Analytic derivation of the inertial range of compressible turbulence

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An analytic model for steady state turbulence is employed to obtain the inertial range power spectrum of compressible turbulence. We assume that for homogeneous turbulence, the timescales controlling the energy injected at a given wavenumber from all smaller wave-numbers, are equal for each spatial component. However, the longitudinal component energy is diverted into compression, so the rate controlling the energy that is transferred to all larger wave-numbers by the turbulent viscosity is reduced. The resulting inertial range is a power law with index -2. Indeed such power spectra were observed in various astrophysical settings and also in numerical simulations.

Compressible supersonic turbulence characterized by a 1D power-law index of -2 has been observed in molecular clouds\textsuperscript{1,2}, in interstellar medium of galaxies\textsuperscript{4,5}, in a shocked nebula near the Galactic center\textsuperscript{6}. It has also been obtained in numerical simulations\textsuperscript{7,8}. Recently, it was found\textsuperscript{11} that the intensity of \(\gamma\)-Ray emission from the Large Magellanic Cloud exhibits such a power spectrum.

In this Letter, we employ a simple analytic model for steady turbulence\textsuperscript{12} to find the power spectrum of the inertial range of supersonic compressible turbulence. We obtain an inertial range of the 1D power spectrum which is proportional to \(k^{-2}\). Details follow.

The model\textsuperscript{12} is an extension of a previous work\textsuperscript{13}. It describes homogeneous isotropic steady state turbulence. Its basic equations are

\[
\begin{align*}
n_s(k) + \frac{y(k)}{n_c(k)} &= \nu_t(k) k^2 \quad \text{(1)} \\
y(k) &= \int_k^\infty F(k) k^2 dk \quad \text{(2)} \\
\nu_t(k) &= \int_k^\infty \frac{F(K)}{n_c(k)} \quad \text{(3)} \\
\nu_t(k) k^2 &= \gamma n_c(k) \quad \text{(4)}
\end{align*}
\]

with \(F(k) = 4\pi k^2 \Phi(k)\) where \(k = |\vec{k}|\), and \(\Phi(\vec{k})\) is the 3D power spectrum, of the turbulent velocity, \(y(k)\) is the k-space mean square vorticity at wavenumber \(k\), \(n_s(k)\) is the rate controlling the energy input from the source at \(k\), \(n_c(k)\) is the inverse of the eddy correlation timescale and \(\nu_t(k)\) is the turbulent kinematic viscosity at wavenumber \(k\) exerted by all the eddies with larger wavenumbers. Eq. (4) relates the turbulent viscosity and the eddy correlation rate, with \(\gamma\) being a dimensionless constant. The model has been quite successful in obtaining turbulence spectra for both the large scale spatial scales as well as the inertial range. An extension of the model\textsuperscript{14} dealing with stellar turbulent convection was very successful and has been widely cited.

In the large-eddy range \(n_s(k)\) is positive and dominates over \(y(n_c(k))\). In the dissipation range (small spatial scales) \(n_s(k) = -\nu k^2\), with \(\nu\) denoting the microscopic kinematic viscosity. Depending on \(n_s(k)\) and \(\nu\), there usually exists a mid wavenumber range—the inertial range, in which both energy input from the source and energy dissipation are very small compared to the rate of energy transfer. Thus, in this range

\[
y(k) = \gamma n_c(k)^2 \quad \text{(5)}
\]

from which

\[
y'(k) = 2\gamma n_c(k) n'_c(k) \quad \text{(6)}
\]

From Eq.(2) Eq.(3) and Eq.(4) follows that

\[
y'(k) = -k^2 \nu'_t(k) = -\gamma k^2 (n_c(k) k^{-2})' \quad \text{(7)}
\]

Eqs.(6) and (7) yield

\[
n_c(k) = A k^{2/3} \quad \text{(8)}
\]

with \(A\) a constant. Since \(F(k) = y'(k) k^{-2}\) one gets

\[
F(k) = \frac{4}{3} \gamma A^2 k^{-5/3} \quad \text{(9)}
\]

Thus, the model yields for the inertial range a power spectrum of the Kolmogorov\textsuperscript{15} form. This is not surprising as in this model all the energy injected from smaller wave-numbers is transferred by the turbulent viscosity to the larger wave-numbers.

However, in compressible turbulence the situation is different: not all the energy available from the wavenumbers smaller than \(k\) is cascaded to smaller scales, since energy is also being diverted into compression. The longitudinal component of the turbulence is the one involved in the energy diversion. For an homogeneous turbulence it is conceivable that the timescales controlling the input from the wave-numbers smaller than \(k\) are equal. Thus, if all its energy is diverted to compression one gets instead of Eq.(1)

\[
n_s(k) + \frac{2}{3} \frac{y(k)}{n_c(k)} = \nu_t(k) k^2 = \gamma n_c(k) \quad \text{(10)}
\]

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The assumption that all the energy of the longitudinal component is diverted into compression, seems justified for supersonic turbulence, which is ubiquitous in astrophysical turbulence. From Eq.(10) follows the rate equation the inertial range:

\[ y(k) = \frac{3}{2} \gamma n_c(k)^2 \]  

(11)

Eq.(7) is unchanged while differentiation of Eq.(11) yields

\[ y'(k) = 3\gamma n_c(k)n'_c(k) \]  

(12)

Thus, the solution is now

\[ n_c(k) = Bk^{1/2} \]  

(13)

with \( B \) a constant, leading to

\[ F(k) = \frac{3}{2} \gamma B^2 k^{-2} \]  

(14)

Which is the expected form of the inertial range of compressible turbulence. The present derivation suggests that for sonic turbulence not all of the longitudinal energy will be diverted. In this case, one may expect that the inertial range logarithmic slope will be intermediate between \(-5/3\) and \(-2\).

I. DATA AVAILABILITY

This is a theoretical paper. It includes no data.

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