WEIGHTED MULTILINEAR SQUARE FUNCTIONS BOUNDS

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ABSTRACT. In this work we study boundedness of Littlewood-Paley-Stein square functions associated to multilinear operators. We prove weighted Lebesgue space bounds for square functions under relaxed regularity and cancellation conditions that are independent of weights, which is a new result even in the linear case. For a class of multilinear convolution operators, we prove necessary and sufficient conditions for weighted Lebesgue space bounds. Using extrapolation theory, we extend weighted bounds in the multilinear setting for Lebesgue spaces with index smaller than one.

1. INTRODUCTION

Given a function $\psi : \mathbb{R}^n \to \mathbb{C}$, define $\psi_t(x) = t^{-n} \psi(t^{-1}x)$ and the associated Littlewood-Paley-Stein type square function

$$g_\psi(f) = \left( \int_0^\infty |\psi_t * f|^2 \frac{dt}{t} \right)^{1/2}. \tag{1.1}$$

These convolution type square functions were introduced by Stein in the 1960’s, see e.g. [40] or [41], and have been studied extensively since then, including classical works by Stein [40], Kurtz [32], Duoandikoetxea-Rubio de Francia [16], and more recently Duoandikoetxea-Seijo [17], Cheng [5], Sato [37], Duoandikoetxea [14], Wilson [42], Lerner [33], and Cruz-Uribe-Martell-Perez [11]. Of particular interest of these, [32], [17], [37], [42], [11], and [33] prove bounds for $g_\psi$ on weighted Lebesgue spaces under various conditions on $\psi$. Non-convolution variants of (1.1) were studied by Carleson [4], David-Journé-Semmes [13], Christ-Journé [7], Semmes [38], Hofmann [28, 29], and Auscher [2] where they replaced the convolution $\psi_t * f(x)$ with

$$\Theta_t f(x) = \int_{\mathbb{R}^n} \Theta_t(x,y)f(y)dy. \tag{1.2}$$

In [13] and [38], the authors proved $L^p$ bounds for square Littlewood-Paley-Stein square functions associated to $\Theta_t$ when $\Theta_t(b) = 0$ for some para-accretive function $b$. In [28, 29], this type of mean zero assumption is replaced by a local cancellation testing condition on dyadic cubes. In [4], [7], and [2], the authors replace mean zero assumption with a Carleson measure condition for $\theta_t$ to prove $L^2$ bounds for the square function. The work of Carleson in [4] was phrased as a characterization of $BMO$ in terms of Carleson measures, but non-convolution type square function bounds are implicit in his work.
In all of the works studying $g_w$ cited above, the authors assume that $\psi$ has mean zero. In fact, if $g_w$ is bounded on $L^2$, then $\psi$ must have mean zero, but in the non-convolution setting, the mean zero condition is no longer a strictly necessary one, as demonstrated in [41], [28], [29], and [2]. This phenomena persists in the multilinear square function setting, and in this work we explore subtle cancellation conditions for multilinear convolution and non-convolution type square function and their interaction with weighted Lebesgue space estimates.

The non-convolution form of the kernel $\theta_t(x,y)$ allows for a natural extension to the multilinear setting. Define for appropriate $\Theta_t : \mathbb{R}^{(m+1)n} \rightarrow \mathbb{C}$

\[
S(f_1, \ldots, f_m)(x) = \left( \int_0^\infty |\Theta_t(f_1, \ldots, f_m)(x)|^2 \frac{dt}{t} \right)^{\frac{1}{2}},
\]

where we use the notation $d\vec{y} = dy_1 \cdots dy_m$. When $m = 1$, i.e. in the linear setting, this is the operator $\Theta_t$ mentioned above, so we use the same notation for it. We wish to find cancellation conditions on $\Theta_t$ that imply boundedness $S$, given that $\Theta_t$ also satisfies some size and regularity estimates. In particular, we assume that $\Theta_t$ satisfies

\[
|\Theta_t(x, y_1, \ldots, y_m)| \lesssim \prod_{i=1}^m \frac{t^{-n}}{(1 + t^{-1}|x - y_i|)^N}
\]

and

\[
|\Theta_t(x, y_1, \ldots, y_m) - \Theta_t(x, y_1', \ldots, y_m')| \lesssim t^{-mn} (1 - |y_i - y_i'|)^\gamma
\]

for all $x, y_1, \ldots, y_m, y_1', \ldots, y_m' \in \mathbb{R}^n$ and $i = 1, \ldots, m$ and some $N > n$ and $0 < \gamma \leq 1$. Note that we do not require any regularity for $\Theta_t(x, y_1, \ldots, y_m)$ in the $x$ variable. Square functions associated to this type of operators have been studied in a number of recent works. In Maldonado [34] and Maldonado-Naibo [35], the authors introduce the operators (1.3), and making the natural extension of Semmes’s point of view in [33] to prove bounds for a Besov type relative of the square function $S$ (1.2).

\[
(f_1, \ldots, f_m) \mapsto \left( \int_0^\infty \||\Theta_t(f_1, \ldots, f_m)|^2 \frac{dt}{t} \right)^{\frac{1}{2}}.
\]

When $p = 2$ this Besov type square function agrees with the square function (1.2). In [26], [22], and [20], Hart, Grafakos-Oliveira, and Grafakos-Lui-Maldonado-Yang proved boundedness results for different versions of the square function $S$ in Lebesgue spaces under various cancellation and regularity conditions on $\Theta_t$. That is, in each of these works the authors proved bounds of the form $||S(f_1, \ldots, f_m)||_{L^p} \lesssim ||f_1||_{L^{p_1}} \cdots ||f_m||_{L^{p_m}}$, for minor modifications of $S$ in various ranges of indices $p, p_1, \ldots, p_m$. The first goal of this work includes proving a weighted version of these results,

\[
||S(f_1, \ldots, f_m)||_{L^p(w^p)} \lesssim \prod_{i=1}^m ||f_i||_{L^{p_i}(w^{p_i})}
\]

for appropriate $1 < p_1, \ldots, p_m < \infty$, $w_i^{p_i} \in A_{p_i}$ and $w = w_1 \cdots w_m$. More generally, the main result of this work is the following theorem.

**Theorem 1.1.** Assume $\Theta_t$ satisfies (1.4) and (1.5). Then the following cancellation conditions are equivalent

i. $\Theta_t$ satisfies the strong Carleson condition,

ii. $\Theta_t$ satisfies the Carleson and two cube testing conditions.
Furthermore, if the equivalent conditions (i) and (ii) hold, then \( S \) satisfies (1.6) for all \( w_i^{p_i} \in A_{p_i} \) where \( w = w_1 \cdots w_m, 1 < p_1, \ldots, p_m < \infty \) satisfying \( \frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_m} \), and \( f_i \in L^{p_i}(w_i^{p_i}) \).

For the definitions of the Carleson, strong Carleson, and two-cube testing conditions, see Section 3. For now we only note that conditions quantify some cancellation of indeces estimates and Carleson type bounds adapted to a weighted setting, and extend bounds to provide a classical Calderón-Zygmund type analogue for square functions: If following are equivalent bounds of multiconvolution operators. We prove necessary and sufficient cancellation conditions for Theorem 1.2.

We organize the article in the following way: In Section 2, we prove the some convergence results and boundedness results for \( S \) when \( \Theta_t(1, \ldots, 1) = 0 \). In Section 3, prove various properties relating the Carleson, strong Carleson, and two cube testing conditions to each other and some bounds for \( S \). Finally in section 4, we prove Theorems 1.1 and 1.2.

2. A REDUCED T1 THEOREM FOR SQUARE FUNCTIONS ON WEIGHTED SPACES

It is well-known that (1.4) implies that \( |\Theta_t(f_1, \ldots, f_m)(x)| \lesssim Mf_1(x) \cdots Mf_m(x) \), where \( M \) is the Hardy-Littlewood maximal function, and hence

\[
\sup_{t > 0} ||\Theta_t(f_1, \ldots, f_m)||_{L^p} \lesssim \prod_{i=1}^m ||f_i||_{L^{p_i}}
\]
when \(1 < p_1, \ldots, p_m < \infty\) satisfy the H"older type relationship

\[
\frac{1}{p} = \sum_{i=1}^{m} \frac{1}{p_i}.
\]

So it is natural to expect that \(p_1, \ldots, p_m\) satisfy this relationship for square function bounds of the form \((\mathbf{1.6})\). For the remainder of this work, we will assume that \(1 < p_1, \ldots, p_m < \infty\) and \(p\) is defined by \((\mathbf{2.1})\).

When we are in the linear setting, with a convolution operator \(\Theta_t(x, y) = \psi_t(x-y) = t^{-n} \psi(t^{-1}(x-y))\), we use the notation \((\mathbf{1.1})\) to avoid confusion with the square function \(S\), and to emphasize that we are using the known Littlewood-Paley theory.

**Definition 2.1.** Let \(w\) be a non-negative locally integrable function. For \(p > 1\) we say that \(w\) is an \(A_p = A_p(\mathbb{R}^n)\) weight, written \(w \in A_p\), if

\[
[w]_{A_p} = \sup_Q \left( \frac{1}{|Q|} \int_Q w(x)dx \right) \left( \frac{1}{|Q|} \int_Q w(x)^{1-p'}dx \right)^{p-1} < \infty
\]

where the supremum is taken over all cubes \(Q \subset \mathbb{R}^n\) with side parallel to the coordinate axes.

The following lemma states that approximation to the identity operators have essentially the same convergence properties in weighted \(L^p\) spaces as unweighted. This result is well-known (an explicit proof is available for example in the work of Wilson \((\mathbf{42})\)), but for the reader’s convenience we state the results precisely and give a short proof.

**Lemma 2.2.** Let \(P_tf = \varphi_t * f\) where \(|\varphi(x)| \lesssim \frac{1}{(1+|x|)^N}\) for some \(N > n\) with \(\widehat{\varphi}(0) = 1\) and \(w \in A_p\) for some \(1 < p < \infty\),

i. If \(f \in L^p(w)\), then \(P_tf \to f\) in \(L^p(w)\) as \(t \to 0\).

ii. If \(f \in L^p(w)\) and there exists a \(1 \leq q < \infty\) such that \(f \in L^q\), then \(P_tf \to 0\) in \(L^p(w)\) as \(t \to 0\).

**Proof.** We first prove (i) by estimating

\[
||P_tf - f||_{L^p(w)} \leq \int_{\mathbb{R}^n} |\varphi(y)| ||f(\cdot-ty) - f(\cdot)||_{L^p(w)}dy.
\]

The integrand \(|\varphi(y)||f(\cdot-ty) - f(\cdot)||_{L^p(w)}\) is controlled by \(2||f||_{L^p(w)}||\varphi(y)||\) which is an integrable function. So by dominated convergence

\[
\lim_{t \to 0} ||P_tf - f||_{L^p(w)} \leq \int_{\mathbb{R}^n} |\varphi(y)| \lim_{t \to 0} ||f(\cdot-ty) - f(\cdot)||_{L^p(w)}dy = 0.
\]

Therefore (i) holds. Now for (ii), suppose that \(f \in L^p(w) \cap L^q(\mathbb{R}^n)\) for some \(1 \leq q < \infty\). Then it follows that for all \(x \in \mathbb{R}^n\)

\[
|P_tf(x)| \leq ||\varphi_t||_{L^q} ||f||_{L^q}
\]

\[
\lesssim t^{-n/q} \left( \int_{\mathbb{R}^n} \frac{dx}{(1+|x|)^Nq} \right)^{1/q} ||f||_{L^q}
\]

which tends to 0 as \(t \to \infty\). So \(P_tf \to 0\) a.e. in \(\mathbb{R}^n\). Furthermore \(|P_tf(x)| \lesssim Mf(x)\) where \(Mf \in L^p(w)\) since \(f \in L^p(w)\) and \(1 < p < \infty\). Then by dominated convergence, we have

\[
\lim_{t \to \infty} \int_{\mathbb{R}^n} |P_tf(x)|^p w(x)dx = \int_{\mathbb{R}^n} \lim_{t \to \infty} |P_tf(x)|^p w(x)dx = 0.
\]

So we have \(P_tf \to 0\) in \(L^p(w)\) as \(t \to \infty\). \(\square\)
Lemma 2.3. Suppose \( \Theta \) satisfies (1.4), \( P \beta f = \varphi \ast f \) where \( \varphi \in C_0^\infty \) with \( \tilde{\varphi}(0) = 1 \), \( w_0 \in A_p \) for \( 1 < p, p_1, \ldots, p_m < \infty \) satisfying (2.1). Define \( w = w_1 \cdots w_m \). Then for \( f_j \in L^{p_j}(w_0^{p_j}) \cap L^{p_j} \)

\[
\Theta_f(f_1, \ldots, f_m) = \sum_{j=1}^m \int_0^\infty \Theta_j \Pi_{j,k}(f_1, \ldots, f_m) \frac{ds}{s}
\]

where the convergence holds in \( L^p(w) \) and \( \Pi_{j,k} \) is defined by

\[
\Pi_{j,k}(f_1, \ldots, f_m) = P_j f_1 \cdots P_j f_{j-1} \otimes Q_j f_j \otimes P_j f_{j+1} \cdots \otimes P_j f_m,
\]

\( Q_j f = \varphi \ast f \), and \( \psi = \frac{\partial}{\partial t}(\varphi \ast \varphi) \). Furthermore there exist \( Q^k \beta f = \psi^k \ast f \) where \( \psi^k \in C_0^\infty \) have mean zero for \( i, 1, 2 \) and \( k = 1, \ldots, n \) and

\[
Q_i = \sum_{k=1}^n Q^k_i Q_i^2^k.
\]

Proof. We note that since \( f_j \in L^{p_j}(w_0^{p_j}) \cap L^{p_j} \), by Lemma 2.2 \( P_j f_j \to f_j \) as \( t \to 0 \) and \( P_j^2 f_j \to 0 \) as \( t \to \infty \) in \( L^p(w_0^{p_j}) \). Then it follows that

\[
\left\| \Theta_f(f_1, \ldots, f_m) - \sum_{j=1}^m \int_0^{1/t} \Theta_j \Pi_{j,k}(f_1, \ldots, f_m) \frac{ds}{s} \right\|_{L^p(w)}
\]

\[
= \left\| \Theta_f(f_1, \ldots, f_m) - \sum_{j=1}^m \int_0^{1/t} \Theta_j (P_j^2 f_1, \ldots, P_j^2 f_m) \frac{ds}{s} \right\|_{L^p(w)}
\]

\[
\leq \left\| \Theta_f(f_1, \ldots, f_m) - \sum_{j=1}^m \left| \Theta_j (P_j^2 f_1, \ldots, P_j^2 f_m) \right|_{L^p(w)} + \left| \Theta_j (P_j^2 f_1, \ldots, P_j^2 f_m) \right|_{L^p(w)} \right\|_{L^p(w)}
\]

\[
\leq \sum_{j=1}^m \left| \left| M f_1 \cdots M f_{j-1}(f_j - P_k f_j) f_{j+1} \cdots f_m \right|_{L^p(w)} + \left| MP_j^2 f_1 \cdots MP_j^2 f_m \right|_{L^p(w)} \right|_{L^p(w)}
\]

As \( \varepsilon \to 0 \), the above expression tends to zero. Therefore we have (2.2) where the convergence is in \( L^p(w_0^{p_j}) \). One can verify that \( \psi^k \beta(x) = -2\partial_x \varphi(x) \) and \( \psi^2 \beta(x) = x_k \varphi(x) \) satisfy the conditions given above. For details, this decomposition of \( Q_j \) was done in the linear one dimensional case by Coifman-Meyer in [3] and in the \( n \) dimensional case by Grafakos in [19].

Lemma 2.4. Let \( P_i, Q_i, Q_i^{k,i}, \Pi_{j,k} \) be as in Lemma 2.2. Then for all \( f_j \in L^{p_j}(w_0^{p_j}) \cap L^{p_j} \), \( s > 0, j = 1, \ldots, m \) and \( x \in \mathbb{R}^n \)

\[
| \Theta_i \Pi_{j,k}(f_1, \ldots, f_m)(x) | \lesssim \frac{1}{s^{t/\gamma}} \sum_{i=1}^{n} M_Q^{2^k} f_j(x) \prod_{i \neq j} M f_i(x)
\]

for some \( 0 < \gamma' \leq \gamma \) where \( u \wedge v = \min(u, v) \) for \( u, v > 0 \).

This lemma is a pointwise result that was proved in the discrete bilinear setting in [26]. We make the appropriate modifications here to prove this multilinear continuous version.

Proof. For this proof, we define for \( M, t > 0 \) and \( x \in \mathbb{R}^n \)

\[
\Phi^M_t(x) = \frac{t^{-n}}{(1 + t^{-1} |x|)^M}.
\]
It follows immediately that $\Phi_M^{d+t} \leq \Phi_M^d$ for any $d \geq 0$, and there is a well known almost orthogonality result, for any $M, L > n$ and $s, t > 0$

\[(2.4) \quad \int_{\mathbb{R}^n} \Phi_M^d(x-u)\Phi_s^d(u-y)du \lesssim \Phi_M^{d+s}(x-y) + \Phi_M^{d+s}(x-y).
\]

Note also that if we take $\eta = \frac{N-n}{2(N+\gamma)}$, $\gamma' = \eta \gamma$, and $N' = (1-\eta)N - \gamma'$, then using a geometric mean with weights $1-\eta$ and $\eta$ of estimates (1.4) and (1.5) it follows that

\[
|\Theta_t(x,y_1,...,y_m) - \Theta_t(x,y'_1,...,y_m)| \lesssim t^{-\eta mn} (t^{-1}|y_1-y'_1|)^{\eta} \left( \prod_{i=2}^{m} \Phi_N^s(x-y_j) \right)^{1-\eta} \\
\quad \times \left( \Phi_N^s(x-y_1) + \Phi_N^s(x-y'_1) \right)^{1-\eta} \\
\quad = (t^{-1}|y_1-y'_1|)^{\gamma'} \left( \Phi_N^{N'+\gamma'}(x-y_1) + \Phi_N^{N'+\gamma'}(x-y'_1) \right) \prod_{j=2}^{m} \Phi_N^{N'+\gamma'}(x-y_j)
\]

It is a direct computation to show that $0 < \gamma' = \gamma \frac{N-n}{2(N+\gamma)} < \gamma$ and $n < N' = \frac{N+n}{2} \leq N - \gamma'$. We will first look at the kernel of $\Theta_t(Q_1^{s,k}, P_s \cdot, ..., P_s \cdot)$ for $k = 1,..., m$, which is

\[
\sum_{k=1}^{n} \int_{\mathbb{R}^{mn}} \Theta_t(x,u_1,...,u_m)\psi_s^{1,k}(u_1-y_1) \prod_{j=2}^{m} \phi_s(u_i-y_i)du.
\]

The goal here is to bound this kernel by a product of $\Phi_N^{N'}(x-y_j) + \Phi_N^{N'}(x-y_j)$. So in the following computations, whenever possible we pull out terms of the form $\Phi_N^{N'}(x-y_j)$. There will also appear terms of the form $\Phi_N^{N'}(x-u_j)$ and $\Phi_N^{N'}(u-y_j)$, for which we will use (2.4) and bound by appropriate functions $\Phi$ depending on $s$, $t$, and $x-y_j$. We estimate the kernel for a fixed $k = 1,..., m$ and simplify notation

\[
\lambda_s(y_1,...,y_m) = \psi_s^{1,k}(y_1) \prod_{j=2}^{m} \phi_s(y_i).
\]
Then for $s < t$, it follows using that $\lambda_s(y_1, \ldots, y_m)$ has mean zero in $y_1$ (since $\psi^{1,k}_s$ has mean zero), $\psi^{1,k}, \varphi \in C_0^\infty$, and $\theta_i$ satisfies (1.4) and (1.5) that

$$
\left| \int_{\mathbb{R}^m} \theta_i(x, u_1, \ldots, u_m) \lambda_s(u_1 - y_1, \ldots, u_m - y_m) d\vec{u} \right|
\lesssim \int_{\mathbb{R}^m} |\theta_i(x, u_1, \ldots, u_m) - \theta_i(x, y_1, u_2, \ldots, u_m)| \left( \prod_{j=1}^m \Phi^{N'+\gamma'}(u_j - y_j) \right) d\vec{u}
\lesssim \int_{\mathbb{R}^m} (t^{-1} |u_1 - y_1|)^\gamma \Phi^{N'+\gamma'}(x - y_1) \Phi^{N'+\gamma'}(u_1 - y_1)
\times \prod_{j=2}^m \left( \Phi^{N'+\gamma'}(x - u_j) \Phi^{N'+\gamma'}(u_j - y_j) \right) d\vec{u}
\leq \frac{s^{N'}\gamma'}{t^\gamma} \Phi^{N'+\gamma'}(x - y_1) \int_{\mathbb{R}^m} \Phi^{N'}(u_1 - y_1) \prod_{j=2}^m \left( \Phi^{N'+\gamma'}(x - u_j) \Phi^{N'+\gamma'}(u_j - y_j) \right) d\vec{u}
\leq \frac{s^{N'}\gamma'}{t^\gamma} \prod_{j=1}^m \left( \Phi^{N'}(x - y_j) + \Phi^{N'}(x - y_j) \right)
(2.5)
$$

Note that we use the computation $(t^{-1} |u_1 - y_1|)^\gamma \Phi^{N'+\gamma'}(u_1 - y_1) \leq \frac{s^{N'}\gamma'}{t^\gamma} \Phi^{N'}(u_1 - y_1)$. Now for $s > t$, we use the assumptions $\Theta_1(1, \ldots, 1) = 0$, $\Theta_i$ satisfies (1.4), and that $\psi^{1,k}, \varphi_s \in C_0^\infty$ for the following estimate

$$
\left| \int_{\mathbb{R}^m} \theta_i(x, u_1, \ldots, u_m) \lambda_s(u_1 - y_1, \ldots, u_m - y_m) d\vec{u} \right|
\lesssim \int_{\mathbb{R}^m} \prod_{j=1}^m \Phi^{N'+\gamma'}(x - u_j) |\lambda_s(u_1 - y_1, \ldots, u_m - y_m) - \lambda_s(x - y_1, \ldots, x - y_m)| d\vec{u}
(2.6)
$$

Next we work to control the second term in the integrand on the right hand side of (2.6). Adding and subtracting successive terms, we get

$$
|\lambda_s(u_1 - y_1, \ldots, u_m - y_m) - \lambda_s(x - y_1, \ldots, x - y_m)|
\leq \sum_{\ell=1}^m [\lambda_s(u_1 - y_1, \ldots, u_{\ell-1} - y_{\ell-1}, x - y_\ell, \ldots, x - y_m)
- \lambda_s(u_1 - y_1, \ldots, u_\ell - y_\ell, x - y_{\ell+1}, \ldots, x - y_m)]
\leq \sum_{\ell=1}^m (s^{-1} |x - u_\ell|)^\gamma \left( \prod_{r=1}^{\ell-1} \Phi^{N'+\gamma'}(u_r - y_r) \right) \Phi^{N'+\gamma'}(u_\ell - y_\ell) + \Phi^{N'+\gamma'}(x - y_\ell)
\times \prod_{r=\ell+1}^m \Phi^{N'+\gamma'}(x - y_r)
$$
Here we use the convection that $\prod_{j=1}^{0} A_j = \prod_{j=m+1}^{m} A_j = 1$ to simplify notation. Then (2.6) is bounded by

\[
\sum_{\ell=1}^{m} \int_{\mathbb{R}^m} \left( \prod_{j=1}^{m} \Phi^{N+\ell}_{y_j} (x - u_j) \right) (x^{-1}\cdot|x - u|)^{\gamma} \left( \prod_{r=1}^{\ell-1} \Phi^{N+\ell}_{y_r} (u_r - y_r) \right) \times \left( \Phi^{N+\ell}_{y}(u_r - y_r) + \Phi^{N+\ell}_{y}(x - y_r) \right) \left( \prod_{r=\ell+1}^{m} \Phi^{N+\ell}_{y}(x - y_r) \right) d\bar{u} 
\]

\[
\leq \frac{\ell!}{s!} \sum_{\ell=1}^{m} \int_{\mathbb{R}^m} \left( \prod_{j=1}^{m} \Phi^{N}_{y_j} (x - u_j) \right) (x^{-1}\cdot|x - u|)^{\gamma} \left( \prod_{r=1}^{\ell-1} \Phi^{N}_{y_r} (u_r - y_r) \right) \times \left( \Phi^{N}_{y}(u_r - y_r) + \Phi^{N}_{y}(x - y_r) \right) \left( \prod_{r=\ell+1}^{m} \Phi^{N}_{y}(x - y_r) \right) d\bar{u} 
\]

\[
\leq \frac{\ell!}{s!} \prod_{\ell=1}^{m} \left( \Phi^{N}_{y}(x - y_r) + \Phi^{N}_{y}(x - y_r) \right) .
\]

Then using (2.5) and (2.7), it follows that

\[
\left| \int_{\mathbb{R}^m} \Theta_t(x, u_1, \ldots, u_m) \psi^{1,k}_{y} (u_1 - y_1) \prod_{j=2}^{m} \Phi_j (u_j - y_j) d\bar{u} \right| 
\leq \left( \frac{\ell!}{s!}\right)^{\gamma} \prod_{j=1}^{m} \left( \Phi^{N}_{y}(x - y_j) + \Phi^{N}_{y}(x - y_j) \right) .
\]

Then since $|\Phi^{N}_{y} f(x)| \lesssim Mf(x)$ uniformly in $t$ and $\Theta_t \Pi_{s,1} = \sum_{k=1}^{n} \Theta(Q^{2,k}_c, P^{2,k}_c, P^{2,k}_s)$, it follows that

\[
|\Theta_t \Pi_{s,1}(f_1, \ldots, f_m)(x)| \lesssim \left( \frac{\ell!}{s!}\right)^{\gamma} \sum_{k=1}^{n} M^{2,k} f_1(x) \prod_{j=2}^{m} Mf_j(x).
\]

By symmetry, this completes the proof. 

Next we work to set the square function results of [26], [22] and [20] in weighted Lebesgue spaces. This is a type of reduced T(1) Theorem for $L^2(\mathbb{R}, \frac{dt}{t})$-valued singular integral operators, where we assume that $\Theta_t (1, \ldots, 1) = 0$ for all $t > 0$. We now state and prove a reduced T(1) Theorem for square functions on weighted spaces.

**Theorem 2.5.** Let $\Theta_t$ and $S$ be defined as in (1.3) and (1.2) where $\Theta_t$ satisfies (1.4) and (1.5). If $\Theta_t (1, \ldots, 1) = 0$ for all $t > 0$, then $S$ satisfies (1.6) for all $w^p_t \in A_p$, $1 < p < \infty$. 

$p, p_1, \ldots, p_m < \infty$ satisfying (2.1), where $w = \prod_{i=1}^m w_i$, and $f_i \in L^{p_i}(w_i^{p_i}) \cap L^{p_i}$. Furthermore, the constant for this bound is at most a constant independent of $w_1, \ldots, w_m$ times

$$\prod_{j=1}^m \left(1 + [w_j^{p_j}]_{A_{p_j}}^{\max(1, p_j'/p_j) + \max(\frac{1}{2}, p_j'/p_j)}\right).$$

**Proof.** Let $P_t, Q_t, \ldots$ be defined as in Lemma 2.3 $f_i \in L^{p_i}(w_i^{p_i}) \cap L^{p_i}$ and $h_t \in L^{\infty}_{w}$ for all $t > 0$ such that

$$\left\| \left( \int_0^t |h_t| \frac{ds}{s} \right)^{\frac{1}{2}} \right\|_{L^{p'}(w^{p})} \leq 1.$$

Recall that the dual of $L^{p'}(w^{p})$ can be realized as $L^{p'}(w^{p})$ if we take the measure space to be $\mathbb{R}^n$ with measure $w(x)^{p'}dx$. We estimate (1.6) by duality making use of Lemmas 2.3 and 2.4

$$\begin{align*}
\left| \int_{\mathbb{R}^n} \int_0^\infty \Theta_i(f_1, \ldots, f_m)(x) h_t(x) \frac{dt}{t} w(x)^p dx \right| \\
= \left| \int_{\mathbb{R}^n} \int_0^\infty \sum_{j=1}^m \Theta_i \Pi_{f_j}(f_1, \ldots, f_m)(x) w(x) h_t(x) w(x)^{p'/p} \frac{ds}{s} \frac{dt}{t} dx \right| \\
\leq \sum_{j=1}^m \left| \left( \int_{[0,\infty)^2} \left( \frac{s}{l} \land \frac{t}{s} \right)^{p'/p} (\Theta_i \Pi_{f_j}(f_1, \ldots, f_m)(x))^{\frac{dt}{s}} \frac{ds}{s} \right)^{\frac{1}{2}} \right|_{L^{p}(w^{p})} \\
\times \left| \left( \int_{[0,\infty)^2} \left( \frac{s}{l} \land \frac{t}{s} \right)^{p'/p} |h_t| \frac{ds}{s} \frac{dt}{t} \right)^{\frac{1}{2}} \right|_{L^{p'}(w^{p})} \\
\leq \sum_{j=1}^m \sum_{k=1}^n \left| \left( \int_{[0,\infty)^2} \left( \frac{s}{l} \land \frac{t}{s} \right)^{p'/p} (MQ_{k}^{\land k} f_j)^{\frac{dt}{s}} \frac{ds}{s} \right)^{\frac{1}{2}} \prod_{i \neq j} M f_i \right|_{L^{p}(w^{p})} \\
\leq \sum_{j=1}^m \sum_{k=1}^n [w_j^{p_j}]_{A_{p_j}}^{\max(1, p_j'/p_j) + \max(\frac{1}{2}, p_j'/p_j)} \|g_{MQ^{\land k}}(f_j)\|_{L^{p_j}(w_j^{p_j})} \prod_{i \neq j} \|M f_i\|_{L^{p_j}(w_j^{p_j})} \\
\leq \sum_{j=1}^m \sum_{k=1}^n [w_j^{p_j}]_{A_{p_j}}^{\max(1, p_j'/p_j) + \max(\frac{1}{2}, p_j'/p_j)} \|f_j\|_{L^{p_j}(w_j^{p_j})} \prod_{i \neq j} \left[ w_i^{p_i} \right]_{A_{p_i}}^{\frac{1}{p_i}} \|f_i\|_{L^{p_i}(w_i^{p_i})} \\
\leq \prod_{i=1}^m \left(1 + [w_j^{p_j}]_{A_{p_j}}^{\max(1, p_j'/p_j) + \max(\frac{1}{2}, p_j'/p_j)}\right) \|f_i\|_{L^{p_i}(w_i^{p_i})}.
\end{align*}$$

Here we have used the weighted bound for the Hardy-Littlewood maximal function, the Fefferman-Stein vector-valued maximal function bound proved originally by Anderson-John [11] and proved with the sharp dependence on the weight constant by Cruz-Uribe-Martell-Perez [11]. We also used the weighted square function estimate for $g_{MQ^{\land k}}$ for $k = 1, \ldots, m$ originally proved by Kurtz [32] and proved with sharp dependence on the weight constant by Lerner in [33].

\[\square\]
Although we use sharp estimates to track the weight constant dependence, we are not claiming that this bound on $S$ is sharp. In the above argument, once we have bounded the dual pairing by products of maximal functions and $g_\theta$ functions, the estimates may be sharp, but there is no evidence provided here that the estimates up to that point are sharp. We track the constant so that we can explicitly apply the extrapolation theorem of Grafakos-Martell [21].

3. CARLESON AND STRONG CARLESON MEASURES

This section is dedicated to defining the cancellation conditions that we will use for $\Theta$, and proving some properties about them. We start with a discussion to motivate these definitions and describe the role that they will play in the theory.

As discussed in the introduction, in the linear setting $S$ is bounded on $L^2$ if and only if $|\Theta(1)(x)|^2 \frac{d\mu}{dx}$ is a Carleson measure, although this may not in general be sufficient for $S$ to be bounded for all $1 < p < \infty$. We will define the strong Carleson condition for $\Theta_i$ and prove that it does imply bounds for all $1 < p < \infty$. There is a stronger notion of Carleson measure defined by Journé in [30] that is related to some of the Carleson conditions in this work. We will discuss this in a little more depth in Section 4.

**Definition 3.1.** A positive measure $d\mu(x,t)$ on $\mathbb{R}^{n+1}_+ = \{(x,t) : x \in \mathbb{R}^n, t > 0\}$ is a Carleson measure if

$$\|d\mu\|_C = \sup_Q \frac{1}{|Q|} \int_Q d\mu(T(Q)) < \infty,$$

where the supremum is taken over all cubes $Q \subset \mathbb{R}^n$, $|Q|$ denotes the Lebesgue measure of the cube $Q$, $T(Q) = Q \times \{0, \ell(Q)\}$ denotes the Carleson box over $Q$, and $\ell(Q)$ is the side length of $Q$.

Suppose $\mu$ is a non-negative measure on $\mathbb{R}^{n+1}_+$ defined by

$$d\mu(x,t) = F(x,t) d\tau(t) dx$$

for some $F \in L^1_{loc}(\mathbb{R}^{n+1}_+, d\tau(t)dx)$. We say that $\mu$ is a strong Carleson measure if

$$\|\mu\|_{SC} = \sup_Q \sup_{x \in Q} \int_0^{\ell(Q)} F(x,t) d\tau(t) < \infty.$$

Given an operator $\Theta_i$ with kernel satisfying (1.2), we say that $\Theta_i$ satisfies the Carleson condition, respectively strong Carleson condition, if $|\Theta_i(1,...,1)(x)|^2 \frac{d\mu}{dx}$ is a Carleson measure, respectively strong Carleson measure.

In [7] and [2], Christ-Journé and Auscher define a Carleson function to be a function $G : \mathbb{R}^{n+1}_+ \to \mathbb{C}$ such that $|G(x,t)|^2 \frac{d\mu}{dx}$ is a Carleson measure. So our definition of the Carleson condition for $\Theta_i$ is exactly that $G(x,t) = \Theta_i(1,...,1)(x)$ is a Carleson function in the language of Christ-Journé and Auscher. We state this definition with a general measure $d\tau(t)$ instead of just $\frac{d\mu}{t}$ because the results in Section 4 can be applied to the discrete case where $d\tau(t) = \delta_{2^{-k}}(t)$, like the ones in [16], [35], [26], [20], and many others.
It is trivial to see that if a non-negative measure $d\mu(x,t) = F(x,t)d\tau(t)dx$ is a strong Carleson measure, then it is a Carleson measure and $||\mu||_C \leq ||\mu||_{SC}$, but we can also prove a partial converse to this for non-negative measures of the form $|\Theta_t(1,...,1)|^2 \frac{dt}{t} dx$ for $\Theta_t$ satisfying (1.4) and (1.5). In Propositions 3.4 and 3.5, we prove that $\Theta_t$ satisfies the two-cube and the Carleson conditions if and only if it satisfies the strong Carleson condition. We first define the two-cube testing condition.

**Definition 3.2.** Let $\Theta_t$ satisfy (1.4) and $\Theta_t$ be defined as in (1.3). We say that $\Theta_t$ satisfies the **two-cube testing condition** if

\[(3.4) \quad \sup_{R \subset Q} \frac{1}{|R|} \int_R \int_{\ell(R)} |\Theta_t(\chi_{2R},\ldots,\chi_{2R})|(x) - \Theta_t(\chi_{2Q},\ldots,\chi_{2Q})(x)|^2 \frac{dt}{t} dx < \infty,\]

where the supremum is taken over all cubes $R$ and $Q$ with $R \subset Q$.

In the linear case, the two-cube condition for $\Theta_t$ becomes

\[(3.5) \quad \sup_{x \subset R^n} |\Theta_t(\chi_{E_1},\ldots,\chi_{E_m})(x)| \lesssim t^{-n} \min(|E_1|,\ldots,|E_m|).\]

**Lemma 3.3.** Suppose $\Theta_t$ satisfies (1.4). Then we have the following

i. Suppose $E_1,\ldots,E_m \subset \mathbb{R}^n$, then

\[(3.6) \quad \sup_{x \subset \mathbb{R}^n} |\Theta_t(\chi_{E_1},\ldots,\chi_{E_m})(x)| \lesssim t^{-n} \ell(Q)^{-m-n}\]

Proof. For $E_1,\ldots,E_m \subset \mathbb{R}^n$ and $x \subset \mathbb{R}^n$, using (1.4) we have

\[|\Theta_t(\chi_{E_1},\ldots,\chi_{E_m})(x)| \lesssim \prod_{j=1}^m \int_{\mathbb{R}^n} \frac{t^{-n}}{(1 + t^{-1}|x-y_j|)^N} \chi_{E_j}(y_j) dy_j \lesssim t^{-n}|E_i|\]

for each $i = 1,\ldots,m$. For (ii), for $x \subset Q \subset 2Q \subset \mathbb{R}^n \setminus E_i$, it follows that $|x-y_j| > \ell(Q)$ for all $y_j \subset E_i$. Then using (1.4), it follows that

\[|\Theta_t(\chi_{E_1},\ldots,\chi_{E_m})(x)| \lesssim \prod_{j=1}^m \int_{\mathbb{R}^n} \frac{t^{-n}}{(1 + t^{-1}|x-y_j|)^N} \chi_{E_j}(y_j) dy_j \]

\[\lesssim \int_{E_i} \frac{t^{-n}}{t^{-1}|x-y_j|} dy_j \lesssim t^{-n} \int_{|x-y_j| > \ell(Q)} \frac{1}{|x-y_j|^N} dy_j \lesssim t^{-n} \ell(Q)^{-(N-n)}.\]

**Proposition 3.4.** Suppose $\Theta_t$ satisfies (1.4) and (1.5). If $\Theta_t(x)$ satisfies the Carleson and the two cube testing conditions, then $\Theta_t$ satisfies the strong Carleson condition.
Proof. We first prove a multilinear result analog of the result of Carleson and Christ-Journé mentioned above, that $\Theta$ satisfies the Carleson condition implies that $S$ satisfies the un-weighted bound (1.3) for $p = 2$. That is $d\mu(x,t) = |(\Theta(1, \ldots, 1)(x)|^2 \frac{dt}{t}$ is a Carleson measure implies for all $1 < p_1, \ldots, p_m < \infty$ satisfying (2.1) with $p = 2$, $S$ is bounded from $L^{p_1} \times \cdots \times L^{p_m}$ into $L^2$. To prove this we adapt a familiar technique from Coifman-Meyer, see e.g. [2] or [10]. Decompose $\Theta_t = (\Theta_t - M_{\Theta_t(1, \ldots, 1)}F_t) - M_{\Theta_t(1, \ldots, 1)}F_t = R_t + U_t$ where

\begin{equation}
F_t(f_1, \ldots, f_m) = \prod_{i=1}^m P_i f_i
\end{equation}

and $P_i$ is a smooth approximation to the identity. The operator $R_t$ satisfies the conditions of Theorem 2.3 and hence the square function associated to $R_t$ is bounded on the appropriate spaces. The second term is bounded as well using the following Carleson measure bound

\[
\left( \int_0^T |U_t(f_1, \ldots, f_m)|^2 \frac{dt}{t} \right)^{\frac{1}{2}} \leq \prod_{i=1}^m \left( \int_{\mathbb{R}^n} |P_i f_i(x)|^p d\mu(x,t) \right)^{\frac{1}{p}} \lesssim \prod_{i=1}^m ||f_i||_{L^p}.
\]

We use a bound proved by Carleson [11], that $\{ P_i \}_{i \geq 0}$ is bounded from $L^q(\mathbb{R}^n)$ into $L^q(\mathbb{R}^{n+1}, d\mu)$ for all $1 < q < \infty$ whenever $d\mu(x,t)$ is a Carleson measure. We now move on to estimate 8.3, so take a cube $Q \subset \mathbb{R}^n$ and define

\[G_Q(x) = \chi_Q(x) \int_0^{\ell(Q)} d\mu(x,t).
\]

To prove that $\mu$ is a strong Carleson measure, it is sufficient to show that $|G_Q|_{L^1} \lesssim 1$ where the constant is independent of $Q \subset \mathbb{R}^n$. Since $d\mu$ is locally integrable in $\mathbb{R}^{n+1}$ and $d\mu$ is a Carleson measure, it follows that $G_Q \in L^1(\mathbb{R}^n)$. Then we have that $G_Q(x) \lesssim MG_Q(x)$ for almost every $x \in \mathbb{R}^n$. So we estimate $||MG_Q||_{L^1}$

\[MG_Q(x) = \sup_{R \ni x} \frac{1}{|R|} \int_R \int_0^{\ell(Q)} |\Theta(1, \ldots, 1)(y)|^2 \chi_Q(y) \frac{dt}{t} dy \]

\[= \sup_{R \ni x} \frac{1}{|R|} \int_R \int_0^{\ell(Q)} |\Theta(1, \ldots, 1)(y)|^2 \frac{dt}{t} dy \]

\[\leq \sup_{R \ni x} \frac{1}{|R|} \int_R \int_0^{\ell(Q)} |\Theta(1, \ldots, 1)(y)|^2 \frac{dt}{t} dy \]

\[+ \sup_{R \ni x} \sum_{F \in A} \frac{1}{|R|} \int_R \int_0^{\ell(Q)} |\Theta(F, \ldots, \chi_{F_m})(y)|^2 \frac{dt}{t} dy \]

\[+ \sup_{R \ni x} \sum_{F \in A} \frac{1}{|R|} \int_R \int_0^{\ell(Q)} |\Theta(F, \ldots, \chi_{F_m})(y)|^2 \frac{dt}{t} dy \]

\[= I + II + III.
\]

where

\[\Lambda = \{ \bar{F} = (F_1, \ldots, F_m) : F_i = 2R \text{ or } F_i = (2R)^c \} \setminus \{(2R, \ldots, 2R)\}.
\]
Note that we may make the reduction to cubes \( R \subset Q \) since \( \supp(G_Q) \subset Q \) and \( G_Q \geq 0 \). For each cube \( R \subset Q \subset \mathbb{R}^n \), we estimate \( I \) using that boundedness of \( S \)

\[
\frac{1}{|R|} \int_R \int_0^{\ell(Q)} |\Theta_t(\chi_{2R}, \ldots, \chi_{2R})(y)|^2 \frac{dt}{t} dy \leq \frac{1}{|R|} \int_R \int_0^\infty |\Theta_t(\chi_{2R}, \ldots, \chi_{2R})(y)|^2 \frac{dt}{t} dy \lesssim \frac{1}{|R|} \prod_{i=1}^m \|\chi_{2R}\|_{L^1}^2 \lesssim 1.
\]

Therefore \( I \) is bounded independent of \( x \) and \( Q \). We bound the second term there exists at least one \( F_i = (2R)^c \). Then using (3.6) from Lemma 3.3 we have

\[
\frac{1}{|R|} \int_R \int_0^{\ell(Q)} |\Theta_t(\chi_{F_i}, \ldots, \chi_{F_m})(y)|^2 \frac{dt}{t} dy \lesssim \frac{1}{|R|} \int_R \int_0^\infty t^{2(N-n)} \frac{dt}{t} dy \lesssim 1.
\]

Since \( |\Lambda| = 2^m - 1 \), this is sufficient to bound \( II \). Now for the term \( III \), we first take \( \vec{F} \in \Lambda \) such that at least one component \( F_i = 2R \). Then by (3.5) from Lemma 3.3 we have

\[
\frac{1}{|R|} \int_R \int_0^{\ell(Q)} |\Theta_t(\chi_{F_i}, \ldots, \chi_{F_m})(y)|^2 \frac{dt}{t} dy \lesssim \frac{1}{|R|} \int_R \int_0^\infty |\chi_{2R}|^2 \frac{dt}{t} dy \lesssim 1.
\]

This bounds all but one term for \( III \). It remains to bound the term where \( \vec{F} = ((2R)^c, \ldots, (2R)^c) \). We do this using (3.6) from Lemma 3.3 and the two cube condition (3.4)

\[
\frac{1}{|R|} \int_R \int_0^{\ell(Q)} |\Theta_t(\chi_{(2R)^c}, \ldots, \chi_{(2R)^c})(y)|^2 \frac{dt}{t} dy \leq \frac{1}{|R|} \int_R \int_0^{\ell(Q)} |\Theta_t(\chi_{(2R)^c}, \ldots, \chi_{(2R)^c})(y)|^2 \frac{dt}{t} dy + \frac{1}{|R|} \int_R \int_0^{\ell(Q)} |\Theta_t(\chi_{(2R)^c}, \ldots, \chi_{(2R)^c})(y)|^2 \frac{dt}{t} dy \lesssim \frac{1}{|R|} \int_R \int_0^{\ell(Q)} t^{2(N-n)} \frac{dt}{t} dy + 1 \lesssim 1.
\]

Therefore \( \|MG_Q\|_{L^\infty} \leq I + II + III \lesssim 1 \) for all \( Q \subset \mathbb{R}^n \) where the constant is independent of \( Q \). Now we can verify that \( d\mu \) satisfies the strong Carleson condition

\[
\sup_{Q \subset \mathbb{R}^n} \sup_{x \in Q} \int_0^{\ell(Q)} |\Theta_t(1, \ldots, 1)(x)|^2 \frac{dt}{t} \leq \sup_{Q \subset \mathbb{R}^n} \|G_Q\|_{L^\infty} \leq \sup_{Q \subset \mathbb{R}^n} \|MG_Q\|_{L^\infty} \lesssim 1.
\]

This completes the proof. 

\[\square\]

**Proposition 3.5.** If \( \Theta_t \) satisfies (1.4), (1.5) and \( \Theta_t \) satisfies the strong Carleson condition, then \( \Theta_t \) satisfies the two cube condition (3.4).
Proof. We estimate (3.4) for $R \subset Q \subset \mathbb{R}^n$

\[
\frac{1}{|R|} \int_R \int_{(Q)} \left| \Theta_r (\chi(2R), \ldots, \chi(2R)) (x) - \Theta_r (\chi(2Q), \ldots, \chi(2Q)) (x) \right|^2 \frac{dt}{t} \, dx \leq \frac{1}{|R|} \int_R \int_{(Q)} \left| \Theta_r (\chi(2R), \ldots, \chi(2R), \chi(2Q), 2R) (x) \right|^2 \frac{dt}{t} \, dx
\]

\[
\leq \frac{1}{|R|} \int_R \int_{(Q)} \left| \Theta_r (\chi(2R), \ldots, \chi(2R), \chi(2Q), 2R) (x) \right|^2 \frac{dt}{t} \, dx + \sum_{j=1}^{m-1} \frac{1}{|R|} \int_R \int_{(Q)} \left| \Theta_r (\chi(2R), \ldots, \chi(2R), 1 - \chi(2Q), 2R) (x) \right|^2 \frac{dt}{t} \, dx
\]

\[
\leq \frac{1}{|R|} \int_R \int_{(Q)} \left| \Theta_r (1, \ldots, 1) (x) - \Theta_r (\chi(2R), \ldots, \chi(2R), \chi(2Q), 2R) (x) \right|^2 \frac{dt}{t} \, dx + \sum_{j=1}^{m-1} \frac{1}{|R|} \int_R \int_{0}^{t^{(Q)}} t^{2(n-1)} \left( Q \right)^{-2(n-1)} \frac{dt}{t} \, dx
\]

Here the middle term is bounded by the assumption that $|\Theta_r (1, \ldots, 1) (x)|^2 \frac{dt}{t} \, dx$ is a strong Carleson measure. Now we bound

\[
\left| \Theta_r (1, \ldots, 1) (x) - \Theta_r (\chi(2R), \ldots, \chi(2R), \chi(2Q), 2R) (x) \right|
\]

\[
\leq \sum_{j=1}^{m-1} \left| \Theta_r (\chi(2R), \ldots, \chi(2R), 1, \ldots, 1) (x) \right| + \left| \Theta_r (\chi(2R), \ldots, \chi(2R), 1, \chi(2Q), 2R) (x) \right|
\]

\[
\leq \sum_{j=1}^{m-1} t^{-n}|R| + \left| \Theta_r (\chi(2R), \ldots, \chi(2R), 1, \chi(2Q), 2R) (x) \right|
\]

\[
\leq t^{-n}|R| + \left| \Theta_r (\chi(2R), \ldots, \chi(2R), \chi(2Q), 2R) (x) \right| + \left| \Theta_r (\chi(2R), \ldots, \chi(2R), \chi(2Q), 2R) (x) \right|
\]

\[
\leq t^{-n}|R| + t^{N-n} \left( Q \right)^{-N-n}.
\]

In the second to last line we bound the last term by $t^{-n}|R|$ and absorb it into the first term of the last line. Therefore we have that

\[
\frac{1}{|R|} \int_R \int_{(Q)} \left| \Theta_r (1, \ldots, 1) (x) - \Theta_r (\chi(2R), \ldots, \chi(2R), \chi(2Q), 2R) (x) \right|^2 \frac{dt}{t} \, dx
\]

\[
\leq \frac{1}{|R|} \int_R \int_{0}^{t^{(Q)}} t^{-2n}|R| \frac{dt}{t} \, dx + \frac{1}{|R|} \int_R \int_{0}^{t^{(Q)}} t^{2(n-1)} \left( Q \right)^{-2(n-1)} \frac{dt}{t} \, dx \lesssim 1,
\]

and hence $\Theta_r$ satisfies the two cube condition (3.4). \qed

We also prove that if $S$ is bounded from $L^{p_1} \times \cdots \times L^{p_m}$ into $L^p$ for some $1 < p_1, \ldots, p_m < \infty$ and $2 \leq p < \infty$ satisfying (2.1), then $\Theta_r$ satisfies the Carleson condition. A partial converse to this was proved within the proof of Proposition 3.4. If $\Theta_r$ satisfies the Carleson condition, then $S$ is bounded from $L^{p_1} \times \cdots \times L^{p_m}$ into $L^2$ for all $1 < p_1, \ldots, p_m < \infty$. 

Proposition 3.6. Assume $\Theta_i$ satisfies (1.4) and $S$ is bounded from $L^{p_1} \times \cdots \times L^{p_m}$ into $L^p$ for some $1 < p_1, \ldots, p_m < \infty$ and $2 \leq p < \infty$ satisfying (2.1). Then it follows that $\Theta_i$ satisfies the Carleson condition.

Proof. Fix a cube $Q \subset \mathbb{R}^n$ and we estimate
\[
\frac{1}{|Q|} \int_Q \int_0^{t(Q)} |\Theta_i(\chi_{2Q}, \ldots, \chi_{2Q})(x)|^2 \frac{dt}{t} dx \leq \frac{1}{|Q|} \int_Q \int_0^{t(Q)} |\Theta_i(\chi_{2Q}, \ldots, \chi_{2Q})(x)|^2 \frac{dt}{t} dx + \sum_{F \in \Lambda} \frac{1}{|Q|} \int_Q \int_0^{t(Q)} |\Theta_i(\chi_{F_1}, \ldots, \chi_{F_m})(x)|^2 \frac{dt}{t} dx
\]
(3.8)
where
\[\Lambda = \{ F = (F_1, \ldots, F_m) : F_i = 2Q \text{ or } F_i = (2Q)^c \} \setminus \{(2Q, \ldots, 2Q)\} .\]

For each cube $Q \subset \mathbb{R}^n$, we estimate $I$
\[
\frac{1}{|Q|} \int_Q \int_0^{t(Q)} |\Theta_i(\chi_{2Q}, \ldots, \chi_{2Q})(x)|^2 \frac{dt}{t} dx \leq \frac{1}{|Q|} \int_Q S(\chi_{2Q}, \ldots, \chi_{2Q})(x)^p dx \leq \left( \frac{1}{|Q|} \int_{\mathbb{R}^n} S(\chi_{2Q}, \ldots, \chi_{2Q})(x)^p dx \right)^{\frac{1}{p}} \lesssim |Q|^{-2/p} \prod_{i=1}^m ||\chi_{2Q}||_{L^p}^2 \lesssim 1.
\]

Now for the second term $II$, we fix $F \in \Lambda$, which has at least one component $F_i = (2Q)^c$. Then by (3.6) from Lemma 3.3 we have
\[
\frac{1}{|Q|} \int_Q \int_0^{t(Q)} |\Theta_i(\chi_{F_1}, \ldots, \chi_{F_m})(x)|^2 \frac{dt}{t} dx \lesssim \frac{1}{|Q|} \int_Q \int_0^{t(Q)} t^{2(N-n)} \ell(Q)^{-2(N-n)} \frac{dt}{t} dx \lesssim 1.
\]
Now noting that $|\Lambda| = 2^n - 1$, it follows that $II \lesssim 1$ as well. So $\Theta_i$ satisfies the Carleson condition.

In fact, this proves that if $\Theta_i$ satisfies (1.4), (1.5) and $\Theta_i$ satisfies the Carleson condition, then $\Theta_i$ satisfies the strong Carleson condition if and only if $\Theta_i$ satisfies the two cube testing condition (3.4). We conclude this section with a few examples of various Carleson measures obtained from operators $\Theta_i$ satisfying (1.4) and (1.5). In Example 3.7 we define operators that give rise to strong Carleson measures, and in Example 3.8 we define operators that give rise to operators that are Carleson measures, but not strong Carleson measures. For the examples, let $P_1$ be a smooth approximation to the identity and $P_2$ be as defined in (3.7).

Example 3.7. Suppose $\psi \in L^1$ with integral zero satisfying $|\psi(x)| \lesssim \frac{1}{(1 + |x|)^N}$
(3.9)
\[
\sup_{\xi \neq 0} \int_0^\infty |\hat{\psi}(i\xi)|^2 \frac{dt}{t} < \infty,
\]
and define $Q_i f = \psi \ast f$. Let $b \in L^q$ for some $1 \leq q < \infty$ with $|b(x) - b(x')| \leq L|x - x'|^\alpha$ where $0 < \alpha < N - n$, $\beta \in L^\infty(\mathbb{R}^n_{\omega+1})$, and define $D_0(f_1, \ldots, f_m)(x) = \beta(x)Q_0 b(x) \hat{P}_2 (f_1, \ldots, f_m)(x)$. It follows that the kernels of $D_0$, which for $t > 0$
\[
d_t(x, y_1, \ldots, y_m) = \beta(x)Q_t b(x) \prod_{i=1}^m \Phi_t(x - y_i),
\]
satisfy (1.4) and (1.5). We also have that \( \Theta_t(1, \ldots, 1) = \beta(x, t)Q_t b \), so we estimate

\[
|Q_t b(x)| = \left| \int_{\mathbb{R}^n} \psi_t(x-y)(b(y) - b(x))dy \right| \leq L \int_{\mathbb{R}^n} |\psi_t(x-y)||x-y|^\alpha dy \\
\leq t^\alpha \int_{\mathbb{R}^n} \frac{t^{-n}}{1 + t^{-1}|x-y|^{N-\alpha}} dy \lesssim t^\alpha.
\]

Also we have that

\[
|Q_t b(x)| \leq \|\psi_t\|_{L^{p'}} \|b\|_{L^p} \lesssim t^{-n/q}.
\]

Then it follows that

\[
\int_0^t \left| \Theta_t(1, \ldots, 1)(x) \right|^2 \frac{dt}{t} \lesssim \|\beta\|^2_{L^2(\mathbb{R}^n_{+1})} \int_0^1 t^{2\alpha} \frac{dt}{t} + \|\beta\|^2_{L^2(\mathbb{R}^n_{+1})} \int_1^\infty t^{-2\alpha} \frac{dt}{t} \lesssim 1.
\]

Therefore with this selection of \( b \) and \( \beta \), it follows that \( D_t \) satisfies the strong Carleson condition. So by Theorem 1.1 it follows that \( D_t \) satisfies the strong Carleson condition. So by Theorem 1.1 it follows that \( D_t \) satisfies the strong Carleson condition.

**Example 3.8.** The purpose of this example is to construct an operator \( \Theta_t \) satisfying (1.4) and (1.5) such that \( \Theta_t \) satisfies the Carleson condition, but not the strong Carleson condition. Define \( \psi(x) = \chi_{(0,1)}(x) - \chi_{(-1,0)}(x) \), \( Q_t f = \psi_t \ast f \), \( b(x) = \chi_{(0,1)}(x) \), and like above \( D_t(f_1, \ldots, f_m)(x) = Q_t b(x) \mathbb{P}_t(f_1, \ldots, f_m)(x) \). As above, we have that \( D_t(1, \ldots, 1) = Q_t b \). It is a quick computation to show that

\[
\hat{\psi}(\xi) = 2 \frac{1 - \cos(\xi)}{\xi_2^2}
\]

with the appropriate modification when \( \xi = 0 \). It follows then that \( |\hat{\psi}(\xi)| \lesssim \min(|\xi|, |\xi|^{-1}) \), and that

\[
|D_t(1, \ldots, 1)(x)| \gtrsim \frac{dt}{t} dx = |\psi_t \ast b(x)| \frac{dt}{t} dx.
\]
is a Carleson measure. Now we show that $D_t$ does not satisfy the strong Carleson condition. Let $Q = [-1,0], x \in [-1,0) \subset Q$, and we estimate (3.3) with the following computation

$$
\int_0^{\ell(Q)} |D_t 1(x)|^2 \frac{dt}{t} = 2 \int_0^{1/2} \left( \int_{-1/2}^{1/2} \psi_t(y) \chi_{(0,1)}(x-y) dy \right)^2 \frac{dt}{t} \\
\geq \int_{-1/2}^{1/2} \left( \int_{-1/2}^{1/2} \psi_t(y) dy \right)^2 \frac{dt}{t} \\
= \int_{-1/2}^{1/2} \left( \int_{-1/2}^{1/2} \frac{(x+y)^2}{t^2} \frac{dt}{t} \\
= x^2 \int_{-1/2}^{1/2} \frac{dt}{t^2} + 2x \int_{-1/2}^{1/2} \frac{dt}{t} + \int_{-1/2}^{1/2} \frac{dt}{t} \\
\geq x^2 \int_{-1/2}^{1/2} dt - 2x - 2 - \log(-x) \\
\geq -\log(-x) - 2.
$$

Therefore

$$
\sup_{x \in [-1,0)} \int_0^{\ell(Q)} |D_t 1(x)|^2 \frac{dt}{t} \geq \sup_{x \in [-1,0)} -\log(-x) - 2 = \infty,
$$

and hence $D_t$ satisfies the Carleson condition, but not the strong Carleson condition.

4. A FULL WEIGHTED T1 THEOREM FOR SQUARE FUNCTIONS FOR $L^2$

In this section, we develop some classical Carleson measure results in a weighted setting with strong Carleson measures. With these new tools, we can apply some familiar arguments to complete the proof of Theorems 1.1 and 1.2. More precisely, Lemmas 4.1, 4.2 and Proposition 4.3 are weighted versions of results proved by Carleson in [4] where we use assume strong Carleson in place of Carleson conditions.

**Lemma 4.1.** If $\mu$ is a strong Carleson measure, then for any locally integrable function $w \geq 0$ and $E \subset \mathbb{R}^n$

$$
\mu_w(E) \leq ||\mu||_{SC} w(E)
$$

where $d\mu_w(x,t) = w(x)d\mu(x,t)$ and $\hat{E} = \{ (x,t) \in \mathbb{R}^n_{+1} : B(x,t) \subset E \}$.

In [30], Journé says that $d\mu_w$ is a Carleson measure with respect to $w \in A_2$ if it satisfied (4.1). He uses this definition to prove that measures that satisfy this estimate also verify weighted analogs of Carleson measure bounds. In particular, Journé proves

**Proof.** Let $Q$ be the Calderón-Zygmund decomposition of $\chi_{E}$ at height $\frac{1}{2}$. Then

$$
E \subset \bigcup_j Q_j \quad \text{and} \quad |E| \leq \sum_j |Q_j| \leq 2|E|.
$$

Let $Q'_j$ be the dyadic cube with double the side length of $Q_j$ containing $Q_j$ and take $(x,t) \in \hat{E}$. Since $B(x,t) \subset E$ and $Q'_j \not\subset E$, it follows that $B(x,t) \subset B(x,3\sqrt{n}\ell(Q_j))$. Then

$$
\hat{E} \subset \bigcup_j Q_j \times (0,2\sqrt{n}\ell(Q_j))
$$
Now $d\mu(x,t) = F(x,t)d\tau(t)dx$ for some non-negative $F \in L^1_{loc}(\mathbb{R}_+^{n+1})$. So using that $\mu$ is a strong Carleson measure, it follows that

$$
\mu_w(\bar{E}) \leq \sum_j \mu_w((E \cap Q_j) \times (0, 2\sqrt{\mu}(Q_j)))
$$

$$
= \sum_j \int_{E \cap Q_j} \int_0^{2\sqrt{\mu}(Q_j)} F(x,t)d\tau(t)w(x)\chi_{Q_j}(x)dx
$$

$$
\leq ||\mu||_{SC} \sum_j \int_{E \cap Q_j} w(x)dx
$$

$$
\leq ||\mu||_{SC} w(E).
$$

In the last line, we use that $E \cap Q_j$ are disjoint. \hfill \Box

**Lemma 4.2.** Suppose $d\mu(x,t) = F(x,t)d\tau(t)dx$ is a strong Carleson measure and $|\phi(x)| \lesssim \frac{1}{(1+|x|)^N}$ for some $N > n$. Then for all $w \in A_p$ for $1 < p < \infty$,

$$
(4.2) \quad \left( \int_{\mathbb{R}_+^n} |\phi \ast f(x)|^p w(x)d\mu(x,t) \right)^\frac{1}{p} \lesssim ||\mu||_{SC}^{1/p} [w]^{1/(p-1)}_{A_p} ||f||_{L^p(w)}.
$$

**Proof.** Define the non-tangential maximal function

$$
M_\phi f(x) = \sup_{t > 0} \sup_{|x-y| < t} |\phi \ast f(t)|.
$$

For $\lambda > 0$, define

$$
E_\lambda = \{x \in \mathbb{R}^n : M_\phi f(x) > \lambda\}
$$

$$
\hat{E}_\lambda = \{(x,t) \in \mathbb{R}^{n+1} : B(x,t) \subset E_\lambda\}.
$$

It follows from Lemma 4.1 that $\mu_w(\hat{E}_\lambda) \leq ||\mu||_{SC} w(E_\lambda)$ where again $d\mu_w(x,t) = w(x)d\mu(x,t)$. Therefore

$$
\int_{\mathbb{R}_+^{n+1}} |\phi \ast f(x)|^p w(x)d\mu(x,t) = p \int_0^\infty \lambda^p \mu_w(\{(x,t) \in \mathbb{R}_+^{n+1} : |\phi \ast f(x)| > \lambda\}) \frac{d\lambda}{\lambda}
$$

$$
\leq p \int_0^\infty \lambda^p \mu_w(\hat{E}_\lambda) \frac{d\lambda}{\lambda}
$$

$$
\leq p ||\mu||_{SC} \int_0^\infty \lambda^p w(\hat{E}_\lambda) \frac{d\lambda}{\lambda}
$$

$$
= ||\mu||_{SC} \int_{\mathbb{R}^n} M_\phi f(x)^p w(x)dx
$$

$$
\lesssim ||\mu||_{SC} [w]^{p/(p-1)}_{A_p} ||f||_{L^p(w)}.
$$

Here we use as before that $|\phi \ast f(x)| \lesssim Mf(x)$ and $||Mf||_{L^p(w)} \lesssim [w]^{1/(p-1)}_{A_p} ||f||_{L^p(w)}$. \hfill \Box

**Proposition 4.3.** Suppose $\theta_i$ satisfies (1.4) and (1.5). If $\Theta_i$ satisfies the strong Carleson condition, then $S$ is satisfies (1.6) for all $w_i^{p_i} \in A_{p_i}$ and $1 < p_1, \ldots, p_m < \infty$ satisfying (2.1) with $p = 2$ where $w = w_1 \cdots w_m$. Furthermore, the constant for this bound is at most a
constant independent of \( w_1, \ldots, w_m \) times

\[(4.3) \quad C_{m,n,w_1,\ldots,w_m,p_1,\ldots,p_m} = \prod_{i=1}^{m} \left( 1 + \left[ w_i^{p_i} |A_{p_i}| \right]^{\max(1, p_i'/p_i) + \max(1/2, p_i'/p_i)} \right) + \| \mu \|_{SC}^{m} \prod_{i=1}^{m} |w_i^{p_i} |A_{p_i}| . \]

**Proof.** Define \( R_t = \Theta_t - M_{\Theta_t(1,\ldots,1)} \mathbb{P}_t \) and \( U_t = M_{\Theta_t(1,\ldots,1)} \mathbb{P}_t \). Then \( R_t \) satisfies (1.4), (1.5), and in addition \( R_t(1,\ldots,1) = 0 \) for all \( t > 0 \). Then by Theorem 2.5 it follows that

\[ \left( \int_0^\infty |R_t(f_1,\ldots,f_m)|^2 \frac{dt}{t} \right)^{1/2} \lesssim \prod_{i=1}^{m} \| f_i \|_{L^p_i} (w_i) . \]

Now we turn to the \( U_t \) term. For any \( w_i^{p_i} \in A_{p_i} \) for \( 1 < p_1, \ldots, p_m < \infty \) satisfying (2.1) with \( p = 2 \), take \( d\mu(x,t) = |\Theta_t(1,\ldots,1)|^2 \frac{dt}{dx} \) if it follows that

\[ \left( \int_0^\infty |U_t(f_1,\ldots,f_m)|^2 \frac{dt}{t} \right)^{1/2} \lesssim \int_{\mathbb{R}^m} \left( \prod_{i=1}^{m} \| f_i \|_{L^p_i} (w_i) \right)^2 d\mu(x,t) \]

\[ \lesssim \prod_{i=1}^{m} \left( \int_{\mathbb{R}^m} |f_i(x)|^{p_i} w_i(x)^{p_i} d\mu(x,t) \right)^{1/p_i} \]

\[ \lesssim \| \mu \|_{SC} \prod_{i=1}^{m} |w_i^{p_i} |A_{p_i}| , \]

The final inequality holds by Lemma 4.2. The first term in the constant (4.3) is from the bound of \( R_t \) by Theorem 2.5 and the second term is from the bound of \( U_t \) above. \( \square \)

These results almost complete the proof of Theorem 1.1 except for dealing with a density issue with \( f_i \in L^p_i(w_i^{p_i}) \cap L^p_i \) and applying weight extrapolation. Propositions 3.4 and 3.5 verify the equivalence of (i) and (ii) from Theorem 1.1. By Proposition 3.4 (i) implies that \( S \) satisfies (1.6) for all \( w_i^{p_i} \in A_{p_i} \) with \( 1 < p_1, \ldots, p_m \) and \( p = 2 \) for \( f_i \in L^p_i(w_i^{p_i}) \cap L^p_i \). In order to conclude boundedness for all \( L^p_i(w_i^{p_i}) \), we make a short density argument in following and apply the extrapolation theorem of Grafakos-Martell (21) to complete the proof of Theorem 1.1. We will use a lemma to prove this.

**Lemma 4.4.** If \( w \in A_p \) and \( 1 < p < \infty \), then \( \frac{1}{(d+|x_0-\cdot|)^n} \in L^p(w) \) for any \( x_0 \in \mathbb{R}^n \) and \( d > 0 \).

**Proof.** We start by noting that for any \( x \in \mathbb{R}^n \)

\[ M_{\chi_{B(x_0,d)}}(x) \geq \frac{1}{|B(x_0,d)|} \int_{B(x_0,d)} \chi_{B(x_0,d)}(x) dx \]

\[ = \frac{|\chi_{B(x_0,d)}(x)|}{|B(x_0,d)|} \frac{d^n}{(d+|x-x_0|)^n} . \]

Then it follows that

\[ \left( \int_{\mathbb{R}^n} \frac{1}{(d+|x-x_0|)^{np}} w(x) dx \right)^{1/2} \leq d^{-n} \| M_{\chi_{B(x_0,d)}} \|_{L^p(w)} \lesssim \| \chi_{B(x_0,d)} \|_{L^p(w)} < \infty . \]

Here we use the Hardy-Littlewood maximal operator bound on \( L^p(w) \) and that \( w \in L^1_{loc} \). \( \square \)
Proof. First we restrict to the case $p = 2$ and take $f_i \in L^p(w_1^{p_i})$ and $f_{i,k} \in L^p(w_{1,k}^{p_i}) \cap L^{p_i}$ with $f_{i,k} \rightarrow f_i$ in $L^p(w_1^{p_i})$ as $k \rightarrow \infty$. It follows that $f_{i,k} \otimes \cdots \otimes f_{m,k} \rightarrow f_i \otimes \cdots \otimes f_m$ as $k \rightarrow \infty$ in the weighted product Lebesgue space $L^p(w_1^{p_1}) \cdots L^m(w_1^{p_m})$. For all $x \in \mathbb{R}^n$

$$|\Theta_i(f_1, \ldots, f_m)(x) - \Theta_i(f_{1,k}, \ldots, f_{m,k})(x)|$$

$$\leq \int_{\mathbb{R}^m} \left| \Theta_i(x, y_1, \ldots, y_m) \right| |f_1(y_1) \cdots f_m(y_m) - f_{1,k}(y_1) \cdots f_{m,k}(y_m)| dy$$

$$\leq \prod_{i=1}^{m} \int_{\mathbb{R}^{n-p}} \left( \int_{\mathbb{R}^n} \frac{w_i(y_i)^{-\theta_i} dy_i}{(t + |x - y_i|)^{\theta_i p_i}} \right)^{\frac{1}{\theta_i}} ||f_1 \otimes \cdots \otimes f_m - f_{1,k} \otimes \cdots \otimes f_{m,k}||_{L^p(w_1^{p_1}) \cdots L^m(w_1^{p_m})}^p,$$

which tends to zero as $k \rightarrow \infty$ almost everywhere since $w_i^{p_i} \in A_{p_i}$ implies that $w_i^{-\theta_i} \in A_{\theta_i p_i}$ and so the first term is finite almost everywhere by Lemma 4.4. Therefore $\Theta_i(f_1, \ldots, f_m) \rightarrow \Theta_i(f_{1,k}, \ldots, f_{m,k})$ pointwise as $k \rightarrow \infty$ a.e. $x \in \mathbb{R}^n$. Then by Fatou’s lemma we have that

$$||S(f_1, \ldots, f_m)||_{L^2(w)}^2 = \int_{\mathbb{R}^n} \int_{0}^{\infty} \limsup_{k \rightarrow \infty} \left| \Theta_i(f_1, \ldots, f_m)(x) \right|^2 \frac{dt}{t} w(x)^2 dx$$

$$\leq \liminf_{k \rightarrow \infty} \int_{\mathbb{R}^n} \int_{0}^{\infty} \left| \Theta_i(f_1, \ldots, f_m)(x) \right|^2 \frac{dt}{t} w(x)^2 dx$$

$$\leq C_{n,m,w_1,\ldots,w_m,p_1,\ldots,p_m} \liminf_{k \rightarrow \infty} \prod_{i=1}^{m} ||f_i||_{L^p(w_1^{p_i})}^{\frac{p_i}{p_1}}$$

$$= C_{n,m,w_1,\ldots,w_m,p_1,\ldots,p_m} \prod_{i=1}^{m} ||f_i||_{L^p(w_1^{p_i})}^{\frac{p_i}{p_1}}.$$

Therefore $S$ satisfies (1.6) for all $1 < p_1, \ldots, p_m < \infty$ satisfying (2.1) with $p = 2$, for all $w_i^{p_i} \in A_{p_i}$, and for all $f_i \in L^p(w_1^{p_i})$. We complete the proof by applying the multilinear extrapolation theorem of Grafakos-Martel (21), which we state now.

**Theorem 4.5** (Grafakos-Martell (21)). Let $1 \leq q_1, \ldots, q_m < \infty$ and $1/m \leq q < \infty$ be fixed indices that satisfy (2.1) and $T$ be an operator defined on $L^q(w_1^{q_1}) \times \cdots \times L^q(w_1^{q_m})$ for all tuples of weights $w_i^{q_i} \in A_{q_i}$. We suppose that for all $B > 1$, there is a constant $C_0 = C_0(B) > 0$ such that for all tuples of weights $w_i^{q_i} \in A_{q_i}$ with $[w_i^{q_i}]_{A_{q_i}} \leq B$ and all functions $f_i \in L^p(w_i^{q_i})$, $T$ satisfies

$$||T(f_1, \ldots, f_m)||_{L^q(w)} \leq C_0 \prod_{i=1}^{m} ||f_i||_{L^q(w_i^{q_i})}^q.$$

Then for all indices $1 < p_1, \ldots, p_m < \infty$ and $1/m < p < \infty$ that satisfy (2.1), all $B > 1$, and all weights $w_i^{p_i} \in A_{p_i}$ with $[w_i^{q_i}]_{A_{q_i}} < B$, there is a constant $C = C(B)$ such that for all $f_i \in L^p(w_i^{q_i})$

$$||T(f_1, \ldots, f_m)||_{L^p(w)} \leq C \prod_{i=1}^{m} ||f_i||_{L^p(w_i^{p_i})}^q.$$

We may take, for example, $q_1 = \cdots = q_m = 2m$ and hence $q = 2$. Then we have just proved that for all $B > 1$ and $w_i^{q_i} \in A_{q_i}$ with $[w_i^{q_i}]_{A_{q_i}} \leq B$ that

$$||S(f_1, \ldots, f_m)||_{L^2(w)} \leq C_{n,m,q_1,\ldots,q_m} \prod_{i=1}^{m} ||f_i||_{L^q(w_i^{q_i})}^q.$$
with $B$ in (4.3) times a constant independent of the weights,

$$C_0(B) = C_{n,m,q_1,...,q_m} \left[ \prod_{i=1}^{m} 2B^{\max(1/(q_i-1),1/2,1/(q_i-1))} + |\mu|^{m/2} \prod_{i=1}^{m} B^{1/(q_i-1)} \right].$$

which verifies the hypotheses of Theorem 4.5 for $S$. Therefore for all $B > 1$, there exists $C$ depending on $B,n,m,q_1,...,q_m$ such that

$$||S(f_1,...,f_m)||_{L^P(w^p)} \leq C \prod_{i=1}^{m} ||f_i||_{L^P(w^p)}$$

for all $1 < p_1,...,p_m < \infty$, $w_i^{p_i} \in A_{w_i}$ with $|w_i^{p_i}|_{A_{w_i}} \leq B$, and $f_i \in L^p(w_i^{p_i})$.

We now prove Theorem 1.2.

**Proof.** The implications $(iv) \Rightarrow (iii) \Rightarrow (ii) \Rightarrow (i)$ have already been proved in a more general context. So it is sufficient to show that $(i) \Rightarrow (iv)$. Since $\Theta_i(x,y_1,...,y_m) = t^{-m}w^p \int_{(x-y_1),...,(x-y_m)}|\Psi|c_{0} \frac{dt}{t}$, it follows that $\Theta_i(1,...,1)(x)$ is constant in $x$. For all $x \in \mathbb{R}^n$

$$\Theta_i(1,...,1)(x) = \int_{\mathbb{R}^m} t^{-m}w^p \int_{(x-y_1),...,(x-y_m)}|\Psi|c_{0} \frac{dt}{t}$$

where the last line here we take as the definition of $F$. But we have assumed that $\Theta_i$ satisfies the Carleson condition, and hence $|F(t)|^2 \frac{dt}{t} dx$ is a Carleson measure. So the strong Carleson condition follows: For all cubes $Q \subset \mathbb{R}^n$

$$\int_{0}^{t(Q)} |\Theta_i(1,...,1)(x)|^2 \frac{dt}{t} = \frac{1}{|Q|} \int_{0}^{t(Q)} |F(t)|^2 \frac{dt}{t} dx \lesssim 1.$$

If we assume also that $\Psi = \Psi^t$ is constant in $t$, then it follows that $F(t) = c_0$ is a constant function. But then $|c_0|^2 \frac{dt}{t} dx$ is a Carleson measure, and hence integrable on $Q \times (0, \ell(Q)]$ for all cubes $Q \subset \mathbb{R}^n$. Then it follows that $c_0 = 0$ when $\Psi^t$ is constant in $t$, which completes the proof. 

**References**

1. K. Andersen, and R. John, Weighted inequalities for vector-valued maximal functions and singular integrals, Studia Math. 69, no. 1, 1931, (1980/81).
2. P. Auscher, Lectures on the Kato square root problem, Proc. Centre Math. Appl. Austral. Nat. Univ., 40, Austral. Nat. Univ., Canberra, (2002).
3. P. Auscher, A. McIntosh, S. Hofmann, M. Lacey, and P. Tchamitchian, The solution of the Kato Square Root Problem for Second Order Elliptic Operators on $\mathbb{R}^n$, Ann. of Math. 156 (2002), 633–654.
4. L. Carleson, An Interpolation Problem for Bounded Analytic Functions, American Journal of Mathematics, 80, 4, (1958), 921-930.
5. L. Cheng, On Littlewood-Paley functions, Proc. Amer. Math. Soc. 135, 10, (2007), 3241-3247.
6. M. Christ, A $T(b)$ theorem with remarks on analytic capacity and the Cauchy integral, Colloq. Math. 60/61, no. 2, 601-628, (1990)
7. M. Christ, J. L. Journé, Polynomial growth estimates for multilinear singular integral operators, Acta Math., 159 1-2 (1987) 51-80.
8. R. R. Coifman and Y. Meyer, A simple proof of a theorem by G. David and J.-L. Journé on singular integral operators, Probability Theory and Harmonic Analysis, pp. 61–65, J. Chao and W. Woyczyński (eds.), Marcel Dekker, New York, 1986.
9. R. Coifman and Y. Meyer, Nonlinear harmonic analysis, operator theory and P.D.E., Ann. of Math. Stud., 112, Princeton Univ. Press, Princeton, NJ, 1986.
10. R. Coifman and Y. Meyer, *A simple proof of a theorem by G. David and J.-L. Journé on singular integral operators*, Probability theory and harmonic analysis (1983), 61-65.

11. D. Cruz-Uribe, J. Martell, and C. Perez, *Sharp weighted estimates for classical operators*, Adv. Math. 229, no. 1 (2012), 408-441.

12. G. David and J. L. Journé, *A boundedness criterion for generalized Calderón-Zygmund operators*, Ann. of Math., 120 (1984) 371-397.

13. G. David, J. L. Journé, and S. Semmes, *Operateurs de Calderon-Zygmund*, fonctions para-accretives et Interpolation, Rev. Mat. Iberoam., 1 (1985) 1-56.

14. J. Duoandikoetxea, *Sharp L_p boundedness for a class of square functions*, (preprint).

15. J. Duoandikoetxea, *Extrapolation of weights revisited: New proofs and sharp bounds*, Journal of Functional Analysis, 260, 1886-1901, (2011).

16. J. Duoandikoetxea and J. L. Rubio de Francia, *Maximal and singular integral operators via Fourier*, Invent. Math., 84, 3, (1986), 541-561.

17. J. Duoandikoetxea and E. Seijo, *Weighted inequalities for rough square functions through extrapolation*, Studia Math. 149, no. 3 (2002), 239-252.

18. C. Fefferman, E. Stein, *H_p spaces of several variables*, Acta Math., 129, 137-193 (1972).

19. L. Grafakos, *Classical and Modern Fourier Analysis*, Springer.

20. L. Grafakos, L. Liu, D. Maldonado, D. Yang *Multilinear analysis on metric spaces*, (preprint).

21. L. Grafakos, J. Martell, *Extrapolation of weighted norm inequalities for multivariable operators and applications*, J. Geom. Anal. 14, no. 1, (2004), 19-46

22. L. Grafakos, L. Oliveira, *Carleson measures associated with families of multilinear operators*, Submitted.

23. L. Grafakos, R.H. Torres, *On multilinear singular integrals of Calderón-Zygmund type*, Publ. Mat., (2002) 57-91.

24. L. Grafakos and R. H. Torres, *Multilinear Calderón-Zygmund theory*, Adv. in Math. 165 (2002), 124–164.

25. Y. Han, *Calderón-type Reproducing Formula and the Tb Theorem*, Rev. Mat. Iberoam., 10 1 (1994) 51-91.

26. J. Hart, *bilinear square functions and vector-valued Calderón-Zygmund operators*, J. Fourier Anal. Appl., to appear (2012).

27. J. Hart, *a new proof of the bilinear T(1) theorem*, submitted.

28. S. Hofmann, *A local T(b) theorem for square functions*, Proc. Sympos. Pure Math. 79 (2008), 175–185

29. S. Hofmann, * Tb theorems and applications in PDE*, International Congress of Mathematicians. Vol. II, Eur. Math. Soc., Zrich, (2006), 1375-1392.

30. J. L. Journé, *Calderón-Zygmund operators, pseudodifferential operators and the Cauchy integral of Calderón*, Lecture Notes in Mathematics, 994, Springer-Verlag, (1983).

31. C. Kenig, E. Stein, *Multilinear estimates and fractional integration*, Math. Res. Lett. 6, no. 3-4, 467, (1999).

32. D. Kurtz, *Littlewood-Paley and multiplier theorems on weighted L_p spaces*, Trans. Amer. Math. Soc. 259, no. 1, 235254, (1980).

33. A. Lerner, *Sharp weighted norm inequality for Littlewood-Paley operators and singular integrals*, Adv. Math. 226, no. 5, (2011), 3912-3926.

34. D. Maldonado, *Multilinear Singular Integrals and Quadratic Estimates*, Doctoral Dissertation, University of Kansas, (2005).

35. D. Maldonado, V. Naibo, *On the boundedness of bilinear operators on products of Besov and Lebesgue spaces*, J. Math. Anal. Appl., 352, 591-603 (2009).

36. J. Peetre, *On convolution operators leaving L^{p,\lambda} invariant*, Ann. Mat. Pura Appl., 72, 295-304 (1966).

37. S. Sato, *Estimates for Littlewood-Paley functions and extrapolation*, Integral Equations Operator Theory 62, no. 3 (2008), 429/440.

38. S. Semmes, *Square Function Estimates and the Tb Theorem*, Proc. of the AMS, 110, 3, (1990).

39. S. Spanne, *Sur l’interpolation entre les espaces L^p,*, Ann. Scuola Norm. Sup. Pisa, 20, 625-648, (1966).

40. E. Stein, *On the functions of Littlewood-Paley, Lusin, and Marcinkiewicz*, Trans. Amer. Math. Soc. 88, (1958), 430-466.

41. E. Stein, *Singular integrals, harmonic functions, and differentiability properties of functions of several variable*, in Calderón, 316-335, (1967).

42. M. Wilson, *Weighted Littlewood-Paley theory and exponential-square integrability*, Lecture Notes in Mathematics, 1924. Springer, Berlin, (2008).
