The Van der Waals interactions and the photoelectric effect in noncommutative quantum mechanics

K. Li\textsuperscript{1,*} and N. Chamoun\textsuperscript{2,3†}

\textsuperscript{1} Department of Physics, Hangzhou Normal University, Hangzhou 310036, China
\textsuperscript{2} The Abdus Salam ICTP, P.O. Box 586, 34100 Trieste, Italy.
\textsuperscript{3} Physics Department, HIAST, P.O.Box 31983, Damascus, Syria.

Abstract

We calculate the long-range Vanderwaals force and the photoelectric cross section in a noncommutative set up. While we argue that non-commutativity effects could not be discerned for the Vanderwaals interactions, the result for the photoelectric effect shows deviation from the usual commutative one, which in principle can be used to put bounds on the space-space non-commutativity parameter.

Keywords: noncommutative space, Vanderwaals forces, photoelectric effect

1 Introduction

A large amount of research work has been devoted to the study of physics on noncommutative space-times (for a review see, e.g., [1]). This was motivated by the discovery in string theory that the low energy effective theory of D-brane in the background of NS-NS $B$ field lives on noncommutative space [2]-[5].

In the noncommutative space, the coordinate and momentum operators verify

$$ [\hat{x}_i, \hat{x}_j] = i\theta_{ij}, \quad [\hat{p}_i, \hat{p}_j] = 0, \quad [\hat{x}_i, \hat{p}_j] = i\hbar\delta_{ij}, \quad (1) $$

where $\hat{x}_i$ and $\hat{p}_i$ are the coordinate and momentum operators, and where we assume the time coordinate is commutative. The parameter $\{\theta_{ij}\}$ is an antisymmetric matrix representing the non-commutativity of the space, and is of dimension $(\text{length})^2$. In many proposals to test the hypothetical spacetime noncommutativity, one does not need the exact quantum field theory, but only its quantum mechanical approximation, and many simple quantum mechanics (QM) problems were treated on noncommutative spaces. For instance, the Hydrogen atom spectrum and the Lamb shift in noncommutative quantum
electrodynamics (QED) were first treated in [6] (look also at [7, 8]). The noncommutative version of QED has been examined in [9]. A method for formulating the non-abelian noncommutative field theories has been discussed in [10] and using these ideas noncommutative version of the standard model (SM) has been proposed [11]. In this noncommutative version, there are several new features and interactions, like triple gauge boson vertices, that appear and some of the related phenomenological aspects of these have been investigated [12]. Also, the questions of $t \to bW$ scattering in noncommutative SM and the PCT theorem in noncommutative fields theory were treated in [13]. Fractional quantum Hall effect could be obtained using noncommutative rank 1 Chern-Simons theory [14], and noncommutative massive Thirring model was treated in [15]. The QM problems could be treated equally in noncommutative phase space [16, 17], where momenta are also noncommutative enabling, thus, to incorporate an additional background magnetic field and to maintain Bose-Einstein statistics [18, 19].

However, although noncommutative QM has been extensively studied, there are still some problems which were not treated in the noncommutative set up. To our knowledge, the phenomenological implications of noncommutativity on the Vanderwaals interaction between molecules and on the photoelectric effect were not examined, and the subject of this letter is just to present such an analysis.

For the Vanderwaals forces, we find that the noncommutative effects can not be determined since they are far smaller than the next term to the dominant perturbative one proportional to $1/R^6$ where $R$ is the interatomic distance. As for the photoelectric effect, we find that the non-commutativity introduces a phase into the corresponding transition matrix element proportional to the parameter $\theta$ of space-space non-commutativity. Thus, the photoelectric cross section gets multiplied by a factor proportional to $\theta^2$, which can be used to put bounds on this parameter.

\section{Analysis}

To start, we note that the action for field theories on noncommutative spaces is obtained from the usual commutative action by replacing each usual product of fields $(f \cdot g)$ by the star-product:

\begin{equation}
(f \ast g)(x) = \exp\left(i \frac{\theta_{\mu\nu} \partial_x \mu \partial_y \nu}{2}\right) f(x)g(y) \big|_{x=y} ,
\end{equation}

where $f$ and $g$ are two arbitrary infinitely differentiable functions on $R^{3+1}$.

Alternatively, one can change the noncommutative problems into problems of familiar commutative spaces using the new noncommutative variables $({\hat{x}}$ and $\hat{p}$) defined in terms of the commutative ones ($x$ and $p$) by:

\begin{align*}
{\hat{x}}_i &= x_i - \frac{1}{2\hbar} \theta_{ij} p_j \\
{\hat{p}}_i &= p_i
\end{align*}

and the Hamiltonian corresponding to the noncommutative problem is obtained by replacing $(x$ and $p)$ in the commutative Hamiltonian by $({\hat{x}}$ and $\hat{p})$. In three dimensions, we
can define the vector $\theta$ by $\theta_{ij} = \frac{1}{2}\epsilon_{ijk}\theta^k$, so that the noncommutative coordinate becomes

$$\hat{x} = x - \frac{1}{4\hbar} \hat{p} \times \theta$$

(4)

2.1 Vanderwaals forces

We discuss here the known problem of the long-range interaction between two Hydrogen atoms in their ground states. We assume the nuclei of the two atoms are fixed in space a distance $R$ apart and we choose the $z$ axis parallel to the line between them. Let $\bf{r}_1$ be the vector displacement of the electron 1 from the nucleus $A$ and $\bf{r}_2$ be the vector displacement of the electron 2 from the nucleus $B$, then the ‘ordinary’ hamiltonian for the two electrons can be written

$$H^c = H^0 + H'^c$$

(5)

$$H^0 = -\frac{\hbar^2}{2m} \left( \nabla_1^2 + \nabla_2^2 \right) - \frac{e^2}{r_1} - \frac{e^2}{r_2}$$

(6)

$$H'^c = \frac{e^2}{R} + \frac{e^2}{r_{12}} - \frac{e^2}{r_{1B}} - \frac{e^2}{r_{2A}}$$

(7)

Expanding $H'^c$ in powers of $r/R$, where $r \sim r_1 \sim r_2$ of order of $a_0$ the Bohr radius, we find [20]

$$H'^c = H'^c_3 + \mathcal{O} \left( \frac{r^3}{R^4} \right)$$

(8)

$$H'^c_3 = \frac{e^2}{R^3} (x_1x_2 + y_1y_2 - 2z_1z_2)$$

(9)

The last term $H'^c_3$ represents the interaction energy of two electric dipoles that correspond to the instantaneous configurations of the two atoms, while the higher order terms denote the quadrupole-dipole, quadrupole-quadrupole and higher order interactions. The unperturbed hamiltonian $H^0$ has the solution

$$u(\bf{r}_1, \bf{r}_2) = u_{100}(\bf{r}_1) u_{100}(\bf{r}_2)$$

(10)

for two interacting hydrogen atoms in their ground state.

In the noncommutative space, and where we assume $R$ as a fixed parameter, we then have

$$H \equiv H^0 (\hat{r}, \hat{p}) + H'^c (\hat{r}, \hat{p})$$

$$= H^0 (r, p) + H' (r, p)$$

(11)

$$H' (r, p) = H'^c + H'^{NC}$$

(12)

$$H'^{NC} = H'^{NC}_1 + H'^{NC}_2$$

(13)

where $H'^{NC}_1 (H'^{NC}_2)$ comes from the non-commutativity effects on $H^0 (H'^c)$

$$H'^{NC}_1 = H^0_1 + \mathcal{O} \left( \frac{\theta^2}{r^5} \right)$$

(15)

$$H'^{NC}_2 = H^0_2 + \mathcal{O} \left( \frac{\theta r}{R^4}, \frac{\theta^2}{R^3 r^2} \right)$$

(16)
with

\[
H_1^\theta = -\frac{e^2}{4\hbar} \cdot \left( \frac{L_1}{r_1^3} + \frac{L_2}{r_2^3} \right) \tag{17}
\]

\[
H_2^\theta = -\frac{e^2}{4\hbar R^3} \left[ \theta^2 \left( y_1 p_{2z} + y_2 p_{1z} + 2z_1 p_{2y} + 2z_2 p_{1y} \right) + \theta^2 \left( x_1 p_{2y} + x_2 p_{1y} - y_1 p_{2x} - y_2 p_{1x} \right) \right] \tag{18}
\]

Here we assumed, without loss of generality, that the vector \( \theta \) has components along the \( z \) and \( x \) axes only.

It is clear that the expectation value of the leading terms \( (H_3^c, H_1^\theta, H_2^\theta) \) for the state \( u(r_1, r_2) \) is zero. This is because \( u_0 \) is an even function of \( r_1 \) and \( r_2 \) while \( H_3^c, H_2^\theta \) are odd functions of \( r_1 \) and \( r_2 \) separately, and \( H_1^\theta \) applied to \( u_0 \) would give zero. One can check, neglecting the terms proportional to \( \theta^2 \), that all the neglected higher order in the perturbation \( H' \) have zero expectation value for \( u_0 \). Thus the leading term in the interaction energy is the second-order perturbation of the dipole-dipole which is proportional to \( (H_3^c)^2 \) and hence varies like \( 1/R^6 \). This is the well known nature of the Vanderwaals force.

The non-commutativity would cause a shift in the interaction energy estimated by:

\[
\Delta E = \sum_{n \neq 0} \frac{|<0|H^{NC}|n>|^2}{E_n - E_0} + 2 \sum_{n \neq 0} \frac{|<0|H^c|n>|<n|H^{NC}|0>|}{E_n - E_0} \tag{19}
\]

We see here that the perturbation \( H_1^\theta \) does not contribute to the matrix element \( <0|H^{NC}|n> \), since \( L|0> = 0 \), and thus does not lead to any energy shift in the Vanderwaals interaction. We would like to mention here that in the case of excited states, the perturbation \( H_1^\theta \) can be perceived only for very ‘dilute’ gas. The reason of this is as follows. Since \( H_1^\theta \sim \frac{\theta}{r^4} \), then keeping \( H_1^\theta \) while dropping the next term to \( H_3^c \) in the expansion of \( H^c \) which is proportional to \( \frac{\theta^2}{r^6} \) implies \( \frac{\theta}{r^4} \leq \frac{\theta}{r^2} \). Equally dropping the next term to \( H_1^\theta \) in the expansion of \( H_1^{NC} \) proportional to \( \frac{\theta^2}{r^6} \) while keeping \( H_3^c \sim \frac{\theta^2}{R^6} \) implies \( \frac{\theta}{r^2} \leq \left( \frac{R}{r} \right)^{\frac{3}{2}} \). Thus we should have

\[
\left( \frac{R}{r} \right)^4 < \frac{\theta}{r^2} < \left( \frac{R}{r} \right)^{\frac{3}{2}} \tag{20}
\]

The bound \( \theta < 10^{-8} \text{ GeV}^{-2} \) [6] (look at [21] for tighter bounds) with the ordinary value of \( r \sim a_0 \sim 10^{-10}m \), would give a bound \( \frac{\theta}{r^2} < 10^{-20} \). Hence, if noncommutative effects for the term \( H_1^\theta \) would be tangible in the Vanderwaals interaction between two ‘excited’ atoms, the gas of molecules should be ‘dilute’ enough satisfying \( \frac{R}{r} < 10^{-5} \).

As to the term \( H_2^\theta \), we should determine to which order \( \mathcal{O} \left( \frac{r^{n-1}}{R^{n}} \right) \) we should expand \( H^c \) so that not to drop \( H_2^\theta \). For the value \( \frac{r}{R} \sim 10^{-5} \) and the bound \( \theta < 10^{-8} \text{ GeV}^{-2} \), we see that we should expand \( H^c \) till the order \( n = 7 \), while for normal gases \( \frac{r}{R} \sim 10^{-1} \), we see that we can not feel \( H_2^\theta \) unless we expand \( H^c \) till \( n = 23 \). We conclude that non-commutativity effects can not be determined for the Vanderwaals interactions.
2.2 The photoelectric effect

We now consider the photoelectric effect—that is, the ejection of an electron when an atom is placed in the radiation field. The basic process is considered to be the transition from an atomic (bound) state to a continuum state \( E > 0 \). For the final state \( |f^0 > \), we must use a positive energy eigenstate of the Coulomb Hamiltonian

\[
H^0 = \frac{p^2}{2m} - \frac{e^2}{r}
\]

However, if the ejected electron is not too slow, then one can ignore the pull of the proton on it and, with negligible error, approximate the continuum state with a plane wave state \( |p_f > \):

\[
|f^0 >= |p_f > + \cdots
\]

an approximation which, moreover, assumes that \( |p_f > \) is dominating the higher order terms in \( |f^0 > \) when evaluating matrix elements\(^{[22]}\).

We consider now the Hydrogen atom in its ground state \( |u_{100} > \) on which is incident the electromagnetic wave

\[
A(r, t) = A_0 e^{i(k \cdot r - \omega t)} \tag{21}
\]

We would like to calculate the rate for the process of liberating a bound electron using the Fermi’s golden rule:

\[
R_{i \to f} = \text{rate of transition } i \to f = \frac{2\pi}{\hbar} |< f^0 | H^i | i^0 > |^2 \delta \left( E_f^0 - E_i^0 - \hbar \omega \right) \tag{22}
\]

where, as we said, the final state \( |f^0 > \) is the plane wave \( |p_f > \) while the initial state \( |i^0 > \) is the ground state \( |100 > \).

The ‘ordinary’ perturbation, in the Coulomb gauge \( \nabla \cdot A = 0 \), is \(^{[23]}\)

\[
H^{ie}(t) = \frac{e}{2mc} e^{ik \cdot r} e^{-i\omega t} A_0 \cdot p = H^{1e} e^{-i\omega t} \tag{23}
\]

Now, we introduce the non-commutativity so to obtain for the perturbation:

\[
H^1 = \frac{e}{2mc} e^{ik \cdot (r - \frac{1}{2}p \times \theta)} A_0 \cdot p \tag{24}
\]

We can proceed now to evaluate the transition matrix element in the coordinate basis

\[
H^1_{fi} = N \int e^{-ip_f \cdot r/\hbar} e^{ik \cdot (r - \frac{1}{2}p \times \theta)} A_0 \cdot (-i\hbar \nabla) e^{-r/a_0} d^3r \tag{25}
\]

\[
N = \frac{e}{2mc \left(2\pi \hbar \right)^{3/2}} \left( \frac{1}{\pi a_0^2} \right)^{1/2} \tag{26}
\]

where \( a_0 \) is the Bohr radius for the ground state Hydrogen atom.

Notice here, that we take the factor \( e^{i(k \cdot r)} \) exactly into account, i.e. we do not assume the electric dipole approximation. Integrating by part and using \( A_0 \cdot k = 0 \), we get up to first order in \( \theta \):

\[
H^1_{fi} = N \frac{(8\pi/a_0)^2}{[(1/a_0)^2 + (p_f/\hbar - k)^2]^{3/2}} A_0 \cdot p_f \left[ 1 - \frac{i}{4\hbar} (\theta \times k) \cdot p_f \right] \tag{27}
\]
Like the ordinary case, we see that the rate depends on the magnitude of the applied field $A_0$ and on its angle with the outgoing momentum, and also on the frequency of the radiation. The electron likes to come parallel to $A_0$, but is also biased towards $k$. However, ejecting the electron parallel to $k$ minimizes the denominator in equation (27), but nullifies the noncommutative effect. The noncommutative effect consists of introducing an imaginary part into the transition matrix element. If our detector counts how many electrons come in the cone of solid state $d\Omega$, then we can associate with the atom a photoelectric differential cross section
\[
\frac{d\sigma}{d\Omega} = \frac{8\pi c}{|\mathbf{A}_0|^2 w^2} \cdot \hbar w \cdot R_{\mathbf{i}\rightarrow\mathbf{f}}
\] (28)
with the final result:
\[
\left(\frac{d\sigma}{d\Omega}\right)^{NC} = \left(\frac{d\sigma}{d\Omega}\right)^c \left[1 + \frac{1}{16\hbar^2} \left((\theta \times k) \cdot \mathbf{p}_f\right)^2\right]
\] (29)
This formula can be used to put bounds on the parameter $\theta$. However, since the correction is in $\theta^2$ (quadratic), the deviation between the experimental data and the ordinary cross section $\left(\frac{d\sigma}{d\Omega}\right)^c$ will not lead to stronger bounds on $\theta$ than the Lamb shift bound ($\theta < 10^{-8} GeV^{-2}$) where it enters linearly [6].

3 Conclusions
We have studied the problems of the noncommutative Vandrwaals interactions and photoelectric effect. For the Vanderwaals force, the noncommutative effect can not be determined experimentally since the errors coming from neglecting higher orders in the ‘ordinary’ perturbative expansion are far larger than the noncommutative corrections. For the photoelectric effect, the cross section is multiplied by a factor proportional to $(1 + \theta^2)$ allowing to put bounds on the noncommutative parameter $\theta$.

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References
[1] N. Seiberg, E. Witten, String Theory and Noncommutative Geometry, JHEP 09 (1999) 032, and references therein.
[2] A. Connes, M. R. Douglas and A. Schwarz, “Noncommutative geometry and matrix theory: Compactification on tori,” JHEP9802, 003 (1998) [hep-th/9711162].
[3] M. R. Douglas and C. M. Hull, “D-branes and the noncommutative torus,” JHEP 9802, 008 (1998) [hep-th/9711165].
[4] C. Chu and P. Ho, “Constrained quantization of open string in background B field and noncommutative D-brane,” Nucl. Phys. B 568, 447 (2000) [hep-th/9906192].

[5] N. Seiberg and E. Witten, “String theory and noncommutative geometry,” JHEP 9909, 032 (1999) [hep-th/9908142].

[6] M. Chaichian, M. M. Sheikh–Jabbari, A. Tureanu, Phys. Rev. Lett. 86, 2716 (2001)

[7] P.-M. Ho and H.-C. Kao, Phys. Rev. Lett. 88 (15), 151602 (2002)

[8] M. Chaichian, M. M. Sheikh–Jabbari, A. Tureanu, Eur. Phys. J. C36 (2004) 251

[9] M. Hayakawa, [hep-th/9912167]; I. F. Riad and M. M. Sheikh-Jabbari, JHEP 008, 045 (2000); N. Chair and M. M. Sheikh-Jabbari, Phys. Lett. B 504, 141 (2001); M. Chaichian, M. M. Sheikh-Jabbari and A. Tureanu, Phys. Rev. Lett. 86, 2716 (2001); H. Arfaei and M. H. Yavartanoo, [hep-th/0010244]; N. t. Binh, [hep-th/0301084]; F. T. Brandt and A. Das, Phys. Rev D 65, 085017 (2002); X. J. Wang, M. L. Yan, JHEP 0203, 047 (2002); K. B. Eom, S. S. Kang, B. H. Lee and C. Park, [hep-th/0205093]; T. Mariz, C. A. de S. Pires and R. F. Ribeiro, [hep-th/0211416]; J. L. Hewett, F. J. Petriello and T. G. Rizzo, Phys. Rev. D 64, 075012 (2001); S. Baek, D. K. Ghosh, X. G. He and W. Y. Hwang, Phys. Rev. D 64, 056001; N. Mahajan, [hep-ph/0110148]; J. L. Hewett, F. J. Petriello and T. G. Rizzo, Phys. Rev. D 66, 036001 (2002); C. E. Carlson, C. D. Carone and R. F. Lebed, Phys. Lett. B 518, 201 (2001); C. E. Carlson, C. D. Carone and R. F. Lebed, Phys. Lett. B 549, 337 (2002).

[10] J. Madore, S. Schraml, P. Schupp and J. Wess, Eur. Phys. J. C 16, 161 (2000); B. Jurco, S. Schraml, P. Schupp and J. Wess, Eur. Phys. J. C 17, 521 (2000); B. Jurco, P. Schupp and J. Wess, Nucl. Phys. B 604, 148 (2001); B. Jurco, L. Moller, S. Schraml, P. Schupp and J. Wess, Eur. Phys. J. C 21, 383 (2001).

[11] X. Calmet, B. Jurco, P. Schupp, J. Wess and M. Wohlgenannt, Eur. Phys. J. C 23, 363 (2002).

[12] X. G. He, [hep-ph/0202223]; W. Behr, N. G. Deshpande, G. Duplancic, P. Schupp, J. Trampetic and J. Wess, [hep-ph/0202121]; E. Iltan, JHEP 0211, 029 (2002); E. Iltan, New J. Phys. 4, 54 (2002); E. Iltan, Phys. Rev. D 66, 034011 (2002); Z. Cang and Z. Xing, Phys. Rev. D 66, 056009 (2002).

[13] N. Mahajan, Phys. Rev. D. 68, (2003) 095001
N. Mahajan, Phys.Lett. B569 (2003) 85

[14] A. P. Polychronakos, JHEP04, 011 (2001)

[15] T. Mariz, J. R. Nascimento and R. F. Ribeiro, Phys. Rev. D (68), 087701 (2003)

[16] A. E. F. Djemai and H. Smail, Commun.Theor.Phys. 41 (2004) 837-844.
[17] K. Li and S. Dulat, hep-th/0508060
    K. Li and S. Dulat, hep-th/0508193

[18] M. R. Douglas, N. A. Nekrasov, Rev. Mod. Phys. (2001) 73

[19] Kang Li, Jianhua Wang, Chiyi Chen, Mod. Phys. Lett. A Vol. 20, No. 28(2005) 2165-2174.

[20] See for e.g.: L. I. Schiff, “Quantum mechanics”, 3rd ed, New York, McGraw-Hill [1968]

[21] I. Mocioiu, M. Pospelov and R. Roiban, Phys. Lett. B489 (2000) 390

[22] See for e.g.: H. bethe and E. Salpeter, “Quantum mechanics of one and two electron atoms”, New York, NY, Plenum Press, 1977.

[23] See for e.g.: R. Shankar, “Principles of quantum mechanics”, 2nd ed, New York, NY, Plenum Press, 1994.