POLYAKOV CONJECTURE AND 2+1 DIMENSIONAL GRAVITY\textsuperscript{a}

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After briefly reviewing the hamiltonian approach to 2+1 dimensional gravity in absence of matter on closed universes, we consider 2+1 dimensional gravity coupled to point particles in an open universe. We show that the hamiltonian structure of the theory is the result of a conjecture put forward by Polyakov in a different context. A proof is given of such a conjecture. Finally we give the exact quantization of the two particle problem in open space.

1 Introduction

Gravity in 2 + 1 dimensions\textsuperscript{1} has attracted considerable interest in the past few years. Solutions in 2 + 1 dimensions are also solution of 3 + 1 dimensional gravity in presence of a space-like Killing vector field and for this reason they describe cosmic strings in some special configuration.

The simplifying feature of gravity in 2 + 1 dimensions is that the Riemann tensor is a linear function of the Ricci tensor

\[ R_{\mu\nu}^{\lambda\rho} = \epsilon^{\mu\nu\sigma} \epsilon_{\lambda\rho\sigma} (R^\sigma - \delta^\sigma_2 R) \]  

and as such it vanishes outside the sources; there are no gravitational waves in 2 + 1 dimensions. Despite that, the theory is far from trivial especially if we couple it to matter, and it retains several features of the theory in higher dimensions.

We shall see in this paper how the theory is strictly related to very interesting mathematical structures, like Liouville theory and the Riemann- Hilbert problem. The hamiltonian structure of the theory will be related to a conjecture put forward by Polyakov in the context of two dimensional quantum gravity and we shall give a proof of such a conjecture. It is possible to solve explicitly the quantum problem in some special cases and this results can throw light in the far more complicated 3 + 1 dimensional case.  

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In addition to the Hamiltonian approach there are several other approaches to the problem. In this paper we shall summarize some old results and some recent developments with regard to the Hamiltonian formulation of the theory. We shall insist on the conceptual side so that the paper can work as a simple introduction to the subject; the technical details can be found in the published papers and reports.

2 Gravity in 2 + 1 dimensions in absence of matter

In absence of matter the only degrees of freedom are given by the moduli of the space sections; thus we can deal only with genus \( g \geq 1 \).

In absence of boundaries the action of the gravitational field reduces to the Einstein-Hilbert term which can be put in Hamiltonian form as

\[
S_H = \int dt \int_{\Sigma} d^Dx \left[ \pi^{ij} \dot{g}_{ij} - N^i H_i - NH \right]
\] (2)

where we used the standard ADM metric

\[
ds^2 = -N^2 dt^2 + 2g_{z\bar{z}}(dz + N^z dt)(d\bar{z} + N^{\bar{z}} dt).
\] (3)

The choice of the gauge is of crucial importance in dealing with the problem. The well known York gauge in which the time slices are provided by the \( D \) (in our case \( D = 2 \)) dimensional surfaces with \( K = \text{const.} \), being \( K \) the trace of the extrinsic curvature tensor is particularly powerful as this gauge decouples the solution of the diffeomorphism constraint from the Hamiltonian constraint. Exploiting this feature it is possible to solve the diffeomorphism constraint and to provide the Hamiltonian on the reduced phase space given by the moduli and their conjugate momenta. The number of moduli are \( 6g - 6 \) for genus \( g \) larger than 1 and 2 for the torus topology. The explicit computation of the Hamiltonian can be performed only in the simplest case of torus topology. It is given by

\[
H = \sqrt{\tau_1^2 + \tau_2^2},
\] (4)

where the \( p_j \) are the momenta conjugate to the moduli \( \tau_j \). Hamilton equations for the Hamiltonian eq.(4) can be exactly solved to obtain

\[
\tau_1 = c + a \tanh(t - t_0)
\]

\[
\tau_2 = \frac{a}{\cosh(t - t_0)}
\] (5)

i.e. in the coordinate space the point moves along a semicircle lying in the upper half plane with diameter on the real axis; such semicircles are the geodesic...
of the Poincaré half-plane. Quantization proceeds by replacing the canonical variables with operators according to the corresponding principle. The ordering problem always subsists; the most natural ordering translates the classical Hamiltonian into the square root of the Maass Laplacian giving rise to the Schrödinger equation

\[ i\frac{\partial \psi(x, y, t)}{\partial t} = \sqrt{-y^2(\partial^2_x + \partial^2_y)} \psi(x, y, t). \]  

(6)

The Maass Laplacian has been widely investigated by mathematicians. The classical as well the quantum Hamiltonians are invariant under modular transformations which in the case of the torus are given by the group \( SL(2, \mathbb{Z}) \) and thus the solutions should be invariant under such modular transformations. The eigenvalue problem under this condition is not trivial; nevertheless it has been thoroughly studied, the general properties of the spectrum is known even if it is not known in closed analytical form and the eigenfunctions cannot be simply expressed in terms of familiar functions. Such an approach gives a complete quantum treatment of universes without matter with torus topology in the York gauge. Higher genera are difficult to treat as we do not know an explicit form of the Hamiltonian for \( g \geq 2 \) and probably here one will be able to assert only qualitative features of the problem.

### 3 Gravity in 2 + 1 dimensions in presence of matter

Obviously one can consider the case of closed and open universes. The case of closed universes in presence of matter appears more difficult to attack as in addition to the particle positions and their conjugate momenta we have to deal with the moduli of the space surface and their conjugate momenta. This however does not appear to be the main difficulty. The favorable feature of working in an open universe is that here the maximally slicing gauge (or Dirac gauge) can be employed, which simplifies notably both the diffeomorphism and the Hamiltonian constraint.

In presence of point particles the action has to be supplemented by the matter action

\[ S_m = \int dt \sum_n \left( P_{n1} \dot{q}_n^i + N^i(q_n) P_{ni} - N(q_n) \sqrt{P_{ni} P_{nj} g^{ij}(q_n) + m_n^2} \right) \]  

(7)

and for open universes by the boundary terms which play a fundamental role. These are given by

\[ S_B = -\int dt H_B \]  

(8)
with

\[ -H_B = 2 \int_{B_t} d^{(D-1)}x \sqrt{\sigma_B} N \left( K_{B_t} + \frac{\eta}{\cosh \eta} D_\alpha v^\alpha \right) - 2 \int_{B_t} d^{(D-1)}x r_\alpha \pi^{\alpha \beta}_{(\sigma_B)} N_\beta. \]  

(9)

The really important term turns out to be the first i.e. \( \sqrt{\sigma_B} N K_{B_t} \) where \( K_{B_t} \) is the extrinsic curvature of the \( D - 1 \) dimensional boundary (in our case one-dimensional) of the time slices as a sub-manifold embedded in the \( D \) dimensional time slices and \( d^{(D-1)}x \sqrt{\sigma_B} \) is the volume form induced by the space metric on the \( D - 1 \) dimensional boundary.

We recall that the action \( S_H + S_B + S_m \) is so constructed as to provide the correct equations of motion (i.e. Einstein’s equations) when one computes a stationary point of the action by keeping the values of the metric fixed on the boundary. Such a procedure is equivalent to the weaker requirement of keeping fixed the intrinsic metric of the boundary.

One could in principle adopt also here the York slicing. However the equations for the diffeomorphism and hamiltonian constraints are not at all trivial. In particular the hamiltonian constraint gives rise to an equation more complex than the inhomogeneous sinh-Gordon equation. Progress has been achieved by the introduction of the instantaneous York gauge\(^{17,18,19,20,21}\), or maximally slicing gauge. This is defined by all time slices having \( K = 0 \).

Simple application of the Gauss- Bonnet theorem shows that such a gauge can be applied only to open universes, or universes with the topology of the sphere\(^{18,20}\). A closer inspection shows that for the sphere topology such a gauge can describe only the simple stationary case\(^{20}\). Thus application of the \( K = 0 \) gauge is practically restricted to open universes, but here it proves very powerful. The technical reason is the immediate solution of the diffeomorphism constraint given by

\[ \pi^{\bar{z}}_z = -\frac{1}{2\pi} \sum_n \frac{P_n}{z - z_n} \equiv -2 \prod_B (z - z_B) \frac{P(z)}{P(z)}, \]  

(10)

where \( z_n \) and \( P_n \) are the complex positions and canonical momenta of the particles. The \( P_n \) are subject to the constraint \( \sum_n P_n = 0 \)\(^{22}\) which is related to the fact that translations are not symmetries of the problem but simply gauge transformations\(^{23,24}\). The hamiltonian constraint reduces to an inhomogeneous Liouville equation, given by

\[ 2\Delta \tilde{\sigma} = -e^{-2\tilde{\sigma}} - 4\pi \sum_n \delta^2(z - z_n)(\mu_n - 1) - 4\pi \sum_B \delta^2(z - z_B), \]  

(11)
to which powerful mathematical methods apply. Here $\sigma$ is defined by

\[ e^{2\sigma} = 2\pi \bar{z} z e^{2\bar{\sigma}}, \quad (12) \]

and is related to the space metric by $g_{ij} = \delta_{ij} e^{2\sigma}$. In the above equation the sources are given by the particles and in addition by the zeros of eq. (10) here denoted by $z_B$; due to the constraint $\sum_n P_n = 0$ they are $N - 2$ in number. They are known in the mathematical literature as apparent singularities. The strength of the particle sources are given by the particle rest masses and that of the apparent singularities by the constant $4\pi$. We shall rewrite eq. (11) more generally as follows

\[ 4\partial_z \partial_{\bar{z}} \phi = e^{\phi} + 4\pi \sum_n g_n \delta^2(z - z_n). \quad (13) \]

In a series of papers at the turn of the past century Picard proved that eq. (13) for real $\phi$ with asymptotic behavior at infinity

\[ \phi(z) = -g_{\infty} \ln(z \bar{z}) + O(1) \quad (14) \]

and $-1 < g_n$, $1 < g_{\infty}$ (which excludes the case of punctures) and $\sum_n g_n + g_{\infty} < 0$ admits one and only one solution (see also 26). Picard achieved the solution of (13) through an iteration process exploiting Schwarz alternating procedure. The same problem has been considered recently with modern variational techniques by Troyanov, obtaining results which include Picard’s findings.

The above inequalities on the $g_n$ are all satisfied in gravity with the following meaning: $1 + g_n = \mu_n = m_n/4\pi$ being $m_n$ the rest mass of the $n$-th particle which is constrained in the limits $0 < m_n < 4\pi$. For $m_n > 4\pi$ the conical defect exceeds its maximum value. $2 - g_{\infty} = \mu = M/4\pi$ being $M$ the total energy of the system which is also constrained to $0 < M < 4\pi$.

From eq. (13) one can easily prove that the function $Q(z)$ defined by

\[ e^{\sigma} \partial_z^2 e^{-\sigma} = -Q(z) \quad (15) \]

is analytic i.e. $Q(z)$ is given by the analytic component of the energy momentum tensor of a Liouville theory. $Q(z)$ is meromorphic with poles up to the second order i.e. of the form

\[ Q(z) = \sum_n \frac{-g_n(g_n + 2)}{4(z - z_n)^2} + \frac{\beta_n}{2(z - z_n)}. \quad (16) \]
All solutions of eq. (13) can be put in the form
\[ e^\phi = \frac{8f'\bar{f}'}{(1-f\bar{f})^2} = \frac{8|w_{12}|^2}{(y_2\bar{y}_2 - y_1\bar{y}_1)^2}, \quad f(z) = \frac{y_1}{y_2} \tag{17} \]
being \( y_1, y_2 \) two properly chosen, linearly independent solutions of the fuchsian equation
\[ y'' + Q(z)y = 0. \tag{18} \]
\( w_{12} \) is the constant wronskian. This is a variant of the Riemann-Hilbert problem as we are not given directly with the monodromies but with the following information on them: all monodromies belong to the \( SU(1,1) \) group, otherwise the conformal factor \( \tilde{\sigma} \) is not single valued and as such cannot solve the Liouville equation eq. (13); in addition we are given with the conjugation classes of the monodromies around the singularities and the conjugation class of the monodromy at infinity.

The conformal factor \( \tilde{\sigma} = -\phi/2 \) is the key quantity in all the subsequent developments. In fact a secondary constraint following from the primary gauge constraint \( K = 0 \) is
\[ \Delta N = e^{-2\tilde{\sigma}} N \tag{19} \]
and such \( N \) can be computed from the knowledge of \( \tilde{\sigma} \). The reason is that the solution of the inhomogeneous Liouville equation \( \tilde{\sigma} \) contains the free parameter \( \mu \) which is related to the behavior at infinity of the conformal factor i.e. \( \exp(2\sigma) \approx s^2(z\bar{z})^{-\mu} \). As the sources do not depend on \( \mu = M/4\pi \) a solution of eq. (19) is given by
\[ N = \frac{\partial(-2\tilde{\sigma})}{\partial M} \tag{20} \]
and one easily proves such a solution to be unique.\footnote{See reference for details.} The other secondary constraint on \( N^z \) is
\[ \partial_z N^z = -\pi^z e^{-2\sigma} N \tag{21} \]
solved by
\[ N^z = -\frac{2}{\pi^z(z)} \partial_z N + g(z). \tag{22} \]
Here \( g(z) \) is a meromorphic function whose role is to cancel the poles occurring in the first member on the r.h.s. due to the zeros of \( \pi^z \). The expression of \( g(z) \) in \( N^z \) is\footnote{See reference for details.}
\[ g(z) = \sum_B \frac{\partial B}{\partial M} \frac{1}{z - z_B} \frac{P(z_B)}{\prod_{C \neq B}(z_B - z_C)} + p_1(z). \tag{23} \]
$p_1(z)$ is a first order polynomial related to the motion of the frame at infinity, more properly to the transformation of translations, rotations and dilatations which leave invariant the conformal structure of the space metric and leave fixed the point at infinity. In the following we shall examine the case $p_1(z) = 0$; this is no limitation as the two structures are related by a canonical transformation. In conclusion the metric is obtained in a straightforward way from $\tilde{\sigma}$.

The equations of motion are extracted from the action; for $p_1(z) = 0$ they can be written as

$$z'_n = - \sum_B \frac{\partial \beta_B}{\partial \mu} \frac{\partial z'_B}{\partial P'_n} \quad \text{and} \quad \dot{P}'_n = \frac{\partial \beta_n}{\partial \mu} + \sum_B \frac{\partial \beta_B}{\partial \mu} \frac{\partial z'_B}{\partial P'_n} \quad (24)$$

where $z'_n = z_n - z_1$ and $P'_n = P_n$ for $n = 2 \ldots N$.

The problem is now to show that such system is of hamiltonian nature and possibly to write down the hamiltonian. The hamiltonian nature of eqs. is expected as we obtained the above equations by reduction of a hamiltonian system. Despite that, it is of interest to have a direct proof of it and an expression of the hamiltonian.

In the simpler case of three body, where we have a single accessory singularity one can prove the hamiltonian nature of the equations of motion by exploiting Garnier equations which give a constraint on the evolution of the accessory parameters under isomonodromic deformations. The fact that our deformations are isomonodromic is a consequence of the constancy in 2+1 dimensional gravity of the monodromies around the particle world lines.

The problem with four or more particles, when we are in presence of two or more accessory singularities is more difficult and it is related to an interesting conjecture due to Polyakov about the accessory parameters of the $SU(1, 1)$ Riemann-Hilbert problem. Such a conjecture states that the regularized Liouville action is computed on the solution of the Liouville equation eq.(13) is the generating function of the accessory parameters in the sense that

$$\beta_n = - \frac{1}{2\pi} \frac{\partial S_P}{\partial z_n} \quad (26)$$
where \( S_P = \lim_{\epsilon \to 0} S_\epsilon \). Polyakov conjecture originated in the context of 2-dimensional quantum gravity. Let \( T(z) \) be the analytic component \( T_{zz} \) of the energy momentum tensor and let \( V_\alpha \) be a primary field of weights \((\Delta_\alpha, \bar{\Delta}_\alpha)\) i.e. such that under a dilatation \( V_\alpha \) transforms like

\[
V_\alpha'(kz) = k^{-\Delta_\alpha}k^{-\bar{\Delta}_\alpha}V_\alpha(z). 
\]  

(27)

The Ward identity reads as

\[
T(z)V_\alpha(w) = \frac{\Delta_\alpha}{(z-w)^2}V_\alpha(w) + \frac{1}{(z-w)}\partial_w V_\alpha(w) + \cdots 
\]  

(28)

Given the field \( V_\alpha = e^{\alpha \phi} \) with classical conformal weights \((\alpha, \alpha)\) we have

\[
\langle V_{\alpha_1}(z_1)V_{\alpha_2}(z_2)\ldots V_{\alpha_N}(z_N) \rangle = \int D[\phi]e^{-\frac{S_P}{\pi h}} 
\]  

(29)

where \( S_P \) is given by eq.(25) in the limit \( \epsilon \to 0 \), with \( g_n = -h\alpha_n \). From the three point function one finds for the quantum weights

\[
\Delta_\alpha = -\frac{h\alpha_n^2}{2} + \alpha_n = \frac{1 - \mu_n^2}{2h} 
\]  

(30)

and

\[
\langle T(z)V_{\alpha_1}(z_1)V_{\alpha_2}(z_2)\ldots V_{\alpha_N}(z_N) \rangle = \int D[\phi]\frac{1}{h}(\partial^2 \phi - \frac{1}{2}(\partial \phi)^2)e^{-\frac{S_P}{\pi h}}. 
\]  

(31)

In the classical limit i.e. for \( h \to 0 \), \( \phi \) becomes the solution of Liouville equation

\[
e^\phi \partial^2 e^{-\frac{\phi}{2}} = -\frac{1}{2}\partial^2 \phi + \frac{1}{4}(\partial \phi)^2 = -Q(z) 
\]  

(32)

which inserted in the previous equation gives again

\[
\Delta_\alpha = \frac{1 - \mu_n^2}{2h} 
\]  

(33)

and the new result

\[
\beta_n = \frac{1}{2\pi} \frac{\partial S_P}{\partial \alpha_n} 
\]  

(34)

which is the content of Polyakov conjecture.

Zograf and Takhtajan provided a proof of eq.(34) for parabolic singularities using the technique of mapping the quotient of the upper half-plane.
by a fuchsian group to the Riemann surface and exploiting certain properties of the harmonic Beltrami differentials. In addition they remark that the same technique can be applied when some of the singularities are elliptic of finite order. The case of only parabolic singularities is of importance in the quantum Liouville theory as such singularities provide the sources from which to compute the correlation functions. On the other hand in $2 + 1$ gravity one is faced with general elliptic singularities and here the mapping technique cannot be applied. As a matter of fact we shall see that the case of elliptic singularities is more closely related to the theory of elliptic non linear differential equations (potential theory) than to the theory of fuchsian groups.

We shall now briefly outline a derivation of Polyakov conjecture for general elliptic singularities. We define $s_\infty^2$ and $s_n^2$ as the constant coefficients in the asymptotic expansion of $\phi$ at the singularities i.e.

$$\phi = -g_\infty \ln(z \bar{z}) - \ln s_\infty^2 + O\left(\frac{1}{|z|}\right) \quad (35)$$

and

$$\phi = g_n \ln((z - z_n)(\bar{z} - \bar{z}_n)) - \ln s_n^2 + O(|z - z_n|). \quad (36)$$

One then considers the derivative with respect to $z_n$ of eq.(13): in a domain which excludes the sources we have

$$4\partial_z \partial_{\bar{z}} \frac{\partial \phi}{\partial z_n} = e^\phi \frac{\partial \phi}{\partial z_n} \quad (37)$$

i.e. outside the sources $\frac{\partial \phi}{\partial z_n}$ obeys a linear elliptic differential equation. Similarly we have

$$4\partial_z \partial_{\bar{z}} \frac{\partial \phi}{\partial g_\infty} = e^\phi \frac{\partial \phi}{\partial g_\infty}. \quad (38)$$

From the previous equations it follows

$$\frac{\partial \phi}{\partial z_n} \partial_z \partial_{\bar{z}} \frac{\partial \phi}{\partial g_\infty} = \frac{\partial \phi}{\partial g_\infty} \partial_z \partial_{\bar{z}} \frac{\partial \phi}{\partial z_n}. \quad (39)$$

If we integrate this expression over a domain $D$ chosen to be a disk of radius $R$ which includes all the singularities, from which disks of radius $\epsilon \to 0$ around each singularity have been removed, we can apply Green’s theorem getting only border contributions. Letting $R \to \infty$ and $\epsilon \to 0$ the only contributions which survive are that coming from the circle at infinity, which gives

$$- \frac{\partial \ln s_\infty}{\partial z_n} \quad (40)$$
and the contribution around $z_n$ which gives

$$\frac{\partial \beta_n}{\partial g_{\infty}}. \tag{41}$$

The last equation is due to the fact that one is able to express the leading term of $\frac{\partial \beta}{\partial g_{\infty}}$ and the linear term in $z - z_n$ of $\frac{\partial \phi}{\partial g_{\infty}}$ around $z = z_n$ simply from the local analysis of the solution of the differential equation eq. (18). Thus one reaches

$$\frac{\partial \beta_n}{\partial g_{\infty}} = \frac{\partial \ln s_{\infty}}{\partial z_n}. \tag{42}$$

Such a result is sufficient to assure the hamiltonian structure of the equations eqs. (24). In fact taking into account that $\mu = 2 - g_{\infty}$ we see that the hamiltonian is simply given by $\ln s_{\infty}^2$. This result also paves the way to the proof of Polyakov conjecture. In fact equation (42) can be generalized to

$$\frac{\partial \beta_n}{\partial g_m} = \frac{\partial \ln s_m}{\partial z_n}. \tag{43}$$

Moreover from the structure of $S_P$ keeping in mind that the solution of eq. (13) is a stationary point of the action, we have

$$-\frac{1}{2\pi} \frac{\partial S_P}{\partial g_m} = \ln s_m, \tag{44}$$

from which we have

$$-\frac{1}{2\pi} dS_P = \sum_n \frac{\partial \beta_n}{\partial g_m} dz_n + \text{c.c.} \tag{45}$$

This is a weak form of Polyakov conjecture which can be rewritten as

$$-\frac{1}{2\pi} dS_P = \sum_n \beta_n dz_n + \sum_n F_n(z_1 \ldots z_N)dz_n + \text{c.c.} \tag{46}$$

where the $F_n$ do not depend on the $g_m$. In order to understand the nature of the functions $F_n$ we consider the limit of $g_1 \to 0$. In this case $S_P$ becomes independent of $z_1$ and at the same time $\beta_1 \approx \text{const} \times g_1 \to 0$. Thus $F_1$ must be identically zero. Repeating the reasoning for the other singularities we have all the $F_n \equiv 0$.

A more direct proof which exploits the expression of the Polyakov action in term of a background field has been given in [35].
4 Quantization: the two particle case

We shall now quantize the two particle system in the reference frame which does not rotate at infinity. In this case there are no apparent singularities and the hamiltonian is given by

\[ H = \ln(Pz \bar{P} \bar{z}) + (\mu - 1) \ln(z \bar{z}) = \ln(Pz^\mu) + \ln(\bar{P} \bar{z}^\mu) = h + \bar{h} \] (47)

with \( P = P_x^\mu \) and \( z = z_x^\mu \). \( h \) and \( \bar{h} \) are separately constant of motion and if we combine them with the generalized conservation law \[ Pz = (1 - \mu)(t - t_0) - ib \] we obtain the solution for the motion

\[ z = \text{const}[(1 - \mu)(t - t_0) - ib]^{\frac{1}{1 - \mu}}. \] (48)

\( H \) can be rewritten as

\[ H = \ln((x^2 + y^2)^\mu((P_x)^2 + (P_y)^2)). \] (49)

Keeping in mind that with our definitions \( P \) is the momentum multiplied by \( 16\pi G_N/c^3 \), applying the correspondence principle we have

\[ [x, P_x] = [y, P_y] = il_P \] (50)

where \( l_P = 16\pi G_N \hbar/c^3 \), all the other commutators equal to zero. \( H \) is converted into the operator

\[ \ln[-(x^2 + y^2)^\mu\Delta] + \text{constant}. \] (51)

The argument of the logarithm is minus the Laplace-Beltrami \( \Delta_{LB} \) operator on the metric \( ds^2 = (x^2 + y^2)^{-\nu}(dx^2 + dy^2) \). Following an argument similar to the one presented in \[36\] one easily proves that if we start from the domain of \( \Delta_{LB} \) given by the infinite differentiable functions of compact support \( C_0^\infty \) which can also include the origin, then \( \Delta_{LB} \) has a unique self-adjoint extension in the Hilbert space of functions square integrable on the metric \( ds^2 = (x^2 + y^2)^{-\nu}(dx^2 + dy^2) \) and as a result since \( -\Delta_{LB} \) is a positive operator, \( \ln(-\Delta_{LB}) \) is also self-adjoint.

Deser and Jackiw\[37\] considered the quantum scattering of a test particle on a cone both at the relativistic and non relativistic level. Most of the techniques developed there can be transferred here. The main difference is the following: instead of the hamiltonian \((x^2 + y^2)^\mu(p_x^2 + p_y^2)\) which appears in their non relativistic treatment, we have now the hamiltonian \( \ln[(x^2 + y^2)^\mu(p_x^2 + p_y^2)] \).

The partial wave eigenvalue differential equation

\[ (r^2)^\mu[-\frac{1}{r} \frac{\partial}{\partial r}r \frac{\partial}{\partial r} + \frac{n^2}{r^2}]\phi_n(r) = k^2 \phi_n(r) \] (52)
with $\mu = 1 - \alpha$ is solved by

$$
\phi_n(r) = J_{|n|}(\frac{k}{\alpha^\alpha} r^\alpha)
$$

(53)
to obtain for the Green function

$$
G(z, z', t) = \frac{2}{\alpha \Gamma(\frac{|n|}{\alpha}) r r'} \left( \frac{r^\alpha + r'^\alpha}{2} \right)^{2ict/lP} \\
\sum_n \frac{e^{i(n\phi - n'\phi')}}{2\pi} \frac{\Gamma(\frac{|n|}{\alpha} + \frac{1 - ict}{lP})}{\Gamma(\frac{|n|}{\alpha} + 1)} \rho^{\frac{|n|}{\alpha} + 1}F_1(\frac{|n|}{\alpha} + 1 - \frac{ict}{lP} ; \frac{|n|}{\alpha} + \frac{1}{2}; \frac{2}{\alpha} |n| + 1; 4\rho)
$$

(54)

where

$$
\rho = \frac{r^\alpha r'^\alpha}{r^\alpha + r'^\alpha}
$$

(55)

The Green function $G$ gives the solution of the two particle quantum problem.

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