Uniform continuity of entropy rate with respect to the $\bar{f}$-pseudometric

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Abstract

Assume that a sequence $x = x_0, x_1, \ldots$ is frequency-typical for a finite-valued stationary stochastic process $X$. We prove that the function associating to $x$ the entropy-rate $\bar{H}(X)$ of $X$ is uniformly continuous when one endows the set of all frequency-typical sequences with the $\bar{f}$ pseudometric. As a consequence, we obtain the same result for the $\bar{d}$ pseudometric. We also give an alternative proof of the Abramov formula for the Kolmogorov-Sinai entropy of the induced measure-preserving transformation.

1 Introduction

Assume that $\Lambda$ stands for a finite set and we are given two $\Lambda$-valued stationary stochastic processes, $X = (X_i)_{i=0}^\infty$ and $Y = (Y_i)_{i=0}^\infty$. Let $x = (x_i)_{i=0}^\infty$ and $y = (y_i)_{i=0}^\infty$ be frequency-typical realisations (samples) of, respectively $X$ and $Y$. Under what conditions on $x$ and $y$ can we conclude that the entropy rates $\bar{H}(X)$ and $\bar{H}(Y)$ are close?

Note that the above question focuses on properties of individual frequency-typical trajectories to determine some global characteristics of stationary processes. This point of view was popularised by Shields [18] and Weiss [23], who presented central issues in ergodic theory and information theory in that “sample path” spirit. Here, we are interested in measurements of distortion determining a pseudometric such that endowing the space of all frequency-typical sequences with that pseudometric turns the entropy rate of the generated processes into a continuous function. Our problem is motivated by recent results in the dynamical systems theory, where the following construction obtains specific invariant measures: In the first step, one finds a sequence of frequency-typical orbits converging in an appropriate sense. In the second step, one demonstrates that the limit of the approximating sequence is a frequency-typical orbit generating the sought measure. The question, whether the Kolmogorov-Sinai entropy of the limiting measure is the limit of entropies of measures generated by frequency-typical orbits in the approximating sequence reduces to the question stated in the first paragraph of our paper. By the nature of the construction, we work with concrete realisations (individual samples of the random processes) and pseudometrics $\bar{d}$ and $\bar{f}$ described below. We stress that we are looking for results valid for any frequency-typical sample, while the so far existing results consider almost all samples and often assume also ergodicity of the processes. We find the information-theoretic formulation a natural one for our problem.

By the entropy rate of a stationary $\Lambda$-valued process $Z = (Z_i)_{i=0}^\infty$, we mean

$$ \bar{H}(Z) = \lim_{n \to \infty} \frac{1}{n} \sum_{\lambda_1, \ldots, \lambda_n \in \Lambda^n} \eta\left( \frac{\mu \left( Z_0 = \lambda_1, \ldots, Z_{n-1} = \lambda_{n-1}, Z_n = \lambda_n \right)}{n} \right), \quad (1) $$
where $\eta$ stands for the entropy function given by $\eta(0) = 0$ and $\eta(t) = -t \log t$ for $t > 0$. For more details (in particular, for the justification that the limit exists) see [5] Lemma 3.8] or [13] Sec. I.6.b) or [3].

To measure the distortion between sequences $x = (x_i)_{i=0}^\infty$ and $y = (y_j)_{j=0}^\infty$ over a common alphabet $\Lambda$ we first introduce a fidelity criterion, that is a sequence $(\rho_n)_{n=1}^\infty$ of distortion measures with $\rho_n$ defined on $\Lambda^n \times \Lambda^n$ for $n = 1, 2, \ldots$ and then take the limit superior as $n \to \infty$ of average (per-symbol) distortion obtaining

$$
\tilde{\rho}(x, y) = \limsup_{n \to \infty} \frac{1}{n} \rho_n(x_0 x_1 \ldots x_{n-1}, y_0 y_1 \ldots y_{n-1}).
$$

This idea goes back to Shannon [19, 20], see also [5] Sec. 5.2, p. 120.

The simplest and most common example of the fidelity criterion is based on a distortion measure provided by the (additive) Hamming distance given for $n \in \mathbb{N}$ and words $x_0 x_1 \ldots x_{n-1}, y_0 y_1 \ldots y_{n-1}$ in $\Lambda^n$ by

$$
d_n(x_0 x_1 \ldots x_{n-1}, y_0 y_1 \ldots y_{n-1}) = \left| \{ 0 \leq j < n : x_j \neq y_j \} \right|,
$$

which is the number of coordinates in which the sequences $x_0 x_1 \ldots x_{n-1}$ and $y_0 y_1 \ldots y_{n-1}$ differ. Note that the Hamming distance is indeed additive in the following sense: for every $n, m \in \mathbb{N}$ and $x_1 x_2 \ldots x_{n+m}, y_1 y_2 \ldots y_{n+m}$ in $\Lambda^{n+m}$ it holds

$$
d_{n+m}(x_i y_i | i=1, j=1 | i=n+1, j=n+1) = d_n(x_i y_i | i=1, j=1 | i=n, j=n) + d_m(x_i y_i | i=n+1, j=n+1).
$$

(2)

A less known fidelity criterion is based on a distortion measure provided by the sequence $(f_n)_{n=1}^\infty$ of $f$-distances (cf. [7] p. 94]). For $n \in \mathbb{N}$ the $f_n$-distance between sequences $x_0 x_1 \ldots x_{n-1}, y_0 y_1 \ldots y_{n-1}$ in $\Lambda^n$ is simply the number of letters one must remove from each sequence so that the remaining words match, that is,

$$
f_n(x_0 x_1 \ldots x_{n-1}, y_0 y_1 \ldots y_{n-1}) = n - k,
$$

where $k$ is the largest integer such that for some $0 \leq i(1) < i(2) < \ldots < i(k) < n$ and $0 \leq j(1) < j(2) < \ldots < j(k) < n$ it holds $x_{i(s)} = y_{j(s)}$ for $s = 1, \ldots, k$. The $f$-distance is sometimes called an edit distance, since it depends on the number of edits (character deletions) that have to be performed in order to obtain matching sequences. Note that other functions are also known as edit distances for example the Levenshtein distance. The family of $f$-distances lacks the additivity property [2] which, vaguely speaking, is a source of difficulties when working with $f$.

The $\tilde{d}$ ($\bar{d}$-bar) pseudometric between sequences $x = (x_i)_{i=0}^\infty$ and $y = (y_j)_{j=0}^\infty$ in $\Lambda^\mathbb{N}$ (see also [6] [12]) is given by

$$
\tilde{d}(x, y) = \limsup_{n \to \infty} \tilde{d}_n(x_0 x_1 \ldots x_{n-1}, y_0 y_1 \ldots y_{n-1}),
$$

(3)

where $\tilde{d}_n$ is the average (or per-letter) Hamming distance given by

$$
\tilde{d}_n(x_0 x_1 \ldots x_{n-1}, y_0 y_1 \ldots y_{n-1}) = \frac{1}{n} d_n(x_0 x_1 \ldots x_{n-1}, y_0 y_1 \ldots y_{n-1}).
$$

Note that $\tilde{d}$ also appears in [5] Section 5.3, p. 121] under the name of sequence distortion, where it is denoted by $\rho_\infty$.
Similarly, replacing $\tilde{d}$ in (3) by the average (or per-letter) $f$-distance $\tilde{f}_n = \frac{1}{n} f_n$ we obtain the $\tilde{f}$ ($f$-bar) pseudometric on $\Lambda^\infty$ defined (see also [7] p. 92]) for sequences $x = (x_i)_{i=0}^\infty$ and $y = (y_i)_{i=0}^\infty$ in $\Lambda^\infty$ as

$$\tilde{f}(x, y) = \limsup_{n \to \infty} \tilde{f}_n(x_0 x_1 \ldots x_{n-1}, y_0 y_1 \ldots y_{n-1}).$$

(4)

Clearly, for every pair of sequences $x = (x_i)_{i=0}^\infty$ and $y = (y_i)_{i=0}^\infty$ in $\Lambda^\infty$ we have

$$\tilde{f}(x, y) \leq \tilde{d}(x, y).$$

(5)

It is also easy to see that the pseudometrics given in (3) and (4) are not equivalent, because taking $\Lambda = \{0, 1\}$ and $x = (01)^\infty$, $y = (10)^\infty$ it holds that $\tilde{d}(x, y) = 1$, while $\tilde{f}(x, y) = 0$.

The pseudometrics $\tilde{d}$ and $\tilde{f}$ can be seen as sample sequence versions of the metrics between random processes, which unfortunately are also denoted by $\tilde{d}$ and $\tilde{f}$ (see [5] Thm. 5.1, [7] Def. 334, Def. 454], [13] Def. 2.4, [15] Def. 7.3, [18] p. 92]). In order to resolve this notational conflict, we will henceforth denote these distances between processes as $\tilde{d}_M$ and $\tilde{f}_M$. The definition of $\tilde{d}_M$ is a variant of the construction of the Kantorovich (or the Kantorovich-Rubinstein, or the Wasserstein vel Vasershtein) optimal transport metric between two processes, where available transportation plans are shift-invariant (stationary) joinings of the processes (see [15, 21, 22]). Ornstein’s $\tilde{d}_M$ metric plays a prominent role in the study of classification problem of Bernoulli (or the Kantorovich-Rubinstein, or the Wasserstein vel Vasershtein) optimal transport metric between two processes and there exist frequency-typical sample sequences as $\tilde{d}_M$ and $\tilde{f}_M$ as well as their topological counterparts (known as the Besicovitch and Feldman-Katok pseudometrics), proved to be very useful in constructions and exploration of stationary processes and invariant measures for continuous maps on compact metric spaces (see [10, 11] and references therein).

Our main result states that the function which takes a frequency Typical sequence $z = (z_i)_{i=0}^\infty$ and associates to $z$ the entropy rate $\hat{H}(Z)$ of the stationary process $Z = (Z_i)_{i=0}^\infty$ generated by $z$ turns out to be uniformly continuous when we endow the set of all frequency Typical sequences in $\Lambda^\infty$ with the pseudometric $\tilde{f}$.

Theorem 1.1. For every finite alphabet $\Lambda$ and $\epsilon > 0$ there is $\delta > 0$ such that if $X$ and $X'$ are $\Lambda$-valued stationary processes and there exist frequency Typical sample sequences $x$ of $X$ and $x'$ of $X'$ satisfying $\tilde{f}(x, x') < \delta$, then $|\hat{H}(X) - \hat{H}(X')| < \epsilon$.

As a corollary of the inequality (5) we immediately obtain an analogous result for $\tilde{d}$.

Theorem 1.2. For every finite alphabet $\Lambda$ and $\epsilon > 0$ there is $\delta > 0$ such that if $X$ and $X'$ are $\Lambda$-valued stationary processes and there exist frequency Typical sample sequences $x$ of $X$ and $x'$ of $X'$ satisfying $\tilde{d}(x, x') < \delta$, then $|\hat{H}(X) - \hat{H}(X')| < \epsilon$.

We were unable to find Theorem 1.1 in the present form in the literature, that is, without any assumptions on the ergodicity of the processes and assuming only the existence of a pair of $f$-close frequency Typical sequences. Note that the continuity of
the entropy-rate function as the function on the space of stationary processes endowed with $d_M$ is well-known, see [5, Corollary 6.1], [7, Theorem 385], or [15, Thm. 7.9] (although often stated only for ergodic processes, omitting uniform continuity as in [18, Thm. 1.9.16]). The analogous statement for $f_M$ is known only for ergodic processes, see [7, Theorem 455] or [13, Prop. 3.4] and both sources use Abramov’s formula for the Kolmogorov-Sinai entropy of induced transformation. To obtain Theorem 1.1 or Theorem 1.2 from the existing results about $d_M$ or $f_M$, one has to show that for every $\epsilon > 0$ there is $\delta > 0$ such that the existence of two frequency-typical sequences that are $\delta$ apart with respect to $f_M$ (respectively, $d_M$) metric on the space of processes. This is known for $d$ and $d_M$, see [15, Theorem 1.9.10], even without ergodicity [15, Thm. 7.10], but known only for ergodic processes for $f$ and $f_M$ (see [13, Prop. 2.6 & 2.7]). Furthermore, the proof in [13] uses the Shannon-McMillan-Breiman theorem for ergodic measures in a crucial way, hence works only for ergodic processes.

In contrast, our demonstration of Theorem 1.1 works for not necessarily ergodic processes, contains the $d$ case as a particular case, and it is more direct even for $d$ (it does not involve the use of auxiliary metrics $d_M$ or $f_M$ on processes, does not require the Shannon-McMillan-Breiman theorem, nor the Abramov formula and conditional expectations). We use only the elementary properties of entropy. Our proof also immediately implies uniform continuity of the entropy rate function when the space of processes is endowed with the metric $d_M$. Again, we do not have to assume ergodicity.

**Theorem 1.3.** For every finite alphabet $\Lambda$ and $\epsilon > 0$ there is $\delta > 0$ such that if $X$ and $X'$ are $\Lambda$-valued stationary processes and $d_M(X, X') < \delta$, then $|H(X) - H(X')| < \epsilon$.

This holds because for any two processes $X$ and $Y$ we can always find realisations $x$ and $x'$ such that $d_M(X, X') = d(x, x')$ (this is easy for ergodic processes, see [13], for not necessarily ergodic processes it follows from [11, Thm. 2.10] and joining characterisation of $d_M$). Similarly, we obtain a new proof of the uniform continuity of the entropy rate function when the space of ergodic processes is endowed with the metric $f_M$. We have to restrict to ergodic processes, because the existence of sample sequences $x$ and $x'$ such that $f(x, x') \leq f_M(X, X')$ is known only for ergodic processes, see [13, Prop. 2.5].

**Theorem 1.4.** For every finite alphabet $\Lambda$ and $\epsilon > 0$ there is $\delta > 0$ such that if $X$ and $X'$ are ergodic $\Lambda$-valued stationary processes and $f_M(X, X') < \delta$, then $|H(X) - H(X')| < \epsilon$.

As a by-product of our approach, we obtain a new proof of the Abramov formula for the entropy of the induced transformation in general, not necessarily ergodic case (see Theorem 5.1). The result in such a generality (attributed to Scheller in [9]) is usually presented in the literature with an additional ergodicity assumption. Our demonstration requires only basic properties of the entropy conditioned on a countable partition. The usual proof uses conditional expectation and conditioning on $\sigma$-algebras.

## 2 Basic Facts and Notation

### Partitions and Names

A **measure preserving system** is a quadruple $(X, \mathcal{X}, \mu, T)$, where $(X, \mathcal{X}, \mu)$ is a standard probability space and $T : X \to X$ preserves $\mu$. Let $\mathcal{P} = \{P_\alpha : \alpha \in \Lambda\}$ be a
measurable partition of $X$ with $\Lambda \subseteq \mathbb{N}_0$. For $a \in \Lambda$ we write $[a]$ to denote $P_a$ and refer to it as a cell of the partition $P$. The join of two partitions $P, Q$ of $X$ is the partition $P \lor Q = \{ [a] \cap [b] : [a] \in P, [b] \in Q \}$. Since $\lor$ is associative, we can define the join of any finite collection of partitions (cf. (6)). We write $Q \supseteq P$ if for every $Q \subseteq P$ there is $P \in P$ such that $Q \subseteq P$. The partition distance $\|P - Q\|$ between $P$ and $Q$ is defined by

$$\|P - Q\| = \sum_{a \in \mathbb{N}_0} \mu(P_a \triangle Q_a)$$

(we extend the alphabets if necessary by adding empty cells). The (full) $P$-name of $x \in X$ is a $\Lambda$-valued sequence $(x_n)_{n \in \mathbb{N}_0}$ such that for every $n \in \mathbb{N}_0$ we have that $x_n = \alpha$ if, and only if, $T_n(x) \in P_\alpha$. Given $S \subseteq \mathbb{R}$ with $S \cap \mathbb{N}_0$ finite, where $S = \mathbb{Z}$ if $T$ is invertible, and $\mathbb{S} = \mathbb{N}_0$ otherwise, we define

$$P^S = \bigvee_{j \in S \cap \mathbb{N}_0} T^{-j}P = \left\{ \bigcap_{j \in S \cap \mathbb{N}_0} T^{-j}([a_j]) : (a_j)_{j \in S \cap \mathbb{N}_0} \subseteq P \right\}. \quad (6)$$

For $n \in \mathbb{N}$ we denote by $P^n$ the partition $P^{[0,n]} = P^{[0,1,\ldots,n-1]}$. Note that $P^1 = P$. Cells of $P^n$ correspond to finite $\Lambda$-valued strings of length $n$, hence for $n \geq 1$ and $a_0, a_1, \ldots, a_{n-1} \in \Lambda$ we write

$$[a_0a_1 \ldots a_{n-1}] = [a_0] \cap T^{-1}([a_1]) \cap T^{-2}([a_2]) \cap \cdots \cap T^{-n+1}([a_{n-1}]) \in P^n.$$ 

Similarly, the cells of $P^{[1,n]}$ (for $n \geq 1$) consist of points sharing $P$-name for entries from 1 to $n$, hence we denote them as

$$[\ast a_1 \ldots a_n] = \{ x : T^j(x) \in [a_j] \text{ for } j = 1, \ldots, n \}, \text{ where } a_1, \ldots, a_n \in \Lambda.$$ 

We have used “$\ast$” to stress that we do not know which symbol appears at the 0 coordinate in the $P$-name of a point from a cell of $P^{[1,n]}$. We clearly have

$$[a_0a_1 \ldots a_n] \cap [\ast a_1 \ldots a_n] = [a_0a_1 \ldots a_n] \quad \text{for all } a_0, a_1, \ldots, a_n \in \Lambda.$$ 

We will also consider partitions of $X$ according to the entries in the $P$-names of points over blocks of varying length. Assume that $\xi : \mathbb{N} \rightarrow \mathbb{N}$ is a measurable function with $\int_X \xi \, d\mu < \infty$, and $P$ is a finite partition of $X$. We define $P^{[1,\xi]}$ to be the partition obtained as follows. First, we partition $X$ into level sets of $\xi$, that is we take $\Xi = \{ \xi^{-1}(n) : n \in \mathbb{N} \}$. Second, for every $n \geq 1$ we further partition the set $\xi^{-1}(n)$ of $\Xi$ according to $P^{[1,n]}$. Each cell of $P^{[1,\xi]}$ gathers points sharing the $P$-name from time 1 to $n$ where $n$ is the common value of $\xi$ for all these points. That is,

$$P^{[1,\xi]} = \bigcup_{n=1}^\infty \{ P \cap \xi^{-1}(n) : P \in P^{[1,n]} \}.$$ 

Equivalently, for $x \in X$ the cell of $P^{[1,\xi]}$ containing $x$ coincides with the cell of $P^{[1,\xi^{-1}(\xi(x))]}$ containing $x$. We can extend this notation in an obvious way, and define $P^{(1,\xi,0]}$.

**Entropy and Conditional Entropy of a Partition**

Let $(X, \mathcal{F}, \mu)$ be a probability space and $P, Q, R$ be countable measurable partitions of $X$. The entropy of $P$ is

$$H_\mu(P) = - \sum_{P \in P} \mu(P) \log \mu(P).$$
The conditional entropy of $\mathcal{P}$ given $Q$ is defined by

$$H_\mu(P|Q) = \sum_{Q \in Q} \mu(Q)H_{\mu_Q}(P),$$

where $\mu_Q$ is the conditional probability measure on $Q$ (that is the measure obtained by restricting $\mu$ to $Q$ and normalizing it). Clearly, $H_\mu(P) = H_\mu(P|\{X\})$. We note the following monotonicity properties of the entropy (see [18 Lemma I.6.6]):

$$P \geq Q \Rightarrow H_\mu(Q|R) \leq H_\mu(P|R), \quad (7)$$
$$P \geq Q \Rightarrow H_\mu(R|P) \leq H_\mu(R|Q). \quad (8)$$

### Stationary Processes and Measure Preserving Systems

By a ($\Lambda$-valued) random variable we mean a measurable function from a standard probability space $(\Omega, \mathcal{B}, \nu)$ to a set $\Lambda \subseteq \mathbb{N}_0 = \{0, 1, 2, \ldots\}$ endowed with the power set $\sigma$-algebra $\mathcal{B}(\Lambda)$. We also refer to $\Lambda$ as to an alphabet. A $\Lambda$-valued process is a sequence of $\Lambda$-valued random variables $X = (X_i)_{i \in \mathbb{S}}$, where $\mathbb{S} = \mathbb{Z}$ or $\mathbb{S} = \mathbb{N}_0$ is an index set, such that the domain of each $X_i$ is a common standard probability space $(\Omega, \mathcal{B}, \nu)$. The process $X$ is stationary if for every $n \geq 1$, every $\lambda_1, \ldots, \lambda_n \in \Lambda$, and any $s \in \mathbb{S}$ we have

$$\nu\left(\{\omega \in \Omega : X_{i-1}(\omega) = \lambda_j \text{ for } j = 1, \ldots, n\}\right) =$$
$$= \nu\left(\{\omega \in \Omega : X_{s+j-1}(\omega) = \lambda_j \text{ for } j = 1, \ldots, n\}\right).$$

We call $\nu$ the law of $X$.

Processes and probability preserving systems are closely connected, see [18 Sec. 1.2] or [5 Sec. 1.3]. We briefly recall that connection.

Given a $\Lambda$-valued stationary process $X = (X_i)_{i \in \mathbb{S}}, n \geq 1, \lambda_1, \ldots, \lambda_n \in \Lambda$, and $t(1), \ldots, t(n) \in \mathbb{S}$ we set

$$\mu\left(\{x \in \Lambda^\mathbb{S} : x_{t(i)} = \lambda_j \text{ for } j = 1, \ldots, n\}\right) =$$
$$= \nu\left(\{\omega \in \Omega : X_{t(i)}(\omega) = \lambda_j \text{ for } j = 1, \ldots, n\}\right), \quad (9)$$

where $(\Omega, \mathcal{B}, \nu)$ is the standard probability space on which all $X_i$'s are defined. We easily see that $\mu$ extends to a probability measure on a $\mu$-completion $\mathcal{C}$ of the product $\sigma$-algebra on $\Lambda^\mathbb{S}$. Furthermore, since $X$ is stationary, we conclude that $\mu$ is invariant for the shift transformation $\sigma : \Lambda^\mathbb{S} \to \Lambda^\mathbb{S}$ given for $x = (x_i)_{i \in \mathbb{S}}$ by $\sigma(x)_i = x_{i+1}$ for every $i \in \mathbb{S}$. It follows that $(\Lambda^\mathbb{S}, \mathcal{C}, \mu, \sigma)$ is a probability preserving system, called a shift system. Furthermore, $\sigma$ is invertible provided $\mathbb{S} = \mathbb{Z}$.

On the other hand, assume we have a measure preserving system $(X, \mathcal{B}, \mu, T)$ and a measurable partition $\mathcal{P} = \{P_\alpha : \alpha \in \Lambda\}$ of $X$ where $\Lambda \subseteq \mathbb{N}_0$. We tacitly ignore $\mu$-null cells. Given $x \in X$ we write $P(x) = \alpha$ if, and only if, $x \in [\alpha]$. We construct a process $(T, \mathcal{P})$ by defining random variables $X_i = P \circ T^i$ for each $i \in \mathbb{S}$, where $\mathbb{S} = \mathbb{Z}$ if $T$ is invertible, and $\mathbb{S} = \mathbb{N}_0$ otherwise. Then all $X_i$'s are $\Lambda$-valued random variables defined over the standard probability space $(X, \mathcal{B}, \mu)$, and the sequence $X = (X_i)_{i \in \mathbb{S}}$ is a stationary process. The entropy rate of the process $(T, \mathcal{P})$ is denoted by $H_\mu(T, \mathcal{P})$, that is $H_\mu(T, \mathcal{P}) = \hat{H}(X)$, where $\hat{H}(X)$ is given by (1). The quantity $H_\mu(T, \mathcal{P})$ appears as (dynamical) entropy of $\mathcal{P}$ in ergodic theory literature. Note that

$$\hat{H}_\mu(T, \mathcal{P}) = \lim_{n \to \infty} \frac{1}{n} H_\mu(P^n) = \lim_{n \to \infty} H_\mu(\mathcal{P}[P^{[1,n]}]), \quad (10)$$

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where the first equality follows immediately from definitions and the second follows from [5, Lemma 3.17].

In particular, if a measure preserving system \((X, \mathcal{X}, \mu, T)\) is a shift system over the alphabet \(A\), that is, \(X = A^S\) for some finite set \(A\), \(\mathcal{X}\) is the \(\mu\)-completion of the product \(\sigma\)-algebra on \(A^S\), and \(\mu\) is a \(\sigma\)-invariant probability measure on \(A^S\), then the associated process is obtained by taking the partition \(\mathcal{L} = \{[\lambda] : \lambda \in \Lambda\}\), where \([\lambda] = \{x \in A^S : x_0 = \lambda\}\) for \(\lambda \in \Lambda\). The entropy rate of the process \((\sigma, \mathcal{L})\) is simply denoted as \(h_\mu(\sigma)\). Note that a Cartesian product of a pair of shift systems is a shift system over an Cartesian product of their alphabets. Nevertheless, if \(\zeta\) is an invariant measure of a shift system whose alphabet is a Cartesian product, we will write \(h_{\zeta}(\sigma \times \sigma)\) for its entropy rate.

Frequency-typical sequences

Let \((k_n)_{n=1}^\infty\) be an increasing sequence in \(\mathbb{N}_0\). A \(\Lambda\)-valued sequence \(z = (z_i)_{i \in S}\) is frequency-typical along \((k_n)_{n=1}^\infty\) if for every \(m \in \mathbb{N}\) and \(\lambda_1 \ldots \lambda_m \in \Lambda^m\) the sequence

\[
\frac{1}{k_n} \sum_{0 \leq j \leq k_n - m} z_j = \lambda_1, \ldots, z_{j+m-1} = \lambda_m
\]

converges as \(n \to \infty\). We say that \(z\) is frequency-typical (along \((k_n)_{n=1}^\infty\)) for or generates (along \((k_n)_{n=1}^\infty\)) a stationary \(\Lambda\)-valued stochastic process \(Z = (Z_i)_{i \in S}\) with the law \(\nu\) if for every \(m \geq 1\) and for every \(\lambda_1, \ldots, \lambda_m \in \Lambda\) the sequence in (11) converges to \(\nu\) \((Z_0 = \lambda_1, \ldots, Z_{m-1} = \lambda_m)\) as \(n \to \infty\). In that case, we also say that \(z\) is frequency-typical (along \((k_n)_{n=1}^\infty\)) for or generates (along \((k_n)_{n=1}^\infty\)) the \(\sigma\)-invariant measure \(\mu\) on \(A^S\) obtained from \(\nu\) by \([13]\). Every frequency-typical sequence generates a unique stationary process and every stationary process has a frequency-typical sequence generating it. Furthermore, frequency-typical sequences have full measures for ergodic processes (for definition, see \([13]\) Sec. 1.2.a) and form a null set for non-ergodic processes. We skip “(along \((k_n)_{n=1}^\infty\)” whenever one can take \(k_n = n\) for every \(n \in \mathbb{N}\) in the above definitions. When we want to say that \(z\) generates (along \((k_n)_{n=1}^\infty\)) a \(\sigma\)-invariant measure \(\mu\) on \(A^S\) without mentioning \((k_n)_{n=1}^\infty\) explicitly, we simply say that \(z\) quasi-generates \(\mu\).

We will use the following observation without further reference: Given an increasing sequence \((k_n)_{n=1}^\infty\) in \(\mathbb{N}_0\) and \(z \in \Lambda^S\) we can always find a subsequence of \((k_n)_{n=1}^\infty\) such that \(z\) is frequency-typical along that subsequence for a \(\sigma\)-invariant measure on \(\Lambda^S\).

Induced measure preserving systems

Given a measure preserving system \((X, \mathcal{X}, \mu, T)\) and \(E \subseteq X\) with \(\mu(E) > 0\) we write \(\mathcal{X}_E\) for the trace \(\sigma\)-algebra on \(E\), that is \(\mathcal{X}_E = \{A \cap E : A \in \mathcal{X}\}\). The induced measure on \(E\) is \(\mu_E\), where \(\mu_E(A) = \mu(A)/\mu(E)\) for \(A \in \mathcal{X}_E\). The first return time to \(E\) is the measurable function \(r_E : E \to \mathbb{N} = \mathbb{N} \cup \{\infty\}\) defined by \(r_E(x) = \min\{n \geq 1 : T^n(x) \in E\}\). By the Poincaré recurrence theorem \(r_E(x) < \infty\) for \(\mu\)-a.e. \(x \in E\).

Setting \(R_n = \{x \in E : r_E(x) = n\}\) for \(n \in \mathbb{N}\), we obtain the return time partition \(\mathcal{R}\) of \(E\). The Kac lemma (see \([16]\) for a proof in this generality) says that

\[
\int_E r_E \, d\mu = \sum_{n \in \mathbb{N}} n \mu(E)(R_n) = \frac{\mu\left(\bigcup_k T^{-k}(E)\right)}{\mu(E)}.
\]

The induced map is the map \(T_E : E \to E\) such that \(T_E(x) = T^r_E(x)\) for \(x \in E\) (we ignore \(\mu\)-null set of points where \(T_E\) is not well-defined). The induced system is the
Assume that \( P \) is a countable measurable partition of \( X \) such that \( H_\mu(P) < \infty \), \( X \setminus E \in P \), and \( P_E \not\equiv E \), then \( \mu(E) h_{\mu_E}(T_E, P_E) = h_\mu(T, P) \).
Consider a finite partition $Q = \{Q_s : s \in \Lambda\}$ of $X$, where $\Lambda \subseteq \mathbb{N}$. We assume that $Q$ contains the information whether the orbit is in or outside $E$, that is, $Q \supseteq \{E, X \setminus E\}$. We want to find a partition of $E$ such that the process generated by the partition and $T_E$ encodes the full information about the process $(T, Q)$. We achieve this by adding to the cells of $Q$ contained in $E$ the information about entry times.

It is now customary to think of a cell $[\alpha] = P_\alpha \in Q$ contained in $E$ as represented or indexed by a starred symbol $\alpha^*$, while the cells of $Q$ outside $E$ are indexed as before by $\alpha \in \Lambda$. At least one starred and at least one non-starred symbol should index a nonempty set. Since $\mu$-almost every $x \in X$ visits $E$ infinitely many times, the $Q$-name $(x_\mu)_{\mu \in \mathbb{Z}}$ of $\mu$ almost every $x \in X$ can be divided into blocks $x_{[n_k-1,n_k]}$ where $n_0$ is chosen so that $n_0$ is the time of the first visit of $x$ to $E$. Each word $x_{[n_k-1,n_k]}$ consists of some number (possibly zero when $E \cap T(E) \neq \emptyset$) of non-starred symbols followed by a single starred one. Now, if we consider the words $x_{[n_k-1,n_k]}$ as symbols of a new alphabet, we obtain a countable partition $Q_0^{(-\xi,0)}$ of $E$, where $\xi = r_E \circ T_E^{-1}$. Recall that cells of $Q_0^{(-\xi,0)}$ are defined by taking the entry-time partition $E$ of $E$ and refining each $E_n \in E$ according to $Q_0^{(-n,0)}$. We will show that $Q_0^{(-\xi,0)} = [Q_0^{(-\xi,0)} \cup \{X \setminus E\}]$ is a partition of $X$ satisfying the assumptions of Proposition 3.1, which yields $H_{\mu_k}(T, Q_0^{(-\xi,0)}) = \mu(E)\bar{H}_{\mu_k}(T_E, Q_0^{(-\xi,0)})$. On the other hand, the processes $(T, Q)$ and $(T, Q_0^{(-\xi,0)})$ are isomorphic by a code which turns a $Q_0^{(-n_k-1,0)}$-name $x_{[n_k-1,n_k]}$ into $Q_0^{(-\xi,0)}$-name $0^{n_k-n_k-1}w$ with $w = x_{[n_k-1,n_k]}$ treated as a symbol of $Q_0^{(-\xi,0)}$.

**Proposition 3.2.** Assume that $(X, \mathcal{X}, \mu, T)$ is an invertible probability measure preserving system and $E \in \mathcal{X}$ sweeps out $X$. If $Q$ is a finite partition of $X$ such that $Q \supseteq \{E, X \setminus E\}$, then the partition $Q_0^{(-\xi,0)}$ of $E$, where $\xi = r_E \circ T_E^{-1}$ satisfies $H_{\mu_k}(Q_0^{(-\xi,0)}) < \infty$ and $\mu(E)\bar{H}_{\mu_k}(T_E, Q_0^{(-\xi,0)}) = \bar{H}_{\mu_k}(T, Q)$.

## 4 Proofs

**Proof of Proposition 3.2.** For $x \in X$ and $N \in \mathbb{N}$ we inductively define an auxiliary function we call the time of the $N$-th return to $E$ and denote $v^N(x)$. Let $v^1(x) = \min\{n \geq 1 : T^n(x) \in E\}$ (we agree that $\min \emptyset = \infty$). Given $N > 1$ and $v^{N-1}(x) \neq \infty$, we set $v^N(x) = \min\{n > v^{N-1}(x) : T^n(x) \in E\}$. By convention, we do not define $v^N(x)$ if $v^M(x) = \infty$ for some $M < N$. Note that by (13) for $\mu$-almost every $x \in X$ we have that for each $N \in \mathbb{N}$ it holds $N \leq v^N(x) < \infty$. Furthermore, $v^1$ coincides with $r_E$ on $E$.

Fix $N \in \mathbb{N}$. For $n \geq N$ we have $P[1,v^N] \geq P[1,n] \geq P[1,(v^N \wedge n)]$, where $(v^N \wedge n)(x) = \min\{v^N(x), n\}$ for $x \in X$. Using (9) we obtain

$$H_{\mu_k}(P[1,v^N]) \leq H_{\mu_k}(P[1,n]) \leq H_{\mu_k}(P[1,(v^N \wedge n)]).$$

(14)

Note that $P[1,v^N]$ and $P[1,(v^N \wedge n)]$ coincide outside the set $\{x \in X : v^N(x) > n\}$, whose $\mu$-measure goes to 0 as $n \to \infty$. It follows that the partition distance $|P[1,v^N] - P[1,(v^N \wedge n)]|$ approaches 0 as $n \to \infty$. By (3) Fact 1.7.10, $H_{\mu_k}(P[\cdot])$ is continuous on the space of countable partitions endowed with the partition distance, so

$$\lim_{n \to \infty} H_{\mu_k}(P[1,(v^N \wedge n)]) = H_{\mu_k}(P[1,v^N]).$$

(15)
Let \( n \to \infty \) in (14), then we use (15), and finally we let \( N \to \infty \) to get

\[
\lim_{n \to \infty} H_\mu(P|P^{[1,x^n]}) = \lim_{n \to \infty} H_\mu(P|P^{[1,n]}).
\]

(16)

By the definition of the conditional entropy we have

\[
H_\mu(P|P^{[1,x^n]}) = \sum_{W \in P^{[1,x^n]}} \mu(W) \sum_{A \in P} \eta(\mu(A \cap W)/\mu(W)).
\]

(17)

Below, we label the cells of \( P_E \subseteq P \) with boldface letters (recall that \([0] = X \setminus E \subseteq P\).

Note that \( P_E \supseteq E \), so if \([a] \in P_E \), then every occurrence of \( a \) in the \( P \)-name is preceded by \( 0^{[a]-1} \). We write \( \mathbf{a} \) for blocks of “correct” number of \( 0 \)'s followed by \( a \), that is, \( \mathbf{a} \) stands for \( 0^{[a]-1}a \). Cells of \( P^{[1,x^n]} \) consist of points sharing \( P \)-names from position 1 to the place where the \( n \)-th bold symbol occurs. In particular, if \( W \in P^{[1,x^n]} \) then

\[
W = \{ \mathbf{a}_1 \mathbf{a}_2 \ldots \mathbf{a}_n \} \subseteq P^{[1,x^n]}, \text{ where } \ell < |a_1|.
\]

(18)

We have two cases: either \( \ell = |a_1| - 1 \) or \( \ell < |a_1| - 1 \). In the former case, we call \( W \) a full cell and note that \( W \subseteq E \), so \( \mu([0] \cap W) = 0 \). In the latter case, \( W \not\subseteq X \setminus E \)\n
\[
\mu([0] \cap W) = 1. \text{ Let } P_n = \text{ the set of full cells in } P^{[1,x^n]} \text{. We rewrite (17) as}
\]

\[
H_\mu(P|P^{[1,x^n]}) = \sum_{\mathbf{a}_1 \mathbf{a}_2 \ldots \mathbf{a}_n} \mu(\{\mathbf{a}_1 \mathbf{a}_2 \ldots \mathbf{a}_n\}) \sum_{\mathbf{a}_0} \eta(\mu(\mathbf{a}_0 \mathbf{a}_1 \mathbf{a}_2 \ldots \mathbf{a}_n)/\mu(\{\mathbf{a}_0 \mathbf{a}_1 \mathbf{a}_2 \ldots \mathbf{a}_n\})).
\]

(19)

Note that a full cell \( \{\mathbf{a}_1 \mathbf{a}_2 \ldots \mathbf{a}_n\} \subseteq P^{[1,x^n]} \) equals \( \{\mathbf{a}_1 \mathbf{a}_2 \ldots \mathbf{a}_n\} \subseteq P^{[1,n]} \), where \( P^{[1,n]} = T_E^{-1}(P_E) \cap \cdots \cap T_E^{n-1}(P_E) \) and \( \{\mathbf{a}_0 \mathbf{a}_1 \mathbf{a}_2 \ldots \mathbf{a}_n\} = \mathbf{a}_0 \mathbf{a}_1 \mathbf{a}_2 \ldots \mathbf{a}_n \subseteq P^{[1,n]} \) for every \( \mathbf{a}_0 \subseteq P_E \). Since \( \mu(B) = \mu(E)\mu(B)/\mu(E) \) for \( B \subseteq E \), the right hand side of (19) equals

\[
\sum_{\{\mathbf{a}_1 \mathbf{a}_2 \ldots \mathbf{a}_n\} \subseteq \mathbf{a}_0 \subseteq P^{[1,n]}} \mu(\{\mathbf{a}_1 \mathbf{a}_2 \ldots \mathbf{a}_n\}) \sum_{\mathbf{a}_0} \eta(B/E) \mu(E) = \mu(E)H_\mu(P|P^{[1,n]}).
\]

Letting \( n \to \infty \), invoking (10) and (16) we obtain

\[
\mu(E)H_\mu(P|P_E) = \lim_{n \to \infty} \mu(E)H_\mu(P|P^{[1,n]}) = \lim_{n \to \infty} H_\mu(P|P^{[1,x^n]}) = H_\mu(P|P_E).
\]

which completes the proof.

Proof of Proposition 3.2

Since every partition \( P = \{ P_1, P_2, \ldots \} \) with \( \sum \mu(P_n) < \infty \) has finite entropy, so \( H_\mu(P(E)) = H_\mu(P(E)) < \infty \). Let \( \xi = \lim_{T_E \to \cdot} T_E^{-1} \). Observe that for every \( n \) there are at most \( |Q| \cdot n \) atoms of \( Q^{(-\xi,0)} \) with nonempty intersection with \( E_n \), that is, atoms corresponding to \( Q \)-names with the last symbol starred. It follows that for every \( n \geq 1 \) the entropy of the partition \( Q^{(-\xi,0)} \) with respect to the measure \( \mu(E) \cap E_n/\mu(E) \) is at most \( n \log |Q| \). It follows directly from the way we defined \( Q^{(-\xi,0)} \) that \( Q^{(-\xi,0)} \geq \xi \), hence \( Q^{(-\xi,0)} \vee E = Q^{(-\xi,0)} \) and

\[
H_\mu(E) = H_\mu(Q^{(-\xi,0)} \vee E) + H_\mu(E) \leq \sum_{n=1}^{\infty} \mu(Q) n \log |Q| + H_\mu(E).
\]
Since both \( H_{\mu_E}(E) \) and \( \log |Q| \sum_{n=1}^{\infty} n \mu_E(E_n) \) are finite we see that \( H_{\mu_E}(Q^{(-\xi,0)}_E) < \infty \).

By adding the set \( P_0 = X \setminus E \) as a cell to \( Q^{(-\xi,0)}_E \) we obtain a partition \( Q^{(-\xi,0)}_0 \) of \( X \) satisfying the assumptions of Proposition 5.1. Applying that result we get

\[
\mu(E) \hat{H}_{\mu_E}(T_E, Q^{(-\xi,0)}_E) = \hat{H}_{\mu}(T, Q^{(-\xi,0)}_0).
\]

Recall that for \( \mu \)-almost every \( x \in X \) there exists a strictly increasing sequence \( (n_k)_{k \in \mathbb{Z}} \) of integers satisfying \( n_{k+1} < n_k \) such that the \( Q \)-name \( (x_k)_{k \in \mathbb{Z}} \) of \( x \) can be divided into \( x_{(n_{k-1}, n_k)} \) for \( k \in \mathbb{Z} \), where each block ends with a single starred symbol preceded by some number (possibly zero) of non-starred symbols. Therefore, given such a \( Q \)-name \( x = (x_k)_{k \in \mathbb{Z}} \) and \( j \in \mathbb{Z} \) we find \( k \in \mathbb{Z} \) such that \( n_{k-1} < j \leq n_k \) and we define

\[
y_j = \begin{cases} x_{(n_{k-1}, n_k)}, & \text{if } j = n_k, \\ 0, & \text{otherwise.} \end{cases}
\]

(20)

Since every block \( x_{(n_{k-1}, n_k)} \) corresponds to a cell of \( Q^{(-\xi,0)}_0 \), the resulting sequence \( (y_j)_{j \in \mathbb{Z}} \) is a valid \( Q^{(-\xi,0)}_0 \)-name. The transformation \( (x_j)_{j \in \mathbb{Z}} \mapsto (y_j)_{j \in \mathbb{Z}} \) given by (20) is clearly an isomorphism of the processes \( (T, \hat{H}^{Q}_0) \) and \( (T, \hat{Q}) \), since given a \( Q^{(-\xi,0)}_0 \)-name, where nonzero blocks correspond to the blocks \( x_{(n_{k-1}, n_k)} \in Q^{(-\xi,0)}_0 \) we can easily reconstruct \( Q \)-name. Thus \( \hat{H}_{\mu}(T, Q^{(-\xi,0)}_0) = \hat{H}_{\mu}(T, Q) \).

For the proof of Theorem 1.1 it will be convenient to replace \( \hat{f} \) by a uniformly equivalent pseudometric \( \hat{f} \). For \( u = (u_i)_{i \in \mathbb{Z}}, v = (v_j)_{j \in \mathbb{Z}} \in \mathbb{A}^5 \), where \( S = \mathbb{N}_0 \) or \( S = \mathbb{Z} \) and strictly increasing sequences \( I = (i(r))_{r \in \mathbb{N}}, I' = (i'(r))_{r \in \mathbb{N}} \in \mathbb{N}_0 \) we write \( u|_I = w|_{I'} \) if \( u_{i(r)} = w_{i'(r)} \) for every \( r \in \mathbb{N} \). We define \( \hat{f}(u, v) \) as

\[
\hat{f}(u, v) = \inf \{ \varepsilon > 0 : u|_I = w|_{I'} \text{ for some strictly increasing sequences } I = (i(r))_{r \in \mathbb{N}}, I' = (i'(r))_{r \in \mathbb{N}} \in \mathbb{N}_0 \text{ with } d(I) \geq 1 - \varepsilon, d(I') \geq 1 - \varepsilon \},
\]

where \( d \) denotes the lower asymptotic density, that is, for \( A \subseteq \mathbb{N}_0 \) we set

\[
d(A) = \liminf_{n \to \infty} \frac{1}{n} |A \cap \{0, 1, \ldots, n - 1\}|.
\]

**Lemma 4.1** \([13]\). The pseudometrics \( \hat{f} \) and \( \hat{f} \) are uniformly equivalent on \( \mathbb{A}^5 \).

**Proof of Theorem 1.1** By Lemma 4.1 it is enough to consider the pseudometric \( \hat{f} \), which is a pseudometric on \( \mathbb{A}^5 \) in both cases, \( S = \mathbb{N}_0 \) and \( S = \mathbb{Z} \). Theorem 1.1 follows immediately from the analogous statement for invertible processes by considering the natural extension, so from now on we assume that \( S = \mathbb{Z} \). Let \( \Lambda = \{1, 2, \ldots, l\} \) (\( \Lambda \) deliberately does not contain 0). Let \( x, x' \in \mathbb{A}^Z \) satisfy \( \hat{f}(x, x') < \varepsilon \). We also assume that \( x, x' \) are frequency-typical for measures \( \mu, \mu' \), respectively. We wish to prove that \( |h_{\mu}(\sigma) - h_{\mu'}(\sigma)| \to 0 \) as \( \varepsilon \to 0 \). By assumption, there exist sets \( A = \{a(1), a(2), \ldots\} \subseteq \mathbb{N}_0 \) and \( A' = \{a'(1), a'(2), \ldots\} \subseteq \mathbb{N}_0 \) such that \( x_{a(n)} = x'_{a'(n)} \) for each \( n \in \mathbb{N} \), and both \( d(A) \) and \( d(A') \) are bounded below by \( 1 - \varepsilon \). We define \( \kappa = (\kappa_n)_{n=0}^\infty \) by \( \kappa_n = x_{a(n+1)} = x'_{a'(n+1)} \) for \( n \in \mathbb{N}_0 \). Let \( y[0, 1]^Z \) be the characteristic function of a set \( A \).

\[^{1}\text{It is even a finitary code, that is, it has the following property: to determine a value of any entry in an output sequence, one should only examine finitely many coordinates of the source sequence, this finite number of coordinates depending upon the input sequence under consideration (see [13]).}\]
Figure 1: An example of sequences $x, x' \in \{1, 2\}^\mathbb{Z}$ and their common subsequence $\kappa$. Underlined entries in $x$ and $x'$ mark the positions in $A$ and $A'$.

In the following, we will repeatedly choose $\sigma$-invariant measures quasi-generated by various finite valued sequences without specifying sequences along which these measures are generated. Each time we pass to a subsequence, we choose it to be a subsequence of the sequence along which the point used in the previous step quasi-generated a measure. For example, $x$ is frequency-typical, so it generates $\mu$ along the sequence consisting of all nonnegative integers, while $x'$ is quasi-generated along a subsequence of the sequence along which $x$ is generated.

Let $\mu$ be a $\Lambda \times \Lambda$-invariant measure on $\mathbb{Z} \times \{0, 1\}$ quasi-generated by the pair $(x,y)$. Let $\nu$ on $\{0, 1\}^\mathbb{Z}$ be the marginal distribution of $\mu$ on the second coordinate.

Then $\eta = \mu \times \nu$ is a joining of $\mu$ and $\nu$, and $\bar{\nu}$ is quasi-generated by $y$, that is, $\bar{\nu}$ is the marginal of $\eta$ with respect to the projection from $\Lambda \times \{0, 1\}$ to the first (respectively, the second) coordinate. Note that the entropy rate of $\bar{\nu}$ satisfies

$$h_{\bar{\nu}}(\sigma) = h_{\nu}(\sigma \times \sigma) \leq h_{\mu}(\sigma) + \eta(\epsilon) + \eta(1 - \epsilon).$$

Therefore $|h_{\mu}(\sigma) - h_{\bar{\nu}}(\sigma \times \sigma)| \to 0$ as $\epsilon \to 0$.

Figure 2: Sequences $y, \kappa, z, \bar{z}$ constructed as in the proof of Theorem 1.1 given $x, x'$ as above.

Let $z = xy \in (\Lambda \cup \{0\})^\mathbb{Z}$ be the pointwise product of $x$ and $y$ and let $\bar{z} = x(1 - y)$. Note that $z$ (respectively, $\bar{z}$) coincides with $x$ along $A$ (respectively, along $A' = \mathbb{N}_0 \setminus A$). The finite sliding-block code of length 1 given by $\Phi(a, b) = (ab, a(1 - b))$ yields an isomorphism between the measure $\bar{\nu}$ quasi-generated by $(x, y)$ and a measure $\bar{\xi}$ quasi-generated by $(z, \bar{z})$ along the same sequence as $\xi$. The marginal distributions of $\bar{\nu}$ are $\sigma$-invariant measures $\zeta$ and $\bar{\xi}$ quasi-generated by $z$ and $\bar{z}$, respectively. Hence

$$h_{\bar{\xi}}(\sigma) \leq h_{\bar{\nu}}(\sigma \times \sigma) = h_{\xi}(\sigma \times \sigma) \leq h_{\xi}(\sigma) + h_{\bar{\xi}}(\sigma).$$

(21)

Note that $\bar{\zeta}$ is generated by $\bar{z}$ and the symbol 0 appears in $\bar{z}$ with the lower asymptotic density bounded below by $1 - \epsilon$, so $\bar{\zeta}(\{0\}) \geq 1 - \epsilon$. Similar reasoning shows that $\zeta(\{0\}) \leq \epsilon$. It follows that $h_{\bar{\zeta}}(\sigma) \leq \eta(1 - \epsilon) + \eta(\epsilon) + \epsilon \log l$, where the right hand-side of the inequality is the entropy of the probability vector $(1 - \epsilon, \epsilon/l, \epsilon/l, \ldots, \epsilon/l)$, with $l = |\Lambda|$. This, together with (21), imply that $|h_{\zeta}(\sigma) - h_{\bar{\zeta}}(\sigma \times \sigma)| \to 0$ as $\epsilon \to 0$. 

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It remains to estimate $h_\varepsilon(\sigma)$. Consider the measure preserving system $(X, \mathcal{F}, \zeta, \sigma)$ where $X = (\Lambda \cup \{0\})^\mathbb{Z}$, $\mathcal{F}$ is the $\zeta$-completion of the product $\sigma$-algebra, and $\sigma$ is the shift transformation. Let $E = X \setminus \{0\}$ be the set of sequences $(x_n)_{n \in \mathbb{Z}}$ with $x_0 \neq 0$. Note that $\zeta([0]) \leq \varepsilon$ implies $\zeta(E) \geq 1 - \varepsilon$. We consider the induced system $(E, \mathcal{F}_E, \zeta_E, \sigma_E)$. Let $Q = \{[a] : a \in \Lambda \cup \{0\}\}$ be the partition of $X$ into cylinder sets of length 1. Then $Q_E = \{[a] : a \in \Lambda\}$, $\overline{Q}_E = Q \cup \{0\}$, and $\overline{Q} = \{E, X \setminus E\}$. We want to apply Proposition 3.2, but $E$ need not to sweep out $X$ with respect to $\zeta$. In that case, there are $0 < a < 1$ and a shift invariant measure $\tilde{\zeta}$ such that $\zeta = (1 - a)\tilde{\zeta} + a\delta_0$, where $\delta_0$ is the Dirac measure concentrated on the fixed point $\tilde{\theta} = \ldots 0.00 \ldots$ and $\zeta$ and $\delta_0$ are mutually singular. It follows that $\tilde{\zeta}([\emptyset]) = 0$, hence $E$ sweeps out $X$ with respect to $\zeta$.

Furthermore, $a \leq \tilde{\zeta}([\emptyset]) \leq \varepsilon$ and

$$\tilde{\zeta}(E) = \frac{\zeta(E)}{1 - a} \geq \frac{1 - \varepsilon}{1 - a}.$$ 

By affinity of the entropy rate [5] Lemma 3.9] we get $h_{\tilde{\zeta}}(\sigma) = (1 - a)h_{\zeta}(\sigma) + ah_{\delta_0}(\sigma) = (1 - a)h_{\zeta}(\sigma)$. It follows that $|h_{\tilde{\zeta}}(\sigma) - h_{\zeta}(\sigma)| \to 0$ as $\varepsilon \to 0$.

Let $Q_{E,0}^\varepsilon$ be the partition of $E$ obtained in Proposition 3.2. Since $X \setminus E \in Q$, we easily see that the processes $(T_E, Q_E^\varepsilon)$ and $(T_f, R)$ where $R = Q_E \vee \mathcal{E}$ are isomorphic. Applying Proposition 3.2, we obtain $H_{T_E}^\varepsilon(R) < \infty$ and

$$\tilde{\zeta}(E)H_{T_E}^\varepsilon(\sigma_E, R) = H_{\tilde{T}_E}^\varepsilon(\sigma, R \cup \{0\}) = H_{\tilde{T}_E}^\varepsilon(\sigma, Q) = h_{\tilde{\zeta}}(\sigma).$$

It follows that $|H_{T_E}^\varepsilon(\sigma_E, R) - h_{\zeta}(\sigma)| \to 0$ as $\varepsilon \to 0$.

Consider the entropy rate of the process $(\sigma_E, E)$ generated on $(E, \mathcal{F}_E, \tilde{T}_E, \sigma_E)$. Setting $p_n = \tilde{\zeta}(E_n) = \tilde{\zeta}(\{x \in E : T_E(\sigma_E(x)) = n\})$ we get a probability vector $p = (p_n)_{n \in \mathbb{N}}$ on $\mathbb{N}$ with the expected value

$$\mathbb{E}(p) = \sum_{n=1}^\infty np_n = 1/\tilde{\zeta}(E) \leq 1/\zeta(E).$$

Since it is well-known that among all the distributions on $\mathbb{N}$ with expected value $1/p$, the largest entropy is $(n(p_n + (1 - p_n))/p$ we get $H_{\tilde{T}_E}^\varepsilon(\sigma_E, \mathcal{E}) \leq (n(p_n + (1 - p_n))/(1 - \varepsilon)$.

Since $R = Q_E \vee \mathcal{E}$ we have $H_{\tilde{T}_E}^\varepsilon(\sigma_E, Q_E) \leq H_{\tilde{T}_E}^\varepsilon(\sigma_E, R)$ and

$$H_{\tilde{T}_E}^\varepsilon(\sigma_E, R) \leq H_{\tilde{T}_E}^\varepsilon(\sigma_E, Q_E) + H_{\tilde{T}_E}^\varepsilon(\sigma_E, \mathcal{E}) \leq H_{\tilde{T}_E}^\varepsilon(\sigma_E, Q_E) + \frac{\eta(p_n + (1 - p_n))}{1 - \varepsilon}.$$ 

Thus $|H_{\tilde{T}_E}^\varepsilon(\sigma_E, Q_E) - H_{\tilde{T}_E}^\varepsilon(\sigma_E, R)| \to 0$ as $\varepsilon \to 0$. Summing up, $|H_{\tilde{T}_E}^\varepsilon(\sigma_E, Q_E) - h_{\mu}(\sigma)|$ approaches 0 as $\varepsilon \to 0$.

Repeating the same steps, but starting with $x'$ in place of $x$ and generating only along subsequences of the sequence along which $\zeta$ was generated, we produce “primed versions” of all objects defined so far. In particular, we have a measure $\tilde{\zeta}'$ such that taking the same set $E$ as above and the partition $Q_E$ of $E$ into cylinders of non-zero symbols $|H_{\tilde{T}_E}^\varepsilon(\sigma_E, Q_E) - h_{\mu}'(\sigma)| \to 0$ as $\varepsilon \to 0$. It remains to compare $H_{\tilde{T}_E}^\varepsilon(\sigma_E, Q_E)$ with its primed variant $H_{\tilde{T}_E}^\varepsilon(\sigma_E, Q_E)$.

We clearly have $\tilde{\zeta}' = \tilde{\zeta}^\varepsilon$ and $\tilde{\zeta}' = \tilde{\zeta}^\varepsilon$. Furthermore, $\varepsilon$ restricted to nonzero entries with nonnegative indices, which coincides with $\kappa$ defined above, gives us a $Q_E$-name of a frequency-typical sequence for the induced process $(\sigma_E, Q_E)$. But $\varepsilon'$ restricted to nonzero entries also yields $\kappa$ and is frequency-typical for the process $(\sigma_E, Q_E)$, hence $H_{\tilde{T}_E}^\varepsilon(\sigma_E, Q_E) = H_{\tilde{T}_E}^\varepsilon(\sigma_E, Q_E)$, and the proof is complete. □
Remark 4.2. The above proof indicates also the modulus of the uniform continuity of the entropy function with respect to $\hat{f}$, namely if two frequency-typical sequences are $\varepsilon$-close with respect to $\hat{f}$, then the entropy rates of the processes generated by these sequences differ by at most

$$2 \left( 2\eta(\varepsilon) + 2\eta(1 - \varepsilon) + 4\varepsilon \log l + \frac{\eta(\varepsilon) + \eta(1 - \varepsilon)}{1 - \varepsilon} \right).$$

Note that this number depends on the cardinality $l$ of the alphabet $\Lambda$.

5 Appendix: The Abramov Formula

As a by-product of our considerations we present an elementary proof of a general version of the Abramov formula, which attributes to Scheller. Let $\text{Part}(D)$ (respectively, $\text{Part}_e(D)$) stand for the set of all finite measurable partitions of $D \in \mathcal{X}$ (respectively, all countable measurable partitions $P$ of $D$ with $H_P(P) < \infty$). Recall that the Kolmogorov-Sinai entropy $h_\mu(T)$ of a measure preserving system $(X, \mathcal{X}, \mu, T)$ is the supremum of entropy rates of all processes generated from the system by taking $P \in \text{Part}(X)$, equivalently, by taking $P \in \text{Part}_e(X)$, that is,

$$h_\mu(T) = \sup_{P \in \text{Part}(X)} \tilde{H}_\mu(T, P) = \sup_{P \in \text{Part}_e(X)} \tilde{H}_\mu(T, P). \quad (22)$$

Theorem 5.1 (The Abramov Formula). Let $(X, \mathcal{X}, \mu, T)$ be a probability measure preserving system and let $E \in \mathcal{X}$ sweep out $X$. Then $h_\mu(T) = \mu(E) h_{\mu_E}(T_E)$.

Proof. Using the natural extension the proof reduces to the invertible case. For every $Q \in \text{Part}(E)$, the partition $Q^{(E)} = (Q \lor E) \cup \{X \setminus E\} \in \text{Part}_e(X)$ satisfies the assumptions of Proposition 3.1 and $Q^{(E)} \geq Q$, so $\mu(E) \tilde{H}_{\mu_E}(T_E, Q) \leq h_\mu(T, Q^{(E)})$. Using (22) we get

$$\mu(E) \tilde{H}_{\mu_E}(T_E) = \mu(E) \sup_{Q \in \text{Part}(E)} \tilde{H}_{\mu_E}(T_E, Q) \leq \sup_{Q \in \text{Part}(E)} \tilde{H}_\mu(T, Q^{(E)}) \leq h_\mu(T).$$

For $Q \in \text{Part}(X)$ we set $\hat{Q} = Q \lor \{E, X \setminus E\}$. Let $Q^{(-\varepsilon,0)}_E$ be the partition of $E$ obtained in Proposition 5.2. We have

$$h_\mu(T) = \sup_{\hat{Q} \in \text{Part}(X)} \tilde{H}_\mu(T, \hat{Q}) = \mu(E) \sup_{\hat{Q} \in \text{Part}(X)} \tilde{H}_{\mu_E}(T_E, \hat{Q}^{(-\varepsilon,0)}_E) \leq \mu(E) h_{\mu_E}(T_E).$$

where the first equality uses that $\hat{Q} \geq Q$ for every $Q \in \text{Part}(X)$, the second equality follows from Proposition 5.2 and the inequality follows from (22). \qed

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