Littlewood–Paley decompositions and Besov spaces on Lie groups of polynomial growth

Giulia Furioli, Camillo Melzi and Alessandro Veneruso

Abstract

We introduce a Littlewood–Paley decomposition related to any sub-Laplacian on a Lie group $G$ of polynomial volume growth; this allows us to prove a Littlewood–Paley theorem in this general setting and to provide a dyadic characterization of Besov spaces $B^{s,q}_p(G)$, $s \in \mathbb{R}$, equivalent to the classical definition through the heat kernel.

1 Introduction

Littlewood–Paley decompositions are a powerful tool in investigating deep properties of function spaces of distributions. Let us recall briefly the classical construction in $\mathbb{R}^n$. Let $\varphi \in C^\infty(\mathbb{R})$ be an even function such that $0 \leq \varphi \leq 1$, $\varphi = 1$ in $[0, \frac{1}{4}]$ and $\varphi = 0$ in $[1, \infty)$. Let $\psi(\lambda) = \varphi(\frac{\lambda}{4}) - \varphi(\lambda)$, so that supp $\psi \subset \{\frac{1}{4} \leq |\lambda| \leq 4\}$. We have the following partition of unity on the frequency space of the Fourier transform:

$$1 = \varphi(|\xi|^2) + \sum_{j=0}^{\infty} \psi(2^{-2j}|\xi|^2), \quad \xi \in \mathbb{R}^n.$$ 

This gives the identity in $\mathcal{S}'(\mathbb{R}^n)$:

$$\hat{u} = \varphi(|\cdot|^2)\hat{u} + \sum_{j=0}^{\infty} \psi(2^{-2j}|\cdot|^2)\hat{u}, \quad u \in \mathcal{S}'(\mathbb{R}^n)$$

and, denoting by $S_0u$ e $\Delta_j u$ respectively

$$\hat{S_0u} = \varphi(|\cdot|^2)\hat{u}, \quad \hat{\Delta_j u} = \psi(2^{-2j}|\cdot|^2)\hat{u},$$

we obtain the Littlewood–Paley decomposition in $\mathcal{S}'(\mathbb{R}^n)$:

$$(1) \quad u = S_0u + \sum_{j=0}^{\infty} \Delta_j u, \quad u \in \mathcal{S}'(\mathbb{R}^n).$$

The following fundamental theorem holds (see e.g. [St]):

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Theorem 1 (Littlewood–Paley)
Let $1 < p < \infty$ and $u \in S'(\mathbb{R}^n)$. Then $u \in L^p(\mathbb{R}^n)$ if and only if $S_0 u \in L^p(\mathbb{R}^n)$ and
\[
\left( \sum_{j=0}^{\infty} |\Delta_j u|^2 \right)^{\frac{1}{2}} \in L^p(\mathbb{R}^n).
\]
Moreover there exists a constant $C_p > 1$, which depends only on $p$, such that
\[
C_p^{-1} \|u\|_{L^p(\mathbb{R}^n)} \leq \|S_0 u\|_{L^p(\mathbb{R}^n)} + \left\| \left( \sum_{j=0}^{\infty} |\Delta_j u|^2 \right)^{\frac{1}{2}} \right\|_{L^p(\mathbb{R}^n)} \leq C_p \|u\|_{L^p(\mathbb{R}^n)}, \quad u \in L^p(\mathbb{R}^n).
\]

The proof of this theorem is based on the classical Hörmander–Mihlin $L^p$-multiplier theorem ([H]) and on the uniform estimates for the norms of the convolution operators $\Delta_j$ on $L^p(\mathbb{R}^n)$. The main purpose of this paper is to prove a Littlewood–Paley theorem on Lie groups of polynomial volume growth, with respect to any sub-Laplacian.

In view of extending the previous construction to a general Lie group of polynomial growth it is more convenient to see the decomposition (1) in terms of multipliers of the Laplacian $\Delta = -\sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}$. Starting from the spectral decomposition of the Laplacian
\[
\Delta = \int_0^\infty \lambda dE_\lambda
\]
and from the functions $\varphi, \psi \in L^\infty(\mathbb{R}^n)$ previously introduced, we consider the multiplier operators
\[
\varphi(\Delta) = \int_0^\infty \varphi(\lambda) dE_\lambda,
\]
\[
\psi(2^{-2j} \Delta) = \int_0^\infty \psi(2^{-2j} \lambda) dE_\lambda.
\]
So we have the identifications between operators
\[
\varphi(\Delta) f = S_0 f, \quad \psi(2^{-2j} \Delta) f = \Delta_j f, \quad f \in L^2(\mathbb{R}^n).
\]
If we denote by $\Psi_j$ the convolution kernel of the operator $\psi(2^{-2j} \Delta)$, due to the dilation structure of $\mathbb{R}^n$ we have the scaling formula
\[
(2) \quad \Psi_j(x) = 2^{nj} \Psi_0(2^j x), \quad x \in \mathbb{R}^n, j \in \mathbb{N}
\]
that allows us to easily obtain the estimates for the norms of the operators $\Delta_j$ only from the operator $\Delta_0$. If we now consider a stratified Lie group $G$, endowed with its natural dilation structure, and if $\Delta$ is a sub-Laplacian on $G$ invariant with respect to the family of dilations, the scaling formula (2) still holds, where $n$ is the homogeneous dimension of $G$. In this case the way to prove a Littlewood–Paley theorem through a Hörmander–Mihlin multiplier theorem is based on classical techniques ([FS], [DM], [MM]). In the particular case of the Heisenberg group $\mathbb{H}_n$, we can deduce a Littlewood–Paley theorem also for the full Laplacian from the results by Müller, Ricci and Stein [MRS1], [MRS2]; but their techniques, based on the Fourier transform on $\mathbb{H}_n$, do not fit to the case of Lie groups of polynomial growth. Alexopoulos
in [A2] proved a Hörmander–Mihlin multiplier theorem for any sub-Laplacian in the general setting of Lie groups of polynomial growth; nevertheless, such result does not provide directly the uniform estimates for the norms of the operators $\Delta_j$ we need to deduce a Littlewood–Paley theorem.

The main result of this paper is Proposition 6 which allows us to deduce the uniform estimates for the norms of the operators $\Delta_j$ and to prove a Littlewood–Paley decomposition in $S'(G)$ (Proposition 8) and a Littlewood–Paley theorem (Theorem 10). As an application of such decomposition, we finally provide a dyadic characterization of Besov spaces $B_{p,q}^s(G)$, $s \in \mathbb{R}$, equivalent to the classical definition through the heat kernel (Proposition 12).

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2 Notation and preliminaries

In this paper $\mathbb{N}$ denotes the set of nonnegative integers, $\mathbb{Z}_+$ the set of positive integers and $\mathbb{R}_+$ the set of positive real numbers. For $p \in [1, \infty]$ we denote by $p'$ the conjugate index of $p$, such that $\frac{1}{p} + \frac{1}{p'} = 1$.

In this section we recall some basic facts about Lie groups of polynomial growth. For the proofs and further information, see e.g. [VSC] and the references given therein.

Let $G$ be a connected Lie group, and let us fix a left-invariant Haar measure $dx$ on $G$. We will denote by $|A|$ the measure of a measurable subset $A$ of $G$ and by $\chi_A$ its characteristic function.

We assume that $G$ has polynomial volume growth, i.e., if $U$ is a compact neighbourhood of the identity element $e$ of $G$, then there is a constant $C > 0$ such that $|U^n| \leq Cn^C$, $n \in \mathbb{Z}_+$. Then $G$ is unimodular. Furthermore, there exists $D \in \mathbb{N}$, which does not depend on $U$, such that

$$|U^n| \sim n^D \quad \text{for} \ n \to \infty. \quad (3)$$

For instance, every connected nilpotent Lie group has polynomial volume growth.

The convolution of two functions $f$ and $g$ on $G$ is defined by

$$f \ast g(x) = \int_G f(y)g(y^{-1}x) \, dy, \quad x \in G$$

and satisfies the Young’s inequality (where $1 + \frac{1}{r} = \frac{1}{p} + \frac{1}{q}$)

$$\|f \ast g\|_{L^r(G)} \leq \|f\|_{L^p(G)}\|g\|_{L^q(G)}. \quad (4)$$

The space $\mathcal{D}(G)$ of test functions and the space $\mathcal{D}'(G)$ of distributions are defined in the usual way (see [E]). The convolution of $\varphi \in \mathcal{D}(G)$ and $u \in \mathcal{D}'(G)$ is defined as usual:

$$\langle \varphi \ast u, \psi \rangle = \langle u, \check{\varphi} \ast \psi \rangle, \quad \psi \in \mathcal{D}(G)$$

where

$$\check{\varphi}(x) = \varphi(x^{-1}), \quad x \in G.$$
Let $X_1, \ldots, X_k$ be left-invariant vector fields on $G$ which satisfy the Hörmander’s condition, i.e., they generate, together with their successive Lie brackets $[X_{i_1}, \ldots, X_{i_n}] \cdots$, the Lie algebra of $G$. For $I = (i_1, \ldots, i_\beta) \in \{1, \ldots, k\}^{\beta} (\beta \in \mathbb{N})$ we put $|I| = \beta$ and $X^I = X_{i_1} \cdots X_{i_\beta}$, with the convention that $X^I = \text{id}$ if $\beta = 0$.

To $X_1, \ldots, X_k$ is associated, in a canonical way, the control distance $\rho$, which is left-invariant and compatible with the topology on $G$. For any $x \in G$ we put $|x| = \rho(e, x)$. The properties of $\rho$ imply that $|xy| \leq |x| + |y|$ for any $x, y \in G$. Furthermore, for any $r > 0$ we put $V(r) = |B(e, r)|$ where $B(e, r) = \{x \in G : |x| < r\}$. By (3) we have

$$V(r) \sim r^D \quad \text{for } r \to \infty.$$ 

On the other hand, there exists $d \in \mathbb{N}$ such that

$$V(r) \sim r^d \quad \text{for } r \to 0.$$ 

These estimates imply the “doubling property”: there exists $K > 0$ such that

$$V(2r) \leq KV(r), \quad r > 0.$$ 

We consider the sub-Laplacian

$$\mathcal{L} = -\sum_{j=1}^k X_j^2$$

which is a positive self-adjoint operator, having as domain of definition the space of all functions $f \in L^2(G)$ such that $\mathcal{L}f \in L^2(G)$. So, by the spectral theorem, for any bounded Borel function $m$ on $[0, \infty)$ we can define the operator $m(\mathcal{L})$ which is bounded on $L^2(G)$. Since the point 0 may be neglected in the spectral resolution of $\mathcal{L}$ (see [C], [A2]), we consider that the function $m$ is defined on $\mathbb{R}_+$. Furthermore, the operator $m(\mathcal{L})$ admits a kernel $M \in \mathcal{D}'(G)$ which satisfies $m(\mathcal{L})f = f \ast M$ for any $f \in \mathcal{D}(G)$. We recall the following well-known results:

**Theorem 2 ([A2])**

Put $N = 1 + \max\{[\frac{d}{2}], [\frac{D}{2}]\}$. If $m \in C^N(\mathbb{R}_+)$ and \(\sup_{\lambda > 0} \lambda^r |m(\lambda)| < \infty\) for any $r \in \{0, \ldots, N\}$, then $m(\mathcal{L})$ extends to a bounded operator on $L^p(G)$, $1 < p < \infty$.

**Proposition 3**

Let $\{m_n\}_{n \in \mathbb{N}}$ be a sequence of bounded Borel functions on $\mathbb{R}_+$ which converges at every point to a bounded Borel function $m$. Suppose also that the sequence $\{|m_n|_{L^\infty(\mathbb{R}_+)}\}_{n \in \mathbb{N}}$ is bounded. Then the sequence $\{M_n\}_{n \in \mathbb{N}}$, where $M_n$ is the kernel of the operator $m_n(\mathcal{L})$, converges in $\mathcal{D}'(G)$ to the kernel $M$ of the operator $m(\mathcal{L})$.

**Proof:** By the spectral theorem $m_n(\mathcal{L})f \to m(\mathcal{L})f$ in $L^2(G)$ for $n \to \infty$ for every $f \in L^2(G)$. In particular, $f \ast M_n \to f \ast M$ in $\mathcal{D}'(G)$ for $n \to \infty$ for every $f \in \mathcal{D}(G)$. Fix $\varphi \in \mathcal{D}(G)$. By [DiM] Théorème 3.1 the function $\varphi$ can be written as a finite sum $\varphi = \sum_{j=1}^r \psi_j \ast \chi_j$ with $\psi_j, \chi_j$ in $\mathcal{D}(G)$. So by (1)

$$\langle M_n, \varphi \rangle = \sum_{j=1}^r \langle \psi_j \ast M_n, \chi_j \rangle \overset{n \to \infty}{\longrightarrow} \sum_{j=1}^r \langle \psi_j \ast M, \chi_j \rangle = \langle M, \varphi \rangle.$$
We introduce the Schwartz space $S(G)$ and its dual space $S'(G)$ as in [S1], [S2]. The definition does not depend on $X_1, \ldots, X_k$. However, a useful characterization of $S(G)$ is the following: a function $f \in C^\infty(G)$ is in $S(G)$ if and only if all the seminorms

$$p_{\alpha,I}(f) = \sup_{x \in G} (1 + |x|)^\alpha |X_I f(x)|$$

$(\alpha \in \mathbb{N}, I \in \bigcup_{\beta \in \mathbb{N}} \{1, \ldots, k\}^\beta)$ are finite. The space $S(G)$ endowed with this family of seminorms is a Fréchet space. It is easy to show that $S(G) \subset S'(G) \subset L^p(G)$ for $1 \leq p \leq \infty$.

The heat kernel $p_t$, i.e. the kernel of the operator $e^{-tL}$ $(t > 0)$, is a positive $C^\infty$ function which satisfies $\int_G p_t(x) \, dx = 1$. Moreover, for any $I \in \bigcup_{\beta \in \mathbb{N}} \{1, \ldots, k\}^\beta$ there exists $C > 0$ such that the following estimates hold:

$$p_t(x) \leq CV(\sqrt{t})^{-1} e^{-\frac{|x|^2}{Ct}}, \quad x \in G, \ t > 0; \quad (6)$$

$$|X_I p_t(x)| \leq Ct^{-\frac{d + |I|}{2}} e^{-\frac{|x|^2}{Ct}}, \quad x \in G, \ 0 < t \leq 1. \quad (7)$$

In particular, estimate (7) implies that $p_t \in S(G)$ for $t \in (0, 1]$. Since $p_{t_1 + t_2} = p_{t_1} * p_{t_2}$ for any $t_1, t_2 > 0$, it follows that $p_t \in S(G)$ for any $t > 0$. Furthermore, estimates (6) and (7) yield the following

**Proposition 4**

For any $\alpha \in \mathbb{N}, I \in \bigcup_{\beta \in \mathbb{N}} \{1, \ldots, k\}^\beta$ and $p \in [1, \infty]$ there exists $C > 0$ such that the following estimates hold:

$$\| (1 + | \cdot |)^\alpha p_t(\cdot) \|_{L^p(G)} \leq C(1 + \sqrt{t})^\alpha V(\sqrt{t})^{-\frac{1}{p}}, \quad t > 0; \quad (8)$$

$$\| (1 + | \cdot |)^\alpha X_I p_t(\cdot) \|_{L^p(G)} \leq Ct^{-\frac{d + |I|}{2} + \frac{|I|^2}{4p}}, \quad 0 < t \leq 1. \quad (9)$$

**Proof:** In this proof we will denote by $C$ a positive constant which will not be necessarily the same at each occurrence, with the convention that $C$ can depend only on $G$ and on $\alpha, I, p$.

Fix $t > 0$. First we note that

$$\sup_{\rho \geq 0} (1 + \rho)^\alpha e^{-\frac{x^2}{Ct}} \leq C(1 + \sqrt{t})^\alpha \quad (10)$$

as is easy to verify by calculating the maximum of the function $\rho \mapsto (1 + \rho)^\alpha e^{-\frac{x^2}{Ct}}$ in $[0, \infty)$. For $p = \infty$ estimates (8) and (9) follow directly by (10) and by (6) and (7), respectively. For $1 \leq p < \infty$ we use the fact that

$$\int_G e^{-\frac{|x|^2}{Ct}} \, dx \leq CV(\sqrt{t}) \quad (11)$$
Fix $\delta > 0$, $n \in \mathbb{Z}_+$ and $h \in C^n(\mathbb{R})$ with compact support. Then there is an even function $g \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R}) \cap C(\mathbb{R})$ such that $\text{supp} \hat{g} \subset [-\delta, \delta]$ and
\[
\sup_{\lambda \in \mathbb{R}} |h(\lambda) - h * g(\lambda)| \leq C\delta^{-n} \sup_{\sigma \in \mathbb{R}} |h^{(n)}(\sigma)|
\]
where $C$ is a positive constant which depends only on $n$ but not on $\delta, h, g$.

Throughout this section we will use the following notation: for any $n \in \mathbb{N}$ and for any $m \in C^n(\mathbb{R}_+)$ we put
\[
\|m\|_{(n)} = \sup_{0 \leq r \leq n, \lambda > 0} (1 + \lambda)^n |m^{(r)}(\lambda)|.
\]
Moreover, the constants $d$ and $K$ are the same which have been introduced in Section 2.

**Proposition 6**

Fix $\alpha \in \mathbb{N}$, $I \in \bigcup_{\beta \in \mathbb{N}} \{1, \ldots, k\}^\beta$ and $p \in [1, \infty]$. There exist $C > 0$ and $n \in \mathbb{Z}_+$ such that for any $m \in C^n(\mathbb{R}_+)$ with $\|m\|_{(n)} < \infty$ the kernel $M_t$ of the operator $m(t\mathcal{L})$, $t > 0$, satisfies the following estimates:
\[
\begin{align*}
&\|(1 + |\cdot|)^\alpha M_t(\cdot)\|_{L^p(G)} \leq C(1 + \sqrt{t})^{\alpha} V(\sqrt{t})^{-\frac{\beta}{p}} \|m\|_{(n)}, \quad t > 0; \\
&\|(1 + |\cdot|)^\alpha X^I M_t(\cdot)\|_{L^p(G)} \leq C t^{-\left(\frac{\beta}{2p} + \frac{\alpha - \beta}{2}ight)} \|m\|_{(n)}, \quad 0 < t \leq 1.
\end{align*}
\]
Remark: The case where $\alpha = |I| = 0$ and $p = 1$ is particularly interesting: it simply reads
$$\|M_t\|_{L^1(G)} \leq C\|m\|_n, \quad t > 0.$$  

Proof: In this proof we will denote by $C$ a positive constant which will not be necessarily the same at each occurrence, with the convention that $C$ can depend only on $G$ and on $\alpha, I, p$.

The proof consists of some steps.

Step 1. We prove (12) for $p = 1$, with the additional assumption that $m = 0$ in $[2, \infty)$.

Fix $t > 0$ and fix $n \in \mathbb{Z}_+$ which will be chosen later. We consider the function $h_t$ on $\mathbb{R}$ defined by
$$h_t(\sigma) = e^{t\sigma^2} m(t\sigma^2), \quad \sigma \in \mathbb{R}.$$  

By the assumptions on $m$ we have
$$\|h_t\|_{L^\infty(\mathbb{R})} \leq e^{2\|m\|_0}.$$  

Moreover
$$m(t\lambda) = e^{-t\lambda} h_t(\sqrt{\lambda}), \quad \lambda > 0.$$  

So by the spectral theorem
$$M_t = h_t(\sqrt{\mathcal{L}}) p_t \in L^2(G)$$  

and
$$\|M_t\|_{L^2(G)} \leq \|h_t\|_{L^\infty(\mathbb{R})} \|p_t\|_{L^2(G)} \leq CV(\sqrt{t})^{-\frac{1}{2}} \|m\|_0$$  

by (8) and (14). Then
$$\int_{|x| < \sqrt{t}} (1 + |x|)^{\alpha} |M_t(x)| \, dx \leq \left( \int_{|x| < \sqrt{t}} (1 + |x|)^{2\alpha} \, dx \right)^{\frac{1}{2}} \left( \int_{|x| < \sqrt{t}} |M_t(x)|^2 \, dx \right)^{\frac{1}{2}} \leq V(\sqrt{t})^{\frac{1}{2}} (1 + \sqrt{t})^\alpha \|M_t\|_{L^2(G)} \leq C(1 + \sqrt{t})^\alpha \|m\|_0.$$  

On the other hand it follows from (15) that
$$\int_{|x| \geq \sqrt{t}} (1 + |x|)^{\alpha} |M_t(x)| \, dx = \sum_{j=0}^\infty \left( \int_{A_{t,j}} (1 + |x|)^{\alpha} |M_t^{(1)}(x)| \, dx + \int_{A_{t,j}} (1 + |x|)^{\alpha} |M_t^{(2)}(x)| \, dx \right)$$  

where:
$$A_{t,j} = \{ x \in G : 2^j \sqrt{t} \leq |x| < 2^{j+1} \sqrt{t} \};$$
$$M_t^{(1)} = h_t(\sqrt{\mathcal{L}}) (p_t \chi_{\{y \in G : |y| < 2^{j-1} \sqrt{t} \}});$$
$$M_t^{(2)} = h_t(\sqrt{\mathcal{L}}) (p_t \chi_{\{y \in G : |y| \geq 2^{j-1} \sqrt{t} \}}).$$
For every $j \in \mathbb{N}$ and for $i = 1, 2$ we have

$$\int_{A_{t,j}} (1 + |x|)^\alpha |M_{t,j}^{(i)}(x)| \, dx \leq \left( \int_{A_{t,j}} (1 + |x|)^{2\alpha} \, dx \right)^{\frac{1}{2}} \left( \int_{A_{t,j}} |M_{t,j}^{(i)}(x)|^2 \, dx \right)^{\frac{1}{2}} \leq V (2^{j+1} \sqrt{t})^{\frac{1}{2}} (1 + 2^{j+1} \sqrt{t})^\alpha \|M_{t,j}^{(i)}\|_{L^2(A_{t,j})}.$$  

The fact that $(1 + 2^{j+1} \sqrt{t})^\alpha \leq C 2^{j\alpha}(1 + \sqrt{t})^\alpha$ and the doubling property [5] imply

$$\int_{A_{t,j}} (1 + |x|)^\alpha |M_{t,j}^{(i)}(x)| \, dx \leq C K^\frac{1}{2} V (\sqrt{t})^\frac{1}{2} 2^{j\alpha}(1 + \sqrt{t})^\alpha \|M_{t,j}^{(i)}\|_{L^2(A_{t,j})}. \tag{18}$$

In order to estimate $\|M_{t,j}^{(i)}\|_{L^2(A_{t,j})}$, first of all we suppose $n \geq 2$ and we note that

$$h_t(\sqrt{\lambda}) = \frac{1}{2\pi} \int_{\mathbb{R}} \hat{h}_t(s) \cos(s\sqrt{\lambda}) \, ds, \quad \lambda > 0$$

since $h_t$ is an even function in $L^1(\mathbb{R})$ whose Fourier transform is in $L^1(\mathbb{R})$. Then for a.e. $x \in A_{t,j}$

$$M_{t,j}^{(i)}(x) = \frac{1}{2\pi} \int_{\mathbb{R}} \hat{h}_t(s) \left( \cos(s\sqrt{\lambda}) (p_t \chi_{\{y \in G : |y| < 2^{j-1} \sqrt{t}\}}) \right)(x) \, ds. \tag{19}$$

Now we use the fact that the kernel $G_s$ of the operator $\cos(s\sqrt{\lambda})$, $s \in \mathbb{R}$, satisfies the following property (see [Me]):

$$\text{supp} \, G_s \subset \{y \in G : |y| \leq |s|\}.$$  

So, for $x \in A_{t,j}$ and $|s| \leq 2^{j-1} \sqrt{t}$ we have

$$\left( \cos(s\sqrt{\lambda})(p_t \chi_{\{y \in G : |y| < 2^{j-1} \sqrt{t}\}}) \right)(x) = 0. \tag{20}$$

By Lemma 5 we can take an even function $\hat{g}_{t,j} \in L^1(\mathbb{R}) \cap L^\infty(\mathbb{R}) \cap C(\mathbb{R})$ such that $\text{supp} \, \hat{g}_{t,j} \subset [-2^{j-1} \sqrt{t}, 2^{j-1} \sqrt{t}]$ and

$$\sup_{\lambda \in \mathbb{R}} |h_t(\lambda) - h_t \ast g_{t,j}(\lambda)| \leq C 2^{-jn} t^{-\frac{n}{2}} \sup_{s \in \mathbb{R}} |h_t^{(n)}(s)|. \tag{21}$$

The support property of $\hat{g}_{t,j}$ and property (20) imply that for a.e. $x \in A_{t,j}$

$$\int_{\mathbb{R}} \hat{h}_t(s) \hat{g}_{t,j}(s) \left( \cos(s\sqrt{\lambda})(p_t \chi_{\{y \in G : |y| < 2^{j-1} \sqrt{t}\}}) \right)(x) \, ds = 0.$$

So formula (19) can be rewritten in the following way: for a.e. $x \in A_{t,j}$

$$M_{t,j}^{(1)}(x) = \frac{1}{2\pi} \int_{\mathbb{R}} \left( h_t(s) - \hat{h}_t(s) \hat{g}_{t,j}(s) \right) \left( \cos(s\sqrt{\lambda})(p_t \chi_{\{y \in G : |y| < 2^{j-1} \sqrt{t}\}}) \right)(x) \, ds = \left( (h_t - h_t \ast g_{t,j})(\sqrt{\lambda})(p_t \chi_{\{y \in G : |y| < 2^{j-1} \sqrt{t}\}}) \right)(x)$$

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since \( h_t - h_t * g_{t,j} \) is an even function in \( L^1(\mathbb{R}) \cap L^\infty(\mathbb{R}) \cap C(\mathbb{R}) \) whose Fourier transform is in \( L^1(\mathbb{R}) \). Then
\[
\| M^{(1)}_{t,j} \|_{L^2(A_{t,j})} \leq \left\| (h_t - h_t * g_{t,j})(\sqrt{t}) (pt \chi_{\{y \leq 2^{j-1} \sqrt{t} \}}) \right\|_{L^2(G)} \\
\leq \| h_t - h_t * g_{t,j} \|_{L^\infty(\mathbb{R})} \| pt \chi_{\{y \leq 2^{j-1} \sqrt{t} \}} \|_{L^2(G)}.
\]
We apply estimate (24) to the first factor and we estimate the second factor simply by (4) and (5), so that
\[
\| pt \chi_{\{y \leq 2^{j-1} \sqrt{t} \}} \|_{L^2(G)} \leq CV(\sqrt{t})^{-1} V(2^{j-1} \sqrt{t})^{\frac{1}{2}} \\
\leq CK^{\frac{1}{2}} V(\sqrt{t})^{-\frac{1}{2}}
\]
and then
\[
(22) \quad \| M^{(1)}_{t,j} \|_{L^2(A_{t,j})} \leq C 2^{-jn} t^{-\frac{1}{2}} K^{\frac{1}{2}} V(\sqrt{t})^{-\frac{1}{2}} \| h_t^{(n)} \|_{L^\infty(\mathbb{R})}.
\]
We still have to estimate \( h_t^{(n)} \|_{L^\infty(\mathbb{R})} \). Note that
\[
h_t^{(n)}(\sigma) = t^{\frac{n}{2}} h_1^{(n)}(t^{\frac{1}{2}} \sigma), \quad \sigma \in \mathbb{R}
\]
and
\[
h_1^{(n)}(\lambda) = \sum_{r=0}^{\infty} \left( \begin{array}{c} n \\ r \end{array} \right) \left( \frac{d^{n-r}}{d\lambda^{n-r}} \right) (e^{-\lambda^2}) \left( \frac{d}{d\lambda} \right) (m(\lambda^2)), \quad \lambda \in \mathbb{R}.
\]
So
\[
(23) \quad \| h_t^{(n)} \|_{L^\infty(\mathbb{R})} = t^{\frac{n}{2}} \| h_1^{(n)} \|_{L^\infty(\mathbb{R})} \leq Ct^{\frac{n}{2}} \| m \|_{(n)}.
\]
Inequalities (22) and (23) give
\[
(24) \quad \| M^{(1)}_{t,j} \|_{L^2(A_{t,j})} \leq C 2^{-jn} K^{\frac{1}{2}} V(\sqrt{t})^{-\frac{1}{2}} \| m \|_{(n)}.
\]
Now we choose \( n > \alpha + \log_2 K \). Then estimates (18) and (24) yield
\[
(25) \quad \sum_{j=0}^{\infty} \left( \int_{A_{t,j}} (1 + |x|)^\alpha |M_t^{(1)}(x)| \, dx \right) \leq C (1 + \sqrt{t})^\alpha \| m \|_{(n)}.
\]
In order to estimate \( \| M^{(2)}_{t,j} \|_{L^2(A_{t,j})} \), first of all we use the properties of \( p_t \) to prove that
\[
\int_{|y| \geq 2^{j-1} \sqrt{t}} (p_t(y))^2 \, dy \leq \left( \sup_{|y| \geq 2^{j-1} \sqrt{t}} p_t(y) \right) \int_{G} p_t(y) \, dy \\
\leq CV(\sqrt{t})^{-1} e^{-\frac{2^j}{C}}
\]
and then
\[
\| M^{(2)}_{t,j} \|_{L^2(A_{t,j})} \leq \| M^{(2)}_{t,j} \|_{L^2(G)} \\
\leq C \| p_t \chi_{\{|y| \geq 2^{j-1} \sqrt{t} \}} \|_{L^2(G)} \| m \|_{(0)} \\
\leq CV(\sqrt{t})^{-\frac{1}{2}} e^{-\frac{2^j}{C}} \| m \|_{(0)}.
\]

This estimate and estimate (18) yield

\[
\sum_{j=0}^{\infty} \left( \int_{A_{t,j}} (1 + |x|)^{\alpha}|M_{t,j}(x)| \, dx \right) \leq C(1 + \sqrt{t})^{\alpha}\|m\|_{(0)}.
\]

So both terms of the right-hand side of (17) can be estimated by (25) and (26), respectively. Thus, taking into account also (16), we obtain

\[
\int_{G} (1 + |x|)^{\alpha}|M_{t}(x)| \, dx \leq C(1 + \sqrt{t})^{\alpha}\|m\|_{(0)}.
\]

Step 2. We prove (12) and (13) for any \( p \in [1, \infty] \), with the additional assumption that \( m = 0 \) in \([2, \infty)\). Fix \( t > 0 \). We consider the function \( f \) on \( \mathbb{R}_{+} \) defined by

\[
f(\lambda) = e^{\lambda}m(\lambda), \quad \lambda > 0.
\]

Then \( \|f\|_{(n)} \leq C\|m\|_{(n)} \). Moreover \( M_{t} = F_{t} \ast p_{t} \), where \( F_{t} \) the kernel of the operator \( f(t\mathcal{L}) \). So \( M_{t} \in C^{\infty}(G) \) and \( X^{I}M_{t} = F_{t} \ast X^{I}p_{t} \). Moreover, for every \( x \in G \) we have

\[
(1 + |x|)^{\alpha}|X^{I}M_{t}(x)| \leq C \left( \int_{G} (1 + |y|)^{\alpha}|F_{t}(y)||X^{I}p_{t}(y^{-1}x)| \, dy + \int_{G} |F_{t}(y)|(1 + |y^{-1}x|)^{\alpha}|X^{I}p_{t}(y^{-1}x)| \, dy \right).
\]

Then

\[
\| (1 + | \cdot |)^{\alpha}X^{I}M_{t}(\cdot)\|_{L^{p}(G)} \leq C \left( \| (1 + | \cdot |)^{\alpha}F_{t}(\cdot)\|_{L^{1}(G)}\|X^{I}p_{t}\|_{L^{p}(G)} + \|F_{t}\|_{L^{1}(G)}\|(1 + | \cdot |)^{\alpha}X^{I}p_{t}(\cdot)\|_{L^{p}(G)} \right).
\]

If \(|I| = 0\) we apply (8) and (27) to both terms of (28) and we obtain (12). If \( 0 < t \leq 1 \) we apply (9) and (27) and we obtain (13).

Step 3. We prove (12) dropping the additional assumption on \( m \). Fix \( t > 0 \) and a non-increasing function \( \varphi \in C^{\infty}(\mathbb{R}_{+}) \) such that \( \varphi = 1 \) in \((0, \frac{1}{2})\) and \( \varphi = 0 \) in \([1, \infty)\). Set

\[
\psi(\lambda) = \varphi\left(\frac{\lambda}{2}\right) - \varphi(\lambda), \quad \lambda > 0.
\]

So \( 0 \leq \psi \leq 1 \) and \( \text{supp} \psi \subset \left[\frac{1}{2}, 2\right] \). Moreover we observe that

\[
\varphi(\lambda) + \sum_{j=0}^{\infty} \psi(2^{-j}\lambda) = 1, \quad \lambda > 0.
\]

So

\[
m(\lambda) = \tilde{m}(\lambda) + \sum_{j=0}^{\infty} m_{j}(\lambda), \quad \lambda > 0
\]

where

\[
\tilde{m}(\lambda) = m(\lambda)\varphi(\lambda), \quad \lambda > 0
\]
and
\[ m_j(\lambda) = m(\lambda)\psi(2^{-j}\lambda), \quad \lambda > 0. \]
By (29) we have \( \sum_{j=0}^{\infty} |m_j| \leq |m| \). So, if we denote by \( \tilde{M}_t \) the kernel of \( \tilde{m}(t\mathcal{L}) \) and by \( M_{j,t} \) the kernel of \( m_j(t\mathcal{L}) \), by Proposition 3 we have
\[(30) \quad M_t = \tilde{M}_t + \sum_{j=0}^{\infty} M_{j,t} \quad \text{in } \mathcal{D}'(G).\]
We observe that
\[ m_j(t\lambda) = h_j(2^{-j}t\lambda), \quad j \in \mathbb{N}, \; \lambda > 0 \]
where
\[(31) \quad h_j(\sigma) = m(2^j\sigma)\psi(\sigma), \quad j \in \mathbb{N}, \; \sigma > 0.\]
Since \( h_j = 0 \) in \([2, \infty)\), by the previous steps
\[(32) \quad \|(1 + |\cdot|)^{\alpha} M_{j,t}(\cdot)\|_{L^p(G)} \leq C(1 + 2^{-\frac{j}{3}}\sqrt{t})^{\alpha} V(2^{-\frac{j}{3}}\sqrt{t})^{-\frac{1}{p'}} \|h_j\|_{(n)}.\]
We observe that the doubling property (5) implies
\[(33) \quad V(2^{-\frac{j}{3}}\sqrt{t})^{-\frac{1}{p'}} \leq K \frac{1}{2^j} V(\sqrt{t})^{-\frac{1}{p'}}.\]
Moreover we estimate \( \|h_j\|_{(n)} \) by means of (31): for \( r \in \{0, \ldots, n\} \) we have
\[(34) \quad h_j^{(r)}(\sigma) = \sum_{l=0}^{r} \binom{r}{l} 2^l m(l)(2^j\sigma)\psi^{(r-l)}(\sigma), \quad j \in \mathbb{N}, \; \sigma > 0.\]
Fix an integer \( n' > n + \frac{\log_2 K}{2p} \). By (34), if \( m \in C^{n'}(\mathbb{R}_+) \) with \( \|m\|_{(n')} < \infty \) then
\[(35) \quad \|h_j\|_{(n)} \leq C2^{j(n-n')} \|m\|_{(n')}\]
It follows from (32), (34) and (35) that
\[(36) \quad \|(1 + |\cdot|)^{\alpha} M_{j,t}(\cdot)\|_{L^p(G)} \leq C(1 + \sqrt{t})^{\alpha} V(\sqrt{t})^{-\frac{1}{p'}} 2^{j(n-n'+\frac{\log_2 K}{2p})} \|m\|_{(n')}\]
On the other hand
\[(37) \quad \|(1 + |\cdot|)^{\alpha} \tilde{M}_t(\cdot)\|_{L^p(G)} \leq C(1 + \sqrt{t})^{\alpha} V(\sqrt{t})^{-\frac{1}{p'}} \|m\|_{(n)}\]
since \( \|\tilde{m}\|_{(n)} \leq C \|m\|_{(n)} \). By (30), (36) and (37) and by the assumption made on \( n' \) we obtain
\[ \|(1 + |\cdot|)^{\alpha} M_t(\cdot)\|_{L^p(G)} \leq C(1 + \sqrt{t})^{\alpha} V(\sqrt{t})^{-\frac{1}{p'}} \|m\|_{(n')}\]

**Step 4.** The proof of (13) without the additional assumption on \( m \) is analogous to Step 3: we observe that equality (30) implies
\[ X^I M_t = X^I \tilde{M}_t + \sum_{j=0}^{\infty} X^I M_{j,t} \quad \text{in } \mathcal{D}'(G).\]
and then we follow the proof of Step 3: we obtain
\[
\|(1 + | \cdot |)^\alpha X^I M_t(\cdot)\|_{L^p(G)} \leq C t^{-\left(\frac{d}{2p'} + \frac{|I|}{2}\right)} \|m\|_{(n'')}, \quad 0 < t \leq 1
\]
where \(n'' > n + \frac{d}{2p'} + \frac{|I|}{2}\).

An immediate consequence of Proposition 6 is the following result, which also generalizes the analogous result for stratified groups (see [Hu], [M]):

**Corollary 7**

Let \(m\) be the restriction on \(\mathbb{R}_+\) of a function in \(S(\mathbb{R})\). Then the kernel \(M\) of the operator \(m(L)\) is in \(S(G)\).

**4 Littlewood–Paley decomposition**

Fix a non-increasing function \(\varphi \in C^\infty(\mathbb{R}_+)\) such that \(\varphi = 1\) in \((0, \frac{1}{4})\) and \(\varphi = 0\) in \([1, \infty)\). Set
\[
\psi(\lambda) = \varphi\left(\frac{\lambda}{4}\right) - \varphi(\lambda), \quad \lambda > 0.
\]
So \(0 \leq \psi \leq 1\) and \(\text{supp} \, \psi \subset \left[\frac{1}{4}, 4\right]\). Moreover we observe that
\[
\varphi(\lambda) + \sum_{j=0}^N \psi(2^{-2j} \lambda) = \varphi(2^{-2(N+1)} \lambda), \quad N \in \mathbb{N}, \ \lambda > 0
\]
and so
\[
\varphi(\lambda) + \sum_{j=0}^\infty \psi(2^{-2j} \lambda) = 1, \quad \lambda > 0.
\]
By Corollary 7 for any \(j \in \mathbb{N}\) the kernels of the operators \(S_j = \varphi(2^{-2j}L)\) and \(\Delta_j = \psi(2^{-2j}L)\) are in \(S(G)\), so the operators \(S_j\) and \(\Delta_j\) can be viewed as continuous operators on \(S'(G)\). By the spectral theorem, any \(f \in L^2(G)\) can be decomposed as \(f = S_0 f + \sum_{j=0}^\infty \Delta_j f\) in \(L^2(G)\). The following proposition shows that such decomposition holds also in \(S(G)\) and in \(S'(G)\).

**Proposition 8**

For any \(f \in S(G)\) and \(u \in S'(G)\) we have:

\[
f = S_0 f + \sum_{j=0}^\infty \Delta_j f \quad \text{in } S(G);
\]
\[
u = S_0 u + \sum_{j=0}^\infty \Delta_j u \quad \text{in } S'(G).
\]

**Proof:** We only have to prove (39), since (40) follows by duality. By (38) we have to prove that \(S_j f \to f\) in \(S(G)\) for \(j \to \infty\). So we fix \(\alpha \in \mathbb{N}\) and \(I \in \bigcup_{\beta \in \mathbb{N}} \{1, \ldots, k\}^\beta\) and we want to
prove that \( p_{\alpha,j}(f - S_j f) \to 0 \) for \( j \to \infty \). Put \( N = \max\{n, 1 + \frac{|I|}{2}\} \), where \( n \) is the integer which appears in Proposition \( 13 \) in the case \( p = 1 \). Then
\[
(41) \quad f - S_j f = 2^{-2jN} m(2^{-2j} \mathcal{L}) \mathcal{L}^N f, \quad j \in \mathbb{N}
\]
where
\[
(42) \quad m(\lambda) = \frac{1 - \varphi(\lambda)}{\lambda^N}, \quad \lambda > 0.
\]
Let \( M_j \) be the kernel of the operator \( m(2^{-2j} \mathcal{L}) \). Since by \( 12 \) \( m \in C^\infty(\mathbb{R}_+) \) and \( \|m\|_{(N)} < \infty \), Proposition \( 6 \) gives
\[
(43) \quad \| (1 + |\cdot|)^\alpha X^I M_j(\cdot) \|_{L^1(G)} \leq C 2^{|I|}, \quad j \in \mathbb{N}
\]
where \( C \) does not depend on \( j \). Then, by \( 11 \) and \( 13 \) and since \( f \in \mathcal{S}(G) \), for every \( x \in G \) we have
\[
(1 + |x|^\alpha|X^I(f - S_j f)(x)|
= 2^{-2jN} (1 + |x|^\alpha|\mathcal{L}^N f \ast X^I M_j(x)|)
\leq 2^{-2jN} \left( \int_G (1 + |y|^\alpha |\mathcal{L}^N f(y)| |X^I M_j(y^{-1} x)| dy + \int_G |\mathcal{L}^N f(y)|(1 + |y^{-1} x|^\alpha |X^I M_j(y^{-1} x)|) dy \right)
\leq C' 2^{|I|-2N}
\]
where \( C' \) does not depend on \( j \) or \( x \). Since \( 2N > |I| \), we have that \( p_{\alpha,j}(f - S_j f) \to 0 \) for \( j \to \infty \).

**Remark:** For \( 1 \leq p < \infty \), since \( \mathcal{S}(G) \) is dense in \( L^p(G) \) and the operators \( \Delta_j \) are uniformly bounded on \( L^p(G) \) by Proposition \( 13 \), it follows from \( 39 \) that any \( f \in L^p(G) \) can be decomposed as \( f = S_0 f + \sum_{j=0}^{\infty} \Delta_j f \) in \( L^p(G) \).

One gets from Proposition \( 13 \) an extension of the classical Bernstein’s inequalities for the dyadic blocks (see e.g. \( 12, 11 \); see also \( 11 \) Proposition 3.2). In what follows, \( d \) is the local dimension of the group introduced in Section \( 2 \).

**Proposition 9**

For \( I \in \bigcup_{\beta \in \mathbb{N}} \{1, \ldots, k\}^\beta \), \( 1 \leq p \leq q \leq \infty \), \( j \in \mathbb{N} \) and \( u \in \mathcal{S}'(G) \) we have:
\[
\| X^I(\sqrt{\mathcal{L}})^\sigma S_j u \|_{L^q(G)} \leq C 2^{j(|I|+\sigma+d(\frac{1}{p}-\frac{1}{2}))} \| S_j u \|_{L^p(G)}, \quad \sigma \geq 0,
\]
\[
\| X^I(\sqrt{\mathcal{L}})^\sigma \Delta_j u \|_{L^q(G)} \leq C 2^{j(|I|+\sigma+d(\frac{1}{p}-\frac{1}{2}))} \| \Delta_j u \|_{L^p(G)}, \quad \sigma \in \mathbb{R},
\]
where \( C \) is a positive constant which depends only on \( I, p, q, \sigma \) but not on \( j \) or \( u \).

**Proof:** We can consider the functions \( \tilde{\varphi}(\lambda) = \varphi(\frac{\lambda}{4}) \) and \( \tilde{\psi}(\lambda) = \varphi(\frac{\lambda}{16}) - \varphi(4\lambda) \), so that \( \tilde{\varphi}(\lambda) \varphi(\lambda) = \varphi(\lambda) \) and \( \tilde{\psi}(\lambda) \psi(\lambda) = \psi(\lambda) \) for all \( \lambda > 0 \). The proof of the proposition follows therefore from Young’s inequality, Proposition \( 13 \) and the identities:
\[
S_j u = \tilde{\varphi}(2^{-2j} \mathcal{L}) S_j u;
\]
\[
\Delta_j u = \tilde{\psi}(2^{-2j} \mathcal{L}) \Delta_j u.
\]
We are now in position to set out the Littlewood–Paley theorem related to the decomposition (40):

**Theorem 10**

Let \( 1 < p < \infty \) and \( u \in S'(G) \). Then \( u \in L^p(G) \) if and only if \( S_0 u \in L^p(G) \) and 
\[
\left( \sum_{j=0}^{\infty} |\Delta_j u|^2 \right)^{\frac{1}{2}} \in L^p(G).
\]
Moreover there exists a constant \( C_p > 1 \), which depends only on \( p \), such that 
\[
C_p^{-1} \| u \|_{L^p(G)} \leq \| S_0 u \|_{L^p(G)} + \left( \sum_{j=0}^{\infty} (2^{js} \| \Delta_j u \|_{L^p(G)})^q \right)^{\frac{1}{q}} \leq C_p \| u \|_{L^p(G)}, \quad u \in L^p(G).
\]

**Proof:** Once we have Theorem 2 and equality (40), the proof of the Littlewood–Paley theorem in \( \mathbb{R}^n \) given for instance in [St] works also in our case.

5 Besov spaces

Besov spaces \( B^s_{p,q}(G) \) for sub-Laplacians in Lie groups were studied by many authors. The usual definition ([F], [S]) has been given by means of the heat kernel associated to the sub-Laplacian. Though in [S], in the case of stratified groups, such spaces are defined with \( s \in \mathbb{R} \), most applications (see e.g. [CS], [Sa], [MV]) have concerned essentially the case \( s > 0 \) where \( B^s_{p,q}(G) \subset L^p(G) \). In the Euclidean case there are many equivalent characterizations of Besov spaces: a useful reference is given by the books by Triebel [T1], [T2]. In particular, the characterizations by the atomic decomposition in the space variable ([FJ]) and by the dyadic decomposition of the frequency space of the Fourier transform ([P]) are often used in the applications. In the case of unimodular Lie groups, Skrzypczak in [S2] has given an atomic characterization of Besov spaces with \( s \in \mathbb{R} \). In a more abstract setting, Galé in [G] has determined a sufficient condition for a positive self-adjoint operator on a Hilbert space which allows to characterize the corresponding Besov spaces with \( s > 0 \) by a dyadic decomposition on the spectrum of the operator. In our setting, Galé’s condition is equivalent to the uniform boundedness of the norms of the operators \( \Delta_j : L^p(G) \rightarrow L^p(G) \), which amounts to the uniform estimate of the norm in \( L^1(G) \) of the convolution kernel of \( \Delta_j \). This is precisely what was pointed out in the remark after Proposition 6.

In this section, we define on a Lie group of polynomial growth Besov spaces with \( s \in \mathbb{R} \) associated to any sub-Laplacian by means of the Littlewood–Paley decomposition obtained in Section 4.

**Definition 11**

Let \( s \in \mathbb{R} \), \( 1 \leq p, q \leq \infty \). We define
\[
B^s_{p,q}(G) = \{ u \in S'(G) : \| u \|_{B^s_{p,q}(G)} = \| S_0 u \|_{L^p(G)} + \left( \sum_{j=0}^{\infty} (2^{js} \| \Delta_j u \|_{L^p(G)})^q \right)^{\frac{1}{q}} < \infty \}
\]
with obvious modifications in the case \( q = \infty \).

This definition is equivalent to the classical one through the heat kernel. In fact, once we have Proposition 6 and the decomposition (40), we can repeat the proof given for instance in [L, Theorem 5.3] in the Euclidean setting and we obtain the following

**Proposition 12**

For \( s \in \mathbb{R}, 1 \leq p, q \leq \infty, m \geq 0 \) such that \( m > s \) and \( u \in S'(G) \), the following assertions are equivalent:

i) \( u \in B^{s,0}_p(G) \);

ii) for all \( t > 0 \), \( e^{-tL}u \in L^p(G) \) and \( \int_0^1 (t^{-s/2}) \| (tL)^{m/2} e^{-tL}u \|_{L^p(G)}^{q \frac{dt}{t}} < \infty \).

Moreover, the norms \( \| e^{-L}u \|_{L^p(G)} + (\int_0^1 (t^{-s/2}) \| (tL)^{m/2} e^{-tL}u \|_{L^p(G)}^{q \frac{dt}{t}})^{1/q} \) and \( \| u \|_{B^{s,0}_p(G)} \) are equivalent.

**Remark:** If \( s > 0 \), one can replace the condition \( e^{-tL}u \in L^p(G) \) by the equivalent \( u \in L^p(G) \); in fact \( B^{s,0}_p(G) \subset B^{0,1}_p(G) \subset L^p(G) \). Furthermore, if \( s > 0 \) one can replace the condition \( \int_0^1 (t^{-s/2}) \| (tL)^{m/2} e^{-tL}u \|_{L^p(G)}^{q \frac{dt}{t}} < \infty \) by the equivalent \( \int_0^\infty (t^{-s/2}) \| (tL)^{m/2} e^{-tL}u \|_{L^p(G)}^{q \frac{dt}{t}} < \infty \), due to the convergence of the integral at infinity.

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Dipartimento di Ingegneria Gestionale e dell’Informazione, Università di Bergamo, Viale Marconi 5, I–24044 Dalmine (BG), Italy
E-mail: gfurioli@unibg.it

Dipartimento di Scienze Chimiche, Fisiche e Matematiche, Università dell’Insubria, Via Valleggio 11, I–22100 Como, Italy
E-mail: melzi@uninsubria.it

Dipartimento di Matematica, Università di Genova, Via Dodecaneso 35, I–16146 Genova, Italy
E-mail: veneruso@dima.unige.it