A new perspective of paramodulation complexity by solving massive 8 puzzles

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Abstract—A sliding puzzle is a combination puzzle where a player slide pieces along certain routes on a board to reach a certain end-configuration. In this paper, we propose a novel measurement of complexity of massive sliding puzzles with paramodulation which is an inference method of automated reasoning. It turned out that by counting the number of clauses yielded with paramodulation, we can evaluate the difficulty of each puzzle. In experiment, we have generated 100 * 8 puzzles which passed the solvability checking by counting inversions. By doing this, we can distinguish the complexity of 8 puzzles with the number of generated with paramodulation. For example, board [2,3,6,1,7,8,5,4, hole] is the easiest with score 3008 and board [6,5,8,7,4,3,2,1, hole] is the most difficult with score 48653. Besides, we have succeeded to observe several layers of complexity (the number of clauses generated) in 100 puzzles. We can conclude that proposal method can provide a new perspective of paramodulation complexity concerning sliding block puzzles.

I. INTRODUCTION

A sliding puzzle (also called as sliding block puzzle) is a combination puzzle where a player slide pieces along certain routes on a board to reach a certain end-configuration (state). The pieces are usually numbered, and sometimes may be imprinted with colors, patterns and sections of a large picture.

In nature, sliding puzzles are two-dimensional even if the sliding is facilitated by encaged marbles or three-dimensional tokens.

| Initial State | Goal State |
|---------------|------------|
| 1 2 3         | 1 2 3      |
| 8 4           | 4 5 6      |
| 7 6 5         | 7 8        |

Fig. 1. Initial state and goal state of 8 puzzle.

In sliding puzzles, a player is prohibited to lift any piece off the board. This constraint separates sliding puzzles from rearrangement puzzles. Consequently, discovering routes opened up by each move with the two-dimensional confines of the board is interesting point of solving sliding block puzzles.
of support. The usable list are the clauses which were once in the set of support but have been already picked up as the focus of attention for deducing additional clauses. More technically, in detail, OTTER maintains four lists of clauses in reasoning process.

1) Usable. This list works as a rule by keeping clauses which are available to make inferences.
2) SoS. Clauses is regarded as facts. Set of support are not used to make inferences. They are kept to participate in the search.
3) Passive. They are specified to be used only for forward subsumption and unit conflict. The passive list does not participate in the search. The passive list does not change from the start of reasoning process as fixed input.
4) Demodulators. Demodulators are used to rewrite newly inferred clauses with equalities.

In this paper, particularly, we focus on the size of set of support list. Set of support is important indicator for introspecting the reasoning process.

B. Given Clause algorithm

OTTER adopts given-clause algorithm in which the program attempts to use any and all combinations from axioms in given clause. In other words, the combinations of clause are generated from given clauses which has been focused on.

Algorithm 1 Given clause algorithm

Input: SOS, Usable List
Output: Proof

1: while until SoS is empty do
2: \quad choose a given clause G from SoS;
3: \quad move the clause g to Usable List;
4: \quad while c_1, ..., c_n in Usable List do
5: \quad \quad while R(c_1, ..., c_n) \in Usable List do
6: \quad \quad \quad A \leftarrow R(c_1, ..., c_n, G, c_{i+1}, ..., c_n);
7: \quad \quad \quad if A is the goal then
8: \quad \quad \quad \quad report the proof;
9: \quad \quad \quad \quad stop
10: \quad \quad \quad else \{A is new odd\}
11: \quad \quad \quad \quad add A to SoS X
12: \quad \quad \quad end if
13: \quad \quad \end while
14: \quad \end while
15: \end while

At line 2, given clause G is extracted from SoS (Set of Support). Line 4 and 5 is a loop to use any and all combinations of given clause and Usable List. In detail, \cite{12} discuss the basic framework of given clause algorithm. To put it simply, given clause algorithm consists of the following steps.

1) Pick up a clause (called the given clause) from the set of support.
2) Add the given clause to the usable list.
3) Applying the inference rule or rules in the effect, infer all clauses which are generated from the given clause (one parent) and the usable list (other parents).
4) Process newly inferred clause.
5) Append each inferred new clause to the SoS. These clause is not discarded as a result of processing. Exactly, this is done in the course of processing the newly generated clause.

In a nutshell, the reasoning program chooses a clause from the clauses which is focused on in the set of support. The selected clause is called as focal clause or given clause.

Definition 1. Definition of given clause. The reasoning program chooses a clause on which to focus from among those in the set of support, where the choice is based on various criteria such as the weight of the clause. The chosen clause is based on various criteria from among those in the clause. The chosen clause is called the “focal clause” (formerly the “given clause”). The algorithm under discussion permits the focal clause to be considered by whatever inference rules are being used, where the remaining clauses required by the inference rule is selected from the usable list, but not from the set of support. An equality literal is a literal whose predicate is to be interpreted as meaning “equal”. The inference rule yields the clause C from the clauses A and B that are assumed to have no variables in common when A contains a positive equality literal and B contains a term which unifies with one of the arguments of that equality literal.

III. Paramodulation

Paramodulation is powerful method of equational reasoning. It is the method based on resolution refutations which includes the equality. In the view of equational reasoning, paramodulation is a generalization of equality substitution. For example, if the equality of \( s = t \) and \( s \) occurs in the sentence \( S \), paramodulation can replace \( s \) in any of the occurrences. Also, on the rule of \( s = t \), if \( t \) occurs in \( S \), then reasoning program can replace \( s \) with \( t \).

\[
\frac{C \lor AD \lor \neg B}{(C \lor A) \sigma} \quad \text{if } \sigma = \text{mgu}(A, B) \quad (3)
\]

here, mgu (A, B) denotes a most general unifier of A and B, and factoring:

\[
\frac{C \lor A \lor B}{(C \lor A) \sigma} \quad \text{if } \sigma = \text{mgu}(A, B) \quad (4)
\]

Definition 2. Definition of paramodulation. An equality means if a literal whose predicate is to be represented as equal. The inference rule of paramodulation yields the clause \( C \) between the clause A and B which are assumed to have no variables in common. Also, if A contains a positive equality literal and B contains a term which unifies with one of the arguments, C is yielded by paramodulation.

In the equations above (3)(4), Clause A is called the from clause, clause B is called the into clause, and clause C a
paramodulant. Corresponding to the syntax of OTTER, for example, given the following two clauses, from the first into the second

\[ \text{EQUAL}(a,b) \].
\[ Q(a) . \]
the clause \[ Q(b) . \] is yielded by adopting paramodulation. For a second example, given the following two clauses, from the first into the second

\[ \text{EQUAL}(\text{sum}(x,0), x) . \]
\[ P(\text{sum}(\text{sum}(a,0), b),c) . \]
Paramodulation yields
\[ P(\text{sum}(a,b),c) . \]

as a paramodulant, from
\[ Q(g(f(g(x)))) . \]
\[ \text{EQUAL}(g(a),b) . \]
the clause
\[ Q(g(f(b))) . \] is deducible.

In general, paramodulation is intended to be utilized, along with resolution, for theorem proving in first-order theories with equality.

Concerning the implementation of OTTER, in paramodulation, two parents and a child are processed. The parent clauses contain the equality applied for the replacement. The parent clauses are divided into two: from parent and from clause. If the equality comes from the literal, the side of the equality unifies with the term which is replaced with the from term. The replaced term is called as the into term. The literal containing the replaced term is also called as the into literal. Also the parent containing the replaced term is called as the into parent or into clause.

IV. METHODOLOGY

A. Setting OTTER’s rule set

As we discussed before, the basic inference mechanism of OTTER is based on the given-clause algorithm. Given-clause algorithm can be viewed as a simple implementation of the set of support strategy. OTTER maintains four lists of clauses: usable, SoS, demodulator and passive. In our case, we cope with two kinds of clauses: usable and SoS.

Horizontal sliding from row[i] to row[i+4] is represented as follows.

\[ \text{list(usable).} \]
\[ \text{EQUAL}(l(\text{hole},l(n(x),y)),l(n(x),l(\text{hole},y))) . \]
\[ \text{end_of_list.} \]

Vertical sliding from row[i] to row[i+4] is represented as follows.

\[ \text{list(usable).} \]
\[ \text{EQUAL}(l(\text{hole},l(x,l(y,l(z,l(u,l(n(w),v)))))), l(n(w),l(x,l(y,l(z,l(u,l(\text{hole},v))))))). \]
\[ \text{end_of_list.} \]

\[ \sigma = \left\{ \begin{array}{cccc}
1 & \text{hole} & 2 & 3 \\
1 & 2 & 3 & \text{hole} \\
6 & 7 & 8 & 9
\end{array} \right\} \] (5)

B. Checking the solvability of N puzzles

In general, to check the solvability of N puzzles, the number of inversions of each number of N slots is calculated. For example, if we have the board configuration board [2,3,6,1,7,8,5,4, hole] (5,2,8,4,1,7, hole, 3,6), the number of inversions as follows:

1) 2 precedes 1 - 1 inversions
2) 3 precedes 1 - 1 inversion
3) 6 precedes 1, 5, 4 - 3 inversions
4) 1 precedes none - 0 inversions
5) 7 precedes 5, 4 - 2 inversions
6) 8 precedes 5, 4 - 2 inversions
7) 5 precedes 4 - 1 inversions
8) 4 precedes none - 0 inversions

Total inversions 1+1+3+0+2+2+1+0 = 10 (Even Number)
So this puzzle configuration is solvable. On the other hand, it is not possible to solve an instance of 8 puzzle if number of inversions is odd in the input state.

Algorithm 2 shows the procedure for checking the solvability of N puzzles. At line 2 to 9, the number of inversions of each slots is counted. These figures are counted up at line 11 to 14. Finally, the sum is checked if it is even or odd number at line 15 to 19.

C. Incrementing the number of generated clauses

The main loop for inferring and processing clauses and searching for a refutation operates mainly on the lists usable and SoS.

1) Choose appropriate given clause in SoS;
2) Move given clause from list(SoS) to list(usable)
3) Infer and process new clauses using the inference rules set.
4) Newly generated clause must have the given clause.
5) Do the retention test on new clauses and append those to list(SoS).

Main loop is depicted in Algorithm 3. At line 9, the number of generated clauses is incremented. After line 8 of picking up the clause from set of support, we can record the current size of set of support. By doing this, we can obtain the plot with # puzzles and the number of generated clauses of Y-axis as shown in the next section.
Algorithm 2 Checking the solvability of N puzzles

Input: Board\([x_1, x_2, ..., x_n, \text{hole}]\)

Output: SOLVABLE or UNSOLVABLE

1. Board\([X|XS] = Board[x_1, x_2, ..., x_n, \text{hole}]\)
2. while \(XS\) in Board\([X|XS]\) is empty do
3. for \(i\) in \(XS\) do
4. statements...
5. if \((X \neq XS[i])\) then
6. \(\text{counter}[i]++\)
7. end if
8. end for
9. end while
10. line = check(Board[...] \(\subseteq\) hole)
11. sum = 0
12. for \(i\) to \(n\) do
13. \(\text{sum}+ = \text{counter}[i]\)
14. end for
15. if \((\text{line} + \text{sum} \% 2 == 0)\) then
16. \(\text{flag} =\) SOLVABLE
17. else
18. \(\text{flag} =\) UNSOLVABLE
19. end if

Algorithm 3 Incrementing the number of generated clauses

1. while given clause is NOT NULL do
2. \(\text{index}_\text{lits}\_\text{clash}(\text{giv}_\text{cl});\)
3. \(\text{append}_\text{cl}(\text{Usable, giv}_\text{cl});\)
4. if splitting() then
5. \(\text{possible}_\text{given}_\text{split}(\text{giv}_\text{cl});\)
6. end if
7. \(\text{infer}_\text{and}_\text{process}(\text{giv}_\text{cl});\)
8. \(\text{giv}_\text{cl} = \text{extract}_\text{given}_\text{clause}();\)
9. \(\text{track}(\text{the number of generated clauses});\)
10. end while

V. EXPERIMENTAL RESULTS

In experiment, we have generated 100 sliding puzzles with size 8 * 8. All generated configurations of 8 puzzle are solvable. For each puzzle, we have measured the number of generated clauses with the procedures shown in Algorithm 2. For simplicity, we have generated the configuration of first 8 slots with random integers ranging from 1 to 8 and fixed 9th slot to hole, as shown in the left side of Table I.

The number of generated clauses with paramodulation ranges from 3008 (2,3,6,1,7,8,5,4, hole) to 468453 (6,5,8,7,4,3,2,1, hole). In the view of complexity of reasoning process, the configuration \([6,5,8,7,4,3,2,1, \text{hole}]\) is 155.73 times harder to solve than the configuration \([2,3,6,1,7,8,5,4, \text{hole}]\).

Table I

| Initial state | Clauses generated |
|---------------|-------------------|
| 2,3,6,1,7,8,5,4, hole | 3008 (easiest) |
| 2,4,3,8,7,6,1, hole | 31344 |
| 2,5,3,8,6,1,7,4, hole | 272413 |
| 6,5,8,7,4,3,2,1, hole | 468453 (the most difficult) |

Curiously, in sorted graph in Figure 3, the number of clauses is not increasing linearly. Instead, the number of clauses generated with paramodulation is increased drastically around X-axis 22, 38, 62, 79 and 97. Consequently, we can conclude that there are several layers of complexity in Figure 3. Also, in each layer, the number of clauses generated is increasing linearly.

VI. RELATED WORK

Archer [1] firstly discusses an algorithmic analysis of 15 puzzle. In [1], a summary of all possible permutations of slots attained by moving the black block from cell i to cell j effecting the permutation of \(\sigma_i, j\). Howe [2] proposes two approaches in the two kinds of viewpoints: the properties of permutations and graph theory. Ariyanto [3] proposes the
new sliding puzzle made with several additional rules from M13 puzzle. Calabro [4] proposes $O(n^2)$ time algorithm for deciding the time when tie initial configuration of the n * n puzzle game is solvable. Conrad [5] discusses 15 puzzle and rubik cube as permutation puzzle. Bischoff [6] adopts reinforcement learning to solve 15-puzzle. Ando [14] applies hot list strategy [15] for faster paramodulation-based viral code detection. Takefuji proposes the application of paramodulation to translator of Common Lisp [17]. Ando and Takefuji applies hot list strategy based on paramodulation for faster graph coloring [16].

Paramodulation originated as a development of resolution [18], one of the main computational methods in first-order logic, see [19]. For improving resolution-based methods, the study of the equality predicate has been particularly important, since reasoning with equality is well-known to be of great importance of mathematics, logic and computer science.

Dan Carson and Larry Wos developed a resolution based theorem prover they called P1 which stands for “Program 1”. P1 is the founder of OTTER and includes basic strategy of OTTER including the set-of-support strategy [13], unit preference [20] and paramodulation. P1 is the first implementation of theorem prover where Wos’s invention of the paramodulation inference rule [21] is experimented. RW1 which stands for “Robinson-Wos 1” which is the product from the collaboration of Wos and George Robinson. RW1 adopts the paramodulation inference rule as well as demodulation [21]. Also, RW1 is based on the concept by Knuth and Bendix, who independently formulated paramodulation and demodulation in the view of a complete set of reductions in their 1970 paper [22].

VII. CONCLUSION

In this paper we propose a novel method for providing new perspective of paramodulation complexity by solving 100 sliding block puzzles (8 puzzle). Paramodulation is designed based on the concept of generalization of a substitution rule for equality. We have counted the number of clauses generated with paramodulation as the complexity of each sliding block puzzle as shown in Algorithm 3. As a result, a wide range of complexity of 100 solvable 8 puzzles have been measured. For example, board [2,3,6,1,7,8,5,4, hole] is the easiest with score 3008 and board [6,5,8,7,4,3,2,1, hole] is the most difficult with score 48653.

There have been many research efforts on the measurement and evaluation of the complexity of sliding block puzzles. However, the method for coping with complexity of the puzzle in the aspect of the computation cost in automated reasoning has never been proposed. We have succeeded to figure out more computational method for the comparison of difficulty of 100 * 8 puzzles with the help of automated reasoning. Besides, we have observed several layers of complexity (the number of clauses generated) in 100 puzzles as shown in Figure 3. We can conclude that proposal method can provide a new perspective of paramodulation complexity concerning sliding block puzzles.