Elementary coordinatization of finitely generated nilpotent groups

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Abstract

This paper has two main parts. In the first part we develop an elementary coordinatization for any nilpotent group $G$ taking exponents in a binomial PID $A$. In case that the additive group $A^+$ of $A$ is finitely generated we prove using a classical result of Julia Robinson that one can obtain a central series for $G$ where the action of the ring of integers $\mathbb{Z}$ on the quotients of each of the consecutive terms of the series is interpretable in $G$ except one very specific gap, called the special gap. Then we use a refinement of this central series to give a criterion for elementary equivalence of finitely generated nilpotent groups in terms of the relationship between group extensions and the second cohomology group.

Keywords: Nilpotent group, Elementary Equivalence, Largest Ring of a Bilinear Map, Coordinatization, Abelian Deformation

1 Introduction

This paper continues the authors efforts [12, 13], in providing a comprehensive and uniform approach to various model-theoretical questions on nilpotent groups and algebras. We introduce our main techniques for approaching the following fundamental problems (stated here for nilpotent groups).

Problem 1. Explain in algebraic terms when and why two finitely generated groups are elementarily equivalent.

Problem 2. Find an algebraic characterization of all groups elementarily equivalent to a given arbitrary finitely generated nilpotent group.

Problem 3. Find algebraic description of nilpotent groups whose elementary theory is $\omega$-stable or of finite Morely rank.

Problem 4. Given a finitely generated nilpotent group $G$, describe in algebraic terms a set of axioms for the elementary theory $Th(G)$ of $G$.

The main idea of our approach is that almost all first-order properties of a given finitely generated nilpotent group $G$ are completely determined by the structure of $G$ as a “non-commutative module” over the maximal ring of scalars.
$A(G)$. This module structure can be rigorously formalized either via P. Hall notion of nilpotent groups taking exponents in the binomial ring $A$ or as a group of $A$-points of an affine nilpotent algebraic group defined over rationals. The core results of our technique show that this module $A(G)$-structure is interpretable in $G$ uniformly with respect to the first-order theory $Th(G)$, except for a special gap, which is the main cause of deviation of the elementarily equivalence from the isomorphism in the case of finitely generated nilpotent groups. The influence the special gap may have on the elementary equivalence and related questions is described completely in terms of second cohomology and abelian deformation of groups. The process of recovering the $A(G)$-structure from the multiplication on $G$ is termed a coordinatization of $G$.

In the first part of the paper (sections 1-5) we describe the coordinatization of a finitely generated nilpotent group $G$ and show that (with exception of the special gap) this coordinatization is first order interpretable in $G$ uniformly with respect to $Th(G)$. In fact, the argument works as well for finitely generated nilpotent $A$-groups for an arbitrary binomial principle ideal domain $A$. It takes a particularly nice form if the additive group $A^+$ of $A$ is finitely generated. Besides we describe elementary equivalence of modules in the two-sorted language (one sort for the abelian group and another for the ring of scalars) and bilinear maps with the same conditions on the rings of scalars $A$ as above in the case of groups. In particular, we give description of elementarily equivalent finite dimensional $A$-algebras for such a ring $A$. In the second part (sections 6-10) of the paper we address in full detail the first of the problems above, leaving the others for the future. We state the main results in Section 1.1.

The problem of elementarily equivalence of finitely generated nilpotent groups goes back to the results of Tarski where he showed that two free nilpotent groups of finite rank are elementarily equivalent if and only if they are isomorphic. In 1969, during his lectures at Novosibirsk University, Kargapolov asked weather this is the case for all finitely generated nilpotent groups. In 1971 Zilber solved this problem in the negative constructing two particular examples of two finitely generated nilpotent groups of class 2 which are elementarily equivalent but not isomorphic [18]. In 1984 the first author showed that elementary equivalence implies isomorphism for a wide class of nilpotent groups, the so-called regular groups [4] (see also [9]). The first algebraic characterization of elementary equivalence of finitely generated (finite-by-) nilpotent groups was found by F. Oger [14]. He proved that two such groups $G$ and $H$ are elementarily equivalent if and only if $G \times \mathbb{Z} \cong H \times \mathbb{Z}$, where $\mathbb{Z}$ refers to the infinite cyclic group. This is a very nice result which shows that in general elementary equivalence is not very far from isomorphism in the case of finitely generated nilpotent groups. Notice that this description does not reveal the algebraic reasons for the elementary equivalence of the groups $G$ and $H$. Indeed, this criterion enables one to check algorithmically (since the isomorphism problem is decidable here) if given $G$ and $H$ are elementarily equivalent, but it does not provide means to construct such an $H$ given a group $G$, or to check if such $H$ non-isomorphic to $G$ even exists. The techniques developed in this paper allows us to answer this question completely in terms of second cohomologies and abelian deformations of groups.
As a corollary we also give an independent proof of the results on regular nilpotent groups mentioned above. The techniques, approaches and aims (Our main focus is on the Problems 1-4.) are very different from Oger’s [14] and they are more closer to the ones developed in [7] for unipotent groups over fields and finite dimensional algebras.

1.1 Our approach and the main results

In this section we formulate our main results. In the rest of the paper we provide details of the constructions and the proofs.

Given a finitely generated nilpotent group $G$ our first aim is to construct a central series

$$G = G_1 > G_2 > \ldots > G_m > 1,$$

where each $G_i$ is first-order definable in $G$ and each quotient $G_i/G_{i+1}$ satisfies exactly one of the following mutually exclusive conditions:

(a) $G_i/G_{i+1}$ is finite,
(b) $G_i/G_{i+1}$ is infinite cyclic and interprets the ring of integers $\mathbb{Z}$,
(c) $G_i/G_{i+1}$ is infinite and no definable infinite subgroup $K/G_{i+1}$ of it interprets $\mathbb{Z}$.

Let us call such a central series a maximally refined series. In some sense this definition is too strict. In particular it should be awfully hard to actually prove that a gap $G_i > G_{i+1}$ that seems to satisfy condition (c) actually satisfies it. But let us keep constructing such a series as an ideal goal. With this goal in mind let us describe a few intermediate constructions which will get us there.

Let

$$G = R_1 > R_2 > \ldots > R_c > R_{c+1} = 1, \quad (R),$$

be an arbitrary central series of the nilpotent group $G$. Let

$$R^u_i = \{ x \in G : [x, G] \subseteq [R_i, G] \}, \quad 1 \leq i \leq c,$$

and

$$R^l_i = R^u_i$$

and

$$R^l_i = [R_i, G], \quad 2 \leq i \leq c + 1.$$

Each $R^u_i$ and $R^l_i$ is a subgroup of $G$, and

$$R^u_i \supseteq R_i \supseteq R^l_i, \quad 1 \leq i \leq c,$$

and

$$[R^u_i, G] = [R_i, G] = R^l_i, \quad 1 \leq i \leq c - 1.$$

Hence

$$G = R_1^u > R_2^u > \ldots > R_c^u = Z(G), \quad (R^u)$$

$$G = R_1^l > R_2^l = G' > R_3^l > \ldots > R_{c+1}^l = 1, \quad (R^l)$$

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are central series for $G$. We call the series $(R^u)$ and $(R^l)$ the upper and lower series associated with $(R)$, respectively. The operation of commutation induces well-defined full bilinear maps

$$f_i : G/G' \times R^u_i/R^u_{i+1} \to R^l_{i+1}/R^l_{i+2}, \quad 1 \leq i \leq c - 1$$

each of which is non-degenerate with respect to the second variable. The mappings constructed form a bundle $S_R = \{f_1, \ldots, f_{c-1}\}$ of bilinear mappings associated with the series $(R)$.

Let

$$R^u = R^u_1/R^u_2 \oplus \ldots \oplus R^u_{c-1}/R^u_c,$$
$$R^l = R^l_2/R^l_3 \oplus \ldots \oplus R^l_c/R^l_{c+1}.$$  

Proposition 1.1. The bundle $S_R$ induces a full non-degenerate with respect to both variables bilinear map

$$F_R : G/VR \times R^u \to R^l,$$

for some subgroup $V_R \geq G'$.

The details of the construction and the proof of the proposition can be found in Section 2.2.

Now one can associate a ring $P_R = P(F_R)$, called the largest ring of scalars of the mapping $F_R$ (see Section 2.1) to $F_R$. Then one can define a $P_R$-module structure on the abelian groups $G/VR$, $R^u_i/R^u_{i+1}$ and $R^l_{i+1}/R^l_{i+2}$, $1 \leq i \leq c - 1$ such that $F_R$ will be $P_R$-bilinear.

The associated series $(R^u)$ and $(R^l)$ possess a considerable shortage. The ring $P_R$ acts exactly on all the quotients of the upper series except the lowest gap $R^u_c = Z(G) > 1$, and on all the quotients of the lower series except on the quotient of the upper gap $G > G' = R^l_2$.

In Section 2.3 we shall introduce refinements $(U(R))$ and $(U(L))$ of $(R^l)$ and $(R^u)$, respectively, that minimize this shortage. We shall also construct a ring $A_R(G)$ that acts simultaneously on all the gaps of the two new series except possibly, the one gap, which we call the special gap. In the case of $(U(R))$ the gap is $Z(G) > Z(G) \cap G'$ and in the case of $(L(R))$ the gap is $Z(G) \cdot G'/G'$. Obviously the gaps, in case that they exist, are isomorphic.

The importance of all these constructions for studying the elementary theory of $G$ is reflected in the following proposition.

Proposition 1.2. Let $G$ be a finitely generated $A$-exponential nilpotent group and

$$G = R_c > R_{c-1} > \ldots R_1 > R_0 = 1 \quad (R)$$
a central series where each term is uniformly with respect to $Th(G)$ definable in $G$. Then :

1. all the terms of the upper completed series $(U(R))$ associated with $(R)$ are interpretable in $G$ uniformly with respect to $Th(G)$,
2. all the terms of the lower completed series \((L(R))\) associated with \((R)\) are interpretable in \(G\) uniformly with respect to \(\text{Th}(G)\),

3. the ring \(A_R(G)\) is interpretable in \(G\) uniformly with respect to \(\text{Th}(G)\), moreover:

   (a) the action of \(A_R(G)\) in all quotients of the upper completed associated series \((U(R))\) (maybe except the gap \(Z(G) \geq G' \cap Z(G)\)) is interpretable in \(G\) uniformly with respect to \(\text{Th}(G)\),

   (b) the action of \(A_R(G)\) on all quotients of the lower completed associated series \((L(R))\) (maybe except the gap \(Z(G) \cdot G'/G'\)) is interpretable in \(G\) uniformly with respect to \(\text{Th}(G)\).

The constructions are introduced in Section 2.3. The logical properties of the constructions are studied in Section 3.

Remark 1.3. As starting point for building a maximally refined series for \(G\) one can choose any central series with good logical properties, such as the lower central series or the upper central series for \(G\). However the existence and form of the gap:

\[
Z(G)/Z(G) \cap G' \cong Z(G) \cdot G'/G'
\]

appearing in 3(a) and 3(b) of Proposition 1.2 is independent of the original series \((R)\). That is why we call it the special gap.

We would have already achieved the goal of constructing a maximally refined series if we could somehow interpret (define) the ring of integers \(\mathbb{Z}\) in the ring \(A_R(G)\). This is the topic we discuss in Section 4. The ring \(A_R(G)\) is a subring of a direct product of finitely many endomorphism rings of finitely generated abelian groups. Therefore it is an example of a ring \(A\) with finitely generated additive group \(A^+\). It is also a Noetherian ring. Let us recall that any such ring admits a decomposition of zero

\[
0 = p_1 \cdot p_2 \cdots p_m, \quad (P)
\]

where the \(p_i\) are certain prime ideals of \(A\).

**Proposition 1.4.** For any Noetherian associative commutative ring \(A\) with a unit, there exists an interpretable decomposition of zero into a product of prime ideals, where the interpretation is uniform with respect to \(\text{Th}(A)\).

Finally the following proposition completes the picture. Its statement is technical, however it implies that any quotient of the terms of \((U(R))\) and \((L(R))\) that interpret the action of \(A_R(G)\) does also interpret the action of \(Z\), moreover all the interpretation are uniform with respect to finite dimensional models of \(\text{Th}(A_R(G))\).

**Proposition 1.5.** Let \(M\) be a finitely generated \(A\)-module, where \(A\) is an integral domain with finitely generated \(A^+\). Let the ring \(A\) and the collection of prime ideals \(\mathcal{P} = \{p_1, \ldots, p_m\}\), \(p_i = id(\tilde{a}_i)\), satisfy the conditions of
Proposition 4.14. Assume $M$ has a $\mathbb{Z}$-basis $\bar{u} = (u_1, \ldots, u_n)$ of period $\bar{e} = (e_1, \ldots, e_n)$ associated with the $\mathfrak{P}$-series of $M$. Then there exists a formula $\phi_{\mathfrak{P}, n}(x_1, \ldots, x_n, \bar{y}_1, \ldots, \bar{y}_m)$ defining in the two-sorted structure $\langle M, A, \bar{u}_1, \ldots, \bar{u}_m \rangle$ the set of all $\mathbb{Z}$-bases of period $\bar{e}$ associated with the $\mathfrak{P}$-series for $M$, uniformly for all similar two-sorted structures $\langle N, B, \bar{b}_1, \ldots, \bar{b}_m \rangle$ where $B^+$ is finitely generated and $\bar{b}_1, \ldots, \bar{b}_n$ satisfy the formula $D_\lambda$ from Lemma 4.4.

Incidentally we also obtain a result on elementary equivalence of finitely generated modules over finite dimensional commutative rings. For us an $A$-module $M$ is a two sorted structure $\langle M, A, s \rangle$, where $M$ is an abelian group, $A$ is a ring and $s$ is the predicate describing the action of $A$ on $M$. We denote the language by $L_2$. For such a ring $A$, $r(A)$ denotes the dimension of the $\mathbb{Q}$ vector space $\mathbb{Q} \otimes A$.

Theorem 1.6. Let $A$ be an associative commutative ring with unit, $r(A) \leq n$, and $M$ be a finitely generated $A$-module. Then there exists a sentence $\psi_A$ of the language $L_2$ such that for any ring $B$ with finitely generated $B^+$ and any $B$-module $N$ we have

$$\langle N, B \rangle \models \psi_A \iff \langle N, B \rangle \equiv \langle M, A \rangle.$$ 

Another direct corollary is the following theorem.

Theorem 1.7. Assume $B$ (D) is a finite dimensional $A$-algebra ($C$-algebra) where $A$ ($C$) is associative commutative and $A^+$ ($C^+$) is finitely generated. Then $B \equiv D$ as rings if and only if $\langle A, B \rangle \equiv \langle C, D \rangle$ as two sorted algebras.

In Section 5 we present a coordinatization theorem for finitely $K$-generated groups taking exponents in a binomial principal ideal domain $K$. Recall that elements in a finitely generated torsion free nilpotent group can be represented as tuples in $\mathbb{Z}^m$, where $m$ is the Hirsch number of $G$. By a classical theorem of P. Hall product and exponentiation are computed by certain polynomials with rational coefficients and integer values on integer entries. Moreover these polynomials are uniquely determined by the so-called structure constants. Here we do not assume that the groups are torsion free. So we need more complicated but basic tools and language to describe such a group as a tuple in $K^m$ where its isomorphism type is determined by structure constants and periods of (relative) torsion elements.

Section 6 introduces terminology related to the “special gap” and also a class of finitely generated nilpotent groups where isomorphism and elementary equivalence meet. In Section 6 we actually deal with a somewhat bigger class of groups (taking exponents in a binomial PID), but for now we just restrict our attention to finitely generated nilpotent groups. So let $G$ be such a group. If $N$ is a subgroup of $G$, By $Is(N)$ we mean the isolator of $N$ in $G$, i.e. the subgroup

$$Is(N) = \{ x \in G : \exists n \neq 0 \in \mathbb{N} \ (x^n \in N) \}.$$ 

Define

$$I(G) = Is(G') \cap Z(G).$$

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The special gap \[ Z(G) \geq G' \cap Z(G) \]
will be called tame, if \( Z(G) = I(G) \).

Let us refine the special gap above with the help of \( I(G) \):

\[ Z(G) \geq I(G) \geq G' \cap Z(G). \]

The quotient \( I(G)/(G' \cap Z(G)) \) is a torsion finitely generated abelian group, and the quotient \( Z(G)/I(G) \) is a free abelian group of finite rank. Indeed \( Z(G) \) splits over \( I(G) \) and therefore there exists a free abelian group of finite rank \( G_0 \leq Z(G) \), such that \( Z(G) \cong G_0 \oplus I(G) \).

**Definition 1.8.** Any subgroup \( G_0 \leq G \) such that \( Z(G) = G_0 \oplus I(G) \) is called an addition of \( G \) and the quotient group \( G_f = G/G_0 \) is called a foundation of \( G \), associated with the addition \( G_0 \).

**Definition 1.9.** A group is called regular if \( G \cong G/G_0 \) or equivalently if \( Is(G' \cdot Z(G)) = Is(G' \cdot Z(G)) \).

The proof of the fact that the two conditions given in the above definition are equivalent is given by Proposition 6.5.

Now let us describe the main results of Section 7. This section reflects the meticulous study of what having a “maximally refined central series” for a finitely generated nilpotent group \( G \) means in understanding the structure of a finitely generated group \( H \) elementarily equivalent to it. The main auxiliary results in the section are Lemma 7.6 and Theorem 7.8. The statements are technical and long, so we do not state them here. Some of the main consequences are summarized in the following theorem.

**Theorem 1.10.** Assume \( G \equiv H \) are finitely generated nilpotent groups. Then

(a) there exists a monomorphism \( \phi : G \to H \) of groups with \( \text{im}(\phi) \) a finite index subgroup of \( H \),

(b) for any addition \( G_0 \) of \( G \) there exists an addition \( H_0 \) of \( H \) so that \( G_0 \cong H_0 \) and \( G/G_0 \cong H/H_0 \),

(c) \( Is(G') \cong Is(H') \)

(d) \( G/Is(G') \cong H/Is(H') \)

(e) \( Is(G' \cdot Z(G))/(Is(G') \cdot Z(G)) \cong Is(H' \cdot Z(H))/(Is(H') \cdot Z(H)) \).

The most immediate consequence of the Theorem 1.10 is the following theorem on elementary equivalence of regular groups.

**Theorem 1.11.** If \( G \) is finitely generated regular group and \( H \) is a finitely generated group such that \( G \equiv H \), then \( G \cong H \).
In Section 8 we give a cohomological account of the results in Section 7. Let us set $N(G) = Is(G') \cdot Z(G) = Is(G') \oplus G_0$, $M(G) = Is(G' \cdot Z(G))$ and $\bar{G} = G/N(G)$. We look at $G$ as the following extension where $\mu$ is the inclusion and $\pi$ is the canonical surjection.

$$1 \to N(G) \xrightarrow{\mu} G \xrightarrow{\pi} \bar{G} \to 1.$$ 

Recall that all the equivalence classes of the extensions of $N(G)$ by $\bar{G}$ with the same coupling, $\chi : \bar{G} \to Out(N(G))$, as this extension are in one-one correspondence with elements of the second cohomology group $H^2(\bar{G}, Z(N(G)))$ where the center $Z(N(G))$ of $N(G)$ is a $G$-module via any lifting of $\chi$ to $Aut(N(G))$. We notice that picking an addition $G_0$ for $G$, one can write $Z(N(G)) = G_0 \oplus N_1(G)$, as a direct sum of $G$-modules, where $N_1(G)$ is a subgroup of $Z(Is(G'))$. We will argue that

$$H^2(\bar{G}, Z(N(G))) \cong H^2(\bar{G}, N_1(G)) \oplus Ext\left(\frac{M(G)}{N(G)}, G_0\right)$$

$$[f] \mapsto [f_1] \oplus [f_2],$$

where $f$, $f_1$ and $f_2$ are 2-cocycles and $[f]$ denotes the class of $f$ in the corresponding cohomology group. This splitting as well as the main results from Section 7 imply that a finitely generated elementarily equivalent copy $H$ of $G$ is realized as an extension of $N(G)$ by $\bar{G}$ with the same coupling as the extension above. If the 2-cocycle defining $G$ has the form $f_1 + f_2$ then the one corresponding to $H$ has the form $f_1 + f_2'$, where the symmetric 2-cocycles $f_2$ and $f_2'$ belong to possibly different classes of $Ext(M(G)/N(G), G_0)$. We can actually say more about the 2-cocycle $f_2'$.

**Definition 1.12** (Cohomological definition of finitely generated abelian deformations). Assume $G$ is a finitely generated nilpotent group corresponding to an element

$$[f_1] \oplus [f_2] \in H^2(\bar{G}, N_1(G)) \oplus Ext\left(\frac{M(G)}{N(G)}, G_0\right) \cong H^2(\bar{G}, Z(N(G))),$$

realizing a coupling $\chi : G \to Out(N(G))$. Assume

$$\frac{M(G)}{N(G)} = \langle u_{i_0+1}^{e_{i_0}} N(G) \rangle \oplus \cdots \oplus \langle u_{i_1}^{e_{i_1}} N(G) \rangle \cong \frac{\mathbb{Z}}{e_{i_0+1} \mathbb{Z}} \oplus \cdots \oplus \frac{\mathbb{Z}}{e_{i_1} \mathbb{Z}}$$

is the invariant factor decomposition of the finite abelian group $M(G)/N(G)$. Set $c = e_{i_0+1} \cdots e_{i_1}$ and $n = i_1 - i_0$. Moreover let

$$G_0 = \langle u_{i_1+1}, \ldots, u_{i_1+n}, \ldots, u_{i_2} \rangle \cong \mathbb{Z}^p,$$

where $\mathbb{Z}$ denotes the infinite cyclic group and $p = i_2 - i_1$. Assume we are given integers $d_{i_1+1}, \ldots, d_{i_1+n}$ and an $n \times n$ matrix $(c_{ij})$ of integers such that

(a) $gcd(d, c) = 1$ where $d = d_{i_1+1} \cdots d_{i_1+n}$
Define the group \( \text{Abdef}(G, \bar{g}, \bar{c}) \) as the extension of \( N(G) \) by \( \bar{G} \) with the same coupling \( \chi : G \to \text{Out}(N(G)) \) defining \( G \), corresponding to \([f_1] \oplus [f_2']\), where

\[
f'_2 = \sum_{i=0}^{t_1} f'_{2t}, \quad f'_{2t} \in S^2(\langle u_i N(G) \rangle, G_0),
\]

and

\[
f_{2t}(u^s N(G), u^t N(G)) = \begin{cases} 1 \prod_{k=i_1+1}^{i_1+n} c^k & \text{if } s + t < e_i \\ \prod_{k=i_1+1}^{i_1+n} c^k & \text{if } s + t \geq e_i \end{cases}
\]

Any group \( H \) isomorphic to \( \text{Abdef}(G, \bar{d}, \bar{c}) \) is called an abelian deformation of \( G \).

Theorem 1.13 (Characterization Theorem). Assume \( G \) is a finitely generated nilpotent group and \( H \) is finitely generated group such that \( G \equiv H \). Then there exist tuples of integers \( \bar{d} \) and \( \bar{c} \), satisfying the conditions in Definition 1.12 such that

\[ H \cong \text{Abdef}(G, \bar{d}, \bar{c}). \]

The proof of the theorem is included in Section 8. The converse is proven in Section 9.

Finally in Section 10 we analyze an example given by B. Zilber [13] of two finitely generated 2-nilpotent groups which are elementarily equivalent but not isomorphic. We will show that they can be looked as abelian deformations of one another.

2 Bilinearization of nilpotent groups

In the this section we define a non-degenerate bilinear mapping \( F_R \), which is canonically associated with an arbitrary central series \((R)\) of a nilpotent group \( G \). In a certain sense, the mapping \( F_R \) is a generalization of the Lie ring associated with the group \( G \). The sense of the bilinearization consists in the fact that though the mapping \( F_R \) is arranged much simpler than the initial nilpotent group \( G \), it nevertheless reflects some important properties of the group \( G \) deeply enough. In particular, reduction from \( G \) to \( F_R \) plays an important role in the model theoretical study of nilpotent group \( G \). Furthermore, with the help of \( F_R \) we construct a commutative associative ring \( A_R \) with unit, acting on quotients of a new central series of the group \( G \) which is associated with \((R)\) in a certain way. It will be shown below that the ring \( A_R \) is an important invariant of \( G \), not only an algebraic one but also a logical one as well.
2.1 Largest ring of a bilinear map

Let us first state a result due to A. Myasnikov [10].

**Theorem 2.1** ([10]). Let $R$ be a commutative ring with unit. Assume $M_1$, $M_2$ and $N$ are exact $R$-modules. Let $f : M_1 \times M_2 \to N$ be a non-degenerate full bilinear mapping of finite type. Assume

$$\mathfrak{U}_R(f) = \langle R, M_1, M_2, N, \delta, s_{M_1}, s_{M_2}, s_N \rangle,$$

where the predicate $\delta$ describes $f$ and $s_{M_1}$, $s_{M_2}$ and $s_N$ describe the actions of $R$ on the modules $M_1$, $M_2$ and $N$ respectively, and

$$\mathfrak{U}(f) = \langle M_1, M_2, N, \delta \rangle.$$

Then there is the largest ring $P(f)$ with respect to which $f$ remains bilinear and the structure $\mathfrak{U}_P(f)$ is absolutely interpretable in $\mathfrak{U}(f)$. Moreover the formulas involved in the interpretation depend only on the type of $f$.

Recall that an $R$-bilinear mapping $f : M_1 \times M_2 \to N$ is called non-degenerate in the first variable if $f(x, M_2) = 0$ implies $x = 0$. Non-degeneracy with respect to the second variable is defined similarly. The mapping $f$ is called non-degenerate in both variables if it is non-degenerate with respect to first and second variables. We call the bilinear map $f$, a full bilinear mapping if $N$ is generated by $f(x, y)$, $x \in M_1$ and $y \in M_2$. Let $f : M_1 \times M_2 \to N$ be a non-degenerate full $R$-bilinear mapping for some commutative ring $R$. The mapping $f$ is said to have finite width if there is a natural number $s$ such that for every $u \in N$ there are $x_i \in M_1$ and $y_i \in M_2$ such that

$$u = \sum_{i=1}^{s} f(x_i, y_i).$$

The least such number, $w(f)$, is the width of $f$.

A set $E_1 = \{e_1, \ldots, e_n\}$ is a left complete system for a non-degenerate mapping $f$ if $f(E_1, y) = 0$ implies $y = 0$. The cardinality of a minimal left complete system for $f$ is denoted by $c_1(f)$. A right complete system and the number $c_2(f)$ are defined correspondingly.

The type of a bilinear mapping $f$, denoted by $\tau(f)$, is the triple

$$(w(f), c_1(f), c_2(f)).$$

The mapping $f$ is said to be of finite type if $w(f)$, $c_1(f)$ and $c_2(f)$ are all finite numbers. If $f, g : M_1 \times M_2 \to N$ are bilinear maps of finite type we say that the type of $g$ is less than the type of $f$ and write $\tau(g) \leq \tau(f)$ if $w(g) \leq w(f)$, $c_1(g) \leq c_1(f)$ and $c_2(g) \leq c_2(f)$. The ring $P(f)$ has the following description. Let $\text{End}_R(M)$ denote the ring of $R$-endomorphisms of an $R$-module $M$. We shall identify $P(f)$ with the subring $S \leq \text{End}_R(M_1) \times \text{End}_R(M_2) \times \text{End}_R(N)$ of all triples $(\phi_1, \phi_2, \phi_0)$ such that,

$$f(\phi_1(x), y) = f(x, \phi_2(y)) = \phi_0(f(x, y)) \ \forall (x, y) \in M_1 \times M_2. \ \ (1)$$
We would like also to remark that the definition in [10, 11] of $P(f)$ does not appear to be the one given above, but it can be easily checked that the two definitions are indeed equivalent.

2.2 Construction of $F_R$

Let $G = R_1 > R_2 > \ldots > R_c > R_{c+1} = 1$, $(R)$,

be an arbitrary central series of the nilpotent group $G$. Let

$$R^u_i = \{ x \in G : [x,G] \subseteq [R_i,G]\}, \quad 1 \leq i \leq c,$$

and

$$R^l_i = G \text{ and } R^l_i = [R_{i-1},G], \quad 2 \leq i \leq c + 1.$$

Clearly each $R^u_i$ and $R^l_i$ is a subgroup of $G$, and

$$R^u_i \geq R_i \geq R^l_i, \quad 1 \leq i \leq c,$$

and

$$[R^u_i,G] = [R_i,G] = R^l_i, \quad 1 \leq i \leq c - 1.$$

Hence

$$G = R^u_1 > R^u_2 > \ldots > R^u_c = Z(G) \quad (R^u),$$

$$G = R^l_1 > R^l_2 = G' > R^l_3 > \ldots > R^l_{c+1} = 1 \quad (R^l),$$

are central series for $G$.

**Definition 2.2.** The series $(R^u)$ and $(R^l)$ are called the upper and lower series associated with $(R)$.

The operation of commutation induces non-degenerate with respect to the second variable full bilinear mappings,

$$f_i : G/G' \times R^u_i/R^u_{i+1} \rightarrow R^l_i/R^l_{i+2}, \quad 1 \leq i \leq c - 1.$$ 

Each $f_i$, $1 \leq i \leq c - 1$ is well defined due to the inclusions:

$$[G', R^u_i] = [[G, G], R^u_i] \subseteq [[R^u_i, G], G]$$

$$= [[G, R^u_i], G] = [[R_i, G], G] \subseteq [R_{i+1}, G]$$

$$= R^l_{i+2}.$$ 

Non-degenerateness is obvious due to the definition of $R^u_{c+1}$. The mappings constructed form a bundle $S_R = \{f_1, \ldots, f_{c-1}\}$ of bilinear mappings, with the series $(R)$.

Let

$$V_i = \{ x \in G : [x, R^u_i] \subseteq R^l_{i+2} \}$$

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be the kernel of $f_i$ with respect to the first variable. Let

$$V_R = \{ \cap_{i=1}^{c-1} V_i \geq G' \cdot Z(G) \},$$

$$R^u = R_1^u / R_2^u \oplus \ldots \oplus R_{c-1}^u / R_c^u,$$

$$R^l = R_2^l / R_3^l \oplus \ldots \oplus R_{c-1}^l / R_c^l.$$

We now introduce a full non-degenerate with respect to both variable bilinear mapping

$$F_R : G/V_R \times R^u \to R^l,$$

which canonically corresponds to the bundle $S_R$ and is defined according to the rule:

$$F_R(x, \sum_{i=1}^{c-1} x_i) = \sum_{i=1}^{c-1} f_i(x_0, x_i),$$

where $x \in G/V_R$, $x_i \in R_i^u / R_{i+1}^u$ and $x_0$ is an arbitrary pre-image of $x$ in $G/G'$.

Let $P_R = P(F_R)$ be the largest ring of scalars of the mapping $F_R$ (see Section 2.1). Then one can define a $P_R$-module structure on the abelian groups $G/V_R$, $R_i^u / R_{i+1}^u$ such that $F_R$ will be $P_R$-bilinear.

**Definition 2.3.** $F_R$ will be called the mapping associated with the series $(R)$.

**Example 2.4.** Consider the lower central series

$$G = \Gamma_1(G) > \ldots > \Gamma_c(G) / 1 = \Gamma_c(G), \quad (\Gamma)$$

of the group $G$. Then we have $\Gamma_i'(G) = \Gamma_i(G)$. So the action of the ring $P_{\Gamma}$ on all the quotients

$$\Gamma_2(G) / \Gamma_3(G), \ldots, \Gamma_c(G) / 1 = \Gamma_c(G)$$

of the lower central series is defined. Note that $G / \Gamma_2(G)$ is excluded from the list above.

**Example 2.5.** For the upper central series

$$Z_c(G) > \ldots > Z_1(G) = Z(G) > 1, \quad (Z)$$

we have $Z_i^u(G) = Z_i(G)$, and the action of the ring $P_Z$ on the quotients

$$Z_c / Z_{c-1}, \ldots, Z_2 / Z_1$$

is defined i.e. on all the quotients of the upper central series except the center $Z(G)$.

**Example 2.6.** Let $G$ be 2-nilpotent.

1. From the upper central series

$$G = Z_2 > Z_1 = Z(G) > 1$$

we obtain $Z_c^u = Z_c, \ Z_1^l = G', \ V_Z = Z(G)$. Therefore $F_Z$ is the standard bilinear mapping,

$$F_Z : G/Z(G) \times G/Z(G) \to G'.$$
2. For the lower central series

\[ G = \Gamma_1(G) > \Gamma_2(G) = G' > 1 \]

we obtain \( \Gamma'_i(G) = \Gamma_i(G), \Gamma''_i(G) = Z(G) \) and \( V_T = Z(G) \), hence

\[ F_T : G/Z(G) \times G/Z(G) \to G' \]

and \( F_T = F_Z \).

Note that in both cases above the action of the ring \( P_T = P_Z \) is defined on all the quotients except the quotient \( Z(G)/G' \) (if it is non-trivial). The gap \( Z(G) \geq G' \) is a special one and its role will be studied in the next section.

### 2.3 Coordinatization of the action of the ring of scalars

Assume \( A \) is a binomial domain. For a nilpotent \( A \)-group \( G \) the action of the ring \( A \) in various quotients of \( G \) is coordinated, i.e. if \( H_1 \triangleleft H_2 \) are \( A \) subgroups of \( G \), then the canonical epimorphism \( H_2 \to H_2/H_1 \) is an \( A \)-epimorphism. In this section we build subrings of the ring \( P_R \), which possess the property of coordinatization on the quotients where the action of \( P_R \) is defined. For a definition of groups admitting exponents in a binomial ring \( R \), or \( R \)-groups for short we refer the reader to [3].

Denote by \( L_R \) the set of canonical homomorphisms \( e_i : R^l_i/R^l_{i+1} \to R^u_i/R^u_{i+1} \), induced by the inclusions \( R^l_i \to R^u_i \). There exists the largest subring \( PL_R \) of the ring \( P_R \) (with respect to inclusion) such that all homomorphisms of \( L_R \) are \( PL_R \)-linear.

Finally let us build a ring which acts simultaneously on quotients of some more central series, e.g., of the upper and lower central series. Let

\[ G = Q_1 > \ldots > Q_c > 1, \quad (Q) \]

be one more central series of the group \( G \), and let \( S_Q \) be the corresponding bundle of bilinear mappings. Consider the bundle \( S_R \cup S_Q \), the bilinear mapping \( F_{R\cup Q} \) corresponding to this bundle and its largest ring of scalars \( P_{R\cup Q} \). According to the construction the ring \( P_{R\cup Q} \) acts simultaneously in the quotients of series associated with \((R)\) and \((Q)\).

The two constructions introduced above can be easily generalized to an arbitrary set \( C \) of central series of the group \( B \) and an arbitrary set \( L \) of homomorphisms between the quotients of series associated with \( C \). In fact, let \( S_R \) be the bundle of mappings corresponding to the series \((R)\) from \( C \). Let \( S_C = \bigcup_{R \in C} S_R \) and \( F_C \) be the corresponding bilinear mapping. Denote by \( PL_C \leq P_C \) the largest subring (with respect to inclusion) such that all homomorphisms from \( L \) are \( PL_C \)-linear. The ring \( PL_C \) is a convenient tool because it allows one to pass freely in reasoning from one series to another (in \( C \)), to refine one series by means of another and so on.
Proposition 2.7. Let $G$ be finitely generated nilpotent $A$-group over a Noetherian ring $A$, $C$ be an arbitrary set of central $A$-series of $G$ and $L$ an arbitrary set of homomorphisms between quotients of series associated with $C$. Then $PL_C$ is a finite dimensional commutative associative $A$-algebra with a unit.

Proof. According to the construction (see Subsection 2.1) $PL_C$ is an $A$-subalgebra of the algebra of $A$-endomorphisms $End_A(G/V_C)$ of the $A$-module $G/V_C$. According to the construction $V_C \geq G'$ and according to the condition $G/G'$ is a finitely generated $A$-group. Consequently the $A$-module $G/V_C$ is finitely generated so is the $A$-module $End_A(G/V_C)$. Moreover this module is Noetherian since $A$ is a Noetherian ring. Hence $PL_C$ is a finitely generated $A$-module.

2.4 Maximally refined series

We note that the ring $P_R$ acts exactly on all the quotients of the upper series except on the one corresponding to the lowest gap $R^u_c = Z(G) > 1$, and on all the quotients of the lower series except on the quotient of the upper gap $G > G' = R^l_1$. On the other hand, $P_R$ acts on the quotient $G/V_R$. Since $V_R \geq G' \cdot Z(G)$, then in the case of $Z(G) \nsubseteq G'$ there always remains a gap, $G' \cdot Z(G) > G'$, on the quotient of which the ring $P_R$ does not act. Similarly in a 2-nilpotent group $G$ if the gap $Z(G) \geq G'$ is nontrivial ($Z(G) \neq G'$), then the action of the ring $P_R$ on the quotient $Z(G)/G'$ is undefined (see Example 2.6).

In this section we introduce a simple construction which allows one to maximally refine the series $(R^u)$ and $(R^l)$ so that the ring $P_R$ acts on all the quotients of the refined series, except the fixed special gap (if it is not trivial). For the refinement of $(R^u)$ the special gap is $Z(G) \geq Z(G) \cap G'$, and for the refinement of $(R^l)$ it is $G' \cdot Z(G) \geq G'$. Note that the quotients of special gaps are isomorphic in the two cases. The special gaps are essential and there is no way to get rid of them.

It was shown earlier that the ring $P_R$ acts on the quotients of the associated central series $(R^u)$ and $(R^l)$. We refine the series $(R^u)$ in the last term $Z(G) > 1$ with the help of the intermediate terms $R^l_i \cap Z(G)$, $i = 2, \ldots, c$, which form a series

$$Z(G) \geq G' \cap Z(G) = R^l_2 \cap Z(G) \geq \ldots \geq R^l_c \cap Z(G) = 1.$$  

So we obtain the refined upper series $(U(R))$ associated with $(R)$:

$$G = R^l_1 \geq R^l_2 \geq \ldots \geq Z(G) \geq G' \cap Z(G) \geq \ldots \geq 1 \quad (U(R)).$$

Consider the set $E = \{\epsilon_i : i = 2, \ldots, c\}$ of the canonical monomorphisms:

$$\epsilon_i : T_i = (R^l_i \cap Z(G))/(R^l_{i+1} \cap Z(G)) \rightarrow R^l_i/R^l_{i+1} = S_i,$$

and denote by $AE_R$ the largest subring of $PL_R$, which leaves all the images $\epsilon_i(T_i)$ in the $PL_R$-module $S_i$ invariant. Then the action of $AE_R$ on $S_i$ induces an action of $AE_R$ on $T_i$ with the help of the monomorphisms $\epsilon_i$ according to the rule:
\[ \alpha x = \epsilon^{-1}(\alpha \epsilon(x)), \quad \alpha \in AE_R, x \in T_i. \]

In this way, the action the ring \( AE_R \) is defined on all the quotients of the series \( (U(R)) \), except the special gap \( Z(G) \geq Z(G) \cap G' \), if it is non-trivial.

Dually starting from the lower series \( (R^l) \) associated with \( (R) \), we construct the refined lower series \( (L(R)) \) associated with \( (R) \). Namely we refine the series

\[ G > V_R \geq G' = R^l_2 \geq \ldots \geq R^l_c = 1 \]

in the gap \( V_R \geq G' \) by means of the upper associated series, i.e. we insert the series

\[ V_R = (V_R \cap R^u_1) \cdot G' \geq (V_R \cap R^u_2) \cdot G' \geq \ldots \geq (V_R \cap R^u_c) \cdot G' = Z(G) \cdot G'. \]

As a result \( (L(R)) \) is of the form

\[ G \geq V_R \geq (V_R \cap R^u_2) \cdot G' \geq \ldots \geq Z(G) \cdot G' \geq G' = R^l_2 > \ldots > R^l_c = 1 \quad (L(R)). \]

Note that \( ((V_R \cap R^u_i) \cdot G')/(V_R \cap R^u_{i+1}) \cdot G' \cong (V_R \cap R^u_i)/(V_R \cap R^u_{i+1}) \) and consider the set of monomorphisms \( \Delta = \{ \delta_i : i = 1, \ldots, c \} \):

\[ \delta_i : X_i = ((V_R \cap R^u_i) \cdot G')/(V_R \cap R^u_{i+1}) \cdot G' \rightarrow R^u_i/R^u_{i+1} = Y_i, \]

and denote by \( AD_R \) the largest subring of \( PL_R \), which leaves all submodules \( \delta_i(X_i) \) in \( Y_i \) invariant. Like \( AE_R \), the ring \( AD_R \) acts on all the quotients of the series \( (L(R)) \), except the special gap \( Z(G) \cdot G' \geq G' \), if it is non-trivial.

Let \( A_R = AE_R \cap AD_R \). Then the ring \( A_R \) acts on all the quotients of the series \( (U(R)) \) and \( (L(R)) \) except the special gaps. Moreover in both cases the special gaps give the same quotients,

\[ (Z(G) \cdot G')/G' \cong Z(G)/(Z(G) \cap G') \]

up to isomorphism.

We close the section with stating three propositions. We omit the proofs as they are all clear.

**Proposition 2.8.** Let \( G \) be a finitely generated nilpotent \( A \)-group over a Noether ring \( A \) and \( (R) \) be a central series of \( A \)-subgroups of \( G \). Then \( A_R \) is an associative commutative finite dimensional \( A \)-algebra with a unit.

**Proposition 2.9.** Let \( G \) be a nilpotent \( A \)-group and \( (R) \) be an arbitrary central series of \( A \)-subgroups of \( G \). Then \( A_R \) is an associative commutative \( A \)-algebra with a unit.

**Proposition 2.10.** If \( G \) is a finitely generated \( A \)-group, and \( A \) is Noetherian then,

1) \( A_R \) is a finite dimensional \( A \)-algebra.

2) If \( C \) is the set of all central \( A \)-series of the group \( G \), then there exists a finite subset \( C_0 = \{ R_1, \ldots, R_n \} \) of \( C \) such that \( A_{C_0} = A_{C_0} \) and \( A_{C_0} \) is a finite dimensional \( A \)-algebra.
3 Some logical invariants of finitely generated nilpotent groups

In this section we study the logical properties of the constructions introduced in the previous section. In particular we will show that all the constructions associated with the bilinearization of \( G \), are definable by formulas of the language of groups.

3.1 Uniform interpretability of the bilinearization

In this subsection it will be shown that bilinearization of an arbitrary finitely generated nilpotent group \( G \) is absolutely interpretable in the group \( G \) by a system of formulas of the signature of groups. Besides, the same formulas absolutely interpret bilinearization in any group \( H \) which is elementary equivalent to \( G \). In other words, bilinearization in the group \( G \) is a logical invariant of \( G \).

Moreover, this is true for arbitrary finitely generated \( A \)-groups over a binomial ring \( A \), e.g. for finitely generated nilpotent pro-\( p \)-groups, unipotent \( k \)-groups over a field \( k \) of zero characteristic. Most of the arguments here are quite well-known and standard.

Let us agree on the terminology. Let \( T \) be a theory of language \( \sigma \). suppose that \( S : \text{Mod}(T) \to K \) is a functor defined on the class \( \text{Mod}(T) \) of all models of the theory \( T \) (a category with isomorphisms) into a certain category \( K \) of structures of language \( \sigma \). We shall say that \( S \) is \textit{absolutely interpretable} in \( T \), if there exists a system of formulas \( \Phi \) of the language \( \sigma \), which absolutely interprets the system \( S(U) \) in any model \( U \) of the theory \( T \). In this case we shall also say that \( S(U) \) is \textit{absolutely interpretable in} \( U \) \textit{uniformly with respect to} \( T \).

For example, the center \( Z(G) \) is interpretable (or in this case definable) in \( G \) uniformly with respect to the theory of groups. In particular, it is a logical invariant of \( G \). On the other hand, the commutator subgroup \( G' \), generally speaking, is not interpretable in \( G \) uniformly with respect to the theory of groups. However, \( G' \) is a logical invariant of an arbitrary finitely generated nilpotent group \( G \). Interpretability of \( G' \) and other verbal subgroups of a finitely generated \( A \)-exponential nilpotent group \( G \) is related to the notion of finite width. Let a subgroup \( N \) of the group \( G \) be generated by the set \( X \). We say that \( N \) is of \( width S \) (with respect to \( X \)), if any element of \( N \) can be represented as a product of no more than \( S \) elements of the set \( X \) and their inverses, \( S \) being the minimal number with this property. The width of the subgroup \( N = \langle X \rangle \) will be denoted by \( S_X(N) \) or simply \( S(N) \).

Let \( v(x_1, \ldots, x_n) = v(\bar{x}) \) be a group word. The set
\[
V = \{ v(\bar{g}) : \bar{g} \in G^n \}
\]
is called a \textit{value set} of \( v \) in \( G \). The subgroup \( v(G) \) generated by the set \( V \) is called a \textit{verbal subgroup}, defined by the word \( v \). By the width of the subgroup \( v(G) \) we mean the width with respect to the value set \( V \).
Let $\varphi(x)$ be a formula of the language of groups. Consider the functor $f_\varphi$ which associates to any group $G$ its subgroup $f_\varphi(G)$, generated by the definable subset 

$$\varphi(G) = \{ g \in G : G \models \varphi(g) \}.$$

**Proposition 3.1.** Let $G$ be a group. Then if the subgroup $f_\varphi(G)$ is of finite width, then $f_\varphi(G)$ is definable in $G$ uniformly with respect to $\text{Th}(G)$.

**Proof.** Let $s$ be the width of the subgroup $f_\varphi(G)$ with respect to $\varphi(G)$. Then the subgroup $\varphi(G)$ is defined in $G$ by the formula:

$$\Phi_s(x) = \exists x_1, \ldots, \exists x_s(x = \prod_{i=1}^s x_i \land \bigwedge_{i=1}^s (\varphi(x_i) \lor \varphi(x_i^{-1}))).$$

The group $G$ and hence any group $H \cong G$ satisfies the sentence

$$\forall x (\Phi_{s+1}(x) \to \Phi_s(x)),$$

which allows to contract any product of elements of the set $\varphi(G) \cup \varphi(G)^{-1}$ to a product of no more than $s$ factors of $\varphi(G) \cup \varphi(G)^{-1}$. Consequently, if $H \cong G$, then the width of $f_\varphi(G)$ is not greater than $s$, and hence it is exactly $s$. Therefore $\Phi_s(x)$ defines $f_\varphi(H)$ in all models $H$ of $\text{Th}(G)$. The proposition is proved.

**Corollary 3.2.** Any verbal subgroup $v(G)$ of finite width is definable in $G$ uniformly with respect to $\text{Th}(G)$.

**Lemma 3.3.** Let $G$ be a $c$-nilpotent $A$-exponential group generated by the set $X = \{x_1, \ldots, x_n\}$. let $\varphi(G)$ be a definable normal subgroup of $G$. Then the subgroup $[\varphi(G), G]$ is of width not greater than $n(c - 1)$ with respect to the set of generators

$$[\varphi(G), X] = \{ [y, x] : y \in \varphi(G), x \in X \}.$$

**Proof.** Let $R_0 = \varphi(G), R_i = \varphi(G), G, \ldots, G$, $i = 1, \ldots, c$, and consider the central series of subgroups

$$\varphi(G) = R_0 \geq [\varphi(G), G] = R_1 \geq R_2 \geq \ldots \geq R_c = 1.$$

Since $[R_i, G] = R_{i+1}$, then for $y \in R_i$, $g_1, g_2 \in G$ and $a \in A$ we have

$$[y, g_1 g_2] R_{i+2} = [y, g_1][y, g_2] R_{i+2}$$

$$[y, g_1^a] R_{i+2} = [y^n, g_1] R_{i+2}. \quad (2)$$

Hence for $y_k \in \varphi(G), g_k \in G$, decomposing $g_k$ with the help of generators from $X$ and using congruences $2$, we obtain:

$$\prod_{k=1}^n [y_k, g_k] = \prod_{k=1}^n [y_k(0), x_k] r_2,$$
where \( y_k(0) \in R_0 = \varphi(G) \), \( r_2 \in R_2 \). Continuing this process for \( r_2 \), we obtain by induction

\[
\prod_{k=1}^{n} [y_k, g_k] = \prod_{j=1}^{c-1} \prod_{i=1}^{n} [y_i(j), x_i],
\]

where \( y_i(j) \in R_j \leq \varphi(G) \). Consequently, the width of \([\varphi(G), G]\) with respect to the set of generators \([\varphi(G), G]\) does not exceed \((c - 1)n\).

\[ \square \]

**Corollary 3.4.** Assume \( G \) is a finitely generated A-exponential nilpotent group. Then each term of the lower central series of \( G \) is a subgroup of finite width.

**Proposition 3.5.** Assume \( G \) is a finitely generated A-exponential group. Then:

1. each term of the upper central series

\[
G = Z_0(G) > Z_1(G) > \ldots > Z(G) > 1 \quad (Z)
\]

is definable in \( G \) uniformly with respect to \( Th(G) \),

2. each term of the lower central series

\[
G = \Gamma_1(G) > \Gamma_2(G) > \ldots > \Gamma_c(G) > 1
\]

is definable in \( G \) uniformly with respect to \( Th(G) \).

**Proof.** The formulas \( \Phi_i(x) \), \( i = 0, 1, \ldots, c \), defining the subgroups \( Z_i(G) \) in \( G \), can be defined recursively as follows.

\[
\Phi_i(x) =_{df} \forall y \Phi_{i-1}([x, y]), \quad \Phi_0(x) =_{df} (x = 1),
\]

Consequently,

\[
\Phi_1(x) = \forall y_1, \ldots, y_1([x, y_1, \ldots, y_1] = 1)
\]

and \( Z_i(G) \) are definable \( G \) uniformly with respect to \( Th(G) \). Statement 2 follows from Corollary 3.4 and Proposition 3.1.

\[ \square \]

**Lemma 3.6.** Assume \( N_i \trianglelefteq G_i \), \( i = 1, 2, \ldots, n \), for some natural number \( n \), are uniformly definable subgroups of a group \( G \). Then,

1. all quotient groups \( G_i/N_i \) are interpretable in \( G \) uniformly with respect to \( Th(G) \),

2. The direct sum \( \bigoplus_{i=1}^{n} G_i/N_i \) is interpretable in \( G \) uniformly with respect to \( Th(G) \).

Proof of the lemma is an elementary exercise in model theory so it is omitted.
Proposition 3.7. Assume $G$ is a finitely generated $A$-exponential nilpotent group, and any term of the series

\[ G = R_c > R_{c-1} > \ldots > R_1 > R_0 = 1 \quad (R) \]

is interpretable in $G$ uniformly with respect to $\text{Th}(G)$ . Then:

1. all the terms $R_u^i$ of the upper series associated with $(R)$ are definable in $G$ uniformly with respect to the $\text{Th}(G)$,
2. all the terms $R_l^i$ of the lower series associated with $(R)$ are definable in $G$ uniformly with respect to the $\text{Th}(G)$;
3. the system $\mathbf{M}(F_R)$ corresponding to the bilinear mapping $F_R$ is interpretable in $G$ uniformly with respect to the $\text{Th}(G)$,
4. the rings $P_R$ and $PL_R$ and their actions on the quotients $R_u^i/R_u^{i-1}$ and $R_l^i/R_l^{i-1}$ are interpretable in $G$ uniformly with respect to the $\text{Th}(G)$.

Proof. Following the construction of bilinearization from Subsection 2.2 let us verify statements 1 - 4. Items 1 and 2 immediately follow from Lemma 3.3 and Proposition 3.1. Statement 3 follows easily from 1, 2, Lemma 3.6 and the fact that we use only commutators $[x,y] = x^{-1}y^{-1}xy$ to define the bilinear map $F_R$. Item 4 is true due to Theorem 2.1 and interpretability of canonical homomorphisms

\[ e_i : R_u^i/R_u^{i-1} \rightarrow R_l^i/R_l^{i-1}. \]

The proposition is proved.

Proof of Proposition 3.2. The proof is similar to that of Proposition 3.7.

4 Interpretations in finite dimensional commutative algebras and elementary equivalence

In this section we describe by first order formulas some algebraic invariants of any associative commutative ring $A$ with finitely generated additive group $A^+$. After that we apply our results to the problem of elementary equivalence of such rings and modules over them.

4.1 Interpretability of decomposition of zero into the product of prime ideals with fixed characteristic

Let $A$ be an arbitrary associative commutative ring with a unit. Suppose that we have a decomposition of zero into the product of prime finitely generated ideals:

\[ 0 = p_1 \cdot p_2 \cdots p_m, \quad (\mathcal{P}) \]

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Let $Char(p_i) = \lambda_i$ be the characteristic of the integral domain $A/p_i$ and \\
$Char(\mathfrak{P}) = (\lambda_1, \ldots, \lambda_m)$.

The purpose of this subsection is to obtain a formula interpreting the decomposition of type $P$ in $A$ with the fixed characteristic $Char(\mathfrak{P})$, where the interpretation is uniform with respect to $Th(A)$.

A sequence of lemmas will follow. We omit some proofs as they are obvious.

**Lemma 4.1.** Consider the formula \\
$Id(x, \bar{y}) = \exists z_1, \ldots, \exists z_n (x = y_1z_1 + \ldots + y_nz_n)$.

For any tuple $\bar{a} = (a_1, \ldots, a_n) \in A^n$ the formula $Id(x, \bar{a})$ defines in $A$ the ideal $id(\bar{a})$, generated by the elements $a_1, \ldots, a_n$.

**Lemma 4.2.** The formula \\
$P(\bar{y}) = \forall x_1, \forall x_2 (Id(x_1x_2, \bar{y}) \rightarrow (Id(x_1, \bar{y}) \lor Id(x_2, \bar{y})))$

is true for the tuple $\bar{a}$ of elements of the ring $A$ if and only if the ideal $id(\bar{a})$ is prime.

**Lemma 4.3.** The formula: \\
$D(\bar{y}_1, \ldots, \bar{y}_m) = \forall x (\bigwedge_{i=1}^m Id(x, \bar{y}_i) \land \bigwedge_{i=1}^m P(\bar{y}_i) \rightarrow x = 0)$

is true for tuples $\bar{a}_1, \ldots, \bar{a}_m$ if and only if the ideals $p_i = Id(\bar{a}_i)$ satisfy the decomposition $(P)$.

**Lemma 4.4.** The formula \\
$D_{\Lambda}(\bar{y}_1, \ldots, \bar{y}_m) = D(\bar{y}_1, \ldots, \bar{y}_m) \land \bigwedge_{i=1}^m \forall x Id(\lambda_i x, \bar{y}_i) \land \bigwedge_{i=1}^m \exists z \neg Id(z, \bar{y}_i)$

where $\Lambda = (\lambda_1, \ldots, \lambda_m) = Char(\mathfrak{P})$ is true for tuple $\bar{a}_1, \ldots, \bar{a}_m$ of elements of $A$ if and only if the ideals $p_i = Id(\bar{a}_i)$ satisfy the decomposition $(\mathfrak{P})$, if $\lambda_i > 0$ then $Char(A/p_i) = \lambda_i$ and integral domains $A/p_i$ are non-zero.

Denote by $0(\mathfrak{P})$ the number of zeros in the tuple $(\lambda_1, \ldots, \lambda_m)$.

**Proposition 4.5.** Let $\mathfrak{P} = (p_1, \ldots, p_m)$ be a collection of finitely generated prime ideals of the ring $A$, satisfying the decomposition $(\mathfrak{P})$ and possessing the least number $0(\mathfrak{P})$ among all such decompositions. Then for any ring $B$ if $A \equiv B$, then the formula $D_{\Lambda}(\bar{y}_1, \ldots, \bar{y}_m)$ is true in $B$ on tuples $\bar{b}_1, \ldots, \bar{b}_m$, if and only if:

1. $Id(x, \bar{b}_i)$ defines the prime ideal $q_i = id(\bar{b}_i)$,
2. \( 0 = q_1 \cdot q_2 \cdots q_m \),

3. \( \text{Char}(B/q_i) = \lambda_i, \; i = 1, \ldots, m \).

**Proof.** Items 1 and 2 follow from Lemma 4.3. If \( \lambda_i > 0 \) then \( \text{Char}(B/q_i) = \lambda_i \) according to Lemma 4.4. Consequently, \( 0(\mathfrak{P}) \geq 0(\Omega) \) where \( \Omega = (q_1, \ldots, q_m) \). If \( 0(\mathfrak{P}) > 0(\Omega) \) then starting from \( \Omega \) we construct the formula \( D_\mu, \mu = \text{char}(\Omega) \). From \( A \equiv B \) and Lemma 4.4 we obtain that there exists a tuple \( \mathfrak{P}' = (p'_1, \ldots, p'_m) \) such that \( 0(\mathfrak{P}') < 0(\mathfrak{P}) \) which contradicts the choice of \( \mathfrak{P} \). Consequently \( 0(\mathfrak{P}) = 0(\Omega) \) and hence \( \text{Char}(\mathfrak{P}) = \text{Char}(\Omega) \). The proposition is proved.

**Remark 4.6.** Any Noetherian commutative associative ring with a unit possesses a decomposition of zero \( 0 = p_1 \cdots p_m \), satisfying the assumptions of Proposition 4.5.

**Proof of Proposition 4.4** The proposition is a direct corollary of Proposition 4.5 and Remark 4.6.

4.2 The case of a ring with a finitely generated additive group

Let \( A \) be a commutative associative ring with unit and a finitely generated additive group \( A^+ \). We shall denote by \( r(A) \) the dimension of \( \mathbb{Q} \otimes A \) as \( \mathbb{Q} \)-vector space. In case that \( M \) is a finitely generated \( A \)-module, \( r(M) \) is the dimension of \( \mathbb{Q} \otimes M \) as a \( \mathbb{Q} \) vector space.

Let us start with stating and proving a well-known statement regarding the uniform interpretability of the ring of integers in a ring \( A \) as above.

**Lemma 4.7.** Let \( A \) be a commutative associative ring with unit, \( \text{char}(A) = 0 \) and \( r(A) = n \neq 0 \) for some natural number \( n \). there exists a formula \( R_n(x) \) defining the subring \( \mathbb{Z} \cdot 1 \) in any integral domain \( B \equiv A \) with \( \text{char}(B) = 0 \) and the rank \( r(B) \leq n \).

**Proof.** We just need to note that the field of fractions \( F \) of \( A \) is an extension of rationals \( \mathbb{Q} \) with dimension \( n \) over \( \mathbb{Q} \). So \( F \) is a field of algebraic numbers of finite degree over \( \mathbb{Q} \). Moreover \( F \) is uniformly interpretable in \( A \). Now the statement follows from the celebrated theorem of Julia Robinson on undecidability of algebraic number fields of finite degree over \( \mathbb{Q} \). (See Theorem on page 956 of [16]).

**Lemma 4.8.** There exists a formula \( R_{n,\lambda}(x, \bar{y}) \) such that for any ring \( A \) with \( r(A) \leq n \) and for any prime ideal \( \mathfrak{p} = \text{id}(\bar{a}) \) of \( A \) if \( \text{char}(A/\mathfrak{p}) = \lambda \) then the formula \( R_{n,\lambda}(x, \bar{a}) \) defines subring

\[ Z \cdot 1 + \mathfrak{p} = \{ z \cdot 1 + x : z \in \mathbb{Z}, x \in \mathfrak{p} \}, \]

in \( A \).
Proof. Indeed the ideal \( \mathfrak{p} = id(\bar{a}) \) is defined in \( A \) by the formula \( Id(x, \bar{a}) \). Consequently the ring \( A/\mathfrak{p} \) and the canonical epimorphism \( A \to A/\mathfrak{p} \) are interpretable in \( A \). Therefore to obtain \( R_{n,\lambda}(x, \bar{y}) \) it is sufficient to define the subring \( Z \cdot 1 \) in \( A/\mathfrak{p} \). In the case of \( \text{Char}(A/\mathfrak{p}) = 0 \) we use the formula \( R_n(x) \) mentioned in the proof of Lemma 4.7. As for the case of \( \text{char}(A/\mathfrak{p}) > 0 \) the set \( Z \cdot 1 + \mathfrak{p} \) is finite in \( A/\mathfrak{p} \) and hence definable in \( A/\mathfrak{p} \).

Lemma 4.9. There exists a first-order formula \( R_{n,\Lambda}(x, \bar{y}_1, \ldots, \bar{y}_m) \) where \( \Lambda = (\lambda_1, \ldots, \lambda_m) \), such that for any ring \( A \) such that \( r(A) \leq n \) and for any prime ideals \( \mathfrak{p}_i = id(\bar{a}_i) \) if \( \text{char}(A/\mathfrak{p}_i) = \lambda_i, i = 1, \ldots, n \), then the formula \( R_{n,\Lambda}(x, \bar{a}_1, \ldots, \bar{a}_m) \) defines in \( A \) the subring

\[
A_\mathfrak{p} = \bigcap_{i=1}^{m}(Z \cdot 1 + \mathfrak{p}_i),
\]

where

\[ 0 = \mathfrak{p}_1 \cdots \mathfrak{p}_m, \quad (\mathfrak{p}), \]

is a given decomposition of 0 in \( A \).

Proof. For each \( 1 \leq i \leq m \) consider the formula \( R_{n,\lambda_i}(x, \bar{y}_i) \) introduced in Lemma 4.9. So we can set

\[
R_{n,\Lambda}(x, \bar{a}_1, \ldots, \bar{a}_m) = \bigcap_{i=1}^{m} R_{n,\lambda_i}(x, \bar{y}_i).
\]

Lemma 4.10. There exists a formula \( \text{id}_i(x, \bar{y}_1, \ldots, \bar{y}_i) \), such that for any tuples \( \bar{a}_1, \ldots, \bar{a}_i \) the formula \( \text{Id}_i(x, \bar{a}_1, \ldots, \bar{a}_i) \) defines in \( A \) the ideal \( \mathfrak{p}_1 \cdots \mathfrak{p}_i \) where \( \mathfrak{p}_k = id(\bar{a}_k) \).

Indeed the ideal \( \mathfrak{p}_1 \cdots \mathfrak{p}_i \) is generated by all the products of the form \( y_1 \cdots y_i \) where \( y_k \) is an element of the tuple \( \bar{a}_k \) and the number of such products is finite.

Let \( M \) be an arbitrary \( A \)-module. The \( \mathfrak{p} \)-series of the ring \( A \) induces a series of \( A \)-modules

\[
M \geq \mathfrak{p}_1 M \geq \mathfrak{p}_1 \mathfrak{p}_2 M \geq \cdots \geq \mathfrak{p}_1 \cdots \mathfrak{p}_m M = 0,
\]

which will also be called a \( \mathfrak{p} \)-series of the module \( M \) or a special series for \( M \).
Lemma 4.11. There exists a formula $\phi_1(x, \bar{y}_1, \ldots, \bar{y}_l)$ such that if $p_1 \cdots p_m = 0$ is a decomposition of zero in the ring $A$ and $p_k = id(\bar{a}_k)$, then $\phi_1(x, \bar{a}_1, \ldots, \bar{a}_l)$ defines the submodule $M_i = p_1 \cdots p_i M$ in the two sorted model $(M, A)$.

Lemma 4.12. There exists a sentence $ch_\lambda$ of the language of rings such that for any integral domain $A$ with finitely generated additive group $A^+$:

$$\text{char}(A) = \lambda \iff A \models ch_\lambda.$$  

Proof. To prove the claim notice that if $\lambda$ is a prime then $ch_\lambda =_{df} \forall x(\lambda x = 0)$. For $\lambda = 0$ it is enough to note that for the integral domain $A$, $\text{char}(A) = 0$ if and only if $2 \neq 0$ and $1/2 \notin A$. In fact if $\text{char}(A) = 0$ then $2 \neq 0$ and if $1/2 \in A$ then $A \cong \mathbb{Z}[1/2]$ but $\mathbb{Z}[1/2]$ is not finitely generated. Contradicting with the assumption that $A^+$ is finitely generated. Conversely if $\text{char}(A) = p \neq 2$. Then $pA = 0$. So $A$ contains the finite field $\mathbb{F}_p$ and so $1/2 \in A$.

Let $L_2$ be the sorted language for the two sorted structure $(M, A, S)$ where $M$ is an abelian group, $A$ a ring and $S$ is the two-sorted predicate specifying the action of $A$ on $M$. Usually We drop $S$ from our notation and denote the structure only by $(M, A)$.

Lemma 4.13. Let $A$ be an integral domain with finitely generated additive group $A^+$ and $M$ a finitely generated $A$-module. Then for each $n \geq 1$ there exists a sentence $\varphi_n$ of the language $L_2$ such that $(M, A) \models \varphi_n$ and for any module $(N, B)$,

$$(N, B) \models \varphi_n \iff r(N) \leq n.$$  

Proof. Let $x_1, \ldots, x_m$ be a set of generators of $M$ over $A$ and $a_1, \ldots, a_n$ be a set of generators of the abelian group $A^+$. Then $r(M) \leq ms$. If $\text{char}(A) \neq 0$ then $M$ and $A$ are finite and $\varphi_n$ will say that $M$ doesn’t have more than $n$ elements where $n = mn^s$. If $\text{char}(A) = 0$ then $r(A^+) = s$ if and only if $|A^+/2A^+| = 2^s$. Therefore the number $s$, as well as number $m$, are definable in the language $L_2$ by some formulas. This proves the lemma.

Proposition 4.14. Let $A$ be an associative commutative ring with unit, $r(A) \leq n$, $\mathfrak{P} = (p_1, \ldots, p_m)$ be collection of prime ideals of $A$ such that $p_1 \cdots p_m = 0$. Then for any ring $B$ with finitely generated additive group $B^+$, and any $B$-module $N$ such that $B \models R_{n, A}$ and $(N, B) \models \varphi_n$, and also for any tuples $\bar{b}_1, \ldots, \bar{b}_m$ of elements of $B$ satisfying the formula $D_\lambda$ from Lemma 4.4 we have:

1. $r(B) \leq n$;
2. $Id(x, \bar{b}_i)$ defines in $B$ the prime ideal $q_i = id(\bar{b}_i)$, $Q = (q_1, \ldots, q_m)$ and $\text{char}(\mathfrak{P}) = \text{char}(Q)$;
3. $0 = q_1 \cdots q_m$;
4. $\phi_i(x, \bar{b}_1, \ldots, \bar{b}_i)$ defines the $i$-th term $q_1 \cdots q_i N$ in $(N, B)$;
5. The formula $R_{n, \Lambda}(x, \bar{b}_1, \ldots, \bar{b}_m)$, $\Lambda = \text{char}(\mathcal{B})$, defines the subring

$$B_{\bar{b}} = \bigcap_{i=1}^{m}(\mathbb{Z} \cdot 1 + q_i),$$

in $B$.

**Proof.** The proof follows from Proposition 4.5, Lemma 4.9, Lemma 4.11, Lemma 4.12 and Lemma 4.13.

**Corollary 4.15.** In the conditions of Proposition 4.14 the models $\langle M_i, A_P \rangle$, where $M_i$ is the $i$-th term of a $\mathcal{B}$-series of the $A$-module $M$, are definable in $\langle M, A \rangle$ uniformly with respect to $n$, $\text{Th}(\langle M, A \rangle)$ and the formula $D_{\Lambda}$ defining the constants used in the interpretation of $\langle M_i, A_P \rangle$ in $\langle M, A \rangle$.

**Proof of Proposition 1.5.** It suffices to find a formula $\phi_i$ for each $i$, defining a basis for $M_i/M_{i+1}$ of fixed period $\bar{e}_i$. According to the Corollary 4.15 the model $\langle M_i, A_P \rangle$ is interpretable in $\langle M, A \rangle$ with the help of any tuples of generating elements $\bar{a}_1, \ldots, \bar{a}_m$ satisfying the formula $D_{\Lambda}$.

Since $M_{i+1} = p_{i+1}M_i$, the model $\langle \overline{M}_i, A_i \rangle$ where $\overline{M}_i = M_i/M_{i+1}$ and $A_i = A_P/\langle p_{i+1} \cap A_P \rangle$ is obviously interpretable in $\langle M, A \rangle$ with the help of $\bar{a}_{i+1}$. In the view of the fact that $A_i$ is either $\mathbb{Z}$ or the finite field $\mathbb{Z}/p\mathbb{Z}$ and the action of $A_i$ on $\overline{M}_i$ is interpretable in $\langle M, A \rangle$ it is easy to write out a formula defining all bases of $\overline{M}_i$ of given period $\bar{e}_i$ and thus to construct the desired formula $\phi_i$.

**Proof of Theorem 1.6.** From Proposition 1.5 we have the formula $\varphi_{\mathcal{B}, n}$ which defines $\mathbb{Z}$-bases of $N$ in $\langle N, B \rangle$ that satisfy $\varphi_{\mathcal{B}, n}$. We need only to describe the structural constant of some fixed base $\bar{e} = (e_1, \ldots, e_n)$ of $M$ satisfies $\varphi_{\mathcal{B}}$ in $\langle M, A \rangle$. It can be done by the formulas of the language $L_2$ because these are integers.

The following corollary is an obvious consequence of the Theorem 1.6. In turn it will imply Theorem 1.7. We omit the proofs.

**Corollary 4.16.** For any associative commutative ring $A$ with unit and finitely generated additive group $A^+$ there exists formula $\psi_A$ such that for any ring $B$ with finitely generated $B^+$ we have

$$B \models \psi_A \iff A \cong B.$$

**Corollary 4.17.** Let $\mathcal{K}$ be the class of all associative commutative rings with finitely generated additive group. Then any $A$ from $\mathcal{K}$ is finitely axiomatizable inside $\mathcal{K}$. 

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5 Coordinatization of finitely generated nilpotent groups admitting exponents in a binomial principal ideal domain

In this section we generalize Philip Hall’s theorem (see [3], Section 6) on coordinatization of finitely generated torsion free nilpotent groups to finitely generated $K$-groups, where $K$ is a binomial PID. In the case that $G$ is a torsion free $K$-group by $K$-coordinatization we mean that one can choose a Mal’cev (canonical) basis $u_1, \ldots, u_n$, $u_i \in G$, with respect to which any element $g \in G$ can be uniquely written as the product

$$g = u_1^{t_1(g)} \ldots u_n^{t_n(g)}, \quad t_i(g) \in K,$$

where $t_i(g)$ is called the $i$-th coordinate of $g$ and the coordinates of the product $gh$ are computed by the coordinates of $g$ and $h$ with the help of some polynomials depending only on the group $G$ and the chosen basis $(u_1, \ldots, u_m) = \bar{u}$.

In this section we introduce the notion of a pseudo-basis for an arbitrary finitely generated group over a binomial PID $K$, which allows one to introduce $K$-coordinatization in the general case (even when $G$ has torsion). Moreover, which is specially important, some of such coordinatizations are logical invariants of the group $G$. In particular pseudo-bases play an important role in the model theory of nilpotent groups. We would like to point out that in [5] such coordinatization was given for finitely generated nilpotent pro-$p$ groups, i.e. when $K = \mathbb{Z}_p$.

Throughout this section $K$ is a binomial PID, and all the groups, subgroups and homomorphisms are $K$-groups, $K$-subgroups and $K$-homomorphisms, respectively, i.e. we work in the category of $K$-groups. Recall that $K$ is a binomial domain, if it is a characteristic zero domain so that for any $k \in K$ and $0 \neq n \in \mathbb{Z} - 1 \subseteq K$ there is a unique element of $K$ denoted by $(\binom{k}{n})$ such that

$$(n!) \binom{k}{n} = d, k(k-1) \ldots (k-n+1).$$

Let $A$ be a finitely generated abelian $K$-group, (i.e. a finitely generated module over $K$). Then $A$ admits a decomposition $A = K\alpha_1 \oplus \ldots \oplus K\alpha_n$ into the direct sum of cyclic $K$-groups. Let $Ann(a_i) = \{ \alpha \in K : \alpha a_i = 0 \}$ be ideal annihilating $a_i$ and $e_i$ be a fixed generator of $Ann(a_i)$. Then $e_i$ will be called a period of $a_i$ and denoted by $e(a_i)$ (if $Ann(a_i) = 0$ we let $e(a_i) = \infty$).

**Definition 5.1.** The tuple $\bar{a} = (a_1, \ldots, a_n)$ of elements of the abelian $K$-group $A$ is called a base of period $e(\bar{a}) = (e(a_1), \ldots, e(a_n))$ of $A$.

For any ideal $Ann(a_i)$ we fix an element $d_i(x_0)$ in each coset $x_0 + Ann(a_i)$ and define a function (a section) $d_i : K \rightarrow K$, which is constant on each coset of $Ann(a_i)$, i.e. if $x - x_0 \in Ann(a_i)$ then $d_i(x) = d_i(x_0)$. In particular if $e_i = \infty$ then $Ann(a_i) = 0$ and so $d_i(x) = x$. The tuple of sections $d(\bar{a}) = (d_1, \ldots, d_m)$ will be called a a section of the base $\bar{a}$.)
Remark 5.2. The basis $\bar{a}$ and the section $d(\bar{a})$ give a $K$-coordinatization of $A$, because any element $g \in A$ admits the unique decomposition

$$g = a_1^{t_1(g)} \ldots a_n^{t_n(g)},$$

(in the multiplicative notation), where $t_i(g) = d_i(\alpha_i)$ for a certain element $\alpha_i \in K$, $i = 1, \ldots, n$, defined uniquely modulo $Ann(\alpha_i)$.

Definition 5.3. Let $G$ be an arbitrary finitely generated $K$-group. Let

$$G = G_1 \geq \ldots \geq G_{c+1} = 1$$

be a certain central series of $K$-subgroups of $G$. Let $\bar{v}_i$ be the tuple of elements of $G_i$, the images of which form a basis of period $e(\bar{v}_i)$ with the section $d(\bar{v}_i)$ of the finitely generated $K$-group $G_i/G_{i+1}, i = 1, \ldots, c$. Then the tuple $\bar{u} = \bar{v}_1 \cup \ldots \cup \bar{v}_c$ will be called a pseudo-basis of $G$ associated with the series (3), and the tuples $e(\bar{u}) = e(\bar{v}_1) \cup \ldots \cup e(\bar{v}_c)$ and $d(\bar{u}) = d(\bar{v}_1) \cup \ldots \cup d(\bar{v}_c)$ are the period and the section of the pseudo-basis $\bar{u}$.

Proposition 5.4. Let $G$ be a finitely generated $K$-group and $\bar{u} = (u_1, \ldots, u_m)$ a pseudo-basis for $G$ of period $e(\bar{u})$ with the sections $d(\bar{u})$. Then

1. any element $g \in G$ can be represented uniquely in the form

$$g = u_1^{t_1(g)} \ldots u_m^{t_m(g)},$$

where $t_i(g) = d_i(\alpha_i)$ for a certain element $\alpha_i \in K$, $i = 1, \ldots, m$, defined uniquely modulo the ideal $Ann(u_i) = Ke(u_i)$;

2. The $K$-subgroups $G_i = \langle u_i, \ldots, u_m \rangle_K$ form a central series

$$G = G_1 > \ldots > G_m > 1$$

in $G$.

Proof. The proof is by induction on $m$. The case $m = 1$ is trivial. It is clear from the definition of a pseudo-basis that $G_m = \langle u_m \rangle_K$ lies in the center of $G$ and the images $\bar{u}_1, \ldots, \bar{u}_{m-1}$ form a pseudo-basis of the group $G/G_m$ of period $(e(u_1), \ldots, e(u_{m-1}))$ with the section $(d_1, \ldots, d_{m-1})$. By induction $G/G_m$ satisfies conditions 1. and 2. of the statement of the proposition. Now one can easily prove 1. and 2. for the entire group $G$.

Fix an arbitrary pseudo-basis $\bar{u} = (u_1, \ldots, u_m)$ with the section $d(\bar{u}) = (d_1, \ldots, d_m)$ of period $e(\bar{u}) = (e_1, \ldots, e_m)$. The elements $t_i(g) = d_i(\alpha_i)$ from decomposition (1) are called the coordinates of $g$ in the pseudo-basis $\bar{u}$. We shall also write equation (1) in the vector form:

$$g = \bar{u}^{t(g)}$$

where $t(g) = (t_1(g), \ldots, t_m(g))$. 26
Definition 5.5. the set of coordinates

$$\Gamma(\bar{u}) = \{t_k([u_i, u_j]), t_k(u_i^n) : 1 \leq i, j, k \leq m\}$$

is called the set of structural constants of the pseudo-basis $\bar{u}$ and the subring $K(\bar{u}) = \mathbb{Z}(\Gamma(\bar{u})) \leq K$, generated over $\mathbb{Z}$ by the set $\Gamma(\bar{u})$ is called the definition ring of the basis $\bar{u}$.

From now on we shall represent $\Gamma(\bar{u})$ in an arbitrary but fixed way in the form of a tuple $\Gamma(\bar{u}) = (\gamma_1, \ldots, \gamma_n)$ so that by the index $i$ one can uniquely determine which coordinate of which element $\gamma_i$ is.

For any cross-section $d_i : K \to K$ and for a fixed period $e_i$ ($e_i$ is a generator of $Ann(u_i)$) we define the function $r_i : K \to K$ from the equality $x = e_ir_i(x) + d_i(x)$. Since $K$ has no zero divisors $r_i(x)$ is defined uniquely.

Let $R_i, D_i, B_i, i = 1, \ldots, m$, be unary function symbols, $+$ and $\cdot$ be binary function symbols and $c_n$ and $b_\gamma$ be constant symbols for $n \in \mathbb{N}$ and $\gamma \in \Gamma(\bar{u})$. Consider the signature

$$\mathfrak{F}_m = \langle R_i, D_i, B_i, +, \ldots, c_n, b_\gamma : n \in \mathbb{N}, \gamma \in \Gamma(\bar{u}), 1 \leq i \leq m\rangle.$$  

Denote by $\mathfrak{F}_m$ the set of all possible terms over $\mathfrak{F}_m$.

Definition 5.6. We shall say that a term $t(x_1, \ldots, x_n) \in \mathfrak{F}_m$ computes a function $f : K^n \to K$ with respect to a pseudo-basis $\bar{u}$ of period $e(\bar{u})$ with the section $d(\bar{u})$ and the set of structural constants $\Gamma(\bar{u})$ if after the substitutions: $R_i(x) \to r_i(x), D_i(x) \to d_i(x), B_n(x) \to (\bar{x})$, $x + y \to x + y, xy \to xy, c_n \to n$ and $b_\gamma \to \gamma$ the obtained operation $t^n(x_1, \ldots, x_n) : K^n \to K$ is equal to $f$.

Theorem 5.7 (Uniform Coordinatization). For any natural number $m$ the following statements are true.

1. There exist terms $M_1, \ldots, M_m \in \mathfrak{F}_m$ computing respectively the multiplication coordinate functions $t_1(gh), \ldots, t_m(gh)$ with respect to any pseudo-basis $\bar{u} = (u_1, \ldots, u_m)$ of any finitely generated $K$-group over any binomial PID, $K$.

2. There exist terms $D_1, \ldots, D_m \in \mathfrak{F}_m$, computing respectively the exponentiation coordinate functions $t_1(g^\alpha), \ldots, t_m(g^\alpha)$ with respect to any pseudo-basis $\bar{u} = (u_1, \ldots, u_m)$ of any finitely generated $K$-group $G$ over any binomial PID, $K$.

Before we proceed to the proof of the theorem we state a definition. Let $W(x_1, \ldots, x_m, y_1, \ldots, y_m, z_1, \ldots, z_m) = W(\bar{x}, \bar{y}, \bar{z})$ be an arbitrary group word. Let $g = W(u_1^{a_1}, \ldots, u_m^{a_m}, u_1^{b_1}, \ldots, u_m^{b_m}, u_1, \ldots, u_m) = W(\bar{u}^\alpha, \bar{u}^\beta, \bar{u})$ be an element of a $K$-group $G$ with the pseudo-basis $\bar{u}$. It is clear that the coordinates $t_i(g)$ in the basis $\bar{u}$ are functions $f_i(\alpha, \beta)$ of $\alpha = (\alpha_1, \ldots, \alpha_m)$ and $\beta = (\beta_1, \ldots, \beta_m)$. We shall say that coordinates of $g$ in the basis $\bar{u}$ are computed $m$-universally, and the word $W$ itself is $m$-universal, if there exist
Lemma 5.8. The following statements are true:

1. if the word $W_0(\bar{u}, \bar{y})$ is $m$-universal then any group word $W(\bar{x}_1, \ldots, \bar{x}_n)$ of the language $L$ is $m$-universal,

2. if the words $W_0(\bar{x}, \bar{y})$ and $W_1(\bar{x}, \lambda)$ are $m$-universal then any word $W(\bar{x}_1, \ldots, \bar{x}_n, \lambda_1, \ldots, \lambda_k)$ of the language $L_1$ is $m$-universal.

Proof. The proof is by induction on the number of letters in the word $W$.

Proof of Theorem 5.7. We proceed with induction on the length of the pseudo-basis $\bar{u} = (u_1, \ldots, u_m)$. Let $\alpha = (\alpha_1, \ldots, \alpha_m)$ and $\beta = (\beta_1, \ldots, \beta_m)$ be the coordinates of the elements $x$ and $y$:

$$x = u_1^{\alpha_1} \cdots u_m^{\alpha_m}, \quad y = u_1^{\beta_1} \cdots u_m^{\beta_m},$$

in vector notation $x = \bar{u}^\alpha$, $y = \bar{u}^\beta$. Then for $m = 1$ we obtain

$$t_1(xy) = d_1(\alpha_1 + \beta_1), \quad t_1(x^\alpha) = d_1(\alpha_1).$$

Now let $\bar{u} = u_1 \cup \bar{v}$, where $\bar{v} = (u_2, \ldots, u_m)$. We have

$$xy = u_1^{\alpha_1} \cdots u_m^{\alpha_m} u_1^{\beta_1} \cdots u_m^{\beta_m} = u_1^{\alpha_1 + \beta_1}(u_1^{-\beta_1} u_2 u_1^{\beta_1})^{\alpha_2} \cdots (u_1^{-\beta_1} u_m u_1^{\beta_1})^{\alpha_m} = u_1^{\alpha_1 + \beta_1} \prod_{i=2}^m (u_1^{-\beta_i} u_i u_1^{\beta_i})^{\alpha_i} \prod_{i=2}^m u_i^{\beta_i}$$

In the case $e_1 \neq \infty$ it is necessary to rewrite the first factor in the following way:

$$u_1^{\alpha_1 + \beta_1} = u_1^{d_1(\alpha_1 + \beta_1)}(\bar{u}^{\beta_1}_1)_{\gamma_1(\alpha_1 + \beta_1)} = u_1^{d_1(\alpha_1 + \beta_1)}(\bar{v}^{\gamma_1})_{\gamma_1(\alpha_1 + \beta_1)}$$

where $\gamma$ is the tuple of structural constants associated to $u_1^{\beta_1}$ from $\Gamma(\bar{u})$. For proving the theorem it suffices to verify the $m$-universality of factors of the type $(\bar{v}^{\gamma})_{\gamma_1(\alpha_1 + \beta_1)}$ and $u_1^{-\beta_1} u_i u_1^{\beta_i}$. For the first one this is true by induction, since $|\bar{v}| = m - 1$, $\gamma$ is just a tuple of structure constants computable by terms of $\mathcal{T}_m$ and the function $r_1(\alpha_1 + \beta_1)$ is computed by a term from $\mathcal{T}_m$. 

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It remains to consider the case of $u_1^{-\beta}u_i\beta$, $i \geq 2$. For any $x \in K$ we have,

$$u_1^{-x}u_1^{-\beta} = u_1^{-x}u_1^{-1}u_1^{-\beta},$$

Then,

$$(u_1u_i^{-1})^x = (u_1[u_1, u_i^{-1}])^x = u_1^x[u_1, u_i^{-1}x\tau_m(x)\tau_2(2)]u_i^{-1},$$

by the Hall-Petresco formula, where $\tau_k$ is a fixed product of simple commutators of length no less than $k$ of elements $u_1$ and $[u_1, u_i^{-1}]$. Consequently,

$$u_1^{-x}u_i^{-1} = [u_1, u_i^{-1}]x\tau_m(x)\tau_2(2)u_i^{-1}u_1,$$

Now it is sufficient to prove that the elements $[u_1, u_i^{-1}]$, $\tau_1, \ldots, \tau_m$ are $m$-universal. We have

$$[u_1, u_i^{-1}] = u_1[u_1, u_i^{-1}]u_i^{-1} = u_1\delta u_i^{-1},$$

where $\delta$ is a tuple of structural constants from $\Gamma(\bar{u})$. By induction the word $u_1\delta u_i^{-1}$ is $(m-1)$-universal and hence $[u_1, u_i^{-1}]$ is $m$-universal.

Since $\tau_1$ is a fixed product of simple commutators of the elements $u_1$ and $[u_1, u_i]$ it suffices to show a way of uniform elimination of all occurrences of $u_1$ into an arbitrary commutator $[g_1, \ldots, g_n]$, $g_i$ being either $u_1$ or a fixed $(m-1)$-universal element from $\langle u_2, \ldots, u_m \rangle tK$. For this purpose it is sufficient to prove that if $g = \bar{v}(t)$ then the coordinates of the commutator $[g, u_i]$ (as functions of $t(g)$) are computed by some terms from $\mathcal{T}\mathfrak{F}_m$. Let the decomposition of $g$ by the base $\bar{v}$ begin with $u_i^{-x}$, $i \geq 2$, i.e. $g = u_i^{-x}f$. Then

$$[g, u_i] = [u_i^{-x}f, u_i] = [u_i^{-x}, u_i][u_i^{-x}, f][f, u_i].$$

Then

$$[u_i^{-x}, u_i] = u_i^{-x}u_i^{-1}u_i^{-x}u_1 = u_i^{-x}(u_i^{-1}u_i)^{-x} = [u_i, u_1x\tau_2(u_i^{-1}, u_i^{-1}u_i)\tau_2(2)\ldots\tau_m(u_i^{-1}, u_i^{-1}u_i)\tau_m(x)]u_i^{-1}u_i^{-1},$$

The coordinates of the elements $[u_i, u_1]$ and $u_i^{-1}u_1u_i = u_i[u_1, u_1]$ are computed by the terms from $\mathcal{T}\mathfrak{F}_m$, then by induction (everything is computed in the basis $\bar{v}$) the coordinates of each factor in the product appeared in the last line of (5), and hence the coordinates of the whole product are computed by terms from $\mathcal{T}\mathfrak{F}_m$. Now it remains to notice that the decomposition of $[f, u_1]$ is obtained by induction. As the final result we obtain the decomposition of $xy$ into the product of $m$-universal elements lying in the group $G_2 = \langle u_2, \ldots, u_m \rangle K$. By induction their product is computed in the base $\bar{v}$ $(m-1)$-universally, which proves part 1. of the statement of the theorem.
Now consider exponentiation:

\[(u_1^{\alpha_1} \ldots u_m^{\alpha_m})^\lambda = u_1^{\alpha_1 \lambda} \ldots u_m^{\alpha_m \lambda} \tau_2 \ldots \tau_m^{-(\lambda)},\]

\(\tau_i\) being a product of fixed commutators of weight \(\geq i\) of the elements \(u_1^{\alpha_1}, \ldots, u_m^{\alpha_m}\).

According to part 1, already proved, the elements \(\tau_1, \ldots, \tau_m\) are \(m\)-universal and lie in the group \(G_2\). Then by induction their powers are \((m - 1)\)-universal and again according to part 1 the entire product is \(m\)-universal, which proves part 2 of the theorem. The theorem is proved.

As a corollary of the theorem we immediately obtain the following theorem.

**Theorem 5.9.** Two finitely generated \(K\)-groups \(G\) and \(H\) are \(K\)-isomorphic if and only if they have pseudo-bases of the same period with the same set of structural constants.

**Proof.** According to the theorem above \(G\) and \(H\) have pseudo-bases \(\bar{u}\) and \(\bar{v}\) of the same period with the same multiplication coordinate functions. Consequently the mapping:

\[u_1^{\alpha_1} \ldots u_m^{\alpha_m} \mapsto v_1^{\alpha_1} \ldots v_m^{\alpha_m}\]

is a \(K\)-isomorphism of \(G\) onto \(H\).

\[\square\]

6 Foundations and additions and regular groups

The aim of this section is to find means of struggling against the special gap. In many cases for a given group \(G\) it is possible to define a subgroup \(G_0 \leq Z(G)\), which is called an addition of \(G\), and the quotient group \(G/G_0 = G_f\), which is called a foundation of \(G\). Moreover, this can be done so that \(G_f\) is the “closest” group to \(G\) group with a tame special gap, and \(G_0\) is the measure of deviation of \(G_f\) from \(G\). In this way the general situation can be reduced to the study of groups which are central extensions of free abelian groups by groups tame special gap. A special part is played by groups with the condition \(G \cong G_f \oplus G_0\), which are called regular groups. The class of regular groups is quite wide. For example, any exponential group over a field is regular.

6.1 Foundations and additions

Let \(K\) be a binomial principal ideal domain, \(G\) be an arbitrary finitely generated \(K\)-group. For a \(K\)-subgroup \(N \leq G\) we shall denote by \(Is(N)\) the isolator of \(N\) in \(G\), i.e. a \(K\)-subgroup of the type:

\[Is(N) = \{x \in G : \exists \alpha \neq 0 \in K \ (x^\alpha \in N)\}.\]

Define

\[I(G) = Is(G') \cap Z(G).\]
The special gap

\[ Z(G) \geq G' \cap Z(G) \]  \hspace{1cm} (6)

will be called tame, if \( Z(G) = I(G) \).

Let us refine the special gap \((6)\) with the help of \( I(G) \):

\[ Z(G) \geq I(G) \geq G' \cap Z(G). \]

The quotient \( I(G)/(G' \cap Z(G)) \) is a finite direct sum of periodic cyclic \( K \)-modules, and the quotient \( Z(G)/I(G) \) is a free \( K \)-module of finite rank. Since \( K \) is a principal ideal domain, there exists a free \( K \)-module of finite rank \( G_0 \leq Z(G) \), such that \( Z(G) \cong G_0 \oplus I(G) \).

**Definition 6.1.** Any subgroup \( G_0 \leq G \) such that \( Z(G) = G_0 \oplus I(G) \) is called an addition of \( G \) and the quotient group \( G_f = G/G_0 \) is called a foundation of \( G \), associated with the addition \( G_0 \).

The facts stated above allow us to formulate the following proposition

**Proposition 6.2.** Any finitely generated \( K \)-group \( G \) over the principal ideal domain \( K \) possesses an addition and a foundation. Besides,

1. the addition \( G_0 \) is a free \( K \)-module of finite rank,
2. the foundation \( G_f \) has a tame special gap.

### 6.2 Regular groups

**Definition 6.3.** A nilpotent \( K \)-group is called regular, if \( G = H \oplus G_0 \), where \( G_0 \) is an addition of \( G \) and \( I(H') \geq Z(H) \).

It has been proved in \([5, 6]\) that if \( K \) is a field, then any \( K \)-group is regular. In the case of \( K = \mathbb{Z} \) it is not true as the following example shows.

**Example 6.4.** let \( G \) be a finitely generated group defined in the variety of 2-nilpotent groups by the presentation

\[ G = \langle a, b, c : [c^3, a] = 1, [c^3, b] = 1 \rangle. \]

Then \( G_0 = \langle c^3 \rangle \cong \mathbb{Z}^+ \), \( G_f = \langle a, b, c : c^3 = 1 \rangle \) and it is clear that \( G \not\cong G_f \oplus G_0 \), therefore (see the proposition below) \( G \) is not regular.

**Proposition 6.5.** Let \( G \) be a finitely generated group over the principal ideal domain \( K \). Then the following conditions are equivalent.

1. \( G \) is a regular group.
2. \( G \cong G_f \oplus G_0 \) for a certain addition \( G_0 \) and the corresponding foundation \( G_f \).
3. \( G \cong G_f \oplus G_0 \) for any addition \( G_0 \) and the foundation \( G_f \), connected with it.
4. $\text{Is}(Z(G) \cdot G') = \text{Is}(G') \cdot Z(G)$.

Proof. The implication $2 \Rightarrow 1$ is obvious for if $G = H \oplus G_0$ is the decomposition from the definition of regular group, then $G_0$ is an addition of $G$ and $H \cong G_f = G/G_0$. The inverse implication $1 \Rightarrow 2$ follows from [10.2]. Let us prove $2 \Rightarrow 4$. So we can assume $G = G_f \oplus G_0$. Since $\text{Is}(G' \cdot Z(G)) = \text{Is}(G' \cdot G_0)$ and $G' \leq G_f$, then

$$\text{Is}(G' \cdot G_0) = \text{Is}(G' \oplus G_0) = \text{Is}(G') \oplus G_0 = \text{Is}(G') \cdot Z(G).$$

To prove $4 \Rightarrow 3$ assume $\text{Is}(G' \cdot Z(G)) = \text{Is}(G') \cdot Z(G)$. The $K$-group $G/\text{Is}(G' \cdot Z(G))$ is a finitely generated abelian $K$-group without $K$-torsion, i.e. a free $K$-module. Consequently,

$$\text{Ab}(G) = G/G' = \pi(\text{Is}(G' \cdot Z(G))) \oplus G_1,$$

where $\pi : G \to G/G'$ is the canonical epimorphism and $G_1$ is a certain direct $K$-complement of $\pi(\text{Is}(G' \cdot Z(G)))$ in $\text{Ab}(G)$. In turn, for an arbitrary addition $G_0$ we have

$$\pi(\text{Is}(G' \cdot Z(G))) = \pi(\text{Is}(G') \cdot Z(G)) = \pi(\text{Is}(G')) \oplus G_0.$$

Hence

$$\text{Ab}(G) = G_1 \oplus \pi(\text{Is}(G')) \oplus G_0.$$

Let $H$ be the complete pre-image of the subgroup $G_1 \oplus \pi(\text{Is}(G'))$ in $G$. Then $G = H \cdot G_0$, $H \cap G_0 \leq G' \cap G_0 = 1$ and $G' \leq H$. Consequently $G \cong H \oplus G_0$ and $H \cong G_f$. The implication $3 \Rightarrow 2$ is obvious. The proposition is proved. \hfill \Box

In the sequel, if $G = H \oplus G_0$ is a regular group, then we shall call the subgroup $H$ a foundation of $G$ as well, identifying $H \cong G/G_0$.

Corollary 6.6. Any finitely generated 2-nilpotent $K$-torsion-free $K$-group is regular.

Actually if the group $G$ is $K$-torsion free then $Z(G)$ is an isolated subgroup of $G$. Therefore

$$\text{Is}(G' \cdot Z(G)) = \text{Is}(Z(G)) = Z(G) = \text{Is}(G') \cdot Z(G),$$

and according to the proposition, $G \cong G_f \oplus G_0$.

Proposition 6.7. Let $G$ be a regular group. Then:

1. $G \cong G_f \oplus G_0$ (maybe not associated with each other);

2. all foundations of $G$ are isomorphic to each other, and so are additions.

Proof. To prove 1 Let $G_0$ be an arbitrary addition of $G$, $H$ be a certain foundation of $G$ which maybe not associated with $G_0$. It is clear from the definition of an addition and a foundation that $H \geq \text{Is}(G')$ and $Z(G) = (H \cap Z(G)) \oplus G_0$. Hence $G = H \cdot Z(G) = H \cdot G_0$ and $H \cap G_0 = 1$, i.e. $G \cong H \oplus G_0$. Statement 2 follows from 1. \hfill \Box
7 Elementary equivalence of finitely generated nilpotent groups

In this section we give a structural and algebraic characterization for elementary equivalence of finitely generated nilpotent groups. We proceed by introducing some terminology and proving a few lemmas.

Lemma 7.1. Assume $G$ is a finitely generated nilpotent group then there is a formula of $L$ that defines $\text{Is}(G')$ uniformly with respect to $\text{Th}(G)$.

Proof. By Proposition 3.5 $G'$ is definable in $G$ uniformly with respect to $\text{Th}(G)$ by a formula $\phi$ of $L$. Since $\text{Is}(G')/G'$ is a a finite group, $g \in \text{Is}(G')$ if and only if $g^m \in G'$, where $m = |\text{Is}(G')/G'|$. So if $\phi(x)$ defines $G'$ in $G$, then $\phi(x^m)$ defines $\text{Is}(G')$ in $G$. Now assume $H \equiv G$ and set

$$H_1 = \{ x \in H : H \models \phi(x^m) \}.$$ 

Obviously $H_1 \subseteq \text{Is}(H')$. Pick $x \in H$ so that $x^n \in H'$ for some $n \in \mathbb{N} \setminus \{0\}$. Note that

$$G \models \forall x(\phi(x^n) \rightarrow \phi(x^m)).$$ 

So the same sentence is true in $H$ implying that $H_1 = \text{Is}(H')$ which concludes the proof.

Lemma 7.2. Assume $G \equiv H$ are finitely generated nilpotent groups. Then $G_0 \cong H_0$ for any respective additions of $G$ and $H$.

Proof. It follows from lemma above and uniform definability of $Z(G)$ that $I(G)$ is uniformly definable in $G$. So the quotient $Z(G)/I(G) \cong G_0$ is uniformly interpretable in $G$. Thus we conclude that $G_0 \cong Z(G)/I(G) \equiv Z(H)/I(H) \cong H_0$. But $G_0$ and $H_0$ are free abelian groups of finite rank, hence $G_0 \cong H_0$.

Corollary 7.3. The subgroups $M(G) = \text{Is}(G' \cdot Z(G))$ and $N(G) = \text{Is}(G') \cdot Z(G)$ are uniformly definable in $G$. In particular the finite abelian quotient

$$\text{Is}(G' \cdot Z(G))/\text{Is}(G') \cdot Z(G)$$ 

is interpretable in $G$ uniformly with respect to $\text{Th}(G)$.

Lemma 7.4. There exists a central series

$$G = G_1 > G_2 > \ldots > G_p > \text{Is}(G' \cdot Z(G)) > \text{Is}(G') \cdot Z(G) > \text{Is}(G')$$

$$> G' > \ldots > G_t > G_{t+1} = 1. \quad (7)$$

so that each quotient of the consecutive terms of the series except possibly the quotient $\text{Is}(G') \cdot Z(G) > \text{Is}(G')$ is either finite or admits an interpretable action of $\mathbb{Z}$. 

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Proof. Consider the lower central series

\[ G = \Gamma_1(G) > \Gamma_2(G) > \ldots > \Gamma_c(G) > 1 \quad (\Gamma), \]

of \( G \). Recall the completed series \((L(\Gamma))\) associated to \((\Gamma)\) (see Section 2.4). Then by Proposition 1.2 there is a ring \( A_\Gamma \) whose action on each quotient of \((L(\Gamma)), \) except possibly on the quotient of the gap \( G' \cdot Z(G) > G' \), is interpretable in \( G \) uniformly. By Proposition 2.8 the ring \( A_\Gamma \) is a commutative associative \( \mathbb{Z} \)-algebra with unit which has a finitely generated additive group. So each quotient of the \((L(\Gamma)), \) except the one corresponding to the gap, is a finitely generated \( A_\Gamma \)-module. So by Proposition 1.5 and its proof there exists a refinement \((L'(\Gamma))\)

\[ \bullet \] each term of \((L'(\Gamma))\) is definable in \( G \) uniformly with respect to \( Th(G), \)

\[ \bullet \] each corresponding quotient (except the quotient corresponding to the gap) is either finite or is infinite and the action of \( \mathbb{Z} \) on it is interpretable in \( G \) uniformly with respect to \( Th(G). \)

Since the quotients \( M(G)/N(G) \) and \( Is(G')/G' \) are finite and appropriate terms of \((L'(\Gamma))\) can be used to fill in the gap \( G' > 1 \) we only need to find a refinement of \( G > Is(G' \cdot Z(G)) \) satisfying the condition. Again by Proposition 1.5 there is a series

\[ G = G_1 > G_2 > \ldots > G_i > G_i \cdot Z(G) \cdot G' = G_{i_0+1}, \]

for some \( i_0 \in \mathbb{N} \) each quotient of which has the desired property. So consider the series

\[ G = G_1 > G_2 \cdot Is(G' \cdot Z(G)) > G_3 \cdot Is(G' \cdot Z(G)) > \ldots > G_{i_0+1} \cdot Is(G' \cdot Z(G)). \]

For each \( i \) consider the obvious epimorphism

\[ \phi_i : G_i/G_{i+1} \to (G_i \cdot Is(G' \cdot Z(G)))/(G_{i+1} \cdot Is(G' \cdot Z(G))). \]

Each \( \phi_i \) is interpretable in \( G \) and thus can be used in an obvious manner to interpret the action of \( Z \) on \((G_i \cdot Is(G' \cdot Z(G)))/(G_{i+1} \cdot Is(G' \cdot Z(G))). \) This finishes the proof.

\[ \square \]

From now on we start working with a finitely generated nilpotent group \( G \) which is not regular. Then the essential gap is not tame, i.e. \( Is(G') \cap Z(G) < Z(G) \) and \( M(G)/N(G) \neq 1. \) It will be clear from discussions that analysis of the regular case is actually much easier and follows by making proper modifications in the arguments that follow.

Lemma 7.5. Consider the series \((7)\) obtained in Lemma 7.2. Then there exists a pseudo-basis \( \bar{u} \in G^m \) adapted to \((7)\) and natural numbers \( 1 < i_0 < i_1 < i_2 < m \) so that:

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1. \( (u_1 M(G), u_2 M(G), \ldots, u_{i_0} M(G) \) is pseudo-basis of \( G/M(G) \),
2. \( (u_{i_0 + 1} N(G), u_{i_0 + 2} N(G), \ldots, u_{i_1} N(G) ) \) is pseudo-basis of \( M(G)/N(G) \),
3. \( (u_{i_1 + 1} \text{Is}(G'), u_{i_1 + 2} \text{Is}(G'), \ldots, u_{i_2} \text{Is}(G') ) \) is a pseudo-basis of \( N(G)/\text{Is}(G') \),
4. \( (u_{i_2 + 1}, u_{i_2 + 2}, \ldots, u_m ) \) a pseudo-basis of \( \text{Is}(G') \).
Moreover if \( K_i = \langle u_i, \ldots, u_m \rangle \), then the series
\[
G = K_1 > K_2 > \ldots > K_m > 1 \quad (K),
\]
is a central series and unless \( i_1 + 1 < i \leq i_2 \) each subgroup \( K_i \) is definable in \( G \) with constants \( (u_i, \ldots, u_m) \).

**Proof.** Most of the statement is clear. We shall just comment on definability of the \( K_i \), \( i \neq i_1 + 2, \ldots, i_2 \). To show this we proceed by induction on \( i \). Indeed either \( K_m \) is infinite cyclic and \( K_m = u_m^\mathbb{Z} \) where the action of \( \mathbb{Z} \) on \( u_m \) is interpretable in \( G \) or it is finite cyclic. So assume \( i > i_2 \) and assume the statement is true for \( i + 1 \). Then
\[
K_i = \{ xy : \exists a \in \mathbb{Z} (xK_{i+1} = (u_i K_{i+1})^a \land y \in K_{i+1}) \},
\]
where the right hand side describes a definable subset by induction hypothesis and the fact that \( \langle u_i K_{i+1} \rangle \) is interpretable in \( G \) as it admits an interpretable action of \( \mathbb{Z} \). If \( i = i_1 + 1 \) then \( K_i = \text{Is}(G') \cdot Z(G) \) which is definable uniformly by Corollary 7.3 and the result follows by an induction on \( i \) for \( i \leq i_1 \).

\[\Box\]

**Lemma 7.6.** Given \( \bar{u} \in G^m \) and \( K_i \leq G \) as in Lemma 7.3 and Lemma 7.7 assume that \( e_i \) are the periods of the \( u_i \) and \( t_k([u_i, u_j]) \) and \( t_k(u_i^s) \) are the structure constants associated to \( \bar{u} \). Assume also \( n = i_1 - i_0 \) and \( p = i_2 - i_1 \). Then there exists a first order formula \( \Phi(x_1, x_2, \ldots, x_m) \) in \( L \) so that
\[
G \models \Phi(\bar{u}),
\]
and \( \Phi(\bar{u}) \) expresses (or implies) that:

1. for all \( x \in G \) there exists a unique tuple
\[
(a_1, a_2, \ldots, a_{i_1}, a_{i_2+1}, a_{i_2+2}, \ldots, a_m) \in \mathbb{Z}^{m-p},
\]
and a tuple
\[
(g_1, g_2, \ldots, g_{i_1}, g_{i_2+1}, g_{i_2+2}, \ldots, g_m) \in G^{m-p},
\]
and an element \( w_0 \in (Z(G) \setminus \text{Is}(G')) \cup 1 \) unique modulo \( \text{Is}(G') \cap Z(G) \), so that
\[
x = \prod_{\substack{1 \leq i \leq m \\atop i \neq i_1+1, \ldots, i_2}} g_i w_0,
\]
and \( g_i K_{i+1} = (u_i K_{i+1})^{a_i} \).
2. for all \(1 \leq i, j \leq m\)

\[
[u_i, u_j] = \prod_{k=i_2+1}^{m} u_k^{t_k([u_i, u_j])},
\]

3. if \(e_i \neq \infty\) then for all \(i\) except \(i = i_0 + 1, \ldots, i_1\)

\[
u^{e_i} = \prod_{k=i_2+1}^{m} u_k^{t_k(u^{e_i})},
\]

4. for each \(i = i_0 + 1, \ldots, i_1\),

\[
u^{e_i} = \prod_{k=i_2+1}^{m} u_k^{t_k(u^{e_i})},
\]

5. the set \(\{u_{i_1+1}Is(G'), u_{i_1+2}Is(G'), \ldots, u_{i_2}Is(G')\}\) is a maximal linearly independent subset over \(\mathbb{Z}\) of \((Is(G') \cdot Z(G))/Is(G')\) and therefore the rank of the addition \(G_0\) containing these \(u_i\)'s (and therefore the rank of any addition) is \(p = i_2 - i_1\).

**Proof.** Pick a pseudo-basis \(\bar{u}\) like the one in Lemma 7.5. Then 1 is true for \(\bar{u}\) by Lemma 7.5 and uniform definability of \(Z(G)\) and \(Is(G')\). Statements 2, 3 and 4 which are clearly true for \(\bar{u}\) by choice and are first order in an extension \((L, \bar{u})\) of \(L\). Note that \(t_k\) and \(e_i\) are fixed known integers. Also note that

\begin{itemize}
  \item \(t_k([u_i, u_j]) = 0\) for any \(i\) and \(j\), if \(k \leq i_2\),
  \item \(t_k(u^{e_i}) = 0\) if \(k = i_1 + 1, \ldots, i_2\) for any \(i\) except \(i = i_0 + 1, \ldots, i_1\).
\end{itemize}

To state Statement 5 in \(L\) first recall that \(G_0 \cong Is(G' \cdot Z(G))/Is(G') = P\) where the right hand side is interpretable in \(G\) uniformly. So given any integer \(e \geq 2\) we can write a sentence of \(L\) that says that the image of

\[
\{u_{i_1+1}, u_{i_1+2}, \ldots, u_{i_2}\}
\]

in \(P/eP\) is a basis of \(P/eP\) which implies 5.

\(\square\)

Let us make the following agreement on the choice of elements \(u_i\), \(i_0 + 1 \leq i \leq i_2\).

**Remark 7.7.** One can choose the pseudo-basis \(\bar{u}\) of \(G\) so that for \(i_0 + 1 \leq i \leq i_1\) and \(i_1 + 1 \leq k \leq i_2\),

\begin{itemize}
  \item \(t_k(u^{e_i}) = 1\) if \(k = i + n\),
  \item \(t_k(u^{e_i}) = 0\) if \(k \neq i + n\)
\end{itemize}
where $n = i_1 - i_0$.

**Proof.** The abelian group $(Is(G') \cdot Z(G))/Is(G')$ is a finite index subgroup of the free abelian group $Is(G') \cdot Z(G))/Is(G')$. So by structure theory of finitely generated abelian groups one can choose a basis $w_1Is(G'), \ldots, w_pIs(G')$, where $p = i_2 - i_1$ is the rank of $(Is(G') \cdot Z(G))/Is(G')$, of $Is(G') \cdot Z(G))/Is(G')$ so that $w_1e_{i_0+1}Is(G'), \ldots, w_pe_{i_0+p}Is(G')$ is a basis of $(Is(G') \cdot Z(G))/Is(G')$. To make the number $i_1 - i_0 = n$ minimal we can make the choice so that we get the invariant factor decomposition:

$$
\frac{Is(G') \cdot Z(G)}{Is(G') \cdot Z(G)} \cong \frac{\mathbb{Z}}{e_{i_0+1}\mathbb{Z}} \oplus \cdots \oplus \frac{\mathbb{Z}}{e_{i_1}\mathbb{Z}},
$$

where $e_{i_0+1}|e_{i_0+2}| \ldots |e_{i_1}$, and for each $i$, $|e_i| > 1$.

Now set:

- $u_{i_0+j} = w_j$ for $1 \leq j \leq n$,
- $u_{i_1+j} = w_{i_0+j}^e$ for $1 \leq j \leq n$, and
- $u_{i_1+j} = w_j$ for $n + 1 \leq j \leq p$

to get the desired result.

Indeed what Lemma 7.6 states is that the length and periods associated to the pseudo-basis $\tilde{u}$ are captured uniformly in $Th(G)$. Therefore all the structure constants of $\tilde{u}$ are captured in $Th(G)$ except those $t_k(u_i^e)$ where $i_0 + 1 \leq i \leq i_1$ and $i_1 + 1 \leq k \leq i_2$. This is mainly due to our failure in expressing in $L$ that $\{u_{i_1+1}, u_{i_1+2}, \ldots, u_{i_2}\}$ is a generating set for $G_0$ despite our success in expressing that this set is a maximal linearly independent set.

**Theorem 7.8.** Assume $G \equiv H$ are finitely generated nilpotent groups and $\tilde{u} \in G^m$ is the pseudo-basis of $G$ appearing in Lemma 7.6 and Remark 7.7. Then there exists a polycyclic central series $(L)$ of $H$

$$
H = L_1 > \ldots > L_{i_0+1} = Is(H' \cdot Z(H)) > L_{i_0+2} > \ldots > L_{i_1+1} = Is(H')Z(H)
$$

$$
> L_{i_1+2} > \ldots > L_{i_2+1} = Is(H') > L_{i_2+2} > \ldots > L_m > 1,
$$

where each term $L_i$, except possibly when $i_1 + 1 < i \leq i_2$, is definable in $H$ using the same formula that defines $K_i$ in $G$. Moreover there exists a pseudo-basis $\tilde{v} \in H^m$ adapted to the series such that

1. for all $1 \leq i \leq m$

$$
[v_i, v_j] = \prod_{k=i+1}^{m} t_k[u_i,u_j],
$$

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2. if \( e_i \neq \infty \) then for all \( i \) except \( i = i_0 + 1, \ldots, i_1 \)
\[
v^{c_i}_i = \prod_{k=i_2+1}^m v_k^{t_k(u_i^{c_i})},
\]

3. There exists a basis \( \{v_iIs(H') : i_1 + 1 \leq i \leq i_2\} \) of \( Is(H') \cdot Z(H)/Is(H') \),
integers \( d_k, i_1 + 1 \leq k \leq i_1 + n \), and an \( n \times n \) matrix of integers \( c_{ik} \), \( i_0 + 1 \leq i \leq i_1, i_1 + 1 \leq k \leq i_1 + n \) such that
\[
\begin{align*}
(a) & v^{c_i}_i = \left( \prod_{k=i_1+1}^{i_1+n} s^d_{ik} c_{ik} \right) \left( \prod_{k=i_2+1}^m v_k^{t_k(u_i^{c_i})} \right), \text{ for each } i_0 + 1 \leq i \leq i_1, \\
(b) & |\text{det}(c_{ik})| = 1, \\
(c) & \gcd(d, e) = 1 \text{ where } d = d_{i_1+1} \cdots d_{i_1+n} \text{ and } e = e_{i_0+1} \cdots e_{i_1}.
\end{align*}
\]

Proof. Let \( \Phi \) be the formula obtained in Lemma \( \ref{lemma:structure_theory} \) Then \( H = \exists \vec{x} \Phi(\vec{x}) \) and so we may pick a tuple
\[
(v_1, v_2, \ldots, v_1, w_{i_1+1}, \ldots, w_{i_2}, v_{i_2+1}, v_{i_2+2}, \ldots, v_m) \in H^m,
\]
that satisfies \( \Phi \). Set \( L_i \) as the subgroup defined by the formula defining \( K_i \) in \( G \).
So indeed \( L_i = \langle v_i, v_{i+1}, \ldots, v_m \rangle \) if \( i > i_2 \) and \( L_i = \langle v_i, \ldots, v_{i_1}, Is(H') \cdot Z(H) \rangle \) if \( i \leq i_1 \).
The elements \( w_{i_1+1}Is(H'), \ldots, w_{i_2}Is(H') \) form a maximal linearly independent subset of the free abelian group \( Is(H') \cdot Z(H)/Is(H') \cong H_0 \) of rank \( p = i_2 - i_1 \). Considering Remark \( \ref{remark:central_series} \) Item 4 of Lemma \( \ref{lemma:structure_theory} \) implies for \( i_0 + 1 \leq i \leq i_1 \) that:
\[
v^{c_i}_i = w_{i+n} \prod_{k=i_2+1}^m u_k^{t_k(u_i^{c_i})}.
\]

Again by structure theory of finitely generated abelian groups there exists a basis \( \{v_kIs(H') : i_1 + 1 \leq k \leq i_2\} \) for \( Is(H') \cdot Z(H)/Is(H') \) and integers \( d_k \) as in the statement where
\[
H_1 = \langle v^{d_i}_i Is(H') : i_0 + 1 \leq i \leq i_0 + n \rangle = \langle w_iIs(H') : i_0 + 1 \leq i \leq i_0 + n \rangle,
\]
as subgroups of \( (Is(H') \cdot Z(H))/Is(H') \).
To make sure that the above change of basis does not affect \( t_k(u_i^{c_i}) \) in \( \ref{eq:change_of_basis} \) we need to pick \( v_1, \ldots, v_{i_1+n} \) discussed above from the addition \( H_0 \) that contains the \( w_i \).

Hence there exists a central series as in the statement and a pseudo-basis \( \bar{v} \) of \( H \) of the same length as \( \bar{u} \) and also the same periods. Notice that by our choice of \( \bar{v} \) for any \( 1 \leq k, i, j \leq m, t_k([v_i, v_j]) = t_k([u_i, u_j]) \) which verifies Item 1. Item 2 can be verified in a similar way.
Items 3-(a) and 3-(b) should be clear once it is noted that the matrix \( (c_{ik}) \) is the change of basis matrix corresponding to the bases obtained for \( H_1 \) above. To see why 3-(c) is true note that by Item 5 of Lemma \( \ref{lemma:structure_theory} \) for the chosen integer \( e \geq 2 \) the index \( l \) of \( \langle w_iIs(H') : i_1 + 1 \leq i \leq i_2 \rangle \) in \( Is(H') \cdot Z(H)/Is(H) \) should be relatively prime to \( e \). By the choice of the \( d_i \) and \( d \) above, \( d|l \), and so if \( e \) is chosen as in the statement then \( \gcd(d, e) = 1 \).

\[\square\]
Remark 7.9. The number $p - n \geq 0$ reflects the rank of the free abelian subgroup of $Z(G)$ (or $G_0$) that splits from $G$. Clearly, from the arguments above this number remains the same when we move from $G$ to a finitely generated elementarily equivalent copy $H$ of it, i.e. the number $p - n$ is an elementary invariant of $G$ with respect to finitely generated models of $Th(G)$.

Before stating some immediate corollaries of this result let us give a name to the construction above. This naming will be justified in the next section.

**Definition 7.10** (Finitely Generated Abelian Deformations). Given a group $G$ presented as in Lemma 7.6 and integers $d_i$ and $c_{ik}$ satisfying 3(b) and 3(c) of Theorem 7.8 the group $\bar{H}$ constructed thus is called an abelian deformation of $H$.

Theorem 1.10 summarizes some of important implications of the results above.

**Proof of Theorem 1.10.** Pick $\bar{u}$ in $G$ and $v_1, \ldots, v_{i_1}, w_{i_1+1}, \ldots, w_{i_2}, v_{i_2+1}, \ldots, v_m$ in $H$ as in Theorem 7.8. Then $u_i \mapsto v_i$ if $i \neq i_1 + 1, \ldots, i_2$ and $u_i \mapsto w_i$ if $i = i_1 + 1, \ldots, i_2$ extends to a monomorphism $\phi$ of groups. Statements (a)-(d) are corollaries of the existence and structure of the monomorphism $\phi$. Statement (e) follows from the uniform interpretability (and has already been used in the proofs of the main results here) of this quotient in $G$.

**Theorem 7.11.** If $G$ is a finitely generated nilpotent group with tame special gap, i.e. $Z(G) \leq Is(G')$, and $H$ is a finitely generated group elementarily equivalent to $G$ then $G \cong H$.

**Proof.** With $\bar{u}$ and $\bar{v}$ chosen above if $p = i_2 - i_1 = 0$ then the two pseudo-bases have the same length, periods and structure constants. So $G \cong H$.

**Proof of Theorem 1.11.** By part (e) of Theorem 1.10 and Proposition 6.5, $H$ is regular. Now the statement follows from part (b) of Theorem 1.10.

8 Elementary equivalence of finitely generated nilpotent groups and the second cohomology group

Our aim here is to prove the cohomological form, Theorem 1.13 of Theorem 7.8. To start we briefly review a few concepts from the relationship between extensions with non-abelian kernel and the second cohomology group. Readers can refer to ([15], Chapter 11) or ([1], Chapter IV) for details.

Assume

$$1 \to N \xrightarrow{\mu} E \xrightarrow{\rho} G \to 1,$$
is a short exact sequence of groups, called an extension of $N$ by $G$. The group $N$ is often called the kernel of the extension. A transversal function $\eta : G \to E$ so that $\epsilon \circ \eta = id_G$, gives a function $\chi_\eta : G \to Aut(N)$ induced by the action of $\eta(G)$ on $\mu(N)$ by conjugation. As usual $Aut(N)$ denotes the group of automorphisms of $N$. This function induces a homomorphism $\chi : G \to Out(N)$ called a coupling associated to the extension which is independent of $\eta$, where $Out(N)$ denotes the group of outer-automorphisms of $N$. It is a known fact that there are couplings which can not be realized by any extensions. However given a coupling $\chi$ and knowing that there exists at least one extension realizing it, we may lift $\chi$ to a set map with target $Aut(N)$, then indeed this induces a well-defined homomorphism from $G$ to $Aut(Z(N))$, that is the center $Z(N)$ of $N$ is actually a $G$-module. Let us denote by $H^2(G, Z(N))$ the usual second cohomology group, i.e. the group of 2-cycles from $G$ into $Z(N)$ factored out by the subgroup of 2-coboundaries. Let us state the following well known fact:

**Fact 1.** There is a bijection between the equivalence classes of extensions of $N$ by $G$ with coupling $\chi$ and the group $H^2(G, Z(N))$, providing that such extensions exist, in which case $Z(N)$ is a $G$-module via any lifting of $\chi : G \to out(N)$ to $Aut(N)$.

Here we follow the notation introduced in Lemma [6]. Let us also recall the following notation

$$N(G) = Is(G') \cdot Z(G) = Is(G') \oplus G_0$$

$$M(G) = Is(G' \cdot Z(G))$$

$$\tilde{G} = \frac{G}{N(G)}$$

We look at $G$ as the following extension, where $\mu$ is inclusion and $\pi$ is the canonical surjection:

$$1 \to N(G) \xrightarrow{\mu} G \xrightarrow{\pi} \tilde{G} \to 1. \tag{9}$$

Assume $\chi$ is the coupling associated to the extension. Now by Fact [1] we know that $G$ corresponds to a unique element in $H^2(\tilde{G}, Z(N(G)))$. Note that $N(G) = Is(G') \oplus G_0$ and thus $Z(N(G)) = N_1(G) \oplus G_0$ where $N_1(G) = Z(Is(G'))$. Since both $N_1(G)$ and $G_0$ are normal in $G$ the action of $G$ by conjugation on $N(H)$ fixes both of the summands. So the decomposition $Z(N(G)) = N_1(G) \oplus G_0$ is a direct sum of $\tilde{G}$-submodules. This implies that a 2-cocycle $f \in Z^2(\tilde{G}, Z(N(G)))$ can be uniquely written as $f_1 \oplus f_2$, modulo 2-coboundaries, where $f_1 \in Z^2(\tilde{G}, N_1(G))$ and $f_2 \in Z^2(\tilde{G}, G_0)$. We note that $f_2$ is the 2-cocycle whose class determines the extension:

$$1 \to G_0 \xrightarrow{G} \frac{G}{Is(G')} \to \tilde{G} \to 1, \tag{10}$$

so indeed $f_2$ is a symmetric 2-cocycle. So we have shown here that

$$H^2(\tilde{G}, Z(N(G))) \cong H^2(\tilde{G}, N_1(G)) \oplus Ext(\tilde{G}, G_0) \tag{11}$$
Moreover,

\[ \bar{G} = \frac{G}{N(G)} \cong \frac{G}{M(G)} \oplus \frac{M(G)}{N(G)}. \]

Recall that \( Ext(A, B) = 0 \) whenever \( A \) is a free abelian group, so

\[
Ext(\bar{G}, G_0) \cong Ext\left( \frac{G}{M(G)}, G_0 \right) \oplus Ext\left( \frac{M(G)}{N(G)}, G_0 \right)
\cong Ext\left( \frac{G}{M(G)}, G_0 \right) \oplus Ext\left( \frac{M(G)}{N(G)}, G_0 \right) \tag{12}
\]

So (11) actually becomes

\[
H^2(\bar{G}, Z(N(G))) \cong H^2(\bar{G}, N_1(G)) \oplus Ext\left( \frac{M(G)}{N(G)}, G_0 \right) \tag{13}
\]

Where \([f] \mapsto [f_1] \oplus [f_2] \),

Now considering (13) and Theorem 7.8 the reader might suspect that a finitely generated group \( H \equiv G \) is isomorphic to an extension of \( N(G) \) by \( \bar{G} \) whose associated element in \( H^2(\bar{G}, Z(N(G))) \) is the class of a 2-cocycle \( f_1 \oplus f'_2 \)

where \( f_1 \) is the \( f_1 \) picked above but \( f'_2 \in S^2\left( \frac{M(G)}{N(G)}, G_0 \right) \) might belong to a different class compared to \( f_2 \).

We have not yet explicitly described how structure constants, which clearly define extensions, relate to 2-cocycles, which are other means of defining extensions. This correspondence is precisely the so-called correspondence between covering group theory and cohomology theory which is discussed in detail in [15], Chapter 11 and we won’t need to delve into that theory in detail here. However in the case of abelian-by-(finite cyclic) extensions the correspondence can be described rather easily and that is all we need here. Indeed the reader might suspect all we need is a description of the 2-cocycle \( f_2 \) (or \( f'_2 \)) in [13] in terms
of the structure constants defining an abelian extension of $G_0$ by $\bar{G}$. By (14) this reduces to abelian-by-finite cyclic extensions. Indeed by (14), $f_2$ can be uniquely written (modulo the 2-coboundaries) as,

$$f_2 = \sum_{i=i_0+1}^{i_1} f_{2i}, \quad f_{2i} \in S^2((u_iN(G)), G_0).$$

(15)

The following lemma is a special case of exercise 11.1.5 in [15].

**Lemma 8.1.** Given an abelian group $A$ freely generated by the $\{u_1, \ldots, u_p\}$, integers $c_1, \ldots, c_p$ and an integer $e > 2$, the abelian group presented by

$$E = \langle u_1, u_2, \ldots, u_p, g : g^e = u_1^{c_1} \cdots u_p^{c_p} \rangle$$

can be described as an extension

$$1 \to A \to E \to \langle gA \rangle \to 1,$$

whose associated element in $Ext(\langle gA \rangle, A)$ is represented by $f$ defined below.

$$f(g^sA, g^tA) = \begin{cases} 1 & \text{if } s + t < e \\ u_1^{c_1} \cdots u_p^{c_p} & \text{if } s + t \geq e \end{cases}$$

**Lemma 8.2.** The group $Abdef(G, \bar{d}, \bar{c})$ given in Definition 1.12 is well-defined and defines the same object as the one introduced in Definition 7.10, and therefore if $H$ is finitely generated group $H \equiv G$ then $H \cong Abdef(G, \bar{d}, \bar{c})$ for some integers $d_i$ and $c_{ik}$ as in the definition.

**Proof.** Well-definedness follows from considerations resulting in (13) and Lemma 8.1. The isomorphism $H \cong Abdef(G, \bar{d}, \bar{c})$ and the claim that the two definitions meet both are direct consequences of Theorem 7.8 and Lemma 8.1.

**Proof of Theorem 1.13.** Obvious from the above lemma.

It is already known that up to isomorphism there are only finitely many finitely generated groups elementarily equivalent to a given finitely generated nilpotent group $G$, say as a corollary of the fact that elementarily equivalent finitely generated nilpotent groups have isomorphic finite quotients and Pickel’s theorem that there only finitely many non-isomorphic such groups with isomorphic finite quotient. However from our results this can be seen quite easily.

**Corollary 8.3.** Given a finitely generated nilpotent group $G$, presented an in Lemma 7.6 there are, up to isomorphism, at most $e^p$ finitely generated groups elementarily equivalent to it, where $e = \left| \frac{M(G)}{N(G)} \right|$, and $p$ is the rank of $G_0$.
Proof. Indeed by Theorem 1.13 there are at most $|\text{Ext}(\frac{M(G)}{N(G)}, G_0)|$ such groups up to isomorphism. But by (14)

$$\text{Ext}(\frac{M(G)}{N(G)}, G_0) \cong \bigoplus_{i=i_0+1}^{i_1} \text{Ext}(\langle u_iN(G) \rangle, G_0)$$

$$\cong \bigoplus_{i=i_0+1}^{i_1} \text{Ext}(\frac{\mathbb{Z}}{d_i\mathbb{Z}}, \mathbb{Z}^p)$$

$$\cong \bigoplus_{i_{0+1}}^{i_1} \mathbb{Z}^p_{e_i\mathbb{Z}^p}$$

So there are at most $e^p$ number of non-isomorphic groups as such.

\[\square\]

9 The converse of the characterization theorem

In this subsection we prove the converse of Characterization Theorem. Again we stick to the notation introduced in Lemma 7.6 and Theorem 7.8.

**Theorem 9.1.** Assume $G$ is a finitely generated nilpotent group presented as in Lemma 7.6 and Remark 7.7. Consider an abelian deformation $H = \text{Abdef}(G, d, \bar{c})$ with the corresponding pseudo-basis $\bar{v}$. Then

$$G \equiv H.$$ 

**Proof.** We shall prove the statement using saturated ultrapowers and the structure theory of saturated abelian groups.

Recall that $e = e_{i_0+1} \cdots e_{i_1}$, $d = d_{i_1} \cdots d_{i_1+n}$ and $\gcd(d, e) = 1$. Assume $\pi$ denotes the set of all prime numbers and that, $\pi_k$ is the set of all prime numbers $p$, such that $p|d_k$, $i_1+1 \leq k \leq i_1+n$. Let us denote the $j$th prime number in $\pi \setminus \pi_k$ by $p_{kj}$ and the product of the first $j$ primes in $\pi \setminus \pi_k$ by $q_{kj}$.

By construction the map $\phi$ defined as follows extends to a monomorphism of groups.

$$\phi(u_i) = \begin{cases} v_i & \text{if } i \neq i_1+1, \ldots i_1+n \\ \prod_{k=i_1+1}^{i_1+n} c_k d_k & \text{if } i = i_1+1, \ldots i_1+n \end{cases}$$

Actually $\phi$ defined above is the same as in Proposition 1.10 while the $w_j$ are written with respect to the new basis of $H_0$ found in the proof Theorem 7.8.

For each $j \in \mathbb{N}$ one could twist the monomorphism $\phi: G \to H$ to get a new one.
denoted by \( \phi_j : G \to H \) and defined by:

\[
\phi_j(u_i) = \begin{cases} 
  v_i & \text{if } i \neq i_0 + 1, \ldots, i_1 + n \\
  v_i \prod_{k=i_1+1}^{i_1+n} v_k^{q_{kj}^* e_i c_{ik}} & \text{if } i = i_0 + 1, \ldots, i_1 \\
  \prod_{k=i_1+1}^{i_1+n} v_k^{d_{ik} e_i + q_{kj}^* e_i c_{ik}} & \text{if } i = i_1 + 1, \ldots, i_1 + n
\end{cases}
\]

where \( e_i = e/e_i \). Again note that \( G \) and \( \text{im}(\phi_j) \) are generated by the pseudo-bases of the same lengths, periods and structure constants.

In order to prove the statement we prove that ultra-powers \( G^* = G^N/D \) and \( H^* = H^N/D \) of \( G \) and \( H \) over any \( \omega_1 \)-incomplete ultra-filter \((\mathbb{N}, \mathcal{D})\) are isomorphic. Note that \( G^* \) and \( H^* \) are both \( \omega_1 \)-saturated. The homomorphism \( \phi^* : G^* \to H^* \) induced by \( (\phi_j)_{j \in \mathbb{N}} \) is injective by construction. We need only prove the surjectivity. Let us identify the \( u_i \) and \( v_i \) with their images under the respective diagonal mappings of \( G \) and \( H \) into \( G^* \) and \( H^* \). The homomorphism \( \phi^* \) induces \( \hat{\phi}^* : \text{Ab}(G^*) \to \text{Ab}(H^*) \). We note that if \( \hat{\phi}^* \) is surjective so is \( \phi^* \). Now by definition of \( \hat{\phi}^* \) this reduces to checking whether the map \( \psi^* : G^*/(\text{IS}(G^*)) \to H^*/(\text{IS}(H^*))^* \) induced by \( \phi^* \) is surjective. So it is enough to prove that all \( v_i(\text{IS}(H^*))^*, i = i_1 + 1, \ldots, i_1 + n \) are in the image of this map. Note that for each \( j \in \mathbb{N} \) and for all \( i_0 + 1 \leq i \leq i_1 \)

\[
\prod_{k=i_1+1}^{i_1+n} v_k^{c_{ik}(d_k + q_{kj}^*)} \in \text{im}(\phi_j).
\]

Now if \( q_{kj}^* \) denotes the class of \( (q_{kj})_{j \in \mathbb{N}} \) in the ring \( \mathbb{Z}^* \) the above implies that for all \( i_0 + 1 \leq i \leq i_1 \) the elements

\[
w_i = \prod_{k=i_1+1}^{i_1+n} v_k^{c_{ik}(d_k + q_{kj}^*)} (\text{IS}(H^*))^*
\]

are in the image of \( \psi^* \). Treating coordinates of each \( w_i \) as a row vector in \((\mathbb{Z}^*)^n\) we get an \( n \times n \) matrix \( \mathcal{M} \) where

\[
det(\mathcal{M}) = det(c_{ik}) \prod_{k=i_1+1}^{i_1+n} (d_k + q_{kj}^*) = \prod_{k=i_1+1}^{i_1+n} (d_k + q_{kj}^*).
\]

Now we claim that for any prime \( p \) and for any \( k = i_1 + 1, \ldots, i_1 + n \), \( p \nmid (d_k + q_{kj}^*) \) in \( \mathbb{Z}^* \). To prove the claim recall that \( q_{kj} = p_{k_1} \cdots p_{k_j} \) where the \( p_{k_1}, \ldots, p_{k_j} \) are the first \( j \) primes that do not divide \( d_k \). Pick a prime \( p \). If \( p \in \pi_d \), i.e. \( p | d_k \) and \( p | (d_k + q_{kj}^*) \), then \( p | q_{kj}^* \) which contradicts the choice of \( q_{kj} \) and the fact that \( \text{gcd}(d_k, e) = 1 \). So for such \( p \), \( p \nmid (d_k + q_{kj}^*) \). Now pick a prime \( p \in \pi \setminus \pi_d \), i.e. \( p \nmid d_k \). Then \( p = p_{kt} \) for some \( t \in \mathbb{N} \), meaning that \( p \) is a factor of \( q_{kj} \). Therefore, for every such \( j \) if \( p | (d_k + q_{kj}^*) \) then \( p | q_{kj}^* \) which is impossible. So for every \( j \geq t \), \( p \nmid d_k + q_{kj}^* \). So indeed for any prime \( p \), \( p \nmid (d_k + q_{kj}^*) \).
By the structure theory of saturated abelian groups (see either of [17] or [2]) each \((d_k + q^2e)\) is a generator of \(\mathbb{Z}^*\) as a \(\mathbb{Z}^*\)-module. In particular these elements are units in the ring \(\mathbb{Z}^*\). This implies that \(\text{det}(\mathcal{M})\) is a unit in \(\mathbb{Z}^*\) so \(\mathcal{M}\) is an invertible matrix as a matrix with entries in \(\mathbb{Z}^*\). So indeed for each \(i_1 + 1 \leq i \leq i_1 + n\)
\[
v_i(\text{Is}(H'))^* \in \langle w_i : i_1 + 1 \leq i \leq i_1 + n \rangle.
\]
By Keisler-Shelah theorem, this prove that
\[G \equiv H.\]

\[\square\]

10 Zilber’s example

Here we present an example due to B. Zilber [18] which was the first example of two elementarily equivalent finitely generated nilpotent groups which are not isomorphic. We shall show that they can be looked at as abelian deformations of one another.

Consider the groups \(G\) and \(H\) presented (in the category of 2-nilpotent groups) below.
\[
G = \langle a_1, b_1, c_1, d_1 | a_1^5 \text{ is central}, [a_1, b_1][c_1, d_1] = 1 \rangle,
\]
\[
H = \langle a_2, b_2, c_2, d_2 | a_2^5 \text{ is central}, [a_2, b_2]^2[c_2, d_2] = 1 \rangle.
\]
B. Zilber [18] proved that \(G \equiv H\) but \(G \not\equiv H\). Let us first apply a Titze transformation to both \(G\) and \(H\) to get
\[
G \cong \langle a_1, b_1, c_1, d_1, f_1 | f_1 \text{ is central, } a_1^5f_1^{-1} = 1, [a_1, b_1][c_1, d_1] = 1 \rangle
\]
\[
H \cong \langle a_2, b_2, c_2, d_2, f_2 | f_2 \text{ is central, } a_2^5f_2^{-1} = 1, [a_2, b_2]^2[c_2, d_2] = 1 \rangle.
\]

Now we are going to show that \(H\) is an abelian deformation of \(G\). So define a group \(K\) by
\[
K = \langle a_3, b_3, c_3, d_3, f_3 | f_3 \text{ is central, } a_3^5f_3^{-2} = 1, [a_3, b_3][c_3, d_3] = 1 \rangle.
\]
Note that we can choose \(G_0 = \langle f_1 = a_1^5 \rangle\) and \(K_0 = \langle f_3 \rangle\). We also have \(\text{Is}(G'G_0) = G'\text{Is}(G_0), \text{Is}(K'K_0) = K'\text{Is}(K_0)\) and so \(\text{Is}(G'G_0)/\text{Is}(G')G_0 = \text{Is}(G_0)/G_0 = \langle a_1 | a_1^5 = 1 \rangle\) and \(\text{Is}(K'K_0)/\text{Is}(K')K_0 = \text{Is}(K_0)/K_0 = \langle a_3 | a_3^5 = 1 \rangle\). Indeed using notation of Definition [18] \(K = \text{Abdef}(G, d, c)\) where \(d = 1\) and \((c)\) is \(1 \times 1\) matrix \((1)\). Indeed we only changed the structure constant \(t_{f_1}(a_1^5)\) to deform \(G\) to \(K\). However in \(K\)
\[
(a_3^3f_3^{-1})^2 = a_3^5f_3^{-2} = a_3(a_3^5f_3^{-2}) = a_3.
\]
Now apply the corresponding Titze transformation to the presentation of \(K\) to get
\[
K = \langle a_3^3f_3^{-1}, b_3, c_3, d_3, f_3 | f_3 \text{ is central, } (a_3^3f_3^{-1})^5f_3^{-1} = 1, [a_3^3f_3^{-1}, b_3]^2[c_3, d_3] = 1 \rangle.
\]

Obviously \(H \cong K\).
References

[1] K. S. Brown, Cohomology of Groups, Springer-Verlag New York, 1982.

[2] P. C. Eklof, and R. F. Fischer, The elementary theory of abelian groups, Ann. Math. Logic, 4(2) (1972) 115-171.

[3] P. Hall, Nilpotent Groups, Notes of lectures given at the Canadian Mathematical Congress, University of Alberta, August 1957.

[4] A. G. Myasnikov, Elementary equivalence of finitely-generated nilpotent groups, 9th All-Union Symposium on Group Theory, Thesis Report, Lenin Moscow State Pedagogical Institute, Moscow (1984) 46-47.

[5] A.G. Myasnikov and V.N. Remeslennikov, Definability of the set of Mal’cev bases and elementary theories of finite-dimensional algebras I, Sibirsk. Math. Zh., 23(5) (1982) 152-167. English transl., Siberian Math. J. 23 (1983) 711-724.

[6] A.G. Myasnikov and V.N. Remeslennikov, Definability of the set of Mal’cev bases and elementary theories of finite-dimensional algebras II, Sibirsk. Math. Zh. 24(2) (1983) 97-113. English transl., Siberian Math. J. 24 (1983) 231-246.

[7] A. G. Myasnikov, Elementary theories and abstract isomorphisms of finite-dimensional algebras and unipotent groups, Dokl. Akad. Nauk SSSR, 297(2) (1987) 290-293.

[8] A. G. Myasnikov, V. N. Remeslennikov, Recursive $p$-adic numbers and elementary theories of finitely generated pro-$p$-groups. (Russian) Izv. Akad. Nauk SSSR Ser. Mat. 51 (1987), no. 3, 613–634; translation in Math. USSR-Izv. 30 (1988), no. 3, 577-597.

[9] A. G. Myasnikov, V. Remeslennikov, Model-theoretic questions in group theory. (Russian) Problems in Algebra, Minsk (1989) no. 4, 16-32.

[10] A. G. Myasnikov, Definable invariants of bilinear mappings, (Russian) Sibirsk. Mat. Zh. 31(1) (1990) 104-115, English trans. in: Siberian Math. J. 31(1) (1990) 89-99.

[11] A. G. Myasnikov, The theory of models of bilinear mappings, (Russian) Sibirsk. Mat. Zh. 31(3) (1990) 94-108, English trans. in Siberian Math. J. 31(3) (1990) 439-451.

[12] A. G. Myasnikov and M. Sohrabi, Groups elementarily equivalent to a free 2-nilpotent group of finite rank, Alg. and Logic, 48(2) (2009), 115-139

[13] A. G. Myasnikov, M. Sohrabi, Groups elementarily equivalent to a free nilpotent group of finite rank, Ann. Pure Appl. Logic 162(11) (2011), 916-933
[14] F. Oger, Cancellation and elementary equivalence of finitely generated finite-by-nilpotent groups, J. London Math. Society 44(2) (1991) 173-183.

[15] D. J. S. Robinson, A Course in the Theory of Groups, 2nd edn. Springer-Verlag New York, 1996.

[16] Julia Robinson, The undecidability of algebraic rings and fields, Proc. Amer. Math. Soc. 10 (1959) 950-957.

[17] W. Szmielew, Elementary properties of abelian groups, Fund. Math. 41 (1955) 203-271.

[18] B. Zilber, An example of two elementarily equivalent, but not isomorphic finitely generated nilpotent groups of class 2, Alg. and Logic, 10(3) (1971) 173-188.