Use of various versions of Schwarz method for solving the problem of contact interaction of elastic bodies

M P Galanin*, V V Lukin, and A S Rodin
Keldysh Institute of Applied Mathematics of the Russian Academy of Sciences, Moscow, Russia
Bauman Moscow State Technical University, Moscow, Russia
E-mail: *galan@keldysh.ru

Abstract. A definition of a sufficiently common problem of mechanical contact interaction in a system of elastic bodies is given. Various versions of realization of the Schwarz method for solving the contact problem numerically are described and the results of solution of a number of problems are presented. Special attention is paid to calculations where the grids in the bodies significantly differ in steps.

1. Introduction
Taking the contact interaction between various structural elements of constructions into account is an important component of assessment of the strain stress distribution (SSD) in bodies. Analytical solutions of contact problems have been received for a very restricted number of contact interactions and forms of the contacting surfaces. In almost all important situations, it is necessary to apply numerical methods among which, for solving the problem of the deformable solid mechanics (DSM), the leading position is held by the finite element method (FEM) [1–4].

2. Definition of the problem of contact interaction in a system of thermo-elastic bodies
In the three-dimensional space $R^3$ with a Cartesian coordinate system $Ox_1x_2x_3$, we consider a group of bodies occupying the area $G = \bigcup_\alpha G_\alpha$ with a piecewise smooth border $\partial G = \bigcup_\alpha \partial G_\alpha$. We assume that the relation between temperatures and deformation can be neglected. Therefore, the problem of heat conductivity can be solved separately, and the received temperature profile can be applied at the solution of a quasi-static problem of equilibrium of bodies.

We use the classical problem definition of heat conductivity taking into account the boundary conditions of second and third kinds [3, 4]. To solve the contact problem, we assume that the temperature at the corresponding points in the neighborhood of contacting bodies is the environment temperature on the site boundary where the heat exchange condition is set.

The mathematical statement of a quasi-static unified problem of DSM in the considered thermo-elastic statement includes the following ratios (for each body $G_\alpha$, $i,j = 1,3$):

the equilibrium equations

$$\sigma_{ji,j}(u, T) + Q_i(x, t) = 0, \quad x \in G_\alpha;$$

(1)
the boundary conditions (kinematic and stress)

\[ u(x, t)|_{S_{1}^\alpha} = u^0(x, t), \quad x \in S_1^\alpha \subset \partial G_A, \quad (2) \]

\[ \sigma_j(u, T)n_j|_{S_2^\alpha} = p_i(x, t), \quad x \in S_2^\alpha \subset \partial G_A; \quad (3) \]

and the governing equations (in this case, Hooke’s law) for stress tensor components

\[ \sigma_{ij} = C_{ijkl}(\varepsilon_{kl} - \varepsilon_{kl}^{th}), \quad (4) \]

where \( u(x, t) \) is the displacement vector of the point determined by the radius vector \( x = x_i e_i \), \( u^0(x, t) \) is the displacement vector of a point located on a surface \( S_{1}^\alpha \), \( Q(x, t) = Q_i(x, t)e_i \) is the vector of mass forces, \( p_i(x, t) = p_i(x, t)e_i \) is the vector of the external loading acting on a surface \( S_{2}^\alpha \), \( \varepsilon_{kl} \) are components of a tensor of the total strain (set by Cauchy relations), \( \varepsilon_{kl}^{th} \) are components of a tensor of temperature strain, \( C_{ijkl} \) are components of a tensor of elastic coefficients.

When solving the contact problem on the contact surfaces of bodies, it is necessary to satisfy the contact conditions for interaction between displacements and stresses. For simplicity, we consider only the case of two bodies with one couple of contact surfaces. Let us consider two thermo-elastic contacting bodies \( A \) and \( B \) occupying the areas \( G_A \) and \( G_B \) with piecewise smooth boundaries \( \partial G_A \) and \( \partial G_B \) in the space \( R^3 \).

On the surface of contact \( S_k = S_k^A = S_k^B \), the following conditions must be satisfied: on the displacements (kinematic conditions)

\[ u_n^A(x, t) - u_n^B(x, t) = \delta_n(x), \quad x \in S_k, \quad (5) \]

and on the stresses (stress condition)

\[ \sigma_n^A(x, t) = \sigma_n^B(x, t) \leq 0, \quad x \in S_k, \quad (6) \]

where \( u_n^A \) and \( u_n^B \) are projections of the displacement vectors of boundary points on the direction of the external normal \( n_A \) to the boundary of a body \( L \), \( \delta_n \) is the initial distance (gap) along the normal between boundary points of bodies \( A \) and \( B \). \( \sigma_n^A \) and \( \sigma_n^B \) are projections of the stress vectors \( \sigma^A \) and \( \sigma^B \) on the external normals \( n_A \) and \( n_B \) respectively, \( S_0^a \cup S_2^a \cup S_b^\alpha = \partial G_A \), \( \alpha \in \{A, B \} \).

The tangential contact stresses \( \tau_n^\alpha = \sigma_n^\alpha \cdot \tau_n \) (\( \tau_n \) is tangent to the contact boundary of the corresponding body) are calculated by the formula (Coulomb’s law)

\[ |\sigma_n^\alpha| = \mu|\sigma_n^\alpha|, \quad (7) \]

where \( \mu \) is the friction coefficient (sliding friction).

If we pose adhesion conditions on the contact boundary, then not only normal components of the vectors of displacements and stresses, but all components of these vectors, appear in formulas (5), (6).

The set of relations (1)–(7) makes the mathematical statement of a contact problem of DSM. It is supposed that all functions entering this formulation have sufficient smoothness.

The multi-contact character of the considered problem is determined, first of all, by the geometry of the interacting bodies. Two bodies can interact but have several incoherent surfaces of contact. On the other hand, several bodies also can participate in interaction. These circumstances need to be considered when developing an algorithm for solving problem (1)–(7) numerically.
3. Application of the alternating Schwarz method for solving the contact problem numerically

The stated problem is solved numerically by FEM [1–4]. In particular, the applied version of FEM for this problem is described in [3, 5]. As a result of FEM application, the problem of MDSB (1)–(7) reduces to solving the linear matrix equation [2, 4]

\[ [K]\{U\} = \{R\}. \tag{8} \]

Here the following notation is used: \([K]\) is global stiffness matrix, \{\(U\)\} is global displacements vector, \{\(R\)\} is global vector of loading.

Various iterative methods are used to solve the contact problems, for example, the penalty method, the Lagrange multiplier method, the combined method of penalties and Lagrange, the pseudo-medium method, an alternating Schwarz method and others [2, 4, 5]. In this work, the application of Schwarz method is considered (a version of decomposition method – see [6, p. 412] and [7] for a general case).

The essence of the method is as follows: at the first step, on the contact surfaces of the bodies, the initial approximation for components of the vector of displacements is set (the approximation is chosen from the range of expected values for the region of contact interaction). After solving this task, kinematic condition (5) on the contact surface is satisfied, but the calculated contact pressures on the opposite contact surfaces of the interacting bodies are not equal (condition (6) is violated). At the next step, by means of in a specially executed correction, we can obtain equal contact stresses but the obtained displacements do not satisfy condition (5). Further, at the next iteration step, the corrected kinematic conditions are again applied (combining the contacting surfaces). The stress and kinematic iterations are alternated to attain the convergence when this task, kinematic condition (5) on the contact surface is satisfied, but the calculated contact pressures on the opposite contact surfaces of the interacting bodies are not equal (condition (6) is violated). At the next step, by means of in a specially executed correction, we can obtain equal contact stresses but the obtained displacements do not satisfy condition (5). Further, at the next iteration step, the corrected kinematic conditions are again applied (combining the contacting surfaces). The stress and kinematic iterations are alternated to attain the convergence when both kinematic (5) and stress conditions (6) on the contact surface are satisfied with a prescribed accuracy. This method is described in more detail in [5, 8, 9]. Thus, the alternating Schwarz method is an iterative method and its essence within the finite element technology is as follows.

At even iterations, the components of vectors of displacements of contact clusters \{\(U_k\)\}(A) and \{\(U_k\)\}(B) of bodies A and B is corrected. For body A, the correcting expressions have a form

\[ \{U_k\}^{2n}_{(A),m} = \{U_k\}^{2n-1}_{(A),m} + \alpha^{2n-1}_{(A),m}(\{U_k\}^{2n-1}_{(B),s} - \{U_k\}^{2n-1}_{(A),m}), \quad n = 1, 2, \ldots, \tag{9} \]

for some initial displacement. Here \(\alpha^{2n-1}_{(A),m}\) is an iteration parameter, \(m (1 \leq m \leq M_A)\) is the number of the current node lying on the contact surface \(S_k^A\) of body A, \(M_A\) is the number of contact nodes on a surface \(S_k^A\). \(\{U\}^{2n-1}_{(B),s}\) is the displacement vector of the corresponding point \(s\) lying on the contact surface \(S_k^B\) of body B.

Similar relations are used to correct the components of displacement vectors of contact nodes of body B. Further, we solve two similar tasks of the elastic theory where the vectors \{\(U_k\)\}(A) and \{\(U_k\)\}(B) also play the role of additional kinematic boundary conditions.

For the Schwarz method used for odd iterations, the vector of global loading is the sum of two parts: the vector \{\(R_{nk}\)\}, which describes the impact of all forces on a body except for the contact ones, and the vector of nodal contact forces \{\(R_k\)\} in which only the components corresponding to the clusters on the contact surface are nonzero.

The calculation of components of nodal forces \{\(R_k\)\}(A) and \{\(R_k\)\}(B), arising at contact nodes of bodies A and B, is also based on the assumption that, for any kinematic restriction imposed on the surface site, there is a stress loading such that it makes an equivalent impact on the body. Therefore, after carrying out the kinematic iteration with number \(2n\), the values \{\(R_k\)\}^{2n}_{(A),m} and \{\(R_k\)\}^{2n}_{(B),m} can also be calculated by the formula (\(j\) is the global number of the variable corresponding to the node on the contact interface with number \(m\)):

\[ \{R_k\}^{2n}_{j} = [K]_j\{U\}^{2n} - \{R_{nk}\}^{2n}_{j}. \tag{10} \]
In the simplest case of the stress conditions, corrections of nodal vectors of forces \( \{R_k\}_{(A)} \) and \( \{R_k\}_{(B)} \) are calculated. For body \( A \), the correction has the form

\[
\{R_k\}_{(A),m}^{2n+1} = \{R_k\}_{(A),m}^{2n} - \alpha_{(A),m}^{2n} (\{R_k\}_{(B),s}^{2n} + \{R_k\}_{(A),m}^{2n}), \quad n = 0, 1, 2, \ldots \quad (11)
\]

Here \( \alpha_{(A),m}^{2n} \) is the iteration parameter, \( m \) (\( 1 \leq m \leq M_A \)) is the number of the current node on the contact surface \( S_k^A \) of body \( A \), \( \{R_k\}_{(B),s}^{2n} \) is the vector of contact nodal force at the point \( s \) corresponding to the node \( m \) and lying on the contact surface \( S_k^B \) of body \( B \). A similar relation is used to correct the components of vectors of the nodal forces arising at contact nodes of body \( B \).

The vectors \( \{R_k\}_{(A)}^{2n+1} \) and \( \{R_k\}_{(B)}^{2n+1} \) are also used to form the global vectors of nodal loading \( \{R\}_{(A)} \) and \( \{R\}_{(B)} \) for bodies \( A \) and \( B \). Then matrix equation (8) is solved for each of the two considered bodies.

Convergence of the described iteration scheme is considered in [8, 9].

The use of relations (11) for grids whose nodes on the contact surfaces of the two bodies coincide or are close to each other results in rather good results. But in the case of distinct nodes, especially when the grid steps are significantly different for the two bodies, the immediate application of (11) significantly worsens the convergence of the repetition process (when the second or higher order FE are used, the situation becomes even worse).

Because, in reality, not the concentrated forces carrying to grid clusters but the distributed contact forces operating on all contact boundaries act on the contacting sites of the surfaces of two bodies (by analogy with the vector of surface forces), the following expression can be written for the \( j \)th component of the vector \( \{R_k\} \):

\[
\{R_k\}_j = \int_{S_k} N_j^S p_k(x) \, dS,
\]

where \( N_j^S \) is the basic function defined on a surface and corresponding to the node with number \( j \) and \( p_k(x) \) is the contact pressure.

It is obvious that, when using distinct grids for stress iterations, it is necessary to correct not the values of nodal forces \( \{R_k\}_{(A)} \) and \( \{R_k\}_{(B)} \), which experience not only the action of pressure, and integrals of basic functions, but the values of contact pressure \( p_k^{(A)} \) and \( p_k^{(B)} \) in the corresponding clusters of the grid.

Further, we consider various realizations of the Schwarz method, which differ only in relations for stress iterations (for kinematic ones, the displacements are calculated by formula (9)).

The first realization of the Schwarz method is based on the following steps:

1. We replace the unknown contact pressure in (12) with an approximation based on use of basic functions \( N_j^S \) which are falling into clusters of the surface grid (for the first-order FE, they are piecewise linear functions)

\[
p_k(x) \approx \sum_{m=1}^M p_{k,m} N_m^S, \quad (13)
\]

where \( M \) is number of the nodes which are on the considered surface of contact.

2. After each kinematic iteration with number \( 2n \) (\( n = 0, 1, 2, \ldots \)), we determine the values of components of the vector of nodal forces \( \{R_k\}_{m}^{2n}, m = 1, \ldots, M \).
3. Substituting (13) in (12), we obtain a system of \( M \) equations for \( M \) unknown pressures

\[
\int_{S_k} [N^S]^T [N^S] dS \{p_k\}^{2n} = \{R_k\}^{2n}.
\] (14)

Solving (14), we obtain the values of contact pressures after kinematic iteration. These pressures do not satisfy condition (6).

4. We carry out corrections of the received pressures by a formula similar to (11):

\[
\{p_k\}^{2n+1} = \{p_k\}^{2n} - \sigma^{2n} \{p_k\}^{2n} \{p_k\}^{2n}(B)_{s} + \{p_k\}^{2n}(A)_{m}.
\] (15)

5. Knowing the corrected pressures, we calculate new values of nodal forces:

\[
\{R_k\}^{2n+1} = \int_{S_k} [N^S]^T [N^S] dS \{p_k\}^{2n+1}.
\] (16)

6. Considering the obtained vector of nodal forces in the vector of global loading, we solve matrix equation (8) for each body.

Further, the method defined by points 1–6 will be called the “first method.”

In [10], a generalization of the given version of the Schwarz method to the case of multi-contact interaction is considered. The influence of the choice of an initial approximation on the speed of convergence of the iteration process is investigated. For a particular class of problems, a modification of the Schwarz method is proposed which allows one to receive an initial approximation by solving a number of auxiliary problems by direct consideration of various loadings acting on the bodies.

4. Application of the Schwarz method to the problem of contact of a large number of bodies

The numerical algorithms described above are realized as computational procedures which, as a component of the central computing core, enter the prototype of the integrated program platform TEMETOS for carrying out computing experiments in complex problems of mathematical model operation [12].

The authors executed a number of calculations in which some thermo-mechanical effects occurring in a fuel element (TVEL) [10, 11] are simulated.

For example, for calculating the SSD parameters in TVEL at the exit to a rated duty, the following problem is considered: there is a thick-walled pipe (TVEL cladding) containing a column of identical cylinders put over each other; the cylinders have an internal opening and flats on both end faces (fuel pellets). In the cylinders, the uniform thermal emission obeying the following law is assumed: there is a linear rise in power of thermal emission in time till the limit value of \( q_{lmax} \), and then the power remains constant. It is necessary to maintain the temperature of the external surface of the pipe at a constant value \( T_1 \). Between the external surfaces of the cylinders and the internal surface of the pipe, there is a heat exchange. The lower end face of the pipe and the lower end face of the lower cylinder are fixed. On the external surface of the pipe, the constant pressure \( p_1 \) is set. On the upper face of the top cylinder, the constant pressure \( p_2 \) is set.

The definition of the problem allows one to use a modified version of the Schwarz method. In the course of solving the problem, the dynamic temperature problem is solved first, and then the received temperature fields are used to solve a quasi-stationary equilibrium equation.

In [10], a series of calculations in the axially symmetric statement, where the number of cylinders varies from several tens to several hundreds (as contact conditions, the sliding
conditions without sliding friction are chosen) is described. At an initial instant, there was a gap between the cylinders and pipe. Then the increase in the body temperature strains was a result of the contact between the cylinders and the pipe. The iterations stopped when the maximal relative change of displacements (compared with the two next iterations) did not exceed 1%. To achieve a similar accuracy, it is required, on the average, 10–15 iterations at one step in time (for any number of cylinders).

Let us note that, in these problems, the total system of linear equations for each body was solved self-contained which allowed one significantly to reduce the calculation time.

A similar problem is also solved in the three-dimensional statement (the number of cylinders varied from two to ten). In [11], the results of three-dimensional calculations for 4 pellets are given. The case of an axially symmetric loading where the calculation area corresponds to the sector $90^\circ$ is therefore considered. As the contact conditions, the sliding condition with sliding friction is chosen.

The obtained results are compared with the results of two-dimensional (axisymmetric) calculations by means of the ANSYS code by using the Lagrange multiplier method. The comparison showed a rather good qualitative and quantitative compliance between the considered sizes.

Figure 1 shows the distribution of radial and axial stresses for the three-dimensional calculation area consisting of two cylinders and a pipe at one of the time instants.

At the moment, the used version of the Schwarz method has an essential drawback: it cannot be applied directly if there are cells of the surface grid which are only in partial contact (this will be called “partial contact of a cell”). In this case, for some clusters located on the boundaries of the contact surface, it is more difficult to interpret the concept of “corresponding point,”

---

**Figure 1.** Radial (a) and axial (b) stresses.
which is of key significance for formulas (9), (11). In the course of heating, considerable axial displacements are observed on the cylinders, and similar situations with partial contact of cells arise constantly. To resolve this problem, a local reorganization of the grid in the pipe near the boundaries of the surface of contact between the pipe and each of cylinders was carried out at each iteration step. Because of such reorganizations of the grid, it was possible to avoid problems with partial contact of cells, but there were situations where the grid steps in special subareas from the cylinders and from the pipe differed significantly (sometimes by more than twice).

At the same time, in all carried-out calculations (three-dimensional and two-dimensional), there were stress concentrators near the contact of the corners of the cylinder facet and the pipe. The numerical values of contact pressure in these areas are very sensitive to the used calculation grid, and therefore such differences in the grid steps led to oscillations on the curves of contact pressure and decelerated the convergence of the iteration process.

These specified problems led to the comprehension that it is necessary to develop a more perfect realization of the Schwarz method. The main ideas which are the cornerstone of new version are explained below.

5. New version of the Schwarz method
The disadvantages of the previous version of the Schwarz method can be explained by the fact that when the grid is too rough for describing the contact pressure, the total force applied to the contact surface of the first body may differ from the total force applied to the contact surface of the second body, which leads to deceleration of the iteration process convergence.

To avoid this situation, it is possible to supplement the applied algorithm with the requirement that the forces acting on the contacting surfaces coincide. This restriction can be considered as follows: we divide the contact surface of each body into non-intersecting sites and, for the power iteration, demand that the equality hold not for the values of pressure at grid nodes but for the values of contact forces obtained after integration of the pressure on each of such sites for one body and for the corresponding site of the other body.

Let us note that a similar approach potentially allows one to solve a problem with partial contact of cells if, instead of the term “corresponding point,” one uses the more universal concept “corresponding site.”

Let us describe one of possible realizations of the given algorithm:

1. We replace the unknown contact pressure in (12) with an approximation based on use of basic functions \( \chi^S \) which fall into the surface grid but optionally coincide with the basic functions \( N^S \):

\[
p_k(x) \approx \sum_{m=1}^L p_{k,m} \chi^S_m,
\]

(17)

where \( L \) is the number of introduced basic functions. We see that the contact surface of the considered body consists of \( L \) non-intersecting sites \( S_k = \bigcup_{j=1}^L S_{k,j} \).

2. After each kinematic iteration with number \( 2n \) \((n = 0, 1, 2, \ldots)\), we find the values of components of the vector of nodal forces \( \{R_k\}_m^{2n}, m = 1, \ldots, M \).

3. We obtain the system of \( M \) equations for \( L \) unknown pressures

\[
\int_{S_k} [N^S]^T [\chi^S] dS \{p_k\}^{2n} = \{R_k\}^{2n}.
\]

(18)

Solving (18), we find the values of the contact pressure after kinematic iteration (if \( L \) does not coincide with \( M \), then it is necessary to use the methods focused on solving systems with a rectangular matrix). The obtained pressure does not satisfy the stress condition (6).
4. We calculate the contact forces corresponding to each site $S_{k,j}$:

$$P_{2n}^{k,j} = \sum_i \int_{S_{k,j}} \chi_i^S dS p_{2n}^{k,i}. \quad (19)$$

5. We adjust the received contact forces by the formula similar to (11):

$$\{P_k\}^{2n+1}_{j} = \{P_k\}^{2n}_{j} - \alpha^{2n}_{j} \{P_k\}^{2n}_{s} \{P_k\}^{2n}_{j} + \{P_k\}^{2n}_{j}. \quad (20)$$

Here $\{P_k\}^{2n}_{s}$ is contact force calculated on the site of the surface of body $B$ which corresponds to the considered part $S_{k,j}$ of body $A$.

6. After calculation of the contact forces, we determine the contact pressure. For this, it is necessary to solve a set of equations:

$$\sum_i \int_{S_{k,j}} \chi_i^S dS p_{2n+1}^{k,i} = P_{2n+1}^{k,j}, \quad j = 1, \ldots, L. \quad (21)$$

7. When the corrected pressures are known, we calculate new values of the nodal forces:

$$\{R_k\}^{2n+1} = \int_{S_k} [N^S]^T \chi^S dS \{p_k\}^{2n+1}. \quad (22)$$

8. Considering the obtained vector of nodal forces as the vector of global loading, we solve for each body the matrix equation (8) for each body.

It is possible to specify functions, piecewise constant on the surface elements, as one of possible choices of basic functions $\chi^S$. Then $L$ is the number of surface elements which are in contact ($L$ differs from $M$, which is the number of nodes in contact).

For further application, we consider the case $L = M$. As non-intersecting sites of the contact surface $S_{k,j}$, we take the Dirichlet cells corresponding to nodes of the surface grid. As basic functions $\chi^S$, we choose the functions $N^S$ (for FE of first order, they are piecewise linear functions).

Further, the method given by points 1–8 is called the “second method.”

6. Comparison of various versions of the Schwarz method for a test problem

Let us analyze the results of calculations with application of two realizations of the Schwarz method described above for the following two-dimensional test problem used as an example: the second body of width 6 and height 2 rests on the first body of height 4 and width 9. On the left side, the bodies are fixed across, and the lower body is fixed down. The upper body is affected by the distributed load given by the formula $p(x) = p_0 [1 - \cos(2\pi x/l)]$, where $p_0 = 10$, $l = 3$.

As the conditions on the contact surface, we choose the adhesion condition, and therefore the solution of the considered contact problem must coincide with the solution of the problem where the considered design is a uniform body. Therefore, to estimate the accuracy of the obtained results, it is advisable to carry out comparison with the calculations corresponding to the case of one body under the same loading.

In all calculations, the FE of first order were applied. All values are given in dimensionless form.

The chosen initial approximation can affect the speed of convergence of the Schwarz method. But for the problems considered in this work, this influence is rather restrictive, and therefore, in all calculations, the following values of displacements on all surfaces of contact were chosen as the initial approximation: $u_x = 0$, $u_y = -10^{-4}$. 

8
Let us consider the numerical results obtained for the solution by using two methods on various grids. The maximal relative errors of the numerical solution of the contact problem (compared with the case of one body) after 50 iterations (stress iteration) are presented in table 1. Calculations with the following steps of the grid are considered: \( h_1 = 0.25 \) and \( h_2 = 0.25 \), \( h_1 = 0.125 \) and \( h_2 = 0.25 \) (the grid steps differ by a factor of two), and \( h_1 = 0.125 \) and \( h_2 = 0.375 \) (the grids steps differ by a factor of three).

To estimate the accuracy of the numerical solution of the contact problem, the following values were considered: the stresses \( \sigma_{xx} \) and \( \sigma_{yy} \) obtained near the contact boundary (at the center of the cells located along the contact surface of the first body), and the displacements \( (u_x, u_y) \) at the nodes located on the contact boundary (for both bodies). The number of a body is given in brackets. These values were compared with the corresponding values obtained in calculations for one body with \( h \) step (for the similar section).

For calculations where a uniform grid with an identical step was used in both bodies, the results obtained by different methods are almost identical. They coincide with the values obtained in calculations for one body with a relative accuracy of about \( 10^{-5} \). When the grid steps differ rather strongly, then the results of calculations become different. In calculations on similar grids, all considered sizes are calculated with a significantly smaller accuracy. Moreover, one can observe that a larger discrepancy of displacements on the contact surfaces of two bodies is essential (after the power iteration): the previously obtained relative error of displacements (when comparing the values on the surfaces of the first and second body) was approximately \( 10^{-5} \), but the displacements obtained now differ by percents. At the same time, the further decrease in the accuracy is not observed: at the power and kinematic iterations, we have two sufficiently steady solutions, which do not coincide with each other. At the power iterations, the nodal values of the contact pressure coincide, but there is an overlap of the displacements. At the kinematic iterations, the displacements coincide, but there is a gap of the contact pressure.

For descriptive reasons, we present the curves of displacements and stresses after 50 iterations for the maximal difference of steps of grids \( (h_1 = 0.125, h_2 = 0.375) \). Figure 2 shows the curves of displacement of \( u_x \) and \( u_y \) on the contact surfaces of both bodies calculated by various methods (curves with markers) and calculated for one body (the grid step is \( h = 0.125 \), curves without markers). The curves show that, for the first method, on the given grids at the considered power iteration, there is a very big divergence of displacements (for \( u_x \), the discrepancy attains 25%, and for \( u_y \), 7%). For the second method, the gap of displacements at the power iteration, is significantly less.
Figure 2. Components of the displacement vector: (a), (b) — “first method”; (c), (d) — “second method”.

Figure 3 shows the curves of stresses ($\sigma_{xx}$, $\sigma_{yy}$) near the contact boundary of the first body for calculations by various contact methods (curves with markers) and for calculations for one body (the grid step is $h = 0.125$, curves without markers). For the considered problem, on the contact surface, there is a singular point, the edge of the upper body. As the grid becomes finer, the contact pressure grows at this point without bounds. Therefore, the stresses calculated in a neighborhood of this point are most sensitive to the grid steps. Therefore, in cases where grids with various steps are constructed, the greatest divergence of the solutions is observed.

Proceeding from the above analysis of the obtained data, it is possible to conclude that, for the considered test problem where significantly different grids were used, the second method allows one to obtain more precise results. Its application for the multi-contact problem considered in section 3 is planned in the near future.
Figure 3. Components of the stress tensor: (a), (b) — “first method”; (c), (d) — “second method”.

Conclusions
An algorithm for numerical solution of the contact problem for a system of interacting elastic bodies by using the alternating Schwarz method is explained. The results of application of this method for solving a multidimensional problem of contact interaction between a large number of bodies are given. A new version of the Schwarz method realization, which is focused on the case where the grid steps are significantly different in different bodies is proposed. An example of a two-dimensional test problem of contact is used to compare two methods for a series of calculations with more and more differing grids. It is shown that the new method allows one to obtain better results compared with the first.

Acknowledgments
The work was partially supported by the Russian Foundation for Basic Research (project No. 15-01-03073).

The authors thank professor I. V. Stankevich for great help in formulating the problem and choosing methods for solving it.
References

[1] Bourago N G and Kukudzhanov V N 2005 A review of contact algorithms Mech. Solids 40 (1) 35–71
[2] Zienkiewicz O 1975 The Finite Element Method in Engineering Science (New York: McGraw-Hill)
[3] Zarubin V S and Stankevich I V 2005 Computations of Thermal Stress Structures (Moscow: Mashinostroenie) p 352 [in Russian]
[4] Bathe K-J 1996 Finite Element Procedures (New York: Prentice Hall)
[5] Stankevich I V, Yakovlev M E, and Si Tu Khtet 2011 Development of contact interaction algorithm on the basis of Schwarz alternating method Vestnik MGTU im. N.E. Baumana. Estestv. Nauki No. S 134–41
[6] Marchuk G I 1989 Methods of Numerical Mathematics (Moscow: Nauka) p 608 [in Russian]
[7] Toselli A and Widlund O 2005 Domain Decomposition Methods – Algorithms and Theory (Berlin - Heidelberg - New York: Springer) p 450
[8] Tsvik L B 1978 Principle of alternating continuity in piecewise solutions of problems in field theory Sov. Phys. Dokl. 23 824–6
[9] Tsvik L B 1980 Principle of alternation in problems of the conjugation and contact of solid deformable bodies Int. Appl. Mech. 16 (1) 9–13
[10] Galanin M P, Lukin V V, Rodin A S, and Stankevich I V 2015 Application of the Schwarz alternating method for simulating the contact interaction of a system of bodies Comput. Math. Math. Phys. 55 (8) 1393–406
[11] Galanin M P, Krupkin A V, Kuznetsov V I, et al. 2016 Modeling of contact interaction of a thermoelastic body system using Schwartz method for a multidimensional case Izv. VUZov. Mashinostroenie No. 12 9–20
[12] Galanin M P, Gorbunov-Possadov M M, Ermakov A V, et al. 2014 Prototype of an integrated software platform for tracking computer simulations to solve complex problems of mathematical modeling Trudy ISP RAN 26 (3) 51–68