JOIN-CONTINUITY + HYPERCONTINUITY = PRIME CONTINUITY

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Abstract. A remarkable result due to Kou, Liu & Luo states that the condition of continuity for a dcpo can be split into quasi-continuity and meet-continuity. Their argument contained a gap, however, which is probably why the authors of the monograph Continuous Lattices and Domains used a different (and fairly sophisticated) sequence of lemmas in order to establish the result. In this note we show that by considering the Stone dual, that is, the lattice of Scott-open subsets, a straightforward proof may be given. We do this by showing that a complete lattice is prime-continuous if and only if it is join-continuous and hypercontinuous. A pleasant side effect of this approach is that the characterisation of continuity by Kou, Liu & Luo also holds for posets, not just dcpos.

1. Introduction

The notion of continuity can be said to be the very foundation of the whole of domain theory. The pioneering class of domains, continuous lattices, introduced by Dana Scott in [18] was intended for applications in theoretical computer science [19]. In these applications, the phenomenon of approximation can be formalized in any partial order by using the way-below relation ≪. A non-empty subset D of a poset P is directed if two elements in D always have an upper bound in D. For any x, y ∈ P, x ≪ y if for any directed set D, ∨ D ≥ y implies D ∩ ↑x ≠ ∅ whenever ∨ D exists. Roughly speaking, one may view x ≪ y as ‘x is an approximation of y’. We say that a poset P is continuous if for each x ∈ P, there are enough elements approximating it in the sense that ↓x := {p ∈ P | p ≪ x} is directed and ∨ ↓x = x. Researchers in continuous lattices soon extended their study to more general classes of partial orders, ranging from directed complete posets (dcpos, for short) [19]) to just posets (see, for example, [20]), and hence the birth of the term domain

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which is meant to include all partially ordered structures that are equipped with some form of approximation.

In recent years the development of domain theory saw the evolution of various kinds of continuous structures. One thread of generalisation involves the replacement of directed subsets by other kinds of subsets (called Z-sets) so that most of the existing results in domain theory carry over to a more general setting, initiated by the work of [21] and followed by later works such as [3, 2, 15, 4, 5]. In fact, Raney’s characterization ([17]) of completely distributive lattices as supercontinuous complete lattices (also called prime continuity in [1, p.107]; see also [9, Exercise 8.3.15]) is a forerunner of this generalisation to Z-subset systems. Connections have been made with mainstream domain theory; for instance when directed sets are replaced by Scott-closed sets, an order-theoretic characterization of the Hoare powerdomain was obtained using the notion of C-continuity ([12]).

Another distinctive thread of generalisation began with the invention of quasicontinuous domains by Gierz, Lawson, and Stralka in [10]. Instead of changing the directed subsets, the idea was to extend the way-below relation between two points in a poset to that between two finite subsets. More precisely, for any two nonempty subsets \( F \) and \( G \) of a poset \( P \), define \( F \ll G \) if whenever an existing supremum of a directed set \( D \) is in \( \uparrow F \), then \( D \cap \uparrow G \neq \emptyset \). A poset \( P \) is said to be quasicontinuous if for all \( x \in P \), \( \text{fin}(x) := \{ \uparrow F \mid F \text{ is a finite subset of } P, F \ll \{x\}\} \) is a directed family of subsets of the poset with respect to reverse inclusion, and \( \bigcap \text{fin}(x) = \uparrow x \). Unlike the Z-generalisation, this current trend in domain theory to develop a more complete understanding of quasicontinuity has had a powerful impact on the development of domain theory itself. We highlight three important instances of this: (1) The Scott topology of quasicontinuous domains are exactly the hypercontinuous lattices; also, the theory of quasicontinuous domains makes connection with the Scott and Lawson topologies, [7, 10]. (2) Besides hypercontinuity, meet-continuity for dcpos is yet another example of a relatively novel variant of continuity. Invented initially as a generalisation from complete lattices to dcpos, this new notion turns out to have close connections with Hausdorff separation, quasicontinuity, continuity and Scott-filter bases, [13]. (3) A recent and significant milestone is the introduction of QRB domains (quasi-retracts of bifinite domains) by Jean Goubault-Larrecq in his attempt to make progress with the Jung-Tix problem, [8]. Of course, one expects fusion of the Z and ‘quasi’ approaches as already witnessed by [22, 23].

With the prolific emergence of new kinds of continuity in domain theory, it is important to understand relationships among them. Here are some examples of known connections:

1. continuity + C-continuity = prime continuity (= complete distributivity) holds for complete lattices ([12, Theorem 3.11])
2. meet continuity + quasicontinuity = continuity holds for dcpos
3. prime continuity \( \implies \) continuity \( \implies \) meet continuity
4. prime continuity \( \implies \) hypercontinuity \( \implies \) continuity

This paper is specifically about Equation (2), first stated in [13, p. 122, Theorem 2.5].

We ask the following two questions:

1. Is it possible to provide a proof of (2) that exploits our knowledge of the structure of the lattice of Scott-open subsets?
2. Is the statement still true when we extend it to the class of all posets?

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\(^1\)Unfortunately, the proof given in [13] contains a faulty argument as we shall explain in Section 3.
Indeed, below we establish the following for complete lattices:

\[
\text{join continuity} + \text{hypercontinuity} = \text{prime continuity} \tag{1.1}
\]

Based on this result, we will be able to answer affirmatively our two questions.

2. Preliminaries

We gather here all the definitions and results that we need in Section 3, leaving out all proofs. However, we take extra care to ensure that none of these make use, directly or indirectly, of Lemma III-2.10 and Proposition III-2.10 in [6] so as to make this note as self-contained as possible.

A subset \( U \) is upper if \( U = \uparrow U \), where \( \uparrow U := \{ p \in P \mid \exists u \in U. u \leq p \} \). We call a subset \( U \) of a poset \( P \) Scott open if (i) \( U \) is upper and (ii) whenever an existing supremum of a directed set \( D \) is in \( U \), then already \( D \cap U \neq \emptyset \). The collection of Scott opens of \( P \), denoted by \( \sigma(P) \), defines a topology on it, termed as the Scott topology. A subset \( C \) of a poset \( P \) is Scott-closed if \( P \setminus C \in \sigma(P) \). We use \( \Gamma(P) \) (or \( \sigma^\text{op}(P) \)) to denote the collection of Scott-closed sets of \( P \). Both \( \sigma(P) \) and \( \Gamma(P) \), when ordered by set inclusion, become complete lattices, and we overload the symbols \( \sigma(P) \) and \( \Gamma(P) \) to refer to these complete lattices.

A completely distributive lattice \( L \) is a complete lattice in which the following, so-called complete distributive law, is satisfied: for all families \( (u_i^j)_{j \in J_i} \), one for each \( i \in I \),

\[
\bigwedge_{i \in I} \bigvee_{j \in J_i} u_i^j = \bigvee_{f \in \prod_{i \in I} J_i} \bigwedge_{i \in I} u_i^{f(i)}.
\]

We have the following well-known result:

**Theorem 2.1.** ([11], Theorem 3.8) The following statements are equivalent for a poset \( P \):

1. \( P \) is a continuous poset.
2. \( \sigma(P) \) is a completely distributive lattice.

Define on a complete lattice \( L \) the way-way-below relation \( \triangleleft \) as follows: \( u \triangleleft v \) if for any \( S \subseteq L \), whenever \( \bigvee S \geq v \) then \( u \in \downarrow S \). A complete lattice \( L \) is said to be prime continuous if every element in \( L \) is the least upper bound of all elements way-way-below it. It is straightforward to show that a complete lattice \( L \) is prime continuous if and only if it is completely distributive, and this was first established in [17], albeit using a very different formulation. In view of the focus of this paper, we use the term ‘prime continuity’ in preference to ‘complete distributivity’.

Besides prime continuity, meet continuity is the property that one encounters very often in domain theory. A complete lattice \( L \) is meet continuous if for all \( x \in P \) and all directed subsets \( D \) of \( P \), it holds that \( x \land \bigvee D = \bigvee \{ x \land d \mid d \in D \} \). This property can be characterised by the Scott topology as \( x \in \text{cl}_\sigma(\downarrow x \cap \downarrow D) \) whenever \( x \leq \bigvee D \). Since the meet operator is not involved, this topological property of meet continuity can be used to give a natural extended meaning to meet continuity in the more general setting of dcpos. Dcpos which enjoy meet continuity are called meet continuous dcpos\(^2\). This definition quickly generalises to meet continuous posets where the phrase ‘all directed subsets of \( P \)’ is replaced by ‘all directed subsets of \( P \) whose suprema exist’ ([16]).

It is very natural to ask if the meet continuity of a poset \( P \) can be recognized from the properties of \( \sigma(P) \). The answer is yes:

**Theorem 2.2.** ([16], Theorem 3.8) The following statements are equivalent for a poset \( P \):

1. \( P \) is meet continuous.

\(^2\)This definition was first proposed in [13].
(2) $\sigma(P)$ is join continuous.

(3) $\sigma^{\text{op}}(P)$ is a frame.

Here, a complete lattice $L$ is said to be join continuous if for all $x \in L$ and all $S \subseteq L$, we have $x \lor \bigwedge S = \bigwedge \{ x \lor s | s \in S \}$. A frame is just the order dual of a join continuous complete lattice. Since prime continuity is equivalent to complete distributivity, it is immediate that

$$\text{prime continuity } \implies \text{join continuity} \quad (2.1)$$

and

$$\text{prime continuity } \implies \text{frame}. \quad (2.2)$$

A third type of continuity that is central to our present discussion is hypercontinuity. Analogous to continuity, this concept is defined via a certain auxiliary relation on a complete lattice $L$: $x \prec y$ if whenever the intersection of a nonempty collection of upper sets is contained in $\uparrow y$, then the intersection of finitely many is contained in $\uparrow x$. A complete lattice $L$ is called hypercontinuous if for all $y \in L$, we have $y = \bigvee \{ x \in L | x \prec y \}$. The following sup-inf characterizations of continuity, hypercontinuity and prime continuity give an immediate insight into the relations among these different notions of continuity:

**Theorem 2.3.** Let $L$ be a complete lattice.

1. $L$ is continuous if and only if for all $x \in L$,

$$x = \bigvee \{ \bigwedge U | x \in U \in \sigma(L) \}.$$

2. $L$ is hypercontinuous if and only if for all $x \in L$,

$$x = \bigvee \{ \bigwedge (L \downarrow M) | M \text{ is a finite subset of } L, x \notin \downarrow M \}.$$

3. $L$ is prime continuous if and only if for all $x \in L$,

$$x = \bigvee \{ \bigwedge (L \downarrow y) | x \notin \downarrow y \}.$$

**Proof.** (1) follows directly from the basic definitions of Scott-open set and way-below relation. The proofs for (2) and (3) can be found at [9, p.509, Proposition VII-3.3] and [17], respectively.

Hence we have the following chain:

$$\text{prime continuity } \implies \text{hypercontinuity } \implies \text{continuity}. \quad (2.3)$$

Hypercontinuity and quasicontinuity are connected via the following crucial result:

**Theorem 2.4.** ([16]) The following are equivalent for a poset $P$:

1. $P$ is a quasicontinuous poset.
2. $\sigma(P)$ is a hypercontinuous lattice.
3. Main results

**Lemma 3.1.** Let $L$ be a join-continuous complete lattice. Then for any finite set $M = \{m_1, \ldots, m_n\} \subseteq L$, the following equation holds:

$$\bigwedge(L \downarrow M) = \bigvee_{k=1}^{n} \bigwedge(L \downarrow m_k).$$

**Proof.** Note that the statement holds in the case that $M$ is empty as both sides then reduce to the least element of $L$. If $M$ is nonempty then we use induction on $n$. For $n = 1$ the equation is trivially true, so assume that the equation holds for all nonempty finite sets with $n$ elements; we must show that

$$\bigwedge(L \downarrow \{m_1, \ldots, m_n, m_{n+1}\}) = \bigvee_{k=1}^{n+1} \bigwedge(L \downarrow m_k).$$

By the induction hypothesis,

$$\bigvee_{k=1}^{n} \bigwedge(L \downarrow m_k) = \bigwedge(L \downarrow \{m_1, \ldots, m_n\}).$$

Thus, we have:

$$\bigvee_{k=1}^{n+1} \bigwedge(L \downarrow m_k) = \left(\bigwedge(L \downarrow m_{n+1})\right) \lor \bigwedge(L \downarrow \{m_1, \ldots, m_n\})$$

$$= \bigwedge\left\{\left(\bigwedge(L \downarrow m_{n+1})\right) \lor s \mid s \in \bigcap_{i=1}^{n}(L \downarrow m_i)\right\}$$

(Note: $L \downarrow \{m_1, \ldots, m_n\} = \bigcap_{i=1}^{n}(L \downarrow m_i)$.)

$$= \bigwedge\left\{\bigwedge\{r \lor s \mid r \in (L \downarrow m_{n+1})\} \mid s \in \bigcap_{i=1}^{n}(L \downarrow m_i)\right\}$$

$$= \bigwedge\left\{r \lor s \mid r \in (L \downarrow m_{n+1}) \text{ and } s \in \bigcap_{i=1}^{n}(L \downarrow m_i)\right\}$$

where join continuity is applied twice to obtain the second and third equalities. We finish the proof by showing that the set $X$ over which the infimum is taken in the last term is the same as $Y = L \downarrow \{m_1, \ldots, m_{n+1}\}$. Indeed, an element of $X$ is by construction not below any of the elements $m_1, \ldots, m_{n+1}$, so we have $X \subseteq Y$. On the other hand, for any element $t \in Y$, $t$ is in both $L \downarrow m_{n+1}$ and $\bigcap_{i=1}^{n}(L \downarrow m_i)$, so $t = t \lor t$ also belongs to $X$. Thus, $X = Y$.

**Theorem 3.2.** The following statements are equivalent for a lattice $L$:

1. $L$ is join continuous and hypercontinuous.
2. $L$ is prime continuous.

**Proof.** By Theorem 2.3 and Theorem 2.1, we have (2) $\implies$ (1). So, it remains to show that (1) $\implies$ (2). To this end, by virtue of Theorem 2.3(3), we only need to show that
for any \( x \in L \), we have \( x = \bigvee \{ \land (L \downarrow y) \mid x \notin \downarrow y \} \). Since \( L \) is hypercontinuous, we have \( x = \bigvee \{ \land (L \downarrow M) \mid M \) is a finite subset of \( P \), \( x \notin \downarrow M \} \). But for each finite set \( M \) with \( x \notin \downarrow M \), by Lemma 3.1 we can write \( \land (L \downarrow M) \) as the supremum of terms of the form \( \land (L \downarrow m) \) with \( m \in M \). Hence
\[
x = \bigvee \{ \land (L \downarrow M) \mid M \) is a finite subset of \( P \), \( x \notin \downarrow M \} = \bigvee \{ \land (L \downarrow y) \mid x \notin \downarrow y \}.
\]

**Theorem 3.3.** The following statements are equivalent for a poset \( P \):

1. \( P \) is meet continuous and quasicontinuous.
2. \( P \) is continuous.

**Proof.** By Theorem 2.2, \( P \) is meet continuous if and only if \( \sigma (P) \) is join continuous, and by Theorem 2.4, \( P \) is quasicontinuous if and only if \( \sigma (P) \) is hypercontinuous. Thus, by Theorem 3.2, (1) is equivalent to \( \sigma (P) \) being prime continuous, which, by Theorem 2.1, is equivalent to \( P \) being continuous.

**Remark 3.4.** In Kou’s original proof of the dcpo version of the above theorem, it was argued that the lattice of Scott-closed sets \( \Gamma (P) \) is continuous if \( P \) is meet continuous. But this is not true in general. Construct a non-continuous frame of opens \( P := \mathcal{O}(X) \) for your favourite non-locally compact space (see [6, p. 417, Theorem V-5.5]). Now, were it the case that \( \Gamma (P) \) is continuous for this choice of \( P \), then by Theorem 3.11 of [12] it would follow that \( \Gamma (P) \) is prime continuous. This would imply, by Theorem 2.1, that \( P \) is continuous, a contradiction.

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