \) and \( L_1^4 \) should be omitted, and \( L_1^4, L_2^4 \) should be replaced by \( 4L_1^4, \) and \( 4L_2^4 \) respectively (see Appendix F).

Comparing \( \Theta_2 \) with \( R_2 \) in (3.69), using the large-\( \ell \) behavior of the shape coefficients from [11] one finds that

\[
\lim_{\ell \to \infty} \left( R_1 + \frac{N}{6} \ell \right) = \Theta_1 ,
\]

\[
\lim_{\ell \to \infty} R_2 = \Theta_2 .
\]

For the susceptibility calculated in \( \chi \)PT for the long cylinder geometry this gives a remarkably simple result,

\[
L_s^2 \chi = \frac{\Theta}{2L_s^2} - \frac{N L_s}{L_s^2} + \mathcal{O} \left( \frac{1}{L_s^2} \right) .
\]

In the \( O(n) \) model for \( \ell \to \infty \) one obtains \( L_s^2 R_2 = \text{const} (n - 2)(n - 4) \ell^2 + \mathcal{O} (1) \), in contrast to the \( \text{SU}(N) \times \text{SU}(N) \) model. It is interesting to observe that in the cases of \( n = 2 \) and \( n = 4 \) the manifold \( S_{n-1} \) on which the system is moving is a group manifold, \( \text{U}(1) \) and \( \text{SU}(2) \) with symmetries \( \text{U}(1) \times \text{U}(1) \) and \( \text{SU}(2) \), correspondingly. While for general \( O(n) \) the expansion parameter for large \( \ell \) is \( \ell/(F^2 L_s^4) \), in these special cases the expansion parameter seems to be \( 1/(F^2 L_s^4) \) (see eq. (3.6) of ref. [11]).

Eq. (4.50) is obtained assuming \( L_s \ll L \ll F^2 L_s^3 \). This is a high-temperature expansion for the spatially constant modes and at the same time a low-temperature expansion for the \( \mathbf{p} \neq 0 \) modes. The leading term, \( \Theta/2L_s^4 \) is the classical result. The second one is the leading quantum correction; it appears both for \( O(n) \) and for \( \text{SU}(N) \times \text{SU}(N) \), and does not depend on the dynamics. Note that \( L_s^2 \chi \propto \langle T_s^2 \rangle = (C_2)/(N^2 - 1) \) where \( C_2 \) is the quadratic Casimir invariant, hence in the more natural choice \( (N^2 - 1)L_s^2 \chi \) the curvature of the \( \text{SU}(N) \) manifold, \( N(N^2 - 1)/12 \) appears.

\(^{10}\)The expression in the square brackets above converges exponentially fast for \( \ell \to \infty \), already at \( \ell = 4 \) it agrees to 9 digits with the limiting value.
A specific feature of the SU($N$) × SU($N$) case is that in the χPT result (4.50) the \( \propto L_s/\Theta \sim 1/(F^2L_s^2) \) term is absent, i.e. the LEC’s to NNL order are hidden in the first term alone. Related to this observation, there is a strong evidence that in the SU($N$) × SU($N$) rotator approximation (describing the contribution of the spatially constant modes) there are no power-like corrections to the first two terms in (4.50) for general $N$ (cf. [18]). For the SU(2) × SU(2) \( \simeq \text{O}(4) \) case this can be shown analytically; writing

\[
\log(z_0(u)) = \log(\sqrt{\pi}/4) - \frac{3}{2} \log u + u + \phi(u),
\]

from (A.38) of [11] it follows that the correction term \( \phi(u) \) decreases faster than any power of \( u \). In fact it is extremely small already at \( u = 0.1 \); one has \( \phi(0.1) = -5.4 \times 10^{-41} \). For \( N = 3, 4 \) and 5 this was shown numerically [18].

A derivation of the susceptibility from a SU($N$) × SU($N$) rotator (for general $N$) will be presented in a separate paper [18]. Suffice it here to say that in this scenario we have numerically shown absence of power-like corrections for \( N = 3, 4 \) and 5.

The absence of a \( \sim 1/\Theta^2 \) term in the SU(3) × SU(3) rotator approximation does however not necessarily mean that a term \( \mathcal{O}(F^{-4}L_s^{-4}) \) cannot be present in (4.50), since the simple rotator model requires modifications in order to match χPT at higher orders.

In (4.48) the limit \( \ell \to \infty \) is reached exponentially fast, while in (4.49) apart from the exponentially small corrections there are \( \propto 1/\ell \) corrections as well. This gives for the deviation of the susceptibility \( \chi_{\text{rot}} \) calculated for the standard rotator\footnote{with the standard Hamiltonian proportional to the quadratic Casimir invariant $C_2$.} from the χPT result \( \chi \) in the NNL order

\[
F^4L_s^4 \chi - \chi_{\text{rot}} = \frac{16}{N\ell} \left[ (2N^2 - 3)(L_0^1 + L_3^1) + N(N^2 + 1)(L_1^1 + L_2^1) \right]
+ \frac{5N^2}{16\pi^2\ell} \left[ \log(\bar{c}L_sM_\pi) + \frac{1}{2} \alpha_0(3)(1) - \frac{1}{3} \right] + \ldots .
\]  

(4.52)

The omitted terms at this order are vanishing exponentially as \( \ell \to \infty \). The \( 1/\ell \) term given above should come from the distortion of the rotator spectrum in the region of energies \( E \ll 1/L_s \), much below the threshold for the \( p \neq 0 \) modes. In other words, the true rotator Hamiltonian differs from that of the standard rotator in higher order. A similar situation was found in [6] for the case of the O($n$) model. The corresponding correction for the SU($N$) × SU($N$) case is discussed in [18].

Finally we make a few remarks concerning the sensitivity of the observables on the 4-derivative couplings $L_i^1$. The sensitivity of the isospin susceptibility at \( \ell = 1 \) (hypercubic lattice) is obtained from (3.69) (observing that \( \gamma^{(4)}(1) = 0 \))

\[
\frac{\delta L \chi}{\chi} = \frac{1}{F^4L_s^4} \left( 136\delta L_1^1 + 52\delta L_2^1 + 52\delta L_3^1 \right), \quad (N = 3, \, \ell = 1) .
\]

(4.53)

For a long cubic tube, \( \ell \gg 1 \), the sensitivity of the susceptibility and of the mass gap
on $L_i^r$ are
\[
\frac{\delta L^r}{\chi} = \frac{\delta L^r m_1}{m_1} = -\frac{16}{3F^4L_s^4} \rho (6\delta L_1^r + 27\delta L_2^r + 7\delta L_3^r),
\]
\[
= \frac{1}{F^4L_s^4} (-26.8\delta L_1^r - 120.6\delta L_2^r - 31.3\delta L_3^r), \quad (N = 3, \; \ell \gg 1).
\]  
(4.54)

Note that all coefficients change sign as $\ell$ varies from 1 to $\infty$; this feature can be used to select optimal values of $\ell$ for certain purposes e.g. to reduce the influence on the uncertainty of the $L_i^r$’s on determination of $F$.

A SU($N$) Gell-Mann matrices

The $N \times N$ Gell-Mann hermitian $\lambda$–matrices satisfy
\[
\text{tr} \lambda^a = 0, \quad (A.1)
\]
\[
\text{tr} \left( \lambda^a \lambda^b \right) = 2\delta_{ab}, \quad (A.2)
\]
\[
\lambda^a \lambda^b = \frac{2}{N} \delta_{ab} + (d_{abc} + if_{abc}) \lambda^c, \quad (A.3)
\]
where $f_{abc}$ is totally anti-symmetric and $d_{abc}$ is totally symmetric and
\[
\sum_a d_{aac} = 0. \quad (A.4)
\]

Note the identities
\[
f_{abc}f_{cde} + f_{dbc}f_{ace} + f_{ecb}f_{ade} = 0, \quad (A.5)
\]
\[
f_{abc}d_{cde} + f_{dbc}d_{ace} + f_{ecb}d_{ade} = 0, \quad (A.6)
\]
\[
f_{abc}f_{dec} = \frac{2}{N} (\delta_{ad}\delta_{be} - \delta_{ae}\delta_{bd}) + d_{ade}d_{bec} - d_{acc}d_{bcd}, \quad (A.7)
\]
and
\[
f_{abc}f_{dbc} = N \delta_{ad}, \quad (A.8)
\]
\[
d_{abc}d_{dbc} = \frac{(N^2 - 4)}{N} \delta_{ad}. \quad (A.9)
\]

Completeness reads
\[
\sum_a \lambda^a_1 \lambda^a_{kl} = 2\delta_{il}\delta_{jk} - \frac{2}{N}\delta_{ij}\delta_{kl}. \quad (A.10)
\]

From this we immediately get
\[
\sum_a \lambda^a \lambda^a = \frac{2N_1}{N} 1, \quad \sum_a \lambda^a \lambda^b \lambda^a = -\frac{2}{N} \lambda^b, \quad (A.11)
\]
and
\[
\sum_a \text{tr} (\lambda^a A) \text{tr} (\lambda^a B) = 2 \text{tr}(AB) - \frac{2}{N} \text{tr}(A) \text{tr}(B), \quad (A.12)
\]
\[
\sum_a \text{tr} (\lambda^a A \lambda^a B) = 2 \text{tr}(A) \text{tr}(B) - \frac{2}{N} \text{tr}(AB). \quad (A.13)
\]
For $N = 2$, $\lambda^a = \sigma^a$, the Pauli matrices. Also for an SU(2) matrix

$$U = \exp\left( i \sum_{a=1}^{3} v_a \sigma^a \right) = \cos(\sqrt{v^2}) + i \frac{\sin(\sqrt{v^2})}{\sqrt{v^2}} \sum_{a=1}^{3} v_a \sigma^a .$$  \hspace{1cm} (A.14)

Note for $N = 3$ we have the extra identity [16]

$$d_{abc}d_{cde} + d_{bde}d_{ace} + d_{eac}d_{ade} = \frac{1}{3} (\delta_{ab}\delta_{de} + \delta_{ad}\delta_{be} + \delta_{ae}\delta_{bd}) . \hspace{1cm} (A.15)$$

A.1 Group factors appearing in the perturbative computation

$$Z_1 \equiv - \sum_{a,b} \text{tr} \left( \left[ \lambda^a, \lambda^b \right] \left[ \lambda^a, \lambda^b \right] \right) = 8NN_1 , \hspace{1cm} (A.16)$$

$$Z_2 \equiv \text{tr} \left( \left[ \lambda^a, \left[ \lambda^b, \lambda^c \right] \right] \left[ \lambda^a, \left[ \lambda^b, \lambda^c \right] \right] \right) = 32f_{bec}f_{aeg}f_{bad}f_{ced} = 32N^2N_1 , \hspace{1cm} (A.17)$$

$$Z_3 \equiv \text{tr} \left( \left[ \lambda^a, \left[ \lambda^b, \lambda^c \right] \right] \left[ \lambda^c, \left[ \lambda^a, \lambda^b \right] \right] \right) = 32f_{bec}f_{aeg}f_{bad}f_{ced} = 16N^2N_1 , \hspace{1cm} (A.18)$$

$$Z_4 \equiv \text{tr} \left( \left[ \lambda^a, \left[ \lambda^a, \lambda^b \right] \right] \left[ \lambda^c, \left[ \lambda^a, \lambda^b \right] \right] \right) = 32N^2N_1 . \hspace{1cm} (A.19)$$

$$Z_5 \equiv \text{tr} \left( \left[ \lambda^a, \lambda^b \right] \left[ \lambda^c, \left[ \lambda^a, \lambda^b \right] \right] \right) + \left[ \lambda^a, \left[ \lambda^c, \lambda^b \right] \right] = -Z_2 - Z_3 - Z_4 = -80N^2N_1 . \hspace{1cm} (A.20)$$

$$Z_6 \equiv \text{tr} \left( \left[ \lambda^a, \lambda^b \right] \left[ \lambda^c, \lambda^d \right] \right) \text{tr} \left( \left[ \lambda^a, \lambda^b \right] \left[ \lambda^c, \lambda^d \right] \right) = 64f_{abe}f_{cde}f_{abc}f_{ced} = 64N^2N_1 , \hspace{1cm} (A.21)$$

$$Z_7 \equiv \text{tr} \left( \left[ \lambda^a, \lambda^b \right] \left[ \lambda^c, \lambda^d \right] \right) \text{tr} \left( \left[ \lambda^a, \lambda^c \right] \left[ \lambda^b, \lambda^d \right] \right) = 64f_{abe}f_{cde}f_{abc}f_{ced} = 32N^2N_1 , \hspace{1cm} (A.22)$$

$$Z_8 \equiv \sum_{a,b,c} \text{tr} \left( \lambda^a\lambda^b\lambda^c\lambda^d \right) = \frac{8N_1(N^2+1)}{N^2} . \hspace{1cm} (A.23)$$

B Proof of eq. (2.9)

We start from the trivial identity

$$\text{tr}(ABAB) = \frac{1}{2} \text{tr}((A,B)^2) - \text{tr}(A^2B^2) . \hspace{1cm} (B.1)$$

Let $A, B$ be traceless SU($N$) matrices $A = A_a \lambda^a, B = B_a \lambda^a$; we have

$$\text{tr}(A^2B^2) = A_a A_b B^b B^d \left[ \frac{4}{N} \delta_{ab}\delta_{cd} + 2d_{abe}d_{cde} \right] , \hspace{1cm} (B.2)$$

$$\text{tr}(ABAB) = -\text{tr}(A^2B^2) + 2A_a B_b A^b B^d \left[ \frac{4}{N} \delta_{ab}\delta_{cd} + 2d_{abe}d_{cde} \right] . \hspace{1cm} (B.3)$$
So

\[ \text{tr}(ABAB) + 2 \text{tr}(A^2B^2) = 2 \left[ 2A^a B^b A^c B^d + A^a A^b B^c B^d \right] \left[ \frac{2}{N} \delta_{ab} \delta_{cd} + \delta_{ade} \delta_{cde} \right] \] (B.4)

\[ = 2A^a A^b B^c B^d \left[ d_{abe} d_{cde} + d_{ace} d_{bde} + d_{ade} d_{bce} + \frac{2}{N} (\delta_{ab} \delta_{cd} + \delta_{ac} \delta_{bd} + \delta_{ad} \delta_{bc}) \right]. \] (B.5)

For SU(3) we have using (A.15)

\[ \text{tr}(ABAB) + 2 \text{tr}(A^2B^2) = 2A^a A^b B^c B^d (\delta_{ab} \delta_{cd} + \delta_{ac} \delta_{bd} + \delta_{ad} \delta_{bc}) \] (B.6)

\[ = \frac{1}{2} \text{tr}(A^2) \text{tr}(B^2) + \text{tr}^2(AB), \quad N = 3. \] (B.7)

Now consider \( U(x) \) slowly varying in \( x \). Define \( U(x) = U(0)V(x) \); then close to \( x = 0 \), \( V(x) \) is close to the identity matrix

\[ V(x) = 1 + ix \mu A_\mu + \ldots \] (B.8)

where \( A_\mu \) are traceless hermitian matrices. Then the \( \mathcal{L}_i \) at \( x = 0 \) can be computed by replacing \( \partial_\mu U \) by \( iU(0)A_\mu \). The factors involving \( U(0) \) cancel and then (2.9) follows using (B.6).

C Redundancy of \( \text{tr} (\square U^\dagger \square U) \)

For the SU(2) case the 4-derivative operator

\[ \mathcal{L}_2^{(X)} = \text{tr} \left( \square U^\dagger \square U \right) \] (C.1)

corresponding to \( (\square S \cdot \square S) \) in O(4), turns out to be redundant: it can be eliminated by changing the integration variable \( U(x) \) in the path integral. Below we show that this remains true for general SU\((N)\).

Consider the change of variables

\[ U \to U e^{\alpha F} = U \left( 1 + \alpha F + \mathcal{O} (\alpha^2) \right) \] (C.2)

where \( F \) is a traceless anti-hermitian matrix.

By choosing

\[ F = \frac{1}{2} \left( U^\dagger \square U - \square U^\dagger U \right), \] (C.3)

one has

\[ U \to U + \alpha UF = U + \frac{\alpha}{2} \left( \square U - U \square U^\dagger U \right) + \mathcal{O} (\alpha^2). \] (C.4)

For the SU(2) case this corresponds to the change of variables

\[ S \to S + \alpha \left[ \square S - S(S \cdot \square S) \right] + \mathcal{O} (\alpha^2), \] (C.5)

which is the transformation used to show the redundancy of the operator \( \text{tr}(\square S \square S) \).
We have still to show that $F$ is indeed traceless. One has
\[ \text{tr} F = \frac{1}{2} \text{tr} \left( U^\dagger \Box U - \Box U^\dagger U \right) = \text{Im} \, \text{tr} \left( U^\dagger \Box U \right). \] (C.6)

Further we can write
\[ \text{Im} \, \text{tr} \left( U^\dagger (x) \Box U(x) \right) \bigg|_{x=0} = \text{Im} \, \text{tr} \left( \Box \left( U^\dagger (0) U(x) \right) \right) \bigg|_{x=0}. \] (C.7)

One has
\[ W(x) \equiv U^\dagger (0) U(x) = \exp \left( i x^\mu A_\mu + \frac{1}{2} i x^\mu x_\nu B_{\mu \nu} + \mathcal{O} \left( x^3 \right) \right), \] (C.8)

where $A_\mu$ and $B_{\mu \nu}$ are traceless hermitian matrices. From this it follows that
\[ \text{Im} \, \Box W(x) \bigg|_{x=0} = \text{Im} \, \text{tr} \left( - A_\mu A_\mu + i B_{\mu \mu} \right) = 0. \] (C.9)

Therefore we can conclude that the operator $\text{tr}(\Box U^\dagger \Box U)$ can be transformed away by a field redefinition.

A similar discussion to that presented above has been given by Leutwyler in eq. (11.6) and the following paragraph of his article [12].

D Faddeev–Popov trick for the zero modes

Consider the SU($N$) partition function formally given by
\[ Z = \int \left[ \prod_x dU(x) \right] e^{-A(U)}. \] (D.1)

We parameterize $U$ as in (2.19). The integral over the constant $u$ factors out for this consideration. The action and measure are invariant under global SU($N$) transformations
\[ U(x) \to VU(x), \] (D.2)

which induces a change
\[ \xi_a(x) \to \xi_a^V(x). \] (D.3)

Define a la Faddeev–Popov $\Phi[\xi]$ through the integral
\[ 1 = \Phi[\xi] \int \text{d}V \prod_a \delta \left( \int_x \xi_a^V(x) \right). \] (D.4)

Now the action, measure and also $\Phi[\xi]$ are invariant under SU($N$) transformations, so inserting 1 in the form of the rhs of (D.4) in the partition function we obtain
\[ Z = \int \text{d}V \int \prod_x \left[ \frac{d\xi(x)}{M[\xi(x)]} \right] e^{-A(\xi)} \Phi[\xi] \prod_{a=1}^{N^2-1} \delta \left( \int_x \xi_a(x) \right). \] (D.5)

The group volume is an irrelevant factor. Also for DR we set $M[\xi(x)] = 1, \forall x$. 

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Now we only need $\Phi[\xi]$ near the surface $\int_x \xi(x) = 0$ and we can consider an infinitesimal transformation
\[ V = 1 + i\alpha_a \lambda^a + \mathcal{O}(\alpha^2). \] (D.6)

This induces a change
\[ g_0\xi(x)\lambda^a = g_0\xi(x)\lambda^a + \alpha_a t^a(x) + \mathcal{O}(\alpha^2), \] (D.7)

with $t(x)$ obtained by solving (the argument $x$ understood)
\[ (1 + i\alpha_a \lambda^a + \mathcal{O}(\alpha^2)) e^{ig_0\xi(x)\lambda^b} = e^{ig_0\xi(x)\lambda^a + i\alpha_a t^a(x)} + \mathcal{O}(\alpha^2), \] (D.8)

thereby yielding
\[ t^a = \lambda^a + g_0 \left( \frac{i}{2} [\lambda^a, \xi] - g_0^2 \frac{1}{12} [\xi, [\xi, \lambda^a]] \right) \]
\[ - g_0^4 \frac{1}{720} [\xi, [\xi, [\xi, [\xi, \lambda^a]]]] + \ldots \] (D.9)

with
\[ T_{ab} = \frac{1}{2} \text{tr} \left( \lambda^b t^a \right) \]
\[ = \delta_{ab} + g_0 f_{abc} \xi_c - g_0^2 \frac{1}{24} \text{tr} \left( \lambda^b [\xi, [\xi, \lambda^a]] \right) \]
\[ - g_0^4 \frac{1}{1440} \text{tr} \left( \lambda^b [\xi, [\xi, [\xi, [\xi, \lambda^a]]]] \right) + \ldots \] (D.10)

So
\[ \Phi^{-1}[\xi] = \int \prod_a \left( \text{d}\alpha_a \delta(\alpha_b T_{ab}[\xi]) \right) \]
\[ = (\text{det} T[\xi])^{-1}, \] (D.13)

with (setting $\int_x \xi_a(x) = 0$)
\[ \overline{T}_{ab}[\xi] = \delta_{ab} - g_0^2 T^{(1)}_{ab} - g_0^4 T^{(2)}_{ab} + \ldots \] (D.15)

where
\[ T^{(1)}_{ab} = \frac{1}{24 V_D} \int_x \text{tr} \left( \lambda^b [\xi(x), [\xi(x), \lambda^a]] \right), \] (D.16)
\[ T^{(2)}_{ab} = \frac{1}{1440 V_D} \int_x \text{tr} \left( \lambda^b [\xi(x), [\xi(x), [\xi(x), [\xi(x), \lambda^a]]]] \right). \] (D.17)

The zero mode action is then given by
\[ A_{\text{zero}} = - \ln \Phi[\xi] \]
\[ = - \text{tr} \ln (T[\xi]) \]
\[ = -(N^2 - 1) \ln V_D + g_0^2 \text{tr} T^{(1)} + g_0^4 \text{tr} \left( T^{(2)} + \frac{1}{2} T^{(1)} T^{(1)} \right) + \ldots \] (D.18)

We have in particular
\[ \text{tr} \overline{T}^{(1)} = \frac{N}{3 V_D} \int_x \sum_a \xi_a(x)^2. \] (D.21)
E Some integrals over SU($N$) matrices

Integrals with Haar measure over SU($N$) matrices $u$ (see e.g. [17]):

$$
\int du = 1. \tag{E.1}
$$

$$
\int du u_{ij}(u^\dagger)_{kl} = \frac{1}{N} \delta_{ij} \delta_{jk}. \tag{E.2}
$$

It follows

$$
\int du \text{tr}(uAu^\dagger B) = \frac{1}{N} \text{tr}(A) \text{tr}(B), \tag{E.3}
$$

$$
\int du \text{tr}(uA) \text{tr}(u^\dagger B) = \frac{1}{N} \text{tr}(AB). \tag{E.4}
$$

$$
\int du u_{i_1j_1} \ldots u_{i_Nj_N} = \frac{1}{N!} \epsilon_{i_1 \ldots i_N} \epsilon_{j_1 \ldots j_N}, \tag{E.5}
$$

where $\epsilon_{i_1 \ldots i_N}$ is the totally antisymmetric tensor with $\epsilon_{1 \ldots N} = 1$.

Note for $N = 2$ and an SU(2) matrix $V$:

$$
\epsilon_{ik} \epsilon_{jl} V_{kl} = V^*_{ij} = \left(V^\dagger\right)_{ji}, \tag{E.6}
$$

(the conjugate of the fundamental representation is equivalent to the fundamental representation for SU(2)). So

$$
\int du \text{tr}(uA) \text{tr}(uB) = \frac{1}{2} \text{tr}(AB^\dagger), \quad N = 2. \tag{E.7}
$$

F Expressions involving the 4-derivative terms

The 4-derivative terms $B^{(i)}_4$, $C^{(i)}_4$ appearing in (3.4) are given by:

$$
B^{(0)}_4 = -i \int_x \text{tr} \left( \left[ \lambda^3, U^\dagger(x) \right] \partial_\mu U(x) \partial_\nu U^\dagger(x) \partial_\mu U(x) \right) + \text{h.c.} \tag{F.1}
$$

$$
C^{(0)}_4 = \frac{1}{4} \int_x \text{tr} \left\{ \partial_\mu U^\dagger(x) \left[ \lambda^3, U(x) \right] \partial_\nu U^\dagger(x) \right\} \left[ \lambda^3, U(x) \right] \left[ \lambda^3, U^\dagger(x) \right] + \partial_\mu U(x) \partial_\nu U^\dagger(x) \partial_\mu U(x) \left[ \lambda^3, U^\dagger(x) \right] \left[ \lambda^3, U(x) \right] + \text{h.c.}. \tag{F.2}
$$

$$
B^{(1)}_4 = -i \int_x \text{tr} \left( \partial_\mu U^\dagger(x) \left[ \lambda^3, U(x) \right] \right) \text{tr} \left( \partial_\nu U^\dagger(x) \partial_\mu U(x) \right) + \text{h.c.}, \tag{F.3}
$$

$$
C^{(1)}_4 = \frac{1}{4} \int_x \left\{ \text{tr} \left( \left[ \lambda^3, U^\dagger(x) \right] \left[ \lambda^3, U(x) \right] \right) \text{tr} \left( \partial_\mu U^\dagger(x) \partial_\mu U(x) \right) \right\} + \text{h.c.}. \tag{F.4}
$$
Using (A.12), (A.13), (E.3) and (E.4) the average \[ C^{(0)}_4 \] in (3.49) is given by
\[
C^{(0)}_4 = \frac{1}{4} \int_x \int du \left\{ \partial_\mu \overline{U}^\dagger(x) u^\dagger \left[ \lambda^\mu, u \overline{U}(x) \right] \partial_\mu U^\dagger(x) u^\dagger \left[ \lambda^\mu, u \overline{U}(x) \right] + u \partial_\mu \overline{U}(x) \partial_\mu U^\dagger(x) u^\dagger \left[ \lambda^\mu, u \overline{U}(x) \right] \right\} \right\} + \text{h.c.}
\] (F.9)

Similarly for \[ C^{(i)}_4, i = 1, 2, 3 \] one obtains
\[
C^{(1)}_4 = -2 \int_x \left\{ N_1 \text{tr} \left( \partial_\mu \overline{U}(x) \partial_\mu U^\dagger(x) \right) + 2 \text{tr} \left( \partial_\mu \overline{U}(x) \partial_\mu U^\dagger(x) \right) \right\},
\] (F.12)
\[
C^{(2)}_4 = -2 \int_x \left\{ N^2 \text{tr} \left( \partial_\mu \overline{U}(x) \partial_\mu U^\dagger(x) \right) + \text{tr} \left( \partial_\mu \overline{U}(x) \partial_\mu U^\dagger(x) \right) \right\},
\] (F.13)
\[
C^{(3)}_4 = -\frac{2}{N} \int_x \left\{ (N^2 - 2) \text{tr} \left( \partial_\mu \overline{U}(x) \partial_\mu U^\dagger(x) \right) + N_1 \text{tr} \left( \partial_\mu \overline{U}(x) \partial_\mu U^\dagger(x) \right) \right\},
\] (F.14)
G Some relations for the O(4) couplings

Some relations between different conventions for the O(4) couplings to connect with those used in ref. [6]

\[ l_i = l_i^r + \frac{w_i}{16\pi^2} \left( \frac{1}{D - 4} + \log(\tau M) \right) \]  

where \( w_1 = n/2 - 5/3, \ w_2 = 2/3 \) and choosing the scale \( \mu = M \), the mass of the charged pion.

Further

\[ l_i = \frac{w_i}{16\pi^2} \left( \frac{1}{D - 4} + \log(\tau \Lambda_i) \right) . \]  

From here

\[ l_i^r = \frac{w_i}{16\pi^2} \log (\Lambda_i/M) = \frac{w_i}{32\pi^2} \gamma_i \]  

since \( \gamma_i = \log (\Lambda_i^2/M^2) \).

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