Decidable problems in substitution shifts

Marie-Pierre Béal, Dominique Perrin
Université Gustave Eiffel, CNRS, LIGM, F-77454 Marne-la-Vallée, France,
and Antonio Restivo
Università di Palermo

June 13, 2023

Abstract
In this paper, we investigate the structure of the most general kind of substitution shifts, including non-minimal ones, and allowing erasing morphisms. We prove the decidability of many properties of these morphisms with respect to the shift space generated by iteration, such as aperiodicity, recognizability and (under an additional assumption) irreducibility, or minimality.

Contents
1 Introduction 2
2 Shift spaces 4
3 Morphisms 7
4 Growing letters 17
5 Fixed points 21
6 Periodic points 32
7 Erasable letters 37
8 Recognizability 40
9 Irreducible substitution shifts 42
10 Minimal substitution shifts 43
1 Introduction

Substitution shifts are an important class of symbolic dynamical systems which have been studied extensively. However, most results have been formulated for primitive substitution shifts, which are generated by primitive morphisms or, slightly more generally, for minimal substitution shifts.

In this paper, we investigate the more general situation, allowing in particular the morphisms to be erasing. Our work is a continuation of the investigations of [41] (who pleasantly writes in his introduction that ‘the complement of primitive morphisms is uncharted territory’) and of [38]. Our work is also related with early results on the so-called D0L-systems, initiated by Lindemayer (see [34]). It has also important connections with the study of numeration systems and automatic sequences (see [1]).

We investigate the structure of these shift spaces in terms of periodic points, fixed points or subshifts. A striking feature is that all these properties (such as, for example, the existence of periodic points) are found to be decidable.

We show that, if the shift \( X(\sigma) \) generated by an endomorphism \( \sigma \) is non-empty, it contains fixed points of some power of \( \sigma \) (Theorem 5.14) and we describe them. We show that there is a finite and computable number of orbits of fixed points. We extend the notion of fixed point to that of quasi-fixed point, which corresponds to the orbits stable under the action of the morphism. We generalize to quasi-fixed points the results concerning fixed points (Proposition 5.21). These results play a role in the characterization of periodic morphisms developed below.

We also investigate periodic points in a substitution shift. We show that their existence is decidable (Theorem 6.5). This generalizes a result of Pansiot [32] which implies the same result for primitive morphisms. We show that both the property of being aperiodic (no periodic points) (Theorem 6.5), and the property of being periodic (all points periodic) (Theorem 6.8) are decidable.

Recognizability is an important property of morphisms. A morphism \( \sigma : A^* \rightarrow B^* \) is recognizable in a shift space \( X \) for a point \( y \in B^Z \) if there is at most one pair \( (k,x) \) with \( 0 \leq k < |\sigma(x_0)| \) and \( x \in X \) such that \( y = S^k(\sigma(x)) \). It is shown in [3] that every endomorphism \( \sigma \) is recognizable for aperiodic points on \( X(\sigma) \) (see Section 8 for the background of this result). We show here that it is decidable whether \( \sigma \) is fully recognizable in \( X(\sigma) \) (Theorem 8.2).

Two basic notions concerning shift spaces (and, more generally topological dynamical systems) are irreducibility and minimality.

Concerning irreducibility, we prove a partial result, which implies its decidability under an additional hypothesis (Theorem 9.1).

We next prove that, under an additional assumption, it is decidable whether a substitution shift is minimal (Theorem 10.3). The proof uses a characterization of minimal substitution shifts (Theorem 10.1) proved in [12]. We also generalize to arbitrary morphisms several results known for more restricted classes. In particular, we prove that every minimal substitution shift is conjugate to a primitive one (Theorem 10.4), a result proved in [27] for non-erasing morphisms, and also a particular case of a result of [17].
We investigate in the last section the notion of quasi-minimal shifts which are such that the number of subshifts is finite. We prove, generalizing a result proved in [4] for non-erasing morphisms, that every substitution shift is quasi-minimal (Proposition 10.8).

Our results show that all properties that we have considered are decidable for a substitution shift. This raises two questions. The first one is the possibility of natural undecidable problems for substitution shifts. Actually, it is not known if the equality or the conjugacy of two substitution shifts is decidable (see [17] for the proof of the decidability of conjugacy for minimal substitution shifts). The second question is whether there is a decidable logical language in which these properties can be formulated. Such a logical language exists for automatic sequences (which are defined using fixed points of constant length morphisms). It is an extension of the first-order logic of integers with addition (also known as Presburger arithmetic) with the function $V_p(x)$ giving the highest power of $p$ dividing $x$ (see [6]). This logic is decidable and thus all properties which can be expressed in this logic are decidable. This has been applied in the software Walnut (see [30] and [36]).

Our paper is organized as follows. In a first section (Section 2), we recall the basic notions of symbolic dynamics. In Section 3 we focus on substitution shifts. We define the $k$-th higher block presentation of a non-erasing morphism and prove that it defines a shift which is conjugate to the original one (Proposition 3.7). We prove several lemmas that are used in the sequel (Lemma 3.11 and 3.13). We finally prove that the language of a substitution shift is decidable (Proposition 3.19).

In Section 4, we prove an important property of substitution shifts, namely that there is a finite and computable set of orbits of points formed of non-growing letters (Proposition 4.3). This result is used several times below.

In Section 5, we prove several results concerning fixed points of an endomorphism $\sigma$. We first describe the one-sided fixed points in the one-sided shift associated to $X(\sigma)$. We prove that there is a finite and computable number of orbits of these points (Proposition 5.3) and we give a complete description of these fixed points (Proposition 5.13). We next prove the analogous results for two-sided fixed points (Theorem 5.14 and Proposition 5.18). We finally introduce the notion of quasi-fixed point, which corresponds to the stability of orbits under the action of the morphism $\sigma$. We characterize the quasi-fixed points in the shift $X(\sigma)$ and show, using a result from [1], that there is a finite and computable number of orbits of these points (Proposition 5.19).

In Section 6, we investigate periodic points in substitution shifts. We prove the decidability of the existence of fixed points (Theorem 6.5) as well as that of the existence of non-periodic points, using the notion of quasi-fixed point (Theorem 6.8). This generalizes a result proved in [32] for primitive morphisms.

In Section 7, we prove a normalization result concerning morphic sequences $x = \phi(\tau^* \omega)$ image by a morphism $\phi$ of a fixed point of a morphism $\tau$. We show that one may always replace the pair $(\tau, \phi)$ by a pair $(\sigma, \theta)$ such that $\sigma$ is non-erasing and $\theta$ is alphabetic (Theorem 7.1). This is essentially a result of Cobham [11] (see also [31], [9]).
In Section 8, we show that it is decidable whether a morphism \( \sigma \) is fully recognizable in the shift \( X(\sigma) \) (Theorem 8.2).

In Section 9, we characterize the substitution shifts which are irreducible (Theorem 9.1), under the additional assumption that \( \mathcal{L}(X(\sigma)) = \mathcal{L}(\sigma) \).

In Section 10, we show that, under an additional assumption, it is decidable whether a substitution shift is minimal (Theorem 10.4). This uses a characterization of minimal substitution shifts which had been proved in [12]. We also prove, generalizing the result obtained in [4] for non-erasing morphisms, that every substitution shift has a finite number of subshifts (Proposition 10.8).

Acknowledgments The authors thank Jeffrey Shallit and Fabien Durand for reading a draft of the manuscript and providing additional references. They are also grateful to Herman Goulet-Ouellet for reading the following version with great care and making many useful comments. The referee is also thanked for helping us to improve the final version.

This work was supported by the Agence Nationale de la Recherche (ANR-22-CE40-0011).

2 Shift spaces

We refer to the classical sources, such as [25] for the basic definitions concerning symbolic dynamics. Concerning words, we refer to [26].

Given a finite alphabet \( A \), we let \( A^* \) denote the set of words on \( A \). For a word \( w \in A^* \), let \( |w| \) denote its length and, for \( a \in A \), by \( |w|_a \) the number of occurrences of \( a \) in \( w \). The alphabet \( A \) is assumed to be finite. We let \( \varepsilon \) denote the empty word and \( A^+ \) the set of nonempty words on \( A \). A nonempty word is primitive if it is not a power of a shorter word. We let \( w^* \) denote the submonoid generated by a word \( w \).

For a two-sided infinite sequence \( x \in A^\mathbb{Z} \), let \( x = \cdots x_{-1}x_0x_1 \cdots \) with \( x_i \in A \). For \( i < j \), we denote \( x_{[i,j]} = x_i \cdots x_j \) and \( x_{(i,j)} = x_i \cdots x_{j-1} \). Similarly, \( x_{(-\infty,j]} = \cdots x_{j-1}x_j \) with all obvious variants.

The set \( A^\mathbb{Z} \) is a metric space for the distance defined for \( x, y \in A^\mathbb{Z} \) by 
\[
d(x, y) = 2^{-r(x,y)}
\]
where \( r(x, y) = \min \{ n \geq 1 \mid x_{[-n,n]} \neq y_{[-n,n]} \} \). Since \( A \) is finite, the topological space \( A^\mathbb{Z} \) is compact.

We let \( S_A \) (or simply \( S \)) denote the shift on \( A^\mathbb{Z} \). It is defined by \( y = S(x) \) if
\[
y_n = x_{n+1}
\]
for all \( n \in \mathbb{Z} \). It is a homeomorphism from \( A^\mathbb{Z} \) onto itself. We also let \( S_A \) denote the shift transformation defined on \( A^\mathbb{N} \) by (2.1) for all \( n \geq 0 \). For \( x \in X \), we write sometimes \( Sx \) instead of \( S(x) \).

A shift space on \( A \) is a subset \( X \) of \( A^\mathbb{Z} \) which is closed for the topology and invariant under the shift. An isomorphism (also called a conjugacy) from a shift space \( X \) on \( A \) onto a shift space \( Y \) on \( B \) is a homeomorphism \( \varphi \) from \( X \) onto \( Y \) such that \( \varphi \circ S_A = S_B \circ \varphi \).
For \( x \in A^Z \), we let \( x^+ \in A^N \) denote the right-infinite sequence \( x_0x_1 \cdots \) and by \( x^- \in A^{-N} \) the left-infinite sequence \( \cdots x_{-2}x_{-1} \). Thus \( x^+ = (x_n)_{n \geq 0} \) and \( x^- = (x_{n-1})_{n \leq 0} \). Conversely, given \( y \in A^{-N} \) and \( z \in A^N \), we let \( x = y \cdot z \) denote the two-sided infinite sequence such that \( x^- = y \) and \( x^+ = z \).

For a shift space \( X \), we denote \( X^+ = \{ x^+ \mid x \in X \} \). A set \( Y \subset A^N \) is called a one-sided shift space if \( Y = X^+ \) for some shift space \( X \). Equivalently, \( Y \) is a one-sided shift space if it is closed and such that \( S(Y) = Y \).

Note that \( X^+ \) determines \( X \) since

\[
X = \{ x \in A^Z \mid x_nx_{n+1} \cdots \in X^+ \text{ for every } n \in \mathbb{Z} \}. \quad (2.2)
\]

The orbit of a point \( x \) in a shift space \( X \) is the set

\[
\mathcal{O}(x) = \{ S^n(x) \mid n \in \mathbb{Z} \}.
\]

When \( X \) is a one-sided shift space, the orbit of \( x \) is \( \{ y \in X \mid S^n(x) = S^n(y) \text{ for some } m, n \geq 0 \} \). The positive orbit of a point \( x \) is the set \( \mathcal{O}^+(x) = \{ S^n(x) \mid n \geq 0 \} \).

A sequence \( x \in A^Z \) is periodic if there is an \( n \geq 1 \) such that \( S^n(x) = x \). The integer \( n \) is called a period of \( x \). For a nonempty word \( w \), we let \( w^\omega \) denote the right-infinite sequence \( w\omega\cdots \) and by \( \omega w \) the left-infinite sequence \( \cdots w\omega \). We denote \( w^\infty = \omega w \cdot w^\omega \). A periodic sequence is of the form \( w^\infty \) for a unique primitive word \( w \). Its minimal period is then \(|w|\).

**Example 2.1** Let \( Y \) be the orbit of \( \omega a \cdot b^\omega \) and let \( X \) be the topological closure of \( Y \). Then \( X = Y \cup \{ a^\infty \} \cup \{ b^\infty \} \) and \( X^+ = \{ a^n b^\omega \mid n \geq 0 \} \cup \{ a^\omega \} \).

A shift space is periodic if it is formed of periodic sequences. It is aperiodic if it contains no periodic sequence.

We let \( \mathcal{L}(X) \) denote the language of a shift space \( X \), which is the set of all factors of the sequences \( x \in X \). We let \( \mathcal{L}_n(X) \) (resp. \( \mathcal{L}_{\geq n}(X) \)) denote the set of words of length \( n \) (resp. \( \geq n \)) in \( \mathcal{L}(X) \). We also let \( \hat{\mathcal{L}}(x) \) denote the set of factors of a sequence \( x \). Thus \( \mathcal{L}(X) = \bigcup_{x \in X} \hat{\mathcal{L}}(x) \).

A language \( L \) is factorial if it contains the factors of its elements. A word \( u \in L \) is extendable in \( L \) if there are letters \( a, b \) such that \( au \in L \). A language \( L \) is extendable if every word in \( L \) is extendable. For every shift space \( X \), the language \( \mathcal{L}(X) \) is factorial and it is extendable. Conversely, for every factorial extendable language \( L \), there is a unique shift space \( X \) such that \( \mathcal{L}(X) = L \), which is the set of sequences \( x \) such that \( \mathcal{L}(x) \subset L \). In particular, \( \mathcal{L}(X) \) determines \( X \).

A word \( w \in \mathcal{L}(X) \) is called right-special if there is more than one letter \( a \in A \) such that \( wa \in \mathcal{L}(X) \). Symmetrically, \( w \in \mathcal{L}(X) \) is left-special if there is more than one letter \( a \in A \) such that \( aw \in \mathcal{L}(X) \).

A nonempty shift space \( X \) is irreducible if for every \( u, v \in \mathcal{L}(X) \) there is some \( w \) such that \( uvw \in \mathcal{L}(X) \).

The following is classical (see [18 Proposition 2.1.3]).

---

1We use \( \subset \) everywhere instead of \( \subseteq \).
Proposition 2.2 A shift space $X$ is irreducible if and only if there is some $x \in X$ with a dense positive orbit.

A nonempty shift space is minimal if it does not contain properly a nonempty shift space. A shift space is minimal if and only if the orbit of every point is dense.

A minimal shift is either aperiodic or periodic (and in this case formed of one periodic orbit). It is aperiodic if and only if the set of right-special (resp. left-special) words is infinite.

A nonempty shift space is uniformly recurrent if for every $w \in \mathcal{L}(X)$, there is an $n \geq 1$ such that every word in $\mathcal{L}_n(X)$ contains $w$. The following result is classical (see [33 Proposition 5.2]).

Proposition 2.3 A shift space is minimal if and only if it is uniformly recurrent.

Every minimal shift $X$ is irreducible. Indeed, if $X$ is minimal, let $u, v \in \mathcal{L}(X)$. There is an $n \geq 1$ such that every word in $\mathcal{L}_n(X)$ contains $u$ and $v$. Then every word in $\mathcal{L}_{2n}(X)$ contains a word of the form $uuv$. The converse is not true since, for example, the full shift is irreducible but not minimal.

Given a shift space $X$ and $w \in \mathcal{L}(X)$, a return word to $w$ is a nonempty word $u$ such that $wu$ is in $\mathcal{L}(X)$ and has exactly two occurrences of $w$, one as a prefix and the other one as a suffix. An irreducible shift space is minimal if and only if the set of return words to $w$ is finite for every $w \in \mathcal{L}(X)$. Indeed, if this condition is satisfied, the shift $X$ is clearly uniformly recurrent.

Let $X$ be a shift on $A$ and let $k \geq 1$ be an integer. Consider an alphabet $A_k$ in one-to-one correspondence with $\mathcal{L}_k(X)$ via a bijection $f_k : \mathcal{L}_k(X) \rightarrow A_k$.

The map $\gamma_k : X \rightarrow A_k^\mathbb{Z}$ defined for $x \in X$ by $y = \gamma_k(x)$ if for every $n \in \mathbb{Z}$

$$y_n = f_k(x_n \cdots x_{n+k-1})$$

is the $k$-th higher block code on $X$ (see Figure 2.1). The set $X^{(k)} = \gamma_k(X)$ is a shift space on $A_k$, called the $k$-th higher block presentation of $X$ (one also uses the term of coding by overlapping blocks of length $k$).

Figure 2.1: The $k$-th higher block code.

The following result follows easily from the definitions.

Proposition 2.4 The higher block code is an isomorphism of shift spaces and the inverse of $\gamma_k$ is the map $y \mapsto x$ such that, for all $n$, $x_n$ is the first letter of the word $u$ such that $y_n = f_k(u)$. 
Example 2.5 Let $X = \{a, b\}^\mathbb{Z}$. Set $A_2 = \{r, s, t, u\}$ with $f_2: r \mapsto aa, s \mapsto ab, t \mapsto ba, u \mapsto bb$. Then $X^{(2)}$ is the set of labels of bi-infinite paths in the graph of Figure 2.2.

![Figure 2.2: The 2nd-higher block presentation of $\{a, b\}^\mathbb{Z}$.](image)

3 Morphisms

A morphism $\sigma: A^* \to B^*$ is a monoid morphism from the free monoid $A^*$ to the free monoid $B^*$. We allow a letter $a \in A$ to be erased, that is, to have $\sigma(a) = \varepsilon$. We say that $\sigma$ is non-erasing if $\sigma(a) \neq \varepsilon$ or all $a \in A$.

For $x = x_0x_1 \cdots \in A^\mathbb{N}$, set $\sigma(x) = \sigma(x_0)\sigma(x_1)\cdots$ if the right hand side is infinite. Since an infinity of the $x_i$ can possibly be erased, it is a finite or right-infinite sequence. Similarly, if $x = \cdots x_{-1}x_0 \in A^{-\mathbb{N}}$, $\sigma(x) = \cdots \sigma(x_{-1})\sigma(x_0)$ is finite or left-infinite. Next, for $x \in A^\mathbb{Z}$, $\sigma(x) = \sigma(x^-) \cdot \sigma(x^+)$ is either finite, left-infinite, right-infinite, or two-sided infinite.

Thus, a morphism $\sigma: A^* \to B^*$ extends to a map from $A^\mathbb{N}$ to $B^* \cup B^\mathbb{N}$. If $\sigma(a_n) = \varepsilon$ for $n \geq k$, then $\sigma(a_0a_1\cdots) = \sigma(a_0\cdots a_{k-1})$. Otherwise $\sigma(a_0a_1\cdots)$ is the right-infinite sequence $\sigma(a_0)\sigma(a_1)\cdots$. Similarly, it extends also to a map from $A^\mathbb{Z}$ to $B^* \cup B^{-\mathbb{N}} \cup B^\mathbb{N} \cup B^\mathbb{Z}$. For $x \in A^\mathbb{Z}$, if $\sigma(x)$ is in $B^\mathbb{Z}$ (in particular if $\sigma$ is non-erasing), one has $\sigma(x) = \sigma(x^-) \cdot \sigma(x^+)$. If $\sigma: A^* \to B^*$ is a morphism, we denote $|\sigma| = \sum_{a \in A} |\sigma(a)|$ and define the size of $\sigma$ as $|\sigma| + \text{Card}(A)$.

Let $\sigma: A^* \to B^*$ be a morphism. A $\sigma$-representation of $y \in B^\mathbb{Z}$ is a pair $(x, k)$ of a sequence $x \in A^\mathbb{Z}$ and an integer $k$ such that

$$y = S^k(\sigma(x)).$$ (3.1)

The $\sigma$-representation $(x, k)$ is centered if $0 \leq k < |\sigma(x_0)|$.

Note that, in particular, a centered $\sigma$-representation $(x, k)$ is such that $\sigma(x_0) \neq \varepsilon$.

The notion of centered representation is a normalization. Indeed, if $y = S^k(\sigma(x))$, there is a unique centered $\sigma$-representation $(x', k')$ of $y$ such that $x'$ is a shift of $x$, namely $(S^n x, k)$ with $k = |\sigma(x_0 \cdots x_{n-1})| + k'$.

Endomorphisms We now define notions specific to an endomorphism of $A^*$, that is, a morphism $\sigma$ from $A^*$ into itself. In this case, one may iterate $\sigma$.

Given an endomorphism $\sigma: A^* \to A^*$, we let $\mathcal{L}(\sigma)$ denote the language of $\sigma$. It is formed of all factors of the words $\sigma^n(a)$ for $n \geq 0$ and $a \in A$. We let $\mathcal{L}_n(\sigma)$ (resp. $\mathcal{L}_{\geq n}(\sigma)$) denote the set of words of length $n$ (resp. $\geq n$) in $\mathcal{L}(\sigma)$.
The shift $X(\sigma)$ associated to $\sigma$ is the set $X(\sigma)$ of two-sided infinite sequences $x \in A^\mathbb{Z}$ with all their factors in $L(\sigma)$. Such a shift space is called a substitution shift.

**Example 3.1** The morphism $\sigma: a \mapsto ab, b \mapsto a$ is called the Fibonacci morphism. The associated shift $X(\sigma)$ is called the Fibonacci shift.

We can also associate to $\sigma$ the set $X^+(\sigma)$ of right-infinite sequences having all their factors in $L(\sigma)$.

**Example 3.2** Let $\sigma: a \mapsto ab, b \mapsto b$. Then $X^+(\sigma) = ab^\omega \cup b^\omega$ but, since $X(\sigma) = b^\infty$, we have $X(\sigma)^+ = b^\omega$.

We have the inclusion $L(X(\sigma)) \subseteq L(\sigma)$ but the converse is not always true. This (unpleasant) phenomenon occurs, for example, when $\sigma$ is the identity, since then $L(\sigma) = A$ but $X(\sigma)$ (and thus $L(X(\sigma))$) is empty.

A letter $a \in A$ is erasable if there is some $n \geq 1$ such that $\sigma^n(a) = \varepsilon$ (the term mortal is also used, see [39] or [1]). A word is erasable if all its letters are erasable.

The mortality exponent of an erasable word $w$, denoted $\text{mex}(w)$, is the least integer $n$ such that $\sigma^n(w) = \varepsilon$. Note that $\text{mex}(w) \leq \text{Card}(A)$, that is $\sigma^{\text{Card}(A)}(w) = \varepsilon$. The mortality exponent of $\sigma$, denoted $\text{mex}(\sigma)$, is the maximal value of the mortality exponents of erasable letters.

A morphism is non-erasing if no letter is erasable.

A word $w \in A^*$ is growing if the sequence $|\sigma^n(w)|$ is unbounded. A word is growing if some of its letters is growing. The morphism $\sigma$ itself is said to be growing if all letters are growing.

An endomorphism $\sigma: A^* \rightarrow A^*$ is primitive if there is an $n \geq 1$ such that for every $a, b \in A$, the letter $b$ appears in $\sigma^n(a)$.

For a primitive morphism $\sigma$, except the trivial case $A = \{a\}$ and $\sigma(a) = a$, every letter is growing and $L(\sigma) = L(X(\sigma))$ (see [18] for example).

An endomorphism $\sigma: A^* \rightarrow A^*$ is minimal if $X(\sigma)$ is a minimal shift space. The following statement is well known.

**Proposition 3.3** Every primitive endomorphism not reduced to the identity on a one-letter alphabet is minimal.

For a proof, see [39, Proposition 5.5] or [18, Proposition 2.4.16].

An endomorphism $\sigma$ is aperiodic (resp. periodic) if the shift $X(\sigma)$ is aperiodic (resp. periodic).

**Example 3.4** The Fibonacci morphism $\sigma: a \mapsto ab, b \mapsto a$ is primitive. It is also aperiodic. Indeed, it is easy to verify by induction on $n$ that $\sigma^n(a)$ is left-special and thus that the number of left-special words is infinite. Since $X(\sigma)$ is minimal, this implies that it is aperiodic.
Example 3.5 The Thue-Morse morphism \( \sigma : a \mapsto ab, b \mapsto ba \) is also primitive and aperiodic (see [13] for example). The shift \( X(\sigma) \) is called the Thue-Morse shift. It is well known that the language \( \mathcal{L}(\sigma) \) does not contain cubes, that is words of the form \( www \), with \( w \) nonempty.

Example 3.6 The morphism \( \sigma : a \mapsto a, b \mapsto bab \) is not primitive. It is periodic and minimal because \( \sigma(ab) = abab \) and thus \( X(\sigma) = \{(ab)^\infty, (ba)^\infty\} \).

We now present a classical construction (see [33]) which, for every \( k \geq 1 \) allows one to replace a non-erasing morphism \( \sigma \) by a morphism \( \sigma_k \) acting on \( k \)-blocks.

Let \( \sigma : A^* \to A^* \) be a non-erasing endomorphism. For every integer \( k \geq 1 \), let \( u \in \mathcal{L}_k(\sigma) \mapsto \langle u \rangle \in A_k \) be a bijection from \( \mathcal{L}_k(\sigma) \) onto an alphabet \( A_k \).

We define an endomorphism \( \sigma_k : A_k^* \to A_k^* \) as follows. Let \( u \in \mathcal{L}_k(\sigma) \) and let \( a \) be the first letter of \( u \). Set \( s = |\sigma(a)| \). To compute \( \sigma_k(\langle u \rangle) \), we first compute the word \( \sigma(u) = b_1b_2\cdots b_k \). Note that, since \( \sigma \) is non-erasing, we have \( \ell \geq |\sigma(a)| + k - 1 = s + k - 1 \). We define

\[
\sigma_k(\langle u \rangle) = \langle b_1b_2\cdots b_k \rangle \langle b_2b_3\cdots b_{k+1} \rangle \cdots \langle b_{s} \cdots b_{s+k-1} \rangle. \tag{3.2}
\]

The morphism \( \sigma_k \) is called the \( k \)-th higher block presentation of \( \sigma \).

Let \( \pi_k : A_k^* \to A^* \) be the morphism defined by \( \pi_k(\langle u \rangle) = a \) where \( a \) is the first letter of \( u \). Then we have for each \( n \geq 1 \) the following commutative diagram which expresses the fact that \( \sigma_k^n \) is the counterpart of \( \sigma^n \) for \( k \)-blocks.

\[
\begin{array}{ccc}
A_k^* & \xrightarrow{\sigma_k^n} & A_k^* \\
\downarrow \pi_k & & \downarrow \pi_k \\
A^* & \xrightarrow{\sigma^n} & A^*
\end{array}
\tag{3.3}
\]

Indeed, for every \( \langle u \rangle \in A_k \), let \( a = \pi_k(\langle u \rangle) \) and let \( s = |\sigma(a)| \). Set \( \sigma(u) = b_1b_2\cdots b_k \). We have by definition of \( \sigma_k \),

\[
\pi_k \circ \sigma_k(\langle u \rangle) = b_1b_2\cdots b_s = \sigma(a) = \sigma \circ \pi_k(\langle u \rangle).
\]

Since \( \pi_k \circ \sigma_k \) and \( \sigma \circ \pi_k \) are morphisms, this proves that the diagram (3.3) is commutative for \( n = 1 \) and thus for all \( n \geq 1 \).

Proposition 3.7 Let \( \sigma \) be a non-erasing endomorphism and \( k \) a positive integer. Then \( X(\sigma_k) = X(\sigma)^{(k)} \).

Proof. Since the diagram (3.3) is commutative, \( \pi_k \) is a conjugacy from \( X(\sigma_k) \) onto \( X(\sigma) \). On the other hand, by Proposition 2.4, it is a conjugacy from \( X(\sigma)^{(k)} \) onto \( X(\sigma) \). \( \blacksquare \)

Example 3.8 Let \( \sigma : a \mapsto ab, b \mapsto a \) be the Fibonacci morphism. We have \( \mathcal{L}_2(X) = \{aa, ab, ba\} \). Set \( A_2 = \{x, y, z\} \) and let \( \langle aa \rangle = x, \langle ab \rangle = y, \langle ba \rangle = z \). We have \( \sigma_2(x) = \sigma_2(\langle aa \rangle) = \langle ab \rangle \langle ba \rangle = yz \). Similarly, we have \( \sigma_2(y) = yz \) and \( \sigma_2(z) = x \).
**Ordered trees** An ordered tree is a graph $T$ in which the vertices are labeled. The end $t$ of an edge starting in a vertex $s$ is called a child of $s$ and $s$ is the parent of $t$. Each vertex has a label and the set of children of each vertex is totally ordered. There is a particular vertex called the root of $T$ and there is a unique path from the root to each vertex. The level of a vertex $s$ is the length of the path from the root to $s$. A leaf of a tree is a vertex without children.

Let $\sigma : A^* \rightarrow A^*$ be a morphism. For each $a \in A$ and $n \geq 0$, we define an ordered tree $T_\sigma(a,n)$ labeled by $A$, called the derivation tree of $a$ at order $n$. The root of $T_\sigma(a,n)$ is labeled by $a$ and has no child if $n = 0$. If $n \geq 1$, set $\sigma(a) = b_1b_2\cdots b_k$ with $b_i \in A$. The root of $T_\sigma(a,n)$ has $k$ children $r_1, r_2, \ldots, r_k$ which are the roots of trees $T_i$ with $1 \leq i \leq k$ and each $T_i$ is isomorphic to $T(b_i, n-1)$. The order on the children is $r_1 < r_2 < \ldots < r_k$.

**Example 3.9** Let $\sigma : a \mapsto ab, b \mapsto a$ be the Fibonacci morphism. The tree $T_\sigma(a,2)$ is represented in Figure 3.1.

![Figure 3.1: The tree $T_\sigma(a,2)$.](image)

**Some useful lemmas** The following three lemmas deal with questions that concern non-growing or erasable letters.

We associate to an endomorphism $\sigma : A^* \rightarrow A^*$ the multigraph $G(\sigma)$ on $A$ with $|\sigma(a)|_b$ edges from $a$ to $b$. We let $M(\sigma)$ denote the adjacency matrix of $G(\sigma)$, that is, the matrix $M$ such that $M_{a,b} = |\sigma(a)|_b$.

It is decidable in linear time whether a letter is erasable or growing. To see this, we build the multigraph $G(\sigma)$. A strongly connected component of $G(\sigma)$ is trivial if it has one vertex and no edges.

A letter $a$ is erasable if all paths going out of $a$ end only in trivial strongly connected components of $G(\sigma)$.

A letter $a$ is growing if and only if either there is a path from $a$ in $G(\sigma)$ to a non-trivial strongly connected component that is not reduced to a cycle or there is a path from $a$ to a non-trivial strongly connected component from which there is also a path to another non-trivial strongly connected component.

**Example 3.10** Let $\sigma : a \mapsto ab, b \mapsto a$ be the Fibonacci morphism. The graph $G(\sigma)$ is shown in Figure 3.2.

The matrix $M(\sigma)$ is

$$M(\sigma) = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$$
Figure 3.2: The graph $G(\sigma)$.
Since it is strongly connected and not reduced to a cycle, $\sigma$ is growing.

Lemma 3.11 Let $\sigma : A^* \to A^*$ be an endomorphism. If $u \in A^*$ is growing, then $\lim_{n \to +\infty} |\sigma^n(u)| = +\infty$.

Proof. We may assume that $u = a$ is a growing letter. We argue by induction on $\text{Card}(A)$. The statement is true if $\text{Card}(A) = 1$. If the strongly connected component in $G(\sigma)$ of $a$ contains strictly a cycle, then $\lim_{n \to +\infty} |\sigma^n(a)| = +\infty$.
Next, if the strongly connected component $C(a)$ of $a$ is reduced to a cycle, we have $\sigma^i(a) = uvaw$ for some $i \geq 1$ and $v, w$ containing no letter of $C(a)$. Either $v$ or $w$, say $v$, has to be growing. Thus the result holds by induction hypothesis applied to $v$.
Finally, if the strongly connected component of $a$ is trivial, the result holds by induction hypothesis.

The following result is proved in [3, Lemma 2.2].

Proposition 3.12 For every endomorphism $\sigma$ there is a finite number of words formed of erasable letters in $L(\sigma)$.

Lemma 3.13 Let $\sigma : A^* \to A^*$ be a morphism. For every $x$ in $X(\sigma)$, the sequence $\sigma(x)$ is in $X(\sigma)$. The map $x \mapsto \sigma(x)$ is continuous on $X(\sigma)$.

Proof. We have to prove that $\sigma(x)$ is two-sided infinite. By Proposition 3.12, $x$ has an infinite number of non-erasable letters on the left and on the right and $\sigma(x)$ is two-sided infinite. By Proposition 3.12 again, there is an integer $k \geq 1$ such that every word in $L_k(\sigma)$ contains a non-erasable letter. This implies that $\sigma(L_{kn}(\sigma))$ is contained in $L_{\geq n}(\sigma)$, whence the second statement.

Note that, since $X(\sigma)$ is compact, the map $\sigma$ is actually uniformly continuous on $X(\sigma)$.

We prove the following concerning $\sigma$-representations.

Proposition 3.14 Let $\sigma : A^* \to A^*$ be a morphism. Every point $y$ in $X(\sigma)$ has a $\sigma$-representation $y = S^k(\sigma(x))$ with $x$ in $X(\sigma)$.

Proof. Let $k = |\sigma|$ and let $y$ be in $X(\sigma)$. For every $n \geq 2k$, there is an integer $m \geq 1$ such that $y_{[-n,n]}$ is a factor of $\sigma^m(a)$ for some letter $a \in A$.

By a compactness argument, there is an integer $0 \leq i < k$ such that for every $n > 2k$, there are words $u_n, v_n$ with $u_nv_n \in L(\sigma)$ such that $y_{[-n+k,-i]}$ is
The language $L(\sigma)$ A language $L \subset A^*$ is recognizable if it can be recognized by a finite automaton. The following result appears in [35, Lemma 3] (see also [3]). We reproduce the proof for the sake of completeness.

Lemma 3.15 Let $\sigma : A^* \to A^*$ be a morphism. It is decidable, for a given recognizable language $L$ whether $L \cap L(\sigma)$ is empty. In particular, $L(\sigma)$ is decidable. If $L$ is moreover factorial, it is decidable whether $L \cap L(\sigma)$ is finite.

Proof. Since $L$ is recognizable, so is $A^*LA^*$. Let $A = (Q, I, T)$ be a finite automaton recognizing $A^*LA^*$. For $w \in A^*$, denote by $\varphi(w)$ the binary relation on $Q$ defined by

$$\varphi(w) = \{(p, q) \in Q \times Q \mid p \xrightarrow{w} q\}.$$ 

The map $w \mapsto \varphi(w)$ is a morphism from $A^*$ into the finite monoid of binary relations on $Q$. Let $\psi_n = \varphi \circ \sigma^n$. Since $\sigma$ and $\varphi$ are morphisms, so is each $\psi_n$. Since $\varphi(A^*)$ is a finite set, there is a finite number of morphisms $\psi_n$ and thus there are $n < m$ such that $\psi_n = \psi_m$. Note that $\psi_n = \psi_m$ implies $\psi_{n+r} = \psi_{m+r}$ because $\psi_{n+r} = \psi_n \circ \sigma^r = \psi_m \circ \sigma^r = \psi_{m+r}$. Then $L \cap L(\sigma) \neq \emptyset$ if and only if there is some $k \leq m$ such that $(i, t) \in \psi_k(A)$ for some $i \in I$ and $t \in T$.

Assume now that $L$ is factorial. We consider this time an automaton recognizing $L$, in which we may assume that $I = T = Q$. We define $\varphi$ and $\psi_n$ as above. Then $L \cap L(\sigma)$ is infinite if and only if there is a growing letter $a$ such that $\psi_k(a) \neq \emptyset$ for every $k < m$.

Finally, one can decide if $w$ is in $L(\sigma)$ since it is equivalent to $\{w\} \cap L(\sigma) \neq \emptyset$. 

The language $L(X(\sigma))$ The following results deal with substitution shifts that fail to satisfy the condition $L(X(\sigma)) = L(\sigma)$. We have already seen that the equality holds for primitive morphisms with $\text{Card}(A) \geq 2$. More generally, one has $L(X(\sigma)) = L(\sigma)$ if and only if $L(\sigma)$ is extendable, which for a non-erasing morphism, occurs if and only if every letter is extendable.

In the general case, let us first note the following property.

Proposition 3.16 Let $\sigma : A^* \to A^*$ be a morphism. One has $L(\sigma) = L(X(\sigma))$ if and only if every letter $a \in A$ is in $L(X(\sigma))$.

Proof. Assume that the condition is satisfied. Every $w \in L(\sigma)$ is a factor of some $\sigma^n(a)$ for $a \in A$ and $n \geq 0$. Let $x \in X(\sigma)$ be such that $a \in L(x)$. By Lemma 3.13 we have $\sigma^n(x) \in X(\sigma)$. Since $w \in L(\sigma^n(x))$, we obtain $w \in L(X(\sigma))$. The converse is obvious.
The following lemma shows that, for a morphism $\sigma : A^* \to A^*$, it is decidable whether a word is a factor of some $\sigma^n(a)$ for $a \in A$ for an infinite number of $n$. Its proof uses the technique of the proof of Lemma 3.15.

**Lemma 3.17** Let $\sigma : A^* \to A^*$ be a morphism and let $u \in A^*$. It is decidable whether there an infinite number of integers $n \geq 0$ such $u$ is a factor of some $\sigma^n(a)$ with $a \in A$.

**Proof.** Let $u \in A^*$ be a word. Let $A = (Q, I, T)$ be a finite automaton recognizing $A^*uA^*$. For $w \in A^*$, denote by $\varphi(w)$ the binary relation on $Q$ defined by

$$\varphi(w) = \{(p, q) \in Q \times Q \mid p \xrightarrow{w} q\}.$$ 

The map $w \mapsto \varphi(w)$ is a morphism from $A^*$ into the finite monoid of binary relations on $Q$. Let $\psi_n = \varphi \circ \sigma^n$.

Thus $\psi_n(w) = \{(p, q) \in Q \times Q \mid p \xrightarrow{\sigma^n(w)} q\}$.

There are integers $m < m'$ such that $\psi_{m+r} = \psi_{m'+r}$ for any nonnegative integer $r$ (see the proof of Lemma 3.15).

Then $u$ is factor of an infinite number of some $\sigma^n(a)$ with $a \in A$ if and only is there is some $m \leq k \leq m'$ such that $(i, t) \in \psi_k(A)$ for some $i \in I$ and $t \in T$.

**Lemma 3.18** Let $\sigma : A^* \to A^*$ be a morphism. There is a constant $K$ such that a word $u$ belongs to $\mathcal{L}(X(\sigma))$ if and only if there are words $w, z$ of length $K$ such that $wуз$ is a factor of words $\sigma^n(a)$ with $a \in A$ for an infinite number of $n \geq 0$.

**Proof.** Define

$$N = \max\{|\sigma^n(a)| \mid a \text{ non-growing}\}, \text{ and } M = \max\{|\sigma^n(a)| \mid a \text{ erasing}\}.$$

Let $r = \max\{|\sigma^i| \mid 0 \leq i \leq \text{Card}(A)^2\}$. We set $K = (N + M)r$.

Let $u$ be a word for which there are words $w, z$ of length $K$ such that $уз$ is factor of words $\sigma^n(a)$ with $a \in A$ for an infinite number of $n \geq 0$. We set $\sigma^n(a) = w(n)уз(z(n))$ with $w(n)$ as short as possible.

Let $e_n$ (resp. $f_n$) be the first (resp. last) growing letter of $\sigma^n(a)$. For $n$ large enough, there are integers $0 \leq k \leq k' \leq \text{Card}(A)^2$ such that $e_k = e_{k'} = e$ and $f_k = f_{k'} = f$. This implies that $e_{k+i} = e_{k'+i}$ and $f_{k+i} = f_{k'+i}$ for any integer $i$.

Let $\sigma^k(a) = vesft$, $\sigma^{k'-k}(e) = v'e'v''$ and $\sigma^{k'-k}(f) = t'f't''$ with $v, t, v', t''$ non-growing (see Figure 3.13) or, in the case where $e, f$ coincide, $\sigma^k(a) = vet$, $\sigma^{k'-k}(e) = v'e'v''$ with $v, t, v', t''$ non-growing.

Since $v, t$ are non-growing, we have $|\sigma^{n-k}(v)| \leq Nr$ and $|\sigma^{n-k}(t)| \leq Nr$. We claim that there is an infinite sequence $(i_n)$ such that the lengths of $w(i_n)$ and $z(i_n)$ tend to infinity. We first prove it for $w(n)$.

Case 1. If $\sigma^{n-k}(v)w$ is a prefix of length at most $|w(n)uw| of w(n)уз(z(n))$ for an infinite number of $n$, then, since $e$ is growing, $\sigma^{n-k}(esft)$ (or $\sigma^{n-k}(et)$
in the case \( e = f \) \( s \) equal to \( w'(n)uz'(n) \) with \( w'(n) \) suffix of \( w(n) \), \( z'(n) \) prefix of \( zz(n) \) and \( |w'(n)| \) going to the infinity.

Case 2. Let us assume that \( \sigma^{n-k}(ve) \) is a prefix \( w(n)wu(n) \) of \( w(n)wu(n)z(n) \) for an infinite number of \( n \).

Case 2(i). If \( v' \) is non-erasable, then again \( \sigma^{n-k}(esft) \) (or \( \sigma^{n-k}(et) \)) is equal to \( w'(n)uz'(n) \) with \( w'(n) \) suffix of \( w(n) \), \( z'(n) \) prefix of \( zz(n) \) and \( |w'(n)| \) going to the infinity.

Case 2(ii). Let us assume that \( v' \) and \( v'' \) are erasable. Since \( |\sigma^{n-k}(v')| \leq Mr \) and \( |\sigma^{n-k}(v'')| \leq Mr \), this case is impossible.

Case 2(iii). If \( v' \) erasable and \( v'' \) is non-erasable, since \( |\sigma^{n-k}(v')| \leq Mr \), \( \sigma^{n-k}(e) = w'(n)wu(n)z'(n) \) with \( w'(n) \) suffix of \( w(n) \), \( w'(n) \) prefix of \( zz(n) \) and \( |w'(n)| \) going to the infinity.

This proves that there is an infinite sequence of integers \( (i_n) \) such that the size of \( w(i_n) \) tends to infinity.

Similarly there is a infinite subsequence \( (j_n) \) of \( (i_n) \) such that the size of \( z(j_n) \) tends to infinity. This proves the claim. It implies that \( u \) belongs to \( \mathcal{L}(X(\sigma)) \).

**Proposition 3.19** Let \( \sigma: A^* \rightarrow A^* \) be a morphism. The language \( \mathcal{L}(X(\sigma)) \) is decidable.

**Proof.** Let \( K = (N + M)r \) be defined as in the proof of Lemma 3.18. By Lemma 3.18, a word \( u \) belongs to \( \mathcal{L}(X(\sigma)) \) if and only if there are words \( w, z \) of length \( K \) such that there is an infinite number of \( n \) for which \( wz \) is factor of some \( \sigma^n(a) \) with \( a \) in \( A \). We thus may check whether there is a word \( wz \) of length \( 2K + |u| \) having this property by Lemma 3.17.

Proposition 3.19 implies that it is decidable whether a letter belongs to \( \mathcal{L}(X(\sigma)) \). The following proposition shows that this can be decided in linear time.
Proposition 3.20 Let \( \sigma \) be a morphism. It is decidable in linear time in the size of \( \sigma \) whether a letter belongs to \( \mathcal{L}(X(\sigma)) \).

Proof. We use the graph \( G(\sigma) \). We will define letters of type 1, 2, 3 as follows.

The letters of type 1 are the letters accessible in \( G(\sigma) \) from non-trivial strongly connected components not reduced to a cycle. Letters of type 1 belong to \( \mathcal{L}(X(\sigma)) \). Indeed, if \( a \) belongs to a non-trivial strongly connected component of \( G(\sigma) \) which is not reduced to a cycle, there is an integer \( k \) such that \( \sigma^k(a) \) contains at least two occurrences of \( a \). It follows that \( X(\sigma) \) contains a sequence with an infinite number of \( a \) on the right and on the left. Letters accessible from \( a \) have the same property.

The letters of type 2 can be of type 2' or 2''. The letters of type 2' are the letters accessible from a non-trivial strongly connected component \( C \) reduced to a cycle such that there are a letter \( a \) in a trivial component, a letter \( b \) in \( C \), integers \( k, p \) such that \( \sigma^k(a) = ubv, \sigma^p(b) = wbz \) with \( u, v, w, z \) satisfying the two following conditions.

(i) either \( u \) is growing or \( w \) is non-erasable

(ii) either \( v \) is growing or \( z \) is non-erasable.

The letters of type 2'' are the letters accessible from a non-trivial strongly connected component \( C \) reduced to a cycle such that there is a letter \( b \) in \( C \), an integer \( p \) such that \( \sigma^p(b) = wbz \) and \( w \) and \( z \) non-erasable. The letters of type 2 belong to \( \mathcal{L}(X(\sigma)) \). Indeed, if the conditions for the type 2' are satisfied, \( \sigma^{k+n_p}(a) = \sigma^n(u)\sigma^p(w)\cdots wbv\cdots \sigma^n(v)\sigma^p(z) \), making \( b \) a letter of \( \mathcal{L}(X(\sigma)) \).

Next, a letter of type 2'' is accessible from a letter \( b \) which is in \( \mathcal{L}(X(\sigma)) \) and it is therefore in \( \mathcal{L}(X(\sigma)) \).

The letters of type 3 are the letters accessible from a non-trivial strongly connected component \( C \) itself accessible from another strongly component \( C \) which is a cycle. The letters of type 3 belong to \( \mathcal{L}(X(\sigma)) \). Indeed, there is a letter \( a \) in \( C \), a letter \( b \) in \( C' \) and an integer \( p \) such that \( \sigma^p(a) = uvabv \) and \( \sigma^p(b) = wbz \), where \( b \) is a letter of \( u \) or of \( v \). Let us assume that \( b \) is a letter of \( v \). Then \( \sigma^p(a) = \sigma^{(n-1)p}(u)\cdots uvb\sigma^p(v)\cdots \sigma^{(n-1)p}(v) \). Each \( \sigma^p(v) \) contains the letter \( b \). Thus there is an infinite number of words \( xby \) with \( |x| = |y| = n \) in \( \mathcal{L}(\sigma) \) and \( b \) belongs to some point in \( X(\sigma) \). Note that letters may have several types.

We now show that if a letter is neither a letter of type 1 to 3, then it is not in \( \mathcal{L}(X(\sigma)) \). Let \( c \) be a letter which is neither of type 1 to 3. Assume that \( c \in \mathcal{L}(X(\sigma)) \). Let \( x \) be a point of \( X(\sigma) \) containing the letter \( c \). We may assume that \( x_0 = c \).

Since \( x \in X(\sigma) \), for each integer \( n \), the word \( x_{[-n,n]} \) is a factor of some \( \sigma^{k_n}(a) \) for some fixed letter \( a \). Let \( T_\sigma(a, k_n) \) be the derivation tree at order \( k_n \) rooted at \( a \). For \( n \) large enough, there are integers \( 0 \leq m < m' \leq \text{Card}(A) \) such that the unique path \( \pi \) going from \( a \) to \( x_0 \) in \( T_\sigma(a, k_n) \) goes through a vertex labeled by \( b \) at the level \( m \) and at the level \( m' \). Thus \( \sigma^m(a) = ubv \) and \( \sigma^p(b) = wbz \) with \( p = m' - m \). Since \( c \) is not of type 1 the letter \( b \) belongs to a non-trivial component \( C \) reduced to a cycle.
If $a$ is in a trivial component, since $c$ is not of type 2', we have either $u$ non-growing and $w$ erasable, or $v$ non-growing and $z$ erasable.

Assume that $u$ is non-growing and $w$ is erasable, the other case being symmetrical. There is a constant $K$ such that the lengths of all $\sigma^n(u)$ are bounded by $K$ and a constant $M$ such that $\sigma^M(w) = \varepsilon$.

Thus for a large enough $n$, there is an integer $r_n, i_n$ with $0 \leq i_n < p$ such that $\chi_{[-n,0]}$ is a factor of $\sigma^n(u) \sigma^i \sigma^M(w) \ldots \sigma^p(w) \sigma^p(w)w$, a contradiction.

If $a$ is in a non-trivial component, this component is $C$, since $c$ is not of type 3, and we may assume that $a = b$. Since $c$ is not of type 2', we have either $w$ or $z$ erasable. Assume that $w$ is erasable. As above, $x_{[-n,0]}$ is a factor of $\sigma^n(u) \sigma^i \sigma^M(w) \ldots \sigma^p(w) \sigma^p(w)w$ for some integers $r_n, i_n$ with $0 \leq i_n < p$, a contradiction.

We now investigate the complexity of testing whether a letter is in $L(X(\sigma))$. The graph $G(\sigma)$ can be built in linear time in the size of $\sigma$. Letters of type 1, 2, or 3 can be computed in linear time in the size of $\sigma$. Indeed, it is clear for types 1 and 3.

For type 2, one can compute for each component $C$ reduced to a cycle whether there is an integer $p$ such that $\sigma^p(a) = uav$ with $a \in C$, and $u$ erasable (resp. $v$ erasable). This can be done in linear time in the size of $\sigma$ for all components. Thus the computation of letters of type $2''$ can be done in linear time.

The growing letters are computed in linear time in the size of $G(\sigma)$. Now finding letters $a$ in trivial components with a path from $a$ to a component reduced to a cycle containing a letter $b$ such that $\sigma^k(a) = ubv$ with $u$ growing (resp. $v$ growing) can be done in linear time. Thus the computation of letters of type $2'$ can be done in linear time.

**Example 3.21** Let $\sigma : a \mapsto ab, b \mapsto bc, c \mapsto cc$. The graph $G(\sigma)$ is represented in Figure 3.4 on the left. Then $c$ is a letter of type 1 and 3 (since it is accessible from $b$) and $b$ is a letter of type 3. The letter $a$ does not belong to $L(X(\sigma))$.

**Example 3.22** Let $\sigma : a \mapsto bac, b \mapsto b, c \mapsto c$. The graph $G(\sigma)$ is represented in Figure 3.4 on the right. Then $a, b, c$ are letters of type $2''$. The point $\cdots bbb \cdots$ belongs to $X(\sigma)$. 

16
Example 3.23 Let $\sigma: e \mapsto cbd, c \mapsto cc, d \mapsto dd, b \mapsto ba, a \mapsto \varepsilon$. The graph $G(\sigma)$ is represented in Figure 3.5 on the right. Then $a, b$ are letters of type $2'$ and $c, d$ are of type $1$. The point $\cdots cccc : baddddd \cdots$ belongs to $X(\sigma)$.

Note that it is not possible to characterize the letters of $\mathcal{L}(X(\sigma))$ on the multigraph $G(\sigma)$. Indeed if $\sigma$ is the morphism of Example 3.23 and $\tau: e \mapsto bcd, c \mapsto cc, d \mapsto dd, b \mapsto ba, a \mapsto \varepsilon$, then $\sigma$ and $\tau$ have the same multigraph. But $b \in \mathcal{L}(X(\sigma))$ while $b \notin \mathcal{L}(X(\tau))$.

It follows from Proposition 3.20 and Proposition 3.16 that the property $\mathcal{L}(\sigma) = \mathcal{L}(X(\sigma))$ is decidable in linear time.

4 Growing letters

Morphisms $\sigma$ having points in $X(\sigma)$ with only non-growing letters are called wild in [27]. The other ones are called tame. A characterization of tame morphisms is given in [27, Theorem 2.9].

A finite fixed point of a morphism $\sigma: A^* \rightarrow A^*$ is a word $w \in A^*$ such that $\sigma(w) = w$ (we shall come back to fixed points in the next section).

Proposition 4.1 Let $\sigma: A^* \rightarrow A^*$ be a morphism. If, for some $a \in A$, one has $\sigma(a) = uav$ with $u, v$ erasable, then $w = \sigma^{\text{mex}(\sigma)}(a)$ is a finite fixed point of $\sigma$.

Proof. Since $u, v$ are erasable, we have $\sigma^{\text{mex}(\sigma)}(u) = \sigma^{\text{mex}(\sigma)}(v) = \varepsilon$. Since

$$w = \sigma^{\text{mex}(\sigma)-1}(u) \cdots \sigma(u)uav\sigma(v) \cdots \sigma^{\text{mex}(\sigma)-1}(v)$$

we have $\sigma(w) = w$.

Proposition 4.2 Let $\sigma: A^* \rightarrow A^*$ be a morphism. There are computable integers $i \geq 0$ and $p \geq 1$ depending only on $\sigma$ such that $\sigma^i(a) = \sigma^{i+p}(a)$ for every non-growing letter $a \in A$.

Proof. Let $a \in A$ be a non-growing and non-erasable letter. Let $i(a)$ be the maximal length of a path from $a$ to a cycle in $G(\sigma)$. Let $p(a)$ be the least common
multiple of the lengths of cycles accessible from \( a \). Then \( \sigma^{i(a)+p(a)} = u\sigma^{i(a)}(a)v \) with \( u, v \) erasable.

This shows that for every non-growing and non-erasable letter \( a_i \), by definition of \( i(a) \), one has \( \sigma^{i(a)}(a_i) = u_0a_0u_1\cdots a_ku_k \) with the component of \( a_i \) on a cycle and \( u \) erasable. Each \( a_i \) is such that \( \sigma^{p(a)}(a_i) = v_ia_iw_i \) with \( v_i, w_i \) erasable. By Proposition 4.1 applied to \( \sigma^{p(a)} \), this implies that \( \sigma^{p(a)\text{mex}(\sigma)(a_i)} \) is a finite fixed point of \( \sigma^{p(a)} \). Then

\[
\sigma^{i(a)+p(a)+p(a)\text{mex}(\sigma)}(a) = \sigma^{p(a)+p(a)\text{mex}(\sigma)}(u_0a_1u_1\cdots a_ku_k)
= \sigma^{p(a)\text{mex}(\sigma)}(u_0a_1u_1\cdots a_ku_k)
= \sigma^{i(a)+p(a)\text{mex}(\sigma)}(a)
\]

The statement follows, choosing

\[
i = \max\{i(a) + p(a)\text{mex}(\sigma) \mid a \text{ non-growing and non-erasable}\} + \text{mex}(\sigma)
\]

and \( p \) equal to the least common multiple of the \( p(a) \) for a non-growing.

The following statement describes a special type of fixed points of morphisms that arises only when there are non-growing letters. It bears some similarity with the result of \cite{27} but proves additionally the existence of periodic points without growing letters.

**Proposition 4.3** Let \( \sigma : A^* \to A^* \) be a morphism. If \( x \) is a point of \( X(\sigma) \) which has only non-growing letters, then there is an integer \( k \) such that \( x = S^k(uv) \), where \( u, v, w \) are finite words of lengths bounded by a computable integer depending only on \( \sigma \). One can further choose the words \( u, v, w \) such that they are fixed points of some power of \( \sigma \). The finite set of orbits of these points is effectively computable.

**Proof.** Let \( x \in X(\sigma) \) be a point without growing letters. Let \( i, p \) be the constants of Lemma 4.1. Let \( p' \) be a multiple of \( p \) larger than \( i \). Hence for each non-growing word \( u, \sigma^{kp'}(u) = \sigma^{p'}(u) \) for every \( k \geq 1 \).

For every \( n \geq 1 \), there is some \( N \geq 1 \) and \( a \in A \) such that \( x_{[-n,n]} \) is a factor of \( \sigma^N(a) \). We may assume that \( a \) is growing.

We distinguish three cases. We choose \( n \) large enough so that \( N > p' \text{ Card}(A)^2 \) (the exponent 2 will be needed only in Case 3).

Case 1. We have, for \( N \) large enough, \( \sigma^N(a) = qbr \) with \( q \) non-growing \( b \in A \) growing and \( x_{[-n,n]} \) a factor of \( q \).

We can write \( N = N_1 + kN_2 + N_3 \) with \( N_1, N_2, N_3 \leq p' \text{ Card}(A), N_1, N_2 \) multiple of \( p' \), \( N_2 > 0 \) and

\[
\sigma^{N_1}(a) = p_1cq_1, \quad \sigma^{N_2}(c) = p_2cq_2, \quad \sigma^{N_3}(c) = p_3bq_3, \quad (4.1)
\]

with \( p_1, p_2, p_3 \) non-growing. Note that, since \( b \) is growing, \( c \) is growing and is the first growing letter of \( \sigma^{N_1}(a) \) and of \( \sigma^{N_2}(c) \). Thus it is uniquely determined.
Since \( p_1, p_2 \) are non-growing and since \( N_2 \) is multiple of \( p' \), by Lemma 4.2, we have
\[
\sigma^{kN_2}(p_1) = p''_2
\]
with \( p''_2 = \sigma^{N_2}(p_1) \) and
\[
\sigma^{kN_2}(c) = p'_{k-1}^2 p_2 c q''_2
\]
with \( p''_2 = \sigma^{N_2}(p_2) \) (see Figure 4.1).

Thus the word \( x_{[-n,n]} \) is a factor of \( p = p_3'' p_3 p_3 \) with \( p''_3 = \sigma^{N_3}(p'_3) \) and
\( p'_3 = \sigma^{N_3}(p''_3 p_3) \). This implies that \( x_{[-n,n]} \) is, up to a prefix and a suffix of bounded length, a word of period \( \sigma^{N_3}(p''_2) \). Thus \( x \) is a periodic point of the form \( x = w^{\infty} \) where \( w \) is a fixed point of a power of \( \sigma \) (actually of \( \sigma^{N_2} \)) of bounded length.

Case 2. We have, for \( N \) large enough, \( \sigma^N(a) = pbq \) with \( q \) non-growing, \( b \in A \) growing and \( x_{[-n,n]} \) a factor of \( q \). This case is symmetric to Case 1. We find that \( x_{[-n,n]} \) is a factor of \( q = q_3 q_3'' q''_3 \) with \( q''_3 = \sigma^{N_3}(q_3) \), \( q''_3 = \sigma^{N_2}(q_2) \), \( q'_3 = \sigma^{N_3}(q_2 q''_3) \) and \( q''_3 = \sigma^{N_3}(q''_3) \) (see Figure 4.2). This implies, as in Case 1, that \( x \) is periodic point of the form \( x = w^{\infty} \) where \( w \) is a fixed point of a power of \( \sigma \) (actually of \( \sigma^{N_2} \)) of bounded length.

Case 3. We have, for \( N \) large enough, \( \sigma^N(a) = pbqcr \) with \( q \) non-growing, \( b, c \in A \) growing and \( x_{[-n,n]} \) a factor of \( q \).
Figure 4.3: Case 3.

For $N$ large enough, we have $N = N_1 + kN_2 + N_3$ with $N_1, N_2, N_3 \leq p' \text{Card}(A)^2$, $N_1, N_2$ multiple of $p'$, $N_2 > 0$, and

$$\sigma^{N_1}(a) = p_1dq_1er_1,$$
$$\sigma^{N_2}(d) = p_2dq_2, \quad \sigma^{N_2}(q_1) = s_2, \quad \sigma^{N_2}(e) = q''_2cr_2,$$

where $d, e \in A$ are growing letters. The words $q_1, q_2, s_2, q''_2$ are non-growing.

We have (see Figure 4.3)

$$\sigma^{kN_2}(d) = p'_2dq_2(q''_2)^{k-1}, \quad \sigma^{kN_2}(q_1) = s_2, \quad \sigma^{kN_2}(q''_2) = (q''_2)^{k-1}q''_2cr_2',$$
$$\sigma^{N_3}(d) = p_3bq_3, \quad \sigma^{N_3}(q_2(q''_2)^{k-1}s_2(q''_2)^{k-1}q''_2) = q'_3, \quad \sigma^{N_3}(e) = q''_3cr_3$$

with $q''_2 = \sigma^{N_2}(q_2), q''_3 = \sigma^{N_2}(q''_2)$. The word $q_3, q'_3, q''_3$ are non-growing.

Thus $x|_{-n,n}$ is a factor of $q = q'_3q''_3$. This implies that $x$ is a shift of $vuvwv^\omega$ with $v = \sigma^{N_3}(q'_2), v = \sigma^{N_3}(q''_2)$ and $w = \sigma^{N_3}(q''_2)$. Each of these words is a fixed point of $\sigma^{N_2}$.

\[ \begin{array}{c}
\sigma^{N_1} \\
\downarrow \\
\sigma^{kN_2} \\
\downarrow \\
\sigma^{N_3} \\
\downarrow \\
p_1 d q_1 e r_1 \\
p_2 d q_2 s_2 q''_2 e r_2 \\
p_3 b q_3 q'_3 q''_3 r_3 \\
\end{array} \]

Example 4.4 Let $\sigma: a \mapsto abb, b \mapsto b$ (see Example 5.11). We have $X(\sigma) = \{b^\infty\}$ and $b^\infty$ is a periodic point having only non-growing letters.

The following corollary of Proposition 4.3 appears in [5, Proposition 5.5] (and also in [27, Corollary 2.11]) in the case of non-erasing morphisms.

Corollary 4.5 If a morphism $\sigma$ is aperiodic, every point in $X(\sigma)$ contains an infinite number of growing letters on the left and on the right.

We will also use the following result describing the points with a finite number of growing letters on the left or on the right.

Lemma 4.6 Let $\sigma: A^* \rightarrow A^*$ be a morphism. If $x$ is a point of $X(\sigma)$ which has a growing letter at some position $i$ such that all letters at smaller positions are non-growing, then there is an integer $k$ such that $x_{(-\infty,i)} = vuv^kz$ where $u, v, w, z$ are finite words of lengths bounded by a value depending only on $\sigma$. 

20
implies that either $x_{[-n,i)}$ is a factor of $\sigma^N(a)$. We distinguish three cases as in the proof of Proposition 4.3.

Case 1. We have, for $N$ large enough, $\sigma^N(a) = pbq$ with $p, q$ non-growing $b \in A$ growing and $x_{[-n,i)}$ a factor of $p$. We define $N_k, p_k, q_k, p'_k, q'_k$ as in Case 1 of the proof of Proposition 4.3. Thus $x_{[-n,i)}$ is a factor of $\sigma^N_3(p'_2)\sigma^N_3((p'_2)^{k-1}p_2)$. Since there is a finite number of possible $p_2, p'_2, p''_2, N_3$ for all $n$, there is an infinite number of $n$ such that $x_{[-n,i)}$ is a factor of $\sigma^N_3(p'_2)\sigma^N_3((p'_2)^{k-1}p_2)$ for some fixed $p_2, p'_2, p''_2, N_3$. This implies that $\sigma^N_3(p_2)$ is non-empty and $x_{(-\infty,i)} = \omega\sigma^N_3(p'_2)s_2$ where $s_2$ is some prefix of $\sigma^N_3(p_2)p_3$.

Case 2. We have, for $N$ large enough, $\sigma^N(a) = pbq$ with $p, q$ non-growing $b \in A$ growing and $x_{[-n,i)}$ a factor of $q$. We define $N_k, p_k, q_k, p'_k, q'_k$ as in Case 2. Thus $x_{[-n,i)}$ is a factor of $q_2\sigma^N_3(q_2)\sigma^N_3(q_2)^{k-1}q''_3$. This implies that $\sigma^N_3(q'_2)$ is non-empty and $x_{(-\infty,i)} = \omega\sigma^N_3(q'_2)s_2$ where $s_2$ is some factor of $\sigma^N_3(q'_2)$.

Case 3. We have, for $N$ large enough, $\sigma^N(a) = pbqcr$ with $q$ non-growing, $b, c \in A$ growing and $x_{[-n,i)}$ a factor of $q$. We define $N_1, N_2, N_3, p_k, q_k, p'_k, q'_k, q''_k, r_k, r'_k$ as in Case 3. We get that $x_{[-n,i)}$ is a factor of

$$q = \sigma^N_3(q_2(q''_2)^{k-1}q'_1(q'_{2''})^{k-1}q''_2)$$

for all $n$ with a finite number of possible $q_1, q_2, q'_2, q''_2, q''_3, N_3$. Then $x_{[-n,i)}$ is a factor of $q$ for some fixed $N_3, q'_1, q_2, q'_2, q''_2, q''_3$ for an infinite number of $n$. This implies that either $x_{[-n,i)} = \omega\sigma^N_3(q'_2)s_2$ with $s_2$ a prefix of $q'_2$, or $x_{(-\infty,i)} = \omega\sigma^N_3(q'_2)r_2$ with $r_2$ a prefix of $\sigma^N_3(q'_2(q'_{2''})^{k-1}q''_2)$ for some non-negative integer $k$.

A symmetric version of Lemma 4.6 holds for points with no growing letter after some position $i$. The following example illustrates Lemma 4.6.

**Example 4.7** Let $\sigma: a \mapsto bac, b \mapsto d, d \mapsto b, c \mapsto c$. Then $x = \omega(db) \cdot ac\omega$ is in $L(X(\sigma))$ and has no growing letter at negative indices. Thus $x^-$ has the form indicated in Lemma 4.6 with $u = db$ and $v = w = \varepsilon$. We have no example with $|vw^kz|$ unbounded.

## 5 Fixed points

We investigate fixed points of morphisms. Beginning with finite fixed points, we will next turn to one-sided infinite and then to two-sided infinite ones.

**Finite fixed points** Let $\sigma: A^* \to A^*$ be a morphism. As we have already seen before, a finite fixed point of $\sigma$ is a word $w \in A^*$ such that $\sigma(w) = w$.

Set

$$A(\sigma) = \{ a \in A \mid \sigma(a) = uav \text{ with } u, v \text{ erasable} \}.$$ 

The following is from [20] (see also [1] Theorem 7.2.3).
Proposition 5.1 The set of finite fixed points of \( \sigma: A^* \to A^* \) is a submonoid of \( A^* \) generated by the finite set
\[
F(\sigma) = \{ \sigma^{\operatorname{Card}(A)}(a) \mid a \in A(\sigma) \}.
\]

Infinite fixed points Let \( \sigma: A^* \to A^* \) be a morphism. A right-infinite fixed point of \( \sigma \) is a right-infinite sequence \( x \in A^N \) such that \( \sigma(x) = x \). Symmetrically, a left-infinite fixed point of \( \sigma \) is a left-infinite sequence \( x \in A^{-N} \) such that \( \sigma(x) = x \). A two-sided infinite fixed point is a two-sided infinite sequence \( x \in A^\infty \) such that \( \sigma(x) = x \).

A morphism \( \sigma: A^* \to A^* \) is right-prolongable on \( u \in A^+ \) if \( \sigma(u) \) begins with \( u \) and \( u \) is growing. In this case, there is a unique right-infinite sequence \( x \in A^N \) which has each \( \sigma^n(u) \) as a prefix. We denote \( x = \sigma^\omega(u) \).

Symmetrically, \( \sigma \) is left-prolongable on \( v \in A^+ \) if \( \sigma(v) \) ends with \( v \) and \( v \) is growing. In this case, there is a unique left-infinite sequence \( y \in A^{-N} \) which has all \( \sigma^n(v) \) as a suffix. We denote \( y = \sigma^\omega(v) \).

We begin with an elementary result concerning periodic fixed points.

Lemma 5.2 For every \( u \in A^+ \), the following conditions are equivalent.

(i) The right-infinite sequence \( u^\omega \) is a right-infinite fixed point of \( \sigma \).

(ii) The left-infinite sequence \( \omega u \) is a left-infinite fixed point of \( \sigma \).

(iii) The two-sided infinite sequence \( u^\infty \) is a two-sided infinite fixed point of \( \sigma \).

(iv) One has \( \sigma(u) = u^n \) for some \( n \geq 1 \).

Proof. Assume that (i) holds. We may assume that \( u \) is primitive. Set \( u' = \sigma(u) \). Then \( u^\omega = u'^\omega \), which implies \( u' = u^n \) since \( u \) is primitive. Thus (i) implies (iv). Symmetrically, (ii) implies (iv). The other implications are clear.

We shall come back to periodic fixed points in the next section.

We now describe more precisely the words \( u \) such that a morphism is right-prolongable on \( u \).

Lemma 5.3 If some power \( \sigma^k \) of a morphism \( \sigma: A^* \to A^* \) is right-prolongable on \( u \in A^+ \), there is a growing letter \( a \) such that \( u = praq \) with \( \sigma^k(p) = p \), the word \( r \) erasable, \( \sigma^k \) right-prolongable on \( ra \) and \( \sigma^{k\omega}(u) = \sigma^{k\omega}(pra) \). Moreover, if \( u \in \mathcal{L}(\sigma) \), such an integer \( k \) can be bounded in terms of \( \sigma \).

Proof. Let \( a \) be the first growing letter of \( u \) and set \( u = saq \). Then \( a \) is also the first growing letter of \( \sigma^k(a) \). Set \( \sigma^k(a) = vaw \). Since we have \( s = \sigma^{nk}(s)\sigma^{(n-1)k}(v)\cdots\sigma(v)v \) for all \( n \geq 1 \), the word \( v \) is erasable and \( r = \sigma^k(\operatorname{mex}(\sigma)^{-1})(v)\cdots\sigma^k(v)v \) is such that \( \sigma^k \) is right-prolongable on \( ra \). Set \( t = w\sigma^k(\cdots\sigma^k(\operatorname{mex}(\sigma)^{-1})(w) \) and \( p = \sigma^k(\operatorname{mex}(\sigma)(s)) \). Then
\[
\sigma^k(\operatorname{mex}(\sigma)(u) = prat\sigma^k(\operatorname{mex}(\sigma)(q)).
\]
Thus \( s = pr \) and \( \sigma^{ka}(u) = \sigma^{ka}(pra) \).

There remains to show that \( k \) can be bounded. We prove separately that one can find a bounded integer \( k \) such that \( \sigma^k(p) = p \) and, provided \( u \in \mathcal{L}(\sigma) \), such that \( \sigma^k \) is right-prolongable on \( ra \). If \( \sigma^k(p) = p \), then, by Proposition 5.4, we have \( p \in F(\sigma^k)^* \). Since \( A(\sigma^k) \subset A(\mex(\sigma))! \), we can choose \( k \leq \mex(\sigma)! \). Next, if \( \sigma^k \) is right-prolongable on \( ra \) with \( r \in \mathcal{L}(\sigma) \) erasable, the length of \( r \) can be bounded in terms of \( \sigma \) by \([3, \text{Lemma 2.2}]\). Since \( a \) is growing, there is a bounded integer \( k \) such that \( |\sigma^k(ra)| \geq |ra| \). Then \( \sigma^k \) is right-prolongable on \( ra \).

The following result is due to \([22]\) (see also \([1, \text{Theorem 7.3.1}]\)). Other proofs or variations are also given in \([37]\) or \([13, \text{Theorem 1}]\).

**Proposition 5.4** Every right-infinite fixed point of a morphism \( \sigma : A^* \to A^* \) is either in \( F(\sigma)^\omega \) or of the form \( \sigma^\omega(u) \) where \( \sigma \) is right-prolongable on \( u \in A^+ \).

A right-infinite fixed point of a morphism \( \sigma \) is admissible if it is in \( X(\sigma)^+ \).

**Proposition 5.5** Let \( \sigma : A^* \to A^* \) be a morphism. The number of orbits of right-infinite admissible fixed points of a power of \( \sigma \) is finite and effectively bounded in terms of \( \sigma \). It is nonzero provided \( X(\sigma) \) is non-empty.

The proof of Proposition 5.5 uses properties of non-negative matrices. To such a matrix \( M \), we associate the multigraph \( G \) such that \( M \) is the adjacency matrix of \( G \). The matrix is irreducible if \( G \) is strongly connected. The period of a graph is the greatest common divisor of the lengths of its cycles. The matrix \( M \) is primitive if it is irreducible and the period of \( G \) is 1. By a well-known bound due to Wielandt \([10]\), if \( M \) is a primitive matrix of dimension \( k \), then \( M^n \) is positive for \( n = (k - 1)^2 + 1 \) and thus for all \( n \geq (k - 1)^2 + 1 \) (see also \([1, \text{Exercise 8.7.8}]\)). It follows that if \( M \) is irreducible and \( G \) has period \( p \), then \( M^p \) is a block diagonal matrix formed of \( p \) primitive diagonal blocks. Thus \( M^{mp} \) is block diagonal with \( p \) positive diagonal blocks for \( m = (k/p - 1)^2 + 1 \).

**Lemma 5.6** Let \( \sigma : A^* \to A^* \) be a morphism. If \( X(\sigma) \) is non-empty, there are disjoint subsets \( B, C, D, E \) of \( A \) with \( B \) non-empty and an integer \( n \leq \text{Card}(A)! \) such that

\[
\begin{align*}
\sigma^n(B) & \subset (B \cup C \cup D \cup E)^*, \\
\sigma^n(C) & \subset (D \cup E)^*, \\
\sigma^n(d) & \subset E^*dE^* \text{ for every } d \in D, \\
\sigma^n(E) & = \{\varepsilon\},
\end{align*}
\]

and, moreover, such that every letter in \( B \) is growing and appears in every word \( \sigma^n(b) \) for every \( n \geq \text{Card}(A)^2 \) and \( b \in B \).
Proof. We use the multigraph $G(\sigma)$. Consider the set $\mathcal{S}$ of non-trivial strongly connected components of $G(\sigma)$. Let $\mathcal{G}$ be the graph on $\mathcal{S}$ with edges $(S,T)$ if there is a path from a vertex of $S$ to a vertex of $T$. The leaves of $\mathcal{G}$ are the elements of $\mathcal{S}$ from which no other non-trivial component can be reached. We consider two cases.

Case 1. All leaves of $\mathcal{G}$ are reduced to a cycle. Since $X(\sigma)$ is not empty, not all elements of $\mathcal{S}$ can be leaves of $\mathcal{G}$. Considering a path of maximal length in $\mathcal{G}$ ending in a leaf of $\mathcal{G}$, it follows that there is an element $H = (V_H, E_H)$ of $\mathcal{S}$ which is not a leaf in $\mathcal{G}$ and which can be followed only by leaves in $\mathcal{G}$.

Let $p$ be the period of $H$ and let $B \subset V_H$ be the set of vertices of a strongly connected component of $G(\sigma^p)$. We choose for $D$ the set of vertices of all leaves of $G$ accessible from $B$. We choose for $C$ the set of vertices of $G$ forming a trivial component on a path from $B$ to $D$ in $G$. Finally, we choose for $E$ the union of vertices of trivial strongly connected components not belonging to $C$ (and thus formed of erasable letters) accessible from $B$ in $G$.

Set $n = \text{Card}(A)!$. Then $n$ is a multiple of the periods of all components $S$ and thus $M(\sigma^n)$ is the identity on the components $S$ in $D$. We have

\[
((\text{Card}(B) - 1)^2 + 1)p \leq \text{Card}(B)!p \leq \text{Card}(A)!.
\]

Thus, by Wielandt’s bound, the matrix $M(\sigma^n)$ is positive on $B$. We also have $\sigma^n(E) = \{\varepsilon\}$ and also $\sigma^n(C) \subset (D \cup E)^*$.

Case 2. There exist a leaf $K = (V_K, E_K)$ of $\mathcal{G}$ which is not reduced to a cycle. Let $p$ be the period of $K$ and let $B$ be the set of vertices in $V_K$ of a strongly connected component of $G(\sigma^p)$. Since $((\text{Card}(B) - 1)^2 + 1)p \leq \text{Card}(A)!$ as above, $M^n$ is positive on $B$. We choose $C = D = \emptyset$ and $E$ is the set of erasable letters accessible from $B$. Since $n \geq \text{Card}(E)$, we have also $\sigma^n(E) = \{\varepsilon\}$. 

Note that, as a reformulation of the conditions of Lemma 5.6, the restriction $M'$ to $B \cup C \cup D \cup E$ of the adjacency matrix of $G(\sigma^n)$ has the form

\[
M' = 
\begin{bmatrix}
B & C & D & E \\
N & * & * & * \\
0 & 0 & * & * \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0
\end{bmatrix}
\]

where $N > 0$ is a primitive matrix with $M_{b,d} \neq 0$ for at least one $b \in B$ and $d \in D$ (Case 1) or $N$ not the identity matrix of dimension 1 and $C = D = \emptyset$ (Case 2). We illustrate the first case of the proof in the following example.

**Example 5.7** Let $\sigma: a \mapsto bab, b \mapsto cd, c \mapsto c, d \mapsto d$. The graph $G(\sigma)$ is represented in Figure 5.1. In this case, we choose $n = 1$, $B = \{a\}$, $C = \{b\}$, $D = \{c, d\}$ and $E = \emptyset$. 

24
The following example illustrates the second case in the proof of Lemma 5.6.

Example 5.8 Let \( \sigma : a \mapsto baab, b \mapsto \varepsilon \). The graph \( G(\sigma) \) is represented in Figure 5.2. We choose this time \( n = 1, B = \{ a \}, C = D = \emptyset \) and \( E = \{ b \} \).

In the next lemma, we use the notation \( \sigma_i^\top(u) = \sigma_p(u) \cdots \sigma(u)u \).

Lemma 5.9 Let \( \sigma : A^* \to A^* \) be a morphism and let \( a \in A \) be a growing letter such that \( \sigma(a) = uav \) with \( u \) non-growing. Let \( i, p \) be such that \( \sigma_i^\top(u) = \sigma_i^p(u) \) with \( p > 0 \). If \( \sigma_i(u) = \varepsilon \), then \( \sigma_i \) is right-prolongable on \( \sigma_i^\top(u)a \). Otherwise \( w^\omega \), with \( w = \sigma_i^p(u) \cdots \sigma_i(u) \sigma_i(u) \) is an admissible right-infinite fixed point of \( \sigma^p \).

Proof. If \( \sigma_i(u) \) is empty, using the notation \( w \geq w' \) if \( w' \) is a prefix of \( w \), we have \( \sigma_i(a) \geq \sigma_i^\top(u)a \). Moreover \( \sigma^n(a) \neq \sigma_i^\top(u)a \) because \( a \) is growing. Set \( t = \sigma_i^\top(u)a \). Since \( \sigma_i(t) \geq \sigma_i(a) \geq t \), this implies that \( \sigma_i \) is right-prolongable on \( t \) and thus the conclusion in this case.

Otherwise, we have for every \( k \geq 1 \),

\[
\sigma^{i+kp}(a) \geq \sigma_i^{i+kp-1}(u)a \\
\geq (\sigma_i^{i+p-1}(u) \cdots \sigma_i(u))^{k-1}\sigma_i^{i+p-1}(u)a \\
\geq w^{k-1}\sigma_i^{i+p-1}(u)a
\]

with \( w = \sigma_i^{i+p-1}(u) \cdots \sigma_i(u) \). This shows that \( w^\omega \) is in \( X(\sigma)^+ \) and is a right-infinite fixed point of \( \sigma^p \).

There is a symmetric version of Lemma 5.9. Assume that that \( \sigma(a) = uav \) with \( v \) non-growing. Let \( i, p \) be such that \( \sigma_i^{i+p}(v) = \sigma_i(v) \) with \( p > 0 \). If
\(\sigma^i(v) = \varepsilon\), then \(\sigma^i\) is left-prolongable on \(av\sigma(v) \cdots \sigma^{i-1}(v)\). Otherwise, \(\omega w\) with \(w = \sigma^i(v)\sigma^{i+1}(v) \cdots \sigma^{i+n-1}(v)\) is a left-infinite fixed point of \(\sigma\).

**Proof of Proposition 5.3.** We first show that the number of orbits is nonzero. Let \(B, C, D, E\) and \(u\) be as in Lemma 5.9. Taking the restriction of \(\sigma^n\) to \(B\) and changing \(\sigma\) for \(\sigma^n\), we may assume that \(A = B \cup C \cup D \cup E\), with \(\sigma(C) \subset (D \cup E)^*\), \(\sigma(D) \subset E^*DE^*\) and \(\sigma(E) = \{\varepsilon\}\) and every letter \(a \in B\) appears in every \(\sigma(b)\) for \(b \in B\).

Changing again \(\sigma\) for some power, we may assume that there is a letter \(a \in B\) such that \(\sigma(a) = uaw\) with \(u \in (C \cup D \cup E)^*\). If \(u\) is non-erasable, the elements of \(C, D, E\) are non-growing and there is a periodic admissible fixed point by Lemma 5.9. If \(u\) is erasable, some power \(\sigma^n\) of \(\sigma\) is right-prolongable on \(w = \sigma^n(u)a\) and thus \(\sigma^\omega (w)\) is a right-infinite fixed point of \(\sigma^n\) (recall that \(\sigma^\omega = (\sigma^n)^\omega\)). If \(v\) is growing, it contains a letter of \(B\) and thus \(a\) is in \(\mathcal{L}(X(\sigma))\), which implies that the right-infinite fixed point \(\sigma^\omega(w)\) is admissible. Otherwise, since \(v\) cannot be erasable, we obtain by a symmetric version of Lemma 5.9 a periodic admissible left-infinite fixed point, and thus also a periodic right-infinite fixed point.

There remains to show that the number of orbits of admissible right-infinite fixed points of a power of \(\sigma\) is finite. Let \(x\) be such a fixed point. By Proposition 5.4, either \(x\) is in \(F(\sigma^n)^\omega\) or of the form \(\sigma^n(u)\) where \(\sigma^n\) is right-prolongable on \(u \in A^*\). If \(x\) is in \(F(\sigma^n)^\omega\), it is formed of non-growing letters and there is a finite number of orbits of such points by Proposition 5.3. Assume that \(x = \sigma^n(u)\) with \(\sigma^n\) right-prolongable on \(u\). By Lemma 5.3, there is a growing letter \(a\) such that \(u = p\sigma q\) with \(\sigma(p) = p\), the word \(r\) erasable, \(\sigma^n\) right-prolongable on \(ra\) and \(x = \sigma^\omega(ra)\). Then \(x\) belongs to the orbit of the admissible right-infinite fixed point \(y = \sigma^\omega(ra)\). By Proposition 3.12, there is a finite number of erasable words \(r\) and thus a finite number of points \(y\).

**Example 5.10** Let \(\sigma\) : \(a \mapsto baab, b \mapsto \varepsilon\) as in Example 5.8. Then \(\sigma(baab) = (baab)^2\). Thus \(\sigma\) is right-prolongable on \(ba\) and \(\sigma^\omega(baab) = (baab)^\omega\) is an admissible one sided fixed point of \(\sigma\).

**Example 5.11** Let \(\sigma\) : \(a \mapsto abb, b \mapsto b\). Then \(X(\sigma) = \{b^\omega\}\) and \(b^\omega\) is an admissible right-infinite fixed point of \(\sigma\). The sequence \(ab^\omega\) is also a right-infinite fixed point but it is not admissible.

The following example shows that the number of admissible right-infinite fixed points of \(\sigma\) need not be finite.

**Example 5.12** Let \(\sigma\) : \(a \mapsto bc, b \mapsto bd, c \mapsto ec, d \mapsto d, e \mapsto e\). Then \(d^n e^\omega\) is an admissible right-infinite fixed point of \(\sigma\) for every \(n \geq 1\).

We now add a complement to Proposition 5.4 which characterizes right infinite admissible fixed points.
Proposition 5.13 Let $\sigma : A^* \to A^*$ be a morphism. A right-infinite sequence $x \in A^\omega$ is an admissible right-infinite fixed point of $\sigma$ if and only if one of the two following conditions is satisfied.

(i) $x = uv\omega$ with $\sigma(u) = u$, $\sigma(v) = v$ and $uv\omega \in \mathcal{L}(X(\sigma))$.

(ii) $x = \sigma^\omega(u)$ with $u \in \mathcal{L}(X(\sigma))$ and $\sigma$ right-prolongable on $u$.

Proof. The conditions are clearly sufficient.

Conversely, if $x$ contains a growing letter, let $u$ be the shortest prefix of $x$ containing a growing letter. Then $u$ is growing and $\sigma$ is right-prolongable on $u$. Thus (ii) is satisfied. Otherwise, by Lemma 5.13 $x$ is eventually periodic. On the other hand, by Proposition 5.4 we have $x \in F(\sigma)^\omega$. Thus there are $u, v \in F(\sigma)^+$ such that $x = uv\omega$. Thus condition (i) is satisfied.

Two-sided fixed points Let us now consider two-sided infinite fixed points. Let $\sigma : A^+ \to A^+$ be a morphism. Recall that a two-sided infinite fixed point of $\sigma$ is a sequence $x \in A^\omega$ such that $\sigma(x) = x$.

Assume that $\sigma$ is right-prolongable on $u \in A^+$ and let $x = \sigma^\omega(u)$. Assume next that $\sigma$ is left-prolongable on $v \in A^+$. Denote by $y = \sigma^\omega(v)$ the left-infinite sequence having all $\sigma^n(v)$ as suffixes. Let $z \in A^\omega$ be the two-sided infinite sequence such that $x = z^+$ and $y = z^-$. Then $z$ is a two-sided infinite fixed point of $\sigma$ denoted $\sigma^\omega(v \cdot u)$. It can happen that $\sigma^\omega(v \cdot u)$ does not belong to $X(\sigma)$. In fact, $\sigma^\omega(v \cdot u)$ belongs to $X(\sigma)$ if and only if $vu \in \mathcal{L}(\sigma)$.

The description of two-sided infinite fixed points follows directly from Proposition 5.13. Indeed, $x \in A^\omega$ is a fixed point of $\sigma : A^+ \to A^+$ if and only if $x^-$ and $x^+$ are one sided infinite fixed points of $\sigma$. Thus, there are four cases according to whether none of $x^-$, $x^+$ is in $\mathcal{F}(\sigma) \cup F(\sigma)^\omega$ or one of them or both.

An admissible two-sided infinite fixed point of $\sigma$ is a sequence $x \in X(\sigma)$ which is a two-sided fixed point. Thus a fixed point of the form $\sigma^\omega(v \cdot u)$ with $vu \in \mathcal{L}(\sigma)$ is admissible.

In the following result, the existence of admissible two-sided infinite fixed points is a modification of [IN Proposition 1.4.13] where it is stated for a non-erasing morphism $\sigma$ such that $X(\sigma)$ is minimal.

Theorem 5.14 A morphism $\sigma : A^+ \to A^+$ such that $X(\sigma)$ is non-empty has a power $\sigma^k$ with an admissible two-sided infinite fixed point. The set of orbits of these fixed points is a finite computable set and the exponent $k$ can be bounded in terms of $\sigma$.

The proof uses the following lemma.

Lemma 5.15 Let $\sigma : A^+ \to A^+$ be a morphism. There is a finite number of words $auvb \in \mathcal{L}(X(\sigma))$ with $u, v$ non-growing, $a, b$ growing, such that for some $k \geq 1$, $\sigma^k$ is left-prolongable on $au$ and right-prolongable on $vb$. Their length is bounded effectively in terms of $\sigma$.  

27
Proof. Consider \(a, u, v, b\) such that for some integer \(k\) the conditions are satisfied. By Lemma 5.3, the least \(k\) such that this is true is bounded in terms of \(\sigma\). Thus we may replace \(\sigma^k\) by \(\sigma\).

Since \(auvb \in \mathcal{L}(X(\sigma))\), there is, for every large enough integer \(n\), a letter \(e \in A\) such that \(auvb\) is a factor of \(\sigma^n(e)\). For each \(e \in A\) and \(n \geq 0\), consider the derivation tree \(T_\sigma(e, n)\) of \(e\) at order \(n\). In this tree, there is a unique path from the root to each of the two vertices labeled \(a\) and \(b\) at level \(n\). Let us say that the tree \(T_\sigma(e, n)\) is special if the nodes at level 1 on these paths are distinct.

We distinguish two cases.

Case 1. The integers \(n\) such that the tree \(T_\sigma(e, n)\) is special are bounded by \(\text{Card}(A)^2\). Then, the length of \(auvb\) is bounded by \(|\sigma| \cdot \text{Card}(A)^2\) and thus the conclusion holds.

Case 2. There is a special tree \(T_\sigma(e, n)\) with \(n > \text{Card}(A)^2\). Since \(a\) and \(b\) are growing, all vertices on these paths are labeled by a growing letter. On the path from the root to \(a\), the successor of a vertex labeled by \(c\) distinct of the root is the last growing letter \(f\) of \(\sigma(c)\) and is thus uniquely determined. Similarly, on the path from the root to \(b\), the successor of a vertex labeled by \(d\) distinct of the root is the first growing letter \(g\) of \(\sigma(d)\) and is thus uniquely determined (see Figure 5.3 on the left).

Since \(n > \text{Card}(A)^2\), we can write \(n = n_1 + kn_2 + n_3\) with \(n_1, n_2, n_3 \leq \text{Card}(A)^2\), \(n_2 > 0\) and \(k \geq 1\), such that there is a path of length \(n_1\) from the root to a node labeled by \(c\) (resp. \(d\)), \(k\) paths of length \(n_2\) from \(c\) to \(c\) (resp. \(d\) to \(d\)) and a path of length \(n_3\) from \(c\) to \(a\) (resp. from \(d\) to \(b\)) (see Figure 5.3).

Since \(T_\sigma(e, n)\) is special, the nodes labeled by \(c\) and \(d\) at level \(n_1\) are distinct. Since \(\sigma\) is left-prolongable on \(au\), the letter \(a\) is the last growing letter of \(\sigma(a)\). This forces \(c = a\). Similarly \(d = b\). Consequently, we can choose \(n_2 = 1\) and \(n_3 = 0\).

Thus we have

\[
\sigma^{n_1}(e) = p_1aq_1br_1, \quad \sigma^k(a) = p_2aq_2, \quad \sigma^k(q_1) = q_2', \quad \sigma^k(b) = q_2''br_2
\]
and \( uv = q_2 q'_2 q''_2 \) (see Figure 5.4). If \( q_2 \) is non-erasable, then \( \sigma \) is not left-prolongable on \( au \), a contradiction. Thus \( q_2 \) is erasable and similarly \( q''_2 \) is erasable. This shows that the length of \( uv \) is bounded. One has actually

\[
|uv| = |q_2 q'_2 q''_2| = |q_2| + |\sigma^k(q_1)| + |q''_2|.
\]

Set \( K = \max\{|\sigma^m(a)| \mid m \geq 1, \text{ and } a \in A \text{ non-growing}\} \). Let also \( L \) be the maximal length of erasable words in \( L(\sigma) \) (see Proposition 3.12). We have \( |\sigma^k(q_1)| \leq K|\sigma|^n \leq K|\sigma|^{\text{Card}(A)^2} \) and since \( q_2, q''_2 \) are erasable, \( |q_2|, |q''_2| \leq L \). Thus

\[
|uv| \leq K|\sigma|^{\text{Card}(A)^2} + 2L
\]

showing that \( |uv| \) is bounded in terms of \( \sigma \).

\[ \square \]

**Proof of Theorem 5.14.** Let \( B, C, D, E \) and \( n \) be as in Lemma 5.6. We change \( \sigma \) for \( \sigma^n \). Suppose first that the graph induced by \( G(\sigma) \) on \( B \) is reduced to a loop on \( a \in A \) and thus that \( \sigma(a) = uav \) with \( u, v \) non-growing. Since \( a \) is growing, we cannot have \( u \) and \( v \) erasable. We may assume that \( u \) is non-erasable. Then, by Lemma 5.9 some power \( \sigma^m \) of \( \sigma \) has an admissible periodic right-infinite fixed point \( w_\omega \) and thus the admissible two-sided infinite fixed point \( w_\infty \).

Assume next that graph induced on \( B \) is not reduced to a loop on one vertex. Then for every \( a \in B \), considering the tree \( T_{\sigma}(a, n) \) for \( n \) large enough (as in Figure 5.4), we can then find \( m \geq 1 \) (at most equal to the least integer \( k \) such that \( M(\sigma)^k \) restricted to \( B \times B \) is positive) and \( b, c \in B \) such that

\[
\sigma^m(a) = pbqcr,
\]

\[
\sigma^m(b) = sbt,
\]

\[
\sigma^m(c) = ucv
\]

with \( q, t, u \in (C \cup D \cup E)^* \). If \( u \) or \( t \) is non-erasable, then by Lemma 5.9 or its symmetric version, some power of \( \sigma \) has a periodic admissible one sided fixed point and thus a periodic admissible two-sided fixed point. Otherwise, assume that \( \sigma^n(u) = \sigma^n(t) = \varepsilon \). Then, by Lemma 5.9 \( \sigma^n \) is right-prolongable on \( w = \sigma^n(u)c \) and, symmetrically, it is left-prolongable on \( z = bts(t) \cdots \sigma^{n-1}(t) \).
Since \( q \) is non-growing, we may assume that \( \sigma^{2n}(q) = \sigma^n(q) \). We can change \( \sigma \) for \( \sigma^n \). Then \( \sigma \) is left-prolongable on \( z\sigma(q) \), right-prolongable on \( w \) and \( z\sigma(q)w \) is in \( L(\sigma) \). Therefore, \( \sigma^\omega(z\sigma(q) \cdot w) \) is an admissible two-sided fixed point of \( \sigma \).

This proves the existence of the fixed point. Let now \( x \) be an arbitrary admissible two-sided fixed point of \( \sigma^n \). Then \( x^+ \) and \( x^- \) are one-sided fixed points. If \( x^+ \) is in \( F(\sigma^n)^\omega \) and \( x^- \) in \( ^\omega F(\sigma^n) \), then \( x \) is formed of non-growing letters and there is a finite number of orbits of such points by Proposition 4.13. If \( x^- \) is formed of non-growing letters and \( x^+ \) contains growing letters, then by Proposition 5.4 we have \( x^+ = \sigma^n\omega(u) \) where \( \sigma^n \) is right-prolongable on \( u \). By Proposition 5.5 there is a finite number of orbits of points of this type which are admissible fixed points of a power of \( \sigma \). The case where \( x^- = \sigma^n\omega(v) \) and \( x^+ \) is formed of non-growing letters is symmetrical. Finally, assume that \( x = \sigma^n\omega(u \cdot v) \), where \( \sigma^n \) is left-prolongable on \( u \) and right-prolongable on \( v \). Set \( u = paq \) and \( v = rbs \) with \( q,r \) non-growing. Then \( \sigma^n \) is left-prolongable on \( aq \) and right-prolongable on \( rb \). Then \( x = \sigma^n\omega(aq \cdot rb) \) and by Lemma 5.15 there is a finite number of points of this form.

**Example 5.16** Let \( A = \{a, b\} \) and let \( \sigma \) be morphism \( a \mapsto a, b \mapsto baabab \). Then \( \sigma^\omega(ba \cdot b) \) is an admissible two-sided fixed point.

Note that, as for one-sided fixed points, the number of admissible two-sided fixed points need not be finite, as shown in the following example.

**Example 5.17** Let \( \sigma : a \mapsto bc, b \mapsto bd, c \mapsto ec, d \mapsto d, e \mapsto e \), as in Example 5.12. Then, for any \( n \geq 0 \), the sequences \( \omega de^n \cdot e^\omega \) and \( ^\omega d \cdot d^n e^\omega \) are admissible two-sided fixed points of \( \sigma \).

In the same way as we did for one-sided fixed points (Proposition 5.13), we now give the following characterization of two-sided admissible fixed points.

**Proposition 5.18** Let \( \sigma : A^* \rightarrow A^* \) be a morphism. A sequence \( x \in X(\sigma) \) is a two-sided infinite fixed point of \( \sigma \) if and only if one of the following conditions is satisfied.

(i) \( x = \omega rs \cdot tu^\omega \) with \( r,s,t,u \in F(\sigma)^* \) and \( r^*stu^* \subset L(X(\sigma)) \).

(ii) \( x = \omega rs \cdot \sigma^\omega(t) \) with \( r,s \in F(\sigma)^* \), \( \sigma \) right-prolongable on \( t \) and \( r^*st \subset L(X(\sigma)) \).

(iii) \( x = \sigma^\omega(r) \cdot st^\omega \) with \( \sigma \) left-prolongable on \( r \), \( s,t \in F(\sigma)^* \) and \( rst^* \subset L(X(\sigma)) \).

(iv) \( x = \sigma^\omega(r) \cdot \sigma^\omega(s) \) with \( \sigma \) left-prolongable on \( r \) and right-prolongable on \( s \) and \( rs \in L(\sigma) \).

**Proof.** The result follows applying Proposition 5.13 to \( x^- \) and \( x^+ \).
Quasi-fixed points Let us say that a two-sided infinite sequence $x$ is a quasi-fixed point if its orbit is stable by $\sigma$, that is $\sigma(\mathcal{O}(x)) \subset \mathcal{O}(x)$. Thus $x$ is a quasi-fixed point if and only if $\sigma(x) = S^k x$ for some $k \in \mathbb{Z}$. A quasi-fixed point $x$ is admissible if $x$ is in $X(\sigma)$. For two-sided infinite sequences $x, y$, we write $x \sim y$ if $x = S^k y$ with $k \in \mathbb{Z}$, that is, if $x, y$ are in the same orbit. If $a$ is a letter such that $\sigma(a) = uav$ with $u, v$ non-erasable and $i = |ua|$, we denote

$$\sigma^{\omega,i}(a) = \cdots \sigma^2(u)\sigma(u)a\sigma(v)\sigma^2(v) \cdots$$

Note that $\sigma^{\omega,i}$ is defined for every $a \in A$ such that $\sigma(a) = uav$ with $u, v$ non-erasable and $i = |ua|$. Otherwise, it is undefined. The following result is from [37] (see also [1, Theorem 7.4.3]).

**Proposition 5.19** Let $\sigma : A^* \to A^*$ be a morphism. A two-sided sequence $x \in A^\mathbb{Z}$ is a quasi-fixed point of $\sigma$ if and only if one of the following conditions is satisfied.

(i) $x$ is a shift of a two-sided infinite fixed point of $\sigma$.

(ii) $x \sim \sigma^{\omega,i}(a)$ for some $a \in A$ such that $\sigma(a) = uav$ with $u, v$ non-erasable and $i = |ua|$.

(iii) $x = (uv)^\infty$ for some non-empty words $u, v$ such that $\sigma(uv) = vu$.

The following example illustrates case (ii).

**Example 5.20** Let $\sigma : a \mapsto bab, b \mapsto b$. Then $\sigma^{\omega,2}(a) = \omega b \cdot ab\omega$ is a quasi-fixed point of $\sigma$.

Let us now characterize admissible quasi-fixed points.

**Proposition 5.21** Let $\sigma : A^* \to A^*$ be a morphism. A two-sided infinite sequence $x \in X(\sigma)$ is a quasi-fixed point of $\sigma$ if and only if one of the following conditions is satisfied.

(i) $x$ is a shift of an admissible two-sided infinite fixed point of $\sigma$.

(ii) $x \sim \sigma^{\omega,i}(a)$ for some $a \in A$ such that $\sigma(a) = uav$ with $u, v$ non-erasable and $i = |ua|$.

(iii) $x = (uv)^\infty$ for some non-empty words $u, v$ such that $\sigma(uv) = vu$.

There is a finite and computable number of orbits of admissible quasi-fixed points of powers of $\sigma$.

**Proof.** The conditions are clearly sufficient.

Conversely, let $x \in X(\sigma)$ be a quasi-fixed point of $\sigma$. By Proposition 5.19, one of the three conditions (i), (ii) or (iii) holds. If condition (i) holds, we have $x \sim y$ where $\sigma(y) = y$. Since $X(\sigma)$ is closed under the shift, we have also
y ∈ X(σ) and thus y is admissible. Thus (i') holds. Next (ii) is the same as (ii') and (iii) is the same as (iii').

Finally, by Theorem 6.14, there is a finite and computable number of orbits of quasi-fixed points of type (i'). The number of orbits of quasi-fixed points of type (ii') is at most Card(A). Finally, the quasi-fixed points of type (iii') are made of non-growing letters and there is a finite number of orbits of points of this type by Proposition 1.3. □

6 Periodic points

We now investigate the periodic points in substitution shifts.

We will need the following technical lemma.

Proposition 6.1 Let σ: A* → A* be a morphism. There is an alphabet B containing A and a morphism τ: B* → B* such that X(σ) = X(τ) and X(τ^n) = X(τ) for every n ≥ 1. Moreover σ^n and τ^n have the same finite and infinite fixed points for every n ≥ 1.

Proof. Let p be the least common multiple of the periods of the strongly connected components of the graph G(σ). Let B = A∪A×{1,...,p−1}. For every a ∈ A, set τ(a) = σ(a), τ(a,1) = a and τ(a,i) = (a,i−1) for every 2 ≤ i ≤ p−1.

Let us show that X(τ^n) = X(τ). Consider a word w ∈ L(X(τ)). There are arbitrary large integers k such that w is a factor of τ^k(a) for some a ∈ A. If k is large enough, we have k = k_1 + k_2 + k_3 and b ∈ A such that b appears in τ^{k_1}(a) and in τ^{k_2}(b) and that w is a factor of τ^{k_3}(b). By the definition of p, this implies that there exists l such that w is a factor of τ^{ℓ+p}(a) for every q ≥ 0. Let t be such that tn ≥ ℓ and set tn−ℓ = qp+r with q ≥ 0 and 0 ≤ r < p. Then w is a factor of τ^{tn}(a,r) = τ^{ℓ+p}(a,r) = τ^{ℓ+p}(a) and thus w is in L(τ^n).

It is clear that σ^n and τ^n have the same finite and infinite fixed points for n ≥ 1. □

The following result relates periodic points to fixed points. It is a complement to Proposition 1.3.

Proposition 6.2 Let σ: A* → A* be a morphism. Every periodic point x ∈ X(σ) is a fixed point of a power of σ.

Proof. By Proposition 6.1, we may assume that X(σ) = X(σ^n) for all n ≥ 1. Let x ∈ X(σ) be a periodic point. We have seen in Proposition 1.3 that the result is true if x has only non-growing letters. Let us assume that x contains growing letters. Let p be the minimal period of x. Set y^{(0)} = x. Since x is in X(σ), there is a sequence (y^{(n)}, k_n)_{n≥1} such that (y^{(n+1)}, k_{n+1}) is a σ-representation of y^{(n)} for n ≥ 0 with y^{(n)} ∈ X(σ) (by Proposition 3.14). Thus

\[ y^{(n)} = S^{k_{n+1}}σ(y^{(n+1)}). \] (6.1) 32
Let $A_n$ be the set of letters appearing in the sequence $y^{(n)}$. Since there is a finite number of possible sets $A_n$, there is an infinity of $n$ such that the sets $A_n$ are equal to the same set $B$. This forces the sequence $(A_n)$ to be periodic. Changing $\sigma$ for some power, we may assume that $A_0 = B$ and $\sigma(B) \subset B^*$. Then $\sigma(B) \subset B^*$ and every letter of $B$ appears in some $\sigma(b)$ with $b \in B$. Let $\tau$ be the restriction of $\sigma$ to $B^*$.

Since every letter of $B$ appears in every $y^{(n)}$, every word of $\mathcal{L}(\tau)$ appears in $x$. Since $x$ is periodic, every growing letter $a$ of $x$ appears with bounded gaps in $\mathcal{L}(\tau)$ and every letter $b \in B$ appears in some $\tau^n(a)$. Since $x$ is in $X(\tau)$ (we use here the fact that $X(\tau) = X(\tau^n)$) This shows By Theorem 10.1 that the shift $X(\tau)$ is formed of the shifts of $x$ and thus $x$ is a fixed point of a power of $\tau$ (since $\sigma$ acts as a permutation on the shifts of $x$).

A morphism $\sigma : A^* \to C^*$ is elementary if for every decomposition $\sigma = \alpha \circ \beta$ with $\beta : A^* \to B^*$ and $\alpha : B^* \to C^*$, one has $\text{Card}(B) \geq \text{Card}(A)$. An elementary morphism is clearly non-erasing.

This notion has been introduced in [19] where it is shown that, if $\sigma$ is elementary, then $\sigma$ defines an injective map from $A^\mathbb{N}$ to $C^\mathbb{N}$.

The proof of the following result follows an argument from [32, Lemma 2] (where the result is actually stated for admissible fixed points of morphisms), see also [18, Exercise 2.38]. It shows that one can decide whether an elementary morphism is aperiodic. Note that the property of being elementary is decidable. Indeed, if $\sigma : A^* \to C^*$ is a morphism consider the finite family $\mathcal{F}$ of sets $U \subset C^*$ such that $\sigma(A) \subset U^* \subset C^*$ with every $u \in U$ being a factor of some $\sigma(a)$ for $a \in A$. Then $\sigma$ is elementary if and only if $\text{Card}(U) \geq \text{Card}(A)$ for every $U \in \mathcal{F}$.

**Lemma 6.3** Let $\sigma : A^* \to A^*$ be an elementary morphism having a periodic point $x \in X(\sigma)$. The minimal period of $x$ can be effectively bounded in terms of the morphism $\sigma$.

**Proof.** Let $x \in X(\sigma)$ be a periodic. By Proposition 6.2, it is a fixed point of a power of $\sigma$.

If $x$ has no growing letter, its period can be bounded by Proposition 4.3. Assume that $x$ contains at least one growing letter.

We first note that since $x$ is periodic, there are no right-special words in $\mathcal{L}(x)$ of length larger than the period of $x$. We claim that there is no word in $\mathcal{L}(x)$ which is right-special with respect to $\mathcal{L}(x)$ and which begins with a growing letter.

Assume the contrary. Let $u = u_0 \in \mathcal{L}(x)$ be a right-special factor of $x$ beginning with a growing letter $a$. Let $b, c \in A$ be distinct letters such that $ub, uc \in \mathcal{L}(x)$. Since $\sigma$ is elementary, it is injective on $A^\mathbb{N}$ and thus there exist $v, w \in \mathcal{L}(x)$ such that $\sigma(bv) \neq \sigma(cw)$. Let $u_1$ be the longest common prefix of $\sigma(ubv)$ and $\sigma(ucw)$. Then $u_1$ is again a right-special factor of $x$ beginning with $\sigma(a)$. Continuing in this way, we build a sequence $u_0, u_1, u_2, \ldots$ of right-special factors of $x$ beginning with $\sigma^n(a)$. Since $a$ is growing, their lengths go to $+\infty$ and this contradicts the fact that $x$ is periodic. This proves the first claim.
We now claim that the length of the factors of $x$ formed of non-growing letters is bounded. To prove this, let $a$ be a growing letter of $x$. Since $x$ is periodic, we may assume that $a = x_0$. We may also assume that $x$ is a fixed point of $\sigma$ and thus that $x = w^\infty$ with $\sigma(w) = w^n$. Since $x$ has some growing letters, we have $n \geq 2$. We do not change $w$ by taking the restriction of $\sigma$ to the set of letters accessible from $a$.

Let $\ell(w)$ be the row vector with components $|w|_b$ for $b \in A$. We have then $\ell(w)M(\sigma) = n\ell(w)$, that is, $\ell(w)$ is a non-zero left eigenvector of $M(\sigma)$ for the eigenvalue $n$. We may assume that all letters are accessible from $a$. Also, since, for some $m \geq 1$, $|\sigma^m(a)| > |w|$, the letter $a$ is accessible from all growing letters. Thus the matrix $M(\sigma)$ is irreducible. Since, for an irreducible matrix, the dominant eigenvalue is the only eigenvalue having a nonnegative eigenvector, the eigenvalue $n$ is the dominant eigenvalue of $M$ and the eigenvalue $n$ is simple. The coefficients of a corresponding eigenvector can be effectively bounded in terms of $\sigma$.

For every eigenvector $z$ of $M(\sigma)$ relative to $n$, consider the ratio

$$r(z) = \frac{\sum_{b \in A} z_b}{\sum_{b \in A} z_b},$$

where $A_i$ is the set of growing letters and $A_f$ is its complement. Since $r(z) = r(\lambda z)$, this ratio is a constant $\rho$ depending only on $\sigma$. Now consider an integer $e$ such that $w^e$ has more than $\text{Card}(A)$ growing letters. We have a factorization $w^e = u_1 u_2 \cdots u_t s$ where each $u_i$ has $\text{Card}(A) + 1$ growing letters and $s$ has less than $\text{Card}(A) + 1$ growing letters. If, for some integer $k$, every $u_i$ has a run of $k(\text{Card}(A) + 1)$ non-growing letters, we have

$$\rho = r(\ell(w)) = r(\ell(w^e)) = \frac{\sum_{i=1}^{t} \ell_f(u_i) + \ell_f(s)}{\sum_{i=1}^{t} \ell_i(u_i) + \ell_i(s)} \geq \frac{\sum_{i=1}^{t} \ell_f(u_i)}{\sum_{i=1}^{t} \ell_i(u_i) + (\text{Card}(A) + 1)} \geq \frac{k(\text{Card}(A) + 1)t}{(\text{Card}(A) + 1)(t + 1)} \geq k/2$$

with $\ell_f(u) = \sum_{b \in A_f} |u|_b$ and $\ell_i(u) = \sum_{b \in A_i} |u|_b$. We conclude, taking $k = 2\rho + 1$, that $w^e$ has a factor with $\text{Card}(A) + 1$ growing letters separated by at most $(2\rho + 1)(\text{Card}(A) + 1) - 1$ non-growing letters. Such a factor has a factor of the form $ubu$ where $b$ is a growing letter. By the first claim, $ub$ is the only return word to $b$ since otherwise, there would be a right-special factor of $x$ beginning with $b$. As a consequence, $|ub|$ is the minimal period of $x$.

Since $|ub| \leq (2\rho + 1)(\text{Card}(A) + 1)$, this shows that the minimal period of $x$ is bounded by $(2\rho + 1)(\text{Card}(A) + 1)$.

When $\sigma$ is growing, the above proof shows that the period of $x$ is bounded by $\text{Card}(A)$. Note the following subtle point. We know, by Theorem 5.14, that the
number of orbits of admissible fixed points of $\sigma$, and thus the number of orbits of its periodic points, is effectively bounded. However, this does not imply that the period of such points is effectively bounded and thus that the problem of deciding whether a fixed point is periodic is decidable. Lemma 6.3 implies that it is decidable for an elementary morphism and Theorem 6.5 below shows that it is true for every morphism.

Example 6.4 The morphism $\sigma: a \mapsto a, b \mapsto baab$, is elementary. Since $\sigma(aab) = (aab)^2$, the sequence $x = (aab)^\infty$ is a fixed point of $\sigma$.

The following was proved in [32] and also [21] for a primitive morphism (the result of [32] and [21] is formulated in terms of a fixed point, see the comments after the proof).

Theorem 6.5 It is decidable whether a morphism $\sigma: A^* \rightarrow A^*$ is aperiodic. The set of periodic points in $X(\sigma)$ is finite and their periods are effectively bounded in terms of $\sigma$.

Proof. By Proposition 6.2, it is enough to prove that one may decide whether a morphism $\sigma$ has a power with a periodic admissible fixed point. From Proposition 4.3, it is decidable whether a morphism has a periodic point with non-growing letters.

Thus it is enough to decide whether $\sigma$ has a power with a periodic admissible fixed point with some growing letters.

If $\sigma$ is elementary, it is enough to check whether $\sigma$ acts as a permutation of a set of points of period $p$ where, by Lemma 6.3 $p$ is bounded in terms of $\sigma$. To check this, we consider the set $F$ of primitive words of length dividing $p$ and look for cycles in the graph with edges $(u,v)$ with $u,v \in F$ and $\sigma(u)$ a power of $v$. Since these points are assumed to have at least one growing letter, such points will be in $X(\sigma)$. Indeed, if $a$ is such a letter, all factors of $x$ are factors of $\sigma^p(a)$.

Otherwise, write $\sigma = \alpha \circ \beta$ with $A^* \xrightarrow{\beta} B^* \xrightarrow{\alpha} A^*$ and $\text{Card}(B) < \text{Card}(A)$. Using induction on $\text{Card}(A)$, we may assume that we can check whether $\tau = \beta \circ \alpha$ is aperiodic. Let $x$ be a fixed point of $\sigma$ and set $y = \beta(x)$. Then

$$\tau(y) = \beta \circ \sigma(x) = \beta(x) = y$$

and thus $y$ is a fixed point of $\beta \circ \alpha$. Moreover, $x$ is periodic if and only if $y$ is periodic (because $y = \beta(x)$ and $x = \alpha(y)$) and $x$ has a growing letter for $\sigma$ if and only if $y$ has a growing letter for $\tau$ (because $\beta \circ \sigma^n = \tau^n \circ \beta$ for all $n \geq 1$). Thus $x \in X(\sigma)$ if and only if $y \in X(\tau)$. Since, by induction hypothesis, we can test the existence of $y$, the same is true for $x = \alpha(y)$. Using again induction on $\text{Card}(A)$, we conclude that the number of periodic points is finite and that their periods are bounded in terms of $\sigma$.

Example 6.6 Let $\sigma: a \mapsto ab, b \mapsto ac, c \mapsto ac$. Then $\sigma^\infty(c \cdot a) = (abac)^\infty$, which is periodic of period $4 > \text{Card}(A)$. The morphism $\sigma$ is not elementary since
σ = α \circ β with α: d \mapsto ab, e \mapsto ac and β: a \mapsto d, b \mapsto e, c \mapsto e. The point y = β((abac)^\infty) is (de)^\infty.

We note the following corollary, concerning the case of an admissible fixed point.

**Corollary 6.7** It is decidable whether a one-sided (resp. two-sided) admissible fixed point of a morphism σ: A^* → A^* is periodic.

**Proof.** This follows from Theorem 6.5 since the period of such a fixed point is bounded in terms of σ.

Note that when the substitution is primitive, Corollary 6.7 is equivalent to Theorem 6.5 since, the shift being in this case minimal, it is periodic if and only if every point is periodic. The result of [31] and [21] is actually formulated as the decidability of the eventual periodicity of an admissible one-sided fixed point σ^\omega(a) of a primitive morphism. Note also that this statement was generalized in [14] as the decidability of the eventual periodicity of the image φ(σ^\omega(u)) by a morphism φ of an admissible one-sided fixed point of a morphism σ.

We now prove the following complement to Theorem 6.5.

**Theorem 6.8** It is decidable whether the shift X(σ) generated by a morphism σ is periodic.

To prove Theorem 6.8 we use the following characterization of periodic substitution shifts.

**Lemma 6.9** The shift X(σ) is periodic if and only if every admissible quasi-fixed point of a power of σ is periodic.

**Proof.** The condition is clearly necessary. Conversely, let σ: A^* → A^* be such that X(σ) is not periodic. Let x ∈ X(σ) be a sequence that is not periodic. We distinguish three cases.

Case 1. Suppose that x is formed of non-growing letters. Since there is a finite number of orbits of sequences formed of non-growing letters, and in particular of non-periodic ones, there exists at least one such orbit stable under a power of σ. Thus there is a quasi-fixed point of a power of σ which is not periodic.

Case 2. There is exactly one growing letter a in x. If some σ^n(x) has more than one growing letter, we change x for σ(x) and go to Case 3. Otherwise, replacing σ by some power, a by another growing letter, and x by its image by this power, we may assume that σ(a) = uav with u, v non-growing. We may also assume that for every n ≥ 1, x[−n,n] is a factor of some σ^m(b) with b a growing letter. By choosing n large enough and replacing x by some σ^n(x), we may assume that b = a. This implies that u, v are both non-erasable. Then σ^{n+1}(a) is, for i = |ua|, a non-periodic quasi-fixed point of a power of σ.

Case 3. There are several growing letters in x. Let aub be a factor of x such that a, b are growing letters and u a non-growing word. We may assume,
replacing $\sigma$ by some power $x$ by some $\sigma^n(x)$ and $a, b$ by other growing letters, that $\sigma(a) = paq$, $\sigma(q) = q$, $\sigma(u) = u$, $\sigma(b) = rbs$ and $\sigma(r) = r$. Then $\sigma^n(aq\cdot urb)$ is a non-periodic fixed point of a power of $\sigma$.

Proof of Theorem 6.8. By Theorem 5.21 there is a finite and computable number of orbits of admissible quasi-fixed points of a power of $\sigma$. Since the period of those which are periodic is effectively bounded by Theorem 6.5, the result follows from Lemma 6.9.

7 Erasable letters

Let $\sigma : B^* \to B^*$ and $\phi : B^* \to A^*$ be two morphisms. We say that the pair $(\sigma, \phi)$ is admissible if $\phi(x)$ is a two-sided infinite sequence for every $x \in X(\sigma)$. When $(\sigma, \phi)$ is admissible, we let $X(\sigma, \phi)$ denote the closure under the shift of $\phi(X(\sigma))$.

A shift $X(\sigma, \tau)$, for an admissible pair $(\sigma, \tau)$, is called a morphic shift. A substitution shift $X(\sigma)$ is also called a purely morphic shift.

As a closely related notion, a morphic sequence is a right-infinite sequence $\tau = \phi(\sigma^\omega(u))$ where $\sigma : B^* \to B^*$ is a morphism right-prolongable on $u$ and $\phi : B^* \to A^*$ is a morphism.

The following statement is a result essentially due to Cobham [11] (see also [31], [9], [14], [24] and [10] and [18, Exercise 7.38]). It is also a variant of a result known as Rauzy Lemma (see Proposition 10.5 below). It is a normalization result, showing that, for a shift generated by a morphic sequence, one can replace a pair $(\tau, \phi)$ by an equivalent pair $(\zeta, \theta)$ with $\zeta$ non-erasing and $\theta$ letter-to-letter (see Figure 7.1). Our statement is slightly more general than the previous ones, as the fixed point is of the form $\sigma^\omega(u)$ for $u$ a word and not a letter.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure7.1.pdf}
\caption{The pairs $(\tau, \phi)$ and $(\zeta, \theta)$}
\end{figure}

**Theorem 7.1** Let $\tau : B^* \to B^*$ be a morphism right-prolongable on $u \in B^+$ and let $y = \tau^\omega(u)$. Let $\phi : B^* \to A^*$ be such that $x = \phi(y)$ is an infinite sequence. There exist an alphabet $C$, a morphism $\gamma : B^* \to C^*$, a non-erasing morphism $\zeta : C^* \to C^*$, right-prolongable on $v = \gamma(u)$ and a letter-to-letter
morphism $\theta : C^* \to A^*$ such that

$$\phi \circ \tau^m = \theta \circ \gamma$$

(7.1)

and

$$\zeta \circ \gamma = \gamma \circ \tau^n$$

(7.2)

for some $m, n \geq 1$. As a consequence, $x = \theta(\zeta^a(v))$.

For this, we use the following lemma (see [9]). It is not a constructive statement since the sequence $(n_i)$ is not shown to be computable. This weakness was overcome in [14] and [24] independently.

**Lemma 7.2** Let $B$ be a finite set and for each $b \in B$, let $(\ell_n(b))_{n \geq 0}$ be a sequence of natural integers with $(\ell_n(a))$ unbounded for some $a \in B$. There is an infinite sequence $n_0 < n_1 < \ldots$ such that for each $b \in B$, the sequence $(\ell_{n_i}(b))$ is either strictly increasing or constant, with $(\ell_{n_i}(a))$ strictly increasing.

**Proof.** The proof is by induction on $\text{Card}(B)$. The statement is clear if $B = \{a\}$. Assuming that it holds for $B \setminus \{b\}$, either the sequence $(\ell_{n_i}(b))$ is bounded and we can find a subsequence $m_i$ of the $n_i$ such that $(\ell_{m_i}(b))$ is constant, or $(\ell_{n_i}(b))$ is unbounded and we can find a subsequence $m_i$ of the $n_i$ such that $(\ell_{m_i}(b))$ is strictly increasing. \[\blacksquare\]

**Proof of Theorem 7.1** By Lemma 5.3 and changing $\tau$ for some of its powers, we may assume that $u = \text{pra}$ with $\tau(p) = p$ and $\tau(r) = \varepsilon$ and $a \in B$ growing.

We first prove that we can change the pair $(\tau, \phi)$ into a pair $(\tau', \phi')$ such that

$$|\phi' \circ \tau'(b)| \geq |\phi'(b)|$$

(7.3)

for every $b \in B$ and with strict inequality when $b = a$.

Since, by Lemma 5.11 \(\lim_{n \to +\infty} |\tau^n(a)| = +\infty\), we can apply Lemma 7.2 to the sequence $\ell_n(b) = |\phi \circ \tau^n(b)|$. Thus, there are $m, n \geq 1$ such that $|\phi \circ \tau^n(b)| \leq |\phi \circ \tau^{m+n}(b)|$ for every $b \in B$ with strict inequality if $b = a$. Set $\tau' = \tau^n$, $\phi' = \phi \circ \tau^n$. Then

$$|\phi' \circ \tau'(b)| = |\phi \circ \tau^m \circ \tau^n(b)| = |\phi \circ \tau^{m+n}(b)|$$

$$\geq |\phi \circ \tau^m(b)| = |\phi'(b)|$$

for every $b \in B$ with strict inequality when $b = a$. This proves (7.3).

We define an alphabet $C = \{b_p \mid b \in B, 1 \leq p \leq |\phi'(b)|\}$, a map $\theta : C \to A$ by $\theta(b_p) = (\phi'(b))_p$ where $(\phi'(b))_p$ denotes the $p$-th letter of $\phi'(b)$ and a map $\gamma : B \to C^*$ by $\gamma(b) = b_1 b_2 \cdots b_{|\phi'(b)|}$ with $\gamma(b) = \varepsilon$ if $\phi'(b) = \varepsilon$. In this way, we have $\theta \circ \gamma = \phi'$ and thus (7.4).

For every $b \in B$, we have

$$|\gamma \circ \tau'(b)| = |\phi' \circ \tau'(b)| \geq |\phi'(b)|$$
Then we define a non-erasing morphism $\zeta: C^* \to C^*$ by $\zeta(b_p) = w_p$. We have by construction $\zeta \circ \gamma = \gamma \circ \tau'$ and thus (7.2). Since $\zeta \circ \gamma(y) = \gamma \circ \tau'(y) = \gamma(y)$, the sequence $\gamma(y)$ is a fixed point of $\zeta$. Since $\zeta(v) = \zeta \circ \gamma(u) = \gamma \circ \tau'(u)$, the sequence $\zeta(v)$ begins with $v$. Moreover, since $|\phi'(r)| \leq |\phi \circ \tau'(r)| = 0$, we have

$$|\zeta(v)| = |\gamma \circ \tau'(u)| = |\phi' \circ \tau'(\text{pra})| = |\phi'(p)| + |\phi' \circ \tau'(a)|$$

Thus $\zeta$ is right-prolongable on $v$. Finally, we have

$$\theta(\zeta^\omega(v)) = \theta(\zeta^\omega(\gamma(u))) = \theta \circ \gamma(\tau^\omega(u)) = \phi'(y) = \phi \circ \tau^m(y) = \phi(y) = x.$$ 

We illustrate the proof of Theorem 7.1 with the following example.

**Example 7.3** Consider the morphism $\tau: a \mapsto abc, b \mapsto ac, c \mapsto \varepsilon$ and the morphism $\phi$ which is the identity on $A = B = \{a, b, c\}$. The sequence $\tau^\omega(a)$ is the Fibonacci sequence with letters $c$ inserted at the division points.

We first replace the pair $(\tau, \phi)$ by the pair $(\tau, \tau)$ in such a way that $|\tau \circ \sigma(e)| \geq |\tau(e)|$ for every letter $e \in B$. We define $C = \{a_1, a_2, a_3, b_1, b_2\}$, $\gamma(a) = a_1a_2a_3$, $\gamma(b) = b_1b_2$ and $\gamma(c) = \varepsilon$. The map $\theta: C \to A$ is defined as shown below.

| b | a | a | a | b | b |
|---|---|---|---|---|---|
| b | a | a | a | b | b |

Next, we set $\zeta(a_1) = \zeta(b_1) = a_1a_2$ and $\zeta(a_2) = a_3b_1$, $\zeta(b_2) = a_3$, $\zeta(a_3) = b_2$. Then $\zeta \circ \gamma = \gamma \circ \tau$ as one may verify.

Note that a more economical solution is $\zeta: a \mapsto ab, b \mapsto ca$ and $c \mapsto c$, since $\tau^\omega(a) = \zeta^\omega(a)$ as one may verify easily.

**Corollary 7.4** For every morphic shift $X = \mathcal{X}(\sigma, \phi)$ such that $\mathcal{X}(\sigma)$ is minimal, there is an admissible pair $(\zeta, \theta)$, with $\zeta$ non-erasing and $\theta$ letter-to-letter, such that $X = \mathcal{X}(\zeta, \theta)$.

**Proof.** Let $y$ be an admissible one-sided fixed point of a power of $\sigma$. By Proposition 5.13 either $y$ is eventually periodic or $y = \sigma^n u$ where $\sigma^n$ is right-prolongable on $u$. If $y$ is eventually periodic, it has to be periodic since $\mathcal{X}(\sigma)$ is minimal. Then $\phi(y)$ is periodic and there exists a non-erasing morphism $\zeta$ such that $\mathcal{X}(\zeta) = X$. Otherwise, By Theorem 7.1, there exists a pair $(\zeta, \theta)$ with $\zeta$ non-erasing and $\theta$ letter-to-letter such that $\phi(y) = \theta(\zeta^\omega(v))$. Since $\mathcal{X}(\sigma)$ is minimal, $X = \mathcal{X}(\sigma, \phi)$ is also minimal and thus $X$ is generated by $\phi(y)$, which implies our conclusion. 


It is possible to prove Corollary 7.4 without the hypothesis that \( X(\sigma) \) is minimal (see [2]).

The following example is from [9]. It shows that, in Theorem 7.1, even when the morphism \( \theta \) is the identity, one cannot dispense with the morphism \( \phi \).

**Example 7.5** Let \( \sigma: a \mapsto abccc, b \mapsto baccc, c \mapsto \varepsilon \) and let \( x = \sigma^2(a) \). We will show that there is no non-erasing morphism \( \sigma' \) such that \( X(\sigma') = X(\sigma) \).

Assume the contrary. Let \( \chi: \{a, b, c\}^* \rightarrow \{a, b\}^* \) be the morphism which erases \( c \), that is, \( \chi: a \mapsto a, b \mapsto b, c \mapsto \varepsilon \). Then \( \chi \circ \sigma = \mu \circ \chi \) where \( \mu: a \mapsto ab, b \mapsto ba \) is the Thue-Morse morphism.

For every \( u \in \mathcal{L}(\sigma) \), the word \( \sigma'(u) \) is in \( \mathcal{L}(\sigma) \) and thus \( \chi \circ \sigma'(u) \) is a factor of the Thue-Morse sequence. Set \( A = \{a, b, c\} \). Since \( \sigma'(\mathcal{L}(\sigma)) \subseteq \mathcal{L}(\sigma) \), we have \( \sigma'(A) \subseteq A^* \).

Let us first show that \( \sigma'(c) = c \). Indeed, since \( \chi \circ \sigma'(ccc) = (\chi \circ \sigma'(c))^3 \) and since \( \mathcal{L}(\mu) \) does not contain cubes (see for example [25, Corollary 2.2.4]), we have \( \chi \circ \sigma'(c) = \varepsilon \). Since \( \mathcal{L}(\sigma) \cap c^2 = \{\varepsilon, cc, ccc\} \), this forces \( \sigma'(c) = c \).

Next, \( \sigma'(a) \) (resp. \( \sigma'(b) \)) cannot begin or end with \( c \). Indeed, if for example, \( \sigma'(a) = cw \), then \( \sigma'(ccca) \) begins with \( c^4 \), which is impossible.

One of \( \sigma'(a), \sigma'(b) \) contains \( c \) since otherwise \( \{\sigma'(ab), \sigma'(ba)\} = \{ab, ba\} \) which contradicts the fact that some power of \( \sigma' \) has a two-sided infinite fixed point (since \( X(\sigma) = X(\sigma') \) is aperiodic, this fixed point is aperiodic and thus \( \sigma' \) has at least one growing letter).

If \( \sigma'(a) \) does not contain \( c \), then \( \sigma'(a) = a \) or \( \sigma'(a) = b \). In the first case, \( \sigma'(b) \) begins with \( bc \) and ends with \( cc \) and thus \( \sigma'^2(b) \) has a factor \( cccc \), which is impossible. The second case is similar.

Similarly, \( \sigma'(b) \) contains \( c \).

Thus, either \( \sigma'(a) \) and \( \sigma'(b) \) are both in \( \{a, b\}ccc\{a, b\}A^* \cap A^*\{a, b\}ccc\{a, b\} \) or both in \( \{a, b\}^2cccA^* \cap A^*ccc\{a, b\}^2 \). In the first case,

\[
\sigma'^2(a) \in \sigma'\{a, b\}ccc\{a, b\}A^* \subseteq \sigma'\{a, b\}cccA^* \subseteq A^*\{a, b\}ccc\{a, b\}cccA^*,
\]

which is impossible since two factors \( c^2 \) must be separated by a word of length 2. In the second case, \( \sigma'(ab) \in A^*\{a, b\}ccc\{a, b\}ccc\{a, b\}A^* \), which is also impossible. Thus, such a \( \sigma' \) cannot exist.

We do not know, for an endomorphism \( \tau \), whether it is decidable if there is a non-erasing morphism \( \zeta \) such that \( X(\tau) = X(\zeta) \).

### 8 Recognizability

Let \( \sigma: A^* \rightarrow B^* \) be a morphism and let \( X \) be a shift space on \( A \). The morphism \( \sigma \) is **recognizable** in \( X \) for a point \( y \in B^2 \) if \( y \) has at most one centered \( \sigma \)-representation \( (x, k) \) with \( x \in X \). It is **fully recognizable** in \( X \) if it is recognizable in \( X \) for every point \( y \in B^2 \).

The following result is proved in [3].
Theorem 8.1 Every morphism \( \sigma \) is recognizable in \( X(\sigma) \) for aperiodic points.

This result has a substantial history. It was first proved by Mossé [28, 29] that every aperiodic primitive morphism is recognizable in \( X(\sigma) \). Later, this important result was generalized in [5] where it is shown that every aperiodic non-erasing morphism \( \sigma \) is recognizable in \( X(\sigma) \). As a further extension, it is proved in [4] that a non-erasing morphism \( \sigma \) is recognizable in \( X(\sigma) \) at aperiodic points. Finally, the last result was extended in [3] to arbitrary morphisms (see also the more recent [7] for a further extension).

We prove the following result.

Theorem 8.2 A morphism \( \sigma \) is fully recognizable in \( X(\sigma) \) if and only if every periodic point in \( X(\sigma) \) is formed of non-growing letters. Consequently, it is decidable whether \( \sigma \) is fully recognizable in \( X(\sigma) \).

We first prove the following lemma, which does not seem to have been proved before.

Lemma 8.3 Let \( \sigma : A^* \to A^* \) be a morphism. For every aperiodic point \( x \in X(\sigma) \), the point \( \sigma(x) \) is aperiodic.

Proof. Assume that for some \( x \in X(\sigma) \), the point \( y = \sigma(x) \) is periodic. By Proposition 6.2, \( y \) is a fixed point of a power of \( \sigma \). We may assume that it is a fixed point of \( \sigma \).

Set \( x = x^{(1)} \). For every \( n \geq 1 \), let \( (k_n, x^{(n+1)}) \) be a \( \sigma \)-representation of \( x^{(n)} \) (its existence follows from [3, Proposition 5.1]).

Since \( \sigma^n(x^{(n)}) \) is a shift of \( y \), we have for every \( k, n \geq 1 \)
\[
\sigma^n(L_k(x^{(n)})) \subset L(y). \tag{8.1}
\]

But since \( \sigma(y) = y \), we have also \( \sigma(L(y)) \subset L(y) \). Thus, we have
\[
\sigma^m(L_k(x^{(n)})) \subset L(y). \tag{8.2}
\]

for all \( m \geq n \). The set
\[
R_k = \bigcup_{n \geq 1} L_k(x^{(n)}) \tag{8.3}
\]
is finite. Thus, for every \( k \), there is an integer \( m(k) \) such that
\[
\sigma^n(R_k) \subset L(y) \tag{8.4}
\]
for every \( n \geq m(k) \).

Now for every \( w \in L_k(x) \) and every \( n \geq 1 \), there is some \( w_n \in L(x^{(n+1)}) \) such that \( w \) is a factor of \( \sigma^n(w_n) \). We may assume that each \( w_n \) is chosen of minimal length.

The lengths of the words \( w_n \) are bounded. Indeed, if \( |w_n| \geq k \), then, since \( |w| = k \) and since \( w_n \) is chosen of minimal length, the word \( w_n \) has at least \( |w_n| - k \) erasable letters. But a word of \( L(\sigma) \) has a bounded number of consecutive erasable letters because otherwise there would exist an infinite sequence.
of erasable letters in $X(\sigma)$, which is impossible. Thus we obtain that $|w_n| \leq K$ for some constant $K \geq 1$.

For $n \geq m(K)$, we have by (8.3), $\sigma^n(w_n) \subset \mathcal{L}(y)$. This shows that $w$ is in $\mathcal{L}(y)$. Since $\mathcal{L}_k(x) \subset \mathcal{L}(y)$ for all $k$, we conclude that $x$ is periodic. □

Proof of Theorem 8.2. By Theorem 8.1 $\sigma$ is fully recognizable in $X(\sigma)$ if and only if it is recognizable in $X(\sigma)$ for every periodic point of $X(\sigma)$. By Proposition 6.2, every periodic point in $X(\sigma)$ is a fixed point of a power of $\sigma$. By Theorem 6.14 there is a finite number of orbits of fixed points of a power of $\sigma$ and these orbits form a computable set. Thus, there is a finite number of periodic points in $X(\sigma)$, which form a computable set. Changing $\sigma$ for some of its powers, we may assume that every periodic point is a fixed point of $\sigma$ (the exponent is bounded in terms of the size of $\sigma$ by Lemma 8.11). Let $x$ be one of them. Set $x = w^\infty$ with $w$ a primitive word (the length of $w$ is bounded by Theorem 6.5). Then $\sigma(w) = w^n$ for some $n = n(x) \geq 1$ by Lemma 8.2. Let $(y, k)$ be a $\sigma$-representation of $x$ with $y \in X(\sigma)$. By Lemma 8.3, $y$ is periodic and thus $\sigma(y) = y$. This forces $k$ to be a multiple of $|w|$. Thus the only possible centered $\sigma$-representations $(y, k)$ of $x$ with $y \in X(\sigma)$ are of the form $(x, k)$ with $k = 0, |w|, \ldots, (n-1)|w|$. Thus, $\sigma$ is recognizable in $X(\sigma)$ if and only if $n(x) = 1$ for every periodic point $x \in X(\sigma)$. Since $n(x) = 1$ if and only if $x$ is formed of non-growing letters, this proves the statement. □

9 Irreducible substitution shifts

Let us say that a morphism $\sigma$ is irreducible if $X(\sigma)$ is irreducible.

The following result shows that it is decidable whether a morphism such that $\mathcal{L}(\sigma) = \mathcal{L}(X(\sigma))$ is irreducible.

Theorem 9.1 A morphism $\sigma: A^* \rightarrow A^*$ such that $\mathcal{L}(\sigma) = \mathcal{L}(X(\sigma))$ is irreducible if and only if there is a letter $a \in A$ such that

(i) its strongly connected component in $G(\sigma)$ is a non-trivial aperiodic graph not reduced to a cycle.

(ii) There is an $n \geq 1$ such that $|\sigma^n(a)|_b \geq 1$ for every letter $b \in A$.

Proof. Set $X = X(\sigma)$. If the condition is satisfied, there are words in $\mathcal{L}(X)$ with an arbitrary large number of occurrences of $a$ and thus there is some $t \in \mathcal{L}(X)$ such that $a$ appears twice in $t$. If $u, v \in \mathcal{L}(X)$ there are $b, c \in A$ such that $u, v$ appear in $\sigma^n(b), \sigma^n(c)$. By condition (ii), $b$ and $c$ appear in $\sigma^n(a)$ for all $n$ large enough. Then a word of the form $uwv$ appears in $\sigma^{n+q}(t)$ for $q \geq m, p$ and thus there is a word $w$ such that $uwv$ is in $\mathcal{L}(X)$. We conclude that $\sigma$ is irreducible.

Conversely, assume $\sigma$ irreducible.

Let $a, b$ be two letters of $\mathcal{L}(\sigma)$. Since $\sigma$ is irreducible, there is a point in $X(\sigma)$ containing a factor $w$ containing both the letters $a$ and $b$. There is an integer
n such that w is a factor of some $\sigma^n(c)$ for some letter c. Thus a and b are connected in $G(\sigma)$. Thus we may assume that $G(\sigma)$ is connected.

Let a be a letter such that the strongly connected component $C(a)$ of a in $G(\sigma)$ is minimal (that is, a is only accessible from an element of $C(a)$).

The component $C(a)$ cannot be trivial since otherwise a is not in $L(X)$. It cannot either be reduced to a cycle. Indeed, in this case, the words of $L(\sigma)$ have at most one occurrence of a. But since $L(X) = L(\sigma)$, a is in $L(X)$. Since X is irreducible, $L(X)$ contains some $aua$, a contradiction.

Thus we can assume that $C(a)$ is not reduced to a cycle. The component $C(a)$ is aperiodic. Indeed, suppose that $C(a) = C_0 \cup C_1 \cup \ldots \cup C_{p-1}$ with all edges going from $C_i$ to $C_{i+1}$ with the indices taken modulo p. For $b,c \in C(a)$, there is a word u such that $buc \in L(X)$. This implies that there is some $d \in C(a)$ such that $buc$ appears in $\sigma^n(d)$ for some $n \geq 1$. But then b,c are in the same $C_i$, which shows that $p = 1$.

If b is another letter such that its strongly connected component $C(b)$ is also minimal, since $aub$ is also in $L(X)$ for some word u, this forces $C(b) = C(a)$. Thus every letter is accessible from a. Since $C(a)$ is aperiodic, there is a path from a to a given letter of all lengths large enough. Thus conditions (i) and (ii) are satisfied.

Note that all conditions of Theorem 9.1 are decidable. Indeed, the condition $L(\sigma) = L(X(\sigma))$ is decidable by Lemma 3.20. Conditions (i) and (ii) are easily verified on the graph $G(\sigma)$.

The following example shows that Theorem 9.1 is false when $L(\sigma) \neq L(X(\sigma))$. We have no proof that irreducibility is decidable in this more general case.

**Example 9.2** The morphism $\sigma : a \mapsto ab, b \mapsto b$ is irreducible and even minimal since $X(\sigma) = b^\infty$ but the strongly connected components of a, b are both reduced to a cycle.

## 10 Minimal substitution shifts

The following statement was proved in [12] (see also [18] Proposition 6.3).

**Theorem 10.1** A morphism $\sigma : A^* \to A^*$ such that $L(X(\sigma)) = L(\sigma)$ is minimal if and only if there is a growing letter $a \in L(\sigma)$ which appears with bounded gaps in $L(\sigma)$ and for every $b \in L(\sigma)$ there is $n \geq 1$ such that $|\sigma^n(a)|_b \geq 1$.

**Proof.** The condition is sufficient. Consider indeed $w \in L(\sigma)$. Let $b \in A$ be such that $w$ appears in $\sigma^m(b)$.

Now there is $n \geq 1$ such that $b$ appears in $\sigma^n(a)$. Thus $w$ appears in $\sigma^{n+m}(a)$. Since $a$ occurs with bounded gaps in $L(\sigma)$, the same is true for $w$. This shows that $X(\sigma)$ is uniformly recurrent and thus minimal.
Conversely, assume that $\sigma$ is minimal. Let $a \in L(\sigma)$ be a growing letter. Since $X(\sigma)$ is minimal, it is uniformly recurrent and thus every $b \in A$ appears with bounded gaps in $L(\sigma)$. Thus, the condition is satisfied.

We give an example illustrating Theorem 10.1.

**Example 10.2** The morphism $\sigma$: $0 \mapsto 0010, 1 \mapsto 1$ is known as the binary Chacon morphism. The condition in Theorem 10.1 is clearly satisfied by the letter $0$ and thus $\sigma$ is minimal.

Theorem 10.1 allows us to prove the following result. It is actually related to the result of [15] which states that it is decidable whether the image by a morphism $\phi$ of an admissible one-sided fixed point of a morphism $\sigma$ is uniformly recurrent.

**Theorem 10.3** It is decidable whether a morphism $\sigma$ such that $L(\sigma) = L(X(\sigma))$ is minimal.

Proof. For each growing letter $a \in L(\sigma)$, condition is decidable. Indeed, one has $|\sigma^n(a)|_b \geq 1$ for some $n$ if and only if $b$ is accessible from $a$ in $G(\sigma)$ and $a$ appears with bounded gaps in $L(\sigma)$ if and only if for every $b \in A$, the language $L(\sigma) \cap (A \setminus \{a\})^*$ is finite, which is decidable by Lemma 3.15.

Let us analyze the time complexity of the above algorithm.

For each growing letter $a \in L(\sigma)$, the condition of Theorem 10.1 is decidable. Indeed, one has $|\sigma^n(a)|_b \geq 1$ for some $n$ if and only if $b$ is accessible from $a$ in $G(\sigma)$. A letter $a \in L(\sigma)$ does not appear with bounded gaps in $L(\sigma)$ if and only if $L(\sigma) \cap (A \setminus \{a\})^*$ is infinite. This is the case if and only if there is a growing letter $b$ such that there is no path from $b$ to $a$ in $G(\sigma)$. Thus, the condition can be checked in linear time in the size of $\sigma$.

**Primitive morphisms** The following was proved in [27, Theorem 2.1] for non-erasing morphisms. We show that the proof can be adapted to arbitrary morphisms. The result below is actually also a particular case of a more general result from [17, Theorem 3.1] which states that every shift $X(\sigma, \phi)$ is conjugate to a primitive substitution shift. We give nevertheless the proof in our case, which is substantially simpler (and already known to Fabien Durand).

**Theorem 10.4** If $\sigma$ is minimal, then $X(\sigma)$ is conjugate to $X(\tau)$ with $\tau$ primitive.

The proof uses the following variant of Theorem 7.1 which is called Rauzy Lemma in [18, Proposition 7.2.10] (see also [16, Proposition 23]). For the sake of completeness, we reproduce the proof. A non-erasing morphism $\phi: B^* \rightarrow A^*$ is said to be circular if the morphism $\phi$ is fully recognizable in $B^*$ (see [3] for an alternative definition).
Proposition 10.5  Let $\sigma : B^* \to B^*$ be a primitive morphism and let $\phi : B^* \to A^*$ be a circular morphism. Let $X = X(\sigma, \phi)$ be the closure of $\phi(X(\sigma))$ under the shift. There exist an alphabet $C$, a primitive morphism $\zeta : C^* \to C^*$ and a letter-to-letter morphism $\theta : C^* \to A^*$ such that $\theta$ is a conjugacy from $X(\zeta)$ onto $X$.

Proof. We may assume that we are not in the trivial case where $B = \{b\}$ and $\sigma(b) = b$. Then, since $\sigma$ is primitive, it is growing and, by substituting $\sigma$ by a power of itself, we may assume that $|\sigma(b)| \geq |\phi(b)|$ for all $b \in B$. As in the proof of Theorem 7.1 we define

$$C = \{b_p \mid b \in B, \ 1 \leq p \leq |\phi(b)|\}.$$ 

Next $\gamma(b) = b_1b_2 \cdots b_{|\phi(b)|}$ and $\theta(b_p) = \phi(b)_p$, where $\phi(b)_p \in A$ denotes the $p$-th letter of $\phi(b)$. Since $\phi$ is non-erasing, we have for every $b \in B$, $|\gamma \circ \sigma(b)| \geq |\sigma(b)| \geq |\phi(b)|$.

For $b \in B$ and $1 \leq p \leq |\phi(b)|$, we define

$$\zeta(b_p) = \begin{cases} \gamma(\sigma(b)_p) & \text{if } 1 \leq p < |\phi(b)| \\ \gamma(\sigma(b)|_{[|\phi(b)|, |\sigma(b)|]} & \text{if } p = |\phi(b)| \end{cases}$$

We have then by definition for every $b \in B$,

$$\zeta \circ \gamma(b) = \zeta(b_1 \cdots b_{|\phi(b)|}) = \gamma(\sigma(b)_1) \gamma(\sigma(b)_2) \cdots \gamma(\sigma(b)|_{[|\phi(b)|-1, |\sigma(b)|]} \gamma(\sigma(b)|_{[|\phi(b)|, |\sigma(b)|]} = \gamma \circ \sigma(b).$$

Thus $\zeta \circ \gamma = \gamma \circ \sigma$.

We claim that $\zeta$ is primitive. Let $n$ be an integer such that $b$ occurs in $\sigma^n(a)$ for all $a, b \in B$. Let $b_p$ and $c_q$ be in $C$. By construction, $\zeta(b_p)$ contains $\gamma(\sigma(b)_p)$ as a factor, thus $\zeta^{n+1}(b_p)$ contains $\zeta^n(\gamma(\sigma(b)_p)) = \gamma(\sigma^n(\sigma(b)_p))$ as a factor. By the choice of $n$, the letter $c$ occurs in $\sigma^n(\sigma(b)_p)$, thus $\gamma(c)$ is a factor of $\gamma(\sigma^n(\sigma(b)_p))$, and also of $\zeta^{n+1}(b_p)$. Since $c_q$ is a letter of $\gamma(c)$, $c_q$ occurs in $\zeta^{n+1}(b_p)$ and our claim is proved.

![Figure 10.1: The pairs $(\sigma, \phi)$ and $(\zeta, \theta)$](image)

Changing $\sigma$ for some of its powers, we may assume that some $x \in X(\sigma)$ is an admissible fixed point of $\sigma$. Since $\zeta \circ \gamma(x) = \gamma \circ \sigma(x) = \gamma(x)$, $y = \gamma(x)$ is a fixed point of $\zeta$. Since $\zeta$ is primitive, $X(\zeta)$ is minimal and thus $X(\zeta)$ is generated by $y$. Since $\theta(y) = \theta \circ \gamma(x) = \phi(x)$, we have $\theta(X(\zeta)) \subset X(\sigma, \phi)$. Since $X(\sigma, \phi)$ is minimal, this implies $\theta(X(\zeta)) = X(\sigma, \phi)$ (see Figure 10.1).
There remains to show that \( \theta \) is injective. Consider \( x \in X \) and \( z \in X(\zeta) \) such that \( x = \theta(z) \). Set \( z_0 = b_{p+1} \) with \( b \in B \) and \( 0 \leq p < |\phi(b)| \). Then \( b_0 = z_{-p-1} \) and thus we have \( z = S^p \gamma(y) \) for some \( y \in B^\mathbb{Z} \) and
\[
x = \theta(z) = \theta(S^p(\gamma(y))) = S^p(\theta(\gamma(y))) = S^p(\phi(y)).
\]
Since \( \phi \) is circular, the pair \((y,p)\) is unique and thus \( \theta \) is injective.

**Proof of Theorem 10.4.** If \( X(\sigma) \) is periodic, we have \( X(\sigma) = X(\tau) \) where \( \tau \) sends every letter to the primitive word \( w \) such that \( X(\sigma) = w^\infty \). Since \( \tau \) is primitive, the result is true.

Otherwise, since \( X(\sigma) \) is minimal, \( X(\sigma) \) is aperiodic. By Theorem 5.14 there is an admissible fixed point of the form \( x = \sigma^\omega(r \cdot \ell) \) where \( r, \ell \) are non-empty and \( r \ell \in L(\sigma) \). Let \( R(r \cdot \ell) \) be the set of non-empty words \( u \in L(\sigma) \) such that the word \( r u \ell \)

(i) begins and ends with \( r \ell \),

(ii) has only two occurrences of \( r \ell \), and

(iii) is in \( L(\sigma) \).

Note that every word \( u \) such that \( r u \ell \) begins and ends with \( r \ell \) and is in \( L(\sigma) \) belongs to \( R(r \cdot \ell)^* \), that is, is a concatenation of words of the set \( U = R(r \cdot \ell) \).

Since \( X(\sigma) \) is minimal, it is uniformly recurrent. It implies that the set \( U \) is finite. Moreover, it is a circular code. This means (see [18]) that every two-sided infinite sequence \( x \) has at most one factorization in terms of words of \( U \). Let \( \phi: B \to U \) be a bijection from an alphabet \( B \) onto \( U \). We extend \( \phi \) to a morphism from \( B^* \) into \( A^* \). Since \( X(\sigma) \) is minimal, there is an infinity of occurrences of \( r \ell \) in \( x \) at positive and at negative indices. Since \( U \) is a circular code, there is a unique \( y \in B^\mathbb{Z} \) such that \( \phi(y) = x \).

For every \( b \in B \), the word \( r(\sigma \circ \phi(b))\ell \) begins and ends with \( r \ell \). This implies that \( \sigma \circ \phi(b) \) is in \( U^* \) and thus there is a unique \( w \in B^* \) such that \( \sigma \circ \phi(b) = \phi(w) \). Set \( \tau(b) = w \). This defines a morphism \( \tau: B^* \to B^* \) characterized by
\[
\phi \circ \tau = \sigma \circ \phi
\]

Let \( k > 0 \) be an occurrence of \( r \cdot \ell \) (in the sense that \( x_{[0,k]} \ell \) ends with \( r \ell \)) large enough so that every word \( w \in R(r \cdot \ell) \) appears in the decomposition of \( x_{[0,k]} \), that is, every \( b \in B \) occurs in the finite word \( u \in B^+ \) defined by \( \phi(u) = x_{[0,k]} \). Let \( n \) be large enough so that \( |\sigma^n(\ell)| > k \). Let \( b, c \in B \). As above, \( x_{[0,k]} \) is a prefix of \( \sigma^n(\ell) \), which is a prefix of \( \sigma^n(\phi(b)) = \phi(\tau^n(b)) \). Thus \( u \) is a prefix of \( \tau^n(b) \), and \( c \) occurs in \( \tau^n(b) \), hence \( \tau \) is primitive.

Clearly, \( X(\sigma) \) is the closure under the shift of \( \phi(X(\tau)) \). Thus, by Proposition 10.3 \( X(\sigma) \) is conjugate to a primitive substitution shift.

A shift space \( X \) is called **linearly recurrent** if there is a constant \( K \) such that every word of \( L_n(\sigma) \) appears in every word of \( L_{Kn}(\sigma) \). It is well known that a primitive substitution shift is linearly recurrent (see [18 Proposition 2.4.24]).
A linearly recurrent shift is clearly minimal. The converse is true for substitution shifts by a result from \[12\] (see also \[18\] Proposition 7.3.2) in the case of a substitution \(\sigma\) such that \(L(X(\sigma)) = L(\sigma)\). A proof in the general case was given by \[38\]. We show how this can be derived from Theorem \[10.3\].

**Theorem 10.6** A non-empty substitution shift is minimal if and only if it is linearly recurrent.

**Proof.** Let \(\sigma: A^* \rightarrow A^*\) be a morphism. If \(X(\sigma)\) is minimal, it is conjugate to a primitive substitution shift \(X(\tau)\) by Theorem \[10.4\]. Thus \(X(\sigma)\) is linearly recurrent.

Conversely, if \(X(\sigma)\) is linearly recurrent then it is clearly uniformly recurrent and thus minimal.  

### Quasi-Minimal shifts

We now investigate the question of whether a substitution shift has a finite number of subshifts. Following the terminology introduced in \[35\] these shifts are called *quasi-minimal*.

First of all, observe that for every shift space \(X\) on the alphabet \(A\) the following properties are equivalent.

(i) \(X\) is quasi-minimal.

(ii) The number of distinct languages \(L(x)\) for \(x \in X\) is finite.

Indeed, for every subshift \(Y\) of \(X\), one has \(L(Y) = \bigcup_{y \in Y} L(y)\) and thus (ii) implies (i). Conversely, for every \(x \in X\), let \(\alpha(x) = \{y \in A^Z \mid L(y) \subset L(x)\}\). Then \(\alpha(x)\) is a subshift of \(X\) and \(\alpha(x) = \alpha(x')\) implies \(L(x) = L(x')\). Thus (i) implies (ii).

Next, if \(X\) is quasi-minimal, it has a finite number of minimal subshifts.

The following example shows that a shift may have a finite number of minimal subshifts without being quasi-minimal.

**Example 10.7** Let, for \(n \geq 1\),

\[ x_n = \cdots ba^{n+2}ba^{n+1}b \cdot a^nbna^{n+1}ba^{n+2}b \cdots \]

and let \(O(x_n)\) denote the orbit of \(x_n\). Let \(X = (\cup_{n \geq 1} O(x_n)) \cup a^\infty\). Then \(a^\infty\) is the only minimal subshift of \(X\) but each set \(O(x_n) \cup a^\infty\) is a subshift of \(X\).

The shifts with only one minimal subshift, as in the above example, have been called *essentially minimal* in \[23\].

The following result is proved in \[31\] Proposition 5.14] for non-erasing morphisms and was proved before in \[5\] Proposition 5.6] for the case of aperiodic non-erasing morphisms. It is interesting to note that, in this case, the number of minimal subshifts is at most Card(\(A\)).

**Proposition 10.8** Let \(\sigma: A^* \rightarrow A^*\) be a morphism. The shift \(X(\sigma)\) is quasi-minimal.
We use the following lemma (see [4] for the non-erasing case).

**Lemma 10.9** Let $\sigma : A^* \rightarrow A^*$ be a morphism and $x$ a point in $X(\sigma)$ such that $x$ has a $\sigma^n$-representation $(x^{(n)}, 0)$ with $x^{(n)} \in X(\sigma)$ for all $n \geq 0$. Then $x$ is a fixed point of some power $r$ of $\sigma$, where $r$ is bounded by a value depending only on $\sigma$.

**Proof.** Let us assume that $x$ contains a growing letter $a$ at some index $i_0 \geq 0$. By [3, Lemma 2], $L(\sigma)$ contains a finite number of erasable words. Thus there is an integer $K$ such that $x^{(n)}_{[0,K]}$ contains at least $i_0 + 1$ non-erasable letters for all $n$. It follows that $x^{(n)}_{[0,K]}$ contains a growing letter for all $n$. As a consequence, there is an index $0 \leq j \leq K$ and an infinite set of integers $E$ such that, for all $n \in E$, $x_j^{(n)} = b$, where $b$ is a growing letter, and $x_{[0,j)}^{(n)} = u$, where $u$ is a non-growing word. Let $i, p$ be the constants of Lemma 4.2. We can choose $p'$ a multiple of $p$ larger than $i$. Hence for each non-growing word $z$, $\sigma^{kp'}(z) = \sigma^{p'}(z)$ for every $k \geq 1$.

Hence we have for any $k \geq 1$, $\sigma^{kp'}(u) = \sigma^{p'}(u)$, $\sigma^{p'}(b) = wcw$, $\sigma^{p'}(c) = v'cw'$, $\sigma^{kp'}(v) = \sigma^{p'}(v)$, $\sigma^{kp'}(v') = \sigma^{p'}(v')$, where $c$ is a growing letter, $v, v'$ are non-growing words.

It follows that $v'$ is erasing and thus $u'$ is non-erasing. Hence $\sigma^{p'}(v') = \varepsilon$. Let $u' = \sigma^{p'}(u)$, $v'' = \sigma^{p'}(v)$.

Considering an infinite number of integers $n$ of $E$ such that $n = kp' + r$ for some fixed integer $0 \leq r < p'$, we get that $x_{[0,\infty)} = \sigma^{r}(u'v''v')\sigma^{r}(\sigma^{r}(c))$ and thus $x_{[0,\infty)}$ is a fixed point of $\sigma^{p'}$. Similarly, if $x$ contains a growing letter at some negative index $x_{(\infty,0)}$ is a fixed point of $\sigma^{p'}$.

Let us now assume that $x$ contains only non-growing letters at non-negative indices. Then $x^{(n)}$ contains only non-growing letters at non-negative indices for all $n$. Thus by Lemma 1.6 there are finite non-growing words $z, t^{(n)}$ such that for an infinite number of $n$, $x^{(n)} = \varepsilon$. We have $\sigma^{p'}(z) = z', \sigma^{p'}(z') = z', \sigma^{p'}(t^{(p')}) = t^{(p')}$ and $\sigma^{p'}(t^{(p')}) = t^{(p')}$.

Then $\sigma^{p'}(x_{[0,\infty)}) = x_{[0,\infty)}$. Similarly if $x$ contains only non-growing letters at negative indices, $\sigma^{p'}(x_{(-\infty,0)}) = x_{(-\infty,0)}$.

**Proof of Proposition 10.8** Assume $X(\sigma) \neq \emptyset$. We will define recursively two finite sequences of words $w^{(k)} \in L(\sigma)$ and shifts $Y_k$ for $0 \leq k \leq K$, and for each such $k$ and all $n \geq 0$, an infinite sequence of alphabets $A^{(k)}_n$ as follows. We set $w^{(0)} = \varepsilon$. For $k \geq 0$, when $w^{(k)}$ is defined, we define

$A^{(k)}_n = \{ a \in A \mid w^{(k)} \text{ is not a factor of } \sigma^n(a) \}$

$Y_k = \{ x \in X(\sigma) \mid x \text{ has a } \sigma^n\text{-representation in } X(\sigma) \cap (A^{(k)}_n)^\mathbb{Z} \text{ for all } n \geq 0 \}$

Thus $A^{(0)}_n = \emptyset, Y_0 = \emptyset$. Assume that $w^{(k)}, A^{(k)}_n$ and $Y_k$ are defined, we construct $w^{(k+1)}$ when $\operatorname{Card}\{L(x) \mid x \in X(\sigma) \setminus Y_k \} \geq 2$ as follows.
We choose \(x, y \in X(\sigma) \setminus Y_k\) such that \(L(x) \neq L(y)\). Up to an exchange of \(x, y\), we may assume that \(L(y)\) is not contained in \(L(x)\). We choose a word \(w^{(k+1)} \in L(y) \setminus L(x)\). Since \(y \in X(\sigma) \setminus Y_k\), there is an integer \(n \geq 0\) such that \(y\) has a \(\sigma^n\)-representation \((z, i)\) where \(z\) is not in \((A^{(k)}_n)^\mathbb{Z}\). Indeed every point in \(X(\sigma)\) has for every \(n \geq 0\), a \(\sigma^n\)-representation in \(X(\sigma)\) by \([2]\) Proposition 5.1. Hence, for some \(n, y\) has a \(\sigma^n\)-representation \((z, i)\) such that for some letter \(\alpha\) of \(z\), the word \(w^{(k)}\) is a factor of \(\sigma^n(\alpha)\) and thus \(y\) contains the factor \(w^{(k)}\). Thus we may choose \(w^{(k+1)} \in L(y) \setminus L(x)\) such that \(w^{(k)}\) is a factor of \(w^{(k+1)}\). As a consequence, we obtain \(A^{(k)}_n \subseteq A^{(k+1)}_n\).

Since \(x \in X(\sigma) \setminus Y_k\), there is an integer \(m \geq 0\) such that it has a \(\sigma^m\)-representation in \(X(\sigma)\) which is not in \((A^{(k)}_m)^\mathbb{Z}\). Then there exists a letter \(\alpha \in A \setminus A^{(k)}_m\) such that \(\sigma^m(\alpha) \in L(x)\). Hence, \(\sigma^m(\alpha)\) does not contain \(w^{(k+1)}\) as a factor and \(\alpha \in A^{(k+1)}\). Moreover, \(A^{(k)}_n \subseteq A^{(k+1)}\).

Now for any point \(z \in X(\sigma) \setminus Y_k\), there is an \(n \geq 0\) such that none of the \(\sigma^n\)-representation of \(z\) in \(X(\sigma)\) is in \((A^{(k)}_n)^\mathbb{Z}\). This holds for all sufficiently large \(n\) since \(\sigma(A^{(k)}_n)^\mathbb{Z} \subseteq (A^{(k)}_n)^\mathbb{Z}\).

It follows that for sufficiently large \(n\), we have \(A^{(k)}_n \subseteq A^{(k+1)}\). Moreover, since \(w^{(k+1)} \in L(y)\), we have \(A^{(k+1)}_n \subseteq A\) for all large \(n\). Hence for all \(n\) large enough, we have

\[
A^{(0)}_n \subseteq A^{(1)}_n \subseteq \cdots \subseteq A^{(k-1)}_n \subseteq A^{(k)}_n \subseteq A.
\]

As a consequence there is a value \(k = K \geq 0\) such that \(\text{Card}\{L(x) \mid x \in X(\sigma) \setminus Y_K\} \leq 1\).

Let us show that the sequence \(A^{(K)}_n\) is eventually periodic. Following the construction of Proposition 5.15 we define a finite automaton \(A = (Q, I, T)\) recognizing \(A^*w(K)A^*\) and the binary relation \(\varphi(w)\) on \(Q\) by

\[
\varphi(w) = \{(p, q) \in Q \times Q \mid p \xrightarrow{w} q\}.
\]

Let \(\psi_n = \varphi \circ \sigma^m\). There are positive integers \(m, p\) such \(\psi_m = \psi_{m+p}\). Then \(\psi_{m+r} = \psi_{m+p+r}\) for all \(r \geq 0\). Now a letter \(a\) belongs to \(A^{(K)}_m\) if and only if \(\psi_n(a)\) contains no pair \((i, t)\) with \(i \in I, t \in T\). Thus the sequence \(A^{(K)}_n\) is eventually periodic.

Let \(m, p \geq 1\) be such that \(A^{(K)}_m = A^{(K)}_{m+p}\). We may choose \(m\) multiple of \(p\). We define \(\sigma': A^{(K)}_m \to (A^{(K)}_m)^+\) as the restriction of \(\sigma^p\) to \(A^{(K)}_{m+p} = A^{(K)}_m\).

Let \(x \in Y_K \setminus X(\sigma')\). The point \(x\) has a \(\sigma^m\)-representation \((x^{(pm)}, k_{pn})\) in \(X(\sigma') \cap (A^{(K)}_m)^\mathbb{Z}\) for every \(n\) such that \(pm \geq m\). Since \(x \notin X(\sigma')\), there is an integer \(\ell\) such that \(x_{(-\ell, 0]}\) is not a factor of \(\sigma^r(\alpha)\) for all \(r \geq 0\) and all \(a \in A^{(k)}_m\). Thus there is a shift \(S^h(x)\) with \(-\ell \leq h \leq \ell\) such that \(S^h(x)\) has a \(\sigma^m\)-representation \((x^{(pm)}, 0)\) in \(X(\sigma') \cap (A^{(K)}_m)^\mathbb{Z}\) for \(n\) in some infinite set of positive integers \(E\). Thus \(S^h(x)\) has a \(\sigma^m\)-representation \((x^{(pm)}, 0)\) in \(X(\sigma)\) for all \(n \geq 0\).

By Lemma 10.9 \(x' = S^h(x)\) is a fixed point of \(\sigma^r\) for some \(r\) bounded by a value depending on \(\sigma\).
By Theorem 5.14 there is a finite number of orbits of points $x$ such that $\sigma^{pr}(x) = x$. Thus there is a finite number of languages $\mathcal{L}(x)$ corresponding to such points.

As a consequence there is a finite number $N_1$ of sets $\mathcal{L}(x)$ for $x \in Y_K \setminus X(\sigma')$. We thus have

$$\text{Card}\{\mathcal{L}(x) \mid x \in X(\sigma)\} \leq \text{Card}\{\mathcal{L}(x) \mid x \in Y_K\} + \text{Card}\{\mathcal{L}(x) \mid x \in X(\sigma')\}$$

Assume by induction hypothesis that $\text{Card}\{\mathcal{L}(x) \mid x \in X(\sigma')\}$ is a finite number $N_2$, then $\text{Card}\{\mathcal{L}(x) \mid x \in X(\sigma)\} \leq N_1 + N_2 + 1$. Thus $\text{Card}\{\mathcal{L}(x) \mid x \in X(\sigma)\}$ is finite.

**References**

[1] Jean-Paul Allouche and Jeffrey Shallit. *Automatic Sequences*. Cambridge University Press, Cambridge, 2003.

[2] Marie-Pierre Béal, Fabien Durand, and Dominique Perrin. *Substitution shifts*. 2023. in preparation.

[3] Marie-Pierre Béal, Dominique Perrin, and Antonio Restivo. Recognizability of morphisms. *Ergodic Theory Dynam. Systems*, page 1–25, 2023.

[4] Valérie Berthé, Wolfgang Steiner, Jörg M. Thuswaldner, and Reem Yassawi. Recognizability for sequences of morphisms. *Ergodic Theory Dynam. Systems*, 39(11):2896–2931, 2019.

[5] Sergey Bezuglyi, Jan Kwiatkowski, and Konstantin Medynets. Aperiodic substitution systems and their Bratteli diagrams. *Ergodic Theory Dynam. Systems*, 29(1):37–72, 2009.

[6] Véronique Bruyère, Georges Hansel, Christian Michaux, and Roger Villemaire. Logic and $p$-recognizable sets of integers. *Bulletin of the Belgian Mathematical Society - Simon Stevin*, 1(2):191 – 238, 1994. Corrigendum, *Bull. Belg. Math. Soc.* 1 (1994), 577.

[7] Marie-Pierre Béal, Dominique Perrin, Antonio Restivo, and Wolfgang Steiner. Recognizability in S-adic shifts. 2023. arXiv:2302.06258.

[8] Olivier Carton and Wolfgang Thomas. The monadic theory of morphic infinite words and generalizations. *Inf. Comput.*, 176(1):51–65, 2002.

[9] Julien Cassaigne and François Nicolas. Quelques propriétés des mots substitutifs. *Bull. Belg. Math. Soc. Simon Stevin*, 10:661–676, 2003.

[10] Émilie Charlier, Julien Leroy, and Michel Rigo. Asymptotic properties of free monoid morphisms. *Linear Algebra Appl.*, 500:119–148, 2016.
[11] Alan Cobham. On the Hartmanis-Stearns problem for a class of tag machines. In *IEEE Conference Record of Seventh Annual Symposium on Switching and Automata Theory*, 1968.

[12] David Damanik and Daniel Lenz. Substitution dynamical systems: characterization of linear repetitivity and applications. *J. Math. Anal. Appl.*, 321(2):766–780, 2006.

[13] Volker Diekert and Dalia Krieger. Some remarks about stabilizers. *Theoretical Computer Science*, 410(30):2935–2946, 2009. A bird’s eye view of theory.

[14] Fabien Durand. Decidability of the HD0L ultimate periodicity problem. *RAIRO Theor. Inform. Appl.*, 47(2):201–214, 2013.

[15] Fabien Durand. Decidability of uniform recurrence of morphic sequences. *Int. J. Found. Comput. Sci.*, 24(1):123–146, 2013.

[16] Fabien Durand, Bernard Host, and Christian Skau. Substitutional dynamical systems, Bratteli diagrams and dimension groups. *Ergodic Theory Dynam. Systems*, 19(4):953–993, 1999.

[17] Fabien Durand and Julien Leroy. Decidability of the isomorphism and the factorization between minimal substitution subshifts. *Discrete Analysis*, 2022.

[18] Fabien Durand and Dominique Perrin. *Dimension Groups and Dynamical Systems*. Cambridge University Press, 2022.

[19] Andrew Ehrenfeucht and Gregorz Rozenberg. Elementary homomorphisms and a solution of the DOL sequence equivalence problem. *Theoret. Comput. Sci.*, 7(2):169–183, 1978.

[20] David Hamm and Jeffrey Shallit. Characterization of finite and one-sided infinite fixed points of morphisms on free monoids. Technical report, School of Computer science, Univerisity of Waterloo, 1999. Technical Report CS-99-17.

[21] Tero Harju and Matti Linna. On the periodicity of morphisms on free monoids. *RAIRO Inform. Théor. Appl.*, 20(1):47–54, 1986.

[22] Tom Head and Barbara Lando. Fixed and stationary $\omega$-words and $\omega$-languages. In Gregorz Rozenberg and Arto Salomaa, editors, *The Book of L*, pages 147–156. Springer-Verlag, 1986.

[23] Richard H. Herman, Ian F. Putnam, and Christian F. Skau. Ordered Bratteli diagrams, dimension groups and topological dynamics. *Internat. J. Math.*, 3(6):827–864, 1992.

[24] Juha Honkala. On the simplification of infinite morphic words. *Theoretical Computer Science*, 410(8):997 – 1000, 2009.

51
[25] Douglas Lind and Brian Marcus. An Introduction to Symbolic Dynamics and Coding. Cambridge University Press, Cambridge, 1995. Second edition, 2021.

[26] M. Lothaire. Combinatorics on Words. Cambridge University Press, second edition, 1997. (First edition 1983).

[27] Gregory R. Maloney and Dan Rust. Beyond primitivity for one-dimensional substitution subshifts and tiling spaces. Ergodic Theory Dynam. Systems, 38:1086–1117, 2018.

[28] Brigitte Mossé. Puissances de mots et reconnaissabilité des points fixes d’une substitution. Theoret. Comput. Sci., 99(2):327–334, 1992.

[29] Brigitte Mossé. Reconnaissabilité des substitutions et complexité des suites automatiques. Bull. Soc. Math. France, 124(2):329–346, 1996.

[30] Hamoon Mousavi. Automatic theorem proving in Walnut, 2016.

[31] Jean-Jacques Pansiot. Hiérarchie et fermeture de certaines classes de tagsystèmes. Acta Informatica, 20:179–196, 1983.

[32] Jean-Jacques Pansiot. Decidability of periodicity for infinite words. RAIRO Inform. Théor. Appl., 20(1):43–46, 1986.

[33] Martine Queffélec. Substitution Dynamical Systems—Spectral Analysis, volume 1294 of Lecture Notes in Mathematics. Springer-Verlag, Berlin, second edition, 2010.

[34] Grzegorz Rozenberg and Arto Salomaa. The Mathematical Theory of L-systems. Academic Press, 1980.

[35] Ville Salo. Decidability and universality of quasiminimal subshifts. Journal of Computer and System Sciences, 89:288–314, 2017.

[36] Jeffrey Shallit. The Logical Approach To Automatic Sequences: Exploring Combinatorics on Words with Walnut, volume 482 of London Mathematical Society Lecture Notes. Cambridge, 2023.

[37] Jeffrey O. Shallit and Ming-wei Wang. On two-sided infinite fixed points of morphisms. Theor. Comput. Sci., 270(1-2):659–675, 2002.

[38] Takashi Shimomura. A simple approach to substitution minimal subshifts. Topology and its Applications, 260:203–214, 2019.

[39] Paul M. B. Vitányi. On the size of D0L languages. In Grzegorz Rozenberg and Arto Salomaa, editors, L Systems, pages 78–92. Springer Berlin Heidelberg, Berlin, Heidelberg, 1974.

[40] Helmut Wielandt. Unzerlegbare, nicht negative Matrizen. Math. Z., 52:642–648, 1950.
[41] Hisatoshi Yuasa. Invariant measures for the subshifts arising from non-primitive substitutions. *Journal d’Analyse Mathématique*, 102(1):143–180, 2007.