O. Dovhopiatyi, E. Sevost’yanov

ON COMPACT CLASSES OF SOLUTIONS OF DIRICHLET PROBLEM IN SIMPLY CONNECTED DOMAINS

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The article is devoted to compactness of solutions of the Dirichlet problem for the Beltrami equation in some simply connected domain. In terms of prime ends, we have proved corresponding results for the case when the maximal dilatations of these solutions satisfy certain integral constraints. The first section is devoted to a presentation of well-known definitions that are necessary for the formulation of the main results. In particular, here we have given a definition of a prime end corresponding to Năkki’s concept. The research tool that was used to establish the main results is the method of moduli for families of paths. In this regard, in the second section we study mappings that satisfy upper bounds for the distortion of the modulus, and in the third section, similar lower bounds. The main results of these two sections include the equicontinuity of the families of mappings indicated above, which is obtained under integral restrictions on those characteristics. The proof of the main theorem is done in the fourth section and is based on the well-known Stoilow factorization theorem. According to this, an open discrete solution of the Dirichlet problem for the Beltrami equation is a composition of some homeomorphism and an analytic function. In turn, the family of these homeomorphisms is equicontinuous (Section 2). At the same time, the equicontinuity of the family of corresponding analytic functions in composition with some (auxiliary) homeomorphisms reduces to using the Schwartz formula, as well as the equicontinuity of the family of corresponding inverse homeomorphisms (Section 3).

1. Introduction. In our recent joint publication [22], we proved the compactness theorem of the classes of solutions of the Dirichlet problem for the Beltrami equation in a simply connected Jordan domain whose characteristics satisfy the constraints of the integral type. In this article we are talking about solutions defined in an arbitrary simply connected domain. Since such domains do not have to be Jordanian, this is a slight relaxation of conditions compared to [22]. Note that, such solutions of the Dirichlet problem exist at this case (see, e.g., [15]). These solutions, generally speaking, do not have a homeomorphic extension to the boundary of the domain in the usual sense, since simply connected domains do not have to be locally connected on their boundary. However, such an extension exists in terms of the so-called prime ends.

Let $D$ be a domain in $\mathbb{C}$. In what follows, a mapping $f : D \to \mathbb{C}$ is assumed to be sense-preserving, moreover, we assume that $f$ has partial derivatives almost everywhere.

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Put  
\[ f_z^+ = \frac{(f_x + if_y)}{2}, \quad f_z^- = \frac{(f_x - if_y)}{2}. \]

The complex dilatation of \( f \) at \( z \in D \) is defined as follows  
\[ \mu(z) = \mu_f(z) := \begin{cases} \frac{f_z^+}{f_z^-}, & \text{for } f_z \neq 0; \\ 0, & \text{otherwise.} \end{cases} \]

The maximal dilatation of \( f \) at \( z \) is the following function  
\[ K_\mu(z) = K_{\mu_f}(z) = \frac{1 + |\mu(z)|}{1 - |\mu(z)|}, \quad (1) \]

Note that the Jacobian of \( f \) at \( z \in D \) is calculated by the formula  
\[ J(z, f) = |f_z|^2 - |f_z|^2. \]

Since we assume that the map \( f \) is sense preserving, the Jacobian of this map is nonnegative at all points of its differentiability.

Let \( \mathbb{D} = \{ z \in \mathbb{C} : |z| < 1 \} \), and let \( \mu : D \to \mathbb{D} \) be a Lebesgue measurable function. Without reference to some mapping, we define the maximal dilatation corresponding to its complex dilatation \( \mu \) by (1).

It is easy to see that  
\[ K_\mu_f(z) = \frac{|f_z^+| + |f_z^-|}{|f_z^+| - |f_z^-|} \]

whenever partial derivatives of \( f \) exist at \( z \in D \) and, in addition, \( J(z, f) \neq 0 \).

Let \( D \) be a domain in \( \mathbb{R}^n \), \( n \geq 2 \) and \( \omega \) be an open set in \( \mathbb{R}^k \), \( k \in \{1, \ldots, n-1\} \). Recall some definitions (see, for example, [6], [8]).

A continuous mapping \( \sigma : \omega \to \mathbb{R}^n \) is called a \( k \)-dimensional surface in \( \mathbb{R}^n \). A surface is an arbitrary \((n-1)\)-dimensional surface \( \sigma \) in \( \mathbb{R}^n \). A surface \( \sigma \) is called a Jordan surface, if \( \sigma(x) \neq \sigma(y) \) for \( x \neq y \). Below we will use \( \sigma \) instead of \( \sigma(\omega) \subset \mathbb{R}^n \), \( \overline{\sigma} \) instead of \( \overline{\sigma(\omega)} \) and \( \partial \sigma \) instead of \( \partial(\sigma(\omega)) \).

A Jordan surface \( \sigma : \omega \to D \) is called a cut of \( D \), if \( \sigma \) separates \( D \), that is \( D \setminus \sigma \) has more than one component, \( \partial \sigma \cap D = \emptyset \) and \( \sigma \cap \partial D \neq \emptyset \).

A sequence of cuts \( \sigma_1, \sigma_2, \ldots, \sigma_m, \ldots \) in \( D \) is called a chain, if:
(i) the set \( \sigma_{m+1} \) is contained in exactly one component \( d_m \) of the set \( D \setminus \sigma_m \), wherein \( \sigma_{m-1} \subset D \setminus (\sigma_m \cup d_m) \);
(ii) \( \bigcap_{m=1}^\infty d_m = \emptyset \).

Two chains of cuts \( \{\sigma_m\} \) and \( \{\sigma'_k\} \) are called equivalent, if for each \( m \in \{1, 2, \ldots\} \) the domain \( d_m \) contains all the domains \( d'_k \), except for a finite number, and for each \( k \in \{1, 2, \ldots\} \) the domain \( d'_k \) also contains all domains \( d_m \), except for a finite number.

The end of the domain \( D \) is the class of equivalent chains of cuts in \( D \). Let \( K \) be the end of \( D \) in \( \mathbb{R}^n \), then the set \( I(K) = \bigcap_{m=1}^\infty d_m \) is called the impression of the end \( K \). Let us to show that  
\[ I(K) = \bigcap_{m=1}^\infty \overline{d_m} \subset \partial D. \quad (2) \]
Let $\xi_0 \in \partial d_m$ for any $m \geq m_0$ and some $m_0 \in \mathbb{N}$.

Indeed, in the contrary case, $\xi_0 \in d_m$ for some increasing sequence $m_k, k \in \{1, 2, \ldots \}$. Now $\xi_0 \in \bigcap_{k=1}^{\infty} d_m$, and, since $d_m \supset d_n$ for $n < m$, we have $\xi_0 \in \bigcap_{m=1}^{\infty} d_m$, as well. The latter contradicts with the definition (ii) $\bigcap_{m=1}^{\infty} d_m = \emptyset$. Thus, $\xi_0 \in \partial d_m$ for any $m \geq m_0$. By the same reasons, $\xi_0 \in \partial d_m$ for any $m \in \mathbb{N}$, not only for $m \geq m_0$. On the other hand, by the definition of domains $d_m$, we have that $\partial d_m \cap D \subset \sigma_m$ for any $m \in \mathbb{N}$. Since $\sigma_{m-1} \subset D \setminus (\sigma_m \cup d_m)$, we have that $\sigma_m \cap \sigma_n = \emptyset$ for $m \neq n$. It follows from this, that the situation $\xi_0 \in \partial d_0 \cap D$ is possible at most for one $l_0 \in \mathbb{N}$. Thus, $\xi_0 \in \partial d_m \cap \partial D$ for any $m \in \{1, 2, \ldots \}$. Therefore, the relation (2) holds, as required.

Throughout the paper, $\Gamma(E, F, D)$ denotes the family of all paths $\gamma: [a, b] \to \mathbb{R}^n$ such that $\gamma(a) \in E$, $\gamma(b) \in F$ and $\gamma(t) \in D$ for every $t \in [a, b]$.

In what follows, $M$ denotes the modulus of a family of paths, and the element $dm(x)$ corresponds to the Lebesgue measure in $\mathbb{R}^n$, $n \geq 2$, see [26]. Following [13], we say that the end $K$ is a prime end, if $K$ contains a chain of cuts $\{\sigma_m\}$ such that $\lim_{m \to \infty} M(\Gamma(C, \sigma_m, D)) = 0$ for some continuum $C$ in $D$.

In the following, the following notation is used: the set of prime ends corresponding to the domain $D$, is denoted by $E_D$, and the completion of the domain $D$ by its prime ends is denoted $\overline{D}_p$.

Consider the following definition, which goes back to Nåkki [13]. We say that the boundary of the domain $D$ in $\mathbb{R}^n$ is locally quasiconformal, if each point $x_0 \in \partial D$ has a neighborhood $U$ in $\mathbb{R}^n$, which can be mapped by a quasiconformal mapping $\varphi$ onto the unit ball $B^n \subset \mathbb{R}^n$ so that $\varphi(\partial D \cup U)$ is the intersection of $B^n$ with the coordinate hyperplane.

For the sets $A, B \subset \mathbb{R}^n$ we set, as usual,

$$\text{diam } A = \sup_{x,y \in A} |x - y|, \quad \text{dist } (A, B) = \inf_{x \in A, y \in B} |x - y|.$$ 

Sometimes we also write $d(A)$ instead of diam $A$ and $d(A, B)$ instead of dist $(A, B)$, if no misunderstanding is possible.

The sequence of cuts $\sigma_m, m \in \{1, 2, \ldots \}$, is called regular, if $\overline{\sigma_m} \cap \overline{\sigma_{m+1}} = \emptyset$ for $m \in \mathbb{N}$ and, in addition, $d(\sigma_m) \to 0$ as $m \to \infty$. If the end $K$ contains at least one regular chain, then $K$ will be called regular.

We say that a bounded domain $D$ in $\mathbb{R}^n$ is regular, if $D$ can be quasiconformally mapped to a domain with a locally quasiconformal boundary whose closure is a compact in $\mathbb{R}^n$, and, besides that, every prime end in $D$ is regular. Note that space $\overline{D}_p = D \cup E_D$ is metric, which can be demonstrated as follows. If $g: D_0 \to D$ is a quasiconformal mapping of a domain $D_0$ with a locally quasiconformal boundary onto some domain $D$, then for $x, y \in \overline{D}_p$ we put:

$$\rho(x, y) := |g^{-1}(x) - g^{-1}(y)|, \quad (3)$$

where the element $g^{-1}(x), x \in E_D$, is to be understood as some (single) boundary point of the domain $D_0$. The specified boundary point is unique and well-defined, see e.g. [6, Theorem 2.1, Remark 2.1], cf. [13, Theorem 4.1]. It is easy to verify that $\rho$ in (3) is a metric on $\overline{D}_p$, and that the topology on $\overline{D}_p$, defined by such a method, does not depend on the choice of the map $g$ with the indicated property.

We say that a sequence $x_m \in D, m \in \{1, 2, \ldots \}$, converges to a prime end of $P \in E_D$ as $m \to \infty$, write $x_m \to P$ as $m \to \infty$, if for any $k \in \mathbb{N}$ all elements $x_m$ belong to $d_k$ except for a
finite number. Here $d_k$ denotes a sequence of nested domains corresponding to the definition of the prime end $P$. Note that for a homeomorphism of a domain $D$ onto $D'$, the end of the domain $D$ uniquely corresponds to some sequence of nested domains in the image under the mapping.

Consider the following Cauchy problem:

$$f_z = \mu(z) \cdot f_z, \quad (4)$$

$$\forall P \in E_D: \lim_{\zeta \to P} \Re f(\zeta) = \varphi(P), \quad (5)$$

where $\varphi : E_D \to \mathbb{R}$ is a predefined continuous function. In what follows, we assume that $D$ is some simply connected domain in $\mathbb{C}$.

The solution of the problem (4)–(5) is called regular, if one of two conditions is fulfilled:

or $f(z) = \text{const}$ in $D$, or $f$ is an open discrete $W^{1/1}_{\text{loc}}(D)$-mapping such that $J(z, f) \neq 0$ for almost any $z \in D$.

Given $z_0 \in D$, a function $\varphi : E_D \to \mathbb{R}$, a function $\Phi : \mathbb{R}^+ \to \mathbb{R}^+$ and a function $M(\Omega)$ of open sets $\Omega \subset D$, we denote by $\mathfrak{F}_{\varphi, \Phi, z_0}(D)$ the class of all regular solutions $f : D \to \mathbb{C}$ of the Cauchy problem (4)–(5) that satisfy the condition $\Im f(z_0) = 0$ and, in addition,

$$\int_{\Omega} \Phi(K_\mu(z)) \cdot \frac{dm(z)}{(1 + |z|^2)^2} \leq M(\Omega) \quad (6)$$

for any open set $\Omega \subset D$. The following statement generalizes [4, Theorem 2] to the case of arbitrary simply connected domains.

**Theorem 1.** Let $D$ be some simply connected domain in $\mathbb{C}$, and let $\Phi : \mathbb{R}^+ \to \mathbb{R}^+$ be a continuous increasing convex function which satisfies the condition

$$\int_\delta^\infty \frac{d\tau}{\tau \Phi^{-1}(\tau)} = \infty$$

for some $\delta > \Phi(0)$. Assume that the function $M$ is bounded, and the function $\varphi$ in (5) is continuous. Then the family $\mathfrak{F}_{\varphi, \Phi, z_0}(D)$ is compact in $D$.

2. Convergence theorems for mappings with upper estimates for modulus distortion. The proof of the main result is based on the theorems on the global behavior of mappings satisfying the weight Poletsky inequality. Results of a similar type in some other situations have been obtained earlier, see, for example, [23]. The case considered below concerns regular domains and mappings with one normalization condition. This case is considered for the first time in this degree of generality.

Given $p \geq 1$, $M_p$ denotes the $p$-modulus of a family of paths, and the element $dm(x)$ corresponds to a Lebesgue measure in $\mathbb{R}^n$, $n \geq 2$, see [26]. In what follows, we usually write $M(\Gamma)$ instead of $M_q(\Gamma)$. Let $x_0 \in \overline{D}$, $x_0 \neq \infty$,

$$S(x_0, r) = \{x \in \mathbb{R}^n : |x - x_0| = r\}, S_i = S(x_0, r_i), \quad i \in \{1, 2\},$$

$$A = A(x_0, r_1, r_2) = \{x \in \mathbb{R}^n : r_1 < |x - x_0| < r_2\}.$$ 

(7)
Everywhere below, unless otherwise stated, the closure $\overline{A}$ and the boundary $\partial A$ of the set $A$ are understood in the topology of the space $\mathbb{R}^n = \mathbb{R}^n \cup \{\infty\}$. Let $Q : \mathbb{R}^n \to [0, \infty]$ be a Lebesgue measurable function satisfying the condition $Q(x) \equiv 0$ for $x \in \mathbb{R}^n \setminus D$, and let $p \geq 1$. Given sets $E$ and $F$ and a given domain $D$ in $\mathbb{R}^n = \mathbb{R}^n \cup \{\infty\}$, we denote by $\Gamma(E, F, D)$ the family of all paths $\gamma : [0, 1] \to \mathbb{R}^n$ joining $E$ and $F$ in $D$, that is, $\gamma(0) \in E$, $\gamma(1) \in F$ and $\gamma(t) \in D$ for all $t \in (0, 1)$. According to [11, Chap. 7.6], a mapping $f : D \to \mathbb{R}^n$ is called a ring $Q$-mapping at the point $x_0 \in \overline{D} \setminus \{\infty\}$ with respect to $p$-modulus, if the condition

$$M_p(f(\Gamma(S_1, S_2, D))) \leq \int_{A \cap D} Q(x) \cdot \eta^p(|x - x_0|) \, dm(x)$$

holds for all $0 < r_1 < r_2 < d_0 := \sup_{x \in D} |x - x_0|$ and all Lebesgue measurable functions $\eta : (r_1, r_2) \to [0, \infty]$ such that

$$\int_{r_1}^{r_2} \eta(r) \, dr \geq 1.$$  (9)

A mapping $f : D \to \mathbb{R}^n$ is called a ring $Q$-mapping in $\overline{D} \setminus \{\infty\}$ with respect to $p$-modulus if (8) holds for any $x_0 \in D \setminus \{\infty\}$. This definition can also be applied to the point $x_0 = \infty$ by inversion: $\varphi(x) = \frac{x}{|x|^2}$, $\infty \mapsto 0$. In what follows, $h$ denotes the so-called chordal metric defined by the equalities

$$h(x, y) = \frac{|x - y|}{\sqrt{1 + |x|^2} \sqrt{1 + |y|^2}}, \quad x \neq \infty \neq y, \quad h(x, \infty) = \frac{1}{\sqrt{1 + |x|^2}}.$$  (10)

For a given set $E \subset \mathbb{R}^n$, we set

$$h(E) := \sup_{x, y \in E} h(x, y).$$  (11)

The quantity $h(E)$ in (11) is called the chordal diameter of the set $E$.

For given sets $A, B \subset \mathbb{R}^n$, we put

$$h(A, B) = \inf_{x \in A, y \in B} h(x, y),$$

where $h$ is a chordal metric defined in (10).

Let $I$ be a fixed set of indices and let $D_i$, $i \in I$, be some sequence of domains. Following [14, Sect. 2.4], we say that a family of domains $\{D_i\}_{i \in I}$ is equi-uniform with respect to $p$-modulus if for any $r > 0$ there exists a number $\delta > 0$ such that the inequality

$$M_p(\Gamma(F^*, F, D_i)) \geq \delta$$

holds for any $i \in I$ and any continua $F, F^* \subset D$ such that $h(F) \geq r$ and $h(F^*) \geq r$.

Given a Lebesgue measurable function $Q : \mathbb{R}^n \to [0, \infty]$ and a point $x_0 \in \mathbb{R}^n$ we set

$$q_{x_0}(t) = \frac{1}{\omega_{n-1} t^{n-1}} \int_{S(x_0, t)} Q(x) \, d\mathcal{H}^{n-1},$$

where $\mathcal{H}^{n-1}$ denotes $(n-1)$-dimensional Hausdorff measure. The following lemma was proved in [21, Lemma 2.1].
Lemma 1. Let \( 1 \leq p \leq n, \) and let \( \Phi : [0, \infty] \to [0, \infty] \) be a strictly increasing convex function such that the relation
\[
\int_{\delta_0}^{\infty} \frac{d\tau}{\tau [\Phi^{-1}(\tau)]^{p-1}} = \infty
\]  
(14)
holds for some \( \delta_0 > \tau_0 := \Phi(0). \) Let \( \Omega \) be a family of functions \( Q : \mathbb{R}^n \to [0, \infty] \) such that
\[
\int_D \Phi(Q(x)) \frac{dm(x)}{(1 + |x|^2)^n} \leq M_0 < \infty
\]  
(15)
for some \( 0 < M_0 < \infty. \) Now, for any \( 0 < r_0 < 1 \) and for every \( \sigma > 0 \) there exists \( 0 < r_* = r_*(\sigma, r_0, \Phi) < r_0 \) such that
\[
\int_{\varepsilon}^{r_0} \frac{dt}{t^{p-1} q^{p-1}_\sigma(t)} \geq \sigma, \quad \varepsilon \in (0, r_*),
\]
for any \( Q \in \Omega. \)

Consider some another auxiliary family of mappings.

For \( p \geq 1, \) a given number \( 0 < M_0 < \infty, \) a domain \( D \subset \mathbb{R}^n, \) \( n \geq 2, \) and a strictly increasing convex function \( \Phi : \mathbb{R}^+ \to \mathbb{R}^+ \) denote by \( A_{\Phi,p,M_0}(D) \) the family all open discrete mappings \( f : D \to \mathbb{R}^n \) satisfying relations (8)–(9) with some \( Q = Q_f \) in \( D \) with respect to \( p\)-modulus. The following statement is true (see [21, Theorem 1.2]).

Lemma 2. Let \( p \in (n-1, n], \) and let \( \delta_0 > \tau_0 := \Phi(0) \) be such that the condition (14) holds. Now the family \( A_{\Phi,p,M_0}(D) \) is equicontinuous in \( D. \)

Here the equicontinuity of the family of mappings \( A_{\Phi,p,M_0}(D) \) should be understood with respect to the spaces \( (D, d) \) and \( (\mathbb{R}^n, d), \) where \( d \) is the Euclidean metric.

Given \( p \geq 1, \) numbers \( \delta > 0, \) \( 0 < M_0 < \infty, \) a domain \( D \subset \mathbb{R}^n, \) \( n \geq 2, \) a point \( a \in D \) and a strictly increasing convex function \( \Phi : \mathbb{R}^+ \to \mathbb{R}^+ \) denote by \( F_{\Phi,p,\delta,M_0}(D) \) the family of all homeomorphisms \( f : D \to \mathbb{R}^n \) satisfying (8)–(9) in \( D \) for some \( Q = Q_f \) such that
\[
h(f(a), \partial f(D)) \geq \delta, \quad h(\mathbb{R}^n \setminus f(D)) \geq \delta
\]
and, in addition, (15) holds.

Theorem 2. Let \( p \in (n-1, n], \) let \( D \) be regular, and let \( D'_f = f(D) \) be bounded domains with a locally quasiconformal boundary which are equi-uniform with respect to \( p\)-modulus over all \( f \in F_{\Phi,a,p,\delta,M_0}(D). \) If there is \( \delta_0 > \tau_0 := \Phi(0) \) such that (14) holds, then any \( f \in F_{\Phi,a,p,\delta,M_0}(D) \) has a continuous extension \( \overline{f} : \overline{D}_P \to \mathbb{R}^n \) and, in addition, the family \( F_{\Phi,a,p,\delta,M_0}(D) \) of all extended mappings \( \overline{f} : \overline{D}_P \to \mathbb{R}^n \) is equicontinuous in \( \overline{D}_P. \)

Remark 1. In Theorem 2, the equicontinuity should be understood in the sense of mappings acting between the spaces \( (X, d) \) and \( (X', d'), \) where \( X = \overline{D}_P \) is the replenishment of the domain \( D \) by its prime ends, and \( d \) is one of the possible metrics that correspond to the topological space \( \overline{D}_P \) in (3). In addition, \( X' = \mathbb{R}^n \) and \( d' = h \) is a chordal (spherical) metric.

An example of a family of plane mappings \( f_n(z) = z^n, \) \( n \in \{1, 2, \ldots\}, \) \( z \in \mathbb{D}, \) indicates the inaccuracy of Theorem 2 for mappings with branching, in particular, this theorem is not true under the normalization condition \( f_n(0) = 0, \) \( n \in \{1, 2, \ldots\}. \)
Proof of Theorem 2. Put \( f \in \mathfrak{F}_{\Phi,A,p,\delta,M_0}(D) \) and \( Q = Q_f(x) \). Given \( x \in \mathbb{R}^n \) we set

\[
Q'(x) = \begin{cases} 
Q(x), & x \in D, Q(x) \geq 1; \\
1, & x \in D, Q(x) < 1; \\
1, & x \notin D.
\end{cases}
\]

Observe that the function \( Q'(x) \) satisfies the relation (15) up to some constant. Indeed,

\[
\int_D \Phi(Q'(x)) \frac{dm(x)}{(1 + |x|^2)^{\alpha}} = \int_{\{x \in D : Q(x) < 1\}} \Phi(Q'(x)) \frac{dm(x)}{(1 + |x|^2)^{\alpha}} + \int_{\{x \in D : Q(x) \geq 1\}} \Phi(Q'(x)) \frac{dm(x)}{(1 + |x|^2)^{\alpha}} \leq M_0 + \Phi(1) \int_{\mathbb{R}^n} \frac{dm(x)}{(1 + |x|^2)^{\alpha}} = M'_0 < \infty.
\]

In this case, by Lemma 1

\[
\int_\varepsilon^{r_0} \frac{dt}{t^{\frac{n-1}{2}} q_0(t)} \to \infty \tag{16}
\]

as \( \varepsilon \to 0 \) for any \( 0 < r_0 < 1 \) and \( \varepsilon \to 0 \), where \( q'_0(t) = \frac{1}{\omega_{n-1} t^{n-1}} \int_{s(x_0,t)} Q'(x) d\mathcal{H}^{n-1} \). Besides that,

\[
\int_\varepsilon^{r_0} \frac{dt}{t^{\frac{n-1}{2}} q_0(t)} < \infty
\]

for any \( \varepsilon \in (0,r_0) \), because \( q'_0(t) \geq 1 \) for almost any \( t \in (0,r_0) \). Observe that, equi-uniform domains with respect to \( p \)-modulus have strongly accessible boundaries with respect to \( p \)-modulus, as well (see [23, Remark 1]). In this case, the condition (16) directly implies that \( f \in \mathfrak{F}_{\Phi,A,p,\delta,M_0}(D) \) has a continuous extension to \( \overline{D}_p \) (see [20, Theorem 3]).

Observe that \( f \in \mathfrak{F}_{\Phi,A,p,\delta,M_0}(D) \) does not equal to infinity for \( p \neq n \) (see, e.g., [5, Lemmas 2.6 and 3.1]). Now, the equicontinuity of \( \mathfrak{F}_{\Phi,A,p,\delta,M_0}(D) \) inside \( D \) follows by Theorem 4.1 in [17] for \( p = n \) and Lemma 2 for \( p < n \).

We prove the equicontinuity of the family \( \mathfrak{F}_{\Phi,A,p,\delta,M_0}(D) \) in \( E_D := \overline{D}_p \setminus D \). Let us assume the opposite, namely that there are \( \varepsilon_* > 0 \), \( P_0 \in E_D \), a sequence \( x_m \in \overline{D}_p \), \( x_m \to P_0 \) as \( m \to \infty \), and a mapping \( f_m \in \mathfrak{F}_{\Phi,A,p,\delta,M_0}(D) \) such that

\[
h(f_m(x_m), f_m(P_0)) \geq \varepsilon_*, \quad m \in \{1,2,\ldots\}. \tag{17}
\]

Since \( f_m \) as a continuous extension at \( P_0 \), we may assume that \( x_m \in D \) and, in addition, there is a sequence \( x'_m \in \overline{D}_p \), \( x'_m \to P_0 \) as \( m \to \infty \), such that

\[
h(f_m(x_m), f_m(x'_m)) \geq \varepsilon_*/2, \quad m \in \{1,2,\ldots\}. \tag{18}
\]

Let \( d_m, m \in \{1,2,\ldots\}, \) be a sequence of cuts \( \sigma_m \) corresponding to \( P_0 \). By [8, Lemma 2] the sequence \( \sigma_m \) may be chosen such that \( \sigma_m \subset S(x_0,r_m) \), where \( x_0 \in \partial D \) and \( r_m \to 0 \) as \( m \to \infty \).

By the definition of the convergence of the sequence \( x_m \to P_0 \) as \( m \to \infty \), there exists \( m_1 \in \mathbb{N} \) such that \( x_{m_1} \in d_1 \). Similarly, there exists \( m_2 > m_1 \) such that \( x_{m_2} \in d_2 \). Etc. Given
Since the domain $D$ is regular, the space $\overline{D}_P$ contains at least two prime ends $P_1$ and $P_2 \in E_D$. Let $P_1 \subset E_D$ be a prime end that does not coincide with $P_0$. Suppose that $G_m$, $m \in \{1, 2, \ldots \}$, is a sequence of domains that corresponds to a prime end $P_1$. Since the mapping $f_m$ has a continuous extension on $\overline{D}_P$ for any $m \in \{1, 2, \ldots \}$, we may choose a sequence $\zeta_m \in G_m$, $\zeta_m \rightarrow P_1$ as $m \rightarrow \infty$, such that $h(f_m(\zeta_m), f_m(P_1)) \rightarrow 0$ as $m \rightarrow \infty$. Note that
\[
h(f_m(a), f_m(\zeta_m)) \geq h(f_m(a), f_m(P_1)) - h(f_m(\zeta_m), f_m(P_1)) \geq \delta/2,
\]
for any $m \geq m_0$ and some $m_0 \in \mathbb{N}$. We construct a sequence of continua $K_m$, $m \in \{1, 2, \ldots \}$, as follows. We join the points $\zeta_1$ and $a$ by an arbitrary path in $D$, which we denote by $K_1$. Next, we join the points $\zeta_2$ and $\zeta_1$ by a path $K'_1$, in $G_1$. Combining the paths $K_1$ and $K'_1$, we obtain a path $K_2$, joining the points $a$ and $\zeta_2$. And so on. Suppose that at some step we have a path $K_m$, that join the points $\zeta_m$ and $a$. Join the points $\zeta_{m+1}$ and $\zeta_m$ with a path $K'_m$, which lies in $G_m$. Combining the paths $K_m$ and $K'_m$, we obtain a path $K_{m+1}$. We show that there is a number $m_1 \in \mathbb{N}$ such that
\[
\forall m \geq m_1: \quad d_m \cap K_m = \emptyset.
\] (20)

We prove this from the opposite, namely, suppose that (20) does not hold. Then there is an increasing sequence of numbers $m_k \rightarrow \infty$, $k \rightarrow \infty$, and points $\xi_k \in K_{m_k} \cap d_{m_k}$, $m \in \{1, 2, \ldots \}$. Then $\xi_k \rightarrow P_0$ as $k \rightarrow \infty$.

Note that two cases are possible: either all elements $\xi_k$ belong to $D \setminus G_1$ for $k \in \{1, 2, \ldots \}$, or there is a number $k_1$ such that $\xi_{k_1} \in G_1$. In the second case, consider the sequence $\xi_k$, $k > k_1$. Note that two cases are possible: or $\xi_k$ for $k > k_1$ belong to $D \setminus G_2$, or there is $k_2 > k_1$ such that $\xi_{k_2} \in G_2$. In the second case, consider the sequence $\xi_k$, $k > k_2$, and so on. Assume that the element $\xi_{k_{l-1}} \in G_{l-1}$ is already constructed. Note that two cases are possible: either $\xi_k$ belong to $D \setminus G_l$ for $k > k_{l-1}$, or there is a number $k_l > k_{l-1}$ such that $\xi_{k_l} \in G_l$, and etc. This procedure can be both finite or infinite, depending on which we have two possible situations:

1) or there are numbers $n_0 \in \mathbb{N}$ and $l_0 \in \mathbb{N}$ such that that $\xi_k \in D \setminus G_{n_0}$ for all $k > l_0$;
2) or for each there is an element $\xi_{k_l}$ such that $\xi_{k_l} \in G_l$, and the sequence $k_l$ is increasing by $l \in \mathbb{N}$.

Consider each of these cases separately and show that in both of them we come to a contradiction. Let situation 1) holds. Observe that all elements of the sequence $\xi_k$ belong to $K_{n_0}$, hence there exists a subsequence $\xi_{k_r}$, $r \in \{1, 2, \ldots \}$, convergent as $r \rightarrow \infty$ to some point $\xi_0 \in D$. However, $\xi_k \in d_{m_k}$, i.e., $\xi_0 \in \bigcap_{m=1}^{\infty} d_m$. Due to (2), $\xi_0 \in \partial D$. The obtained contradiction indicates the impossibility of the case 1). Suppose that case 2) holds, then simultaneously $\xi_k \rightarrow P_0$ and $\xi_k \rightarrow P_1$ as $k \rightarrow \infty$. Since space $\overline{D}_P$ is metric with a metric $\rho$ in (3), by the triangle inequality it follows that $P_1 = P_0$, which contradicts the choice of $P_1$. The obtained contradiction indicates the validity of the relation (20).
where $\varepsilon_0$ is some positive number which may be chosen as $\varepsilon_0 := r_1$. Thus

$$f_m(\Gamma(|\gamma_m|, K_m, D)) > f_m(\Gamma(S(x_0, r_m), S(x_0, \varepsilon_0), D)),$$

whence, by the definition of the class $\mathcal{F}_{\Phi,a,p,\delta,M_0}(D)$

$$M_p(f_m(\Gamma(|\gamma_m|, K_m, D))) \leq M_p(f_m(\Gamma(S(x_0, r_m), S(x_0, \varepsilon_0), D)) \leq \int_{A_n D} Q_m(x) \cdot |x - x_0| \, dm(x), \quad (21)$$

where $\eta$ is any Lebesgue measurable function satisfying (9) for $r_1 \mapsto r_m$ and $r_2 \mapsto \varepsilon_0$, in addition, $Q_m := Q_{f_m}$ corresponds to the function $Q$ in (8). Let us to prove the inequality

$$M_p(f_m(\Gamma(S(x_0, r_m), S(x_0, \varepsilon_0), D)) \leq \frac{\omega_{n-1}}{I_m}, \quad (22)$$

where $I_m = \int_{r_m}^{\varepsilon_0} \frac{dr}{\frac{n-1}{p-1} q_{mx_0}(r)}$, $q_{mx_0}(t) = \frac{1}{\omega_{n-1} r^n} \int_{S(x_0,t)} Q_m(x) \, d\mathcal{H}^{n-1}$ and $Q_m := Q_{f_m}$ (we set $Q_m(x) \equiv 1$ for $x \notin D$). To do this, we will reason similarly to the proof of Lemma 1 in [18]. We may assume that $I \neq 0$, since (22) is obviously in this case. We may also assume that $I \neq \infty$, because otherwise we may consider $Q(x) + \delta$ instead of $Q(x)$ in (22), and then go to the limit as $\delta \to 0$. Let $0 \neq I \neq \infty$. Then $q_{x_0}(r) \neq 0$ for $r \in (r_m, \varepsilon_0)$. Put

$$\psi(t) = \begin{cases} 1/[t^{\frac{p-1}{n-1}} q_{x_0}(t)], & t \in (r_m, \varepsilon_0); \\ 0, & t \notin (r_m, \varepsilon_0). \end{cases}$$

By the Fubini theorem

$$\int_A Q_m(x) \cdot |x - x_0| \, dm(x) = \omega_{n-1} I_m, \quad (23)$$

where $A = A(r_m, \varepsilon_0, x_0)$ is defined in (7). Observe that the function $\eta(t) = \psi(t)/I$, $t \in (r_m, \varepsilon_0)$, satisfies (9). Now, by (8) and (23) we obtain relation (22), as required.

Finally, from (21), (22) and from Lemma 1 it follows that

$$M_p(f_m(\Gamma(|\gamma_m|, K_m, D))) \leq \frac{\omega_{n-1}}{\int_{r_m}^{\varepsilon_0} \frac{dr}{\frac{n-1}{p-1} q_{mx_0}(r)}} \to 0, \quad m \to \infty, \quad (24)$$

where $q_{mx_0}(t) = \frac{1}{\omega_{n-1} r^n} \int_{S(x_0,t)} Q_m(x) \, d\mathcal{H}^{n-1}$ and $Q_m := Q_{f_m}$ corresponds to the function $Q$ in (13). The relation (21) contradicts the equi-uniformity of the sequence of domains $D'_m := f_m(D)$. Indeed,

$$h(f_m(K_m)) \geq \delta/2$$

according to (19), and

$$h(f_m(|\gamma_m|)) > \varepsilon_*/2$$
by the relation (18). Hence, since the sequence of domains $D_m' := f_m(D)$ is equi-uniform, we obtain that

$$M_p(f_m(\Gamma(|\gamma_m|, K_m, D))) = M_p(\Gamma(f_m(|\gamma_m|), f_m(K_m), f_m(D))) \geq \delta_\ast > 0$$

for some $\delta_\ast > 0$ and any $m \in \{1, 2, \ldots\}$, which contradicts the relation (24). The obtained contradiction indicates the incorrectness of the assumption in (17).

☐

3. Equicontinuity of families of mappings with inverse Poletsky inequality. Let $f : D \to \mathbb{R}^n,$ $n \geq 2,$ and let $Q : \mathbb{R}^n \to [0, \infty]$ be a Lebesgue measurable function such that $Q(x) \equiv 0$ for $x \in \mathbb{R}^n \setminus f(D).$ Let $A = A(y_0, r_1, r_2)$ and let $\Gamma_f(y_0, r_1, r_2)$ denotes the family of all paths $\gamma : [a, b] \to D$ such that $f(\gamma) \in \Gamma(S(y_0, r_1), S(y_0, r_2), A(y_0, r_1, r_2))$, i.e., $f(\gamma(a)) \in S(y_0, r_1),$ $f(\gamma(b)) \in S(y_0, r_2),$ and $\gamma(t) \in A(y_0, r_1, r_2)$ for any $a < t < b.$ We say that $f$ satisfies the inverse Poletsky inequality at $y_0 \in f(D)$ if the relation

$$M(\Gamma_f(y_0, r_1, r_2)) \leq \int_{A(y_0, r_1, r_2) \cap f(D)} Q(y) \cdot \eta^n(||y - y_0||) \, dm(y)$$

holds for any Lebesgue measurable function $\eta : (r_1, r_2) \to [0, \infty]$ such that

$$\int_{r_1}^{r_2} \eta(r) \, dr \geq 1.$$  

(26)

Given domains $D, D' \subset \mathbb{R}^n,$ points $a \in D, b \in D'$ and a number $M_0 > 0$ denote by $\mathcal{S}_{a,b,M_0}(D, D')$ the family of all paths $\gamma : [a, b] \to D$ such that $\gamma(t) \in A(y_0, r_1, r_2)$ for any $y_0 \in f(D)$ such that $f(a) = b.$

**Theorem 3.** Assume that $D$ has a weakly flat boundary, none of the components of which degenerates into a point, and $D'$ is regular. Then any $f \in \mathcal{S}_{a,b,M_0}(D, D')$ has a continuous extension $\overline{f} : \overline{D} \to \overline{D'},$ while $\overline{f}(\overline{D}) = \overline{D'}$ and, in addition, the family $\mathcal{S}_{a,b,M_0}(\overline{D}, \overline{D'})$ of all extended mappings $\overline{f} : \overline{D} \to \overline{D'}$ is equicontinuous in $\overline{D}.$

Theorem 3 was proved in [24, Theorem 7.1] in the case of a fixed function $Q$ (cf. Theorem 4.2 in [22]). The proof of it may be found in [22, Theorem 4.2].

4. Compactness of families of solutions of the Dirichlet problem.

**Proof of Theorem 1.** In general, we will use the scheme of proving Theorem 1.2 in [22].

I. Let $f_m \in \mathfrak{F}_{\varphi, \Phi, \mu}(D),$ $m \in \{1, 2, \ldots\}.$ By Stoilow’s factorization theorem (see, e.g., [25, 5(III).V]) a mapping $f_m$ has a representation

$$f_m = \varphi_m \circ g_m,$$  

(27)

where $g_m$ is some homeomorphism, and $\varphi_m$ is some analytic function. By Lemma 1 in [19], the mapping $g_m$ belongs to the Sobolev class $W^{1,1}_{\text{loc}}(D)$ and has a finite distortion. Moreover, by [1, (1).C, Ch. I]

$$f_{mz} = \varphi_{mz}(g_m(z))g_{mz}, \quad f_{m\overline{z}} = \varphi_{m\overline{z}}(g_m(z))g_{m\overline{z}}$$

(28)
for almost all \( z \in D \). Therefore, by the relation (28), \( J(z, g_m) \neq 0 \) for almost all \( z \in D \), in addition, \( K_{\mu_{gm}}(z) = K_{\mu_{gm}}(z) \).

II. We prove that \( \partial g_m(D) \) contains at least two points. Suppose the contrary. Then either \( g_m(D) = \mathbb{C} \), or \( g_m(D) = \mathbb{C} \setminus \{a\} \), where \( a \in \mathbb{C} \). Consider first the case \( g_m(D) = \mathbb{C} \). By Picard’s theorem \( \varphi_m(g_m(D)) \) is the whole plane, except perhaps one point \( \omega_0 \in \mathbb{C} \). On the other hand, for every \( m \in \{1, 2, \ldots\} \) the function \( u_m(z) := \text{Re} \, f_m(z) = \text{Re} \, (\varphi_m(g_m(z))) \) is continuous on the compact set \( \overline{D} \) under the condition (5) by the continuity of \( \varphi \). Therefore, there exists \( C_m > 0 \) such that \( |\text{Re} \, f_m(z)| \leq C_m \) for any \( z \in D \), but this contradicts the fact that \( \varphi_m(g_m(D)) \) contains all points of the complex plane except, perhaps, one. The situation \( g_m(D) = \mathbb{C} \setminus \{a\} \), \( a \in \mathbb{C} \), is also impossible, since the domain \( g_m(D) \) must be simply connected in \( \mathbb{C} \) as a homeomorphic image of the simply connected domain \( D \).

Therefore, the boundary of the domain \( g_m(D) \) contains at least two points. Then, according to Riemann’s mapping theorem, we may transform the domain \( g_m(D) \) onto the unit disk \( \mathbb{D} \) using the conformal mapping \( \psi_m \). Let \( z_0 \in D \) be a point from the condition of the theorem. By using an auxiliary conformal mapping

\[
\tilde{\psi}_m(z) = \frac{z - (\psi_m \circ g_m)(z_0)}{1 - z(\psi_m \circ g_m)(z_0)}
\]

of the unit disk onto itself we may consider that \( (\psi_m \circ g_m)(z_0) = 0 \). Now, by (27) we obtain that

\[
f_m = \varphi_m \circ g_m = \varphi_m \circ \psi^{-1}_m \circ \psi_m \circ g_m = F_m \circ G_m, \quad m \in \{1, 2, \ldots\},
\]

where \( F_m := \varphi_m \circ \psi^{-1}_m \), \( F_m : \mathbb{D} \to \mathbb{C} \), and \( G_m = \psi_m \circ g_m \). Obviously, a function \( F_m \) is analytic, and \( G_m \) is a regular Sobolev homeomorphism in \( D \). In particular, \( \text{Im} \, F_m(0) = 0 \) for any \( m \in \mathbb{N} \).

III. We prove that the \( L^1 \)-norms of the functions \( K_{\mu_{gm}}(z) \) are bounded from above by some universal positive constant \( C > 0 \) for all \( m \in \{1, 2, \ldots\} \). Indeed, by the convexity of the function \( \Phi \) in (6) and by [3, Proposition 5, I.4.3], the slope \( [\Phi(t) - \Phi(0)]/t \) is a non-decreasing function. Hence there exist constants \( t_0 > 0 \) and \( C_1 > 0 \) such that

\[
\Phi(t) \geq C_1 \cdot t \quad \forall \ t \in [t_0, \infty).
\]

Fix \( m \in \mathbb{N} \). By (6) and (29), we obtain that

\[
\int_D K_{\mu_{gm}}(z) \, dm(z) = \int_{\{z \in D : K_{\mu_{gm}}(z) < t_0\}} K_{\mu_{gm}}(z) \, dm(z) + \int_{\{z \in D : K_{\mu_{gm}}(z) \geq t_0\}} K_{\mu_{gm}}(z) \, dm(z) \leq
\]

\[
\leq t_0 \cdot m(D) + \frac{1}{C_1} \int_D \Phi(K_{\mu_{gm}}(z)) \, dm(z) \leq
\]

\[
\leq t_0 \cdot m(D) + \frac{\sup_{z \in D} (1 + |z|^2)^2}{C_1} \int_D \Phi(K_{\mu_{gm}}(z)) \cdot \frac{1}{(1 + |z|^2)^2} \, dm(z) \leq
\]

\[
\leq t_0 \cdot m(D) + \frac{\sup_{z \in D} (1 + |z|^2)^2}{C_1} \mathcal{M}(D) < \infty,
\]
IV. We prove that each map $G_m$, $m \in \{1, 2, \ldots \}$, has a continuous extension to $E_D$, in addition, the family of extended maps $\overline{G}_m$, $m \in \{1, 2, \ldots \}$, is equicontinuous in $\overline{D}_p$. Indeed, as proved in item III, $K_{\mu_m} \in L^1(D)$. By [7, Theorem 3] (see also [9, Theorem 3.1]) each $G_m$, $m \in \{1, 2, \ldots \}$, is a ring $Q$-homeomorphism in $\overline{D}$ for $Q = K_{\mu_m}(z)$, where $\mu$ is defined in (4), and $K_{\mu}$ my be calculated by the formula (1). Note that the unit disk $\mathbb{D}$ is a uniform domain as a finitely connected flat domain at its boundary with a finite number of boundary components (see, for example, [12, Theorem 6.2 and Corollary 6.8]). Then the desired conclusion follows by Theorem 2.

V. Let us prove that the inverse homeomorphisms $G_m^{-1}$, $m \in \{1, 2, \ldots \}$, have a continuous extension $\overline{G}_m^{-1}$ to $\partial \mathbb{D}$ in terms of prime ends in $D$, and $\{\overline{G}_m^{-1}\}_{m=1}^\infty$ is equicontinuous in $\overline{D}$ as a family of mappings from $\overline{D}$ to $\overline{D}_p$. Since by the item IV mappings $G_m$, $m \in \{1, 2, \ldots \}$, are ring $K_{\mu_m}(z)$-homeomorphisms in $D$, the corresponding inverse mappings $G_m^{-1}$ satisfy (25) (in this case, $D$ corresponds the unit disk $\mathbb{D}$ in (26), $f \rightarrow G_m$, $Q \rightarrow K_{\mu_m}(z)$, and $f(D) \rightarrow D$).

Since $G_m(0) = z_0$ for any $m \in \{1, 2, \ldots \}$, the possibility of a continuous extension of $G_m^{-1}$ to $\partial \mathbb{D}$, and the equicontinuity of $\{\overline{G}_m^{-1}\}_{m=1}^\infty$ as mappings $\overline{G}_m^{-1} : \mathbb{D} \rightarrow \overline{D}_p$ follow by Theorem 3.

VI. Since, as proved above the family $\{G_m\}_{m=1}^\infty$ is equicontinuous in $D$, by Arzela-Ascoli criterion there exists an increasing subsequence of numbers $m_k$, $k \in \{1, 2, \ldots \}$, such that $G_{m_k}$ converges locally uniformly in $D$ to some continuous mapping $G : D \rightarrow \overline{C}$ as $k \rightarrow \infty$ (see, e.g., [26, Theorem 20.4]). By [22, Lemma 2.1], either $G$ is a homeomorphism with values in $\mathbb{R}^n$, or a constant in $\mathbb{R}^n$. Let us prove that the second case is impossible. Let us apply the approach used in proof of the second part of Theorem 21.9 in [26]. Suppose the contrary: let $G_{m_k}(x) \rightarrow c = \text{const}$ as $k \rightarrow \infty$. Since $G_{m_k}(z_0) = 0$ for every $k \in \{1, 2, \ldots \}$, we have that $c = 0$. By item V, the family of mappings $G_m^{-1}$, $m \in \{1, 2, \ldots \}$, is equicontinuous in $\mathbb{D}$. Then

$$h(z, G_{m_k}^{-1}(0)) = h(G_{m_k}^{-1}(G_{m_k}(z)), G_{m_k}^{-1}(0)) \rightarrow 0$$

as $k \rightarrow \infty$, which is impossible because $z$ is an arbitrary point of $D$. The obtained contradiction refutes the assumption made above. Thus, $G : D \rightarrow \overline{C}$ is a homeomorphism.

VII. According to V, the family of mappings $\{\overline{G}_m^{-1}\}_{m=1}^\infty$ is equicontinuous in $\overline{D}$. By the Arzela-Ascoli criterion (see, e.g., [26, Theorem 20.4]) we may consider that $\overline{G}_m^{-1}(y)$, $k \in \{1, 2, \ldots \}$, converges to some mapping $\overline{F} : \overline{D} \rightarrow \overline{D}$ as $k \rightarrow \infty$ uniformly in $\overline{D}$. Let us to prove that $\overline{F} = \overline{G}^{-1}$. For this purpose, we show that $G(D) = \overline{D}$. Fix $y \in \overline{D}$. Since $G_{m_k}(D) = \overline{D}$ for every $k \in \{1, 2, \ldots \}$, we obtain that $G_{m_k}(x_k) = y$ for some $x_k \in D$. Since $D$ is regular, the metric space $(\overline{D}_p, \rho)$ is compact. Thus, we may assume that $\rho(x_k, x_0) \rightarrow 0$ as $k \rightarrow \infty$, where $x_0 \in \overline{D}_p$. By the triangle inequality and the equicontinuity of $\{\overline{G}_m\}_{m=1}^\infty$ in $\overline{D}_p$ (see IV), we obtain that

$$|\overline{G}(x_0) - y| = |\overline{G}(x_0) - \overline{G}_{m_k}(x_k)| \leq |\overline{G}(x_0) - \overline{G}_{m_k}(x_0)| + |\overline{G}_{m_k}(x_0) - \overline{G}_{m_k}(x_k)| \rightarrow 0$$

as $k \rightarrow \infty$. Thus, $\overline{G}(x_0) = y$. Observe that $x_0 \in D$, because $G$ is a homeomorphism. Since $y \in \overline{D}$ is arbitrary, the equality $G(D) = \overline{D}$ is proved. In this case, $G_{m_k}^{-1} \rightarrow G^{-1}$ locally uniformly in $\overline{D}$ as $k \rightarrow \infty$ (see, e.g., [16, Lemma 3.1]). Thus, $\overline{F}(y) = G^{-1}(y)$ for every $y \in \overline{D}$.

Finally, since $\overline{F}(y) = G^{-1}(y)$ for any $y \in \overline{D}$ and, in addition, $\overline{F}$ has a continuous extension on $\partial \mathbb{D}$, due to the uniqueness of the limit at the boundary points we obtain that
\( \tilde{F}(y) = G^{-1}(y) \) for \( y \in \overline{D} \). Therefore, we have proved that \( G_{m_k}^{-1} \rightarrow G^{-1} \) uniformly in \( \overline{D} \) with as \( k \rightarrow \infty \) with respect to the metrics \( \rho \) in \( \overline{D}_P \).

VIII. By VII, for \( y = e^{i\theta} \in \partial \mathbb{D} \)

\[
\text{Re } F_{m_k}(e^{i\theta}) = \varphi \left( G_{m_k}^{-1}(e^{i\theta}) \right) \rightarrow \varphi \left( G^{-1}(e^{i\theta}) \right) \tag{30}
\]

as \( k \rightarrow \infty \) uniformly on \( \theta \in [0, 2\pi) \). Here we have used that \( \varphi \) is continuous in \( \overline{D}_P \). Since by the construction \( \text{Im } F_{m_k}(0) = 0 \) for any \( k \in \{1, 2, \ldots \} \), by the Schwartz formula (see, e.g., [2, relation (66), sect. 6.3, ch. IV]) the analytic function \( F_{m_k} \) is uniquely restored by its real part, namely,

\[
F_{m_k}(y) = \frac{1}{2\pi i} \int_{S(0,1)} \varphi \left( G_{m_k}^{-1}(t) \right) \frac{t + y}{t - y} \frac{dt}{t}. \tag{31}
\]

Set

\[
F(y) := \frac{1}{2\pi i} \int_{S(0,1)} \varphi \left( G^{-1}(t) \right) \frac{t + y}{t - y} \frac{dt}{t}. \tag{32}
\]

Let \( K \subset \mathbb{D} \) be an arbitrary compact set, and let \( y \in K \). By (31) and (32) we obtain that

\[
|F_{m_k}(y) - F(y)| \leq \frac{1}{2\pi} \int_{S(0,1)} |\varphi(G_{m_k}^{-1}(t)) - \varphi(G^{-1}(t))| \left| \frac{t + y}{t - y} \right| |dt|. \tag{33}
\]

Since \( K \) is compact, there is \( 0 < R_0 = R_0(K) < \infty \) such that \( K \subset B(0, R_0) \). By the triangle inequality \( |t + y| \leq 1 + R_0 \) and \( |t - y| \geq |t| - |y| \geq 1 - R_0 \) for \( y \in K \) and any \( t \in S^1 \). Thus

\[
\left| \frac{t + y}{t - y} \right| \leq \frac{1 + R_0}{1 - R_0} := M = M(K).
\]

Put \( \varepsilon > 0 \). By (30), for a number \( \varepsilon' := \frac{\varepsilon}{M} \) there is \( N = N(\varepsilon, K) \in \mathbb{N} \) such that

\[
|\varphi(G_{m_k}^{-1}(t)) - \varphi(G^{-1}(t))| < \varepsilon'
\]

for any \( k \geq N(\varepsilon) \) and \( t \in S^1 \). Now, by (33)

\[
|F_{m_k}(y) - F(y)| < \varepsilon \quad \forall \ k \geq N. \tag{34}
\]

It follows from (34) that the sequence \( F_{m_k} \) converges to \( F \) as \( k \rightarrow \infty \) in the unit disk locally uniformly. In particular, we obtain that \( \text{Im } F(0) = 0 \). Note that \( F \) is analytic function in \( \mathbb{D} \) (see comments before the relation (66) in [2, sect. 6.3, ch. IV]), and

\[
\text{Re } F(re^{i\psi}) = \frac{1}{2\pi} \int_0^{2\pi} \varphi \left( G^{-1}(e^{i\theta}) \right) \frac{1 - r^2}{1 - 2r \cos(\theta - \psi) + r^2} d\theta
\]

for \( z = re^{i\psi} \). By [2, relation (66), sect. 6.3, ch. IV]

\[
\lim_{\zeta \rightarrow z} \text{Re } F(\zeta) = \varphi(G^{-1}(z)) \quad \forall \ z \in \partial \mathbb{D}. \tag{35}
\]
Observe that $F$ either is a constant or open and discrete (see, e.g., [25, Ch. V, I.6 and II.5]). Thus, $f_{m_k} = F_{m_k} \circ G_{m_k}$ converges to $f = F \circ G$ locally uniformly as $k \to \infty$, where $f = F \circ G$ either is a constant or open and discrete. Moreover, by (35)

$$
\lim_{\zeta \to P} \Re f(\zeta) = \lim_{\zeta \to P} \Re F(G(\zeta)) = \varphi(G^{-1}(G(P))) = \varphi(P).
$$

IX. Since by VI $G$ is a homeomorphism, by [10, Theorem 1] $G$ is a regular solution of the equation (4) for some function $\mu : \mathbb{C} \to \mathbb{D}$. Since the set of points of the function $F$, where its Jacobian is zero, consist only of isolated points (see [25, Ch. V, 5.II and 6.II]), $f$ is regular solution of the Dirichlet problem (4)–(5) whenever $F \not\equiv \text{const}$. Note that the relation (6) holds for the corresponding function $K_\mu = K_{\mu_f}$ (see e.g. [10, Lemma 1]).

Therefore, $f \in \mathcal{M}_{\varphi, \Phi, z_0}^\mu (D)$.

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Zhytomyr Ivan Franko State University
Zhytomyr, Ukraine
alex dov1111111@gmail.com

Zhytomyr Ivan Franko State University
Zhytomyr, Ukraine
Institute of Applied Mathematics and Mechanics of NAS of Ukraine
Slovyansk, Ukraine
esevostyanov2009@gmail.com

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