NORM INFLATION FOR THE DERIVATIVE NONLINEAR SCHRÖDINGER EQUATION

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Abstract. In this note, we study the ill-posedness problem for the derivative nonlinear Schrödinger equation (DNLS) in the one-dimensional setting. More precisely, by using a ternary-quinary tree expansion of the Duhamel formula we prove norm inflation in Sobolev spaces below the (scaling) critical regularity for the gauged DNLS. This ill-posedness result is sharp since DNLS is known to be globally well-posed in $L^2(\mathbb{R})$ [14]. The main novelty of our approach is to control the derivative loss from the cubic nonlinearity by the quintic nonlinearity with carefully chosen initial data.

1. Setup of the problem

1.1. The derivative nonlinear Schrödinger equation. We consider the derivative nonlinear Schrödinger equation (DNLS) defined on $\mathbb{R}$:

$$\begin{cases}
i\partial_t u + \partial_x^2 u = i\partial_x(|u|^2u) \\
u|_{t=0} = u_0,
\end{cases} \quad (x,t) \in \mathbb{R}^2,$$

(1.1)

where $u = u(t,x)$ is a complex-valued function. The equation (1.1) was derived in the plasma physics literature [23] and has been extensively studied from a theoretical perspective. It is known that (1.1) is completely integrable and admits infinitely many conservation laws [20]. See [24, 21] and reference therein for recent developments.

If $u(x,t)$ solves (1.1) on $\mathbb{R}$, then, for any $\lambda > 0$,

$$u^\lambda := \lambda^{\frac{1}{2}} u(\lambda^2 t, \lambda x)$$

also solves (1.1) with scaled initial data $\phi^\lambda := \lambda^{\frac{1}{2}} \phi(\lambda x)$. This scaling invariance heuristically suggests that the critical Sobolev regularity of DNLS (1.3) is given by $s_{\text{crit}} := 0$.

Therefore, it is natural to conjecture the following:

Conjecture 1.1. The DNLS equation (1.1) is well-posed in $H^s(\mathbb{R})$ for $s \geq 0$, and ill-posed for $s < 0$.

Regarding well-posedness, Conjecture 1.1 has seen some recent progress culminating in the breakthrough work [14] which proves global well-posedness of (1.1) in $L^2(\mathbb{R})$. We also mention the following works on the well-posedness theory for (1.1) [30, 31, 18, 29, 21, 15, 14]. On the other hand, Biagioni and Linares showed a mild form of ill-posedness for (1.1): they showed that the data-to-solution map fails to be uniformly continuous (strictly) below $H^\frac{1}{2}(\mathbb{R})$.

2020 Mathematics Subject Classification. 35Q55, 35R25.

Key words and phrases. derivative nonlinear Schrödinger equation; ill-posedness; norm inflation.
In the study of (1.1), the gauge transform plays an important role. Define the nonlinear map \( G : L^2(\mathbb{R}) \to L^2(\mathbb{R}) \) by
\[
Gf(x) := e^{-i \int_0^x |f(y)|^2 dy} f(x).
\] (1.2)

Then, a smooth function \( u \) satisfies (1.1) if and only if \( v = G(u) \) satisfies
\[
\begin{cases}
i \partial_t v + \partial_x^2 v = -iv \partial_x v - \frac{1}{3} |v|^4 v \\
v|_{t=0} = \phi,
\end{cases}
\] (1.3)
with \( \phi = G(u_0) \).

Note that if \( u \) solves (1.1) in \( L^2(\mathbb{R}) \) then \( v = G(u) \) solves (1.3) in \( L^2(\mathbb{R}) \). Hence, we can transfer any well-posedness result for (1.1) in \( L^2(\mathbb{R}) \) to a well-posedness result for (1.3) in \( L^2(\mathbb{R}) \), and vice-versa.

The aim of this note is to study the ill-posedness problem of (1.3) in Sobolev spaces \( H^s \).

One way of showing ill-posedness is to show the discontinuity of the solution map, which can be done through norm inflation. More precisely, given \( s < 0 \), we say that (1.3) exhibits norm inflation in \( H^s(\mathbb{R}) \) if for any \( \varepsilon > 0 \) and \( \phi \in H^s(\mathbb{R}) \), there exists a solution \( v \) to (1.3) on \( \mathbb{R} \) and \( t \in (0, \varepsilon) \) such that
\[
\|v(0) - \phi\|_{H^s(\mathbb{R})} < \varepsilon \quad \text{and} \quad \|v(t)\|_{H^s(\mathbb{R})} > \varepsilon^{-1}.
\] (1.4)

Note that if (1.3) exhibits norm inflation in \( H^s(\mathbb{R}) \), then it not possible to define a continuous data-to-solution map. We invite the reader to consult [9, 10, 11, 4, 7, 16, 19, 22, 26, 27, 28, 32, 5, 6] and references therein for more information on the norm inflation phenomena.

The main purpose of this note is to show norm inflation of the gauged DNLS equation (1.3). Our main result is the following:

**Theorem 1.1.** Suppose \( s < 0 \). Fix \( \phi \in H^s(\mathbb{R}) \). Then, given any \( \varepsilon > 0 \), there exist a global solution \( v_\varepsilon \) to the gauged equations (1.3) on \( \mathbb{R} \) and \( t \in (0, \varepsilon) \) such that (1.4) holds.

To the best of our best knowledge, Theorem 1.1 is the first ill-posedness result of (1.3) in terms of discontinuity of the flow map. We prove Theorem 1.1 via a Fourier analytic method. In [19] Iwabuchi-Ogawa, showed norm inflation (1.4) with \( u_0 = 0 \) for quadratic nonlinear Schrödinger equations. Building upon the work of Bejenaru and Tao [1], their approach is based on based on a Picard iteration scheme to show norm inflation using the high-to-low energy transfer in the first Picard iterate. It turns out that this method is widely applicable and works particularly well with power type nonlinearities [7, 16, 22, 26, 27, 28, 32, 5, 6] and references therein for more information on the norm inflation phenomena.

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The main difficulty when one tries to apply these methods to (1.3) comes from the presence of derivative in (1.3). Okamoto [28] pointed out that Iwabuchi-Ogawa’s argument is applicable for some dispersive equations where a derivative appears in the nonlinearity. In particular, he proved norm inflation for the Kawahara equation:
\[
\partial_t u - \partial_x^5 u + \partial_x(u^2) = 0,
\]
in \( H^s(\mathbb{R}) \) with \( s < -2 \), which is sharp in the sense that the well-posedness is known in \( H^s \) when \( s > -2 \).

In [28], the strong dispersion of the Kawahara equation plays a crucial rule in absorbing the derivative loss along the Picard iteration. Unfortunately, the dispersion of (1.3) is not strong enough to handle the derivative loss. In order to achieve norm inflation below the
critical scaling exponent $s < 0$, we need to further exploit the cubic-quintic structure of the nonlinearity. Since the derivative loss in (1.3) only appears in the cubic part of the nonlinearity, we observe that by carefully choosing our initial data, the cubic part can be controlled by the quintic component of the nonlinearity. Therefore, the problem then essentially reduces to showing norm inflation for the quintic nonlinear Schrödinger equation below $L^2(\mathbb{R})$, which is already known in the literature, see [22].

Another difficulty comes from the nonlinearity, which consists of a derivative cubic term and a non-derivative quartic term. This non-homogeneous structure makes the series expansion of a solution complicated, see (2.6) below. To track these multilinear expressions, we introduce a notion of ternary-quinary trees generalising the ternary trees in [27]. We note that Kishimoto [22] had to deal with this issue with a polynomial nonlinearity (consisting of several terms) without any derivative loss.

1.2. Further remarks. We conclude this section with several remarks.

Remark 1.2. Since (1.1) is globally well-posed in $L^2(\mathbb{R})$ [14], norm inflation cannot occur in $H^s(\mathbb{R})$ for $s \geq 0$ in view of the continuity of the Gauge transform (1.2) on smooth functions (i.e. from $C([0,T];L^2(\mathbb{R}))$ to itself, for any $T > 0$, see [17] for a proof in the periodic setting). Therefore, Theorem 1.1 is sharp up to the endpoint $s = 0$.

Remark 1.3. Regarding the periodic setting, Theorem 1.1 still holds, i.e. we can prove norm inflation for the gauged DNLS on $\mathbb{T}$ in $H^s(\mathbb{T})$ for $s < 0$ (note that the gauged equation (1.3) is modified on the torus, see [18, 12]. The proof is a minor modification of the real line case. We also note that in the setting of Fourier-Lebesgue spaces, (1.1) was shown to be locally well-posed in [12] in the whole subcritical regime (i.e. in Fourier-Lebesgue spaces that scale like $H^s(\mathbb{T})$, for any $s > 0$).

Remark 1.4. Using standard norm inflation techniques, we can also prove norm inflation for (1.1) in $H^s(\mathcal{M})$ for $\mathcal{M} = \mathbb{R}$ or $\mathbb{T}$ for $s < -1$. The $-1 \leq s < 0$ case in Conjecture 1.1 remains however open.

Remark 1.5. On the torus, another notion of probabilistic criticality was developed in [13] giving rise to a probabilistic critical exponent $s_p$. More precisely, let $\{g_n\}_{n \in \mathbb{Z}}$ be an i.i.d. family of standard complex Gaussian random variables and define for any $s \in \mathbb{R}$, the function

$$u_0^s(\omega) := \sum_{n \in \mathbb{Z}} \frac{g_n(\omega)}{|n|^{s+\frac{1}{2}}} e^{inx}.$$ 

Then the exponent $s_p$ is defined to be the smallest $s \in \mathbb{R}$ such that the second Picard iterate of (1.1) (after some appropriate frequency truncation) with initial data given by $u_0^s$ stays bounded in $H^s(\mathbb{T})$; see [13] for more details. A computation shows that for (1.1), we have $s_p = s_c = 0$. Hence, it is unlikely that any random data theory (see for instance [3, 13, 25]) would allow us to go beyond the $L^2(\mathbb{T})$ well-posedness threshold.

The rest of the paper is organized as follows. We introduce the notion of ternary-quinary trees and establish multilinear estimates in Section 2. In Section 3, we prove Theorem 1.1.
2. Preliminary analysis

2.1. Power series expansion indexed by trees. We define two Duhamel integral operators \( J \) and \( K \) by

\[
J[v_1, v_2, v_3](t) := -i \int_0^t S(t - t') v_1(t') v_2(t') \partial_x \overline{v}_3(t') dt',
\]

\[
K[v_1, v_2, v_3, v_4, v_5](t) := -\frac{1}{2} \int_0^t S(t - t') v_1(t') \overline{v}_2(t') \overline{v}_3(t') \overline{v}_4(t') v_5(t') dt',
\]

where \( S(t) := e^{it\partial_x^2} \) denotes the linear propagator associated with (1.3). We also use the following shorthand notations:

\[
J^3[v] := J[v, v, v],
\]

\[
K^5[v] := K[v, v, v, v, v].
\]

In what follows, a solution \( v \) to (1.3) with \( v|_{t=0} = \phi \) is a distribution which satisfies the Duhamel formulation

\[
v(t) = S(t) \phi + J^3[v](t) + K^5[v](t).
\]

It is expected that the following Picard iteration

\[
P_0(\phi) = S(t) \phi \quad \text{and} \quad P_j(\phi) = S(t) \phi + J^3[P_{j-1}(\phi)] + K^5[P_{j-1}(\phi)], \quad j \in \mathbb{N},
\]

converges to a solution \( v \) to (1.3) at least for short times. If this is the case, then \( v \) can be expressed as a series of multilinear terms in \( \phi \):

\[
v(t) = \sum_{j \in \mathbb{N}} H_j[\phi](t),
\]

where \( H_j[\phi] \) consists of all homogeneous multilinear terms in \( \phi \) of degree \( j \). For instance,

\[
H_0[\phi](t) := S(t) \phi;
\]

\[
H_3[\phi](t) := J^3[S(t) \phi];
\]

\[
H_5[\phi](t) := K^5[S(t) \phi];
\]

\[
H_6[\phi](t) := K^3[J^3[S(t) \phi]];
\]

\[
H_8[\phi](t) := K^5[J^3[S(t) \phi]] + J^3[K^5[S(t) \phi]];
\]

\ldots

Since the nonlinearity of (1.3) consists of both cubic and quintic nonlinearities, the series expansion (2.5) and (2.6) gets more involved as \( j \) gets larger. Inspired by [8] [27], we shall use trees to index the series (2.5). To this purpose, we introduce the following notion of ternary-quinary trees:

**Definition 2.1.** (i) Given a partially ordered set \( T \) with partial order \( \leq \), we say that \( b \in T \) with \( b \leq a \) and \( b \neq a \) is a child of \( a \in T \), if \( b \leq c \leq a \) implies either \( c = a \) or \( c = b \). If the latter condition holds, we also say that \( a \) is the parent of \( b \).

(ii) A tree \( T \) is a finite partially ordered set, satisfying the following properties:

- Let \( a_1, a_2, a_3, a_4 \in T \). If \( a_4 \leq a_2 \leq a_1 \) and \( a_4 \leq a_3 \leq a_1 \), then we have \( a_2 \leq a_3 \) or \( a_3 \leq a_2 \).
- A node \( a \in T \) is called terminal, if it has no child. A non-terminal node \( a \in T \) is a node with exactly three or five children.
• There exists a maximal element \( r \in \mathcal{T} \) (called the root node) such that \( a \leq r \) for all \( a \in \mathcal{T} \).

• \( \mathcal{T} \) consists of the disjoint union of \( \mathcal{T}^0 \) and \( \mathcal{T}^\infty \), where \( \mathcal{T}^0 \) and \( \mathcal{T}^\infty \) denote the collections of non-terminal nodes and terminal nodes, respectively.

Given a tree \( \mathcal{T} \), we denote by \( n_3(\mathcal{T}) \) (resp. \( n_5(\mathcal{T}) \)) the number of non-terminal nodes which have three (resp. five) children.

We also denote the collection of trees in the \((k,p)\)-th generation (i.e. with \( k \) parental nodes with three children and \( p \) parental nodes with five children) by \( \mathbf{T}^{3,5}(k,p) \):

\[
\mathbf{T}^{3,5}(k,p) := \{ \mathcal{T} : \mathcal{T} \text{ is a tree with } (n_3(\mathcal{T}), n_5(\mathcal{T})) = (k,p) \}. \tag{2.7}
\]

Note that the number \( |\mathcal{T}| \) of nodes in a tree \( \mathcal{T} \in \mathbf{T}^{3,5}(k,p) \) is \( 3k + 5p + 1 \) for \( k, p \in \mathbb{N} \cup \{0\} \).

In particular, the number of non-terminal nodes is \( |\mathcal{T}^0| = k + p \) and the number of terminal nodes is \( |\mathcal{T}^\infty| = 2k + 4p + 1 \).

**Remark 2.2.** The ternary-quinary trees defined in Definition 2.1 generalise the ternary trees introduced in [27] by allowing some parental nodes to have five children. It is easy to see that \( \mathbf{T}^{3,5}(k,0) \) and \( \mathbf{T}^{3,5}(0,p) \) consist of only ternary trees and quinary trees respectively.

**Remark 2.3.** Let \( \mathcal{T} = \{ a \}_{a \in \mathcal{T}} \) be a tree. Then, for every \( a \in \mathcal{T} \), we denote by \( \mathcal{T}_a \) the sub-tree whose root node is \( a \). \( \mathcal{T}^0_a \) and \( \mathcal{T}^\infty_a \) are the associated sets of non-terminal and terminal nodes respectively. Let \( r \in \mathcal{T} \) be the root node of \( \mathcal{T} \), then

\[
\mathcal{T} = \mathcal{T}_r.
\]

And similarly, \( \mathcal{T}^0 = \mathcal{T}^0_r \) and \( \mathcal{T}^\infty = \mathcal{T}^\infty_r \).

We have the following bound on the number of trees in \( \mathbf{T}^{3,5}(k,p) \):

**Lemma 2.4.** Let \( \mathbf{T}^{3,5}(k,p) \) be as in (2.7). Then, there exists \( C > 0 \) such that

\[
\# \mathbf{T}^{3,5}(k,p) \leq C^{k+p},
\]

for all \((k,p) \in (\mathbb{N} \cup \{0\})^2\).

**Proof.** We observe that

\[
\# \mathbf{T}^{3,5}(k,p) \leq \# \mathbf{T}^{3,5}(0,k+p),
\]

where \( \mathbf{T}^{3,5}(0,k+p) \) consists of all quinary trees in the \((k+p)\)-th generation. Then the bound follows from the same argument as Lemma 2.3 in [27]. \( \square \)

Next, we associate operators to trees in the following manner. Fix \( \phi \) a function. Let \( \mathcal{T} \in \mathbf{T}^{3,5}(k,p) \) for \((k,p) \in \mathbb{N}^2 \cup \{(0,0)\} \) be a tree. Given functions \( \phi_1, \cdots, \phi_{2k+4p+1} \), we formally associate \( \Psi(\mathcal{T}, \phi_1, \cdots, \phi_{2k+4p+1}) \), a multilinear operator, by the following rules:

• Replace a non-terminal node by the Duhamel integral operator \( \mathcal{J} \) (resp. \( \mathcal{K} \)) defined in (2.1) with its three (resp. five) children as arguments \( u_1, u_2 \) and \( u_3 \) (resp. \( u_1, u_2, u_3, u_4, u_5 \)).

• Replace a terminal node by the linear solution \( S(t)\phi_j, j = 1, \cdots, 2k+4p+1 \).

In the following, we set \( \Psi_\phi(\mathcal{T}) = \Psi(\mathcal{T}, \phi, \cdots, \phi) \). Therefore, \( \Psi_\phi \) denotes a mapping from \( \bigcup_{k,p \geq 0} \mathbf{T}^{3,5}(k,p) \) to \( \mathcal{D}'(\mathcal{M} \times (-T,T)) \). Note that, if \( \mathcal{T} \in \mathbf{T}^{3,5}(k,p) \), then \( \Psi_\phi(\mathcal{T}) \) is \((2k+4p+1)\)-linear in \( \phi \). At times, we might identify the multilinear expression \( \Psi_\phi(\mathcal{T}) \) with its associated tree \( \mathcal{T} \) when the base function \( \phi \) is fixed.
For \( j \geq 0 \), we define \( \Xi_{0,0}(t) = S(t)\phi \) and
\[
\Xi_{3,5}^{k,p}(\phi) := \sum_{T \in T^{3,5}(k,p)} \Psi_{\phi}(T),
\]
for \( k + p \geq 1 \). Finally, we can rewrite (2.5) the series expansion of the solution \( v \) to (2.3) as:
\[
v = \sum_{k,p \geq 0} \Xi_{3,5}^{k,p}(\phi),
\]
where \( \Xi_{3,5}^{k,p}(\phi) \) consists of homogeneous multilinear terms in \( \phi \) of degree \( 2k + 5p + 1 \). We group all the \( j \)-th generation of the Picard iterations in \( \Xi_{j}(\phi) \), i.e.
\[
\Xi_{j}(\phi) := \sum_{j=k+p}^{\infty} \Xi_{3,5}^{k,p}(\phi),
\]
Then (2.9) can be further written as
\[
v = \sum_{j=0}^{\infty} \Xi_{j}(\phi).
\]
In the following, we shall show the convergence of the series (2.11) with some specific initial data \( \phi \).

2.2. Multilinear estimates. In this subsection, we establish some multilinear estimates with special initial data. Fix \( N \gg 1 \) (to be chosen later). We define \( \phi \) by setting
\[
\hat{\phi}(\xi) = R\{1_{2N+Q_{A}}(\xi) + 1_{3N+Q_{A}}(\xi)\},
\]
where \( Q_{A} = [-\frac{A}{2}, \frac{A}{2}] \), \( R = R(N) \) is a real parameter, and \( A = A(N) \gg 1 \), satisfying \( A \ll N \), is to be chosen later. Note that we have
\[
\|\phi\|_{H^s} \sim N^sRA^3 \quad \text{and} \quad \|\phi\|_{F^{L^1}} \sim RA
\]
for any \( s \in \mathbb{R} \).

Now we are ready to state our key multilinear estimates.

Lemma 2.5. Let \( \phi \) be as in (2.12). We have for \( t \geq 0 \),
\[
\|\Xi_{k,p}^{3,5}(\phi)(t)\|_{F^{L^1}} \leq (Ct)^{k+p}N^k(RA)^{2k+4p+1},
\]
(2.14)
\[
\|\Xi_{k,p}^{3,5}(\phi)(t)\|_{F^{L^{\infty}}} \leq (Ct)^{k+p}N^k(RA)^{2k+4p}R,
\]
(2.15)
\[
\|\partial_x(\Xi_{k,p}^{3,5}(\phi))(t)\|_{F^{L^{\infty}}} \leq (Ct)^{k+p}N^{k+1}(RA)^{2k+4p}R,
\]
(2.16)
for all \( (k,p) \in (N \cup \{0\})^2 \) and \( T \in T^{3,5}(k,p) \).

Proof. By Lemma 2.4 and (2.5), we only need to prove (2.14), (2.15), and (2.16) with \( \Xi_{3,5}^{k,p}(\phi) \) replaced by \( \Psi_{\phi}(T) \).

To prove (2.14), we proceed by induction on \( n = |T| \). If \( |T| = 1 \), i.e. the tree \( T \) only has a single node (and thus \( \Psi_{\phi}(T)(t) = \Xi_{0,0}(t) = S(t)\phi \)), then (2.14) follows from (2.13). Fix \( n \geq 1 \) and assume that (2.14) holds for all trees \( T \in T^{3,5} \) with \( |T| \leq n \). Let \( T \in T^{3,5}(k,p) \) with \( |T| = n + 1 \), i.e. \( n + 1 = 3k + 5p + 1 \) for some \( k \) and \( p \). Since \( \Psi_{\phi}(T) \) is \( |T^{\infty}| \)-linear in \( \phi \), we observe from (2.12) that the function \( F[\Psi_{\phi}(T)] \) is supported on
\{ \xi \in \mathbb{R} : |\xi| \leq 3|T^{\infty}|(N + A) \}. We divide our argument into two cases depending on the number of children of the root node.

**Case 1:** the root node \( a \) of \( T \) has three children. We denote these three children by \( a_s, s \in \{1, 2, 3\} \). By the notations in Remark 2.3, let \( j_s := |T_{a_s}| =: 3k_s + 5p_s + 1 \) for \( s \in \{1, 2, 3\} \). Then it follows that

\[
    k_1 + k_2 + k_3 + 1 = k, \quad p_1 + p_2 + p_3 = p
\]

and

\[
    \Psi_\phi(T)(t) = -i \int_0^t S(t - t') \left( \Psi_\phi(T_{a_1})(t') \Psi_\phi(T_{a_2})(t') \partial_x \overline{\Psi_\phi(T_{a_3})}(t') \right) dt'.
\]

By unitarity of the linear propagator \( S(t) \) in \( FL^1 \), Young’s inequality and the induction hypothesis, we have

\[
    \|\Psi_\phi(T)(t)\|_{FL^1} \leq \int_0^t \|\Psi_\phi(T_{a_1})(t')\|_{FL^1} dt'
\]

\[
    \leq 3 \cdot |T_{a_3}^{\infty}|(N + A) \int_0^t 3 \prod_{i=1}^3 \|\Psi_\phi(T_{a_i})(t')\|_{FL^1} dt'
\]

\[
    \leq 3 \cdot |T_{a_3}^{\infty}|(N + A) \int_0^t \prod_{i=1}^3 (Ct)^{k_i + p_i} N^{k_i}(RA)^{2k_i + 4p_i + 1} dt'
\]

\[
    = 3 \cdot |T_{a_3}^{\infty}|(N + A) \int_0^t (Ct')^{k/p - 1} N^{k-1}(RA)^{2(k-1)+4p+3} dt'
\]

\[
    \leq 6 |T_{a_3}^{\infty}| C^{k/p - 1} N^{k}(RA)^{2k+4p+1},
\]

where we used \( N \geq A \) in the last inequality. Then the desired estimate follows by choosing \( C \) large enough and noting that \( |T^{\infty}| = 2k + 4p + 1 \leq 5(k + p) \).

**Case 2:** the root node \( a \) has five children. We denote them by \( a_s \) for \( s \in \{1, 2, 3, 4, 5\} \). Let \( j_s := |T_{a_s}| =: 3k_s + 5p_s + 1 \) for \( s \in \{1, 2, 3, 4, 5\} \). We notice that

\[
    k_1 + k_2 + k_3 + k_4 + k_5 = k, \quad p_1 + p_2 + p_3 + p_4 + p_5 + 1 = p,
\]

and

\[
    \Psi_\phi(T)(t) = -i \int_0^t S(t - t') \left( \Psi_\phi(T_{a_1}) \overline{\Psi_\phi(T_{a_2})} \Psi_\phi(T_{a_3}) \overline{\Psi_\phi(T_{a_4})} \Psi_\phi(T_{a_5}) \right) dt'.
\]

As in the previous case, we bound

\[
    \|\Psi_\phi(T)(t)\|_{FL^1} \leq \int_0^t \|\Psi_\phi(T_{a_1}) \overline{\Psi_\phi(T_{a_2})} \Psi_\phi(T_{a_3}) \overline{\Psi_\phi(T_{a_4})} \Psi_\phi(T_{a_5})\|_{FL^1} dt'
\]

\[
    \leq \int_0^t 5 \prod_{i=1}^5 \|\Psi_\phi(T_{a_i})(t')\|_{FL^1} dt'
\]

\[
    \leq \int_0^t 5 \prod_{i=1}^5 (Ct)^{k_i + p_i} N^{k_i}(RA)^{2k_i + 4p_i + 1} dt'
\]

\[
    \leq \int_0^t (Ct')^{k/p - 1} N^{k-1}(RA)^{2(k-1)+4p+5} dt'
\]

\[
    \leq C^{k/p - 1} N^{k}(RA)^{2k+4p+5},
\]
which again gives (2.14) provided $C \geq 1$. This finishes the proof of (2.14). The proofs of (2.15) and (2.16) follow from the similar arguments and we omit details.

□

Lemma 2.6. Let $\phi$ be as in (2.12). Then we have for $t \geq 0$ and $s < 0$,

$$\|\Xi_{k,p}^3(\phi)(t)\|_{H^s} \leq C^{k+p}f_s(A)t^{k+p}N^k(RA)^{2k+4p}R. \tag{2.19}$$

Here, $f_s(A)$ is given by

$$f_s(A) = \begin{cases} 1, & \text{if } s < -\frac{1}{2}, \\ (\log A)^{\frac{1}{2}}, & \text{if } s = -\frac{1}{2}, \\ A^{\frac{1}{2} + s}, & \text{if } s > -\frac{1}{2}. \end{cases} \tag{2.20}$$

Proof. Recall that $|T| = 3k + 5p + 1$ for $T \in \mathcal{T}^{3,5}(k, p)$. In view of Lemma 2.4, (2.19) follows from the bound

$$\|\Psi_\phi(T)(t)\|_{H^s} \leq C^{k+p}f_s(A)t^{k+p}N^k(RA)^{2k+4p}R, \tag{2.21}$$

for all $(k, p) \in (\mathbb{N} \cup \{0\})^2$ and $T \in \mathcal{T}^{3,5}(k, p)$. We note that $\Psi_\phi(T)$ is $|T^\infty|$-linear in $\phi$, and thus the support of $F(\Psi_\phi(T))$ is contained in at most $2^{|T^\infty|} \cdot A$. Furthermore, since $(\xi)^s$ is a decreasing function in $|\xi|$ for $s < 0$, we thus have

$$\| (\xi)^s \|_{L^2(\text{supp } F[\Psi_\phi(T)(t)])} \leq 2 \frac{|T^\infty|}{|T^\infty|} f_s(|T^\infty|A), \tag{2.22}$$

uniformly in $t \geq 0$.

If $|T| = 1$, then the claimed result follows from (2.13). We now assume that $|T| > 1$ so that $k + p \geq 1$. Let us first assume that the root node of $T$ has three children $a_s$ for $s \in \{1, 2, 3\}$. Then by (2.22), Lemma 2.5 and (2.17), we get

$$\|\Psi_\phi(T)(t)\|_{H^s} \leq \| (\xi)^s \|_{L^2(\text{supp } F[\Psi_\phi(T)(t)])} \int_0^t \|\Psi_\phi(T_{a_1})\|F_{L^1}\|\Psi_\phi(T_{a_2})\|F_{L^1}\|\partial_x\Psi_\phi(T_{a_3})\|F_{L^1}\ dt'$$

$$\leq 2 \frac{|T^\infty|}{|T^\infty|} f_s(|T^\infty|A) \int_0^t \|\Psi_\phi(T_{a_1})\|F_{L^1}\|\Psi_\phi(T_{a_2})\|F_{L^1}\|\partial_x\Psi_\phi(T_{a_3})\|F_{L^1}\ dt'$$

$$\leq 2 \frac{|T^\infty|}{|T^\infty|} f_s(|T^\infty|A) \int_0^t \prod_{i=1}^2 (Ct)^{k_i+p_i}N^{k_i}(RA)^{2k_i+4p_i+1}$$

$$\times (Ct)^{k_3+p_3}N^{k_3+1}(RA)^{2k_3+4p_3}R dt'$$

$$\leq 2 \frac{|T^\infty|}{k + p} f_s(A)(Ct)^{k+p}N^k(RA)^{2k+4p}R$$

$$\leq C^{k+p}f_s(A)t^{k+p}N^k(RA)^{2k+4p}R,$$

for $C > 0$ large enough. Note that in the last inequality we used $|T^\infty| = 2k + 4p + 1 \leq 5(k + p)$. 

We then consider the case when the root node of $T$ has five children $a_s$ for $s \in \{1, 2, 3, 4, 5\}$. Then by (2.22), Lemma 2.5 and (2.18), we get

$$
\|\Psi_\phi(T)(t)\|_{H^s} \leq \|\langle x \rangle^s \|_{L^2(\text{supp}\, F[\Psi_\phi(T)(t)])} \int_0^t \left\| \prod_{i=1}^5 \Psi_\phi(T_{a_i}) \right\|_{\mathcal{F}L^\infty} dt'
$$

$$
\leq 2^{\left[\frac{\tau_0}{2}\right]} f_s(|T^\infty| A) \int_0^t \prod_{i=1}^4 \|\Psi_\phi(T_{a_i})\|_{\mathcal{F}L^1} \|\Psi_\phi(T_{a_5})\|_{\mathcal{F}L^\infty} dt'
$$

$$
\leq 2^{\left[\frac{\tau_0}{2}\right]} f_s(|T^\infty| A) \int_0^t \prod_{i=1}^4 (Ct)^{k_i+p_i} N^{k_i}(RA)^{2k_i+4p_i+1}
$$

$$
\times (Ct)^{k_5+p_5} N^{k_5}(RA)^{2k_5+4p_5 R dt'}
$$

$$
\leq 2^{\left[\frac{\tau_0}{2}\right]} \frac{|T^\infty|}{k+p} f_s(A)(Ct)^{k+p} N^{k}(RA)^{2k+4p R}
$$

$$
\leq C^{k+p} f_s(A) t^{k+p} N^{k}(RA)^{2k+4p R},
$$

for $C > 0$ large enough. This shows (2.21) and finishes the proof.

As a consequence of the last lemma, we have the following estimate on the power expansion (2.11). It essentially states that under some choice of parameters the main contribution to the Picard iterates (2.10) comes from the multilinear terms corresponding to the quintic nonlinearity (or trees for which parents nodes all have five children), i.e. $\{\Xi_{0,p}(\phi)\}_{p \geq 1}$.

**Lemma 2.7.** Let $\phi$ be as in (2.12). If $R^2 A^2 \gg N$, then for any $j \geq 1$ we have for $t \geq 0$,

$$
\|\Xi_j(\phi)(t)\|_{H^s} \leq C^j f_s(A) t^j (RA)^{4j} R,
$$

(2.23)

where $\Xi_j$ is given in (2.10).

**Proof.** Fix $j \geq 1$ and $t \geq 0$. By (2.10), it suffices to show

$$
\|\Xi_{k,j}^j(\phi)(t)\|_{H^s} \leq C^j f_s(A) t^j (RA)^{4j} R
$$

(2.24)

for any $T \in T^{3,5}(k,p)$ with $k + p = j$. By Lemma 2.6, it suffices to show

$$
N^{j-p}(RA)^{2(j+p)} \ll (RA)^{4j},
$$

which is a consequence of the assumption $R^2 A^2 \gg N$.

The following lemma shows that the multilinear expressions $\Xi_j$ defined in (2.10) are stable under suitable perturbations.

**Lemma 2.8.** Let $\phi$ be as in (2.12) and $\psi \in F^{-1} C_0^\infty(\mathbb{R})$ with $\|\psi\|_{\mathcal{F}L^1} \lesssim RA$ and $\text{supp}(\tilde{\psi}) \subset [-M, M]$ for some $M \geq 0$. We further assume $R^2 A^2 \gg N$ and $N \gg M$. Then, there exists $C > 0$ such that

$$
\|\Xi_j(\phi + \psi)(t) - \Xi_j(\phi)(t)\|_{L^2} \leq C^j \|\psi\|_{L^2}(t R A^4)^j
$$

(2.25)

for all $j \in \mathbb{N}$. 
Proof. From (2.8) and (2.10), we have
\[
\begin{align*}
\Xi_j(\phi + \psi) - \Xi_j(\phi) &= \sum_{T \in \mathcal{T}^{3.5}(k,p)} \left( (\Psi_{\phi+\psi}(T) - \Psi_{\phi}(T)) \right) \\
&= \sum_{T \in \mathcal{T}^{3.5}(k,p)} \sum_{j=k+p} \Psi(T; \phi_1, \ldots, \phi_{2k+4p+1}),
\end{align*}
\]
where the second summation in \(\phi_1, \ldots, \phi_{2k+4p+1}\) is over all possible combinations of \(\phi_i \in \{\phi, \psi\}\) with at least one occurrence of \(\psi\). Given \(T \in \mathcal{T}^{3.5}(k,p)\) with \((k,p) \in (\mathbb{N} \cup \{0\})^2\), we have \(\text{supp} \mathcal{F}[\Psi(T; \phi_1, \ldots, \phi_{2k+4p+1})] \subset \{\xi \in \mathbb{R} : |\xi| \leq 6(2k+4p+1)N\}\) provided \(A,M \ll N\).

Without loss of generality, we assume \(\phi_1 = \psi\). A similar induction argument as in Lemma 2.5 and the fact \(\|\psi\|_{L^1} \lesssim RA\) yields
\[
\begin{align*}
\|\Psi(T; \phi_2, \ldots, \phi_{2k+4p+1})(t)\|_{L^2} &\leq C^{j}j^3 \|\psi\|_{L^2} N^k \prod_{j=2}^{2k+4p+1} \|\phi_j\|_{L^1} \\
&\leq C^{j}j^3 \|\psi\|_{L^2} N^k \cdot (RA)^{2k+4p} \\
&\leq C^{j}j^3 \|\psi\|_{L^2} \cdot (RA)^{4k+4p},
\end{align*}
\]
provided \(R^2A^2 \gg N\).

To conclude this subsection, we remark that the series
\[
v_1(t) = \sum_{j=0}^{\infty} \Xi_j(\phi + \psi)(t)
\]
converges on \([-T,T]\) under the conditions of Lemma 2.8 as long as \(\|\psi\|_{L^1} \lesssim RA\) and \(T \ll (RA)^{-4} \ll N^{-2} \ll 1\). In particular, the series (2.11) converges as long as \(T \ll (RA)^{-4} \ll N^{-2} \ll 1\). Let us note that the function \(v_1\) solves the initial value problem (1.3) and (2.3) with initial data \(\phi\) replaced by \(\phi + \psi\).

2.3. First Picard iterate. In this subsection, we obtain a lower bound on the second Picard iterate \(\Xi_1(\phi)\) in (2.10). This will allow us to prove later that this term represents the main contribution to the series expansion (2.11) and to the norm inflation phenomena. See Subsection 3.1. To this purpose, we first recall the following elementary bounds on characteristic functions of intervals.

Lemma 2.9. For any \(a,b,c,d,e, \xi \in \hat{\mathcal{M}}\) and \(A \geq 1\), there exists \(c > 0\) such that
\[
1_{a+Q_A} * 1_{b+Q_A} * 1_{c+Q_A} * 1_{d+Q_A} * 1_{e+Q_A}(\xi) \geq cA^4 1_{a+b+c+d+e+Q_A}(\xi).
\]

We now state the lower bound on \(\Xi_1(\phi)\).

Proposition 2.10. Let \(\phi\) be as in (2.12). Then, for \(0 < t \ll N^{-2}\) and \(R^2A^2 \gg N\), we have
\[
\|\Xi_1(\phi)(t)\|_{H^s} \gtrsim f_a(A) \cdot tR^5A^4,
\]
where \(f_a(A)\) is the function defined in (2.20).
Proposition 3.1. From (2.28), we have,
\[ \Xi_1(\phi)(t) := \Xi_{1,0}^5(\phi)(t) + \Xi_{1,1}^5(\phi)(t) = \Psi_{\phi}^\star(T_3) + \Psi_{\phi}(T_5), \] (2.28)
where \( T_3 \) (resp. \( T_5 \)) is the tree with one parent node and three (resp. five) descendents.
Namely, \( \Psi_{\phi}^\star(T_3) = \mathcal{J}^5[S(t)\phi] \) and \( \Psi_{\phi}(T_5) = \mathcal{K}^5[S(t)\phi] \), see (2.22). From Lemma 2.6 we have
\[ \|\Psi_{\phi}(T_3)(t)\|_{H^s} \lesssim f_s(A) \cdot tN(RA)^2R. \] (2.29)
We now turn to the quintic term \( \Psi_{\phi}(T_5) \) and have
\[ \mathcal{F}[\Psi_{\phi}(T_5)(t)](\xi) = -\frac{1}{2}e^{-i|\xi|^2t} \int_{\xi=\xi_1+\cdots+\xi_5} \hat{\phi}(\xi_1)\hat{\phi}(\xi_2)\hat{\phi}(\xi_3)\hat{\phi}(\xi_4)\hat{\phi}(\xi_5)d\xi_1d\xi_2d\xi_3d\xi_4d\xi_5. \] (2.30)
From (2.12), we have \(|\xi_j| \lesssim N\) for \( \xi_j \in \text{supp } \hat{\phi} \). Then, since \( \xi = \xi_1 - \xi_2 + \xi_3 - \xi_4 + \xi_5 \) we have
\[ |\xi_1|^2 - |\xi_2|^2 + |\xi_3|^2 - |\xi_4|^2 + |\xi_5|^2 \lesssim N^2, \]
which implies that,
\[ \text{Re} \left( e^{it'|\xi|^2-|\xi_2|^2+|\xi_3|^2-|\xi_4|^2+|\xi_5|^2} \right) \geq \frac{1}{2} \] (2.31)
holds for all \( 0 < t' \ll N^{-2} \). Thus, by (2.30), (2.31), and Lemma 2.9 we arrive at
\[ |\mathcal{F}[\Psi_{\phi}(T_5)(t)](\xi)| \gtrsim tR^5A^4 \cdot 1_{Q_A}(\xi). \] (2.32)
Noting that \( ||\xi^s||_{L^2_{\xi}(Q_A)} \sim f_s(A) \), we obtain from (2.32),
\[ \|\Psi_{\phi}(T_5)(t)\|_{H^s} \gtrsim f_s(A) \cdot t(RA)^4R. \] (2.33)
Finally, by collecting (2.28), (2.29), (2.33) along with the assumption \( R^2A^2 \gg N \), we obtain (2.27). \( \square \)

3. Proof of Theorem 1.1

In this section, we present the proof of Theorem 1.1. A density argument reduces the proof of Theorem 1.1 to the following statement.

Proposition 3.1. Let \( s < 0 \). Fix \( \psi \in S(M) \) such that \( \hat{\psi} \in C^\infty_0(\mathbb{R}) \). Then, given any \( n \in \mathbb{N} \), there exist a solution \( v_n \) to (1.3) and \( t_n \in (0, \frac{1}{n}) \) such that
\[ \|v_n(0) - \psi\|_{H^s(M)} < \frac{1}{n} \quad \text{and} \quad \|v_n(t_n)\|_{H^s(M)} > n. \] (3.1)

In what follows, we prove Proposition 3.1. Given \( n \in \mathbb{N} \), let \( N_n, A_n, R_n, T_n \gg 1 \) to be chosen later. We omit the dependence in \( n \) of these constants from now on for convenience.
We set \( v_0 = \psi + \phi \) with \( \phi \) as in (2.12).
Denote by \( v = v_n \) the global solution to (1.3) with initial data \( v(0) = v_0 \).
3.1. **Proof of Proposition 3.1.** We now prove Proposition 3.1. We claim that it suffices to show that the following properties hold:

(i) \( N^{s} R A^{\frac{1}{2}} \ll \frac{1}{n} \),

(ii) \( T R^{4} A^{4} \ll 1 \),

(iii) \( n \ll f_{s}(A) \cdot T R^{5} A^{4} \),

(iv) \( N \ll R^{2} A^{2} \),

(v) \( A \ll N \),

(vi) \( T \ll N^{-2} \),

for some \( A, R, T, \) and \( N \gg 1 \), depending on \( n \) and \( \psi \).

We first prove the above claim, i.e. we show how conditions (i)-(vi) imply Proposition 3.1. By Lemma 2.7 and Lemma 2.8, we have the following series expansion representation of solution \( v(t) := \sum_{j=0}^{\infty} \xi_{j}(v_{0})(t) \) \((3.2)\)

provided conditions (ii) and (v) hold. By (2.13), the first condition (i) ensures that the first inequality in (3.1) holds. By (3.2), (2.13), Proposition 2.10, Lemma 2.7, and Lemma 2.8 (which hold under (iv), (v) and (iv)), we have

\[
\|v(T)\|_{H^{s}} \geq \|\xi_{1}(\phi)(T)\|_{H^{s}} - \|\xi_{0}(\phi + \psi)(T)\|_{H^{s}} \\
- \sum_{j=1}^{\infty} \|\xi_{1}(\phi)(T) - \xi_{1}(\psi + \phi)(T)\|_{H^{s}} - \sum_{j=2}^{\infty} \|\xi_{j}(\phi)(T)\|_{H^{s}} \\
\geq f_{s}(A) \cdot T R^{5} A^{4} - (1 + N^{s} R A^{\frac{1}{2}}) - T R^{4} A^{4} - f_{s}(A) \cdot T^{2} R^{9} A^{8} \\
\sim T R^{5} A^{4} \cdot f_{s}(A) \geq n,
\]

where we used (i), (ii) and (iii) in the last inequality. Finally, choosing \( N \) sufficiently large such that \( R^{4} A^{4} \geq n \), together with (ii), imply \( t_{n} = T \in (0, \frac{1}{n}) \). This proves Proposition 3.1.

It thus remains to verify the conditions (i)-(v). In what follows, we consider the following three cases:

- **Case 1:** \( s < -\frac{1}{2} \). We set

\[
A = N^{\frac{\delta}{2}}, \quad R = N^{\frac{1}{2}}, \quad \text{and} \quad T = N^{-2-\delta},
\]

with \( \delta > 0 \) sufficiently small such that \( s + \frac{1}{2} + \frac{\delta}{10} < 0 \). The conditions (ii), (iv), (v) and (vi) are trivially satisfied for \( N \gg 1 \). By choosing \( N \) large enough, we have

\[
N^{s} R A^{\frac{1}{2}} = N^{s+\frac{1}{2}+\frac{\delta}{10}} \ll \frac{1}{n},
\]

\[
T R^{5} A^{4} = N^{\frac{1}{2} - \frac{\delta}{5}} \gg n,
\]

which verify the conditions (i) and (iii) as \( f_{s}(A) = 1 \) in this case.

- **Case 2:** \( s = -\frac{1}{2} \). In this case we have \( f_{s}(A) = (\log A)^{\frac{1}{2}} \). Set
\[ A = (\log N)^{\frac{3}{2}}, \quad R = \frac{N^{\frac{1}{2}}}{\log N}, \quad \text{and} \quad T = N^{-2-\delta}, \]

with \( 0 < \delta \ll 1 \). The conditions (ii), (iv), (v) and (vi) are trivially satisfied for \( N \gg 1 \). By choosing \( N \) large enough, we have

\[ N^{-\frac{1}{4}} RA^{\frac{1}{2}} = (\log N)^{-\frac{1}{4}} \ll \frac{1}{n}, \]

\[ TR^5 A^4 f_s(A) \gtrsim N^\frac{4}{5} \gg n. \]

Thus, the conditions (i) and (iii) are satisfied.

**Case 3:** \(-\frac{1}{2} < s < 0\). In this case, we have \( f_s(A) = A^{s+\frac{1}{2}} \). Choose

\[ A = N^{1+2s+\frac{9}{4} \delta}, \quad R = N^{-\frac{1}{2} - 2s - \frac{9}{4} \delta}, \quad \text{and} \quad T = N^{-2-\delta}, \]

with \( 0 < \delta \ll 1 \) satisfying \( 2s + \frac{9}{4} \delta < 0 \) and \( 2s^2 - \frac{3 \delta}{2} + \frac{9}{2} \delta s > 0 \). We then have

\[ N^s RA^{\frac{1}{2}} = N^{-\delta} \ll \frac{1}{n}, \]

\[ TR^4 A^4 = N^{-\frac{1}{2}} \ll 1, \]

\[ TR^5 A^{s+\frac{9}{2}} = N^{2s^2 - \frac{3 \delta}{2} + \frac{9}{4} \delta s} \gg n, \]

\[ R^2 A^2 = N^{1+\frac{s}{2}} \gg N, \]

\[ A = N^{1+2s+\frac{9}{4} \delta} \ll N, \]

provided \( N \) large enough. This verifies (i) - (vi).

**Remark 3.2.** This framework fails to provide the norm inflation at the endpoint regularity \( s = 0 \), which concurs with Remark 1.2. As a matter of fact, if \( s = 0 \), the condition (i) writes \( R \ll A^{-\frac{1}{2}} \), which implies \( R^2 A^2 \ll A \). This is incompatible with conditions (iv) and (v).

**Acknowledgments.** Y.W. and Y.Z. would like to thank Tadahiro Oh for suggesting this problem. Y.Z. was supported by the European Research Council (grant no. 864138 “SingStochDispDyn”). Y.W. was supported by supported by the EPSRC New Investigator Award (grant no. EP/V003178/1).

**References**

[1] I. Bejenaru, T. Tao, *Sharp well-posedness and ill-posedness results for a quadratic non-linear Schrödinger equation*, J. Funct. Anal. 233 (2006), no. 1, 228–259.

[2] H. A. Biagioni, F. Linares, *Ill-posedness for the derivative Schrödinger and generalized Benjamin-Ono equations*, Trans. Amer. Math. Soc. 353 (2001), no. 9, 3649–3659.

[3] J. Bourgain, *Invariant measures for the 2D-defocusing nonlinear Schrödinger equation*, Comm. Math. Phys. 176 (1996), no. 2, 421–445.

[4] R. Carles, T. Kappeler, *Norm-inflation with infinite loss of regularity for periodic NLS equations in negative Sobolev spaces*, Bull. Soc. Math. France 145 (2017), no. 4, 623–642. (Reviewer: Gabriel Stoltz) 35Q55 (35A01 35B30 81Q05).

[5] I. Chevyrev, *Norm inflation for a non-linear heat equation with Gaussian initial conditions*, arXiv:2205.14350 [math.AP].

[6] I. Chevyrev, T. Oh, Y. Wang, *Norm inflation for the cubic nonlinear heat equation above the scaling critical regularity*, arXiv:2205.14488 [math.AP].

[7] A. Choffrut, O. Pocovnicu, *Ill-posedness of the cubic half-wave equation and other fractional NLS on the real line*, preprint.
[8] M. Christ, Power series solution of a nonlinear Schrödinger equation, Mathematical aspects of nonlinear dispersive equations, 131–155, Ann. of Math. Stud., 163, Princeton Univ. Press, Princeton, NJ, 2007.

[9] M. Christ, J. Colliander, T. Tao, Asymptotics, frequency modulation, and low regularity ill-posedness for canonical defocusing equations, Amer. J. Math. 125 (2003), no. 6, 1235–1293.

[10] M. Christ, J. Colliander, T. Tao, Instability of the Periodic Nonlinear Schrödinger Equation, arXiv:math/0311227v1 [math.AP].

[11] M. Christ, J. Colliander, T. Tao, Ill-posedness for nonlinear Schrödinger and wave equations, arXiv:math/0311048 [math.AP].

[12] Y. Deng, A.R. Nahmod, H. Yue, Optimal local well-posedness for the periodic derivative nonlinear Schrödinger equation, arXiv:1905.04352 [math.AP].

[13] Y. Deng, A.R. Nahmod, H. Yue, Invariant Gibbs measures and global strong solutions for nonlinear Schrödinger equations in dimension two, arXiv:1910.08492 [math.AP].

[14] B. Harrop-Griffiths, R. Killip, M. Ntekoume, M. Visan, Global well-posedness for the derivative nonlinear Schrödinger equation in $L^2(\mathbb{R})$, arXiv:2204.12548 [math.AP].

[15] B. Harrop-Griffiths, R. Killip, M. Visan, Large-data equicontinuity for the derivative NLS, arXiv:2106.13333 [math.AP].

[16] J. Forlano, M. Okamoto, A remark on norm inflation for nonlinear wave equations, Dyn. Partial Differ. Equ. 17 (2020), no. 4, 361–381.

[17] S. Herr, Well-posedness results for dispersive equations with derivative nonlinearities, Ph.D. dissertation.

[18] S. Herr, On the Cauchy problem for the derivative nonlinear Schrödinger equation with periodic boundary condition, Int. Math. Res. Not. 2006, Art. ID 96763, 33 pp.

[19] T. Iwabuchi, T. Ogawa, Ill-posedness for the nonlinear Schrödinger equation with quadratic non-linearity in low dimensions, Trans. Amer. Math. Soc. 367 (2015), no. 4, 2613–2630.

[20] D.J. Kaup, A.C. Newell, An exact solution for a derivative nonlinear Schrödinger equation, J. Mathematical Phys. 19 (1978), no. 4, 798–801.

[21] R. Killip, M. Ntekoume, M. Visan, On the well-posedness problem for the derivative nonlinear Schrödinger equation, arXiv:2102.12274v1.

[22] N. Kishimoto, A remark on norm inflation for nonlinear Schrödinger equations, Commun. Pure Appl. Anal. 18 (2019), no. 3, 1375–1402.

[23] W. Mio, T. Ogino, K. Minami, S. Takeda, Modified nonlinear Schrödinger equation for Alfven waves propagating along the magnetic field in cold plasmas, J. Phys. Soc. Japan 41 (1976), no. 1, 265–271.

[24] R. Mosincat, Global well-posedness of the derivative nonlinear Schrödinger equation with periodic boundary condition in $H^s(\mathbb{R})$, J. Differential Equations 263 (2017), no. 8, 4658–4722.

[25] A.R. Nahmod, T. Oh, L. Rey-Bellet, G. Staffilani, Invariant weighted Wiener measures and almost sure global well-posedness for the periodic derivative NLS, J. Eur. Math. Soc. (JEMS) 14 (2012), no. 4, 1275–1330.

[26] T. Oh, Y. Wang, On the ill-posedness of the cubic nonlinear Schrödinger equation on the circle, An. Științ. Univ. Al. I. Cuza Iași. Mat. (N.S.) 64 (2018), no. 1, 53–84.

[27] T. Oh, A remark on norm inflation with general initial data for the cubic nonlinear Schrödinger equations in negative Sobolev spaces, Funkcial. Ekvac. 60 (2017), no. 2, 259–277.

[28] M. Okamoto, Norm inflation for the generalized Boussinesq and Kawahara equations, Nonlinear Anal. 157 (2017), 44–61.

[29] H. Takaoka, Well-posedness for the one-dimensional nonlinear Schrödinger equation with the derivative nonlinearity, Adv. Differential Equations 4 (1999), no. 4, 561–580.

[30] M. Tsutsumi, I. Fukuda, On solutions of the derivative nonlinear Schrödinger equation. Existence and uniqueness theorem, Funkcial. Ekvac. 23 (1980), no. 3, 259–277.

[31] M. Tsutsumi, I. Fukuda, On solutions of the derivative nonlinear Schrödinger equation. II, Funkcial. Ekvac. 24 (1981), no. 1, 85–94.

[32] B. Xia, Generic ill-posedness for wave equation of power type on 3D torus, arXiv:1507.07179 [math.AP].

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