Reduction of Divisors and the Clebsch System

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Abstract—There are a few Lax matrices of the Clebsch system. Poles of the Baker–Akhiezer function determine the class of equivalent divisors on the corresponding spectral curves. According to the Riemann–Roch theorem, each class has a unique reduced representative. We discuss properties of such a reduced divisor on the spectral curve of a $3 \times 3$ Lax matrix having a natural generalization to the $gl^*(n)$ case.

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Dedicated to the memory of Alexey Borisov

1. INTRODUCTION

The Clebsch system was proposed in 1870 and it represents a specific famous case of the Kirchoff equations which describes the motion of a rigid body in an ideal fluid [13]. A full description of the long history of this system can be found in the book of Borisov and Mamaev [9, pp. 218–220]. We will limit ourselves to listing a few points from this book.

The Clebsch system is isomorphic or belongs to a few families of integrable systems:

- rigid body motion in a central Newtonian field, see the textbooks [3, 9] and the papers by Brun [10], Tisserand [40] and Weber [51] about integrable electrodynamic systems;
- geodesic motion on an ellipsoid [29] and the Kowalevski gyrostat [22];
- the Frahm–Schottky–Manakov system on $so(4) = so(2) \times so(2)$, see [6, 42];
- the Landau–Lifshitz equation and the antisymmetric chiral $O(3)$ field model, see [11, 50] and references in [7];
- integrable systems on $gl^*(n, R)$ obtained by Perelomov, see [33];
- the $n$-cite elliptic Gaudin model on $so(2) \times so(2) \cdots \times so(2)$, see [37] and references in [7];
- quasi-Stäckel integrable systems [27, 28].

So, we have a set of equivalent two-dimensional integrable systems appearing in different physical applications. Let us briefly describe the well-known approaches to the solution of the corresponding equations of motion:

- in 1859 Neumann integrated the equations of motion of a point on a sphere which coincide with Clebsch’s equations of motion in a partial case [32];
- in 1879 Weber solved Clebsch’s equations of motion for the Neumann partial case [51];
- in 1887 Halphan solved Clebsch’s equations in terms of elliptic functions appearing in another partial case [19];
- in 1891 Schottky integrated the Frahm–Schottky–Manakov equations of motion [36].
• in 1892 Kötter solved Clebsch’s equations of motion in the generic case [23];
• in 1895 Kobb obtained quadratures for rigid body motion in a central Newtonian field [21]. Construction of these quadratures involves solution of the fourth-order algebraic equation;
• in 1893 Steklov studied quadratures in generic and particular cases of Clebsch’s system [38, 39];
• in 1900 Chaplygin studied particular solutions of Clebsch’s equations using characteristic function theory [12];
• In 1959 Harlamova obtained quadratures generalizing Chaplygin’s method in the generic case. These quadratures also involve solutions of the fourth-order algebraic equation [20];
• in 1963 and 1974 Arkhangel’skii, Demin and Kiselev studied periodic solutions of Clebsch’s equations under certain restrictions [3, 14];
• in 1987 Bobenko presented theta-function formulae for all the classical tops using finite-gap theory and $2 \times 2$ Lax matrices for the Clebsch and Frahm – Schottky – Manakov systems [7];
• in 1998 Zhivkov and Christov presented theta-functions formulae using finite-gap theory and $4 \times 4$ Lax matrices [52];
• in 1998 Sklyanin and Takebe obtained variables of separation for the elliptic Gaudin magnet [37];
• in 2008 Sokolov and Marikhin obtained quadratures for a pair of quasi-Stäckel Hamiltonians [27, 28]
\[
H_1 = ap_1^2 + 2bp_1p_2 + cp_2^2 + dp_1 + ep_2 + f, \\
H_2 = Ap_1^2 + 2Bp_1p_2 + Cp_2^2 + Dp_1 + Ep_2 + F.
\]

In this case the construction of quadratures involves solutions of the cubic algebraic equation;
• in 2015 Magri and Skrypnyk found quadratures for the complex variables of separation, which are two roots of the cubic algebraic equation [26];
• in 2021 Fedorov, Magri and Skrypnyk solved the Clebsch system in terms of theta functions using complex variables of separation, which consist of one of the eight solutions of the system of quadratic algebraic equations and its derivative [17, 18].

Following Borisov and Mamaev [9], we fully agree with Chaplygin [12] and Magnus [25] that these solutions of equations of motion belong rather to the field of mathematical sport and add nothing to the description of motion. Nevertheless, it is natural to assume that this list will be continued and, one day, we will see unambiguously defined real variables of separation depending on real time, similar to the Kowalevski top [24] and Euler’s two fixed centers problem [5].

In this note we do not study solutions of equations of motion in theta functions obtained via quadratures or without quadratures. We also do not discuss relations between different quadratures. Our aim is to study properties of the Lax matrix for integrable systems on $gl^*(n, R)$ obtained by Perelomov [33]. For all these matrices, the number of degrees of freedom $n$ is more than the genus of the corresponding spectral curve, similar to the Heisenberg and Gaudin magnets.

For instance, when $n = 3$ the corresponding Baker – Akhiezer vector function $\psi$ has three poles $P_1, P_2$ and $P_3$ on the genus one spectral curve. So, there is a chain of equivalent divisors
\[
D = P_1 + P_2 + P_3 \rightarrow D' = P_1' + P_2' \rightarrow \rho(D) = D'' = P''
\]
according to the Riemann – Roch theorem for divisors on algebraic curves [30]. Below we calculate all these points on a spectral curve using the standard Abel reduction algorithm [1] and prove that:

• affine coordinates of points $P_{1,2}'$ are functions on integrals of motion and one variable $X$, which also can be found in the Kobb and Harlamova quadratures;

• affine coordinates of the point $P''$ in the reduced divisor $\rho(D)$ are functions only on integrals of motion.
Thus, we prove that a reduced divisor for the $gl^*(n, R)$ system at $n = 3$ has the same properties as reduced divisors for the harmonic oscillator and the Kepler problem [45], for the Drach systems [47], for the Heisenberg and Gaudin magnets [46], for the Kowalevski top [48] and the Euler two fixed centers problem [49].

The main result is to prove for the $3 \times 3$ Lax matrix of the Clebsch system the nonobvious assumption that a reduced divisor is fixed, i.e., independent of time, when the dimension of the configuration space is more than the genus of the corresponding spectral curve.

1.1. Kirchhoff Equations in the Clebsch Case

In the Clebsch case, the Kirchhoff equations for the motion of a rigid body in an ideal incompressible fluid are equal to

$$\dot{M} = p \times Ap, \quad \dot{p} = p \times M.$$  \hfill (1.1)

Here $p$ and $M$ are three-dimensional vectors, $\times$ denotes the vector product and $A$ is a diagonal matrix

$$A = \begin{pmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{pmatrix}, \quad a_i \in \mathbb{R}.$$  

The vector $M$ is the total angular momentum vector, whereas $p$ represents the total linear momentum of a system [13]. There are two geometric integrals of motion

$$c_1 = p_1^2 + p_2^2 + p_3^2, \quad c_2 = p_1 M_1 + p_2 M_2 + p_3 M_3$$

and two Hamiltonians

$$H = M_1^2 + M_2^2 + M_3^2 + a_1(p_2^2 + p_3^2) + a_2(p_1^2 + p_3^2) + a_3(p_1^2 + p_2^2),$$
$$K = a_1 M_1^2 + a_2 M_2^2 + a_3 M_3^2 + a_2 a_3 p_1^2 + a_1 a_3 p_2^2 + a_1 a_2 p_3^2,$$  \hfill (1.2)

which Poisson commute with respect to the Lie–Poisson bracket on the Lie algebra $e^*(3)$

$$\{ M_i, M_j \} = \varepsilon_{ijk} M_k, \quad \{ M_i, p_j \} = \varepsilon_{ijk} p_k, \quad \{ p_i, p_j \} = 0, \quad (1.3)$$

where $\varepsilon_{ijk}$ is a skew-symmetric tensor. All these definitions are invariant under the cyclic permutation of indices.

We know reductions of these equations of motion (1.1) to quadratures, solutions of these equations in terms of theta functions, the bi-Hamiltonian structure of these equations [41, 43], various Lax pair representations [2, 4, 33, 52], topological invariants [8], the Hirota–Kimura type discretization [34], numerical solutions [35], etc.

2. THE LAX MATRIX AND THE SPECTRAL CURVE

If we know the Lax representation of the original equations of motion (1.1)

$$\frac{d}{dt} \mathcal{L}(x) = [\mathcal{L}(x), \mathcal{A}(x)],$$  \hfill (2.1)

for two $N \times N$ matrix functions $\mathcal{L}(x)$ and $\mathcal{A}(x)$ on phase space depending on the auxiliary spectral parameter $x$, then we can directly integrate equations (2.1) in terms of theta functions by using finite-gap integration theory. Indeed, the time-independent spectral equation

$$\mathcal{L}(x) \psi(x, y) = y \psi(x, y)$$  \hfill (2.2)

allows us to represent the Baker–Akhiezer vector function $\psi$ in terms of the Riemann theta function on a nonsingular compactification of the spectral curve defined by the equation

$$\Gamma : \quad f(x, y) = \det(L(x) - y) = 0.$$
The second equation defines time
\[
\frac{d}{dt} \psi(x, y) = -A(x)\psi(x, y),
\] (2.3)
and the evolution of the original variables \(p_i(t)\) and \(M_i(t)\) with respect to this time, see [6, 52] for details.

For integrable systems on \(gl^*(n, R)\) the Lax representation (2.1) was found by Perelomov [33]. When \(n = 3\) we have the following \(3 \times 3\) matrix:
\[
\begin{pmatrix}
\lambda^2 p_1^2 + a_1 & \lambda^2 p_1 p_2 + M_3 \lambda & \lambda^2 p_1 p_3 - M_2 \lambda \\
\lambda^2 p_1 p_2 - M_3 \lambda & \lambda^2 p_2^2 + a_2 & \lambda^2 p_2 p_3 + M_1 \lambda \\
\lambda^2 p_1 p_3 + M_2 \lambda & \lambda^2 p_2 p_3 - M_1 \lambda & \lambda^2 p_3^2 + a_3
\end{pmatrix},
\] (2.4)
Here \(M \in so^*(3)\) is a skew-symmetric matrix associated with the vector \(M \in R^3\) and one of the corresponding second Lax matrices,
\[
A = -\frac{2}{\lambda} (A + \lambda M) \quad \text{or} \quad A = 2\lambda p \otimes p,
\]
in (2.1) defines equations of motion coinciding with the Kirchhoff equations (1.1).

2.1. Poles of the Baker–Akhiezer Function

Let us emphasize again that we do not discuss equations of motion (1.1) or (2.1) et al., because instead of the evolution of poles of the Baker–Akhiezer function we want to study the reduction of poles of the Baker–Akhiezer function only.

Indeed, the spectral curve \(\Gamma\)
\[
\Gamma : \quad f(\mu, \lambda) = c_2^2 \lambda^4 + (c_1 \mu^2 - H\mu + K)\lambda^2 + \det(A - \mu) = 0 \tag{2.5}
\]
is a 2-fold covering of the elliptic curve \(E\) at \(\lambda^2 = y\)
\[
E : \quad f(\mu, y) = c_2^2 y^2 + (c_1 \mu^2 - H\mu + K)y + \det(A - \mu) = 0. \tag{2.6}
\]
The reduced divisor is a point \(P^m\) on this elliptic curve \(E\) (2.6). Our aim is to calculate this point and to study its affine coordinates.

As a first step in this direction we have to construct a class of linearly equivalent divisors
\[
D = \sum_{i=1}^m P_i,
\]
which are a formal sum of poles \(P_i = (\mu_i, \lambda_i)\) of the Baker–Akhiezer vector function \(\psi\) (2.2) with some fixed normalization \(\vec{\alpha}\)
\[
\vec{\alpha} \cdot \psi = \sum_{i=1}^N \alpha_i \psi_i = 1,
\]
Because
\[
\psi_j = \frac{(L(\lambda) - \mu)^{\wedge}}{\vec{\alpha} \cdot (L(\lambda) - \mu)^{\wedge}}, \quad \forall k = 1, 2, 3,
\]
where the wedge denotes the adjoint or cofactor matrix, and the poles of the Baker–Akhiezer function \(\psi(\lambda, \mu)\) are common zeroes of the four algebraic equations
\[
f(\mu, \lambda) = \det(L(\lambda) - \mu) = 0 \quad \text{and} \quad \vec{\alpha} \cdot (L(\lambda) - \mu)^{\wedge} = 0. \tag{2.7}
\]
For the general normalization \(\vec{\alpha}\) the algebraic equations (2.7) have five solutions \(P_i = (\mu_i, \lambda_i)\), \(i = 1 \ldots 5\), whereas at
\[
\vec{\alpha} = (p_1, p_2, p_3)
\]
there are only three solutions of Eqs. (2.7). Indeed, three solutions of Eqs. (2.7) define three points $P_i = (\mu_i, \lambda_i)$ on the spectral curve $\Gamma$ (2.5) which form the positive or semi-reduced divisor

$$D = P_1 + P_2 + P_3, \quad \deg D = 3$$

on the genus three algebraic curve $\Gamma$, $g(\Gamma) = 3$. According to the Riemann–Roch theorem, the dimension of the linear system $|D|$, which is the set of all the nonnegative divisors linearly equivalent to $D$

$$|D| = \{D' \in \text{Div}(\Gamma) \mid D' \sim D \text{ and } D' > 0\},$$

is equal to

$$\dim|D| = \deg D - \text{genus}(\Gamma) = 3 - 3 = 0,$$

see the textbook [30] for definitions and other details.

In our case, $\dim|D| = 0$ and, therefore, divisor $D$ is a unique reduced divisor in the corresponding class of equivalent divisors. Thus, we cannot reduce this divisor to the prime divisor of degree one on a genus three bielliptic curve $\Gamma$.

On the elliptic curve $E$ three points $P_i = (\mu_i, y_i)$, where $\lambda_i^2 = y_i$, define the semi-reduced divisor on the genus one elliptic curve $E$ (2.6)

$$D = P_1 + P_2 + P_3, \quad \deg D = 3,$$

so that

$$\dim|D| = \deg D - \text{genus}(E) = 3 - 1 = 2.$$ 

According to the Riemann–Roch theorem, we can reduce this semi-reduced divisor $D$ to equivalent divisors $D'$ and $D''$:

$$D \rightarrow D' \rightarrow D'', \quad \dim|D| = 2, \quad \dim|D'| = 1, \quad \dim|D''| = 0.$$ 

Below we study these semi-reduced $D'$ and reduced $D'' = \rho(D)$ divisors on the elliptic curve $E$ (2.6).

### 2.2. Semi-reduced Divisor of Degree Three

For the generic normalization $\tilde{\alpha}$ the last three equations in (2.7) are cubic polynomials in $\lambda$

$$e_i = (\tilde{\alpha} \times p)_i \lambda^3 + e_i^{(2)}(\mu)\lambda^2 + e_i^{(1)}(\mu)\lambda + e_i^{(0)}(\mu) = 0, \quad i = 1, 2, 3,$$

and solutions of these equations for $\lambda$ and $\mu$ are the roots of fifth-order polynomials

$$A_5(\lambda) = 0, \quad B_5(\mu) = 0,$$

which can be easily obtained using modern computer algebra systems and, therefore, here we do not present these polynomials for brevity.

If $\tilde{\alpha} = (p_1, p_2, p_3)$, then the vector $(\tilde{\alpha} \times p) = 0$ is equal to zero and the algebraic equations (2.7) have the following form:

$$f(\mu, \lambda) = c_2^2\lambda^4 + (c_1 \mu^2 - H \mu + K)\lambda^2 + \det(A - \mu) = 0,$$

$$e_1 = M_1c_2\lambda^2 + ((M_2p_3 - M_3p_2)\mu + a_3M_3p_2 - a_2M_2p_3)\lambda + p_1(\mu^2 - (a_2 + a_3)\mu + a_2a_3) = 0,$$

$$e_2 = M_2c_2\lambda^2 + ((M_3p_1 - M_1p_3)\mu + a_1M_1p_3 - a_3M_3p_1)\lambda + p_2(\mu^2 - (a_1 + a_3)\mu + a_1a_3) = 0,$$

$$e_3 = M_3c_2\lambda^2 + ((M_1p_2 - M_2p_1)\mu + a_2M_2p_1 - a_1M_1p_2)\lambda + p_3(\mu^2 - (a_1 + a_2)\mu + a_1a_2) = 0.$$ 

Solutions of these equations for $\lambda$ and $\mu$ are the roots of the cubic polynomials

$$A_3(\lambda) = -b_3c_2\lambda^3 + \left(a_1(p_2M_3 - M_3M_2)(c_1M_1 + c_2p_1) + a_2(p_3M_1 - p_1M_3)(c_1M_2 + c_2p_2) + a_3(p_1M_2 - p_2M_1)(c_1M_3 + c_2p_3)\right)\lambda^2$$

$$- \left(p_1^2 + p_2^2\right)(a_1 - a_3)(a_1 - a_2)p_1M_1 + \left(p_1^2 + p_2^2\right)(a_2 - a_3)(a_2 - a_1)p_2M_2$$

$$+ \left(p_1^2 + p_2^2\right)(a_3 - a_2)(a_3 - a_1)p_3M_3 \lambda - p_1p_2p_3(a_1 - a_2)(a_2 - a_3)(a_3 - a_1)$$

$$\quad (2.8)$$

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are invariant under cyclic permutations of the indices. Thus, we consider a linear combination of

\[ B_3(\mu) = b_3\mu^3 + b_2\mu^2 + b_1\mu + b_0, \]  

(2.9)

where

\[
\begin{align*}
    b_3 &= c_1(M_1^2 + M_2^2 + M_3^2) - c_2^2, \\
    b_2 &= 2c_2(a_1p_1M_1 + a_2p_2M_2 + a_3p_3M_3) - 2c_1(a_1M_1^2 + a_2M_2^2 + a_3M_3^2) \\
    &\quad - a_1(M_2p_3 - M_3p_2)^2 - a_2(M_1p_3 - M_3p_1)^2 - a_3(M_2p_1 - M_1p_2)^2, \\
    b_1 &= (a_1a_2 + a_1a_3 + a_2a_3)b_3 - p_1^2(a_2 + a_3)(a_1a_2 + a_1a_3M_2^2 + (a_1 - a_3)M_3^2), \\
    b_0 &= a_1a_2a_3c_2 - (a_1M_1^2 + a_2M_2^2 + a_3M_3^2)(a_2a_3p_1^2 + a_1a_3p_2^2 + a_1a_2p_3^2).
\end{align*}
\]

The roots of polynomials \( A_3(\lambda) \) and \( B_3(\mu) \) determine the poles \( P_i = (\mu_i, \lambda_i) \) of the Baker–Akhiezer function on the spectral bielliptic curve \( \Gamma \) (2.5).

To determine the corresponding poles \( P_i = (\mu_i, y_i = \lambda_i^2) \) on the elliptic curve \( E \), we can replace three equations depending on \( \mu \) and \( \lambda \)

\[ e_k(\mu, \lambda) = e_k^{(2)}\lambda^2 + e_k^{(1)}\lambda + e_k^{(0)} = 0, \quad k = 1, 2, 3, \]

with three equations depending on \( \mu \) and \( y = \lambda^2 \)

\[
\begin{align*}
    E_{12}(\mu, y) &= e_2^{(1)}e_1 - e_1^{(1)}e_2 = 0, \\
    E_{23}(\mu, y) &= e_3^{(1)}e_2 - e_2^{(1)}e_3 = 0, \\
    E_{31}(\mu, y) &= e_1^{(1)}e_3 - e_3^{(1)}e_1 = 0.
\end{align*}
\]

Equations of motion for the Clebsch system (1.1), Hamiltonians \( H_{1,2} \) (1.2) and polynomial \( B_3(\mu) \) are invariant under cyclic permutations of the indices. Thus, we consider a linear combination of \( E_{ik} \)

\[ g(\mu, y) = \sum \varepsilon_{ijk}p_iE_{jk} = p_1E_{23} + p_2E_{31} + p_3E_{12} = c_2Q(\mu)y - P(\mu) = 0, \]

which is also invariant under permutations. Here \( Q(\mu) \) and \( P(\mu) \) are polynomials of first and second order in \( \mu \)

\[ Q(\mu) = -\mu b_3 + (a_1M_1^2 + a_2M_2^2 + a_3M_3^2)c_1 - c_2X, \quad P(\mu) = P_2\mu^2 + P_1\mu + P_0, \]

where

\[ X = a_1p_1M_1 + a_2p_2M_2 + a_3p_3M_3. \]  

(2.10)

and

\[
\begin{align*}
    P_2 &= (a_1p_1^2 + a_2p_2^2 + a_3p_3^2)c_2 - c_1X, \\
    P_1 &= \left( (a_1 + a_2 + a_3)c_1 - a_1p_1^2 - a_2p_2^2 - a_3p_3^2 \right)X \\
    &\quad - \left( (a_1a_2 + a_1a_3 + a_2a_3)c_1 - (a_1a_2p_3^2 + a_1a_3p_2^2 + a_2a_3p_1^2) \right)c_2, \\
    P_0 &= a_1a_2a_3c_1c_2 - (a_1a_2p_3^2 + a_1a_3p_2^2 + a_2a_3p_1^2)X.
\end{align*}
\]

It is easy to prove that three points \( P_i = (\mu_i, y_i) \) with coordinates

\[ B_3(\mu) = b_3(\mu - \mu_1)(\mu - \mu_2)(\mu - \mu_3) = 0 \quad \text{and} \quad y_i = \frac{P(\mu_i)}{c_2Q(\mu_i)}, \quad i = 1, 2, 3, \]

lie on the elliptic curve \( E \) (2.6). A formal sum of these points

\[ D = P_1 + P_2 + P_3, \quad \text{deg}D = 3, \]

is a semi-reduced divisor of degree three on the elliptic curve \( E \) (2.6).
2.3. Semi-reduced Divisor of Degree Two

Following Abel’s idea [1], let us consider variable points of intersection of the elliptic curve E with a family of curves depending on time

$$\Upsilon : \quad g(\mu, y) = c_2 Q(\mu) y - P(\mu) = 0.$$  

Substituting

$$y = \frac{P(\mu)}{c_2 Q(\mu)}$$

into the elliptic curve Eq. (2.6)

$$f(\mu, y) = c_2^2 y^2 + (c_1 \mu^2 - H \mu + K) y + \text{det}(A - \mu) = 0,$$

we obtain Abel’s polynomial

$$\Psi = \theta \cdot B_3(\mu) \cdot B'_4(\mu)$$

which determines an intersection divisor

$$D + D' + D_\infty = (P_1 + P_2 + P_3) + (P_4 + P_5) + D_\infty = 0,$$

(2.11)

where $D_\infty$ is a linear combination of the points at infinity.

The abscissas of points $P_4$ and $P_5$ are the roots of the polynomial

$$B'_4(\mu) = b'_2(\mu - \mu_1)(\mu - \mu_5) = \left( c_2^2 - c_1 c_2 H + (a_1 + a_2 + a_3)c_1^2 c_2 - c_1^2 X \right) \mu^2$$

$$+ \left( c_1 c_2 K - (a_1 a_2 + a_1 a_3 + a_2 a_3)c_1^2 c_2 - (c_1 H - 2c_2^2) X \right) \mu + a_1 a_2 a_3 c_1^2 c_2 - c_1 K X + c_2 X^2.$$

The ordinates of these points are equal to

$$y_i = \frac{P(\mu_i)}{c_2 Q(\mu_i)}, \quad i = 4, 5.$$

The affine coordinates of points $P_4$ and $P_5$ are functions on the integrals of motion $c_1, c_2, H, K$ and one variable $X$ (2.10) so that

$$\{\mu_4, H\} \{\mu_5, K\} - \{\mu_4, K\} \{\mu_5, H\} = 0, \quad \{\mu_4, \mu_5\} \neq 0,$$

and

$$\mu_4 - \mu_5 = c_1(y_4 - y_5),$$

(2.12)

i.e., points $P_4$ and $P_5$ move along a straight line and the slope of this line is equal to $c_1^{-1}$.

2.4. Reduced Divisor of Degree One

To directly apply the Euler [16] and Abel [1] formulae, we have to rewrite Eq. (2.6) in the form

$$z^2 = a_4 \mu^4 + a_3 \mu^3 + a_2 \mu^2 + a_1 \mu + a_0$$

using the birational transformation

$$y = z - \frac{c_1 \mu^2 - \mu H + K}{2c_2^2}.$$

Then we have to consider various points of intersection of $E$ with a family of curves depending on time

$$\Upsilon' : \quad z = \sqrt{a_4 \mu^2 + b_1 \mu + b_0}, \quad \text{where} \quad a_4 = \frac{c_1^2}{4c_2^2}.$$  

(2.13)

Here $b_1$ and $b_0$ are coefficients of the interpolating polynomial which is defined by the equations

$$z_4 = \sqrt{a_4 \mu^2 + b_1 \mu_5 + b_0}, \quad z_5 = \sqrt{a_4 \mu^2 + b_1 \mu_5 + b_0},$$

\[ \begin{array}{c}
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\end{array} \]
where \((\mu_4, z_4)\) and \((\mu_5, z_5)\) are the abscissas and ordinates of points \(P_4\) and \(P_5\) lying on the auxiliary curve \(\Upsilon'\). The corresponding intersection divisor of curves \(E\) and \(\Upsilon'\) has the following form:

\[
D' + D'' + D_\infty = (P_4 + P_5) + P_6 + D_\infty = 0.
\]

According to [1], the abscissa \(\mu_6\) of point \(P_6\) is equal to

\[
\mu_6 = -\mu_4 - \mu_5 - \frac{2b_0\sqrt{a_4} + b_1^2 - a_2}{2b_1\sqrt{a_4} - a_3} \equiv \frac{\nu}{\nu}, \tag{2.14}
\]

In the Clebsch case, it is a function on the integrals of motion with the numerator

\[
\nu = c_1^6(\mu_4 + \mu_5)K + c_1^4(\mu_4^2(\mu_4 + \mu_5)^2 + H K)
- c_1^3(2(\mu_4 + \mu_5)H - K)c_2 + c_1^2c_2(2c_1^2(\mu_5 + \mu_5) + H^2) - 2c_1c_2^2H + c_2^6
\]

and the denominator

\[
v = c_1^2\left(c_1^4(\mu_4 + \mu_5) + c_1^2(\mu_4 + \mu_5)H + K\right)
- 3c_1c_2^2H + 2c_2^4
\]

Such reduced divisors also appear for various superintegrable systems [45–47] and for the Kowalevski top [48].

Points \(P_4, P_5\) and \(P_6\) lie on parabola \(\Upsilon'\) of a constant size, which leads to the vanishing of the determinant

\[
\begin{vmatrix}
\mu_4 & z_4 & 1 \\
\mu_5 & z_5 & 1 \\
\mu_6 & z_6 & 1 \\
\end{vmatrix} = 0
\]

and to the vanishing of the corresponding Abel’s integral relation [1]

\[
\frac{d\mu_4}{z_4} + \frac{d\mu_5}{z_5} + \frac{d\mu_6}{z_6} = 0.
\]

Because the abscissa \(\mu_6\) (2.14) is a constant of motion, i.e., \(d\mu_6 = 0\), we immediately obtain the following equation for the abscissas of points from the support of the semi-reduced divisor \(D' = P_4 + P_5\) (2.11)

\[
\frac{d\mu_4}{z_4} + \frac{d\mu_5}{z_5} = 0. \tag{2.15}
\]

Because points \(P_1, P_2, P_3\) and \(P_4, P_5\) belong to the intersection divisor (2.11), we also have the equation

\[
\frac{d\mu_1}{z_1} + \frac{d\mu_2}{z_2} + \frac{d\mu_3}{z_3} = 0
\]

involving a regular differential on the elliptic curve \(E\). This equation is a consequence of the fact that the unique reduced divisor in this class of equivalent divisors is a constant of motion.

The variables \(\mu_{4,5}\) depend on the integrals of motion \(c_1, c_2, H, K\) and the one time-dependent variable \(\chi\) (2.10) and, therefore, they satisfy the geometric equation (2.12).

Let us introduce elliptic coordinates \(u_{1,2}\) on \(\epsilon^*(3)\) using the standard definition

\[
\frac{p_1^2}{z - a_1} + \frac{p_2^2}{z - a_2} + \frac{p_3^2}{z - a_3} = \frac{c_1(z - u_1)(z - u_2)}{(z - a_1)(z - a_2)(z - a_3)} = 0,
\]
and the corresponding conjugated momenta, see [41]. These coordinates satisfy Abel’s equations

\[
\begin{align*}
\frac{u_1}{\sqrt{(u_1 - a_1)(u_1 - a_2)(u_1 - a_3)(u_1^2 - Hu_1 + K)}} + \frac{u_2}{\sqrt{(u_2 - a_1)(u_2 - a_2)(u_2 - a_3)(c_1 u_2^2 - Hu_2 + \tilde{K})}} = 0, \\
\frac{u_1 u_1'}{\sqrt{(u_1 - a_1)(u_1 - a_2)(u_1 - a_3)(c_1 u_1^2 - Hu_1 + K)}} + \frac{u_2 u_2'}{\sqrt{(u_2 - a_1)(u_2 - a_2)(u_2 - a_3)(c_1 u_2^2 - Hu_2 + \tilde{K})}} = 4,
\end{align*}
\]

(2.16)

where \( \tilde{K} = K - 2c_2X(t) \) is a function of time. If we know \( X(t) \), we can try to solve these equations for \( u_1 \) and \( u_2 \).

This variable \( X \) also appears in the Kobb [21] and Harlamova [20] calculations. We will discuss this topic in a forthcoming publication.

3. NEUMANN’S PROBLEM

In the special case

\[ c_1 = p_1^2 + p_2^2 + p_3^2 = 1, \quad c_2 = p_1 M_1 + p_2 M_2 + p_3 M_3 = 0 \]

the Clebsch system is equivalent to the well-studied Neumann problem [32] describing the motion of a particle on a unit sphere in the field of a quadratic potential, see the textbooks [15, 31] for an algebro-geometric description of this system.

The sphero-conical coordinates \( u_1 \) and \( u_2 \) on the sphere are defined through the equation

\[ \frac{p_1^2}{z - a_1} + \frac{p_2^2}{z - a_2} + \frac{p_3^2}{z - a_3} = \frac{(z - u_1)(z - u_2)}{(z - a_1)(z - a_2)(z - a_3)} = 0, \]

which implies that \( c_1 = \sum p_i^2 = 1 \). Similar to elliptic coordinates in \( \mathbb{R}^3 \), these coordinates in \( S^2 \) are also orthogonal and only locally defined. They take values in the intervals

\[ a_1 < u_1 < a_2 < u_2 < a_3, \]

so that

\[ p_i = \sqrt{\frac{(u_1 - a_i)(u_2 - a_i)}{(a_j - a_i)(a_m - a_i)}}, \quad i \neq j \neq m. \]

If \( \pi_{1,2} \) are canonically conjugated momenta

\[ \{u_1, \pi_1\} = \{u_2, \pi_2\} = 1 \]

with respect to the Poisson brackets (1.3), then

\[ M_i = \frac{2\varepsilon_{ijm}p_j p_m (a_j - a_m)}{\mu_1 - \mu_2} \left( (a_i - \mu_1)\pi_1 - (a_i - \mu_2)\pi_2 \right). \]

At \( c_1 = 1 \) and \( c_2 = 0 \) the Hamiltonians (1.2) are the second-order polynomials in momenta

\[ H_0 = \frac{4\varphi_1 \pi_1^2}{\mu_1 - \mu_2} + \frac{4\varphi_2 \pi_2^2}{\mu_1 - \mu_2} + \mu_1 + \mu_2, \quad K_0 = \frac{4\mu_2 \varphi_1 \pi_1^2}{\mu_1 - \mu_2} - \frac{4\mu_1 \varphi_2 \pi_2^2}{\mu_1 - \mu_2} + \mu_1 \mu_2, \]

where for brevity we denote

\[ \varphi_k = \det(A - \mu_k) = (a_3 - \mu_k)(a_2 - \mu_k)(a_1 - \mu_k), \quad k = 1, 2. \]

In this partial case, the spectral curve \( \Gamma \) (2.5) is a genus two hyperelliptic curve which can be rewritten in the following form:

\[ \Gamma_0 : \quad \left( \det(\mu - A) \chi \right)^2 = (\mu^2 - H_0 \mu + K_0) \det(\mu - A), \quad \chi = -\lambda^{-1}. \]

Below we will study the evolution of the semi-reduced and reduced divisors on this curve.
At $c_2 = 0$ the cubic polynomial $A_3(\lambda)$ (2.8) becomes a quadratic polynomial on $\chi = \lambda^{-1}$

$$A_3(\chi) = \sqrt{\varphi_1 \varphi_2} (\chi - 2\pi_1)(\chi - 2\pi_2).$$

Two roots of this polynomial define the reduced divisor of degree two

$$D = P_1 + P_2, \quad |D| = 0, \quad P_1 = (u_1, 2\pi_1), \quad P_2 = (u_2, 2\pi_2)$$
on the genus two spectral curve $\Gamma_0$ (3.1).

In this case we have not a third-degree semi-reduced divisor $D$ and, therefore, we cannot apply Abel’s reduction in this case.

### 3.1. Semi-reduced Divisor

At $c_2 = 0$ the cubic polynomial $B_3(\mu)$ (2.9) remains a cubic polynomial on $\mu$

$$B_3(\mu) = b_3(\mu) = (\mu - \mu_1)(\mu - \mu_2)(\mu - \mu_3) = \frac{4(\mu - u_1)(\mu - u_2)^2 \varphi_1 \pi_1^2}{u_1 - u_2} - \frac{4(\mu - u_1)^2 (\mu - u_2) \varphi_2 \pi_2^2}{u_1 - u_2}.$$ (3.2)

Three roots of this polynomial $\mu_1, \mu_2$ and $\mu_3$ define a semi-reduced divisor of poles on the genus two spectral curve $\Gamma_0$ (3.1)

$$D = P_1 + P_2 + P_3, \quad P_1 = (u_1, 2\pi_1), \quad P_2 = (u_2, 2\pi_2), \quad P_3 = (\mu_3, \chi_3),$$

where the affine coordinates of the third pole are functions on the variables $u_{1,2}$ and $\pi_{1,2}$

$$\mu_3 = \frac{K_0 - u_1 u_2}{H_0 - u_1 - u_2}, \quad \chi_3 = \frac{4\pi_1 \pi_2}{H_0 - u_1 - u_2} \sqrt{\frac{\varphi_1 \varphi_2}{\varphi_3}}.$$ (3.3)

The evolution of three poles $P_1, P_2$ and $P_3$ along the curve $\Gamma_0$ (3.1) is determined by Abel’s equations

$$\Omega_1(P_1)\dot{\mu}_1 + \Omega_1(P_2)\dot{\mu}_2 = 0 \quad \Omega_2(P_1)\dot{\mu}_1 + \Omega_2(P_2)\dot{\mu}_2 = 4,$$ (3.4)

and

$$\Omega_1(P_3)\dot{\mu}_3 = \frac{4X}{\sqrt{\varphi_3(H_0 - u_1 - u_2)}},$$

where the regular differentials on $\Gamma_0$ (3.1) have the form

$$\Omega_1 = \frac{1}{\det(\mu - A) \chi}, \quad \Omega_2 = \frac{\mu}{\det(\mu - A) \chi},$$

and the variable $X$ (2.10) at $c_2 = 0$ is equal to

$$X = a_1 p_1 M_1 + a_2 p_2 M_2 + a_2 p_3 M_3 = \frac{2\sqrt{\varphi_1 \varphi_2 (\pi_1 - \pi_2)}}{u_1 - u_2}.$$ (3.5)

The abscissa and the ordinate of the third point $P_3$ are functions of elliptic coordinates and integrals of motion, and, therefore, two Abel's equations (3.4) completely determine the evolution of the semi-reduced divisor of poles

$$D = P_1 + P_2 + P_3, \quad \deg D = 3, \quad \dim |D| = \deg D - g = 3 - 2 = 1.$$ (3.5)

### 3.2. Reduced Divisor

According to the Riemann–Roch theorem, there is a unique reduced divisor $D'$ on the genus two hyperelliptic curve $\Gamma_0$ (3.1)

$$\rho(D) = D' = P_4 + P_5, \quad \deg D' = 2, \quad \dim |D'| = \deg D' - g = 2 - 2 = 0.$$ (3.6)

which is equivalent to the semi-reduced divisor $D$ (3.5).

Following Abel [1], we can identify the reduced divisor $D'$ with a part of the intersection divisor

$$D + D' + D_\infty = 0$$

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of the hyperelliptic curve $\Gamma_0$ with a family of parabolas $\Upsilon$ involving three points $P_1, P_2$ and $P_3$ on a projective plane. The parabola

$$\Upsilon: y = V_0(\mu)$$

is defined by the interpolation polynomial

$$V_0(\mu) = \frac{(\mu - \mu_2)(\mu - \mu_3)}{\mu_1 - \mu_2}(\mu_1 - \mu_3)\chi_1 + \frac{(\mu - \mu_1)(\mu - \mu_3)}{\mu_2 - \mu_1}(\mu_2 - \mu_3)\chi_2 + \frac{(\mu - \mu_1)(\mu - \mu_2)}{\mu_3 - \mu_1}(\mu_3 - \mu_2)\chi_3. \quad (3.6)$$

Substituting $y = V_0(\mu)$ into (3.1), we obtain Abel’s polynomial $\Psi$ defining the abscissas $\mu_{4,5}$ of points $P_4$ and $P_5$

$$\Psi = \theta \cdot B_0(\mu) \cdot B'_0(\mu), \quad B_0 = b_3(\mu - \mu_1)(\mu - \mu_2)(\mu - \mu_3), \quad B'_0 = (\mu - \mu_4)(\mu - \mu_5),$$

see Abel’s original calculations for genus two hyperelliptic curves [1] or [44] and references therein. In our case

$$B'_0(\mu) = \mu^2 + (\mu_1 + \mu_2 + \mu_3 - a_1 - a_2 - a_3 - H_0)\mu + \mu_1^2 + \mu_2^2 + \mu_3^2$$

$$- (a_1 + a_2 + a_3)(\mu_1 + \mu_2 + \mu_3) + (a_1 + a_2 + a_3 - \mu_1 - \mu_2)H_0 + a_1a_2 + a_1a_3 + a_2a_3 + 2\mu_1\mu_2,$$

where

$$\mu_1 = u_1, \quad \mu_2 = u_2, \quad \mu_3 = \frac{K_0 - u_1u_2}{H_0 - u_1 - u_2}.$$ 

The affine coordinates of this reduced divisor $D' = P_4 + P_5$ satisfy the same Abel equations (3.4)

$$\Omega_1(P_4)u_4 + \Omega_1(P_5)u_5 = 0 \quad \text{and} \quad \Omega_2(P_4)\dot{u}_4 + \Omega_2(P_5)\dot{u}_5 = 4,$$

but

$$\{\mu_4, \mu_5\} \neq 0.$$ 

In this case the affine coordinates of the reduced divisor have properties similar to the properties of the reduced divisor on the genus two algebraic curve for Euler’s two fixed centers problem [49].

4. CONCLUSION

We study the properties of the $n \times n$ Lax matrix obtained by Perelomov [33]. When $n = 3$, the Lax equations coincide with the equations of motion for the Clebsch system. We study the reduction of divisors on the corresponding elliptic spectral curve

$$D = P_1 + P_2 + P_3 \to D' = P'_1 + P'_2 \to \rho(D) = P'^n$$

using the standard Abel reduction algorithm [1]. We prove that the reduced divisor is fixed, i.e., independent of time, when the dimension of the configuration space is more than the genus of the corresponding spectral curve.

The affine coordinates of the semi-reduced divisor $D'$ are functions on integrals of motion and one dynamical variable $X$, which is related to quadratures by Kobb [21] and Harlamova [20]. The evolution of this semi-reduced divisor and its relations with known quadratures will be discussed later.

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CONFLICT OF INTEREST

The author declares that he has no conflicts of interest.
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