Uniqueness of the Axionic Kerr Black Hole

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ABSTRACT

Under the axisymmetry and under the invarance of simultaneous inversion of time and azimuthal angle, we show that the axionic Kerr black hole is the unique stationary solution of the minimal coupling theory of gravity and the Kalb-Ramond field, which has a regular event horizon, is asymptotically flat and has a finite axion field strength at event horizon.
1. Introduction

Recently, a lot of interest has been aroused in the theory in which gravity is coupled to the Kalb-Ramond field\(^{[1-5]}\). The only static, spherically symmetric solution to the minimal coupling theory, which has a regular event horizon, is asymptotically flat and has a finite axionic charge \(q\), is the Schwarzschild solution with vanishing axion field strength but nonvanishing Kalb-Ramond field\(^{[1,4]}\). Since the axion field strength vanishes everywhere and the axion charge can be detected not by a point particle but by a string in a process analogous to the Aharonov-Bohm effect, the axionic charge is regarded as a quantum hair of black hole\(^{[6]}\).

Studying the axisymmetric field equation of the theory, we found an axionic Kerr black hole solution with non-vanishing axion field strength for the case of slow rotation \((a^2 < M^2)\)\(^{[5]}\). In addition to mass and angular momentum, the solution indicates that there exists a new classical hair called axionic hair, which can be detected by a test particle directly without resorting to the stringy Aharonov-Bohm process.

In this paper, we show that, under the axial symmetry and under the invariance of simultaneous inversion of time and azimuthal angle, the axionic Kerr black hole is the \textit{unique} stationary solution of the minimal coupling theory of gravity and Kalb-Ramond field. This black hole has a regular event horizon, is asymptotically flat and has a finite axion field strength at event horizon.

2. The stationary axisymmetric Einstein-Kalb-Ramond equation

The minimal coupling theory of Kalb-Ramond field and gravity is described by the action

\[
I = \int d^4x \sqrt{-g} \left( \frac{1}{2\kappa} R - \frac{1}{6} H_{\mu\nu\lambda} H^{\mu\nu\lambda} \right),
\]

in which

\[
H_{\mu\nu\lambda} = \partial_\mu B_{\nu\lambda} + \partial_\nu B_{\lambda\mu} + \partial_\lambda B_{\mu\nu}
\]

is the field strength of Kalb-Ramond field \(B_{\mu\nu}\), and the field equations are the

\[
\frac{1}{2\kappa} R - \frac{1}{6} H_{\mu\nu\lambda} H^{\mu\nu\lambda} = 0.
\]
Einstein-Kalb-Ramond (EKR) equation:

\[ R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = - \kappa \left( H_{\mu\lambda\rho} H^{\lambda\rho} - \frac{1}{6} g_{\mu\nu} H_{\lambda\rho\sigma} H^{\lambda\rho\sigma} \right) \]  

(3)

\[ \partial_{\sigma} \left( \sqrt{-g} H^{\sigma\mu\nu} \right) = 0. \]  

(4)

To describe a stationary axisymmetric spacetime, it is convenient to take the time \( t \) and the azimuthal angle \( \phi \) about the axis of symmetry as two of the coordinates. The stationary and the axisymmetric character of the spacetime require that the metric be independent of \( t \) and \( \phi \)\(^{[7,8]}\). Besides stationarity and axisymmetry, it is also required that the spacetime is invariant under the simultaneous inversion of the time \( t \) and the angle \( \phi \), i.e., to the transformation \( t \rightarrow -t \) and \( \phi \rightarrow -\phi \), for a purely rotation. Hence, under the assumption made, the metric\(^{[7]}\) components

\[ g_{tr} = g_{t\theta} = g_{\phi r} = g_{\phi\theta} = 0, \]

where \( r \) and \( \theta \) are the two remaining spatial coordinates. Moreover, a further reduction in the form of the metric can be achieved by using the theorem — a two dimensional space, with a positive or a negative definite signature, can always be brought to diagonal form by a coordinate transformation. Therefore, the stationary axisymmetric metric is

\[ ds^2 = e^{2A(r,\theta)} dt^2 - e^{2B(r,\theta)} (d\phi - \omega(r, \theta) dt)^2 - e^{2C(r,\theta)} dr^2 - e^{2D(r,\theta)} d\theta^2. \]  

(5)

Here, we introduce the tetrad frame with basis 1-forms

\[ \omega^0 = e^A dt, \omega^1 = e^B (d\phi - \omega dt), \omega^2 = e^C dr, \omega^3 = e^D d\phi. \]

We also impose the stationarity, axisymmetry and simultaneous inversion invariance upon the Kalb-Ramond field. Analogous to the metric components, the

\* We shall use the conventions given in reference\(^{[7]}\).
corresponding non-vanishing tetrad components of $B_{\mu\nu}$ and $H_{\mu\nu\lambda}$ are:

\[
B_{01} = -B_{10} \equiv E(r, \theta) \\
B_{23} = -B_{32} \equiv F(r, \theta),
\]

(6)

\[
H_{012} = e^{-A-B-C}(E(r, \theta)e^{A+B})_r \\
H_{013} = e^{-A-B-D}(E(r, \theta)e^{A+B})_\theta.
\]

(7)

The component $B_{23}$ can be regarded as a gauge freedom that does not contribute the field strength $H_{\mu\nu\lambda}$.

Thus, the stationary axisymmetric EKR equations are

\[
\frac{1}{2}e^{2B-2A}[e^{-2C}(\omega_r)^2 + e^{-2D}(\omega_\theta)^2] \\
= e^{-2C}[A_{,r}r + A_r (A + B - C + D)_r] \\
+ e^{-2D}[A_{,\theta} + A_\theta (A + B + C - D)_\theta]
\]

(8)

\[
(e^{3B-A-C+D}\omega_r)_r + (e^{3B-A+C-D}\omega_\theta)_\theta = 0
\]

(9)

\[
-\frac{1}{2}e^{2B-2A}[e^{-2C}(\omega_r)^2 + e^{-2D}(\omega_\theta)^2] \\
= e^{-2C}[B_{,r}r + B_r (B + A - C + D)_r] \\
+ e^{-2D}[B_{,\theta} + B_\theta (B + A + C - D)_\theta]
\]

(10)

\[
-\frac{1}{2}\omega_r\omega_\theta e^{2B-2A} + (A + B),_r, \theta - (A + B)_\theta C_\theta \\
- (A + B)_\theta D_r + A_{,r} A_\theta + B_{,r} B_\theta \\
= 2\kappa e^{-2A-2B}(Ee^{A+B})_r (Ee^{A+B})_\theta
\]

(11)

\[
e^{-2C}[A_{,r}(B + D)_r + B_{,r} D_r] + \frac{1}{4}e^{2B-2A}[e^{-2C}(\omega_r)^2 - e^{-2D}(\omega_\theta)^2] \\
+ e^{-2D}[(A + B),_\theta + (B + A)_\theta (A - D)_\theta + (B_\theta)^2]
\]

(12)
\[ e^{-2D} [A, \theta (B + C), r + B, \theta C, \theta ] - \frac{1}{4} e^{2B - 2A} [e^{-2C}(\omega, \theta)^2 - e^{-2D}(\omega, \theta)^2] \]
\[ + e^{-2C} [(A + B), r, r + (B + A), r (A - C), r + B, r B, r] \]
\[ = - \kappa [e^{-2A - 2B - 2C}((Ee^{A + B}), r)^2 - e^{-2A - 2B - 2D}((Ee^{A + B}), \theta)^2] \]
\[ (e^{-A - B - C + D} (Ee^{A + B}), r), r + (e^{-A - B + C - D} (Ee^{A + B}), \theta), \theta = 0. \] (14)

**3. Uniqueness proof of the axionic Kerr black hole**

It is quite amazing that eqs.(8)-(10) are identical to eqs.(5)-(7) in the page 274 of reference [7]. Such that, following the same procedure in reference [7], we take a choice of gauge freedom
\[ e^{2A} = \Delta, \] (15)
this is consistent with the existence of an event horizon, and in accordance with the Robinson’s theorem [7,9],
\[ e^{2B} = \frac{\delta \Sigma^2}{\rho^2} \]
\[ \omega = \frac{2aMr}{\Sigma^2}, \] (16)
in which
\[ \Delta = r^2 - 2Mr + a^2 \]
\[ \rho^2 = r^2 + a^2 \mu^2 \]
\[ \Sigma^2 = (r^2 + a^2)^2 - a^2 \Delta \delta \] (17)
\[ \mu = \cos \theta \]
\[ \delta = \sin^2 \theta. \]
is the unique solution of eqs.(8)-(10) which satisfy the conditions of asymptotical flat, regular event horizon and non-singular outside of the horizon. The integration
constants \( M \) and \( a \) parametrize the mass and spin angular momentum per mass of the rotating black hole.

Now, the remaining problem is to solve eqs.(11)-(14). Plugging eq.(16) into eq.(14), we end up with

\[
\frac{1}{\sqrt{\delta}}(\sqrt{\Delta E})_{,rr} + \frac{1}{\sqrt{\Delta}}(\sqrt{\delta E})_{,\mu,\mu} = 0. \tag{18}
\]

In reference[5], we found an asymptotical-flat solution

\[
E(r, \theta) = \frac{\alpha \mu + \beta}{\sqrt{\Delta \delta}}, \tag{19}
\]

in which \( \alpha \) and \( \beta \) are integration constants. Here, we could find the general solution of \( E(r, \theta) \) for the slowly rotating case \( (a^2 < M^2) \), and show that eq.(19) is an unique solution of eq.(18) which satisfies the asymptotical-flat condition.

Replacing \( E(r, \theta) \) by

\[
E(r, \theta) \sim \frac{1}{\sqrt{\Delta \delta}} U(r) \Theta(\mu), \tag{20}
\]

we end up with two separated differential equations

\[
\Delta(U(r))_{,rr} - \ell(\ell + 1)U(r) = 0 , \tag{21}
\]

\[
\delta(\Theta(\mu))_{,\mu,\mu} + \ell(\ell + 1)\Theta(\mu) = 0 . \tag{22}
\]

Using the Frobenius’ method, the regular solutions of eq.(22) are \( \Theta_\ell \) polynomials

\[
\Theta_\ell = \begin{cases} 
\alpha' \mu + \beta' & \text{for } \ell = 0 \\
\sum_{n=0}^{m} a_{2m-1}^{2n} \mu^{2n} & \text{for } \ell = 2m - 1 \quad m \geq 1 \\
\sum_{n=0}^{m} a_{2m}^{2n+1} \mu^{2n+1} & \text{for } \ell = 2m \quad m \geq 1
\end{cases} \tag{23}
\]
where
\[
a^0_{2m-1} = a^1_{2m} = 1
\]
\[
a^{2n}_{2m-1} = \frac{1}{(2n)!} \prod_{k=1}^{n} [(2k-2)(2k-3) - 2m(2m-1)] \quad \text{for} \quad 1 \leq n \leq m
\]
\[
a^{2n+1}_{2m} = \frac{1}{(2n+1)!} \prod_{k=1}^{n} [(2k-1)(2k-2) - (2m+1)2m] \quad \text{for} \quad 1 \leq n \leq m
\]
and \(\alpha', \beta'\) are integration constants of the the trivial solution \(\Theta_0\).

Here, \(\ell\) should be an integer to get convergent series for all \(\mu\) in the regime \([-1, 1]\]. Since negative integer values of \(\ell\) would simply give solutions already obtained for positive \(\ell\)'s, it is customary to restrict \(\ell\) to be non-negative values. For instance, the first few \(\Theta_\ell\) polynomials are
\[
\Theta_0 = \alpha' \mu + \beta',
\]
\[
\Theta_1 = 1 - \mu^2,
\]
\[
\Theta_2 = \mu - \mu^3,
\]
\[
\Theta_3 = 1 - 6\mu^2 + 5\mu^4.
\]

Moreover, we rewrite eq.(21) as
\[
\xi(\xi + \xi_0)U_{\xi,\xi} - \ell(\ell + 1)U = 0 \quad (24)
\]
where
\[
\xi = \frac{(r - r_+)}{r_+},
\]
\[
\xi_0 = \frac{(r_+ - r_-)}{r_+},
\]
and \(r_\pm = M \pm \sqrt{M^2 - a^2}\). The polynomial solutions \(U_\ell\) are
\[
U_0(\xi) = \alpha'' \xi + \beta''
\]
\[
U_\ell(\xi) = \sum_{n=0}^{\ell} b^{n+1}_\ell \frac{\xi^{n+1}}{\xi_0^n} \quad \text{for} \quad \ell \geq 1 \quad (25)
\]
in which

\[ b_\ell^1 = 1 \]
\[ b_\ell^{n+1} = \frac{1}{(n+1)!n!} \prod_{k=1}^{n} [(\ell+1)\ell - k(k - 1)] \quad \text{for} \quad 1 \leq n \leq \ell, \]

and \( \alpha''', \beta''' \) are integration constants. The first few \( U_\ell \) polynomials are

\[ U_0 = \alpha''\xi + \beta''' \]
\[ U_1 = \xi + \frac{\xi^2}{\xi_0} \]
\[ U_2 = \xi + \frac{3}{\xi_0} \xi^2 + \frac{2}{\xi_0} \xi^3 \]
\[ U_3 = \xi + \frac{6}{\xi_0} \xi^2 + \frac{10}{\xi_0} \xi^3 + \frac{5}{\xi_0} \xi^4. \]

Therefore, the general solution is

\[ E(r, \mu) = \sum_\ell \frac{A_\ell}{\sqrt{\Delta \delta}} U_\ell(r) \Theta_\ell(\mu) \quad (26) \]

where \( A_\ell \) are constants.

Since \( \frac{1}{\sqrt{\Delta}} \sim r^{-1} \) and \( U_\ell(r) \sim \xi^{\ell+1} \sim r^{\ell+1} \) as \( r \gg 1 \), the trivial solution eq.(19) is an unique asymptotically-flat solution.

After using the replacement

\[ e^{C+D} = \frac{\rho^2}{\sqrt{\Delta}} f(r, \theta), \quad (27) \]

eqs.(16),(19) and the gauge condition eq.(15), those eqs.(11)-(13) are reduced to two first order coupled differential equations

\[ -\frac{(r-M)}{\Delta} (\ln f)_{,\mu} + \frac{\mu}{\delta} (\ln f)_{,r} = 0, \quad (28) \]
\[(r - M)(\ln f)_r + \mu (\ln f)_\mu = -\frac{2\kappa \alpha^2}{\Delta}. \quad (29)\]

When the asymptotically flat condition \(f(r \to \infty, \theta) \to 1\) is imposed, the solution
\[\begin{align*}
f(r, \theta) &= [\mu^2 + \delta \frac{(r - M)^2}{\Delta}] \frac{\kappa \alpha^2}{\mu^2 - M^2} \quad (30)
\end{align*}\]
is uniquely determined by straightforward calculation. Moreover, eqs. (15) and (27) give the metric components
\[\begin{align*}
e^{2C} &= \frac{\rho^2}{\Delta} f(r, \theta), \\
e^{2D} &= \rho^2 f(r, \theta). \quad (31)
\end{align*}\]

Base on Robinson’s theorem and under the symmetry of simultaneous inversion of time and azimuthal angle, the unique axisymmetric black hole is
\[\begin{align*}
ds^2 &= \frac{\Delta \rho^2}{\Sigma^2} dt^2 - \frac{\delta}{\rho^2 \Sigma^2} (d\phi - \omega dt)^2 - \frac{\rho^2}{\Delta} f dr^2 - \rho^2 f d\theta^2, \quad (32)
\end{align*}\]
and the non-vanishing tetrad components
\[B_{01} = \frac{\alpha \mu + \beta}{\sqrt{\Delta \delta}}, \quad (33)\]
where \(\delta, \rho^2, \Sigma^2, \delta, \omega, f\), have been defined in eqs. (17) and (30). The non-vanishing tetrad components of axionic field strength
\[H_{013} = -\frac{\alpha}{\sqrt{\Delta \rho}} \quad (34)\]
is asymptotically-flat, nonsingular outside the horizon and regular at the event horizon for \(\frac{\kappa \alpha^2}{M^2 - a^2} \geq 1\).
4. Discussion

Some interesting characteristics for the structure of the axionic Kerr black hole were studied in reference[5]. Besides the ring singularity at \( r = 0, \theta = \frac{\pi}{2} \), there exist two extra singularities \( (r_{\pm} = r \pm \mu \sqrt{M^2 - a^2}, \quad -1 < \mu < 1) \) between the inner event horizon \( r_- \) and the outer event horizon \( r_+ \). The extra singularities may be regarded as the sources of axionic field strength. Therefore, if we are only interested in the region outside the source, the axionic Kerr black hole solution is an acceptable physical solution. The non-vanishing axionic field strength could offer observational features which might not require a detail knowledge of the way that the fundamental string theory devolves into the standard model.

It is worthwhile to note that the axisymmetric axion field strength will still induce an axisymmetric non-rotating black hole if \( a = 0 \), and the axionic Kerr black hole will reduce to the Kerr black hole if the new axionic hair is absent \( i.e. \alpha = 0 \). When we set both \( a = 0 \) and \( \alpha = 0 \), the Schwarzschild black hole is reproduced but with vanishing spherical symmetric Kalb-Ramond field. However, it is, under a gauge transformation, consistent with those results in reference [1,2,4] — the unique spherical symmetric black hole solution of EKR equation is Schwarzschild solution with non-vanishing Kalb-Ramond field \( B_{13} = \frac{q}{4\pi r^2} \). In fact, the field strength \( H_{\mu\nu\lambda} \) is invariant under the gauge transformation \( B_{\mu\nu}^{\prime} = B_{\mu\nu} + \partial_{\mu}\Lambda_{\nu} - \partial_{\nu}\Lambda_{\mu} \). Since both of them give the same vanishing field strength \( H_{\mu\nu\lambda} \), we can recover the non-vanishing Kalb-Ramond field \( B_{13} = \frac{q}{4\pi r^2} \) by a gauge transformation \( \Lambda_{t} = \Lambda_{r} = \Lambda_{\theta} = 0 \) and \( \Lambda_{\phi} = -\frac{q}{4\pi} \cos \theta \).

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