TOPOLOGICAL STRUCTURES OF GENERALIZED VOLTERRA-TYPE INTEGRAL OPERATORS

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Abstract. We study the generalized Volterra-type integral and composition operators acting on the classical Fock spaces. We first characterize various properties of the operators in terms of growth and integrability conditions which are simpler to apply than those already known Berezin type characterizations. Then, we apply these conditions to study the compact and Schatten $S_p$ class difference topological structures of the space of the operators. In particular, we proved that the difference of two Volterra-type integral operators is compact if and only if both are compact.

1. INTRODUCTION

Ever since A. Aleman and A. Siskakis published their seminal works [3, 4] on integral operators acting on Hardy and Bergman spaces, Volterra-type integral operators have been studied on many Banach spaces over several domains. For holomorphic functions $f$ and $g$ in a given domain, we define the Volterra-type integral operator $V_g$ and its companion $J_g$ by

$$V_g f(z) = \int_0^z f(w)g'(w)dw \quad \text{and} \quad J_g f(z) = \int_0^z f'(w)g(w)dw.$$ 

Applying integration by parts in any one of the above integrals gives the relation

$$V_g f + J_g f = M_g f - f(0)g(0), \quad (1.1)$$

where $M_g f = gf$ is the multiplication operator of symbol $g$. Studies on these operators have been mainly aiming to describe the connection between their operator-theoretic behaviours with the function-theoretic properties of the inducing symbols $g$. For more information on the subject, we refer to [1, 2, 11] and the related references therein.

In 2008, S. Li and S. Stević extended $V_g$ and $J_g$ to the operators

$$V_{(g,\psi)} f(z) = \int_0^z f(\psi(w))g'(w)dw, \quad \text{and} \quad J_{(g,\psi)} f(z) = \int_0^z f'(\psi(w))g(w)dw, \quad (1.2)$$

and studied some of their operator-theoretic properties in terms of properties of the pairs $(g, \psi)$ on some spaces of analytic functions on the unit disk [9, 10]. For more recent results on this class of operators, one may also consult the materials for instance in [12, 13, 16].

2010 Mathematics Subject Classification. Primary 47B32, 30H20; Secondary 46E22, 46E20, 47B33.

Key words and phrases. Fock spaces, Bounded, Compact, Generalized Volterra-type, Composition operator, Schatten class, Topological structures, Compact difference.
We may note in passing that if we, in particular, set \( \psi(z) = z \), then the operators \( V_{(g, \psi)} \) and \( J_{(g, \psi)} \) become respectively the operators \( V_g \) and \( J_g \). On the other hand, the choice \( g = \psi' \) reduce the operators \( J_{(g, \psi)} \) to the composition operators \( C_\psi \) taking \( f \) to \( f(\psi) \) up to a constant. Owing to this fact, the operators in (1.2) are often referred as the generalized Volterra-type integral operators and generalized composition operators. Such operators have found applications for example in the study of linear isometries of spaces of analytic functions. If \( S^p \) denotes the space of all analytic functions \( f \) in the unit disc for which its derivative \( f' \) belongs to the Hardy space \( H^p \), then it has been shown that for \( p \neq 2 \), any surjective isometry \( T \) of \( S^p \) under the norm \( \| f \|_{S^p} = |f(0)| + \| f' \|_{H^p} \) is of the form

\[
Tf = \lambda f(0) + \lambda J_{(g, \psi)}f
\]

for some unimodular \( \lambda \) in \( \mathbb{C} \), a nonconstant inner function \( \psi \) and a function \( g \) belonging to the space \( H^p \) [7].

Let \( \mathbb{C} \) be the complex plane and \( 0 < p < \infty \). Then the classical Fock spaces \( \mathcal{F}_p \) consist of entire functions \( f \) for which

\[
\| f \|_p^p = \frac{p}{2\pi} \int_\mathbb{C} |f(z)|^p e^{-\frac{z}{2}|z|^2} dA(z) < \infty,
\]

where \( dA \) denotes the Lebesgue area measure. In particular, \( \mathcal{F}_2 \) is a reproducing kernel Hilbert space with kernel and normalized reproducing kernel functions given by the explicit formulas

\[
K_w(z) = e^{\overline{w}z} \quad \text{and} \quad k_w(z) = e^{\overline{w}z - \frac{|w|^2}{2}}.
\]

The kernel function \( K_w \) belongs to all the Fock space \( \mathcal{F}_p \) with norms

\[
\| K_w \|_p = e^{\frac{|w|^2}{2}}
\]

for all \( w \in \mathbb{C} \) and \( 0 < p < \infty \). This follows from a simple computation

\[
\| K_w \|_p^p = \frac{p}{2\pi} \int_\mathbb{C} e^{pR(\overline{w}z)-\frac{p}{2}|z|^2} dA(z) = e^{\frac{p}{2}|w|^2} \frac{p}{2\pi} \int_\mathbb{C} e^\frac{p}{2} \left( 2R(\overline{w}z) - |w||z|^2 \right) dA(z)
\]

\[
= e^{\frac{p}{2}|w|^2} \left( \frac{p}{2\pi} \int_\mathbb{C} e^{-\frac{p}{2}|w-z|^2} dA(z) \right) = e^{\frac{p}{2}|w|^2}.
\]

It follows that \( k_w \) constitutes a unit norm sequence of functions \( \mathcal{F}_p \) and converges to zero in compact subset of \( \mathbb{C} \). This and the norms in (1.3) will be used repeatedly in our further considerations.

In [11, 12], Mengestie studied the operators \( V_{(g, \psi)} \) and \( J_{(g, \psi)} \) on Fock spaces and characterized various properties of the operators in terms of the Berezin type integral transforms

\[
B_{(g|p, \psi)}(w) = \int_\mathbb{C} |k_w(\psi(z))|^p \frac{|g'(z)|^p e^{-\frac{p}{2}|z|^2}}{(1 + |z|)^p} dA(z) \quad \text{and}
\]

\[
\widetilde{B}_{(g|p, \psi)}(w) = \int_\mathbb{C} |k_w(\psi(z))|^p \frac{(|w| + 1)^p |g(z)|^p e^{-\frac{p}{2}|z|^2}}{(1 + |z|)^p} dA(z).
\]
As will be indicated in Section 5, the characterizations in [11, 12] require to estimate the $L^q$, $0 < q \leq \infty$ norms of the integral transforms $B_{[|g|^p,\psi]}$ and $\tilde{B}_{[|g|^p,\psi]}$. Such types of characterizations have been also referred as reproducing kernel thesis properties for the operators. One of the main purposes of this work is to substantially improve these conditions and provide characterizations which are rather simpler to apply. Our new results will be expressed in terms of the functions $M_{(g,\psi)}(z) = \frac{|g'(z)|}{1+|z|} e^{\frac{1}{2}|\psi(z)|^2 - |z|^2}$ and $\tilde{M}_{(g,\psi)}(z) = \frac{|g(z)|(1 + |\psi(z)|)}{1+|z|} e^{\frac{1}{2}|\psi(z)|^2 - |z|^2}$, which are easier to handle with than those class of integral transforms in (1.4).

2. Bounded and Compact $V_{(g,\psi)}$ and $J_{(g,\psi)}$

In this section we characterized the boundedness and compactness properties of the operators $V_{(g,\psi)}$ and $J_{(g,\psi)}$ in terms of the functions $M_{(g,\psi)}$ and $\tilde{M}_{(g,\psi)}$. Our first main result reads as follows.

**Theorem 2.1.** Let $0 < p \leq q < \infty$ and $(g,\psi)$ be pairs of nonconstant entire functions. Then

(i) $V_{(g,\psi)} : \mathcal{F}_p \to \mathcal{F}_q$ is bounded if and only if $M_{(g,\psi)} \in L^\infty(\mathbb{C}, dA)$.
(ii) $V_{(g,\psi)} : \mathcal{F}_p \to \mathcal{F}_q$ is compact if and only if $\lim_{|z| \to \infty} M_{(g,\psi)}(z) = 0$.
(iii) $J_{(g,\psi)} : \mathcal{F}_p \to \mathcal{F}_q$ is bounded if and only if $\tilde{M}_{(g,\psi)} \in L^\infty(\mathbb{C}, dA)$.
(iv) $J_{(g,\psi)} : \mathcal{F}_p \to \mathcal{F}_q$ is compact if and only if $\lim_{|z| \to \infty} \tilde{M}_{(g,\psi)}(z) = 0$.

It should be remarked that all the results above are independent of the Fock exponent $p$ in the domain spaces $\mathcal{F}_p$ apart from the condition that it is bounded by the exponent $q$ in the target space $\mathcal{F}_q$. This is manifested due to the fact that the Fock spaces are nested in the sense that $\mathcal{F}_p \subseteq \mathcal{F}_q$ whenever $p \leq q$ (see Theorem 2.10, [17]). As one would expect, our results are different for the cases $p \leq q$ and $q < p$. For the latter case, we have rather a stronger condition under which the boundedness implies compactness as formulated in our next main result.

**Theorem 2.2.** Let $0 < q < p < \infty$ and $(g,\psi)$ be pairs of nonconstant entire functions. Then

(i) the following statements are equivalent.
   (a) $V_{(g,\psi)} : \mathcal{F}_p \to \mathcal{F}_q$ is bounded;
   (b) $V_{(g,\psi)} : \mathcal{F}_p \to \mathcal{F}_q$ is compact;
   (c) $M_{(g,\psi)} \in L^{\frac{q}{p-q}}(\mathbb{C}, dA)$.
(ii) the following statements are also equivalent.
   (a) $J_{(g,\psi)} : \mathcal{F}_p \to \mathcal{F}_q$ is bounded;
   (b) $J_{(g,\psi)} : \mathcal{F}_p \to \mathcal{F}_q$ is compact;
   (c) $\tilde{M}_{(g,\psi)} \in L^{\frac{q}{p-q}}(\mathbb{C}, dA)$. 
3. **Schatten $S_p$ class membership of $V_{(g,\psi)}$**

The singular values of a compact operator $T$ on a Hilbert space $\mathcal{H}$ are the square roots of the positive eigenvalues of the operator $T^*T$, where $T^*$ denotes the adjoint of $T$. Said differently, these are simply the positive eigenvalues of $(T^*T)^{1/2} = |T|$. For $0 < p < \infty$, the Schatten $p$-class of $\mathcal{H}$ which we denote it by $S_p(\mathcal{H})$ represents the space of all compact operators $T$ on $H$ with $\ell^p$ summable singular value sequences. The space $S_p(\mathcal{H})$ is a two sided ideal in the algebra of all bounded linear operators on the space $\mathcal{H}$. We refer readers to the materials in [6, 17] for a good overview of the subject.

Our next main result characterizes the Schatten $S_p(\mathcal{F}_2)$ membership of the operators $V_{(g,\psi)}$ in terms of $L^p$ integrability condition for $M_{(g,\psi)}$.

**Theorem 3.1.** Let $0 < p < \infty$ and $(g, \psi)$ be pair of entire functions and the map $V_{(g,\psi)} : \mathcal{F}_2 \to \mathcal{F}_2$ be bounded. Then $V_{(g,\psi)}$ belongs to the algebra $S_p(\mathcal{F}_2)$ if and only if $M_{(g,\psi)} \in L^p(\mathbb{C}, dA)$.

The analogous problem for $J_{(g,\psi)}$ has been already studied in [13] where partly a source of inspiration for this paper stems from.

4. **Topological structures of $V_{(g,\psi)}$ and $J_{(g,\psi)}$**

Over the past few decades much effort has been paid to study various operator-theoretic properties of these operators in terms of the function-theoretic properties of the inducing pairs $(g, \psi)$. On the contrary, there has been no effort devoted to understand the topological structures of the space of bounded operators $V_{(g,\psi)}$ and $J_{(g,\psi)}$ equipped with the operator norm topology. It is worth noting that such structures have been studied for many other operators including compact differences of composition operators. We refer to [15] for references and historical remarks in this topic. In this section, we use our results from the proceeding sections and begin a study on such structures and characterize the compact and Schatten $S_p$ class differences of the operators.

**Theorem 4.1.** Let $0 < p \leq q < \infty$, $(g_1, \psi_1)$ and $(g_2, \psi_2)$ be pairs of nonconstant entire functions, and $V_{(g_1,\psi_1)}$, $V_{(g_2,\psi_2)}$, $J_{(g_1,\psi_1)}$ and $J_{(g_2,\psi_2)}$ be bounded operators from $\mathcal{F}_p$ into $\mathcal{F}_q$. Then the operator

(i) $V_{(g_1,\psi_1)} - V_{(g_2,\psi_2)}$ is compact if and only if either both $V_{(g_1,\psi_1)}$ and $V_{(g_2,\psi_2)}$ are compact or $\psi_1 = \psi_2$ and

$$\lim_{|z| \to \infty} M_{(g_1-g_2,\psi_1)}(z) = 0.$$  \hspace{1cm} (4.1)

(ii) $J_{(g_1,\psi_1)} - J_{(g_2,\psi_2)}$ is compact if and only if either both $J_{(g_1,\psi_1)}$ and $J_{(g_2,\psi_2)}$ are compact or $\psi_1 = \psi_2$ and

$$\lim_{|z| \to \infty} \tilde{M}_{(g_1-g_2,\psi_1)}(z) = 0.$$ \hspace{1cm} (4.2)

An interesting future in the study of differences of operators is the important factors arising from the cancellation property. From (4.1) and (4.2), such factors are $|\psi_1 - \psi_2| = 0$, and $|g_1' - g_2'|$ and $|g_1 - g_2|$. The same impact is observed in
the stronger Schatten $S_p$ class difference membership as stated in the next main result.

**Theorem 4.2.** Let $0 < p < \infty$, $(g_1, \psi_1)$ and $(g_2, \psi_2)$ be pairs of nonconstant entire functions, and $V_{(g_1, \psi_1)}$, $V_{(g_2, \psi_2)}$, $J_{(g_1, \psi_1)}$ and $J_{(g_2, \psi_2)}$ be bounded operators on $\mathcal{F}_2$. Then the operator

(i) $V_{(g_1, \psi_1)} - V_{(g_2, \psi_2)}$ belongs to the Schatten $S_p(\mathcal{F}_2)$ class if and only if either both $V_{(g_1, \psi_1)}$ and $V_{(g_2, \psi_2)}$ belong to the $S_p(\mathcal{F}_2)$ class or $\psi_1 = \psi_2 = \psi$ and

$$
\int_C \frac{|g_1(z) - g_2(z)|^p}{(1 + |z|)^p} e^{\frac{\psi(z)^2}{2} - |z|^2} dA(z) < \infty. \tag{4.3}
$$

(ii) $J_{(g_1, \psi_1)} - J_{(g_2, \psi_2)}$ belongs to the Schatten $S_p(\mathcal{F}_2)$ class if and only if either both $J_{(g_1, \psi_1)}$ and $J_{(g_2, \psi_2)}$ belong to the $S_p(\mathcal{F}_2)$ class or $\psi_1 = \psi_2$ and

$$
\int_C \frac{|g_1(z) - g_2(z)|^p}{(1 + |z|)^p} e^{\frac{\psi(z)^2}{2} - |z|^2} dA(z) < \infty. \tag{4.4}
$$

As noted earlier, if we set $\psi(z) = z$, then the operators $V_{(g, \psi)}$ becomes the operators $V_g$. By Theorem 4.1, it turns out that a Volterra-type integral difference is compact if and only if both operators are compact. We record this as part of the following corollary since it is interests of its own.

**Corollary 4.3.**

(i) Let $0 < p < q < \infty$, $g_1 \neq g_2$, and $V_{g_1}$ and $V_{g_2}$ be bounded operators from $\mathcal{F}_p$ into $\mathcal{F}_q$. Then the difference operator $V_{g_1} - V_{g_2}$ is compact if and only if both $V_{g_1}$ and $V_{g_2}$ are compact.

(ii) Let $g_1 \neq g_2$, $0 < p < \infty$, and $V_{g_1} - V_{g_2}$ be compact operators on $\mathcal{F}_2$. Then the operator $V_{g_1} - V_{g_2}$ belongs to the Schatten $S_p(\mathcal{F}_2)$ class if and only if $p > 2$.

(iii) Let $1 \leq p < \infty$, and $V_{g_1}$ and $V_{g_2}$ be bounded operators on $\mathcal{F}_p$. That is $g_1(z) = a_1z^2 + b_1z + c_1$ and $g_1(z) = a_2z^2 + b_2z + c_2$. Then the spectrum of $V_{g_1} - V_{g_2}$ on $\mathcal{F}_p$ is given by

$$
\sigma(V_{g_1} - V_{g_2}) = \left\{ \lambda \in \mathbb{C} : |\lambda| \leq 2|a_1 - a_2| \right\}.
$$

This shows that the difference of two Volterra-type integral operators can not be nontrivially compact. A natural question to ask is whether there exist compact difference of multiplication operators $M_g$ and the other Volterra type operator $J_g$. Clearly, if any two of the operators in (1.1) are bounded, so is the third one. The boundedness and compactness properties of $M_g$ and $J_g$ on Fock spaces were described in [11]. The result there ensures that $M_g$ or $J_g$ is compact if and only if $g = 0$. Thus, a difference of any two of such operators fails to be compact.

5. **Proof of the main results**

In this section we prove our results. We may first give a key proposition that will be used repeatedly in our subsequent considerations.

**Proposition 5.1.** Let $(g, \psi)$ be a pair of nonconstant entire functions and $0 < p < \infty$. Then if any of
(i) $M_{(g,\psi)}$, $\widetilde{M}_{(g,\psi)}$, $B_{(|g|^p,\psi)}$ or $\widetilde{B}_{(|g|^p,\psi)}$ belongs to $L^\infty(\mathbb{C}, dA)$, then $\psi(z) = az + b$ for some $|a| \leq 1$.

(ii) $M_{(g,\psi)}(z)$, $\widetilde{M}_{(g,\psi)}(z)$, $B_{(|g|^p,\psi)}(z)$, or $\widetilde{B}_{(|g|^p,\psi)}(z)$ tends to zero as $|z| \to \infty$, then $\psi(z) = az + b$ with $|a| < 1$.

Proof. The proof of the proposition for the parts $M_{(g,\psi)}$ and $\widetilde{M}_{(g,\psi)}$ follows from a simple variant of the proof of Proposition 2.1 in [8] or Lemma 2.3 of [13]. On the other hand, for the integral transform $B_{\psi}(\|g\|^p)$, we have

$$B_{\psi}(\|g\|^p)(w) = \int_{\mathbb{C}} \frac{|k_w(\psi(z))|^p|g'(z)|^p}{(1 + |z|)^p e^{\frac{p}{2}|z|^2}} dA(z) \geq \int_{D(\zeta, 1)} \frac{|k_w(\psi(z))|^p|g'(z)|^p}{(1 + |z|)^p e^{\frac{p}{2}|z|^2}} dA(z)$$

for all $w, \zeta \in \mathbb{C}$, where $D(\zeta, 1)$ is a disc of radius 1 and center $\zeta$.

Since $1 + z^1 \simeq 1 + \zeta$ for all $z \in D(\zeta, 1)$, we further estimate the integral above as

$$\int_{D(\zeta, 1)} \frac{|k_w(\psi(z))|^p|g'(z)|^p}{(1 + |z|)^p e^{\frac{p}{2}|z|^2}} dA(z) \simeq \frac{1}{(1 + |\zeta|)^p} \int_{D(\zeta, 1)} \frac{|k_w(\psi(z))|^p|g'(z)|^p}{(1 + |z|)^p e^{\frac{p}{2}|z|^2}} dA(z) \simeq \left\{ \frac{1}{(1 + |\zeta|)^p} \right\} k_w(\psi(\zeta)) |g'(\zeta)|^p e^{-\frac{p}{2}|\zeta|^2}$$

for all $\zeta \in \mathbb{C}$. Setting $w = \psi(\zeta)$ in particular and applying (1.3) gives

$$B_{\psi}(\|g\|^p)(\psi(\zeta)) \simeq \left\{ \frac{1}{(1 + |\zeta|)^p} \right\} k_w(\psi(\zeta)) |g'(\zeta)|^p e^{-\frac{p}{2}|\zeta|^2}$$

and the assertion follows from the boundedness condition on $M_{(g,\psi)}$.

Similarly, if $\widetilde{B}_{(|g|^p,\psi)}$ is bounded, then

$$\widetilde{B}_{(|g|^p,\psi)}(w) = \int_{\mathbb{C}} \frac{|k_w(\psi(z))|^p(|w| + 1)^p|g(z)|^p e^{-\frac{p}{2}|z|^2}}{(1 + |z|)^p} dA(z) \geq \int_{D(\zeta, 1)} \frac{|k_w(\psi(z))|^p(|w| + 1)^p|g(z)|^p e^{-\frac{p}{2}|z|^2}}{(1 + |z|)^p} dA(z) \simeq \left\{ \frac{1 + |w|}{(1 + |\zeta|)^p} \right\} k_w(\psi(\zeta)) |g(\zeta)|^p e^{-\frac{p}{2}|\zeta|^2},$$

and setting $w = \psi(\zeta)$ and applying (1.3) again results

$$\widetilde{B}_{(|g|^p,\psi)}(\psi(\zeta)) \simeq \left\{ \frac{1 + |\psi(\zeta)|}{(1 + |\zeta|)^p} \right\} k_w(\psi(\zeta)) |g(\zeta)|^p e^{-\frac{p}{2}|\zeta|^2}$$

and the assertion follows from the boundedness condition on $\widetilde{M}_{(g,\psi)}$ and completes the proof.

Footnote: The notation $U(z) \lesssim V(z)$ (or equivalently $V(z) \gtrsim U(z)$) means that there is a constant $C$ such that $U(z) \leq CV(z)$ holds for all $z$ in the set of a question. We write $U(z) \simeq V(z)$ if both $U(z) \lesssim V(z)$ and $V(z) \lesssim U(z)$. 


To prove our results, we also need the following crucial Littlewood-Paley type estimate for Fock spaces [5]. The estimate asserts that for \( 0 < p < \infty \) and \( f \in \mathcal{F}_p \)
\[
\int_{\mathbb{C}} |f(z)|^p e^{-\frac{2}{p} |z|^2} dA(z) \simeq |f(0)|^p + \int_{\mathbb{C}} \frac{|f'(z)|^p}{(1 + |z|)^p} e^{-\frac{2}{p} |z|^2} dA(z). \tag{5.3}
\]
We may now turn to the proofs of our main results.

5.1. Proof of Theorem 2.1. By Theorem 1 of [12], \( V_{(g, \psi)} : \mathcal{F}_p \to \mathcal{F}_q \) is bounded if and only if the Berezin type integral transforms \( B_{(|g|^p, \psi)} \) is bounded, and compact if and only if \( B_{(|g|^p, \psi)}(z) \to 0 \) as \( |z| \to \infty \). Seemingly, by Theorem 1 of [11], \( J_{(g, \psi)} : \mathcal{F}_p \to \mathcal{F}_q \) is bounded if and only if \( \widetilde{B}_{(|g|^p, \psi)} \) is bounded and compact if and only if \( \widetilde{B}_{(|g|^p, \psi)}(z) \to 0 \) as \( |z| \to \infty \). Due to these, the proof of our theorem will be complete if we prove the following two results which we formulate them as lemmas as they are interest of their own.

Lemma 5.2. Let \((g, \psi)\) be a pair of nonconstant entire functions on \( \mathbb{C} \). Then
\[
(i) \quad M_{(g, \psi)} \in \mathcal{L}^\infty(\mathbb{C}, dA) \text{ if and only if } B_{(|g|^p, \psi)} \in \mathcal{L}^\infty(\mathbb{C}, dA) \text{ for some } 0 < p < \infty,
\]
\[
(ii) \quad M_{(g, \psi)} \in \mathcal{L}^\infty(\mathbb{C}, dA) \text{ if and only if } \widetilde{B}_{(|g|^p, \psi)} \in \mathcal{L}^\infty(\mathbb{C}, dA) \text{ for some } 0 < p < \infty.
\]

Proof. (i). If the Berezin type integral transform \( B_{(|g|^p, \psi)} \) is bounded for some \( 0 < p < \infty \), then the series of estimates in (5.1) implies that \( M_{(g, \psi)} \) is bounded. On the other hand, if \( M_{(g, \psi)} \) is bounded, then by Proposition 5.1, we may set \( \psi(z) = az + b \) and argue
\[
B_{(|g|^p, \psi)}(w) = \int_{\mathbb{C}} |k_w(\psi(z))|^p \frac{|g'(z)|^p}{(1 + |z|)^p} e^{-\frac{2}{p} |z|^2} dA(z)
\]
\[
= \int_{\mathbb{C}} |k_w(\psi(z))|^p e^{-\frac{2}{p} |\psi(z)|^2} \left( \frac{|g'(z)|^p}{(1 + |z|)^p} \right) e^{\frac{2}{p} |\psi(z)|^2 - |z|^2} dA(z)
\]
\[
\leq \left( \sup_{z \in \mathbb{C}} \frac{|g'(z)|^p}{(1 + |z|)^p} e^{\frac{2}{p} |\psi(z)|^2 - |z|^2} \right) \int_{\mathbb{C}} |k_w(\psi(z))|^p e^{-\frac{2}{p} |\psi(z)|^2} dA(z)
\]
\[
= \left( \sup_{z \in \mathbb{C}} M_{(g, \psi)}^p(z) \right) \frac{1}{|a|^2} \int_{\mathbb{C}} |k_w(\psi(z))|^p e^{-\frac{2}{p} |\psi(z)|^2} dA(z).
\]

where the last equality follows by (1.3) and hence the claim.

Similarly, one side of the statement in part (ii) follows from the series of the estimations made leading to (5.2). On the other hand, applying Proposition 5.1 and (1.3) again,
\[
\widetilde{B}_{(|g|^p, \psi)}(w) = \int_{\mathbb{C}} \frac{|k_w(\psi(z))|^p}{e^{\frac{2}{p} |\psi(z)|^2}} \left( \frac{(1 + |w|)^p (1 + |\psi(z)|)^p |g(z)|^p}{(1 + |\psi(z)|)^p (1 + |z|)^p} e^{\frac{2}{p} |\psi(z)|^2 - |z|^2} \right) dA(z)
\]
\[
\leq \left( \sup_{z \in \mathbb{C}} \frac{(1 + |\psi(z)|)^p |g(z)|^p}{(1 + |z|)^p} e^{\frac{2}{p} |\psi(z)|^2 - |z|^2} \right) \int_{\mathbb{C}} |k_w(\psi(z))|^p e^{-\frac{2}{p} |\psi(z)|^2} dA(z)
\]
\[
= \left( \sup_{z \in \mathbb{C}} M_{(g, \psi)}^p(z) \right) \frac{1}{|a|^2} \int_{\mathbb{C}} |k_w(\psi(z))|^p e^{-\frac{2}{p} |\psi(z)|^2} dA(z).
\]
By Proposition 5.1 we estimate the above last integral by
\[
\int_{|z|=|w|} \frac{|k_w(\psi(z))|^p}{(1 + |\psi(z)|)^p} e^{-\frac{p}{2}|\psi(z)|^2} dA(z) = \int_{|z|=|w|} \frac{|k_w(\psi(z))|^p}{e^{\frac{p}{2}|\psi(z)|^2}} M^p_{(g,\psi)}(z) dA(z)
\]
from which we have that
\[
\tilde{B}_{(g|^p,\psi)}(w) \lesssim \sup_{z \in \mathbb{C}} M^p_{(g,\psi)}(z) \|k_w\|^p_p \leq \sup_{z \in \mathbb{C}} \tilde{M}^p_{(g,\psi)}(z),
\]
and completes the proof.

**Lemma 5.3.** Let \((g, \psi)\) be a pair of nonconstant entire functions on \(\mathbb{C}\). Then

(i) \(M_{(g,\psi)}(w) \to 0\) as \(|w| \to \infty\) if and only if \(B_{(g|^p,\psi)}(w) \to 0\) as \(|w| \to \infty\) for some \(0 < p < \infty\).

(ii) \(\tilde{M}_{(g,\psi)}(w) \to 0\) as \(|w| \to \infty\) if and only if \(\tilde{B}_{(g|^p,\psi)}(w) \to 0\) as \(|w| \to \infty\) for some \(0 < p < \infty\).

**Proof.** (i) One side of the statement follows easily from the estimates in (5.1). We shall proceed to show the other side, and assume that \(M_{(g,\psi)}(w) \to 0\) as \(|w| \to \infty\).

Then we estimate
\[
B_{(g|^p,\psi)}(w) = \int_{\mathbb{C}} \frac{|k_w(\psi(z))|^p |g(z)|^p}{(1 + |\psi(z)|)^p} e^{\frac{p}{2}|\psi(z)|^2} dA(z) = \int_{|z|\leq|w|} \frac{|k_w(\psi(z))|^p}{e^{\frac{p}{2}|\psi(z)|^2}} M^p_{(g,\psi)}(z) dA(z)
\]
\[
+ \int_{|z|>|w|} \frac{|k_w(\psi(z))|^p M^p_{(g,\psi)}(z)}{e^{\frac{p}{2}|\psi(z)|^2}} dA(z) \lesssim \left( \sup_{z:|z|\leq|w|} |k_w(\psi(z))|^p \right) \int_{|z|\leq|w|} \frac{M^p_{(g,\psi)}(z)}{e^{\frac{p}{2}|\psi(z)|^2}} dA(z)
\]
\[
+ \left( \sup_{z:|z|>|w|} M^p_{(g,\psi)}(z) \right) \int_{|z|>|w|} |k_w(\psi(z))|^p e^{-\frac{p}{2}|\psi(z)|^2} dA(z)
\]

By Proposition 5.1 and the assumption that \(\psi\) is nonconstant, it follows
\[
\int_{|z|\leq|w|} M^p_{(g,\psi)}(z) e^{-\frac{p}{2}|\psi(z)|^2} dA(z) \leq \left( \sup_{z \in \mathbb{C}} M^p_{(g,\psi)}(z) \right) \int_{|z|\leq|w|} e^{-\frac{p}{2}|\psi(z)|^2} dA(z) < \infty
\]
from which we deduce
\[
B_{(g|^p,\psi)}(w) \lesssim \sup_{z:|z|\leq|w|} |k_w(\psi(z))|^p + \sup_{z:|z|>|w|} M^p_{(g,\psi)}(z)
\]
\[
= \sup_{z:|z|\leq|w|} |k_w(\psi(z))|^p + \sup_{z:|z|>|w|} M^p_{(g,\psi)}(z).
\]
The first term in the last sum converges to zero as \(|w| \to \infty\) since the normalized reproducing kernel converges to zero on compact subsets. On the other hand, by hypothesis, we have
\[
\sup_{z:|z|>|w|} M_{(g,\psi)}(z) \to 0 \text{ as } |w| \to \infty.
\]
and hence the assertion.
(ii) From the inequalities in (5.2), we easily deduce one of the implications for this part as well. On the other hand, if \( \tilde{M}_{(g,\psi)}(w) \to 0 \) as \(|w| \to \infty\), then arguing as above and eventually applying (5.3) and (1.3)

\[
\tilde{B}_{(g^{p},\psi)}(w) = \int_{C} |k_{w}(\psi(z))|^{p} \frac{1 + |w|^{p}|g(z)|^{p}}{1 + |z|^{p}} e^{-\frac{2}{5}|z|^{2}} dA(z)
\]

\[
\approx \int_{|z| \leq |w|} \frac{|k_{w}'(\psi(z))|^{p}}{1 + |\psi(z)|^{p}} e^{-\frac{2}{5}|\psi(z)|^{2}} \tilde{M}_{(g,\psi)}^{p}(z) dA(z)
\]

\[
+ \int_{|z| > |w|} \frac{|k_{w}'(\psi(z))|^{p}}{1 + |\psi(z)|^{p}} e^{-\frac{2}{5}|\psi(z)|^{2}} \tilde{M}_{(g,\psi)}^{p}(z) dA(z)
\]

\[
\lesssim \left( \sup_{z: |z| \leq |w|} |k_{w}'(\psi(z))|^{p} \right) \int_{|z| \leq |w|} \tilde{M}_{(g,\psi)}^{p}(z) e^{-\frac{2}{5}|\psi(z)|^{2}} dA(z)
\]

\[
+ \left( \sup_{z: |z| > |w|} \tilde{M}_{(g,\psi)}^{p}(z) \right) \int_{|z| > |w|} \frac{|k_{w}'(\psi(z))|^{p}}{1 + |\psi(z)|^{p}} e^{-\frac{2}{5}|\psi(z)|^{2}} dA(z)
\]

\[
\lesssim \sup_{z: |z| \leq |w|} |k_{w}'(\psi(z))|^{p} + \left( \sup_{z: |z| > |w|} \tilde{M}_{(g,\psi)}^{p}(z) \right) \|k_{w}\|_{p}^{p}
\]

from which the assertion follows as before since \( k_{w}' \) converges uniformly on compact subsets and \( \sup_{z: |z| > |w|} \tilde{M}_{(g,\psi)}(z) \to 0 \) when \(|w| \to \infty|w| \to \infty\).

5.2. Proof of Theorem 2.2. By Theorem 2 of [12], \( V_{(g,\psi)} : \mathcal{F}_{p} \to \mathcal{F}_{q} \) is bounded (compact) if and only if the integral transform \( B_{(g^{q},\psi)} \) belongs to \( L^{\frac{q}{p}}(C,dA) \). By Theorem 2 of [11], the operator \( J_{(g,\psi)} : \mathcal{F}_{p} \to \mathcal{F}_{q} \) is also bounded (compact) if and only if \( \tilde{B}_{(g^{q},\psi)} \) belongs to \( L^{\frac{q}{p}}(C,dA) \). Thus, our theorem will be proved once we prove the following key lemma which is again interest of its own.

Lemma 5.4. Let \( 0 < q < p < \infty \) and \( (g,\psi) \) be pairs of nonconstant entire functions on \( C \). Then

(i) \( M_{(g,\psi)} \in L^{\frac{q}{p}}(C,dA) \) if and only if \( B_{(g^{q},\psi)} \in L^{\frac{q}{p}}(C,dA) \).

(ii) \( \tilde{M}_{(g,\psi)} \in L^{\frac{q}{p-q}}(C,dA) \) if and only if \( \tilde{B}_{(g^{q},\psi)} \in L^{\frac{q}{p-q}}(C,dA) \).

Proof. (i). Making use of the estimate in (5.1) and Proposition 5.1 namely that \( \psi(z) = az + b \), we estimate

\[
\int_{C} M_{(g,\psi)}^{q}(z) dA(z) \lesssim \int_{C} B_{(g^{q},\psi)}(\psi(z)) dA(z) = \frac{1}{|a|^{2}} \int_{C} B_{(g^{q},\psi)}(\psi(z)) dA(z),
\]

and the necessity of the conditions follows.

To prove the sufficiency, we may consider

\[
\int_{C} B_{(g^{q},\psi)}(w) dA(w) = \int_{C} \left( \int_{C} |k_{w}(\psi(z))|^{q} \frac{|g(\psi(z))|^{q}}{1 + |z|^{q}} e^{-\frac{2}{5}|z|^{2}} dm(z) \right)^{\frac{q}{q}} dA(w).
\]
Applying Hölder’s inequality to the inner integral above gives
\[
H(w) := \left( \int_C |k_w(\psi(z))|^q \frac{|g'(z)|^q}{(1 + |z|)^q} e^{-\frac{q}{2}|z|^2} dA(z) \right)^{\frac{p}{p-q}}
\]
\[
\leq \int_C |k_w(\psi(z))|^q \frac{|g'(z)|^{\frac{qp}{p-q}}}{(1 + |z|)^{\frac{qp}{p-q}}} e^{-\frac{qp}{2(p-q)}|z|^2} e^{\frac{p}{2(p-q)}|\psi(z)|^2} dA(z)
\]
\[
\times \left( \int_C |k_w(\psi(z))|^q e^{-\frac{q}{2}|\psi(z)|^2} dA(z) \right)^{\frac{p}{2(p-q)}}
\]
\[
\lesssim \int_C |k_w(\psi(z))|^q \frac{|g'(z)|}{(1 + |z|)} e^{-\frac{qp}{2(p-q)}|z|^2} e^{\frac{p}{2(p-q)}|\psi(z)|^2} dA(z).
\]
Making use of Fubini’s Theorem and (1.3), we further estimate
\[
\int_C B_{\psi(z)}^p(w) dA(w) = \int_C H(w) dA(w)
\]
\[
= \int_C \left| \frac{g'(z)}{1 + |z|} \right|^{\frac{qp}{p-q}} e^{-\frac{qp}{2(p-q)}|z|^2} e^{\frac{p}{2(p-q)}|\psi(z)|^2} \int_C |K_{\psi(z)}(w)|^q e^{-\frac{q}{2}|w|^2} dA(w) dA(z)
\]
\[
\simeq \int_C \left| \frac{g'(z)}{(1 + |z|)} \right|^{\frac{qp}{p-q}} e^{\frac{p}{2(p-q)}|\psi(z)|^2 - |z|^2} dA(z) = \int_C M_{\psi(z)}^{\frac{p}{p-q}}(z) dA(z) < \infty.
\]
(ii). The necessity of the condition follows from the inequalities in (5.2) and the fact that \(\psi(z) = az + b\) as ensured by Proposition 5.1.

To prove the sufficiency, we may set
\[
T(w) := \left( \int_C |k_w(\psi(z))|^q \frac{(1 + |w|)^q}{(1 + |z|)^q} |g(z)|^q e^{-\frac{q}{2}|z|^2} dA(z) \right)^{\frac{p}{p-q}},
\]
and write
\[
\int_C B_{\psi(z)}^p(w) dA(w) = \int_C T(w) dA(w).
\]
Apply Hölder’s inequality and subsequently Fubini’s Theorem, (5.3), and (1.3) we observe that the above integral is bounded by
\[
\int_C \left| \frac{(1 + |\psi(z)|)}{1 + |z|} \right|^{\frac{qp}{p-q}} e^{-\frac{qp}{2(p-q)}|z|^2} e^{\frac{p}{2(p-q)}|\psi(z)|^2} \times \int_C \left( \frac{k'(\psi(z))}{1 + |\psi(z)|} \right|^q e^{-\frac{q}{2}|w|^2} dA(w) dA(z)
\]
\[
\simeq \int_C \left| \frac{(1 + |\psi(z)|)}{1 + |z|} \right|^{\frac{qp}{p-q}} e^{\frac{p}{2(p-q)}|\psi(z)|^2 - |z|^2} dA(z) = \int_C \tilde{M}_{\psi(z)}^{\frac{p}{p-q}}(z) dA(z),
\]
and completes the proof.

5.3. Proof of Theorem 3.1. By Theorem 4 of [12], the operator \(V_{(g, \psi)} : \mathcal{F}_2 \to \mathcal{F}_2\) belongs to the Schatten \(S_p(\mathcal{F}_2)\) class if and only if the Berezin type integral transform \(B_{(|g|^2, \psi)}\) belongs to \(L^{\frac{p}{2}}(\mathbb{C}, dA)\). Thus, we only need to establish the following lemma.
Lemma 5.5. Let $0 < p < \infty$ and $(g, \psi)$ be a pair of nonconstant entire functions on $\mathbb{C}$. Then $M_{(g, \psi)} \in L^p(\mathbb{C}, dA)$ if and only if $B_{(g^2, \psi)} \in L^2(\mathbb{C}, dA)$.

We remark in passing that Lemma 5.5 does not follow from Lemma 5.4.

Proof. Applying the inequalities in (5.1) and Proposition 5.1 we have

$$\int_{\mathbb{C}} M_{(g, \psi)}^p(z) dA(z) \leq \int_{\mathbb{C}} B_{(g^2, \psi)}^p(\psi(z)) dA(z) = \frac{1}{|a|^2} \int_{\mathbb{C}} B_{(g^2, \psi)}^p(z) dA(z),$$

from which one side of the assertion in the lemma follows.

For the remaining part, we consider two different cases.

Case 1. If $0 < p < 2$, then using (5.3) and the fact that $\mathcal{F}_p \subset \mathcal{F}_2$ for $0 < p \leq 2$,

$$\int_{\mathbb{C}} |k_w(\psi(\zeta))|^2 \frac{|g'(\zeta)|^2 e^{-|\zeta|^2}}{(1 + |\zeta|^2)^2} dA(\zeta)$$

$$\approx \int_{\mathbb{C}} \left( \int_0^z k_w(\psi(\zeta)) g'(\zeta) dA(\zeta) \right)^2 e^{-|z|^2} dA(z)$$

$$\lesssim \left( \int_{\mathbb{C}} \left| \int_0^z k_w(\psi(\zeta)) g'(\zeta) dA(\zeta) \right|^p e^{-\frac{p}{2}|z|^2} dA(z) \right)^{\frac{2}{p}}$$

$$\approx \left( \int_{\mathbb{C}} |k_w(\psi(\zeta))|^p |g'(\zeta)|^p e^{-\frac{p}{2}|\zeta|^2} \frac{1}{(1 + |\zeta|^2)^p} dA(\zeta) \right)^{\frac{2}{p}}.$$

Using this, Fubini’s Theorem and (1.3), we further estimate

$$\int_{\mathbb{C}} B_{(g^2, \psi)}^p(w) dA(w) \lesssim \int_{\mathbb{C}} \int_{\mathbb{C}} |k_w(\psi(\zeta))|^p |g(\zeta)|^p e^{-\frac{p}{2}|\zeta|^2} \frac{1}{(1 + |\zeta|^2)^p} dA(\zeta) dA(w)$$

$$\int_{\mathbb{C}} |g'(\zeta)|^p e^{-\frac{p}{2}|\zeta|^2} \int_{\mathbb{C}} |K_{\psi(\zeta)}(w)|^p e^{-\frac{p}{2}|w|^2} dA(\zeta) dA(w)$$

$$\approx \int_{\mathbb{C}} |g'(\zeta)|^p e^{-\frac{p}{2}|\zeta|^2} \frac{1}{(1 + |\zeta|^2)^p} dA(\zeta) \approx \int_{\mathbb{C}} M_{(g, \psi)}^p(\zeta) dA(\zeta) < \infty.$$

Case 2. If $p > 2$, then applying Hölder’s inequality we get

$$S(w) := \left( \int_{\mathbb{C}} |k_w(\psi(\zeta))|^2 \frac{|g'(\zeta)|^2 e^{-|\zeta|^2}}{(1 + |\zeta|^2)^2} dA(z) \right)^{\frac{2}{p}}$$

$$\leq \left( \int_{\mathbb{C}} |k_w(\psi(\zeta))|^2 \frac{|g'(\zeta)|^2 e^{-\frac{p}{2}|\zeta|^2}}{(1 + |\zeta|^2)^p} e^{\left(\frac{p}{2} - 1\right)|\psi(\zeta)|^2} dA(z) \right)$$

$$\times \left( \int_{\mathbb{C}} |k_w(\psi(\zeta))|^2 e^{-|\psi(\zeta)|^2} dA(z) \right)^{\frac{p-2}{2}}.$$

Making a change of variables and applying (1.3) again yields

$$\int_{\mathbb{C}} |k_w(\psi(\zeta))|^2 e^{-|\psi(\zeta)|^2} dA(z) \approx 1,$$
which implies that
\[ S(w) \lesssim \int_C |k_w(\psi(z))|^2 |g'(z)|^p |e^{-\frac{p}{2}|z|^2} z|^p e^{\frac{p}{2}|\psi(z)|^2} dA(z). \]

From this, Fubini’s Theorem and (1.3) we have the estimate
\[
\int_C B_{(g,\psi)}^p(w) dA(w) = \int_C S(w) dA(w)
\lesssim \int_C \int_C |k_w(\psi(z))|^2 |g'(z)|^p |e^{-\frac{p}{2}|z|^2} z|^p e^{\frac{p}{2}|\psi(z)|^2} dA(z) dA(w).
\]
\[
= \int_C |g'(z)|^p |e^{-\frac{p}{2}|z|^2} z|^p e^{\frac{p}{2}|\psi(z)|^2} \int_C |K_\psi(z)(w)|^2 |e^{-|w|^2} w|^2 dA(w) dA(z)
\lesssim \int_C |g'(z)|^p |e^{\frac{p}{2}(|\psi(z)|^2 - |z|^2)} dA(z) = \int_C M_{(g,\psi)}^p(z) dA(z) < \infty.
\]

5.4. **Proof of Theorem 4.1.** The sufficiency of the condition in the theorem follows from Theorem 2.1. Thus, we may assume that the difference \( V_{(g_1,\psi_1)} - V_{(g_2,\psi_2)} \) is compact and proceed to show the necessity. If one of either \( V_{(g_1,\psi_1)} \) or \( V_{(g_2,\psi_2)} \) is compact, so is the other one which follows from the algebra of compact operators. It follows that either both operators are compact or both are noncompact. Thus, we may assume the later. To this end, by Theorem 2.1, there exist two positive numbers \( \alpha_1 \) and \( \alpha_2 \) such that
\[
\alpha_1 = \limsup_{|z| \to \infty} M_{(g_1,\psi_1)}(z) \quad \text{and} \quad \alpha_2 = \limsup_{|z| \to \infty} M_{(g_2,\psi_2)}(z).
\]

Since \( k_w \) is weakly convergent, compactness of \( V_{(g_1,\psi_1)} - V_{(g_2,\psi_2)} \) implies that
\[
\|(V_{(g_1,\psi_1)} - V_{(g_2,\psi_2)}) k_w\|_q \to 0 \quad \text{as} \quad |w| \to \infty.
\]
(5.4)

On the other hand, using (5.3) and the techniques leading to (5.1), we have
\[
\|(V_{(g_1,\psi_1)} - V_{(g_2,\psi_2)}) k_w\|_q \simeq \left( \int_C |g'_1(z) k_w(\psi_1(z)) - g'_2(z) k_w(\psi_2(z))|^q |e^{-\frac{p}{2}|z|^2} dA(z) \right)^{\frac{1}{q}}
\geq \frac{1}{1 + |z|} |g'_1(z) k_\psi(\psi_1(z)) - g'_2(z) k_\psi(\psi_2(z))| e^{-\frac{p}{2}|z|^2}
\]
(5.5)

after setting \( w = \psi_1(z) \). The right-hand side above can be estimated further as
\[
\frac{1}{1 + |z|} |g'_1(z) k_\psi(\psi_1(z)) - g'_2(z) k_\psi(\psi_2(z))| e^{-\frac{p}{2}|z|^2}
\geq |M_{(g_1,\psi_1)}(z) - M_{(g_2,\psi_2)}(z) e^{-\frac{p}{2}|\psi_1(z) - \psi_2(z)|^2} |
\]
(5.6)

Since \( V_{(g_1,\psi_1)} \) and \( V_{(g_2,\psi_2)} \) are bounded operators, by Theorem 2.1 and Proposition 5.1, \( \psi_1 \) and \( \psi_2 \) have the linear forms \( \psi_1(z) = az + b \) and \( \psi_2(z) = cz + d \) where
$|a| \leq 1$ and $|c| \leq 1$. It follows from this that if $a \neq c$, then
\[
\lim_{|z| \to \infty} e^{-\frac{1}{2}|\psi_1(z) - \psi_2(z)|^2} = 0.
\]
Combining this with (5.4), (5.5), (5.6) and the triangle inequality, we deduce
\[
M_{(g_1, \psi_1)}(z) \leq \frac{M_{(g_2, \psi_2)}(z)}{e^{\frac{1}{2}|\psi_1(z) + \psi_2(z)|^2}} + \left| M_{(g_1, \psi_1)}(z) - \frac{M_{(g_2, \psi_2)}(z)}{e^{\frac{1}{2}|\psi_1(z) + \psi_2(z)|^2}} \right|
\]
which implies
\[
\lim_{|z| \to \infty} M_{(g_1, \psi_1)}(z) = 0.
\]
It follows from this and Theorem 2.1 that $V_{(g_1, \psi_1)}$ is compact which contradicts our assumption. Thus, we must have $a = c$. Taking this into account,
\[
\alpha_1 - \alpha_2 e^{-\frac{1}{2}|b-d|^2} \leq \limsup_{|z| \to \infty} \left| M_{(g_1, \psi_1)}(z) - M_{(g_2, \psi_2)}(z) e^{-\frac{1}{2}|b-d|^2} \right|
\]
\[
\lesssim \limsup_{|z| \to \infty} \| (V_{(g_1, \psi_1)} - V_{(g_2, \psi_2)}) k_{\psi_1(z)} \|_q = 0
\]
from which we get
\[(5.7) \quad \alpha_1 \leq \alpha_2 e^{-\frac{1}{2}|b-d|^2}.
\]
On the other hand, if we repeat the above process by setting $w = \psi_2(z)$, we get
\[
\alpha_2 \leq \alpha_1 e^{-\frac{1}{2}|b-d|^2}.
\]
From this and (5.7), we find
\[
\alpha_1 \leq \alpha_2 e^{-\frac{1}{2}|b-d|^2} \leq \alpha_1 e^{-|b-d|^2} \leq \alpha_1,
\]
which holds only if $b = d$. This shows that $\psi_1 = \psi_2$ and hence the necessity of the condition follows from Theorem 2.1.
The proof of part (ii) follows in a similar fashion.

5.5. **Proof of Theorem 4.2.** Since all Schatten $S_p(\mathcal{F}_2)$ class operators are compact, we can assume that $V_{(g_1, \psi_1)} - V_{(g_2, \psi_2)}$ is compact. Then by Theorem 4.1, the difference is compact if and only if either both $V_{(g_1, \psi_1)}$ and $V_{(g_2, \psi_2)}$ are compact or $\psi_1 = \psi_2 = \psi$ and condition (4.1) holds. If both are compact and the difference is in the $S_p$ class, then either both are in the $S_p(\mathcal{F}_2)$ class or both are not. In the latter case, following the same argument as in the proof of Theorem 4.1, the assumption that $V_{(g_1, \psi_1)} - V_{(g_2, \psi_2)}$ belongs to $S_p$ implies $\psi_1 = \psi_2 = \psi$. On the other hand, since $V_{(g_1, \psi_1)} - V_{(g_2, \psi_2)} = V_{(g_1 - g_2, \psi)}$ is itself a generalized Volterra-type integral operator induced by the pair of symbols $(g_1 - g_2, \psi)$, by Theorem 3.1, it belongs to the $S_p$ class if and only $M_{(g_1 - g_2, \psi)} \in L^p(\mathbb{C}, dA)$. That is
\[
\int_{\mathbb{C}} \frac{|g'(z)|^p}{(1 + |z|)^p} e^{\frac{1}{2}(|\psi(z)|^2 - |z|^2)} dA(z) < \infty
\]
which completes the proof of part (i) in the theorem.
The proof of part (ii) follows in a similar manner.
Proof of Corollary 4.3. Part (i) of the corollary follows once by setting \( \psi_1(z) = \psi_2(z) = z \) in Theorem 4.1. Thus, we shall verify part (ii). By part (i), if the difference \( V_{g_1} - V_{g_2} \) is compact, then both \( V_{g_1} \) and \( V_{g_2} \) are compact and hence Corollary 2 of [12] gives the representation \( g_1(z) = az + b \) and \( g_2(z) = cz + d \). Since the integral is linear, we also have the relation

\[
V_{g_1} - V_{g_2} = V_{g_1 - g_2}.
\]

On the other hand, \( V_{g_1 - g_2} = V_{g_1 - g_2, z} \). Then by Theorem 3.1, the difference operator \( V_{g_1 - g_2} \) belongs to the Schatten \( S_p \) class if and only if

\[
\int_\mathbb{C} \frac{|(g'_1 - g'_2)(z)|^p}{(1 + |z|)^p} dA(z) = \int_\mathbb{C} \frac{|a + b - c - d|^p}{(1 + |z|)^p} dA(z) < \infty. \tag{5.8}
\]

Since \( g_1 \neq g_2 \), we easily see that (5.8) holds only if \( p > 2 \).

Part (iii). By linearity of the integral, \( V_{g_1} - V_{g_2} = V_{g_1 - g_2} = V_{g_3} \) where

\[
g_3(z) = (a_1 - a_2)z^2 + (b_1 - b_2)z + c_1 + c_2.
\]

Then by Theorem 1.3 of [14],

\[
\sigma(V_{g_3}) = \{0\} \cup \left\{ \lambda \in \mathbb{C} \setminus \{0\} : e^{\frac{\pi i}{2}} \in \mathcal{F}_p \right\} = \left\{ \lambda \in \mathbb{C} : |\lambda| \leq 2|a_1 - a_2| \right\} = \sigma(V_{g_1} - V_{g_2}).
\]

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