Abstract

It is well known that any entangled mixed state in $2 \otimes 2$ systems can be purified via infinite copies of the mixed state. But can one distill a pure maximally entangled state from finite copies of a mixed state in any bipartite system by local operation and classical communication? This is more meaningful in practical application. We give a necessary and sufficient condition of this distillability. This condition can be expressed as: there exists distillable-subspaces. According to this condition, one can judge whether a mixed state is distillable or not easily. We also analyze some properties of distillable-subspaces, and discuss the most efficient purification protocols. Finally, we discuss the distillable entanglement of two-quibt system for the case of finite copies.

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Maximally entangled states have many applications in quantum information, such as error correcting code [1], dense coding [2] and teleportation [3], etc. In the laboratory, however, maximally entangled state became a mixed state easily due to the interaction with environment. This results in poor application. It involve a basic question: how to distill a pure entangled state from a mixed state by local operation and classical communication(LOCC)? Bennett et al [4] proposed a entanglement purification scheme for a class of Werner states, then Horodecki et al [5] proved that any inseparable state in $2 \otimes 2$ systems (or two-qubit system) can be distilled into a singlet via this scheme by infinite rounds of purification protocols, and the necessary condition of distillability [6] is negative partial transpose(NPT) [7] for any bipartite systems. This scheme ask only the output states with fidelity $F \rightarrow 1$ under infinite copies of the purified mixed state. This means that, for one thing, one can get some states with desirous fidelity from many copies, and get a near ”perfect” singlet from the infinite number of copies of a mixed state, for another, one may not get a singlet from some ”distillable” states because there are no infinite copies of a mixed state in laboratory. A natural question is : Can one get a pure entangled state from the finite number copies of a mixed state? Recently, it has been shown that no distillation scheme with individual measurement can produce a pure entangled state from a mixed state of $2 \otimes 2$ systems [8,9]. More recently it was proved that many copies of some mixed state, even if which are almost a maximally entangled state, also can not produce a pure entangled state [10]. In this paper, we consider a case of distillability: one can get a pure maximally entangled state (or a pure entangled state, because pure entangled states can be transferred reversibly into maximally entangled states) from the finite number of copies of the mixed states, i.e., ask the output states with fidelity $F = 1$ under finite copies. We prove that the necessary and sufficient condition of the distillability for any bipartite systems is that there exists a distillable-subspace(DSS). We analyze some properties of a DSS, and give an example to demonstrate how to find a DSS. It is not difficult for one to judge a DSS via this condition. From the concept of DSS one can get the most efficient purification protocols. Finally, we discuss the distillable entanglement of mixed states in $2 \otimes 2$ systems for the case of finite copies, and find only a class of states are distillable.

A mixed state $\rho_{AB}$ in $N_A \otimes N_B$ system ($N_A$, $N_B$ is the dimension of system A, B, respectively) can be expressed as:

$$\rho_{AB} = \sum_{i=1}^{k} \lambda_i \langle \Psi_{AB}^i | \Psi_{AB}^i \rangle$$

(1)

here $\{|\Psi_{AB}^i \rangle\}$ are pure states with probability $\{\lambda_i\}$, respectively, $k$ is the rank of $\rho_{AB}$. We say $\rho_{AB}$ is $n$-distillable iff from $n$ copies of $\rho_{AB}, \rho_{AB}^{\otimes n}$, one can get a pure entangled state $|\Psi\rangle$ at least,
Here \( \{ |e_i\rangle_A \} \) and \( \{ |f_i\rangle_B \} \) is a set of orthonormal bases of Hilbert space of A and B system, respectively, and all \( a_i \neq 0, m \geq 2 \). If all \( a_i \neq 0 \) (i = 1, ..., m) in Eq(2) we say the Schmidt number of pure state \( |\Psi\rangle \) is \( m \).

Theorem 1.1: One can distill a pure entangled state \( |\Psi\rangle \) in Eq(2) from \( \rho_{AB}^{\otimes 2} \), i.e., \( \rho_{AB} \) is 1-distillable iff: 1) all of \( |\Psi_{AB}\rangle \)'s include a pure state \( |\Phi\rangle \) (e.g., we say the state \( \frac{1}{\sqrt{3}}(|00\rangle + |11\rangle + |23\rangle) \) includes the state \( (|00\rangle + |11\rangle)/\sqrt{2} \)

\[
|\Psi\rangle = \sum_{i=1}^{m} a_i |e_i\rangle_A |f_i\rangle_B 
\] (2)

where all \( b_i \neq 0, m \geq 2 \). \( \{ |e_i\rangle_A \} \) and \( \{ |f_i\rangle_B \} \) are another set of orthonormal bases of Hilbert space of A and B system, respectively; or 2) some \( |\Psi_{AB}\rangle \)'s include \( |\Phi\rangle \) and the others which do not include \( |\Phi\rangle \) have not component \( \{ |e_i\rangle_A |f_j\rangle_B \} \) (i, j = 1, ..., m).

Proof: If \( \rho_{AB} \) satisfy the condition in theorem 1.1, one can first distill the pure \( |\Phi\rangle \) with nonzero probability by project operation \( P_A \) and \( P_B \):

\[
P_A = \sum_{i=1}^{m} |e_i\rangle_A \langle e_i|_A \\
P_B = \sum_{i=1}^{m} |f_i\rangle_B \langle f_i|_B 
\] (4)

\( P_A \) and \( P_B \) act on Hilbert space of system A and B, respectively. Then, transfer the bases \( |e_i\rangle_A |f_i\rangle_B \) into \( |e_i\rangle_A |f_i\rangle_B \) by local unitary transformation on \( \rho_{AB} \), and get pure \( |\Phi'| \):

\[
|\Phi'| = \sum_{i=1}^{m} b_i |e_i\rangle_A |f_i\rangle_B 
\] (5)

Finally, one transfer \( |\Phi'| \) into \( |\Psi| \) by local filter operation. Conversely, suppose that \( \rho_{AB} \) does not satisfy the condition in the theorem 1.1. This include two cases: 1. there are not \( |\Phi\rangle \) in all pure states \( |\Psi_{AB}\rangle \) under any local unitary transformation, i.e., \( \rho_{AB} \) is separable; 2. Some \( |\Psi_{AB}\rangle \) have component \( |\Phi\rangle \), but the others which do not include \( |\Phi\rangle \) have ”impure component” \( |e_i\rangle_A |f_j\rangle_B \). Obviously, for the first case one cannot distill \( |\Phi\rangle \) from \( \rho_{AB} \). For the second case, one must discard the ”impure component” to get pure entangled state. To achieve this, one should distinguish locally the state \( \Phi \) from a state \( |e_i\rangle_A |f_j\rangle_B \) without destruction of \( |\Phi\rangle \). But this is impossible, because both the state \( |\Phi\rangle \) and the state \( |e_i\rangle_A |f_j\rangle_B \) include \( |e_i\rangle \) of system A and \( |f_j\rangle \) of system B. One cannot distinguish locally the state \( |e_i\rangle_A \) (or \( |f_j\rangle_B \)) in state \( \Phi \) from the state \( |e_i\rangle_A \) (or \( |f_j\rangle_B \)) in state \( |e_i\rangle_A |f_j\rangle_B \). In other words, if one can do so, one can get \( |\Phi\rangle \) with probability
\( \lambda \) from a mixed state \( \rho = \lambda |\Phi \rangle \langle \Phi | + (1 - \lambda) |e_i \rangle_A |f_j \rangle_B, \) and the distillable entanglement of \( \rho, E_D(\rho) = \lambda E(|\Phi \rangle \rangle) \geq E_F(|\Phi \rangle \rangle), \) here \( E(|\Phi \rangle \rangle) \) is entanglement of pure state \(|\Phi \rangle \rangle, E_F \) is formation entanglement \[14\]. This inequality cannot hold obviously \[11\]. So one cannot discard the "impure component" without destruction of \(|\Phi \rangle \rangle. \) Thus we finish the proof of theorem 1.1.

The theorem above is also fit to the case of \( \rho_{AB}^{\otimes^n} \), if we regard \( \rho_{AB}^{\otimes^n} \) as a state in \((N_A)^{\otimes^n} \otimes (N_B)^{\otimes^n} \) systems. It is to say that theorem 1.1 can be generalized into the following theorem:

**Theorem 1.2:** \( \rho_{AB} \) is n-distillable iff all pure states \(|\Psi^{i}_{AB} \rangle \rangle \) in the pure state decomposition of \( \rho_{AB}^{\otimes^n} \) include a pure state \(|\Phi \rangle \rangle \), or some \(|\Psi^{i}_{AB} \rangle \rangle \) include \(|\Phi \rangle \rangle \) and the others which do not include \(|\Phi \rangle \rangle \) have not component \( \{|e'_i \rangle_A |f'_j \rangle_B \rangle\rangle (i,j=1,...,m) \} \) are orthonormal vectors of Hilbert space of \( A^{\otimes n} \) and \( B^{\otimes n} \) systems, respectively, and \(|\Phi \rangle \rangle = \sum_{i=1}^{m} b_i |e'_i \rangle_A |f'_i \rangle_B \), \( b_i \neq 0, m \geq 2 \).

In essence, theorem 1 show that if one can distill a pure entangled state \(|\Psi \rangle \rangle \) in equation (2) from \( \rho_{AB}^{\otimes^n} \), there should exist a subspace \( H_{m\otimes m} \), the dimension of which is \( m \otimes m \). The component of \( \rho_{AB}^{\otimes^n} \) in this subspace is a pure state \(|\Phi \rangle \rangle \) with same Schmidt number as \(|\Psi \rangle \rangle \). The distillation protocol is just to project \( \rho_{AB}^{\otimes^n} \) onto this subspace and project out the pure state \(|\Phi \rangle \rangle \). We define this subspace as distillable-subspace (DSS).

If \( \rho_{AB}^{\otimes^n} \) is distillable, \( \rho_{AB}^{\otimes^n} \) has at least a DSS \( H_{m\otimes m} \) \( (m \geq 2) \). Because the component of \( \rho_{AB}^{\otimes^n} \) in the DSS \( H_{m\otimes m} \) is a pure state \(|\Phi \rangle \rangle \), so if we write down the matrix of \( \rho_{AB}^{\otimes^n} \) under the product bases \( \{|e'_i \rangle_A |f'_j \rangle_B \rangle\rangle (i,j = 1, ..., m) \} \) there are \( m^2 - m(m \geq 2) \) rows zero elements and \( m^2 - m(m \geq 2) \) columns zero elements in the matrix of \( \rho_{AB}^{\otimes^n} \). The rank of \( \rho_{AB}^{\otimes^n} \) is at most \( N_A^n, N_B^n - m^2 + 1 \), for the rank of a DSS \( H_{m\otimes m} \) is one. Thus we can get the following conclusion:

**Theorem 2:** If \( \rho_{AB} \) is n-distillable, the rank of \( \rho_{AB}^{\otimes^n} \) is at most \( N_A^n, N_B^n - m^2 + 1(m \geq 2) \), and \( \rho_{AB}^{\otimes^n} \) can be expressed as a form with \((m^2 - m)(m \geq 2) \) rows elements and \((m^2 - m)(m \geq 2) \) columns elements being zero.

Theorem 2 imply that all mixed states \( \rho_{AB} \) in 2 \( \otimes \) 2 systems cannot be distilled by individual copy, as is acclaimed before \[11\]. Because the dimension of Hilbert space for any DSS is equal to or more than 4, if one can distill a pure entangled state from 2 \( \otimes \) 2 systems, then the whole space of the 2 \( \otimes \) 2 systems will be a DSS and \( \rho_{AB} \) be a pure state. In fact, there are some mixed states in 2 \( \otimes \) 2 systems, even if they are entangled state one cannot distill a pure entangled state from \( \rho_{AB}^{\otimes^n} \) if \( n \) is finite \[10\].

Obviously, the DSS have the following properties.

1. The component of \( \rho_{AB}^{\otimes^n} \) in a DSS is a pure entangled state.
2. From the example in the following, it can be shown that \( \rho_{AB}^{\otimes^n} \) may has many DSS, and a few DSS may be combined into a DSS.
3. Any LOCC cannot produce a new extra DSS without the destruction of existing DSS owing to entanglement non-increasing under LOCC.

Now, we give an example to demonstrate how to judge whether a mixed state $\rho_{AB}$ (or $\rho_{AB}^{\otimes n}$) has a DSS or not. We have a mixed state in $2 \otimes 2$ systems:

$$\rho_{AB} = \begin{bmatrix} \frac{1}{4} & 0 & 0 & \frac{1}{4} \\ 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{1}{4} & 0 & 0 & \frac{1}{4} \end{bmatrix}$$

(6)

One can find the DSS of $\rho_{AB}^{\otimes 2}$ in following steps:

1. Calculate the eigenvalues $|\Psi_{AB}^1\rangle$ and the nonzero eigenvalues $\lambda_i$ of $\rho_{AB}$. In this case, $|\Psi_{AB}^1\rangle = (|\uparrow\rangle |\uparrow\rangle + |\downarrow\rangle |\downarrow\rangle)/\sqrt{2}$, $|\Psi_{AB}^2\rangle = |\uparrow\rangle |\downarrow\rangle$; $\lambda_1 = \lambda_2 = 1/2$.

2. Write down all pure states of $\rho_{AB}^{\otimes 2}$.

$$\lambda_3^2 : (|0\rangle |0\rangle + |1\rangle |1\rangle + |2\rangle |2\rangle + |3\rangle |3\rangle)/2; \quad \lambda_1\lambda_2 : (|0\rangle |2\rangle + |1\rangle |3\rangle)/\sqrt{2}$$

(7)

where $|0\rangle = |\uparrow\rangle\downarrow\rangle, |1\rangle = |\uparrow\rangle\uparrow\rangle, |2\rangle = |\downarrow\rangle\uparrow\rangle, |3\rangle = |\downarrow\rangle\downarrow\rangle$, are similar to binary form.

3. Find the DSS from all pure states above. In this case we find easily a DSS with probability $1/2\lambda_3^2$, in which the state $(|1\rangle |1\rangle + |2\rangle |2\rangle)/\sqrt{2}$ can be distilled. We can write down the pure decomposition of $\rho_{AB}^{\otimes n}$ with the method in step 2 and can find the all DSS of $\rho_{AB}^{\otimes n}$ by symmetry (see Fig.1). For example, the DSS of $\rho_{AB}^{\otimes 3}$ are (we represent the DSS with corresponding distillable pure state from this DSS): $(|1\rangle |1\rangle + |2\rangle |2\rangle)/\sqrt{2}$ with probability $1/4\lambda_3^3$, $(|1\rangle |5\rangle + |2\rangle |6\rangle)/\sqrt{2}$ with probability $1/2\lambda_3^2\lambda_2$; $(|5\rangle |5\rangle + |6\rangle |6\rangle)/\sqrt{2}$ with probability $1/4\lambda_2^2\lambda_1$; $(|3\rangle |3\rangle + |4\rangle |4\rangle)/\sqrt{2}$ with probability $1/4\lambda_1^3$. Or, the four lower dimension DSS above can be combined into two higher dimension DSS: $(|1\rangle |1\rangle + |2\rangle |2\rangle + |4\rangle |4\rangle)/\sqrt{3}$ with probability $3/8\lambda_1^3$ and $(|3\rangle |3\rangle + |5\rangle |5\rangle + |6\rangle |6\rangle)/\sqrt{3}$ with probability $3/8\lambda_2^3$.

Now, we discuss the most efficient protocols for the entanglement purifying. For the case of finite copies and asking the output with unite fidelity, from the proof of theorem 1 and the conception of DSS we can get that the most efficient protocols for entanglement distillation from $\rho_{AB}^{\otimes n}$ is to project out, and not destroy, every independent DSS of $\rho_{AB}^{\otimes n}$ with corresponding probability, this is because LOCC cannot produce a extra DSS, but may destroy the DSS. For example, to get the two DSS of $\rho_{AB}^{\otimes 3}$ in the above example, $(|1\rangle |1\rangle + |2\rangle |2\rangle + |4\rangle |4\rangle)/\sqrt{3}$ with $3/8\lambda_1^3$ probability and $(|3\rangle |3\rangle + |5\rangle |5\rangle + |6\rangle |6\rangle)/\sqrt{3}$ with $3/8\lambda_2^3$ probability, after local unitary transformations one may use three local project operation:

$$P_1 = |1\rangle \langle 1| + |2\rangle \langle 2| + |4\rangle \langle 4|$$

(8)

$$P_2 = |3\rangle \langle 3| + |5\rangle \langle 5| + |6\rangle \langle 6|$$

$$P_3 = |0\rangle \langle 0| + |7\rangle \langle 7|$$
onto A system, then A system sends the output result to B system. After receiving the information from A, B system use the same project operation. Consequently, one get the two DSS with same probability $\frac{3}{8}\lambda^3$, and the other state with probability $(1 - \frac{3}{4}\lambda^3)$. As Ref [1] mentioned, all distillation protocols involve one-way or two-way communication. The efficiency of one-way is higher than two-way. Obviously, the distillation protocol here involve only one-way communication. According to the most efficient protocol one can calculate the distillable entanglement for the case of the finite copies.

Similarly, for the case of infinite copies, the most efficient purifying protocols of $\rho_{AB}$ is to let $n$-copy of $\rho_{AB}$ ($n \to \infty$) spans a bigger Hilbert space, then project out the desirable subspace of $\rho_{AB}^n$ by PostSelection operation [11]. The component of $\rho_{AB}^n$ in the desirable subspace tends to a pure entangled state when $n \to \infty$. The whole purifying process ask only a round of purifying protocols. In the purifying scheme in Ref [4], every round of purifying is also to keep the desirable subspace, the bases of which is $|1\rangle|1\rangle$, $|1\rangle|2\rangle$, $|2\rangle|1\rangle$, $|2\rangle|2\rangle$(these bases have same definition as in Eq[4]), but the whole purifying process needs many rounds of purifying protocols which result in the unavoidable loss of distillable entanglement, so this scheme is not the most efficient one. How to calculate the distillable entanglement for infinite-copy case is beyond the realm of this paper.

Now, we will discuss the distillable entanglement $E_D$ of mixed states in $2 \otimes 2$ systems for the case of finite copies. Here we will show that only a class of states are distillable.

First, a necessary condition of distillability from $\rho_{AB}^n$ is that $\rho_{AB}^n$ is not a quasi-separable state(QSS) [10]. We say a state $\rho$ is a QSS iff one or many new-state of $\rho$ is separable. Any mixed state $\rho$ has infinite sets of pure state decompositions.

$$\rho = \sum_{i=1}^{\infty} p_i |\Psi_i\rangle \langle \Psi_i|$$  \hspace{1cm} (9)

For every decomposition, if one lets the pure state $|\Psi_i\rangle$ unchanged but change the probability $p_i$ of pure state $|\Psi_i\rangle$ in the real numbers realm (0,1), we say one gets a new-state of $\rho$ [10].

Second, a mixed state $\rho$ can be decomposed into [14]

$$\rho = \sum_{i=1}^{l} |x_i\rangle \langle x_i| = \sum_{i=1}^{m} |z_i\rangle \langle z_i| ,$$  \hspace{1cm} (10)

where $|x_i\rangle$, unnormalized, is a complete set of orthogonal eigenvectors corresponding to the nonzero eigenvalues of $\rho$, and $\langle x_i| x_i\rangle$ is equal to the its nonzero eigenvalues. For a state of $2 \otimes 2$ systems, there exist a set of decomposition $|z_i\rangle$ of $\rho$ noted by

$$|z_i\rangle = \sum_{j=1}^{l} u_{ij} |x_j\rangle , \hspace{1cm} i = 1, 2, \cdots, m$$  \hspace{1cm} (11)
where $|z_i\rangle$ is not necessarily orthogonal, the columns of transformation $u_{k\times l}$ are orthonormal vectors, and

$$\langle z_i | \tilde{z}_j \rangle = \lambda_i^j \delta_{ij}, \quad (12)$$

where, $|\tilde{z}_i\rangle = \sigma_y \otimes \sigma_y |z_i^*\rangle$, and $\lambda_i^j > \lambda_j^i > \lambda_j^j > \lambda_i^i \geq 0$. Let us suppose $\lambda_i^1 - \lambda_j^2 - \lambda_j^3 - \lambda_j^4 > 0$, namely $\rho$ is inseparable [14]. If not all $\lambda_i^j(i = 2, 3, 4)$ being zero, one can transfer the state $\rho$ into a separable state by decreasing the probability appearing $|z_1\rangle$, so the state $\rho$ is a QSS and $\rho^\otimes n$ is also a QSS. Thus $\rho$ cannot be distilled, i.e., $E_D(\rho) = 0$.

Third, a mixed state $\rho$ of $2 \otimes 2$ systems can be expressed as [8]:

$$\rho = \lambda |\Psi\rangle \langle \Psi| + (1 - \lambda) \rho_{\text{sep}} \quad (13)$$

$\rho_{\text{sep}}$ is a separable state. $\rho$ surely includes a mixed state $\rho_1$:

$$\rho_1 = \lambda_1 |\Psi\rangle \langle \Psi| + (1 - \lambda_1) |\Phi\rangle \langle \Phi| \quad (14)$$

where

$$|\Psi\rangle = \sin \theta |0\rangle_A |0\rangle_B + \cos \theta |1\rangle_A |1\rangle_B \quad (15)$$

$$|\Phi\rangle = (a_1 |0\rangle_A + a_2 |1\rangle_A) \otimes (b_1 |0\rangle_B + b_2 |1\rangle_B) \quad (16)$$

$|0\rangle_i$ and $|1\rangle_i$ $(i = A$ or $B$) are any orthogonal bases of Alice’s or Bob’s system, respectively. It is to say $\rho$ is a mixture of $\rho_1$ and a separable state. So if $\rho$ is distillable, then $\rho_1$ is also distillable and $\rho_1$ should be not a QSS. If $a_1, a_2, b_1, b_2$ are all nonzero, the matrix of $\rho_1$ and $\rho_1^\otimes n$ have not zero diagonal elements under any product bases (i.e., the bases vectors are not separable states). From the theorem 2 one can follow that $E_D(\rho_1) = 0$. If $|\Phi\rangle = (a_1 |0\rangle_A + a_2 |1\rangle_A) \otimes |0\rangle_B$ or $|\Phi\rangle = |0\rangle_A \otimes (b_1 |0\rangle_B + b_2 |1\rangle_B)$, $\rho_1$ have two nonzero $\lambda_i$ (see Eq(12)). Thus $\rho_1$ is a QSS, and $E_D(\rho_1) = 0$. So only when $|\Phi\rangle = |0\rangle_A |1\rangle_B$ or $|\Phi\rangle = |1\rangle_A |0\rangle_B$, $E_D(\rho_1) \neq 0$. But if $\rho = \lambda_1 |\Psi\rangle \langle \Psi| + \lambda_2 |0\rangle_A |1\rangle_B \langle 0|_A \langle 1|_B + \lambda_3 |1\rangle_A |0\rangle_B \langle 1|_A \langle 0|_B$, $E_D(\rho) \neq 0$ for the matrix of $\rho$ and $\rho^\otimes n$ have not zero diagonal elements under any product bases. So only $\rho$ with following forms have nonzero distillable entanglement:

$$\rho = \lambda_1 |\Psi'\rangle \langle \Psi'| + \lambda_2 |\Phi'\rangle \langle \Phi'| \quad (17)$$

or

$$\rho = \lambda_1 |\Psi''\rangle \langle \Psi''| + \lambda_2 |\Phi''\rangle \langle \Phi''| \quad (18)$$

where $\lambda_1 + \lambda_2 = 1$, $|\Psi'\rangle = \sin \theta |0\rangle_A |0\rangle_B + \cos \theta |1\rangle_A |1\rangle_B$, $|\Phi'\rangle = |01\rangle$ or $|10\rangle$; $|\Psi''\rangle = \sin \theta |0\rangle_A |1\rangle_B + \cos \theta |1\rangle_A |0\rangle_B$, $|\Phi''\rangle = |00\rangle$ or $|11\rangle$.
Now, we would like to discuss the relations between the DSS which emerge in \( n \) copies of uncorrelated pairs and decoherence-free-subspaces which are due to some collective noise. Suppose that each of many pure singlet pairs became the same mixed states owing to interaction with environment. Although each one became a mixed state, there may be some subspaces with pure entangled states in the whole Hilbert space of all pairs. These subspaces are decoherence-free. So in this sense, DSS are decoherence-free-spaces, and somehow allow for perfect error correction.

In summary, one can distill a pure entangled state iff there exists a distillable-subspace in which the component of \( \rho_{AB} \) is a pure state with Schmidt number \( m \geq 2 \). If there exists the distillable-subspace, one can get a pure entangled state by the project operation. It is not different for one to find a distillable-subspace of \( \rho_{AB} \), and to get all distillable-subspace of \( \rho_{AB}^{\otimes n} \) with symmetry. The most efficient distillation protocols (include the case of both infinite copies and finite copies) is to keep the desirous subspace and discard the other subspace by project operation. For the case of finite copies of mixed states in \( 2 \otimes 2 \) system, only a class of state are distillable.

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Figure Captions

Fig. 1. The pure-state decomposition of $\rho_{AB}^\otimes n$. Each pure state is expressed by some binary numbers. According to the symmetry, one can write down all pure state of $\rho_{AB}^\otimes n$. One can also find the independent DSS of $\rho_{AB}^\otimes n$. 
FIGURES