Periodic solution for strongly nonlinear oscillators by He’s new amplitude-frequency relationship

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1 Introduction

Nonlinear vibration arises everywhere in science, engineering and other disciplines, since most phenomena in our world today, are essentially nonlinear and are described by nonlinear equations. It is very important in applications to have a version of the frequency (or period) to have a better understanding of the phenomena modeled through differential equations that contain terms with high nonlinearities, and a simple mathematical method is very useful for practical applications.

Recently many analytical methods have appeared to obtain the approximate solutions of nonlinear systems, such as the parameter-expansion method [1], the harmonic balance method [2, 4, 6], the energy balance method [7, 8], the Hamiltonian approach [10, 12], the use of special functions [13, 14], the max-min approach [15, 16], the variational iteration method [17, 18, 20, 21] and homotopy perturbation [22, 23, 24, 25, 26, 27, 28], and others. An excellent study, in which many of these techniques can be found in detail to solve nonlinear
problems of oscillatory type can be seen in [29].
Recently, In [11] an analytical approximate technique for large and small amplitudes oscillations of a class of conservative single degree-of-freedom systems with odd non-linearity is proposed. In this study, we have applied new method to find the approximate solutions of nonlinear differential equation governing strongly nonlinear oscillators and have made a comparison with the exact solution. The most interesting features of the used method are its simplicity and its excellent accuracy of both period and corresponding periodic solution for the entire range of oscillation amplitude. Finally, four examples are presented to describe the solution methodology and to illustrate the usefulness and effectiveness of the proposed technique.

2 He’s new amplitude-frequency relationship

Consider a one-dimensional, nonlinear oscillator governed by

\[ u'' + f(u) = 0, \] (1)

with the initial conditions

\[ u(0) = A, \quad u'(0) = 0. \] (2)

where a prime denotes differentiation with respect to \( t \) and the nonlinear function \( f(u) \) is odd, i.e. \( f(-u) = -f(u) \) and satisfies \( f(u)/u > 0 \) for \( u \in [-A,A] \), \( u \neq 0 \). It is obvious that \( u = 0 \) is the equilibrium position. The system oscillates between the symmetric bounds \(-A\) and \( A \). The period and corresponding periodic solution are dependent on the oscillation amplitude \( A \).

According to He’s new amplitude-frequency formulation, the approximate frequency as a function of \( A \) can be obtained as follows [11]:

\[ \omega^2(A) = \sum_{i=1}^{N} \omega_i^2(A) \] (3)

with each \( \omega_i^2(A) \) defined by

\[ \omega_i^2(A) = f'(u_i) \] (4)

where \( u_i \) are location points, \( 0 < u_i < A \). Explicitly, \( u_i = iA/N \) for every \( i = 1, 2, \ldots, N - 1 \). The simplest way to calculate the frequency is given by

\[ \omega_i^2(A) = f'(u_i), \] (5)

for some \( 0 < u_i < A \). The accuracy, however, depends greatly upon the location point.

In Table 1 we present the criteria suggested by Ji-huan He in [11] for choosing a suitable location point \( u_i \).

| Conditions | Location point for Eq. (5) |
|------------|---------------------------|
| \( u f''(u) < 0 \) | \( 0 < u_i < A/2 \) |
| \( u f''(u) > 0 \) | \( A/2 \leq u_i < A \) |

Table 1: Criterion for choosing a location point
Therefore, the analytical approximate frequency \( \omega \) as a function of \( A \) is
\[
\omega_{\text{app}}(A) = \sqrt{f'(u_i)}.
\] (6)
From Eq. (6) we obtain the following approximate periodic solution to (1)
\[
u_{\text{app}}(t) = A \cos \left( \sqrt{f'(u_i)} \cdot t \right).
\] (7)

3 Numerical examples

In this section, we will give four examples to illustrate the use and the effectiveness of the present approach.

Example 1
Consider the cubic-quintic Duffing nonlinear oscillator, which is modelled by the following second-order differential equation
\[
u'' + \nu + \nu^3 + \nu^5 = 0,
\] (8)
with initial conditions
\[
u(0) = A, \quad \nu'(0) = 0.
\] (9)
In the present example we have \( f(\nu) = \nu + \nu^3 + \nu^5 \), it is clear that \( f \) is an odd function and satisfies \( f(\nu)/\nu > 0 \).
Calculating we have \( f'(\nu) = 1 + 3\nu^2 + 5\nu^4 \) and \( f''(\nu) = 6\nu + 20\nu^3 \), hence \( uf''(\nu) > 0 \). Now, considering the criterion given in Table 1 we must take the location points \( A/2 \leq u_i < A \). If we take \( u_i = 0.5772A \) and consider the proposed approach in Eq. (6), one can assume for the frequency-amplitude formulation
\[
\omega_{\text{app}}(A) = \sqrt{1 + 3(0.5772)^2A^2 + 5(0.5772)^4A^4}.
\] (10)
We, therefore, obtain the following periodic solution:
\[
u_{\text{app}}(t) = A \cos \left( \sqrt{1 + 3(0.5772)^2A^2 + 5(0.5772)^4A^4} \cdot t \right)
\] (11)
which has a high accuracy (see Figs. 1-2).
The exact frequency for the present example is given by (10):
\[
\omega_{\text{ex}}(A) = \frac{2\pi}{\int_0^{\pi/2} \frac{4d\theta}{\sqrt{1 + \frac{1}{3}(1 + \sin^2 \theta)A^2 + \frac{1}{5}(1 + \sin^2 \theta + \sin^2 \theta)A^4}}}
\] (12)
From Table 2 it can be observed that Eq. (10) yield excellent analytical approximate periods for both small and large values of oscillation amplitude \( A \).
Table 2: Comparison between frequencies \( \omega_{\text{app}}(A) \) and \( \omega_{\text{ex}}(A) \) for different values of \( A \).

| \( A \)  | \( \omega_{\text{app}}(A) \) Eq. (10) | \( \omega_{\text{ex}}(A) \) Eq. (12) | Relative Error (%) |
|---------|--------------------------------------|--------------------------------------|-------------------|
| 1/1000  | 1.0000004997                         | 1.0000003750                         | 0.0000124%        |
| 1/10    | 1.0000499755                         | 1.0000375023                         | 0.0012401%        |
| 10      | 75.171283755                         | 75.177400632                         | 0.0081363%        |
| 50      | 1863.0910920                        | 1867.5739782                        | 0.2400379%        |
| 100     | 7450.3513534                        | 7468.8303066                        | 0.2474142%        |
| 1000    | 744968.72043                       | 746834.68847                        | 0.2498502%        |

Fig. 1: Comparison of analytical approximation (dashed) and exact solution (black) for \( A = 1/10 \) in example 1.

Fig. 2: Comparison of analytical approximation (dashed) and exact solution (black) for \( A = 50 \) in example 1.
Example 2
Consider the nonlinear oscillator
\[ u'' + u + u^5 = 0, \]  \hspace{1cm} (13)
subject to the initial conditions
\[ u(0) = A, \quad u'(0) = 0. \]  \hspace{1cm} (14)

For this problem,
\[ f(u) = u + u^5, \]
it is clear that \( f \) is an odd function and satisfies \( f(u)/u > 0 \).
Derivating we have,
\[ f'(u) = 1 + 5u^4 \quad \text{and} \quad f''(u) = 20u^3, \]
and hence
\[ u f''(u) = 20u^4 > 0. \]
Therefore, considering the criterion given in Table 1 we must take the location points \( A/2 \leq u_i < A \).
If we take \( u_i = 0.5779A \) and consider the proposed approach in Eq. (6), one can assume for the frequency-amplitude formulation
\[ \omega_{app}(A) = \sqrt{1 + 5(0.5779)^4 A^4}. \]  \hspace{1cm} (15)

The exact frequency for the present problem was established in [19] and is given by
\[ \omega_{ex}(A) = \pi \sqrt{A^4 + 3} \left( \int_0^{\pi/2} \frac{1}{\sqrt{1 + \left( \frac{4}{A^4} \right) \left( \sin^2 \theta + \sin^4 \theta \right)}} d\theta \right)^{-1}. \]  \hspace{1cm} (16)

To illustrate and verify accuracy of these approximate analytical approach, a comparison of approximate frequencies \( \omega_{app}(A) \) for different values of amplitude \( A \) and the exact frequencies \( \omega_{ex}(A) \) is presented in Table 3. Note that the approximation is very accurate for small values and large values of \( A \). From Table 3 we can see that

| A     | \( \omega_{app}(A) \) Eq. (15) | \( \omega_{ex}(A) \) Eq. (16) | Relative Error (%) |
|-------|---------------------------------|------------------------------|--------------------|
| 1/100 | 1.00000000000                   | 1.00000000000                | 0.0000000%         |
| 1     | 1.0000000028                    | 1.0000000031                 | 0.0000000%         |
| 1/10  | 1.0000278833                    | 1.0000312493                 | 0.0003365%         |
| 1     | 1.2480683052                    | 1.2647077571                 | 1.3156756%         |
| 10    | 74.684301857                    | 74.690887847                 | 0.0088176%         |
| 100   | 746.77603739                    | 746.83420769                 | 0.0077840%         |
| 500   | 186694.01678                    | 186708.55006                 | 0.0077839%         |
| 1000  | 746776.06710                    | 746834.20022                 | 0.0077839%         |
| 10000 | 7.467760 \times 10^7           | 7.468342 \times 10^7        | 0.0077839%         |

Table 3: Comparison between frequencies \( \omega_{app}(A) \) and \( \omega_{ex}(A) \) for different values of \( A \).

\[ \lim_{A \to 0^+} \frac{\omega_{app}(A)}{\omega_{ex}(A)} = 1 \quad \text{and} \quad \lim_{A \to \infty} \frac{\omega_{app}(A)}{\omega_{ex}(A)} = 0.999922. \]  \hspace{1cm} (17)

Considering the approximation for the frequency obtained in Eq. (15) the approximate solution of Eq. (13) becomes
\[ u_{app}(t) = A \cos \left( \sqrt{1 + 5(0.5779)^4 A^4} \cdot t \right). \]  \hspace{1cm} (18)
For this example we will not show graphs as we did in the previous example, because the high precision would not allow the distinction between them.

**Example 3**
Consider the cubic-quintic Duffing nonlinear oscillator, which is modelled by the following second-order differential equation

\[ u'' + \frac{1}{u} = 0, \quad (19) \]

with initial conditions

\[ u(0) = A, \quad u'(0) = 0. \quad (20) \]

This is an important and interesting nonlinear differential equation since it occurs in the modeling of certain phenomena in plasma physics \[9\].

The exact solution for Eq. (19) as a function of \( A \) was obtained in \[5\] and this is

\[ \omega_{ex}(A) = 2\pi \left[ 2\sqrt{2} A \int_0^1 \frac{ds}{\sqrt{\ln(1/s)}} \right]^{-1}. \quad (21) \]

To use the method presented in the section \[2\] we will consider \( f(u) = \frac{1}{u} \), it is clear that \( f \) is an odd function and satisfies \( f(u)/u > 0 \).

Calculating, we get \( f'(u) = -\frac{1}{u^2} \) and \( f''(u) = \frac{2}{u^3} \), hence \( uf''(u) > 0 \). Now, considering again the criterion given in Table \[1\] we must take the location points \( A/2 \leq u_t < A \). If we take \( u_t = 0.799A \) and consider the proposed approach in Eq. (6), one can assume for the frequency-amplitude formulation

\[ \omega_{app}(A) = \sqrt{\frac{1}{(\frac{799}{1000})^2A^2}} = \frac{1000}{799A}. \quad (22) \]

| \( A \) | \( \omega_{app}(A) \) Eq. (22) | \( \omega_{ex}(A) \) Eq. (21) | Relative Error (%) |
|---|---|---|---|
| 1/1000 | 1251.5644556 | 1253.3141373 | 0.13960% |
| 1/100 | 125.15644556 | 125.33141373 | 0.13960% |
| 1/10 | 12.515644556 | 12.533141373 | 0.13960% |
| 1 | 1.2515644556 | 1.2533141373 | 0.13960% |
| 10 | 0.1251564456 | 0.1253314137 | 0.13960% |
| 100 | 0.0125156446 | 0.0125331414 | 0.13960% |
| 500 | 0.0025031289 | 0.0025066282 | 0.13960% |
| 1000 | 0.0012515644 | 0.0012533141 | 0.13960% |

Table 4: Comparison between frequencies \( \omega_{app}(A) \) and \( \omega_{ex}(A) \) for different values of \( A \).

\[ \lim_{A \to 0^+} \frac{\omega_{app}(A)}{\omega_{ex}(A)} = \lim_{A \to \infty} \frac{\omega_{app}(A)}{\omega_{ex}(A)} = 0.9986. \quad (23) \]

Finally, considering the approximation \( \omega_{app}(A) \), we have obtain the following periodic solution of the Eq. (19)

\[ u_{app}(t) = A \cos \left( \frac{1000}{799A} t \right). \quad (24) \]

The obtained solution is of remarkable accuracy, as shown in Table 4 and Fig. 3.
Example 4
As a last example, we consider the following nonlinear differential equation:

$$u'' + u + \frac{u}{\sqrt{1 + u^2}} = 0, \quad u(0) = A, \quad u'(0) = 0.$$  \hspace{1cm} (25)

Which, \(f(u) = u + \frac{u}{\sqrt{1 + u^2}}\). Its derivatives are:

$$f'(u) = 1 + \frac{1}{\sqrt{(1 + u^2)^3}}, \quad f''(u) = -\frac{3u}{\sqrt{(1 + u^2)^5}}.$$  \hspace{1cm} (26)

From Eq. (26) we have \(uf''(u) < 0\). Considering the criterion given in Table 1 we must take the location points \(A < u_i < A/2\). If we take \(u_i = 0.48A\) and consider the proposed approach in Eq. (6), one can assume for the frequency-amplitude formulation

$$\omega_{app}(A) = \sqrt{1 + \frac{1}{(1 + (0.48A)^2A^2)^{3/2}}}.$$  \hspace{1cm} (27)

The nonlinear oscillator described in Eq. (25) is a conservative system. By integrating Eq. (25) and using the initial conditions, we arrive at

$$\omega_{ex}(A) = \frac{1}{2\pi} \left( \int_0^{\pi} \frac{A\cos \theta}{\sqrt{A^2\cos^2 \theta - 2(\sqrt{1 + A^2} \sin \theta - \sqrt{1 + A^2})}} d\theta \right)^{-1}.$$  \hspace{1cm} (28)

By taking into account our approximation made through He’s frequency-amplitude formulation Eq. (27) and \(\omega_{ex}(A)\) from Eq. (28) we can calculate the Table 5 for small and large values of \(A\).
Also, considering the approximation (27), we have obtained the following periodic solution of the Eq. (25)

\[
\omega_{\text{app}}(A) = A \cos \left( \sqrt{1 + \frac{1}{1 + \frac{48}{100} A^2}} \right) t.
\]

The obtained solution is very acceptable accuracy, as shown in Fig. 4 and Fig. 5. We can conclude that formula (27) is valid for the whole range of values of amplitude of oscillation and its maximum relative error is 5.3% and this is obtained when \( A = 10 \). We can also see that, for very large or very small values of \( A \), we have

\[
\lim_{A \to 0^+} \frac{\omega_{\text{app}}(A)}{\omega_{\text{ex}}(A)} = \lim_{A \to \infty} \frac{\omega_{\text{app}}(A)}{\omega_{\text{ex}}(A)} = 1.
\]

| \( A \)  | \( \omega_{\text{app}}(A) \) Eq. (27) | \( \omega_{\text{ex}}(A) \) Eq. (28) | Relative Error (%) |
|--------|-----------------------------------|-----------------------------------|-------------------|
| 1/1000 | 1.4142134402                      | 1.4142134298                     | 0.0000007%        |
| 1/10   | 1.4142013439                      | 1.4142003049                     | 0.0000734%        |
| 1      | 1.4129946662                      | 1.4128952474                     | 0.0070365%        |
| 10     | 1.0042330178                      | 1.060605289                      | 5.3151037%        |
| 100    | 1.00000045182                     | 1.0063415277                     | 0.6297076%        |
| 1000   | 1.0000000045                      | 1.0006636862                     | 0.0635976%        |
| 10000  | 1.0000000000                      | 1.0000636597                     | 0.0063655%        |

Table 5: Comparison between frequencies \( \omega_{\text{app}}(A) \) and \( \omega_{\text{ex}}(A) \) for different values of \( A \).

Fig. 4: Comparison of analytical approximation (dashed) and exact solution (black) for \( A = 10 \) in example 4.
4 Conclusions

He’s new amplitude-frequency relationship recently established by Ji-Huan He in [11] is proved to be a powerful mathematical tool for use in the search for periodic solutions of nonlinear oscillators. It is simple, straightforward and effective. Moreover the approximate analytical solutions are valid for small as well as large amplitudes of oscillation.

The new method applied in this paper is of potential and can be applied to other strongly nonlinear oscillators with more general restoring forces provided that they meet the requirements established in section 2.

Finally, four examples have been presented to illustrate excellent accuracy of the analytical approximate periods and the corresponding periodic solutions. The technique is very simple in principle, all numerical calculations have been made with the help of the software MATHEMATICA.

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