A matrix generalization of Euler identity $e^{j\varphi} = \cos \varphi + j \sin \varphi$

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Abstract

In this work we present a matrix generalization of the Euler identity about exponential representation of a complex number. The concept of matrix exponential is used in a fundamental way. We define a notion of matrix imaginary unit which generalizes the usual complex imaginary unit. The Euler-like identity so obtained is compatible with the classical one. Also, we derive some exponential representation for matrix real and imaginary unit, and for the first Pauli matrix.

Keywords: Euler identity, matrix exponential, series expansion, matrix unit representation, Pauli matrices representation.

1 The matrix exponential

Let $A$ denotes a generic matrix. Based on the Taylor expansion centered at 0 for the one real variable function $e^x$, the matrix exponential (see e.g. [3]) is formally defined as

$$e^A = \sum_{n=0}^{+\infty} \frac{A^n}{n!}$$

(1)
2 A class of complex matrices

Let it be $j$ the imaginary unit, $j^2 = -1$, $\alpha$, $a$, $b$ real numbers, $\alpha \neq 0$. We consider the class of $2 \times 2$ matrices of the form

$$T = \begin{pmatrix} a & jb \\ j\alpha^2 b & a \end{pmatrix}$$

These matrices are very important in the theory of transfer matrix method for modelization of acoustical transmission in physical structures (see [3], [2]). Note that $\text{Det}(T) = a^2 + \alpha^2 b^2$, so that if $a$ and $b$ are not both zero, $T$ is invertible. In acoustical transfer matrices, usually $\alpha = \frac{S}{c}$, where $S$ is the section of a tube or duct and $c$ is the sound speed in the fluid contained in the tube.

Consider the matrix

$$\Phi = \begin{pmatrix} 0 & j \\ j\alpha^2 & 0 \end{pmatrix}$$

Then, if $I$ is the identity matrix, the following representation for previous $T$ matrices holds:

$$T = aI + b\Phi$$

(2)

Note the analogy with a usual complex number $a + jb$. The analogy is more evident if one consider that, with a simple calculation, $\Phi^2 = -\alpha^2 I$. For this reason, we call $\Phi$ the imaginary unit matrix. Also, note that, for $\alpha = 1$, we obtain $\Phi = j\sigma_1$, where $\sigma_1$ is one of the Pauli matrices of quantum mechanics (see e.g. [4]).

3 A generalization of Euler identity

The Euler identity $e^{j\varphi} = \cos \varphi + j \sin \varphi$ is valid for any real $\varphi$. Usually this formula is proven by use of Taylor expansion of the complex function $e^z$ and of the real functions $\cos \varphi$ and $\sin \varphi$ (see [1]).

We prove a generalization, in the environment of the complex matrices of type (2), of this identity.
Lemma 1 Let it be $\Psi = -j\Phi$. Then, for every natural $n$, the following relation holds:

$$\Psi^n = \alpha^{n-r}\Psi^r$$

(3)

where $r = \text{mod}(n, 2)$.

\textit{Dim.} By induction on $n$. For $n = 0$ and $n = 1$ the thesis is obvious. Note that

$$\Psi = \begin{pmatrix} 0 & 1 \\ \alpha^2 & 0 \end{pmatrix}$$

Let it be $n = 2$. By a simple calculation

$$\Psi^2 = \left( \begin{array}{cc} 0 & 1 \\ \alpha^2 & 0 \end{array} \right)^2 = \left( \begin{array}{cc} \alpha^2 & 0 \\ 0 & \alpha^2 \end{array} \right) = \alpha^2 I = \alpha^2 \Psi^0$$

and the thesis is verified. Then we suppose that the thesis is verified for a generic $n$ too. Using the last inductive step we have

$$\Psi^{n+1} = \Psi^n \Psi = \alpha^{n-r}\Psi^r \Psi$$

(4)

where $r = \text{mod}(n, 2)$. If $\text{mod}(n + 1, 2) = 1$, then $r = 0$, therefore

$$\Psi^{n+1} = \alpha^n \Psi = \alpha^{n+1-s} \Psi^s$$

(5)

with $s = \text{mod}(n + 1, 2) = 1$, and the thesis is true in this case. If $\text{mod}(n + 1, 2) = 0$, then $r = 1$, therefore, using the first inductive step for $n = 2$,

$$\Psi^{n+1} = \alpha^{n-1} \Psi \Psi = \alpha^{n+1} \Psi^0 = \alpha^{n+1-s} \Psi^s$$

(6)

with $s = \text{mod}(n + 1, 2) = 0$, and the thesis is true in this case too. \qed

Now we can prove the matrix generalization of Euler identity:

Theorem 1 For every real $\varphi$, the following formula holds:

$$e^{\varphi \Phi} = \cos(\alpha \varphi) I + \frac{1}{\alpha} \sin(\alpha \varphi) \Phi$$

(7)
If $\varphi = 0$ the formula is obvious. For $\varphi \neq 0$, from the formal definition we can write

$$e^{\varphi \Phi} = \sum_{n=0}^{+\infty} \frac{(\varphi \Phi)^n}{n!} = \sum_{n \text{ odd}} \frac{\varphi^n \Phi^n}{n!} + \sum_{n \text{ even}} \frac{\varphi^n \Phi^n}{n!}$$

(8)

Recall that, if $n$ is even, then $j^n$ alternates $-1$ and $+1$, while if $n$ is odd, then $j^n$ alternates $-j$ and $+j$. Therefore, from $\Phi = j\Psi$, from the usual series expansion for $\cos(\alpha \varphi)$ and $\sin(\alpha \varphi)$, and using the previous Lemma, we have

$$e^{\varphi \Phi} = \sum_{n \text{ even}} j^n \frac{\varphi^n \Psi^n}{n!} + \sum_{n \text{ odd}} j^n \frac{\varphi^n \Psi^n}{n!} =$$

(9)

\[= \left( \sum_{n \text{ even}} j^n \frac{(\alpha \varphi)^n}{n!} \right) \mathbf{I} + \frac{1}{\alpha} \left( \sum_{n \text{ odd}} j^n \frac{(\alpha \varphi)^n}{n!} \right) \Psi = \]

\[= \cos(\alpha \varphi) \mathbf{I} + \frac{1}{\alpha} j \sin(\alpha \varphi) \Psi = \cos(\alpha \varphi) \mathbf{I} + \frac{1}{\alpha} \sin(\alpha \varphi) \Phi \]

that is the thesis. □

Note 1. Let it be $\alpha = 1$, and $\mathbf{I} = [1]$, $\Phi = [j]$ two $1 \times 1$ matrices, so that $\mathbf{I}$ is the usual real unit and $\Phi$ the usual imaginary unit. From (7) we have

$$e^{j\varphi} = \cos(\varphi)[1] + \sin(\varphi)[j] = \cos \varphi + j \sin \varphi$$

(10)

that is the classical Euler identity.

Note 2. If we write in explicite mode the relation (7), we obtain

$$e^{\varphi \Phi} = \begin{pmatrix} \cos(\alpha \varphi) & \frac{j}{\alpha} \sin(\alpha \varphi) \\ j\alpha \sin(\alpha \varphi) & \cos(\alpha \varphi) \end{pmatrix}$$

so that $Det(e^{\varphi \Phi}) = 1$, which is compatible with the fact that for usual complex numbers $|e^{j\varphi}| = 1$.

Note 3. If $\alpha = 1$ and $\varphi = 2m\pi$, with $m$ integer, (7) becomes

$$e^{2m\pi \Phi} = \mathbf{I}$$

(11)
that is a matrix unit representation. The classical analogous formula is
\[ e^{j2m\pi} = 1. \]

**Note 4.** If \( \alpha = 1 \) and \( \varphi = m\frac{\pi}{2} \), with \( m = 1 + 4k \), \( k \) integer, (7) becomes
\[ e^{m\frac{\pi}{2}\Phi} = \Phi \quad (12) \]
that is a matrix imaginary unit representation. The classical analogous formula is \( e^{im\frac{\pi}{2}} = j \). Also, if we multiply previous formula by \(-j\), we have an exponential representation of Pauli matrix \( \sigma_1 \):
\[ \sigma_1 = -je^{m\frac{\pi}{2}\Phi} \quad (13) \]

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