Polynomial growth harmonic functions on finitely generated abelian groups

Bobo Hua · Jürgen Jost · Xianqing Li-Jost

Received: 19 December 2012 / Accepted: 26 March 2013 / Published online: 6 April 2013
© Springer Science+Business Media Dordrecht 2013

Abstract In the present paper, we develop geometric analysis techniques on Cayley graphs of finitely generated abelian groups to study the polynomial growth harmonic functions. We provide a geometric analysis proof of the classical Heilbronn theorem (Heilbronn in Proc Camb Philos Soc 45:194–206, 1949) and the recent Nayar theorem (Nayar in Bull Pol Acad Sci Math 57:231–242, 2009) on polynomial growth harmonic functions on lattices $\mathbb{Z}^n$ that does not use a representation formula for harmonic functions. In the abelian group case, by Yau’s gradient estimate, we actually give a simplified proof of a general polynomial growth harmonic function theorem of (Alexopoulos in Ann Probab 30:723–801, 2002). We calculate the precise dimension of the space of polynomial growth harmonic functions on finitely generated abelian groups by linear algebra, rather than by Floquet theory Kuchment and Pinchover (Trans Am Math Soc 359:5777–5815, 2007). While the Cayley graph not only depends on the abelian group, but also on the choice of a generating set, we find that this dimension depends only on the group itself. Moreover, we also calculate the dimension of solutions to higher order Laplace operators.

Keywords Harmonic functions · Lattices · Discrete geometric analysis · Yau’s gradient estimate

Mathematics Subject Classification (2000) 31C05 · 20K15 · 20F69 · 82B41
1 Introduction

Classically, in 1949 Heilbronn [22] proved the polynomial growth harmonic function theorem on the lattice $\mathbb{Z}^n$ that polynomial growth discrete harmonic functions are polynomials, and calculated the dimension of the space of polynomial growth discrete harmonic functions. Recently, Nayar [44] gave another proof of this theorem by the probabilistic method. Their proofs all depend on the representation formula for discrete harmonic functions. In this paper, we give a geometric analysis proof that can yield more general results.

The study of harmonic polynomials on $\mathbb{R}^n$ is classical, and the precise dimension calculation of harmonic polynomials can be found in [36]. In 1975, Yau [52] proved the Liouville theorem for harmonic functions on Riemannian manifolds with non-negative Ricci curvature. Then, Cheng and Yau [8] used Bochner’s technique to derive the gradient estimate for positive harmonic functions, called Yau’s gradient estimate, which implies that sublinear growth harmonic functions on these manifolds are constant. Then, Yau [53,54] conjectured that the space of polynomial growth harmonic functions of growth order less than or equal to $d$ on Riemannian manifolds with non-negative Ricci is of finite dimension. Li and Tam [37] and Donnelly and Fefferman [18] independently solved the conjecture for manifolds of dimension two. Then, Colding and Minicozzi [10–12] gave the affirmative answer using the volume doubling property and the Poincaré inequality for arbitrary dimension. The simplified argument by the mean value inequality can be found in [13,35] where the dimension estimate is asymptotically optimal. This inspired many generalizations on manifolds [7,28,34,38,48,50,51] and on singular spaces [16,24,25,27,29]. In this paper, we give the precise dimension calculation of polynomial growth harmonic functions on finitely generated abelian groups.

In another direction, Avellaneda and Lin [5] first proved the polynomial growth harmonic function theorem for elliptic differential operators with periodic coefficients in $\mathbb{R}^n$ (see [1,2,4,32,40,41] for more generalizations). Alexopoulos [3] proved a more general polynomial growth harmonic function theorem for groups of polynomial volume growth. By a celebrated theorem of Gromov [20], every finitely generated group $G$ of polynomial growth is virtually nilpotent (i.e. it has a nilpotent subgroup $H$ of finite index). For some torsion-free finite-index subgroup $H'$ of $H$, it can be embedded as a lattice in a simply connected nilpotent Lie group $N$. By considering the exponential coordinate of $N$, $N$ is identified with $\mathbb{R}^n$ (see [45]). Alexopoulos proved that every polynomial growth harmonic function on $G$, when restricted to $H'$, can be extended to a polynomial on $\mathbb{R}^n$. He used the homogenization theory, Krylov–Safonov’s argument and some ideas of Avellaneda and Lin [5]. Instead of doing that, we shall prove Yau’s gradient estimate on abelian groups to simplify the argument in this special case.

From another point of view, Kuchment and Pinchover [30] introduced a method in Floquet theory to study the space of solutions to periodic elliptic equations on the abelian cover of a compact Riemannian manifold or a finite graph. They calculated the dimension of polynomial growth solutions to the standard operators, in particular, the Laplace operators, which covers our results. The aim of this paper is to provide a geometric point of view of this problem. We will use some geometric analysis favored methods and basic linear algebra to investigate this problem which may shed some light on how to solve these problems on general graphs.

We develop some geometric analysis techniques on Cayley graphs of finitely generated abelian groups. First, we prove Yau’s gradient estimate (see Theorem 1) for positive discrete harmonic functions. Note that Kleiner [29] obtained the Poincaré inequality on Cayley graphs of finitely generated (not necessarily abelian) groups (see also [47]). For the abelian case, combining it with the natural volume doubling property, we obtain the uniform Poincaré
inequality (see Lemma 1) from which the mean value inequality follows by the Moser iteration. In addition, for an abelian group, Bochner’s formula can be easily verified, i.e. \(|\nabla u|^2\) is subharmonic for a discrete harmonic function \(u\) which is essentially due to Chung and Yau [9] and Lin and Yau [43] on Ricci flat graphs. Then, these results together imply Yau’s gradient estimate on Cayley graphs of finitely generated abelian groups as in the case of Riemannian manifolds with nonnegative Ricci curvature. Besides the geometric method, this can also be obtained by the local central limit theorem, a probabilistic method (see Theorem 1.7.1 in Lawler [33] for the standard lattice case). By the fundamental theorem of finitely generated abelian group (see [46,49]), any finitely generated abelian group \(G\) is isomorphic to the direct sum \(\mathbb{Z}^m \oplus \oplus_{i=1}^l \mathbb{Z}_{q_i}\), where \(m, l \in \mathbb{N}, q_i = p_i^{a_i}\) for some prime number \(p_i\) and \(a_i \in \mathbb{N}\).

**Theorem 1** (Yau’s gradient estimate) Let \((G, S)\) be the Cayley graph of a finitely generated abelian group with symmetric generating set \(S\) (i.e. \(S = -S\)), and \(G \cong \mathbb{Z}^m \oplus \oplus_{i=1}^l \mathbb{Z}_{q_i}\). Then, there exist constants \(C_1\) and \(C_2\) depending only on \(m\) and \(S\), such that for any \(x \in G, R \geq 1\) and any positive discrete harmonic function \(f\) on \(B_{C_1}(x)\), we have

\[ |\nabla f|(x) \leq \frac{C_2}{R} f(x). \]

(1)

Second, from Yau’s gradient estimate, we know that \(|\nabla f| \leq CR^{d-1}\), on \(B_{R}\) for \(R \geq R_0\) if the discrete harmonic function \(f\) satisfies \(|f| \leq CR^d\) on \(B_{R}\) for \(R \geq R_1\). That is, the growth order decreases when we take derivatives.

This is the key to an induction argument to give a geometric analysis proof of Heilbronn’s theorem. This scheme will, in fact, work for all abelian groups. Let us denote the set of polynomial growth harmonic functions of growth order less than or equal to some prime number \(p\) and \(G\) can also be obtained by the local central limit theorem, a probabilistic method (see Theorem 1.7.1 in Lawler [33] for the standard lattice case). By the fundamental theorem of finitely generated abelian group (see [46,49]), any finitely generated abelian group \(G\) is isomorphic to the direct sum \(\mathbb{Z}^m \oplus \oplus_{i=1}^l \mathbb{Z}_{q_i}\), where \(m, l \in \mathbb{N}, q_i = p_i^{a_i}\) for some prime number \(p_i\) and \(a_i \in \mathbb{N}\).

**Theorem 2** (Generalized Heilbronn’s theorem) Let \((G, S)\) be the Cayley graph of a finitely generated abelian group \(G = G_1 \oplus G_2 \cong \mathbb{Z}^m \oplus \oplus_{i=1}^l \mathbb{Z}_{q_i}\). Then

\[ H^d(G, S) = H^d(G_1, \pi_{G_1} S) \]

moreover, any \(f \in H^d(G, S)\) is a polynomial when it is restricted to \(G_1 \cong \mathbb{Z}^m\), and it is constant on \(G_2\), i.e. \(f(x + w) = f(x)\), for \(\forall x \in G, \forall w \in G_2\).

Nayar [44] proved a strong version of Heilbronn’s theorem. We denote by \(HM^d(G, S) := \text{Span}\{u : G \to \mathbb{R} | L^S u = 0, u(x) = -C(d^S(p, x) + 1)^d\}\) the linear span of one-sided bounded polynomial growth harmonic functions. It is trivial that \(H^d(G, S) \subset HM^d(G, S)\). By the Harnack inequality (6), we give a geometric analysis proof of Nayar’s theorem.

**Theorem 3** (Generalized Nayar’s theorem) Let \((G, S)\) be the Cayley graph of a finitely generated abelian group \(G = G_1 \oplus G_2 \cong \mathbb{Z}^m \oplus \oplus_{i=1}^l \mathbb{Z}_{q_i}\). Then

\[ HM^d(G, S) = H^d(G, S) = H^d(G_1, \pi_{G_1} S) \]
Third, for discrete harmonic polynomials, we calculate the precise dimension by linear algebra. In fact, instead of using the technical lemma (see [19, 23]) from difference equations as Heilbronn [22] did in the lattice case, we apply the dimension comparison argument with harmonic polynomials in $\mathbb{R}^n$. Conversely, our argument provides a proof of this difference equation lemma. Moreover, we can calculate the dimension of the space of polynomial growth harmonic functions on arbitrary Cayley graph of a finitely generated abelian group. It is surprising that the dimension of polynomial growth harmonic functions does not depend on the choice of generating set $S$ for the abelian group $G$. Actually, the graph structures of two Cayley graphs of the abelian group $G$ with two generating set $S_1, S_2$ can be quite different. While the Laplacian operator depends on the generating set, the dimension of polynomial growth harmonic functions does not. We denote by $H P^k(\mathbb{R}^m)$ the space of harmonic polynomials on $\mathbb{R}^m$ with degree less than or equal to $k$, $k \in \mathbb{N} \cup \{0\}$. It is well known (see [36]) that

$$\dim H P^k(\mathbb{R}^m) = \binom{m+k-1}{k} + \binom{m+k-2}{k-1}. $$

**Theorem 4** (Dimension calculation) Let $(G, S)$ be the Cayley graph of a finitely generated abelian group, $G = G_1 \oplus G_2 \cong \mathbb{Z}^m \oplus \bigoplus_{i=1}^l \mathbb{Z} q_i$. Then

$$\dim H M^d(G, S) = \dim H^d(G, S) = \dim H P^d(\mathbb{R}^m).$$

In the last section, we study some higher order operators. The $n$th order Laplace operator is defined as

$$L^{n,S} = L^S \circ L^S \circ \cdots \circ L^S,$$

i.e. $n$-times composition of Laplace operators. These operators correspond to higher order elliptic operators in the continuous setting. For instance, when $n = 2$, $L^{2,S}$ is a discrete generalization of the bi-Laplace operator $\Delta^2$. A function $u$ on $G$ is called discrete $n$-harmonic if $L^{n,S} u = 0$. Let us denote by $H^{n,d}(G, S) := \{ u : G \to \mathbb{R} \mid L^{n,S} u = 0, |u|(x) \leq C(d^S(p, x) + 1)^d \}$ the space of polynomial growth $n$-harmonic functions of growth order not larger than $d$. By the techniques developed before for harmonic functions, we also obtain the precise dimension of $H^{n,d}(G, S)$.

**Theorem 5** Let $(G, S)$ be the Cayley graph of a finitely generated abelian group, $G = G_1 \oplus G_2 \cong \mathbb{Z}^m \oplus \bigoplus_{i=1}^l \mathbb{Z} q_i$. Then

$$H^{n,d}(G, S) = H^{n,d}(G_1, \pi G_1 S),$$

moreover, any $f \in H^{n,d}(G, S)$ is a polynomial on $G_1 \cong \mathbb{Z}^m$, and constant on $G_2$. In addition,

$$\dim H^{n,d}(G, S) = \sum_{i=[d]-2n+1}^{[d]} \binom{m+i-1}{i}.$$ 

Since we do not use the representation formula, this method can be applied in more general settings. In fact, Bochner’s formula is proved for Ricci flat graphs by Chung and Yau [9], but it is not easy to get the volume control or the Poincaré inequality in that case. Lin and Yau [43] proved Bochner’s formula for general graphs, but this does not lead to a version of Yau’s gradient estimate analogous to the case of Riemannian manifolds with non-negative Ricci curvature. In the Cayley graph case, Kleiner [29] obtained the Poincaré inequality,
but for non-abelian groups, Bochner’s formula is unavailable (consider the free group case).
For some special graphs which can be embedded into a surface with non-negative sectional curvature in the sense of Alexandrov, Hua, Jost and Liu [27] proved the volume doubling property and the Poincaré inequality but Bochner’s formula. It seems that Bochner’s formula is sensitive to the local structure, but the volume growth property and the Poincaré inequality are not, c.f. [15]. Hence, the abelian groups are very suitable candidates for the application of Bochner’s formula and Yau’s gradient estimate. In addition, Alexopoulos’ theorem [3] is more general than ours, but it depends on the embedding of the nilpotent subgroup to simply connected Lie group. It seems hard to calculate the precise dimension of polynomial growth harmonic functions on groups of polynomial volume growth. One step back, we give a dimension estimate by the geometric method in [26].

2 Preliminaries and notations

Let \( G \) be an abelian group. It is called finitely generated if it has a finite generating set. In this paper, we assume that any finite generating set \( S = \{s_1, s_2, \ldots, s_2\} \) of \( G \) is symmetric, i.e. \( S = -S \), or more precisely \( s_i = -s_{i+1} \), \( 1 \leq i \leq l \), but we do allow that elements of \( S \) are repeated, that is possibly \( s_i = s_j \) for some \( i \neq j \). We also allow \( 0 \in S \). For any finitely generated abelian group \( G \) with a generating set \( S \), we have the associated Cayley graph \((V, E)\) for which \( V = G \), and \( xy \in E \) (denoted by \( x \sim y \)) if \( y - x \in S \) for \( x, y \in V \). The duplicity of elements in \( S \) produces multiedges between vertices, and \( 0 \in S \) makes self-loops. The vertices \( x \) and \( y \) are called neighbors if \( x \sim y \). The degree of a vertex \( x \) is the number of its neighbors. Note that all vertices in \((G, S)\) have the same degree \( \#S \), the cardinality of \( S \). For the lattice \( \mathbb{Z}^n \) with the standard generating set \( S^0 = \{e_i\}_{i=1}^{2n} \), where \( e_i = (0, \ldots, 1, \ldots, 0) \) is the \( i \)th unit vector and \( e_i = -e_{i+n} \), \( 1 \leq i \leq n \), we obtain the standard integer lattice in \( \mathbb{R}^n \). This is the object Heilbronn [22] and Nayar [44] studied. In this paper, we consider the discrete harmonic functions on Cayley graphs of \( G \) with arbitrary finite generating set \( S \).

The Cayley graph of \((G, S)\) is endowed with a natural metric, called the word metric (c.f. [6]). For any \( x, y \in G \), the distance between them is defined as the length of the shortest path connecting \( x \) and \( y \) (each edge is of length one),

\[
d^S(x, y) := \inf\{k \in \mathbb{N} \mid \exists x = x_0 \sim x_1 \sim \cdots \sim x_k = y\}.
\]

Denote by \( B^S_r(x) := \{y \in G \mid d^S(y, x) \leq r\} \) the closed geodesic ball centered at \( x \) of radius \( r (r > 0) \). The volume of \( B^S_r(x) \) is \( |B^S_r(x)| := \#G \cap B^S_r(x) \), i.e. the number of vertices contained in \( B^S_r(x) \). For the subset \( \Omega \subset G \), \( d^S(x, \Omega) := \inf\{d^S(x, y) \mid y \in \Omega\} \) for any \( x \in G \), \( \partial\Omega := \{z \in G \mid d^S(z, \Omega) = 1\} \), and \( \bar{\Omega} := \Omega \cup \partial\Omega \). For any function \( f : \bar{\Omega} \to \mathbb{R} \), the discrete Laplacian operator is defined on \( \Omega \) as \((x \in \Omega)\)

\[
L^S f(x) = \sum_{y \sim x} (f(y) - f(x)).
\]

The function \( f \) is called discrete harmonic (subharmonic) on \( \Omega \) if \( L^S f(x) = 0 \) (\( \geq 0 \)) for all \( x \in \Omega \). In the lattice case \((\mathbb{Z}^n, S^0)\), the Laplacian operator is

\[
L^{S^0} f(x) = \sum_{i=1}^{2n} (f(x + e_i) - f(x)).
\]
Moreover, the \( n \)th order Laplace operator is defined as

\[
L^{n,S} = L^S \circ L^S \circ \cdots \circ L^S,
\]

i.e. \( n \)-times composition of Laplace operators. These operators are counterparts of higher order elliptic operators in the continuous setting. A function \( f \) is called \( n \)-harmonic if \( L^{n,S} f = 0 \). The gradient of \( f \) at \( x \in \Omega \) is defined as

\[
|\nabla^S f|(x) = \sqrt{\sum_{y \sim x} (f(y) - f(x))^2}.
\]

We also need the partial difference operator (for \( s \))

\[
\delta_s f(x) := f(x + s) - f(x).
\]

For Cayley graphs of \((G, S_1)\) and \((G, S_2)\), it is known (c.f. [20, 31]) that

\[
C_1 d^{S_1}(x, y) \leq d^{S_2}(x, y) \leq C_2 d^{S_1}(x, y),
\]

for any \( x, y \in G \), where \( C_1 \) and \( C_2 \) depend only on \( S_1, S_2 \). They are bi-Lipschitz equivalent in the metric point of view. Hence, for any \( x \in G, r > 0 \),

\[
|B^{S_1}_{r/(c_2)}(x)| \leq |B^{S_2}_{r/c_1}(x)| \leq |B^{S_1}_{r/(c_1)}(x)|.
\]

By the fundamental theorem of finitely generated abelian groups [46, 49], any finitely generated abelian group \( G \) is isomorphic to the direct sum \( \mathbb{Z}^m \oplus \bigoplus_{i=1}^l \mathbb{Z}_{q_i} \), where \( m, l \in \mathbb{N}, q_i = p_i^{a_i} \) for some prime number \( p_i \) and some \( a_i \in \mathbb{N} \). Hence, there exists a generating set \( S^0 = \{e_1, \ldots, e_{2m}, w_1, \ldots, w_{2l}\} \) (\( e_i = -e_{i+m}, w_j = -w_{j+l} \), for \( 1 \leq i \leq m, 1 \leq j \leq l \)) such that \( G \) is identified with \( \mathbb{Z}^m \oplus \bigoplus_{i=1}^l \mathbb{Z}_{q_i} \), where \( \{e_1 \ldots e_{2m}\} \) generates the torsion-free part \( \mathbb{Z}^m \) and \( \{w_1 \ldots w_{2l}\} \) generates the torsion part \( \bigoplus_{i=1}^l \mathbb{Z}_{q_i} \). For \((G, S^0)\), it is easy to see that \( C_1(m, S^0) r^m \leq |B^S_{2r}(x)| \leq C_2(m, S^0) r^m \), and \( |B^{S_0}_{2r}(x)| \leq C_3(m, S^0) |B^S_{2r}(x)| \), for any \( x \in G \) and \( r \geq 1 \). Hence, by the bi-Lipschitz equivalence, for any Cayley graph \((G, S)\), we have

\[
C_1(m, S^0, S) r^m \leq |B^S_{r}(x)| \leq C_2(m, S^0, S) r^m,
\]

\[
|B^{S_0}_{2r}(x)| \leq C_3(m, S^0, S) |B^S_{2r}(x)|.
\]

The volume growth property (2) is called the volume doubling property.

In the sequel, for simplicity, we shall omit \( S \) from our notation, e.g., \( B_r(x) := B^S_{r}(x) \), if it does not cause any confusion. Also, harmonic functions on \( G \) mean discrete harmonic functions. And constants \( C \) may change from line to line.

3 Yau’s gradient estimate

In this section, we prove Bochner’s formula on finitely generated abelian groups and derive Yau’s gradient estimate analogous to the one in Riemannian manifolds with non-negative Ricci curvature.

Kleiner [29] proved the Poincaré inequality on Cayley graphs of finitely generated (not necessarily abelian) groups. Let \((G, S)\) be the Cayley graph of the finitely generated abelian group

\[
G \cong \mathbb{Z}^m \oplus \bigoplus_{i=1}^l \mathbb{Z}_{q_i}.
\]

By the volume doubling property (2), we obtain the uniform Poincaré inequality.
Lemma 1 (Poincaré inequality, [29]) Let \((G, S)\) be the Cayley graph as (3). Then, there exists a constant \(C(m, S)\) such that for any \(p \in G\), \(R > 0\) and any function \(f : B_{3R}(p) \to \mathbb{R}\), we have

\[
\sum_{x \in B_R(p)} (f(x) - f_{BR})^2 \leq CR^2 \sum_{x,y \in B_{5R}(p): x \sim y} (f(x) - f(y))^2,
\]

where \(f_{BR} = \frac{1}{|B_R(p)|} \sum_{x \in B_R(p)} f(x)\).

Note that by the independent works of Delmotte [17] and Holopainen and Soardi [21], the Moser iteration can be carried out for harmonic functions on graphs satisfying the volume doubling property and the Poincaré inequality. First, the Caccioppoli inequality for harmonic functions was obtained for general graphs with bounded degree (c.f. [14, 21, 42]).

Lemma 2 (Caccioppoli inequality) Let \((G, S)\) be the Cayley graph as (3). For any harmonic function \(f\) on \(B_{6R}(p)\), \(R \geq 1\), it holds that

\[
\sum_{x \in B_R(p)} |
abla f|^2(x) \leq \frac{C}{R^2} \sum_{x \in B_{6R}(p)} f^2(x),
\]

where \(C = C(S)\).

The Moser iteration implies the Harnack inequality for positive harmonic functions.

Lemma 3 (Harnack inequality) Let \((G, S)\) be the Cayley graph as (3). Then, there exist constants \(C_1(m, S)\) and \(C_2(m, S)\) such that for any \(p \in G\), \(R \geq 1\) and any positive harmonic function \(f\) on \(B_{C_1R}(p)\), we have

\[
\max_{B_R(p)} f \leq C_2 \min_{B_R(p)} f.
\]

The mean value inequality follows from one part of Moser iteration (c.f. [14, 17, 21]).

Lemma 4 (Mean value inequality) Let \((G, S)\) be the Cayley graph as (3). Then, there exists a constant \(C_1(m, S)\) such that for any \(p \in G\), \(R > 0\) and any non-negative subharmonic function \(f\) on \(B_{R}(p)\), we have

\[
f(p) \leq \frac{C_1}{|B_R(p)|} \sum_{x \in B_R(p)} f(x).
\]

The following Liouville theorem is a corollary of the Harnack inequality (6).

Lemma 5 (Liouville theorem) Let \((G, S)\) be the Cayley graph as (3). Then, any non-negative harmonic function \(f\) on \(G\) is constant.

Bochner’s formula was obtained by Chung-Yau and Lin-Yau on Ricci flat graphs ([9, 43]). For the case of Cayley graphs of finitely generated abelian groups, we present the proof here for the convenience of readers.

Lemma 6 (Bochner’s formula) Let \((G, S)\) be the Cayley graph as (3) and \(f\) be a harmonic function defined on \(B_1(x)\), for \(x \in G\). Then

\[
L^2 |\nabla f|^2(x) \geq 0.
\]
Proof Let us denote $S = \{s_1, s_2, \ldots, s_{2l}\}$, where $s_i = -s_{i+l}$, $1 \leq i \leq l$. Then

$$|
abla f|^2(x) = \sum_{y \sim x} (f(y) - f(x))^2 = \sum_{i=1}^{2l} |\delta_{s_i} f|^2(x).$$

Without loss of generality, it suffices to prove

$$L^S |\delta_{s_1} f|^2(x) \geq 0.$$ 

$$L^S |\delta_{s_1} f|^2(x) = \sum_{y \sim x} (|\delta_{s_1} f|^2(y) - |\delta_{s_1} f|^2(x))$$

$$= \sum_{y \sim x} (f(y + s_1) - f(y))^2 - 2l|\delta_{s_1} f|^2(x)$$

$$\geq \frac{1}{2l} \left[ \sum_{y \sim x} f(y + s_1) - f(y) \right]^2 - 2l|\delta_{s_1} f|^2(x)$$

$$= \frac{1}{2l} \left[ \sum_{i=1}^{2l} f(x + s_i + s_1) - \sum_{y \sim x} f(y) \right]^2 - 2l|\delta_{s_1} f|^2(x)$$

$$= \frac{1}{2l} \left[ \sum_{z \sim (x+s_1)} f(z) - \sum_{y \sim x} f(y) \right]^2 - 2l|\delta_{s_1} f|^2(x)$$

$$= \frac{1}{2l} [2lf(x + s_1) - 2lf(x)]^2 - 2l|\delta_{s_1} f|^2(x)$$

$$= 0,$$

where we used the Cauchy-Swarchz inequality in (9) and the harmonicity of $f$ in (10).

Combining Bochner’s formula with previous results, we obtain Yau’s gradient estimate for positive harmonic functions.

Proof of Theorem 1 We choose $C_1 = 7C_1'$, where $C_1' > 1$ is the constant $C_1(m, S)$ in Lemma 3. Bochner’s formula (8) implies that $|\nabla f|^2$ is a subharmonic function on $B_{C_1 R}(x)$. Then, the theorem follows from the mean value inequality (7), the Caccioppoli inequality (5), the volume doubling property (2) and the Harnack inequality (6),

$$|\nabla f|^2(x) \leq \frac{C}{|B_R(x)|} \sum_{y \in B_R(x)} |\nabla f|^2(y)$$

$$\leq \frac{C}{R^2 |B_R(x)|} \sum_{y \in B_{6R}(x)} f^2(y)$$

$$\leq \frac{C}{R^2 |B_{6R}(x)|} \sum_{y \in B_{6R}(x)} f^2(y)$$

$$\leq \frac{C}{R^2} f^2(x).$$

\square
Corollary 1 Let \((G, S)\) be the Cayley graph as \((3)\). There exist constants \(C_1(m, S)\) and \(C_2(m, S)\) such that for any \(x \in G, R \geq 1\) any harmonic function \(f\) on \(B_{C_1R}(x)\), we have
\[
|\nabla f|(x) \leq \frac{C_2}{R} \text{osc}_{B_{C_1R}(x)} f,
\]
where \(\text{osc}_{B_{C_1R}(x)} f := \max_{B_{C_1R}(x)} f - \min_{B_{C_1R}(x)} f\).

Proof It suffices to choose \(f - \min_{B_{C_1R}(x)} f + \epsilon\) (for small \(\epsilon\)) as the positive harmonic function in Theorem 1.

4 Polynomial growth harmonic functions are polynomials

The fundamental theorem of finitely generated abelian groups implies that for any finitely generated abelian group \(G\), there exists a generating set \(S^0 = \{e_1, \ldots, e_{2m}, w_1, \ldots, w_{2l}\}\) \((e_i = -e_i + m, w_j = -w_{j + l}, \text{ for } 1 \leq i \leq m, 1 \leq j \leq l)\) such that \(G = G_1 \oplus G_2\) is identified with \(\mathbb{Z}^m \oplus \bigoplus_{i = 1}^{l} \mathbb{Z}_{q_i}\), where \(\{e_1, \ldots, e_{2m}\}\) generates the torsion-free part \(G_1 \cong \mathbb{Z}^m\) and \(\{w_1, \ldots, w_{2l}\}\) generates the torsion part \(G_2 \cong \bigoplus_{i = 1}^{l} \mathbb{Z}_{q_i}\). For some fixed \(p \in G\), we denote by \(H^d(G, S) = \{u : G \to \mathbb{R} | L^S u = 0, \exists C \text{ s.t. } |u(x)| \leq C(d^S(p, x) + 1)^d\}\) the space of polynomial growth harmonic functions on \(G\) of growth order less than or equal to \(d\). For fixed \(S^0\), \(G_1\) is identified with \(\mathbb{Z}^m\), then we denote by \(P^d(\mathbb{Z}^m) := P^d(\mathbb{R}^m)\) the space of polynomials in \(\mathbb{R}^m\) restricted to the lattice \(\mathbb{Z}^m\) with degree less than or equal to \(d\).

First, we consider the easy case that \(G\) is torsion-free, i.e. \(G \cong \mathbb{Z}^m\). The following theorem generalizes the classical theorem of Heilbronn [22].

Theorem 6 Let \((G, S)\) be the Cayley graph of a torsion-free finitely generated abelian group, \(G \cong \mathbb{Z}^m\). Then, polynomial growth harmonic functions on \(G\) are polynomials, i.e.
\[
H^d(G, S) \subset P^d(\mathbb{Z}^m).
\]

Proof Let \(S^0 = \{e_1, \ldots, e_{2m}\}\) be the standard basis for \(\mathbb{Z}^m\), \(S = \{s_1, \ldots, s_{2l}\}\) be the generating set for the Cayley graph \((G, S)\). Then
\[
e_i = \sum_{k=1}^{2l} a_i^k s_k,
\]
where \(a_i^k \in \mathbb{Z}\) for \(1 \leq i \leq 2m, 1 \leq k \leq 2l\). Let
\[
\delta_i f(x) := \delta_{e_i} f(x) = f(x + e_i) - f(x),
\]
for \(1 \leq i \leq 2m\). Since \(G\) is abelian, it is easy to show that
\[
L^S \delta_i f = \delta_i L^S f
\]
which implies that \(\delta_i f\) is harmonic if \(f\) is.

We claim that \(\delta_i f \in H^{d-1}(G, S)\) if \(f \in H^d(G, S)\). Although for any \(x \in G\), \(x + e_i\) may not be neighbors in the Cayley graph \((G, S)\), there exists a path from \(x\) to \(x + e_i = x + \sum_{k=1}^{2l} a_i^k s_k\), i.e. \(x = x_0 \sim x_1 \sim \cdots \sim x_t = x + e_i\), whose length is
It suffices to show that

\[
|\delta_i f|(x) = |f(x + e_i) - f(x)| 
\leq |f(x + e_i) - f(x_{i-1})| + |f(x_{i-1}) - f(x_{i-2})| + \cdots + |f(x_1) - f(x)| 
\leq \sum_{j=0}^{i-1} |\nabla f|(x_j) 
\leq \frac{C}{R} \operatorname{osc}_{B_{2R}(x)} f,
\]

for \( R \geq R_1(S^0, S) \), since \( x_j \in B_C(x) \), for 0 \( \leq j \leq t - 1 \). The last inequality follows from Corollary 1. For any \( x \in B_R(p) \), we have \( B_{2R}(x) \subset B_{3R}(p) \). Then

\[
\operatorname{osc}_{B_R(p)} \delta_i f \leq 2 \max_{x \in B_R(p)} |\delta_i f|(x) \leq \frac{C}{R} \operatorname{osc}_{B_{3R}(p)} f \leq CR^{d-1},
\]

for \( R \geq R_1 \). This proves the claim.

Hence by taking finitely many times partial differences and the Liouville theorem (Lemma 5), we obtain

\[
\delta_{k_1} \delta_{k_2} \cdots \delta_{k_{2m}} f = 0,
\]

for any \( k_1 + k_2 + \cdots + k_{2m} \geq [d] + 1 \) and \( f \in H^d(G, S) \), where \([d]\) is the maximal integer not exceeding \( d \). By the basic difference equation theory or Lemma 2.13 in Nayar [44], we conclude that \( f \) is a polynomial. \( \square \)

Then, we can prove the generalized Heilbronn’s theorem.

**Proof of Theorem 2** It suffices to show that \( f \) is constant on \( G_2 \), i.e. \( f(x + w) = f(x) \), for any \( x \in G, w \in G_2 \). An alternative method by covering argument can be found in the last section.

Let \( S^0 = \{e_1, \ldots, e_{2m}, w_1, \ldots, w_{2l}\} \) \( (e_i = -e_{i+m}, w_j = -w_{j+l}, \) for \( 1 \leq i \leq m, 1 \leq j \leq l \) such that \( G = G_1 \oplus G_2 \) is identified with \( \mathbb{Z}^m \oplus \mathbb{Z}^l \mathbb{Z}_{ql} \), where \( \{e_1, \ldots, e_{2m}\} \) generates the torsion-free part \( G_1 \cong \mathbb{Z}^m \) and \( \{w_j, w_{j+l}\} \) generates \( \mathbb{Z}_{ql} \). Let \( \delta w_j f(x) := f(x + w_j) - f(x) \). The same argument as in the proof Theorem 6 implies that \( \delta w_j f \in H^{d-1}(G, S) \) if \( f \in H^d(G, S) \). Then \( \delta w_j^{[d] + 1} f \equiv 0 \), for \( f \in H^d(G, S) \). For fixed \( x \in G \), lifting \( \mathbb{Z}_{ql} \) to \( \mathbb{Z} \), we obtain that \( f \) is a polynomial on \( \mathbb{Z} \). Hence, as a periodic polynomial \( f \), i.e. \( f(x + (r + kq_j)w_j) = f(x + rw_j) \) for any \( k, r \in \mathbb{Z} \), must be constant, i.e. \( f(x + rw_j) = f(x) \), for any \( r \in \mathbb{Z} \). Since this is true for any \( 1 \leq j \leq l \), we obtain \( f(x + w) = f(x) \), for any \( x \in G, w \in G_2 \). Then it is easy to see that \( L^{\pi G_1, S} f(x_1) = 0 \) for any \( x_1 \in G_1 \) if \( L^S f(x) = 0 \) for any \( x \in G \). Hence, \( H^d(G, S) = H^d(G_1, \pi G_1, S) \subset P^{[d]}(\mathbb{Z}^m) \). \( \square \)

By the Harnack inequality, we reprove Nayar’s theorem.

**Proof of Theorem 3** It suffices to show \( HM^d(G, S) \subset H^d(G, S) \). Without loss of generality, for any harmonic function \( f \) satisfying \( f(x) \geq -C(d(p, x) + 1)^d \), we need to prove that \( f(x) \leq C(d(p, x) + 1)^d \), for some \( C \). For simplicity, we assume \( f(p) = 0 \). Let \( C_1 \) be the constant for the Harnack inequality in the Lemma 3. Then for any \( x \in B_R(p), R > 0 \), it is easy to see that \( B_{C_1 R}(x) \subset B_{(C_1 + 1)R}(p) \). Moreover

\[
f(y) \geq -C(d(p, y) + 1)^d \geq -C((C_1 + 1)R + 1)^d \geq -CR^d,
\]
for \( y \in B(C_{1+1}R(p)) \), \( R \geq 1 \). That is \( f(y) + CR^d \geq 0 \) on \( B_{C_1 R}(x) \). The Harnack inequality (6) implies that

\[
f(x) + CR^d \leq C(f(p) + CR^d) = C_2 R^d.
\]

Then we have

\[
f(x) \leq CR^d,
\]

for \( x \in B_R(p) \), \( R \geq 1 \). Hence, there exists a constant \( C \) such that \( f(x) \leq C(d(p, x) + 1)^d \).

\[ \square \]

5 Calculating the dimension

For calculating the dimension, by Theorem 2, it suffices to consider harmonic polynomials on a torsion-free finitely generated abelian group. Let \((G, S)\) be the Cayley graph of \( G \cong \mathbb{Z}^m \). There exists a generating set \( S^0 = \{e_1, \ldots, e_2n\} \) such that \( G \) is identified with \( \mathbb{Z}^m \) in \( \mathbb{R}^m \). For \( k \in \mathbb{N} \cup \{0\} \), we denote by \( P^k(\mathbb{R}^m) \) the space of polynomials on \( \mathbb{R}^m \) of degree less than or equal to \( k \), by \( P^k_m := P^k(\mathbb{Z}^m) := \{u : \mathbb{Z}^m \to \mathbb{R} | u|_{\mathbb{Z}^m} = f|_{\mathbb{Z}^m}, f \in P^k(\mathbb{R}^m)\} \) the space of the restriction of polynomials on \( \mathbb{R}^m \) to \( \mathbb{Z}^m \) of degree less than or equal to \( k \). We denote by \( R^k_m := H P^k(\mathbb{R}^m) \) the space of harmonic polynomials on \( \mathbb{R}^m \) (\( \Delta u = 0 \)) of degree less than or equal to \( k \). For the Cayley graph \((G, S)\) which is identified with \((\mathbb{Z}^m, S)\), we set \( D^k_{S,m} := H P^k(\mathbb{Z}^m, S) := \{u \in P^k(\mathbb{Z}^m) | L^S u = 0\} \). In order to calculate the dimension of discrete harmonic polynomials, we make the dimension comparison between \( D^k_{S,m} \) and \( R^k_m \).

It is well known for harmonic polynomials on \( \mathbb{R}^m \) (see [36]) that

\[
\begin{align*}
\dim R^k_m &= \binom{m + k - 1}{k} + \binom{m + k - 2}{k - 1}, & (k \geq 1, \dim R^0_m = 1) \\
\dim R^k_m &= \dim R^k_{m-1} + \dim R^k_{m-1}, & (k \geq 1) \\
\dim P^k_m &= \dim P^k(\mathbb{R}^m) = \sum_{i=0}^{k} \binom{m + i - 1}{i}, & (k \geq 0) \\
\dim P^k_m &= \dim R^k_m + \dim P^{k-2} m, & (k \geq 2)
\end{align*}
\]

Lemma 7

\[
\dim D^k_{S,1} = \dim R^k_{1} = 2. \tag{12}
\]

for any \( S, k \geq 1 \).

Proof Let \( S = \{a_i\}_{i=1}^{2l} \), \( a_i \in \mathbb{Z} \), and \( a_i = -a_{i+l} \) for \( 1 \leq i \leq l \). Since \( S \) is a generating set, at least one \( a_i \neq 0 \). For any \( f \in D^k_{S,1} \), \( f = b_k x^k + b_{k-1} x^{k-1} + \cdots + b_0 \), where \( b_0, b_1, \ldots, b_k \in \mathbb{R} \). By Taylor expansion, the difference equation \( L^S f = 0 \) reads

\[
\sum_{n=1}^{\infty} \sum_{i=1}^{l} \frac{2^{2n}}{(2n)!} f^{(2n)}(x) = 0.
\]

Comparing the degree of polynomials, we have

\[
(b_k x^k)'' = 0,
\]

for \( x \in \mathbb{Z} \). Hence, \( k \leq 1 \), i.e. \( f \) is linear. \[ \square \]
Lemma 8

\[ \dim D_{S,m}^k \leq \dim R_m^k, \]

for any \( S, k \geq 0, m \geq 1 \).

Proof It is easy to see that \( D_{S,m}^0 = R_m^0 = \text{const} \). We apply an induction argument on \( k \). It suffices to prove (13) for \( k = l \) if it is true for all \( k \leq l - 1 \). We pick \( e_1 \in S^0 \) and define \( \delta_1 f(x) = f(x + e_1) - f(x) \), for any function \( f \) on \( G \). Since \( G \) is abelian, \( \delta_1L^S = L^S\delta_1 \), then \( \delta_1 \) is a well-defined linear operator,

\[ \delta_1 : D_{S,m}^k \rightarrow D_{S,m}^{k-1}. \]

By linear algebra,

\[ \dim D_{S,m}^k = \dim \ker \delta_1 + \dim \text{im} \delta_1, \]

where \( \ker \delta_1 \) and \( \text{im} \delta_1 \) are Kernel and Image of \( \delta_1 \). There is a natural projection \( P : \mathbb{Z}^m \rightarrow \mathbb{Z}^{m-1} \), for any \( x = (x_1, x_2, \ldots, x_m) \in \mathbb{Z}^m \), \( Px = x' \), where \( x' = (x_2, \ldots, x_m) \). For any generating set \( S \) for \( \mathbb{Z}^m \), \( S' = \{s'| s \in S\} \) is the generating set of \( \mathbb{Z}^{m-1} \). Moreover, \( S^{(t)} := P^tS \) is the generating set for \( \mathbb{Z}^{m-t}, 1 \leq t \leq m - 1 \).

For any \( f \in \ker \delta_1 \), i.e. \( \delta_1 f = 0 \), then \( f(x_1, x_2, \ldots, x_m) = g(x_2, \ldots, x_m) \). Hence

\[ 0 = L^S f(x) = \sum_{s \in S} (f(x + s) - f(x)) \]

\[ = \sum_{s \in S} (g(x' + s') - g(x')) = L^{S'} g(x'). \]

That is \( \ker \delta_1 = D_{S',m-1}^k \). Hence (14) implies that

\[ \dim D_{S,m}^k \leq \dim D_{S',m-1}^k + \dim D_{S,m}^{k-1}, \]

for any \( S, k \geq 1, m \geq 1 \). Then it follows that

\[ \dim D_{S,m}^l \leq \dim D_{S',m-1}^l + \dim D_{S,m}^{l-1} \]

\[ \leq \dim D_{S''',m-2}^l + \dim D_{S,m-1}^{l-1} + \dim D_{S,m}^{l-1} \]

\[ \leq \cdots \]

\[ \leq \dim D_{S^{(m-i)},1}^l + \sum_{i=2}^m \dim D_{S^{(m-i)},1}^{l-1} \]

\[ \leq 2 + \sum_{i=2}^m \dim R_i^{l-1} \]

\[ = \dim R_1^l + \sum_{i=2}^m \dim R_i^{l-1} \]

\[ = \dim R_m^l, \]

where we use Lemma 7, the inductive assumption (13) for \( k \leq l - 1 \) and some facts in \( \mathbb{R}^n \).

Now we can prove the main Theorem 4.
**Proof of Theorem 4** It suffices to show that

\[ \dim D_{S,m}^k = \dim R_m^k, \]

for any \( S, k \geq 0 \) and \( m \geq 1 \). We may assume \( k \geq 2 \), otherwise it is trivial. By \( S = -S \), the Laplacian operator \( L_S^k \) is a linear operator,

\[ L^S: P_m^k \to P_m^{k-2}. \tag{15} \]

Since \( \ker L^S = D_{S,m}^k \), we have

\[ \dim P_m^k = \dim \ker L^S + \dim \text{im} L^S \leq \dim D_{S,m}^k + \dim P_m^{k-2} \leq \dim R_m^k + \dim P_m^{k-2} = \dim P_m^k, \]

which follows from (13). Hence, \( \dim D_{S,m}^k = \dim R_m^k \).

\( \square \)

**Remark 1** In the above theorems, all the inequalities for the dimension comparison are actually equalities. Hence, we obtain that \( \delta_1 \) and \( L^S \) are surjective linear operators. In fact, Heilbronn [22] used the technical lemma in difference equation theory (see [19,23]) that \( L_{S^0}^S: P_m^k \to P_m^{k-2} \) is surjective to calculate the dimension. Conversely, by the dimension comparison, we obtain a more general difference equation lemma.

**Corollary 2** Let \((\mathbb{Z}^m, S)\) be the Cayley graph of \( \mathbb{Z}^m \). Then

\[ \delta_1: D_{S,m}^k \to D_{S,m}^{k-1}, \]

and

\[ L^S: P_m^k \to P_m^{k-2} \tag{16} \]

are surjective linear operators.

6 Dimension of higher order harmonic functions

Let \((G, S)\) be the Cayley graph of the finitely generated abelian group with a finite generating set \( S \), \( G \cong \mathbb{Z}^m \oplus \bigoplus_{i=1}^{l} \mathbb{Z}_{q_i} \). We define the \( n \)th Laplace operator as

\[ L^n: = L^S \circ L^S \circ \cdots \circ L^S, \]

i.e. \( n \)-times composition of Laplace operators. A function \( u \) on \( G \) is called \( n \)-harmonic if \( L^n u = 0 \). We use the results for Laplace operators in Theorem 2 and Corollary 2 to obtain the dimension estimate for higher order harmonic functions.

**Proof of Theorem 5** To simplify the argument, we only prove the theorem for \( n = 2 \) which directly applies to general \( n \). For the first assertion, it suffices to show that \( H^2, d(G, S) \subset P^d(\mathbb{Z}^m) \) for any torsion-free \( G \cong \mathbb{Z}^m \) (as in Theorem 6) since we may reduce the general case to this one. For a general \( G = G_1 \oplus G_2 \cong \mathbb{Z}^m \oplus \bigoplus_{i=1}^{l} \mathbb{Z}_{q_i} \) and a generating set \( S \), we may find a torsion-free \( \tilde{G} \cong \mathbb{Z}^{m+l} \cong \mathbb{Z}^m \oplus \bigoplus_{i=1}^{l} \mathbb{Z} \) and a generating set \( \tilde{S} \) such that the Cayley graph \((\tilde{G}, \tilde{S})\) is a natural covering of \((G, S)\). For any \( f \in H^2, d(G, S) \), the lifting function \( \tilde{f} \) of \( f \) on \( \tilde{G} \) is 2-harmonic and of polynomial growth in the sense of \((\tilde{G}, \tilde{S})\). This implies that
\( \tilde{f} \) is a polynomial on \( \tilde{G} \) by the torsion-free case. Since \( \tilde{f} \) is periodic in the factor \( \mathbb{Z}^l \) by the lifting, this yields that as a polynomial \( \tilde{f} \) is constant on \( \mathbb{Z}^l \). Hence, \( f \) is a polynomial on \( \mathbb{Z}^m \) and constant on the \( G_2 \) factor.

From now on, we assume \( G \simeq \mathbb{Z}^m \). For any \( f \in H^{2,d}(\mathbb{Z}^m, S) \),

\[
L^{2,S} f = L^S( L^S f) = 0.
\]

Since \( |L^S f(x)| \leq m \max_{y \sim x} \{|f(y)| + |f(x)|\} \), we have \( L^S f \in H^d(\mathbb{Z}^m, S) \). By Theorem 2 or 6, we know that \( L^S f \) is a polynomial on \( \mathbb{Z}^m \), denoted by \( g := L^S f \). By (16) in Corollary 2, we may find a polynomial, say \( f_1 \), satisfying \( L^S f_1 = g \). Hence \( L^S(f - f_1) = 0 \). The polynomial growth of \( f - f_1 \) implies that \( f - f_1 \) is actually a polynomial. This proves that \( f \) is a polynomial (In fact, \( f \) is a polynomial of degree less than or equal to \( d \) by \( f \in H^{2,d}(\mathbb{Z}^m, S) \)).

Now we calculate the dimension of \( H^{2,d}(\mathbb{Z}^m, S) \). We set \( k = [d] \). By (15), we have a linear operator,

\[
L^{2,S} : P_m^k \to P_m^{k-4}. \quad (k \geq 4)
\]

It is easy to see that \( \ker L^{2,S} = H^{2,k}(\mathbb{Z}^m, S) \). Moreover, (16) implies that \( L^{2,S} \) is surjective. Hence

\[
\dim H^{2,k}(\mathbb{Z}^m, S) = \dim \ker L^{2,S} = \dim P_m^k - \dim P_m^{k-4}.
\]

This proves the theorem. \( \square \)

References

1. Alexopoulos, G.: An application of homogenization theory to harmonic analysis: Harnack inequalities and Riesz transforms on Lie groups of polynomial growth. Canad. J. Math. 44(4), 691–727 (1992)
2. Alexopoulos, G.: An application of homogenization theory to harmonic analysis on solvable Lie groups of polynomial growth. Pacific J. Math. 159(1), 19–45 (1993)
3. Alexopoulos, G.: Random walks on discrete groups of polynomial volume growth. Ann. Probab. 30(2), 723–801 (2002)
4. Alexopoulos, G., Lohoué, N.: Sobolev inequalities and harmonic functions of polynomial growth. J. London Math. Soc. (2) 48(3), 452–464 (1993)
5. Avellaneda, M., Lin, F.H.: Un théorème de Liouville pour des équations elliptiques à coefficients périodiques. C. R. Acad. Sci. Paris Sér. I Math. 309(5), 245–250 (1989)
6. Burago, D., Burago, Yu., Ivanov, S.: A course in metric geometry, Graduate Studies in Mathematics 33. American Mathematical Society, Providence, RI (2001)
7. Chen, R., Wang, J.: Polynomial growth solutions to higher-order linear elliptic equations and systems. Pacific J. Math. 229(1), 49–61 (2007)
8. Cheng, S.Y., Yau, S.T.: Differential equations on Riemannian manifolds and their geometric applications. Comm. Pure Appl. Math. 28, 333–354 (1975)
9. Chung, F.R.K., Yau, S.T.: Logarithmic Harnack inequalities. Math. Res. Lett. 3(6), 793–812 (1996)
10. Colding, T.H., Minicozzi II, W.P.: Harmonic functions with polynomial growth. J. Diff. Geom. 46(1), 1–77 (1997)
11. Colding, T.H., Minicozzi II, W.P.: Harmonic functions on manifolds. Ann. of Math. (2) 146(3), 725–747 (1997)
12. Colding, T.H., Minicozzi II, W.P.: Weyl type bounds for harmonic functions. Invent. Math. 131(2), 257–298 (1998)
13. Colding, T.H., Minicozzi II, W.P.: Liouville theorems for harmonic sections and applications. Comm. Pure Appl. Math. 51(2), 113–138 (1998)
14. Coulhon, T., Grigoryan, A.: Random walks on graphs with regular volume growth. Geom. Funct. Anal. 8(4), 656–701 (1998)
15. Coulhon, T., Saloff-Coste, L.: Variétés riemanniennes isométriques à l’infini. Rev. Mat. Iberoamericana 11(3), 687–726 (1995)
16. Delmotte, T.: Harnack inequalities on graphs, Séminaire de Théorie Spectrale et Géométrie, 16, 217–228 (1997–1998)
17. Delmotte, T.: Inégalité de Harnack elliptique sur les graphes. Colloq. Math. 72(1), 19–37 (1997)
18. Donnelly, H.; Fefferman, C.: Nodal domains and growth of harmonic functions on noncompact manifolds. J. Geom. Anal. 2(1), 79–93 (1992)
19. Duffin, R.J., Shelly, E.P.: Difference equations of polyharmonic type. Duke Math. J. 25, 209–238 (1958)
20. Gromov, M.: Groups of polynomial growth and expanding maps. Inst. Hautes Études Sci. Publ. Math. No. 53, 53–73 (1981)
21. Holopainen, I., Soardi, P.M.: A strong Liouville theorem for p-harmonic functions on graphs. Ann. Acad. Sci. Fenn. Math. 22(1), 205–226 (1997)
22. Heilbronn, H.A.: On discrete harmonic functions. Proc. Cambridge Philos. Soc. 45, 194–206 (1949)
23. Hobson, E.W.: The theory of spherical and ellipsoidal harmonics. Chelsea Publishing Company, New York (1955)
24. Hua, B.: Generalized Liouville theorem in nonnegatively curved Alexandrov spaces. Chin. Ann. Math. Ser. B 30(2), 111–128 (2009)
25. Hua, B.: Harmonic functions of polynomial growth on singular spaces with nonnegative Ricci curvature. Proc. Amer. Math. Soc. 139, 2191–2205 (2011)
26. Hua, B.: Polynomial growth harmonic functions on groups of polynomial volume growth, preprint, arXiv:1201.5238.
27. Hua, B., Jost, J.: Polynomial growth harmonic functions on complete Riemannian manifolds. Math. Z. 223(1), 103–113 (2000)
28. Kleiner, B.: A new proof of Gromov’s theorem on groups of polynomial growth. J. Amer. Math. Soc. 23(3), 815–829 (2010)
29. Lin, Y.H.: Polynomial growth harmonic functions on complete Riemannian manifolds. Rev. Mat. Iberoamericana 20(2), 315–332 (2004)
30. Li, P.: Harmonic sections of polynomial growth. Math. Res. Lett. 4(1), 35–44 (1997)
31. Li, P.: Harmonic functions and applications to complete manifolds (lecture notes), preprint.
32. Li, P.: Harmonic sections of polynomial growth. Math. Res. Lett. 4(1), 35–44 (1997)
33. Li, P., Wang, J.: Mean value inequalities. Indiana Univ. Math. J. 48(4), 1257–1283 (1999)
34. Li, P., Wang, J.: Counting dimensions of L-harmonic functions. Ann. of Math. (2) 152(2), 645–658 (2000)
35. Li, P., Wang, J.: Polynomial growth solutions of uniformly elliptic operators of non-divergence form. Proc. Amer. Math. Soc. 129(12), 3691–3699 (2001)
36. Lin, F.: Asymptotically conic elliptic operators and Liouville type theorems, Geometric analysis and the calculus of variations, pp. 217–238. Int. Press, Cambridge, MA (1996)
37. Lin, Y., Xi, L.: Lipschitz property of harmonic function on graphs. J. Math. Anal. Appl. 366(2), 673–678 (2010)
38. Lin, Y., Zeng, S.: Ricci curvature and eigenvalue estimate on locally finite graphs. Math. Res. Lett. 17(2), 343–356 (2010)
39. Nayar, P.: On polynomially bounded harmonic functions on the $\mathbb{Z}^d$ lattice. Bull. Pol. Acad. Sci. Math., 57(3–4), 231–242 (2009)
40. Raghunathan, M.S.: Discrete subgroups of Lie groups, Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 68. Springer-Verlag, New York-Heidelberg (1972)
41. Robinson, D.J.S.: A course in the theory of groups, second edition, Graduate Texts in Mathematics, 80. Springer-Verlag, New York (1996)
42. Shalom, Y., Tao, T.: A finitary version of Gromov’s polynomial growth theorem. Geom. Funct. Anal. 20(6), 1502–1547 (2010)
43. Sung, C.-J., Tam, L.-F., Wang, J.: Spaces of harmonic functions. J. London Math. Soc. (2) 61(3), 789–806 (2000)
49. Suzuki, M.: Group theory I, Translated from the Japanese by the author, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], 247. Springer-Verlag, Berlin-New York (1982)

50. Tam, L.-F.: A note on harmonic forms on complete manifolds. Proc. Amer. Math. Soc. 126(10), 3097–3108 (1998)

51. Wang, J.: Linear growth harmonic functions on complete manifolds. Comm. Anal. Geom. 4, 683–698 (1995)

52. Yau, S.T.: Harmonic functions on complete Riemannian manifolds. Comm. Pure Appl. Math. 28, 201–228 (1975)

53. Yau, S.T.: Enseign. Math. Nonlinear analysis in geometry. 33(2), 109–158 (1987)

54. Yau, S. T.: Differential Geometry: Partial Differential Equations on Manifolds, Proc. of Symposia in Pure Mathematics, 54, part 1, Ed. by R.Greene and S.T. Yau (1993)