TRANSFORMATION FORMULAS FOR THE HIGHER POWER OF ODD ZETA VALUES AND GENERALIZED EISENSTEIN SERIES

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ABSTRACT. In this article, we obtain a transformation formula for the higher power of odd zeta values, which generalizes Ramanujan’s formula for odd zeta values. We have also investigated many important applications, which in turn provide generalizations of the transformation formula of the Eisenstein series, Dedekind eta function etc.

1. INTRODUCTION

The Riemann zeta function \( \zeta(s) \) is a central object in analytic number theory to study the distribution of primes and has applications in physics, probability theory, and applied statistics. Over the years, the special values of \( \zeta(s) \) and its arithmetic behaviour have drawn the attention of many mathematicians. The story begins with a remarkable discovery of Euler in 1734 about determining the special values of \( \zeta(s) \) at positive even integral arguments. Euler’s formula states that for an odd natural number \( m \),

\[
\zeta(2m) = (-1)^{m+1} \frac{(2\pi)^{2m} B_{2m}}{2(2m)!}.
\]

where \( B_m \) denotes the \( m \)-th Bernoulli number. The above formula readily implies that all even zeta values \( \zeta(2m) \) with \( m \in \mathbb{N} \), are transcendental due to the well-known fact that \( \pi \) is transcendental and \( B_m \) is rational.

The arithmetic nature of zeta values at odd integral arguments is still far from being known. The irrationality of \( \zeta(3) \) was established by Apéry \[1, 2\]. Recently, Zudilin \[25\] has shown that at least one of the four numbers \( \zeta(5), \zeta(7), \zeta(9), \zeta(11) \) is irrational.

In this context, Ramanujan \[20\] p. 173, Ch. 14, Entry 21(i)] obtained an elegant identity for odd zeta values, which precisely states that for any non-zero integer \( m \) and \( \alpha, \beta > 0 \) with \( \alpha \beta = \pi^2 \),

\[
\alpha^{-m} \left\{ \frac{1}{2} \zeta(2m+1) + \sum_{n=1}^{\infty} \frac{n^{-2m-1}}{e^{2\pi n \alpha} - 1} \right\} = (-\beta)^{-m} \left\{ \frac{1}{2} \zeta(2m+1) + \sum_{n=1}^{\infty} \frac{n^{-2m-1}}{e^{2\pi n \beta} - 1} \right\} - 2^{2m} \sum_{j=0}^{m} \frac{(-1)^j B_{2j} B_{2m+2-2j}}{(2j)! (2m+2-2j)!} \alpha^{m-j} \beta^j.
\]

As an immediate application of the above identity, one can consider \( \alpha = \beta = \pi \) and \( m \) to be odd in \( (1.2) \) to obtain Lerch’s identity \[14\], given by,

\[
\zeta(2m+1) + 2 \sum_{n=1}^{\infty} \frac{1}{n^{2m+1} (e^{2\pi n} - 1)} = 2^{2m} \pi^{2m+1} \sum_{j=0}^{m} \frac{(-1)^j B_{2j} B_{2m+2-2j}}{(2j)! (2m+2-2j)!}.
\]

Thus it follows from the above identity that for \( m \) odd, at least one of \( \zeta(2m+1) \) and \( \sum_{n=1}^{\infty} \frac{1}{n^{2m+1} (e^{2\pi n} - 1)} \) is transcendental.

In \( (1.2) \), Ramanujan expressed \( \zeta(2m+1) \) in terms of the Lambert series

\[
\sum_{n=1}^{\infty} \frac{n^a}{e^{ny} - 1} = \sum_{n=1}^{\infty} \sigma_a(n) e^{-ny}
\]

with \( a = -2m-1 \) and \( \text{Re}(y) > 0 \). Here \( \sigma_a(n) \) denotes the general divisor function defined by \( \sigma_a(n) := \sum_{d|n} d^a \).

For \( a = 2m-1 \) with \( m \in \mathbb{N} \) and \( y = -2\pi iz \) with \( z \in \mathbb{H} \), the upper half-plane, either of the above series essentially
represents the Eisenstein series $E_{2m}$ of weight $2m$ on the full modular group $SL_2(\mathbb{Z})$. The transformation formula satisfied by $E_{2m}$ with $m > 1$ over $SL_2(\mathbb{Z})$ under the transformation $z \mapsto -\frac{1}{z}$ are namely, for $\alpha, \beta > 0$ with $\alpha \beta = \pi^2$,

$$
\alpha^m \sum_{n=1}^{\infty} \frac{n^{2m-1}}{e^{2\pi n^2} - 1} - (-\beta)^m \sum_{n=1}^{\infty} \frac{n^{2m-1}}{e^{2\pi n^2} - 1} = (\alpha^m - (-\beta)^m) \frac{B_{2m}}{4m}.
$$

(1.5)

The series $\sum_{n=1}^{\infty} \sigma_1(n)e^{2\pi inz}$ is essentially the quasi-modular form $E_2(z)$, that is, the Eisenstein series of weight 2 over $SL_2(\mathbb{Z})$. The transformation formula of $E_2(z)$ under the transformation $z \mapsto -\frac{1}{z}$ are namely, for $\alpha, \beta > 0$ with $\alpha \beta = \pi^2$,

$$
\alpha \sum_{n=1}^{\infty} \frac{n}{e^{2\pi n} - 1} + \beta \sum_{n=1}^{\infty} \frac{n}{e^{2\pi n} - 1} = \frac{\alpha + \beta}{24} - \frac{1}{4}.
$$

(1.6)

When $a = -2m + 1$, the above series represents the Eichler integral corresponding to the weight $2m$ Eisenstein series (cf. [6, Section 5]). Thus, Ramanujan’s identity (1.2) encapsulates fundamental modular properties of even integral weight Eisenstein series over the full modular group and their Eichler integrals. The identity (1.2) also has an application in theoretical computer science [12], in particular, in the analysis of special data structures and algorithms.

Malurkar [15] first independently obtained the proof of (1.2). Later, the formula was rediscovered by Grosswald [10] [11], where he studied it more generally. Both Euler’s formula (1.1) and Ramanujan’s formula (1.2) follow as a special case of Berndt’s general transformation formula [5, Theorem 2.2], which in turn shows that Euler’s and Ramanujan’s formulas are natural companions of each other. Recently, O’Sullivan [21, Theorem 1.3] studied non-holomorphic analogues of the formulas of Ramanujan, Grosswald, and Berndt containing Eichler integrals of holomorphic Eisenstein series.

Let $d(n) := \sum_{d|n} 1$ be the divisor function. The Dirichlet series associated to the divisor function $d(n)$ is precisely $\sum_{n=1}^{\infty} d(n)n^{-s} = \zeta^2(s)$, which was studied by numerous mathematicians in different directions (cf. [9], [10], [23]) from the point of view of analytic number theory.

The formula for $\zeta^2(2m)$ can be obtained by squaring both sides of (1.1), but the resulting formula for $\zeta^2(2m+1)$ after squaring both sides of (1.2) is extremely intricate and could not be simplified further. Recently, Dixit and Gupta [8, Theorem 2.1] established a transformation formula for $\zeta^2(2m+1), m \in \mathbb{Z} \setminus \{0\}$, which can be considered as an analogue of Ramanujan’s identity (1.2).

Koshliakov [13] studied a function, namely,

$$
\Omega(x) := 2 \sum_{j=1}^{\infty} d(j) \left( K_0 \left( 4\pi\epsilon \sqrt{jx} \right) + K_0 \left( 4\pi\Gamma \sqrt{jx} \right) \right),
$$

(1.7)

where $\epsilon = \exp\left(\frac{1}{4}\right)$ and $K_{\frac{1}{2}}(x)$ denotes the modified Bessel function of the second kind of order $\frac{1}{2}$ (cf. [24, p. 78]), which is defined later in [22]. In the same article, Koshliakov established two beautiful identities [13] (Equations (27), (29)), which derives an infinite series involving $\Omega(x)$ as follows:

$$
\sum_{n=1}^{\infty} n^{4m+1} d(n)\Omega(n) = \frac{B_{2m+2}}{(4m+2)^2} \left[ \log(2\pi) - \sum_{k=1}^{4m+1} \frac{1}{k} - \frac{\zeta'(4m+2)}{\zeta(4m+2)} \right]
$$

and

$$
\sum_{n=1}^{\infty} nd(n)\Omega(n) = \frac{1}{144} \left[ \log(2\pi) - 1 - \frac{6}{\pi^2} \zeta'(2) \right] - \frac{1}{32\pi}.
$$

In [8], Dixit and Gupta generalized the function $\Omega(x)$ in (1.7) by introducing a new parameter $\rho$ as

$$
\Omega_{\rho}(x) := 2 \sum_{j=1}^{\infty} d(j) \left( K_0 \left( 4\rho\epsilon \sqrt{jx} \right) + K_0 \left( 4\rho\Gamma \sqrt{jx} \right) \right)
$$

(1.8)

such that $\Omega_{\pi}(x) = \Omega(x)$ and obtained the transformation formula for $\zeta^2(2m+1)$, which precisely states that for any non-zero integer $m$ and $\alpha, \beta > 0$ with $\alpha \beta = \pi^2$,

$$
\alpha^{-2m} \left[ \zeta^2(2m+1) \left( \gamma + \log \left( \frac{\alpha}{\pi} \right) - \frac{\zeta'(2m+1)}{\zeta(2m+1)} \right) + \sum_{n=1}^{\infty} \frac{d(n)\Omega_{\alpha}(n)}{n^{2m+1}} \right]
$$

where $\gamma$ is the Euler-Mascheroni constant.
\[
= (-1)^{-m} \beta^{-2m} \left[ \zeta^2(2m + 1) \left( \gamma + \log \left( \frac{\beta}{\pi} \right) - \frac{\zeta'(2m + 1)}{\zeta(2m + 1)} \right) + \sum_{n=1}^{\infty} \frac{d(n)\Omega_{\beta}(n)}{n^{2m+1}} \right] \\
- 2^{4m} \pi \sum_{j=0}^{m+1} \frac{(-1)^j B^2_{2j} B^2_{2m+2-2j}}{(2j)!^2 (2m+2-2j)!^2} \alpha^2 j \beta^{2m+2-2j}. \tag{1.9}
\]

Let \( d_k(n) \) denotes the number of ways \( n \) can be written as \( k \) given factors, which is sometimes known as Piltz divisor function. We define an infinite series \( \Psi_{\rho,k}(x) \) involving the Meijer G-function (cf. \(2.3\)). This series includes the Lambert series \((1.4)\) that appeared in Ramanujan’s formula \((1.2)\), and the generalized Koshliakov’s function \((1.8)\) appeared in the identity \((1.9)\) of Dixit and Gupta as a special case. For \( \rho > 0 \), let \( \Psi_{\rho,k}(x) \) be defined as

\[
\Psi_{\rho,k}(x) := \frac{\pi^{\rho k/2-1}}{2^{k-1}} \sum_{j=1}^{\infty} d_k(j) G^{k+1, 0}_{0, 2k} \left( (0), \frac{1}{2} ; (\frac{1}{2})_{k-1} \left| \frac{\rho^{2}j^{2}x^{2}}{2\pi k} \right. \right), \tag{1.10}
\]

where \( k \) is any natural number. In this article, we have generalized Ramanujan’s identity and the identity of Dixit and Gupta by studying the transformation formula for \( \zeta^k(2m + 1) \) with \( m \in \mathbb{Z} \setminus \{0\} \), where \( k \) is any natural number.

**Theorem 1.1.** For any non-zero integer \( m \) and for any \( \alpha, \beta > 0 \) satisfying \( \alpha \beta = \pi^2 \), we have

\[
(\alpha^k)^{-m} \sum_{n=1}^{\infty} \frac{d_k(n)}{n^{2m+1}} \Psi_{(2\alpha)^k,k}(n) - \frac{1}{(k-1)!} \frac{d^{k-1}}{dk-1} \left( \zeta^k(2m + 1 + s) \zeta^k(s) \Gamma^k(s + 1) \cos^{k-1} \left( \frac{\pi s}{2} \right) (2\alpha)^{-ks} \right) \bigg|_{s=0} = (\beta^k)^{-m} \sum_{n=1}^{\infty} \frac{d_k(n)}{n^{2m+1}} \Psi_{(2\beta)^k,k}(n) - \frac{1}{(k-1)!} \frac{d^{k-1}}{dk-1} \left( \zeta^k(2m + 1 + s) \zeta^k(s) \Gamma^k(s + 1) \cos^{k-1} \left( \frac{\pi s}{2} \right) (2\beta)^{-ks} \right) \bigg|_{s=0}
\]

\[
+ (-1)^{km+k+m} \left( \frac{\pi}{2} \right)^{k-1} 2^{km} \sum_{j=0}^{m+1} (-1)^j \frac{B^2_{2m-2j+2} B^2_{2j}}{(2m-2j+2)!^2 (2j)!^2} \alpha^{k(m+1-j)} \beta^j,
\]

where \( k \) is any natural number.

As an immediate application of the above theorem, we obtain the following two results.

**Corollary 1.2.** Ramanujan’s identity \((1.2)\) for odd zeta values is valid.

**Corollary 1.3.** The identity \((1.9)\) of Dixit and Gupta is valid.

In Theorem 1.1, the identity \((1.11)\) can be analytically continued to any complex \( \alpha, \beta \) with \( \text{Re}(\alpha) > 0 \) and \( \text{Re}(\beta) > 0 \). Thus, letting \( \alpha = -\pi iz \) with any complex number \( z \) in upper-half plane and \( m = -m \) with \( m > 0 \), the series in \((1.11)\) namely,

\[
\sum_{n=1}^{\infty} d_k(n)n^{2m-1} \Psi_{(2\alpha)^k,k}(n)
\]

represents the generalization of usual Eisenstein series of weight \( 2m \) over \( SL_2(\mathbb{Z}) \). The following corollary provides the generalization of the transformation formula \((1.3)\) of \( E_{2m}(z) \) for \( m > 1 \) over \( SL_2(\mathbb{Z}) \), which can be obtained directly from \((1.11)\), substituting \( m \) by \(-m\).

**Corollary 1.4.** Let \( m \) be any natural number with \( m > 1 \). Then for any \( \alpha, \beta > 0 \) satisfying \( \alpha \beta = \pi^2 \), we have

\[
(\alpha^k)^{m} \sum_{n=1}^{\infty} d_k(n)n^{2m-1} \Psi_{(2\alpha)^k,k}(n) - (-\beta^k)^{m} \sum_{n=1}^{\infty} d_k(n)n^{2m-1} \Psi_{(2\beta)^k,k}(n)
\]

\[
= \frac{1}{(k-1)!} \frac{d^{k-1}}{dk-1} \left( \zeta^k(1 - 2m + s) \zeta^k(s) \Gamma^k(s + 1) \cos^{k-1} \left( \frac{\pi s}{2} \right) 2^{-ks} (\alpha^{-k(s-m)} - (-1)^m \beta^{-k(s-m)}) \right) \bigg|_{s=0}
\]

The next result generalizes the transformation formula \((1.6)\) of Quasi modular form \( E_2(z) \) over \( SL_2(\mathbb{Z}) \), which can be obtained by inserting \( m = -1 \) in \((1.11)\).
Corollary 1.5. Let \( \alpha, \beta \) be any real number with \( \alpha, \beta > 0 \), which satisfies \( \alpha \beta = \pi^2 \). Then the following identity
\[
\alpha^k \sum_{n=1}^{\infty} nd_k(n) \Psi_{(2\alpha)^k,k}(n) + \beta^k \sum_{n=1}^{\infty} nd_k(n) \Psi_{(2\beta)^k,k}(n)
= \frac{1}{(k-1)!} \left. \frac{d^{k-1}}{ds^{k-1}} \left( \zeta^k(s-1) \zeta^k(s) \Gamma^k(s+1) \cos^{k-1} \left( \frac{\pi s}{2} \right) \left( \alpha^{-k(s-1)} + \beta^{-k(s-1)} \right) \right) \right|_{s=0} - \left( \frac{\pi}{2} \right)^{k-1} 2^{-2k}
\]
holds.

For any complex number \( z \) with \( \text{Im}(z) > 0 \), the Dedekind eta function \( \eta(z) \) can be defined as
\[
\eta(z) := e^{\frac{\pi iz}{2}} \prod_{n=1}^{\infty} (1 - e^{2\pi inz}).
\]

It is a modular form of weight \( \frac{1}{2} \) over the full modular group and has a Fourier series expansion \( \eta(z) = \sum_{n=1}^{\infty} \sigma_1(n) e^{2\pi inz} \). Ramanujan’s formula (1.2) also provides a transformation formula of \( \eta(z) \) under the transformation \( \eta(z) \rightarrow e^{i \theta} \eta(z) \).

Theorem 1.6. For \( \alpha, \beta > 0 \) satisfying \( \alpha \beta = \pi^2 \),
\[
\sum_{n=1}^{\infty} \frac{d_k(n)}{n} \Psi_{(2\alpha)^k,k}(n) - \sum_{n=1}^{\infty} \frac{d_k(n)}{n} \Psi_{(2\beta)^k,k}(n)
= \left( \frac{-1}{2k-1} \right)^{2k-1} \left( \Gamma^k(1+s) \Gamma^k(1-s) \zeta^k(s) \zeta^k(-s) \cos^{2k-1} \left( \frac{\pi s}{2} \right) \left( \frac{\alpha}{\beta} \right)^{-k} \right) \bigg|_{s=0} + \frac{2}{24} \left( \frac{-1}{2k-1} \right)^{k-1} \left( \beta^k - \alpha^k \right)
\]

Clearly, one can obtain the transformation formula of \( \eta(z) \) by putting \( k = 1 \) in the above identity. For \( k = 2 \), the above theorem reduces to [8, Theorem 2.9].

The next result can be considered as a generalization of Lerch’s identity (1.3), which follows directly from Theorem 1.1 by inserting \( \alpha = \beta = \pi^k \) in (1.11) and considering \( m \) to be odd.

Theorem 1.7. Let \( m \) be any non-zero odd integer. Then for any natural number \( k \),
\[
\sum_{n=1}^{\infty} \frac{d_k(n)}{n^{2m+1}} \Psi_{(2\pi)^k,k}(n) = \frac{1}{(k-1)!} \left. \frac{d^{k-1}}{ds^{k-1}} \left( \zeta^k(2m+1+s) \zeta^k(s) \Gamma^k(s+1) \cos^{k-1} \left( \frac{\pi s}{2} \right) \left( 2\pi \right)^{-k} \right) \right|_{s=0}
+ 2^{2km-k} \pi^{2km+2k-1} \sum_{n=0}^{m+1} (-1)^{n+1} \frac{B_{2m-2n+2}^k B_{2n}}{(2m-2n+2)(2n)!}.
\]

In particular, at \( k = 1 \), the above identity reduces to Lerch’s identity (1.3).

The case \( k = 2 \) of the above identity was obtained in [8, Equation (2.4)].

The paper is organized as follows. In \( \S2 \) we collect basic tools, which we have applied throughout. \( \S3 \) provides the proof of the transformation formula of \( \zeta^k(2m+1) \), and its special cases. Finally, in \( \S4 \) we generalized the transformation formula of \( \eta(z) \).

2. Preliminaries

In this section, we collect some basic tools of analytic number theory and complex analysis, which will be applied throughout.
2.1. Gamma function. The Gamma function plays a significant role in this paper. For \( \text{Re}(z) > 0 \), it can be defined via the convergent improper integral

\[
\Gamma(z) = \int_0^\infty e^{-t}t^{z-1}dt.
\]

The function \( \Gamma(z) \) is absolutely convergent for \( \Re(z) > 0 \). It can be analytically continued to the whole complex plane except for simple poles at every non-positive integers. It also satisfies the functional equation [3, Appendix A], namely,

\[
\Gamma(z + 1) = z\Gamma(z).
\]

Two important properties of Gamma function which will be applied frequently is the following:

\[
\Gamma(z)\Gamma(1 - z) = \pi \sin \pi z,
\]

where \( z \notin \mathbb{Z} \) and

\[
\Gamma(z)\Gamma\left(z + \frac{1}{2}\right) = 2^{1-2z}\sqrt{\pi}\Gamma(2z)
\]

for any complex \( z \), which are known as reflection formula and duplication formula respectively. The proof of these properties can be found in [3, Appendix A].

The well-known formula of Stirling for the Gamma function on a vertical strip states that for \( a \leq \sigma \leq b \) and \( t \geq 1 \) (cf. [7, p. 224]),

\[
|\Gamma(\sigma + it)| = (2\pi)^{\frac{1}{2}|t|}\frac{1}{\sigma - \frac{1}{2}}e^{-\frac{1}{2}\pi|t|}\left(1 + O\left(\frac{1}{|t|}\right)\right).
\]

2.2. Riemann zeta function. The Riemann zeta function can be defined by the following series

\[
\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s},
\]

where \( \text{Re}(s) > 1 \). It can be continued analytically to the whole complex plane except for the simple pole at \( s = 1 \). The functional equation of the Riemann zeta function is given by [22, p. 13, Equation (2.1.1)]

\[
\zeta(s) = 2^s\pi^{s-1}\Gamma(1-s)\zeta(1-s)\sin\left(\frac{\pi s}{2}\right).
\]

Taking \( k \)-th exponent on both sides of (2.4), we have for \( \text{Re}(s) > 1 \),

\[
\sum_{n=1}^{\infty} \frac{d_k(n)}{n^s} = \zeta^k(s).
\]

2.3. Special function. The mathematical functions which have more or less established names and notations due to their importance in mathematical analysis, functional analysis, geometry, physics, or other applications are known as special functions. These mainly appear as solutions of differential equations or integrals of elementary functions.

One of the most important families of special functions are the Bessel functions, which are basically the canonical solution of Bessel’s differential equations

\[
x^2 \frac{d^2 y}{dx^2} + x \frac{dy}{dx} + (x^2 - a^2)y = 0
\]

where \( a \) is any arbitrary complex number.

Meijer in 1936 introduced a general special function namely, \( G \)-function (cf. [17, p. 143]), which includes most of the known special functions as particular cases. For any non-zero complex number \( z \) and for integers \( m, n, p, q \) satisfying \( 0 \leq m < p \) and \( 0 \leq n < q \), the Meijer \( G \)-function can be defined as an inverse Mellin transform of quotient of products of gamma factors as

\[
G_{m, n}^{p, q}\left(\begin{array}{c}
\frac{a_1, \ldots, a_p}{b_1, \ldots, b_q}
\end{array}\biggm| z\right) = \frac{1}{2\pi i} \int_{(C)} \frac{\prod_{j=1}^{m} \Gamma(b_j - s) \prod_{j=1}^{n} \Gamma(1 - a_j + s)}{\prod_{j=m+1}^{q} \Gamma(1 - b_j + s) \prod_{j=n+1}^{p} \Gamma(a_j - s)} z^s ds,
\]

(2.7)
where \((C)\) in the integral denotes the vertical line from \(C - i\infty\) to \(C + i\infty\).

Special cases of the \(G\)-function include many other special functions. For instance, there are many formulae which yield relations between the \(G\)-function and the Bessel functions. Two important formulas among them, which we have used are given by \([3\, p.\ 216,\ Equation\ (4)],\ [19\ p.\ 675,\ Equation\ (13)]\)

\[
G_{0,2}^{2,0} \left( \begin{array}{c} - \\ a, b \\ \end{array} \bigg| \frac{z}{2} \right) = 2z^{\frac{1}{a+b}} K_{a-b}(2z^{1/2}),
\]

\[
G_{0,4}^{3,0} \left( \begin{array}{c} - \\ 0, 0, \frac{1}{2}, \frac{1}{2} \\ \end{array} \bigg| \frac{z}{2} \right) = 4\text{Re} \left( K_0 \left( 4z^\frac{1}{4} e^{\frac{iz}{2}} \right) \right).
\]

3. Generalization of Ramanujan’s identity and it’s special cases

In this section, we prove Theorem 1.1 and its special cases at \(k = 1\) and \(2\). We begin with the following lemma.

Lemma 3.1. Let \(\Psi_{\rho,k}(x)\) be defined as in (1.10). Then for \(c = \text{Re}(s) > 1\), we have

\[
\Psi_{\rho,k}(x) = \frac{1}{2\pi i} \int_{(c)} \Gamma^k(s) \zeta^k(s) \cos^{k-1} \left( \frac{\pi s}{2} \right) (\rho x)^{-s} ds.
\]

Proof. The definition of Meijer \(G\)-function in (2.7) yields, for \(\mu < 0\),

\[
G_{0,2k}^{k+1,0} \left( \begin{array}{c} - \\ (0)\ k; \frac{1}{2}; (\frac{1}{2})_{k-1} \\ \frac{\rho^2 j^2 x^2}{2^{2k}} \end{array} \bigg| \right) = \frac{1}{2\pi i} \int_{(\mu)} \frac{\Gamma^k(-s) \Gamma \left( \frac{1}{2} - s \right)}{\Gamma^{k-1} \left( \frac{1}{2} + s \right)} \left( \frac{\rho^2 j^2 x^2}{2^{2k}} \right)^s ds
\]

\[
= \frac{1}{2\pi i} \int_{(-\mu)} \frac{\Gamma^k(s) \Gamma^k \left( \frac{1}{2} + s \right)}{\Gamma^{k-1} \left( \frac{1}{2} + s \right) \Gamma^{k-1} \left( \frac{1}{2} - s \right)} \left( \frac{\rho^2 j^2 x^2}{2^{2k}} \right)^{-s} ds,
\]

where in the last step we make the change of variable \(s \mapsto -s\). Employing the reflection formula (2.1) in the denominator and the duplication formula (2.2) in the numerator of the above integral, we obtain

\[
G_{0,2k}^{k+1,0} \left( \begin{array}{c} - \\ (0)\ k; \frac{1}{2}; (\frac{1}{2})_{k-1} \\ \frac{\rho^2 j^2 x^2}{2^{2k}} \end{array} \bigg| \right) = \frac{2k}{\pi^{k-1} 2\pi i} \int_{(-\mu)} \Gamma^k(2s) \cos^{k-1} \left( \pi s \right) (\rho j x)^{-2s} ds
\]

\[
= \frac{2k-1}{\pi^{k-1} 2\pi i} \int_{(c)} \Gamma^k(s) \cos^{k-1} \left( \frac{\pi s}{2} \right) (\rho j x)^{-s} ds,
\]

where \(c = -2\mu > 0\). Invoking (3.1) into the definition (1.10) of \(\Psi_{\rho,k}(x)\), we obtain

\[
\Psi_{\rho,k}(x) = \frac{1}{2\pi i} \sum_{j=1}^{\infty} d_k(j) \int_{(c)} \Gamma^k(s) \cos^{k-1} \left( \frac{\pi s}{2} \right) (\rho j x)^{-s} ds
\]

We next interchange the order of summation and integration in the above expression for \(c = \text{Re}(s) > 1\), which follows by applying (2.1) on the cosine factor and then (2.3) on Gamma factors. Therefore, for \(c = \text{Re}(s) > 1\), we have

\[
\Psi_{\rho,k}(x) = \frac{1}{2\pi i} \int_{(c)} \Gamma^k(s) \left( \sum_{j=1}^{\infty} \frac{d_k(j)}{j^s} \right) \cos^{k-1} \left( \frac{\pi s}{2} \right) (\rho x)^{-s} ds
\]

\[
= \frac{1}{2\pi i} \int_{(c)} \Gamma^k(s) \zeta^k(s) \cos^{k-1} \left( \frac{\pi s}{2} \right) (\rho x)^{-s} ds,
\]

where in the last step we have applied (2.6). This completes the proof of the lemma. \(\square\)

3.1. Proof of Theorem 1.1. We begin with the following infinite series :

\[
\mathcal{L}_m(\rho) := \sum_{n=1}^{\infty} \frac{d_k(n)}{\eta^{2m+1}} \Psi_{\rho,k}(n), \tag{3.3}
\]
where \( d_k(n) \) is the Piltz divisor function. Applying Lemma 3.1, we have for \( 1 < c < 3 \),

\[
\mathcal{L}_m(\rho) = \frac{1}{2\pi i} \sum_{n=1}^{\infty} \frac{d_k(n)}{n^{2m+1}} \int_{(c)} \Gamma^k(s) \zeta^k(s) \cos^{k-1} \left( \frac{\pi s}{2} \right) (\rho n)^{-s} \, ds
\]

\[
= \frac{1}{2\pi i} \int_{(c)} \Gamma^k(s) \zeta^k(s) \left( \sum_{n=1}^{\infty} \frac{d_k(n)}{n^{2m+1+s}} \right) \cos^{k-1} \left( \frac{\pi s}{2} \right) \rho^{-s} \, ds
\]

\[
= \frac{1}{2\pi i} \int_{(c)} \Gamma^k(s) \zeta^k(s) \zeta^k(2m+1+s) \cos^{k-1} \left( \frac{\pi s}{2} \right) \rho^{-s} \, ds,
\]

where in the penultimate step, the interchange of the order of summation and integration is justified similarly as was done in (3.2) and in the last step we have applied (2.6). Employing functional equation (2.5) of Riemann zeta function into the above integral, we obtain

\[
\mathcal{L}_m(\rho) = \frac{1}{2\pi i} \int_{(c)} F_m(s) \, ds,
\]

where

\[
F_m(s) = \frac{\zeta^k(2m+s+1)\zeta^k(1-s) \left( \frac{\rho}{(2\pi)^k} \right)^{-s}}{\cos \left( \frac{\pi s}{2} \right)}.
\]

We next consider the contour \( C \) determined by the line segments \( [c-iT, c+iT], [c+iT, \lambda+iT], [\lambda+iT, \lambda-iT], [\lambda-iT, c-iT], \) where \( -2m - 3 < \lambda < -2m - 1 \). It can be easily seen that the integrand \( F_m(s) \) in the above integral has poles of order \( k \) at \( s = 0 \) and at \( s = -2m \) due to the poles of each of the zeta functions. The integrand \( F_m(s) \) also has simple poles at \( 1 - 2j \) for \( 0 \leq j \leq m + 1 \) due to the zeros of cosine factor in the numerator. Note that the zeros of the cosine factor at \( 1 - 2j \) for \( j > m + 1 \) do not contribute any poles since it get cancelled with the trivial zeros of \( \zeta(2m+1+s) \).

Letting \( R_a \) be the corresponding residue of \( F_m(s) \) at \( s = a \), the Cauchy residue theorem yields

\[
\frac{1}{2\pi i} \int_{C} F_m(s) \, ds = \frac{1}{2\pi i} \left[ \int_{c-iT}^{c+iT} + \int_{c+iT}^{\lambda+iT} + \int_{\lambda+iT}^{\lambda-iT} + \int_{\lambda-iT}^{c-iT} \right] F_m(s) \, ds
\]

\[
= R_0 + R_{-2m} + \sum_{j=0}^{m+1} R_{1-2j},
\]

We next evaluate the residues on the right hand side of (3.5). The residue at \( s = 0 \) is given by

\[
R_0 = \frac{1}{(k-1)!} \lim_{s \to 0} \frac{d^{k-1}}{ds^{k-1}} \left( \frac{s^k \zeta(2m+1+s) \zeta^k(1-s) \left( \frac{\rho}{(2\pi)^k} \right)^{-s}}{\cos \left( \frac{\pi s}{2} \right)} \right)
\]

\[
= \frac{2^k}{(k-1)!} \lim_{s \to 0} \frac{d^{k-1}}{ds^{k-1}} \left( s^k \zeta^k(2m+1+s) \zeta^k(s) \Gamma^k(s) \cos^{k-1} \left( \frac{\pi s}{2} \right) \rho^{-s} \right)
\]

\[
= \frac{2^k}{(k-1)!} \frac{d^{k-1}}{ds^{k-1}} \left( \zeta^k(2m+1+s) \zeta^k(s) \Gamma^k(s) \cos^{k-1} \left( \frac{\pi s}{2} \right) \rho^{-s} \right) \bigg|_{s=0},
\]

where in the penultimate step we used the functional equation (2.5) of the Riemann zeta function. The residue at \( s = -2m \) evaluates as

\[
R_{-2m} = \frac{1}{(k-1)!} \lim_{s \to -2m} \frac{d^{k-1}}{ds^{k-1}} \left( \frac{(s+2m)^k \zeta^k(2m+1+s) \zeta^k(1-s) \left( \frac{\rho}{(2\pi)^k} \right)^{-s}}{\cos \left( \frac{\pi s}{2} \right)} \right)
\]

\[
= \frac{(-1)^{m+1} 2^k}{(k-1)!} \frac{d^{k-1}}{ds^{k-1}} \left( \zeta^k(2m+1+s) \zeta^k(s) \Gamma^k(s) \cos^{k-1} \left( \frac{\pi s}{2} \right) \left( \frac{\rho}{(4\pi^2)^k} \right)^s \right) \bigg|_{s=0},
\]
where in the penultimate step we made the change of variable \( s \mapsto -s - 2m \) and in the last step we have used (2.5). The residue at \( s = 1 - 2j \) with \( 0 \leq j \leq m + 1 \), can be calculated similarly as the previous residue calculation, which is given by

\[
R_{1-2j} = \lim_{s \to 1-2j} \left( \frac{(s - 1 + 2j)\zeta^k(2m + 1 + s)\zeta(1 - s)}{\cos \left( \frac{\pi s}{2} \right)} \right) \left( \frac{\rho}{(2\pi)^k} \right)^s
\]

\[
= (-1)^{j+1} \frac{2}{\pi} \zeta^k(2m + 2 - 2j)\zeta(2j) \left( \frac{\rho}{(2\pi)^k} \right)^{(2j-1)}
\]

\[
= (-1)^{km+2j+1} \frac{B_{2m-2j+2} B_{2j}}{(2m-2j+2)!k(2j)!k} \left( \frac{\rho}{(2\pi)^k} \right)^{(2j-1)}, \quad (3.8)
\]

where in the last step we applied (1.1) on the zeta factors.

It can be easily shown that the horizontal integrals in (3.3) vanish as \( T \to \infty \) by applying (2.3) on the Gamma factors and an elementary bound on zeta factors in \( F_m(s) \). Therefore, letting \( T \to \infty \), (3.4) and (3.5) together imply

\[
\mathcal{L}_m(\rho) = \frac{1}{2^k} \left[ R_0 + R_{-2m} + \sum_{j=0}^{m+1} R_{1-2j} + \frac{1}{2\pi i} \int (\lambda) F_m(s) \, ds \right]. \quad (3.9)
\]

We next handle the vertical integral on the right-hand side of the above equation. Substituting \( s \) by \(-2m - s\), the integral reduces to

\[
\frac{1}{2\pi i} \int (\lambda) F_m(s) \, ds = (-1)^m \left( \frac{\rho}{(2\pi)^k} \right)^{2m} \frac{1}{2\pi i} \int (c') \zeta^k(1-s)\zeta^k(2m+1+s) \cos \left( \frac{\pi s}{2} \right) \left( \frac{\rho}{(2\pi)^k} \right)^s \, ds
\]

\[
= (-1)^m \left( \frac{\rho}{(2\pi)^k} \right)^{2m} \frac{1}{2\pi i} \int (c') \zeta^k(1-s)\zeta^k(2m+1+s) \cos \left( \frac{\pi s}{2} \right) \left( \frac{(4\pi^2)^k}{\rho} \right)^{-s} \, ds
\]

\[
= (-1)^{m} 2^k \left( \frac{\rho}{(2\pi)^k} \right)^{2m} \mathcal{L}_m \left( \frac{(4\pi^2)^k}{\rho} \right), \quad (3.10)
\]
Letting $\alpha^k = \frac{\alpha}{\pi^2}$ and $\beta^k = \frac{\beta}{\pi^2}$ and multiplying both sides by $\alpha^{-km}$ in the above equation, we arrive at

$$(\alpha^k)^{-m} \left[ \sum_{n=1}^{\infty} \frac{d_k(n)}{n^m + 1} \Psi_{(2\alpha^k,k),k}(n) - \frac{1}{(k-1)!} \frac{d^{k-1}}{ds^{k-1}} \left( \zeta^k(2m + 1 + s) \zeta^k(s) \Gamma^k(s + 1) \cos^{k-1} \left( \frac{\pi s}{2} \right) (2\alpha)^{-ks} \right) \right]_{s=0}$$

$$= (-\beta^k)^{-m} \left[ \sum_{n=1}^{\infty} \frac{d_k(n)}{n^m + 1} \Psi_{(2\beta^k,k),k}(n) - \frac{1}{(k-1)!} \frac{d^{k-1}}{ds^{k-1}} \left( \zeta^k(2m + 1 + s) \zeta^k(s) \Gamma^k(s + 1) \cos^{k-1} \left( \frac{\pi s}{2} \right) (2\beta)^{-ks} \right) \right]_{s=0}$$

$$+ (-1)^{km+k+m} \left( \pi^2 \right)^{k-1} 2^{2km} \sum_{j=0}^{m+1} (-1)^j \frac{B_{2m-2j+2} B_{2j}}{(2m-2j+2)! (2j)!} \alpha^{k(m+1-j)} \beta^j,$$

where we applied the definition \[3.3\] of $\mathcal{L}_m(\rho)$. This completes the proof of our theorem.

We next show that our formula provides Ramanujan’s identity at $k = 1$.

### 3.2. Proof of Corollary 1.2

It follows from the definition \[1.10\] that at $k = 1$, we have

$$\Psi_{\rho,1}(x) = \frac{1}{\sqrt{\pi}} \sum_{j=1}^{\infty} G_{0,2}^{2,0} \left( 0, 0, \frac{1}{2} \mid \frac{\rho^2 j^2 x^2}{2^2} \right)$$

$$= \sqrt{\frac{2}{\pi}} \sum_{j=1}^{\infty} (\rho j x)^{1/2} K_{1/2}(\rho j x),$$

where in the last step we used \[2.8\]. Now applying \[18\] p. 254, Equation (10.39.2)], we obtain

$$\Psi_{\rho,1}(x) = \sum_{j=1}^{\infty} e^{-\rho j x}$$

Therefore invoking the above relation into \[1.11\], we conclude our corollary.

We next prove that the identity \[1.9\] can be obtained as a special case of Theorem 1.1 for $k = 2$.

### 3.3. Proof of Corollary 1.3

Employing \[2.9\] in the definition \[1.10\] at $k = 2$, we obtain

$$\Psi_{\rho,2}(x) = \frac{1}{2} \sum_{j=1}^{\infty} d(j) G_{0,4}^{3,0} \left( 0, 0, 0, \frac{1}{2}, \frac{1}{2} \mid \frac{\rho^2 j^2 x^2}{16} \right)$$

$$= \frac{1}{2} \sum_{j=1}^{\infty} d(j) \text{ Re} \left( K_0 \left( 2\epsilon \sqrt{\rho j x} \right) \right)$$

$$= \sum_{j=1}^{\infty} d(j) \left( K_0 \left( 2\epsilon \sqrt{\rho j x} \right) + K_0 \left( 2\pi \sqrt{\rho j x} \right) \right),$$

where $\epsilon = e^{i \pi}$. We next insert the above relation into \[1.11\] and make the change of variable $j \mapsto m + 1 - j$ in the finite sum on the right-hand side of \[1.11\] to conclude our corollary.

### 4. Transformation of the Generalized Dedekind eta function

In this section we mainly prove Theorem 1.6.

#### 4.1. Proof of Theorem 1.6

Let

$$\mathcal{L}(\rho) := \sum_{n=1}^{\infty} \frac{d_k(n)}{n} \Psi_{\rho,k}(n),$$

where $d_k(n)$ is the Piltz divisor function. We next handle the series $\mathcal{L}(\rho)$ similarly as $\mathcal{L}_m(\rho)$, which was derived in the proof of Theorem 1.1. Proceeding similarly as in \[3.4\], the series $\mathcal{L}(\rho)$ transforms into

$$\mathcal{L}(\rho) = \frac{1}{2^k} \frac{1}{2\pi i} \int_{(c)} \zeta^k(1+s) \zeta^k(1-s) \left( \frac{\rho}{(2\pi)^k} \right)^{-s} ds,$$
where $1 < c < 3$. We next shift the line of integration to $-3 < \lambda < -1$ by constructing a rectangular contour. Thus the integrand has a pole of order $2k$ at $s = 0$ due to the poles of zeta functions in the numerator as well as simple poles at $s = -1$ and $s = 1$ due to the zeros of the cosine factor in the denominator.

Letting $R_a$ be the corresponding residue at $s = a$, the residues at the above poles can be evaluated as

$$R_0 = (-1)^k \frac{2^{2k}}{(2k-1)!} \frac{d^{2k-1}}{ds^{2k-1}} \left( \Gamma^k(1+s) \Gamma^k(1-s) \zeta^k(s) \zeta^k(-s) \cos^{2k-1} \left( \frac{\pi s}{2} \right) \left( \frac{\rho}{(2\pi)^k} \right)^{-s} \right) \bigg|_{s=0},$$

$$R_1 = 2 \frac{(-1)^{k+1} \pi^{k-1}}{6^k \rho}$$

and

$$R_{-1} = 2 \frac{(-1)^k \pi^k \rho}{(24)^k}.$$

Thus applying Cauchy residue theorem and simplifying further as in the proof of Theorem 4.1, we arrive at

$$\mathcal{L}(\rho) - \mathcal{L} \left( \frac{(4\pi^2)^k}{\rho} \right) = \frac{(-1)^k 2^{2k-1}}{(2k-1)!} \frac{d^{2k-1}}{ds^{2k-1}} \left( \Gamma^k(1+s) \Gamma^k(1-s) \zeta^k(s) \zeta^k(-s) \cos^{2k-1} \left( \frac{\pi s}{2} \right) \left( \frac{\rho}{(2\pi)^k} \right)^{-s} \right) \bigg|_{s=0}$$

$$+ 2 \frac{(-1)^{k+1} \pi^{k-1}}{24^k} \left( \frac{\rho}{2\pi} \right)^{2k}.$$  

Finally letting $\alpha^k = \frac{\rho}{2\pi}$, $\beta^k = \frac{2^{2k} \pi^k}{\rho}$ and applying the definition of $\mathcal{L}(\rho)$ in (4.1), we can conclude our theorem.

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