Solitons in Affine and Permutation Orbifolds

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Abstract
We consider properties of solitons in general orbifolds in the algebraic quantum field theory framework and constructions of solitons in affine and permutation orbifolds. Under general conditions we show that our construction gives all the twisted representations of the fixed point subnet. This allows us to prove a number of conjectures; in the affine orbifold case we clarify the issue of “fixed point resolutions”; in the permutation orbifold case we determine all irreducible representations of the orbifold, and we also determine the fusion rules in a non-trivial case, which imply an integral property of chiral data for any completely rational conformal net.

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1 Introduction

Let $\mathcal{A}$ be a completely rational conformal net (cf. §3.5 and def. 3.6 following [21]). Let $\Gamma$ be a finite group acting properly on $\mathcal{A}$ (cf. definition (3.4)). The starting point of this paper is Th. 3.7 proved in [44] which states that the fixed point subnet (the orbifold) $\mathcal{A}^\Gamma$ is also completely rational, and by [21] $\mathcal{A}^\Gamma$ has finitely many irreducible representations which are divided into two classes: the ones that are obtained from the restrictions of a representation of $\mathcal{A}$ to $\mathcal{A}^\Gamma$ which are called untwisted representations, and the ones which are twisted (cf. definition after Th. 3.7). It follows from Th. 3.7 that twisted representation of $\mathcal{A}^\Gamma$ always exists if $\mathcal{A}^\Gamma \neq \mathcal{A}$. The motivating question for this paper is how to construct these twisted representations of $\mathcal{A}^\Gamma$.

It turns out that all representations of $\mathcal{A}^\Gamma$ are closely related to the solitons of $\mathcal{A}$ (cf. §3.3 and Prop. 4.1). Solitons are representations of $\mathcal{A}_0$, the restriction of $\mathcal{A}$ to the real line identified with a circle with one point removed. Every representation of $\mathcal{A}$ restricts to a soliton of $\mathcal{A}_0$, but not every soliton of $\mathcal{A}_0$ can be extended to a representation of $\mathcal{A}$. In §4 we develop general theory of solitons in the case of orbifolds with two main results: Th. 4.5 gives a formula for the index of of solitons obtained from restrictions, and Th. 4.8 clarifies the general structure of the restriction of a soliton. These results are natural extensions of similar results in [33] and [45] in special cases.

The construction of solitons depends on the net $\mathcal{A}$ and the action of $\Gamma$. In the case of affine orbifold, our construction (cf. def. 5.6) is partially inspired by the “twisted representations” of [18], and in fact can be viewed as an “exponentiated version” of the “twisted representations” of [18] (cf. §5.2.1). Combined with the general properties of solitons described above, this construction allows us to clarify the issue of “fixed point” problem in [18] in Th. 5.16, and we also show that our construction gives all the irreducible representations of the fixed point subnet under general conditions in Th. 5.11 and Cor. 5.12, thus answering our motivating question in this case.

In the case of permutation orbifolds (cf. §6), our construction of solitons in (6.5) is a simple generalizations of the construction of solitons in [33] for the case of cyclic orbifolds. Note that the construction of solitons in [33] also leads to structure results such as a dichotomy for any split local conformal net. In Th. 8.1 (resp. Th. 8.5) we show that our construction gives all the irreducible representations of the cyclic orbifold (resp. the permutation orbifold), and in Th. 8.4 (resp. Th. 8.7) we list all the irreducible representations of the cyclic orbifold (resp. the permutation orbifold). These results generalize the results of [33] and prove a claim in [1] which is based on heuristic arguments. Using these results in §9 we determine the fusion rules for the first nontrivial case when $n = 2$ in Th. 9.8 which implies an integral property of the chiral data for any completely rational net (cf. Cor. 9.9), proving a conjecture in the paper [5], that contained the first computations leading to correct fusion rules.

The rest of this paper is organized as follows: §2 and §3 are preliminaries on the algebraic quantum field theory framework where orbifold construction is considered. In these sections we have collected some basic notions that appear in this paper for the convenience of the reader who may not have an operator algebra background.
The results in §2, §3 are known except Prop. 3.8 on extensions of solitons which plays an important role in §7. In §4 we apply the results in §2 and §3 to obtain general properties of solitons under inductions and restrictions, and in particular we prove Th. 4.5 and Th. 4.8. In §5, after recalling basic definitions and properties in affine orbifold from [18], we give the constructions of solitons in §5.2 and in §5.2.1 compare it with the twisted representations in [18]. Th. 5.11 and its Cor. 5.12 are proved in §5.3. In §5.5 we clarify the issue of fixed point resolutions in Th. 5.16, Cor. 5.17. In §5.6 we illustrate the results of §5.5 in an example considered in [18]. In §6 we first recall the construction of solitons from [33] in the cyclic permutation case, and in §6.3 give the general construction of solitons for permutation orbifolds. We prove in §7 the important property of these solitons (Th. 7.1) which so far has no direct proof. In §8 we apply the results of previous sections to prove four theorems which are briefly described above. In §9 after proving some simple properties of S matrix (cf. Lemma 9.1), we determine the fusions of solitons in cyclic orbifold in a special case in Prop. 9.4. In §9.3 we determine the fusion rules for the case n = 2 in Th. 9.8 which implies an integral property in Cor. 9.9.

2 Elements of Operator Algebras and Conformal QFT

For the convenience of the reader we collect here some basic notions that appear in this paper. This is only a guideline and the reader should look at the references for a more complete treatement.

2.1 von Neumann algebras

Let \( \mathcal{H} \) be a Hilbert space that we always assume to be separable to simplify the exposition. With \( B(\mathcal{H}) \) the algebra of all bounded linear operator on \( \mathcal{H} \) a von Neumann algebra \( M \) is a *-subalgebra of \( B(\mathcal{H}) \) containing the identity operator such that \( M = M'' \) (von Neumann density theorem), where the prime denotes the commutant: \( M' \equiv \{ a \in B(\mathcal{H}) : xa = ax \ \forall x \in M \} \).

A linear map \( \eta \) from a von Neumann algebra \( M \) to a von Neumann algebra \( N \) is positive if \( \eta(M_+) \subset N_+ \), where \( M_+ \equiv \{ x \in M : x > 0 \} \) denotes the cone of positive elements of \( M \). \( \eta \) is normal if commutes with the sup operation, namely \( \sup \eta(x_i) = \eta(\sup x_i) \) for any bounded increasing net of elements in \( M_+ \); \( \eta \) is normal if it is weakly (equivalently strongly) continuous on the unit ball of \( M \). \( \eta \) is faithful if \( \eta(x) = 0, x \in M_+ \), implies \( x = 0 \). By a homomorphism of a von Neumann algebra we shall always mean an identity preserving homomorphism commuting with the *-operation, and analogously for isomorphisms and endomorphisms. Isomorphisms between von Neumann algebras are automatically normal. By a representation of \( M \) on a Hilbert space \( \mathcal{K} \) we mean a homomorphism of \( M \) into \( B(\mathcal{K}) \).
A state \( \omega \) on von Neumann algebra \( M \) is a positive linear functional on \( M \) with the normalization \( \omega(1) = 1 \). The relevant states for a von Neumann algebras are the normal states. By the GNS construction, every normal state of \( M \) is given by \( \omega(x) = (\pi(x)\Omega, \Omega) \), where \( \pi \) is a normal representation of \( M \) on a Hilbert space \( K \) and \( \Omega \in K \) is cyclic (i.e. \( \pi(M)\Omega \) is dense in \( K \), see below). Given \( \omega \), the triple \((K, \pi, \Omega)\) is unique up to unitary equivalence.

A factor is a von Neumann algebra with trivial center, namely \( M \cap M' = \mathbb{C} \). We note that a factor is a simple algebra, i.e., the only weakly closed ideal of the factor is either trivial or equal to the factor itself. If \( M \) is a factor (and \( K \) is separable), a representation of \( M \) on \( K \) is automatically normal.

A factor \( M \) is finite if there exists a tracial state \( \omega \) on \( M \), namely \( \omega(xy) = \omega(yx) \), \( x, y \in M \) (automatically normal and unique). Otherwise \( M \) is called an infinite factor.

For a factor \( M \), the following are equivalent:

- \( M \) is infinite;
- \( M \) is isomorphic to \( M \otimes B(K) \), with \( K \) a separable infinite dimensional Hilbert space;
- \( M \) contains a non-unitary isometry (an isometry \( v \) is an operator with the property \( v^*v = 1 \));
- \( M \) contains a non degenerate Hilbert \( H \) space of isometries with arbitrary dimension (but separable).

Here a Hilbert space of isometries \( H \) in \( M \) we mean a norm closed linear subspace \( H \subset M \) such that \( x^*y \in \mathbb{C} \) for all \( x, y \in M \). Thus \( x, y \mapsto y^*x \) is scalar product on \( H \). Then, if \( L \) is a set with \( \{v_\ell, \ell \in L\} \) an orthonormal basis for \( H \), we have \( v_\ell v_\ell^* = \delta_{\ell\ell} \), namely the \( v_\ell \)'s are isometries \( H \) with pairwise orthogonal range projections. \( H \) is non-degenerate if the left support of \( H \) is 1, that is the final projections form a partition of the identity: \( \sum_{\ell \in L} v_\ell v_\ell^* = 1 \).

A factor \( M \) is of type III (or purely infinite) if every non-zero projection \( e \in M \) is equivalent to the identity, namely there exists an isometry \( v \in M \) wit \( vv^* = 1 \). As we shall see, factors appearing in CFT as local algebras are of type III and the reader may focus on this case for the need of this paper.

A semifinite factor is a factor \( M \) isomorphic to \( M_0 \otimes B(K) \) with \( M_0 \) a finite factor and \( K \) a Hilbert space. Semifinite factor are characterized by the existence of a normal, possible unbounded, trace (that we do not define here). A factor is either semifinite or of type III.

A factor \( M \) of type III has only one representation (on a separable Hilbert space) up to unitary equivalence. Namely, if \( \pi : M \to B(K) \) is a representation, there exists a unitary \( U : \mathcal{H} \to K \) such that \( \pi(x) = U x U^* \), \( x \in M \).

Refs: [40].
2.2 Tomita-Takesaki modular theory

Let $M$ be a von Neumann algebra and $\omega$ a normal faithful state on $M$. By the GNS construction, we may assume that $\omega = (\cdot, \Omega)$ with $\Omega$ a cyclic and separating vector ($\mathcal{M}$ acts standardly).

Here a vector $\Omega$ is cyclic if $\mathcal{M}\Omega = \mathcal{H}$ and separating if $x \in M$, $x\Omega = 0$ implies $x = 0$. A vector is cyclic for $\mathcal{M}$ iff it is separating for $\mathcal{M}'$.

The anti-linear operator $x\Omega \mapsto x^*\Omega$, $x \in \mathcal{M}$, is closable and its closure is denoted by $S$. The polar decomposition $S = J\Delta^{1/2}$ gives a antiunitary involution $J$, the modular involution, and a positive non-singular linear operator $\Delta \equiv S^*S$, the modular operator.

We have

\begin{align*}
\Delta^u M \Delta^{-u} &= M \quad (1) \\
JMJ &= M' \quad (2)
\end{align*}

in other words the modular theory associates with $\Omega$ a canonical “evolution”, i.e. a one-parameter group of modular automorphisms of $\mathcal{M}$ $\sigma^\omega_t \equiv \text{Ad}\Delta^u$ and an anti-isomorphism $\text{Ad}J$ of $\mathcal{M}$ with $\mathcal{M}'$.

Let $N \subset M$ be an inclusion of von Neumann algebras. We always assume that $N$ and $M$ have the same identity. A conditional expectation $\varepsilon : M \to N$ is a positive, unital map from $M$ onto $N$ such that $\varepsilon(n_1xn_2) = n_1\varepsilon(x)n_2$, $x \in M$, $n_1, n_2 \in N$.

If $\omega$ is a faithful normal state of $M$, by Takesaki theorem there exists a normal conditional expectation $\varepsilon : M \to N$ preserving $\omega$ (i.e. $\omega \cdot \varepsilon = \omega$) if and only if $N$ is globally invariant under the modular group $\sigma^\omega$ of $M$.

If $\rho$ is an endomorphism of $M$ and $\varepsilon : M \to \rho(M)$ is a conditional expectation, the map $\varphi \equiv \rho^{-1} \cdot \varepsilon$ satisfies $\varphi \cdot \rho = \text{id}$ and is called a left inverse of $\rho$.

Refs: [40].

2.3 Jones index

Let $N \subset M$ an inclusion of factors. The index of $N$ in $M$ can be defined by different point of views: analytic, probabilistic or tensor categorical.

Analytic definition. The index was originally considered by Jones in the setting of finite factors. Assume $M$ to be finite and let $\omega$ be the faithful tracial state $\omega$ on $M$. As above we may assume that $\omega$ is the vector state given by the vector $\Omega$. With $e$ the projection onto $\overline{N\Omega}$, the von Neumann algebra generated by $M$ and $e$

$$M_1 = \{M, e\}'' = J_MN'J_M$$

is a semifinite factor. $N \subset M$ has finite index iff $M_1$ is finite and the index is then defined by $\lambda = \omega(e)^{-1}$ with $\omega$ also denoting the tracial state of $M_1$. Jones theorem shows the possible values for the index:

$$\lambda \in \left\{4\cos^2 \frac{\pi}{n}, n \geq 3 \right\} \cup [4, \infty].$$
If $N \subset M$ is an inclusion of finite factor, there exists a unique trace-preserving conditional expectation $\varepsilon : M \to N$ ($\sigma^\omega$ is trivial in this case).

A definition for the index $[M : N]_\varepsilon$ of an arbitrary inclusion of factors $N \subset M$ with a faithful normal conditional expectation $\varepsilon : M \to N$ was given by Kosaki using Connes-Haagerup dual weights. It depends on the choice of $\varepsilon$. Given $\varepsilon$, choose a normal faithful state $\omega$ of $M$ with $\omega \cdot \varepsilon = \omega$ and $\Omega$ a cyclic vector implementing $\omega$. If $[M : N]_\varepsilon < \infty$, it is possible to define a canonical expectation $\varepsilon' : M_1 \to M$ and then $[M : N]_\varepsilon = \varepsilon'(e)^{-1}$, with $e$ the projection onto $N\Omega$. Jones restriction on the index values holds for $[M : N]_\varepsilon$ as well.

The good properties are shared by the minimal index

$$[M : N] = \inf_\varepsilon [M : N]_\varepsilon = [M : N]_{\varepsilon_0},$$

where $\varepsilon_0$ is the unique minimal conditional expectation.

The analytic point of view will not play an explicit role in this paper.

**Probabilistic definition.** Pimsner and Popa inequality, and its extension to the infinite factor case, shows that $\lambda \equiv [M : N]_\varepsilon^{-1}$ is the best constant such that

$$\varepsilon(x) \geq \lambda x, \quad x \in M^+,$$

where $\varepsilon : M \to N$ a normal conditional expectation (if $M$ is finite-dimensional $\lambda$ is not an optimal bound).

This gives a general way to define the index and a powerful tool to check whether a given inclusion has finite index.

**Tensor categorical definition.** We shall get to this point in a moment.

REFS: [16, 22, 25, 28, 31, 35, 40] and references therein.

### 2.4 Joint modular structure. Sectors

Let $N \subset M$ be an inclusion of infinite factors. We may assume that $N'$ and $M'$ are infinite so $M$ and $N$ have a cyclic and separating vector. With $J_N$ and $J_M$ modular conjugations of $N$ and $M$, the unitary $\Gamma = J_NJ_M$ implements a canonical endomorphism of $M$ into $N$

$$\gamma(x) = \Gamma x \Gamma^*, \quad x \in M.$$

$\gamma$ depends on the choice of $J_N$ and $J_M$ only up to perturbations by an inner automorphism of $M$ associated with a unitary in $N$. The restriction $\gamma|N$ is called the dual canonical endomorphism (it is the canonical endomorphism associated with $\gamma(M) \subset N$). $\gamma$ is canonical as a sector of $M$ as we define now.

Given the infinite factor $M$, the sectors of $M$ are given by

$$\text{Sect}(M) = \text{End}(M)/\text{Inn}(M)$$

namely $\text{Sect}(M)$ is the quotient of the semigroup of the endomorphisms of $M$ modulo the equivalence relation: $\rho, \rho' \in \text{End}(M)$, $\rho \sim \rho'$ iff there is a unitary $u \in M$ such that $\rho'(x) = u\rho(x)u^*$ for all $x \in M$. 

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Sect($M$) is a $\ast$-semiring (there is an addition, a product and an involution) equivalent to the Connes correspondences (bimodules) on $M$ up to unitary equivalence. If $\rho$ is an element of End($M$) we shall denote by $[\rho]$ its class in Sect($M$). The operations are:

**Addition** (direct sum): Let $\rho_1, \rho_2, \ldots, \rho_n \in \text{End}(M)$. Choose a non-degenerate $n$-dimensional Hilbert $H$ space of isometries in $M$ and a basis $v_1, \ldots, v_n$ for $H$. Then

$$\rho(x) \equiv \sum_{i=1}^{n} v_i \rho_i(x) v_i^*, \quad x \in M,$$

is an endomorphism of $M$. The definition of the direct sum endomorphism $\rho$ does not depend on the choice of $H$ or on the basis, up to inner automorphism of $M$, namely $\rho$ is a well-defined sector of $M$.

**Composition** (monoidal product). The usual composition of maps

$$\rho_1 \cdot \rho_2(x) = \rho_1(\rho_2(x)), \quad x \in M,$$

defined on End($M$) passes to the quotient Sect($M$).

**Conjugation.** With $\rho \in \text{End}(M)$, choose a canonical endomorphism $\gamma_{\rho} : M \to \rho(M)$. Then

$$\hat{\rho} = \rho^{-1} \cdot \gamma_{\rho}$$

well-defines a conjugation in Sect($M$). By definition we thus have

$$\gamma_{\rho} = \rho \cdot \hat{\rho} \quad (3)$$

Refs: [14, 22, 26] and references therein.

### 2.5 The tensor category End($M$)

With $M$ an infinite factor, then End($M$) is a strict tensor $C^*$-category, as is already implicit in the previous section.

More precisely define a category End($M$) whose objects are the elements of End($M$) and the arrows Hom($\rho, \rho'$) between the objects $\rho, \rho'$ are

$$\text{Hom}(\rho, \rho') \equiv \{a \in M : a \rho(x) = \rho'(x) a \ \forall x \in M\}.$$

The composition of intertwiners (arrows) is the operator product. Clearly Hom($\rho, \rho'$) is a Banach space and there is a $\ast$-operation $a \in \text{Hom}(\rho, \rho') \mapsto a^* \in \text{Hom}(\rho', \rho)$ with the usual properties and the $C^*$-norm equality $||a^*a|| = ||a||^2$. Thus End($M$) is a $C^*$-category.

Moreover there is a tensor (or monoidal) product in End($M$). The tensor product $\rho \otimes \rho'$ is simply the composition $\rho \rho'$. For simplicity the symbol $\otimes$ is thus omitted in this case: $\rho \otimes \rho' = \rho \rho'$. If $\sigma, \sigma' \in \text{End}(M)$, and $t \in \text{Hom}(\rho, \rho')$, $s \in \text{Hom}(\sigma, \sigma')$, the tensor product arrow $t \otimes s$ is the element of Hom($\rho \otimes \sigma, \rho' \otimes \sigma'$) given by

$$t \otimes s \equiv t \rho(s) = \rho'(s)t.$$
As usual, there is a natural compatibility between tensor product and composition, thus $\text{End}(M)$ is a $C^*$-tensor category. Moreover there is an identity object $\iota$ for the tensor product (the identity automorphism).

So far we have not made much use that $M$ is an infinite factor. This enters crucially for the conjugation in $\text{End}(M)$.

If $\rho$ is irreducible (i.e. $\rho(M)' \cap M = \mathbb{C}$) and has finite index, then $\bar{\rho}$ is the unique sector such that $\rho \bar{\rho}$ contains the identity sector. More generally the objects $\rho, \bar{\rho} \in \text{End}(M)$ are conjugate according to the analytic definition and have finite index if and only if there exist isometries $v \in \text{Hom}(\iota, \rho \bar{\rho})$ and $\bar{v} \in \text{Hom}(\iota, \bar{\rho} \rho)$ such that

$$v^* \otimes 1_{\rho} \cdot 1_{\rho} \otimes v \equiv v^* \rho(v) = \frac{1}{d}, \quad v^* \otimes 1_{\rho} \otimes \bar{v} \equiv v^* \rho(\bar{v}) = \frac{1}{d},$$

for some $d > 0$.

The minimal possible value of $d$ in the above formulas is the dimension $d(\rho)$ of $\rho$; it is related to the minimal index by

$$[M : \rho(M)] = d(\rho)^2$$

(tensor categorical definition of the index) and satisfies the dimension properties

$$d(\rho_1 \oplus \rho_2) = d(\rho_1) + d(\rho_2)$$
$$d(\rho_1 \rho_2) = d(\rho_1)d(\rho_2)$$
$$d(\bar{\rho}) = d(\rho).$$

It follows that the subcategory of $\text{End}(M)$ having finite-index objects is a $C^*$-tensor category with conjugates and direct sums.

Formula (3) shows that given $\gamma \in \text{End}(M)$ the problem of deciding whether it is a canonical endomorphism with respect to some subfactor is essentially the problem of finding a “square root” $\rho$. $\gamma$ is canonical and has finite index iff there exist isometries $t \in \text{Hom}(\iota, \gamma)$, $s \in \text{Hom}(\gamma, \gamma^2)$ satisfying the algebraic relations

$$s^* s^* = s^* \gamma(s^*) \quad (4)$$
$$s^* \gamma(t) \in \mathbb{C} \setminus \{0\}, \quad s^* t \in \mathbb{C} \setminus \{0\}. \quad (5)$$

It is immediate to generalize the notion of $\text{Sect}(M)$ to $\text{Sect}(M,N)$, for a pair of factors $M,N$. They are the homomorphisms of $M$ into $N$ up to unitary equivalence given by a unitary in $N$. If $N \subset M$ is an inclusion of infinite factors, the canonical endomorphism $\gamma : M \to N$ is a well defined element of $\text{Sect}(M, N)$; if $[M : N] < \infty$, the above formula show that $\gamma$ is the conjugate sector of the inclusion homomorphism $\iota_N : N \to M$:

$$\gamma = \iota_N \iota_N, \quad \gamma | N = \iota_N \iota_N.$$

We use $\langle \lambda, \mu \rangle$ to denote the dimension of $\text{Hom}(\lambda, \mu)$; it can be $\infty$, but it is finite if $\lambda, \mu$ have finite index. $\langle \lambda, \mu \rangle$ depends only on $[\lambda]$ and $[\mu]$. Moreover we have if $\nu$ have finite dimension, then $\langle \nu \lambda, \mu \rangle = \langle \lambda, \nu \mu \rangle$, $\langle \lambda \nu, \mu \rangle = \langle \lambda, \mu \nu \rangle$ which follows from Frobenius duality. $\mu$ is a subsector of $\lambda$ if there is an isometry $v \in M$ such that
\(\mu(x) = v^* \lambda(x)v, \forall x \in M.\) We will also use the following notation: if \(\mu\) is a subsector of \(\lambda\), we will write as \(\mu \prec \lambda\) or \(\lambda \succ \mu\). A sector is said to be irreducible if it has only one subsector.

Refs: [7, 29, 32] and references therein.

3 Conformal nets on \(S^1\)

By an interval of the circle we mean an open connected non-empty subset \(I\) of \(S^1\) such that the interior of its complement \(I'\) is not empty. We denote by \(\mathcal{I}\) the family of all intervals of \(S^1\).

A net \(\mathcal{A}\) of von Neumann algebras on \(S^1\) is a map

\[ I \in \mathcal{I} \rightarrow \mathcal{A}(I) \subset B(\mathcal{H}) \]

from \(\mathcal{I}\) to von Neumann algebras on a fixed Hilbert space \(\mathcal{H}\) that satisfies:

A. Isotony. If \(I_1 \subset I_2\) belong to \(\mathcal{I}\), then

\[ \mathcal{A}(I_1) \subset \mathcal{A}(I_2). \]

If \(E \subset S^1\) is any region, we shall put \(\mathcal{A}(E) \equiv \bigvee_{E \supset I \in \mathcal{I}} \mathcal{A}(I)\) with \(\mathcal{A}(E) = \mathbb{C}\) if \(E\) has empty interior (the symbol \(\vee\) denotes the von Neumann algebra generated).

The net \(\mathcal{A}\) is called local if it satisfies:

B. Locality. If \(I_1, I_2 \in \mathcal{I}\) and \(I_1 \cap I_2 = \emptyset\) then

\[ [\mathcal{A}(I_1), \mathcal{A}(I_2)] = \{0\}, \]

where brackets denote the commutator.

The net \(\mathcal{A}\) is called Möbius covariant if in addition satisfies the following properties C,D,E,F:

C. Möbius covariance. There exists a strongly continuous unitary representation \(U\) of the Möbius group \(\text{Möb}\) (isomorphic to \(\text{PSU}(1,1)\)) on \(\mathcal{H}\) such that

\[ U(g)\mathcal{A}(I)U(g)^* = \mathcal{A}(gI), \quad g \in \text{Möb}, \quad I \in \mathcal{I}. \]

Note that this implies \(\mathcal{A}(\bar{I}) = \mathcal{A}(I), \quad I \in \mathcal{I}\) (consider a sequence of elements \(g_n \in \text{Möb}\) converging to the identity such that \(g_n\bar{I} \nearrow I\)).

D. Positivity of the energy. The generator of the one-parameter rotation subgroup of \(U\) (conformal Hamiltonian) is positive.

E. Existence of the vacuum. There exists a unit \(U\)-invariant vector \(\Omega \in \mathcal{H}\) (vacuum vector), and \(\Omega\) is cyclic for the von Neumann algebra \(\bigvee_{I \in \mathcal{I}} \mathcal{A}(I)\).
By the Reeh-Schlieder theorem $\Omega$ is cyclic and separating for every fixed $\mathcal{A}(I)$. The modular objects associated with $(\mathcal{A}(I), \Omega)$ have a geometric meaning

$$\Delta^\text{i}_I = U(\Lambda_I(2\pi t)), \quad J_I = U(r_I).$$

Here $\Lambda_I$ is a canonical one-parameter subgroup of $\text{Möb}$ and $U(r_I)$ is an antiunitary acting geometrically on $\mathcal{A}$ as a reflection $r_I$ on $S^1$.

This implies Haag duality:

$$\mathcal{A}(I)' = \mathcal{A}(I'), \quad I \in \mathcal{I},$$

where $I'$ is the interior of $S^1 \setminus I$.

**F. Irreducibility.** $\bigvee_{I \in \mathcal{I}} \mathcal{A}(I) = B(\mathcal{H})$. Indeed $\mathcal{A}$ is irreducible iff $\Omega$ is the unique $U$-invariant vector (up to scalar multiples). Also $\mathcal{A}$ is irreducible iff the local von Neumann algebras $\mathcal{A}(I)$ are factors. In this case they are $\text{III}_1$-factors in Connes classification of type III factors (unless $\mathcal{A}(I) = \mathbb{C}$ for all $I$).

By a **conformal net** (or diffeomorphism covariant net) $\mathcal{A}$ we shall mean a Möbius covariant net such that the following holds:

**G. Conformal covariance.** There exists a projective unitary representation $U$ of $\text{Diff}(S^1)$ on $\mathcal{H}$ extending the unitary representation of Möb such that for all $I \in \mathcal{I}$ we have

$$U(g)\mathcal{A}(I)U(g)^* = \mathcal{A}(gI), \quad g \in \text{Diff}(S^1),$$

$$U(g)xU(g)^* = x, \quad x \in \mathcal{A}(I), \quad g \in \text{Diff}(I'),$$

where $\text{Diff}(S^1)$ denotes the group of smooth, positively oriented diffeomorphism of $S^1$ and $\text{Diff}(I)$ the subgroup of diffeomorphisms $g$ such that $g(z) = z$ for all $z \in I'$.

Let $G$ be a simply connected compact Lie group. By Th. 3.2 of [8], the vacuum positive energy representation of the loop group $LG$ (cf. [36]) at level $k$ gives rise to an irreducible conformal net denoted by $\mathcal{A}_{G_k}$. By Th. 3.3 of [8], every irreducible positive energy representation of the loop group $LG$ at level $k$ gives rise to an irreducible covariant representation of $\mathcal{A}_{G_k}$.

### 3.1 Doplicher-Haag-Roberts superselection sectors in CQFT

The DHR theory was originally made on the 4-dimensional Minkowski spacetime, but can be generalized to our setting. There are however several important structure differences in the low dimensional case.

A (DHR) representation $\pi$ of $\mathcal{A}$ on a Hilbert space $\mathcal{H}$ is a map $I \in \mathcal{I} \mapsto \pi_I$ that associates to each $I$ a normal representation of $\mathcal{A}(I)$ on $B(\mathcal{H})$ such that

$$\pi_I | \mathcal{A}(I) = \pi_I, \quad I \subset \bar{I}, \quad I, \bar{I} \subset \mathcal{I}.$$
\( \pi \) is said to be Möbius (resp. diffeomorphism) covariant if there is a projective unitary representation \( U_\pi \) of \( \text{M"ob} \) (resp. \( \text{Diff}^{(\infty)}(S^1) \), the infinite cover of \( \text{Diff}(S^1) \) ) on \( \mathcal{H} \) such that
\[
\pi_g(U(g)xU(g)^*) = U_\pi(g)\pi_I(x)U_\pi(g)^*
\]
for all \( I \in \mathcal{I} \), \( x \in \mathcal{A}(I) \) and \( g \in \text{M"ob} \) (resp. \( g \in \text{Diff}^{(\infty)}(S^1) \)). Note that if \( \pi \) is irreducible and diffeomorphism covariant then \( U \) is indeed a projective unitary representation of \( \text{Diff}(S^1) \).

By definition the irreducible conformal net is in fact an irreducible representation of itself and we will call this representation the \textit{vacuum representation}.

Given an interval \( I \) and a representation \( \pi \) of \( \mathcal{A} \), there is an \textit{endomorphism of \( \mathcal{A} \) localized in \( I \)} equivalent to \( \pi \); namely \( \rho \) is a representation of \( \mathcal{A} \) on the vacuum Hilbert space \( \mathcal{H} \), unitarily equivalent to \( \pi \), such that \( \rho_I = \text{id} | \mathcal{A}(I') \).

Fix an interval \( I_0 \) and endomorphisms \( \rho, \rho' \) of \( \mathcal{A} \) localized in \( I_0 \). Then the \textit{composition} (tensor product) \( \rho \rho' \) is defined by
\[
(\rho \rho')_I = \rho_I \rho'_I
\]
with \( I \) an interval containing \( I \). One can indeed define \( (\rho \rho')_I \) for an arbitrary interval \( I \) of \( S^1 \) (by using covariance) and get a well defined endomorphism of \( \mathcal{A} \) localized in \( I_0 \). Indeed the endomorphisms of \( \mathcal{A} \) localized in a given interval form a tensor \( C^* \)-category. For our needs \( \rho, \rho' \) will be always localized in a common interval \( I \).

If \( \pi \) and \( \pi' \) are representations of \( \mathcal{A} \), fix an interval \( I_0 \) and choose endomorphisms \( \rho, \rho' \) localized in \( I_0 \) with \( \rho \) equivalent to \( \pi \) and \( \rho' \) equivalent to \( \pi' \). Then \( \pi \cdot \pi' \) is defined (up to unitary equivalence) to be \( \rho \rho' \). The class of a DHR representation modulo unitary equivalence is a \textit{superselection sectors} (or simply a sector).

Indeed the localized endomorphisms of \( \mathcal{A} \) for a tensor \( C^* \)-category. For our needs, \( \rho, \rho' \) will be always localized in a common interval \( I \).

We now define the statistics. Given the endomorphism \( \rho \) of \( \mathcal{A} \) localized in \( I \in \mathcal{I} \), choose an equivalent endomorphism \( \rho_0 \) localized in an interval \( I_0 \in \mathcal{I} \) with \( I_0 \cap \bar{I} = \emptyset \) and let \( u \) be a local intertwiner in \( \text{Hom}(\rho, \rho_0) \) as above, namely \( u \in \text{Hom}(\rho_I, \rho_0, \bar{I}) \) with \( I_0 \) following clockwise \( I \) inside \( \bar{I} \) which is an interval containing both \( I \) and \( I_0 \).

The \textit{statistics operator} \( \varepsilon := u^* \rho(u) = u^* \rho_I(u) \) belongs to \( \text{Hom}(\rho^2_I, \rho^2_I) \). An elementary computation shows that it gives rise to a presentation of the Artin braid group
\[
\varepsilon_i \varepsilon_{i+1} \varepsilon_i = \varepsilon_{i+1} \varepsilon_i \varepsilon_{i+1}, \quad \varepsilon_i \varepsilon_{i'} = \varepsilon_{i'} \varepsilon_i \quad \text{if } |i - i'| \geq 2,
\]
where \( \varepsilon_i = \rho^{-1}(\varepsilon) \). The (unitary equivalence class of the) representation of the Artin braid group thus obtained is the \textit{statistics} of the superselection sector \( \rho \).

It turns out the endomorphisms localized in a given interval form a \textit{braided \( C^* \)-tensor category} with unitary braiding.

The \textit{statistics parameter} \( \lambda_\rho \) can be defined in general. In particular, assume \( \rho \) to be localized in \( I \) and \( \rho_I \in \text{End}(\mathcal{A}(I)) \) to be irreducible with a conditional expectation \( E : \mathcal{A}(I) \to \rho_I(\mathcal{A}(I)) \), then
\[
\lambda_\rho := E(\varepsilon)
\]
depends only on the superselection sector of \( \rho \).

The statistical dimension \( d_{\text{DHR}}(\rho) \) and the univalence \( \omega_\rho \) are then defined by

\[
d_{\text{DHR}}(\rho) = |\lambda_\rho|^{-1}, \quad \omega_\rho = \frac{\lambda_\rho}{|\lambda_\rho|}.
\]

Refs: [7, 9, 25, 26, 31].

3.2 Index-statistics and spin-statistics relations

Let \( \rho \) be an endomorphism localized in the interval \( I \). A natural connection between the Jones and DHR theories is realized by the index-statistics theorem

\[
\text{Ind}(\rho) = d_{\text{DHR}}(\rho)^2.
\]

Here \( \text{Ind}(\rho) = \text{Ind}(\rho_I) \); namely \( d((\rho_I)) = d_{\text{DHR}}(\rho) \). We will thus omit the suffix DHR in the dimension. Since by duality \( \rho(\mathcal{A}(I)) \subset \mathcal{A}(I) \) coincides with \( \rho(\mathcal{A}(I)) \subset \rho(\mathcal{A}(I'))' \) one may rewrite the above index formula directly in terms of the representation \( \rho \).

The map \( \rho \to \rho_I \) is a faithful functor of \( C^* \)-tensor categories of endomorphism of |\( A \) localized in \( I \) into \( \text{End}(M) \) with \( M \equiv A(I) \). Passing to quotient one obtains a natural embedding

\[
\text{Superselection sectors} \to \text{Sect}(M).
\]

Restricting to finite-dimensional endomorphisms, the above functor is full, namely, given endomorphisms \( \rho, \rho' \) localized in \( I \), if \( a \in \text{Hom}(\rho_I, \rho'_I) \) then \( a \) intertwines the representations \( \rho \) and \( \rho' \) (this is obviously true also in the infinite-dimensional case if there holds the strong additivity property below, but otherwise a non-trivial result).

The conformal spin-statistics theorem shows that

\[
\omega_\rho = e^{i2\pi L_0(\rho)},
\]

where \( L_0(\rho) \) is the conformal Hamiltonian (the generator of the rotation subgroup) in the representation \( \rho \). The right hand side in the above equality is called the univalence of \( \rho \).

Refs: [11, 25].

3.3 Genus 0 \( S, T \)-matrices

Next we will recall some of the results of [37] and introduce notations.

Let \( \{[\lambda], \lambda \in \mathcal{L} \} \) be a finite set of all equivalence classes of irreducible, covariant, finite-index representations of an irreducible local conformal net \( \mathcal{A} \). We will denote the conjugate of \([\lambda]\) by \([\bar{\lambda}]\) and identity sector (corresponding to the vacuum representation) by \([1]\) if no confusion arises, and let \( N_{\mu \nu}^\nu = \langle [\lambda][\mu], [\nu] \rangle \). Here \( \langle \mu, \nu \rangle \) denotes the dimension of the space of intertwiners from \( \mu \) to \( \nu \) (denoted by \( \text{Hom}(\mu, \nu) \)). We will denote by \( \{T_\nu\} \) a basis of isometries in \( \text{Hom}(\nu, \lambda \mu) \). The univalence of \( \lambda \) and
the statistical dimension of (cf. §2 of [10]) will be denoted by $\omega_\lambda$ and $d(\lambda)$ (or $d_\lambda$) respectively.

Let $\varphi_\lambda$ be the unique minimal left inverse of $\lambda$, define:

$$Y_{\lambda\mu} := d(\lambda)d(\mu)\varphi_\mu(\epsilon(\mu, \lambda)^*\epsilon(\lambda, \mu)^*),$$

(6)

where $\epsilon(\mu, \lambda)$ is the unitary braiding operator (cf. [10]).

We list two properties of $Y_{\lambda\mu}$ (cf. (5.13), (5.14) of [37]) which will be used in the following:

**Lemma 3.1.**

$$Y_{\lambda\mu} = Y_{\mu\lambda} = Y_{\lambda\mu}^* = Y_{\lambda\mu}{\bar{\mu}}.$$

$$Y_{\lambda\mu} = \sum_k N_{\lambda\mu}^\nu \frac{\omega^\lambda\omega_\mu}{\omega_\nu} d(\nu).$$

We note that one may take the second equation in the above lemma as the definition of $Y_{\lambda\mu}$.

Define $a := \sum_i d_\rho_i^2 \omega_\rho^{-1}$. If the matrix $(Y_{\mu\nu})$ is invertible, by Proposition on P.351 of [37] $a$ satisfies $|a|^2 = \sum_\lambda d(\lambda)^2$.

**Definition 3.2.** Let $a = |a| \exp(-2\pi i c_0^8)$ where $c_0 \in \mathbb{R}$ and $c_0$ is well defined mod 8Z.

Define matrices

$$S := |a|^{-1} Y, T := C\text{Diag}(\omega_\lambda)$$

(7)

where

$$C := \exp(-2\pi i \frac{c_0}{24}).$$

Then these matrices satisfy (cf. [37]):

**Lemma 3.3.**

$$SS^\dagger = TT^\dagger = \text{id},$$

$$STS = T^{-1}ST^{-1},$$

$$S^2 = \hat{C},$$

$$T\hat{C} = \hat{C}T = T,$$

where $\hat{C}_{\lambda\mu} = \delta_{\lambda\mu}$ is the conjugation matrix.

Moreover

$$N_{\lambda\mu}^\nu = \sum_\delta \frac{S_{\lambda\delta}S_{\mu\delta}S_{\nu\delta}^*}{S_{1\delta}}.$$

(8)

is known as Verlinde formula.

We will refer the $S, T$ matrices as defined above as **genus 0 modular matrices of $\mathcal{A}$** since they are constructed from the fusion rules, monodromies and minimal
indices which can be thought as genus 0 chiral data associated to a Conformal Field Theory.

We note that in all cases $c_0 - c \in 8\mathbb{Z}$, where $c$ is the central charge associated with the projective representations of $\text{Diff}(S^1)$ of the conformal net $\mathcal{A}$ (cf. [17] or [33]). We will prove in Lemma 9.7 that $c_0 - c \in 4\mathbb{Z}$ under general conditions.

The commutative algebra generated by $\lambda$’s with structure constants $N_{\lambda\mu}^\nu$ is called fusion algebra of $\mathcal{A}$. If $Y$ is invertible, it follows from Lemma 3.3, (8) that any nontrivial irreducible representation of the fusion algebra is of the form $\lambda \rightarrow \frac{S_{\lambda\nu}}{S_{\lambda\mu}}$ for some $\mu$.

3.4 The orbifolds

Let $\mathcal{A}$ be an irreducible conformal net on a Hilbert space $\mathcal{H}$ and let $\Gamma$ be a finite group. Let $V : \Gamma \rightarrow U(\mathcal{H})$ be a unitary representation of $\Gamma$ on $\mathcal{H}$. If $V : \Gamma \rightarrow U(\mathcal{H})$ is not faithful, we set $\Gamma' := \Gamma/\ker V$.

**Definition 3.4.** We say that $\Gamma$ acts properly on $\mathcal{A}$ if the following conditions are satisfied:

1. For each fixed interval $I$ and each $g \in \Gamma$, $\alpha_g(a) := V(g)aV(g^*) \in \mathcal{A}(I)$, $\forall a \in \mathcal{A}(I)$;
2. For each $g \in \Gamma$, $V(g)\Omega = \Omega$, $\forall g \in \Gamma$.

We note that if $\Gamma$ acts properly, then $V(g)$, $g \in \Gamma$ commutes with the unitary representation $U$ of $\text{Möb}$.

Define $\mathcal{B}(I) := \{a \in \mathcal{A}(I) | \alpha_g(a) = a, \forall g \in \Gamma\}$ and $\mathcal{A}^{\Gamma}(I) := \mathcal{B}(I)P_0$ on $\mathcal{H}_0$ where $\mathcal{H}_0 := \{x \in \mathcal{H} | V(g)x = x, \forall g \in \Gamma\}$ and $P_0$ is the projection from $\mathcal{H}$ to $\mathcal{H}_0$. Then $U$ restricts to an unitary representation (still denoted by $U$) of $\text{Möb}$ on $\mathcal{H}_0$. Then:

**Proposition 3.5.** The map $I \in \mathcal{I} \rightarrow \mathcal{A}^{\Gamma}(I)$ on $\mathcal{H}_0$ together with the unitary representation (still denoted by $U$) of $\text{Möb}$ on $\mathcal{H}_0$ is an irreducible Möbius covariant net.

The irreducible Möbius covariant net in Prop. 3.5 will be denoted by $\mathcal{A}^{\Gamma}$ and will be called the orbifold of $\mathcal{A}$ with respect to $\Gamma$. We note that by definition $\mathcal{A}^{\Gamma} = \mathcal{A}^{\Gamma'}$.

3.5 Complete rationality

We first recall some definitions from [21]. Recall that $\mathcal{I}$ denotes the set of intervals of $S^1$. Let $I_1, I_2 \in \mathcal{I}$. We say that $I_1, I_2$ are disjoint if $\overline{I_1 \cap I_2} = \emptyset$, where $\overline{I}$ is the closure of $I$ in $S^1$. When $I_1, I_2$ are disjoint, $I_1 \cup I_2$ is called a 1-disconnected interval in [46]. Denote by $\mathcal{I}_2$ the set of unions of disjoint 2 elements in $\mathcal{I}$. Let $\mathcal{A}$ be an irreducible Möbius covariant net as in §2.1. For $E = I_1 \cup I_2 \in \mathcal{I}_2$, let $I_3 \cup I_4$ be the interior of the complement of $I_1 \cup I_2$ in $S^1$ where $I_3, I_4$ are disjoint intervals. Let

$\mathcal{A}(E) := A(I_1) \lor A(I_2), \quad \hat{A}(E) := (A(I_3) \lor A(I_4))'.$
Note that $\mathcal{A}(E) \subset \hat{\mathcal{A}}(E)$. Recall that a net $\mathcal{A}$ is split if $\mathcal{A}(I_1) \vee \mathcal{A}(I_2)$ is naturally isomorphic to the tensor product of von Neumann algebras $\mathcal{A}(I_1) \otimes \mathcal{A}(I_2)$ for any disjoint intervals $I_1, I_2 \in \mathcal{I}$. $\mathcal{A}$ is strongly additive if $\mathcal{A}(I_1) \vee \mathcal{A}(I_2) = \mathcal{A}(I)$ where $I_1 \cup I_2$ is obtained by removing an interior point from $I$.

**Definition 3.6.** [21] $\mathcal{A}$ is said to be completely rational if $\mathcal{A}$ is split, strongly additive, and the index $[\hat{\mathcal{A}}(E) : \mathcal{A}(E)]$ is finite for some $E \in \mathcal{I}_2$. The value of the index $[\hat{\mathcal{A}}(E) : \mathcal{A}(E)]$ (it is independent of $E$ by Prop. 5 of [21]) is denoted by $\mu_\mathcal{A}$ and is called the $\mu$-index of $\mathcal{A}$. If the index $[\hat{\mathcal{A}}(E) : \mathcal{A}(E)]$ is infinity for some $E \in \mathcal{I}_2$, we define the $\mu$-index of $\mathcal{A}$ to be infinity.

A formula for the $\mu$-index of a subnet is proved in [21]. With the result on strong additivity for $\mathcal{A}^\Gamma$ in [44], we have the complete rationality in following theorem.

**Theorem 3.7.** Let $\mathcal{A}$ be an irreducible Möbius covariant net and let $\Gamma$ be a finite group acting properly on $\mathcal{A}$. Suppose that $\mathcal{A}$ is completely rational. Then:

1. $\mathcal{A}^\Gamma$ is completely rational or $\mu$-rational and $\mu_{\mathcal{A}^\Gamma} = |\Gamma'|^2 \mu_{\mathcal{A}}$;
2. There are only a finite number of irreducible covariant representations of $\mathcal{A}^\Gamma$ (up to unitary equivalence), and they give rise to a unitary modular category as defined in II.5 of [39] by the construction as given in §1.7 of [48].

Suppose that $\mathcal{A}$ and $\Gamma$ satisfy the assumptions of Th. 3.7. Then $\mathcal{A}^\Gamma$ has only finite number of irreducible representations $\lambda$ and

$$\sum_{\lambda} d(\lambda)^2 = \mu_{\mathcal{A}^\Gamma} = |\Gamma'|^2 \mu_{\mathcal{A}}.$$  

The set of such $\lambda$’s is closed under conjugation and compositions, and by Cor. 32 of [21], the $Y$-matrix in (6) for $\mathcal{A}^\Gamma$ is non-degenerate, and we will denote the corresponding genus 0 modular matrices by $\hat{S}, \hat{T}$. We note that $d(\lambda)$ is conjectured to be related to the asymptotic dimension of Kac-Wakimoto in [19], and one can find a precise statement of the conjecture and its consequences in [27] and in §2.3 of [50]. Denote by $\hat{\lambda}$ (resp. $\mu$) the irreducible covariant representations of $\mathcal{A}^\Gamma$ (resp. $\mathcal{A}$) with finite index. Denote by $b_{\mu \lambda} \in \mathbb{N} \cup \{0\}$ the multiplicity of representation $\lambda$ which appears in the restriction of representation $\mu$ when restricting from $\mathcal{A}$ to $\mathcal{A}^\Gamma$. The $b_{\mu \lambda}$ are also known as the branching rules. An irreducible covariant representation $\lambda$ of $\mathcal{A}^\Gamma$ is called an untwisted representation if $b_{\mu \lambda} \neq 0$ for some representation $\mu$ of $\mathcal{A}$. These are representations of $\mathcal{A}^\Gamma$ which appear as subrepresentations in the the restriction of some representation of $\mathcal{A}$ to $\mathcal{A}^\Gamma$. A representation is called twisted if it is not untwisted. Note that $\sum_{\lambda} d(\lambda)b_{\mu \lambda} = d(\mu)|\Gamma'|$, and $b_{1 \lambda} = d(\lambda)$. So we have

$$\sum_{\lambda \text{ untwisted}} d(\lambda)^2 \leq \sum_{\mu} \bigg( \sum_{\lambda} d(\lambda)b_{\mu \lambda} \bigg)^2 = |\Gamma'| + \sum_{\mu \neq 1} d(\mu)^2 |\Gamma'|^2 < |\Gamma'|^2 + \sum_{\mu \neq 1} d(\mu)^2 |\Gamma'|^2 = \mu_{\mathcal{A}^\Gamma}$$
if $\Gamma'$ is not a trivial group, where in the last $=$ we have used Th. 3.7. It follows that the set of twisted representations of $\mathcal{A}^{\Gamma}$ is not empty. This fact has already been observed in a special case in [21] under the assumption that $\mathcal{A}^{\Gamma}$ is strongly additive. Note that this is very different from the case of cosets, cf. [47] Cor. 3.2 where it was shown that under certain conditions there are no twisted representations for the coset.

3.6 Restriction to the real line: Solitons

Denote by $\mathcal{I}_0$ the set of open, connected, non-empty, proper subsets of $\mathbb{R}$, thus $I \in \mathcal{I}_0$ iff $I$ is an open interval or half-line (by an interval of $\mathbb{R}$ we shall always mean a non-empty open bounded interval of $\mathbb{R}$).

Given a net $\mathcal{A}$ on $S^1$ we shall denote by $\mathcal{A}_0$ its restriction to $\mathbb{R} = S^1 \setminus \{-1\}$. Thus $\mathcal{A}_0$ is an isotone map on $\mathcal{I}_0$, that we call a net on $\mathbb{R}$. In this paper we denote by $J_0 := (0, \infty) \subset \mathbb{R}$.

A representation $\pi$ of $\mathcal{A}_0$ on a Hilbert space $\mathcal{H}$ is a map $I \in \mathcal{I}_0 \mapsto \pi_I$ that associates to each $I \in \mathcal{I}_0$ a normal representation of $\mathcal{A}(I)$ on $B(\mathcal{H})$ such that

$$\pi_I \upharpoonright \mathcal{A}(I) = \pi_I, \quad I \subset \tilde{I}, \quad I, \tilde{I} \in \mathcal{I}_0.$$  

A representation $\pi$ of $\mathcal{A}_0$ is also called a soliton. As $\mathcal{A}_0$ satisfies half-line duality, namely

$$\mathcal{A}_0(-\infty, a)' = \mathcal{A}_0(a, \infty), a \in \mathbb{R},$$

by the usual DHR argument [7] $\pi$ is unitarily equivalent to a representation $\rho$ which acts identically on $\mathcal{A}_0(-\infty, 0)$, thus $\rho$ restricts to an endomorphism of $\mathcal{A}(J_0) = \mathcal{A}_0(0, \infty)$. $\rho$ is said to be localized on $J_0$ and we also refer to $\rho$ as soliton endomorphism.

Clearly a representation $\pi$ of $\mathcal{A}$ restricts to a soliton $\pi_0$ of $\mathcal{A}_0$. But a representation $\pi_0$ of $\mathcal{A}_0$ does not necessarily extend to a representation of $\mathcal{A}$.

If $\mathcal{A}$ is strongly additive, and a representation $\pi_0$ of $\mathcal{A}_0$ extends to a DHR representation of $\mathcal{A}$, then it is easy to see that such an extension is unique, and in this case we will use the same notation $\pi_0$ to denote the corresponding DHR representation of $\mathcal{A}$.

3.7 A result on extensions of solitons

The following proposition will play an important role in proving Th.7.1.

**Proposition 3.8.** Let $H_1, H_2$ be two subgroups of a compact group $\Gamma$ which acts properly on $\mathcal{A}$, and let $\pi$ be a soliton of $\mathcal{A}_0$. Assume that $\mathcal{A}$ is strongly additive. Suppose that $\pi \upharpoonright \mathcal{A}^{H_i}, i = 1, 2$ are DHR representations. Then $\pi \upharpoonright (\mathcal{A}^{H_1} \vee \mathcal{A}^{H_2})$ is also a DHR representation, where $\mathcal{A}^{H_1} \vee \mathcal{A}^{H_2}$ is an intermediate net with $(\mathcal{A}^{H_1} \vee \mathcal{A}^{H_2})(I) = \mathcal{A}^{H_1}(I) \vee \mathcal{A}^{H_2}(I), \forall I$. 

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**Proof** Let $I$ be an arbitrary interval with $-1 \in I$. It is sufficient to show that $\pi$ has a normal extension to $A_{H_1}(I) \lor A_{H_2}(I)$. Since $\pi$ is a soliton, by choosing a unitary equivalence class of $\pi$ we may assume that $\pi(x) = x, \forall x \in A(I')$. Let $J \supset I$ be an interval sharing a boundary point with $I$ and let $I_0 = J \cap I$. Since $\pi \upharpoonright A_{H_i}$ is a DHR representation, it is localizable on $I_0$. Denote the corresponding DHR representation localized on $I_0$ by $\pi_{a,I_0}$, then we can find unitary $u_i$ such that $u_i\pi_{a,I_0}u_i^* = \pi$ on $A_{H_i}$. It follows that $u_i \in A_{H_i}(J)$ since $\pi$ is localized on $I$, and we have $\pi(x) = u_i x u_i^*, \forall x \in A_{H_i}(I)$. Note that $A_{H_i}(I) \lor A_{H_2}(I) \supset A_{\Gamma}(I)$, hence $u_i^2 u_1 \in A_{\Gamma}(I) \cap A(J)$. Since $A_{\Gamma} \subset A$ is a strongly additive pair (cf. [49]), it follows that $A_{\Gamma}(I) \cap A_{\Gamma}(J) = A(I_0)$, and $u_1 x u_1^* = u_2 x u_2^*, \forall x \in A(I)$. Hence $\text{Ad}_{u_1}$ defines a normal extension of $\pi$ from $A_{H_1}(I)$ to $A_{H_1}(I) \lor A_{H_2}(I)$. Such an extension is also unique by definition. \qed

## 4 Induction and restriction for general orbifolds

Let $\mathcal{A}$ be a Möbius covariant net and $\mathcal{B}$ a sub-net. Given a bounded interval $I_0 \in \mathcal{I}_0$ we fix canonical endomorphism $\gamma_{I_0}$ associated with $A(I_0) \subset A(I)$. Then we can choose for each $I \subset \mathcal{I}_0$ with $I \supset I_0$ a canonical endomorphism $\gamma_I$ of $A(I)$ into $B(I)$ in such a way that $\gamma_I \upharpoonright A(I_0) = \gamma_{I_0}$ and $\lambda_i$ is the identity on $B(I_i)$ if $I_i \subset \mathcal{I}_0$ is disjoint from $I_0$, where $\lambda_i \equiv \gamma_I \upharpoonright B(I)$. We then have an endomorphism $\gamma$ of the $C^*$-algebra $\mathfrak{A} \equiv \bigcup_{I} A(I)$ ($I$ bounded interval of $\mathbb{R}$).

Given a DHR endomorphism $\rho$ of $\mathcal{B}$ localized in $I_0$, the $\alpha$-induction $\alpha_\rho$ of $\rho$ is the endomorphism of $\mathfrak{A}$ given by

$$\alpha_\rho \equiv \gamma^{-1} \cdot \text{Ad}(\rho, \lambda) \cdot \rho \cdot \gamma,$$

where $\varepsilon$ denotes the right braiding unitary symmetry (there is another choice for $\alpha$ associated with the left braiding). $\alpha_\rho$ is localized in a right half-line containing $I_0$, namely $\alpha_\rho$ is the identity on $A(I)$ if $I$ is a bounded interval contained in the left complement of $I_0$ in $\mathbb{R}$. Up to unitarily equivalence, $\alpha_\rho$ is localizable in any right half-line thus $\alpha_\rho$ is normal on left half-lines, that is to say, for every $a \in \mathbb{R}$, $\alpha_\rho$ is normal on the $C^*$-algebra $\mathfrak{A}(-\infty, a) \equiv \bigcup_{I \subset (-\infty, a)} A(I)$ ($I$ bounded interval of $\mathbb{R}$), namely $\alpha_\rho \upharpoonright \mathfrak{A}(-\infty, a)$ extends to a normal morphism of $A(-\infty, a)$. We have the following Prop. 3.1 of [33]:

**Proposition 4.1.** $\alpha_\rho$ is a soliton endomorphism of $A_0$.

### 4.1 Solitons as endomorphisms

Let $\mathcal{A}$ be a conformal net and $\Gamma$ a finite group acting properly on $\mathcal{A}$ (cf. (3.4)). We will assume that $\mathcal{A}$ is strongly additive. Let $\pi$ be an irreducible soliton of $A_0$ localized on $J_0 = (0, \infty)$. Note that the restriction of $\pi$ to $A(J_0)$ is an endomorphism and we denote this restriction by $\pi$ when no confusion arises. Let $\pi_{A'}$ be a soliton of $A'_0$ localized on $J_0$ and unitarily equivalent to $\pi \upharpoonright A'$. Let $\rho_1$ be an endomorphism of $A(J_0)$
such that $\rho_1(\mathcal{A}(J_0)) = \mathcal{A}^F(J_0)$ and $\rho_1\bar{\rho}_1 = \gamma$, where $\gamma$ is the canonical endomorphism from $\mathcal{A}(J_0)$ to $\mathcal{A}^F(J_0)$. Note that $[\gamma] = \bigoplus_{g \in \Gamma'}[g']$, where for simplicity we have used $[g]$ to denote the sector of $\mathcal{A}(J_0)$ induced by the automorphism $\beta_g$. By [31] as sectors of $\mathcal{A}^F(J_0)$ we have $[\pi_{A^F}] = [\gamma \pi \mid \mathcal{A}^F(J_0)]$.

**Definition 4.2.** Define $\Gamma_{\pi} := \{ h \in \Gamma | h\pi h^{-1} = [\pi] \}$. Note that $\ker V$ (cf. definition before (3.4)) is a normal subgroup of $\Gamma_{\pi}$ and let $\Gamma'_{\pi} := \Gamma_{\pi}/\ker V$.

By Frobenius duality we have

$$\langle \pi_{A^F}, \pi_{A^F} \rangle = \langle \lambda, \gamma \lambda \gamma \rangle.$$ 

**Lemma 4.3.** (1) If $g \neq h$, then $\langle \pi, g\pi h^{-1} \rangle = 0$;

(2) $\langle \pi, \gamma \pi \gamma \rangle = |\Gamma'_{\pi}| = \langle \gamma \pi \mid \mathcal{A}^F, \gamma \pi \mid \mathcal{A}^F \rangle$ where $\Gamma'_{\pi} = \Gamma_{\pi}/\ker V$;

(3) $\langle \gamma \pi \mid \mathcal{A}^F, \gamma \pi \mid \mathcal{A}^F \rangle = \langle \gamma_1 \pi \mid \mathcal{A}^{F*}, \gamma_1 \pi \mid \mathcal{A}^{F*} \rangle$

where $\gamma_1$ is the canonical endomorphism from $\mathcal{A}(J_0)$ to $\mathcal{A}^{F*}(J_0)$;

(4) Every irreducible summand of $\pi \mid \mathcal{A}^{F*}_0$ (as soliton of $\mathcal{A}^{F*}_0$) remains irreducible when restricting to $\mathcal{A}^F_0$.

**Proof** Note that $g\pi h^{-1} = g\pi g^{-1}gh^{-1}$, and $g\pi g^{-1}$ is a soliton equivalent to $\pi g^{-1}$ but localized on $J_0$. By Lemma 8.5 of [33] we have proved (1). (2),(3) follows from (1) and the definition of $\Gamma_{\pi}$. (4) follows from (3). ■

**Proposition 4.4.** Let $\pi_1, \pi_2$ be two irreducible solitons of $\mathcal{A}_0$. If there is $g \in \Gamma'$ such that $[\pi_1] = [g\pi_2g^{-1}]$, then $[\gamma \pi_1 \mid \mathcal{A}^F] = [\gamma \pi_2 \mid \mathcal{A}^F]$. Otherwise $\langle \gamma \pi_1 \mid \mathcal{A}^F, \gamma \pi_2 \mid \mathcal{A}^F \rangle = 0$.

**Proof** By Frobenius duality and Lemma 8.5 of [33] we have

$$\langle \gamma \pi_1 \mid \mathcal{A}^F, \gamma \pi_2 \mid \mathcal{A}^F \rangle = \sum_{g \in \Gamma'} \langle \pi_1, g\pi_2g^{-1} \rangle$$

Hence $\langle \gamma \pi_1 \mid \mathcal{A}^F, \gamma \pi_2 \mid \mathcal{A}^F \rangle = 0$ if there is no $g \in \Gamma'$ such that $[\pi_1] = [g\pi_2g^{-1}]$. If there is $g \in \Gamma'$ such that $[\pi_1] = [g\pi_2g^{-1}]$, then

$$\langle \gamma \pi_1 \mid \mathcal{A}^F, \gamma \pi_2 \mid \mathcal{A}^F \rangle = \sum_{h \in \Gamma'_{\pi_1}} \langle \pi_1, hg\pi_2g^{-1}h^{-1} \rangle = |\Gamma'_{\pi_1}|$$

By exchanging $\pi_1$ and $\pi_2$ we get

$$\langle \gamma \pi_1 \mid \mathcal{A}^F, \gamma \pi_2 \mid \mathcal{A}^F \rangle = \langle \gamma \pi_1 \mid \mathcal{A}^F, \gamma \pi_1 \mid \mathcal{A}^F \rangle = \langle \gamma \pi_2 \mid \mathcal{A}^F, \gamma \pi_2 \mid \mathcal{A}^F \rangle$$

It follows that $[\gamma \pi_1 \mid \mathcal{A}^F] = [\gamma \pi_2 \mid \mathcal{A}^F]$. ■
Theorem 4.5. Assume that $\pi$ is irreducible with finite index and $[\beta] = [\gamma \pi \mid A^\Gamma] = \bigoplus_j m_j [\beta_j]$. Then $[\alpha_{\beta_j}] = m_j (\bigoplus [h_i \pi h_i^{-1}])$ where $h_i$ are representatives of $\Gamma / \Gamma_{\pi}$. In particular $d(\beta_j) = m_j d(\pi) \frac{|\Gamma_{\pi}|}{|\Gamma_{\pi}|}$, and $\sum_j d(\beta_j)^2 = \frac{|\Gamma_{\pi}|^2}{|\Gamma_{\pi}|^2} d(\pi)^2$.

Proof By the definition we have $[\gamma \alpha_{\beta}] = [\beta \gamma] = [\gamma \pi \gamma]$. So we have $\langle \gamma \alpha_{\beta}, \pi \rangle = \langle \gamma \pi \gamma, \pi \rangle = |\Gamma_{\pi}|$. By Lemma 8.5 of [33] we have $\langle \gamma \alpha_{\beta}, \pi \rangle = \langle \alpha_{\beta}, \pi \rangle$, and therefore $\alpha_{\beta} \geq |\Gamma_{\pi}|$. By Lemma 8.1 of [33] we have $[h_i \alpha_{\beta_j} h_i^{-1}] = [\alpha_{\beta}]$, so $[\alpha_{\beta}] \geq |\Gamma_{\pi}'| \bigoplus [h_i \pi h_i^{-1}]$. On the other hand $d(\alpha_{\beta_j}) = d(\beta_j) = |\Gamma_{\pi}'| \sum_k d(h_i \pi h_i^{-1})$. It follows that

$$[\alpha_{\beta}] = |\Gamma_{\pi}'|(\bigoplus_i [h_i \pi h_i^{-1}]).$$

Note that by Lemma 8.1 of [33] $[h_i^{-1} \alpha_{\beta_j} h_i] = [\alpha_{\beta_j}]$, hence

$$\langle \alpha_{\beta_j}, h_i \pi h_i^{-1} \rangle = \langle h_i^{-1} \alpha_{\beta_j} h_i, \pi \rangle = \langle \alpha_{\beta_j}, \pi \rangle.$$ 

So we must have $[\alpha_{\beta_j}] = k_j (\bigoplus [h_i \pi h_i^{-1}])$ for some positive integer $k_j$. We note that $k_j = \langle \beta_j, \gamma \pi \mid A^\Gamma \rangle = m_j$ by definitions and Frobenius duality. On the other hand $\sum_j m_j k_j = |\Gamma_{\pi}'| = \sum_j m_j^2$, and we conclude that $k_j = m_j$. Since by definition $\frac{|\Gamma_{\pi}'|}{|\Gamma_{\pi}'|}$, the proof of the theorem follows. ■

4.2 Solitons as representations

In this section we use $\hat{\pi}$ to denote an irreducible soliton of $A_0$ on a Hilbert space $H_\pi$. Let $\pi$ be a soliton unitarily equivalent to $\hat{\pi}$ but localized on $J_0$ as in the previous section. The restriction of $\hat{\pi}$ to $A_0^\Gamma$, denoted by $\hat{\pi} \mid A_0^\Gamma$ is also a soliton. Define $\text{Hom}(\hat{\pi} \mid A_0^\Gamma, \hat{\pi} \mid A_0^\Gamma) := \{ x \in B(H_\pi) \mid x \hat{\pi}(a) = \hat{\pi}(a)x, \forall x \in A_0^\Gamma \}$, and let $\langle \hat{\pi} \mid A_0^\Gamma, \hat{\pi} \mid A_0^\Gamma \rangle = \dim \text{Hom}(\hat{\pi} \mid A_0^\Gamma, \hat{\pi} \mid A_0^\Gamma)$.

Lemma 4.6. (1) $\langle \hat{\pi} \mid A_0^\Gamma, \hat{\pi} \mid A_0^\Gamma \rangle = \langle \gamma \pi \mid A_0^\Gamma, \gamma \pi \mid A_0^\Gamma \rangle$;

(2) $h \in \Gamma_{\pi}$ if and only if $\hat{\pi} \cdot \text{Ad}_h \simeq \hat{\pi}$ as representations of $A_0$.

Proof By [31] $\hat{\pi} \mid A_0^\Gamma$ and $\gamma \pi \mid A_0^\Gamma$ are unitarily equivalent as solitons of $A_0^\Gamma$. Note that $\gamma \pi \mid A_0^\Gamma$ is localized on $J_0$, and (1) follows directly. As for (2), we note that $h^{-1} \pi h$ is localized on $J_0$ and unitarily equivalent to $\hat{\pi} \cdot \text{Ad}_h$, and (2) now follows from the definition (4.2). ■

From (2) of Lemma 4.6 we have for any $h \in \Gamma_{\pi}'$, there is a unitary operator denoted by $\hat{\pi}(h)$ on $H_\pi$ such that $\text{Ad}_{\hat{\pi}(h)} \cdot \hat{\pi} = \hat{\pi} \cdot \text{Ad}_h$ as solitons of $A_0$. Since $\hat{\pi}$ is irreducible, there is a $U(1)$ valued cocycle $c_\pi(h_1, h_2)$ on $\Gamma_{\pi}$ such that $\hat{\pi}(h_1) \hat{\pi}(h_2) = c_\pi(h_1, h_2) \hat{\pi}(h_1 h_2)$. We note that $c_\pi(h_1, h_2)$ is fixed up to coboundaries (cf. §2 of [20]). Hence $h \to \hat{\pi}(h)$ is a projective unitary representation of $\Gamma_{\pi}'$ on $H_\pi$ with cocycle $c_\pi$. Assume that

$$H_\pi = \bigoplus_{\sigma \in E} M_{\sigma} \otimes V_{\sigma}$$

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where $E$ is a subset of irreducible projective representations of $\Gamma'_\pi$ with cocycle $c_\pi$, and $M_\alpha$ is the multiplicity space of representation $V_\sigma$ of $\Gamma_\pi$. Then by definition each $M_\alpha$ is a representation of $A_0^\Gamma$.

**Lemma 4.7.** Fix an interval $I$. Assume that $\hat{\pi}$ is a representation of $A(I)$ (resp. a projective representation of $\Gamma'$ with cocycle $c_\pi$) on a Hilbert space $H$ such that $\hat{\pi}(\beta_t(x)) = \hat{\pi}(h)\hat{\pi}(x)\hat{\pi}(h^*)$, $\forall x \in A(I)$. Let $\sigma_1 \in \Gamma'$ where $\Gamma'$ denotes the set of irreducible representations of $\Gamma'$, and $\sigma_2$ be an irreducible summand of the representation $\hat{\pi}$ of $\Gamma'$. Then:

1. any irreducible summand $\sigma$ of $\sigma_1 \otimes \sigma_2$ appears as an irreducible summand of the projective representation $\hat{\pi}$ of $\Gamma'$ with cocycle $c_\pi$. In particular if $\sigma_2$ is the trivial representation of $\Gamma'$ then all elements of $\Gamma'$ appear as irreducible summand of the representation $\pi$ of $\Gamma'$.

2. Every irreducible projective representation of $\Gamma'$ with cocycle $c_\pi$ appear as an irreducible summand of $\hat{\pi}$, and

$$\sum_{\sigma, \sigma' \text{ has cocycle } c_\pi} \dim(\sigma)^2 = |\Gamma'|^2.$$

**Proof** Ad(1): Since the action of $\Gamma'$ on $A$ is proper, and $A^\Gamma(I)$ is a type III factor, for any $\sigma_1 \in \Gamma'$, by Page 48 of [14] we can find a basis $V(\sigma_1)_i$, $1 \leq i \leq \dim \sigma_1$ in $A(I)$ such that $V(\sigma_1)_i^* V(\sigma_1)_j = \delta_{ij}$, and the linear span of $V(\sigma_1)_i$, $1 \leq i \leq \dim \sigma_1$ forms the irreducible representation $\sigma_1$ of $\Gamma'$. Let $W(\sigma_2)_i \in H, 1 \leq i \leq \dim \sigma_2$ be an orthogonal basis of representation $\sigma_2$. We claim that the vectors $\pi(V(\sigma_1)_i)W(\sigma_2)_i, 1 \leq i \leq \dim \sigma_1, 1 \leq j \leq \dim \sigma_2$ in $H$ are linearly independent. If $\sum_i C_{ij} \pi(V(\sigma_1)_i)W(\sigma_2)_j = 0$ for some complex numbers $C_{ij}$, multiply both sides by $\pi(V(\sigma_1)_i)$ and use the orthogonal property of $V(\sigma_1)_i$’s above we have $\sum_j C_{ij} W(\sigma_2)_j = 0$, and hence $C_{ij} = 0$ since $W(\sigma_2)_j$’s are linearly independent. It follows that the linear span of $\pi(V(\sigma_1)_i)W(\sigma_2)_j, 1 \leq i \leq \dim \sigma_1, 1 \leq j \leq \dim \sigma_2$ gives a tensor product representation of $\Gamma'$ on a subspace of $H$, and the lemma follows.

Ad (2): Let $\sigma_3$ be an irreducible summand of $\pi$, and let $\sigma_4$ be an arbitrary irreducible projective representation of $\Gamma'$ with cocycle $c_\pi$. By definition $\sigma_3 \otimes \sigma_4$ is a representation of $\Gamma'$ ($\sigma_3$ stands for the conjugate of $\sigma_3$), and hence $\sigma_3 \otimes \sigma_4 \succ \sigma_5$ for some $\sigma_5 \in \hat{\Gamma}$, and it follows that $\sigma_4$ appears as an irreducible summand of $\sigma_3 \otimes \sigma_5$, and so by (2) every irreducible projective representation of $\Gamma'$ with cocycle $c_\pi$ appears as an irreducible summand of $\pi$. Note the twisted group algebra $C^{c_\pi}[\Gamma']$ with cocycle $c_\pi$ (cf. P. 85 of [20]) is semisimple, and the equality in (2) follows.

**Theorem 4.8.** (1) $\text{Hom}(\hat{\pi}, A^\Gamma) = \bigoplus_{\sigma \in E} \text{Mat}(\dim(\sigma))$ where $E$ is the set of irreducible projective representations of $\Gamma'_\pi$ with the cocycle $c_\pi$;

2. $\sum_{\sigma \in E} \dim(\sigma)^2 = |\Gamma'|^2$;

3. $M_\sigma$ as defined before Lemma 4.7 is an irreducible representation of $A_0^\Gamma$, and $M_\sigma$ is not unitarily equivalent to $M_{\sigma'}$ if $\sigma \neq \sigma'$. 

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5.1 Conformal nets associated with the affine algebras

5 Solitons in Affine Orbifold

5.1 Conformal nets associated with the affine algebras

Let $G$ be a compact Lie group of the form $G := G^0 \times G^1 \times \cdots \times G^s$ where $G^0 = U(1)^r$, and $G^j, j = 1, ..., s$, are simple simply-connected groups. Let $g^j$ denote the Lie algebra of $G^j, j = 0, ..., s$ and let $L := \{\omega \in g^0| e^{2\pi i \omega} = 1\}$. Note that $G^0 = U(1)^r = \mathbb{R}^r/L$. We assume that $g = \bigoplus_j g^j$ is equipped with a symmetric even negative definite invariant bilinear form. This means that the length square of any $\omega \in ig^j (j = 0, ..., s)$ such that $e^{2\pi i \omega} = 1$ is an even integer. Note that our condition on the bilinear form is slightly stronger than the condition on $P$. 61 of [18] to ensure locality of our nets (cf. Remark 1.1 of [18]). When restricted to a simple $g^j$, the even property means that the bilinear form is equal to $k_j \langle v|v' \rangle$, where $k_j \in \mathbb{N}$ will be identified with the level of the affine Kac-Moody algebra $\hat{g}^j$ and

$$(v|v') = \frac{1}{2g^j} \mathrm{Tr}_{g^j}(\mathrm{Ad}_v \mathrm{Ad}_{v'})$$

($g^j$ is the dual Coxeter number of $g^j$). We will fix $k_0 = 1$.

We will denote by $\bar{L}G$ the central extension of $LG$ whose Lie algebra is the (smooth) affine Kac-Moody algebra $\hat{g}$. For an interval $I \subset S^1$, we denote by $\bar{L}_I G : \{f \in \bar{L}G|f(t) = e, \forall t \in I' \}$ where $e$ is the identity element in $G$, and $\bar{L}_I g : \{p \in \bar{L}g|p(t) = 0, \forall t \in I' \}$. We will write elements of $\bar{L}g$ as $(f, c)$ where $f \in Lg, c \in \mathbb{C}$ and $(0, c)$ is in the center of $Lg$. Denote by $\mathcal{A}_{G_k}$ the conformal net associated with representations of $\bar{L}G$ at level $k = (k_0, ..., k_s)$. The following lemma follows from [41]:

Lemma 5.1. $\mathcal{A}_{G_k}$ is strongly additive.

For simplicity we will denote $\mathcal{A}_{G_k}$ by $\mathcal{A}$ in this chapter. Let $Z_j \subset G^j$ denote the center of $G^j, j = 1, ..., s$, and let $Z^0 = L^*/L$ where $L^* := \{\mu \in g^0| (\mu|\omega) \in \mathbb{Z}, \forall \omega \in L\}$.

The following finite subgroup of $G$ will play an important role:

$$Z(G) := Z^0 \times Z^1 \times \cdots \times Z^s$$
Recall from §4.2 of [18] that an element $g \in G$ is called non-exceptional if there exists $\beta(g) \in i\mathfrak{g}$ such that $g = e^{2\pi i \beta(g)}$ and the centralizer $G_g := \{b' \in G | b'gb'^{-1} = g\}$ of $g$ is the same as $G_{\beta(g)} : = \{b' \in G | b'\beta(g)b'^{-1} = \beta(g)\}$, the centralizer of $\beta(g)$.

Let $\Gamma$ be a finite subgroup of $G$. Then it follows by definition that $\Gamma$ acts properly on $A$. We will be interested in the irreducible representations of $A^\Gamma$. Note that $Z(G)$ acts on $A$ trivially. Hence $A^\Gamma = A^{(\Gamma, Z(G))}$ where $(\Gamma, Z(G))$ is the subgroup of $G$ generated by $\Gamma, Z(G)$. Without losing generality, we will always assume that $\Gamma \supset Z(G)$. By definition before (3.4) we have $\Gamma' = \Gamma/Z(G)$.

The following definition is definition 4.1 of [18]:

**Definition 5.2.** A group $\Gamma$ is called a non-exceptional subgroup of $G$ if for any $g \in \Gamma$ there exists $\zeta \in Z(G)$ such that $\zeta g$ is a non-exceptional element.

Recall from [18] that every element of $Z$ can be written in the form

$$\zeta = (\zeta_0^{(0)}, \ldots, \zeta_j^{(s)}) \in Z_0 \times \cdots \times Z_s, \zeta_j^{(v)} = e^{2\pi i \lambda_j^{(v)}}.$$  

Here $\lambda_j^{(0)}$ generate the finite abelian group $L^*/L$; for each simple component $\mathfrak{g}$ the fundamental weight $\Lambda_j$ belongs to the set $J$ (1.33) of [18]. If both $g$ and $\zeta_j g$ are non-exceptional, we can write

$$\beta(\zeta_j g) = \beta(g) + \Lambda_j + m, [\beta(g), \beta(\zeta_j g)] = 0, e^{2\pi i m} = 1.$$  

Now we define the action of $\zeta_j$ on $\Lambda$. By Lemma 4.1 of [18] the phase factor

$$\sigma_j (b') = e^{2\pi i (k\lambda_j + km)\beta'}, b' = e^{2\pi i \beta'} \in G_g, [\beta', \lambda_j + m] = 0$$  

gives a 1-dimensional representation of $\sigma_j$ of $\Gamma_g$. The transformation $\Lambda \rightarrow \zeta_j(\Lambda)$ of a lattice weight $\Lambda \in L^*$ is given by $\zeta_j(\Lambda) = (\Lambda + \lambda_j) \bmod L$. If $\mathfrak{g}$ is a simple rank $l$ Lie algebra and $\Lambda$ is an integral weight at level $k$, then $\zeta_j(\Lambda) := k\lambda_j + w_j \Lambda$ where $w_j$ is the unique element of the Weyl group of $\mathfrak{g}$ that permutes the set $\{-\theta, \alpha_1, \ldots, \alpha_l\}$ and satisfies $-w_j\theta = \alpha_j$.

**Definition 5.3.** [18] For any $\zeta \in Z$, $\Lambda = \sum_\nu \Lambda_\nu$, we define:

$$\zeta(\Lambda) = \sum_\nu (w_j \Lambda_\nu + k_\nu \lambda_j) \Lambda_\nu.$$  

We will use $\pi_\Lambda$ to denote the irreducible representations of $\tilde{L}G$ on a Hilbert space $\mathcal{H}_\Lambda$ with highest weight $\Lambda$.

Note that $\pi_\Lambda$ gives an irreducible representation of $A_{G_k}$ by §3 of [8] on $\mathcal{H}_\Lambda$. We will write $\zeta = e^{2\pi i \beta(\zeta)}$ with $\beta(\zeta) = (\beta(\zeta_0^{(0)}), \ldots, \beta(\zeta_j^{(s)}))$ and $\beta(\zeta_j^{(v)}) = \Lambda_j^{(v)} + m$ where $m$ is as in (10). Let $P_g : [0, 1] \rightarrow G$ be a map with $P_g(\theta) = e^{2\pi i \beta(g)\theta}$, $0 \leq \theta \leq 1$, $P_g : [0, 1] \rightarrow G$ be a map with $P_g(\theta) = e^{2\pi i \beta(\zeta)\theta}$, $0 \leq \theta \leq 2\pi$, and $P_\zeta : [0, 1] \rightarrow G$ be a map with $P_\zeta(\theta) = e^{2\pi i \beta(\zeta)\theta}$, $0 \leq \theta \leq 1$. We note that $\text{Ad}_{P_{\zeta}}$ is an automorphism of $LG$ since $\zeta$ is in the center of $G$.  

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Lemma 5.4. (1) If $g$ is non-exceptional then $P_g \in Z(G_g)$; 
(2) If $\zeta g$, $g$ are non-exceptional then $P_{\zeta g}P^{-1}_g = P_\zeta$.

Proof If $h \in G_g$, since $g$ is non-exceptional, it follows that
$$he^{2\pi i \beta(g)}h^{-1} = e^{2\pi i \beta(g)}, 0 \leq \theta \leq 1$$
and (1) is proved.
Since $\zeta g$, $g$ are non-exceptional, by (9) $[\beta(\zeta g), \beta] = 0$ and (2) follows immediately.

Lemma 5.5. If $\zeta g$, $g$ are non-exceptional in $\Gamma$, and with notations as above, we have:
(1) $\Ad_{\zeta g}$ lifts to an automorphism denoted by $\Ad_{\zeta}$ of $\tilde{L}G$;
(2) The induced action of $\Ad_{\zeta}$ on $\tilde{L}g$ is given by
$$\Ad_{\zeta}(f, c) = (\Ad_{\zeta} f, k(\zeta f) + c);$$
(3) There is an unitary $U : H_{\zeta(\Lambda)} \to H_{\Lambda}$ such that $U^* \pi_{\zeta(\Lambda)}(\Ad_{\zeta})U = \pi_{\Lambda}$ as representations of $\tilde{L}G$;
(4) $U^* \pi_{\zeta(\Lambda)}(h)\sigma_{\zeta}(h)U = \pi_{\Lambda}(h)$ for any $h \in \Gamma_g$, where $\sigma_{\zeta} = \otimes_\nu \sigma_{j\nu}$ with $\sigma_{j\nu}$ as defined in (10).

Proof We note that the path $P_{\zeta g}P_{\zeta}^*$ is an element of $L(G/Z(G))$. When $G$ is semisimple, (1),(2) follows from Lemma 4.6.5 and equation (4.6.4) of [36]. The proof in §4.6 of [36] also generalizes easily to the proof of (1) and (2) when $G = G_0 = U(1)^r$.

As for (3), first note that $\pi_{\zeta(\Lambda)}(\Ad_{\zeta})$ is an irreducible representation of $\tilde{L}G$, since such irreducible representations are classified (cf. [36] and [17]), we just have to identify it with the known representations. By using Th. 4.2 of [18] for the special case when the group $\Gamma$ is trivial, we conclude that the character of $\pi_{\zeta(\Lambda)} \cdot \Ad_{\zeta}$ is the same as that of $\pi_{\Lambda}(\tilde{L}G)$, and it follows that they are unitarily equivalent as representations of $\tilde{L}G$.

For any $h = e^{2\pi i \beta} \in G_g \subset \tilde{L}G$, by (2) we have
$$\pi_{\Lambda}(\Ad_{\zeta}(h)) = \pi_{\Lambda}(h) \prod_{\nu} e^{2\pi i (\beta' | j_{\nu} + m)} = \pi_{\Lambda}(h)\sigma_{\zeta}(h)$$
Using (3) we have
$$U^* \pi_{\zeta(\Lambda)}(h)\sigma_{\zeta}(h)U = \pi_{\Lambda}(h)$$

5.2 Constructions of solitons

Let $\pi_{\Lambda}$ be an irreducible representation of $\tilde{L}G$ with highest integral weight $\Lambda$. We will denote the net $\mathcal{A}_{G_g}$ simply by $\mathcal{A}$ in this section. For $g \in G$, let $\beta(g)$ be an element in the Lie algebra of $G$ such that $e^{2\pi i \beta(g)} = g$. Define $P_g(\theta) := e^{2\pi i \theta \beta(g)}, 0 \leq \theta \leq 1$. Identify $\mathbb{R}$ with the open interval $(0, 1)$ via a smooth map $\varphi : (-\infty, +\infty) \to (0, 1), \varphi(t) = \frac{1}{\pi} (\tan^{-1}(t) + \frac{\pi}{2})$. For any $I \subset \mathbb{R}$, Let $P_{g,I} \in L_I G$ be a loop localized on $I$ such that $P_{g,I}(t) = P_g(\varphi(t)), \forall t \in I$. 23
Definition 5.6. For any \( x \in \mathcal{A}(I) \), define \( \hat{\pi}_{\Lambda,g,I}(x) := \pi_{\Lambda}(P_{g,I}xP_{g,I}^*) \).

We note that the above definition is independent of the choice of \( P_{g,I} \): if \( \tilde{P}_{g,I} \) is another loop such that \( \tilde{P}_{g,I}(t) = P_{g,I}(t), \forall t \in I \), then \( \tilde{P}_{g,I}(t)P_{g,I}^{-1} \) is a loop with support in \( I' \), and so \( \pi_{\Lambda}(P_{g,I}xP_{g,I}^*) = \pi_{\Lambda}(\tilde{P}_{g,I}x\tilde{P}_{g,I}^*) \), \( \forall x \in \mathcal{A}(I) \). One checks easily that definition (5.6) defines a soliton, and we denote it by \( \hat{\pi}_{\Lambda,g} \).

Fix \( J_0 := (0, \infty) \subset \mathbb{R} \). To obtain a soliton equivalent to \( \hat{\pi}_{\Lambda,g} \) but localized on \( J_0 \), we choose a smooth path \( P_{g,J_0} \in \mathcal{C}^\infty(\mathbb{R}, G) \) which satisfies the following boundary conditions: \( P_{g,J_0}^J(t) = e \), if \(-\infty < t \leq 0\) and \( P_{g,J_0}^J(t) = g \), if \( 1 \leq t < \infty \). For any interval \( I \subset \mathbb{R} \), we choose a loop \( P_{g,J_0} \in LG \) such that \( P_{g,J_0}(t) = P_{g,J_0}^J(t), \forall t \in I \).

Definition 5.7. For any \( x \in \mathcal{A}(I) \), define \( \pi_{\Lambda,g,I} := \Lambda(P_{g,I}xP_{g,I}^*) \) where we use \( \Lambda \) to denote a representation unitarily equivalent to \( \pi_{\Lambda} \) but localized on \( J_0 \).

We denote the soliton in the above definition as \( \pi_{\Lambda,g} \).

Proposition 5.8. The unitary equivalence class of \( \pi_{\Lambda,g} \) is independent of the choice of the path \( P_{g,J_0} \) as long as it satisfies the boundary conditions given as above, and \( \pi_{\Lambda,g} \) is localized on \( J_0 \). Moreover \( \pi_{\Lambda,g} \) is unitarily equivalent to \( \hat{\pi}_{\Lambda,g} \), and \( \pi_{\Lambda,g} \) restricts to a DHR representation of \( \mathcal{A}^{(g)} \) where \( (g) \) denotes the closed subgroup of \( G \) generated by \( g \).

Proof If \( \tilde{P}^J_{g,J_0} \) is another path which satisfies the same boundary condition as \( P^J_{g,J_0} \), then \( \tilde{P}^J_{g,J_0}P^J_{g,J_0}^{-1} \in LG \), and the first statement of the proposition follows by definition. By definition \( \pi_{\Lambda,g,J_0}(x) = x, \forall x \in \mathcal{A}(J_0) \) since \( P^J_{g,J_0}(t) = e \), if \(-\infty < t \leq 0\), and so \( \pi_{\Lambda,g} \) is localized on \( J_0 \). Since \( P^J_{g,J_0}P^J_{g,J_0}^{-1} \) extends to an element in \( LG \), it follows that \( \pi_{\Lambda,g} \) is unitarily equivalent to \( \pi_{\Lambda,g} \). To prove the last statement, let \( I \) be an interval with \(-1 \in I \). It is sufficient to show that \( \pi_{\Lambda,g} \) has a normal extension to \( \mathcal{A}^{(g)}(I) \). Recall from §3.6 that we identify \( \mathbb{R} = S^1 \times \{-1\} \) and \( J_0 = (0, \infty) \subset \mathbb{R} \). Since the net \( \mathcal{A} \) is strongly additive by Lemma 5.1, and so \( \mathcal{A}^{(g)} \) is strongly additive by [49], we can assume that \( \mathcal{A}^{(g)}(I) = \mathcal{A}^{(g)}(-\infty, a) \cup \mathcal{A}^{(g)}(b, \infty) \) where \( a < b \). Let us assume that \( P^J_{g,J_0(-\infty, a)} \) and \( P^J_{g,J_0(b, \infty)} \) are the elements in \( LG \) such that \( \text{Ad}_{\Lambda(P^J_{g,J_0(-\infty, a)})} = \pi_{\Lambda,g,-\infty,a} \) and \( \text{Ad}_{\Lambda(P^J_{g,J_0(b, \infty)})} = \pi_{\Lambda,g,b,\infty} \) as in Definition 5.7. Choose an element \( \tilde{P} \in LG \) so that \( \tilde{P}(t) = gP^J_{g,J_0(-\infty, a)}(t), -\infty < t < a \) and \( \tilde{P}(t) = P^J_{g,J_0(b, \infty)}(t), b < t < \infty \). Then by definition \( \text{Ad}_\Lambda(\tilde{P})(x) = x, \forall x \in \mathcal{A}^{(g)}(-\infty, a) \cup \mathcal{A}^{(g)}(b, \infty) \), and hence \( \text{Ad}_\Lambda(\tilde{P}) \) defines the normal extension of \( \pi_{\Lambda,g} \) to \( \mathcal{A}^{(g)}(I) \).

Proposition 5.9. As sectors of \( \mathcal{A}(J_0) \) we have:

1. \( [\pi_{\Lambda,g}] = [\Lambda[\pi_{1,g}]] \);
2. \( [\pi_{1,g}][\pi_{1,g2}] = [\pi_{1,g1g2}], [h^{\pi_{\Lambda,g}h^{-1}}] = [\pi_{\Lambda,gh^{-1}}] \);
3. Assume that \( \Lambda, \mu \) are irreducible DHR representations of \( \mathcal{A} \). Then \( \langle \Lambda, \mu \pi_{1,gh} \rangle = 1 \) if and only if \( h \in Z(G), g \in Z(G) \) and \( \Lambda = g^{-1}(\mu) \) where the action of the center is as in (5.3). In all other cases \( \langle \Lambda, \mu \pi_{1,gh} \rangle = 0 \);
(4) If $\Lambda_1, \Lambda_2$ are irreducible DHR representations of $\mathcal{A}$, then $\langle \pi_{\Lambda_1,g_1}, \pi_{\Lambda_2,g_2}h \rangle = 1$ if and only if $h \in Z(G)$ and there exists $g \in Z(G)$ such that $g_2 = gg_1$ and $\Lambda_2 = g^{-1}(\Lambda_1)$. In all other cases $\langle \pi_{\Lambda_1,g_1}, \pi_{\Lambda_2,g_2}h \rangle = 0$.

(5) The stabilizer $\Gamma_{\Lambda,g}$ of $\pi_{\Lambda,g}$ (cf. (4.2)) is given by $\Gamma_{\Lambda,g} = \{ h \in \Gamma | hgh^{-1} = g_1g, g_1(\Lambda) = \Lambda, g_1 \in Z(G) \}$.

Proof  (1) and (2) follows directly from definition 5.7. Now assume that $\langle \Lambda, \mu \pi_{\Lambda_1,g}h \rangle = 1$. By lemma 8.5 of [33] we conclude that $[h] = [1]$ and so $h \in Z(G)$, hence $[\Lambda] = [\mu \pi_{\Lambda_1,g}]$, and it follows that $\mu \pi_{\Lambda_1,g}$ is a DHR representation of the net $\mathcal{A}$. In particular $\mu \pi_{\Lambda_1,g}$ is normal on $\mathcal{A}(-\infty,0) \lor \mathcal{A}(1,\infty)$. Choose $\mu$ to be localized on $\mathcal{A}(0,1)$. Since $\mathcal{A}(-\infty,0) \lor \mathcal{A}(1,\infty)$ is a type III von Neumann algebra, there is a unitary $u$ such that $\pi_{1,g}(x) = uxu^*, \forall x \in \mathcal{A}(-\infty,0) \lor \mathcal{A}(1,\infty)$. Since $\pi_{1,g} = id$ on $\mathcal{A}(-\infty,0)$ and $\pi_{1,g} = Ad_g$ on $\mathcal{A}(1,\infty)$, we have $u \in \mathcal{A}(-\infty,0) \lor \mathcal{A}(1,\infty)' \lor \mathcal{A}(1,\infty)'$. By (2) of Lemma 3.6 in [49] the pair $\mathcal{A}^G \subset \mathcal{A}$ is strongly additive (cf. Definition 3.2 of [49] ) since $\mathcal{A}$ is strongly additive by Lemma 5.1, and so $\mathcal{A}(-\infty,0)' \lor \mathcal{A}(1,\infty)' = \mathcal{A}(0,1)$. Therefore $u \in \mathcal{A}(0,1), Ad_u(x) = x, \forall x \in \mathcal{A}(1,\infty)$, and so $g \in Z(G)$. Hence we have $\langle \Lambda, \mu \pi_{\Lambda_1,g} \rangle = 0$. By (3) of Lemma 5.5 and definition of $\pi_{1,g}$ we have $\Lambda = g^{-1}(\mu)$ where the action of the center is defined in (5.3).

As for (4), (by (1) and (2) we have

$$\langle \pi_{\Lambda_1,g_1}, \pi_{\Lambda_2,g_2}h \rangle = \langle \Lambda_1 \pi_{\Lambda_1,g_1}, \Lambda_2 \pi_{\Lambda_1,g_2}h \rangle = \langle \Lambda_1, \Lambda_2 \pi_{\Lambda_1,g_2}h \rangle \pi_{\Lambda_1,g_1}^{-1} \rangle = \langle \Lambda_1, \Lambda_2 \pi_{\Lambda_1,g_2}h^{-1}h \rangle \rangle$$ (11)

and (4) follows from the above equation and (3). (5) follows from definitions and (4).

5.2.1 Comparing solitons with “twisted representations”

Let $e^{2\pi i \beta} = g$ and choose the Cartan subalgebra of $\mathfrak{g}$ which contains $\beta$. In the definition (5.6), if we choose $x = \pi_1(y), y \in \tilde{L}_I G$, then $\pi_{\Lambda_1,g_1}(\pi_1(y)) = \pi_\lambda(P_{g_1} y P_{g_1})$. Note that $Ad_{P_{g_1}}$ is an automorphism of $\tilde{L}_I G$, and induces an automorphism on $\tilde{L}_I \mathfrak{g}$. By Prop. 4.3.2 of [36], if we write elements of $\tilde{L}_I \mathfrak{g}$ as $(f, c)$, where $f \in C^\infty(S^1, \mathfrak{g})$ with support in $I$, and $c \in \mathbb{C}$, then

$$Ad_{P_{g_1}}(f, c) = (Ad_{P_{g_1}}f, c + k(\beta f))$$ (13)

Let us check that (13) agrees with the definition of twisted representation 2.11-2.14 of [18] on $\tilde{L}_I \mathfrak{g}, \forall I \subset \mathbb{R}$. Let $E^\alpha$ be a raising or lowering operator as on Page 64 of [18]. Let $f_1 \in C^\infty(S^1, \mathbb{R})$ be a smooth map such that $f_1(t) = 0, \forall t \in I'$. By the commutation relation $[E^\alpha, \beta] = -(\alpha|\beta)E^\alpha$ we have $Ad_{P_{g_1}}f = z^{-(\alpha|\beta)} E^\alpha f_1$ where $z^{-(\alpha|\beta)} := e^{-2\pi i\theta(\alpha|\beta)}$ as a function on $[0,1]$, and $(\beta|f_1 E^\alpha) = 0$ by definition. By (13) we have

$$Ad_{P_{g_1}}(f_1 E^\alpha, c) = (z^{-(\alpha|\beta)} E^\alpha f_1, c)$$

which is the restriction of (2.11) of [18] to $\tilde{L}_I \mathfrak{g}$. Similarly one can check that (13) agrees with the definition of twisted representation 2.12-2.14 of [18] on $\tilde{L}_I \mathfrak{g}, \forall I \subset \mathbb{R}$.
Hence our soliton representations in Definition 5.6 can be regarded as “exponentiated” version of the twisted representations in §2 of [18]. In the next section we shall see that these soliton representations are important in constructing irreducible DHR representations of $\mathcal{A}^\Gamma$. Motivated by the above observations, we have the following conjecture:

**Conjecture 5.10.** There is a natural one to one correspondence between the set of irreducible DHR representations of $\mathcal{A}^\Gamma$ and the set of irreducible representations of the orbifold chiral algebra as defined on Page 74 of [18] with gauge group $\Gamma$.

We note that this conjecture, together with the results of §5.4 and §5.5, give a prediction on the the set of irreducible representations of the orbifold chiral algebra as defined on Page 74 of [18] with non-exceptional gauge group $\Gamma$.

### 5.3 Completely rational case

Assume that the net $\mathcal{A}$ associated to $G$ has the property that

$$\mu_{\mathcal{A}} = \sum_{\Lambda} d(\Lambda)^2$$

where the sum is over all irreducible projective representations of $LG$ of a fixed level. When $G = SU(N)$ this property is proved by [46]. We show that all irreducible DHR representations of $\mathcal{A}^\Gamma$ are obtained from decomposing the restriction of solitons $\pi_{\Lambda, g}$ to $\mathcal{A}^\Gamma$, answering one of the motivating questions for this paper. By Prop. 4.4 $\pi_{\Lambda_1, g_1} | \mathcal{A}^\Gamma \simeq \pi_{\Lambda_2, g_2} | \mathcal{A}^\Gamma$ iff there exists $h \in \Gamma$ such that $[h \pi_{\Lambda_1, g_1} h^{-1}] = [\pi_{\Lambda_2, g_2}]$. By (2) and (4) of Prop. 5.9 this is true if there is a $g_3 \in Z(G)$ such that $\Lambda_2 = g_3^{-1}(\Lambda_1)$ and $g_2 = h g_3 g_1 h^{-1}$. Define an action of group $Z(G) \times \Gamma$ on the set $\{\Lambda, g\}$ by $(g_3, h)(\Lambda, g) = (g_3^{-1}(\Lambda), h g_3 g_1 h^{-1})$. Denote the orbit of $(\Lambda, g)$ by $\{\Lambda, g\}$. Note that the stabilizer of $(\Lambda, g)$ has the same order as the stabilizer $\Gamma_{\Lambda, g}$ of $\pi_{\Lambda, g}$ by (5) of Prop. 5.9. Hence the orbit $\{\Lambda, g\}$ contains $|Z(G) \times \Gamma|/|\Gamma_{\Lambda, g}|$ elements. Let $[\gamma \pi_{\Lambda, g}] = \sum_i m_i[\beta_i]$ where $\beta_i$ are irreducible DHR representations of $\mathcal{A}^\Gamma$.

By Th. 4.5 $\sum_i d(\beta_i)^2 = |\Gamma|/|\Gamma_{\Lambda, g}| d(\Lambda)^2$. By Prop. 4.4 we get the sum of index of all different irreducible DHR representations of $\mathcal{A}^\Gamma$ coming from decomposing the restriction of $\pi_{\Lambda, g}$ to $\mathcal{A}^\Gamma$ is given by

$$\sum_{\{\Lambda, g\}} |\Gamma|/|\Gamma_{\Lambda, g}| d(\Lambda)^2.$$  

Since the orbit $\{\Lambda, g\}$ contains $|Z(G) \times \Gamma|/|\Gamma_{\Lambda, g}|$ elements, the above sum is equal to

$$\sum_{\Lambda, g} |\Gamma|/|\Gamma| d(\Lambda)^2 = |\Gamma|/|\Gamma| \mu_{\mathcal{A}} = \mu_{\mathcal{A}^\Gamma}$$

where in the last $= \text{ we have used Th. 3.7.}$ By Th. 33 of [21] we have proved the following:
Theorem 5.11. If equation (14) holds, then every irreducible DHR representation of $\mathcal{A}^\Gamma$ is contained in the restriction of $\pi_{\Lambda,g}$ to $\mathcal{A}^\Gamma$ for some $\Lambda, g$ where $\pi_{\Lambda,g}$ is defined as in (5.7).

Let $G = SU(N_1) \times SU(N_2) \times \cdots \times SU(N_m)$ and let level $k = (k_1, ..., k_m)$. Since $\mathcal{A}_{G_k}$ verifies equation (14) by [46], we have the following:

Corollary 5.12. Let $\Gamma \subset G = SU(N_1) \times SU(N_2) \times \cdots \times SU(N_m)$ be a finite subgroup. Then every irreducible DHR representation of $\mathcal{A}_{G_k}$ is contained in the restriction of $\pi_{\Lambda,g}$ to $\mathcal{A}_{G_k}^\Gamma$ for some $\Lambda, g \in \Gamma$ where $\pi_{\Lambda,g}$ is defined as in definition (5.7) and $\mathcal{A}_{G_k}^\Gamma$ is the conformal net associated with the projective representation of LG at level $k = (k_1, ..., k_m)$.

5.4 Identifying representations of $\mathcal{A}^\Gamma$ for non-exceptional $\Gamma$

In this section we assume that $\Gamma$ is a non-exceptional finite subgroup of $G$ (cf. 5.2). Assume that $g \in \Gamma$ is a non-exceptional element in $\Gamma$ with $g = e^{2\pi i \beta}$ and $G_g = G_{\beta}$. We will choose the path $P_g$ as $P_g(\theta) = e^{2\pi i \beta \theta}$, $0 \leq \theta \leq 1$. Let $\sigma$ be an irreducible character of the group $\Gamma_{\beta} := \Gamma \cap G_{\beta} = \Gamma_g$. Let

$$P_{\Lambda,\sigma} := \frac{\sigma(1)}{|\Gamma_g|} \sum_{h \in \Gamma_{\beta}} \sigma^*(h)\pi_{\Lambda}(h)$$

By Lemma 5.15 $P_{\Lambda,\sigma,\pi_{\Lambda,g}}$ is a direct sum of $\sigma(1)$ copies of a DHR representation of $\mathcal{A}^\Gamma$ (on $\mathcal{A}_{\Lambda,\sigma}$) which we denote by $\pi_{\Lambda,g,\sigma}$. We have:

Proposition 5.13. Let $h \in N_G(\Gamma_g) := \{b \in G|b\Gamma_gb^{-1} = \Gamma_g\}$. Then as representation of $\mathcal{A}^{\Gamma_g}$ we have

$$\pi_{\Lambda,g,\sigma} \cdot \text{Ad}_{h^{-1}} \simeq \pi_{\Lambda,hgh^{-1},\sigma^h}$$

where $\sigma^h$ is an irreducible representation of $\Gamma_{hgh^{-1}}$ defined by $\sigma^h(b) = \sigma(h^{-1}bh)$.

Proof By definition (5.6) $\forall x \in \mathcal{A}(I), I \subset \mathbb{R}$ we have

$$\hat{\pi}_{\Lambda,g}(\text{Ad}_{h^{-1}}x) = \pi_{\Lambda}(g) \pi_{\Lambda}(hP_{g,1}h^{-1}xhP_{g,1}^*) \pi_{\Lambda}(h)$$

$$= \pi_{\Lambda}(h) \pi_{\Lambda}(g) \pi_{\Lambda}(hP_{g,1}h^{-1}xhP_{g,1}^*) \pi_{\Lambda}(h)$$

On the other hand from the definition (15) one checks that

$$\pi_{\Lambda}(h) \pi_{\Lambda}(g) \pi_{\Lambda}(hP_{g,1}h^{-1}xhP_{g,1}^*) \pi_{\Lambda}(h) = \pi_{\Lambda}(h) \pi_{\Lambda}(hP_{g,1}h^{-1}xhP_{g,1}^*) \pi_{\Lambda}(h)$$

It follows that $\forall y \in \mathcal{A}^{\Gamma_g}(I)$

$$\pi_{\Lambda,g,\sigma} \cdot \text{Ad}_{h^{-1}}y = \pi_{\Lambda}(g) \pi_{\Lambda,hgh^{-1},\sigma^h}(y) \pi_{\Lambda}(g).$$
Proposition 5.14. For the pair of non-exceptional triples \( X = (\Lambda, g, \sigma) \) and
\[
\zeta(X) := \sum_{\nu}(w_{\nu} \Lambda^\nu + k_{\nu} \Lambda_{j_{\nu}} \zeta g, \sigma \otimes (\otimes_{\nu} \sigma_{j_{\nu}}))
\]
where \( \sigma_{j_{\nu}} \) is defined as in (10), we have \( \pi_X \simeq \pi_{\zeta(X)} \) as DHR representations of \( A^\Gamma_{0,g} \).

Proof For any \( a \in A(\mathcal{I}) \) we have:
\[
\hat{\pi}_{\zeta(\Lambda), \zeta g}(a) = \pi_{\zeta(\Lambda)}(P_{\zeta g} a P_{g}^* P_{g}^* P_{g}^* P_{g}^*) = \pi_{\zeta(\Lambda)}(P_{\zeta g} a P_{g}^* P_{g}^* P_{g}^*)
\]
where we have used (2) of Lemma 5.4. By (3) of Lemma 5.5 There exists a unitary \( U \) such that
\[
\pi_{\zeta(\Lambda)}(P_{\zeta g} a P_{g}^* P_{g}^* P_{g}^*) = U \pi_{\Lambda,g}(a) U^*.
\]
By (4) of Lemma 5.5
\[
\pi_{\zeta(\Lambda)}(h) = U \pi_{\Lambda}(h) \sigma_{\zeta}(h) U^*,
\]
and it follows by definition (15)
\[
P_{\zeta(\Lambda), \sigma \otimes \sigma_{\zeta}} = U P_{\Lambda, \sigma} U^*,
\]
hence the proposition is proved by definition. \( \blacksquare \)

5.5 Details on decomposing solitons: fixed point resolutions

Assume that \( g \in \Gamma \) is a non-exceptional element with \( g = e^{2\pi i \beta} \) and \( G_g = G_{\beta} \). We will choose the path \( P_g \) as \( P_g(\theta) = e^{2\pi i \theta \beta}, 0 \leq \theta \leq 1 \). Let \( \hat{\pi}_{\Lambda,g} | \mathcal{A}^\Gamma \simeq \sum_i \beta_i \) where \( \beta_i \) are irreducible DHR representations of \( \mathcal{A}^\Gamma \). Define \( \Gamma_g := \{ h \in \Gamma | h g = g h \} \). Note that \( \Gamma_g \) is a normal subgroup of \( \Gamma_{\Lambda,g} \) and \( \Gamma_{\Lambda,g} / \Gamma_g = \{ h \in Z(G) | h \Lambda = \Lambda \} \) is an abelian group (cf. (5) of Lemma 5.9).

Lemma 5.15. For all \( x \in A(\mathcal{I}), h \in \Gamma_g \),
\[
\pi_{\Lambda}(h) \hat{\pi}_{\Lambda,g}(x) \pi_{\Lambda}(h)^* = \hat{\pi}_{\Lambda,g}(h x h^*)
\]

Proof Since \( \pi_1(\mathcal{L}_I G) \) generates \( A(\mathcal{I}) \), it is sufficient to check the equation for \( x = \pi_1(y), y \in \mathcal{L}_I G \). As elements in \( LG \) we have
\[
h P_{g,1} y P_{g,1}^{-1} h^{-1} = P_{g,1} h y h^{-1} P_{g,1}^{-1}
\]
where we have used \( h P_g h^{-1} = P_g \) by (1) of Lemma 5.4. It follows by definition (5.6) that
\[
\pi_{\Lambda}(h) \hat{\pi}_{\Lambda,g}(x) \pi_{\Lambda}(h)^* = \hat{\pi}_{\Lambda,g}(h x h^*)
\]
\( \blacksquare \)
Assume that when restricting to $\mathcal{A}^\Gamma$, $\mathcal{H}_\Lambda = \bigoplus_{\sigma \in E} M_{\sigma} \otimes V_\sigma$ where $V_\sigma$ are irreducible representation spaces of $\Gamma_g$, $E \subset \text{Irr} \Gamma_g$ and $M_{\sigma}$ the corresponding multiplicity spaces. By Th. 4.1 of [18], $\sigma$ appears in the above decomposition iff $\sigma|Z(G) = \Lambda|Z(G)$. Applying Th. 4.8 to the pair $\mathcal{A}^\Gamma \subset \mathcal{A}$, each $M_{\sigma}$ with $\sigma|Z(G) = \Lambda|Z(G)$ is an irreducible DHR representation of $\mathcal{A}^\Gamma$. We will denote $M_{\sigma}$ by $\pi_{\Lambda,g,\sigma}$. When $\Gamma_{\Lambda,g}/\Gamma_g$ is nontrivial, the next question is how $\pi_{\Lambda,g,\sigma}$ decomposes when restricting to $\mathcal{A}^{\Gamma_{\Lambda,g}}$. This is the issue of “fixed point resolutions”, since the action of the center has a nontrivial fixed point on the quadruples as described on Page 78 of [18], and the question about the nature of how $\pi_{\Lambda,g,\sigma}$ decomposes as representation of $\mathcal{A}^\Gamma$ is implicitly raised. Assume that $\Gamma_{\Lambda,g}/\Gamma_g = \{ h \in Z(G)| h\Lambda = \Lambda \}$. Then $\mathcal{A}^{\Gamma_{\Lambda,g}} \subset \mathcal{A}^\Gamma$ is the fixed point subnet under the action of $\Gamma_{\Lambda,g}/\Gamma_g$. Note that $\Gamma_{\Lambda,g}/\Gamma_g \simeq \{ \zeta \in Z(G)| \zeta \Lambda = \Lambda \}$ and denote the isomorphism by $h \rightarrow \zeta(h)$. Then we have:

**Theorem 5.16.** (1):

$$\langle \pi_{\Lambda,g,\sigma} \upharpoonright \mathcal{A}^\Gamma, \pi_{\Lambda,g,\sigma} \upharpoonright \mathcal{A}^\Gamma \rangle = \{ h \in \Gamma_{\Lambda,g}/\Gamma_g | \sigma_{\zeta(h)} \simeq \sigma \otimes \sigma_{\zeta} \}$$

where $\sigma_{\zeta(h)}$ is as defined in (4) of Lemma 5.5;

(2): $\pi_{\Lambda,g,\sigma} \upharpoonright \mathcal{A}^\Gamma$ decomposes into irreducible representations of $\mathcal{A}^\Gamma$ which are in one-to-one correspondence with all irreducible projective representation of the group $\Gamma_{\Lambda,g}/\Gamma_g$ with a fixed cocycle.

**Proof** Ad (1): $\mathcal{A}^{\Gamma_{\Lambda,g}} \subset \mathcal{A}^\Gamma$ is the fixed point subnet under the action of $\Gamma_{\Lambda,g}/\Gamma_g$. Applying Lemma 4.3 to the pair $\mathcal{A}^{\Gamma_{\Lambda,g}} \subset \mathcal{A}^\Gamma$, $\langle \pi_{\Lambda,g,\sigma} \upharpoonright \mathcal{A}^\Gamma, \pi_{\Lambda,g,\sigma} \upharpoonright \mathcal{A}^\Gamma \rangle$ is equal to the number of elements $h \in \Gamma_{\Lambda,g}/\Gamma_g$ such that $\pi_{\Lambda,g,\sigma}(\text{Ad}h)$ as representations of $\mathcal{A}^\Gamma$. By Prop.5.13 $\pi_{\Lambda,g,\sigma}(\text{Ad}h) \simeq \pi_{\Lambda,gh^{-1}} = \pi_{\Lambda,\zeta(h)g,\sigma}$, and by Prop. $\pi_{\Lambda,g,\sigma} \simeq \pi_{\Lambda,\zeta(h)g,\sigma} \otimes \sigma_{\zeta(h)}$ as representations of $\mathcal{A}^\Gamma$. It follows that $\pi_{\Lambda,g,\sigma} \simeq \pi_{\Lambda,\zeta(h)g,\sigma} \otimes \sigma_{\zeta(h)}$ as representations of $\mathcal{A}^\Gamma$ iff $\sigma^h \simeq \sigma \otimes \sigma_{\zeta(h)}$. Hence

$$\langle \pi_{\Lambda,g,\sigma} \upharpoonright \mathcal{A}^{\Gamma_{\Lambda,g}}, \pi_{\Lambda,g,\sigma} \upharpoonright \mathcal{A}^{\Gamma_{\Lambda,g}} \rangle = \{ h \in \Gamma_{\Lambda,g}/\Gamma_g | \sigma_{\zeta(h)} \simeq \sigma \otimes \sigma_{\zeta} \}$$

By (4) of Lemma 4.3 (1) is proved. (2) follows by applying Th. 4.8 to the pair $\mathcal{A}^{\Gamma_{\Lambda,g}} \subset \mathcal{A}^\Gamma$ and (4) of Lemma 4.3.

Combine the above theorem with Cor. 4.9 we immediately have:

**Corollary 5.17.** If the group $\{ h \in \Gamma_{\Lambda,g}/\Gamma_g | \sigma^h \simeq \sigma \otimes \sigma_{\zeta(h)} \}$ is cyclic of order $m$, then $\pi_{\Lambda,g,\sigma} \upharpoonright \mathcal{A}^\Gamma$ decomposes into $m$ irreducible pieces.

### 5.6 An example

Here we illustrate Cor. 5.17 in the example 6.4 of [18]. We keep the same notation of [18]. Set $G = SU(2)$ and $\Gamma = H_8$ the quaternion group. $H_8$ has 8 elements, $\{1, \epsilon, q_i, \bar{\epsilon}, \bar{q}_{i}, i = 1, 2, 3, \}$; they obey the multiplication rules $q_{i}^{2} = \epsilon$, $q_{i}q_{j}q_{i}^{-1} = \epsilon q_{j} = \bar{q}_{j}$, $i \neq j$. We note that $q_{i}, \epsilon, q_{i}$ are non-exceptional elements of $SU(2)$. The centralizer of $\Gamma_{q_{i}} \simeq \mathbb{Z}_{4}$, and we will label its irreducible representations by the exponents
\( \sigma = 0, +1, -1, 2. \) There are 5 irreducible representations of \( H_8, \{ \alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_4 \} \) with dimensions 1, 1, 2, 1, 1 respectively. The characters of these representations are given on Page 94 of [18].

Consider the net \( A_{SU(2)2k_1} \). The irreducible DHR representations of \( A_{SU(2)2k_1} \) are labeled by irreducible representations of \( \hat{L}SU(2) \) at level \( 2k_1 \), and we will use integers 0, 1, ..., \( 2k_1 \) to label these representations such that 0 is the vacuum representation. The only representation which is fixed by the action of the center is \( k \). We note that \( \sigma_\epsilon = 2k_1(\text{mod}4). \) When \( k_1 \) is odd, consider the DHR representation \( \pi_{k_1,q_1,1}. \) We have \( \Gamma_{k_1,q_1} = H_8. \) We note that \( \sigma_\epsilon = 2k_1(\text{mod}4), \) and so the stabilizer of \( \pi_{k_1,q_1,\pm1} \) is \( \{ h \in H_8/\mathbb{Z}_4 | \sigma h = \sigma \zeta(h) \} \cong \mathbb{Z}_2. \) Hence by Cor. 5.17, \( \pi_{k_1,q_1,\pm1} \) decomposes into two distinct irreducible DHR representations of \( A_{H_8}^{SU(2)}(2k_1). \) When \( k_1 = 1 \) this is first observed in [18] by identifying \( A_{H_8}^{SU(2)}(2k_1) \) with the tensor products of three “Ising Models” (cf. Page 99 of [18]).

When \( k_1 \) is even, consider the DHR representation \( \pi_{k_1,q_1,0} \) or \( \pi_{k_1,q_1,2}. \) Similar as above the stabilizer of \( \pi_{k_1,q_1,0} \) or \( \pi_{k_1,q_1,2} \) is \( \mathbb{Z}_2, \) and by using Cor. 5.17 again we conclude that \( \pi_{k_1,q_1,0} \) or \( \pi_{k_1,q_1,2} \) decomposes into two distinct irreducible DHR representations of \( A_{H_8}^{SU(2)}(2k_1). \)

6 Constructions of solitons for permutation orbifolds

6.1 Preliminaries on cyclic orbifolds

In the rest of this paper we assume that \( A \) is completely rational. \( D := A \otimes A \cdots \otimes A \) (\( n \)-fold tensor product) and \( B := D_{\mathbb{P}_n} \) (resp. \( D^n \) where \( \mathbb{P}_n \) is the permutation group on \( n \) letters) is the fixed point subnet of \( D \) under the action of cyclic permutations (resp. permutations). Recall that \( J_0 = (0, \infty) \subset \mathbb{R}. \) Note that the action of \( \mathbb{Z}_n \) (resp. \( \mathbb{P}_n \)) on \( D \) is faithful and proper. Let \( v \in D(J_0) \) be a unitary such that \( \beta_g(v) = e^{2\pi i} v \) (such \( v \) exists by P. 48 of [14]) where \( g \) is the generator of the cyclic group \( \mathbb{Z}_n \) and \( \beta_g \) stands for the action of \( g \) on \( D \). Note that \( \sigma := \text{Ad}_v \) is a DHR representation of \( B \) localized on \( J_0. \) Let \( \gamma : D(J_0) \to B(J_0) \) be the canonical endomorphism from \( D(J_0) \) to \( B(J_0) \) and let \( \gamma_B := \gamma \mid B(J_0). \) Note \( [\gamma] = [1] + [g] + ... + [g^{n-1}] \) as sectors of \( D(J_0) \) and \( [\gamma_B] = [1] + [\sigma] + ... + [\sigma^{n-1}] \) as sectors of \( B(J_0). \) Here \( [g^i] \) denotes the sector of \( D(J_0) \) which is the automorphism induced by \( g^i. \) All the sectors considered in the rest of this paper will be sectors of \( D(J_0) \) or \( B(J_0) \) as should be clear from their definitions. All DHR representations will be assumed to be localized on \( J_0 \) and have finite statistical dimensions unless noted otherwise. For simplicity of notations, for a DHR representation \( \sigma_0 \) of \( D \) or \( B \) localized on \( J_0, \) we will use the same notation \( \sigma_0 \) to denote its restriction to \( D(J_0) \) or \( B(J_0) \) and we will make no distinction between local and global intertwiners for DHR representations localized on \( J_0 \) since they are the same by the strong additivity of \( D \) and \( B. \) The following is Lemma 8.3 of [33]:

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Lemma 6.1. Let $\mu$ be an irreducible DHR representation of $\mathcal{B}$. Let $i$ be any integer. Then:

1. $G(\mu, \sigma^i) := \varepsilon(\mu, \sigma^i)\varepsilon(\sigma^i, \mu) \in \mathbb{C}$, $G(\mu, \sigma)^i = G(\mu, \sigma^i)$. Moreover $G(\mu, \sigma)^n = 1$;
2. If $\mu_1 \prec \mu_2 \mu_3$ with $\mu_1, \mu_2, \mu_3$ irreducible, then $G(\mu_1, \sigma^i) = G(\mu_2, \sigma^i)G(\mu_3, \sigma^i)$;
3. $\mu$ is untwisted if and only if $G(\mu, \sigma) = 1$;
4. $G(\mu, \sigma^i) = G(\mu, \sigma^i)$.

6.2 One cycle case

First we recall the construction of solitons for permutation orbifolds in §6 of [33]. Let $h : S^1 \setminus \{-1\} \simeq \mathbb{R} \to S^1$ be a smooth, orientation-preserving, injective map which is smooth also at $\pm\infty$, namely the left and right limits $\lim_{z \to \pm\infty} \frac{dh}{dz}$ exist for all $n$.

The range $h(S^1 \setminus \{-1\})$ is either $S^1$ minus a point or a (proper) interval of $S^1$. With $I \in \mathcal{I}$, $-1 \notin I$, we set

$$\Phi_{h,I} \equiv \text{Ad}U(k),$$

where $k \in \text{Diff}(S^1)$ and $k(z) = h(z)$ for all $z \in I$ and $U$ is the projective unitary representation of $\text{Diff}(S^1)$ associated with $\mathcal{A}$. Then $\Phi_{h,I}$ does not depend on the choice of $k \in \text{Diff}(S^1)$ and

$$\Phi_h : I \mapsto \Phi_{h,I}$$

is a well-defined soliton of $\mathcal{A}_0 \equiv \mathcal{A} \upharpoonright \mathbb{R}$.

Clearly $\Phi_h(\mathcal{A}_0(\mathbb{R}))'' = \mathcal{A}(h(S^1 \setminus \{-1\}))''$, thus $\Phi_h$ is irreducible if the range of $h$ is dense, otherwise it is a type III factor representation. It is easy to see that, in the last case, $\Phi_h$ does not depend on $h$ up to unitary equivalence.

Let now $f : S^1 \to S^1$ be the degree $n$ map $f(z) \equiv z^n$. There are $n$ right inverses $h_i$, $i = 0, 1, \ldots, n-1$, for $f$ (n-roots); namely there are $n$ injective smooth maps $h_i : S^1 \setminus \{-1\} \to S^1$ such that $f(h_i(z)) = z$, $z \in S^1 \setminus \{-1\}$. The $h_i$’s are smooth also at $\pm\infty$.

Note that the ranges $h_i(S^1 \setminus \{-1\})$ are $n$ pairwise disjoint intervals of $S^1$, thus we may fix the labels of the $h_i$’s so that these intervals are counterclockwise ordered, namely we have $h_0(1) < h_1(1) < \cdots < h_{n-1}(1) < h_0(1)$, and we choose $h_j = e^{2\pi i/n}h_0$, $0 \leq j \leq n-1$.

For any interval $I$ of $\mathbb{R}$, we set

$$\pi_{1,\{0,1,\ldots,n-1\},I} \equiv \chi_I \cdot (\Phi_{h_0,I} \otimes \Phi_{h_1,I} \otimes \cdots \otimes \Phi_{h_{n-1},I}), \quad (18)$$

where $\chi_I$ is the natural isomorphism from $\mathcal{A}(I_0) \otimes \cdots \otimes \mathcal{A}(I_{n-1})$ to $\mathcal{A}(I_0) \vee \cdots \vee \mathcal{A}(I_{n-1})$ given by the split property, with $I_k \equiv h_k(I)$. Clearly $\pi_{1,\{0,1,\ldots,n-1\}}$ is a soliton of $\mathcal{D}_0 \equiv \mathcal{A}_0 \otimes \mathcal{A}_0 \otimes \cdots \otimes \mathcal{A}_0$ (n-fold tensor product). Let $p \in \mathbb{P}_n$. We set

$$\pi_{1,\{p(0),p(1),\ldots,p(n-1)\}} = \pi_{1,\{0,1,\ldots,n-1\}} \cdot \beta_p^{-1} \quad (19)$$

where $\beta$ is the natural action of $\mathbb{P}_n$ on $\mathcal{D}$, and $\pi_{1,\{0,1,\ldots,n-1\}}$ is as in (18). The following is part of Prop. 6.1 in [33]:

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Proposition 6.2. (1): $\text{Index}(\pi_1,\{0,1,...,n-1\}) = \mu_A^{n-1}$.

(2): The conjugate of $\pi_1,\{0,1,...,n-1\}$ is $\pi_1,\{0,n-1,n-2,...,1\}$.

Let $\lambda$ be a DHR representation of $\mathcal{A}$. Given an interval $I \subset S^1 \setminus \{-1\}$, we set

Definition 6.3.

$$\pi_{\lambda,}\{p(0),p(1),...,p(n-1)\},I(x) = \pi_{\lambda,I}(\pi_1,\{p(0),p(1),...,p(n-1)\},I(x)),$$ where $\pi_{\lambda,}\{p(0),p(1),...,p(n-1)\},I$ is defined as in (19), and $J$ is any interval which contains $I_0 \cup I_1 \cup ... \cup I_{n-1}$. Denote the corresponding soliton by $\pi_{\lambda,}\{p(0),p(1),...,p(n-1)\}$.

When $p$ is the identity element in $\mathbb{P}_n$, we will denote the corresponding soliton by $\pi_{\lambda,n}$.

The following follows from Prop. 6.4 of [33]:

Proposition 6.4. The above definition is independent of the choice of $J$, thus $\pi_{\lambda,}\{p(0),p(1),...,p(n-1)\},I$ is a well defined soliton of $\mathcal{D}$.

We can localize $\pi_{\lambda,}\{p(0),p(1),...,p(n-1)\}$, $\pi_{\lambda,}\{p(0),p(1),...,p(n-1)\}$ and $\lambda$ on $J_0$. Denote by $\tilde{\pi}, \tilde{\pi}_\lambda$ and $(\lambda, 1, 1, ..., 1) := \lambda \otimes \iota \otimes \iota \cdot \otimes \iota \uparrow \mathcal{D}(J_0)$ respectively the corresponding endomorphisms of $\mathcal{D}(I)$. Then as sectors of $\mathcal{D}(J_0)$ we have

$$[\tilde{\pi}_\lambda] = [\tilde{\pi} \cdot (\lambda, 1, 1, ..., 1)].$$

In particular $\text{Index}(\pi_{\lambda,}\{p(0),p(1),...,p(n-1)\}) = d(\lambda)^2 \mu_A^{n-1}$.

6.3 General case

Let $\psi : \{0,1,...,n-1\} \rightarrow \mathcal{L}$ where $\mathcal{L}$ is the set of all irreducible DHR representations of $\mathcal{D}$. For any $p \in \mathbb{P}_n$ we set $p.\psi(i) := \psi(p^{-1}.i), i = 0,...,n-1$ where $\mathbb{P}_n$ acts via permutation on the $n$ numbers $\{0,1,...,n-1\}$. Assume that $p.\psi = \psi$, and $p = c_1...c_k$ is a product of disjoint cycles. Since $p.\psi = \psi$, $\psi$ takes the same value denoted by $\psi(c_j)$ on the elements $\{a_1,a_2,...a_l\}$ of each cycle $c_j = (a_1...a_l)$. A presentation $f_j$ of the cycle $c_j = (a_1...a_l)$ is a list of numbers $\{b_1,...,b_l\}$ such that $(b_1...b_l) = c_j$ as cycles. The length $l(f_j)$ of $f_j$ is $l$. We note that for a cycle of length $l$ there are $l$ different presentations. For each element $x = x_0 \otimes x_1 \otimes \cdots \otimes x_{n-1} \in \mathcal{D}$, and each cycle $c = (a_1...a_l)$ with a fixed presentation $f = \{b_1,...b_l\}$, we define $x_{c,f} = x_{b_1} \otimes x_{b_2} \otimes \cdots \otimes x_{b_l}$. Now we are ready to define solitons for permutation orbifolds:

Definition 6.5. Assume that $p.\psi = \psi$ and $p = c_1...c_k$ is a product of disjoint cycles as above. For each $c_j$ we fix a presentation $f_j$. Then for any $x = x_0 \otimes x_1 \otimes \cdots \otimes x_{n-1} \in \mathcal{D}(I), I \subset S^1 \setminus \{-1\} = \mathbb{R}$,

$$\pi_{\psi,p} \equiv \pi_{\psi,c_1c_2...c_kf_1...f_k}(x) = \pi_{\psi}(c_1,c_1,f_1) \otimes \pi_{\psi}(c_2,c_2,f_2) \otimes \cdots \otimes \pi_{\psi}(c_k,c_k,f_k)$$

on $\mathcal{H}_{\psi(c_1)} \otimes \mathcal{H}_{\psi(c_2)} \otimes \cdots \otimes \mathcal{H}_{\psi(c_k)}$ where $\pi_{\psi}(c_j, f_j)$ is as in Def. 6.3.

Here and in the following, to simplify notations, we do not put the interval suffix $I$ in a representation, if no confusion arises.
Lemma 6.6. The unitary equivalence class of $\pi_{\psi,p}$ in Definition 6.5 depends only on $p \in \mathbb{P}_n$.

Proof. We have to check that the unitary equivalence class of $\pi_{\psi,p}$ in Definition 6.5 is independent of the order $c_1, \ldots, c_k$ and the presentation of $c_j$. The first case is obvious, and second case follows from (a) of Prop. 6.2 in [33].

Due to the above lemma, for each $p \in \mathbb{P}_n$ we will fix a choice of the order $c_1, \ldots, c_k$ and presentations of $c_1, \ldots, c_k$. For simplicity we will denote the corresponding soliton simply by $\pi_{\psi,p}$.

Proposition 6.7. $\pi_{h,\psi,hph^{-1}} \simeq \pi_{\psi,p} \cdot \beta_{h^{-1}}$ as solitons of $\mathcal{D}_0$, $p, h \in \mathbb{P}_n$.

Proof. Let $p = c_1 \ldots c_k$ be a product of disjoint cycles with $c_j = (a_1 \ldots a_l)$. Then $hph^{-1} = hc_1h^{-1} \ldots hc_kh^{-1}$ with $hc_jh^{-1} = (h(a_1) \ldots h(a_l))$. Note that $h.\psi(h(a_1)) = \psi(a_1) = \psi(c_j)$, and $\beta_{h^{-1}}(x_0 \otimes x_1 \otimes \ldots \otimes x_{n-1}) = x_{h(0)} \otimes x_{h(1)} \otimes \ldots \otimes x_{h(n-1)}$, $\forall x_0 \otimes x_1 \otimes \ldots \otimes x_{n-1} \in \mathcal{D}(I)$. The proposition now follows directly from definition (6.5).

7 Identifying solitons in the permutation orbifolds

The goal in this section is to prove the following:

Theorem 7.1. Let $\pi_{\psi_1,p_1}$, $\pi_{\psi_2,p_2}$ be two solitons as given in definition (6.5). Then $\pi_{\psi_1,p_1} \simeq \pi_{\psi_2,p_2}$ as solitons of $\mathcal{D}_0$ if and only if $\psi_1 = \psi_2, p_1 = p_2$.

We note that even for the first nontrivial case $n = 3$ we do not know a direct proof of the theorem. Our proof is indirect and is divided into the following steps:

7.1 Identifying solitons: Cyclic case

We will first prove Th. 7.1 for the case when both $p_1, p_2$ are one cycle. In this case $\psi_1$ (resp. $\psi_2$) is a constant function with value denoted by $\lambda_1$ (resp. $\lambda_2$). We will denote $\psi_1$ (resp. $\psi_2$) simply by $\lambda_1$ (resp. $\lambda_2$). If $g \in \Gamma$, we will denote by $\mathcal{D}(g)$ the fixed-point subnet of $\mathcal{D}$ under the subgroup generated by $g$.

Proposition 7.2. (1) Let $g_1 = (01, \ldots, n-1)$ and $g_2 = g_1^m$ with $(m, n) = 1$. Then $\pi_{\lambda_1,g_1} \simeq \pi_{\lambda_1,g_2}$ if and only if $\lambda_1 = \lambda_2, g_1 = g_2$;

(2) If $\pi_{\lambda_1,g_1}$ restricts to a DHR representation a subnet $\mathcal{B}$ with $\mathcal{D}(g_1) \subset \mathcal{B} \subset \mathcal{D}$, then $\mathcal{B} = \mathcal{D}(g_1)$.

Proof. Ad (1): It is sufficient to show that if $\pi_{\lambda_1,g_1} \simeq \pi_{\lambda_1,g_2}$, then $\lambda_1 = \lambda_2, g_1 = g_2$.

Since $(m, n) = 1$, there exists $h \in \mathbb{P}_n$ such that $hg_1h^{-1} = g_2$. By Prop. 6.7, we can assume that $\pi_{\lambda_1,g_1} \simeq \pi_{\lambda_2,g_1} \cdot \text{Ad}_h$. As in §8.3 of [33], we denote the $n$ irreducible DHR representations of $\mathcal{D}(g_1)$ of $\pi_{\lambda_1,g_1}$ by $\tau_{\lambda_1}^{(0)}, \ldots, \tau_{\lambda_1}^{(n-1)}$. Since $\pi_{\lambda_1,g_1} \simeq \pi_{\lambda_2,g_1} \cdot \text{Ad}_h$, we must have $\tau_{\lambda_1}^{(0)} \simeq \tau_{\lambda_2}^{(i)} \cdot \text{Ad}_h$ for some $0 \leq i \leq n-1$. By (48) of [33] we have that...
In one cycle of \( g \) \( \langle \lambda_1, 1, \ldots, 1 \rangle \) \( \mathcal{D}(g_1) \tau^{(0)} \), and (2) and (3) of Lemma 6.1 we have \( G(\tau^{(0)}_{\lambda_1}, \sigma^{k(1)}) = G(\tau^{(0)}_1, \sigma^{k(1)}) = e^{\frac{2\pi i}{\ell}} \), where \( 1 \leq k(1) \leq n - 1 \) and \( (k(1), n) = 1 \) (cf. the paragraph after (47)). Similarly \( G(\tau^{(j)}_{\lambda_2}, \sigma^{k(1)}) = e^{\frac{2\pi i}{\ell}} \). On the other hand note that by definition

\[
G(\tau^{(j)}_{\lambda_2} \cdot \text{Ad}_h, \sigma^{k(1)} \cdot \text{Ad}_h) = G(\tau^{(j)}_{\lambda_2}, \sigma^{k(1)}) = e^{\frac{2\pi i}{\ell}}
\]

Since \( [\text{Ad}_h : g] = [g^m] \), we have \( \sigma \cdot \text{Ad}_h \simeq \sigma^m \), and so we have

\[
G(\tau^{(j)}_{\lambda_2} \cdot \text{Ad}_h, \sigma^{k(1)} \cdot \text{Ad}_h) = G(\tau^{(j)}_{\lambda_2}, \sigma^{k(1)} \cdot \text{Ad}_h) = G(\tau^{(0)}_{\lambda_1}, \sigma^{k(1)})^m = e^{\frac{2\pi i}{\ell}}
\]

where in the second = we have used (1) of Lemma 6.1. Hence \( e^{\frac{2\pi i}{\ell}} = e^{\frac{2\pi i}{\ell m}} \) and it follows that \( m = 1 \) since \( (m, n) = 1 \). So we have \( \pi_{\lambda_1, g_1} \simeq \pi_{\lambda_2, g_1} \), and by (2) of Th. 8.6 in [33] we have \( \lambda_1 = \lambda_2 \).

Ad (2): First we note that the subnet \( \mathcal{B} = \mathcal{D}(g_1) \) for some \( 1 \leq l \leq n, n = l l_1 \) by the Galois correspondence (cf. [14]). Also the vacuum representation of \( \mathcal{D}(g_1) \) restricts to \( \bigoplus_{1 \leq i \leq l} \mathcal{D}(g_i) \). If \( \pi_{\lambda_1, g_1} \) restricts to a DHR representation of \( \mathcal{D}(g_i) \), by applying (3) of Lemma 6.1 to the pair \( \mathcal{D}(g_1) \subset \mathcal{D}(g_i) \) we conclude that \( G(\tau^{(0)}_{\lambda_1}, \sigma^{l_1}) = 1 \). Since \( G(\tau^{(0)}_{\lambda_1}, \sigma^{l_1}) = e^{\frac{2\pi i}{\ell}} \), by using (1) of Lemma 6.1 we have

\[
G(\tau^{(0)}_{\lambda_1}, \sigma^{l_1} \cdot \sigma^{k(1)}) = G(\tau^{(0)}_{\lambda_1}, \sigma^{l_1})^{k(1)} = 1 = G(\tau^{(0)}_{\lambda_1}, \sigma^{k(1)})^l = e^{\frac{2\pi i}{\ell}}
\]

Hence \( n | l_1 \) and we conclude that \( l_1 = n, \mathcal{B} = \mathcal{D}(g_1) \).

**Proposition 7.3.** Let \( g_1 \) (resp. \( g_2 \)) be one cycle of length \( n \). Then \( \pi_{\lambda_1, g_1} \simeq \pi_{\lambda_2, g_2} \) if and only if \( \lambda_1 = \lambda_2, g_1 = g_2 \).

**Proof** It is sufficient to show that if \( \pi_{\lambda_1, g_1} \simeq \pi_{\lambda_2, g_2} \), then \( \lambda_1 = \lambda_2, g_1 = g_2 \).

Note that \( \pi_{\lambda_1, g_1} \) (resp. \( \pi_{\lambda_2, g_2} \)) restricts to a DHR representation of \( \mathcal{D}(g_1) \) (resp. \( \mathcal{D}(g_2) \)), it follows that \( \pi_{\lambda_1, g_1} \) restricts to a DHR representation of \( \mathcal{D}(g_2) \). By Prop.3.8 \( \pi_{\lambda_1, g_1} \) restricts to a DHR representation of \( \mathcal{D}(g_1) \lor \mathcal{D}(g_2) \), and by (2) of 7.2 we must have \( \mathcal{D}(g_1) \lor \mathcal{D}(g_2) = \mathcal{D}(g_1) \). It follows that \( \mathcal{D}(g_2) \subset \mathcal{D}(g_1) \) and by Galois correspondence again (cf. [14]) we have \( \langle g_2 \rangle \subset \langle g_1 \rangle \). Exchanging \( g_1 \) and \( g_2 \) we conclude that \( \langle g_2 \rangle = \langle g_1 \rangle \). Hence \( g_2 = g_1^m \) for some integer \( m \) with \( (m, n) = 1 \). By (1) of Prop.7.2 we have proved that \( g_1 = g_2, \lambda_1 = \lambda_2 \).

7.2 Proof of Th.7.1 for general case and its corollary

Assume that \( g_1 = c_1 c_2 \ldots c_k \) and \( g_2 = c_1' \ldots c_k' \) where \( c_j \) (resp. \( c_j' \)) are disjoint cycles. Fix \( 1 \leq j \leq k \) and let \( c_j = (a_1 \ldots a_m) \). Let us first show that \( a_1, \ldots, a_m \) must appear in one cycle of \( g_2 \). Let \( U \) be the unitary such that \( \pi_{\psi_1, g_1} = \text{Ad}_U \cdot \pi_{\psi_2, g_2} \). Choose \( x = x_0 \otimes x_1 \otimes \cdots \otimes x_n \in \mathcal{D}_0 \) such that \( x_{j} = 1 \) if \( i \neq a_j, j = 1, \ldots, m \), and no other constraints. Denote by \( \mathcal{D}_{0,c_j} \) the subalgebra of \( \mathcal{D}_0 \) generated by such elements. We note that \( \pi_{\psi_1, g_1}(\mathcal{D}_{0,c_j}) \) is a type I factor by strong additivity. If \( a_1, \ldots, a_m \) appear in more than one cycle of \( g_2 \), then by definition (18) \( \pi_{\psi_2, g_2}(\mathcal{D}_{0,c_j}) \) will be
tensor products of factors of the form \( \pi_\lambda(A_J) \), where \( J \) is a union of intervals of \( S^1 \), but \( \bar{J} \neq S^1 \), and so \( \pi_{\psi_1,p_1}(D_{0,c_i}) \) will be tensor products of type III factors, contradicting \( \pi_{\psi_1,p_1}(D_{0,c_i}) = U \pi_{\psi_2,p_2}(D_{0,c_i})U^* \). By exchanging the role of \( g_1 \) and \( g_2 \) we conclude that \( a_1, \ldots, a_m \) must be exactly the elements in one cycle \( c'_i \) of \( g_2 \) for some \( 1 \leq i \leq l \), and we have \( \pi_{\psi_1,p_1}(D_{0,c_i}) = U \pi_{\psi_2,p_2}(D_{0,c_i})U^* \). Let \( \mathcal{H} = \mathcal{H}_{\psi_1(c_i)} \otimes \mathcal{H}_p \). We have \( UB(\mathcal{H}_{\psi_1(c_i)})U^* = U_1B(\mathcal{H}_{\psi_2(c'_i)})U_1^* \). Hence \( \pi_{\psi_1(c_i),l(c_i)} = U_1\pi_{\psi_2(c'_i),d(c_i)}U_1^* \) on \( D_{0,c_i} \), and by Prop.7.3 we conclude that \( c_j = c'_i, \psi_1(c_j) = \psi_2(c'_i) \). Since \( j \) is arbitrary, exchanging the roles of \( g_1 \) and \( g_2 \) we have proved \( g_1 = g_2, \psi_1 = \psi_2 \).

**Proposition 7.4.** Assume that \( p = c_1 \ldots c_k \) where \( c_i \) are disjoint cycles. Let \( \psi \) be such that \( p\psi = \psi \). Then:

1. The centralizer (cf. (4.2)) of \( \pi_{\psi,p} \) in \( \Pi_n \) is \( \Gamma_{\psi} = \{ h \in \Pi_n | h.\psi = \psi, hph^{-1} = p \} \);
2. If \( p \in \mathbb{Z}n \), the centralizer (cf. (4.2)) of \( \pi_{\psi,p} \) in \( \mathbb{Z}n \) is \( \Gamma_{\psi,p} = \{ h \in \mathbb{Z}n | h.\psi = \psi, hph^{-1} = p \} \);
3. \( d(\pi_{\psi,p})^2 = \prod_{1 \leq i \leq k} d(\psi(c_i))^2 \mu_A^{-1} \).

**Proof** (1) (2) follows from Prop. 6.7 and Th. 7.1. Assume that each cycle \( c_i \) has length \( m_i, 1 \leq i \leq k \). Then \( \sum_{1 \leq i \leq k} m_i = n \). By definition (6.5), we have \( d(\pi_{\psi,p})^2 = \prod_{1 \leq i \leq k} d(\pi_{\psi(c_i)})^2 \). By Prop.6.4 and (1) of Prop. 6.2 \( d(\pi_{\psi(c_i)})^2 = d(\psi(c_i))^2 \mu_A^{-1} \), hence

\[
d(\pi_{\psi,p})^2 = \prod_{1 \leq i \leq k} d(\psi(c_i))^2 \mu_A^{-1} = \prod_{1 \leq i \leq k} d(\psi(c_i))^2 \mu_A^{-1}.
\]

**8 Identifying all the irreducible representations of the permutation orbifolds**

**8.1 Cyclic orbifold case**

**Theorem 8.1.** Let \( g = (01 \ldots n - 1) \). Then every irreducible DHR representation of \( \mathcal{D}^{\mathbb{Z}n} \) appears as an irreducible summand of \( \pi_{\psi,g} \) for some \( \psi, g' \).

**Proof** By Prop. 4.4 \( \pi_{\psi_1,g_1} \uparrow \mathcal{B} \simeq \pi_{\psi_2,g_2} \uparrow \mathcal{B} \) iff there exists \( h \in \mathbb{Z}n \) such that \( \pi_{\psi_1,g_1} = \pi_{\psi_2,g_2} \), and by Prop. 6.7 and (2) of Prop. 7.4 we have \( h.\psi_1 = \psi_2, h.g_1^{-1} = g_2^{-1} \). Denote the orbit of \( \pi_{\psi_1,g_1} \) under the action of \( \mathbb{Z}n \) by \( \{ \psi_1, g_1 \} \). Note that the orbit \( \{ \psi_1, g_1^{-1} \} \) has length \( \frac{n^2}{|\psi_1,g_1'|} \). By Th. 4.5 the sum of index of the irreducible summands of \( \pi_{\lambda,g} \) is \( \frac{n^2}{|\psi,g'|} d(\pi_{\lambda,g})^2 \). Hence the sum of index of distinct irreducible summands of \( \pi_{\lambda,g} \) for all \( \psi, g \in \mathbb{Z}n \) is given by \( \sum_{\{\psi,g\}} \frac{n^2}{|\psi,g'|} d(\pi_{\lambda,g})^2 \) where the sum is over different orbits. Assume that \( g^i = c_1 \ldots c_k \). Then \( k = (n, i) \) (the greatest common divisor of \( n \) and \( i \)) and each cycle \( c_i \) has length \( \frac{n}{(n,i)} \). For each
element $\psi_2, g^{i_2}$ in the orbit $\{\psi, g^i\}$, by (3) of Prop. 7.4 $d(\pi_{\psi_2, g^{i_2}})^2 = d(\pi_{\lambda, g^i})^2 = \prod_{1 \leq j \leq (n,i)} d(\psi(c_j))^2 n^{-\frac{n}{(n,i)}}$ Hence

$$\sum_{\psi, g^i} \frac{n^2}{|\Gamma_{\psi, g^i}|} d(\pi_{\psi, g^i})^2 = \sum_{\lambda, 0 \leq i \leq n} \frac{1}{|\Gamma_{\psi, g^i}|} \frac{n^2}{|\Gamma_{\psi, g^i}|} d(\pi_{\psi, g^i})^2$$

$$= n \sum_{\psi, 0 \leq i \leq n} \prod_{1 \leq j \leq (n,i)} d(\psi(c_j))^2 \mu_A^{n^{-\frac{n}{(n,i)}}} = n^2 \mu_A \mu_D = (21)$$

where in the last = we have used Th. 3.7. The theorem now follows from Th. 30 of [21].

Let us now decompose $\pi_{\lambda, g}$ into irreducible pieces. In this case $\Gamma_{\lambda, g} = \mathbb{Z}_n$ since $g = (012...n-1)$ (cf. (2) of Prop. 7.4). By definition (18) $\forall \lambda, \psi \in \mathbb{H}, \forall x_0 \otimes x_1 \cdots \otimes x_{n-1} \in \mathcal{D}(H)$,

$$\pi_{\lambda, g} \cdot \text{Ad}_{g^{-1}}(x_0 \otimes x_1 \otimes \cdots \otimes x_{n-1}) = \pi_{\lambda, g}(x_1 \otimes x_2 \otimes \cdots \otimes x_0)$$

$$= \pi_{\lambda}(R(\frac{2\pi}{n}) \pi_{\lambda, g}(x_0 \otimes x_1 \otimes \cdots \otimes x_{n-1}) \pi_{\lambda}(R(\frac{2\pi}{n}))^*)$$

Here $\pi_{\lambda}(R(\cdot))$ denotes the unitary one-parameter rotation subgroup in the representation $\lambda$. Note that $\pi_{\lambda}(R(\frac{2\pi}{n}))^n = \pi_{\lambda}(R(2\pi)) = C_{\lambda} \text{id}$ for some complex number $C_{\lambda}, |C_{\lambda}| = 1$. Let $\Omega_{\lambda} \in \mathcal{H}_{\lambda}$ be a unit vector such that $\pi_{\lambda}(R(\frac{2\pi}{n})) \Omega_{\lambda} = C_{\lambda} \Omega_{\lambda}$ with $(C_{\lambda})^n = C_{\lambda}$.

**Definition 8.2.** $\pi_{\lambda, g}(g) := C_{\lambda}^{-1} \pi_{\lambda}(R(\frac{2\pi}{n}))$, and $\pi_{\lambda, g}(g^i) := \pi_{\lambda}(g)^i$.

Then it follows that $g^i \rightarrow \pi_{\lambda, g}(g^i)$ gives a representation of $\mathbb{Z}_n$ on $\mathcal{H}_{\lambda}$, and $\pi_{\lambda, g}(g^i) \Omega_{\lambda} = \Omega_{\lambda}$. So $\Omega_{\lambda}$ affords a trivial representation of $\mathbb{Z}_n$ on $\mathcal{H}_{\lambda}$. It follows from Lemma 4.7 that all irreducible representations of $\mathbb{Z}_n$ appear in the representation $\pi_{\lambda}$. It follows by Th. 4.8 that $\pi_{\lambda, g^i}, i \in \mathbb{Z}_n$ are distinct irreducible representations.

Note that $g^i = c_1 \cdots c_k$ is a product of $k = (n, i)$ disjoint cycles of the same length $\frac{n}{(n,i)}$. Let $h \in \Gamma_{\psi, g^i}$. Then $\text{Ad}_h$ induces a permutation among the cycles $c_1, \ldots, c_k$. We define an element $h' \in \mathbb{P}_k$ by the formula $h c_i h^{-1} = c_{h(i)}, i = 1, \ldots, k$. We note that in the definition of $\pi_{\psi, g}$ a presentation of $g$ has been fixed. Assume that $h f_{h^{-1}(i)}^{-1} h = h''(i). f_i$ where $h''(i)$ is an element in the cyclic group generated by $c_i$. Define

**Definition 8.3.**

$$\pi_{\psi, g^i}(h) := h' \left( \pi_{\psi(c_1), c_1}(h''(1)) \otimes \cdots \otimes \pi_{\psi(c_k), c_k}(h''(k)) \right)$$

where the action of $h' \in \mathbb{P}_k$ on $\mathcal{H}_{\psi(c_1)} \otimes \cdots \otimes \mathcal{H}_{\psi(c_k)}$ is by permutation of the tensor factors, and $\pi_{\psi(c_i), c_i}(h''(i))$ is as defined in definition (8.2).

One checks easily that Definition 8.3 gives a representation of $\Gamma_{\psi, g^i}$, $\text{Ad}_{\pi_{\psi, g^i}(h)} \pi_{\psi, g^i} = \pi_{\psi, g^i} \text{Ad}_h$, and the vector $\Omega_{\psi(c_1)} \otimes \cdots \otimes \Omega_{\psi(c_k)}$ is fixed by $\pi_{\psi, g^i}(\Gamma_{\psi, g^i})$. It follows by lemma 4.7 and Th. 4.8 that we have proved the following:

**Theorem 8.4.** $\pi_{\psi, g^i, \sigma} \in \Gamma_{\psi, g^i}$ gives all the irreducible summands of $\pi_{\psi, g^i} \mid \mathcal{D}^{\mathbb{Z}_n}$.

We note that Th. 8.1 and Th. 8.4 generalizes the considerations of §8 of [33] for the case $n = 2, 3, 4.$

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8.2 Permutation orbifold case

Theorem 8.5. Every irreducible DHR representation of $\mathcal{D}^E_n$ appears as an irreducible summand of $\pi_{\psi,p}$ for some $\psi,p \in \mathbb{P}_n$.

Proof The proof is similar to the proof of Th. 8.1 with small modifications. By Prop. 4.4 and Th. 7.1 $\pi_{\psi_1,p_1} \triangleright \mathcal{D}^E_n \simeq \pi_{\psi_2,p_2} \triangleright \mathcal{D}^E_n$ iff there exists $h \in \mathbb{P}_n$ such that $h.\psi_1 = \psi_2, h.p_1.h^{-1} = p_2$. Denote the orbit of $\pi_{\psi_1,p_1}$ under the action of $\mathbb{P}_n$ by $\{\psi_1,p_1\}$. Note that the orbit $\{\psi_1,p_1\}$ has length $\frac{n!}{|\Gamma_{\psi,p}|}$ By Prop. 4.5 the sum of index of the irreducible summands of $\pi_{\psi,p}$ is $\frac{n!^2}{|\Gamma_{\psi,p}|} \pi_{\psi,p}$ The sum is over different orbits. Assume that $\psi,p$ is a product of disjoint cycles. For each element $\psi_2,p_2$ in the orbit $\{\psi,p\}$, by Prop. 7.4 $d(\pi_{\psi_2,p_2})^2 = d(\pi_{\psi,p})^2 = \prod_{1 \leq j \leq k} d(\psi_j)^2 \mu_A^{n-k}$ Hence

$$\sum_{\{\psi,p\}} \frac{n!^2}{|\Gamma_{\psi,p}|} d(\pi_{\psi,p})^2 = n! \prod_{1 \leq j \leq k} d(\psi_j)^2 \mu_A^{n-k} = n!^2 \mu_A = \mu_{\mathcal{D}^E_n}$$

where in the last $= \mu_{\mathcal{D}^E_n}$ we have used Th. 3.7. The theorem now follows from Th. 30 of [21].

Let $p = c_1...c_k$ be a product of $k$ disjoint cycles. Let $h \in \Gamma_{\psi,p}$. Then $\text{Ad}_h$ induces a permutation among the cycles $c_1,...,c_k$. We define an element $h' \in \mathbb{P}_k$ by the formula $hc_ih^{-1} = c_{h'(i)}$, $i = 1,...,k$. We note that in the definition of $\pi_{\psi,g}$ a presentation of $g$ has been fixed. Assume that $h_f h^{-1} = h''(i) f_i$ where $h''(i)$ is an element in the cyclic group generated by $c_i$. Define

Definition 8.6.

$$\pi_{\psi,p}(h) := h' \pi_{\psi(c_1),c_1}(h''(1)) \otimes ... \otimes \pi_{\psi(c_k),c_k}(h''(k))$$

where the action of $h' \in \mathbb{P}_k$ on $\mathcal{H}_{\psi(c_1)} \otimes ... \otimes \mathcal{H}_{\psi(c_k)}$ is by permutation of the tensor factors, and $\pi_{\psi(c_i),c_i}(h''(i))$ is as defined in definition (8.2).

One checks easily that Definition 8.6 gives a representation of $\Gamma_{\psi,p}$, $\text{Ad}_{\psi_{\psi,p}} = \pi_{\psi,p} \text{Ad}_h$, and the vector $\Omega_{\psi(c_1)} \otimes ... \otimes \Omega_{\psi(c_k)}$ is fixed by $\pi_{\psi,p}(\Gamma_{\psi,p})$. It follows by Lemma 4.7, Th. 4.8 that $\pi_{\psi,p} \sigma \in \Gamma_{\psi,p}$ gives all the irreducible summands of $\pi_{\psi,p} \triangleright \mathcal{D}^E_n$, and we have proved:

Theorem 8.7. $\pi_{\psi,p} \sigma \in \Gamma_{\psi,p}$ gives all the irreducible summands of $\pi_{\psi,p} \triangleright \mathcal{D}^E_n$.

Note that by Prop. 8.7 and Th. 8.5 the irreducible DHR representations of $\mathcal{D}^E_n$ are labeled by triples $(\psi,p,\sigma)$ with $p.\psi = \psi, \sigma \in \Gamma_{\psi,p}$ with equivalence relation $\sim$, $(\psi,p,\sigma) \sim (\psi_1,p_1,\sigma_1)$ iff there is $h \in \mathbb{P}_n$ such that $\psi_1 = h.\psi, p_1 = hph^{-1}, \sigma_1 = \sigma^h$. In [1], based on heuristic argument it is claimed that the irreducible representations of $\mathcal{D}^E_n$ should be given by the set of pairs $(\psi,\varphi)$ where $\varphi$ is an irreducible representation
of the double \( \mathcal{D}(F_\psi) \) of the stabilizer \( F_\psi = \{ p \in \mathbb{P}_n | p \cdot \psi = \psi \} \) with equivalence relation 
\((\psi, \varphi) \sim (\psi_1, \varphi_1)\) iff there is \( h \in \mathbb{P}_n \) such that \( \psi_1 = h \cdot \psi, \varphi_1 = \varphi^h \). We note that the irreducible representation of the double \( \mathcal{D}(F_\psi) \) are labeled by \((g, \pi) / F_\psi\), where \( g \in F_\psi \), \( \pi \) is an irreducible representation of the centralizer of \( g \) in \( F_\psi \), and the action of \( F_\psi \) on \((g, \pi)\) is given by \( h \cdot (g, \pi) = (hgh^{-1}, \pi^h) \). Hence the labels [1] are exactly the same as the labels we described above, and we have confirmed this claim of [1].

9 Examples of fusion rules

9.1 Some properties of S matrix for general orbifolds

Let \( \mathcal{A} \) be a completely rational conformal net and let \( \Gamma \) be a finite group acting properly on \( \mathcal{A} \). By Th. 3.7 \( \mathcal{A}^\Gamma \) has only finitely many irreducible representations. We use \( \hat{\lambda} \) (resp. \( \mu \)) to label representations of \( \mathcal{A}^\Gamma \) (resp. \( \mathcal{A} \)). We will denote the corresponding genus 0 modular matrices by \( \hat{S}, \hat{T} \) (cf. (7)). Denote by \( \lambda \) (resp. \( \mu \)) the irreducible covariant representations of \( \mathcal{A}^\Gamma \) (resp. \( \mathcal{A} \)) with finite index. Recall that \( b_{\mu \lambda} \in \mathbb{Z} \) denote the multiplicity of representation \( \lambda \) which appears in the restriction of representation \( \mu \) when restricting from \( \mathcal{A} \) to \( \mathcal{A}^\Gamma \). \( b_{\mu \lambda} \) is also known as the branching rules.

Lemma 9.1. (1) If \( \tau \) is an automorphism (i.e., \( d(\tau) = 1 \)) then \( S_{\tau(\lambda)\mu} = G(\tau, \mu)^* S_{\lambda \mu} \) where \( \tau(\lambda) := \tau \cdot \lambda, G(\tau, \mu) = \varepsilon(\tau, \mu) \varepsilon(\mu, \tau) \);

(2) For any \( h \in \Gamma \), let \( h(\lambda) \) be the DHR representation \( \lambda \cdot \text{Ad}_{h^{-1}} \). Then \( S_{\lambda \mu} = S_{h(\lambda)h(\mu)} \);

(3) If \( \lambda \rightarrow z(\lambda) \frac{S_{\lambda \mu}}{S_{1 \mu}} \) gives a representation of the fusion algebra of \( \mathcal{A} \) where \( z(\lambda) \) is a complex-valued function, \( z(1) = 1 \), then there exists an automorphism \( \tau \) such that \( z(\lambda) = S_{\lambda \bar{\lambda}} \frac{S_{\bar{\lambda} \mu}}{S_{1 \mu}} \);

(4) If \([\alpha_\lambda] = [\mu \alpha_\delta]\), then for any \( \lambda_1, \mu_1 \) with \( b_{\lambda_1 \mu_1} \neq 0 \) we have \( \frac{S_{\lambda_1 \lambda_2}}{S_{1 \lambda_2}} = \frac{S_{\mu_1 \mu_2}}{S_{1 \mu_2}} \frac{S_{\lambda_1 \lambda_2}}{S_{1 \lambda_2}} \).

Proof

Ad (1): Since \( \lambda \rightarrow \frac{S_{\lambda \mu}}{S_{1 \mu}} \) is a representation of the fusion algebras, it follows that

\[
\frac{S_{\tau(\lambda)\mu}}{S_{1 \mu}} = \frac{S_{\lambda \mu} S_{\tau \mu}}{S_{1 \mu} S_{1 \mu}}
\]

On the other hand

\[
\frac{S_{\tau \mu}}{S_{1 \mu}} = \frac{\omega_{\tau \mu}}{\omega_{\tau \mu}} = G(\tau, \mu)^*
\]

where the last equation follows from the monodromy equation (cf.[37]) and (1) is proved.

Ad (2) By Lemma 3.1, it is sufficient to show that \( N_{h(\lambda)h(\mu)}^{h(\bar{\delta})} = N_{\lambda \mu}^{\delta} \) and \( \omega_{h(\lambda)} = \omega_{\lambda} \). The first equation follows from the definition. For the second one, we note that \( \omega_{\lambda} = \pi_\lambda(R(2\pi)) \). Since \( h \) commutes with the vacuum unitary representation of \( \text{M"{o}b} \), it follows that \( \omega_{h(\lambda)} = \omega_{\lambda} \).
Ad (3): By assumption \( \lambda \rightarrow z(\lambda) \frac{S_{\lambda\mu}}{S_{1\mu}} \) is a non-trivial representation of the fusion algebra, and so there exists \( \tau \) such that \( z(\lambda) \frac{S_{\lambda\mu}}{S_{1\mu}} = \frac{S_{\lambda\mu}}{S_{1\mu}}, \forall \lambda \). Hence \( |z(\lambda)| \leq 1 \). From

\[
(23) \quad z(\lambda_1) \frac{S_{\lambda_1\mu}}{S_{1\mu}} z(\lambda_2) \frac{S_{\lambda_2\mu}}{S_{1\mu}} = \sum_{\lambda_3} N^{\lambda_3}_{\lambda_1\lambda_2} z(\lambda_3) \frac{S_{\lambda_3\mu}}{S_{1\mu}} \\
= \sum_{\lambda_3} N^{\lambda_3}_{\lambda_1\lambda_2} z(\lambda_3) \frac{S_{\lambda_3\mu}}{S_{1\mu}}
\]

we have

\[
\sum_{\lambda_3} N^{\lambda_3}_{\lambda_1\lambda_2} (z(\lambda_1)z(\lambda_2) - z(\lambda_3)) \frac{S_{\lambda_3\mu}}{S_{1\mu}} = 0.
\]

Using \( N^{\lambda_3}_{\lambda_1\lambda_2} = \sum_{\delta} \frac{S_{\lambda_1\delta} S_{\lambda_2\delta} S_{\lambda_3\delta}}{S_{1\delta}} \) and the orthogonal property of \( S \) matrix in Lemma 3.3 we have

\[
N^{\lambda_3}_{\lambda_1\lambda_2} (z(\lambda_1)z(\lambda_2) - z(\lambda_3)) = 0.
\]

Since \( N^{1}_{\lambda_1\lambda_1} = 1 \) we have \( z(\lambda_1)z(\lambda_1) = 1 \). So we conclude that \( |z(\lambda)| = 1, \forall \lambda, \) and

\[
\left| \frac{1}{S_{1\tau}} \right|^2 \sum_{\lambda} |S_{\lambda\tau}|^2 = \sum_{\lambda} \left| \frac{S_{\lambda\tau}}{S_{1\tau}} \right|^2 = \left| \frac{1}{S_{1\tau}} \right|^2
\]

Hence \( S_{1\tau} = S_{11} \) and \( d(\tau) = 1 \), i.e., \( \tau \) is an automorphism.

Ad (4): By [49] or [4] there is a unit vector \( \psi \) in the vector space spanned by the irreducible components of \( \alpha_{\lambda_2} \), \( \forall \lambda_2 \) such that

\[
\alpha_{\lambda_2} \psi = \frac{S_{\lambda_1\mu_1}}{S_{1\lambda_1}} \psi, \mu \psi = \frac{S_{\mu_1}}{S_{1\mu_1}} \psi, \alpha_{\delta} \psi = \frac{S_{\delta\lambda_1}}{S_{1\lambda_1}} \psi
\]

and (4) follows immediately. \( \blacksquare \)

### 9.2 Fusions of solitons in cyclic orbifolds

Let \( \mathcal{B} \subset \mathcal{D} \) be as in §6.1. Set \( i = 0 \) in Th. 8.4. In this \( \psi \) is a constant function, and we denote it by its value \( \lambda \). For simplicity we will label the representation \( \pi_{\lambda,g^i,j} \) \((g = (01...n-1)) \) by \( (\lambda, g^i, i) \). Define \( (\lambda i) := (\lambda, 1, i) \) where \( i \in \mathbb{Z}_n \simeq \mathbb{Z}_n \).

**Lemma 9.2.** If \( (k, n) = 1 \), then

\[
\sum_{0 \leq j \leq n-1} e^{\frac{2\pi ij}{n}} N_{(\lambda_0(\delta_0), (\lambda_0(\delta_0))}^{(\delta_0)} = N^{\delta}_{\lambda_\mu}
\]

**Proof** Let \( V := \text{Hom}(\delta, \lambda_\mu) \subset \mathcal{A}(J_0) \). Note that \( \mathbb{Z}_n \) acts on \( W := V \otimes V \otimes \cdots \otimes V \) (n-tensor factors) by permutations. Let \( W_j := \{ w \in W | \beta_j(w) = e^{2\pi ij nw} \} \). Note that if \( w \in W_j \), then \( w^j \in \text{Hom}(v^{-j} \delta^{\otimes n} v^j, \lambda^{\otimes n} \mu^{\otimes n}) \cap \mathcal{D}^{\mathbb{Z}_n}(J_0) \), where \( v \) is defined as before

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Lemma 6.1. Hence we have an injective map $w \in W_j \rightarrow wv^j \in \text{Hom}((\delta j), (\lambda 0)(\mu 0))$. By definition the map is also surjective. So we have

$$
\sum_{0 \leq j \leq n-1} e^{\frac{2\pi i k j}{n}} N_{(\lambda 0)(\mu 0)}^{(\delta j)} = \sum_{0 \leq j \leq n-1} e^{\frac{-2\pi i k j}{n}} \dim W_j = \text{Tr}_W(\beta_{g^k})
$$

When $(k, n) = 1$, $g^k$ is one cycle, and it follows that $\text{Tr}_W(\beta_{g^k}) = \dim V = N_{\lambda \mu}^\delta$. Take the complex conjugate of both sides we have proved the lemma.

Lemma 9.3. Let $f_\mu := (\mu, g, 0)$. Then:

1. $G(\sigma, f_\mu) = e^{\frac{2\pi i l_1}{n}}$ for some integer $l_1$ with $(l_1, n) = 1$;
2. $\lambda \rightarrow \frac{S_{(\lambda 0) f_\mu}}{S_{(10) f_\mu}}$ is a representation of the fusion algebra of $A$;
3. There exists an automorphism $\tau$, $[\tau^2] = [1]$ such that $\frac{S_{(\lambda 0) f_\mu}}{S_{(10) f_\mu}} = \frac{S_{\lambda_\tau(\mu)}}{S_{1\tau(\mu)}}$

Proof Ad(1): By the paragraph after (47) in [33] we have $G(\sigma^{k(1)}, f_\mu) = e^{\frac{2\pi i l_1}{n}}$ where $(k(1), n) = 1$. By (1) of Lemma 6.1 we have $G(\sigma, f_\mu)^{k(1)} = e^{\frac{2\pi i l_1}{n}}$. Choose $l_1$ such that $l_1 k(1) = 1 \mod n$, we have $G(\sigma, f_\mu) = e^{\frac{2\pi i l_1}{n}}$ for some integer $l_1$ with $(l_1, n) = 1$.

As for (2) and (3), first we note that by Lemma 6.1, if $\delta \prec (\lambda 0)(\mu 0)$, then $\delta$ is untwisted. Suppose that $\delta$ is an irreducible component of the restriction of $(\delta_1, ..., \delta_n)$ to $D^Z_{\alpha_4}$. We claim that $S_{\delta f_\mu} = 0$ if $\delta_i \neq \delta_j$ for some $i \neq j$. In fact if $\delta_i \neq \delta_j$ for some $i \neq j$, then the stabilizer of $(\delta_1, ..., \delta_n)$ under the action of $Z_n$ is a proper subgroup of $Z_n$, and by Th. 4.5 $\alpha_4$ is reducible, and $[\alpha^k \delta] = [\delta]$ for some $1 \leq k \leq n - 1$. By (1) of Lemma 7 we have $S_{\delta f_\mu} = S_{\sigma^k(\delta) f_\mu} = G(\sigma^k, f_\mu)^* S_{\delta f_\mu}$. Since $G(\sigma^k, f_\mu) = e^{\frac{-2\pi i l_1}{n}}$ with $(l_1, n) = 1$ by (1), $G(\sigma^k, f_\mu)^* \neq 1$, hence $S_{\delta f_\mu} = 0$. So we have

$$
\frac{S_{(\lambda 1) f_\mu}}{S_{(10) f_\mu}} \frac{S_{(\lambda 0) f_\mu}}{S_{(10) f_\mu}} = \sum_{\lambda_3, 0 \leq j \leq n-1} N_{(\lambda_1 0)(\lambda_2 0)}^{(\lambda_3 j)} S_{(\lambda 3 j) f_\mu} S_{(10) f_\mu} \tag{25}
$$

where we have used (1) of Lemma 9.1 and Lemma 9.2 in the second $=$ and third $=$ respectively.

Ad (2): Since $\alpha_{f_\mu} = (\mu, 1, ..., 1) \alpha_{f_1}$ by (48) of [33], by (4) of Lemma 9.1 we have

$$
\frac{S_{f_\mu(\lambda 0)}}{S_{(10)(\lambda 0)}} = \frac{S_{\mu \lambda}}{S_{1\lambda}} \frac{S_{f_1(\lambda 0)}}{S_{(10)(\lambda 0)}}
$$

Combined with (1) it follows that there exists $\tau$ such that the map

$$
\lambda \rightarrow \frac{S_{\lambda \tau}}{d(\lambda) S_{1\tau}} \frac{S_{\lambda \mu}}{S_{1\mu}}
$$
gives a representation of the fusion algebra of \(A\). By (3) of Lemma 9.1 we have that \(\tau\) is an automorphism and
\[
\frac{S_{(\lambda,\alpha)f_1}}{S_{(10)f_1}} = \frac{S_{\lambda\tau}}{S_{1\tau}}
\]
Let \(h \in \mathbb{P}_n\) such that \(hgh^{-1} = g^{-1}\). By definition \(h((\lambda 0)) = (\lambda 0)\). By Prop. 6.2 \([h(f_1)] = [\sigma^j(f_1)]\) for some \(1 \leq j \leq n\), and it follows by Lemma 9.1 that
\[
S_{(\lambda,\alpha)f_1} = S_{h((\lambda 0))h(f_1)} = S_{(\lambda 0)\sigma^j(f_1)} = S_{(\lambda 0)f_1} = S_{(\lambda 0)f_1}^*;
\]
hence \(\frac{S_{\lambda\tau}}{S_{1\tau}} = \frac{S_{\lambda\tau}}{S_{1\tau}}, \forall \lambda\), and so \([\tau] = [\hat{\tau}]\). \(\blacksquare\)

We conjecture that \([\tau] = [1]\) in the above lemma.

Let \(f_1 := (1, g, 0)\) where 0 stands for the trivial representation of \(\mathbb{Z}_n\). In [33] the questions about the nature of \([\alpha_1^k\] = [\pi_{1,g}^k]\) (cf. (44) of [33]) where \(k\) is an integer is raised.

**Proposition 9.4.** When \(n\) is even we have
\[
[\pi_{1,g}^n] = \bigoplus_{\lambda_1,\ldots,\lambda_n} M_{\lambda_1,\ldots,\lambda_n}((\lambda_1, \ldots, \lambda_n))
\]
where \(M_{\lambda_1,\ldots,\lambda_n} := \sum_{\lambda} S_{\lambda_1}^{2-2g} \prod_{1 \leq i \leq n} \frac{S_{\lambda\lambda}}{S_{\lambda\lambda}}\) with \(g = \frac{(n-1)(n-2)}{2}\).

**Proof** We note that by Lemma 6.1 \(\pi_{1,g,0}^{n-1}\) is untwisted, and must be sum of irreducible untwisted representations. It follows that by Cor. 8.4 of [33] that
\[
[\alpha_1^n] = \bigoplus_{\lambda_1,\ldots,\lambda_n} M_{\lambda_1,\ldots,\lambda_n}((\lambda_1, \ldots, \lambda_n))
\]
with \(M_{\lambda_1,\ldots,\lambda_n}\) non-negative integers. Let \(\mu\) be any irreducible subsector of \(\alpha_1^{n-1}\). By the equation above \(\mu \alpha_f > (\lambda_1, \ldots, \lambda_n)\) for some \((\lambda_1, \ldots, \lambda_n)\), and by Frobenius duality \(\mu \prec (\lambda_1, \ldots, \lambda_n)\alpha_f\). By (46) of [33] \((\lambda_1, \ldots, \lambda_n)\alpha_f = \sum_{\lambda} (\lambda_1 \cdots \lambda_n, \lambda)((\lambda, 1, 1, \ldots, 1)\alpha_f)\) and by (48) of [33] each \((\lambda, 1, 1, \ldots, 1)\alpha_f\) is irreducible. Hence \([\mu] = [(\lambda, 1, 1, \ldots, 1)\bar{\alpha}_f]\) for some \(\lambda\). Hence
\[
[\alpha_1^{n-1}] = [\pi_{1,g}^{n-1}] = \bigoplus_{\lambda} m_{\lambda}((\lambda, 1, \ldots, 1)\bar{\pi}_{1,g})
\]
with \(m_{\lambda}\) non-negative integers. By (4) of Lemma 9.1 we have
\[
\frac{(S_{(\lambda,\mu)0})^{n-1}}{S_{(10)(\mu)0}} = \sum_{\lambda} m_{\lambda} \frac{S_{\lambda\mu} S_{(\lambda,\mu)0}}{S_{(1\mu)0} S_{(\lambda,\mu)0}}
\]
Note that \(\frac{1}{S_{(10)(10)}} = \mu \mathcal{D}_n = n^2 \mu \mathcal{D} = n^2 \frac{1}{S_{(11)}}\), hence \(S_{(10)(10)} = \frac{S_{(11)}}{n}\). From \(S_{(\lambda 0)(10)} S_{(10)(\lambda 0)} = d((\lambda 0)) = nd(\lambda)^n\) we have \(S_{(\lambda 0)(10)} = S_{(\lambda 1)}^n\). By (2) of Lemma 9.3 and our assumption that \(n\) is even and hence \([\tau^n] = [1]\), we have
\[
\frac{1}{S_{(1\mu)0} S_{(n-1)(n-2)}} = \sum_{\lambda} m_{\lambda} \frac{S_{\lambda\mu}}{S_{1\lambda}}
\]

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By orthogonal property of $S$ matrix in Lemma 3.3 we have

$$m_{\lambda} = \sum_{\mu} \frac{S_{\lambda\mu}}{S_{1\mu}^{n^2-3n+1}}$$

Combine this with (46) of [33] and (8) the proposition follows. ■

We remark $M_{\lambda_1,...,\lambda_n}$ is the dimension of genus $\frac{(n-1)(n-2)}{2}$ conformal blocks with the insertion of representations $\lambda_1,...,\lambda_n$. Also note that $\frac{(n-1)(n-2)}{2}$ is the genus of an algebraic curve with degree $n$. It may be interesting to give a geometric interpretation of Prop.9.4.

Note that the conjecture $[\tau] = [1]$ at the end of previous section is true, then the above proposition is also true for odd $n$.

9.3 n=2 case

In this section we consider the fusions rules for the simplest non-trivial case $n = 2$. Partial results have been obtained in §8 of [33]. We will confirm the results in §4.6 of [5]. Let us first simply our notations by introducing similar notations in [5]. Let

$$\hat{\lambda}_0 := (\lambda, -1, 0), \hat{\lambda}_1 := (\lambda, -1, 1).$$

Note that by §2 of [33] we can choose

$$\omega(\hat{\lambda}_0) = e^{\frac{2\pi i (\Delta_{\lambda_1} + \frac{c_0}{2})}{2}}, \omega(\hat{\lambda}_1) = e^{\frac{2\pi i (\Delta_{\lambda_1} + 1 + \frac{c_0}{2})}{2}}$$  (28)

where $c$ is the central charge. We also note by definitions

$$\omega(\lambda_0) = e^{4\pi i \Delta_{\lambda}}, \omega(\lambda_1, \lambda_2) = e^{2\pi i (\Delta_{\lambda_1} + \Delta_{\lambda_2})}, \lambda_1 \neq \lambda_2$$  (29)

Lemma 9.5. (1)

$$N^{(\lambda_0)}_{(\lambda_{1\epsilon_1})(\lambda_{2\epsilon_2})} + N^{(\lambda_1)}_{(\lambda_{1\epsilon_1})(\lambda_{2\epsilon_2})} = \sum_{\mu} \frac{S_{\lambda_0\mu}^2 S_{\lambda_1\mu} S_{\lambda_2\mu}}{S_{1\mu}^2}$$

where $\epsilon_1, \epsilon_2 = 0$ or 1;

(2)

$$N^{(\lambda_4\lambda_5)}_{(\lambda_{1\epsilon_1})(\lambda_{2\epsilon_2})} = \sum_{\mu} \frac{S_{\lambda_4\mu} S_{\lambda_5\mu} S_{\lambda_1\mu} S_{\lambda_2\mu}}{S_{1\mu}^2}$$

(3)

$$\sum_{\mu} \frac{S_{\lambda_4\mu} S_{\lambda_5\mu} S_{\lambda_2\mu}}{S_{1\mu}^2} \frac{1}{\omega_{\lambda_3}^2} d((\lambda_30)) + \sum_{\mu, \lambda_4 \neq \lambda_5} \frac{S_{\lambda_4\mu} S_{\lambda_5\mu} S_{\lambda_1\mu} S_{\lambda_2\mu}}{S_{1\mu}} \frac{1}{\omega_{\lambda_3}} d(\lambda_4) d(\lambda_5)) = \frac{S_{\lambda_1\mu} T_{\mu}^2 S_{\lambda_2\mu}}{S_{11}^2} e^{-2\pi ic_0}$$

where $c_0$ is defined as in (3.2).
Proof  Ad (1): By (48) and (3) of Prop. 8.8 in [33] we have
\[ \langle \alpha(\lambda_1, \lambda_2), (\lambda_3, \lambda_3) \rangle = \langle \lambda_1, \lambda_2, \lambda_3, 1 \rangle \]

Note that
\[ \langle \alpha(\lambda_1, \lambda_2), (\lambda_3, \lambda_3) \rangle = N(\lambda_3) \]
and by (8) (1) is proved. (2) is proved in a similar way.

Ad (3):
\[ \sum \frac{S^2_{\lambda_3\mu} S_{\lambda_1\mu} S_{\lambda_2\mu}}{S^2_{1\mu}} d((\lambda_3)) + \sum \frac{S_{\lambda_4\mu} S_{\lambda_5\mu} S_{\lambda_1\mu} S_{\lambda_2\mu}}{S^2_{1\mu}} \frac{1}{\omega_{\lambda_3}} d(\lambda_4) d(\lambda_5) \]

From Lemma 3.3 we have \( S^* T^{-1} S^* = TS^* T \) and so
\[ \sum \frac{S_{\lambda_3\mu} S_{\lambda_3\lambda_2}}{\omega_{\lambda_3}} = S_{1\mu} e^{-\frac{\pi i c_0}{12}} T_{\mu} \]

Substitute into the equations above we have proved (3).

Define matrices \( T^\frac{1}{2} \) such that \( T^\frac{1}{2} = \delta_{\lambda \mu} e^{\pi i (A_{\lambda} - \frac{c_0}{24})} \) and

**Definition 9.6.**
\[ P := T^\frac{1}{2} S T^\frac{1}{2}, \tilde{P} = e^{\frac{2\pi i (c - 6){\zeta}}{8}} P \]

It follows by (6) that
\[ Y(\lambda_1, \lambda_2) = \omega_{\lambda_1} \omega_{\lambda_2} \times (\sum \frac{N(\lambda_3)}{\omega_{\lambda_3}} + \sum \frac{N(\lambda_3)}{\omega_{\lambda_3}}) \frac{1}{\omega_{\lambda_3}} d((\lambda_3)) + \frac{1}{2} \sum_{\lambda_4 \neq \lambda_5} \frac{N(\lambda_4, \lambda_5)}{\omega_{\lambda_4} \omega_{\lambda_5}} d((\lambda_4, \lambda_5)) \]

where in the last = we have used (3) of Lemma 9.5. Note that \( S^2_{(10)(10)} = \frac{1}{4\pi D} = \frac{1}{4\pi A} = \frac{1}{4} S^4_{11} \), and so \( S_{(10)(10)} = \frac{1}{2} S^2_{11} \). It follows by (7) that
\[ S_{(\lambda_1, \lambda_2)} = e^{\pi i (\epsilon_1 + \epsilon_2)} \frac{1}{2} \tilde{P}_{\lambda_1 \lambda_2} \]
Note that by Lemma 9.3 we have
\[ S_{(\lambda_0)(\mu_0)} = S_{(\lambda_0)(\mu_0)} \frac{S_{\lambda \mu}}{S_{1 \mu}} \times \frac{S_{\tau \lambda}}{S_{1 \lambda}} \]

Since \([\tau^2] = [1], \frac{S_{\lambda}}{S_{1 \lambda}} = \pm 1\) and so \(S_{\lambda 0} \hat{\mu} S_{\mu 0} = S_{\lambda 0} \hat{\mu} S_{\mu 0} S_{\lambda \mu} S_{1 \mu} \times S_{\tau \lambda} S_{1 \lambda} S_{1 \mu} S_{1 \mu}\). By (1) of Lemma 9.1 we can choose our labeling \(\tilde{(\lambda_0)}, \tilde{(\lambda_1)}\) such that as a set \(\{\tilde{(\lambda_0)}, \tilde{(\lambda_1)}\}\) is the same as \(\{(\lambda_0), (\lambda_1)\}\) and

\[ S_{(\lambda_0)(\mu_0)} = e^{\pi i \epsilon} S_{(10)(10)} \frac{S_{\lambda \mu}}{S_{1 \mu}} \]

From
\[ [\alpha(\lambda_1, \lambda_2)] = [(\lambda_1, \lambda_2)] + [(\lambda_2, \lambda_1)] \]
and (4) of Lemma 9.1 we have
\[ \frac{S_{(\lambda)(\mu)} S_{(10)(\lambda)}}{S_{(10)(\lambda)}} = \frac{S_{\lambda \lambda}}{S_{1 \lambda}} + \frac{S_{\lambda \lambda}}{S_{1 \lambda}}, \quad \frac{S_{(\lambda)(\mu)} S_{(10)(\lambda)}}{S_{(10)(\lambda)}} = \frac{S_{\lambda \lambda} S_{\lambda \mu}}{S_{1 \lambda} S_{1 \mu}} + \frac{S_{\lambda \lambda} S_{\lambda \mu}}{S_{1 \lambda} S_{1 \mu}} \]

Since
\[ S_{(10)(10)} = \frac{1}{2} S_{11}^2, \]
we get the following on the entries of \(S\)-matrix of \(\mathcal{D}^\mathbb{Z}_2\):
\[ S_{(\lambda)(\lambda)(\mu)(\mu)} = S_{\lambda \lambda} S_{\mu \mu} + S_{\lambda \lambda} S_{\mu \lambda}, S_{(\lambda)(\mu)(\lambda)(\mu)} = S_{\lambda \lambda} S_{\mu \lambda} = S_{\lambda \lambda} S_{\mu \lambda} \]
\[ S_{(\lambda)(\mu)(\lambda)(\mu)} = \frac{1}{2} e^{\pi i \epsilon} S_{\lambda \lambda}, S_{(\lambda)(\mu)(\lambda)(\mu)} = \frac{1}{2} e^{\pi i (\epsilon + \epsilon_1)} P_{\lambda \lambda}, \epsilon, \epsilon_1 = 0, 1 \]

Denote by \(c_0\) the number (well defined mod\(8\)) of \(\mathcal{D}^\mathbb{Z}_2\) (cf. (3.2).

**Lemma 9.7.** (1) \(c_0 - 2c_0 \in 8\mathbb{Z}\);
(2) \(c_0 - c \in 4\mathbb{Z}\).

**Proof** By Lemma 3.3 we have
\[ S T S = T^{-1} S T^{-1} \]

First let us compare the \((10)(10)\) entry of both sides for \(S, T\) matrix of \(\mathcal{D}^\mathbb{Z}_2\). By using the formula before the lemma we have:
\[ \left( \sum_{\lambda} S_{11}^2 e^{2\pi i \lambda} \right)^2 = e^{2\pi i c_0} S_{11}^2. \]

On the other hand comparing the entry 11 of (39) for \(S\)-matrix of \(\mathcal{D}\) we have
\[ \sum_{\lambda} S_{11}^2 e^{2\pi i \lambda} = e^{2\pi i c_0} S_{11}, \]
and (1) follows by combining the two equations. 

As for (2), we compare the entry $\hat{\lambda}(10)$ of both sides of (39). By using the equations before the lemma the $\hat{\lambda}(10)$ entry of the left hand side of (39) is given by $e^{2\pi i (c-c_0)/8} e^{\pi i (c_0)/24}$ multiplied by the $\lambda_1$ entry of the matrix $PT^{-1}S$. By applying (39) to $S, T$ matrix of $D$ we have

$$PT^{-1}S = T^{-1}ST^2S = T^{-1}ST^{-2}.$$ 

Using these equations to compare with the $\hat{\lambda}(10)$ entry of right hand side of (39) we have $e^{2\pi i (c-c_0)/8} = 1$ and (2) if proved. ■

By (8) and (2) of Lemma 9.7 we immediately obtain the following fusion rules:

$$N(\lambda_2 \mu_2)_{(\lambda \mu)(\lambda_1 \mu_1)} = N^{\lambda_2}_{\lambda \lambda_1} N^{\mu_2}_{\mu \mu_1} + N^{\mu_2}_{\lambda \lambda_1} N^{\lambda_2}_{\mu \mu_1} + N^{\mu_2}_\lambda N^{\lambda}_\mu + N^{\lambda_2}_\lambda N^{\lambda_1}_\mu$$

$$N(\lambda_2 \epsilon)_{(\lambda \epsilon)(\lambda_1 \epsilon_1)} = \frac{1}{2} N^{\lambda_2}_{\lambda \lambda_1} (N^{\lambda_2}_{\lambda \lambda_1} + e^{\pi i (\epsilon + \epsilon_1 + \epsilon_2)})$$

$$N(\lambda_2 \epsilon_2)_{(\lambda \epsilon)(\lambda_1 \epsilon_1)} = \sum_{\mu} N^{\mu}_\lambda N^{\mu}_\mu$$

where $\epsilon, \epsilon_1, \epsilon_2 = 0$ or 1. Let us summarize the above equations in the following:

**Theorem 9.8.** The fusion rules of $D^{Z_2}$ are given by the above equations.

From the theorem we immediately have:

**Corollary 9.9.** For any completely rational $A$

$$\frac{1}{2} \sum_{\mu} \frac{S_{\lambda \mu}^2 S_{\lambda_1 \mu} S_{\lambda_2 \mu}}{S_{1 \mu}^2} \pm \frac{1}{2} \sum_{\mu} \frac{S_{\lambda \mu} P_{\lambda_1 \mu} P_{\lambda_2 \mu}}{S_{1 \mu}^2}$$

is a non-negative integer where $P$ is defined in (9.6).

Cor. 9.9 confirmed a conjecture in §4.6 of [5]. We note that even for known examples the direct confirmation of Cor. 9.9 seems to be very tedious.

It will be an interesting question to generalize our results to $n > 2$ cases.

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