ON CHARACTERIZATION
OF INTEGRABLE SESQUILINEAR FORMS

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Dedicated to Prof. David J. Foulis on the occasion of his 80th birthday
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ABSTRACT. We give a necessary and sufficient condition for a sesquilinear form to be integrable with respect to a faithful normal state on a von Neumann algebra.

The fundamental solution to the problem of constructing a noncommutative $L_1(\varphi)$-space associated with a faithful normal semifinite weight $\varphi$ on a von Neumann algebra $\mathcal{M}$ was obtained in 1972–78. This space was realized as a space of “integrable” sesquilinear forms defined on a “lineal of weight” and “affiliated” with $\mathcal{M}$. In the next years this approach was thoroughly developed (see the survey [7] and the monograph [9]). For the other approaches to the integration with respect to weights and states we refer the reader to the surveys [7], [4] and the recent paper [3].

It is well known that a bounded linear operator on a Hilbert space is nuclear if and only if it has finite matrix trace (see for instance [2: Theorem III.8.1]). In the present paper we examine a problem whether a certain analogue of that assertion holds for integrable sesquilinear forms.

In what follows, $H$ is a complex Hilbert space with the scalar product denoted by $\langle \cdot, \cdot \rangle$.

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Let \( \varphi \) be a faithful normal semifinite weight on a von Neumann algebra \( \mathcal{M} \) of operators on \( H \) (see, e.g., [6]), \( m_\varphi^+ = \{ x \in \mathcal{M}^+ : \varphi(x) < +\infty \} \), \( m_\varphi^{sa} = m_\varphi^+ - m_\varphi^+ \).

It is well known that the formula
\[
\| x \|_\varphi \equiv \inf \{ \varphi(x_1 + x_2) : x = x_1 - x_2; \ x_1, x_2 \in m_\varphi^+ \}
\]
determines a norm \( \| \cdot \|_\varphi \) on \( m_\varphi^{sa} \). By \( L_1(\varphi)^{sa} \) we will denote the corresponding completion of \( m_\varphi^{sa} \).

The linear subspace of \( H \)
\[
D_\varphi \equiv \{ f \in H : \exists \lambda > 0 \ \forall x \in \mathcal{M}^+ (\langle xf, f \rangle \leq \varphi(x)) \}
\]
was introduced and called the lineal of weight in [8]. Clearly, if \( \varphi \) is represented in the form
\[
\varphi = \sum_{i \in I} \langle f_i, f_i \rangle, \quad f_i \in H,
\]
then \( f_i \in D_\varphi \ (i \in I) \).

The real Banach space \( L_1(\varphi)^{sa} \) can be realized by hermitian sesquilinear forms defined on \( D_\varphi \). Namely, if \( \tilde{x} \in L_1(\varphi)^{sa} \) and \( (x_n) \) is a Cauchy sequence in the normed space \( (m_\varphi^{sa}, \| \cdot \|_\varphi) \), which determines the element \( \tilde{x} \) of the completion, then the formula
\[
a_{\tilde{x}}(f, g) = \lim_n \langle x_n f, g \rangle, \quad f, g \in D_\varphi,
\]
correctly defines a hermitian sesquilinear form \( a_{\tilde{x}} \). The sequence \( (x_n) \) is called defining for \( a_{\tilde{x}} \). Also, since \( |\varphi(x)| \leq \| x \|_\varphi \) for any \( x \in m_\varphi^{sa} \), the formula
\[
\varphi(a_{\tilde{x}}) = \lim_n \varphi(x_n)
\]
correctly defines the value \( \varphi(a_{\tilde{x}}) \) which is called the integral (or the expectation) of the sesquilinear form \( a_{\tilde{x}} \) with respect to \( \varphi \). Accordingly, such sesquilinear forms are called integrable. Moreover, the main result of [8] (Theorem 2) says that the map \( \tilde{x} \mapsto a_{\tilde{x}} \ (\tilde{x} \in L_1(\varphi)^{sa}) \) is injective (see also [9] Theorem 16.7, [7] Theorem 1)). Thus, \( L_1(\varphi)^{sa} \) is meaningfully described as a real Banach space of integrable sesquilinear forms. The cone \( L_1(\varphi)^+ \) of integrable positive sesquilinear forms induces a natural order structure in \( L_1(\varphi)^{sa} \). The space \( L_1(\varphi) \) is defined as a certain complexification of \( L_1(\varphi)^{sa} \) [9: 16.11], [7: 1.5], and the notion of the integral is extended to sesquilinear forms in \( L_1(\varphi) \). The following proposition gives an “explicit” form of such integral.

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PROPOSITION 1. ([9 Proposition 17.11]) Let

\[ \varphi = \sum_{i \in I} \langle \cdot f_i, f_i \rangle, \quad f_i \in H, \]  

(1)

be a faithful normal semifinite weight on a von Neumann algebra \( \mathcal{M} \), and let a sesquilinear form \( a \) defined on \( D_\varphi \) be integrable with respect to \( \varphi \), i.e. \( a \in L_1(\varphi) \). Then

\[ \varphi(a) = \sum_{i \in I} a(f_i, f_i), \]  

(2)

where the series in (2) converges absolutely and its sum does not depend on the choice of representation of \( \varphi \) in the form (1).

In [9 page 166], the following problem was posed: does the converse to Proposition 1 hold? The theorem below gives an affirmative answer to the question in the special case of normal states.

THEOREM 2. Let \( \varphi \) be a faithful normal state on a von Neumann algebra \( \mathcal{M} \). For a sesquilinear form \( a \) defined on \( D_\varphi \), the following conditions are equivalent:

(i) \( a \in L_1(\varphi) \),

(ii) for any representation \( \varphi = \sum_{i \in I} \langle \cdot f_i, f_i \rangle \), the series \( \sum_{i \in I} a(f_i, f_i) \) converges absolutely and the sum does not depend on the representation of \( \varphi \).

Proof. By virtue of Proposition 1, it suffices to prove (ii) \( \implies \) (i). Moreover, it is clear that we can restrict ourselves to the case when \( a \) is hermitian.

So, let \( \varphi \) be a faithful normal state on \( \mathcal{M} \) and a hermitian sesquilinear form \( a \) on \( D_\varphi \) satisfy (ii).

Denote by \( Y \) the Banach space of hermitian \( \sigma \)-weakly continuous functionals \( \psi \) on \( \mathcal{M} \) such that \( -\lambda \varphi \leq \psi \leq \lambda \varphi \) for some \( \lambda \geq 0 \), supported with the norm

\[ \| \psi \|_\varphi^\varphi = \inf\{ \lambda \geq 0 : -\lambda \varphi \leq \psi \leq \lambda \varphi \}. \]

Observe that if \( -\lambda \varphi \leq \psi \leq \lambda \varphi \) then \( 0 \leq \frac{1}{2}(\lambda \varphi - \psi) \leq \lambda \varphi \), \( 0 \leq \frac{1}{2}(\lambda \varphi + \psi) \leq \lambda \varphi \) and \( \psi = \frac{1}{2}(\lambda \varphi + \psi) - \frac{1}{2}(\lambda \varphi - \psi) \). Therefore the space \( Y \) is generated by its positive part \( Y^+ \). One can verify in a standard way that the restriction operation \( \Psi \mapsto \Psi|_{\mathcal{M}^{sa}} \) determines an isometric and order isomorphism between the Banach conjugate space \( (L_1(\varphi)^{sa})^* \) and \( Y \); and we will identify these spaces.
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Associate with the form \( a \) the linear functional \( F_a \) on \( Y \) in the following way.

a) If \( 0 \leq \psi \leq \lambda \varphi \) and \( \psi = \sum_{i \in I} \langle \cdot, g_i \rangle \) then \( g_i \in D_\varphi \), and we set

\[
F_a(\psi) \equiv \sum_{i \in I} a(g_i, g_i).
\]

The value \( F_a(\psi) \) is defined correctly. Indeed, let \( \psi = \sum_{j \in J} \langle \cdot, h_j \rangle \) be another representation of \( \psi \). Then, assuming that \( \lambda = 1 \) for laying out simplification, we have

\[
\varphi = \sum_{i \in I} \langle \cdot, g_i \rangle + \sum_{k \in K} \langle \cdot, l_k \rangle = \sum_{j \in J} \langle \cdot, h_j \rangle + \sum_{k \in K} \langle \cdot, l_k \rangle
\]

for some \( l_k \in H \). Consequently,

\[
\sum_{i \in I} a(g_i, g_i) + \sum_{k \in K} a(l_k, l_k) = \sum_{j \in J} a(h_j, h_j) + \sum_{k \in K} a(l_k, l_k),
\]

hence, \( \sum_{i \in I} a(g_i, g_i) = \sum_{j \in J} a(h_j, h_j) \).

b) The functional \( F_a \) defined above on \( Y^+ \) is additive and positively homogeneous, therefore it can be uniquely extended to the linear functional on \( Y \).

It is easily seen that \( F_a \) has the property:

\[
\left( \psi, \psi_n \in Y^+ \quad \& \quad \psi = \sum_{n=1}^{\infty} \psi_n \right) \implies F_a(\psi) = \sum_{n=1}^{\infty} F_a(\psi_n). \quad (3)
\]

It follows, in particular, that \( F_a \) is bounded. Indeed, it suffices to prove that

\[
\sup\{ |F_a(\psi)| : 0 \leq \psi \leq \varphi \} < \infty.
\]

If the latter were false, there would exist a sequence \( (\psi_n) \) such that \( 0 \leq \psi_n \leq \varphi \) and \( |F_a(\psi_n)| \geq 2^n \). Consider \( \psi = \sum_{n=1}^{\infty} \frac{\psi_n}{2^n} \). Then \( 0 \leq \psi \leq \varphi \), while the series \( \sum_{n=1}^{\infty} F_a(\frac{\psi_n}{2^n}) \) does not converge, a contradiction.

Thus, \( F_a \in Y^* \).

Now, consider the mapping \( \gamma \) which is the isometric and order isomorphism of \( L_1(\varphi)^{sa} \) onto \( M_*^{sa} \) (see [9: Theorem 17.1, Theorem 17.6], [7: Theorem 2]). Then \( \gamma^* \) is the isometric and order isomorphism of \( (M_*^{sa})^* = M^{sa} \) onto \( (L_1(\varphi)^{sa})^* = Y \) and \( \gamma^{**} \) is the isometric and order isomorphism of \( Y^* \) onto \( (M^{sa})^* \).
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Let us show that the functional $\gamma^{**}(F_a)$ on $\mathcal{M}^{sa}$ is $\sigma$-weakly continuous. Take $x_n, x$ in $\mathcal{M}^+$ such that $x = \sum_{n=1}^{\infty} x_n$ in the sense of $\sigma$-weak topology on $\mathcal{M}^{sa}$, that is equivalent to $x = \sup \sum_{n=1}^{k} x_n$. Then $\gamma^*(x) = \sum_{n=1}^{\infty} \gamma^*(x_n)$ and we have by (3):

$$
\gamma^{**}(F_a)(x) = F_a(\gamma^*(x)) = \sum_{n=1}^{\infty} F_a(\gamma^*(x_n)) = \sum_{n=1}^{\infty} \gamma^{**}(F_a)(x_n).
$$

It follows (cf. [5 Corollary III.3.11]) that $\gamma^{**}(F_a)$ is $\sigma$-weakly continuous, i.e. belongs to $\mathcal{M}^{*sa}$. Therefore we can consider the integrable sesquilinear form $\gamma^{-1}(\gamma^{**}(F_a))$ which coincides with $a$ by uniqueness arguments. □

Remark. In the general case of infinite weight the validity of the implication (ii) \(\implies\) (i) question remains open. However, it follows from results of [1] that the implication holds in the special case of standard trace on the algebra $\mathcal{B}(H)$ of all bounded operators on a Hilbert space $H$ (see also [9 Theorem 5.2]).

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