Hypertoric manifolds and hyperKähler moment maps

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To Simon Salamon on the occasion of his 60th birthday

Abstract We discuss various aspects of moment map geometry in symplectic and hyperKähler geometry. In particular, we classify complete hyperKähler manifolds of dimension 4n with a tri-Hamiltonian action of a torus of dimension n, without any assumption on the finiteness of the Betti numbers. As a result we find that the hyperKähler moment in these cases has connected fibres, a property that is true for symplectic moment maps, and is surjective. New examples of hypertoric manifolds of infinite topological type are produced. We provide examples of non-Abelian tri-Hamiltonian group actions of connected groups on complete hyperKähler manifolds such that the hyperKähler moment map is not surjective and has some fibres that are not connected. We also discuss relationships to symplectic cuts, hyperKähler modifications and implosion constructions.

1 Introduction

A symplectic structure on a (necessarily even-dimensional) manifold is a closed non-degenerate two-form. Several Riemannian and pseudo-Riemannian geometries have been developed over the years which give rise to a symplectic structure as part of their data. The most famous example is that of a hyperKähler structure, where we have a Riemannian metric g and complex structures I, J, K obeying the quaternionic multiplication relations, and
such that \( g \) is Kähler with respect to \( I, J, K \). We therefore obtain a triple \( (\omega_I, \omega_J, \omega_K) \) of symplectic forms.

One of the foundational results of symplectic geometry is the Darboux Theorem, which says that locally a symplectic structure can be put into a standard form \( \omega = \sum_{i=1}^{n} dp_i \wedge dq_i \). Many of the interesting questions in symplectic geometry are therefore global in nature, giving the subject a more topological flavour.

Geometries involving a metric of course do not have a Darboux-type theorem, because the metric contains local information through its curvature tensor. However, there is one area of symplectic geometry, that concerning moment maps where a rich theory has been developed for other geometries by analogy with the symplectic situation. In this paper we shall discuss some aspects of this, especially related to hypertoric manifolds, cutting and implosion.

2 Hypertoric manifolds

Let \( M \) be a hyperKähler manifold \( M \) of dimension \( 4n \). We say that an action of a group \( G \) on \( M \) is tri-symplectic if it preserves each of the symplectic forms \( \omega_I, \omega_J \) and \( \omega_K \). This is equivalent to \( G \) preserving both the metric \( g \) and each of the associated complex structures \( I, J \) and \( K \); so the action is isometric and tri-holomorphic. We will usually assume that \( G \) is connected and that the action is effective.

Because hyperKähler metrics are Ricci-flat, we have that if \( M \) is compact, then any Killing field \( X \) is parallel and so \( G \) is Abelian. As the complex structures are also parallel the distribution \( \mathbb{H}X = \text{Span}_\mathbb{R}\{X, IX, JX, KX\} \) is integrable and flat. Up to finite covers, an \( M \) is a product \( T^{4m} \times M_0 \), with \( G \) acting trivially on \( M_0 \).

Thus the interesting cases are when \( M \) is non-compact. From the Riemannian perspective is now natural to consider complete metrics. Note that by Alekseevskii & Kimel’fel’d, any homogeneous hyperKähler manifold is flat; such a manifold is necessarily complete, so its universal cover is \( \mathbb{R}^{4n} \) with the flat metric. Thus one should consider actions on \( M \) with orbits of dimension strictly less than \( 4n \).

One says that a tri-holomorphic action of \( G \) on \( M \) is tri-Hamiltonian if it is Hamiltonian for each symplectic structure, meaning that there are equivariant moment maps

\[
\mu_I, \mu_J, \mu_K : M \to \mathfrak{g}^*, \quad (1)
\]

\[
d\mu_X = X \omega_A, \quad (2)
\]
where $\mu_X^Y = \langle \mu_A, X \rangle$. Here we write $X$ both for the element of $\mathfrak{g}$ and the corresponding vector field $x \mapsto X_x$ on $M$. Also $\langle \alpha, X \rangle = \alpha(X)$ is the pairing between $\mathfrak{g}^*$ and $\mathfrak{g}$.

As each $X \in \mathfrak{g}$ preserves $\omega_A$, we have $0 = L_X \omega_A = X \cdot d\omega_A + d(X \omega_A) = d(\omega_A)$, so $X \omega_A$ is exact. Thus if $M$ is simply-connected then, equation (2), has a solution $\mu_A^X \in C^\infty(M)$ that is unique up to an additive constant.

### 2.1 Abelian actions

For an Abelian group $G$, equivariance of $\mu_A$ is the just the condition $L_X \mu_A^Y = 0$ for each $X, Y \in \mathfrak{g}$. But $L_X \mu_A^Y = X \cdot d\mu_A^Y = -\omega_A(X, Y)$ and $d(\omega_A(X, Y)) = L_Y (X \omega_A) = 0$. So $L_X \mu_A^Y$ is constant and the action is tri-Hamiltonian only if for each $A$ we have $G \cdot \mathfrak{g}$, where $G_x = \{X \mid X \in \mathfrak{g}\} \subset T_x M$. This last condition is equivalent to $\dim H_{G_x} = 4 \dim G_x$ for each $x \in M$.

**Proposition 1.** Suppose $G$ is a connected Abelian group that has an effective tri-Hamiltonian action on a connected hyperKähler manifold $M$ of dimension $4n$. Then the dimension of $G$ is at most $n$.

**Proof.** For each $x \in M$, the discussion above shows that the tri-Hamiltonian condition gives $\dim G_x \leq n$. We thus need to show that there is some $x \in M$ such that the map $\mathfrak{g} \to \mathcal{G}_x$, $X \mapsto X_x$, is injective.

Fix a point $x \in M$ such that $\dim \text{stab}_G(x)$ is the least possible. Note that $H = \text{stab}_G(x)$ is a compact subgroup of $\text{Sp}(n) \leq \text{SO}(4n)$. We may therefore $H$-invariantly write $T_x M = T_x (G \cdot x) \oplus W$ as an orthogonal direct sum of the tangent space to the orbit through $x$ and its orthogonal complement $W$. Now consider the map $F : G \times W \to M$ given by

$$F(g, w) = g \cdot (\exp_x w) = \exp_{gx} g_x w.$$  

At $(e, 0) \in G \times W$ this has differential $(F_x)_{(e,0)}(X, w) = X_x + w$ and $F(gh, (h_x)^{-1} w) = F(g, w)$ for each $h \in H$. Thus $F$ descends to a diffeomorphism from a neighbourhood of $(e, 0) \in G \times H W$ to a neighbourhood $U$ of $x \in M$ which is equivariant for the action of $\mathfrak{g}$. In particular $\text{stab}_G F(e, w) \subset H$ when $F(e, w) \in U$.

As $G$ acts effectively, we have for each $X \in \mathfrak{g} \setminus \{0\}$ there is some point $y$ with $X_y \neq 0$. But $M$ is Ricci-flat, so the Killing vector field $X$ is analytic, thus the set $\{y \in M \mid X_y \neq 0\}$ is open and dense.

If $\dim H = \dim \text{stab}_G(x)$ is non-zero, then there is a non-zero element $X \in \mathfrak{h}$. Now $X$ is non-zero at some point $y = F(g, w)$ of $U$, and $z = g^{-1} y = F(e, w)$ has $X_z = (g_x)^{-1} X_y \neq 0$ too, since $G$ is Abelian. So $\text{Lie} \text{stab}_G(z)$ is a subspace of $\mathfrak{h}$ not containing $X$. It follows that $\dim \text{stab}_G(z) < \dim \text{stab}_G(x)$, contradicting our choice of $x$. 

We conclude that \( \dim \mathfrak{h} = 0 \), so \( \text{stab}_G(x) \) is finite and \( g \mapsto G_x \) is a bijection. Thus \( \dim g \leq n \). \( \square \)

Any connected Abelian group of finite dimension is of the form \( G = \mathbb{R}^m \times T^k \) for some \( m, k \geq 0 \). If \( M \) is simply-connected then the tri-symplectic \( T^k \)-action is necessarily tri-Hamiltonian: each \( \mu_A^\gamma \) obtains its maximum on each \( T^k \)-orbit, and so \( L_X \mu_A^\gamma = X_\omega \mu_A^\gamma \) is zero at these points, and hence on all of \( M \). If \( \dim G = n \) and the \( G \)-action is tri-Hamiltonian, Bielawski \([5]\) proves that the \( \mathbb{R}^m \) factor acts freely and any discrete subgroup of \( \mathbb{R}^m \) acts properly discontinuously, so a discrete quotient of \( M \) has a tri-Hamiltonian \( T^n \)-action.

In general, a hyperK"ahler manifold of dimension 4 with a tri-Hamiltonian \( T^n \)-action is called hypertoric. Bielawski \([5]\) classified the hypertoric manifolds in any dimension under the assumption that \( M \) has finite topological type, meaning that the Betti numbers of \( M \) are finite. For \( \dim M = 4 \), this classification is extended to general hypertoric \( M \) in \([24]\). Here we wish to provide the full classification of hypertoric manifolds in arbitrary dimension, without any restriction on the topology. First let us recall some of the four-dimensional story.

### 2.2 Dimension four

Let \( M \) be a four-dimensional hyperK"ahler manifold with an effective tri-Hamiltonian \( S^1 \)-action of period \( 2\pi \). Let \( X \) be the corresponding vector field on \( M \). Note that the only special orbits for the action are fixed points: if \( g \in S^1 \) stabilises the point \( x \) and \( X_x \neq 0 \), then \( g \) fixes \( T_x M = \text{Span}\{X_x, IX_x, JX_x, KX_x\} \) and hence a neighbourhood of \( x \), so by analyticity \( g = e \).

The hyperK"ahler moment map

\[
\mu = (\mu_I, \mu_J, \mu_K): M \to \mathbb{R}^3
\]

is a local diffeomorphism away from the fixed point set \( M^X \). Locally on \( M' = M \setminus M^X \), the hyperK"ahler metric may be written as

\[
g = \frac{1}{V} \beta_0^2 + V (\alpha_I^2 + \alpha_J^2 + \alpha_K^2),
\]

where \( \alpha_A = X_\omega A = d\mu_A \), \( V = 1/g(X, X) \) and \( \beta_0 = \alpha_0/\|X\| = g(X, \cdot) V^{1/2} \).

The hyperK"ahler condition is now equivalent to the monopole equation \( d\beta_0 = -*_3 dV \), which implies that locally \( V \) is a harmonic function on \( \mathbb{R}^3 \).

**Theorem 2** \([24]\). Let \( M \) be a complete connected hyperK"ahler manifold of dimension 4 with a tri-Hamiltonian circle action of period \( 2\pi \). Then the hyperK"ahler moment map \( \mu: M \to \mathbb{R}^3 \) is surjective with connected fibres and
induces a homeomorphism \( \pi: M/S^1 \to \mathbb{R}^3 \). The metric on \( M \) is specified by any harmonic function \( V: \mathbb{R}^3 \setminus Z \to (0, \infty) \) of the form

\[
V(p) = c + \frac{1}{2} \sum_{q \in Z} \frac{1}{\|p - q\|}
\]

where \( c \geq 0 \) is constant and \( Z \subset \mathbb{R}^3 \) is finite or countably infinite. \( \square \)

The set \( Z = \mu(M^X) \) is the image of the fixed-point set \( M^X \). The metrics with \( c = 0 \) and \( Z \) finite are the Gibbons-Hawking metrics \([11]\); \( c > 0 \) and \( Z \) finite gives the older multi-Taub-NUT metrics \([16]\). For \( Z \neq \varnothing \), the hyperKähler manifold \( M \) is simply-connected, with \( b_2(M) = |Z| - 1 \); for \( c > 0 \), \( Z = \varnothing \), we have \( M = S^1 \times \mathbb{R}^3 \) with \( \mu \) projection to the second factor. For \( Z \) infinite and \( c = 0 \), the hyperKähler metrics are of type \( A_\infty \) as constructed by Anderson, Kronheimer and LeBrun \([3]\) and Goto \([13]\), concentrating on the case \( Z = \{(n^2, 0, 0) \mid n \in \mathbb{N}\} \), and written down for general \( Z \) in Hattori \([15]\). These are the examples of infinite topological type.

To understand and extend Hattori’s general formulation of these structures, we need to study when (3) gives a finite sum on an open subset of \( \mathbb{R}^3 \). By Harnack’s Principle (see e.g. \([4]\)) if (3) is finite at one point of \( \mathbb{R}^3 \), then it is finite on all of \( \mathbb{R}^3 \setminus Z \). This may be seen in a elementary way via the following result.

**Lemma 3.** Suppose \( (q_n)_{n \in \mathbb{N}} \) is a sequence of points in \( \mathbb{R}^3 \) is given. Then the series \( S_1 = \sum_{n \in \mathbb{N}} \|p - q_n\|^{-1} \) converges at some \( p \in \mathbb{R}^3 \setminus \{q_n \mid n \in \mathbb{N}\} \) if and only if the series \( S_2 = \sum_{n \in \mathbb{N}} (1 + \|q_n\|)^{-1} \) converges.

**Proof.** First note that if there is a compact subset \( C \) of \( \mathbb{R}^3 \) containing infinitely many points of the sequence \( (q_n) \), then neither sum converges: there is some subsequence \( (q_{n_i})_{i \in I} \) that converges to a \( q \in \mathbb{R}^3 \), and so infinitely many terms are greater than some strictly positive lower bound.

Now putting \( c = 1 + \|p\| \), we have \( \|p - q\| \leq \|p\| + \|q\| \leq c(1 + \|q\|) \). It follows that convergence of \( S_1 \) implies convergence of \( S_2 \).

For the converse, we consider \( q \in \mathbb{R}^3 \setminus \overline{B}(0; R) \) for \( R = 1 + 2\|p\| \) and have

\[
\|p - q\| \geq \|q\| - \|p\| = \frac{1}{2}\|q\| + \left(\frac{1}{2}\|q\| - \|p\|\right)
\]

\[
> \frac{1}{2}(\|q\| + 1).
\]

If \( S_2 \) converges, then \( \{n \in \mathbb{N} \mid q_n \in \overline{B}(0; R)\} \) is finite, so the inequality (4) implies convergence of \( S_1 \). \( \square \)

Finally let us remark that scaling the hyperKähler metric \( q \) by a constant \( C \) scales \( V \) as a function on \( M \) by \( C^{-1} \). However the hyperKähler moment map \( \mu \) also scales by \( C \), so the induced function \( V(p) = V(\mu(x)) \) on \( \mathbb{R}^3 \) has the same form, with a new constant term \( c/C \) and the points \( q \) replaced by \( q/C \). On the other hand scaling the vector field \( X \) by a constant, so that the action it
generates is no longer of period \(2\pi\), scales \(V\) on \(M\) and \(\mu\) by different weights. In particular, such a change alters the factors \(1/2\) in (3).

### 2.3 Construction of hypertoric manifolds

Bielawski and Dancer [6] provided a general construction of hypertoric manifolds in all dimensions with finite topological type. Goto [13] gave a particular construction of examples of in arbitrary dimension of infinite topological type. Let us now build on Hattori’s four-dimensional description [15], to combine these two constructions.

Let \(L\) be a finite or countably infinite set. Choose \(\Lambda = (\Lambda_k)_{k \in L} \in \mathbb{H}^L\) and define \(\lambda = (\lambda_k)_{k \in L}\) by \(\lambda_k = -\frac{1}{2} \sum_{i=1}^{d} \Lambda_k i A_k \in \text{Im} \mathbb{H}\). For each \(k \in L\), let \(u_k \in \mathbb{R}^n\) be a non-zero vector and put \(\hat{\lambda}_k = \lambda_k / \|u_k\|\). Suppose

\[
\sum_{k \in L} (1 + |\hat{\lambda}_k|)^{-1} < \infty.
\]

Consider the Hilbert manifold \(M_{\Lambda} = A + L^2(\mathbb{H})\), where

\[
L^2(\mathbb{H}) = \left\{ v \in \mathbb{H}^L \mid \sum_{k \in L} |v_k|^2 < \infty \right\}.
\]

Let \(T_\lambda\) be the Hilbert group

\[
T_\lambda = \left\{ g \in T^L = (S^1)^L \mid \sum_{k \in L} (1 + |\lambda_k|) |1 - g_k|^2 < \infty \right\}.
\]

If \(\|u_k\|\) is bounded away from 0, then \(g \in T_\lambda\) implies \(g_k\) is arbitrarily close to 1 except for a finite number of \(k \in L\). As \(|1 - \exp(it)|^2 = 2 - 2 \cos(t) \leq t^2\) for all \(t \in \mathbb{R}\) and \(|1 - \exp(it)|^2 \geq 2t^2 / \pi^2\) on \((-\pi/2, \pi/2)\), we see that the Lie algebra of \(T_\lambda\) is

\[
t_\lambda = \left\{ t \in \mathbb{R}^L \mid \|t\|_{L^2}^2 = \sum_{k \in L} (1 + |\lambda_k|) |t_k|^2 < \infty \right\}.
\]

Now consider the linear map \(\beta: t_\lambda \to \mathbb{R}^n\) given by \(\beta(e_k) = u_k\), where \(e_i = (\delta^k_i)_{k \in L}\), where \(\delta^k_i \in \{0, 1\}\) is Kronecker’s delta. Supposing \(\beta\) is continuous then we define \(n_\beta = \ker \beta \subset t_\lambda\). If \(u_k \in \mathbb{Z}^n \subset \mathbb{R}^n\) for each \(k \in L\), we may define a Hilbert subgroup \(N_\beta\) of \(T_\lambda\) by

\[
N_\beta = \ker(\exp \circ \beta \circ \exp^{-1}: T_\lambda \to T^n).
\]

This gives exact sequences
Our aim now is to construct hypertoric manifolds of dimension 4n as hyperKähler quotients of $M_A$ by $N_\beta$.

**Remark 4.** The construction of Hattori [15] corresponds to $n = 1$ and $u_k = 1 \in \mathbb{R}$ for each $k$. For general dimension $4n$, Goto’s construction [13] corresponds to $L = (\mathbb{Z} \setminus \{0\}) \amalg \{1, \ldots, n\}$.

\[
A_k = \begin{cases} 
  k\Lambda, & \text{for } k \in \mathbb{Z}_{>0}, \\
  k\Lambda, & \text{for } k \in \mathbb{Z}_{<0}, \\
  0, & \text{for } k \in \{1, \ldots, n\},
\end{cases} \quad \text{with } u_k = \begin{cases}
  e_1, & \text{for } k \in \mathbb{Z} \setminus \{0\}, \\
  \sum_{i=1}^n e_i, & \text{for } k = 1 \in \{1, \ldots, n\}, \\
  -e_r, & \text{for } k = r \in \{2, \ldots, n\}.
\end{cases}
\]

Thus Goto’s construction is for one concrete choice of $(\lambda_k)_{k \in L}$ and only one of the $u_k$’s is repeatedly infinitely many times.

Returning to the general situation, note that the integrality of $u_k$ implies $\|u_k\| \geq 1$, so the convergence condition (5) implies

\[
\sum_{k \in L} (1 + |\lambda_k|)^{-1} < \infty. \tag{8}
\]

The group $T_\lambda$ acts on $M_A$ via $gx = (g_k x_k)_{k \in L}$: indeed for $g \in T_\lambda$ and $x = \Lambda + v \in M_A$, we have $gx = g\Lambda + gv = \Lambda - (1-g)\Lambda + gv$, but $gv \in \mathbb{L}^2(\mathbb{H})$ and $\|(1-g)\Lambda\|^2 = \sum_{k \in L} \frac{1}{2} |\lambda_k||1 - g_k|^2 \leq \frac{1}{2} \sum_{k \in L} (1 + |\lambda_k|)|1 - g_k|^2$, which is finite by the definition of $T_\lambda$, so $(1-g)\Lambda \in \mathbb{L}^2(\mathbb{H})$ too. The action preserves the flat hyperKähler structure with $\mathbb{L}^2$-metric and complex structures obtained by regarding $\mathbb{L}^2(\mathbb{H})$ as a right $\mathbb{H}$-module. Identifying $\mathbb{R}^3$ with $\text{Im } \mathbb{H} = \mathbb{R}i + \mathbb{R}j + \mathbb{R}k$, a corresponding hyperKähler moment map is given by $\langle \mu_A(x), t \rangle = \frac{1}{2} \sum_{k \in L} (\tau_k i \lambda_k x_k - \tau_k t_k A_k)$. The terms $-\frac{1}{2} \tau_k t_k A_k$ ensure that the sum in $\mu_A$ converges, but otherwise are arbitrary linear terms in $t_k$ with values in $\text{Im } \mathbb{H}$. With our definition of $\lambda_k$, we have

\[
\mu_A(x) = \sum_{k \in L} (\lambda_k + \frac{1}{2} \tau_k i \lambda_k) e_k^*.
\]

The hyperKähler moment map for the subgroup $N_\beta$ is then $\mu_\beta = \iota^* \mu_A: M_A \rightarrow \text{Im } \mathbb{H} \otimes n_\beta^\perp$. We define

\[
M = M(\beta, \lambda) = M_A/\|N_\beta = \mu^{-1}_\beta(0)/N_\beta.
\]

Since (6) is exact, we have dually $\ker \iota^* = \text{im } \beta^*$ and hence the following characterisation of $\mu^{-1}_\beta(0)$.
Lemma 5. A point \( x \in M_A \) lies in the zero set of the hyperKähler moment map \( \mu_\beta \) for \( N_\beta \) if and only if there is an \( a \in \text{Im} \mathbb{H} \otimes (\mathbb{R}^n)^* \) with

\[
a(u_k) = \lambda_k + \frac{i}{2}x_k
\]

for each \( k \in \mathbb{L} \), where \( u_k = \beta(e_k) \).

In equation (9), note that \( x_k = 0 \) if and only if \( a(u_k) = \lambda_k \). Indeed, the map \( \mathbb{H} \to \text{Im} \mathbb{H}, v \mapsto iv \), is surjective with \( |iv|^2 = |v|^2 \). As in [6], we define affine subspaces \( H_k \subset \text{Im} \mathbb{H} \otimes (\mathbb{R}^n)^* \) of real codimension 3 by

\[
H_k = H(u_k, \lambda_k) = \{ a \in \text{Im} \mathbb{H} \otimes (\mathbb{R}^n)^* \mid a(u_k) = \lambda_k \}
\]

which we call flats. Note that \( T^n = T_\Lambda/N_\beta \) acts on \( M \) and that if \( M \) is smooth this action preserves induced the hyperKähler structure and has moment map \( \phi: M \to \text{Im} \mathbb{H} \otimes (\mathbb{R}^n)^* \) induced by \( \mu_A \); indeed \( \text{Lie} T^n = t_\Lambda/n_\beta \) implies \( (\text{Lie} T^n)^* = (n_\beta)^0, \) the annihilator of \( n_\beta \) in \( t_\Lambda \), so \( n_\beta^{-1}(0) \) the map \( \mu_A \) takes values in \( (n_\beta)^0 = (\text{Lie} T^n)^* \) and descends to \( M \) as \( \phi \). It follows, as in [6], that \( \phi \) induces a homeomorphism \( M/T^n \to \text{Im} \mathbb{H} \otimes (\mathbb{R}^n)^* \) and that for \( p \in M \), the stabiliser \( \text{stab}_{T^n}(p) \) is the subtorus with Lie algebra spanned by the \( u_k \) such that \( \phi(p) \in H_k \).

Theorem 6. Suppose \( u_k = \beta(e_k) \in \mathbb{Z}^n, k \in \mathbb{L}, \) are primitive and span \( \mathbb{R}^n \). Let \( \lambda_k \in \text{Im} \mathbb{H}, k \in \mathbb{L}, \) be given such that the convergence condition (5) holds and the flats \( H_k = H(u_k, \lambda_k), k \in \mathbb{L}, \) are distinct. Then the hyperKähler quotient \( M = M(\beta, \lambda) \) is smooth if

(a) any set of \( n + 1 \) flats \( H_k \) has empty intersection, and
(b) whenever \( n \) distinct flats \( H_{k(1)}, \ldots, H_{k(n)} \) have non-empty intersection the corresponding vectors \( u_{k(1)}, \ldots, u_{k(n)} \) form a \( \mathbb{Z} \)-basis for \( \mathbb{Z}^n \).

We break the proof in to several steps.

Proposition 7. Suppose \( u_k \in \mathbb{Z}^n, k \in \mathbb{L}, \) are primitive, span \( \mathbb{R}^n \) and satisfy condition (9) of Theorem 6. Then \( \mathcal{U} = \{ u_k \mid k \in \mathbb{L} \} \) is finite.

Proof. Note that if \( u_{k(1)}, \ldots, u_{k(n)} \) are linearly independent then \( \bigcap_{j=1}^n H_{k(j)} \) is a single point. Thus (9) implies that \( \mathcal{U} \) contains a \( \mathbb{Z} \)-basis \( v_1, \ldots, v_n \) for \( \mathbb{Z}^n \). Then matrix \( A \) with columns \( v_1, \ldots, v_n \) is invertible with inverse in \( M_n(\mathbb{Z}) \), so \( A \) lies in \( \text{GL}(n, \mathbb{Z}) = \{ B \in M_n(\mathbb{Z}) \mid \det B = \pm 1 \} \). Multiplying with \( A^{-1} \) we may thus assume for the purpose of this proof that \( \mathcal{U} \) contains the standard basis \( e_1, \ldots, e_n \).

Suppose \( u = (u_1, \ldots, u_n) \in \mathcal{U} \) is different from \( e_i \), for all \( i = 1, \ldots, n \). For \( j \in \{1, \ldots, n\} \), consider the matrix \( A_j \) with columns \( u, e_1, \ldots, e_j, \ldots, e_n, \) so \( e_j \) is omitted. We have \( \det A_j = \pm u_j \). If \( \det A_j \) is non-zero, then its columns are linearly independent and the discussion above gives \( \det A_j = \pm 1 \). It follows that \( u_j \in \{-1, 0, 1\} \) for each \( j \in \{1, \ldots, n\} \). In particular, there are only finitely many such \( u \)’s. \( \square \)
It follows that under condition (b), the set \( \{ \| u_k \| \mid k \in \mathbb{L} \} \) is bounded, so (S) and (T) are equivalent. Now the map \( \beta: \mathbb{L} \to \mathbb{R}^n \) has Riesz representation \( \beta(t) = \langle t, \gamma \rangle_{\lambda, t} = \sum_{k \in \mathbb{L}} (1 + \| \lambda_k \|) t_k \gamma_k \) given by \( \gamma_k = u_k/(1 + \| \lambda_k \|) \). As \( \| \gamma \|_{L, t}^2 = \sum_{k \in \mathbb{L}} (1 + \| \lambda_k \|)^{-1} \| u_k \|^2 \), boundedness of \( \| u_k \| \) and (S) show that \( \gamma \) lies in \( t_\lambda \). Thus \( \beta \) is continuous and it follows that \( N_\beta \) is a Hilbert subgroup of \( T_\lambda \) of codimension \( n \).

**Lemma 8.** If conditions (a) and (b) hold then the group \( N_\beta \) acts freely on \( \mu_\beta^{-1}(0) \).

**Proof.** Given a subset \( \mathbb{K} \subseteq \mathbb{L} \), we define the subgroup \( T_\mathbb{K} \) subgroup of \( T_\lambda \) by

\[
T_\mathbb{K} = \{ (g_k)_{k \in \mathbb{L}} \in T_\lambda \mid ge = 1 \forall k \notin \mathbb{K} \},
\]

so that the Lie algebra of \( T_\mathbb{K} \) is spanned by \( \{ e_k \mid k \in \mathbb{K} \} \). For \( x \in M_\lambda \), the stabiliser \( stab_{T_\mathbb{K}}(x) \) is \( T_\mathbb{K} \) where \( \mathbb{K} = \{ k \mid x_k = 0 \} \).

Now consider \( x \in \mu_\beta^{-1}(0) \). Equation (9) implies that \( x_k = 0 \) if and only if \( \phi(\pi(x)) \in H_k \), where \( \pi: \mu_\beta^{-1}(0) \to M \) is the quotient map. Thus \( stab_{T_\mathbb{K}}(x) = T_{\mathbb{K}(x)} \), where \( \mathbb{K}(x) = \{ k \mid \phi(\pi(x)) \in H_k \} \). Condition (a) implies that \( \mathbb{K}(x) \) contains at most \( n \) elements. The stabiliser of \( x \) under \( N_\beta \) consists of those \( \lambda \) in \( stab_{T_\mathbb{K}}(x) \) that lie in the kernel of the map \( T_\lambda \to T^n \) induced by \( \beta \). But \( \beta(e_k) = u_k \) and implies that \( (g_k)_{k \in \mathbb{L}} \in T_{\mathbb{K}(x)} \) maps to

\[
h = \prod_{k \in \mathbb{K}(x)} g_k \exp(u_k) = \prod_{k \in \mathbb{K}(x)} \exp(i \theta_k u_k) \in T^n,
\]

where \( g_k = e^{i \theta_k} \).

Condition (b) implies that the \( u_k, k \in \mathbb{K}(x) \), are part of a \( \mathbb{Z} \)-basis for \( \mathbb{Z}^n \), so we may change basis via an element of \( GL(n, \mathbb{Z}) \) so that the \( u_k \) become the first \( r \) basis elements. Then \( h \) becomes \( diag(e^{i \theta(1)}, \ldots, e^{i \theta(r)}, 1, \ldots, 1) \), where \( \theta(j) \) is a corresponding relabelling of the \( \theta_k \)'s. It follows that \( \theta_k \in 2\pi \mathbb{Z} \) and so \( g_k = 1 \) for each \( k \in \mathbb{K}(x) \). Thus \( stab_{N_\beta}(x) \) is trivial, as claimed. \( \square \)

As in (13), for \( x \in M_\lambda \), let \( X_x: t_\lambda \to T_x M_\lambda \) be the map sending an element to the corresponding tangent vector at \( x \) generated by the action:

\[
X_x(t) = \left. \frac{d}{ds} (\exp(st)x) \right|_{s=0} = (it_k x_k)_{k \in \mathbb{L}}.
\]

**Lemma 9.** For \( x \in \mu_\beta^{-1}(0) \), the map \( X_x \) induces a linear homeomorphism from \( n_\beta \) to the tangent space \( T_x(N_\beta : x) \) of the \( N_\beta \)-orbit through \( x \).

**Proof.** For general \( x \in M_\lambda \), we have \( \| X_x(t) \|^2 = \|(t_k x_k)_{k \in \mathbb{L}}\|^2 = \sum_{k \in \mathbb{L}} |t_k|^2 |A_k + v_k|^2 \).

Except for finitely many \( k \in \mathbb{L} \), we have \( |A_k| \geq 2|v_k| \), so for these \( k \), we have \( |A_k + v_k|^2 \leq 3|A_k/2|^2 = 9|\lambda_k|/8 \). It follows that there is a constant \( C_r \), independent of \( t \), such that \( \| X_x(t) \| \leq C_r \| t \|_{\lambda, t} \). Thus \( X_x: t_\lambda \to T_x M_\lambda \) is continuous.

Now let \( \mathbb{K}_1 = \mathbb{L} \setminus \{ k \in \mathbb{L} \mid |A_k| \geq 2|v_k| \} \), which is a finite set by (S) and the condition that \( v \in \mathbb{R}^2 \). For \( k \notin \mathbb{K}_1 \), we have \( |\lambda_k| \geq 1/2 \), so \( |x_k|^2 = |A_k + v_k|^2 \geq |A_k/2|^2 = |\lambda_k|/8 \geq (1 + |\lambda_k|)/32 \). It follows that for
that are isometric via a tri-holomorphic map. For $x \in \mu_{\beta}^{-1}(0)$, the set $\mathbb{K}(x)$ coincides with the previous definition $\mathbb{K}(x) = \{k \mid \phi(\pi(x)) \in H_k\}$ and so contains at most $n$ elements. Let $V_x = \text{Span}\{e_k \mid k \in \mathbb{K}(x)\} \leq t_x$ and write $pr^t : t_x \to V_x$ for the orthogonal projection away from $V_x$. Then $\beta$ is injective on $V_x$, so $pr^t$ is a continuous linear bijection $pr_\beta : n_\beta \to pr^t(n_\beta)$. The image is the orthogonal complement to $V_x \oplus \beta^!(\beta(V_x)^\perp)$, where $\beta(V_x)^\perp$ is the orthogonal complement in $\mathbb{R}^n$. As $\beta$ is surjective, its adjoint $\beta^t$ is injective, so $pr^t(n_\beta)$ is of finite codimension and thus a Hilbert subspace of $t_\lambda$. By the Open Mapping Theorem, we conclude that $pr_\beta^{-1}$ is continuous, and note that its norm is non-zero.

Now for $x \in \mu_{\beta}^{-1}(0)$ and $t \in n_\beta$, we have

$$
\|X_x(t)\|^2 = \|(t_k x_k)_{k \in L}\|^2 \geq c_x \sum_{k \in \mathbb{K}(x)} (1 + |\lambda_k|)|t_k|^2 = c_x \|pr_{\beta}(t)\|^2_{\lambda,t} \geq \frac{c_x}{\|pr_{\beta}^{-1}\|^2} \|t\|^2_{\lambda,t},
$$

showing that $X_x$ has continuous inverse on $T_x(N_\beta \cdot x)$. □

It now follows, as in [13], that for $x \in \mu_{\beta}^{-1}(0)$, the differential $d\mu_\beta : T_x M_A \to \text{Im} \mathbb{H} \otimes n_\beta^*$ is split, with right inverse the $\mathbb{R}$-linear map given by $a \otimes \delta = AX_x(t)$, where $\delta = (t, \cdot)_{\lambda,t}$ and $A = a_1 I + a_2 J + a_3 K$ for $a = a_1 i + a_2 j + a_3 k$. This implies that $\mu_{\beta}^{-1}(0)$ is a smooth Hilbert submanifold of $M_A$. On $\mu_{\beta}^{-1}(0)$, Goto’s construction [13] of slices $S_x$ goes through unchanged: one considers the map $F_x : \mu_{\beta}^{-1}(0) \to n_\beta^*$ given by $F_x(x + w)(t) = \langle w, X_x(t) \rangle$ and puts $S_x = F_x^{-1}(0) \cap U$, for a sufficiently small neighbourhood $U$ of $x$. Thus $M(\beta, \lambda) = \mu_{\beta}^{-1}(0)/N_\beta$ is a smooth manifold.

Fix a point $q \in \text{Im} \mathbb{H} \otimes (\mathbb{R}^n)^* \setminus \bigcup_{k \in L} H_k$. For $x \in \mu_{\beta}^{-1}(0)$ with $\phi\pi(x) = q$, we have that $x_k \neq 0$ for all $k \in L$. Thus $T_\lambda$ acts freely on $x_k$. As $e_k \in t_\lambda$ for each $k \in L$, it follows that $e_k \in F = T_x(T_\lambda \cdot x)$, and that $T_x M_A = F \oplus IF \oplus JF \oplus KF$. As $\beta$ is surjective, we conclude that $F/T_x(N_\beta \cdot x)$ is of dimension $n$ and that $M(\beta, \lambda) = \mu_{\beta}^{-1}(0)/N_\beta$ is of dimension $4n$.

The standard considerations of the hyperKähler quotient construction shows that $M(\beta, \lambda)$ inherits a smooth hyperKähler structure, completing the proof of Theorem 6.

Just as in Hattori [15], one may use the $T_\lambda$ action to show that different choices of $(A_k)_{k \in L}$ yielding the same $(\lambda_k)_{k \in L}$ result in hyperKähler structures that are isometric via a tri-holomorphic map.
2.4 Classification of complete hypertoric manifolds

Now suppose that $M$ is an arbitrary complete connected hypertoric manifold of dimension $4n$ and write $G = T^n$. Bielawski \[5\] \S 4 shows that locally $M$ has much of the structure of the hypertoric manifolds constructed above.

Indeed for each $p \in M$, we may find a $G$-invariant neighbourhood of the form $U = G \times_H W$, where $H = \text{stab}_G(p)$ and $W = T_p(G \cdot p)^\perp \subset T_p M$. Now $H$ acts trivially on $W_1 = (\text{Im } H) T_p(G \cdot p)$, and effectively as an Abelian subgroup of $\text{Sp}(r)$ on the orthogonal complement $W_2 = W \cap W_1^\perp \cong \mathbb{H}^r$. Counting dimensions, it follows that $H$ acts as $T^r$ on $W_2$, and hence $H$ is connected. The image of the singular orbits in $U$ is a union of distinct flats $H_k = H(u_k, \lambda_k)$ as in (10), where $u_k$ may be chosen to lie in $\mathbb{Z}^n \subset \mathbb{R}^n = g$ and be primitive vectors. The collection $\{u_k \mid \mu(p) \in H_k\}$ is then part of a $\mathbb{Z}$-basis for $\mathbb{R}^n$ spanning the Lie algebra of $H$. Furthermore, examining the structure of $\mu$ on such a neighbourhood $U$, Bielawski shows that $\mu$ induces a local homeomorphism $M/G \to \text{Im } H \otimes g^* \cong \mathbb{R}^{3n}$. In particular, the hyperKähler moment map $\mu : M \to \text{Im } H \otimes g^*$ is an open map.

Let $\{H_k \mid k \in \mathbb{L}\}$ be the collection of all flats that arise in this way. The index set $\mathbb{L}$ is finite or countably infinite, since $M$ is second countable.

**Lemma 10.** Suppose $\alpha \in (\mathbb{Z}^n)^* \subset (\mathbb{R}^n)^* = g^*$ is non-zero. Let $T_\alpha$ be the subtorus of $G = T^n$ whose Lie algebra is spanned by $\ker \alpha = \{u \in g \mid \alpha(u) = 0\}$. For $a \in \text{Im } \mathbb{H} \otimes g^*$, write $[a]_\alpha = a + \text{Im } \mathbb{H} \otimes \mathbb{R} \alpha$ for the equivalence class of $a$ in $\text{Im } \mathbb{H} \otimes (g^*/\mathbb{R} \alpha)$. Then except for countably many choices of $[a]_\alpha$, the group $T_\alpha$ acts freely on $\mu^{-1}([a]_\alpha)$.

Note that $(\ker \alpha)^* = g^*/\mathbb{R} \alpha$.

**Proof.** Consider the intersection $[a]_\alpha \cap H_k$. A general point of $[a]_\alpha$ is $a + q \otimes \alpha$, $q \in \text{Im } \mathbb{H}$, which lies in $H_k = H(u_k, \lambda_k)$ only if $a(u_k) + q \alpha(u_k) = \lambda_k$. If $\alpha(u_k) \neq 0$ this equation has a unique solution for $q$; if $\alpha(u_k) = 0$ then there is a solution only if $a(u_k) = \lambda_k$ and then $[a]_\alpha \subset H_k$. Thus choosing $a(u_k) \neq \lambda_k$ for each $k \in \mathbb{L}$, ensures that $[a]_\alpha \cap H_k$ is empty for every $k$ for which $u_k \in \ker \alpha$. It follows that $T_\alpha$ acts almost freely on $\mu^{-1}([a]_\alpha)$, but as each stabiliser of the $T_\alpha$-action is connected, we find that $T_\alpha$ acts freely. \hfill $\Box$

**Corollary 11.** For $\alpha, T_\alpha$ and $a$ as in Lemma 10, the hyperKähler quotient $M(a, \alpha) = \mu^{-1}([a]_\alpha)/T_\alpha$ is a complete hyperKähler manifold of dimension four. Furthermore, $M(a, \alpha)$ carries an effective tri-Hamiltonian circle action.

**Proof.** As $T_\alpha$ is compact and acts freely on $\mu^{-1}([a]_\alpha)$, it follows that $M(a, \alpha)$ is hyperKähler \[17\]. Completeness of $M$ implies completeness of the level set $\mu^{-1}([a]_\alpha)$ and hence of the hyperKähler quotient.

As $\mu$ is $T^n$-invariant, the level set $\mu^{-1}([a]_\alpha)$ is preserved by $T^n$ and we get an action of the circle $T^n/T_\alpha$ on the quotient. Identifying $[a]_\alpha$ with $\text{Im } \mathbb{H}$ via $a + q \otimes \alpha \mapsto q$, the restriction of $\mu$ to $\mu^{-1}([a]_\alpha)$ descends to a hyperKähler moment map for this action. \hfill $\Box$
From the four-dimensional classification Theorem 2, we find have that the moment map $\mu$ surjects on to $\text{Im } H$. Interpreting this in terms of the moment map $\mu$ of $M$, we have that $[a]_\alpha$ lies in the image of $\mu$. But on $M$ the moment map $\mu$ is an open map. And, as $\bigcup \{ [a]_\alpha \mid a(u_k) \neq \lambda_k \forall k \in \ker \alpha \}$ is dense in $\text{Im } H \otimes \mathbb{R}^n$, we conclude that $\mu$ is a surjection.

Furthermore, the metric on $M(a, \alpha)$ is given by a potential of the form (3) up to an overall positive scale. The elements of $Z$ are just the intersection points of $[a]_\alpha \cap H_k$, for $u_k \notin \ker \alpha$. These are the points $a_k = a + q_k \otimes \alpha$ with $q_k = (\lambda_k - a(u_k))/\alpha(u_k)$. Now for $p \in \mu^{-1}([a]_\alpha)$, writing $\mu(p) = a + q \otimes \alpha$, we have

$$\mu(p) - a_k = (q - q_k) \otimes \alpha = \frac{q\alpha(u_k) - (\lambda_k - a(u_k))}{\alpha(u_k)} \otimes \alpha$$

$$= (\langle \mu(p), u_k \rangle - \lambda_k) \otimes \frac{\alpha}{\alpha(u_k)}.$$ 

Thus the potential for $M(a, \alpha)$ is proportional to

$$V_\alpha(p) = c + \frac{1}{2} \sum_{k \in L} \frac{1}{\|\langle \mu(p), u_k \rangle - \lambda_k \| \| \alpha \|} |\alpha(u_k)|,$$

where we may include the terms with $u_k \in \ker \alpha$, since they contribute zero, and $\| \alpha \|$ is the norm of $\alpha$ with respect to the standard inner product from the identification $g = \mathbb{R}^n$.

Using this inner product we may identify $g$ with $g^*$. Then the function

$$r_k(b) = \| b(u_k) - \lambda_k \| / \| u_k \| = \| b(\hat{u}_k) - \hat{\lambda}_k \|,$$

where $\hat{u}_k = u_k / \| u_k \|$ and $\hat{\lambda}_k = \lambda_k / \| u_k \|$, corresponds to the distance of $b$ from the flat $H_k$. We may thus write

$$V_\alpha(p) = c + \frac{1}{2} \sum_{k \in L} \frac{|\hat{\alpha}(\hat{u}_k)|}{r_k(\mu(p))},$$

(11)

for $\hat{\alpha} = \alpha / \| \alpha \|$.

Now choose $a$ so that $a(u_k) \neq \lambda_k$ for all $k \in L$. Then for $p \in \mu^{-1}(a)$ we have that $V_\alpha$ in (11) is finite for each non-zero integral $\alpha \in g^*$. As each unit vector $\hat{u} \in g \cong \mathbb{R}^n$ has $\langle \hat{u}, e_i \rangle \geq 1/\sqrt{n}$ for some $i \in \{1, \ldots, n\}$, we conclude that $\sum_{k \in L} 1/r_k(\mu(p))$ converges. In particular the distance of $\mu(p)$ to $H_k$ is bounded below by a uniform constant. It follows that there is an open neighbourhood $U$ of $\mu(p)$ in $\mu(M) \subset \text{Im } H \otimes g^*$ for which $a(u_k) \neq \lambda_k$ for all $a \in U$.

Let $M_U$ be a connected component of $\mu^{-1}(U)$. Then $T^n$ acts freely on $M_U$ and the hyperKähler structure on $M_U$ is uniquely determined via a polyharmonic function $F$ on $U \subset \text{Im } H \otimes g^*$ as follows. The hyperKähler metric is of the form
where for $X_1, \ldots, X_n$ is a basis for $\mathbb{R}^n = \mathfrak{g}$, we have $\alpha_j^k = X_i \omega_{ij}$, $(V^{-1})_{ij} = (g(X_i, X_j))$ and $\beta_0^j, \ldots, \beta_n^j$ are the $C^\infty(M_U)$-linear combinations of $g(X_1, \cdot), \ldots, g(X_n, \cdot)$ such that $\beta_0^j(X_j) = \delta_j^j$. As a result of Pedersen and Poon [22] and the Legendre transform of Lindström and Roček [20], the functions $V_{ij}$ on $U \subset \text{Im } \mathbb{H} \otimes \mathfrak{g}^*$ are polyharmonic meaning that they are harmonic on each affine subspace $a + \text{Im } \mathbb{H} \otimes \mathbb{R} \alpha$, $\alpha \in \mathfrak{g}^* \setminus \{0\}$. Furthermore this matrix of functions is given by a single polyharmonic function $F: U \to \mathbb{R}$ via $V_{ij} = F_{x_i x_j}$, where we choose a unit vector $e \in \text{Im } \mathbb{H}$ and $(x_1, \ldots, x_n)$ are standard coordinates on $\mathbb{R}^n = \text{Re} \otimes \mathfrak{g}^* \subset \text{Im } \mathbb{H} \otimes \mathfrak{g}^*$. We write

$$s_k(b) = \langle e, b(\hat{u}_k) - \hat{\lambda}_k \rangle.$$ 

As $e$ acts on $e^\perp \subset \text{Im } \mathbb{H}$ as a complex structure, we may choose corresponding standard complex coordinates $(z_1, \ldots, z_n)$ on $e^\perp \otimes \mathfrak{g}^*$. A potential $V$ of the form (3) is then $V = F_{xx}$ with

$$F(x, z) = \frac{1}{4}c(2x^2 - |z|^2) + \frac{1}{2} \sum_{k \in \mathbb{L}} (s_k \log(s_k + r_k) - r_k).$$

As in Bielawski [3], we now deduce that for $M_U$ the function $F$ has the form

$$F = \sum_{k \in \mathbb{L}} a_k (s_k \log(s_k + r_k) - r_k) + \sum_{i, j = 1}^n c_{ij} (4x_i x_j - z_i z_j - z_j z_i)$$

for some real constants $a_k, c_{ij}$. As Bielawski explains the $c_{ij}$ terms are from a Taub-NUT deformation of a metric determined by the first sum, that is there is a hypertoric manifold $M_2$ and $M_U = M_2 \times (S^1 \times \mathbb{R}^3)^m // T^m$ with $T^m$ acting effectively on the product of $S^1$-factors, trivially on the $\mathbb{R}^3$-factors and as a subgroup of $T^m$ on $M_2$. By analyticity, the hyperKähler metric on $M_U$ determines the hyperKähler metric on $M$.

Using Bielawski’s techniques and the computations of [6], one may now conclude that $M_2$ comes from the construction of the previous section. In particular, we note that the $a_k$’s are bounded and convergence of $V_\alpha(p)$ in (11) for each non-zero integral $\alpha$ corresponds to the condition (5). To see this first we remark that convergence of the $V_\alpha(p)$’s corresponds to convergence of $R(b) = \sum_{k \in \mathbb{L}} 1/r_k(b)$ for $b = \mu(p)$: this follows from $|\alpha(\hat{u}_k)| \leq 1$ and $\langle \hat{u}_k, e_i \rangle \geq 1/\sqrt{m}$ for some $i$. Now $r_k(b) \leq |b(u_k)| + |\hat{\lambda}_k| \leq (1 + ||b||)(1 + |\hat{\lambda}_k|)$ gives that convergence of $R(b)$ implies (5). Conversely, note that (5) implies that $|\hat{\lambda}_k| < 1 + 2||b||$ for only finitely many $k \in \mathbb{L}$. But for $|\hat{\lambda}_k| \geq 1 + 2||b||$, we have $r_k(b) \geq (|\hat{\lambda}_k| + 1)/2$, as in (4), so we get convergence of $R(b)$.

We have thus proved the following result.
Theorem 12. Let $M$ be a connected hypertoric manifold of dimension $4n$. Then $M$ is a product $M = M_2 \times (S^1 \times \mathbb{R}^3)^m$ with $M_2$ a hypertoric manifold of the type constructed in §2.3, i.e., the hyperKähler quotient of flat Hilbert hyperKähler manifold by an Abelian Hilbert Lie group. The hyperKähler metric in $M$ is either the product hyperKähler metric or a Taub-NUT deformation of this metric. □

From the proof and the construction of the previous section, we have the following properties of the hyperKähler moment map of in this situation.

Corollary 13. If $M$ is a connected complete hyperKähler manifold of dimension $4n$ with an effective tri-Hamiltonian action of $T^n$, then the hyperKähler moment map $\mu: M \to \text{Im} \mathbb{H} \otimes (\mathbb{R}^n)^* \cong \mathbb{R}^{3n}$ is surjective with connected fibres. □

3 Cuts and modifications

Symplectic cutting was introduced by Lerman in 1995 [19]. The construction starts with a symplectic manifold with Hamiltonian circle action, and produces a new manifold of the same type, but with different topology. Explicitly, given $M$ with circle action and associated moment map $\mu$, we form the symplectic quotient at level $\epsilon$,

$$M^\epsilon_{\text{cut}} = (M \times \mathbb{C})//_{\epsilon} S^1$$

where the $S^1$ is the antidiagonal of the product action obtained from the given action on $M$ and the standard rotation on $\mathbb{C}$. Note that the moment map for the action on $\mathbb{C}$ is $\phi: z \mapsto |z|^2$.

The new space $M^\epsilon_{\text{cut}}$ is of the same dimension as $M$ and inherits a circle action from the diagonal action on $M \times \mathbb{C}$. Moreover, as

$$M_{\text{cut}} = \{(m, z) \mid \mu(m) - |z|^2 = \epsilon\}/S^1$$

we see that the points $\{m \mid \mu(m) < \epsilon\}$ are removed. Moreover, because $\phi: z \mapsto |z|^2$ is a trivial circle fibration over $(0, \infty)$ with the circle fibre collapsing to a point at the origin, we see that the region $\{m \mid \mu(m) > \epsilon\}$ remains unchanged, while the hypersurface $\mu^{-1}(\epsilon)$ is collapsed by a circle action.

We can generalise this to the case of torus actions, by replacing $\mathbb{C}$ by a toric variety associated to a polytope $\Delta$. The region $\mu^{-1}(\mathbb{R}^n \setminus (\Delta + \epsilon))$ will be removed, the preimage of the $\epsilon$-translate of the interior of $\Delta$ is unchanged, while collapsing by tori takes place on the preimage of lower-dimensional faces of the translated polytope.

For general geometries, we want to mimic this construction by looking at the appropriate quotient of $M \times N$ by an Abelian group $G$, where $N$ is a
space whose reduction by $G$ is a point. The topological change in $M$ will be controlled by the geometry of the moment map for the $G$ action on $N$.

The simplest example is that of a hyperKähler manifold with circle action. We now explain the hyperKähler analogue of a cut in this situation, which we call a *modification* [10]. The natural choice of $N$ is now the quaternions $\mathbb{H}$. Several new features now emerge, because the hyperKähler moment map $\mu : \mathbb{H} \to \mathbb{R}^3$ has very different properties from that for $\mathbb{C}$. In particular, it is surjective, and is a non-trivial circle fibration over $\mathbb{R}^3 \setminus \{0\}$; in fact $\phi$ is given by the Hopf map on each sphere. This means that in forming the hyperKähler quotient $M_{\text{mod}} = (M \times \mathbb{H})//S^1$ we do not discard any points in $M$ (hence the use of the terminology modification rather than cut!). We still collapse the locus $\mu^{-1}(\epsilon)$ by a circle action, because $\phi$ is injective over the origin. The complements $M \setminus \mu^{-1}(\epsilon)$ and $M_{\text{mod}} \setminus (\mu^{-1}(\epsilon)/S^1)$ can no longer be identified, because of the non-triviality of the Hopf fibration. Instead, the topology has been given a ‘twist’. An example of this is if we start with $M = \mathbb{H}$. Now iterating the above construction generates the Gibbons-Hawking $A_k$ multi-instanton spaces, where the spheres at large distance are replaced by lens spaces $S^3/\mathbb{Z}_{k+1}$.

We can make this more precise by observing that the space

$$M_1 = \{(m, q) \in M \times \mathbb{H} \mid \mu(m) - \phi(q) = \epsilon\}$$

projects onto both $M$ and $M_{\text{mod}}$. The first map is just projection onto the first factor (onto as $\phi$ is surjective), while the second is just the quotient map $M_1 \to M_{\text{mod}}$. Note that the first map is not quite a fibration: it has $S^1$ fibres generically but over $\mu^{-1}(\epsilon)$ the fibres collapse to a point.

On the open sets where both maps are fibrations, note that $M_1 \to M_{\text{mod}}$ is a Riemannian submersion, but that the projection to first factor is not. As shown in [24], the metric $\tilde{g}$ induced on $M$ by $M_1$ has the form

$$\tilde{g} = g + V(\mu)g_{\mathbb{H}},$$

where $g_{\mathbb{H}} = \alpha_0^2 + \alpha_1^2 + \alpha_2^2 + \alpha_3^2$, with $\alpha_0 = g(X, \cdot)$, etc., and $V(\mu) = 1/2\|\mu - \epsilon\|$ is the potential of the flat hyperKähler metric on $\mathbb{H}$.

One may now generalise the hyperKähler modification, by replacing $\mathbb{H}$ by any hypertoric manifold $N$ of dimension 4. The modification changes the metric as in [12] with $V(\mu)$ now the potential function of $N$, so one of the functions $\{\}$ In [24] it is proved that metric changes of the form (12) with $V$ now an arbitrary smooth invariant function on $M$, so called ‘elementary deformations’, only lead to new hyperKähler metrics when $V$ is $\pm V(\mu)$ for some hypertoric $N^4$. The case of negative $V$ corresponds precisely to inverting a modification via a positive $V$.

For general torus actions one can take $N$ to be a hypertoric manifold and we get a similar picture to that above.
4 Non-Abelian moment maps

One can also consider cutting constructions for non-Abelian group actions. Now, because the diagonal and anti-diagonal actions no longer commute, one considers the product of a $K$-manifold $M$ with a space $N$ with $K \times K$ action and then reduces by the antidiagonal action formed from the action on $M$ and (say) the left action on $N$.

In the symplectic case, following Weitsman [25], with $K = U(n)$ one can take $N = \text{Hom}(\mathbb{C}^n, \mathbb{C}^n)$ with $K \times K$ action $A \mapsto UAV^{-1}$. The moment map for the right action is $\mu: A \mapsto iA^*A$, with image $\Delta$ the set of non-negative Hermitian matrices. We have a picture that is quite reminiscent of the Abelian case, essentially because the fibres of $\phi$ are orbits of the left action. The right moment map $\phi$ gives a trivial fibration with fibres $U(n)$ over the interior of the image, while over the lower-dimensional faces of $\Delta$ (corresponding to non-negative Hermitian matrices that are not strictly positive), the fibres are $U(n)/U(n - k)$ where $k$ is the number of positive eigenvalues. This gives a nice non-Abelian generalisation of the toric cuts described above. To form the cut space we remove the complement of $\mu^{-1}(\Delta + \epsilon)$ and perform collapsing by the appropriate unitary groups on the preimages of the lower-dimensional faces of $\Delta + \epsilon$.

In the hyperKähler case life becomes more complicated. An obvious choice of $N$ is $\text{Hom}(\mathbb{C}^n, \mathbb{C}^n) \oplus \text{Hom}(\mathbb{C}^n, \mathbb{C}^n)^*$, with action

$$(U, V): (A, B) \mapsto (UAV^{-1}, VBU^{-1}).$$

The hyperKähler moment map for the right $U(n)$-action is now

$$(A, B) \mapsto \left(\frac{i}{2}(A^*A - BB^*), BA\right) \in \text{Im} \mathbb{H} \otimes u(n)^* \cong u(n) \oplus \mathfrak{gl}(n, \mathbb{C}).$$

In contrast to the symplectic case, or the Abelian hyperKähler case, the fibres of $\mu$ are no longer group orbits in general; in particular the left $U(n)_L$ action need not be transitive and indeed the quotient of a fibre by this action may have positive dimension. The result is that when we perform the non-Abelian hyperKähler modification, blowing up of certain loci occurs.

This is closely related to the phenomenon that the fibres of hyperKähler moment maps over non-central elements may have larger than expected dimension, even on the locus where the group action is free. This is because the kernel of the differential of a moment map $\mu$ is the orthogonal of $I\mathcal{G} + J\mathcal{G} + K\mathcal{G}$, where $\mathcal{G}$, as in §2.1, denotes the tangent space to the orbits of the group action. For the fibre over a central element this sum is direct (because $\mathcal{G}$ is tangent to the fibre so is orthogonal to the sum, and hence the three summands are mutually orthogonal), but over non-central elements this is no longer necessarily true. The dimension of the fibre is now no longer determined by the dimension of $\mathcal{G}$, and hence not determined by the dimension of the stabiliser. Note that in the Kähler situation the kernel of $d\mu$ is just the

$$(U, V): (A, B) \mapsto (UAV^{-1}, VBU^{-1})$$

$$(A, B) \mapsto \left(\frac{i}{2}(A^*A - BB^*), BA\right) \in \text{Im} \mathbb{H} \otimes u(n)^* \cong u(n) \oplus \mathfrak{gl}(n, \mathbb{C}).$$
orthogonal of $\mathcal{I}_G$, so here the dimension of the fibre is completely controlled by the dimension of the stabiliser, even in the non-Abelian case.

Another example of the unexpected behaviour of hyperKähler moment maps over non-central elements is the phenomenon of disconnected fibres. This is in contrast to symplectic moment maps, whose level sets enjoy many connectivity properties, see [23] and the references therein. As a simple hyperKähler example we may consider the $SU(2)$ action on $H^2 = \mathbb{C}^2 \times (\mathbb{C}^2)^*$

$$A: (z, w) \mapsto (Az, wA^{-1})$$

(13)

with hyperKähler moment map

$$\mu = (\mu_R, \mu_C): (z, w) \mapsto \left(\frac{i}{2}(zz^\dagger - w^\dagger w_0), (zw)_0\right),$$

(14)

where $\cdot_0$ denotes trace-free part, and $\cdot^\dagger$ the conjugate transpose. (This calculation arose in discussions with S. Tolman).

We are interested in finding the fibre of $\mu$ over $(\alpha, \beta) \in su(2) \oplus sl(2, \mathbb{C}) \cong Im \mathbb{H} \otimes su(2)^*$. Using the $SU(2)$-equivariance, we may take

$$\beta = \begin{pmatrix} \lambda & \mu \\ 0 & -\lambda \end{pmatrix}.$$ 

We find that the fibre is:

1. empty or a disjoint pair of circles if $\lambda, \mu$ are both non-zero;
2. empty, a circle or a disjoint pair of circles if $\lambda$ is zero and $\mu$ non-zero;
3. empty or a disjoint pair of circles if $\mu$ is zero and $\lambda$ non-zero;
4. a disjoint pair of circles or a point if $\lambda = \mu = 0$ (the point fibre occurs exactly over the origin $\alpha = \beta = 0$.

Now let us turn to the the case of $SU(3)$. This acts on $T^*C^3 = \mathbb{C}^3 \oplus (\mathbb{C}^3)^*$ via (13) and has the same formula (14) for the hyperKähler moment map $\mu: T^*C^3 \to Im \mathbb{H} \otimes su(3)^* \cong su(3) \oplus sl(3, \mathbb{C})$. In particular $\mu_C(z, w)$ has off-diagonal $(i, j)$-entries $z_iw_j$ whilst the diagonal entries are of the form $2z_iw_i - \sum_{k \neq i} z_kw_k$. Similarly, for $j > i$, $(\mu_R(z, w))_{ij} = \frac{i}{2}(z_i\overline{z_j} - \overline{w_i}w_j)$ and

$$(\mu_R(z, w))_{ii} = \frac{i}{6} \left\{2(|z_i|^2 - |w_i|^2) - \sum_{k \neq i} (|z_k|^2 - |w_k|^2)\right\}. $$

We consider the fibre $\mu^{-1}(\alpha, \beta)$, for $\alpha \in su(3)$, $\beta \in sl(3, \mathbb{C})$. Note that $\dim C sl(3, \mathbb{C}) = 8$ is strictly greater than $\dim C T^*C^3 = 6$ so there are now restrictions on $\beta$ to lie in the image of $\mu_C$. In particular, we see that $\mu$ is not surjective.

For more detail, note that the map $\mu$ is $SU(3)$-equivariant and we may use the action to put $\beta$ in to the canonical upper triangular form...
\[
\beta = \begin{pmatrix}
\lambda_1 & \xi_1 & \zeta \\
0 & \lambda_2 & \xi_2 \\
0 & 0 & \lambda_3
\end{pmatrix},
\]

with \(\lambda_1 + \lambda_2 + \lambda_3 = 0\). This gives the three constraints
\[
z_2w_1 = 0, \quad z_3w_1 = 0, \quad z_3w_2 = 0.
\] (15)

Case 1

Both \(\xi_i \neq 0\): we have \(z_1w_2 \neq 0 \neq z_2w_3\) implying that \(z_1, z_2, w_2, w_3\) are non-zero and so \(\zeta \neq 0\). Equation (15) gives \(w_1 = 0 = z_3\). This implies
\[
\lambda_1 = \lambda_3 = -\frac{1}{2} \lambda_2 = -\frac{1}{3} z_2w_2.
\]
Now \(z_1\) determines the remaining variables via
\[
w_2 = \frac{\xi_1}{z_1}, \quad w_3 = \frac{\zeta}{z_1}, \quad z_2 = \frac{\xi_2}{\zeta} z_1
\]
giving the relation
\[
3\lambda_2 \xi_2 = 2\xi_1 \zeta,
\]
so \(\lambda_2 \neq 0\). Constraints on \(z_1\) come from \(\mu_R\) and using the above relations they are seen to only involve linear combinations of \(x = |z_1|^2\) and \(1/x\). Thus there are at most 2 values for \(|z_1|\). However, closer inspection reveals that the entries above the diagonal in \(\mu_R\) are
\[
\begin{pmatrix}
* & p|z_1|^2 & 0 \\
* & * & q/|z_1|^2 \\
* & * & *
\end{pmatrix},
\]
with \(p = \xi_2/\zeta, q = \zeta \xi_1\). Thus \(\alpha_{13} = 0, 4\alpha_{12} \alpha_{23} = -pq = -\xi_2 \xi_1\) and there is at most one solution for \(|z_1|\), which together with \(\beta\) specifies the diagonal entries of \(\alpha\). Thus the fibre is either a circle or empty.

Case 2

\(\xi_1 \neq 0, \xi_2 = 0\): we have \(z_1 \neq 0 \neq w_2\) and thus \(z_3 = 0\). Now \(z_2w_1 = 0 = z_2w_3\) divides into cases.

(a) \(z_2 \neq 0, w_1 = 0 = w_3\): gives \(\lambda_2 = \frac{3}{2} z_2w_2 = -2\lambda_1 = -2\lambda_3 \neq 0\).
\(z_2 = 3\lambda_2/2w_2, z_1 = \xi_1/w_2\). In \(\alpha\), the (1, 2)-entry determines \(z_1 \neq 0\) which specifies \(|w_2|\) uniquely. So the fibre is either a circle or empty.
(b) \( w_1 \neq 0, z_2 = 0 \): gives \( \lambda_1 = \frac{3}{2} z_1 w_1 = -2\lambda_2 = -2\lambda_3 \neq 0, \zeta = z_1 w_3 \). So \( w_1 = 3\lambda/2z_1, w_2 = \xi_1/z_1, w_3 = \zeta/z_1 \). The (1, 2) entry of \( \alpha \) then specifies \(|z_1|\) uniquely and the fibre is a single circle.

(c) \( w_1 = 0 = z_2 \): gives \( \lambda_i = 0 \) for all \( i \) and \( \zeta = z_1 w_3 \), so \( w_2 = \xi_1/z_1, w_3 = \zeta/z_1 \). For \( \zeta \neq 0, |z_1| \) is determined by \( \alpha_{23} \) and the fibre is a circle or empty. For \( \zeta = 0 \), the only non-zero entries are the diagonal ones, with a the first entry \(-2\) times the other two which are equal. These give a single quadratic equation for \(|z_1|\), so the fibre is either 2 circles, 1 circle or empty. For example, the first diagonal entry in \( \mu_R \) is proportional to \( 2|z_1|^2 + |\xi_1|^2/|z_1|^2 \), which attains any sufficiently large positive value at two different values of \(|z_1|\). In particular, the fibre of \( \mu \) can be disconnected.

Case 3

\( \xi_1 = 0 = \xi_2 = \zeta \); there are three types of case:

(a) two \( z \)'s non-zero: \( z_1 \neq 0 \neq z_2 \) implies \( w \equiv 0 \) and \( \beta = 0 \). The off-diagonal entries of \( \alpha \) determine \( z_2 \) and \( z_3 \) in terms of \( z_1 \). If \( \alpha_{23} \neq 0 \), then this determines \(|z_1|\) and the fibre is empty or a circle. Otherwise the diagonal entries lead to a quadratic constraint in \(|z_1|\). The fibres are thus 2 circles, one circle or empty.

(b) one \( z \) and one \( w \) non-zero: then these must have the same index, say \( z_1 \neq 0 \neq w_1 \). So \( w_2 = 0 = w_3 = z_2 = z_3 \). \( \beta \) is diagonal with two repeated eigenvalues, \( w_1 = -3\lambda_1/2z_1 \). \( \alpha \) is necessarily diagonal, the diagonal entries give a quadratic constraint on \(|z_1|\). The fibres are 2 circles, one circle or empty.

(c) one \( z \) or \( w \) non-zero: gives \( \beta = 0 \), \( \alpha \) diagonal with the sign of the entries in \( i\alpha \) determined by whether it is \( z \) or \( w \) that is non-zero. The fibre is either a circle, a point or empty.

5 Implosion

Implosion arose as an abelianisation construction in symplectic geometry \[14\]. Given a Hamiltonian \( K \)-manifold \( M \), one forms a new Hamiltonian space \( M_{\text{impl}} \) with an action of the maximal torus \( T \) of \( K \), such that the symplectic reductions agree

\[
M/\lambda K = M_{\text{impl}}/\lambda T
\]

for \( \lambda \) in the closed positive Weyl chamber. In most cases the implosion is a singular stratified space, even in \( M \) is smooth.

The key example is the implosion \((T^*K)_{\text{impl}} \) of \((T^*K) \) by (say) the right \( K \) action. Because \((T^*K) \) has a \( K \times K \) action, the implosion has a \( K \times T \) action, and in fact we may implose a general Hamiltonian \( K \)-manifold by forming the reduction of \( M \times (T^*K)_{\text{impl}} \) by the diagonal \( K \) action. In this sense \((T^*K)_{\text{impl}} \)
is a universal example for imploding $K$-manifolds. Concretely, $(T^*K)_{impl}$ is obtained from the product of $K$ with the closed positive Weyl chamber $\bar{t}^*$ by stratifying by the face of the Weyl chamber (i.e. by the centraliser $C$ of points in $\bar{t}^*$) and then collapsing by the commutator of $C$.

There is also a more algebraic description of the universal implosion $(T^*K)_{impl}$, as the non-reductive Geometric Invariant Theory (GIT) quotient $K_C//N$, where $N$ is the maximal unipotent subgroup. Note that in general non-reductive quotients need not exist as varieties, due to the possible failure of finite generation for the ring of $N$-invariants. In the above case (and in the hyperKähler and holomorphic symplectic situation described below) it is a non-trivial result that we do have finite generation, so the quotient does exist as an affine variety.

In a series of papers [8, 9, 7] the authors and Kirwan described an analogue of implosion in hyperKähler geometry. There is a hyperKähler metric (due to Kronheimer [18]) with $K \times K$ action on the cotangent bundle $T^*K_C$, and the idea is that the analogue of the universal symplectic implosion should be the complex-symplectic quotient (in the GIT sense) of $T^*K_C$ by $N$. Explicitly, this quotient is $(K_C \times n^0)//N$, which it is often convenient to identify with $(K_C \times b)//N$, where $b$ is the Borel subalgebra.

In the case of $K = SU(n)$ it was shown in [8] that $(K_C \times n^0)//N$ arises, via a quiver construction, as a hyperKähler quotient with residual $K \times T$ action. Moreover the hyperKähler quotients by $T$ may be identified with the Kostant varieties, that is the subvarieties of $t_C$ obtained by fixing the values of a generating set of invariant polynomials. For example, reducing at zero gives the nilpotent variety. For general semi-simple $K$, results of Ginzburg and Riche [12] give the existence of the complex-symplectic quotient $(K_C \times n^0)//N$ as an affine variety, and the complex-symplectic quotients by the $T_C$ action again give the Kostant varieties.

There is a link here with some intriguing work by Moore and Tachikawa [21]. They propose a category $HS$ whose objects are complex semi-simple groups, and where elements of $\text{Mor}(G_1, G_2)$ are complex-symplectic manifolds with $G_1 \times G_2$ action (together with a commuting circle action that acts on the complex-symplectic form with weight $-2$). We compose morphisms $X \in \text{Mor}(G_1, G_2)$ and $Y \in \text{Mor}(G_2, G_3)$ by taking the complex-symplectic reduction of $X \times Y$ by the diagonal $G_2$ action. The Kronheimer space $T^*G$, where $G = K_C$, gives a canonical element of $\text{Mor}(G, G)$—in fact this functions as the identity in $\text{Mor}(G, G)$. The implosion now may be viewed as giving an element of $\text{Mor}(G, T_C)$, where $G = K_C$ and $T_C$ is the complex maximal torus in $G$.

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