Local Existence and Uniqueness of Heat Conductive Compressible Navier–Stokes Equations in the Presence of Vacuum Without Initial Compatibility Conditions

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Abstract. In this paper, we investigate the initial-boundary value problem to the heat conductive compressible Navier–Stokes equations. Local existence and uniqueness of strong solutions is established with any such initial data that the initial density \( \rho_0 \), velocity \( u_0 \), and temperature \( \theta_0 \) satisfy \( \rho_0 \in W^{1,q} \), with \( q \in (3,6) \), \( u_0 \in H^1_0 \), and \( \sqrt{\rho_0} \theta_0 \in L^2 \). The initial density is assumed to be only nonnegative and thus the initial vacuum is allowed. In addition to the necessary regularity assumptions, we do not require any initial compatibility conditions such as those proposed in Cho and Kim (J Differ Equ 228(2):377–411, 2006), which although are widely used in many previous works, put some inconvenient constrains on the initial data. Due to the weaker regularities of the initial data and the absence of the initial compatibility conditions, leading to weaker regularities of the solutions compared with those in the previous works, the uniqueness of solutions obtained in the current paper does not follow from the arguments used in the existing literatures. Our proof of the uniqueness of solutions is based on the following new idea of two-stages argument: (1) showing that the difference of two solutions (or part of their components) with the same initial data is controlled by some power functions of the time variable; (2) carrying out some singular-in-time weighted energy differential inequalities fulfilling the structure of the Grönwall inequality. The existence is established in the Euler coordinates, while the uniqueness is proved in the Lagrangian coordinates first and then transformed back to the Euler coordinates.

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1. Introduction

1.1. The Compressible Navier–Stokes Equations

Let \( \Omega \) be a bounded domain in \( \mathbb{R}^3 \) with suitably smooth boundary \( \partial \Omega \). Consider the following heat conductive compressible Navier–Stokes equations in \( \Omega \times (0,T) \):

\[
\begin{align*}
\rho_t + \text{div}(\rho u) &= 0, \\
\rho(u_t + u \cdot \nabla u) - \mu \Delta u - (\mu + \lambda) \nabla \text{div} u + \nabla P &= 0, \\
c_v \rho (\theta_t + u \cdot \nabla \theta) - \kappa \Delta \theta + P \text{div} u &= Q(\nabla u),
\end{align*}
\]

where the unknowns \( \rho \geq 0, u \in \mathbb{R}^3, \) and \( \theta \geq 0, \) respectively, represent the density, velocity, and absolute temperature, \( P = R \rho \theta, \) with positive constant \( R, \) is the pressure, \( c_v \) is a positive constant, constants \( \mu \) and \( \lambda \) are the bulk and shear viscous coefficients, respectively, satisfying the physical constraints

\[
\mu > 0, \quad 2\mu + 3\lambda \geq 0,
\]

positive constant \( \kappa \) is the heat conductive coefficient, and

\[
Q(\nabla u) = \frac{H}{2} |\nabla u + (\nabla u)^T|^2 + \lambda (\text{div} u)^2,
\]

with \((\nabla u)^T\) being the transpose of \( \nabla u. \)
There has been many works on the mathematical studies on the compressible Navier–Stokes equations. In the absence of vacuum, i.e., in the case that the initial density has a uniform positive lower bound, the uniqueness of solutions was first established by Graffi [16] for the isentropic case, and later extended by Serrin [46] to the general case. Local existence of solutions to the compressible Navier–Stokes equations was first established by Nash [44] in the Sobolev type spaces and later by Itaya [21] in the Hölder type spaces, see also [47,48] for further developments. Global well-posedness of strong solutions in one dimension with arbitrary large initial data was first discovered by Kanel [25] for the isentropic case, and thereafter extended by Kazhikhov–Shelukhin [27] and Kazhikhov [26] to the heat conductive case, see also [2,24,33,50] for some related results. For the multi-dimensional case, global well-posedness of strong solutions was first established by Matsumura–Nishida [39–41] for small perturbed initial data around the equilibrium states in some Sobolev spaces of high order and by Hoff [17] for discontinuous initial data. For the local and global well-posedness of strong solutions in the critical spaces, one refers to [3,4,8,9,11,12] and the references therein.

In the presence of vacuum, i.e., in the case that the initial density vanishes in some region, there has been a considerable number of works on the compressible Navier–Stokes equations. Global existence of weak solutions to the isentropic compressible Navier–Stokes equations was established, with the pressure law given by \( P = a \rho^\gamma \) for a positive constant \( a \) and the adiabatic constant \( \gamma \geq \frac{5}{3} \). This was extended to the case that \( \gamma \geq \frac{3}{2} \) by Feireisl–Novotný–Petzeltová [14], and further by Jiang–Zhang [22,23] to the case that \( \gamma > 1 \) but only for the spherically symmetric or axisymmetric solutions, see Bresch–Jabin [1] for some recent developments where the more general stress tensor and pressure laws are allowed. Global existence of weak solutions to the full compressible Navier–Stokes equations under some structure assumptions on the viscous and heat conductive coefficients as well as the equations of states was established by Feireisl [13]. However, the uniqueness of weak solutions is still an open problem.

Same to the case in the absence of vacuum, one can also establish the local existence and uniqueness of strong solutions in the presence of vacuum, if the initial data have suitably high regularities. In fact, Salvi–Stráskraba [45], Choe–Kim [7], and Cho–Choe–Kim [5] established the local well-posedness of strong solutions to the isentropic compressible Navier–Stokes equations with suitably regular initial data satisfying some compatibility conditions. For the full case, Cho–Kim [6] proved the local well-posedness of strong solutions for the Cauchy problem in \( \mathbb{R}^3 \) with initial data satisfying \( \rho_0 - \rho_\infty \in W^{1,q}(\mathbb{R}^3) \cap W^{1,r}(\mathbb{R}^3) \), with \( q \in (3,6), \rho_\infty \geq 0, (u_0,\theta_0) \in D_0^1(\mathbb{R}^3) \cap D^2(\mathbb{R}^3) \), and the following compatibility conditions

\[
-\mu \Delta u_0 - (\mu + \lambda) \nabla \div u_0 + \nabla (R \rho_0 \theta_0) = \sqrt{\rho_0} g_1, \quad (1.4)
\]

\[
\kappa \Delta \theta_0 + \frac{\mu}{2} |\nabla u_0 + (\nabla u_0)^T|^2 + \lambda (\div u_0)^2 = \sqrt{\rho_0} g_2, \quad (1.5)
\]

for \( (g_1, g_2) \in L^2(\mathbb{R}^3) \), where \( r = 2 \) or \( 3 \) if \( \rho_\infty > 0 \) and \( r = 2 \) if \( \rho_\infty = 0 \). However, the corresponding global existence with general initial data is not expectable, due to the recent breakthrough works Merle–Raphaël–Rodnianski–Szeftel [42,43], where the blowup in finite time results are established to the three-dimensional isentropic compressible Navier–Stokes equations with spherical symmetry, for a class of initial data in the presence of far field vacuum. Global existence of strong solutions of small energy but allowed to have large oscillations was first established by Huang–Li–Xin [20] to the Cauchy problem of the isentropic compressible Navier–Stokes equations in \( \mathbb{R}^3 \); see [19,32,34,49] for some further developments in this direction. Different from the multi-dimensional case, in the one-dimensional case, the global well-posedness of strong solutions can be established for arbitrary large initial data for both heat conductive and non-heat-conductive cases, see [30,31,35–37]. In particular, local and global well-posedness of entropy-bounded solutions was established firstly in [35–37].

It should be pointed out that the compatibility conditions (1.4)–(1.5) or their natural amendments play an essential role in the well-posedness theories established in [6,7,45] and, as a results, they are accepted as standard assumptions to get the well-posedness of the compressible Navier–Stokes equations in the presence of vacuum. Note that the compatibility conditions (1.4)–(1.5) ask for some restrictive constraints on the initial data in the vacuum region and also in the neighborhood of the vacuum-nonvacuum interface.
In fact, by the compatibility conditions (1.4) and (1.5), the initial velocity $u_0$ and temperature $\theta_0$ are destined to obey

$$-\mu \Delta u_0 - (\mu + \lambda) \nabla \text{div} u_0 = 0 \quad \text{and} \quad \kappa \Delta \theta_0 + \frac{\mu}{2} \left| \nabla u_0 + (\nabla u_0)^T \right|^2 + \lambda (\text{div} u_0)^2 = 0$$

in the vacuum region, which however seem not physically relevant. From this point of view, the well-posedness theories established in [6,7,45] does not always match the physical requirements. In particular, it does not always provide the desired well-posedness for any suitably smooth initial data without any extra constrains.

Due to the analysis in the above paragraph, it is both mathematically and physically important to establish an alternative well-posedness theory without requiring any initial compatibility conditions like (1.4) and (1.5). The first study towards this direction was made by the first author of this paper in [28] for the inhomogeneous incompressible Navier–Stokes equations, where the local well-posedness was successfully established without any compatibility conditions on the initial data, see Danchin–Mucha [10] for further developments aiming to relax the smoothness of the initial density. Similar local well-posedness theory without any initial compatibility conditions was later established for the isentropic compressible Navier–Stokes equations by Gong–Li–Liu–Zhang [15] and Huang [18] independently. However, for the full compressible Navier–Stokes equations, to the best of our knowledge, the desired local well-posedness theory without any compatibility conditions on the initial data has not been established, and only part result is available, see Lai–Xu–Zhang [28], where they removed (1.5) but still required (1.4).

The aim of this paper is to establish the desired local well-posedness theory to the full compressible Navier–Stokes equations without any extra compatibility conditions beyond the necessary smoothness conditions on the initial data. We also pay some attention to finding some minimal regularities on the initial data to guarantee the well-posedness. In this paper, we consider the initial-boundary value problem; however, the result and method presented this paper work also for the Cauchy problem.

The initial and boundary conditions read as:

$$\begin{align*}
(\rho, \rho u, \rho \theta)|_{t=0} &= (\rho_0, \rho_0 u_0, \rho_0 \theta_0), \\
\n u|_{\partial \Omega} &= 0, \quad \nabla \theta \cdot n|_{\partial \Omega} = 0,
\end{align*}$$

(1.6)

(1.7)

where $\rho_0$, $u_0$, and $\theta_0$ are given functions.

It should be pointed out that the real values of $u_0$ and $\theta_0$ that we need are only in the non-vacuum region $\Omega_+ := \{ x \in \Omega | \rho_0(x) > 0 \}$ but not in the vacuum region $\Omega_0 := \{ x \in \Omega | \rho_0(x) = 0 \}$. Precisely, denote by $u_0^{\text{real}}$ and $\theta_0^{\text{real}}$, respectively, the initial velocity and temperature in the non-vacuum region $\Omega_+$, and define $\mathcal{J}_{\text{ext}}$ as

$$\mathcal{J}_{\text{ext}} = \left\{ (\tilde{u}_0, \tilde{\theta}_0) | (\tilde{u}_0, \tilde{\theta}_0) = (u_0^{\text{real}}, \theta_0^{\text{real}}) \text{ on } \Omega_+, \tilde{u}_0 \in H_0^1(\Omega), \tilde{\theta}_0 \text{ is measurable and finitely valued a.e. in } \Omega \right\},$$

then any $(u_0, \theta_0) \in \mathcal{J}_{\text{ext}}$ can be chosen as the “initial” velocity and temperature without changing the initial condition (1.6). In fact, for any $(u_0, \theta_0) \in \mathcal{J}_{\text{ext}}$, it is clear that

$$\rho_0 u_0 = \begin{cases} 
\rho_0 u_0^{\text{real}} & \text{in } \Omega_+, \\
0 & \text{in } \Omega_0,
\end{cases} \quad \rho_0 \theta_0 = \begin{cases} 
\rho_0 \theta_0^{\text{real}} & \text{in } \Omega_+, \\
0 & \text{in } \Omega_0.
\end{cases}$$

Due to the above explanation, throughout this paper, we always assume that the “initial” velocity $u_0$ and temperature $\theta_0$ are defined on the whole domain such that $u_0 \in H_0^1(\Omega)$ and that $\theta_0$ is Lebesgue measurable and finitely valued almost everywhere.

Before stating the main results, we first clarify some necessary notations being used throughout this paper and state the definition of solutions to be established.

For $1 \leq q \leq \infty$ and positive integer $m$, we use $L^q = L^q(\Omega)$ and $W^{m,q} = W^{m,q}(\Omega)$ to denote the standard Lebesgue and Sobolev spaces, respectively, and we use $H^m$ to replace $W^{m,2}$. For simplicity, we also use notations $L^q$ and $H^m$ to denote the $N$ product spaces $(L^q)^N$ and $(H^m)^N$, respectively. We always use $\| u \|_q$ to denote the $L^q$ norm of $u$, while the $L^2$ norm is further simplified as $\| \cdot \|$. For shortening
the expressions, we sometimes use \( \| (f_1, f_2, \cdots, f_N) \|_X \) to denote the norm \( \sum_{i=1}^{N} \| f_i \|_X \) or its equivalence \( \left( \sum_{i=1}^{N} \| f_i \|_X^2 \right)^{\frac{1}{2}} \).

The strong solutions to be established in this paper are defined as follows.

**Definition 1.1.** Given a positive time \( T \in (0, \infty) \) and let \( q \in (3, 6) \). Assume that \( \theta_0 \) is nonnegative, Lebesgue measurable, and finitely valued a.e.\ in \( \Omega \), and that
\[
0 \leq \rho_0 \in W^{1,q}(\Omega), \quad u_0 \in H_0^1(\Omega), \quad \sqrt{\rho_0} \theta_0 \in L^2(\Omega).
\]
A triple \((\rho, u, \theta)\) is called a strong solution to system (1.1)–(1.3) in \( \Omega \times (0, T) \), subject to (1.6)–(1.7), if it has the regularities
\[
0 \leq \rho \in C([0, T]; L^2) \cap L^\infty(0, T; W^{1,q}), \quad \rho_t \in L^\infty(0, T; L^2),
\]
\[
\sqrt{\rho} u \in C([0, T]; L^2), \quad u \in L^\infty(0, T; H_0^1) \cap L^2(0, T; H^2) \cap L^1(0, T; W^{2,q}),
\]
\[
\sqrt{\rho} u_t \in L^2(0, T; H_0^1), \quad \sqrt{\rho} u \in L^\infty(0, T; H^2) \cap L^2(0, T; W^{2,q}),
\]
\[
\sqrt{\rho} u_t \in L^2(0, T; H_0^1), \quad \sqrt{\rho} \theta \in C([0, T]; L^2), \quad 0 \leq \theta \in L^2(0, T; H_0^1),
\]
\[
\sqrt{\rho} u_t \in L^2(0, T; H_0^1), \quad \sqrt{\rho} \theta \in L^\infty(0, T; H_0^1) \cap L^2(0, T; H^2), \quad \sqrt{\rho} u_t \in L^2(0, T; L^2),
\]
\[
t_0 \in L^2(0, T; W^{2,0}), \quad \theta_0 \in L^2(0, T; W^{2,0}),
\]
satisfies equations (1.1)–(1.3) a.e.\ in \( \Omega \times (0, T) \), and fulfills the initial condition (1.6).

**Remark 1.1.** By the regularities of \( \rho \) stated in Definition 1.1, it follows from the Gagliardo-Nirenberg inequality that \( \rho \in C([0, T]; C(\overline{\Omega})) \). Thanks to this and recalling that \( \sqrt{\rho} u, \sqrt{\rho} \theta \in C([0, T]; L^2) \), it is clear that \( \rho u, \rho \theta \in C([0, T]; L^2) \). Therefore, the initial values of \( \rho u \) and \( \rho \theta \) are well-defined.

We are now ready to state the main result of this paper.

**Theorem 1.1.** Let \( q \in (3, 6) \). Assume that \( \theta_0 \) is nonnegative, Lebesgue measurable, and finitely valued a.e.\ in \( \Omega \),
\[
0 \leq \rho_0 \in W^{1,q}(\Omega), \quad u_0 \in H_0^1(\Omega), \quad \sqrt{\rho_0} \theta_0 \in L^2(\Omega),
\]
and that \( \rho_0 \) is not identically zero. Then, there exists a positive time \( T_0 \) depending only on \( R, \mu, \lambda, c_0, q, \) and \( \Phi_0 \), such that system (1.1)–(1.3), subject to (1.6)–(1.7), has a unique strong solution in \( \Omega \times (0, T_0) \), where \( \Phi_0 := \| \rho_0 \|_\infty + \| \nabla \rho_0 \|_q + \| (\sqrt{\rho_0} \theta_0, \nabla u_0) \|_2 \).

**Remark 1.2.** (i) No compatibility conditions as those in [5–7, 28, 45] are required in Theorem 1.1. Comparing with the result proved in [28], where compatibility condition (1.5) was removed but (1.4) was still required, in Theorem 1.1, both (1.4) and (1.5) were removed.

(ii) The arguments presented in this paper with slightly modifications work also for the Cauchy problem and similar result as in Theorem 1.1 still holds, with the assumptions on \( u_0 \) and \( \theta_0 \) replaced by \( (u_0, \sqrt{\rho_0} \theta_0) \in D_0^1 \times L^2 \). Note that these assumptions are weaker than those in [6, 28]. In fact, [6] requires \( (u_0, \theta_0) \in D_0^1 \cap D^2 \) while [28] requires \( u_0 \in D_0^1 \cap D^2 \) and \( \theta_0 \in D^1 \).

The key of proving the existence part of Theorem 1.1 is to carry out some suitable a priori estimates of the following quantity
\[
\Phi(t) := \int_0^t \left( \| \sqrt{\rho} u_t \|_2^2 + \| \nabla^2 u \|_2^2 + \| \sqrt{s} \nabla u_t \|_2^2 + \| \nabla \theta \|_2^2 + \| \sqrt{s} \sqrt{\rho} \theta_t \|_2^2 + \| \sqrt{s} \nabla^2 \theta \|_2^2 \right) ds
\]
\[
+ \sup_{0 \leq s \leq t} \left( \| \rho \|_\infty + \| \nabla \rho \|_q + \| \sqrt{\rho} \theta \|_2^2 + \| \nabla u \|_2 + \| \sqrt{s} \nabla \theta \|_2^2 \right) + 1,
\]
for any approximate solution \( (\rho, u, \theta) \) to system (1.1)–(1.3), subject to (1.6)–(1.7). The estimates for \( \Phi \) have to be independent of the compatibility conditions. Roughly speaking, the estimate of \( \Phi(T) \) is achieved based on the following conditional a priori estimate by the continuity argument: it holds that

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We want to derive some Gronwall type structure from the above inequalities. Recalling that one only has
This will then yield a preparing existence result by passing the limit to the approximate solutions; however, the regularities that $\sqrt{\rho u}, \sqrt{\rho \theta} \in C([0, T]; L^2)$ are not guaranteed in this passage, and instead what we have are $\rho u \in C([0, T]; L^2)$ and $\rho \theta \in C_w([0, T]; L^2)$, here $C_w$ represents the weak continuity. As a compensation, we prove that $\|\sqrt{\rho u}\|^2(t) \leq \|\sqrt{\rho_0 u_0}\|^2 + Ct$ and $\|\sqrt{\rho \theta}\|^2(t) \leq \|\sqrt{\rho_0 \theta_0}\|^2 + C\sqrt{t}$, which are employed to prove the continuities with respect to time of $\sqrt{\rho u}$ and $\sqrt{\rho \theta}$ in $L^2$ in the Lagrangian coordinates first and finally transformed back to those in the Euler coordinates.

Due to the lower regularities on the initial data and the absence of the initial compatibility conditions, the regularities of the strong solutions obtained in this paper are weaker than those required in [5,6,28,45] to prove the uniqueness in their ways. We also note that even though the uniqueness was achieved in [10, 15,18,29] for the inhomogeneous incompressible Navier–Stokes equations and the isentropic compressible Navier–Stokes equations with the initial data $(\rho_0, u_0, \theta_0)$ satisfying some similar regularities as in this paper and without any initial compatibility conditions, still the arguments in these works do not apply to the current paper. The main reasons are that the regularities of the initial temperature assumed in this paper are weaker than those of the initial velocity, and even worse that the entropy production term $Q(\nabla u)$ has stronger nonlinearities than the convection terms. As a result, in matter of overcoming the difficulties caused by the low regularities and lack of compatibility conditions on the initial data, the ideas used to deal with the velocity are not sufficient to deal with the temperature.

Our strategies of proving the uniqueness are illustrated as follows. Let $(\tilde{\rho}, \tilde{\theta})$ and $(\rho, u, \theta)$ be two solutions with the same initial data and denote by $(\rho, u, \theta)$ their subtraction. Then, one has some differential inequalities of the form

$$
\frac{d}{dt} \|\sqrt{\rho u}\|^2 + \|\nabla u\|^2 \leq C \|\nabla \tilde{u}_t\|^2 \|\rho\|^2 + \cdots,
$$

(1.9)

$$
\frac{d}{dt} \|\sqrt{\rho \theta}\|^2 + \|\nabla \theta\|^2 \leq C \|\nabla^2 \tilde{\pi}\| \|\nabla u\|^2 + C \|\nabla \tilde{t}_t\|^2 \|\rho\|^2 + \cdots,
$$

(1.10)

$$
\frac{d}{dt} \|\rho\|^2 \leq C \|\nabla \pi\|_\infty \|\rho\|^2 + C \|\nabla u\| \|\rho\|,
$$

(1.11)

where all other quantities that can be dealt with relatively easier are omitted in the suspension points. We want to derive some Gronwall type structure from the above inequalities. Recalling that one only has $t \nabla \tilde{u}_t \in L^2(0, T; L^2)$, the hardest term $\|\nabla \tilde{u}_t\|^2 \|\rho\|^2$ in (1.10) has to be dealt with as $\|t \nabla \tilde{u}_t\|^2 \|\rho\|^2 t^2$ and, as a result, one has to consider the differential inequality for $\|\rho\|^2 t^2$, which can be derived from (1.11) as

$$
\frac{d}{dt} \|\rho\|^2 + \frac{\|\rho\|^2}{t^2} \leq C \|\nabla \pi\|_\infty \|\rho\|^2 + \frac{C \|\nabla u\|^2}{t}.
$$

This motivates us to divide (1.9) with $t$, leading to

$$
\frac{d}{dt} \left(\frac{\|\sqrt{\rho u}\|^2}{t} + \frac{\|\sqrt{\rho \theta}\|^2}{t^2} + \frac{\|\nabla u\|^2}{t} \right) \leq C \|\sqrt{t \nabla \tilde{u}}\|^2 \|\rho\|^2 + \cdots.
$$

(1.12)

Noticing that $\sqrt{t \nabla^2 \tilde{\pi}} \in L^\infty(0, T; L^2)$, one can derive from the above two and (1.10) that

$$
\frac{d}{dt} \left(\|\sqrt{\rho \theta}\|^2 + \frac{\|\sqrt{\rho u}\|^2}{t} + \frac{\|\rho\|^2}{t^2} \right) \leq C \|\sqrt{t \nabla \tilde{u}}\|^2 \|\nabla \pi\|_\infty \|\rho\|^2 + \cdots,
$$

which meets the Gronwall type structure. It remains to guarantee that the quantity with singular weights $\|\sqrt{\rho \theta}\|^2 + \|\sqrt{\rho u}\|^2 + \|\rho\|^2 t^2$ tends to zero when approaching the initial time. This is expected to be verified from (1.9)–(1.11) by using the regularities of the solutions.

It is worth pointing out some technique points in the arguments explained in the above paragraph. First, in deriving the singular $t$-weighted energy inequality (1.12), one also encounters a term of the form

$$
\sqrt{t \nabla^2 \tilde{\pi}} \in L^\infty(0, T; L^2),
$$

and without any initial compatibility conditions, still the arguments in these works do not apply to the current paper. The main reasons are that the regularities of the initial temperature assumed in this paper are weaker than those of the initial velocity, and even worse that the entropy production term $Q(\nabla u)$ has stronger nonlinearities than the convection terms. As a result, in matter of overcoming the difficulties caused by the low regularities and lack of compatibility conditions on the initial data, the ideas used to deal with the velocity are not sufficient to deal with the temperature.

Our strategies of proving the uniqueness are illustrated as follows. Let $(\tilde{\rho}, \tilde{\theta})$ and $(\rho, u, \theta)$ be two solutions with the same initial data and denote by $(\rho, u, \theta)$ their subtraction. Then, one has some differential inequalities of the form

$$
\frac{d}{dt} \|\sqrt{\rho u}\|^2 + \|\nabla u\|^2 \leq C \|\nabla \tilde{u}_t\|^2 \|\rho\|^2 + \cdots,
$$

(1.9)
The aim of this section is to derive some a priori estimates for the strong solutions to system (1.1)–(1.3), to verify this in the Euler coordinates, as the initial condition is (1.6)–(1.7), with initial data satisfying some compatibility conditions. We emphasize that the singular weights used in the Lagrangian coordinates are actually less singular than those in the Euler coordinates. This also reflects another advantage of using the Lagrangian coordinates to remark that the singular weights used in the Lagrangian coordinates are actually less singular than those in the Euler coordinates. This also reflects another advantage of using the Lagrangian coordinates to prove the uniqueness.

As for the second technique point, in the Lagrangian coordinates, the corresponding requirement is that \( \sqrt{\rho_0} u = \sqrt{\rho_0} \hat{u} \) at the initial time of which the proof is given in Proposition 4.4. Finally, we would like to remark that the singular weights used in the Lagrangian coordinates are actually less singular than those in the Euler coordinates. This also reflects another advantage of using the Lagrangian coordinates to prove the uniqueness.

Throughout this paper, we use \( C \), which may vary from place to place, to denote a generic constant depending only on \( R, \mu, \lambda, c_v, q \), and the upper bound of \( \Phi_0 \) unless we clearly specify.

2. A Priori Estimates Independent of Compatibility Conditions

The aim of this section is to derive some a priori estimates for the strong solutions to system (1.1)–(1.3), subject to (1.6)–(1.7), with initial data satisfying some compatibility conditions. We emphasize that although we assume the initial compatibility conditions, the a priori estimates established in this section do not depend on these conditions. This is crucial to finally establish the existence of strong solutions without any compatibility conditions.

We start with the following local well-posedness result which can be proved in the same way as in [6] where the compatibility conditions are required.

**Proposition 2.1.** Let \( q \in (3, 6] \) and assume that \( (\rho_0, u_0, \theta_0) \) satisfies
\[
0 \leq \rho_0 \in W^{1,q}(\Omega), \quad u_0 \in H^1_0(\Omega) \cap H^2(\Omega), \quad 0 \leq \theta_0 \in H^1_0(\Omega) \cap H^2(\Omega),
\]
and the compatibility conditions
\[
-\mu \Delta u_0 - (\mu + \lambda) \nabla \text{div} u_0 + \nabla(R\rho_0 \theta_0) = \sqrt{\rho_0} g_1, \\
\kappa \Delta \theta_0 + \frac{\mu}{2} \nabla u_0 + (\nabla u_0)^T \nabla u_0 + \lambda (\text{div} u_0)^2 = \sqrt{\rho_0} g_2,
\]
for some \( g_1, g_2 \in L^2(\Omega) \).

Then, there exists a positive time \( T_* \) depending on \( R, \mu, \lambda, c_v, q, \|\nabla^2 u_0\|, \|\nabla^2 \theta_0\|, \|g_1\|, \|g_2\| \), and such that system (1.1)–(1.3), subject to (1.6)–(1.7), admits a unique strong solution \( (\rho, u, \theta) \) in \( \Omega \times (0, T_*) \), satisfying
\[
\rho \in C([0, T_*]; W^{1,q}), \quad \rho_t \in C([0, T_*]; L^q), \\
(\rho_t, \theta_t) \in L^2(0, T_*; H^1_0), \quad (\sqrt{\rho_0} \hat{u}_t, \sqrt{\rho_0} \hat{\theta}_t) \in L^\infty(0, T_*; L^2), \\
(u, \theta) \in C([0, T_*]; H^1_0 \cap H^2) \cap L^2(0, T_*; W^{2,q}).
\]

It will be shown in this section that the existence time \( T_* \) in the above proposition can be chosen depending only on \( R, \mu, \lambda, c_v, q \), and the upper bound of
\[ \Phi_0 := \|\rho_0\|_\infty + \|\nabla \rho_0\|_q + \|(\sqrt{\rho_0} \theta_0, \nabla u_0)\|^2. \]

In particular, \(T_*\) can be chosen independent of \(\|\nabla^2 u_0\|, \|\nabla^2 \theta_0\|, \|g_1\|, \) and \(\|g_2\|\). Let \(\Phi\) be the quantity given by (1.8). The main issue of this section is to derive the local in time estimate of \(\Phi\) independent of \(\|\nabla^2 u_0\|, \|\nabla^2 \theta_0\|, \|g_1\|, \) and \(\|g_2\|\), and therefore independent of the initial compatibility conditions.

In the rest of this section until the last proposition, we always assume that \((\rho, u, \theta)\) is a solution to system (1.1)–(1.3), subject to (1.6)–(1.7), in \(\Omega \times (0, T)\), for some positive time \(T \leq 1\), satisfying the regularities in Proposition 2.1 with \(T_*\) there replaced by \(T\). We emphasize again that \(C\), which may vary from place to place, is a generic constant depending only on \(R, \mu, \lambda, c_v, q\), and the upper bound of \(\Phi_0\).

**Proposition 2.2.** Denote \(M_0 := \int_{\Omega} \rho_0 \, dx > 0\). Then, it holds that

\[ \|\theta\|^2 \leq C \left( \|\sqrt{\rho} \theta\|^2 + \|\rho\|^2_{2, \infty} \|\nabla \theta\|^2 \right), \]

for a positive constant \(C\) depending only on \(M_0\) and \(\Omega\).

**Proof.** Using the boundary condition \(u|_{\partial \Omega} = 0\) and integrating (1.1) over \(\Omega\) yield

\[ \int_{\Omega} \rho \, dx = \int_{\Omega} \rho_0 \, dx = M_0, \]

for any \(t \in (0, T)\). Denote by \((\theta^2)_\Omega = \frac{1}{|\Omega|} \int_{\Omega} \theta^2 \, dx\) the average of \(\theta^2\) over \(\Omega\). It follows from the above equality and the Poincaré and Cauchy inequalities that

\[
\frac{M_0}{|\Omega|} \|\theta\|^2 = \int_{\Omega} \rho ((\theta^2)_{\Omega} - \theta^2) \, dx + \int_{\Omega} \rho \theta^2 \, dx
\]

\[ \leq \|\rho\|_{\infty} ((\theta^2)_{\Omega} - \theta^2)_{L^1} + \|\sqrt{\rho} \theta\|^2 \leq C \|\rho\|_{\infty} \|\theta\|_{\infty} \|\nabla \theta\|_{L^1} + \|\sqrt{\rho} \theta\|^2
\]

\[ \leq \frac{M_0}{2|\Omega|} \|\theta\|^2 + C \|\rho\|_{\infty}^2 \|\nabla \theta\|^2 + \|\sqrt{\rho} \theta\|^2, \]

which implies the conclusion. \(\square\)

**Proposition 2.3.** It holds that

\[ \int_0^T \left( \|\nabla u\|_{\infty} + \|\nabla^2 u\|_q \right) \, dt \leq CT^{\frac{6-q}{2q}} \Phi^{\frac{5}{4}}(T). \]

**Proof.** Applying the elliptic estimates to (1.2), one obtains

\[ \|\nabla^2 u\|_q \leq C(\|\rho u\|_q + \|\rho (u \cdot \nabla) u\|_q + \|\nabla P\|_q). \]

It follows from the Hölder, Gagliardo-Nirenberg, Sobolev, Poincaré inequalities, and Proposition 2.2 that

\[ \|\rho u_t\|_q \leq C\|\rho\|^{\frac{1}{2}} \|\sqrt{\rho} u_t\|^{\frac{6-q}{2q}} \|\sqrt{\rho} u_t\|^{\frac{3q-6}{2q}} \leq C\|\rho\|^{\frac{3q-6}{4q}} \|\sqrt{\rho} u_t\|^{\frac{6-q}{2q}} \|\nabla u_t\|^{\frac{3q-6}{2q}}, \]

\[ \|\rho (u \cdot \nabla) u\|_q \leq \|\rho\|_{\infty} \|u\|_{\infty} \|\nabla u\|_q \leq C\|\rho\|_{\infty} \|\nabla u\|^{\frac{1}{2}} \|\nabla^2 u\|_{L^2}, \]

\[ \|\nabla P\|_q \leq C(\|\nabla \rho\|_q \|\theta\|_{\infty} + \|\rho\|_q \|\nabla \theta\|_q) \leq C(\|\nabla \rho\|_q + \|\rho\|_q \|\nabla \theta\|_q) \leq C(\|\nabla \rho\|_q + \|\rho\|_q \|\sqrt{\rho} \theta\| + \|\nabla \theta\|^{\frac{6-q}{2q}} \|\nabla^2 \theta\|^{\frac{3q-6}{2q}}). \]

Integrating the above estimates from 0 and \(T\), one gets

\[ \int_0^T \|\rho u_t\|_q \, dt \leq C \int_0^T \|\rho\|^{\frac{5q-6}{4q}} \|\sqrt{\rho} u_t\|^{\frac{6-q}{2q}} \|\nabla u_t\|^{\frac{3q-6}{2q}} \, dt
\]

\[ \leq C \Phi^{\frac{5q-6}{4q}}(T) \left( \int_0^T \|\sqrt{\rho} u_t\|^2 \, dt \right)^{\frac{6-q}{4q}} \left( \int_0^T \|\nabla u_t\|^2 \, dt \right)^{\frac{3q-6}{2q}} T^{\frac{6-q}{3-q}} \]

\[ \leq CT^{\frac{6-q}{2q}} \Phi^{\frac{7q-6}{4q}}(T), \]

\(\varepsilon\) Birkhäuser
Choosing and the Poincaré inequalities that
\[
\int_0^T \|\rho(u \cdot \nabla)u\|_q \, dt \leq C \int_0^T \|\rho\|_\infty \|\nabla u\|^{\frac{2}{3}} \|\nabla^2 u\|^{\frac{1}{3}} \, dt
\]
\[
\leq C \Phi^{\frac{5}{4}}(T) \left( \int_0^T \|\nabla^2 u\|^2 \, dt \right)^{\frac{1}{2}} \leq CT^{\frac{1}{2}} \Phi^2(T),
\]
as well as
\[
\int_0^T \|\nabla P\|_q \, dt
\]
\[
\leq C \int_0^T (\|\nabla \rho\|_q + \|\rho\|_\infty)(\|\sqrt{\theta}\| + \|\nabla \theta\| + \|\nabla \theta\|^{\frac{6-q}{2q}} \|\nabla^2 \theta\|^{\frac{3q-6}{2q}}) \, dt
\]
\[
\leq CT^{\frac{1}{2}} \Phi^2(T) + C \Phi^2(T) \left( \int_0^T \|\nabla \theta\|^2 \, dt \right)^{\frac{1}{2}} T^{\frac{1}{2}}
\]
\[
+ C \Phi(T) \left( \int_0^T \|\nabla \theta\|^2 \, dt \right)^{\frac{6-q}{2q}} \left( \int_0^T \|\sqrt{\nabla^2 \theta}\|^2 \, dt \right)^{\frac{3q-6}{2q}} T^{\frac{1}{2}}
\]
\[
\leq CT^{\frac{1}{2}} \Phi^2(T).
\]

Therefore, we show
\[
\int_0^T \|\nabla^2 u\|_q \, dt \leq C \left( T^{\frac{6-q}{2q}} \Phi^{\frac{3q-6}{2q}}(T) + T^{\frac{1}{2}} \Phi^2(T) + T^{\frac{1}{2}} \Phi^{\frac{5}{4}}(T) \right) \leq CT^{\frac{6-q}{2q}} \Phi^\frac{5}{4}(T),
\]
where \(\frac{1}{4} \geq \frac{6-q}{2q}\) for \(q \in (3, 6), \ T \leq 1, \) and \(\Phi(T) \geq 1\) were used. Thanks to this, it follows from the Sobolev and the Poincaré inequalities that
\[
\int_0^T \|\nabla u\|_\infty \, dt \leq C \int_0^T \|\nabla^2 u\|_q \, dt \leq CT^{\frac{6-q}{2q}} \Phi^\frac{5}{4}(T).
\]
The proof is complete. \(\square\)

**Proposition 2.4.** It holds that
\[
\sup_{0 \leq t \leq T} \|\rho\|_\infty \leq \|\rho_0\|_\infty \exp \left\{ CT^{\frac{6-q}{2q}} \Phi^\frac{5}{4}(T) \right\},
\]
\[
\sup_{0 \leq t \leq T} \|\rho\|_{W^{1,q}} \leq C \left( 1 + T^{\frac{6-q}{2q}} \Phi^\frac{5}{4}(T) \right) \exp \left\{ CT^{\frac{6-q}{2q}} \Phi^\frac{5}{4}(T) \right\}.
\]

**Proof.** For any given \(x \in \Omega\) and \(s \in [0, T]\), define \(U(x, t; s)\) as
\[
\begin{cases}
\frac{d}{dt} U(x, t; s) = u(U(x, t; s), t), & \forall t \in [0, T],
\end{cases}
\]
\[
U(x, s; s) = x.
\]

Note that \(u \in L^1(0, T; W^{1,\infty})\) guaranteed by Proposition 2.3, \(U\) is well-defined. Then, it follows from (1.1) that
\[
\frac{d}{dt} \rho(U(x, t; s), t) = -\text{div} u(U(x, t; s), t) \rho(U(x, t; s), t), \quad \forall t \in (0, T).
\]

Solving the above ordinary differential equation yields
\[
\rho(U(x, t; s), t) = \rho(x, s) e^{-\int_0^t \text{div} u(U(x, \tau; s), \tau) \, d\tau}, \quad \forall s, t \in [0, T].
\]

Choosing \(t = 0\) in the above, one gets
\[
\rho(x, s) = \rho_0(U(x, 0; s)) e^{-\int_0^s \text{div} u(U(x, \tau; s), \tau) \, d\tau}, \quad \forall (x, s) \in \Omega \times [0, T].
\]
Therefore, by Proposition 2.3, it holds that
\[
\sup_{0 \leq t \leq T} \| \rho \|_\infty(t) \leq \| \rho_0 \|_\infty C T_0^T \| \text{div} u \|_\infty(\tau) d\tau \leq \| \rho_0 \|_\infty C T^{6 - \frac{4}{q} - \frac{5}{q^2}} (T),
\]
proving the first conclusion.

Multiplying (1.1) with \( \rho^{q-1} \) and integrating over \( \Omega \), one deduces by integration by parts that
\[
d \frac{d}{dt} \| \rho \|^q_q(t) \leq C \| \nabla u \|_\infty \| \rho \|^q_q(t).
\]
Applying the operator \( \nabla \) to (1.1) and multiplying the resultant with \( |\nabla \rho|^q \nabla \rho \), it follows from integration by parts that
\[
d \frac{d}{dt} \| \nabla \rho \|^q_q(t) \leq C \int_\Omega (|\nabla u| |\nabla \rho|^q + |\rho| |\nabla \rho|^{q-1} |\nabla^2 u|) dx
\leq C (|\nabla u|_\infty \| \nabla \rho \|^q_q(t) + \| \rho \|_\infty \| \nabla \rho \|_q^{q-1} \| \nabla^2 u \|_q).
\]
Hence,
\[
\frac{d}{dt} \| \rho \|_{W^{1,q}} = \frac{d}{dt} (\| \rho \|_q + \| \nabla \rho \|_q) \leq C (\| \nabla u \|_\infty \| \rho \|_{W^{1,q}} + \| \rho \|_\infty \| \nabla^2 u \|_q),
\]
from which, by the Grönwall inequality, using the first conclusion, and by Proposition 2.3, it follows that
\[
\| \rho \|_{W^{1,q}} \leq \left( \| \rho_0 \|_{W^{1,q}} + \int_0^T \| \nabla^2 u \|_q dt \right) e^{C \int_0^T \| \nabla u \|_\infty dt}
\leq C \left( 1 + T^{6 - \frac{4}{q} - \frac{5}{q^2}} \Phi^\frac{5}{2}(T) \right) e^{CT^{6 - \frac{4}{q} - \frac{5}{q^2}} \Phi^\frac{5}{2}(T),
\]
proving the second conclusion.

As a direct corollary of Proposition 2.4, one obtains:

**Corollary 2.1.** There is a sufficiently small positive constant \( \epsilon_0 \leq 1 \) depending only on \( R, \mu, \lambda, c_v, q, \) and \( \Phi_0 \), such that
\[
\sup_{0 \leq t \leq T} \| \rho \|_\infty(t) \leq 2 \| \rho_0 \|_\infty, \quad \sup_{0 \leq t \leq T} \| \rho \|_{W^{1,q}} \leq C,
\]
as long as
\[
T^{6 - \frac{4}{q} - \frac{5}{q^2}} \Phi^\frac{5}{2}(T) \leq \epsilon_0.
\]

Under the assumption (2.2) and since \( \Phi(T) \geq 1 \), it is easy to check that the following relations hold:
\[
T \Phi^3(T) = \left( T^{6 - \frac{4}{q} - \frac{5}{q^2}} \Phi^\frac{5}{2}(T) \right)^\frac{3}{5} T^{\frac{18}{10} - \frac{12}{10} - \frac{18}{10} - \frac{12}{10}} \leq \epsilon_0 \leq 1,
\]
\[
T^2 \Phi^2(T) = \left( T^{6 - \frac{4}{q} - \frac{5}{q^2}} \Phi^\frac{5}{2}(T) \right)^\frac{1}{2} T^{\frac{7}{10} - \frac{6}{10} - \frac{7}{10} - \frac{6}{10}} \leq \epsilon_0 \leq 1,
\]
\[
T \Phi^\frac{3}{2}(T) = \left( T^{6 - \frac{4}{q} - \frac{5}{q^2}} \Phi^\frac{5}{2}(T) \right)^\frac{3}{5} T^{\frac{18}{10} - \frac{12}{10} - \frac{18}{10} - \frac{12}{10}} \leq \epsilon_0 \leq 1.
\]

These will be frequently used in the rest of this section.

**Proposition 2.5.** Let \( \epsilon_0 \) be the number stated in Corollary 2.1 and assume that (2.2) holds. Then, the following estimate holds
\[
\sup_{0 \leq t \leq T} \| \nabla \theta \|^2 + \int_0^T \| \nabla \theta \|^2 dt \leq C.
\]
Proof. Multiply (1.3) with $\theta$ and integrate it over $\Omega$ to get
\[
\frac{c_v}{2} \frac{d}{dt} \| \sqrt{\rho} \theta \|_2^2 + \kappa \| \nabla \theta \|_2^2 = - \int_\Omega \text{div} u P \theta dx + \int_\Omega Q(\nabla u) \theta dx.
\]
By Proposition 2.2 and Corollary 2.1, terms on the right-hand side are estimated by the Hölder, Sobolev, and Young inequalities as
\[
\begin{align*}
\int_\Omega \text{div} u P \theta dx &\leq R \int_\Omega \rho |\theta|^2 |\nabla u| dx \leq C \| \nabla u \|_\infty \| \sqrt{\rho} \theta \|_2^2, \\
\int_\Omega Q(\nabla u) \theta dx &\leq C \| \nabla u \|_3 \| \theta \|_6 \leq C \| \nabla u \|_2 \| \nabla^2 u \|_4 (\| \sqrt{\rho} \theta \| + \| \nabla \theta \|)
\end{align*}
\]
\[
\leq \frac{\kappa}{2} \| \nabla \theta \|_2^2 + C (\| \sqrt{\rho} \theta \|_2^2 + \| \nabla u \|_3^3 \| \nabla^2 u \|).
\]
Therefore, it follows that
\[
c_v \frac{d}{dt} \| \sqrt{\rho} \theta \|_2^2 + \kappa \| \nabla \theta \|_2^2 \leq C (1 + \| \nabla u \|_\infty) \| \sqrt{\rho} \theta \|_2^2 + C \| \nabla u \|_3^3 \| \nabla^2 u \|,
\]
from which, by the Grönwall inequality, it follows from the Hölder inequality, Proposition 2.3, and Corollary 2.1 that
\[
\sup_{0 \leq t \leq T} c_v \| \sqrt{\rho} \theta \|_2^2 + \kappa \int_0^T \| \nabla \theta \|_2^2 dt 
\leq \left( c_v \| \sqrt{\rho_0} \theta_0 \|_2^2 + C \int_0^T \| \nabla u \|_3^3 \| \nabla^2 u \| dt \right) e^{C \int_0^T (1 + \| \nabla u \|_\infty) dt}
\leq C \left( 1 + T \frac{1}{2} \Phi^2(T) \right) e^{C \left( 1 + T \frac{6 \kappa}{\kappa - \nu} \Phi^2(T) \right)} \leq C,
\]
where in the last step (2.2) and (2.4) were used. This completes the proof. \qed

Proposition 2.6. Under the assumptions of Proposition 2.5, it holds that
\[
\sup_{0 \leq t \leq T} \| \nabla u \|_2^2 + \int_0^T (\| \sqrt{\rho} u_t \|_2^2 + \| \nabla^2 u \|_2^2) dt \leq C.
\]

Proof. By Corollary 2.1, Proposition 2.2, and the Hölder, Gagliardo-Nirenberg, and Young inequalities, one deduces
\[
\| \nabla^2 u \|_2^2 \leq C (\| \rho u_t \|_2^2 + \| \rho (u \cdot \nabla) u \|_2^2 + \| \nabla P \|_2^2)
\leq C (\| \rho \|_\infty \| \sqrt{\rho} u_t \|_2^2 + \| \rho \|_\infty^2 \| u \|_3^2 \| \nabla u \|_3^2 + \| \nabla \rho \|_3^2 \| \theta \|_6^2 + \| \rho \|_\infty^3 \| \nabla \theta \|_2^2)
\leq C (\| \sqrt{\rho} u_t \|_2^2 + \| \nabla u \|_3^3 \| \nabla^2 u \| + \| \sqrt{\rho} \theta \|_2^2 + \| \nabla \theta \|_2^2)
\leq \frac{1}{2} \| \nabla^2 u \|_2^2 + C \left( \| \sqrt{\rho} u_t \|_2^2 + \| \nabla u \|_6^6 + \| \sqrt{\rho} \theta \|_2^2 + \| \nabla \theta \|_2^2 \right)
\]
and, thus,
\[
\| \nabla^2 u \|_2^2 \leq C \left( \| \sqrt{\rho} u_t \|_2^2 + \| \nabla u \|_6^6 + \| \sqrt{\rho} \theta \|_2^2 + \| \nabla \theta \|_2^2 \right). \tag{2.6}
\]
Note that (1.3) implies
\[
P \frac{R}{c_v} \left( Q(\nabla u) + \kappa \Delta \theta - P \text{div} u - c_v \text{div}(\rho u \theta) \right). \tag{2.7}
\]
Thus, by integration by parts, one gets
\[
\int_{\Omega} \nabla P u_t \, dx = -\frac{d}{dt} \int P \nabla u \, dx + \int P_t \nabla u \, dx
\]
\[
= -\frac{d}{dt} \int P \nabla u \, dx + \frac{R}{c_v} \int \nabla u \left( Q(\nabla u) + \kappa \Delta \theta - P \nabla u - c_v \nabla (\rho u \theta) \right) \, dx
\]
\[
= -\frac{d}{dt} \int P \nabla u \, dx + \frac{R}{c_v} \int \nabla u \left( Q(\nabla u) - \frac{\kappa R}{c_v} \int \nabla u \cdot \nabla \theta \, dx \right)
\]
\[
- \frac{R}{c_v} \int P (\nabla u)^2 \, dx + R \int \rho u \cdot \nabla \, dx.
\]

Multiplying (1.2) with \(u_t\), integrating over \(\Omega\), and using the above identity, it follows
\[
\frac{d}{dt} \left( \frac{\mu}{2} \|\nabla u\|^2 + \frac{\mu + \lambda}{2} \|\nabla u\|^2 - \int P \nabla u \, dx \right) + \|\sqrt{\rho} u_t\|^2
\]
\[
= -\int \rho (u \cdot \nabla) u_t \, dx - \frac{R}{c_v} \int \nabla u Q(\nabla u) \, dx + \frac{\kappa R}{c_v} \int \nabla u \cdot \nabla \theta \, dx
\]
\[
+ \frac{R}{c_v} \int P (\nabla u)^2 \, dx - R \int \rho u \cdot \nabla \, dx := \sum_{i=1}^{5} J_i. \tag{2.8}
\]

By Corollary 2.1 and Proposition 2.2, it follows from the Gagliardo-Nirenberg, Sobolev, Poincaré, and Young inequalities that
\[
|J_1| \leq \rho \frac{1}{2} \|\nabla u\|^3 \|\nabla u\|_3 \|\nabla u_t\| \leq C \|\nabla u\|^3 \|\nabla u\|_3 \|\nabla u_t\|
\]
\[
\leq \frac{1}{8} \|\nabla u_t\|^2 + \eta \|\nabla u\|^2 + C_\eta \|\nabla u\|^6,
\]
\[
|J_2| \leq C \int \|\nabla u\|^3 \, dx \leq C \|\nabla u\|^3 \|\nabla u\|^2 \leq \eta \|\nabla u\|^2 + C_\eta \|\nabla u\|^6,
\]
\[
|J_3| \leq C \|\nabla u\|^2 \|\nabla \theta\| \leq \eta \|\nabla u\|^2 + C_\eta \|\nabla \theta\|^2,
\]
\[
|J_4| \leq C \rho \|\nabla \theta\| \|\nabla u\| \|\nabla u\|_3 \leq C (\|\nabla \theta\| + \|\nabla \theta\|) \|\nabla u\|^3 \|\nabla u\|^2 \|\nabla u\|^2
\]
\[
\leq \eta \|\nabla u\|^2 + C_\eta (\|\nabla \theta\|^2 + \|\nabla \theta\|^2 + \|\nabla u\|^6 + \|\nabla u\|^6),
\]
\[
|J_5| \leq C \rho \|\nabla \theta\| \|\nabla u\| \|\nabla \theta\| \|\nabla u\| 
\]
\[
\leq C \|\nabla u\|^2 \|\nabla u\|^2 \|\nabla \theta\|^4
\]
\[
\leq \eta \|\nabla u\|^2 + C_\eta \|\nabla u\|^2 \|\nabla \theta\|^4,
\]

for any positive number \(\eta\). Plugging the above estimates into (2.8), adding the resultant with (2.6) multiplied with a small positive number \(\varepsilon_1\), and choosing \(\eta\) sufficiently small, one obtains
\[
\frac{d}{dt} \left( \frac{\mu}{2} \|\nabla u\|^2 + \frac{\mu + \lambda}{2} \|\nabla u\|^2 - \int P \nabla u \, dx \right) + \frac{1}{2} \|\sqrt{\rho} u_t\|^2 + \frac{\varepsilon_1}{2} \|\nabla u\|^2
\]
\[
\leq C \left( \|\nabla u\|^6 + \|\nabla \theta\|^2 + \|\nabla \theta\|^2 + \|\nabla u\|^2 \|\nabla \theta\|^4 \right).
\]

Integrating the above inequality over \((0, T)\) and using Proposition 2.5, one deduces
\[
\mu \sup_{0 \leq t \leq T} \|\nabla u\|^2 + \int_0^T \left( \|\sqrt{\rho} u_t\|^2 + \varepsilon_1 \|\nabla u\|^2 \right) \, dt
\]
\[
\leq 2 \sup_{0 \leq t \leq T} \left| \int P \nabla u \, dx \right| + C \left( \|\nabla u_0\|^2 + \left| \int P \nabla u_0 \, dx \right| \right)
\]
\[
+ C \int_0^T \left( \|\nabla u\|^6 + \|\sqrt{\rho} \theta\|^2 + \|\nabla \theta\|^2 + \|\nabla u\|^2 \|\sqrt{\rho} \theta\|^4 \right) \, dt.
\]
Proposition 2.7. Under the conditions of Proposition 2.5, it holds that
\[
\sup_{0 \leq t \leq T} \|(\sqrt{t} \nabla \theta, \sqrt{t} \sqrt{\rho} u_t, \sqrt{t} \sqrt{\nabla^2 u})\|^2 + \int_0^T \|(\sqrt{t} \sqrt{\rho} \theta_t, \sqrt{t} \sqrt{\nabla^2 \theta}, \sqrt{t} \sqrt{\nabla u_t})\|^2 \, dt \leq C.
\]

Proof. By Corollary 2.1 and Proposition 2.2, it follows from (1.3) and the Sobolev and Young inequalities that
\[
\|\nabla^2 \theta\|^2 \leq C (\|\rho \theta_t\|^2 + \|\rho (u \cdot \nabla) \theta\|^2 + \|\text{div } P\|^2 + \|Q(\nabla u)\|^2)
\leq C (\|\rho\|_{\infty} \|\sqrt{\rho} \theta_t\|^2 + \|\rho\|_{\infty} \|u\|_{\infty} \|\nabla \theta\|_3^2 + \|\rho\|_{\infty} \|\theta\|_3 \|\nabla u\|_3^3 + \|\nabla u\|_4^4)
\leq C (\|\sqrt{\rho} \theta_t\|^2 + \|\nabla u\|_2 \|\nabla \theta\| \|\nabla^2 \theta\| + \|\nabla u\| \|\nabla^2 u\|_3^3)
+ C (\|\sqrt{\rho} \theta_t\|^2 + \|\nabla \theta\|_2^2) \|\nabla u\| \|\nabla^2 u\|)
\leq \frac{1}{2} \|\nabla^2 \theta\| + C \left[\|\sqrt{\rho} \theta_t\|^2 + \|\nabla u\|^4 \|\nabla \theta\|^2 + \|\nabla u\| \|\nabla^2 u\|^3\right]
+ (\|\sqrt{\rho} \theta_t\|^2 + \|\nabla \theta\|_2^2) \|\nabla u\| \|\nabla^2 u\|,
\]
and, thus,
\[
\|\nabla^2 \theta\|^2 \leq C \left(\|\sqrt{\rho} \theta_t\|^2 + \|\nabla u\|^4 \|\nabla \theta\|^2 + \|\nabla u\| \|\nabla^2 u\|^3\right)
+ C \left(\|\sqrt{\rho} \theta_t\|^2 + \|\nabla \theta\|_2^2\right) \|\nabla u\| \|\nabla^2 u\|.
\]

Testing (1.3) with \(\theta_t\) yields
\[
\frac{\kappa}{2} \frac{d}{dt} \|\nabla \theta\|^2 + c_v \|\sqrt{\rho} \theta_t\|^2 = \int_{\Omega} \left[-c_v \rho (u \cdot \nabla) \theta_t - P \text{div } u_t + Q(\nabla u) \theta_t\right] \, dx.
\]

Terms on the right-hand side of (2.10) are estimated by Corollary 2.1, Proposition 2.2, Proposition 2.5, and the Gagliardo-Nirenberg, Poincaré, and Young inequalities as follows
\[
\left|\int_{\Omega} \rho (u \cdot \nabla) \theta_t \, dx\right| \leq \|\rho\|_{\infty} \|u\|_{\infty} \|\nabla \theta\| \|\sqrt{\rho} \theta_t\| \leq C \|\rho\|_{\infty} \|\nabla u\|^2 \|\nabla \theta\| \|\sqrt{\rho} \theta_t\|
\leq \frac{c_v}{8} \|\sqrt{\rho} \theta_t\|^2 + C \|\nabla u\| \|\nabla^2 u\| \|\nabla \theta\|^2,
\]
\[
\left|\int_{\Omega} P \text{div } u_t \, dx\right| \leq R \int_{\Omega} \rho |\nabla u| |\theta_t| \, dx \leq C \|\rho\|_{\infty} \|\theta_t\|_6 \|\nabla u\|_3 \|\sqrt{\rho} \theta_t\|
\leq C \|\rho\|_{\infty} \|\sqrt{\rho \theta}\| + \|\nabla \theta\| \|\nabla u\|^2 \|\nabla \theta\| \|\sqrt{\rho} \theta_t\|
\leq \frac{c_v}{8} \|\sqrt{\rho} \theta_t\|^2 + C \|\nabla u\| \|\nabla^2 u\| \left(\|\sqrt{\rho} \theta\|^2 + \|\nabla \theta\|^2\right),
\]
\[
\int_{\Omega} Q(\nabla u) \theta_t \, dx = \frac{d}{dt} \int_{\Omega} Q(\nabla u) \theta \, dx - \int_{\Omega} (4\mu D u : D u_t + 2\lambda \text{div } u \text{div } u_t) \theta \, dx
\leq \frac{d}{dt} \int_{\Omega} Q(\nabla u) \theta \, dx + C \|\nabla u\|_3 \|\nabla u_t\| \|\theta\|_6
\leq \frac{d}{dt} \int_{\Omega} Q(\nabla u) \theta \, dx + C \|\nabla u\|^2 \|\nabla^2 u\|^2 \|\nabla u_t\| \left(\|\sqrt{\rho} \theta\| + \|\nabla \theta\|\right).
Plugging (2.11)–(2.13) into (2.10) and adding the resultant with (2.9) multiplied with a small positive number $\epsilon_2$, one obtains
\[
\frac{d}{dt}\left(\frac{\kappa}{2} \|\nabla \theta\|^2 - \int_{\Omega} \mathcal{Q}(\nabla u) \theta \, dx\right) + \frac{c_v}{2} \|\sqrt{\rho} \theta_t\|^2 + \epsilon_2 \|\nabla^2 \theta\|^2 \\
\leq C \left[ \|\nabla u\|^4 \|\nabla \theta\|^2 + \left( \|\nabla \theta\|^2 + \|\nabla \theta\|^2 \right) \|\nabla u\| \|\nabla^2 u\| + \|\nabla u\| \|\nabla^2 u\|^3 \\
+ \|\nabla u\| \frac{1}{2} \|\nabla^2 u\| \frac{1}{2} \|\nabla u_t\| \left( \|\sqrt{\rho} \theta\| + \|\nabla \theta\| \right) \right].
\]

Multiplying the above inequality with $t$ yields
\[
\frac{d}{dt}\left(\frac{\kappa}{2} \|\sqrt{t} \nabla \theta\|^2 - t \int_{\Omega} \mathcal{Q}(\nabla u) \theta \, dx\right) + \frac{c_v}{2} \|\sqrt{t} \sqrt{\rho} \theta_t\|^2 + \epsilon_2 \|\sqrt{t} \nabla^2 \theta\|^2 + \int_{\Omega} \mathcal{Q}(\nabla u) \theta \, dx \\
\leq C \left[ \|\nabla \theta\|^2 + \|\nabla u\|^4 \|\sqrt{t} \nabla \theta\|^2 + \left( \|\sqrt{\rho} \theta\|^2 + \|\sqrt{t} \nabla \theta\|^2 \right) \|\nabla u\| \|\nabla^2 u\| \\
\frac{1}{2} \|\nabla u\| \|\sqrt{t} \nabla u\| \frac{1}{2} \|\nabla u_t\| \left( \|\sqrt{\rho} \theta\| + \|\sqrt{t} \nabla \theta\| \right) \right].
\]

By Corollary 2.1, Proposition 2.2, and Proposition 2.5, it follows from the Gagliardo-Nirenberg inequality and (2.3) that
\[
t \int_{\Omega} \mathcal{Q}(\nabla u) \theta \, dx \leq C t \int_{\Omega} \|\nabla u\|^2 \theta \, dx \leq C t \|\nabla u\| \|\nabla u\|_3 \|\theta\|_6 \\
\leq C T \frac{1}{2} \|\nabla u\|^2 \|\sqrt{t} \nabla^2 u\| \frac{1}{2} \left( \|\sqrt{\rho} \theta\| + \|\sqrt{t} \nabla \theta\| \right) \\
\leq C \left[ T \Phi^3(T) \right] \left( \|\sqrt{\rho} \theta\| + \|\sqrt{t} \nabla \theta\| \right) \|\sqrt{t} \nabla^2 u\| \frac{1}{2} \\
\leq \eta \sup_{0 \leq t \leq T} \|\sqrt{t} \nabla \theta\|^2 + C_\eta \left( \sup_{0 \leq t \leq T} \|\sqrt{t} \nabla^2 u\| + 1 \right),
\]
for any $\eta > 0$. Integrating (2.14) over $(0, T)$ and using (2.15), one deduces by Proposition 2.5 and the Young inequality that
\[
\kappa \sup_{0 \leq t \leq T} \|\sqrt{t} \nabla \theta\|^2 + \int_0^T \left( c_v \|\sqrt{t} \sqrt{\rho} \theta_t\|^2 + 2\epsilon_2 \|\sqrt{t} \nabla^2 \theta\|^2 \right) \, dt \\
\leq \sup_{0 \leq t \leq T} \left( t \int_{\Omega} \mathcal{Q}(\nabla u) \theta \, dx \right) + C \int_0^T \left[ \|\nabla \theta\|^2 + \|\nabla u\|^4 \|\sqrt{t} \nabla \theta\|^2 \\
+ \left( \|\sqrt{\rho} \theta\|^2 + \|\sqrt{t} \nabla \theta\|^2 \right) \|\nabla u\| \|\nabla^2 u\| + \sqrt{t} \|\nabla u\| \|\nabla^2 u\| |\sqrt{t} \nabla^2 u| \\
+ \|\nabla u\| \frac{1}{2} \|\nabla^2 u\| \frac{1}{2} \|\nabla u_t\| \left( \|\sqrt{\rho} \theta\| + \|\sqrt{t} \nabla \theta\| \right) \right] \, dt \\
\leq \eta \sup_{0 \leq t \leq T} \|\sqrt{t} \nabla \theta\|^2 + C_\eta \left( 1 + T \frac{1}{2} \Phi^2(T) \right) \sup_{0 \leq t \leq T} \|\sqrt{t} \nabla^2 u\| \\
+ C_\eta \left( T \Phi^3(T) + T \frac{1}{2} \Phi^2(T) + T \frac{1}{2} \Phi^3(T) + 1 \right),
\]
for any $\eta > 0$, from which, choosing $\eta$ sufficiently small and by (2.3)–(2.5), one gets
\[
\sup_{0 \leq t \leq T} \|\sqrt{t} \nabla \theta\|^2 + \int_0^T \left( \|\sqrt{t} \sqrt{\rho} \theta_t\|^2 + \|\sqrt{t} \nabla^2 \theta\|^2 \right) \, dt \leq C \left( \sup_{0 \leq t \leq T} \|\sqrt{t} \nabla^2 u\| + 1 \right).
\]

Differentiating (1.2) with respect to $t$ yields
\[
\rho(u_{tt} + u \cdot \nabla u_t) + \rho u_t \cdot \nabla u + \rho_t (u_t + u \cdot \nabla u) \\
- \mu \Delta u_t - (\mu + \lambda) \nabla \text{div} u_t + \nabla P_t = 0.
\]
It follows from (2.7) that
\[
\int_\Omega \nabla P_t u_t \, dx = - \int_\Omega P_t \text{div} u_t \, dx \\
= \frac{R}{c_v} \int_\Omega \text{div} u_t (-Q(\nabla u) - \kappa \Delta \theta + P \text{div} u + c_v \text{div}(\rho u \theta)) \, dx.
\]

Testing (2.17) by \(u_t\) and utilizing the above equality yield
\[
\frac{1}{2} \frac{d}{dt} \|\sqrt{\rho} u_t\|^2 + \mu \|\nabla u_t\|^2 + (\mu + \lambda) \|\text{div} u_t\|^2 \\
= \frac{R}{c_v} \int_\Omega \text{div} u_t \left( Q(\nabla u) + \kappa \Delta \theta - P \text{div} u + c_v \text{div}(\rho u \theta) \right) \, dx \\
+ \int_\Omega \text{div}(\rho u)(u_t + (u \cdot \nabla)u) \cdot u_t \, dx - \int_\Omega \rho(u_t \cdot \nabla)u \cdot u_t \, dx.
\]

Multiplying the above identity with \(t\) and integrating over \((0, T)\) lead to
\[
\frac{1}{2} \sup_{0 \leq t \leq T} \|\sqrt{\rho} u_t\|^2 + \mu \int_0^T \|\nabla u_t\|^2 \, dt \\
\leq \frac{1}{2} \int_0^T \|\sqrt{\rho} u_t\|^2 \, dt + C \int_0^T \int_\Omega |\nabla u_t||\nabla u|^2 \, dx \, dt + \frac{\kappa R}{c_v} \int_0^T \int_\Omega \text{div} u_t \Delta \theta \, dx \, dt \\
- \frac{R}{c_v} \int_0^T \int_\Omega \text{div} u_t \text{div}(\rho u \theta) \, dx \, dt - \int_0^T \int_\Omega \rho u \cdot \nabla |u_t|^2 \, dx \, dt - \int_0^T \int_\Omega \rho u \cdot \nabla ((u \cdot \nabla)u) \cdot u_t \, dx \, dt \\
- \int_0^T \int_\Omega \rho(u_t \cdot \nabla)u \cdot u_t \, dx \, dt = \sum_{i=1}^8 K_i. \tag{2.18}
\]

By the Cauchy and Gagliardo-Nirenberg inequalities, it follows from Propositions 2.2 and 2.5–2.7 and Corollary 2.1 that \(K_1 \leq C\),

\[
K_2 \leq C \int_0^T t |\nabla u_t||\nabla u|^2 \, dt \leq C \int_0^T \|\nabla \nabla u_t\|\|\|\nabla u\|^{\frac{1}{2}}\|\nabla \nabla^2 u\|\|\nabla^2 u\|^{\frac{1}{2}} \, dt \\
\leq CT^{\frac{3}{4}} \Phi(T) \sup_{0 \leq t \leq T} \|\nabla \nabla^2 u\|,
\]

\[
K_3 \leq \frac{\mu}{2} \int_0^T \|\nabla \nabla u_t\|^2 \, dt + C \int_0^T \|\nabla \nabla^2 \theta\|^2 \, dt,
\]

\[
K_4 \leq C \int_0^T t |\rho||\theta||\nabla u|_{3}||\nabla u_t| \, dt \\
\leq C \int_0^T \left( \|\sqrt{\rho} \theta| + \|\sqrt{\nabla \nabla \theta} \right) \|\nabla u|^{\frac{1}{2}}\|\nabla \nabla u|^{\frac{1}{2}}\|\nabla \nabla u_t| \, dt \leq CT^{\frac{3}{4}} \Phi^{\frac{3}{2}}(T),
\]

\[
K_5 \leq C \int_0^T \int_\Omega |\nabla u_t| \left( |\nabla |\rho||u| |\theta| + |\rho| |\nabla u| |\theta| + |\rho| |u| |\nabla \theta| \right) \, dx \, dt \\
\leq C \int_0^T \int_\Omega \left( \|\nabla |\rho| |3||u||\theta|| \|\rho||\theta||\nabla u|_{3}||\theta|_{6} + \|\rho||\theta||\nabla u|_{3}||\theta|_{6} + \|\rho||\theta||u|_{6}||\nabla \theta||_{3} \right) \, dt \\
\leq C \int_0^T \|\nabla \nabla u_t\|\|\nabla u\|^{\frac{1}{2}}\|\nabla \nabla u\|^{\frac{1}{2}} \left( \|\nabla \nabla \theta\| + \|\nabla \nabla \theta\| \right) \, dt
\]
Corollary 2.2. Under the assumptions of Proposition 2.5, it holds that

\[ \Phi(T) + \sup_{0 \leq t \leq T} \| (\sqrt{t} \nabla^2 u, \sqrt{t} \nabla u) \|^2 \leq C, \]
that is
\[
\sup_{0 \leq t \leq T} \left( \|\rho\|_\infty + \|\nabla \rho\|_q + \left( \sqrt{\rho} \theta, \nabla u, \sqrt{\rho} \nabla \theta, \sqrt{\rho} \nabla^2 u, \sqrt{\rho} \nabla^2 \theta \right)^2 \right) \\
+ \int_0^T \left( \|\sqrt{\rho} u_t, \nabla u_t, \sqrt{\rho} \nabla \theta_t, \sqrt{\rho} \nabla^2 u_t, \sqrt{\rho} \nabla^2 \theta_t \right)^2 dt \leq C.
\]

**Proposition 2.8.** Under the assumptions of Proposition 2.5, it holds that
\[
\sup_{0 \leq t \leq T} \left( \|\rho_t, t \nabla^2 \theta, t \sqrt{\rho} \theta_t\|_2^2 + \int_0^T \left( \|t \nabla \theta_t\|_2^2 + \|t \nabla^2 \theta\|_6^2 + \|\sqrt{t} \nabla^2 u\|_2^2 \right) dt \leq C.
\]

**Proof.** Using (1.1), it follows from the Hölder and Sobolev inequalities and Corollary 2.2 that
\[
\|\rho_t\| = \|u \cdot \nabla \rho + \text{div} \, u \rho\| \leq \|u\|_6 \|\nabla \rho\|_3 + \|\nabla u\| \|\rho\|_\infty \\
\leq C(\|\nabla \rho\|_3 + \|\rho\|_\infty) \|\nabla u\| \leq C. \tag{2.20}
\]

Differentiating (1.3) with respect to \( t \) yields
\[
c_v \rho(\theta_{tt} + u \cdot \nabla \theta_t) + c_v \rho_t(\theta_t + u \cdot \nabla \theta) + c_v \rho u_t \cdot \nabla \theta - \kappa \Delta \theta_t \\
+ P_t \text{div} \, u + P \text{div} \, u_t = 4\mu D \text{div} \, u : D u_t + 2\lambda \text{div} \, u \text{div} \, u_t.
\]

Multiplying the above equality with \( \theta \) and integrating over \( \Omega \), one gets
\[
\frac{c_v}{2} \frac{d}{dt} \|\sqrt{\rho} \theta_t\|^2 + \kappa \|\nabla \theta_t\|^2 \\
= -c_v \int_\Omega \rho_t |\theta_t|^2 dx - c_v \int_\Omega \rho u \cdot \nabla \theta_t dx - c_v \int_\Omega \rho u_t \cdot \nabla \theta_t dx - \int_\Omega P_t \text{div} \, u \theta_t dx \\
- \int_\Omega P \text{div} \, u_t \theta_t dx + \int_\Omega (4\mu D \text{div} \, u : D u_t + 2\lambda \text{div} \, u \text{div} \, u_t) \theta_t dx =: \sum_{i=1}^6 L_i. \tag{2.21}
\]

Terms on the right-hand side are estimated by Corollary 2.2, Proposition 2.2, (1.1), (2.20), and the Hölder, Gagliardo-Nirenberg, and Young inequalities as follows
\[
|L_1| = c_v \int_{\Omega} \text{div}(\rho u) \theta_t^2 dx \leq C \int_{\Omega} |\rho| |u| |\theta_t| |\nabla \theta_t| dx \\
\leq C \|\rho\|_\infty \|u\|_6 \|\sqrt{\rho} \theta_t\|_3 \|\nabla \theta_t\| \\
\leq C \|\rho\|_\infty \|\nabla u\| \left( \|\sqrt{\rho} \theta_t\| \|\nabla \theta_t\| + \|\sqrt{\rho} \theta_t\| \|\nabla \theta_t\| \right) \leq \frac{\kappa}{8} \|\nabla \theta_t\|^2 + C \|\sqrt{\rho} \theta_t\|^2.
\]

\[
|L_2| = c_v \int_{\Omega} \text{div}(\rho u) u \cdot \nabla \theta_t dx \\
\leq c_v \int_{\Omega} \rho |u| \left( |\nabla u| |\nabla \theta| |\theta_t| + |u| |\nabla^2 \theta| |\theta_t| + |u| |\nabla \theta| |\nabla \theta_t| \right) dx \\
\leq C \|\rho\|_\infty \left( \|u\|_\infty \|\nabla u\|_3 \|\nabla \theta\| \|\theta_t\|_6 + \|u\|_6 \|\nabla^2 \theta\| \|\theta_t\|_6 + \|u\|_6 \|\nabla \theta\|_6 \|\nabla \theta_t\| \right) \\
\leq C \|\rho\|_\infty \left( \|\nabla u\| \|\nabla^2 u\| \|\nabla \theta\| + \|\nabla^2 \theta\| \|\theta_t\| + \|\nabla \theta\| \|\nabla \theta_t\| \right) \left( \|\sqrt{\rho} \theta_t\| + \|\nabla \theta_t\| \right) \\
\leq \frac{\kappa}{8} \|\nabla \theta_t\|^2 + C \left( \|\sqrt{\rho} \theta_t\|^2 + \|\nabla^2 u\|_2 \|\nabla \theta\|^2 + \|\nabla \theta\|^2 + \|\nabla^2 \theta\|^2 \right).
\]

\[
|L_3| \leq c_v \int_{\Omega} |\rho| |u_t| |\nabla \theta| |\theta_t| dx \leq C \|\rho\|_\infty \|\nabla \theta_t\| \|\nabla \theta\| |\theta_t\|_6 \\
\leq C \|\rho\|_\infty \|\sqrt{\rho} \theta_t\| \|\nabla \theta\| \|\nabla \theta_t\| \left( \|\sqrt{\rho} \theta_t\| + \|\nabla \theta_t\| \right) \\
\leq \frac{\kappa}{8} \|\nabla \theta_t\|^2 + C \left( \|\sqrt{\rho} \theta_t\|^2 + \|\nabla \theta_t\| \|\nabla \theta_t\| \|\nabla \theta\| \|\nabla \theta_t\| \right).
\]

\[
|L_4| \leq R \int_{\Omega} \left( |\rho_t| |\nabla u| |\theta_t| + \rho |\theta_t|^2 |\nabla u| \right) dx
\]
Applying the elliptic estimates to (1.2) and by Proposition 2.2 and Corollary 2.2, one obtains from the Integrating (2.22) over \((0,T)\) and using Corollary 2.2 yield

\[
\sup_{0 \leq t \leq T} \|t \sqrt{\rho \theta_t} \|^2 + \int_0^T \|t \nabla \theta_t \|^2 \, dt \leq C.
\] (2.23)

Recalling (2.9), it follows from (2.23) and Corollary 2.2 that

\[
\sup_{0 \leq t \leq T} \|t \Delta \theta \|^2 \leq C \sup_{0 \leq t \leq T} \left( \|t \sqrt{\rho \theta_t} \|^2 + \epsilon \|\nabla u \|^2 \right) \left( \|t \sqrt{\nabla \theta} \|^2 + \sqrt{t} \|\nabla \theta \|^2 \right) \leq C.
\] (2.24)

Applying the elliptic estimates to (1.2) and by Proposition 2.2 and Corollary 2.2, one obtains from the H"older, Sobolev, and Poincaré inequalities that

\[
\|\nabla^2 u\|_q \leq C \left( \|\rho u_t\|_q + \|\rho (u \cdot \nabla) u\|_q + \|\nabla P\|_q \right)
\leq C \left( \|\rho\|_{\infty} \|u_t\|_6 + \|\rho\|_{\infty} \|u\|_{\infty} \|\nabla u\|_6 + \|\nabla \rho\|_q \|\theta\|_{\infty} + \|\rho\|_{\infty} \|\nabla \theta\|_6 \right)
\leq C \left( 1 + \|\nabla u_t\| + \|\nabla^2 u\| + \|\nabla \theta\| + \|\Delta \theta\| \right)
\]

and, thus, by Corollary 2.2, one gets

\[
\int_0^T \|\sqrt{t} \nabla^2 u\|^2 \, dt \leq C \int_0^T \left( \|\sqrt{t} \nabla u_t\|^2 + \|\sqrt{t} \nabla \theta\|^2 \right) \|\nabla^2 u\|^2 \, dt
\]
\[
+ C \int_0^T \left( \|\nabla \theta\|^2 + \|\nabla \theta\|^2 \right) \|\nabla \theta\|^2 \, dt \leq C.
\] (2.25)
Finally, applying the elliptic estimates to (1.3) and using the Sobolev and Poincaré inequalities, one deduces by Corollary 2.2 that
\[
\|\nabla^2 \theta\|_6^2 \leq C \left( \|\rho \theta\|_6^2 + \|\rho(u \cdot \nabla) \theta\|_6^2 + \|\rho \div u\|_6^2 + \|Q(\nabla u)\|_6^2 \right)
\leq C \left( \|\rho\|_\infty^2 \|\theta\|_6^2 + \|\rho\|_\infty^2 \|u\|_6^2 \|\nabla \theta\|_6^2 + \|\rho\|_\infty^2 \|\theta\|_6^2 \|\nabla u\|_6^2 + \|\nabla u\|_6^2 \right)
\leq C \left( \|\sqrt{\rho} \theta\|_2^2 + \|\nabla \theta\|_2^2 + \|\nabla^2 u\| (||\nabla \theta||_2^2 + ||\nabla^2 \theta||_2^2) \right)
+ C \left( 1 + ||\nabla \theta||_2 \right) \|\nabla^2 u\|_6^2 + \|\nabla^2 u\|_6^2 \|\nabla^2 u\|_6^2.
\] (2.26)
Hence, it follows from (2.23), (2.25), and Corollary 2.2 that
\[
\int_0^T \|t \nabla^2 \theta\|_6^2 dt \leq C \int_0^T \left[ \|\sqrt{t} \nabla \rho \theta\|_2^2 + \|t \nabla \theta\|_2^2 + (1 + \|\sqrt{t} \nabla \theta\|_2^2) \|\sqrt{t} \nabla^2 u\|_6^2 \right] dt
+ C \int_0^T \left[ \|\sqrt{t} \nabla^2 u\| (||\sqrt{t} \nabla \theta||_2^2 + ||\sqrt{t} \nabla^2 \theta||_2^2) + \|\sqrt{t} \nabla^2 u\|_6^2 \|\sqrt{t} \nabla^2 u\|_6^2 \right] dt \leq C.
\] (2.27)
Combining (2.20) with (2.23)–(2.27) yields the conclusion. □

As the end of this section, we prove in the next proposition that the existence time $T_0$ depends only on $R, \mu, \lambda, c_v, q$, and the upper bound of $\Phi_0$, but is independent of the quantities $||\nabla^2 u_0||, ||\nabla^2 \theta_0||, ||g_1||$, and $||g_2||$.

**Proposition 2.9.** Let $q \in (3, 6)$ and assume that $(\rho_0, u_0, \theta_0)$ satisfies
\[
\rho \leq \rho_0 \in W^{1,q}(\Omega), \quad u_0 \in H_0^1(\Omega) \cap H^2(\Omega), \quad 0 \leq \theta_0 \in H_0^1(\Omega) \cap H^2(\Omega),
\]
for some positive number $\rho$.

Then, there exist two positive constants $T_0$ and $C$ depending only on $R, \mu, \lambda, c_v, q$, and the upper bound of $\Phi_0$, such that system (1.1)–(1.3), subject to (1.6)–(1.7), admits a unique solution $(\rho, u, \theta)$, in $\Omega \times (0, T_0)$, satisfying
\[
\sup_{0 \leq t \leq T_0} \left( ||\nabla\theta||_\infty + ||\rho||_{W^{1,q}} + ||(\rho, \sqrt{\rho} \theta, \nabla u)||_2 \right) + \int_0^{T_0} \|\nabla \theta, \sqrt{\rho} u_t, \nabla^2 u\|_6^2 dt \leq C
\]
and
\[
\sup_{0 \leq t \leq T_0} \left( \|\sqrt{t} \nabla \theta, t \sqrt{\rho} \theta_t, t \nabla^2 \theta, \sqrt{t} \nabla \rho u_t, \sqrt{t} \nabla^2 u\|_6^2 \right) + \int_0^{T_0} \left( \|\sqrt{t} \nabla^2 u\|_6^2 + ||t \nabla^2 \theta||_6^2 \right) dt
+ \int_0^{T_0} ||(\sqrt{t} \nabla \theta, \sqrt{t} \nabla^2 \theta, t \nabla \theta_t, \sqrt{t} \nabla u_t)||_6^2 dt \leq C.
\]

**Proof.** By Proposition 2.1, there is a unique local strong solution $(\rho, u, \theta)$ on $\Omega \times (0, T_*)$ satisfying the regularities stated in Proposition 2.1. By applying Proposition 2.1 inductively, one can extend the local solution uniquely to the maximal time of existence $T_{\max}$. Then, the following holds
\[
\sup_{T_* \leq t \leq T_{\max}} \left( \frac{1}{|\rho|_\infty} + ||\rho||_{W^{1,q}} + ||u||_{H^2} + ||\theta||_{H^2} \right) = \infty.
\] (2.28)
For any $T \in (0, T_{\max})$, $(\rho, u, \theta)$ satisfies the regularities in Proposition 2.1 with $T_*$ replaced by $T$. Let $\epsilon_0$ be the constant stated in Corollary 2.1, $\Phi$ the function given by (1.8), and set
\[
T_0 = \sup \left\{ T \in (0, T_{\max}) \mid T^\frac{6-q}{4q} \Phi^\frac{2}{q}(T) \leq \epsilon_0 \right\}.
\]
Claim: $T_0 < T_{\max}$. Assume by contradiction that $T_0 = T_{\max}$. Then, by definition
\[
T^\frac{6-q}{4q} \Phi^\frac{2}{q}(T) \leq \epsilon_0, \quad \forall T \in (0, T_{\max}).
\] (2.29)
Since $\Phi(T) \geq 1$, it follows from (2.29) that $T_{\max} \leq \epsilon_0^{\frac{4q}{n-q}}$. Thanks to (2.29), it follows from Corollary 2.2 and Proposition 2.8 that
\[ \Phi(T) + \sup_{0 \leq t \leq T} \| (\sqrt{t} \nabla^2 u, t \nabla^2 \theta) \|^2 \leq C, \quad \forall T \in (0, T_{\max}), \]
for a positive constant $C$ independent of $T \in (0, T_{\max})$. By following the arguments in Proposition 2.3, one deduces by Proposition 2.4 and (2.29) that
\[ \inf_{\Omega \times (0,T)} \rho \geq \rho e^{-\frac{\epsilon_0}{C} \| \nabla u \|_{\infty} dt} \geq \rho e^{-C \epsilon_0}, \quad \forall T \in (0, T_{\max}), \]
for a positive constant $C$ independent of $T \in (0, T_{\max})$. Combining the above two yields
\[ \sup_{r_* \leq t < T_{\max}} \left( \| \rho \|^2 + \| \rho \|_\infty \| u \|_{W^{1,q}} + \| u \|_{H^2} + \| \theta \|_{H^2} \right) \leq C, \]
which contradicts to (2.28). This contradiction proves the claim.

Since $T_0 < T_{\max}$ and noticing that $\Phi(T)$ is continuous on $[0, T_{\max})$, one gets by the definition of $T_0$ that
\[ T_0 \overset{\text{def}}{=} \Phi^\frac{1}{6}(T_0) = \epsilon_0. \quad (2.30) \]
Thanks to this and recalling that $\Phi(T) \geq 1$, it follows from Corollary 2.2 that $T_0 \leq \epsilon_0^{\frac{4q}{n-q}}$ and $\Phi(T_0) \leq C_0$ for a positive constant $C_0$ depending only on $R, \mu, \lambda, c_v, q$, and the upper bound of $\Phi_0$. Therefore, it follows from (2.30) that $T_0 \geq \left( \frac{\epsilon_0}{C_0^\frac{q}{n-q}} \right)^{\frac{n-q}{n}}$. The corresponding estimates follow from Corollary 2.2 and Proposition 2.8. This completes the proof. \hfill \Box

3. A Preparing Existence Result

In this section, we prove the following existence result, which is a preparation of proving the existence part of Theorem 1.1. Note that the uniqueness is not included here.

**Proposition 3.1.** Assume that all the conditions of Theorem 1.1 hold. Denote
\[ \Phi_0 := \| \rho_0 \|_\infty + \| \nabla \rho_0 \|_q + \| (\sqrt{\rho_0} \theta_0, \nabla u_0) \|^2. \]
(i) Then, there exists a positive time $T_0$ depending only on $R, \mu, \lambda, c_v, q$, and $\Phi_0$, such that system (1.1)–(1.3), subject to (1.6)–(1.7), in $\Omega \times (0, T_0)$, admits a solution $(\rho, u, \theta)$, which satisfies all the properties stated in Definition 1.1 except that the regularities $\sqrt{\rho u}, \sqrt{\rho \theta} \in C([0, T_0]; L^2)$ are replaced by
\[ \rho u \in C([0, T_0]; L^2) \quad \text{and} \quad \rho \theta \in C_w([0, T_0]; L^2), \]
where $C_w$ represents the weak continuity.

(ii) Moreover, for any $t \in (0, T_0)$, it holds that
\[ \| \sqrt{\rho u} \|^2(t) \leq \| \sqrt{\rho_0 u_0} \|^2 + Ct, \quad \| \sqrt{\rho \theta} \|^2(t) \leq \| \sqrt{\rho_0 \theta_0} \|^2 + C \sqrt{t}. \]

**Proof.** (i) **Step 1. Construction of the initial data.** Choose $\{u_{0n}\}_{n=1}^\infty \subseteq H_0^1 \cap H^2$ such that $u_{0n} \to u_0$ in $H^1$ as $n \to \infty$. Set $\rho_{0n} = \rho_0 + \frac{1}{n^2}$. Then, it is clear that
\[ \| \rho_{0n} \|_\infty + \| \nabla \rho_{0n} \|_q + \| \nabla u_{0n} \|^2 \leq \| \rho_0 \|_\infty + \| \nabla \rho_0 \|_q + \| \nabla u_0 \|^2 + \frac{1}{2} \]
for large $n$. Put
\[ \theta_{0n}(x) = \begin{cases} 0, & x \in \{ x \in \Omega \mid \rho_0(x) < \frac{1}{n} \}, \\ \theta_0, & x \in \{ x \in \Omega \mid \rho_0(x) \geq \frac{1}{n} \}, \end{cases} \]
and take $\theta_{0n} \geq 0$ such that
\[ \|\theta_{0n} - \bar{\theta}_{0n}\| \leq \frac{1}{n}. \]  
(3.2)

Note that such $\theta_{0n}$ exists. For example, one can take $\theta_{0n} = j_{\varepsilon_n} \ast \bar{\theta}_{0n}$ for sufficiently small positive $\varepsilon_n$, where $j_{\varepsilon_n}$ is the standard mollifier and $\bar{\theta}_{0n}$ is the zero extension of $\bar{\theta}_{0n}$ on $\mathbb{R}^3$, that is, $\bar{\theta}_{0n} = \bar{\theta}_{0n}$ on $\Omega$ and $\bar{\theta}_{0n} = 0$ on $\mathbb{R}^3 \setminus \Omega$.

We want to show
\[ \|\rho_{0n}\|_{\infty} + \|\nabla \rho_{0n}\| + \|\nabla u_{0n}\|^2 + \|\sqrt{\rho_{0n}} \theta_{0n}\|^2 \leq \Phi_0 + 1 \]  
for large $n$, and
\[ \int_{\Omega} \rho_{0n} \theta_{0n} \chi dx \rightarrow \int_{\Omega} \rho_0 \theta_0 \chi dx \quad \text{as} \quad n \rightarrow \infty, \quad \forall \chi \in L^2(\Omega). \]  
(3.4)

The quantity $\|\sqrt{\rho_{0n}} \theta_{0n}\|$ is estimated as follows. By the elementary inequality $\sqrt{a+b} \leq \sqrt{a} + \sqrt{b}$ for $a, b \geq 0$ and recalling the definition of $\bar{\theta}_{0n}$, it follows that
\[
\sqrt{\rho_{0n}} \theta_{0n} = \sqrt{\rho_0 + \frac{1}{n^2} (\theta_{0n} - \bar{\theta}_{0n})} + \sqrt{\rho_0 + \frac{1}{n^2} \bar{\theta}_{0n}}
\leq \left( \sqrt{\rho_0 + \frac{1}{n}} \right) |\theta_{0n} - \bar{\theta}_{0n}| + \sqrt{\rho_0 \bar{\theta}_{0n} + \frac{\bar{\theta}_{0n}}{n}}
\leq \left( \sqrt{\rho_0 + \frac{1}{n}} \right) |\theta_{0n} - \bar{\theta}_{0n}| + \sqrt{\rho_0 \theta_0 + \frac{\bar{\theta}_{0n}}{n}}
\]
and, thus, recalling (3.2), one gets
\[
\|\sqrt{\rho_{0n}} \theta_{0n}\| \leq \left( \|\rho_0\|_{\infty}^\frac{1}{2} + \frac{1}{n} \right) \frac{1}{n} + \|\sqrt{\rho_0} \theta_0\| + \frac{\|\bar{\theta}_{0n}\|}{n}.
\]

With the aid of the above and noticing that
\[
\|\bar{\theta}_{0n}\| = \left( \int_{\Omega \cap \{x|\rho_0(x) \geq \frac{1}{n}\}} \theta_{0n}^2 dx \right)^\frac{1}{2} \leq \sqrt{n} \left( \int_{\Omega \cap \{x|\rho_0 \geq \frac{1}{n}\}} \rho_0 \theta_0^2 dx \right)^\frac{1}{2}
\leq \sqrt{n} \left( \int_{\Omega} \rho_0 \theta_0^2 dx \right)^\frac{1}{2} = \sqrt{n} \|\sqrt{\rho_0} \theta_0\|,
\]  
(3.5)

one obtains
\[
\|\sqrt{\rho_{0n}} \theta_{0n}\| \leq \left( \|\rho_0\|_{\infty}^\frac{1}{2} + \frac{1}{n} \right) \frac{1}{n} + \left( 1 + \frac{1}{\sqrt{n}} \right) \|\sqrt{\rho_0} \theta_0\|.
\]  
(3.6)

This implies
\[
\|\sqrt{\rho_{0n}} \theta_{0n}\|^2 \leq \|\sqrt{\rho_0} \theta_0\|^2 + \frac{1}{2}, \quad \text{for large} \quad n.
\]  
(3.7)

Combining (3.1) with (3.7) leads to (3.3).

Thanks to (3.2) and (3.5), it follows for any $\chi \in L^2(\Omega)$ that
\[
\left| \int_{\Omega} \rho_{0n} \theta_{0n} \chi dx - \int_{\Omega} \rho_0 \theta_0 \chi dx \right|
= \left| \int_{\Omega} [\rho_{0n}(\theta_{0n} - \bar{\theta}_{0n}) + (\rho_{0n} - \rho_0) \bar{\theta}_{0n} + \rho_0(\bar{\theta}_{0n} - \theta_0)] \chi dx \right|
\leq \|\rho_{0n}\|_{\infty} \|\theta_{0n} - \bar{\theta}_{0n}\| \|\chi\| + \frac{1}{n^2} \|\bar{\theta}_{0n}\| \|\chi\| + \int_{\Omega \cap \{x|\rho_0(x) < \frac{1}{n}\}} \rho_0 \theta_0 |\chi| dx
\]
\[ \frac{\| x \|}{n} \left( \| \rho_0 \|_\infty + \frac{1}{n^2} \right) + \frac{\| x \|}{n^2} \| \sqrt{\rho_0} \theta_0 \| + \frac{\| x \|}{\sqrt{n}} \| \sqrt{\rho_0} \theta_0 \|, \]

which implies \((3.4)\).

**Step 2. Approximate solutions and convergence.** Thanks to \((3.3)\) and Proposition 2.9, there are two positive constants \(T_0\) and \(C\) independent of \(n\) such that system \((1.1)-(1.3)\), subject to \((1.6)-(1.7)\), admits a unique solution \((\rho_n, u_n, \theta_n)\), in \(\Omega \times (0, T_0)\), and the following a priori estimates hold

\[
\begin{align*}
&\int_0^{T_0} \| (\nabla \theta_n, \sqrt{\rho_n} \nabla u_n, \nabla^2 u_n) \| dt \leq C, \\
&\sup_{0 \leq t \leq T_0} \left( \| \rho_n \|_\infty + \| \rho_n \|_{W^{1, q}} + \| (\partial_t \rho_n, \sqrt{\rho_n} \theta_n, \nabla u_n) \| \right) \leq C, \\
&\int_0^{T_0} \left( \| (\sqrt{t} \sqrt{\rho_n} \partial_t \theta_n, \sqrt{t} \nabla^2 \theta_n, \sqrt{t} \nabla \partial_t u_n) \| + \| \sqrt{t} \nabla^2 u_n \|_q^2 \right) dt \leq C, \\
&\int_0^{T_0} t^2 \| (\nabla \partial_t \theta_n) \|^2 + \| \nabla \theta_n \|_6^2 \| dt \leq C, \\
&\sup_{0 \leq t \leq T_0} \left( \| (\sqrt{t} \nabla \theta_n, t \sqrt{\rho_n} \partial_t \theta_n, t \nabla^2 \theta_n, \sqrt{t} \sqrt{\rho_n} \partial_t u_n, \sqrt{t} \nabla^2 u_n) \| \right) \leq C,
\end{align*}
\]

for large \(n\). Then, by the Banach-Alaoglu theorem and using the Cantor’s diagonal arguments, there is a subsequence, still denoted by \((\rho_n, u_n, \theta_n)\), and \((\rho, u, \theta)\) satisfying

\[
\begin{align*}
\rho &\rightarrow \rho, \quad \text{in } L^\infty(0, T_0; W^{1, q}), \\
\partial_t \rho_n &\rightarrow \partial_t \rho, \quad \text{in } L^\infty(0, T_0; L^2), \\
u_n &\rightarrow u, \quad \text{in } L^\infty(0, T_0; H^1_0), \\
\partial_t u_n &\rightarrow u_t, \quad \text{in } L^2(\delta, T_0; H^1_0), \\
\theta_n &\rightarrow \theta, \quad \text{in } L^\infty(\delta, T_0; H^1_0), \\
\theta_n &\rightarrow \theta, \quad \text{in } L^2(\delta, T_0; W^{2, 6}), \\
\partial_t \theta_n &\rightarrow \partial_t \theta, \quad \text{in } L^2(\delta, T_0; H^1_0),
\end{align*}
\]

for any \(\delta \in (0, T_0)\). Moreover, since \(W^{1, q} \hookrightarrow C(\bar{\Omega})\) for \(q \in (3, 6)\), and \(H^2 \hookrightarrow H^1 \hookrightarrow L^2\), it follows from the Aubin-Lions lemma and \((3.14)-(3.21)\) that

\[
\begin{align*}
\rho_n &\rightarrow \rho, \quad \text{in } C([0, T_0]; C(\bar{\Omega})), \\
u_n &\rightarrow u, \quad \text{in } C([\delta, T_0]; L^2(\Omega)) \cap L^2(\delta, T_0; H^1_0(\Omega)), \\
\theta_n &\rightarrow \theta, \quad \text{in } C([\delta, T_0]; L^2(\Omega)) \cap L^2(\delta, T_0; H^1_0(\Omega)).
\end{align*}
\]
Due to the convergence (3.18), (3.21), and (3.22)–(3.24), one has the following convergence of the nonlinear terms

\[
\begin{align*}
(\rho_n u_n, \sqrt{\rho_n} u_n, \rho_n \theta_n, \sqrt{\rho_n} \theta_n) & \to (\rho u, \sqrt{\rho} u, \rho \theta, \sqrt{\rho} \theta) \quad \text{in } C([\delta, T_0]; L^2), \\
(\rho_n \partial_t u_n, \sqrt{\rho_n} \partial_t u_n, \rho_n \partial_t \theta_n, \sqrt{\rho_n} \partial_t \theta_n) & \to (\rho u_t, \sqrt{\rho} u_t, \rho \theta_t, \sqrt{\rho} \theta_t) \quad \text{in } L^2(\Omega \times (\delta, T_0)), \\
\rho_n (u_n \cdot \nabla) u_n & \to \rho (u \cdot \nabla) u, \quad \rho_n (u_n \cdot \nabla) \theta_n \to \rho (u \cdot \nabla) \theta, \quad \text{in } L^1(\Omega \times (\delta, T_0)), \\
\rho_n \theta_n \div u_n & \to \rho \theta \div u, \quad Q(\nabla u_n) \to Q(\nabla u), \quad \text{in } L^1(\Omega \times (\delta, T_0)),
\end{align*}
\] (3.25)

for any \( \delta \in (0, T_0) \). By the weakly lower semi-continuity of norms, it follows from (3.8), (3.25), (3.26) that

\[
\int_{\delta}^{T_0} (\|\sqrt{\rho} u_t\|^2 + \|\sqrt{\rho} \theta_t\|^2) dt \leq \lim_{n \to \infty} \int_{\delta}^{T_0} \left( \|\sqrt{\rho_n} \partial_t u_n\|^2 + \|\sqrt{\rho_n} \partial_t \theta_n\|^2 \right) dt \leq C,
\]

for any \( \delta, t \in (0, T_0) \) and for a positive constant \( C \) independent of \( \delta \) and \( t \). Therefore,

\[
\sqrt{\rho} \theta \in L^\infty(0, T_0; L^2) \quad \text{and} \quad \sqrt{\rho} u_t, \sqrt{\rho} \theta_t \in L^2(0, T_0; L^2).
\] (3.29)

The regularity \( u \in L^1(0, T_0; W^{2,q}) \) can be proved in the same as in Proposition 2.3.

**Step 3. The existence.** Thanks to the convergence (3.14)–(3.28), one can take the limit as \( n \to \infty \) to the equations of \((\rho_n, u_n, \theta_n)\) to show that \((\rho, u, \theta)\) satisfies equations (1.1)–(1.3) in the sense of distribution. Due to the regularities (3.9)–(3.13), one can further show that \((\rho, u, \theta)\) satisfies (1.1)–(1.3), a.e. in \( \Omega \times (0, T_0) \). The initial condition \( \rho|_{t=0} = \rho_0 \) is guaranteed by (3.22) by recalling that \( \rho_n|_{t=0} = \rho_0 + \frac{1}{n^2} \).

To complete the proof of (i), one still needs to show the regularities \( \rho u \in C((0, T_0]; L^2) \) and \( \rho \theta \in C_w((0, T_0]; L^2) \), as well as the initial condition \((\rho u, \rho \theta)|_{t=0} = (\rho_0 u_0, \rho_0 \theta_0)\). To this end, noticing that \((\rho u, \rho \theta) \in C((0, T_0]; L^2)\) guaranteed by (3.25), it suffices to show

\[
\rho u \to \rho_0 u_0 \quad \text{in } L^2, \quad \text{as } t \to 0,
\]

(3.30)

\[
\rho \theta \to \rho_0 \theta_0 \quad \text{in } L^2, \quad \text{as } t \to 0.
\] (3.31)

We first verify (3.30). By (3.8), it follows from the Gagliardo-Nirenberg and Hölder inequalities that

\[
\begin{align*}
\int_0^{T_0} \|\partial_t (\rho_n u_n)\|^2 dt & \leq 2 \int_0^{T_0} \left( \|\partial_t \rho_n u_n\|^2 + \|\rho_n \partial_t u_n\|^2 \right) dt \\
& \leq C \int_0^{T_0} \left( \|\partial_t \rho_n\|^2 \|u_n\|^2_{\infty} + \|\rho_n\|_{\infty} \|\sqrt{\rho_n} \partial_t u_n\|^2 \right) dt \\
& \leq C \int_0^{T_0} \|\nabla u_n\| \|\nabla^2 u_n\| dt + C \leq C
\end{align*}
\] (3.32)

for large \( n \). Thanks to this, it follows from the Newton-Leibnitz formula, the Minkowski and Hölder inequalities that

\[
\begin{align*}
\|\rho u(\cdot, t) - \rho_0 u_0\| & \leq \|\rho u - \rho_n u_n\|(t) + \|\rho_n u_n - \rho_0 u_0\|(t) + \|\rho_0 u_0 - \rho_n u_n - \rho_0 u_0 - \rho_0 u_0\| \\
& \leq \|\rho u - \rho_n u_n\|(t) + \int_0^t \|\partial_t (\rho_n u_n)\| d\tau + \|\rho_0 (u_0 - u_0)\| + \frac{C}{n^2} \|u_0\| \\
& \leq \|\rho u - \rho_n u_n\|(t) + C \sqrt{t} + \|\rho_0\|_{\infty} \|u_0 - u_0\| + \frac{C}{n^2} \|u_0\|
\end{align*}
\] (3.33)

for large \( n \), from which, recalling (3.25) and \( u_0 \to u_0 \) in \( H^1 \) as \( n \to \infty \), one gets by taking \( n \to \infty \) that \( \|\rho u - \rho_0 u_0\|(t) \leq C \sqrt{t} \), proving (3.30).
Then, we verify (3.31). Since $\rho \theta \in L^\infty(0, T_0; L^2)$ and $C_c^\infty(\Omega)$ is dense in $L^2$, it suffices to verify
\[
\left( \int_\Omega \rho \theta \phi \, dx \right) (t) \to \int_\Omega \rho_0 \theta_0 \phi \, dx \quad \text{as} \quad t \to 0, \quad \forall \phi \in C_c^\infty(\Omega).
\] (3.34)

Rewrite the equation for $\theta_n$ as
\[
c_v \left[ \int_\Omega \rho_n \theta_n \phi \, dx \right] (t) = \int_0^t \int_\Omega \rho_n \theta_n u_n \cdot \nabla \phi \, dx \, d\tau + \kappa \int_0^t \int_\Omega \Delta \theta_n \phi \, dx \, d\tau
\]
Multiplying the above equation with $\phi \in C_c^\infty(\Omega)$ and integrating over $\Omega \times (0, t)$ yield
\[
c_v \int_0^t \int_\Omega \rho_n \theta_n \phi \, dx \, d\tau = \int_0^t \int_\Omega \rho_n \theta_n \phi \, dx \, d\tau + \int_0^t \int_\Omega Q(\nabla u) \phi \, dx \, d\tau =: \sum_{i=1}^4 M_i.
\]

Terms on the right-hand side are estimated by integration by parts, the H"older inequality, and (3.8) as follows:
\[
|M_1| \leq c_v \int_0^t \int_\Omega \rho_n \frac{1}{2} ||\nabla \rho \theta_n||_3 \|\phi\|_3 \, d\tau \leq Ct,
\]
\[
|M_2| \leq \kappa \int_0^t \int_\Omega ||\nabla \theta_n||_3 \|\phi\|_3 \, d\tau \leq C \left( \int_0^t \int_\Omega ||\nabla \theta_n||^2 \, d\tau \right)^{\frac{1}{2}} \sqrt{t} \leq C \sqrt{t},
\]
\[
|M_3| \leq C \int_0^t \int_\Omega ||\rho_n||_3 \|\nabla \rho \theta_n\| \|\nabla u_n\| \|\phi\|_\infty \, d\tau \leq Ct,
\]
\[
|M_4| \leq C \int_0^t \int_\Omega ||\nabla u_n||^2 \|\phi\|_3 \, d\tau \leq C \int_0^t \int_\Omega ||\nabla u_n||^2 \|\phi\|_3 \, d\tau \leq Ct,
\]
for large $n$. Therefore, for large $n$, it follows
\[
\left| \left( \int_\Omega \rho \theta \phi \, dx \right) (t) - \int_\Omega \rho_0 \theta_0 \phi \, dx \right| \leq C \sqrt{t}, \quad \forall t \in (0, T_0),
\]
for any $\phi \in C_c^\infty(\Omega)$ and for a positive constant $C$ independent of $n$. Thanks to this and recalling (3.4) and (3.25), one gets by taking $n \to \infty$ that
\[
\left| \left( \int_\Omega \rho \theta \phi \, dx \right) (t) - \int_\Omega \rho_0 \theta_0 \phi \, dx \right| \leq C \sqrt{t}, \quad \forall t \in (0, T_0), \quad \forall \phi \in C_c^\infty(\Omega),
\]
verifying (3.31).

(ii) Multiplying equation (1.2) for $(\rho_n, u_n, \theta_n)$ with $u_n$ and integrating over $\Omega$, one gets by integration by parts that
\[
\frac{d}{dt} \int_\Omega \frac{\rho_n}{2} |u_n|^2 \, dx + \mu \int_\Omega \nabla u_n \nabla u_n \, dx + (\mu + \lambda) \int_\Omega \|\phi\|_3 \, dx - \int_\Omega P_n \nabla u_n \, dx = 0.
\]
Integrating the above with respect to $t$, by the H"older inequality, and using (3.8), one deduces for large $n$ that
\[
\|\nabla \rho u_n\|_1^2(t) \leq \|\nabla \rho_0 u_0\|_1^2 + 2 \int_0^t \int_\Omega P_n \nabla u_n \, dx \, d\tau
\]
\[
\leq \|\nabla \rho_0 u_0\|_1^2 + 2R \int_0^t \|\rho_n\|_3 \|\nabla \rho_0 \theta_n\| \|\nabla u_n\|_\infty \, d\tau
\]
\[
\leq \|\nabla \rho_0 u_0\|_1^2 + Ct, \quad \forall t \in (0, T_0),
\]
for a positive constant $C$ independent of $n$. Thanks to the above, recalling (3.25) and noticing that 
\[ \|\sqrt{\rho_n}u_n\|^2(t) \leq \|\sqrt{\rho_0}u_0\|^2 + Ct, \quad \forall t \in (0, T_0). \] (3.35)
Multiplying equation (1.3) for $(\rho_n, u_n, \theta_n)$ with $\theta_n$ and integrating over $\Omega$, one gets by integration by parts, the Sobolev embedding inequality, and (3.8) that
\[ \frac{c_v}{2} \frac{d}{dt} \|\sqrt{\rho_n} \theta_n\|^2 + \kappa \|\nabla \theta_n\|^2 = -\int_{\Omega} \text{div} u_n P_n \theta_n dx + \int_{\Omega} Q(\nabla u_n) \theta_n dx \leq C \left( \|\rho_n\|_{L^{\frac{3}{2}}} \|\sqrt{\rho_n} \theta_n\| \|\nabla u_n\|_3 + \|\nabla u_n\|_3 \|\nabla u_n\|_3 \right) \|\theta_n\|_6 \leq \frac{\kappa}{2} \|\nabla \theta_n\|^2 + C (1 + \|\nabla^2 u_n\|), \]
for large $n$, from which, integrating with respect to $t$, using (3.8) again, and by the H"older inequality, one obtains
\[ \|\sqrt{\rho_n} \theta_n\|^2(t) \leq \|\sqrt{\rho_0} \theta_0\|^2 + C \int_0^t (1 + \|\nabla^2 u_n\|) \, dt \leq \|\sqrt{\rho_0} \theta_0\|^2 + C \sqrt{t}, \quad \forall t \in (0, T_0). \]
Thanks to this and recalling (3.6) and (3.25), one can take $n \to \infty$ to get
\[ \|\sqrt{\rho \theta}\|^2(t) \leq \|\sqrt{\rho_0} \theta_0\|^2 + C \sqrt{t}, \quad \forall t \in (0, T_0). \]
Combining this with (3.35), the conclusion follows.

4. Proof of Theorem 1.1

This section is devoted to the proof of Theorem 1.1. As already explained in the introduction that the existence of strong solutions, which enjoy all the regularities stated in Definition 1.1 except that $\sqrt{\rho_0}u_n \to \sqrt{\rho_0}u_0$ in $L^2$ as $n \to \infty$, one gets by taking $n \to \infty$ that
\[ \|\sqrt{\rho_0}u\|^2(t) \leq \|\sqrt{\rho_0}u_0\|^2 + Ct, \quad \forall t \in (0, T_0). \] (3.35)

4.1. Lagrangian Coordinates and Some Properties

Given a velocity field $u \in L^1(0, T_0; C^1(\hat{\Omega}))$ satisfying $u|_{\partial \Omega} = 0$ and let $x = \varphi(y, t)$ be the corresponding coordinates transform, governed by the velocity field $u$, between the Euler coordinates $(x, t)$ and the Lagrangian coordinates $(y, t)$, that is,
\[
\left\{
\begin{array}{l}
\partial_t \varphi(y, t) = u(\varphi(y, t), t), \quad \forall t \in [0, T_0], \\
\varphi(y, 0) = y.
\end{array}
\right.
\] (4.1)
By the classical theory for ODEs, $\varphi$ is well-defined and $\varphi : \Omega \times [0, T_0] \to \Omega$. Moreover, by the unique solvability of ODEs, for each $t \in [0, T_0]$, $\varphi(\cdot, t) : \Omega \to \Omega$ is bijective. Denote by $y = \psi(x, t)$ the inverse mapping of $x = \varphi(y, t)$ with respect to $y$, which satisfies
\[
\left\{
\begin{array}{l}
\partial_t \psi(x, t) + (u(x, t) \cdot \nabla) \psi(x, t) = 0, \\
\psi(x, t)|_{t=0} = x.
\end{array}
\right.
\] (4.2)
Set
\[
A(y, t) = (a_{ij}(y, t))_{3 \times 3}, \quad a_{ij}(y, t) = \partial_i \psi_j(x, t)|_{x=\varphi(y, t)},
\] (4.3)
\[
J(y, t) = \det A(y, t) = \det \nabla \psi(x, t)|_{x=\varphi(y, t)},
\] (4.4)
Given Proposition 4.1. in the “Appendix”.

\[ B(y,t) = (b_{ij}(y,t))_{3\times3}, \quad b_{ij}(y,t) = \partial_i \varphi_j(y,t). \]  

Then, one can check
\[
\begin{align*}
\partial_t A(y,t) &= -\nabla u(\varphi(y,t),t) A(y,t), \\
A(y,t)|_{t=0} &= I
\end{align*}
\]  

and
\[
\begin{align*}
\partial_t B(y,t) &= B(y,t) \nabla u(\varphi(y,t),t), \\
B(y,t)|_{t=0} &= I,
\end{align*}
\]

here we set \( \nabla u = (\partial_i u_j)_{3\times3} \). Recalling the definition of \( J \), one derives from (4.6) that
\[
\begin{align*}
\partial_t J(y,t) &= -\text{div}(\varphi(y,t),t) J(y,t), \\
J(y,t)|_{t=0} &= 1.
\end{align*}
\]

Some properties of the mapping \( \varphi \) are stated in the next two propositions whose proofs are postponed in the “Appendix”.

**Proposition 4.1.** Given \( u \in L^\infty(0,T_0; H^1_0) \cap L^1(0,T_0; W^{2,q}) \), with \( q \in (3,6) \), and let \( \varphi, \psi, A, B, \) and \( J \) be defined as before.

Then, \( J > 0 \) on \( \Omega \times (0,T_0) \) and the following hold:
\[
\sup_{0 \leq t \leq T_0} \left( \left\| \left( \frac{1}{J}, J, A, B \right) \right\|_\infty^\infty + \left\| (J_t, A_t, B_t) \right\| + \left\| (\nabla J, \nabla A, \nabla B) \right\|_q \right) \leq C, \tag{4.9}
\]

\[
\left\| \nabla [g(\varphi(\cdot,t))] \right\|_{W^{1,\infty}} \leq \left\| \nabla g \right\|_{W^{1,\infty}} \leq \left\| \nabla [g(\psi(\cdot,t))] \right\|_{W^{1,\infty}}, \quad \forall \alpha \in [1,q], \tag{4.10}
\]

\[
\left\| \nabla [g(\varphi(\cdot,t))] \right\|_{W^{1,\infty}} \leq \left\| \nabla g \right\|_{W^{1,\infty}} \leq \left\| \nabla [g(\psi(\cdot,t))] \right\|_{W^{1,\infty}}, \quad \forall \alpha \in [1,\infty], \tag{4.11}
\]

\[
\left\| g(\varphi(\cdot,t)) \right\|_{W^{1,\infty}} \leq \left\| g \right\|_{W^{1,\infty}} \leq \left\| g(\psi(\cdot,t)) \right\|_{W^{1,\infty}}, \quad \forall \alpha \in [1,\infty], \tag{4.12}
\]

for any function \( g \) such that all the relevant quantities are finite, here we denote \( \mathcal{D}_1 \simeq \mathcal{D}_2 \) means \( \frac{\mathcal{D}_1}{\mathcal{D}_2} \leq C \mathcal{D}_1 \) for a positive constant \( C \) depending only on \( \Omega, \alpha, q, T_0, \) and \( \|u\|_{L^\infty(0,T_0; H^{1,\infty}(\Omega))} \cap L^1(0,T_0; W^{2,q}(\Omega)). \)

**Proposition 4.2.** Under the assumptions as in Proposition 4.1, the following hold:
(i) \( h(\varphi(\cdot,t),t) \in C([0,T_0]; L^2) \) if \( h \in C([0,T_0]; L^2) \);
(ii) \( h(\varphi(\cdot,t),t) \in C_w([0,T_0]; L^2) \) if \( h \in C([0,T_0]; L^2) \).

### 4.2. Regularities and Reduced System in the Lagrangian Coordinates

Given initial data \((\rho_0, u_0, \theta_0)\) satisfying the assumptions in Theorem 1.1. Let \((\rho, u, \theta)\) be the solution established in Proposition 3.1 and \( \varphi \) the corresponding mapping defined by (4.1). Set
\[
\begin{align*}
\varrho(y,t) &= \rho(\varphi(y,t),t), \\
v(y,t) &= u(\varphi(y,t),t), \\
\theta(y,t) &= \theta(\varphi(y,t),t).
\end{align*}
\]

As direct corollaries of Proposition 4.1 and Proposition 4.2 and recalling the regularities of \((\rho,u,\theta)\) in Proposition 3.1, one has:
\[
\begin{align*}
\varrho &\in L^\infty(0,T_0; W^{1,q}), \\
v &\in L^\infty(0,T_0; H^1) \cap L^2(0,T_0; H^2) \cap L^1(0,T_0; W^{2,q}), \\
\sqrt{\varrho v} &\in L^\infty(0,T_0; H^2) \cap L^2(0,T_0; W^{2,q}), \\
\sqrt{\varrho \theta} &\in L^\infty(0,T_0; L^2), \\
\sqrt{\varrho \theta} &\in L^\infty(0,T_0; H^1), \\
\varrho v &\in C([0,T_0]; L^2), \\
\varrho \theta &\in C([0,T_0]; L^2).
\end{align*}
\]

Direct calculations show
\[
\partial_t \varrho(y,t) = (\partial_t \rho + u \cdot \nabla \rho)(\varphi(y,t),t),
\]
\[ \partial_t v(y, t) = [\partial_t u + (u \cdot \nabla)u](\varphi(y, t), t), \]
\[ \partial_t \vartheta(y, t) = (\partial_t \theta + u \cdot \nabla \theta)(\varphi(y, t), t). \]

By Proposition 4.1 and recalling the regularities of \((\rho, u, \theta)\) stated in Proposition 3.1, one deduces by the Hölder and Gagliardo-Nirenberg inequalities that

\[
\sup_{0 \leq t \leq T_0} \|\partial_t^{\alpha} \| = \sup_{0 \leq t \leq T_0} \| (\rho_t + u \cdot \nabla \rho)(\varphi(y, t), t) \|
\leq C \sup_{0 \leq t \leq T_0} \| \rho_t + u \cdot \nabla \rho \| \leq C \sup_{0 \leq t \leq T_0} (\| \rho_t \| + \| u \|_\infty \| \nabla \rho \|_3)
\leq C \sup_{0 \leq t \leq T_0} (\| \rho_t \| + \| \nabla u \|_2 \| \nabla \rho \|_q) \leq C,
\]

\[
\int_0^{T_0} \| \sqrt{\rho} v_t \|^2 dt = \int_0^{T_0} \| [\sqrt{\rho} (\partial_t u + (u \cdot \nabla)u)](\varphi(y, t), t) \|^2 dt
\leq C \int_0^{T_0} \| \sqrt{\rho} (\partial_t u + (u \cdot \nabla)u) \|^2 dt
\leq C \int_0^{T_0} (\| \sqrt{\rho} u_t \|^2 + \| u \|^2_\infty \| \nabla u \|^2_2) dt
\leq C \int_0^{T_0} (\| \sqrt{\rho} u_t \|^2 + \| \nabla u \|^3_2 \| \nabla^2 u \|) dt \leq C
\]

and

\[
\int_0^{T_0} t \| \nabla v_t \|^2 dt = \int_0^{T_0} t \| \nabla ((\partial_t u + (u \cdot \nabla)u)(\varphi(y, t), t)) \|^2 dt
\leq C \int_0^{T_0} t \| \nabla (\partial_t u + (u \cdot \nabla)u) \|^2 dt
\leq C \int_0^{T_0} t \left( \| \nabla u_t \|^2 + \| u \|^2_\infty \| \nabla^2 u \|^2 + \| \nabla u \|^4_4 \right) dt
\leq C \int_0^{T_0} t \left( \| \nabla u_t \|^2 + \| \nabla u \|^2_2 \| \nabla^2 u \|^3 \right) dt \leq C.
\]

Similarly,

\[
\int_0^{T_0} t \| \sqrt{\rho} \vartheta_t \|^2 dt = \int_0^{T_0} t \| (\sqrt{\rho} \vartheta_t + \sqrt{\rho} u \cdot \nabla \theta)(\varphi(y, t), t) \|^2 dt
\leq C \int_0^{T_0} t \| \sqrt{\rho} \vartheta_t + \sqrt{\rho} u \cdot \nabla \theta \|^2 dt
\leq C \int_0^{T_0} t(\| \sqrt{\rho} \vartheta_t \|^2 + \| \rho \|_\infty \| u \|^2_\infty \| \nabla \theta \|^2_3) dt
\leq C \int_0^{T_0} t(\| \sqrt{\rho} \vartheta_t \|^2 + \| \nabla u \|^2_2 \| \nabla \theta \| \| \nabla^2 \theta \|) dt \leq C
\]

and

\[
\int_0^{T_0} t^2 \| \nabla \vartheta_t \|^2 dt = \int_0^{T_0} t^2 \| \nabla ((\vartheta_t + u \cdot \nabla \theta)(\varphi(y, t), t)) \|^2 dt
\leq C \int_0^{T_0} t^2 \| \nabla (\vartheta_t + u \cdot \nabla \theta) \|^2 dt
\leq C \int_0^{T_0} t^2 \int_\Omega (|\nabla \vartheta_t|^2 + |u|^2 |\nabla^2 \theta|^2 + |\nabla u|^2 |\nabla \theta|^2) dx dt
\]
\[
\leq C \int_0^{T_0} t^2 (\|\nabla \theta \|^2 + \|u\|^2_\infty \|\nabla^2 \theta \|^2 + \|\nabla u\|_2^2 \|\nabla \theta\|^2 \|\nabla \theta\|_6) \, dt \\
\leq C \int_0^{T_0} t^2 (\|\nabla \theta \|^2 + \|\nabla u\| \|\nabla^2 u\| \|\nabla^2 \theta\|^2) \, dt \\
+ C \int_0^{T_0} t^2 \|\nabla^2 u\|^2 (\|\nabla \theta \|^2 + \|\nabla \theta\|^2 \|\nabla \theta\|) \, dt \leq C.
\]

Therefore,
\[
\varrho_t \in L^\infty(0, T_0; L^2), \quad \sqrt{\varrho} v_t \in L^2(0, T_0; L^2), \quad \sqrt{\varrho} v_t \in L^2(0, T_0; H^1), \\
\sqrt{t} \sqrt{\varrho} \theta_t \in L^2(0, T_0; L^2), \quad t \theta_t \in L^2(0, T_0; H^1).
\] (4.15)

So, we have the following proposition:

**Proposition 4.3.** Given initial data \((\rho_0, u_0, \theta_0)\) satisfying the assumptions in Theorem 1.1. Let \((\rho, u, \theta)\) be the solution established in Proposition 3.1 and \(\varphi\) the corresponding mapping defined by (4.1). Then, \((\varrho, v, \theta)\) defined by (4.13) satisfies (4.14) and (4.15).

Let \(A\) be defined as before in the previous subsection and denote
\[
\nabla_A f := A \nabla f, \quad \text{div}_A v := A : (\nabla v)^T, \quad \nabla v = (\partial_i v_j)_{3 \times 3}.
\]

Then, by direct computations, one can derive from (1.1)–(1.3) and (4.6)–(4.8) that
\[
\varrho_t + \text{div}_A v \varrho = 0, \quad (4.16)
\]
\[
J \rho_0 v_t - \mu \text{div}_A (\nabla_A v) - (\mu + \lambda) \nabla_A (\text{div}_A v) + R \nabla_A (J \rho_0 \theta) = 0, \quad (4.17)
\]
\[
c_v J \rho_0 \theta_t + RJ \rho_0 \theta \text{div}_A v - \kappa \text{div}_A (\nabla_A \theta) = \frac{\mu}{T} \nabla_A v + (\nabla_A v)^T \| \nabla \theta \|^2 + \lambda (\text{div}_A v)^2,
\]
\[
A_t + \nabla_A v A = 0, \quad (4.18)
\]
\[
J_t + \text{div}_A v J = 0.
\] (4.20)

System (4.16)–(4.20) are satisfied a.e. in \(\Omega \times (0, T_0)\). Here in (4.17) and (4.18) we have used the fact that
\[
\frac{\varrho}{J} = \frac{\rho_0}{J_0} = \rho_0
\] (4.21)
to replace \(\varrho\) with \(J \rho_0\), as \(\partial_t \left( \frac{\varrho}{J} \right) = 0\) guaranteed by (4.16) and (4.20). The component form of (4.17) reads as
\[
J \rho_0 \partial_i v_i - \mu a_{ki} \partial_i (a_{km} \partial_m v_i) - (\mu + \lambda) a_{ii} \partial_i (a_{km} \partial_m v_k) + Ra_{ii} \partial_i (J \rho_0 \theta) = 0.
\]

The initial-boundary conditions read as
\[
(\varrho v, \varrho \theta, A, J)|_{t=0} = (\rho_0 u_0, \rho_0 \theta_0, I, 1), \quad (4.22)
\]
\[
v|_{\partial \Omega} = 0, \quad \nabla \varrho \cdot n|_{\partial \Omega} = 0.
\] (4.23)

Since \(\varrho \, v \in C([0, T_0]; L^2)\) and \(\varrho \theta \in C_v([0, T_0]; L^2)\), guaranteed by (4.14), and \(A, J \in C([0, T_0]; L^2)\), guaranteed by Proposition 4.1, the initial condition (4.22) is well-defined.

Finally, we state and prove the continuities of \(\sqrt{\rho_0} v\) and \(\sqrt{\rho_0} \theta\).

**Proposition 4.4.** Under the same assumptions in Proposition 4.3, it holds that
\[
\sqrt{\rho_0} v, \sqrt{\rho_0} \theta \in C([0, T_0]; L^2),
\]
and
\[
(\sqrt{\rho_0} v, \sqrt{\rho_0} \theta) \to (\sqrt{\rho_0} u_0, \sqrt{\rho_0} \theta_0), \quad \text{in} \ L^2, \quad \text{as} \ t \to 0.
\]
Proof. We only give the proof for $\sqrt{\rho_0} \theta$ while that for $\sqrt{\rho_0} \nu$ can be done similarly. Due to (4.14) and (4.15), one has $\sqrt{\rho_0} \theta \in C([0,T_0]; L^2)$. It remains to show

$$
\sqrt{\rho_0} \theta \to \sqrt{\rho_0} \theta_0, \quad \text{in } L^2, \quad \text{as } t \to 0. \tag{4.24}
$$

Noticing that $\inf_{(y,t) \in \Omega \times [0,T_0]} J(y,t) > 0$, $J \in L^\infty([0,T_0]; W^{1,4})$, and $J_t \in L^\infty([0,T_0]; L^2)$, guaranteed by Proposition 4.1, one can verify easily that $\frac{1}{J} \in C([0,T_0]; C(\overline{\Omega}))$. Thanks to this and recalling (4.21), it follows from (4.14) and (4.22) that

$$
\rho_0 \theta = \frac{1}{J} \rho \theta \to \rho_0 \theta_0 \quad \text{in } L^2(\Omega) \quad \text{as } t \to 0. \tag{4.25}
$$

In order to show $\sqrt{\rho_0} \theta \to \sqrt{\rho_0} \theta_0$ in $L^2$ as $t \to 0$, one needs to verify

$$
\lim_{t \to 0} \| \sqrt{\rho_0} \theta \|^2 \leq \| \sqrt{\rho_0} \theta_0 \|^2. \tag{4.26}
$$

To verify (4.26), since $\sqrt{\rho_0} \theta = 0$ on $\Omega_0 \times (0,T_0)$ and $\sqrt{\rho_0} \theta_0 = 0$ on $\Omega_0$, it suffices to show

$$
\sqrt{\rho_0} \theta \to \sqrt{\rho_0} \theta_0 \quad \text{in } L^2(\Omega_+) \quad \text{as } t \to 0,
$$

where $\Omega_+ = \{ y \in \Omega | \rho_0(y) > 0 \}$ and $\Omega_0 = \{ y \in \Omega | \rho_0(y) = 0 \}$. Recalling that $\sqrt{\rho_0} \theta \in L^\infty(0,T_0; L^2)$ and since $C^\infty_c(\Omega_+)$ is dense in $L^2(\Omega_+)$, one only needs to check

$$
\int_{\Omega_+} \sqrt{\rho_0} \theta \chi \, dy \to \int_{\Omega_+} \sqrt{\rho_0} \theta_0 \chi \, dy \quad \text{as } t \to 0, \quad \forall \chi \in C^\infty_c(\Omega). \tag{4.28}
$$

Take arbitrary $\chi \in C^\infty_c(\Omega_+)$ and denote $S := \{ y \in \Omega_+ | \chi(y) \neq 0 \}$. Then $S \subset \Omega \subset \Omega_+$. Since $\rho_0 \in C(\overline{\Omega})$ and $\rho_0 > 0$ on $\Omega_+$, it follows that $\min_{y \in S} \rho_0(y) > 0$ and, thus, $\frac{\chi}{\sqrt{\rho_0}} \in L^2(S)$. So, it follows from (4.25) that

$$
\int_{\Omega_+} \sqrt{\rho_0} \theta \chi \, dy = \int_S \rho_0 \theta \chi \, dy \to \int_S \rho_0 \theta_0 \chi \, dy = \int_{\Omega_+} \sqrt{\rho_0} \theta_0 \chi \, dy \quad \text{as } t \to 0.
$$

Therefore, (4.28) and further (4.26) hold.

We now verify (4.27). First, noticing that

$$
\sqrt{\rho_0} \theta = \frac{1}{\sqrt{J}} \sqrt{\rho_0} \theta, \quad \frac{1}{J} \in C([0,T_0]; C(\overline{\Omega})), \quad J|_{t=0} = 1,
$$

one has $\lim_{t \to 0} \| \sqrt{\rho_0} \theta \|^2 = \lim_{t \to 0} \| \sqrt{\rho \theta} \|^2$. Therefore, it suffices to show

$$
\lim_{t \to 0} \| \sqrt{\rho \theta} \|^2 \leq \| \sqrt{\rho_0} \theta_0 \|^2. \tag{4.29}
$$

Since $\det \nabla \psi(x,t) = J(\psi(x,t),t) > 0$, one deduces by direct calculations that

$$
\| \sqrt{\rho_0} \theta \|^2(t) = \int_{\Omega} \rho(\varphi(y,t),t) \theta^2(\varphi(y,t),t) \, dy = \int_{\Omega} \rho(x,t) \theta^2(x,t) \, dx = \int_{\Omega} \rho(x,t) \theta^2(x,t) J(\psi(x,t),t) \, dx \leq \| \sqrt{\rho_0} \theta \|^2(t) + \| J - I \|_\infty(t) \| \sqrt{\rho_0} \theta \|^2(t).
$$

Thanks to the above and since

$$
\sqrt{\rho_0} \theta \in L^\infty(0,T_0; L^2), \quad J \in C([0,T_0]; C(\overline{\Omega})), \quad J|_{t=0} = 1,
$$

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it follows from Proposition 3.1 that
\[
\lim_{t \to 0} \|\sqrt{\theta}t\|^2(t) \leq \lim_{t \to 0} \|\sqrt{\rho t}\|^2(t) + C \lim_{t \to 0} \|J - 1\|_\infty(t)
\]
\[
= \lim_{t \to 0} \|\sqrt{\rho t}\|^2(t) \leq \|\sqrt{\rho_0\theta_0}\|^2.
\]
This verifies (4.29) and further (4.27).

4.3. Proof of Theorem 1.1

We are now ready to give the proof of Theorem 1.1.

Proof of Theorem 1.1. (i) Existence. By virtue of Proposition 3.1, it remains to show that \(\sqrt{\rho u}, \sqrt{\rho \theta} \in C([0, T_0]; L^2)\). Let \((\rho, v, \theta)\) be given by (4.13). Then, it follows from Proposition 4.4 that \(\sqrt{\rho_0 v}, \sqrt{\rho_0 \theta} \in C([0, T_0]; L^2)\), from which, recalling that \(\rho = J\rho_0\) and noticing that \(J \in C([0, T_0]; C(\overline{\Omega}))\) guaranteed by Proposition 4.1, one gets \(\sqrt{\rho v}, \sqrt{\rho \theta} \in C([0, T_0]; L^2)\). Thanks to these, similarly to the proof of (ii) of Proposition 4.2 (since \(\psi\) has the same properties as those of \(\varphi\), one can then show that \(\sqrt{\rho u}, \sqrt{\rho \theta} \in C([0, T_0]; L^2)\).

(ii) Uniqueness. Let \((\hat{\rho}, \hat{u}, \hat{\theta})\) and \((\check{\rho}, \check{u}, \check{\theta})\) be two solutions to system (1.1)–(1.3), subject to (1.6)–(1.7), in \(\Omega \times (0, T_0)\), with the same initial data \((\rho_0, u_0, \theta_0)\). Let \((\check{\varphi}, \check{\psi}, \check{\varphi}_i, \check{\varphi}_g, \check{\varphi}_v, \check{\varphi}_\theta, \check{\varphi}_A, \check{\varphi}_J)\) and \((\hat{\varphi}, \hat{\psi}, \hat{\varphi}_i, \hat{\varphi}_g, \hat{\varphi}_v, \hat{\varphi}_\theta, \hat{\varphi}_A, \hat{\varphi}_J)\) be the corresponding quantities defined as before and denote
\[
(v, \vartheta, A, J) = (\hat{v}, \hat{\vartheta}, \hat{A}, \hat{J}) - (\check{v}, \check{\vartheta}, \check{A}, \check{J}).
\]
Then, \((\hat{v}, \hat{\vartheta}, \hat{A}, \hat{J})\) and \((\check{v}, \check{\vartheta}, \check{A}, \check{J})\) have the regularities (4.14) and (4.15), satisfy system (4.17)–(4.20) a.e. in \(\Omega \times (0, T_0)\), and fulfill the initial-boundary conditions (4.22)–(4.23). One can check by direct calculations that \((v, \vartheta, A, J)\) satisfies:

\[
\begin{align*}
\rho_0 \hat{J} v_t - \mu \nabla \hat{A} (\nabla \hat{v}) - (\mu + \lambda) \nabla \hat{A} (\nabla \hat{v}) &= -\rho_0 J \check{v}_t \\
+ \mu \nabla \hat{A} (\nabla \check{v}) + \mu \nabla \hat{A} (\nabla \check{v}) + (\mu + \lambda) \nabla \hat{A} (\nabla \check{v}) + (\mu + \lambda) \nabla \hat{A} (\nabla \check{v}) \\
- R \nabla \hat{A} (\rho_0 \hat{\vartheta} + \rho_0 J \hat{\vartheta}) &= R \nabla \hat{A} (\rho_0 \check{\vartheta}), \tag{4.30}
\end{align*}
\]

\[
\begin{align*}
c_v \rho_0 \check{J} \hat{\vartheta}_t - \kappa \nabla \hat{A} (\nabla \hat{\vartheta}) &= -c_v \rho_0 \hat{J} \check{\vartheta}_t + \kappa \nabla \hat{A} (\nabla \check{\vartheta}) + \kappa \nabla \hat{A} (\nabla \check{\vartheta}) \\
- R \rho_0 \left( \hat{J} \check{\vartheta} \nabla \hat{A} v + \hat{J} \check{\vartheta} \nabla \hat{A} \check{v} + \hat{J} \check{\vartheta} \nabla \hat{A} v + \hat{J} \check{\vartheta} \nabla \hat{A} \check{v} \right) \\
+ \mu \left( \nabla \hat{A} v + \nabla \hat{A} \check{v} + \nabla \hat{A} v + \nabla \hat{A} \check{v} \right) \\
+ \lambda \left( \nabla \hat{A} v + \nabla \hat{A} \check{v} \right) (\nabla \hat{A} v + \nabla \hat{A} \check{v}) \\
A_t + \nabla \hat{A} v A + \nabla \hat{A} v \check{A} + \nabla \hat{A} v \check{A} = 0, \tag{4.32}
\end{align*}
\]

\[
J_t + \nabla \hat{A} v J + \nabla \hat{A} \check{v} J + \nabla \hat{A} \check{v} J = 0. \tag{4.33}
\]

For any vector field \(W\) and function \(f\) such that either \(W|_{\partial \Omega} = 0\) or \(f|_{\partial \Omega} = 0\), by Lemma 5.1, it follows from integration by parts that

\[
\int_{\Omega} \frac{1}{J} \nabla \hat{A} f \cdot W \, dy = \int_{\Omega} \frac{1}{J} \hat{a}_{ij} \hat{\vartheta} f W_i \, dy \\
= - \int_{\Omega} \left( \partial_i \left( \frac{\hat{a}_{ij}}{J} \right) W_i + \frac{1}{J} \hat{a}_{ij} \partial_i W_i \right) f \, dy = - \int_{\Omega} \frac{1}{J} \div \hat{A} W f \, dy. \tag{4.34}
\]
Step 1. Energy inequalities. Multiplying \((4.30)\) with \(\frac{\nu}{J}\) and using \((4.34)\), one gets

\[
\frac{1}{2} \frac{d}{dt} \| \rho_0 v \|_2^2 + \mu \left\| \frac{\nabla \hat{A} v}{\sqrt{J}} \right\|_2^2 + (\mu + \lambda) \left\| \frac{\text{div} \bar{v} \cdot \nabla v}{\sqrt{J}} \right\|_2^2 \\
= - \int_{\Omega} \rho_0 J \hat{v}_i \dot{v}_i dy - \frac{1}{J} \int_{\Omega} \left[ \mu \nabla \hat{A} \cdot \nabla v + (\mu + \lambda) \text{div} \bar{v} \cdot \nabla v \right] dy \\
+ \int_{\Omega} \frac{1}{J} \left[ \mu \text{div} \hat{A} (\nabla \hat{A} \cdot v) + (\mu + \lambda) \nabla \text{div} \bar{v} \cdot v \right] dy - \frac{R}{J} \int_{\Omega} \nabla \hat{A} (\rho_0 \dot{\rho} \dot{\hat{A}}) dy \\
+ R \int_{\Omega} \left( \rho_0 \dot{\hat{A}} \theta + \rho_0 \dot{\rho} \dot{\hat{A}} \right) \text{div} \bar{v} \cdot v dy =: \sum_{i=1}^{5} N_i. \tag{4.35}
\]

By Proposition 4.1, it follows

\[
\sup_{0 \leq t \leq T_0} \left( \left\| \frac{1}{J} \right\|_\infty + \|(\hat{J}, \hat{J}, \hat{A})\|_\infty + \|\nabla \hat{A}, \nabla J\|_3 \right) \leq C.
\]

Thanks to this, terms \(N_i, i = 1, 2, \cdots, 5\), are estimated by the Hölder, Gagliardo-Nirenberg, Sobolev, and Young inequalities as follows:

- \(N_1 \leq C \|J\| \|\rho_0 \hat{v}_i\|_3 \|v\|_6 \leq C \|J\| \|\sqrt{\rho_0 \hat{v}_i}\|_2 \|\hat{v}_i\|_6 \|\nabla v\| \leq \frac{\mu}{8} \left\| \frac{\nabla \hat{A} v}{\sqrt{J}} \right\|_2^2 + C \|\sqrt{\rho_0 \hat{v}_i}\|_2 \|\nabla \hat{v}_i\|_2 \|J\|_2^2; \]

- \(N_2 = - \int_{\Omega} \frac{1}{J} \left[ \mu \nabla \hat{A} \cdot \nabla v + (\mu + \lambda) \text{div} \bar{v} \cdot \nabla v \right] dy \leq \frac{\mu}{8} \left\| \frac{\nabla \hat{A} v}{\sqrt{J}} \right\|_2^2 + C \|\nabla \hat{v}\|_\infty \|A\| \leq \frac{\mu}{8} \left\| \frac{\nabla \hat{A} v}{\sqrt{J}} \right\|_2^2 + C \|\nabla \hat{v}\|_\infty \|A\|_2^2; \]

- \(N_3 = \int_{\Omega} \frac{1}{J} \left[ \mu \text{div} \hat{A} (\nabla \hat{A} \cdot v) + (\mu + \lambda) \nabla \text{div} \bar{v} \cdot v \right] dy \leq \frac{\mu}{8} \left\| \frac{\nabla \hat{A} v}{\sqrt{J}} \right\|_2^2 \leq C \int_{\Omega} \left[ A \|\nabla^2 \hat{v}\|_3 + |\nabla \hat{A}| \|\nabla \hat{v}\|_6 \right] dy \leq C \|A\| \|\nabla^2 \hat{v}\|_3 + \|\nabla \hat{A}\|_3 \|\nabla \hat{v}\|_6 \|v\|_6 \leq \frac{\mu}{8} \left\| \frac{\nabla \hat{A} v}{\sqrt{J}} \right\|_2^2 + C \|\nabla \hat{v}\|_\infty \|A\|_2^2; \]

- \(N_4 = - R \int_{\Omega} a_{ii} \partial_i (\rho_0 \dot{J} \dot{\rho}) \frac{v_i}{J} dy \leq C \int_{\Omega} \left[ A \right] \left| |\nabla \rho_0| \cdot |J| \cdot |\dot{\rho}| + \rho_0 \left| \nabla J \right| |\dot{\rho}| + \rho_0 \left| J \right| |\nabla \dot{\rho}| \cdot \frac{|v|}{J} \right] dy \leq C \|A\| \left( \|\nabla \rho_0\|_3 \|\dot{\rho}\|_\infty + \|\nabla J\|_3 \|\dot{\rho}\|_\infty + \|\nabla \dot{\rho}\|_3 \|v\|_6 \right) \leq \frac{\mu}{8} \left\| \frac{\nabla \hat{A} v}{\sqrt{J}} \right\|_2^2 + C \left( 1 + \|\nabla \dot{\rho}\|_2 \right) \|\nabla^2 \dot{\rho}\| \|\hat{A}\|_2^2. \)
\[ N_5 \leq \frac{\mu + \lambda}{4} \left\| \frac{\text{div} \tilde{A} v}{\sqrt{J}} \right\|^2 + C \left( \| \sqrt{\rho_0} \tilde{\vartheta} \|^2 + \| \tilde{\vartheta} \|^2_{\infty} \| J \|^2 \right) \]

\[ \leq \frac{\mu + \lambda}{4} \left\| \frac{\text{div} \tilde{A} v}{\sqrt{J}} \right\|^2 + C \left( \| \sqrt{\rho_0} \tilde{\vartheta} \|^2 + (1 + \| \nabla \tilde{\vartheta} \|) \| \nabla^2 \tilde{\vartheta} \| \| J \|^2 \right). \]

Substituting these estimates into (4.35) yields
\[
\frac{d}{dt} \| \sqrt{\rho_0} \tilde{\vartheta} \|^2 + \mu \left\| \frac{\nabla \tilde{A} v}{\sqrt{J}} \right\|^2 \\
\leq C \| \sqrt{\rho_0} \tilde{\vartheta} \|^2 + C \left( \| \sqrt{\rho_0} \tilde{v}_t \| \| \nabla \tilde{v}_t \| + (1 + \| \nabla \tilde{\vartheta} \|) \| \nabla^2 \tilde{\vartheta} \| + \| \nabla^2 \tilde{\vartheta} \|^2 \right) \| A, J \|^2. \tag{4.36}
\]

Testing (4.31) with \( \frac{\vartheta}{J} \) and using (4.34), one gets
\[
c_v \frac{d}{dt} \| \sqrt{\rho_0} \vartheta \|^2 + \kappa \left\| \frac{\nabla \tilde{A} \vartheta}{\sqrt{J}} \right\|^2 = \frac{c_v}{2} \frac{d}{dt} \| \sqrt{\rho_0} \vartheta \|^2 + \kappa \left\| \frac{\nabla \tilde{A} \vartheta}{\sqrt{J}} \right\|^2 \\
- c_v \int_{\Omega} \rho_0 J \tilde{\vartheta} \frac{\partial \tilde{J}}{\sqrt{J}} \frac{d y}{J} - \kappa \int_{\Omega} \nabla \tilde{A} \vartheta \cdot \frac{\nabla \tilde{A} \vartheta}{\sqrt{J}} \frac{d y}{J} + \kappa \int_{\Omega} \text{div}_A \left( \nabla \tilde{A} \vartheta \right) \frac{\partial \tilde{J}}{\sqrt{J}} \frac{d y}{J} \\
- R \int_{\Omega} \rho_0 \left( \tilde{J} \frac{\partial \text{div}_A \tilde{A} v} {\sqrt{J}} + \tilde{J} \frac{\partial \text{div}_A \tilde{A} \tilde{v}} {\sqrt{J}} \right) \frac{d y}{J} + \frac{\partial \tilde{A} \tilde{v}} {\sqrt{J}} \frac{d y}{J} \\
+ \frac{\mu}{2} \int_{\Omega} \left( \nabla^2 \tilde{v} + \nabla^2 \tilde{\vartheta} + \nabla^2 \tilde{\vartheta} + \nabla^2 \tilde{\vartheta} \right) \left( \nabla^2 \tilde{v} + \nabla^2 \tilde{\vartheta} + \nabla^2 \tilde{\vartheta} + \nabla^2 \tilde{\vartheta} \right) \frac{d y}{J} \\
+ \lambda \int_{\Omega} \left( \text{div}_A \tilde{v} + \text{div}_A \tilde{\vartheta} \right) \left( \text{div}_A \tilde{v} + \text{div}_A \tilde{\vartheta} \right) \frac{d y}{J} =: \sum_{i=1}^{6} O_i.
\]

Similar to \( N_1, N_2, \) and \( N_3, \) one has the following estimates for \( O_1, O_2, \) and \( O_3: \)
\[
O_1 \leq \frac{\kappa}{8} \left\| \frac{\nabla \tilde{A} \vartheta}{\sqrt{J}} \right\|^2 + C \left( \| \sqrt{\rho_0} \tilde{\vartheta}_t \|^2 + \| \sqrt{\rho_0} \tilde{\vartheta}_t \| \| \nabla \tilde{\vartheta}_t \| \| J \|^2 \right),
\]
\[
O_2 \leq \frac{\kappa}{8} \left\| \frac{\nabla \tilde{A} \vartheta}{\sqrt{J}} \right\|^2 + C \| \nabla \tilde{\vartheta} \|^2 \| A \|^2,
\]
\[
O_3 \leq \frac{\kappa}{8} \left\| \frac{\nabla \tilde{A} \vartheta}{\sqrt{J}} \right\|^2 + C \left( \| \nabla^2 \tilde{\vartheta} \|^2 + \| \nabla \tilde{\vartheta} \|^2 \| A \|^2 \right).
\]

By the Hölder, Sobolev, and Young inequalities, one deduces
\[
O_4 \leq C \int_{\Omega} \rho_0 |\tilde{\vartheta}||\tilde{\vartheta}| |\nabla \tilde{A} v| + |\tilde{v}||A||\nabla \tilde{v}| + |\tilde{\vartheta}||\nabla \tilde{v}| + |J||\tilde{\vartheta}||\nabla \tilde{v}| \frac{d y}{J} \\
\leq C \left( \| \sqrt{\rho_0} \tilde{\vartheta} \| \| \nabla \tilde{\vartheta} \| \right) \left( \| \nabla \tilde{v} \| \right) \left( \| A \| \right) \left( \| \nabla \tilde{v} \| \right) \left( \| J \| \right) \\
\leq C \left( \| \tilde{\vartheta} \|_{\infty} + \| \tilde{\vartheta} \|_{\infty} \right) \| \nabla \tilde{v} \| \sqrt{\rho_0 \tilde{\vartheta}} \left( \| A \| + \| J \| \right) \\
+ C \| \tilde{\vartheta} \|_{\infty} \left\| \frac{\nabla \tilde{A} v}{\sqrt{J}} \right\| \sqrt{\rho_0 \tilde{\vartheta}} \| \nabla \tilde{v} \|^2,
\]
\[
O_5 + O_6 \leq C \int_{\Omega} \left( \| \nabla \tilde{v} \| + \| \nabla \tilde{v} \| \right) \left( \| \nabla \tilde{A} v \| + |A||\nabla \tilde{v}| \right) d y.
\]
\[
\leq C \| \nabla (\hat{v}, \dot{v}) \|_3 \left\| \frac{\nabla \hat{A}^T}{\sqrt{J}} \right\| \| \dot{\vartheta} \|_6 + C \| \nabla (\hat{v}, \dot{v}) \|_6^2 \| A \| \| \dot{\vartheta} \|_6
\]
\[
\leq \frac{\kappa}{8} \left( \frac{\nabla \hat{A} \dot{\vartheta}}{\sqrt{J}} \right)^2 + C \left( \| \sqrt{\rho_0} \dot{\vartheta} \|^2 + \| \nabla (\hat{v}, \dot{v}) \|_6^2 \| \frac{\nabla \hat{A} \dot{v}}{\sqrt{J}} \|^2 + \| \nabla^2 (\hat{v}, \dot{v}) \|_4^4 \| A \|^2 \right).
\]

Thus, combining the above yields
\[
c_v \frac{d}{dt} \| \sqrt{\rho_0} \dot{\vartheta} \|^2 + \kappa \left( \frac{\nabla \hat{A} \dot{\vartheta}}{\sqrt{J}} \right)^2 \leq C \left( \| \sqrt{\rho_0} \dot{t} \|^2 + \| \sqrt{\rho_0} \dot{t} \| \| \nabla \dot{t} \| \| J \|^2 \right) + C \left( \| \nabla \dot{\vartheta} \|_\infty^2 + \| \nabla^2 \dot{\vartheta} \|_3^2 + \| \nabla^2 (\hat{v}, \dot{v}) \|_4^4 \| A \|^2 \right)
\]
\[
+ C \| \nabla \dot{v} \|_\infty \left( \| \dot{\vartheta} \|_\infty + \| \dot{\vartheta} \|_\infty \right) \| \sqrt{\rho_0} \dot{\vartheta} \| (\| A \| + \| J \|).
\]

**(Step 2. Growth estimates.** We proceed to consider the growth estimates of \( A \) and \( J \). Testing (4.32) and (4.33), respectively, with \( A \) and \( J \), and summing the resultants up, one obtains after some straightforward computations that
\[
\frac{d}{dt} (\| A \|^2 + \| J \|^2) \leq C \| \nabla (\hat{v}, \dot{v}) \|_\infty (\| A \| + \| J \|)^2 + C \| \nabla v \| (\| A \| + \| J \|).
\]

and thus,
\[
\frac{d}{dt} \sqrt{\| A \|^2 + \| J \|^2} \leq C \| \nabla (\hat{v}, \dot{v}) \|_\infty \sqrt{\| A \|^2 + \| J \|^2} + C \| \nabla v \|.
\]  
(4.39)

Since \( \hat{v}, \dot{v} \in L^1(0, T_0; W^{2,q}) \cap L^\infty(0, T_0; H^1) \) and \( W^{1,q} \hookrightarrow L^\infty \) for \( q \in (3, 6) \), one has \( \| \nabla (\hat{v}, \dot{v}) \|_\infty \in L^1((0, T_0)) \) and \( \| \nabla v \| \in L^\infty((0, T_0)) \). Thanks to these and applying the Grönwall inequality to (4.39), one deduces
\[
\sqrt{\| A \|^2 + \| J \|^2} \leq Ae^{C \int_0^{t_0} \| \nabla (\hat{v}, \dot{v}) \|_\infty ds} \int_0^t \| \nabla v \| ds \leq Ct, \quad \forall t \in (0, T_0).
\]  
(4.40)

Recalling that
\[
\sqrt{\rho_0} \hat{v} t \in L^2(0, T_0; L^2), \quad \sqrt{\rho_0} \dot{v} t \in L^2(0, T_0; H^1), \quad \hat{\vartheta} \in L^2(0, T_0; H^1),
\]
\[
\sqrt{t} \hat{\vartheta} \in L^2(0, T_0; H^2), \quad \sqrt{t} \dot{\vartheta} \in L^2(0, T_0; W^{2,q}),
\]
one gets
\[
\omega_1(t) \triangleq t (\| \sqrt{\rho_0} \hat{v} t \|_\infty \| \nabla \hat{v} t \| + (1 + \| \nabla \dot{\vartheta} \|) \| \nabla^2 \dot{\vartheta} \| + \| \nabla^2 \dot{v} \|_\infty^2) \in L^1((0, T_0)).
\]

Since \( \sqrt{\rho_0} \hat{\vartheta}, \sqrt{\rho_0} \dot{v} \in C([0, T_0]; L^2) \) (guaranteed by Proposition 4.4) and \( \sqrt{\rho_0} v \big|_{t=0} = 0 \), integrating (4.36) with respect to \( t \) and using (4.40) yield
\[
\| \sqrt{\rho_0} \dot{v} \|^2(t) + \mu \int_0^t \left( \frac{\nabla \hat{A} \dot{v}}{\sqrt{J}} \right)^2 ds \leq Ct + C \int_0^t \omega_1 ds \leq Ct, \quad \forall t \in (0, T_0).
\]

Combining this with (4.40) leads to
\[
\sup_{0 \leq t \leq T_0} (\| A \| + \| J \| + \| \sqrt{\rho_0} \dot{v} \|^2) + \int_0^t \| \nabla v \|^2 ds \leq Ct, \quad \forall t \in (0, T_0).
\]  
(4.41)
Step 3. Singular $t$-weighted energy inequalities and uniqueness. Multiplying (4.38) by $t^{-\frac{3}{2}}$ yields

$$
\frac{d}{dt} \left( \frac{\|A\|^2}{t^{\frac{3}{2}}} + \frac{\|J\|^2}{t^{\frac{3}{2}}} \right) + \frac{3}{2} \left( \frac{\|A\|^2}{t^{\frac{3}{2}}} + \frac{\|J\|^2}{t^{\frac{3}{2}}} \right) 
\leq C \|\nabla(\bar{\nu}, \bar{v})\|_\infty \left( \frac{\|A\|^2}{t^{\frac{3}{2}}} + \frac{\|J\|^2}{t^{\frac{3}{2}}} \right) + C \|\nabla v\|_1 \left( \frac{\|A\|^2}{t^{\frac{3}{2}}} + \frac{\|J\|^2}{t^{\frac{3}{2}}} \right)
$$

and, thus,

$$
\frac{d}{dt} \left( \frac{\|A\|^2}{t^{\frac{3}{2}}} + \frac{\|J\|^2}{t^{\frac{3}{2}}} \right) \leq C \|\nabla(\bar{\nu}, \bar{v})\|_\infty \left( \frac{\|A\|^2}{t^{\frac{3}{2}}} + \frac{\|J\|^2}{t^{\frac{3}{2}}} \right) + C \left( \frac{\nabla A v}{\sqrt{J}} \right)^2 .
$$

Multiplying (4.36) with $\frac{1}{\sqrt{t}}$ and recalling the definition of $\omega(t)$ yield

$$
\frac{d}{dt} \left( \frac{\|\sqrt{\rho_0} v\|^2}{\sqrt{t}} \right) + \frac{\|\sqrt{\rho_0} v\|^2}{2t^{\frac{3}{2}}} + \frac{\nabla A v}{\sqrt{J}} \right)^2 
\leq C \left( \frac{1}{\sqrt{t}} + \omega(t) \right) \left( \|\sqrt{\rho_0} v\|^2 + \frac{\|A\|^2}{t^{\frac{3}{2}}} + \frac{\|J\|^2}{t^{\frac{3}{2}}} \right).
$$

Denote

$$
\omega_1(t) \triangleq t^{\frac{3}{2}} \left( \|\sqrt{\rho_0} \bar{\phi}_t\|^2 + \|\sqrt{\rho_0} \bar{\phi}_t\| \|\nabla \bar{\phi}_t\| \right) ,
$$

$$
\omega_2(t) \triangleq t^{\frac{3}{2}} \left( \|\nabla \bar{\phi}\|_\infty^2 + \|\nabla^2 \bar{\phi}\|_3^2 \|\nabla^2 (\bar{\nu}, \bar{v})\|^4 \right) ,
$$

$$
\omega_3(t) \triangleq \sqrt{t} \|\nabla (\bar{\nu}, \bar{v})\|_3^2 , \quad \omega_4(t) \triangleq \sqrt{t} \|\bar{\phi}\|_\infty^2 , \quad \omega_5(t) \triangleq 1 + \|\nabla \bar{v}\|_\infty ,
$$

$$
\omega_6(t) \triangleq t^{\frac{3}{2}} \|\nabla \bar{v}\|_\infty \|\bar{\phi}\|_\infty .
$$

Recalling the regularities of $(\bar{\nu}, \bar{\phi})$ and $(\bar{\nu}, \bar{v})$, we have

$$
\omega_1(t) \leq \left( \frac{\|\sqrt{t} \sqrt{\rho_0} \bar{\phi}_t\|^2}{\sqrt{t}} + \frac{\|\sqrt{t} \sqrt{\rho_0} \bar{\phi}_t\| \|t \nabla \bar{\phi}_t\|} \right) \in L^1((0, T_0)),
$$

$$
\omega_2(t) \leq C \|\nabla (\bar{\nu}, \bar{v})\|_\infty \|\sqrt{t} \nabla^2 (\bar{\nu}, \bar{v})\| \in L^\infty((0, T_0)),
$$

$$
\omega_3(t) \leq C \|\nabla \bar{\phi}\|_\infty \|\sqrt{t} \nabla^2 (\bar{\nu}, \bar{v})\| \in L^1((0, T_0)) , \quad \omega_4(t) = 1 + \|\nabla \bar{v}\|_\infty \in L^1((0, T_0)) ,
$$

$$
\omega_5(t) \leq C \|\sqrt{t} \nabla^2 (\bar{\nu}, \bar{v})\|_\infty \left( 1 + \|\nabla (\bar{\nu}, \bar{\phi})\|_3^2 \right) \|\sqrt{t} \nabla^2 (\bar{\nu}, \bar{v})\|_2 \frac{1}{2} \in L^1((0, T_0)).
$$

For $\omega_2$, by Proposition 4.1, it follows from the Gagliardo-Nirenberg inequality that

$$
\omega_2(t) \leq C t^{\frac{3}{2}} \left( \|\nabla \bar{\phi}\|_\infty^2 + \|\nabla^2 \bar{\phi}\|_3^2 \right) + C \sqrt{t} \|\nabla \bar{\phi}\|_\infty \|\sqrt{t} \nabla^2 (\bar{\nu}, \bar{v})\|_2^2 \leq C \left( \|\nabla \nabla \bar{\phi}\| + \|\sqrt{t} \nabla^2 \bar{\phi}\| \right) \|t \nabla^2 \bar{\phi}\|_\infty + C \sqrt{t} \|\nabla^2 (\bar{\nu}, \bar{v})\|_2 \|\sqrt{t} \nabla^2 (\bar{\nu}, \bar{v})\|_2^2 \in L^1((0, T_0)).
$$

In terms of $\omega_i, i = 1, 2, \cdots, 6$, one gets from (4.37) by the Young inequality that

$$
cv \frac{d}{dt} \|\sqrt{\rho_0} \bar{\phi}\|^2 + \kappa \left( \frac{\nabla A \bar{\phi}}{\sqrt{J}} \right)^2 \leq C \left( \frac{\omega_1(t)}{t^{\frac{3}{2}}} + \frac{\omega_2(t)}{t^{\frac{3}{2}}} + \frac{\omega_3(t)}{t^{\frac{3}{2}}} + \frac{\omega_4(t)}{t^{\frac{3}{2}}} \right).
\begin{align*}
+ C \sqrt{\omega_2}(t) \left( \frac{1}{\sqrt{t}} \left\| \frac{\nabla A^v v}{\sqrt{\mathcal{J}}} \right\| \right)^2 \left\| \sqrt{\rho_0} \vartheta \right\| + C \omega_2(t) \left\| \sqrt{\rho_0} \vartheta \right\|^2 \\
+ C \omega_2(t) \left( \left\| \sqrt{\rho_0} \vartheta \right\|^2 + \frac{\| A \|^2}{t^2} + \frac{\| J \|^2}{t^2} \right),
\end{align*}
from which, by the Young inequality and recalling that \( \omega_2 \in L^\infty((0, T_0)) \), one gets
\begin{align*}
c_v \frac{d}{dt} \left\| \sqrt{\rho_0} \vartheta \right\|^2 + \kappa \left\| \frac{\nabla A^v}{\sqrt{\mathcal{J}}} \right\|^2 \\
\leq C(\omega_2(t) + 1) \frac{1}{\sqrt{t}} \left\| \frac{\nabla A^v v}{\sqrt{\mathcal{J}}} \right\|^2 + C \omega_2(t) \left( \left\| \sqrt{\rho_0} \vartheta \right\|^2 + \frac{\| A \|^2}{t^2} + \frac{\| J \|^2}{t^2} \right) \\
\leq C \frac{1}{\sqrt{t}} \left\| \frac{\nabla A^v v}{\sqrt{\mathcal{J}}} \right\|^2 + C \omega_2(t) \left( \left\| \sqrt{\rho_0} \vartheta \right\|^2 + \frac{\| A \|^2}{t^2} + \frac{\| J \|^2}{t^2} \right),
\end{align*}
where
\( \omega_2 := \omega_{21} + \omega_{22} + \omega_{24} + \omega_{25} + \omega_{26} \in L^1((0, T_0)) \).

Multiplying (4.42) and (4.44) with a small positive number \( \zeta \) and adding the resultants with (4.43), one obtains
\begin{align*}
\frac{d}{dt} \left[ \frac{\| \sqrt{\rho_0} v \|^2}{\sqrt{t}} + \zeta \left( \frac{\| A \|^2}{t^2} + \frac{\| J \|^2}{t^2} + c_v \left\| \sqrt{\rho_0} \vartheta \right\|^2 \right) \right] + \mu \frac{1}{2 \sqrt{t}} \left\| \frac{\nabla A^v v}{\sqrt{\mathcal{J}}} \right\|^2 + \kappa \zeta \left\| \frac{\nabla A^v}{\sqrt{\mathcal{J}}} \right\|^2 \\
\leq C \left( \frac{1}{\sqrt{t}} + \omega_1(t) + \omega_2(t) + \| \nabla (\tilde{v}, \tilde{v}) \|_\infty \right) \left( \left\| \frac{A}{t^2} \right\|^2 + \frac{\| J \|^2}{t^2} + \frac{\| \sqrt{\rho_0} \vartheta \|^2}{t^2} \right).
\end{align*}
By Proposition 4.4 and recalling (4.41), it follows that
\begin{align*}
\lim_{t \to 0} \left[ \frac{\| \sqrt{\rho_0} v \|^2}{\sqrt{t}} + \zeta \left( \frac{\| A \|^2}{t^2} + \frac{\| J \|^2}{t^2} + c_v \left\| \sqrt{\rho_0} \vartheta \right\|^2 \right) \right](t) = 0.
\end{align*}
Thanks to this and by the Grönnwall inequality, one gets
\begin{align*}
\left( \frac{\| \sqrt{\rho_0} v \|^2}{\sqrt{t}} + \frac{\| A \|^2}{t^2} + \frac{\| J \|^2}{t^2} + \frac{\| \sqrt{\rho_0} \vartheta \|^2}{t^2} \right)(t) + \int_0^t \left( \frac{1}{\sqrt{t}} \left\| \frac{\nabla A^v}{\sqrt{\mathcal{J}}} \right\|^2 + \left\| \frac{\nabla A^v}{\sqrt{\mathcal{J}}} \right\|^2 \right) \, dt = 0,
\end{align*}
which implies \( A = J = v = \vartheta = 0 \).

Recalling that \( \tilde{\vartheta} = \tilde{J} \rho_0 \) and \( \tilde{\vartheta} = \tilde{J} \rho_0 \), it follows \( \tilde{\vartheta} = \check{\vartheta} \). Noticing that \( \partial_t \check{\varphi}(y, t) = \check{v}(y, t) \) and \( \partial_t \check{\varphi}(y, t) = \check{v}(y, t) \), one has \( \check{\varphi} = \check{\varphi} \) and further that \( \check{v} = \check{v} \). Then, it follows
\begin{align*}
\check{u}(x, t) = \check{v}(\check{\varphi}(x, t), t) = \check{v}(\check{\varphi}(x, t), t) = \check{u}(x, t),
\end{align*}
that is \( \check{u} \equiv \check{u} \). Similarly, one has \( \theta \equiv \check{\theta} \) and \( \rho \equiv \check{\rho} \). This proves the uniqueness. \( \square \)

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5. Appendix

In this “Appendix”, we give the proof of Proposition 4.1 and Proposition 4.2, as well as a lemma used in (4.34).

Proof of Proposition 4.1. Solving (4.8) yields

\[ J(y, t) = e^{-\int_0^t \text{div} u(\varphi(y, s), s) ds} \]

and, thus,

\[ \frac{1}{C_*} \leq J(y, t) \leq C_*, \quad \forall (y, t) \in \Omega \times [0, T_0], \quad (5.1) \]

where \( C_* := e^{\int_0^{T_0} \|\text{div} u\|_{\infty} dt} \). Since \( \det \nabla \psi(x, t) = J(\psi(x, t), t) \), it holds that

\[
\| g(\varphi(\cdot, t)) \|_\alpha = \left( \int_{\Omega} |g(\varphi(y, t))|^{\alpha} dy \right)^{\frac{1}{\alpha}} = \left( \int_{\Omega} |g(x)|^{\alpha} |\det \nabla \psi(x, t)| dx \right)^{\frac{1}{\alpha}} \\
= \left( \int_{\Omega} |g(x)|^{\alpha} J(\psi(x, t), t) dx \right)^{\frac{1}{\alpha}},
\]

from which, by (5.1), one gets

\[ C_*^{-\frac{1}{\alpha}} \|g\|_\alpha \leq \|g(\varphi(\cdot, t))\|_\alpha \leq C_*^{\frac{2}{\alpha}} \|g\|_\alpha, \quad \alpha \in [1, \infty). \]

Letting \( \alpha \to \infty \) in the above leads to the estimate for \( \alpha = \infty \). Therefore,

\[ \|g\|_\alpha \simeq \|g(\varphi(\cdot, t))\|_\alpha, \quad \alpha \in [1, \infty]. \quad (5.2) \]

Applying the above to \( g(\psi(x, t)) \) leads to

\[ \|g\|_\alpha \simeq \|g(\psi(\cdot, t))\|_\alpha, \quad \alpha \in [1, \infty]. \quad (5.3) \]

Combining (5.2) with (5.3) leads to (4.12).

It follows from (4.7) that

\[ \partial_t |B|^2 = 2B : (B \nabla u(\varphi, t)) \leq C \|\nabla u\|_{\infty} |B|^2 \]

and, thus,

\[ |B|^2(y, t) \leq 3e^{C \int_0^t \|\nabla u\|_{\infty} ds}, \quad \forall (y, t) \in \Omega \times [0, T_0]. \]

Therefore

\[ \sup_{0 \leq t \leq T_0} \|B\|_{\infty}(t) \leq C e^{C \int_0^{T_0} \|\nabla u\|_{\infty} dt} \leq C. \quad (5.4) \]
Similarly, one gets from (4.6) that
\[
\sup_{0 \leq t \leq T_0} \| A \|_\infty \leq C. \tag{5.5}
\]
Thanks to (5.1), (5.4), and (5.5), it follows from (4.6)–(4.8) and (5.2) that
\[
\sup_{0 \leq t \leq T_0} \| (J_t, A_t, B_t) \| \leq C \sup_{0 \leq t \leq T_0} \| \nabla u(\cdot, t) \| \leq C \sup_{0 \leq t \leq T_0} \| \nabla u \| \leq C. \tag{5.6}
\]
One gets from (4.7) that
\[
\partial_t \partial_i B = \partial_i B \nabla u(\cdot, t) + B \nabla \partial_i u(\cdot, t) \partial_i \varphi_t = \partial_i B \nabla u(\cdot, t) + B \nabla \partial_t u(\cdot, t) b_{il}
\]
and, thus, by (5.4), it follows that
\[
\partial_t |\nabla B|^2 = 2 \partial_t B : (\partial_i B \nabla u(\cdot, t) + B \nabla \partial_i u(\cdot, t) b_{il}) \\
\leq C(\| \nabla u \|_\infty |\nabla B|^2 + |\nabla B| |\nabla^2 u(\cdot, t)|) \\
\leq C(\| \nabla u \|_\infty |\nabla B|^2 + |\nabla^2 u(\cdot, t)| |\nabla B|),
\]
and further that
\[
\partial_t |\nabla B|^q \leq C(\| \nabla u \|_\infty |\nabla B|^q + |\nabla^2 u(\cdot, t)| |\nabla B|^{q-1}).
\]
Integrating the above over \( \Omega \), it follows from the Hölder inequality and (5.2) that
\[
\frac{d}{dt} \| \nabla B \|_q \leq C (\| \nabla u \|_\infty \| \nabla B \|_q + \|\nabla^2 u(\cdot, t)\|_q \|\nabla B\|_{q-1}) \\
\leq C (\| \nabla u \|_\infty \| \nabla B \|_q + \|\nabla^2 u(\cdot, t)\|_q \|\nabla B\|_{q-1})
\]
and, thus,
\[
\frac{d}{dt} \| \nabla B \|_q \leq C (\| \nabla u \|_\infty \| \nabla B \|_q + \|\nabla^2 u(\cdot, t)\|_q).
\]
Applying the Grönwall inequality to the above yields
\[
\sup_{0 \leq t \leq T_0} \| \nabla B \|_q \leq C e^{C \int_0^{T_0} \| \nabla u \|_\infty dt} \int_0^{T_0} \|\nabla^2 u(\cdot, t)\|_q dt \leq C. \tag{5.7}
\]
Similarly, one derives from (4.6) and (4.8) that
\[
\sup_{0 \leq t \leq T_0} (\| \nabla A \|_q + \| \nabla J \|_q) \leq C. \tag{5.8}
\]
Conclusion (4.9) follows from (5.1) and (5.4)–(5.8).

Fix \( t_0 \in [0, T_0] \) and denote
\[
G(y) := g(\varphi(y, t_0)), \quad \forall y \in \Omega.
\]
Then, it is clear that
\[
g(x) = G(\psi(x, t_0)), \quad \forall x \in \Omega.
\]
By direct calculations and recalling the definitions of \( a_{ij}(y, t) \) and \( b_{ij}(y, t) \), one has
\[
\partial_t G(y) = b_{il}(y, t_0) \partial_i g(\varphi(y, t_0)), \quad \partial_i g(x) = a_{il}(\psi(x, t_0), t_0) \partial_i G(\psi(x, t_0)).
\]
Therefore, it follows from (5.4) and (5.5) that
\[
|\nabla G(y)| \leq C|\nabla g(\varphi(y, t_0))|, \quad |\nabla g(x)| \leq C|\nabla G(\psi(x, t_0))|. \tag{5.9}
\]
Thanks to (5.9), it follows from (5.2) and (5.3) that
\[
\| \nabla G \|_\alpha \leq C \| \nabla g(\cdot, t_0) \|_\alpha \leq C \| \nabla g \|_\alpha, \quad 1 \leq \alpha \leq \infty, \tag{5.10}
\]
\[
\| \nabla g \|_\alpha \leq C \| \nabla G(\cdot, t_0) \|_\alpha \leq C \| \nabla G \|_\alpha, \quad 1 \leq \alpha \leq \infty. \tag{5.11}
\]
As a result, recalling the definition of $G$, one gets
\[\|\nabla [g(\varphi(\cdot, t_0))]\|_\alpha \simeq \|\nabla g\|_\alpha, \quad 1 \leq \alpha \leq \infty,\]
which applied to $g(\psi(x, t_0))$ yields
\[\|\nabla g\|_\alpha \simeq \|\nabla [g(\psi(\cdot, t_0))]\|_\alpha, \quad 1 \leq \alpha \leq \infty.\]
Therefore (4.11) holds.

By direct calculations and recalling the definitions of $a_{ij}(y, t)$ and $b_{ij}(y, t)$, one has
\[
\partial^2_{ij} G(y) = \partial_i b_{j}(y, t_0) \partial_j g(\varphi(y, t_0)) + b_{ij}(y, t_0) \partial^2_{ij} g(\varphi(y, t_0)),
\]
\[
\partial^2_{ij} g(x) = a_{jm}(\psi(x, t_0), t_0) \partial_{m} a_{il}(\psi(x, t_0), t_0) \partial_i G(\psi(x, t_0))
+ a_{il}(\psi(x, t_0), t_0) a_{jm}(\psi(x, t_0), t_0) \partial^2_{jm} G(\psi(x, t_0)).
\]
Then, it follows from (5.4) and (5.5) that
\[
|\nabla^2 G(y)| \leq C |\nabla B(y, t_0)| |\nabla g(\varphi(y, t_0))| + C |\nabla^2 g(\varphi(y, t_0))|,
\]
\[
|\nabla^2 g(x)| \leq C |\nabla A(\psi(x, t_0), t_0)| |\nabla G(\psi(x, t_0))| + C |\nabla^2 G(\psi(x, t_0))|.
\]
Thanks to these, it follows from the Hölder and Sobolev inequalities, (5.7)–(5.8), and (5.2)–(5.3) that: for $1 \leq \alpha < 3$,
\[
\|\nabla^2 G\|_\alpha \leq C \left( \|\nabla B\|_3 \|\nabla g(\varphi(\cdot, t_0))\|_3 + 3 \|\nabla^2 g(\varphi(\cdot, t_0))\|_\alpha \right)
\leq C \left( \|\nabla B\|_q \|\nabla g\|_{\frac{3\alpha}{3-\alpha}} + \|\nabla^2 g\|_\alpha \right) \leq C \|\nabla g\|_{W^{1,\alpha}},
\]
\[
\|\nabla^2 g\|_\alpha \leq C \left( \|\nabla A(\psi(\cdot, t_0), t_0)\|_3 \|\nabla G(\psi(\cdot, t_0))\|_{\frac{3\alpha}{3-\alpha}} + \|\nabla^2 G(\psi(\cdot, t_0))\|_\alpha \right)
\leq C \left( \|\nabla A\|_3 \|\nabla G\|_{\frac{3\alpha}{3-\alpha}} + \|\nabla^2 G\|_\alpha \right) \leq C \|\nabla G\|_{W^{1,\alpha}};
\]
for $\alpha = 3$,
\[
\|\nabla^2 G\|_3 \leq C \left( \|\nabla B\|_q \|\nabla g(\varphi(\cdot, t_0))\|_3 + \|\nabla^2 g(\varphi(\cdot, t_0))\|_3 \right)
\leq C \left( \|\nabla g\|_{\frac{3\alpha}{3-\alpha}} + \|\nabla^2 g\|_3 \right) \leq C \|\nabla g\|_{W^{1,3}},
\]
\[
\|\nabla^2 g\|_3 \leq C \left( \|\nabla A(\psi(\cdot, t_0), t_0)\|_q \|\nabla G(\psi(\cdot, t_0))\|_{\frac{3\alpha}{3-\alpha}} + \|\nabla^2 G(\psi(\cdot, t_0))\|_3 \right)
\leq C \left( \|\nabla A\|_q \|\nabla G\|_{\frac{3\alpha}{3-\alpha}} + \|\nabla^2 G\|_3 \right) \leq C \|\nabla G\|_{W^{1,3}};
\]
and for $3 < \alpha \leq q$,
\[
\|\nabla^2 G\|_\alpha \leq C \left( \|\nabla B\|_q \|\nabla g(\varphi(\cdot, t_0))\|_\infty + \|\nabla^2 g(\varphi(\cdot, t_0))\|_\alpha \right)
\leq C \left( \|\nabla g\|_\infty + \|\nabla^2 g\|_\alpha \right) \leq C \|\nabla g\|_{W^{1,\alpha}},
\]
\[
\|\nabla^2 g\|_\alpha \leq C \left( \|\nabla A(\psi(\cdot, t_0), t_0)\|_q \|\nabla G(\psi(\cdot, t_0))\|_\infty + \|\nabla^2 G(\psi(\cdot, t_0))\|_\alpha \right)
\leq C \left( \|\nabla A\|_q \|\nabla G\|_\infty + \|\nabla^2 G\|_\alpha \right) \leq C \|\nabla G\|_{W^{1,\alpha}}.
\]
Therefore, for any $1 \leq \alpha \leq q$, it holds that
\[
\|\nabla^2 G\|_\alpha \leq C \|\nabla g\|_{W^{1,\alpha}}, \quad \|\nabla^2 g\|_\alpha \leq C \|\nabla G\|_{W^{1,\alpha}}.
\]
Thanks to these, by (5.10)–(5.11), and recalling the definition of $G$, it follows that
\[
\|\nabla [g(\varphi(\cdot, t_0))]\|_{W^{1,\alpha}} \leq C \|\nabla g\|_{W^{1,\alpha}}, \quad \|\nabla g\|_{W^{1,\alpha}} \leq C \|\nabla [g(\varphi(\cdot, t_0))]\|_{W^{1,\alpha}},
\]
for any $1 \leq \alpha \leq q$. This proves
\[
\|\nabla [g(\varphi(\cdot, t_0))]\|_{W^{1,\alpha}} \simeq \|\nabla g\|_{W^{1,\alpha}}, \quad \forall 1 \leq \alpha \leq q.
\]
which, applied to $g(\psi(x, t))$, yields further
\[ \|\nabla g\|_{W^{1,\alpha}} \lesssim \|\nabla [g(\psi(x, t_0))]\|_{W^{1,\alpha}}, \quad \forall 1 \leq \alpha \leq q. \]
Therefore, (4.10) holds.

\[ \square \]

**Proof of Proposition 4.2.** By Proposition 4.1, it holds that $\|h(\varphi(\cdot, t), t)\| \leq C\|h\|$ for any $t \in [0, T_0]$ and, thus,
\[ \|h(\varphi(\cdot, t), t)\|_{L^\infty(0, T_0; L^2)} \leq C\|h\|_{L^\infty(0, T_0; L^2)}. \]

Fix $t_0 \in [0, T_0]$ and take arbitrary $\varepsilon > 0$. Choose $\xi \in C^\infty_c(\Omega)$ such that
\[ \|\xi - h(\cdot, t_0)\|_{L^2} \leq \varepsilon. \]

(i) By the H"older inequality and by Proposition 4.1, one deduces
\[
\|h(\varphi(\cdot, t), t) - h(\varphi(\cdot, t_0), t_0)\|
\leq \|h(\varphi(\cdot, t), t) - h(\varphi(\cdot, t_0), t_0)\| + \|h(\varphi(\cdot, t), t_0) - \xi(\varphi(\cdot, t_0))\|
+ \|\xi(\varphi(y, t_0)) - \xi(\varphi(y, t_0))\| + \|\xi(\varphi(y, t_0)) - h(\varphi(\cdot, t), t_0)\|
\leq C(\|h(\cdot, t) - h(\cdot, t_0)\| + \|h(\cdot, t_0) - \xi\| + \|\xi(\varphi(\cdot, t)) - \xi(\varphi(\cdot, t_0))\|)
\leq C(\|h(\cdot, t) - h(\cdot, t_0)\| + \varepsilon + \|\xi(\varphi(\cdot, t)) - \xi(\varphi(\cdot, t_0))\|) \tag{5.12}
\]
for any $t \in [0, T_0]$. Recalling (4.1) and by Proposition 4.1, it follows from the Gagliardo-Nirenberg and H"older inequalities that
\[
\|\xi(\varphi(\cdot, t)) - \xi(\varphi(\cdot, t_0))\| = \left\| \int_{t_0}^t \nabla \xi(\varphi(\cdot, s)) \cdot \partial_t \varphi(\cdot, s) ds \right\|
\leq \left\| \int_{t_0}^t \nabla \xi(\varphi(\cdot, s)) \cdot u(\varphi(\cdot, s), s) ds \right\| \leq \left| \int_{t_0}^t \|\nabla \xi(\varphi(\cdot, s))\||u|_\infty ds \right|
\leq C \left| \int_{t_0}^t \|\nabla u\|^2 \|\nabla^2 u\|^{\frac{2}{3}} ds \right| \leq C|t - t_0|^\frac{2}{3}, \quad \forall t \in [0, T_0]. \tag{5.13}
\]
Plugging this estimate into (5.12) leads to
\[ \|h(\varphi(\cdot, t), t) - h(\varphi(\cdot, t_0), t_0)\| \leq C(\|h(\cdot, t) - h(\cdot, t_0)\| + \varepsilon + |t - t_0|^\frac{2}{3}), \tag{5.14} \]
which implies $\|h(\varphi(\cdot, t), t) - h(\varphi(\cdot, t_0), t_0)\| \to 0$ as $t \to t_0$, for any $t_0 \in [0, T_0]$. Therefore, $h(\varphi(\cdot, t), t) \in C([0, T_0]; L^2)$.

(ii) Note that
\[
\int_{\Omega} [h(\varphi(y, t), t) - h(\varphi(y, t_0), t_0)]\chi(y)dy
= \int_{\Omega} [h(\varphi(y, t), t) - h(\varphi(y, t_0), t_0)]\chi(y)dy + \int_{\Omega} [h(\varphi(y, t), t_0) - \xi(\varphi(y, t))]\chi(y)dy
+ \int_{\Omega} [\xi(\varphi(y, t)) - \xi(\varphi(y, t_0))]\chi(y)dy + \int_{\Omega} [\xi(\varphi(y, t_0)) - h(\varphi(y, t_0), t_0)]\chi(y)dy
=: R_1 + R_2 + R_3 + R_4.
\]
Since $\det \nabla \psi(x, t) = J(\psi(x, t), t) > 0$, one deduces
\[
R_1 = \int_{\Omega} [h(\varphi(y, t), t) - h(\varphi(y, t), t_0)]\chi(y)dy
= \int_{\Omega} [h(x, t) - h(x, t_0)]\chi(\psi(x, t))|\det \nabla \psi(x, t)|dx
= \int_{\Omega} [h(x, t) - h(x, t_0)]\chi(\psi(x, t))J(\psi(x, t), t)dx
\]
\[
= \int_{\Omega} [h(x, t) - h(x, t_0)] [\chi(\psi(x, t))J(\psi(x, t), t) - \chi(\psi(x, t_0))J(\psi(x, t_0), t_0)] \, dx \\
+ \int_{\Omega} [h(x, t) - h(x, t_0)] \chi(\psi(x, t_0))J(\psi(x, t_0), t_0) \, dx \\
=: R_{11} + R_{12}.
\]

Since \(\chi(\psi(x, t_0))J(\psi(x, t_0), t_0) \in L^2\), guaranteed by Proposition 4.1, one has
\[
R_{12} \to 0, \quad \text{as } t \to t_0. \tag{5.15}
\]

Recalling (4.2) and (4.8), one has
\[
\chi(\psi(x, t))J(\psi(x, t), t) - \chi(\psi(x, t_0))J(\psi(x, t_0), t_0)
\]
\[
= \int_{t_0}^{t} [\chi(\psi(x, s))(\partial_s J(\psi(x, s), s) + \nabla J(\psi(x, s), s) \cdot \partial_t \psi(x, s)) \\
+ \nabla \chi(\psi(x, s)) \cdot \partial_t \psi(x, s)J(\psi(x, s), s)] \, ds
\]
\[
= - \int_{t_0}^{t} (u(x, s) \cdot \nabla) \psi(x, s) \cdot [\chi(\psi(x, s)) \nabla J(\psi(x, s), s) \\
+ \nabla \chi(\psi(x, s))J(\psi(x, s), s)] \, ds - \int_{t_0}^{t} \chi(\psi(x, s)) \text{div} u(x, s)J(\psi(x, s), s) \, ds
\]
\[
= - \int_{t_0}^{t} u(x, s) \cdot A(\psi(x, s), s) \cdot [\chi(\psi(x, s)) \nabla J(\psi(x, s), s) \\
+ \nabla \chi(\psi(x, s))J(\psi(x, s), s)] \, ds
\]
\[
- \int_{t_0}^{t} \chi(\psi(x, s)) \text{div} u(x, s)J(\psi(x, s), s) \, ds
\]

and, thus, by Proposition 4.1, it follows
\[
\|\chi(\psi(\cdot, t))J(\psi(\cdot, t), t) - \chi(\psi(\cdot, t_0))J(\psi(\cdot, t_0), t_0)\|
\]
\[
\leq \left| \int_{t_0}^{t} \|u\|_{\infty} \|A(\psi(\cdot, s), s)\| \|\nabla \chi\|_{\infty} J(t)_{\infty} + \|u\|_{\infty} \|A\|_{\infty} \|\chi\|_{\infty} \|\nabla J(\psi(\cdot, s), s)\| \, ds \right|
\]
\[
+ \left| \int_{t_0}^{t} \|\chi\|_{\infty} \|\text{div} u\|_{\infty} J(t)_{\infty} \, ds \right| \leq C \left| \int_{t_0}^{t} \left( \|\nabla u\|_{\frac{3}{4}} \|\nabla^2 u\|_{\frac{3}{4}} \right) \, ds \right| + C|t - t_0| \leq C|t - t_0|^\frac{3}{4}. \tag{5.16}
\]

Thanks to this, one has
\[
|R_{11}| \leq \|h(\cdot, t) - h(\cdot, t_0)\| \|\chi(\psi(\cdot, t))J(\psi(\cdot, t), t) - \chi(\psi(\cdot, t_0))J(\psi(\cdot, t_0), t_0)\|
\]
\[
\leq C|t - t_0|^\frac{3}{4}. \tag{5.16}
\]

Using the Hölder inequality and by Proposition 4.1, it follows
\[
|R_{2} + R_{4}| \leq C\|\chi\| \|h(\varphi(\cdot, t), t_0) - \xi(\varphi(\cdot, t))\| + \|h(\varphi(\cdot, t_0), t_0) - \xi(\varphi(\cdot, t_0))\|
\]
\[
\leq C\|\chi\| \|h(\cdot, t) - \xi\| \leq C\varepsilon. \tag{5.17}
\]

For \(R_{3}\), it follows from the Hölder inequality and (5.13) that
\[
|R_{3}| \leq \|\chi\| \|\xi(\varphi(y, t)) - \xi(\varphi(y, t_0))\| \leq C|t - t_0|^\frac{3}{2}. \tag{5.18}
\]

Combining (5.16)–(5.18), one gets
\[
\int_{\Omega} \left| h(\varphi(y, t), t) - h(\varphi(y, t_0), t_0) \right| \chi(y) \, dy \leq C|t - t_0|^\frac{3}{2} + C\varepsilon + |R_{12}|.
\]

With the aid of this and recalling (5.15), one derives
\[
\int_{\Omega} \left| h(\varphi(y, t), t) - h(\varphi(y, t_0), t_0) \right| \chi(y) \, dy \to 0, \quad \text{as } t \to t_0,
\]

which implies that \( h(\varphi(\cdot, t), t) \) is weakly continuous in \( L^2(\Omega) \) at any \( t_0 \in [0, T_0] \). Therefore, \( h(\varphi(\cdot, t), t) \in C_w([0, T_0]; L^2) \).

Finally, we prove the following lemma which is used in (4.34) during the proof of the uniqueness in the previous section.

\textbf{Lemma 5.1.} Given a bounded domain \( \Omega \) in \( \mathbb{R}^3 \). Let \( \Phi \in W^{2,q} \) with \( q \in (3, 6) \) be a bijective mapping on \( \Omega \). Denote \( x = \Phi(y) \) and \( y = \Psi(x) \). Then, it holds that

\[
\text{div}_y \left( \frac{\partial x_i}{\text{det}\nabla x y} \right) = 0 \quad \text{and} \quad \text{div}_x \left( \frac{\partial y_i}{\text{det}\nabla y x} \right) = 0.
\]

\textbf{Proof.} We only give the proof of the first identity while the second one can be proved in the same way. In the proof of this lemma, for a matrix \( A = (a_{ij})_{3 \times 3} \), we use \( R_i(A) \), \( M_{ij}(A) \), and \( A_{\text{adj}} \), respectively, to denote the \( i \)-th row of \( A \), the minor of the entry \( a_{ij} \), and the classical adjoint of \( A \) that is, \( A_{\text{adj}} = ((-1)^{i+j}M_{ij})^T \). Denote \( \nabla_x y = (\partial x_i y_j)_{3 \times 3} \) and \( \nabla_y x = (\partial y_i x_j)_{3 \times 3} \). Then, the chain rule gives \( \nabla_x y \nabla_y x = I \). Thanks to this, one deduces

\[
\frac{\nabla x y}{\text{det}\nabla x y} = \frac{(\nabla x y)^{-1}}{\text{det}\nabla x y} = \frac{(\nabla x y)_{\text{adj}}}{\text{det}\nabla y x \text{det}\nabla x y} = (\nabla x y)_{\text{adj}}.
\]

Therefore,

\[
\text{div}_y \left( \frac{\partial x_i}{\text{det}\nabla x y} \right) = \text{div}_y \left( R_i((\nabla x y)_{\text{adj}}) \right).
\]

It remains to show that \( \text{div}_y \left( R_i((\nabla x y)_{\text{adj}}) \right) = 0 \) for \( i = 1, 2, 3 \). We only prove the case \( i = 1 \), the proofs for \( i = 2, 3 \) are the same. By definition, one has

\[
\text{div}_y \left( R_i((\nabla x y)_{\text{adj}}) \right) = \begin{vmatrix}
\partial y_1 & \partial y_2 & \partial y_3
\partial y_2 & \partial y_2 x_2 & \partial y_2 x_3
\partial y_3 & \partial y_3 x_2 & \partial y_3 x_3
\end{vmatrix},
\]

where the determinant is understood by expanding along the first column. By direct calculations, one can verify that the above determinant is identically zero and, thus, the conclusion holds. \( \square \)

\textbf{References}

[1] Bresch, D., Jabin, P.-E.: Global existence of weak solutions for compressible Navier–Stokes equations: thermodynamically unstable pressure and anisotropic viscous stress tensor. Ann. Math. (2) \textbf{188}(2), 577–684 (2018)

[2] Chen, G.-Q., Hoff, D., Trivisa, K.: Global solutions of the compressible Navier–Stokes equations with large discontinuous initial data. Commun. Partial Differ. Equ. \textbf{25}(11–12), 2233–2257 (2000)

[3] Chen, Q., Miao, C., Zhang, Z.: Global well-posedness for compressible Navier–Stokes equations with highly oscillating initial velocity. Commun. Pure Appl. Math. \textbf{63}(9), 1173–1224 (2010)

[4] Chikami, N., Danchin, R.: On the well-posedness of the full compressible Navier–Stokes system in critical Besov spaces. J. Differ. Equ. \textbf{258}(10), 3435–3467 (2015)

[5] Cho, Y., Choe, H.J., Kim, H.: Unique solvability of the initial boundary value problems for compressible viscous fluids. J. Math. Pures Appl. (9) \textbf{83}(2), 243–275 (2004)

[6] Cho, Y., Kim, H.: Existence results for viscous polytropic fluids with vacuum. J. Differ. Equ. \textbf{228}(2), 377–411 (2006)

[7] Choe, H.J., Kim, H.: Strong solutions of the Navier–Stokes equations for isentropic compressible fluids. J. Differ. Equ. \textbf{190}(2), 504–523 (2003)

[8] Danchin, R.: Global existence in critical spaces for compressible Navier–Stokes equations. Invent. Math. \textbf{141}(3), 579–614 (2000)

[9] Danchin, R.: Local theory in critical spaces for compressible viscous and heat-conductive gases. Commun. Partial Differ. Equ. \textbf{26}(7–8), 1183–1233 (2001)

[10] Danchin, R., Mucha, P.B.: The incompressible Navier–Stokes equations in vacuum. Comm. Pure Appl. Math. \textbf{72}(7), 1351–1385 (2019)

[11] Danchin, R., Xu, J.: Optimal decay estimates in the critical \( L^p \) framework for flows of compressible viscous and heat-conductive gases. J. Math. Fluid Mech. \textbf{20}(4), 1641–1665 (2018)

[12] Fang, D., Zhang, T., Zi, R.: Global solutions to the isentropic compressible Navier–Stokes equations with a class of large initial data. SIAM J. Math. Anal. \textbf{50}(5), 4983–5026 (2018)
[13] Feireisl, E.: On the motion of a viscous, compressible, and heat conducting fluid. Indiana Univ. Math. J. 53(6), 1705–1738 (2004)
[14] Feireisl, E., Novotný, A., Petzeltová, H.: On the existence of globally defined weak solutions to the Navier–Stokes equations. J. Math. Fluid Mech. 3(4), 358–392 (2001)
[15] Gong, H., Li, J., Liu, X., Zhang, X.: Local well-posedness of isentropic compressible Navier–Stokes equations with vacuum. Commun. Math. Sci. 18(7), 1891–1909 (2020)
[16] Graffi, D.: Il teorema di unicità nella dinamica dei fluidi compressibili. J. Ration. Mech. Anal. 2, 99–106 (1953)
[17] Hoff, D.: Discontinuous solutions of the Navier–Stokes equations for multidimensional flows of heat-conducting fluids. Arch. Ration. Mech. Anal. 139(4), 303–354 (1997)
[18] Huang, X.D.: On local strong and classical solutions to the three-dimensional barotropic compressible Navier–Stokes equations with vacuum. Sci. China Math. 64(8), 1771–1788 (2021)
[19] Huang, X., Li, J.: Global classical and weak solutions to the three-dimensional full compressible Navier–Stokes system with vacuum and large oscillations. Arch. Ration. Mech. Anal. 227(3), 995–1059 (2018)
[20] Huang, X., Li, J., Xin, Z.: Global well-posedness of classical solutions with large oscillations and vacuum to the three-dimensional isentropic compressible Navier–Stokes equations. Commun. Pure Appl. Math. 65(4), 549–585 (2012)
[21] Itaya, N.: On the Cauchy problem for the system of fundamental equations describing the movement of compressible viscous fluid. Kodai Math. Sem. Rep. 23, 60–120 (1971)
[22] Jiang, S., Zhang, P.: On spherically symmetric solutions of the compressible isentropic Navier–Stokes equations. Commun. Math. Phys. 215(3), 559–581 (2001)
[23] Jiang, S., Zhang, P.: Axisymmetric solutions of the 3D Navier–Stokes equations for compressible isentropic fluids. J. Math. Pures Appl. (9) 82(8), 949–973 (2003)
[24] Jiang, S., Zlotnik, A.: Global well-posedness of the Cauchy problem for the equations of a one-dimensional heat-conducting gas with Lebesgue initial data. Proc. R. Soc. Edinb. Sect. A 134(5), 939–960 (2004)
[25] Kanel, J.I.: A model system of equations for the one-dimensional motion of a gas. Differencial’nye Uravnenija 4, 721–734 (1968)
[26] Kazhikhov, A.V.: On the Cauchy problem for the equations of a viscous gas. Sibirsk. Mat. Zh. 23(1), 60–64 (1982)
[27] Kazhikhov, A.V., Shelukhin, V.V.: Unique global solution with respect to time of initial-boundary value problems for one-dimensional equations of a viscous gas. J. Appl. Math. Mech. 41(2), 273–282 (1977). translated from Prikl. Mat. Meh. 41 (1977), no. 2, 282–291
[28] Lai, S., Xu, H., Zhang, J.: Well-posedness and exponential decay for the Navier–Stokes equations of viscous compressible heat-conducting fluids with vacuum. Math. Models Methods Appl. Sci. 32(9), 1725–1784 (2022)
[29] Li, J.: Local existence and uniqueness of strong solutions to the Navier–Stokes equations with nonnegative density. J. Differ. Equ. 263(10), 6512–6536 (2017)
[30] Li, J.: Global well-posedness of the one-dimensional compressible Navier–Stokes equations with nonnegative density. SIAM J. Math. Anal. 51(5), 3666–3693 (2019)
[31] Li, J.: Global well-posedness of non-heat conductive compressible Navier–Stokes equations in 1D. Nonlinearity 33(5), 2181–2210 (2020)
[32] Li, J.: Global small solutions of heat conductive compressible Navier–Stokes equations with vacuum: smallness on scaling invariant quantity. Arch. Ration. Mech. Anal. 237(2), 899–919 (2020)
[33] Li, J., Liang, Z.: Some uniform estimates and large-time behavior of solutions to one-dimensional compressible Navier–Stokes system in unbounded domains with large data. Arch. Ration. Mech. Anal. 220(3), 1195–1208 (2016)
[34] Li, J., Xin, Z.: Global well-posedness and large time asymptotic behavior of classical solutions to the compressible Navier–Stokes equations with vacuum. Ann. PDE 5(1), 37 pp (2019)
[35] Li, J., Xin, Z.: Entropy bounded solutions to the one-dimensional compressible Navier–Stokes equations with zero heat conduction and far field vacuum. Adv. Math. 361, 106923, 50 pp (2020)
[36] Li, J., Xin, Z.: Entropy-bounded solutions to the one-dimensional heat conductive compressible Navier–Stokes equations with far field vacuum. Commun. Pure Appl. Math. 75(11), 2393–2445 (2022)
[37] Li, J., Xin, Z.: Local and global well-posedness of entropy-bounded solutions to the compressible Navier–Stokes equations in multi-dimensions. Sci. China Math. (2022). https://doi.org/10.1007/s11425-022-2047-0
[38] Lions, P.-L.: Mathematical topics in fluid mechanics. Vol. 2, Oxford Lecture Series in Mathematics and its Applications, 10, The Clarendon Press, Oxford University Press, New York (1998)
[39] Matsumura, A., Nishida, T.: The initial value problem for the equations of motion of compressible viscous and heat-conductive fluids. Proc. Jpn. Acad. Ser. A Math. Sci. 55(9), 337–342 (1979)
[40] Matsumura, A., Nishida, T.: The initial value problem for the equations of motion of viscous and heat-conductive gases. J. Math. Kyoto Univ. 20(1), 67–104 (1980)
[41] Matsumura, A., Nishida, T.: Initial-boundary value problems for the equations of motion of general fluids, in Computing methods in applied sciences and engineering, V (Versailles,: 389–406. North-Holland, Amsterdam (1981)
[42] Merle, F., Raphaël, P., Rodnianski, I., Szeftel, J.: On the implosion of a compressible fluid I: smooth self-similar inviscid profiles. Ann. Math. (2) 196, 567–778 (2022)
[43] Merle, F., Raphaël, P., Rodnianski, I., Szeftel, J.: On the implosion of a compressible fluid II: singularity formation. Ann. Math. (2) 196, 779–889 (2022)
[44] Nash, J.: Le problème de Cauchy pour les équations différentielles d’un fluide général. Bull. Soc. Math. France 90, 487–497 (1962)
[45] Salvi, R., Straškraba, I.: Global existence for viscous compressible fluids and their behavior as $t \to \infty$. J. Fac. Sci. Univ. Tokyo Sect. IA Math. 40(1), 17–51 (1993)

[46] Serrin, J.: Mathematical principles of classical fluid mechanics, in Handbuch der Physik (herausgegeben von S. Flügge), Bd. 8/1, Strömungsmechanik I (Mitherausgeber C. Truesdell), 125–263, Springer-Verlag, Berlin

[47] Solonnikov, V.A.: Solvability of the problem of the motion of a viscous incompressible fluid that is bounded by a free surface. Izv. Akad. Nauk SSSR Ser. Mat. 41(6), 1388–1424 (1977)

[48] Vol’pert, A.I., Hudjaev, S.I.: The Cauchy problem for composite systems of nonlinear differential equations. Mat. Sb. (N.S.) 87(129), 504–528 (1972)

[49] Wen, H., Zhu, C.: Global solutions to the three-dimensional full compressible Navier–Stokes equations with vacuum at infinity in some classes of large data. SIAM J. Math. Anal. 49(1), 162–221 (2017)

[50] Zlotnik, A.A., Amosov, A.A.: Stability of generalized solutions of the one-dimensional motion of a viscous heat-conducting gas. (Russian), Math. Notes 63(5–6), 736–746 (1998); translated from Mat. Zametki 63(6), 835–846 (1998)

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