Classifying Leavitt path algebras up to involution preserving homotopy

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Abstract
We prove that the Bowen–Franks group classifies the Leavitt path algebras of purely infinite simple finite graphs over a regular supercoherent commutative ring with involution where 2 is invertible, equipped with their standard involutions, up to matricial stabilization and involution preserving homotopy equivalence. We also consider a twisting of the standard involution on Leavitt path algebras and obtain partial results in the same direction for purely infinite simple graphs. Our tools are \( K \)-theoretic, and we prove several results about (Hermitian, bivariant) \( K \)-theory of Leavitt path algebras, such as Poincaré duality, which are of independent interest.

1 Introduction

A directed graph \( E \) consists of a set \( E^0 \) of vertices and a set \( E^1 \) of edges together with source and range functions \( r, s : E^1 \to E^0 \). This article is concerned with the Leavitt path algebra \( L(E) \) of a directed graph \( E \) over a commutative ring \( \ell \) with involution [4]. When \( \ell = \mathbb{C} \), with complex conjugation as involution, \( L(E) \) is a normed \( \ast \)-algebra; its completion is the graph \( \mathbb{C} \)-algebra \( \mathbb{C}^\ast(E) \). A graph \( E \) is called finite if both \( E^0 \) and \( E^1 \) are; a finite graph \( E \) is purely infinite simple if and only if \( \mathbb{C}^\ast(E) \) is, which in turn is equivalent to a combination of graph-theoretic conditions on \( E \) [4, Section 5.6]. A result of Cuntz and Rørdam [27, Theorem 6.5] says that \( \mathbb{C}^\ast \)-algebras of finite purely infinite simple graphs, i.e. purely infinite simple Cuntz–Krieger algebras, are classified
up to (stable) isomorphism by the Bowen–Franks group $\mathcal{B}_\mathfrak{G}$ of the corresponding graph, defined as

$$\mathcal{B}_\mathfrak{G}(E) = \text{Coker}(I - A_E^t).$$

Here $A_E$ is the incidence matrix of $E$. It is an open question whether a similar result holds for Leavitt path algebras [6]. Here we consider the question for the Leavitt path algebra $L(E)$ over a fixed commutative ring $\ell$ with involution, viewed as a $*$-algebra over $\ell$ by means of its usual $\ell$-semilinear involution $*$.

The following classification theorem follows from Theorem 14.2 (see Example 14.3); the vocabulary and notations therein are explained right after the theorem.

**Theorem 1.1** Let $E$ and $F$ be finite, purely infinite simple graphs and let $\ell$ be regular supercoherent such that 2 is invertible and $-1$ is positive in $\ell$. Let $\xi_0 : \mathcal{B}_\mathfrak{G}(E) \rightarrow \mathcal{B}_\mathfrak{G}(F)$ be a group isomorphism. Then there exist $*$-homomorphisms $\phi : LE \rightarrow LF$ and $\psi : LF \rightarrow LE$, compatible with $\xi_0$ and $\xi_0^{-1}$, both very full, and such that $\psi \circ \phi \sim_{M_{2,2}}^* \text{id}_{LE}$ and $\phi \circ \psi \sim_{M_{2,2}}^* \text{id}_{LF}$. If furthermore $\xi_0([1]_E) = [1]_F$, then $\phi$ and $\psi$ can be chosen to be unital.

In the theorem above,

$$[1]_E = \sum_{v \in E^0} [v] \in \mathcal{B}_\mathfrak{G}(E).$$

(1.1)

A ring $R$ is regular if every $R$-module admits a projective resolution of finite length, coherent if the category of finitely presented $R$-modules is abelian, and supercoherent if $R[t_1, \ldots, t_n]$ is coherent for all $n \geq 0$. The hypothesis that $-1$ be positive in $\ell$ means that there exist $n \geq 1$ and $x_1, \ldots, x_n \in \ell$ such that

$$-1 = \sum_{i=1}^n x_i x_i^*.$$  

(1.2)

Hypothesis (1.2) is satisfied, for example, when $\ell = \mathbb{C}$ with trivial involution, but fails for $\mathbb{C}$ with complex conjugation as involution. A $*$-homomorphism is an involution preserving homomorphism between $*$-algebras over $\ell$. A projection $p$ in a unital $*$-algebra $R$ is very full if there is $x \in pR$ such that $x^* x = 1$. If $E$ is purely infinite simple and finite and $R$ is a unital $*$-algebra, a $*$-homomorphism $\phi : L(E) \rightarrow R$ is very full if $\phi(1)$ is very full. The compatibility condition above is made explicit in Theorem 14.2 below. We write $M_{2,2} R$ for the algebra of $2 \times 2$ matrices equipped with the conjugate transpose involution, and $M_{\pm 2} R$ for the same algebra but with the following involution

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^* = \begin{bmatrix} a^* & -c^* \\ -b^* & d^* \end{bmatrix}.$$  

(1.3)

Consider the upper left hand corner inclusions $\iota_1 : R \rightarrow M_{2,2} R$ and $\iota_+ : R \rightarrow M_{\pm 2} R$. We write $f \sim_{M_{\pm 2}}^* g$ to indicate that two $*$-homomorphisms $f, g : A \rightarrow R$ become homotopic via an involution preserving homotopy upon composing both of them with $\iota_+ \iota_1$.}

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In the absence of the positivity hypothesis (1.2), we need to stabilize further. Theorem 14.1 implies the following (see Example 14.3).

**Theorem 1.2** Let $E$ and $F$ be purely infinite simple, finite graphs. Assume that $2$ is invertible in $\ell$. Let $\xi_0 : \mathcal{B}_{\widetilde{\mathcal{G}}}(E) \sim \to \mathcal{B}_{\widetilde{\mathcal{G}}}(F)$ be an isomorphism. Then there are $*$-homomorphisms $\phi : L(E) \to M_{\pm} L(F)$ and $\psi : L(F) \to M_{\pm} L(E)$ compatible with $\xi_0$ and $\xi_0^{-1}$, both very full, and such that $M_{\pm}(\psi) \circ \phi \sim_{M_{\pm}}^{s} \iota_{\pm}^{2} : L(E) \to M_{\pm} M_{\pm} L(E)$ and $M_{\pm}(\phi) \circ \psi \sim_{M_{\pm}}^{s} \iota_{\pm}^{2} : L(F) \to M_{\pm} M_{\pm} L(F)$.

Here $\sim_{M_{\pm}}^{s}$ refers to stable $M_{\pm}*$-homotopy, defined in Sect. 8, right before (8.3). For very full $*$-homomorphisms it agrees with $M_{\pm}^2$-homotopy.

Theorems 1.1 and 1.2 complement [13, Theorem 6.1] where a similar classification result was obtained for Leavitt path algebras over a field, up to not necessarily involution preserving homotopy. In this paper we also prove a version of the latter result that is valid for Leavitt path algebras over any regular supercoherent ring, Theorem 14.4.

We also consider the problem of the existence of homomorphisms connecting the Leavitt path algebras $L_2$ of the graph $\mathcal{R}_2$ consisting of one vertex and two loops, and $L_2^-$ of its Cuntz splice. It is well-known that if $\ell$ is a field, then there is no isomorphism $L_2 \sim \to L_2^-$ that is homogeneous with respect to the standard $\mathbb{Z}$-grading. Associated to this $\mathbb{Z}$-grading, we have a $\mathbb{Z}/2\mathbb{Z}$, even-odd grading. We show in Proposition 5.2 that if $\ell$ is regular supercoherent and such that the canonical map $\mathbb{Z} \to K_0(\ell)$ is an isomorphism (e.g. if $\ell = \mathbb{Z}$ or a field) then there is no $\mathbb{Z}/2\mathbb{Z}$-homogeneous unital $\ell$-algebra homomorphism from $L_2$ to $L_2^-$ nor in the opposite direction. R. Johansen and A. Sørensen have shown in [21] that in the case $\ell = \mathbb{Z}$, $L_2$ and $L_2^-$, equipped with their standard involutions, are not isomorphic as $*$-rings. Here we consider, for any graph $E$ the *signed* involution

$$
: L(E) \to L(E)^{\text{op}}, \quad \bar{v} = v, \quad \bar{e} = -e^*, \quad (v \in E^0, \ e \in E^1). \quad (1.4)
$$

We write $\overline{L(E)}$ for $L(E)$ equipped with the signed involution. Proposition 5.2 also shows that if $\ell = \mathbb{C}$ equipped with complex conjugation, then there is no unital $*$-homomorphism connecting $\overline{L_2}$ and $\overline{L_2^-}$.

The classification theorem for graph $C^*$-algebras of [27] uses Kasparov’s bivariant $C^*$-algebra $K$-theory, which takes involutions into consideration. The classification result for Leavitt path algebras in [13] uses the bivariant algebraic $K$-theory of [15], which is defined on the category of all algebras. Here we use the hermitian bivariant $K$-theory $kk^h$ introduced in [16]; it is a bivariant $K$-theory for $*$-algebras over a commutative ring $\ell$ with involution. The ring $\ell$ is assumed to satisfy the following.

\textbf{$\lambda$-assumption 1.3} The ground ring $\ell$ contains an element $\lambda$ such that $\lambda + \lambda^* = 1$.

For such $\ell$ there are a triangulated category $kk^h$ and a functor $j^h : \text{Alg}^{\text{op}}_{\ell} \to kk^h$ from the category of $*$-algebras which is homotopy invariant, matricially stable, hermitian stable, and satisfies excision and is universal with those properties. We write $[−1]$ for
The suspension in $\text{kk}^h$ and set

$$\text{kk}^h_n(A, B) = \text{hom}_{\text{kk}^h}(j^h_n(A), j^h_n(B)[n]), \quad \text{kk}^h(A, B) = \text{kk}^h_0(A, B). \quad (1.5)$$

The main technical result about $\text{kk}^h$ that we prove in this paper, which is key to the proofs of Theorems 1.1 and 1.2, is Theorem 13.2, which we reproduce in part below as Theorem 1.3. We call a $*$-algebra $R$ strictly properly infinite if it contains two orthogonal isometries. A ring $R$ is regular with respect to a functor $F$ if $F(R) \to F(R[t_1, \ldots, t_n])$ is an isomorphism for all $n \geq 1$. We write $K^h_0(R)$ for the Witt–Grothendieck group of $R$ (see Sect. 7). We write $\mathcal{L}(E), R$ for the set of $M_{\pm 2}$-homotopy classes of very full $*$-homomorphisms; if $R$ is strictly properly infinite, this set has a natural semigroup structure (see Remark 9.2).

**Theorem 1.3** Let $E$ be a finite, purely infinite simple graph and $R \in \text{Alg}^*_\ell$ a $K^h_0$-regular, strictly properly infinite $*$-algebra over a ring $\ell$ satisfying the $\lambda$-assumption. Assume that $-1$ is positive in $R$. Then the map

$$j^h : [\mathcal{L}(E), R]_{M_{\pm 2}}^{f} \to \text{kk}^h(\mathcal{L}(E), R) \quad (1.6)$$

is a semigroup isomorphism.

We show in Corollary 8.5 that if $F$ is finite purely infinite simple then $L(F)$ is strictly properly infinite. If $\ell$ is regular supercoherent, then $L(F)$ is $K_0$-regular, and even $K^h_0$-regular if 2 is invertible in $\ell$ (Lemma 4.3). Thus if $-1$ is positive in $\ell$, Theorem 1.3 reduces the proof of Theorem 1.1, to showing that an isomorphism $BF(E) \sim \Rightarrow BF(F)$ lifts to an isomorphism $j^h(\mathcal{L}(E)) \sim \Rightarrow j^h(\mathcal{L}(F))$. If $-1$ is not positive in $\ell$, then it cannot be positive in $L(E)$ (Lemma 2.15), hence we cannot apply Theorem 1.3 with $R = L(F)$. Observe however that $-1$ is positive in $M_{\pm}$ (see Example 2.7); the proof of Theorem 1.2 uses Theorem 1.3 applied to $R = M_{\pm}L(F)$ and the fact that $\iota_\pm$ is an isomorphism in $\text{kk}^h$.

If $F$ is finite and regular, then $-1$ is positive in $\overline{L(F)}$ (see (2.19)). We show in Corollary 8.5 that if $F$ is finite and purely infinite simple, then $\overline{L(F)}$ is strictly properly infinite, hence Theorem 1.3 applies to $R = \overline{L(F)}$. In order to obtain a classification result similar to those of Theorems 1.1 and 14.1 for Leavitt path algebras of finite purely infinite simple graphs equipped with the involution (1.4), we would need to have an analogue of Theorem 1.3 with $\overline{L(E)}$ substituted for $L(E)$. The proof of the surjectivity of (1.6) uses Theorem 9.3 which in turn relies on the fact that the edges of $E$ are partial isometries of $L(E)$. This is no longer the case in $\overline{L(E)}$, as $e\overline{e}e = -e$ for $e \in E^1$, so our argument does not apply. However we do manage to show, in Theorem 13.4, that for $E$ and $R$ as in Theorem 1.3, the map

$$j^h : [\overline{L(E)}, R]_{M_{\pm 2}}^{f} \to \text{kk}^h(\overline{L(E)}, R) \quad (1.7)$$

is injective. The proof of this injectivity result as well as of the injectivity of (1.6) uses Poincaré duality for Leavitt path algebras, established in Theorem 11.2, which says
that if $E$ is finite and has no sinks and no sources and $E_\ell$ is the dual graph, then for any two $\ast$-algebras $R$ and $S$ there are isomorphisms

$$kk^h(R \otimes L(E), S) \cong kk^h_1(R, S \otimes L(E_\ell)), \quad kk^h(R \otimes \overline{L(E)}, S) \cong kk^h_1(R, S \otimes \overline{L(E_\ell)}).$$

(1.8)

This is a purely algebraic version of the analogue statement for graph $\mathbb{C}^*$-algebras established by Kaminker and Putnam in [22].

The rest of this article is organized as follows. Section 2 contains some basic material about $\ast$-algebras, bivariant hermitian $K$-theory and Cohn and Leavitt path algebras. In particular we recall from [16] that $kk^h_\ast(\ell, R) = KH^h_\ast(R)$ is a Weibel-style [30] homotopy invariant version of hermitian $K$-theory $K^h$, and is related to the latter via a comparison map $K^h_\ast(R) \to KH^h_\ast(R)$ that is an isomorphism when $R$ is $K^h$-regular. For example we show in Lemma 4.3 that if $R$ is regular supercoherent and $E$ is countable, then $L(E) \otimes R$ is $K$-regular, and even $K^h$-regular if furthermore 2 is invertible in $R$. In Lemma 2.15 we show that if $E^0$ is finite then $-1$ is positive in $L(E)$ if and only if it is positive in $\ell$, in which case $L(E) \cong \overline{L(E)}$.

The category $kk^h$ is enriched over $KH^h_\ast(\ell) = kk^h_\ast(\ell, \ell)$, which is an algebra over $\mathbb{Z}[\sigma]$, the group ring of the group with two elements. In Sect. 3 we prove Theorem 3.2, which says that if $T$ is a triangulated category and $X : \text{Alg}^\ast_\ell \to T$ is a homotopy invariant, matricially and hermitian stable and excisive which commutes with direct sums of at most $|E^0|$ summands, then there are distinguished triangles in $T$

$$X(\ell)(\text{reg}(E)) \xrightarrow{I - A_{E^\ell}} X(\ell)(E^0) \xrightarrow{} X(L(E))$$

(1.9)

$$X(\ell)(\text{reg}(E)) \xrightarrow{I - \sigma A_{E^\ell}} X(\ell)(E^0) \xrightarrow{} X(\overline{L(E)}).$$

(1.10)

In particular this applies to $j^h$ when $E^0$ is finite (Theorem 3.3). Both triangles above are obtained from the standard presentation $L(E) = C(E)/K(E)$ as a quotient of the Cohn algebra by an ultramatricial ideal. The key calculation is done in Theorem 3.1, where we show that $j^h(C(E)) \cong j^h(C(E)) \cong j^h(\ell,E^0)$.

In Sect. 4 we introduce, for a graph $E$, the $\mathbb{Z}[\sigma]$-module

$$\overline{\mathcal{B}_3}(E) = \text{Coker}(I - \sigma A_{E^\ell}).$$

We use the results of the previous section, and $K^h$-regularity Lemma 4.3, to make some hermitian $K$-theory computations in Corollaries 4.4 and 4.5. For example, the latter corollary proves that if $\ell$ is a field with char($\ell$) $\neq 2$ and $E$ is countable, then

$$K^h_0(L(E)) = \mathcal{B}_3(E) \otimes_{\mathbb{Z}} K^h_0(\ell), \quad K^h_0(\overline{L(E)}) = \overline{\mathcal{B}_3}(E) \otimes_{\mathbb{Z}[\sigma]} K^h_0(\ell).$$

(1.11)

We also show, in Lemma 4.7, that if $E^0$ is finite then $j^h(L(E)) = 0$ if and only if $\mathcal{B}_3(E) \otimes KH^h_0(\ell) = 0$ and that $j^h(\overline{L(E)}) = 0$ if and only if $\overline{\mathcal{B}_3}(E) \otimes_{\mathbb{Z}[\sigma]} KH^h_0(\ell) = 0$. In Example 4.8 we exhibit a purely infinite simple finite graph $\Upsilon$ such that $j^h(L(\Upsilon)) = j^h(\overline{L(\Upsilon)}) = 0$. 
In Sect. 5 we observe that if $\ell$ is regular supercoherent and $E^0$ is finite, then for the $K$-theory of $\mathbb{Z}/2\mathbb{Z}$-graded projective modules,

$$K_0^{gr}(L(E)) = \overline{\mathcal{B}_3}(E) \otimes_{\mathbb{Z}} K_0(\ell).$$  \hspace{1cm} (1.12)

The results about non-existence of unital homomorphisms connecting $L_2$ and $L_2$—mentioned above are proved in Proposition 5.2. The proof combines (1.11), (1.12) and the computation

$$\overline{\mathcal{B}_3}(R_2) = \mathbb{Z}/2\mathbb{Z}, \quad \overline{\mathcal{B}_3}(R_2-) = \mathbb{Z}/7\mathbb{Z}.$$  

In Sect. 6 we classify $j^h(L(E))$ and $j^h(\overline{L(E)})$ up to isomorphism. For example, we show in Theorem 6.2 that if $E$ and $F$ have finitely many vertices and the same number of singular vertices, then any group isomorphism $\mathcal{B}_3(E) \sim \mathcal{B}_3(F)$ lifts to an isomorphism $j^h(L(E)) \sim j^h(L(F))$; this is key to the strategy, explained above, of the proof of Theorem 13.2. We also show, in Theorem 6.8, that if furthermore $\text{Ker}(I - \sigma A_E^i) = \text{Ker}(I - \sigma A_F^i) = 0$, then any $\mathbb{Z}[\sigma]$-module isomorphism $\overline{\mathcal{B}_3}(E) \sim \overline{\mathcal{B}_3}(F)$ lifts to an isomorphism $j^h(L(E)) \sim j^h(L(F))$. In Proposition 6.9 and Remark 6.11, we also characterize, under some hypothesis on $\ell$, those finite graphs $E$ such that $\mathcal{B}_3(E)$ and $\mathcal{B}_3(F)$ are finite and isomorphic, an show that for such $E$, $j^h(L(E))$ is a direct summand of $j^h(L(E))$.

In Sect. 7 we consider, for a unital $*$-algebra $R$, the group completion $K_0(R)^*$ of the monoid $\mathcal{V}_{\infty}(R)^*$ of Murray–von Neumann equivalences classes of projections in $\mathcal{M}_R$. If $R$ is a $C^*$-algebra, $K_0(R)^*$ is just $K_0(R)$. For any unital $*$-algebra $R$, the hermitian Witt–Grothendieck group of $R$ is $K_0^h(R) = K_0(\mathcal{M}_R)$. The inclusion $\iota_+ : R \to K_0^h(R)$ induces a natural transformation $K_0(R)^* \to K_0^h(R)$, which is an isomorphism if $-1$ is positive in $R$. We consider the concept of a strictly full projection and show in Lemma 7.1 that if $p \in R$ is strictly full, then the inclusion $p R p \subset R$ induces an isomorphism in $\mathcal{V}_{\infty}(\cdot)^*$ and $K_0(\cdot)^*$. For example if $-1$ is positive in $R$ then $\iota_+(1) \in \mathcal{M}_R$ is strictly full, and so the natural map $K_0(R)^* \to K_0^h(R)$ is an isomorphism. Remark 7.2 introduces, for a graph $E$ with finite $E^0$, a canonical group homomorphism $\mathcal{B}_3(E) \to K_0(L(E))^*$.

In Sect. 8 we prove several technical lemmas concerning projections and strictly proper infinite algebras. For example we show in Corollary 8.5 that if $E$ is finite and purely infinite simple, then $L(E)$ and $\overline{L(E)}$ are strictly proper infinite. This, together with the $K^h$-regularity result from Sect. 2 mentioned above (Lemma 4.3) allows us to apply Theorem 1.3 to $R = L(E)$, $L(F)$ in the proof of Theorem 1.1 and to $R = \mathcal{M}_L, \mathcal{M}_L(F)$ in that of Theorem 1.2.

In Sect. 9 we prove Theorem 9.3, which says that if $E$ is a countable graph and $R$ a strictly proper infinite $*$-algebra, then any group homomorphism $\xi : \mathcal{B}_3(E) \to K_0(R)^*$ lifts to a very full $*$-homomorphism $\phi : L(E) \to R$ such that $K_0(\phi)^* \circ \text{can'} = \xi$. This result is used later on, in the proof of Theorem 13.2, to show that the map (1.6) is onto.

In Sect. 10 we define the group $K_1(R)^*$ of a unital $*$-algebra $R$ as the abelianization of the infinite unitary group $U_{\infty}(R) = \bigcup_{n=1}^{\infty} U(M_n R)$. We show that $K_1(\cdot)^*$ is
invariant under passing to strictly full corners (Lemma 10.1). The usual hermitian $K_1$ is obtained as $K^+_1(R) = K_1(M_+(R)^*)$, and the natural map $\iota_+: K_1(R)^* \to K^+_1(R)$ is an isomorphism whenever $-1$ is positive in R (Remark 10.2). We also consider a homotopy invariant, Karoubi–Villamayor style [23, 24] version of $K_1(-)^*$, which we call $KV_1(-)^*$. There is a canonical surjective map $K_1(R)^* \to KV_1(R)^*$ which is an isomorphism if $-1$ is positive in $R$ and $R$ is $K^+_1$-regular. Proposition 10.3 and Lemma 10.4 describe $K_1(R)^*$ and $KV_1(R)^*$ for $R$ strictly properly infinite, as quotients of $U(R)$. This description is later used in the proofs of Theorems 13.2 and 13.4.

Section 11 is concerned with Poincaré duality (1.8), proved in Theorem 11.2. In Sect. 12 we use the triangles (1.9) and (1.10) to prove Theorem 12.1, which says that if $E$ is a graph with $|E^0| < \infty$ and $R$ is a $\ast$-algebra, then for

\[ \overline{\mathcal{B}^\vee_\mathcal{S}}(E) = \text{Coker}(I^\iota - \sigma A_E) \quad \text{and} \quad \mathcal{B}^\vee_\mathcal{S}(E) = \overline{\mathcal{B}^\vee_\mathcal{S}}(E) \otimes_{\mathbb{Z}[\sigma]} \mathbb{Z}, \]

we have exact sequences

\[
0 \to K^+_1(R) \otimes \overline{\mathcal{B}^\vee_\mathcal{S}}(E) \to kk^h(L(E), R) \to \text{hom}(\overline{\mathcal{B}^\vee_\mathcal{S}}(E), K^+_1(R)) \to 0,
\]

\[
0 \to K^+_1(R) \otimes_{\mathbb{Z}[\sigma]} \overline{\mathcal{B}^\vee_\mathcal{S}}(E) \to kk^h(L(E), R) \nonumber \to \text{hom}_{\mathbb{Z}[\sigma]}(\overline{\mathcal{B}^\vee_\mathcal{S}}(E), K^+_1(R)) \to 0. \tag{1.13}
\]

If $E$ is finite with no sinks and no sources, then $\mathcal{B}^\vee_\mathcal{S}(E) = \mathcal{B}^\vee_\mathcal{S}(E_t)$ and $\overline{\mathcal{B}^\vee_\mathcal{S}}(E) = \overline{\mathcal{B}^\vee_\mathcal{S}}(E_t)$. Combining this with Poincaré duality we obtain a description of the injective maps in (1.13) that is key in the proofs of Theorems 13.2 and 13.4.

Theorem 1.3 is proved in Sect. 13 as Theorem 13.2. Theorem 13.4 shows that the map (1.7) is injective. The proofs use material from all of the previous sections.

Section 14 uses Theorem 1.3 to prove Theorems 1.1 and 1.2, which are respectively Theorem 14.2 and Theorem 14.1. They classify Leavitt path algebras of finite, purely infinite simple graphs up to involution preserving homotopy and matricial stabilization. Theorem 14.4 gives an analogous result under not necessarily involution preserving homotopy.

2 Preliminaries

2.1 Algebras and involutions

A commutative unital ring $\ell$ with involution $\ast$ is fixed throughout the article. A $\ast$-algebra is an $\ell$-algebra $R$ equipped with an involution $\ast: R \to R^{\text{op}}$ that is semilinear with respect to the $\ell$-module action; $(\lambda a)^* = \lambda^* a^*$ for all $\lambda \in \ell$ and $a \in R$. We use the term $\ast$-ring for a $\ast$-$\mathbb{Z}$-algebra. A $\ast$-homomorphism between $\ast$-algebras is an algebra homomorphism which commutes with involutions. We write $\text{Alg}_\ell$ for the category of algebras and algebra homomorphisms and $\text{Alg}_\ell^\ast$ for the subcategory of $\ast$-algebras and $\ast$-homomorphisms.

Tensor products of algebras are taken over $\ell$; we write $\otimes$ for $\otimes_\ell$. We also use $\otimes$ for tensor products of abelian groups, e.g. $K_0(R) \otimes K_0(S) = K_0(R) \otimes_{\mathbb{Z}} K_0(S)$. If $R$ and
$S$ are $\ast$-algebras, we regard $R \otimes S$ as a $\ast$-algebra with the tensor product involution $(a \otimes b)\ast = a\ast \otimes b\ast$. If $L$ is a $\ast$-algebra and $A \in \text{Alg}_R^\ast$, we shall often write $LA$ for $L \otimes A$.

**Example 2.1** Let $A$ be a ring. Put $\text{inv}(A) = A \oplus A^{\text{op}}$ for the $\ast$-ring with the coordinate-wise operations and involution $(a, b) \mapsto (b, a)$. If $B$ is a $\ast$-ring, then $\text{inv}(A) \otimes_{\mathbb{Z}} B \rightarrow \text{inv}(A \otimes_{\mathbb{Z}} B)$ $(a_1 \otimes b_1, a_2 \otimes b_2) \mapsto (a_1 \otimes b_1, a_2 \otimes b_2^\ast)$ is a $\ast$-isomorphism. If $A$ is an algebra over a ground ring $\ell$, then $\text{inv}(A)$ is a $\ast$-$\text{inv}(\ell)$ algebra; this gives rise to a category equivalence $\text{inv} : \text{Alg}_\ell \rightarrow \text{Alg}_{\text{inv}(\ell)}^\ast$.

The polynomial ring $B[t] = B \otimes_{\mathbb{Z}} \mathbb{Z}[t]$ with coefficients in a $\ast$-algebra is equipped with the tensor product involution, where $\mathbb{Z}[t]$ has the trivial involution.

An elementary $\ast$-homotopy between two $\ast$-homomorphisms $f_0, f_1 : A \rightarrow B$ is a $\ast$-homomorphism $H : A \rightarrow B[t]$ such that $\text{ev}_i \circ H = f_i$ ($i = 0, 1$). We say that two $\ast$-homomorphisms $f, g : A \rightarrow B$ are $\ast$-homotopic, and write $f \sim^\ast g$, if there is a finite sequence $f = f_0, \ldots, f_n = g$ such that for each $i$ there is an elementary $\ast$-homotopy between $f_i$ and $f_{i+1}$. We write $[A, B]^\ast$ for the set of $\ast$-homotopy classes of $\ast$-homomorphisms $A \rightarrow B$.

A $\ast$-ideal $I$ in a $\ast$-algebra $R$, denoted $I \lhd R$, is an $\ell$-$\ast$-subalgebra $I \subset R$ which is also a two-sided ideal. An element $a$ in a $\ast$-algebra $R$ is self-adjoint if $a^\ast = a$.

Let $X$ be a set; consider Karoubi’s cone algebra $\Gamma_X$ of square matrices indexed by $X$ with a finite coefficient set and a global bound on the size of the support of its rows and columns. Viewing $X \times X$-matrices with coefficients in $\ell$ as functions $X \times X \rightarrow \ell$, the elements of $\Gamma_X$ are all the functions $a : X \times X \rightarrow \ell$ that satisfy the following conditions:

$$|\text{Im}(a)| < \infty \text{ and } (\exists N) \max(|\text{supp}(a(x, -)), |\text{supp}(a(-, x))| : x \in X) \leq N.$$  

We equip $\Gamma_X$ with the product of matrices, or what is the same, with the convolution product of functions. Observe that $\Gamma_X$ contains the algebra of finitely supported matrices as an ideal

$$M_X = \{a : X \times X \rightarrow \ell : |\text{supp}(a)| < \infty\} \lhd \Gamma_X.$$  

The standard involution on $\Gamma_X$ is

$$\ast : \Gamma_X \rightarrow \Gamma_X, a^\ast(x, y) = a(y, x)^\ast (x, y \in X).$$  

Observe that $M_X$ is a $\ast$-ideal with respect to the standard involution. Karoubi’s suspension algebra is the quotient $\ast$-algebra

$$\Sigma_X = \Gamma_X/M_X.$$  

If $(x, y) \in X \times X$, write $\epsilon_{x,y}$ for the matrix unit and $t_x : R \rightarrow M_X R$ for the corner embedding $t_x(a) = \epsilon_{x,x} a$. Observe that $t_x$ is a $\ast$-homomorphism for the standard involution on $M_X$ as well as for any other involution which makes $\epsilon_{x,x}$ self-adjoint.
When $X = \{1, \ldots, n\}$, we write $M_n$ for $M_X$. If $R$ is a $\ast$-algebra, we write $\Gamma_X R$ and $M_X R$ for $\Gamma_X \otimes R$ and $M_X \otimes R$.

### 2.2 $\mathbb{Z}/2\mathbb{Z}$-gradings

Let $G$ be an abelian group and $A = \bigoplus_{g \in G} A_g$ a $G$-graded algebra; the opposite algebra $A^{\text{op}}$ is also $G$-graded with $A_g^{\text{op}} = A_{-g}$. By a $G$-graded $\ast$-algebra we understand a $G$-graded algebra equipped with an involution that $\ast : A \rightarrow A^{\text{op}}$ is homogeneous of degree 0. If $A = A_0 \oplus A_1$ is a $\mathbb{Z}/2\mathbb{Z}$-graded $\ast$-algebra, then

$$\tau : A \rightarrow A, \quad \tau(a_0 + a_1) = a_0 - a_1$$

is a $\ast$-automorphism. Composing it with the involution, we obtain a new involution

$$\cdot : A \rightarrow A^{\text{op}}, \quad a_0 + a_1 = a_0^* - a_1^*.$$  

Write $\overline{A}$ for $A$ equipped with the involution $\cdot$. If $B = B_0 \oplus B_1$ is another $\mathbb{Z}/2\mathbb{Z}$-graded $\ast$-algebra and $f : A \rightarrow B$ is a homogeneous $\ast$-homomorphism of degree 0, then $f(\overline{a}) = \overline{f(a)}$ for all $a \in A$. Hence the function $f$ defines a $\ast$-homomorphism

$$\overline{f} : \overline{A} \rightarrow \overline{B}, \quad \overline{f}(a) = f(a).$$

Note also that $A \mapsto \overline{A}$ commutes with tensor products of graded $\ast$-algebras; we have

$$A \otimes B = \overline{A} \otimes \overline{B}.$$  

**Example 2.2** The algebra $M_2$ admits a $\mathbb{Z}/2\mathbb{Z}$-grading, where $|\epsilon_{i,j}| \equiv i - j \mod 2$. We write $M_{\pm} = \overline{M}_2$. Thus $M_{\pm}$ is the algebra of $2 \times 2$-matrices with coefficients in $\ell$, equipped with the involution (1.3). We write $\iota_+$ and $\iota_-$ for the upper left and lower right corner inclusions $\ell \rightarrow M_{\pm}$, respectively. For a $\ast$-algebra $R$, we put $M_{\pm} R = M_{\pm} \otimes R$ for the matrix algebra with the tensor product involution; we also abuse notation and write $\iota_{\pm} : R \rightarrow M_{\pm} R$, for $\iota_{\pm} \otimes \text{id}_R$. It follows from (2.4) that

$$\overline{M_{\pm} R} = M_{\pm} \overline{R}.$$  

**Example 2.3** If $X$ is a set then any function $l : X \rightarrow \{0, 1\}$ induces a $\mathbb{Z}/2\mathbb{Z}$-grading on $\Gamma_X$ that makes $M_X$ a homogeneous ideal, where

$$|\epsilon_{x,y}| \equiv l(x) + l(y) \mod 2.$$  

The grading of $M_2$ in Example 2.2 is particular case of this. Observe also that, by passage to the quotient, we also obtain a $\mathbb{Z}/2\mathbb{Z}$-grading on $\Sigma_X$.  

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Example 2.4 If $A$ is any $\mathbb{Z}/2\mathbb{Z}$-graded $\ast$-algebra, then so are $M_2A$ and $M_{\pm}A$, with the tensor product grading. The involutions of $M_2A$ and $M_{\pm}A$ agree on the common $\ast$-subalgebra

$$\hat{A} = \begin{bmatrix} A_0 & A_1 \\ A_1 & A_0 \end{bmatrix}$$

We have a $\ast$-homomorphism

$$A \to \hat{A}, \ a_0 + a_1 \mapsto \begin{bmatrix} a_0 & a_1 \\ a_1 & a_0 \end{bmatrix}.$$  

(2.7)

Composing with the inclusions $\hat{A} \subset M_{\pm}A$ and $\hat{A} \subset M_2A$ we get $\ast$-homomorphisms

$$\Delta_A : A \to M_{\pm}A, \ \Delta'_A : A \to M_2A.$$  

(2.8)

A calculation shows that the following diagrams commute

(2.9)

Remark 2.5 Let $\sigma$ be the generator of $\mathbb{Z}/2\mathbb{Z}$, written multiplicatively. Then

$$\mathbb{Z}[\sigma] = \mathbb{Z}[\mathbb{Z}/2\mathbb{Z}]$$

is a Hopf ring, and a $\mathbb{Z}/2\mathbb{Z}$-graded ring is the same thing as a comodule-algebra over $\mathbb{Z}[\sigma]$. One checks that the algebra $\hat{A}$ of (2.6) is the crossed product of $A$ with $\mathbb{Z}/2\mathbb{Z}$ under this coaction, as defined for example in [8, Definition 2.1]. In particular, $\hat{A}$ is equipped with a $\mathbb{Z}/2\mathbb{Z}$ action; the generator $\sigma$ acts by the automorphism

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \mapsto \begin{bmatrix} d & c \\ b & a \end{bmatrix}.$$  

Observe that (2.7) is an isomorphism from $A$ onto the fixed ring under the automorphism above. Let $M = M_0 \oplus M_1$ be a graded $A$-module; regarding an element $m_0 + m_1 \in M_0 \oplus M_1$ as a column vector and using the matricial product, one obtains a $A$-module $\hat{M}$ with the same underlying $\ell$-module $M$. Next assume that $A$ has graded local units in the sense of [8, Section 2.1], and let $\text{Gr}_{\mathbb{Z}/2\mathbb{Z}} \text{Mod} A$ and $\text{Mod} \hat{A}$ be the categories of graded and ungraded modules that are unital in the sense of loc.cit. By [8, Proposition 2.5], the functor

$$\text{Gr}_{\mathbb{Z}/2\mathbb{Z}} \text{Mod} A \to \text{Mod} \hat{A}, \ M \mapsto \hat{M}$$

(2.10)

is an isomorphism of categories, and maps the shift functor $M_\ast \mapsto M_{\ast+1}$ to the action of $\sigma$.  

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2.3 Positivity

Let \( a \in A \in \text{Alg}_\ell^* \) and \( n \geq 1 \); we call \( a \) \( n \)-positive if \( a \neq 0 \) and can be written as a sum \( a = \sum_{i=1}^{n} x_i x_i^* \) for some \( x_1, \ldots, x_n \in A \), and positive if it is \( n \)-positive for some \( n \). We call \( a \) negative if \( -a \) is positive. In a general \( * \)-algebra it can happen that an element is positive and negative at the same time.

**Example 2.6** Assume that \( \ell \) is a field of char \((k) \neq 2 \). Then every self-adjoint element in \( \ell \) can be written as a difference of two 1-positive elements. Hence \( -1 \) is positive if and only if every self-adjoint element of \( \ell \) is positive.

**Example 2.7** The element

\[
x = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \in M_{\pm}
\]

is self-adjoint and satisfies \( x^2 = -1 \). Hence if \( R \) is any unital \( * \)-algebra, then \( -1 \) is 1-positive in \( M_{\pm} R \).

**Example 2.8** Let \( L_1 = \ell[t, t^{-1}] \) be the Laurent polynomials, with involution \( t \mapsto t^{-1} \). A \( \mathbb{Z} \)-graded algebra \( A = \bigoplus_{n \in \mathbb{Z}} A_n \) is an \( L_1 \)-comodule algebra, and the comultiplication map \( A \to A \otimes L_1 \) which sends a homogeneous element \( a \) to \( at^{\|a\|} \) is a \( * \)-homomorphism. Now regard \( A \) and \( L_1 \) as \( \mathbb{Z}/2\mathbb{Z} \)-graded, via their even/odd gradings. Then comultiplication defines a \( * \)-homomorphism

\[
c : \mathbb{A} \to A \otimes L_1.
\]

If now \( R \) is a unital \( * \)-algebra and \( x \in R \) a central element such that \( xx^* = -1 \), then \( \mu_x : L_1 \otimes R \to R, t \otimes a \mapsto xa \) is a \( * \)-homomorphism. One checks that

\[
\theta_x := (A \otimes \mu_x) \circ (c \otimes R) : \mathbb{A} \otimes R \to A \otimes R
\]

is a \( * \)-isomorphism with inverse \( \theta_{x^{-1}} \). Similarly, the map

\[
\theta^x : \text{Hom}_{\text{Alg}_\ell^*}(A, R) \to \text{Hom}_{\text{Alg}_\ell^*}(\mathbb{A}, R), \quad \theta^x(f) = \mu_x \circ (f \otimes L_1) \circ c
\]

is bijective with inverse \( \theta^{x^{-1}} \).

**Remark 2.9** The hypothesis that \( x \) be central—which is not satisfied in Example 2.7— is essential in Example 2.8. For example \( M_{\pm} \mathbb{A} \) and \( M_{\pm} A \) are not isomorphic in general. Indeed, they have the same Hermitian \( K_0 \)-groups \( K_0^h \) as \( \mathbb{A} \) and \( A \), respectively, but in general, \( K_0^h(A) \not\cong K_0^h(\mathbb{A}) \); see Example 4.6.

**Lemma 2.10** Let \( S \) be a set; equip \( \Gamma_S \) with the standard involution. Then \( -1 \) is positive in \( \Gamma_S \) if and only if it is positive in \( \ell \).

**Proof** The if direction is clear. Assume conversely that \( -1 \) is positive in \( \Gamma_S \). Then there are elements \( y(1), \ldots, y(n) \in \Gamma_S \) such that \( -1 = \sum_{i=1}^{n} y(i)^* y(i) \). Hence for \( N = \max\{ |\text{supp}(y(i))_{*1}| : 1 \leq i \leq n \} \), we have the following identity between elements of \( \ell \)
An excisive homology theory

In this subsection we assume that \( \ell \) satisfies the \( \lambda \)-assumption 1.3.

An extension of \( \ast \)-algebras is a sequence of \( \ast \)-homomorphisms

\[
(E) \quad A \xrightarrow{i} B \xrightarrow{p} C
\]

with \( i \) injective and \( p \) surjective and which is exact as a sequence of \( \ell \)-modules. An extension is semi-split if it is split as a sequence of \( \ell \)-modules. Under our standing Assumption 1.3, if (2.11) is semisplit, then there exists an involution preserving linear map \( s : C \to B \) such that \( p \circ s = \text{id}_C \). Let \( \mathcal{T} \) be a triangulated category; write \([-n]\) for the \( n \)-fold suspension in \( \mathcal{T} \). Let \( \mathcal{E} \) be the class of all semi-split extensions (2.11). An excisive homology theory on \( \text{Alg}^\ast_{\ell} \) with values in \( \mathcal{T} \) is a functor \( \mathcal{H} : \text{Alg}^\ast_{\ell} \to \mathcal{T} \) together with a family of maps \( \{\partial_E : \mathcal{H}(C)[1] \to \mathcal{H}(A) | E \in \mathcal{E}\} \) such that for every \( E \in \mathcal{E} \),

\[
\mathcal{H}(C)[1] \xrightarrow{\partial_E} \mathcal{H}(A) \xrightarrow{\mathcal{H}(i)} \mathcal{H}(B) \xrightarrow{\mathcal{H}(p)} \mathcal{H}(C)
\]

is a triangle in \( \mathcal{T} \), and such that \( \partial \) is compatible with maps of extensions in the sense of [15, Section 6.6]. Let \( \mathcal{H} : \text{Alg}^\ast_{\ell} \to \mathcal{T} \) be an excisive homology theory, \( X \) an infinite set, \( x \in X \) and \( A \in \text{Alg}^\ast_{\ell} \). Consider the natural evaluation and corner inclusion maps \( e_0 : A[t] \to A \) and \( \iota_+: A \to M_+A, \iota_\times : A \to M_XA \). We say that a \( \mathcal{H} \) is homotopy invariant, \( \iota_+\)-stable and \( M_X\)-stable if for every \( A \in \text{Alg}^\ast_{\ell} \), the maps \( \mathcal{H}(e_0), \mathcal{H}(\iota_+) \) and \( \mathcal{H}(\iota_\times) \) are isomorphisms in \( \mathcal{T} \). By [16, Lemma 2.4.1], \( M_X \)-stability is independent of the choice of the element \( x \) in the previous definition. It was proved in [16, Proposition 6.2.7] that for any infinite set \( X \), there exists an excisive, homotopy invariant, \( \iota_+\)-stable and \( M_X\)-stable homotopy theory \( j^h : \text{Alg}^\ast_{\ell} \to \text{kk}^h \), depending on \( X \), such that if \( \mathcal{H} : \text{Alg}^\ast_{\ell} \to \mathcal{T} \) is any other excisive, homotopy invariant, \( \iota_+\)-stable and \( M_X\)-stable homology theory, then there exists a unique triangulated functor \( \mathcal{H} : \text{kk}^h \to \mathcal{T} \) such that \( \mathcal{H} \circ j^h = \mathcal{H} \). We fix such an \( X \) and for \( A, B \in \text{Alg}^\ast_{\ell} \) and \( n \in \mathbb{Z}, \) we write

\[
\text{kk}^h_n(A, B) = \text{hom}_{\text{kk}^h}(j^h(A), j^h(B)[n]), \quad \text{kk}^h(A, B) = \text{kk}^h_0(A, B).
\]

The suspension in \( \text{kk}^h \) is represented by the Karoubi suspension; for any infinite set \( Y \) with \( |Y| \leq |X| \) and any \( A \in \text{Alg}^\ast_{\ell} \), we have

\[
j^h(A)[-1] = j^h(\Sigma_Y A).
\]
The inverse suspension is obtained by tensoring with $\Omega = (1-t)t[1]$; for all $A \in \text{Alg}_\ell^*$ we have

$$j^h(A)[+1] = j^h(\Omega A).$$

It was shown in [16, Proposition 8.1] that $kk^h$ recovers Weibel-style homotopy algebraic Hermitian $K$-theory; we have

$$kk^h_n(\ell, B) = KH^h_n(B). \quad (2.12)$$

For a definition of $KH^h$ and its relation to the more standard Hermitian $K$-theory defined by Karoubi (sometimes called Grothendieck–Witt theory) see [16, Section 3]; $K^h_0$ is discussed in Sect. 7. There is a natural map

$$K^h_n(B) \to KH^h_n(B). \quad (2.13)$$

Recall that if $F$ is a functor defined on a subcategory $\mathcal{C} \subset \text{Alg}_\ell$ closed under tensoring with $\ell[t]$ and $A \in \mathcal{C}$, then $A$ is $F$-regular if for every $n \geq 1$, $F$ maps the inclusion $A \subset A[t_1, \ldots, t_n]$ to an isomorphism. $A$ is $K^h$ or $K$-regular if it is $K^h_m$ or $K_m$-regular for every $m$. The map (2.13) is an isomorphism for all $n$ whenever $B$ is $K^h$-regular. If $B = \text{inv}(B_0)$ for some unital ring $B_0$, then $B$ is $K^h$-regular if and only if it is $K$-regular. If multiplication by 2 is invertible in $B$ and $B$ is $K$-regular and unital then it is $K^h$-regular [16, Lemma 3.8]. The last two assertions hold more generally for $B$ $K$-excisive in the sense of [16, Section 3].

It follows from (2.12) that for all $A, B \in \text{Alg}_\ell^*$, the groups $kk^h(A, B)$ are modules over the ring $kk^h(\ell, \ell) = KH^h_0(\ell)$, and thus over $K^h_0(\ell)$ via the canonical map $K^h_0(\ell) \to KH^h_0(\ell)$. As will be recalled in Sect. 7 below, $K^h_0(\ell)$ is the group completion of the monoid of equivalence classes of self-adjoint idempotent finite matrices over $M_+$. There is a unital ring homomorphism $\mathbb{Z}[\sigma] \to K^h_0(\ell)$ mapping 1 to the class of $\iota_+(1)$ and $\sigma$ to the class of $\iota_-(1)$. Thus $kk^h(A, B)$ is a $\mathbb{Z}[\sigma]$-module for all $A, B \in \text{Alg}_\ell^*$. If $-1$ is 1-positive in $\ell$, then $\sigma$ acts trivially in $kk^h$; $\sigma \xi = \xi$ for all $\xi \in kk^h(A, B)$ and all $A, B \in \text{Alg}_\ell^*$.

**Remark 2.11** Let $\ell_0$ be a commutative ring and let $j : \text{Alg}_{\ell_0} \to kk(\ell_0)$ be the universal homotopy stable, $M_X$-stable and excisive homology theory of [15]. Let $\ell = \text{inv}(\ell_0)$ and let $j^h : \text{Alg}_\ell^* \to kk^h(\ell)$ be the universal homotopy stable, $M_X$-stable, hermitian stable and excisive homology theory. Then by [16, Example 6.2.11], the category equivalence $\text{inv} : \text{Alg}_{\ell_0} \to \text{Alg}_\ell^*$ of Example 2.1 induces a triangle equivalence $kk(\ell_0) \to kk^h(\ell)$. Thus $kk$ is a particular case of $kk^h$.

**Lemma 2.12** Assume that $\ell$ satisfies the $\lambda$-assumption 1.3. Let $X$ be an infinite set and let $j^h : \text{Alg}_\ell^* \to kk^h$ be the universal excisive, homotopy invariant, $\iota_+$-stable and $M_X$-stable homology theory. Let $l : X \to \{0, 1\}$ be a function and let $x \in X$. Equip $M_X$ with the $\mathbb{Z}/2\mathbb{Z}$-grading induced by $l$, as in Example 2.3. Consider the corner embedding $\iota_x : \ell \to \overline{M}_X$, $\iota_x(a) = \epsilon_{x,a}$. Then $j^h(\iota_x)$ is an isomorphism for every $x \in X$, and if $l(x) = l(y)$, then $j^h(\iota_x) = j^h(\iota_y)$. \[ Springer \]
Proposition 2.13 Let $A$ be a $\Delta_1$-algebra and let $X_i = \ell^{-1}(i)$ and let $\text{inc} : M_{X_i} \to M_X$ be the map induced by the inclusion $X_i \subset X$. We have a commutative diagram

$$
\begin{array}{ccc}
\overline{M_X} & \xrightarrow{\text{inc}} & M_{X_i} \\
\downarrow{\tau_x} & & \downarrow{\text{inc}} \\
\ell & \xrightarrow{i_x} & M_{X_i}.
\end{array}
$$

The map $j^h(i_x)$ is an isomorphism by $M_X$-stability; by [16, Lemma 2.4.1] it is independent of $x \in X_i$. The map $j^h(\text{inc})$ is an isomorphism by [16, Lemma 2.4.3].

Proposition 2.13 Let $A$ be a $\mathbb{Z}/2\mathbb{Z}$-graded $*$-algebra and let $\Delta = \Delta_A : A \to M_{\pm \overline{A}}$ and $\Delta' = \Delta'_A : A \to M_{2A}$ be as in (2.8). Set $\Delta''_A = j^h(i_1)^{-1} \circ j^h(\Delta_A) \in kk^h(A, \overline{A})$ and $\Delta''_A = j^h(i_1)^{-1} \circ j^h(\Delta'_A) \in kk^h(A, A)$. Then

$$
\Delta''_A \circ \Delta_A = (1 + \sigma) \circ \Delta'_A, \quad \Delta_A \circ \Delta'_A = 2\Delta'_A.
$$

If furthermore $2$ is invertible and $1$-positive in $\ell$, then for $\tau$ as in (2.1), we have $\Delta' = 1 + j^h(\tau)$.

Proof The displayed identities follow from the commutative diagrams (2.9) and naturality of $\Delta$, using Lemma 2.12. Next assume that $2 = xx^*$ for some invertible $x \in \ell$, and consider

$$
u = \begin{bmatrix} 1/x & 1/x \\ 1/x^* & -1/x^* \end{bmatrix}.
$$

A calculation shows that $\text{ad}(\nu) \circ \Delta' = i_1 + i_2 \tau$, whence $\Delta' = 1 + j^h(\tau)$.

In the next corollary and elsewhere we write $kk^h[1/2]$ for the idempotent completion of the Verdier quotient of $kk^h$ by the full subcategory of those objects $C$ such that $j^h(\text{id}_C)$ is $2$-torsion, $j^h[1/2] : \Alg^*_\ell \to kk^h \to kk^h[1/2]$ for the composite of the canonical functors, and $kk^h[1/2](A, B) = \text{hom}_{kk^h[1/2]}(j^h[1/2](A), j^h[1/2](B))$ for $A, B \in \Alg^*_\ell$.

Corollary 2.14 Let $p_+ = (1 + \sigma)/2 \in \mathbb{Z}[1/2, \sigma]$. If $A$ is $\mathbb{Z}/2\mathbb{Z}$-graded then $\Delta_A/2$ and $\Delta_A/2$ induce inverse isomorphisms $\text{Im}(p_+ \circ (\Delta'_A/2)) \cong \text{Im}(p_+ \circ (\Delta''_A/2))$.

2.5 Cohn and Leavitt path algebras

A (directed) graph is a quadruple $E = (s, r : E^1 \Rightarrow E^0)$ consisting of sets $E^0$ and $E^1$ of vertices and edges and source and range maps $s$ and $r$. A vertex $v \in E^0$ is a sink
if \( s^{-1}([v]) = \emptyset \), a source if \( r^{-1}([v]) = \emptyset \), an infinite emitter if \( s^{-1}([v]) \) is infinite and a singular vertex it is either a sink or an infinite emitter. Vertices which are not singular are called regular. We write sink \((E)\), sour \((E)\), inf \((E)\) for the sets of sinks, sources, and infinite emitters, and sing \((E)\) and reg \((E)\) for those of singular and regular vertices. We say that \(E\) is regular if \(E^0 = \text{reg}(E)\). A graph \(E\) is countable or finite if both \(E^0\) and \(E^1\) are. The reduced incidence matrix of a graph \(E\) is the matrix \(A_E\) with nonnegative integer coefficients, indexed by reg \((E) \times E^0\), whose \((v, w)\) entry is the number of edges with source \(v\) and range \(w\):

\[
(A_E)_{v,w} = |s^{-1}(v) \cap r^{-1}(w)|.
\]

Our conventions are such that we will mainly deal with the transpose \(A_E^t\). We abuse notation and write \(I\) for the \(E^0 \times \text{reg}(E)\)-matrix obtained from the identity matrix of \(M_{E^0,\mathbb{Z}}\) upon removing the columns corresponding to the singular vertices. Thus \(I - A_E^t\) is a well-defined integral matrix indexed by \(E^0 \times \text{reg}(E)\).

We write \(C(E)\) and \(L(E)\) for the Cohn and Leavitt path algebras over \(\ell\) [4, Definitions 1.5.1 and 1.2.3]. Each of these carries a standard \(\ell\)-semilinear involution \(a \mapsto a^{\ast}\) which fixes the vertices and maps each edge \(e\) to the corresponding phantom edge \(e^{\ast}\). Let \(\mathcal{P} = \mathcal{P}(E)\) be the set of all finite paths in \(E\) [4, Definitions 1.2.2] we write \(|\alpha|\) for the length of a path \(\alpha \in \mathcal{P}(E)\). For \(v \in E^0\), set

\[
\mathcal{P}_v = \{\mu \in \mathcal{P} \mid r(\mu) = v\}, \quad \mathcal{P}^v = \{\mu \in \mathcal{P} \mid s(\mu) = v\}. \tag{2.14}
\]

Let

\[
\rho : C(E) \to \Gamma_{\mathcal{P}(E)},
\]

\[
\rho(v) = \sum_{\alpha \in \mathcal{P}_v} \epsilon_{\alpha,\alpha}, \quad \rho(e) = \sum_{\alpha \in \mathcal{P}_{\ell}(e)} \epsilon_{e\alpha,\alpha}, \tag{2.15}
\]

\[
\rho(e^{\ast}) = \sum_{\alpha \in \mathcal{P}_{\ell}(e)} \epsilon_{\alpha,e\alpha}, \quad (v \in E^0, e \in E^1).
\]

Observe that \(\rho\) is a \(\ast\)-homomorphism for the standard involutions on \(C(E)\) and \(\Gamma_{\mathcal{P}(E)}\). Recall that \(C(E)\) carries a natural \(\mathbb{Z}\)-grading, where \(C(E)_n\) is generated by all \(\alpha \beta^{\ast}\) with \(|\alpha| - |\beta| = n\). Hence we may regard \(C(E)\) as \(\mathbb{Z}/2\mathbb{Z}\)-graded, via the even/odd grading. The twisted involution \(\overset{\sim}{\cdot}\) of (2.2) is the algebra homomorphism

\[
\overset{\sim}{:} C(E) \to C(E)^{\text{op}}, \quad \overset{\sim}{v} = v, \quad \overset{\sim}{\alpha} = -e^{\ast}, \quad \overset{\sim}{e^{\ast}} = -e, \quad (v \in E^0, e \in E^1). \tag{2.16}
\]

Similarly, the length modulo 2 induces a \(\mathbb{Z}/2\mathbb{Z}\)-grading on \(\Gamma_{\mathcal{P}(E)}\), and \(\rho\) is homogeneous for this grading. Because \(\rho\) is homogeneous and a \(\ast\)-homomorphism for the standard involution, it defines a \(\ast\)-homomorphism \(\bar{\rho} : \overset{\sim}{C(E)} \to \Gamma_{\mathcal{P}(E)}\) as in (2.3).

Let \(K(E) = \text{Ker}(C(E) \to L(E))\) be the kernel of the canonical surjection. We have semi-split extensions of \(\ast\)-algebras..
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\[0 \to \mathcal{K}(E) \to C(E) \to L(E) \to 0\]  
(2.17)

\[0 \to \mathcal{K}(E) \to \mathcal{C}(E) \to \mathcal{L}(E) \to 0.\]  
(2.18)

An important feature of the involution \(a \mapsto \overline{a}\) is that if \(E\) is finite and regular, then the following identity holds in \(L(E)\)

\[-1 = \sum_{e \in E^1} e \overline{e}.\]  
(2.19)

This says that \(-1\), which is clearly negative, is also positive in \(\mathcal{L}(E)\). By contrast, \(-1\) is positive in \(L(E)\) if and only if this happens already in \(\ell\), as shown by the next lemma.

**Lemma 2.15** Let \(E\) be a graph with finitely many vertices. Then \(-1\) is positive in \(L(E)\) if and only if it is positive in \(\ell\). If \(-1\) is \(1\)-positive in \(\ell\), then \(\mathcal{L}(E) \cong \mathcal{L}(E)\) in \(\text{Alg}_{\ell}^*\).

**Proof** The if direction is clear. To prove the converse, it suffices, in view of Lemma 2.10, to find a set \(X\) and a unital \(*\)-homomorphism \(f : \mathcal{L}(E) \to \Gamma_X\). Let \(S(E) = \{\alpha \beta^* : \alpha, \beta \in \mathcal{P}(E)\}\) be the inverse semigroup associated to \(E\). Let \(X\) be an infinite set of cardinality \(|X| \geq |E^1|\) and \(\mathcal{I}(X)\) the inverse semigroup of all partially defined injections \(X \supset \text{Dom}(f) \to X\). Proceed as in the proof of [14, Proposition 4.11] to find a semigroup homomorphism \(\mu : S(E) \to \mathcal{I}(X)\) such that the associated action of \(S(E)\) on \(X\) is tight in the sense of Exel [14, Section 3]. By [14, Lemma 3.1] and [1, Lemma 6.1], \(\mu\) induces an algebra homomorphism \(\mathcal{L}(E) \to \Gamma_X\). One checks further that \(\mu\) is a \(*\)-homomorphism. This completes the proof of the first assertion. The second assertion is immediate from Example 2.8. \(\square\)

**3 Leavitt path algebras in \(kk^h\)**

This section is concerned with establishing the triangles (1.9) and (1.10). The general strategy for proving these results is similar to that used in [13] to establish analogous ones in \(kk\), and ultimately goes back to the pioneering work of J. Cuntz on the computation of \(K\)-theory of Cuntz–Krieger algebras [17, 18]. However some technical difficulties appear in that we need all maps and homotopies to preserve involutions.

Throughout this section, we assume that \(\ell\) satisfies the \(\lambda\)-assumption 1.3.

For a set \(X\) and a \(*\)-algebra \(R\), we write \(R^X\) for the \(*\)-algebra of all functions \(X \to R\) with pointwise operations and pointwise involution, and \(R^{(X)} \subset R^X\) for the \(*\)-ideal of finitely supported functions. If \(x \in X\) and \(a \in R\), we write \(a \chi_x\) for the function supported in \(\{x\}\) which maps \(x \mapsto a\).

Let \(E\) be a graph and \(C(E)\) the Cohn algebra of \(E\). The assignment

\[\ell(E^0) \to C(E), \quad \chi_v \mapsto v\]  
(3.1)

defines \(*\)-homomorphisms \(\phi : \ell(E^0) \to C(E)\) and \(\overline{\phi} : \ell(E^0) \to \mathcal{C}(E)\).

We shall say that a homology theory is \(E\)-stable if it is stable with respect to a set \(X\) of cardinality \(|E^0 \sqcup E^1 \sqcup \mathbb{N}|\).
Theorem 3.1 Assume that $\ell$ satisfies the $\lambda$-assumption 1.3. Let $j^h : \text{Alg}_{\ell}^* \to kk^h$ be the universal homotopy invariant, excisive, Hermitian stable and $E$-stable homology theory. Let $E$ be a graph and let $\phi$ be as in (3.1). Then $j^h(\phi)$ and $j^h(\overline{\phi})$ are isomorphisms. In particular, $j^h(C(E)) \cong j^h(C(\overline{E}))$.

Proof The analogue statement for the universal homotopy invariant, excisive and $E$-stable homology theory $j : \text{Alg}_{\ell} \to kk$ was proved in [12, Theorem 4.2]. The same proof goes through here with minor adjustments and works for both choices of involution on $C(E)$. The adjustments are in the homotopies occurring in [12, Lemma 4.17] and [12, Lemma 4.21], which are not $*$-homomorphisms. In both cases the problem is fixed by applying the trick of [16, Lemma 5.4], as we shall explain presently. Lemma 4.17 of [12] says that a certain map $\tau_\alpha$ is an isomorphism, so it suffices to prove it when $|\alpha| = 0. In the proof of [12, Lemma 4.17], a vertex $w \in E^0$ is fixed and elements $A_v, B_v \in M_{\mathcal{P}(E)}C(E)$ are defined for each $v \in E^0$. With notation as in [16, Lemma 5.4], put $C_v = c(A_v, B_v)$ and let $H : C(E) \to M_{\mathcal{P}(E)}C(E)[r]$ be the homomorphism determined by $H(v) = C_v\ell_+(\epsilon_{e,v} \otimes v)C^*_v$, $H(e) = C_{s(e)}\ell_+(\epsilon_{s(e),r(e)} \otimes e)C^*_{r(e)}, H(e^*) = H(e)^*$. One checks that $H$ is a $*$-algebra homomorphism for both choices of involution. It follows that $H$ is an elementary $*$-homotopy between the composite of $\ell_+$ with the maps $\hat{\imath}_\tau$ and $\hat{\imath}_w$ of [12, Lemma 4.17], again for both choices of involution. Next we pass to the analogue of [12, Lemma 4.21]. With notations as in loc.cit., for each $e \in E^1$ consider the following elements of $\mathfrak{A}[1]$

$$U_e = \epsilon_{s(e),s(e)}(1 - t^2)ee^* + \epsilon_{e,s(e)}te^*, \quad V_e = \epsilon_{s(e),s(e)}(1 - t^2)ee^* + \epsilon_{s(e),e}(2t - t^3).$$

One checks that the homotopy $H^+ : C(E) \to D[r]$ defined in the proof of [12, Lemma 4.21] satisfies the following identity for each $e \in E^1$.

$$H^+(e) = (em_{r(e)}, U_e\epsilon_{s(e),e}(r(e)), H^+(e^*) = (m_{r(e)}e^*, \epsilon_{r(e),s(e)}e^*V_e).$$

Put $W_e = c(U_e, V_e)$. Let

$$H : C(E) \to M_{\mathcal{P}(E)}D[r],$$

$$H(e) = \ell_+(em_{r(e)}, 0) + (0, W_e)(0, \ell_+(\epsilon_{s(e),r(e)}e)),$$

$$H(e^*) = H(e)^*, \quad H(v) = (m_v, 0) \quad (v \in E^0, \ e \in E^1).$$

One checks that $H$ is a $*$-algebra homomorphism for both choices of involution, so that for both choices of involution, $H$ is a $*$-homotopy between the maps $\psi_0$ and $\psi_{1/2}$ of [12, Lemma 4.21]. This finishes the proof. □

Let $S$ be a set, $\mathcal{T}$ a triangulated category, and $\mathcal{H} : \text{Alg}_{\ell}^* \to \mathcal{T}$ an excisive homology theory. We say that $\mathcal{H}$ is $S$-additive if direct sums of at most $|S|$ factors exist in $\mathcal{T}$, and
for any set $T$ with cardinality $|T| \leq |S|$ and any family of $*$-algebras $\{A_t : t \in T\}$, the canonical map $\bigoplus_{t \in T} \mathcal{H}(A_t) \to \mathcal{H}(\bigoplus_{t \in T} A_t)$ is an isomorphism.

**Theorem 3.2** Assume that $\ell$ satisfies the $\lambda$-assumption 1.3. Let $X : \text{Alg}^*_E \to \mathcal{F}$ be an excisive, homotopy invariant, Hermitian stable, $E$-stable and $E^0$-additive homology theory and let $R \in \text{Alg}^*_E$. Then (2.17) and (2.18) induce the following distinguished triangles in $\mathcal{F}$

$$
\begin{align*}
X(R)(\text{reg}(E)) \xrightarrow{I-A_E^t} X(R)(E^0) \xrightarrow{} X(L(E) \otimes R) \\
X(R)(\text{reg}(E)) \xrightarrow{I-\sigma A_E^t} X(R)(E^0) \xrightarrow{} X(L(E) \otimes R).
\end{align*}
$$

**Proof** It follows from the universal property of $j^h$ that $\otimes R$ induces a triangulated functor in $kk^h$. Hence upon replacing $X$ by $X(- \otimes R)$ if necessary, we may assume that $R = \ell$. Let $\text{inc} : K(E) \to C(E)$ and $\text{inc} : K(E) \to C(E)$ be the inclusions. For each vertex $w \in E^0$, let $p_w : \ell(E^0) \twoheadrightarrow \ell : \chi_w$ be the projection onto the $w$-coordinate and the inclusion into the $w$-summand. If $v \in \text{reg}(E)$, write $m_v = \sum_{s(e) = v} ee^* \in C(E)$ and let $q : \ell(\text{reg}(E)) \to K(E), v \mapsto q_v = v - m_v$. We shall abuse notation and write $q_v$ also for the map $\ell \to K(E), v \mapsto q_v$. In view of Theorem 3.1 and the additivity hypothesis on $X$, it suffices to identify, for each pair $(v, w) \in \text{reg}(E) \times E^0$, the composite $j^h(p_w)j^h(\phi)^{-1}j^h(\text{inc}q_v) \in KH^h_0(\ell)$, and similarly for $\bar{\phi}$, $\bar{\text{inc}}$ and $\bar{q}_v$. In the case of the standard involution, for each $e \in E^1$ the projection $ee^*$ is $M$-$\nu\Lambda$ equivalent to $e^*e = r(e)$ and the same calculation as in [12, Proposition 5.2] goes through. However in the case of the twisted involution, this is no longer true. In fact, taking into account (1.3) and (2.16) and writing $\ast$ for the involution of $M_\pm C(E)$, we obtain the following identities

$$
t_+(ee^*) = \begin{bmatrix} 0 & e \\ 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & 0 \\ e^* & 0 \end{bmatrix} = \begin{bmatrix} 0 & e \\ 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & e^* \\ 0 & 0 \end{bmatrix} \quad (3.2)
$$

$$
t_-(r(e)) = \begin{bmatrix} 0 & e^* \\ 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & e \\ 0 & 0 \end{bmatrix}
$$

It follows that $[ee^*] = \sigma[r(e)]$ in $KH^h_0(\overline{C(E)})$, so from the orthogonal sum $v = q_v + m_v$ we get that $j^h(\text{inc}q_v) = j^h(\bar{\phi}\chi_v) - \sum_{s(e) = v} \sigma j^h(\bar{\phi}r(e))$. Hence $j^h(\bar{\phi})^{-1} \circ j^h(\text{inc}) \circ j^h(\bar{q}) = I - \sigma A_E^t$. 

**Theorem 3.3** Let $E$ be a graph. Assume that $|E^0| < \infty$ and that $\ell$ satisfies the $\lambda$-assumption 1.3. Then there are distinguished triangles in $kk^h$.
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\[ j^h(\ell^{\text{reg}}(E)) \xrightarrow{I - A_{\ell}'} j^h(\ell^0) \xrightarrow{\partial} j^h(LE) \]

\[ j^h(\ell^{\text{reg}}(E)) \xrightarrow{I - \sigma A_{\ell}'} j^h(\ell^0) \xrightarrow{\partial} j^h(LE). \]

**Proof** Apply Theorem 3.2 to \( X = j^h. \)

**Corollary 3.4** Let \( E \) be as in Theorem 3.3 \( \tau : L(E) \to L(E) \) as in (2.1) and \( \bar{\tau} \) as in (2.3). Then

\[ 2(j^h(\tau) - j^h(\text{id}_{L(E)})) = 2(j^h(\bar{\tau}) - j^h(\text{id}_{L(E)})) = 0. \]

**Proof** The restrictions of \( \tau : C(E) \to C(E) \) to the images of \( \phi : \ell^0 \to C(E) \) and \( q : \ell^{\text{reg}}(E) \to K(E) \subset C(E) \) are the identity maps. Hence writing 1 for all identity maps, we have a map of triangles

\[
\begin{align*}
  j^h(\ell^{\text{reg}}(E)) & \xrightarrow{I - A_{\ell}'} j^h(\ell^0) \xrightarrow{p} j^h(LE) \xrightarrow{\partial} j^h(\ell^{\text{reg}}(E))[-1] \\
  j^h(\ell^{\text{reg}}(E)) & \xrightarrow{I - \sigma A_{\ell}'} j^h(\ell^0) \xrightarrow{p} j^h(LE) \xrightarrow{\partial} j^h(\ell^{\text{reg}}(E))[-1]
\end{align*}
\]

From the exact sequences obtained by applying \( kk^h(L(E), -) \) and \( kk^h(-, LE) \) to the triangles above we obtain factorizations \( 1 - j^h(\tau) = p \circ \xi = \eta \circ \partial \). It follows that

\[ 0 = \eta \circ \partial \circ p \circ \xi = (1 - j^h(\tau))^2 = 2(1 - j^h(\tau)). \]

The same argument shows that \( 2(1 - j^h(\bar{\tau})) = 0. \)

**Corollary 3.5** Let \( E \) be a finite graph and \( B_E \) and \( J \) as in (3.3) and (3.4). Then there are distinguished triangles in \( kk^h \)

\[
\begin{align*}
  (B_E)_{e,x} & = \begin{cases} 
    \delta_{r(e),x(s(x))} & x \in E^1 \\
    \delta_{r(e),x} & x \in \text{sink}(E)
  \end{cases} \\
  J_{x,e} & = \begin{cases} 
    \delta_{x(s(x)),r(e)} & x \in E^1 \\
    \delta_{x,r(e)} & x \in \text{sink}(E)
  \end{cases}
\end{align*}
\]

Also let \( J \in \mathbb{Z}^{(E^1 \times \text{sink}(E)) \times E^1}, \)

**Corollary 3.5** Let \( E \) be a finite graph and \( B_E \) and \( J \) as in (3.3) and (3.4). Then there are distinguished triangles in \( kk^h \)
\[ \ell E^1 \xrightarrow{J - B'_E} \ell E^1 \cup_{\text{sink}} (E) \xrightarrow{} LE \]

\[ \ell E^1 \xrightarrow{J - \sigma B'_E} \ell E^1 \cup_{\text{sink}} (E) \xrightarrow{} \overline{LE}. \]

**Proof** As observed in [12, Remark 5.7], \( B_E = A_{E_s} \) for the out-split graph \( E_s \) of [4, Definition 6.3.23]. An explicit \( \mathbb{Z} \)-graded \( * \)-algebra isomorphism \( f : LE \xrightarrow{} L(E_s) \) is constructed in the proof of [5, Theorem 2.8]; one checks that \( f(a) = \overline{f}(a) \) for all \( a \in L(E) \). Given all this, the corollary is immediate from Theorem 3.3.

\[ \square \]

### 4 Hermitian K-theory and Bowen–Franks groups

The **Bowen–Franks** group of a graph \( E \) is

\[ \mathfrak{B} \bar{F}(E) = \text{Coker}(I - A'_E). \]

We shall also consider the following \( \mathbb{Z} \sigma \)-module

\[ \mathfrak{B} \bar{F}(E) = \text{Coker}(I - \sigma A'_E). \]

**Remark 4.1** Let \( E \) be a regular graph and let \( E^2 \) be the graph with the same vertices and where an edge is a path of length 2 in \( E \). Then we have a group isomorphism

\[ \mathfrak{B} \bar{F}(E) \cong \text{Coker}(I - (A'_E)^2) = \mathfrak{B} \bar{F}(E^2). \]

Under the isomorphism above, the action of \( \sigma \) becomes multiplication by \( A'_E \).

**Theorem 4.2** Let \( E \) be a graph and \( R \in \text{Alg}^*_\ell \). Then there are exact sequences

\[ 0 \to \mathfrak{B} \bar{F}(E) \otimes K H^h_n(R) \to K H^h_n(L(E) \otimes R) \to \text{Ker}((I - A'_E) \otimes K H^h_{n-1}(R)) \to 0 \]

\[ 0 \to \mathfrak{B} \bar{F}(E) \otimes \mathbb{Z} \sigma \otimes K H^h_n(R) \to K H^h_n(L(E) \otimes R) \to \text{Ker}((I - \sigma A'_E) \otimes K H^h_{n-1}(R)) \to 0 \]

**Proof** Apply Theorem 3.2 to the functor \( K H^h \) from \( \text{Alg}^*_\ell \) to the homotopy category of spectra that sends \( A \in \text{Alg}^*_\ell \) to the homotopy Hermitian K-theory spectrum \( K H^h(A) \); then take homotopy groups.

\[ \square \]

**Lemma 4.3** Let \( E \) be a countable graph and \( R \) a unital algebra. If \( R \) is regular supercoherent, then \( L(E) \otimes R \) is K-regular. If moreover 2 is invertible in \( \ell \), then \( L(E) \otimes R \) is \( K^h \)-regular.

\[ \square \]
Proof The hypothesis on \( R \) implies that \( R[t_1, \ldots, t_n] \) is regular supercoherent for all \( n \), by the argument of [7, beginning of Section 7]. Hence \( L_E(Z) \otimes_Z R = L(F) \otimes R \) is \( K \)-regular for every regular supercoherent \( R \) and every row-finite graph \( E \), by [7, Theorem 7.6]. In the general case, because \( E \) is countable, we may choose a desingularization \( E_\delta \) of \( E \) as in [3, Section 5]. Theorem 5.6 of [3] says that if \( \ell \) is a field, then \( L(E) \) and \( L(E_\delta) \) are Morita equivalent as rings with local units; the proof actually works over arbitrary commutative \( \ell \), and in particular for \( \ell = \mathbb{Z} \). We claim that \( M_\infty L_E(Z) \cong M_\infty L_E(Z_\delta) \). Since \( L(E) \otimes R = L(Z_\delta) \otimes_Z R \), the claim together with the row-finite case already proved shows that \( L(E) \otimes R \) is \( K \)-regular. By [2, Theorem 5 and Remark 1], two rings \( A \) and \( B \) with countable systems of local units which are topologically projective in the sense of [2, Definition on page 409] are Morita equivalent if and only if \( M_\infty A \cong M_\infty B \). Hence it suffices to show that \( L(Z) \) is topologically projective for any graph \( E \). Let \( U = \bigoplus_{v \in E_0} LEv \) and let \( \pi : U \to LE \) be the sum of the canonical inclusions. Let \( s : L(E) \to U \), \( s(a)v = av \). For \( a \in L(E) \), put \( E \supseteq F = \text{supp}(s(a)) \) and \( p = \sum_{v \in F} v \). Then \( s(a + b - bp)v = s(a)v \) for all \( b \in L(E) \) and all \( v \in F \). This proves that \( L(E) \) is topologically projective, concluding the proof of the first assertion of the lemma. The second assertion follows from the first using [16, Lemma 3.8]. \( \square \)

Corollary 4.4 Let \( R \in \text{Alg}^e_\ell \) be unital and \( E \) a countable graph. Assume that \( 2 \) is invertible in \( R \) and that \( R \) is regular supercoherent. Then the maps \( K^h_0(L(E) \otimes R) \to KH^h_0(L(E) \otimes R) \) and \( K^h_0(L(F) \otimes R) \to KH^h_0(L(F) \otimes R) \) of (2.13) are isomorphisms and the exact sequences of Theorem 4.2 also hold for \( L(E) \otimes R \) and \( L(F) \otimes R \) with \( KH^h \) replaced by \( K^h \).

Proof Because \( R \) is unital, \( L(E) \otimes R \) has local units and therefore \( K \)-excisive in the sense of [16, Section 3]. Now apply [16, Lemma 3.8]. \( \square \)

Corollary 4.5 Assume that \( E \) is countable and that \( \ell \) is a field of characteristic \( \text{char}(\ell) \neq 2 \). Then

\[
K^h_0(L(E)) = \mathcal{B}_h(E) \otimes_{\mathbb{Z}} K^h_0(\ell), \quad K^h_0(L(F)) = \mathcal{B}_h(E) \otimes_{\mathbb{Z}[\sigma]} K^h_0(\ell).
\]

Proof In view of Corollary 4.4, it suffices to show that \( K^h_{-1}(\ell) = 0 \). By [29, Theorems 7.1 and 8.1] and [28, Proposition 6.3] \( K^h_{-1}(\ell) \) agrees with Ranicki’s \( U_{-1}(\ell) \) which vanishes by [26, Proposition]. \( \square \)

Example 4.6 Let \( \ell \) be a field of \( \text{char}(\ell) \neq 2 \). Assume that the canonical map \( \mathbb{Z}[\sigma] \to KH^h_0(\ell) \) is an isomorphism. It follows from Corollary 4.5 that \( K^h_0(L(E)) = \mathcal{B}_h(E) \) and \( K^h_0(L(F)) = \mathcal{B}_h(E) \otimes_{\mathbb{Z}[\sigma]} \). This is the case, for example, when \( \ell = \mathbb{C} \) equipped with complex conjugation as involution.

Lemma 4.7 Let \( E \) and \( \ell \) be as in Theorem 3.3. Then

(i) \( j^h(L(E)) = 0 \iff \mathcal{B}_h(E) \otimes_{\mathbb{Z}[\sigma]} KH^h_0(\ell) = 0 \).

(ii) \( j^h(L(F)) = 0 \iff \mathcal{B}_h(E) \otimes KH^h_0(\ell) = 0 \).

(iii) \( j^h(L(F)) = 0 \Rightarrow j^h(L(E)) = 0 \).

\( \square \) Springer
(iv) If \( j^h(L(E)) = 0 \) then \( E \) is regular.

**Proof** By Theorem 3.3, \( j^h(L(E)) = 0 \) if and only if the image of \( I - A_E^t \) in \( KH_0^h(\ell)^{E_0 \times \text{reg}(E)} \) under the map induced by \( \mathbb{Z} \subset \mathbb{Z}[\sigma] \rightarrow KH_0^h(\ell) \) is an invertible matrix. Because \( KH_0^h(\ell) \) is a commutative ring, the latter condition implies that \( \text{reg}(E) = E_0^0 \) and is equivalent to having \( \mathcal{B}\mathfrak{S}(L(E)) \otimes KH_0^h(\ell) = 0 \). Similarly, \( j^h(L(E)) = 0 \) is equivalent to \( \mathcal{B}\mathfrak{S}(L(E)) \otimes \mathbb{Z}[\sigma] \otimes KH_0^h(\ell) = 0 \) and implies that \( j^h(L(E)) = 0 \), since by definition \( \mathcal{B}\mathfrak{S}(E) \) is a quotient of \( \mathcal{B}\mathfrak{S}(E) \). \( \square \)

**Example 4.8** Let \( \Upsilon \) be the following graph

\[
\begin{array}{c}
\circ \\
\circ \\
\circ
\end{array}
\]

Then \( \det(I - \sigma A^t_\Upsilon) = -\sigma \), so \( j^h(L(\Upsilon)) = j^h(L(\Upsilon)) = 0 \).

## 5 Hermitian \( K \)-theory of \( \hat{L}(E) \) vs. \( \mathbb{Z}/2\mathbb{Z} \)-graded \( K \)-theory of \( L(E) \)

Let \( E \) be a graph and let \( \hat{E} \) be the graph with \( \hat{E} = E^i \times \mathbb{Z}/2\mathbb{Z}, (i = 0, 1) \) with source and range functions \( s(e, i) = (s(e), i), r(e, i) = (r(e), i + 1) \). Observe that \( \mathbb{Z}/2\mathbb{Z} \) acts on \( \hat{E} \) by translation on the second component. We have an isomorphism of \( \mathbb{Z}[\sigma] \)-modules \( \mathbb{Z}(\hat{E})^0 \cong \mathbb{Z}[\sigma] \otimes \mathbb{Z}(E)^0 \) which restricts to an isomorphism \( \mathbb{Z}(\text{reg}(\hat{E})) \cong \mathbb{Z}[\sigma] \otimes \mathbb{Z}(\text{reg}(E)) \). Under these identifications, \( A_\hat{E} \) becomes \( \sigma A_E \). Hence we have an isomorphism of \( \mathbb{Z}[\sigma] \)-modules

\[
\mathcal{B}\mathfrak{S}(\hat{E}) \cong \mathcal{B}\mathfrak{S}(E).
\] (5.1)

Hence by [8, Corollary 5.3] and Remark 2.5, for \( \hat{L}(E) \) as in Example 2.4, we have a *-isomorphism

\[
\hat{L}(E) \cong \hat{L}(E).
\] (5.2)

Write \( K_0^\text{gr} \) for the graded \( K \)-theory of \( \mathbb{Z}/2\mathbb{Z} \)-rings. It follows from (5.2), Remark 2.5 and Theorem 4.2 applied to \( \text{inv}(\ell) \), that if \( E \) has finitely many vertices and \( \ell \) is regular supercoherent, then

\[
K_0^\text{gr}(L(E)) = \mathcal{B}\mathfrak{S}(E) \otimes K_0(\ell).
\] (5.3)

In particular, if \( \ell \) is as in Example 4.6, then we have

\[
K_0^h(\hat{L}(E)) \cong K_0^\text{gr}(L(E)).
\] (5.4)

**Remark 5.1** It follows from Example 2.4 that \( L(\hat{E}) \) is a *-subalgebra of \( M_\pm L(E) \).

Write \text{inc} for inclusion map; we have a commutative diagram

\[
\begin{array}{ccc}
\mathcal{B}\mathfrak{S}(E) \otimes KH_0^h(\ell) & \longrightarrow & KH_0^h(\hat{L}(E)) \\
\downarrow & & \downarrow \quad \text{KH}_0^h(\text{inc}) \\
\mathcal{B}\mathfrak{S}(E) \otimes \mathbb{Z}[\sigma] \otimes KH_0^h(\ell) & \longrightarrow & KH_0^h(L(E)).
\end{array}
\] \( \square \) Springer
Here the rows are the monomorphisms of Theorem 4.2 and the left column is the canonical surjection. In particular, $K H_0^h (\text{inc})$ is not injective in general.

In the next proposition and elsewhere, we write $\mathcal{R}_n$ for the graph consisting of a single vertex and $n$ loops, $\mathcal{R}_n^-$ for its Cuntz splice [6, Definition 2.11],

$$L_n = L(\mathcal{R}_n) \text{ and } L_n^- = L(\mathcal{R}_n^-). \quad (5.5)$$

**Proposition 5.2**  Assume that $\ell$ is regular supercoherent and such that the canonical map $\mathbb{Z} \to K_0(\ell)$ is an isomorphism. Then there is no $\mathbb{Z}/2\mathbb{Z}$-homogeneous unital $\ell$-algebra homomorphism from $L_2$ to $L_2^-$ nor in the opposite direction. If furthermore, $\ell$ is a field with $\text{char}(\ell) \neq 2$ and $\mathbb{Z}[\sigma] \to K_0^h (\ell)$ is an isomorphism, then there is no unital $*$-algebra homomorphism $\bar{L}_2 \to \bar{L}_2^-$ nor in the opposite direction.

**Proof**  By (5.3) and the hypothesis on $\ell$, we have $K_0(\ell(E)) = \mathcal{B}\mathcal{F}(E)$ for any graph $E$. Next we compute (e.g. using Remark 4.1) that

$$\mathcal{B}\mathcal{F}(\mathcal{R}_2) = \mathbb{Z}/3\mathbb{Z}, \quad \mathcal{B}\mathcal{F}(\mathcal{R}_2^-) = \mathbb{Z}/7\mathbb{Z}. \quad (5.6)$$

By (5.1) and Remark 2.5, any unital, $\mathbb{Z}/2\mathbb{Z}$-homogeneous algebra homomorphism between $L_2$ and $L_2^-$ would induce a homomorphism between $\mathbb{Z}/3\mathbb{Z}$ and $\mathbb{Z}/7\mathbb{Z}$ mapping $\bar{1} \mapsto 1$, but there is no such homomorphism. This proves the first assertion. If $\ell$ is as in the the second assertion, then (5.4) and (5.6) together imply

$$K_0^h (\bar{L}_2) = \mathbb{Z}/3\mathbb{Z}, \quad K_0^h (\bar{L}_2^-) = \mathbb{Z}/7\mathbb{Z}. \quad (5.7)$$

The proof is now immediate. \hfill $\square$

**6 Structure theorems for Leavitt path algebras in $kk^h$**

Let $n_0, n_1 \geq 1$, $M \in \mathbb{Z}^{n_0 \times n_1}$, and

$$j^h (\ell)^{n_1} \overset{M}{\longrightarrow} j^h (\ell)^{n_0} \longrightarrow j^h (R) \quad (6.1)$$

a distinguished triangle in $kk^h$. Applying $kk^h (\ell, -)$ we obtain a monomorphism $\text{Coker}(M) \otimes K H_0^h (\ell) \to K H_0^h (R)$; composing it with $- \otimes [1] : \text{Coker}(M) \to \text{Coker}(M) \otimes K H_0^h (\ell)$ we obtain a canonical map

$$\text{can} : \text{Coker}(M) \to K H_0^h (R). \quad (6.2)$$

Hence for every $S \in \text{Alg}^*_\ell$, there is an evaluation map

$$\text{ev} : kk^h (R, S) \to \text{hom}(\text{Coker}(M), K H_0^h (S)), \quad \text{ev}(\xi) = K H_0^h (\xi) \circ \text{can}. \quad (6.3)$$
Lemma 6.1 Let $E$ be a graph with $|E^0| < \infty$, and let $n_0, n_1 \geq 1$, $M \in \mathbb{Z}^{n_0 \times n_1}$, and $R$ be as in (6.1). Assume that $\text{rk}(\text{Ker} M) = \text{rk}(\text{Ker}(I - A'_E))$. Let $\xi_0 : \mathcal{B}(E) \xrightarrow{\sim} \text{Coker}(M)$ be a group isomorphism. Then there exists an isomorphism $\xi : j^h(L(E)) \xrightarrow{\sim} j^h(R)$ such that $\text{ev}(\xi) = \text{can} \circ \xi_0$.

**Proof** Put $m_0 = |E^0|, m_1 = |\text{reg}(E)|$. Because the free abelian groups $\text{Ker}(M)$ and $\text{Ker}(I - A'_E)$ have the same rank, there is an isomorphism $\xi_1 : \text{Ker}(I - A'_E) \xrightarrow{\sim} \text{Ker}(M)$. Because $\text{Im}(I - A'_E)$ and $\text{Im}(M)$ are free, the surjections $\mathbb{Z}^{m_1} \twoheadrightarrow \text{Im}(I - A'_E)$ and $\mathbb{Z}^{n_1} \twoheadrightarrow \text{Im}(M)$ admit sections $s$ and $t$. Let $f_0 : \mathbb{Z}^{m_0} \to \mathbb{Z}^{n_0}$ be any lift of $\xi_0$; put $f_1(x) = \xi_1(x - s(I - A'_E)x) + t(f_0((I - A'_E)x)).$ (6.4)

Then $\mathbb{Z}^{m_1} \xrightarrow{I - A'_E} \mathbb{Z}^{m_0} \xrightarrow{\xi_1} \mathbb{Z}^{n_1} \xrightarrow{M} \mathbb{Z}^{n_0}$ is a quasi-isomorphism $f$ with $H_i(f) = \xi_i$ ($i = 0, 1$). Using the canonical map $\mathbb{Z} \to KH^h_0(\ell)$ we can associate to $f$ a commutative solid arrow diagram

\[
\begin{array}{cccccc}
\mathbb{Z}^{m_1} & \xrightarrow{I - A'_E} & \mathbb{Z}^{m_0} & \xrightarrow{\xi_1} & \mathbb{Z}^{n_1} & \xrightarrow{M} & \mathbb{Z}^{n_0}
\end{array}
\]

Because $kk^h$ is triangulated there exists a filler $\xi$ as above. By construction, the following diagram commutes

\[
\begin{array}{cccccc}
\mathcal{B}(E) \otimes KH^h_0(\ell) & \xrightarrow{\cup} & KH^h_0(L(E)) & \xrightarrow{\xi_0 \otimes 1} & KH^h_0(\xi) \\
\downarrow & & \downarrow & & \\
\text{Coker}(M) \otimes KH^h_0(\ell) & \xrightarrow{\cup} & KH^h_0(R)
\end{array}
\]

It follows that $\text{ev}(\xi) = \text{can} \xi_0$. It remains to show that $\xi$ is an isomorphism. Proceeding as above, we obtain a quasi-isomorphism $g : \mathbb{Z}^{n_1} \to \mathbb{Z}^{m_1}$ with $H_*(g) = \xi_*^{-1}$, such that for all $y \in \mathbb{Z}^{n_1}$,

\[g_1(y) = \xi_1^{-1}(y - tMy) + s(g_0(My)).\] (6.5)
As before, the chain map $g$ induces a map of triangles

$$
\begin{array}{ccccccccc}
\llap{$M$}j^h(\ell)^{n_1} & \rightarrow & \llap{$g_1$}j^h(\ell)^{n_0} & \rightarrow & \llap{$g_0$}j^h(R) & \rightarrow & \llap{$\eta$}j^h(\ell)[1] \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\llap{$I - A_E^t$}j^h(\ell)^{m_1} & \rightarrow & \llap{$p$}j^h(L(E)) & \rightarrow & \llap{$\delta$}j^h(\ell)^{m_1}[1].
\end{array}
$$

It follows from (6.4) and (6.5) that $g_1 f_1$ restricts to the identity on $\text{Ker}(I - A_E^t)$ and that there is a homomorphism $h : \mathbb{Z}^{m_0} \rightarrow \mathbb{Z}^{n_1}$ with $h \circ (I - A_E^t) = \text{id} - g_1 f_1$ and $(I - A_E^t) \circ h = \text{id} - g_0 f_0$. Hence

$$
\begin{align*}
\partial (1 - \eta \xi) &= (1 - g_1 f_1[-1]) \partial = (h \circ (I - A_E^t))[1] \circ \partial = 0 \\
(1 - \eta \xi) p &= p(1 - f_0 g_0) = p(1 - A_E^t) h = 0.
\end{align*}
$$

Therefore there exist $\xi_0 \in k h(L(E), \ell^{m_0})$ and $\xi_1 \in k h(L(E))$ such that $1 - \eta \xi = p \xi_0 = \xi_1 \partial$. In particular, $(1 - \eta \xi)^2 = \xi_1 \partial p \xi_0 = 0$, and therefore $\eta \xi$ is an isomorphism. Similarly, $\xi \eta$ is an isomorphism, and so $\xi$ is an isomorphism, concluding the proof.

**Theorem 6.2** Let $E$ and $F$ be graphs with finitely many vertices and such that $|\text{sing}(E)| = |\text{sing}(F)|$. Let $\xi_0 : \mathcal{B}\mathfrak{F}(E) \sim \mathcal{B}\mathfrak{F}(F)$ be an isomorphism. Assume that $\ell$ satisfies the $\lambda$-assumption 1.3. Then there exists an isomorphism $\xi : j^h(L(E)) \sim j^h(L(F))$ such that $\text{ev}(\xi) = \text{can} \circ \xi_0$.

**Proof** Apply Lemma 6.1 with $M = I - A_F^t$ and $R = L(F)$.

**Remark 6.3** One may ask whether a $K H_0^h(\ell)$-module isomorphism $\xi_0 : \mathcal{B}\mathfrak{F}(E) \otimes K H_0^h(\ell) \sim \mathcal{B}\mathfrak{F}(F) \otimes K H_0^h(\ell)$ lifts to a $k h$-isomorphism $\xi : j^h(L(E)) \sim j^h(L(F))$ if $|\text{sing}(E)| = |\text{sing}(F)|$. The argument of Lemma 6.1 proves that this is indeed the case if we additionally assume that $\text{Tor}_1^E(\mathcal{B}\mathfrak{F}(E), K H_0^h(\ell)) = 0$.

Let $E$ be a graph with $|E|^0 < \infty$. Then $\mathcal{B}\mathfrak{F}(E)$ is finitely generated. Thus there are $r, n \geq 0$ and $2 \leq d_1, \ldots, d_n$ with $d_i \setminus d_{i+1}$ for all $i$ such that

$$
\mathcal{B}\mathfrak{F}(E) = \mathbb{Z}^r \oplus \bigoplus_{i=1}^n \mathbb{Z}/d_i \mathbb{Z}. \quad (6.6)
$$

**Theorem 6.4** Let $E$ be a graph such that $E^0$ is finite. Assume that $\ell$ satisfies the $\lambda$-assumption 1.3. Let $r, n$ and $d_1, \ldots, d_n$ be as in (6.6) and let $s = |\text{sing}(E)|$ be the number of singular vertices. Then

$$
j^h(L(E)) \cong j^h \left( L_0^s \oplus L_1^{r-s} \oplus \bigoplus_{i=1}^n L_{d_i+1} \right).
$$
Proof Apply Lemma 6.1 with $M$ the Smith normal form of $I - A_i'$ and $R = L_0' \oplus L_1^{r-s} \oplus \bigoplus_{i=1}^{r} L_{d_i+1}$.

As previously recalled from [12] and [16], for any graph $E$ we have $E$-stable variants of the universal homology theories $j : \text{Alg}_\ell \to kk$ and $j^h : \text{Alg}_\ell \to kk^h$. In the following corollary, as well as in any other statement involving two graphs $E$ and $F$ with possibly infinitely many edges, $j$ and $j^h$ are understood to be the $E \sqcup F$-stable ones.

**Theorem 6.5** Let $E$ and $F$ be graphs with finitely many vertices. Assume that $\ell$ satisfies the $\lambda$-assumption 1.3, that $KH_{-1}(\ell) = 0$ and that the canonical map $\mathbb{Z} \to KH_0(\ell)$ is an isomorphism. Then the following are equivalent.

(i) $j(LE) \cong j(LF)$ in $kk$.
(ii) $j^h(LE) \cong j^h(LF)$ in $kk^h$.

**Proof** If (ii) holds, then the forgetful functor $kk^h \to kk$ sends the isomorphism $j^h(LE) \cong j^h(LF)$ to an isomorphism $j(LE) \cong j(LF)$. If (i) holds, then by [12, Corollary 6.11], $\mathfrak{B}_\ell(E) \cong \mathfrak{B}_\ell(F)$ and $|\text{sing}(E)| = |\text{sing}(F)|$, so $j^h(LE) \cong j^h(LF)$ by Theorem 6.2.

**Remark 6.6** The analogue of Theorem 6.5 with $\overline{LE}$ and $\overline{LF}$ substituted for $LE$ and $LF$ does not hold. Indeed, for any $\ell$, $j(L_2) = j(L_{2^-}) = 0$ but for $\ell$ as in Example 4.6, $j^h(\overline{L_2}) \not\cong j^h(\overline{L_{2^-}})$, by (5.7).

The proof of Lemma 6.1 does not work for the analogue of the lemma with a matrix $M$ with coefficients in $\mathbb{Z}[\sigma]$ and $\mathfrak{B}_\ell(E)$ substituted for $\mathfrak{B}_\ell(E)$. This is because a submodule of a free $\mathbb{Z}[\sigma]$-module need not be free or even projective. However the problem disappears if $\text{Ker}(M) = \text{Ker}(I - \sigma A_i') = 0$, and we have the following.

**Lemma 6.7** Let $E$ be a graph with $\left|E^0\right| < \infty$, and let $n_0, n_1 \geq 1$, $M \in \mathbb{Z}[\sigma]^{n_0 \times n_1}$, and $R$ as in (6.1). Assume that $\text{Ker}M = \text{Ker}(I - \sigma A_i') = 0$. Let $\xi_0 : \overline{\mathfrak{B}_\ell(E)} \to \text{Coker}(M)$ be a group isomorphism. Then there exists an isomorphism $\xi : j^h(\overline{L(E)}) \cong j^h(R)$ such that $\text{ev}(\xi) = \text{can} \circ \xi_0$.

**Proof** The argument of the proof of Lemma 6.1 shows this.

**Theorem 6.8** Let $E$ and $F$ be graphs with finitely many vertices and such that $\text{Ker}(I - \sigma A_i') = \text{Ker}(I - \sigma A_i') = 0$. Let $\xi_0 : \overline{\mathfrak{B}_\ell(E)} \to \overline{\mathfrak{B}_\ell(F)}$ be an isomorphism. Assume that $\ell$ satisfies the $\lambda$-assumption 1.3. Then there exists an isomorphism $\xi : j^h(\overline{L(E)}) \cong j^h(L(F))$ such that $\text{ev}(\xi) = \text{can} \circ \xi_0$.

**Proof** Apply Lemma 6.7 with $M = I - \sigma A_i'$ and $R = \overline{L(F)}$.

**Proposition 6.9** Assume that $2$ is invertible and $1$-positive in $\ell$. Let $E$ be a finite graph such that the $\mathbb{Z}[\sigma]$-modules $\overline{\mathfrak{B}_\ell(E)}$ and $\overline{\mathfrak{B}_\ell(E)}$ are finite and isomorphic. Let $\widehat{\Delta}$ be as in Proposition 2.13. Then $2 \text{id}_{L(E)}$ and $2 \text{id}_{\overline{L(E)}}$ are invertible elements of $kk^h(L(E), L(E))$ and $kk^h(\overline{L(E)}, L(E))$, and we have

$$\left(\frac{\Delta}{2}\right) \circ \left(\frac{\Delta}{2}\right) = \frac{j^h(\text{id}_{\overline{L(E)}})}{2}.$$

\[ \square \]
In particular, $j^h(L(E))$ is a retract of $j^h(L(E))$. Moreover the following sequence is exact

$$0 \longrightarrow j^h(L(E)) \xrightarrow{\Delta_{L(E)}} j^h(L(E)) \xrightarrow{1-\sigma} j^h(L(E)).$$

**Proof** If $\mathcal{B}(E)$ is finite, then $E$ is regular, so by Remark 4.1, $\mathcal{B}(E)$ is isomorphic as a $\mathbb{Z}[\sigma]$-module to $M = \text{Coker}(I - (A'_E)^2)$ where $\sigma$ acts as multiplication by $A'_E$. Then $M \cong \mathcal{B}(E)$ as $\mathbb{Z}[\sigma]$-modules if and only if $A'_E$ acts trivially on $M$, in which case $I + A'_E$ descends to multiplication by 2, and we have

$$\text{Im}(I - A'_E) = \text{Im}((I - (A'_E)^2) = (I + A'_E)(\text{Im}(I - A'_E)). \quad (6.7)$$

Hence $I + A'_E$ restricts to an injection on $\text{Im}(I - A'_E)$, which has rank $|E^0|$ since $\mathcal{B}(E)$ is finite. Thus $\det(I + A'_E) \neq 0$, which combined with (6.7) implies that $(I + A'_E)x$ goes to zero in $M$ if and only if $x$ does. It follows that multiplication by 2 on $M$ is injective and therefore bijective since $M$ is finite. Hence $I + A'_E$ is invertible and $n = |M|$ is odd. Write $n = 2q - 1$; then by the argument of Lemma 6.1, $n^2 j^h(\text{id}_{L(E)}) = n^2 j^h(\text{id}_{L(E)}) = 0$, so $2j^h(\text{id}_{L(E)})$ and $2j^h(\text{id}_{L(E)})$ are isomorphisms. The same argument but with $\mathbb{Z}[\sigma]$ substituted for $\mathbb{Z}$, shows that

$$2(1 - \sigma) \text{id}_{L(E)} = (1 - \sigma)^2 \text{id}_{L(E)} = 0,$$

which by what we have already seen implies that $(1 - \sigma) \text{id}_{L(E)} = 0$. The proposition now follows from Proposition 2.13 and Corollary 3.4. \qed

**Remark 6.10** Let $E$ and $\ell$ be as in Proposition 6.9. Observe that $\sigma$ acts as $1 \otimes \sigma$ on $\mathcal{B}(E) \otimes K_{H_0^h}(\ell) \subset K_{H_0^h}(L(E))$; this action is nontrivial in general, so in particular $0 \neq 1 - \sigma \in \text{kk}(L(E), L(E))$. For example $\ell = \mathbb{C}$ satisfies the hypothesis of the proposition, and $\sigma$ acts on $\mathcal{B}(E) \oplus \sigma \mathcal{B}(E) = \mathcal{B}(E) \otimes K_{H_0^h}(\mathbb{C})$ by interchanging the summands.

**Remark 6.11** The proof of Proposition 6.9 shows that if a regular finite graph $E$ with incidence matrix $A$ satisfies the hypothesis of the proposition and $\mathcal{B}(E) \neq 0$, then

$$\det(I + A) \in \{\pm 1\} \text{ and } \det(I - A) \in \mathbb{Z} \setminus \{0, \pm 1\}. \quad (6.8)$$

The converse is also true; if (6.8) holds, then $I + A' : \mathcal{B}(E) \rightarrow \mathcal{B}(E^2) \cong \mathcal{B}(E)$ is a $\mathbb{Z}[\sigma]$-module isomorphism. A concrete example is the graph with incidence matrix

$$A_E = \begin{bmatrix} 1 & 3 \\ 1 & 1 \end{bmatrix}.$$
7 $K_0$ invariants for $*$-algebras

Let $A$ be a $*$-algebra and $x \in A$. A projection in $A$ is a self-adjoint idempotent element; we write $\mathfrak{Proj}(A)$ for the set of all projections in $A$. If $p, q \in \mathfrak{Proj}(A)$, we write $p \geq q$ whenever the identities $q = pq = qp$ hold. Two projections $p, q \in A$ are (Murray–von Neumann) equivalent—and we write $p \sim q$—if there is an element $x \in A$ such that $x^*x = p$, $xx^* = q$. Such an $x$ can be chosen to be in $pAq$, in which case we call it an $(\text{MvN})$ equivalence and write $x : p \sim q$. All the basic properties about equivalence of idempotents proved in [10, Section 4.2] hold verbatim for projections, with the same proofs. A partial isometry is an element $x \in A$ such that $xx^*x = x$. Let $R$ be a unital $*$-algebra. An isometry in $R$ is an element $u$ such that $u^*u = 1$; a unitary is an invertible isometry. For $1 \leq n \leq \infty$, put $\mathfrak{Proj}_n(R) = \mathfrak{Proj}(M_n R)$ and consider the set of equivalence classes $\mathcal{V}_n(R)^* = \mathfrak{Proj}_n(R)/\sim$. If $n < m$, then the map $\mathcal{V}_n(R)^* \to \mathcal{V}_m(R)^*$ is injective, and $\mathcal{V}_\infty(R)^* = \bigcup_n \mathcal{V}_n(R)^*$. The set $\mathcal{V}_\infty(R)^*$ forms an abelian monoid under orthogonal sum; if $x, y \in \mathfrak{Proj}_\infty(R)$ then there are orthogonal projections $p, q \in \mathfrak{Proj}_\infty(R)$ with $x = [p]$ and $y = [q]$ and $x + y := [p + q]$. This is well-defined by the projection analogue of [10, Proposition 4.2.4]. A projection $p \in \mathfrak{Proj}(R)$ is strictly full if there exist $n \geq 1$ and $(x_1, \ldots, x_n) \in R^n$ such that $\sum_{i=1}^n x_i^*px_i = 1$.

**Lemma 7.1** Let $R$ be a unital $*$-algebra and $p \in \mathfrak{Proj}(R)$ a strictly full projection. Then the inclusion $pRp \subset R$ induces an isomorphism $\mathcal{V}_\infty(pRp)^* \sim \mathcal{V}_\infty(R)^*$.

**Proof** Consider the category $\mathfrak{P}(R)$ whose set of objects is $\mathfrak{Proj}_\infty(R)$ and where a homomorphism $q_1 \to q_2$ is an element $x \in M_\infty R$ such that $x^*x = q_1$ and $xx^* \leq q_2$. We have a functor to abelian monoids $F : \mathfrak{P}(R) \to \text{AbMon}$ which sends $q \mapsto \mathcal{V}_1(q M_\infty Rq)^*$. Note that any two homomorphisms $q_1 \to q_2$ in $\mathfrak{P}(R)$ induce the same monoid homomorphism upon applying $F$. Hence if $\mathcal{C} \subset \mathfrak{P}(R)$ is a subcategory such that for every $q \in \mathfrak{P}(R)$ there is $q \to q'$ with $q' \in \mathcal{C}$, then $\text{colim}_\mathcal{C} F = \text{colim}_\mathfrak{P}(R) F$. Apply this to the subcategory $\mathfrak{I}(R) \subset \mathfrak{P}(R)$ whose objects are the identity matrices $1_n$ and where a homomorphism $1_n \to 1_m$ is a matrix $x \in R^{m \times n}$ whose columns are $n$ consecutive vectors of the canonical basis of $R^m$. Then $\text{colim}_\mathfrak{P}(R) F = \text{colim}_\mathfrak{I}(R) \mathcal{V}_n(R)^* = \mathcal{V}_\infty(R)^*$. Similarly, if $p \in \mathfrak{Proj}(R)$ is strictly full, then $\mathcal{V}_\infty(pRp)^* = \text{colim}_\mathfrak{P}(pRp) F \sim \text{colim}_\mathfrak{P}(R) F = \mathcal{V}_\infty(R)^*$.

Consider the group completion

$$K_0(R)^* := (\mathcal{V}_\infty(R)^*)^+$$

Now assume that the center of $R$ satisfies the $\lambda$-assumption 1.3. The Hermitian Witt–Grothendieck group $K_0^h(R)$ is defined as the group completion of the monoid

$$\mathcal{V}_\infty^h(R) = \mathcal{V}_\infty(M_\pm R)^*$$

thus we have

$$K_0^h(R) = K_0(M_\pm(R))^*.$$

\(\square\) Springer
The inclusion \( \iota_+: R \to M_\pm R \) induces a canonical homomorphism \( V_\infty^h(R)^* \to V_\infty^h(R) \). We may also regard \( V_\infty^h(R) \) as the monoid of unitary isomorphism classes of all pairs \( [(P, \phi)] \) consisting of a finitely generated projective right module equipped with a nondegenerate Hermitian form \( \phi \). In the same fashion, the monoid \( V_\infty(R)^* \) consists of the unitary classes of those \( P = (P, \phi) \) for which there is an \( n \geq 1 \) such that \( P \) embeds as an orthogonal direct summand of the free module of rank \( n \) equipped with the standard Hermitian form

\[
\langle x, y \rangle = \sum_{i=1}^{n} x^*_i y_i.
\]

Observe that \(-1\) is positive in \( R \) if and only if there is some \( n \geq 1 \) such that the equation

\[
\langle x, x \rangle = -1 \quad (7.3)
\]

has a solution \( x \in R^n \). In this case \( p : R^n \to R^n, p(y) = -x \langle x, y \rangle \) is an orthogonal projection onto \( xR \). In particular the hyperbolic module of rank \( 2, H(R) \), embeds as an orthogonal summand in \( R^{n+1} \), and \( M_\pm R \cong (1 \oplus p)M_{n+1}R(1 \oplus p) \). One checks that

\[
\sum_{1 \leq i, j \leq n} \epsilon_{i,j} p \epsilon_{i,j}^* = \sum_{i=1}^{n} \sum_{j=1}^{n} -x_j x_j^* = I_n
\]

Hence \( p \) is a strictly full projection of \( M_nR \), and therefore \( 1 \oplus p \) is a strictly full projection of \( M_{n+1}R \). Thus by Lemma 7.1 the canonical map \( V_\infty(R)^* \to V_\infty^h(R) \) is an isomorphism, as is the induced map \( K_0(R)^* \to K_0^h(R) \).

**Remark 7.2** Let \( R \) be a unital \(*\)-algebra. The involution defines an action of \( \mathbb{Z}/2\mathbb{Z} \) on \( K_0(R) \). With no assumptions on \( \ell \), for any unital \(*\)-algebra \( R \), we have a canonical forgetful map \( \text{forg} : K_0(R)^* \to K_0(R)^{\mathbb{Z}/2\mathbb{Z}} \) which maps the class of a projection modulo \( \text{Mv-N} \) equivalence of projections to its class modulo \( \text{Mv-N} \) equivalence of idempotents. If \( E \) is a graph with finitely many vertices and \( v \in \text{reg}(E) \), then the identity \([v] = \sum_{s(e)=v} [r(v)]\) holds in \( K_0(LE)^* \). Hence we have a canonical group homomorphism

\[
\text{can}' : \mathcal{B}\mathcal{G}(E) \to K_0(LE)^*.
\]

If \( \ell \) is regular supercoherent and \( \mathbb{Z} \to K_0(\ell) \) is an isomorphism, then \( \text{forg} \circ \text{can}' \) is the well-known isomorphism \( \mathcal{B}\mathcal{G}(E) \sim K_0(LE) \). In particular, \( \text{can}' \) is a split monomorphism in this case.

**Remark 7.3** The identities (3.2) show that if \( E \) is a graph with finitely many vertices, then \([ee^*] = \sigma[r(e)] \in K_0^h(LE) \) for every \( e \in E^1 \). Hence there is a canonical \( \mathbb{Z}[\sigma] \)-module homomorphism

\[
\text{can} : \mathcal{B}\mathcal{G}(E) \to K_0^h(LE).
\]
8 Strictly properly infinite $\ast$-algebras, purely infinite simple graphs, and their $K_0$ invariants

Let $C_n$ be the Cohn path algebra of $R_n$ $(0 \leq n \leq \infty)$; observe that $C\infty = L\infty$, the Leavitt path algebra of $R\infty$. A unital algebra $R$ is properly infinite there is a unital algebra homomorphism $C_2 \rightarrow R$, or equivalently, an algebra homomorphism $L\infty \rightarrow R$. (Properly infinite algebras were called sum algebras in [20] and renamed $C_2$-algebras in [12, Section 2]; we further rename them to conform to standard $C^\ast$-algebra terminology.) A $\ast$-algebra $R$ is strictly properly infinite if there is a unital $\ast$-homomorphism $C_2 \rightarrow R$, or equivalently, a unital $\ast$-homomorphism $L\infty \rightarrow R$. Observe that if $R$ is strictly properly infinite and $\phi : R \rightarrow S$ is a unital $\ast$-homomorphism, then $S$ is strictly properly infinite too. Thus if $R$ and $S$ are unital $\ast$-algebras and $R$ is strictly properly infinite, so is $R \otimes S$. In particular, $L\infty \otimes R$ is strictly properly infinite for every unital $\ast$-algebra $R$, and if $R$ is strictly properly infinite, then $M_nR$ and $M_{\pm n}R$ are strictly properly infinite for all $1 \leq n < \infty$. A projection $p \in R$ is strictly properly infinite if $pRp$ is strictly properly infinite. Equivalently $p$ is strictly properly infinite if there are nonzero orthogonal projections $p_1, p_2 \in pRp$, such that $p \sim p_1 \sim p_2$.

**Remark 8.1** The sum of orthogonal strictly properly infinite projections in a unital $\ast$-algebra is again strictly properly infinite. In particular, if $R$ contains orthogonal strictly properly infinite projections $p_1, \ldots, p_n$ such that $\sum_{i=1}^n p_i = 1$, then $R$ is strictly properly infinite.

We say that a graph $E$ is *simple* if it is cofinal [4, Definitions 2.9.4] and every cycle in $E$ has an exit [4, Definitions 2.2.2]. A simple graph having at least one cycle is called *purely infinite simple*. If $\ell$ is a field, then $L(E)$ is (purely infinite) simple if and only if $E$ is [4, Theorems 2.9.1 and 3.1.10 and Lemma 2.9.6].

**Example 8.2** The graph $\Upsilon$ of Example 4.8 is purely infinite simple.

**Lemma 8.3** Let $E$ be a finite graph and $v \in \text{sour}(E) \setminus \text{sink}(E)$. Let $E/v$ be the source elimination graph of $E$ [4, Definition 6.3.26]. Then

(i) The element $p = 1 - v \in L(E)$ is a strictly full projection.
(ii) The inclusion $E/v \subset E$ induces $\ast$-isomorphisms $L(E/v) \rightarrow pL(E)p$ and $L(\overline{E/v}) \rightarrow pL(\overline{E})p$.
(iii) $E$ is (purely infinite) simple if and only if $E/v$ is.

**Proof** The argument of the proof of [6, Proposition 1.4] shows that the canonical homomorphism $L(E/v) \rightarrow L(E)$ induced by the inclusion, which is a $\ast$-homomorphism with respect to both the standard and the twisted involution, corestricts to an isomorphism onto $pL(E)p$; this proves (ii). Because $v$ is a source but not a sink, we have $v = \sum_{s(e)=v} epe^\ast$, whence $1 = p + \sum epe^\ast$ and therefore $p$ is strictly full, proving i). If $\ell$ is a field, the (pure infinite) simplicity of $E$ is equivalent to that of $L(E)$, which by (ii), is Morita equivalent to $L(E/v)$. The proof of (iii) and of the lemma is concluded by using that simplicity and purely infinite simplicity are preserved by Morita equivalence [9, Corollary 1.7]. □
Lemma 8.4 Let $E$ be a finite, purely infinite simple graph. Then every vertex of $E$ is a strictly properly infinite projection of $L(E)$ and $\overline{L(E)}$.

Proof If $v \in E^0$ is in a cycle, then there is a cycle $\alpha_i^1$ based at $v$. Let $\alpha_i^2$ be a closed path starting at $v$, following $\alpha_i^1$ up to an exit, taking the exit, then coming back to the cycle—as is possible due to cofinality of $E$—and following $\alpha_i^1$ again until $v$. Upon replacing $\alpha_i^1$ and $\alpha_i^2$ by their squares, if necessary, we may assume that their lengths are even, so that $\alpha_i^1 = (\alpha_i^2)^*$ for $i = 1, 2$. Then $(\alpha_i^1)^* \alpha_i^1 = \delta_{ii} v$, and $v$ is a strictly properly infinite projection of both $L(E)$ and $\overline{L(E)}$. If sour($E$) = $\emptyset$, every vertex is in a cycle, and the lemma follows. Otherwise, we can proceed by source elimination until we arrive to a purely infinite simple graph without sources. At each step, the source eliminated is equal to a sum of projections of the form $ee^*$ with $e \in E^1$ and $r(e) \in E^0_{/v}$. If $x_1, x_2 \in r(e)L(E)r(e)$ are orthogonal isometries for either of the involutions, then $ex_1 e^*, ex_2 e^* \in ee^* L(E) ee^*$ are again orthogonal isometries, for the same involution. Since the sum of orthogonal strictly properly infinite projections is again strictly properly infinite, we get that every vertex is a strictly properly infinite projection.

Corollary 8.5 If $E$ is finite and purely infinite simple, then $L(E)$ and $\overline{L(E)}$ are strictly properly infinite.

Proof Immediate from Lemma 8.4 and Remark 8.1.

Example 8.6 Let $R, S \in \text{Alg}^s_1$ be unital and $p \in \mathfrak{Proj}(R)$. Following [10, Section 6.11] we say that $p$ is very full if there exists $q \in \mathfrak{Proj}(R)$ such that $p \geq q \sim 1$. Observe that any projection equivalent to a very full one is again very full. We write $\mathfrak{Proj}_f(R) \subset \mathfrak{Proj}(R)$ for the subset of very full projections.

Lemma 8.7 Let $E$ be a finite, purely infinite simple graph and let $v \in E^0$. Then $v$ and $1 \otimes v$ are very full projections of $L(E)$ and $\overline{L(E)}$.

Proof Let $v \in E^0$. By Lemma 8.4 we may choose an element $x_w \in vL(E)v$ for each $w \in E^0 \setminus \text{sour}(E)$ such that $x_w x_{w'} = \delta_{w,v} v$. Because $E$ is purely infinite simple, for every $w \in E^0 \setminus \text{sour}(E)$, there is a path $\alpha_w$ from $v$ to $w$. Then

$$x = \sum_{w \in E^0 \setminus \text{sour}(E)} x_w \alpha_w$$

(8.1)

satisfies $xx^* \leq v$ and $p := x^* x = \sum_{w \in E^0 \setminus \text{sour}(E)} w$. By Lemma 8.4 and Remark 8.1 $R = pL(E)p$ is strictly properly infinite. Hence we may choose a family of orthogonal isometries $R \supset \{ s(e) : s(e) \in \text{sour}(E) \} \cup \{ y_w : w \in E^0 \setminus \text{sour}(E) \}$; put $y = \sum_{s(e) \in \text{sour}(E)} y e e^* + \sum_{w \in E^0 \setminus \text{sour}(E)} y_w w$. Then $yy^* \leq p$ and $y^* y = 1$; hence $p$ is very full and therefore $v$ is a very full projection of $L(E)$. It follows that $1 \otimes v \in M_+ \overline{L(E)}$ is very full since it is the image of $v$ under the unital $*$-homomorphism $\Delta$ of (2.8).
Let $R$ be a unital, strictly pure infinite $*$-algebra. Consider the set of equivalence classes

$$\mathcal{V}_1(R) \supset \mathcal{V}_f(R) = \{ [p] : p \in R \text{ very full} \}.$$ 

**Proposition 8.8** (cf. [10, Theorem 6.11.7]) *Let $R$ be a unital, strictly properly infinite $*$-algebra. Then the orthogonal sum makes $\mathcal{V}_f(R)$ into a group, canonically isomorphic to $K_0(R)^*$.***

**Proof** The argument of the proof of [10, Theorem 6.11.7] shows this. □

**Corollary 8.9** *Let $R$ be as in Proposition 8.8. Further assume that the center of $R$ satisfies the $\lambda$-assumption 1.3. Then $\mathcal{V}_f(M_{\pm}R)$ is a group, canonically isomorphic to $K_0^h(R)$.***

**Proof** Combine Proposition 8.8 and (7.2). □

For the rest of this section we assume that $\ell$ satisfies the $\lambda$-assumption 1.3.

**Lemma 8.10** *Let $\ell_i : \ell \to M_2$, $\ell_i(x) = \epsilon_{i,j}x$. We have $\ell_+ \circ \ell_1 \sim^* \ell_+ \circ \ell_2$.***

**Proof** By [15, Section 3.4], there is a matrix $g(t) \in \text{GL}_2(\ell)$ such that $g(0) = 1$, $g(1)$ is unitary, and $\ell_2 = \text{ad}(g(1)) \circ \ell_1$. Hence $\ell_+ \circ \ell_1 \sim^* \ell_+ \ell_2$, by [16, Lemma 5.4]. □

Let $A, B \subseteq \text{Alg}^*_\ell$; two $*$-homomorphisms $\phi, \psi : A \to B$ are $M_{\pm}^2$-$*$-homotopic—and we write $\phi \sim^*_{M_{\pm}^2} \psi$—if $\ell_+ \circ \ell_1 \circ \phi$ and $\ell_+ \circ \ell_1 \circ \psi$ are $*$-homotopic. We write $[A, B]_{M_{\pm}^2}^*$ for the set of $M_{\pm}^2$-$*$-homotopy classes of $*$-homomorphisms.

**Lemma 8.11** *Let $A, B \subseteq C$ $*$-subalgebras and $\text{inc}_A$ and $\text{inc}_B$ the inclusion maps. Let $x \in C$ such that $x Ax^* \subseteq B$ and $ax^* xa' = a a'$ for all $a, a' \in A$. Then $\text{ad}(x) : A \to B$, $\text{ad}(x)(a) = x a x^*$ is a $*$-homomorphism and $\text{inc}_B \text{ad}(x) \sim^*_{M_{\pm}^2} \text{inc}_A$. If moreover $A = B$ and $A x \subseteq A$, then $\text{ad}(x) \sim^*_{M_{\pm}^2} \text{id}_A$.***

**Proof** Combine the argument of [12, Lemma 2.3] with Lemma 8.10 above. □

**Lemma 8.12** *Let $R$ be a unital $*$-algebra and $p \in R$ a strictly full projection and let $\ell_p : pRp \to R$ be the inclusion. Then $j^h(\ell_p)$ is an isomorphism in $kk^h$.***

**Proof** Because $p$ is strictly full, there are $n \geq 1$ and $x \in p R^n$ such that $\sum_{i=1}^n x_i^* x_i = 1$. Set $y = \sum_{i=1}^n \epsilon_{i,j} x_j \in M_n p R \subseteq M_n R$ and $z = yp \in M_n(pRp)$. Then $\phi : R \to M_n(pRp)$, $\phi(a) = y\ell_1(a)y^* = \sum_{i,j} \epsilon_{i,j} x_j x_i^*$ is a $*$-homomorphism. Let $\ell_1 = \ell_1^{pRp} : pRp \to M_n p Rp$, $\ell_1(a) = \epsilon_{1,1} a$. Then $\phi \circ \text{inc}_p = \text{ad}(z) \circ \ell_1$, which by Lemma 8.11 is $M_{\pm}^2$-$*$-homotopic to $\ell_1$, so $j^h(\phi \circ \text{inc}_p)$ is an isomorphism in $kk^h$. Similarly, writing $\ell_1$ now for $\ell_1^{R}$, $M_n(\text{inc}_p) \circ \phi = \text{ad}(x) \circ \ell_1$ is $M_{\pm}^2$-$*$-homotopic to $\ell_1$ and thus $j^h(M_n(\text{inc}_p) \circ \phi)$ is an isomorphism too. Hence $j^h(\phi)$ is an isomorphism, which by what we have already proved implies that $j^h(\text{inc}_p)$ is an isomorphism, concluding the proof. □
Let $R$ be a strictly properly infinite $\ast$-algebra. By definition, there are $s_1, s_2 \in R$ such that $s_i^* s_j = \delta_{i,j}$. Let

$$\boxplus : R \oplus R \to R, \quad a \boxplus b = s_1a s_1^* + s_2 b s_2^*.$$ 

Let $\phi, \psi : A \to R$ be $\ast$-homomorphisms. Put

$$\phi \boxplus \psi : A \to R, \quad (\phi \boxplus \psi)(a) = \phi(a) \boxplus \psi(a). \quad (8.2)$$

**Lemma 8.13** Let $A$ and $R \in \text{Alg}_{\ell}^*$, with $R$ strictly properly infinite. Then $(8.2)$ makes $[A, R]^\ast_{\mathbb{M}_{\pm 2}}$ into an abelian monoid.

**Proof** Combine Lemma 8.11 with the argument of [12, Lemma 2.5].

Let $A$ and $R$ be as in Lemma 8.13 and let $\phi_0, \phi_1 : A \to R$ be $\ast$-homomorphisms; we say that $\phi_0$ and $\phi_1$ are *stably $M_{\pm 2}$-homotopic*, and write $\phi_0 \sim^s_{M_{\pm 2}} \phi_1$, if there exists a $\ast$-homomorphism $\psi : A \to R$ such that

$$\phi_0 \boxplus \psi \sim^s_{M_{\pm 2}} \phi_1 \boxplus \psi. \quad (8.3)$$

In other words, $\phi_0 \sim^s_{M_{\pm 2}} \phi_1$ means that the $M_{\pm 2}$-homotopy classes of $\phi_0$ and $\phi_1$ go to the same element in the group completion

$$[A, R]^\ast_{\mathbb{M}_{\pm 2}} \to ([A, R]^\ast_{M_{\pm 2}})^+ \quad (8.4)$$

**Remark 8.14** Assume $\ell = \text{inv}(\ell_0)$ for some commutative ring $\ell_0$. Then every $R \in \text{Alg}_{\ell}^*$ is of the form $\text{inv}(R_0)$ for some $R_0 \in \text{Alg}_{\ell_0}$, projections in $R$ correspond to idempotents in $R_0$, MVN equivalence and very and strict fullness of projections to very fullness and fullness of idempotents, and $R$ is strictly properly infinite if and only if $R_0$ is properly infinite. We claim that, furthermore, $\iota_+$ can be dropped in Lemma 8.10. For $a \in M_{\ell_0}$, let $a^*$ be the transpose matrix. Then $M_2 = M_2 \ell_0 \oplus M_2 \ell_0$ equipped with the involution $(a, b)^* = (b^*, a^*)$. Let $g(t) \in GL_2(\ell_0)$ be as in the proof of Lemma 8.10 and let $h(t) = (g(t), (g(t))^*)^{-1}$. Then $h(t) \in U_2(\ell)$, $h(0) = (1, 1)$ and $\text{ad}(h(1)) \circ \iota_1 = \iota_2$, and so we have $\iota_1 \sim \iota_2$, proving the claim. It follows that in 8.11 and 8.13 we may replace $M_{\pm 2}$-homotopy by $M_2$-$\ast$-homotopy, defined in the obvious way, which under the category equivalence $\text{inv} : \text{Alg}_{\ell_0} \to \text{Alg}_{\ell}^*$ corresponds to $M_2$-homotopy as defined in [12, Section 2]. Thus the lemmas above specialize to [12, Lemmas 2.1, 2.3 and 2.5].

9 Lifting $K_0$-maps to $\ast$-algebra maps

Let $E$ be a graph, $R$ a strictly properly infinite unital $\ast$-algebra and $\phi : L(E) \to R$ an algebra homomorphism. We say that $\phi$ is *very full* if

$$\{\phi(ee^*) : e \in E^1\} \cup \{\phi(v) : v \in \text{sing}(E)\} \subset \mathcal{P}_{\text{proj}}(R). \quad (9.1)$$
Example 9.1 Let $E$ be a finite, purely infinite graph, let $\phi : L(E) \to R$ be $*$-homomorphism and put $p = \phi(1)$. By Lemma 8.7, for every element $q$ in the left hand side of the inclusion (9.1) there is a projection $q' \leq q$ such that $q' \sim p$. Hence $\phi$ is very full if and only if $p \in \mathcal{P}\text{roj}_f(R)$.

Remark 9.2 If $\phi, \psi : LE \to R$ are very full $*$-homomorphisms, then so is their sum (8.2). Thus the subset

$$[LE, R]^*_M \supset [LE, R]^{\bar{f}}_M = \{[\phi] : \phi \text{ is very full}\}$$

is a subsemigroup.

Theorem 9.3 Let $E$ be a countable graph and $R$ a unital $*$-algebra. Assume that $R$ is strictly properly infinite. Let $\xi : \mathcal{B}_\mathcal{G}(E) \to K_0(R)^*$ be a group homomorphism and let $\text{can'} : \mathcal{B}_\mathcal{G}(E) \to K_0(LE)^*$ be the canonical map of Remark 7.2. Then there is a very full $*$-homomorphism $\phi : L(E) \to R$ such that $K_0(\phi)^* \circ \text{can'} = \xi$. If furthermore $E^0$ is finite, $[1]_E$ is as in (1.1) and $p \in \mathcal{P}\text{roj}_f(R)$ is such that $\xi([1]_E) = [p]$, then $\phi$ can be chosen so that, in addition to the above properties, also satisfies $\phi(1) = p$.

Proof Because $R$ is strictly properly infinite by assumption, it has a sequence of orthogonal projections equivalent to 1. Hence in view of Proposition 8.8 and the countability assumption on $E$, there are orthogonal very full projections $\{p_e : e \in E^1\} \cup \{p_v : v \in \text{sing}(E)\} \subset \mathcal{P}\text{roj}_f(R)$ such that, in $V_f(R) = K_0(R)^*$, $\xi[v] = [p_v]$ and $\xi[ee^*] = [p_e]$ for all $v \in \text{sing}(E)$ and $e \in E^1$. If $E$ is row-finite, then proceeding as in the proof of [13, Theorem 3.1] we obtain a $*$-homomorphism $\phi : LE \to R$ as required. For general countable $E$, we may choose a desingularization $E_\delta$; there is a canonical $*$-homomorphism $\iota_\delta : L(E) \to L(E_\delta)$ [3, Proposition 5.5] which maps vertices to vertices and edges to paths. Hence if $\psi : L(E_\delta) \to R$ is a very full $*$-homomorphism, so is $\phi \circ \iota_\delta$ and the diagram below commutes

$$\begin{array}{ccc}
K_0(L(E))^* & \xrightarrow{K_0(\phi)^*} & K_0(L(E_\delta))^* \\
\text{can'} \uparrow & & \uparrow \text{can'} \\
\mathcal{B}_\mathcal{G}(L(E)) & \xrightarrow{\mathcal{B}_\mathcal{G}(\iota_\delta)} & \mathcal{B}_\mathcal{G}(E_\delta).
\end{array}$$

The proof of [19, Lemma 2.3] shows that the bottom row in the diagram above is an isomorphism. Thus the general countable case reduces to the row-finite case. Finally if $E^0$ is finite, $p \in \mathcal{P}\text{roj}_f(R)$ and $\xi([1]) = [p]$, then by what we have just seen there is a very full $*$-homomorphism $\psi : LE \to R$ such that $K_0^*(\psi) \circ \text{can'} = \xi$. Let $q = \psi(1)$; choose an MvN equivalence $y : q \sim p$. Consider the $*$-homomorphism $\text{ad}(y) : qRq \to pRp \subset R$.

Let $\phi := \text{ad}(y) \circ \psi : L(E) \to R$; then $\phi$ is very full and $\phi(1) = p$. Moreover, $K_0(\phi)^* = K_0(\psi)^*$, hence we also have $K_0(\phi)^* \circ \text{can'} = \xi$. \hfill \Box

Remark 9.4 One may ask whether in the situation of Theorem 9.3, if we further require that $E$ be finite and that $-1$ be positive in $R$, any homomorphism of $\mathbb{Z}[\sigma]$-modules...
\[ \mathcal{B}_\mathcal{F}(E) \to K_0(R)^* \] can be lifted to a very full \(*\)-homomorphism \(L(E) \to R\). The argument of 9.3 does not work for this purpose, since it uses the fact that the edges of \(E\) are partial isometries in \(L(E)\), and this is no longer true in \(L(E)\). Note however that it follows from Theorem 9.3 and the isomorphism (5.1) that any group homomorphism \(\mathcal{B}_\mathcal{F}(E) \to K_0(R)^*\) lifts to a \(*\)-homomorphism \(\phi : L(E) \to R\).

**Corollary 9.5** Let \(E, R\) and \(\ell\) be as in Theorem 9.3. Further assume that \(\ell\) satisfies the \(\lambda\)-assumption 1.3. Let \(\xi : \mathcal{B}_\mathcal{F}(E) \to K_0(R)\) be a group homomorphism. Then there is a very full \(*\)-homomorphism \(\psi : L(E) \to M_{\pm}^\ell R\) such that \(K_0^h(\psi) \circ \text{can} = K_0^h(\xi)\). If furthermore \(E^0\) is finite and \(\xi([1]) = [p]\) for some \(p \in \text{Proj}_f(M_{\pm}^\ell R)\), then we can choose \(\psi\) so that, in addition to the above properties, also satisfies \(\psi(1) = p\).

**Proof** By Theorem 9.3 there is a \(*\)-homomorphism \(\psi : L E \to M_{\pm}^\ell R\) –which can be chosen so that \(\psi(1) = p\)– such that \(K_0(\psi)^\ast \circ \text{can}' = \xi\). Hence

\[
K_0^h(\psi) \circ \text{can} = K_0(M_{\pm}^\ell \psi)^\ast \circ K_0(\xi)^\ast \circ \text{can}' \]

\[= K_0(\xi)^\ast \circ K_0(\psi)^\ast \circ \text{can}' = K_0(\xi)^\ast \circ \xi.\]

Let

\[
u = \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix} \in M_{\pm}M_{\pm}
\]

One checks that \(u\) is unitary and that \(\text{ad}(u) \circ \xi = M_{\pm}^\ell \xi\). Hence \(K_0(\xi)^\ast = K_0^h(\xi)\) and \(K_0(\xi)^\ast \circ \text{can}' = K_0(\xi)^\ast \circ \xi\). In the next two lemmas and elsewhere, if \(E\) is a finite graph, we write \(DL(E)\) for the diagonal subalgebra of \(L(E)\),

\[
DL(E) = \left( \bigoplus_{v \in \text{sink}(E)} \ell v \right) \oplus \left( \bigoplus_{e \in E^1} \ell ee^* \right).
\]

The following lemmas will be used later on, in the proofs of Theorems 13.2 and 13.4.

**Lemma 9.6** Let \(E\) be finite graph, \(R \in \text{Alg}_f^\ast\) strictly properly infinite and \(\phi, \psi : L(E) \to R\) very full \(*\)-homomorphisms. If \(K_0(\phi)^\ast \circ \text{can}' = K_0(\psi)^\ast \circ \text{can}'\) then there exists a very full \(*\)-homomorphism \(\psi' : L(E) \to R\) such that \(K_0(\phi)^\ast \circ \text{can}' = K_0(\psi')^\ast \circ \text{can}'\), \(\psi' \sim_{M_{\pm}^\ell} \psi\) and \(\psi'|_{DL(E)} = \phi|_{DL(E)}\).

**Proof** For every \(\alpha \in \Theta = \text{sink}(E) \cup E^1\), choose an MvN equivalence \(x_\alpha : \psi(\alpha \alpha^*) \sim \phi(\alpha \alpha^*)\). Put \(x = \sum_{\alpha \in \Theta} x_\alpha\), one checks that \(x\) is an MvN equivalence \(p = \psi(1) \sim \phi(1)\), that \(\text{ad}(x) : pRp \sim qRq\) is a \(*\)-homomorphism and, using Lemma 8.11, that \(\psi' = \text{ad}(x) \circ \psi\) satisfies the requirements of the lemma. \qed
Lemma 9.7 Let \( E \) and \( R \in \text{Alg}_\ell^* \) be as in Lemma 9.6; further assume that \(-1\) is positive in \( R \). Let \( \phi, \psi : L(E) \to R \) be very full \(*\)-homomorphisms. If \( K_0^h(\phi) \circ \text{can} = K_0^h(\psi) \circ \text{can} \) then there exists a very full \(*\)-homomorphism \( \psi' : L(E) \to R \) such that \( K_0^h(\phi) \circ \text{can}' = K_0^h(\psi') \circ \text{can}, \psi' \sim_{M \pm 2} \psi \) and \( \psi'_{DL(E)} = \phi_{DL(E)} \).

**Proof** Let \( e \in E^1 \); by (3.2), \([ee^*] = \sigma[r(e)] \in K_0^h(L(E))\). Thus the identity \( K_0^h(\phi) \circ \text{can} = K_0^h(\psi) \circ \text{can} \) implies that \( K_0^h(\phi)[ee^*] = K_0^h(\psi)[ee^*] \). Hence we can proceed as in the proof of Lemma 9.6 above. \( \square \)

Remark 9.8 Assume \( \ell = \text{inv}(\ell_0) \) for some commutative ring \( \ell_0 \). Then in the proof of Lemma 9.6, \( \psi' = \text{ad}(x) \circ \psi \) is \( M_2\)-homotopic to \( \psi \), by Remark 8.14. Hence by the same remark, Lemma 9.6 recovers [13, Proposition 3.5].

## 10 Unitary \( K_1 \) of a strictly properly infinite \(*\)-algebra

Let \( R \) be a unital \(*\)-algebra. For \( n \geq 1 \), write \( \mathcal{U}_n R = \mathcal{U}(M_n R) \) for the group of unitary elements; set \( \mathcal{U}_\infty(R) = \colim_n \mathcal{U}_n(R) \). For \( 1 \leq n \leq \infty \), let \( \mathcal{U}_n^h(R) = \mathcal{U}_n(M_{\pm R}) \). Put

\[
K_1(R)^* = \mathcal{U}_\infty(R)_{ab}, \quad K_1^h(R) = K_1(M_{\pm R})^* = \mathcal{U}_\infty^h(R)_{ab}. \tag{10.1}
\]

**Lemma 10.1** Let \( R \) be a unital \(*\)-algebra and \( p \in R \) a strictly full projection. Then the inclusion \( pRp \to R \) induces an isomorphism \( K_1(pRp)^* \sim K_1(R)^* \).

**Proof** The proof is similar to that of Lemma 7.1 once we observe that \( K_1(R)^* = \colim_\mathbb{P}(R) \mathcal{U}(pRp)_{ab} \). \( \square \)

**Remark 10.2** If \( R \in \text{Alg}_\ell^* \) is unital and \(-1\) is positive in \( R \), then, as explained in Sect. 7 after Lemma 8.12, \( M_{\pm R} \) is \(*\)-isomorphic to a strictly full corner of \( M_{n+1} R \), so \( K_1(R)^* \to K_1^h(R) \) is an isomorphism by Lemma 10.1.

**Proposition 10.3** Let \( R \) be a strictly properly infinite \(*\)-algebra. Let \( \mathcal{N}(R) \subset \mathcal{U}(R) \) be the smallest normal subgroup containing the subset \( \{u^{-1}(ux^* + 1 - xx^*): u \in \mathcal{U}(R), x^* x = 1\} \). Then \( K_1(R)^* = \mathcal{U}(R)/\mathcal{N}(R) \).

**Proof** We keep the notation as in the proof of Lemma 10.1. Because \( R \) is strictly properly infinite, the full subcategory \( \mathbb{I} \) of \( \mathcal{P}(R) \) whose only object is the identity element \( 1 \in R \) is cofinal. Hence

\[
K_1(R)^* = \colim_1 \mathcal{U}(R)_{ab} = \mathcal{U}(R)/[\mathcal{U}(R) : \mathcal{U}(R)] \cdot \mathcal{N}(R).
\]

It remains to show that \( \mathcal{N}(R) \supset \mathcal{U}(R) \). Let \( s_1, s_2 \) be orthogonal isometries and let \( u, v \in \mathcal{U}(R) \). Modulo \( \mathcal{N}(R) \),

\[
uv \equiv (s_1us_1^* + (1 - s_1s_1^*)) \cdot (s_2vs_2^* + (1 - s_2s_2^*))
\]

\[
= s_1us_1^* + s_2vs_2^* + 1 - \sum_{i=1}^2 s_is_i^*
\]
\[ (s_2us_2^* + (1 - s_2s_2^*)) (s_1us_1^* + (1 - s_1s_1^*)) \equiv uu. \]

\[ KV_1(R)^* = \text{Coker}(ev_0 - ev_1 : K_1(R[t])^* \to K_1(R)^*), \quad KV_1^h(R) = KV_1(M_{\pm R})^*. \] (10.2)

One checks that \( KV_1^h \) as defined above agrees with that of [24]. Let

\[ U(R) = \{ u \in U(R) | (\exists U \in U(R[t])) U(0) = 1, U(1) = u \}, \]

\[ U_n(R) = U(M_n R), \quad U_n^h(R) = U_n(M_{\pm R})_0 (n \geq 1). \]

**Lemma 10.4** Let \( R \in \text{Alg}_\ell^* \) be strictly properly infinite and such that \(-1\) is positive in \( R \). Regard \( U(R) \subset U_2^h(R) \) through the group monomorphism induced by \( \iota_+ \circ \iota_1 \).

Then

\[ KV_1^h(R) = U(R)/U(R) \cap U_2^h(R)_0. \]

**Proof** By (10.2), Proposition 10.3 and Remark 10.2, the inclusion \( \iota_+ : U(R) \subset U_1^h(R) \) induces an isomorphism

\[ KV_1(R)^* = U(R)/N(R) \cdot (U(R) \cap U_\infty(R)_0) \xrightarrow{\sim} KV_1^h(R). \]

In particular,

\[ U(R) \cap U_2^h(R)_0 \subset N(R) \cdot (U(R) \cap U_\infty(R)_0). \]

Let \( x \in R \) be an isometry, \( g(t) \in \text{GL}_2(R[t]) \) as in the proof of Lemma 8.10 and \( 1_n \) the \( n \times n \)-identity matrix. Let \( y(t) = \text{ad}(g(t)) (x \oplus 1), z(t) = \text{ad}(g(t)) (x^* \oplus 1) \in M_2 R; \)

then \( h(t) = 1_2 - y(t) z(t) + y(t) \iota_1(u) z(t) \in \text{GL}_2(R[t]) \) connects \( \iota_1(u) = u \oplus 1 \) with \( \iota_1(1 - xx^* + xux^*) \) in \( \text{GL}_2(R) \). Let \( c(t) = c(h(t), h(t)^{-1}) \) be as in [16, Lemma 5.4]; then \( c(t) \in U_2^h(R) \), and connects \( \iota_+(\iota_1(u)) \) with \( \iota_+(\iota_1(1 - xx^* + xux^*)) \). It follows that \( N(R) \subset U(R) \cap U_2^h(R)_0 \), since \( U(R) \cap U_2^h(R)_0 \subset U(R) \) is a normal subgroup. It remains to show that \( U_2^h(R)_0 \subset U(R) \cap U_\infty(R)_0 \).

Let \( u \in U(R) \) and suppose that for some \( n \geq 2 \) there exists \( U(t) \in U_n(R[t]) \) such that \( U(0) = 1 \) and \( U(1) = u_n(u) = u \oplus 1_{n-1} \). Because \( R \) is strictly properly infinite, there exists \( x = (x_1, \ldots, x_n) \in R^{1 \times n} \) such that \( xx^* = 1 \in R \) and \( x^* x = 1_n \in M_n R \). Then

\[ V(t) = 1 - xx^* + xU(t)x^* \in U(R[t]) \]

satisfies \( V(0) = 1 \) and

\[ V(1) = 1 - xx^* + x(u \oplus 1_{n-1})x^* = 1 - x_1x_1^* + x_1u_1x_1^*. \]

By the first part of the proof, \( \iota_+(\iota_1(V(1))) \) is connected to \( \iota_+(\iota_1(u)) \) by a path \( W(t) \in U_2^h(R[t]) \); hence \( W(t)(\iota_+ \circ \iota_1)(V(1 - t)^{-1}) \) connects \( \iota_+(\iota_1(1)) \) to \( \iota_+(\iota_1(u)) \). \( \square \)
Consider the particular case of Lemma 10.4 when \( \ell = \text{inv}(\ell_0) \). By Remark 8.14 \( R = \text{inv}(S) \) for some properly infinite \( \ell_0 \)-algebra \( S \), \( KV^h_1(R) \) is Karoubi–Villamayor’s \( KV_1(S) \) [23], and one may replace \( g(t) \) in the proof of the lemma by a unitary \( h(t) \), obtaining

\[
KV_1(S) = KV^h_1(R) = U_1(R)/U_1(R) \cap U_2(R) = GL_1(S)/GL_1(S) \cap GL_2(S).
\]

## 11 Poincaré duality

Let \( E \) be a finite graph. The dual graph \( E^\text{dual} \) is the graph with \( E^i_j = E^j_i \) for \( i = 0, 1 \) and with source and range maps \( s_t = r \) and \( r_t = s \). Write \( e_t \) for an edge \( e \in E^1 \) regarded as an edge of \( E^\text{dual} \). The purpose of this section is to prove Theorem 11.2, which is an algebraic version of a similar result for graph \( C^* \)-algebras [22]. First we need the following lemma.

**Lemma 11.1** Let \( \pi : R \to S \) be a surjective, unital homomorphism of \( * \)-algebras, set \( I = \text{Ker}(\pi) \) and let \( \partial : K^h_1(S) \to K^h_0(I) \) be the connecting map. Let \( u \in U_n(S) \). Assume that there exists a partial isometry \( \hat{u} \in M_n R \) such that \( \pi(\hat{u}) = u \). Then \( \partial u^+[u] = u^+[1 - \hat{u}^*\hat{u}] - [1 - \hat{u}\hat{u}^*] \).

**Proof** For every pair of elements \( \hat{u}, \hat{v} \in M_n R \) such that \( \pi(\hat{u}) = u \) and \( \pi(\hat{v}) = v^* \), we can lift \( \text{diag}(u, u^*) \in U(M_{2n} S) \) to an elementary matrix \( h = h(\hat{u}, \hat{v}) \in E_{2n} R \); a formula for this matrix is given in [11, Formula (17)]. One checks that if \( \hat{u} \) is a partial isometry, then \( h(\hat{u}) := h(\hat{u}, \hat{u}^*) \in U_{2n} R \). Thus if \( p \in M_n R \) is the identity matrix, we have \( \partial (u^+[u]) = \iota_+([\text{ad}(h(\hat{u}))(p)] - [p]) \), and one computes that \( [\text{ad}(h(\hat{u}))(p)] - [p] = [1 - \hat{u}^*\hat{u}] - [1 - \hat{u}\hat{u}^*] \). \( \square \)

Recall from [16, Example 6.12] that if \( B \) is any \( * \)-algebra, then the functor \( \sim \otimes B : \text{Alg}^*_\ell \to \text{Alg}^*_\ell \) induces a functor \( kk^h \to kk^h \) which we again name \( \sim \otimes B \), such that \( j^h \circ (\sim \otimes B) = (\sim \otimes B) \circ j^h \).

Following [4, Definition 6.3.11 (iii)] we call a graph \textit{essential} if it contains no sources and no sinks. Thus a finite graph \( E \) is essential if and only if both \( E \) and \( E^\text{dual} \) are regular.

**Theorem 11.2** Let \( E \) be a finite essential graph. Then the functors \( \sim \otimes \Omega L(E) \) and \( \sim \otimes \Omega L(E) : kk^h \to kk^h \) are right adjoint to the functors \( \sim \otimes LE \) and \( \sim \otimes LE : kk^h \to kk^h \). Thus for every \( R, S \in \text{Alg}^*_\ell \) there are natural isomorphisms of \( KH^h_0(\ell) \)-modules

\[
kk^h(R \otimes LE, S) \sim kk^h_1(R, S \otimes L(E)),
\]

\[
kk^h(R \otimes LE, S) \sim kk^h_1(R, S \otimes L(E)).
\]

**Proof** Let \( P \) be the set of finite paths in \( E \) and \( P_{\geq 1} \subset P \) the subset of paths of positive length; let \( X = (P_{\geq 1})_+ \) be the pointed set obtained by adding a basepoint \( \bullet \). Let \( \pi : \Gamma_X \to \Sigma_X \) be the projection. With notations as in (2.14), for each \( e \in E^1 \), put
\[ \rho_1(e_t) = \pi \left( \sum_{\alpha \in \mathcal{P}_t(e)} \epsilon_{ae, \alpha} \right), \quad \rho_2(e) = \pi \left( \sum_{\alpha \in \mathcal{P}_t(e)} \epsilon_{e\alpha, \alpha} \right). \]

One checks (as in [22, Proposition 4.2]) that the assignments \( e_t \mapsto \rho_1(e_t) \) and \( e \mapsto \rho_2(e) \) extend to unital \( * \)-homomorphisms \( \rho_1 : L(E_t) \to \Sigma_X \) and \( \rho_2 : L(E) \to \Sigma_X \) and that \( \rho_1(a) \) and \( \rho_2(b) \) commute for every \( a \in L(E_t) \) and \( b \in L(E) \). Hence for \( \mathcal{E} = L(E_t) \otimes L(E) \) we have a \( * \)-homomorphism \( \rho : \mathcal{E} \to \Sigma_X \), which defines a class \( \kappa := j^h(\rho) \in kk^h(L(E_t) \otimes L(E), \Sigma_X) = kk^h_{-1}(L(E_t) \otimes L(E), \mathcal{E}). \) Using it, we obtain a homomorphism of \( KH^0_1(\mathcal{E}) \)-modules

\[ kk^h(R, S \otimes L(E_t)) \to kk^h(R \otimes L(E, S), \xi \mapsto (S \otimes \kappa) \circ (\xi \otimes L(E)). \] (11.1)

Consider the elements \( p = \sum_{v \in E_0} v \otimes v \) and \( u = \sum_{e \in E_1} e \otimes e^* \) of \( \mathcal{E}' = L(E) \otimes L(E_t) \). One checks that \( u \in \mathcal{U}(p \mathcal{E}' \rho). \) Hence \( u_1 = u + 1 - p \in \mathcal{U}(\mathcal{E}'); \) write \( \nabla \) for the image of the class \( [u_1] \in K_1(\mathcal{E}')^* \) under the composite of canonical maps

\[ K_1^*(\mathcal{E}') \to K_1^1(\mathcal{E}') \to KH^1_1(\mathcal{E}') = kk^h(\ell, \mathcal{E}'). \]

In fact, we can describe \( \nabla \) explicitly as follows. Let \( L_1 = \ell[s, s^{-1}] \) be as in (5.5) and consider the kernel \( I = \text{Ker}(ev_1 : L_1 \to \ell) \) of the evaluation map. Because \( ev_1 \circ \phi = \text{id}_\mathcal{E} \), it follows from Theorem 3.1 that \( kk^h(I, \mathcal{E}') = kk^h(I, \ell) \), and that \( \nabla \) is the class of the restriction to \( I \) of the \( * \)-homomorphism \( \theta : L_1 \to \mathcal{E}' \), \( \theta(s) = u_1 \).

We use \( \nabla \) to define another \( KH^0_1(\ell) \)-linear homomorphism

\[ kk^h(R \otimes L(E), S) \to kk^h_1(R, S \otimes L(E_t)), \quad \eta \mapsto ((\eta \otimes L(E_t))[+1]) \circ (R \otimes \nabla). \] (11.2)

To prove that (11.1) and (11.2) are isomorphisms, it suffices to show that both \( (\kappa \otimes L(E_t)) \circ (L(E_t) \otimes \nabla) \) and \( (L(E) \otimes \kappa) \circ (\nabla \otimes L(E)) \) are. In the first case, this boils down to showing that the class of

\[ \zeta : L_1 \otimes L(E_t) \to \Sigma_X L(E_t), \quad \zeta(s \otimes 1) = \zeta_s := (\rho_2 \otimes 1)(u_1), \quad \zeta(1 \otimes x) = \rho_1(x) \otimes 1 \] (11.3)

restricts to a \( kk^h \) equivalence \( I \otimes L(E_t) \to \Sigma_X L(E_t) \). The second case reduces to the first upon switching \( E \) and \( E_t \) and permuting tensor factors.

Consider the \( * \)-homomorphism

\[ \partial = \sum v \in E_0 \partial_v : L(E_t) \to \bigoplus_{v \in E_0} \Sigma p_v, \quad \partial_v(e_t) = \sum_{a \in \mathcal{P}_t(e)} \epsilon_{ae, \alpha}. \]

One checks that \( \partial \) lifts to a \( * \)-homomorphism \( C(E_t) \to \bigoplus_{v \in E_0} \Gamma p_v \) which restricts to the canonical isomorphism \( \mathcal{K}(E_t) \cong \bigoplus_{v \in E_0} Mp_v \). Thus the \( kk^h \) class of \( \partial \) corresponds to the boundary map \( L(E_t) \to \mathcal{K}(E_t)[-1] \) associated to the Cohn extension.
Thus, because composing the projection $C(E_i) \to L(E_i)$ with the $kk^h$-isomorphism $\phi : \ell^E \to C(E_i)$ of Theorem 3.1 gives the inclusion $\text{inc} : \ell^E_0 \subseteq L(E_i)$, we have a distinguished triangle in $kk^h$

$$
\ell^E_0 \xrightarrow{\text{inc}} L(E_i) \xrightarrow{\partial} \bigoplus_{v \in E^0} \Sigma P_v
$$

We are going to complete $\zeta$ to a map of triangles as follows

$$
I \otimes \ell^E_0 \xrightarrow{I \otimes \text{inc}} I \otimes L(E_i) \xrightarrow{I \otimes \partial} I \otimes \bigoplus_{v \in E^0} \Sigma P_v
$$

(11.4)

For each $v \in E^0$ consider the following elements of $\Sigma X L(E_i)$

$$
p_v = \sum_{\alpha \in P, r(\alpha) = v} \epsilon_{\alpha, \alpha} \otimes r(\alpha), \quad u_v = \sum_{e \in E^1, \alpha \in P, r(e) = s(\alpha), r(\alpha) = v} \epsilon_{e, \alpha, \alpha} \otimes e^*.
$$

(11.5)

A calculation shows that

$$
\zeta(s \otimes v) = \zeta_v := \sum_{\alpha \in P_v} \epsilon_{\alpha, \alpha} \otimes 1 - p_v + u_v.
$$

(11.6)

Hence the restriction of $\zeta$ to $I \otimes v$ corresponds in

$$
kk^h(I \otimes v, \Sigma X L(E_i)) = KH^h_1(\Sigma X L(E_i)) = KH^h_0(L(E_i))
$$

to the index of the unitary

$$
1 \otimes 1 - \sum_{\alpha \in P_v} \epsilon_{\alpha, \alpha} \otimes 1 + \zeta(s \otimes v) = 1 \otimes 1 - p_v + u_v.
$$

(11.7)

A calculation shows that the index of (11.7) is the class of $v$. We have just built the vertical isomorphism on the left of (11.4). Next we build the isomorphism on the right of (11.4). Consider the following elements in $\Sigma X C(E_i)$.

$$
\hat{p} = \sum_{\alpha \in P_{\geq 1}} \epsilon_{\alpha, \alpha} \otimes s(\alpha), \quad \hat{u} = \sum_{e \in E^1, \alpha \in P, r(e) = s(\alpha)} \epsilon_{e, \alpha, \alpha} \otimes e^*,
$$

$$
\hat{\zeta}_s = 1 \otimes 1 - \hat{p} + \hat{u}.
$$

One checks that $\hat{\theta}_s$ is a partial isometry. Hence the restriction of (11.3) to $L_1 = L_1 \otimes 1 \subseteq L_1 \otimes L(E_i)$ lifts to a $*$-homomorphism $C_1 \to \Sigma X C(E_i)$ mapping $s \mapsto \hat{\theta}_s$. 

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Because the components to the left of the tensors in $\hat{\rho}$ and $\hat{u}$ are in the image of $\rho_2$, and because the images of $\rho_1$ and $\rho_2$ commute, it follows that $\zeta$ lifts to a $\ast$-homomorphism

$$\hat{\zeta} : C_1 \otimes L(E_t) \rightarrow \Sigma_X C(E_t), \quad \hat{\zeta}(s \otimes 1) = \hat{\zeta}_s, \quad \hat{\zeta}(1 \otimes x) = \rho_1(x) \otimes 1.$$ 

Let $\pi : C_1 \rightarrow L_1$ be the projection; by Theorem 3.1, $\pi^{-1}(I)$ is zero in $\mathcal{K}$. Hence $(\Sigma_X \partial) \circ \zeta$ is equivalent to the delooping $\hat{\zeta}[-1]$ of the restriction of $\hat{\zeta}$ to $\text{Ker} \pi$, which is generated by $q = 1 - ss^\ast$, and we have an isomorphism $\text{Ker} \pi \cong M_\infty$ that takes $q \mapsto \epsilon_{1,1}$. A computation shows that for $a \in L(E_t)$ and $q_v = v - \sum_{e \in E \cap r(e) = v} e_t e_t^\ast$, we have

$$\hat{\zeta}(q \otimes a) = (\rho_2(a) \otimes 1) \sum_v \left( \sum_{\alpha \in \mathcal{P}^v} \epsilon_{\alpha, \alpha} \otimes q_v \right). \quad (11.8)$$

Let $\text{inc}_v : \mathcal{P}^v \subset X$ be the inclusion that sends $v \mapsto \ast$ and fixes the paths of length $\geq 1$; $\text{inc}_v$ induces a map of extensions

$$\begin{array}{ccc}
M_{\mathcal{P}^v} & \xrightarrow{\text{inc}_v \circ \partial_v} & \Gamma_{\mathcal{P}^v} \\
\downarrow & & \downarrow \\
M_X & \xrightarrow{\partial_v} & \Sigma_X
\end{array}$$

The vertical map on the left is a $\mathcal{K}$ equivalence by matrix stability; since the middle terms of both extensions are zero in $\mathcal{K}$, it follows that the vertical map on the right is a $\mathcal{K}$ equivalence. Abusing notation and writing $\text{inc}_v$ for the latter map, we see it follows from (11.8) that we have a commutative diagram in $\mathcal{K}$

$$\begin{array}{ccc}
L(E_t) & \xrightarrow{\sum_v \text{inc}_v \circ \partial_v} & \bigoplus_v \Sigma_X \\
q \otimes L(E_t) & \xrightarrow{\sum_v \Sigma_X \otimes q_v} & \Sigma_X \mathcal{K}(E_t) \\
M_\infty L(E_t) & \xrightarrow{\hat{\zeta}_1} & \Sigma_X \mathcal{K}(E_t)
\end{array}$$

We have thus obtained the vertical isomorphism on the right of (11.4). Thus the first adjunction of the theorem is proved. Observe that the homomorphisms $\rho_1$ and $\rho_2$ defined above are also $\ast$-homomorphisms for the involutions of $L(E)$, $L(E_t)$ and $\Sigma_X$. Notice also that the elements $p$ and $v$ are a projection and a unitary also for the involution of $L(E) \otimes L(E_t)$. Hence the argument above also proves the second adjunction of the theorem. 

\begin{corollary}
Let $E$ and $S$ be as in Theorem 11.2. Assume that $S$ is unital and contains a central element $x$ such that $xx^* = -1$. There there is a natural isomorphism of $\mathcal{K}H_0(\ell)$-modules

$$\mathcal{K} \cong kk^h(L(E), S).$$
\end{corollary}
Proof Using Theorem 11.2 at the first and third steps and Example 2.8 at the second step, we obtain natural isomorphisms
\[ kk^h(L(E), S) \sim K H^1_h(S \otimes L(E_i)) \sim K H^1_h(S \otimes L(E_i)) \sim kk^h(L(E), S). \]

\[ \square \]

Remark 11.4 Let \( E \) be a finite essential graph. Theorem 11.2 and the equivalence of Remark 2.11 together imply that \( L(E) \) and \( L(E_i) \) are Poincaré dual not only in \( kk^h \) but also in \( kk \).

12 Universal coefficient theorem

Let \( E \) be a graph. Put
\[ \mathcal{B} \mathcal{S}^\vee(E) = \text{Coker}(I - \sigma A_E), \quad \mathcal{B} \mathcal{S}^\vee(E) = \mathcal{B} \mathcal{S}^\vee(E) \otimes \mathbb{Z}[\sigma] \mathbb{Z}. \]

Let \( R \in \text{Alg}_\mathbb{Z}^* \); write \( \text{hom} \) for homomorphisms of abelian groups. Consider the maps
\[ \text{ev} : kk^h(L E, R) \to \text{hom}(\mathcal{B} \mathcal{S}(E), K H^0(R)), \]
\[ \text{ev} : kk^h(L E, R) \to \text{hom}_{\mathbb{Z}[\sigma]}(\mathcal{B} \mathcal{S}(E), K H^0(R)) \]
\[ \xi \mapsto \xi_0 = K H^0_0(\xi) \circ \text{can}. \]

Theorem 12.1 Let \( E \) be a graph such that \( |E^0| < \infty \) and let \( R \in \text{Alg}_\mathbb{Z}^* \). Then the maps (12.1) are surjective and fit into exact sequences
\[ 0 \to K H^1_1(R) \otimes \mathcal{B} \mathcal{S}^\vee(E) \to kk^h(L(E), R) \]
\[ \to \text{hom}(\mathcal{B} \mathcal{S}(E), K H^0(R)) \to 0, \]
\[ 0 \to K H^1_1(R) \otimes \mathcal{B} \mathcal{S}^\vee(E) \to kk^h(L(E), R) \]
\[ \to \text{hom}_{\mathbb{Z}[\sigma]}(\mathcal{B} \mathcal{S}(E), K H^0(R)) \to 0. \]

Proof Applying \( kk^h(-, R) \) to the triangle of Theorem 3.3, and using that for any finite set \( X \),
\[ kk^h(\ell^X, R) \equiv \text{hom}(\ell^X, K H^0(R)) = \text{hom}_{\mathbb{Z}[\sigma]}(\ell^X, K H^0(R)) \]
one obtains exact sequences as in the theorem. One checks that the surjections therein agree with the evaluation maps (12.1).

\[ \square \]

Remark 12.2 Let \( E \) be a finite essential graph. Then \( A_{E_i} = A_{E}^t \), and
\[ \mathcal{B} \mathcal{S}^\vee(E) = \mathcal{B} \mathcal{S}(E_i), \quad \mathcal{B} \mathcal{S}^\vee(E) = \mathcal{B} \mathcal{S}(E_i). \]
Lemma 12.3 Let $E$ and $R$ be as in Theorem 11.2. Then the following diagrams, where the horizontal maps are as in Theorem 11.2, the slanted ones as in Theorem 4.2 and the vertical ones as in Theorem 11.2, commute.

\[
\begin{array}{ccc}
\mathcal{B}(E_t) \otimes KH^h_1(R) & \xrightarrow{12.1} & kk^h(LE, R) \\
\downarrow{4.2} & & \downarrow{11.2} \\
KH^h_1(R \otimes L(E_t)) & & \\
\end{array}
\]

\[
\begin{array}{ccc}
\mathcal{B}(E_t) \otimes \mathbb{Z} \sigma & \xrightarrow{12.1} & kk^h(L \sigma, R) \\
\downarrow{4.2} & & \downarrow{11.2} \\
KH^h_1(R \otimes L(E_t)) & & \\
\end{array}
\]

**Proof** Consider the first diagram first. The composite of the slanted arrow with the surjection $p: \mathbb{Z}E^0 \otimes KH^h_1(R) \rightarrow \mathcal{B}(E_t) \otimes KH^h_1(R)$ maps an element $\xi = \chi_v \otimes \xi_v$ to $\sum_{v \in E^0[v]} \chi_v \otimes \xi_v$. Let $[u_1] \in KH^h_1(LE \otimes LE_t)$ be the class of the element $u_1$ appearing in the proof of Theorem 11.2. Consider the algebra extension $\mathcal{E}$ of $L(E) \otimes L(E_t)$ which results by applying $\otimes L(E_t)$ to the Cohn extension (2.17). Let $\eta = \partial_{\mathcal{E}}([u_1]) \in KH^h_0(\ell E^0 \otimes L(E_t))$ be the index of $[u_1]$. The composite of the horizontal and vertical arrows with $p$ sends $\chi_v \otimes \xi$ to

\[
(\xi \otimes LE_t) \circ \eta \in kk^h_1(\ell E^0 \otimes L(E_t), R \otimes L(E_t)).
\]

A straightforward calculation, using Lemma 11.1, shows that $\eta = [\sum_{v \in E^0} \chi_v \otimes v]$. It follows that the first diagram commutes. The same argument, substituting $L(E)$ and $L(E_t)$ for $L(E)$ and $L(E_t)$ and the extension (2.18) for (2.17), proves that also the second diagram commutes. $\square$

**Remark 12.4** It follows using the equivalence of Remark 2.11, that the first exact sequence of Theorem 12.1 and the first commutative diagram of Lemma 12.3 still hold if we remove the superscript $h$ everywhere.

### 13 Lifting $kk^h$-maps to algebra maps

Let $E$ be a finite graph such that $\text{sink}(E) = \emptyset$. Set

\[
L^0(E) = L(E), \quad L^1(E) = L(E).
\]  \hspace{1cm} (13.1)

For $\epsilon \in \{0, 1\}$, let $\phi : L^\epsilon(E) \rightarrow R$ be a unital $\ast$-algebra homomorphism with $R$ strictly properly infinite. Assume that $\phi$ is very full. Set

\[
R_\phi = \{ x \in R : \phi(\epsilon e^* x) = x \phi(\epsilon e^*), \text{ for all } e \in E^1 \} \quad (13.2)
\]
Because $\phi$ is very full, $\phi(gee^*) \in \text{Proj}_f(R)$ for all $e \in E^1$, whence the inclusion $\phi(gee^*)R\phi(gee^*) \subset R$ induces an isomorphism in $K_1^R$, by Lemma 10.1. It follows that the direct sum $R_\phi \subset R^{E^1}$ of those inclusions induces an isomorphism

$$K_1(R_\phi)^* = \bigoplus_{e \in E^1} K_1(\phi(gee^*)R\phi(gee^*))^* \xrightarrow{\sim} (K_1(R)^*)^{E^1}. \quad (13.3)$$

Let $f : X \to Y$ be a map between finite sets and $M$ an abelian group. We write $f^* : \mathcal{U}(X) \to \mathcal{U}(Y)$, $f^*(\chi_x) = \chi_{f(x)}$; we shall abuse notation and also write $f^*$ for $f^* \otimes M$. In particular the source map $s : E^1 \to E^0$ induces a homomorphism $s^* : K_1(R)^{E^1} \to K_1(R)^{E^0}$. Consider the composite

$$\partial : K_1(R_\phi)^* \xrightarrow{\sim} (K_1(R)^*)^{E^1} \xrightarrow{s^*} (K_1(R)^*)^{E^0} \to K_1(R)^{E^0} \to \text{kh}(L^\ell(E), R) \quad (13.4)$$

Let

$$u = (u_e)_{e \in E^1} \in \mathcal{U}(R_\phi) = \bigoplus_{e \in E^1} \mathcal{U}(\phi(gee^*)R\phi(gee^*)); \quad (13.5)$$

consider the $*$-homomorphism

$$\phi_u : L^\ell(E) \to R, \phi_u(e) = u\phi(e). \quad (13.6)$$

**Lemma 13.1** Let $E$ be a finite graph, $R$ a $*$-algebra and $\phi : L^\ell(E) \to R$ a unital $*$-homomorphism. Assume that $E$ is purely infinite simple and that $R$ is strictly properly infinite. Then

$$j^h(\phi_u) = j^h(\phi) + \partial([u]).$$

**Proof** Let $n = |\text{sour}(E)|$; we shall prove the lemma by induction on $n$. First we assume that $n = 0$. By Lemma 12.3, the composite of $\partial$ with the isomorphism (11.2) sends $[u]$ to the class in $KH_1^R(R \otimes L^\ell(E))$ of the element

$$\prod_{e \in E^1} \left(1 - \phi(gee^*) + u_e \right) \otimes s(e) + \sum_{v \neq s(e)} 1 \otimes v \right).$$

Let $\psi : L^\ell(E) \to R$ be a $*$-homomorphism such that $\phi(gee^*) = \psi(gee^*)$ for all $e \in E^1$. Then (11.2) sends $j^h(\psi)$ to the $KH_1^R$-class of

$$\xi(\psi) := 1 \otimes 1 - \sum_{v \in E^0} \phi(v) \otimes v + \sum_{f \in E^1} \psi(f) \otimes f_i^*.$$
We shall abuse notation and write $u_e$ for the element of $\bigoplus_{f \in E^1} U(\phi(\epsilon e^*) R \phi(\epsilon e^*))$ whose $f$-coordinate is $u_{\delta^{f,e}} \epsilon (f f^*)$. One checks that

$$\left(1 - \phi(\epsilon e^*) + u_e \otimes s(e) + \sum_{v \neq s(e)} 1 \otimes v\right) \xi(\psi) = \xi(\psi_{u_e}).$$

Starting with $\psi = \phi$ and applying the identity above repeatedly we obtain that the isomorphism (11.2) sends the two sides of the identity of the lemma to the same element of $KH_1^h(R \otimes L^e(E_t))$. This concludes the proof of the case $n = 0$. Next let $n \geq 0$, assume that the lemma holds for graphs with at most $n$ sources, and let $E$ be a finite graph without sinks and with $n + 1$ sources. Let $v \in sour(E)$ and let $F = E \setminus v$ be the source elimination graph. Let $\pi: R = \bigoplus_{e \in E^1} \phi(\epsilon e^*) R \phi(\epsilon e^*) \to S = \bigoplus_{e \in E^1, s(e) \neq v} \phi(\epsilon e^*) R \phi(\epsilon e^*)$

be the projection. The map $\pi$ together with the corner inclusions $inc_p$ above and $inc_q: S \subset R$ induce a commutative diagram

By induction, the composite of the vertical map on the left with the horizontal map at the bottom sends the class of $u = (u_e)$ to $j^h(\phi'(\pi(u))) - j^h(\phi')$. Composing with the upward slanted arrow, we obtain $j^h(\phi''(\pi(u)) \circ inc_p) - j^h(\phi'' \circ inc_p)$. Because the downward slanted map is an isomorphism, it follows that the top horizontal arrow maps $[u] \mapsto j^h(\phi''(\pi(u))) - j^h(\phi')$. □

**Theorem 13.2** Let $E$ be a finite, purely infinite simple graph and $R \in \text{Alg}_\ell^*$ a $K_0^h$-regular, strictly properly infinite $*$-algebra over a ring $\ell$ satisfying the $\lambda$-assumption. Assume that $-1$ is positive in $R$. Then the map

$$j^h: [L(E), R]_{M_{\pm 2}}^f \to kk^h(L(E), R)$$

(13.7)
is a semigroup isomorphism. For each $p \in \Psi_{\text{proj}}(R)$, we have

$$ev^{-1}([p]) = j^h([\phi] : \phi(1) = p)).$$

(13.8)

**Proof** Let $\xi \in kk^h(L(E), R)$. Because by assumption, $-1$ is positive in $R$, the map $K_0(R)^* \to K_0^h(R)$ is an isomorphism, as explained in Sect. 7. Hence by Theorem 9.3 there is a very full $*$-homomorphism $\phi : L(E) \to R$ such that $K_0^h(\phi) \circ \text{can} = ev(\xi)$. Let $p = \phi(1)$ and $\text{inc} : pRp \subset R$ the inclusion. By Lemma 8.12, there exists $\xi' \in kk^h(L(E), pRp)$ such that $\text{inc}_*(\xi') = \xi$. By Lemma 13.1, there exists $u \in U((pRp)_\phi)$ such that $\xi' = j^h(\phi_u)$. Hence $\xi = j^h(\text{inc} \circ \phi_u)$ and the map of the theorem is surjective. Next let $\phi, \psi : LE \to R$ be very full $*$-homomorphisms such that $j^h(\phi) = j^h(\psi)$. Then $K_0^h(\phi) \circ \text{can} = K_0^h(\psi) \circ \text{can}$; because $-1$ is positive in $R$, this implies that $K_0(\phi)^* \circ \text{can}' = K_0(\psi)^* \circ \text{can}'$. Hence by Lemma 9.6, we may assume that $\psi_1|DL(E) = \phi_1|DL(E)$. Thus there is $p \in V_f(R)$ with $\psi(1) = \phi(1) = p$; by Lemma 7.1, we may assume that $p = 1$. For each $e \in E^1$, let

$$u_e = \psi(e)\phi(e^*) \in \mathcal{U}(\phi(\phi^*) R\phi(\phi^*)) = \mathcal{U}(R\phi).$$

A calculation shows that $\psi = \phi_u$. Hence $\partial([u]) = 0$, by Lemma 13.1. As in [27, Section 5], we consider the $*$-homomorphism $\lambda : R_\phi \to R_\phi, \lambda(a) = \sum_{e \in E^1} \phi(e)a\phi(e^*)$. Because $R$ is $K_0^h$-regular by assumption, $KH_1^h(R) = KV_1^h(R)$. Let $B = B_E$ be as in (3.3); using that, by Proposition 10.3, $\mathcal{N}(R) \subset \mathcal{U}(R)$ maps to the trivial subgroup in $KV_1^h(R)$, one checks that the following diagram commutes

$$\begin{array}{ccc}
KV_1^h(R_\phi) & \xrightarrow{\lambda} & KV_1^h(R_\phi) \\
\downarrow & & \downarrow \\
KV_1^h(R)E^1 & \xrightarrow{B_\phi} & KV_1^h(R)E^1.
\end{array}$$

(13.9)

Thus identifying $[u] = \sum_{e \in E^1} [u_e]$, and using that $\partial([u]) = 0$, it follows that there exists $v \in U(R_\phi)$ such that $[u] = [v\lambda(v)^{-1}]$. Hence by Lemma 10.4 there is $U(\iota) \in U_2^h(R_\phi[t])$ with $U(0) = v\lambda(v)^{-1}, U(1) = u$. Thus by Lemma 8.11, for the inclusion $\text{inc} : R_\phi \subset R$, we obtain

$$\psi = \phi_{u^*} \sim_{M_{\pm 2}} \phi_{v\lambda(v)^{-1}} = \text{inc} \circ \text{ad}(v) \circ \phi \sim_{M_{\pm 2}} \phi.$$

This proves the first assertion of the theorem. Next let $p \in \Psi_{\text{proj}}(R)$ and let $\xi \in kk^h(L(E), R)$ such that $ev(\xi) = p$. By Theorem 9.3 there is a very full $*$-homomorphism $\phi : LE \to R$ such that $\phi(1) = p$ and $ev(j^h(\phi)) = ev(\xi)$, and by Lemma 13.1 there is $u \in U(R_\phi)$ such that $j^h(\phi_u) = \xi$. This finishes the proof, since $\phi_u(1) = \phi(1) = p$. \hfill $\square$

In the next corollary we refer to the stable $M_{\pm 2}$-homotopy relation $\sim^s_{M_{\pm 2}}$ defined in (8.3).
Corollary 13.3 For every class \([\phi] \in [L(E), R]^\ast_{M \pm 2}\) there exists a unique class \([\phi^f] \in [L(E), R]^f_{M \pm 2}\) such that \(\phi \sim_{M \pm 2}^s \phi^f\). The map

\[ [L(E), R]^\ast_{M \pm 2} \to [L(E), R]^f_{M \pm 2}, \quad [\phi] \mapsto [\phi^f] \]

is the group completion (8.4).

Proof By Theorem 13.2 there is a very full \(\ast\)-homomorphism \(0^f : L(E) \to R\) such that \(j^h(0^f) = 0\). For a \(\ast\)-homomorphism \(\phi : L(E) \to R\), set \(\phi^f = \phi \oplus 0^f\). Then \(\phi^f\) is very full and \(\phi^f \sim_{M \pm 2}^s \phi\). It is straightforward to check that \([\phi] \mapsto [\phi^f]\) is well-defined and has the universal property of group completion. \(\square\)

Theorem 13.4 Let \(E, R\) and \(\ell\) be as in Theorem 13.2. Then the map

\[ j^h : [L(E), R]^f_{M \pm 2} \to kk^h(L(E), R) \quad (13.10) \]

is a semigroup monomorphism. For each \(p \in \text{Proj}_{\ell}(R)\), we have

\[ \text{ev}^{-1}([p]) = j^h([\{\phi : \phi(1) = p\}]). \]

If furthermore \(\text{sour}(E) = \emptyset\) and \(R\) contains a central element \(x\) such that \(xx^\ast = -1\), then (13.10) is an isomorphism.

Proof Proceed as in the proof of the injectivity part of Theorem 13.2, substituting Lemma 9.7 for Lemma 9.6. One checks that the analogue of (13.9) holds with \(\sigma B^t\) substituted for \(B^t\); the rest of the proof of the first assertion of the current theorem follows as in Theorem 13.2. To prove the second assertion, begin by observing that the bijection \(\theta^x\) of Example 2.8 passes down to a homomorphism between the monoids of \(M_{\pm 2}\)-\(\ast\) homotopy classes. Hence it induces a map \([L(E), R]^f_{M \pm 2} \to [L(E), R]^f_{M \pm 2}\), which, together with the isomorphism of Corollary 11.3 and the maps (13.7) and (13.10), fits into a diagram

\[ [L(E), R]^f_{M \pm 2} \xrightarrow{(13.10)} kk^h(L(E), R) \quad (13.10) \]

\[ \downarrow \theta^x \quad (11.3) \downarrow \]

\[ [L(E), R]^f_{M \pm 2} \xrightarrow{(13.7)} kk^h(L(E), R) \]

A straightforward calculation shows that the diagram above commutes; this finishes the proof. \(\square\)

Remark 13.5 Let \(E, R\) and \(\ell\) be as in Theorems 13.4 and 13.2, \(\epsilon \in \{0, 1\}\) and \(L^\epsilon(E)\) as in (13.1), and let \(\phi, \psi : L^\epsilon(E) \to R\) be very full \(\ast\)-homomorphisms. Put \(p = \phi(1)\) and \(S = pRp\). The common argument for the proof of injectivity of the map \(j^h\) in both theorems shows that \(j^h(\phi) = j^h(\psi)\) if and only if there exist:
• an $MvN$ equivalence $R \ni y_e : \psi(\varepsilon e^*) \to \phi(\varepsilon e^*)$ for every $\varepsilon \in E^1$ and
• elements $U(t) \in U^h(S_{\phi}[t])$, and $u, v \in U(S_{\phi})$ such that $t_+t_1(u) \oplus 1_3 = U(1)$, $U(0) = t_+(t_1(\nu_{\lambda}(v)^{-1})) \oplus 1_3$ and such that for $y = \bigoplus_{\varepsilon \in E^1} y_\varepsilon$, we have

$$\text{ad}(y) \circ \psi = \phi_u.$$  

**Example 13.6** Let $F$ be a graph, $\varepsilon \in \{0, 1\}$ and $L^\varepsilon F$ as in (13.1). If $F^0$ is finite, then $L^\varepsilon(F) \otimes L_\infty$ is strictly properly infinite. If furthermore $E$ is finite and $\ell$ is regular supercoherent, then $L(F)$ is regular supercoherent, and thus $L(F) \otimes L_\infty$ is $K^h$-regular, by Lemma 4.3. If in addition 2 is invertible in $\ell$, then $L^\varepsilon(F) \otimes L_\infty$ is $K^h$-regular, again by Lemma 4.3. Moreover $j^h(L^\varepsilon(F) \otimes L_\infty) \cong j^h(L^\varepsilon(F))$, by Theorem 3.3. Hence by Theorem 13.2, if $E$ is purely infinite simple and finite we have

$$kk^h(L(E), \overline{L(F)}) = [L(E), \overline{L(F)} \otimes L_\infty]_{M_{1,2}}$$  

(13.11)

$$kk^h(L(E), L(F)) = [L(E), M_\pm L(F)]_{M_{1,2}}.$$  

(13.12)

The identities (13.11) and (13.12) describe $kk^h$-groups in terms of homotopy classes, and thus provide an algebraic analogue, limited to Leavitt path algebras, of the description of $C^*-K K$-groups in terms of homotopy classes of [25, Theorem 4.1.1]. The latter theorem further describes the $K K$-groups in terms of approximate unitary equivalence of asymptotic *-homomorphisms; this is somewhat akin to the description of Remark 13.5.

**Example 13.7** Let $E$ be a finite regular graph, $R$ a unital algebra and $\phi : L(E) \to R$ be an algebra homomorphism. We say that $\phi$ has property (P) if for every $\varepsilon \in E^1$, $\phi(\varepsilon e^*)$ is a very full idempotent of $R$. Let

$$[L(E), R]_{M_2} \supset [L(E), R]_{M_2}^P = \{[\phi] : \phi \text{ has property (P)}\}.$$  

Next assume $R$ is $K_0$-regular and properly infinite, and that $E$ is purely infinite simple. By Remarks 2.11 and 10.5, the proof of Theorem 13.2 in the case of $\text{inv}(\ell)$-algebras gives a semigroup isomorphism

$$[L(E), R]_{M_2}^P \sim \text{kk}(L(E), R).$$  

The analogue of (13.8) holds verbatim. The analogue of the description in Remark 13.5 holds, and $U(t)$ can be taken in $GL_2(R[t])$. If furthermore $\phi$ is unital and $R$ is purely infinite simple, then so is $R_\phi$, and by [13, Proposition 2.8] we may take $U(t) \in GL_1(R_\phi[t])$. If $\ell$ is a field, then $R = L(F)$ is purely infinite simple if and only if $F$ is. If $\ell$ is not a field and $I \triangleleft \ell$ is a proper ideal, then $IL(F)$ is a proper ideal, so $L(F)$ is not simple.
14 Classification theorems

**Theorem 14.1** Let $E$ and $F$ be purely infinite finite graphs. Assume that $\ell$ satisfies the $\lambda$-assumption 1.3 and that $L(E)$ and $L(F)$ are $K^0_\ell$-regular. Let $\xi_0 : \mathcal{B}(E) \rightarrow \mathcal{B}(F)$ be an isomorphism and let $\psi$ be as in (6.3). Then there are $*$-homomorphisms $\phi : L(E) \rightarrow M_\pm L(F)$ and $\psi : L(F) \rightarrow M_\pm L(E)$ with property (P) such that $\psi(\xi) = \xi_0 \circ \xi_0^{-1}$, $M_\pm(\psi) \circ \phi \sim_{M_{\pm2}^2} \xi_0^2 : L E \rightarrow M_\pm M_\pm L(E)$ and $M_\pm(\phi) \circ \psi \sim_{M_{\pm2}^2} \xi_0^2 : L F \rightarrow M_\pm M_\pm L(F)$.

**Proof** Because $E$ and $F$ are purely infinite simple, sink($E$) = sink($F$) = $\emptyset$. Hence $\xi_0$ lifts to an isomorphism $\xi : j^h(L(E)) \cong j^h(L(F))$ such that $\psi(\xi) = \xi_0 \circ \xi_0^{-1}$, by Theorem 6.2. By Theorem 13.2, there are full $*$-homomorphisms $\phi : L(E) \rightarrow M_{\pm}(L(F))$ and $\psi : L(F) \rightarrow M_{\pm}(L(E))$ such that $j^h(\phi) = j^h(\xi_0)\xi_0^{-1}$ and $j^h(\psi) = j^h(\xi_0)\xi_0^{-1}$. Omitting $j^h$ for ease of notation, we have constructed the following commutative diagram in $kk^h$

![Diagram](image)

Hence $j^h(\xi_0)\xi_0^{-1} = j^h((M_\pm(\psi)\phi)$, and therefore $\xi_0^2 \sim_{M_{\pm2}^2} (M_\pm(\psi)\phi)$, by Corollary 13.3. (In fact, $(M_\pm(\psi)\phi$ is very full, by Examples 8.6 and 9.1, so $[(M_\pm(\psi)\phi] = [(\xi_0^2)^{-}]$.) Similarly, $\xi_0^2 \sim_{M_{\pm2}^2} (M_\pm(\phi)\psi$.

**Theorem 14.2** Let $E$, $F$, $\xi_0$ and $\ell$ be as in Theorem 14.1. Further assume that $-1$ is positive in $\ell$. Then there exist very full $*$-homomorphisms $\phi : L(E) \rightarrow L(F)$ and $\psi : L(F) \rightarrow L(E)$ such that $\psi(\xi) = \xi_0$, $\psi(\xi_0^{-1}) = \xi_0^{-1}$, $\psi \circ \phi \sim_{M_{\pm2}^2} \mathrm{id}_{L(E)}$ and $\phi \circ \psi \sim_{M_{\pm2}^2} \mathrm{id}_{L(F)}$. If furthermore $\xi_0([1]_E) = [1]_F$, then $\phi$ and $\psi$ can be chosen to be unital.

**Proof** The proof is essentially the same as in Theorem 14.2; the only difference is that because $-1$ is positive in $L(E)$ and $L(F)$, one can apply Theorem 6.2 directly, without going through $M_{\pm}(L(E)$ and $M_{\pm}(L(F)$; since the identity maps of $L(E)$ and $L(F)$ are very full, we get strict rather than stable $M_{\pm2}^2$-homotopy equivalence.

**Example 14.3** The hypothesis of Theorem 14.1 are satisfied, for example, when $\ell$ is regular supercoherent and $2$ is invertible in $\ell$; if in addition $-1$ is positive in $\ell$, then also Theorem 14.2 applies.

**Theorem 14.4** Let $E$ and $F$ be finite, purely infinite graphs, let $\ell$ be regular supercoherent and let $\xi_0 : \mathcal{B}(E) \rightarrow \mathcal{B}(F)$ be an isomorphism. Then there exist $\ell$-algebra $\phi : L(E) \rightarrow L(F)$ and $\psi : L(F) \rightarrow L(E)$ such that $\phi(\xi^2_0) = \psi(\xi^2_0)$.
are very full idempotents for every \( e \in E^1 \), \( \psi \circ \phi \sim_{M_2} \text{id}_{LE} \) and \( \phi \circ \psi \sim_{M_2} \text{id}_{LF} \). If furthermore \( \xi_0([1]_E) = [1]_F \), then \( \phi \) and \( \psi \) can be chosen to be unital.

**Proof** The proof follows as in Theorem 14.2, using Remark 13.7. \( \square \)

**Remark 14.5** Theorem 6.1 of [13] shows that if \( \ell \) is a field, then in the last part of Theorem 14.4 above, the unital homomorphisms \( \phi : L(E) \rightarrow L(F) : \psi \) can be chosen so that \( \phi \circ \psi \) and \( \psi \circ \phi \) are not just \( M_2 \)-homotopic, but strictly homotopic to the identity maps. The proof uses the fact, straightforward from Example 13.7, that a unital endomorphism of the Leavitt path algebra of a finite purely infinite simple graph over a field goes to the corresponding identity map in \( kk \) if and only if it is homotopic to an inner automorphism.

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