Abstract

In this study, we show that the energy eigenvalues and the eigenfunctions of the Schrodinger equation for the power-law and the logarithmic potential can be easily obtained by using variation technique for special type wave functions. The results are in very good agreement with exact numerical results.

1 Introduction

A large number of important physical problems require solving the Schrodinger equation for spherical symmetric potential to determine the energy eigenvalues and the eigenfunctions. It is known that for very limited potentials, Schrodinger equation is exactly solvable. In general one has to resort to numerical techniques or approximation schemes, Most popular approximation methods like $1/N$ expansion, WKB method, perturbation theory are widely used for this purpose. But some of these methods have drawbacks in application. Although some methods give simple relations for the eigenvalues, they give very complicated relations for the eigenfunction. The aim of present work is to give a simple way for finding both eigenvalues and the eigenfunctions of Schrodinger and Schrodinger-like equations for power-law and logarithmic potentials, which are very important in particle physics[1-10].

This paper is organized as follows. In the first section, the eigenvalues are obtained for the $\text{sgn}(v)A r^\nu (\nu > -2)$ type potential by using variation technique for the special form
of eigenfunction. In section 2, the same technique is applied to the logarithmic potential.

In the last section, we give some concluding remarks.

2 The solution of the Schrodinger equation for the power-law potentials

The radial part of Schrodinger equation for $sgn(v)A r^\nu (\nu > -2)$ type potential is written as

$$\left[ -\frac{d^2}{dr^2} + sgn(v)Ar^{\nu} + \frac{l(l+1)}{r^2} \right] g(r) = Eg(r) \quad (1)$$

Using substitution $r = \left(\frac{1}{A}\right)^{\frac{1}{\nu+2}} \rho$, Eq. (1) can be written in the following form

$$\left[ -\frac{d^2}{d\rho^2} + sgn(v)\rho^{\nu} + \frac{l(l+1)}{\rho^2} \right] g(\rho) = \epsilon g(\rho) \quad (2)$$

where $\epsilon = E.(\frac{1}{A})^{\frac{2}{\nu+2}}$. It is well known that Eq. (2) have exact analytical solutions for $\nu = -1$ and $\nu = 2$ and the forms of the solutions are:

for $\nu = -1$ (Coulomb potential)

$$g(\rho) \approx \rho^{l+1}Exp(-x\rho) L_n^{2l+1}(2x\rho), \quad (3)$$

for $\nu = 2$ (Harmonic potential)

$$g(\rho) \approx \rho^{l+1}Exp\left(-\left(x\rho\right)^2\right) L_n^{2\nu+1}(2(x\rho)^2). \quad (4)$$

where $L$ is Laguerre polynomials. From the similarity of these solutions, the following solution can be proposed for arbitrary $\nu$

$$g(\rho) \approx \rho^{l+1}Exp\left(-(x\rho)^d\right) L_n^{2d+1}(2(x\rho)^d) \quad (5)$$

where $x$ and $d$ variation parameters and they can be obtained by minimizing of $\epsilon$ in Eq. (2) with respect to these parameters. So;

$$\frac{\partial \epsilon}{\partial x} = 0, \quad (6)$$
\[
\frac{\partial \varepsilon}{\partial d} = 0. \quad (7)
\]

Using Eqs. (2) and (5), for \( \varepsilon \), we get

\[
\varepsilon_{nl}(x,d) = cx^2 + \frac{b}{x^\nu} \quad (8)
\]

and from Eq. (6), we have \( x = \left( \frac{b\nu}{2c} \right)^\frac{1}{\nu+2} \), thus, we have found \( \varepsilon_{nl} \) as the following form

\[
\varepsilon_{nl}(d) = (v + 2) \left( \frac{c}{v} \right)^\frac{v}{\nu+2} \left( \frac{b}{2} \right)^\frac{2}{\nu+2}, \quad (9)
\]

where;

\[
c = 2^\frac{2-\nu}{d} \frac{\sum_{k=0}^n \sum_{m=0}^n a_k a_m s \Gamma(k + m + \frac{2l+1}{d})}{\sum_{k=0}^n \sum_{m=0}^n a_k a_m \Gamma(k + m + \frac{2l+3}{d})}, \quad (10)
\]

\[
b = \text{sgn}(v) 2^\frac{-\nu}{d} \frac{\sum_{k=0}^n \sum_{m=0}^n a_k a_m \Gamma(k + m + \frac{2l+\nu+3}{d})}{\sum_{k=0}^n \sum_{m=0}^n a_k a_m \Gamma(k + m + \frac{2l+3}{d})}, \quad (11)
\]

\[
s = (2l + 1)(2l + d + 1) + (k + m - (k - m)^2)d^2, \quad (12)
\]

where \( a_j \) are the coefficients of the generalized Laguerre polynomials. The parameter \( d \) can be obtained from Eq. (7), but unfortunately this equation cannot be solved analytically. However, for \( n = 0, l = 0 \) case, the value of \( d \) which minimized of \( \varepsilon \) can be found easily.

In Fig. 1, we present the dependence \( d \) on \( \nu \). Behavior of the curve is similar to \( \sqrt{\nu + 2} \) function. But, this function does not fit exactly to the curve. Therefore, a correction factor is necessary. According to the our assumption, the correction factor must be equal to 1 at \( v = -1 \) and \( v = 2 \). Thus, we chose the correction factor in the following form;

\[
w = (1 + tp)^h \quad (13)
\]
where \( p = \frac{(\nu+1)(2-\nu)}{a_1 \nu^2 + a_2 \nu + a_3} \). So, we write

\[
d = \sqrt{v + 2w}
\] (14)

Fitting this equation to the curve in Fig. 1, \( t, a_1, a_2, a_3 \) and \( h \) constants are obtained as, 0.2075, 0.1381, 1.05, 2.484 and 0.08104 respectively. So, when any values of \( v \) is given, \( d \) values corresponding to \( v \) can be easily calculated by using Eq. (14). In Tables 1 and 2, we present results of our calculations for the eigenvalues at different \( \nu \). For comparison in these tables, we also present existing numerical solutions in literature. In Table 2, the calculations done using eq. (9) are multiplied by \( 2^{\frac{\nu}{2}} \) factor because of consistency with results of ref[11].

When we study on linear potentials, Eq. (2) has the following form for S states

\[
\left[ -\frac{d^2}{dz^2} + z \right] g(z) = 0
\] (15)

where, \( z = \rho - \epsilon \). The equation given above is known as Airy equation and has exact solution in terms of Airy functions given as the following form

\[
g_n^{\text{exact}}(\rho) = Ai(\rho - \epsilon_n)
\] (16)

Where \( \epsilon_n \) are the zeros of the Airy function and as given in Table 3. For S states, our prediction for eigenfunctions has the following form from Eq. (5)

\[
g_n^{\text{our}}(\rho) \approx \rho^1 \exp\left(-\left(\frac{\rho}{d}\right)^d\right) L_n^{\frac{1}{d}}(2(x\rho)^d)
\] (17)
In Figures 2 and 3, our wave function and the exact wave function are presented together for \( n = 0, l = 0 \) and \( n = 4, l = 0 \) states respectively. We see that the agreement between two solutions is excellent. In addition to the wave functions, in Table 3, eigenvalues of the linear potential for S states \( (n = 0, 1, 2, \ldots \text{ and } l = 0) \) are given and compared with the exact results which are well-known zeros of the Airy function. Similarly, in Table 4, eigenvalues of \( r^{0.5} \) are calculated and compared with exact numerical results. The calculated results are in good agreement with exact numerical results and better than given in ref[17].

3 The logarithmic potential case

Let us consider \( V(r) = \log(r) \) potential which is very important in particle physics. The radial part of Schrodinger equation for logarithmic potential is written as

\[
\left[-\frac{d^2}{dr^2} + \log(r) + \frac{l(l+1)}{r^2}\right] g(r) = Eg(r).
\] (18)

The function \( \log(r) \) at \( \nu \approx 0 \) can be written in following form

\[
\log(r) \approx \frac{1}{\nu} [r^\nu - 1].
\] (19)

Substitute Eq. (15) into Eq. (14) and then apply the method presented in previous section to Eq. (14) for \( d \) we found the value 1.43203 at \( \nu = 0.00001 \) (see Eq. (13)). Thus, the eigenvalues of Eq. (14) are written as

\[
E_{nl} = \frac{\epsilon_{nl}}{\nu^{1/2}} - \frac{1}{\nu}
\] (20)

where \( \epsilon_{nl} \) is given as in Eq. (8). The calculated results for \( \epsilon_{nl} \) are presented in Table 4. In this table we also give the results of numerical solutions. The eigenfunctions of logarithmic potential can be written from Eq. (5) as

\[
g(\rho) \approx \rho^{l+1} Exp \left(- (x \rho)^d \right) L_n^{2l+1} (2(x \rho)^d)
\] (21)
where, $\rho = \frac{\nu}{\nu^2 + 1}$. It is obvious that the eigenvalues obtained for the logarithmic potential are in good agreement with the results obtained from numerical solution. Also, in order to show the validity of the wavefunctions given in Eq. (17) obtained for the logarithmic potential, first of all, some energy levels are chosen. Their wavefunctions are obtained by Numerov’s Method. Finally, these results are given in Figures 4 and 5 together with predictions.

4 Conclusion

In this paper, we have calculated eigenvalues and eigenfunctions of power-law and logarithmic potentials by using variational techniques for special type wave functions. Our results are in good agreement with existing exact numerical ones. However, for the higher values of $\nu$, there is some difference between two approaches. However this differences not larger than that one coming from $1/N$ expansion. Moreover, we obtained that present method predicts not only the eigenvalues as well as the eigenfunctions of given potentials. The correspondence of wavefunctions for lower energy levels is very good agreement with numerical results as expected the except case $n$ and $l$ are large. The method used in this study present an easy way of calculating both eigenfunctions and eigenvalues of power-law and logarithmic potentials. In spite of its simple structure, the method is very practical and the results of the method are in good agreement with exact results.

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6 References

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Table 1. Comparison of this study results for the ground state \((n = 0, l = 0)\) of various power-law potentials.

| \(V(r)\)       | This Work | Numerical[11] | \(V(r)\)       | This work | Numerical[11] |
|-----------------|-----------|---------------|-----------------|-----------|---------------|
| \(-r^{-1.5}\)   | -0.29703  | -0.29609      | \(r^2\)        | 3         | 3             |
| \(-r^{-1.25}\)  | -0.22027  | -0.22029      | \(r^3\)        | 3.45110   | 3.45056       |
| \(-r^{-1}\)     | -0.25     | -0.25         | \(r^4\)        | 3.80241   | 3.79967       |
| \(r^{0.9}\)     | 1         | 1             | \(r^5\)        | 4.09626   | 4.33801       |
| \(r^{0.15}\)    | 1.32798   | 1.32795       | \(r^6\)        | 4.35243   | 4.54690       |
| \(r^{0.5}\)     | 1.83352   | 1.83339       | \(r^7\)        | 4.58158   | 4.71772       |
| \(r^{0.75}\)    | 2.10829   | 2.10814       | \(r^8\)        | 4.79013   | 4.92220       |
| \(r^{1.5}\)     | 2.70816   | 2.70809       | \(r^{10}\)     | 5.16092   | -             |
Table 2: Eigenvalues of $-2^{1.7}r^{-0.2}$ and $-2^{0.8}r^{-0.8}$ for different $nl$

| $n$ | $l$ | This work | numerical [11-15] | $n$ | $l$ | This work | numerical [11-15] |
|-----|-----|-----------|------------------|-----|-----|-----------|------------------|
| 0   | 0   | -2.6859   | -2.686           | 0   | 0   | -1.2186   | -1.218           |
| 1   | 0   | -2.2530   | -2.253           | 1   | 0   | -0.4622   | -0.462           |
| 2   | 0   | -2.0440   | -2.044           | 2   | 0   | -0.2648   | -0.265           |
| 0   | 1   | -2.3449   | -2.345           | 0   | 1   | -0.5004   | -0.500           |
| 1   | 1   | -2.1006   | -2.101           | 1   | 1   | -0.2806   | -0.281           |
| 2   | 1   | -1.9504   | -1.951           | 2   | 1   | -0.1873   | -0.187           |
| 0   | 2   | -2.1562   | -2.156           | 0   | 2   | -0.2947   | -0.295           |
| 1   | 2   | -1.9900   | -1.990           | 1   | 2   | -0.1949   | -0.195           |
| 2   | 2   | -1.8749   | -1.875           | 2   | 2   | -0.1420   | -0.142           |
| 0   | 3   | -2.0291   | -2.029           | 0   | 3   | -0.2019   | -0.202           |
| 1   | 3   | -1.9049   | -1.905           | 1   | 3   | -0.1463   | -0.146           |
| 2   | 3   | -1.8124   | -               | 2   | 3   | -0.1128   | -               |
Table 3: Eigenvalues of linear potential for different $n$ and $l = 0$.

| $n$ | $l$ | This work | numerical\([16]\) |
|-----|-----|-----------|----------------|
| 0   | 0   | 2.33825   | 2.33810       |
| 1   | 0   | 4.08918   | 4.08795       |
| 2   | 0   | 5.52132   | 5.52056       |
| 3   | 0   | 6.78614   | 6.78671       |
| 4   | 0   | 7.94189   | 7.94413       |
| 5   | 0   | 9.01859   | 9.02265       |
Table 4: Eigenvalues of $r^{0.5}$ for different $nl$ together with exact values and other researcher results, with percentage errors.

| $n$ | $l$ | This work | Numerical[17] | Ref[17] | %   |
|-----|-----|-----------|---------------|---------|-----|
| 0   | 0   | 1.83352   | 1.83339       | 1.83375 | 0.007 |
| 1   | 0   | 2.55152   | 2.55065       | 2.55142 | 0.03 |
| 2   | 0   | 3.05177   | 3.05118       | 3.05224 | 0.019 |
| 3   | 0   | 3.45197   | 3.45213       | 3.45341 | 0.005 |
| 4   | 0   | 3.79233   | 3.79336       | 3.79482 | 0.027 |
| 0   | 1   | 2.30056   | 2.30050       | 2.30073 | 0.003 |
| 1   | 1   | 2.85473   | 2.85434       | 2.85486 | 0.014 |
| 2   | 1   | 3.28666   | 3.28583       | 3.28659 | 0.025 |
| 3   | 1   | 3.64838   | 3.64739       | 3.64835 | 0.027 |
| 4   | 1   | 3.96361   | 3.96268       | 3.96382 | 0.023 |
| 0   | 2   | 2.65760   | 2.65756       | 2.65775 | 0.002 |
| 1   | 2   | 3.12048   | 3.12033       | 3.12077 | 0.005 |
| 2   | 2   | 3.50296   | 3.50245       | 3.50309 | 0.015 |
| 3   | 2   | 3.83338   | 3.83254       | 3.83336 | 0.022 |
| 4   | 2   | 4.12686   | 4.12581       | 4.12678 | 0.025 |
| 0   | 3   | 2.95448   | 2.95445       | 2.95461 | 0.001 |
| 1   | 3   | 3.35764   | 3.35759       | 3.35798 | 0.001 |
| 2   | 3   | 3.70299   | 3.70270       | 3.70327 | 0.008 |
| 3   | 3   | 4.00796   | 4.00737       | 4.00810 | 0.015 |
| 4   | 3   | 4.28282   | 4.28196       | 4.28283 | 0.020 |
| 0   | 4   | 3.21236   | 3.21233       | 3.21247 | 0.001 |
| 1   | 4   | 3.57275   | 3.57275       | 3.57310 | 0.000 |
| 2   | 4   | 3.88913   | 3.88898       | 3.88950 | 0.004 |
| 3   | 4   | 4.17308   | 4.17268       | 4.17335 | 0.010 |
| 4   | 4   | 4.43196   | 4.46131       | 4.43164 | 0.015 |
Table 5: Eigenvalues of logarithmic Potential for different \( nl \)

| \( n \) | \( l \) | This work | numerical [11-15] | \( n \) | \( l \) | This work | numerical [11-15] |
|-------|-------|-----------|-----------------|-------|-------|-----------|-----------------|
| 0     | 0     | 1.0445    | 1.0443          | 3     | 0     | 2.5957    | 2.5957          |
| 0     | 1     | 1.6412    | 1.6430          | 3     | 1     | 2.7465    | 2.7440          |
| 0     | 2     | 2.0134    | 2.0150          | 3     | 2     | 2.8801    | 2.8800          |
| 0     | 3     | 2.2842    | 2.2860          | 3     | 3     | 2.9996    | 2.9990          |
| 1     | 0     | 1.8485    | 1.8474          | 3     | 4     | 3.1071    | 3.1070          |
| 1     | 1     | 2.1513    | 2.1510          | 4     | 0     | 2.8293    | 2.8299          |
| 1     | 2     | 2.3875    | 2.3880          | 4     | 1     | 2.9498    | 2.9480          |
| 1     | 3     | 2.5798    | 2.5810          | 4     | 2     | 3.0592    | 3.0600          |
| 2     | 0     | 2.2903    | 2.2897          | 4     | 3     | 3.1592    | 3.1590          |
| 2     | 1     | 2.4917    | 2.4910          | 4     | 4     | 3.2512    | 3.2510          |
| 2     | 2     | 2.6629    | 2.6630          | 6     | 0     | 3.1770    | 3.1791          |
| 2     | 3     | 2.8106    | -               | 10    | 0     | 3.6411    | 3.6427          |
Figure captions

Figure 1. The dependence of \( d \) on \( \nu \).

Figure 2. Comparison of our wavefunction and Airy function for linear potential \( (n = 0, \ l = 0) \). Dot lines represent Airy function,

Figure 3. Comparison of our wavefunction and Airy function for linear potential \( (n = 4, \ l = 0) \). Dot lines represent Airy function,

Figure 4. Comparison of our wavefunction and corresponding numerical wavefunction for logarithmic potential
\( (n = 0, \ l = 0) \)

Figure 5. Comparison of our wavefunction and corresponding numerical wavefunction for logarithmic potential
\( (n = 4, \ l = 4) \)
n=0, l=0 (Logarithmic potential)
This work

Numerical

n=4, l=4 (Logarithmic potential)