REMARKS ON EIGENSPECTRA OF ISOLATED SINGULARITIES

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Abstract. We introduce a simple calculus, extending a variant of the Steenbrink spectrum, for describing Hodge-theoretic invariants of (smoothings of) isolated singularities with (relative) automorphisms. After computing these “eigenspectra” in the quasi-homogeneous case, we give three applications to singularity bounding and monodromy of VHS.

INTRODUCTION

Recent work of the third author and R. Laza on the Hodge theory of degenerations [KL1, KL2] re-examined the mixed Hodge theory of the Clemens-Schmid and vanishing-cycle sequences, with an emphasis on applications to limits of period maps and compactifications of moduli. When a degenerating family of varieties has a finite group $G$ acting on its fibers, these become exact sequences in the category of mixed Hodge structures with $G \times \mu_k$-action, where $k$ is the order of $T_{ss}$ (the semisimple part of monodromy). These kinds of situations often show up in generalized Prym or cyclic-cover constructions; for instance, instead of using the period map attached to a family of varieties, one may want to use the “exotic” period map arising from a cyclic cover branched along the family.

In this note we explain how to encode the contributions of isolated singularities with $G$-action to the vanishing cohomology in terms of $G$-spectra. These are formal sums (with positive integer coefficients) of triples in $\mathbb{Z} \times \mathbb{Q} \times \mathfrak{R}$, where $\mathfrak{R}$ is the set of irreducible representations of $G$. The term eigenspectrum refers to the specific case of a cyclic group $\langle g \rangle$ with fixed generator. In §A this formalism emerges naturally from the general setting of a proper morphism of 1-parameter degenerations over a disk, by specializing the morphism to an automorphism $g \in \text{Aut}(\mathcal{X}/\Delta)$ fixing a singularity $x \in X_0$. The eigenspectrum $\sigma_{p,x}^g$ simply records the dimensions of simultaneous eigenspaces of $g^*$ and $T_{ss}$ in the $(p, q)$-subspaces of $V_x$ (Defn. A.12). We give a general computation in

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In the case of a quasi-homogeneous singularity, in terms of a monomial basis for the associated Jacobian ring (Cor. B.7).

In the remaining sections, we give three applications. The first, in §C, is to bounding the number of nodes on Calabi-Yau hypersurfaces in weighted projective spaces (Thm. C.6) by passing to cyclic covers. In particular, this simplifies the proof for Varchenko’s bound of 135 for quintic hypersurfaces in $\mathbb{P}^3$. We also obtain the curious result that a CY hypersurface in $\mathbb{P}^{n+1}$ with isolated singularities and symmetric under $\mathfrak{S}_{n+2}$ cannot contain a node whose $\mathfrak{S}_{n+2}$-orbit has cardinality $(n + 2)!$ (Thm. C.11).

The other two applications concern codimension-one monodromy phenomena for VHSs over moduli of configurations of points and hyperplanes. In §D, the moduli space is $M_{0,2n}$, with the VHS arising from cyclic covers of $\mathbb{P}^1$ branched along the $2m$ ordered points. Propositions D.5-D.6 describe the eigenspectra for boundary strata of certain compactifications $\overline{M}^{H}_{0,2n}$ due to Hassett [Ha], also closely related to work of Deligne and Mostow [DM]; this generalizes a computation of [GKS].

Our other main example, treated in §E, is the VHS $\mathcal{H} \to S$ on the moduli space of general configurations of $(2n + 2)$ hyperplanes in $\mathbb{P}^n$, arising from the middle (intersection) cohomology of a 2:1 cover $X \to \mathbb{P}^n$ branched along these hyperplanes. By passing to a smooth complete intersection $2^{2n}$-cover of $X$ and applying the Cayley trick, we replace $X$ by a smooth hypersurface $Y \subset \mathbb{P}(\mathcal{O}_{\mathbb{P}^{2n+1}}(2)^{\oplus(n+1)})$ with automorphisms by a group of order $2^{2n}$. In codimension-one in moduli, $Y$ acquires nodes, and a variant of Schoen’s result in [Sc] ensures that these produce nontrivial symplectic transvections for $\mathcal{H}$ when $n$ is odd. This gives an easy proof that the geometric monodromy group of $\mathcal{H}$ is maximal (for all $n$), and the period map “non-classical”, a fact first proved by [GSVZ] for $n = 3$ and by [SXZ] in general.

Notation. In this paper “MHS” stands for $\mathbb{Q}$-mixed Hodge structure. We shall make frequent use of the Deligne bigrading on a MHS $V$ [De1]. This is (by definition) the unique decomposition $V_C = \bigoplus_{p,q \in \mathbb{Z}} V^{p,q}$ with the properties that $F^k V_C = \bigoplus_{p,q \geq k} V^{p,q}$, $W^\ell V_C = \bigoplus_{p,q \leq \ell} V^{p,q}$, and $V^{q,p} \equiv V^{p,q} \mod \bigoplus_{b<q} V^{a,b}$. We shall make free use of standard multi-index notation (for $n$-tuples of variables or field-elements) to simplify formulas, viz. $\mathbf{z} = (z_1, \ldots, z_n)$, $\mathbb{C}[\mathbf{z}] = \mathbb{C}[z_1, \ldots, z_n]$, $\mathbf{z}^m = \Pi_i z_i^m$, $\mathbf{m} \cdot \mathbf{w} = \sum_i m_i w_i$, $|\mathbf{m}| = \sum_i m_i$, $e^{(i)} = i^{th}$ standard basis vector, etc.

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A. G-spectra and eigenspectra

We begin in the general setting of a proper morphism

\[ Y \xrightarrow{\pi} X \]

of complex analytic spaces over a disk, which we assume is the restriction to \( \Delta \) of a proper morphism of quasi-projective varieties over an algebraic curve. (In particular, at the level of fibers we have that \( \pi_t: Y_t \to X_t \) is a proper algebraic morphism of quasi-projective varieties.) Let \( K^\bullet \in D^b\text{MHM}(X) \) and \( L^\bullet \in D^b\text{MHM}(Y) \) be given, with a morphism \( \rho: K^\bullet \to R\pi^*L^\bullet \). Writing \( i: X_0 \hookrightarrow X \) for the inclusion, the vanishing cycle triangle

\[ \begin{array}{c}
  i^* \xrightarrow{sp} \psi_f \xrightarrow{can} \phi_f \xrightarrow{[+1]} \\
  \downarrow \rho \quad \downarrow \rho \quad \downarrow \rho \\
  i^* \xrightarrow{sp} \psi_{f'} \xrightarrow{can} \phi_{f'} \xrightarrow{[+1]}
\end{array} \]

consists of functors from \( D^b\text{MHM}(X) \) to \( D^b\text{MHM}(X_0) \), with natural transformations between them; moreover, monodromy \( T = T_{ss} e^N \) induces natural automorphisms of \( \psi_f \) and \( \phi_f \). By proper base-change and faithfulness of \( \text{rat}: D^b\text{MHM}(X_0) \to D^b\text{MHM}(X) \), \( R\pi_*: D^b\text{MHM}(Y_0) \to D^b\text{MHM}(X_0) \) intertwines the corresponding triangle (and monodromy actions) for \( (Y, f') \). Taking hypercohomology on \( X_0 \) yields:

A.3. Proposition. We have the commutative diagram

\[ \begin{array}{c}
  H(X_0, i^*K^\bullet) \xrightarrow{sp} H(X_0, \psi_f K^\bullet) \xrightarrow{can} H(X_0, \phi_f K^\bullet) \xrightarrow{\delta} H(X_0, i^*K^\bullet) \\
  \downarrow \rho \quad \downarrow \rho \quad \downarrow \rho \\
  H(Y_0, i^*L^\bullet) \xrightarrow{sp} H(Y_0, \psi_{f'} L^\bullet) \xrightarrow{can} H(Y_0, \phi_{f'} L^\bullet) \xrightarrow{\delta} H(Y_0, i^*L^\bullet)
\end{array} \]

with rows the vanishing-cycle (long-exact) sequences, in which all arrows are morphisms of MHS. Moreover, the diagram intertwines the actions of \( T_{ss} \) (by automorphisms of MHS) and \( N \) (by nilpotent \((-1, -1)\)-endomorphisms of MHS), which are trivial (Id resp. 0) on the end terms.
A.4. Remark. If $f, f'$ are themselves projective (hence proper), and $K^\bullet, L^\bullet$ semisimple with respect to the perverse $t$-structure (e.g. $K^\bullet = IC^\bullet_{\mathcal{X}}, L^\bullet = IC^\bullet_{\mathcal{Y}}$), then the Decomposition Theorem applies, yielding Clemens-Schmid sequences (cf. [KL1, §5]) which are then automatically compatible under $\rho$. The main consequence is that the local invariant cycle theorem holds, i.e. $sp$ surjects onto the $T$-invariants.

Next, assume $\mathcal{X}, \mathcal{Y}, \{X_t\}_{t \neq 0}$, and $\{Y_t\}_{t \neq 0}$ are smooth, and take $L^\bullet = \mathbb{Q}_Y$ and $K^\bullet = \mathbb{Q}_X$; then the diagram in Prop. A.3 becomes

\begin{equation}
(A.5) \quad H^k(X_0) \xrightarrow{sp} H^k_{\lim}(X_t) \xrightarrow{can} H^k_{\text{van}}(X_t) \xrightarrow{\delta} H^{k+1}(X_0) \xrightarrow{} \\
\downarrow \pi^* \quad \quad \downarrow \pi^* \quad \quad \downarrow \pi^* \quad \quad \downarrow \pi^* \\
H^k(Y_0) \xrightarrow{sp} H^k_{\lim}(Y_t) \xrightarrow{can} H^k_{\text{van}}(Y_t) \xrightarrow{\delta} H^{k+1}(Y_0) \xrightarrow{} .
\end{equation}

Now if $n = \dim X_0$ and $\Sigma := \text{sing}(X_0)$ is finite, then $H^k_{\text{van}}(X_t) = \{0\}$ for $k \neq n$ and, defining $V_x := H^0_{\text{ss}} \phi f^* \mathbb{Q}_X[n]$, $H^n_{\text{van}}(X_t) \cong \bigoplus_{x \in \Sigma} V_x$

as MHS. We have of course $\pi^{-1}(\Sigma) \subset \tilde{\Sigma} := \text{sing}(Y_0)$, and if $\dim Y_0 = n$ and $|\tilde{\Sigma}| < \infty$ then, writing $\tilde{V}_y := H^0_{\text{ss}} \phi f^* \mathbb{Q}_Y[n]$ $(y \in \tilde{\Sigma})$, $\pi^*$ restricts to morphisms

\begin{equation}
(A.7) \quad [\pi^*]_x: V_x \rightarrow \bigoplus_{y \in \pi^{-1}(x)} \tilde{V}_y
\end{equation}

of $T$-MHS — i.e. morphisms of MHS intertwining $T$ (hence $T_{\text{ss}}$ and $N$). These are local invariants.

Recall that $T_{\text{ss}}$ acts through finite cyclic groups on each $V_x$ (and $\tilde{V}_y$), and let $\kappa$ be the $\text{lcm}$ of their orders. Write $\zeta_{\kappa} := e^{2\pi i/\kappa}$ and $V_{p,q}^{x,\kappa} = \mathbb{C}$ for the $e^{2\pi i/\kappa}$-eigenspace of $T_{\text{ss}}$ in $V_{p,q}^x \subset V_{x,\mathbb{C}}$. The explicit choice of $\zeta_{\kappa} \in \mathbb{C}$ is needed to make the following

A.8. Definition. The mixed spectrum $\sigma_{f,x}$ of the isolated singularity $x \in \Sigma$ is the element $\sum_{\alpha, w} m_{\alpha, w} f_{\alpha, w}(\alpha, w)$ of the free abelian group $\mathbb{Z}(\mathbb{Q} \times \mathbb{Z})$, where $m_{\alpha, w} = \dim(V_{x,\mathbb{C}}^{e(\alpha), w-\{\alpha\}})$.

Evidently (A.7) must be compatible with the decompositions recorded by the mixed spectra.

Now let $G \leq \text{Aut}(\mathcal{X}/\Delta)$, with $\mathcal{X}$ and $\{X_t\}_{t \neq 0}$ smooth and $|\Sigma| < \infty$. Applying the foregoing results with $\mathcal{Y} = \mathcal{X}$, $f = f'$, and $\pi := g \in G$, together with [KL1] Prop. 5.5(i), yields

\footnote{Here $\lfloor \cdot \rfloor$ is the greatest integer (floor) function; note also that $e(\alpha)$ is equivalent to taking the fractional part $\{\alpha\} := \alpha - \lfloor \alpha \rfloor$ of $\alpha$.}
A.9. **Corollary.** The vanishing-cycle sequence

\[(A.10) \quad 0 \to H^n(X_0) \xrightarrow{sp} H^n_{\lim}(X_t) \xrightarrow{\text{can}} \bigoplus_{x \in \Sigma} V_x \xrightarrow{\delta} H^{n+1}_{\text{ph}}(X_0) \to 0 \]

is an exact sequence of $G \times \mu_\kappa$-MHS\(^2\) where the $\langle T_{ss} \rangle \cong \mu_\kappa$-action on the end terms is trivial. If $\mathcal{X}/\Delta$ is proper, then $H^{n+1}_{\text{ph}}(X_0) := \ker(\text{sp}) \subseteq H^{n+1}(X_0)$ is pure of weight $n + 1$.

The decomposition of terms in \[(A.10)\] into irreps for $G \times \mu_\kappa$ only becomes useful if we understand the action on the vanishing cohomology $\bigoplus_{x \in \Sigma} V_x$ for a given collection of singularities. In particular, if $gx = x$ then we need to further refine the spectrum under the resulting automorphism $g^*: V_x \to V_x$ of $T$-MHS.

A.11. **Definition.** Write $G \leq \text{stab}(x) \leq \mathcal{G}$, and $\mathcal{R}_G$ for the set of complex irreducible representations of $G$. The $G$-spectrum $\sigma^G_{f,x}$ of $x$ is the element $\sum_{(\alpha, w, U)} m^{f, x, G}_{\alpha, w, U}(\alpha, w, U)$ of the free abelian group $\mathbb{Z}(\mathbb{Q} \times \mathbb{Z} \times \mathcal{R}_G)$, where (for each $(\alpha, w)$) $V^{[\alpha], w-\lceil \alpha \rceil}_{x, e(\alpha)} \cong \bigoplus_{U \in \mathcal{R}_G} U^{\oplus m^{f, x, G}_{\alpha, w, U}}$ as $G$-representations.

In the special case where $G = \langle g \rangle \cong \mu_\ell$ is cyclic, the $\mathbb{C}$-irreps are characters indexed by the power $\zeta^e_\ell = e^{2\pi i \ell}$ of $\zeta_\ell$ to which $g$ is sent.

A.12. **Definition.** The eigenspectrum of an isolated singularity $x$ with automorphism $g$ is the element

$$\sigma^g_{f,x} = \sum_{(\alpha, w, \gamma)} m^{f, x, g}_{\alpha, w, \gamma}(\alpha, w, \gamma) \in \mathbb{Z}(\mathbb{Q} \times \mathbb{Z} \times \mathbb{Q}/\mathbb{Z}),$$

where $m^{f, x, g}_{\alpha, w, \gamma}$ is the dimension of the eigenspace $(V^{[\alpha], w-\lceil \alpha \rceil}_{x, e(\alpha)})^{e(\gamma)} \subseteq V^{[\alpha], w-\lceil \alpha \rceil}_{x, e(\alpha)}$ for $g^*$ with eigenvalue $e(\gamma) = e^{2\pi i \gamma}$.

A.13. **Remark.** For $\mathcal{X}/\Delta$ proper (with hypotheses as in Cor. [A.9]), $H^n(X_t)$ is a VHS on $\Delta^*$ whose automorphism group contains $\mathcal{G}$. For any field extension $K/\mathbb{Q}$, this decomposes as $K$-VHS into a direct sum of $G$-isotypical components, corresponding to $K$-irreps of $\mathcal{G}$. The $\mathcal{G}$-action on and decomposition of $H^n_{\lim}(X_t)$ obtained by taking limits are the same as those arising from the $\mathcal{G}$-MHS structure on $H^n_{\lim}(X_t)$ in Cor. [A.9].

We now turn to the explicit computation of these eigenspectra in the simplest case.
B. Quasihomogeneous singularities with automorphism

Let \( F \in \mathbb{C}[z_1, \ldots, z_{n+1}] \) (with \( n > 0 \)) be a quasi-homogeneous polynomial with an isolated singularity at the origin \( 0 \). That is, choosing a weight vector \( \underline{w} = (w_1, \ldots, w_{n+1}) \in \mathbb{Q}_{>0}^{n+1} \) and setting \( M_{\underline{w}} := \{ m \in \mathbb{Z}_{\geq 0}^{n+1} \mid m \cdot \underline{w} = 1 \} \), we have

\[
F = \sum_{m \in M_{\underline{w}}} a_m z^m
\]

for some \( a_m \in \mathbb{C} \). We recall that the degree \( \kappa_F \) of \( F \) is the least integer such that \( \kappa_F w_i \in \mathbb{N} \) for \( i = 1, \ldots, n+1 \); define \( w_i := \kappa_F w_i \) and set \( \underline{\kappa} := (\kappa_1, \ldots, \kappa_{n+1}) \).

Next recall the setting of Defn. A.8, where \( f: X \to \Delta \) is a holomorphic map with quasi-projective fibers and smooth total space, with \( X_t \) smooth for \( t \neq 0 \) and \( \text{sing}(X_0) =: \Sigma \) finite. A singularity \( x \in \Sigma \subset X_0 \) is quasi-homogeneous if \( f \) can be locally analytically identified with (B.1) for some \( \underline{w} \). In that case, \( V_x \) and \( \sigma_{f,x} \) identify with the vanishing cohomology

\[
V_F := H^0_{\underline{0}} \phi_F \mathbb{Q}_{\mathbb{C}^{n+1}}
\]

of \( F: \mathbb{C}^{n+1} \to \mathbb{C} \) at \( \underline{0} \), and its mixed spectrum \( \sigma_F \). These were first computed by Steenbrink in [St], and we briefly review the treatment from [KL2 §2] before passing to eigenspectra.

Writing \( J_F := (\frac{\partial F}{\partial z_1}, \ldots, \frac{\partial F}{\partial z_{n+1}}) \subset \mathbb{C}[\underline{z}] \) for the Jacobian ideal, let \( B \subset \mathbb{Z}_{\geq 0}^{n+1} \) be chosen so that the monomials \( \{z^{\beta}\}_{\beta \in B} \) provide a basis of \( \mathbb{C}[\underline{z}]/J_F \). Write \( \mu_F := |B| \) for the Milnor number of \( F \), and \( \ell(\underline{\beta}) := \frac{1}{\kappa_F} \sum_{i=1}^{n+1} \kappa_i (\beta_i + 1) = \sum_{i=1}^{n+1} w_i (\beta_i + 1) \).

B.3. Proposition. We have \( \mu_F = \dim V_F \) for the Milnor number and

\[
\sigma_F = \sum_{\beta \in B} (\alpha(\underline{\beta}), w(\underline{\beta})) \in \mathbb{Z}(\mathbb{Q} \times \mathbb{Z})
\]

for the mixed spectrum, where \( \alpha(\underline{\beta}) := n+1 - \ell(\underline{\beta}) \) and \( w(\underline{\beta}) := n \) [resp. \( n+1 \)] if \( \alpha(\underline{\beta}) \notin \mathbb{Z} \) [resp. \( \in \mathbb{Z} \)].

Sketch. Perform a base-change followed by weighted blow-up at \( 0 \)

\[
\begin{array}{ccc}
\mathbb{C}^{n+1} & \xrightarrow{\mathbb{X}} & \mathbb{Y} \\
F \downarrow & & \downarrow F \\
C & \xrightarrow{\Delta} & \mathbb{F} \\
\end{array}
\]

\[
t^F \quad \xrightarrow{t}
\]
with exceptional divisor $E = \{ T^{\kappa_F} = F(Z) \} \subset \mathbb{WP}[1:K] =: P$ (in weighted homogeneous coordinates $T, Z_1, \ldots, Z_{n+1}$). The singular fiber $\mathcal{Y}_0 := \hat{E}^{-1}(0)$ is the union of $E$ and the proper transform $\hat{X}_0$ of $X_0 := F^{-1}(0) = \hat{F}^{-1}(0)$, meeting in

$$E := \mathcal{E} \cap \hat{X}_0 = \{ F(Z) = 0 \} \subset H := \{ T = 0 \} (\cong \mathbb{WP}[K]) \subset P.$$ 

The claim is then that $V_F \cong H^n(\mathcal{E} \setminus E)$, which can be checked using (A.5) with $\pi = Bl_x$. Since $E$ [resp. $\hat{E}$] is a deformation retract of $\mathcal{Y}_0$ [resp. $X_0$], while $\mathcal{Y}_t = X_t$ for $t \neq 0$, and $\phi_{F Q} : Q \simeq t^E Q_E(-1)[−1]$ (cf. [KL1, 6.3 and 8.3-4]), the diagram becomes

$$0 \longrightarrow H^n_{\lim}(X_t) \xrightarrow{\cong} V_F \xrightarrow{\|} 0$$

$$H^{n-2}(E)(−1) \xrightarrow{H^n} H^n(\mathcal{E}) \xrightarrow{H^n_{\lim}} H^{n-1}(\mathcal{Y}_t) \xrightarrow{H^{n-1}(E)(−1)} H^{n+1}(\mathcal{E})$$

whence the result.

Next, one constructs a basis of $H^n(\mathcal{E} \setminus E)$ from $B$, using residue theory. Writing (with $T := Z_0$)

$$\Omega_P = \sum_{j=0}^{n+1} (-1)^j Z_j dZ_0 \wedge \cdots \wedge dZ_j \wedge \cdots \wedge dZ_{n+1},$$

for each $\beta \in B$ we set (with $Z^\beta = Z_1^{\beta_1} \cdots Z_{n+1}^{\beta_{n+1}}$)

$$\Omega_\beta := \frac{T^{\kappa_F} Z^\beta \Omega_P}{T(F(Z) − T^{\kappa_F})^{\lfloor \ell(\beta) \rfloor}} \in \Omega^{n+1}(P \setminus \mathcal{E} \cap H) \tag{B.5}$$

and $\omega_\beta := \text{Res}_{\mathcal{E} \setminus E}(\Omega_\beta) \in H^n(\mathcal{E} \setminus E)$. We refer to [KL2, Thm. 2.2] for the proof that this has $(p, q)$-type $(|\alpha(\beta)|, |\ell(\beta)|)$, and [St] Thm. 1 for the assertion that the $\{\omega_\beta\}$ give a basis. Note that $|\alpha(\beta)| + |\ell(\beta)| = w(\beta)$.

Finally, the action of $T_{ss}$ is computed by $T \mapsto \zeta^{\kappa_F} T$, or equivalently (in weighted projective coordinates) by $Z_i \mapsto \zeta^{−\kappa_i} Z_i = e^{−2\pi i w_i} Z_i$. Clearly the effect of this on (B.5) is to multiply it by $e^{2\pi i \sum w_i (\beta_i + 1)} = e^{2\pi i \alpha(\beta)}$, as desired. \hfill $\square$

Now given a finite group $G \leq \text{Aut}(\mathcal{X}/\Delta)$ fixing $x \in \Sigma$, we can always choose local holomorphic coordinates on which the action is linear [Ca]. So for a given $g \in G$, we can choose coordinates to make the action diagonal, through roots of unity. Accordingly, we shall compute the eigenspectrum in the case where $g \in \text{Aut}(\mathbb{C}^{n+1}, \Omega)$ is given by

$$g(z_1, \ldots, z_{n+1}) := (\zeta^{c_1}_\ell z_1, \ldots, \zeta^{c_{n+1}}_\ell z_{n+1}) \tag{B.6}$$
and $F \in \mathbb{C}[z]^{(g)}$ is a $g$-invariant quasi-homogeneous polynomial. In fact, taking $\mathcal{B} \subset \mathbb{Z}_{\geq 0}^{n+1}$ as above, we have the

**B.7. Corollary.** The eigenspectrum $\sigma_{\mathcal{F}}^g$ is given by

$$
\sum_{\beta \in \mathcal{B}} (\alpha(\beta), w(\beta), \gamma(\beta)) \in \mathbb{Z}(\mathbb{Q} \times \mathbb{Z} \times \mathbb{Q}/\mathbb{Z}),
$$

where $\gamma(\beta) := \frac{1}{\ell} \sum_{i=1}^{n+1} c_i(\beta_i + 1)$.

**Proof.** We only need to compute the action of $g^*$ on $\omega_{\beta}$, which is to say the effect of $Z_i \mapsto \zeta_i^\ell Z_i$ on $\mathbb{Z}[\Omega]$. Clearly this is just multiplication by $\zeta_\ell^{c_i(\beta_i + 1)} = e^{2\pi i \gamma(\beta)}$. \hfill $\square$

**B.8. Example.** For a Brieskorn-Pham singularity $F = \sum_{i=1}^{n+1} z_i^{\lambda_i}$ ($\lambda_i = \frac{1}{w_i} = \frac{m_i}{n_i}$), we have $\mathcal{B} = \times_{i=1}^{n+1} \{Z_i \cap [0, d_i - 2]\}$. Hence, writing $\Gamma_m = \sum_{j=1}^{m-1} (\frac{j}{m})$ in the group ring $\mathbb{Z}[(\mathbb{Q})]$ (with product $*$), we have $\sum_{\beta \in \mathcal{B}} [\alpha(\beta)] = \Gamma_{\lambda_1} * \cdots * \Gamma_{\lambda_{n+1}}$. This extends to $\sum_{\beta \in \mathcal{B}} [\alpha(\beta), \gamma(\beta)] = \tilde{\Gamma}_{\lambda_1}(\frac{\ell}{\ell}) * \cdots * \tilde{\Gamma}_{\lambda_{n+1}}(\frac{\ell}{\ell})$ in the group ring $\mathbb{Z}[\mathbb{Q} \times (\mathbb{Q}/\mathbb{Z})]$ if we write $\tilde{\Gamma}_m(\frac{\ell}{\ell}) = \sum_{j=1}^{m-1} [(\frac{m-j}{m}, \frac{j}{m})]$.\hfill $\square$

**B.9. Remark.** The eigenspectrum of a $\mu$-constant (semi-quasi-homogeneous) deformation of $(\mathcal{F}, \gamma)$ remains constant. Even in the more general case of [KL2] §5.2, one can in principle still use the action of $\gamma^*$ on the (local) Jacobian ring $\mathcal{O}_{n+1}/J_{\mathcal{F}}$ to refine $\sigma_{\mathcal{F}}$ to $\sigma_{\mathcal{F}}^g$. But Corollary B.7 (and quasi-homogeneous deformations of Example B.8) will suffice for our purposes below.

### C. Bounding nodes on Calabi-Yau hypersurfaces

Let $\mathbb{W} = \mathbb{W} \mathbb{P}[e_0 : \cdots : e_{n+1}]$ be a weighted projective $(n + 1)$-space with finitely many singularities.\footnote{We may assume (without loss of generality) that no $n + 1$ of the $e_i$ have a common factor.} Suppose we want to bound (numbers and types of) singularities on a hypersurface $X_0 = \{F_0(\mathbb{W}) = 0\} \subset \mathbb{W}$ of degree $d$, where a smooth such hypersurface would have Hodge numbers $h = (h^{0,0}, h^{n-1,1}, \ldots, h^{0,n})$. Write $d_i = \frac{d}{e_i}$ for $i = 0, \ldots, n + 1$.

We shall assume that the singularities of $X_0$ are all isolated. Taking a general deformation $F_t = F_0 + tG$ to produce a family of $f : \mathcal{X} \to \Delta$ with smooth total space, the vanishing-cycle sequence

$$(C.1) \quad 0 \to H^n(X_0) \to H^n_{\lim}(X_t) \to \bigoplus_{x \in \Sigma} V_x \to H_{ph}^{n+1}(X_0) \to 0$$
of nodes on a quartic surface (cf. Example C.8).

Moreover, the mixed spectrum \( \sigma \) \cite{Do}, a basis for the automorphism. By Dolgachev's extension of Griffiths's residue theory giving one for odd \( n \), \( \sum_{i=1}^{d}(\frac{n}{2}+1) \) in \( \Gamma_{d_{n+1}} \) as a bound, which while better than nothing is rather weak.

C.4. Example. The simplest nontrivial case is \( W = \mathbb{P}^{3} \) \( (n = 2) \) and \( (d_{0} = d_{1} = d_{2} = d_{3} = d) = 4 \), where \( \Gamma_{n}^{4} = (\lfloor \frac{1}{1} \rfloor + \lfloor \frac{1}{2} \rfloor + \lfloor \frac{3}{4} \rfloor)^{4} \).

\[ \text{(C.5) } [1] + 4[\frac{1}{2}] + 10[\frac{3}{4}] + 16[\frac{5}{8}] + 19[2] + 16[\frac{7}{8}] + 10[\frac{9}{16}] + 4[\frac{11}{16}] + [3] \]

correctly gives \( 19 = h_{pr}^{1,3}(X_{t}) \). This is also a poor bound for the number of nodes on a quartic surface (cf. Example C.8).

However, there is a simple trick which improves the bound while also giving one for odd \( n \):

C.6. Theorem. The number of nodes on \( X_{0} \) is bounded by the coefficient, in \( \Gamma_{d_{0}} * \Gamma_{d_{1}} * \cdots * \Gamma_{d_{n+1}} \), of \( \lfloor \frac{n+1}{2} \rfloor + \lfloor \frac{1}{d} \rfloor \) if \( n \) is even and \( d \) is odd, or of \( \lfloor \frac{n+1}{2} \rfloor + \frac{1}{d} \) otherwise.

Prove. Let \( Y_{t} = \{ F_{t}(W) + W_{n+2}^{d} = 0 \} \subset \mathbb{WP}[e] \): \( \tilde{W} \) be the cyclic \( d;1 \)-cover of \( W \) branched over \( X_{t} \), with \( g: W_{n+2} \mapsto \zeta_{d}W_{n+2} \) the cyclic automorphism. By Dolgachev's extension of Griffiths's residue theory \cite{Do}, a basis for the \( g^{*} \)-eigenspace \( H^{3-n,q+1,q}(Y_{t})C^{j}_{d_{t}} \) \( (t \neq 0, 0 \leq j < d) \) is given by the Poincaré residue classes

\[ \text{Res}_{Y_{t}} \left( \frac{W^{k_{i}}W_{n+2}^{-d-j-1} \Omega_{\tilde{W}}}{(F_{t} + W_{n+2}^{d})^{q+1}} \right) \]

with \( k_{i} \in \mathbb{Z} \cap (0, d_{i}) \) \( (i = 0, \ldots, n+1) \) and weights of numerator and denominator equal: that is, \( \sum_{i=0}^{n+1} e_{i}k_{i}+(d-j) = (q+1)d \), or equivalently

\[ \text{Res}_{Y_{t}} \left( \frac{W^{k_{i}}W_{n+2}^{-d-j-1} \Omega_{\tilde{W}}}{(F_{t} + W_{n+2}^{d})^{q+1}} \right) \]

for the same residue theory as used in the proof of Theorem C.6 below. The notation '*' is from Example B.8.

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4This is by the same residue theory as used in the proof of Theorem C.6 below. The notation '*' is from Example B.8.
which case the middle entry is $n_{168}$. If $h$ that nodes. The bounds here are the coefficients of $n$ and $q$ nodes, applying (C.1)-(C.2) to $Y$ and refining by $g^*$-eigenspaces therefore yields $h^{p_j,q_j}(Y)_{\tilde{\sigma}} \geq r$ (for $0 < j < d$), where $p_j = \lceil \frac{n+1}{2} + \frac{j}{d} \rceil$ and $q_j = n + 1 - p_j$. Taking $j = 1$ if $n$ is odd and $j = \lceil \frac{d+1}{2} \rceil$ if $n$ is even (so that $p_j = \frac{n+1}{2} \text{ resp. } \frac{n+1}{2}+1$) yields $q_j + \frac{j}{d} = \frac{n+1}{2} + \frac{1}{d} \text{ resp. } \frac{n}{2} + \frac{1}{d} \lceil \frac{d+1}{2} \rceil$, hence the claimed bound. □

C.7. Remark. When $\mathcal{W} = \mathbb{P}^{n+1}$, this recovers the bound conjectured by Arnol’d [Ar] and proved by Varchenko by applying his semicontinuity theorem to the Bruce deformation (see also [vS2]), While Varchenko also uses the “cyclic-cover trick”, our approach avoids the use of deformations and semicontinuity.

C.8. Example. For CY hypersurfaces on $\mathbb{P}^{n+1}$ ($d = n+2$), Thm. C.6 yields the bounds 3, 16, 135, 1506, and 20993 for $n = 1, 2, 3, 4, 5$, the first two of which are sharp. (This is also better than what (C.3) yields for $n = 2$ and $4$, namely 19 and 1751.) It is still not known whether 135 is sharp for quintic 3-folds. The well-known Fermat pencil has fiber $W^5_0 + \cdots + W^5_4 = 5W_0 \cdots W_4$, with 125 $= |(\mathbb{Z}/5\mathbb{Z})^5|$ nodes, while the example of van Straten [vS1] with 130 nodes remains the record.

C.9. Remark. For $n = 2$, the following bound by Miyaoka [Mi] sometimes yields better results. If $X$ is any smooth projective surface which is smooth except at $r$ nodes, and $K_X$ is nef, then $r \leq 8\chi(\mathcal{O}_X) - \frac{2}{3}K_X^2$.

(a) For $X \subset \mathbb{P}^2$ a surface of degree $d$, this yields the bound $\frac{2}{3}(d-1)(d-2)(d-3) + 8 - \frac{8}{9}d(d-2)^2 = \frac{4}{9}d(d-1)^2$, which is better than Thm. C.6 for $d \geq 6$ even or $d \geq 15$ odd. A case in point is $d = 6$, where (C.3) gives 85, the Theorem 68, and Miyaoka 66; this was further reduced to 65 (which is sharp) by a clever use of coding theory [JR]. Another is $d = 8$, where we get $r \leq 174$.

(b) As a weighted projective example, one can consider surfaces $X$ of degree 10 in $\mathbb{P}\mathbb{P}[1:1:1:2]$. We have $\chi(\mathcal{O}_X) = 1 + h^2(\mathcal{O}_X) = 35$ and $(K_X \cdot K_X)_X = (X \cdot (X + K_W)^2)_W = \frac{100(10-5)^2}{1+1+1+2} = 125$, hence $r \leq \lfloor \frac{1520}{9} \rfloor = 168$.

5The union of 3 lines in $\mathbb{P}^2$ has 3 nodes, and a Kummer quartic $K3$ in $\mathbb{P}^4$ has 16 nodes. The bounds here are the coefficients of $\lfloor \frac{n+1}{2} + \frac{1}{n+2} \rfloor$ in $\Gamma_{n+2}^{(n+2)}$; e.g., 16 is the coefficient of $\left[ \frac{2}{7} \right]$ in (C.6).
C.10. Examples. We consider some CY 3-fold hypersurfaces with $r$ nodes in weighted projective 4-folds.

(i) $X_0 \subset WP[1:1:1:2]$ of degree 6: the Theorem yields $r \leq 137$, while the “Fermat pencil” type example $W^6_0 + \cdots + W^3_2 + W^4_4 = 3 \cdot 2^4 W_0 \cdots W_4$ has $|((Z/6Z)^3 \times Z/3Z)/(Z/6Z)| = 108$ nodes.

(ii) $X_0 \subset WP[1:1:1:4]$ of degree 8: the Theorem yields $r \leq 180$, while $W^8_0 + \cdots + W^3_3 + W^4_4 = 4W_0 \cdots W_4$ has $|((Z/8Z)^3 \times Z/2Z)/(Z/8Z)| = 128$ nodes. Here we can improve both the bound and example, since $X_0$ is (by the quadratic formula) a double-cover of $\mathbb{P}^3$ branched along an $r$-nodal octic surface. So Rem. \textit{C.9(a)} gives $r \leq 174$, while Endraß’s example \textit{[En]} has $r = 168$.

(iii) $X_0 \subset WP[1:1:2:5]$ of degree $d = 10$: the Theorem yields $r \leq 169$, but because these are double covers of $WP[1:1:1:2]$ branched along an $r$-nodal dectic surface, Rem. \textit{C.9(b)} reduces the bound to 168. The standard example is $W^{10}_0 + W^{10}_1 + W^{10}_2 + W^5_3 + W^2_4 = 2^4 \cdot 5^4 W_0 \cdots W_4$, but this has only 100 nodes. One can do somewhat better by taking the preimage of a Togliatti quintic $\textit{[Be]}$ (with 31 nodes avoiding the coordinate axes) under $WP[1:1:1:2] \overset{1:2}{\rightarrow} WP[1:1:2:2] \overset{1:2}{\rightarrow} WP[1:2:2:2] \cong \mathbb{P}^3$, to get $4 \cdot 31 = 124$.

In the case of a symmetric hypersurface $X_0 \subset \mathbb{P}^{n+1}$, cut out by $F_0 \in \mathbb{C}[W]^{G_{n+2}}$ (homogeneous of degree $d$), one can consider the family $\mathcal{Y} \to \Delta$ of $d$-fold cyclic covers branched along an $G_{n+2}$-invariant smoothing $\mathcal{X} \to \Delta$. A full accounting of this story gets into $G$-spectra ($G \cong \mu_d \times \text{stab}_{G_{n+2}}(x)$) of the resulting $A_{d-1}$ singularities of $Y_0$. This leads to constraints, via character theory of $G_{n+2}$, on how $\Sigma$ can be built out of $G_{n+2}$-orbits. (However, it does not, for example, rule out the possibility of 135 nodes on an $G_{5}$-symmetric quintic threefold.) Here we shall only give the simplest result in this direction:

C.11. Theorem. A symmetric CY hypersurface in $\mathbb{P}^{n+1}$ (of degree $d = n + 2$) with isolated singularities cannot contain a node with trivial stabilizer in $G_{n+2}$.

Proof. Suppose otherwise; then $Y_0$ has a set of $(n + 2)!$ $A_{n+1}$ singularities with eigenspectra $\sum_{j=1}^{n+1} (\frac{n+1}{2} + \frac{1}{n+2}, n+1, \frac{j}{n+2})$. This contributes a subspace $V$ of dimension $(n+2)!$ to the $g^*$-eigenspace $H^{n+1}_{\text{can}}(Y_t)_{\mathfrak{C}_{n+2}}$. It is closed under the action of $G_{n+2}$, and the triviality of the stabilizers of these $A_{n+1}$ singularities means that the trace of any $\sigma \in G_{n+2} \setminus \{1\}$ is zero. So $V$ is a copy of the regular representation of $G_{n+2}$, which

\text{As before, $g: W_{n+2} \mapsto \zeta_{n+2} W_{n+2}$ denotes the cyclic automorphism of $Y_t$.}
belongs to \( \ker(\delta) \subseteq H^{n+1}_{\text{van}}(Y)_{c_n+2} \). By the compatibility of the vanishing-cycle sequence for \( Y \) with \( g^* \) and \( S_{n+2} \), this forces a copy of the regular representation in \( H^{n+1}_{\text{lim}}(Y)_{c_n+2} \), hence \( H^{n+1}_{\text{lim}}(Y)_{c_n+2} \) for \( t \neq 0 \) (as \( S_{n+2} \) acts on the VHS, compatibly with taking limits, cf. Remark A.13).

Now \( U := H^{n+1}_{\text{lim}}(Y)_{c_n+2} \) has a basis of the form

\[ \eta_k := \text{Res}_{Y_t} \left( \frac{W_k - 1}{(F_0(W) + W_{n+2}^{n+2})^{n+3/2}} \right), \]

where \( 0 < k_i < n + 2 \) (for \( i = 0, \ldots, n + 1 \)) and (for equality of weights of numerator and denominator) \( (\sum_{i=0}^{n+1} k_i) + 1 = \frac{n+3}{2}(n+2) \). Here \( S_{n+2} \) acts trivially on the denominator, through the sign representation \( \chi \) on \( \Omega_{p_{n+2}} \), and by permutations of the \( W_i \) on \( W_k^{n+1} \). We claim that \( U \) contains no copy of the trivial representation, a fortiori of the regular representation, furnishing the desired contradiction.

Clearly it is equivalent to show that the representation of \( S_{n+2} \) on the \( \mathbb{C} \)-span \( \bar{U} (\cong U \otimes \chi) \) of the monomials \( \{W_k^{n+1}\}_{k \text{ as above}} \) contains no copy of \( \chi \). Let \( o := S_{n+2} \cdot W_k \) be an orbit and \( \bar{U}_o \subseteq \bar{U} \) its span. By Burnside’s Lemma, \( \frac{1}{(n+2)!} \sum_{g \in S_{n+2}} |o^g| = 1 \). On the other hand, \( k = (k_0, \ldots, k_{n+1}) \) contains a repeated entry since there are only \( n + 1 \) choices for each \( k_i \); hence for some transposition \( \tau \), \( |o^\tau| \neq 0 \). Since \( \text{sgn}(\tau) = -1 \), this forces \( \frac{1}{(n+2)!} \sum_{g \in S_{n+2}} \text{sgn}(g) |o^g| \), which computes the number of copies of \( \chi \) in \( \bar{U}_o \), to be zero. □

For \( n = 1 \) or \( 2 \) this result is obvious (since \( 6 > 3 \) and \( 24 > 16 \)), but for \( n = 3, 4, \) or \( 5 \) it is less so (as \( 120 < 135, 720 < 1506, \) and \( 5040 < 20993 \)). In particular, since the examples of quintic 3-folds with 125 and 130 nodes are \( S_5 \)-symmetric, and the latter has a 60-node orbit, it is interesting that a 120-node orbit is impossible.

### D. Cyclic covers of \( \mathbb{P}^1 \)

In the final two sections we turn to “codimension-one” monodromy phenomena for period maps arising from cyclic covers. We begin with a story that generalizes elliptic curves and goes back to Deligne and Mostow [DM]. Given distinct points \( t_1, \ldots, t_{2m} \in \mathbb{P}^1 \) (with projective coordinates \( [S_i : T_i] \)), define

\[ C_{\omega} := \{ [Z_0 : Z_1 : Z_2] \in \mathbb{P}[1:1:2] \mid Z_2^m = \prod_{i=1}^{2m} (S_i Z_1 - T_i Z_0) \}, \]

This is nothing but Cor. A.9 with \( G = \langle g \rangle \times S_{n+2} \).
with automorphism $g([Z_0:Z_1:Z_2]) := [Z_0:Z_1:z_mZ_2]$. For $m = 2, 3, 4,$ or $6,$ the sum of $g^*$-eigenspaces $H^1(C)\mathbf{C}^m \oplus H^1(C)\mathbf{C}^m$ produces a \$\Xi\$-VHS over $M_{0,2m}$, hence a period map to a ball quotient $\Gamma\backslash \mathbb{B}_{2m-3}$. This turns out to be injective\footnote{Recall that $M_{0,n}$ parametrizes ordered $n$-tuples of distinct points on $\mathbb{P}^1$ modulo the action of $\text{PSL}_2(\mathbb{C})$.} and extend to an isomorphism between GIT resp. Hassett/KSBA compactifications of $M_{0,2m}$ and Baily-Borel resp. toroidal compactifications of the ball quotient \cite{DM, GKS}. 

Here we will not be concerned with the ball quotient or even the period map \textit{per se}, but only with

- the \$\Xi\$-VHS $\mathcal{V}$ over $M_{0,2m}$ arising from $H^1(C)$,
- its sub-$\mathcal{C}$-VHSs $\mathcal{V}^\mathcal{C}_m := \ker(g^* - \zeta^j_m \mathbb{I})$ ($1 \leq j \leq m-1$), and
- their limiting behavior along the boundary of the Hassett compactifications $\overline{\mathcal{M}}_{0,1}$.\cite{Ha}

The point is that \textit{these can be considered for all $m \geq 2$, not just $m = 2, 3, 4,$ and 6.}

To begin with, in affine coordinates $x = \frac{Z_0}{Z_2},$ $y = \frac{Z_1}{Z_2},$ $C$ takes the form $y^m = f(x) := \prod_{i=1}^{2m} (x-t_i)$ [resp. $\prod_{j \neq i} (x-t_i)$, if $t_j = \infty$]. While there are three possibilities for the Newton polytope $\Delta$, they all have the same interior integer points

$$(\Delta \setminus \partial \Delta) \cap \mathbb{Z}^2 = \{(i,j) \mid 1 \leq j \leq m-1, \ 1 \leq i \leq 2(m-j) - 1\},$$

which provide a basis of $\Omega^1(C)$ via

$$\omega_{(i,j)} := \text{Res}_{C} \left( \frac{x^{i-1} y^{j-1} dx \wedge dy}{y^m - f(x)} \right).$$

Since $g^* \omega_{(i,j)} = \zeta^j_m \omega_{(i,j)},$ we find that

\begin{equation}
\text{rk}(\mathcal{V}^\mathcal{C}_m)^{1,0} = 2(m-j) - 1, \quad \text{rk}(\mathcal{V}^\mathcal{C}_m)^{0,1} = 2j - 1
\end{equation}

\begin{equation}
\text{rk}\mathcal{V}^\mathcal{C}_m = 2m - 2, \quad \text{and} \quad \text{rk}\mathcal{V} = (m-1)^2.
\end{equation}

Next, we need the following:

D.2. Definition (\cite{Ha}). A \textit{weighted stable rational curve} for the weight $\mu := (\mu_1, \ldots, \mu_n) \in \{(0,1) \cap \mathbb{Q}\}^n$ is a pair $\mathcal{C} \ (\sum \mu_i p_i)$ with:

- $\mathcal{C}$ a nodal connected projective curve of arithmetic genus $0$;
- each $p_i$ a smooth point of $\mathcal{C}$;
- if $p_{i_1} = \cdots = p_{i_r}$, then $\mu_{i_1} + \cdots + \mu_{i_r} \leq 1$; and
- the $\mathbb{Q}$-divisor $K_C + \sum \mu_i p_i$ is ample (i.e. on each irreducible component, the sum of weights plus number of nodes is $> 2$).

\footnote{\textit{For $m = 6$ one has to quotient $M_{0,12}$ by $\mathfrak{S}_{12}$; see \cite{GKS}.}}

\footnote{\textit{Despite the sum notation, the order of points with equal weights is retained.}}
We will write \((\mu, \ldots, \mu) =: [\mu]_n\) for repeated weights.

D.3. **Theorem** ([Ha]). (i) There exists a smooth projective fine moduli space \(\overline{M}_{0,\mu}\) parametrizing \(\mu\)-weighted stable rational curves, and containing \(\overline{M}_{0,n}\) as a Zariski-open subset.

(ii) Given weights \(\mu = (\mu_1, \ldots, \mu_n)\) and \(\tilde{\mu} = (\tilde{\mu}_1, \ldots, \tilde{\mu}_n)\) with \(\mu_i \leq \tilde{\mu}_i\) \((\forall i)\), there exists a birational reduction morphism \(\pi_{\tilde{\mu},\mu}: \overline{M}_{0,\tilde{\mu}} \to \overline{M}_{0,\mu}\) contracting all components which violate the ampleness property in (D.2) for the weight \(\tilde{\mu}\).

D.4. **Remark.**

(a) \(\overline{M}_{0,[1]}_n\) reproduces the Deligne-Mumford-Knudsen compactification \(\overline{M}_{0,n}\).

(b) Although the ampleness property forces \(\sum \mu_i > 2\), if for \(|\mu| = 2\) we define \(\overline{M}_{0,\mu}\) to be the GIT quotient \((\mathbb{P}^1)^n \sslash_{\mu} \text{SL}_2\), then (D.3)(ii) extends to this case; and if we take \(\tilde{\mu}_i = \mu_i + \epsilon\) \((\epsilon \in \mathbb{Q}, 0 < \epsilon \ll 1)\) then \(\pi_{\tilde{\mu},\mu}\) is Kirwan’s partial desingularization.

Our interest henceforth is in the equal-weight Hassett compactification \(\overline{M}_{0,2m}^H := \overline{M}_{0,[1_m]_{2m}}\) and its morphism \(\pi\) to \(\overline{M}_{0,2m}^{\text{GIT}} := \overline{M}_{0,[1_m]_{2m}}\).

As the reader may easily check, the irreducible components of \(\overline{M}_{0,2m}^H \setminus \overline{M}_{0,2m}\) are of two types, parametrizing \(^1\) stable weighted curves as shown (up to reordering of the \(\{p_i\}\)):

![Diagram](image)

It is also clear that \(\pi\) preserves the type (A) strata whilst contracting the type (B) ones to a (strictly semistable) point parametrizing the object

\[ p_1 = \ldots = p_m \quad p_{m+1} = \ldots = p_{2m} \]

\(^1\)More precisely, it is a dense open subset of each component that parametrizes the displayed objects.
The $\mathbb{C}$-VHSs $\mathcal{V}^{e,j}_m$ admit canonical extensions across the smooth part of $\overline{M}_{0,2m} \setminus M_{0,2m}$, and we shall now compute the LMHS types there.

**D.5. Proposition.** Along type (A) strata:

- $\mathcal{V}^{e,j}_m$ is pure of weight 1, with $h^{1,0} = 2m - 2j - 1$ and $h^{0,1} = 2j - 1$, unless $j = \frac{m}{2}$;
- if $j = \frac{m}{2}$, then $h^{1,1} = h^{0,0} = 1$, $h^{1,0} = h^{0,1} = m - 1$, and $T = e^N$ (with $N$ an isomorphism from the $(1,1)$ to $(0,0)$ part); and
- if $j > \frac{m}{2}$ [resp. $< \frac{m}{2}$], then we have the decomposition $\mathcal{V}^{e,j}_m = \mathcal{V}^{e,j}_{m,1} \oplus \mathcal{V}^{e,j}_{m,2j}$ into $T = T_{ss}$-eigenspaces, where $\mathcal{V}^{e,j}_{m,2j}$ is 1-dimensional of type $(0,1)$ [resp. $(1,0)$].

**Proof.** Begin by locally modeling (the effect on $C\ell$ of) the collision of two points by $y^m + z^2 = s$, as $s \to 0$. This has eigenspectrum

$$\sum_{j=1}^{m-1} \left( \frac{3}{2} - \frac{j}{m}, w(j), \frac{j}{m} \right),$$

where $w(j) = 2$ if $j = \frac{m}{2}$ and 1 otherwise. Next, we apply the vanishing-cycle sequence (with $H^2_{ph} = \{0\}$ since the degenerate curve remains irreducible) to compute the LMHS. Finally, we perform a base-change by $s \mapsto s^2$ to preserve ordering of points, which squares the eigenvalues of the $T_{ss}$-action; in other words, we replace $\frac{3}{2} - \frac{j}{m}$ by $\{2(\frac{3}{2} - \frac{j}{m})\} + [\frac{3}{2} - \frac{j}{m}]$ ($\{\cdot\}$ denoting the fractional part), which gives the result. \(\square\)

**D.6. Proposition.** Along the type (B) strata, for each $1 \leq j \leq m - 1$, $\mathcal{V}^{e,j}_m$ has Hodge numbers $h^{1,1} = h^{0,0} = 1$, $h^{1,0} = 2m - 2j - 2$, and $h^{0,1} = 2j - 2$; $N$ is an isomorphism from the $(1,1)$ to $(0,0)$ part, and $T = e^N$ is unipotent.

**Proof.** In the GIT compactification for unordered points, the degeneration is locally modeled by two copies of $y^m + x^m = s$, each with eigenspectrum

$$\sum_{j=1}^{m-1} (1, 2, \frac{j}{m}) + \sum_{j=2}^{m-1} \sum_{k=1}^{j-1} (\frac{k+m-j}{m}, 1, \frac{j}{m}) + \sum_{j=1}^{m-2} \sum_{k=j+1}^{m-1} (\frac{k+m-j}{m}, 1, \frac{j}{m}).$$

At this point one applies the vanishing-cycle sequence to deduce the form of the LMHS, noting that the degenerate curve is a union of $m$ $\mathbb{P}^1$s and $H^2_{ph} \cong \mathbb{Q}(-1)^{\oplus m-1}$. For $\overline{M}^H_{0,2m}$, one then applies the base-change by $s \mapsto s^m$, which trivializes $T_{ss}$, allowing the extension class to vary along the type (B) stratum. \(\square\)

**D.7. Example.** Combining \([D.1]\) with the two Propositions, $\mathcal{V}^{e,m}$ has Hodge-Deligne diagrams
\[ T_{ss} = \zeta_m^2 \subset 1 \] 

\[ \text{type (A)} \quad \lim \quad \text{type (B)} \]

For \( m = 4 \) resp. 6, the monodromy in type (A) is thus given by a complex reflection resp. “triflection”.

D.8. Remark. For any \( m \), \( V_{\overline{\zeta}_m} (\oplus V_{\zeta_m}) \) induces a map from the universal cover \( \tilde{M}_{0,2m} \) to a ball \( B_{2m-3} \). Moreover, both LMHS types have \( 2m-4 \) complex moduli. However, for \( m \) different from 2, 3, 4, or 6, this does not lead to a tidy extended period map: as the projection of the monodromy to \( U(1, 2m-3) \) is not discrete [Mo], the quotient of \( B_{2m-3} \) by this is not Hausdorff. To circumvent this problem, we must replace \( B_{2m-3} \) by its product with other (non-ball) symmetric domains, which receives the image of the period map for the \( \mathbb{Q} \)-VHS \( \oplus_{(j,m)=1} V_{\zeta_j^m} \).

E. Hyperplane configurations and Dolgachev’s conjecture

Let \( L_0, \ldots, L_{2n+1} \subset \mathbb{P}^n \) be hyperplanes defined by linear forms \( \ell_i \), in general position in the sense that \( \cup L_i \) is a normal crossing divisor. Consider the 2:1 cover \( X \rightarrow \mathbb{P}^n \) branched along \( \cup L_i \), and the rank-1 \( \mathbb{Q} \)-local system \( L \) on \( U = \mathbb{P}^n \setminus (\cup L_i) \xrightarrow{j} \mathbb{P}^n \) with monodromy \( -1 \) about each \( L_i \). Since \( X \) has finite quotient singularities, we have \( \text{IC}_X^* = \mathbb{Q}_X[n] \) and

\[
H := H^n_{\text{pr}}(X) := \frac{H^n(X)}{\pi^* H^n(\mathbb{P}^n)} \cong H^n(\mathbb{P}^n, j_* L) \cong \text{IH}^n(\mathbb{P}^n, L)
\]

is a pure HS of weight \( n \). By [DK] Lemma 8.2, it has Hodge numbers

\[
h^{p,n-p}_{\text{pr}}(X) = \binom{n}{p}^2 \implies h^{n}_{\text{pr}}(X) = \binom{2n}{n}.
\]

It is polarized by the intersection form \( Q \), which presents no difficulties as \( X \) has a smooth finite cover.

Taking \( \mathcal{S} \subset (\mathbb{P}^n)^{2n+2} / \text{PGL}_{n+1}(\mathbb{C}) =: \overline{\mathcal{S}} \) to be the \((n^2\text{-dimensional})\) moduli space of \( 2n + 2 \) ordered hyperplanes in \( \mathbb{P}^n \) in general position, this construction yields a \( \mathbb{Z} \)-PVHS \( \mathcal{H} \rightarrow \mathcal{S} \) of CY-\( n \) type with \( H \) as reference fiber. Let \( \rho: \pi_1(\mathcal{S}) \rightarrow \text{Aut}(H, Q)^\circ =: \text{M}_{\text{max}} \) be the monodromy.

---

\[12\text{See [HTT] Prop. 8.2.30] for the statement that } \text{IC}_{\mathbb{P}^n}^* L = j_* L[n].\]
representation of $\mathcal{H}$ its geometric monodromy group, and $M$ its Hodge (special Mumford-Tate) group. Here $\Pi$ is the identity connected component of $\tilde{\Pi} := \overline{\rho(\pi_1(S))}^{\text{Q-Zar}}$, and $\Pi \leq M \leq M_{\text{max}}$. A conjecture attributed by [SXZ] to Dolgachev states that the period map for $\mathcal{H}$ factors through a locally symmetric variety (also $n^2$-dimensional) of type $I_{n,n}$ which would imply that $m_\mathbb{R} \cong \mathfrak{su}(n,n)$. This is equivalent to saying that,

\[
\text{(E.3) \quad \mathcal{H} \text{ is the } n^{th} \text{ wedge power of a VHS of weight 1 and rank } 2n.}
\]

The conjecture does hold for $n = 1$ and $n = 2$, but this merely reflects exceptional isomorphisms of Lie groups in low rank, namely $\text{SU}(1,1) \cong \text{SL}_2(\mathbb{R})$ and $\text{SU}(2,2) \cong \text{Spin}(2,4)^+$. That is, in both of these cases we also have $\Pi \cong M_{\text{max}}$ (= SL$_2$ resp. SO(2,4)). For $n \geq 3$, in contrast, the conjecture would have $\Pi < M_{\text{max}}$ a proper algebraic subgroup. In [op. cit.] (and earlier works [GSVZ, GSZ, GSZ2]), it was shown via quite computationally involved differential methods that in fact the monodromy is maximal for all $n$, and the conjecture fails for $n \geq 3$:

E.4. Theorem. $\Pi = M = M_{\text{max}} \forall n \geq 1$.

In the remainder of this section, we explain how asymptotic methods provide a much simpler approach to these results. First we will give a careful argument disproving the conjecture for $n \geq 3$ odd, which \textit{a priori} is a weaker statement than the Theorem in that case. (The relation to the main theme of his paper — specifically, to the setting of Cor. A.9 — enters when we pass to the smooth finite cover $\tilde{X}$ of $X$.) Then we sketch a proof of Theorem E.4 using a more topological and monodromy-theoretic approach.

Disproof of (E.3) for $n$ odd. Most of the analysis that follows works for all $n$, though the last step is inconclusive for even $n$.

To begin, consider a pencil $\mathbb{P}^1 \xrightarrow{\varepsilon} \mathfrak{S}$ of hyperplane configurations given by fixing $L_0, \ldots, L_{2n}$ (in general position) and letting $L_{2n+1} := L_s$ vary along a line in $\mathbb{P}^{2n}$ (chosen to avoid linear spans of any $n-2$ $L_i$ in $\mathbb{P}^{2n}$).\footnote{Here $(\cdot)^\circ$ means the identity component as algebraic group (i.e. SO($H$) instead of O($H$) if $n$ is even).} Writing $\Sigma = \varepsilon^{-1}(\mathfrak{S}\setminus\mathcal{S})$, we have $|\Sigma| = \binom{2n+1}{n}$; and degenerations.

\footnote{Note that the “tautological VHS” over $I_{n,n}$ is already geometrically realized by the $n^{th}$ primitive cohomology of a universal family of Weil abelian $2n$-folds.}

\footnote{It already follows from Zariski’s theorem [Vo, Thm. 3.22] that $\rho(\pi_1(\mathbb{P}^1 \setminus \Sigma)) = \rho(\pi_1(S))$, but we won’t need this.}
\( \mathcal{X}_\sigma \rightarrow \Delta_\sigma \) of our double-covers at \( \sigma \in \Sigma \) are locally modeled (with \( t = s - \sigma \)) by

\[
(E.5) \quad w^2 = x_1 \cdots x_n(t - x_1 - \cdots - x_n)
\]

after a \( \text{PGL}_{n+1}(\mathbb{C}) \)-action. Accordingly, writing \( X_0, \ldots, X_n \) for projective coordinates on \( \mathbb{P}^n \), we take \( \ell_i = X_i \) for \( 0 \leq i \leq n \) and \( \ell_{2n+1} = tX_0 - \sum_{i=1}^n X_i \), and \( \ell_{n+1}, \ldots, \ell_{2n} \) “general”.

Let \( \mathbb{P}^n \hookrightarrow \mathbb{P}^{2n+1} \) denote the linear embedding \( [X_0: \cdots : X_n] \mapsto [\ell_0(X): \cdots : \ell_{2n+1}(X)] \), and \( \phi : \mathbb{P}^{2n+1} \rightarrow \mathbb{P}^{2n+1} \) denote the map sending \( [Z_0: \cdots : Z_{2n+1}] \mapsto [Z_0^2: \cdots : Z_{2n+1}^2] \). Then \( \hat{X} := \phi^{-1}(\mathbb{P}^n) \subset \mathbb{P}^{2n+1} \) is a smooth complete intersection on which \( [E.10] \ A := (\mathbb{Z}/2\mathbb{Z})^{2n+2}/\Delta(\mathbb{Z}/2\mathbb{Z}) \) acts via \( \xi \mapsto \{Z_i \mapsto -Z_i \} \), with quotient \( \mathbb{P}^n \); explicitly, we have

\[
(E.6) \quad \hat{X} = \cap_{k=0}^n \{0 = F_k(Z) := -Z_{n+k+1}^2 + \ell_{n+k+1}(Z_0^2, \ldots, Z_n^2) \}.
\]

Write \( \chi \in X^*(A) \) for the character sending each \( \xi \mapsto -1 \), \( A^c := \ker(\chi) \leq A \), and \( \eta : \hat{X} \rightarrow X \) for the quotient by \( A^c \); then \( H \cong q^*H^*_p(X) \cong H^n(\hat{X})^\chi \). Since \( F_0(Z) = tZ_0^2 - \sum_{i=1}^{n+1} Z_i^2 \), we have thus replaced our original non-isolated degeneration \( (E.5) \) by a nodal one.

Next, we use the “Cayley trick” to replace the complete intersection \( \hat{X} \) by a hypersurface

\[
(E.7) \quad Y := \{0 = F := \sum_{k=0}^n Y_k F_k(Z) \} \subset \mathbb{P}((\mathbb{P}^{2n+1}(2)^{\oplus n+1}) =: P
\]

of dimension 3n. We have an \( A \)-equivariant isomorphism \( H^{3n}(Y)(n) \cong H^n(\hat{X}) \) of HSs, so that \( H \cong H^{3n}(Y)^\chi(n) \). Notice that in affine coordinates \( (z_1, \ldots, z_{2n+1}; y_1, \ldots, y_n) \), \( F = 0 \) becomes\(^{16}\)

\[
(E.8) \quad 0 = -t + z_1^2 - \cdots - z_{2n+1}^2 + \sum_{k=1}^n y_k (b_k - z_{n+k+1})(b_k + z_{n+k+1}) + \text{h.o.t.}
\]

where \( b_k := \sqrt{F_k(1,0,\ldots,0)} \). So at \( t = 0 \), the singular fiber \( Y_\sigma \) has 2\( n \) nodes at \( (Z_0; Z_1; \ldots; Z_{n+1}; Z_{n+2}; \cdots, Z_{2n+1}; Y_0; Y_1; \ldots, Y_n) = (1; 0, \ldots, 0; (-1)^n b_1, \ldots, (-1)^n b_n; 1; 0, \ldots, 0) \), \( \mathfrak{a} \in (\mathbb{Z}/2\mathbb{Z})^n \), and the degeneration \( \mathcal{Y}_\sigma \rightarrow \Delta_\sigma \) has smooth total space. The mixed spectrum of each node is \( \left[ (\frac{3n+1}{2}, 3n + 1) \right] \) for \( n \) odd and \( \left[ (\frac{3n+1}{2}, 3n) \right] \) for \( n \) even; so \( T_\sigma \) acts through multiplication by \( (-1)^{n+1} \) on

\[
(E.9) \quad H_{\text{van}}^{3n}(Y_\sigma) \cong \mathbb{Q}((-\frac{3n+1}{2})^{\oplus 2n}).
\]

Moreover, since the summands of \( (E.10) \) are represented by \( \eta_\mathfrak{a} = (-1)^{\mathfrak{a}}(dz_1 \wedge \cdots \wedge dz_{2n+1} \wedge dy_1 \wedge \cdots \wedge dy_n)/F^{\left[\frac{3n+1}{2}\right]} \) near the nodes \( (E.9) \)

\(^{16}\)Here \( \Delta \) denotes the diagonal embedding.

\(^{17}\)Here “h.o.t.” means terms vanishing to order 3 at the nodes.
(in the sense of [KL2, §2]), it has a 1-dimensional subspace (generated by $\eta := \sum (-1)^{|a|} \eta_a$) on which $A$ acts through $\chi$.

Taking $\chi$-eigenspaces of the vanishing-cycle sequence for $Y_\sigma \to \Delta_\sigma$ and twisting by $Q(n)$ now yields

(E.11) 

$$0 \to H^3(Y_\sigma)^\chi(n) \xrightarrow{sp^X} H^3_{\text{lim}}(Y_t)^\chi(n) \xrightarrow{\text{can}^X} \mathbb{Q}(-\lfloor \frac{a+1}{2} \rfloor) \xrightarrow{\delta^X} H^{3n+1}_{\text{ph}}(Y_\sigma)^\chi(n) \to 0$$

We claim that $\delta = 0$. For $n$ even, this is clear, since $T_\sigma$ acts trivially on $H^{3n+1}_{\text{ph}}(Y_\sigma)$ and by $-1$ on $\mathbb{Q}(-\lfloor \frac{a+1}{2} \rfloor)$. So we conclude that $T_\sigma$ acts on $H_{\text{lim}}$ via an orthogonal reflection. This doesn’t factor through $\wedge^n$ of any automorphism of $\mathbb{C}^{2n}$, but because it is finite (of order 2), this does not (yet) disprove the conjecture.

On the other hand, for $n$ odd, it is not automatic that $\delta = 0$. (This is a well-known problem with nodal degenerations in odd dimensions, cf. [KL2, §2.2]; and as we saw in the proof of Lemma E.5, our degenerations are finite quotients of nodal ones.) But if we can show $\delta = 0$, then the conjecture is immediately disproved (for odd $n \geq 3$). Here is why: by (E.6), $H_{\text{lim}}$ then has a class of type $(n+1,n+1)$, which must go to an $(n,n)$ class by $N_\sigma$, forcing $\text{rk}(N_\sigma) = 1$ (rather than 0). (In different terms, each $T_\sigma$ is a nontrivial symplectic transvection.) But this is impossible for $\wedge^n$ of a nilpotent endomorphism of $\mathbb{C}^{2n}$.

To complete the (dis)proof, then, we apply [KL2, Thm. 2.9]: for a nodal degeneration $Y \rightsquigarrow Y_\sigma$ of an odd-dimensional hypersurface of a smooth projective variety $P$ satisfying Bott vanishing, the rank of $\delta$ is the number $m$ of nodes minus the rank of the map

$$\text{ev}: H^0(P, K_P(\frac{3n+1}{2}Y_\sigma)) \to \mathbb{C}^m$$

given by evaluation at the nodes. The proof in [loc. cit.] is equivariant in $A$, and so we find that $\delta^X = 0 \iff \text{ev}$ is nonzero on $H^0(P, K_P(\frac{3n+1}{2}Y_\sigma))^\chi$, which can be checked at any node. Writing $\vec{e}_1 := \sum_{i=0}^n Y_i \frac{\partial}{\partial Y_i}$, $\vec{e}_2 := \sum_{j=0}^{2n+1} Z_j \frac{\partial}{\partial Z_j} - 2\vec{e}_1$, and $\Omega := \langle \vec{e}_2, \langle \vec{e}_1, dZ \wedge dY \rangle \rangle$, ...
one checks that
\[(E.12) \quad Y_0Z^2_0\Omega/(F_{t=0})^{3n+1}
\]
is a well-defined section of \(K_F(3n+1)Y_\sigma\), (cf. [Kh §4.5]); and evidently \(A\) acts on it through \(\chi\).\(^\text{18}\) Clearly, it is nonzero on the fiber of \(K_F(3n+1)Y_\sigma\) at any of the nodes \((E.9)\).

**Sketch of proof of Theorem E.4.** Returning to the local picture \((E.5)\), we now seek a more concrete topological description of the orthogonal reflections \((n\ \text{even})\) and symplectic transvections \((n\ \text{odd})\) through which \(T_\sigma\) acts on \(H\). So let \(U_0 \subset \mathbb{A}^n\) be the complement of the hyperplanes \(x_1 = 0, \ldots, x_n = 0\) and \(x_1 + \cdots + x_n = 1\), and \(\mathbb{L}_0\) the rank-1 local system on \(U_0\) with monodromies \(-1\) about each of them. While the singularity \(x_\sigma \mapsto X_\sigma\) “at \(0\)” in \((E.5)\) isn’t isolated, the vanishing-cycle complex \(\phi_!\mathbb{Q}_X\) is nothing but \(\zeta V[-n]\), where \(V := IH^n(\mathbb{A}^n, \mathbb{L}_0)\) (as MHS). We begin with a local analogue of the covering argument just seen.

**E.13. Lemma.** (i) \(IH^n(\mathbb{A}^n, \mathbb{L}_0) \cong \mathbb{Q}(-\lfloor \frac{n+1}{2} \rfloor)\).

(ii) Local monodromy \(T_\sigma\) acts on \(V\) through multiplication by \((-1)^{n+1}\).

(iii) The canonical map \(\pi_0 : H_\lim \to V\) is onto.

**Proof.** Define maps
- \(f_0 : \mathbb{A}^n \hookrightarrow \mathbb{A}^{n+1}\) by \(x \mapsto (x, 1 - \sum_{i=1}^n x_i)\), and
- \(\phi_0 : \mathbb{A}^{n+1} \to \mathbb{A}^{n+1}\) by squaring all coordinates \(z_i\);

then \(\hat{X}_0 := \phi_0^{-1}(f_0(\mathbb{A}^n)) \subset \mathbb{A}^{n+1}\) is the quadric hypersurface \(\sum_{i=1}^{n+1} z_i^2 = 1\). The group \(A_0 := (\mathbb{Z}/2\mathbb{Z})^{n+1}/\Delta(\mathbb{Z}/2\mathbb{Z})\) acts on \(\hat{X}_0\) (multiplying coordinates by \(\pm 1\)), with quotient \(\mathbb{A}^n\). The quotient \(q_0 : \hat{X}_0 \to X_0\) by the augmentation subgroup \(A_0^0\) yields the obvious 2:1 branched cover of \(\mathbb{A}^n\), with \(H^n(X_0) \cong IH^n(\mathbb{A}^n, \mathbb{L}_0)\).

By the localization sequence for \(\hat{X}_0\) (relative to its closure \(\overline{\hat{X}_0} \subset \mathbb{P}^{n+1}\)) and weak Lefschetz, one easily shows that \(H^j(\hat{X}_0) = 0\) for \(j \neq n\)\(^\text{18}\) and \(H^n(\hat{X}_0) \cong \mathbb{Q}(-\lfloor \frac{n+1}{2} \rfloor)\). (Writing \(\partial \hat{X}_0 = \hat{X}_0 \setminus \hat{X}_0\), this is \(H^n(\hat{X}_0)/H^{n-2}(\partial \hat{X}_0)(-1)\) for \(n\) even, and \(\ker(H^{n-1}(\partial \hat{X}_0)(-1) \to H^{n+1}(\hat{X}_0))\) for \(n\) odd.) A generator for the dual group \(H^n_c(\hat{X}_0)\) is given by the real (vanishing) \(n\)-sphere \(S^n_1 := \{\sum z_i^2 = 1\} \cap \mathbb{R}^{n+1}\), whose class is invariant under \(A_0^0\) hence comes from \(H^n_c(X_0)\). This gives (i).

The degeneration is modeled by replacing \(\sum z_i^2 = 1\) by \(\sum z_i^2 = t\); as the spectrum of \(\sum z_i^2\) is \([\frac{n+1}{2}]\), the monodromy is as described in (ii).
Finally, (iii) follows from the last subsection since can\(\sigma\) identifies with can\(x\) in (E.11).

The vanishing sphere \(S^n\) := \(\{\sum z_i^2 = t\}\) \(\cap \mathbb{R}^{n+1}\) in \(\hat{X}_0\) has image in \(X_0\) (by \(q_0\)) given by the double cover of \((\cap_{i=1}^n\{x_i \geq 0\}) \cap \{\sum x_i \leq t\}\). Let its image in \(X\) (essentially via can\(x\)) : \(H^n_\chi(X_0) \to H^n(X)\) be denoted by \(\nu_\sigma\); this is the vanishing cycle at \(\sigma\), a “double simplex” branched along \(\mathcal{H}_s\) and \(n\) additional hyperplanes. It follows from (iii) that \(T_\sigma\) is a transvection/reflection in \(\nu_\sigma\). More precisely, rescaling \(Q\) to have

\[
Q(\nu_\sigma, \nu_\sigma) = \frac{1+(-1)^n}{2},
\]

(E.14)

\[
T_\sigma(u) = u - 2Q(u, \nu_\sigma)\nu_\sigma
\]

for \(u \in H\).

Now consider the general setting where \(L_{2n+1} = L_s, L_0 = \{X_0 = 0\}\), and the remaining \(L_i\) are in general position. An easy extension of (E.11) gives \(H \cong \Pi^n_0(\mathbb{A}^n, \mathbb{L}) \cong H^n_\pi(X \setminus L_0)\), whence \(H^n_\pi(X)\) is spanned by double simplices branched along \(n + 1\) of the \(L_i\geq 0\). Obviously all of these can be rewritten as \(\mathbb{Z}\)-linear combinations of double simplices branched along \(L_s\) and \(n\) of the \(\{L_i\}_{1 \leq i \leq 2n}\); call these \(\nu_I\), where \(I \subset \{1, \ldots, 2n\}\) with \(|I| = n\). Since \(\text{rk}H = \binom{2n}{n}\) and there are \(\binom{2n}{n}\) of these vanishing cycles, they form a \(\mathbb{Q}\)-basis of \(H = H^n_\pi(X)\). Write \(T_I\) for the corresponding monodromies, and \(\Gamma \leq \text{Aut}(H, Q)\) for the smallest \(\mathbb{C}\)-algebraic group containing them; clearly \(\Gamma \leq \tilde{\Pi}_\mathbb{C}\). Moreover, we note that if \(|I \cap I'| = n - 1\), then \(Q(\nu_I, \nu_{I'}) = \pm 1\) (rescaling as above, compatibly with (E.14)).

Suppose then that \(|I \cap I'| = n - 1\). If \(n\) is odd, then \(T_I(\nu_{I'}) = \nu_I \pm \nu_{I'} = \pm T_{I'}^{-1}(\nu_I)\), whence \(\nu_{I'}\) is in the \(\Gamma\)-orbit of \(\nu_I\); so all the \(\nu_I\) are in the \(\Gamma\)-orbit of \(\nu_I\). If \(n\) is even, then reasoning as in [De2, §4.4] [19] \(T_I T_{I'}^{-1}\) is a transvection and its Zariski closure a \(G_a\) including transformations which send \(\nu_I \mapsto \nu_{I'}\) and vice-versa; once again, all the \(\nu_I\) are in the \(\Gamma\)-orbit of a single \(\nu_I\).

Let \(R := \Gamma.\nu_I\) denote this orbit. Obviously it spans \(H_\mathbb{C}\). Furthermore, for any \(\delta \in R, \Gamma\) contains the transvection/reflection \(T_\delta\): writing \(\delta = \gamma.\nu_I\ (\gamma \in \Gamma)\), we have \(T_\delta = T_{\gamma.\nu_I} = \gamma T_I \gamma^{-1} \in \Gamma\). So \(\Gamma\) is in fact the \(\mathbb{C}\)-algebraic closure of the \(\{T_\delta\}_{\delta \in R}\), and we are exactly in the situation of [De2, Lemme 4.4.2]. Conclude that \(\Gamma = \text{Aut}(H, Q)\), hence \(\tilde{\Pi} = \text{Aut}(H, Q)\), and thus \(\Pi = \text{Aut}(H, Q)^o\).

E.15. Remark. After writing this paper we encountered the article by [Xu] which treats the more general setting of \(r\)-covers of \(\mathbb{P}^n\) branched along hyperplanes by considering local monodromies (as we have just

[19]See the paragraph after Lemme 4.4.3°.
done). The argument is necessarily more complicated and technical than ours. However, in the case \( r = 2 \) (i.e., our setting) it appears to be incomplete.

If \( r = 2 \) and \( n \) is odd, [Xu, Prop. 3.4] does not actually establish that \( \epsilon(1) \) (in the notation of [loc. cit.]) is nonzero; this is exactly the issue regarding possible nonvanishing of \( \delta \) dealt with above. One could read [Xu, Prop. 4.2] as confirming this in retrospect, but this makes the argument quite convoluted.

If \( r = 2 \) and \( n \) is even, the proof of [Xu, Prop. 4.2] is wrong, as it makes use of the (false) statement that \( \text{Sp}_{2n}(\mathbb{R}) \) “does not admit any nontrivial one-dimensional invariant subspace” in its action on \( \Lambda^n \mathbb{R}^{2n} \).

References

[Ar] V. Arnol’d, Some problems in singularity theory, Proc. Indian Acad. of Science 90 (1981), No. 1, 1-9.

[Be] A. Beauville, Sur le nombre maximum de points doubles d’une surface dans \( \mathbb{P}^3 \) (\( \mu(5) = 31 \)), Journées de Géometrie Algébrique d’Angers, Sijthoff & Noordhoff, 1980, pp. 207-215.

[Ca] H. Cartan, Quotient d’une variété analytique par un groupe discret d’automorphismes, Séminaire Henri Cartan 6 (1953-4), Exposé No. 12, 1-13.

[Cs] B. Castor, Bounding projective hypersurface singularities, Washington University Ph.D. thesis, 2021.

[De1] P. Deligne, Théorie de Hodge II, Publ. IHES 40 (1971), 5-57.

[De2] ———, La conjecture de Weil II, Publ. IHES 52 (1980), 137-252.

[DM] P. Deligne and G. Mostow, Monodromy of hypergeometric functions and nonlattice integral monodromy, Publ. IHES 63 (1986), 5-89.

[Do] I. Dolgachev, Weighted projective varieties, in “Group actions and fields (Vancouver, B.C., 1981)”, 34-71, Lecture Notes in Math. 956, Springer, Berlin, 1982.

[DK] I. Dolgachev and S. Kondo, Moduli of K3 surfaces and complex ball quotients, in “Arithmetic and geometry around hypergeometric functions”, 43-100, Progress in Math. 260, Birkhäuser, Basel, 2007.

[En] S. Endraß, A projective surface of degree eight with 168 nodes, J. Alg. Geom. 6 (1997), no. 2, 325-334.

[GKS] P. Gallardo, M. Kerr, and L. Schaffler, Geometric interpretation of toroidal compactifications of moduli of points in the line and cubic surfaces, Adv. Math. 381 (2021), Paper No. 107632, 48 pp.

[GSZ] R. Gerkmann, M. Sheng, and K. Zuo, Disproof of modularity of the moduli space of CY 3-folds of double covers of \( \mathbb{P}^3 \) ramified along 8 planes in general position, arXiv:0709.1051

[GSZ2] ———, Computational details of the disproof of modularity, arXiv:0709.1054

[GSVZ] R. Gerkmann, M. Sheng, D. van Straten, and K. Zuo, On the monodromy of the moduli space of Calabi-Yau threefolds coming from eight planes in \( \mathbb{P}^3 \), Math. Ann. 348 (2010), no. 1, 211-236.
REMARKS ON EIGENSPECTRA OF ISOLATED SINGULARITIES

[Ha] B. Hassett, *Moduli spaces of weighted pointed stable curves*, Adv. Math. 173 (2003), no. 2, 316-352.

[HTT] R. Hotta, K. Takeuchi and T. Tanizaki, “D-modules, perverse sheaves, and representation theory” (trans. Takeuchi), Progress in Math. 236, Birkhäuser Boston, Inc., Boston, 2008.

[JR] D. Jaffe and D. Ruberman, *A sextic surface cannot have 66 nodes*, J. Alg. Geom. 6 (1997), no. 1, 151-168.

[Ke] M. Kerr, “Geometric construction of regulator currents with applications to algebraic cycles”, Princeton University Ph.D. thesis, 2003.

[KL1] M. Kerr and R. Laza, *Hodge theory of degenerations, (I): consequences of the decomposition theorem*, with an appendix by M. Saito, Selecta Math. 27 (2021), no. 4, Paper No. 71, 48 pp.

[KL2] M. Kerr and R. Laza, *Hodge theory of degenerations, (II): vanishing cohomology and geometric applications*, preprint, 2020, arXiv:2006.03953.

[Mi] Y. Miyaoka, *The maximal number of quotient singularities on surfaces with given numerical invariants*, Math. Ann. 268 (1984), 159-171.

[Mo] G. Mostow, *On discontinuous action of monodromy group on the complex n-ball*, J. Amer. Math. Soc. 1 (1988), no. 3, 555-586.

[Sa] M. Saito, *Mixed Hodge modules*, Publ. Res. Inst. Math. Sci. 26 (1990), 221-333.

[Sc] C. Schoen, *Algebraic cycles on certain desingularized nodal hypersurfaces*, Math. Ann. 270 (1985), no. 1, 17-27.

[SXZ] M. Sheng, J. Xu and K. Zuo, *The monodromy groups of Dolgachev’s moduli spaces are Zariski dense*, Adv. Math. 272 (2015), 699-742.

[St] J. Steenbrink, *Intersection form for quasi-homogeneous singularities*, Compos. Math. 34 (1977), no. 2, 211-223.

[vS1] D. van Straten, *A quintic hypersurface in $\mathbb{P}^4$ with 130 nodes*, Topology 32 (1993), no. 4, 857-864.

[vS2] ———, *The spectrum of hypersurface singularities*, arXiv:2003.00519.

[Va] A. Varchenko, *On the semicontinuity of the singularity spectrum and an upper bound for the number of singular points in projective hypersurfaces*, Doklady Ak. Nauk 270 (1983), no. 6, 1294-1297.

[Vo] C. Voisin, “Hodge theory and complex algebraic geometry II” (translated by L. Schneps), Cambridge Univ. Press, Cambridge, 2003.

[Xu] J. Xu, *Zariski density of monodromy groups via Picard-Lefschetz type formula*, IMRN 2018 (2018), 3556-3586.

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