RANKIN-SELBERG INTEGRAL WITH NON-UNIQUE MODEL FOR THE
STANDARD $\mathcal{L}$-FUNCTION OF $G_2$

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Abstract. Let $\mathcal{L}^S(\pi, s, st)$ be a partial $\mathcal{L}$-function of degree 7 of a cuspidal automorphic representation $\pi$ of the exceptional group $G_2$. Here we construct a Rankin-Selberg integral for representations having certain Fourier coefficient.

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References

1. Introduction

Until the late 1980’s it was believed that a Rankin-Selberg integral must unfold to a unique model of the representation in order to be factorizable. By unique model we mean that the space of functionals on the representation space with certain invariance properties is one-dimensional. The most common is the Whittaker model, but other unique models such as Bessel model were also used.

In their pioneering work [13] I. Piatetski-Shapiro & S. Rallis interpreted an integral, earlier considered by A. Andrianov[1], as an adelic integral that unfolds to a non-unique model. Although the functional is not factorizable, the integral is, since the local integral produces the same $\mathcal{L}$-factor for any functional with the same invariance properties applied to a spherical vector.

There are many examples of adelic integrals that unfold with non-unique models. Only few of them were shown to represent $\mathcal{L}$-functions. Some more examples are detailed in [2] and [3]. All the examples rely on the knowledge of the generating function for the considered $\mathcal{L}$-function.
In this paper we consider a new Rankin-Selberg integral on the exceptional group $G_2$ and prove that it represents the standard $L$-function $L^S(\pi, s, st)$ of degree 7 for cuspidal representations having certain Fourier coefficient along the Heisenberg unipotent subgroup. The candidate global integral was suggested by Dihua Jiang in the course of the work on [7] and he also performed the unfolding. However, since the generating function for the $L$-function has not been known, the unramified computation was not completed. It is only now that we found a way to overcome this difficulty. To the best of our knowledge this is the first time the unramified computation is performed without explicit knowledge of the generating function.

The integral introduced here binds the analytic behaviour of $L^S(\pi, s, st)$ with that of a degenerate Eisenstein series of Spin$_8$ which was studied in [7]. In the last section we use information on the poles of this Eisenstein series to show that for a cuspidal representation $\pi$ having certain Fourier coefficient along the Heisenberg unipotent subgroup, the non-vanishing of the theta lift of $\pi$ to the finite group scheme $S_3$ is equivalent to the $L$-function having a double pole at $s = 2$.

2. Preliminaries

Let $k$ be a number field and $\mathcal{P}$ be its set of places. For any $\nu \in \mathcal{P}$ denote by $k_{\nu}$ the local field associated to $\nu$. If $\nu < \infty$ denote by $\mathcal{O}_{\nu}$ the ring of integers of $k_{\nu}$ and by $q_{\nu}$ the cardinality of the residue field of $k_{\nu}$.

2.1. The group $G_2$. Let $G$ be the split simple algebraic group of the exceptional type $G_2$ defined over $k$ with torus $T$ and Borel subgroup $B$. Fix a root system of $G$ and denote by $\alpha$ and $\beta$ the short and the long simple roots respectively. The Dynkin diagram of $G$ has the form

$$
\alpha \quad \beta
$$

and the set of positive roots is

$$
\Phi^+ = \{ \alpha, \beta, \alpha + \beta, 2\alpha + \beta, 3\alpha + \beta, 3\alpha + 2\beta \}.
$$

The fundamental weights are denoted by

$$
\omega_1 = 2\alpha + \beta, \quad \omega_2 = 3\alpha + 2\beta.
$$

For any root $\gamma$ fix a one-parametric subgroup $x_\gamma : \mathbb{G}_a \to G$. For any simple root $\gamma$ denote by $w_\gamma$ the simple reflection with respect to it, that is an element of the Weyl group of $G$. Define also the coroot subgroups $h_\gamma : \mathbb{G}_m \to G$ such that for any root $\epsilon$

$$
\epsilon (h_\gamma (t)) = t^{<\epsilon, \gamma^\vee>}.
$$

2.2. The partial $L$-function. The dual group of $G$ is isomorphic to $G_2(\mathbb{C})$. Denote the seven-dimensional complex representation of $G_2(\mathbb{C})$ by $st$. For an irreducible cuspidal representation $\pi = \otimes_{\nu} \pi_{\nu}$, unramified outside of a finite set of places $S$, the standard partial $L$-function of $\pi$ is defined by:

$$
L^S(s, \pi, st) = \prod_{\nu \in S} \frac{1}{\det (I - st(t_{\pi_{\nu}})q_{\nu}^{-s})}.
$$
Here \( t_{\pi_v} \) is the Satake parameter of \( \pi_v \).

2.3. **Fourier coefficients.** The group \( G \) contains a Heisenberg parabolic subgroup \( P = M \cdot U \). The Levi part \( M \) is isomorphic to \( GL_2 \) generated by the simple root \( \alpha \), while \( U \) is a five dimensional Heisenberg group. We denote the elements of \( U \) by

\[
\left( \begin{array}{cc}
 r_1 & r_2 \\
 r_3 & r_4 \\
 r_5 & r_6
\end{array} \right),
\]

where \( r_1 \) and \( r_5 \) are real numbers, \( r_2 \) and \( r_6 \) are imaginary numbers, \( r_3 \) and \( r_4 \) are complex numbers.

The group \( M \) acts naturally on \( U \) and hence on \( \text{Hom}(U, \mathbb{C}) \). It was shown in [11] that for any field \( F \) of characteristic zero the \( M(F) \)-orbits of \( \text{Hom}(U(F), F) \) are naturally parametrized by isomorphism classes of cubic \( F \)-algebras.

Fixing an additive complex unitary character \( \psi = \otimes_v \psi_v \) of \( k \backslash \mathbb{A} \) this give rise to the correspondence between \( M(k) \)-orbits of complex characters of \( U(k) \backslash U(\mathbb{A}) \) and cubic algebras over \( k \). Let us denote by \( \Psi \), the character corresponding to the split cubic algebra \( k \times k \times k \) and call it the split character. More explicitly,

\[
\Psi(u(r_1, r_2, r_3, r_4, r_5)) = \psi(r_2 + r_3).
\]

Its stabilizer \( S_{\Psi} \) in \( M(k) \) is isomorphic to \( S_3 \) and is generated by \( w_\alpha \) and \( h_\alpha (-1) x_\alpha (-1) x_{-\alpha} (1) \).

Denote by \( \mathcal{A}(G) \) the space of automorphic forms on \( G \). For any form \( \varphi \) in \( \mathcal{A}(G) \) and complex character \( \Psi \) of \( U(k) \backslash U(\mathbb{A}) \) define the Fourier coefficient of \( \varphi \) with respect to \( (U, \Psi) \) by

\[
L_{\Psi}(\varphi)(g) = \int_{U(k) \backslash U(\mathbb{A})} \varphi(ug) \overline{\Psi(u)} \, du.
\]

For any \( g \in G \) \( L_{\Psi}(\cdot)(g) \) defines a functional in \( \text{Hom}_{U(\mathbb{A})}(\mathcal{A}(G), \mathbb{C}) \).

For an automorphic representation \( \pi \) of \( G(\mathbb{A}) \) we say that \( \pi \) supports a \( (U, \Psi) \) coefficient if there exists a function \( \varphi \) from the underlying space of \( \pi \) such that \( L_{\Psi}(\varphi) \neq 0 \).

It was shown in [13] Theorem 3.1 that for any cuspidal representation \( \pi \) there exists an étale cubic algebra such that \( \pi \) supports a Fourier coefficient with respect to this algebra. Conversely, in [12] it was shown that for any étale cubic algebra there exists a cuspidal representation supporting the Fourier coefficient corresponding to it. In this paper we consider only representations that support the split Fourier coefficient.

For a finite \( v \in \mathcal{P} \) denote by \( K_v \) the maximal compact subgroup \( G(\mathcal{O}_v) \) of \( G(k_v) \) and by \( \mathcal{H}_v \) the spherical Hecke algebra. Given a complex character \( \Psi \) of \( U(k_v) \) define

\[
\mathcal{M}_\Psi = \{ f : G(k_v) \to \mathbb{C} \mid f(ug) = \Psi(u) f(g) \forall u \in U(k_v), k \in K_v \}
\]

\[
\mathcal{M}_\Psi^0 = \{ f : G(k_v) \to \mathbb{C} \mid f(sugk) = \Psi(su) f(g) \forall u \in U(k_v), s \in S_\Psi, k \in K_v \}.
\]

For \( f \in \mathcal{H}_v \) define its Fourier transform \( f^\Psi \) with respect to the character \( \Psi \) by

\[
f^\Psi(g) = \int_{U(k_v)} f(ug) \overline{\Psi(u)} \, du.
\]

Obviously \( f^\Psi \) belongs to \( \mathcal{M}_\Psi^0 \).

2.4. **The group Spin_8.** Let \( H \) be a simply connected algebraic group of type \( D_4 \). We label its simple roots according to the following diagram.
The group of outer automorphisms of $H$ is isomorphic to $S_3$. Fixing one-parametric subgroups $x_\gamma : G_a \to H$ defines a splitting of the sequence

$$1 \to H^{ad} \to \text{Aut} (H) \to \text{Out} (H) \to 1 .$$

In particular the semidirect product $H \rtimes S_3$ can be formed. It is well known that the centralizer of $S_3$ in $H \rtimes S_3$ is the group $G$. We identify $G$ with a subgroup of $H$ in this way. The group $H$ contains a maximal Heisenberg parabolic $P_H = M_H U_H$ such that $P = P_H \cap G$ given by

$$M_H \simeq \{(g_1, g_2, g_3) \in GL_2 \times GL_2 \times GL_2 \mid \det (g_1) = \det (g_2) = \det (g_3)\} .$$

The modulus character of $P_H$ is given by $\delta_{P_H} (g_1, g_2, g_3) = | \det (g_1) |^5$.

2.5. The Eisenstein series. Consider the induced representation $I_H (s) := \text{Ind}_{P_H}^{H (A)} \delta_{P_H}^s$. All induced representations in this paper are not normalized. For any $K$-finite standard section $f_s$ define an Eisenstein series

$$E (g, f_s) = \sum_{\gamma \in P_H (F) \backslash H (F)} f_s (\gamma g) .$$

It has a meromorphic continuation to the whole complex plane. The behaviour at $s = 4/5$ was studied in [1].

**Proposition 2.1** ([7] Proposition 9.1). For any standard section $f_s$, the Eisenstein series $E (g, f_s)$ has at most a double pole at $s = \frac{4}{5}$. The double pole is attained by the spherical section $f^0$. Also, the space

$$\text{Span}_C \left\{ \left( s - \frac{4}{5} \right)^2 E (g, f_s) \bigg|_{s = \frac{4}{5}} \right\} ,$$

is isomorphic to the minimal representation $\Pi$ of $H$.

It is customary to define the normalized Eisenstein series

$$E^* (g, f_s) = j (s) E (g, f_s) ,$$

where $j (s) = \zeta (5s) \zeta (5s - 1)^2 \zeta (10s - 4)$.

3. The Zeta Integral

Let $\pi = \bigotimes \pi_v$ be an irreducible cuspidal representation of $G (A)$. For $\varphi \in \pi$ and $f_s \in I_H (s)$ we consider the following integral,

$$Z (s, \varphi, f) = \int_{G (A) \backslash G (K)} \varphi (g) E^* (g, f_s) \, dg .$$

Since $\varphi$ is cuspidal, and hence rapidly decreasing, the integral defines a meromorphic function on the complex plane. Our main result is:

**Theorem 3.1.** Let $\pi = \bigotimes \pi_v$ be an irreducible cuspidal representation supporting the split Fourier coefficient. Let $\varphi = \bigotimes \varphi_v \in \pi$, $f_s = \bigotimes f_s, v \in I_H (s)$ be factorizable data. Let $S \subset \mathcal{P}$ be a finite set such that if $v \notin S$ then

- $v \not\in \{ 2, 3, \infty \}$
- $\varphi_v$ is spherical
- $f_s, v$ is spherical.

Then

$$Z (s, \varphi, f) = \mathcal{L}^S (s, \pi, \mathfrak{a}) d_S (s, \varphi_S, f_S) .$$

Moreover for any $s_0$ there exist vectors $\varphi_S, f_S$ such that $d_S (s, \varphi_S, f_S)$ is analytic in a neighbourhood of $s_0$ and $d_S (s_0, \varphi_S, f_S) \neq 0$. 
In particular the partial \( L \)-function \( L^S(s, \pi, st) \) admits a meromorphic continuation.

**Remark 3.1.** If \( \pi \) does not support the split Fourier coefficient, the zeta integral vanishes identically. However if \( \pi \) supports a Fourier coefficient corresponding to an étale cubic algebra \( E \) there is a similar integral, using an Eisenstein series on the inner form of \( \text{Spin}_8 \) corresponding to \( E \), that is expected to represent the same \( L \)-function. We plan to study these integrals in the near future.

The proof of the theorem will occupy the rest of the paper. We will explain main ideas, deferring the technical part to later sections and appendices.

**Theorem 3.2 (Unfolding).** For \( \Re (s) >> 0 \) we have

\[
Z(s, \varphi, f) = \int_{U(\mathbb{A}) \cap G(\mathbb{A})} L_{\Psi^s}(\varphi)(g) F^*(g, s) \, dg,
\]

where

\[
F^*(g, s) = j(s) \int_{\mathbb{A}} f_s(w_2 w_3 x_{-\alpha_1})(1) x_{\alpha_+}(r) \psi(r) \, dr.
\]

This computation was performed by Dihua Jiang, but since his proof was never published we include it in section 4.

The function \( F^*(g, s) \) is factorizable whenever the involved section \( f_s \) is. In particular

\[
F^*(g, s) = \Pi_v F^*_v(g_v, s), \quad \text{where} \quad F^*_v(g, s) = j_v(s) \int_{k_v} f_{s, v}(\mu x_{\alpha_+}(r) g) \psi_v(r) \, dr,
\]

and for almost all places \( f_{s, v} = f_{s, v}^0 \in \text{Ind}_{H(k_v)}^{H(k_v)} \delta_{P_{\mathbb{H}}}^v \) is a spherical vector with \( f_{s, v}^0(1) = 1 \).

Note that as the space \( \text{Hom}_{U(\mathbb{A})}(\pi, \mathbb{C} \Psi_v) \) is usually infinite dimensional the functional \( L_{\Psi^s} \) is not necessarily factorizable. Nevertheless it will be shown that the integral \( Z(\varphi, f, s) \) is factorizable. The factorizability of the integral follows from the next surprising local statement, that replaces the unramified computation.

**Theorem 3.3 (Unramified Computation).** Let \( \pi_v \) be an irreducible unramified representation of \( G(k_v) \) and let \( v^0 \) be a fixed spherical vector in \( \pi_v \). Assume that \( \text{Hom}_{U(k_v)}(\pi_v, \mathbb{C} \Psi_{v, v}) \neq 0 \). There exists \( s_0 \in \mathbb{R} \) such that for any \( \Re s > s_0 \) and any \( l \in \text{Hom}_{U(k_v)}(\pi_v, \mathbb{C} \Psi_{s, v}) \) it holds

\[
\int_{U(k_v) \backslash G(k_v)} l(\pi_v(g)v^0) F^*_v(g, s) \, dg = L(5s - 2, \pi_v, st) l(v^0),
\]

where \( F^*_v(g, s) \) is the function corresponding to the normalized spherical section \( f_{s, v}^0 \).

The identity in the main theorem follows from equation 3.2 using standard argument as in [13]. For the sake of completeness of presentation the argument is included in section 5. This argument also defines \( d_{S}(s, \varphi, s, f_s) \) explicitly.

The proof of theorem 3.3 is the most non-trivial part of the paper and can be found in section 6. In fact the proof is quite amusing. Following the ideas of [13] it boils down to proving the identity between \( F_s \) and a Fourier transform of the generating function \( \Delta \) of \( L(s, \pi, st) \). We could not find the explicit formula for \( \Delta \), which must be very complicated. Instead we have proven that the two functions become equal after being convolved with a third function. Both sides are evaluated explicitly (Appendix A) and (Appendix B). Finally we show in proposition 6.2 that the latter convolution is in fact an invertible operation.
Theorem 3.4 (Ramified Computation). For any $s_0 \in \mathbb{C}$ there exist datum $\varphi_s$ and $f_s$ such that $d_s(s_0, \varphi_s, f_s)$ is holomorphic and non-vanishing in a neighbourhood of $s_0$.

This theorem is proven in section 7.

4. Unfolding

The proof of Theorem 3.2 is fairly standard. First we introduce some more notations that will be used in this section and also in section 7.

Denote by $Q = LV$ the maximal parabolic subgroup of $G$ other than $P$. The Levi part $L \simeq GL_2$ is generated by the root $\beta$. The unipotent radical of the Borel subgroup of $L$ will be denoted by $N_\beta$.

The following fact will be used ([14, Theorem 5]):

\[ \int_{R(k) \backslash R(A)} \varphi(rg) \, dr = \sum_{\nu \in N_3(k) \backslash L(k)} W_\psi(\varphi)(\nu g), \]

where $W_\psi(\varphi)$ is the standard Whittaker coefficient.

There are five $G(k)$-orbits of $P_H(k) \backslash H(k)$. The representatives of the orbits and their stabilizers are given in the next Lemma [12, Lemma 2.1]:

**Lemma 4.1.** The following is a list of representatives of $G(k)$-orbits in $P_H(k) \backslash H(k)$ and their stabilizers:

1. $\mu = 1$, and the stabilizer $G^\mu = P$.
2. $\mu = w_2w_1, w_2w_3, w_2w_4$, and the stabilizer $G^\mu = LR$.
3. $\mu = w_2w_3x_{-\alpha_1}(1)$ is a representative of the open orbit. The stabilizer of $P_H(k)\mu G(k)$ is $G^\mu = T^\mu \cdot U^\mu$ where $T^\mu = \{ h_3 \alpha_2 \beta(t) \mid t \in k^\times \}$, $U^\mu = \{ u(r_1, r_2, r_3, r_4, r_5) \mid r_i \in k \}$

**Proof of Theorem 3.2.** For $9\Re(s) >> 0$ it holds

\[ \int_{G(k) \backslash G(\mathbb{A})} \varphi(g) E(g, f_s) \, dg = \int_{G(k) \backslash G(\mathbb{A})} \varphi(g) \sum_{\gamma \in P_H(k) \backslash H(k)} f_s(\gamma g) \, dg = \sum_{\mu \in P_H(k) \backslash H(k) / G(k)} I_\mu(\varphi, f_s), \]

where

\[ I_\mu(\varphi, f_s) = \int_{G^\mu(k) \backslash G(\mathbb{A})} \varphi(g) f_s(\mu g) \, dg. \]

Next we show that $I_\mu(\varphi, f_s) = 0$ unless $\mu$ is a representative of the open orbit.

1. $\mu = 1$. Then

\[ I_\mu(\varphi, f_s) = \int_{P(k) \backslash G(\mathbb{A})} \varphi(g) f_s(g) \, dg = \int_{M(k) \backslash U(k) \backslash G(\mathbb{A})} f_s(g) \int_{U(k) \backslash U(\mathbb{A})} \varphi(ug) \, du \, dg = 0, \]

since $\varphi$ is cuspidal.

2. $\mu = w_2w_1, w_2w_3, w_2w_4$. Then

\[ I_\mu(\varphi, f_s) = \int_{L(k) R(k) \backslash G(\mathbb{A})} \varphi(g) f_s(\mu g) \, dg = \int_{L(k) R(k) \backslash G(\mathbb{A})} f_s(\mu g) \int_{R(k) \backslash R(\mathbb{A})} \varphi(rg) \, dr \, dg. \]
Proof of theorem 3.1. 

By definition (5.1)

\[ G \]

where

(4.3)

\[ \Re s > s \]

Expanding the function given by an inner integral along the root \( \alpha + \beta \) and collapsing the sum with the outer integration the above equals

(4.2)

\[ \int_{\Omega} \int_{U(\mu)} \phi(ug) \Psi_s(u) du \int_{\lambda} f_s(\mu x_{\alpha+\beta}(r) g(r) dr dg \]

Now let us compute the contribution from the open orbit. For \( \mu = w_2 w_3 x_{-\alpha_1/2} \) it holds

\[ I_\mu(\phi, f) = \int T^\mu(k) U(\mu) U(k) U(\mu) \int_{\lambda} f_s(\mu x_{\alpha+\beta}(r) g(r) dr dg \]

Expanding the function given by an inner integral along the root \( \alpha + \beta \) and collapsing the sum with the outer integration the above equals

(4.3)

\[ \int_{\Omega} \int_{U(\mu)} \phi(ug) \Psi_s(u) du \int_{\lambda} f_s(\mu x_{\alpha+\beta}(r) g(r) dr dg \]

Since \( U = U^\mu \cdot x_{\alpha+\beta} \) we bring the integral to its final form

\[ \int_{\Omega} \int_{U(\mu)} \phi(ug) \Psi_s(u) du \int_{\lambda} f_s(\mu x_{\alpha+\beta}(r) g(r) dr dg \]

5. Derivation of The Main Theorem from Theorems 3.3 and 3.4

Proof of Theorem 3.1. By definition

(5.1)

\[ Z(s, \phi, f) = \lim_{S \subset \Omega \subset \mathbb{C}} \int_{U(\mu)} L_{\Psi_s}(\phi, g) F^*(g, s) dg \]

where \( G(\lambda) = \prod_{\nu \in \Omega} G(k_\nu) \). Fix \( s_0 \in \mathbb{R} \) such that the right hand side of Equation 3.1 converges for \( \Re s > s_0 \). The integrals of the right hand side of Equation 5.1 must also converge there. Also fix \( s_1 \in \mathbb{R} \) such that Equation 3.2 holds for \( \Re s > s_1 \). For a finite set \( S \subset \Omega \) and \( \nu \notin \Omega \) we have

\[ \int_{U(\mu)} L_{\Psi_s}(\phi, g) F^*(g, s) dg = \]

\[ = \int_{U(\mu)} F^*(g, s) \int_{U(k_\nu)} L_{\Psi_s}(\phi, g g_\nu) F^*(g g_\nu, s) dg_\nu dg = \]

\[ = \int_{U(\mu)} F^*(g, s) \int_{U(\mu)} L_{\Psi_s}(\phi, g g_\nu) F^*(g g_\nu, s) dg_\nu dg = \]

\[ = \mathcal{L}(5s - 2, \pi_\nu, g) \int_{U(\mu)} L_{\Psi_s}(\phi, g) F^*(g, s) dg , \]

where \( \mathcal{L}(\cdot) \) is a function of interest.
where the last equality is due to [Theorem 3.3] A priori the last equality holds only for \( \Re s > \max \{s_0, s_1\} \), but since \( \mathcal{L} (5s - 2, \pi_\nu, \mathfrak{st}) \) is a meromorphic function the equality actually holds for \( \Re s > s_0 \). Plugging this into Equation 5.1 we get

\[
Z (s, \varphi, f) = \lim_{S \subset \mathfrak{c} \subset \mathfrak{p}} \int_{U (\mathfrak{a}) \backslash \mathcal{G} (\mathfrak{a})} L_{\varphi_\pi} (\varphi) (g) F^* (g, s) \, dg = \\
= \lim_{S \subset \mathfrak{c} \subset \mathfrak{p}} \prod_{\nu \in \Omega (\mathfrak{c})} \mathcal{L} (5s - 2, \pi_\nu, \mathfrak{st}) \int_{U (\mathfrak{a}) \backslash \mathcal{G} (\mathfrak{a})_S} L_{\varphi_\pi} (\varphi) (g) F^* (g, s) \, dg = \\
= \mathcal{L}^S (5s - 2, \pi, \mathfrak{st}) \int_{U (\mathfrak{a}) \backslash \mathcal{G} (\mathfrak{a})_S} L_{\varphi_\pi} (\varphi) (g) F^* (g, s) \, dg .
\]

We finish the proof by fixing our datum according to [Theorem 3.4] and taking

\[
d_s (s, \varphi, f_s) = \int_{U (\mathfrak{a}) \backslash \mathcal{G} (\mathfrak{a})_S} L_{\varphi_\pi} (\varphi) (g) F^* (g, s) \, dg .
\]

\( \square \)

6. Unramified Computation

Let \( F = k_\nu \) with the ring of integers \( \mathcal{O} \) and uniformizer \( \varpi \) for some \( \nu \notin S \). By abuse of notations we denote in this section, and in Appendix B and Appendix A \( \pi \) for \( \pi_\nu, \psi \) for \( \psi_\nu \) etc. In this section we prove [Theorem 3.3] Recall that \( \mathcal{G} (F) \) contains the maximal compact subgroup \( K = \mathcal{G} (\mathcal{O}) \). We fix on \( G \) the Haar measure \( \mu \) such that \( \mu (K) = 1 \).

Recall that the Satake isomorphism is an isomorphism of \( \mathbb{C} \)-algebras \( \mathcal{H} \cong \text{Rep} (L^2 G) \). Denote by \( A_j \in \mathcal{H} \) the elements corresponding to \( \text{Sym}^j (\mathfrak{st}) \) by the Satake isomorphism. In particular for any unramified representation \( \pi \) and a \( K \)-invariant vector \( v^0 \in \pi \) it holds

\[
(6.1) \quad \int_G A_j (g) \pi (g) v^0 \, dg = \text{tr} (\text{Sym}^j (\mathfrak{st}) (t_\pi)) (v^0),
\]

where \( t_\pi \) is the Satake parameter of \( \pi \).

For any unramified representation \( \pi \) the Satake isomorphism induces an algebra homomorphism that sends \( f \in \mathcal{H} \) to the complex number \( \hat{f} (\pi) \) such that

\[
\int_G f (g) \pi (g) v^0 \, dg = \hat{f} (\pi) v^0.
\]

In particular for \( f_1, f_2 \in \mathcal{H} \) it holds \( \hat{f_1} * \hat{f_2} = \hat{f_1} \cdot \hat{f_2} \). The homomorphism \( f \to \hat{f} (\pi) \) can be extended linearly to a map \( \mathcal{H} [\![q^{-s}]\!] \to \mathbb{C} [\![q^{-s}]\!] \).

**Lemma 6.1** (Poincaré identity). There exists a generating function \( \Delta (g, s) \in \mathcal{H} [\![q^{-s}]\!] \) such that for any unramified representation \( \pi \) with a spherical vector \( v^0 \) and any functional \( l \) on \( \pi \) it holds

\[
(6.2) \quad \int_G \Delta (g, s) l (\pi (g) v^0) \, dg = \mathcal{L} (s, \pi, \mathfrak{st}) l (v^0)
\]

for \( \Re s >> 0 \).

**Proof.** We must show that there exists \( \Delta \) with \( \hat{\Delta} (\pi, s) = \mathcal{L} (s, \pi, \mathfrak{st}) \). The construction is formal.
The relation between Proposition 6.1. There exists \( P \) such that

\[
\mathcal{L}(s, \pi, \mathfrak{s}t) = \frac{1}{\det(1 - q^{-s} \mathfrak{s}t(t_\pi))} = \prod_{i=1}^{7} (1 - q^{-s} \mathfrak{s}t(t_\pi)_{ii})^{-1} = \prod_{i=1}^{7} \sum_{j=0}^{\infty} (q^{-s} \mathfrak{s}t(t_\pi)_{ii})^j = \sum_{j=0}^{\infty} \text{tr} (\text{Sym}^j (\mathfrak{s}t(t_\pi))) q^{-js},
\]

where \( t_\pi \) is the Satake parameter of \( \pi \). The series converge absolutely for \( \Re s >> 0 \). Plugging Equation 6.1 into the previous equality gives

\[
\mathcal{L}(s, \pi, \mathfrak{s}t) l(v^0) = \int_G \left( \sum_{j=0}^{\infty} A_j q^{-js} \right) (g) l(g \cdot v^0) dg.
\]

The assertion holds for \( \Delta(\cdot, s) = \sum_{j=0}^{\infty} A_j q^{-js} \) for any unramified representation \( \pi \). Uniqueness follows from the fact that the action of the spherical functions of unramified representations gives rise to a spectral decomposition of \( \mathcal{H} \).

For any \( l \in \text{Hom}_{L(F)}(\pi, C_{\Psi_s}) \) one has

\[
\mathcal{L}(s, \pi, \mathfrak{s}t) l(v^0) = \int_G \int_{V \setminus G} l(\pi(g)v^0) \Delta(g, s) dg = \int_{V \setminus G} l(\pi(g)v^0) \Delta^{\Psi_s}(g, s) dg.
\]

Thus, in order to prove that for all unramified \( \pi \) and all \( l \in \text{Hom}_{L(F)}(\pi, \Psi_s) \) Equation 3.2 holds, it is enough to show the basic identity

\[
(6.3) \quad \Delta^{\Psi_s}(g, 5s - 2) = F^s(g, s).
\]

We prove this equality a priori only for \( \Re s >> 0 \) but since for any \( g \in G \) we know the right hand side to define a meromorphic function then this equality holds for all \( s \in \mathbb{C} \). While the right hand side is given explicitly, we do not have an explicit formula for the generating function \( \Delta(g, s) \).

To overcome that difficulty we introduce the new function \( D \in \mathcal{H}([q^{-s}]) \). Recall the Cartan decomposition \( G = KT^+K \) where

\[
T^+ = \{ t \in T \mid |\gamma(t)| \leq 1 \forall \gamma \in \Phi^+ \}.
\]

The function \( D \) is bi-\( K \)-invariant and is defined on the torus \( T^+ \) by

\[
D(t, s) = |\omega_1(t)|^{5s+1} \forall t \in T^+
\]

The relation between \( D \) and \( \Delta \) can be seen from the following proposition.

**Proposition 6.1.** There exists \( P(\cdot, s) \in \mathcal{H}[q^{-s}] \) and \( s_0 \in \mathbb{R} \) such that for \( \Re s > s_0 \) it holds

\[
D(\cdot, s) = \Delta(\cdot, 5s - 2) * P(\cdot, s).
\]

More precisely

\[
(6.4) \quad P(\cdot, s) = \frac{P_0(q^{2-5s}) A_0 - P_1(q^{2-5s}) A_1}{\zeta(5s-1) \zeta(5s+1) \zeta(5s-2)},
\]

where

\[
P_0(z) = \zeta(z) q^2 + \left( \frac{1}{q^2} + \frac{1}{q} \right) z^3 + \frac{z^2}{q} + \left( \frac{1}{q} + 1 \right) z + 1, \quad P_1(z) = \frac{z^2}{q}.
\]
Proposition 6.2. Let \( \omega_\pi \) be the normalized spherical function associated to \( \pi \). For any functional \( l \) of \( \pi \) one has

\[
\int_G D(g,s) l(\pi(g)\nu^0) \, dg = l(\nu^0) \int_G D(g,s) \omega_\pi(g) \, dg.
\]

Using Macdonald’s formula \cite{4} Theorem 4.2] for \( \omega_\pi \) this turns to a sum of geometric progressions that converges for \( \Re s > 0 \). A direct computation yields

\[
(6.5) \quad \hat{D}(\pi,s) = \int_G D(g,s) \omega_\pi(g) \, dg = \mathcal{L}(5s-2,\pi,\text{st}) \cdot Q(s,\pi).
\]

Here

\[
(6.6) \quad Q(\pi,s) = \frac{P_0(q^{2-5s}) - P_1(q^{2-5s}) \text{tr}(\delta t)(t_\pi)}{\zeta(5s-1)\zeta(5s+1)\zeta(5s-2)}.
\]

On the other hand \( \mathcal{L}(5s-2,\pi,\text{st}) = \hat{\mathcal{D}}(\pi,5s-2) \) and obviously \( Q(\pi,s) = \hat{\mathcal{P}}(\pi,s) \). Since Equation 6.5 holds for any unramified \( \pi \) the proposition follows. \( \square \)

Since the Fourier transform is a map of \( \mathcal{H} \) modules it follows:

**Corollary 6.1.**

\[D^{\Psi_s}(s) = \Delta^{\Psi_s}(5s-2) * P(s).
\]

The basic identity 6.3 will follow once we prove

\[
(6.7) \quad D^{\Psi_s} = F^* * P
\]

and

**Proposition 6.2.** There exists \( s_0 \) such that whenever \( \Re s > s_0 \) \( f \ast P(\cdot,s) = 0 \) implies \( f = 0 \) for any \( f \in \mathcal{M}_{\Psi_s} \).

Indeed from Equation 6.7 we get

\[
(\Delta^{\Psi_s}(\cdot,5s-2) - F^*(\cdot,s)) \ast P(\cdot,s) = (D^{\Psi_s} - F^*) \ast P = 0
\]

and hence by Proposition 6.2 we have \( \Delta^{\Psi_s} = F^* \) for \( \Re s > 0 \). As already mentioned this equality is actually true for all \( s \in \mathbb{C} \). The only restriction on the convergence of the integral in Theorem 3.3 is the domain of convergence of the Poincaré identity. We now turn to prove Proposition 6.2 and Equation 6.7

The following observation is useful for the proof of Proposition 6.2

**Remark 6.1.** We note that \( \mathcal{H} \) can be completed into a \( C^* \)-algebra \( \hat{\mathcal{H}} \) as a closed subspace of the reduced group \( C^* \)-algebra of \( G \). One way to do this is to use the action of \( \mathcal{H} \) on \( L^2(K \setminus G/K) \) by convolution. This is a separable Hilbert space and \( \mathcal{H} \) admits an embedding into \( B(L^2(K \setminus G/K)) \) in which we complete it with respect to the operator norm. In fact, for our needs we only need to know that a \( C^* \)-norm and such a completion exist.

**Proof of Proposition 6.2.** We will show a stronger statement: there exists \( s_0 \) such that for any \( \Re(s) > s_0 \) the element \( P(\cdot,s) \) is invertible in \( \hat{\mathcal{H}} \). For \( \Re(s) > 0 \) this is equivalent to showing that

\[
x := A_0 - \frac{P_1(q^{2-5s})}{P_0(q^{2-5s})} A_1
\]

is invertible. Since \( \hat{\mathcal{H}} \) is a \( C^* \)-algebra it will suffice to show that \( \left\| \frac{P_1(q^{2-5s})}{P_0(q^{2-5s})} A_1 \right\| < 1 \). We have

\[
\left\| \frac{P_1(q^{2-5s})}{P_0(q^{2-5s})} A_1 \right\| = \frac{\left\| P_1(q^{2-5s}) \right\|}{\left\| P_0(q^{2-5s}) \right\|} \left\| A_1 \right\|< 1.
\]
Lemma 6.2. \( \Gamma : G \) where \( GL \) viewed it as a subgroup of convolution. This tedious, but quite straightforward computation is performed in Appendix A.

Theorem 6.1. Both \( D^{\psi_s} \) and \( F^* * P \) vanish outside \( S_{\psi_s} UTK \). For \( t = h_\alpha (t_1) h_\beta (t_2) \) such that \( t_1, t_2 \in O \) it holds

\[
D^{\psi_s} (t, s) = (F^* * P) (t, s) = \begin{cases} 
\frac{1 + q^{1-5s}}{\zeta (5s+1)} \frac{t_2}{t_1} |t_1|^{5s} , & \frac{t_2}{t_1} < 1 \\
\frac{1 + q^{1-5s}}{\zeta (5s+1)} \frac{t_1}{t_2} |t_2|^{5s} , & \frac{t_1}{t_2} > 1 \\
\frac{1 + 2q^{1-5s}}{\zeta (5s+1)} |t_1|^{5s+1} , & \frac{t_1}{t_2} = 1
\end{cases}
\]

For the right hand side we first compute explicitly the function \( F_s = \frac{e^s}{\zeta (s)} \) and then perform the convolution. This tedious, but quite straightforward computation is performed in Appendix A. In Appendix B we give a realization of this map. Define a function \( \Gamma : G (F) \to \mathbb{R} \) by

\[
\Gamma (g) = \max_{1 \leq i, j \leq 7} |(g)_{i,j}| .
\]

The following result is easily checked.

Lemma 6.2. \( \Gamma \) is a bi-\( K \)-invariant function and for \( t \in T^+ \)

\[
\Gamma (t) = |\omega_1 (t)|^{-1} .
\]

Thus \( D (g, s) = \sum_{k=0}^{\infty} D_k (g) q^{-(5s+1)k} \), where

\[
D_k (g) = \begin{cases} 
1 , & \Gamma (g) = q^k \\
0 , & \text{otherwise}
\end{cases}
\]

For any \( g \in G \) define \( U_k (g) = \{ u \in U : \Gamma (ug) \leq q^k \} \). Then obviously

\[
D^{\psi_s} (g, s) = \sum_{k=0}^{\infty} (E_k (g) - E_{k-1} (g)) q^{-(5s+1)k} ,
\]

where

\[
E_k (g) = \int_{U_k (g)} \Psi_s (u) du .
\]

The computation of \( E_k (g) \) can be further reduced to a calculation of volumes of certain sets. For a given \( g \) there is at most two values of \( k \) for which \( E_k (g) \neq 0 \). The detailed computation is performed in Appendix B.
7. Ramified Computation

Fix a vector \( \varphi = \otimes v_\nu \) in \( \pi \) such that \( L_{\Psi_s}(\varphi) \neq 0 \) and \( v_\nu \) is spherical outside of \( S \). Recall from [Theorem 3.2] that for the representative of the open orbit \( \mu \)

\[
d_S(s, \varphi_S, f_s) = \int_{U_S \setminus G_S} L_{\Psi_s}(\varphi)(g) F^*(g, s) \, dg = \int_{U_S^\nu \setminus G_S} L_{\Psi}(\varphi)(g) f(\mu g, s) \, dg .
\]

7.1. Non-Archimedean case.

**Proposition 7.1.** Let \( \nu \) be a finite prime. There exists \( v \in \pi_\nu \) and a section \( f \in \text{Ind}^H_{\rho_H(\kappa)} \delta^s_{\rho_H} \) such that for any \( t \in \text{Hom}_{U(k_\nu)}(\pi, \mathbb{C}) \) and \( s \in \mathbb{C} \) it holds

\[
\int_{U_S^\nu \setminus G_\nu} l(\pi_\nu(g) v) f(\mu g, s) \, dg = l(v) .
\]

**Proof.** Start with \( v = v_\nu \), its stabilizer contains a congruence compact subgroup \( K_m \subset K(\mathcal{O}_\nu) \) for some \( m \). Denote \( \chi_s(p) = \delta_{\rho_H}(p^\nu) \). Since \( \mu \) generates the open orbit, the map

\[
i : \text{Ind}^H_{\rho_H} \delta_{\rho_H} \hookrightarrow \text{ind}_{G_{\nu}}^{G_k} \chi_s ,
\]

defined by \( i(f)(g) = f(\mu g, s) \) is an isomorphism. By induction by stages the latter representation equals \( \text{Ind}^G_{\rho_G} \text{ind}^{T_{\pi} U_\nu} \chi_s \).

Any \( g \in G(k_\nu) \) can be represented as \( g^\nu h_{\beta}(t) x_{\alpha}(r_1) x_{\alpha+\beta}(r_2) k \) with \( g^\nu = u^\mu t^\mu \in G^\nu, k \in K \). We fix a section \( f \) such that

\[
f(g) = \chi_s(g^\nu) 1_{T^\nu \cap K_m}(t) 1_{U \cap K_m}(r_1, r_2) 1_{K_m}(k) .
\]

Then

\[
\int_{U_S^\nu \setminus G_\nu} l(\pi_\nu(g) v) f(\mu g, s) \, dg = \int_{T_\nu^\nu} l(\pi(t^\nu) v) \chi_s(t_\mu) \, dt_\mu = \int_{T_\nu^\nu} l(\pi(h_{w_2}(t)) \cdot v) |t|^{5s} \, d^{\times} t .
\]

For any \( \phi \in S(k_\nu) \) define

\[
\phi * v = \int_{k_\nu} \phi(r) \pi(x_{2\alpha+\beta}(r)) v \, dr ,
\]
then

\[
l(h_{2\alpha+\beta}(t) \phi * v) = \dot{\phi}(t) l(h_{2\alpha+\beta}(t) v) .
\]

In particular, taking \( \phi \) such that \( \dot{\phi} = 1_{1+\pi=\mathcal{O}} \) one has

\[
\int_{k_\nu^\nu} l(t(\phi * v)) |t|^{5s} \, dt = l(v) .
\]

\[\square\]

Using the proposition above the computation of \( d_S \) is reduced to the computation of \( d_\infty \). Indeed, for a finite \( \nu \in S \) denote \( S' = S \setminus \{ \nu \} \). Having \( \varphi_S \) be a pure tensor vector \( \otimes_{\nu \in S'} v_\nu \otimes v_\nu \) with \( v_\nu \) as
Recall from [15, 2.6] that any Schwartz function.

Denote

Proof. 

Proposition 7.2. For any \( s \) non-zero in a neighbourhood of \( \Pi \), the minimal representation

\[
\{ \Theta \}
\]

The theta correspondence \( \Theta \) in proposition 7.1

As in the non-Archimedean case it holds

\[
d_s (s, \varphi_S f_s) = \int_{U_s^\nu \backslash G_S} L_{\Psi_s} (\varphi) (g) f (\mu g, s) \, dg = \\
= \int_{U_s^\nu \backslash G_S} \int_{U_s^\nu \backslash G_{\nu}} L_{\Psi_s} (g_{S^\nu} \otimes g_{\nu} v_\nu) f_\nu (\mu g_\nu, s) \, dg_\nu f_{S^\nu} (\mu g, s) \, dg = \\
= \int_{U_s^\nu \backslash G_S} L_{\Psi_s} (g_{S^\nu} v) f_{S^\nu} (\mu g_{S^\nu}, s) \, dg_\nu f (v_\nu) .
\]

By induction the integral equals (up to a non-zero constant)

\[
d_\infty (\varphi_\infty, f_{s, \infty}, g) = \int_{U_\infty^\nu \backslash G_\infty} L_{\Psi_s} (\varphi) (g) f (\mu g, s) \, dg .
\]

7.2. Archimedean place.

Proposition 7.2. For any \( s_0 \in \mathbb{C} \) there exists \( \varphi_\infty \) and \( f_\infty \) so that \( d_{S_\infty} (\varphi_\infty, f_{s, \infty}, g) \) is analytic and non-zero in a neighbourhood of \( s_0 \).

Proof. Denote \( \varphi_\infty = \otimes_{\nu | \infty} v_\nu \), arguing as in the non-Archimedean case it holds

\[
d_\infty (s, \varphi_\infty, f_{s, \infty}) = \int_{k_\infty} L_{\Psi_s} (h_{\omega_2} (t) \varphi_\infty) |t|^{5s} \, dt .
\]

Recall from [15, 2.6] that any Schwartz function \( \phi \in S (k_\infty) \) acts on \( \pi_\infty \) by

\[
\phi \ast v = \int_{k_\infty} \phi (r) \pi_\infty (x_{2\alpha + \beta} (r) v) \, dr .
\]

As in the non-Archimedean case it holds

\[
d_{\infty} (s, \phi \ast \varphi_\infty, f_{s, \infty}) = \int_{k_\infty} \hat{\phi} (t) L_{\Psi_s} (h_{\omega_2} (t) \varphi_\infty) |t|^{5s} \, dt .
\]

Recall that \( L_{\Psi_s} (\varphi) \neq 0 \). The function \( |t|^{5s} \) is \( C^\infty \) in \( t \) and analytic in \( s \) on compact sets of the form \( \{ t \mid |t - 1| < \epsilon \} \times \{ s \mid |s - s_0| < \epsilon \} \). For \( \phi \in S (k_\infty) \) such that \( \hat{\phi} \) is non-zero with compact support in \( \{ t \mid |t - 1| < \epsilon \} \) the function \( d_\infty (s, \phi \ast \varphi_\infty, f_{s, \infty}) \) is analytic in \( \{ s \mid |s - s_0| < \epsilon \} \). One can choose \( \epsilon \) and \( \phi \) such that also \( d_\infty (s_0, \phi \ast \varphi_\infty, f_{s, \infty}) \neq 0 \). \( \square \)

8. Application - \( \Theta \)-Lift For The Dual Pair \( (S_3, G_2) \)

The theta correspondence \( \Theta_H \) for the dual pair \( (S_3, G_2) \) in the group \( H \rtimes S_3 \) has been studied in [7]. 

The minimal representation \( \Pi \) of \( H \) can be extended to the group \( H \rtimes S_3 \). A cuspidal representation \( \pi \) belongs to the image of \( \Theta_H \) if

\[
\int_{G(k) \backslash G(A)} \varphi (g) F (g) \, dg \neq 0
\]

for \( \varphi \) in the space \( \pi \) and \( F \) in the space of the minimal representation \( \Pi \). It was proven in [7] that any such representation \( \pi \) supports the split Fourier coefficient. Besides, \( \pi \) is a non-tempered representation and \( L^S (\pi, s, \mathfrak{g}) \) has a double pole at \( s = 2 \). Taking the residue (of depth 2) at \( s = 2 \)
for the main equality we obtain the converse, i.e. the double pole of the standard $L$-function at $s = 2$ characterizes the image of $\Theta_H$. In other words

**Theorem 8.1.** For a cuspidal representation $\pi$ of $G(\mathbb{k})$ that supports the split Fourier coefficient the following statements are equivalent

1. $L^\mathbb{S}(s, \pi, \mathfrak{sl})$ has a double pole at $s = 2$.
2. $\Theta_H(\pi) \neq 0$.

## Appendices

### Appendix A. Computing $F(\cdot, s) \ast P(\cdot, s)$

Recall that

$$P(s) = P_0 A_0 + P_1 A_1$$

and by [10]

$$A_0 = \mathbb{1}_K, \quad A_1 = q^{-3} (\mathbb{1}_K + \mathbb{1}_{K\omega(\mathfrak{w})K}) ,$$

hence

$$F^*(\cdot, s) \ast P(\cdot, s) = j(s) \left( P_0(s) - q^{-3} P_1(q^{-s}) \right) F(\cdot, s) - q^{-3} P_1(q^{2-5s}) F(\cdot, s) \ast 1_{K\omega(\mathfrak{w})K}(\cdot).$$

We shall compute each summand separately. In this section we prove the following result.

**Proposition A.1.** The following holds for $(F(\cdot, s) \ast P(\cdot, s))$.

1. $(F^*(\cdot, s) \ast P(\cdot, s)) \in M_{\Psi_s}$.
2. $(F^*(\cdot, s) \ast P(\cdot, s))(g) = 0$ unless $g \in S_{\Psi_s} U TK$.
3. Let $t = h_\alpha(t_1) h_\beta(t_2) \in T$. If $t_1, \frac{t_2}{t_1} \in O$ it holds

$$F^*(\cdot, s) \ast P(\cdot, s)(t) = \begin{cases} \frac{1+q^{1-5s}}{\zeta(5s+1)} & \frac{t_2}{t_1} \bigg| t_1 \bigg|^{5s}, \quad \frac{t_2}{t_1} \frac{t_2}{t_1} < 1, \\ \frac{1+q^{1-5s}}{\zeta(5s+1)} & \frac{t_2}{t_1} \bigg| t_1 \bigg|, \quad \frac{t_2}{t_1} \frac{t_2}{t_1} > 1, \\ \frac{1+2q^{1-5s}}{\zeta(5s+1)} & \frac{t_2}{t_1} \bigg| t_1 \bigg|^{5s+1}, \quad \frac{t_2}{t_1} \frac{t_2}{t_1} = 1 \end{cases}$$

otherwise $(F^*(\cdot, s) \ast P(\cdot, s))(t) = 0$.

### A.1. The spaces $M_{\Psi_s}, M_{\psi_s}$

In this subsection we list some properties of $M_{\Psi_s}$ and $M_{\psi_s}$ which will be used in this section and in [Appendix B]. By Iwasawa decomposition any function $H \in M_{\Psi_s}$ is determined by the values it attains on $B_M/(B_M \cap K)$, i.e. on the elements

$$g = h_\alpha(t_1) h_\beta(t_2) x_\alpha(d) ,$$

where $d \in F/O$. In this appendix if $d \in O$ we choose a representative $d = 1$.

Note that for any positive root $\gamma$ and $|d| \geq 1$ one has

$$x_{-\gamma}(d) = x_{\gamma}(d^{-1}) h_\gamma(d^{-1}) k$$

(A.3) $$x_{\gamma}(d) = h_\gamma(d) x_{-\gamma}(d) k',$$

for some $k, k' \in K$.

Using the invariance properties one easily checks the following lemma.

**Lemma A.1.** Let $g = h_\alpha(t_1) h_\beta(t_2) x_\alpha(d)$.
(1) Let $H \in \mathcal{M}_\Phi$. Then $H(g) = 0$ unless

\[ t_1, \ t_1 + \frac{t_2}{t_1}, \ 2d \frac{t_2}{t_1} + d^2 t_1 \in \mathcal{O}. \tag{A.4} \]

(2) Let $H \in \mathcal{M}_0^\Phi$. Then $H(g) = 0$ unless

\[ t_1, t_1, \ t_1, \ t_1, \ t_1, d^2 t_1, d \frac{t_2}{t_1} \in \mathcal{O}. \tag{A.5} \]

The following lemma will be useful in the computation of the second summand of \textbf{Equation A.1}.

\textbf{Lemma A.2.} Let $H \in \mathcal{M}_\Phi$, $t \in T$ and $\mathbb{1}_{KtK}$ be a characteristic function of the double coset $KtK$. Then

\[ H \ast \mathbb{1}_{KtK} (g) = \sum_i H(gb_i^{-1}) , \]

where $KtK = \bigsqcup Kb_i$. Note that the representatives $b_i$ can be taken in the Borel subgroup $B$ of $G$.

\section*{A.2. Computation of $F_\ast$.}

\textbf{Proposition A.2.} Assume that $g = h_\alpha (t_1) h_\beta (t_2) x_\alpha (d) \in M$ satisfy \textbf{Equation A.3}. It holds

\[ F(g, s) = \begin{cases} \frac{\zeta(5s)}{\zeta(3s)} |t_1|^{5s} \left| \frac{t_2}{t_1} \right| \left( 1 - \frac{d^2 t_1^2 + d}{t_1} 5s - 1 \right), & \left| d^2 t_2^2 + d \right| \leq 1 \\ \frac{\zeta(5s)}{\zeta(3s)} |t_1|^{5s} \left| \frac{t_2}{t_1} \right| d^2 t_2^2 + d \left( 1 - \frac{d^2 t_1^2 + d}{t_1} 5s - 1 \right), & \left| d^2 t_2^2 + d \right| \geq 1 \end{cases} . \]

For $g \in M$ violating \textbf{Equation A.3} we have $F(g, s) = 0$.

\textbf{Proof.} We recall that

\[ F(g, s) = \int_F f_s \left( \mu x_{\alpha + \beta} (r) g \right) \psi (r) \ dr , \]

where $f_s$ here the spherical section such that $f_s (1) = 1$. For $g$ as above we have

\[ F(g, s) = \int_F f_s \left( w_2 w_3 x_{-\alpha_1} (1) x_{\alpha + \beta} (r) h_\alpha (t_1) h_\beta (t_2) x_\alpha (d) \right) \psi (r) \ dr = \]

\[ = |t_1|^{5s} \int_F f_s \left( w_2 w_3 x_{-\alpha_1} \left( \frac{t_1}{t_2} \right) x_{\alpha_2 + \alpha_3} \left( \frac{t_1}{t_2} \right) x_\alpha (d) x_\alpha (d) \right) \psi (r) \ dr . \]

Making a change of variables $r' = \frac{t_1}{t_2}$ and conjugating $w_3$ to the right we get

\[ F(g, s) = |t_1|^{5s} \frac{t_1}{t_2} \int_F f_s \left( w_2 x_{-\alpha_1} \left( \frac{t_1}{t_2} t_2 \right) x_{\alpha_2 + \alpha_3} \left( \frac{t_1}{t_2} \right) x_\alpha (d) x_\alpha (d) \right) \psi \left( \frac{t_2}{t_1} r' \right) dr' . \]

Due to \textbf{Equation A.3} we have

\[ F(m, s) = |t_1|^{5s} \frac{t_2}{t_1} \int_F f_s \left( w_2 x_{-\alpha_1} \left( \frac{t_1}{t_2} t_2 \right) x_{\alpha_2 + \alpha_3} \left( \frac{t_1}{t_2} \right) x_\alpha (d) x_\alpha (d) h_\alpha (d) \right) \psi \left( \frac{t_2}{t_1} r' \right) dr' . \]
Conjugating the elements associated with $\alpha_3$ to the left and using a similar equality for $\alpha_1$ yields

$$F(g, s) = \frac{t_1}{d} \left| \frac{t_2}{t_1} \right| \int_F f_s \left( w_2 x_{\alpha_1} \left( \frac{t_2}{t_1} \right) x_{\alpha_1} (d) \right) \psi \left( \frac{t_2}{t_1} \right) \, dr' =$$

$$= \frac{t_1}{d} \left| \frac{t_2}{t_1} \right| \int_F f_s \left( w_2 x_{\alpha_1} \left( \frac{t_2}{t_1} \right) x_{\alpha_1} (d) \right) \psi \left( \frac{t_2}{t_1} \right) \, dr' =$$

$$= \left| t_1 \right| \left| \frac{t_2}{t_1} \right| \int_F f_s \left( w_2 x_{\alpha_1} (r') x_{\alpha_1} \left( \frac{d t_2}{t_1} + d \right) \right) \psi \left( \frac{t_2}{t_1} \right) \, dr' .$$

If \( |d^2 \frac{t_2}{t_1} + d| \leq 1 \) we have

$$F(g, s) = \left| t_1 \right|^{5s} \left| \frac{t_2}{t_1} \right| \int_F f_s \left( w_2 x_{\alpha_2} (r') \right) \psi \left( \frac{t_2}{t_1} \right) \, dr' .$$

The integral is evaluated by separation to $\mathcal{O}$ and $F \setminus \mathcal{O}$ and once again using Equation A.3. It holds

$$\int_{F \setminus \mathcal{O}} f_s \left( w_2 x_{\alpha_2} (r') \right) \psi \left( \frac{t_2}{t_1} \right) \, dr' = \int_{F \setminus \mathcal{O}} f_s \left( w_2 x_{\alpha_2} (r') \right) \psi \left( \frac{t_2}{t_1} \right) \, dr' + \int_{F \setminus \mathcal{O}} f_s \left( w_2 x_{\alpha_2} (r') \right) \psi \left( \frac{t_2}{t_1} \right) \, dr' =$$

$$= 1 + \int_{F \setminus \mathcal{O}} |r'|^{-5s} \psi \left( \frac{t_2}{t_1} \right) \, dr' = 1 + \sum_{1 < q^k < \left| \frac{t_2}{t_1} \right|} q^{-5ks} \int_{|r|=q^k} \psi (r) \, dr - \left| \frac{t_2}{t_1} \right| \int_{|r|=\left| \frac{t_2}{t_1} \right|} \psi (r) \, dr$$

$$= \zeta (5s - 1) \left( 1 - \left| \frac{t_2}{t_1} \right|^{5s-1} \right) .$$

And hence

$$F(g, s) = \frac{\zeta (5s - 1)}{\zeta (5s)} \left| t_1 \right|^{5s} \left| \frac{t_2}{t_1} \right| \left( 1 - \left| \frac{t_2}{t_1} \right|^{5s-1} \right) .$$

Assume now that \( |d^2 \frac{t_2}{t_1} + d| > 1 \) and denote \( p = d^2 \frac{t_2}{t_1} + d \). It holds

$$(r'' = \frac{t_2}{p}) \quad  \left| \frac{t_1}{p} \right| \left| \frac{t_2}{t_1} \right| \int_F f_s \left( w_2 x_{\alpha_2} \left( \frac{t_2}{p} \right) \right) \psi \left( \frac{t_2}{t_1} \right) \, dr' =$$

$$= \left| \frac{t_1}{p} \right| \left| \frac{t_2}{t_1} \right| \int_F f_s \left( w_2 x_{\alpha_2} \left( \frac{t_2}{p} \right) \right) \psi \left( \frac{t_2}{t_1} \right) \, dr' =$$

$$= \left| \frac{t_1}{p} \right| \left| \frac{t_2}{t_1} \right| \left| \frac{t_2}{t_1} \right| \int_F f_s \left( w_2 x_{\alpha_2} \left( \frac{t_2}{p} \right) \right) \psi \left( \frac{t_2}{t_1} \right) \, dr'' .$$

If \( \left| \frac{t_2}{t_1} \right| \leq 1 \) then as in the previous case

$$F(g, s) = \frac{\zeta (5s - 1)}{\zeta (5s)} \left| t_1 \right|^{5s} \left| \frac{t_2}{t_1} \right| \left| d^2 \frac{t_2}{t_1} + d \right|^{1-5s} \left( 1 - \left| \frac{t_2}{t_1} \right| \left( d^2 \frac{t_2}{t_1} + d \right) \left| t_2 \right|^{5s-1} \right) .$$
If on the other hand $|p_{\frac{t_2}{t_1}}| > 1$ in a similar way to the previous cases it holds

$$
\int \mathcal{F}_s (w_2 x_{\alpha_2} (r)) \psi \left( \frac{t_2}{t_1} pr \right) dr = \int \psi \left( \frac{t_2}{t_1} pr \right) dr + \int_{\mathcal{F}' \mathcal{O}} |r|^{-5s} \psi \left( \frac{t_2}{t_1} pr \right) dr = 0 + 0,
$$

since $\psi$ is of conductor $\mathcal{O}$. Note that when $|p_{\frac{t_2}{t_1}}| > 1$ Equation A.3 is violated. \( \square \)

**Remark A.1.** Direct check shows that $F (q, s)$ is $S_{\Phi, s}$-invariant and hence $F (\cdot, s) \in \mathcal{M}_{\Phi, s}^0$.

A.3. **Decomposition of** $K \omega_1 (\varpi) K$ **into left** $K$ **cosets.** We recover the list of left $K$ cosets in $K \omega_1 (\varpi) K$ from [8 Proposition 13.3 and Proposition 14.2]. The decomposition of $K \omega_1 (\varpi) K = \bigsqcup b_i' K$ as a union of right $K$ cosets is given described there, after listing them we will make them into left cosets. Here $b_i' = n_i t_i$ belong to Borel subgroup $B = NT$. Fix $Y$ to be Teichmüller representatives in $\mathcal{O}$ of $\mathcal{O} / (\varpi)$ (or any other set of representatives) and $Z$ to be a set of representatives in $\mathcal{O}$ of $\mathcal{O} / (\varpi^2)$.

| Class mod $M (\mathcal{O})$ | $\# \text{ cosets}$ | Representatives |
|-----------------------------|----------------------|------------------|
| $h_{-\omega_2} (\varpi)$    | 1                    | $h_{-\omega_2} (\varpi)$ |
| $h_{-\beta} (\varpi)$      | $q^6$                | $u (r_1, r_2, r_3, r_4, r_5) h_{\omega_2} (\varpi)$ |
|                            |                      | $r_1, r_2, r_3, r_4 \in Y, r_5 \in Z$ |
| $h_\alpha (\varpi) h_\beta (\varpi)$ | $q (q + 1)$ | $u (r_1, 0, 0, 0, 0) h_{-\alpha} (\varpi) h_{-\beta} (\varpi)$ |
|                            |                      | $r_1 \in Y$ |
|                            |                      | $u (0, 0, 0, r_4, 0) x_\alpha (z) h_{-\beta} (\varpi)$ |
|                            |                      | $r_4, z \in Y$ |
| $h_{\omega_2} (\varpi)$    | $q^4 (q + 1)$        | $u (r_1, r_2, 0, 0, r_5) h_{\beta} (\varpi)$ |
|                            |                      | $r_2, r_5 \in Y, r_1 \in Z$ |
|                            |                      | $u (0, 0, r_3, r_4, r_5) x_\alpha (z) h_\alpha (\varpi) h_\beta (\varpi)$ |
|                            |                      | $r_3, r_5, z \in Y, r_4 \in Z$ |
| 1                           | $q^3 - 1$            | $u \left( \frac{r_4}{r_5}, 0, 0, 0, \frac{r_3}{r_5} \right)$ |
|                            |                      | $r_5 \in Y, r_5 \neq 0$ |
|                            |                      | $u \left( \frac{r_3}{r_5}, \frac{r_4}{r_5}, 0, 0, \frac{r_5}{r_5} \frac{r_3}{r_5}, \frac{r_5}{r_5} \frac{r_2}{r_5} \right)$ |
|                            |                      | $r_1, r_5, y \in Y, r_1 \neq 0$ |

We need now to make the right coset representatives $\{b_i'\}$ into left coset representatives. Let $w_0 = w_\alpha w_\beta w_\alpha w_\beta w_\alpha w_\beta \in K$ be the longest element in the Weyl group of $G$. Recall that $w_0$ send $\gamma$ to $-\gamma$ for all $\gamma \in \Phi$. Also note that $\{w_0 b_i'\}$ is also a full set of representatives of right cosets.
Denote by $\theta$ the Cartan antiinvolution, fixing the torus $T$ such that $\theta(x_\gamma(r)) = x_{-\gamma}(r)$. Let $w_0 \in K$ be a lifting to $G$ of the longest Weyl group element such that $\theta(w_0) = w_0$. Then

$$(A.6) \quad K\omega (w) K = \theta(K\omega (w) K) = \theta \left( \prod_i w_0 b_i' K \right) = \prod_i K\theta(b_i') w_0 = \prod_i Kt_i^{-1}n_i .$$

Fixing $b_i = t_i^{-1}n_i$ gives a set of left coset representatives.

A.4. **Convolution.** Combining [Equation A.1][proposition A.2] and [Equation A.6] the computation of the convolution is straightforward. We shall present the computation for toral elements only. The case of non-toral elements is dealt similarly.

By [lemma A.2] we have

$$(F (\cdot, s) * 1_{K\lambda_1 K} (g) = \sum_i F (gb_i^{-1}, s) .$$

Assume that $g = t = h_\alpha (t_1) h_\beta (t_2)$. By $S_{\Psi_\cdot}$-invariance we may assume that $|\frac{t_1^2}{t_2^2}| \leq 1$. The case $|\frac{t_1^2}{t_2^2}| > 1$ follows by symmetry from $|\frac{t_1^2}{t_2^2}| < 1$, since $F (\cdot, s) \in M_{\Psi_\cdot}$. We can write $\left(F (\cdot, s) * 1_{K\lambda_1 K} \right) (t)$ as follows

$$\left(F (\cdot, s) * 1_{K\omega_1 (w) K} \right) (t) = F (h_{-\omega_1 (w)} t, s) + q^6 F (h_{\omega_1 (w)} t, s) + q F (h_{-\alpha (w)} h_{-\beta (w)} t, s) +$$

$$+ q \sum_{y \in \mathcal{O} / (w)} F (h_{-\beta (w)} t x_\alpha (-\frac{y}{\omega}), s) + q^4 F (h_{\beta (w)} t, s) +$$

$$+ q^4 \sum_{y \in \mathcal{O} / (w)} F (h_{\alpha (w)} h_{\beta (w)} t x_\alpha (-\frac{y}{\omega}), s) +$$

$$+ \left( q \sum_{r, y \in \mathcal{O} / (w)} \psi \left( -\frac{y}{\omega}, \left( \frac{t_2}{t_1} t y + t_1 \right) r \right) - 1 \right) F (t, s) .$$

(A.7)

We separate this computation into four cases depending on the absolute value of $t_1$ and $t_2$. All the following results follow by applying [proposition A.2] to the summands in [Equation A.7]. Denote $|t_1| = q^{-n}$ and $|t_2| = q^{-m}$.

1. Assume $|t_2| = |t_1| = 1$:

$$\left(F (\cdot, s) * 1_{K\lambda_1 K} \right) (t) = \left( q^6 - 10s + q^{5-5s} + 3q^{1-5s} + 2q^2 - 1 \right) \frac{\zeta (5s)}{\zeta (5s)} .$$

and also

$$F (1, s) = \frac{1}{\zeta (5s)} .$$

Plugging this into [Equation A.1] yields

$$\left(F^* (\cdot, s) * P (\cdot, s) \right) (t) = \frac{1 + 2q^{1-5s}}{\zeta (5s + 1)} .$$
(2) Assume $|t_2| < |t_1|$ and $\frac{|t_2|}{|t_1|} = 1$:

$$(F(\cdot, s) * \mathbb{1}_{K_{\lambda_1, K}})(t) = \frac{\zeta(5s - 1)}{\zeta(5s)} \left(\frac{q^{1+5s-n-5ns}}{\zeta (5ns)} + \frac{q^{5-5s-n-5ns}}{\zeta (5(n+2)s)} + \frac{q^{2-n-5ns}}{\zeta (5ns)} + \frac{q^{4-n-5(n+1)s}}{\zeta (5(n+1)s)} + (q^3 - 1) \frac{q^{-n-5ns}}{\zeta (5(n+1)s)} + q \left(\frac{2q^{1-n-5ns}}{\zeta (5ns)} + (q - 2) \frac{q^{2-n-5(n+1)s}}{\zeta (5(n-1)s)}\right) + +q^4 \left(\frac{2q^{-n-5(n+1)s}}{\zeta (5(n+1)s)} + (q - 2) \frac{q^{1-n-5(n+2)s}}{\zeta (5ns)}\right) \right)$$

and also

$$F(t, s) = \frac{\zeta (5s - 1)}{\zeta (5s)} \frac{q^{-n-5ns}}{\zeta (5(n+1)s)}.$$ 

Plugging this into Equation A.1 yields

$$(F^* (\cdot, s) \ast P (\cdot, s)) (t) = \frac{1 + 2q^{1-5s}}{\zeta (5s + 1)} |t_1|^{5s+1}.$$ 

(3) Assume $|t_2| < |t_1|$ and $\frac{|t_2|}{|t_1|} < 1$:

$$(F(\cdot, s) * \mathbb{1}_{K_{\lambda_1, K}})(t) = \frac{\zeta(5s - 1)}{\zeta(5s)} \left(\frac{q^{1+5s-m-5ns}}{\zeta (5(m-n)s)} + \frac{q^{5-5s-m-5ns}}{\zeta (5(m-n+2)s)} + \frac{q^{1-m-5ns+m+5s}}{\zeta (5(m-n+1)s)} + +\frac{q^{3-5ns-m+n}}{\zeta (5(m-n+2)s)} + q \left(\frac{q^{1-5ns-m+n}}{\zeta (5(m-n)s)} + (q - 1) \frac{q^{2-5s-m+n-5ns}}{\zeta (5(m-n-1)s)}\right) + +q^4 \left(\frac{q^{n-5s-5ns}}{\zeta (5(m-n+1)s)} + (q - 1) \frac{q^{1-10s-m+n-5ns}}{\zeta (5(m-n)s)}\right) + + (q^3 - 1) \frac{q^{-n-5ns}}{\zeta (5(m-n+1)s)} \right)$$

and also

$$F(t, s) = \frac{\zeta (5s - 1)}{\zeta (5s)} \frac{q^{-m-5ns}}{\zeta (5(m-n+1)s)}.$$ 

Plugging this into Equation A.1 yields

$$(F^* (\cdot, s) \ast P (\cdot, s)) (t) = \frac{1 + q^{1-5s}}{\zeta (5s + 1)} |t_2| |t_1|^{5s}.$$ 

(4) Assume $|t_2| = |t_1|$ and $\frac{|t_2|}{|t_1|} < 1$:

$$(F(\cdot, s) * \mathbb{1}_{K_{\lambda_1, K}})(t) = \frac{\zeta(5s - 1)}{\zeta(5s)} \left(\frac{q^{5-5s-5ns}}{\zeta (10s - 2)} + \frac{q^{1+5s-5ns}}{\zeta (5s - 1)} + \frac{q^{3-5ns}}{\zeta (5s - 1)} + \frac{q^{4-5ns}}{\zeta (5s - 1)} + (q^2 - 1) \frac{q^{-5ns}}{\zeta (5s - 1)} \right)$$

and also

$$F(t, s) = \frac{\zeta (5s - 1)}{\zeta (5s)} \frac{q^{-5ns}}{\zeta (5s - 1)}.$$ 

Plugging this into Equation A.1 yields

$$(F^* (\cdot, s) \ast P (\cdot, s)) (t) = \frac{1 + q^{1-5s}}{\zeta (5s + 1)} |t_1|^{5s}.$$
Appendix B. Computation of $D\Psi_s$

Recall that our aim is to compute

$$E_k (g) = \int_{U_k(g)} \overline{\Psi_s (u)du}.$$  

We treat first the case where $g \in S\Psi_{\mathcal{U}\mathcal{T}K}$ and then the case where $g \notin S\Psi_{\mathcal{U}\mathcal{T}K}$.

We note the following helpful fact that will be used repeatedly throughout this section.

**Lemma B.1.** For $a, b, c \in \mathbb{N}$ with $a + b \geq c$ it holds

$$\mu \left\{ (x, y) \mid |x| \leq q^a, |y| \leq q^b, |xy| \leq q^c \right\} = q^{c} \left( 1 + (a + b - c)(1 - q^{-1}) \right).$$

**B.1. Toral elements.** For $t = h_\alpha(t_1)h_\beta(t_2)$ and $u = u(r_1, r_2, r_3, r_4, r_5)$, the matrix $\iota (ut)$ has form

$$(B.1) \begin{pmatrix} 1 & 0 & r_2 & \frac{r_1}{2} & \frac{r_1 r_2 + r_5}{2} & \frac{r_2 r_4 - r_5^2}{2} \\ 0 & 1 & r_1 & \frac{r_1}{2} & \frac{r_1 r_2 - r_5^2}{2} & r_4 - r_5 \\ 0 & 0 & 1 & 0 & \frac{r_3}{2} & \frac{r_3}{2} \\ 0 & 0 & 0 & 1 & -r_2 & -r_3 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} t_1 \\ t_2 \\ t_3 \\ t_4 \\ t_5 \\ t_1 \end{pmatrix}.$$ 

Consider an element $g \in T$ and denote it by $t = h_\alpha(t_1)h_\beta(t_2)$. Denote $|t_1| = q^{-n}$ and $|t_2| = q^{-m}$. By **Equation A.3**, $E_k (t) = 0$ unless $|t_1|, \frac{t_2}{t_1} \leq 1$. Since $E_k \in \mathcal{M}_k^{0}$, and $w_\alpha \in S\Psi_{\mathcal{U}\mathcal{T}K} \subset K$ one has

$$E_k (t) = E_k (w_\alpha tw_\alpha^{-1}).$$

In particular we can assume $|\alpha (t)| = \left| \frac{t_2}{t_1} \right| \leq 1$ or $|t_1| \leq \left| \frac{t_2}{t_1} \right|$. Also, $U_k (t) = \emptyset$ unless $|t_1| \geq q^{-k}$. To sum up we have to compute $E_k (t)$ where

$$q^{-k} \leq |t_1| \leq \left| \frac{t_2}{t_1} \right| \leq 1.$$ 

We may exchange integration over $U_k(t)$ to integration over a smaller and simpler set, namely

**Lemma B.2.**

$$E_k (t) = \int_{\overline{U_k(t)}} \overline{\Psi_s (u)} du,$$

where

$$\overline{U_k(t)} = \left\{ u (r_1, r_2, r_3, r_4, r_5) \in U_k (t) \mid |r_2|, |r_3| \leq q \right\}.$$  

**Proof.** For any $x, y \in F$ define

$$U_k^{(x,y)} (t) = \left\{ u (r_1, r_2, r_3, r_4, r_5) \in U_k (t) \mid r_2 = x, r_3 = y \right\}$$

and note that for $s_1, s_2 \in \mathcal{O}^\times$

$$U_k^{(s_1,x,s_2,y)} (t) = h (s_1, s_2) U_k^{(x,y)} (t) h^{-1} (s_1, s_2),$$

also

$$E_k (t) = \int_{\overline{U_k(t)}} \overline{\Psi_s (u)} du,$$  

where

$$\overline{U_k(t)} = \left\{ u (r_1, r_2, r_3, r_4, r_5) \in U_k (t) \mid |r_2|, |r_3| \leq q \right\}.$$
where \( h(s_1, s_2) = h_\beta(s_1) h_{2\alpha+\beta}(s_2) \). Since \( \delta_F(h(s_1, s_2)) = 1 \) it follows that \( \mu\left(U_k(t)\right) = \mu\left(U_k^{i.x,y}(t)\right) \) which means it depends only on \( t, |x| \) and \( |y| \). In particular if \( |x| = q^i, |y| = q^j \) we denote \( \mu\left(U_k^{i.x,y}(t)\right) \) by \( \mu\left(U_k^{i,j}(t)\right) \).

Thus

\[
E_k(t) = \int_{U_k(t)} \Psi(u) du = \int_{F \times F} \mu\left(U_k^{i.x,y}(t)\right) \psi(x + y) \, dx \, dy =
\]

\[
= \sum_{i,j = -\infty}^{\infty} \mu\left(U_k^{i,j}(t)\right) \int_{|x| = q^i} \psi(x) \, dx \int_{|y| = q^j} \psi(y) \, dy .
\]

Since \( \int_{|z| = q^k} \psi(z) \, dz = 0 \) for \( k > 1 \), the proposition follows. \( \square \)

**Remark B.1.** We can describe \( U_k(t) \) by a short list of inequalities. Namely \( u \in U_k(t) \) if and only if

\[
k \geq n \quad |r_2|, |r_3| \leq q \quad |r_1|, |r_2|, |r_3|, |r_2r_3 + r_5|, |r_1r_3 - r_2^2| \leq q^{k+n-m} \quad |r_2|, |r_3|, |r_4|, |r_2r_4 - r_2^2|, |r_1r_4 - 2r_2r_3 - r_5| \leq q^{k-n} .
\]

**Corollary B.1.**

\[
E_k(t) = \mu\left(U_k^{0,0}(t)\right) - \mu\left(U_k^{0,1}(t)\right) - \mu\left(U_k^{1,0}(t)\right) + \mu\left(U_k^{1,1}(t)\right) .
\]

**Proof.** We first note that for every \( i, i', j \leq 0, j' \leq 1 \) and \( k \) it holds

\[
\mu\left(U_k^{i,j}(t)\right) = \mu\left(U_k^{i',j}(t)\right) , \quad \mu\left(U_k^{j,i}(t)\right) = \mu\left(U_k^{j,i}(t)\right) .
\]

We also recall that

\[
\int_{|r| \leq 1} \psi(r) = 1, \quad \int_{|r| = q} \psi(r) = -1 .
\]

The claim then follows by a simple computation

\[
E_k(t) = \sum_{i,j = -\infty}^{\infty} \mu\left(U_k^{i,j}(t)\right) \int_{|x| = q^i} \psi(x) \, dx \int_{|y| = q^j} \psi(y) \, dy =
\]

\[
= \mu\left(U_k^{0,0}(t)\right) \int_{|x| \leq 1} \psi(x) \, dx \int_{|y| \leq 1} \psi(y) \, dy - \mu\left(U_k^{0,1}(t)\right) \int_{|x| \leq 1} \psi(x) \, dx \int_{|y| = q} \psi(y) \, dy -
\]

\[
- \mu\left(U_k^{1,0}(t)\right) \int_{|x| = q} \psi(x) \, dx \int_{|y| \leq 1} \psi(y) \, dy + \mu\left(U_k^{1,1}(t)\right) \int_{|x| = q} \psi(x) \, dx \int_{|y| = q} \psi(y) \, dy =
\]

\[
= \mu\left(U_k^{0,0}(t)\right) - \mu\left(U_k^{0,1}(t)\right) - \mu\left(U_k^{1,0}(t)\right) + \mu\left(U_k^{1,1}(t)\right) .
\]

\( \square \)

**Proposition B.1.** For \( t \) as above, with \( |t_1| = q^{-n} \), it holds

1. \( E_k(t) = 0 \) for \( k \neq n, n + 1 \).
\[ E_n(t) = \begin{cases} 1 & |\alpha(t)| = 1 \\ |\alpha(t)|^{-1} & |\alpha(t)| < 1 \end{cases}, \quad E_{n+1}(t) = \begin{cases} 2q^2 & |\alpha(t)| = 1 \\ 2q^2 |\alpha(t)|^{-1} & |\alpha(t)| < 1 \end{cases}. \]

**Proof.** We separate the proof according to the absolute value of \( \alpha(t) \).

- Assume that \(|\alpha(t)| = 1\), i.e. \( \frac{t^2}{r^2} = 1 \).
  1. Assume \( k = n \), then \( u \in U_k(t) \) if and only if \(|r_1|, |r_2|, |r_3|, |r_4|, |r_5| \leq 1\).
    
    In this case \( U_k(t) \), \( U_k(t) \), \( U_k(t) \) = 0 and \( U_k(t) = \mathcal{O}^3 \). Hence
    \[ E_n(g) = 1 \]
  2. Assume \( k = n + 1 \), then \( u \in U_k(t) \) if and only if
     \[ |r_1|, |r_2|, |r_3|, |r_4| \leq q \]
     \[ |r_2r_3 + r_5|, |r_1r_3 - r_2^2|, |r_2r_4 - r_3^2|, |r_1r_4 - 2r_2r_3 - r_5| \leq q. \]

We demonstrate the measurement of \( U_k(t) \) in this case as an example to the calculation held in all other cases. Assume that \(|r_2|, |r_3| \leq 1\), then \( u \in U_k(t) \) if and only if
\[ |r_1|, |r_4|, |r_5|, |r_1r_4| \leq q \]

and hence, by [Lemma B.1](#)
\[ \mu \left( U_k(t) \right) = q(q + (q - 1)) = 2q^2 - q. \]

Assume \(|r_2| = q\) and \(|r_3| \leq 1\). Then \(|r_1r_4| \leq q\) but also \(|r_1r_4 - r_2^2| \leq q\) which contradicts the fact that \(|r_2^2| = q^2\). Hence \( U_k(t) = \emptyset\), by a similar argument \( U_k(t) = \emptyset\).

Assume that \(|r_2|, |r_3| = q\). Let us parametrize \( U_k(t) \) in the following way
\[ r_1 = \frac{x + r_2^2}{r_3}, \quad r_4 = \frac{y + r_3^2}{r_2}, \quad r_5 = z - r_2r_3. \]

The domain of integration for the new variables is \(|x|, |y|, |z| \leq q\). Also
\[ dr_1 = \frac{dx}{q}, \quad dr_4 = \frac{dy}{q}, \quad dr_5 = dz. \]

Note that now
\[ |r_1r_4 - 2r_2r_3 - r_5| = \left| \frac{x + r_2^2}{r_3} \cdot \frac{y + r_3^2}{r_2} - r_2r_3 - z \right| = \left| \frac{xy + x r_3^2 + y r_2^2}{r_2r_3} - z \right| \leq q. \]

Hence
\[ \mu \left( U_k(t) \right) = \int_{q^{-1}0} dx \int_{q^{-1}0} dy \int_{q^{-1}0} dz = q. \]

Combining the computed \( \mu \left( U_k(t) \right) \) yields
\[ E_{n+1}(t) = \mu \left( U_k(t) \right) - \mu \left( U_k(t) \right) - \mu \left( U_k(t) \right) - \mu \left( U_k(t) \right) = (2q^2 - q) - 0 - 0 + q = 2q^2. \]
(3) Assume $k > n + 1$, then $u \in \overline{U_k(t)}$ if and only if
\[
|r_2|, |r_3| \leq q
\]
\[
|r_1|, |r_4|, |r_2r_4|, |r_1r_3|, |r_1r_4| \leq q^{k-n}.
\]
Hence, according to Lemma B.1
\[
\mu \left( U_{k,0}^0 \right(t \right) = q^{2(k-n)} \left( 1 + (k-n) \left( 1 - q^{-1} \right) \right)
\]
\[
\mu \left( U_{k,1}^0 \right(t \right) = \mu \left( U_{k,0}^1 \right(t \right) = q^{2(k-n)} \left( 1 + (k-n-1) \left( 1 - q^{-1} \right) \right)
\]
\[
\mu \left( U_{k,1}^1 \right(t \right) = q^{2(k-n)} \left( 1 + (k-n-2) \left( 1 - q^{-1} \right) \right),
\]
and then
\[
E_k(g) = 0.
\]
Evaluating $D^{\Psi^*}$ at $t$ yields
\[
D^{\Psi^*}(t) = q^{-n} E_n(t) + q^{-n-1} (E_{n+1}(t) - E_n(t)) + q^{-n-2} E_{n+1}(t) = \frac{1 + 2q^{1-5s}}{\zeta(5s+1)} |t_1|^{5s+1}.
\]
- Assume that $|\alpha(t)| < 1$, i.e. $\left| \frac{r^2}{r_2} \right| < 1$.
  
  (1) Assume $k = n$, then $u \in \overline{U_k(t)}$ if and only if
  \[
  |r_2|, |r_3|, |r_4|, |r_2r_4 - r_3| \leq 1
  \]
  \[
  |r_1|, |r_5| \leq q^{2n-m}.
  \]
  By making a change of variables $r_5 = x + r_1 r_4$ this is equivalent to
  \[
  |r_2|, |r_3|, |r_4|, |x| \leq 1
  \]
  \[
  |r_1|, |r_1 r_4| \leq q^{2n-m}.
  \]
  Hence, according to Lemma B.1
  \[
  \mu \left( t_{k,0}^{0,0} \right(t \right) = q^{2n-m}
  \]
  \[
  \mu \left( t_{k,0}^{1,0} \right(t \right) = \mu \left( t_{k,0}^{0,1} \right(t \right) = \mu \left( t_{k,1}^{1,1} \right(t \right) = 0,
  \]
  and then
  \[
  E_n(t) = q^{2n-m}.
  \]
(2) Assume $k = n+1$, then $u \in \overline{U_k(t)}$ if and only if
\[
|r_2|, |r_3|, |r_4|, |r_2r_4 - r_3^2|, |r_1r_4 - 2r_2r_3 - r_5| \leq q
\]
\[
|r_1|, |r_5|, |r_1 r_3| \leq q^{2n-m+1}.
\]
Hence, according to Lemma B.1 and arguments similar to case 2 with $|\alpha(t)| = 1$,
\[
\mu \left( t_{k,0}^{0,0} \right(t \right) = q^{2n-m+2} \left( 1 + (1 - q^{-1}) \right),
\]
\[
\mu \left( t_{k,0}^{1,0} \right(t \right) = \mu \left( t_{k,0}^{0,1} \right(t \right) = \mu \left( t_{k,1}^{1,1} \right(t \right) = q^{2n-m+2},
\]
and then
\[
E_n(t) = q^{2n-m+2}.
\]
(3) Assume $k > n + 1$, then $u \in U_k(g)$ if and only if

$$
\begin{align*}
|r_2|, |r_3| &\leq q \\
|r_1|, |r_5|, |r_1 r_3| &\leq q^{k+n-m} \\
|r_4|, |r_2 r_4|, |r_1 r_4 - r_5| &\leq q^{k-n}.
\end{align*}
$$

By making a change of variables $r_5 = x + r_1 r_4$ this is equivalent to

$$
\begin{align*}
|r_2|, |r_3| &\leq q \\
|r_1|, |r_1 r_4|, |r_1 r_3| &\leq q^{k+n-m} \\
|r_4|, |r_2 r_4|, |x| &\leq q^{k-n}.
\end{align*}
$$

Hence, according to Lemma B.1

$$
\begin{align*}
\mu\left(U_{k,0}^0(t)\right) &= q^{k-n}q^{k+n-m}(1 + (k-n)(1 - q^{-1})) \\
\mu\left(U_{k,0}^1(t)\right) &= q^{k-n}q^{k+n-m}(1 + (k-n-1)(1 - q^{-1})) \\
\mu\left(U_{k,1}^0(t)\right) &= q^{k-n}q^{k+n-m}(1 + (k-n-1)(1 - q^{-1})) \\
\mu\left(U_{k,1}^1(t)\right) &= q^{k-n}q^{k+n-m}(1 + (k-n-2)(1 - q^{-1}))
\end{align*}
$$

and then

$$
E_n(t) = 0.
$$

Evaluating $D_{\Psi_s}$ at $t$ yields

$$
D_{\Psi_s}(t) = 1 + \frac{q^{1-5s}}{\zeta(5s+1)} \frac{t_2}{t_1} |t_1|^{5s}.
$$

□

B.2. **Non-toral case.** This case is technically more involved than the case of the toral elements, but all the ideas for the toral elements can be carried to this case as well. We will prove the following result.

**Proposition B.2.** $E_k(g) = 0$ for $g \notin S_{\Psi_s}UTK$.

Let $g = tx_\alpha(d)$, where $t = h_\alpha(t_1) h_\beta(t_2)$ and $|d| > 1$. Since $g \notin S_{\Psi_s}UTK$ it holds $|d^2 \alpha(t) + d| \geq 1$. By Equation A.5 $E_k(t) = 0$ unless

$$
t_1, \frac{t_2}{t_1}, d^2 t_1, \frac{d^2 t_2}{t_1} \in \mathcal{O}.
$$

Since $E_k \in \mathcal{M}_{\Psi_s}^0$ and $w_\alpha \in S_{\Psi_s} \subset K$ one has $E_k(g) = E_k(w_\alpha gw_\alpha^{-1})$. Hence it is enough to compute $E_k(g)$ when $|d\alpha(t)| = \frac{|d^2 t_2|}{|t_2|} \leq 1$. 
The matrix \( \iota(x_\alpha(d)) \) has the form
\[
\begin{pmatrix}
1 & d & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & -d & 0 & 0 \\
0 & 0 & 0 & 1 & d & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
\end{pmatrix}.
\]
Denote \( |t_1| = q^{-n} \), \( |t_2| = q^{-m} \) and \( |d\alpha(t)| = q^l \). Under this notations \( U_k(g) = \emptyset \) when \( k < n \) and so we may assume that \( k \geq n \).

We now reduce the domain of integration. The proof of this lemma is similar to the proof of [lemma B.2](#) and is omitted.

**Lemma B.3.**
\[
E_k(t) = \int_{\widehat{U}_k(g)} \Psi_s(u)du,
\]
where
\[
\widehat{U}_k(g) = \{u(r_1, r_2, r_3, r_4, r_5) \in U_k(g) \mid |r_2 + r_3| \leq q\}.
\]

**Remark B.2.** Denote \( b = d^2 \). When \( |d\alpha(t)| \leq 1 \), we have \( u \in \widehat{U}_k(g) \) if and only if
\[
k \geq n
\]
\[
|r_2 + r_3| \leq q,
\]
\[
|r_1|, |r_2|, |r_3|, |r_2r_3 + r_5|, |r_1r_3 - r_2^2| \leq q^{k+n-m}
\]
\[
|br_1 - r_2|, |br_2 - r_3|, |br_3 - r_4| \leq q^{k-n}
\]
\[
|r_2r_4 - r_3^2 - br_2r_3 - br_5|, |r_1r_4 - 2r_2r_3 + br_2^2 - br_1r_3 - r_5| \leq q^{k-n}
\]

We are now ready to prove proposition B.2.

**Proof.**
- Assume that \( |b| < 1 \) i.e. \( l < 0 \). Note that under this assumption
\[
\widehat{U}_k(g) = U_k(g)
\]
and thus
\[
E_k(g) = \mu(U_k^{0,0}(g)) - \mu(U_k^{0,1}(g)) - \mu(U_k^{1,0}(g)) + \mu(U_k^{1,1}(g)).
\]

(1) Assume \( k = n \), then \( u \in \widehat{U}_k(g) \) if and only if
\[
|r_2| \leq q
\]
\[
|r_1|, |r_3| \leq q^{2n-m},
\]
\[
|r_3|, |r_4|, |br_1 - r_2|, |r_2r_4 - br_3|, |r_1r_4 - 2r_2r_3 + br_2^2 - br_1r_3 - r_5| \leq 1.
\]
Hence, according to Lemma B.1 and arguments performed in the toral case,

\[
\mu \left( \hat{U}_k^0 (g) \right) = q^{-2l}, \quad \mu \left( \hat{U}_k^1 (g) \right) = q^{-2l}, \quad \mu \left( U_k^0 (g) \right) = 0, \quad \mu \left( U_k^1 (g) \right) = 0,
\]

and then

\[
E_k (g) = 0.
\]

(2) Assume \( k = n + 1 \), then \( u \in \hat{U}_k (g) \) if and only if

\[
|r_1| \leq q^{1-l} \quad |r_2|, |r_3|, |r_4|, |r_2r_4 - r_3^2 - br_5|, |r_1r_4 - 2r_2r_3 - br_1r_3 - r_5| \leq q.
\]

Hence, according to Lemma B.1 and arguments performed in the toral case,

\[
\mu \left( \hat{U}_k^0 (g) \right) = q^{2-l} \left( 1 + (1 - q^{-1}) \right), \quad \mu \left( \hat{U}_k^1 (g) \right) = q^{2-l} \left( 1 + (1 - q^{-1}) \right)
\]

\[
\mu \left( U_k^0 (g) \right) = (1 - l) q^{2-l} \left( q^{-1} \right), \quad \mu \left( U_k^1 (g) \right) = (1 - l) q^{2-l} \left( q^{-1} \right),
\]

and then

\[
E_k (g) = 0.
\]

(3) Assume \( k > n + 1 \) and denote \( x = r_2 + r_3 \). Then \( u \in \hat{U}_k (g) \) if and only if

\[
|x| \leq q \quad |x|, |r_5 - r_3^2|, |r_1r_3 - r_3^2| \leq q^{k+n-m} \\
|r_4|, |br_1|, |br_5 + r_3r_4 - (b + 1) r_3^2|, |r_1 (r_4 - br_3) + (b + 2) r_3^2 - 2xr_3 - r_5| \leq q^{k-n}.
\]

The set \( \hat{U}_k (g) \) is invariant under the change of variables

\[
(r_1, x, r_3, r_4, r_5) \mapsto (r_1, x + \omega^{-1}, r_3, r_4, r_5 + 2r_3 \omega^{-1})
\]

Making this change of variables in the integral yields

\[
E_k (g) = \psi \left( \omega^{-1} \right) E_k (g),
\]

hence

\[
E_k (g) = 0.
\]

- When \( |b| = \left| \frac{a^2}{2a} \right| = 1 \) the calculation is more involved and is omitted. Nonetheless \( E_k \) vanishes on such elements and hence \( D^\Psi \) also vanishes.

\[\square\]

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