On bistochastic Kadison-Schwarz operators on $M_2(\mathbb{C})$

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Abstract. In this paper we describe bistochastic Kadison-Schwarz operators acting on $M_2(\mathbb{C})$. Such a description allows us to find positive, but not Kadison-Schwarz operators. Moreover, by means of that characterization we are able to construct Kadison-Schwarz operators, which are not completely positive.

1. Introduction

It is known that the theory of quantum dynamical systems provides a convenient mathematical description of irreversible dynamics of an open quantum system (see [3]) investigation of various properties of such dynamical systems have had a considerable growth. In a quantum setting, the matter is more complicated than in the classical case. Some differences between classical and quantum situations are pointed out in [13]. This motivates an interest to study dynamics of quantum systems (see [13]). One of the main objects of this theory is mapping (or channel) defined on matrix algebras. One of the main constraints to such a mapping is positivity and complete positivity. There are many papers devoted to this problem (see for example [4, 9, 17, 18]). In the literature the most tractable maps, the completely positive ones, have proved to be of great importance in the structure theory of $C^\ast$-algebras. However, general positive (order-preserving) linear maps are very intractable [9, 10, 11]. It is therefore of interest to study conditions stronger than positivity, but weaker than complete positivity. Such a condition is called Kadison-Schwarz (KS) property, i.e. a map $\phi$ satisfies the KS property if $\phi(a)^\ast\phi(a) \leq \phi(a^\ast a)$ holds for every $a$. Note that every unital completely positive map satisfies this inequality, and a famous result of Kadison states that any positive unital map satisfies the inequality for self-adjoint elements $a$. But KS-operators no need to be completely positive. In [16] relations between $n$-positivity of a map $\phi$ and the KS property of certain map is established (see also [2]). Some ergodic properties of the Kadison-Schwarz maps were investigated in [8, 5, 15].

In this paper we are going to describe KS-operators which are unital, trace preserving linear mappings (i.e. bistochastic operators) defined on the algebra of 2 by 2 matrices $M_2(\mathbb{C})$. In Section 3 we provide a characterization of KS-operators form $M_2(\mathbb{C})$ to $M_2(\mathbb{C})$. In section 4, we give a sufficient condition for a class of bistochastic mappings from $M_2(\mathbb{C})$ to $M_2(\mathbb{C})$ $\otimes M_2(\mathbb{C})$ to be KS-operator. Such a description allows us to find positive, but not Kadison-Schwarz operators. Moreover, by means of that conditions we are able to construct KS-operators, which are not completely positive. Note that trace-preserving maps arise naturally in quantum information theory [6, 7, 13, 14] and other situations in which one wishes to restrict attention to a quantum system that should properly be considered a subsystem of a larger system with...
which it interacts.

2. Preliminaries
Let $M_n(\mathbb{C})$ be the algebra of $n \times n$ matrices over the complex field $\mathbb{C}$. Recall that a linear mapping $\Phi : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ is called
(i) positive if $\Phi(x) \geq 0$ whenever $x \geq 0$;
(ii) unital if $\Phi(\mathbf{1}) = \mathbf{1}$;
(iii) trace preserving if $\tau(\Phi(x)) = \tau(x)$, where $\tau$ is the normalized trace;
(iv) bistochastic if $\Phi$ is unital and trace preserving;
(v) $n$-positive if the mapping $\Phi_n : M_n(A) \to M_n(B)$ defined by $\Phi_n(a_{ij}) = (\Phi(a_{ij}))$ is positive.
Here $M_n(A)$ denotes the algebra of $n \times n$ matrices with $A$-valued entries;
(vi) completely positive if it is $n$-positive for all $n \in \mathbb{N}$;
(vii) Kadison-Schwarz operator (KS-operator), if one has
\[ \Phi(x)^* \Phi(x) \leq \Phi(x^*x) \text{ for all } x \in A. \] (1)

It is clear that any KS-operator is positive. Note that every unital 2-positive map is KS-operator, and a famous result of Kadison states that any positive unital map satisfies the inequality (1) for all self-adjoint elements $x \in A$.

By $\mathcal{KS}(M_n, M_m)$ we denote the set of all KS-operators mapping from $M_n(\mathbb{C})$ to $M_m(\mathbb{C})$.

Theorem 2.1 ([12]) The following assertions hold true:
(i) Let $\Phi, \Psi \in \mathcal{KS}(M_n, M_m)$, then for any $\lambda \in [0, 1]$ the mapping $\Gamma = \lambda \Phi + (1 - \lambda)\Psi$ belongs to $\mathcal{KS}(M_n, M_m)$. This means $\mathcal{KS}(M_n, M_m)$ is convex;
(ii) Let $U, V$ be unitaries in $M_n(\mathbb{C})$ and $M_m(\mathbb{C})$, respectively, then for any $\Phi \in \mathcal{KS}(M_n, M_m)$ the mapping $\Psi_{U,V}(x) = U\Phi(VxV^*)U^*$ belongs to $\mathcal{KS}(M_n, M_m)$.

3. Kadison-Schwarz operators from $M_2(\mathbb{C})$ to $M_2(\mathbb{C})$
In this section we are going to provide description of bistochastic mappings from $M_2(\mathbb{C})$ to $M_2(\mathbb{C})$.

It is known (see [3, 6]) that the identity and the Pauli matrices $\{\mathbf{1}, \sigma_1, \sigma_2, \sigma_3\}$ form a basis for $M_2(\mathbb{C})$, where
\[ \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \]

Every matrix $a \in M_2(\mathbb{C})$ can be written in this basis as $a = w_0 \mathbf{1} + w \cdot \sigma$ with $w_0 \in \mathbb{C}, w = (w_1, w_2, w_3) \in \mathbb{C}^3$, here by $w \cdot \sigma$ we mean the following
\[ w \cdot \sigma = w_1 \sigma_1 + w_2 \sigma_2 + w_3 \sigma_3. \]

The following facts holds (see [14]):
(a) a matrix $a \in M_2(\mathbb{C})$ is self-adjoint if and only if $w_0$ and $w$ are real;
(b) a matrix $a \in M_2(\mathbb{C})$ is positive if and only if $\|w\| \leq w_0$, where
\[ \|w\| = \sqrt{|w_1|^2 + |w_2|^2 + |w_3|^2}; \]
(c) a matrix $a \in M_2(\mathbb{C})$ is normal if and only if $[w, \overline{w}] = [\overline{w}, w]$ for every $w \in \mathbb{C}^3$, where $[\cdot, \cdot]$ stands for the cross product of vectors in $\mathbb{C}^3$. 

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Every $\Phi : M_2(\mathbb{C}) \to M_2(\mathbb{C})$ linear mapping can also be represented in this basis by a unique $4 \times 4$ matrix $F$. It is trace preserving if and only if $F = \begin{pmatrix} 1 & 0 \\ t & T \end{pmatrix}$ where $T$ is a $3 \times 3$ matrix and $0$ and $t$ are row and column vectors respectively so that

$$\Phi(\omega_0 \mathbb{I} + w \cdot \sigma) = \omega_0 \mathbb{I} + (\omega_0 t + T w) \cdot \sigma. \quad (2)$$

When $\Phi$ is also positive then it maps the subspace of self-adjoint matrices of $M_2(\mathbb{C})$ into itself, which implies that $T$ is real. A linear mapping $\Phi$ is unital if and only if $t = 0$. So, in this case we have

$$\Phi(\omega_0 \mathbb{I} + w \cdot \sigma) = \omega_0 \mathbb{I} + (T w) \cdot \sigma. \quad (3)$$

Hence, any bistochastic mapping $\Phi : M_2(\mathbb{C}) \to M_2(\mathbb{C})$ has a form (3). Now we are going to give a characterization bistochastic KS-operators.

**Theorem 3.1** ([12]) *Any bistochastic mapping $\Phi : M_2(\mathbb{C}) \to M_2(\mathbb{C})$ is KS-operator if and only if one has*

$$
\begin{align*}
\| T w \| & \leq \| w \|, \quad T w = T \mathbb{w} \quad (4) \\
\| T[w, \mathbb{w}] - [T w, \overline{T w}] \| & \leq \| w \|^2 - \| T w \|^2 \quad (5)
\end{align*}
$$

*for all $w \in \mathbb{C}^3$.*

Let $\Phi$ be a bistochastic KS-operator on $M_2(\mathbb{C})$, then it can be represented by (3). Following [6] let us decompose the matrix $T$ as follows $T = RS$, here $R$ is a rotation and $S$ is a self-adjoint matrix (see [6]). Define a mapping $\Phi_S$ as follows

$$\Phi_S(\omega_0 \mathbb{I} + w \cdot \sigma) = \omega_0 \mathbb{I} + (S w) \cdot \sigma. \quad (6)$$

Every rotation is implemented by a unitary matrix in $M_2(\mathbb{C})$, therefore there is a unitary $U \in M_2(\mathbb{C})$ such that

$$\Phi(x) = U \Phi_S(x) U^*, \quad x \in M_2(\mathbb{C}). \quad (7)$$

On the other hand, every self-adjoint operator $S$ can be diagonalized by some unitary operator, i.e. there is a unitary $V \in M_2(\mathbb{C})$ such that $S = V D_{\lambda_1, \lambda_2, \lambda_3} V^*$, where

$$D_{\lambda_1, \lambda_2, \lambda_3} = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}, \quad (8)$$

where $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}$.

Consequently, the mapping $\Phi$ can be represented by

$$\Phi(x) = \tilde{U} \Phi_{D_{\lambda_1, \lambda_2, \lambda_3}}(x) \tilde{U}^*, \quad x \in M_2(\mathbb{C}) \quad (9)$$

for some unitary $\tilde{U}$. Due to Theorem 2.1 the mapping $\Phi_{D_{\lambda_1, \lambda_2, \lambda_3}}$ is also KS-operator. Hence, all bistochastic KS-operators can be characterized by $\Phi_{D_{\lambda_1, \lambda_2, \lambda_3}}$ and unitaries. In what follows, for the sake of shortness by $\Phi_{(\lambda_1, \lambda_2, \lambda_3)}$ we denote the mapping $\Phi_{D_{\lambda_1, \lambda_2, \lambda_3}}$. It is clear to observe from (4) that $|\lambda_k| \leq 1, k = 1, 2, 3$.

Let us characterize KS operators of the form $\Phi_{(\lambda_1, \lambda_2, \lambda_3)}$. 


Theorem 3.2 ([12]) If

\[
(1 + \lambda_1^2)(3 + \lambda_2^2 + \lambda_3^2 - \lambda_1^2) \leq 4(1 + \lambda_1\lambda_2\lambda_3); \quad (10)
\]
\[
(1 + \lambda_2^2)(3 + \lambda_1^2 + \lambda_3^2 - \lambda_2^2) \leq 4(1 + \lambda_1\lambda_2\lambda_3); \quad (11)
\]
\[
(1 + \lambda_3^2)(3 + \lambda_1^2 + \lambda_2^2 - \lambda_3^2) \leq 4(1 + \lambda_1\lambda_2\lambda_3); \quad (12)
\]
\[
\lambda_1^2 + \lambda_2^2 + \lambda_3^2 \leq 1 + 2\lambda_1\lambda_2\lambda_3. \quad (13)
\]

are satisfied, then a map \( \Phi(\lambda_1, \lambda_2, \lambda_3) \) is a KS-operator.

The last theorem allows us to construct lots of KS-operators, which are not completely positive.

4. A class of Kadison-Schwarz operators from \( M_2(\mathbb{C}) \) to \( M_2(\mathbb{C}) \otimes M_2(\mathbb{C}) \)

In this section we are going to provide description of a class of bistochastic mappings from \( M_2(\mathbb{C}) \) to \( M_2(\mathbb{C}) \otimes M_2(\mathbb{C}) \).

First we need the following auxiliary

Lemma 4.1 Let \( x = w_0 \mathbf{1} \otimes \mathbf{1} + w \cdot \sigma \otimes \mathbf{1} + \mathbf{1} \otimes r \cdot \sigma \). Then the following statements hold true:

(i) \( x \) is self-adjoint if and only if \( w_0 \in \mathbb{R} \) and \( w, r \in \mathbb{R}^2 \);

(ii) \( x \) is positive if and only if \( w_0 > 0 \) and \( \|w\| + \|r\| \leq w_0 \).

Now consider the following operator \( T : M_2(\mathbb{C}) \to M_2(\mathbb{C}) \otimes M_2(\mathbb{C}) \) given by

\[
T(w_0 \mathbf{1} + w \cdot \sigma) = w_0 \mathbf{1} \otimes \mathbf{1} + A \cdot \sigma \otimes \mathbf{1} + \mathbf{1} \otimes Cw \cdot \sigma \quad (14)
\]

where \( A, C \) are linear operators on \( \mathbb{C}^2 \).

Theorem 4.2 The mapping \( T \) given by (14) is positive if and only if

\[
\|Aw\| + \|Cw\| \leq 1,
\]

for all \( w \in \mathbb{R}^3 \) with \( \|w\| = 1 \).

The proof immediately follows from Lemma 4.1.

Corollary 4.3 Let \( A = C \) then \( T \) is positive if and only if \( \|A\| \leq \frac{1}{2} \).

Let us define the following mappings

\[
\Phi(x) = w_0 \mathbf{1} + 2A \cdot \sigma, \quad \Psi(x) = w_0 \mathbf{1} + 2C \cdot \sigma \quad (15)
\]

Then one finds

\[
T(x) = \frac{1}{2} \left( \Phi(x) \otimes \mathbf{1} + \mathbf{1} \otimes \Psi(x) \right). \quad (16)
\]

Theorem 4.4 Let \( T \) be a mapping given by (16). If one has

\[
\|w\|^2 - 2\|Aw\|^2 - 2\|Cw\|^2 \geq 0 \quad (17)
\]

\[
\|A[w, w] - 2[Aw, Aw]\| + \|C[w, w] - 2[Cw, Cw]\| \leq \|w\|^2 - 2\|Aw\|^2 - 2\|Cw\|^2 \quad (18)
\]

Then \( T \) is a Kadison-Schwarz operator.
According to Lemma 4.4, we have

\[ T(x^*x) - T(x)^*T(x) = \frac{1}{2} \left( (\Phi(x^*x) - \Phi(x)^*\Phi(x)) \otimes I + I \otimes (\Psi(x^*x) - \Psi(x)^*\Psi(x)) \right) + \frac{1}{4} \left( I \otimes (\Phi(x) - \Phi(x) \otimes I) \right)^* \left( I \otimes (\Psi(x) - \Phi(x) \otimes I) \right). \]  

(19)

Now taking into account the following formula

\[ x^*x = (|w_0|^2 + \|w\|^2) I + (w_0 \bar{w} + \bar{w}_0 w - i[w, \bar{w}]) \cdot \sigma \]

from (15) we have

\[ \Phi(x^*x) - \Phi(x)^*\Phi(x) = (||w||^2 - ||2Aw||^2) I - 2i(A|w, \bar{w}| - 2[Aw, A\bar{w}])\sigma, \]

\[ \Psi(x^*x) - \Psi(x)^*\Psi(x) = (||w||^2 - ||2Cw||^2) I - 2i(C|w, \bar{w}| - 2[Cw, C\bar{w}])\sigma. \]

Therefore, one gets

\[ \left( \Phi(x^*x) - \Phi(x)^*\Phi(x) \right) \otimes I + I \otimes \left( \Psi(x^*x) - \Psi(x)^*\Psi(x) \right) \]

\[ = \left( (||w||^2 - 4||Aw||^2) I - 2i(A|w, \bar{w}| - 2[Aw, A\bar{w}])\sigma \right) \otimes I \]

\[ + \quad I \otimes \left( (||w||^2 - 4||Cw||^2) I - 2i(C|w, \bar{w}| - 2[Cw, C\bar{w}])\sigma \right) \]

\[ = \left( 2||w||^2 - 4||Aw||^2 - 4||Cw||^2 \right) I \otimes I \]

\[ - 2i(A|w, \bar{w}| - 2(Aw, A\bar{w})\sigma \otimes I - I \otimes 2i(C|w, \bar{w}| - 2[Cw, C\bar{w}])\sigma \]

According to Lemma 4.1 we conclude that the last expression is positive if and only if (17) and (18) are satisfied. Consequently, from (19) we infer that under the last conditions the mapping \( T \) is a KS operator. This completes the proof.

**Remark 4.5** We should stress that the conditions (17), (18) are sufficient to be KS-operator.

**Corollary 4.6** If the mappings \( \Phi \) and \( \Psi \) are KS operators, then \( T \) is also KS operator.

The proof immediately follows from (19).

**Remark 4.7** We have to stress that if \( T \) is KS operator, then the mappings \( \Phi \) and \( \Psi \) no need to be KS.

Let us consider a more concrete case, namely, we assume that \( Aw = \lambda w \) and \( Cw = \mu w \). Then by \( T_{\lambda,\mu} \) we denote the corresponding operator (see (14)). Then one can see that \( \Phi(2\lambda,2\lambda,2\lambda) \) and \( \Psi(2\mu,2\mu,2\mu) \) are the corresponding mappings. Due to Theorem 3.1 one can find that \( \Phi(2\lambda,2\lambda,2\lambda) \) is a KS-operator if and only if

\[ 2|\lambda|1 - 2\lambda||[w, \bar{w}]|| \leq (1 - 4\lambda^2) \|w\|^2. \]

From \(|[w, \bar{w}]| \leq \|w\|^2 \) (if we choose \( w = (0,1,i) \), one gets \(||[w, \bar{w}]|| = \|w\|^2 \) one finds

\[ 2|\lambda|(1 - 2\lambda) \leq 1 - 4\lambda^2. \]
The solution of the last inequality is \( \lambda \in \left[ -\frac{1}{4}; \frac{1}{2} \right] \).

Similarly, one finds that \( \Psi_{(2\mu, 2\mu, 2\mu)} \) is a KS-operator if and only if \( \mu \in \left[ -\frac{1}{4}; \frac{1}{2} \right] \).

From Corollary 4.6 we immediately conclude that if \( \lambda, \mu \in \left[ -\frac{1}{4}; \frac{1}{2} \right] \) then \( T_{\lambda, \mu} \) is a KS-operator.

Next we want to provide for other values of \( \lambda \) and \( \mu \) for which \( T_{\lambda, \mu} \) is Kadison-Schwarz.

**Theorem 4.8** Let \( T_{\lambda, \mu} : M_2(\mathbb{C}) \to M_2(\mathbb{C}) \otimes M_2(\mathbb{C}) \) be given by (14). If

\[
|\lambda||1 - 2\lambda| + |\mu||1 - 2\mu| \leq 1 - 2\lambda^2 - 2\mu^2
\]

is satisfied, then the map \( T_{\lambda, \mu} \) is KS-operator.

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