PT symmetry in quasi-integrable models

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Abstract

Observations of almost stable scattering in nonintegrable models have been reinforced and a framework is proposed to describe quasi-integrability in terms of PT symmetry. This new mechanism can be used to regard PT symmetry in classical field theories as a guiding principle to also select relevant systems when it comes to integrability properties. It turns out that if a deformed Lax pair is invariant under this symmetry, corresponding to the unbroken PT-symmetric regime, quasi-integrable excitations are produced with asymptotically conserved charges. A generic nonlinear field equation is used in order to verify the validity of the assumptions but results for a specific non-integrable class of models are also presented. A set of quasi-integrable excitations is investigated and shown to have spectral functions with appropriate properties, which might lead to the determination of the almost conserved charges.

Keywords: PT-symmetry, quasi-integrability, deformed sine-Gordon

1. Introduction

Since it was first conjectured [1], the use of PT symmetry as a guiding principle to identify potentially relevant physical systems has been applied to various quantum settings. There have been also investigations on the effect of such a symmetry at the classical level. A vast range of classical models have been studied, from point particle systems, e.g. [2–6], to classical field theories [7–11], just to name a few. One interesting observation was that the requirement of symmetry under parity together with time reversal of Calogero-like particles would imply that a class of solitons of the Boussinesq equation would have real energies [12]. At that time it was noticed that most classical field theories of interest were described by equations which were PT-symmetric, in some sense. There are many relevant works investigating PT-symmetric integrable models, e.g. [13–22]. However, this property has not been scrutinised thoroughly enough and there are aspects which remain underexplored. For
instance, the transition between the broken and the unbroken phases of the $\mathcal{PT}$ symmetry has not been fully understood.

In a quantum system, it was shown [1] that the invariance of the Hamiltonian with respect to both parity and time reversal is not enough to guarantee the reality of energies. In order to have real eigenvalues it is necessary to have invariant eigenstates as well. If this is the case, eigenenergies are real, otherwise they would come in complex conjugate pairs. Classically, what are the effects of being in the unbroken or broken phases of a $\mathcal{PT}$-symmetric model? It turns out that this symmetry imposes restrictions on the phase space which can mimic those of integrable systems giving rise to almost stable solitonic solutions. Quasi-integrable models have been studied independently, as in [23], since a particular generalisation of the sine-Gordon model was proposed [25]. Initially just a numerical observation, the appearance of quasi-integrable behaviour has been more carefully investigated and a deeper comprehension of the reasons why charges were almost conserved was pursued. This led to an interesting algebraic description of such a phenomenon in [23], where the authors identified the intriguing role played by $\mathcal{PT}$.

Here it is shown how one can understand the quasi-integrability of unbroken $\mathcal{PT}$-symmetric problems in terms of the flat gauge connections in the Lax formulation of integrable models. In the appendix, we motivate the present investigation by showcasing a series of well known integrable nonlinear partial differential equations with some important solutions and call attention to the space–time symmetries of those. In the next section 2, we present the basic properties of a classical $\mathcal{PT}$-symmetric field theory and establish a distinction between the broken and unbroken regimes. Then in section 3 we review the construction of conserved charges based on a path-independent solution of Lax’s auxiliary problem and describe the effect of parity and time reversal operations on those objects. This formalism allows for a general comment on the quasi-integrability of $\mathcal{PT}$-symmetric models in the unbroken phase. These tools are used in section 4, where a particular nonintegrable class of models is investigated and shown to possess almost $\mathcal{PT}$-symmetric excitations corresponding to the quasi-conservation of integrals of motion. We also employ a numerical procedure to verify the approximate constancy of the spectral determinant far from the interaction region, which leads to a family of almost conserved charges. We finish by stating our final remarks in section 5.

2. $\mathcal{PT}$ symmetry in field theories

The presence of $\mathcal{PT}$ symmetry in a quantum Hamiltonian theory defined by

$$ H|\psi_n\rangle = E_n|\psi_n\rangle $$

provides a good indication that the theory admits a real spectrum. This happens because, if both the Hamiltonian and its eigenstates are invariant under $\mathcal{PT}$ transformations, both of the following expressions [1] are valid,

$$ [H, \mathcal{PT}] = 0 \quad \text{and} \quad \mathcal{PT}|\psi_n\rangle = \varepsilon|\psi_n\rangle, $$

with $\varepsilon = \pm 1$. Then one has, due to the anti-linearity of $\mathcal{PT}$,

$$ E_n|\psi_n\rangle = H|\psi_n\rangle = \varepsilon H\mathcal{PT}|\psi_n\rangle = \varepsilon \mathcal{PT}H|\psi_n\rangle = \varepsilon E_n^\ast \mathcal{PT}|\psi_n\rangle = E_n^\ast|\psi_n\rangle. $$

The validity of the first property above, in (2), is simple to verify; it is enough to check that $H$ remains invariant under the combined action of
Unfortunately, on the other hand, the same is not true for the second one, which does not necessarily hold if the first one does. In the case where the eigenstates of $H$ are also invariant under $\mathcal{PT}$, so that (2) are valid, a situation known as unbroken $\mathcal{PT}$ symmetry, the reality of the eigenvalues is guaranteed. Otherwise, if $\mathcal{PT}$ is a symmetry only of the Hamiltonian but not of the eigenstate, one is said to be in the broken $\mathcal{PT}$ symmetry phase, and eigenvalues come in complex conjugate pairs.

Thus, the $\mathcal{PT}$ symmetry of a Hamiltonian system is not enough to guarantee the reality of its spectrum; instead one should also impose a restriction on the eigenvectors. In practice, one can only make a definite statement about the system being in the broken or unbroken regime if the problem has been solved. However, even without completely characterizing the system one can ensure the reality of the spectrum if a solution of a particular type exists. It turns out that this concept can be transported to ensure that the energy of a specific field configuration in classical mechanics is real [8] and the role of the unbroken phase can be elucidated.

In a similar way to how $\mathcal{PT}$ symmetry can be imposed in a quantum system by the requirements (2) one can have classical Hamiltonian functions invariant under the linear parity action and the anti-linear time reversal operation. The $\mathcal{PT}$ invariance of a classical theory of the field, $\phi(x, t) = \phi_\tau(x, t) + i \phi_\bar{t}(x, t)$, complex in general, described by a Hamiltonian density $\mathcal{H}$, reads

$$\mathcal{PT}[\phi(x, t); x, t] = \mathcal{H}^*[\phi(-x, -t)^*; -x, -t] = \mathcal{H}[\phi(x, t); x, t],$$

and the unbroken phase corresponds to

$$\mathcal{PT}[\phi(x, t); x, t] = \mathcal{H}^*[\varepsilon \phi(x, t); -x, -t] = \mathcal{H}[\phi(x, t); x, t].$$

In general one has

$$\mathcal{H}[\phi(x, t); x, t] = \mathcal{H}_r[\phi_\tau(x, t), \phi_\bar{t}(x, t); x, t]$$

$$+ i \mathcal{H}_i[\phi_\tau(x, t), \phi_\bar{t}(x, t); x, t],$$

so that the invariance occurs whenever

$$\mathcal{H}_r(x, t) = \mathcal{H}_r[\phi_\tau(-x, -t), \phi_\bar{t}(-x, -t); -x, -t]$$

$$= + \mathcal{H}_r[\phi_\tau(x, t), \phi_\bar{t}(x, t); x, t] = + \mathcal{H}_r(-x, -t),$$

$$\mathcal{H}_i(x, t) = \mathcal{H}_i[\phi_\tau(-x, -t), \phi_\bar{t}(-x, -t); -x, -t]$$

$$= - \mathcal{H}_i[\phi_\tau(x, t), \phi_\bar{t}(x, t); x, t] = - \mathcal{H}_i(-x, -t),$$

and the Hamiltonian is then symmetric,

$$\mathcal{PT}[\mathcal{H}(x, t); x, t] = \mathcal{H}(-x, -t)^* = \mathcal{H}(x, t).$$

Because the imaginary part of the Hamiltonian density is odd, in these circumstances, it does not contribute when integrated on a symmetric interval and the energy is real. In fact, the mass of the travelling excitation $\phi(x, t) = f(x - vt)$ can be calculated in the rest reference frame, where $v = 0$, or equivalently at $t = 0$, and is real, $mc^2 = E(0) = \mathcal{E}^*(0)$, since
\[ E(t) = \int_{-L}^{L} dx \mathcal{H}(x, t) = -\int_{-L}^{L} dx \mathcal{H}(-x, t) = \int_{-L}^{L} dx \mathcal{H}(-x, t) = \int_{-L}^{L} dx \mathcal{H}^{*}(x, -t) = \int_{-L}^{L} dx \mathcal{H}^{*}(x, -t) = E^{*}(-t) \] (12)

is real if the energy is conserved. Moreover, we can use this symmetry to ensure the reality of the action function,

\[ S = \int_{-L}^{L} dx \int_{-T}^{T} dt \mathcal{L}[\phi(x, t)] = \int_{-L}^{L} dx \int_{-T}^{T} dt \mathcal{L}[\phi(-x, -t)]. \] (13)

If the system is invariant under $\mathcal{P}\mathcal{T}$,

\[ \mathcal{P}\mathcal{T}\mathcal{L}[\phi(x, t)] = \mathcal{L}[\phi(-x, -t)]^{*} = \mathcal{L}[\phi(x, t)] \] (14)

then $S = S^{*}$.

The unbroken symmetry can only be checked once the solution $\phi(x, t)$ is constructed and it suggests the existence of broken and unbroken $\mathcal{P}\mathcal{T}$ symmetry for classical models. In fact, it has been found [3, 4] that in some complexified dynamical systems the breaking of $\mathcal{P}\mathcal{T}$ symmetry can be felt as the corresponding real energy classical trajectories, which are closed and periodic in the unbroken phase, become unconfined and eventually run off to infinity.

It turns out that important models of classical field theory are associated to invariant equations under time reversal and parity operations. We revisit in the appendix some completely integrable systems and highlight their behaviour under $\mathcal{P}\mathcal{T}$ to further motivate the study of the behaviour of this symmetry with respect to deviations from the integrable domain. More than that, we see that their most relevant solutions, namely their solitonic excitations, are also invariant. Thus, the regime of stable particle-like waves which can scatter indefinitely, preserving the form of their profiles and conserving energy, coincide with the so called unbroken phase of $\mathcal{P}\mathcal{T}$ symmetry. Our observations suggest that some systems which are not exactly integrable but which satisfy an invariance of this type can behave as quasi-integrable, in the sense that they possess families of asymptotically preserved charges. In the next section we explain how the presence of unbroken $\mathcal{P}\mathcal{T}$ symmetry implies quasi-integrability properties for a general system.

### 3. Path-independence and quasi-conservation laws for $\mathcal{P}\mathcal{T}$-symmetric models

Once we have appreciated the role of $\mathcal{P}\mathcal{T}$ symmetry in classical field theories, we now turn our attention to an attempt to understand the origin of important associated properties, more notably the stability of solutions. In order to do so, we must start from a formulation of integrable systems in which the space and time coordinates arise naturally. In fact, an integrable theory can be formulated in terms of Wilson loops in space–time and the non-abelian Stokes theorem [26],

\[ W(\Gamma) \equiv \hat{\mathcal{P}} \exp \left( -\oint_{\Gamma} A_{\mu} dx^{\mu} \right) = \hat{\mathcal{P}} \exp \left( -\int_{\Sigma} dx^{\mu} dx^{\nu} W^{-1} F_{\mu\nu} W \right) \equiv V(\Sigma), \] (15)

where $\hat{\mathcal{P}}$ indicates a path-order exponential, $A_{\mu}$ is a gauge potential and $F_{\mu\nu}$ the associated curvature, defined on surface $\Sigma$ with boundary $\Gamma$. One can have closed paths starting (and finishing) at different points: for instance, using the four vertices $(-L, -\tau), (+L, -\tau), (+L, +\tau), (-L, +\tau)$—of figure 1, one can write, for a closed contour around the point $(-L, -\tau)$,
We note that the object \( W \) in (15) will depend on
\[
\int dt A_t(L, t) + \int dt A_t(-L, t) = \int dt A_t(L, t) - \int dt A_t(-L, -t)
\]
and
\[
\int dx A_x(x, -\tau) + \int dx A_x(x, \tau) = -\int dx A_x(-x, -\tau) + \int dx A_x(x, \tau),
\]
which will vanish if these gauge potentials are invariant under parity and time reversal operations,
\[
A_t(L, t) = A_t(-L, -t),
A_x(x, \tau) = A_x(-x, -\tau).
\]
Indeed, then we have, respectively,
\[
U(t) = W(\Gamma_{-L}) = W(\Gamma_{-L})^{-1},
\]
\[
S(x) = W(\Gamma_{+\tau}) = W(\Gamma_{-\tau})^{-1},
\]
and consequently
\[
W(\Gamma_C) = U(t)^{-1}S(x)U(t)S(x)^{-1},
\]
can be associated to a conserved object, since \( \det W(\Gamma_C) = 1 \). Assuming one has an auxiliary vector \( \Psi(x, t) \) evolving in the following form
\[
\Psi(x, t) = W(\Gamma)\Psi_0,
\]
where the nonintegrable phase of this path-ordered exponential is dictated by the object \( E_{\Gamma}(x, t) \), then its value will heavily rely on the chosen curve \( \Gamma \).

An important result is that if the curvature \( E_{\Gamma}(x, t) \) vanishes, the system becomes integrable as the phase trivialises and the auxiliary vector satisfies

Figure 1. We suppose to have an auxiliary state \( \Psi \) that evolves in space and time through the operators \( W(\Gamma) \), defined on a certain path connecting starting and ending points. A closed path will in general produce a phase shift in the state, as in (24), due to the presence of the potentials \( A_\mu \) in that region of space. However, if these gauge potentials present certain behaviour under parity reflection and time reversal then the integral in (15) can be path-independent.

\[
W(\Gamma_C) = W(\Gamma_{-L})W(\Gamma_{+\tau})W(\Gamma_{+L})W(\Gamma_{-\tau}).
\]

We note that the object \( W \) in (15) will depend on
\[ \begin{align*} 
[\partial_t + A_t(x, t)]\psi(x, t) &= 0, \\
[\partial_x + A_x(x, t)]\psi(x, t) &= 0, 
\end{align*} \]

and the object \( F_{\mu} \) can be identified with

\[ F_{\mu} \equiv [\partial_t + A_t, \partial_x + A_x] = 0. \]

Nevertheless, if one is concerned with closed paths \( \Gamma_c \) there is another way of obtaining a trivial integrated phase, \( W(\Gamma_c) = 1 \), and that is if the integrand is symmetric with respect to space and time along a certain integration contour, mimicking the effects of a zero curvature condition \( F_{\mu} = 0 \). That means that, albeit not being necessarily integrable, a model might present excitation configurations that resemble in some senses those of an integrable theory.

Taking specifically the example of the sine-Gordon model as a reference integrable model, one can write the gauge fields associated to the laboratory coordinates as in \([23]\). For our purposes here, it will be more convenient to express the problem in terms of space and time coordinates, rather than light-cone variables. Starting from a standard form of a more symmetric Lax pair, we express it in terms of a generic potential \( V(\phi) \). The result are the following gauge fields,

\[ \begin{align*} 
A_t(x, t) &= \frac{1}{2i}B\phi_t(x, t)\sigma_+ + i\alpha_+, \\
A_x(x, t) &= \frac{1}{2i}B\phi_t(x, t)\sigma_- + i\alpha_-, 
\end{align*} \]

with

\[ \alpha_\pm = \frac{1}{2\sqrt{2}} \left( \pm \frac{1}{\lambda} \left( \lambda \pm \frac{1}{\lambda} \right) \frac{V'(\phi(x, t))}{\sqrt{V(\phi(x, t))}} \sigma_\pm \mp B \left( \lambda \mp \frac{1}{\lambda} \right) \sqrt{V(\phi(x, t))} \sigma_\mp \right). \]

where \( B \) is the coupling constant of the theory, \( \lambda \) is the spectral parameter and \( \sigma_\pm \) denote the \( 2 \times 2 \) Pauli matrices. making the \( A_\mu \) potentials invariant under the operations of space–time reversal, \([19]\) and \([20]\), if so are the fields, or, in other words if one is in the unbroken regime of \( \mathcal{PT} \) symmetry. As a matter of fact, these properties also render the equations \((25)\) and \((26)\) \( \mathcal{PT} \)-symmetric if one considers the parity transformation to act as \( \mathcal{P} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \).

\[ \begin{align*} 
\mathcal{P} \sigma_\pm \mathcal{P} &= \sigma_\mp, \\
\mathcal{T} \sigma_\pm \mathcal{T} &= \sigma_\pm, \\
\mathcal{PT} \sigma_\pm \mathcal{PT} &= \sigma_\mp, \\
\mathcal{P} \sigma_\mp \mathcal{P} &= -\sigma_\pm, \\
\mathcal{T} \sigma_\mp \mathcal{T} &= -\sigma_\pm, \\
\mathcal{PT} \sigma_\mp \mathcal{PT} &= -\sigma_\pm. 
\end{align*} \]

The unbroken \( \mathcal{PT} \) symmetry of the classical field is thus not directly related to the unbroken phase of the auxilar vector \( \Psi(x, t) \), but with the \( \mathcal{PT} \) symmetry of the classical field \( \phi(x, t) \). The gauge connections above correspond to an anomalous curvature,

\[ F_{\mu} = (\phi_n(x, t) - \phi_\nu(x, t) + V'(\phi(x, t)))\sigma_\pm + \delta, \]

clearly associated to a generalized Klein–Gordon equation, with potential \( V(\phi) \), up to an anomaly term.
\[
\delta = \frac{1}{2\sqrt{2}B}\left[\frac{1}{2}\left(\lambda - \frac{1}{\lambda}\right)\phi_j(x, t) + \frac{1}{2}\left(\lambda + \frac{1}{\lambda}\right)\phi_i(x, t)\right]
\times \left(4B^2\sqrt{V(\phi(x, t))} + \frac{2V''(\phi(x, t))}{\sqrt{V(\phi(x, t))}} - \frac{V'(\phi(x, t))^2}{V(\phi(x, t))^{3/2}}\right)\sigma_i,
\]

(33)

responsible for breaking the integrability of the system.

So far we have referred the reader to the appendix, where we present various examples of integrable systems, real and complex, highlighting their \(\mathcal{PT}\)-symmetric properties. In what follows we somehow reverse the logic and present some examples of nonintegrable models and argue, following the reasoning presented, that if \(\mathcal{PT}\) symmetry is present then we can appreciate the survival of a few features which are reminiscent of integrable systems. To achieve this we have chosen a particular deformation of an integrable model which was introduced in [25] and further explored in [23], where stability and symmetry properties have been spotted. It is important to stress the power of equation (23) which imposes restrictions on the effective curvature and the evolution of the system.

4. The quasi-integrability of \(\mathcal{PT}\)-symmetric models

The Zakharov–Shabat zero curvature condition [24] is a standard framework used to describe integrable models, and since we are dealing with systems that are almost integrable it is a natural procedure to investigate the consequences of the breaking of integrability on such a scheme. The results do not rely on any particular way of deforming an integrable equation. In fact, a generic potential function \(V(\phi)\) has been used, which leads to the integrable sine–Gordon case for a very specific choice, but that otherwise provides a more general equation. The form of the \(A_\mu\) potentials in (28)–(29) is convenient as it allows one to treat nonintegrable extensions, such as [25]

\[
V(\phi) = \frac{32}{N^2} \frac{(4M)^2}{(2B)^2} \tan^2\left(\frac{B\phi}{2}\right)\left(1 - \sin^N\left(\frac{B\phi}{2}\right)\right)^2,
\]

(34)
depending on a real deformation parameter \(\varepsilon = N - 2\), which vanishes for the sine-Gordon case, with \(N = 2\) and \(M\) being a mass term.

The arguments used regarding the \(\mathcal{PT}\) symmetry of the gauge potentials are very general and do not rely on its particular form, so should still be valid if deformations of other integrable equations are attempted. A certain choice of potential \(V\) above has been made only in order to study explicitly a class of models and verify numerically that the general arguments hold true. It is interesting to note that, again, the zero curvature condition (32) depends not only on the model—as it is the case for integrable systems—but also on the solution. However, in order to have a vanishing anomaly one needs constant solutions, \(\phi_i(x, t) = \phi_j(x, t) = 0\), which necessarily occurs asymptotically for models with degenerate vacua. Unless, of course, if \(V = V_{SG}\), when the Lax pairs are those commonly found in the literature, e.g. [27], and the second factor of \(\delta\) is identically satisfied.

In order to verify the existence of unbroken \(\mathcal{PT}\)-symmetric excitations associated to this class of models we present here some numerical analysis to investigate the scattering of solitary waves which present approximately \(\mathcal{PT}\) symmetry. For that we let two localized wave packets, whose dynamics are governed by a generalised sine–Gordon interaction of the type (34), found in [25], travel and collide against each other. The numerical evolution of the system is shown in figure 2, a density plot of the field excitations as time evolves along the horizontal axis, as colliding solitary waves seen from above. There one also observes that as
the deformation parameter $\varepsilon = N - 2$ increases, the less stable the scattering tends to be, and some radiation waves start to be observed after the collision, shown in the gray areas of the figure, as described in [23], breaking the time reversal symmetry. There is also a slightly better behaviour of positive $N$ compared to odd values. These excitations, nonetheless, cannot be identified with unstable particles [28–30], as the latter are present in integrable systems and the former constitute a sign of the breaking of integrability.

Having the excitations an approximate $\mathcal{PT}$ symmetry we can check for the $\mathcal{PT}$-invariance on the gauge potentials $A_x(x, t)$ and $A_t(x, t)$. Therefore, we show in figure 3 the behaviour of the diagonal and off-diagonal components of these connections and confirm the desired properties are observed, at least in a first approximation. In fact, it becomes evident, however, that as the deformation parameter, $\varepsilon = N - 2$, moves away form the integrable...
situation, $N = 2$, the scattering symmetry tends to deteriorate. In all of these cases, one could also plot the behaviour of the anomaly $\delta$ coming from the nonflatness of the curvature $F_{\mu\nu}$ and it is possible to verify that its values is relatively small and bounded in this region we are interested.

Thus, if one is concerned only with the in- and out-states in a remote past and a far away future, respectively, there is the possibility of constructing a great deal of objects which are conserved in a broader sense. In order to so, it is essential to investigate the behaviour of the spectral data, $a(\lambda)$ and $b(\lambda)$. We can expand at $x \to \infty$

$$
\Psi_+ (x, \lambda) = a(\lambda) \overline{\Psi} (x, \lambda) + b(\lambda) \Psi_- (x, \lambda),
$$

where $\overline{\Psi} = \sigma, \Psi^*$, and apply to it the time evolution. An important property of the spectral coefficients $a(\lambda)$ and $b(\lambda)$ is that the action-angle variables are constructed in terms of them. This implies one can expand the action variable in powers of the spectral parameter $\lambda$, so that if action variable is constant then one can generate infinitely-many conserved charges. Relaxing this latter condition slightly, if the action variable changes slowly with time, keeping its final state equal to its initial state, then one has an infinite number of quantities whose values are asymptotically the same.

As a consequence, one can show that $a(\lambda)$ keeps its original value, $a_0(\lambda)$, if one is interested in asymptotic states only. On the other hand, as can be seen in more detail in [31], the other spectral coefficient $b(\lambda)$ evolves according to the potential $V(\phi)$, evaluated at the initial vacua configuration,

$$
a(\lambda, t) = a_0(\lambda), \quad b(\lambda, t) = b_0(\lambda) e^{i\phi} e^{\Gamma t},
$$

with an oscillating phase, and an exponential decay/growth term. The latter does not contribute if the minimum of the potential, achieved by the field asymptotically, can be set to vanish by a simple shift. In this case the $\Gamma$ factor does not affect the time evolution and one only has the oscillating behaviour left. In fact, in the event of unbroken $\mathcal{PT}$ symmetry, the auxiliary vectors, and consequently the spectral coefficients, tend to recover their forms, for suitable boundary conditions. Since the conserved charges can be obtained as the coefficients in a power series of the spectral parameter $\lambda$ for a function of $a(\lambda)$ only, then one can argue that the charges themselves must also be asymptotically conserved in time.

Finally, aiming at confirming the asymptotic conservation laws for this class of models we present in figure 4 the associated spectral curves $\Delta(\lambda) = \text{Re} (a(\lambda))$, the trace of the scattering matrix, at various times. This numerical evaluation shows that, for the model proposed by Bazeia et al, the spectral curve at different instants indicate a small nontrivial time evolution of the scattering data $a(\lambda)$. However, the figures show that, despite the existence of a deviation from the completely conservative regime, the spectral curve, and consequently the conserved charges, tend to recover their values asymptotically. Not surprisingly, the greater the deviation from the integrable case the more the spectral curves vary. It could happen that for other deformations the $\mathcal{PT}$ invariance under may be less, or more, evident. Also, other initial conditions might lead to a better, or worse, crossing symmetry; that is, the collision might or not destroy the $\mathcal{PT}$ symmetry of original colliding field. Note that the classical unbroken $\mathcal{PT}$ symmetry is not easy to investigate beforehand, anyway, just like in the quantum unbroken $\mathcal{PT}$ symmetry scenario. The message is that the more $\mathcal{PT}$-symmetric the interaction is, the more it behaves as a (quasi-) integrable theory. This seems to be the roles played by the classical unbroken $\mathcal{PT}$ symmetry. The explicit computation of the charges remain a challenge.
5. Concluding remarks

The space and time coordinates play a key role in the zero curvature formulation based on the nonabelian Stokes theorem. In this work it was shown that investigating the $\mathcal{P}\mathcal{T}$ symmetry of an integrable model at that level might prove interesting as one considers deformations which in principle break the integrability of the system. Because a complete understanding of the quasi-integrable system still does not exist, despite recent works, the most effective description is not settled and debate is still open. The present contribution was to identify the $\mathcal{P}\mathcal{T}$ symmetry of the Lax pair as a mechanism capable of mimicking integrable properties even if the model is not integrable (and the curvature does not vanish). This is particularly useful as one notes that most models of physical interest have a lack of completely integrability properties. We have introduced a general scheme and a particular example where asymptotic charges may be considered to be preserved even if the model does not admit a compatible Lax pair. This is general, although in the manuscript some representations have been adopted in order to allow for graphical visualization, through the confection of figures. The numerical studies carried out also emphasize the connection between $\mathcal{P}\mathcal{T}$ symmetry and the so called quasi-integrability. This opens possibilities of integrable-like behaviour in a wider class of models than the fewer integrable ones.

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Appendix A. \textit{PT} symmetry in integrable field theories

A.1. Korteweg–de Vries

The KdV equation, first introduced at the end of the 19th century, corresponds to a well established model used to describe the undulatory behaviour of water on shallow regions. It is an example of an exactly solvable partial differential equation, allowing for solutions through analytical methods, and it has been well studied also in the realm of non-Hermitian Hamiltonians. In [7], it was shown that a certain $\mathcal{PT}$-symmetric deformation admits the existence of certain conservation laws and in [8] it emerged that a family of Hamiltonian flows can be associated to a particular deformation of the KdV equation. Further generalisations were considered in [10] and in [11] compactons were investigated.

But one does not have to impose $\mathcal{PT}$-symmetric extensions of the KdV in order to appreciate the effects of these simultaneous space–time transformations. In fact, taking the KdV equation,

\[ \phi_y + a \phi \phi_{xx} + b \phi_{xxx} = 0, \]  

arising form the Hamiltonian density

\[ \mathcal{H} = -\frac{a}{6} \phi^3 + \frac{b}{2} \phi_x^2. \]

Usually $a = -6$ and $b = 1$ and following the standard definition of the model, with $a, b \in \mathbb{R}$, one can see that the Hamiltonian remains invariant if we replace $x \rightarrow -x$, $t \rightarrow -t$ and $\phi \rightarrow \phi$ at once. Note that taking $a = i \alpha$, $b = i \beta \in \mathbb{C}$ to be pure imaginary we have an imaginary Hamiltonian density which is symmetric with respect to $x \rightarrow -x, t \rightarrow -t, i \rightarrow -i$ and $\phi \rightarrow -\phi$. According to our framework, however, there is no need that the equation of motion be symmetric.

This model admits multi-solitons excitations, and a two-soliton solution is known to be

\[ \phi(x, t) = \frac{12 \alpha \beta \epsilon \epsilon (c_2 \sinh^2 (\xi_1) + c_1 \cosh^2 (\xi_2))}{\left( \frac{c_1}{\sqrt{b}} - \frac{c_2}{\sqrt{b}} \right) \cosh (\xi_1 + \xi_2) + \left( \frac{c_1}{\sqrt{b}} + \frac{c_2}{\sqrt{b}} \right) \cosh (\xi_1 - \xi_2)} \],

where we have abbreviated $\xi_i = \frac{1}{2 \sqrt{b}} (x - c_i t)$. The action of the symmetry operation into these objects gives $\mathcal{PT}[\xi_i] = -\xi_i$ making this solitonic solution $\phi(x, t)$, which is even in $\xi_i$, invariant, i.e. $\mathcal{PT}[\phi(x, t)] = \phi(-x, -t) = \phi(x, t)$, as required. These properties can be immediately identified if one plots the evolution surface for this KdV solitons collision, which is shown in figure A1. There is evidently an axis of symmetry with respect to which the simultaneous reflections on the space and time coordinates produce an identity operation, be it a repulsion or an attraction.

Like the KdV model, there are many other examples of integrable field equations which not only present the parity-time reversal symmetry but also admit $\mathcal{PT}$-symmetric stable solutions. In what follows we discuss a few more illustrations.

A.2. Sinh-Gordon

The sinh-Gordon models is described by a Lorentz-invariant equation,

\[ \phi_{tt} - \phi_{xx} + \frac{c^2}{\beta} \sinh \beta \phi = 0, \]

where
In the latter, the derivative terms are invariant under space–time reflections and all terms remain unchanged if solutions belong to the unbroken $\mathcal{PT}$ symmetry phase, $\mathcal{PT}\phi = \pm \phi$.

Remarkable solutions for this equation present a travelling wave form, stable under collision, as depicted below. Some of these complex waves can be cast in the following form

\[
\phi(x, t, \omega, \mu) = \frac{1}{\beta} \log \left( \frac{1 - 2 \cos(\omega) e^{i(\mu \sinh(\theta) - x \cosh(\theta))} + e^{2i(\mu \sinh(\theta) + x \cosh(\theta))}}{1 + 2 \cos(\omega) e^{i(\mu \sinh(\theta) + x \cosh(\theta))} + e^{2i(\mu \sinh(\theta) - x \cosh(\theta))}} \right) \\
+ \frac{2i}{\beta} \arctan \left( \frac{2 \sin(\omega) e^{i(\mu \sinh(\theta) + x \cosh(\theta))}}{e^{2i(\mu \sinh(\theta) - x \cosh(\theta))} - 1} \right),
\]

so the $\mathcal{PT}$-invariance of the equation of motion above, as $x \to -x$ and $t \to -t$, reads

\[
\phi(x, t, \omega, \mu) = \phi(-x, -t, -\omega, \mu) = \phi(-x, -t, \omega, \mu)^\ast,
\]
in accordance with the complex conjugation that comes along with time-reversal in quantum, and complex, field theories (see figure A2 for the real and imaginary parts of the excitation). Here the rapidity variable \( q = - \tanh v \) has been used, with \( c = 1 \).

**A.3. Sine-Gordon**

By complexifying the coupling constant \( \beta \rightarrow i \gamma \) of the previous model, one obtains the sine-Gordon Hamiltonian density,

\[
\mathcal{H} = \frac{1}{2} (\phi_x^2 + \phi_y^2) - \frac{m^2}{\gamma^2} (\cos \gamma \phi - 1),
\]

and its equation of motion,

\[
\phi_{tt} - \phi_{xx} + \frac{m^2}{\gamma} \sin \gamma \phi = 0,
\]

with localized solutions,

\[
\phi(x, t) = \frac{4}{\gamma} \arctan (e^{m \cosh \theta (x - t \tanh \theta)}),
\]

describing kinks \((\epsilon = +1)\) and anti-kinks \((\epsilon = -1)\), depicted in figure A3. In order to appreciate its \( \mathcal{PT} \) properties one should note that

\[
4(\arctan(e^\gamma) - \arctan(-e^{-\gamma})) = 0 \mod 2\pi.
\]

An interesting feature is therefore that these two fundamental configurations, the kink and anti-kink, from which one can construct other excitations (breathers, bound states), are connected via \( \mathcal{PT} \), up to a topological equivalence. The system composed of a single (anti)-kink is not explicitly \( \mathcal{PT} \)-invariant but if one wants to investigate their stability one can create a collision between a kink and an anti-kink and this process is similar to the one shown in figure A2 (on the right). Other non-symmetric collisions are also possible but in those cases topological arguments are to be used.

**A.4. The Boussinesq equation and Calogero scattering**

Finally, we illustrate the good \( \mathcal{PT} \) properties of integrable models by presenting one last example, that of the Boussinesq equation. In [12] an explicit construction of a class of solutions depending on dynamical movable poles has been achieved. Interestingly, the poles satisfy an interaction dictated by the Calogero Hamiltonian [32]. In that occasion it was
concluded that $PT$-symmetric Calogero particles would guarantee that the solutions to the Boussinesq equation have real energies.

Below we depict the evolution of the real and imaginary parts of the Boussinesq $PT$-symmetric solution associated to the existence of three Calogero-like poles. One can verify from figure A4 that the fields are indeed invariant under space and time reversal: the real part at two points $t$ and $-t$ correspond to a simple reflection $x \to -x$, but the imaginary part also suffers a sign inversion, associated to the complex conjugation. The poles appearing in this solution interact as shown in figure A5. If one decomposes the scattering in two-bodies collisions, the trajectories of the particles are symmetric with respect to space–time inversions relative to the centre of mass reference frame.

Although constructed from a classical starting point, these ideas can be extended to quantum scattering processes. The classical collisions above resemble the decomposition property of the scattering matrix. From figure A6 one sees the effect of space–time reversal and it becomes evident that indeed $PT^2 = 1$.

In the case of quasi-integrable systems, despite the asymptotic stability of the excitations one observes the presence of radiation near the interaction region. It has been noticed [28–30] that unstable particles can be produced and annihilated without being observed asymptotically in a scattering process. These particles are associated to complex poles in the scattering amplitude and could be related to the present observations.
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