NONLINEAR CHOQUARD EQUATIONS: DOUBLY CRITICAL CASE

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ABSTRACT. Consider nonlinear Choquard equations

\[ \begin{cases} -\Delta u + u = (I_\alpha * F(u))F'(u) & \text{in } \mathbb{R}^N, \\ \lim_{|x| \to \infty} u(x) = 0, \end{cases} \]

where \( I_\alpha \) denotes Riesz potential and \( \alpha \in (0, N) \). In this paper, we show that when \( F \) is doubly critical, i.e., \( F(u) = \frac{1}{|\alpha|} |u|^{\alpha+2} + \frac{2}{\alpha} |u|^{\frac{2N+2}{N-2}} \), the nonlinear Choquard equation admits a nontrivial solution if \( N \geq 5 \) and \( \alpha + 4 < N \).

Keywords: semilinear elliptic; Choquard equation; critical exponent; variational method

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1. Introduction

Let \( N \geq 3 \), \( \alpha \in (0, N) \). We are concerned with the nonlinear Choquard equation:

\[ \begin{cases} -\Delta u + u = (I_\alpha * F(u))F'(u) & \text{in } \mathbb{R}^N, \\ \lim_{|x| \to \infty} u(x) = 0, \end{cases} \]

(1.1)

where \( I_\alpha \) is Riesz potential given by

\[ I_\alpha(x) = \frac{\Gamma(\frac{N-\alpha}{2})}{\Gamma(\frac{\alpha}{2})\pi^\frac{N}{2}|x|^{N-\alpha}}. \]

and \( \Gamma \) denotes the Gamma function. It is the Euler-Lagrange equation of the functional

\[ J_\alpha(u) = \frac{1}{2} \int_{\mathbb{R}^N} |\nabla u|^2 + u^2 \, dx - \frac{1}{2} \int_{\mathbb{R}^N} (I_\alpha * F(u))F(u) \, dx. \]

Physical motivation of (1.1) comes from the case that \( F \) is out of subcritical, i.e., \( 1 < p \leq 2 \) or \( p \geq 2N/(N-2) \). In this case, the equation (1.1) is called the Choquard-Pekar equation \([9, 18]\), Hartree equation \([6, 8]\) or Schrödinger-Newton equation \([13, 24]\), depending on its physical backgrounds and derivations. The existence of a ground state in this case is studied in \([9, 11, 12]\) via variational arguments.

The functional \( J_\alpha \) can be considered as a nonlocal perturbation of the fairly well-studied functional consisting of only local terms:

\[ J_0(u) = \frac{1}{2} \int_{\mathbb{R}^N} |\nabla u|^2 + u^2 \, dx - \int_{\mathbb{R}^N} G(u) \, dx \]

since as \( \alpha \to 0 \), \( J_\alpha \) approaches to \( J_0 \) with \( G(u) = \frac{1}{2} F'(u) \). A critical point of \( J_0 \) is a solution to the stationary nonlinear Schrödinger equation:

\[ -\Delta u + u = G'(u). \]

(1.2)

The power type function \( \frac{1}{p} |u|^p \) is a standard choice for nonlinearity \( G(u) \) (and also \( F(u) \)). By Sobolev inequality, it can be shown that the functional \( J_\alpha \) is a well-defined \( C^1 \) functional on \( H^1(\mathbb{R}^N) \) if \( G(u) = \frac{1}{p} |u|^p \) and \( p \in [2, \frac{2N}{N-2}] \). It is a classical result that it admits a nontrivial critical point of ground state level in the subcritical range \( p \in (2, \frac{2N}{N-2}] \) \([3, 22]\). Moreover, the standard application of Pohozaev’s identity says that if \( p \) is out of subcritical, i.e., \( 1 < p \leq 2 \) or \( p \geq 2N/(N-2) \), the equation (1.2) does not admit any nontrivial finite energy solution. In case of \( J_\alpha \), Hardy-Littlewood-Sobolev inequality (Proposition 2.2 below) replaces Sobolev inequality to see that \( J_\alpha \) with \( F(u) = \frac{1}{p} |u|^p \) is well-defined and is continuously differentiable on \( H^1(\mathbb{R}^N) \) if \( p \in (\frac{2N}{N+2}, \frac{2N}{N-2}) \). Two numbers \( \frac{2N}{N+2} \) and \( \frac{2N}{N-2} \) play roles of lower and upper critical exponents for existence. It is proved by Moroz and Van Schaftingen \([15]\) that for every \( \alpha \in (0, N) \), there exists a nontrivial ground state solution if \( p \) is in the subcritical range, i.e., \( p \in (\frac{2N}{N+2}, \frac{2N}{N-2}) \) and there is no nontrivial finite energy solution if \( p \) is outside of subcritical, i.e., \( 1 < p \leq \frac{2N}{N-2} \) or \( p \geq \frac{2N}{N-2} \). This result is compatible with the existence of the limit equation (1.2). Observe the existence range \( p \in (\frac{2N}{N+2}, \frac{2N}{N-2}) \) tends to \( p \in (1, \frac{2N}{N-2}) \) and the nonlinear term \( (I_\alpha * |u|^\alpha)|u|^\alpha \) tends to \( |u|^{2p} \) as \( \alpha \to 0 \). We recall that the equation (1.2) with \( G(u) = \frac{1}{2} F'(u) = \frac{1}{2p} |u|^{2p} \) admits a nontrivial finite energy solution if and only if \( p \in (0, \frac{N}{N-2}) \). Furthermore, we have \( H^1 \) convergence of ground states. For any \( p \in (1, \frac{N}{N-2}) \), choose a small \( \alpha_0 > 0 \) that \( p \) belongs to the segment \( (\frac{2N}{N+2}, \frac{2N}{N-2}) \) for every \( \alpha \in (0, \alpha_0) \) so that a radial positive ground state \( u_0 \) to (1.1) with \( F(u) = \frac{1}{p} |u|^p \) exists. Then it is possible to show that as \( \alpha \to 0 \), \( u_\alpha \) converges in \( H^1 \) sense to a ground state \( u_0 \) of the corresponding functional \( J_0 \). See \([19, 20]\).

For general nonlinearity \( G \), Berestycki and Lions prove in their celebrated paper \([4]\) that (1.2) admits a ground state solution when \( G \) is \( C^2(\mathbb{R}) \) and satisfies the following:

(G1) there exists a constant \( C > 0 \) such that for every \( s \in \mathbb{R} \),

\[ |s G(s)| \leq C(|s|^2 + |s|^\frac{2N}{N-2}), \]

(G2) \( \lim_{s \to -\infty} \frac{G(s)}{|s|^\frac{2N}{N-2}} = 0 \) and \( \lim_{s \to 0} \frac{G(s)}{|s|^\frac{2N}{N-2}} = 0 \),

(G3) there exists a constant \( s_0 \in \mathbb{R} \setminus \{0\} \) such that \( G(s_0) > \frac{\alpha}{2} \).
In the same spirit, it is proved in [14] that there exists a ground state solution to (1.1) under the following conditions for the nonlinearity function $F \in C^{1}(\mathbb{R})$:

(F1) (growth) there exists a constant $C > 0$ such that for every $s \in \mathbb{R}$,
$$|sF'(s)| \leq C(|s|^{\frac{N+\alpha}{N}} + |s|^{\frac{N}{N-2}}),$$

(F2) (subcriticality) $\lim_{s \to 0} \frac{F(s)}{|s|^{q}} = 0$ and $\lim_{s \to \infty} \frac{F(s)}{|s|^{p}} = 0$.

(F3) (nontriviality) there exists a constant $s_0 \in \mathbb{R} \setminus \{0\}$ such that $F(s_0) \neq 0$.

Interestingly, note that the condition (F3) is inconsistent with (G3) of limit equation (1.2) while we have seen the consistency between (1.1) and (1.2) of power type.

In this paper, we are interested in some choice of $F$ that violates the subcriticality condition (F2). The setting that we can naturally choose would be

$$F(u) := \frac{1}{p} |u|^p + \frac{1}{q} |u|^q, \quad N + \alpha < q \leq \frac{N + \alpha}{N - 2},$$

and either $p = \frac{N+\alpha}{N}$ or $q = \frac{N}{N-2}$. With $G(u) = \frac{1}{p} |u|^p + \frac{1}{q} |u|^q$, it is proved in [11] that the limit equation (1.2) admits a nontrivial solution if $2 < p < q < \frac{2N}{N-2}$ for $N \geq 4$ and $4 < p < q$ for $N = 3$. In this setting, considering a doubly critical choice of $p$ and $q$, i.e., $p = 2$, $q = \frac{N}{N-2}$ does not seem appropriate because the equation is just reduced to the Lane-Emden equation $-\Delta u = |u|^p - |u|^q u$. The situation is however different when we consider the nonlinear Choquard equation (1.1) with a pair of lower and upper critical exponents: $p = \frac{N+\alpha}{N}$ and $q = \frac{N}{N-2}$. The purpose of this paper is to study this case. We shall prove that there exists a nontrivial solution under some restrictions on $N$ and $\alpha$.

**Theorem 1.1.** Let $N \geq 5$ and $F(u) = \frac{1}{p} |u|^p + \frac{1}{q} |u|^q$. Then, there exists a nontrivial solution $u \in H^1(\mathbb{R}^N)$ to (1.1) if $p = \frac{N+\alpha}{N}$, $q = \frac{N}{N-2}$ and $N > 4 + \alpha$.

For the related critical problems involving only a single critical exponent, we refer to [2, 3, 5, 16, 21]. When we approach by variational methods to prove Theorem 1.1, the main difficulty we encounter is to deal with two different types of loss of compactness. By expanding the nonlinear term $\int_{\mathbb{R}^N} (I_{\alpha} * |u|^\frac{N+\alpha}{N}) |u|^{\frac{N}{N-2}}$ and $\int_{\mathbb{R}^N} (I_{\alpha} * |u|^\frac{N}{N-2}) |u|^{\frac{N}{N-2}}$, which are invariant under dilations $\lambda^{N/2}u(\lambda x)$ and $\lambda^{N/2-2}u(\lambda x)$ respectively. These two dilations are noncompact group actions on $H^1(\mathbb{R}^N)$, each of which prevents a general $(PS)$ sequence of $J_\alpha$ from being relatively compact. The following two inequality, that is special cases of Hardy-Littlewood-Sobolev inequality, play significant roles to resolve this difficulty. The first one is

$$S_1 \left( \int_{\mathbb{R}^N} (I_{\alpha} * |u|^\frac{N+\alpha}{N}) |u|^{\frac{N}{N-2}} \right)^{\frac{N}{N-2}} \leq \int_{\mathbb{R}^N} u^2 \, dx \quad (1.3)$$

whose extremal functions are

$$u(x) = C \frac{\lambda^{N/2}}{(\lambda^2 + |x|^2)^{N/2}}.$$

The second one is

$$S_2 \left( \int_{\mathbb{R}^N} (I_{\alpha} * |u|^\frac{N}{N-2}) |u|^{\frac{N}{N-2}} \right)^{\frac{N}{N-2}} \leq \int_{\mathbb{R}^N} |\nabla u|^2 \, dx \quad (1.4)$$

whose extremal functions are

$$u(x) = C \frac{\lambda^{N/2}}{(\lambda^2 + |x|^2)^{N/2}}.$$

A key step to prove the existence of a solution is a characterization of level sets at which a $(PS)$ sequence of $J_\alpha$ converges. These levels are given in terms of two best constants $S_1$ and $S_2$ of inequalities (1.3) and (1.4). More precisely, we shall obtain the following proposition. We say a sequence $(u_j) \in H^1(\mathbb{R}^N)$ is a $(PS)$ sequence of $J_\alpha$ at level $c$ if

$$J_\alpha(u_j) \to 0 \quad \text{in} \quad H^{-1}(\mathbb{R}^N) \quad \text{and} \quad J_\alpha(u_j) \to c \quad \text{as} \quad j \to \infty.$$

**Proposition 1.1.** Assume $p = \frac{N+\alpha}{N}$, $q = \frac{N}{N-2}$. Let $(u_j) \subset H^1(\mathbb{R}^N)$ be a $(PS)$ sequence of $J_\alpha$ at level $c$. Then it is relatively compact in $H^1(\mathbb{R}^N)$ if

$$c < \min \left\{ \frac{1}{2} \left( 1 - \frac{1}{p} \right) \frac{p}{p-1} S_1^{\frac{1}{p}} \right\} \leq \frac{1}{2} \left( 1 - \frac{1}{q} \right) S_2^{\frac{1}{q}}.$$

**Proposition 1.1** shall be proved in Section 3. Then, the remaining work is to show the mountain pass energy level $c$ of $J_\alpha$ satisfies the condition of Proposition 1.1. This shall be done in Section 4 by testing two aforementioned families of extremal functions to the functional $J_\alpha$. In Section 2, we collect various useful inequalities and estimate required when we prove Proposition 1.1 and Theorem 1.1.

2. Auxiliary tools

In this section, we prepare some auxiliary tools for proving our main theorem. The well-known Hardy-Littlewood-Sobolev inequality is stated as follows.
Proposition 2.1 (Hardy-Littlewood-Sobolev inequality (10)). Let $p, r > 1$ and $0 < \alpha < N$ be such that
\[
\frac{1}{p} + \frac{1}{r} = 1 + \frac{\alpha}{N}.
\]
Then there exists $C > 0$ depending only on $N, \alpha, p$ such that for any $f \in L^p(\mathbb{R}^N)$ and $g \in L^r(\mathbb{R}^N)$
\[
\left| \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{f(x)g(y)}{|x-y|^{N-\alpha}} \, dx \, dy \right| \leq C(N, \alpha, p)\|f\|_{L^p(\mathbb{R}^N)}\|g\|_{L^r(\mathbb{R}^N)}.
\]
Hardy-Littlewood-Sobolev inequality has a dual form called the Riesz potential estimate.

Proposition 2.2 (Riesz potential estimate (11)). Let $1 \leq p < s < \infty$ and $0 < \alpha < N$ be such that
\[
\frac{1}{p} - \frac{1}{s} = \frac{\alpha}{N}.
\]
Then there exists $C > 0$ depending only on $N, \alpha$ such that for any $f \in L^p(\mathbb{R}^N)$,
\[
\left\| \frac{1}{|\cdot|^{N-\alpha}} * f \right\|_{L^s(\mathbb{R}^N)} \leq C\|f\|_{L^p(\mathbb{R}^N)}.
\]

We denote by $H^s_0(\mathbb{R}^n)$ the space of radial functions in $H^s(\mathbb{R}^N)$. Non-invariance of $H^s$ norm of a function $u \in H^s_0(\mathbb{R}^N)$ by translations induces the compact embedding to $L^s(\mathbb{R}^N)$ for subcritical $p$. See (22).

Proposition 2.3. The Sobolev embedding $H^s_0(\mathbb{R}^N) \hookrightarrow L^q(\mathbb{R}^N)$ is compact if $2 < p < 2N/(N-2)$.

By combining Hardy-Littlewood-Sobolev inequality (Proposition 2.1) and the compact Sobolev embedding, a standard analysis shows the following convergences hold. We refer to (20) for details.

Proposition 2.4. Let $\alpha \in (0, N)$ and $\{u_j\} \subset H^s_0(\mathbb{R}^N)$ be a sequence converging weakly to some $u_0 \in H^s_0(\mathbb{R}^N)$ in $H^s(\mathbb{R}^N)$ as $j \to \infty$.

(i) If $\frac{N+s}{N} < p \leq q < \frac{N+s}{N-2}$, then
\[
\int_{\mathbb{R}^N} \left( \frac{1}{|\cdot|^{N-\alpha}} * |u_j|^p \right) |u_j|^q \, dx \to \int_{\mathbb{R}^N} \left( \frac{1}{|\cdot|^{N-\alpha}} * |u_0|^p \right) |u_0|^q \, dx;
\]

(ii) If $\phi \in H^s_0(\mathbb{R}^N)$, $\frac{N+s}{N} \leq p \leq q \leq \frac{N+s}{N-2}$, then
\[
\int_{\mathbb{R}^N} \left( \frac{1}{|\cdot|^{N-\alpha}} * |u_j|^p \right) |u_j|^{p-2} \phi \, dx \to \int_{\mathbb{R}^N} \left( \frac{1}{|\cdot|^{N-\alpha}} * |u_0|^p \right) |u_0|^{p-2} \phi \, dx.
\]

The following version of Brezis-Lieb lemma for the Riesz potential is useful for our analysis. We refer to (13) for a proof.

Proposition 2.5. Let $\alpha \in (0, N)$ and $p \in [\frac{N+s}{N}, \frac{N+s}{N-2}]$ be given. If $\{u_n\}$ be a bounded sequence in $L^{\frac{N+s}{N}}(\mathbb{R}^N)$ such that $u_n \to u$ almost everywhere as $n \to \infty$ for some function $u$, then $u \in L^{\frac{N+s}{N}}(\mathbb{R}^N)$ and
\[
\lim_{n \to \infty} \left( \int_{\mathbb{R}^N} (I_s * |u_n|^p)|u_n|^q - \int_{\mathbb{R}^N} (I_s * |u_n-u|^p)|u_n-u|^q \right) = \int_{\mathbb{R}^N} (I_s * |u|^p)|u|^q.
\]

3. Proof of Proposition 1.1

In this section, we prove Proposition 1.1. We first show that $\{u_j\}$ is bounded in $H^s(\mathbb{R}^N)$. Indeed, from the definition of $(PS)$ sequence,
\[
c + o(1) = J_s(u_j) = \frac{1}{2} ||u_j||_{H^s}^2 - \frac{1}{2} \int_{\mathbb{R}^N} (I_s * \left( \frac{1}{p} |u_j|^p + \frac{1}{q} |u_j|^q \right))(\frac{1}{p} |u_j|^p + \frac{1}{q} |u_j|^q) \, dx,
\]
\[
o(1)||u_j||_{H^s} = J'_s(u_j) = ||u_j||_{H^s}^2 - \int_{\mathbb{R}^N} (I_s * \left( \frac{1}{p} |u_j|^p + \frac{1}{q} |u_j|^q \right))(\frac{1}{p} |u_j|^p + \frac{1}{q} |u_j|^q) \, dx.
\]
Then,
\[
\frac{1}{2} ||u_j||_{H^s}^2 \leq c + o(1) + \frac{1}{2p} \int_{\mathbb{R}^N} (I_s * \left( \frac{1}{p} |u_j|^p + \frac{1}{q} |u_j|^q \right))(\frac{1}{p} |u_j|^p + \frac{1}{q} |u_j|^q) \, dx
\]
\[
= \frac{1}{2p} \left( ||u_j||_{H^s}^2 + o(1)||u_j||_{H^s} \right) + c + o(1).
\]
Since $p > 1$, this shows $||u_j||_{H^s}$ is bounded.

Now, up to a subsequence, $\{u_j\}$ weakly converges to some $u_0 \in H^s_0(\mathbb{R}^N)$. Using (ii) of Proposition 2.4 it is standard to show that $u_0$ is a weak solution of (1.1). Let $w_j := u_j - u_0$. From Proposition 2.5 and Proposition 2.4 we see that
\[
||w_j||_{H^s}^2 = ||u_j - u_0||_{H^s}^2 = ||u_j||_{H^s}^2 - ||u_0||_{H^s}^2 + o(1)
\]
\[
= \int_{\mathbb{R}^N} (I_s * \left( \frac{1}{p} |u_j|^p + \frac{1}{q} |u_j|^q \right))(\frac{1}{p} |u_j|^p + \frac{1}{q} |u_j|^q) \, dx + o(1)||u_j||_{H^s}^2
\]
\[
- \int_{\mathbb{R}^N} (I_s * \left( \frac{1}{p} |u_0|^p + \frac{1}{q} |u_0|^q \right))(\frac{1}{p} |u_0|^p + \frac{1}{q} |u_0|^q) \, dx + o(1)
\]
\[
= \frac{1}{p} \int_{\mathbb{R}^N} (I_s * |w_j|^p)|w_j|^q \, dx + \frac{1}{q} \int_{\mathbb{R}^N} (I_s * |w_j|^q)|w_j|^p \, dx + o(1).
\]

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Combining inequalities (1.3) and (1.4) with this,
\[
S_1 \left( \int_{\mathbb{R}^N} (I_u * |w|^p)|w|^p \right) + S_2 \left( \int_{\mathbb{R}^N} (I_u * |w|^p)|w|^p \right)
\leq \frac{1}{p} \int_{\mathbb{R}^N} (I_u * |w|^p)|w|^p \, dx + \frac{1}{q} \int_{\mathbb{R}^N} (I_u * |w|^p)|w|^p \, dx + o(1).
\]
We define
\[
x := \limsup_{j \to \infty} \int_{\mathbb{R}^N} (I_u * |w|^p)|w|^p, \quad y := \limsup_{j \to \infty} \int_{\mathbb{R}^N} (I_u * |w|^p)|w|^p,
\]
both of which are finite since \(||w||_{H^1} \) is bounded. Passing to a limit, we have
\[
S_1 x^p + S_2 y^p \leq \frac{1}{p} x + \frac{1}{q} y. \tag{3.6}
\]
We claim that \(x = y = 0\). We prove this by getting rid of any other possibilities: (1) \(x = 0, y \neq 0\); (2) \(x \neq 0, y = 0\); and (3) \(x \neq 0, y \neq 0\). Suppose first the case (1). Then one has \(S_2 y^p \leq \frac{1}{q} y\), which implies \(y \geq \frac{q}{2} \). In the case (2), we have \(x \geq (pS_1)^{\frac{1}{2p}}\). In the case (3), one has either \(y \geq (qS_2)^{\frac{1}{2q}}\) or \(x \geq (pS_1)^{\frac{1}{2p}}\) because, if \(y < (qS_2)^{\frac{1}{2q}}\) and \(x < (pS_1)^{\frac{1}{2p}}\), then
\[
\frac{1}{p} x + \frac{1}{q} y - S_1 x^p - S_2 y^p = x^p \left( \frac{1}{p} x^{-\frac{1}{p}} - S_1 \right) + y^p \left( \frac{1}{q} y^{-\frac{1}{q}} - S_2 \right) < 0,
\]
so that (3.6) does not hold. Thus we conclude that in any case, \(y \geq (qS_2)^{\frac{1}{2q}}\) or \(x \geq (pS_1)^{\frac{1}{2p}}\). Now we again use Proposition 2.5, Proposition 2.4 and 3.5 to deduce
\[
J_x(u_0) = J_x(u_0) + \frac{1}{2} ||w||_{H^1}^2 - \frac{1}{2p} \int_{\mathbb{R}^N} (I_u * |w|^p)|w|^p \, dx
\]
\[
= \frac{1}{2p} \int_{\mathbb{R}^N} (I_u * |w|^p)|w|^p \, dx
\]
\[
+ \frac{1}{2q} \int_{\mathbb{R}^N} (I_u * |w|^p)|w|^p \, dx + o(1).
\]
Taking a limit and using the fact \(J_x(u_0) \geq 0\), we conclude that either \(c \geq (\frac{1}{2p} - \frac{1}{2q})(pS_1)^{\frac{1}{2p}}\) or \(c \geq (\frac{1}{2q} - \frac{1}{2p})(qS_2)^{\frac{1}{2q}}\) but this contradicts with the assumption of the proposition. So the claim is proved.

Now we are ready to complete the proof. Since \(x = y = 0\), the equality 3.4 says that \(||w||_{H^1} \to 0\) as \(j \to \infty\) up to a subsequence. Therefore \(u_j \to u_0\) in \(H^1\) as \(j \to \infty\) up to a subsequence.

4. Proof of Theorem 1.1

We first show that \(J_x\) satisfies the mountain pass geometry on \(H^1(\mathbb{R}^N)\). In other words, we show that there exist \(r_0 > 0\) and \(u_0 \in H^1(\mathbb{R}^N)\) such that
\[
(i) \quad \inf_{\|w\|_{H^1} = r_0} J_x(u) \geq 0 \quad \text{and} \quad \inf_{\|w\|_{H^1} \to 0} J_x(u) > 0,
\]
\[
(ii) \quad J_x(u_0) < 0 \quad \text{(and thus \(\|w_0\|_{H^1} > r_0\)).}
\]
The assertion (ii) immediately follows from the Hardy-Littlewood-Sobolev inequality (Proposition 2.1). Also, for any given \(u \in H^1(\mathbb{R}^N)\), the function \(f(t) := J_x(tu)\) takes the form \(A^2 - Br^{2p} + Cr^{2q} - D^{p+q} t\) so that \(f(0) = 0\) and \(\lim_{t \to \infty} f(t) = -\infty\).

On the interval \((0, \infty)\), we can see that \(f'(t) = 0\) if and only if \(2pBr^{2p-2} + 2qCr^{2q-2} + (p + q)D^{p+q-2} = 2A\). Define \(g(t) := 2pBr^{2p-2} + 2qCr^{2q-2} + (p + q)D^{p+q-2}\). Observe \(g\) is strictly increasing on \((0, \infty)\), \(g(0) = 0\) and \(\lim_{t \to \infty} g(t) = \infty\). This shows \(f\) admits a unique critical point \(t_0\) on \((0, \infty)\) such that \(f\) takes the maximum at \(t = t_0\), \(f\) is strictly increasing on \((0, t_0)\) and \(f\) is strictly decreasing on \((t_0, \infty)\), and the assertion (ii) follows.

Let \(\Gamma\) denote the set of every continuous paths \(\gamma : [0, 1] \to H^1(\mathbb{R}^N)\) satisfying \(\gamma(0) = 0\) and \(J_x(\gamma(1)) < 0\). We define
\[
c_0 = \inf_{\gamma \in C(0, 1)} J(\gamma(t)).
\]
Since \(J_x\) satisfies the mountain pass geometry, the standard deformation lemma (see [23, 25]) says that there exists a (PS) sequence \(\{u_j\} \subset H^1(\mathbb{R}^N)\) of \(J_{\|u\|_{L^2}}\) at level \(c_0\), i.e.,
\[
J'_{\|u\|_{L^2}}(u_j) \to 0 \quad \text{in} \quad H^{-1}(\mathbb{R}^N) \quad \text{and} \quad J_{\|u\|_{L^2}}(u_j) \to c_0 \quad \text{as} \quad j \to \infty.
\]
We claim that
\[
c_0 < \min \left( \frac{1}{2} \left( 1 - \frac{1}{p} \right) p^{\frac{1}{p}} S_1^{\frac{1}{p}} \quad \text{and} \quad \frac{1}{2} \left( 1 - \frac{1}{q} \right) q^{\frac{1}{q}} S_2^{\frac{1}{q}} \right).
\]
If this is shown, it follows that \(c_0\) is a critical level of \(J_{\|u\|_{L^2}}\) by Proposition 1.1. Since the mountain pass geometry of \(J_x\) implies \(c_0 \geq 0\), we get a nontrivial critical point of \(J_{\|u\|_{L^2}}\). By the principle of symmetric criticality by Palais [12], this is also a nontrivial critical point of \(J_x\) on \(H^1(\mathbb{R}^N)\), which is a solution to (1.1).
which are the extremal functions of the inequalities (4.3) and (4.4) respectively. The constants $A$ and $B$ are chosen to satisfy
\[
\int_{\mathbb{R}^N} |u_1|^2 \, dx = \int_{\mathbb{R}^N} (I_a + |u_1|^p) |u_1|^p \, dx, \quad \int_{\mathbb{R}^N} |\nabla u_1|^2 \, dx = \int_{\mathbb{R}^N} (I_a + |v_1|^q) |v_1|^q \, dx.
\]
Since $N \geq 5$, one has $\mu_s, \nu_s \in H^1_0(\mathbb{R}^N)$. Let $t_0 > 0$ and $s_0 > 0$ be two values satisfying
\[
J_{s_0}(t_0 \mu_s) = \max_{t \geq 0} J_t(\mu_s), \quad J_{s_0}(s_0 \nu_s) = \max_{t \geq 0} J_t(\nu_s).
\]
Let $\tilde{t}_j$ and $\tilde{s}_j$ be the numbers that satisfy $J_{s_0}(\tilde{t}_j \mu_s) < 0$ and $J_{s_0}(\tilde{s}_j \nu_s) < 0$. We have seen $\tilde{t}_j > t_j$ and $\tilde{s}_j > s_j$ should hold. Then by defining $\gamma(t) := t \mu_s$ and $\nu(t) := \tilde{t}_j \nu_s$, we see that
\[
c_0 \leq \min_{t \in (0,1]} \max_{t \in \mathbb{R}} J_t(\gamma_t(t)), \quad \max_{t \in \mathbb{R}} J_t(\nu(t)) = \min\{J_{s_0}(t \mu_s), J_{s_0}(s_0 \nu_s) \}.
\]
We compute
\[
0 = \frac{d}{dt} \bigg|_{t=t_j} J_t(\mu_s)
= t_j \int_{\mathbb{R}^N} |\nabla \mu|^2 \, dx + t_j \int_{\mathbb{R}^N} |\mu|^2 \, dx - \frac{p}{2} \int_{\mathbb{R}^N} (I_a + |\mu|^p) |\mu|^p \, dx - \frac{2^{p-1}}{q} \int_{\mathbb{R}^N} (I_a + |\mu|^q) |\mu|^q \, dx
- \frac{(p+q)p^{p-1}}{pq} \int_{\mathbb{R}^N} (I_a + |\mu|^p) |\mu|^p \, dx
= t_j A - \frac{p}{2} \int_{\mathbb{R}^N} |\nabla \mu|^2 \, dx + t_j \int_{\mathbb{R}^N} |\mu|^2 \, dx - \frac{2^{p-1}}{q} \int_{\mathbb{R}^N} (I_a + |\mu|^q) |\mu|^q \, dx
- \frac{(p+q)p^{p-1}}{pq} \int_{\mathbb{R}^N} (I_a + |\mu|^p) |\mu|^p \, dx
\]
Let $t_m := \limsup_{j \to \infty} t_j$. Suppose that $t_m = \infty$. Then dividing the both side of (4.7) by $t_j$ and taking a limit $\lambda \to \infty$, we get a contradiction and thus $t_m < \infty$. We again pass to a limit $\lambda \to \infty$ in (4.7) to obtain
\[
0 = t_m \int_{\mathbb{R}^N} |\mu|^2 \, dx - t_j \left( \frac{p-2}{2} \right) \int_{\mathbb{R}^N} (I_a + |\mu|^p) |\mu|^p \, dx - \frac{2^{p-1}}{q} \int_{\mathbb{R}^N} (I_a + |\mu|^q) |\mu|^q \, dx,
\]
which implies $t_m = p^{1/(2p-2)}$.

Now, observe
\[
J_t(t \mu_s) = \frac{t_j A}{2} \int_{\mathbb{R}^N} |\nabla \mu|^2 \, dx + \frac{t_j}{2} \int_{\mathbb{R}^N} |\mu|^2 \, dx - \frac{2^{p-1}}{q} \int_{\mathbb{R}^N} (I_a + |\mu|^q) |\mu|^q \, dx
- \frac{(p+q)p^{p-1}}{pq} \int_{\mathbb{R}^N} (I_a + |\mu|^p) |\mu|^p \, dx
\leq \left( \frac{t_j}{2} - \frac{2^{p-1}}{2p^2} \right) \int_{\mathbb{R}^N} |\mu|^2 \, dx
- \frac{(p+q)p^{p-1}}{pq} \int_{\mathbb{R}^N} (I_a + |\mu|^p) |\mu|^p \, dx.
\]
Note that the curve $f(t) := (\frac{t_j}{2} - \frac{2^{p-1}}{2p^2}) \int_{\mathbb{R}^N} |\mu|^2 \, dx$ attains its maximum at $t = t_m$. This shows
\[
\left( \frac{t_j}{2} - \frac{2^{p-1}}{2p^2} \right) \int_{\mathbb{R}^N} |\mu|^2 \, dx \leq \frac{1}{2} \left( 1 - \frac{1}{p} \right) \mu_s^p S_1^{\frac{p}{2}}
\]
Since $\frac{p}{2} (p + q) - (N + \alpha) < 2$ if and only if $4 + \alpha < N$, we deduce that for sufficiently large $\lambda > 0$
\[
J_{s_0}(t \mu_s) < \frac{1}{2} \left( 1 - \frac{1}{p} \right) \mu_s^p S_1^{\frac{p}{2}}.
\]
Similarly we have
\[
0 = \frac{d}{dt} \bigg|_{t=t_j} J_t(\nu_j)
= s_j \int_{\mathbb{R}^N} |\nabla \nu|^2 \, dx + s_j \int_{\mathbb{R}^N} |\nu|^2 \, dx - \frac{p}{p \nu_j^2 (N-2p+2)} \int_{\mathbb{R}^N} (I_a + |\nu|^p) |\nu|^p \, dx - \frac{2^{p-1}}{q} \int_{\mathbb{R}^N} (I_a + |\nu|^q) |\nu|^q \, dx
- \frac{(p+q)p^{p-1}}{pq} \int_{\mathbb{R}^N} (I_a + |\nu|^p) |\nu|^p \, dx.
\]
from which we conclude that $s_0 := \limsup_{p \to 0} s_1 = q/(2q-2)$ by arguing similarly. Then we see that

$$J_\alpha(s_1 v_1) \leq \left( \frac{s_1^2}{2} - \frac{s_1^2}{2q} \right) \int_{\mathbb{R}^N} |\nabla v_1|^2 \, dx + \frac{s_1^2 l}{2} \int_{\mathbb{R}^N} |v_1|^2 \, dx$$

$$- (p + q) \frac{s_1^{p+q-1}}{p q} \alpha \lambda^{\frac{p+q}{2}} \frac{\alpha}{\sqrt{s_1}} \int_{\mathbb{R}^N} (L_\alpha + |v_1|^p) \, dx$$

$$\leq \frac{1}{2} \left( 1 - \frac{1}{q} \right) q^{-1} \int_{\mathbb{R}^N} |\nabla v_1|^2 \, dx$$

for sufficiently small $\lambda$ since $-\frac{q}{s_0^2} (p + q) + N + \alpha = \frac{N}{s_0} < 2$. This completes the proof.

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