A geometrical dual to relativistic Bohmian mechanics - the multi particle case

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In this article it is shown that the fundamental equations of relativistic Bohmian mechanics for multiple bosonic particles have a dual description in terms of a classical theory of curved space-time. We further generalize the results to interactions with an external electromagnetic field, which corresponds to the minimally coupled Klein-Gordon equation.

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I. INTRODUCTION

Understanding quantum mechanics as it is usually taught is a mayor challenge for many physics students. For most of them the problem is not the introduction of new mathematical tools, but the understanding of the new concepts like observables. In standard quantum mechanics observables and the corresponding uncertainty are promoted to a fundamental principle, which can not be further understood. But it was shown by David Bohm that this does not necessarily have to be the case [1,2]. In the de Broglie-Bohm (dBB) interpretation it is explained that the uncertainty “principle” and the description by means of operators can be understood in terms of uncontrollable initial conditions and non-local interactions between an additional field (the “pilot wave”) and the measuring apparatus. This theory was further generalized to relativistic quantum mechanics and quantum field theory with bosonic and fermionic fields [3,4,5,6]. It is well known that (due to its non-locality) the dBB theory is not in contradiction to the Bell inequalities [7]. Due to its contextuality, the dBB theory is also not affected by the Kochen-Specker theorem [8].

One mayor drawback of the dBB theory is that the pilot wave and the corresponding “quantum potential” have to be imposed by hand without further justification. In a previous work [9,10] it was shown that the relativistic dBB theory for a single particle is dual to a scalar theory of curved space-time. In this dual theory the ominous “pilot wave” can be readily interpreted as a well known physical quantity, namely a space-time dependent conformal factor of the metric. This work on the single particle has many features in common with other publications on the subject [11,12,13,14,15]. However, having a duality for the single particle case is not enough, because the dBB interpretation only is a consistent quantum theory when it also has the many particle case. The many particle theory is for example crucial for understanding the quantum uncertainty and for evading the non existence theorems [7,8]. Therefore, we will generalize the previous results and present a dual for the relativistic many particle dBB theory.

The paper is organized as follows: In the second section a summary of the ingredients of the many particle dBB theory will be given. In the third section we will define a theory of curved space-time and proof that it is dual to the many particle dBB theory. In the fourth section it is shown, how both theories can be coupled to an external electromagnetic field. The fifth section contains the non-relativistic limit (the Schrödinger equation). It is further discussed what this limit corresponds to in the dBB theory and in its geometrical dual. In the sixth section issues like non-locality, differences to general relativity, and the universality of the given matching conditions are addressed. In the seventh section the results are summarized.

II. RELATIVISTIC BOHMIAN MECHANICS FOR MANY PARTICLES

In this section we shortly list the ingredients for the interpretation of the many particle quantum Klein-Gordon equation in terms of Bohmian trajectories. For a detailed description of subsequent topics in the dBB theory like particle creation, the theory of quantum measurement, many particle states, and quantum field theory the reader is referred to [6]. Let |0⟩ be the vacuum and |n⟩ be an arbitrary n-particle state. The corresponding n-particle wave function is [4]

$$\psi(x_1; \ldots ; x_n) = \frac{P_s}{\sqrt{n!}} (0|\hat{\Phi}(x_1)\ldots\hat{\Phi}(x_n)|n) \quad , \quad (1)$$

where the $\hat{\Phi}(x_j)$ are scalar Klein-Gordon field operators and the symbol $P_s$ denotes symmetrization over all positions $x_j$. For free fields, the wave function [11] satisfies the equation

$$\left( \sum_j \partial^2_j + \frac{M^2}{\hbar^2} \right) \psi(x_1; \ldots ; x_n) = 0 \quad . \quad (2)$$

The mass of a single particle is given by $M$. The index $j$ indicates which one of the $n$ particles is affected and the index $m$ is the typical space-time index in four flat dimensions. The use of the Latin indices in contrast to
the usual Greek indices will allow in the next section allow a distinction between coordinates in curved space-time (Greek) and coordinates in flat space-time (Latin). The wave function (11) allows further for the construction of a conserved current

$$\sum_j^n \partial_j^m (\psi^* \partial_j^m \psi) = 0 \quad .$$

This is still standard quantum mechanics, in order to arrive at the dBB interpretation one rewrites the wave function \( \psi = Pe^{iS/\hbar} \) by introducing a real amplitude \( P(x_1; \ldots ; x_n) \) and a real phase \( S(x_1; \ldots ; x_n) \). Doing this, the complex equation (2) splits up into two real equations

$$2MQ = \sum_j^n (\partial_j^m S)(\partial_j^m m) - nM^2 \quad ,$$

$$0 = \sum_j^n \partial_j^m (P^2(\partial_j^m S)) \quad ,$$

where \( Q(x_1; \ldots ; x_n) \) is the quantum potential, and \( P(x_1; \ldots ; x_n) \) is the pilot wave. The first equation can be interpreted as a classical Hamilton-Jacobi equation with the additional potential \( Q \) and the second equation takes the form of a conserved current. The quantum potential in equation (11) is given from \( \hbar \), the particle mass, and the pilot wave by

$$Q = \sum_j^n \frac{\hbar^2}{2M} \frac{\partial_j^m \partial_j^m P}{P} \quad .$$

This is the only way that the \( \hbar \) enters into the dBB theory. In the dBB interpretation one postulates the existence of particle trajectories \( x_j^m(s) \) whose momentum \( p_j^m \) satisfies the relation

$$p_j^m = M \frac{dx_j^m}{ds} \equiv -\partial_j^m S \quad .$$

Now one can derive this expression with respect to \( ds \) and use the identity

$$\frac{d}{ds} = \sum_j^n \frac{dx_j^m}{ds} \partial_j^m \quad .$$

This gives the equation of motion for all \( n \) relativistic particles in the dBB interpretation

$$\frac{d^2x_j^m}{ds^2} = \sum_i^n (\partial_i^m S)(\partial_i^m \partial_j^m S) M^2 \quad .$$

By using equation (11) this can be further simplified to

$$M \frac{d^2x_j^m}{ds^2} = \partial_j^m Q \quad .$$

The infinitesimal parameter \( s \) is not necessarily time, because every single particle carries its own reference frame. For convenience one might try to choose \( s \) as the eigen-time of the particle which is finally subject to a measurement. The non-local nature of the dBB theory becomes obvious in the above equations of motion. The trajectory of the particle \( j \) is determined from the potential \( Q(x_1; \ldots ; x_n) \), which depends on the positions of all other particles in the system.

The equations (11) are the building blocks of the many particle dBB theory. The functions \( P, S, \) and \( Q \) that appear in those equations depend on the \( 4 \times n \) coordinates \( x_j^m \). It is therefore possible to introduce a single \( 4n \) dimensional coordinate

$$x^L = (x_1^0, x_1^1, x_1^2, x_1^3; \ldots ; x_n^0, x_n^1, x_n^2, x_n^3) \quad ,$$

which has a capital Latin index and contains the space-time positions of all \( n \) particles. One further observes that in all equations every summation over a particle index \( j \) is accompanied by a summation over the space-time index \( m \) of the corresponding particle. This allows to replace \( \partial_j^m \rightarrow \partial_L \) and \( \partial_j^m \rightarrow \partial_L \). Thus, one can rewrite the equations for the many particle case (12) as

$$2MQ = (\partial_L^S)(\partial_L P) - nM^2 \quad with \quad \begin{align*}
Q & = \frac{\hbar^2}{2M} \frac{\partial_L P}{P} \\
0 & = \partial_L (P^2(\partial_L S)) \\
p^L & = \frac{M}{\partial_L} \frac{dx^L}{ds} = -\partial_L S \\
\frac{d^2x^L}{ds^2} & = \frac{(\partial_N^S)(\partial_L \partial_N S)}{M^2} \quad with \quad \begin{align*}
\frac{d}{ds} & \equiv \frac{dx^L}{ds} \partial_L 
\end{align*}
\end{align*}$$

### III. A \( 4 \times n \) Dimensional Theory of Curved Space-Time

We will now show that the equations of the many particle dBB theory (1215) have a dual description in a scalar theory of curved space-time in \( 4 \times n \) dimensions. As a generalization of the previous single particle approach (10) we will define a setup where the momentum of every particle is defined in the particles own four dimensional space-time. Such a \( n \)-particle theory is therefore defined in a \( 4 \times n \) dimensional space-time. Following the notation in (11) the coordinates in curved space-time will be denoted as

$$\hat{x}^A = (\hat{x}_1^0, \hat{x}_1^1, \hat{x}_1^2, \hat{x}_1^3; \ldots ; \hat{x}_n^0, \hat{x}_n^1, \hat{x}_n^2, \hat{x}_n^3) \quad .$$

The theory can be formulated by starting from the n-particle action

$$S[\hat{g}, \hat{\Lambda}, \hat{\alpha}] = \left\{ \int d^4x \sqrt{|g|} \alpha \left( \hat{\mathcal{R}} + \kappa \hat{\mathcal{L}}_M - \hat{P}_s(\hat{\mathcal{R}} + \kappa \hat{\mathcal{L}}_M) \right) \right\} \quad .$$
Here, \( \hat{g} \) is the determinant of the metric, \( \alpha \) is a Lagrange multiplier, \( \mathcal{P}_\alpha \) is a symmetrization operator between different particles \( x^\alpha_j \) and \( x^\lambda_j \), \( \hat{R} \) is the Ricci scalar, \( \hat{\mathcal{L}}_M \) is the matter Lagrangian, and \( \kappa \) is the coupling constant of this theory. A variation of \( [17] \) with respect to \( \alpha \) ensures that only the part of \( \hat{R} + \kappa \hat{\mathcal{L}}_M \) which is symmetric under the exchange of two particle coordinates \( x^\alpha_i \) and \( x^\lambda_j \) survives

\[
\mathcal{P}_\alpha \left[ \hat{R} + \kappa \hat{\mathcal{L}}_M \right] = \hat{R} + \kappa \hat{\mathcal{L}}_M . \tag{18}
\]

This means that that the fields that are a solution of the problem have to be symmetric under exchange of two particle coordinates \( x^\alpha_i \) and \( x^\lambda_j \). It also means that all the different particles agree on their definition of what is the \( x^\mu \) direction. Therefore, if one wants to perform a coordinate transformation in a single four dimensional subspace, one has to perform the same transformation in all other four dimensional subspaces. After having derived the symmetry relation \( [18] \) the remaining action is

\[
S[\hat{g}_{\Lambda\Delta}] = \int d\hat{x}^A \sqrt{|\hat{g}|} \left( \hat{R} + \kappa \hat{\mathcal{L}}_M \right) . \tag{19}
\]

In order to describe the local conformal part of this theory separately one splits the metric \( \hat{g} \) up into a conformal function \( \phi(x) \) and a non-conformal part \( g \)

\[
\hat{g}_{\Lambda\Gamma} = \phi^{2n-4} g_{\Lambda\Gamma} . \tag{20}
\]

The index notation for tensors used here is explained in table \( [1] \) and it allows to write the equations in \( D = 4 \times n \) dimensions either with one index (capital letters), or with two indices (lower case letters). A further distinction is made between indices that are shifted by the metric \( \hat{g} \) (Greek) and indices that are shifted by the metric \( g \) (Latin). In this section we will only use the single index notation, but all results can be immediately translated into the double index notation used in the previous section. The inverse of the metric \( [20] \) is

\[
\hat{g}^{\Lambda\Gamma} = \phi^{-\frac{2}{2n-4}} g^{\Lambda\Gamma} . \tag{21}
\]

Indices with a lower Greek and a lower Roman index can be identified \( \partial_\lambda \equiv \partial_L \). From this follows for example that the adjoint derivatives are not identical, in both notations

\[
\hat{\partial}^\Lambda = \hat{g}^{\Lambda\Sigma} \partial_\Sigma = \phi^{-\frac{2}{2n-4}} g^{L\Sigma} \partial_\Sigma = \phi^{-\frac{2}{2n-4}} \partial^L . \tag{22}
\]

A. The geometrical dual to the first dBB equation

The definition of the metric \( [20] \) allows to reformulate the action in terms of the separate functions \( \phi \) and \( g_{\Lambda\Delta} \). Keeping in mind the symmetrization condition \( [18] \) the action \( [17] \) reads

\[
S[\phi, g_{\Lambda\Delta}] = \int d\hat{x}^A \sqrt{|\hat{g}|} \left[ \frac{2(4n-1)}{1-2n} (\partial^L \phi)(\partial_L \phi) + \phi^2 (R + \kappa \mathcal{L}_M) \right] . \tag{23}
\]

We are primarily interested in studying a flat Minkowski background space-time \( g_{\Lambda\Delta} = \eta_{\Lambda\Delta} \). This can be achieved by adding an additional Lagrange condition term to the action \( [17] \) that demands a vanishing Weyl curvature

\[
C_{\Sigma\Lambda\Xi\Delta} = R_{\Sigma\Lambda\Xi\Delta} - \frac{1}{2n-1} (g_{\Sigma\Xi} R_{\Lambda\Delta} - g_{\Lambda\Xi} R_{\Delta\Sigma})
+ \frac{1}{(4n-1)(2n-1)} R g_{\Sigma\Xi} g_{\Delta\Lambda} = 0 . \tag{24}
\]

Such a condition also appears in the scalar theories of curved space-time suggested by Gunnar Nordström \( [17, 18, 19] \). Like in standard general relativity one further imposes that the metric has a vanishing covariant derivative

\[
(\nabla \phi)_{\Lambda\Xi} = 0 . \tag{25}
\]

From the conditions \( [23, 25] \) it follows that the metric \( g_{\Lambda\Delta} \) has only \( \pm 1 \) on the diagonal, while all other entries vanish

\[
g_{\Lambda\Delta} = \eta_{\Lambda\Delta} . \tag{26}
\]

Thus, \( |g| = 1, R = 0 \), and therefore the action \( [23] \) simplifies to

\[
S[\phi] = \int d\hat{x}^A \left[ \frac{2(4n-1)}{1-2n} (\partial^L \phi)(\partial_L \phi) + \kappa \phi^2 \mathcal{L}_M \right] . \tag{27}
\]

The Euler-Lagrange equation for this action is

\[
\frac{2(4n-1)}{1-2n} \frac{\partial^L \partial_L \phi}{\phi} = \kappa \mathcal{L}_M . \tag{28}
\]

An Extension of the Hamilton Jacobi matter Lagrangian \( \mathcal{L}_M \) can be defined by subtracting a mass term \( M^2 \) for every particle on finds

\[
\hat{\mathcal{L}}_M = \hat{\mathcal{P}}^\Lambda \hat{p}_\Lambda - n \hat{M}_G^2 
= (\hat{\partial}^\Lambda S_H)(\hat{\partial}_\Lambda S_H) - n \hat{M}_G^2
= \phi^{-\frac{2}{2n-4}} ((\partial^L S_H)(\partial_L S_H) - n M^2) 
= \phi^{-\frac{2}{2n-4}} \mathcal{L}_M . \tag{29}
\]
The Hamilton principle function \( S_H \) defines the local momentum \( \hat{p}^\Lambda = M_G \hat{x}^\Lambda / \hat{s} = -\hat{\partial}^\Lambda S_H \). Plugging this Lagrangian into equation (28) gives

\[
\frac{2(4n - 1)}{\kappa(1 - 2n)} \frac{\partial^\Lambda \partial^\Lambda \phi}{\phi} = (\partial^\Lambda S_H)(\partial^\Lambda S)H) - nM_G^2 \tag{30}
\]

Now one can see that this is exactly the first dBB equation [12] if one identifies

\[
\phi(x) = P(x) , \tag{31} \\
S_H(x) = S(x) , \\
\kappa = \frac{2(4n - 1)}{1 - 2n}/\hbar^2 , \\
M = M_G^2 .
\]

Note that the matching conditions demand like in the single particle case [10] a negative coupling \( \kappa \).

**B. The geometrical dual to the third dBB equation**

The stress-energy tensor corresponding to the matter Lagrangian (29) is

\[
\hat{T}^{\Lambda\Delta} = -2(\hat{\partial}^{\Lambda} S_H)(\hat{\partial}_{\Delta} S_H) + \hat{g}^{\Lambda\Delta} \left( (\partial S_H)^2 - nM_G^2 \right) . \tag{32}
\]

According to the Hamilton-Jacobi formalism the derivatives of the Hamilton principle function \( S_H \) define the momenta

\[
\hat{p}_\Lambda = -\left( \hat{\partial}_{\Lambda} S_H \right) . \tag{33}
\]

Therefore, with the prescription (22) and the matching condition (31) one sees immediately that the third Bohmian equation (14) is fulfilled.

**C. The geometrical dual to the second dBB equation**

In order to find the dual to the second Bohmian equation one has to exploit that the stress-energy tensor (32) is covariantly conserved

\[
\nabla_\Lambda \hat{T}^{\Lambda\Delta} = 0 . \tag{34}
\]

This is true if the following relations are fulfilled

\[
\nabla_\Lambda (\hat{\partial}^{\Lambda} S_H) = 0 , \tag{35} \\
(\hat{\partial}^{\Lambda} S_H) \nabla^\Delta (\hat{\partial}_{\Delta} S_H) = 0 , \tag{36} \\
(\hat{\partial}^{\Lambda} S_H) \nabla_\Lambda (\hat{\partial}_{\Lambda} S_H) = 0 . \tag{37}
\]

In addition to the covariant conservation of momentum (35) and the conservation of squared momentum (36) the tensor nature of (32) also demands (37), which will be important in the next subsection. In order to calculate the covariant derivatives in (35,37), one needs to know the Levi Civita connection

\[
\Gamma^\Sigma_{\Delta\Lambda} = \frac{1}{2} g^{\Sigma\Xi} \left( \partial_{\Lambda} g_{\Delta\Xi} + \partial_{\Delta} g_{\Xi\Lambda} - \partial_{\Xi} g_{\Lambda\Delta} \right) \tag{38}
\]

or, equivalently, the equation of motion in the Minkowski coordinates \( x^L \)

\[
\frac{d^2 x^L}{d\hat{s}^2} = \frac{(\hat{\partial}^N S_H)(\hat{\partial}_L \hat{\partial}_{\Lambda} S_H)}{M^2} . \tag{42}
\]

Using (31) one sees that the equation of motion (42) is dual to the equation of motion of relativistic Bohmian mechanics (15). This is however almost a triviality because the equations (15), (11), and (12) are all derived from the same mathematical prescription (8). But in curved space-time there is another equation of motion in addition to the equation of motion (41), the geodesic equation

\[
\frac{d^2 \hat{x}^L}{d\hat{s}^2} + \hat{\Gamma}^\Lambda_{\Delta\Sigma} \frac{d\hat{x}^\Lambda}{d\hat{s}} \frac{d\hat{x}^\Sigma}{d\hat{s}} = \hat{p}_L \cdot f(\hat{x}) . \tag{43}
\]

Here, \( f(\hat{x}) \) is some arbitrary scalar function. For this equation of motion it is on the contrary not trivial to show that it is consistent with (15). To prove this, one can proceed as follows: First one uses on the left hand side of (43) the equations (11) and (38). By using the relations (35) and (37) one can show that the result is some scalar function times \( (\hat{\partial}^\Lambda S_H) \). This proofs that the geodesic equation (43) is consistent with equation (11). Since it was already shown that (11) is dual to (15), one can conclude that also (43) is dual to (15).

Thus, all the equations of the dBB theory (12,15) have a dual description (30,33,39, and 43) in the presented theory of higher dimensional curved space-time.


IV. INTERACTION WITH AN EXTERNAL FIELD

Now, the results of the previous sections will be generalized to interactions with an external electromagnetic field.

A. An external field in the dBB theory

Coupling $n$ bosonic particles \( \text{2} \) with charge $e$ to an external electromagnetic field $A_m$ is achieved by replacing the partial derivative with a gauge covariant derivative $\partial_m \rightarrow \partial_m + ieA_m/\hbar$ in the Klein-Gordon Lagrangian. The resulting equation of motion is

\[
\sum_j^n \left[ \left( \partial_j^m \partial_j m + \frac{M}{\hbar^2} \left( -e^2 A^2 \right) \right) \psi + \frac{i\partial_j^m (\psi^2 A_j m)}{\hbar} \right] = 0 ,
\]

where $A_j^m$ is the electromagnetic potential $A^m$ evaluated at the position of particle $j$. By again rewriting the $n$-particle wave function $\psi = P \exp(iS)$ the two equations \([4, 9] \) generalize to

\[
2MQ \equiv \sum_j^n (\partial_j^m S + eA_j^m)(\partial_j m S + eA_j m) - nM^2 , \quad (45) \\
0 \equiv \sum_j^n \partial_j m (P^2(\partial_j^m S + eA_j^m)) , \quad (46)
\]

where the quantum potential $Q$ in the first equation is given by \([10] \). In the presence of an external force, the dBB definition for the particles momentum \([7] \) now contains the canonical momentum $\pi_j^m$ instead of the normal momentum $p_j^m$

\[
\pi_j^m = M \frac{d^m x_j}{d\tau} \equiv -(\partial_j^m S + eA_j^m) . \quad (47)
\]

Thus, using \([8, 45, 47] \) and the relation $\partial_{km} A_j^m = \delta_{kj} \partial_m A_j m$ one finds the equation of motion for all $n$ particles in an external field $A^m$

\[
M \frac{d^m x_j}{d\tau} = \partial_j^m Q + e\pi_j m F_{mn} . \quad (48)
\]

Here, the field strength tensor is $F_{mn} = \partial_m A_n - \partial_n A_m = \delta_{km} F_{mn}$ with $F^m n = \partial_m A_n - \partial_n A_m$. On the RHS of equation \((48) \) appear two terms. The first term is the quantum potential also present in equation \((10) \) and the second term is the Lorentz force which is familiar from classical electrodynamics.

Now on can apply again the formal rewriting of coordinates \([11] \) and the interacting equations \([45, 48] \) read

\[
2MQ \equiv (\partial_j^L S + eA_j^L)(\partial_j L S + eA_L) - nM^2 , \quad (49) \\
0 \equiv \partial_j L (P^2(\partial_j^L S + eA_L)) , \quad (50) \\
p_j^L \equiv M \frac{d^L x_j}{d\tau} \equiv -(\partial_j^L S + eA_L) , \quad (51) \\
M \frac{d^L x_j}{d\tau} = \partial_j^L Q + e\pi_K F_{LK} . \quad (52)
\]

B. An external field in the $4 \times n$ dimensional theory of curved space-time

In the interacting case, the classical $4 \times n$ dimensional theory of curved space-time is analogous to the discussion in section \([11] \) The only difference appears in the definition of the canonical momentum

\[
\hat{\pi}^A = \hat{M}_G \frac{d^A \hat{x}}{d\tau} = -(\hat{\partial}^A S_H + e\hat{A}^A) , \quad (53)
\]

instead of the free momentum $\hat{p}_A$. With this replacement the equations \([30, 39, 57] \) and \([33] \) transform to

\[
\frac{2(4n - 1)}{\kappa(1 - 2n)} \frac{d^L \partial_j \phi}{d\tau} \equiv (\partial_j^L S_H + eA_j^L)(\partial_j L S_H + eA_L) - nM^2 \hat{\partial}_j L , \quad (54) \\
0 \equiv \phi \frac{2M}{\kappa} \partial_j L (P^2(\partial_j^L S_H + eA_j^L)) . \quad (55)
\]

One immediately sees that with the identifications \([31] \) the above equations \([53, 57] \) are dual to the equations \([49, 47] \). In order to check the equation of motion \([52] \) we use \([30] \) and find

\[
\frac{d^2 \hat{\pi}^A}{d\tau^2} = \phi \frac{2M}{\kappa} \frac{d^2 \hat{x}^L}{d\tau^2} + \hat{\pi}^L . \quad (56)
\]

Using this and the identity \([57] \) in its canonical version $\hat{\pi}^A \hat{\nabla} \hat{\pi}^A = 0$ one can verify that the geodesic equation

\[
\frac{d^2 \hat{\pi}^A}{d\tau^2} + \hat{\nabla}^A \hat{\pi}^A \frac{\hat{\pi}^A}{\hat{M}_G} \hat{M}_G = \hat{\pi}^A . \quad (57)
\]

is consistent with \([52] \).

V. THE NON-RELATIVISTIC LIMIT

The dBB theory was originally developed for non relativistic quantum mechanics. It will now be shown how this limit can be obtained from the interacting relativistic version and its dual. Since in this limit spatial and temporal derivatives are treated differently it is instructive to return to the two-index notation. Nevertheless translation to the one-index notation is always possible with the help the index convention \([11] \). Starting from the relativistic and interacting many particle equations \([45, 47] \) the non-relativistic limit can be obtained from a number of low energy approximations. At first, one wants to
achieve that the quantum phase depends only on one single time coordinate (which will be denoted \( t_1 \)) instead of the \( n \) time coordinates \( t_j \). Therefore, the quantum phase \( S \) (or equivalently the Hamilton principal function \( S_H \)) is redefined by

\[
S(t_j, \vec{x}_j) = -M t_j + \tilde{S}(t_1, \vec{x}_j) \quad .
\]  

(58)

For the vector potential \( A^m \) one chooses the Coulomb gauge and imposes a similar condition for the time coordinates and indices

\[
A_j^0(t_k, \vec{x}_k) = A_0^0(t_1, \vec{x}_k) \quad \text{and} \quad A_j^j = 0 \quad .
\]

(59)

For the pilot wave, the condition for the time coordinate is analogously

\[
P(t_j, \vec{x}_j) \equiv P(t_1, \vec{x}_j) \quad .
\]

(60)

Now the equations \((55)\) \((60)\) read

\[
Q(P(t_1, \vec{x}_j)) = -\dot{\tilde{S}} + \frac{\dot{\tilde{S}}^2}{2M} - e A_0 + \frac{e^2 A_0^2}{2M} - \sum_{j=1}^{n} \frac{(\partial_j \tilde{S})^2}{2M} \quad ,
\]

\[
0 = (\partial_0 P^2) \left( 1 - \frac{e A_0}{M} - \frac{\dot{S}}{M} \right) \quad ,
\]

\[
0 = -P_2 \frac{\dot{A}_0 + \ddot{S}}{M} + \sum_{j=1}^{n} \partial_j \left( \frac{P_2 \partial_j \tilde{S}}{M} \right) \quad ,
\]

where derivatives with respect to \( t_1 \) are denoted with a dot. The non-relativistic limit consists now in assuming that the mass \( M \) is a much larger quantity than all the time components of the four vectors

\[
M \gg \dot{\tilde{S}} , \; M \gg e A_0 , \; M \gg \frac{\partial_0 P}{P} \quad .
\]

(63)

In this limit the above equations give

\[
0 = \dot{\tilde{S}} + e A_0 + \sum_{j=1}^{n} \left( \frac{(\partial_j \tilde{S})^2}{2M} - \frac{\hbar^2}{2M} \partial_j^2 P^2 \right) \quad ,
\]

\[
0 = \partial_0 (P^2) + \sum_{j=1}^{n} \partial_j \left( \frac{P_2 \partial_j \tilde{S}}{M} \right) \quad .
\]

(64)

(65)

By replacing \( \psi = P e^{i \tilde{S}/\hbar} \) one sees that this is the many particle Schrödinger equation with the potential \( e A_0 \). Finally, with the same relations \((58)\) \((60)\) gives the original momentum definition in the Bohmian interpretation

\[
\vec{p}_j = \partial_j \tilde{S} \quad .
\]

(66)

Due to the matching conditions \((51)\), in the geometrical dual the non-relativistic limit means the following: First, the Hamilton principle function \( S_H \) is split up into a linear part containing the mass and a remaining part \( \tilde{S}_H \) \((58)\). Second, all the time coordinates \( t_j \) in the functions \( \tilde{S}_H , \; \phi \), and \( A \) are synchronized \((55)\) \((56)\) and the Coulomb gauge is chosen. Third, the time components of the remaining four vectors are taken to the limit where they are much smaller than the mass \( M_G \) \((53)\).

In this limit one finds in the geometrical theory that the function \( \phi \) plays a double role. On the one hand it is a conformal field of the metric which interacts with matter. On the other hand its square \( \phi^2(t, \vec{x}_j) \) gives the probability density of finding the particles of the system in the positions \( \vec{x}_j \) at the time \( t \). When one intends to find normalizable solutions (for instance of a central potential \( A_0 \sim 1/r \)) usually one imposes the boundary condition \( \lim_{x \to \infty} \phi^2 = 0 \). This limit means that in the dual theory the metric \( \hat{g}_{\Delta \Delta} \) as to vanish asymptotically for far regions.

VI. DISCUSSIONS

In this section some interpretational issues are addressed. This aims to develop a physical understanding of the presented duality.

A. Locality

Quantum mechanics in the dBB interpretation is a theory which allows to talk about particle position at the cost of non-local interactions. The non-locality becomes obvious by looking at the equation of motion for a free particle in the dBB theory \((10)\). The movement of this “free” particle is not free but it is governed by the quantum potential \( Q \), which simultaneously depends on the positions of all the other particles of the system. It is often argued that such a non-locality can not have any dual in a (by construction) local theory such as a geometrical theory of curved space-time. Therefore the first question is:

“How can a non-local theory have a dual local theory?”

The reason is that in the presented theory of curved space-time, every single particle is living in its own four dimensional space-time. Therefore, the positions of \( n \) different particles correspond to one single point in the \( 4 \times n \) dimensional space-time. In some sense the introduction of additional dimensions seems to be a common feature of a classical understanding of quantum theories (see the AdS/CFT dualities or the many worlds interpretation). This higher dimensional construction helps around the non-locality argument but it brings a new problem:

B. Four dimensional space-time

“How does the \( 4 \times n \) dimensional toy model cope with the fact that every accepted physical theory is formulated in four dimensions?”
We don’t know the answer to this question yet. Therefore, it is up to now only proven that the presented duality exists mathematically, but a better physical understanding is certainly desirable. Part of the answer might lie in the symmetrization condition $P_s$, in the action (17), which corresponds to the symmetrization in bosonic many particle quantum mechanics. Since all solutions have to be invariant under the exchange of two sub-coordinates $x_i^\mu$ and $x_j^\mu$ it is clear that all those sub-coordinates $x_i^\mu$ have to have the same interpretation. This symmetry forbids for instance to rotate the $x$ and $y$ coordinates for a single particle without rotating the coordinates for the other particles. In this sense the interpretation of common sense coordinates in 4 dimensions is unique.

C. Gravity

“How is this geometrical theory related to THE geometrical theory - general relativity?” It is very tempting to think about such a relation because the action (17) is very similar to the Einstein Hilbert action and the metric condition (25) is the same as in general relativity. However the presented theory in this higher dimensional form can NOT describe four dimensional gravity. The possible connection between both geometrical theories still has to be explored.

D. The matching conditions

The matching conditions (31) are not unique, but they were chosen in order to have the most direct connection between three functions. Those are the quantum phase $S$, connected to the Hamilton principle function $S_H$, the quantum amplitude $P$ connected to the conformal metric function $\phi$, and the quantum mass $M$ connected to the mass $M_G$. Fixing those relations forces a relation between the particle number $n$, the Planck constant $\hbar$ and the coupling of the geometrical theory $\kappa$

$$\kappa = \frac{2(4n-1)}{1-2n} / \hbar^2.$$ 

This has the consequence that if one gives $\hbar$ a fixed numerical value, then the coupling of the geometrical theory $\kappa$ runs from $-6/\hbar^2$ to $-4/\hbar^2$, depending on the number of particles ($n = 1, \ldots, \infty$) which are part of the system. Reversely demanding a fixed geometrical coupling would result in a running Planck constant $\hbar$. This behavior seems to be the only unexpected (and undesired) peculiarity of the chosen matching conditions (31).

More optimistically, one might also see the scale dependence of the coupling $\kappa$ as a feature of the toy model which it shares with effective quantum field theories and even some approaches to gravity [20].

VII. SUMMARY

In this paper we showed that the equations of the free relativistic dBB theory for many particles [12-15] have a dual description in a $4 \times n$ dimensional theory of curved space-time. For the translation between the two theories a single set of matching conditions was defined (31). The result was then generalized to interactions with an external electromagnetic field. Before discussing interpretational issues, the limit of the non-relativistic Schrödinger equation is derived. The important question whether such dualities can also be found for, self-interacting theories, fermions, or quantum field theory will be subject of future studies.

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