Manifestly duality-invariant interactions in diverse dimensions

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Abstract

As an extension of the Ivanov-Zupnik approach to self-dual nonlinear electrodynamics in four dimensions [1,2], we reformulate U(1) duality-invariant nonlinear models for a gauge \((2p-1)\)-form in \(d = 4p\) dimensions as field theories with manifestly U(1) invariant self-interactions. This reformulation is suitable to generate arbitrary duality-invariant nonlinear systems including those with higher derivatives.
1 Introduction

As an extension of the seminal work by Gaillard and Zumino [3], the general formalism of duality-invariant models for nonlinear electrodynamics in four dimensions was developed in the mid-1990s [4–7]. The Gaillard-Zumino-Gibbons-Rasheed (GZGR) approach was generalised to off-shell $\mathcal{N} = 1$ and $\mathcal{N} = 2$ globally [8, 9] and locally [10, 11] supersymmetric theories. In particular, the first consistent perturbative scheme to construct the $\mathcal{N} = 2$ supersymmetric Born-Infeld action was given in [9] (this approach was further pursued in [12]). The GZGR formalism was also extended to higher dimensions [13–15].

Nonlinear electrodynamics with U(1) duality symmetry is described by a Lorentz invariant Lagrangian $L(F_{ab})$ which is a solution to the self-duality equation

$$\tilde{G}^{ab}G_{ab} + \tilde{F}^{ab}F_{ab} = 0 ,$$

where

$$\tilde{G}^{ab}(F) := \frac{1}{2} \epsilon^{abcd} G_{cd}(F) = 2 \frac{\partial L(F)}{\partial F_{ab}} .$$

In the case of theories with higher derivatives, this scheme is generalised in accordance with the two rules given in [8]. Firstly, the definition of $\tilde{G}$ is replaced with

$$\tilde{G}^{ab}[F] = 2 \frac{\delta S[F]}{\delta F_{ab}} .$$

Secondly, the self-duality equation (1.1) is replaced with

$$\int d^4x \left( \tilde{G}^{ab}G_{ab} + \tilde{F}^{ab}F_{ab} \right) = 0 .$$

Duality-invariant theories with higher derivative theories naturally occur in $\mathcal{N} = 2$ supersymmetry [9]. Further aspects of duality-invariant theories with higher derivatives were studied in, e.g., [16–19].

Self-duality equation (1.1) is nonlinear, and therefore its general solutions are difficult to find. In the early 2000s, Ivanov and Zupnik [1, 2] proposed a reformulation of duality-invariant electrodynamics involving an auxiliary antisymmetric tensor $V_{ab}$, which is equivalent to a symmetric spinor $V_{\alpha\beta} = V_{\beta\alpha}$ and its conjugate $\bar{V}^{\dot{\alpha}\dot{\beta}}$. The new Lagrangian $L(F, V)$ is at most quadratic in the electromagnetic field strength $F_{ab}$, while the self-interaction is described by a nonlinear function of the auxiliary variables, $L_{\text{int}}(V_{ab})$,

$$L(F_{ab}, V_{ab}) = \frac{1}{4} F^{ab}F_{ab} + \frac{1}{2} V^{ab}V_{ab} - V^{ab}F_{ab} + L_{\text{int}}(V_{ab}) .$$

This approach was inspired by the structure on the $\mathcal{N} = 3$ supersymmetric Born-Infeld action proposed in [20].
The original theory \( L(F_{ab}) \) is obtained from \( L(F_{ab}, V_{ab}) \) by integrating out the auxiliary variables. In terms of \( L(F_{ab}, V_{ab}) \), the condition of U(1) duality invariance was shown \([1,2]\) to be equivalent to the requirement that the self-interaction

\[
L_{\text{int}}(V_{ab}) = L_{\text{int}}(\nu, \bar{\nu}) \ , \quad \nu := V^{\alpha\beta} V_{\alpha\beta} \tag{1.6}
\]
is invariant under linear U(1) transformations \( \nu \to e^{i\varphi} \nu \), with \( \varphi \in \mathbb{R} \), and thus

\[
L_{\text{int}}(\nu, \bar{\nu}) = f(\nu \bar{\nu}) \ ,
\]
where \( f \) is a real function of one real variable. The Ivanov-Zupnik (IZ) approach \([1,2]\) has been used by Novotný \([21]\) to establish the relation between helicity conservation for the tree-level scattering amplitudes and the electric-magnetic duality.

The above discussion shows that the IZ approach is a universal formalism to generate U(1) duality-invariant models for nonlinear electrodynamics. Some time ago, there was a revival of interest in duality-invariant dynamical systems \([17,22,23]\) inspired by the desire to achieve a better understanding of the UV properties of extended supergravity theories. The authors of \([22]\) have put forward the so-called “twisted self-duality constraint,” which was further advocated in \([17,23]\), as a systematic procedure to generate manifestly duality-invariant theories. However, these approaches have been demonstrated \([24]\) to be variants of the IZ scheme \([1,2]\) developed a decade earlier.

The IZ approach has been generalised to off-shell \( \mathcal{N} = 1 \) and \( \mathcal{N} = 2 \) globally and locally supersymmetric theories \([25,26]\). In this note we provide a generalisation of the approach to higher dimensions, \( d = 4p \). In even dimensions, \( d = 2n \), the maximal duality group for a system of \( k \) gauge \((n - 1)\)-forms depends on the dimension of spacetime. The duality group is U\((k)\) if \( n \) is even, and O\((k) \times O(k)\) if \( n \) is odd \([14]\) (see, e.g, section 8 of \([9]\) for a review). This is why we choose \( d = 4p \). The fact that the maximal duality group depends on the dimension of space-time was discussed in the mid-1980s \([27,28]\) and also in the late 1990s \([29,30]\).

## 2 New formulation

In Minkowski space of even dimension \( d = 4p \equiv 2n \), with \( p \) a positive integer, we consider a self-interacting theory of a gauge \((n - 1)\)-form \( A_{a_1...a_{n-1}} \) with the property that the Lagrangian, \( L = L(F) \), is a function of the field strengths \( F_{a_1...a_n} = n \partial_{[a_1} A_{a_2...a_n]} \)\footnote{We follow the notation and conventions of \([9]\).} We
assume that the theory possesses U(1) duality invariance. This means that the Lagrangian is a solution to the self-duality equation

$$\tilde{G}^{a_1...a_n}G_{a_1...a_n} + \tilde{F}^{a_1...a_n}F_{a_1...a_n} = 0 \ ,$$

(2.1)

where we have introduced

$$\tilde{G}^{a_1...a_n}(F) = n! \frac{\partial L(F)}{\partial F_{a_1...a_n}} \ .$$

(2.2)

As usual, the notation $\tilde{F}$ is used for the Hodge dual of $F$,

$$\tilde{F}^{a_1...a_n} = \frac{1}{n!} \epsilon^{a_1...a_n b_1...b_n} F_{b_1...b_n} \ .$$

(2.3)

We now introduce a reformulation of the above theory. Along with the field strength $F_{a_1...a_n}$, our new Lagrangian $L(F, V)$ is defined to depend on an auxiliary rank-$n$ antisymmetric tensor $V_{a_1...a_n}$ which is unconstrained. We choose $L(F, V)$ to have the form

$$L(F, V) = \frac{1}{n!} \left\{ \frac{1}{2} F \cdot F + V \cdot V - 2V \cdot F \right\} + L_{\text{int}}(V) \ ,$$

(2.4)

where we have denoted

$$V \cdot F := V^{a_1...a_n} F_{a_1...a_n} \ .$$

(2.5)

The last term in (2.4), $L_{\text{int}}(V)$, is at least quartic in $V^{a_1...a_n}$. It is assumed that the equation of motion for $V$,

$$\frac{\partial}{\partial V_{a_1...a_n}} L(F, V) = 0 \ ,$$

(2.6)

allows one to integrate out the auxiliary field $V$ to result with $L(F)$.

It may be shown that the self-duality equation (2.1) is equivalent to the following condition on the self-interaction in (2.4)

$$\tilde{V}^{a_1...a_n} \frac{\partial}{\partial V_{a_1...a_n}} L_{\text{int}}(V) = 0 \ .$$

(2.7)

Introducing (anti) self-dual components of $V$,

$$V^{a_1...a_n}_\pm = \frac{1}{2} \left( V^{a_1...a_n} \pm i\tilde{V}^{a_1...a_n} \right) \ , \quad \tilde{V}_\pm = \mp iV_\pm \ , \quad V = V_+ + V_- \ ,$$

(2.8)

the above condition turns into

$$\left( V^{a_1...a_n}_+ \frac{\partial}{\partial V_{a_1...a_n}^+} - V^{a_1...a_n}_- \frac{\partial}{\partial V_{a_1...a_n}^-} \right) L_{\text{int}}(V_+, V_-) = 0 \ .$$

(2.9)
This means that $L_{\text{int}}(V_+, V_-)$ is invariant under U(1) phase transformations,

$$L_{\text{int}}(e^{i\varphi}V_+, e^{-i\varphi}V_-) = L_{\text{int}}(V_+, V_-), \quad \varphi \in \mathbb{R}.$$ \hspace{1cm} (2.10)

In four dimensions, the most general solution to this condition is given by eq. (1.7). Similar solutions exist in higher dimensions, $L_{\text{int}}(V_+, V_-) = f(V_+ V_+ V_- V_-)$, with $f(x)$ a real function of one variable. However more general self-interactions become possible beyond four dimensions.

It is worth pointing out that an infinitesimal U(1) duality transformation

$$\delta \begin{pmatrix} G \\ F \end{pmatrix} = \begin{pmatrix} 0 & -\lambda \\ \lambda & 0 \end{pmatrix} \begin{pmatrix} G \\ F \end{pmatrix}$$ \hspace{1cm} (2.11a)

leads to the following transformation of $V$

$$\delta V = \lambda \tilde{V}.$$ \hspace{1cm} (2.11b)

Equation (2.7) tells us that $L_{\text{int}}(V)$ duality invariant.

There are several interesting generalisations of the construction described. They include (i) coupling to gravity; (ii) coupling to a dilaton with enhanced SL(2, $\mathbb{R}$) duality; (iii) duality-invariant systems with higher derivatives; and (iv) U($k$) duality-invariant systems of $k$ gauge ($2p - 1$)-forms in $d = 4p$ dimensions.

Recently, U(1) duality-invariant theories of a gauge ($2p - 1$)-form in $d = 4p$ dimensions have been described [31] within the Pasti-Sorokin-Tonin approach [32, 33]. It was argued in [31] that the approach of [32, 33] is the most efficient method to determine all possible manifestly U(1) duality invariant self-interactions provided Lorentz invariance is kept manifest. Our analysis has provided an alternative formalism.

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