Linear and nonlinear stability analysis of rotating stall in a wide vaneless diffuser

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Abstract: A two dimensional stability analysis is introduced to study the characteristics of rotating stall in a wide radial vaneless diffuser. The prediction of the critical flow angle and propagation velocity given by a linear stability model are given and compared to existing results in literatures. The growth rate of the different competitive mode are used do try to determinate the dominant stall mode. The equations used for a weakly nonlinear analysis which focusing on a small distance from the critical condition are then proposed.

Nomenclature

| Symbol | Description                        |
|--------|------------------------------------|
| $V$    | Stable velocity                    |
| $R$    | Diffuser radius ratio              |
| $h$    | Width of diffuser                  |
| $P$    | Pressure                           |
| $n$    | Number of cells                    |
| $Q$    | Flow rate                          |
| $I$    | Circulation                        |
| $i$    | Imaginary unit                     |
| $t$    | Time                               |
| $u$    | Perturbation velocity              |
| $p$    | Perturbation pressure              |
| $r$    | Radius                             |
| $\theta$ | Angle                          |
| $z$    | Axial direction                    |
| $\alpha$ | Flow angle                        |
| $\zeta$ | Vorticity                          |
| $\omega$ | Angular velocity                  |

Subscripts

| Symbol | Description                        |
|--------|------------------------------------|
| $n$    | Number of cells                    |
| $Q$    | Flow rate                          |
| $I$    | Circulation                        |
| $i$    | Imaginary unit                     |
| $t$    | Time                               |
| $u$    | Perturbation velocity              |
| $p$    | Perturbation pressure              |
| $r$    | Radius                             |
| $\theta$ | Angle                          |
| $z$    | Axial direction                    |

Superscripts

| Symbol | Description                        |
|--------|------------------------------------|
| $-$    | Dimensional variables              |
| $\sim$ | Perturbation variables             |

Greek letters

| Symbol | Description                        |
|--------|------------------------------------|
| $\alpha$ | Flow angle                        |
| $\zeta$ | Vorticity                          |
| $\omega$ | Angular velocity                  |

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1. Introduction

Rotating stall in pumps and compressors is one of the phenomena which appear at partial flow rates and which is a limit in the operating range of the machines. Rotating stall can be found in different components of the turbomachineries, and the present study is focusing on the rotating stall in the vaneless diffuser of radial pumps or compressors. This kind of diffuser is placed downstream to a radial impeller. By decreasing the flow rate, rotating stall can occur when the critical stall condition is reached, it is characterized by several stall cells which propagate around the diffuser at a speed which is a fraction of the impeller speed (Dazin et al [1, 2]), as shown in figure 1.

Figure 1. Sketch of the vaneless diffuser. Figure 2. Flow in the wide vaneless diffuser (Ljevar [3]).

Depending on the diffuser width ratio, different types of vaneless diffuser rotating stall have been concluded (Abdelhamid and Bertrand [4]; Gao et al [5]; Dou [6]; Shin et al [7]; Ljevar [3]). For wide vaneless diffusers ($h/r_2 > 0.1$), the space is large enough to have a two dimensional core flow between boundary layers and rotating stall is triggered by the two dimensional core flow instability, as shown in figure 2. This kind of diffuser rotating stall has been studied by Jansen [8], Senoo et al [9], Tsujimoto et al [10], Abdehamid [11], Moore [12], Ljevar et al [3], Dazin et al [1, 2], Pavesi et al [13] and it has been showed that a core flow instability develops when a critical inlet flow angle $\alpha_c$ is reached.

In the first part of the present study, rotating stall in a wide vaneless diffuser is studied by a two dimensional linear stability analysis. The present study is clearly based on the approach proposed by Tsujimoto et al [10]. But whereas Tsujimoto’s work was limited to the study of the instability at critical conditions, the present work extends the analysis to unstable conditions. The ability and limits of such an approach are discussed. Then, to describe the nonlinear dynamic behavior of rotating stall, the equations are expanded up to the third order. A new temporal scale is introduced by focusing on a small distance from the critical condition.

2. Theoretical model

2.1 Basic equations

In this analysis, several assumptions were made: the flow is two dimensional, and the fluid is incompressible and inviscid, axisymmetric boundary conditions (uniform static pressure at diffuser outlet and imposed velocity magnitude and angle at inlet) are assumed. Under these conditions, the streamline in the vaneless diffuser is a logarithmic spiral. Starting the analysis from dimensional form (quantities with overbar), the continuity equation in cylindrical coordinate system is

$$\frac{\partial (\bar{r} \bar{V}_r)}{\partial \bar{r}} + \frac{\partial \bar{V}_\theta}{\partial \bar{\theta}} = 0$$

If the gravity is neglected, the momentum equation is

$$\bar{\rho} \left( \frac{\partial \bar{V}}{\partial \bar{t}} + (\bar{V} \cdot \nabla) \bar{V} \right) = -\nabla \bar{P}$$

The above expression in radial and tangential directions are:
applying the curl operator to equation (2), we have the following vorticity equation:

\[
\frac{\partial \bar{\zeta}}{\partial t} + (\bar{\nabla} \cdot \bar{V}) \bar{\zeta} - (\bar{\zeta} \cdot \bar{\nabla}) \bar{V} = 0
\]  

where \( \bar{V} \) and \( \bar{\zeta} \) can be decomposed as:

\[
\bar{V} = \bar{V}_r \hat{r} + \bar{V}_\theta \hat{\theta}; \quad \bar{\zeta} = \bar{\zeta}(\bar{r}, \bar{\theta}) \bar{\zeta}
\]

As the flow is two dimensional, the only equation component of interest in equation (5) is the axial one and the equation can be expressed in the following form:

\[
\frac{\partial \bar{\zeta}(\bar{r}, \bar{\theta})}{\partial \bar{t}} + \bar{V}_r \frac{\partial \bar{\zeta}(\bar{r}, \bar{\theta})}{\partial \bar{r}} + \frac{\bar{V}_\theta}{\bar{r}} \frac{\partial \bar{\zeta}(\bar{r}, \bar{\theta})}{\partial \bar{\theta}} = 0
\]  

one can also note that:

\[
\bar{\zeta} = \frac{1}{\bar{r}} \frac{\partial \bar{V}_\theta}{\partial \bar{r}} - \frac{1}{\bar{r}} \frac{\partial \bar{V}_r}{\partial \bar{\theta}}
\]  

Then a set of basic equations can be collected from (1), (3), (4), (6) and (7). The basic solutions of a steady flow, for this equation set are:

\[
\bar{V}_r = Q/2\pi \bar{r}, \quad \bar{V}_\theta = \Gamma / 2\pi \bar{r}, \quad \bar{\zeta} = 0 \quad \text{and} \quad \bar{P}_b = -(\bar{\rho}/8\pi^2) \left(Q^2 + \Gamma^2\right) \left(1/\bar{r}^2 - 1/\bar{r}_1^2\right)
\]

2.2 Dimensionless Equations

The quantities \( \bar{r}_1, \frac{2\pi r_1^2}{Q}, \frac{Q}{2\pi r_1^2}, \frac{Q}{2\pi r_1^2}, \text{ and } \frac{\bar{\rho}Q^2}{(2\pi r_1)^2} \) are used to scale the quantities of length, time, velocity, vorticity and pressure respectively, then the following dimensionless quantities are obtained:

\[
r = \frac{\bar{r}}{\bar{r}_1}, \quad t = \frac{Q}{2\pi \bar{r}_1^2}, \quad \bar{V} = \frac{2\pi \bar{r}_1^2}{Q} \bar{V}, \quad \bar{\zeta} = \frac{2\pi \bar{r}_1^2}{Q} \bar{\zeta}, \quad \bar{P} = \frac{(2\pi \bar{r}_1)^2}{\bar{\rho}Q^2} \bar{P}
\]

where \( 1 \leq r \leq R = \bar{r}_1 / \bar{r}_1 \), and the corresponding dimensionless basic steady solutions of the basic equations become:

\[
V_r = 1/r, \quad V_\theta = (1/r)(\Gamma / Q), \quad \zeta = 0 \quad \text{and} \quad \bar{P}_b = -0.5 \left(1 + \Gamma^2 / Q^2\right) \left(1/r^2 - 1/R^2\right)
\]

We then define \( \mu = \Gamma / Q \) to represent \( 1/\tan \alpha \) in the following text, where \( \alpha \) is the absolute flow angle in the diffuser.

2.3 Linearization and normal modes analysis

The flow in the vaneless diffuser is then represented by the sum of the steady flow and a small unsteady disturbance (\( \varepsilon << 1 \)), as follows:

\[
V_r = 1/r + \varepsilon u_r(r, \theta, t), \quad V_\theta = \mu/r + \varepsilon u_\theta(r, \theta, t), \quad \zeta = 0 + \varepsilon \zeta(r, \theta, t), \quad \bar{P} = \bar{P}_b + \varepsilon p(r, \theta, t)
\]

Introducing the above expansion into the basic equations and keeping only the terms with first order of \( \varepsilon \), we are able to obtain the linear equations for \( u_r, u_\theta, \zeta, \text{ and } p \). The solutions for disturbances in the form of normal modes can be written as follow:

\[
\{u_r, u_\theta, \zeta, p\}(r, \theta, t) = \{\bar{u}_r, \bar{u}_\theta, \bar{\zeta}, \bar{P}\}(r)e^{i\omega x - i\phi t}
\]  

where \( \bar{V}_r \) and \( \bar{V}_\theta \) are the radial and tangential velocity components.

The quantity of vorticity is defined as: \( \bar{\zeta} = \text{rot}(\bar{V}) \), applying the curl operator to equation (2), we have the following vorticity equation:

\[
\frac{\partial \bar{\zeta}}{\partial t} + (\bar{\nabla} \cdot \bar{V}) \bar{\zeta} - (\bar{\zeta} \cdot \bar{\nabla}) \bar{V} = 0
\]  

The solutions for disturbances are the radial and tangential velocity components.
Where \( n \) is the number of modes, which could represent in our case, the number of rotating stall cells. The complex pulsation \( \omega \) could be divided in a real and complex part: \( \omega = \omega_{\text{real}} - i\sigma \), in which the real part \( \omega_{\text{real}} \) is the angular frequency and \( \sigma \) is the growth rate of the instability, and the flow is unstable (stable) when \( \sigma > 0 \) (\( \sigma < 0 \)). The neutral stability state is obtained by setting \( \sigma = 0 \). Introducing the normal mode solutions, the final linear model is then obtained:

\[
\frac{\partial(r \tilde{u}_r)}{\partial r} - in \tilde{u}_\theta = 0 \tag{9a}
\]

\[
i \omega \tilde{u}_r - \frac{1}{r^2} \tilde{u}_r + \frac{1}{r} \frac{\partial \tilde{u}_r}{\partial r} - \frac{\mu}{r^2} \tilde{u}_r - \frac{2\mu}{r^2} \tilde{u}_\theta = - \frac{\partial \tilde{p}}{\partial r} \tag{9b}
\]

\[
i \omega \tilde{u}_\theta + \frac{1}{r} \frac{\partial \tilde{u}_\theta}{\partial r} + \frac{1}{r^2} \tilde{u}_\theta - \frac{i \mu}{r^2} \tilde{u}_r = \frac{1}{r} \tilde{p} \tag{9c}
\]

\[
i \omega \tilde{\zeta} + \frac{1}{r} \frac{\partial \tilde{\zeta}}{\partial r} - \frac{i \mu}{r^2} \tilde{\zeta} = 0 \tag{9d}
\]

\[
\tilde{\zeta} = \frac{1}{r} \frac{\partial (r \tilde{u}_\theta)}{\partial r} + \frac{in}{r} \tilde{u}_r \tag{9e}
\]

The above system can be solved with boundary conditions: \( \tilde{u}_r = \tilde{u}_\theta = 0 \) at \( r = 1 \); \( \tilde{p} = 0 \) at \( r = R \) and the solutions for disturbances \( \tilde{\zeta}, \tilde{u}_r, \tilde{u}_\theta \) are:

\[
\tilde{\zeta} = Ce^{\frac{in}{2}[(r^{-1} - r_0^{-1}) + \mu \ln r]} , \quad \tilde{u}_r = -iC \{ F + G - r^{n-1}G(r = 1) \} , \quad \tilde{u}_\theta = C \{ F - G + r^{n-1}G(r = 1) \}
\]

where \( F \) and \( G \) are defined by

\[
F(r) = \frac{1}{2} \int_1^r e^{\frac{in}{2}[\frac{1}{r} - \frac{1}{r_0} + \mu \ln \frac{r}{r_0}]} \left( \frac{r}{r_0} \right)^{n-1} dr_0, G(r) = \frac{1}{2} \int_r^1 e^{\frac{-in}{2}[\frac{1}{r} - \frac{1}{r_0} + \mu \ln \frac{r}{r_0}]} \left( \frac{r_0}{r} \right)^{n-1} dr_0
\]

When \( r = R \), equation (9c) can be simplified by using equation (9e) to obtain the dispersion equation,

\[
i \omega \tilde{u}_\theta (R) + (1/R) \tilde{\zeta} (R) - (in / R^2) \left[ \tilde{u}_r (R) + \mu \tilde{u}_\theta (R) \right] = 0 \tag{10}
\]

### 2.4 Theoretical results

At the critical stability condition \( (\sigma = 0) \), the dispersion equation (10) allows to determine \( \omega_{\text{real}} \) and \( \mu \) for fixed values of \( R \) and \( n \). It must be reminded that this kind of linear analysis was first proposed by Tsujimoto et al [10], thus the results obtained in this study have been compared to the results given by Tsujimoto et al [10] in order to validate the present linear model, as shown in figure 3. When the flow angle below the critical value of one stall mode, the corresponding mode becomes unstable and should be observed in the experiment. According to the comparisons in figure 3, there is no doubt that the same results are obtained by the present model for the presented modes \( n = 1, 2 \) and 3. They are depending on the number of stall modes \( n \) in addition to the diffuser radius ratio \( R \). A different behavior for the critical angle versus diffuser radius ratio has been obtained when the number of stall cells is 4. This has already been reported by Ljevar [3].

In addition to Tsujimoto’s results, the experimental results presented in a previous experimental study (Heng et al [14]) are to determine the abilities and limits of the present linear stability analysis. The experiments were performed with a radial impeller which was coupled with a wide vaneless diffuser \( (h/l_s = 0.15) \) downstream, the diffuser radius ratio: \( R = 1.5 \). Through the spectrum analysis, rotating stall was identified at five partial flow rates: \( Q/Q_d = 0.26, 0.36, 0.47, 0.56 \) and 0.58, the details of the identified stall modes are given in table 1, and the dominant stall mode is the one with the largest amplitude in the spectrum (more details of the experiments, see Heng et al [14]).
(a) Critical flow angle: $\alpha = \arctan \left( \frac{1}{\mu} \right)$.

(b) Propagation velocity: $V_p/V_{\theta2} = \omega_{\text{real}}(n\mu)$.

Figure 3. Comparisons between present study and Tsujimoto et al [10].

Table 1. Different stall modes at each flow rate in experiment (Heng et al [14]).

| $Q/Q_d$ | Dominant | Second | Third | Fourth |
|---------|----------|--------|-------|--------|
| 0.26    | $n = 3$  | $n = 2$ | no    | no     |
| 0.36    | $n = 2$  | $n = 3$ | no    | no     |
| 0.47    | $n = 4$  | $n = 2$ | $n = 3$ | no     |
| 0.56    | $n = 4$  | no     | no    | no     |
| 0.58    | $n = 4$  | no     | no    | no     |

With the decrease of the flow rate, the first stall mode which appears is the one with 4 cells which is coherent with the theoretical prediction in figure 3(a). At $Q/Q_d = 0.26, 0.36$ and $0.47$, different stall modes have been identified, this is also reflected in figure 3(a): more and more modes will become unstable if the flow angle keeps decreasing. One can notice that the theory suggests that the lower of the flow rate is, the more unstable modes are developing, but this result cannot be observed so clearly in the experiment. It is believed that there exist interactions between different stall modes which results in the disappearance of some stall modes in the experiment.

Compared to Tsujimoto’s study, what is new in the present linear model is that the research can be extended to unstable conditions. With the dispersion equation (10), if the pair $(R, n)$ is given, the growth rate of the perturbation - $\sigma$ and the angular velocity of the stall mode - $\omega_k$ can be calculated as a function of the flow angle $\alpha$. Since the critical flow angle is determined, if we decrease the flow angle from the critical value $\alpha_c$, physically, it means the flow condition is changing from the critical stall condition to unstable/stall conditions. Then, the characteristics of rotating stall can be plotted as a function of the flow angle. The comparison of the propagation velocity between present linear model and previous experiment is shown in figure 4.
It can be seen that the magnitude of the predicted propagation velocities is of the same order than then experimental data. Moreover, the decrease of the propagation velocity with the flow angle is observed in experimental and numerical results with similar slopes. However, the mode with more cells propagates faster in the theory which is not observed in the experiment. This discrepancy has also been reported by Tsujimoto et al [10].

The evolution of the predicted growth rate $\sigma$ with the flow angle is plotted in figure 5. An attempt to identify the dominant mode observed in the experience to the mode with the greatest growth rate given by the stability analysis has been done. Unfortunately, it can be seen (in table 1) that, in the experiment, there is a switch of the dominant mode from $n = 4$ to $n = 2$ and $n = 3$ when the flow rate is decreased whereas the predicted mode with the highest growth rate is always the mode $n = 4$ (figure 5). When several competitive modes arise, the physic is probably more complex than the one proposed by the linear stability analysis and nonlinear interaction between modes have to be taken into account. This motives the nonlinear analysis whose equations are proposed in the following paragraph.

3. Weakly nonlinear stability analysis

3.1 Nonlinear governing equations

In section 2.3, the diffuser flow is divided into a steady flow plus a very small perturbation, and the linear equations are obtained by keeping only the terms with first order of $\epsilon$. To perform the nonlinear stability analysis, the perturbations will be rewritten as following:

$$V_\alpha = 1 + u_\alpha(r, \theta, t), V_\beta = 1 + u_\beta(r, \theta, t), \xi = 0 + \xi(r, \theta, t), P = P_0 + p(r, \theta, t)$$

Introducing above form into the non-dimensional basic equations, the nonlinear governing equations are obtained,

$$\frac{\partial (ru_\alpha)}{\partial r} + \frac{\partial u_\beta}{\partial \theta} = 0$$

$$\frac{\partial u_\alpha}{\partial t} + \frac{u_\alpha}{r} + \frac{1}{r^2} \frac{\partial u_\alpha}{\partial r} + \frac{\mu}{r^2} \frac{\partial u_\alpha}{\partial \theta} + \frac{2}{r} \frac{\partial u_\alpha}{\partial \theta} + \frac{\partial P}{\partial r} = - \frac{u_\alpha}{r} \frac{\partial u_\alpha}{\partial r} - \frac{u_\alpha}{r} \frac{\partial u_\beta}{\partial r} - \frac{u_\beta}{r}$$

$$\frac{\partial u_\beta}{\partial t} + \frac{u_\beta}{r} + \frac{1}{r^2} \frac{\partial u_\beta}{\partial r} + \frac{\mu}{r^2} \frac{\partial u_\beta}{\partial \theta} + \frac{1}{r^2} \frac{\partial P}{\partial r} = - \frac{u_\alpha}{r} \frac{\partial u_\alpha}{\partial r} - \frac{u_\alpha}{r} \frac{\partial u_\beta}{\partial r} - \frac{u_\beta}{r}$$

$$\xi = \frac{1}{r} \frac{\partial (ru_\alpha)}{\partial r} - \frac{1}{r} \frac{\partial u_\beta}{\partial \theta}$$

$$\frac{\partial \xi}{\partial t} + \frac{1}{r} \frac{\partial \xi}{\partial r} + \frac{\mu}{r^2} \frac{\partial \xi}{\partial \theta} = - \frac{u_\alpha}{r} \frac{\partial \xi}{\partial r} - \frac{u_\alpha}{r} \frac{\partial \xi}{\partial \theta}$$

3.2 Amplitude equation of the perturbation

Since the linear stability analysis is validated at the critical condition, a detailed analysis is possible if the flow angle is only slightly smaller than the critical value $\alpha_c$. In that case, the weakly nonlinear
dynamics of the linear unstable modes can be described by an amplitude equation. In order to derive the amplitude equation, the perturbations are expanded in power series of \( \varepsilon \):
\[
\{u_1, u_2, \xi, p\} (r, \theta, t, t_2) = \varepsilon \{u_1^{(1)}, u_2^{(1)}, \xi^{(1)}, p^{(1)}\} + \varepsilon^2 \{u_1^{(2)}, u_2^{(2)}, \xi^{(2)}, p^{(2)}\} + \ldots + \varepsilon^3 \{u_1^{(3)}, u_2^{(3)}, \xi^{(3)}, p^{(3)}\} + o(\varepsilon^3)
\]

In solving weakly nonlinear hydrodynamic problems with this method for a non-self-adjoint linear operator, we must use the corresponding adjoint operator. Then we perform inner product using both, the solutions of the homogeneous linear operator and of the adjoint homogeneous linear operator. The corresponding amplitude equation is then obtained from a solvability condition, known as the Fredholm alternative, which states the fact that the inhomogeneity must be orthogonal to the solution of the adjoint problem. The main steps and results are presented in the following.

As the weakly nonlinear stability analysis is focusing on a small distance from the critical condition. Instead the flow angle \( \alpha \), the new control parameter \( \mu \) will be used:
\[
\mu = \mu_\alpha + \varepsilon^2 \mu_2
\]

As \( \mu - \mu_\alpha = O(\varepsilon^2) \), the modulation is at slow temporal scale,
\[
t_2 = \varepsilon^2 t
\]

The temporal derivative is then replaced by
\[
\frac{\partial}{\partial t} \rightarrow \frac{\partial}{\partial t} + \varepsilon^2 \frac{\partial}{\partial t_2}
\]

Applying the conditions (12) – (15) to the nonlinear system (11a - e), and collecting coefficient of \( \varepsilon \), a set of equations is obtained. For example, equation (11e) becomes:
\[
\left[ \left( \frac{\partial}{\partial t} + \varepsilon^2 \frac{\partial}{\partial t_2} \right) \right] + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial}{\partial \theta} \left[ \varepsilon^2 \mu_1 + \varepsilon^2 \mu_2 \frac{\partial}{\partial \theta} \right] \left[ \varepsilon^2 \xi^{(1)} + \varepsilon^2 \xi^{(2)} + \varepsilon^3 \xi^{(3)} + \ldots \right]
\]
\[
= -\left[ \varepsilon u_1^{(1)} + \varepsilon^2 u_1^{(2)} + \varepsilon^3 u_1^{(3)} + \ldots \right] \frac{\partial}{\partial r} \left[ \varepsilon^2 \xi^{(1)} + \varepsilon^2 \xi^{(2)} + \varepsilon^3 \xi^{(3)} + \ldots \right]
\]
\[
- \left[ \varepsilon u_2^{(1)} + \varepsilon^2 u_2^{(2)} + \varepsilon^3 u_2^{(3)} + \ldots \right] \frac{\partial}{\partial \theta} \left[ \varepsilon^2 \xi^{(1)} + \varepsilon^2 \xi^{(2)} + \varepsilon^3 \xi^{(3)} + \ldots \right]
\]

3.2.1 First order solutions. At first order of \( \varepsilon \), the solutions are exactly the same as the ones obtained in the linear model.

3.2.2 Second order solutions. At order \( \varepsilon^2 \), the equation (16) leads to the nonhomogeneous problem for the vorticity field \( \xi^{(2)} \),
\[
\frac{\partial \xi^{(2)}}{\partial t} + \frac{1}{r} \frac{\partial \xi^{(2)}}{\partial r} + \mu \frac{\partial \xi^{(2)}}{\partial \theta} = -u_1^{(1)} \frac{\partial \xi^{(1)}}{\partial r} - u_2^{(1)} \frac{\partial \xi^{(1)}}{\partial \theta}
\]

Without loss of generality and in order to avoid a new amplitude different from the amplitude \( \xi^{(1)} \) of the most unstable mode, the homogeneous solution \( \xi_h^{(2)} \) of (17) is chosen to be orthogonal to the solution of the first order \( \xi^{(1)} \). In the present case, the appropriate scalar product is
\[
\langle \xi_h^{(2)}, \xi^{(1)} \rangle = \frac{1}{R} \frac{n}{2\pi} \int_0^{2\pi} \int_0^{\pi} \xi_h^{(2)}(\rho) \xi^{(1)}(\rho) r dr d\rho dt
\]

here, in the nonlinear analysis, the overbar \( \xi^{(1)} \) means the complex conjugate. The condition of the orthogonality leads to \( \xi_h^{(2)} = 0 \).

The right side of equation (18) suggests a particular inhomogeneous solution in the following form:
\[
\xi^{(2)} = A^2 (t_2) \hat{\xi}^{(2,2)} (r) e^{2i\omega t_2} + A(t_2)^2 \hat{\xi}^{(2,0)} (r) + C.C
\]
By introducing the above form into equation (17), $\tilde{\zeta}^{(2,2)}$ and $\tilde{\zeta}^{(2,0)}$ are respectively the solutions of equations:

$$2i\omega_t \tilde{\zeta}^{(2,2)} + \frac{1}{r} \frac{\partial}{\partial r} \tilde{\zeta}^{(2,2)} - 2i m \frac{\mu_c}{r^2} \tilde{\zeta}^{(2,2)} = -\tilde{u}_r^{(1)} \frac{\partial}{\partial r} \tilde{\zeta}^{(1)} + \frac{\mu_c}{r} \tilde{u}_r^{(1)} \tilde{\zeta}^{(1)}$$

(20)

$$\frac{1}{r} \frac{\partial}{\partial r} \frac{1}{r} \frac{\partial}{\partial r} = -\tilde{u}_r^{(1)} \frac{\partial}{\partial r} \tilde{\zeta}^{(1)} + \frac{\mu_c}{r} \tilde{u}_r^{(1)} \tilde{\zeta}^{(1)}$$

(21)

and the analytical solutions are:

$$\tilde{\zeta}^{(2,2)}_s (r) = r \tilde{u}_r^{(1)} \tilde{\zeta}^{(1)}_s, \quad \tilde{\zeta}^{(2,0)}_s (r) = -r \tilde{u}_r^{(1)} \tilde{\zeta}^{(1)}_s$$

Once $\tilde{\zeta}^{(2)}$ is known, the solutions for $\{\tilde{u}_r^{(2)}, \tilde{u}_\theta^{(2)}\}$ can be determined. At order $\epsilon^2$, equations (11a) and (11d) become

$$\frac{\partial (ru_r^{(2)})}{\partial r} + \frac{\partial u_r^{(2)}}{\partial \theta} = 0$$

(22)

$$\frac{1}{r} \frac{\partial (ru_r^{(2)})}{\partial r} - \frac{1}{r} \frac{\partial u_r^{(2)}}{\partial \theta} = \zeta^{(2)}$$

(23)

with boundary conditions: $u_r^{(2)} = u_\theta^{(2)} = 0$ at $r = 1$

In the same manner, $\{u_r^{(2)}, u_\theta^{(2)}\}$ suggest the solutions in the following form:

$$(u_r^{(2)}, u_\theta^{(2)}) = A^2 (t_c) (\tilde{u}_r^{(2,2)}(r), \tilde{u}_\theta^{(2,2)}(r)) e^{2i\omega - 2m \theta} + [A(t_c)]^2 (\tilde{u}_r^{(2,0)}(r), \tilde{u}_\theta^{(2,0)}(r)) + C.C$$

(24)

and the analytical solutions can be written as:

$$\tilde{u}_r^{(2,2)}(r) = -i \left\{ F_2 + G_2 - r^{2n-1} G_2 (r = 1) \right\}, \tilde{u}_\theta^{(2,2)}(r) = F_2 - G_2 + r^{2n-1} G_2 (r = 1)$$

$$\tilde{u}_r^{(2,0)} = 0, \tilde{u}_\theta^{(2,0)} = \frac{1}{r} \int_{r_0}^r \tilde{\zeta}^{(2,0)}(t_0) d t_0$$

where $F_2$ and $G_2$ are defined by

$$F_2(r) = \frac{1}{2} \int_1^r \tilde{\zeta}^{(2,2)}(r_0) r_0^{2n+1} d r_0, \quad G_2(r) = \frac{1}{2} \int_1^r \tilde{\zeta}^{(2,2)}(r_0) \left( \frac{r}{r_0} \right)^{2n+1} d r_0$$

3.2.3 Third order solvability condition. At order $\epsilon^3$, the following system for $(u_r^{(3)}, u_\theta^{(3)}, \zeta^{(3)})$ is obtained,

$$\frac{\partial (ru_r^{(3)})}{\partial r} + \frac{\partial u_r^{(3)}}{\partial \theta} = 0$$

$$\frac{1}{r} \frac{\partial (ru_r^{(3)})}{\partial r} - \frac{1}{r} \frac{\partial u_r^{(3)}}{\partial \theta} - \zeta^{(3)} = 0$$

$$\frac{\partial \zeta^{(3)}}{\partial t} + \frac{1}{r} \frac{\partial \zeta^{(3)}}{\partial r} + \frac{\mu_c}{r} \frac{\partial \zeta^{(3)}}{\partial \theta} = \text{RHS}$$

(25)

$$\text{RHS} = -\frac{\partial \tilde{\zeta}^{(1)}}{\partial t_c} - \frac{\mu_c}{r} \frac{\partial \tilde{\zeta}^{(1)}}{\partial \theta} - u_r^{(1)} \frac{\partial \tilde{\zeta}^{(2)}}{\partial r} - u_r^{(1)} \frac{\partial \tilde{\zeta}^{(2)}}{\partial r} - u_\theta^{(1)} \frac{\partial \tilde{\zeta}^{(2)}}{\partial r} - u_\theta^{(1)} \frac{\partial \tilde{\zeta}^{(2)}}{\partial r} - u_\theta^{(1)} \frac{\partial \tilde{\zeta}^{(2)}}{\partial r}$$

We use RHS to represent all terms on the right hand side in the last expression, and the above equation has a solution if and only if a solvability condition, known as the Fredholm alternative, is satisfied. This condition states that the right hand side of the above equations must be orthogonal to the linear solution of the adjoint problem. As the solution of the adjoint problem is

$$\tilde{\zeta}^{(1)^*}(r, \theta, t) = \tilde{\zeta}(r) e^{i\omega t - im \theta} + c.c$$

(26)
where “*” represents the adjoint problem and “c.c” is the corresponding complex conjugate.

The solvability condition is obtained by performing the inner product:

\[
\langle \text{RHS}, \zeta^{(1)*} \rangle = \frac{1}{R} \int_{0}^{2\pi} \int_{0}^{2\pi} \int_{0}^{2\pi} \text{RHS} \cdot \overline{\zeta^{(1)}} \, rd\theta dt
\]  

(27)

The above condition (27) yields, the behavior of the amplitude \( A(t) \) at the time scale \( t_2 \) is obtained:

\[
dA/dt = (\lambda_\epsilon + i\lambda_\iota) (\mu - \mu_c) A - (\gamma_\epsilon + i\gamma_\iota) A |A|^2
\]

(28)

with \( t_2 = \epsilon^2 \) and \( \mu_2 = (\mu - \mu_c)/\epsilon^2 \), the above amplitude equation can be rewritten as following form by substituting \( \epsilon A \) by \( A \),

\[
dA/dt = (\lambda_\epsilon + i\lambda_\iota) (\mu - \mu_c) A - (\gamma_\epsilon + i\gamma_\iota) A |A|^2
\]

(29)

Writing the complex amplitude \( A \) in the norm-phase form,

\[
A = |A|e^{i\omega_2 t}
\]

(30)

and substituting it in equation (29), we get the following equations,

\[
d|A|/dt = \lambda_\epsilon (\mu - \mu_c) |A| - \gamma_\iota |A|^2, \omega_2 = \lambda_\epsilon (\mu - \mu_c) - \gamma_\iota |A|^2
\]

(31)

The physical meaning of these coefficients is: \( \lambda_\epsilon (\mu - \mu_c) \) is the linear growth rate of the instability above its threshold, \( \lambda_\epsilon (\mu - \mu_c) \) and \( -\gamma_\iota |A|^2 \) represent respectively the linear and the nonlinear correction of the frequency in the supercritical regime, and the sign of \( \gamma_\iota \) determines the nature of the bifurcation.

The coefficients \( \gamma_\iota \) and \( \gamma_\iota \) which give nonlinear corrections have not been numerically solved yet. However, the solved linear coefficients \( \lambda_\epsilon \) and \( \lambda_\iota \) are used to compare the prediction of this model with the experimental results to have a first evaluation of this weakly nonlinear stability analysis. It should be noticed that the experimental growth rate of rotating stall has never been measured, which means that the comparison of the instability growth rate \( \lambda_\epsilon \) cannot be made. On the other hand, the experimental angular frequencies of rotating stall are available in Heng et al [14], and because the weakly nonlinear analysis restricts the flow condition very close to the critical stall condition, in order to ensure the distance as small as possible, we take the first two points available experimentally to determine the experimental slope of the angular frequency, as shown in figure 6. Then, the experimental and theoretical results are compared in table 2.

| Table 2. Comparison of the slope of angular frequency. |
|-----------------------------------------------|
| Stall mode | Experiment (Heng et al [14]) | Theory (\( \lambda_\epsilon \)) |
| n = 2 | 0.98 | 1.12 |
| n = 3 | 1.51 | 1.79 |
| n = 4 | 1.64 | 2.08 |

Figure 6. Slopes of the experimental angular velocity of rotating stall (Heng et al [14]).

It can be seen that good agreements are obtained in the comparison: the experimental and theoretical results are very close to each other, and the slope of the angular frequency is larger for modes with more number of cells. The theoretical results are slightly higher than the experimental ones, but this may be corrected by the nonlinear contribution which is not taken into account in this comparison. A completely comparison will be made when the nonlinear coefficients have been numerically solved in the future.

4. Conclusions

The present paper has presented a stability analysis conducted to predict the arising and characteristics of rotating stall in a wide vaneless diffuser. The calculated critical flow angle and the propagation
velocity are validated by Tsujimoto’s results, and have been compared to a previous experiment. The comparisons showed that the linear stability analysis is able to predict the order of magnitude of the critical angle and propagation velocity of the instability. But some discrepancies exist in the comparison with experimental results. Moreover, the use of the growth rate is unable to predict the most dominant mode which is experimentally observed. It is certainly due to the interactions between modes which are ignored by the linear analysis.

To take these effects into account a weakly nonlinear stability analysis, the nonlinear equations up to the third order are presented in the paper. The amplitude equation is obtained which is able to describe the nonlinear grow rate and propagation velocity of rotating stall. Although the nonlinear coefficients have not been numerically solved yet, the predicted evolution of the instability velocity close to the critical condition is coherent with experimental data. The calculated results are slightly higher than the experimental data, but it could be corrected by the nonlinear part which will be available in a next future.

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