Internal tide generation from isolated seamounts and continental shelves

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Abstract

We model linear, inviscid non-hydrostatic internal tides generated by the interaction of a barotropic tide with variable topography in two dimensions. We first derive an asymptotic solution for the non-uniform barotropic flow over the topography that serves as forcing for the baroclinic equations. The resulting internal-tide generation problem is reformulated as a Coupled-Mode System (CMS) by means of a series decomposition of the baroclinic stream function in terms of vertical basis functions. We solve this CMS numerically and also provide a method for estimating the sea-surface signature of internal tides. We consider several seamounts and shelf profiles and perform calculations for a wide range of (topographic) heights and slopes. For subcritical topographies, the energy flux as a function of height exhibits local maxima, separated by cases of weakly- or even non-radiating topographies. For supercritical topographies, the energy flux generally increases with height and criticality. Our calculations agree with the Weak Topography Approximation only for very small heights. Perhaps more surprisingly, they agree with the Knife Edge model only for moderately supercritical topographies. We also compare the effect of the adjusted barotropic tide on the energy flux and the local properties of the baroclinic field with other semi-analytical methods based on a uniform barotropic tide. We observe significant differences in the flow field near the topographies only.

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1 Introduction

Oceanic internal tides (ITs) are waves generated in a stratified ocean (internal waves) through the interaction of the astronomically induced barotropic tidal flow with the seafloor. They are ubiquitous oscillatory baroclinic perturbations of the fluid velocity and density at the tidal frequency, which propagate away from bottom irregularities such as seamounts, continental shelves, or ridges (Garrett & Kunze 2007). ITs are considered to be one of the main sinks of energy for the barotropic tide, and an important contributor to ocean mixing at a global scale (Garrett 2003, Wunsch & Ferrari 2004, de Lavergne et al. 2016). An accurate description of the IT flow and a good prediction of barotropic-to-baroclinic energy conversion are necessary for the reliable ocean and climate modelling. Climate models often implement internal-wave driven mixing through a parametric formula involving an idealized estimation of the local energy conversion rate at generation sites. Moreover, the dissipation of ITs is also parametrized by using a modal description of the flow. We refer to MacKinnon et al. (2017) for a detailed overview of these considerations. Various analytical or semi-analytical approaches have been developed in the past decades to model ITs, mostly in two-dimensions. As discussed in Garrett & Kunze (2007), these approaches may be classified in two categories in terms of the type of the barotropic forcing: (i) the boundary-forcing approach and (ii) the body-forcing approach. Topographic features are commonly classified into two regimes, subcritical or supercritical, depending on whether the maximum slope value is smaller or larger than the characteristic slope of internal waves at the tidal frequency. Both of the aforementioned approaches can be used to study ITs generated in any of the two topographical regimes.

In the boundary-forcing approach, a homogeneous internal wave equation is solved with a non-homogeneous bottom boundary condition, which implies an oscillating barotropic velocity of constant amplitude in the vertical direction. It was first developed in conjunction with either of two strong and opposing assumptions. The first one is the “weak topography approximation” (WTA) which allows the linearization of the bottom boundary condition around a flat bottom and therefore approximates the flow over a topography of small height but arbitrary horizontal shape (Bell 1975, Llewellyn Smith & Young 2002, Khatiwala 2003, Grisouard & Bühler 2012). The second one is the “knife-edge” approximation (Robinson 1969, Llewellyn Smith & Young 2003), where the topography is assumed of zero elevation everywhere except at a single point, where it is arbitrarily large. These idealizations make the problem analytically tractable, i.e., analytical methods such as Fourier/eigenfunction expansions and Green’s function methods can lead to useful expressions for the far-field energy flux or conversion rate (Llewellyn Smith & Young 2002, Llewellyn Smith & Young 2003, St. Laurent et al. 2003, Nycander 2005, Falahat et al. 2014). Semi-analytical methods that consider the full bottom boundary condition have also been developed based on the Fourier representation of a periodic topography in an infinitely deep ocean (Balmforth et al. 2002) and on the Green’s function method (Pétrélis et al. 2006, Balmforth & Peacock 2009, Echeverri & Peacock 2010, Mercier et al. 2012, Mathur et al. 2016).

In the body-forcing formulation, the governing wave equation has a homogeneous bottom boundary condition but is forced by a source term that represents a non-uniform or topographically adjusted barotropic flow. This formulation was introduced by Baines (1973), who also proposed a technique based on ray theory and an integral-equation reformulation for the calculation of the far-field energy flux (see also Baines 1982). Garrett & Gerkena (2007) noted that the body forcing term in Baines (1973) is inconsistent with non-hydrostatic conditions and derived a consistent formulation. Maas (2011) used appropriate transformations of the underlying equations, establishing the existence of a class of topographies that do not lead to far-field internal-tide generation (non-radiating topographies). To the best of our knowledge, general solution procedures that take into account the non-uniformity of the barotropic flow over varying topography are less developed in comparison with the uniform barotropic forcing case, a gap we propose to address here.

In this work, we adopt the body-forcing formulation and develop a semi-analytical model that describes the internal-tide generation process. The analytical step is the exact reformulation of the problem as a Coupled-Mode System (CMS) of equations that can be solved numerically in an efficient way. This is accomplished by means of a modal or eigenfunction expansion of the unknown stream function, in terms of appropriate basis functions and unknown modal amplitudes. This approach, also called coupled-mode theory, has been applied to the solution of various non-uniform waveguide problems in acoustics (Brekhovskikh & Godin 1992, Desaubies & Dysthe 1995, Maurel et al. 2014), geophysics (Maupin 1988, He et al. 2019), water waves (Porter & Staziker 1995, Athanassoulis & Belibassakis 1999, Papoutsellis et al. 2018) among other disciplines. In the context of IT-generation within the body-forcing formulation, Griffiths & Grimshaw (2007) derived a CMS from the velocity formulation of the problem, with the hydrostatic approximation, and calculated 2D ITs over continental shelves. The CMS we develop
here differs from this work in two ways. First we remove the hydrostatic approximation, and, second, our modal decomposition is applied to the stream function formulation with a different set of vertical basis functions allowing for faster convergence of the numerical solution.

To our knowledge, although several approaches to study IT generation are available, some questions remain that we seek to address in this paper. First, how does the nature of the barotropic forcing influence the nature of the generated ITs? Indeed, the boundary-forcing approach is associated with a barotropic flow that is uniform with depth, whereas the body forcing approach allows for a topographically adjusted barotropic flow which satisfies the correct bottom boundary-condition. The comparison of the two approaches has not been clearly addressed in the literature. Second, how valid are the analytical predictions of the conversion rate based on the WTA or the knife-edge approximation? This type of predictions are often used in global ocean models (Vic et al. 2019, Lahaye et al. 2020), or for comparisons with estimates made at large supercritical topographic features when investigating specific locations (Gerkema et al. 2004, Qian et al. 2010, Rainville et al. 2010). To our knowledge, the limit of validity of these predictions in terms of topographic features (height, slope) has not been sufficiently quantified (Zhang et al. 2017). A third question relates to the recent results of Maas (2011) concerning specially constructed non-radiating topographies. They also require more investigations in order to assess if such a peculiar phenomenon is more generally observed for any type of topography. Here we discuss all these aspects, based on a description of the barotropic forcing that consistently takes into account large bottom variations and a modal description of the baroclinic response.

The paper is organized as follows. In section 2, we begin with a review of the basic governing equations, in which we incorporate an arbitrary volume forcing, and we present an asymptotic solution for the barotropic flow. We then formulate the internal tide generation problem in section 3 and derive the associated energy equation and conversion rates. The coupled-mode reformulation of this problem is presented in section 4. In section 5, we present numerical convergence results of our method and give some examples of calculated flow fields. In section 6, we present our results on the conversion rates for various topographies in comparison with existing approaches. Finally, in section 7, we present our conclusions.

## 2 Governing Equations

### 2.1 Posing the problem

We consider a two-dimensional Cartesian coordinate system $Oxz$, with the vertical $z$ axis pointing upwards, describing a horizontally infinite layer of a density stratified fluid bounded from above by a “rigid lid” $\{z = 0\}$, and from below by the bottom surface $\{z = -h_0 + h(x)\}$ with $0 < h < h_0$. Specifically, we consider bottom topographies $h(x)$ of characteristic height $\Lambda$ and characteristic horizontal scaling length $L$. We do not limit ourselves to small heights, i.e., $\Lambda$ does not have to be much smaller than $h_0$ or any other characteristic vertical length scale. However, we require the topography to be isolated or asymptotically flat, that is, its slope must vanish at infinity, $\lim_{x \to \pm \infty} |\partial_x h| = 0$, and we define $\lim_{x \to \pm \infty} h = h_{\pm}$. The latter requirement allows us to take into account fluid domains with different depths at infinity, e.g., continental slopes and shelves.

In static equilibrium the fluid velocity is zero and the background density stratification is $\rho_0(z)$, weakly departing from a constant reference density $\overline{\rho_0}$ such that the Boussinesq approximation applies. Furthermore, the static density profile decreases linearly with height, such that the Brunt-Väisälä frequency $N = \sqrt{(g/h_0)\, \partial \rho_0 \, / \partial z}$ is constant. The hydrostatic pressure $p_0(z)$ is then defined by its vertical gradient $p_{0,z} \equiv \partial p_0 / \partial z = -\rho_0 g$. The hydrodynamic problem is posed on the $f$-plane, with $f$ the Coriolis parameter. Our aim is to find perturbations of this state driven by the interaction of an ambient tidal flow and the bottom topography. This barotropic tidal forcing is characterized by an angular frequency $\omega \in (f, N)$ for $N > f$ and a constant volume flux $Q$.

Under the Boussinesq approximation, our setup is characterized by seven dimensionless parameters ($h_0$, $L$, $\Lambda$, $f$, $N$, $\omega$, and $Q$, and we defer discussion of $h_{\pm}$ to later), which we summarize in figure 1, measured in combinations of meters and seconds. A complete dynamical description of our system therefore requires five non-dimensional numbers. Garrett & Kunze (2007) highlight a procedure to choose them, from which we deviate somewhat. First, we introduce a “funning” ratio,

$$\delta = \frac{\max(h)}{h_0} \sim \frac{\Lambda}{h_0},$$

(1)

that measures the reduction in cross-section of the flow. The second and third parameters are the
non-dimensional frequency $\omega/f$ and the internal wave characteristic slope,

$$\mu^{-1} = \sqrt{\frac{\omega^2 - f^2}{N^2 - \omega^2}} = \tan \theta,$$

(2)

$\theta$ being the angle of the free internal wave group velocity with respect to the horizontal plane, which derives from the hyperbolic operator that describes the linear internal-wave dynamics (see section 22.2).

Our fourth non-dimensional parameter is the relative steepness,

$$\varepsilon = \mu \max \{|\partial_x h|\} \sim \frac{\mu \Lambda}{L},$$

(3)

and we shall use the first, more accurate version of its definition above (see Section 5). This parameter’s first purpose is to measure the criticality of the topography, with $\varepsilon < 1$ ($> 1$) corresponding to the subcritical (supercritical) regime. Fifth and finally, we have the tidal excursion,

$$\tau = \frac{Q}{(h_0 - \Lambda)\omega L},$$

(4)

which compares the typical displacement amplitude of a water parcel above the topography, $Q/|(h_0 - \Lambda)\omega|$, with the horizontal scale $L$. Should this parameter be finite, the curvature of the particle trajectories at the bottom would generate internal waves of frequencies, other that $\omega$ (Bell 1975). To ensure monochromatic disturbances, we therefore assume $\tau \ll 1$. In the ocean, for the lunar semi-diurnal tide $M_2$, the tidal excursion over flat bottom $Q/(\omega h_0)$ is $O(100 \text{ m})$ (Bell 1975), and therefore, even a moderate topographic width $L = O(10 \text{ km})$ would satisfy this so-called “acoustic limit”.

Of the five parameters defined above, $\omega/f = O(1)$ corresponds to a tidal component at mid-latitudes, and we will hold $\mu^{-1}$ fixed to a small value (for purposes of illustration, $f = 10^{-4} \text{ s}^{-1}$ is the value around latitude $45^\circ \text{N}$, $\omega/f = 1.4$ for the $M_2$ component and $N = 1.5 \times 10^{-3} \text{ s}^{-1}$, meaning that $\mu \approx 15.2$). Starting in Section 5, the parameters $\varepsilon$ and $\delta$ will be the parameters we vary primarily. Finally, for the seamount (respectively shelf) cases, we adopt a standard fixed value for $Q = 120 \text{ m}^2\text{s}^{-1}$ (resp. $Q = 134 \text{ m}^2\text{s}^{-1}$), corresponding e.g. to a barotropic velocity amplitude at $x \to -\infty$ of $U_0 = Q/h_0 = 4 \text{ cm/s}$ (resp. $U_0 = 4.46 \text{ cm/s}$) and a depth $h_0 = 3 \text{ km}$. With the variations of $\varepsilon$ and $\delta$, the value of $\tau$ will therefore vary across calculations while remaining small. From these five parameters, we can derive other non-dimensional parameters that will prove useful in characterizing the barotropic response. These are

$$\mu_0^{-1} = \sqrt{1 - \frac{f^2}{\omega^2}}$$

and

$$\sigma = \frac{h_0^2}{L^2} \sim \left(\frac{\varepsilon}{\mu_0}\right)^2. $$

(5)

the former being a frequency parameter appearing in the barotropic elliptical operator, and the latter representing the aspect ratio to be used for the perturbative computation of the barotropic response in subsection 22.3 and Appendix A.

With the parameters defined above, we can discuss the linearity of our equations. A first way to define it is to estimate the susceptibility of the radiated internal waves to undergo instabilities, which, in the
\( \tau \ll 1, \varepsilon \ll 1 \) regime, is small when \( \varepsilon \tau \ll 1 \) (e.g., Balmforth et al. 2002, Garrett & Kunze 2007). In other words, the \( \tau \ll 1, \varepsilon \ll 1 \) regime is linear by construction. However, no matter the value of \( \tau \), linearity breaks down as \( \varepsilon \) increases, implying that this parameter is a better measure of linearity, or absence thereof (Garrett & Kunze 2007, Bühler & Muller 2007, Grisonard & Bühler 2012). In this article, we adopt the common approach of letting \( \varepsilon \) be significantly supercritical in some cases, while always solving a linear set of equations (Pétrélis et al. 2006, Griffiths & Grimshaw 2007, Balmforth & Peacock 2009, Echeverri & Peacock 2010, Mathur et al. 2016). Indeed, we are primarily interested in predicting conversion from a given large-scale barotropic forcing to a topography-scale response. The subsequent non-linear evolution of this response, which could take the form of different instabilities (see Dauxois et al. 2018, for a review), could be addressed by separate parameterization procedures such as that of Muller & Bühler (2009), but would represent a layer on top of the radiation scheme we present in this article.

### 2.2 Equations of motion and stream function formulation

Based on the discussion above, we neglect nonlinear effects, as well as diffusion of momentum and buoyancy. The flow may be approximated by the linearized, inviscid Boussinesq equations. Introducing the buoyancy \( b = -g(\rho / \overline{\rho}_0 - 1) \), where \( \rho \equiv \rho(x,z,t) \) is the total density field, the governing equations are

\[
\begin{align*}
    u_t - f v &= -p_x, \quad v_t + f u = 0, \quad \text{(6a)} \\
    w_t &= -p_z + b, \quad \text{(6b)} \\
    b_t + N^2w &= 0, \quad \text{(6c)} \\
    u_x + w_z &= 0, \quad \text{(6d)}
\end{align*}
\]

where \((u, v, w) \equiv (u(x, z, t), v(x, z, t), w(x, z, t))\) are the velocity components and \( p \equiv p(x, z, t) \) the pressure divided by \( \overline{\rho}_0 \), and subscripts denote partial derivatives. Additionally, we invoke the non-penetration boundary condition on \( z = -h_0 + h \) and the rigid-lid approximation on \( z = 0 \) by requiring that

\[
-h_x u(x, -h_0 + h) + w(x, -h_0 + h) = 0, \quad w(x, 0) = 0. \quad \text{(7a, b)}
\]

Eq. (6d) allows to introduce a stream function \( \psi(x, z, t) \) such that

\[
    u = -\psi_z, \quad w = \psi_x, \quad \text{(8)}
\]

and elimination of \( v, p \) and \( b \) from (6a)–(6c) results in a single partial differential equation on \( \psi \),

\[
    \psi_{xzt} + \psi_{zxt} + f^2 \psi_{xz} + N^2 \psi_{xx} = 0. \quad \text{(9)}
\]

The rigid-lid condition (7b) becomes \( \psi_z(x, 0, t) = (\psi(x, 0, t))_x = 0 \), which implies that \( \psi(x, 0, t) \) depends solely on time, i.e., \( \psi(x, 0, t) = c_1(t) \). Similarly, from (7a) we have

\[
    h_x \psi_z(x, -h_0 + h) + \psi_z(x, -h_0 + h) = (\psi(x, -h_0 + h, t))_z = 0
\]

therefore \( \psi(x, -h, t) = c_2(t) \). In order to prescribe \( c_1 \) and \( c_2 \), we first note that we may take \( c_1(t) = 0 \) since we can always redefine \( \psi \) as \( \psi + c_1 \). Next, we observe that the volume flux at a vertical section is

\[
    \int_{-h_0 + h}^0 u dz = -\int_{-h_0 + h}^0 \psi_x dz = c_2(t) - c_1(t) = c_2(t).
\]

In our case, \( \psi \) should account for an oscillating flow, hence we choose \( c_2(t) = Q \exp(-i\omega t) \) where \( Q \) is a constant magnitude and \( \omega \) the frequency of the oscillation. Eq. (9) is thus supplemented by the boundary conditions \( \psi(x, 0, t) = 0 \) and \( \psi(x, -h_0 + h, t) = Q \exp(-i\omega t) \). Looking for time harmonic solutions in the form

\[
    \psi(x, z, t) = \Re \{ \phi(x, z) \exp(-i\omega t) \}, \quad \text{(10)}
\]

where \( \phi \) is a complex time independent amplitude and \( \Re \) stands for the real part, we obtain the following boundary value problem (BVP)

\[
    \mathcal{L}_\mu \phi = 0, \quad \phi(x, 0) = 0, \quad \phi(x, -h_0 + h(x)) = Q, \quad \text{(11)}
\]
where
\[ \mathcal{L}_\mu := \partial^2_x - \mu^{-2} \partial^2_z, \]
with \(\mu\) defined in (2). The BVP (11) can be equivalently written as a BVP with a homogeneous boundary condition on \(z = -h_0 + h\) by introducing another unknown function \(\phi^\#(x, z)\) and a known function \(\chi(x, z)\), yet to be determined, satisfying the conditions
\[
\phi^\# = \phi - \chi(x, z) \phi(x, -h_0 + h) = \phi - \chi(x, z) Q, \\
\chi(x, 0) = 0, \quad \text{and} \quad \chi(x, -h_0 + h) = 1. \tag{13}
\]
The advantage of solving for \(\phi^\#\) rather than for \(\phi\) is that we are now solving for a volume-forced problem with homogeneous Dirichlet boundary conditions, i.e., \(\phi^\#\) satisfies
\[
\mathcal{L}_\mu \phi^\# = -Q \mathcal{L}_\mu \chi, \quad \phi^\#(x, 0) = 0, \quad \phi^\#(x, -h_0 + h) = 0, \tag{14}
\]
and a straightforward representation of \(\phi^\#\) in terms of basis functions (eigenfunctions) that vanish on \(z = 0, -h_0 + h\) can be invoked (Section 4). In order to eventually find the baroclinic response governed by the BVP (14), the function \(\chi\) must be determined so that it accounts for an appropriate barotropic flow. Such a flow is considered in the following subsection.

### 2.3 Barotropic flow

We now compute the background barotropic flow that is established in the fluid domain as a response to the external tidal forcing. Later, in section 3.1.1, we will use this barotropic flow as a volume forcing for the baroclinic component. The barotropic flow is solution to the system
\[
\begin{align*}
U_t - fV &= -P_x, \quad V_t + fU = 0, \tag{15a} \\
W_t &= -P_z, \tag{15b} \\
B_t + N^2 W &= 0, \tag{15c} \\
U_x + W_z &= 0, \tag{15d}
\end{align*}
\]
where \((U, V, W)\), \(P\) and \(B\) denote the velocity components, the scaled pressure, and the buoyancy perturbations induced by the oscillating barotropic flow, respectively (see e.g. (Baines 1973, Sec. 8) or (Garrett & Gerkema 2007)), together with the boundary conditions
\[
-h_x U(x, -h_0 + h) + W(x, -h_0 + h) = 0, \quad W(x, 0) = 0. \tag{16a,b}
\]
Introducing a barotropic stream function \(\Psi\) such that \(U = -\Psi_z\) and \(W = \Psi_x\), we obtain from the system (15) a single equation on \(\Psi\),
\[
\Psi_{ztt} + \Psi_{zxtt} + f^2 \Psi_{zz} = 0.
\]
In terms of \(\Psi\), the boundary conditions (16) may be written as \(\Psi(x, 0, t) = 0\) and \(\Psi(x, -h_0 + h, t) = Q \exp(-i\omega t)\). Assuming that \(\Psi = \Re\{\Phi e^{-i\omega t}\}\) for some time independent function \(\Phi\), we obtain the following BVP for the barotropic flow
\[
\Phi_{xx} + \mu_0^{-2} \Phi_{zz} = 0, \quad \Phi(x, 0) = 0, \quad \Phi(x, -h_0 + h) = Q, \tag{17}
\]
with \(\mu_0\) defined in (5). Note that the partial differential equation in (17) is elliptic (cf. Eq. (11)). The above problem is solved in a perturbative way in terms of the ratio of the water depth and the horizontal scale of the topography, \(\sigma = h_0^2 / L^2\). This is done in detail in Appendix A. The net result is that the barotropic flow quantities can be approximated by an expression of the form
\[
\Xi = \Xi^{(0)} + \sum_{i=1}^{K} \Xi^{(i)}, \tag{18}
\]
for some order \(K \geq 1\), where \(\Xi\) is a placeholder for any of \(\Phi, \Psi, U, V, W, B\) and \(P\). In particular, \(\{\Phi^{(i)}\}_{i=0}^{K}\) are determined recursively (see Eq. (62)); the first two orders are
\[
\begin{align*}
\Phi^{(0)} &= \frac{z}{-h_0 + h}, \quad \Phi^{(1)} = -Q \mu_0^2 \left( \frac{1}{-h_0 + h} \right) \frac{1}{6} \left( z^3 - (-h_0 + h)^2 z \right), \tag{19}
\end{align*}
\]
and for \(i \geq 0\) we have
\[
U^{(i)} = -\Psi_x^{(i)} = -\Phi_x^{(i)} \cos \omega t, \quad W^{(i)} = \Psi_x^{(i)} = \Phi_x^{(i)} \cos \omega t \tag{20a, b}
\]
\[
V^{(i)} = \frac{f}{\omega} \Phi_x^{(i)} \sin \omega t, \quad B^{(i)} = -\frac{N^2}{\omega} \Phi_x^{(i)} \sin \omega t. \tag{21a, b}
\]
As shown in Appendix A, the above solution provides a decomposition of the barotropic flow into a hydrostatic part \(\Xi^{(0)}\) and a residual non-hydrostatic part, denoted by \(\Xi' \equiv \sum_{i=1}^{K} \Xi^{(i)}\). Note that the hydrostatic part depends on the topography \(h(x)\) and that the non-hydrostatic part vanishes when \(h_x = 0\).

3 The internal tide generation problem

3.1 Formulation

Owing to the solution of the barotropic flow developed in the previous section, we proceed by making the internal-tide generation problem (14) explicit. Choosing \(\chi(x, z)Q = \Phi^{(0)} + \Phi'\), the BVP (14) takes the form
\[
L_\mu \phi^# = -L_\mu (\Phi^{(0)} + \Phi'), \quad \phi^#(x, 0) = 0 \quad \phi^#(x, -h_0 + h) = 0, \tag{22a, b, c}
\]
This shows that the barotropic flow, \(\Phi^{(0)} + \Phi'\), forces the baroclinic response \(\phi^#\). Alternatively, exploiting the linearity of \(L_\mu\), the BVP (22) can be equivalently stated in the form
\[
L_\mu \phi^\dagger = -L_\mu \Phi^{(0)}, \quad \phi^\dagger(x, 0) = \phi^\dagger(x, 0) = 0, \quad \phi^\dagger(x, -h_0 + h) = 0, \tag{23a, b, c}
\]
with \(\phi^\dagger = \phi^# + \Phi'\), implying that the hydrostatic part of the barotropic flow, \(\Phi^{(0)}\), forces a baroclinic response as well as a non-hydrostatic barotropic one (Garrett & Gerkema 2007). Here, we shall proceed with the latter formulation for two reasons. First, the non-hydrostatic barotropic flow does not propagate and therefore does not influence the conversion calculations, making \(\phi^#\) and \(\phi^\dagger\) equally useful for this purpose. Second, it is easier to implement numerically because the right-hand side of Eq. (23a), \(L_\mu \Phi^{(0)}\), contains second order derivatives of the topography \(h\), whereas the right-hand side of Eq. (22a), \(L_\mu \Phi'\), contains derivatives of \(h\) of order \(2(K + 1)\) with \(K \geq 1\).

The above set of equations must be supplemented with radiation conditions for \(|x| \to \infty\) ensuring that the waves generated in the interior of the domain propagate outward at infinity as plane waves. Plane internal waves can be expressed through separation of variables of \(L_\mu \phi^\dagger = 0\) on a domain of constant depth \(h_\pm\) as
\[
\phi^\dagger(x, z) = \sum_{n=1}^{\infty} c_n^\pm \exp(\pm ik_n^\pm x) \sin(k_n^\pm z), \quad \text{as} \quad x \to \pm \infty, \tag{24}
\]
where \(c_n^\pm \in \mathbb{C}\) are constants,
\[
k_n^\pm = \frac{n\pi}{-h_0 + h_\pm}, \quad k_n^\pm = \frac{k_n^\pm}{\mu}, \quad n = 1, 2, \ldots, \tag{25}
\]
and \(h \to h_\pm\) as \(x \to \pm \infty\).

Assuming that the solution \(\phi^\dagger\) has been found, the purely baroclinic response is \(\phi^# = \phi^\dagger - \Phi'\). In Section 5, we calculate the flow fields by using \(\Phi' = \Phi^{(1)}\), where \(\Phi^{(1)}\) is given in (19). This choice is justified since \(\sigma < 0.01\) in the presented solutions and keeping a higher order in \(\Phi'\) does not result in a significant correction of the baroclinic response.

It will later prove useful to write down equations for the rest of the response fields. Let us first note that, in view of the first equation in (13) and the analysis in subsection 22.3, the total field \(\phi\) is written as \(\phi = \Phi^{(0)} + \Phi' + \phi^# = \Phi^{(0)} + \phi^\dagger\). Analogously, we introduce the decomposition
\[
\xi = \Xi^{(0)} + \Xi' + \xi^# = \Xi^{(0)} + \xi^\dagger, \tag{26}
\]
Table 1: Various notations for the fields, we solve for. The symbol \( \xi \) stands for \( u, v, w, b, p, \psi \) or \( \phi \), and \( \Xi \) for their capitalized versions.

| Notation | Comments | Related definition(s) |
|----------|----------|-----------------------|
| \( \xi \) | Solution to the full problem (6)–(7) | \( \xi = \Xi + \xi^\# = \Xi^{(0)} + \xi^\dagger \) |
| \( \Xi \) | Barotropic component, solution to (15)–(16) | |
| \( \xi^\# \) | Baroclinic component | See eqns. (13) and (22) |
| \( \Xi^{(n)}, n \in \mathbb{N} \) | Asymptotic components of \( \Xi \), ordered by \( \sigma \) | \( \Xi = \sum_{n=0}^{\infty} \sigma^n \Xi^{(n)} \) |
| \( \Xi^r \) | Higher-order barotropic response, non-hydrostatic | \( \xi^\dagger = \xi^\# - \Xi^r \); see eqn. (23) |

where, like for the barotropic response, \( \xi \) is a placeholder for any of \( \phi, u, v, w, b \) or \( p \). Plugging the second equality of (26) into Eqs. (6)–(7) and taking into account Eqs. (74)–(77) with \( i = 0 \), we obtain

\[
\begin{align*}
    u^i_t - f v^i &= -p^i_z, & v^i_t + f u^i &= 0, \\
    w^i_t &= -p^i_z + b^i + B^{(0)} - W_t^{(0)}, \\
    b^i_z + N^2 w^i &= 0, \\
    u^i_x + w^i_z &= 0, \\
    -h_x u^i(x, -h_0 + h) + w^i(x, -h_0 + h) &= 0, \quad w^i(x, 0) = 0.
\end{align*}
\]  

(27a–29)

Linearity of the problem allows us to assume that all daggered variables in Eqs. (27)–(30) have the same time-periodicity as \( \phi^\dagger \), e.g., \( u^\dagger = \mathbb{R}\{\phi^*_z e^{-i\omega t}\} \). We can therefore deduce from the previous statement and Eqs. (27)–(29) the standard polarization for internal waves, i.e.,

\[
(u^*, v^*, w^*, b^*) = \mathbb{R}\left\{(-\phi^*_z, \frac{f}{\omega} \phi^*_z, \frac{iN^2}{\omega} \phi^*_z, -i\frac{N^2}{\omega} \phi^*_z) e^{-i\omega t}\right\},
\]

(31)

where the superscript * stands for either \( \dagger \) or \( \# \), i.e., the above polarization relations apply to both the combined baroclinic and non-hydrostatic response (\( \dagger \)), and to the baroclinic response alone (\( \# \)). For easy reference, we summarize the notations for the flow fields that have been defined in Table 1.

### 3.2 Energy equation and conversion rate

Finally, we derive the energy equation associated with the internal-tide generation problem formulated in the previous subsection, to be used later for validation purposes (Section 55.2) and to define conversion rates and energy fluxes. Taking the dot product of Eqs. (27)–(28) with the state vector \( (u^\dagger, b^\dagger/N^2) \equiv (u^1, v^1, w^1, b^1/N^2) \) and using Eq. (29), we obtain

\[
\mathcal{E}_t^\dagger + \nabla \cdot (p^\dagger u^\dagger) = S,
\]

(32)

where \( \nabla = (\partial_x, \partial_z) \), \( \mathcal{E}^\dagger(x, z, t) \) is the energy density associated with the baroclinic and non-hydrostatic barotropic flows, and \( S(x, z, t) \) is a source term due to the conversion from the hydrostatic-barotropic flow, i.e.,

\[
\mathcal{E}^\dagger = \frac{1}{2} (u^\dagger)^2 + \frac{1}{2} (b^\dagger)^2 \quad \text{and} \quad S = \left(B^{(0)} - W_t^{(0)}\right) w^\dagger.
\]

Integrating Eq. (32) over the domain \( \Omega = [x_L, x_R] \times [-h_0 + h(x), 0] \), we introduce the instantaneous energy

\[
E^\dagger(t) = \int_{\Omega} \mathcal{E}^\dagger d\Omega = \int_{x_L}^{x_R} \int_{-h_0 + h}^{0} \mathcal{E}^\dagger dx dz,
\]

7
and invoking the divergence theorem for the flux term, and using the boundary conditions in (30), we obtain

$$E^t_\ell - \int_{-h_0 + h(x_L)}^{0} p^t u^t dz + \int_{-h_0 + h(x_R)}^{0} p^t u^t dz = \int_{\Omega} \left( B^{(0)} - W^{(0)}_t \right) w^t d\Omega,$$

which expresses the energy balance over \( \Omega \). Applying the phase average operator \( \langle \cdot \rangle = 1/T \int_0^T \cdot dt \), with \( T = 2\pi/\omega \), on the above equation and taking into account the periodicity of \( E^t(t) \), yields

$$- \int_{-h_0 + h(x_L)}^{0} \langle p^t u^t \rangle dz + \int_{-h_0 + h(x_R)}^{0} \langle p^t u^t \rangle dz = \int_{\Omega} \left( \langle B^{(0)} - W^{(0)}_t \rangle \right) w^t d\Omega. \tag{34}$$

The two terms in the left-hand side of the above equation represent the mean energy flux crossing the vertical boundaries of \( \Omega \), at \( x_L \) and \( x_R \). At the far-field \( (x_L \to \infty \text{ and } x_R \to \infty) \), these terms represent the rate at which energy is radiated away from the topography and are called conversion rates. They are defined here by

$$C_{\pm} = \lim_{x \to \pm \infty} \left[ \pm \int_{-h_0 + h(x)}^{0} \langle p^t u^t \rangle dz \right]. \tag{35}$$

In order to calculate \( C_{\pm} \), we first note that the integral in (35) can be expressed in terms of \( p^t \) by using integration by parts and the fact that \( \psi \) vanishes on \( z = 0 \), \( -h_0 + h(x) \):

$$\int_{-h_0 + h(x)}^{0} \langle p^t u^t \rangle dz = -\int_{-h_0 + h(x)}^{0} \langle p^t \psi^t \rangle dz = \int_{-h_0 + h(x)}^{0} \langle p^t \psi^t \rangle dz.$$

Using the expression for \( p^t \) obtained from eqs. (31) and (27b), we obtain

$$\langle p^t \psi^t \rangle = \frac{\omega^2 - N^2}{\omega} \left\langle \Re \{ \phi^t e^{-i\omega t} \} \Re \{ \phi^t e^{-i\omega t} \} + \Phi^{(0)}_z \sin \omega t \Re \{ \phi^t e^{-i\omega t} \} \right\rangle. \tag{36}$$

Writing \( e^{-i\omega t} = \cos \omega t - i \sin \omega t \) and noting that \( \langle \cos \omega t \sin \omega t \rangle = 0 \), \( \langle \cos^2 \omega t \rangle = \langle \sin^2 \omega t \rangle = 1/2 \) and \( \lim_{|x| \to \infty} \Phi^{(0)}_z = 0 \) yields

$$C_{\pm} = \frac{N^2 - \omega^2}{2\omega} \int_{-h_0 + h_{\pm}}^{0} \Im \left( \phi^t \phi^t \right) dz, \tag{37}$$

where the overline denotes the complex conjugate, \( \Im \) denotes the imaginary part, and where we recall that \( \lim_{|x| \to \infty}[h] = h_{\pm} \).

## 4 Modal decomposition

### 4.1 The Coupled-Mode System

In order to solve the problem (23)–(24), we follow a modal decomposition approach. The unknown function \( \phi^t(x, z) \) is represented as a series of the form

$$\phi^t(x, z) = \sum_{n=1}^{\infty} \phi_n(x) Z_n(z; x), \tag{38}$$

where \( \{\phi_n(x)\}_{n=0}^{\infty} \) are the unknown complex modal amplitudes to be determined and \( \{Z_n(z; x)\}_{n=0}^{\infty} \) are prescribed vertical basis functions with a parametric dependence on \( x \). In order for the expansion in eq. (38) to be exact, the set \( \{Z_n(z; x)\}_{n=0}^{\infty} \) must be complete. Such a set is obtained as the set of local eigenfunctions associated with a Sturm-Liouville problem parametrized by \( x \), also called “reference waveguide” (Brekhovskikh & Godin 1992). In our case, the reference waveguide reads

$$Z_{n, z} + k_0^2 Z_n = 0, \quad -h_0 + h(x) \leq z \leq 0,$$

$$Z_n = 0, \quad z = 0, \tag{39b}$$

$$Z_n = 0, \quad z = -h_0 + h(x). \tag{39c}$$
The corresponding local eigenfunctions and eigenvalues are given by
\[ Z_n(z; x) = \sin (\tilde{\kappa}_n z), \quad \tilde{\kappa}_n = \tilde{\kappa}_n(x) = \frac{n\pi}{-h_0 + h(x)}, \quad (40a, b) \]
and satisfy the orthogonality relation \[ \int_{-h_0 + h}^0 Z_n Z_m dz = (h_0 - h)\delta_{nm}/2, \] where \( \delta_{nm} \) is the Kronecker delta. It follows that each \( \phi_n \) is defined by
\[ \phi_n(x) = \frac{2}{h_0 - h(x)} \int_{-h_0 + h(x)}^0 \phi^\dagger(x, z) Z_n(z; x) dz. \quad (41) \]
Note that this eigenfunction expansion is consistent with the boundary conditions in (23). Furthermore, provided that \( \phi^\dagger \) is sufficiently smooth, a repeated integration by parts in (41) yields
\[ \phi_n(x) = \frac{2}{(h_0 - h)\tilde{\kappa}_n} \left( [\tilde{\phi}^\dagger_{xx} Y_n]_0^{0} - \int_{-h_0 + h}^0 \tilde{\phi}^\dagger_{xxx} Y_n dz \right), \quad (42) \]
with \( Y_n = \cos(\tilde{\kappa}_n z) \). This shows that \( \| \phi_n \|_\infty := \max |\phi_n| = O(n^{-3}) \) and that the eigenfunction expansion converges uniformly. Similar estimates are obtained for the decay of \( \|\phi_{n,x}\|_\infty \) and \( \|\phi_{n,xx}\|_\infty \) by adapting the procedure developed in Athanasoulis & Papoutsellis (2017, Section 4). These estimates suffice to establish the term-wise differentiability of (38) and justify its use in conjunction with Eq. (23).

We proceed by projecting Eqs. (23)–(24) onto (40). Substituting (38) into the differential equation in Eq. (23), multiplying with \( Z_n \) and integrating over the interval \([-h_0 + h(x), 0]\), we find that the \( \phi_n(x) \) are solutions to the Coupled-Mode System (CMS)
\[ \sum_{n=1}^\infty A_{mn}(h)\phi_{n,x} + B_{mn}(h)\phi_{n,x} + C_{mn}(h)\phi_n = F_m(h), \quad m \geq 1, \quad (43) \]
where the matrix coefficients \( A_{mn} = A_{mn}(h) \), \( B_{mn} = B_{mn}(h) \), \( C_{mn} = C_{mn}(h) \) and the sequence \( F_m = F_m(h) \) are provided in Appendix B. The CMS (43) must be supplemented with appropriate radiation conditions on the modal amplitudes \( \phi_n \). In view of Eq. (38), the radiation conditions (24) are written
\[ \sum_{n=1}^\infty \phi_n(x)Z_n(x, z) = \sum_{n=1}^\infty c_n \exp(\pm ik_n^\pm x) \sin(\kappa_n^\pm z), \quad \text{as} \quad x \to \pm \infty. \quad (44) \]
Multiplying both sides by \( Z_q, \quad q \in \{1, 2, \ldots\} \), integrating over \([-h_0 + h, 0]\) and taking into account that \( Z_n \to \sin(\kappa_n^\pm z) \) as \( x \to \pm \infty \), we obtain
\[ \forall n \in \{1, 2, \ldots\}, \quad \phi_n = c_n \exp(\pm ik_n^\pm z), \quad \text{as} \quad x \to \pm \infty. \]
and therefore
\[ \partial_x \phi_n \pm ik_n^\pm \phi_n = 0, \quad \text{as} \quad x \to \pm \infty. \quad (45) \]
Thus, the IT generation problem has been exactly reformulated as the CMS (43) together with the radiation conditions (45). A numerical method for its solution is presented in the following subsection.

### 4.2 Numerical solution

In order to solve numerically the infinite CMS Eqs. (43)-(45), a finite number \( M \) is chosen as the truncation order and the infinite domain is replaced by a finite interval \([x_L, x_R]\). In practice, \( x_L, x_R \) are chosen so that \( h_x(x_L), h_x(x_R) \) are negligible. The truncated CMS reads
\[ \sum_{n=1}^M A_{mn}\phi_{n,x} + B_{mn}\phi_{n,x} + C_{mn}\phi_n = F_m, \quad x \in [x_L, x_R], \quad (46a) \]
\[ \phi_{m,x} + ik_m\phi_m = 0, \quad \text{at} \quad x = x_L, \quad (46b) \]
\[ \phi_{m,x} - ik_m\phi_m = 0, \quad \text{at} \quad x = x_R. \quad (46c) \]
with \( m = 1, \ldots, M \). For the numerical solution of the above problem, we discretize the interval \([x_L, x_R]\) using a uniform grid of spacing \( \delta x \), \( \{x_i, \quad i = 1, N_X\} \), and compute the grid values \( \phi_n^i \) that approximate the solution \( \phi_n(x_i) \). The derivatives in (46) are approximated with 4th-order central finite differences except
for the boundary points and their neighboring points where one-sided and asymmetric finite-differences are used in order to preserve the 4-th-order accuracy. The corresponding formulae can be found in (Papoutsellis et al. 2019, Appendix C). Using these expressions, we obtain a sparse square linear system of dimension \((N X M)^2\) for the determination of \(\{ \phi_n^i, n \in [1, M], i \in [1, N X] \}\). The system’s matrix is comprised by \(M^2\) blocks of dimension \(N^2 X\) and every block has an almost five-diagonal structure. Its solution is then obtained by means of a \(LU\) decomposition.

Once the solution is found, the baroclinic fields \(u^#, v^#, w^#\) and \(b^#\) are constructed using a truncated version of (31). Similarly, from Eq. (37) the conversion rate is computed by

\[
C_{\pm} = \frac{N^2 - \omega^2}{2\omega} \frac{h_0 - h_{\pm}}{2} \sum_{n=1}^{M} \Im \left[ \phi_n \overline{\phi}_{n,x} \right]_{x=x_R,x_L}.
\]

Finally, we also provide here an estimation of the free-surface elevation due to the internal-tide motion given by \(\eta = [p^#]_{z=0}/g\), where \([p^#]_{z=0}\) is the pressure induced on \(z = 0\) by the baroclinic flow. Details are given in Appendix C.

5 Cases studied and numerical convergence

5.1 Topographic profiles and numerical values

In the following section, we calculate ITs for three bathymetries, namely the “witch of Agnesi”, the “Gaussian” and what we will call the “bump”, which are given by

\[
h_W = \frac{\Lambda}{1 + \frac{x^2}{L^2}}, \quad h_G = \Lambda \exp \left(-\frac{x^2}{2L^2}\right), \quad h_B = \Lambda \exp \left(1 - \frac{1}{1 + \frac{x^2}{L^2}}\right) \mathbb{1}_{(-L,L)},
\]

with \(\mathbb{1}_{(-L,L)} = 1\) in \((-L,L)\) and \(\mathbb{1}_{(-L,L)} = 0\) otherwise. Note that the “bump” profile has a compact support in contrast with the other two profiles that tend to zero at infinity; see figure 2. We shall also consider the case of a shelf connecting two different constant depths. The shelf profile is the same as in Griffiths & Grimshaw (2007, Section 5), hereafter referred to as GG07:

\[
h = \begin{cases} 
  h_-, & x \leq 0 \\
  h_- + (h_+ - h_-) \sin^2(\pi x/2L), & 0 \leq x \leq L \\
  h_+, & x \geq L 
\end{cases}
\]

We recall from subsection 22.1 that two important parameters will be considered, namely, the criticality of the bottom slope and the relative topography height

\[
\varepsilon = \frac{\max\{|h_x|\}}{\tan \theta} \quad \text{and} \quad \delta = \frac{\max\{h\}}{h_0}.
\]

We also recall that, for illustration purposes, \((\omega, U_0)\) correspond to a typical \(M_2\) tide, i.e., \((\omega, U_0) = (1.4 \times 10^{-4} \text{ s}^{-1}, 0.04 \text{ m s}^{-1})\) and that the Brunt-Väisälä and Coriolis frequencies are \(N = 0.0015 \text{ s}^{-1}\) and \(f = 10^{-4} \text{ s}^{-1}\) respectively. For all seamounts, the depth far from the topography is \(h_0 = 3000 \text{ m}\).
5.2 Truncation and numerical convergence

We first examine the rate of decay of the modal amplitudes with respect to mode number obtained by solving the proposed CMS (46). For this calculation, the “bump” profile is considered with $\lambda = 1500$ m ($\delta = 0.5$) and $\varepsilon = 0.9$, 1 and 1.5. Results on the uniform norm $\| \cdot \|_\infty$ of $\phi_n$, $\phi_{n,x}$ and $\phi_{n,xx}$ are shown in the first row of figure 3. For $\varepsilon = 0.9$, $\| \phi_n \|_\infty$ decays like $n^{-4}$ with $s \approx 3$ for $n > 10$ while $s \approx 3.8$ and 4.0 for $\phi_{n,x}$ and $\phi_{n,xx}$ in the same range. Similar decay rates are observed for $\varepsilon = 1$, though in this case they take effect after $n > 30$. For $\varepsilon = 1.5$, $\| \phi_n \|_\infty$ decays more slowly, reaching a minimum at $n_{\text{min}} = 66$. In this case, a linear fit gives $\| \phi_n \|_\infty \sim n^{-5}$ with $s \approx 1.5$ for $n \in [30, n_{\text{min}}]$. The first derivative decays with a smaller rate, $s \approx 0.5$ for $n \in [30, n_{\text{min}}]$, while the second derivative clearly diverges. This change in the decay rate is related with the formation of singularities for $\varepsilon > 1$ and is also reported in the modal solution of GG07. In fact, as shown in Section 44.1, $\| \phi_n \| = O(n^{-2})$ if $\phi^i$ is smooth, in accordance with our numerical calculations when $\varepsilon < 1$. On the other hand, a decay rate $n^{-s}$, with $s = 3/2$ suggests a formation of a singularity in $z$ takes place (Salem 1939, Raisbeck 1955).

For all the considered topographies in this paper, the rapid decay rate $n^{-s}$, with $s \approx 3$, of $\| \phi_n \|_\infty$ holds for $\varepsilon \leq 1$ and $\delta > 0.1$. We observe that for smaller $\delta$ and sufficiently large $\varepsilon$ ($\varepsilon \gtrsim 0.2$), the decay rate approaches the one obtained with $\varepsilon > 1$, namely, $s \approx 1.5$, showing that the solution becomes singular. In this case. For $\varepsilon > 1$, the slow convergence $s \approx 1.5$ holds for all $\delta$.

Next, in order to assess the accuracy of the CMS approach and the numerical scheme, we compare the energy radiated away from the topography with the energy conversion in the entire domain. To do so, we solve Eqs. (46) for a wide range of the parameters $(M, \delta x)$ and we monitor the decay of the corresponding absolute error of the averaged energy equation Eq. (34),

$$\text{Err.} = \left| -C_- + C_+ - \int_\Omega \left( B^{(0)} w^1 - W_{\tau}^{(0)} w^1 \right) d\Omega \right|,$$

(50)

where $C_\pm$ are given by Eq. (37). We use $M = 10$ to 120 modes and choose to resolve each case with $\delta x = L_M/r$ where $L_M$ is the horizontal wavelength of the $M^{\text{th}}$ free internal wave mode over the summit of the seamount, i.e. $L_M = 2\mu(h_0 - \Lambda)/M$, and $r = 4, 6, 8, 10$ and 12. Results for $h = h_0(x)$ with $h_0 = 1500$ m ($\delta = 0.5$), and $\varepsilon = 0.9$, 1 and 1.5 are shown in figure 3. The expected $r^{-4}$ rate is obtained for all $\varepsilon$ verifying the fourth-order accuracy of the present spatial discretisation scheme. For $\varepsilon = 0.9$ and $\varepsilon = 1$, the error decays rapidly as $M^{-4}$ and $M^{-3.5}$ respectively. For $\varepsilon = 1.5$, the error decays as $M^{-1}$ showing the slow convergence of the modal solution when the underlying field becomes singular.

At this point, some general remarks are in order. For the subsequent analysis, the calculations are performed with $M = 50$ and $M = 100$ when $s \approx 3$ and $s \approx 1.5$ respectively, while $\delta x$ is at least six times smaller than $L_M$ in order for the last calculated mode to be sufficiently resolved. After extensive numerical investigation, we have observed that increasing the resolution $(M, r)$ further than the aforementioned values does not significantly change the energy flux nor the first thirty modes, thus these values may be considered a good compromise between efficiency and accuracy. It must be stressed that our rates of decay are consistent with the $n^{-2}$ and $n^{-1/2}$ rates reported by GG07 for the modal amplitudes of the horizontal velocity; indeed, with our notation, $u_n \sim \phi_n \kappa_n \cos(\kappa_n z) \sim \phi_n n \cos(\kappa_n z)$. Nevertheless, if 2D solutions are sought, it is more efficient to work with the stream function formulation.

5.3 Examples of flow fields

In the remainder of this section, we consider converged solutions and the corresponding magnitude of the baroclinic velocity field $|u|^\# = \sqrt{(u^\#)^2 + (v^\#)^2 + (w^\#)^2}$, calculated by Eq. (31) and normalized by $U_0$, isodensity curves and free-surface perturbation calculated by Eq. (81). In figure 4, we plot the results for two characteristic instants at $t = 0$ and $t = T/4$, for the bump profile introduced above. We describe below the main features of the obtained solutions. The full evolution can be found in the supplemental material.

For $\varepsilon = 0.9$, the isodensity curves have a smooth wavelike form away from the seamount and follow the two low-amplitude beams which are visible in the velocity field. Over the seamount, the isopycnals are more perturbed and the beams emanating from the sides are clearly visible. The free-surface perturbation is a smooth, low-amplitude wave except over the seamount where it is steeper.

For $\varepsilon = 1$, a single beam is clearly formed over the seamount and the isodensity curves assume a cusp-like structure. This beam connects the two points of the topography for which the slope becomes equal to $1/\mu$. The free-surface has and has a larger wave height in comparison with the previous case.
Figure 3: (a)-(c): Decay of the uniform norms of the modal amplitudes, $\|\phi_n\|_\infty$, $\|\phi_{n,x}\|_\infty$, $\|\phi_{n,xx}\|_\infty$, for the case of a bump topography with $\delta = 0.5$ and $\varepsilon = 0.9, 1, 1.5$. (d)-(f): Absolute error $\delta_{50}$ as a function of $(M,r)$.

For $\varepsilon = 1.5$, two fully developed beams are visible throughout the entire domain. They emanate from the sides of the seamount, and they intersect over its summit. The isopycnals have a cusp-like form for one beam and a step-like form for the other, and the free-surface follows a similar trend. Their vanishing widths and diverging amplitudes are physical manifestations of the singularity discussed in the previous subsection.

Results concerning the shelf profile with $h_-/h_+ = 0.5$ and $\varepsilon = 2$ are shown in figure 5. Fully developed beams are also visible. The shoreward beams appear more intense in comparison with the oceanward ones. This calculation is qualitatively similar with the one found in figure 6 of GG07. The free-surface perturbation alternates between singular cusp-like forms and smooth steep forms, assuming larger amplitudes towards the deeper direction.

It must be noted that for $\varepsilon > 1$, small-amplitude high-frequency oscillations are visible. These are attributed to the Gibbs-like phenomenon occurring due to the formation of singularities in the solution and are also present in other supercritical solutions (GG07; Echeverri & Peacock 2010). Following GG07, we attenuate this phenomenon when plotting the fields by adjusting the Fourier-like summation (38), after the solution of the CMS, with the multiplication of a filter function of the form $\sin(n\pi/M)M/(n\pi)$ when $\varepsilon > 1$. Finally, we also note that the cusp-like features of the isopycnal contours as $\varepsilon$ increases physically correspond to statically unstable density profiles, which would be regularized in the ocean via gravitational or shear instabilities. This is however outside the scope of this article.

6 Conversion rates and comparison with previous approaches

6.1 Energy conversion rate

We first examine the far-field energy flux or energy conversion rate for the topographies presented in section 5. Our primary objective is to compare our calculations with existing analytical predictions, or past numerical computations. We also provide quantitative estimates of the limits of applicability for
Figure 4: Baroclinic velocity amplitude $|u^*|/U_0$, isodensity curves (full black lines) and free surface perturbation $2.5 \times 10^3 \times \eta$ (magenta lines) at $t = 0, T/4$ for the bump profile with $\delta = 0.5$ and $\varepsilon = 0.9, 1, 1.5$ ($L \approx 55, 49, 32$ km). The dashed black lines correspond to the rigid lid. The vertical magenta lines corresponds to a 2 cm amplitude.
analytical results, as well as a parametric formula for estimating energy flux radiated by supercritical topographies.

### 6.1.1 Subcritical seamounts

We examine the subcritical ($\varepsilon < 1$) and critical ($\varepsilon = 1$) cases first. In the regime $\varepsilon \ll 1$ and $\delta \ll 1$ (vanishing topographic slope and height), the WTA may be combined with a uniform barotropic tidal flow along the fluid domain. Using these assumptions together with the hydrostatic approximation, Llewellyn Smith & Young (2002) derived a simple expression for the far-field energy flux in terms of the topography and the physical parameters of the problem. In the two-dimensional non-hydrostatic case, it is written as (St. Laurent et al. 2003)

$$\mathcal{C}[h] = F_0 \sum_{n=1}^{\infty} \kappa_n \hat{h}(k_n)\hat{\eta}(k_n)\delta k,$$

with $F_0 = \frac{1}{2\pi} \frac{\left[ (N^2 - \omega^2)(\omega^2 - f^2) \right]^{1/2}}{\omega} U^2 h_0^3$, (51)

where $\hat{h}(\xi) = \int_{-\infty}^{\infty} \exp(-ix\xi)h(x)dx$ is the Fourier transform of $h$, $k_n = n\pi/(\mu h_0)$ and $\delta k = k_n/n = \pi/(\mu h_0)$. From Eq. (51) with $h = h_W$ and $h = h_G$, we obtain

$$\mathcal{C}[h_W] = F_0 \frac{\pi^2}{4} \delta^2 \frac{c^2 e^{-c}}{(1-e^{-c})^2}, \quad \mathcal{C}[h_G] = 2\pi F_0 \delta^2 \left( \frac{L_\pi}{h_0 \mu} \right)^2 \sum_{n=1}^{\infty} n \exp \left( -\frac{n\pi}{\mu h_0} \right)^2$$

with $c = (3^{3/2}/4)(h_0 \delta/2L)$. For $h = h_B$, we calculated $\mathcal{C}[h_B]$ numerically using the Gauss-Kronrod quadrature. The total conversion rate in Eq. (47) calculated by means of the present CMS for $\varepsilon = 0.1$, 0.5 and 1, and $\delta \in [0.05, 9]$, is compared with the predictions of Eq. (51) in figure 6.

Considering first the “witch of Agnesi” and “Gaussian” profiles in figure 6(a) and (b), we observe that for every $\varepsilon$, both the calculated and analytical conversion rates increase proportionally to $\delta^2$ up to a global maximum value attained at a value $\delta_c(\varepsilon)$. The values of the maximum conversion rate and of $\delta_c$ both increase with $\varepsilon$. The WTA predictions based on Eq. (51) are in adequate agreement with our calculations only for small values of $\delta$. Arguably the most striking difference between our semi-analytical calculation and the analytical prediction is the accurate prediction, in our case, of “non-radiating topographies” (Pétrélis et al. 2006, Maas 2011). These topographies manifest themselves as abrupt drops in the conversions rate for discrete values of $\delta$, up to $\delta \approx 1$, which indicate a sharp transition from weakly radiating to non-radiating. We must note however that while the phenomenon happens for all topographies we tested, the conversion rate never strictly drops to zero (due to numerical uncertainties). In the case of the bump profile (figure 6(c)), oscillating patterns are more regular and a similar mismatch with analytical predictions occurs as $\delta$ increases, but some local minima cannot be considered as zeros and a small amount of energy escapes to infinity.

A more thorough comparison of our calculations with WTA predictions as a function of $\delta$ and $\varepsilon$ is shown in the left panels of figure 8 by mapping the quantity $E_\gamma = [(C - \mathcal{C}[h_\gamma])/C]$, $\gamma = W, G, B$, for

![Figure 5: Baroclinic velocity amplitude $|u^\theta|/U_0$, isodensity curves (full black lines) and free surface perturbation $2.5 \times 10^3 \times \eta$ (magenta lines) at $t = 0, T/4$ for the shelf profile with $h_-/h_+ = 0.5$ and $\varepsilon = 2$. The dashed black lines correspond to the rigid lid. The vertical magenta line corresponds to a 2 cm amplitude.](image)
Figure 6: Energy conversion rate as a function of $\delta$ for the (a) witch of Agnesi, (b) Gaussian and (c) bump profiles with $\varepsilon = 0.1, 0.5, 1$. Full lines correspond to the calculated conversion rates by the present method and dashed lines to the predictions by Eq. (51). Their corresponding maxima are shown with full circles and crosses.

the three different topographies given in Eq. (48). A visual indication (black dashed lines) indicates our suggestion for the limit of applicability of analytical predictions when the differences between our computations and predictions are greater than 100%. Eq. (51) should be used only for values of $\delta$ smaller than $\alpha \varepsilon$ with $\alpha$ being 0.8 for the “witch”, 0.7 for the “Gaussian”, and 0.4 for the “bump” profiles.

6.1.2 Supercritical seamounts

We now turn our attention to supercritical seamounts ($\varepsilon > 1$). In this case, an analytical formula for seamounts of general smooth shape is lacking. Instead, we consider here the energy conversion rate due to knife-edge ridges, i.e., finite-height, zero-width vertical barriers. Indeed, using a Green’s function approach, Llewellyn Smith & Young (2003) derived an expression for the energy flux, namely, (see also Garrett & Kunze 2007)

$$C_{LSY}^{\text{knife}} = 2\pi F_0 \int_0^\delta Z \left(1 - \cos \pi Z \cos \pi \delta\right)^{1/2} dZ.$$  \hspace{1cm} (53)

where the integral in Eq. (53) is computed here using the Gauss-Kronrod quadrature. At the same time, St. Laurent et al. (2003) using a matching of modal solutions at the knife edge, obtained

$$C_{\text{StL}}^{\text{knife}} = F_0 \sum_{n=1}^\infty \frac{1}{n^2 a_n^2},$$  \hspace{1cm} (54)

where $a_n$ are constant modal amplitudes obtained by the solution of a linear algebraic system. In order to compare our results with the above predictions, we calculate the energy flux for the three considered seamounts with $\varepsilon = 2, 5, 10$ and $\delta \in [0.1, 0.9]$.

Figure 7 shows that the conversion rate for supercritical topographies has a simpler dependence on $\delta$, in comparison with the subcritical case. It is in general increasing with $\delta$ with no points of low conversion rates. The two knife-edge predictions, Eqs. (53) and (54), differ slightly and give a more or less accurate prediction for $\varepsilon = 2$. For larger values of $\varepsilon$ our calculations differ significantly from the knife-edge predictions for all seamounts; for example, $C$ becomes up to four times as large as $C_{LSY}^{\text{knife}}$ for $\varepsilon = 10$ and $\delta > 0.5$. For the “bump” profile, a small hump in the conversion rate is observed for $\varepsilon = 2$ and $0.4 < \delta < 0.7$ and only weak changes in the slope of $C(\delta)$ are visible for $\varepsilon = 5$ and 10. It must be noted that similar observations for the supercritical radiated energy flux have been observed with other approaches for generation (Nycander 2006, Balmforth & Peacock 2009, Zhao et al. 2015) and scattering problems (Mathur et al. 2014).

We empirically define a parametric radiated energy flux $C_{\text{param}}$, prescribed as

$$C_{\text{param}}/F_0 = p_0 \varepsilon^{p_1} \delta^{p_2},$$  \hspace{1cm} (55)

for values of $\delta$ between 0 and 1 and $\varepsilon$ between 1.1 and 10. The values for the parameters $p_0$, $p_1$, $p_2$ are summarized in Table 2. The influence of the topographic height (via $\delta$) increases with a power law.
Figure 7: Energy conversion rates versus $\delta$ for the supercritical seamounts ($\varepsilon = 2, 5, 10$), for (a) witch of Agnesi, (b) Gaussian and (c) "bump". The full and pointed black lines correspond to the knife-edge predictions $C_{\text{knife}}^{\text{LSY}}$ and $C_{\text{knife}}^{\text{StL}}$, respectively.

| seamount | $p_0$   | $p_1$   | $p_2$   |
|----------|---------|---------|---------|
| witch    | $8.19 \times 10^3$ | 1.02    | 2.29    |
| Gaussian | $8.60 \times 10^3$ | 0.97    | 2.16    |
| bump     | $7.16 \times 10^3$ | 0.93    | 2.03    |
| mean     | $7.8 \times 10^3$  | 1.00    | 2.10    |

Table 2: Values for the parameters in eq. (55) for different topographies.

slightly larger than the WTA model ($p_2 > 2$), and the radiated energy flux is approximately a linear function of $\varepsilon$. Like before, we compare our calculations with this parametric prediction as a function of $\delta$ and $\varepsilon$ by mapping the quantity $E_\gamma = |(C - C_{\text{param}})/C|$ in the right panels of figure 8. On average, conversion rates for topographies with $\delta > 0.3$ are well prescribed by this parametric approach. Trends in the conversion rates are not adequately reproduced for $\varepsilon \gg 1$ and $\delta < 0.3$. Similar results are obtained with the parametric model using the values of the parameters for each topography or the mean values in Table 2, which are thus a convenient set of values for any type of seamount.

6.1.3 Shelf profile

In order to compare with the calculations of GG07, we take $h_-/h_+ = 0.5$ and calculate the shoreward and oceanward non-dimensional energy fluxes given by $C'_- = C_-/Q^2$ and $C'_+ = C_+/Q^2$ for $\varepsilon \in [0.1, 4]$. Our results are plotted in figure 9, together with results digitized from Fig. 1 of GG07. For $\varepsilon \lesssim 0.4$, our calculations are in very good agreement capturing the oscillating pattern of $C'_\pm$ and the appearance of local extrema. In the region $0.4 < \varepsilon < 0.8$, we obtain the same increasing pattern, with our calculations giving slightly smaller values. Differences are more significant for $\varepsilon > 0.8$. In our calculations, the shoreward and oceanward energy fluxes start to significantly diverge from each other at about $\varepsilon = 1.2$ while in GG07’s results this happens earlier, at about $\varepsilon = 0.8$. Right after these points, the shoreward energy flux for both methods attains a local maximum. In our calculation, the shoreward energy flux starts increasing after a while, whereas GG07’s calculation decreases. The oceanward energy flux increase rapidly in a small region ($\varepsilon \in [0.8, 1]$ for GG07 and $\varepsilon \in [1.1, 2]$ for the present CMS) and then follows an increasing pattern of smaller rate. This difference is possibly due to the hydrostatic approximation invoked in GG07 for the baroclinic equations, which is absent here.
6.2 Comparisons with the Green’s function method and discussion on non-radiating topographies

As stated before, semi-analytical models based on the Green’s function method consider a uniform barotropic flow (Pétrélis et al. 2006, Echeverri & Peacock 2010), whereas in the present body-forcing approach we consider a topographically adjusted barotropic flow. Therefore, it is natural to assess the effect of the body-forcing approach by comparing our calculations with the ones based on the Green’s function method, implemented here by means of the tool iTides (Mercier et al. 2012).

In figure 10(a), we plot the normalized baroclinic velocity amplitude $|\mathbf{u}^B|/U_0$ for a subcritical Gaussian topography with $\varepsilon = 0.5$ and $\delta = 0.3350$, for which a maximum energy flux is obtained. Differences are observed in the region over the summit of the seamount, with solution from iTides resulting in larger velocity amplitudes. In both calculations, we see a single wave beam on either side of the seamount that is of the same far-field velocity amplitude and parallel to characteristic lines. The difference between the calculated conversion rates is less than 2%.

The supercritical calculations ($\varepsilon = 1.5$, $\delta = 0.5$) shown in figure 10(b) feature stronger differences in the velocity fields. Our CMS calculation produces two beams, respectively initially upward and downward at the critical slope point, of clearly different far-field velocity amplitudes. On the other hand, iTides yields two beams of comparable far-field velocity amplitudes. In the CMS calculation, the beams emanating over the summit do not follow a straight path towards the sea surface and seabed. They slightly deviate from the characteristics and align with them only when $h_x$ becomes negligible. This peculiar observation is due to the spatial dependency of the eigenvectors $\tilde{\kappa}_n(x)$ in the modal decomposition of the CMS problem (see Eq. (40)). Indeed, the deviation from straight characteristics is more pronounced when $h_x$ is large. Consequently, beams in the two calculations follow a visibly different path away from the topography. Nevertheless, the difference between the calculated far-field energy fluxes does not exceed 10%.

For a non-radiating case ($\delta = 0.7577$, $\varepsilon = 0.5$) shown in figure 11, the CMS calculation yields a trapped baroclinic response, corresponding to a three-cell structure of the stream function in accordance with the results of Maas (2011, figure 3b). On the other hand, the response calculated by the Green’s function
method has a simpler shape, the corresponding velocity being almost constant along the $z$-direction. Surprisingly, the solution from iTides is non-radiating but cannot be seen as solely the baroclinic response; the wavefield shown resembles the adjusted hydrostatic barotropic field $\sqrt{W^{(0)} + V^{(0)} + U^{(0)}}$ of section 23.1 (not shown here). In both calculations, the normalized far-field energy flux is extremely small, of the order of $10^{-4}$, and the solutions are trapped around the topography.

More generally, our computations on the energy conversion rate have shown that all considered topographies exhibit non- or weakly-radiating behavior for discrete values of $\delta$, at a fixed value of $\varepsilon$. Such a behavior was noticed by Pétrélis et al. (2006) for the cases of a triangular and polynomial ridges. For these profiles, these so-called null points of the parameter space are already present within the WTA and their calculations show that they also exist for the complete problem. However, in the case of the asymptotically flat Gaussian and witch of Agnesi profiles, the WTA formula (51) does not predict null-points in contrast with our calculations. For the bump profile, the WTA formula yields points of low energy conversion rate (no null points) but, as we have shown, these are different from the solutions of our CMS. Maas (2011) constructed non-radiating topographies, arguing that the absence of tidal conversion may be common. He also predicted that non-radiation is only possible for subcritical topographies with sufficiently large values of $\delta$. Our calculations confirm these facts. It must be noticed that we did not find a valid approach to predict the values of the parameters $\varepsilon$ and $\delta$ for non-radiating cases, without solving the IT-generation problem. GG07 have discussed the weakly-radiating cases for the continental shelf in terms of an approximate phase variation of the first internal wave mode along the topography, but we tested their predictions with no success. The only robust result we observed is that a non-radiating topography is also a non-scattering one. Indeed, we can solve the problem of mode-1 scattering by a Gaussian non-radiating topography ($\delta = 0.7577$, $\varepsilon = 0.5$), and our computations have shown that the transmitted wave field is only a mode-1 of same amplitude, while the reflected field is zero.

Finally, it is worth noting that our CMS computations can treat domains of different depths at infinity (shelves) for which a treatment with Green’s function is currently lacking. The numerical efficiency has not been compared in detail with the tool iTides, but our implementation provides solutions faster.

7 Conclusions

A new semi-analytical model is developed that describes the generation of linear internal tides due to the interaction of the barotropic tide with arbitrary topography in two dimensions. The main novelty of the present approach is that it takes into account a barotropic flow that consistently varies over the bottom topography. The hydrodynamical problem is reformulated as a Coupled-Mode System (CMS) based on a local eigenfunction expansion of the unknown baroclinic response in terms of prescribed local vertical eigenfunctions and unknown horizontal modal amplitudes. Numerically solving the CMS yields the modal amplitudes, which are then used to determine the response fields and the energy conversion rate.
Figure 10: Gaussian profile with (a) $\varepsilon = 0.5$, $\delta = 0.3350$ and (b) $\varepsilon = 1.5$, $\delta = 0.5$. Left panel: Baroclinic velocity amplitude $|u^*|/U_0$ and isodensity curves calculated by the present CMS. Right panel: baroclinic velocity amplitude calculated by the iTides-Green’s function method. Characteristic lines are shown in orange.

A method to calculate the pressure-induced free-surface elevation is also proposed. The streamfunction formulation of our new CMS shows good convergence properties, being more efficient than approaches based on the velocity formulation, such as that of Griffiths & Grimshaw (2007).

We have compared our calculations with existing simplified predictions (WTA, knife-edge) of the radiated energy flux for three seamount profiles: two commonly used asymptotically flat topographies, the Gaussian and the witch of Agnesi, and a compactly supported one, the bump, studied here for the first time. The agreement between our calculations and analytical predictions is adequate provided $\delta$ is small enough for the WTA, and $\varepsilon$ close to 1 for the knife-edge approach. For $\varepsilon \leq 1$, our calculations show that there exist specific values of $\delta$ for which tidal conversion is either almost lacking for the Gaussian and witch of Agnesi profiles, or it is very weak for the bump profile. This effect is not captured by the WTA predictions. In the supercritical regime, for fixed $\varepsilon$ the energy flux is increasing with $\varepsilon$, in contrast with the knife-edge predictions. The poor agreement between analytical predictions and the present calculations suggests that the use of the former in global ocean models for the parameterization of energy conversion should be reviewed. For practical purposes, we propose a simplified dependence of the energy conversion rate in the supercritical regime, approximately proportional to $\varepsilon \delta^2$. We have also calculated the energy flux radiated due to a shelf profile. In this case, we have compared our calculations with the coupled-mode approach of Griffiths & Grimshaw (2007). We have observed strong differences between the two models, especially for $\varepsilon > 1$, which are likely due to the hydrostatic approximation invoked in the latter work. Another set of comparisons was performed with the Green’s function method, in which a uniform barotropic flow is considered. The CMS approach provides a more realistic flow above the topography, but yields more or less the same energy conversion rate as the Green’s function method. Nevertheless, our approach is more versatile in the sense that it can treat shelf-like domains for which a Green’s function tool such as iTides is currently lacking. The treatment of trenches is also possible.

The present CMS can be extended in several interesting directions. The assumption of constant stratification can be lifted, at the cost of using a set of local basis functions which are calculated numerically, given a smooth stratification $N(z)$. Similarly, the set of local basis functions can be modified in order to take into account the linearized free-surface condition, instead of using the rigid-lid approximation. In this case, the added complexity comes from the fact that the local wavenumbers and eigenfunctions
Figure 11: Gaussian profile with $\delta = 0.7577$ and $\varepsilon = 0.5$. (a) Baroclinic velocity amplitude $|u^\#|/U_0$ calculated by the present CMS. (b) Baroclinic velocity amplitude calculated by iTides (Green’s function method).
are determined in terms of a local transcendental equation (dispersion relation) (see e.g. Kelly 2016), which can nevertheless be solved efficiently (Papathanasiou et al. 2018). Finally, we note that following (Papoutsellis 2016, Chapter 2.1), it can be shown that the rapid convergence of the eigenfunction expansion remains after these extensions, thus we anticipate that realistic topographies and stratification will be efficiently taken into account. For realistic oceanic configurations, comparing satellite observations with calculated free-surface elevations will provide further validations of the method.

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Data statement

Data available upon request.

A Asymptotic analysis of the barotropic equations

Here, we derive an asymptotic solution of the barotropic flow problem introduced in §22.3. We start by introducing the dimensionless parameters

\[ \sigma = \frac{h_0^2}{L^2} \quad \text{and} \quad \beta = \frac{\Lambda}{h_0}, \]

where \( L \) and \( \Lambda \) are the horizontal and vertical scales of the topography, respectively. We shall treat first the BVP (17) by introducing the scaling,

\[ \tilde{z} = \frac{z}{h_0}, \quad \tilde{x} = \frac{x}{L}, \quad \tilde{h} = \frac{h}{\Lambda}, \quad \tilde{\Phi} = \frac{\Phi}{Q}, \]

see figure 1. With these definitions, the dimensionless version of the BVP (17) is written as follows, after dropping the tildes,

\[ \sigma \Phi_{xx} + \mu_0^{-2} \Phi_{zz} = 0, \quad \Phi(x, 0) = 0, \quad \Phi(x, -1 + \beta h) = 1. \] (58)

If \( \sigma \ll 1 \) and \( 0 < \beta < 1 \), an approximate solution of (58) may be written in the form of the asymptotic expansion

\[ \Phi^{\text{app}}(x, z) = \sum_{i=0}^{K} \sigma^i \Phi^{(i)}(x, z), \] (59)

for some order of truncation \( K \geq 0 \). Substitution of Eq. (59) in the first equation of (58) yields

\[ \sum_{j=0}^{K} \sigma^j \left( \Phi^{(j-1)}_{xx} + \mu_0^{-2} \Phi^{(j)}_{zz} \right) = O(\sigma^{K+1}). \] (60)

where the convention \( \Phi^{(-1)} = 0 \) is used. The residual \( O(\sigma^{K+1}) \) is canceled, if

\[ \Phi^{(j)}_{zz} = -\mu_0^2 \Phi^{(j-1)}_{xx}, \quad j = 0, 1, 2, ..., K. \]

Taking into account the boundary conditions in Eq. (58), the following recurrence relation can be formulated:

\[ (j = 0): \quad \Phi^{(0)}_{zz} = 0, \quad \Phi^{(0)}(x, 0) = 0, \quad \Phi^{(0)}(x, -1 + \beta h) = 1 \]
\[ (1 \leq j \leq K): \quad \Phi^{(j)}_{zz} = -\mu_0^2 \Phi^{(j-1)}_{xx}, \quad \Phi^{(j)}(x, 0) = 0, \quad \Phi^{(j)}(x, -1 + \beta h) = 0, \] (61)
or, equivalently,
\[
\Phi^{(0)} = \frac{z}{1 + \beta h},
\]
\[
\Phi^{(j)} = -\frac{z}{1 + \beta h} \mu_0^2 \int_{-1 + \beta h}^{0} \int_{-1 + \beta h}^{z'} \Phi_{xx}^{(j-1)} dz'' dz' + \mu_0^2 \int_{z}^{0} \int_{-1 + \beta h}^{z'} \Phi_{xx}^{(j-1)} dz'' dz'.
\] (62)

Performing the computation for \( j = 1 \) and using Eqs. (56) and (57) we readily arrive to the expressions in Eq. (19).

We proceed now with Eqs. (15)–(16). Introducing the scaling \( \hat{t} = \omega t \) and
\[
\hat{U} = \frac{U}{U_0}, \quad \hat{V} = \frac{V}{U_0}, \quad \hat{W} = \frac{W}{W_0}, \quad \hat{P} = \frac{P}{\omega U_0 L}, \quad \hat{B} = \frac{B}{g},
\] (63)
with \( U_0 = \sqrt{gh_0} \) and \( W_0 = U_0 h_0 / L \), Eqs. (15)–(16) become, after dropping the tildes,
\[
U_t - \frac{f}{\omega} V = -P_x, \quad V_t + \frac{f}{\omega} U = 0,
\] (64a)
\[
\sigma W_t = -P_x,
\] (64b)
\[
B_t + \frac{N^2}{\omega g} W = 0,
\] (65)
\[
U_x + W_z = 0,
\] (66)
\[
-\beta h_x U(x, -\beta h) + W(x, -\beta h) = 0, \quad W(x, 0) = 0.
\] (67)

Plugging the asymptotic expansion
\[
(U_{\text{app}}, V_{\text{app}}, W_{\text{app}}, B_{\text{app}}, P_{\text{app}}) = \sum_{i=0}^{K} \sigma^i \left( U^{(i)}, V^{(i)}, W^{(i)}, B^{(i)}, P^{(i)} \right),
\] (68)
into Eqs. (64a), (65)–(67) we obtain
\[
U_{t}^{(i)} - \frac{f}{\omega} V^{(i)} = -P_x^{(i)}, \quad V_{t}^{(i)} + \frac{f}{\omega} U^{(i)} = 0,
\] (69)
\[
B_{t}^{(i)} + \frac{N^2}{\omega g} W^{(i)} = 0,
\] (70)
\[
U_{x}^{(i)} + W_{z}^{(i)} = 0,
\] (71)
\[
-\beta h_x U^{(i)}(x, -\beta h) + W^{(i)}(x, -\beta h) = 0, \quad W^{(i)}(x, 0) = 0.
\] (72)

On the other hand, Eq. (64b) gives
\[
\sum_{i=0}^{K} \sigma^{i+1} W_{x}^{(i)} + \sigma^i P_{x}^{(i)} = 0,
\]
\[
\sum_{i=0}^{K-1} \sigma^{i+1} W_{x}^{(i)} + \sum_{i=0}^{K} \sigma^i P_{x}^{(i)} = O(\sigma^{K+1}).
\]

Using the convention \( W^{(-1)} = 0 \), the latter equation is written (cf. Eq. (60))
\[
\sum_{i=0}^{K} \sigma^i \left( W_{t}^{(i-1)} + P_{z}^{(i)} \right) = O(\sigma^{K+1}),
\]
therefore the residual \( O(\sigma^{K+1}) \) is canceled if
\[
W_{t}^{(i-1)} + P_{z}^{(i)} = 0.
\] (73)
Then, Eqs. (69)–(73) in dimensional form read

\[ U_{t}^{(i)} - fV^{(i)} = -P_{x}^{(i)}, \quad V_{t}^{(i)} + fU^{(i)} = 0, \]
\[ W_{t}^{(i+1)} = -P_{z}^{(i)}, \] (74a)
\[ B_{x}^{(i)} + N^2W^{(i)} = 0, \]
\[ U_{x}^{(i)} + W_{z}^{(i)} = 0, \] (75)
\[ -h_x U^{(i)}(x, -h_0 + h) + W^{(i)}(x, -h_0 + h) = 0, \quad W^{(i)}(x, 0) = 0. \] (76)

For \( i = 0 \), Eq. (74b) is \( P_{x}^{(0)} = 0 \), thus, at leading order the flow is hydrostatic. By definition of the stream function we obtain Eqs. (20). Finally, integrating the second equation of (74a) and Eq. (75) in time by using Eq. (20) and assuming that \( V(x, z, 0) = B(x, z, 0) = 0 \), we obtain Eqs. (21).

### B Matrix coefficients of the CMS

The matrix coefficients \( A_{mn} = A_{mn}(h), B_{mn} = B_{mn}(h), C_{mn} = C_{mn}(h) \) and the sequence \( F_m = F_m(h) \) of the CMS (43) are defined by

\[ A_{mn} = \int _{-h_0 + h}^{0} Z_n Z_m dz, \] (78a)
\[ B_{mn} = 2 \int _{-h_0 + h}^{0} Z_n x Z_m dz, \] (78b)
\[ C_{mn} = \int _{-h_0 + h}^{0} \left( Z_{n,xx} - \frac{1}{\mu^2} Z_{n,zz} \right) Z_m dz, \] (78c)
\[ F_m = Q \left( \frac{1}{h_0 - h} \right) \frac{1}{z} \int _{-h_0 + h}^{0} z Z_m dz. \] (78d)

Performing the calculations involved in (78) using Eq. (40) we obtain

\[ A_{mn} = \begin{cases} 0, & m \neq n \\ \frac{1}{2} (h_0 - h), & m = n \end{cases} \] (79a)
\[ B_{mn} = \begin{cases} 2 \frac{(-1)^{m+n} mn}{(m+n)(-m+n)} h_x, & m \neq n \\ -\frac{1}{2} h_x, & m = n \end{cases} \] (79b)
\[ C_{mn} = \begin{cases} -2(1)^{m+n} mn (m^2 + n^2) h_x^2 - \frac{(-1)^{m+n} mn (m^2 - n^2)}{(m-n)^2 (m+n)^2} h_{xx}, & m \neq n \\ (3 + 2n^2 \pi^2) h_x^2 - \frac{1}{12} \frac{h_x}{h_0 - h} + \frac{1}{\mu^2} \frac{n^2 \pi^2}{2(h_0 - h)}, & m = n \end{cases} \] (79c)
\[ F_m = Q \left( \frac{1}{h_0 - h} \right) \frac{2(-1)^{m} (h_0 - h)}{m \pi}. \] (79d)

### C Modal reconstruction of free-surface pressure and elevation

The proper calculation of the free-surface perturbation induced by the internal tides would require the replacement of the rigid-lid approximation, invoked in this work, by the linearized kinematic surface condition, \( \eta_t = w(x, 0) \), \( \eta \) being the free-surface elevation. For consistency reasons, this would also require the use of different basis functions, similar to the ones used, for example, in (Griffiths & Grimshaw 2007) or (Kelly 2016). However, in the present context an estimation of the free-surface is still possible since
the baroclinic surface pressure on \( z = 0 \), \( [p^#]_{z=0} \), is non-zero. This is done here by first evaluating \( p^#_x \) and \( p^#_{xx} \) on \( z = 0 \),

\[
[p^#_x]_{z=0} = - \left( [u^#]_{z=0} - f [v^#]_{z=0} \right) = \left( \frac{\omega^2 - f^2}{\omega} \right) \exists \left\{ \left[ \phi^#_x \right]_{z=0} e^{-i\omega t} \right\} := g(x,t). \quad (80a)
\]

\[
[p^#_{xx}]_{z=0} = - \left( [u^{#xx}]_{z=0} - f [v^#]_{z=0} \right) = \left( \frac{\omega^2 - f^2}{\omega} \right) \exists \left\{ \left[ \phi^#_{xx} \right]_{z=0} e^{-i\omega t} \right\} := F(x,t), \quad (80b)
\]

and solving numerically the problem

\[
[p^#_{xx}]_{z=0} = F(x,t), \quad p^#_x(0,x_L) = g(x_L,t), \quad p^#_x(0,x_R) = g(x_R,t). \quad (81)
\]

Then, the induced surface elevation is reconstructed by \( \eta = [p^#]_{z=0}/g \).

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