Baryon as dyonic instanton

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Abstract

We discuss the baryon in the holographic QCD framework and focus on the role of the bifundamental scalar field, realizing chiral symmetry breaking. We suggest the interpretation of a baryon as a dyonic instanton within the Atiyah-Manton-like approach for the flavor gauge group in the peculiar cylindrical ansatz in five-dimensional theory. Our approach provides the new mechanism of the stabilization of the baryon size.
I. INTRODUCTION

The topological interpretation of the baryon has been found long time ago in the Skyrme model \[1\]. Later it was recognized that the holography provides very transparent geometrical realization of topological nature of the baryon. The baryonic vertex has been found in \[2\] and its realization as the instanton in the flavor gauge group has been clarified in \[3\]. The relation between the Skyrmions and the instantons in five dimensions has been noticed in \[4\] where the Skyrme field gets interpreted as a kind of monodromy in the instanton background. The Atiyah-Manton interpretation has been realized dynamically in the model with the domain wall localized in the fifth dimension \[5\].

The interpretation of the baryonic charge as the instanton charge in \(D=5\) \[6\] or difference of the Chern numbers \[7\] works perfectly however some characteristics of the nucleon are harder to obtain. In particular the nucleon mass is determined by the instanton action and the mechanism of stabilization of the instanton size is required. In the conventional Skyrme model the baryon size is stabilized by the higher derivative terms and the holographic mechanisms were discussed in the literature as well \[6, 8, 9\]. However there is no satisfactory size stabilization pattern which would be free from drawbacks. There also exist the other approach to address the baryon physics in holographic models by introducing the fermion field in the bulk \[10\], but it is not related to the present study, because it doesn’t account for the size of the baryon at all.

On the other hand there is the mysterious Ioffe’s formula \[11\] derived via QCD sum rules

\[ m_N^3 = -8\pi^2 < 0|q\bar{q}|0 > . \quad (1) \]

Its status was questionable for decades since there was no alternative derivation of this suggestive relation. It certainly implies that the nucleon size and mass are fixed by some stabilization mechanism involving the chiral symmetry breaking.

Motivated by these unsolved problems we shall look for another solution to the
equations of motion in D=5 holographic dual with the proper quantum numbers. The proper candidate is the dyonic instanton which generalizes the conventional instanton in the D=5 theory with the scalar field in the adjoint representation. Such solution has been found in [12] and it carries the additional charge quantum number as well as the angular momentum. The key point is that the instanton size is fixed by the vacuum expectation value (VEV) of the scalar $v$ and the electric charge $Q_e$ of solution.

$$\rho = \sqrt{\frac{Q_e}{4\pi^2 v}}$$

If the Chern-Simons term is added to the 5D action the moduli space of the dyonic instanton gets modified [13].

We shall use such solution in the holographic QCD framework when the gauge group $SU_L(N_F) \times SU_R(N_F)$ on the flavor branes is broken to the diagonal one by the chiral condensate [14]. It is convenient to describe the condensate via the scalar field in the bifundamental representation with respect to the bare gauge group. Upon the chiral symmetry breaking we find ourselves in the situation where the dyonic instantons could be look for.

Comparing to the Lambert-Tong solution [12] there are some essential differences. First, instead of the flat space we should start with the instanton in $AdS_4$ geometry. Secondly our dual theory involves the scalar in the bifundamental representation hence we need to clarify the physical meaning of the corresponding charge and VEV which would stabilize the dyonic instanton size. We shall work in the chiral limit hence according to the holographic dictionary the boundary behavior in the radial coordinate of the VEV of the bifundamental yields the chiral condensate. On the other hand the scalar interacts with the axial gauge field hence the proper second charge of the dyonic instanton is identified with the axial charge in the physical space. This identification fits with the non-vanishing axial charge of the nucleon determined by the matrix element of the axial current

$$<N|A|N> \sim g_A$$

We shall look for the dyonic instanton type solution in the simplest holographic
hard-wall model [14] in convenient cylindrical ansatz [15] which describes the $SO(3)$ symmetric solution in the physical space. It will be found that the Atiyah-Manton type approach [4] can be generalized in the theory with an additional scalar field. The domain wall type solution in $AdS_4$ geometry is described and its second charge (similar to the electric charge of dyonic instanton) is identified with the axial charge of the baryon. Hence we obtain the new mechanism for the baryon size stabilization where both the chiral condensate and the baryon axial charge play the crucial role. This qualitatively fits with the Ioffe’s formula for the baryon mass.

The paper is organized as follows. In Section 2 we remind the Lambert-Tong solution and its key features. In Section 3 we present the generalization of dyonic instanton solution for the holographic QCD. Section 4 is devoted to the comments on the brane interpretation of the solution found. Summary of our findings and the open questions can be found in the Conclusion.

II. ON DYONIC INSTANTONS

The dyonic instanton [12] is solution to the conventional $D = 5$ equation of motion

$$F_{\mu\nu} = *F_{\mu\nu}, \quad D_\mu \phi = E_\mu, \quad D_0 \phi = 0,$$

where $\phi$ is the scalar field and $E_\mu$ is electric field in $R^4$. The BPS formula yields for its mass

$$M = \frac{4\pi^2}{g^2} |I| + |vQ_e|,$$

where $|I|$ is the instanton charge and $v$ is the VEV of the scalar. The BPS-ness is provided by the combined effect of the scalar, electric field and the ”running” of the instantons.

There are dyonic instantons with the different symmetries with respect to the space rotations. In what follows we shall discuss the solution with the spherical symmetry however the most transparent physical picture is realized by the tubular D2
brane stretched between two parallel D4 branes \[16\] which we shall briefly describe in this Section. To identify all charges it is useful to discuss the worldvolume Lagrangian of the tubular D2 brane of constant radius \( R \) in flat Euclidean space-time \[16\] 

\[
L = -\sqrt{R^2(1 - E^2) + B^2},
\]

where \( E, B \) are the worldvolume electric and magnetic fields. The corresponding canonical momentum reads as

\[
\Pi = \frac{\partial L}{\partial E} = \frac{R^2 E}{\sqrt{R^2(1 - E^2) + B^2}}
\]

and Hamiltonian density is

\[
\mathcal{H} = R^{-1}\sqrt{\Pi^2 + R^2}(B^2 + R^2)
\]

There are two quantum numbers

\[
Q_F = \frac{1}{2\pi} \oint d\phi \Pi, \quad Q_0 = \frac{1}{2\pi} \oint d\phi B
\]

corresponding to the F1 and D0 conserved charges per unit length carried by the D2 tube. The tension reads as

\[
T = \frac{1}{2\pi} \oint d\phi \mathcal{H},
\]

which equals at the solution to the equation of motion to

\[
T = |Q_F| + |Q_0|
\]

Therefore the D2 brane tension does not contribute and the cross section of the tube behaves as the "tensionless" object which explains the arbitrary form of the tube cross section.

The solution carries crossed electric and magnetic fields that is the Poynting vector does not vanish and yields

\[
M_\phi = \Pi B
\]

5
Therefore there is non-vanishing angular momentum of dyonic instanton per unit length \[ L = Q_F Q_0 \] (13) directed along the axis of the cylinder. One can calculate it exactly at the cross-section of the supertube \[ L = \oint ds \left( x_3 \frac{\partial x_4}{\partial s} - x_4 \frac{\partial x_3}{\partial s} \right) \] (14) It is this angular momentum which provides the stabilization of the radius of the tubular D2 brane stretched between two D4 branes.

The conventional dyonic instanton solution involves the physical time hence upon the analytic continuation to the Minkowski space it corresponds to some tunneling process. The key point concerns the existence of the negative mode in the fluctuation spectrum at the top of the solution which would provide a sort of instability in the Minkowski space. Such negative mode responsible for the expanding of the radius of the solution has been found for the large radii in \[ R_4 \] (18). Hence at least at large radius of the dyonic instanton fixed by its angular momentum there is negative mode in \[ R_4 \] which supports the bounce interpretation. However at small values of the angular momentum there are no negative modes and the solution behaves as instanton.

III. DYONIC INSTANTON IN HOLOGRAPHIC QCD

To realize the dyonic instanton type solution in holographic framework we take the simplest “hard-wall” model of AdS/QCD (14). It posses two gauge fields, corresponding to the QCD flavor group of \( SU(2)_L \times SU(2)_R \) and a bifundamental scalar \( X \), responsible for the chiral symmetry breaking (it is actually crucial for construction of dyonic instanton [12]). The action of the model is simple

\[
S = \int d^5 x \; dt \; dz \left\{ \frac{1}{2} \left( 1 - \frac{1}{4 g_5^2} \right) (F_L^2 + F_R^2) + \frac{\Lambda^2}{z^3} (DX)^2 + \frac{\Lambda^2}{z^5} 3|X|^2 \right\},
\] (15)

where \( D^L = \partial^L - i L^L + i \cdot R \) is an appropriate covariant derivative, \( g_5 \) and \( \Lambda \) are the 5D coupling constant and normalization constant of scalar field, which are fixed by the
comparison of 2-point correlation functions of vector and scalar currents, computed in the model and in QCD sum rules [14, 19, 20]. The model is embedded in the AdS space with a hard wall, placed at certain radial coordinate:

$$ds^2 = \frac{1}{z^2}(-dz^2 - dx_i^2 + dt^2), \quad z < z_m$$  \hspace{1cm} (16)

As was shown in [2, 6], generally the Skyrmion in holographic model is represented by the topologically charged 4D field configuration in $t = \text{const}$ slice of $AdS_5$, and the baryon number identified with the topological charge of this “$AdS_4$ instanton”. For the particular model with two gauge fields the baryon number was defined in [7–9].

$$Q_B = \frac{1}{32\pi^2} \int d^3x \int_0^{z_m} dz \langle F_L \tilde{F}_L - F_R \tilde{F}_R \rangle,$$  \hspace{1cm} (17)

where $\tilde{F}$ is a dual field strength tensor and angle brackets denote the trace over flavor (gauge group) indices.

In what follows we will describe a classical solution in the model (15) with finite action and nonzero charge (17), incorporating nontrivial dynamics of the scalar field $X$, which can be named “holographic dyonic instanton”. The similar problem was considered in [21] and we will point out the difference later. First of all we will take the “cylindrical” ansatz, proposed in [15] and used in holographic context in [7–9, 21]:

$$A_j^a = 1 + \frac{\varphi_2(r, z)}{r} \epsilon_{jaking} \frac{x_k}{r} + \frac{\varphi_1(r, z)}{r} \left( \delta_{ja} - \frac{x_j x_a}{r^2} \right) + A_r(r, z) \frac{x_j x_a}{r^2}$$  \hspace{1cm} (18)

$$A_z^a = A_2(r, z) \frac{x_a}{r}$$  \hspace{1cm} (19)

for $A^L$ and $A^R$, and

$$X = \chi_0(r, z) \frac{1}{2} + i\chi_1(r, z) \frac{\tau^a x^a}{r}$$  \hspace{1cm} (20)

for bifundamental scalar field. Here $r^2 = x_1^2 + x_2^2 + x_3^2$ is a spacial radial coordinate, $\tau^a$ are the $SU(2)$ group generators, $a, b, \cdots = 1 \ldots 3$ are the group indices and $i, j, \cdots = 1 \ldots 3$ – the spacial ones. Moreover, the P-parity conditions (see [9])

$$A_t^a(x, z) = -A_t^a(-x, z), \quad L_z(x, z) = R_z(-x, z)$$  \hspace{1cm} (21)
force additional constraints on the new fields

\[
\varphi_1^L = -\varphi_1^R, \quad \varphi_2^L = \varphi_2^R, \quad A_r^L = -A_r^R, \quad A_z^L = -A_z^R, \quad (22)
\]

and we will use only left fields below, omitting the index \(L\). This ansatz reduces our model to the 2-dimensional Abelian gauge model with two charged scalars

\[
\varphi = \varphi_1 + i\varphi_2 \equiv \varphi e^{i\alpha}, \quad (23)
\]

\[
\chi = \chi_0 + i\chi_1 \equiv \chi e^{i\beta},
\]

with the action

\[
S = 4\pi \int dt \int dr dz \left\{ -\frac{1}{2g_5^2} \left[ \frac{2}{z} |D_a \varphi|^2 + \frac{1}{2} (F_{ab})^2 + \frac{1}{r^4} (1 - |\varphi|^2)^2 \right] - \frac{\Lambda r^2}{z^5} \left[ \frac{1}{2} |D_a \chi|^2 + \frac{1}{2r^2} (|\chi|^2 + |\varphi|^2 - |\varphi + \chi|^2) \right] + \frac{\Lambda^2 r^2}{z^5} \frac{3}{2} |\chi|^2 \right\}, \quad (24)
\]

where \(a, b = (z, r)\), \(D_a = \partial_a + iA_a\) and \(F_{ab} = \partial_a A_b - \partial_b A_a\). Apart of the holographic mass term for the \(X\) field, there are two interesting potential terms in the action. The third term in the first line is the usual potential of the Abelian Higgs model. It defines, that in the vacuum states

\[
\varphi \bigg|_{\text{vac}} = 1. \quad (25)
\]

Note however, that one does not need to impose this condition at spatial infinity \((r \to \infty)\) in order to get the finite action because of the factor \(1/r^2\) in front of the corresponding potential term.

The solution interpolating between different vacuum states is an instanton in the Abelian Higgs model. Interestingly (see \([9, 15]\)), the baryon number \((17)\) expressed in terms of 2D fields \((18)\)

\[
Q_B = \frac{1}{2\pi} \int dr \int dz \epsilon_{ab} \left( \partial_a (-i \varphi D_b \varphi) - \text{h.c.} \right) + F_{ab} \quad (26)
\]
coincides with the topological charge of 2D instanton. Rewriting \( \partial_a (\partial^a \phi^b \phi^b - h.c.) = 2 \partial_a (\phi^2 \partial^a \alpha) \) we see, that the nonzero charge is related to the nontrivial behavior of the phase \( \alpha \).

The second potential term in the action \([24]\) is the last one in the second line. It is convenient to rewrite it in terms of moduli and phases of the scalar fields:

\[
\frac{\Lambda^2}{2z^3}(|\chi|^2 + |\phi|^2 - |\phi + \chi|^2) = \frac{\Lambda^2}{x^3} \chi^2 \phi^2 \cos(\alpha - \beta)^2
\]

We see, that the vacuum state is now defined by the relative phase of scalar fields.

In what follows we will use the special notion for it

\[
\gamma = \alpha - \beta - \frac{\pi}{2}.
\]

In the vacuum state one has

\[
\gamma\big|_{\text{vac}} = \pi N, \quad N \in \mathbb{Z}.
\]

This is in fact the manifestation of the additional topological quantum number \( N \), which counts these discrete vacua and appears in the theory, when the scalar field is included. The aim of this work is to describe the classical solution possessing an additional charge \( Q_5 \neq 0 \), related to this quantum number, which is presumably connected to the axial charge of the baryon. Note also, that the potential term under consideration is not suppressed at infinity, so in order to have finite action we require

\[
\chi^2 \phi^2 \sin(\gamma)^2\big|_{r \to \infty} = 0.
\]

Following the usual ideology of constructing the instanton solution, we would like to have a field configuration, which interpolates between different vacua at different boundaries of the space. Our next step is to study by looking at the asymptotics of equations of motion whether the nontrivial behavior of the phases along the spacial
boundary is possible. The equations of motion following from the action \([24]\) are

\[
\varphi : \quad \partial_z \frac{1}{z} \partial_z \varphi + \frac{1}{z} \partial_r^2 \varphi = \frac{1}{z} \varphi \left[ (\partial_z \gamma + \partial_z \beta)^2 + (\partial_r \gamma + \partial_r \beta + A_r)^2 \right] \\
\quad - \frac{1}{r^2 z} \varphi (1 - \varphi^2) + 12 \frac{1}{z^3} \chi^2 \varphi \sin(\gamma)^2,
\]

\[
\chi : \quad \partial_z \frac{r^2}{z^3} \partial_z \chi + \partial_r \frac{r^2}{z^3} \partial_r \chi = \frac{r^2}{z^3} \chi \left[ (\partial_z \beta)^2 + (\partial_r \beta + A_r)^2 \right] \\
\quad + 8 \frac{1}{z^3} \chi \varphi^2 \sin(\gamma)^2 - 3 \frac{r^2}{z^5} \chi,
\]

\[
\beta : \quad \partial_z \frac{2}{z} \varphi^2 (\partial_z \gamma + \partial_z \beta) + \partial_r \frac{2}{z} \varphi^2 (\partial_r \gamma + \partial_r \beta + A_r) \\
\quad + \partial_z \frac{3r^2}{z^3} \chi^2 \partial_z \beta + \partial_r \frac{3r^2}{z^3} \chi^2 (\partial_r \beta + A_r) = 0,
\]

\[
\gamma : \quad \partial_z \frac{1}{z} \varphi^2 (\partial_z \gamma + \partial_z \beta) + \partial_r \frac{1}{z} \varphi^2 (\partial_r \gamma + \partial_r \beta + A_r) = -\frac{6}{z^3} \chi^2 \varphi^2 \sin(2\gamma),
\]

\[
A_r : \quad \partial_z \frac{1}{z} \partial_r A_r = \frac{4}{z} \varphi^2 (\partial_r \gamma + \partial_r \beta + A_r) + \frac{6r^2}{z^3} \chi^2 (\partial_r \beta + A_r),
\]

where we used phases \(\beta\) and \(\gamma\) as independent fields and fixed the gauge \(A_z = 0\).

Let us start from the boundary \(\{r = 0, z \in (0, z_m)\}\). The requirement of the finiteness of the action \([24]\) at this boundary leads to the condition

\[
\varphi(r, z) \bigg|_{r=0} = 1.
\]

As the equations of motion are singular, we need to impose more requirements on boundary conditions in order to avoid singularities in the solutions. The straightforward way to figure out these requirements is to expand the expressions for the differential operators of the second order, appearing in the equations of motion, near the corresponding boundary. In other words, we require the second derivative, transverse to the boundary, to be finite on this boundary. At \(r = 0\) this gives:

\[
\gamma(0, z) = 0, \quad \partial_r \phi(0, z) = 0, \quad \partial_r \chi(0, z) = 0
\]

We see, that although \(\gamma\) is fixed along the boundary, \(\beta\) is unconstrained, so two phases \(\alpha\) and \(\beta\) can change simultaneously, keeping their difference constant (see fig\([2\text{d}])\).

The change of \(\alpha\) will contribute to the topological charge \([26]\). One can check, that in order to get the smooth solution for the gauge field in ansatz \([18]\) we need to require
\[ \alpha = -\frac{\pi}{2}, \text{ corresponding to } \varphi_2 = -1, \text{ but as was pointed out in } [15], \text{ provided } \varphi = 1 \text{ a singular gauge transformation can be found, which drives a singular solution to the smooth one.} \]

The \( \{r \in (0, \infty), z = 0\} \) boundary can also be analyzed. In order to keep the action finite we have to demand

\[ \varphi(r, z) \bigg|_{z=0} = 1, \quad \chi(r, z) \bigg|_{z=0} \sim \sigma z^3, \quad A_r(r, z) \bigg|_{z=0} \sim z^2 \]  

Looking to the asymptotic behavior of the equations of motion, we see the following constraints

\[ \partial_r \alpha(r, \epsilon) \equiv \partial_r \gamma(r, \epsilon) + \partial_r \beta(r, \epsilon) = -A_r(r, \epsilon) \sim \epsilon^2 \bigg|_{\epsilon=0}, \]  
\[ \partial_z \alpha(r, \epsilon) \equiv \partial_z \gamma(r, \epsilon) + \partial_z \beta(r, \epsilon) = 0. \]  

These conditions prohibit the shift of \( \alpha \), but again leave two other phases free. We will consider a flow of \( \gamma \) along the \( z = 0 \) boundary from 0 to \( -\pi \) to get the solution with nontrivial second topological number \[ [29] \]. Accordingly, the phase \( \beta \) will flow from 0 to \( \pi \) (see fig.[2(a)]). Note, that this behavior of \( X \) is just the same as expected for usual Skyrmion solution in the boundary theory, where the simplest pion field configuration is:

\[ U(r) = 1 \sin(\theta(r)) + \pi^a \tau^a \cos(\theta(r)), \]  
where \( \theta(0) = 0 \) and \( \theta(\infty) = \pi \). We will discuss the connection between the running of \( \beta \) and the axial charge below.

The hard wall boundary \( \{r \in (0, \infty), z = z_m\} \) in our system is a bit special, because the boundary conditions on it are not dictated by the singular properties of action or equations of motion, but are imposed by hand according to the basic features of the hard-wall model (see \[ [9, 14] \]). In order to be in agreement with \[ [14] \], while keeping in mind, that the “hard-wall” is not the only possible AdS/QCD model and one can consider the models with infinite IR radius \( z_m \to \infty \) \[ [22] \], we impose the conditions at \( z = z_m \), which forbid any dynamics and demand the action to vanish.
We argue, that these boundary conditions will lead to the finite action in any AdS/QCD model.

\[ \partial_r \alpha(r, z_m) = \partial_r \beta(r, z_m) = \partial_r \gamma(r, z_m) = 0, \quad (41) \]
\[ \varphi(r, z_m) = 1. \]

The only boundary left unexplored is the radial infinity \( \{ r \to \infty, z \in (0, z_m) \} \). The finiteness of the action requires

\[ \varphi^2 \chi^2 \sin(\gamma)^2 \bigg|_{r \to \infty} = 0, \quad (42) \]

but note, that we are not forced to impose \( \varphi = 1 \) condition here. This fact opens an interesting possibility to fulfill (42) keeping the nontrivial dynamics of \( \gamma \).

FIG. 1: Boundary value of the field \( \varphi \) at \( r \to \infty \). The field vanishes at \( z = z_0 \).

On the boundary \( \varphi(r, z) \) must obey the equation of motion

\[ \partial_z^2 \varphi(r, z) - \frac{1}{z} \varphi(r, z) = 0, \quad r \to \infty \quad (43) \]

and we can consider the boundary value (\( \theta \) is a Heaviside step function)

\[ \varphi(r, z) \bigg|_{r \to \infty} = \theta(z_0 - z) \left( 1 - \frac{z^2}{z_0^2} \right) + \theta(z - z_0) \left( \frac{z^2}{z_0^2} - \frac{z_m^2}{z_0^2} \right) \quad (44) \]

which has proper values \( \varphi = 1 \) at the boundaries \( z = 0 \) and \( z = z_m \), but falls to zero at \( z = z_0 \) (see fig.2). At this point the boundary value of \( \gamma(r, z) \) can perform a jump.
from $-\pi$ to 0 (see fig. 2(b))

$$\gamma(r,z)\bigg|_{r=\infty} = -\pi + \theta(z - z_0)\pi,$$

thus closing the contour of boundary values. While this arrangement seems to be singular, it doesn’t lead to the divergence of the action. Indeed, in the kinetic term the derivative of $\alpha$, which diverges at $z = z_0$ in our solution, is multiplied by the modulus $\varphi$, which is zero exactly in this point. Moreover, as the behavior of $\varphi$ is not smooth at $z = z_0$, its second derivative diverges, but in the action only first derivative of $\varphi$ enters the kinetic term and this derivative is finite everywhere in the vicinity of $z_0$. We should point out also, that as the action is a two dimensional integral over $z$ and $r$, even a simple pole in the action density at point $\{z = z_0, r \to \infty\}$ would not produce the divergence of the whole action.

It should be underlined, that our solution differs substantially from a constant one, which was considered in [21] within the similar framework.

FIG. 2: The qualitative boundary behavior of the phases $\alpha$ (dotted), $\beta$ (dashed) and $\gamma$ (solid). (Horizontal axis of (c) and (d) reversed.)

We note, that the solutions for $\alpha(z)$ and $\varphi(z)$ respect the singular equations of motion at the boundary. The spectacular feature of these boundary values is that they represent a domain wall, located at $z = z_0$. The position of the wall governs mostly the effective mass of our instanton solution (see [6]) and we note, that the
instanton can not “fall” on the hard wall, because of the boundary value of the field \( \varphi(r,z) \) at \( z = z_m \). That means that the potential (27) at spatial infinity \( (r \to \infty) \) stabilizes the mass and the size of the baryon. We stress, that one do not need to consider any additional mechanisms of the stabilization of instanton size \([6, 8, 9]\), as it is fixed by the presence of the second topological charge of the solution itself. This effect is somewhat inherited from the feature of the original dyonic instanton \([12]\) whose size is stabilized by its quantum numbers \([2]\).

At the end of the day we obtained an interesting picture of asymptotics of our “dyonic instanton” \((37, 39, 41, 44, 45)\). The phase \( \gamma \) is 0 at the \( r = 0 \) boundary, then it flows to \(-\pi\) along \( z = 0 \) and jumps back to zero on \( r \to \infty \), thus interpolating between different vacua \([29]\), realized at \( r = 0 \) and point \( r \to \infty, z = 0 \). The phase \( \alpha \) flows from \( \frac{3\pi}{2} \) to \( \frac{\pi}{2} \) along \( r = 0 \), than stays constant at \( z = 0 \) and jumps with \( \gamma \) back to \( \frac{3\pi}{2} \) on \( r \to \infty \). It realizes the nontrivial topological configuration with nonzero charge \([26]\), showing that our solution is indeed a baryon.

A few words are in order concerning the phase \( \beta \). In the holographic action \([15]\) the axial field couples to the field \( X \) via the kinetic term (see also \([20]\)). Consequently the \( X \) field can be a source to the axial current (dual to the axial field), namely

\[
J_A^\mu \sim i(\partial_\mu XX^\dagger - X\partial_\mu X^\dagger).
\]

In the 2D fields notation, this expression for non-singlet current looks as

\[
J_r^A \sim 2\chi^2 \partial_r \beta(r,z) \tau.
\]

That means, that the phase \( \beta \) governs the coupling of our solution with external axial current, and thus the axial charge of the baryon \([3]\).

IV. ON THE BRANE PICTURE

In this Section we shall make few comments concerning the brane interpretation of the solution under consideration. The dyonic instantons can be considered as the
instantons (D0 branes) with attached fundamental strings [12, 16, 17]. More precisely the D0 branes are localized at the D4 branes and fundamental string connects two parallel D4 branes with the geometry \( R^4 \times I \) where \( D = 5 \) gauge theory lives on. The dyonic instanton size is fixed by the asymptotic distance between D4 branes.

Let us mention the similarities between the "color" dyonic instanton [12, 17] and the "flavor" solution we have considered. It is useful to have in mind the Sakai-Sugimoto type geometry [23], where the 5D gauge theory is defined at the worldvolume of \( N_f \) D8-\( \bar{D}8 \) branes wrapped around the internal \( S^4 \) and extended along the radial coordinate on the cigar. The branes and antibranes are localized at the different points in the periodic coordinate on the cigar. Baryons are identified with the D4 branes wrapped around \( S^4 \) [2, 6].

To keep close to the standard dyonic instanton [16] we assume that the brane configuration involves also the fundamental string attached to the D4 instanton and connecting left and right stacks of \( D8 \) branes in the Sakai-Sugimoto framework [24, 25]. In our case similarly the VEV of the bifundamental scalar identified with the chiral condensate is related to the distance between the stacks like in [24].

More general dyonic instanton solution corresponds to the (D0+ F1) state blown into the tubular D2 branes with the electric field and the instanton charge [17]. That is in the \( D = 4 \) space-time the point-like instanton becomes the circular monopole loop which carries the topological charge due to D0 density and electric field due to the fundamental string. Our spherically symmetric Skyrmion discussed above is interpreted as a particle representing the baryon. However in more general solution we could expect that D4 brane gets blown into the D6 brane [16, 17]. If such D6 brane is extended in one space coordinate the solution looks like a closed string similar to the monopole loop in the original Lambert-Tong case [17]. The total magnetic charge of the closed string vanishes but its dipole magnetic moment survives. It would be very interesting to compare the natural loop structure of such more general dyonic instanton solution with the loop structure observed numerically for the Skyrmions with baryonic charge \( B \geq 2 \) [26] long time ago.
We have mentioned above that in some range of parameters there is negative mode around the dyonic instanton configuration. The mode analysis in $AdS_4$ case has to be performed and we restrict ourselves by one remark. It is known that monopole induces the baryon decay via Rubakov-Callan effect \[27\]. It has been realized holographically in \[28\] where the process has been described as the blow up of D4 brane into D6 brane somewhat similar to our analysis. We hope to discuss these issues elsewhere.

V. CONCLUSION

In this paper we have discussed the realization of the dyonic instanton solution within the framework of holographic QCD as baryon. We have showed, that considering the holographic model with bifundamental scalar field, dual to the chiral condensate of quarks, one can figure out two different topological numbers of the solution, corresponding to the baryonic and the axial charges of QCD baryon.

The solution, discussed in this work has a peculiar form of domain wall, whose thickness, while being zero at radial infinity, rises closer to the core, covering the whole distance between holographic boundaries in the center of instanton. First of all this shape reminds the connection between Skyrmions and instantons inside a domain wall, pointed out by Atiyah and Manton and developed by Tong at al. \[4, 5\]. Second, the domain wall is forced to be located between two holographic boundaries, resolving the known problem of holographic instantons “falling” on a hard wall. Indeed, we do no need to consider special dynamics on the IR brane \[8\] or the back reaction of the vector Abelian field, sourced by $Q_B$ on the solution \[9\], which turns out to be suppressed in certain models by the string scale (see discussion in \[9\] and \[6\]). Instead, our solution by the construction has finite size, which is determined solely by the dynamics. The position of the domain wall, related to the mass scale of the baryon is governed by the value of the chiral condensate $\sigma$, being the only tunable parameter of the model. That means, that our approach of describing baryons in
holographic models provides a solid ground for the Ioffe relation (1), which origin was otherwise rather mysterious.

While the qualitative study of some features of the holographic dyonic instanton can be based on the consideration of the asymptotic behavior of solution, obtaining of the full solution to the equations of motion with above mentioned asymptotics is highly desirable and we leave it for future work. The other interesting developments in this direction could be the consideration of the time components of gauge fields, giving rise to 4D electrical field strength and study of possible solutions, absent in the cylindrical anztatz, used here. It would be very interesting to clarify the issue of an angular momentum and its role in the stabilization of the configuration. The possible magnetic dipole nature of the solution deserves the additional investigation together with the possible negative modes at large quantum numbers.

The separate issue concerns the temperature dependence of the solution and its behavior under the chiral phase transition. If there would be nontrivial ”holonomy” along the radial $AdS_4$ coordinate one could expect caloron-like solution involving ”dressed monopoles” in the flavor group with the fractional baryon numbers. Such states are familiar in the theories with the compact dimensions. Certainly all these issues deserve further investigation.

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