Bott-Cattaneo-Rossi invariants for long knots in asymptotic homology $\mathbb{R}^3$

David Leturcq

Abstract

In this article, we express the Alexander polynomial of null-homologous long knots in punctured rational homology 3-spheres in terms of integrals over configuration spaces. To get such an expression, we use a previously established formula, which gives generalized Bott-Cattaneo-Rossi invariants in terms of the Alexander polynomial and vice versa, and we relate these Bott-Cattaneo-Rossi invariants to the perturbative expansion of Chern-Simons theory.

Keywords: Knot theory, Configuration spaces, Alexander polynomial, Perturbative expansion of the Chern-Simons theory.

MSC: 55R80, 57K10, 57K14, 57K16.

1 Introduction

Knot invariants defined as combinations of integrals over configuration spaces or, equivalently, as combinations of algebraic counts of diagrams, emerged after the seminal work of Witten [Wit89] on the perturbative expansion of the Chern-Simons theory. Knot invariants defined from spatial configurations of unitrivalent graphs were formally defined by Guadagnini, Martellini and Mintchev [GMM90], Bar-Natan [BN95a], Altschüler and Freidel [AF97], Bott and Taubes [BT94], and others, for knots in $\mathbb{R}^3$. These invariants can be unified in an invariant $Z = (Z_k)_{k \in \mathbb{N}}$ defined by Altschüler and Freidel in [AF97] for knots in $\mathbb{R}^3$, and called the perturbative expansion of the Chern-Simons theory. The invariant $Z$ takes its values in a vector space $\mathcal{A} = \prod_{k \in \mathbb{N}} \mathcal{A}_k$, spanned by classes of Jacobi unitrivalent diagrams, precisely described in Definition 2.3. Altschüler and Freidel proved that $Z$ is a universal Vassiliev invariant for knots in $\mathbb{R}^3$. The Kontsevich integral $Z^K = (Z^K_k)_{k \in \mathbb{N}}$ described by Bar-Natan [BN95a], and defined using integrals over spaces of planar...
configurations is another universal Vassiliev invariant. An article of Lescop [Les02] connects $Z$ to $Z^K$, up to the Bott and Taubes anomaly $\alpha$, and implies that the two invariants are equivalent.

In [BNG96], Bar-Natan and Garoufalidis defined a linear form $w_C$ on $\bigoplus_{k \in \mathbb{N}} A_k$ (see Definition 2.6), called the Conway weight system, and they expressed the Alexander polynomial $\Delta_\psi$ for knots in $\mathbb{R}^3$ as

$$\Delta_\psi(e^h) = \frac{2 \sinh \left( \frac{h}{2} \right)}{h} \sum_{k \geq 0} (w_C \circ Z^K_k)(\psi) h^k.$$ 

Both hands of the formulas are also well-defined for the long knots in $\mathbb{R}^3$, and

$$\Delta_\psi(e^h) = \sum_{k \geq 0} (w_C \circ Z^K_k)(\psi) h^k.$$ 

The perturbative expansion of Chern-Simons theory $(Z_k)_{k \in \mathbb{N}}$ extends to long knots in rational asymptotic homology $\mathbb{R}^3$, as in [Les15, Les20]. In this article, we prove the following result (Corollary 2.17).

**Theorem.** For any null-homologous long knot $\psi$ of an asymptotic rational homology $\mathbb{R}^3$,

$$\Delta_\psi(e^h) = \sum_{k \geq 0} (w_C \circ Z_k)(\psi) h^k.$$ 

In Proposition 2.18, we use the relation of Lescop [Les02] between $Z$ and $Z^K$ to notice that our theorem in terms of the perturbative expansion of the Chern-Simons theory is equivalent to the formula of Bar-Natan and Garoufalidis in terms of the Kontsevich integral for long knots of $\mathbb{R}^3$. The proof of our more general theorem relies on completely different methods, even for long knots of $\mathbb{R}^3$, and our theorem holds in the wider setting of null-homologous long knots in asymptotic rational homology $\mathbb{R}^3$. Our proof uses the direct computations of integrals over configuration spaces of our article [Let20]. For codimension two null-homologous long knots of any asymptotic homology $\mathbb{R}^{n+2}$, these computations allowed us to express the generalized Bott-Cattaneo-Rossi (BCR for short) invariants $(Z_{BCR,k})_{k \in \mathbb{N} \setminus \{0,1\}}$, which we defined in [Let19], in terms of the Reidemeister torsion (or Alexander polynomials).

These generalized BCR invariants generalize invariants defined by Bott [Bot96], and Cattaneo and Rossi [CR05] for (codimension 2) long knots in odd-dimensional Euclidean spaces $\mathbb{R}^{n+2}$ with $n + 2 \geq 5$. The BCR invariant $Z_{BCR,k}$ is a combination of integrals over configuration spaces associated with some diagrams with $2k$

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1 See Section 2.1 for a definition of long knots, in a wider setting.

2 These spaces are defined in Section 2.1.

3 The definition given for $n = 1$ in Section 2.1 easily adapts to any $n \geq 1$. 

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vertices of two kinds and $2k$ edges of two kinds, called BCR diagrams. In [Let19], given a parallelized asymptotic homology $\mathbb{R}^{n+2}$ $(M^0, \tau)$, we defined some particular forms on the two-point configuration spaces of $\mathbb{R}^n$ or of $M^0$, called propagators of $(M^0, \tau)$. The generalized BCR invariant $Z_{BCR,k}$ maps the data of a parallelized asymptotic homology $\mathbb{R}^{n+2}$ $(M^0, \tau)$, a long knot $\psi$, and a family of propagators $F$ of $(M^0, \tau)$ to a real number. In [Let19], we proved that this number depends only of the diffeomorphism class of $(M^0, \psi)$ when $n \geq 3$. For $n = 1$, the definition of [Let19] still makes sense, but $Z_{BCR,k}$ might depend on the choice of the parallelization or of the propagators, and might not be an invariant. In this article, we prove the formula

$$Z_{BCR,k}(\psi) = -(w_C' \circ Z_k)(\psi),$$

for any long knot $\psi$ of a rational asymptotic homology $\mathbb{R}^3$ and for the weight system $w_C'$ defined in Lemma 2.8 where the invariant $Z_{BCR,k}$ can be computed with any set of propagators. In particular, Formula 1 and the results of [Les20] on the perturbative expansion of Chern-Simons theory imply that for any long knot $\psi$ of an asymptotic homology $\mathbb{R}^3$, the number $Z_{BCR,k}(\psi)$ does not depend on the choice of the propagators or of the parallelization, and that it is invariant under ambient diffeomorphism: this is Corollary 2.15. The proof of Formula 1 only relies on combinatorics of BCR diagrams and Jacobi untrivalent diagrams.

The weight system $w_C'$ coincides with the Conway weight system $w_C$ defined by Bar-Natan and Garoufalidis in [BNG96] on the non-empty connected untrivalent diagrams. It vanishes on trivalent diagrams, and on non-trivial products of diagrams. It satisfies the formula

$$\sum_{k \geq 0} (w_C \circ Z_k)(\psi) h^k = \exp \left( \sum_{k \geq 1} (w_C' \circ Z_k)(\psi) h^k \right),$$

for any long knot of an asymptotic rational homology $\mathbb{R}^3$, as noticed in Theorem 2.13.

For any $n \geq 1$, our flexible definition of [Let19] for the BCR invariants allows us to compute them with arbitrary propagators. In [Let20], we present an explicit computation based on so-called admissible propagators, which yields exact formulas for $Z_{BCR,k}(\psi)$ in terms of Alexander polynomials. For a null-homologous long knot of an asymptotic rational homology $\mathbb{R}^3$, these formulas reduce to

$$\Delta_\psi(e^h) = \exp \left( - \sum_{k \geq 2} Z_{BCR,k}(\psi) h^k \right).$$

Formulas 1, 2 and 3 imply the theorem stated in the beginning of this introduction.

In Section 2 we review the definitions of the two invariants $Z_k$ and $Z_{BCR,k}$ of this article, and we state the forementioned results in Theorems 2.14 (Formula 1)
above) and \(\text{(Formula 3 above)}\). Section 3 is devoted to the proof of Theorem 2.14.

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2 Bott-Cattaneo-Rossi invariants and perturbative expansion of the Chern-Simons theory

2.1 Long knots in asymptotic homology \(\mathbb{R}^3\)

Let \(M\) be a smooth oriented compact connected 3-manifold with the rational homology of \(S^3\). Fix a point \(\infty\) in \(M\) and a closed ball \(B_\infty(M)\) around \(\infty\), and set \(M^o = M \setminus \{\infty\}\). Identify the punctured ball \(B^o_\infty(M) = B_\infty(M) \setminus \{\infty\}\) of \(M^o\) with the complement \(B^o_\infty\) of the open unit ball in \(\mathbb{R}^3\), and let \(B(M)\) denote the closure of \(M^o \setminus B^o_\infty\). The manifold \(M^o\) together with the decomposition \(M^o = B(M) \cup B^o_\infty\) is called an asymptotic rational homology \(\mathbb{R}^3\).

A parallelization of an asymptotic rational homology \(\mathbb{R}^3\) is a bundle isomorphism \(\tau: M^o \times \mathbb{R}^3 \rightarrow TM^o\) that coincides with the canonical trivialization of \(T\mathbb{R}^3\) on \(B^o_\infty \times \mathbb{R}^3\). Such a parallelization always exists: see for example [Les20, Proposition 5.5].

A long knot in an asymptotic rational homology \(\mathbb{R}^3\) is a smooth embedding \(\psi: \mathbb{R} \hookrightarrow M^o\) such that, for any \(x \in \mathbb{R}\),

- if \(x \in [-1, 1]\), \(\psi(x) \in B(M)\),
- if \(x \not\in [-1, 1]\), \(\psi(x) = (0, 0, x) \in B^o_\infty \subset \mathbb{R}^3\).

In the following, we let \((M^o, \tau)\) be a fixed parallelized rational asymptotic homology \(\mathbb{R}^3\).

2.2 BCR diagrams

We recall the definition of BCR diagrams, as introduced in [Let19, Section 2.2]. In all the following, if \(k\) is a positive integer, \(\overline{k}\) denotes the set \(\{1, \ldots, k\}\).

**Definition 2.1.** A BCR diagram is a non-empty oriented connected graph \(\Gamma\), defined by a set \(V(\Gamma)\) of vertices, decomposed into \(V(\Gamma) = V_i(\Gamma) \sqcup V_e(\Gamma)\), and a set \(E(\Gamma)\) of ordered pairs of distinct vertices, decomposed into \(E(\Gamma) = E_i(\Gamma) \sqcup E_e(\Gamma)\), whose elements are called edges\(^4\) where the elements of \(V_i(\Gamma)\) are called internal

\(^4\)Note that this implies that our graphs have neither loops nor multiple edges with the same orientation.
vertices, those of $V_e(\Gamma)$ external vertices, those of $E_i(\Gamma)$ internal edges, and those of $E_e(\Gamma)$ external edges, and such that, for any vertex $v$ of $\Gamma$, one of the five following properties holds:

1. $v$ is external, with two incoming external edges and one outgoing external edge, and exactly one of the incoming edges comes from a univalent vertex.
2. $v$ is internal and trivalent, with one incoming internal edge, one outgoing internal edge, and one incoming external edge, which comes from a univalent vertex.
3. $v$ is internal and univalent, with one outgoing external edge.
4. $v$ is internal and bivalent, with one incoming external edge and one outgoing internal edge.
5. $v$ is internal and bivalent, with one incoming internal edge and one outgoing external edge.

In the following, internal edges are depicted by solid arrows, external edges by dashed arrows, internal vertices by black dots, and external vertices by white dots, as in Figure 1, where the five behaviors of Definition 2.1 appear.

![Figure 1: An example of a BCR diagram of degree 6](image)

Definition 2.1 implies that any BCR diagram consists of one cycle with some legs attached to it, where legs are external edges that come from a (necessarily internal) univalent vertex, and where the graph is a cyclic sequence of pieces as in Figure 2 with as many pieces of the first type than of the second type. In particular, a BCR diagram has an even number of vertices, and this number is also the number of its edges.

![Figure 2](image)
The degree of a BCR diagram is the integer $\text{deg}(\Gamma) = \frac{1}{2}\text{Card}(V(\Gamma))$. A numbering of a degree $k$ BCR diagram is an injection $\sigma : E_e(\Gamma) \rightarrow 3k$.

2.3 Jacobi unitrivalent diagrams

In this section, we recall the definition of unitrivalent diagrams, widely used in the theory of Vassiliev invariants.

**Definition 2.2.** A Jacobi diagram is a graph $\Gamma$, given by a set $V(\Gamma)$ of vertices, decomposed into $V(\Gamma) = V_i(\Gamma) \sqcup V_e(\Gamma)$, a set $E(\Gamma)$ of unordered pairs of distinct vertices called edges, such that the vertices of $V_e(\Gamma)$ are trivalent, the vertices of $V_i(\Gamma)$ are univalent, and the set $V_i(\Gamma)$ of univalent vertices is totally ordered.

The degree $\text{deg}(\Gamma)$ of such a diagram $\Gamma$ is half its number of vertices. A numbering of a degree $k$ Jacobi diagram is an injection $j : E(\Gamma) \rightarrow 3k$.

An orientation of a trivalent vertex $v$ is the choice of a cyclic order on the three half-edges adjacent to $v$. A vertex-orientation of a Jacobi diagram $\Gamma$ is the choice of an orientation of any trivalent vertex. A vertex-oriented Jacobi diagram is a Jacobi diagram together with a vertex orientation.

An edge-orientation of a Jacobi diagram is the choice of an orientation for each edge. An edge-oriented Jacobi diagram is a Jacobi diagram with a given edge-orientation. A bioriented Jacobi diagram is a Jacobi diagram with both a vertex-orientation and an edge-orientation.

In the following, Jacobi diagrams will be depicted by a planar immersion of $\Gamma \cup \mathbb{R}$, where $\mathbb{R}$ is a plain vertical line, on which the univalent vertices of $\Gamma$ lie, and the edges of $\Gamma$ are dashed lines. The order on the univalent vertices is given by the vertical direction on the plain vertical line $\mathbb{R}$ from bottom to top. When dealing with vertex-oriented Jacobi diagrams, the vertex-orientation will be given by the counterclockwise order in the plane. An example of Jacobi diagram is given in Figure 3.

![Figure 3: An example of a degree 7 Jacobi diagram.](image-url)

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5In this article, numberings are valued in $3k$ rather than in $2k$ as in [Let19] or [Let20], and only the external edges are numbered.
**Definition 2.3.** Let $k$ be a nonnegative integer. The space $A_k$ is the real vector space spanned by the degree $k$ vertex-oriented Jacobi diagrams up to the equivalence relation spanned by the three following rules:

- **AS relation:** if $\Gamma$ is obtained from $\Gamma$ by reversing the orientation of one trivalent vertex, $[\Gamma] = -[\Gamma]$,

- **IHX relation:** if $\Gamma_1$, $\Gamma_2$ and $\Gamma_3$ can be represented by planar immersions that coincide outside a disk and are as in the second row of Figure 4 inside this disk, $[\Gamma_1] + [\Gamma_2] + [\Gamma_3] = 0$.

- **STU relation:** if $\Gamma$, $\Gamma_1$ and $\Gamma_2$ can be represented by planar immersions that coincide outside a disk and are as in the third row of Figure 4 inside this disk, $[\Gamma] = [\Gamma_1] - [\Gamma_2]$.

We also denote the vector subspace of $A_k$ generated by the classes of diagrams such that any connected component contains a univalent vertex by $\bar{A}_k$. Set $A = \prod_{k \geq 0} A_k$ and $\bar{A} = \prod_{k \geq 0} \bar{A}_k$. Note that $A_0 = \bar{A}_0 = \mathbb{R}.[\emptyset]$, where $[\emptyset]$ is the class of the empty diagram.

**Figure 4:** The three relations of Definition 2.3.

**Definition 2.4.** The **product** of two Jacobi diagrams $\Gamma_1$ and $\Gamma_2$ is the Jacobi diagram $\Gamma$ such that

- the set of univalent vertices of $\Gamma$ is $V_i(\Gamma) = V_i(\Gamma_1) \sqcup V_i(\Gamma_2)$,

- the set of trivalent vertices of $\Gamma$ is $V_e(\Gamma) = V_e(\Gamma_1) \sqcup V_e(\Gamma_2)$,

- the set of edges of $\Gamma$ is $E(\Gamma) = E(\Gamma_1) \sqcup E(\Gamma_2)$,

- the order on $V_i(\Gamma)$ is the unique order compatible with the injections $\left(V_i(\Gamma_j) \hookrightarrow V_i(\Gamma)\right)_{j=1,2}$ such that any element of $V_i(\Gamma_1)$ is before any element of $V_i(\Gamma_2)$.
Note that the data of vertex-orientations (resp. edge-orientations) for $\Gamma_1$ and $\Gamma_2$ induce a natural vertex-orientation (resp. edge-orientation) for their product $\Gamma$.

The product of Jacobi diagrams is compatible with the relations of Definition 2.3. Bar-Natan proved in [BN95a, Theorem 7] that the induced graded algebra structure on $\prod_{k \in \mathbb{N}} A_k$ is commutative. This allows us to define the associated exponential map $\exp : \prod_{k \in \mathbb{N}} A_k \to \prod_{k \in \mathbb{N}} A_k$.

### 2.4 The Conway weight system

Let us first recall the definition of the Conway weight system of [BNG96, Section 3.1].

**Definition 2.5.** Let $\Gamma$ be a Jacobi diagram with only univalent vertices (such a diagram is called a chord diagram). Use the edges of $\Gamma$ to do the surgeries on the line $\mathbb{R}$ as in Figure 5. The obtained manifold is the disjoint union of one line and $c$ circles. The Conway weight system $w_C$ is defined as

$$w_C(\Gamma) = \begin{cases} 1 & \text{if } c = 0, \\ 0 & \text{otherwise.} \end{cases}$$

![Figure 5: Surgeries involved in the definition of $w_C$](image)

Figure 5: Surgeries involved in the definition of $w_C$

![Figure 6: Computation of $w_C(\Gamma)$ for two chord diagrams.](image)

Figure 6: Computation of $w_C(\Gamma)$ for two chord diagrams.

Bar-Natan’s theorem proves that $\prod_{k \in \mathbb{N}} A_k$ is a commutative and cocommutative Hopf algebra, for a given coproduct, but we do not use the coproduct in this article.
In [BNG96], Bar-Natan and Garoufalidis proved that this definition determines a linear form \( w_C : \bigoplus_{k \in \mathbb{N}} \mathcal{A}_k \rightarrow \mathbb{R} \). Set \( w_C(\Gamma) = 0 \) for diagrams with at least one component without univalent vertices. The form \( w_C \) naturally extends to \( w_C : \bigoplus_{k \in \mathbb{N}} \mathcal{A}_k \rightarrow \mathbb{R} \) by sending diagrams with a non-empty trivalent connected component to zero. Chmutov [Chm98, p. 9] proved that \( w_C \) is determined by the following properties.

**Lemma 2.6.** The Conway weight system \( w_C \) is the unique linear form \( w_C : \bigoplus_{k \in \mathbb{N}} \mathcal{A}_k \rightarrow \mathbb{R} \) such that

- \( w_C \) vanishes on \( \mathcal{A}_1 \) and maps \([\emptyset]\) to 1,
- for any integer \( k \geq 2 \), if \( \Gamma_k \) denotes the diagram depicted in Figure 7, \( w_C([\Gamma_k]) = -1 - (-1)^k \),
- if the number of trivalent vertices of \( \Gamma \) is greater than its degree, then \( w_C([\Gamma]) = 0 \),
- if \( \Gamma \) is the product of two diagrams \( \Gamma_1 \) and \( \Gamma_2 \), then \( w_C([\Gamma]) = w_C([\Gamma_1])w_C([\Gamma_2]) \).

In particular, \( w_C \) vanishes on odd-degree diagrams.

![Figure 7: The degree k Jacobi diagram \( \Gamma_k \)](image)

The following result directly follows from [Les20, Corollary 6.36].

**Lemma 2.7.** For any \( k \geq 1 \), let

- \( \mathcal{P}_k \) denote the subspace of \( \mathcal{A}_k \) spanned by the classes of connected diagrams with at least one univalent vertex,
- \( \mathcal{N}_k \) denote the subspace of \( \mathcal{A}_k \) spanned by the classes of non-trivial products, which are products of two non-empty diagrams,
- \( \mathcal{T}_k \) denote the subspace of \( \mathcal{A}_k \) spanned by the classes of degree \( k \) diagrams with at least one trivalent connected component,

\(^7\)A trivalent graph is a graph with only trivalent vertices.
and set \( P_0 = T_0 = \{0\} \) and \( N_0 = \mathbb{R}.[0] \). For any \( k \geq 0 \), the space \( A_k \) splits into \( A_k = P_k \oplus N_k \oplus T_k \). This yields a natural projection \( p^c: \bigoplus_{k \geq 0} A_k \to \bigoplus_{k \geq 0} P_k \).

Define the logarithmic Conway weight system \( w'_C \) as \( w'_C = w_C \circ p^c \). It is characterized as follows.

**Lemma 2.8.** The map \( w'_C \) is the unique linear form \( w'_C: \bigoplus_{k \in \mathbb{N}} A_k \to \mathbb{R} \) such that

- \( w'_C \) vanishes on \( A_0 \oplus A_1 \),
- for any \( k \geq 2 \), \( w'_C([\Gamma_k]) = -1 - (-1)^k \),
- if the number of trivalent vertices of \( \Gamma \) is greater than its degree, then \( w'_C([\Gamma]) = 0 \),
- if \( \Gamma \) is a non-trivial product of Jacobi diagrams, then \( w'_C([\Gamma]) = 0 \).

**Proof.** By Lemma 2.6, \( w'_C \) satisfies these properties, and they characterize it on the summand \( \bigoplus_{k \in \mathbb{N}} P_k \) of \( \bigoplus_{k \in \mathbb{N}} A_k \).

### 2.5 Configuration spaces

#### 2.5.1 For BCR diagrams

**Definition 2.9.** Let \( \psi: \mathbb{R} \hookrightarrow M^o \) be a long knot. Let \( \Gamma \) be a BCR diagram. The open configuration space associated to \( \Gamma \) and \( \psi \) is

\[
C^0_{\Gamma, BCR}(\psi) = \{c: V(\Gamma) \hookrightarrow M^o \mid \text{There exists } c_i: V_i(\Gamma) \hookrightarrow \mathbb{R}, c|_{V_i(\Gamma)} = \psi \circ c_i\}.
\]

This space is oriented as follows. Let \( \Gamma \) be a BCR diagram. For any internal vertex \( v \), let \( dt_v \) denote the coordinate of \( c_i(v) \). For any external vertex \( v \), let \( (dX^i_v)_{i \in \{1,2,3\}} \) denote the coordinates of \( c(v) \) in an oriented chart of \( M^o \). Split any external edge \( e \) into two half-edges \( e_- \) (the tail) and \( e_+ \) (the head). For any external half-edge \( e_\pm \), define a form \( \Omega_\pm \) as follows:

- for the head \( e_+ \) of an edge that is not a leg, going to an external vertex \( v \), \( \Omega_{e_+} = dX^1_v \),
- for the head \( e_+ \) of a leg going to an external vertex \( v \), \( \Omega_{e_+} = dX^2_v \),
- for the tail \( e_- \) of an edge coming from an external vertex \( v \), \( \Omega_{e_-} = dX^3_v \),
- for any external half-edge \( e_\pm \) adjacent to an internal vertex \( v \), \( \Omega_{e_\pm} = dt_v \),

and set \( \varepsilon(\Gamma) = (-1)^{\text{card}(E_v(\Gamma)) + N_T(\Gamma)} \) where \( N_T(\Gamma) \) is the number of trivalent vertices of \( \Gamma \). With these notations, the manifold \( C^0_{\Gamma, BCR}(\psi) \) is oriented by the form \( \Omega(\Gamma) = \varepsilon(\Gamma) \wedge_{e \in E_v(\Gamma)} \left( \Omega_{e_-} \wedge \Omega_{e_+} \right) \).
2.5.2 For Jacobi diagrams

**Definition 2.10.** Let \( \psi: \mathbb{R} \to M^o \) be a long knot. Let \( \Gamma \) be a Jacobi diagram. The open configuration space associated to \( \Gamma \) and \( \psi \) is

\[
C^0_{\Gamma,J}(\psi) = \left\{ c: V(\Gamma) \to M^o \mid \text{There exists an increasing map } c_i: V_i(\Gamma) \to \mathbb{R} \text{ such that } c|_{V_i(\Gamma)} = \psi \circ c_i \right\}.
\]

Let us fix a bioriented Jacobi diagram \( \Gamma \) and orient the manifold \( C^0_{\Gamma,J}(\psi) \). We use the edge-orientation to split any edge \( e \) of \( \Gamma \) in two half-edges \( e^- \) and \( e^+ \) as above and, for each trivalent vertex, we fix an order on the set of its three adjacent half-edges, compatible with the cyclic order given by the vertex-orientation of the diagram. For any half-edge \( e_\pm \) of \( \Gamma \), define forms \( \Omega_{e_\pm} \) such that,

- if \( e^- \) is adjacent to a univalent vertex \( v \), \( \Omega_{e^-} = dt_v \),
- if \( e^+ \) is the \( i \)-th half-edge adjacent to a trivalent vertex \( v \), \( \Omega_{e^+} = dX^i_v \),

and set \( \Omega(\Gamma) = \bigwedge_{e \in E(\Gamma)} (\Omega_{e^-} \wedge \Omega_{e^+}) \). Note that this form does not depend of the "compatible with the vertex-orientation" choice of the orders of the half-edges around each trivalent vertex. Note that the orientation form \( \Omega(\Gamma) \) is multiplied by \(-1\) if we change the orientation of one edge or of one trivalent vertex.

2.6 Propagators and configuration space integrals

Here, we review the definition of \( C_2(M^o) \) in [Les15, Section 2.2]. Let \( C_2(M^o) \) denote the space obtained from \( M^2 \) after the differential blow-up\(^8\) of \( \{ \infty, \infty \} \) and of the closures of \( M^o \times \{ \infty \}, \{ \infty \} \times M^o \) and \( \Delta = \{(x, x) \mid x \in M^o \} \) in the obtained manifold. The manifold with boundary and corners \( C_2(M^o) \) is a compactification of \( C^0_2(M^o) = (M^o)^2 \setminus \Delta \). Let \( G_\tau: \partial C_2(M^o) \to S^2 \) denote the Gauss map as defined in [Les15, Proposition 2.3] or [Les20, Proposition 3.7]. The map \( G_\tau \) is an analogue of the map \( G: C_2(\mathbb{R}^3) \to S^2 \) that extends \( (x, y) \in C^0_2(\mathbb{R}^3) \mapsto \frac{x-y}{\|x-y\|} \in S^2 \), but \( G_\tau \) is only defined on the boundary of the two-point configuration space. It depends on the parallelization \( \tau \) of Section 2.1.

A form \( \omega \) on \( C_2(M^o) \) is *antisymmetric* if \( T^*(\omega) = -\omega \), where \( T: C_2(M^o) \to C_2(M^o) \) is the smooth extension of \( (x, y) \in C^0_2(M^o) \mapsto (y, x) \in C_2(M^o) \). Let us recall the definition of the external propagators of [Let19, Section 2.5].\(^9\

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\(^8\)For example, see [Les15, Section 2.2] for more details on these blow-ups.

\(^9\)It is called \( p_\tau \) in these sources.

\(^{10}\)Here, we require antisymmetric propagators in order to get a simpler formula in Theorem...
**Definition 2.11.** A *propagator* of \( (M^o, \tau) \) is a closed antisymmetric 2-form \( \omega \) on \( C_2(M^o) \) such that there exists a closed 2-form \( \delta_\omega \) on \( S^2 \) with total volume 1 such that \( \omega|_{\partial C_2(M^o)} = G^*_\tau(\delta_\omega) \).

A \( k \)-family of propagators of \( (M^o, \tau) \) is the data of \( 3k \) propagators \( (\omega_i)_{i \in 3k} \) of \( (M^o, \tau) \).

In \( \mathbb{R}^3 \) with its canonical parallelization, the pull-back of the \( SO(3) \)-invariant form on \( S^2 \) with total volume 1 under the Gauss map \( G \) is a propagator.

**Definition 2.12.** For any edge-oriented Jacobi (resp. BCR) diagram \( \Gamma \), define the following maps on the configuration space \( C_{0,J}(\psi) \) (resp. \( C_{0,BCR}(\psi) \)):

- for any edge (resp. external edge) \( e = (v, w) \) of \( \Gamma \), \( p_e \) denotes the map \( (c \mapsto (c(v), c(w))) \) from the configuration space to \( C_2(M^o) \),
- for any pair \( (v, w) \) of distinct univalent (resp. internal) vertices, \( \varepsilon_{v,w} \) denotes the map \( (c \mapsto \text{sign}(c_i(w) - c_i(v))) \) from the configuration space to \( \{-1, 1\} \).

Note that the map \( p_e \) depends on the chosen edge-orientation of a Jacobi diagram.

For any degree \( k \) numbered BCR diagram \( (\Gamma, \sigma) \) and any \( k \)-family \( F = (\omega_i)_{i \in 3k} \) of propagators of \( (M^o, \psi, \tau) \), define a form \( \omega^F(\Gamma, \sigma, \psi) \) on \( C_{0,BCR}(\psi) \) as

\[
\omega^F(\Gamma, \sigma, \psi) = \frac{(-1)^{N_i^-(\Gamma, c)}}{2\text{Card}(E_i(\Gamma))} \bigwedge_{e \in E_i(\Gamma)} p_e^*(\omega_{\sigma(e)}),
\]

where for any \( c \in C_\Gamma(\psi) \), \( N_i^-(\Gamma, c) \) is the number of internal edges from a vertex \( v \) to a vertex \( w \) such that \( \varepsilon_{v,w}(c) < 0 \). Note that the form \( \omega^F(\Gamma, \sigma, \psi) \) is the analogue of the form of the same name defined in [Let19, Section 2.6], when all the internal propagators of this article are equal to the 0-form \( \alpha \) on \( C_2(\mathbb{R}) \) that extends \( (x, y) \in C_2(\mathbb{R}) \mapsto \text{sign}(y - \frac{x}{2}) \in \mathbb{R} \).

For any degree \( k \) numbered edge-oriented Jacobi diagram \( (\Gamma, j) \) and any \( k \)-family \( F = (\omega_i)_{i \in 3k} \) of propagators of \( (M^o, \tau) \), define a form \( \omega^F(\Gamma, j, \psi) \) as

\[
\omega^F(\Gamma, j, \psi) = \bigwedge_{e \in E(\Gamma)} p_e^*(\omega_{j(e)}).
\]

Now, for any numbered BCR diagram, set

\[
I^F(\Gamma, \sigma, \psi) = \int_{C_{0,BCR}(\psi)} \omega^F(\Gamma, \sigma, \psi),
\]

and for any numbered bioriented Jacobi diagram, set

\[
I^F(\Gamma, j, \psi) = \int_{C_{0,J}(\psi)} \omega^F(\Gamma, j, \psi).
\]
These two integrals converge because of the existence of compactifications of the configuration spaces, to which the above forms extend.\footnote{Such compactifications were first introduced by Axelrod and Singer in \cite{AS94} Section 5 for Jacobi diagrams and by Rossi in his thesis \cite{Ros02} Section 2.5 for BCR diagrams. See also \cite{Les20} Section 8 for an extensive description of these compactifications for Jacobi diagrams.}

### 2.7 The perturbative invariant $Z$

Let $\tilde{D}_k$ denote the set of degree $k$ (unoriented) numbered Jacobi diagrams. Let $(\Gamma, j) \in \tilde{D}_k$, and let $F = (\omega_i)_{i \in 3k}$ be a family of propagators of $(M^0, \tau)$. Orient both the edges and the trivalent vertices of $\Gamma$ arbitrarily. Since the propagators are antisymmetric, the orientation of $C^0_{\Gamma,j}(\psi)$ and the sign of $\omega^F(\Gamma, j, \psi)$ both change when changing the orientation of one edge of $\Gamma$, and $w'_C(\Gamma)$ remains unchanged. When the orientation of one trivalent vertex changes, the sign of $w'_C([\Gamma])$ and the orientation of $C^0_{\Gamma,j}(\psi)$ both change and $\omega^F(\Gamma, j, \psi)$ remains unchanged.

Therefore, $I^F(\Gamma, j, \psi)w'_C([\Gamma])$ depends neither of the vertex-orientation, nor of the edge-orientation, and is thus well-defined for $(\Gamma, j) \in \tilde{D}_k$.

Let $j_\emptyset$ denote the only numbering of the empty diagram, let $F_\emptyset$ denote the empty family of propagators, and set $I^{F_\emptyset}(\emptyset, j_\emptyset, \psi) = 1$.

**Theorem 2.13.** Let $(M^0, \tau)$ be a parallelized asymptotic rational homology $\mathbb{R}^3$. Fix a long knot $\psi$ of $M^0$. Fix an integer $k \geq 0$, and a $k$-family $F = (\omega_i)_{i \in 3k}$ of propagators of $(M^0, \tau)$, and set

\[
\left( w_C \circ Z_k^F \right)(\psi, \tau) = \sum_{(\Gamma, j) \in \tilde{D}_k} \frac{(3k - \text{Card}(E(\Gamma)))!}{(3k)!} I^F(\Gamma, j, \psi)w_C([\Gamma]),
\]

\[
\left( w'_C \circ Z_k^F \right)(\psi, \tau) = \sum_{(\Gamma, j) \in \tilde{D}_k} \frac{(3k - \text{Card}(E(\Gamma)))!}{(3k)!} I^F(\Gamma, j, \psi)w'_C([\Gamma]).
\]

- The quantity $w'_C \circ Z_k^F$ does not depend on the choice of the $k$-family $F$ of propagators of $(M^0, \tau)$.
- The quantity $w'_C \circ Z_k = w'_C \circ Z_k^F$ does not depend on the choice of the parallelization $\tau$ of $M^0$.
- The quantity $w'_C \circ Z_k$ only depends on the diffeomorphism class of $(M^0, \psi)$.
- We have the following equality in $\mathbb{R}[[h]]$:

\[
\sum_{k \geq 0} (w_C \circ Z_k)(\psi)h^k = \exp \left( \sum_{k \geq 0} (w'_C \circ Z_k)(\psi)h^k \right).
\]
Proof. \cite{Les20} Theorem 12.32], which is a mild generalization for long knots of
\cite{Les20} Theorem 7.20], implies that
\[
Z^F_{k}(\psi, \tau) = \sum_{(\Gamma, j) \in \tilde{\mathcal{G}}_k} \frac{(3k - \text{Card}(E(\Gamma)))!}{(3k)!} I^F(\Gamma, j, \psi)[\Gamma]
\]
is independent of the choice of the family $F$ of propagators of $(M^o, \tau)$. \cite{Les20, Theorem 12.32}, the last assertion of \cite{Les20, Theorem 12.18} and \cite{Les20, Theorem 6.37} imply that
\[
3_k(\psi) = p^c \circ Z^F_{k}(\psi, \tau) - \frac{1}{4} p_1(\tau) \beta_k - I_\theta(\psi, \tau) \alpha_k
\]
does not depend on the parallelization and is invariant under diffeomorphism, where $\frac{1}{4} p_1(\tau)$ and $I_\theta(\psi, \tau)$ are some real numbers, and where $\alpha_k$ and $\beta_k$ are two elements of $A_k$, called anomalies. \cite{Les20, Propositions 10.14 & 10.20} imply that $\alpha_k$ and $\beta_k$ vanish if $k$ is even. Since $w_C$ vanishes on odd-degree diagrams, this proves that $w^C_k \circ Z^F_{k}$ as defined in the theorem coincides with $w^C_k \circ 3_k$. In particular, this implies the first three assertions of the theorem.

Set $Z = \sum_{k \geq 0} Z_k h^k$ in $\mathcal{A}[[h]]$. Since $w_C$ is multiplicative, $\exp \circ w_C = w_C \circ \exp$. The last assertion of \cite{Les20, Theorem 12.18} and \cite{Les20, Theorem 6.37} imply that $\exp \circ p^c \circ Z = Z$. Therefore,
\[
w_C \circ Z = w_C \circ \exp \circ p^c \circ Z = \exp \circ w_C \circ p^c \circ Z = \exp \circ w'_C \circ Z.
\]

2.8 The BCR invariants

The following theorem is the main result of this article. It is proved in Section 3.

**Theorem 2.14.** Fix an integer $k \geq 2$, a null-homologous long knot $\psi$ of a parallelized asymptotic rational homology $\mathbb{R}^3 (M^o, \tau)$, and a $k$-family $F = (\omega_i)_{i \in \mathbb{N}}$ of propagators of $(M^o, \tau)$. Set
\[
Z^F_{\text{BCR}, k}(\psi, \tau) = \sum_{(\Gamma, \sigma) \in \widetilde{\mathcal{G}}_k} \frac{(3k - \text{Card}(E(\Gamma)))!}{(3k)!} I^F(\Gamma, \sigma, \psi),
\]
where $\widetilde{\mathcal{G}}_k$ denote the set of degree $k$ numbered BCR diagrams up to numbered graph isomorphisms. We have
\[
Z^F_{\text{BCR}, k}(\psi, \tau) = -(w^C_k \circ Z)(\psi).
\]
Note that the coefficients \( \frac{1}{(3k)!} \) in the definition of \( Z_{BCR,k}^F \) replace the \( \frac{1}{(2k)!} \) in the definition of \[ \text{Let19, Theorem 2.10} \] since we allowed numberings to take value in \( 3k \) and since the internal edges are not numbered anymore. The above theorem and Theorem 2.13 imply the following corollary.

**Corollary 2.15.** The quantity \( Z_{BCR,k}^F(\psi) = Z_{BCR,k}^F(\psi, \tau) \) does not depend on the choice of the propagators of \( (M^o, \tau) \), nor of the parallelization, and is invariant under ambient diffeomorphism.

### 2.9 Relation with the Alexander polynomial

Since the propagators defined in [Let20, Section 4] are dual to propagators in the sense of Definition 2.11, [Let20, Theorem 2.31] implies the following theorem.

**Theorem 2.16.** Let \( M^o \) be an asymptotic rational homology \( \mathbb{R}^3 \), and let \( \psi: \mathbb{R} \hookrightarrow M^o \) be a null-homologous long knot of \( M^o \). If \( \Delta_\psi(t) \) denotes the Alexander polynomial of \( \psi \), then

\[
\Delta_\psi(e^h) = \exp \left( - \sum_{k \geq 2} Z_{BCR,k}^F(\psi) h^k \right).
\]

The above theorem and Theorems 2.13 and 2.14 imply the following.

**Corollary 2.17.** If \( M^o \) is an asymptotic rational homology \( \mathbb{R}^3 \), and if \( \psi: \mathbb{R} \hookrightarrow M^o \) is a null-homologous long knot of \( M^o \), then

\[
\Delta_\psi(e^h) = \sum_{k > 0} (w_C \circ Z_k)(\psi) h^k.
\]

### 2.10 Compatibility with the formula in terms of the Kontsevich integral

In this section, we prove that Corollary 2.17 can be stated equivalently in terms of the Kontsevich integral or of the perturbative expansion of the Chern-Simons theory, for long knots of \( \mathbb{R}^3 \).

**Proposition 2.18.** For any long knot \( \psi \) of \( \mathbb{R}^3 \),

\[
(w_C \circ Z_k)(\psi) = (w_C \circ Z_k^K)(\psi)
\]

**Proof.** Let \( \gamma = (\gamma_k)_{k \in \mathbb{N} \setminus \{0\}} \), where \( \gamma_k \) is a combination of degree \( k \) diagrams with exactly two univalent vertices, and \( \gamma_1 \neq 0 \). Lescop [Les02, Definition 2.1] proved the existence of a well-defined morphism \( \Psi(\gamma): \mathcal{A} \to \mathcal{A} \) such that for any Jacobi diagram \( \Gamma \), \( \Psi(\gamma)([\Gamma]) \) is obtained as follows:
• Write $\gamma_k = \sum_{i \in I_k} b(\Gamma_{i,k})[\Gamma_{i,k}]$ for any $k \geq 1$, and set $I = \{\Gamma_{i,k} \mid k \geq 1, i \in I_k\}$.

• In each connected component $\Gamma_j$ of $\Gamma$, fix $\deg(\Gamma_j)$ edges, and let $X$ denote the set of the chosen edges.

• For any map $\xi : X \to I$, let $\Psi^0(\gamma)(\Gamma, \xi)$ denote the graph obtained from $\Gamma$ after replacing each edge $e$ of $X$ with $\xi(e)$.

• Set

$$
\Psi(\gamma)([\Gamma]) = \sum_{\xi : X \to I} \left( \prod_{e \in X} b(\xi(e)) \right) [\Psi^0(\gamma)(\Gamma, \xi)].
$$

With these notations, [Les02, Theorem 2.3] states that $Z = \Psi(2\alpha)(Z^K)$ where $\alpha$ is the Bott-Taubes anomaly. [Les20, Proposition 10.20] implies that $2\alpha_1$ is the class of the diagram $\Gamma_\theta$ with one edge between two univalent vertices.

Proposition 2.18 follows from the following lemma.

**Lemma 2.19.** $w_C \circ \Psi(2\alpha) = w_C$.

**Proof.** Set $w = w_C \circ \Psi(2\alpha)$, and let us check that $w$ satisfies the four properties of Lemma 2.6.

• The first property is immediate.

• For any $k \geq 3$, $2\alpha_k$ is a combination of diagrams with $2k - 2 > k$ trivalent vertices. Therefore, for any $p \geq 2$ and any $\xi : X \to I$ as above, if $\xi$ maps at least one edge to an element different from $\Gamma_\theta$, then the number of trivalent vertices of $\Psi^0(2\alpha)(\Gamma_p, \xi)$ is greater than its degree. This implies that $w([\Gamma_p]) = w_C([\Gamma_p])$.

• If $\Gamma$ contains more than $\deg(\Gamma)$ trivalent vertices, then the same argument implies that $w([\Gamma]) = 0$.

• If $\Gamma$ is the product of two non-empty diagrams $\Gamma_1$ and $\Gamma_2$, set $X_i = X \cap E(\Gamma_i)$ for $i \in \{1, 2\}$. For any $\xi : X \to I$, set $\xi_i = \xi_{|X_i}$ for $i \in \{1, 2\}$. Note that $\Psi^0(2\alpha)(\Gamma, \xi)$ is the product of $\Psi^0(2\alpha)(\Gamma_1, \xi_1)$ and $\Psi^0(2\alpha)(\Gamma_2, \xi_2)$. Therefore, since $w_C$ is multiplicative,

$$
w([\Gamma]) = \sum_{\xi : X \to I} \left( \prod_{e \in X} b(\xi(e)) \right) w_C \left( [\Psi^0(2\alpha)(\Gamma, \xi)] \right)
$$

$$
= \sum_{\xi_1 : X_1 \to I, \xi_2 : X_2 \to I} \left( \prod_{e \in X_1} b(\xi_1(e)) \right) \left( \prod_{e \in X_2} b(\xi_2(e)) \right) w_C \left( [\Psi^0(2\alpha)(\Gamma_1, \xi_1)] \right) w_C \left( [\Psi^0(2\alpha)(\Gamma_2, \xi_2)] \right)
$$

$$
= w([\Gamma_1])w([\Gamma_2]).
$$

\qed
3 Proof of Theorem 2.14

3.1 Definition of a weight system for the BCR invariants

From now on, we fix a parallelized asymptotic rational homology \( \mathbb{R}^3 (M^o, \tau) \), a long knot \( \psi: \mathbb{R}^n \hookrightarrow M^o \), an integer \( k \geq 2 \), and a \( k \)-family \( F = (\omega_i)_{i \in \mathbb{M}} \) of propagators of \( (M^o, \tau) \).

For any BCR diagram \( \Gamma \), let \( \mathcal{D}(\Gamma) \) denote the set of total orders on \( V_i(\Gamma) \). Represent an element of \( \mathcal{D}(\Gamma) \) by the unique increasing bijection \( \rho: V_i(\Gamma) \rightarrow \{1, \ldots, \text{Card}(V_i(\Gamma))\} \). For any degree \( k \) BCR diagram \( \Gamma \), the configuration space \( C^0_{\Gamma, \text{BCR}}(\psi) \) splits into a disjoint union of connected components \( \bigsqcup_{\rho \in \mathcal{D}(\Gamma)} C^0_{\Gamma, \text{BCR}, \rho}(\psi) \) where

\[
C^0_{\Gamma, \text{BCR}, \rho}(\psi) = \left\{ c \in C^0_{\Gamma, \text{BCR}}(\psi) \bigg| \text{for any distinct internal vertices } v, w, \rho(v) < \rho(w) \iff \varepsilon_{v,w}(c) > 0 \right\}.
\]

Any \( \rho \in \mathcal{D}(\Gamma) \) induces a bioriented Jacobi unitrivalent diagram \( \Gamma_{\rho} \) with the same degree as follows. The vertices of \( \Gamma_{\rho} \) are all the vertices of \( \Gamma \) and the edges of \( \Gamma_{\rho} \) are the external edges of \( \Gamma \), so that the internal (resp. external) vertices of \( \Gamma_{\rho} \) yield the univalent (resp. trivalent) vertices of \( \Gamma_{\rho} \). The order of the univalent vertices of \( \Gamma_{\rho} \) is given by \( \rho \). It remains to orient the trivalent vertices of \( \Gamma_{\rho} \), i.e. to fix a cyclic order on the half-edges adjacent to any external vertex \( v \). Let \( c \) denote the external edge of the cycle going to \( v \), let \( \ell \) denote the leg going to \( v \), and let \( f \) denote the external edge of the cycle coming from \( v \). The orientation of \( v \) is given by the cyclic order \((e_+, \ell_+, f_-)\). Note that any numbering \( \sigma \) of \( \Gamma \) yields a canonical numbering \( j_{\rho, \sigma} \) of \( \Gamma_{\rho} \). With these notations, the following lemma is immediate.

**Lemma 3.1.** For any BCR diagram \( \Gamma \), any order \( \rho \in \mathcal{D}(\Gamma) \), and any numbering \( \sigma \) of \( \Gamma \),

\[
\int_{C^0_{\Gamma, \text{BCR}, \rho}(\psi)} \omega^F(\Gamma, \sigma, \psi) = \frac{\varepsilon(\Gamma)\varepsilon_2(\Gamma, \rho)}{2\text{Card}(E_i(\Gamma))} I^F(\Gamma_{\rho}, j_{\rho, \sigma}, \psi),
\]

where \( \varepsilon(\Gamma) \) is defined in Section 2.5.1 and \( \varepsilon_2(\Gamma, \rho) = \prod_{(v,w) \in E_i(\Gamma)} \text{sgn}(\rho(w) - \rho(v)) \in \{\pm 1\} \).

Let us introduce the following notations.

**Notation 3.2.** For any degree \( k \) vertex-oriented numbered Jacobi diagram \( \Gamma_{J, j} \), let \( G(\Gamma_{J, j}, j) \) denote the set of ordered and numbered BCR diagrams \((\Gamma, \sigma, \rho)\) such that \((\Gamma_{\rho}, j_{\rho, \sigma})\) coincides with \((\Gamma_{J, j}, j)\), up to the orientation of the trivalent vertices and after forgetting the orientation of the edges. For such a numbered and ordered BCR diagram, let \( \varepsilon_3(\Gamma_{J, j}, \Gamma, \rho) \in \{\pm 1\} \) be such that \([\Gamma_{\rho}] = \varepsilon_3(\Gamma_{J, j}, \Gamma, \rho)[\Gamma_{J, j}] \). Set

\[
w_{\text{BCR}}(\Gamma_{J, j}) = \sum_{(\Gamma, \sigma, \rho) \in G(\Gamma_{J, j}, j)} \frac{\varepsilon(\Gamma)\varepsilon_2(\Gamma, \rho)\varepsilon_3(\Gamma_{J, j}, \Gamma, \rho)}{2^{2k - \text{Card}(E(\Gamma_{J, j}))}},
\]
and note that \( w_{BCR}(\Gamma_J, j) \) does not depend on \( j \). Denote it by \( w_{BCR}(\Gamma_J) \).

We are going to prove the following proposition.

**Proposition 3.3.** For any Jacobi diagram \( \Gamma_J \), \( w_{BCR}(\Gamma_J) = -w'_C([\Gamma_J]) \).

Note that the above proposition and Lemma 3.1 imply Theorem 2.14. We now prove Proposition 3.3 until the end of this article. In the next subsections, we check that \((-w_{BCR})\) induces a linear map \( A_k \rightarrow \mathbb{R} \) that satisfies the properties of Lemma 2.8. Note that it is immediate that \((-w_{BCR})\) vanishes on degree 0 and 1 diagrams, since a BCR diagram is non-empty, and since the only degree 1 BCR diagram is counted with opposite signs when the order of its two internal vertices is changed. The following lemma gives an example of diagrams with non-trivial \( w_{BCR} \) and proves the second and third property of Lemma 2.8.

**Lemma 3.4.**

1. If \( \Gamma_J \) is a degree \( k \) Jacobi diagram and has more than \( k \) trivalent vertices, then \( w_{BCR}(\Gamma_J) = 0 \).

2. If \( \Gamma_k \) is the graph of Figure 7, then \( w_{BCR}(\Gamma_k) = 1 + (-1)^k \).

**Proof.** The first point is immediate since any external vertex of a BCR diagram has a univalent neighbour, so that there cannot be more than \( k \) external vertices in a degree \( k \) BCR diagram.

For the second point, given a numbering \( j \) of \( \Gamma_k \), there are exactly two elements in \( G(\Gamma_k, j) \), and they are given by \( (\Gamma_{BCR}^k, \rho_a, j_a) \) and \( (\Gamma_{BCR}^k, \rho_b, j_b) \), as depicted in Figure 8 where \( v_i \) denotes the vertex \( \rho^{-1}(i) \), and the numberings \( j_a \) and \( j_b \) are uniquely determined by \( j \).

![Figure 8](image-url)

Note that \( \varepsilon_2(\Gamma_{BCR}^k, \rho_a) = \varepsilon_2(\Gamma_{BCR}^k, \rho_b) = 1 \), and that \( \varepsilon(\Gamma_{BCR}^k) = (-1)^k \). The graph \( (\Gamma_{BCR}^k)_{\rho_a} \) has exactly the same vertex-orientation than \( \Gamma_k \), and \( (\Gamma_{BCR}^k)_{\rho_b} \) has the opposite orientation at each trivalent vertex, so that \((-1)^k[(\Gamma_{BCR}^k)_{\rho_b}] = [(\Gamma_{BCR}^k)_{\rho_a}] = [\Gamma_k] \). Therefore, \( w_{BCR}(\Gamma_k) = 1 + (-1)^k \). \( \square \)
3.2 Linear extension of $w_{BCR}$ to $\mathcal{A}_k$

**Lemma 3.5.** The map $w_{BCR}$ induces a linear form on $\mathcal{A}_k$.

**Proof.** It suffices to prove that $w_{BCR}$ is compatible with the relations of Definition 2.3. The compatibility with the AS relation is immediate. For diagrams with only trivalent vertices, the IHX relation is immediate since $w_{BCR}$ is zero on the three involved diagrams. Bar-Natan [BN95a, Theorem 6] proved that the IHX relation is a consequence of the STU relation for diagrams of $\mathcal{A}_k$.

Let us now prove the STU relation. Let $\Gamma_j$, $\Gamma_j^{(1)}$ and $\Gamma_j^{(2)}$ be three Jacobi diagrams connected by the STU relation of Definition 2.3, as in Figure 9. We are going to prove that $w_{BCR}(\Gamma_j) = w_{BCR}(\Gamma_j^{(1)}) - w_{BCR}(\Gamma_j^{(2)})$. Let the vertices $v$, $w$, $t$ and $u$ and the edges $e$, $f$ and $h$ be as in Figure 9.

![Figure 9](image)

We have a natural identification $E(\Gamma_j^{(1)}) \cong E(\Gamma_j^{(2)}) \cong E(\Gamma_j) \setminus \{h\}$. Fix a numbering $j_1$ of $\Gamma_j^{(1)}$, and let $j_2$ the associated numbering of $\Gamma_j^{(2)}$. Fix $i_0 \in 3k \setminus j_1(E(\Gamma_j))$ and let $j$ denote the numbering of $\Gamma_j$ such that $j(h) = i_0$ and that induces $j_1$ on $\Gamma_j^{(1)}$.

For $i \in \{1, 2\}$, split $\mathcal{G}(\Gamma_j^{(i)}, j_i)$ into $\mathcal{G}_1(\Gamma_j^{(i)}, j_i)$ and $\mathcal{G}_2(\Gamma_j^{(i)}, j_i)$, where

$$\mathcal{G}_1(\Gamma_j^{(i)}, j_i) = \{(\Gamma, \sigma, \rho) \in \mathcal{G}(\Gamma_j^{(i)}, j_i) \mid \text{There is exactly one internal edge between } v \text{ and } w\},$$

$$\mathcal{G}_2(\Gamma_j^{(i)}, j_i) = \mathcal{G}(\Gamma_j^{(i)}, j_i) \setminus \mathcal{G}_1(\Gamma_j^{(i)}, j_i).$$

For any $(\Gamma, \sigma, \rho) \in \mathcal{G}(\Gamma_j^{(i)}, j_i)$, let $\rho^*$ denote the ordering $\rho \circ \rho_{v,w}$, where $\rho_{v,w}$ is the transposition of $v$ and $w$. This induces a bijection $((\Gamma, \sigma, \rho) \in \mathcal{G}(\Gamma_j^{(i)}, j_i) \mapsto (\Gamma, \sigma, \rho^*) \in \mathcal{G}(\Gamma_j^{(2)}, j_{2i}))$, which preserves the above decomposition $\mathcal{G}(\Gamma_j^{(i)}, j_i) = \mathcal{G}_1(\Gamma_j^{(i)}, j_i) \sqcup \mathcal{G}_2(\Gamma_j^{(i)}, j_i)$. Note that $\varepsilon_3(\Gamma_j^{(i)}, \Gamma, \rho) = \varepsilon_3(\Gamma_j^{(2)}, \Gamma, \rho^*)$ and that

$$\varepsilon_2(\Gamma, \rho^*) = \begin{cases} -\varepsilon_2(\Gamma, \rho) & \text{if } (\Gamma, \sigma, \rho) \in \mathcal{G}_1(\Gamma_j^{(1)}, j_1), \\ \varepsilon_2(\Gamma, \rho) & \text{if } (\Gamma, \sigma, \rho) \in \mathcal{G}_2(\Gamma_j^{(1)}, j_1). \end{cases}$$
This yields \( w_{BCR}(\Gamma_{j}^{(1)}) - w_{BCR}(\Gamma_{j}^{(2)}) = 2 \sum_{(\Gamma, \sigma, \rho) \in \mathcal{G}_{1}(\Gamma_{j}^{(1)}, j_{1})} \frac{\varepsilon(\Gamma)\varepsilon_{2}(\Gamma, \rho)\varepsilon_{3}(\Gamma_{j}^{(1)}, \Gamma, \rho)}{2^{2k - \text{Card}(E(\Gamma_{j}^{(1)}))}}. \)

Let \( \mathcal{G}_{b}^{a}(\Gamma_{j}^{(1)}, j_{1}) \) denote the set of ordered and numbered BCR diagrams \((\Gamma, \sigma, \rho) \in \mathcal{G}_{1}(\Gamma_{j}^{(1)}, j_{1})\) such that \( v \) and \( w \) are both trivalent in \( \Gamma \), and set \( \mathcal{G}_{b}^{a}(\Gamma_{j}^{(1)}, j_{1}) = \mathcal{G}_{1}(\Gamma_{j}^{(1)}, j_{1}) \setminus \mathcal{G}_{1}(\Gamma_{j}^{(1)}, j_{1}) \). For any \((\Gamma, \sigma, \rho) \in \mathcal{G}_{b}^{a}(\Gamma_{j}^{(1)}, j_{1})\), let \( x \) and \( y \) denote the univalent vertices respectively adjacent to \( v \) and \( w \) as in Figure 10, and set \( \rho^{*} = \rho \circ \rho_{x,y} \circ \rho_{e,w} \) and \( \sigma^{*} = \sigma \circ \rho_{e,f} \). Since there is only one internal edge from \( v \) to \( w \), there are internal edges \((v', v)\) and \((w, w')\) where \( v' \) and \( w' \) are neither \( v \) nor \( w \), and \( \varepsilon_{2}(\Gamma, \rho^{*}) = -\varepsilon_{2}(\Gamma, \rho) \). The orientation of the possible external vertices did not change, so \( \varepsilon_{3}(\Gamma_{j}^{(1)}, \Gamma, \rho) = \varepsilon_{3}(\Gamma_{j}^{(1)}, \Gamma, \rho^{*}) \). Since \( (\Gamma, \sigma, \rho) \in \mathcal{G}_{b}^{a}(\Gamma_{j}^{(1)}, j_{1}) \mapsto (\Gamma, \sigma^{*}, \rho^{*}) \in \mathcal{G}_{b}^{a}(\Gamma_{j}^{(1)}, j_{1}) \) is a bijection, this yields

\[ \sum_{(\Gamma, \sigma, \rho) \in \mathcal{G}_{b}^{a}(\Gamma_{j}^{(1)}, j_{1})} \frac{\varepsilon(\Gamma)\varepsilon_{2}(\Gamma, \rho)\varepsilon_{3}(\Gamma_{j}^{(1)}, \Gamma, \rho)}{2^{2k - \text{Card}(E(\Gamma_{j}^{(1)}))}} = 0. \]

Figure 10: Notations for a graph of \( \mathcal{G}_{a}^{a}(\Gamma_{j}^{(1)}, j_{1}). \)

We now define a bijection

\[ (\Gamma, \sigma, \rho) \in \mathcal{G}_{b}^{a}(\Gamma_{j}^{(1)}, j_{1}) \mapsto (\Gamma^{*}, \sigma^{*}, \rho^{*}) \in \mathcal{G}(\Gamma_{j}, j) \].

For any \((\Gamma, \sigma, \rho) \in \mathcal{G}_{b}^{a}(\Gamma_{j}^{(1)}, j_{1})\), let \( \Gamma^{*} \) denote the BCR diagram defined as follows:

- if \( v \) and \( w \) are both bivalent, \( \Gamma^{*} \) is obtained as in Figure 11.
• if $v$ is trivalent and $w$ bivalent, $\Gamma^*$ is obtained as in Figure 12

\begin{figure}[h]
\centering
\includegraphics[width=0.8\textwidth]{figure12}
\caption{Figure 12}
\end{figure}

\begin{figure}[h]
\centering
\includegraphics[width=0.8\textwidth]{figure13}
\caption{Figure 13}
\end{figure}

• if $v$ is bivalent and $w$ trivalent, $\Gamma^*$ is obtained as in Figure 13

The ordering $\rho$ induces a natural ordering $\rho^*$ of $\Gamma^*$. Note that $E_e(\Gamma^*) \setminus \{h\} \cong E_e(\Gamma)$. Let $\sigma^*$ denote the numbering of $\Gamma^*$ that coincide with $\sigma$ on $E_e(\Gamma^*) \setminus \{h\}$ and such that $\sigma^*(h) = i_0$, so that $(\Gamma^*, \sigma^*, \rho^*) \in \mathcal{G}(\Gamma, J)$. Let us check that $\varepsilon(\Gamma)\varepsilon_2(\Gamma, \rho)\varepsilon_3(\Gamma^{(1)}_J, \Gamma, \rho) = \varepsilon(\Gamma^*)\varepsilon_2(\Gamma^*, \rho^*)\varepsilon_3(\Gamma_J, \Gamma^*, \rho^*)$.

• If $v$ and $w$ are bivalent, since $\Gamma^*$ has one more external edge and one more trivalent vertex than $\Gamma$, $\varepsilon(\Gamma) = \varepsilon(\Gamma^*)$.

  – If there is an internal edge from $v$ to $w$ as in the first row of Figure 11, $\varepsilon_2(\Gamma^*, \rho^*) = \varepsilon_2(\Gamma, \rho)$ since $\rho(w) - \rho(v)$ is positive. The orientation of the vertex $t$ is given by the cyclic order $(e, h, f)$ as in $\Gamma_J$, and $\varepsilon_3(\Gamma_J, \Gamma^*, \rho^*) = \varepsilon_3(\Gamma^{(1)}_J, \Gamma, \rho)$.  

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Lemma 3.6. Let \( \Gamma \) be a non-trivial product of diagrams, then \( w_{BCR}(\Gamma) = 0 \).

Proof. Let \( \Gamma \) be a non-trivial product, and let \( \Gamma^{(1)} \) and \( \Gamma^{(2)} \) be two Jacobi diagrams such that \( \Gamma = \Gamma^{(1)} \circ \Gamma^{(2)} \). Let \( j \) be a numbering of \( \Gamma \). We are going to define an involution \( ((\Gamma, \sigma, \rho) \in \mathcal{G}(\Gamma, j) \mapsto (\Gamma^*, \sigma^*, \rho^*) \in \mathcal{G}(\Gamma, j)) \).

For any \( (\Gamma, \sigma, \rho) \in \mathcal{G}(\Gamma, j) \), let \( V(\Gamma) = V_1 \cup V_2 \) be the partition of \( V(\Gamma) \) such that the vertices of \( V_1 \) correspond to the vertices of \( \Gamma^{(i)} \) in \( \Gamma \). Let \( \Gamma' \) denote the graph obtained from \( \Gamma \) by keeping only the vertices of \( V_2 \) and the edges between such vertices, and let \( \Gamma_c \) denote the connected component of \( \Gamma' \) that contains

3.3 Restriction to connected diagrams

\[ w_{BCR}(\Gamma) = \frac{\varepsilon(\Gamma)\varepsilon(\Gamma^*)\varepsilon(\Gamma^{(1)}, \Gamma, \rho)}{2^{2k-1-\text{Card}(E(\Gamma^{(1)}))}} = \frac{\varepsilon(\Gamma^*)\varepsilon(\Gamma^*)\varepsilon(\Gamma^{(2)}, \Gamma^*, \rho^*)}{2^{2k-\text{Card}(E(\Gamma))}} \]

This yields \( w_{BCR}(\Gamma^{(1)}) - w_{BCR}(\Gamma^{(2)}) = w_{BCR}(\Gamma) \) and concludes the proof of Lemma 3.5. \( \square \)
the external edge with minimal $\sigma$. Let $V_3$ denote the set of vertices of $\Gamma_c$. By construction of BCR diagrams, there is exactly one internal edge $e_1 = (v, w)$ from $V_1$ to an internal vertex $w$ of $V_3$. Note that any element of $\rho(V_1 \cap V_i(\Gamma))$ is before any element of $\rho(V_3 \cap V_i(\Gamma))$.

Let $\Gamma^*$ denote the BCR diagram defined as follows.

- If $w$ is bivalent in $\Gamma$, and if we have an external edge $e = (w, w_2)$ from $w$ to another bivalent vertex $w_2$, then $w_2$ is in $V_3$ since $\Gamma_c$ is connected. Set $w_1 = w$. The graph $\Gamma^*$ is obtained from $\Gamma$ after replacing the internal edge $e_1 = (v, w_1)$ with an internal edge from $v$ to $w_2$ as in Figure 14, and $\rho^*$ and $\sigma^*$ are naturally deduced from $\rho$ and $\sigma$. Since $w_1$ and $w_2$ are in $V_3$ and $v$ in $V_1$, $\rho(w_2) - \rho(v)$ and $\rho(w_1) - \rho(v)$ have the same sign, and $\varepsilon_2(\Gamma, \rho^*) = \varepsilon_2(\Gamma, \rho)$. Furthermore, we have $\varepsilon(\Gamma^*) = -\varepsilon(\Gamma)$ since $\Gamma^*$ has the same external edges as $\Gamma$ but one more (internal) trivalent vertex. Since nothing changed around the external trivalent vertices, $\varepsilon_3(\Gamma, \Gamma^*, \rho^*) = \varepsilon_3(\Gamma, \Gamma, \rho)$.

- If $w$ is trivalent in $\Gamma$, set $w_2 = w$. There is a leg $e = (w_1, w_2)$ from a univalent vertex $w_1$ to $w_2$. Since $\Gamma_c$ is connected, $w_1$ is in $V_3$. In this case, $\Gamma^*$ is the graph obtained from $\Gamma$ after replacing the internal edge $e_1 = (v, w)$ with $(v, w_1)$ as in Figure 14, and $\rho^*$ and $\sigma^*$ are naturally determined by $\rho$ and $\sigma$. As above, we have $(\varepsilon(\Gamma^*), \varepsilon_2(\Gamma^*, \rho^*), \varepsilon_3(\Gamma, \Gamma^*, \rho^*)) = (-\varepsilon(\Gamma), \varepsilon_2(\Gamma, \rho), \varepsilon_3(\Gamma, \Gamma, \rho))$.

- Otherwise, $\Gamma_c$ is as in Figure 15 and $w$ is connected to an external trivalent vertex $t$, where a leg from a univalent vertex $x$ arrives. In this case, $\Gamma^* = \Gamma$ and $\rho^* = \rho \circ \rho_{x,w}$, where $\rho_{x,w}$ is the transposition of $x$ and $w$. We have $\rho^*(w) = \rho(w_2) - \rho(v)$ and $\rho(x) - \rho(v)$. This expression has the same (positive) sign as $\rho(w) - \rho(v)$ since $x$ and $w$ are in $V_3$ and $v$ is in $V_1$. Therefore, $\varepsilon_2(\Gamma, \rho^*) = \varepsilon_2(\Gamma, \rho)$. Since we only changed the order of two internal vertices adjacent to the trivalent vertex $t$, we have $\varepsilon_3(\Gamma, \Gamma^*, \rho^*) = \varepsilon_3(\Gamma, \Gamma, \rho)$.
This yields an involution \(((\Gamma, \sigma, \rho) \in G(\Gamma_J, j) \mapsto (\Gamma^*, \sigma^*, \rho^*) \in G(\Gamma_J, j))\) as announced and we have \(\varepsilon(\Gamma^*) \varepsilon_2(\Gamma^*, \rho^*) \varepsilon_3(\Gamma_J, \Gamma^*, \rho^*) = -\varepsilon(\Gamma) \varepsilon_2(\Gamma, \rho) \varepsilon_3(\Gamma_J, \Gamma, \rho)\) for any \((\Gamma, \sigma, \rho) \in G(\Gamma_J, j)\). This concludes the proof of the lemma.

Lemmas 3.4, 3.5 and 3.6 and Lemma 2.8 conclude the proof of Proposition 3.3 so that Theorem 2.14 is proved.

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