SOME SUBCLASSES OF ANALYTIC FUNCTIONS WITH NEGATIVE COEFFICIENTS FOR OPERATORS ON HILBERT SPACE

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Abstract: The main object of the present paper is to investigate some results concerning a sufficient and necessary condition, coefficient estimates and distortion theorem for the class $T_\lambda^\delta(\alpha, A)$. Furthermore, some applications of the fractional calculus for operator on Hilbert space are also considered.

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1. Introduction and definitions

Let $A$ denote the class of functions $f(z)$ of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k,$$

which are analytic in the open unit disk $U = \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$. Also let $S$ denote the class of functions in $A$ which are univalent in the unit disk $U$.

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Then a function \( f(z) \in S \) is said to be starlike of order \( \alpha \) \((0 \leq \alpha < 1)\) in \( \mathbb{U} \) if and only if
\[
\text{Re} \left( \frac{zf'(z)}{f(z)} \right) > \alpha \quad (0 \leq \alpha < 1; z \in \mathbb{U}).
\] (2)

We denote by \( S^*(\alpha) \) the class of all functions in \( S \) which are starlike of order \( \alpha \) in \( \mathbb{U} \).

A function \( f(z) \in S \) is said to be convex of order \( \alpha \) \((0 \leq \alpha < 1)\) in \( \mathbb{U} \) if and only if
\[
\text{Re} \left( 1 + \frac{zf''(z)}{f'(z)} \right) > \alpha \quad (0 \leq \alpha < 1; z \in \mathbb{U}).
\] (3)

We denote by \( K(\alpha) \) the class of all functions in \( S \) which are convex of order \( \alpha \) in \( \mathbb{U} \).

Let \( a, b \) and \( c \) be complex numbers with \( c \neq 0, -1, -2, \cdots \). Then the Gaussian/classical hypergeometric function \( _2F_1(a, b; c; z) \) is defined by
\[
_2F_1(a, b; c; z) = \sum_{k=0}^{\infty} \frac{(a)_k(b)_k z^k}{(c)_k k!},
\]
where \((\eta)_k\) is the Pochhammer symbol defined, in terms of the Gamma function, by
\[
(\eta)_k = \frac{\Gamma(\eta + k)}{\Gamma(\eta)} = \begin{cases} 
1 & (k = 0) \\
\eta(\eta + 1) \cdots (\eta + k - 1) & (k \in \mathbb{N}).
\end{cases}
\]
The hypergeometric function \( _2F_1(a, b; c; z) \) is analytic in \( \mathbb{U} \) and if \( a \) or \( b \) is a negative integer, then it reduces to a polynomial.

For functions \( f_j(z) \in A \), given by
\[
f_j(z) = z + \sum_{k=2}^{\infty} a_{k,j} z^k \quad (j = 1, 2),
\]
we define the Hadamard product (or convolution) of \( f_1(z) \) and \( f_2(z) \) by
\[
(f_1 * f_2)(z) = z + \sum_{k=2}^{\infty} a_{k,1} a_{k,2} z^k = (f_2 * f_1)(z) \quad (z \in \mathbb{U}).
\]

For the purpose to define the Srivastava-Attiya transform, we recall here the general Hurwitz-Lerch Zeta function, which is defined in [15] by the following series:
\[
\Phi(z, \lambda, \delta) := \frac{1}{\delta^\lambda} + \sum_{k=1}^{\infty} \frac{z^k}{(k+\delta)^\lambda}
\]
(δ ∈ C \ Z₀⁻ = \{0, -1, -2, ...\}; λ ∈ C when z ∈ U;
Re(λ) > 1 when |z| = 1).

For the properties and characteristics of the Hurwitz-Lerch Zeta function and other related special functions, see for example [4], [9] and [16].

Recently, Srivastava and Attiya [14] have introduced the linear operator
L_λ,δ : A → A, defined in terms of the Hadamard product by

L_λ,δ f(z) = G_λ,δ(z) * f(z) \quad (δ ∈ C \ Z₀⁻; λ ∈ C; z ∈ U), \quad (4)

where

G_λ,δ(z) = (1 + δ) \lambda \left[ Φ(z, λ, δ) - δ^{-λ} \right] \quad (z ∈ U). \quad (5)

The operator L_λ,δ is now popularly known in the literature as the Srivastava-Attiya operator. Various class-mapping properties of the operator L_λ,δ (and its variants) are discussed in the recent works of Srivastava and Attiya [14], Liu [8], Murugusundaramoorthy [10], Yuan and Liu [19], Yunus et al. [20] and others.

It is easy to observe from (1) and (4) that

L_λ,δ f(z) = z + \sum_{k=2}^{∞} \left( \frac{1 + δ}{k + δ} \right)^{λ} a_k z^k. \quad (6)

We note that:
(i) L_{0,0} f(z) = f(z);
(ii) L_{1,0} f(z) = L f(z) = \int_{0}^{z} \frac{f(t)}{t} \, dt \quad (f ∈ A) (see Alexander [1]);
(iii) L_{m,1} f(z) = T^m f(z) \quad (m ∈ N₀ = N \cup \{0\} = \{0, 1, 2, 3, ...\}) (see Flett [5]);
(iv) L_{γ,1} f(z) = Q^γ f(z) \quad (γ > 0) (see Jung et al. [6]);
(v) L_{m,0} f(z) = L^m f(z) \quad (m ∈ N₀) (see Sălăgean [12]).

Let T denote the subclass of S consisting of functions whose nonzero coefficients are negative. That is, an analytic and univalent function f is in T if it can be expressed as

f(z) = z - \sum_{k=2}^{∞} a_k z^k \quad (a_k ≥ 0). \quad (7)

We also denote by T^*(α) and C(α) the subclasses of T that are, respectively, starlike of order α and convex of order α. See Silverman [13] for further information on them. And let T^λ_δ(α) denote the class of functions of the form (7) satisfying the condition:

Re \left( \frac{z(L_λ,δ f)'(z)}{L_λ,δ f(z)} \right) > α \quad (λ ∈ R; δ > -1; 0 ≤ α < 1; z ∈ U). \quad (8)
Clearly, we have $\mathcal{T}_0^0(\alpha) = \mathcal{T}^*(\alpha)$ and $\mathcal{T}_0^{\alpha-1}(\alpha) = \mathcal{C}(\alpha)$ ($0 \leq \alpha < 1$).

Finally, let $A$ be a bounded linear operator on a complex Hilbert space $\mathcal{H}$. For a complex valued function $f$ analytic on a domain $E$ of the complex plane containing the spectrum $\sigma(A)$ of $A$ we denote $f(A)$ as Riesz-Dunford integral [2, p.568], that is,

$$f(A) := \frac{1}{2\pi i} \int_C f(z)(zI - A)^{-1}dz,$$

where $I$ is the identity operator on $\mathcal{H}$ and $C$ is positively oriented simple closed rectifiable contour containing $\sigma(A)$. Also $f(A)$ can be defined by the series $f(A) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} A^k$ which converges in the norm topology, [3]. If $f(z)$ is defined by (1), we also have

$$\mathcal{L}_{\lambda, \delta} f(A) = \sum_{k=1}^{\infty} \left( \frac{1 + \delta}{k + \delta} \right)^\lambda a_k A^k \quad (a_1 = 1).$$

Throughout this paper, $A^*$ shall always denote the conjugate operator of $A$.

By using arguments similar to [13, Theorem 2] with (8), we prove the following lemma.

**Lemma 1.** Let $f(z)$ of the form (7) be analytic in $\mathbb{U}$, $\lambda \in \mathbb{R}$, $\delta > -1$, and $0 \leq \alpha < 1$. Then the following statements are equivalent:

(i) $f \in \mathcal{T}_\delta^\lambda(\alpha)$;

(ii) $\left| \frac{z(\mathcal{L}_{\lambda, \delta} f(z))'}{\mathcal{L}_{\lambda, \delta} f(z)} - 1 \right| \leq 1 - \alpha \quad (z \in \mathbb{U})$;

(iii) $\sum_{k=2}^{\infty} \frac{k - \alpha}{1 - \alpha} B_k(\lambda, \delta) a_k \leq 1$,

where

$$B_k(\lambda, \delta) = \left( \frac{1 + \delta}{k + \delta} \right)^\lambda.$$

**Proof.** In view of the definition of $\mathcal{T}_\delta^\lambda(\alpha)$, we obtain

$$f \in \mathcal{T}_\delta^\lambda(\alpha) \Leftrightarrow \mathcal{L}_{\lambda, \delta} f \in \mathcal{T}^*(\alpha),$$

and so, Lemma 1 follows immediately from the result [13, Theorem 2].

From Lemma 1, we define a new class $\mathcal{T}_\delta^\lambda(\alpha, A)$ as following.

**Definition 1.** Let $\mathcal{T}_\delta^\lambda(\alpha, A)$ denote the class of functions of the form (7) satisfying the condition
\[ \|A(\mathcal{L}_{\lambda,\delta} f)'(A) - \mathcal{L}_{\lambda,\delta} f(A)\| \leq (1 - \alpha)\|\mathcal{L}_{\lambda,\delta} f(A)\|, \]  
(12)

where \( \lambda \in \mathbb{R} \), \( \delta > -1 \), \( 0 \leq \alpha < 1 \), and all operators \( A \) with \( \|A\| < 1 \), \( A \neq \theta \) (\( \theta \) denotes the zero operator on \( \mathcal{H} \)).

The following definition is given below for some operators of generalized fractional calculus defined by Kim et al. [7] (see also [11] and [17]).

**Definition 2.** For an invertible operator \( A \), the fractional integral operator \( \mathcal{I}_{0,A}^{a,b,c} \) is defined by

\[ \mathcal{I}_{0,A}^{a,b,c} f(A) = \frac{1}{\Gamma(a)} \int_0^1 A^{-b} F_1(a + b, -c; a; 1 - t) f(tA)(1 - t)^{a-1} dt, \]  
(13)

where \( a > 0 \) and \( b, c \in \mathbb{R} \).

The fractional derivative operator \( \mathcal{D}_{0,A}^{a,b,c} \) is defined by

\[ \mathcal{D}_{0,A}^{a,b,c} f(A) = \frac{1}{\Gamma(1 - a)} g'(A), \]  
(14)

where

\[ g(z) = \int_0^1 z^{-b} F_1(b - a + 1, -c; 1 - a; 1 - t) f(tz)(1 - t)^{-a} dt, \]

\( 0 < a < 1 \) and \( b, c \in \mathbb{R} \). In both (13) and (14), \( f(z) \) is an analytic function in a simply-connected region of the \( z \)-plane containing the origin with the order

\[ f(z) = \mathcal{O}(|z|^\epsilon) \quad (z \to 0) \]

for \( \epsilon > \max\{0, b - c\} - 1 \), and the multiplicity of \( (1 - t)^{a-1} \) is in (13) (and that of \( (1 - t)^{-a} \) in (14)) removed by requiring \( \log(1 - t) \) to be real when \( 1 - t > 0 \).

In this article, we provide some results concerning a sufficient and necessary condition, coefficient estimates and the distortion theorem for the class \( \mathcal{T}_\delta^\lambda(\alpha, A) \). Also, we consider several applications of fractional calculus for operators on Hilbert space.

### 2. Some results for the class \( \mathcal{T}_\delta^\lambda(\alpha, A) \)

We begin by proving an equivalent condition for the class \( \mathcal{T}_\delta^\lambda(\alpha, A) \) due to Lemma 1 as following.

**Lemma 2.** Let \( f(z) \) be in the class \( \mathcal{T}_\delta^\lambda(\alpha, A) \) for all proper contraction \( A \) with \( A \neq \theta \) if and only if
\[
\sum_{k=2}^{\infty} \frac{k - \alpha}{1 - \alpha} B_k(\lambda, \delta)a_k \leq 1,
\] (15)

where \(B_k(\lambda, \delta)\) is given by (11), \(\lambda \in \mathbb{R}, \delta > -1\) and \(0 \leq \alpha < 1\). The result is sharp for the function

\[
f(z) = z - \frac{1 - \alpha}{(k - \alpha)B_k(\lambda, \delta)} z^k \quad (k \geq 2).
\] (16)

**Proof.** Assume that the inequality (15) holds. By using (10) and (11), we have

\[
\| A(L_{\lambda, \delta} f)'(A) - L_{\lambda, \delta} f(A) \| - (1 - \alpha) \| L_{\lambda, \delta} f(A) \|
\]

\[
= \| A - \sum_{k=2}^{\infty} k B_k(\lambda, \delta)a_k A^k - A + \sum_{k=2}^{\infty} B_k(\lambda, \delta)a_k A^k \|
\]

\[
- (1 - \alpha) \| A - \sum_{k=2}^{\infty} B_k(\lambda, \delta)a_k A^k \|
\]

\[
= \| \sum_{k=2}^{\infty} (k - 1) B_k(\lambda, \delta)a_k A^k \| - (1 - \alpha) \| A - \sum_{k=2}^{\infty} B_k(\lambda, \delta)a_k A^k \|
\]

\[
\leq \sum_{k=2}^{\infty} (k - 1 + 1 - \alpha) B_k(\lambda, \delta)a_k - (1 - \alpha) \leq 0.
\]

Hence \(f(z) \in T_{\delta}^{\lambda}(\alpha, A)\). For the converse, assume that

\[
\| A(L_{\lambda, \delta} f)'(A) - L_{\lambda, \delta} f(A) \| \leq (1 - \alpha) \| L_{\lambda, \delta} f(A) \|.
\]

Then

\[
\| \sum_{k=2}^{\infty} (k - 1) B_k(\lambda, \delta)a_k A^k \| \leq (1 - \alpha) \| A - \sum_{k=2}^{\infty} B_k(\lambda, \delta)a_k A^k \|.
\]

Choose \(A = eI\) \((0 < e < 1)\). We obtain

\[
\frac{\sum_{k=2}^{\infty} (k - 1) B_k(\lambda, \delta)a_k e^k}{e - \sum_{k=2}^{\infty} B_k(\lambda, \delta)a_k e^k} \leq 1 - \alpha.
\] (17)

Upon clearing the denomination in (17) and letting \(e \to 1\), we have

\[
\sum_{k=2}^{\infty} (k - 1) B_k(\lambda, \delta)a_k \leq (1 - \alpha) \{ 1 - \sum_{k=2}^{\infty} B_k(\lambda, \delta)a_k \},
\]

which yields the required condition (15). It is evident that the function (16) is an extreme one for Lemma 2. \(\square\)
Corollary 1. Let \( f(z) \) be in the class \( T_\delta^\lambda(\alpha, A) \). Then
\[
a_k \leq \frac{1 - \alpha}{(k - \alpha)B_k(\lambda, \delta)} \quad (k \geq 2),
\]
where \( B_k(\lambda, \delta) \) is given by (11), \( \lambda \in \mathbb{R}, \delta > -1 \) and \( 0 \leq \alpha < 1 \). The result is sharp for the function
\[
f(z) = z - \frac{1 - \alpha}{(k - \alpha)B_k(\lambda, \delta)} z^k \quad (k \geq 2).
\]

Theorem 1. Let \( f_1(z) = z \) and
\[
f_k(z) = z - \frac{1 - \alpha}{(k - \alpha)B_k(\lambda, \delta)} z^k \quad (k \geq 2).
\]
Then \( f(z) \in T_\delta^\lambda(\alpha, A) \) if and only if it can be expressed in the form
\[
f(z) = \sum_{k=1}^{\infty} \mu_k f_k(z), \tag{18}
\]
where \( \mu_k \geq 0 \) \( (k \geq 1) \) and \( \sum_{k=1}^{\infty} \mu_k = 1 \).

Proof. If we set
\[
f(z) = \sum_{k=1}^{\infty} \mu_k f_k(z),
\]
then
\[
f(z) = z - \sum_{k=2}^{\infty} \mu_k \frac{1 - \alpha}{(k - \alpha)B_k(\lambda, \delta)} z^k.
\]
Thus we obtain
\[
\sum_{k=2}^{\infty} \frac{(k - \alpha)B_k(\lambda, \delta)}{1 - \alpha} \mu_k \frac{1 - \alpha}{(k - \alpha)B_k(\lambda, \delta)} = \sum_{k=2}^{\infty} \mu_k = 1 - \mu_1 \leq 1.
\]
Hence \( f(z) \in T_\delta^\lambda(\alpha, A) \). For the converse, we assume that \( f(z) \) given by (7) is in the class \( T_\delta^\lambda(\alpha, A) \). From Corollary 1 we have
\[
a_k \leq \frac{1 - \alpha}{(k - \alpha)B_k(\lambda, \delta)} \quad (k \geq 2).
\]
We may set
\[
\mu_k = \frac{(k - \alpha)B_k(\lambda, \delta)}{1 - \alpha} a_k \quad (k \geq 2)
\]
and \( \mu_1 = 1 - \sum_{k=2}^{\infty} \mu_k \). Hence we conclude that
\[
f(z) = \sum_{k=1}^{\infty} \mu_k f_k(z),
\]
which completes the proof of Theorem 1. \( \square \)
**Theorem 2.** If $f(z) \in T_{\delta}^\lambda(\alpha, A)$ for $\lambda \geq 0$, $\delta > -1$, $0 \leq \alpha < 1$ and $\|A\| < 1$, $A \neq \theta$, then

$$\|A\| - \frac{1 - \alpha}{2 - \alpha} \left(\frac{1 + \delta}{2 + \delta}\right)^\lambda \|A\|^2 \leq \|f(A)\| \leq \|A\| + \frac{1 - \alpha}{2 - \alpha} \left(\frac{1 + \delta}{2 + \delta}\right)^\lambda \|A\|^2$$

and

$$1 - \frac{2(1 - \alpha)}{2 - \alpha} \left(\frac{1 + \delta}{2 + \delta}\right)^\lambda \|A\| \leq \|f'(A)\| \leq 1 + \frac{2(1 - \alpha)}{2 - \alpha} \left(\frac{1 + \delta}{2 + \delta}\right)^\lambda \|A\|.$$ 

**Proof.** From Lemma 2, we see that

$$\frac{2 - \alpha}{1 - \alpha} \left(\frac{1 + \delta}{2 + \delta}\right)^{-\lambda} \sum_{k=2}^{\infty} k - \frac{\alpha}{1 - \alpha} \sum_{k=2}^{\infty} k - \frac{\alpha}{1 - \alpha} B_k(-\lambda, \delta) a_k \leq 1$$

for $\lambda \geq 0$ and $\delta > -1$ which gives

$$\sum_{k=2}^{\infty} a_k \leq \frac{1 - \alpha}{2 - \alpha} \left(\frac{1 + \delta}{2 + \delta}\right)^\lambda.$$ 

Therefore we have

$$\|f(A)\| \geq \|A\| - \|A\|^2 \sum_{k=2}^{\infty} a_k \geq \|A\| - \frac{1 - \alpha}{2 - \alpha} \left(\frac{1 + \delta}{2 + \delta}\right)^\lambda \|A\|^2$$

and

$$\|f(A)\| \leq \|A\| + \|A\|^2 \sum_{k=2}^{\infty} a_k \leq \|A\| + \frac{1 - \alpha}{2 - \alpha} \left(\frac{1 + \delta}{2 + \delta}\right)^\lambda \|A\|^2.$$ 

By noting the relation

$$\frac{k(2 - \alpha)}{2(1 - \alpha)} \left(\frac{1 + \delta}{2 + \delta}\right)^{-\lambda} \leq \frac{k - \alpha}{1 - \alpha} B_k(-\lambda, \delta) \quad (k \geq 2),$$

we have

$$\sum_{k=2}^{\infty} k(2 - \alpha) \left(\frac{1 + \delta}{2 + \delta}\right)^{-\lambda} a_k \leq \sum_{k=2}^{\infty} \frac{k - \alpha}{1 - \alpha} B_k(-\lambda, \delta) a_k \leq 1,$$

that is

$$\sum_{k=2}^{\infty} k a_k \leq \frac{2(1 - \alpha)}{2 - \alpha} \left(\frac{1 + \delta}{2 + \delta}\right)^\lambda.$$ 

Thus

$$\|f'(A)\| \geq 1 - \|A\| \sum_{k=2}^{\infty} k a_k \geq 1 - \frac{2(1 - \alpha)}{2 - \alpha} \left(\frac{1 + \delta}{2 + \delta}\right)^\lambda \|A\|$$

and
\[ \|f'(A)\| \leq 1 + \frac{2(1-\alpha)}{2-\alpha} \left( \frac{1+\delta}{2+\delta} \right)^\lambda \|A\|.\]

This completes the proof of Theorem 2. \(\Box\)

3. Some results for fractional calculus operators

By using Definition 2, we prove the following theorem.

**Theorem 3.** Let \(\max\{b-c,b,-c-a\} < 2\), \(2a > b(a+c)\), \(\lambda \geq 0\) and \(\delta > -1\). If \(f(z) \in T^\lambda_{\alpha}(\alpha, A)\) (\(0 \leq \alpha < 1\)), then

\[ \|I_{a,b,c}^A f(A)\| \leq \frac{\Gamma(2-b+c)}{\Gamma(2-b)\Gamma(a+2+c)} \|A\|^{1-b} \]

\[ + \frac{(1-\alpha)\Gamma(2-b+c)}{(2-\alpha)\Gamma(2-b)\Gamma(a+2+c)} \left( \frac{1+\delta}{2+\delta} \right)^\lambda \|A\|^{2-b} \]

and

\[ \|I_{a,b,c}^A f(A)\| \geq \frac{\Gamma(2-b+c)}{\Gamma(2-b)\Gamma(a+2+c)} \|A\|^{1-b} \]

\[ - \frac{(1-\alpha)\Gamma(2-b+c)}{(2-\alpha)\Gamma(2-b)\Gamma(a+2+c)} \left( \frac{1+\delta}{2+\delta} \right)^\lambda \|A\|^{2-b} \]

for \(a > 0\), \(b,c \in \mathbb{R}\) and all invertible operator \(A\) with \((A^q)^*A^q = A^q(A^q)^*\) \((q \in \mathbb{N})\), \(\|A\| \leq 1\) and \(r_{sp}(A)r_{sp}(A^{-1}) \leq 1\), where \(r_{sp}(A)\) is the radius of spectrum of \(A\).

**Proof.** Consider the function

\[ F(A) = \frac{\Gamma(2-b)\Gamma(a+2+c)}{\Gamma(2-b-c)} A^b I_{a,b,c}^A f(A) \]

\[ = A - \sum_{k=2}^{\infty} \frac{\Gamma(k+1-b+c)\Gamma(k+1)\Gamma(2-b)\Gamma(a+2+c)}{\Gamma(k+1-b)\Gamma(a+k+1+c)\Gamma(2-b+c)} a_k A^k \]

\[ = A - \sum_{k=2}^{\infty} b_k A^k ,\]

where

\[ b_k = \frac{\Gamma(k+1-b+c)\Gamma(k+1)\Gamma(2-b)\Gamma(a+2+c)}{\Gamma(k+1-b)\Gamma(a+k+1+c)\Gamma(2-b+c)} a_k.\]
Further, for convenience, we put
\[ \Phi(k) = \frac{\Gamma(k + 1 - b + c)\Gamma(k + 1)\Gamma(2 - b)\Gamma(a + 2 + c)}{\Gamma(k + 1 - b)\Gamma(a + k + 1 + c)\Gamma(2 - b + c)} \quad (k \geq 2). \]

Then, by the constraints of the hypotheses, we see that \( \Phi(k) \) is non-increasing for integers \( k \geq 2 \) and we have \( 0 < \Phi(k) < 1 \). By virtue of Lemma 2, we obtain
\[
\frac{2 - \alpha}{1 - \alpha} \left( \frac{1 + \delta}{2 + \delta} \right)^{-\lambda} \sum_{k=2}^{\infty} b_k \leq \sum_{k=2}^{\infty} \frac{k - \alpha}{1 - \alpha} B_k(-\lambda, \delta) b_k \leq \sum_{k=2}^{\infty} \frac{k - \alpha}{1 - \alpha} B_k(-\lambda, \delta) a_k \leq 1,
\]
which gives
\[
\sum_{k=2}^{\infty} b_k \leq \frac{1 - \alpha}{2 - \alpha} \left( \frac{1 + \delta}{2 + \delta} \right)^{\lambda} \quad \text{and} \quad F(z) \in \mathcal{T}_{\delta}^{\lambda}(\alpha, A).
\]

Therefore we have
\[
\|T_{0,A}^{a,b,c} f(A)\| \leq \frac{\Gamma(2 - b + c)}{\Gamma(2 - b)\Gamma(a + 2 + c)} \|A\| \|A^{-b}\| \quad (20)
\]
\[
+ \frac{(1 - \alpha)\Gamma(2 - b + c)}{(2 - \alpha)\Gamma(2 - b)\Gamma(a + 2 + c)} \left( \frac{1 + \delta}{2 + \delta} \right)^{\lambda} \|A\|^2 \|A^{-b}\|
\]
and
\[
\|T_{0,A}^{a,b,c} f(A)\| \geq \frac{\Gamma(2 - b + c)}{\Gamma(2 - b)\Gamma(a + 2 + c)} \|A\| \|A^{-b}\| \quad (21)
\]
\[
- \frac{(1 - \alpha)\Gamma(2 - b + c)}{(2 - \alpha)\Gamma(2 - b)\Gamma(a + 2 + c)} \left( \frac{1 + \delta}{2 + \delta} \right)^{\lambda} \|A\|^2 \|A^{-b}\|.
\]
By the equation (7) of [18, p.307],
\[
\|A^b\| = \|A\|^b \quad (b > 0).
\]
Since \( A^*A = AA^* \), \( \|A\| = r_{sp}(A) \). So,
\[
1 = \|AA^{-1}\| \leq \|A\| \|A^{-1}\| = r_{sp}(A)r_{sp}(A^{-1}) \leq 1.
\]
Thus \( \|A^{-1}\| = \|A\|^{-1} \). Therefore
\[
\|A^b\| = \|A\|^b \quad (22)
\]
for all real \( b \). By applying (20), (21) and (22), we evidently completes the proof of Theorem 3. \( \square \)
Theorem 4. Let \( \max\{b-c-1, b, -2-c+a\} < 1 \), \( c+1 < (1-b)(2-a+c) \), \( b(2-a+c) \leq 2(1-\alpha) \), \( \lambda \geq 0 \) and \( \delta > -1 \). If \( f(z) \in T^\lambda_\delta(\alpha, A) \) (\( 0 \leq \alpha < 1 \)), then

\[
\|D_{0,A}^{a,b,c} f(A)\| \leq \frac{\Gamma(2-b+c)}{\Gamma(1-b)\Gamma(3-a+c)}\|A\|^{-b} \noindent + \frac{2(1-\alpha)\Gamma(2-b+c)}{(2-\alpha)\Gamma(1-b)\Gamma(3-a+c)} \left( \frac{1+\delta}{2+\delta} \right) \lambda \|A\|^{1-b}
\]

and

\[
\|D_{0,A}^{a,b,c} f(A)\| \geq \frac{\Gamma(2-b+c)}{\Gamma(1-b)\Gamma(3-a+c)}\|A\|^{-b} \noindent - \frac{2(1-\alpha)\Gamma(2-b+c)}{(2-\alpha)\Gamma(1-b)\Gamma(3-a+c)} \left( \frac{1+\delta}{2+\delta} \right) \lambda \|A\|^{1-b}
\]

for \( a > 0 \), \( b, c \in \mathbb{R} \) and all invertible operator \( A \) with \( (A^{\frac{1}{q}})^* A^{\frac{1}{q}} = A^\frac{1}{q} (A^\frac{1}{q})^* \) (\( q \in \mathbb{N} \)), \( \|A\| \leq 1 \) and \( r_{sp}(A)r_{sp}(A^{-1}) \leq 1 \), where \( r_{sp}(A) \) is the radius of spectrum of \( A \).

Proof. Consider the function

\[
G(A) = \frac{\Gamma(1-b)\Gamma(3-a+c)}{\Gamma(2-b-c)} A^{b+1} D_{0,A}^{a,b,c} f(A)
\]

\[
= A - \sum_{k=2}^{\infty} \frac{\Gamma(k+1-b+c)\Gamma(k+1)\Gamma(1-b)\Gamma(3-a+c)}{\Gamma(k-b)\Gamma(k+2-a+c)\Gamma(2-b+c)} a_k A^k
\]

\[
= A - \sum_{k=2}^{\infty} c_k A^k,
\]

where

\[
c_k = \frac{\Gamma(k+1-b+c)\Gamma(k+1)\Gamma(1-b)\Gamma(3-a+c)}{\Gamma(k-b)\Gamma(k+2-a+c)\Gamma(2-b+c)} a_k.
\]

Further, for convenience, we put

\[
\Psi(k) = \frac{\Gamma(k+1-b+c)\Gamma(k)\Gamma(1-b)\Gamma(3-a+c)}{\Gamma(k-b)\Gamma(k+2-a+c)\Gamma(2-b+c)} \quad (k \geq 2).
\]

Then, by the constraints of the hypotheses, we see that \( \Psi(k) \) is non-increasing for the integers \( k \geq 2 \) and we have \( 0 < \Psi(k) < 1 \), that is,

\[
0 < \frac{\Gamma(k+1-b+c)\Gamma(k+1)\Gamma(1-b)\Gamma(3-a+c)}{\Gamma(k-b)\Gamma(k+2-a+c)\Gamma(2-b+c)} < k.
\]
Therefore, by applying (19) and Lemma 2, we obtain
\[
\frac{2 - \alpha}{2(1 - \alpha)} \left( \frac{1 + \delta}{2 + \delta} \right)^{-\lambda} \sum_{k=2}^{\infty} c_k = \sum_{k=2}^{\infty} \frac{2 - \alpha}{2(1 - \alpha)} \left( \frac{1 + \delta}{2 + \delta} \right)^{-\lambda} k\Psi(k)a_k \\
\leq \sum_{k=2}^{\infty} \frac{k - \alpha}{1 - \alpha} B_k(-\lambda, \delta)\Psi(k)a_k \\
\leq \sum_{k=2}^{\infty} \frac{k - \alpha}{1 - \alpha} B_k(-\lambda, \delta)a_k \leq 1,
\]
which gives
\[
\sum_{k=2}^{\infty} c_k \leq \frac{2(1 - \alpha)}{2 - \alpha} \left( \frac{1 + \delta}{2 + \delta} \right)^{\lambda}.
\]
Hence, by using same arguments with the proof of Theorem 3, we have
\[
\|D_{a,b,c}^0 f(A)\| &= \frac{\Gamma(2 - b + c)}{\Gamma(1 - b)\Gamma(3 - a + c)}\|A\|^{-b} + \frac{\Gamma(2 - b + c)}{\Gamma(1 - b)\Gamma(3 - a + c)}\|A\|^{-b}\sum_{k=2}^{\infty} c_k \\
&\leq \frac{\Gamma(2 - b + c)}{\Gamma(1 - b)\Gamma(3 - a + c)}\|A\|^{-b} \\
&\quad + \frac{2(1 - \alpha)\Gamma(2 - b + c)}{(2 - \alpha)\Gamma(1 - b)\Gamma(3 - a + c)} \left( \frac{1 + \delta}{2 + \delta} \right)^{\lambda} \|A\|^{-b}
\]
and
\[
\|D_{a,b,c}^0 f(A)\| \geq \frac{\Gamma(2 - b + c)}{\Gamma(1 - b)\Gamma(3 - a + c)}\|A\|^{-b} \\
- \frac{2(1 - \alpha)\Gamma(2 - b + c)}{(2 - \alpha)\Gamma(1 - b)\Gamma(3 - a + c)} \left( \frac{1 + \delta}{2 + \delta} \right)^{\lambda} \|A\|^{-b}
\]
This evidently completes the proof of Theorem 4. \[\square\]

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