An ill-posed problem in hydrodynamic stability of multi-layer Hele-Shaw flow

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Abstract

An useful approximation for the displacement of two immiscible fluids in a porous medium is the Hele-Shaw model. We consider several liquids with different constant viscosities, inserted between the displacing fluids. The linear stability analysis of this model leads us to an ill-posed problem. The growth rates (in time) of the perturbations exist iff some compatibility conditions on the interfaces are verified. We prove that these conditions cannot be fulfilled.

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1. Introduction

The Stokes flow in a rectangular Hele-Shaw cell is studied in [1], [4], [5]. In [9] has been shown that the sharp interface, which exists between two immiscible fluids in a Hele-Shaw cell (or an equivalent porous medium), becomes unstable if the displacing fluid is less viscous.

Some experimental and numerical results (see [6] and references therein) have shown that an intermediate liquid with a continuous variable increasing viscosity in the flow direction, inserted between the displacing fluids, can minimize the Saffman - Taylor instability. A theoretical model for this three-layer flow was first considered in [2], [3]. An optimal intermediate viscosity, which minimizes the instability, was obtained by using a numerical procedure.

In a large number of papers was considered the multi-layer Hele-Shaw flow: the injection of a sequence of fluids, with different viscosities, in a

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homogeneous equivalent porous medium. Some instability control seems be obtained by using this flow model - see [7], [8] and references therein.

We consider the following problem: can we use a sequence of liquids with constant viscosities, inserted between the initial fluids, in order to minimize the Saffman-Taylor instability?

In this paper we give a negative answer for the above problem. The analysis of the linear stability of this model leads us to an ill-posed problem. The growth rates (in time) of the perturbations are obtained by using usual boundary conditions on the interfaces, based on the Laplace-Young law. We show that these conditions, related also to the amplitude of the velocity perturbations in each layer, cannot be fulfilled. Thus the growth rates can not exist.

2. The flow model

In [2], [3] was proposed an “optimal policy” in order to minimize the Saffman-Taylor instability, which appears when a less viscous fluid is displacing a more viscous one in a Hele-Shaw cell. The point is to consider an intermediate region, filled by a given amount of polymer-solute, with a variable continuous viscosity. This “three-layer” structure is moving with the velocity, say, $U$, of the displacing fluid far upstream. The cell is parallel with the plane $\pi Oy$ and the flow is in the positive direction $Ox$. A basic solution is considered, with two straight interfaces. The length of intermediate region (between interfaces) is constant. The viscosity $\mu(\pi)$ of the intermediate liquid is invertible with respect to the concentration of the polymer-solute. The continuity equation for the polymer-solute is used to get the equation of the (a priori) unknown viscosity $\mu(\pi)$ in the intermediate region:

$$\mu_t + u\mu_\pi + v\mu_y = 0,$$

where $(u, v)$ are the velocity components and the indices $t, \pi, y$ denote the partial derivatives with respect to time and $\pi, y$. In [2], [3] were considered a viscosity jump and a surface tension only on the second interface, between the intermediate region and the displaced fluid. On the first interface, the viscosity was continuous and the surface tension was missing. A linear stability analysis of the basic slolution was performed and an “optimal” viscosity was obtained by using a numerical procedure. This optimal viscosity gives us the minimum values of the growth rates (in time) of perturbations.
If the intermediate region is large enough, the viscosity is continuous and
surface tensions are missing on both interfaces, then there exist a variable
viscosity which leads us to an arbitrary small (positive) growth rates of per-
turbations - see [6].

In this paper we consider a basic solution with several constant interme-
diate viscosities. The surface tensions exist on all interfaces. As in [2], [3], the
Laplace-Young law is used to get the boundary conditions on the interfaces
(which depends also on the amplitudes of the perturbations of the horizon-
tal velocity). We show that these conditions cannot be fulfilled. Thus the
growth rates can not exist.

We recall below the model given in [2], [3], with variable intermediate
viscosity and use it to get the flow model with constant intermediate viscosity.

The following three-layer basic flow with the intermediate region \( \bar{x} \in (Ut + a, Ut + b) \) is considered

\[
\begin{align*}
    u &= U, \quad v = 0; \quad \bar{x}_L = Ut + a, \quad \bar{x}_R = Ut + b; \\
    P_x &= -\mu_G U; \quad P_y = 0; \\
    \mu_G &= \mu_L, \quad \bar{x} < \bar{x}_L, \quad \mu_G = \mu_R, \quad \bar{x} > \bar{x}_R, \\
    \mu_G &= \mu(\bar{x} - Ut), \quad \bar{x} \in (\bar{x}_L, \bar{x}_R),
\end{align*}
\]

(2)

where \( P \) is the basic pressure, \( \mu_G \) is the basic viscosity, \( \mu_L, \mu_R \) are the constant
viscosities of the displacing and displaced fluids. The last relation is obtained
by using the relations (1) with \( u = U, v = 0 \). We use the moving reference
frame \( x = \bar{x} - Ut \). Thus the intermediate region becomes the segment \( (a, b) \),
in which \( \mu_G = \mu(x) \). Without loss of generality we can consider \( b < 0 \).

The stability system for small perturbations, obtained in [2], is linear.
Thus the following perturbation \( u' \) of the horizontal velocity is considered

\[
u(x, y, t) = f(x)[\cos(ky) + i \sin(ky)]e^{\sigma t}, \quad k \geq 0, \quad i = \sqrt{-1},
\]

(3)

where \( f(x) \) is the amplitude, \( \sigma \) is the growth constant and \( k \) are the wave
numbers. The cross derivation of the perturbed pressures leads us to the
amplitude equation (see the relation (2.17) of [2], where \( f \) is denoted by \( \psi \)
and \( \mu \) is denoted by \( \mu_0 \))

\[
- (\mu f_x)_x + k^2 \mu f = \frac{1}{\sigma} U k^2 f \mu_x, \quad \forall x \notin \{a, b\},
\]

(4)
The viscosity is constant outside the intermediate region, thus it follows

\[-f_{xx} + k^2 f = 0, \quad x \notin (a, b);\]

\[-(\mu f_x)_x + k^2 \mu f = \frac{1}{\sigma} U k^2 f \mu_x, \quad \forall x \in (a, b).\]  \hspace{1cm} (5)

The perturbations must decay to zero in the far field and \( f \) is continuous, therefore we get

\[f(x) = f(a)e^{k(x-a)}, \quad \forall x \leq a; \quad f(x) = f(b)e^{-k(x-b)}, \quad \forall x \geq b.\]  \hspace{1cm} (6)

As we mentioned above, the boundary conditions are obtained by using the Laplace - Young law. It means the pressure jump (across the interfaces) should equal the surface tension \( T \) times the curvature and the horizontal velocity should be continuous.

Let’s consider two surface tensions \( T(a), T(b) \) and two viscosity jumps at \( x = a \) and \( x = b \). Thus \( f_x \) could be discontinuous in \( a, b \). The right and left limit values at the points \( a, b \) are denoted by the superscripts \( +, - \). Just like in [2], page 84 formula (2.27), we get:

\[\mu^-(a)f^-_x(a) - \mu^+(a)f^+_x(a) = \frac{kE(a)}{\sigma} f(a),\]

\[\mu^-(b)f^-_x(b) - \mu^+(b)f^+_x(b) = \frac{kE(b)}{\sigma} f(b),\]

\[E(a) := kU[\mu^+(a) - \mu_L] - k^3 T(a),\]

\[E(b) := kU[\mu_R - \mu^-(b)] - k^3 T(b).\]  \hspace{1cm} (7)

In both relations (7) we have the same eigenvalue(s), then the unknown amplitude \( f \) must verify the compatibility relation

\[\frac{kE(a)f(a)}{\mu^-(a)f^-_x(a) - \mu^+(a)f^+_x(a)} = \frac{kE(b)f(b)}{\mu^-(b)f^-_x(b) - \mu^+(b)f^+_x(b)}.\]  \hspace{1cm} (8)

The stability of the basic solution (2) is governed by the system (5) - (8).
3. The non existence of the growth rates

We consider a constant intermediate viscosity \( \mu = \mu_1 \) s.t.

\[
0 < \mu_L < \mu_1 < \mu_R,
\]

\[
\mu^-(a) = \mu_L, \quad \mu^+(a) = \mu_1 = \mu^-(b), \quad \mu^+(b) = \mu_R. \tag{9}
\]

Thus both viscosity jumps are positive in the flow direction. As \( \mu \) is constant inside the intermediate region, from (5) we obtain

\[
-f_{xx} + k^2 f = 0, \quad \forall x \in (a, b). \tag{10}
\]

The stability system for the flow with constant viscosity (9) is given by the equations (6)-(10).

A crucial difference from the Gorell and Homsy model exists: the solution of (10) is independent of \( \sigma \). This property leads us to the non-existence of the eigenvalues. We recall (6) - (9) and we introduce the notations

\[
VL := \frac{kE(a)f(a)}{\mu_L kf(a) - \mu_1 f^+(a)}, \quad VR := \frac{kE(b)f(b)}{\mu_1 f^-(b) + k\mu_R f(b)}, \tag{11}
\]

Therefore the compatibility condition (8) is equivalent with \( VL = VR \).

**Proposition 1.** The growth constants of the problem (6) - (10) can not exist. To this end, we prove that the condition (8) cannot be fulfilled.

Proof. Suppose that there exist growth rates of the considered problem. The solution of (10) is \( f(x) = Ae^{kx} + Be^{-kx} \), where \( A, B \) can depend on \( k \) and \( |f(x)| < \infty, \quad \forall k \). Thus the relations (7) become

\[
\mu_L(Ae^{ka} + Be^{-ka}) - \mu_1(Ae^{ka} - Be^{-ka}) = \frac{E_a}{\sigma}(Ae^{ka} + Be^{-ka}),
\]

\[
\mu_1(Ae^{kb} - Be^{-kb}) + \mu_R(Ae^{kb} + Be^{-kb}) = \frac{E_b}{\sigma}(Ae^{kb} + Be^{-kb}),
\]

and we get the system

\[
Ae^{ka}(\mu_L - \mu_1 - \frac{E_a}{\sigma}) + Be^{-ka}(\mu_L + \mu_1 - \frac{E_a}{\sigma}) = 0,
\]

\[
Ae^{kb}(\mu_1 + \mu_R - \frac{E_b}{\sigma}) - Be^{-kb}(\mu_1 - \mu_R - \frac{E_b}{\sigma}) = 0. \tag{12}
\]
A solution \((A, B) \neq (0, 0)\) exists if the following condition is verified:

\[
e^{k(a-b)}(\mu_L - \mu_1 - \frac{E_a}{\sigma})(\mu_1 - \mu_R - \frac{E_b}{\sigma}) + \\
e^{k(b-a)}(\mu_1 + \mu_R - \frac{E_b}{\sigma})(\mu_L + \mu_1 - \frac{E_a}{\sigma}) = 0.
\]  \hspace{1cm} (13)

We study the values of the growth rates when \(k\) is large enough, such that \(e^{2k(a-b)} \approx 0\) (recall \(a - b < 0, k > 0\)). Thus

\[
(\mu_1 + \mu_R - \frac{E_b}{\sigma})(\mu_L + \mu_1 - \frac{E_a}{\sigma}) = 0.
\]  \hspace{1cm} (14)

The relation \(14\) is a second order equation for \(\sigma\) and we have two real roots:

\[
\sigma_1 = \frac{E_b}{\mu_1 + \mu_R}, \quad \sigma_2 = \frac{E_a}{\mu_L + \mu_1}.
\]  \hspace{1cm} (15)

We insert \(\sigma_1\) in \((12)\) and get \(2\mu_R B e^{-kb} = 0\). We have \(\mu_R > 0\), then it follows \(B = 0\). So actually the solution to the equation \(10\) is \(f(x) = A e^{kx}\).

We use the notations \(11\), we impose the condition \(VL = VR\), then we get

\[
\frac{kU(\mu_1 - \mu_L) - k^3T(a)}{\mu_L - \mu_1} = \frac{kU(\mu_R - \mu_1) - k^3T(b)}{\mu_1 + \mu_R}.
\]  \hspace{1cm} (16)

We equate the coefficients of \(k, k^3\) and get

\[
\mu_R = 0, \quad \mu_1[T(a) + T(b)] = \mu_L T(b).
\]  \hspace{1cm} (17)

From the relation \(16\) with large \(k\) we obtain

\[
\frac{T(a)}{T(b)} = \frac{\mu_L - \mu_1}{\mu_1 + \mu_R} \Rightarrow \mu_L > \mu_1.
\]  \hspace{1cm} (18)

The relationship \(17\) and \(18\) contradict the hypothesis \(9\).

We insert \(\sigma_2\) in \((12)\) and get \(2\mu_1 A e^{ka} = 0\). The viscosity \(\mu_1\) must be strictly positive, thus \(A = 0\) and the solution of \(10\) becomes \(f(x) = B e^{-kx}, \ x < 0, k > 0\). The values of \(f(x)\) must be finite, thus we impose the condition \(\max_k B e^{-kx} < \infty\). We use the notations \(11\) with \(f(x) = B e^{-kx}\), we impose the condition \(VL = VR\), then it follows

\[
\frac{kU(\mu_1 - \mu_L) - k^3T(a)}{\mu_L + \mu_1} = \frac{kU(\mu_R - \mu_1) - k^3T(b)}{-\mu_1 + \mu_R}.
\]  \hspace{1cm} (19)
We equate the coefficients of \(k, k^3\) and get
\[
\mu_L = 0, \quad \mu_1[T(a) + T(b)] = \mu_R T(a).
\] (20)

The condition (19) for large enough \(k\) gives us
\[
\frac{T(a)}{T(b)} = \frac{\mu_L + \mu_1}{\mu_R - \mu_1},
\] (21)

which is a restriction on the viscosities. The relations (20), (21) contradict the hypothesis (9). The bottom line is: the condition (8) is not satisfied, thus the problem (6) - (10) has no eigenvalues.

\[ \square \]

**Remark 1.** We consider now a constant intermediate viscosity \(\mu = \mu_1\) s.t.
\[
0 < \mu_L > \mu_1, \quad \mu_1 < \mu_R.
\]

We get the same relationship (17) - (18) and (20) - (21). This time, the relationship \(\mu_L > \mu_1\) is not a contradiction, but \(\mu_R = 0\) and \(\mu_L = 0\) contradicts the hypothesis (9). For large \(k\), the relations (18) and (21) represent restrictions on the viscosity \(\mu_1\), which also contradict the assumption (9).

\[ \square \]

We use the above results for the case of \(N\) intermediate layers. Consider \((N + 1)\) interfaces \(x_i\) and \(N\) constant viscosities \(\mu_i\) in each layer \((x_{i-1}, x_i)\):
\[
x_0 = a < x_1 < x_2 < \ldots < x_N = b,
\]
\[
0 < \mu_L \equiv \mu_0 < \mu_1 < \mu_2 < \ldots < \mu_i \ldots < \mu_N < \mu_R.
\] (22)

The surface tensions, amplitudes, viscosities and limit values in the points \(x_i\) are denoted by
\[
T_i = T(x_i); \quad f_i = f(x_i);
\]
\[
f_{x}^{+,-}(i) = f_{x}^{+,-}(x_i); \quad \mu^{+,-}(i) = \mu^{+,-}(x_i);
\]
\[
\mu(x) = \mu_i, \quad \forall x \in (x_{i-1}, x_i); \quad \mu_i = \mu^+(i - 1) = \mu^-(i).
\] (23)

The stability of the flow model with viscosities (22) is governed by the relations (6) and the system (24)-(27) below:
\[
-f_{xx} + k^2 f = 0, \quad x \in (a, b), \quad x \neq x_i;
\] (24)
\[ \mu^-(i)f_x^+(i) - \mu^+(i)f_x^+(i) = \frac{kE_if_i}{\sigma}, \]
\[ E_i = kU[\mu^+(i) - \mu^-(i)] - k^3T_i; \quad (25) \]
\[ V_0 = V_1 = ... = V_i = ... = V_N; \quad (26) \]
\[ V_i \equiv \frac{kE_if_i}{\mu^-(i)f_x^+(i) - \mu^+(i)f_x^+(i)}; \quad (27) \]

**Proposition 2.** The above conditions (26) cannot be fulfilled.

Proof. i) The solutions of the equations (24) in the intervals \((x_{i-1}, x_i)\) could be
\[ f_i(x) = A_i e^{kx} + B_i e^{-kx}, \]
where \(A_i, B_i\) are absolute constants. However, for large \(k\), the terms \(|B_i e^{-kx}|\) become very large. Thus \(|u'|\) also becomes very large and we exceed the frame of small perturbations. The linear stability analysis makes no sense. For this reason, we consider only the solutions \(f_i(x) = A_i e^{kx}\), where \(A_i\) are absolute constants. The amplitude \(f\) is continuous at the points \(x_i\). That means
\[ A_i \exp(kx_i) = A_{i+1} \exp(kx_i), \quad \forall k \Rightarrow A_i = A_{i+1}, \quad \forall i. \]
Therefore \(A_i\) are the same for all \(x \in [a, b]\) and exists a constant, say, \(A\) such that \(A_i = A, \quad \forall i.\)

In the following, we show that the compatibility relation \(V_0 = V_N\) cannot be fulfilled. We will highlight some restrictions on the viscosities \(\mu_i\) which contradict the hypothesis (22).

We impose the condition \(V_0 = V_N\) with \(f(x) = A e^{kx}\), thus
\[ \frac{kU(\mu_1 - \mu_L)}{\mu_L - \mu_1} = \frac{kU(\mu_R - \mu_N)}{\mu_N + \mu_R}, \quad (28) \]
We equate the coefficients of \(k, k^3\) and get \(\mu_R = 0, \quad \mu_N T_0 + \mu_1 T_N = \mu_L T_N, \)
in contradiction with (22). The equation (28) with large enough \(k\) gives us \(T_0/T_N = (\mu_L - \mu_1)/(\mu_N + \mu_R), \) so \(\mu_L > \mu_1, \) which also contradict the hypothesis (22).

ii) The most general solution of (24) is
\[ f(x) = A_i e^{kx} + B_i e^{-kx}, \quad x \in (x_{i-1}, x_i), \]
\[ A_i = A_i(k), \quad B_i = B_i(k), \quad 1 \leq i \leq N. \quad (29) \]

In order to remain in the frame of small perturbations, the maximal values and the limits of \( A_i e^{kx}, B_i e^{-kx} \) (for large \( k \)) must be finite. It seems natural to impose the following conditions:

\[ \max_k \{ A_i e^{kx} \} < \infty, \quad \lim_{k \to \infty} A_i e^{kx} = AI < \infty; \]
\[ \max_k \{ B_i e^{-kx} \} < \infty, \quad \lim_{k \to \infty} B_i e^{-kx} = BI < \infty. \quad (30) \]

We have to prove \( V_0 \neq V_N \). To this end, we introduce the notations

\[ A = A_1, \quad B = B_1, \quad C = A_N, \quad D = B_N, \]
\[ f(x) = A(k)e^{kx} + B(k)e^{-kx}, \quad x \in (x_0, x_1); \]
\[ f(x) = C(k)e^{kx} + D(k)e^{-kx}, \quad x \in (x_{N-1}, x_N). \]

Suppose \( V_0 = V_N \), then

\[ \frac{E_0}{E_N} = \frac{\mu_L - \mu_1 X}{\mu_N Y + \mu_R}, \quad (31) \]
\[ X = \frac{A e^{ka} - B e^{-ka}}{A e^{ka} + B e^{-ka}}, \quad Y = \frac{C e^{kb} - D e^{-kb}}{C e^{kb} + D e^{-kb}}, \]
\[ F_1 := \lim_{k \to \infty} X = \frac{A_1 - B_1}{A_1 + B_1}, \quad F_N := \lim_{k \to \infty} Y = \frac{A_N - B_N}{A_N + B_N}. \]

We consider large values of \( k \) in (31), the above relations gives us

\[ \frac{T_0}{T_N} = \frac{\mu_L - \mu_1 F_1}{\mu_N F_N + \mu_R}. \]

This is a restriction on the viscosities \( \mu_i \) which contradicts the hypothesis (22). If \( B_1 = B_N = 0 \), then we recover the formula (18).

Our conclusion is following: the conditions (26) cannot be fulfilled, thus the eigenvalues of the problem (24)-(27) cannot exist. We use Remark 1 and get the same result for negative viscosity jumps in the flow direction.

\[ \square \]

**Remark 2.** In the last part of [6] was studied the linear stability of the flow with an intermediate liquid with continuous viscosity and without surface tensions; we obtained \( \mu^+(a) = \mu_L, \mu^-(b) = \mu_R, T(a) = T(b) = 0 \). The growth rates appear only in the equation (5) and not in the boundary conditions (7). Both terms in (5) become zero.

\[ \square \]
References

[1] J. Bear, Dynamics of Fluids in Porous Media, Elsevier, New York, 1972.

[2] S.B. Gorell and G.M. Homsy, A theory of the optimal policy of oil recovery by secondary displacement process, *SIAM J. Appl. Math.* 43(1983), 79-98.

[3] S.B. Gorell and G. M. Homsy, A theory for the most stable variable viscosity profile in graded mobility displacement process, *AIChE Journal*, 31(1985), 1598-1503.

[4] H. S. Hele-Shaw, Investigations of the nature of surface resistance of water and of streamline motion under certain experimental conditions, Inst. Naval Architects Transactions 40(1898), 21-46.

[5] H. Lamb, Hydrodynamics, Dower Publications, New York, 1933.

[6] G. Pasa, A paradox in Hele-Shaw displacements, Ann. Univ. Ferrara, 66(1)(2020), 99-111.

[7] P. Daripa and X. Ding, Selection Principle of Optimal Profiles for Immiscible Multi-Fluid Hele-Shaw Flows and Stabilization, Transp. Porous. Media, 96(2) (2013), 353-367.

[8] P. Daripa and X. Ding, Universal stability properties for multi-layer Hele-Shaw flows and application to instability control, SIAM J. Appl. Math., 72(5) (2012), 667-1685.

[9] P.G. Saffman and G.I. Taylor, The penetration of a fluid in a porous medium or Helle-Shaw cell containing a more viscous fluid, *Proc. Roy. Soc. A*, 245(1958), 312-329.