Unruh acceleration effect on the precision of parameter estimation

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Abstract

The dynamics of Fisher information for an accelerated system initially prepared in the $X$-state is discussed. An analytical solution, which consists of three parts: classical, the average over all pure states and a mixture of pure states is derived for the general state and for Werner state. It is shown that, the Unruh acceleration has a depleting effect on the Fisher information. This depletion depends on the degree of entanglement of the initial state settings. For the $X$-state, for some intervals of Unruh acceleration, the Fisher information remains constant, irrespective to the Unruh acceleration. In general, the possibility of estimating the state’s parameters decreases as the acceleration increases. However, the precision of estimation can be maximized for certain values of the Unruh acceleration. We also investigate the contribution of the different parts of the Fisher information on the dynamics of the total Fisher information.

Keywords: Estimation, Fisher information, Unruh acceleration

1 Introduction

Quantum Fisher information, $\mathcal{F}_I$ represents one of the most important measures in the context of estimation theory [1], where it measures the sensitivity of a given state with respect to changes in one of its parameters. It is considered as a resource to detect the entanglement between two particles [2]. Quantum Fisher information, is introduced as a quantitative measure of information flow between the quantum system and its surroundings [3]. There are several studies that have been devoted to investigate the dynamics of the Fisher information for different quantum systems. For example, Zhong et. al [4] have introduced an analytical form of $\mathcal{F}_I$ for a single qubit in terms of Bloch vectors. The problem of parameter estimation in a qubit system under different non-markovian conditions is studied in [6]. Quantum Fisher information for a system consists of two entangled qubits interacts with its surrounding is discussed by Q. Zheng et. al [5]. Wang et. al [7] have investigated the dynamics of Fisher information for separable, as well as, entangled two qubit system, where each qubit interacts independently with its environment. The dynamics of $\mathcal{F}_I$ for W-state in the presence of noisy channel is examined in [8]. The quantum Fisher information for the Greenberger-Horne-Zeilinger (GHZ) state in the presence of decoherence is investigated by Jian Ma et. al [9]. Enhancing the quantum teleportation of Fisher information by using partial measurement is discussed by Xiao et. al [10]. Recently an experimental scheme is proposed by Fröwis et. al [11] to quantify the lower bounds of Fisher Information.

Investigating the dynamics of Fisher information in non-inertial frame is limited. Recently, the effect of the Unruh acceleration on the dynamics of $\mathcal{F}_I$ for a single Dirac qubit is discussed by Banerjee et. al [12]. The dynamics of Fisher information for a two qubit system in non-inertial frame is discussed by Yao et. al [13]. Here, we are motivated to study the dynamics of Fisher information for a system of two qubits initially prepared in the $X$-state or its spacial class which is known by Werner state. In this current study, we assume that,
only one qubit is accelerated uniformly while the other stays in the inertial frame. The Fisher information with respect to the output parameters is quantified, where we investigate the effect of the acceleration and the initial state settings on the Fisher information.

The paper is organized as follows: In Sec. 2, the relations between Minkowski and Rindler spaces are reviewed and the mathematical form of the quantum Fisher information is introduced. The suggested model and its dynamics in the Rindler space are discussed in Sec. 3, where analytical solutions for the Fisher information for different parameters are given for the X-state and Werner state. Finally, a summary and discussion on the results are provided in Sec. 4.

2 Unruh acceleration and Fisher information

In the following subsection (2.1) we review the relation between inertial and non-inertial frames and effect of Unruh acceleration on Dirac qubits. In subsection (2.2), we review the main aspects of quantum Fisher Information and its mathematical forms.

2.1 Rindler and Minkowski Spaces

Let us assume that, Alice’s qubit (subsystem a) will be accelerated, while Bob’s qubit (subsystem b) will stay in the inertial frame. Therefore, if the coordinate of a particle is defined by \((t, z)\) in Minkowski space, then in Rindler space it is defined by \((\tau, x)\), where

\[
\tau = r \tanh(t/z), \quad x = \sqrt{t^2 - z^2},
\]

and \(-\infty < r < \infty, -\infty < x < \infty, \tan r_b = \text{Exp}[-\pi \omega c a_b], 0 \leq r_b \leq \pi/4, -\infty \leq a \leq \infty, \omega\) is the frequency, \(c\) is the speed of light, and \(\phi\) is the phase space which can be absorbed in the definition of the operators [14]. These coordinates define two regions in the Rindler space, I and II, where the accelerated particle will be in the first region I and the anti-accelerated particle will be in the second region II. In the computational basis \(|0_k\rangle\) and \(|1_k\rangle\) can be written as [15, 16],

\[
|0_k\rangle = \cos r_a |0_k\rangle_I |0_{-k}\rangle_{II} + \sin r_a |1_k\rangle_I |1_{-k}\rangle_{II}, \quad |1_k\rangle = |1_k\rangle_I |0_k\rangle_{II}.
\]

2.2 Fisher Information

Quantum Fisher information represents the central role in the estimation theory, where it can be used to quantify some parameters that cannot be quantified directly [17]. In this subsection, we review the mathematical form of Fisher information. Let \(\kappa\) be the parameter to be estimated. Let the spectral decomposition of the density operator is given by \(\rho_\kappa = \sum_{j=1}^{n} \lambda_j |\theta_j\rangle \langle \theta_j|\), where \(\lambda_j\) and \(|\theta_j\rangle\) are the eigenvalues and the corresponding eigenvectors of the state \(\rho_\kappa\). The Fisher information with respect to the parameter \(\kappa\) is defined as [18]

\[
\mathcal{F}_\kappa = \text{tr}\{\rho_\kappa L_\kappa^2\}, \quad L_\kappa = \frac{1}{2}(\rho_\kappa L_\kappa + L_\kappa \rho_\kappa), \quad L_\kappa = \frac{\partial \rho_\kappa}{\partial \kappa}.
\]

Using the spectral decomposition and the identity \(\sum_{j=1}^{n} |\theta_j\rangle \langle \theta_j|\) in Eq.(3), one can obtain the final form of the Fisher information as [10, 13, 19],

\[
\mathcal{F}_I = \mathcal{F}_c + \mathcal{F}_p - \mathcal{F}_m.
\]
between Alice and Bob can be described as [21],

In this treatment, it is assumed that Alice’s qubit will be accelerated with a uniform acceleration $a$.

3.1 The X-type [21, 20]. In this manuscript, the users Alice and Bob share a two-qubit state of $X$-type [21, 20]. In the computational basis it can be written as,

$$\rho_X = |0\rangle_a \langle 0|\left(B_{11}|0\rangle_b \langle 0| + B_{22}|1\rangle_b \langle 1|\right) + |1\rangle_a \langle 1|\left(B_{33}|0\rangle_b \langle 0| + B_{44}|1\rangle_b \langle 1|\right)$$

$$+ |0\rangle_a \langle 1|\left(B_{23}|1\rangle_b \langle 0| + B_{14}|0\rangle_b \langle 1|\right) + |1\rangle_a \langle 0|\left(B_{32}|0\rangle_b \langle 0| + +B_{41}|1\rangle_b \langle 0|\right),$$

where

$$B_{11} = B_{44} = \frac{1}{4}(1 + z), \quad B_{22} = B_{33} = \frac{1}{4}(1 - z),$$

$$B_{23} = B_{32} = \frac{1}{4}(x + y), \quad B_{14} = B_{41} = \frac{1}{4}(x - y),$$

and $x = tr\{\rho_X \sigma_z^{(1)} \sigma_z^{(2)}\}$, $y = tr\{\rho_X \sigma_y^{(1)} \sigma_y^{(2)}\}$ and $z = tr\{\rho_X \sigma_z^{(1)} \sigma_z^{(2)}\}$, $\sigma_k^{(j)} = 1, 2, k = x, y$ and $z$ are the pauli matrices for the first and the second qubit, respectively. The subscripts ”$a$” and ”$b$” indicate to Alice and Bob qubits. Since, we are interesting in the dynamics of the shared state between the users in Rindler space, it is important to review the relation between Minkowski space and Rindler space. The next sub-section is devoted for this aim.

3.1 The X-State

In this treatment, it is assumed that Alice’s qubit will be accelerated with a uniform acceleration $a$ while Bob’s qubit remains in the inertial frame. By using the initial state (3), the transformation (2), and tracing out the states of the mode $I$, the final accelerated state between Alice and Bob can be described as [21].

$$\rho_{ab}^{acc} = \rho_{11}|00\rangle \langle 00| + \rho_{14}|00\rangle \langle 11| + \rho_{22}|01\rangle \langle 01| + \rho_{23}|01\rangle \langle 10|$$

$$+ \rho_{32}|10\rangle \langle 01| + \rho_{33}|10\rangle \langle 10| + \rho_{41}|11\rangle \langle 00| + \rho_{44}|11\rangle \langle 11|,$$
where,

\[ g_{11} = B_{11}c^2, \quad g_{14} = B_{14}c, \quad g_{22} = B_{22}c^2, \quad g_{23} = B_{23}c, \]
\[ g_{32} = B_{32}c, \quad g_{33} = B_{11}s^2 + B_{33}, \quad g_{41} = B_{41}c, \quad g_{44} = B_{22}s^2 + B_{44}, \]

where \( c = \cos r \) and \( s = \sin r \). The eigenvalues of the accelerated state (8) are given by,

\[ \lambda_{1,2} = \frac{1}{16} \left\{ 4 + 2(1 + \cos 2r)z \mp \sqrt{2}k_1(x, y) \right\}, \]
\[ \lambda_{3,4} = \frac{1}{8} \left\{ 2 - (1 + \cos 2r)z \pm 2k_2(x, y) \right\}, \]

where

\[ k_1(x, y) = \sqrt{4(x - y)^2 - 4[1 - (x - y)^2] \cos 2r + \cos 4r + 3}, \]
\[ k_2(x, y) = \sqrt{(x + y)^2 \cos 2r + \sin^4 r}. \]

and the corresponding eigenvectors are defined as,

\[ |\psi_1\rangle = \frac{1}{\sqrt{\mu_1^2 + 1}} \left( -\mu_1, 0, 0, 1 \right), \quad |\psi_2\rangle = \frac{1}{\sqrt{\mu_2^2 + 1}} \left( \mu_2, 0, 0, 1 \right), \]
\[ |\psi_3\rangle = \frac{1}{\sqrt{\mu_3^2 + 1}} \left( 0, -\mu_3, 1, 0 \right), \quad |\psi_4\rangle = \frac{1}{\sqrt{\mu_4^2 + 1}} \left( 0, \mu_4, 1, 0 \right), \]

where

\[ \mu_{1,2} = \pm \sec r \frac{\sqrt{2}k_1(x, y)}{4(x - y)} \left\{ 2(1 - \cos 2r) \mp \sqrt{2} \right\}, \]
\[ \mu_{3,4} = \pm \sec r \frac{\sqrt{2}k_3(x, y)}{4(x + y)} \left\{ 2(1 - \cos 2r) \pm \sqrt{2} \right\}, \]

with \( k_3(x, y) = \sqrt{4(x + y) - 4[1 - (x + y)^2] \cos 2r + \cos 4r + 3} \).

It is evident that, the final accelerated state (8) is a function of the parameters \( x, y, z \) and \( r \). Therefore, we can estimate these parameters by means of Fisher information. In what follows, we calculate \( F_i^z \), \( F_i^x \) and \( F_i^y \).

- **Fisher information with respect to z, \( F_i^z \):**
  It is noted that, the eigenvectors don’t depend on the parameter \( z \), therefore, there is no any contribution from \( F_q \) and \( F_m \). However, the total Fisher information with respect to the parameter \( z \) depends on the classical part, which can be written explicitly as

\[ F^z_i = \frac{(1 + \cos 2r)^2}{4} \left[ \frac{8 + 4(1 + \cos 2r)z}{(4 + 2(1 + \cos 2r)z)^2 - 2k_1^2(x, y)} \right] \]
\[ + \frac{2 - (1 + \cos 2r)z}{(2 - (1 + \cos 2r)z)^2 - 4k_2^2(x, y)}. \]

- **Fisher information with respect to x, \( F_i^x \):**
  Using Eqs.(10-13) together with (5), we can write the components of the Fisher information as,

\[ F^x_c = \frac{(x - y)(1 + \cos 2r)^2}{k_1^2(x, y)} \left\{ \frac{2 + (1 + \cos 2r)z}{[4 + 2(1 + \cos 2r)z]^2 - 2k_1^2(x, y)} \right\} \]
Figure 1: (a) Fisher information, $\mathcal{F}_z^I$ for an accelerated system initially prepared the X-state with $x = -0.3$, $y = -0.6$ and $z = -0.3, -0.5, -0.6$ dash-dot and solid curves, respectively. (b) Fisher information with respect to $x$, $\mathcal{F}_x^I$, where we set $y = -0.6$, $z = -0.5$ and $x = -0.4, -0.5$ and $-0.7$ for the dash, dot and solid curves, respectively.

\[
\begin{align*}
\mathcal{F}_p^x &= 4 \sum_{i=1}^{4} \lambda_i \frac{\mu_i^2}{(1 + \mu_i^2)^2}, \quad i = 1...4, \quad \mu_i' = \frac{d\mu_i}{dx}, \\
\mathcal{F}_m^x &= \frac{8\lambda_1\lambda_2}{(\lambda_1 + \lambda_2)(1 + \mu_1^2)(1 + \mu_2^2)} \left\{ \frac{\mu_1^2}{(1 + \mu_1^2)^2} + \frac{\mu_2^2}{(1 + \mu_2^2)^2} \right\} + \frac{8\lambda_3\lambda_4}{(\lambda_3 + \lambda_4)(1 + \mu_3^2)(1 + \mu_4^2)} \left\{ \frac{\mu_3^2}{(1 + \mu_3^2)^2} + \frac{\mu_4^2}{(1 + \mu_4^2)^2} \right\},
\end{align*}
\]

The dynamics of Fisher information with respect to the parameters $z$ and $x$ is shown in Fig.(1). Different initial state settings of $X-$ state are considered. Fig.(1a) shows the behavior of $\mathcal{F}_z^I$, for different values of $z$, where we fix the values of $x(= -0.3)$ and $y(= -0.6)$. The general behavior shows that the Fisher information decreases as the acceleration increases. The decreasing rate depends on the initial state settings, where as one increases $|z|$, the upper bounds of Fisher information increase. This means that as one increases the entanglement of the initial state, the Fisher information with respect to $z$ increases.

The behavior of $\mathcal{F}_x^I$ for different initial state settings is depicted in Fig.(1b). There are a three behaviors manifested of Fisher information: for small values of the acceleration, Fisher Information $\mathcal{F}_x^I$ remains constant. For further values of $r$, the Fisher information with respect to $x$, increases suddenly to reach its maximum value at a certain finite value of $r$. However for larger values of the acceleration, $\mathcal{F}_x^I$ decreases gradually to reach its minimum values at $r \to \infty$. The upper and lower values of $\mathcal{F}_x^I$ depend on the initial value of the parameter $x$, where as one increases $|x|$, the upper bounds increase.

Generally speaking, larger acceleration leads to smaller values of the Fisher information, namely, less precision of estimation on the $z$ and $x$ parameters. The possibility.
Figure 2: The components of Fisher information $F^x_I$ for a system is initially prepared in $X$-state with $x = -0.5$, $y = -0.6$ and $z = -0.5$. The solid, dot and dash curves for $F^x_c$, $F^x_p$ and $F^x_m$, respectively.

of estimating these parameters increases for systems initially that has a large degree of entanglement. Another feature is depicted, Fisher information with respect to $x$ has a maximum value at a certain value of the acceleration, while that with respect to $z$ decreases monotonously with respect to $r$.

In Fig.(2), we plot the three different type of Fisher information, $F^x_c$, $F^x_p$ and $F^x_m$, where it is assumed that the system is initially prepared in the $x$-state with $x = -0.5$, $y = -0.6$ and $z = -0.5$. It is evident that, at $r = 0$, the total Fisher information, $F^x_I = F^x_c$, while there is no any contribution from $F^x_p$ and $F^x_m$. As the Unruh acceleration increases, the classical component of Fisher information decreases, while the pure and mixed parts increase. However each of $F^x_p$ and $F^x_m$ reaches their maximum values at a certain value of the Unruh acceleration. This displays that, there are some pure states that are generated at $r \in [0, 0.4]$. For further values of $r \in [0.4, 0.6]$, the number of the generated pure states decreases.

To explain the behavior of the Fisher information, it is important to plot the behavior of the probabilities distribution of the different eigenvectors, where $\mathcal{P}_{ix} = \lambda_i^2 / \lambda$, $\lambda = \sum_{i=1}^{4} \lambda_i^2$. Therefore, we consider the case for $F^x_I$ with $x = -0.5$, $y = -0.6$ and $z = -0.5$. Fig.(3) describes the behaviors of $\mathcal{P}_i$, $i = 1.4$. It is evident that, the travelling state can be written in a different structure in the interval $r \in [0, 0.8]$. For example for $r \in [0, 0.05]$, the final density operator can be written as, $\rho_1 = \kappa_1 |\psi_{1x}\rangle \langle \psi_{1x}| + |\psi_{1x}\rangle \langle \psi_{1x}| + \kappa_2 |\psi_{2x}\rangle \langle \psi_{2x}| + \kappa_3 |\psi_{4x}\rangle \langle \psi_{4x}|$. However, for $r \in [0.05, 0.4]$ another form can be written as $\rho_2 = \sum_j \kappa_j |\psi_{ix}\rangle \langle \psi_{ix}|$, $j = 1.4$. For $r = 0.4$ a third structure can be depicted of the travelling state, $\rho_3$ which is similar to that shown for $\rho_1$ but with different values of $\kappa_j$. For further values of $r \in [0.4, 0.8]$ the travelling state can be written as $\rho_2$ but with a different $\kappa_j$’s. From these forms, it is seen that the possibility of generating pure states is large. This explain why the contribution from $F^x_p$ is the largest one.

- **Fisher information with respect to $r$, $F^r_I$:**
Figure 3: The dynamics of the populations, $P_{\psi_1 x}$ (dot-curve), $P_{\psi_2 x}$ (solid curve), $P_{\psi_3 x}$ (dash curve), $P_{\psi_4 x}$ (dash-dot curve). The small scale Figure display the behavior of $P_{\psi_1 x}$ and $P_{\psi_4 x}$.

In this case, the classical part of Fisher information $F_c^r$ is given by

$$F_c^r = \frac{1}{2\lambda_1\kappa_1^2(x, y)} \left[ \sqrt{2}z\kappa_1(x, y) \sin 2r + 2[1 - (x - y)^2] \sin 2r - \sin 4r \right]^2$$

$$+ \frac{1}{2\lambda_2\kappa_1^2(x, y)} \left[ \sqrt{2}z\kappa_1(x, y) \sin 2r - 2[1 - (x - y)^2] \sin 2r + \sin 4r \right]^2$$

$$+ \frac{\sin 2r^2}{\lambda_3\kappa_2^2(x, y)} \left[ (x + y)^2 - 2\sin^2 r + 2z\kappa_2(x, y) \right]^2$$

$$+ \frac{\sin 2r^2}{\lambda_4\kappa_2^2(x, y)} \left[ (x + y)^2 - 2\sin^2 r - 2z\kappa_2(x, y) \right]^2. \tag{16}$$

The other components $F_p^r$ and $F_m^r$ are similar to those shown in Eqs.(15) but the derivative will be with respect to the parameter $r$, namely, we replace $\frac{\partial}{\partial x}$ by $\frac{\partial}{\partial r}$.

The dynamics of Fisher information with respect to the Unruh acceleration parameter $F_t^r$ is displayed in Fig.(4). Different initial state settings are considered. The behavior shows that at $r = 0$, $F_t^r = 0$. As soon as the particle is accelerated, the Fisher information increases as Unruh acceleration increases sharply for systems that possess large degree of entanglement. The Fisher information reaches their maximum values at a certain value of acceleration. An important remark that, systems that possess a small degree of entanglement reaches their upper bounds at larger values of Unruh acceleration. For further values of $r$, the Fisher information, $F_t^r$ decays faster for systems that have a large degree of entanglement. As $r \to \infty$, the upper bounds of Fisher information depends on the degree of entanglement of the initial accelerated systems.

From this figure, it is seen that, it is possibly to estimate the Unruh acceleration parameter with high precision if the accelerated systems have a large degree of entanglement.
Figure 4: Fisher Information, $\mathcal{F}_f^r$ for an accelerated system initially prepared the X-state with $y = -0.6$, $z = -0.5$ and $x = -0.2, -0.4, -0.5$ dash, dot and solid curves, respectively.

at smaller accelerations.

### 3.2 Werner state

Here, we consider the users share initially a two qubit state of Werner type ($\rho_w$). This means that, we set $x = y = z$ in (7). According to the previous suggestion i.e., only Alice’s qubit will be accelerated, then at the end of the process the users will obtain an accelerated Werner state $\rho^{\text{acc}}_w$ with the following eigenvalues,

$$
\lambda_{1w} = \frac{c^2}{4} (1 + x), \quad \lambda_{2w} = \frac{1}{4} (1 + s^2 + c^2 x),
$$

$$
\lambda_{3w,4w} = \frac{1}{4} \left(1 - x \cos^2 r \pm \frac{1}{2 \sqrt{2}} \sqrt{16 x^2 - 4(1 - 4x^2) \cos 2r + \cos 4r + 3}\right), \quad (17)
$$

and their corresponds eigenvectors,

$$
|\psi_{1w}\rangle = \left(1, 0, 0, 0\right), \quad |\psi_{2w}\rangle = \left(0, 0, 0, 1\right),
$$

$$
|\psi_{3w}\rangle = \frac{1}{\sqrt{1 + \mu_{3w}}^2}\left(0, -\mu_{3w}, 1, 0\right), \quad |\psi_{4w}\rangle = \frac{1}{\sqrt{1 + \mu_{4w}^2}}\left(0, -\mu_{4w}, 1, 0\right), \quad (18)
$$

where $\mu_{3w,4w} = \frac{1}{8x \cos r} \left[2 - 2 \cos 2r + \gamma\right]$, $\gamma = \sqrt{16 x^2 - 4(1 - 4x^2) \cos 2r + \cos 4r + 3}$.

- **Estimating $x$-parameter:**

Now, we have all details to evaluate Fisher information for the parameter $x$ and the parameter $R$. Inserting (17), (18) in the definition of Fisher information (5), one gets $\mathcal{F}_f^w = \mathcal{F}_x^w + \mathcal{F}_p^w - \mathcal{F}_{xm}^w$ explicitly where,
Figure 5: (a) Fisher information, $\mathcal{F}_{xI}^w$, for an accelerated system initially prepared in Werner states. The solid, dot, dash, dash-dot, and dash-dot-dot curves for $x = -0.9, -0.8, -0.7, -0.6, -0.5$, respectively. (b) $\mathcal{F}_{xJ}^w$ represents the contributions from the classical $\mathcal{F}_{xc}^w$ (dot curve), the pure part, $\mathcal{F}_{xp}^w$ (dash-curve) and from the mixed part $\mathcal{F}_{xm}^w$ (dash-dot curve), while the solid curve represents the total Fisher information, $\mathcal{F}_{xI}^w$.

\[
\mathcal{F}_{xc}^w = \frac{\cos^2 r}{4(1 + x)} + \frac{\cos^4 r}{4(1 + \sin^2 r + x \cos^2 r)} + \left(\frac{\gamma \cos^2 r + 8x(1 + \cos 2r)}{\gamma^2 \left(1 - x \cos^2 r - \frac{1}{2\gamma^2}\right)}\right)^2
\]

\[
\mathcal{F}_{xp}^w = 4\left[\lambda_{3w} \mu_{3w}^2 (1 + \mu_{3w}^2)^2 + \lambda_{4w} \mu_{4w}^2 (1 + \mu_{4w}^2)^2\right],
\]

\[
\mathcal{F}_{xm}^w = \frac{8\lambda_{3w} \lambda_{4w}}{\lambda_{3w} + \lambda_{4w}} \left[\frac{(\mu_{3w} + \mu_{4w})^2}{(1 + \mu_{3w}^2)(1 + \mu_{4w}^2)}\right] \left\{ \frac{\mu_{3w}^2}{(1 + \mu_{3w}^2)^2} + \frac{\mu_{4w}^2}{(1 + \mu_{4w}^2)^2} \right\},
\]

where

\[
\mu_{3w}' = \frac{\partial \mu_{3w}}{\partial x} = -\frac{1}{\cos r} \left[\frac{4}{\sqrt{2\gamma}}(1 + \cos 2r) - \frac{2 - 2 \cos 2r + \sqrt{2\gamma}}{8x^2}\right],
\]

\[
\mu_{4w}' = \frac{\partial \mu_{4w}}{\partial x} = \frac{1}{\cos r} \left[\frac{4}{\sqrt{2\gamma}}(1 + \cos 2r) - \frac{2 - 2 \cos 2r - \sqrt{2\gamma}}{8x^2}\right].
\]

In Fig.(5a), we showed the behavior of the Fisher information ($\mathcal{F}_{xI}^w$ for a system initially prepared in a two qubit state of Werner type). This state affect by Unruh effect, where only the first qubit (Alice’s qubit) undergoes a uniform acceleration, while the second qubit (Bob’s qubit) remains in an inertial frame. Due to this effect, the Fisher information decreases as the acceleration increases. The decay rate of $\mathcal{F}_{xI}^w$ depends on the initial state settings and the acceleration value. It is clear that, for higher entangled state, the Fisher information is larger than that depicted for less entangled states. For small value of acceleration, $\mathcal{F}_{xI}^w$ is almost a stable.
From the definition of the Fisher information, it consists of three parts. One part represents the classical contribution, the second one represents the contributions of all pure states and the third part represents the mixture of all pure states. In Fig. (5b), we investigate the behavior of the three types of these information where we assume that the system is initially prepared in a Werner state with \( x = -0.6 \).

- Estimating the acceleration parameter, \( r \):
  Using the eigenvalues (17) and the normalized eigenvectors (18) in (5), one obtains a similar expression to that given by (19), where
  \[
  F_w^{rI} = F_w^{rc} + F_w^{rq} - F_w^{rm},
  \]
  The expressions of \( F_w^{rq} \) and \( F_w^{rm} \) are similar to that obtained in (19), but the derivatives with respect to the parameter \( r \). Namely we replace \( \partial \mu_{iw} / \partial x \) and \( \partial \mu_{iw} / \partial r \) by \( \partial \mu_{iw} / \partial r \) and \( \partial \mu_{iw} / \partial r \), respectively.

  \[
  \frac{\partial \mu_{iw}}{\partial r} = \pm \sec \frac{r}{2x} \left[ \sin 2r \pm \frac{1}{\sqrt{2}\gamma} \left(2(1-4x^2) \sin 2r - \sin 4r\right) \right. \\
  \left. \pm \frac{1}{2} (1-\cos 2r) - \frac{1}{2\sqrt{2}\gamma} \tan r \right].
  \]
  where \( i = 3, 4 \), respectively.

  While the expression of \( F_m^r \) takes the following form,

  \[
  F_w^{rc} = (1+x) \sin^2 r + \frac{1-x}{4} \sin 2r + \frac{1}{2\gamma^2} \left[ \left(\sqrt{2x}\gamma \sin 2r + 2(1-4x^2) \sin 2r + \sin 4r\right)^2 \right. \\
  \left. \left. 4 - 4x \cos^2 r - \sqrt{2}\gamma \right] \\
  + \frac{1}{2\gamma^2} \left[ \left(\sqrt{2x}\gamma \sin 2r + 2(1-4x^2) \sin 2r - \sin 4r\right)^2 \right. \\
  \left. \left. 4 - 4x \cos^2 r + \sqrt{2}\gamma \right].
  \]

Figure 6: Fisher information, \( F_w^{rI} \) for an accelerated system initially prepared in Werner states. (a) The solid, dot, dash, dash-dot, and dash-dot-dot curves for \( x = -0.9, -0.8, -0.7, -0.6 \), respectively. (b) The total Fisher information \( F_w^{rI} \) (solid curve) \( F_c \) (dot curve), \( F_w^{rp} \) (dash-curve) and \( F_w^{rm} \) (dash-dot curve), where \( x = -0.6 \).
In Fig.(6a), the behavior of $F_{wrI}$ is investigated for different initial state settings. It is clear that, the Fisher information increases as the acceleration increases. The increasing rate depends on the initial entanglement of the accelerated state. However, for less initial entanglement state, the increasing rate of $F_{wrI}$ is larger than that displayed for higher entangled state. This shows that we can measure the parameter $r$ with high precision.

Fig.(6b) shows the contributions from the different types of the Fisher information, where we assume that the system is initially prepared in Werner state described by $x = -0.6$. It is clear that all $F_{wJ}$, $J = c, p, m$ increase as $r$ increases, but with a different increasing rate. However, the contribution from the classical part is the largest one, while the pure part, $F_{rp}$ represent the smallest contribution. This shows that the possibility of obtaining classical correlated system is very large. On the other hand, this possibility decreases for obtaining pure states.

4 Conclusion

The behavior of Fisher information with respect to the parameters of an accelerated $X$-state is discussed, where we assume only one particle is uniformly accelerated and the other partial remains in the initial frame. A detailed exposition of the analytical form of the Fisher information is provided for the general state and its special version (Werner state). The effect of the Unruh acceleration on the dynamics of the Fisher information is investigated for different initial state settings. This study helps in revealing the sensitivity of the suggested system to the change of the state parameters. The contribution of the components of the Fisher information, classical, average pure and mixture on the total Fisher information is inspected.

The results show that, Fisher information has different behaviors depending on the initial state settings and the estimated parameter. In general, Fisher information decays as the Unruh acceleration increases. This decay may be steeper or gradually depending on the value of the acceleration and the values of the initial parameters. Three different behaviors are depicted for the Fisher information namely, constant, sudden increasing and sudden decay. The upper bounds of Fisher information are large for systems that have large degree of entanglement and consequently the precision of estimating the parameters increases. On the other hand, as the acceleration goes to infinity the lower bounds of the Fisher information are non zero.

Fisher information is quantified with respect to the $z, x$ and $r$ parameters for the $X$-state and with respect to $x$ and $r$ for Werner state. For Werner state, Fisher information decays as the acceleration increases and the maximum contribution on the total Fisher information comes from the classical part. This shows that, the amount of classical information increases as the Unruh acceleration increases and consequently the travelling state loses its entangled behavior and may turn into a separable state. However, for the $X$- state, the contributor part depends on the estimated parameter. For example, if we estimate the $x$ parameter, we can see that the contribution from the pure part is the largest one, which indicates that the number of the generated pure state is large.

In conclusion: the precision of estimating the parameters of the accelerated state depends on the degree of entanglement of the initial accelerated system and the value of the acceleration. Generally speaking the estimation precision decreases as Unruh acceleration.
increases for Werner state, while for $X$-state, there are values of Unruh acceleration that maximize the estimated value.

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