OPEN GROMOV-WITTEN THEORY ON CALABI-YAU MANIFOLDS AND SYMPLECTIC CUTTING

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ABSTRACT. In this paper we will study moduli space of J-holomorphic discs in an almost Calabi-Yau $X$ of real dimension $2n$ with boundary on Lagrangian submanifolds which are either diffeomorphic to $S^n$ or $\mathbb{R}P^n$. Our main tool will be symplectic-cut technique. As result will prove rationality of numbers defined in [FO-C] and non-displacability of Lagrangian spheres in dimension bigger than two. In the case that the Lagrangian $L$ is in the fixed point set of some anti-symplectic involution, we show that if $L$ is diffeomorphic to $S^3$ then all open invariants defined using the symmetry are zero and if $L$ is diffeomorphic to $\mathbb{R}P^3$, then there is some relation between open invariants of odd classes in $X$ and closed invariants of another almost Calabi-Yau 3-fold $X_{cut}$ constructed from $X$.

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Date: January 2010.
1. Introduction

Over the following two decades, mathematicians including Ruan-Tian [RT], McDuff-Salamon [MS1], Li-Tian [LT] and Fukaya-Ono [FO], successfully established a rigorous foundation of Gromov-Witten invariants for symplectic manifolds. Existence of open Gromov-Witten invariants counting pseudoholomorphic maps in a symplectic manifold with boundary on a Lagrangian manifold, have predicted by physicists [F1, F2, F3, F4]. These invariants would naturally generalize classical Gromov-Witten invariants to include maps from Riemann surfaces with boundary. But any attempt to use techniques used in classical case directly in this case fails due to existence of boundary for moduli space of discs (or generally any domain with boundary).

Let \((X, w, L, J)\) be a tuple consisting of a symplectic manifold \(X\) with symplectic 2-form \(w\), an embedded Lagrangian submanifold \(L\) and an almost complex structure \(J\) compatible with \(w\) and \(L\). Also consider a 2nd relative homology class \(\beta \in H_2(X, L)\) with \(w(\beta) > 0\) and maslov index \(\mu(\beta)\). We then define \(M_{k,l}(X, L, J, \beta)\) to be stable compactification of moduli space of all maps \(u : D^2 \to X\) with \(\partial u : \partial D^2 \to L\) and \(\bar{\partial}_J u = 0\), having \(k\) marked point on boundary and \(l\) marked point in the interior of \(D^2\). These moduli spaces have boundary coming from two possible type of degenerations:

1. Type (I) : A \(J\)-holomorphic disc might bubble off at a boundary point in limit, when considering a sequence of \(J\)-holomorphic discs.
2. Type (II) : In the case that \(\beta\) is in the image of map \(i : H_2(X) \to X_2(X, L)\), the boundary may shrink to a point in the limit.

GW invariants are obtained by integration on moduli spaces, so due to the existence of boundary for these moduli spaces, the calculation in open case will depend on actual differential forms and also on \(w, J, \cdots\) and so we can’t get invariants in this way.

Also existence of boundary is an obstruction for defining Flore homology of Lagrangian intersections. This problem is studied in the book [FOOO] and also the papers [FO1, FO2, FO3]. We also have orientation problem in open case. While moduli spaces of closed \(J\)-holomorphic curves have a canonical orientation coming from almost complex structure, an orientation in open case might not exist and if it exists it depends on orientation and a relative spin or pin structure on \(L\). See [FOOO] chapter 8 for orientation coming from relative spin structure and Jake’s thesis [Jake] for orientation coming from relative pin-structure.

There has been some attempts to define open GW-invariants by various people including Liu [L], Welschinger, Jake Solomon and Fukaya. In [W1, W2, W3] Welschinger defined invariants using real curves in complex algebraic varieties. Jake Solomon [Jake] extended this idea by considering Lagrangians which are fixed points of an anti-symplectomorphic involution. Using the involution and after studying orientation problem he identified different boundary components to make a space without boundary and integrate on it. Later on and in a collaboration with R.Pandhripandeh and Walcher [JRW] they calculated some open GW invariants relative to fixed point of some involution in Quinitic, using localization techniques. In [FO-C] Fukaya defined some numbers relative to spherical Lagrangians in Calabi-Yau 3-folds. These numbers are real and depend on choice of almost complex
structure via a wall-crossing formula. In this paper and as a corollary of Theorem 1.1, we’ll prove the rationality of these numbers conjectured there. His works can be extended to $\mathbb{R}P^3$ case but it will produce real numbers again, unlike in [JRW, Jake] where they get rational numbers.

The story is by now well understood, in the case that the oriented Lagrangian 3-fold $L$ is in the fix point set of some anti-symplectic involution $\phi$. Lets define $\Pi_2(X, L)$ to be quotient of $H_2(X, L)$ with respect to equivalence relation

$$\beta_1 \sim \beta_2 \iff w(\beta_1) = w(\beta_2) \text{ and } \mu(\beta_1) = \mu(\beta_2)$$

It is shown in [Jake] section 4 and also in [FO3] section 6.3, that as long as we restrict ourself to almost complex structures compatible with $\phi$, we can define some open invariants, invariant under symmetric deformations, virtually counting pseudo holomorphic discs in class $\beta \in \Pi_2(X, L)$. We will call these invariants $N_{sym}^\beta$ where "sym" means we are in symmetric case.

In this paper we will study moduli space of $J$-holomorphic discs on almost Calabi-Yau manifolds using symplectic cut-sum technique, where by an almost Calabi-Yau, I mean a symplectic manifold with trivial canonical bundle. In general case where there is not necessarily an anti symplectic involution we will prove:

**Theorem 1.1.** Consider an almost Calabi-Yau manifold $(X, w)$ and an embedded Lagrangian submanifold $L \subset X$. Assume that $L$ is diffeomorphic to $S^n$ and $n > 2$. Then for any positive real number $E > 0$ there is an open set $U_E$ in set of all almost complex structures compatible with $w$ and $L$, say $J_{comp}$, such that for any $J \in U_E$, all moduli spaces $M(X, L, J, \beta)$ with $w(\beta) < E$ are empty, simultaneously!

**Corollary 1.1.** Invariants defined in [FO-C] for Lagrangian spheres in Calabi-Yau 3-folds are rational.

Infact starting from an almost complex structure given above we can show that invariants defined there for an arbitrary almost complex structure comes from wall-crossing (degenerations of type (II) above) and so are rational.

**Corollary 1.2.** Under conditions of theorem 1.1, the Lagrangian is not displaceable and infact for any $E > 0$ there exist $J_E$ with respect to which:

$$HF(L, J_E)^{mod \sim T_E} H^*(L)$$

We then try to use same ideas with some modification to prove the same results in symmetric case and we get:

**Theorem 1.2.** If $X$ is an almost Calabi-Yau 3-fold admitting an anti-symplectic involution $\phi$ and $L \subset Fix(\phi)$ is a Lagrangian sphere, then all open Gromov-Witten invariants $N_{sym}^\beta$, $\beta \in \Pi_2(X, L)$, are zero!

This shows that there is no interesting open GW theory with respect to Lagrangian spheres in Calabi-Yau 3-folds and in a similar manner one can get same results for higher dimensions.

But the story is totally different in the case that Lagrangian is diffeomorphic to real projective space $\mathbb{R}P^3$ and I believe that after some consideration which will
be discussed later and using some kind of sum-formula one can prove following conjecture:

**Conjecture 1.1.** Let $X$ be an almost Calabi-Yau 3-fold and $L \cong \mathbb{R}P^3 \subset X$ be an embedded Lagrangian. Then for odd classes $\beta \in \Pi_2(X, L)$, the open GW invariants $N_{\beta}^{sym}$ can be expressed as a function of ordinary GW invariants of another almost Calabi-Yau $X_{out}$, constructed from $X$, using the homology classes $F_{\beta', k} \in H_2(X_{out}, \mathbb{Z})$ constructed from $\beta'$.

Also considering one of the following situations:

- $L \cong S^2$ and $X$ is a $K3$ surface or
- $L \cong \mathbb{R}P^3$ and $X$ is Calabi-Yau 3-fold.

Then something interesting happens. From $X$ and after a surgery given in 2.2 we can construct another symplectic manifold $X_{out}$ with $c_1^{X_{out}} = 0$, replacing Lagrangian with a symplectic divisor. So it enables us to produce lots of almost Calabi-Yau 3-folds.

As we discuss the symmetric case we will observe that we can produce many examples of symplectic manifolds with an anti-symplectic involution without fixed point.

The main idea in whole paper is as follows:

We start by a triple $(X, w, L)$ in general and $(X, w, L, \phi)$ in the case there is an anti-symplectic involution and then from that and using symplectic cut in a neighborhood of $L$ corresponding to some intrinsic Hamiltonian $S^1$ action, we construct a singular symplectic manifold $X_0 = X_{in} \cup_D X_{out}$ which is made of two smooth symplectic manifolds intersecting along a common symplectic divisor. Then using symplectic sum, we construct a fiberation $\pi : Z \rightarrow \Delta$ over unit disk in $\mathbb{C}$ which has central fiber equal to $X_0$, and smooth symplectic fibers isotopic to $X$ elsewhere. We then try to relate the modules on smooth fiber to modules in $X_0$. The main observation would be the fact that $X_{out}$ has positive canonical bundle and so it will help us to prove some emptiness theorems.

The organization of paper is as follows. We first review the definition of symplectic cut-sum surgeries and apply it to the special case we need, in section 2. In section 3 we investigate the effect of this surgery on the topology of manifold, which can be skipped in first time reading. We then use the material of chapter 2 and a little bit of chapter 3 in chapter 4 to prove the theorems stated above. Chapter 5 is devoted to give a better picture of some algebraic terms used to define invariants in $\mathbb{FOC}$ and will not be used anywhere else in this paper. We finish in chapter 6 by some remarks and questions regarding this surgery. And finally the appendix gives a quick reference to topics around Lagrangians with vanishing maslov class, which are one of the main ingredients of this paper.

1.1. **Acknowledgements.** I would like to thank my advisor, professor Gang Tian for suggesting to me this problem and for his continuous encouragement and support. In addition, I would like to thank K.Fukaya, J.Solomon, R.Pandripandeh, D.McDuff for many helpful conversations.
In this section first we review the symplectic-cut and symplectic-sum surgeries. We then consider the situation that we have a symplectic manifold \((X, w)\) and a Lagrangian \(L \subset X\) which is a homological sphere and from this we construct a singular symplectic manifold \(X_0\) made of two symplectic manifolds \((X_{in}, w_{in})\) and \((X_{out}, w_{out})\) intersecting along a symplectic divisor \(D\) whose normal bundle in \(X_{in}\) and \(X_{out}\) are dual to each other. In next sections we will use \(X_0\) to study the moduli space of \(J\)-holomorphic discs in \(X\).

2. Symplectic Cut-Sum

2.1. Review. Symplectic-cut is a surgery technique on symplectic manifolds, which using that we can decompose a given manifold into two pieces. There is an inverse operation, the symplectic-sum, that glues two manifolds together into one. The symplectic cut was introduced in 1995 by Eugene Lerman [LL] as a generalization of symplectic-blowup [MS2], who used it to study the symplectic quotient and other operations on manifolds.

Let \((X^{2n}, w)\) be a symplectic manifold and \(V^{2n-1} \subset X\) be a smooth submanifold of \(X\) and suppose we have a Hamiltonian \(S^1\) action on some open neighborhood \(U\) of \(V\) with moment map \(h : U \to \mathbb{R}\). Now assume \(a \in \mathbb{R}\) is a regular value for \(h\) and \(V\) is equal to \(V_a = h^{-1}(a)\) in \(X\). Then \(V_a\) is invariant under \(S^1\) action, so we can construct the quotient space \(D^{2n-2}_a = V_a / S^1\) which inherits a symplectic structure from \(X\).

Now again suppose we are in the previous situation and lets decompose \(X\) into two pieces with equal boundaries \(V_a\), say \(X_{h \geq a}\) and \(X_{h \leq a}\). What we want to do is to construct two smooth symplectic manifolds \((X_{out}, w_{out})\) and \((X_{in}, w_{in})\) from above pieces by contracting two boundaries with respect to \(S^1\) action as we did above such that \(D_a\) gives a symplectic divisor in each of them and also such that open set \(X_{out} \setminus D_a\) (resp. \(X_{in} \setminus D_a\)) is symplectomorphic to open set \(X_{h > a}\) (resp. \(X_{h < a}\)). We proceed as follows:

Consider the symplectic manifold \(X \times \mathbb{C}\) with symplectic structure \(w \oplus w_0\) where \(w_0\) is standard symplectic structure on \(\mathbb{C}\). Also consider open set \(U' = U \times \mathbb{C}\) and extend the \(S^1\) action on \(U\) to an \(S^1\) action on \(U'\) which is equal to multiplication by \(e^{i\theta}\) on \(\mathbb{C}\) part. Again this is a Hamiltonian action with moment map \(h_{out} = h - \frac{1}{2}|z|^2\) and \(a\) is a regular value of \(h'\). So we can consider \(V'_{a} = h'^{-1}(a)\) and its quotient \(h'^{-1}(a)/S^1\) which inherits a symplectic structure from \(X \times \mathbb{C}\) and we will call it \(X_{out}'\). Similarly we’ll define \(X_{in}'\) considering moment map \(h + \frac{1}{2}|z|^2\). Then the subset \(V'_{a} \cap (X \times \{0\}) \subset V'_{a}\) gives a copy of \(D_a\) in \(X_{out}'\). We can also prove that the normal bundle of \(D_a\) in \(X_{out}'\) is dual to normal bundle of \(D_a\) in \(X_{in}'\).

Symplectic-sum is reverse of this procedure obtained by gluing to symplectic manifold along a common symplectic divisor. We start with two symplectic manifolds \(X_{in}\) and \(X_{out}\) and two co-dimension 2 symplectic submanifolds \(D_{in}\) and \(D_{out}\) in them symplectomorphic to each other such that the symplectic normal bundles of \(D_D = D_{in} = D_{out}\) in \(X_{in}\) and \(X_{out}\) are dual to each other. We then construct a family of symplectic manifolds \(\pi : \mathcal{Z} \to \Delta\) depending on a complex parameter \(\lambda \in \Delta\), where \(\Delta\) is a neighborhood of 0 \(\in \mathbb{C}\) with following properties:

1. The total space \(\mathcal{Z}\) is a smooth symplectic manifold (a symplectic fibration)
(2) The Fiber over $0$, $X_0$ is singular and is a normal crossing composed of two components $X_{in}$ and $X_{out}$ having intersection along their common divisor $D$.

(3) Over $\Delta - \{0\}$ the fibers are smooth symplectic manifolds $X_\lambda$ symplectically isotopic to one another; each is a model of the symplectic sum. (we write $X_\lambda = X_{in} \#_D X_{out}$).

(4) If we perform symplectic-cut on a symplectic manifold $X$ and then consider the symplectic sum of resulting manifolds; we get a fibration where each smooth fiber is symplectically isotopic with the starting symplectic manifold $X$ and so in this way we can consider the symplectic-sum as a reverse of symplectic-cut.

See [IP2] section 2 for details of construction. Construction of symplectic sum is a little bit harder than symplectic cut, but it has an easier description in the following situation which we need through the paper:

If we start from a symplectic manifold $X$ and perform symplectic cut along a hypersurface $V$ as above, we get a singular symplectic manifold composed of two symplectic manifolds $X_{in}$ and $X_{out}$ for which we can do symplectic sum and construct the family $\mathcal{Z}$ containing $X_0 = X_{in} \cup_D X_{out}$ as a central fiber. Although the construction of symplectic sum in general is a little bit hard but the composition of these two surgeries is easier to describe and in what follows we will show how one can construct $\mathcal{Z}$ from $X$ directly:

Let’s again come back to the situation where we have a symplectic manifold $(X, w)$ and a Hamiltonian $S^1$ action on an open neighborhood $U$ of hypersurface $V \subset X$ with moment map $h : U \to \mathbb{R}$, where $V = h^{-1}(a)$ for some regular value $a \in \mathbb{R}$. We may assume that $a = 0$, $U = h^{-1}((-1/2, 1/2))$, and all values in $(-1/2, 1/2)$ are regular value for $h$. We will construct the symplectic fibration $\mathcal{Z}$ from three patches. $X \setminus V$ is made of two open sets. Lets $U_{out}$ be the one which is in the positive side of $h$ and $U_{in}$ be the one which is in the negative side of $h$. Then three patches which we need are:

1. $End_{in} = U_{in} \times \Delta_\epsilon$, where $\Delta_\epsilon$ is disk of radius $\epsilon$ around $0 \in \mathbb{C}$.
2. $End_{out} = U_{out} \times \Delta_\epsilon$.
3. and we construct the third one (The neck) as follows:

Consider $U \times \Delta_1 \times \Delta_1$ with symplectic form $w \oplus w_0 \oplus w_0$ and $S^1$ action given on it by:

$e^{i\theta}(x, z_1, z_2) = (e^{i\theta} \cdot x, e^{i\theta} z_1, e^{-i\theta} z_2)$

with moment map $\mu(x, z_1, z_2) = h(x) - \frac{|z_1|^2}{2} + \frac{|z_2|^2}{2}$.

There is a complex valued function on $U \times \Delta_1 \times \Delta_1$, invariant under $S^1$ action, given by:

$\lambda(x, z_1, z_2) = z_1 \cdot z_2$

Again zero is a regular value for moment map $\mu$ and so we can consider the quotient:

$neck = \mu^{-1}(0)/S^1$
which is an open symplectic manifold, inheriting it’s symplectic structure from $U \times \Delta \times \Delta$. Since $\lambda$ is $S^1$ invariant it induces a function, still called $\lambda$, on neck. Lets $\rho$ denote the projection from $\mu^{-1}(0)$ to neck. Having all these three patches in hand we construct $Z$ as follows.

To construct $Z$, we glue three open sets $\text{End}_{\text{in}}, \text{End}_{\text{out}}$ and neck using following maps:

$$\psi_{\text{out}} : \text{End}_{\text{out}} \cap (U \times \Delta) \to \text{neck}$$

is given by:

$$\psi_{\text{out}}(x, re^{i\theta}) = \rho(x, \sqrt{h(x) + \sqrt{h(x)^2 + |r|^2}}, \frac{re^{i\theta}}{\sqrt{h(x) + \sqrt{h(x)^2 + |r|^2}}}$$

and

$$\psi_{\text{in}} : \text{End}_{\text{in}} \cap (U \times \Delta) \to \text{neck}$$

is given by:

$$\psi_{\text{in}}(x, re^{i\theta}) = \rho(x, \frac{-re^{i\theta}}{\sqrt{-h(x) + \sqrt{h(x)^2 + |r|^2}}, \sqrt{-h(x) + \sqrt{h(x)^2 + |r|^2}}}$$

It is easy to see that $\lambda$ induces a function, still called $\lambda$ on whole $Z$. Also it is easy to see that $\lambda^{-1}(0)$ is nothing but the singular manifold $X_0 = X_{\text{in}} \cup X_{\text{out}}$ we had before and for non zero numbers $\alpha \in \Delta$, the manifolds $\lambda^{-1}(\alpha)$ are all diffeomorphic to $X$. We just need to put a symplectic structure on $Z$ in a correct way. From the equation of $\psi_{\text{out}}$ and the fact that :

$$\rho^*(w_{\text{neck}}) = (w_X \oplus w_0 \oplus w_0) \big|_{\mu^{-1}(0)}$$

we get :

$$\psi_{\text{out}}^* w_{\text{neck}} = w_X + \frac{1}{2} d\left(\frac{r^2}{g} d\theta\right)$$

where

$$g(x, r) = h(x) + \sqrt{h(x)^2 + |r|^2}$$

Note that the image of $\psi_{\text{out}}$ and $\psi_{\text{in}}$ have no overlap since the first component comes from disjoint sets in $X$. Now for $\delta$ small enough and for $x \in U$ with $h(x) \in \left(\frac{1}{2} - \delta, \frac{1}{2}\right)$, $g(x)$ is very close to one and so $d\left(\frac{r^2}{g} d\theta\right)$ is very close to $d(r^2 d\theta)$ and actually using a cut of function $\beta = \beta(h(x))$, we can merge $d\left(\frac{r^2}{g} d\theta\right)$ into $d(r^2 d\theta)$ and so we can merge $\psi_{\text{out}}^* w_{\text{neck}}$ into $w_{\text{End}_{\text{out}}}$, to get a symplectic form which is equal to $w_{\text{End}_{\text{out}}}$ on $\text{End}_{\text{out}} \setminus U \times \Delta$, and is equal to $w_{\text{neck}}$ for small values of $h$. We can do the same thing for $\text{End}_{\text{in}}$ and so in this way we get a symplectic form $w_Z$ on $Z$ with required properties.
There is one more thing which I would like to mention at this point. Suppose that in previous situation, we have one more data which is an anti-symplectic involution \( \phi \) on \( X \) compatible with the \( S^1 \) action in following sense:

1. \( \phi \) maps each level set of \( h \) to itself.
2. \( \phi \circ e^{i\theta} = e^{-i\theta} \circ \phi \)

Then we can extend this anti-symplectic involution to anti-symplectic involutions on \( \text{End}_{in}, \text{End}_{out}, U \times \Delta_c \times \Delta_c \) and the neck by considering complex conjugation on each disk \( \Delta \) and then it is easy to see it induces an anti-symplectic involution \( \phi_Z \) on \( Z \) with following properties:

1. \( \phi_Z^* w_Z = -w_Z \).
2. \( \phi_Z \) maps fiber over \( \lambda \) to fiber over \( \bar{\lambda} \).
3. \( \phi_Z \) maps \( X_0, X_{in} \) and \( X_{out} \) and so \( D_0 \) to themselves, and \( \phi_Z |_{X_{out}} \) has no fixed point.

We will need this result for the proof of theorem 1.2 and discussion of conjecture 1.1 in section 4.

We end up this section by two remarks about symplectic cut and sum techniques.

Remark 2.1. Symplectic cut-sum technique is the main ingredient for defining relative GW invariants and the sum formula relating relative invariants of \( X_{in} \) and \( X_{out} \) to the ordinary GW invariants of \( X \), where one can obtain GW-invariants of \( X \) from relative GW-invariants of \( X_{in} \) and \( X_{out} \) via some complicated sum-formula. Relative GW invariants and the sum-formula from the point of view of symplectic geometry are developed in two papers of Ionel-Parker [IP1] [IP2].

Remark 2.2. Let \( SM \) be the category of all smooth symplectic manifolds of various dimension, whose objects are smooth symplectic manifolds and morphisms are symplectomorphisms. \( SM \) is a monoid under disjoint union and let \( SM^+ \) be the free abelian group generated by \( SM \). Now consider the subset \( RS \subset SM^+ \) of all double point relations:

\[ [X] - [X_{out}] - [X_{in}] + [\mathbb{P}(N_{X_{out/in}}^D)] \]

where \( \mathbb{P}(N_{X_{out/in}}^D) \) is the projectivization of normal bundle of \( D \) in \( X_{out} \) or \( X_{in} \) (they are equal). It’s conjectured in [LR] that \( SM^+/RS \) can be generated by set of projective spaces.

2.2. Surgery near Lagrangian submanifolds. In this subsection we apply methods of section 2.1 to a special case and using that we construct a surgery which given a symplectic manifold \( (X, w) \) and a homological sphere Lagrangian \( L \) in it, gives another symplectic manifold \( X_{out} \) which is equal to changing the Lagrangian in \( X \) by a symplectic divisor.

Suppose we have a symplectic manifold \( (X^{2n}, w) \) and a smooth Lagrangian submanifold \( L \) of \( X \), then by a theorem of Weinstein [MS2] we know that a neighborhood of \( L \) in \( X \) say \( U_L \) is symplectomorphic to a neighborhood of \( L_0 \subset T^*L \), considering \( T^*L \) with its canonical symplectic form \( w_L \) and \( L_0 \) to be it’s zero section.
From now on we restrict our self to those Lagrangians which are either $S^n$ or $\mathbb{R}P^n$. Consider the pair $(L, g)$ where $g$ is the round metric on $L$. It induces a metric on vector bundle $T^*L$ and I’ll define the function $h : T^*L \to \mathbb{R}$ to be the length function of this metric, which is a smooth function outside $L_0$. So restricting to $T^*L \setminus L_0$ we can consider it’s differential form $dh$ and corresponding Hamiltonian vector field $dh := -i_{X_h} w_L$. We then consider the Hamiltonian flow of $h$ given by vector field $X_h$ say $\phi_h(t) : T^*L \setminus L_0 \to T^*L \setminus L_0$ for which we have following theorem.

**Theorem 2.1.** The flow of $X_h$ gives an Hamiltonian $S^1$ action on $T^*L \setminus L_0$ with following properties:

1. Trajectories of this flow are identical to co-geodesic flow after a reparametrization
2. Each level set $V_a = h^{-1}(a)$ for $a > 0 \in \mathbb{R}$ is invariant under flow (and so $S^1$ action) and is a contact sum manifold of $T^*L$ with contact form $\lambda |_{V_a}$ where $\lambda$ is the canonical one form on $T^*L$ with $d\lambda = -w_L$.

**Proof.** Renormalize $g$ such that all geodesics have length one. Let $x = (x_1, ..., x_n)$ be a set of local coordinates on an open set $U \subset L$. Then we get local coordinates $(x, y) = (x_1, ..., x_n; y_1, ..., y_n)$ on $T^*L$, where $y_i$’s are coefficients for a differential 1-form $\alpha$ in the basis given by $dx_i$’s:

$$\alpha = \sum_i y_i dx_i$$

In this coordinate we have:

$$w_L = \sum_i dx_i \wedge dy_i$$

and

$$\lambda = \sum_i y_i dx_i$$

Consider two functions $H$ and $h$, given in this local coordinate by $H = \frac{1}{2} \sum_i g^{ij} y_i y_j$ and $h = \sqrt{2H}$. The Hamiltonian flow of $H$ is simply the co-geodesic flow and since $dh = \frac{dH}{dH}$ we get $X_h = \frac{X_H}{\sqrt{2H}}$ and flow lines of $\phi_H(t)$ are identical to flow lines of $\phi_h(t)$ up to a reparametrization by some factor depending on trajectory. If $\gamma$ is a trajectory of $\phi_H(t)$ starting at the point $(x, y)$ with $\|y\| = c$ then $\gamma$ has period $\frac{c}{\sqrt{2H}}$ and so considering $\gamma$ as a trajectory of $\phi_h(t)$, it would have period $\frac{c}{\sqrt{2H}} \cdot \frac{\|X_H\|}{\|X_h\|} = \frac{1}{2} \times c = 1$. So the flow of $\phi_h(t)$ is periodic with a constant period equal to one and so gives a Hamiltonian $S^1$ action. \hfill \square

We will use the result of previous theorem and that of subsection 2.1 in the following way:

Consider the triple $(X^{2n}, w, L)$ as above and a neighborhood $UL$ of $L$ in $X$ and identify it with a neighborhood $U'$ of $L_0 \subset T^*L$, for which we have moment map $h$ defined on $U' \setminus L_0$ equal to length function as above. Suppose for all numbers $a < C$ the level sets $V_a = h^{-1}(a)$ are contained in $U'$. Then considering the symplectic-cut construction on symplectic manifold $X$ with respect to contact hypersurface $V_a$, $a < C$, and Hamiltonian $S^1$ action with moment map $h$ in a neighborhood of $V_a$, we get following theorems:
**Theorem 2.2.** Suppose $L$ is diffeomorphic to $S^n$. After doing symplectic-cut along $V_a$ with respect to Hamiltonian $S^1$ action with moment map $h$, we obtain two symplectic manifolds $(X_{in}, w_{in})$ and $(X_{out}, w_{out})$ with following properties:

1. There is a copy of symplectic manifold $D_a = V_a/S^1$ in each of them such that symplectic normal bundles of $D_a$ in each copy is dual to the other one.

2. $(X_{in}; D_a; w_{in})$ is symplectomorphic to $(Q^n; Q^{n-1}; \delta_a \cdot w_{FS})$ where $Q^n$ is Quadratic hypersurface in $\mathbb{C}P^{n+1}$ given by equation:
   
   \[ X_0^n - (\sum_{i=1}^{n+1} X_i^2) = 0 \] in $\mathbb{C}P^{n+1}$, and $Q^{n-1} = Q^n \cap (X_0 = 0)$.
   
   $w_{FS}$ is the Fubini-Study metric of $\mathbb{C}P^n$ restricted to $Q^n$ and $\delta_a$ is some constant depending on parameter $a$.

3. If $X$ is a Calabi-Yau n-fold , then $c_1^{X_{out}} = -(n-2) \cdot \text{PoincareDual}([D_a])$

**Theorem 2.3.** Suppose $L$ is diffeomorphic to $\mathbb{R}P^n$. After doing symplectic-cut along $V_a$ with respect to Hamiltonian $S^1$ action with moment map $h$, we obtain two symplectic manifolds $(X_{in}, w_{in})$ and $(X_{out}, w_{out})$ with following properties:

1. There is a copy of symplectic manifold $D_a = V_a/S^1$ in each of them such that symplectic normal bundles of $D_a$ in each copy is dual to the other one.

2. $(X_{in}; D_a; w_{in})$ is symplectomorphic to $(\mathbb{C}P^n; Q^{n-1}; \delta_a \cdot w_{FS})$ where $Q^{n-1}$ is Quadratic hypersurface in $\mathbb{C}P^n$ given by equation:
   
   \[ \sum_{i=1}^{n} X_i^2 = 0 \] in $\mathbb{C}P^n$, $w_{FS}$ is the Fubini-Study metric of $\mathbb{C}P^n$ and $\delta_a$ is some constant depending on parameter $a$.

3. If $X$ is a Calabi-Yau n-fold, $n > 1$, then $c_1^{X_{out}} = -\frac{n-3}{2} \cdot \text{PoincareDual}([D_a])$

**Remark 2.3.** Under above procedure we get a copy of lagrangian $L$ inside $X_{in}$ disjoint from $D_a$.

**Remark 2.4.** From 2.2.3 and 2.3.3 we get:

(a) If $X$ is a $K_3$ surface and $L \cong S^2$ is a Lagrangian in it , then after surgery we get another almost Calabi-Yau symplectic manifold ($c_1^{X_{out}} = 0$).

(b) If $X$ is a Calabi-Yau 3-fold and $L \cong \mathbb{R}P^3$ is a Lagrangian in it , then after surgery we get another almost Calabi-Yau 3-fold. It would be an interesting question to find whether $X_{out}$ has a Kahler structure or not.

**Proof.** The first two parts of these theorems are proved in Michle Audin [Audin 4.4]. We just give a proof for the statement on the first chern class of tangent bundle of $X_{out}$:

proof of 2.2.3: We have $H_2(Q^n, S^n, \mathbb{Z}) \cong H_2(Q^n, \mathbb{Z}) = \mathbb{Z}$. Let's call it's generator $\beta$. Then $\beta$ has intersection number one with $Q^{n-1}$ in an interior point. Now consider a Homology Class $\alpha \in H_2(X_{out}, \mathbb{Z})$ and lets $A$ be a representative of $\alpha$ intersecting $D_a \cong Q^{n-1}$ transversally at $s$ points. There is a representative of...
class $s\beta$ in $H_2(Q^n, S^n)$, say $B$ intersecting $Q^{n-1}$ with same intersection pattern. Considering symplectic sum-construction introduced in section 2.1 we can glue two Homology classes $A$ and $B$ to get a homology class $\delta = A \# B \in H_2(X, L)$ as follows: first cut a neighborhood of intersection points in $A$ and $B$ to get cycles with boundaries say $A'$ and $B'$, then push them to a regular fiber $X_\lambda$ for small $\lambda \in \mathbb{C}$ and then glue the boundary circles of $A'$ and $B'$ corresponding to each intersection point by small cylinders (c.f [IP1] section 5). The maslov index of $\delta$ is zero by our assumption. We also have following formula relating maslov index of $\delta$ in $X$ and those of $A$ and $B$ in $X_{out}$ and $X_{in}$ respectively (c.f [IP1] Lemma 2.2 for its proof) where maslov index($A$) by definition is twice its chern number:

$$0 = \mu_X(\delta) = \mu_{X_{out}}(A) + \mu_{X_{in}}(B) - 4 \cdot s$$

$$= 2c_1^{X_{out}}(A) + 2n \cdot s - 4s \tag{1}$$

Here we use the fact that $\mu_{X_{in}}(B) = 2c_1^{Q^n}(B) = 2n \cdot s$.

So we get $c_1^{X_{out}}(A) = -(n - 2) \cdot s$, which gives the desired result.

2.3.3 can be proved in a similar way. □

The factor $\frac{1}{2}$ in theorem 2.3.3 seems to be weird because we know that first chern class is an integral cohomology class. Actually in next section we will prove that there is no compact almost Calabi-Yau n-fold $X$ with first betti number zero, containing an embedded Lagrangian $\mathbb{R}P^n$, when $n$ is even. This is a good news because manifolds $\mathbb{R}P^{2k}$ are not orientable and it is harder to put orientation on moduli space of $J$-Holomorphic discs with non orientable Lagrangian boundary condition. In non orientable case one needs to use pin structure instead of spin structure.

Remark 2.5. In proof of theorem 1.2 and also in conjecture 1.1 we work with those triples $(X, w, L)$, where the Lagrangian is in fixed point set of some anti-symplectic involution $\phi$. If in a neighborhood of Lagrangian, $\phi$ be compatible with $S^1$ action in the following sense :

1. $e^{i\theta} \circ \phi = \phi \circ e^{-i\theta}$
2. $\phi$ maps each level set of moment map to itself.

then as we saw in the construction of symplectic sum in section 2, $\phi$ induces an anti-symplectic involution on the fiberation $Z$ obtained from $X$ and $L$, with following set of properties:

1. $\phi^*Z = -w_z$.
2. $\phi_Z$ maps fiber over $\lambda$ to fiber over $\lambda$.
3. $\phi_Z$ maps $X_0$, $X_{in}$ and $X_{out}$ and so $D_\alpha$ to themselves.
4. For real $\lambda$, $\phi_Z$ maps the fiber to itself fixing the Lagrangian.
5. $\phi_Z$ restricted to $X_{out}$ has no fixed point.

But in general $\phi$ might not be compatible with $S^1$ action and we can not expect to get an anti-symplectic involution on $Z$. In the next theorem we prove that given an arbitrary anti-symplectic involution $\phi$, then there exist a Hamiltonian isotopy
deforming it to another one which is compatible with action. We will later use this theorem and its consequence on \( Z \) mentioned above in proof of theorem 1.2 and in the formulation of conjecture 1.1.

**Theorem 2.4.** Suppose \( L \subseteq X \) is a lagrangian submanifold of a symplectic manifold with \( H^1(L) \cong 0 \), and suppose \( L \) is (a component of) the fixed point set of some anti symplectic involution \( \phi \) on \( X \). Then there is a Hamiltonian isotopy \( \psi_t \) on \( X \), \( t \in [0, 1] \), with following properties:

- \( \psi_t \) is equal to identity outside some neighborhood of \( L \).
- \( \psi_0 \) is identity on whole \( X \).
- \( \psi_t \circ \phi \) is an anti symplectic involution with same fix points as \( \phi \).
- \( \psi_1 \circ \phi \) is equal to standard anti symplectic involution on \( T^*L \) in some neighborhood of \( L \) where the standard anti-symplectic involution on \( T^*L \) is one given by sending any vector \( v^* \in T^*L \) to \( -v^* \).

**Proof.** By Weisstein theorem, consider a neighborhood \( UL \) of \( L \) in \( T^*L \) symplectomorphic to a neighborhood of \( L \) in \( X \) and consider \( \phi \) as a map from \( UL \) to \( T^*L \). Let \( \tau \) be the standard symplectic involution on \( T^*L \) given by \( (x, v^*) \rightarrow (x, -v^*) \) for \( (x, v^*) \in T^*_x L \). So \( \psi = \tau \circ \phi : UL \rightarrow T^*L \) is a symplectic map fixing \( L \). For \( (x, 0) \in T^*L \) we know that \( D\phi : T_{(x, 0)}T^*L \rightarrow T_{(x, 0)}T^*L \) is an anti-symplectomorphic involution and also we have the decomposition \( T_{(x, 0)}T^*L \cong T^*_x L \oplus T_x L \), so we will see that the linearization of \( \phi \) along \( L \) is of the form:

\[
D\phi(x) = \begin{pmatrix} I & A \\ 0 & -I \end{pmatrix}
\]

where \( A : T^*_x L \rightarrow T_x L \) is a linear map satisfying:

\[
v_1^*(A(v_2^*)) = v_2^*(A(v_1^*)) \quad \forall v_1^*, v_2^* \in T^*_x L
\]

We also get:

\[
D\psi(x) = \begin{pmatrix} I & A \\ 0 & I \end{pmatrix}
\]

We prove the theorem in two steps: In first step we will try to construct a Hamiltonian isotopy localized near \( L \) changing \( \phi \) to another anti-symplectic involution \( \tilde{\phi} \) whose linearization along \( L \) is equal to that of \( \tau \). At the second step we will construct another Hamiltonian isotopy localized near \( L \) changing \( \tilde{\phi} \) to \( \tau \) itself.

In both steps we first construct a local Hamiltonian isotopy and then using following lemma we change it to a global one localized near \( L \).

**Lemma 2.1.** A Hamiltonian isotopy \( \psi_t \) satisfies:

\[
(\phi \circ \psi_t) \circ (\phi \circ \psi_t) = \text{id} \quad \text{and} \quad \psi_t \text{ fixes } L
\]

iff the corresponding Hamiltonian function can be chosen in a way, that it satisfies:

\[
H_t \circ \delta_t = H_t \quad \text{and} \quad H_t |_L = 0
\]

where \( \delta_t = \psi_t \circ \phi \)

Note that then \( \delta_t \) is an anti symplectic involution.
Proof. If \( \psi_t \) is a Hamiltonian isotopy with Hamiltonina function \( H_t \) then \( \psi_t^{-1} \) is a Hamiltonian isotopy with Hamiltonian \(-H_t \circ \psi_t\). Also Let \( \varphi \) be a symplectomorphism and \( \phi \) be an anti-symplectomorphism then \( \varphi^{-1} \circ \psi_t \circ \varphi \) and \( \phi^{-1} \circ \psi_t \circ \phi \) are Hamiltonian isotopies with Hamiltonian funtion \( H_t \circ \varphi \) and \(-H_t \circ \phi \) respectively. If \( \psi_t \) and \( \varphi_t \) are two Hamiltonian isotopies with Hamiltonian functions \( H_t \) and \( G_t \) then \( \psi_t \circ \varphi_t \) is also a Hamiltonian isotopy with Hamiltonian function \( H_t + G_t \circ \psi_t^{-1} \).

(see section 10 of [MS2] for proofs)

So by calculating Hamiltonian functions of both sides of

\[
\phi \circ \psi_t \circ \phi = \psi_t^{-1}
\]

we see:

\[
(\phi \circ \psi_t) \circ (\phi \circ \psi_t) = \text{id} \quad \text{and} \quad \psi_t \text{ fixes } L \quad \Leftrightarrow \quad \phi \circ \psi_t \circ \phi = \psi_t^{-1} \quad \text{and} \quad \psi_t \text{ fixes } L
\]

and so we get the desired result. \( \square \)

We come back to proof of the theorem:

Step 1: Consider the function

\[
H_A(x, v^*) = \frac{\alpha}{2} v^*(A(v^*))
\]

defined on \( T^*L \) where \( \alpha \) is some constant and define \( H = H_A + H_A \circ \phi \). Let \( \varphi_t \) be the Hamiltonian flow of \( H_A \) then Hamiltonian flow of \( H \) is given by

\[
\psi_t = \varphi_t \circ \phi \circ \varphi_t^{-1} \circ \phi
\]

and define \( \phi_t \) to be \( \phi \circ \psi_t \). Then we have \( H = H \circ \phi = H \circ \psi_t \) because \( H \) is symmetric with respect to \( \phi \) and is constant along flow of \( \psi_t \). So conditions of previous lemma are satisfied and \( \phi_t \) is an anti-symplectic involution. Since \( \psi_t \) and \( \varphi_t \) fix \( L \) so their linearization for each time \( t \) gives a linear map at each point of \( L \), given by the equation:

\[
D\varphi_1(x, 0) = \begin{pmatrix} I & \alpha A \\ 0 & I \end{pmatrix}
\]

and because

\[
D\psi_1(x, 0) = D\phi_1 \circ D\phi \circ D\varphi_1^{-1} \circ D\phi
\]

we get:

\[
D\phi_1(x, 0) = \begin{pmatrix} I & 2\alpha A \\ 0 & -I \end{pmatrix}
\]

Putting \( \alpha = \frac{1}{2} \) we get the desired result.

For moving to global picture we modify \( H \) to \( H_t \) such that \( H_t \) in some neighborhood of \( L \) is equal to \( H \) and outside some neighborhood is zero and also \( H_t \circ \psi_t \circ \phi = H_t \) everywhere. From these we see that the Hamiltonian flow of \( H_t \) is localized along \( L \) and deforms \( \phi \) to anti-symplectic involution \( \phi \) such that linearization of \( \phi \) along \( L \) is equal to linearization of \( \tau \).

Step 2: Lets \( \psi \) be the symplectomorphism defined in a neighborhood of \( L \) by \( \psi = \phi \circ \tau \). Then \( \psi \) is equal to identity outside a neighborhood of \( L \), fixes \( L \) and its linearization along \( L \) is equal to identity.

We first construct a symplectic isotopy \( \psi_t : UL \rightarrow T^*L \) with following properties:

1. \( \psi_t \) fixes \( L \).
2. \( \psi_t \circ \tau \) is an anti symplectic involution.
3. \( \psi_0 = \text{identity} \) and \( \psi_1 = \psi \).
We define $\psi_t$ in following way: Let $S_r : UL \to T^*L$ be the map given by $(x, v^*) \to (x, rv^*)$ for constant number $r > 0 \in \mathbb{R}$. Define $\psi_t := R_t \circ \psi \circ R_t$ (for left hand side to be well defined one might need to reduce a little bit to a smaller domain). One can easily check that it has required properties. Note that because $D(x)\psi = id$ for each $x \in L$ we get $\psi_t \to id$ as $t \to 0$.

Now because $H^3(L) = 0$ we know that this isotopy is a Hamiltonian isotopy given by some Hamiltonian function $H_t$.

We know finish the proof of theorem by moving to global picture as follows:

1. From lemma 2.1 we know that $H_t \circ \delta_t = H_t$ for $\delta_t = \psi_t \circ \tau$.
2. Consider a sequence of neighborhoods $V_1 \subset V_2 \subset UL$ and modify $H_t$ to another Hamiltonian function $G_t$ such that $G_t$ is equal to $H_t$ in some compact neighborhood of $L$ inside $V_1$. It is equal to zero outside $V_2$ and $G_t \circ \delta_t = G_t$ everywhere.
3. Let $\psi_t$ be the Hamiltonian isotopy given by $G_t$ and consider $\phi_t := \psi_t^{-1} \circ \tilde{\tau}$. Then $\phi_t$ is equal to standard anti-symplectic involution $\tau$ in a neighborhood of $L$ and has the required properties, i.e. it can be obtained from $\phi$ via a Hamiltonian isotopy.

3. Effect of surgery on topology of 3-folds

We devote this section to understand how does the topology of the symplectic manifold $X$ changes when we perform the surgery given in previous section near a homological sphere Lagrangian. We will restrict ourself to Lagrangians of dimension three. The results of this section are not used in the proof of theorems given in the rest of the paper and so one can skip this section in the first time reading.

In this section we will have following assumptions on the triple $(X, w, L)$:

$(X, w)$ will be simply connected compact symplectic manifold of real dimension 6, unless mentioned. $L \subset X$ is an embedded smooth Lagrangian diffeomorphic to either $\mathbb{R}P^3$ or $S^3$.

3.1. 3-dimensional sphere Lagrangians. Consider the hypersurface $Q^3$ in $\mathbb{C}P^4$ given by equation

$$\left\{-Z_0^2 + Z_1^2 + Z_2^2 + Z_3^2 + Z_4^2 = 0 \mid [Z_0, ..., Z_4] \in \mathbb{C}P^4\right\}$$

then the set of real points of this equation is a $S^3$ Lagrangian

$$\left\{ l = [1, x_1, ..., x_4] \mid x_i \in \mathbb{R} \text{ and } \sum x_i^2 = 1 \right\}.$$  

and we have a copy of $Q^2$ inside that given by the set of points $Z_0 = 0$, which does not intersect with $S^3$. Let’s consider a neighborhood of $S^3 \subset Q^3$ and a contact hypersurface $V_a$ as in section 2.3. Cutting along this hypersurface we get two sets with boundaries equal to $V_a$, say $UL$ which contains $S^3$ and $UQ^2$ which contains $Q^2$. Then topologically the construction of $X_{out}$ from $X$ can be described by removing a neighborhood $UL$ of $L$ in $X$ similar to that of $S^3$ above with boundary $V_a$ and filling it’s place with a copy of $UQ^2$ by matching corresponding points along the boundary. First let’s look at the topology of $Q^3$, $Q^2$ and $V_a$:

1. For $Q^3$ we have: It has the same Betti numbers as $\mathbb{C}P^3$ and $H_2(Q^3, \mathbb{Z})$ is generated by class of a line in $\mathbb{C}P^4$.  

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(2) For \( Q^2 \) we have: \( Q^2 \cong \mathbb{C}P^1 \times \mathbb{C}P^1 \) and its second homology class is generated by class of two set of lines on \( Q^2 \) say \([l_0]\) and \([l_1]\) with \( l_0 \cdot l_1 = 1 \) and \( l_i \cdot l_i = 0 \).

(3) For \( V_a \) we have: using the fact that tangent bundle of \( S^3 \) is trivial or using the Gysin Sequence for \( S^1 \) bundle \( \pi : V_a \to Q^2 \) we get \( H_2(V_a, \mathbb{Z}) = \mathbb{Z} \) and is generated by class of \([l_0] - [l_1]; H_3(V_a, \mathbb{Z}) = \mathbb{Z} \) generated by a lift of \( S^3 \) to \( V_a \).

Now lets write the Mayer-Vitroies sequence for decomposition \( X = UL \cup (X \setminus L) \). \( UL \) retracts to \( L \), \( UL \cap (X \setminus L) \) retracts to \( V_a \) and if we set \( A := (X \setminus L) \) we get:

\[
0 \to H_6(V_a, \mathbb{Z}) \xrightarrow{(i_6,j_6)} H_6(S^3, \mathbb{Z}) \oplus H_6(A, \mathbb{Z}) \xrightarrow{k_6\cdot i_6} H_6(X, \mathbb{Z})
\]

\[
\delta_5 : H_5(V_a, \mathbb{Z}) \xrightarrow{(i_5,j_5)} H_5(S^3, \mathbb{Z}) \oplus H_5(A, \mathbb{Z}) \xrightarrow{k_5\cdot i_5} H_5(X, \mathbb{Z})
\]

\[
\delta_4 : H_4(V_a, \mathbb{Z}) \xrightarrow{(i_4,j_4)} H_4(S^3, \mathbb{Z}) \oplus H_4(A, \mathbb{Z}) \xrightarrow{k_4\cdot i_4} H_4(X, \mathbb{Z})
\]

\[
\delta_3 : H_3(V_a, \mathbb{Z}) \xrightarrow{(i_3,j_3)} H_3(S^3, \mathbb{Z}) \oplus H_3(A, \mathbb{Z}) \xrightarrow{k_3\cdot i_3} H_3(X, \mathbb{Z})
\]

\[
\delta_2 : H_2(V_a, \mathbb{Z}) \xrightarrow{(i_2,j_2)} H_2(S^3, \mathbb{Z}) \oplus H_2(A, \mathbb{Z}) \xrightarrow{k_2\cdot i_2} H_2(X, \mathbb{Z})
\]

\[
\delta_1 : H_1(V_a, \mathbb{Z}) \xrightarrow{(i_1,j_1)} H_1(S^3, \mathbb{Z}) \oplus H_1(A, \mathbb{Z}) \xrightarrow{k_1\cdot i_1} H_1(X, \mathbb{Z}) \to 0
\]

Assume we know: \( H_2(X, \mathbb{Z}) \cong H_3(X, \mathbb{Z}) \cong \mathbb{Z}^a \) and \( H_3(X, \mathbb{Z}) = \mathbb{Z}^b \) which in Calabi-Yau case it means it has following Hodge-Diamond:

\[
\begin{array}{cccc}
0 & 0 & 0 & 0 \\
0 & a & 0 & 0 \\
1 & c & a & 1 \\
0 & a & 0 & 0 \\
0 & 0 & 0 & 0 \\
1 & & & \\
\end{array}
\]

where \( b = 2 + 2c \). Also assume that \( L \) is non-trivial in \( H_3(X) \). Considering these assumptions we calculate Homology groups of \( A \). Since \( \delta_5 \) is surjective, and after substitution of known elements in first two rows of \((4)\) we get: \( H_6(A, \mathbb{Z}) \cong H_5(A, \mathbb{Z}) \cong 0 \). Also by our assumption \( i_3 \) is surjective, so \((4)\) splits and we get:

\[
0 \to H_4(A, \mathbb{Z}) \xrightarrow{i_4} H_4(X, \mathbb{Z}) \xrightarrow{\delta_4} 0
\]

and

\[
0 \to H_3(A, \mathbb{Z}) \xrightarrow{i_3} H_3(X, \mathbb{Z}) \xrightarrow{\delta_3} H_2(V_a, \mathbb{Z}) \xrightarrow{(j_2)} H_2(A, \mathbb{Z}) \xrightarrow{i_2} H_2(X, \mathbb{Z}) \to 0
\]

which means:

\[
H_4(A, \mathbb{Z}) \cong H_4(X, \mathbb{Z}) \cong \mathbb{Z}^a
\]

and

\[
0 \to H_3(A, \mathbb{Z}) \xrightarrow{i_3} \mathbb{Z}^b \xrightarrow{\delta_2} \mathbb{Z} \xrightarrow{(j_2)} H_2(A, \mathbb{Z}) \xrightarrow{i_2} \mathbb{Z}^a \to 0
\]

To understand the kernel of \((j_2)\) we have to understand image of \( j_2 \). Since \( L \) is non-trivial in \( H_3(X) \) and has 0 intersection by itself there should be another
element in $H_2(X)$ say $L'$ (not necessarily Lagrangian) having non-zero intersection by $L$ say $L \cdot L' = m \neq 0$. This means image of $\delta_2$ is not zero, so $j_2$ can produce at most torsion elements in $H_2(A)$ so we get:

$$H_2(A, \mathbb{Z})_{\text{non-torsion}} \cong H_2(X, \mathbb{Z})$$

$$H_3(A, \mathbb{Z}) \cong \mathbb{Z}^{b-1}. \quad (9)$$

Note that we have a trivial map $H_2(X, \mathbb{Z}) \to H_2(A, \mathbb{Z})$, because by dimension reason any element in $H_2(X, \mathbb{Z})$ has a representative not intersecting $L$.

Now we move to $X_{out}$ and write the Mayer-Vities decomposition for $X_{out} = UQ^2 \cup (X_{out} \setminus Q^2)$. $UQ^2$ retracts to $Q^2$, $(X_{out} \setminus Q^2) \cong A$ and their intersection retracts to $V_a$ so we get:

$$0 \to H_6(V_a, \mathbb{Z}) \xrightarrow{(i_a, j_4)} H_6(S^3, \mathbb{Z}) \oplus H_6(A, \mathbb{Z}) \xrightarrow{k_6-l_6} H_6(X_{out}, \mathbb{Z})$$

$$\delta_2 \xrightarrow{} H_5(V_a, \mathbb{Z}) \xrightarrow{(i_a, j_4)} H_5(Q^2, \mathbb{Z}) \oplus H_5(A, \mathbb{Z}) \xrightarrow{k_5-l_5} H_5(X_{out}, \mathbb{Z})$$

$$\delta_3 \xrightarrow{} H_4(V_a, \mathbb{Z}) \xrightarrow{(i_4, j_2)} H_4(Q^2, \mathbb{Z}) \oplus H_4(A, \mathbb{Z}) \xrightarrow{k_4-l_4} H_4(X_{out}, \mathbb{Z})$$

$$\delta_4 \xrightarrow{} H_3(V_a, \mathbb{Z}) \xrightarrow{(i_3, j_2)} H_3(Q^2, \mathbb{Z}) \oplus H_3(A, \mathbb{Z}) \xrightarrow{k_3-l_3} H_3(X_{out}, \mathbb{Z})$$

$$\delta_5 \xrightarrow{} H_2(V_a, \mathbb{Z}) \xrightarrow{(i_2, j_2)} H_2(Q^2, \mathbb{Z}) \oplus H_2(A, \mathbb{Z}) \xrightarrow{k_2-l_2} H_2(X_{out}, \mathbb{Z}) \to 0 \quad (10)$$

Substituting Known ones in (10) we get:

$$H_5(X_{out}) = 0$$

$$0 \to \mathbb{Z} \oplus \mathbb{Z} \xrightarrow{k_4-l_4} H_4(X_{out}, \mathbb{Z}) \xrightarrow{\delta_3} \mathbb{Z} \xrightarrow{(j_3)} \mathbb{Z}^{b-1} \xrightarrow{-l_3} H_3(X_{out}, \mathbb{Z})$$

$$\delta_2 \xrightarrow{} \mathbb{Z} \xrightarrow{(i_2, j_2)} \mathbb{Z}^2 \oplus \mathbb{Z}^a \oplus \text{torsion} \xrightarrow{k_2-l_2} H_2(X_{out}, \mathbb{Z}) \to 0 \quad (11)$$

From (11) we get two inequalities on rank of $H_2$ and $H_4$:

(1) $\text{rank } H_2(X_{out})_{\text{non-torsion}} \geq a + 1$

(2) $\text{rank } H_4(X_{out})_{\text{non-torsion}} \leq a + 1$

and since they are equal we get:

$$H_2(X_{out})_{\text{non-torsion}} \cong H_4(X_{out})_{\text{non-torsion}} \cong \mathbb{Z}^{a+1} \quad (12)$$

So at least at the level of non-torsion elements we have:

$$H_2(X_{out}, \mathbb{Z})_{nt} \cong H_4(X_{out}, \mathbb{Z})_{nt} \cong \mathbb{Z}^{a+1}$$

$$H_3(X_{out}, \mathbb{Z})_{nt} \cong \mathbb{Z}^{b-2} \quad (13)$$

which means under this surgery the second and forth Betti number increase by one and the 3rd Betti number decrease by two, so if $X_{out}$ has a Kahler structure it’s Hodge diamond will be:
\[
\begin{array}{cccccc}
1 & 0 & 0 \\
0 & a + 1 & 0 \\
1 & c - 1 & c - 1 & 1 \\
0 & a + 1 & 0 \\
0 & 0 \\
1 & 0 & 0 \\
\end{array}
\]

That is because the canonical bundle \( K_{X_{\text{out}}} \) is positive and so \( h^{3,0} \) won’t be zero.

Also analyzing \( H_2(X_{\text{out}}) \) we see:

\( H_2(X_{\text{out}}, \mathbb{Z})^{\text{non-tor}} \cong H_2(X, \mathbb{Z}) \oplus \mathbb{Z} \cdot [l] \)

which means non-torsion part of 2nd homology of \( X_{\text{out}} \) is generated by two type of elements:

1. Those coming from \( X \) which have a representative not intersecting \( Q^2 \subset X_{\text{out}} \)
2. One coming from the class of line (Both Class \([l_0]\) and \([L_1]\) induce same class \([l]\)) with \( [l] \cdot Q^2 = -1 \)

In the case which \( L \) is trivial in 3rd Homology of \( X \) we can do similar calculations but we are only interested in this case.

### 3.2. Real projective space Lagrangians

In this case and going through the same procedure we will get similar results, i.e. the change in Betti numbers will be same and also if there exist a Kahler structure on \( X_{\text{out}} \) then Hodge-diamond changes similarly with this difference that \( X_{\text{out}} \) will be a Calabi-Yau 3-fold. But some thing is different in this case which leads to following theorem:

**Theorem 3.1.** For even integer \( n = 2k \), there is no compact smooth Calabi-Yau manifold \( X \) of real dimension \( 2n \), with \( H_1(X, \mathbb{Z})^{\text{torsion}} = 0 \), containing an embedded \( \mathbb{R}P^n \) as a Lagrangian manifold.

**Proof.** Let \( L \cong \mathbb{R}P^n \) be a Lagrangian submanifold in \( X \). From theorem 2.3.3 we know that the first chern class of tangent bundle of almost Calabi-Yau manifold \( X_{\text{out}} \) obtained from \( X \) by surgery near \( L \) is:

\( c_1^{X_{\text{out}}} = \frac{n-3}{2} \cdot PD(D_a) \)

So it would be enough to find an integral Homology class \( F \in H_2(X_{\text{out}}) \) having an odd intersection number with \( D_a \), i.e. \( F \cdot D_a = odd \), because then we get \( c_1^{X_{\text{out}}}(F) \notin \mathbb{Z} \) which is impossible since \( c_1^{X_{\text{out}}} \) is an integral Cohomology class.

To achieve this consider the long exact sequence corresponding to pair \((X, L)\):

\[
\cdots \rightarrow H_2(X, \mathbb{Z}) \rightarrow H_2(X, L) \rightarrow H_1(L, \mathbb{Z}) \rightarrow H_1(X, \mathbb{Z}) \cdots
\]

Substituting \( H_1(L) = \mathbb{Z}_2 \) and using the fact that \( H_1(X, \mathbb{Z})^{\text{torsion}} = 0 \) we get

\[
\cdots \rightarrow H_2(X, \mathbb{Z}) \rightarrow H_2(X, L) \rightarrow \mathbb{Z}_2 \rightarrow 0
\]

Let’s define an odd class in \( H_2(X, L) \) to be one which has non zero image in \( H_1(L) \), and let’s \( \beta \in H_2(X, L) \) be an odd class. From \( \beta \) we will construct a Homology class \( F_{\beta, k} \in H_2(X_{\text{out}}, \mathbb{Z}) \) intersecting \( D_a \) in \( k \) number of points where \( k \) is an odd number.
As before we can decompose $X_{\text{out}}$ as a union of two open sets $UD_a$ and $A$ where $A = X_{\text{out}} \setminus D_a = X \setminus L$ and $UD_a$ is a neighborhood of $D_a$ which retracts to $D_a$. We know $UD_a \cap A$ retracts to $V_a$ where $V_a$ is the $S^1$ bundle over $D_a$ corresponding to normal vectors of unit length. The odd class $\beta \in H_2(X, L)$ gives a relative homology class $\beta \in H_2(A, V_a) \cong H_2(X, L)$ whose boundary $\partial \beta \in H_1(V_a) \cong \mathbb{Z}_2$ is the non trivial element in $\mathbb{Z}_2$. But $\partial UD_a = V_a$ and $H_1(D_a) = 0$, so there is a disc in $UD_a$ filling $\partial \beta$. The disc can be chosen such that it intersects $D_a$ in one point, infact $UD_a$ is diffeomorphic to a neighborhood of quadratic hypersurface $Q^{n-1}$ in $\mathbb{C}P^n$, so if we consider the class of line $[l]$ in $\mathbb{C}P^n$, it has intersectin number two with $Q^{n-1}$. We then can split this projective line into two pieces where each of them is a disc intersecting $Q^{n-1}$ at one point and this gives the disc we want. Attaching this disc to $\beta$ along the boundary gives a Homology class in $H_2(X_{\text{out}}, \mathbb{Z})$ which we call it $F_{\beta,1}$ and has intersection number equal to 1 with $D_a$ in $X_{\text{out}}$. For any other $k = 2s + 1$ we define $F_{\beta,k}$ to be the class of $F_{\beta,1} - s \cdot [l]$ where $[l]$ is the class of line in $D_a \cong Q^{n-1} \subset X_{\text{out}}$. Note that when $n = 3$ there are two different class of lines in $D_a$ which are the same, when considered as a class in $H_2(X_{\text{out}}, \mathbb{Z})$.

4. Proofs and discussion of main results

In this section we will prove the theorems 1.1 and 1.2 first and at the end of section we give a short explanation of conjecture 1.1. Although the essence of both theorems 1.1 and 1.2 is the same but we will need different techniques for the proof of each one. In the first one we will simply use the classical techniques discussed in chapter 3 of the book of McDuff and Salamon [MS1], but for the second one we will use Kuranishi structure on moduli spaces which is the base ingredient of the book [FOOO]. One can find the appendix in [Jake] as a quick reference to study Kuranishi structures. It is also possible to prove theorem 1.2 using perturbed Cauchy-Riemann equations, but we prefer to use more general notion of virtual class and Kuranishi structure here.

4.1. Proof of theorem 1.1

Proof. In this Section we will prove the Theorem 1.1 in two main steps. Consider the situation of Theorem 1.1. Following section 2.2 from the triple $(X^{2n}, w, L^n)$ we can construct a singular symplectic manifold $X_0 = X_{\text{in}} \cup_{D_a} X_{\text{out}}$ which is made of two smooth symplectic manifolds $(X_{\text{in}}, w_{\text{in}})$ and $(X_{\text{out}}, w_{\text{out}})$ intersecting along a common divisor $D_a$; and also a smooth symplectic fibration $\pi : Z \to \Delta$ whose fiber over zero is $X_0$. By the result of Theorems 2.2.3 we know that $c_1^{X_{\text{out}}} = -(n - 2) \cdot PD(D_a)$ where $PD(D_a)$ means Poincare dual of symplectic divisor $D_a$. Consider $X_{\text{out}}$ and let $J_{1,\text{comp}}^{X_{\text{out}}}$ to be the set of almost complex structures $J$ of class $C^1$ on $X_{\text{out}}$ compatible with $D_a$ and $Z$, where compatible means $J$ maps the tangent space of $D_a$ to itself and can be extended to an almost complex structure $J_Z$ on $Z$ compatible with fibration. It is shown in [IP2] lemma 2.3 that this space is non empty; and more than that, if we deform such an almost complex structure to another one, outside a neighborhood of singular locus, we again can get an almost complex structure in this space, which means we can extend it to the fiberation. Pick $J \in J_{1,\text{comp}}$ and $A \in H_2(X_{\text{out}}, \mathbb{Z})$ and let $M^*(X_{\text{out}}, J, A)$ be the moduli space of all $J$-holomorphic spheres representing class $A \in H_2(X_{\text{out}}, \mathbb{Z})$, where $*$ means we consider only those $J$-holomorphic curves which are somewhere
injective (not multiple cover) and intersect $D_a$ at a discrete set of points (they are not map into $D_a$). But if $[A] \cdot [D_a] = s > 0$ and $n > 2$, we get:

\[
\text{virtual dimension of } \mathcal{M}^*(X_{out}, J, A) = 2(n - 3) + -2(n - 2) \cdot s < 0
\]

So it would be reasonable to expect $\mathcal{M}^*(X_{out}, J, A)$ to be empty for a generic $J$. Actually we prove (step 1):

**Theorem 4.1.** There is a subset of second category in $\mathcal{J}_{l,\text{comp}}$, say $\mathcal{J}_{reg}$ such that for any $J \in \mathcal{J}_{reg}$ and any class $A \in H^2(X_{out}, \mathbb{Z})$ with $[A] \cdot [D_a] = s > 0$ and $n > 2$, we get:

\[
\text{virtual dimension of } \mathcal{M}^*(X_{out}, J, A) = 2(n - 3) + -2(n - 2) \cdot s < 0
\]

So it would be reasonable to expect $\mathcal{M}^*(X_{out}, J, A)$ to be empty for a generic $J$.

**Proof.** We will essentially follow the proof of transversality theorem in [MS1]. Let’s first consider the moduli space $\mathcal{M}^*(X_{out}, J_{l,\text{comp}}, A)$ of all maps $u : \mathbb{C}P^1 \to X$ which are $J$-holomorphic for some $J \in \mathcal{J}_{l,\text{comp}}$ and represent class $[A]$. We will prove that linearization of Cauchy-Riemann equation is transversal for this moduli space.

**Lemma 4.1. Linearization of Cauchy-Riemann operator :**

\[
D_{u,J} : W^{k,p}(\mathbb{C}P^1, u^*TX) \times T_J\mathcal{J}_{l,\text{comp}} \to W^{k-1,p}(\mathbb{C}P^1, u^*TX \otimes J^0 \Omega^1)
\]

given by equation:

\[
D_{u,J}(\xi, Y) = L_{u,J}(\xi) + \frac{1}{2} Y(u) du \circ j
\]

\[
L_{u,J}(\xi) = \nabla \xi + J \nabla \xi \circ j
\]

is surjective for any $(u, J) \in \mathcal{M}^*(X_{out}, \mathcal{J}_{l,\text{comp}}, A)$.

**Proof.** $D_{u,J}$ is Fredholm so it suffices to prove it has a dense image. We may assume $k = 1$, so if it’s image is not dense, then there exist:

\[
\eta \in L^q(u^*TX_{out} \otimes \Omega^0 \Omega^1) \quad \frac{1}{p} + \frac{1}{q} = 1
\]

such that:

\[
\int < \eta, D_{u,J}(\xi, Y) > dvol_{\mathbb{C}P^1} = 0
\]

so we get:

\[
(1) \quad \int < \eta, L_{u,J}(\xi) > dvol_{\mathbb{C}P^1} = 0
\]

and

\[
(2) \quad \int < \eta, Y(u) du \circ j > dvol_{\mathbb{C}P^1} = 0.
\]

By regularity theorems for elliptic operators and from (1) we get $\eta$ is of class $W^{1,p}$ (so continuous) and $L^*_{u,J} \eta = 0$.

Since $u$ is some-where injective so the set of injective points are open dense in $\mathbb{C}P^1$. Also we know that $u$ has finit intersection with $D_a$, so we can find a dense open set of injective points $z_0 \in \mathbb{C}P^1$ such that:

\[
u(z_0) \not\in D_a; \quad du(z_0) \neq 0; \quad u^{-1}(u(z_0)) = z_0
\]
We will prove that \( \eta \) vanishes at such \( z_0 \), which then proves \( \eta \) is almost everywhere zero and so is zero. Suppose \( \eta(z_0) \neq 0 \) then by Lemma 3.2.2 in [MS2] we can construct an endomorphism \( Y_0 \in \text{End}(T_{u(z_0)}X_{out}) \) such that:

1. \( YJ + JY = 0 \)
2. \( w_{u(z_0)}(Y\xi, \zeta) + w_{u(z_0)}(\xi, Y\zeta) = 0 \quad \forall \xi, \zeta \in T_{u(z_0)}X_{out} \)
3. \( \langle \eta(z_0), Y_d u(z_0) \circ j(z_0) \rangle > 0 \)

Now extend \( Y_0 \) to \( Y \in \Gamma(\text{End}(TX_{out})) \) in a neighborhood \( U_{u(z_0)} \) of \( u(z_0) \in X_{out} \) satisfying the conditions (1) and (2) above (i.e. \( Y \) is tangent to \( T\mathcal{J} \)) such that \( Y(u(z_0)) = Y_0 \) and \( U_{u(z_0)} \cap D_\lambda = \emptyset \). So by reducing \( U_{u(z_0)} \) to a smaller neighborhood \( U_{u(z_0)}' \) we may assume:

- \( u^{-1}(U_{u(z_0)}') \) is a small neighborhood of \( z_0 \) in \( CP^1 \). (Since \( z_0 \) is injective point)
- \( \langle \eta(z), Y_0 du(z) \circ j(z) \rangle > 0 \) for any \( z \in u^{-1}(U_{u(z_0)}') \)

Multiplying such \( Y \) with a bump function supported in \( U_{u(z_0)}' \) we get a \( Y \in T\mathcal{J}_{\text{comp}} \). That is because support of \( Y \) is disjoint from \( D_\lambda \) and so does not affect the compatibility conditions of \( J \). This contradicts the assumption in equation (15) and so we see \( D_{u,J} \) is surjective.

Now consider the map \( \pi : \mathcal{M}^*(X_{out}, \mathcal{J}_{\text{comp}}, A) \rightarrow \mathcal{J}_{\text{comp}} \). \( d\pi \) is Fredholm and its kernel and cokernel are equal to kernel and cokernel of \( L_{u,J} \). Hence by the Sard-Smale theorem whenever \( l - 2 \geq 0 \geq \text{index}(L_{u,J}) \), the set of regular values of \( \pi \) is of second category. But a regular value \( J \in \mathcal{J}_{\text{reg}} \) for \( d\pi \), means \( L_{u,J} \) is surjective and so for such \( J \) \( \mathcal{M}^*(X_{out}, J, A) \) will be a negative dimensional smooth moduli space and so empty. Intersecting along all curve classes \( A \in H_2(X_{out}, \mathbb{Z}) \) we will find a dense set of compatible almost complex structures for which all the moduli spaces \( \mathcal{M}^*(X_{out}, J, A) \) with \( [A] \cdot [D_\lambda] = s > 0 \) are empty.

**Step 2:** We use the theorem [3.1] to prove theorem [4.1] in the following way:

Pick an almost complex structure \( J_{out} \) in \( \mathcal{J}_{X_{out}}^{\text{reg}} \) and extend it to an almost complex structure \( J_Z \) on \( Z \), compatible with fiberation, which means \( J_Z \) induces an almost complex structure \( J_\lambda \) on each fiber \( X_\lambda = \pi^{-1}(\lambda) \). For \( \lambda \rightarrow 0 \), \( J_\lambda \) can be viewed as a sequence of almost complex structures on \( X \) converging to singular almost complex structure \( J_0 \).

Fix some positive constant \( E \) as in theorem [4.1] and consider all the moduli spaces of \( J_\lambda \)-holomorphic discs with boundary on \( L, \mathcal{M}(X_\lambda, L, J_\lambda, \beta) \), for all \( \beta \in H_2(X, L) \) with \( w_\lambda(\beta) < E \). Suppose that there is sequence \( \{\lambda_i\}_{i=1}^\infty \) such that for at least one \( \beta_i \) with \( w_\lambda(\beta_i) < E \), the the moduli space \( \mathcal{M}(X_{\lambda_i}, L, J_{\lambda_i}, \beta_i) \) is non-empty and has at least one element \( u_{\lambda_i} \). Note that \( u_{\lambda_i} \) is not requested to be smooth and is an element in compactification of moduli space. This sequence can be considered as a sequence of \( J_Z \)-holomorphic discs in \( Z \) and we have a energy bound for symplectic area of sequence \( u_{\lambda_i} \). Although \( Z \) is not compact but the sequence \( u_{\lambda_i} \) lives in a compact neighborhood in \( Z \) and so we may apply Gromov-Compactness theorem to the sequence \( u_{\lambda_i} \) as a sequence of \( J_Z \)-holomorphic discs in \( Z \) and get:

**Theorem 4.2.** (Gromov-Compactness)

(a) Given any sequence \( u_{\lambda_i} : \Sigma_i \rightarrow Z \) of \( J_\lambda \)-holomorphic discs in \( Z \), with \( E(u_{\lambda_i}) < E \), one can pass to a subsequence and find:
(1) a bubble domain $B$ with resolution $r : \Sigma \to B$, and
(2) diffeomorphisms $\psi_i$ of $\Sigma_i$ preserving the orientation,
so that the modified subsequence $(u_\lambda_i \circ \psi_i)$ converges to a limit

$$\Sigma \xrightarrow{\gamma} B \xrightarrow{u_0} X_0$$

where $u_0$ is a stable $J_0$-holomorphic map inside $X_0$. This convergence is in $C^0$, in $C^i$ on compact sets not intersecting the collapsing curves $\gamma$ of the resolution $r$, and the energy is preserved in the limit.

Furthermore:

(b) Any component of $u_0$ has image either in $X_{in}$ or $X_{out}$ (and may be in $D_0$).
(c) There is at least one component mapped inside $X_{out}$ intersecting $D_0$ in a non empty discrete set.

Proof. Part (a) is just the standard Gromov-Compactness theorem for a sequence inside $Z$ (c.f [MS1] Section 4 and 5).

Part (b): Consider a component of $u_0$, say $u_{0i}$, which is either a $J_0$-holomorphic sphere or a disc with boundary on $L \subset X_{in}$ $(L \cap D_0 = \emptyset)$. Suppose that $u_{0i}$ is not mapped into $D_0$ so it is mapped entirely in one of $X_{in}$ or $X_{out}$ or it intersects $D_0$ in some curve $\gamma \subset D_0$, but the later one is impossible since tangent bundle of $D_0$ is invariant under $J_0$.

Part (c): Each $u_\lambda_i$ has non empty intersection with $L$ so $u_0$ has non empty intersection with $L \subset X_{in}$ $(L \cap D_0 = \emptyset)$ which means there is a non-zero component $u_{in}$ of $u_0$ mapped into $X_{in}$. Consider contact hypersurfaces $V_{a+\epsilon}$ for $\epsilon > 0$ in $X$. After performing symplectic cut along $V_a$ as in section 2.1 we get a copy of contact hypersurfaces $V_{a+\epsilon}$ in $X_{out}$ which is the boundary of normal disc bundle of $D_0 \subset X_{out}$. Each $u_\lambda_i$ has non empty intersection with $V_{a+\epsilon}$ because the symplectic form inside the neighborhood surrounded by $V_a$ is exact and so there is no $J_\lambda_i$-holomorphic disc completely inside that. This shows that the limit curve $u_0$ has also non empty intersection with $V_{a+\epsilon} \subset X_{out}$ and so it must have a component in $X_{out}$ not embedded in $D_0 \subset X_{out}$. This component intersect $D_0$ in non empty discrete set of points due to the fact that image is connected and has one component in $X_{in}$. Actually we are using the known fact that near intersection point, the behavior is similar to that of real holomorphic objects.

Coming back to proof of theorem 1.1 we see that part (c) of previous theorem gives a contradiction because by choice of $J_0$ there is no $J_0$-holomorphic sphere in $X_{out}$ intersecting $D_0$ in non empty set of points. So our assumption was wrong and for some $\lambda'$, $\mathcal{M}(X, L, J_{\lambda'}, \beta)$ is empty for all $\beta$ with $w(\beta) < E$. Again because of Gromov Convergence theorem this also holds for some neighborhood $U_E$ of $J_{\lambda'} \in \mathcal{J}_X$. This finishes the proof of theorem 1.1.

4.2. Proof of theorem 1.2. In this case we have one more information which is the anti-symplectic involution. So we have the tuple $(X, w, L, \phi)$, where $(X, w)$ is an almost Calabi-Yau symplectic manifold of real dimension six. $L$ is a Lagrangian in $X$ diffeomorphic to $S^3$ and $\phi$ is an anti-symplectic involution on $X$ with $L \subset \text{fix}(\phi)$. As a result of theorem 2.4 we may assume that $\phi$ is equal to standard anti-symplectic involution near $L$ and so from the discussion in remark 2.5 we know that we can
construct a smooth symplectic fibration $\pi: Z \to \Delta$ which admits an anti-symplectic involution $\phi_Z$ mapping fiber over $\lambda$ to fiber over $\bar{\lambda}$. Also $\phi_0 = \phi_Z|_{X_0}$ induces anti-symplectic involutions on each component $X_{in}$ and $X_{out}$, say $\phi_{in}$ and $\phi_{out}$, which both map $D_a$ to itself and $\phi_{out}$ has no fixed point.

If we try to prove this theorem in the same way we proved theorem 1.1, we fail, because we cannot prove the transversality at those $J_{out}$-holomorphic curves $u: \mathbb{C}P^1 \to X_{out}$, whose image is invariant under action of $\phi_{out}$, because we are allowed to use only symmetric almost complex structures. So we need a new method. We will use the Kuranishi structure on moduli spaces and multi sections to construct the virtual class. A good source for studying Kuranishi structure is the appendix A1 in the book [FOOO].

For the proof we use the fibration above with centeral fiber $X_0$, and we fix a complex structure $J_Z$ on $Z$ which is compatible with anti-symplectic involution $\phi_Z$ and fibration. So in this way and if we just look at $\pi^{-1}([0, 1])$ (assuming $\Delta$ is the whole unit disk), we get a family of symplectic manifolds $X_t$ for $t \in [0, 1]$, where for $t \neq 0$, the fiber is smooth and symplectically isotopic to $X$ and fiber over zero is the singular symplectic manifold $X_0$. Also for each $t$, we have an anti-symplectic involution $\phi_t$ on $X_t$ fixing the Lagrangian such that for non zero $t$ it is the original anti-symplectic involution we had on $X$ and an almost complex structure $J_t$ compatible with that.

Fix some class $\beta \in H_2(X, L)$ and let $\mathcal{M}(\beta, \{J_t\}, [0, 1])$ be the moduli space of all $J_t$-holomorphic discs in $X_t$ for $t \in (0, 1]$ and in class $\beta$. This is a moduli space with virtual dimension one, and each slice is a moduli space of virtual dimension zero.

We first need to compactify this moduli space at time zero. This compactification is provided by Gromov Compactness theorem 4.2. Where each limit curve can be modeled over a bubble domain with components either in $X_{in}$, $X_{out}$, or inside $D_a$.

In this section and in construction of Kuranishi structure and also in the formulation of conjecture 1.1 we need a more precise version of theorem 4.2 for gluing which will be discussed later.

So up to this point we have the compact moduli space $\mathcal{M}(\beta, \{J_t\}, [0, 1])$ which has two kind of boundaries:

1. Boundary at $t = 0$ and $t = 1$.
2. Boundary components at the middle which corresponds to two types of degeneration at co-dimension one of type (I) and (II) discussed before.

In What follows we first discuss the Kuranishi Structure on $\mathcal{M}(\beta, \{J_t\}, [0, 1])$. We then get rid of boundary components in the middle and construct another moduli space $\mathcal{M}(\beta, \{J_t\}, [0, 1])$, which is obtained by identifying various boundary components at the middle. So $\mathcal{M}(\beta, \{J_t\}, [0, 1])$ would have just boundaries at $t = 0, 1$. We then construct the virtual fundamental class of this moduli space with boundaries at $t = 0, 1$. The last step is to show that for generic perturbation the boundary at $t = 0$ will be empty and so virtual class $\mathcal{M}(\beta, J_1)^{virt}$ is a boundary which means $N_{\beta}^{sym} = 0$. 

-Kuranishi structure, review:

Let $\mathcal{M}$ be a compact space. A Kuranishi chart is $(V_\alpha, E_\alpha, \Gamma_\alpha, \psi_\alpha, s_\alpha)$ which satisfies following:

1. $V_\alpha$ is a smooth manifold, possibly with boundaries and corners and $\Gamma_\alpha$ is a finite group acting on $V_\alpha$ effectively.
2. $\pi_\alpha : E_\alpha \rightarrow V_\alpha$ is a finite dimensional vector bundle. $\Gamma_\alpha$ acts on it and $\pi_\alpha$ is equivariant.
3. $s_\alpha$ is a $\Gamma_\alpha$-equivariant section of $E_\alpha$.
4. $\psi_\alpha : s_\alpha^{-1}(0)/\Gamma_\alpha \rightarrow \mathcal{M}$ is a homeomorphism to an open subset of $\mathcal{M}$. For $p \in \psi_\alpha(s_\alpha^{-1}(0)/\Gamma_\alpha)$, $(V_\alpha, E_\alpha, \Gamma_\alpha, \psi_\alpha, s_\alpha)$ is said to be a Kuranishi neighborhood of $p$.

Then if $(V_\alpha, E_\alpha, \Gamma_\alpha, \psi_\alpha, s_\alpha)$ are two neighborhoods of a point $p$, we have a Kuranishi change of coordinates, which is the set of data:

5. $h_{\alpha_{1\alpha_2}} : \Gamma_\alpha \rightarrow \Gamma_{\alpha_2}$, which is an injective homomorphism.
6. $\phi_{\alpha_{1\alpha_2}} : V_{\alpha_1\alpha_2} \rightarrow V_{\alpha_2}$, which is an equivariant smooth embedding and induces an injective map $\tilde{\phi}_{\alpha_{1\alpha_2}} : V_{\alpha_1\alpha_2}/\Gamma_\alpha \rightarrow V_{\alpha_2}/\Gamma_{\alpha_2}$.
7. $(\phi_{\alpha_{1\alpha_2}}, \tilde{\phi}_{\alpha_{1\alpha_2}})$, which is an $h_{\alpha_{1\alpha_2}}$ equivalent embedding of vector bundles $E_{\alpha_1}|_{V_{\alpha_1\alpha_2}} \rightarrow E_{\alpha_2}$ and
8. $\hat{\phi}_{\alpha_{1\alpha_2}} \circ s_\alpha = s_\alpha \circ \phi_{\alpha_{1\alpha_2}}$.
9. $\psi_\alpha = \psi_{\alpha_2} \circ \phi_{\alpha_{1\alpha_2}}$.
10. The map $h_{\alpha_{1\alpha_2}}$ restricts to an isomorphism $(\Gamma_\alpha)_x \rightarrow (\Gamma_{\alpha_2})_{\phi_{\alpha_{1\alpha_2}}(x)}$ for any $x \in V_{\alpha_2}$.
11. $\psi_\alpha(s_\alpha^{-1}/\Gamma_\alpha) \cap \psi_{\alpha_2}(s_\alpha^{-1}/\Gamma_{\alpha_2}) = \psi_{\alpha_2}((s_\alpha^{-1}\cap V_{\alpha_1\alpha_2})/\Gamma_{\alpha_2})$

Then a Kuranishi structure on $\mathcal{M}$, is made of a system of kuranishi charts (similar to definition of manifold) and transition maps between them.

**Definition 4.1.** A Kuranishi structure on $\mathcal{M}$ assigns a Kuranishi chart $(V_\alpha, E_\alpha, \Gamma_\alpha, \psi_\alpha, s_\alpha)$ for each $p \in \mathcal{M}$ and a coordinate change $(\phi_{pq}, \phi_{qp}, h_{pq})$ for each $q \in \psi_p(s_p^{-1}(0)/\Gamma_p)$ such that the following holds:

1. $\dim V_p \cdot \text{rank} E_p$ is independent of $p$ (=virtual dimension of $\mathcal{M}$).
2. For another point $r \in \psi_q((V_q \cap s_q^{-1}(0))/\Gamma_q)$, there exists $\gamma_{pqr} \in \Gamma_p$ such that:

$$h_{pq} \circ h_{qr} = \gamma_{pqr} \cdot h_{pr} \cdot \gamma_{pqr}^{-1}, \quad \phi_{pq} \circ \phi_{qr} = \gamma_{pqr} \cdot \phi_{pr}, \quad \hat{\phi}_{pq} \circ \hat{\phi}_{qr} = \gamma_{pqr} \cdot \hat{\phi}_{pr}.$$  

In the case $V_p$ has boundary or corner, we say Kuranishi structure has boundary or corner.

For $\mathcal{M}$ with Kuranishi structure and two Kuranishi charts as above, lets consider the normal bundle $N_{\phi_{\alpha_{2\alpha_1}}(V_{\alpha_1})}V_{\alpha_2}$. The fiber derivative of the Kuranishi map $s_{\alpha_2}$ gives a homomorphism:

$$d_{fiber}s_{\alpha_2} : N_{\phi_{\alpha_{2\alpha_1}}(V_{\alpha_1})}V_{\alpha_2} \rightarrow E_{\alpha_2} |_{\text{im}\phi_{\alpha_{2\alpha_1}}}$$

which is $h_{\alpha_{1\alpha_2}}$ equivariant. A space with Kuranishi structure is said to have tangent bundle if $d_{fiber}s_{\alpha_1}$ gives a bundle isomorphism:

$$N_{\phi_{\alpha_{2\alpha_1}}(V_{\alpha_1})}V_{\alpha_2} \cong \frac{E_{\alpha_2}}{\phi_{\alpha_{2\alpha_1}}(E_{\alpha_1})}.$$
A Kuranishi space by definition is then a space with a Kuranishi structure which has tangent bundle.

Also we call a Kuranishi space orientable if $V_\alpha$ and $E_\alpha$ are oriented and the isomorphism given above pereserves the orientation.

When constructing Kuranishi structure for moduli spaces, one can think of points in $M$ as $J$-holomorphic curves; $s_p$ will be the Cauchy-Riemann equation $\bar{\partial}_J$; $E_p$, an enlargement of cokernel of linearization of $\bar{\partial}_J$ and $V_p$ as a set of maps which are almost $J$-holomorphic.

**Definition 4.2.** For a Kuranishi space $M$ as above, and a manifold $Y$. A strongly continuous map $f : M \to Y$ means a sequence of maps $\{f_\alpha\}$ of $\Gamma_\alpha$-equivariant maps $f_\alpha : V_\alpha \to Y$ such that:

$$f_{\alpha_1} \circ \phi_{\alpha_1,\alpha_2} = f_{\alpha_2}$$
on $V_{\alpha_1,\alpha_2}$. When $Y$ is a smooth manifold, a strongly continuous map $f : M \to Y$ is smooth if all $f_\alpha$ are smooth and is said to be weakly submersive if each of $f_\alpha$ is a submersion.

We know explain the construction of virtual fundamental class for a Kuranishi space:

The idea is to perturb the kuranishi map $s$ to another one $s$ which is transversal and then glue different components of $s^{-1}$ together to get the fundamental chain. For this purpose we need the notion of multisections. Let $(V, E, \Gamma, s)$ be one of $(V_\alpha, E_\alpha, \Gamma_\alpha, s_\alpha)$ we had before. Define $E^n$ to be direct sum of $n$ copy of $E$. The symmetric group of order $n$, $S_n$ acts on it and let $S^n(E)$ be the quotient space $E^n/S_n$. Action of $\Gamma$ induces an action on quotient space and the map $E^n \to E^{nm}$ defined by:

$$(x_1, \cdots, x_n) \to (x_1, \cdots, x_1, x_2, \cdots, x_2, \cdots, x_n, \cdots, x_n)$$

induces a $\Gamma$-equivariant map $S^n(E) \to S^{nm}(E)$.

**Definition 4.3.** An $n$-multisection $s$ of $\pi : E \to V$ is a $\Gamma$-equivariant map $V \to S^n(E)$. It is said to be liftable if it can be lifted to a section $\tilde{s} : V \to E^n$, not necessarily equivariant. If $s$ is an $n$-multisection then it can also be considered as a $nm$-multisection via map given above. An $n$-multisection and an $m$-multisection are said to be equivalent if their corresponding $nm$-multisections coincide to each other. An equivalence class of multisections is said to be a multisection.

A liftable multisection is said to be transversal to zero if each of its branch is transversal to zero.

A family of multisections $s_\epsilon$ is said to converge to $s$ as $\epsilon \to 0$ if there exists $n$ such that $s_\epsilon$ is represented by an $n$-multisection $s^n_\epsilon$ and $s^n_\epsilon$ converges to a representative of $s$.

Then intuitively the foundamental class of $M$ is constructed by considering the zero section of a set of transversal multisections $s^n_\epsilon$, $\epsilon$-close to $s_\alpha$, which are compatible with change of coordinates and gluing them together.

The next thing which will be very important in the proof is the fiber product of Kuranishi spaces. Let $M^1$ and $M^2$ be two Kuranishi spaces and $f_i : M^i \to Y$ be...
two smooth strongly continuous submersive maps. We can then consider the fiber product
\[ M^1 \times_Y M^2 \]
given by set of points \( \{(p_1, p_2) \mid f_1(p_1) = f_2(p_2)\} \). We can define a Kuranishi structure on \( M^1 \times_Y M^2 \) as follows:

**Definition 4.4.** Let \( \bar{p} = (p_1, p_2) \in M^1 \times_Y M^2 \) and \( V^i_{p_i} \) be Kuranishi charts around \( p_i \). We put:

\[ V_{\bar{p}} := \left\{ (x_1, x_2) \in \prod_i V^i_{p_i} \mid f^1_{p_1}(x_1) = f^2_{p_2}(x_2) \right\}. \]

Since \( f^i_{p_i} \) are submersions, it follows that \( V_{\bar{p}} \) is a smooth manifold for each \( \bar{p} \). We put \( E_{\bar{p}} := \oplus_i E_{p_i} \), and \( \Gamma_{\bar{p}} := \prod_i \Gamma^i_{p_i} \). We define \( s_{\bar{p}} \) and \( \psi_{\bar{p}} \) in a similar way.

Then it is true that if the Kuranishi structures on \( M_i \) have tangent bundle, so does the Kuranishi structure on \( M^1 \times_Y M^2 \), and also if \( Y \) and \( M_i \) are oriented, so is \( M^1 \times_Y M^2 \).

We can define a Kuranishi structure on fiber product of more than two spaces in a similar way.

We next define the action of a finite group on a space with Kuranishi structure. Let \( \varphi : M \to M \) be a homeomorphism of Kuranishi space \( M \). We say \( \varphi \) induces an automorphism of Kuranishi structure of the following holds:

- for \( q = \varphi(p) \) and two Kuranishi charts \( (V_p, E_p, \Gamma_p, \psi_p, s_p) \) and \( (V_q, E_q, \Gamma_q, \psi_q, s_q) \) around \( p \) and \( q \) respectively, there exists \( \rho_p : \Gamma_p \to \Gamma_q \), \( \varphi_p : V_p \to V_q \), and \( \hat{\varphi}_p : E_p \to E_q \) such that:
  1. \( \rho_p \) is an isomorphism of groups,
  2. \( \varphi_p \) is a \( \rho_p \) equivariant diffeomorphism,
  3. \( \hat{\varphi}_p \) is a \( \rho_p \) equivariant bundle isomorphism covering \( \varphi_p \).
  4. \( s_q \circ \varphi_p = \hat{\varphi}_p \circ s_p \).
  5. the restriction of \( \varphi_p \) to \( s^{-1}_p(0) \) induces a homeomorphism \( s^{-1}_p(0)/\Gamma_p \to s^{-1}_q(0)/\Gamma_q \), which we call it \( \hat{\varphi}_p \) and we have \( \psi_q \circ \hat{\varphi}_p = \varphi \circ \psi_p \).

We also assume that \( (\rho_p, \varphi_p, \hat{\varphi}_p) \) are compatible with coordinate changes of Kuranishi structure in following sense:

For \( q \in \psi_p(s^{-1}_p(0)/\Gamma_p) \) and \( q' \in \psi_{p'}(s^{-1}_{p'}(0)/\Gamma_{p'}) \) such that \( \varphi(p) = p' \) and \( \varphi(q) = q' \)

\[
\begin{align*}
(6) & \quad \rho_p \circ \psi_{pq} = \gamma_{pq} \cdot (h_{pq'} \circ \rho_q) \cdot \gamma^{-1}_{pq} \\
(7) & \quad \varphi_p \circ \psi_{pq} = \gamma_{pq} \cdot (\theta_{pq'} \circ \varphi_q) \\
(8) & \quad \hat{\varphi}_p \circ \phi_{pq} = \gamma_{pq} \cdot (\hat{\theta}_{pq'} \circ \varphi_q).
\end{align*}
\]

In practice we will see a group action on Kuranishi structure of moduli spaces coming from the anti-symplectic involution.

After this short introduction to subject we will continue by stating main theorems we need to prove theorem 1.2

**Proposition 4.3.** \( \mathcal{M}(\beta, \{ J_i \}, [0, 1]) \) has a topology with respect to which it is compact and Hausdorff. It has an oriented Kuranishi structure with boundary of virtual
dimension one with respect to which the map \( \pi : \mathcal{M}(\beta, \{J_t\}, [0, 1]) \to [0, 1] \) is a smooth strongly continuous and is weakly submersive.

Furthermore, the boundary of \( \mathcal{M}(\beta, \{J_t\}, [0, 1]) \) except \( \pi^{-1}(0) \) and \( \pi^{-1}(1) \) corresponds to moduli spaces

\[
\mathcal{M}_1(\beta_1, \{J_t\}, (0, 1))_{\text{ev}} \times_{\text{ev}} \mathcal{M}_1(\beta_2, \{J_t\}, (0, 1)) \quad \beta_1 + \beta_2 = \beta
\]

and

\[
\mathcal{M}^\text{closed}_1(\tilde{\beta}, \{J_t\}, (0, 1))_{\text{ev}} \times L
\]

via gluing maps

\[
\text{glue} : \mathcal{M}_1(\beta_1, \{J_t\}, (0, 1))_{\text{ev}} \times_{\text{ev}} \mathcal{M}(\beta, \{J_t\}, (0, 1)) \to \partial \mathcal{M}(\beta, \{J_t\}, [0, 1])
\]

and

\[
\text{clop} : \mathcal{M}^\text{closed}_1(\tilde{\beta}, \{J_t\}, (0, 1))_{\text{ev}} \times L \to \partial \mathcal{M}(\beta, \{J_t\}, [0, 1])
\]

such that the Kuranishi structure on \( \mathcal{M}_1(\beta_1, \{J_t\}, (0, 1))_{\text{ev}} \times_{\text{ev}} \mathcal{M}(\beta, \{J_t\}, (0, 1)) \) and \( \mathcal{M}^\text{closed}_1(\tilde{\beta}, \{J_t\}, (0, 1))_{\text{ev}} \times L \) coincides with the pull-back of induced Kuranishi structure on \( \partial \mathcal{M}(\beta, \{J_t\}, [0, 1]) \).

Note that in the above fiber products and in the first one we are considering evaluation maps at boundary marked points, while in the second one we are considering moduli space of \( J \)-holomorphic spheres in class \( \tilde{\beta} \) where \( \tilde{\beta} \) is the corresponding second homology class to \( \beta \) via the map \( H_2(X) \to H_2(X, S^3) \to 0 \) and the evaluation map is evaluation at a marked point on \( \mathbb{C}P^1 \).

**Proof.** The statement and the proof of this proposition is identical to that of propositions 7.1.1 and 7.2.2 in [FOOO], except some modification which we will discuss below.

Proof of propositions 7.1.1 and 7.1.2 in [FOOO] is composed of two parts. They first consider the subset \( \mathcal{M}^{\text{main}}(\beta, \{J_t\}, [0, 1]) \) of \( \mathcal{M}(\beta, \{J_t\}, [0, 1]) \) which is composed of smooth maps and construct Kuranishi neighborhoods for them. Then they consider the points in \( \mathcal{M}(\beta, \{J_t\}, [0, 1]) \setminus \mathcal{M}^{\text{main}}(\beta, \{J_t\}, [0, 1]) \) which is a union of different strata of singular \( J \)-holomorphic discs and using a gluing theorem, construct Kuranishi neighborhoods near these points.

The only additional terms that we have here and we have to take care of them are elements in \( \pi^{-1}(0) \). We first describe these elements in detail and then after borrowing a gluing theorem from [IP2] we can construct Kuranishi neighborhoods near points of \( \pi^{-1}(0) \) in a similar way it is constructed in the proof of propositions 7.1.1 and 7.1.2 in [FOOO].

From the the compactness theorem [4.2] in proof of theorem [1.1] we know that the elements of \( \pi^{-1}(0) \) are nodal \( J \)-holomorphic discs including components both in \( X_{\text{in}} \) and \( X_{\text{out}} \) and may be in \( D_a = X_{\text{in}} \cap X_{\text{out}} \). But not any element of this form can be limit of a sequence of \( J \)-holomorphic discs. For this reason we need a better understanding of limit curves in \( X_0 \).

A limit curve in \( X_0 \) is called regular if it has no component in \( D_a \). To get a better understanding of non-regular curves we will look at limiting procedure again. Consider a \( J_0 \)-holomorphic disc in limit, \( u_0 \), then \( u_0 \) is a limit of \( J_t \)-holomorphic discs \( u_t \). For non-trivial \( J_0 \)-holomorphic discs mapped into \( D_a \), there is some lower bound on the energy, say \( \alpha_D \). It is shown in section 3 of [IP2] that if the amount
of energy of maps $u_t$, concentrated in the neck is less than $\alpha_D$ then the limit curve has no component in $D_a$. In the same way that we defined $X_0$ we can define similar singular symplectic manifolds, say $X_0^{a_1, \ldots, a_{k+1}}$ which is obtained by cutting $X$ along $k + 1$ hypersurfaces $V_{a_1}, \ldots, V_{a_{k+1}}$. So $X_0$ is simply $X_0^0$. Such singular symplectic manifolds are composed of $k + 2$ components where two of them are simply $X_{in}$ and $X_{out}$, which we had before, and the other $k$ components in the middle are all symplectic manifolds similar to $\mathbb{P}(N_{Q^2} \oplus \mathbb{C})$, where $\mathbb{P}(N_{Q^2} \oplus \mathbb{C})$ is the projectivization of normal bundle of a two dimensional quadratic hypersurface inside the three dimensional quadratic hypersurface inside $\mathbb{CP}^1$. This is a $\mathbb{CP}^1$ fiberation over $Q^2$. We will simply denote $\mathbb{P}(N_{Q^2} \oplus \mathbb{C})$ by $\mathbb{P}_D$ and define $X_0^k$ to be $X_{in} \cup \mathbb{P}_D \cup X_{out}$. It is shown in section 6 of [IP2] that for any limit curve there is some minimum $k$ such that it can be represented by a regular nodal $J_0$-holomorphic map inside $X_0^k$ where here again regular means there is no component mapped into the intersection divisors. Such map is called to be in layer $k$. In this way one can stratify the space of limit maps depending on their layer structure and intersection pattern $s$, along the divisor $D_a$. From what we have said we can conclude:

**Proposition 4.4.** We denote by $M(\beta, J_0, X_0)$ the space of nodal $J_0$ holomorphic discs in class $\beta$ which arise as limit of $J_1$-holomorphic discs in same class. This space can be stratified by the layer structure and intersection pattern along common divisor and we can write

$$M(\beta, J_0, X_0) = \bigcup M^{s,k}(\beta, J_0, X_0)$$

where each $M^{s,k}(\beta, J_0, X_0)$ can be written as a union of fiber product of moduli spaces in $X_{in}, X_{out}$ and $P_{D_a}$ and there is at least one component in each of $X_{in}$ and $X_{out}$ involved, intersecting $D_a$ discretely. i.e

$$M^{s,k,reg}(\beta, J_0, X_0) = \bigcup_i M^{reg}(A_i, X_{out}, D_a) \times \times \prod_j v(\bigcup_i M^{reg}(B_j, P_{D_a})) \times \cdots \times (\bigcup_i M^{reg}(C_i, X_{in}, D_a))$$

where all moduli spaces involved there are moduli space of $J_0$-holomorphic spheres except one which is a moduli of $J_0$-holomorphic discs in $X_{in}$ with boundary on $S^3 \subset X_{in}$. Again here "reg" means that we are considering those curves which are not mapped into intersection divisor.

Then to construct Kuranishi structure on $M(\beta, J_0, X_{out})$, we can first construct Kuranishi structures on individual components involved in fiber product and then consider the Kuranishi structure on $M(\beta, J_0, X_{out})$ as a fiber product of Kuranishi structures.

Next job is to construct Kuranishi neighborhoods for limit curves in $M(\beta, J_0, X_{out})$ as elements in $M(\beta, \{J_t\}, [0, 1])$ which are elements in $\pi^{-1}(0)$. For this goal we need a gluing theorem to extend Kuranishi neighborhoods of $M(\beta, J_0, X_{out})$ to Kuranishi neighborhoods in $M(\beta, \{J_t\}, [0, 1])$. This gluing theorem is provided by proposition 9.4 and 10.1 in [IP2], recalling that gluing theorem we have:

**Proposition 4.5.** The Kuranishi structure on $M(\beta, \{J_t\}, [0, 1])$ can be constructed in a way such that the boundary at $\pi^{-1}(0)$ corresponds to $|s|$-covering of moduli.
spaces
\[ \mathcal{M}^{s,k}(\beta, J_0, X_0) \]
via a gluing map
\[ \xi_{\text{glue}} : \mathcal{M}^{s,k}(\beta, J_0, X_0) \to \partial \mathcal{M}(\beta, \{J_t\}, [0, 1]) \]
\[ \mathcal{M}^{s,k}(\beta, J_0, X_0) \]
such that the pull-back Kuranishi structure on
\[ \mathcal{M}^{s,k}(\beta, J_0, X_0) \]
via covering map, coincides with the pull-back of induced Kuranishi structure on
\[ \partial \mathcal{M}(\beta, \{J_t\}, [0, 1]) \].

Here \(|s|\) means the product of multiplicities at each intersection point along \(D_a\).

Remark 4.1. It is also easy to show that Kuranishi structure on \(\mathcal{M}(\beta, \{J_t\}, [0, 1])\)
can be chosen in way to be compatible with boundary component at \(t = 1\). And so up to this point we have a Kuranishi structure on \(\mathcal{M}(\beta, \{J_t\}, [0, 1])\) which is compatible with Kuranishi structures on its boundary components.

Before we bring the anti-symplectic involution into the story we would like to mention the key ingredient of proof of theorem [2.2] at this point.

Following proposition is proved in section 7.2 of [FOOO] and corollary 5.1 of [FO-CYC].

**Proposition 4.6.** The family of transversal multisections \(s_\varepsilon\) for \(\mathcal{M}(\beta, \{J_t\}, [0, 1])\) and other moduli spaces involved in the fiber products of moduli spaces above can be chosen in a way such that \(s_\varepsilon\) coincides with the fiber product of family of multisections on fiber product of components of fiber product.

From this we get an important result:

lets \([\mathcal{M}(\beta, \{J_t\}, [0, 1])]^{s_\varepsilon}\) be the fundamental class constructed from \(s_\varepsilon\). This is a 1-cycle with boundaries at \(t = 0, 1\) and also at the middle. I want to show that boundary at \(t = 0\) is empty. But the boundary points \(s_\varepsilon|_{t=0}\) correspond to zero locus of family of multisections on \(\mathcal{M}^{s,k}(\beta, J_0, X_0)\) for various \(s\) and \(k\). Each of these moduli spaces is a fiber product of various moduli spaces where one of them is a moduli space of the form \(\mathcal{M}(A, J_0, X_{\text{out}})\) for some second homology class \(A \in H_2(X_{\text{out}}, \mathbb{Z})\) with \(A \cdot D_a > 0\). As before this means that virtual dimension of this component is negative and so the zero locus of transversal family of multisections on this component is empty and by compatibility we get that the zero locus of family of multisections on \(\mathcal{M}^{s,k}(\beta, J_0, X_0)\) is empty and so \([\mathcal{M}(\beta, \{J_t\}, [0, 1])]^{s_\varepsilon}|_{t=0}\) is empty. So \([\mathcal{M}(\beta, \{J_t\}, [0, 1])]^{s_\varepsilon}\) has boundary components only at the middle and \(t = 1\).

The next step is to get rid of boundary components at the middle by using the anti-symplectic involution we have on \(Z\) and the ideas discussed in [Jake], chapter 5.
From now on we will restrict ourself to relative homology classes $\beta \in \Pi_2(X, S^3)$. Note that $S^3$ is an oriented 3-dimensional manifold and so spin. We fix an orientation and spin structure on it. This gives an orientation on moduli spaces $\mathcal{M}_{k,l}(\beta, X)$. We denote by $\mathcal{M}^{main}(\beta, \{J_t\}, [0, 1])$ to be the subset of smooth maps. For a map $u : D^2 \to \mathcal{Z}$ in $\mathcal{M}^{main}(\beta, \{J_t\}, [0, 1])$ lets $\phi_* u$ be the map given by $\phi_\mathcal{Z} \circ u(\bar{z})$. This is an element of $\mathcal{M}^{main}(\beta, \{J_t\}, [0, 1])$. Because $S^3 \subset X$ is in the fixed point set and so the map $\phi_*$ preserves the spin structure and so both $u$ and $\phi_*u$ are in the same moduli spaces with same orientaions. Then we have following theorem:

**Theorem 4.7.** (Thm 4.9 in [FO3])

The map $\phi_* : \mathcal{M}^{main}_{k,l}(\beta, \{J_t\}, [0, 1]) \to \mathcal{M}^{main}_{k,l}(\beta, \{J_t\}, [0, 1])$ extends to a map $\phi_*$, denoted by the same symbol:

$$\phi_* : \mathcal{M}_{k,l}(\beta, \{J_t\}, [0, 1]) \to \mathcal{M}_{k,l}(\beta, \{J_t\}, [0, 1])$$

It preserves the orientation if and only if $k + l$ is even. Furthermore it can be regarded as an involution on the space $\mathcal{M}_{k,l}(\beta, \{J_t\}, [0, 1])$ with Kuranishi structure.

In this sense we can consider moduli spaces $\mathcal{M}_{k,l}(\beta, \{J_t\}, [0, 1])$ as Kuranishi spaces with group action on them, and also the family of multisections can be chosen in a compatible way. Now consider following maps sending middle boundary components to middle boundary components of $\mathcal{M}(\beta, \{J_t\}, [0, 1])$.

As we discussed before there two types of middle boundaries:

- $\mathcal{M}_1(\beta_1, \{J_t\}, (0, 1))_{ev} \times_{ev} \mathcal{M}_1(\beta_2, \{J_t\}, (0, 1))$ \(\beta_1 + \beta_2 = \beta\) and
- $\mathcal{M}^{closed}_{1}(\beta, \{J_t\}, (0, 1))_{ev} \times L$

For the first one we consider the map $\phi_{bubble}$ which is given by identity on the first component and $\phi_*$ on the second component. For the second one we consider $\phi_*$ itself. Then as result of theorem above we see that both these maps are orientation reversing mapping boundary components to boundary components and so we can define:

$$\tilde{\mathcal{M}}(\beta, \{J_t\}, [0, 1]) := \mathcal{M}(\beta, \{J_t\}, [0, 1]) / \sim \text{ given by } \phi_{bubble} \text{ and } \phi_* \text{ above.}$$

So we get the moduli space $\tilde{\mathcal{M}}(\beta, \{J_t\}, [0, 1])$ with boundaries only at $t = 0, 1$. This is the method used in [Jake], (section 5), to define invariants. But we showed that for a general compatible choice of family of multisections $s_\epsilon$, the boundary at $t = 0$ is empty so from this we can construct a fundamental 1-cycle $[\mathcal{M}(\beta, \{J_t\}, [0, 1])]^{s_\epsilon}$ which has boundary points only at $t = 1$. But by compatability we have

$$N^{sym}_\beta = \text{signed sum of points in } [\tilde{\mathcal{M}}(\beta, \{J_t\}, [0, 1])]^{s_\epsilon}_{t=1}$$

and so

$$N^{sym}_\beta = \partial[\tilde{\mathcal{M}}(\beta, \{J_t\}, [0, 1])]^{s_\epsilon} = 0$$

This finishes the proof of theorem [1,2].
4.3. Conjecture 1.1  The whole idea again is to use the symplectic sum fibration \( \pi : Z \to \Delta \) we constructed before which borrows an anti-symplectic involution from \( X \), and mix it with a sum-formula similar to one in [IP2] relating the GW invariants of \( X_{in} \) and \( X_{out} \) to that of \( X \). Note that, only one component of a nodal \( J_0 \)-holomorphic discs in \( X_0 \) is a \( J_0 \)-holomorphic disc and it lies in \( X_{in} = \mathbb{C}P^3 \) with boundary on \( \mathbb{R}P^3 \) which is disjoint from intersection divisor. So the only modulies in \( X_{out} \) which will be involved correspond to close curves. To drive such formula one has to:

- determine the possible patterns of nodal \( J_0 \)-holomorphic discs in \( X_0 \) which contribute. These patterns can be modeled over trees with some labelings.
- calculate the relevant open and closed GW invariants in \( \mathbb{C}P^3 \) which contribute to sum-formula. These numbers appear as coefficients in the formula.
- Find the correct formula for open GW invariants \( N_{\beta}^{sum} \) in terms of numbers calculated in previous step and GW invariants of \( X_{out} \).

The homology classes in \( H_2(X_{out}, \mathbb{Z}) \) which will contribute are those elements \( F_{\beta,k} \) which we constructed in the proof of theorem 3.1.

Let me finish by giving a simple example:

Suppose \( X \) is Quintic and let \( X_{out} \) be the corresponding almost Calabi-Yau. Then for degree one holomorphic discs we know \( n_1^X = 30 \) (see [Jake]) where \( n_1^X \) is the open GW invariant counting number of degree one \( J \)-holomorphic discs in Quintic. The only possible pattern for limit of a degree one disc is a nodal \( J_0 \)-holomorphic disc inside \( X_0 \) composed of two components intersecting along a single point on \( D_a \), where the component in \( X_{in} \) is in class of generator of \( H_2(\mathbb{C}P^3, \mathbb{R}P^3) = \mathbb{Z} \) and the component in \( X_{out} \) is in class \( F_{1,1} \). Here \( F_{1,1} \in H_2(X_{out}, \mathbb{Z}) \) is the homology class in \( X_{out} \), corresponding to generator of \( H_2(X, \mathbb{R}P^3) = \mathbb{Z} \), having intersection number one with \( D_a \). Consider the moduli \( \mathcal{M}(F_{1,1}, X_{out}) \) which is of virtual dimension zero. Since \( F_{1,1} \) is a primitive class we have \( n_{F_{1,1}} := \# \mathcal{M}(F_{1,1}, X_{out}) \), is an integer number. Any element inside \( \mathcal{M}(F_{1,1}, X_{out}) \) has a unique intersection point with \( D_a \) and can be completed with a unique degree one holomorphic disc (half of a line) inside \( \mathbb{C}P^3 \) intersecting at the same point so we get:

\[
n_{F_{1,1}}^{X_{out}} = n_1^X = 30
\]

which is a simple relation between an ordinary GW invariant of \( X_{out} \) and an open invariant of \( X \). But for higher dimensions many complicated patterns happen which needs much work to build a relation, specially because one of the moduli spaces involved is a moduli space of holomorphic discs in \( \mathbb{C}P^3 \) and so we have to consider the orientation problem and \cdots.

We postpone the proof and a complete calculation of this conjecture to a sequel paper.

5. Disc bubble and linking number

The main goal of this section is to give some geometric intuition for the value of superpotential used in [FO-C] to compensate the effect of type (I) degenerations
(disc bubble). Let's first explain what is that superpotential and how is that constructed.

In [FOOO], for a Lagrangian \( L \) in a symplectic manifold \((X, w)\), an almost complex structure \( J \) on \( X \) and using all moduli of \( J \)-holomorphic discs \( M_{k+1}(\beta, J, X) \) at one place, they construct a filtered \( A_\infty \) algebra which is a sequence of operators \( m_{k, \beta}^J \) defined on the \( k \)-th tensor power of chain or cochain complex of \( L \) over Novikov ring. i.e
\[
m_{k, \beta}^J : B_k(\bar{C}[1]) \to \bar{C}[1]
\]
A solution of \( A_\infty \) version of Maurer-Cartan equation of this filtered \( A_\infty \) algebra associated to the \( L \) is called a bounding cochain. For a bounding cochain \( b \) the superpotential function is defined by :
\[
\Psi(b, J) = \sum_{k=0}^{\infty} \sum_{\beta} T \langle \frac{m_{k, \beta}^J(b, ..., b), b}{k+1} \rangle
\]
where \( \langle \cdot \rangle \) is some inner product and \( T \) is the parameter of Novikov ring. Other than \( (b, J) \) the value of \( \Psi(b, J) \) also depends of multi section used in the Kuranishi structure and some other data, but it is shown there that total function:
\[
\sum \tilde{N}_{\beta, J} T^{w(\beta)} = \sum N_{\beta, J/s} T^{w(\beta)} + \Psi(b, J)
\]
is independent of various choices made, but it still depends on choice of \( J \) due to existence of wall-crossing (degenerations of type (II)). Note that here \( N_{\beta, J/s} \) means the Euler characteristic of Kuranishi space \( \mathcal{M}(\beta, J, X, s) \) which can be considered virtually as number of \( J \)-holomorphic discs. Let's forget type (II) degenerations for a moment. Then the above statement means that although because of existence of type (I) degenerations the function \( \sum \tilde{N}_{\beta, J} T^{w(\beta)} \) changes by change of \( J \) (and also change of \( s ) \) but the superpotential also changes in a way that it compensates the change of that.

Our Goal is to give some geometric interpretation for change of superpotential as almost complex structure changes and so it would help to get a better picture of what is really going on geometrically.

From now on we will assume that \( X \) is a Calabi-Yau 3-fold and \( L \) is either sphere or real projective plane. Since both of them are orientable real 3-dimensional manifolds so they are spin, and using a spin structure and a choice of orientation on \( L \) we can put orientation on moduli spaces \( \mathcal{M}(X, L, \beta, J_t) \) for various \( \beta \in H^2(X, L) \). For a full discussion of orientation problem we refer the reader to chapter 8 of the book [FOOO].

Consider two almost complex structures \( J_0 \) and \( J_1 \) on \( X \) and a path of almost complex structures \( J_t, t \in [0, 1] \), connecting \( J_0 \) and \( J_1 \). Let \( \mathcal{M}(X, L, \beta, \{J_t\}) \) be the moduli space of \( J_t \)-holomorphic discs in class \( \beta \) for some \( t \in [0, 1] \). So \( \mathcal{M}(X, L, \beta, J_0) \) and \( \mathcal{M}(X, L, \beta, J_1) \) appear as two boundary components of \( \mathcal{M}(X, L, \beta, \{J_t\}) \) at \( t = 0 \) and \( t = 1 \). \( \mathcal{M}(X, L, \beta, \{J_t\}) \) has virtual dimension 1 and each section \( \mathcal{M}(X, L, J_t) \) has virtual dimension zero. Other than two boundary components at \( t = 0, 1 \), \( \mathcal{M}(X, L, \beta, \{J_t\}) \) has other boundary components at the middle which correspond to degenerations of type (I) and (II). So suppose as we travel from \( t = 0 \) to \( t = 1 \) a degeneration of type (I) happens which means we will see a nodal \( J_t \) disc \( u_{t_0} \) made of two components \( u_{t_0}^{\beta_0} \) and \( u_{t_0}^{\beta_1} \) in classes \( \beta_0 \) and \( \beta_1 \) respectively,
intersecting transversally at a boundary point \( p \in L \) such that \( \beta_0 + \beta_1 = \beta \). For simplicity assume that moduli space is transversal at both of these \( J_{t_0} \)-holomorphic discs which means changing \( t \) in some interval \([t_0 - \epsilon, t_0 + \epsilon]\) we can deform \( u^\beta_{t_0} \) to \( J_t \) holomorphic discs \( u^\beta_t \) for \( t \in [t_0 - \epsilon, t_0 + \epsilon] \). Consider the image of boundaries of \( u^\beta_t \) as a subset of \( L \times [t_0 - \epsilon, t_0 + \epsilon] \) which gives us two 2-cycles say \( A_0 \) and \( A_1 \) in \( L \times [t_0 - \epsilon, t_0 + \epsilon] \). Then the intersection point \( p \) corresponds to the intersection point \((p, t_0)\) of \( A_0 \) and \( A_1 \) in \( L \times [t_0 - \epsilon, t_0 + \epsilon] \) (for simplicity we are assuming that they intersect transversally). \( L \times [t_0 - \epsilon, t_0 + \epsilon] \) has an orientation induced by orientation of \( L \) and the canonical orientation of \( \mathbb{R} \), so we can assign a sign to the intersection at \((p, t_0)\) which is either \(-1\) or \(+1\). Two chains \( A_i \) can be considered as a subset of \( ev(\mathcal{M}_1(X, L, \beta_i, \{J_i\})) \) where \( \mathcal{M}_1(X, L, \beta_i, \{J_i\}) \) means the moduli space with one boundary marked point and \( ev \) is the evaluation map at marked point to \( L \times [t_0 - \epsilon, t_0 + \epsilon] \). Consider the curves \( \gamma^{1, t} = \partial u^\beta_t \subseteq L \) for each \( t \). So as \( t \) moves from \( t_0 - \epsilon \) to \( t_0 + \epsilon \) these curves intersect each other at time \( t_0 \) and their linking number changes by \( \pm 1 \).

**Lemma 5.1.** The change in the linking number:

\[
\text{Link}(\gamma^{0, t_0 + \epsilon}, \gamma^{1, t_0 + \epsilon}) - \text{Link}(\gamma^{0, t_0 - \epsilon}, \gamma^{1, t_0 - \epsilon})
\]

is equal to \( \text{sign}(A_0 \cdot A_1) \).

The proof is easy and we leave it to reader.

Now we try to find another interpretation for \( \text{sign}(A_0 \cdot A_1) \) based on calculations of \([\text{FO-CYC}]\).

Using the moduli spaces \( \mathcal{M}_{k+1}(X, L, \beta, \{J_t\}) \), one defines similar maps

\[
\hat{m}_{k, \beta} : B_k(\hat{C}(L \times I)[1]) \to \hat{C}(L \times I)[1]
\]

such that \( \hat{m}_{0, \beta} = m_{k, \beta}^t + c_{k, \beta}^t dt \) is a differential 2-form on \( L \times I \). (see remark 8.3 in \([\text{FO-CYC}]\)). \( m_{k, \beta}^t \) and \( c_{k, \beta}^t \) are differential 2-forms and 1-forms respectively on \( L \times I \) not involving \( dt \). Then its proved in theorem 3.1 of \([\text{FO-C}]\), that the change in the superpotential from time \( t = 0 \) to \( t = 1 \) is proportional to

\[
\sum_{\beta_0 + \beta_1 = \beta} \langle \hat{m}_{0, \beta_0}^t, \hat{m}_{0, \beta_1}^t \rangle
\]

where

\[
\langle \hat{m}_{0, \beta_0}^t, \hat{m}_{1, \beta_1}^t \rangle = \int_{L \times I} \langle \hat{m}_{0, \beta_0}^t, \hat{m}_{0, \beta_1}^t \rangle = \int_{L \times I} (m_{k, \beta_0}^t \wedge c_{k, \beta_1}^t dt + m_{k, \beta_1}^t \wedge c_{k, \beta_0}^t dt)
\]

But Geometrically the contribution of \( u^\beta_{t_0} \) to \( \hat{m}_{0, \beta}^{\{J_t\}} \) corresponds to the Poincare dual of \( ev(\mathcal{M}_1(X, L, \beta, \{J_t\})) \subset L \times I \) and so the equation above is equal to \( A_0 \cdot A_1 \).

From this we can present the following statement:

Observation:
"Although the number of $J$-holomorphic discs changes as we move from $t = 0$ to $t = 1$ but the quantity given by:
Number of curves in class $\beta$ - linking number between boundary of curves in subdivisions $\beta_0 + \beta_1 = \beta$
is a constant quantity”.

In this way we can interpret the numbers $n_{\beta,J}$ defined in [FO-C] by:

$$
\sum N_{\beta,J} T^{w(\beta)} + \Psi(b, J) = \sum_{d \in \mathbb{Z}} d^{-2} n_{\beta/d,J}
$$
as conjectrally integer numbers where for generic $J$ are equal to:

$$
n_{\beta,J} = \text{signed sum of individual smooth simple curves in class $\beta$} - \text{sum over linking number between boundary of $J$-holomorphic discs in classes $\beta_1$ and $\beta_2$ with $\beta = \beta_1 + \beta_2$}.
$$

Although what we said in previous statement is not rigrous and even not mathematically defined, but it helps to get a better picture of what should we looking for. Also note that we still have the wall-crossing problem which some from moduli of $J$-holomorphic spheres intersecting the Lagrangian.

6. Final remarks and questions

In this section focusing on Calabi-Yau 3-folds and K-3 surfaces, we will discuss some interesting phenomenona and questions. We won’t prove any thing in this chaper and just give some ideas for a future work. A typical example of a Calabi-Yau 3-fold is the a hypersurface of degree 5 in $\mathbb{C}P^4$ (Quintic) and a typical example of a K3 surface is a degree 4 hypersurface in $\mathbb{C}P^3$ (Quartic). While it is very easy to find a lot of $\mathbb{R}P^3$ Lagrangians in Quintic as fixed points of some anti-symplectic involution, it is not an easy job to construct other types of vanishing maslov class Lagrangians. You can see [RLB] for an example of construction of a Lagrangian tori. We can also find lots of 2-sphere Lagrangians in Quartic.

If we perform the surgery used in this paper for a $\mathbb{R}P^3$ Lagrangian in Quintic or a sphere lagrangian in Quartic we get new symplectic manifolds $X_{out}$ replacing Lagrangian with a divisor which has an interesting property:

$$
c_1^{X_{out}} = 0
$$

So from this way we can construct a big list of symplectic manifolds with trivial canonical bundle.

The first interesting question is :

Question 1 : " Whether $X_{out}$ has a Kahler structure ?"

If the Lagrangian $\mathbb{R}P^3$ we are using is non-trivial in 3rd Homology of $X$ then as a result of section[X] we see that the change in Betti numbers from $X$ to $X_{out}$ is as follows:

$$
h^3(X) - h^3(X_{out}) = 2 \quad h^2(X) - h^2(X_{out}) = 1 \quad h^4(X) - h^4(X_{out}) = 1
$$
which means we lose two Homology class at middle dimension and instead we get two \((p,p)\) class coming from the divisor and line class in it. In the case the answer to first question is yes it means the change in Hodge Diamonds is as follows:

\[
\begin{array}{cccc}
1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & a & 0 & 0 \\
1 & c & c & 1 \\
0 & a & 0 & 0 \\
0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 \\
\end{array}
\]

Changes to \(\Rightarrow\)

\[
\begin{array}{cccc}
1 & c - 1 & c - 1 & 1 \\
0 & a + 1 & 0 & 0 \\
0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 \\
\end{array}
\]

The second interesting question is:

**Question 2:** "How many times we can perform this surgery?"

Trivially the \(h^3(X)/2\) is an upper bound for this question. Let’s discuss this problem for the specific example of Quintic in \(\mathbb{CP}^4\) with \(h^{1,1} = 1\) and \(h^{1,2} = 101\). Then we can ask both questions for this specific example. For a moment suppose we can do this procedure 100 times and the answer to first question at each step is yes so after 100 times we get Calabi-Yau \(Y\) whose hodge diamond is mirror two previous one. Furthermore we will now an exact relation between middle dimension homology classes in \(X\) and \((p,p)\) classes in the resulting Calabi-Yau \(Y\). This would be an amazing relation (for some body interesting in mirror symmetry) between pairs \((X,Y)\) and also pairs obtained after \(i\)-times and \((100-i)\)-times of doing this surgery. They will all have mirror hodge diamonds with a relation between middle dimension homology classes of one with \((p,p)\) classes of the other one. At this moment I don’t know any answer for those two questions even for this specific example.

7. **Appendix: Lagrangians with vanishing Maslov class in C-Y manifolds**

We will first start by a linear algebra discussion:

Consider the vector space \(\mathbb{R}^{2n} \cong \mathbb{C}^n\) with standard basis \(\langle e_1, ..., e_n; e_{n+1}, ..., e_{2n} \rangle\). Let \((x,y) = (x_1, ..., x_n; y_1, ..., y_n)\) be coordinate of vectors in \(\mathbb{R}^{2n}\) with respect to this base. Now consider the following objects on \(\mathbb{R}^{2n} \cong \mathbb{C}^n\):

\[
w_0 = \sum_i dx_i \wedge dy_i,
\]

\[
J_0 : \mathbb{R}^{2n} \rightarrow \mathbb{R}^{2n},
\]

\[
\Omega_0 = dz_1 \wedge ... \wedge dz_n.
\]

where \(w_0\) is the standard symplectic structure on \(\mathbb{C}^n\), \(J_0\) is the standard complex structure given by \(J_0(e_i) = e_{n+i} \); \(J(e_{n+i}) = -e_i\) and \(\Omega_0\) is the complex volume n-form on \(\mathbb{C}^n\) where \(z_a = x_a + iy_a\). The subgroup of \(GL(2n, \mathbb{R})\) preserving all of these three objects is \(SU(n)\), so if we have a 2n-dimensional Riemannian manifold \((X^{2n}, g)\) such that the Holonomy group of \(g\), \(Holg\), is a subgroup of \(SU(n)\), then we can extend these three object defined over a tangent space of a fixed point \(x \in X\) to all of \(TX\) to get a tuple \((X, g, J, w, \Omega)\) where \(w\) is a symplectic 2-form, \(J\) is an integrable complex structure and \(\Omega\) is a non zero complex differential n-form, all
compatible with each other satisfying:

\[ \Omega \wedge \bar{\Omega} = \frac{2^n n!}{w_0^n} \]

Such manifold is called a Calabi-Yau manifold. The most important feature of a Calabi-Yau manifold in this paper is triviality of its canonical bundle which implies \( c_1^X = 0 \). We call a symplectic manifold almost Calabi-Yau if its canonical bundle is trivial.

Backing to linear model again and considering the subspace \( L_0 = \langle e_1, \ldots, e_n \rangle \subset \mathbb{R}^{2n} \) we will see that \( w \mid_{L_0} = 0 \) and so it is a Lagrangian subspace. The Lagrangian Grassmanian in \( \mathbb{R}^{2n} \) is equal to \( U(n)/O(n) \) where each Lagrangian subspace is obtained from \( L_0 \) via action of a matrix in \( U(n) \).

Suppose that \( X \) is Calabi-Yau as above and \( L \) is embedded oriented Lagrangian submanifold of \( X \) then we have:

**Lemma 7.1.** \( \Omega \mid_L \) is equal to \( f \cdot dvol_L \) where \( f : L \to U(1) \) is a function on \( L \) and \( dvol_L \) is the volume form induced by metric \( g \) on \( L \).

**Proof.** Start from a fix point \( x \in X \) and consider the linear model at \( T_x X \) as we did above and extend the triple \( (w_0, J_0, \Omega_0) \) to all of \( X \) by parallel translation. Also consider the \( L_0 \subset T_x L \) and extend it all over \( X \) via parallel translation to get lagrangian subspaces at each point of \( X \). This extension depends on actual path and does not result in a unique Lagrangian subspace at each point, but there is something which is independent of any path:

Consider an arbitrary point \( y \in X \) and an arbitrary Lagrangian subspace \( l \subset T_y L \). Also consider two different paths connection \( x \) and \( y \) say \( \gamma_0 \) and \( \gamma_1 \) and let \( l_0 \) and \( l_1 \) be corresponding Lagrangian subspaces at \( y \) obtained by parallel translation of \( L_0 \) along these paths. Then there are two matrices \( A_0 \) and \( A_1 \) in \( U(n) \) for which \( l = A_i \cdot l_i \). Since holonomy is a subgroup of \( SU(n) \) we get:

\[ f(l) := \text{Det}(A_0) = \text{Det}(A_1) \]

is independent of path chosen and only depends on initial space \( L_0 \) and \( l \). Now consider the situation of lemma, following the discussion above and starting from fixed point \( x \) and \( L_0 \) we will see that \( \Omega \mid_L(y) = f(T_y L) \cdot dvol \) where \( f \) is the function constructed above. \( \square \)

The function \( f \) in previous lemma has another interesting property:

**Lemma 7.2.** For any element \( \beta \in \pi_2(X, L) \) with boundary loop \( \alpha \in \pi_1(L) \) we have:

\[ \mu(\beta) = \text{winding number of } f^2 \mid_{\alpha \cdot S^1} : S^1 \to S^1. \]

**Proof.** Consider the loop \( \alpha \) an a trivialization of tangent space \( TL \) along \( \alpha \). Start from the point \( \alpha(1) \) and extend the unitary Lagrangian frame at this point to Lagrangian frames at each point \( \alpha(e^{it}) \). At each point \( \alpha(e^{it}) \) consider the matrix \( A(t) \) in \( U(n) \) which transforms the tangent plane at \( \alpha(e^{it}) \) to translated one. By definition maslov index is equal to winding number of square of determinant of matrix loop \( A(t) \) which by definition is nothing but winding number of \( f^2 \) along \( \alpha \). \( \square \)
Corollary 7.1. A lagrangian submanifold $L$ in a Calabi-Yau manifold $X$ has vanishing maslov class iff the corresponding function $f_L : L \to U(1)$ can be lifted to a function $\tilde{f}_L : L \to \mathbb{R}$. (and so $f_L = e^{2\pi i \tilde{f}_L}$).

Note that Lagrangian submanifolds with vanishing maslov class in Calabi-Yau 3-folds are those ones we work with and so above lemma gives a criterion to find which Lagrangians have vanishing maslov class. In common literature a Lagrangian submanifold with $f_L \equiv 1$ is called a special Lagrangians. But lets modify this definition a little bit and define:

Definition 7.1. We call $L$ to be a special Lagrangian if we have $f_L \equiv c$ for some constant $c \in U(1)$.

Special Lagrangians easily arise as fixed points of complex conjugations and by above criterion they have vanishing maslov class. But it is interesting to know which Lagrangians with vanishing maslov class can be deformed to a Special one. By a work of McLean [ML] we know that if $L$ is a special lagrangian then the moduli of special Lagrangian submanifold near that has dimension equal to first Betti number of $L$ and there is a correspondence between infinitesimal deformations and harmonic 1-forms on $L$. In What follows we briefly review a natural possible way of answering above question using the associated function $f_L$ of $L$.

Suppose we have a Lagrangian embedding $\iota : L \to X$ with vanishing maslov class. Using a function $h : L \to \mathbb{R}$ we can deform our Lagrangian embedding $\iota$ to another one in direction of exact 1-form $dh$, by considering path of Lagrangians given by graph of $d(h(t)), t \in \mathbb{R}$, in $T^*L$. Lets denote this path by $\iota_t(h) : L \to X$ and its image by $L^h_t$. The first thing we like to know is how does $\tilde{f}_L^h : L \to \mathbb{R}$ changes by change of $t$.

Let's define $D_L(h)$ to be infinitesimal change of $\tilde{f}_L$ in direction of $dh$. Then following the argument in the proof of theorem 3.6 in [ML] we get:

Theorem 7.1. \[ D_L(g) = \Delta h \]
Where $\Delta h$ is the laplacian of $h$ with respect to induced metric on $L$.

For an arbitrary function $f : L \to \mathbb{R}$ define $f^{avg}$ to be:

\[ f^{avg} = f - \frac{\int_L f \cdot dvol}{volume(L)} \]

which is the average free part of $f$. The big goal is to deform $\iota : L \to X$ to one for which the associated function $\tilde{f}_L$ is constant. So the above theorem suggests that the correct direction to deform $\iota : L \to X$ is the direction given by exact 1-form $dh$ satisfying:

\[ \Delta h = -\tilde{f}_L^{avg} \]

which certainly has a solution. For a function $g$ on $L$, define $k(g)$ to be the function which satisfies $\Delta k(g) = g^{avg}$. Then above theorem tells us that correct flow to consider is

\[ \frac{\partial}{\partial t} \iota_t = \zeta(k(f_t)) \]
where $f_t$ is the function $f$ associated to Lagrangian $\iota_t \subset X$ and $\zeta(k(f_t))$ is the vector field normal to Lagrangian which corresponds to function $k(f)$ by relation

$$d(k(g)) = -\iota_{\zeta} w$$

There has been many attempts to analyse this flow and find a solution to the question above, but there is no result yet.

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