CELL-LIKE MAPS AND SURGERY

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Abstract. We show that for any \( n \geq 6 \), for any \( m \), we can find \( m \) non-homeomorphic \( n \)-manifolds that can be mapped by cell-like maps onto the same space \( X \).

1. Introduction

Cell-like maps constitute a natural useful class of maps for many reasons. Suppose that a sequence of homeomorphisms \( h_n : M \to N \) of closed manifolds converges to a continuous map \( h : M \to N \), then \( h \) is not necessarily a homeomorphism. It is the next best thing - a cell-like map. By definition a proper map \( f : X \to Y \) is called cell-like if it has cell-like point preimages \( f^{-1}(y) \) for all \( y \). When \( \dim X < \infty \) this condition means that \( f^{-1}(y) \) can be presented as the intersection of a nested sequence of closed cells in some euclidian space. Since the empty set is not cell-like, every cell-like map is surjective.

Cell-like maps of a manifold \( M \) can be constructed by means of upper semi-continuous decomposition of \( M \) into cell-like sets. Many interesting cell-like maps of lower dimensional manifolds were constructed that way by Bing and his school [2]. If \( q : M \to X \) is the quotient map for such a decomposition and if \( X \) is a manifold, then \( X \) is homeomorphic to \( M \) and \( q \) can be approximated by homeomorphisms. This is due to the Siebenmann Approximation Theorem [11]. In the case where \( X \) is not a manifold, Lacher, in 1977, asked the following question: Can one find two nonhomeomorphic closed manifolds \( M_1 \) and \( M_2 \) that map onto the same \( X \) by cell-like maps?

The image \( X \) of a cell-like map \( f : M \to X \) of a manifold still carries manifold features, in particular, it is a homology manifold. A typical example of such is a manifold with certain singularities. Hence, it’s quite natural to call the domain \( M \) of a cell-like map \( f : M \to X \) a resolution of \( X \). In the late 70s, Quinn proved the uniqueness of resolution theorem [22], Proposition 3.2.3, which implies that if \( \dim X < \infty \), then the answer to Lacher’s question is negative.

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Thus, a positive answer to Lacher’s question is possible only for \( \dim X = \infty \). Whether cell-like maps can raise the dimension to infinity was a big open problem in the area since the 50s. In the late 70s, R. Edwards proved that this problem is equivalent to the even older (early 30s) Alexandroff problem of \textit{whether the covering dimension coincides with the cohomological dimension for compact metric spaces}. He showed that every infinite dimensional compact metric space with finite cohomological dimension is the cell-like image of a finite dimensional compact metric space and, in the 80s, Dranishnikov exhibited these spaces [25]. In 2006, Dranishnikov and Ferry announced an example of two closed non-homeomorphic 7-dimensional manifolds that can be mapped by cell-like maps onto the same space [8]. In 2020 [9], Dranishnikov, Ferry, and Weinberger gave an example, Corollary 2.15, in dimension 6.

In this paper we answer the following version of Lacher’s question: \textit{How many pairwise nonhomeomorphic manifolds can be mapped by cell-like maps onto the same space?} Ferry proved [13] that we cannot have infinitely many such manifolds. In this paper we show that:

1.1. \textbf{Theorem.} For any \( n \geq 6 \), for any \( m \), there are \( m \) closed non-homeomorphic \( n \)-manifolds that can be mapped by cell-like maps onto the same space \( X \).

This result is based on the work in [9].

We recall that the topological structure group \( S(M) \) on a manifold \( M \) consists of classes of (simple) homotopy equivalences \( f : N \to M \). In [9], the authors studied the subgroup \( S^{CE}(M) \) of \( S(M) \) which is defined by homotopy equivalences that factor through cell-like maps. This means that for every \([f : N \to M] \in S^{CE}(M)\), there is \( X \) and and cell-like maps \( q_1 : N \to X \) and \( q_2 : M \to X \) such that the diagram

\[
\begin{array}{ccc}
N & \xrightarrow{f} & M \\
\downarrow{q_1} & & \downarrow{q_2} \\
X & & X
\end{array}
\]

homotopy commutes.

In this paper, we show that often (but not always) a common space \( X \) can be chosen for all elements of \( S^{CE}(M) \).

1.2. \textbf{Theorem.} Let \( M \) be an \( n \)-dimensional closed connected topological manifold, \( n \geq 6 \) and \( G \) be a finitely generated subgroup of \( S^{CE}(M) \),
then there exists a space $X$ such that for any $[N,f] \in G$, $f : N \to M$ factors through cell-like maps to $X$.

In particular, when $M$ is simply connected, we can find one $X$ that works for all the classes in $S^{CE}(M)$:

1.3. **Proposition.** [9] If $L_{n+1}(\mathbb{Z} \pi_1(M))$ has finitely generated odd torsion, then $S^{CE}(M)$ is finite.

Also, we note that there are closed manifolds with infinitely generated $S^{CE}(M)$ [9].

2. **Preliminaries**

All manifolds in this paper are closed, orientable, connected, topological manifolds of dimension $\geq 6$. Since ordinary homology does not behave well for non-ANRs, we will use the Steenrod extension of a generalized homology theory $H_\ast$,[19],[12], [4], [10], which satisfies the usual Eilenberg-Steenrod axioms for (generalized) homology theories, together with the union axiom. We denote by $\tilde{H}_\ast$ its reduced version and we define it for pairs by setting $\tilde{H}_\ast(X,Y) = \tilde{H}_\ast(X/Y) = H_\ast(X,Y)$.

Recall the axioms of a reduced Steenrod homology theory:

(a) Exactness: For a compact metrizable pair $(X,Y)$ there's a long exact sequence

$$\ldots \to \tilde{H}_i(Y) \to \tilde{H}_i(X) \to \tilde{H}_i(X/Y) \to \tilde{H}_{i-1}(Y) \to \ldots$$

(b) Milnor’s additivity axiom: Given a countable collection $X_i$ of pointed compact metric spaces and letting $\vee X_i \subset \prod X_i$ be the null wedge, we have an isomorphism

$$\tilde{H}_\ast(\vee X_i) \cong \prod \tilde{H}_\ast(X_i).$$

We need the following results [1],[3],[26]. Denote by $M(p)$ the $\mathbb{Z}_p$ Moore spectrum.

2.1. **Theorem.** If $p > 1$ is an integer and $n \geq 3$, $\tilde{H}_\ast(K(\pi,n);KO \wedge \mathbb{Z}_p) = 0$ for any group $\pi$. If $\pi$ is torsion, $\tilde{H}_\ast(K(\pi,n);KO) = 0$, for $n = 2$.

For odd $p$, we have a chain of homotopy equivalences of spectra:

$$KO \wedge M(p) \sim \tilde{KO}[\frac{1}{2}] \wedge M(p) \sim \mathbb{L}[\frac{1}{2}] \wedge M(p) \sim \mathbb{L} \wedge M(p).$$
We recall the Universal Coefficient Formula for coefficients in $\mathbb{Z}_p$ for an extraordinary homology theory given by a spectrum $E$ of CW complexes.

$$0 \to H_n(K; \mathbb{E}) \otimes \mathbb{Z}_p \to H_n(K; E \wedge M(p)) \to \text{Tor}(H_{n-1}(K; \mathbb{E}), \mathbb{Z}_p) \to 0 \to 0$$

where $\text{Tor}(H, \mathbb{Z}_p) = \{ c \in H | pc = 0 \}$.

2.2. **Definition.** Given a CW-complex $K$, denote by $P_2(K)$ the second stage of the Postnikov tower of $K$. It is the CW-complex obtained by attaching to $K$ cells in dimensions 4 and higher to kill the homotopy groups of $K$ in dimensions 3 and above. Thus, $P_2(K) - K$ consists of cells of dimension $\geq 4$ and $\pi_n(P_2(K)) = 0$ for $n \geq 3$.

2.3. **Definition.** A compact subset $X$ of an ANR $M$ is cell-like if for each neighborhood $U$ of $X$ the inclusion $i : X \hookrightarrow U$ is nullhomotopic.

Cell-likeness is an intrinsic property of the space $X$. It can be easily shown that if $X$ is a compact ANR, then being cell-like is equivalent to being contractible. A classical example of a space that is cell-like but not contractible is the topologist’s sine curve. For expositions on cell-like maps, see [11], [16], [17].

2.4. **Definition.** A map $f : X \to Y$ between compact metric spaces is cell-like provided that each point inverse is cell-like.

It follows from the definition of cell-like spaces that a cell-like map is surjective.

2.5. **Definition.** A homotopy equivalence $f : N \to M$ between closed manifolds factors through cell-like maps if there is a space $X$ and cell-like maps $c_1 : N \to X$ and $c_2 : M \to X$ such that the following diagram

$$\begin{array}{ccc}
N & \xrightarrow{f} & M \\
\downarrow{c_1} & & \downarrow{c_2} \\
X & & \\
\end{array}$$

homotopy commutes. We say that $N$ and $M$ are $CE - related$.

2.6. **Remark.** Note that if $c_1 : N \to X$ and $c_2 : M \to X$ are cell-like maps, then there exists a homotopy equivalence $f : N \to M$ so that $c_2 \circ f \sim c_1$. This follows from [17] Lemma 2.3.

We recall the definition of the simple topological structure set of a manifold $M$ and the algebraic surgery exact sequence.
2.7. Definition. A simple topological structure \((N, f)\) on an \(n\)-dimensional manifold \(M\) is an \(n\)-dimensional manifold \(N\) together with a simple homotopy equivalence \(f : N \to M\).

2.8. Definition. The simple topological structure set, for which we will omit decorations and denote here by \(S(M)\), of an \(n\)-dimensional manifold \(M\) is the set of equivalence classes of simple manifold structures on \(M\). That is, \((N, f) \sim (N', f')\) if there is a homeomorphism \(h\) so that the diagram

\[
\begin{array}{ccc}
N & \xrightarrow{h} & N' \\
\downarrow{f} & & \downarrow{f'} \\
M & \xrightarrow{=} & M'
\end{array}
\]

homotopy commutes.

Denote by \(S^{CE}(M)\) the set of topological structures that factor through cell-like maps. All homotopy equivalences in \(S^{CE}(M)\) are simple \([9]\), Proposition 2.3. The structure set of a closed topological \(n\)-manifold, \(n \geq 5\) fits into the Sullivan-Wall surgery exact sequence \([24]\) which is isomorphic to the 1-connective algebraic surgery exact sequence of abelian groups \([23]\), Theorem 18.5:

\[
\cdots \to L_{n+1}(\mathbb{Z}\pi_1(M)) \to S(M) \xrightarrow{\theta} [M, G/TOP] \to L_n(\mathbb{Z}\pi_1(M)) \\
\cdots \to L_{n+1}(\mathbb{Z}\pi_1(M)) \to S_{n+1}(M) \to H_n(M; \mathbb{L}) \xrightarrow{A} L_n(\mathbb{Z}\pi_1(M))
\]

The homomorphism \(\theta\) is called the surgery obstruction and the homomorphism \(A\) is called the assembly map.

In this paper, we will make use of the following version of the algebraic surgery exact sequence \([9]\) in which the index of the structure set is shifted by one:

\[
\cdots \to L_{n+1}(\mathbb{Z}\pi_1(M)) \to S_n(M) \xrightarrow{\psi} H_n(M; \mathbb{L}) \xrightarrow{\Delta'} L_n(\mathbb{Z}\pi_1(M)) \to \cdots
\]

where \(H_n(M; \mathbb{L}) = H^0(M; \mathbb{L}) = [M, G/TOP \times \mathbb{Z}], S(M) \subset S_n(M)\).

For \(M\) closed and connected, there’s a split monomorphism \([9]\)

\[
0 \to S(M) \xrightarrow{i} S_n(M) \to \mathbb{Z}.
\]

So, \(S_n(M)\) differs from \(S(M)\) by at most a \(\mathbb{Z}\).
2.9. **Proposition.** [25] Let $E$ be a CW complex with trivial homotopy groups $\pi_i(E) = 0$, $i \geq k$ for some $k$, and $q : X \to Y$ a cell-like map between compacta. Then $q$ induces a bijection of the homotopy classes $q^* : [Y, E] \to [X, E]$.

Let $q : M \to X$ be a cell-like map and $j : M \to \mathbb{P}^2(M)$ the inclusion map. Then by the above proposition, there is a map $g : X \to \mathbb{P}^2(M)$ such that $g \circ q \sim j$. Denote by $i$ the induced map on their mapping cylinders, $i : M_q \to M_j$ where $i|_M = \text{id}|_M$ and $i|_X = g$, and by $i_* : H_*(M_q, M; \mathbb{L}) \to H_*(P_2(M), M; \mathbb{L})$ the induced homomorphism on the Steenrod $\mathbb{L}$-homology groups [12][14].

We state the main results of [9].

Let $M$ be a closed $n$-manifold, there’s a map

$$\delta : H_{n+1}(P_2(M), M; \mathbb{L}) \cong S_{n+1}(P_2(M), M) \xrightarrow{\partial} S_n(M) \xrightarrow{p} S(M).$$

where $p$ is any splitting of $i$. For details on the maps above, refer to [9], section 2.

2.10. **Theorem.** ([9], Theorems 2.4 and 2.7) Let $M^n$ be a closed topological manifold, $n \geq 6$. Let $T_{\text{odd}}(H_{n+1}(P_2(M), M; \mathbb{L}))$ be the odd torsion subgroup of $H_{n+1}(P_2(M), M; \mathbb{L})$.

Then $S^{CE}(M) = \delta(T_{\text{odd}}(H_{n+1}(P_2(M), M; \mathbb{L})))$. In particular, $S^{CE}(M)$ is a subgroup of the odd torsion subgroup of $S(M)$.

Moreover, if $M$ is simply connected with finite $\pi_2(M)$, then $S^{CE}(M)$ is the odd torsion subgroup of $S(M)$.

3. **Main Results**

In [9] Corollary 2.15 the authors constructed a closed simply connected 6-dimensional manifold $M$ with $H_6(M; \mathbb{L}) = \mathbb{Z}_2 \oplus \mathbb{Z}_p \oplus \mathbb{Z}$ and showed, using the Browder-Novikov-Wall surgery exact sequence and a bundle theoretic argument, that for any element $[N, f] \in S^{CE}(M)$, $N$ is not homeomorphic to $M$ (We note that in [9], in the computation of $H_6(M; \mathbb{L})$, the $\mathbb{Z}$ and $\mathbb{Z}_2$ summands were mistakenly omitted. However, that does not change the final result since $S^{CE}(M) = \mathbb{Z}_p$ regardless).

The manifold $M$ was constructed by PL-embedding the Moore complex $P = S^4 \cup_p B^2$ in $\mathbb{R}^6$, then suspending to obtain $P' = S^2 \cup_p B^3$ and taking $M = \partial W$, where $W$ is the regular neighborhood of $P' = S^2 \cup_p B^3$ in $\mathbb{R}^7$. We state this result for completeness.

3.1. **Theorem** ([9] Corollary 2.15). There are closed non-homeomorphic 6-dimensional manifolds $M$ and $N$ which are CE-related.
In the next corollary, we generalize the authors’ construction to any $n$-dimensional, $n \geq 6$, manifolds $M$ and $N$. First, we need the following fact.

3.2. Proposition. ([23], Proposition 20.3) For $n \geq 4$ the structure set $S(M)$ of a simply connected $n$-dimensional topological manifold $M$ is such that $S(M) = S_{n+1}(M) = \ker(A : H_n(M; \mathbb{L}) \to L_n(\mathbb{Z}))$ if $n \equiv 0 \mod 2$, and $S(M) = S_{n+1}(M) = H_n(M; \mathbb{L})$ if $n \equiv 1 \mod 2$.

3.3. Remark. Note that in this paper we use $H_n(M; \mathbb{L})$ and $S_n(M)$ which differ from $H_n(M; \mathbb{L})$ and $S_{n+1}(M)$ by at most a $\mathbb{Z}$ (See preliminaries and [9], Proposition 4.5). Also, recall that $L_n(\mathbb{Z}) = \mathbb{Z}$ if $n = 4k$, $L_n(\mathbb{Z}) = \mathbb{Z}_2$ if $n = 4k + 2$, and $L_n(\mathbb{Z}) = 0$ for $n$ odd.

3.4. Corollary. For any $n \geq 6$, for any $m$, there are $m$ closed non-homeomorphic simply connected $n$-manifolds $M$ and $N$ which are CE-related.

Proof. Let $p$ be a prime number such that $p \geq m$ and $p$ does not divide order($x_i$) for every $x_i \in \pi_i(BG)$, $i \leq n + 1$. This is possible since the groups $\pi_i(BG)$ are finite. Let $n \neq 7, 8$. By general position, the Moore complex $P = S^1 \cup_p B^2$ can be PL-embedded in $\mathbb{R}^6$. Suspending $n - 5$ times embeds $P' = S^{n-4} \cup_p B^{n-3}$ in $\mathbb{R}^{n+1}$. Let $W$ be a regular neighborhood of $P'$ in $\mathbb{R}^{n+1}$ and let $\partial W = M$. The manifold $M$ is stably parallelizable because it is a closed codimension one submanifold of euclidean space $\mathbb{R}^{n+1}$ and it is simply connected. By Lefschetz duality, $H_{n-4}(W; M) = H^5(W) = H^5(P') = 0$ and $H_{n-3}(W; M) = H^4(W) = H^4(P') = 0$. Hence, the exact sequence of the pair $(W, M)$ implies that $H_{n-4}(M) = \mathbb{Z}_{7}$. By the Atiyah-Hirzebruch spectral sequence $H_{n-4}(M; \mathbb{L}) \cong L_n(\mathbb{Z}) \oplus \mathbb{Z}_p \oplus \mathbb{Z}$. Therefore, $H_{n}(M; \mathbb{L}) \cong L_n(\mathbb{Z}) \oplus \mathbb{Z}_p \oplus \mathbb{Z}$. For $n = 7$, from the exact sequence of pairs $(W, M)$, we get $H_3(M) = \mathbb{Z}_p \oplus \mathbb{Z}_p$. By the Atiyah-Hirzebruch spectral sequence $H_{3}(M; \mathbb{L}) \cong \mathbb{Z}_p \oplus \mathbb{Z}_p \oplus \mathbb{Z}$. Hence, $H_7(M; \mathbb{L}) \cong \mathbb{Z}_p \oplus \mathbb{Z}_p \oplus \mathbb{Z}$. For $n = 8$, consider the PL-embedding of the Moore complex $P = S^1 \cup_p B^2$ in $\mathbb{R}^7$ and suspend twice to embed $P' = S^3 \cup_p B^4$ in $\mathbb{R}^9$. As before, let $M = \partial W$, where $W$ is the regular neighborhood of $P'$ in $\mathbb{R}^9$. By a similar argument, we get $H_3(M) = \mathbb{Z}_p$ and $H_4(M; \mathbb{L}) \cong H_8(M; \mathbb{L}) \cong \mathbb{Z} \oplus \mathbb{Z}_p \oplus \mathbb{Z}$. Hence, by Proposition 3.2, Remark 3.3, and Theorem 2.11, $S^{CE}(M) = \mathbb{Z}_p$ for $n \neq 7$ and $S^{CE}(M) = \mathbb{Z}_p \oplus \mathbb{Z}_p$ for $n = 7$. 
We get the following commutative diagram from the Sullivan-Wall and the Quinn-Ranicki exact sequences.

\[
\begin{array}{c}
L_{n+1}(\mathbb{Z}) \longrightarrow S(M) \longrightarrow [M; G/TOP] \longrightarrow L_n(\mathbb{Z}) \\
\downarrow = \downarrow \subset \downarrow \downarrow \downarrow \\
L_{n+1}(\mathbb{Z}) \longrightarrow S_n(M) \longrightarrow \eta' \longrightarrow H_n(M, L) \longrightarrow L_n(\mathbb{Z})
\end{array}
\]

Choose a nontrivial \( p \)-torsion element \( \beta = [N, f] \in S^{CE}(M) \). We next show that \( N \) is not homeomorphic to \( M \) by showing that \( N \) has a nontrivial topological stable normal bundle. Let \( [\gamma] = \eta(\beta) \). The class \([\gamma]\) represents the difference between topological stable normal bundles on \( M \) and \( N \) which are defined by two lifts \( \nu_M : M \to BTOP \) and \( \sigma : M \to BTOP \) of the Spivak map \( M \to BG \) with respect to the fibration \( j : BTOP \to BG \). Here \( \nu_M \) denotes a classifying map for the topological stable normal bundle on \( M \). Note that \( \nu_N = \sigma \circ f \). Thus, the lifts \( \nu_M \) and \( \sigma \) are not fiberwise homotopic. We need to show that \( \nu_M \) and \( \sigma \) are not homotopic in \( BTOP \).

Since the stable normal bundle of \( M \) is trivial, the map \( \nu_M : M \to BTOP \) is nullhomotopic. Note that the map \( \sigma \) is homotopic to \( i \circ \gamma \) where \( i : G/TOP \to BTOP \) is the inclusion of the fiber into the total space of the fibration \( j \). The homotopy exact sequence of the fibration \( j \) implies that after inverting by finitely many primes \( p_1, p_2, ..., p_k \), where \( p_i \neq p \), we get that the inclusion \( i \) is an \((n + 1)\)-equivalence. In particular, \( i_* : [M, G/TOP] \to [M, BTOP] \) is a bijection. Therefore, the map \( i \circ \gamma \) is not nullhomotopic.

Thus, \( \nu_N \) is not nullhomotopic, the topological stable normal bundle of \( N \) is nontrivial and, hence, \( N \) is not homeomorphic to \( M \). \( \square \)

3.5. Definition. We define the reduced group \( \tilde{S}^{CE}(M) \) of \( S^{CE}(M) \) to be the quotient group \( S^{CE}(M)/\langle [N, f] - [N, \psi \circ f] \rangle \) where \( f : N \to M \) and \( \psi : M \to M \) are orientation preserving homotopy equivalences that factor through cell-like maps.

The above definition makes sense since \( h_1 \circ h_2 \) factors through cell-like maps whenever \( h_1 \) and \( h_2 \) factor through cell-like maps [8], Corollary 2.11. Note that two different classes in \( \tilde{S}^{CE}(M) \) contain nonhomeomorphic manifolds, for let \( [N, f] \) and \( [N, g] \) be elements of \( S^{CE}(M) \) then \( g = \psi \circ f \), where \( \psi = gf^{-1}, \psi : M \to M \).

3.6. Corollary. Let \( M \) be as in Corollary 3.4. Any two classes \([N,f]\) and \([P,g]\) in \( S^{CE}(M) \) are such that \( N \) and \( P \) are non-homeomorphic.

Proof. From Corollary 3.4, we have \( S^{CE}(M) = \mathbb{Z}_p \) for \( n \neq 7 \) and \( S^{CE}(M) = \mathbb{Z}_p \oplus \mathbb{Z}_p \) for \( n = 7 \) and nontrivial classes of \( S^{CE}(M) \) are...
classes of manifolds not homeomorphic to $M$. Hence, for any self homotopy equivalence $\psi$ of $M$, $[M, \psi] = [M, id]$ and therefore $\psi$ is homotopic to a homeomorphism. Therefore, $[N, f] = [N, \psi \circ f]$ and it follows that $S^{CE}(M) = \widetilde{S}^{CE}(M)$. □

Next, we generalize a result due to Dranishnikov ([7], Theorem 7.2) which we need for the proof of Proposition 4.1. Dranishnikov showed that, under some assumptions, for a compact polyhedral pair $(P, L)$ and a non-zero element $\alpha \in \widetilde{H}_*(P, L)$, there is a compact metric space $Y \supset L$ and a map $f : (Y, L) \to (P, L)$ such that $\alpha \in \text{Im}(f_*)$.

3.7. Definition. The cohomological dimension of a compactum $X$ with coefficients in the group $G$, denoted $\dim_G(X)$, is the largest integer $n$ such that $H^n(X, A; G) \neq 0$ for some closed subset $A$ of $X$.

3.8. Definition. A map $f : (X, L) \to (Y, L)$ is called strict if $f(X - L) = Y - L$ and $f|_L = id_L$.

3.9. Theorem. [7](Theorem 7.2) Let $H_*$ be a generalized homology theory. Suppose that $\widetilde{H}_*(K(G, n)) = 0$. Then for any finite polyhedron pair $(K, L)$ and any element $\alpha \in H_*(K, L)$ there is a compactum $Y \supset L$ and a strict map $f : (Y, L) \to (K, L)$ such that
(a) $\dim_G(Y - L) \leq n$
(b) $\alpha \in \text{Im}(f_*)$

3.10. Lemma. Let $H_*$ a generalized homology theory. Suppose that $\widetilde{H}_*(K(G, n)) = 0$. Then for any finite polyhedron pair $(K, L)$ and $\alpha_1, \ldots, \alpha_k \in H_*(K, L)$ there is a compactum $Y \supset L$ and a strict map $f : (Y, L) \to (K, L)$ such that
(a) $\dim_G(Y - L) \leq n$
(b) $\alpha_1, \ldots, \alpha_k \in \text{Im}(f_*)$

Proof. Let $\alpha_1, \ldots, \alpha_k \in H_*(K, L)$. By the above theorem, there are compacta $Y_i \supset L$ and strict maps $f_i : (Y_i, L) \to (K, L)$ such that $\dim_G(Y_i - L) \leq n$ and $\alpha_i \in \text{Im}(f_{i*})$, $i \in \{1, \ldots, k\}$.

Let $Y = \bigsqcup_i Y_i$, the spaces $Y_i$ attached along $L$ via the identity maps $id : L \subset Y_i \to Y \subset Y_j$. Consider $f : (Y, L) \to (K, L)$, where $f|_{Y_i} = f_i$.

Clearly, $f$ is a strict map. Since $Y - L = \bigsqcup_i (Y_i - L)$ we obtain $\dim_G(Y - L) \leq n$.

For part (b), we have $H_*(Y, L) = \widetilde{H}_*(Y/L) = \widetilde{H}_*(\vee(Y_i/L)) = \oplus \widetilde{H}_*(Y_i/L) = \oplus H_*(Y_i, L)$ and the induced map on homology $f_* : H_*(Y, L) = \oplus H_*(Y_i, L) \to H_*(K, L)$ is such that $f_*|_{H_*(Y_i, L)} = f_{i*}$. □
In what follows we show that for $S^C_\mathbb{L}(M)$ finite, there exists a space $X$ and a cell-like map $q : M \to X$ such that for any element $\gamma \in S^C_\mathbb{L}(M)$, $\gamma$ can be traced back to $H_{n+1}(M_q, M; \mathbb{L})$ by the following sequence:

$$H_{n+1}(M_q, M; \mathbb{L}) \xrightarrow{i_*} H_{n+1}(P_2(M), M; \mathbb{L}) \xrightarrow{\delta} S(M)$$

3.11. Proposition. Let $M^n$ be a closed connected topological manifold, $n \geq 6$, $p$ odd and $\alpha_1, \ldots, \alpha_k \in H_*(P_2(M), M; \mathbb{L} \wedge M(p))$, where $\mathbb{L} \wedge M(p)$ is $\mathbb{L}$-theory with coefficients in $\mathbb{Z}_p$. Then there is a cell-like map $q : M \to X$ and elements $\hat{\alpha}_1, \ldots, \hat{\alpha}_k \in H_*(M_q, M; \mathbb{L} \wedge M(p))$ such that $i_*(\hat{\alpha}_i) = \alpha_i$.

Proof. Let $n \geq 7$ and $\alpha_1, \ldots, \alpha_k \in H_*(P_2(M), M; \mathbb{L} \wedge M(p))$. Then there is a finite complex $K$, $M \subset K \subset P_2(M)$ and elements $\gamma_1, \ldots, \gamma_k \in H_*(K, M; \mathbb{L} \wedge M(p))$ such that $\gamma_i$ is taken to $\alpha_i$ by the inclusion homomorphism. By Theorem 1.1 $H_*(K(\mathbb{Z}, 3); \mathbb{L} \wedge M(p)) = 0$, hence it follows from Lemma 3.8 that there is a compactum $Y \supset M$ and a strict map $f : (Y, M) \to (K, M)$ such that $\dim_{\mathbb{Z}}(Y - M) \leq 3$ and $\alpha_1, \ldots, \alpha_k \in \text{Im}(f_*)$. The rest of the proof and the case $n = 6$ is exactly like the proof of Proposition 3.6 in [9].

3.12. Theorem. Let $M^n$ be a closed connected topological manifold, $n \geq 6$. If $\beta_1, \ldots, \beta_k \in H_*(P_2(M), M; \mathbb{L})$ are odd torsion elements of order $\beta_i = p_i$, then there exist a cell-like map $q : M \to X$ and elements $\hat{\beta}_1, \ldots, \hat{\beta}_k \in H_*(M_q, M; \mathbb{L})$ such that $i_*(\hat{\beta}_i) = \beta_i$.

Proof. Let $l = p_1p_2 \cdots p_k$. Consider the following commutative diagram of universal coefficient formulas

$$
\begin{array}{ccc}
H_{*+1}(M_q, M; \mathbb{L} \wedge M(l)) & \xrightarrow{\phi'} & \text{Tor}(H_*(M_q, M; \mathbb{L}), \mathbb{Z}_l) \xrightarrow{\subseteq} H_*(M_q, M; \mathbb{L}) \\
\downarrow i_* & & \downarrow i_* \\
H_{*+1}(P_2(M), M; \mathbb{L} \wedge M(l)) & \xrightarrow{\phi} & \text{Tor}(H_*(P_2(M), M; \mathbb{L}), \mathbb{Z}_l) \xrightarrow{\subseteq} H_*(P_2(M), M; \mathbb{L})
\end{array}
$$

where $\phi$ and $\phi'$ are epimorphisms. Let $\beta_1, \ldots, \beta_k \in H_*(P_2(M), M; \mathbb{L})$, thus $\beta_1, \ldots, \beta_k \in \text{Tor}(H_k(P_2(M), M; \mathbb{L}), \mathbb{Z}_l)$. Pick $\alpha_i \in \phi^{-1}(\beta_i)$ for each $i$. By Proposition 3.9, there is a cell-like map $q : M \to X$ and $\hat{\alpha}_1, \ldots, \hat{\alpha}_k \in H_{*+1}(M_q, M; \mathbb{L} \wedge M(l))$ such that $i_*(\hat{\alpha}_i) = \alpha_i$. It follows from the commutativity of the diagram that $i_*(\hat{\beta}_i) = \beta_i$ where $\hat{\beta}_i = \phi'(\hat{\alpha}_i)$. 

3.13. Definition. Let $f : (Y, L) \to (X, L)$ be a strict map. A homotopy $f_1 : Y \to X$ which is strict at each level is called strict if the homotopy $f_1 : (Y, L) \to (X, L)$ is continuous.
3.14. **Definition.** [9] Let $Y$ be an open manifold and let $\bar{Y}$ be a compactification of an end of $Y$ by $X$, $Y = \bar{Y} - X$. A strict homotopy equivalence near $X$ is a strict map $\bar{f} : (\bar{W}, X) \to (\bar{Y}, X)$, where $\bar{W}$ is a compactification of an end of $W$ by $X$, such that there are neighborhoods $\bar{U} \supseteq \bar{V}$ of $X$ in $\bar{W}$ and $\bar{U}' \supseteq \bar{V}'$ of $X$ in $\bar{Y}$ such that $\bar{f}(\bar{U}) \subset \bar{U}'$ and there is a strict map $\bar{g} : (\bar{U}', X) \to (\bar{U}, X)$ such that

(a) $\bar{g} \circ \bar{f}|_{\bar{V}}$ is strict homotopic in $\bar{U}$ to $id_{\bar{V}}$.
(b) $\bar{f} \circ \bar{g}|_{\bar{V}'}$ is strict homotopic in $\bar{U}'$ to $id_{\bar{V}'}$.

3.15. **Definition.** [9] The set of germs of continuously controlled structures on $Y$ at $X$, denoted $S^{cc}(\bar{Y}, X)_{\infty}$, is the set of equivalence classes of strict homotopy equivalences of manifolds near $X$. Two strict homotopy equivalences near $X$, $\bar{f} : (\bar{W}, X) \to (\bar{Y}, X)$ and $\bar{f}' : (\bar{W}', X) \to (\bar{Y}, X)$ are equivalent if there exist a neighborhood $\bar{V}$ of $X$ in $\bar{W}$ and a strict map $\bar{h} : (\bar{V}, X) \to (\bar{W}', X)$ which is an open imbedding and $\bar{f} \circ \bar{h} : \bar{V} \to \bar{Y}$ is strict homotopic to $\bar{f}|_{\bar{V}}$.

Let $q : M \to X$ be a cell-like map of a closed connected manifold $M$ and $M_q$ its mapping cylinder. Let $\hat{M}_q = M_q - M \times \{0\}$.

3.16. **Remark.** (refer to [9], Corollary 4.6 and section 5): Let $[\bar{g}]$ be an element of $S^{cc}(\hat{M}_q, X)_{\infty}$. Then $\bar{g} : (\bar{W}, X) \to (\hat{M}_q, X)$ is a strict homotopy equivalence near $X$. We can assume that $W = N \times (0, 1)$ and $\bar{W} = \hat{M}_p$, where $p : N \to X$ is cell-like. The forgetful map $\phi : S^{cc}(\hat{M}_q, X)_{\infty} \to S(M)$ sends $[\bar{g}]$ to $[f]$, where $f : N \to M$ is a homotopy equivalence that factors through the cell-like maps $p$ and $q$.

3.17. **Proposition.** ([9], Proposition 5.4) Let $M$ be a closed connected $n$-manifold and $q : M \to X$ a cell-like map. Then the forgetful map $\phi : S^{cc}(\hat{M}_q, X)_{\infty} \to S(M)$ factors as

$$S^{cc}(\hat{M}_q, X)_{\infty} \xrightarrow{j} H_{n+1}(M_q, M; \mathbb{L}) \xrightarrow{i} H_{n+1}(P_2(M), M; \mathbb{L}) \xrightarrow{\delta} S(M)$$

where $j$ is a monomorphism with cokernel $\mathbb{Z}$ or 0.

3.18. **Theorem.** Let $M$ be an $n$-dimensional closed connected topological manifold, $n \geq 6$ and $G$ be a finitely generated subgroup of $S^{CE}(M)$. Then there exists a space $X$ such that for any $[N, f] \in G$, $f : N \to M$ factors through cell-like maps to $X$.

**Proof.** It follows from Theorem 2.11 that $G$ is finite. Let $\gamma_1, ..., \gamma_k \in G$. Then there are odd torsion elements $\gamma'_1, ..., \gamma'_k$ in $H_{n+1}(P_2(M), M; \mathbb{L})$ such that $\partial(\gamma_i) = \gamma'_i$. By Theorem 3.12, there exist a cell-like map $q : M \to X$ and elements $\beta_1, ..., \beta_k \in H_* (M_q, M; \mathbb{L})$ such that $i_*(\beta_i) = \gamma'_i$. The result follows from Proposition 3.17. $\square$
3.19. **Theorem.** For $n \geq 6$ and any $m$, there are $m$ closed non-homeomorphic $n$-manifolds that can be mapped by cell-like maps onto the same space $X$.

**Proof.** Let $p$ and $M$ be as in Corollary 3.4. Then, by Corollary 3.6, any two classes in $S^CE(M)$ contain non-homeomorphic manifolds. The result follows from Theorem 3.19. □

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