TOWARD A CONJECTURE OF TAN AND TU ON FIBERED GENERAL TYPE SURFACES

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Abstract. Given a semistable non-isotrivial fibered surface $f : X \to \mathbb{P}^1$ it was conjectured by Tan and Tu that if $X$ is of general type, then $f$ admits at least 7 singular fibers. In this paper we prove this conjecture in several particular cases, i.e. assuming $f$ is obtained from blowing-up the base locus of a transversal pencil on an exceptional minimal surface $S$ or assuming that $f$ is obtained as the blow-up of the base locus of a transversal and adjoint pencil on a minimal surface.

1. Notation

We work on the complex field number $\mathbb{C}$. All the considered varieties will be assumed irreducible and projective. Through the paper we shall use the following notation:

- $X$ will be a general type surface and $S$ its minimal model. $\pi : X \to S$ will be the associated chain of blowing-downs.
- $f : X \to \mathbb{P}^1$ will be a semi-stable, non-isotrivial fibration, and $F$ the general fibre. We set $g =$ the genus of $F$. By $C$ we will denote the image of $F$ under $\pi$ and

  $\Lambda : S \dashrightarrow \mathbb{P}^1$,

  the pencil induced by $f$. We denote by $s$ the number of singular fibers.
- We shall say that $\Lambda$ is transversal if its general member $C \in \Lambda$ is non-singular and intersects transversally any other general member $C' \in \Lambda$.
- We’ll freely use the standard notation in surfaces’ theory. In particular $q = h^1(X, \mathcal{O}_X)$ will be the irregularity of $X$ and $p_g = h^0(X, K_X)$ its geometric genus. By $e(X)$ will be denote the topological Euler characteristic.
- Given divisors $D_1$ and $D_2$ in an algebraic surface we denote as usual $D_1 \equiv D_2$ for the numeric equivalence and $D_1 \sim D_2$ for the linear one. Most of the time we will be working on regular surfaces and in this case we indistinctly use both symbols.
- The number $m$ will be:

  $m := K_S^2 - K_X^2 = e(X) - e(S)$. 

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Note that in general $m \leq C^2$. Adjunction Formula gives the important inequality $C.K_S + m \leq 2(g-1)$ with equality holding if $\Lambda$ is transversal.

2. Introduction

Let $f : X \to \mathbb{P}^1$ be a non-isotrivial semistable fibered surface. It is a classical result that such a fibration admits a certain number of singular fibers (in contrast to the case when the base of the fibration is not rational or elliptic [1]). In the seminal paper [2] it was proved that this number $s$ must be at least 4. Subsequently the bound have been sharpened to $s \geq 5$ if $g \geq 2$ and $s \geq 6$ if the surface $X$ is not birationally ruled ([11] [12], [13]). It was conjectured by Tan and Tu, in a preprint previous to [13] that this bound must raise to 7 if $X$ is of general type (Tan-Tu conjecture for what follows). They also proved the conjecture for genus $2 \leq g \leq 4$ and characterized fibrations of genus 5 with $s = 6$ on a general type surface as those obtained from blowing up the base locus of a transversal pencil on a Horikawa surface. Using this characterization the proof of the conjecture for $g = 5$ was completed in [15].

Roughly speaking the proof of these bounds are based, in case $g \geq 2$ on the canonical class inequality:

$$K_f^2 < 2(g-1)(2gB - 2 + s),$$

for any non iso-trivial semi-stable fibration $f : X \to B$ of genus $g \geq 2$. Here $K_f$ is the relatively canonical divisor, $K_f := K_X - f^*K_B$, which turns out to be $K_X(-2F)$ if $B = \mathbb{P}^1$. The bounds for $s$ are obtained from the positivity properties of $K_f$ and $K_f(-F)$.

Unfortunately, this approach is useless for proving Tan-Tu conjecture, since in this case the only relevant information the inequality provides is that $K_X^2 < 0$ if $s = 6$. However for most of the cases a fibered surface (of general type or not) satisfies $K_X^2 < 0$. Indeed, such a surface is obtained by blowing up the base locus of some pencil $\Lambda$ on a minimal surface $S$.

In this paper we deal with Tan-Tu conjecture in some particular cases. We first impose on the minimal model $S$ of $X$ the conditions of being exceptional in the sense that either $K_S^2 = 2$, $p_g = 3$ or $K_S^2 = 1$ and $p_g = 2$ (see [11], VII.8) and assuming that $f$ is obtained as the blowing-up of the base locus of a transversal pencil $\Lambda$ in $S$. In these cases we are able to prove the conjecture using explicit descriptions of these surfaces as double coverings of rational surfaces, by means of the canonical or bi-canonical map. This is the content of Section 3, Theorems 4.1 and 4.2. In this sense there is some hope of extending the result to a wider class of surfaces.

Next, in section 5 we prove the conjecture assuming that the pencil $\Lambda$ is adjoint, i.e. $C = B + K_S$ with $B$ a big and nef divisor in $S$ and $K_S^2 \geq 3$ and for $K_S^2 \leq 2$ assuming not only that $\Lambda$ is adjoint, but also transversal (Theorem 5.3). The case $K_S^2 = p_g = 1$ is the subtler and is stated and proved in Proposition 5.2.

We list below the cases in which Tan-Tu conjecture have been proved in this article:

- If $\chi(O_S) = 1$ with no extra assumption on $S$ or $\Lambda$.
- If either $K_S^2 = 1$, $p_g = 2$ or $K_S^2 = 2$, $p_g = 3$ assuming $\Lambda$ is transversal.
- If $\Lambda$ is transversal and adjoint and $K_S^2 \leq 2$ or if $\Lambda$ is merely adjoint and $K_S^2 \geq 3$. 

3. SOME GENERAL FACTS AND RESULTS

The following inequality will be systematically used: given a semi-stable non-isotrivial fibration of genus \( g \geq 2 \), \( f : X \rightarrow B \), for any integer \( e \geq 2 \):

\[
\frac{1}{3} e^2 (K_X^2 - 2(g - 1)(6(gB - 1) + s - s/e)) \leq e_f.
\]

Original formulation involves the number of \((-2)\) vertical curves in \( X \), but for our purposes this version will be sufficient. The proof can be found in [12] and is based on successive changes of the base \( B \) of the fibration. We call this Tan’s inequality. In particular if \( B = \mathbb{P}^1 \) (our interest’s case) and \( s = 6 \) we obtain:

\[
(1) \quad \frac{1}{3} e(K_X^2 e + 12(g - 1)) \leq e_f.
\]

Useful forms of this inequality are collected in the following:

**Lemma 3.1.** Let \( f : X \rightarrow \mathbb{P}^1 \) be semistable, non-isotrivial of genus \( g \geq 2 \). If \( s = 6 \), then evaluating (1) we obtain:

i) \[ K^2_S + C.K_S \leq 3\chi(O_S) \text{ if } e = 3, \]

ii) \[ 19K^2_S + 18C.K_S \leq m + 36\chi(O_S) \text{ if } e = 4, \]

iii) \[ 7K^2_S + 6C.K_S \leq m + 9\chi(O_S) \text{ if } e = 5. \]

**Proof.** Evaluate (1) at the indicated value of \( e \) and substitute:

\[
m + C.K_S \leq 2(g - 1),
\]

\[
e(X) = 12\chi(O_X) - K^2_X \text{ (Noether’s Formula)},
\]

\[
K^2_X = K^2_S - m,
\]

and

\[
e_f = 4(g - 1) + e(X) \text{ (because } f \text{ is semistable).}
\]

\[\square\]

We start by sharpening the bound for \( m \) obtained in [13], inequality (3.2) (compare with the proof of Theorem 2.1(4) in [13]).

**Lemma 3.2.** Assume \( f : X \rightarrow \mathbb{P}^1 \) is a semistable fibration and the minimal model \( S \) of \( X \) is a general type surface, then:

\[
m \leq C^2 \leq \frac{4(g - 1) + K^2_S - \sqrt{8(g - 1)K^2_S + (K^2_S)^2}}{2}.
\]
Proof. From Index Hodge Theorem applied to $K_S$ and $C$ we get:

$$mK_S^2 \leq C^2K_S^2 \leq (C.K_S)^2.$$ 

Adjoint formula gives $C.K_S \leq 2(g-1) - m$, therefore

$$0 \leq m^2 - (4(g-1) + K_S^2)m + 4(g-1)^2.$$ 

Consider the right hand term of the previous inequality as a polynomial in $m$. Its discriminant turns out to be

$$\Delta = 8(g-1)K_S^2 + (K_S^2)^2.$$ 

Thus its roots are:

$$m_{\pm} = \frac{4(g-1) + K_S^2 \pm \sqrt{\Delta}}{2}.$$ 

It follows that either $m \leq m_-$ or $m \geq m_+$. We claim that $m \leq m_-$ is the only possible case. Indeed, if

$$\frac{4(g-1) + K_S^2 + \sqrt{\Delta}}{2} \leq m,$$

then $C.K_S \leq 2(g-1) - m \leq 0$ that is impossible. $\square$

As a consequence of Lemma 3.2 we obtain our first general fact concerning the number $s$:

**Proposition 3.3.** Let $S$ be of general type. If $s = 6$, then

$$K_S^2 + \sqrt{8(g-1)K_S^2 + (K_S^2)^2} \leq 6\chi(\mathcal{O}_S).$$

In particular, if $g \geq 6$ and $\chi(\mathcal{O}_S) = 1$, then $s \geq 7$.

Proof. Assume $s = 6$, by Tan’s inequality:

$$\frac{1}{3}e(K_S^2e + 12(g-1)) \leq e_f,$$

for any natural number $e \geq 2$. Evaluating in $e = 3$ we obtain:

(2) $$3K_S^2 + 12(g-1) \leq e_f.$$ 

Since $f$ is semistable $e(X) = -4(g-1) + e_f$ and (2) becomes:

(3) $$3K_X^2 + 12(g-1) \leq e_f = e(X) + 4(g-1).$$

By definition $m = K_S^2 - K_X^2 = e(X) - e(S)$. This, combined with Noether Formula $e(S) = 12\chi(\mathcal{O}_S) - K_S^2$ leads to:

$$K_S^2 + 2(g-1) \leq 3\chi(\mathcal{O}_S) + m.$$ 

The desired inequality follows after applying Lemma 3.2 $\square$

The importance of this Proposition is that given a family of general type surfaces with given invariants $K_S^2$ and $\chi(\mathcal{O}_S)$ only a finite numbers of values of $g$ must be discharged in order to conclude that a fibered surface birational to $S$ has at least 7 singular fibers. This principle will be illustrated in the next section.

Note that in Proposition 3.3 the hypothesis of being $S$ of general type is essential ([2], Example 2).
4. Fibrations obtained from exceptional surfaces

Surfaces satisfying either $K_S^2 = 2$ and $p_g = 3$ or $K_S^2 = 1$ and $p_g = 2$ are called exceptional because of the behavior of the tri-canonical map (Theorem VII 8.3 in [1]). In this section we study fibrations in these surfaces by means of the canonical and bi-canonical map, respectively.

**Theorem 4.1.** Assume $S$ satisfies $K_S^2 = 2$ and $p_g = 3$. Let $f : X \to \mathbb{P}^1$ be obtained as the blowing-up of a transversal pencil $\Lambda$ in $S$. Then, $s \geq 7$.

**Proof.** By Debarre’s Inequality ([6]) we know that $q = 0$ and therefore $\chi(O_S) = 4$.

Assume $s = 6$, by Proposition 3.3 it is sufficient to consider:

$$6 \leq g \leq 31.$$  

In this case the canonical map $\phi_{K_S}$ defines a 2 : 1 covering:

$$\phi_{K_S} : S \to \mathbb{P}^2,$$

ramified along $R = 4K_S$ ([8]). Consider the restriction $\phi := \phi_{K_S}|C$.

First, we consider the case $\phi$ is a 2 : 1 covering. Denote by $G \subset \mathbb{P}^2$ the image of $C$ under $\phi$ and $d$ for its degree. We have $C = \phi^*G$ and therefore:

$$m = C^2 = (\phi^*G)^2 = 2G^2 = 2d^2.$$  

On the other hand, taking into account that $\phi^*H = K_S$, with $H$ a hyperplane section (i.e. the divisor associated with $O_{\mathbb{P}^2}(1)$) we obtain:

$$d = \frac{C.K_S}{2} \quad \text{and} \quad m = \frac{(C.K_S)^2}{2}.$$  

From this we get $2m = (2(g - 1) - m)^2$. The possible values of $g$ satisfying such a relation in the range $6 \leq g \leq 31$ are: $g = 7, 13, 21, 31$. The corresponding values of $m$ and $C.K_S$ are listed below:

| $g - 1$ | $C.K_S$ | $m$ |
|---------|---------|-----|
| 6       | 4       | 8   |
| 12      | 6       | 18  |
| 20      | 8       | 32  |
| 30      | 10      | 50  |

If $g - 1 = 20$ or 30 we use the inequalities of Lemma 3.1(ii), in order to obtain a contradiction.

Assume $g - 1 = 6$. In this case the number of singular points in the fibers of $f$ will be:

$$E_f = e(X) + 4(g - 1)$$  
$$= -K_X^2 + 12\chi(O_X) + 4(g - 1)$$  
$$= 6 + 48 + 24 = 78.$$  

Since $s = 6$ there exists at least a singular fiber $F_0$ of $f$ containing $\sigma_0 = 13 = 78/6$ singular points. But then, denoting $F_0 = F_1 + ... + F_l$ for the decomposition into irreducible components:
\[ 6 = (g - 1) = \sum_{i=1}^{l} (g_i - 1) + \sigma_0 \]
\[ \geq \sum_{i=1}^{l} (g_i - 1) + 13, \]
with \(g_i\) standing for the geometric genus of \(F_i\) and \(\sigma_0\) for the number of singular points of \(F_0\). We have:
\[ \sum_{i=1}^{l} (g_i - 1) \leq -7. \]

In particular, \(F_0\) have at least 7 irreducible rational components. Being \(\phi\) a 2 : 1 covering and \(C\) be applied under \(\phi\) to a plane curve \(G\) of degree \(d = C.K_S/2 = 2\) we have that the number \(l\) of irreducible components of \(F_0\) is at most 4. In this way we get a contradiction with the assumption \(s = 6\).

The case \(g - 1 = 12\) follows after similar considerations, this time taking into account that the covering \(\phi\) sends \(C\) onto a curve \(G\) of degree 3.

Consider now the case \(\phi\) restricted to \(C\) is 1 : 1. In this case there exists a curve \(C'\) such that \(C + C' = \phi^* G\).

Moreover, since the ramifications of \(\phi\) over \(C\) occurs exactly on the intersections of \(C\) and \(C'\) we have \(C.C' = C.R = C.4K_S\). Similarly, we conclude that \(C.K_S = C'.K_S\). Also, we have that:
\[ 2C.K_S = \phi^* G.K_S = 2d, \]
i.e. \(d = C.K_S\).

It follows that \(\phi^* G = \phi^* ((C.K_S)H) = (C.K_S)K_S\). From
\[ C + C' = (C.K_S)K_S, \]
we obtain, after intersecting with \(C\):
\[ 4K_S.C = C.C' = (C.K_S)^2 - m. \]

The possibilities for such a relationship are listed below:

| \((g - 1)\) | \(C.K_S\) | \(m\) |
|----------|----------|------|
| 5        | 5        | 5    |
| 9        | 6        | 12   |
| 14       | 7        | 21   |
| 20       | 8        | 32   |
| 27       | 9        | 45   |

Using Lemma 3.1 we obtain a contradiction in the following cases: \(g - 1 = 20\) and \(27\) when evaluating at \(e = 4\), and \(g - 1 = 5, 9\) when evaluating at \(e = 5\).

Finally, we must analyze the case \(g - 1 = 14\). In this case,
\[ e_f = 4(g - 1) - K_X^2 + 12\chi(O_X) \]
\[ = 123. \]
Assuming \( s = 6 \) there must to exist a singular fiber \( F_0 \) with its number of nodes \( \sigma_0 \geq 21 \). Assume \( F_0 = F_1 + \ldots + F_l \) is its decomposition into irreducible components. Denoting by \( g_i \) the geometric genus of \( F_i \) we have:

\[
14 = \sum_{i=1}^{l} (g_i - 1) + \sigma_0 \geq \sum_{i=1}^{l} (g_i - 1) + 21.
\]

From this we get \( l \geq 7 \). Now, the image of \( F_0 \) in \( \mathbb{P}^2 \) (under the composition \( \phi \circ \pi \)) is a degree 7 curve \( G_0 \). It follows that \( l = 7 \), \( G_0 = L_1 + \ldots + L_7 \) is the sum of seven lines \( L_i \), and \( C_0 := \pi(F_0) \) must be the sum of seven irreducible components \( C_0 = C_1 + \ldots + C_7 \) (\( C_i = \pi(F_i) \)). We have, moreover, \( \phi^* L_i = C_i + C_i' \equiv K_S \), \( C_i^2 = C_i'^2 = -3 \) and \( C_i C_i' = 4 \).

A simple cohomological computation shows that \( h^0(\mathcal{O}_S(C_i + C_i + C_i)) = 1 \) for any indexes \( i_1, i_2, i_3 \in \{1, \ldots, 7\} \) and \( h^0(\mathcal{O}_S(C_1 + \ldots + C_4)) = 2 \). This, together with \( (C_1 + \ldots + C_4)^2 = 0 \) means that \( |C_1 + \ldots + C_4| \) is a base point free pencil. Call \( \Delta := C_1 + \ldots + C_4 \). We have, \( C_i^2 \Delta = 0 \) for \( i = 5, 6, 7 \), thus \( C_i^2 + C_6^2 + C_7^2 \), being connected, must be a vertical divisor with respect to \( |\Delta| \). We conclude the existence of an effective divisor \( D' \) such that \( \Delta \sim C_5^2 + C_6^2 + C_7^2 + D' \). It is easy to deduce that \( D' \) is a rational \((-3)\)-curve and \( \phi(D') \) is a line in \( \mathbb{P}^2 \).

From this relation and

\[
(C_1 + C_1') + \ldots + (C_7 + C_7') \sim 7K_S,
\]

it follows that:

\[
C \sim C_1 + \ldots + C_7 \sim 3K_S + D'.
\]

This gives a contradiction, as the image \( \phi(C) \) is a degree 7 curve and the image \( \phi(3K_S + D') \) is a degree 4 curve.

\[\square\]

**Theorem 4.2.** Assume \( f : X \to \mathbb{P}^1 \) is obtained as a blow up of a transversal pencil \( \Lambda \) on a minimal surface \( S \) with \( K_S^2 = 1 \) and \( p_g = 2 \). Then \( s \geq 7 \).

**Proof.** By Proposition 3.3 we must consider only the values \( 6 \leq g \leq 37 \).

Considering the classical Horikawa’s construction and notation (14), let \( \Pi : \tilde{S} \to S \) be the blowing up centered in the base point \( p \) of \( |K_S| \), denote by \( E \) its exceptional divisor and consider the ramified double covering:

\[\phi_2 : \tilde{S} \to \mathbb{P}^2.\]

The map \( \phi_2 \) is given as follow: the bicanonical map of \( S \) determines a double cover on the singular quadric \( Q \subset \mathbb{P}^3 \), the singular point being the image of \( p \). \( \phi_2 \) is the induced map on \( \tilde{S} \) after considering the desingularization \( \mathbb{F}_2 \) of the quadric. The locus branch of \( \phi_2 \) is the divisor \( B = 6\Delta + 10\Gamma \), with \( \Delta \) and \( \Gamma \) denoting respectively the class of the \((-2)\)-section and the class of the fiber in \( \mathbb{F}_2 \) of the structural morphism and the ramification divisor \( R = 5K_S + E \). Here and in what follows given \( D \) any divisor in \( S \) we just write \( D \) for the divisor \( \Pi^*D \) in \( \tilde{S} \).

Denote by \( \lambda \) the induced pencil on \( \tilde{S} \). Depending on whether \( p \) is a base point of \( \Lambda \) or not we have
(4) \[ \mathcal{C} = \begin{cases} \Pi^*C & \text{if } p \notin \Lambda \\ \Pi^*C - E & \text{if } p \in \Lambda. \end{cases} \]

Let \( G \) be the image of \( \mathcal{C} \) under \( \phi_2 \). If we denote \( G = a\Delta + b\Gamma \) and considering that \( \phi^*\Delta = 2E, \phi^*\Gamma = K_S - E \) we have

(5) \[ \phi^*G = bK_S + (2a - b)E. \]

Let be \( \phi := \phi_2|_C : \bar{C} \to G \) with \( \deg \phi = n = 1 \) or 2.

We analyze the two cases \( n = 1 \) or 2 and within each one the subcases \( p \in \Lambda \) or not.

**Case 1** \( \phi \) is 2:1. In this case \( \phi^*G = \bar{C} \).

Assume first that \( p \notin \Lambda \).

By (4) and (5) we have that \( 2a - b = 0 \), using this \( \bar{C}^2 = m = b^2 \). On the other hand \( m = bC.K_S \), therefore \( b = C.K_S \). The next table shows the possible values of \( m, C.K_S \) and \( g - 1 \).

| C.K_S | a | m   | g - 1 |
|-------|---|-----|-------|
| 4     | 2 | 16  | 10    |
| 6     | 3 | 36  | 21    |
| 8     | 4 | 64  | 36    |

Assume \( g - 1 = 10 \). In this case the number of singular points in the fibers of \( f \) will be:

\[ e_f = 12\chi(\mathcal{O}_X) - K_X^2 + 4(g - 1) = 36 + 15 + 4(10) = 91 \]

Since \( s = 6 \) there exists at least a singular fiber \( F_0 \) of \( f \) containing 16 singular points. Let \( F_0 = F_1 + \ldots + F_l \) be the decomposition into irreducible components, then:

\[ 10 = g - 1 \geq \sum_{i=1}^{l} (g_i - 1) + \sigma_0 \]
\[ \geq \sum (g_i - 1) + 16 \]

(6)

where \( g_i \) denotes the geometric genus of \( F_i \). This imply that \( l \geq 6 \). Moreover there are at least 6 of this components that are rational curves and are mapping onto rational components of \( G_0 \).

From [7] (Corollary V.5.18) we know that the possible irreducible curves in \( F_2 \) are: \( \Gamma, \Delta \) and \( \alpha \Delta + \beta \Gamma \) with \( \alpha > 0, \beta \geq 2a \). Let \( G_0 = G_1 + \ldots + G_s \) be the decomposition into irreducible components.

Note that even being \( \Delta \) a rational curve, is not a possibility for any of the \( G_i \)'s, that because if \( G_1 = \Delta \) then

\[ \bar{C}_1 = \phi^*G_0 = 2E \]

that contradicts the semistability of \( f \).

With respect to the components \( G_i = a_i\Delta + b_i\Gamma, a_i > 0, b_i \geq 2a, G_i \) is rational if and only if \( a_i = 1 \) and \( b_i \geq 2 \). Since \( a = \sum_{i=1}^{s} a_i \) and \( b = \sum_{i=1}^{s} b_i \) the only possible
decomposition is $G_0 = G_1 + G_2$ with $G_i = \Delta + 2\Gamma_i$, $i = 1, 2$, i.e. we can’t have the 6 needed rational components.

If $g - 1 = 21, 36$ we use similar arguments with $\sigma_0 = 26, 41$ respectively. In both cases there exist at least 5 rational components and this is not possible because $a < 5$.

Now consider the case $p \in \Lambda$.

By (4) and (5) $b = 2a + 1$, so $m - 1 = \bar{C}^2 = 4a(a + 1)$ and $m = b^2$. Moreover,

$$\bar{C}.K = C.K + 1 = b + 1$$

Therefore $m = (C.K)^2$. Keeping in mind the previous notation, we get the next possible values

| $C.K$ | $a$ | $m$ | $g - 1$ | $e_f$ | $\sigma_0$ | $l \geq$ |
|-------|-----|-----|--------|-------|-----------|--------|
| 3     | 1   | 9   | 6      | 68    | 12        | 6      |
| 5     | 2   | 25  | 15     | 120   | 20        | 5      |
| 7     | 3   | 49  | 28     | 196   | 33        | 5      |

Observe that $l$ is, as before, the minimal number of rational components in $G_0$, so analogous to the case $p \notin \Lambda$, all possibilities in the table can’t occur because of $a < l$.

Case 2 $\phi$ is 1:1. In this case there exists a divisor $\bar{C}'$ such that $\phi^*G = \bar{C} + \bar{C}'$.

Denote as before by $F_0$ a singular fiber of $f$, $C_0$ its image under $\pi$. If $F_0 = F_1 + \ldots + F_k$ is the decomposition of $F_0$ into irreducible components, we denote by $C_0 = C_1 + \ldots + C_k$ the corresponding decomposition for $C_0$ and by $\bar{C}_0 = \bar{C}_1 + \ldots + \bar{C}_k$ the corresponding curves and decomposition in $\bar{S}$ and by $G_0 = G_1 + \ldots + G_k$ their images in $\mathbb{P}^2$.

We begin by stating the following:

**Lemma 4.3.** In the previous situation, let $G_1$ be any irreducible component of $G_0$, then neither $G_1 \sim \Delta$, nor $G_1 \sim \Gamma$ nor $G_1 \sim \Delta + \Delta + 2\Gamma$. In particular, if $G_0 \sim a\Delta + b\Gamma$, then the number of rational irreducible components of $G_0$ is least or equal than $a$.

**Proof.** Let $G_1$ be equivalent to $\Delta$. Then $\phi^*(G_1) \sim 2E$. Note that there exists a divisor $C_1'$ such that $C_1 + C_1' = \phi^*(G_1)$. This implies $C_1 = E$, which is impossible by the definition of $C$.

Now, assume $G_1 \equiv \Gamma$, then $\phi^*G_1 \equiv K_S - E = C_1 + C_1'$. From this, intersecting with $K_S$, and using that $K_S$ is nef we obtain that either $C_1, K_S = 0$ or $K_S, C_1' = 0$. It follows that there exists a curve $D$ on $S$ with $K_S.D = 0$, which contradicts that ampleness of $|K_S|$.

Finally, suppose that $G_1 = \Delta + 2\Gamma$, in this case $\phi^*G_1 \equiv 2K_S$.

Note that for any decomposition $2K_S = A + A'$ we must have that both, $A$ and $A'$ must be irreducible and equivalent to $K_S$ and $A.A' = 1$. Indeed, assume $A$ irreducible, that from $2K_S = A + A'$ with easy it follows that $A.K_S = A'.K_S = 1$. Therefore, $A^2 = 1$, because $2K_S$ is 1-connected, $A \equiv K_S$ follows from HIT.

One established this fact, just note that if $2K_S = C_1 + C_1'$, then $\phi : C_1 \to G_1$ can not be 1 : 1, because $C_1$ is a genus 2 curve.

The last assertion follows from the fact that the only irreducible rational curves on $\mathbb{P}^2$ are equivalent to either $\Delta, \Gamma$ or $\Delta + b\Gamma$ with $b \geq 2$. $\square$
Continuing the proof of the Theorem, assume first that \( p \notin \Lambda \). We have the commutative diagram:

\[
\begin{array}{ccc}
S & \xrightarrow{\phi_2} & \mathbb{P}_2 \\
\downarrow & & \downarrow \\
S & \xrightarrow{\phi_{2K}} & Q
\end{array}
\]

therefore \( \phi_{2K}(\Pi(\bar{C}')) \) contains the singular point of \( Q \) and from this it follows that \( \bar{C}' = \Pi^*C' \), for some effective divisor \( C' \) in \( S \). By (5) \( b = 2a \) and \( \bar{C} + \bar{C}' \sim bK_S \).

Therefore,

\[
C.K_S + C'.K_S = (\bar{C} + \bar{C}')K_S = (bK_S)(K_S + E) = b.
\]

The ramifications of \( \phi_2 \) occurring on \( \bar{C} \) are given by the intersections of \( \bar{C} \) and \( \bar{C}' \), so we have \( \bar{C}.\bar{C}' = \bar{C}.R = \bar{C}'.R \). In particular, the right hand term of the previous equation implies that \( C.K_S = C'.K_S \) and \( b = 2C.K_S \).

Moreover,

\[
bC.K_S = \phi^*G.\bar{C} = (\bar{C} + \bar{C}').\bar{C} = m + \bar{C}.\bar{C}' = m + 5C.K_S.
\]

We conclude that \( m = C.K_S(2C.K_S - 5) \) and we get the possible values (keeping in mind the previous notation introduced for \( \sigma_0 \) and \( l \)):

| \( b \) | \( C.K_S \) | \( m \) | \( g - 1 \) |
|---|---|---|---|
| 8 | 4 | 12 | 8 |
| 10 | 5 | 25 | 15 |
| 12 | 6 | 42 | 24 |
| 14 | 7 | 63 | 35 |

The values \( g - 1 = 24, 35 \) are impossible because of the Hodge Index Theorem.

If \( g - 1 = 8 \) we have that \( e_f = 79 \) and then there must exists a singular fiber \( F_0 \) with at least 14 singular points. It follows that \( G_0 \) has 6 or more rational components. Using that \( a = 4 \) and Lemma 4.3, we get a contradiction. The case \( g - 1 = 15 \) follows after similar considerations.

It remains to analyze the case \( p \in \Lambda \).

As in the previous case, like \( \bar{C} = C - E \) also \( \bar{C}' = C' - E \). From this we get \( 2a - b = -2 \). Therefore we have:

\[
\phi^*_2G = C + C' - 2E = bK_S - 2E.
\]

Moreover, \( \bar{C}.R = \bar{C}'.R \) and then \( C.K_S + 1 = \bar{C}.K_S = \bar{C}'.K_S \)

We get the next formulas

\[
\phi^*_2G.K_S = b + 2 = 2C.K_S + 2
\]

and therefore

\[
m + 5C.K_S = \phi^*_2G.\bar{C} = (bK_S - 2E)(C - E) = bC.K_S - 2
\]

We conclude that \( b = 2C.K_S \) and \( m = 2(C.K_S)^2 - 5C.K_S - 2 \). The table of possible values is:
If \( g - 1 = 23, 34 \) we get a contradiction by Hodge Index Theorem. If \( g - 1 = 7 \), then \( c_f = 73 \), therefore there exists a singular fiber \( F_0 \) with at least 13 singular points and at least 6 rational components. Taking in consideration that \( a = 3 \) and Lemma 4.3 we obtain a contradiction. The case \( g - 1 = 14 \) is similar. □

5. The adjoint case

In this section we consider fibrations \( f : X \rightarrow \mathbb{P}^1 \) satisfying the property that \( C \) is an adjoint linear system, i.e., \( C \equiv B + K_S \) with \( B \) a big and nef divisor. The typical example for bearing in mind is \( C \equiv nK_S \), i.e. the fibration \( f \) is obtained after blowing up the base locus of a generic pencil of curves \( \Lambda \subset |nK_S| \).

We collect, for further use, some general elemental facts in the following:

Lemma 5.1. Assume \( C \equiv B + K_S \) with \( B \) a big and nef divisor, denoting by \( g_B \) the arithmetic genus of \( B \), we have:

\begin{itemize}
  \item[i)] \( 2(g_B - 1) = B^2 + B.K_S \).
  \item[ii)] \( m = (g_B - 1) + (g - 1). \)
  \item[iii)] \( (g - 1) = (g_B - 1) + B.K_S + K_S^2. \)
  \item[iv)] \( 2 \leq B.K_S \), and if \( g \geq 5 \), then \( g + 1 \leq m. \)
\end{itemize}

Proof. Assertions i)-iii) follow immediately from adjunction formula. As for iv), note that it is enough to prove that \( 2 \leq B.K_S \), because then using i) \( g_B - 1 \geq 2 \) and the desired inequality follows from ii). Now, if \( B.K_S = 1 \), by Index Hodge Theorem \( B^2 = K_S^2 = 1 \) and applying from \( m = B^2 + 2B.K_S + K_S^2 \) we have \( m = 4 \). On the other hand, from i) and ii) \( m = 1 + (g - 1) = g \geq 5. \) □

We start by studying the case \( K_S^2 = 1 \), which is similar in nature to Theorems 4.1 and 4.2

Proposition 5.2. Let \( f : X \rightarrow \mathbb{P}^1 \) be a fibration obtained as the blowing up of the base locus of a transversal and adjoint pencil \( \Lambda \) on a minimal surface with \( K_S^2 = 1 \). Then, \( s \geq 7. \)

Proof. By Noether Inequality \( p_g \leq 2 \), and by Proposition 4.3 and Theorem 4.2 we can assume \( p_g = 1 \). It is well known that for a surface with such invariants the bicanonical map \( \phi_{2K_S} \) defines a \( 4 : 1 \) morphism onto \( \mathbb{P}^2 \) ( [9], [10], [11] ):

\[ \phi_{2K_S} : S \rightarrow \mathbb{P}^2, \]

ramified along a divisor \( R \equiv 7K_S \). We consider, as before, the restriction of this map to \( C \):
In this case \( n \) is a divisor of 4, \( G \subset \mathbb{P}^2 \) is the image of \( C \) and \( j \) denotes its normalization. Denote by \( d \) the degree of \( G \).

If we assume \( s = 6 \) we only need, according to Proposition 3.3 to consider \( 6 \leq g \leq 16 \).

We start by analyzing the case \( n = 4 \): in this case we have \( C = \phi^* G \) and therefore:

\[
m = C^2 = (\phi^* G)^2 = 4G^2 = 4d^2.
\]

On the other hand, taking into account that \( \phi^* H \equiv 2K_S \), with \( H \) hyperplane section (i.e the divisor associated with \( O_{\mathbb{P}^2}(1) \)) we obtain:

\[
C.K_S = \phi^* G.K_S = \phi^* (dH).K_S = 2d.
\]

Adjunction formula gives \( 2(g - 1) = 2d(2d + 1) \). The only value of \( g \) that satisfies the relation in the range \( 6 \leq g \leq 16 \) is \( g - 1 = 10 \) with \( d = 2 \) and \( m = 16 \). Evaluating in Tan’s inequality for \( e = 4 \) (Lemma 3.1 ii)) we obtain a contradiction.

If \( n < 4 \), then there exists an effective divisor \( C' > 0 \) such that:

\[
C + C' \equiv \phi^* K_S G \equiv 2dK_S.
\]

Note that:

\[
C'.K_S = 2d - C.K_S,
\]

and

\[
C'^2 = 4d^2 - 4dC.K_S + C^2.
\]

Moreover, since \( h^1(C) = h^2(C) = 0 \),

\[
h^0(C) = \frac{C^2 - C.K_S}{2} + 2.
\]

Next, the only possibility for \( h^2(C') = h^0(K_S - C') \neq 0 \) is \( C' \equiv K_S \), because \( p_g = 1 \). This would imply \( C \equiv K_S \), which is impossible.

It is easy to prove that \( H^0(C) \simeq H^0(C') \) and by Riemann-Roch we get:

\[
h^0(C) = h^0(C') \geq \frac{C'^2 - C'.K_S}{2} + 2,
\]

and substituting the values of \( C'^2 \) and \( C'.K_S \):

\[
h^0(C) \geq h^0(C) + \frac{4d^2 - (4d - 2)C.K_S - 2d}{2}.
\]

This implies,

\[
2d^2 - (2d - 1)C.K_S - d \leq 0,
\]

that is equivalent to \( d \leq C.K_S \).

Now, assume \( n = 1 \). The linear system \( |2K_S| |_C \) defines a base point free linear system on \( C \) and the associated map a \( 1 : 1 \) cover onto a plane degree \( d \) curve.
Thus, we have $d = 2C.K_S$ and we obtain a contradiction with the just obtained bound $d \leq C.K_S$.

The case $n = 2$ remains to be analyzed: in this case we have $C.K_S = d$. Note that the intersections of $C$ and $C'$ gives place to ramifications points of $\phi_{2K_S}$. Therefore:

$$2dC.K_S - m = C.C' \leq C.R = 7C.K_S.$$ 

From this we get:

$$d(2d - 7) \leq m \leq 2d^2 - 2.$$ 

It follows that $d \leq 5$. In general we have that

$$e_f = e(X) + 4(g - 1) = 24 - (1 - m) + 4(g - 1),$$

$$= 23 - d + 6(g - 1).$$

Thus, assuming $s = 6$ and $d \leq 3$, there must exits a singular fiber $F_0$ of $f$ having at least $(g - 1) + 4$ nodes. Call $\sigma_0$ the number of nodes of $F_0$. Note that the number of nodes of $C_0 = \pi(F_0)$ is also $\sigma_0$. Then, $\sigma_0 \geq 9$. On the other hand, the plane curve $G_0 = \phi_{2K_S}(C_0)$, being of degree $d \leq 3$ admits at most 3 nodes. In this way we get the contradiction $\sigma_0 \leq 6$.

Similar argumentations lead to contradictions for the cases $d = 4, 5$. Indeed, if $d = 4$, then $\sigma_0 \geq g + 3$. We have:

$$g - 1 = \sum_{i=1}^{l} (g_i - 1) + \sigma_0 \geq -l + g + 3,$$

with $g_i$ standing for the geometric genus of the components $F_i$ of $F_0$. It follows that $l \geq 4$ and therefore $F_0$ (and in consequence $C_0$) has at least 4 rational components. From this it follows that $G_0$ has at least 2 irreducible rational components. Taking in account that $G_0$ is a degree 4 curve we have that, either $G_0$ contains a line as an irreducible component or it is the product of two irreducible conics. If $G_0$ is the product of two irreducible conics then it has only 4 nodes and we get, as before a contradiction, in any other case, if $L$ is an irreducible component of $G_0$ and $C_0 = C_1 + ... + C_l$, then

$$\phi_{2K_S}^*L = C_i + C'_i,$$

for $C_i$ some rational components of $C_0$. But then:

$$2K_S \equiv C_i + C'_i,$$

and it follows ([11] Lemma 1, page 181) that $C_i = \Delta$, the only effective divisor in $|K_S|$. This give a contradiction, since $\Delta$ is a curve of geometric genus 2.

Finally the case $d = 5$ follows after similar considerations. In this case $C_0$ admits at least 3 irreducible rational components and $G_0$ at least 2 irreducible rational components. The only subtle case to be treated careful being the possibility that $G_0 = Q + E$, with $Q$ an irreducible conic and $E$ a singular irreducible cubic. But in this case either $Q$ or $E$ must satisfies that it pull back under $\phi_{2K_S}$ is the sum of two irreducible components $C_i + C_j$ of $C_0$.

Note that $\phi_{2K_S}^*E = C_i + C_j$ is impossible, because then
and then \( K_S.(C_i + C_j) = 6 \), that contradicts \( K_S.C_0 = 5 \). On the other hand

\[
4K_S = \phi^*_2 K_S Q = C_i + C_j
\]

implies that \( C_0 \) has exactly 3 irreducible components: \( C_0 = C_1 + C_2 + C_3 \). Suppose \( i = 1 \) and \( j = 2 \), then \( C_3.K_S = 1 \) and

\[
\phi_{2K_S} : C_3 \to E,
\]

must be a 2 : 1 map onto a degree cubic and we obtain the contradiction

\[
2K_S.C_3 = 6.
\]

\[ \square \]

Finally we have:

**Theorem 5.3.** Let \( f : X \to \mathbb{P}^1 \) a semi-stable non-isotrivial fibration obtained as the blow-up of the base locus of an adjoint pencil \( \Lambda \) on the minimal surface \( S \). Then if \( K^2_S \geq 3 \) the number \( s \) of singular fibers of \( f \) is at least 7. If \( K^2_S \leq 2 \) and \( \Lambda \) is also transversal, then \( s \geq 7 \).

**Proof.** We assume \( s = 6 \) and \( 2 \leq K^2_S \). From Noether’s inequality:

\[
p_g \leq \frac{K^2_S}{2} + 2,
\]

we have:

\[
\chi(\mathcal{O}_S) \leq \frac{K^2_S}{2} + 3.
\]

Applying Lemma 3.3

\[
K^2_S + 8K^2_S(g - 1) + (K^2_S)^2 \leq 3K^2_S + 18,
\]

which implies

\[
8K^2_S(g - 1) + (K^2_S)^2 \leq 4(K^2_S)^2 + 72K^2_S + 18^2.
\]

Substituting \( 2(g - 1) \geq C.K_S + m = B.K_S + K^2_S + m \), we get:

\[
m \leq \frac{72 - 4B.K_S}{4} + \frac{18^2 - (K^2_S)^2}{4K^2_S}.
\]

Now, combine the previous bound for \( m \) with Lemma 3.1 ii), in order to deduce:

\[
19K^2_S + 19B.K_S \leq 18 + \frac{18^2 - (K^2_S)^2}{4K^2_S} + 108.
\]

Using again the adjoint hypothesis, this amount to:

\[
\frac{(K^2_S)}{4} + 19C.K_S \leq 166,
\]

that is,

\[
C.K_S \leq 8.
\]
Now, use Hodge Index Theorem:

\[ m \leq \frac{(C.K_S)^2}{K_S^2} \leq \frac{64}{K_S^2}, \]

and apply one more time Lemma 3.1 ii):

\[ K_S^2 + 18C.K_S \leq \frac{64}{K_S^2} + 108, \]

\[ 19K_S^2 + 18B.K_S \leq \frac{64}{K_S^2} + 108. \]

Finally, using \( 2 \leq B.K_S, (\text{Lemma 5.1 iv}) \) we arrive to:

\[ 19K_S^2 \leq \frac{64}{K_S^2} + 72. \]

This implies \( K_S^2 \leq 4 \). Thus, only the cases \( K_S^2 = 2, 3, 4 \) remains to be discharged. This is easy and essentially is a reproduction of the previous argument.

For instance, for case \( K_S^2 = 2 \) we have by Lemma 5.2 and Proposition 3.3 that \( g \leq 16 \) and \( m \leq 26 \), moreover, assuming \( \Lambda \) is transversal, we can apply Theorem 4.1 and assume that \( \chi(O_S) \leq 3 \). Evaluating Tan’s Inequality at \( e = 5 \) (Lemma 3.1 iii)) we get:

\[ 7K_S^2 + 6C.K_S \leq m + 9\chi(O_S), \]

that, under our fixed values becomes \( C.K_S \leq 6 \). Using Hodge Index Theorem we obtain \( m \leq 18 \). Evaluating again Tan’s Inequality at \( e = 4 \) we have \( C.K_S \leq 4 \) and \( m \leq 8 \). Finally, we evaluate once again Tan’s Inequality at \( e = 5 \) and get the final contradiction \( C.K_S \leq 3 \) and \( g + 1 \leq m \leq 4 \).

Cases \( K_S^2 = 3, 4 \) are quite analogous, only that in these cases we don’t need Theorem 4.1 and in consequence the transversality hypothesis can be avoided. \( \square \)

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