Versions of Gradient Temporal Difference Learning
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Abstract—Sutton, Szepesvári and Maei introduced the first gradient temporal-difference (GTD) learning algorithms compatible with both linear function approximation and off-policy training. The goal of this paper is (a) to propose some variants of GTDs with extensive comparative analysis and (b) to establish new theoretical analysis frameworks for the GTDs. These variants are based on convex-concave saddle-point interpretations of GTDs, which effectively unify all the GTDs into a single framework, and provide simple stability analysis based on recent results on primal-dual gradient dynamics. Finally, numerical comparative analysis is given to evaluate these approaches.

Index Terms—Reinforcement learning (RL), temporal-difference (TD) learning, optimization, saddle-point problem, convergence

I. INTRODUCTION

Temporal-difference (TD) learning [1] is one of the most popular reinforcement learning (RL) algorithms [2] for policy evaluation problems. However, its main limitation lies in its inability to accommodate both off-policy learning and linear function approximation for convergence guarantees, which has been an important open problem for decades. In 2009, Sutton, Szepesvári, and Maei [3], [4] introduced the first TD learning algorithms compatible with both linear function approximation and off-policy training based on gradient estimations, which are thus called gradient temporal-difference learning (GTD).

The goal of this paper is to propose some variants of GTDs with comparative numerical analysis and new theoretical convergence analysis. The main pathways to these developments are based on convex-concave saddle-point interpretations of GTDs, which were first introduced in [5] based on the Lagrangian duality and in [6] based on the Fenchel duality [7]. In particular, GTD2, proposed in [4], can be interpreted as a stochastic primal-dual gradient dynamics (PDGD) of a convex-concave saddle-point problem, and hence, its convergence analysis can be approached from a different angle using optimization theories [5], [6], [8]. These interpretations were subsequently applied to distributed RL problems in [8]–[11].

On the other hand, [12] recently developed control theoretic frameworks for stability analysis of continuous PDGD, where the saddle-point problem considered in [12] corresponds to a Lagrangian function of equality constrained convex optimization. This new result can potentially open new opportunities in analysis of algorithms based on PDGD of convex-concave saddle-point problems, especially the saddle-point frameworks of GTDs.

Although the saddle-point perspectives can provide unified viewpoints and greater flexibilities in analysis & design of GTDs and RLs, to the authors’ knowledge, their potentials have not been fully investigated yet. Motivated by this insight, we develop three GTD versions which are unified with GTD2 [4] in a single framework through saddle-point perspectives. The main contributions of this paper are summarized as follows:

1) Algorithm developments: Three versions of GTDs are proposed, which are named GTD3, GTD4, and GTD5. From simulation experiments, their convergence and performance are evaluated.

2) Unified saddle-point analysis: The proposed versions can be interpreted in a unified way based on saddle-point interpretations, which are derived from slightly different angles from those in [5], [6]. Moreover, the proposed framework allows simple and unified stability analysis in ODE methods [13] based on the recent results [12] for PDGD.

3) Comparative analysis: Comprehensive numerical experiments are given to compare convergence of the proposed GTDs and GTD2 in [4]. It turns out that the proposed GTD4 and GTD5 tend to converge faster than the other methods for the randomly generated 5000 environments.

Related previous works are briefly summarized as follows: As mentioned before, saddle-point perspectives of GTDs and RLs were introduced in [5], [6] based on the Lagrangian duality [5] and Fenchel duality [6]. These ideas were applied to distributed RL problems in [8]–[11]. Even though they and the proposed saddle-point framework lead to the same algorithm, the latter one is derived from a slightly different way based on a simple constrained convex optimization formulation, which are compatible with techniques in [12]. In addition, we note that GTD5 proposed in this paper can be interpreted as GTD2 with a quadratic regularization term, which was also used in the distributed RLs in [8]–[10]. Compared to them, GTD5 focuses on the single agent case, has different algorithmic structures, and uses diminishing weights on the regularization term in the comparative analysis. The so-called TD with Regularized Corrections (TDRC) was introduced in [14], which adds an additional term to TDC updates in [4] corresponding to l2 regularization, and [15] extends the ideas of GTDs to nonlinear function approximations.

II. PRELIMINARIES

A. Markov decision process

A Markov decision process (MDP) is characterized by a quadruple $\mathcal{M} := (\mathcal{S}, \mathcal{A}, P, r, \gamma)$, where $\mathcal{S}$ is a finite state-space, $\mathcal{A}$ is a finite action space, $P(s'|s,a)$ represents the (unknown) state transition probability from state $s$ to $s'$ given action $a$, $r : \mathcal{S} \times \mathcal{A} \times \mathcal{S} \to \mathbb{R}$ is the reward function, and $\gamma \in (0,1)$ is the discount factor. In particular, if action $a$ is selected with the current state $s$, then the state transits to $s'$ with probability $P(s'|s,a)$ and incurs a reward $r(s,a,s')$. The stochastic policy is a map $\pi : \mathcal{S} \times \mathcal{A} \to [0,1]$ representing the probability, $\pi(a|s)$, of selecting action $a$ at the current state $s$. $P^\pi$ denotes the transition matrix under policy $\pi$, and $d^\pi : \mathcal{S} \to \mathbb{R}$ denotes the stationary distribution of the state $s \in \mathcal{S}$ under $\pi$. We also define $R^\pi(s)$ as the expected reward given the policy $\pi$ and the current state $s$. The infinite-horizon discounted value function with policy $\pi$ is

$$J^\pi(s) := \mathbb{E} \left[ \sum_{k=0}^{\infty} \gamma^k r(s_k, a_k, s_{k+1}) \bigg| s_0 = s \right],$$

where $\mathbb{E}$ stands for the expectation taken with respect to the state-action trajectories under $\pi$. Given pre-selected basis (or feature) functions $\phi_1, \ldots, \phi_q : \mathcal{S} \to \mathbb{R}$, the matrix, $\Phi \in \mathbb{R}^{\mathcal{S} \times q}$, called the feature matrix, is defined as a full column rank matrix whose $s$-th row vector is $\phi(s) := [\phi_1(s) \ldots \phi_q(s)]$.

Assumption 1. Throughout the paper, we assume that $\Phi \in \mathbb{R}^{\mathcal{S} \times q}$. 

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B. Basics of nonlinear system theory

We will briefly review basic nonlinear systems used in the ODE method introduced soon. Consider the nonlinear system
\[
d\frac{dx}{dt} = f(x), \quad x_0 \in \mathbb{R}^n, \quad t \geq 0,
\]
where \( x_t \in \mathbb{R}^n \) is the state, \( t \geq 0 \) is the time, \( x_0 \in \mathbb{R}^n \) is the initial state, and \( f : \mathbb{R}^n \to \mathbb{R}^n \) is a nonlinear mapping. For simplicity, we assume that the solution to (1) exists and is unique. In fact, this holds true so long as the mapping \( f \) is globally Lipschitz continuous.

Lemma 1 ([16, Thm. 3.2]). Consider the nonlinear system (1), and assume that \( f \) is globally Lipschitz continuous, i.e.,
\[
\| f(x) - f(y) \| \leq l \| x - y \|, \quad \forall x, y \in \mathbb{R}^n,
\]
for some \( l > 0 \) and norm \( \| \cdot \| \). Then, it admits a unique solution \( x_t \) for all \( t \geq 0 \) and \( x_0 \in \mathbb{R}^n \).

An important concept in dealing with nonlinear systems is the equilibrium point. A point, \( x_\infty \in \mathbb{R}^n \), in the state-space is said to be an equilibrium point of (1) if whenever the state of the system starts at \( x_\infty \), it will remain at \( x_\infty \) [16]. For (1), the equilibrium points are the real roots of the equation \( f(x) = 0 \). The equilibrium point \( x_\infty \) is said to be globally asymptotically stable if for any initial state \( x_0 \in \mathbb{R}^n \), \( x_t \to x_\infty \) as \( t \to \infty \).

C. ODE-based stochastic approximation

Due to its generality, the convergence analyses of many RL algorithms rely on the ODE (ordinary differential equation) approach [13, 17]. It analyzes convergence of general stochastic recursions by examining stability of the associated ODE model based on the fact that the stochastic recursions with diminishing step-sizes approximate the corresponding ODEs in the limit. One of the most popular approach is based on the Borkar and Meyn theorem [18]. We now briefly introduce the Borkar and Meyn’s ODE approach for analyzing convergence of the general stochastic recursions
\[
\theta_{k+1} = \theta_k + \alpha_k (f(\theta_k) + \epsilon_{k+1})
\]
where \( f : \mathbb{R}^n \to \mathbb{R}^n \) is a nonlinear mapping. Basic technical assumptions are given below.

Assumption 2.

1) The mapping \( f : \mathbb{R}^n \to \mathbb{R}^n \) is globally Lipschitz continuous and there exists a function \( f_\infty : \mathbb{R}^n \to \mathbb{R}^n \) such that
\[
\lim_{c \to \infty} \frac{f(cx)}{c} = f_\infty(x), \quad \forall x \in \mathbb{R}^n.
\]
2) The origin in \( \mathbb{R}^n \) is an asymptotically stable equilibrium for the ODE \( \dot{\theta}_t = f_\infty(\theta_t) \).
3) There exists a unique globally asymptotically stable equilibrium \( \theta_\infty \in \mathbb{R}^n \) for the ODE \( \dot{\theta}_t = f(\theta_t) \), i.e., \( \theta_t \to \theta_\infty \) as \( t \to \infty \).
4) The sequence \( \{ \epsilon_k, \theta_k, k \geq 1 \} \) with \( \theta_k = \sigma(\theta, \epsilon, i_k \leq k) \) is a Martingale difference sequence. In addition, there exists a constant \( C_0 < \infty \) such that for any initial \( \theta_0 \in \mathbb{R}^n \), we have
\[
\mathbb{E}[(\| \epsilon_{k+1} \|^2)] \leq C_0 (1 + \| \theta_k \|^2), \quad \forall k \geq 0. \quad \text{Here,} \quad \| \cdot \| \quad \text{denotes the standard Euclidean norm.}
\]
5) The step-sizes satisfy \( \alpha_k > 0 \), \( \sum_{k=0}^{\infty} \alpha_k = \infty \), \( \sum_{k=0}^{\infty} \alpha_k^2 < \infty \).

Lemma 2 ([18, Borkar and Meyn theorem]). Suppose that Assumption 2 holds. Then, the following statements hold true:

1) For any initial \( \theta_0 \in \mathbb{R}^n \), \( \sup_{k \geq 0} \| \theta_k \| < \infty \) with probability one.

2) In addition, \( \theta_k \to \theta_\infty \) as \( k \to \infty \) with probability one.

The Borkar and Meyn theorem states that under Assumption 2, the stochastic process \( \{ \theta_k \}_{k=0}^{\infty} \) generated by (3) is bounded and converges to \( \theta_\infty \) almost surely. It will be used to prove convergence of various algorithms throughout the paper.

D. Saddle-point problem

In this subsection, we briefly review the saddle-point problem [12, 19]. Consider a convex-concave function \( L : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R} \).

Definition 1 (Saddle point). A saddle point is defined as a vector pair \( (\theta^*, \lambda^*) \) that satisfies
\[
L(\theta^*, \lambda^*) \leq L(\theta^*, \lambda) \leq L(\theta, \lambda^*), \quad \forall (\theta, \lambda) \in \mathbb{R}^n \times \mathbb{R}^n
\]
Saddle point problems arise in a number of areas such as constrained optimization duality, zero-sum games, and general equilibri um theory [19]. Moreover, it is also known to be a solution of the min-max problem.

Problem 1 (Min-max problem). Solve
\[
\min_{\theta \in \mathbb{R}^n} \max_{\lambda \in \mathbb{R}^n} L(\theta, \lambda) = \min_{\lambda \in \mathbb{R}^n} \max_{\theta \in \mathbb{R}^n} L(\theta, \lambda)
\]
Moreover, \( (\theta, \lambda) \) is a saddle-point if and only if
\[
\nabla_{\theta} L(\theta^*, \lambda^*) = \nabla_{\lambda} L(\theta^*, \lambda^*) = 0.
\]
The so-called primal-dual gradient method [19] is a popular method for solving Problem 1:
\[
\theta_{k+1} = \theta_k - \alpha_k \nabla_{\theta} L(\theta_k, \lambda_k),
\]
\[
\lambda_{k+1} = \lambda_k + \alpha_k \nabla_{\lambda} L(\theta_k, \lambda_k),
\]
where \( (\alpha_k)_{k=0}^{\infty} \) is a step-size. This iteration will be called the discrete-time primal-dual gradient dynamics (PDGD) throughout the paper. Its continuous-time counterpart is
\[
\dot{\theta}_t = -\nabla_{\theta} L(\theta_t, \lambda_t), \quad \dot{\lambda}_t = \nabla_{\lambda} L(\theta_t, \lambda_t),
\]
and is called the continuous-time PDGD [12]. Both PDGDs converge to a saddle-point under some mild assumptions [12, 19]. If the gradients are not accessible, but only their stochastic approximations are available, then the stochastic counterpart is also possible:
\[
\theta_{k+1} = \theta_k - \alpha_k \nabla_{\theta} (L(\theta_k, \lambda_k) + v_k),
\]
\[
\lambda_{k+1} = \lambda_k + \alpha_k \nabla_{\lambda} (L(\theta_k, \lambda_k) + w_k),
\]
where \( (v_k, w_k) \in \mathbb{R}^n \times \mathbb{R}^n \) is i.i.d. noise with zero mean. In this paper, it will be called the stochastic PDGD. Stochastic PDGD also converges to a saddle-point in probabilistic senses [20, 21].

As an application, let us consider the constrained convex optimization problem.

Problem 2. Solve
\[
\min_{x \in \mathbb{R}^n} \ f(x) \quad \text{s.t.} \quad Ax = b,
\]
where \( f \) is convex and continuously differentiable. The corresponding (convex-concave) Lagrangian function is defined as
\[
L(\theta, \lambda) = f(\theta) + \lambda^T (A \theta - b),
\]
where \( \lambda \in \mathbb{R}^n \) is the Lagrangian multiplier. Using the standard results in convex optimization theories, if Problem 2 satisfies the Slater’s condition [7, Chap. 5], then its solution can be found by solving the saddle-point problem in Problem 1.
III. REVIEW OF GTD ALGORITHM

In this section, we briefly review the gradient temporal difference (GTD) learning developed in [4], which tries to solve the policy evaluation problem. Roughly speaking, the goal of the policy evaluation is to find the weight vector \( \theta \) such that \( \Phi \theta \) approximates the true value function \( J^\pi \). This is typically done by minimizing the so-called mean-square Bellman error loss function [4]. The overall problem is summarized below.

**Problem 3.** Solve
\[
\min_{\theta \in \mathbb{R}^q} \text{MSPBE}(\theta) := \frac{1}{2} \| \Pi(R^s + \gamma P^s \Phi \theta - \Phi \theta) \|^2_{\mathbb{R}^q},
\]
where \( R^s \in \mathbb{R}^{\mathcal{S}} \) is a vector enumerating all \( R^s(s), s \in \mathcal{S}, q \) is the number of feature functions, \( D^q \) is a diagonal matrix with positive diagonal elements \( d^q(s), s \in \mathcal{S} \), and \( \| x \|_D := \sqrt{x^T D x} \) for any positive-definite \( D \).

The GTD in [4] considers another equivalent form of Problem 3 using the mean-square projected Bellman error loss function.

**Problem 4.** Solve
\[
\min_{\theta \in \mathbb{R}^q} \text{MSPBE}(\theta) := \frac{1}{2} \| \Pi(R^s + \gamma P^s \Phi \theta - \Phi \theta) \|^2_{\mathbb{R}^q},
\]
where \( \Pi \) is the projection onto the range space of \( \Phi \), denoted by \( R(\Phi) \): \( \Pi(x) := \arg \min_{x' \in R(\Phi)} \| x - x' \|_{\mathbb{R}^q} \). The projection can be performed by the matrix multiplication: we write \( \Pi(x) := I x \), where \( \Pi := \Phi(\Phi^T D \Phi)^{-1} \Phi^T D \Phi \). Note that in the objective of Problem 4, \( D^q \) depends on the behavior policy, \( \beta \), while \( P^s \) and \( \Phi \) depend on the target policy, \( \pi \), that we want to evaluate. This structure allows us to obtain an off-policy learning algorithm. It can be done by the importance sampling [22] or sub-sampling techniques [3]. Some properties related to Problem 4 are summarized below for convenience and completeness. Their proofs are given in Appendix A.

**Lemma 3.** The following statements hold true:

1. A solution of Problem 4 exists and unique.
2. \( \Phi^T D^3(\gamma P^s - 1) \Phi \) is nonsingular, where \( 1 \) denotes the identity matrix with an appropriate dimension.
3. The solution of Problem 4 is given by
\[
\theta^* = -(\Phi^T D^3(\gamma P^s - 1) \Phi)^{-1} \Phi^T D^3 R^s
\]

Based on this objective function, [4] developed GTD2. The reader is referred to [4] for more details. After [3], some different interpretations were developed based on saddle-point perspectives. They are briefly presented before proceeding to different versions of GTDs.

A. First approach: dual representation

A saddle-point perspective of GTD2 was introduced in [5]. The main idea is to convert Problem 4 into the equivalent quadratic constrained optimization problem
\[
\begin{align*}
\min_{\theta, w \in \mathbb{R}^q} & \quad \frac{1}{2} w^T (\Phi^T D^3 \Phi)^{-1} w \\
\text{s.t.} & \quad w = \Phi^T D^3 (R^s + \gamma P^s \Phi \theta - \Phi \theta)
\end{align*}
\]
where \( w \in \mathbb{R}^q \) is a newly introduced vector variable. Introducing the Lagrangian function
\[
L(\theta, w, \lambda) = \frac{1}{2} w^T (\Phi^T D^3 \Phi)^{-1} w + \lambda^T (\Phi^T D^3 (R^s + \gamma P^s \Phi \theta - \Phi \theta) - w),
\]
where \( \lambda \in \mathbb{R}^q \) is the Lagrangian multiplier, the dual problem [7] is
\[
\min_{\lambda \in \mathbb{R}^q} \frac{1}{2} \lambda^T \Phi^T D^3 \Phi \lambda - \lambda^T \Phi^T D^3 R^s
\]
s.t. \( 0 = \lambda^T \Phi^T D^3 (\gamma P^s \Phi - \Phi) \)

The main reason to consider the dual problem instead of the primal problem is that the dual formulation removes the matrix inverse in the objective. Next, we can again construct the corresponding Lagrangian function for the dual problem as follows:
\[
L(\theta, \lambda) = \frac{1}{2} \lambda^T \Phi^T D^3 \Phi \lambda - \lambda^T \Phi^T D^3 R^s + \lambda^T \Phi^T D (\gamma P^s \Phi - \Phi) \theta
\]
where \( \theta \in \mathbb{R}^q \) is the Lagrangian multiplier. Then, it turned out that GTD2 is identical to a stochastic PDGD for solving the saddle-point problem, \( \min_{\lambda \in \mathbb{R}^q} \max_{\theta \in \mathbb{R}^q} L(\theta, \lambda) \). For more details, the reader is referred to [5].

B. Second approach: Fenchel duality

GTD2 can be also interpreted in a different direction using the Fenchel dual to Problem 4 as shown in [6]. In particular, using the Fenchel duality, the conjugate form of MSPBE(\( \theta \)):
\[
\text{MSPBE}(\theta) = \frac{1}{2} \| \Pi(R^s + \gamma P^s \Phi \theta - \Phi \theta) \|^2_{\mathbb{R}^q}
\]
is given by
\[
\text{MSPBE}(\theta) = \max_{\lambda \in \mathbb{R}^q} L(\theta, \lambda)
\]
\[
:= \lambda^T \Phi^T D (R^s + \gamma P^s \Phi \theta - \Phi \theta) - \frac{1}{2} \lambda^T \Phi^T D \Phi \lambda.
\]

Therefore, Problem 4 can be represented by the convex-concave saddle-point problem, \( \min_{\theta \in \mathbb{R}^q} \max_{\lambda \in \mathbb{R}^q} \text{MSPBE}(\theta) = \min_{\lambda \in \mathbb{R}^q} \max_{\theta \in \mathbb{R}^q} L(\theta, \lambda) \). Then, GTD2 is identical to a stochastic primal-dual algorithm for solving the above saddle-point problem. In the next, we introduce an alternative saddle-point approach to derive GTD2, which is similar to those in the last subsections, but in a slightly different direction.

IV. THIRD APPROACH

In this section, we introduce a slightly different approach to derive GTD2. To this end, consider the following constrained optimization problem.

**Problem 5.** Solve
\[
\begin{align*}
\min_{\theta \in \mathbb{R}^q} \quad & 0 = \Phi^T D^3 (R^s + \gamma P^s \Phi \theta - \Phi \theta) \\
\text{s.t.} & \quad 0 = \Phi^T D^3 (R^s + \gamma P^s \Phi \theta - \Phi \theta)
\end{align*}
\]
Note that in Problem 5, we introduce a null objective, \( f \equiv 0 \), to fit the problem into an optimization form. We can prove that the optimization admits a unique solution, which is identical to the solution of Problem 4.

**Proposition 1.** A solution of Problem 5 exists and is unique given by \( \theta^* \).

**Proof.** The equality constrain in (5) can be equivalently written as
\[
0 = \Phi^T D^3 (R^s + \gamma P^s \Phi \theta - \Phi \theta)
\]
\[
\Leftrightarrow \Phi^T D^3 \Phi \theta = \Phi^T D^3 (R^s + \gamma P^s \Phi \theta)
\]
\[
\Leftrightarrow \theta = (\Phi^T D^3 \Phi)^{-1} \Phi^T D^3 (R^s + \gamma P^s \Phi \theta)
\]
which completes the proof. \( \square \)

To formulate Problem 5 into a min-max saddle-point problem, we introduce the Lagrangian function for Problem 5 as follows:
\[
L(\theta, \lambda) := \lambda^T \Phi^T D^3 (R^s + \gamma P^s \Phi \theta - \Phi \theta).
\]
Introducing a regularization term to make it strongly concave in \( \lambda \), we obtain the modification
\[
L(\theta, \lambda) := \lambda^T \Phi^T D^\beta (R^\pi + \gamma P^\pi \Phi \theta - \Phi \theta) - \frac{1}{2} \lambda^T \Phi^T D^\beta \Phi \lambda.
\] (5)

The corresponding saddle-point problem of (5) is then given as follows.

**Problem 6.** Solve
\[
\min_{\theta \in \mathbb{R}^n} \max_{\lambda \in \mathbb{R}^n} L(\theta, \lambda) := \lambda^T \Phi^T D^\beta (R^\pi + \gamma P^\pi \Phi \theta - \Phi \theta) - \frac{1}{2} \lambda^T \Phi^T D^\beta \Phi \lambda
\]

Since Problem 6 is modified, a natural question is if the original equality constrained optimization in Problem 5 can be solved by addressing the saddle-point problem in Problem 6 for the regularized Lagrangian function (5). We can conclude that the solutions of Problem 6 is indeed identical to those of Problem 4.

**Proposition 2.** A solution of Problem 6 exists and is unique given by \( \theta = \theta^* \) and \( \lambda = 0 \).

**Proof.** Since \( \Phi^T D^\beta (\gamma P^\pi - I) \Phi \) is nonsingular from Lemma 3, \( \nabla_\lambda L(\theta, \lambda) = \gamma P^\pi \Phi \theta - \Phi \theta + \theta \) implies \( \lambda = 0 \). On the other hand, \( \nabla_\theta L(\theta, \lambda) = \Phi^T D^\beta (R^\pi + \gamma P^\pi \Phi \theta - \Phi \theta - \Phi \theta) = 0 \) with \( \lambda = 0 \) leads to the desired conclusion. \( \square \)

Now, let’s turn out attention to its continuous-time PDGD [12]:
\[
\dot{\theta}_t = -\nabla_\theta L(\theta_t, \lambda_t), \quad \dot{\lambda}_t = \nabla_\lambda L(\theta_t, \lambda_t) = \Phi^T D^\beta (R^\pi + \gamma P^\pi \Phi \theta - \Phi \theta - \Phi \theta) = 0
\]

(6)

Considering \( s_k \sim d^\pi, a_k \sim \pi(\cdot|s_k), \) and \( s'_k \sim P(\cdot|s_k, a_k) \), the corresponding stochastic PDGD can be obtained as follows:
\[
\begin{align*}
\theta_{k+1} &= \theta_k + \alpha_k (\gamma e_{s_k} e_{s_k}^T \Phi - \Phi) e_{s_k} e_{s_k}^T \Phi \lambda_k, \\
\lambda_{k+1} &= \lambda_k + \alpha_k \beta (\delta_k - \delta_k^T \lambda_k)
\end{align*}
\]

This recursion is identical to GTD2, which is summarized in Algorithm 1 for completeness of presentations.

**Algorithm 1 GTD2**

1. Set the step-size sequence \( \{\alpha_k\}_{k=0}^\infty \).
2. Initialize \( (\theta_0, \lambda_0) \).
3. for \( k \in \{0, \ldots\} \) do
4. Observe \( s_k \sim d^\pi, a_k \sim \beta(\cdot|s_k), \) and \( s'_k \sim P(\cdot|s_k, a_k) \),
5. Update parameters according to
\[
\begin{align*}
\theta_{k+1} &= \theta_k + \alpha_k (\gamma e_{s_k} e_{s_k}^T \Phi - \Phi) e_{s_k} e_{s_k}^T \Phi \lambda_k, \\
\lambda_{k+1} &= \lambda_k + \alpha_k \beta (\delta_k - \delta_k^T \lambda_k)
\end{align*}
\]

where \( \phi_k := \phi(s_k), \phi'_k := \phi(s'_k) \), \( \rho_k := \frac{\pi(a|s_k)}{\pi(a|s'_k)} \), and \( \delta_k = \rho_k e_{s_k} + \rho_k (\phi'_k)^T \theta_k - \phi_k^T \theta_k \).
6. end for

Note that in Algorithm 1, an importance sampling ratio, \( \rho_k := \frac{\pi(a_k|s_k)}{\pi(a_k|s'_k)} \), is introduced for off-policy learning [22]. In particular, to obtain a stochastic approximation of \( P^\pi \) and \( R^\pi \) from the samples under the behavior policy, \( \beta \), we use the fact
\[
P^\pi(s'|s) = \sum_{a \in A} P(s'|s, a) \pi(a|s)
\]

and
\[
R^\pi(s) := \sum_{a \in A} \sum_{s' \in S} \pi(a|s) P(s'|s, a) r(s, a, s')
\]

\[
= \sum_{a \in A} \sum_{s' \in S} \beta(a|s) P(s'|s, a) \frac{\pi(a|s)}{\beta(a|s)} r(s, a, s').
\]

Although the convergence of GTD2 was given in [4], we will provide another approach based on recent results in [12] in the next section.

**Remark 1.** A different algorithm can be obtained with the following Lagrangian function
\[
L(\theta, \lambda) := \lambda^T \Phi^T D^\beta (R^\pi + \gamma P^\pi \Phi \theta - \Phi \theta) - \frac{1}{2} \lambda^T \Phi^T D^\beta \Phi \lambda
\]

which has different convergence properties. In general, the corresponding algorithm performs better with smaller step-sizes, while in general, GTD2 converges faster.

**V. CONVERGENCE OF GTD2**

In this section, we will provide an alternative approach to the convergence of GTD2 based on the recent results in [12] in combination with the constrained optimization perspective of GTD2 in the previous section. Before proceeding, some results of [12] are briefly summarized.

**Lemma 4** (Thm. 1. [12]), Consider the equality constrained optimization in Problem 2, and suppose that \( f \) is twice differentiable, \( \mu \)-strongly convex, and I-smooth, i.e., for all \( x, y \in \mathbb{R}^n \)
\[
\|x - y\|^2 \leq (\nabla f(x) - \nabla f(y))^T (x - y) \leq l \|x - y\|^2.
\]

Moreover, suppose that \( A \) is full rank. Consider the corresponding Lagrangian function (4). Then, the corresponding saddle-point \( (\theta^*, \lambda^*) \) is unique, and the corresponding continuous-time PDGD,
\[
\dot{x}_t = -\nabla_\theta L(\theta_t, \lambda_t), \quad \dot{\lambda}_t = \nabla_\lambda L(\theta_t, \lambda_t) = A \theta_t - b,
\]

exponentially converges to \( (\theta^*, \lambda^*) \).

In the sequel, we will apply Lemma 4 to prove the convergence of GTD2, especially, for the global asymptotic stability of its ODE model. The main difficulty in applying Lemma 4 to Problem 5 is that Problem 5 has a null objective, \( f \equiv 0 \), which does not satisfy the strong convexity assumption of the objective function in Lemma 4. To resolve this problem, we will consider the dual problem of Problem 5 instead of its original form.

**Problem 7** (Dual problem). Solve
\[
\begin{align*}
\max_{\lambda \in \mathbb{R}^q} \lambda^T \Phi^T D^\beta R^\pi - \frac{1}{2} \lambda^T \Phi^T D^\beta \Phi \lambda \\
\text{s.t.} \quad \Phi^T (\gamma P^\pi - I) D^\beta \Phi \lambda = 0
\end{align*}
\]

**Proposition 3.** Problem 7 is the dual problem of Problem 5.

**Proof.** Consider the Lagrangian function (5) and the corresponding min-max problem in Problem 6. The Lagrangian function can be written by \( L(\theta, \lambda) = \lambda^T \Phi^T D^\beta R^\pi - \frac{1}{2} \lambda^T \Phi^T D^\beta \Phi \lambda + (\lambda^T \Phi^T D^\beta \gamma P^\pi \Phi - \lambda^T \Phi^T D^\beta \Phi) \theta \). If we fix \( \lambda \), then the problem \( \min_{\lambda \in \mathbb{R}^q} L(\theta, \lambda) \) has a finite optimal value, when \( \lambda^T \Phi^T D^\beta \gamma P^\pi \Phi - \lambda^T \Phi^T D^\beta \Phi = 0 \). Therefore, the dual problem \( \max_{\lambda \in \mathbb{R}^q} \min_{\theta \in \mathbb{R}^n} L(\theta, \lambda) \) is Problem 7. \( \square \)

Now, Problem 7 can be equivalently written as
\[
\min_{\lambda \in \mathbb{R}^q} \frac{1}{2} \lambda^T \Phi^T D^\beta \Phi \lambda - \lambda^T \Phi^T D^\beta R^\pi
\]


The corresponding Lagrangian function is
\[
L(\theta, \lambda) = \frac{1}{2} \lambda^T \Phi^T D^\beta \Phi \lambda - \lambda^T \Phi^T D^\beta R^\pi + \lambda^T \Phi^T D^\beta \Phi - \lambda^T \Phi^T D^\beta \gamma P^\pi \Phi \Theta,
\]
and the corresponding continuous-time PDGD is
\[
\dot{\lambda}_t = - \nabla_{\lambda} L(\theta_t, \lambda_t) = - \Phi^T D^\beta (\gamma P^\pi \Phi \Theta_t - \Phi \theta_t - R^\pi + \Phi \lambda_t),
\]
\[
\dot{\theta}_t = \nabla_{\theta} L(\theta_t, \lambda_t) = \left( \Phi - \gamma P^\pi \Phi \right)^T D^\beta \Phi \lambda_t.
\]
We can easily check that the PDGD is identical to that of Problem 5, given in (6). Therefore, Lemma 4 can be applied to Problem 5 in place of Problem 7.

**Proposition 4.** Consider the trajectory \((\theta_t, \lambda_t)\) of the PDGD in (6). Then, \((\theta_t, \lambda_t) \rightarrow (\theta^*, 0)\) as \(t \rightarrow \infty\).

**Proof.** Note that (7) has a strongly convex, smooth, and twice differentiable objective function. Moreover, \(\Phi^T (I - \gamma P^\pi)^T D^\beta \Phi\) is nonsingular by Lemma 3, and hence is full row rank. The other assumptions are also met. Therefore, we can apply Lemma 4 to obtain the desired conclusion.

Now, we can easily apply the Borkar and Meyn theorem with Proposition 4 to complete the proof. Details of the remaining parts can be found in [4].

**Lemma 5 (Thm. 1, [4]).** Consider Algorithm 1, and assume that the step-sizes satisfy
\[
\alpha_k > 0, \quad \sum_{k=0}^{\infty} \alpha_k = \infty, \quad \sum_{k=0}^{\infty} \alpha_k^2 < \infty.
\]
Then, \(\theta_k \rightarrow \theta^*\) and \(\lambda_k \rightarrow 0\) with probability one.

**Problem 8.** Solve
\[
\min_{\theta \in \mathbb{R}^n} \frac{1}{2} \theta^T \Phi^T D^\beta \Phi \theta
\]
\[
s.t. \quad 0 = \Phi^T D^\beta (R^\pi + \gamma P^\pi \Phi \theta - \Phi \Theta)
\]

Compared to Problem 5 for GTD2, the main difference is that it has a quadratic objective instead of the null objective. A natural question is if this optimization admits the identical solution to Problem 5. The answer is indeed positive.

**Proposition 5.** A solution of Problem 8 exists and is unique given by \(\theta = \theta^*\).

**Proof.** It is clear from Proposition 2 that the inequality constraint has a unique feasible point \(\theta = \theta^*\). Therefore, the optimal solution is also uniquely determined by \(\theta^*\). This completes the proof.

To derive a saddle-point formulation again, consider the Lagrangian function for Problem 8
\[
L(\theta, \lambda) = \frac{1}{2} \theta^T \Phi^T D^\beta \Phi \theta + \lambda^T \Phi^T D^\beta (R^\pi + \gamma P^\pi \Phi \theta - \Phi \Theta)\quad (9)
\]

Note that it is concave in \(\lambda\) and strongly convex in \(\theta\). Compared to the Lagrangian function of GTD2 in (5), the regularization term, \(\frac{1}{2} \theta^T \Phi^T D^\beta \Phi \theta\), which is strongly concave in \(\lambda\), is replaced with the regularization term, \(\frac{1}{2} \theta^T \Phi^T D^\beta \Phi \theta\), which is strongly convex in \(\theta\). The corresponding min-max saddle-point formulation is given as follows.

**Problem 9.** Solve
\[
\min_{\theta \in \mathbb{R}^n} \max_{\lambda \in \mathbb{R}^n} L(\theta, \lambda) := \frac{1}{2} \theta^T \Phi^T D^\beta \Phi \theta + \lambda^T \Phi^T D^\beta (R^\pi + \gamma P^\pi \Phi \theta - \Phi \Theta).
\]

Again, we can conclude that the solutions of Problem 9 is also identical to those of Problem 4.

**Proposition 6.** Problem 9 admits a unique solution given by \(\theta = \theta^*\) and \(\lambda = \lambda^*\), where \(\theta^*\) is given in Lemma 3, and
\[
\lambda^* := (\Phi^T (\gamma P^\pi - I)^T D^\beta \Phi)^{-1} \Phi^T D^\beta \Phi \Theta - \Phi \Theta^*.
\]

The results can be easily obtained by solving \(\nabla_{\lambda} L(\theta, \lambda) = 0\) and \(\nabla_{\theta} L(\theta, \lambda) = 0\). Therefore, the detailed proof is omitted here. Similar to the previous section, the continuous-time PDGD of Problem 8 (or equivalently, Problem 9) is
\[
\dot{\theta}_t = - \Phi^T D \Phi \theta_t - (\gamma P^\pi \Phi - \Phi)^T D \Phi \lambda_t,
\]
\[
\dot{\lambda}_t = \Phi^T D (R^\pi + \gamma P^\pi \Phi \theta_t - \Phi \Theta_t),
\]
and its discrete-time counterpart is
\[
\theta_{k+1} = \theta_k - \alpha_k (\Phi^T D \Phi \theta_k + (\gamma P^\pi \Phi - \Phi)^T D \Phi \lambda_k),
\]
\[
\lambda_{k+1} = \lambda_k + \gamma P^\pi \Phi \theta_k - \Phi \Theta_k.
\]

With the samples \(s_k \sim d^\beta, a_k \sim \pi(\cdot|s_k),\) and \(s_{k}^* \sim P(\cdot|s_k, a_k),\) a stochastic approximation [20], [21] of the discrete-time counterpart is given as
\[
\theta_{k+1} = \theta_k - \alpha_k (\Phi^T e_{s_k} e_{s_k}^T \Phi \theta_k + \Phi^T (\gamma e_{s_k} e_{s_{k+1}} - I)^T \Phi \lambda_k),
\]
\[
\lambda_{k+1} = \lambda_k + \alpha_k \Phi^T (e_{s_k} e_{s_k} e_{r_k} + e_{s_k} e_{s_{k+1}} \Phi \theta_k - e_{s_k} e_{s_k}^T \Phi \theta_k).
\]

The proposed algorithm summarized in Algorithm 2 is the update in (12) with the importance sampling [22].

**Algorithm 2 GTD3**

1. Set the step-size sequence \((\alpha_k)_{k=0}^{\infty}\).
2. Initialize \((\theta_0, \lambda_0)\).
3. for \(k \in \{0, \ldots\}\) do
4. Observe \(s_k \sim d^\beta, a_k \sim \beta(\cdot|s_k),\) and \(s_{k}^* \sim P(\cdot|s_k, a_k),\)
5. \(r_k := r(s_k, a_k, s_k^*)\).
6. Update parameters according to
\[
\theta_{k+1} = \theta_k + \alpha_k [\phi_k - \gamma \rho_k \phi_k (\phi_k \lambda_k - \phi_k (\phi_k \theta_k))],
\]
\[
\lambda_{k+1} = \lambda_k + \alpha_k \delta_k \phi_k,
\]
where \(\phi_k := \phi(s_k, a_k), \phi_k^* := \phi(s_k^*), \rho_k := \frac{\pi_k |s_k|}{\pi_k |s_k|^*},\) and \(\delta_k = \rho_k r_k + \gamma \rho_k (\phi_k^* \theta_k - \phi_k^* \theta_k).
7. end for

Note that Algorithm 2 is different from GTD2, linear TD with gradient correction (TDC) [4], and the original GTD in [3]. Moreover, the optimization problem corresponding to Algorithm 2, Problem 8, already has the required structures in Lemma 4. Therefore, Lemma 4 can be directly applied to prove the global stability of the corresponding PDGD in (11), which is also the ODE model of Algorithm 2.
Proposition 7. Consider the trajectory $(\theta_t, \lambda_t)$ of the PDGD in (11). Then, $(\theta_t, \lambda_t) \to (\theta^*, \lambda^*)$ as $t \to \infty$.

Proof. Problem 8 has a strongly convex, smooth, and twice differentiable objective function. Moreover, $\Phi^T \Phi = (I - \gamma P^r)^{-1}D^\delta \Phi$ is nonsingular, and hence is full rank. The other assumptions are also met. Therefore, the primal-dual gradient dynamic of Problem 8, given in (11), is globally asymptotically stable, and converges to its unique equilibrium point $(\theta^*, \lambda^*)$, where $\lambda^*$ is defined in (10). This completes the proof. \hfill \Box

Based on Proposition 7, convergence of Algorithm 2 can be proved using the Bokar and Mayn theorem.

Theorem 1. Consider Algorithm 2, and assume that the step-sizes satisfy (8). Then, $\theta_k \to \theta^*$ with probability one.

The proof of Theorem 1 is given in Appendix B.

Remark 2. A different algorithm can be obtained with the following Lagrangian function

$$L(\theta, \lambda) = \frac{1}{2} \theta^T \Phi^T D^\delta \Phi \theta + \lambda^T \Phi^T D^\delta (R^\pi + \gamma P^\pi \Phi \theta - \Phi \theta),$$

which has different convergence properties. In general, the corresponding algorithm performs better with smaller step-sizes, while in general, GTD3 converges faster.

In this section, we proposed a new version of GTD based on a new saddle-point formulation in Problem 9. In the next section, we propose the second version of GTD based on the Lagrangian function which has more symmetric form.

VIII. GTD4 & 5

Let us recall the Lagrangian functions (5) for GTD2 and (9) for GTD3. The Lagrangian function in (9) replaces $\frac{1}{2} \theta^T \Phi^T D^\delta \Phi \theta$ in (5) with $\frac{1}{2} \lambda^T \Phi^T D^\delta \Phi \lambda$. In this section, we will consider the function

$$L(\theta, \lambda) = \frac{1}{2} \sigma^2 \theta^T \Phi^T D^\delta \Phi \theta + \lambda^T \Phi^T D^\delta (R^\pi + \gamma P^\pi \Phi \theta - \Phi \theta) - \frac{1}{2} \lambda^T \Phi^T D^\delta \Phi \lambda,$$

where $\sigma \geq 0$ is a design parameter. Note that with $\sigma = 0$, (13) is reduced to (5). The function includes both $\frac{1}{2} \lambda^T \Phi^T D^\delta \Phi \lambda$ and $\frac{1}{2} \sigma^2 \theta^T \Phi^T D^\delta \Phi \theta$, and hence, more symmetric than (5) and (9). Moreover, it is strongly convex in $\theta$ and strongly concave in $\lambda$. The corresponding min-max saddle-point problem is summarized below for convenience.

Problem 10. Solve

$$\min_{\theta \in \mathbb{R}^d} \max_{\lambda \in \mathbb{R}^J} L(\theta, \lambda)$$

$$= \frac{1}{2} \sigma^2 \theta^T \Phi^T D^\delta \Phi \theta + \lambda^T \Phi^T D(R^\pi + \gamma P^\pi \Phi \theta - \Phi \theta) - \frac{1}{2} \lambda^T \Phi^T D^\delta \Phi \lambda.$$

Solving the equations $\nabla_{\theta} L(\theta, \lambda) = 0$ and $\nabla_{\lambda} L(\theta, \lambda) = 0$, we obtain the following solution of Problem 10 for the $\theta$-coordinate:

$$\dot{\theta}_\sigma := - (\Phi^T D^\delta (\gamma P^r - I) \Phi + \sigma E)^{-1} \Phi^T D^\delta R^\pi$$

where $E := \Phi^T D^\delta \Phi (\Phi^T (\gamma P^r - I)^T D^\delta \Phi)^{-1} \Phi^T D^\delta \Phi$

From the result, it turns out that the solution of Problem 10 for the $\theta$-coordinate is not exactly identical to $\theta^*$, but includes a bias term $\sigma E$. However, since $\theta_\sigma \to \theta^*$ as $\sigma \to 0$, one can control the degree of the error by adjusting $\sigma$. Moreover, it is expected that larger $\sigma$ can more stabilize the final algorithm, and speed up its convergence because it improves the degree of the strong concavity of (13). Therefore, there may exist some trade-off between stability and error degree in choosing $\sigma$. Besides, the solution is formally stated in the following proposition for convenience.

Proposition 8. The unique saddle-point of Problem 10 for $\theta$-coordinate is given by

$$\dot{\theta}_\sigma := - (\Phi^T D^\delta (\gamma P^r - I) \Phi + \sigma E)^{-1} \Phi^T D^\delta R^\pi$$

where $E := \Phi^T D^\delta \Phi (\Phi^T (\gamma P^r - I)^T D^\delta \Phi)^{-1} \Phi^T D^\delta \Phi$.

The proof of Proposition 8 is a direct calculation, and so omitted here for brevity. Similar to the previous section, the continuous-time PDGD is

$$\dot{\theta}_t = - \nabla_{\theta} L(\theta_t, \lambda_t) = - \Phi^T D \Phi \theta_t - (\gamma P^r \Phi - \Phi^T) D \Phi \lambda_t,$$

$$\dot{\lambda}_t = - \nabla_{\lambda} L(\theta_t, \lambda_t) = \Phi^T D (R^\pi + \gamma P^\pi \Phi \theta_t - \Phi \theta_t) - \sigma \Phi^T D \Phi \lambda_t,$$

(14)

We will first prove that the PDGD in (14) is globally asymptotically stable.

Proposition 9. Consider the trajectory $(\theta_t, \lambda_t)$ of the PDGD in (14), and let $(\theta_\sigma, \lambda_\sigma)$ be the unique saddle-point. Then, $(\theta_t, \lambda_t) \to (\theta_\sigma, \lambda_\sigma)$ as $t \to \infty$.

Proof. The PDGD in (14) can be rewritten by

$$\frac{d}{dt} \begin{bmatrix} \theta_t - \theta_\sigma \\ \lambda_t - \lambda_\sigma \end{bmatrix} = \begin{bmatrix} - \nabla_{\theta} L(\theta_t, \lambda_t) \\ - \nabla_{\lambda} L(\theta_t, \lambda_t) \end{bmatrix} = \begin{bmatrix} - \Phi^T D \Phi \\ - (\gamma P^r \Phi - \Phi^T) D \Phi \end{bmatrix} \begin{bmatrix} \theta_t - \theta_\sigma \\ \lambda_t - \lambda_\sigma \end{bmatrix} \begin{bmatrix} - \sigma \Phi^T D \Phi \\ -2 \Phi^T D \Phi \\ 0 \\ -2 \sigma \Phi^T D \Phi \end{bmatrix} \begin{bmatrix} \theta_t - \theta_\sigma \\ \lambda_t - \lambda_\sigma \end{bmatrix}$$

For all $\theta_t - \theta_\sigma \neq 0$, $\lambda_t - \lambda_\sigma \neq 0$. Therefore, by the Lyapunov theorem [16], the system is globally asymptotically stable. This completes the proof. \hfill \Box

With the samples $s_k \sim d^\pi$, $a_k \sim \pi(\cdot|s_k)$, and $s'_{k+1} \sim P(\cdot|s_k, a_k)$, a stochastic PDGD corresponding to (14) is given as

$$\theta_{k+1} = \theta_k - \alpha_k (\Phi^T e_{sk} e_{sk}^T \Phi \theta_k + (\gamma e_{sk} e_{sk}^T \Phi - \Phi) e_{sk} e_{sk}^T \Phi \lambda_k),$$

$$\lambda_{k+1} = \lambda_k + \alpha_k (\Phi^T e_{sk} e_{sk}^T (e_{sk} e_{sk}^T + \gamma e_{sk} e_{sk}^T \Phi \theta_k - \Phi \theta_k) - \sigma \Phi^T e_{sk} e_{sk}^T \Phi \lambda_k),$$

which is the second proposed algorithm, called GTD4 in this paper. The overall algorithm with the importance sampling for off-policy learning is summarized in Algorithm 3.

With Proposition 9, one can easily prove the convergence of Algorithm 3 using the ODE method in Lemma 2.

Theorem 2. Consider Algorithm 3, and assume that the step-sizes satisfy (8). Then, $\theta_k \to \theta_\sigma$ with probability one.
Proposition 9

Theorem 1

2. The PDGD is globally asymptotically stable.

(a) depicts results for step-size

(b) for step-size

shows error evolutions for different GTDs, GTD2 performs slightly

2.5 provides another instance where GTD2 performs slightly

3.5

16 shows error evolutions for different GTDs, GTD2

Algorithm 3 GTD4

1: Set the step-size sequence \((\alpha_k)_{k=0}^\infty\).
2: Initialize \((\theta_0, \lambda_0)\).
3: for \(k \in \{0, \ldots\}\) do
4: Observe \(s_k \sim d^\beta\), \(a_k \sim \beta(|s_k|\), and \(s_k' \sim P(\cdot|s_k, a_k)\),
\(r_k := r(s_k, a_k, s_k')\).
5: Update parameters according to
\[ \begin{align*}
\theta_{k+1} &= \theta_k + \alpha_k [\phi_k - \gamma \rho_k \phi_k'] (\phi_k^T \lambda_k) - \lambda_k \phi_k \theta_k] \\
\lambda_{k+1} &= \lambda_k + \alpha_k (\delta_k - \sigma \phi_k^T \lambda_k) \phi_k \\
\end{align*} \]
where \(\phi_k := \phi(s_k), \phi_k' := \phi(s_k')\), and \(\delta_k = \rho_k \tau_k + \gamma \rho_k (\phi_k')^T \theta_k - \phi_k^T \theta_k\).
6: end for

Proof. By Proposition 9, the PDGD is globally asymptotically stable. The remaining parts of the proof are almost identical to those of Theorem 1, and hence, are omitted here for brevity.

Finally, a modification of (13) leads to GTD5

\[
L(\theta, \lambda) = \frac{1}{2} \theta^T \theta + \lambda^T \Phi^T D^\beta (R^\pi + \gamma P^\pi \Phi \theta - \Phi \theta) - \frac{1}{2} \lambda^T \Phi^T D^\beta \Phi \lambda,
\]
(15)

where \(\frac{1}{2} \theta^T \theta + \lambda^T \Phi^T D^\beta \Phi \lambda\) in (13) is replaced with \(\frac{1}{2} \theta^T \theta\) in (15). The algorithm is summarized in Algorithm 4.

Algorithm 4 GTD5

1: Set the step-size sequence \((\alpha_k)_{k=0}^\infty\).
2: Initialize \((\theta_0, \lambda_0)\).
3: for \(k \in \{0, \ldots\}\) do
4: Observe \(s_k \sim d^\beta\), \(a_k \sim \beta(\cdot|s_k)\), and \(s_k' \sim P(\cdot|s_k, a_k)\),
\(r_k := r(s_k, a_k, s_k')\).
5: Update parameters according to
\[ \begin{align*}
\theta_{k+1} &= \theta_k + \alpha_k [\phi_k - \gamma \rho_k \phi_k'] (\phi_k^T \lambda_k) - \lambda_k \phi_k \theta_k] \\
\lambda_{k+1} &= \lambda_k + \alpha_k (\delta_k - \sigma \phi_k^T \lambda_k) \phi_k \\
\end{align*} \]
where \(\phi_k := \phi(s_k), \phi_k' := \phi(s_k')\), and \(\delta_k = \rho_k \tau_k + \gamma \rho_k (\phi_k')^T \theta_k - \phi_k^T \theta_k\).
6: end for

Since \(\sigma > 0\) leads to biases in solutions, a reasonable heuristic approach is to diminish \(\sigma\), i.e., \(\sigma \to 0\) as \(k \to \infty\). A comparative analysis of several GTDs will be given in the next section.

VIII. COMPARATIVE ANALYSIS

We randomly generate MDPs with 100 states and 10 actions, target, and behavior policies, and set \(\gamma = 0.9\). The reward function is generated such that \(r(s, a, s')\) is uniformly distributed over \([-1, 1]\). Then, to make it sparse, elements with \(|r(s, a, s')| \leq 0.2\) are set to be zero. Similarly, 10 feature functions are generated such that each element uniformly distributed over \([-1, 1]\), and \(\Phi\) is full column rank. Figure 1 shows error evolutions for different GTDs, GTD2 (blue line), GTD3 (red line), GTD4 (green line), GTD5 (magenta line), in a logarithmic scale. Figure 1(a) depicts results for step-size \(\alpha_k = 5/(k+5)\), and Figure 1(b) for step-size \(\alpha_k = 10/(k+10)\). For GTDs 4 and 5, we used a diminishing \(\sigma\): \(\sigma_k = 100/(k+10)\). The step-sizes were selected such that all the algorithms perform reasonably well. The results show an instance where GTD4 and GTD5 overcome GTD2, and GTD3. GTD3 converges slightly faster than GTD2 in this example.

Figure 1. First instance: (a) Evolution of error, \(\|\theta_k - \theta^*\|\), for step-size \(\alpha_k = 5/(k+5)\); (b) Evolution of error, \(\|\theta_k - \theta^*\|\), for step-size \(\alpha_k = 10/(k+10)\). The figure illustrates error evolutions for GTD2 (blue), GTD3 (red), GTD4 (green), GTD5 (magenta) in a logarithmic scale. For GTD4 and GTD5, we used a diminishing \(\sigma\): \(\sigma_k = 100/(k+10)\).

Figure 2 provides another instance where GTD2 performs slightly better than or equal to the other approaches. From our experiences, GTD4 and GTD5 overcome the other two approaches more frequently. For a fair and more comprehensive analysis, we ranked the four approaches based on the performance index

\[
\sum_{k=0}^{\tau} \frac{\|\theta_k - \theta^*\|}{1000}
\]
(16)

for 5000 randomly generated MDPs, where \(N\) is the total number of iterations set to be \(\tau = 250000\) in this example. In addition to the random generating scheme used in the previous two MDP instances, we also randomly select the number of states and number of actions uniformly distributed in \([3, 4, \ldots, 100]\) and \([2, 3, \ldots, 30]\), respectively. The number of feature functions is chosen such that it is around \(1/10\) of the state size. Rankings of the different GTDs for 5000 MDP instances are summarized in Figure 3, where each bar implies the number of MDP instances where the corresponding ranking is achieved by each method in terms of the performance index (16). We used the step-size \(\alpha_k = 5/(k+5)\) and diminishing weight \(\sigma_k = 100/(k+100)\) for this experiment. GTD5 takes the first place most frequently (3316 times over 5000 trials), and GTD4 takes the second-best places most frequently (3016 times over 5000 trials). GTD2 and GTD3 are comparable to each other. The results suggest that GTD5 and GTD4 overcome the other approaches in most cases.


In this paper, we proposed variants of GTDs based on convex-concave saddle-point interpretations of GTDs, which allow new stability analysis based on recent results [12] on stability of PDGD. Performance of the GTDs was evaluated through numerical experiments, which suggest that GTD4 and GTD5 overcome the other methods for randomly generated 5000 MDPs. Therefore, we can conclude that the use of regularization terms with diminishing weights can potentially improve the convergence speed. Possible future works include rigorous convergence rate analysis of the proposed GTDs.

**IX. CONCLUSION**

In this paper, we proposed variants of GTDs based on convex-concave saddle-point interpretations of GTDs, which allow new stability analysis based on recent results [12] on stability of PDGD. Performance of the GTDs was evaluated through numerical experiments, which suggest that GTD4 and GTD5 overcome the other methods for randomly generated 5000 MDPs. Therefore, we can conclude that the use of regularization terms with diminishing weights can potentially improve the convergence speed. Possible future works include rigorous convergence rate analysis of the proposed GTDs.

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**APPENDIX A**

**PROOF OF LEMMA 3**

1) Define the Bellman operator, $T^*(x) := R^x + \gamma P^x x$, which is known to be a contraction. Moreover, the projection $\Pi$ is known to be nonexpansive. Therefore, the operator $\Pi T^*$ is a contraction as well. By the Banach fixed point theorem, its fixed point satisfying $x = \Pi T^* x$ exists and is unique. Let $x^*$ be the fixed point, i.e., $x^* = \Pi T^* (x^*)$. Due to the projection, $x^*$ should lie in the range of $\Phi$, i.e., there exists $\theta^*$ such that
where

\[ x^* = \Phi \theta^* \] Therefore, we have \( \Phi \theta^* = \Pi T^R(\Phi \theta^*) \). Such \( \theta^* \)

is unique because \( \Phi \) is assumed to be full rank. Therefore, 
\( \min_{\theta \in \mathbb{R}^n} \| \Phi \theta - \Pi T^R(\Phi \theta) \|_2^2 \) admits a unique minimizer, \( \theta^* \).

This completes the proof of the first statement.

2) The equation, \( \Pi (R^* + \gamma P^s \Phi \theta) - \Phi \theta = 0 \), can be equivalently written as

\[ \Pi (R^* + \gamma P^s \Phi \theta) - \Phi \theta = 0 \]

\[ \Leftrightarrow \Phi (\Phi^T D^3 \Phi)^{-1} \Phi^T D^3 (R^* + \gamma P^s \Phi \theta) - \Phi \theta = 0 \]

\[ \Leftrightarrow \Phi^T D^3 (R^* + \gamma P^s \Phi \theta) - \Phi^T D^3 \Phi \theta = 0 \]

\[ \Leftrightarrow \Phi^T D^3 (\gamma P^s - I) \Phi \theta = -\Phi^T D^3 R^* \quad (17) \]

For the last equation to admit a unique solution, \( \Phi^T D^3 (\gamma P^s - I) \Phi \) should be full column rank. Since it is a square matrix, it is nonsingular.

3) Since \( \Phi^T D^3 (\gamma P^s - I) \Phi \) is nonsingular, (17) admits the solution given in the statements of Lemma 3.

APPENDIX B

PROOF OF THEOREM 1

The algorithm in (12) can be written as

\[ \begin{bmatrix} \theta_{k+1} \\ \lambda_{k+1} \end{bmatrix} = \begin{bmatrix} \theta_k \\ \lambda_k \end{bmatrix} + \alpha_k \left( f \left( \begin{bmatrix} \theta_k \\ \lambda_k \end{bmatrix} \right) + \varepsilon_{k+1} \right) , \quad (18) \]

where

\[ f \left( \begin{bmatrix} \theta \\ \lambda \end{bmatrix} \right) := \begin{bmatrix} \Phi^T D^3 \Phi & \Phi^T (\gamma P^s - I)^T D^3 \Phi \\ D^3 \Phi & 0 \end{bmatrix} \begin{bmatrix} \theta \\ \lambda \end{bmatrix} + \begin{bmatrix} 0 \\ \Phi^T D^3 R^* \end{bmatrix} , \]

and

\[ \varepsilon_{k+1} = \begin{bmatrix} \Phi^T e_{s_k} e_{s_k}^T \Phi \theta_k + \Phi^T (\gamma e_{s_k} e_{s_k}^T - I)^T \Phi \lambda_k \\ \Phi^T (e_{s_k} e_{s_k}^T r_k + \gamma e_{s_k} e_{s_k}^T \Phi \theta_k - e_{s_k} e_{s_k}^T \Phi \theta_k) \end{bmatrix} - f \left( \begin{bmatrix} \theta_k \\ \lambda_k \end{bmatrix} \right) \]

The proof is completed by examining all the statements in Assumption 2.

1) To prove the first statement of Assumption 2, we have

\[ \lim_{c \to \infty} f \left( c \begin{bmatrix} \theta \\ \lambda \end{bmatrix} \right) / c = \begin{bmatrix} \Phi^T D^3 \Phi & \Phi^T (\gamma P^s - I)^T D^3 \Phi \\ D^3 \Phi & 0 \end{bmatrix} \begin{bmatrix} \theta \\ \lambda \end{bmatrix} + \begin{bmatrix} 0 \\ \Phi^T D^3 R^* \end{bmatrix} \]

Moreover, since \( f \) is affine in its argument, it is globally Lipschitz continuous.

2) The second statement of Assumption 2: The PDGD of Problem 8 can be written as

\[ \frac{d}{dt} \begin{bmatrix} \theta_t \\ \lambda_t \end{bmatrix} = \begin{bmatrix} -\Phi^T D^3 \Phi & -(\gamma P^s - \Phi)^T D^3 \Phi \\ \Phi^T D^3 (\gamma P^s - \Phi) & 0 \end{bmatrix} \begin{bmatrix} \theta_t - \theta_\infty \\ \lambda_t - \lambda_\infty \end{bmatrix} , \]

whose origin is the globally asymptotically stable equilibrium point. Now, one can observe that it is identical to the ODE

\[ \frac{d}{dt} \begin{bmatrix} \theta - \theta_\infty \\ \lambda - \lambda_\infty \end{bmatrix} = \begin{bmatrix} \theta_t - \theta_\infty \\ \lambda_t - \lambda_\infty \end{bmatrix} . \]

Therefore, its origin is the globally asymptotically stable equilibrium point.

3) The third statement of Assumption 2: The ODE, \( \frac{d}{dt} \begin{bmatrix} \theta_t \\ \lambda_t \end{bmatrix} = f \left( \begin{bmatrix} \theta_t \\ \lambda_t \end{bmatrix} \right) \), is identical to the PDGD of Problem 8. Therefore, it admits a unique globally asymptotically stable equilibrium point by Lemma 4.

4) Next, we prove the remaining parts. Recall that the GTD update defines the history \( \mathcal{G}_k := (\varepsilon_k, \varepsilon_k, \ldots, \varepsilon_1, \theta_k, \theta_{k-1}, \ldots, \theta_0, \lambda_k, \lambda_{k-1}, \ldots, \lambda_0) \), and the process \( (M_k)_{k=0}^{\infty} \) with \( M_k := \sum_{i=1}^{k} \varepsilon_i \). Then, we can prove that \( (M_k)_{k=0}^{\infty} \) is a Martingale. To do so, we first prove

\[ \mathbb{E}[\varepsilon_{k+1} | \mathcal{G}_k] = \mathbb{E} \left[ \begin{bmatrix} \Phi^T e_{s_k} e_{s_k}^T \Phi \theta_k + \Phi^T (\gamma e_{s_k} e_{s_k}^T - I)^T \Phi \lambda_k \\ \Phi^T (e_{s_k} e_{s_k}^T r_k + \gamma e_{s_k} e_{s_k}^T \Phi \theta_k - e_{s_k} e_{s_k}^T \Phi \theta_k) \end{bmatrix} \bigg| \mathcal{G}_k \right] \]

\[ \mathbb{E} \left[ f \left( \begin{bmatrix} \theta_k \\ \lambda_k \end{bmatrix} \right) \bigg| \mathcal{G}_k \right] \]

where the second equality is due to the i.i.d. assumption of samples. Using this identity, we have

\[ \mathbb{E}[M_{k+1} | \mathcal{G}_k] = \mathbb{E} \sum_{i=1}^{k+1} \varepsilon_i | \mathcal{G}_k = \mathbb{E}[\varepsilon_{k+1} | \mathcal{G}_k] + \mathbb{E} \sum_{i=1}^{k} \varepsilon_i | \mathcal{G}_k \]

\[ = \mathbb{E} \sum_{i=1}^{k} \varepsilon_i | \mathcal{G}_k = k \varepsilon_i = M_k. \]

Therefore, \( (M_k)_{k=0}^{\infty} \) is a Martingale sequence, and \( \varepsilon_{k+1} = M_{k+1} - M_k \) is a Martingale difference. Moreover, it can be easily proved that the second statement of the fourth condition of Assumption 2 is satisfied by algebraic calculations. Therefore, the fourth condition is met.