Lax pairs and Fourier analysis: The case of sine-Gordon and Klein-Gordon equations

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Abstract. In this paper we construct a new Lax pair for the Klein-Gordon equation. The structure algebra of this Lax pair is the algebra $T_{\mathfrak{a}}^{2}$ of upper triangular Toeplitz block matrices with $\mathfrak{u}(2)$ blocks. For the suitable choice of the values of the spectral parameter, the integrals of motion, obtained from the holonomy of the spatial part of the Lax pair, have simple expressions in terms of the Fourier data. We compare these integrals to the corresponding integrals of the sine-Gordon system.

1. Introduction

The inverse scattering method is often referred to as the non-linear version of the strategy of solving the partial differential equations by means of the Fourier transform. Often this comparison is given in somewhat vague terms and it is not even meant to be taken too literally. On the other hand, there exists a substantial body of deep and important results on non-linear versions of Fourier transform, applied to the study of integrable systems (see e.g. [4], [5], [6], [7]). In this note we will describe a case of a simple and concrete connection between the zero-curvature condition and Fourier analysis. We shall consider the periodic sine-Gordon equation and its linearization at the vacuum solution, the periodic Klein-Gordon equation. These equations are equations of motion of two Hamiltonian systems $(H_{sg}, \{ -, - \}, \mathcal{M})$ and $(H_{kg}, \{ -, - \}, \mathcal{M})$. The phase space of both systems is

$$\mathcal{M} = \{(q(x), p(x)) ; q(x), p(x) : S^1 \to \mathbb{R}\}$$

and the Poisson bracket is given by

$$\{F, G\} = \int_0^{2\pi} \left( \frac{\delta F}{\delta q(x)} \frac{\delta G}{\delta p(x)} - \frac{\delta F}{\delta p(x)} \frac{\delta G}{\delta q(x)} \right) dx.$$ 

The Hamiltonian $H_{sg}$ of the sine-Gordon system is given by

$$H_{sg}(q(x), p(x)) = \int_0^{2\pi} \left( \frac{1}{2} p^2(x) + \frac{1}{2} q_x^2(x) - (\cos q(x) - 1) \right) dx$$

and the one for the Klein-Gordon system is

$$H_{kg}(q(x), p(x)) = \int_0^{2\pi} \left( \frac{1}{2} p^2(x) + \frac{1}{2} q_x^2(x) + q^2(x) \right) dx.$$
The sine-Gordon system is a well-known integrable system. In particular, the sine-Gordon equation is equivalent to the zero-curvature condition for the Lax pair \((L(z), A(z))\) given in (6) below. Here \(z\) denotes the spectral parameter.

The main results presented in this paper are the following:

(i) By means of a perturbation method we construct a new Lax pair with the spectral parameter \((L(z), A(z))\) for the Klein-Gordon system. The structure group of this Lax pair is the Lie group, corresponding to the Lie algebra \(T\mathfrak{a}_2\) of the upper-triangular block Toeplitz matrices with \(su(2)\) blocks. Our Lax pair is given in formula (11).

(ii) The integrals of the Klein-Gordon system, yielded by the Lax pair \((L(z), A(z))\), have simple expressions in terms of the Fourier coefficients of the elements from the phase space \(\mathcal{M}\).

More concretely, let \(H(q(x), p(x); z)\) be the holonomy of \(L(q(x), p(x))(z)\) evaluated along the loop \(x \mapsto (x, 0)\). Let \(\mathcal{F}\) be an \(Ad\)-invariant function on the Lie algebra \(T\mathfrak{a}_2\). Then, as usual, the function
\[
F_z : \mathcal{M} \longrightarrow \mathbb{R}
\]
given by
\[
F_z(q(x), p(x)) = \mathcal{F}(H(q(x), p(x); z))
\]
(1)
is a conserved quantity of the Klein-Gordon system for every value of the spectral parameter \(z\).

Now, let us develop \(q(x)\) and \(p(x)\) into the Fourier series
\[
q(x) = \sum_{n=-\infty}^{\infty} \alpha_n e^{i n x}, \quad p(x) = \sum_{n=-\infty}^{\infty} \beta_n e^{i n x}.
\]
For every integer \(n \in \mathbb{Z}\) let the value \(z_n\) of the spectral parameter be given by
\[
z_n = -n + \sqrt{n^2 + 1}.
\]
Then from (1) we get the integrals
\[
F_{z_n}(q(x), p(x)) = |i \omega_n \alpha_n + \beta_n|^2, \quad \text{where} \quad \omega_n = \sqrt{n^2 + 1}.
\]
(2)

These results are proved in sections 2 and 3. In section 4 we describe a relationship between the integrals \(F_{z_n}\) of the Klein-Gordon system and the corresponding integrals of the sine-Gordon system in some detail. Sections 2 and 3 provide a shorter and more easily readable presentation of the results that appeared in [10]. The discussion in section 4 is new.

2. Jet sine-Gordon system and its Lax pair
The key ingredient in the proof of the above claims is the construction of the so-called jet sine-Gordon system. Let
\[
q(x, t; s) : S^1 \times [0, T] \times (-\epsilon, \epsilon) \longrightarrow \mathbb{R}
\]
be a path of solutions of the periodic sine-Gordon equation
\[
\begin{align*}
qu_t - q_{xx} = -\sin q \\
q(x + 2\pi, t) = q(x, t), \quad q_x(x + 2\pi, t) = q_x(x, t).
\end{align*}
\]
(3)
Suppose that we have \( q(x, t; 0) \equiv 0 \), that is, the path starts at the vacuum solution. Consider the power series development
\[
q(x, t; s) = \sum_{n=1}^{k} \frac{s^n}{k!} q^{(n)}(x, t) + \mathcal{O}(k + 1).
\]

Then the coefficients \( q^{(n)}(x, t) \) are a solution of the system of partial differential equations
\[
(q^{(n)})_{tt} - (q^{(n)})_{xx} = -q^{(n)} + f_n(q^{(1)}, \ldots, q^{(n-2)}), \quad n = 1, \ldots, k, \tag{4}
\]
where the functions \( f_n \) are given by
\[
\sin \left( sx_1 + \frac{s^2}{2!} x_2 + \ldots + \frac{s^k}{k!} x_k \right) = \sum_{n=1}^{k} (x_n - f_n(x_1, \ldots, x_{n-2})) s^n + \mathcal{O}(k + 1).
\]

The system (4) will be called the \( k \)-jet sine-Gordon system. We will show that this system is equivalent to a zero-curvature condition for a certain Lax pair with spectral parameter.

Let \( J^k_0 \mathcal{M} = \{(q^{(1)}(x), p^{(1)}(x)), \ldots, (q^{(k)}(x), p^{(k)}(x))\} \) be the space of \( k \)-jets of paths \( s \mapsto (q(x; s), p(x; s)) \) at \( (0, 0) \) in the phase space \( \mathcal{M} \). The equations (4) can be rewritten in the form of the system
\[
\begin{pmatrix}
(q^{(n)})_t \\
p^{(n)}
\end{pmatrix}_t = \begin{pmatrix}
(q^{(n)})_{xx} - q^{(n)} + f_n(q^{(1)}, \ldots, q^{(n-2)}) \\
p^{(n)}
\end{pmatrix}, \quad n = 1, \ldots, k \tag{5}
\]
whose solutions are paths \( t \mapsto j^k(q, p)(t) = [(q^{(1)}(x, t), p^{(1)}(x, t)), \ldots, (q^{(k)}(x, t), p^{(k)}(x, t))] \) in the jet space \( J^k_0 \mathcal{M} \).

Recall that the equation of motion (3) for the sine-Gordon system \( (H_{sg}, \{-,-\}, \mathcal{M}) \) is equivalent to the zero-curvature condition
\[
\left( L(q, p)(z) \right)_t - \left( A(q, p)(z) \right)_x + \left[ L(q, p)(z), A(q, p)(z) \right] = 0
\]
for the Lax pair
\[
L(q, p)(z) = i \begin{pmatrix}
\frac{z^2 + 1}{4z} \cos \frac{z}{2} & \frac{z^2 - 1}{4z} \sin \frac{z}{2} \\
\frac{z^2 - 1}{4z} \sin \frac{z}{2} & \frac{z^2 + 1}{4z} \cos \frac{z}{2}
\end{pmatrix}
\]
\[
A(q, p)(z) = i \begin{pmatrix}
\frac{z^2 - 1}{4z} \cos \frac{z}{2} & \frac{z^2 + 1}{4z} \sin \frac{z}{2} \\
\frac{z^2 + 1}{4z} \sin \frac{z}{2} & \frac{z^2 - 1}{4z} \cos \frac{z}{2}
\end{pmatrix}
\]  \tag{6}

This Lax pair is slightly different from the usual one, found for instance in [3]. It can be derived from the Lax pair for the Maxwell-Bloch equation, given in [8] and [9]. Consider now a path \( t \mapsto j^k(q, p)(t) = [(q^{(1)}(x, t), p^{(1)}(x, t)), \ldots, (q^{(k)}(x, t), p^{(k)}(x, t))] \) in the jet space \( J^k_0 \mathcal{M} \). Choose a map
\[
(t; s) \mapsto (q(s), p(s)) = (q(x, t; s), p(x, t; s)) \in \mathcal{M}, \quad (q(x, t; 0), p(x, t; 0)) \equiv (0, 0)
\]
such that \( \frac{d^n}{ds^n}|_{s=0}(q(x, t; s), p(x, t; s)) = (q^{(n)}(x, t), p^{(n)}(x, t)) \) for \( n = 1, \ldots, k \). For every \( s_0 \) formulae (6) assign the Lax pair \( (L(q(s_0), p(s_0))(z), A((q(s_0), p(s_0)))(z)) \) to the path \( t \mapsto (q(x, t; s_0), p(x, t; s_0)) \) in \( \mathcal{M} \). Let us denote
\[
L^{(n)}(z) = \frac{d^n}{ds^n}|_{s=0} L(q(s), p(s))(z), \quad A^{(n)}(z) = \frac{d^n}{ds^n}|_{s=0} A(q(s), p(s))(z), \quad n = 1, \ldots, k.
\]
We now arrange the above matrices into two upper triangular block-Toeplitz matrices with $su(2)$ blocks:

$$
\mathcal{L} = \begin{pmatrix}
L^{(0)} & L^{(1)} & \cdots & \frac{1}{k!} L^{(k)} \\
0 & L^{(0)} & \cdots & \frac{1}{(k-1)!} L^{(k-1)} \\
0 & 0 & \cdots & \frac{1}{(k-2)!} L^{(k-2)} \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & L^{(0)}
\end{pmatrix},
A = \begin{pmatrix}
A^{(0)} & A^{(1)} & \cdots & \frac{1}{k!} A^{(k)} \\
0 & A^{(0)} & \cdots & \frac{1}{(k-1)!} A^{(k-1)} \\
0 & 0 & \cdots & \frac{1}{(k-2)!} A^{(k-2)} \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & A^{(0)}
\end{pmatrix}.
$$

(7)

The above construction associates a pair $(\mathcal{L}, A)$ to every path $t \mapsto j^k(q(x, t), p(x, t))$ in the jet space $J^k_0 \mathcal{M}$. It is easily seen that upper triangular block-Toeplitz matrices with $su(2)$ blocks form a Lie algebra. We shall denote this Lie algebra by $\mathcal{T}A_k$. The proof of the following proposition is rather straightforward.

**Proposition 1** A path $t \mapsto j^k(q, p)$ in the jet space $J^k_0 \mathcal{M}$ is a solution of the system (5) if and only if the associated Lax pair $(\mathcal{L} j^k(q, p)(z), A j^k(q, p)(z))$ satisfies the zero-curvature condition

$$
(\mathcal{L} j^k(q, p)(z))_t - (A j^k(q, p)(z))_x + [\mathcal{L} j^k(q, p)(z), A j^k(q, p)(z)] = 0.
$$

When a system of differential equations has a zero-curvature formulation, one expects it to have a large number of conserved quantities. These stem from the $Ad$-invariant functions of the structure group. In our case the structure group is the Lie group $\mathcal{T}G_k$ of the Lie algebra $\mathcal{T}A_k$. The group $\mathcal{T}G_k$ is obtained from the algebra $\mathcal{T}A_k$ by exponentiation. Therefore it is clear that the nontrivial $Ad$-invariant functions on $\mathcal{T}G_k$ do not come from the spectra of the elements. The following proposition is proved in [10].

**Proposition 2** For every $n = 1, \ldots, k$ the function

$$
\varphi_n: \mathcal{T}G_k \rightarrow \mathbb{R}
$$

given by

$$
\varphi_n(g) = n! \sum_{j=0}^n \text{tr}(g_j \cdot g_{n-j}),
$$

where

$$
g = \begin{pmatrix}
g_0 & g_1 & g_2 & \cdots & g_k \\
0 & g_0 & g_1 & \cdots & g_{k-1} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & g_0
\end{pmatrix} \in \mathcal{T}G_k,
$$

(8)

is an $Ad$-invariant function.

The essential ingredient of the proof is the fact that the matrix Lie algebra $\mathcal{T}A_k$ is isomorphic to the algebra $J^k su(2)$ whose elements are paths in $su(2)$ of the form

$$
j^k \alpha = \alpha^{(0)} + s \alpha^{(1)} + \ldots + \frac{s^k}{k!} \alpha^{(k)}, \alpha^{(n)} \in su(2)
$$

and whose bracket is given by the truncated polynomial multiplication combined with the bracket on $su(2)$,

$$
[j^k \alpha, j^k \beta] = \sum_{n=0}^k \frac{s^k}{k!} \left( \sum_{l=0}^n \binom{n}{l} [\alpha^{(l)}, \beta^{(n-l)}] \right).
$$
Proposition 2 enables us to compute the conserved quantities of the jet sine-Gordon system quite easily. Let \( j^k(q,p) \in J^k_0M \) be an arbitrary element in the jet space. The holonomy \( H_{j^k(q,p)}(z) \) of \( \mathcal{L}_{j^k(q,p)}(z) \) is given by \( H_{j^k(q,p)}(z) = \Phi(2\pi; z) \), where \( \Phi(x; z) \) is the solution of the initial problem
\[
(\Phi(z))_x = \mathcal{L}_{j^k(q,p)}(z) \cdot \Phi(z), \quad \Phi(0; z) = Id.
\]
Let the blocks of the upper triangular Toeplitz block matrix \( \Phi(x; z) \) be denoted by \( \Phi^{(n)}(x; z) \), \( n = 1, \ldots, k \). In terms of blocks the above initial problem is given by the system
\[
(\Phi^{(n)})_x = L(0)\Phi^{(n)} + \sum_{j=1}^{n} \binom{n}{j} L^{(j)} \cdot \Phi^{(n-j)}, \quad \Phi^{(0)}(0; z) = Id, \quad \Phi^{(n)}(0; z) = 0, \quad n \geq 1. \quad (9)
\]
This is a system of non-homogeneous linear equations. They all have the same homogeneous part \( \Phi_x = L(0)\Phi \) with the constant coefficient matrix \( L(0) \). The non-homogeneity of \( n \)-th equation depends only on the solutions of the previous \((n-1)\) equations. Therefore, we can successively compute the blocks \( \Phi^{(n)}(x) \) and, by evaluating at \( x = 2\pi \), we obtain all the blocks \( H^{(n)}(z) \) of the holonomy \( H_{j^k(q,p)}(z) \). From proposition 2 it then follows that for every value \( z \) of the spectral parameter the functions
\[
F_z : J^k_0M \longrightarrow \mathbb{R},
\]
given by
\[
F_z(j^k(q,p)) = \varphi_n(H_{j^k(q,p)}(z)) = \sum_{j=1}^{n} \binom{n}{j} \text{tr}(H^{(j)}(z) \cdot H^{n-j}(z)), \quad n = 1, \ldots, k, \quad (10)
\]
are integrals of the jet sine-Gordon system.

Even though the jet sine-Gordon system has many conserved quantities, it is not difficult to see that it is not integrable. Namely, the conserved quantities do not form a complete system.

3. Lax pair and integrals of the Klein-Gordon system
Let us now consider the jet sine-Gordon system \( (5) \) on \( J^k_0M \), that is, on the space of 2-jets. The equation of motion of this system is given simply by two uncoupled Klein-Gordon equations
\[
(q^{(1)})_{tt} - (q^{(1)})_{xx} = -q^{(1)},
\]
\[
(q^{(2)})_{tt} - (q^{(2)})_{xx} = -q^{(2)}.
\]
The 2-jet sine-Gordon system is the Hamiltonian system \( (K, \{ - , - \}_p, M \times M) \), where \( \{ - , - \}_p \) is the usual product Poisson bracket on the Cartesian product phase space \( M \times M \), and the Hamiltonian \( K \) is given by
\[
K(\{(q^{(1)}, p^{(1)}), (q^{(2)}, p^{(2)})\}) = H_{kg}(q^{(1)}, p^{(1)}) + H_{kg}(q^{(2)}, p^{(2)}).
\]
A path \( t \mapsto [(q^{(1)}(x,t), p^{(1)}(x,t)), (q^{(2)}(x,t), p^{(2)}(x,t))] \) in \( J^k_0M = M \times M \) is a solution of the 2-jet system if and only if \( (q^{(i)}(x,t), p^{(i)}(x,t)) \) are solutions of the Klein-Gordon system for \( i = 1, 2 \). In particular, the path \( t \mapsto [(q^{(1)}(x,t), p^{(1)}(x,t)), (0, 0)] \) is a solution of the 2-jet system precisely when \( (q^{(1)}(x,t), p^{(1)}(x,t)) \) is a solution of the Klein-Gordon system.

The above remarks lead to the formulation of the following theorem.

**Theorem 1** Let \( t \mapsto (q(x,t), p(x,t)) \) be a path in the phase space \( M \). Let us associate to this path the \( \mathfrak{su}(2) \) matrix functions
\[
L_r^{(0)} = \begin{pmatrix} \frac{1 - z^2}{4z} & 0 \\ -i \frac{1 - z^2}{4z} & 0 \end{pmatrix}, \quad L_r^{(1)} = \begin{pmatrix} 0 & i \frac{1 - z^2}{8z} q + \frac{p}{4} \\ i \frac{1 - z^2}{8z} q - \frac{p}{4} & 0 \end{pmatrix}, \quad L_r^{(2)} = \begin{pmatrix} -i \frac{1 - z^2}{16z} q^2 & 0 \\ 0 & i \frac{1 - z^2}{16z} q^2 \end{pmatrix},
\]
\[ A_r^{(0)} = \begin{pmatrix} \frac{i + z^2}{4} & 0 \\ 0 & -\frac{i + z^2}{4} \end{pmatrix}, \quad A_r^{(1)} = \begin{pmatrix} 0 & \frac{i - z^2}{8} - \frac{q_p}{4} \\ \frac{i - z^2}{8} + \frac{q_p}{4} & 0 \end{pmatrix}, \quad A_r^{(2)} = \begin{pmatrix} -\frac{i + z^2}{16}q^2 & 0 \\ 0 & \frac{i + z^2}{16}q^2 \end{pmatrix}. \]

Let the block Toeplitz matrices \( \mathcal{L}(x,t;z) = \mathcal{L}(q(x,t),p(x,t))(z) \) and \( \mathcal{A}(x,t;z) = \mathcal{A}(q(x,t),p(x,t))(z) \) be given by

\[ \mathcal{L}(x,t;z) = \begin{pmatrix} L_r^{(0)} & L_r^{(1)} & \frac{1}{2} L_r^{(2)} \\ 0 & L_r^{(0)} & L_r^{(1)} \\ 0 & 0 & L_r^{(0)} \end{pmatrix}, \quad \mathcal{A}(x,t;z) = \begin{pmatrix} A_r^{(0)} & A_r^{(1)} & \frac{1}{2} A_r^{(2)} \\ 0 & A_r^{(0)} & A_r^{(1)} \\ 0 & 0 & A_r^{(0)} \end{pmatrix}. \] (11)

Then the path \( t \mapsto (q(x,t),p(x,t)) \) solves the Klein-Gordon equation if and only if the associated Lax pair \( (\mathcal{L},\mathcal{A}) \) defined above satisfies the zero-curvature condition

\[ \mathcal{L}_t - \mathcal{A}_x + [\mathcal{L},\mathcal{A}] = 0. \]

**Proof:** Let \((t,s) \mapsto (Q(x,t,s),P(x,t,s)) \in \mathcal{M} \) be a map such that \((Q(x,t,0),P(x,t,0)) = (0,0)\) and such that is satisfies the conditions

\[ \frac{d}{ds}\big|_{s=0}(Q(x,t,s),P(x,t,s)) = (q(x,t),p(x,t)), \quad \frac{d^2}{ds^2}\big|_{s=0}(q(x,t,s),p(x,t,s)) = (0,0), \]

where \((q(x,t),p(x,t))\) is the path in \( \mathcal{M} \) considered in the statement of the theorem. Matrices \( L_r^{(i)} \) and \( A_r^{(i)} \) for \( i = 0,1,2 \) are obtained by taking derivatives

\[ L_r^{(i)}(x,t;z) = \frac{d^i}{ds^i}\big|_{s=0}L(Q(x,t,s),P(x,t,s))(z), \quad A_r^{(i)}(x,t;z) = \frac{d^i}{ds^i}\big|_{s=0}A(Q(x,t,s),P(x,t,s))(z), \]

where \((L(z),A(z))\) is the Lax pair of the sine-Gordon system, evaluated at \((q(x,t,s),p(x,t,s))\).

It now follows from proposition 1 that \((\mathcal{L},\mathcal{A})\) given above is a Lax pair for the 2-jet sine-Gordon system, restricted the subspace \( \mathcal{N} \subset J^2_0 \mathcal{M} \) and given by

\[ \mathcal{N} = \{((q(x),p(x),(0,0))) \} \subset J^2_0 \mathcal{M}. \]

But we have seen that this restriction is precisely the Klein-Gordon system.

If we restrict the Lax pair (11) to the functions, constant with respect to \( x \), we get a Lax pair for the harmonic oscillator. This Lax pair is indeed new and clearly different from the usual one, found e.g. in [1].

Let now \( \mathcal{F} : J^2_0 \mathcal{M} = \mathcal{M} \times \mathcal{M} \to \mathbb{R} \) be an integral of the 2-jet system. Then the map

\[ F : \mathcal{M} \to \mathbb{R} \]

given by

\[ F(q,p) = \mathcal{F}([(q,p),(0,0)]) \]

is an integral of the Klein-Gordon system. But at the end of section 2 we have seen that the conserved quantities of jet sine-Gordon systems are explicitly computable. We shall now perform the calculation for the restricted 2-jet system and this will give us the conserved quantities of the Klein-Gordon system.

The expression (10) gives

\[ F_z(q(x),p(x)) = \mathcal{F}_z([(q(x),p(x),(0,0)]) = 2 \text{tr}(\mathcal{H}^1(z) \cdot \mathcal{H}^1(z)) + 2 \text{tr}(\mathcal{H}^0(z) \cdot \mathcal{H}^2(z)), \] (12)
where $H^{(i)} = \Phi^{(i)}(2\pi)$ and $\Phi^{(i)}(x)$ are given by the system (9). In the case of the restricted 2-jet system we have

\[
\begin{align*}
(\Phi^{(0)})_x &= L_r^{(0)} \cdot \Phi^{(0)} \\
(\Phi^{(1)})_x &= L_r^{(0)} \cdot \Phi^{(1)} + L_r^{(1)} \cdot \Phi^{(0)} \\
(\Phi^{(2)})_x &= L_r^{(0)} \cdot \Phi^{(2)} + 2L_r^{(1)} \cdot \Phi^{(1)} + L_r^{(2)} \cdot \Phi^{(0)},
\end{align*}
\]

where the matrices $L^{(i)}$, $i = 0, 1, 2$ are given in theorem 1. The integration gives us

\[
\begin{align*}
\Phi^{(0)}(x) &= \exp (xL_r^{(0)}) \\
\Phi^{(1)}(x) &= \Phi^{(0)}(x) \cdot \int_0^x (\Phi^{(0)})^{-1}(\xi) \cdot L_r^{(1)}(\xi) \cdot (\Phi^{(0)})(\xi) \, d\xi \\
\Phi^{(2)}(x) &= 2\Phi^{(0)}(x) \cdot \int_0^x (\Phi^{(0)})^{-1}(\xi) \cdot L_r^{(1)}(\xi) \cdot (\Phi^{(0)})(\xi) \cdot \int_0^\xi (\Phi^{(0)})^{-1}(\eta) \cdot L_r^{(1)}(\eta) \cdot (\Phi^{(0)})(\eta) \, d\eta \, d\xi \\
&\quad + \Phi^{(0)}(x) \cdot \int_0^x (\Phi^{(0)})^{-1}(\xi) L_r^{(2)}(\xi) \cdot (\Phi^{(0)})(\xi) \, d\xi.
\end{align*}
\]

The first equation above gives

\[
\Phi^{(0)}(x) = \begin{pmatrix}
e^{i\frac{\kappa x}{2}} & 0 \\ 0 & e^{-i\frac{\kappa x}{2}} \end{pmatrix}
\]

and the second

\[
\Phi^{(1)}(x) = \Phi^{(0)} \cdot \begin{pmatrix} 1 & 0 \\ \frac{1}{4} \int_0^x (-p(x) + i\omega q(x)) e^{-i\kappa x} \, dx & 1 \end{pmatrix},
\]

where $\kappa = \kappa(z) = \frac{1-\omega^2}{2}$ and $\omega = \frac{1+\omega^2}{2}$. Evaluation of $\Phi^{(0)}(x)$ at $x = 2\pi$ will yield the first term in the expression (12). For the second term we have

\[
\begin{align*}
\text{tr}(\mathcal{H}^0 \cdot \mathcal{H}^2) &= 2 \text{tr} \left( \int_0^{2\pi} \text{Ad}(\Phi^{(0)})^{-1}(x) L_r^{(1)}(x) \cdot \int_0^x \text{Ad}(\Phi^{(0)})^{-1}(\xi) L_r^{(1)}(\xi) \, d\xi \, dx \right) \\
&\quad + \text{tr} \left( \int_0^{2\pi} \text{Ad}(\Phi^{(0)})^{-1}(x) L_r^{(2)}(x) \, dx \right).
\end{align*}
\]

The function $L_r^{(2)}$ takes values in the traceless Lie algebra $su(2)$, therefore the second term in the above expression is equal to zero. If on the first term we perform integration by parts and take into account the relation $\text{tr}(ab) = \text{tr}(ba)$, we get

\[
\text{tr}(\mathcal{H}^0 \cdot \mathcal{H}^2) = \text{tr}(\mathcal{H}^{(1)} \cdot \mathcal{H}^{(1)}),
\]

and finally,

\[
F_2(q(x), p(x)) = 4 \text{tr} \left( \mathcal{H}^{(1)}_{(q(x), p(x))}(z) \cdot \mathcal{H}^{(1)}_{(q(x), p(x))}(z) \right).
\]
Theorem 2 Let the elements \((q(x), p(x)) \in \mathcal{M}\) of the phase space be given by the Fourier series
\[
q(x) = \sum_{n=-\infty}^{\infty} \alpha_n e^{inx}, \quad p(x) = \sum_{n=-\infty}^{\infty} \beta_n e^{inx}.
\]
If for every \(n \in \mathbb{Z}\) we choose the value of the spectral parameter to be
\[
z_n = -n + \sqrt{n^2 + 1}
\]
and set \(\omega_n = \sqrt{n^2 + 1}\), then the corresponding values of the integrals of motion of the Klein-Gordon system are given by the formula
\[
F_{z_n}(q(x), p(x)) = \frac{1}{2} |i \omega_n \alpha_n + \beta_n|^2, \quad n \in \mathbb{Z}.
\]
\(\square\)

4. On integrals of the sine-Gordon system

We now take a short look at the application of the scheme, presented above, to the evaluation of the integrals of the periodic sine-Gordon system. Not surprisingly, it turns out that the expressions of the sine-Gordon integrals involve infinitely many Fourier coefficients of \(q(x)\) and \(p(x)\). Therefore these expressions are not easily manageable. Nevertheless, they give some insight by providing the explicit comparison with the integrals of the Klein-Gordon system. In particular, let the phase space \(\mathcal{M}\) be replaced by the space
\[
\mathcal{M}_P = \{(q(x), p(x)) : q(x), p(x), q_x(x), p_x(x) \text{ periodic with period } P\}.
\]
We shall find an approximation of the sine-Gordon integrals on \(\mathcal{M}_P\) whose quality increases when \(P\) approaches to zero.

Let \(\Phi(x; z)\) be the solution of the initial problem
\[
\Phi_x(x; z) = L_{(q(x), p(x))}(z) \cdot \Phi(x; z), \quad \Phi(0; z) = Id,
\]
where \(L_{(q,p)}(z)\) is given by (6). Denote \(\mathcal{H}(z) = \Phi(2\pi; z)\). Then, for every value of the spectral parameter \(z\), the function
\[
G_z(q(x), p(x)) = \text{tr}(\mathcal{H}(z)) : \mathcal{M} \rightarrow \mathbb{R}
\]
is an integral of the sine-Gordon system. We shall consider lines of the form \(s \mapsto (sq(x), sp(x))\) in \(\mathcal{M}\) and the expansion of \(G_z(sq(x), sp(x))\) with respect to the perturbation parameter \(s\). We can think of a line \(s \mapsto (sq(x), sp(x))\) as of a line of initial conditions of the path of solutions \(s \mapsto (q(x, t; s), p(x, t; s))\) of the sine-Gordon system with initial conditions
\[
(q(x, 0; s), p(x, 0; s)) = (sq(x), sp(x)).
\]
If we derive the equation
\[
\Phi_x(x; s; z) = L_{(sq(x), sp(x))} \cdot \Phi(x; s; z)
\]
consecutively with respect to \(s\) and evaluate at \(s = 0\), we obtain
\[
(\Phi^{(n)})_x = L^{(0)} \Phi^{(n)} + \sum_{j=1}^{n} \binom{n}{j} L^{(j)} \cdot \Phi^{(n-j)}, \quad \Phi^{(0)}(0; z) = Id, \quad \Phi^{(n)}(0; z) = 0, \quad n \geq 1,
\]
where we denoted $\Phi^{(n)} = \frac{d^n}{dx^n}|_{x=0}\Phi(s)$ and $L^{(n)} = \frac{d^n}{dx^n}|_{x=0}L_{(sq,sp)}$. Note that the above equation is the same as (9). The usual integration of a non-homogeneous ordinary differential equation with constant coefficients gives us

$$\Phi^{(n)}(x) = \sum_{l=1}^{n} \binom{n}{l} \Phi^{(0)} \int_{0}^{x} (\Phi^{(0)}(\xi))^{-1} L^{(l)}(\xi) \Phi^{(n-l)}(\xi) \, d\xi;$$

where, as before,

$$\Phi^{(0)}(x) = \exp xL^{(0)} = \left(\begin{array}{cc} e^{-\frac{ix}{2}} & 0 \\ 0 & e^{\frac{ix}{2}} \end{array}\right).$$

Recursive application of the above formula to itself eventually gives us the expression

$$\Phi^{(n)}(x) = \Phi^{(0)}(x) \left[ \int_{0}^{x} \tilde{L}_n(\xi) \, d\xi + \sum_{k_1+k_2=n} \binom{n}{k_1,k_2} \int_{\triangle_2(x)} \tilde{L}_{k_1}(\xi) \tilde{L}_{k_2}(\xi) \, d\xi_2 d\xi_1 \right. \\
+ \sum_{k_1+\ldots+k_l=n} \binom{n}{k_1,\ldots,k_l} \int_{\triangle_l(x)} \tilde{L}_{k_1}(\xi_1) \tilde{L}_{k_2}(\xi_2) \ldots \tilde{L}_{k_l}(\xi_l) \, d\xi_1 \ldots d\xi_l + \left. \frac{n!}{\prod_{k=1}^{l} k!} \right] \int_{\triangle_n(x)} \tilde{L}_1(\xi_1) \tilde{L}_1(\xi_2) \ldots \tilde{L}_1(\xi_n) \, d\xi_n \ldots d\xi_1,$$

where

$$\tilde{L}_k(x) = (\Phi^{(0)}(x))^{-1} \cdot L^{(k)}(x) \cdot \Phi^{(0)}(x) = Ad_{\Phi^{(0)}(x)}^{-1} L^{(k)}(x),$$

and

$$\binom{n}{k_1,k_2,\ldots,k_l} = \frac{n!}{k_1! k_2! \ldots k_l!}, \quad \text{for} \quad k_1 + k_2 + \ldots + k_l = n$$

is the multinomial symbol. Integrals are taken over the simplices

$$\triangle_l(x) = \{ (\xi_1, \xi_2, \ldots, \xi_l); x \geq \xi_1 \geq \ldots \geq \xi_l \geq 0 \}. $$

The integrals $G_z$, evaluated at $(sq(x), s(p(x)) \in \mathcal{M}$, are given by

$$G_z(sq(x), sp(x)) = \text{tr} \left( \mathcal{H}_{(sq(x), sp(x))} (z) \right) = \sum_{n=0}^{\infty} \frac{s^{2n}}{(2n)!} \text{tr}(\Phi^{(2n)}(2\pi)).$$

(Note that traces of $\Phi^{(n)}$ for odd $n$ are equal to zero.) By means of the expression (16) the above sum, evaluated at $s = 1$, can be rearranged in the form

$$G_z(q(x), p(x)) = \int_{\triangle_2} \mathcal{L}_2(q(x), p(x)) + \sum_{m=2}^{\infty} \int_{\triangle_{2m}} \mathcal{L}_{2m}(q(x), p(x)), \quad (17)$$

where we denoted $\triangle_k = \triangle_k(2\pi)$. The terms are given by

$$(\mathcal{L}_m)_{(sq(x),sp(x))} (\xi_1, \ldots, \xi_m) = \sum_{n=m}^{\infty} \frac{s^n}{n!} \text{tr} \left[ \sum_{k_1+\ldots+k_m=n} \binom{n}{k_1,\ldots,k_m} \tilde{L}_{k_1}(\xi_1) \tilde{L}_{k_2}(\xi_2) \ldots \tilde{L}_{k_m}(\xi_m) \right].$$

The leading term of the development (17) is given by the integrations over the 2-simplex. Evaluation of $\mathcal{L}_2$ gives

$$(\mathcal{L}_2)_{(q,p)} (\xi_1, \xi_2) = \text{tr} \left( (\tilde{L}_{(q(\xi_1),p(\xi_1))} - I) \cdot (\tilde{L}_{(q(\xi_2),p(\xi_2))} - I) \right).$$
Since this is a symmetric function of $(\xi_1, \xi_2)$, we get

$$\int_{\triangle_2} (\mathcal{L}_2)_{(q,p)}(\xi_1, \xi_2) = \frac{1}{2} \text{tr} \left( \int_0^{2\pi} (\tilde{\mathcal{L}}_{(q(x),p(x))}) - I \right) dx)^2.$$ 

For the values of the spectral parameter to $z_n$, given by (15), this term is equal to

$$\int_{\triangle_2} (\mathcal{L}_2)_{(q,p)} = \frac{1}{2} \left( i \omega_n \gamma_n + \beta_n \right)^2 + \left( \int_0^{2\pi} (\cos(q(x)) - 1) dx \right)^2,$$

where, as before, $\omega_n = \sqrt{n^2 + 1}$, while $\gamma_n$ and $\beta_n$ are the coefficients in

$$\sin(q(x)) = \sum_{n=-\infty}^{\infty} \gamma_n e^{inx}, \quad p(x) = \sum_{n=-\infty}^{\infty} \beta_n e^{inx}.$$ 

The terms belonging to higher dimensional simplices, when evaluated at $(sq, sp)$, involve powers of $s$, higher than 2. Therefore we clearly have

$$\frac{d^2}{ds^2}|_{s=0} G_{z_n}(sq, sp) = \frac{d^2}{ds^2}|_{s=0} \int_{\triangle_2} (\mathcal{L}_2)_{(q,p)} = F_{z_n}(q, p),$$

where $F_{z_n}$ are the integrals of the Klein-Gordon system, constructed in the previous section.

Consider now the functionals

$$G_{z_n} : \mathcal{M}_P \to \mathbb{R}, \quad \text{where } z_n = \frac{2\pi n}{P} + \sqrt{\left( \frac{2\pi n}{P} \right)^2 + 1}$$

which are the integrals of the sine-Gordon system on $\mathcal{M}_P$ defined above. The quality of the approximation $\int_{\triangle_2} \mathcal{L}_2(q,p)$ for the integrals $G_{z_n}(q, p)$ is better than the approximation by the Klein-Gordon integrals $F_{z_n}$ and it increases as $P$ approaches zero. The reason for this is the fact that the volume of the simplex $\triangle_m(P)$ is equal to $\frac{P^m}{m!}$. Therefore the absolute values of the higher-order terms in (17), which are integrals over $\triangle_m(P)$ for $m \geq 4$, decrease with $P$ faster that the leading term.

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