Scaling Behavior of the Hirsch Index for Paper Citations, Failure Avalanches and Percolation Clusters

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A popular measure for the success or citation inequalities of individual scientists have been the Hirsch index ($h$). If the number $n_c$ of citations are plotted against the number $n_p$ of papers (having those citations), then $h$ corresponds to the fixed point (where $n_c = h = n_p$) of the above-mentioned citation function. There have been theoretical as well as numerical studies suggesting $h \sim \sqrt{N_c} \sim \sqrt{N_p}$, for any author having $N_p = \sum n_p$ total papers and $N_c = \sum n_c$ total citations. With extensive data analysis here, we show that $h \sim \sqrt{N_c/\log N_c} \sim \sqrt{N_p/\log N_p}$. Our numerical study also shows that the $h$-index values for size distribution of avalanches in equal load-sharing fiber bundle models (of materials failure), consisting of $N$ fibers, also scale as $\sqrt{N/\log N}$. We however observe a notable discrepancy in the scaling relation of $h$ for the size distribution of clusters near the percolation point of square lattice (indicating that the scaling behavior depends on the dimension). We also show that if the number $N_s$ of the members of parliaments or national assemblies of different countries of the world are identified effectively as their Hirsch index $h$, then $N_s$ indeed scales with the population $N$ as $\sqrt{N/\log N}$ for the respective countries and this observation may help comprehending the discrepancies with $\sqrt{N}$ relationship, reported in a very recent analysis.

I. INTRODUCTION

Monotonically and nonlinearly decaying inequality functions are ubiquitous. When the papers by any author (or an institution) are arranged from highest cited one to the lowest, it becomes a monotonically decaying nonlinear function (see e.g., [1]). The same is true for avalanches in materials failure or in earthquakes (see e.g., [2]), cluster size distributions in percolation problems (see e.g., [3]), etc.. Large avalanches, strong quakes, or big size clusters come or occur in small numbers, while smaller or weaker the avalanches or quakes, or clusters, larger are their abundance in occurrence.

How does one measure statistically these inequalities in appreciations or occurrence frequencies of citations or avalanche or cluster sizes? Obviously, the corresponding distribution functions for inequalities in citations or sizes would contain the entire statistics. However, they are not convenient to handle. One can consider the citation number of the best cited paper (as some times done for some unique awards, etc), or study the statistical (self-similar) structure of the biggest avalanche or the largest (percolating) cluster (as in statistical physics [3]). Hirsch proposed [1] another (social) measure of these inequalities, by locating the fixed point of the nonlinear inequality function. The Hirsch index ($h$) corresponds to the citation number or occurrence frequency which is commensurate in magnitude with the number of publications size avalanche cluster.

How does the $h$ index scale with total number of publications ($N_p$) by the author (institution) or the total number ($N_c$) of citations received by the author (institution)? Young [4] suggested analytically that $h$-index value should scale with the total number of citations $N_c$ as $\sqrt{N_c}$ asymptotically. In
ref. [5], a brief analysis of the Google Scholar data indicated independently that $h \sim \sqrt{N_p}$ for large values of $N_p$, where $N_p$ denotes the total number of publications by the author. Note that if both these relationships are valid then, statistically speaking, for a prolific author, the total number of citations would be linearly proportional to the total number of papers published and the proportionality constant would perhaps be determined by the size of the existing author network in the subject (suggesting an effective Dunbar number [6, 7], for the community of authors).

In a recent Monte Carlo study [8] on the avalanche sizes and their numbers in the Fiber Bundle Models (FBM) of materials failure (see e.g., [8, 9]) due to increasing stress on such bundles, the numerical analysis of the data for the nonlinearly decaying numbers of avalanches with their sizes (or released elastic energies) suggested $h \sim \sqrt{N/\log N}$.

Our detailed study of Google Scholar data, numerical study of avalanches in FBMs and of size distributions for clusters near the percolation point of square lattices, all indicate here that the Hirsch index $h$ indeed scales with $\sqrt{N/\log N}$, where $N$ corresponds to the number of papers or citations or fibers (in FBM) or lattice size. This observation (of the log correction) for the author citation $h$-index also indicates that the effective Dunbar number for the author networks are not strictly defined.

Also we find, if the number $N_s$ of the members of parliaments or national assemblies of different countries of the world are identified effectively as the Hirsch index $h$, then the above scaling relation of $h$ suggest that $N_s$ scale as $\sqrt{N/\log N}$ with population $N$ of the respective countries of the world. This can help to comprehend the discrepancies (c.f. [10]) with the $\sqrt{N}$ relationship, reported in a very recent analysis [11].

II. DATA ANALYSIS & NUMERICAL STUDIES

In order to indicate how the $h$ index value is obtained, say, for an individual scientist, we give in Fig. 1 a cartoon of the citation function of a typical scientist. The $h$-index is given by its fixed point or the intersection point of the 45 degree line with the citation or paper axes. At that intersection point, the citation number equals the number of papers and both becomes equal to $h$. At the intersection...
point, the number \( n_c \) of citations becomes equal to the number \( n_p \) of papers (having those citations) and both become equal to \( h \). We will study here the scaling behavior of \( h \) with \( N_c (\equiv \Sigma n_c) \), the total citations, and with \( N_p (\equiv \Sigma n_c) \), the total number of papers, by a scientist.

Similarly, for the mechanical failure in non-brittle materials (see e.g., [2, 9]), there occur precursor avalanches or bursts of elastic energy (usually measured using the ultrasonic emission spectroscopy). Here also the big sizes avalanches or bursts are lower in numbers while smaller the avalanche sizes, the avalanches come larger in numbers. Hence, to estimate the avalanche \( h \)-index for material failure, citations can be replaced by the avalanche sizes and the paper numbers are replaced by the the numbers of such avalanches. The size distribution of clusters in the percolation problem is an well studied one (see e.g., [3]). Here also the largest or percolating cluster is unique and come alone, while smaller clusters appear larger in numbers. To extract the \( h \)-index then, citations in Fig. 1 can be similarly replaced by the cluster sizes and the paper numbers are replaced by the numbers of such clusters in a pre-percolating system.

We first analyze the data for \( h \)-index and its scaling with the total number of publications \( N_p \) and of citations \( N_c \) for the one hundred scientists (in mathematics, physics, chemistry, medicine, biology, economics, sociology; including those of twenty Nobel Laureates in those subjects), given in ref. [5]. Next, we collected (in May-June, 2021) the same kind of data for one thousand scientists (mostly physicists) in all the above-mentioned subjects from Google Scholar. These data (for \( h, N_c \) and \( N_p \)) were then fitted to the scaling forms: power law (a) \( h \sim N^\gamma \) and that with log correction (b) \( h \sim N^\gamma / \log N \), where \( N \) represents either \( N_c \) or \( N_p \) of the individual scientists. We then calculated the standard deviations \( \sigma = \sqrt{\Sigma_i (h_d(i) - h_f(i))^2/\Sigma_i 1} \), where \( i \) represents the individual scientists with the corresponding \( h \) value from the data set (\( h_d \)) or that obtained using the scaling fit (\( h_f \)). We look for the best fit values of prefactors to both the scaling forms with different values of the exponent \( \gamma \). We then search for the value of \( \gamma \) to get the most reasonably good fit scaling form. These are shown in Figs. 2 and 3(a, b), where the main figure gives the best fit and the inset gives the \( \sigma \) values.

\footnote{\textsuperscript{1} The data will be available on request to corresponding author}
for different values of $\gamma$.

Using similar approach, we show here (see also [8]) that the breaking point avalanche size $h$-index values in a $N$ fiber equal-load-sharing Fiber bundle Model (FBM) having uniform distribution of fiber breaking thresholds [9], scale better (see Fig. 4) with the $h \sim \sqrt{N_c}/\log N_c$ scaling than with power law scaling. The cluster size distribution in the percolation problem has also been studied for estimating the $h$-index scaling with the total number $N$ of lattice sites at the percolation threshold of site percolation on square lattice. We of course find here the best fit scaling form to be $h \sim N^{0.54}/\log N$.
FIG. 4. Scaling behavior for the average $h$-index (in the range $4 \leq h \leq 20$) at the breaking point of the bundles with total number $N$ (in the range $200 \leq N \leq 20000$) of the fibers in the equal-load-sharing FBM (with uniform distribution of fiber breaking thresholds) considered here (cf. [8]). The Fig. shows the the best fit of $h$ to $\sqrt{N_c}/\log N_c$. The inset shows the standard deviations $\sigma$ of the $h$-data fits to power law $N^\gamma$ and that with log correction $N^\gamma/\log N$.

FIG. 5. Scaling behavior for the $h$-index (in the range $20 \leq h \leq 4421$) with the total number (or size) $N$ of sites (of the square lattice) at the site percolation point ($p_c = 0.5927$); lattice size $N$ ranges from $10^4$ to $10^9$. The Fig. shows the the best fit of $h$ to $N^{0.54}/\log N_c$. The inset shows the standard deviations $\sigma$ of the $h$-data fits to power law $N^\gamma$ and that with log correction $N^\gamma/\log N$. It indicates, the value of the exponent $\gamma$ depends on the effective dimension of the system considered and while its value remains 0.50 for all the other long-range or infinite dimensional systems considered here, it value changes to about 0.54 for such two dimensional (nearest-neighbor or short-range) percolation problem.
FIG. 6. Scaling behavior for the number $N_s$ (in the range $15 \leq N_s \leq 3040$) of the representatives in National assemblies or Parliaments of different countries of the world (1972 data from the original study [10]; see also [11]) with total population $N$ (in the range $19 \times 10^4 \leq N \leq 70.5 \times 10^7$) of the respective countries. Here $N_s$ can be identified as the $h$ index for the countries, and the Fig. shows the best fit of $N_s$ to $\sqrt{N}/\log N$. The inset shows the standard deviations $\sigma$ of the $h$-data fits to power law $N^\gamma$ and that with log correction $N^\gamma/\log N$. This resolves the discrepancy noted in the analysis of the same data in ref. [11].

(see Fig. 5). The inset shows the standard deviations $\sigma$ of the $h$-data fits to power law $N^\gamma$ and that with log correction $N^\gamma/\log N$. We believe, this short-range two dimensional percolation considered here corresponds to this different (from $\gamma = 0.5$) scaling exponent, as the effective dimension of the citation networks, as well as that of the equal-load-sharing FBM are infinite.

Finally, we note that if the number $N_s$ of the representatives in the National Assemblies or in the Parliaments of different countries of the world are identified as the $h$-index for respective country, having population $N$, then we find (see Fig. 6) that the data analyzed in ref [10] shows the best fit of $N_s$ to $\sqrt{N}/\log N$. This also resolves the discrepancy noted in the analysis if the same data ([11] and references therein).

III. SUMMARY AND CONCLUSION

As discussed already, the Hirsch index ($h$) has become an important social measure of the inequalities in success (through citation in the subsequent literature) of individual scientists. The citation function, giving the number $n_c$ of citations plotted against the number $n_p$ of papers having those citations, is an well documented nonlinear and monotonically decaying function (cf. Zipf law [12]); $n_p$ becomes higher, lower the citations $n_c$. The Hirsch index $h$ corresponds to the fixed point (where $n_c = h = n_p$) of these nonlinear citation functions. Analytical [4] as well as numerical (see e.g., [5]) studies suggested $h \sim \sqrt{N_c} \sim \sqrt{N_p}$, for any author having $N_p = \Sigma n_p$ total papers and $N_c = \Sigma n_c$ total citations. With extensive data analysis here (see section II, Figs. 2 a, b)), we show that a much better fit (with considerably smaller standard deviations, see the insets in the respective figures) is $h \sim \sqrt{N_c}/\log N_c \sim \sqrt{N_p}/\log N_p$.

The data extracted from this paper can be found in https://sciencehistory.epfl.ch/physics-and-sociology/
It may be noted that while some earlier analysis with small data sets (e.g., in [13, 14]) indicated a fit to $h \sim N^\gamma$, with $\gamma = 0.5$, analysis of a much larger data set [15] indicated significantly lower value ($\simeq 0.42$) of $\gamma$. As may be seen from the analysis (of $\sigma$ values) in Figs. 2 and 3 this is probably due to the missing log correction term in the scaling relation and we believe the data in [15] would fit well with $h \sim \sqrt[4]{N}/logN$.

Next, in order to check the validity of such a scaling relation for the $h$-index in physical systems, we studied numerically the effective $h$ index for the avalanche statistics of the equal load-sharing fiber bundle model (FBM) of materials failure, where the frequencies avalanches also increase similarly with decreasing avalanche sizes. In such bundles, $N$ fibers or Hooke springs, having identical spring constant but random breaking thresholds, support a gradually increasing external load. As some elements fail, the extra load coming from the failure of the broken fiber(s) are distributed equally among the surviving ones and if this extra share of load causes further failure(s), the avalanche continues. The estimated $h$-index values from such size distributions of avalanches are seen (see Fig. 3) to scale as $\sqrt[4]{N}/logN$. We however observe a notable discrepancy (see Fig. 4 in section II) in the scaling relation of $h$ for the size distribution of clusters near the percolation point of square lattice. This probably indicates that the scaling behavior depends on the dimension, as the effective dimensions of the author collaboration or citation network as well as that of the equal load sharing FBM is infinity, while the percolation statistics on square lattice studied here is for two dimension.

We also show that when the number $N_s$ of the members of national assemblies or of parliaments for different countries of the world are identified effectively as their Hirsch index $h$, then $N_s$ indeed scales (see Fig. 4 in section II) with the total population $N$ as $\sqrt[4]{N}/logN$ for the respective countries. This observation should help comprehending the discrepancies with the proposed $N_s \sim \sqrt[4]{N}$ relationship, reported in a very recent analysis [11].

In this paper we explored the best fit of the Hirsch index values $h$ with its system size $N$ scaling form $N^\gamma$ or that with log correction $N^\gamma/logN$, essentially based on the calculation of the error or standard deviation $\sigma$ in fitting them with the data. That too, with the exponent $\gamma = 1/2$ as a very forceful and analytically guided conjecture (see e.g., [4]). However, we also note the fact that a few of the outliers in the data set often appear to contradict any of these choices of scaling form. For example, the minimum values of $\sigma$ in Figs. 3 and 4 suggest a strict power law fit with $\gamma$ values far away from $1/2$. However, with less than one percent sacrifice in the errors ($\sigma$), fit of $h$ to $\sqrt[4]{N}/logN$ appears very reasonable even in these cases.

We thus find that in contradiction to the analytical [4] and numerical findings regarding the square root variation of the Hirsch index $h$ with the total number of citations $N_c$ or the total number of publications $N_p$ by the scientists, a log correction term is needed: $h \sim \sqrt[4]{N_c}/logN_c \sim \sqrt[4]{N_p}/logN_p$ (see Figs. 2 and 3). The same kind of behavior is observed to be valid for the variations of the $h$-index for the avalanche sizes at the breaking threshold of the equal-load-sharing (infinite range/dimensions) Fiber Bundle Models with $N$ fibers: $h \sim \sqrt[4]{N}/logN$ (see Fig. 3). This is because the citation networks of the scientists are effectively infinite dimensional. In contrast, for the $h$-index values for nearest neighbor clusters on square lattice (two dimension) shows (see Fig. 5) a similar scaling with the lattice size $N$, but with different exponent value: $h \sim N^\gamma/logN$ with $\gamma \simeq 0.54$. We also find a remarkable resolution of the variation of the number $N_s$ of the members of the National Assemblies or Parliaments of different countries with their respective sizes ($N$) of populations (see refs. [10] and [11], and the references therein) if we identify $N_s$ as an effective $h$-index for the country, and use then the scaling relation $N_s \sim \sqrt[4]{N}/logN$, which fits remarkably (see Fig. 6) and helps resolving the controversy [11].

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