Cent $U(n)$ and a construction of Lipsman–Wolf

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Abstract: Let $G$ be a complex simply-connected semisimple Lie group and let $\mathfrak{g} = \text{Lie } G$. Let $\mathfrak{g} = \mathfrak{n}_- + \mathfrak{h} + \mathfrak{n}$ be a triangular decomposition of $\mathfrak{g}$. The authors in [LW] introduce a very nice representation theory idea for the construction of certain elements in center $U(n)$. A key lemma in [LW] is incorrect but the idea is in fact valid. In our paper here we modify the construction so as to yield the desired elements in center $U(n)$.

Key words: cascade of orthogonal roots, invariant theory

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1. Introduction

1.1. Let $G$ be a complex simply-connected semisimple Lie group and let $\mathfrak{g} = \text{Lie } G$. Let $\mathfrak{g} = \mathfrak{n}_- + \mathfrak{h} + \mathfrak{n}$ be a triangular decomposition of $\mathfrak{g}$. If $\mathfrak{s}$ is a complex Lie algebra, then $U(\mathfrak{g})$ and $S(\mathfrak{g})$ will denote respectively the enveloping algebra and symmetric algebra of $\mathfrak{g}$.

Some time ago I introduced what is presently referred to as the cascade of orthogonal roots. Using the cascade, Anthony Joseph and I independently obtained, with very different methods, a number of structure theorems of $S(\mathfrak{n})^n$ (or equivalently cent $U(n)$). The cascade is also used in [LW]. The present paper deals with a neat, interesting representation-theoretic idea in [LW] for constructing certain elements in $S(\mathfrak{n})^n$. Basically the construction begins with the linear functional $f$ on $U(\mathfrak{g})$ obtained, as a matrix unit, involving the highest and lowest weight vectors of the irreducible representation $\pi_\nu$ of $G$ with highest weight $\nu$. Without unduly detracting from the idea we point out here that a key lemma in [LW] (Lemma 3.7) is incorrect as it stands.
A counterexample is given in our present paper. Next we show that the construction can be modified so as to produce a correct result. The modification is of independent interest in that it introduces the notion of what we call the codegree of a linear functional like \( f \). If the codegree of \( f \) is \( k \), then one obtains a harmonic element \( f_{(k)} \) of degree \( k \) in \( S(\mathfrak{g}) \). We then go on to show that \( f_{(k)} \) is the desired element in \( S(\mathfrak{n})^n \). The main result is Theorem 2.8.

2. Lipsman–Wolf construction

2.1. Let \( \mathfrak{g} \) be a complex semisimple Lie algebra and let \( G \) be a simply-connected complex Lie group such that \( \mathfrak{g} = \text{Lie} \, G \). Let \( \ell = \text{rank} \, \mathfrak{g} \) and let \( \mathcal{B} \) be the Killing form \( (x, y) \) on \( \mathfrak{g} \). If \( u \) is a complex vector space, \( S(u) \) will denote the symmetric (graded) algebra over \( u \). \( \mathcal{B} \) extends to a nonsingular symmetric bilinear form, still denoted by \( \mathcal{B} \), on \( S(\mathfrak{g}) \) where, if \( x, y \in \mathfrak{g} \) and \( m, n \in \mathbb{N} \), then \( (x^n, y^m) = 0 \) if \( m \neq n \) and

\[
(x^n, y^n) = n!(x, y)^n.
\]

One may then identify \( S(\mathfrak{g}) \) with the algebra of polynomial functions on \( \mathfrak{g} \) where

\[
x^n(y) = (x, y)^n.
\]

Also \( u \mapsto \partial_u \) defines an isomorphism of \( S(\mathfrak{g}) \) with the algebra of differential operators on \( \mathfrak{g} \) with constant coefficients. If \( u, v, w \in S(\mathfrak{g}) \) one readily has

\[
(u, vw) = (\partial_v u, w).
\] (2.1)

The symmetric algebra \( S(\mathfrak{g}) \) becomes a degree-preserving \( G \)-module by extending, as a group of automorphisms, the adjoint action of \( G \) on \( \mathfrak{g} \). Let \( J = S(\mathfrak{g})^G \). Then one knows (Chevalley) that \( J = \mathbb{C}[p_1, \ldots, p_\ell] \) is a polynomial ring with homogeneous generators \( p_1, \ldots, p_\ell \). A polynomial \( f \in S(\mathfrak{g}) \) is called harmonic if \( \partial_{p_i} f = 0, i = 1, \ldots, \ell \). Let \( H \subset S(\mathfrak{g}) \) be the graded subspace of all harmonic polynomials in \( S(\mathfrak{g}) \). Then \( H \) is a
G-submodule of $S(\mathfrak{g})$ and one knows

$$S(\mathfrak{g}) = J \otimes H$$

is a $G$-module decomposition of $S(\mathfrak{g})$. Explicitly, for any $k \in \mathbb{N}$, $H^k$ is given by

$$H^k = \text{The } \mathbb{C} \text{ span of } \{w^k \mid \text{ where } w \in \mathfrak{g} \text{ is nilpotent.}\}$$

2.2. Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$ and let $\Delta$ be the set of roots of $(\mathfrak{h}, \mathfrak{g})$. Also for any $\varphi \in \Delta$ let $e_\varphi \in \mathfrak{g}$ be a corresponding root vector. If $\mathfrak{s} \subset \mathfrak{g}$ is any subspace which is stable under $\text{ad} \mathfrak{h}$ let $\Delta(\mathfrak{s}) = \{\varphi \in \Delta \mid e_\varphi \in \mathfrak{s}\}$. Let $\mathfrak{b}$ be a Borel subalgebra of $\mathfrak{g}$ which contains $\mathfrak{h}$ and let $\mathfrak{n}$ be the nilradical of $\mathfrak{b}$. A system of positive roots $\Delta_+$ in $\Delta$ is chosen so that $\Delta_+ = \Delta(\mathfrak{n})$. Let $\mathfrak{b}_-$ be the Borel subalgebra containing $\mathfrak{h}$ which is “opposite” to $\mathfrak{b}$. One then has a triangular decomposition

$$\mathfrak{g} = \mathfrak{n}_- + \mathfrak{h} + \mathfrak{n}$$

where $\mathfrak{n}_-$ is the nilradical of $\mathfrak{b}_-$. If $\Delta_- = \Delta(\mathfrak{n}_-)$, then of course $\Delta_- = -\Delta_+$.

Let $L$ be the lattice of integral linear forms on $\mathfrak{h}$ and let $L_o$ be the sublattice of integral linear forms which are also in the root lattice. If $M$ is any locally finite $G$-module and $\mu \in L$, then $M(\mu)$ will denote the $\mu$-weight space. Let $\text{Dom}(L)$ (resp. $\text{Dom}(L_o)$) be the set of dominant elements in $L$ (resp. $L_o$). For any $\lambda \in \text{Dom}(L)$ let $\pi_\lambda : G \to \text{Aut} V_\lambda$ be some fixed irreducible finite-dimensional representation with highest weight $\lambda$. One knows that

$$V_\lambda(0) \neq 0 \iff \lambda \in \text{Dom}(L_o).$$

(2.4)

If $M$ is a locally finite $G$-module, let $M(\lambda)$ be the primary component of $M$ corresponding to $\lambda$. It is obvious that if $\mu \in L$ and $S(\mathfrak{g})(\mu) \neq 0$, then $\mu \in L_o$. In particular if $\lambda \in \text{Dom}(L)$ and $S(\mathfrak{g})(\lambda) \neq 0$, then $\lambda \in \text{Dom}(L_o)$. On the other hand if $\lambda \in \text{Dom}(L_o)$, then (2.2) readily implies

$$S(\mathfrak{g})(\lambda) = J \otimes H(\lambda).$$

(2.5)
For any $\lambda \in \text{Dom}(L_o)$ let
\[ \ell(\lambda) = \dim V_\lambda(0). \] (2.6)

Then one knows
\[ \dim \text{Hom}_G(V_\lambda, H) = \text{Hom}_G(V_\lambda, H(\lambda)) = \ell(\lambda). \] (2.7)

Let $\sigma_i, i = 1, \ldots, \ell(\lambda)$, be a basis of $\text{Hom}_G(V_\lambda, H(\lambda))$ and put $H_{\lambda,i} = \sigma_i(V_\lambda)$ so that one has a complete reduction of $H(\lambda)$,
\[ H(\lambda) = \sum_{i=1}^{\ell(\lambda)} H_{\lambda,i} \] (2.8)

into a sum of irreducible components. Furthermore we can choose the $\sigma_i$ so that $H_{\lambda,i}$ is homogeneous for all $i$. In fact there is a unique nondecreasing sequence of integers, $m_i(\lambda), i = 1, \ldots, m_{\ell(\lambda)}(\lambda)$, which are referred to as generalized exponents such that
\[ H_{\lambda,i} \subset H^{m_i(\lambda)}. \] (2.9)

Moreover the maximal generalized exponent, $m_{\ell(\lambda)}(\lambda)$, occurs with multiplicity 1. That is
\[ m_i(\lambda) < m_{\ell(\lambda)}(\lambda) \] (2.10)
for any $i < \ell(\lambda)$ and the maximal generalized exponent is explicitly given by
\[ m_{\ell(\lambda)}(\lambda) = \sum_{i=1}^{\ell} k_i \] (2.11)

where $\lambda = \sum_{i=1}^{\ell} k_i \alpha_i$ and
\[ \{\alpha_i\}, i = 1, \ldots, \ell, \] (2.12)
is the set of simple positive roots.

Let $\min m(\lambda)$ be the minimal value of $m_i(\lambda)$ for $i = 1, \ldots, m_{\ell(\lambda)}$. If $\ell(\lambda) > 1$, note that
\[ \min m(\lambda) < m_{\ell(\lambda)}(\lambda) \] (2.13)
by (2.10).

Clearly $H_{\lambda,i}(\lambda)$ is the highest weight space of $H_{\lambda,i}$ and hence we also note, by (2.5), that

\[
(S(\mathfrak{g})(\lambda))^n = \sum_{i=1}^{\ell(\lambda)} J \otimes H_{\lambda,i}(\lambda).
\]  

(2.14)

Of course the left side of (2.14) is graded. It follows immediately from (2.14) that if $S^k(\mathfrak{g})(\lambda)^n \neq 0$, then

\[
k \geq \min m(\lambda).
\]  

(2.15)

2.3. The universal enveloping algebra of a Lie algebra $\mathfrak{s}$ is denoted by $U(\mathfrak{s})$. Since we will be dealing with multiplication in both $U(\mathfrak{g})$ and $S(\mathfrak{g})$, when $x \in \mathfrak{g}$, we will on occasion to avoid confusion write $\tilde{x}$ for $x$ when $x$ is to be regarded as an element of $U(\mathfrak{g})$ and not $S(\mathfrak{g})$. Of course $U(\mathfrak{g})$ is a $G$-module by extension of the adjoint representation. By PBW one has a $G$-module isomorphism

\[
\tau : S(\mathfrak{g}) \rightarrow U(\mathfrak{g}) \text{ where for any } k \in \mathbb{N} \text{ and } x \in \mathfrak{g} \text{ one has } \tau(x^k) = \tilde{x}^k.
\]  

(2.16)

Since $\tau$ is a $\mathfrak{g}$-module isomorphism the restriction of (2.16) to $S(\mathfrak{g})^n$ and to $S(\mathfrak{n})^n$ readily yields linear isomorphisms

\[
\tau : S(\mathfrak{g})^n \rightarrow U(\mathfrak{g})^n
\]  

(2.17)

and

\[
\tau : S(\mathfrak{n})^n \rightarrow \text{cent } U(\mathfrak{n}).
\]  

(2.18)

If $\mathfrak{s} \subset \mathfrak{g}$ is a Lie subalgebra and $k \in \mathbb{N}$ the image, under $\tau$, of $S^k(\mathfrak{s})$, will be denoted by $U^{(k)}(\mathfrak{s})$, and one readily notes the direct sum

\[
U(\mathfrak{s}) = \sum_{k=0}^{\infty} U^{(k)}(\mathfrak{s}).
\]  

(2.19)
If $u \in U(s)$ then, by abuse of terminology, we will say that $u$ is homogeneous of degree $k$ if $u \in U^{(k)}(s)$. We are particularly interested in the case where $s = n$ and $n_-$.

Since $\mathfrak{h}$ normalizes $n$, both sides of (2.17) and (2.18) are bigraded by degree and $(\mathfrak{h})$ weight and clearly (2.17) and (2.18) preserve the bigrading. The proof of the following theorem uses results in [J] where cent $U(n)$ is denoted by $Z(n)$.

**Theorem 2.1.** (T. Joseph) Let $\lambda \in \text{Dom}(L_\alpha)$ and assume that $S(n)^{n}(\lambda) \neq 0$. Then $S(n)^{n}(\lambda)$ is homogeneous of degree $\min m(\lambda)$.

**Proof.** By (2.18) one has cent $U(n)(\lambda) \neq 0$. By (iii), p. 260, in the Theorem of §4.12 of [J] and (iii), p. 261, in the lemma of §4.13 of [J] one has that cent $U(n)(\lambda)$ is homogeneous of degree $j$ for some $j \in \mathbb{N}$. But then $S(n)^{n}(\lambda)$ is homogeneous of degree $j$ by (2.18). Consequently

$$j \geq \min m(\lambda) \quad (2.20)$$

by (2.15). Assume $j > \min m(\lambda)$. Then there exists $i \in \{1, \ldots, \ell(\lambda)\}$ such that

$$m_i(\lambda) < j. \quad (2.21)$$

But $H_{\lambda,i}(\lambda) \subset (S^{m_i(\lambda)}(g))^{n}(\lambda)$. It follows then that $(U^{(m_i(\lambda))}(g))^{\lambda}(\lambda) \neq 0$ by (2.17). But this and (2.21) contradict (iii), p. 261, in the lemma of §4.13 of [J]. Thus $j = \min m(\lambda)$. QED

2.4. Let $e = \sum_{i=1}^{\ell} e_{\alpha_i}$ so that $e$ is a principal nilpotent of $g$. Let $h \in \mathfrak{h}$ be fixed so that $\alpha_i(h) = 2$ for all simple roots $\alpha_i$ so that

$$[h, e] = 2e. \quad (2.22)$$

Then as one knows there exists $c_i \in \mathbb{C}^\times$, $i = 1, \ldots, \ell$, such that if $e_- = \sum_{i=1}^{\ell} c_i e_{-\alpha_i}$, then $\{h, e, e_-\}$ is an $Sl(2)$-triple and spans a principal TDS $\mathfrak{a}$ in $g$. Let $\xi \in \text{Dom}(L)$. Let $\xi^* \in \text{Dom}(L)$ be the highest weight of the $G$-module $V_{\xi}^*$ dual to $V_{\xi}$. We retain
this $*$ notation throughout. Then as $G$-modules

$$V_\xi \otimes V_\xi^* \cong \text{End} V_\xi.$$  

Let $\nu = \xi + \xi^*$ so that $V_\nu$ identifies with the Cartan product of $V_\xi$ and $V_\xi^*$. Since the weights of $V_\xi^*$ are the negatives of the weights of $V_\xi$ it follows that $\nu \in \text{Dom}(L_0)$. Furthermore since $tr AB$ defines a nonsingular invariant symmetric bilinear form on $\text{End} V_\xi$ it follows immediately that the corresponding bilinear form on $V_\xi \otimes V_\xi^*$ restricts to a nonsingular $G$-invariant bilinear form $(u, v)$ on $V_\nu$. Thus the highest and lowest weights in $V_\nu$ are respectively $\nu$ and $-\nu$. Consider the action of the principal TDS $\mathfrak{a}$ on $V_\nu$. Clearly, by the dominance of $\nu$ and the regularity of $h$, the maximal (resp. minimal) eigenvalue of $\pi_\nu(h)$ on $V_\nu$ is $\nu(h)$ (resp. $-\nu(h)$) and these eigenvalues have multiplicity 1. Thus if $0 \neq v_\nu$ and $0 \neq v_{-\nu}$ are respectively highest and lowest $\mathfrak{g}$-weight vectors there exists an irreducible $\mathfrak{a}$-module $M$ in $V_\nu$ having $v_\nu$ and $v_{-\nu}$ respectively as highest ($\mathbb{C} h$) weight vectors. Hence for some $z \in \mathbb{C}^\times$ one has

$$\left(\pi_\nu(e_-)^{\nu(h)}v_\nu\right) v_{-\nu} = z v_{-\nu}.$$  

(2.23)

Let $\lambda = 2\nu$ so that certainly $\lambda \in \text{Dom} L_0$. Recalling (2.11) and the 2 in (2.22) one has

$$\nu(h) = m_{\ell(\lambda)}(\lambda).$$  

(2.24)

But clearly $(v_{-\nu}, v_\nu) \neq 0$ (because of multiplicity 1). Thus one has

$$\left(\pi_\nu(e_-^{m_{\ell(\lambda)}(\lambda)})v_\nu, v_\nu\right) \neq 0.$$  

(2.25)

For the tilde notation see (2.16).

In the next section we will give a simple condition guaranteeing that $\ell(\lambda) > 1$.

2.5. We noted the following result, Proposition 2.2, a long time ago. However it is likely that the result was well known even then but we are unable to find published
references to it so, for completeness, we will give a proof here. The applications, Theorems 2.4 and 2.5, of Proposition 2.2, are recent with me. Let \( \beta, \gamma \in \text{Dom}(L) \).

Then, as one knows, the Cartan product \( V_{\beta+\gamma} \) occurs with multiplicity one in the tensor product \( V_{\beta} \otimes V_{\gamma} \) so that there exists a unique \( G \)-invariant projection

\[
\Gamma : V_{\beta} \otimes V_{\gamma} \rightarrow V_{\beta+\gamma}.
\]  

(2.26)

**Proposition 2.2.** Let \( 0 \neq u \in V_{\beta} \) and \( 0 \neq w \in V_{\gamma} \). Then

\[
\Gamma(u \otimes w) \neq 0.
\]  

(2.27)

**Proof.** Let \( 0 \neq v_\beta \) (resp. \( 0 \neq v_\gamma \)) be a highest weight vector in \( V_{\beta} \) (resp. \( V_{\gamma} \)) so that \( v_\beta \otimes v_\gamma \) is a highest weight vector in the Cartan product \( V_{\beta+\gamma} \). Consequently taking into account the action of \( G \),

\[
\{ g \cdot v_\beta \otimes g \cdot v_\gamma \mid g \in G \} \text{ spans } V_{\beta+\gamma}.
\]  

(2.28)

Let \( K \) be a maximal compact subgroup of \( G \) (so that \( G \) is the complexification of \( K \)) and let \( \{ y, z \}_\beta \) (resp. \( \{ y, z \}_\gamma \)) be a \( K \)-invariant Hilbert space structure on \( V_{\beta} \) (resp. \( V_{\gamma} \)). These induce a natural \( K \)-invariant Hilbert space structure \( \{ y, z \}_{\beta,\gamma} \) on \( V_{\beta} \otimes V_{\gamma} \). Furthermore it is immediate that \( \Gamma \) is a Hermitian projection with respect to the latter inner product. Thus to prove the proposition it suffices to show that there exists \( g_o \in G \) such that

\[
\{ g_o \cdot v_\beta \otimes g_o \cdot v_\gamma, u \otimes w \}_{\beta,\gamma} \neq 0.
\]  

(2.29)

But now since \( V_{\beta} \) (resp. \( V_{\gamma} \)) is the span of \( \{ g \cdot v_\beta \mid g \in G \} \) (resp. \( \{ g \cdot v_\gamma \mid g \in G \} \)) it follows immediately that the function \( F_\beta \) (resp. \( F_\gamma \)) on \( G \) given by \( F_\beta(g) = \{ g \cdot v_\beta, u \}_\beta \) (resp. \( F_\gamma(g) = \{ g \cdot v_\gamma, w \}_\gamma \)) is nonvanishing and analytic. Thus there exists \( g_o \in G \) such that \( F_\beta(g_o)F_\gamma(g_o) \neq 0 \). But the left side of (2.29) equals \( F_\beta(g_o)F_\gamma(g_o) \). This proves (2.29).
As a corollary one has

**Proposition 2.3.** Let the notation be as in Proposition 2.2. Assume that \( s \) is any subspace of \( V_\beta \). Then the map

\[
s \to V_{\beta + \gamma}, \ x \mapsto \Gamma(x \otimes w)
\]

is linear and injective. Furthermore if \( s \subset V_\beta(\mu) \) for some \( \mu \in L \) and \( w \in V_\gamma(\delta) \), for some \( \delta \in L \), then the image of (2.30) is contained in \( V_{\beta + \gamma}(\mu + \delta) \).

**Proof.** Obviously (2.30) is linear. The injectivity is immediate from Proposition 2.2. The second conclusion follows from the fact that \( \Gamma \) is, among other things, an \( \mathfrak{h} \)-map. QED

We now have the following information about a 0-weight space.

**Theorem 2.4.** Let \( \beta \in \text{Dom}(L) \) and let \( \beta^* \) be the highest weight of the contra-
gredient module to \( V_\beta \). Let \( d \) be the maximal value of all the multiplicities of weights in \( V_\beta \). Then

\[
\dim V_{\beta + \beta^*}(0) \geq d.
\]

**Proof.** We retain the notation of Propositions 2.2 and 2.3. Choose \( \gamma = \beta^* \) and let \( \mu \) be any weight of \( V_\beta \). Choose \( \delta = -\mu \). Then the image of (2.30) is contained in \( V_{\beta + \beta^*}(0) \) by Proposition 2.3. But the dimension of the image equals \( \dim V_\beta(\mu) \) by the injectivity of (2.30). Since \( \mu \) is arbitrary this of course implies (2.31). QED

A similar argument leads to the following monotonicity result of weight multiplicities.
**Theorem 2.5.** Let \( \beta \in \text{Dom}(L) \) and \( \gamma \in \text{Dom}(L_o) \). Then for any weight \( \mu \) of \( V_\beta \) one has
\[
\dim V_{\beta+\gamma}(\mu) \geq \dim V_\beta(\mu).
\] (2.32)

**Proof.** Again we use the notation and result in Propositions 2.3. Since \( \gamma \in \text{Dom}(L_o) \) one knows \( V_\gamma(0) \neq 0 \). Let \( \delta = 0 \) in Proposition 2.3 and let \( \mathfrak{s} = V_\beta(\mu) \) so that the image of (2.30) is contained in \( V_{\beta+\gamma}(\mu) \). But then (2.32) follows from the injectivity of (2.30). QED

We return to the notation of §2.4. Recall that \( \xi \in \text{Dom}(L) \), \( \nu = \xi + \xi^* \) and \( \lambda = 2\nu \) so that \( \nu, \lambda \) are in \( \text{Dom}(L_o) \). Let \( d \) be the maximal value of all weight multiplicities of \( V_\xi \).

**Theorem 2.6.** Assume
\[
d > 1.
\] (2.33)

Then \( \dim V_\lambda(0) > 1 \) so that
\[
\min m(\lambda) < m_{\ell(\lambda)}(\lambda).
\] (2.34)

Furthermore \( \xi \) can be chosen so that (2.33) is satisfied if and only if there exists a simple component of \( \mathfrak{g} \) which is not of type \( A_1 \).

**Proof.** One has \( \dim V_\nu(0) > 1 \) by Theorem 2.4. But then \( \dim V_\lambda(0) > 1 \) by Theorem 2.5. The statement (2.34) is then just (2.13). If all the simple components of \( \mathfrak{g} \) are of type \( A_1 \), then clearly \( d = 1 \) for any \( \xi \in \text{Dom}(L) \). However, if not, then \( d > 1 \) for the adjoint representation of a component, not of type of \( A_1 \), when extended trivially to the other components. QED

**2.6.** In [LW] the authors introduce a very neat idea for constructing certain elements in \( S(n)^n \) (or equivalently in \( \text{cent} \ U(n) \)) using representation theory. The
statement of this idea, Lemma 3.7 in [LW], however, is not correct as it stands. We
give a counterexample in this section. Nevertheless the statement of this lemma in
[LW] can be modified so as to establish that this very interesting technique in [LW]
does indeed yield elements in $S(n)^n$. We do this in the section that follows this one.

The bilinear form $B$ on $S(g)$ clearly defines a nonsingular pairing of $S(n_-)$ and
$S(n)$ with $S^i(n_-)$ orthogonal to $S^j(n)$ when $i \neq j$ and $S^i(n) \simeq S^i(n_-)^*$. Let the
following notation be as in §2.4 and let $f_\nu \in S(n)$ be defined so that if $\Xi \in S(n_-)$,
then
\[
(f_\nu, \Xi) = (\pi_\nu(\tau(\Xi)) v_\nu, v_\nu)
\]  
where $\tau$ is defined as in (2.16). One notes that (5.35) vanishes if $\Xi \in S^j(n_-)$ for $j > \dim V_\nu$ so that $f_\nu \in S(n)$ is well defined. Lemma 3.7 in [LW] asserts that
\[
f_\nu \in S(n)^n(\lambda).
\]  
But (2.35) is not 0 if $\Xi = (e_-)^{\ell(\lambda)}(\lambda)$ by (2.25). On the other hand $S(n)^n(\lambda) \subset S^{\min \, m(\lambda)}(g)$
by Theorem 2.1 and $(e_-)^{\ell(\lambda)}(\lambda) \in S^{\min \, m(\lambda)}(g)$. But one has (2.34) if $\xi$ and $g$ are
chosen as in the last statement in Theorem 2.6. Such a choice leads to a contradiction
since (2.34) implies $S^{\min \, m(\lambda)}(g)$ is orthogonal to $S^{\min \, m(\lambda)}(g)$.

2.7. Let $U_n(g)$, $n \in \mathbb{N}$, be the standard filtration of $U(g)$. A nonzero linear
functional $f$ on $U(g)$ will be said to have codegree $k$ if $k \in \mathbb{N}$ is maximal such that $f$
vanishes on $U_{k-1}(g)$ (putting $U_{-1}(g) = 0$). Assume $f$ has codegree $k$ and $k \geq 1$. Note
that if $x_i \in g$, $i = 1, \ldots, k$, then for any permutation $\sigma$ of $\{1, \ldots, k\}$ one has
\[
f(\tilde{x}_1 \cdots \tilde{x}_k) = f(\tilde{x}_{\sigma(1)} \cdots \tilde{x}_{\sigma(k)})
\]  
using the notation of (2.16).

Now let $f_{(k)}$ be the linear functional on $S^k(g)$ defined by the restriction $f \circ \tau$ on
$S^k(g)$. Using $B$ on $S(g)$ we may regard $f_{(k)}$ as an element in $S^k(g)$ so that by (2.37)
one has
\[ f(\bar{x}_1 \cdots \bar{x}_k) = (f_{(k)}, x_1 \cdots x_k) \] (2.38)
for any \( x_i \in \mathfrak{g}, i = 1, \ldots, k \). One also notes that if \( k = i + j \), where \( i, j \in \mathbb{N} \), and \( v \in S^i(\mathfrak{g}), w \in S^j(\mathfrak{g}) \), then
\[ f(\tau(v)\tau(w)) = (f_{(k)}, vw) \] (2.39)
since, clearly \( \tau(v)\tau(w) - \tau(vw) \in U_{k-1}(\mathfrak{g}) \).

Taking a clue from [LW] we will choose \( f \) so that it arises from a matrix entry of a \( U(\mathfrak{g}) \)-module. In fact assume \( M \) is a \( U(\mathfrak{g}) \)-module (not necessarily finite dimensional) with respect to a representation \( \pi : U(\mathfrak{g}) \to \text{End} M \). We recall that \( \pi \) has an infinitesimal character \( \chi : \text{cent} U(\mathfrak{g}) \to \mathbb{C} \) if for any \( z \in \text{cent}, U(\mathfrak{g}) \) one has \( \pi(z) = \chi(z) \text{Id}_M \). Then one has

**Theorem 2.7.** Assume \( \pi \) is a representation of \( U(\mathfrak{g}) \) on a vector space \( M \) and that \( \pi \) has an infinitesimal character \( \chi \). Assume also if \( s \in M \) and \( s' \in M^* \) are such that \( f \in U(\mathfrak{g})^* \), defined by \( f(u) = s'(\pi(u)s) \), for any \( u \in U(\mathfrak{g}) \), is nonvanishing with codegree \( k \geq 1 \). Then \( f_{(k)} \) is harmonic. That is \( f_{(k)} \in H^k \).

**Proof.** Let \( i \in \mathbb{N} \) where \( i \geq 1 \). We must show that if \( r \in S^i(\mathfrak{g})^g \), then
\[ \partial_r f_{(k)} = 0. \] (2.40)
Obviously one has (2.40) if \( i > k \) so that we can assume that \( i \leq k \). Let \( j = k - i \), so that \( j < k \), and let \( w \in S^j(\mathfrak{g}) \) be arbitrary so that it suffices to show that
\[ (\partial_r f_{(k)}, w) = 0. \] (2.41)
But by (2.1) the left side of (2.41) equals \( (f_{(k)}, rw) \). But clearly \( \tau(r) \in \text{cent} U(\mathfrak{g}) \). Let \( c = \chi(\tau(r)) \). Then by (2.39)
\[ (f_{(k)}, rw) = f(\tau(r)\tau(w)) \]
\[ = cf(\tau(w)). \]
But $\tau(w) \in U_{k-1}(\mathfrak{g})$ since $j < k$. Hence $f(\tau(w)) = 0$. Thus $(f(k), rw) = 0$. This establishes (2.40). QED

The main result, to follow, will use notations of §2.4 but with basically fewer restrictive conditions. We eliminate $\xi$ and $0 \neq \nu \in \text{Dom}(L)$ is now arbitrary. Put $\lambda = \nu + \nu^*$. Also $V_{\nu^*}$ will be identified with the dual space $V_{\nu}^*$ to $V_{\nu}$. Let $0 \neq \nu$ (resp. $0 \neq \nu^*$) be a highest weight vector of $V_{\nu}$ (resp. $V_{\nu^*}$). Let $f \in U(\mathfrak{g})^*$ be defined by

$$f(u) = v_{\nu^*}(\pi_{\nu}(u)v_{\nu})$$  
(2.42)

for any $u \in U(\mathfrak{g})$. Expressed another way regard $v_{\nu} \otimes v_{\nu^*}$ as a rank 1 linear operator on $V_{\nu}$ where for any $v \in V_{\nu}$ one has $v_{\nu} \otimes v_{\nu^*}(v) = v_{\nu^*}(v) v_{\nu}$. Then

$$f(u) = \text{tr} \pi_{\nu}(u) v_{\nu} \otimes v_{\nu^*}. \quad (2.43)$$

Now let $k$ be the codegree of $f$. But $v_{\nu^*}(v_{\nu}) = 0$ since $0 \neq \nu$. Thus $k \geq 1$. It is then immediate from (2.19) and (2.43) that

$$f(k) \in S(\mathfrak{g})^n(\lambda). \quad (2.44)$$

The main point is to show that we may replace $\mathfrak{g}$ by $\mathfrak{n}$ in (2.44).

Let $u = S^k(\mathfrak{g}) \cap (\mathfrak{h} + \mathfrak{n})S(\mathfrak{g})$. Clearly one has a direct sum

$$S^k(\mathfrak{g}) = S^k(\mathfrak{n}_-) \oplus u. \quad (2.45)$$

Furthermore it is immediate that $u$ is the $\mathcal{B}$ orthocomplement of $S^k(\mathfrak{n})$ in $S^k(\mathfrak{g})$. Thus to prove that

$$f(k) \in S(\mathfrak{n})^n(\lambda) \quad (2.46)$$

it suffices to show that $f(k)$ is $\mathcal{B}$ orthogonal to $u$. Clearly any element in $u$ is a sum of elements of the form $yw$ where $w \in \mathfrak{h} + \mathfrak{n}$ and $y \in S^{k-1}(\mathfrak{g})$. However given such
an element there exists $a \in \mathbb{C}$ such that $\pi_\nu(\tau(w))v_\nu = av_\nu$. Also $f(\tau(y)) = 0$ since $f$ vanishes on $U_{k-1}(g)$. But then by (2.39)

\[
(f(k), yw) = f(\tau(y)\tau(w))
= v_{\nu^*}(\pi_\nu(\tau(y)\tau(w))v_\nu)
= av_{\nu^*}(\pi_\nu(\tau(y))v_\nu)
= af(\tau(y))
= 0.
\]

This proves (2.46). That is, we have proved

**Theorem 2.8.** Let $0 \neq \nu \in \text{Dom}(L)$ and let $\lambda = \nu + \nu^*$. Let $f \in U(g)^*$ be defined by (2.42) or equivalently (2.43). Let $k$ be the codegree of $f$. Then $k \geq 1$ and

\[
f(k) \in S(n)^n(\lambda).
\]

(2.47)

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