Interferometric distillation and determination of unknown two-qubit entanglement

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We propose a scheme for both distilling and quantifying entanglement, applicable to individual copies of an arbitrary unknown two-qubit state. It is realized in a usual two-qubit interferometry with local filtering. Proper filtering operation for the maximal distillation of the state is achieved, by erasing single-qubit interference, and then the concurrence of the state is determined directly from the visibilities of two-qubit interference. We compare the scheme with full state tomography.

PACS numbers: 03.65.Ud, 03.67.Mn, 85.35.Ds

Introduction.— Multiparticle interference is a striking phenomenon connecting with quantum entanglement. For pure states, the connection is rather straightforward. In a two-particle interferometry [1, 2], the interference visibility gives the concurrence [3, 4], a widely used entanglement measure, of the two-particle pure state [5]. In a multiparticle Aharonov-Bohm interferometry [6], the visibility can be used to prove the quantum nonlocality of Greenberger-Horne-Zeilinger entanglement [7]. For mixed states, however, multiparticle interference comes from a mixture of entanglement and classical correlation, and it is hard to distinguish the two different correlations. It is interesting to find a way to extract entanglement from the interference, which is the aim of this work.

In quantum information research, there are strong demands of distilling and quantifying entanglement [8]. Currently available schemes are of two types, one using multiple copies of a target state and the other using individual copies. Since the multiple copies are harder to prepare in laboratory in general, it may be necessary to explore further the latter type. The distillation of the latter type has been done using local filtering, for a known two-qubit state [9] or after full state tomography [10]. And no scheme of the latter type has been proposed for directly quantifying an entanglement measure, such as concurrence, of an arbitrary mixed state in experiments; note that the existing schemes of the former type for determining concurrence have not been realized [11] or provide a lower bound of concurrence [12] for mixed states, while concurrence was recently determined in experiments by using two copies of a pure state [13]. Therefore, it is valuable to find a scheme of the latter type for distilling and directly determining entanglement of an unknown state (without full state reconstruction).

In this work, we propose an interferometric scheme for both distilling and determining entanglement, applicable to individual copies of an arbitrary unknown two-qubit state. It can be realized in a two-qubit interferometry with local filtering [10, 11]. The maximal distillation (the normal form [12]) of the state is first achieved, by iteratively erasing single-qubit interference, and then the concurrences of both the initial and the distilled states are directly determined from the visibilities of two-qubit interference. This quantification is based on our important findings that the two-qubit interference shows three different "local" extrema (visibilities) in general and that when the single-qubit interference is fully erased, the three extrema give the Lorentz singular values [13], a linear combination of which gives the concurrences. Our scheme is conceptually different from full state tomography and practically useful.

Two-qubit interferometry with local filtering.— We introduce the interferometry (Fig. 1). The source generates individual copies of an arbitrary unknown state of two qubits A and B, each represented by pseudospins $\uparrow_j$ and $\downarrow_j$, $j = A, B$. Qubit A flies to the detectors (DA) of Alice, passing through three beam splitters (BS), while B to Bob (DB); see solid arrows. The state is transformed into its maximally distilled state in the distillation parts, and then its concurrence is determined by measuring the visibilities of two-qubit interference in the detection parts.

![FIG. 1. Two-qubit interferometry with local filtering. It consists of a source, distillation parts, and detection parts. The source generates individual copies of an arbitrary unknown state of two qubits A and B, each represented by pseudospins $\uparrow_j$ and $\downarrow_j$, $j = A, B$. Qubit A flies to the detectors (DA) of Alice, passing through three beam splitters (BS), while B to Bob (DB); see solid arrows. The state is transformed into its maximally distilled state in the distillation parts, and then its concurrence is determined by measuring the visibilities of two-qubit interference in the detection parts.](http://example.com/fig1.png)
The trace over the degrees of freedom of qubit \( \bar{j} \) states ending in neither D
\( 1 \) is the reflection amplitude of BS
\( 3 \). Then the single-qubit states (\( \rho_j = \text{Tr}(\hat{\rho} \sigma_j) \)) and BS
\( 3 \) is filtered off or not is certified at detector D
\( 1 \). It transforms the initial state \( \hat{\rho} \) into its normal form, 

\[
\hat{\rho}_{\text{dis}} = (D_A \otimes D_B) \hat{\rho} (D_A \otimes D_B)^\dagger.
\]

The local filtering \( \hat{F}(f_j) \) and the rotation \( \hat{U}(\theta, \phi) \) of qubit j are supported by two beam splitters, BS
\( j \), and BS
\( 1 \), respectively, and represented as

\[
\hat{F}(f) = \begin{pmatrix} 1 & 0 \\ 0 & f \end{pmatrix}, \quad \hat{U}(\theta, \phi) = \begin{pmatrix} \cos \frac{\theta}{2} & i \sin \frac{\theta}{2} e^{-i\phi} \\ -i \sin \frac{\theta}{2} e^{i\phi} & \cos \frac{\theta}{2} \end{pmatrix}.
\]

Here, 0 ≤ \( f \) ≤ 1 is the filtering parameter controlled by the reflection amplitude of BS
\( j \), \( \theta_{\text{dis},j} \in [0, \pi] \) parameterizes the transmission at BS
\( 1 \), and \( \phi_{\text{dis},j} \in [0, 2\pi] \) is the phase shift. In the filtering operation \( \hat{F}(f_j) \), particle \( j \) is abandoned with probability \( 1 - f_j^2 \) when it flies along the lower path after scattering by BS
\( 1 \). Whether qubit \( j \) is filtered off or not is certified at detector D
\( 3 \). The two beam splitters constitute the minimal setup for the distillation. This is understood from the fact that each local operation on qubit \( j \) corresponds to a Lorentz transformation of the 4-vector of qubit \( 1 \). \( \hat{F} \) and \( \hat{U} \) correspond to the Lorentz boost and the spatial rotation, respectively. We emphasize that the Lorentz boost mathematically introduced in Ref. [15] is physically realized here by \( \hat{F} \) using beam splitter BS
\( 2 \). We will see later how \( \hat{D}_3 \) is efficiently found for an unknown state \( \hat{\rho} \).

In the detection parts, Alice and Bob count the number \( n_{ij} \) of the particles \( j \) arriving at detector D
\( i \), \( i = 1, 2 \), during measurement time, long enough to get the statistical average of coincidence correlation \( \langle n_{1B} n_{2B} \rangle \) for a given setting of all the beam splitters. They tune BS
\( 3A \) and BS
\( 3B \) to see single- and two-qubit interferences in \( \langle n_{ij}(n_{1B} + n_{2B}) \rangle \) and \( \langle n_{1A} n_{1B} \rangle \equiv \langle n_{1A}(n_{1B} + n_{2B}) \rangle \), \( n_{1B}(n_{1A} + n_{2A}) \). The other correlations involving D
\( 3 \) contain the same information. Here, the number \( n_{ij} \) is normalized by the total number \( N \) of states ending in neither D
\( 3A \) nor D
\( 3B \), and \( n_{ij} \) means the average number of qubit \( j \) in the events where the other qubit \( j \) (≠ \( j \)) is not filtered (does not end in D
\( 3j \)). The qubit rotation at BS
\( j \) is represented by

\[
\hat{U}(\theta_{\text{det},j}, \phi_{\text{det},j}) = \hat{U}(\theta_{\text{det},j}, \phi_{\text{det},j}), \quad \theta_{\text{det},j} \in [0, \pi] \quad \text{and} \quad \phi_{\text{det},j} \in [0, 2\pi] \text{.}
\]

The visibility of single- and two-qubit interferences are defined, respectively, as

\[
V_{j=A,B} = W[\langle n_{1j}(n_{1j} + n_{2j}) \rangle], \quad V_{AB} = W[\langle n_{1A} n_{1B} \rangle + 1/4],
\]

where \( W[x] = \max\{\max[x] - \min[x], 0\} \) and \( W[x] = \max\{\min[x], 0\} \). The local filtering \( \hat{U}_n \equiv \text{Tr}(\hat{\rho} \sigma_n) = \theta_n/\sqrt{2} \). Then the single-qubit states \( \langle \delta \rangle \) and BS
\( 3A \) given setting of all the beam splitters. They tune BS
\( 3B \). Then the single-qubit states \( \rho_n \) is filtered off or not is certified at detector D
\( 3 \). The local filtering \( \hat{F}(\theta, \phi) \equiv \hat{F}(\theta)U(\theta, \phi) \) on qubit \( j = A \) (B). It transforms the initial state \( \hat{\rho} \) into its normal form.

Note that the phase accumulation of particle \( j \) along its path is absorbed in the rotation angles \( \phi_{\text{dis},j} \) and \( \phi_{\text{det},j} \).
a compact form of the cross-correlation, \( \langle \delta n_{1A} \delta n_{1B} \rangle = \frac{1}{2} \vec{v}_A Q \vec{v}_B^* \), where \( \vec{v}_j \) is the rotation vector of BS\(_{3j} \), the column vector \( \vec{v}_B^* \) is the transpose of \( \vec{v}_B \). \( Q \) is the \( 3 \times 3 \) matrix defined by \( Q_{j'}^{l'} = R_{j'l'}/R_{00} - R_{j'0}R_{0l'}/(R_{00})^2 \), \( l, l' = 1, 2, 3 \), and \( R \) is the real parametrization of \( \rho' \) in Eq. (1). The number normalization by \( N \) gives the factors \( 1/R_{00} \) and \( 1/(R_{00})^2 \) in \( Q_{j'l'} \). From this compact form and the fact that \( \vec{v}_j = A, B \) spans over the surface of unit sphere, it is easy to see that \( (\delta n_{1A} \delta n_{1B}) \) has three pairs of “local” extrema \( \pm \lambda_l \)’s (\( \lambda_1 \geq \lambda_2 \geq \lambda_3 \geq 0 \)), i.e., \( V_{AB} \) has the three values \( \lambda_l \)’s, and that \( \lambda_l \)’s are identical to the singular values of \( Q \) up to sign factor. Here, \( \lambda_1 \) is the global maximum of \( (\delta n_{1A} \delta n_{1B}) \), while \( \lambda_2, \lambda_3 \) is the maximum over the space of \( \vec{v}_A \) and \( \vec{v}_B \) orthogonal to \( \vec{v}_{A,1} \) and \( \vec{v}_{B,1} \) (\( \vec{v}_{A,2}, \vec{v}_{A,2}, \vec{v}_{B,2} \)), where \( \vec{v}_{j,l} \) is the rotation vector of BS\(_{3j} \) at which \( (\delta n_{1A} \delta n_{1B}) \) shows \( \lambda_l \).

The above finding becomes very useful when \( \rho_{\text{dis}} \) is achieved in the distillation parts (\( \rho' = R_{\text{dis}} \)) In this case, the visibilities \( \lambda_l \)’s give the Lorentz singular values \( \lambda_l A, B \) of the initial state as

\[
s_0 = \frac{N}{M}, \quad s_1 = s_0 \lambda_1, \quad s_2 = s_0 \lambda_2, \quad s_3 = s_0 \lambda_3, \quad (5)
\]

since they are equal to the singular values of \( R_{\text{dis}}/(f_A f_B) \), \( R_{\text{dis},0} = R_{\text{dis},0} = 0 \), and \( R_{\text{dis},00} = N/M \). Here, \( M \) is the total number of injection of \( \rho \) from the source, \( q = \text{Det}(\vec{v}_{A,1}, \vec{v}_{A,2}, \vec{v}_{A,3}) \text{Det}(\vec{v}_{B,1}, \vec{v}_{B,2}, \vec{v}_{B,3}) \) is the sign factor guaranteeing the correct singular value decomposition, \( \text{Det}(\cdots) \) means matrix determinant, and \( (\vec{v}_{j,1}, \vec{v}_{j,2}, \vec{v}_{j,3}) \) is the matrix whose columns are \( \vec{v}_{j,l} \)’s. Using the relation \( \lambda_l A, B \) between concurrence and Lorentz singular values, we find an important result that the concurrences \( C \) of \( \rho \) and \( \rho_{\text{dis}} \) are directly obtained from \( V_{AB} \),

\[
C(\rho) = s_0 C(\rho_{\text{dis}}), \quad C(\rho_{\text{dis}}) = \max[0, \frac{1}{2}(-1 + \lambda_1 + \lambda_2 - q \lambda_3)]. \quad (6)
\]

**Examples.** In Fig. 2 the concurrence is determined at each k-th iteration step, for two examples of \( \rho \), using \( V_{AB} \) and Eq. (6). For typical cases of \( \rho \) (non-asymptotic case) [Fig. 2(a)], \( V_{AB}^{(k)} \) vanishes rapidly within a few steps, and the determined value \( C(k)(\rho) \) approaches to the exact value \( C(\rho) \) more rapidly. In this case, the deviation of \( C(k)(\rho) \) from \( C(\rho) \) is estimated [20] as \( |C(\rho) - C(k)(\rho)| \propto (V_{AB}^{(k)})^2 \) for small \( V_{AB}^{(k)} \). Thus, one can determine a precise value of \( C \) even before the complete distillation. The properties of particular types of \( \rho \) are given below.

(i) When \( \rho \) is pure, only one distillation step is necessary, since \( V_{AB}^{(k)} = V_{AB}^{(k=0)} \) for all \( k \) due to the complementarity [3]. Note that \( (V_{AB}^{(k=0)})^2 + C^2(\rho) = 1 \) for \( f_A = f_B = -1 \).

(ii) When \( \rho \) is separable and uncorrelated, \( \hat{\rho} = \hat{\rho}_A \otimes \hat{\rho}_B \), the local properties of \( A \) and \( B \) are independent. Therefore, only two steps are required, \( V_{AB} = 0 \), and \( C(\hat{\rho}) = 0 \). Particularly, when either \( \hat{\rho}_A \) or \( \hat{\rho}_B \) is pure, its single-qubit viscosity is one at \( k = 0 \), and \( \hat{\rho} \) cannot be distilled as \( \rho_{\text{dis}} \) vanishes. When \( \hat{\rho} \) is separable but has classical correlations, on the other hand, more than two steps are necessary, \( V_{AB} \neq 0 \), and \( C(\hat{\rho}) = 0 \).

(iii) When \( \rho \) is a Werner state or a Bell-diagonal state [3, 21], the distillation is not necessary.

(iv) There is the so-called asymptotic case [13], where large number of steps are necessary [Fig. 2(b)] and most states are abandoned by the filtering (\( f_{j \rightarrow k+\infty} \rightarrow 0 \)).

Below, we propose an optimal way of determining concurrence. After the distillation, the first step determines all the rotation vectors \( \vec{v}_{j,l} \), by observing \( (\delta n_{1A} \delta n_{1B}) \) in nine different settings of BS\(_{3j} \) and BS\(_{3B} \) and comparing the results with the compact form of \( (\delta n_{1A} \delta n_{1B}) \) (derived before). Then, one measures \( (\delta n_{1A} \delta n_{1B}) \) at the determined setting of BS\(_{3j} = A(B) \) at \( \vec{v}_{A,B,j,l} \). We emphasize that a crude determination of \( \vec{v}_{j,l} \) is enough for a precise detection of \( \lambda_l \) and \( C \). It is because \( \lambda_l \) is a local maximum, around which a small error \( (\sim \delta) \) in the direction \( \vec{v}_{j,l} \) for \( \lambda_l \) causes only a much smaller error \( (\sim \delta^2) \) in the value of \( \lambda_l \). This makes our scheme efficient. Table I shows that for states not much filtered, our scheme is as efficient as the tomography [22] for the quantification of the initial state. For the distillation and the quantification together, it can be more efficient than previous tomographic schemes, e.g., in Ref. [11], the previous schemes require roughly 2-4 times larger number of state copies.
In the distillation, the copies are used to achieve and to settings for the three maxima $\lambda$ and four detectors. We test six representative states considered here the most efficient one with nine measurement settings for the determination of $\vec{v}_j$, and three settings for the three maxima $\lambda_i$; the parentheses show $k_{\text{dis}}$. In the distillation, the copies are used to achieve and to check $V_j^{(k_{\text{dis}})} < 0.1$; for the tested states, this condition of $V_j^{(k_{\text{dis}})}$ is enough to obtain $C$ within the $\pm 0.01$ error. On the other hand, among available tomography schemes, we consider here the most efficient one with nine measurement settings and four detectors. We test six representative states usually tested in entanglement detection [9,10], a Bell state $|\psi_0\rangle = \frac{1}{2}(|\uparrow\uparrow\rangle + |\downarrow\downarrow\rangle)$, a Werner state $\hat{\rho}_W = \frac{1}{2}(\hat{\rho}_0 + \frac{1}{4}I)$, $1/4, \hat{\rho}_0 = |\psi_0\rangle\langle\psi_0| + \frac{1}{4}(|\uparrow\uparrow\rangle\langle\uparrow\uparrow| + |\downarrow\downarrow\rangle\langle\downarrow\downarrow|)$, and $\hat{\rho}_{\lambda,\epsilon}$ (introduced in Fig.2) with two different sets of ($\epsilon, \lambda$). Note that the efficiency of our scheme strongly depends on $\hat{\rho}$ (as the tomography) and becomes worse for states more filtered (those with smaller $N/M$); $N/M = 1$ (no distillation; $k_{\text{dis}} = 0$) for the first four states, 0.95 for $\hat{\rho}_{0.9,0.6}$, and 0.42 for $\hat{\rho}_{0.5,0.8}$.

Table I. Monte Carlo simulation [9] of the minimum number of individual copies of a given state $\hat{\rho}$ that need to be used to determine its concurrence [or $s_0(-1 + \lambda_1 + \lambda_2 - q\lambda_3)/2$ in Eq. (3)] within $\pm 0.01$ statistical error in our scheme (fourth column) and by full state tomography [22] (fifth). In our scheme, it is the sum of the number of necessary copies for the distillation (second column) with $k_{\text{dis}}$ iterative steps, and that for the quantification (third) consisting of nine measurement settings for the determination of $\vec{v}_j$, and three settings for the three maxima $\lambda_i$; the parentheses show $k_{\text{dis}}$. The results of the tomography process are shown in the last column.

| State | Distillation | Quantification | Total Tomography |
|-------|-------------|----------------|------------------|
| Bell  | 2400 (0)    | 9 $\times$ 100 + 3 $\times$ 200 | 3900 | 360 |
| Werner | 2400 (0)    | 9 $\times$ 500 + 3 $\times$ 4000 | 18900 | 38700 |
| $1/4$ | 2400 (0)    | 9 $\times$ 100 + 3 $\times$ 7800 | 26700 | 24300 |
| $\rho_{0,0}$ | 2400 (0) | 9 $\times$ 300 + 3 $\times$ 3800 | 16500 | 30600 |
| $\rho_{0.9,0.6}$ | 7200 (1) | 9 $\times$ 500 + 3 $\times$ 6800 | 32100 | 54900 |
| $\rho_{0.5,0.8}$ | 48000 (5) | 9 $\times$ 600 + 3 $\times$ 9600 | 75000 | 38700 |

Conclusion.— We have proposed a “quantum entanglement concentrator”, in which the entanglement of an arbitrary unknown two-qubit state is distilled and determined. We remark the following meaningful features.

First, our scheme is within experimental reach and applicable to generic types of qubits, as it has only local operations using a tunable beam splitter, currently available [11]. Second, we show that even for mixed states, concurrence and Lorentz singular values are directly and experimentally accessible, interestingly from the extrema of two-qubit interference; concurrence has been determined experimentally only for a pure state [14]. This motivates to study the features of the singular values [24]. Third, entanglement quantification can be closely related with distillation [11,25,26]. In our scheme, the former can be done after the latter. Finally, our scheme may be practically useful (e.g., for teleportation [27]), as it achieves the distillation and the quantification within one framework. It would be valuable to generalize our scheme to larger systems of multiple qubits, where tomography error estimation becomes less feasible.

We thank J. B. Altepeter, N. Gisin, Hee Su Park, and Tzu-Chieh Wei for valuable discussions, and especially the group of P. G. Kwiat for the numerical code for the tomography. This work was supported by KAIST-HRHRP.

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