Numerical solution to the Falkner-Skan equation: a novel numerical approach through the new rational $a$-polynomials∗

S. ABBASBANDY†, J. HAJISHAFIEIHA

Department of Applied Mathematics, Faculty of Science, Imam Khomeini International University, Qazvin 34148-96818, Iran
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Abstract The new rational $a$-polynomials are used to solve the Falkner-Skan equation. These polynomials are equipped with an auxiliary parameter. The approximated solution to the Falkner-Skan equation is obtained by the new rational $a$-polynomials with unknown coefficients. To find the unknown coefficients and the auxiliary parameter contained in the polynomials, the collocation method with Chebyshev-Gauss points is used. The numerical examples show the efficiency of this method.

Key words Falkner-Skan equation, rational Chebyshev polynomial, mapping parameter, collocation method, singular Sturm-Liouville problem

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1 Introduction

Nonlinear ordinary differential equations play an essential role in scientific, technical, economic, and mathematical modeling. Applications of nonlinear ordinary differential equations include the calculation of trajectories of space gliders and airplanes, forecast calculations, spread of AIDS, and automated parking maneuvering based on mechanical systems.[1]. Boundary layer equations are an important class of nonlinear ordinary differential equations that have many applications in physics and fluid mechanics[2–3]. Applications of boundary layer equations include metal plate cooling in a cooling bath, aerodynamic extrusion of plastic sheets, plastic film design, metal spinning, metallic plates, insulation materials, and glass and polymer studies[4]. One class of these boundary layer equations is the stationary Falkner-Skan boundary layer equation. The problem is very old and has already been addressed with many different numerical methods. Furthermore, computational fluid dynamics have already evolved in more advanced topics. Efficient numerical methods for boundary layer equations were developed 50 years ago, and finding a new computational method will be useful for solving new problems. In this paper, the authors aim to present a new method for the Falkner-Skan equation, which may be applicable to other problems.

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† Corresponding author, E-mail: abbasbandy@ikiu.ac.ir
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The Falkner-Skan equation plays an important role in fluid mechanics, aerospace, and heat transfer. In fluid dynamics, the Falkner-Skan boundary layer equation describes the two-dimensional laminar boundary layer formed on a wedge, where the plate is not parallel to the flow. The Falkner-Skan equation is the generalized state of the Blasius boundary layer equations.

Different methods have been used to solve the Falkner-Skan equation, involving exponential second kind Chebyshev\cite{5-6}, rational Chebyshev functions\cite{7-8}, iterative method\cite{9}, Chebyshev cardinal functions\cite{10}, shifted Chebyshev collocation method\cite{11}, piecewise linear functions, Legendre rational polynomials\cite{13}, Chebyshev finite difference method\cite{14}, Adomian’s decomposition method\cite{15}, homotopy analysis method\cite{16}, Sinc-collocation method\cite{17}, finite-difference method\cite{18}, rational Legendre functions\cite{19}, spline functions\cite{20}, group invariance theory\cite{21}, Gegenbaer neural network\cite{22}, and very recently the wavelet method\cite{23}.

In this paper, new $a$-polynomials equipped with parameter $a$ are used to solve this equation. These polynomials were used in Ref. [24] to solve the Benjamin-Bona-Mahony-Burgers (BBMB) equations and in Ref. [25] to solve an inverse problem.

2 Problem formulation

In this section, we explain how to obtain the Falkner-Skan ordinary differential equation. Assume the following boundary layer equations for steady incompressible flow with constant viscosity and density\cite{26}:

\[
\begin{align*}
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} &= 0, \\
\frac{\partial u}{\partial x} + v \frac{\partial v}{\partial y} &= U \frac{\partial U}{\partial x} + \nu \frac{\partial^2 u}{\partial y^2} - \frac{\sigma B_o^2}{\rho} (u - U) \tag{1}
\end{align*}
\]

with the conditions

\[
\begin{align*}
&u(x, 0) = 0, \quad v(x, 0) = 0, \\
&u = U(x), \quad y \to \infty, \\
&U(x) = ax.
\end{align*}
\]

Here, the coordinate system is chosen with $x$ pointing parallel to the plate in the direction of the flow and $y$-coordinate pointing towards the free stream. $u$ and $v$ are the $x$- and $y$-velocity components, respectively. $U$ is the inherent characteristic velocity. $\sigma$ is the electrical conductivity. $\rho$ is the fluid density. $\nu$ is the kinematic viscosity.

We can generalize the Blasius boundary layer by considering a wedge at an angle of attack $\pi \beta /2$ from some uniform velocity field $U_0$. We then estimate the outer flow to be of the form

\[
u^m_U(x) = U_0 \left( \frac{x}{l} \right)^m,
\]

where $l$ is a characteristic length, and $m$ is a dimensionless constant. In the Blasius solution, $m = 0$, corresponding to an angle of attack of zero radians. Thus, we can write

\[
\beta = \frac{2m}{2m + 1}.
\]

In the Blasius solution, we use a similarity variable $\eta$ to solve the boundary layer equations. We will obtain\cite{27}

\[
\eta = y \sqrt{\frac{U_0 (m + 1)}{2 \nu l}} \left( \frac{x}{l} \right)^{(m-1)/2}, \quad \psi = \sqrt{\frac{2 \nu U_0}{m + 1}} \left( \frac{x}{l} \right)^{(m+1)/2} f(\eta), \quad u = U_0 f', \quad v = -\sqrt{\frac{\nu U_0}{x}} f.
\]
Therefore, Eqs. (1) and (2) can be transformed into the following third-order nonlinear Falkner-Skan differential equation:

$$\frac{d^3f}{d\eta^3} + f\frac{d^2f}{d\eta^2} + \beta \left(1 - \left(\frac{df}{d\eta}\right)^2\right) - M^2 \left(\frac{df}{d\eta} - 1\right) = 0$$  \hspace{1cm} (3)

with the boundary conditions

$$f(0) = f'(0) = 0, \quad f'(\infty) = 1,$$

in which $M^2 = \sigma B_0^2/(\rho a)$. No exact analytical solution is known for Eq. (3).

3 Rational $a$-polynomials

In this section, the concept of a new class of rational polynomials is introduced. Also, by changing the coordinate, the unbounded domain is considered.

**Definition** \cite{28} Assume that $\alpha$ is a constant parameter and $A_0(\eta) = 1$. The $a$-polynomials as a combination of the Chebyshev polynomials of the second kind, $U_n(\eta)$, are defined by

$$A_n(z) = \alpha z U_{n-1}(z) + U_n(z), \quad z \in [-1, 1], \quad n \geq 1.  \hspace{1cm} (4)$$

The following equations are also established:

$$A_{n+1}(z) = 2zA_n(z) - A_{n-1}(z), \quad n \geq 1,  \hspace{1cm} (5)$$

$$A_n(z) = \left(1 + \frac{\alpha}{2}\right) U_n(z) + \frac{\alpha}{2} U_{n-2}(z), \quad n \geq 2.  \hspace{1cm} (6)$$

See Refs. [24] and [28] for more properties. By changing the variable $z = \frac{\eta - L}{\eta + L}$ for $\eta \in [0, +\infty)$, in which $L$ is an arbitrary large positive number and will be chosen later, the following new rational $a$-polynomials are obtained from Eq. (4):

$$AL_n(\eta) = \alpha \frac{\eta - L}{\eta + L} U_{n-1}\left(\frac{\eta - L}{\eta + L}\right) + U_n\left(\frac{\eta - L}{\eta + L}\right), \quad \eta \in [0, +\infty), \quad n \geq 1  \hspace{1cm} (7)$$

with $AL_0(\eta) = 1$. The constant $L$ is called the mapping parameter. By using

$$R_n(\eta) = U_n\left(\frac{\eta - L}{\eta + L}\right),  \hspace{1cm} (8)$$

according to Eq. (6), we have

$$AL_n(\eta) = \left(1 + \frac{\alpha}{2}\right) R_n(\eta) + \frac{\alpha}{2} R_{n-2}(\eta), \quad n \geq 2.  \hspace{1cm} (9)$$

**Proposition 1** The following recursive relation holds for the rational $a$-polynomials:

$$AL_{n+1}(\eta) = 2 \frac{\eta - L}{\eta + L} AL_n(\eta) - AL_{n-1}(\eta), \quad n \geq 1.$$

**Proposition 2** Considering the inner product

$$(f, g) = \int_0^\infty \omega(\eta) f(\eta) g(\eta) d\eta,  \hspace{1cm} (10)$$

in which $\omega(\eta)$ is a weight function.
where $\omega(\eta) = \frac{4L\sqrt{\eta L}}{(\eta + L)^2}$ is the positive weight function, we have

\[
(AL_n, AL_m)_\omega = \begin{cases} 
\frac{\pi}{2}, & n = m = 0, \\
\frac{1}{8}(2 + a)^2 \pi, & n = m = 1, \\
\frac{1}{4}(2 + 2a + a^2) \pi, & n = m \geq 2, \\
a \pi, & |n - m| = 2, \quad nm = 0, \\
\frac{1}{8}a(2 + a) \pi, & |n - m| = 2, \quad nm \neq 0, \\
0, & \text{otherwise}
\end{cases}
\]

and

\[
(R_n, R_m)_\omega = \frac{\pi}{2} \delta_{n,m},
\]

where $\delta_{n,m}$ is the delta Kronecker number.

**Proposition 3** The system $\{AL_n\}_{n=0}^\infty$ is an orthogonal system with respect to the inner product defined in Proposition 2.

**Proposition 4** $U_n$ is the eigenfunction of the singular Sturm-Liouville problem

\[
\left( (1 - \eta^2)^{-1/2} \frac{d}{d\eta} \left( (1 - \eta^2)^{3/2} \frac{d}{d\eta} \right) + n(n+2) \right) U_n(\eta) = 0,
\]

and $R_n$ is the eigenfunction of the singular Sturm-Liouville problem

\[
\left( \frac{(\eta + L)^3}{4L\sqrt{\eta L}} \frac{d}{d\eta} \left( \frac{4\eta\sqrt{L\eta}}{(\eta + L)} \frac{d}{d\eta} \right) + n(n+2) \right) R_n(\eta) = 0, \quad n = 0, 1, 2, \ldots.
\]

**Proposition 5** Suppose that $\omega_1(\eta) = \frac{4\eta\sqrt{L\eta}}{\eta + L}$. Then, we have

\[
\int_0^\infty \frac{dR_n(\eta)}{d\eta} \frac{dR_m(\eta)}{d\eta} \omega_1(\eta)d\eta = \frac{1}{2} n(n+2) \delta_{n,m}.
\]

**Remark 1**

\[
\lim_{\eta \to \infty} R_n(\eta) = n + 1, \quad \lim_{\eta \to \infty} \frac{dR_n(\eta)}{d\eta} = 0, \quad \lim_{\eta \to \infty} \left( \frac{4\eta\sqrt{L\eta}}{\eta + L} \right) \frac{dR_n(\eta)}{d\eta} = 0.
\]

4 Convergence and stability theorems

4.1 The convergence theorem

Assume $\Lambda = [0, +\infty)$, $\omega$ is the weight function in Proposition 2, and $L^2(\Lambda)$ is the function Hilbert space with the inner product of Eq. (10). Let $N$ be the positive integer, and we will consider

\[
\mathcal{R}_N = \text{span} \{ R_0, R_1, \ldots, R_N \},
\]

which is the subspace of $L^2(\Lambda)$. We define the $L^2(\Lambda)$-orthogonal projection as follows:

\[
P_N : L^2(\Lambda) \to \mathcal{R}_N,
\]

\[
(P_N u)(\eta) = \sum_{i=0}^{N} c_i R_i(\eta)
\]
and \( \| \) is obtained from Eqs. (18) and (19).

By parts, we obtain

\[
q \equiv \frac{\eta}{L} \frac{\eta + 3}{2} \frac{d^2 v}{d \eta^2} = \frac{\eta + 3}{2} \frac{d}{d \eta} \left( \frac{d}{d \eta} v(\eta) \right)
\]

\[
\| \|_{r, \omega, Q} = \left( \sum_{i=0}^{r} \left( \eta + L \right)^2 \frac{d^i u}{d \eta^i} \right)^{1/2}.
\]

**Theorem 1**  For any real \( r > 0 \), \( v \in H_{r, Q} \), \( v = \sum_{n=0}^{\infty} \hat{v}_n R_n(\eta) \), and \( c \in \mathbb{R} \), there is

\[
\| P_N v - v \|_{r, \omega, Q} \leq c N^{-r} \| v \|_{r, \omega, Q}.
\]

**Proof**  First, we assume that \( r = 2m \). Due to Eqs. (11), (13), and (17) and the integration by parts, we obtain

\[
\hat{v}_n = \frac{2}{\pi} \int_{\Lambda} v(\eta) R_n(\eta) \omega(\eta) d\eta = \frac{2}{\pi n(n+2)} \int_{\Lambda} v(\eta) Q R_n(\eta) \omega(\eta) d\eta
\]

\[
= -\frac{2}{\pi n(n+2)} \int_{\Lambda} v(\eta) \frac{d}{d \eta} \left( \frac{4\eta \sqrt{L \eta}}{\eta + L} d \eta \right) R_n(\eta) d\eta
\]

\[
= \frac{2}{\pi n(n+2)} \int_{\Lambda} \frac{4\eta \sqrt{L \eta}}{\eta + L} v(\eta) \frac{d}{d \eta} R_n(\eta) d\eta
\]

\[
= \frac{2}{\pi n(n+2)} \int_{\Lambda} Q v(\eta) R_n(\eta) \omega(\eta) d\eta
\]

\[
= \cdots = \frac{2}{\pi n(n+2)^m} \int_{\Lambda} Q^m v(\eta) R_n(\eta) \omega(\eta) d\eta.
\]

Now, according to Eqs. (19) and (21) and the definition of \( H_{r, Q} \), we have

\[
\| P_N v - v \|_{r, \omega}^2 = \sum_{n=N+1}^{\infty} \hat{v}_n \| R_n \|_{\omega}^2
\]

\[
\leq c N^{-4m} \sum_{n=N+1}^{\infty} \left( \int_{\Lambda} Q^m v(\eta) R_n(\eta) \omega(\eta) d\eta \right)^2 \| R_n \|_{\omega}^2
\]

\[
\leq c N^{-4m} \| Q^m v \|_{\omega}^2 \leq c N^{-4m} \| v \|_{r, \omega, Q}^2.
\]
Next, we put \( r = 2m + 1 \). With Eq. (13) and the integration by parts, we have

\[
\hat{v}_n = \frac{2}{\pi n^m(n+2)^m} \int_\Lambda Q^m v(\eta) R_n(\eta) \omega(\eta) d\eta \\
= -\frac{2}{\pi n^{m+1}(n+2)^{m+1}} \int_\Lambda Q^m v(\eta) \frac{d}{d\eta} \left( \frac{4\eta \sqrt{L \eta}}{\eta + L} \right) R_n(\eta) d\eta \\
= -\frac{2}{\pi n^{m+1}(n+2)^{m+1}} \int_\Lambda \frac{d}{d\eta} (Q^m v(\eta)) \frac{d}{d\eta} R_n(\eta) \omega_1(\eta) d\eta. \tag{22}
\]

Now, with Eqs. (14) and (19), we obtain

\[
\| P_N v - v \|_\omega^2 = \sum_{n=N+1}^\infty \hat{v}_n^2 \| R_n \|_\omega^2 \\
= \sum_{n=N+1}^\infty \frac{4}{\pi^2 (n(n+2))^{2m+2}} \left( \int_\Lambda \frac{d}{d\eta} (Q^m v(\eta)) \frac{d}{d\eta} R_n(\eta) \omega(\eta) d\eta \right)^2 \\
= \sum_{n=N+1}^\infty \frac{4}{\pi^2 (n(n+2))^{2m+2}} \left( \int_\Lambda \frac{d}{d\eta} (Q^m v(\eta)) \frac{d}{d\eta} R_n(\eta) \omega(\eta) d\eta \right)^2 \left\| \frac{d}{d\eta} R_n \right\|_{\omega_1}^2 \\
\leq c N^{-2(2m+1)} \sum_{n=N+1}^\infty \left( \int_\Lambda \frac{d}{d\eta} (Q^m v(\eta)) \frac{d}{d\eta} R_n(\eta) \omega(\eta) d\eta \right)^2 \left\| \frac{d}{d\eta} R_n \right\|_{\omega_1}^2 \\
\leq c N^{-2(2m+1)} \left\| \frac{d}{d\eta} (Q^m v) \right\|_{\omega_1}^2 \leq c N^{-2(2m+1)} \left\| \frac{d}{d\eta} (\tilde{v}_N) (\eta + L)^{7/2} \right\|_{\omega_1}^2 \\
\leq c N^{-2(2m+1)} \| v \|_{r,\omega,Q}^2.
\]

The general result follows from the previous results and space interpolation.

Now, we get the approximation error of \( f = \sum_{i=0}^\infty c_i A L_i(\eta) \) by \( f_N(\eta) = \sum_{i=0}^N c_i A L_i(\eta) \).

**Theorem 2** For any real \( r > 0 \) and \( f \in H^r_{\omega,Q}(\Lambda) \), we have

\[
\| f_N - f \|_\omega \leq \tilde{c}(N - 2)^{-r} \| f \|_{r,\omega,Q}. \tag{23}
\]

**Proof** Using Eq. (9) and Theorem 1, we get the following inequality:

\[
\| f_N - f \|_\omega = \left\| \sum_{i=N+1}^\infty c_i A L_i(\eta) \right\|_\omega = \left\| \sum_{i=N+1}^\infty c_i \left( \frac{1 + a}{2} \right) R_i(\eta) + \frac{a}{2} R_{i-2}(\eta) \right\|_\omega \tag{Eq. (9)} \\
\leq \left| \frac{1 + a}{2} \right| \left\| \sum_{i=N+1}^\infty c_i R_i(\eta) \right\|_\omega + \left| \frac{a}{2} \right| \left\| \sum_{i=N+1}^\infty c_i R_{i-2}(\eta) \right\|_\omega \tag{Eq. (20)} \\
\leq \left| \frac{1 + a}{2} \right| c' N^{-r} \| f \|_{r,\omega,Q} + \left| \frac{a}{2} \right| c'' (N - 2)^{-r} \| f \|_{r,\omega,Q} \leq \tilde{c}(N - 2)^{-r} \| f \|_{r,\omega,Q},
\]

where \( \tilde{a} = \max\{|1 + \frac{a}{2}|, |\frac{a}{2}|\} \), and \( c = \max\{c', c''\} \).

This theorem shows that the rational \( a \)-polynomial approximation has exponential convergence.
4.2 The stability theorem

Put \( L = 1 \). Then, we have \( z = \frac{n-1}{2} \). Suppose that \( \sigma_{N,j} = \cos(2j\pi/(2N+1)) \) are \( N+1 \) Chebyshev-Gauss points, and

\[
\zeta_{N,j} = (1 + \sigma_{N,j})(1 - \sigma_{N,j})^{-1}, \quad \omega(\eta) \frac{d\eta}{dz} = \sqrt{1 - z^2}, \quad \rho(z) = \sqrt{1 - z^2}.
\]

The Chebyshev-Gauss formula implies that

\[
\int_{\Lambda} \varphi(\eta) \omega(\eta) d\eta = \int_{I} \varphi\left(\frac{1+z}{1-z}\right) \rho(z) dz = \sum_{j=0}^{N} \varphi(\zeta_{N,j}) \omega_j, \quad \forall \varphi \in \mathcal{R}_N,
\]

where \( \omega_0 = \pi/(2N+1) \) and \( \omega_j = \pi/(N+1) \) for \( 1 \leq j \leq N \). The discrete inner product and norm of the Chebyshev-Gauss rational interpolation points are

\[
(u, v)_{\omega,N} = \sum_{j=0}^{N-1} u(\zeta_{N,j}) v(\zeta_{N,j}) \omega_j, \quad \|v\|_{\omega,N} = (v, v)_{\omega,N}^{1/2}.
\]

According to Eq. (24), we have

\[
(\varphi, \psi)_{\omega,N} = (\varphi, \psi)_{\omega}, \quad \forall \varphi, \psi \in \mathcal{R}_N.
\]

The Chebyshev-Gauss rational interpolation operator \( I_N f(\eta) : C(\Lambda) \to \mathcal{R}_N \) is

\[
I_N f(\zeta_{N,j}) = f(\zeta_{N,j}), \quad 0 \leq j \leq N.
\]

The following theorem is related to the stability of the Chebyshev-Gauss rational interpolation.

**Theorem 3** For any \( f \in H^1_{\omega,Q} \), we have

\[
\|I_N f\|_{\omega} \leq c\left(\|f\|_{\omega, N}^{-1}\|\eta + 1\|^{1/4}(\eta + 1)^{1/4}\partial_{\eta} f\|_{\omega}\right).
\]

**Proof** Let \( \eta = (1 + \cos \theta)/(1 - \cos \theta) \), \( \hat{f}(\theta) = f((1 + \cos \theta)/(1 - \cos \theta)) \), and

\[
\frac{d\eta}{d\theta} = -\frac{2\sin \theta}{(1 - \cos \theta)^2} = \frac{1}{2}(\eta + 1)\sqrt{\eta}.
\]

Put \( K_j = (2j\pi/(2N+1), (2j+1)\pi/(2N+1)) \) for \( j = 0, 1, \cdots, N \). Then, by Eq. (25), there is

\[
\|I_N f\|^2_{\omega} = \|I_N f\|^2_{\omega,N} = \sum_{j=0}^{N} f^2(\zeta_{N,j}) \omega_j
\]

\[
= \sum_{j=0}^{N} \hat{f}^2(\theta_{N,j}) \omega_j \leq \frac{\pi}{N+1} \sum_{j=0}^{N-1} \sup_{\theta \in K_j} \hat{f}^2(\theta).
\]

By the following inequality (13.7 in Ref. [29]):

\[
\max_{a \leq \eta \leq b} |g(\eta)| \leq c \left(\frac{1}{\sqrt{b-a}} \|g\|_{L^2(a,b)} + \sqrt{b-a} \|\partial_\eta g\|_{L^2(a,b)}\right), \quad \forall g \in H^1(a, b),
\]

applying it to each interval \( K_j \), yields

\[
\|I_N f\|^2_{\omega} \leq c \sum_{j=0}^{N} \left(\|\hat{f}(\theta)\|^2_{L^2(K_j)} + N^{-2} \|\partial_\theta \hat{f}(\theta)\|^2_{L^2(K_j)}\right)
\]

\[
\leq c\left(\|\hat{g}(\theta)\|^2_{L^2(0,\pi)} + N^{-2} \|\partial_\theta \hat{f}(\theta)\|^2_{L^2(0,\pi)}\right)
\]

\[
\leq c\left(\|\hat{f}(\eta)\|^2_{L^2(\Lambda)} + N^{-2} \|\eta + 1\|^{1/4}(\eta + 1)^{1/4}\partial_{\eta} \hat{f}(\eta)\|^2_{L^2(\Lambda)}\right),
\]

which implies the desired result.
5 Method implementation

The solution to the Falkner-Skan equation is obtained by using the present method. To find the solution, the approximate solution to the equation is estimated as

\[ f_N(\eta) = \eta + \sum_{i=0}^{N} c_i A L_i(\eta), \]  

(26)
due to Remark 1 and the boundary condition \( f'(\infty) = 1 \). Considering the Chebyshev-Gauss points, the collocation points of the present method are defined as

\[ z_j = \cos(\frac{j\pi}{N}) \text{ for } j = 0, 1, \ldots, N. \]

Now, we consider the collocation method for the residual of Eq. (3) with the collocation points \( \eta_j (j = 0, 1, \ldots, N) \), where \( \eta_j = \frac{\eta - L}{\eta_j + L} \). Therefore, the unknown coefficients \( \{c_j\}_{j=0}^{N} \) and the parameters \( a \) and \( L \) are obtained by solving the following nonlinear system of equations:

\[
\begin{align*}
R(\eta_j) &= 0, \quad j = 0, 1, \ldots, N, \\
f(0) &= 0, \\
f'(0) &= 0,
\end{align*}
\]

(27)

where

\[
R(\eta) = \frac{d^3 f_N}{d\eta^3} + f_N \frac{d^2 f_N}{d\eta^2} + \beta \left(1 - \left(\frac{df_N}{d\eta}\right)^2\right) - M^2 \left(\frac{df_N}{d\eta} - 1\right).
\]

In Table 1, the results of the method are compared with those of the spline function method\cite{20}, the Chebyshev polynomial approximation\cite{30}, the Chebyshev finite difference method\cite{14}, and the piecewise linear functions method\cite{12}. The results show good approximation for the solution to the equation.

**Table 1** Comparison of \( f''(0) \) for different methods at \( f(0) = 0 \) and \( M = 0 \)

| \( \beta \) | \( N \) | \( L \) | \( a \) | Present method | Ref. [20] | Ref. [30] | Ref. [14] | Ref. [12] |
|---|---|---|---|---|---|---|---|---|
| 0  | 20 | 2.0000 | 0.5016 | 0.469675 01 | 0.469600 00 | 0.469600 00 | 0.469600 12 | 0.469600 |
| 0.3| 16 | 2.2426 | 2.0491 | 0.774758 92 | 0.774782 77 | 0.774783 00 | 0.774782 74 | -- |
| 0.5| 16 | 2.0077 | 2.4937 | 0.927680 32 | 0.927805 46 | 0.927805 46 | 0.927805 39 | 0.927680 |
| 1  | 16 | 1.7691 | 2.0433 | 1.232588 18 | -- | 1.232588 | -- | 1.232588 |
| 2  | 16 | 1.7814 | 2.4100 | 1.687218 67 | 1.687225 60 | 1.687226 | 1.687225 86 | 1.687222 |
| 10 | 16 | 2.1498 | 3.8498 | 3.675234 03 | 3.675213 00 | 3.675234 | 3.675234 31 | -- |
| 15 | 16 | 2.0645 | 3.6235 | 4.491486 86 | 4.491463 00 | 4.491487 | 4.491486 88 | -- |

In the nonlinear system of Eq. (27), instead of the initial condition \( f(0) = 0 \), put a more general condition \( f(0) = \alpha \). Table 2 provides the results of the method for the initial condition \( f(0) = \alpha \) compared with the homotopy analysis method (HAM)\cite{16}. The results are in good approximation for the solution to the equation.

**Table 2** Comparison of \( f''(0) \) for different methods at \( f(0) = \alpha \)

| \( \beta \) | \( N \) | \( L \) | \( \alpha \) | Present method | Ref. [20] | Ref. [30] | Ref. [14] | Ref. [12] |
|---|---|---|---|---|---|---|---|---|
| 0  | 20 | 2.0000 | 0.5016 | 0.469675 01 | 0.469600 00 | 0.469600 00 | 0.469600 12 | 0.469600 |
| 0.3| 16 | 2.2426 | 2.0491 | 0.774758 92 | 0.774782 77 | 0.774783 00 | 0.774782 74 | -- |
| 0.5| 16 | 2.0077 | 2.4937 | 0.927680 32 | 0.927805 46 | 0.927805 46 | 0.927805 39 | 0.927680 |
| 1  | 16 | 1.7691 | 2.0433 | 1.232588 18 | -- | 1.232588 | -- | 1.232588 |
| 2  | 16 | 1.7814 | 2.4100 | 1.687218 67 | 1.687225 60 | 1.687226 | 1.687225 86 | 1.687222 |
| 10 | 16 | 2.1498 | 3.8498 | 3.675234 03 | 3.675213 00 | 3.675234 | 3.675234 31 | -- |
| 15 | 16 | 2.0645 | 3.6235 | 4.491486 86 | 4.491463 00 | 4.491487 | 4.491486 88 | -- |

In Table 3, we can conclude that the results are acceptable compared with other methods. In Table 4, the results of the present method are compared for \( \beta = -3 \) (\( m = -\frac{1}{2} \)) by using the HAM\cite{31}, the Sinc-collocation method\cite{17}, and the Gegenbaer neural network method\cite{22}. According to Table 3, we can conclude that the results are acceptable compared with other methods. In Table 4, the results of the present method are compared for \( \beta = -3 \) (\( m = -\frac{1}{2} \)) by using the HAM\cite{31}, the Sinc-collocation method\cite{17}, and the Gegenbaer neural network method\cite{22}.
To verify the accuracy and convergence of the method, \( \|R\|_2 \) is used. According to Table 5, the results of the present method are more accurate than those of the other methods. Moreover, the increase in the collocation points increases the accuracy of the solution and reduces \( \|R\|_2 \). The results of Table 5 confirm the convergence Theorem 2.

In Table 6, the results of \( f''(0) \) for the \( a \)-polynomial method at \( f(0) = \alpha \) and \( M = 0 \) are displayed. In this table, the results of the residual error of the equation are compared in rational \( a \)-polynomials and \( a \)-polynomial methods, and it can be seen that the superiority of the rational \( a \)-polynomial method is evident. Besides, CPU time of the mentioned methods is acceptable.
In Fig. 1, the $f'(\eta)$ curves are plotted for the values $\beta = 0, 1, 2$ under the initial condition $f(\eta) = 0$ and $M = 0$. It shows that the results are easily observable and comparable. Figure 2 depicts the $f'(\eta)$ curves for $\beta = 1$ under the initial conditions $f(0) = \alpha = 0, 1, 2, 3$ and $M = 0$. In Fig. 3, the $f'(\eta)$ curves are plotted for the values $M = 2, 5, 10, 50$ under the initial condition $f(0) = 0$ and $\beta = \frac{1}{2} \ (m = 2)$. In Fig. 4, the $f'(\eta)$ curves are plotted for $M = 5, 10, 15, 20$ under the initial condition $f(0) = 0$ and $\beta = -3 \ (m = -\frac{3}{5})$.

Table 6: Results of $f''(0)$ for $\alpha$-polynomial method at $f(0) = \alpha$ and $M = 0$ and comparison of $\|R\|_2$

| $\beta$ | $\alpha$ | $N$ | $\alpha$ | $f''(0)$ | $\alpha$-polynomial method | Present method |
|--------|--------|-----|--------|--------|----------------------|----------------|
|        |        |     |        |        | $\|R\|_2$ (CPU time/s) | $\|R\|_2$ (CPU time/s) |
| 0      | 2      | 20  | 2      | 1.41820 $\times 10^6$ | 1.283635 | 5.80249 $\times 10^{-5}$ (2) | 7.92882 $\times 10^{-6}$ (6) |
| 0      | 2      | 20  | 9.40653 $\times 10^7$ | 2.194530 | 2.93219 $\times 10^{-4}$ (2) | 3.36447 $\times 10^{-6}$ (6) |
| 0      | 3      | 20  | 1.92207 $\times 10^9$ | 3.145199 | 2.43328 $\times 10^{-4}$ (2) | 1.06890 $\times 10^{-5}$ (6) |
| 1      | 1      | 16  | 9.12232 $\times 10^2$ | 1.889286 | 1.94615 $\times 10^{-3}$ (1) | 9.47916 $\times 10^{-5}$ (2) |
| 1      | 2      | 16  | 1.87921 $\times 10^4$ | 2.670021 | 7.08507 $\times 10^{-4}$ (1) | 1.17158 $\times 10^{-5}$ (2) |
| 1      | 3      | 16  | 8.97651 $\times 10^4$ | 3.526436 | 5.65119 $\times 10^{-3}$ (1) | 1.37253 $\times 10^{-5}$ (2) |
| 2      | 1      | 16  | 3.28932 $\times 10^4$ | 2.369973 | 9.31681 $\times 10^{-4}$ (1) | 1.53240 $\times 10^{-5}$ (2) |
| 2      | 2      | 16  | 2.61471 $\times 10^5$ | 3.037975 | 2.52814 $\times 10^{-2}$ (1) | 2.15517 $\times 10^{-5}$ (2) |
| 2      | 3      | 16  | 6.52909 $\times 10^5$ | 3.833574 | 9.51944 $\times 10^{-2}$ (1) | 9.04868 $\times 10^{-7}$ (2) |

Fig. 1  $f'(\eta)$ for different values of $\beta$ at $\alpha = 0$ and $M = 0$ (color online)

Fig. 2  $f'(\eta)$ for different values of $\alpha$ at $\beta = 1$ and $M = 0$ (color online)

Fig. 3  $f'(\eta)$ for different values of $M$ at $\alpha = 0$ and $\beta = \frac{1}{2} \ (m = 2)$ (color online)

Fig. 4  $f'(\eta)$ for different values of $M$ at $\alpha = 0$ and $\beta = -3 \ (m = -\frac{3}{5})$ (color online)
6 Conclusions

A novel method is presented for solving the classical Falkner-Skan equation. It uses two parameters to optimize and improve the approximation accuracy of the solution. The results are acceptable in comparison with other methods. In fact, in many cases, the results obtained by the proposed method are better than the findings of other papers. The present method includes all types of the Falkner-Skan equation. According to the theorems proved, the convergence of the method is guaranteed.

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References

[1] HERMANN, M. and SARAVI, M. Nonlinear Ordinary Differential Equations, Springer, India (2016)
[2] ZHANG, J., FANG, T. G., and ZHONG, Y. F. Analytical solution of magnetohydrodynamic sink flow. *Applied Mathematics and Mechanics (English Edition)*, 32(10), 1221–1230 (2011) https://doi.org/10.1007/s10483-011-1495-9
[3] SU, X. H. and ZHENG, L. C. Approximate solutions to MHD Falkner-Skan flow over permeable wall. *Applied Mathematics and Mechanics (English Edition)*, 32(10), 1221–1230 (2011) https://doi.org/10.1007/s10483-011-1425-9
[4] BAKODAH, H. O., EBAID, A., and WAZWAZ, A. M. Analytical and numerical treatment of Falkner-Skan equation via a transformation and Adomian’s method. *Romanian Reports in Physics*, 70, 1–17 (2018)
[5] RAMADAN, M. A., RASLAN, K. R., EL-DANAF, T. S., and ABD-EL-SALAM, M. A. An exponential Chebyshev second kind approximation for solving high-order ordinary differential equations in unbounded domains with application to Dawson’s integral. *Journal of the Egyptian Mathematical Society*, 25(2), 197–205 (2017)
[6] RAMADAN, M. A., RASLAN, K. R., EL-DANAF, T. S., and ABD-EL-SALAM, M. A. On the exponential Chebyshev approximation in unbounded domains: a comparison study for solving high-order ordinary differential equations. *International Journal of Pure and Applied Mathematics*, 105(3), 399–413 (2015)
[7] ABBASBANDY, S., HAYAT, T., GHEHSAREH, H. R., and ALSAEDI, A. MHD Falkner-Skan flow of Maxwell fluid by rational Chebyshev collocation method. *Applied Mathematics and Mechanics (English Edition)*, 34(8), 921–930 (2013) https://doi.org/10.1007/s10483-013-1717-7
[8] SEZER, M., GÜLSU, M., and TANAY, B. Rational Chebyshev collocation method for solving higher-order linear ordinary differential equations. *Numerical Methods for Partial Differential Equations*, 27(5), 1130–1142 (2011)
[9] ZHANG, J. and CHEN, B. An iterative method for solving the Falkner-Skan equation. *Applied Mathematics and Computation*, 210(1), 215–222 (2009)
[10] LAKESTANI, M. Numerical solution for the Falkner-Skan equation using Chebyshev cardinal functions. *Acta Universitatis Apulensis*, 27, 229–238 (2011)
[11] KAJANI, M. T., MALEKI, M., and ALLAME, M. A numerical solution of Falkner-Skan equation via a shifted Chebyshev collocation method. *AIP Conference Proceedings*, 1629, 381–386 (2014)
[12] ASAITHAMBI, A. Numerical solution of the Falkner-Skan equation using piecewise linear functions. *Applied Mathematics and Computation*, 159(1), 267–273 (2004)
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[13] GUO, B. Y., SHEN, J., and WANG, Z. Q. A rational approximation and its applications to differential equations on the half line. *Journal of Scientific Computing*, 15(2), 117–147 (2000)

[14] ELBARBARY, E. M. Chebyshev finite difference method for the solution of boundary-layer equations. *Applied Mathematics and Computation*, 160(2), 487–498 (2005)

[15] ABBASBANDY, S. A numerical solution of Blasius equation by Adomian’s decomposition method and comparison with homotopy perturbation method. *Chaos, Solitons and Fractals*, 31(1), 257–260 (2007)

[16] YAO, B. Approximate analytical solution to the Falkner-Skan wedge flow with the permeable wall of uniform suction. *Communications in Nonlinear Science and Numerical Simulation*, 14(8), 3320–3326 (2009)

[17] PARAND, K., DEHGHAN, M., and PIRKHEDRRI, A. The use of Sinc-collocation method for solving Falkner-Skan boundary-layer equation. *International Journal for Numerical Methods in Fluids*, 68(1), 36–47 (2012)

[18] ASAITHAMBI, A. A finite-difference method for the Falkner-Skan equation. *Applied Mathematics and Computation*, 92(2-3), 135–141 (1998)

[19] PARAND, K., PAKNIAT, N., and DELAFKAR, Z. Numerical solution of the Falkner–Skan equation with stretching boundary by collocation method. *International Journal of Nonlinear Science*, 11(3), 275–283 (2011)

[20] EL-HAWARY, H. M. A deficient spline function approximation for boundary layer flow. *International Journal of Numerical Methods for Heat and Fluid Flow*, 11(3), 227–236 (2001)

[21] FAZIO, R. The Falkner-Skan equation: numerical solutions within group invariance theory. *Calcolo*, 31(1-2), 115–124 (1994)

[22] PARAND, K., HASHEMI, S., and GHADERI, A. Application of Gegenbaer neural network to solve the MHD Falkner-Skan flow. *The Second National Conference on Meta-Heuristic Algorithms and Their Applications in Engineering and Science*, Payame Noor University, Najafabad (2017)

[23] KARKERA, H., KATAGI, N. N., and KUDENATTI, R. B. Analysis of general unified MHD boundary-layer flow of a viscous fluid—a novel numerical approach through wavelets. *Mathematics and Computers in Simulation*, 168, 135–154 (2020)

[24] HAJISHAFIEIHA, J. and ABBASBANDY, S. A new class of polynomial functions for approximate solution of generalized Benjamin-Bona-Mahony-Burgers (gBBMB) equations. *Applied Mathematics and Computation*, 367, 124765 (2020)

[25] HAJISHAFIEIHA, J. and ABBASBANDY, S. A new method based on polynomials equipped with a parameter to solve two parabolic inverse problems with a nonlocal boundary condition. *Inverse Problems in Science and Engineering*, 28(5), 739–753 (2020)

[26] ABBASBANDY, S., NAZ, R., HAYAT, T., and ALSAEDI, A. Numerical and analytical solutions for Falkner-Skan flow of MHD Maxwell fluid. *Applied Mathematics and Computation*, 242, 569–575 (2014)

[27] ROSENHEAD, L. *Laminar Boundary Layers: An Account of the Development, Structure, and Stability of Laminar Boundary Layers in Incompressible Fluids, Together with a Description of the Associated Experimental Techniques*, Clarendon Press, Oxford (1963)

[28] ABBASBANDY, S. A new class of polynomial functions equipped with a parameter. *Mathematical Sciences*, 11, 127–130 (2017)

[29] BERNARDI, C. and MADAY, Y. Spectral method. In *Handbook of Numerical Analysis*, (eds. CIARLET, P. G. and LIONS, L. L.). Vol. 5 (Part 2). Elsevier, Amsterdam (1997)

[30] NASR, H., HASSANIEL, I. A., and EL-HAWARY, H. M. Chebyshev solution of laminar boundary layer flow. *International Journal of Computer Mathematics*, 33(1-2), 127–132 (1990)

[31] ABBASBANDY, S. and HAYAT, T. Solution of the MHD Falkner-Skan flow by homotopy analysis method. *Communications in Nonlinear Science and Numerical Simulation*, 14(9-10), 3591–3598 (2009)