Induced vacuum current and magnetic field in the background of a cosmic string

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Abstract
Vacuum polarization effects in the cosmic string background are considered. We find that a current is induced in the vacuum of the quantized massive scalar field and that the current circulates around the string which is generalized to a \((d-2)\)-brane in locally flat \((d+1)\)-dimensional spacetime. As a consequence of the Maxwell equation, a magnetic field strength is also induced in the vacuum and is directed along the cosmic string. The dependence of the current and the field strength on the string flux and tension is comprehensively analysed. Both the current and the field strength are holomorphic functions of the space dimension, decreasing exponentially with the distance from the string. In the case of \(d = 3\) we show that, due to the vacuum polarization, the cosmic string is enclosed in a tube of the magnetic flux lines if the mass of the quantized field is less than the inverse of the transverse size of the string core.

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1. Introduction
Cosmic strings are topological defects which are formed as a result of phase transitions with spontaneous breakdown of symmetries in the early universe, see, e.g., reviews in [1, 2]. Starting with a random tangle, the cosmic string network evolves into two distinct sets: the stable one which consists of several long, approximately straight strings spanning the horizon volume and the unstable one which consists of a variety of string loops decaying by gravitational radiation. A straight infinitely long cosmic string in its rest frame is characterized by the stress–energy tensor with only two nonvanishing components \(T_{00} = -T_{33}\), where the third
coordinate axis is chosen to be directed along the string. The appropriate global characteristic of the string is the tension (or linear density of mass)

\[ \mu = \int_{\text{core}} d^2x \sqrt{g} T_{00}, \]  

(1)

where the integration is over the transverse section of the core of the string, and in what follows we use units \( c = \hbar = 1 \) and the metric conventions of [3]. According to general relativity, the stress–energy tensor is a source of gravity, and, consequently, the spacetime region corresponding to the string core possesses positive scalar curvature \( R = 16\pi G T_{00} \) (\( G \) is the gravitational constant), since the energy density \( T_{00} \) is positive inside this region. In the case of a cosmic string associated with spontaneous breakdown of a continuous symmetry, tension (1) is related to the mass \( m_H \) of the Higgs scalar field, \( \mu \sim m_H^2 \), and, in addition, the string is also characterized by gauge field with strength \( B^3 = \varepsilon^{ii} \partial_i V_\perp \) (\( V \) is the gauge field vector potential and \( V_0 = 0 \)) which is directed along the string and is nonvanishing inside its core [4]. The appropriate global characteristic is the flux of this gauge field strength

\[ \Phi = \int_{\text{core}} d^2x \sqrt{g} B^3; \]  

(2)

note that the transverse size of the string core is of the order of \( m_H^{-1} \). The spacetime metric outside the string core is defined by squared length element

\[ ds^2 = -dr^2 + (1 - 4G \mu)^{-1}d\vec{r}^2 + (1 - 4G \mu)d\varphi^2 + dx^2 = -dr^2 + dr^2 + r^2 d\varphi^2 + dx^2, \]

(3)

where

\[ \vec{r} = r \sqrt{1 - 4G \mu}, \quad 0 \lesssim \varphi < 2\pi, \quad 0 \lesssim \varphi < 2\pi (1 - 4G \mu). \] 

(4)

A surface which is transverse to the string is isometric to the surface of a cone with a deficit angle equal to \( 8\pi G \mu \). Such spacetimes were known a long time ago (M Fierz, unpublished, see footnote in [5]) and were studied in detail in [6, 7]. In the present context, as cosmological objects appearing after phase transitions and under the name of cosmic strings, they were introduced in seminal works of Kibble [8, 9] and Vilenkin [10, 11] (see also [12, 13]). The interest to this subject has been recently revived owing to the finding that many supergravity (superstring) models of inflation predict the production of cosmic strings at the end of inflationary phase [14–16]. Cosmic strings may also arise in other approaches such as string gas cosmology [17] and dark matter strings [18]. The nonvanishing of string tension (1) leads to various cosmological consequences and, among them, to a very distinctive gravitational lensing effect [7, 10] (see [19] for current discussion) and to a specific form of discontinuities in the temperature of the cosmic microwave background radiation [20]. Namely the latter effect imposes the upper bound on the value of tension: \( \mu \lesssim 10^{-7} G^{-1} \) (see [21]).

As already mentioned, the flux \( \Phi \) is nonvanishing for the strings corresponding to spontaneous breakdown of local (continuous) symmetries. If tension vanishes (\( \mu = 0 \)), then such a string becomes similar to a magnetic string, i.e. a tube of the magnetic flux lines in flat space. If the tube is impenetrable for quantum matter, then quantum effects outside the tube may depend on the flux \( \Phi \) periodically with period \( 2\pi e^{-1} \) (\( e \) is the coupling constant—charge of the matter particle). This is known as the Bohm–Aharonov effect [22], which has no analogue in classical physics, since the classical motion of charged particles cannot be affected by the magnetic flux from the impenetrable for the particles region. The natural question is how the nonvanishing string tension (\( \mu \neq 0 \)) influences quantum effects
of the string. Thus, the subject of cosmic strings, in addition to tantalizing phenomenological applications, acquires a certain conceptual importance.

Quantum-mechanical motion of charged particles in the background of a cosmic string was considered in [23–26]. In the present paper, we are interested in the effects of the second-quantized theory and consider the vacuum polarization which is induced by a cosmic string in quantum matter. The vacuum energy–momentum tensor in the cosmic string background was considered in [27–30], and our concern will be about the vacuum current in this background.

We shall assume that the effects of the string core structure are negligible, since the transverse size of the core is estimated to be of several orders less than the size of a proton and can be neglected; thus, our consideration will be restricted to idealized (infinitely thin) cosmic strings and some of the transverse size effects will be discussed afterwards. On the other hand, in view of the significance of high-dimensional spacetimes in various aspects, we shall deal with a generalization of a cosmic string to spaces of arbitrary dimensions. Quantum matter will be represented by the charged massive scalar field.

2. Generalization to spacetimes of arbitrary dimensions

The generalization of an idealized cosmic string to a $d$-dimensional space is given by a $(d - 2)$-brane in the conical $(d + 1)$-dimensional spacetime with squared length element

$$ds^2 = -dt^2 + dr^2 + (1 - 4G\mu)^2 r^2 d\varphi^2 + dx_{d-2}^2,$$

where $r$ and $\varphi$ are polar coordinates of the conical surface, and $x_{d-2}$ are Cartesian coordinates of flat $(d - 2)$-dimensional space; thus, the spatial curvature is nonvanishing and singular in the $(d - 2)$-brane (i.e. point in the $d = 2$ case, line in the $d = 3$ case, plane in the $d = 4$ case and $(d - 2)$-hypersurface in the $d > 4$ case). The gauge field strength (bundle curvature) is generalized to be directed along the brane, i.e. bundle connection $(V_r, V_\varphi, V_{d-2})$ is taken in the form

$$V_r = 0, \quad V_\varphi = \Phi/2\pi, \quad V_{d-2} = 0,$$

and, appropriately, bundle curvature $B^{3 \cdots d}(r, \varphi, x_{d-2})$ is nonvanishing and singular in the brane:

$$B^{3 \cdots d}(r, \varphi, x_{d-2}) = \Phi \frac{\delta(r)}{(1 - 4G\mu)r} \Delta(\varphi).$$

Here $\Phi$ is the total flux of the bundle curvature and is given by (2) with $B^{3 \cdots d}$ substituted for $B^3$; $\Delta(\varphi) = (2\pi)^{-1} \sum_{n \in \mathbb{Z}} e^{in\varphi}$ is the delta function for a compact (angular) variable, and $\mathbb{Z}$ is the set of integer numbers.

The operator of a second-quantized charged scalar field is presented in the form (see, e.g., [31])

$$\Psi(t, x) = \sum_\lambda \frac{1}{\sqrt{2E_\lambda}} \left[ e^{-iE_\lambda t} \psi_\lambda(x) a_\lambda + e^{iE_\lambda t} \psi_{\lambda'}(x) b_\lambda^{\dagger} \right].$$

Here $a_\lambda$ and $a_\lambda^{\dagger}$ ($b_\lambda$ and $b_\lambda^{\dagger}$) are the scalar particle (antiparticle) creation and destruction operators satisfying commutation relations

$$[a_\lambda, a_\lambda^{\dagger}]_\lambda = [b_\lambda, b_\lambda^{\dagger}]_\lambda = \langle \lambda | \lambda' \rangle;$$

$\lambda$ is the set of parameters (quantum numbers) specifying the state; $E_\lambda = E_{-\lambda} > 0$ is the energy of the state; symbol $\sum_\lambda$ denotes summation over discrete and integration (with a certain
measure) over continuous values of \( \lambda \); the ground state (vacuum) is defined conventionally by the relationship
\[
a_\lambda |\text{vac}\rangle = b_\lambda |\text{vac}\rangle = 0. \tag{10}
\]
Starting with the lagrangian for the complex scalar field \( \psi \) in the form
\[
L = -[(\partial_\mu - i\varepsilon V_\mu)\psi^* \gamma^\mu [\partial_\mu - i\varepsilon V_\mu] \psi] - (m^2 + \xi R)\psi^* \psi,
\]
where \( m \) is the mass of the scalar field and the coupling \( \varepsilon \), in general, differs from electromagnetic coupling \( e \), and assuming the ultrastaticity of the metric \((g_{00} = 1 \text{ and } \partial_0 g_{ij} = 0) \text{ and the bundle (}V_0 = 0 \text{ and } \partial_0 V_\mu = 0)\), we define functions \( \psi_\lambda(x) \) forming a complete set of solutions to the stationary Klein–Gordon equation
\[
[-g^{-1/2}(\partial_\mu - i\varepsilon V_\mu)g^{1/2}g^{\mu\nu}(\partial_\nu - i\varepsilon V_\nu) + m^2 + \xi R] \psi_\lambda(x) = E^2_\lambda \psi_\lambda(x), \tag{11}
\]
and obeying the orthonormalization condition
\[
\int d^d x \sqrt{\gamma} \psi_\lambda^*(x) \psi_{\lambda'}(x) = (\lambda |\lambda'), \tag{12}
\]
where \( g = \det g_{\mu\nu} \).

In the case of spacetime (5) with bundle connection (6), the stationary Klein–Gordon equation outside the brane (at \( r \neq 0 \)) takes the form
\[
\left[-r^{-1}\partial_r \partial_r - (1 - 4G\mu)^{-2}r^{-2} \left( \partial_\rho - \frac{i\varepsilon \Phi}{2\pi} \right)^2 - \Delta_{d-2} + m^2 \right] \psi_{\text{exp}}(r, \rho, x_{d-2}) = (p^2 + k^2 + m^2) \psi_{\text{exp}}(r, \rho, x_{d-2}), \tag{13}
\]
where \( 0 < k < \infty, n \in \mathbb{Z}, -\infty < p^j < \infty (j = 1, d - 2) \). Imposing the Dirichlet boundary condition on the brane\(^3\),
\[
\psi_{\text{exp}}(0, \rho, x_{d-2}) = 0, \tag{14}
\]
and normalizing in consistency with (12), one gets the solution to (13):
\[
\psi_{\text{exp}}(r, \rho, x_{d-2}) = \frac{(2\pi)^{(1-d)/2}}{\sqrt{1 - 4G\mu}} J_{|n-\varepsilon \Phi/2\pi|/(1-4G\mu)^{-1}}(kr) e^{i\varepsilon \Phi x_{d-2}}, \tag{15}
\]
where \( J_n(u) \) is the Bessel function of order \( n \). As a consequence of (14), there is no overlap between the region where the matter is quantized \((r \neq 0) \) and the region where the spatial and the bundle curvatures are nonvanishing \((r = 0) \). In this sense, the brane is impenetrable for quantum matter.

### 3. Vacuum current

The vacuum current of the scalar field is given by the expression
\[
J(x) = \frac{1}{4i} \langle \text{vac} | [\{ \Psi^\dagger(t, x), \nabla \Psi(t, x) \}]_e - [\{ \nabla \Psi^\dagger(t, x), \Psi(t, x) \}]_e |\text{vac}\rangle, \tag{16}
\]
where symbol \([\ldots, \ldots, \ldots]\) stands for the anticommutator, and \( \nabla \) is the covariant derivative over spatial coordinates. Using (8) and (15), we get \( j_\rho = j_{d-2} = 0 \) and
\[
j_\mu(r) = (2\pi)^{1-d} \int d^{d-2} p \int_0^{\infty} dk \ k (p^2 + k^2 + m^2)^{-1/2}
\times \sum_{n \in \mathbb{Z}} \langle n - \varepsilon \Phi/2\pi \rangle \frac{1}{1 - 4G\mu} J^2_{|n-\varepsilon \Phi/2\pi|/(1-4G\mu)^{-1}}(kr). \tag{17}
\]
\(^3\) The most general boundary condition implies that the scalar field is divergent but square integrable at \( r = 0 \). This condition involves at least four additional parameters entailing a contact interaction with the brane (see [32, 33] for the case of \( d = 2 \text{ and } \mu = 0 \)) and will be considered elsewhere. Condition (14) corresponds to a particular choice of the parameter values, when the contact interaction vanishes.
As a manifestation of the Bohm–Aharonov effect, the vacuum current is a periodic function of the flux $\Phi$ with a period equal to $2\pi \tilde{e}^{-1}$, i.e. it depends on the quantity

$$F = \frac{\tilde{e} \Phi}{2\pi} \mod \frac{\tilde{e} \Phi}{2\pi}.$$  

where $\|u\|$ denotes the integer part of the quantity $u$ (i.e. the integer which is less than or equal to $u$). Note that relation (18) can be rewritten as

$$F = \frac{\tilde{e}}{\tilde{e}_H} \mod \frac{\tilde{e}}{\tilde{e}_H}.$$  

where $\tilde{n} \in \mathbb{Z}$ and $\tilde{e}_H$ is the coupling of the Higgs scalar field to bundle connection (6), since the values of the flux are quantized as $\Phi = 2\pi \tilde{n} \tilde{e}_H^{-1}$ [4].

Using the relation

$$J_\omega^2(u) = \frac{1}{2\pi} \left[ I_\omega(-iu) K_\omega(-iu) - I_\omega(iu) K_\omega(iu) \right]$$

($I_\omega(u)$ and $K_\omega(u)$ are the modified Bessel functions with the exponential increase and decrease at large real positive values of their argument), we get

$$j_\phi(r) = \frac{1}{2\pi} (2\pi)^{1-d} \int d^{d-2} p \int_{-\infty}^{\infty} dk \frac{k (p^2 + k^2 + m^2)^{-1/2}}{\kappa \kappa (\kappa^2 - p^2 - m^2)^{-1/2}} \times \sum_{n \in \mathbb{Z}} \nu(n - F) I_{|n-F|}(-ikr) K_{|n-F|}(-ikr),$$  

where the notation

$$\nu = (1 - 4G\mu)^{-1}$$

is used for brevity. The integral over real $k$ can be transformed into the integral over a contour circumventing a part of the imaginary axis in the complex $k$-plane, see figure 1. As a result, we get

$$j_\phi(r) = \frac{4}{(2\pi)^d} \int d^{d-2} p \int_{p^2+m^2}^{\infty} dk \frac{k (k^2 - p^2 - m^2)^{-1/2}}{\kappa \kappa (\kappa^2 - p^2 - m^2)^{-1/2}} \sum_{n \in \mathbb{Z}} \nu(n - F) I_{|n-F|}(-\kappa r) K_{|n-F|}(-\kappa r)$$

$$= \frac{2}{(2\pi)^d} \int d^{d-2} p \int_{p^2+m^2}^{\infty} dk \frac{k (k^2 - p^2 - m^2)^{-1/2}}{\kappa \kappa (\kappa^2 - p^2 - m^2)^{-1/2}} \int_0^{\infty} \frac{dy}{y} \exp\left(-\frac{\kappa^2 r^2}{2y} - y\right)$$

$$\times \sum_{n \in \mathbb{Z}} \nu(n - F) I_{|n-F|}(y),$$  

(22)
where in the last step the integral representation for the product of modified Bessel functions is used (see, e.g., [34]). Using the Schl"afli contour integral representation

\[ I_\nu(y) = \frac{1}{2\pi i} \int_{C_+} dz \, e^{y \cosh z - \omega z} = -\frac{1}{2\pi i} \int_{C_-} dz \, e^{y \cosh z + \omega z}, \]

we perform summation over \( n \) and get

\[
\sum_{n \in \mathbb{Z}} v(n - F) I_{\nu(n - F)}(y) = -\frac{y}{4\pi i} \int_{C_0} dz \, e^{y \cosh z} \sinh \frac{(F - \frac{1}{2})vz}{\sinh(vz/2)} = -\frac{y}{\pi} \int_{0}^{\infty} du \, e^{-y \cosh u} \sinh u \Lambda \left( \frac{u}{2}; F, v \right),
\]

(23)

where the contours \( C_+, C_- \) and \( C_0 \) in the complex \( z \)-plane are shown in figure 2, and

\[
\Lambda(v; F, v) = \frac{\sin(Fv\pi) \sinh[2(1 - F)v \arccosh v] - \sin[(1 - F)\nu\pi] \sinh(2Fv \arccosh v)}{\cosh(2Fv \arccosh v) - \cos(v\pi)}.
\]

(24)

Substituting the last line of (23) into (22), integrating over the variables \( \kappa \) and \( y \), we get

\[
\mathbf{j}_\nu(r) = -\frac{2}{(2\pi)^d} \sqrt{\frac{r}{\pi}} \int dp \left[ (p^2 + m^2)^{d/2} \right]^{3/4} \int_{0}^{\infty} du K_{d/2} \left( 2r \sqrt{p^2 + m^2} \cosh \frac{u}{2} \right)
\]

\[
\times \frac{\sinh \frac{u}{2}}{\cosh \frac{u}{2}} \Lambda \left( \frac{u}{2}; F, v \right).
\]

Integrating over \( p \) and changing the variable \( u \) to \( v = \cosh \frac{u}{2} \), we get the final expression for the vacuum current

\[
\mathbf{j}_\nu(r) = -\frac{32}{(4\pi)^{d+3/2}} \frac{m^{(d+1)/2}}{r^{(d-3)/2}} \int_{1}^{\infty} dv \left[ v^{(1-d)/2} K_{(d+1)/2} (2mr) \Lambda(v; F, v) \right].
\]

(25)
The vacuum current vanishes in the case of vanishing flux \((\Phi = 0)\), while in the case of vanishing tension \((\mu = 0)\) we get
\[
\Lambda(v; F, v) = -v^{-1} \sin(F \pi) \sinh(2F - 1) \arccosh v,
\]
(26)
and the result of [35] is recovered:
\[
j_\phi(r) = \frac{32 \sin(F \pi)}{(4\pi)^{(d+1)/2}} m^{(d+1)/2} \int_1^\infty dv \, v^{-(d+1)/2} K_{(d+1)/2}(2mv) \sinh[(2F - 1) \arccosh v], \quad v = 1.
\]
(27)

Using the asymptotics of the Macdonald function at large values of the argument,
\[
K_\nu(u) \approx e^{-\pi \frac{\pi}{2\nu}} [1 + O(u^{-1})],
\]
we get the asymptotics of the vacuum current at large distances from the brane:
\[
j_\phi(r) = \frac{v(F \sin(1 - F)v \pi) - (1 - F) \sin(F \nu \pi)}{(4\pi)^{(d+1)/2} \sin^2(\nu \pi/2)} e^{-2m r m^{(d-3)/2} r^{-(d+1)/2}} \left[1 + O[(mr)^{-1}].\right]
\]
(28)

4. Vacuum magnetic field

Assuming the validity of the Maxwell equation
\[
\frac{1}{(d-2)!} \nabla_i e^{i\theta_1 \cdots \theta_d} B_{\theta_1 \cdots \theta_d}^{(d-2)} = e j^i,
\]
(29)
one can deduce that, owing to the induced angular component of the vacuum current, a magnetic field strength is also induced in the vacuum outside the brane, being directed along it:
\[
B^{3, d}_i (r) = \int_r^\infty dr \frac{v}{r} e j_\phi(r);
\]
(30)

\[\text{note that the coupling constant } e \text{ possesses dimension } m^{(3-d)/2} \text{ in } (d + 1)\text{-dimensional spacetime. Substituting (25) into (30), we get the expression}
\]
\[
B^{3, d}_i (r) = -\frac{16ev}{(4\pi)^{(d+3)/2}} \left(\frac{m}{r}\right)^{(d-1)/2} \int_1^\infty dv \, v^{-(d+1)/2} K_{(d-1)/2}(2mr) \Lambda(v; F, v),
\]
(31)

which decreases at large distances from the brane as
\[
B^{3, d}_i (r) \approx \frac{e v^2 [F \sin(1 - F)v \pi) - (1 - F) \sin(F \nu \pi)]}{2(4\pi)^{(d+1)/2} \sin^2(\nu \pi/2)} e^{-2m r m^{(d-3)/2} r^{-(d+1)/2}} \times \left[1 + O[(mr)^{-1}].\right]
\]
(32)

Consequently, the magnetic flux is induced in the vacuum outside the brane:
\[
\Phi^{(1)} = \int_0^{2\pi} d\phi \int_0^\infty dv \frac{v}{v} B^{3, d}_i (r)
\]
\[\approx -\frac{8em^{(d-1)/2}}{(4\pi)^{(d+1)/2}} \int_1^\infty dv \, v^{-(d+1)/2} \Lambda(v; F, v) \int_0^\infty dr \, r^{(3-d)/2} K_{(d-1)/2}(2mr v).
\]
(33)

In the case of \(Re \, d < 3\), we get (see the next section for details and compare with relation (41))
\[
\Phi^{(1)} = \frac{e m^{d-3} \Gamma\left(\frac{3-d}{2}\right)}{3(4\pi)^{(d-1)/2}} \left(F - \frac{1}{2}\right) F(1 - F)v^2.
\]
(34)
where $\Gamma(u)$ is the Euler gamma function; this result can be analytically extended to a meromorphic function on the whole complex $d$-plane, which in the physically meaningful domain $\text{Re } d \geq 2$ has poles at real odd values of $d$ and is finite at real even values of $d$.

However, it seems more reasonable to restrict the validity of (34) to the case of $d = 2$,

$$\Phi^{(l)} = \frac{e}{6\pi} \left( F - \frac{1}{2} \right) F(1 - F) v^2, \quad d = 2,$$

while in the cases of $d = 3, 4, \ldots$ cutoff $r_0$ at small distances to the brane should be introduced:

$$\Phi^{(l)} = \frac{e}{6\pi} \left( F - \frac{1}{2} \right) F(1 - F) v^2 (-\ln m r_0), \quad d = 3,$$

$$\Phi^{(l)} = -\frac{2e \Gamma \left( \frac{d-1}{2} \right)}{(4\pi)^{(d+1)/2}} \int_0^\infty \! v^{-d} \Lambda(v; F, v), \quad d > 3;$$

the integral in (37) can be explicitly taken in the case of real odd values of $d$, see the next section.

5. Massless quantum matter

Using the asymptotics of the Macdonald function at small values of the argument, we get the vacuum current for the massless scalar field:

$$j_\psi(r)|_{m=0} = -\frac{16}{(4\pi)^{(d+3)/2}} e^{-d+1} \int_1^\infty \! v^{-d} \Lambda(v; F, v). \quad (38)$$

In the case of vanishing tension, the integral in (38) is taken, yielding [35]

$$j_\psi(r)|_{m=0} = \frac{4 \sin(F\pi)}{(4\pi)^{(d+2)/2}} e^{-d+1} \left( F - \frac{1}{2} \right) \frac{\Gamma \left( \frac{d+1}{2} + F \right) \Gamma \left( \frac{d+1}{2} - F \right)}{\Gamma \left( \frac{d}{2} + 1 \right)}, \quad v = 1. \quad (39)$$

In the case of nonvanishing tension, the integral in (38) is explicitly taken for odd values of $d$ only [28]. To see this, let us use the contour integral representation given by the first line in (23) and present (38) in another way:

$$j_\psi(r)|_{m=0} = \frac{\Gamma \left( \frac{d+1}{2} \right)}{(2\pi)^{(d+1)/2}} e^{-d+1} \frac{1}{4\pi i} \int_C \! dz \frac{\sin z}{(1 - \cosh z)^{(d+1)/2}} \frac{\sin \left[ \left( F - \frac{1}{2} \right) vz \right]}{\sinh(vz/2)}. \quad (40)$$

If $d$ is odd, then the integrand in (40) has a pole at $z = 0$ and the contour $C_0$ is deformed to encircle this pole, yielding

$$j_\psi(r)|_{m=0} = \frac{1}{6\pi^2} e^{-d} \left( F - \frac{1}{2} \right) F(1 - F) v^2, \quad d = 3,$$

$$j_\psi(r)|_{m=0} = \frac{1}{12\pi^3} e^{-d} \left( F - \frac{1}{2} \right) F(1 - F) v^2 \left\{ \frac{1}{3} + \frac{4}{7} \left[ \frac{1}{3} + F(1 - F) \right] v^2 \right\}, \quad d = 5, \quad (42)$$

$$j_\psi(r)|_{m=0} = \frac{1}{120\pi^5} e^{-d} \left( F - \frac{1}{2} \right) F(1 - F) v^2 \times \left\{ \frac{4}{3} + \left[ \frac{1}{3} + F(1 - F) \right] v^2 + \frac{1}{7} \left[ \frac{1}{3} + F(1 - F) + F^2(1 - F) \right] v^4 \right\}, \quad d = 7, \quad (43)$$
\[
\begin{align*}
\left. j_\nu(r) \right|_{m=0} &= \frac{1}{560\pi^5} r^{-8} \left( F - \frac{1}{2} \right) F(1-F)v^2 \\
&\times \left\{ 4 + \frac{49}{15} \left[ \frac{1}{3} + F(1-F) \right] v^2 + \frac{2}{3} \left[ \frac{1}{3} + F(1-F) + F^2(1-F)^2 \right] v^4 \\
&\quad + \frac{1}{15} \left[ \frac{1}{3} + F(1-F) + \frac{10}{9} F^2(1-F)^2 + \frac{5}{9} F^3(1-F)^3 \right] v^6 \right\}, \\
&d = 9, \quad (44)
\end{align*}
\]
and so on; note that (41) was first obtained in [36]. If \( d \) is even, then the integrand in (40) has a branch point singularity at \( z = 0 \).

Using (30), we get the expression for the vacuum magnetic field strength:
\[
\left. B_{\nu}^{1, d}(r) \right|_{m=0} = -\frac{8\nu\Gamma\left(\frac{d-1}{2}\right)}{(4\pi)^{(d+1)/2}} r^{-d+1} \int_1^\infty dv \, v^{-d} \Lambda(v; F, v). \quad (45)
\]
Thus, the induced vacuum flux is divergent either at small or at large distances from the brane, or both. In the case of \( d = 2 \), introducing cutoff \( r_\infty \) at large distances from the brane, we get
\[
\left. \Phi^{(1)} \right|_{m=0} = -\frac{e}{2\pi} r_\infty \int_1^\infty dv \, v^{-2} \Lambda(v; F, v), \quad d = 2. \quad (46)
\]
In the cases of \( d = 4, 5, \ldots \), introducing cutoff \( r_0 \) at small distances to the brane, we get relation (37). In the case of \( d = 3 \), the vacuum flux is logarithmically divergent both at small and at large distances from the brane, and introducing both cutoffs, we get
\[
\left. \Phi^{(1)} \right|_{m=0} = \frac{e}{6\pi} \left( F - \frac{1}{2} \right) F(1-F)v^2 \ln \frac{r_\infty}{r_0}, \quad d = 3. \quad (47)
\]

### 6. Discussion

In the present paper, we have obtained integral representation (25) for the induced vacuum current of the quantized charged massive scalar field in the cosmic string background. All the dependence on the cosmic string parameters (tension \( \mu \) and flux \( \Phi \)) is described by the function \( \Lambda \) (24), whereas the dependence on the spatial dimension \( d \) is factored otherwise. The Bohm–Aharonov effect is manifested in the periodic dependence of the vacuum current on the flux \( \Phi \) with period \( 2\pi \hat{e}^{-1} \), i.e. expression (25) depends on the fractional part of \( e\Phi/2\pi \) (18) rather than on the quantity \( e\Phi/2\pi \) itself. Expression (25) vanishes at \( F = 1/2 \) (i.e. at \( \hat{n} \hat{e} = (n + 1/2)\hat{e}_H \), where \( n \in \mathbb{Z} \) and \( \hat{e}_H \in \mathbb{Z} \)), being negative and convex downwards in the interval \( 0 < F < 1/2 \), while positive and convex upwards in the interval \( 1/2 < F < 1 \); positions of the minimum and the maximum are symmetric with respect to the point \( F = 1/2 \). Expression (25) can be analytically extended to complex values of the space dimension yielding a holomorphic function of the complex variable \( d \).

According to the Maxwell equation a magnetic field is generated by a direct current, and thus the magnetic field strength is also induced in the vacuum, being directed along the cosmic string, see (31). The dependence of the field strength on the cosmic string parameters and on the spatial dimension is qualitatively similar to that of the current. Both the current and the field strength decrease exponentially at large distances from the cosmic string, see (28) and (32). Both the current and the field strength increase as \( r^{-d+1} \) in the vicinity of the cosmic string, see (38) and (45). Thus, the induced vacuum magnetic flux is finite in the case of \( d = 2 \) only, see (35), whereas divergent otherwise, and a small vicinity of the cosmic string is eliminated by cutoff \( r_0 \) in the cases of \( d > 2 \), see (36) and (37); note that the flux is hence
independent of mass $m$ in the cases of $d > 3$. In the case of $d = 3$, by identifying the cutoff with the transverse size of the core of the cosmic string, $r_0 \simeq m_H^{-1}$, we get

$$\Phi^{(1)} = \frac{e}{6\pi} \left( F - \frac{1}{2} \right) F(1 - F) v^2 \ln \frac{m_H}{m}, \quad d = 3. \quad (48)$$

One can conclude that, if the mass of the quantized scalar is smaller than the mass of the Higgs scalar, $m \ll m_H$, then the cosmic string is enclosed in a tube of the magnetic flux lines with the transverse size of the order of $(2m)^{-1}$.

It should be noted that the vacuum current and, consequently, the vacuum magnetic field are odd under charge conjugation. The production of cosmic magnetic fields during cosmological phase transitions is currently discussed in the context of fundamental CP violation in particle physics, see [37]. As follows from our consideration, topological defects (cosmic strings) appearing in the aftermath of phase transitions can produce magnetic fields via the vacuum polarization.

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