On some problems related to the Hilbert-Smith conjecture

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Abstract. The Hilbert-Smith conjecture claims that if a compact group $G$ acts freely on a manifold, then it is a Lie group. For a finite-dimensional orbit space a reduction of the Hilbert-Smith conjecture to certain other problems in geometric topology is presented; in these the key problem is the existence of an essential sequence of lens spaces of increasing dimension.

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§1. Introduction

Hilbert’s fifth problem [23] has an as yet unresolved extension known as the Hilbert-Smith conjecture [43], [50].

Conjecture 1.1 (Hilbert-Smith conjecture). If a compact group $G$ acts effectively (freely) on a connected manifold, then $G$ is a Lie group.

It is known that the Hilbert-Smith conjecture is equivalent to the question of whether the group of $p$-adic integers $A_p$ can act effectively (freely) on a manifold [44].

The conjecture has been proved for $n$-dimensional manifolds with $n \leq 2$ [34] and $n = 3$ [35]. For arbitrary $n$ the Hilbert-Smith conjecture has been proved for smooth actions on a smooth manifold [34], for Lipschitz actions on Riemannian manifolds [38], for Hölder actions [30] and for quasi-conformal actions [31].

A quite deep but as yet unsuccessful line of research into the Hilbert-Smith conjecture is known as the orbit space method. For an effective action of a compact group $G$ on a space $X$ the formula for the dimension of the orbit space

$$\dim X/G = \dim X - \dim G$$

seems quite natural. Since $A_p$ is homeomorphic to the Cantor set and hence $\dim A_p = 0$, one would expect the dimension of the orbit space $M/A_p$ of an effective action of an $n$-manifold would equal $n$. However, in 1940 Smith found that this dimension is not $n$ [43]. Later Yang proved [51] that the cohomological dimension $\dim_\mathbb{Z} M/A_p$ of the orbit space of an effective $p$-adic action on an $n$-manifold $M$ equals $n + 2$. Therefore, by Alexandroff’s theorem [1], [49], which
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sas that cohomological and covering dimensions coincide in the case when the latter is finite, either \( \dim M/A_p = n + 2 \) or \( \dim M/A_p = \infty \). Other surprising properties of the orbit spaces of a hypothetical \( p \)-adic action on a manifold can be found in [6], [36], [37], [50] and [52]. There is still a remote hope that these bizarre properties of the orbit space \( M/A_p \) might lead to a contradiction and prove the Hilbert-Smith conjecture.

In this paper we consider the Hilbert-Smith conjecture under the assumption that the dimension of the orbit space is finite. We consider only free actions.

**Conjecture 1.2** (Weak Hilbert-Smith conjecture). If a compact group \( G \) acts freely on a manifold \( M \) and the dimension of the orbit space \( M/G \) is finite, then \( G \) is a Lie group.

We reduce the weak Hilbert-Smith conjecture to two problems which we call the **essential lens sequence problem** (Problem 3.1) and the **injectivity conjecture** (Conjecture 5.6). The reduction is based on an idea due to Williams to use the infinite product in the \( K \)-theory of some special version of a classifying space \( BA_p \) for the group \( A_p \) [50]. We should warn the reader that in the definition of classifying spaces for \( A_p \) in [50] the order of direct and inverse limit must be exchanged.

The essential lens sequence problem is a quest for a compact ‘classifying space’ (here we call it a **rough classifying space**) which has an infinite product in \( K \)-theory and hence is infinite dimensional. A finite dimensional compact rough classifying space for \( A_p \) was constructed by Floyd [50]. The existence of an infinite dimensional rough classifying space, together with Borel’s construction, would lead to examples of cell-like maps which are Hurewicz fibrations, but have nontrivial cokernels in \( K \)-theory. This looks surprising but does not contradict any known facts about cell-like maps. For instance, it is known that cell-like maps of manifolds can have nontrivial kernels in homological \( K \)-theory [13].

The injectivity conjecture is a technical statement about Hurewicz fibrations with fibre a closed manifold \( M \) over a fixed base \( B \). It states that for a fixed nonzero generalized cohomology class \( \alpha \in h^*(B \times M) \), for a sufficiently close approximation \( f : E \to B \times M \) of the trivial fibration \( \pi : B \times M \to B \) by a Hurewicz fibration \( p : E \to B \) with fibre \( M \) the map \( f \) takes \( \alpha \) to a nonzero element \( f^*(\alpha) \). We apply the injectivity conjecture when \( h^* \) is the reduced \( K \)-theory. We note that for general spaces \( B \) the concept of Hurewicz fibration is a peculiar one, since \( B \) does not necessarily support interesting homotopies. There is a seemingly weaker notion of a completely regular map which does not appeal to homotopy in its definition. It is not yet known whether every completely regular map is a Hurewicz fibration; this is the Hurewicz fibration problem. We conclude the paper by reducing the injectivity conjecture first to the Hurewicz fibration problem and then to ANR type properties of the classifying space for completely regular maps with a given manifold fibre.

**Theorem 1.3.** If both the essential lens sequence problem and the Hurewicz fibration problem have a positive solution, then the weak Hilbert-Smith conjecture for closed aspherical manifolds holds.

**Theorem 1.4.** If the essential lens sequence problem has a positive solution and the classifying space for completely regular maps with a given compact \( Q \)-manifold
is an absolute neighbourhood extensor for compact metric spaces, then the weak Hilbert-Smith conjecture holds true.

Our approach to the Hilbert-Smith conjecture does not exclude the possibility of a $p$-adic action on a manifold with an infinite-dimensional orbit space. On the other hand there are still no known examples of $p$-adic actions on a finite dimensional compact space with an infinite dimensional orbit space. However, there are examples of such actions of Cantor groups [15], [28].

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§2. Preliminaries

$p$-Adic actions. We denote the cyclic group of order $m$ by $\mathbb{Z}_m = \mathbb{Z}/m\mathbb{Z}$. Let $p$ be a prime number. The $p$-adic integers is a topological group defined as the inverse limit of the sequence

$$\mathbb{Z}_p \leftarrow \mathbb{Z}_p^2 \leftarrow \mathbb{Z}_p^3 \leftarrow \cdots,$$

where each bonding map $\mathbb{Z}_p^{k+1} \to \mathbb{Z}_p^k$ is the reduction mod $p^k$. Note that every closed subgroup of $\mathbb{A}_p$ has the form $p^k \mathbb{A}_p$. If the group $\mathbb{A}_p$ acts on a compact metric space $X$, then $X$ can be presented as the limit space of the inverse sequence

$$Y_0 \leftarrow Y_1 \leftarrow Y_2 \leftarrow Y_3 \leftarrow \cdots,$$

where $Y_0 = X/\mathbb{A}_p$, each space $Y_k$ is equal to the orbit space $X/p^k \mathbb{A}_p$ of the action of the subgroup $p^k \mathbb{A}_p$, each bonding map $q_k^{k+1}$ is the projection onto the orbit space of a $\mathbb{Z}_p$-action, and every composition

$$q_k^{k+i} = q_k^{k+1} \circ q_{k+1}^{k+2} \circ \cdots \circ q_{k+i-1}^{k+i} : Y_{k+i} \to Y_k$$

is the projection onto the orbit space of the action of the quotient group

$$\mathbb{Z}_{p^i} = p^k \mathbb{A}_p / p^{k+i} \mathbb{A}_p.$$

**Proposition 2.1.** Suppose that the group of $p$-adic integers $\mathbb{A}_p$ acts freely on a connected and locally connected compact metric space $X$ with an orbit space $Y$. Then $X$ and $Y$ can be represented as the inverse limit of sequences of simplicial complexes such that the diagram

$$K_0 \xleftarrow{\varphi_0} K_1 \xleftarrow{\varphi_1} K_2 \xleftarrow{\varphi_2} K_3 \xleftarrow{\cdots} X$$

is commutative, where each $p_k$ is the projection onto the orbit space of a free action of $\mathbb{Z}_{p^k}$ and each bonding map $\varphi_{k-1}^k$ is $\mathbb{Z}_{p^k}$-equivariant.
Proof. We take $Y = Y_0$ from (1) and construct the inverse sequence $\{L_i, \psi^i_{i-1}\}$ using nerves of a sequence of finite open covers $\{\mathcal{U}_i\}$ with mesh $\mathcal{U}_i \to 0$. Since $Y$ is locally connected, we may assume that all sets in each $\mathcal{U}_i$ are connected. Thus, $L_i = \text{Nerve}(\mathcal{U}_i)$. Let $\psi_i: Y \to L_i$ denote the projection onto the nerve. We recall that it is defined by means of a partition of unity subordinated to $\mathcal{U}_i$.

We may assume that each $U \in \mathcal{U}_i$ admits a section of $q_i$. Thus, the preimage $(q_0^i)^{-1}(U)$ is a disjoint union of $p^i$ copies of $U$. These copies of $U$ define a finite open cover $\mathcal{V}_i$ of $Y_i$. Let $K_i = \text{Nerve}(\mathcal{V}_i)$. Note that there exists a simplicial map $q_i: K_i \to L_i$ which is the projection onto the orbit space of the $\mathbb{Z}_{p^i}$-action. Moreover, there exists a map $\psi^i_i: Y_i \to K_i$ which defines a pull-back diagram

$$
\begin{align*}
K_i & \xleftarrow{\psi^i_i} Y_i \\
\downarrow{p_i} & \downarrow{q^i_0} \\
L_i & \xleftarrow{\psi_i} Y
\end{align*}
$$

It is easy to show that the multi-valued upper semi-continuous map $F: K_i \to K_{i-1}$ defined by $F(x) = \psi^i_{i-1}(\psi^i_i)^{-1}(x)$ is in fact single-valued. Thus $F$ defines a continuous map $\varphi^i_{i-1}$. Now we show that the square diagrams in our sequence are commutative, that is $p_{i-1} \varphi^i_{i-1} = \psi^i_{i-1} p_i$:

$$
p_{i-1} \varphi^i_{i-1} = p_{i-1} F(x) = p_{i-1} \psi^i_{i-1} ((\psi^i_i)^{-1}(x)) = \psi_{i-1} q^i_0 ((\psi^i_i)^{-1}(x)) = \psi_{i-1} (\varphi^i_{i-1}(x)) = \psi^i_{i-1} p_i(x).
$$

Clearly, $\lim \{K_i, \varphi^i_{i-1}\} = X$.

The proposition is proved.

**Borel construction.** Let a group $G$ act on spaces $X$ and $E$ with the projections $q_X: X \to X/G$ and $q_E: E \to E/G$ onto the orbit spaces. Let $q_{X \times E}: X \times E \to X \times_G X = (X \times E)/G$ denote the projection onto the orbit space of the diagonal action of $G$ on $X \times E$. Then there is a commutative diagram, called the Borel construction [4]:

$$
\begin{align*}
X & \xleftarrow{pr_X} X \times E \xrightarrow{pr_E} E \\
\downarrow{q_X} & \downarrow{q_{X \times E}} & \downarrow{q_E} \\
X/G & \xleftarrow{p_E} X \times_G E \xrightarrow{p_X} E/G
\end{align*}
$$

Suppose that $G$ is compact and the actions are free. Then if $q_E$ is a locally trivial bundle, so is $p_X$. In particular, this holds true for free actions of compact Lie groups. Since the projection of the limit space of an inverse sequence whose bonding maps are locally trivial fibrations onto the first space of the sequence is a Hurewicz fibration, using the approximation of a compact group by compact Lie groups we see that in the case of free $G$-actions all projections in the Borel construction are Hurewicz fibrations. The fibre $p_X^{-1}(y)$ is homeomorphic to $X/I_z$, where $I_z = \{g \in G \mid g(z) = z\}$ is the isotropy group of $z \in q_E^{-1}(y)$. 


§ 3. Essential sequences of lens spaces

For an integer $m$ we denote the standard $(2n-1)$-dimensional lens space mod $m$ by

$$L^n(m) = S^{2n-1}/\mathbb{Z}_m,$$

where the $\mathbb{Z}_m$-action on the $(2n-1)$-sphere $S^{2n-1} \subset \mathbb{C}^n$ is obtained from a rotation through $2\pi/m$ in every coordinate plane $\mathbb{C}$ in $\mathbb{C}^n$.

Note that the classifying space $B\mathbb{Z}_m = K(\mathbb{Z}_m, 1)$ can be represented as the increasing union $\bigcup_n L^n(m)$.

We call a map between lens spaces $q: L^m(p^k) \to L^n(p^\ell)$ essential if it induces an epimorphism of the fundamental groups. A sequence of mappings of spheres

$$S^{k_0} \leftarrow f_0^1 S^{k_1} \leftarrow f_1^2 S^{k_2} \leftarrow f_2^3 \cdots$$

is called inessential if for every $i$ there exists $j$ such the $f_i^{i+1} \circ \cdots \circ f_i^{i+j}$ is null-homotopic.

**Problem 3.1** (Essential lens sequence problem). Given an odd prime $p$, does there exist an infinite sequence of lens spaces

$$L^{n_0}(p^{k_0}) \leftarrow q_0^1 L^{n_1}(p^{k_1}) \leftarrow q_1^2 L^{n_2}(p^{k_2}) \leftarrow q_2^3 \cdots,$$  \hspace{1cm} (2)

with essential bonding maps, $k_i \to \infty$, and $n_i > p^{k_i-k_0}$, such that the sequence of covering spheres

$$S^{2n_0-1} \leftarrow \bar{q}_0^3 S^{2n_1-1} \leftarrow \bar{q}_1^2 S^{2n_2-1} \leftarrow \bar{q}_2^1 \cdots$$  \hspace{1cm} (3)

is inessential?

It is known (see § 4) that $d(c)$, the dimension of a lens space in an essential sequence as a function of the cardinality of its fundamental group, can grow at most linearly. Floyd’s example [50] gives us an essential sequence of lens spaces with constant function $d(c) = 3$. Using ideas from [12] it is possible to construct an essential sequence with $d(c) \sim \log c$. It turns out that for our applications to the Hilbert-Smith conjecture we need $d(c)$ to be linear. This requirement is spelled out in the condition $n_i > p^{k_i-k_0}$.

The ELS problem can be stated for concrete values of $k_i$ and $n_i$:  

1) $k_i = i + 1$ and $n_i = p^i + 1$;
2) $k_i = i + 1$ and $n_i = p^i + 2$;
3) $k_i = 2^i$ and $n_i = p^{2^{i-1} + 1}$.

Suppose that the required sequence in Problem 3.1 does exist. Then the maps $q_i^{i+1}: L^{n_i+1}(p^{k_i+1}) \to L^{n_i}(p^{k_i})$ define maps $\bar{q}_i^{i+1}: S^{2n_i+1-1} \to S^{2n_i-1}$ between the universal covers. Denote the corresponding inverse limits by

$$E = \lim\downarrow \{S^{2n_i-1}, \bar{q}_i^{i+1} \}$$

and

$$B = \lim\downarrow \{L^{n_i}(p^{k_i}), q_i^{i+1} \}.$$  \hspace{1cm} (4)

Then the following proposition is straightforward.
Proposition 3.2. The compact set $E$ is cell-like and there is a free $A_p$-action on $E$ with the orbit space $B$.

Using Ferry’s theorem [18] we can modify the bonding maps (possibly with stabilizations) in the inverse sequence $\lim\{L^n_i(p^k_i), q^{i+1}_i\}$ to $UV^0$-maps. Then we can assume that $B$ and $E$ are path connected and locally path connected.

$K$-theory of lens spaces. For any $r$ the actions of the groups $\mathbb{Z}_{pr} \subset \mathbb{Z}_{pr+1} \subset S^1$ on $S^\infty$ and $S^{2n-1}$ form a commutative diagram

$$
\begin{array}{cccc}
S^\infty & \longrightarrow & B\mathbb{Z}_{pr} & \longrightarrow & B\mathbb{Z}_{pr+1} & \longrightarrow & \mathbb{C}P^\infty \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
S^{2n-1} & \longrightarrow & L^n(p^r) & \longrightarrow & L^n(p^{r+1}) & \longrightarrow & \mathbb{C}P^n
\end{array}
$$

The canonical line bundle $\eta$ over $\mathbb{C}P^\infty$ defines the line bundles over all spaces in the diagram. This bundle defines an element in the $K$-theory of the Eilenberg-MacLane space $B\mathbb{Z}_{pk}$ which will be denoted by $\eta_k$.

Atiyah computed that the $K$-theory ring of $B\mathbb{Z}_{pk}$ equals the completion of the representation ring $R\mathbb{Z}_{pk}$ of $\mathbb{Z}_{pk}$ (see [2]). We recall that $R\mathbb{Z}_{pk} = \mathbb{Z}[\eta]/(1 - \eta^{pk})$ where $\eta$ is the class of complex representation of $\mathbb{Z}_{pk}$ generated by the group embedding $\eta: \mathbb{Z}_{pk} \to S^1$. Note that $\eta_k$ is obtained from $\eta$ by passing to the map of classifying spaces $\eta: B\mathbb{Z}_{pk} \to BS^1 = \mathbb{C}P^\infty$ and pulling back the canonical complex line bundle. Thus, taking the completion we obtain $K^0(B\mathbb{Z}_{pk}) = (R\mathbb{Z}_{pk})^\wedge = \mathbb{Z}[[\eta_k]]/(1 - \eta_k^{pk})$, where $A[[x]]$ denotes the ring of formal series with the variable $x$ and coefficients in $A$. Note that the mod $p^k$ reduction homomorphism $\mathbb{Z}_{pk+r} \to \mathbb{Z}_{pk}$ takes the generator $\eta_k$ to $\eta_{k+r}$.

Let $\eta_k$ denote the restrictions of these classes to the $(2n - 1)$-dimensional lens space $L^n(p^k)$. The $K$-theory of this lens space was computed in [29] and [24]:

$$K^0(L^n(p^k)) = \mathbb{Z}[[\eta_k]]/(1 - \eta_k^{pk}, (\eta_k - 1)^n).$$

We note that the ideal generated by $\eta_k - 1$ in the above ring is isomorphic to the reduced $K$-theory of $L^n(p^k)$ (see [25]).

Proposition 3.3. For any positive integers $k > l$ and $m < p^l$, where $p$ is prime, the polynomial $(x^{p^k-l} - 1)^m$ does not belong to the ideal $\langle x^{p^k} - 1, (x - 1)^n \rangle$ of the polynomial ring $\mathbb{Z}[x]$, provided that $mp^{k-l} < n$.

Proof. We make the change of variable $y = x - 1$. Thus, we need to show that $((y + 1)^{p^k-l} - 1)^m$ does not belong to $\langle (y + 1)^{p^k} - 1, y^n \rangle$. The mod $p^k-l$ reduces this problem to whether $y^{mp^{k-l}}$ belongs the ideal $\langle y^{p^k}, y^n \rangle$. Since $mp^{k-l} < \min\{p^k, n\}$, the answer to the question is negative and the result follows.

Proposition 3.4. Let $q_i: B \to \mathbb{L}^{n_i}(p^{k_i})$ be the projection of the limit in the inverse system (4). Suppose that $m < p^{k_i-k_0}$. Then the induced homomorphism in the reduced $K$-theory

$$q_i^*: \mathbb{K}^0(\mathbb{L}^{n_i}(p^{k_i})) \to \mathbb{K}^0(B)$$

takes $(\eta_{k_i} - 1)^m$ to a nonzero element.
We apply Proposition 3.3 with $k = k_{i+j}$, $l = k_i$ and $n = n_{i+j}$ and obtain
$$(\eta_{k_{i+j}} - k_i - 1)^m \neq 0,$$
that is, the assumptions of Proposition 3.3 are satisfied. The proposition is proved.

**Corollary 3.5.** The reduced $K$-theory cup length of $B$ is unbounded.

§ 4. The lens sequence problem and the Schwarz genus

We consider the level functions defined in [32]

$$v_{p,k}(m) := \min \{ n : \exists f : L^m(p^{k-1}) \to S^{2n-1} \},$$

where $f$ is a $\mathbb{Z}_p$-equivariant map with respect to the standard free actions. Lower bounds for these functions were given by Vick [48], rediscovered by Bartsch [3], and formulated in this way by Meyer [32]:

$$v_{p,k}(m) \geq \left\lceil \frac{m-1}{p^{k-1}} \right\rceil + 1.$$

Meyer also showed that $v_{p,2}(m) = \frac{m-2}{p} + 2$ for odd $p$ and $m = 2 \mod p$ [32].

**Remark 4.1.** The existence of a sequence of lens spaces as in Problem 3.1 does not contradict the above estimates of the level functions. Indeed, for $i > j$ the composition $q_{j+1} \circ \cdots \circ q_{i-1}$ induces a $\mathbb{Z}_p$-equivariant map $L^{n_i}(p^{k_i-k_j}) \to S^{2n_j-1}$. Therefore, we have the inequality

$$n_j \geq v_{p,k_i-k_j+1}(n_i) \geq \left\lceil \frac{n_i-1}{p^{k_i-k_j}} \right\rceil + 1 \geq \frac{p^{n_i-k_0}}{p^{k_i-k_j}} + 1 > p^{k_j-k_0},$$

which is consistent with the conditions on $k_i$ and $n_i$.

**Schwarz genus.** We recall that the Schwarz genus $\text{Sg}(f)$ of a fibration $f : E \to B$ is the minimal $k$ such that $B$ can be covered by open sets $A_1, \ldots, A_k$ such that $p$ admits a section on each $A_i$ [41].

**Proposition 4.2** (see [41]). $\text{Sg}(f) \leq n$ if and only if $\ast^n f : \ast^n B E \to B$ admits a section.

Free action of $\mathbb{Z}_r$ on $S^1$ determines a free $\mathbb{Z}_r$-action on $S^{2m-1} = \ast^m S^1$, and free action of $\mathbb{Z}_p$ on $S^{2m-1}$ determines a free $\mathbb{Z}_r$-action on $L^m(p)$. In the question below we consider the free $\mathbb{Z}_{p^k}$-actions on $L^m(p)$ and $S^1$ determined in this way.

Let $\pi^m_k = p_{S^1} : W^m_k = L^m(p) \times_{\mathbb{Z}_{p^k}} S^1 \to L^m(p^{k+1})$ be the $S^1$-bundle from the Borel construction for $\mathbb{Z}_{p^k}$-actions on $S^1$ and $L^m(p)$.

**Theorem 4.3.** There is a map $q : L^m(p^{k+1}) \to L^n(p^k)$ that induces an epimorphism of the fundamental groups if and only if $\text{Sg}(\pi^m_k) \leq n$. 
Proof. Such a map $q$ exists if and only if there is a $\mathbb{Z}_{p^k}$-equivariant map $q': L^m(p) \to S^{2n-1}$ for free actions. This is equivalent to the existence of a section of the locally trivial $S^{2n-1}$-bundle $\pi: L^m(p) \times_{\mathbb{Z}_{p^k}} S^{2n-1} \to L^m(p^{k+1})$ from the Borel construction.

Since the sphere $S^{2n-1} = \ast_{i=1}^n S^1$ is the join product of $n$ circles and the action comes from a free $\mathbb{Z}_{p^k}$-action on $S^1$, the bundle $\pi$ is the fibrewise join product of $n$ copies of the $S^1$-bundle $\pi^m_k$. Then the result follows from Proposition 4.2. The theorem is proved.

**Corollary 4.4.** The inequality $\text{Sg}(\pi^m_k) \geq \lceil \frac{m-1}{p} \rceil + 1$ holds.

For odd $p$, $\text{Sg}(\pi^m_k) = \frac{m-2}{p} + 2$ provided $m = 2 \mod p$.

**Proof.** Suppose that $\text{Sg}(\pi^m_k) = n$. Then there exists a map $q: L^m(p^{k+1}) \to L^n(p^k)$ that induces an epimorphism of the fundamental groups. Going over to $\mathbb{Z}_{p^k}$-covers we obtain a $\mathbb{Z}_{p^k}$-equivariant map $f: L^m(p) \to S^{2n-1}$. Thus, the map $f$ is also $\mathbb{Z}_p$-equivariant, as $\mathbb{Z}_p \subset \mathbb{Z}_{p^k}$. Hence $v_{p,2}(m) \leq n$. Thus, we have proved the inequality $\text{Sg}(\pi^m_k) \geq v_{p,2}(m)$. The results due to Vick, Bartsch and Meyer, cited above, imply that $\text{Sg}(\pi^m_k) \geq \lceil \frac{m-1}{p} \rceil + 1$. The corollary is proved.

**Problem 4.5.** 1) Given the $S^1$-bundle

$$\pi^m_k: L^m(p) \times_{\mathbb{Z}_{p^k}} S^1 \to L^m(p^{k+1})$$
from the Borel construction, what is its Schwarz genus?

2) In particular, if $m = 2 \mod p$, is $\text{Sg}(\pi^m_k) = \frac{m-2}{p} + 2$ for all $k$?

**Corollary 4.6.** If Problem 4.5, 2) has a positive answer, then so does Problem 3.1.

**Proof.** We take $k_i = i$ and $n_i = 3p^i - \sum_{s=1}^{i-1} p^s + 1$. Clearly, the condition $n_i > p^{k_i-k_0}$ is satisfied. Note that $m = n_{i+1} + 1 = 2 \mod p$ and

$$\frac{m-2}{p} + 2 = 3p^i - \sum_{s=1}^{i-1} p^s - 1 \leq n_i.$$
If $\text{Sg}(\pi^m_{i_1}) = \frac{m-2}{p} + 2$, then by Theorem 4.3 there exists an essential map $f_{i+1}^i: L^m(p^{i+1}) \to L^{n_1}(p^i)$. We define $\varphi_{i+1}^i: L^{n_{i+1}}(p^{i+1}) \to L^{n_1}(p^i)$ to be the restriction of $f_{i+1}^i$ to $L^{n_{i+1}}(p^{i+1}) \subset L^m(p^{i+1})$. Then the corresponding sequence of spheres is inessential. The corollary is proved.

§ 5. **A reduction of the weak Hilbert-Smith conjecture**

**Dimension, LS-category and cup-length.** We recall that the Lusternik-Schnirelmann category of a topological space $X$ (LS-category for short) satisfies $\text{cat}(X) \leq n$ if there is an open cover $U_0, \ldots, U_n$ of $X$ by $n+1$ sets contractible in $X$. It is known that in the case of ANR we can take closed or even arbitrary $U_i$’s [45]. We refer to [11] for the general properties of the LS-category. The basic properties are that $\text{cat}(X)$ is a homotopy invariant, is bounded above by the dimension $\text{dim } X$ and is bounded below by the length of a nonzero cup product of nonzero dimensional elements in cohomology. It is known that the cohomology in this cup product can be generalized or even 0-dimensional if we use a reduced cohomology theory defined by means of a spectrum [46].
Proposition 5.1. The LS-category of a finite connected complex is greater than or equal to the cup-length for any reduced generalized cohomology theory $\tilde{h}^*$.

Proof. This proposition can be deduced from [40]. We give an alternative short proof. Assume that $w = \alpha_1 \cup \cdots \cup \alpha_k \neq 0$ in $\tilde{h}^*(X)$. Suppose that $\text{cat}(X) \leq k - 1$. Then there is a cover $U_1, \ldots, U_k$ of $X$ by subcomplexes contractible in $X$ (for some subdivision). The long exact sequence of pair $(X, U_i)$ for the reduced $h$-cohomology and the fact that $\tilde{h}^*(X) \rightarrow \tilde{h}^*(U_i)$ are 0-homomorphisms imply that the homomorphisms $\tilde{h}^*(X; U_i) \rightarrow \tilde{h}^*(X)$ are isomorphisms in all dimensions. Let $\bar{x}_i$ denote the corresponding elements in $\tilde{h}^*(X; U_i)$. Then the product $w = \alpha_1 \cup \cdots \cup \alpha_k$ is distinct from zero. On the other hand, $w \in \tilde{h}^*(X; U_1 \cup \cdots \cup U_k) = \tilde{h}^*(X; X) = 0$. We have a contradiction. The proposition is proved.

Corollary 5.2. The cup-length of a finite connected complex for any generalized reduced cohomology theory $\tilde{h}^*$ does not exceed the dimension of the complex.

Theorem 5.3. For a finite dimensional compact metric connected space $X$, the cup-length for any reduced generalized cohomology theory does not exceed $\text{dim } X$.

Proof. Let $\text{dim } X = n$. Then $X$ can be presented as the inverse limit of a sequence of $n$-dimensional polyhedra $X = \limleftarrow L_m$. If

$$\alpha_1 \cup \cdots \cup \alpha_k \neq 0$$

in $\tilde{h}^*(X)$, then there exists $m$ such that $\alpha_i = p_m^*(\beta_i)$, $i = 1, \ldots, k$, and

$$\beta_1 \cup \cdots \cup \beta_k \neq 0,$$

where $p_k: X \rightarrow L_k$ is the projection in the inverse system. By Corollary 5.2, $k \leq n$. The theorem is proved.

Injectivity conjecture. The Chapman-Ferry $\alpha$-approximation theorem [9] has several versions. For instance Theorems 1–4 in [20] are all variations of it. One of the weakest versions states that for a fixed metric on a closed manifold $M$ for each $\delta > 0$ there is $\varepsilon > 0$ such that each $\varepsilon$-map $f: M \rightarrow M$ is a $\delta$-homotopy equivalence. The following conjecture is a parametrized version of this version of the $\alpha$-approximation theorem with a fixed space of parameters $B$. Since we do not assume that $B$ is nice, in our conjecture we replace the homotopy equivalence by a shape equivalence.

Let $F$ be a compact metric space. We say that a fibration $p: E \rightarrow B$ is a fibration with isometric fibres $F$ if there exists a metric on $E$ such that all fibres $p^{-1}(x)$, $x \in B$, are isometric to $F$.

Conjecture 5.4 (Parametrized $\alpha$-approximation conjecture). For every compact manifold $M$ (or $Q$-manifold) with a fixed metric on it and any connected and locally connected compact space $B$ there exists $\varepsilon > 0$ such that for any Hurewicz fibration $p: E \rightarrow B$ with isometric fibres $M$ every fibrewise $\varepsilon$-map $f: E \rightarrow M \times B$ is a shape equivalence. In particular, it induces an isomorphism of the generalized cohomology groups.

The $\alpha$-approximation conjecture seems out of reach. For applications to the Hilbert-Smith conjecture it suffices to prove the following.
Conjecture 5.5 (Injectivity conjecture). For each closed manifold (or $Q$-manifold) $M$ with a fixed metric on it, any connected and locally connected compact space $B$ and any nonzero element $\alpha \in h^*(B)$ for some generalized cohomology theory $h^*$ there exists $\varepsilon > 0$ such that for every Hurewicz fibration $p: E \to B$ with isometric fibres $M$ and every fibrewise $\varepsilon$-map $f: E \to M \times B$ the image $f^*\pi^*(\alpha)$ is distinct from zero, where $\pi: M \times B \to B$ is the projection.

We note that the $\alpha$-approximation theorem holds true for Hilbert cube manifolds ($Q$-manifolds). Thus, it makes sense to extend the injectivity conjecture to $Q$-manifolds as well. We note that the injectivity conjecture for $Q$-manifolds implies the injectivity conjecture for ordinary manifolds via multiplication by the Hilbert cube.

Let $G$ be a compact metrizable topological group. The orbit space of a free $G$-action on a Peano continuum of a trivial shape will be called a rough classifying space for $G$.

Corollary 5.6 (A consequence of the injectivity conjecture). Let $B$ be a rough classifying space for $A_p$. Suppose that $A_p$ acts on a compact manifold (or $Q$-manifold) $M$. Then for any nonzero $\alpha \in K^0(B)$ there exists $k$ such that $p^*_M(\alpha) \neq 0$ where $p_M: M \times A_p E \to B$ is the projection from the Borel construction for the action of the subgroup $p^k A_p \cong A_p$ on $M$.

Proof. Let $\mu$ be an invariant measure on $A_p$. Integrating a metric $d$ on $M$ over $A_p$ with respect to $\mu$ gives an $A_p$-invariant metric on $M$:

$$
\rho(x, y) = \int_{A_p} d(gx, gy) \, d\mu.
$$

We take any metric $d'$ on $E$ and consider the $\ell_1$-product metric $\rho + d'$ on $M \times E$. This defines the quotient metric on each of the orbit spaces $E_k = M \times A_p E$ for the diagonal action of $A_p$ where for the action on $M$ the group $A_p$ is identified with $p^k A_p$. Thus, $p_M: E_k \to B$ is a fibration with isometric fibres $M$.

We claim that for large $k$ the total space $E_k = M \times A_p E$ of the Borel construction of the action of $p^k A_p$ on $M$ admits a retraction onto a fibre $M$, whose restriction to any other fibre is an $\varepsilon_k$-map with $\varepsilon_k \to 0$. There are several ways to prove this. We leave the proof to the reader. One approach would be to argue that the composition of the inverse to the quotient map $M \times E \to E_k$ followed by a retraction $r: M \times E \to M$ to a fibre defines a multivalued retraction of the required kind with the diameter $d_k$ of the images of points tending to 0. This and the fact that $M$ is ANR would be sufficient to derive our claim.

Now the injectivity conjecture implies the required result.

The corollary is proved.

The following is the main result of the paper.

Theorem 5.7. Suppose that the injectivity conjecture holds true and there exists an infinite sequence of lens spaces as in the essential lens sequence problem. Then there is no free $A_p$-action on a closed manifold with a finite dimensional orbit space.

Proof. Assume that a free $A_p$-action on an $n$-manifold $M$ with $\dim M/A_p < \infty$ exists. Then by the Yang’s theorem $\dim M/A_p = n + 2$. Let $B$ and $E$ be as
in Proposition 3.2. We apply the injectivity conjecture to the product of desired length
\[ \alpha = \alpha_1 \cdots \alpha_{n+3} \in \tilde{K}^0(B), \]
defined by Corollary 3.5. Then \( p^*_M(\alpha) \neq 0 \) for the action of \( p^kA_p \) for some \( k \).

By Proposition 3.2 the projection \( p_E \) from the Borel construction for this action is a cell-like map. Since \( \dim M/p^kA_p < \infty \), it is a shape equivalence \([27]\) and hence it induces an isomorphism in \( K \)-theory. Hence for each \( i \) there is \( \beta_i \in K^0(M/p^kA_p) \) such that \( p^*_E(\beta_i) = p^*_M(\alpha_i) \). Hence \( \beta_1 \cdots \beta_{n+3} \neq 0 \). Now we look at the commutative diagram
\[
\begin{array}{ccc}
M & \xrightarrow{pr_n} & M \times E \\
qu_M & & \downarrow{q_M \times E} & \downarrow{q_E} \\
M/p^kA_p & \xrightarrow{p_E} & M \times A_p \xrightarrow{p_M} B
\end{array}
\]
Since \( \dim M/(p^kA_p) = n + 2 \), we obtain a contradiction to Theorem 5.3.

The theorem is proved.

§ 6. The Hurewicz fibration problem

**Completely regular maps.** Dyer and Hamstrom introduced the concept of a completely regular map in [16]. We recall that a continuous surjection \( p: E \to B \) between metric spaces is called **completely regular** if for all \( b \in B \) and \( \varepsilon > 0 \) there exists \( \delta(b, \varepsilon) > 0 \) such that if \( d_B(b, b') < \delta \), then there exists a homeomorphism \( h: p^{-1}(b) \to p^{-1}(b') \) with \( d_E(x, h(x)) < \varepsilon \) for all \( x \in p^{-1}(b) \). It is known that the complete regularity of a map does not depend on the choice of metrics \( d_B \) and \( d_E \).

Using Michael’s selection theorem [33], Dyer and Hamstrom proved the following theorem [16].

**Theorem 6.1.** Suppose that for a compact \( F \) the space \( \text{Homeo}(F) \) is locally contractible. Then every completely regular map \( p: E \to B \) with fibre \( F \) and a finite-dimensional \( B \) is a locally trivial fibration.

The mistake in the proof presented in [16] was corrected in [22]. A detailed proof can be found in [39]. Similar or stronger related results were later proven in [26], [42], [8] and [19].

Ferry’s \( \alpha \)-approximation theorem (Theorem 1 from [20]) admits the following variation.

**Theorem 6.2.** Let \( M \) be a closed \( n \)-manifold, \( n \geq 5 \), with a fixed metric. Then for any \( \varepsilon > 0 \) there is \( \delta > 0 \) such that for every \( \delta \)-map \( g: M \to N \) onto a closed \( n \)-manifold \( N \) there exists a homeomorphism \( h: N \to M \) such that the composition \( h \circ g \) is \( \varepsilon \)-close to the identity \( 1_M \).

Like the proof of Ferry’s original theorem, the proof of this variation uses the consecutive application of Theorems 2, 3 and 4 in [20]. We use the following weaker version of Theorem 6.2 for compact \( Q \)-manifolds.
Theorem 6.3. Let $M$ be a compact $Q$-manifold with a fixed metric. Then for any $\varepsilon > 0$ there is $\delta > 0$ such that for every $\delta$-map $g: M \to M$ there exists a homeomorphism $h: M \to M$ such that the composition $h \circ g$ is $\varepsilon$-close to the identity $1_M$.

Proof. First we observe that for every compact metric ANR $M$, for each $\beta > 0$ there is $\delta > 0$ such that each $\delta$-map $g: M \to M$ is a $\beta$-homotopy equivalence. Let $f: M \to M$ denote the corresponding homotopy inverse map for $g$. Then to obtain the required homeomorphism $h$ we apply the $\alpha$-approximation theorem for $Q$-manifolds to $f$ (Theorem 3.1, [21]); this states that given a compact $Q$-manifold $M$ and $\alpha > 0$, there is $\beta > 0$ such that any $\beta$-equivalence $f: M \to M$ is $\alpha$-close to a homeomorphism. The theorem is proved.

We recall that a map $f: X \to Y$ of a subset $X$ of a metric space $(Y, d)$ is called an $\varepsilon$-move if $d(x, f(x)) < \varepsilon$ for all $x \in X$.

Proposition 6.4. Let $M$ be a compact ANR with a fixed metric. Then given $n \in \mathbb{N}$ and $\varepsilon_0 > 0$, there is $\delta_0 > 0$ such that for every isometric embedding $M \subset X$ for any $n$-dimensional compact set $Z \subset N_{\delta_0}(M)$ in the $\delta_0$-neighbourhood of $M$ there exists a continuous $\varepsilon_0$-move $r: Z \to M$.

Proof. By the Lefschetz criterion for ANRs (see [5], Theorem 8.1), for any $\varepsilon > 0$ there is $\delta > 0$ such that for every map of the vertices $f: K^{(0)} \to M$ of a $n$-dimensional simplicial complex $K$, satisfying $d(f(v), (v')) < \delta$ for every edge $[v, v'] \subset K$, there exists an extension $\overline{f}: K \to M$ with $\text{diam} \overline{f}(\Delta) < \varepsilon$ for each simplex $\Delta \subset K$.

We prove the proposition when $Z$ is a polyhedron. The general case (not needed in this paper) can be obtained by approximating $Z$ by nerves of small open covers.

We take $\varepsilon < \varepsilon_0/3$ to obtain $\delta < \varepsilon$ from the Lefschetz criterion. Take $\delta_0 < \delta/4$ and consider a triangulation of $Z$ with the mesh $\delta'$ satisfying $\delta' < \delta - 2\delta_0$. We can assume that $\delta' < \varepsilon_0 - \varepsilon - \delta_0$. We define $r$ on the vertices of $Z$ by sending each vertex $v$ to a nearest point of $M$. Thus, $d(v, r(v)) < \delta_0$. Then for any edge $[v, v']$ we obtain $d(r(v), r(v')) < 2\delta_0 + \delta' < \delta$. Let $r: Z \to M$ be an extension given by the Lefschetz criterion. Then for each $z \in Z$ we consider a simplex $\Delta \subset Z$ that contains $z$ and fix a vertex $v \in \Delta$. By the triangle inequality

$$d(z, r(z)) \leq d(z, v) + d(v, r(v)) + d(r(v), r(z)) < \delta' + \delta_0 + \varepsilon < \varepsilon_0.$$

The proposition is proved.

Theorem 6.5. Let $\varphi: X \to Y$ be a continuous map between compact metric spaces such that all point preimages $\varphi^{-1}(y)$ are isometric to a closed $n$-manifold $M$, $n \geq 5$. Then $\varphi$ is a completely regular map.

Proof. Let $y \in Y$ and $\varepsilon > 0$ be given. Proposition 6.4 implies that there is a neighbourhood $U(y)$ of $y \in Y$ such that for every $y' \in U(y)$ there exists a $\delta/2$-move $r: \varphi^{-1}(y') \to \varphi^{-1}(y)$, where we set $\delta$ to be $\varepsilon/2$ in the notation of Theorem 6.2. Let $r': \varphi^{-1}(y) \to \varphi^{-1}(y')$ be a similar map back. We can assume that $r' \circ r$ is homotopic to the identity. Therefore, we can assume that $r$ is surjective. Let $i: M \to \varphi^{-1}(y')$ be an isometry. Note that $r \circ i$ is a $\delta$-map. Hence there exists
a homeomorphism $h: \varphi^{-1}(y) \to M$ with $d_M(hri(z), z) < \varepsilon/2$. Note that the homeomorphism $i \circ h$ is an $\varepsilon$-move:

$$d_X(ih(x), x) = d_X(ihri(z), ri(z)) \leq d_X(ihri(z), i(z)) + d_X(i(z), ri(z))$$

$$= d_M(hri(z), z) + d_X(i(z), ri(z)) < \varepsilon/2 + \frac{\delta}{2} < \varepsilon.$$  

Here $z \in M$ is such that $ri(z) = x$. Such a $z$ exists in view of the surjectivity of $r$.

The theorem is proved.

We note that Theorem 6.5 holds true for $Q$-manifolds as well. For the proof we use the following version of Proposition 6.4.

**Proposition 6.6.** Let $M$ be a compact $Q$-manifold with a fixed metric. Then given $n \in \mathbb{N}$ and $\varepsilon_0 > 0$, there is $\delta_0 > 0$ such that for all isometric embeddings $j: M \to X$ and $i: M \to N_{\delta_0}(M)$ there exists a continuous $\varepsilon_0$-move $\psi: i(M) \to j(M)$.

**Proof.** By the triangulation theorem for $Q$-manifolds [7] there exists a finite polyhedron $K \subset M$ and an $\varepsilon_0/2$-retraction $r_0: M \to K$. We apply Proposition 6.4 to $M$ with $n = \dim K$ and $\varepsilon_0/2$ to obtain an $\varepsilon_0/2$-move $r: K \to j(M)$. Then $\psi = r \circ i \circ r_0 \circ i^{-1}: i(M) \to j(M)$ is an $\varepsilon_0$-move. The proposition is proved.

**Question 6.7** (Hurewicz fibration problem). Is every completely regular map with a manifold fibre a Hurewicz fibration?

In view of Theorem 6.1, when the base is infinite-dimensional this is an open problem. It is known that a completely regular map is a Serre fibration. We refer to [14] for further discussion of the fibration problem.

**Completely regular maps in the Borel construction.** Suppose that a compact group $G$ acts freely on a metric space $E$ with the orbit space $B$. Suppose that it also acts on a compact space $F$. We can assume that $G$ acts by isometries.

**Proposition 6.8.** The projection $p_F: F \times_G E \to B$ in the Borel construction is completely regular.

**Proof.** Fix $y \in q_E^{-1}(x)$. Since $q_E$ is open, any sequence $x_n$ converging to $x$ in $B$ admits a lift $y_n$ converging to $y$ in $E$. Then

$$(q_F \times E)(1_F \times c_i)((q_F \times E)|_{F \times \{y\}})^{-1}: p_F^{-1}(x) \to p_F^{-1}(x_k)$$

is a sequence of homeomorphisms converging to the identity $id: p_F^{-1}(x) \to p_F^{-1}(x)$ where $c_k: y \to y_k$ is the map of one-point spaces and $q_F \times E: F \times E \to F \times_G E$ is the orbit map of the diagonal action. The proposition is proved.

We use the notation Homeo($M$) for the group of homeomorphisms of a manifold $M$ with the compact-open topology. By Homeo$_0(M)$ we denote the subgroup of homeomorphisms isotopic to the identity. For manifolds with boundary we let Homeo($M, \partial M$) denote the group of homeomorphisms of $M$ which are the identity on $\partial M$. 

Question 6.9. Let $M$ be a manifold and let $G \subset \text{Homeo}_0(M)$ be a compact subgroup that admits a deformation $H : G \times [0, 1] \to \text{Homeo}(M)$ to the identity element 1. Does there exist such a deformation $H$ through homomorphisms, that is, such that $h_t = H(-, t) : G \to \text{Homeo}(M)$ is a group homomorphism for every $t$?

Theorem 6.10. Suppose that Questions 6.7 and 6.9 have positive answers. Then the injectivity conjecture is true.

Proof. Let $A_p$ act freely on a manifold $F$ (or $Q$-manifold). Let $h_t$ be a deformation of $A_p$ to the identity in $\text{Homeo}(F)$ by virtue of a family of subgroups $h_t(A_p) \subset \text{Homeo}(M)$. We define an $A_p$-action on $F \times [0, 1]$ by letting the group $h_t(A_p)$ act on $F \times \{t\}$. Let $B$ be a rough classifying space for $A_p$ with the universal covering $q_E : E \to B$.

The projection $p_{F \times [0, 1]}$ in the Borel construction factors through the map

$$p : (F \times [0, 1]) \times_{A_p} E \to B \times [0, 1]$$

with the fibre $F$. We will show that $p$ is completely regular. Taking an invariant metric on $F \times [0, 1]$ we can assume that $p$ has isometric fibres over $B \times t$ for every $t$.

Using Proposition 6.8 (or Theorem 6.5 for $n \geq 5$) we see that $p$ is completely regular over each $B \times t$. Thus, it suffices to prove that $p$ is completely regular over $b \times [0, 1]$ uniformly in $b \in B$ in the following sense: the number $\delta((b, t), \varepsilon)$ from the definition of complete regularity can be chosen independent of $b$. For $(b, t)$ and $(b, t')$ we define a homeomorphism of fibres $h_{t, t'}$ by fixing $e \in E$ with $q_E(e) = b$ and identifying $p^{-1}(b, t)$ with $F \times t \times e$ by means of the inverse of the projection onto the orbit space $q : F \times I \times E \to (F \times I) \times_{A_p} E$, then translating it to $F \times t' \times e$ and projecting by $q$ to $p^{-1}(b, t')$. This homeomorphism does not depend on the choice of $e$. The translation of $F \times t \times e$ to $F \times t' \times e$ is an $\varepsilon$-move, where $\varepsilon$ depends only on $t$ and $|t - t'|$, and $\varepsilon \to 0$ as $t' \to t$. Thus, $h_{t, t'}$ is an $\varepsilon$-move for all $b \in B$.

If Question 6.7 has an affirmative answer, then $p$ is a Hurewicz fibration. Then the identification $F \times \{1\} \times B = p^{-1}(B \times \{1\})$ extends to a fibrewise map $F \times [0, 1] \times B \to (F \times [0, 1]) \times_{A_p} E$ over $B \times [0, 1]$. The restriction of this fibrewise map over $B \times \{0\}$ yields a splitting of the fibrewise map in the injectivity conjecture.

The theorem is proved.

Proposition 6.11. Suppose that $A_p \subset \text{Homeo}(D^n, \partial D^n)$. Then $A_p$ admits a deformation $h_t : A_p \to \text{Homeo}(D^n, \partial D^n)$ to the identity such $h_t$ is a group homomorphism for every $t$.

Proof. This follows from Alexander’s trick. Let $tD^n$ denote the image of $D^n$ under multiplication by $t \leq 1$. Extending the identity homeomorphism of the boundary $\partial tD^n$ to $D^n \setminus tD^n$ using the identity defines an embedding

$$h_t : \text{Homeo}(tD^n, \partial tD^n) \to \text{Homeo}(D^n, \partial D^n),$$

of topological groups with the image of $h_t$ converging to the identity element as $t \to \infty$. By precomposing this embedding with the given embedding $A_p \to \text{Homeo}(D^n, \partial D^n)$ and the isomorphism

$$(L_t)_* : \text{Homeo}(D^n, \partial D^n) \to \text{Homeo}(tD^n, \partial tD^n),$$
where \( L_t : D^n \to tD^n \) is multiplication by \( t \) we obtain the desired deformation. The proposition is proved.

**Corollary 6.12.** If every completely regular map with fibre \( D^n \) is a Hurewicz fibration, then the injectivity conjecture holds true for any uniformly bounded \( A_p \)-action on \( \mathbb{R}^n \).

We call a \( G \)-action on a metric space \( X \) *uniformly bounded* if there is an upper bound for the diameter of orbits. Note that an action of \( A_p \) on a closed aspherical manifold defines a uniformly bounded action on its universal cover.

Perhaps Edwards-Kirby’s theorem would allow one to extend Proposition 6.11 to all manifolds. We recall that by the Edwards-Kirby theorem [17] (which goes back to the proof of Chernavsky’s theorem on the local contractibility of \( \text{Homeo}(M) \) [10]) every homeomorphism \( h : M \to M \) of a closed manifold which is homotopic to the identity can be presented as a finite composition \( h = h_n \circ \cdots \circ h_1 \) of homeomorphisms fixing the complement to a ball.

### §7. The moduli space of topological manifolds

Let \( F \) be a compact metric space. We set

\[
\text{Emb}(F) = \{ \varphi : F \to s \}
\]

to be the space of all topological embeddings of \( F \) into the pseudo-interior \( s = (0, 1)\omega \) of the Hilbert cube \( Q = [0, 1]\omega \) with the supremum metric:

\[
d(\varphi_1, \varphi_2) = \sup \{ \| \varphi_1(x) - \varphi_2(x) \| \mid x \in F \}.
\]

The pseudo-interior is chosen so that we only need deal with tame embeddings.

The following theorem implies that the space \( \text{Emb}(F) \) is an absolute neighbourhood extensor for compact metrizable spaces, that is, it is in the class \( \text{ANE(')} \). In particular, it implies that \( \text{Emb}(F) \) is \( n \)-connected and locally \( n \)-connected for all \( n \).

**Theorem 7.1** (see [7]). Let \( (A, A_0) \) be a compact pair and let \( f : A \to s \) be a given map. Then for each \( \varepsilon > 0 \) there exists an \( \varepsilon \)-close map \( g : A \to s \) that agrees with \( f \) on \( A_0 \) and is an embedding on \( A \setminus A_0 \).

We note that the group of homeomorphisms of \( F \), endowed with the compact-open topology, \( H = \text{Homeo}(F) \), acts on \( \text{Emb}(F) \) from the right by composition:

\( \varphi \to \varphi \circ h, h \in H \). Thus \( \varphi_1, \varphi_2 \in \text{Emb}(F) \) are in the same orbit if and only if \( \text{im} \varphi_1 = \text{im} \varphi_2 \). Note that \( H \) acts on \( \text{Emb}(F) \) by isometries: \( d(\varphi_1, \varphi_2) = d(\varphi_1 \circ h, \varphi_2 \circ h) \).

We call the orbit space of this action the *moduli space* of \( F \) and denote it by \( \mathcal{M}(F) \).

Let \( q_F : \text{Emb}(F) \to \mathcal{M}(F) = \text{Emb}(F)/H \) be the projection onto the orbit space. We consider the quotient metric \( \rho \) on \( \mathcal{M}(F) \):

\[
\rho(\varphi_1 H, \varphi_2 H) = \inf \{ d(\varphi_1 \circ h, \varphi_2) \mid h \in H \}.
\]

We will check that \( \rho \) is a metric. It is symmetric, since \( d(\varphi_1 \circ h, \varphi_2) = d(\varphi_1, \varphi_2 \circ h^{-1}) \). If \( \varphi_1 H \neq \varphi_2 H \), then \( \text{im} \varphi_1 \neq \text{im} \varphi_2 \). Then for any \( h_1, h_2 \in H \),

\[
d(\varphi_1 \circ h_1, \varphi_2 \circ h_2) \geq d^Q_H(\text{im} \varphi_1, \text{im} \varphi_2) > 0,
\]
where \( d_H^Q \) is the Hausdorff distance between closed subsets of \( Q \). For \( i = 1, 2 \) let \( h_i \) be such that \( d(\varphi_i h_i, \varphi_3) - \rho(\varphi_i H, \varphi_3 H) < \varepsilon/2 \). Then the triangle inequality follows as \( \varepsilon \to 0 \):

\[
\rho(\varphi_1 H, \varphi_2 H) \leq d(\varphi_1 h_1, \varphi_2 h_2) \leq d(\varphi_1 h_1, \varphi_3) + d(\varphi_2 h_2, \varphi_3) < \rho(\varphi_1 H, \varphi_3 H) + \rho(\varphi_3 H, \varphi_2 H) + \varepsilon.
\]

We will identify each orbit \( \varphi H \in \mathcal{M}(F) \) with the subset \( \varphi(F) \) of the Hilbert cube.

**Proposition 7.2.** For \( F_1, F_2 \in \mathcal{M}(F) \), \( \rho(F_1, F_2) < \varepsilon \) if and only if there exists a homeomorphism \( g : F_1 \to F_2 \) with the displacement

\[
D_g = \max\{\|g(x) - x\| \mid x \in F_1\} < \varepsilon.
\]

**Proof.** Let \( F_i = \varphi_i(F), \varphi_i \in \text{Emb}(F), i = 1, 2 \).

If \( \rho(F_1, F_2) < \varepsilon \) then \( d(\varphi_1, \varphi_2 h) < \varepsilon \) for some \( h \in H \), and so \( D_g < \varepsilon \) if \( g = \varphi_2 h \varphi_1^{-1} \).

Conversely, if \( D_g < \varepsilon \) then \( d(\varphi_1, \varphi_2 h) < \varepsilon \) for \( h = \varphi_2^{-1} g \varphi_1 \).

The proposition is proved.

Let \( \mathcal{E}(F) = \{(F', x) \in \mathcal{M}(F) \times s \mid x \in F'\} \subset \mathcal{M}(F) \times s \) and let \( \nu_F : \mathcal{E}(F) \to \mathcal{M}(F) \) be the restriction of the projection onto the first factor. We note that \( \nu_F \) is completely regular.

**Proposition 7.3.** For each continuous map \( f : X \to \mathcal{M}(F) \) the pull-back \( f^*(\nu_F) \) is completely regular.

For each completely regular map \( p : X \to Y \) between compact metric spaces with fibre \( F \) there exists a continuous map \( f : Y \to \mathcal{M}(F) \) such that \( p = f^*(\nu_F) \).

**Proof.** Any embedding \( j : X \to s \) defines a map \( f : Y \to \mathcal{M}(F) \) by \( f(y) = j(p^{-1}(y)) \).

Let \( y_k \to y \) be a convergent sequence in \( Y \). Then there is a sequence of homeomorphisms \( h_k : j(p^{-1}(y)) \to j(p^{-1}(y_k)) \) with \( D_{h_k} \to 0 \). By Proposition 7.2 \( \rho(f(y), f(y_k)) \to 0 \). Therefore, \( f \) is continuous. Clearly, \( p \) is isomorphic to \( f^*(\nu_F) \).

The proposition is proved.

**The moduli space of \( Q \)-manifolds.**

**Proposition 7.4.** Let \( F \) be such that \( \text{Homeo}(F) \) is locally contractible. Then

1) \( \mathcal{M}(F) \) is locally path connected.

2) \( q_F : \text{Emb}(F) \to \mathcal{M}(F) \) is a Serre fibration with fibre \( q_F^{-1}(y) \cong \text{Homeo}(F) \) for all \( y \in \mathcal{M}(F) \).

**Proof.** 1) Let \( F_1 \) and \( F_2 \) be elements of \( \mathcal{M}(F) \) at a distance \( \rho(F_1, F_2) < \delta \). Thus, \( F_1, F_2 \subset s \) and there exists a homeomorphism \( h : F_1 \to F_2 \) with displacement \( D_h < \delta \).

Let \( f : F \to F_1 \) be a homeomorphism. We consider a linear homotopy \( H : F \times I \to s \) between \( f \) and \( h \circ f \). By Theorem 7.1 there exists a \( \delta \)-approximation \( H' : F \times I \to s \) of \( H \) by an embedding that coincides with \( H \) on \( F \times \{0, 1\} \). This defines a path from \( F_1 \) to \( F_2 \) in \( \mathcal{M}(F) \) of diameter \( < 2\delta \).

2) Let \( H : I^n \times I \to \mathcal{M}(F) \) and \( h : I^n \times \{0\} \to \text{Emb}(F) \) with \( q_F h = H|_{I^n \times \{0\}} \).

In view of Theorem 6.1, \( H^*(\nu_F) \) is a locally trivial fibre bundle with fibre \( F \). The map \( h \) defines a trivialization of \( H^*(\nu_F) \) over \( I^n \times \{0\} \). The projection \( I^n \times I \to I^n \) defines an extension of this trivialization to a trivialization over the whole of \( I^n \times I \). This trivialization defines a lift \( \overline{H} \) of \( H \) that extends \( h \).

The proposition is proved.
We use the standard notation $\text{LC}^n$ for the class of locally $n$-connected spaces.

**Theorem 7.5** (Ungar [47]). Let $p: E \to B$ be a Serre fibration of metric spaces, let $E$ lie in $\text{LC}^n$, $p^{-1}(b)$ in $\text{LC}^{n-1}$ for all $b \in B$ and let $B$ lie in $\text{LC}^0$. Then $B$ lies in $\text{LC}^n$.

**Corollary 7.6.** Suppose that $H = \text{Homeo}(F)$ is a locally contractible. Then $\mathcal{M}(F)$ is in $\text{LC}^n$ for all $n$.

**Proof.** We apply Theorem 7.5 to the map $q_F: \text{Emb}(F) \to \mathcal{M}(F)$. Using the fact that $\text{Emb}(F) \in \text{ANE}$, in view of Proposition 7.4 we find that $\mathcal{M}(F)$ is $\text{LC}^n$. The corollary is proved.

We denote the class of absolute neighbourhood extensors for $n$-dimensional compact metric spaces by $\text{ANE}(n)$. Note that Kuratowski’s theorem characterizes $\text{ANE}(n)$ spaces as $\text{LC}^n$ spaces.

**Theorem 7.7.** For each compact $Q$-manifold $F$, $\mathcal{M}(F) \in \text{ANE}(n)$ for all $n$.

**Proof.** We apply Ferry’s theorem [21], which states that $\text{Homeo}(F)$ is an ANE for a $Q$-manifold $F$, and use Corollary 7.6 and Kuratowski’s characterization of $\text{ANE}(n)$. The theorem is proved.

**Problem 7.8.** Let $F$ be a compact $Q$-manifold. Is the space $\mathcal{M}(F)$ an absolute neighbourhood extensor for compact metric spaces?

Since $\mathcal{M}(F)$ is the orbit space of an action by isometries of an ANE group upon an ANE($\mathcal{C}$) space, it would not be a big surprise if the above problem has a positive answer.

**Theorem 7.9.** An affirmative answer to Problem 7.8 implies the injectivity conjecture.

**Proof.** Assume that the injectivity conjecture fails to hold for $M$. Then there exists a compact space $B$, nonzero $\alpha \in h^*(B)$, a sequence of Hurewicz fibrations $p_k: E_k \to B$ with isometric fibres $M$ and a sequence of fibrewise $\varepsilon_k$-maps $f_k: E_k \to M \times B$, $\varepsilon \to 0$, such that $f_k^*\pi_B^*(\alpha) = 0$ for all $k$. The latter implies that $p_k^*(\alpha) = 0$ for all $k$. Here $\pi_B$ and $\pi_M$ denote the projections of the product $B \times M$ onto its factors.

We define the compactification $X$ of $\bigsqcup E_k$ by $M$ to be the subspace

$$X = \bigsqcup G_k \cup \{a\} \times M \subset \alpha\left(\bigsqcup_k E_k\right) \times M$$

of the product of the one-point compactification of the union of $E_k$ and $M$, where $G_k \subset E_k \times M$ is the graph of the composition $\pi_M \circ f_k$ and $a$ is the compactifying point in $\alpha(\bigsqcup_k E_k)$. Proposition 6.8 and Theorem 6.3 imply that the union of the $p_k$ defines a completely regular map $p: X \to \alpha(\bigsqcup B_k)$ with fibre $M$, where each $B_k$ is homeomorphic to $B$. By Proposition 7.3, $p = f^*(\nu_M)$ for some continuous map $f: \alpha(\bigsqcup B_k) \to \mathcal{M}(M)$.

We present $B = \varprojlim \{L_i, \varphi_j^i\}$ as the limit of an inverse sequence of compact polyhedra. Let $T$ be the natural compactification of $\bigsqcup L_i$ by $B$. If $\mathcal{M}(M)$ is
an ANE for compact metric spaces, then there exists an extension $\overline{f} : W \to \mathcal{M}(M)$ to a neighbourhood of $\alpha(\bigcup B_k)$ in $\alpha(\bigcup T_k)$. We note that there exists a $k$ such that $T_k \subset W$ and the restriction of $\overline{f}$ to $T_k$ is null-homotopic. Let $\overline{p}_k : Z_k \to T_k$ be the restriction of $\overline{f}|_{\nu M}$ over $T_k$.

For sufficiently large $i$, there is an element $\alpha_i \in h^*(L_i)$ that maps to $\alpha \in h^*(B)$. Let $U_i \subset T$ be the compactification of $L_i \sqcup L_{i+1} \sqcup \cdots$ by $B$. The bonding map $B \to L_i$ factors through $U_i$; let $\alpha'_i \in h^*(U_i)$ be the image of $\alpha_i$. Let $V_i \subset Z_k$ be the preimage in $Z_k$ of the copy of $U_i$ in $T_k$. Let $\beta'_i \in h^*(V_i)$ be the image of $\alpha'_i$. Since $\alpha'_i$ maps to $\alpha$, the image of $\beta'_i$ in $h^*(E_k)$ is the same as the image of $\alpha$, which is trivial by assumption. Hence the image of $\beta'_i$ in $h^*(V_j)$ is trivial for some $j > i$. If $\alpha'_j \in h^*(U_j)$ is the image of $\alpha'_i$, then the image of $\alpha'_j$ in $h^*(V_j)$ is trivial. Finally, if $\alpha_j \in h^*(L_j)$ is the image of $\alpha_i$, then $\alpha_j$ maps to $\alpha'_j$ under the map $U_j \to L_j$, which in turn goes to 0 under the restriction $V_j \to U_j$ of $\overline{p}_k$. The composition $\overline{p}_k^{-1}(L_j) \to V_j \to U_j \to L_j$ coincides with the restriction $(\overline{p}_k)^{-1}(L_j) \to L_j$ of $\overline{p}_k$, and therefore $\alpha_j$ goes to zero under the latter. By Theorem 6.1, $\overline{p}_k$ is a locally trivial bundle over $L_j$. Since $\overline{f}|_{L_j}$ is null-homotopic, we obtain that $\overline{p}_k$ is a trivial bundle. This brings a contradiction.

The theorem is proved.

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