Rotating strings and particles in AdS:
Holography at weak gauge coupling and without
conformal symmetry

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Abstract

We consider gauge/gravity correspondence between maximally supersymmetric Yang-Mills theory in $(p + 1)$ dimensions and superstring theory on the near-horizon limit of the $Dp$-brane solution. The string-frame metric is $\text{AdS}_{p+1} \times S^{8-p}$ times a Weyl factor, and there is no conformal symmetry except for $p = 3$. In this paper, we consider states which have angular momenta in the AdS directions. We first show that Gubser, Klebanov and Polyakov’s solution, in which a folded string is rotating near the center of AdS, can be recast into a form which connects two points on the boundary. Transition amplitudes of such strings can be interpreted as gauge theory correlators, whether or not there is conformal symmetry. Then, we consider the case of zero gauge coupling, assuming the string worldsheet consists of discrete bits. We reproduce the free-field correlators from string theory, extending the previous result obtained for a special operator.

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1 Introduction

Since Maldacena’s proposal of AdS/CFT correspondence [1], numerous examples of gauge/gravity correspondence have been proposed. This has led to exciting developments in the studies of quantum gravity and of strongly coupled quantum systems.

In spite of such developments, in our opinion, there are unclear issues associated to the following aspects of gauge/gravity correspondence: the correspondence without conformal symmetry and/or at weak gauge coupling. A purpose of this paper is to shed new light on these aspects. We shall first explain our motivation, and then describe the concrete problem studied in this paper.

1.1 Conformal symmetry

Conformal symmetry has been extremely helpful in formulating and testing gauge/gravity correspondence. As we briefly remind the readers below.

Maldacena’s proposal of AdS/CFT correspondence [1] is a concrete realization of the holographic principle [2, 3], which states that a theory of quantum gravity should be described by a theory without gravity defined with respect to degrees of freedom localized on the spatial boundary. The biggest clue for this highly non-trivial proposal was that both sides have the same symmetry: The isometry group of \(d+1\) dimensional Anti-de-Sitter (AdS) space and the conformal group in \(d\) dimensions are isomorphic.

The prescription relating the correlation functions for gauge theory on the boundary of AdS and the partition functions for gravity in AdS has been proposed by Gubser, Klebanov and Polyakov [4] and by Witten [5] (GKPW). The dictionary between the bulk fields and the boundary operators, namely, the coupling between them in the GKPW prescription, is determined from the requirement that they should belong to the same representations of the (super) conformal group (see e.g., [6]).

In theories with superconformal symmetry, there are powerful non-renormalization theorems. The scaling dimensions of the so-called BPS operators are not renormalized from their free-field values, even in the strong coupling; this follows from the representation theory of superconformal algebra. The BPS operators correspond to supergravity modes in the bulk. Their correlation functions, calculated by GKPW prescription by using the tree-level supergravity (a low-energy approximation to string theory) are supposed to give results for strong gauge coupling. The fact that they agree with the free-field values provided important consistency checks of AdS/CFT correspondence.
There have been highly non-trivial tests of AdS/CFT correspondence from calculations of quantities interpolating between weak and strong couplings. In such analyses, conformal symmetry played essential technical roles. Examples include the calculation of the expectation values of Wilson loops by using conformal transformations of the shape of the loops (see e.g., [7]) and the calculation of cusp anomalous dimensions using integrability (see e.g., [8]).

**The cases without conformal symmetry**

Although holographic principle should be a concept independent of the conformal symmetry, gauge/gravity correspondence without conformal symmetry is much less understood than the cases with conformal symmetry.

There have been indeed many examples of gauge/gravity correspondence in which conformal symmetry is broken. Even though many of them involve highly non-trivial and interesting ideas, one could also say that most of them are associated to theories with conformal symmetry, in the sense that conformal symmetry is restored in some limit. To give just a few examples of such interesting theories: there have been attempts to realize QCD-like theories (which are non-supersymetric and non-conformal) by compactifying one spatial dimension in the conformally invariant $\mathcal{N} = 4$ super Yang-Mills (SYM) theory, starting from the work by Witten [9]; there have been studies to construct geometries which interpolate between two AdS regions to understand the the renormalization (RG) flows in gauge theories [10, 11, 12]; in applications of gauge/gravity correspondence to nuclear or condensed matter physics (see e.g., [13, 14] for reviews), central attention is on the the vicinity of the quantum phase transition points at which conformal symmetry is realized.

One of the very few examples of gauge/gravity correspondence that does not have conformal symmetry from the outset is the case associated with the two descriptions of $Dp$-branes. Namely, the equivalence between maximally supersymmetric $SU(N)$ Yang-Mills theories in $(p+1)$ dimensions and superstring theories on the near-horizon limit of the $Dp$-brane solutions [15]. The former is an open-string description on the worldvolume of $Dp$-branes, and the latter is a closed string description treating the $Dp$-branes as classical solutions in supergravity. The geometry is $\text{AdS}_{p+2} \times S^{8-p}$ times a Weyl factor. The $p = 3$ case is conformal invariant, and is the most typical example of the AdS/CFT correspondence [1]. For $p \neq 3$, the gauge coupling (the only independent coupling in these theories with high degree of supersymmetry) has non-zero dimension, thus the theory is not conformally invariant. Without conformal symmetry, the analysis for $p \neq 3$ is not
easy, but the correspondence is as well-motivated as in the $p = 3$ case. In view of the fact that there is no proof of gauge/gravity correspondence, we expect the $p \neq 3$ cases to yield useful data which will help us understand how gauge/gravity correspondence works. In this paper, we will study this example of gauge/gravity correspondence.

**Worldsheet analysis**

A major drawback of the cases without conformal symmetry is that it is not clear how to perform bulk analysis based on the string worldsheet.

For theories with conformal symmetry, one can take the global coordinates in the Lorentzian AdS, and identify the bulk energy with the scaling dimension of the corresponding gauge-theory operator. This identification is based on the isomorphism of the symmetry groups: the translation with respect to the global time in AdS corresponds to the dilatation in gauge theory. Although the superstring action on AdS has complicated non-linear interactions and is difficult to solve, there has been significant progress based on semi-classical approximations, especially in the case of AdS$_5 \times S^5$: (a) In the limit of large angular momenta along $S^5$, Berenstein, Maldacena and Nastase (BMN) found the spectrum of quadratic fluctuations around a point-like classical configuration of the string moving along $S^5$. From this analysis, the scaling dimensions of the corresponding operators (the so-called BMN operators) was obtained, not only those corresponding to supergravity modes, but also those corresponding to higher string excitations, whose dimensions are not protected [20]. (b) In the limit of large angular momenta along AdS$_5$, Gubser, Klebanov and Polyakov (GKP) identified classical solutions of strings with such momenta, which are folded and rotating in AdS, and obtained the scaling dimensions of the corresponding operators in gauge theories. This result is expected to capture the properties of non-supersymmetric theory as well [21].

For the cases without conformal symmetry, one cannot apply the above identification of the energy and the scaling dimension. To the best of our knowledge, the only worldsheet analyses performed so far are those based on the formulation by Dobashi, Shimada and Yoneya (DSY) [22]. In the original paper of DSY, the BMN operators (those with large angular momenta along $S^5$) in the conformally invariant $p = 3$ case (AdS$_5 \times S^5$) was studied. They consider Euclidean AdS, since the GKPW prescription based on supergravity has been formulated with the Euclidean signature. There is a geodesic in Euclidean AdS which

\[^{1}\text{Also, in other cases such as AdS}_3/CFT_2 \, [16, 17, 18] \, \text{and D1-D5 system (see e.g., [19]), detailed worldsheet analyses have been performed.}\]
connects two points on the boundary. By performing a semi-classical approximation of superstring along this geodesic, one can compute transition amplitudes of the Euclidean string. An amplitude is interpreted as the correlation function of the corresponding BMN operator. The results thus obtained are consistent with the ones obtained by BMN [20]; furthermore, some puzzles have been solved (see [22, 23, 24]). This formalism does not use conformal symmetry. Also, the relation between the bulk and boundary is intuitively clear. Asano, Yoneya and one of the present authors have applied this to the case of general p [25, 26]. So far this formalism has been formulated only for the BMN operators (those with large angular momenta along $S^{8-p}$). One of the purposes of the present paper is to clarify how to apply it to more general operators, namely to the GKP operators, which have angular momenta in the AdS directions.

1.2 Weak gauge coupling

Another purpose of this paper is to understand gauge/gravity correspondence at weak gauge coupling.

The weak gauge coupling limit corresponds to the limit where the string length is much smaller than the AdS radius. In the gauge/gravity correspondence for the Dp-branes that we consider, the radius of $S^{8-p}$ is of the same order as the AdS radius. Thus, the energies of the supergravity modes, including the Kaluza-Klein (KK) modes from $S^{8-p}$ are of order the inverse AdS radius. Thus, in the weak gauge coupling limit, the energies of the string excited modes are smaller than those of supergravity modes, and cannot be ignored. Thus, one need to use the string worldsheet theory, not the supergravity approximation. String theory in this limit, the tensionless string, is known to be difficult to analyze. (See [28] for recent developments from an approach somewhat different from ours.)

Previous work: angular momentum along $S^{8-p}$

In a previous paper by one of the present authors [29], the BMN operators in the $(p + 1)$-dimensional super Yang-Mills theory have been studied at zero gauge coupling from string theory. The basic idea is as follows: The string states with $J$ units of angular momenta are represented by single trace operators, $\text{Tr}(Z^J)$, where $Z$ is a complex combination of two scalar fields in gauge theory. Now we assume the spatial direction of the worldsheet for the

\[\text{Furthermore, in this limit, stretched strings within a patch with the AdS size cannot be ignored. This fact may provide a clue for understanding of the “sub-AdS locality,” known to be a difficult problem in gauge/gravity correspondence [27].}\]
corresponding string is discretized into \( J \) bits. At weak gauge coupling, the interactions among these bits are weak. The reasons for considering the string bits are twofold: First, on the gauge theory side, if one wants to represent the string worldsheet by the cyclic sequence of the fields inside the trace \([20]\), we can only represent \( J \) sites. Second, on the string theory side, to represent a state with angular momentum, one usually inserts creation operators which have such angular momenta, and smear them on the worldsheet. Before smearing, each operator inserted on the worldsheet can be regarded as a particle (bit) with a unit angular momentum, interacting with other bits via strings connecting them. At weak gauge coupling limit, the unit of angular momentum is large, and the interactions among the bits are weak. Thus, it would be appropriate to treat these bits as discrete objects.

In the maximally supersymmetric Yang-Mills theories in \((p + 1)\) dimensions, from the supergravity analysis based on the GKPW prescription, it is known that the operators corresponding to supergravity modes have power-law correlators at strong gauge coupling \([30, 31, 25, 26]\), even though there is no conformal invariance for \( p \neq 3 \). The power is in general a fractional number, and is different from the free-field value. This power law has not been proven analytically in gauge theory, but for some operators in \( p = 0 \), it has been confirmed to a high precision by the Monte-Carlo simulation in gauge theory \([32, 33]\). On the other hand, it has not been known how to obtain the free-field value from the bulk analysis. However, in the previous paper \([29]\), the case of zero coupling was studied by treating the bits as non-interacting. Then, the amplitude for the collection of bits was computed by properly taking into account the zero-point energies of each bit. By rewriting the amplitude in the form of gauge theory correlator following the DSY prescription \([22, 25]\), the free-field result of the \((p + 1)\)-dimensional field theory was reproduced.

**Aim of the present work: angular momentum along \( \text{AdS}_{p+2} \)**

In this paper, in an attempt to generalize the intriguing result of \([29]\) to more general class of states, we consider the states with angular momenta along \( \text{AdS} \). At strong coupling, they will be described by strings which are folded and rotating in the bulk \([21]\). At weak gauge coupling, the unit of angular momentum along \( \text{AdS}_{p+2} \), as well as along \( S^{8-p} \), is large. As in the previous paper, we assume a bit has a unit angular momentum along \( S^{8-p} \). We consider the zero coupling case, and ignore interactions among the bits. Both for \( p = 3 \) and for \( p \neq 3 \), we will find the free-field results (the scaling dimension is given by the sum of the free-field dimension of the scalar fields plus the number of derivatives
in the operator). We regard this as an indication for the validity of the approach initiated in [29].

1.3 Organization of this paper

This paper is organized as follows. In Section 2, we will briefly review gauge/gravity correspondence for Dp-branes. Basic features of the near-horizon limit of the Dp-brane solution will be described. In Section 3, we will consider the case of strong gauge coupling. We will first review the solution found by Gubser, Klebanov and Polyakov, then, rotating the coordinates in the imaginary directions, we obtain a string configuration which connects two points on the boundary. We will show that the string amplitude can be interpreted as the gauge theory correlator. Our analysis here is based on the $p = 3$ case, but this formalism is applicable to the cases without conformal symmetry. In Section 4, we consider weak gauge coupling. We study the zero coupling case, by considering the string bits without interactions among them. We first describe the analysis for $p = 3$ using global time. From the energy of the rotating particle, we obtain the free-field result in gauge theory. We then study the case of general $p$. We compute the amplitude for particles along a trajectory connecting two points on the boundary, and show that this reproduces the free-field result in gauge theory. In Section 5, we conclude, and mention directions for future research.

2 Gauge/gravity correspondence

Gauge/gravity correspondence associated to the Dp-branes has been proposed by Itzhaki, Maldacena, Sonnenschein and Yankielowicz [15]. Here we will review only the basic facts, and refer the readers to the original papers for details: for supergravity analysis based on the GKPW prescription, see [30, 31, 37]; for worldsheet analysis, see [25, 26, 38]; for the tests by Monte Carlo simulations in gauge theory, see [32, 33]; for the correspondence at weak gauge coupling, see [29].

The metric and the dilaton for the zero-temperature Dp-brane solution in the string frame is given by

$$
\begin{align*}
    ds^2 &= H^{-1/2} \left(-dt^2 + dx_a^2\right) + H^{1/2} \left(dr^2 + r^2d\Omega_{8-p}^2\right), \\
    e^\phi &= g_s H^{\frac{3-p}{2}}, \quad H = 1 + \frac{q}{r^{7-p}}
\end{align*}
$$

3See also [34, 35, 36] for “generalized conformal symmetry,” which motivated the following analyses.
where \( a = 1, \ldots, p \) and \( q = \tilde{c}_p g_s N_s^{7-p} \) with \( \tilde{c}_p = 2^{6-p} \pi^{(5-p)/2} \Gamma(7-p)/2 \). The integer \( N \) denotes the number of the Dp-branes, \( \ell_s \) is the string length, and \( g_s \) is related to the Yang-Mills coupling by \( g_{YM}^2 = (2\pi)^{p-2} g_s \ell_s^{p-3} \). We consider the near-horizon limit \( r \ll q^{1/(7-p)} \), and take \( H \to q/r^{7-p} \).

For \( p = 3 \), the near-horizon geometry is \( \text{AdS}_5 \times S^5 \). For \( p \neq 3 \), it is related to \( \text{AdS}_{p+2} \times S^{8-p} \) by a Weyl transformation:

\[
\begin{align*}
    ds^2 &= H^{1/2} r^2 \left[ \left( \frac{2}{5-p} \right)^2 \left( \frac{dt^2 + dx_a^2 + dz^2}{z^2} \right) + d\Omega_{8-p}^2 \right].
\end{align*}
\]

The Weyl factor \( H^{1/2} r^2 \) is constant for \( p = 3 \). The radial variable \( z \) in the Poincaré coordinates for \( \text{AdS}_{p+2} \) is defined by

\[
    z = \frac{2}{5-p} (g_s N)^{1/2} (l_s^{(7-p)/2})^{p-(5-p)/2} = \frac{2}{5-p} H^{1/2} r.
\]

For \( p \neq 3 \), there is no AdS isometry, since the Weyl factor, dilaton, and the gauge fields do not have such a symmetry.

The string coupling and the curvature of the background depend on the position for \( p \neq 3 \). For \( p < 3 \), we have strong coupling near the center \( r \to 0 \), and strong curvature near the boundary \( r \to \infty \). To use the supergravity approximation, we will need the region of weak string coupling and weak curvature to cover a large part of the near horizon region. As described in \cite{30}, this is achieved if we take \( N \to \infty \) with 't Hooft coupling fixed but large. In this paper, we will assume \( N \) to be infinite, and assume this is satisfied.

Gauge theory which corresponds to superstring theory on the above background is the maximally supersymmetric \( SU(N) \) Yang-Mills theory in \( (p+1) \) dimensions. This theory is obtained by dimensional reduction from the super Yang-Mills theory in \( (9+1) \) dimensions, and has \( (9-p) \) scalar fields, \( \phi_{p+1} \ldots \phi_9 \), in addition to the gauge field in \( (p+1) \) dimensions. We will consider a complex combination of two scalar fields, say, \( Z \equiv \phi_8 + i\phi_9 \), and a complex combination of two gauge covariant derivatives, say, \( D \equiv D_1 + iD_2 \).

### 3 Rotating strings (strong gauge coupling)

In this section, we will consider the \( p = 3 \) case. We start from the solution found by Gubser, Klebanov and Polyakov (GKP) \cite{21}. After briefly reviewing GKP’s analysis in

\[\text{For } p > 4, \text{ the inequalities are reversed: We have strong curvature near the center } r \to 0, \text{ and strong string coupling near the boundary } r \to \infty.\]
the Lorentzian signature, we will rotate some coordinates to imaginary, and find a solution which connects two points on the boundary. We will see that the Euclidean amplitude for this string can be written in the form of gauge theory correlator.

### 3.1 Lorentzian signature (GKP)

We write the $\text{AdS}_5 \times S^5$ using the global coordinates for AdS,
\[
ds^2 = L^2 \left\{ -\cosh^2 \rho dt^2 + d\rho^2 + \sinh^2 \rho \left( \cos^2 \theta d\psi^2 + d\theta^2 + \sin^2 \theta d\phi^2 \right) + \cos^2 \tilde{\theta} d\tilde{\psi}^2 + d\tilde{\theta}^2 + \sin^2 \tilde{\theta} d\Omega_3^2 \right\}, \tag{4}
\]
since the symmetry is manifest in this coordinate system.

We consider the bosonic part of the string action,
\[
I = \frac{1}{4\pi \alpha'} \int dt \int_0^{2\pi \alpha'} d\sigma \sqrt{-h} \partial_\alpha X^\mu \partial_\beta X^\nu g_{\mu\nu}. \tag{5}
\]
In addition to the equation of motion for $X^\mu$, there is a constraint obtained by varying the action with respect to the worldsheet metric $h^{\alpha\beta}$,
\[
-\frac{1}{2} h^{\alpha\beta} \partial_\gamma X^\mu \partial_\gamma X^\nu g_{\mu\nu} + \partial_\alpha X^\mu \partial_\beta X^\nu g_{\mu\nu} = 0. \tag{6}
\]
In this section we will take the conformal gauge, $\sqrt{-h}h^{\alpha\beta} = \eta^{\alpha\beta}$, in which the above constraint becomes
\[
\dot{X}^\mu \dot{X}^\nu g_{\mu\nu} = -X^\mu X^\nu g_{\mu\nu}, \tag{7}
\]
\[
\dot{X}^\mu X^\nu g_{\mu\nu} = 0. \tag{8}
\]

On the background (4), the momenta corresponding to the translations in $t$, $\psi$ and $\tilde{\psi}$, are conserved. We will call them $E$, $S$ and $J$, respectively,
\[
E \equiv P_t = \frac{\delta S}{\delta \dot{t}} = \frac{L^2}{2\pi \alpha'} \int_0^{2\pi \alpha'} d\sigma \cosh^2 \rho \dot{t}, \tag{9}
\]
\[
S \equiv P_\psi = -\frac{\delta S}{\delta \dot{\psi}} = \frac{L^2}{2\pi \alpha'} \int_0^{2\pi \alpha'} d\sigma \sinh^2 \rho \cos^2 \theta \dot{\psi}, \tag{10}
\]
\[
J \equiv P_{\tilde{\psi}} = -\frac{\delta S}{\delta \dot{\tilde{\psi}}} = \frac{L^2}{2\pi \alpha'} \int_0^{2\pi \alpha'} d\sigma \cos^2 \tilde{\theta} \dot{\tilde{\psi}}. \tag{11}
\]
Following GKP [21] (also allowing the motion along $S^5$ [39, 40, 41]), we take the following ansatz for classical solutions (with $\theta = \tilde{\theta} = 0$),

$$t = \tau, \quad \psi = \omega \tau, \quad \rho = \rho(\sigma), \quad \tilde{\psi} = \tilde{\omega} \tau. \tag{12}$$

Since $t$ and $\psi$ are functions of $\tau$ only, and $\rho$ is a function of $\sigma$ only, the constraint (8) is satisfied. The other constraint (7) determines $\rho$ as a function of $\sigma$,

$$d\sigma = \frac{d\rho}{\sqrt{\cosh^2 \rho - \tilde{\omega}^2 - \omega^2 \sinh^2 \rho}}. \tag{13}$$

The string is folded and stretched: the points $\sigma = 0$ and $\sigma = \pi \alpha'$ on the worldsheet are at the center of the string, $\rho = 0$; the points $\sigma = \pi \alpha'/2$ and $\sigma = 3\pi \alpha'/2$ are at the end, $\rho = \rho_0$.

From (9), (10) and (11), one can find the relation among $E$, $S$, $J$. For example, as described in [21] (for $J = 0$), for small $S$ one has $E^2 \sim S$; for large $S$ one has $E - S \sim \ln S$. By identifying the energy with the scaling dimension, $\Delta \sim E$, strong coupling results of gauge theory have been obtained [21].

### 3.2 Euclidean signature

We would like to make contact with the GKPW prescription based on supergravity, defined on Euclidean AdS. Thus, we replace $t \rightarrow it_E$ in (11). To study strings (or particles) on this background, we will take the worldsheet (or worldline) time imaginary also, $\tau \rightarrow i \tau_E$. In this Euclidean calculation, the angular momenta $J$ and $S$ should be kept real, since these are quantum numbers that specify the representation of the symmetry group, and have direct meanings in gauge theory. Accordingly, as we see from (10) and (11), the angular variables $\psi$ and $\tilde{\psi}$ should be taken imaginary, since $\tau$ is now imaginary. The Euclidean solution takes the form,

$$t_E = \tau_E, \quad \psi = i \omega \tau_E, \quad \rho = \rho(\sigma), \quad \tilde{\psi} = i \tilde{\omega} \tau_E, \tag{14}$$

with $\omega$ and $\tilde{\omega}$ being real; these are the same as the angular velocities for the Lorentzian solution.

By solving the constraint (7), which now becomes

$$\partial_{\tau_E} X^\mu \partial_{\tau_E} X^\nu g_{\mu \nu} = X^{\mu'} X^{\nu'} g_{\mu \nu}, \tag{15}$$

We regard the momentum representation to be fundamental for some variables such as $\psi$ and $\tilde{\psi}$ here. Thus, we do not particularly pursue physical meanings for the imaginary values of $\psi$ and $\tilde{\psi}$.

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we obtain the relation between $\rho$ and $\sigma$. It is of the same form as (13).

The Euclidean version of $E$, which we will call $H$, is

$$H = \frac{L^2}{2\pi \alpha'} \int_{0}^{2\pi \alpha'} d\sigma \cosh^2 \rho \frac{dE}{d\tau} = \frac{2L^2}{\pi \alpha'} \int_{0}^{\rho_0} d\rho \frac{\cosh^2 \rho}{\sqrt{\cosh^2 \rho - \tilde{\omega}^2 - \omega^2 \sinh^2 \rho}},$$

(16)

and the angular momentum along $\psi$ is

$$S = -i \frac{L^2}{2\pi \alpha'} \int_{0}^{2\pi \alpha'} d\sigma \sinh^2 \rho \frac{d\psi}{d\tau} = \frac{2L^2}{\pi \alpha'} \omega \int_{0}^{\rho_0} d\rho \frac{\sinh^2 \rho}{\sqrt{\cosh^2 \rho - \tilde{\omega}^2 - \omega^2 \sinh^2 \rho}}.$$

(17)

**Poincaré coordinates**

We now rewrite the above solution in the Poincaré coordinates, so that the correspondence with gauge theory becomes clear. For definiteness, we present the coordinate transformations using the embedding coordinates explicitly. We will write the formulas for general $p$ in this subsection.

The Euclidean AdS$_{p+2}$ is represented as a hyperboloid,

$$-X_{p+2}^2 + X_0^2 + \sum_{a=1}^{p+1} X_a^2 = -L^2,$$

(18)

in the $p + 3$ dimensional flat space, $ds^2 = -dX_{p+2}^2 + dX_0^2 + \sum_{a=1}^{p+1} dX_a^2$. The hyperboloid is parametrized by the global coordinates as

$$X_{p+2} = L \cosh \rho \cosh t_E,$$

$$X_0 = L \cosh \rho \sinh t_E,$$

$$X_a = L \sinh \rho \Omega_a,$$

(19)

where $\sum_{a=1}^{p+1} \Omega_a^2 = 1$.

The solution (14) effectively has angular variable $\psi$ imaginary. If we choose $\psi$ to parametrize the rotation in the $X_1$-$X_2$ plane as follows,

$$X_1 = L \sinh \rho \cos \theta \cos \psi, \quad X_2 = L \sinh \rho \cos \theta \sin \psi,$$

(20)

the coordinate $X_2$ effectively becomes imaginary if $\psi$ is imaginary. In the following, $X_2$, and also $x_2$ defined below, are understood to be imaginary.$^6$

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$^6$If we define a real variable $\hat{x}_2$ by $x_2 = i \hat{x}_2$, the expression $\sum_i x_i^2$ below means $x_1^2 - \hat{x}_2^2 + x_3^2 + \cdots$. 

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The Poincaré coordinates are defined as
\[
X_{p+2} = \frac{z}{2} \left( 1 + \frac{L^2 + \sum x_i^2 + t_i^2}{z^2} \right),
\]
\[
X_0 = L \frac{t_P}{z}, \quad X_i = L \frac{x_i}{z},
\]
\[
X_{p+1} = \frac{z}{2} \left( 1 + \frac{-L^2 + \sum x_i^2 + t_i^2}{z^2} \right). \tag{21}
\]

Equating (19) and (21), we obtain
\[
t_P = \tilde{\ell} \tanh \tau_E, \tag{22}
\]
\[
z = \frac{\tilde{\ell}}{\cosh \rho \cosh \tau_E}, \tag{23}
\]
\[
x_i^2 = z^2 \sinh^2 \rho = \frac{\tilde{\ell}^2 \tanh^2 \rho}{\cosh^2 \tau_E}, \tag{24}
\]
where \(\tilde{\ell}\) is an arbitrary constant.

From (24), we see that the spatial extent of the string reduces to a point in terms of the coordinate \(x_i\), as the string approaches the boundary \(z \to 0\) (or \(\tau \to \pm \infty\)), in the following sense: the string occupies \(0 \leq \rho \leq \rho_0\) (with \(\rho_0\) being a finite constant), and the center of the string, \(\rho = 0\), is at \(x_i = 0\); eq. (24) shows that the value of \(x_i^2\) corresponding to the endpoint of the string, \(\rho = \rho_0\), goes to zero\(^7\) as \(z \to 0\) (or \(|\tau| \to \infty\)). This fact is in comfort with the fact that we represent this state as a local operator in gauge theory.

**Gauge theory correlator**

By substituting \(\tau_E = \pm \infty\) in (22), we see that the coordinate distance between the two points, \(t_i\) and \(t_f\) (both with \(x_i = 0\)), on the boundary is given by
\[
|t_f - t_i| = 2\tilde{\ell}. \tag{25}
\]

In fact, (24) shows that the center of mass of the string follows the same trajectory as the trajectory of a particle which has an angular momentum along \(S^{8-p}\) (and not along \(\text{AdS}_{p+2}\)). The latter was first studied for \(p = 3\) by Dobashi, Shimada and Yoneya [22], and

\(^7\)If \(x_i\) were all real, this would really mean that the string reduces to a single point at the boundary. In our case where \(x_2 = i\tilde{x}_2\) is imaginary, string is really stretched in the “lightlike” direction in the \(x_1-\tilde{x}_2\) plane. However, as mentioned in a previous footnote, we do not attach a particular physical meaning to imaginary values of the coordinates. Thus, if the invariant distance \(\sum_i x_i^2\) is zero, we interpret it as a single point.
then generalized to \( p \neq 3 \) by Asano, Sekino and Yoneya [25]. It will be also described in Sec. 4 of this paper.

The Euclidean amplitude for a string which propagates from a point on the boundary to another point on the boundary is interpreted as the two-point function of gauge theory. In the classical approximation, the bulk amplitude is

\[
e^{-S_{cl}} = e^{-\int_{-T}^{T} d\tau E H(\tau_E)}, \tag{26}\]

where \( H(\tau_E) \) is the Hamiltonian at Euclidean worldsheet time \( \tau_E \). This is just the Euclidean version of the global energy given in (16), since the worldsheet time is equal to the global time, in our classical solution (14).

We introduce a cutoff \( T \) for the worldsheet time, \(-T \leq \tau \leq T\). We also introduce a cutoff for the radial coordinate, \( z \geq 1/\Lambda \). This IR cutoff in the bulk is interpreted as a UV cutoff in gauge theory. The relation between the two cutoffs can be read off from (22) and (23) in the \( |T| \rightarrow \infty \) limit as

\[
2\tilde{\ell} \Lambda = e^T. \tag{27}\]

For the solution (14), the Hamiltonian is independent of the worldsheet time. Therefore, it can be taken out of the integral, and amplitude can be written as

\[
e^{-\int_{-T}^{T} d\tau E H(\tau_E)} = e^{-2HT} = \frac{1}{(\Lambda |t_f - t_i|)^{2H}}. \tag{28}\]

The worldsheet Hamiltonian gives the scaling dimension, and we have recovered the result of GKP.

This formalism will be applicable to the \( p \neq 3 \) case without conformal symmetry. In that case, the worldsheet Hamiltonian will not be time independent in general, thus the above integral has to be evaluated explicitly. In addition, finding the string solution is more challenging than \( p = 3 \), since the AdS isometry is not the symmetry of the string due to the position dependent Weyl factor. One approach would be to study the limit of short string, by using the approximate form of the geometry near the center of the string, which follows the similar trajectory as for the \( p = 3 \) case. This subject is under study [42], and will be reported elsewhere.

4 Rotating particles (weak gauge coupling)

Let us now consider weak gauge coupling. In this paper, we will concentrate on the case where the gauge coupling is strictly zero. On the string theory side, this corresponds to the
case where string tension is zero. If the spatial direction of the worldsheet is discretized into bits, as proposed in the previous paper [29], the interactions among the bits can be ignored in this limit. We assume that one bit carries a single angular momentum along $S^{8-p}$. The previous paper [29] considered only the state with angular momentum along $S^{8-p}$, but here we will extend the analysis to the states which have angular momentum also along $\text{AdS}_{p+2}$.

We first study the $p = 3$ case using the Lorentzian formulation, by identifying the AdS energy in terms of the global time with the scaling dimension. We then study the case of general $p$ using the Euclidean formulation which does not rely on conformal symmetry.

### 4.1 $p = 3$

A bit is a massless particle in ten dimensional spacetime [29], namely, $\text{AdS}_5 \times S^5$ in this case. We will take the momentum representation in the $S^5$ direction. For the classical analysis performed in this subsection, we can perform Kaluza-Klein reduction, and treat a bit as a massive particle on $\text{AdS}_5$. It has mass $m = J/L$, where $J$ is an integer, and $L$ is the radius of $S^5$, which is equal to the radius of $\text{AdS}_5$. One bit has $J = 1$ [29], but we will keep $m$ in the formulas below for the time being.

The action for a single bit is given by

$$I_{\text{bit}}[X^\mu] = \frac{1}{2} \int d\tau \left[ \frac{1}{\eta(\tau)} g_{\rho\sigma}(X^{\mu}(\tau)) \dot{X}^\rho(\tau) \dot{X}^\sigma(\tau) - \eta(\tau) m^2 \right],$$  \hspace{1cm} (29)

and the corresponding EOM is

$$g_{\rho\sigma} \ddot{X}^\rho \ddot{X}^\sigma = -\eta^2 m^2,$$  \hspace{1cm} (30)

$$\dot{X}^\rho \partial_\rho \dot{X}^\mu = -\Gamma^\mu_{\rho\sigma} \dot{X}^\rho \dot{X}^\sigma.$$  \hspace{1cm} (31)

For $\text{AdS}_5$ in the global coordinates [41], the single-bit action becomes

$$I_{\text{bit}}[X^\mu] = \frac{1}{2} \int d\tau \left[ \frac{L^2}{\eta} \left( -\cosh^2 \rho \dot{t}^2 + \dot{\rho}^2 + \sinh^2 \rho \dot{\psi}^2 + \text{(remaining angular part)} \right) - \eta m^2 \right].$$  

The following canonical momenta are conserved, corresponding to the isometry of the background,

$$E \equiv -p_t = \frac{L^2}{\eta} \cosh^2 \rho \dot{t},$$  \hspace{1cm} (32)

$$S \equiv p_\psi = \frac{L^2}{\eta} \sinh^2 \rho \dot{\psi}.$$  \hspace{1cm} (33)
Let us take $\rho = \text{const.}$ as an ansatz for a classical solution. Setting $\dot{\rho} = 0$ in the constraint (30), we find

$$-\frac{E^2}{\cosh^2 \rho} + \frac{S^2}{\sinh^2 \rho} = -m^2 L^2. \quad (34)$$

In addition, in order to have $\ddot{\rho} = 0$, the EOM for $\rho$,

$$\ddot{\rho} = \cosh \rho \sinh \rho \left[ -\dot{t}^2 + \dot{\psi}^2 \right]$$
$$= \cosh \rho \sinh \rho \frac{\eta^2}{L^4} \left[ -\frac{E^2}{\cosh^4 \rho} + \frac{S^2}{\sinh^4 \rho} \right], \quad (35)$$

indicates

$$-\frac{E^2}{\cosh^2 \rho} + \frac{S^2}{\sinh^4 \rho} = 0. \quad (36)$$

Eqs. (34) and (36) lead to

$$E^2 = m^2 L^2 \cosh^4 \rho, \quad S^2 = m^2 L^2 \sinh^4 \rho$$
$$\iff E = mL \cosh^2 \rho, \quad |S| = mL \sinh^2 \rho. \quad (37)$$

Thus,

$$E = |S| + mL \quad (38)$$

One bit has a single unit of angular momentum on $S^5$, and $m = 1/L$, so its energy is

$$E_{\text{bit}} = |S| + 1. \quad (39)$$

Now consider a collection of $n$ non-interacting bits. Its total energy $E = \sum_{i=1}^{n} E_i$ can be written as

$$E = \sum_{i=1}^{n} |S_i| + n, \quad (40)$$

where $|S_i|$ is the magnitude of the angular momentum along AdS$_5$ (which is integer in our convention) carried by the $i$-th bit. The $n$ bits could have different directions of angular momenta. One bit has a single unit of angular momentum along $S^5$, and contributes 1

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8The energy takes only positive values $E \geq 0$, but the angular momentum $S$ could be positive or negative.
to the energy; summing this over the \( n \) bits, we obtain \( n \) in the last term of (40). In the special case where all the bits have angular momenta in the same direction, both for AdS\(_5\) and S\(_5\), the total energy is

\[
E = |S| + J,
\]

where \(|S|\) and \(J\) are the magnitudes of the total angular momenta along AdS\(_5\) and S\(_5\), respectively.

The above results for the non-interacting bits correctly reproduce the free-field results in gauge theory, if we identify the global energy \(E\) with the scaling dimension \(\Delta\) of the corresponding operator. A single bit is assumed to correspond to a single scalar field inside the trace, which contributes 1 to the scaling dimension in the free theory in (3+1) dimensions. The \(i\)-th bit with angular momentum \(S_i\) along AdS\(_5\) can be realized by applying \(|S_i|\) gauge covariant derivatives on the \(i\)-th scalar field. In the free theory, covariant derivative is just a partial derivative, which contributes 1 to the scaling dimension. Thus, (40) gives the correct free-field result under the identification \(\Delta = E\). We will often consider operators of the form \(\text{Tr}(Z^{J-k}D^SZ^k)\), which corresponds to the state where the directions of angular momenta for all the bits are the same, whose scaling dimension is given by (41).

### 4.2 General \(p\)

Let us now consider the case of general \(p\). The background geometry is conformal to the AdS\(_{p+2} \times S^{8-p}\) spacetime\(^9\),

\[
g = L^2 r^{-\frac{3-p}{2}} \left[ \left(\frac{2}{5-p}\right)^2 \frac{1}{z^2} \left\{ -dt^2 + dz^2 + dx_a^2 \right\} + (d\theta^2 + \cos^2 \theta d\psi^2 + \sin^2 \theta d\Omega^2_{6-p}) \right],
\]

where \(a = 1, \ldots, p\). The radius of AdS\(_{p+2}\) is \(\frac{2}{5-p}\) times the radius of \(S^{8-p}\), as we see from the factor on the first term. For general \(p\), there is no conformal symmetry (or AdS isometry), and we cannot use the identification \(\Delta = E\), so we follow the DSY prescription \([22, 25, 26]\). As explained in Section 3, gauge theory correlators are obtained by calculating the transition amplitude from the path integral in the multiply Wick rotated background:

\[
\langle t^E_f, J, \Theta_f | t^E_i, J, \Theta_i \rangle = \langle it_f, J, \Theta_f | it_i, J, \Theta_i \rangle = \int DX e^{-(I+J\psi-f-J\psi+\Theta f-S\Theta_i)}.
\]

\(^9\)Up to now we have put tilde on the angles in \(S^{8-p}\), but in this subsection, we will omit tilde and denote them \(\theta, \psi\), to simplify the notations.
As in the last subsection, we will study the case of zero gauge coupling by ignoring interactions among the bits. The action for a single bit on the background (42) is

\[ I = \int_{-T}^{T} d\tau \, \frac{\alpha}{2} L^2 \left[ \left( \frac{2}{5 - p} \right)^2 \frac{1}{z^2} \left\{ \dot{t}^2 + \dot{z}^2 + \dot{R}^2 - R^2 \dot{\Theta}^2 + \cdots \right\} + (\dot{\theta}^2 - \cos^2 \theta \dot{\psi}^2 + \cdots) \right] . \]  

(44)

Here, we have absorbed the Weyl factor in the metric (42) by a suitable choice of the einbein; as a result, the action (44) is formally the same as the one in AdS_{p+2} \times S^{8-p}. The factor \( \tilde{\alpha} \) is a constant to be determined later. The coordinates \( R \) and \( \Theta \) are the radius and angle defined from two of the coordinates \( x_a \).

The equations of motion are

\[ \frac{d}{d\tau} \left( \frac{\dot{t}}{z^2} \right) = 0, \quad \frac{d}{d\tau} \left( \frac{\dot{z}}{z^2} \right) = -\frac{1}{z^3} \{ \dot{t}^2 + \dot{z}^2 + \dot{R}^2 - R^2 \dot{\Theta}^2 + \cdots \}, \quad \frac{d}{d\tau} \left( \frac{\dot{R}}{z^2} \right) = -\frac{R}{z^2} \dot{\Theta}^2, \]

\[ \frac{d}{d\tau} \left( \frac{R^2}{z^2} \dot{\Theta} \right) = 0, \quad \frac{d}{d\tau} \dot{\theta} = -\sin \theta \cos \theta \dot{\psi}^2, \quad \frac{d}{d\tau} \dot{\psi} = 0, \]

and the constraint is

\[ \left( \frac{2}{5 - p} \right)^2 \frac{1}{z^2} \left\{ \dot{t}^2 + \dot{z}^2 + \dot{R}^2 - R^2 \dot{\Theta}^2 + \cdots \right\} + (\dot{\theta}^2 - \cos^2 \theta \dot{\psi}^2 + \cdots) = 0 . \]

(46)

**The S = 0 case**

When \( S = 0 \), the solution has been obtained by Asano, Sekino and Yoneya [25, 26];

\[ t = \tilde{\ell} \tanh \tau, \quad z = \frac{\tilde{\ell}}{\cosh \tau}, \quad R = \Theta = 0, \]

\[ \theta = 0, \quad \psi = \frac{2}{5 - p} \tau, \]

(47)

where \( \tilde{\ell} \) parametrizes the separation between \( t_f \) and \( t_i \), that is,

\[ |t_f - t_i| = 2\tilde{\ell} . \]

(48)

The solution (47) is represented as a half sphere \( t^2 + z^2 = \tilde{\ell}^2 \) in the coordinate space \( (t, z) \).

As in the last section, we introduce the IR cutoff at \( z = \frac{1}{\Lambda} \), which is related to the worldsheet cutoff \( T \) as

\[ 2\tilde{\ell} \Lambda = e^T . \]

(49)
The transition amplitude at the zero-loop level on the worldsheet becomes
\[
\langle t_f, J, 0 | t_i, J, 0 \rangle \simeq e^{-(I + J(\psi_f - \psi_i))} = e^{-\frac{4}{5-p}JT} = \frac{1}{\Lambda^{\frac{4}{5-p}} t_f - t_i} \frac{1}{\frac{4}{5-p} J} \quad \text{(at zero loop)} \tag{50}
\]
where we have used (48) and (49) in the last line to rewrite the amplitude in terms of the variables in gauge theory. The expression (50) can be regarded as the leading part in the large $J$ limit of the correlator of $\text{Tr}(Z^J)$ at strong gauge coupling [25, 26].

Let us consider the one-loop contribution on the worldsheet, following [29]. The action of a single bit at the quadratic level of the bosonic fluctuations\(^{10}\) around the classical trajectory (47) is [25, 26]
\[
I^{(2)} = \frac{\tilde{\alpha}}{2} \int_{-T}^{T} d\tau \{ \dot{x}_a^2 + m_x^2 x_a^2 + \dot{y}_i^2 + m_y^2 y_i^2 \} \tag{51}
\]
where $m_x = 1$, $m_y = \frac{2}{5-p}$.

In [29], the modification to the correlator (50) for $\text{Tr}(Z^J)$ due to the one-loop contribution on the worldsheet has been obtained by an operator method, by including the zero-point energies of the bosonic and fermionic fluctuations (which are harmonic oscillators). Here we will derive the same result from the Euclidean path integral of harmonic oscillators with the boundary condition $x(T) = x(-T) = 0$.

Since the complete set of basis may be given by
\[
\left\{ \cos \frac{\pi}{2T} (2\bar{k} + 1) \tau, \sin \frac{\pi}{2T} (2\tilde{k}) \tau \right\}_{\bar{k}, \tilde{k}} \tag{52}
\]
the eigenvalues of the operator $\left(-\frac{d^2}{d\tau^2} + m^2\right)$ are given by
\[
\lambda_k = \frac{\pi^2}{4T^2} k^2 + m^2 \quad (k \in \mathbb{Z}_{\geq 0}) \tag{53}
\]
Therefore, the path integral is
\[
Z^{(2)} = \int D_x e^{-\frac{1}{2} \int_{-T}^{T} d\tau \{ \dot{x}_a^2 + m_x^2 x_a^2 \}} = \prod_{\bar{k}, \tilde{k}} \int dA \int dB e^{-A^2 \left( \frac{x_a^2}{m_x^2} (2\bar{k} + 1)^2 + m^2 \right) T} e^{-B^2 \left( \frac{x_i^2}{m_y^2} (2\tilde{k})^2 + m^2 \right) T} = \prod_{k} \sqrt{\frac{\pi}{\lambda_k T}} \tag{54}
\]
\(^{10}\)For the quadratic action including fermionic fluctuations, see [26, 29].
that is,

$$\log Z^{(2)} = -\frac{1}{2} \sum_k \log \lambda_k + \text{(divergent part)}. \tag{55}$$

Here, we apply the zeta function regularization method. Redefine $Z^{(2)}$ by

$$\log Z^{(2)} \equiv \frac{1}{2} \frac{d\zeta(s)}{ds} \bigg|_{s=0} \tag{56}$$

where the generalized zeta function is given by

$$\zeta(s) \equiv \sum_k \lambda_k^{-s}. \tag{57}$$

Now make $k$ continuous by considering large $T$, i.e.,

$$\zeta(s) = \sum_k \left( \frac{\pi^2}{4T^2} k^2 + m^2 \right)^{-s} = \frac{2T}{\pi} \sum_K (K^2 + m^2)^{-s} \Delta K \notag$$

$$\rightarrow \frac{2T}{\pi} \int_0^\infty dK (K^2 + m^2)^{-s} \tag{58}$$

where we changed the variable $K \equiv \frac{\pi}{2T} k$ and $\Delta K = \frac{\pi}{2T}$. Then,

$$\zeta(s) = \frac{2T}{\pi} \int_0^\infty dK (K^2 + m^2)^{-s} = \frac{T}{\sqrt{\pi}} m^{1-2s} \frac{\Gamma(s - \frac{1}{2})}{\Gamma(s)} \tag{59}$$

$$\simeq -2mTs + 4mT(-1 + \log 2 + \log m)s^2 + \cdots \tag{60}$$

Eventually,

$$Z^{(2)} = e^{\frac{1}{2} \frac{d\zeta(s)}{ds} \bigg|_{s=0}} = e^{-mT}. \tag{61}$$

Note that this is independent of the overall factor of the harmonic oscillator action. Therefore, the one-loop contribution of the bosonic part is

$$Z_{J \text{ bit (boson)}}^{(2)} = e^{-J((p+1)m_x + (7-p)m_y)T} = e^{-\frac{p^2+2p+9}{5-2p}JT}. \tag{62}$$

Similarly, the one-loop contribution of the fermionic part is

$$Z_{J \text{ bit (fermion)}}^{(2)} = e^{8m_fT} = e^{\frac{28-4p}{5-2p}JT}. \tag{63}$$

Then

$$Z_{J \text{ bit}}^{(2)} = Z_{J \text{ bit (boson)}}^{(2)} Z_{J \text{ bit (fermion)}}^{(2)} = e^{\frac{(p-3)^2}{5-2p}JT}. \tag{64}$$
Combining the zero- and one-loop contributions on the worldsheet, the amplitude becomes

\[
\langle t_f, J, 0|t_i, J, 0 \rangle \simeq e^{-(I+J(\psi_f-\psi_i))}Z^{(2)}_{J_{\text{bit}}}
= e^{-(p-1)JT}
= \frac{1}{\Lambda^{(p-1)J}} \frac{1}{|t_f-t_i|(p-1)J}.
\]  

This reproduces the free-field result for the two-point function of Tr($Z^J$) in the $(p+1)$-dimensional gauge theory \[29\]: from dimensional analysis, a scalar field has dimension $(p-1)/2$, and the operator consists of $J$ scalar fields.

**The $S \neq 0$ case**

In this case, the solution is

\[
t = \bar{\ell} \tanh \tau, \quad z = \sqrt{\bar{\ell}^2 - B^2} \cosh \tau, \quad R = \frac{B}{\cosh \tau}, \quad \Theta = \tau,
\]

\[
\theta = 0, \quad \psi = \frac{2}{5-p} \tau.
\]  

Similarly to the solution (47) for $S = 0$, this is represented as a half sphere, now given by $t^2 + z^2 + R^2 = \bar{\ell}^2$ in the coordinate space $(t,z,R)$. Since $J$ and $S$ are given by

\[
J = \tilde{\alpha} L^2 \psi = \tilde{\alpha} L^2 \frac{2}{5-p},
\]

\[
S = \tilde{\alpha} L^2 \left( \frac{2}{5-p} \right)^2 \frac{R^2}{z^2} \Theta = \tilde{\alpha} L^2 \left( \frac{2}{5-p} \right)^2 \frac{B^2}{\bar{\ell}^2 - B^2},
\]  

the constants $\tilde{\alpha}$ and $B$ are related to $J$, $S$ and $\bar{\ell}$ via

\[
\tilde{\alpha} = \frac{5-p}{2} \frac{J}{L^2},
\]

\[
B = \sqrt{\frac{S(5-p)}{2J+S(5-p)}} \bar{\ell}.
\]  

We introduce the IR cutoff at $z = \frac{1}{\Lambda}$. The relation between $\Lambda$ and $T$ can be read off from (66) in the $|T| \to \infty$ limit, and becomes

\[
2\bar{\ell} \Lambda \sqrt{\frac{2J}{2J+S(5-p)}} = e^T.
\]
At the zero-loop level on the worldsheet, the transition amplitude is given by

\[ \langle t_f, J, S | t_i, J, S \rangle \simeq e^{-\left( I + J(\psi_f - \psi_i) + S(\Theta_f - \Theta_i) \right)} \]

\[ = e^{-\left( \frac{4}{\sqrt{p}} J + 2S\right) T} \]

\[ = \left( \frac{2J + S(5 - p)}{2J} \right)^{\frac{2}{\sqrt{p}} J + S} \frac{1}{\Lambda^{\frac{1}{2}} p J + 2S} \frac{1}{|t_f - t_i|^{\frac{1}{2}} p J + 2S}. \]  \tag{72} \]

The amplitude including the one-loop contribution can be obtained in the same manner as in the \( S = 0 \) case, and becomes\[^{[11]}\]

\[ \langle t_f, J, S | t_i, J, S \rangle \simeq e^{-\left( I + J(\psi_f - \psi_i) + S(\Theta_f - \Theta_i) \right)} Z^{(2)}_{J\text{-bit}} \]

\[ = e^{-\left( (p-1) J + 2S\right) T} \]

\[ = \left( \frac{2J + S(5 - p)}{2J} \right)^{\frac{p-1}{2} J + S} \frac{1}{\Lambda^{(p-1)/2} p J + 2S} \frac{1}{|t_f - t_i|^{(p-1)/2} J + 2S}. \]  \tag{73} \]

This agrees with the free field result of the correlator for the operator of the form \( \text{Tr}(Z^{J-k} D^S Z^k) \):

At zero coupling, the covariant derivative \( D \) is just a partial derivative. Inserting one derivative in the operator increases two powers of the coordinate distance in the two-point function.

### 5 Conclusions

In this paper, we considered gauge/gravity correspondence between maximally supersymmetry Yang-Mills theories in \((p+1)\)-dimensions and superstrings on the near horizon limit of the \( Dp \)-brane solutions. We computed two-point functions of operators with angular momentum along the AdS directions from the string worldsheet theory.

First, we considered the conventional continuum string theory, which should correspond to the strongly-coupled gauge theory. We considered the conformally invariant case of \( p = 3 \), and started from the the folded and rotating string solution found by Gubser, Klebanov and Polyakov \[21\]. We rotated some coordinates to imaginary, and obtained a string configuration which connects two points on the boundary. From the string amplitude, we obtained gauge-theory correlators. This is not based on the identification of the energy with the global time, and can be applied to theories without conformal invariance. We

\[^{[11]}\text{Here we assumed the quadratic action for the bosonic and fermionic fluctuations around the solution (66) is the same as the one around (47). This seems plausible, but may need a rigorous proof.}\]
will defer the analysis of $p \neq 3$ for future study. This is an interesting technical challenge, since the isometry of $\text{AdS}_{p+2} \times S^{8-p}$ is not a symmetry of string action, since the position dependent Weyl factor contributes.

Then, we considered the opposite limit of zero gauge coupling. As in the previous paper [29], we assumed the string is made of bits, each of which has a single unit of angular momentum along $S^{8-p}$. In the limit of weak gauge coupling, the string tension is small compared with the scale of angular momenta. At zero coupling, we computed the amplitude by ignoring the interactions among bits. We confirmed that we obtain the free-field correlator of gauge theory. We regard this result as a strong indication that the approach of [29] is valid in general.

There are two main directions for future research. One direction is towards understanding of the weak-coupling limit of gauge/gravity correspondence [42]. We will be able to incorporate the interactions among the string bits perturbatively. The angular momentum (mass after reduction on $S^{8-p}$) of a bit is inversely proportional to the AdS radius. When AdS radius is small relative to the string scale, the mass of a bit is large. Also in this limit, the string tension is effectively small. This limit corresponds to weak gauge coupling. It is a highly important problem whether perturbative expansion in string tension (the coupling among the bits) agrees with the perturbative expansion in gauge theory. If they agree, this can be regarded as a proof of gauge/gravity correspondence.

Another direction is towards understanding of gauge theory without conformal invariance at strong gauge coupling. The string solution along the lines of Sec. 3 for $p \neq 3$ is under study [42]. It is important to study the large $S$ behavior of that solution. For $p = 3$ there is a characteristic large $S$ behavior in the scaling dimensions of the form $\log S$. In gauge theory, this comes from gauge fields propagating in the internal lines. Large $S$ behavior for $p \neq 3$ will give new piece of information for the structure of these gauge theories.

Apart from the above two problems, it would be interesting to extend the analysis in this paper to more general backgrounds. Our formalism of computing the gauge-theory correlator from the string amplitude, described in Sec. 3, would be applicable to general backgrounds without conformal symmetry. Also it is important to see how general the string bit picture, described in Sec. 4 and in [29], is valid. There would be many backgrounds where one can ignore interactions among the bits in some limits. One could pursue interpretations of these limits in terms of gauge theories.
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References

[1] Juan Martin Maldacena. The Large N limit of superconformal field theories and supergravity. *Adv. Theor. Math. Phys.*, 2:231–252, 1998.

[2] Leonard Susskind. The World as a hologram. *J. Math. Phys.*, 36:6377–6396, 1995.

[3] Christopher R. Stephens, Gerard ’t Hooft, and Bernard F. Whiting. Black hole evaporation without information loss. *Class. Quant. Grav.*, 11:621–648, 1994.

[4] S. S. Gubser, Igor R. Klebanov, and Alexander M. Polyakov. Gauge theory correlators from noncritical string theory. *Phys. Lett. B*, 428:105–114, 1998.

[5] Edward Witten. Anti-de Sitter space, thermal phase transition, and confinement in gauge theories. *Adv. Theor. Math. Phys.*, 2:505–532, 1998.

[6] Sergio Ferrara, Christian Fronsdal, and Alberto Zaffaroni. On N=8 supergravity on AdS(5) and N=4 superconformal Yang-Mills theory. *Nucl. Phys. B*, 532:153–162, 1998.

[7] J. K. Erickson, G. W. Semenoff, R. J. Szabo, and K. Zarembo. Static potential in N=4 supersymmetric Yang-Mills theory. *Phys. Rev. D*, 61:105006, 2000.

[8] Niklas Beisert, Burkhard Eden, and Matthias Staudacher. Transcendentality and Crossing. *J. Stat. Mech.*, 0701:P01021, 2007.

[9] Edward Witten. Anti-de Sitter space and holography. *Adv. Theor. Math. Phys.*, 2:253–291, 1998.

[10] D. Z. Freedman, S. S. Gubser, K. Pilch, and N. P. Warner. Renormalization group flows from holography supersymmetry and a c theorem. *Adv. Theor. Math. Phys.*, 3:363–417, 1999.
[11] Igor R. Klebanov and Matthew J. Strassler. Supergravity and a confining gauge theory: Duality cascades and chi SB resolution of naked singularities. *JHEP*, 08:052, 2000.

[12] Joseph Polchinski and Matthew J. Strassler. Hard scattering and gauge / string duality. *Phys. Rev. Lett.*, 88:031601, 2002.

[13] Jorge Casalderrey-Solana, Hong Liu, David Mateos, Krishna Rajagopal, and Urs Achim Wiedemann. *Gauge/String Duality, Hot QCD and Heavy Ion Collisions*. Cambridge University Press, 2014.

[14] Sean A. Hartnoll, Andrew Lucas, and Subir Sachdev. Holographic quantum matter. 12 2016.

[15] Nissan Itzhaki, Juan Martin Maldacena, Jacob Sonnenschein, and Shimon Yankielowicz. Supergravity and the large N limit of theories with sixteen supercharges. *Phys. Rev. D*, 58:046004, 1998.

[16] Juan Martin Maldacena and Hirosi Ooguri. Strings in AdS(3) and SL(2,R) WZW model 1.: The Spectrum. *J. Math. Phys.*, 42:2929–2960, 2001.

[17] Juan Martin Maldacena, Hirosi Ooguri, and John Son. Strings in AdS(3) and the SL(2,R) WZW model. Part 2. Euclidean black hole. *J. Math. Phys.*, 42:2961–2977, 2001.

[18] Juan Martin Maldacena and Hirosi Ooguri. Strings in AdS(3) and the SL(2,R) WZW model. Part 3. Correlation functions. *Phys. Rev. D*, 65:106006, 2002.

[19] Justin R. David, Gautam Mandal, and Spenta R. Wadia. Microscopic formulation of black holes in string theory. *Phys. Rept.*, 369:549–686, 2002.

[20] David Eliecer Berenstein, Juan Martin Maldacena, and Horatiu Stefan Nastase. Strings in flat space and pp waves from N=4 superYang-Mills. *JHEP*, 04:013, 2002.

[21] S. S. Gubser, I. R. Klebanov, and Alexander M. Polyakov. A Semiclassical limit of the gauge / string correspondence. *Nucl. Phys. B*, 636:99–114, 2002.

[22] Suguru Dobashi, Hidehiko Shimada, and Tamiaki Yoneya. Holographic reformulation of string theory on AdS(5) x S**5 background in the PP wave limit. *Nucl. Phys. B*, 665:94–128, 2003.
[23] Suguru Dobashi and Tamiaki Yoneya. Resolving the holography in the plane-wave limit of AdS/CFT correspondence. Nucl. Phys. B, 711:3–53, 2005.

[24] Suguru Dobashi. Impurity Non-Preserving 3-Point Correlators of BMN Operators from PP-Wave Holography. II. Fermionic Excitations. Nucl. Phys. B, 756:171–206, 2006.

[25] Masako Asano, Yasuhiro Sekino, and Tamiaki Yoneya. PP wave holography for Dp-brane backgrounds. Nucl. Phys. B, 678:197–232, 2004.

[26] Masako Asano and Yasuhiro Sekino. Large N limit of SYM theories with 16 supercharges from superstrings on Dp-brane backgrounds. Nucl. Phys. B, 705:33–59, 2005.

[27] Leonard Susskind and Edward Witten. The Holographic bound in anti-de Sitter space. 5 1998.

[28] Matthias R. Gaberdiel and Rajesh Gopakumar. String Dual to Free $N = 4$ Supersymmetric Yang-Mills Theory. Phys. Rev. Lett., 127(13):131601, 2021.

[29] Yasuhiro Sekino. Evidence for weak-coupling holography from the gauge/gravity correspondence for Dp-branes. PTEP, 2020(2):021B01, 2020.

[30] Yasuhiro Sekino and Tamiaki Yoneya. Generalized AdS / CFT correspondence for matrix theory in the large N limit. Nucl. Phys. B, 570:174–206, 2000.

[31] Yasuhiro Sekino. Super currents in matrix theory and the generalized AdS / CFT correspondence. Nucl. Phys. B, 602:147–171, 2001.

[32] Masanori Hanada, Jun Nishimura, Yasuhiro Sekino, and Tamiaki Yoneya. Monte Carlo studies of Matrix theory correlation functions. Phys. Rev. Lett., 104:151601, 2010.

[33] Masanori Hanada, Jun Nishimura, Yasuhiro Sekino, and Tamiaki Yoneya. Direct test of the gauge-gravity correspondence for Matrix theory correlation functions. JHEP, 12:020, 2011.

[34] Antal Jevicki and Tamiaki Yoneya. Space-time uncertainty principle and conformal symmetry in D particle dynamics. Nucl. Phys. B, 535:335–348, 1998.
[35] Antal Jevicki, Yoichi Kazama, and Tamiaki Yoneya. Quantum metamorphosis of conformal transformation in D3-brane Yang-Mills theory. *Phys. Rev. Lett.*, 81:5072–5075, 1998.

[36] Antal Jevicki, Yoichi Kazama, and Tamiaki Yoneya. Generalized conformal symmetry in D-brane matrix models. *Phys. Rev. D*, 59:066001, 1999.

[37] Ingmar Kanitscheider, Kostas Skenderis, and Marika Taylor. Precision holography for non-conformal branes. *JHEP*, 09:094, 2008.

[38] Masako Asano. Stringy effect of the holographic correspondence for dp-brane backgrounds. *JHEP*, 12:029, 2004.

[39] S. Frolov and Arkady A. Tseytlin. Semiclassical quantization of rotating superstring in AdS(5) x S**5. *JHEP*, 06:007, 2002.

[40] S. Frolov and Arkady A. Tseytlin. Multispin string solutions in AdS(5) x S**5. *Nucl. Phys. B*, 668:77–110, 2003.

[41] S. Frolov and Arkady A. Tseytlin. Quantizing three spin string solution in AdS(5) x S**5. *JHEP*, 07:016, 2003.

[42] Tomotaka Kitamura, Shoichiro Miyashita, and Yasuhiro Sekino. work in progress.