CHEREDNIK ALGEBRAS FOR ALGEBRAIC CURVES

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Abstract. For any algebraic curve $C$ and $n \geq 1$, P. Etingof introduced a ‘global’ Cherednik algebra as a natural deformation of the cross product $D(C^n) \rtimes S_n$, of the algebra of differential operators on $C^n$ and the symmetric group. We provide a construction of the global Cherednik algebra in terms of quantum Hamiltonian reduction. We study a category of character $D$-modules on a representation scheme associated to $C$ and define a Hamiltonian reduction functor from that category to category $\mathcal{O}$ for the global Cherednik algebra.

In the special case of the curve $C = \mathbb{C}^\times$, the global Cherednik algebra reduces to the trigonometric Cherednik algebra of type $A_{n-1}$, and our character $D$-modules become holonomic $D$-modules on $GL_n(\mathbb{C}) \times \mathbb{C}^n$. The corresponding perverse sheaves are reminiscent of (and include as special cases) Lusztig’s character sheaves.

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1. Introduction

1.1. Associated with an integer $n \geq 1$ and an algebraic curve $C$, there is an interesting family, $H_{\kappa, \psi}$, of sheaves of associative algebras on $C^{(n)} = C^n/S_n$, the $n$-th symmetric power of $C$. The algebras in question, referred to as global Cherednik algebras, see [EG], are natural deformations of the cross-product $D_\psi(C^n) \rtimes S_n$, of the sheaf of (twisted) differential operators on $C^n$ and the symmetric group $S_n$ that acts on $D_\psi(C^n)$. The algebras $H_{\kappa, \psi}$ were introduced by P. Etingof, [E], as ‘global counterparts’ of rational Cherednik algebras studied in [GG].

The global Cherednik algebra $H_{\kappa, \psi}$ contains an important spherical subalgebra $eH_{\kappa, \psi}e$, where $e$ denotes the symmetriser idempotent in the group algebra of the group $S_n$, and prove that the algebra $eH_{\kappa, \psi}e$ may be obtained as a quantum Hamiltonian reduction of $D_{n \kappa, \psi}(\text{rep}_{C}^{n} \times \mathbb{P}^{n-1})$, a sheaf of twisted differential operators on $\text{rep}_{C}^{n} \times \mathbb{P}^{n-1}$, cf. Theorem 3.3.3.

Our result provides a strong link between categories of $D_{n \kappa, \psi}(\text{rep}_{C}^{n} \times \mathbb{P}^{n-1})$-modules and $H_{\kappa, \psi}$-modules, respectively. Specifically, following the strategy of [GG], §7, we construct an exact functor

$$\mathbb{H} : D_{n \kappa, \psi}(\text{rep}_{C}^{n} \times \mathbb{P}^{n-1})\text{-mod} \longrightarrow H_{\kappa, \psi}\text{-mod},$$

called the functor of Hamiltonian reduction.

1.2. In mid 80’s, G. Lusztig introduced an important notion of character sheaf on a reductive algebraic group $G$. In more detail, write $\mathfrak{g}$ for the Lie algebra of $G$ and use the Killing form to identify $\mathfrak{g}^* \cong \mathfrak{g}$. Let $\mathcal{N} \subset \mathfrak{g}^*$ be the image of the set of nilpotent elements in $\mathfrak{g}$, and let $G \times \mathcal{N} \subset G \times \mathfrak{g}^* = T^*G$ be the nil-cone in the total space of the cotangent bundle on $G$.

Recall further that, associated with any perverse sheaf $M$ on $G$, one has its characteristic variety $SS(M) \subset T^*G$. A character sheaf is, by definition, an $\text{Ad}G$-equivariant perverse sheaf
trigonometric Cherednik algebra with parameter $\kappa$ of the above parameters $D\kappa$ and $\psi$.

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1.4. Convention. The trigonometric Cherednik algebra depends on one complex parameter, to be denoted $\kappa \in \mathbb{C}$. Such an algebra corresponds, via the quantum Hamiltonian reduction construction, to a sheaf of twisted differential operators on $SL_n \times \mathbb{P}^{n-1}$, which is also labelled by one complex parameter, to be denoted $c \in \mathbb{C}$. Throughout the paper, we will use the normalization of the above parameters $\kappa$ and $c$, such that the sheaf of TDO with parameter $c$ gives rise to the trigonometric Cherednik algebra with parameter

$$\kappa = c/n.$$  

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2. A representation scheme

2.1. In this paper, we work over \( \mathbb{C} \) and write \( \text{Hom} = \text{Hom}_\mathbb{C} \), \( \text{End} = \text{End}_\mathbb{C} \), \( \otimes = \otimes_\mathbb{C} \).

Let \( Y \) be a scheme with structure sheaf \( \mathcal{O}_Y \) and coordinate ring \( \mathcal{O}(Y) \).

**Definition 2.1.1.** Let \( V \) be a finite dimensional vector space, \( \mathcal{F} \) a finite length (torsion) \( \mathcal{O}_Y \)-sheaf, and \( \vartheta : V \rightarrow \Gamma(Y, \mathcal{F}) \), a vector space isomorphism. Such a data \( (V, \mathcal{F}, \vartheta) \) is called a representation of \( \mathcal{O}_Y \) in \( V \).

**Proposition 2.1.2.** (i) For each \( n \geq 1 \), there is a \( GL_n(\mathbb{C}) \)-scheme \( \text{rep}^n_\mathbb{C} \), of finite type, that parametrizes representations of \( \mathcal{O}_Y \) in \( \mathbb{C}^n \).

(ii) If \( Y \) is affine then we have \( \text{rep}^n_\mathbb{C} = \text{rep}^n \mathcal{O}(Y) \), is the affine scheme that parametrizes algebra homomorphisms \( \mathcal{O}(Y) \rightarrow \text{End} \mathbb{C}^n \).

**Proof.** The scheme \( \text{rep}^n_\mathbb{C} \) may be identified with an open subscheme of Grothendieck’s Quot-scheme that parametrizes surjective morphisms \( \mathbb{C}^n \otimes \mathcal{O}_Y \rightarrow \mathcal{F} \) such that the composition \( \mathbb{C}^n \hookrightarrow \Gamma(Y, \mathbb{C}^n \otimes \mathcal{O}_Y) \rightarrow \Gamma(Y, \mathcal{F}) \) is an isomorphism. The group \( GL_n \) acts on \( \text{rep}^n_\mathbb{C} \) by base change transformations, that is, by changing the isomorphism \( \mathbb{C}^n \cong \Gamma(Y, \mathcal{F}) \).

Below, we fix \( n \geq 1 \), let \( V = \mathbb{C}^n \), and \( GL(V) = GL_n(\mathbb{C}) \). Given a representation \( (\mathcal{F}, \vartheta) \) in \( V \), the isomorphism \( \vartheta : V \rightarrow \Gamma(Y, \mathcal{F}) \) makes the vector space \( V \) an \( \mathcal{O}(Y) \)-module. We often drop \( \vartheta \) from the notation and write \( \mathcal{F} \in \text{rep}^n_\mathbb{C} \).

**Example 2.1.3.** For \( Y := \mathbb{A}^1 \), we have \( \mathcal{O}(Y) = \mathbb{C}[t] \), hence \( \text{rep}^n_\mathbb{C} = \text{rep}^n \mathbb{C}[t] = \text{End} V \).

For \( Y := \mathbb{C}^\times \), we have \( \mathcal{O}(Y) = \mathbb{C}[t, t^{-1}] \), hence \( \text{rep}^n_\mathbb{C} = \text{rep}^n \mathbb{C}[t, t^{-1}] = GL(V) \). In these examples, the action of the group \( GL(V) \) is the conjugation-action.

2.2. Factorization property. From now on, we will be concerned exclusively with the case where \( Y = C \) is a smooth algebraic curve (either affine or complete).

Let \( S_n \) denote the symmetric group. Write \( C^n \) and \( C^{(n)} = C^n / S_n \) for the \( n \)-th cartesian and symmetric power of \( C \), respectively. Taking support cycle of a length \( n \) coherent sheaf on \( C \) gives a natural map

\[
\varpi : \text{rep}^n_\mathbb{C} \rightarrow C^{(n)} , \quad \mathcal{F} \mapsto \text{Supp} \mathcal{F} .
\]

(2.2.1)

It is straightforward to see that the map \( \varpi \) is an affine and surjective morphism of schemes and that the group \( GL(V) \) acts along the fibers of \( \varpi \).

The collection of morphisms \( \varpi \) for various values of the integer \( n \) enjoy an important factorization property. Specifically, let \( k, m \), be a pair of positive integers and \( D_1 \in C^{(k)} \), \( D_2 \in C^{(m)} \), \( |D_1 \cap D_2| = \emptyset \), a pair of divisors with disjoint supports, that is, a pair of unordered collections of \( k \) and \( m \) points of \( C \), respectively, which have no points in common. The factorization property says that there is a natural isomorphism

\[
\varpi^{-1}(D_1 + D_2) \cong GL_{k+m} \times GL_{k+m} \to (\varpi^{-1}(D_1) \times \varpi^{-1}(D_2)).
\]

The factorization property also holds in families. To explain this, let \( C^{(k, m)} \subset C^{(k)} \times C^{(m)} \) be the open subset formed by all pairs of divisors with disjoint support. There is a natural composite projection \( C^{(k, m)} \rightarrow C^{(k)} \times C^{(m)} \rightarrow C^{(k+m)} \).

The family version of the factorization property reads

\[
\text{rep}^{k+m}_{C^{(k+m)}} \times C^{(k, m)} \cong \left( GL_{k+m} \times GL_{k+m} \to \left( \text{rep}^k_{C^{(k)}} \times \text{rep}^m_{C^{(m)}} \right) \right) \times C^{(k+m)} .
\]

(2.2.2)
2.3. Tangent and cotangent bundles. Let \( \Omega_C \) be the sheaf of Kähler differentials on \( C \). Given a finite length sheaf \( \mathcal{F} \) we let \( \mathcal{F}^\vee := \mathcal{F}^\dagger(\mathcal{F}, \Omega_C) \), denote the Grothendieck-Serre dual of \( \mathcal{F} \). Let \( \gamma : C \hookrightarrow C \) be an open imbedding of \( C \) into a complete curve. Then we have natural isomorphisms

\[
\Gamma(C, \mathcal{F}^\vee) = \text{Ext}^1(\mathcal{F}, \Omega_C) \cong \text{Hom}(\mathcal{O}_C, \mathcal{F})^* = \Gamma(C, \mathcal{F})^*,
\]

(2.3.1)

where we write \( \mathcal{F} = \gamma_\ast \mathcal{F} \) and where the isomorphism in the middle is provided by the Serre duality. In particular, for any \( \mathcal{F} \in \text{rep}_C^b \), one has canonical isomorphisms

\[
\Gamma(C^2, \mathcal{F} \boxtimes \mathcal{F}^\vee) = \Gamma(C, \mathcal{F}) \otimes \Gamma(C, \mathcal{F}^\vee) \xrightarrow{\partial V \otimes V^*} V \otimes V^* = \text{End} V.
\]

(2.3.2)

Let \( \Delta \subset C^2 := C \times C \) denote the diagonal divisor. Thus, on \( C^2 \), we have a triple of sheaves \( \mathcal{O}_{C^2}(-\Delta) \hookrightarrow \mathcal{O}_{C^2} \hookrightarrow \mathcal{O}_{C^2}(\Delta) \). For any \( \mathcal{F} \in \text{rep}_C^b \), let \( \text{Ker} \mathcal{F} \) denote the kernel of the composite morphism \( V \otimes \mathcal{O}_{C} \twoheadrightarrow \Gamma(C, \mathcal{F}) \otimes \mathcal{O}_{C} \twoheadrightarrow \mathcal{F} \).

Lemma 2.3.3. The scheme \( \text{rep}_C^b \) is smooth. For the tangent, resp. cotangent, space at a point \( \mathcal{F} \in \text{rep}_C^b \), there are canonical isomorphisms:

\[
\begin{align*}
T_{\mathcal{F}}(\text{rep}_C^b) & \cong \text{Hom}(\text{Ker} \mathcal{F}, \mathcal{F}) \cong \Gamma(C^2, (\mathcal{F} \boxtimes \mathcal{F}^\vee)(\Delta)), \quad \text{resp.} \quad T_{\mathcal{F}}^\vee(\text{rep}_C^b) \cong \text{Ext}^1(\mathcal{F}, \text{Ker} \mathcal{F} \otimes \mathcal{O}_C) \cong \Gamma(C^2, (\mathcal{F} \boxtimes \mathcal{F}^\vee)(-\Delta)).
\end{align*}
\]

Proof. It is well known that the Zariski tangent space to the scheme parametrizing surjective morphisms \( V \otimes \mathcal{O}_{C} \rightarrow \mathcal{F} \) is equal to the vector space \( \text{Hom}(\text{Ker} \mathcal{F}, \mathcal{F}) \). This proves the first isomorphism. From this, by Serre duality one obtains \( T_{\mathcal{F}}^\vee(\text{rep}_C^b) \cong \text{Ext}^1(\mathcal{F}, \text{Ker} \mathcal{F} \otimes \mathcal{O}_C) \).

To complete the proof, it suffices to prove the second formula for the tangent space \( T_{\mathcal{F}}(\text{rep}_C^b) \); the corresponding formula for \( T_{\mathcal{F}}^\vee(\text{rep}_C^b) \) would then follow by duality.

To prove the formula for \( T_{\mathcal{F}}(\text{rep}_C^b) \), we may assume that \( C \) is affine and put \( \mathcal{A} = \mathcal{O}(C) \). Also, let \( V = \Gamma(C, \mathcal{F}) \) and \( K = \Gamma(C, \text{Ker} \mathcal{F}) \). Thus, we have \( \text{Hom}(\text{Ker} \mathcal{F}, \mathcal{F}) = \text{Hom}_{\mathcal{A}}(K, V) \).

The sheaf extension \( \text{Ker} \mathcal{F} \hookrightarrow V \otimes \mathcal{O}_{C} \twoheadrightarrow \mathcal{F} \) yields a short exact sequence of \( \mathcal{A} \)-modules

\[
0 \rightarrow K \rightarrow A \otimes V \xrightarrow{\text{act}} V \rightarrow 0,
\]

where \( \text{act} \) is the action-map.

The action of \( A \) on \( V \) makes the vector space \( \text{End} V \) an \( A \)-bimodule. We write \( \text{Der}_{\mathcal{A}}(A, \text{End} V) \) for the space of derivations of the algebra \( A \) with coefficients in the \( A \)-bimodule \( \text{End} V \). Further, given an \( A \)-module map \( f : K \rightarrow V \), for any element \( a \in A \) we define a linear map

\[
\delta_f(a) : V \rightarrow V, \quad v \mapsto f(a \otimes v - 1 \otimes av).
\]

It is straightforward to see that the assignment \( a \mapsto \delta_f(a) \) gives a derivation \( \delta_f \in \text{Der}_{\mathcal{A}}(A, \text{End} V) \). Moreover, this way, one obtains a canonical isomorphism

\[
\text{Hom}_{\mathcal{A}}(K, V) \xrightarrow{\sim} \text{Der}_{\mathcal{A}}(A, \text{End} V), \quad f \mapsto \delta_f.
\]

The above isomorphism provides a well known alternative interpretation of the tangent space to a representation scheme in the form \( T_{\mathcal{F}}(\text{rep}_C^b) \cong \text{Der}_{\mathcal{A}}(A, \text{End} V) \).

We observe next that, for any \( a \in A \), one has \( a \otimes 1 - 1 \otimes a \in \Gamma(C^2, \mathcal{O}_{C^2}(-\Delta)) \subset A \otimes A \). Further, for any section \( s \in \Gamma(C^2, (\mathcal{F} \boxtimes \mathcal{F}^\vee)(\Delta)) \), the map

\[
A \rightarrow \Gamma(C^2, \mathcal{F} \boxtimes \mathcal{F}^\vee) = V^* \otimes V = \text{End} V, \quad a \mapsto \xi^s(a) := s \cdot (a \otimes 1 - 1 \otimes a)
\]

is easily seen to be a derivation. Moreover, one shows that any derivation has the form \( \xi^a \) for a unique section \( a \).

Thus, combining all the above, we deduce the desired canonical isomorphisms

\[
T_{\mathcal{F}}(\text{rep}_C^b) \cong \text{Hom}_{\mathcal{A}}(K, V) \cong \text{Der}_{\mathcal{A}}(A, \text{End} V) \cong \Gamma(C^2, (\mathcal{F} \boxtimes \mathcal{F}^\vee)(\Delta)).
\]

In particular, for any \( \mathcal{F} \in \text{rep}_C^b \), we find that

\[
\dim T_{\mathcal{F}}(\text{rep}_C^b) = \dim \Gamma(C^2, (\mathcal{F} \boxtimes \mathcal{F}^\vee)(\Delta)) = \dim(V \otimes V^*) = n^2,
\]

is independent of \( \mathcal{F} \). This implies that the scheme \( \text{rep}_C^b \) is smooth. \( \square \)
2.4. Example: formal disc. Given a formal power series \( f = f(t) \in \mathbb{C}[[t]] \), we define the difference-derivative of \( f \) as a power series \( Df(t', t'') := \frac{f(t') - f(t'')}{t' - t''} \in \mathbb{C}[[t', t'']] \).

For any \( X \in \text{End} V \), write \( L_X, R_X : \text{End} V \to \text{End} V \), for the pair of linear maps of left, resp. right, multiplication by \( X \) in the algebra \( \text{End} V \). These maps commute, hence, for any polynomial \( f \in \mathbb{C}[t] \), there is a well-defined linear operator \( Df(L_X, R_X) : \text{End} V \to \text{End} V \). For instance, in the special case where \( f(t) = t^m \), we find \( Df(L_X, R_X)(Y) = X^{m-1}Y + X^{m-2}YX + \ldots + XYX^{m-2} + YX^{m-1} \).

Let \( C_x \) be the completion of a smooth curve \( C \) at a point \( x \in C \), a formal scheme. A choice of local parameter on \( C_x \) amounts to a choice of algebra isomorphism \( \mathcal{O}(C_x) = \mathbb{C}[[t]] \). Thus, for the corresponding representation schemes, we obtain \( \text{rep}^n \mathcal{O}(C_x) \cong \text{rep}^n \mathbb{C}[[t]] = \hat{N} \), the completion of \( \text{End} V \) along the closed subscheme \( N \subset \text{End} V \), of nilpotent endomorphisms.

The tangent space at a point of \( \text{rep}^n \mathcal{O}(C_x) = \hat{N} \) may be therefore identified with the vector space \( \text{End} V \). More precisely, the tangent bundle \( T(\text{rep}^n \mathcal{O}(C_x)) \) is the completion of \( \text{End} V \times \text{End} V \) along \( \hat{N} \times \text{End} V \). Abusing the notation slightly, we may write a point in the tangent bundle as a pair \( (X, Y) \in \text{End} V \times \text{End} V \), where \( X \in \text{rep}^n \mathcal{O}(C_x) = \hat{N} \), and \( Y \) is a tangent vector at \( X \). Thus, associated with such a point \( (X, Y) \) and any formal power series \( f \in \mathbb{C}[[t]] \), there is a well-defined linear operator \( Df(L_X, R_X) \), acting on an appropriate completion of \( \text{End} V \).

Now, let \( t \) and \( u \) be two different local parameters on the formal scheme \( C_x \). Thus, one can write \( u = f(t) \), for some \( f \in \mathbb{C}[[t]] \). We have

**Lemma 2.4.1.** The differential of \( f \) acts on the tangent bundle \( T(\text{rep}^n \mathcal{O}(C_x)) \) by the formula

\[
df : (X, Y) \mapsto (f(X), Df(L_X, R_X)(Y)).
\]

**Remark 2.4.2.** In the special case where \( f(t) = e^t \), writing \( \Phi(t) := (e^t - 1)/t \), we compute

\[
\frac{f(t') - f(t'')}{t' - t''} = \frac{e^{t'} - e^{t''} - 1}{t' - t''} = e^{t''} \Phi(t' - t'') - e^{t''}.
\]

Therefore, for the differential of the exponential map \( X \mapsto \exp X \), using the lemma and the notation \( \text{ad} X(Y) := XY - YX = (L_X - R_X)(Y) \), one recovers the standard formula

\[
d\exp : (X, Y) \mapsto (\exp X, Df(L_X, R_X)(Y)) = (e^X, \text{ad} X(Y) \cdot e^X). \]

2.5. Étale structure. The assignment \( C \mapsto \text{rep}^n C \) is a functor. Specifically, for any morphism of curves \( p : C_1 \to C_2 \), the push-forward of coherent sheaves induces a \( \text{GL}_n \)-equivariant morphism \( \text{rep}^n p : \text{rep}^n C_1 \to \text{rep}^n C_2 \), \( F \mapsto p_* F \).

The following result says that, for any smooth curve \( C \), the representation scheme \( \text{rep}^n C \) is locally isomorphic, in étale topology, to \( \text{rep}^n \mathbb{A}^1 = \mathfrak{gl}_n \), the corresponding scheme for the curve \( C = \mathbb{A}^1 \).

**Proposition 2.5.1.** For any smooth curve \( C \) and \( n \geq 1 \), one can find a finite collection of open sub-curves \( C_s \subset C, s = 1, \ldots, r \), such that the following holds:

For each \( s = 1, \ldots, r \), there is a Zariski open subset \( U_s \subset \text{rep}^n C_s \), and a diagram of morphisms of curves

\[
C \leftarrow \text{rep} q_s : C_s \to \mathbb{A}^1, \quad s = 1, \ldots, r,
\]

that gives rise to a diagram of representation schemes

\[
\text{rep} C \leftarrow \text{rep} q_s : \text{rep} C_s \supset U_s \to \text{rep} p_s : \text{rep} \mathbb{A}^1 = \mathfrak{gl}_n.
\]

The above data satisfies, for each \( s = 1, \ldots, r \), the following properties:

- The map \( q_s \) is an open imbedding, and \( \text{rep} C \cong (\text{rep} q_1)(U_1) \cup \ldots \cup (\text{rep} q_r)(U_s) \) is an open cover;
- The map \( p_s \) is étale and the restriction of the morphism \( \text{rep} p_s \) to \( U_s \) is étale as well.

This proposition is an immediate consequence of two lemmas below. Fix a smooth curve \( C \).
Lemma 2.5.2. Given a collection of pairwise distinct points $c_1, \ldots, c_N \in C$ there is an open subset $C' \subset C$ and an étale morphism $p : C' \to \mathbb{A}^1$ such that $c_1, \ldots, c_N \in C'$ and all the values $p(c_1), \ldots, p(c_N)$ are pairwise distinct.

Proof. Pick a point $x \in C \setminus \{c_1, \ldots, c_N\}$. Then $C \setminus \{x\}$ is affine. Choose an embedding $\gamma : C \setminus \{x\} \hookrightarrow \mathbb{A}^m$. Then the composition of $\gamma$ with a general linear projection $\mathbb{A}^m \to \mathbb{A}^1$ is the desired $p$; while $C'$ is the complement of the set of ramification points of $p$ in $C \setminus \{x\}$. \hfill $\Box$

Given an étale morphism of curves $p : C \to \mathbb{A}^1$ define

$$0C^{(n)} := \{(c_1, \ldots, c_n) \in C^{(n)} \mid p(c_i) = p(c_j) \text{ iff } c_i = c_j, i, j = 1, \ldots, n\}.$$ 

This is clearly a Zariski open subset in $C^{(n)}$. Let $\omega^{-1}(0C^{(n)})$ be its preimage in $\text{rep}_{\mathbb{A}}^{[n]}$, cf. (2.2.1).

Lemma 2.5.3. The morphism below induced by the map $p : C \to \mathbb{A}^1$ is étale,

$$\text{rep } p : \omega^{-1}(0C^{(n)}) \to \text{rep}_{\mathbb{A}}^{[n]} = \mathfrak{gl}_n.$$ 

Proof. To simplify notation, write $U := \omega^{-1}(0C^{(n)})$ and $p_* := \text{rep } p$.

We must check that the differential of $p_*$ on the tangent spaces is invertible. Let $C = (k_1c_1 + \ldots + k_ic_i) \in 0C^{(n)}$ (so that $c_1, \ldots, c_i$ are all distinct). Then $p(c_1), \ldots, p(c_i)$ are all distinct. If $F \in U$, and $\omega F = C$, then $p_* F \in \text{rep}_{\mathbb{A}}^{[n]}$, and $\omega(p_* F) = (k_1p(c_1) + \ldots + kp(c_i))$. As we know, $T_F \text{rep}_{\mathbb{A}}^{[n]} = \text{Hom}(\text{Ker}_F, F)$.

Let $C_{c_i}$ (resp. $\mathbb{A}^{k(c_i)}_c$) stand for the completion of $C$ at $c_i$ (resp. $\mathbb{A}^1$ at $p(c_i)$). Let $(\text{Ker}_F)_{c_i}$ (resp. $(\mathcal{F})_{c_i}$) denote the restriction of $\text{Ker}_F$ (resp. $\mathcal{F}$) to $C_{c_i}$. Then the restriction to these completions induces an isomorphism

$$\text{Hom}(\text{Ker}_F, \mathcal{F}) = \bigoplus_{1 \leq i \leq l} \text{Hom}((\text{Ker}_{c_i} F), (\mathcal{F})_{c_i}).$$ 

Similarly, we have

$$\text{Hom}(\text{Ker}_{p_* F}, p_* F) = \bigoplus_{1 \leq i \leq l} \text{Hom}((\text{Ker}_{p_* c_i} F), (p_* F)_{p(c_i)}).$$ 

The map $p$ being étale at $c_i$, it identifies $C_{c_i}$ with $\mathbb{A}^{k(p(c_i))}_c$. Under this identification $(\mathcal{F})_{c_i}$ gets identified with $(p_* F)_{p(c_i)}$, and $(\text{Ker}_F)_{c_i}$ gets identified with $(\text{Ker}_{p_* F})_{p(c_i)}$. Observe that the latter identification is provided by the differential of the morphism $\text{rep } p$ at the point $F \in \text{rep}_{\mathbb{A}}^{[n]}$. The lemma follows. \hfill $\Box$

3. Cherednik algebras associated to algebraic curves

3.1. Global Cherednik algebras. For any smooth algebraic curve $C$, P. Etingof defined in [E], 2.19, a sheaf of Cherednik algebras on the symmetric power $C^{(n)}$, $n \geq 1$.

To recall Etingof’s definition, for any smooth variety $Y$, introduce a length two complex of sheaves $\Omega_{Y}^{1,2} := [\Omega_Y^1 \to (\Omega_Y^2)_{\text{closed}}]$, concentrated in degrees 1 and 2. The sheaves of algebraic twisted differential operators (TDO) on $Y$ are known to be parametrized (up to isomorphism) by elements of $H^2(Y, \Omega_{Y}^{1,2})$, the second hyper-cohomology group, cf. eg. [K].

Remark 3.1.1. For an affine curve $C$, we have $H^2(C, \Omega_{C}^{1,2}) = 0$; for a projective curve $C$, we have $H^2(C, \Omega_{C}^{1,2}) \cong C$, where a generator is provided by the first Chern class of a degree 1 line bundle on $C$.

Now, let $C$ be a smooth algebraic curve. Given a class $\psi \in H^2(C, \Omega_{C}^{1,2})$, write $\psi^\otimes n \in H^2(C^n, \Omega_{C^n}^{1,2})$ for the external product of $n$ copies of $\psi$. Pull-back via the projection $C^n \to C^{(n)}$ induces an isomorphism $H^2(C^{(n)}, \Omega_{C^{(n)}}^{1,2}) \cong H^2(C^n, \Omega_{C^n}^{1,2})^\otimes n \subset H^2(C^n, \Omega_{C^n}^{1,2})$. We let $\psi_n \in H^2(C^{(n)}, \Omega_{C^{(n)}}^{1,2})$ denote the preimage of the class $\psi^\otimes n$ via the above isomorphism.

For any $n \geq 1$, $\kappa \in \mathbb{C}$, and $\psi \in H^2(C, \Omega_{C}^{1,2})$, Etingof defines the sheaf of global Cherednik algebras, as follows, cf. [E], 2.9. Let $\eta$ be a 1-form on $C^n$ such that for the cohomology class $\eta \in H^2(C^n, \Omega_{C^n}^{1,2})$ one has $d\eta = \psi^\otimes n$. Further, for any $i, j \in [1, n]$, let $\Delta_{ij} \subset C^n$ be the corresponding
(ij)-diagonal, with equal i-th and j-th coordinates. Thus, $\Delta_n = \cup_{ij} \Delta_{ij}$, is the big diagonal, and the image of $\Delta_n$ under the projection $C^n \to C^{(n)}$ is the discriminant divisor, $D \subset C^{(n)}$.

Given a vector field $v$ on $C^n$, for each pair $(i,j)$, choose a rational function $f_{ij}$ on $C^n$ whose polar part at $\Delta_{ij}$ corresponds to $v$, as explained in loc. cit. 2.4. Associated to such a data, Etingof defines in [E], 2.9 the following Dunkl operator

$$D_v := \text{Lie}_v + (v, \eta) + \kappa \cdot n \cdot \sum_{i \neq j} (s_{ij} - 1) \otimes f_{ij}^n \in \mathcal{D}_\psi_n(C^n \setminus \Delta_n) \times \mathbb{S}_n.$$ 

Here $\text{Lie}_v$ stands for the Lie derivative with respect to the vector field $v$ and the cross-product algebra $\mathcal{D}_\psi_n(C^n \setminus \Delta_n) \times \mathbb{S}_n$ on the right is viewed as a sheaf of associative algebras on $C^{(n)} \setminus D$.

The sheaf of Cherednik algebras is defined as a subsheaf of the sheaf $j_* \mathcal{D}_\psi_n(C^n \setminus \Delta_n) \times \mathbb{S}_n$, where $j : C^{(n)} \setminus D \to C^{(n)}$ stands for the open imbedding.

Specifically, following Etingof, we have

**Definition 3.1.2.** Let $H_{\kappa, \psi, n}$, the sheaf of Cherednik algebras, be the sheaf, on $C^{(n)}$, of associative subalgebras, generated by all regular functions on $C^n$ and by the Dunkl operators $D_v$, for all vector fields $v$ and $f_{ij}$ as above.

Next, let $e \in \mathbb{C}[\mathbb{S}_n]$ be the idempotent projector to the trivial representation. The subalgebra $eH_{\kappa, \psi, n}e \subset H_{\kappa, \psi, n}$ is called spherical subalgebra. This is a sheaf of associative algebras on $C^{(n)}$ that may be identified naturally with a subsheaf of $\mathcal{D}_\psi_n(C^{(n)})$.

The following is a global analogue of a result due to Bezrukavnikov and Etingof [BE], and Gordon and Stafford [GS], Theorem 3.3, in the case where $C = \mathbb{A}^1$.

**Proposition 3.1.3.** For any $\kappa \in \mathbb{C} \setminus [-1, 0)$, the functor below is a Morita equivalence

$$H_{\kappa, \psi, n} \cdot \text{mod} \longrightarrow eH_{\kappa, \psi, n}e \cdot \text{mod}, \quad M \mapsto eM.$$

**Proof.** It is a well known fact that the Morita equivalence statement is equivalent to an equality $H_{\kappa, \psi, n} = H_{\kappa, \psi, n} \cdot e \cdot H_{\kappa, \psi, n}$. In any case, on $C^{(n)}$, one has an exact sequence of sheaves

$$0 \to H_{\kappa, \psi, n} \cdot e \cdot H_{\kappa, \psi, n} \to H_{\kappa, \psi, n} \to H_{\kappa, \psi, n} / H_{\kappa, \psi, n} \cdot e \cdot H_{\kappa, \psi, n} \to 0. \tag{3.1.4}$$

Proving that the sheaf on the right vanishes is a ‘local’ problem. Thus, one can restrict (3.1.3) to an open subset in $C^{(n)}$. Then, we are in a position to use the above cited result of Gordon and Stafford saying that Proposition 3.1.3 holds for the curve $C = \mathbb{A}^1$. In effect, by Proposition 2.6.1 each point in $C^{(n)}$ is contained in an open subset $U$ with the following property. There exists an étale morphism $f : U \to (\mathbb{A}^1)^{(n)}$ such that the pull-back via $f$ of the sheaf on the right of (3.1.3) for the curve $\mathbb{A}^1$ is equal to the corresponding sheaf for the curve $C$.

The sheaf $H_{\kappa, \psi, n}$ comes equipped with an increasing filtration arising from the standard filtration by the order of differential operator, cf. [EG], [E]. The filtration on $H_{\kappa, \psi, n}$ induces, by restriction, an increasing filtration, $F_*(eH_{\kappa, \psi, n}e)$, on the spherical subalgebra. Etingof proved a graded algebra isomorphism, cf. [E],

$$\text{gr}^F(eH_{\kappa, \psi, n}e) \cong p_* \mathcal{O}_Y, \quad Y := (T^*C)^{(n)} = (T^*(C^n)) / \mathbb{S}_n, \quad \tag{3.1.5}$$

where $p : (T^*C)^{(n)} \to C^{(n)}$ denotes the natural projection.

### 3.2. The determinant line bundle

We fix a curve $C$, an integer $n \geq 1$, and let $V = C^n$. Set $X_n := \text{rep}_C^n \times V$, a smooth variety. Let $X^\text{cyc}_n \subset X_n = \text{rep}_C^n \times V$ be a subset formed by the triples $(F, \vartheta, v)$ such that $v$ is a cyclic vector, i.e., such that the morphism $\mathcal{O}_C \to F$, $f \mapsto f v$ is surjective. It is clear that the set $X^\text{cyc}_n$ is a $GL(V)$-stable Zariski open subset of $X_n$.

Choose a basis of $V$, and identify $V = C^n$, and $GL(V) = GL_n$, etc. Assume further that our curve $C$ admits a global coordinate $t : C \hookrightarrow \mathbb{A}^1$. Then, associated with each pair $(F, \vartheta, v) \in X^\text{cyc}_n$, there is a matrix $g(F, \vartheta, v) \in GL_n$, whose $k$-th row is given by the $n$-tuple of coordinates of the vector $t^{k-1}(v) \in C^n = V = \Gamma(C, F)$, $k = 1, \ldots, n$. 

Lemma 3.2.1. (i) The map \( \varpi : X^\text{cyc}_n \to C(n) \), cf. [2.2.1], makes the scheme \( X^\text{cyc}_n \) a \( GL(V) \)-torsor over \( C(n) \).

(ii) Given a basis in \( V \) and a global coordinate \( t : C \hookrightarrow \mathbb{A}^1 \), the map

\[
g \times \varpi : X^\text{cyc}_n \cong GL_n \times C(n), \quad (F, \vartheta, v) \mapsto (g(F, \vartheta, v), \text{Supp} F),
\]

provides a \( GL(V) \)-equivariant trivialization of the \( GL(V) \)-torsor from (i).

Proof. The action of \( GL(V) \) on \( \text{rep}^V_n \times V \) is given by \( g(F, \vartheta, v) = (F, \vartheta \circ g^{-1}, gv) \).

To prove (i), we observe that the quotient \( GL(V) \backslash X^\text{cyc}_n \) is the moduli stack of quotient sheaves of length \( n \) of the structure sheaf \( \mathcal{O}_C \). However, this stack is just the Grothendieck Quot scheme \( \text{Quot}^n_{\mathcal{O}_C} \) isomorphic to \( C(n) \). Part (i) follows. Part (ii) is immediate.

On \( X_n = \text{rep}^V_n \times V \), we have a trivial line bundle \( \text{det} \) with fiber \( \wedge^n V \). This line bundle comes equipped with the natural \( GL(V) \)-equivariant structure given, for any \( a \in \wedge^n V \), by \( g(F, \vartheta, v; a) = (F, \vartheta \circ g^{-1}, gv; (\text{det} g) \cdot a) \). We define a determinant bundle to be the unique line bundle \( L \) on \( C(n) \) such that \( \varpi^* L \), the pull-back of \( L \) via the projection \( X^\text{cyc}_n \to C(n) \), is isomorphic to \( \text{det} |_{X^\text{cyc}_n} \).

The square \( L^2 \), of determinant bundle, has a canonical rational section \( \delta \), defined as follows. Let \( (c_1, \ldots, c_n) \) be pairwise distinct points of \( C \), and \( F = \mathcal{O}_C/\mathcal{O}_C(-c_1 - \cdots - c_n) = \mathcal{O}_{c_1} \oplus \cdots \oplus \mathcal{O}_{c_n} \), a point in \( X^\text{cyc}_n \). The fiber of \( L^2 \) at the point \( \varpi(F) \in C(n) \) is identified with the vector space \( L^2_{\varpi} = (\mathcal{O}_{c_1} \oplus \cdots \oplus \mathcal{O}_{c_n})^\otimes_2 \). We define \( \delta(c_1, \ldots, c_n) := (1_{c_1} \wedge \cdots \wedge 1_{c_n})^2 \), where 1_{c_i} \in \mathcal{O}_{c_i}, \) stands for the unit element.

We have the finite projection map \( p : C^n \to C(n) \), the (big) diagonal divisor, \( \Delta_n \subset C^n \), and the discriminant divisor, \( D \subset C(n) \). Observe that, set theoretically, we have \( |D| = p(|\Delta_n|) \); however, for divisors, one has an equation \( p^* D = 2 \Delta_n \).

Next, we put \( X^\text{reg}_n := X^\text{cyc}_n \cap \varpi^{-1}(C(n) \setminus D) \). Clearly, we have \( X^\text{reg}_n \subset X^\text{cyc}_n \subset X_n \). Given a global coordinate \( t : C \to \mathbb{A}^1 \), we write

\[
\pi(c) := \prod_{1 \leq i < j \leq n} (t(c_i) - t(c_j)), \quad \forall c = (c_1, \ldots, c_n) \in C^n.
\]

for the Vandermonde determinant. Thus, \( \pi^2(c) \) is a regular function on \( C(n) \).

Proposition 3.2.3. (i) For any smooth connected curve \( C \), we have

\[
X_n \setminus X^\text{reg}_n = \varpi^{-1}(D) \cup (X_n \setminus X^\text{cyc}_n),
\]

is a union of two irreducible divisors.

(ii) There exists a regular function \( f \in \mathcal{O}(X^\text{reg}_n) \) such that

- it has a zero of order 2 at the divisor \( X_n \setminus X^\text{cyc}_n \), and a pole of order 1 at the divisor \( \varpi^{-1}(D) \);
- it is a \( GL_n \)-semi-invariant, specifically, we have

\[
f(g \cdot x) = (\text{det} g)^2 \cdot f(x), \quad \forall g \in GL_n, \ x \in X^\text{reg}_n.
\]

(iii) The function \( f \) is defined uniquely up to multiplication by the pull back of an invertible function on \( C(n) \). Furthermore, in a trivialization as in Lemma 3.2.1(ii), one can put

\[
f(x) := (\text{det} g(x)) \cdot (\varpi(x))^2, \quad x \in X^\text{reg}_n.
\]

Proof. The uniqueness statement follows from the semi-invariant property of \( f \). To prove the existence, recall the definition of the line \( L \). By that definition, a section of \( L \) is the same thing as a semi-invariant section of the trivial bundle with fiber \( \wedge^n V \), on \( X^\text{cyc}_n \). We let \( f \) be the rational section of the latter bundle corresponding, this way, to the rational section \( \delta \) of \( L \). A choice of base in \( V \) gives a basis in \( \wedge^n V \), hence allows to view \( f \) as a semi-invariant rational function.

The order of zero of \( f \) at \( X_n \setminus X^\text{cyc}_n \) is a local question, so we may assume the existence of a local coordinate \( t \) on \( C \), trivializing our torsor as in [3.2.1(ii)]. Then the divisor \( X_n \setminus X^\text{cyc}_n \) is given by an equation \( \text{det}(g(x)) = 0 \). So we see that the zero of \( f \) at \( X_n \setminus X^\text{cyc}_n \) indeed has order 2. The statement that \( f \) has a pole of order 1 at \( \varpi^{-1}(D) \) immediately follows from the next lemma. In the presence of a local coordinate \( t \), the explicit formula for the function \( g \) then follows from the uniqueness statement. □
Lemma 3.2.4. There is a canonical isomorphism $L^{g \otimes 2} \simeq O_{C(n)}(D)$.

Proof. We have to construct an $S_n$-equivariant section of $p^*L^{g \otimes 2}$ which is regular nonvanishing on $C^n - \Delta_n$, and has a second order pole at (each component of) $\Delta_n$. Clearly, $\delta$ is an $S_n$-invariant regular nonvanishing section of $p^*L^{g \otimes 2}|_{C^n - \Delta_n}$. To compute the order of the pole of $\delta$ at the diagonal $c_1 = c_2$ it suffices to consider the case where $C = \mathbb{A}^1$, $n = 2$. In the latter case, a straightforward calculation shows that $\delta$ has a second order pole. \qed

3.3. Quantum Hamiltonian reduction. The goal of this section is to provide a construction of the sheaf of spherical Cherednik subalgebras in terms of quantum Hamiltonian reduction, in the spirit of [GG].

Let $c_1(L) \in H^2(C^n, \Omega_{C(n)}^{1,2})_{S_n}$ denote the image of the first Chern class of the determinant line bundle $L$. For a complex number $\kappa$, we will sometimes denote the class $\kappa \cdot c_1(L) \in H^2(C^n, \Omega_{C(n)}^{1,2})_{S_n}$ simply by $\kappa$, if there is no risk of confusion. Let $\psi \in H^2(C, \Omega^{1,2}_{C})$. Note that if $C$ is affine, then $\psi = 0$, and if $C$ is projective, then $\psi$ is proportional to the first Chern class $c_1(L)$ of a degree one line bundle $L$ on $C$, so that $\psi = k \cdot c_1(L)$. The line bundle $L^{g \otimes 2}$ on $C^n$ is $S_n$-equivariant. Let $L^{(n)}$ denote the subsheaf of $S_n$-invariants in $p_*L^{g \otimes 2}$, the direct image sheaf on $C^{(n)}$.

We set $\psi_n = k \cdot c_1(L^{(n)}) \in H^2(C^{(n)}, \Omega_{C(n)}^{1,2})$, and let $\omega^*(\psi_n) \in H^2(C^{(n)}, \Omega_{C(n)}^{1,2})$, be its pull-back via the support-morphism (2.2.1), cf. [BB], 2.9. Let $D_\psi(X_n)$ denote the sheaf on $X_n = \text{rep}_C^\circ \times V$, of twisted differential operators associated with the class $\omega^*(\psi_n)$.

We have a $GL_n$-action along the fibers of the support-morphism. Therefore, the TDO associated with a pull-back class comes equipped with a natural $GL_n$-equivariant structure. More precisely, [BB], Lemma 1.8.7, implies that the pair $(D_\psi(X_n), GL_n)$ has the canonical structure of a Harish-Chandra algebra on $X_n$, in the sense of [BB], §1.8.3. It follows, in particular, that there is a natural Lie algebra morphism $gl_n \to D_\psi(X_n)$, $u \mapsto \tilde{u}$, compatible with the $GL_n$-action on $X_n$.

For any $\kappa \in C$, the assignment $u \mapsto \tilde{u} - \kappa \cdot tr(u) \cdot 1$ gives another Lie algebra morphism $gl_n \to D_\psi(X_n)$. Let $g_\kappa \subset D_\psi(X_n)$ denote the image of the latter morphism. Thus, $g_\kappa$ is a Lie subalgebra of first order twisted differential operators, and we write $D_\psi(X_n)g_\kappa$ for the left ideal in $D_\psi(X_n)$ generated by the vector space $g_\kappa$.

Let $D_\psi(X_n^{cyc})$ be the restriction of the TDO $D_\psi(X_n)$ to the open subset $X_n^{cyc} \subset X_n$. Let $\omega_\kappa D_\psi(X_n^{cyc})$ denote the sheaf-theoretic direct image, a sheaf of filtered associative algebras on $C^{(n)}$, equipped with a $GL_n$-action. We have the left ideal $D_\psi(X_n^{cyc})g_\kappa$, in $D_\psi(X_n^{cyc})$, and the corresponding $GL_n$-stable left ideal $\omega_\kappa D_\psi(X_n^{cyc})g_\kappa \subset \omega_\kappa D_\psi(X_n^{cyc})$.

The TDO $D_\psi(X_n)$ may be identified with a subsheaf of the direct image of $D_\psi(X_n^{cyc})$ under the open embedding $X_n^{cyc} \hookrightarrow X_n$. This way, one obtains a restriction morphism

$$r : \omega_\kappa D_\psi(X_n)/\omega_\kappa D_\psi(X_n^{cyc})g_\kappa \to \omega_\kappa D_\psi(X_n^{cyc})/\omega_\kappa D_\psi(X_n^{cyc})g_\kappa$$

We are now going to generalize [GG] formula (6.15), and construct, for any $\kappa \in C$, $\psi \in H^2(C^n, \Omega_{C(n)}^{1,2})$, a canonical ‘radial part’ isomorphism

$$\text{rad} : (\omega_\kappa D_\psi(X_n^{reg})/\omega_\kappa D_\psi(X_n^{reg})g_\kappa)^{GL_n} \simeq D_\psi(C^{(n)} \smallsetminus D), \quad u \mapsto \tilde{u}_\kappa, \quad (3.3.1)$$

of sheaves of filtered associative algebras on $X_n^{reg}/GL_n = C^{(n)} \smallsetminus D$.

To this end, assume first that the class $\psi = c_1(L)$, is the first Chern class of a line bundle $L$ on $C$. Then, we have $\psi_n = c_1(L^{(n)})$. Further, we have the pull-back $\omega^*\cdot L^{(n)}$, a line bundle on $X_n^{cyc}$ equipped with a natural $GL_n$-equivariant structure. Write $D(C^{(n)}, L^{(n)})$, resp. $D(X_n^{cyc}, \omega^*\cdot L^{(n)})$ for the sheaf of TDO on $C^{(n)}$, acting in the line bundle $L^{(n)}$, resp. TDO on $X_n^{cyc}$, acting in the line bundle $\omega^*\cdot L^{(n)}$.

It is clear that, on $X_n^{reg}$, one has a sheaf isomorphism

$$L^{(n)} \simeq \tilde{L}^{(n)}, \quad s \mapsto f \cdot \omega^*(s), \text{ where } \tilde{L}^{(n)} := \{ \tilde{s} \in \omega_\kappa \omega^*\cdot L^{(n)} \mid g(\tilde{s}) = (\det g) \cdot \tilde{s}, \forall g \in GL_n \}$$. 

Observe further that, for any $GL_n$-invariant twisted differential operator $u \in \mathcal{D}(X_{\text{reg}}, \varpi^* \mathcal{L}(n))$, we have $u(\mathcal{D}(n)) \subset \mathcal{L}(n)$. We deduce that, for any $\kappa \in \mathbb{C}$, and any $GL_n$-invariant twisted differential operator $u \in \mathcal{D}(X_{\text{reg}}, \varpi^* \mathcal{L}(n))$ the assignment

$$
\hat{u}_\kappa : \mathcal{L}(n) \to \mathcal{L}(n), \quad s \mapsto f^{-\kappa} \cdot u(f^\kappa \cdot s),
$$

is well-defined and, moreover, it is given by a uniquely determined twisted differential operator $\hat{u}_\kappa \in \mathcal{D}_{\psi_n}(C(n) \setminus D)$.

The above construction of the map $u \mapsto \hat{u}_\kappa$ may be adapted to cover the general case, where the class $\psi$ is not necessarily of the form $c_1(\mathcal{L})$ for a line bundle $\mathcal{L}$. The map $u \mapsto \hat{u}_\kappa$ is, by definition, the radial part homomorphism $\hat{\kappa}$ that appears in (3.3.1).

The group $S_n$ acts freely on $C_n \setminus \Delta_n$, and we have $(C_n \setminus \Delta_n)/S_n = C(n) \setminus D$. Hence, $\mathcal{D}_{\psi_n}(C(n) \setminus D) \simeq \mathcal{D}_{\psi_{S_n}}(C_n \setminus \Delta_n)^{\mathbb{C}_n}$. The lift of the determinant line bundle $\mathcal{L}$ to $C_n$ has a canonical rational section $\delta^{1/2}$. The restriction of this section to $C_n \setminus \Delta_n$ is invertible, and we let $\text{twist} : \mathcal{D}_{\psi_n}(C(n) \setminus D) \to \mathcal{D}_{\psi_{n+1}}(C(n) \setminus D)$ be an isomorphism of TDO induced via conjugation by $\delta^{1/2}$.

Combining all the above, we obtain the following morphism of sheaves of filtered algebras on $C(n)$, where $j$ stands for the open embedding $C(n) \setminus D \hookrightarrow C(n)$:

$$
\text{twist} \circ \text{rad} \circ r : \left( \varpi, \mathcal{D}_{\psi_n}(X_n) / \varpi, \mathcal{D}_{\psi_n}(X_n)_\mathfrak{g}_n \right)^{GL_n} \to j_\ast \mathcal{D}_{\psi_{n+1}}(C(n) \setminus D).
$$

Here, the algebra on the left is the quantum Hamiltonian reduction of $\mathcal{D}_{\psi}(X_{\text{reg}})$ at the point $\kappa \cdot r \in \mathfrak{g}_n^\ast$.

Our main result about quantum Hamiltonian reduction reads

**Theorem 3.3.3.** The image of the composite morphism in (3.3.2) is equal to the spherical Cherednik subalgebra. Moreover, this composite yields a filtered algebra isomorphism

$$
(\varpi, \mathcal{D}_{\psi_n}(X_n) / \varpi, \mathcal{D}_{\psi_n}(X_n)_\mathfrak{g}_n)^{GL_n} \simeq \mathcal{E}_{H_{n,\psi,n+1}},
$$

as well as the associated graded algebra isomorphism

$$
\text{gr}(\varpi, \mathcal{D}_{\psi_n}(X_n) / \varpi, \mathcal{D}_{\psi_n}(X_n)_\mathfrak{g}_n)^{GL_n} \simeq \text{gr}(\mathcal{E}_{H_{n,\psi,n+1}}).
$$

### 3.4. Proof of Theorem 3.3.3

It will be convenient to introduce the following simplified notation. Let $\mathcal{H}_{n,\psi} = j_\ast \mathcal{D}_{\psi_{n+1}}(C(n) \setminus D)$ be the image of the composite morphism in (3.3.2) (note the shift $\psi_n + 1$ on the right); also, for the spherical subalgebra, write

$$
\mathcal{A}_{n,\kappa,\psi} := \mathcal{E}_{H_{n,\psi,n+1}}.
$$

We will use the factorization isomorphism (2.2.2) for $k + m = n$. Write $j : \mathcal{C}_{C(k,m)} \hookrightarrow C(k,m) := C(k) \times C(m)$ for the open imbedding and $pr : \mathcal{C}_{C(k,m)} \to C(k) \to C(n)$ for the composite projection.

A natural factorization property for the determinant bundle gives rise to a canonical isomorphism of sheaves of TDO,

$$
pr^\ast \mathcal{D}_{\psi_{n+1}}(C(n)) \simeq j^\ast (\mathcal{D}_{\psi_{k+1}}(C(k)) \boxtimes \mathcal{D}_{\psi_{m+1}}(C(m))).
$$

**Lemma 3.4.2.** The above isomorphism restricts to the isomorphism of subalgebras $pr^\ast \mathcal{H}_{n,\kappa,\psi} \simeq j^\ast (\mathcal{H}_{k,\kappa,\psi} \boxtimes \mathcal{H}_{m,\psi,n})$.

**Proof.** Let $(\mathcal{F}_1, v_1) \in X_k$ and $(\mathcal{F}_2, v_2) \in X_m$ be such that the sheaves $\mathcal{F}_1$ and $\mathcal{F}_2$ have disjoint supports. Then, the stabilizer in $GL_n$ of the point $(\mathcal{F}_1 \oplus \mathcal{F}_2, v_1 \oplus v_2) \in X_n$, $n = k + m$, is equal to the product of the stabilizers of $(\mathcal{F}_1, v_1)$ and $(\mathcal{F}_2, v_2)$ in $GL_k$ and $GL_m$, respectively. This yields the following factorization isomorphism, cf. (2.2.2),

$$
\left(GL_{k+m}^{GL_k \times GL_m}(X_k \times X_m) \right)^{C(k) \times C(m)} \simeq X_k \times X_n \times C(m) \quad \text{C}(k,m),
$$

$$
(g, \mathcal{F}_1, v_1, \mathcal{F}_2, v_2) \mapsto (\mathcal{F}_1 \oplus \mathcal{F}_2, (v_1 \oplus v_2) \circ g^{-1}, g(v_1 \oplus v_2)).
$$
Now, the desired isomorphism of subalgebras in the statement of the lemma is a particular case of the following situation. We have a subgroup $G' \subset G''$ (in our case $G' = GL_k \times GL_m$, $G'' = GL_n$), and a $G'$-variety $X'$ with a TDO $\mathcal{D}'$ equipped with an action of $G'$ (in our case $X' = (X'_{k, m} \times C'(m, n)$, and $\mathcal{D}' = J^*(\mathcal{D}_\psi \otimes \mathcal{D}_\varphi)$). We have a $G''$-invariant linear functional $\chi''$ on the Lie algebra $\mathfrak{g}''$ whose restriction to $\mathfrak{g}'$ is denoted by $\chi'$ (in our case $\chi'' = (k - 1) \cdot \text{tr}$).

We set $X'' = G'' X''$, it is equipped with the TDO $\mathcal{D}''$ lifted from $X'$, acted upon by $G''$ (in our case $X'' = X_n \times C(n)$, and $\mathcal{D}'' = \text{pr}^* \mathcal{D}_\psi$). Then, it is easy to verify that one has $(\mathcal{D}' / \mathcal{D}''_{\chi''})^{G''} \simeq (\mathcal{D}' / \mathcal{D}''_{\chi'})^{G''}$.

\[\text{Lemma 3.4.3.} \quad \text{Let } U \subset C^{(n)} \text{ be an open subset and let } D_1, D_2 \in \Gamma(U, H_{n, \kappa, \psi}) \subset \mathcal{D}_{\psi+1}(U), \text{ resp. } D_2 \in \Gamma(U, A_{n, \kappa, \psi}), \text{ be a pair of second order twisted differential operator on } U \text{ with equal principal symbols.}

Then, the difference $D_1 - D_2$ is a zero order differential operator, more precisely, it is the operator of multiplication by a regular function on $U$.

\[\text{Proof.} \quad \text{It is enough to prove the statement in the formal neighbourhood of each point } \mathfrak{c} = (k_1 c_1 + \ldots + k_l c_l) \in U \subset C^{(n)}, \text{ where } c_1, \ldots, c_l \text{ are pairwise distinct and } k_1 + \ldots + k_l = n. \text{ By induction in } n, \text{ we are reduced to the diagonal case } \mathfrak{c} = (nc). \text{ Then it suffices to take the formal disk around } c \text{ for } C. \text{ In the latter case, our claim follows from the explicit calculation in } \text{GGS, §5, and the proof of Proposition 2.18 in } \text{ES}. \]

\[\text{Proof of Theorem 3.3.3.} \quad \text{We equip the sheaf } H_{n, \kappa, \psi}, \text{ defined above (3.1.5), with the standard increasing filtration induced by the standard increasing filtration on } \mathcal{D}_\psi(X_n), \text{ by the order of differential operator. Let } \text{gr } H_{n, \kappa, \psi} \text{ denote the associated graded sheaf. We also have the increasing filtration on the algebra } A_{n, \kappa, \psi}, \text{ cf. (3.1.4). Further, Lemma 3.4.3 yields } F_2 A_{n, \kappa, \psi} = F_2 H_{n, \kappa, \psi}. \]

Next, we observe that the spherical Cherednik subalgebra $A_{n, \kappa, \psi}$ is generated by the subsheaf $F_2 A_{n, \kappa, \psi}$. Indeed, it is enough to prove the equality of $A_{n, \kappa, \psi}$ with the subsheaf generated by $F_2 A_{n, \kappa, \psi}$ in the formal neighbourhood of any point $\mathfrak{c} \in C^{(n)}$. Arguing by induction in $n$ as in the proof of the lemma, it suffices to consider the case $\mathfrak{c} = (nc)$. Then we take to the formal disk around $c$ for $C$. In such a case, the desired statement is proved in Section 10 of \text{EG}.

Thus, we have filtered algebra morphisms

$$A_{n, \kappa, \psi} \hookrightarrow H_{n, \kappa, \psi} \twoheadrightarrow (\text{gr } \mathcal{D}_\psi(X_n) / \mathcal{D}_\psi(X_n) \mathfrak{g}_\kappa)^{GL_n}. \]

To prove that these morphisms are isomorphisms, it suffices to show that the induced morphisms of associated graded algebras,

$$\text{gr } A_{n, \kappa, \psi} \to \text{gr } H_{n, \kappa, \psi} \twoheadrightarrow (\text{gr } \mathcal{D}_\psi(X_n) / \mathcal{D}_\psi(X_n) \mathfrak{g}_\kappa)^{GL_n}, \]

are isomorphisms. Reasoning as above, we may further reduce the proof of this last statement to the case where $C$ is the formal disk around $c$. The latter case follows from \text{GG}, page 40. The theorem is proved.

4. Character sheaves

4.1. The moment map. Fix a smooth curve $C$ and let $\Delta \hookrightarrow C^2 = C \times C$ be the diagonal.

Recall the smooth scheme $X_n = \text{rep}_C^n \times V$, and let $T^* X_n = (T^* \text{rep}_C^n) \times V \times V^*$ denote the total space of the cotangent bundle on $X_n$. We will write a point of $T^* X_n$ as a quadruple

$$(F, y, i, j) \quad \text{where } F \in \text{rep}_C^n, \ y \in \Gamma(C^2, (F \boxtimes F^\vee)(-\Delta)), \ i \in V, \ j \in V^*. \]

The group $GL(V)$ acts diagonally on $\text{rep}_C^n \times V$. This gives a Hamiltonian $GL(V)$-action on $T^* X_n$, with moment map $\mu$. Given $F \in \text{rep}_C^n$, we will also use the map

$$\nu_F : \Gamma(C^2, (F \boxtimes F^\vee)(-\Delta)) \hookrightarrow \Gamma(C^2, F \boxtimes F^\vee) \cong \text{End } V, \quad \text{(4.1.1)} \]

induced by the sheaf embedding $\mathcal{O}_{C^2}(\Delta) \hookrightarrow \mathcal{O}_{C^2}$, cf. (3.3.2).
Lemma 4.1.2. The moment map \( \mu \) is given by the formula, cf. (4.1.4),
\[
\mu : (T^*\text{rep}_C^n) \times V \times V^* \to \text{End} V = \text{Lie GL}(V), \quad (F, y, i, j) \mapsto \nu_F(y) + i \otimes j.
\]

We leave the proof to the reader.

Example 4.1.3. In the special case \( C = \mathbb{A}^1 \), we have \( \text{rep}_C^n = \text{End} V \cong \mathfrak{gl}_n \). In this case, the moment map reads, see [GG],
\[
\mu : \mathfrak{gl}_n \times \mathfrak{gl}_n \times V \times V^* \to \mathfrak{gl}_n, \quad (x, y, i, j) \mapsto [x, y] + i \otimes j.
\]

Similarly, in the case \( C = \mathbb{C}^* \), the moment map reads
\[
\mu : \text{GL}_n \times \mathfrak{gl}_n \times V \times V^* \to \mathfrak{gl}_n, \quad (x, y, i, j) \mapsto xyx^{-1} - y + i \otimes j.
\]

4.2. A categorical quotient. The goal of this subsection is to construct an isomorphism \( (T^*C)^{(n)} \cong \mu^{-1}(0)//\text{GL}_n \) (the categorical quotient). To stress the dependence on \( n \) we will sometimes write \( \mu_n \) for the moment map \( \mu \).

Note that we have the direct sum morphism
\[
X_k \times X_m \to X_{k+m}, \quad (F, \vartheta_1, \vartheta_2, v_1, v_2) \mapsto (F \oplus G, \vartheta_1 \oplus \vartheta_2, v_1 \oplus v_2).
\]

This morphism induces a similar direct sum morphism \( T^*X_k \times T^*X_m \to T^*X_{k+m} \).

\[
\Gamma(F \boxtimes F'(-\Delta)) \times \Gamma(G \boxtimes G'(-\Delta)) \ni (y_1, y_2) \mapsto y_1 \oplus y_2 \in \Gamma(F \oplus G) \boxtimes (F \oplus G') \boxtimes (-\Delta)),
\]

where we have used simplified notation \( \Gamma(-) = \Gamma(C^2, -) \).

Clearly, we have \( T^*X_1 = T^*C \times \mathbb{A}^1 \times (\mathbb{A}^1)^* \). Iterating the direct sum morphism \( n \) times, we obtain a morphism \( (T^*C \times \mathbb{A}^1 \times (\mathbb{A}^1)^n) \to T^*X_n \).

Restricting this map further to the product of \( n \) copies of the subset \( T^*C \times \{1\} \times \{0\} \subset T^*C \times \mathbb{A}^1 \times (\mathbb{A}^1)^* \), we obtain a morphism \( (T^*C)^n \to T^*X_n \). The image of the latter morphism is clearly contained in \( \mu^{-1}_n(0) \). It is also clear that the composite morphism \( (T^*C)^n \to T^*X_n \to T^*X_n//\text{GL}_n \) factors through the symmetrization projection \( (T^*C)^n \to (T^*C)^{(n)} \).

This way, we have constructed a morphism \( \varepsilon : (T^*C)^{(n)} \to \mu_n^{-1}(0)//\text{GL}_n \).

Lemma 4.2.1. \( \varepsilon : (T^*C)^{(n)} \to \mu_n^{-1}(0)//\text{GL}_n \) is an isomorphism.

Proof. Use the factorization and the “local” result for \( C = \mathbb{A}^1 \) proved in [GG] 2.8. □

Notation 4.2.2. We let \( \pi_n \) denote the natural projection \( \mu_n^{-1}(0) \to \mu_n^{-1}(0)//\text{GL}_n \cong (T^*C)^{(n)} \). If the value of \( n \) is clear from the context, we will simply write \( \pi : \mu^{-1}(0) \to (T^*C)^{(n)} \).

4.3. Flags. Fix a smooth curve \( C \) and a sheaf \( F \in \text{rep}_C^n \). Let \( F_* : 0 = F_0 \subset F_1 \subset \ldots \subset F_n = F \) be a complete flag of subsheaves, length\( F_r = r \). Thus, for each \( r = 1, 2, \ldots, n \), we have a sheaf imbedding \( i_r : F_r \to F \), and also the dual projection \( i_r^* : F^\vee \to F_r^\vee \).

Thus, for the spaces of global sections of sheaves on \( C^2 \), we have a diagram
\[
\Gamma((F_{r-1} \boxtimes F_r^\vee)(-\Delta)) \xrightarrow{i_r^{-1}} \Gamma((F \boxtimes F_r^\vee)(-\Delta)) \xrightarrow{i_r^*} \Gamma((F \boxtimes F^\vee)(-\Delta)).
\]

Definition 4.3.1. (a) Let \( \widetilde{\text{rep}}_C^\nu \) be the moduli scheme of pairs \( (F, F_*) \), where \( F \in \text{rep}_C^n \), and \( F_* \) is a flag of subsheaves.

(b) Fix \( F \in \text{rep}_C^n \), and an element \( y \in \Gamma(C^2, (F \boxtimes F^\vee)(-\Delta)) \). A flag \( F_* \) is said to be a nil-flag for the pair \( (F, y) \) if we have
\[
i_r^*(y) \in i_r(\Gamma(C^2, (F_{r-1} \boxtimes F_r^\vee)(-\Delta))), \quad \forall r = 1, 2, \ldots, n.
\]

We have a natural forgetful morphism \( \widetilde{\text{rep}}_C^\nu \to \text{rep}_C^n \), a \( GL(V) \)-equivariant proper morphism which is an analogue of Grothendieck simultaneous resolution, cf. [La] 3.2.

We further extend the above morphism to a map \( \phi : \text{rep}_C^n \times V \to X_n = \text{rep}_C^n \times V \), identical on the second factor \( V \). For the corresponding cotangent bundles, one obtains a standard diagram
\[
\begin{array}{ccc}
T^*X_n & \xrightarrow{p_1} & \phi^*(T^*X_n) \\
& \xrightarrow{p_2} & T^*(\widetilde{\text{rep}}_C^\nu \times V),
\end{array}
\]
where the map \( p_2 \) is a natural closed imbedding, and \( p_1 \) is a proper morphism. Thus, \( \phi^*(T^*X_n) \) is the smooth variety parametrizing quintuples \( (\mathcal{F}, \mathcal{F}_*, y, i, j) \).

Let \( Z \subset T^*(\text{rep}^n_{\mathbb{A}^1} \times V) \) be a closed subscheme formed by the quintuples \( (\mathcal{F}, \mathcal{F}_*, y, i, j) \) such that \( \mathcal{F}_* \) is a nil-flag for \( y \). Now, the set \( p_1^{-1}(\mu^{-1}(0)) \) is clearly a closed algebraic subvariety in \( \phi^*T^*(\text{rep}^n_{\mathbb{A}^1} \times V) \), and so is \( p_1^{-1}(\mu^{-1}(0)) \cap p_2^{-1}(Z) \). The map \( p_1 : p_1^{-1}(\mu^{-1}(0)) \cap p_2^{-1}(Z) \to T^*X_n \) is a proper projection.

Observe next that the zero section embedding \( C \hookrightarrow T^*C \) induces an embedding \( C^{(n)} \hookrightarrow (T^*C)^{(n)} \), and recall the map \( \pi \) from Notation 4.2.2.

**Proposition 4.3.2.** For a quadruple \( (\mathcal{F}, y, i, j) \in \mu^{-1}(0) \) the following are equivalent
- The pair \( (\mathcal{F}, y) \) has a nil-flag;
- \( (\mathcal{F}, y, i, j) \) belongs to the image of the projection \( p_1 : p_1^{-1}(\mu^{-1}(0)) \cap p_2^{-1}(Z) \to T^*X_n \).
- \( \pi(\mathcal{F}, y, i, j) \in C^{(n)} \subset (T^*C)^{(n)} \).

This proposition will be proved in the next section.

### 4.4. A Lagrangian subvariety

We define a nil-cone \( \mathbb{M}_{\text{nil}}(C) \subset T^*X_n \) to be the set of points satisfying the equivalent conditions of Proposition 4.3.2 with the natural structure of a reduced closed subscheme of \( T^*X_n \) arising from the third condition of the proposition.

In the special case of the curve \( C = \mathbb{A}^1 \), we have \( \text{rep}^n_{\mathbb{A}^1} = \text{End} V \cong \mathfrak{gl}_n \), and the corresponding variety \( \mathbb{M}_{\text{nil}}(C) \) is nothing but the Lagrangian subvariety introduced in [GG], (1.3). In more detail, write \( N \subset \mathfrak{gl}_n \) for the nilpotent variety. Then, one has

**Lemma 4.4.1.** (i) Let \( C = \mathbb{A}^1 \). Then, we have
\[
\mathbb{M}_{\text{nil}}(\mathbb{A}^1) = \{(x, y, i, j) \in \mathfrak{gl}_n \times \mathfrak{gl}_n \times V \times V^* \mid [x, y] + i \otimes j = 0 \ & \ y \in N\}. \tag{4.4.2}
\]

(ii) Similarly, in the case \( C = \mathbb{C}^\times \), we have
\[
\mathbb{M}_{\text{nil}}(\mathbb{C}^\times) \simeq \{(x, y, i, j) \in GL_n \times \mathfrak{gl}_n \times V \times V^* \mid xyx^{-1} = y + i \otimes j = 0 \ & \ y \in N\}.
\]

**Proof.** In the case \( C = \mathbb{A}^1 \), the morphism \( \phi : \text{rep}^n_{\mathbb{A}^1} \to \text{rep}^n_{\mathbb{A}^1} = \mathfrak{gl}_n \) becomes the Grothendieck simultaneous resolution, cf. [La1] (3.2). In this case Proposition 4.3.2 is known. Hence we only have to check that for any quadruple \((x, y, i, j)\) as in (4.4.2) there exists a complete flag \( V \) such that \( xV_k \subset V_k \) and \( yV_k \subset V_k \), for any \( k = 1, \ldots, n \). But this is also well known, see eg. [EC], Lemma 12.7. Part (i) follows. Part (ii) is proved similarly. \( \square \)

**Proposition 4.4.3.** \( \mathbb{M}_{\text{nil}}(C) \) is the reduced scheme of a Lagrangian locally complete intersection subscheme in \( T^*(\text{rep}^n_{\mathbb{A}^1} \times V) \).

We recall that it has been proved in [GG], Theorem 1.2, that the right hand side in (4.4.2) is the reduced scheme of a lagrangian complete intersection in \( \mathfrak{gl}_n \times \mathfrak{gl}_n \times V \times V^* \).

**Proof of Proposition 4.4.3.** We are going to reduce the statement of the proposition to the above mentioned special case of \( C = \mathbb{A}^1 \), see (4.4.2).

To this end, fix a curve \( C' \). Proposition 2.5.1 implies that there exists a finite collection of Zariski open subsets \( C \subset C' \) and, for each subset \( C \), an étale morphism \( p : C \to \mathbb{A}^1 \) and a Zariski open subset \( U \subset \text{rep}^n_{\mathbb{A}^1} \times V \), such that the following holds:
- The scheme \( \text{rep}^n_{\mathbb{A}^1} \times V \) is covered by the images in \( \text{rep}^n_{\mathbb{A}^1} \) of the subsets \( U \), corresponding to the curves \( C \) from our collection;
- For each \( C \), there is an étale morphism of curves \( p : C \to \mathbb{A}^1 \) such that the induced morphism below is also étale,
\[
\text{rep} p \times \text{Id}_V : \text{rep}^n_{\mathbb{A}^1} \times V \to X := \text{rep}^n_{\mathbb{A}^1} \times V = \mathfrak{gl}(V) \times V.
\]
Then, we have $T^*U = U \times_X T^*X$, hence, the map $\text{rep}_p \times \text{Id}_V$ induces a $GL(V)$-equivariant étale morphism $p_k : T^*U \to T^*X$. The moment map $T^*U \to \text{End } V$ factors through $T^*U \to T^*X \to \text{End } V$, and $M_{\text{nil}}(C) \cap T^*U$ is the preimage of $M_{\text{nil}}(A_1)$ in $T^*U$. Thus, $M_{\text{nil}}(C) \cap T^*U$ is the reduced scheme of a lagrangian locally complete intersection in $T^*U$. The proposition follows.

\[ \square \]

4.5. Projectivization. One can perform the quantum hamiltonian reduction of Theorem 3.3.3 in two steps: first with respect to the central subgroup $\mathbb{C}^* \subset GL_n$, of scalar matrices, and then with respect to the subgroup $SL_n \subset GL_n$. The subgroup $\mathbb{C}^* \subset GL_n$ acts trivially on $\text{rep}^n_C$; it also acts naturally on $V = \mathbb{C}^n$, by dilations. We put

$$ V^0 \overset{\text{def}}{=} V \setminus \{0\}; \quad \mathbb{P} \overset{\text{def}}{=} \mathbb{P}(V) = V^0/\mathbb{C}^*; \quad \mathcal{X}_n = \text{rep}^n_C \times \mathbb{P}. $$

We have a natural Hamiltonian $\mathbb{C}^*$-action on $T^*V$, a symplectic manifold, and Hamiltonian reduction procedure replaces $T^*V$ by $T^*\mathbb{P}$.

Similarly, the diagonal $\mathbb{C}^*$-action on $\text{rep}^n_C \times V^0$ gives rise to a Hamiltonian $\mathbb{C}^*$-action on $T^*(\text{rep}^n_C \times V^0)$, with moment map $\mu$. It is clear that the space

$$ T^*\mathcal{X}_n = \mu^{-1}(0)/\mathbb{C}^* = \{(x, y, i, j) \in SL_n \times \mathfrak{sl}_n \times \mathbb{C}^* \times \mathbb{C}^* \mid \langle j, i \rangle = 0\}/\mathbb{C}^*, $$

is a Hamiltonian reduction of the symplectic manifold $T^*(\text{rep}^n_C \times V^0)$.

One may also consider the Hamiltonian reduction of the Lagrangian subscheme $M_{\text{nil}} \subset T^*X_n$. This way we get a closed Lagrangian subscheme $M_{\text{nil}}(C) \overset{\text{def}}{=} (M_{\text{nil}}(C) \cap \mu^{-1}(0))/\mathbb{C}^* \subset T^*\mathcal{X}_n$.

Recall that, by Hodge theory, there are natural isomorphisms $H^2(\mathbb{P}, \Omega^2_{\mathbb{P}}) \cong H^1(\mathbb{P}, \mathbb{C}) = \mathbb{C}$. Therefore, for any $\psi \in H^2(C, \Omega^2_C)$ and $c \in \mathbb{C} = H^2(\mathbb{P}, \Omega^2_{\mathbb{P}})$, there is a well defined class $(\psi, c) \in H^2(\mathcal{X}_n, \Omega^2_{\mathcal{X}_n})$, and the corresponding TDO $\mathcal{D}_{\psi, c}(\mathcal{X}_n)$, on $\mathcal{X}_n$.

We have a natural algebra isomorphism

$$ \mathcal{D}_{\psi, c}(\mathcal{X}_n) \cong \left( \mathcal{D}_{\psi, n}(\mathcal{X}_n) / \mathcal{D}_{\psi, n}(\mathcal{X}_n) \cdot (\mathfrak{eu} - c) \right) \mathbb{C}^*, \quad (4.5.1) $$

where $\mathfrak{eu}$ is the Euler vector field on $V$ that corresponds to the action of the identity matrix $\text{Id} \in \mathfrak{gl}(V)$. The algebra on the right hand side of $(4.5.1)$ is a quantum hamiltonian reduction with respect to the group $\mathbb{C}^*$, at the point $\kappa = c/n$.

Definition 4.5.2. An $SL(V)$-equivariant twisted $\mathcal{D}_{\psi, c}(\mathcal{X}_n)$-module is called a character $\mathcal{D}$-module if its characteristic variety is contained in $M_{\text{nil}}(C) = (M_{\text{nil}}(C) \cap \mu^{-1}(0))/\mathbb{C}^*$.

Let $\mathcal{C}_{\psi, c}$ denote the (abelian) category of character $\mathcal{D}_{\psi, c}(\mathcal{X}_n)$-modules.

It is clear, by Proposition 4.3.2, that any object of the category $\mathcal{C}_{\psi, c}$ is a holonomic $\mathcal{D}$-module; in particular, such an object has finite length.

Remark 4.5.3. For a general curve $C$ and a general pair $(\psi, c) \in H^2(C, \Omega^2_C) \times \mathbb{C}$, the category $\mathcal{C}_{\psi, c}$ may have no nonzero objects at all. It is an interesting open problem, to analyze which nonzero objects of the category $\mathcal{C}_{0,0}$ admit (at least formal) deformation in the direction of some $(\psi, c) \neq (0, 0)$.

In the special case of the curve $C = \mathbb{C}^*$, however, we have $H^2(C, \Omega^2_C) = 0$. Thus, there is only one nontrivial parameter, $c \in \mathbb{C}$. It turns out that, for any $c \in \mathbb{C}$, there is a lot of nonzero character $\mathcal{D}$-modules on $GL_n \times \mathbb{P}$; moreover, all these $\mathcal{D}$-modules have regular singularities. \[ \square \]

4.6. Hamiltonian reduction functor. We now introduce a version of category $\mathcal{O}$ for $A_{\kappa, \psi} = eH_{\kappa, \psi, n+1}$, the spherical Cherednik algebra associated with a smooth curve $C$, see (3.4.1).

Definition 4.6.1. Let $\mathcal{O}(A_{\kappa, \psi})$ be the full subcategory of the abelian category of left $A_{\kappa, \psi}$-modules whose objects are coherent as $\mathcal{O}_{C(n)}$-modules.

For any $SL(V)$-equivariant $\mathcal{D}_{\psi, c}(\mathcal{X}_n)$-module $\mathcal{F}$, one has the sheaf-theoretic push-forward $\varpi_! \mathcal{F}$, a sheaf on $C(n)$, cf. (2.2.1). The latter sheaf comes equipped with a natural locally finite (rational) $SL(V)$-action, and we write $(\varpi_! \mathcal{F})^{SL(V)} \subset \varpi_! \mathcal{F}$, for the $\mathcal{O}_{C(n)}$-subsheaf of $SL(V)$-fixed sections.
Thanks to Theorem \[3.3.3\] one can apply the general formalism of Hamiltonian reduction, as outlined in [GG], §7, to the spherical Cherednik algebra. Specifically, we have the following result

**Proposition 4.6.2.** (i) One has the following exact functor of Hamiltonian reduction:

\[ \mathbb{H} : \mathscr{C}_{\psi,c} \rightarrow \mathcal{O}(\mathcal{A}_{n,\psi}), \quad \mathcal{F} \mapsto \mathbb{H}(\mathcal{F}) = \mathcal{F}(\mathcal{A}_{c}^{*}V) \quad \kappa = c/n. \]

Moreover, this functor induces an equivalence \[ \mathcal{C}_{\psi,c}/\ker \mathbb{H} \cong \mathcal{O}(\mathcal{A}_{n,\psi}). \]

(ii) The functor \[ \mathbb{H} : \mathcal{O}(\mathcal{A}_{n,\psi}) \rightarrow \mathcal{C}_{\psi,c}, \quad M \mapsto (\mathcal{D}_{\psi,c}(\mathcal{X}_{n})/\mathcal{D}_{\psi,c}(\mathcal{X}_{n})\mathfrak{g}_{n}) \otimes_{\mathcal{A}_{n,\psi}} M. \]

Moreover, for any \[ M \in \mathcal{O}(\mathcal{A}_{n,\psi}), \] the canonical adjunction \[ \mathbb{H} \circ \mathbb{H}(M) \rightarrow M \] is an isomorphism.

To prove Proposition 4.6.2 first recall the projection \[ T^{*}C_{n} \rightarrow (T^{*}C)^{(n)}. \] Abusing the language we will refer to the image of the zero section of \[ T^{*}C_{n} \] under the projection as the `zero section' of \[ (T^{*}C)^{(n)}. \]

Recall that the spherical algebra \[ \mathcal{A}_{n,\psi} \] has an increasing filtration. Therefore, given a coherent \[ \mathcal{A}_{n,\psi} \]-module \[ M, \] one has a well-defined notion of characteristic variety, \[ SS(M) \subset \text{Spec}(\mathfrak{gr}^{n} \mathcal{A}_{n,\psi}). \] We may (and will) use isomorphism \[ (3.1.5) \] and view \[ \mathfrak{g}_{n} \] as a graded object, as a coherent sheaf on \[ (T^{*}C)^{(n)}. \]

Then, the following is clear

**Lemma 4.6.3.** A coherent \[ \mathcal{A}_{n,\psi} \]-module \[ M \] is an object of \[ \mathcal{O}(\mathcal{A}_{n,\psi}) \] if and only if \[ SS(M) \] is contained in the zero section of \[ (T^{*}C)^{(n)}. \] □

**Proof of Proposition 4.6.2** The assignment \[ \mathbb{H} : \mathcal{F} \mapsto (\mathcal{A}_{n,\psi})^{*}V \] clearly gives a functor from the category of \[ SL(V) \]-equivariant coherent \[ \mathcal{D}_{\psi,c}(\mathcal{X}_{n}) \]-modules to the category of quasi-coherent \[ \mathcal{O}_{C_{n}}-\text{modules}. \] This functor is exact since the support-morphism \[ \mathcal{A}_{n,\psi} \] is affine and \[ SL(V) \] is a reductive group acting rationally on \[ \mathcal{A}_{n,\psi}. \] Thus, our Theorem \[ 3.3.3 \] combined with [GG], Proposition 7.1, imply that \[ (\omega, \mathcal{F})^{*}V \] has a natural structure of coherent \[ \mathcal{A}_{n,\psi} \]-module.

We may use Lemma \[ 4.2.4 \] and the map \[ \pi \] introduced after the lemma, to obtain a diagram

\[ \text{Spec} \left( \mathfrak{gr}^{n} \mathcal{D}_{\psi,c}(\mathcal{X}_{n}) \right) = T^{*}X_{n} \supset \mu^{-1}(0) \xrightarrow{\pi} (T^{*}C)^{(n)} = \text{Spec} \left( \mathfrak{gr}^{n} \mathcal{A}_{n,\psi} \right). \] (4.6.4)

We remark that the statement of Lemma \[ 4.2.4 \] involves the space \[ X_{n} \] rather than \[ \mathcal{X}_{n}; \] the above diagram is obtained from a similar diagram for the subset \[ \text{rep}_{c}^{\mathfrak{g}} \times V^{o} \subset \mathcal{X}_{n}, \] by Hamiltonian reduction with respect to the \[ C^{\times} \]-action on \[ T^{*}(\text{rep}_{c}^{\mathfrak{g}} \times V^{o}) \] induced by the natural action of the group \[ C^{\times} \subset GL(V) \] on \[ \text{rep}_{c}^{\mathfrak{g}} \times V^{o}. \] Further, from Lemma \[ 4.2.1 \] we deduce that the map \[ \pi \] in \[ 4.6.4 \] induces an isomorphism \[ \mu^{-1}(0)/SL(V) \cong (T^{*}C)^{(n)}. \]

Now let \[ \mathcal{F} \] be an \[ SL(V) \]-equivariant coherent \[ \mathcal{D}_{\psi,c}(\mathcal{X}_{n}) \]-module. Choose a good increasing filtration on \[ \mathcal{F} \] by \[ SL(V) \]-equivariant \[ \mathcal{O}_{\mathcal{X}_{n}} \]-coherent subsheaves and view \[ \mathfrak{gr}^{n} \mathcal{F} \], the associated graded object, as a coherent sheaf on \[ T^{*}X_{n}. \] It follows that \[ SS(\mathcal{F}) = \text{Supp}(\mathfrak{gr}^{n} \mathcal{F}) \subset \mu^{-1}(0). \]

The functors \[ \omega, \] and \[ (\cdot)^{SL(V)} \], each being exact, we deduce \[ \mathfrak{gr}(\omega, \mathcal{F})^{SL(V)} \cong (\pi, \mathfrak{gr}^{n} \mathcal{F})^{SL(V)}. \]

Moreover, the isomorphism \[ (T^{*}C)^{(n)} \cong \mu^{-1}(0)/SL(V) \] insures that \[ (\pi, \mathfrak{gr}^{n} \mathcal{F})^{SL(V)} \] is a coherent sheaf on \[ (T^{*}C)^{(n)}. \] Hence, the filtration on \[ \mathbb{H}(\mathcal{F}) = (\mathcal{A}_{n,\psi})^{*}V \] induced by the one on \[ \mathcal{F} \], is a good filtration, that is, \[ \mathfrak{gr}^{n} \mathcal{F} \] is a coherent \[ \mathfrak{gr}^{n} \mathcal{A}_{n,\psi} \]-module, cf. (4.6.4). We conclude that \[ SS(\mathbb{H}(\mathcal{F})) \subset \pi(\text{Supp}\mathcal{F})]. \]

The above implies that, for any character \[ \mathcal{D} \]-module \[ \mathcal{F}, \] one has \[ \text{Supp}(\mathcal{F}) \subset \pi(\text{Supp}(\mathcal{F})) \subset \pi(\mathbb{M}_{\mathfrak{g}}(C)) = \text{zero section of } (T^{*}C)^{(n)}. \]

The first claim of part (i) of the proposition follows from these inclusions and Lemma \[ 4.6.3 \]

Part (ii) is proved similarly, using that, for any coherent \[ \mathcal{A}_{n,\psi} \]-module \[ M, \] one has \[ SS(\mathbb{H}(M)) \subset \pi^{-1}(\mathbb{M}(M)). \] (4.6.5)

At this point, the second claim of part (i) is a general consequence of the existence of a left adjoint functor, cf. [GG], Proposition 7.6, and we are done. □

**Corollary 4.6.6.** Let \[ \mathcal{F} \] be a simple character \[ \mathcal{D} \]-module such that \[ \mathbb{H}(\mathcal{F}) \neq 0. \] Then, we have

\[ \Gamma(C^{(n)}, \mathbb{H}(\mathcal{F})) \prec \infty \iff \omega(\text{Supp}\mathcal{F}) \text{ is a finite subset of } C^{(n)}. \]
Proof. The space of global sections of a coherent $O_{C(n)}$-module is finite dimensional if and only if the module has finite support. Further, we know that $\mathbb{H}(\mathcal{F})$ is a coherent $O_{C(n)}$-module and, moreover, it is clear that $\text{Supp} \mathbb{H}(\mathcal{F}) \subset \mathcal{F}$. This gives the implication ‘⇒’.

To prove the opposite implication, let $M := \mathbb{H}(\mathcal{F})$. We have a non-zero morphism $\mathbb{T}(M) = \mathbb{T}(\mathbb{H}(\mathcal{F})) \rightarrow \mathcal{F}$, that corresponds to $\text{Id}_{\mathbb{H}(\mathcal{F})}$ via the adjunction isomorphism

$$\text{Hom}_{\mathbb{H}(\mathcal{F})}(\mathbb{T}(\mathbb{H}(\mathcal{F})), \mathcal{F}) = \text{Hom}_{O_{\mathbb{A}^n,\mathbb{R}}}(\mathbb{H}(\mathcal{F}), \mathbb{H}(\mathcal{F})).$$

This yields the following inclusions

$$\mathcal{F} \subset \text{Supp} \mathbb{H}(\mathcal{F}) \subset \mathcal{F}(\mathbb{T}(M)) \subset \mathcal{F}(\mathbb{T}(M)) = \text{Supp} M = \text{finite set}.$$

Here, the leftmost inclusion follows since $\mathcal{F}$ is simple, hence the map $\mathbb{T}(M) \rightarrow \mathcal{F}$ is surjective, and the rightmost equality is due to our assumption that the sheaf $\mathbb{H}(\mathcal{F})$ has finite support. The implication ‘⇒’ follows.

\[\square\]

5. The trigonometric case

5.1. Trigonometric Cherednik algebra. From now on, we consider a special case of the curve $C = \mathbb{C}^*$. Thus, we have $\text{rep}_{\mathbb{C}^*} \cong GL_n$.

Let $H \subset GL_n$ be the maximal torus formed by diagonal matrices, and let $\mathfrak{h}$ be the Lie algebra of $H$. Let $\{x_1, \ldots, x_n\}$ and $\{x_1, \ldots, x_n\}$ be dual bases of $\mathfrak{h}$ and $\mathfrak{h}^*$, respectively. The coordinate ring $\mathbb{C}[H]$, of the torus $H$, may be identified with $\mathbb{C}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$, the Laurent polynomial ring in the variables $x_i = \exp(x_i)$. Given an $n$-tuple $\nu = (\nu_1, \ldots, \nu_n) \in \mathbb{Z}^n$, we write $x^\nu := x_1^{\nu_1} \cdots x_n^{\nu_n} \in \mathbb{C}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$, for the corresponding monomial.

Let $T = SL_n \cap H$ be a maximal torus in $SL_n$, and let $t := \text{Lie } T$. The coordinate ring $\mathbb{C}[T]$, of the subtorus $T \subset H$, is the quotient ring of $\mathbb{C}[H]$ by the relation $x_1 \cdots x_n = 1$. The diagonal Cartan torus $H^0$ of $PGL_n$ is the quotient torus of $H$, and $\mathbb{C}[H^0] \subset \mathbb{C}[H]$ is the subring generated by products $x_i x_j^{-1}$, $1 \leq i, j \leq n$.

Let $P = \text{Hom}(H, \mathbb{C}^*)$, resp. $P_0 = \text{Hom}(T, \mathbb{C}^*)$, and $P^0 = \text{Hom}(H^0, \mathbb{C}^*)$, be the weight lattice of the group $GL_n$, resp. of the group $SL_n$, and $PGL_n$. Thus, $P^0 \subset P_0$, and we put $\Omega := P_0/P_0 \cong \mathbb{Z}/n\mathbb{Z}$. We form semi-direct products $W^c = P \times W$ (an extended affine Weyl group), resp. $W^\alpha := P^0 \times W$, and $W^\alpha_0 := P_0 \times W$. One has an isomorphism $W^\alpha_0 = W^\alpha \rtimes \Omega$.

We fix $\kappa \in \mathbb{C}$. The *trigonometric Cherednik algebra* $\mathbb{H}_{\kappa}^{\text{trig}}(GL_n)$ of type $GL_n$ is generated by the subalgebras $\mathbb{C}[W^\alpha] := \mathbb{C}[H] \rtimes \mathbb{C}[S_n]$ and $\text{Sym}(\mathfrak{h}) = \mathbb{C}[x_1, \ldots, x_n]$, with relations, see e.g. [AST], 1.3.7, or [ST], §2:

$$s_i \cdot y - s_i(y) \cdot s_i = -\kappa(x_i - x_{i+1}, y), \quad \forall y \in \mathfrak{h}, \ 1 \leq i < n;$$
$$[y_i, x_j] = \kappa x_j \cdot s_{ij}, \quad 1 \leq i \neq j \leq n;$$
$$[y_k, x_k] = x_k - \kappa x_k \cdot \sum_{i \in [1, n] \setminus \{k\}} s_{ik}, \quad 1 \leq k \leq n.$$

Recall that $\mathbb{P} := \mathbb{P}(V)$, where we put $V = \mathbb{C}^n$. According to [CG], (6.13), one has a canonical isomorphism

$$\mathbb{D}(GL_n \times V)/\mathbb{D}(GL_n \times V)_{\mathfrak{g}_n}^{GL_n} \simeq (\mathbb{D}_{\mathfrak{n}}(GL_n \times \mathbb{P})/\mathbb{D}_{\mathfrak{n}}(GL_n \times \mathbb{P})_{\mathfrak{g}_n})^{SL_n},$$

where $\mathbb{D}_{\mathfrak{n}}(GL_n \times \mathbb{P}) = \mathbb{D}(GL_n \times \mathbb{P}) \otimes \mathbb{D}_{\mathfrak{n}}(\mathbb{P})$ stands for the sheaf of twisted differential operators on $GL_n \times \mathbb{P}$, with a twist $\mathfrak{n} \cdot \kappa$ along $\mathbb{P}$.

Corollary 5.1.1. The spherical trigonometric Cherednik subalgebra $\mathbb{e} \mathbb{H}_{\kappa}^{\text{trig}}(GL_n)$ is isomorphic to the quantum Hamiltonian reduction $(\mathbb{D}_{\mathfrak{n}}(GL_n \times \mathbb{P})/\mathbb{D}_{\mathfrak{n}}(GL_n \times \mathbb{P})_{\mathfrak{g}_n})^{SL_n}$.

Proof. Recall that $\rho := \frac{1}{2} \sum_{\alpha \in \mathbb{R}^+} \alpha \in \mathfrak{h}^*$. For $y \in \mathfrak{h}$ let $\partial_y$ denote the corresponding translation invariant vector field on $H$.

Introduce the *Dunkl-Cherednik operator* $T^\kappa_y$ as an endomorphism of $\mathbb{C}[H]$ defined as follows:

$$T^\kappa_y := \partial_y - \kappa(\rho, y) + \kappa \sum_{1 \leq i < j \leq n} \frac{(x_i - x_j, y)}{1 - x_i^{-1} x_j} (1 - s_{ij}) \quad (5.1.2)$$
Let $A(GL_n)$ be a subalgebra of $\operatorname{End}(\mathbb{C}[H])$ generated by the operators corresponding to the action of elements $w \in S_n$, by the commutative algebra $\mathbb{C}[H]$, of multiplication operators, and by all the operators $T^x_y$, $y \in \mathfrak{h}$.

The following key result is due to Cherednik; for a nice exposition see e.g. [Op], 3.7:

The assignment

$$x^\nu w \mapsto \exp(\nu w), \quad \mathfrak{h} \ni y \mapsto T^x_y, \quad \nu \in \mathbb{Z}^n, \quad w \in S_n, \quad y \in \mathfrak{h},$$

extends to an algebra isomorphism $H^{\text{trig}}_n(GL_n) \cong A(GL_n)$.

To complete the proof, observe that, the curve $(\mathbb{C}^\times)^{\langle n \rangle}$ being affine, one can replace the sheaf of Cherednik algebras by the corresponding algebra of global sections. Moreover, since $H^2(\mathbb{C}^\times, \Omega_{\mathbb{C}^\times}^{1,2}) = 0$, the parameter $\psi$ in $H_{\mathbb{C}}\psi$ vanishes. Thus, Theorem 5.2.7 says that the Hamiltonian reduction algebra $(\mathcal{D}_{nk}(GL_n \times \mathcal{P})/\mathcal{D}_{nk}(GL_n \times \mathcal{P})\mathfrak{g}_k)^{SL_n}$, is isomorphic to the algebra $A(GL_n)$.

5.2. The trigonometric Cherednik algebra $H^{\text{trig}}_n(PGL_n)$ of $PGL_n$-type is defined as a subalgebra in $H^{\text{trig}}_n(GL_n)$ generated by $\mathbb{C}[W^n]$ and $\operatorname{Sym}(t)$. Equivalently, $H^{\text{trig}}_n(PGL_n)$ is generated by the subalgebras $\mathbb{C}[W^n]$ and $\operatorname{Sym}(t)$ with relations

\begin{equation}
\begin{aligned}
s_i \cdot y - s_i(y) \cdot s_i &= -\kappa(x_i - x_{i+1}, y) \quad \forall y \in t, \quad 1 \leq i < n \\
[y, x] &= \langle \eta, y \rangle x - \kappa \sum_{1 \leq i < j \leq n} \langle \alpha_{ij}, y \rangle \frac{x - s_{ij}(x)}{1 - x_i^{-1}x_j} s_{ij} \quad \forall \ y \in t, \ x = \exp(\eta), \ \eta \in P^0 \subset \mathbb{C}[W^n]
\end{aligned}
\end{equation}

(5.2.1)

The trigonometric Cherednik algebra $H^{\text{trig}}_n(SL_n)$ of $SL_n$ is generated by the subalgebras $\mathbb{C}[W^n]$ and $\operatorname{Sym}(t)$ with relations

\begin{equation}
\begin{aligned}
s_i \cdot y - s_i(y) \cdot s_i &= -\kappa(x_i - x_{i+1}, y) \quad \forall y \in t, \quad 1 \leq i < n \\
\omega \cdot y - \omega(y) \cdot \omega &\quad \forall y \in t, \ \omega \in \Omega \\
[y, x] &= \langle \eta, y \rangle x - \kappa \sum_{1 \leq i < j \leq n} \langle \alpha_{ij}, y \rangle \frac{x - s_{ij}(x)}{1 - x_i^{-1}x_j} s_{ij} \quad \forall \ y \in t, \ x = \exp(\eta), \ \eta \in P_0 \subset \mathbb{C}[W^n]
\end{aligned}
\end{equation}

(5.2.2)

(5.2.3)

(5.2.4)

(5.2.5)

For an equivalent definition see e.g. [Op], 3.6.

The algebras $H^{\text{trig}}_n(SL_n)$, $H^{\text{trig}}_n(PGL_n)$, $H^{\text{trig}}_n(GL_n)$ are closely related to each other. To formulate the relations more precisely, let $\mathcal{D}(\mathbb{C}^\times)$ be the algebra of differential operators on $\mathbb{C}^\times$. Let $t$ be a coordinate on $\mathbb{C}^\times$, and $\eta = t\partial_t$. Then $\mathcal{D}(\mathbb{C}^\times)$ is generated by $t^{\pm 1}$, $\eta$ with the relation $[\eta, t] = t$.

We have an embedding $\mathbb{C}^\times \hookrightarrow H$ as the scalar (central) matrices in $GL_n$. Taking product with the embedding $T \hookrightarrow H$, we obtain a morphism (n-fold covering) $T \times \mathbb{C}^\times \hookrightarrow H$. At the level of functions we have an embedding $\mathbb{C}[H] \hookrightarrow \mathbb{C}[T] \otimes \mathbb{C}[\mathbb{C}^\times]$. In coordinates, we have $x_i \mapsto x_i \otimes x_i$, $1 \leq i \leq n$. The isogeny $T \times \mathbb{C}^\times \rightarrow H$ induces an isomorphism of the Lie algebras of our tori. The inverse isomorphism $\mathfrak{h} \rightarrow t \otimes \mathbb{C}$ sends $y_i \rightarrow (y_i - \frac{1}{n} \sum_{k=1}^n y_k, \frac{1}{n}\eta)$ (recall from the previous paragraph that $\mathbb{C} = \text{Lie} \mathbb{C}^\times$ is equipped with the base $\{\eta\}$).

We consider the tensor product algebra $H^{\text{trig}}_n(SL_n) \otimes \mathcal{D}(\mathbb{C}^\times)$ together with the following elements:

$$\Xi(x_i) := x_i \otimes t; \quad \Xi(y_i) := (y_i - \frac{1}{n} \sum_{k=1}^n y_k) \otimes 1 + 1 \otimes \frac{1}{n}\eta; \quad 1 \leq i \leq n.$$

The following is a straightforward corollary of definitions.

**Corollary 5.2.6.** (i) There is a natural isomorphism $H^{\text{trig}}_n(SL_n) \cong H^{\text{trig}}_n(PGL_n) \times \Omega$.

(ii) The map $x_i \mapsto \Xi(x_i), \ y_i \mapsto \Xi(y_i), \ s_{ij} \mapsto s_{ij}; \ 1 \leq i \neq j \leq n$, defines an injective homomorphism $\Xi : H^{\text{trig}}_n(GL_n) \hookrightarrow H^{\text{trig}}_n(SL_n) \otimes \mathcal{D}(\mathbb{C}^\times)$.

**Corollary 5.2.7.** For any $c \in \mathbb{C} \setminus [-1, 0]$, the functor below is a Morita equivalence

$$H^{\text{trig}}_n(SL_n)_{\text{mod}} \longrightarrow \mathfrak{e}H^{\text{trig}}_n(SL_n)_{\text{e-mod}}, \ M \mapsto eM.$$
Proof. We need to prove that $H^\text{trig}(SL_n) = H^\text{trig}(GL_n) \cdot e \cdot H^\text{trig}(SL_n)$. The similar statement for the trigonometric Cherednik algebra $H^\text{trig}(GL_n)$ follows from Proposition 3.1.3 (for $C = \mathbb{C}^\times$). Therefore, we have an equality

$$1 = \sum_{k=1}^{r} u_k \cdot v_k, \quad u_1, \ldots, u_r, v_1, \ldots, v_r \in H^\text{trig}(GL_n).$$

(5.2.8)

This implies, via the imbedding $H^\text{trig}(GL_n) \hookrightarrow H^\text{trig}(SL_n) \otimes \mathcal{D}(\mathbb{C}^\times)$, of Corollary 5.2.6(ii), that a similar equality holds in the algebra $H^\text{trig}(SL_n) \otimes \mathcal{D}(\mathbb{C}^\times)$. Thus, in $H^\text{trig}(SL_n) \otimes \mathcal{D}(\mathbb{C}^\times)$ we have

$$1 \otimes 1 = \sum_{k=1}^{r} (u_k \cdot v_k) \otimes w_k, \quad u_k, v_k \in H^\text{trig}(SL_n), \quad w_k \in \mathcal{D}(\mathbb{C}^\times).$$

(5.2.9)

Choose a $\mathbb{C}$-linear countable basis $\{q_v\}_{v \in \mathbb{N}}$ of the vector space $\mathcal{D}(\mathbb{C}^\times)$ such that $q_1 = 1$. Expanding each of the elements $w_k \in \mathcal{D}(\mathbb{C}^\times)$ in this basis and equating the corresponding terms in (5.2.9), one deduces from (5.2.8) an equation of the form $1 = \sum \alpha \cdot e \cdot b$, where $a, b \in H^\text{trig}(SL_n)$. Thus, we have shown that

$$H^\text{trig}_\kappa(SL_n) = H^\text{trig}_\kappa(SL_n) \cdot e \cdot H^\text{trig}_\kappa(SL_n),$$

and the Morita equivalence for the algebra $H^\text{trig}_\kappa(SL_n)$ follows. □

5.3. We have the $S_n$-equivariant product morphism $H = (\mathbb{C}^\times)^n \to \mathbb{C}^\times$ with the kernel $T \subset H$. We can consider the restriction of the sheaf of Cherednik algebras $H_\kappa$ to the closed subvariety $T/S_n \subset H/S_n$. Abusing the notation, we also write $H_\kappa$ for the corresponding algebra of global sections.

The proof of the following result copies the proof of Proposition 5.1.1

Corollary 5.3.1. The spherical trigonometric Cherednik subalgebra $e H^\text{trig}_\kappa(SL_n) e$ is isomorphic to the quantum Hamiltonian reduction $(\mathcal{D}_{\kappa}(SL_n \times \mathbb{P})/\mathcal{D}_{\kappa}(SL_n \times \mathbb{P})_{\mathbb{D}})_{SL_n}$.

An important difference between the algebras $e H^\text{trig}_\kappa(GL_n) e$ and $e H^\text{trig}_\kappa(SL_n) e$ is that the latter may have finite dimensional representations, while the former cannot have such representations. In view of this, it is desirable to have a version of Hamiltonian reduction functor that would relate $\mathcal{D}$-modules on $SL_n \times \mathbb{P}$ (rather than on $GL_n \times \mathbb{P}$) with $e H^\text{trig}_\kappa(SL_n) e$-modules. To this end, one needs first to introduce a Lagrangian nil-cone in $T^*(SL_n \times \mathbb{P})$, and the corresponding notion of character $\mathcal{D}$-module on $SL_n \times \mathbb{P}$.

To define the nil-cone, we intersect $M_{\text{nil}}(\mathbb{C}^\times) \subset GL_n \times \mathfrak{g}_{\mathbb{C}^\times} \otimes V \otimes V^* = T^*(\mathfrak{g}_{\mathbb{C}^\times} \otimes V^*)$, the nil-cone for the group $GL_n$, with $SL_n \times \mathfrak{sl}_n \otimes V \otimes V^*$. Let $M_{\text{nil}} \subset T^*(SL_n \times \mathbb{P})$ be the Hamiltonian reduction of the resulting variety with respect to the Hamiltonian $\mathbb{C}^\times$-action on the factor $T^*V = V \otimes V^*$.

We write $\mathcal{D}_\kappa := \mathcal{D}(SL_n) \otimes \mathcal{D}_\kappa(\mathbb{P})$. An $SL_n$-equivariant $\mathcal{D}$-module provided its characteristic variety is contained in $M_{\text{nil}}$.

5.4. A Springer type construction. We are going to introduce certain analogues of Springer resolution in our present setting. The constructions discussed below work more generally, in the framework of an arbitrary smooth curve $C$. However, to simplify the exposition, we restrict ourselves to the case $C = \mathbb{C}^\times$.

Write $B$ for the flag variety, that is, the variety of complete flags $F = (0 = F_0 \subset F_1 \subset \ldots \subset F_{n-1} \subset F_n = V)$, where $\dim F_k = k, \forall k = 0, \ldots, n$. In the trigonometric case, the variety $\mathcal{D}_\kappa$, introduced in [4.3] reduces to $\hat{SL}_n = \{(g, F) \in SL_n \times B \mid g(F_k) \subset F_k, \forall k = 0, \ldots, n\}$. The first projection yields a proper morphism $\hat{SL}_n \to SL_n$ known as the Grothendieck-Springer resolution.

Now, fix an integer $1 \leq m \leq n$. We define a closed subvariety $\mathfrak{X}_{n,m} \subset \mathfrak{X}_n$ as follows

$$\mathfrak{X}_{n,m} := \{(g, \ell) \in SL_n \times \mathbb{P} \mid \dim(C[g]\ell) \leq m\},$$

where $C[g]\ell \subset V$ denotes the minimal $g$-stable subspace in $V$ that contains the line $\ell \subset V$. 18
Next, we put \(\tilde{X}_n = SL_n \times \mathbb{P}\) and define a closed subvariety \(\tilde{X}_{n,m} \subset \tilde{X}_n\) as follows

\[
\tilde{X}_{n,m} := \{(g, F, \ell) \in SL_n \times B \times \mathbb{P} \mid \ell \in F_m, \ & g(F_k) \subset F_k, \forall k = 0, \ldots, n\}.
\]

We have a diagram

\[
\begin{array}{ccc}
SL_n \times \mathbb{P} & \xrightarrow{\pi_{n,m}} & \tilde{X}_{n,m} & \xrightarrow{(F, \ell) \mapsto (g, F, \ell)}
\end{array}
\]

We observe that:

- The projection \(\tilde{X}_{n,m} \to B \times \mathbb{P}\), \((g, F, \ell) \mapsto (F, \ell)\) makes \(\tilde{X}_{n,m}\) a locally trivial fibration over the base \(\{(F, \ell) \in B \times \mathbb{P} \mid \ell \subset F_m\}\). Both the fiber and the base are smooth.

  Thus, \(\tilde{X}_{n,m}\) is smooth.

- The image of the projection \(\tilde{X}_{n,m} \to SL_n \times \mathbb{P}\) is contained in \(X_{n,m}\), hence the projection gives a well defined morphism:

\[\pi_{n,m} : \tilde{X}_{n,m} \rightarrow X_{n,m}, \quad (g, F, v) \mapsto (g, v).\]

Let \(X^\text{reg}_{n,m}\) be an open subset of \(X_{n,m}\) formed by the pairs \((g, \ell)\) such that \(g\) is a matrix with \(n\) pairwise distinct eigenvalues. Let \(\tilde{X}^\text{reg}_{n,m} = X^\text{reg}_{n,m} \sqcup \tilde{X}_{n,m}\), an open subset in \(\tilde{X}_{n,m}\).

**Proposition 5.4.1.** (i) The map \(\pi_{n,m} : \tilde{X}_{n,m} \to X_{n,m}\) is a dominant proper morphism, which is small in the sense of Goresky-MacPherson.

(ii) The restriction \(\pi_{n,m} : \tilde{X}^\text{reg}_{n,m} \to X^\text{reg}_{n,m}\) is a Galois covering with the Galois group \(S_m \times S_{n-m}\).

**Proof.** Clearly, \(\pi_{n,m}\) is proper and has a dense image. Hence, it is dominant.

Let \(Z := \tilde{X}_{n,m} \times_{X_{n,m}} \tilde{X}_{n,m}\), so we have a projection \(Z \to X_{n,m}\). To prove that \(\pi_{n,m}\) is small, one must check that \(\dim Z \leq n^2 + m\). To prove that \(\pi_{n,m}\) is small one must show in addition that each irreducible component of \(Z\), of dimension \(n^2 + m\), dominates \(X_{n,m}\). The argument is very similar to the standard proof of smallness of the Grothendieck-Springer resolution.

In more detail, for \(w \in S_n\) we denote by \(Z_w\) the locally closed subvariety of \(Z\) formed by the quadruples \((g, F, F', \ell)\) such that the flags \(F\) and \(F'\) are in relative position \(w\) (and such that \(\ell \subset F_m \cap F'_m\), and \(g(F) = F, g(F') = F'\)). Then we have \(Z = \bigcup_{w \in S_n} Z_w\).

We may view \(Z_w\) as a fibration over an \(SL_n\)-orbit in \(B \times B\), the cartesian square of flag variety. We see immediately that \(\dim Z_w \leq n^2 + m\) with an exact equality if and only if \(F_m = F'_m\). The latter equality holds if and only if \(w \in S_m \times S_{n-m} \subset S_n\). For such a \(w\) it is clear that \(Z_w\) dominates \(X_{n,m}\). This completes the proof of (i).

Now part (ii) follows easily from the above description of irreducible components. \(\square\)

Let \(\Sigma_N\) denote the set of partitions of an integer \(N\). For any partition \(\lambda \in \Sigma_m\), resp. \(\mu \in \Sigma_{n-m}\), write \(L_\lambda\), resp. \(L_\mu\), for an irreducible representation of the Symmetric group \(S_m\), resp. \(S_{n-m}\), associated with that partition in a standard way. Thus, \(L_\lambda \boxtimes L_\mu\) is an irreducible representation of the group \(S_m \times S_{n-m}\).

Let \(C_{\tilde{X}^\text{reg}_{n,m}}\) be the constant sheaf on \(\tilde{X}^\text{reg}_{n,m}\). According to Proposition 5.4.1(ii), we have

\[
(\pi_{n,m})_* C_{\tilde{X}^\text{reg}_{n,m}} = \bigoplus_{(\lambda, \mu) \in \Sigma_m \times \Sigma_{n-m}} (L_\lambda \boxtimes L_\mu) \otimes \mathcal{L}_{\lambda, \mu},
\]

where \(\mathcal{L}_{\lambda, \mu}\) is an irreducible local system on \(X^\text{reg}_{n,m}\) with monodromy \(L_\lambda \boxtimes L_\mu\).

Now, let \(C_{\tilde{X}_{n,m}}[\dim \tilde{X}_{n,m}]\) be the constant sheaf on \(\tilde{X}_{n,m}\), with shift normalization as a perverse sheaf. Then, from part (i) of Proposition 5.4.1 using the definition of an intersection cohomology complex, we deduce
Corollary 5.4.2. There is a direct sum decomposition
\[
(\pi_{n,m})*C_{\widehat{x}_{n,m}}[\dim \widehat{x}_{n,m}] = \bigoplus_{(\lambda,\mu) \in \Sigma_m \times \Sigma_m} (L_\lambda \boxtimes L_\mu) \otimes IC(L_{\lambda,\mu}).
\]

Here, IC(\(-\)) denotes the intersection cohomology extension of a local system.

One can also translate the statement of the corollary into a \(\mathcal{D}\)-module language. To this end, write \(i_{n,m}: \widehat{x}_{n,m} \rightarrow \widehat{\mathcal{X}} = \mathcal{S}_L \times \mathbb{P}\) for an obvious closed embedding. For any integer \(c \in \mathbb{Z}\), one has a \(\mathcal{D}_{0,c}\)-module \(\mathcal{O}(c)_{n,m} \coloneqq i_{n,m}^{\ast}(\mathcal{O}_{\mathcal{S}_L} \boxtimes \mathcal{O}_p)[m - n]\), on \(\widehat{x}_{n,m}\).

Corollary 5.4.2 yields the following result.

Corollary 5.4.3. The direct image \((\pi_{n,m})*\mathcal{O}(c)_{n,m}\) is a semisimple \(\mathcal{D}_{0,c}\)-module on \(\widehat{x}_{n,m}\) and one has a direct sum decomposition
\[
(\pi_{n,m})*\mathcal{O}(c)_{n,m} = \bigoplus_{(\lambda,\mu) \in \Sigma_m \times \Sigma_m} (L_\lambda \boxtimes L_\mu) \otimes \mathcal{F}_{\lambda,\mu},
\]
where \(\mathcal{F}_{\lambda,\mu}\) is an irreducible character \(\mathcal{D}_{0,c}\)-module on \(\widehat{x}_n\). \(\square\)

5.5. Cuspidal \(\mathcal{D}\)-modules. The goal of this subsection is to describe character \(\mathcal{D}\)-modules which have finite dimensional Hamiltonian reduction. These \(\mathcal{D}\)-modules turn out to be closely related to cuspidal character sheaves on \(SL_n\).

In more detail, write \(Z(SL_n)\) for the center of the group \(SL_n\). Thus, \(Z(SL_n)\) is a cyclic group, the group of scalar matrices of the form \(z \cdot 1\mathbb{I}_d\), where \(z \in \mathbb{C}\) is an \(n\)-th root of unity.

Let \(U \subset SL_n\) be the unipotent cone, and let \(j: U^\text{reg} \hookrightarrow U\) be an open imbedding of the conjugacy class formed by the regular unipotent elements. The fundamental group of \(U^\text{reg}\) may be identified canonically with \(Z(SL_n)\). For each integer \(p = 0, 1, \ldots, n - 1\), there is a group homomorphism \(Z(SL_n) \rightarrow \mathbb{C}^\times, z \cdot 1\mathbb{I}_d \mapsto z^p\). Let \(L_p\) be the corresponding rank one \(SL_n\)-equivariant local system, on \(U^\text{reg}\), with monodromy \(\theta = \exp(\frac{2\pi i}{n})\).

From now on, we assume that \((p, n) = 1\), i.e., that \(\theta\) is a primitive \(n\)-th root of unity. Then, the local system \(L_p\) is known to be clean, that is, for \(\mathcal{D}\)-modules on \(SL_n\), one has \(j_!L_p \cong j_*L_p = j^*L_p\), cf. [L] or [Os]. Given a central element \(z \in Z(SL_n)\), we have the conjugacy class \(zU^\text{reg} \subset SL_n\), the \(z\)-translate of \(U^\text{reg}\), and we let \(zj_!L_p\) denote the corresponding translated \(\mathcal{D}\)-module supported on the closure of \(zU^\text{reg}\). According to Lusztig [L], \(zj_!L_p\) is a cuspidal character \(\mathcal{D}\)-module on the group \(SL_n\).

Further, for any integer \(c \in \mathbb{Z}\), we may form \(zL_{p,c} \coloneqq (zj_!L_p) \boxtimes \mathcal{O}(c)\), a twisted \(\mathcal{D}\)-module on \(SL_n \times \mathbb{P}\). Thus, \(zL_{p,c}\) is a simple character \(\mathcal{D}\)-module.

Theorem 5.5.1. (i) Let \(c\) be a nonnegative real number and let \(\mathcal{F}\) be a nonzero simple character \(\mathcal{D}\)-module. Then, the following properties are equivalent:

1. The support of \(\mathcal{F}\) is contained in \((Z(SL_n) \cdot U) \times \mathbb{P}\);
2. We have \(\mathcal{F} \cong zL_{p,c}\), for some \(z \in Z(SL_n)\) and some integers \(p, c\) such that \(p\) is prime to \(n\), and \(0 < p < n\).

(ii) For a simple character \(\mathcal{D}\)-module \(\mathcal{F}\), we have
\[
\mathbb{H}(\mathcal{F}) \neq 0 \quad \text{and} \quad \dim \mathbb{H}(\mathcal{F}) < \infty \iff (1) \cdot (2) \text{ hold and, we have } n|(c - p).
\]

(iii) Let \(\kappa = c/n\), where \(c\) and \(n\) are mutually prime integers. Then, the functor \(\mathbb{H}\) yields a one-to-one correspondence between simple character \(\mathcal{D}_{\kappa}(SL_n \times \mathbb{P})\)-modules of the form \(zL_{p,c}\), with \(n|(c - p)\), and finite dimensional irreducible \(e\mathfrak{H}_\text{rig}(SL_n)e\)-modules, respectively.

Proof. The implication (2) \(\Rightarrow\) (1) of part (i) is clear. We now prove that (1) \(\Rightarrow\) (2).

The group \(GL_n\) acts on \(U \times V^\circ\) with finitely many orbits, see [GG]. Corollary 2.2. The open orbit \(\mathcal{O}_0\) is formed by the pairs \((X, v)\) where \(X \in \mathcal{U}_{\text{reg}}\), and \(v \in V \setminus \Im(X - 1)\). The action of \(GL_n\) on \(\mathcal{O}_0\) is free (and transitive). Let \(\mathcal{O}\) be a unique \(GL_n\)-orbit open in the support of \(\mathcal{F}\). The singular support of \(\mathcal{F}\) contains the conormal bundle \(T^*_\mathcal{O}(SL_n \times V^\circ)\). According to [GG], Theorem 4.3, if \(\mathcal{O}\) lies in \(U \times V^\circ\), and \(T^*_\mathcal{O}(SL_n \times V^\circ)\) lies in \(\mathcal{M}_\text{nil} \coloneqq M_{\text{nil}}(\mathbb{C}^\times) \cap T*(SL_n \times V^\circ)\), then \(\mathcal{O} = \mathcal{O}_0\).
We see that $\mathcal{F}$ must be the minimal extension of an irreducible local system on $\mathcal{O}_0$. Since $\mathcal{O}_0$ is a $GL_n$-torsor, its fundamental group is $\mathbb{Z}$, and an irreducible local system is 1-dimensional with monodromy $\vartheta$; it is $GL_n$-monodromic with monodromy $\vartheta$. We will denote such local system by $L_\vartheta$. It remains to prove that $\vartheta = \theta$. Note that the boundary of $\mathcal{O}_0$ contains a codimension 1 orbit $\mathcal{O}_1$ formed by the pairs $(X,v)$ such that $X \in U^\text{reg}$, $v \in \text{Im}(X-1) \setminus \text{Im}(X-1)^2$. The monodromy of $L_\vartheta$ around $\mathcal{O}_1$ equals $\theta^n$. Thus if $\vartheta^n \neq 1$, the singular support of the minimal extension of $L_\vartheta$ necessarily contains $T^*_\mathcal{O}_1(SL_n \times V^\circ)$. As we have just seen, the latter is not contained in $\mathcal{M}_{\mathcal{O}_1}$; a contradiction. Hence $\vartheta^n = 1$.

Now the monodromy of $L_\vartheta$ along lines in $V^\circ$ is also equal to $\theta^n = 1$. Hence $L_\vartheta$ is a pullback to $U^\text{reg} \times V^\circ$ of a local system on $U^\text{reg}$ with monodromy $\vartheta$. We conclude that the minimal extension of this latter local system must be a classical character sheaf on $SL_n$, so $\vartheta$ must be a primitive root of unity $\theta$. This completes the proof of part (i) of the theorem.

We prove part (ii). It follows from [CEE], Theorem 9.19, that $\mathbb{H}(L_{p,c}) \neq 0$ iff the integer $c$ is prime to $n$, and $p$ is the residue of $c$ modulo $n$. Now the first statement is immediate from Corollary 4.6.6. Let $M$ be a finite dimensional $\mathfrak{sl}_n$-module and let $\mathcal{F} := \mathbb{H}(M)$. Then, $\mathcal{F}$ is a character $\mathcal{D}$-module by Proposition 4.6.2. Thus, $\mathcal{F}$ has finite length. Let $\mathcal{F}_1, \ldots, \mathcal{F}_r$ be the collection of simple subquotients of $\mathcal{F}$, counted with multiplicities. The inclusion in 4.6.5 combined with Corollary 4.6.6 imply that each of the $\mathcal{D}$-modules $\mathcal{F}_i$ satisfies the equivalent conditions (1)-(3) of part (i) of the theorem, hence, has the form $\mathcal{F}_i = z_i L_{p_i,c_i}$, for some integers $c_i$ prime to $n$ with residues $c_i$ modulo $n$, and some $z_i \in Z(SL_n)$.

On the other hand, the functor of Hamiltonian reduction is exact and we know that $\mathbb{H}(\mathbb{H}(M)) = M$, by Proposition 4.6.2. We deduce that $\mathcal{F}_i = 0$ for all $i$ except one. Thus, $\mathcal{F}$ is a simple character $\mathcal{D}$-module and part (ii) follows. The statements of part (iii) are now clear. 

\[ \square \]

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