Complex Geometry and Dirac Equation

Stefano De Leo*,b, Waldyr A. Rodrigues, Jr.†b and Jayme Vaz, Jr.‡b

a Dipartimento di Fisica, Università degli Studi Lecce,
Istituto Nazionale di Fisica Nucleare, INFN, Sezione di Lecce
via Arnesano, CP 193, 73100 Lecce, Italia
and
b Instituto de Matemática, Estatística e Computação Científica, IMECC-UNICAMP
CP 6065, 13081-970, Campinas, S.P., Brasil
(November 18, 2018)

Complex geometry represents a fundamental ingredient in the formulation of the Dirac equation by the Clifford algebra. The choice of appropriate complex geometries is strictly related to the geometric interpretation of the complex imaginary unit $i = \sqrt{-1}$. We discuss two possibilities which appear in the multivector algebra approach: the $\sigma_{123}$ and $\sigma_{21}$ complex geometries. Our formalism permits to perform a set of rules which allows an immediate translation between the complex standard Dirac theory and its version within geometric algebra. The problem concerning a double geometric interpretation for the complex imaginary unit $i = \sqrt{-1}$ is also discussed.

PACS: 02.10.R, 03.65.P

I. INTRODUCTION

In this paper we present a set of rules for passing back and forth between the standard (complex) matrix-based approach to spinors in 4 dimensions and the geometric algebra formalism. This “translation” is only partial, consistent with the fact that the Hestenes formalism [1] provides additional geometrical interpretations. In a pure translation nothing can be predicted which is not already in the original theory. In the new version of Dirac’s equation some assumptions appear more natural, some calculations more rapid and new geometric interpretations for the complex imaginary unit $i = \sqrt{-1}$ appear in the translated version for the first time.

The matrix form of spinor calculus and the vector calculus formulated by Gibbs can be replaced by a single mathematical system, called multivector algebra, with which the tasks of theoretical physics can be carried out more efficiently. The multivector algebra derives its power from the fact that both the elements and the operations of the algebra are subject to direct geometric interpretation [2]. The geometric algebra is surely the most powerful and general language available for the development of mathematical physics [3,4]. The central result is a representation of the Dirac wave function which reveals a geometric structure, hidden in the conventional formulation [5].

“The projection of the Dirac equation into the Pauli algebra eliminates redundancies, simplifying our task to solve this equation, since in the Pauli algebra we work in an eight dimensional space over the real numbers, while in the standard formulation we have to do with a 32-dimensional space over the reals, the space of $4 \times 4$ complex matrix $\mathcal{C}_4$” - Zeni [6].

“The imaginary unit appearing in the Dirac equation and the energy-momentum operator represents the bivector generator of rotations in a space-like plane corresponding to the direction of the electron spin” - Hestenes [7].

We wish to clarify these statements. We agree with fact that in the Pauli algebra (isomorphic to the even part of the space/time algebra $\mathcal{C}_{1,3}^+$) we have only 8 real parameters in defining the Dirac spinors, but in defining the most general operator which acts on them, how many real parameters do we need? The imaginary unit $i$ is identified by the bivector $\sigma_{21} \in \mathcal{C}_{3,0}$. Is this the only opportunity? What about the possibility to identify the complex imaginary unit by the pseudoscalar $\sigma_{123} \in \mathcal{C}_{3,0}$?

In formulating the Dirac equation by the Pauli algebra we can start from the standard matrix formulation and use the ideal approach to spinors to make a clear translation to the Clifford algebra $\mathcal{C}_{4,1}$ which is isomorphic to $M_4(C)$. The following step is to reduce the formulation of the Dirac equation to an algebra of smaller dimension, the space-time algebra, $\mathcal{C}_{1,3}$. Finally, we get a projection of the Dirac equation in the Pauli algebra $\mathcal{C}_{3,0}$ [6].

*E-Mail: deleos@le.infn.it, deleo@ime.unicamp.br
†E-Mail: walrod@ime.unicamp.br
‡E-Mail: vaz@ime.unicamp.br
In this paper we shall perform a different approach. We give a set of rules which allow to immediately write the Dirac equation by using the Pauli algebra. The fundamental ingredients of this translation are the direct identification of the complex imaginary unit \(i = \sqrt{-1}\) by elements of the Pauli algebra and the introduction of the concept of “complex” geometry.

The standard (complex) 4-dimensional spinor

\[
\Psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix} \equiv \begin{pmatrix} \varphi_1 + i\eta_1 \\ \varphi_2 + i\eta_2 \\ \varphi_3 + i\eta_3 \\ \varphi_4 + i\eta_4 \end{pmatrix} \quad \varphi_m, \eta_m \in \mathbb{R}, \quad m = 1, 2, 3, 4 ,
\]

is characterized by 8 real parameters, which can be settled in the following 8-dimensional Clifford algebras

\[ Cl_{3,0} \sim M_2(\mathbb{C}), \quad Cl_{1,2} \sim M_2(\mathbb{C}), \quad Cl_{0,3} \sim \mathcal{H} \oplus \mathcal{H}, \quad Cl_{2,1} \sim M_2(\mathbb{R}) \oplus M_2(\mathbb{R}) .\]

The natural choice is \( Cl_{3,0} \sim M_2(\mathbb{C}) \), the algebra of the three-dimensional space. Such algebra allows an immediate geometric interpretation for the Pauli matrices:

| \( Cl_{3,0} \) | scalar 1 |
| --- | --- |
| vectors \( \sigma_1, \sigma_2, \sigma_3 \) |
| bivectors \( \sigma_2\sigma_1, \sigma_2\sigma_3, \) \( \sigma_3\sigma_1 \) |
| trivector \( \sigma_1\sigma_2\sigma_3 \) |

The Pauli algebra can be also represented by the complexified quaternionic ring \[10,11\]:

| \( \mathcal{H}_c \) |
| --- |
| \( 1 \) |
| \( \iota I, \iota J, \iota K \) |
| \( I, J, K \) |

In the following, we prefer to use the vectors \( \vec{\sigma} \in Cl_{3,0} \), in order to avoid confusion in the identification of the standard (complex) imaginary unit \(i = \sqrt{-1}\) by elements of the Pauli algebra. By identifying the complex imaginary unit \(i = \sqrt{-1}\) by elements of \( Cl_{3,0} \), we must recognize two possibilities

\[ i = \sqrt{-1} \rightarrow \sigma_{21} \equiv \sigma_2\sigma_1 \text{ (bivector)} \quad \text{or} \quad \sigma_{123} \equiv \sigma_1\sigma_2\sigma_3 \text{ (volume element)} ,\]

in fact

\[ \sigma_{21}^2 = \sigma_{123}^2 = -1 .\]

Consequently, \( \varphi_m + i\eta_m \) can be respectively translated by

\[ \varphi_m + \sigma_{21}\eta_m \quad \text{or} \quad \varphi_m + \sigma_{123}\eta_m \quad m = 1, \ldots, 4 .\]

We propose in this paper a discussion concerning these two different possibilities of translation for the standard complex Dirac theory. These two possibilities are strictly related to the use of two different “complex” geometries, namely

the \( \sigma_{123} \) and \( \sigma_{21} \) complex geometries.

In our formalism the standard physical results are soon reproduced. The possibility of choosing two different “complex” geometries in performing our translations will give an embarrassing situation: two different geometric interpretations for the complex imaginary unit \(i = \sqrt{-1}\), namely

bivector or volume element.
II. PROBABILITY AMPLITUDES AND COMPLEX GEOMETRY

The noncommutativity of the element of \( Cl_{3,0} \) algebra requires to specify whether our Hilbert space, \( V_{Cl_{3,0}} \), is to be performed by right or left multiplication of vectors by scalars. We will follow the usual choice and work with a linear vector space under right multiplication by scalars \([10,12–17]\). In quantum mechanics, probability amplitudes, rather than probabilities, superimpose, so we must determine what kinds of number system can be used for the probability amplitudes \( A \). We need a real modulus function \( N(A) \) such that

\[
\text{Probability} = |N(A)|^2.
\]

The first four assumptions on the modulus function are basically technical in nature

\[
\begin{align*}
N(0) &= 0, \\
N(A) &> 0 \text{ if } A \neq 0, \\
N(rA) &= |r|N(A), \ r \text{ real}, \\
N(A_1 + A_2) &\leq N(A_1) + N(A_2).
\end{align*}
\]

A final assumption about \( N(A) \) is physically motivated by imposing the correspondence principle in the following form: We require that in the absence of quantum interference effects, probability amplitude super-imposition should reduce to probability super-imposition. So we have an additional condition on \( N(A) \):

\[
N(A_1A_2) = N(A_1)N(A_2).
\]

A remarkable theorem of Albert shows that the only algebras over the reals, admitting a modulus functions with the previous properties are the reals \( \mathbb{R} \), the complex \( \mathbb{C} \), the (real) quaternions \( \mathbb{H} \) and the octonions \( \mathbb{O} \). The previous properties of the modulus function seem to constrain us to work with division algebras (which are finite dimensional algebras for which \( a \neq 0, b \neq 0 \) imply \( ab \neq 0 \)), in fact

\[ A_1 \neq 0, \ A_2 \neq 0 \]

implies

\[ N(A_1A_2) = N(A_1)N(A_2) \neq 0 \]

which gives

\[ A_1A_2 \neq 0. \]

A simple example of non-division algebra is provided by the algebra \( Cl_{3,0} \) since

\[
(1 + \sigma_3)(1 - \sigma_3) = 0
\]

guarantees that there are nonzero divisors of zero. So, if the probability amplitudes are assumed to be element of \( Cl_{3,0} \), we cannot give a satisfactory probability interpretation. Nevertheless, we know that probability amplitudes are connected to inner products, thus, we can overcome the above difficulty by defining an appropriate scalar product.

We have four possibilities:

We can define a binary mapping \( \langle \Psi | \Phi \rangle \) of \( V_{Cl_{3,0}} \times V_{Cl_{3,0}} \) into the scalar(S)/bivectorial(BV) part of \( Cl_{3,0} \), we recall that \( V_{Cl_{3,0}} \) represents the Hilbert space with elements defined in the Pauli algebra,

\[
\langle \Psi | \Phi \rangle_{(S,BV)} = \left( \int d^3x \Psi^\dagger \Phi \right)_{(S,BV)}.
\]

Note that the algebra \( (1, \sigma_{21}, \sigma_{23}, \sigma_{31}) \) is isomorphic to the quaternionic algebra. Thus, we have the mapping

\[
V_{Cl_{3,0}} \times V_{Cl_{3,0}} \rightarrow Cl_{0,2} \sim \mathcal{H}.
\]

We can also adopt the more restrictive “scalar” projection \( \langle \Psi | \Phi \rangle_S \):

\[
V_{Cl_{3,0}} \times V_{Cl_{3,0}} \rightarrow Cl_{0,0} \sim \mathcal{R}.
\]
The grade involution is an automorphism
\[ \langle \Psi | \Phi \rangle_{1(1;\sigma_{21})} \quad \text{and} \quad \langle \Psi | \Phi \rangle_{1(1;\sigma_{123})} . \]

In these case we define the following binary mappings
\[ V_{Cl_{3,0}} \times V_{Cl_{3,0}} \to C_{V_{0,1}^{123}} \equiv C_{1(1;\sigma_{21})} , \]
\[ V_{Cl_{3,0}} \times V_{Cl_{3,0}} \to C_{V_{0,1}^{123}} \equiv C_{1(1;\sigma_{123})} . \]

In the standard definition of inner product we find the operation of transpose conjugation, \( \Psi^\dagger \). How can we translate the transpose conjugation in the geometric algebra formalism?

The Clifford algebra \( Cl_{3,0} \) has three involutions similar to complex conjugation. Take an arbitrary element
\[ E = E_0 + E_1 + E_2 + E_3 \quad \text{in} \quad Cl_{3,0} , \]
written as a sum of a scalar \( E_0 \), a vector \( E_1 \), a bivector \( E_2 \) and a volume element \( E_3 \). We introduce the following involutions
\[ E^\bullet = E_0 - E_1 + E_2 - E_3 \quad \text{grade involution} , \]
\[ E^* = E_0 - E_1 - E_2 + E_3 \quad \text{conjugation} , \]
\[ E^\dagger = E_0 + E_1 - E_2 - E_3 \quad \text{reversion} . \]

The grade involution is an automorphism
\[ (E_a E_b)^\bullet = E_a^\bullet E_b^\bullet , \]
while the reversion and the conjugation are anti-automorphism, that is,
\[ (E_a E_b)^* = E_b^* E_a^* , \]
\[ (E_a E_b)^\dagger = E_b^\dagger E_a^\dagger . \]

\[ E^\dagger \equiv E^{\bullet*} \equiv E^{**} . \] We shall show that the reversion can be used to represent the hermitian conjugation.

Let us analyze the following products: \( \Psi^\star \Psi \), \( \Psi^\bullet \Psi \), \( \Psi^\dagger \Psi \), which involve the three involutions defined within the Clifford algebra \( Cl_{3,0} \). We must consider the two possibilities due to the identification of the complex imaginary unit
\[ i = \sqrt{-1} \] by \( \sigma_{21} \) and \( \sigma_{123} \). Let us perform a real projection of these products,
\[ \langle \Psi^\star \Psi \rangle_S =_{(i \equiv \sigma_{21})} \left\{ \langle \phi_1 + \sigma_{21} \eta_1 + \sigma_{23} \phi_2 + \sigma_{13} \eta_2 \rangle - \sigma_{123} \langle \phi_3 + \sigma_{21} \eta_3 + \sigma_{23} \phi_4 + \sigma_{13} \eta_4 \rangle \right\} \times \]
\[ \left\{ \langle \phi_1 + \sigma_{21} \eta_1 + \sigma_{23} \phi_2 + \sigma_{13} \eta_2 \rangle + \sigma_{123} \langle \phi_1 + \sigma_{21} \eta_3 + \sigma_{23} \phi_4 + \sigma_{13} \eta_4 \rangle \right\} \times \]
\[ \eta_1 - \eta_2 - \eta_3 - \eta_4 , \]
\[ \langle \Psi^\bullet \Psi \rangle_S =_{(i \equiv \sigma_{123})} \left\{ \langle \phi_1 + \sigma_{21} \eta_1 - \sigma_{23} \phi_2 - \sigma_{13} \eta_2 \rangle + \sigma_{123} \langle \phi_3 - \sigma_{21} \eta_3 - \sigma_{23} \phi_4 - \sigma_{13} \eta_4 \rangle \right\} \times \]
\[ \left\{ \langle \phi_1 + \sigma_{21} \eta_1 - \sigma_{23} \phi_2 + \sigma_{13} \eta_2 \rangle + \sigma_{123} \langle \phi_1 + \sigma_{21} \eta_3 - \sigma_{23} \phi_4 + \sigma_{13} \eta_4 \rangle \right\} \times \]
\[ \eta_1 - \eta_2 + \eta_3 - \eta_4 , \]
\[ \langle \Psi^\dagger \Psi \rangle_S =_{(i \equiv \sigma_{21})} \left\{ \langle \phi_1 - \sigma_{21} \eta_1 - \sigma_{23} \phi_2 - \sigma_{13} \eta_2 \rangle - \sigma_{123} \langle \phi_3 - \sigma_{21} \eta_3 - \sigma_{23} \phi_4 - \sigma_{13} \eta_4 \rangle \right\} \times \]
\[ \left\{ \langle \phi_1 - \sigma_{21} \eta_1 - \sigma_{23} \phi_2 + \sigma_{13} \eta_2 \rangle + \sigma_{123} \langle \phi_1 - \sigma_{21} \eta_3 - \sigma_{23} \phi_4 + \sigma_{13} \eta_4 \rangle \right\} \times \]
\[ \eta_1 + \eta_2 + \eta_3 + \eta_4 . \]
The first conclusion should be the use of the involution $\dagger$ and the assumption of a “real” geometry. Thus, we should translate

$$\left(\psi^* \psi^* \psi^* \psi^* \right) \equiv \sum_{m=1}^{4} (\varphi_m^2 + \eta_m^2)$$

by

$$\langle \psi^\dagger \psi \rangle_S.$$  

Nevertheless, this real projection of inner products gives an undesired orthogonality between $1, \sigma_{21}$ and $\sigma_{123}$. We know that the complex imaginary unit, $i = \sqrt{-1}$, represents a phase in the standard quantum mechanics, thus if we wish to adopt the identifications $i = \sqrt{-1} \rightarrow \sigma_{21}$ or $\sigma_{123}$, we must abandon the “real” geometry. We have another possibility. Let us rewrite $\Psi$ as follows

$$\Psi = h_1 + \sigma_{123}h_2 \quad h_{1,2} \in \mathcal{H}(1, \sigma_{21}, \sigma_{23}, \sigma_{31}),$$

the full $\Psi^\dagger \Psi$ product is given by

$$\Psi^\dagger \Psi = \left(h_1^\dagger - \sigma_{123}h_2^\dagger\right) \left(h_1 + \sigma_{123}h_2\right) = |h_1|^2 + |h_2|^2 + \sigma_{123} \left(h_1^\dagger h_2 - \text{h.c.}\right).$$

and so

$$\Psi^\dagger \Psi = \text{Real Part} + \text{Vectorial Part}.$$  

Consequently,

$$\langle \Psi^\dagger \Psi \rangle_S \equiv \langle \Psi^\dagger \Psi \rangle_{(1, \sigma_{21})} \quad \sigma_{21}\text{-complex geometry},$$

$$\langle \Psi^\dagger \Psi \rangle_S \equiv \langle \Psi^\dagger \Psi \rangle_{(1, \sigma_{123})} \quad \sigma_{123}\text{-complex geometry}.$$  

Now, $(1, \sigma_{21})$ and $(1, \sigma_{123})$ do not represent orthogonal states, and our spinor $\Psi$ have four complex orthogonal states, the complex orthogonality freedom degrees needed to connect a general element of the Pauli algebra to the 4-dimensional Dirac spinor

**III. BARRED OPERATORS**

We justify the choice of a complex geometry by noting that although there is the possibility to define an anti-self-adjoint operator, $\tilde{\partial}$, with all the properties of a translation operator, imposing a non-complex geometry, there is no corresponding self-adjoint operator with all the properties expected for a momentum operator. We can overcome such a difficulty by using a complex scalar product and defining as the appropriate momentum operator

$$\sigma_{21}\text{-complex geometry} \quad \bar{p} \equiv -\sigma_{21} \tilde{\partial},$$

$$\sigma_{123}\text{-complex geometry} \quad \bar{p} \equiv -\sigma_{123} \tilde{\partial},$$

where $1 \mid \sigma_{21}$ indicates the right action of the bivector $\sigma_{21}$. For $\sigma_{123}$, it is not important to distinguish between left and right action because $\sigma_{123}$ commutes with all the elements in $\mathcal{Cl}_{3,0}$. Note that the choice $\bar{p} \equiv -\sigma_{21} \tilde{\partial}$ still gives a self-adjoint operator with the standard commutation relations with the coordinates, but such an operator does not commute with the Hamiltonian, which will, in general, be an element of $\mathcal{Cl}_{3,0}$. Obviously, in order to write equations that are relativistically covariant, we must treat the space components and time in the same way, hence we are obliged to modify the standard “complex” equations by the following substitutions.
σ_{21}\text{-complex geometry} \quad i\partial^\mu \rightarrow \partial^\mu \mid_{\sigma_{21}},

σ_{123}\text{-complex geometry} \quad i\partial^\mu \rightarrow \sigma_{123}\partial^\mu .

Let us now introduce the complex/linear barred operators. Due to the non-commutative nature of the elements of $Cl_{3,0}$, we must distinguish between left and right action of $σ_{21}$, $σ_{23}$, $σ_{31}$. Explicitly, we write

$$1 \mid σ_{21} , \ 1 \mid σ_{23} , \ 1 \mid σ_{31} , \quad (2)$$

to identify the right multiplication of $σ_{21}$, $σ_{23}$, $σ_{31}$.

Note that the right action of $σ_{1}, σ_{2}, σ_{3}$ can be immediately obtained from the operators in (2) by $σ_{123}$ multiplication.

In rewriting the Dirac equation, we need to work with “complex” linear barred operators. Here, we must distinguish between $σ_{21}$ and $σ_{123}$ complex geometry. In fact, by working with a $σ_{123}$-complex geometry it is immediate to prove that

$$1 \mid σ_{21} , \ 1 \mid σ_{23} , \ 1 \mid σ_{31} ,$$

represent $σ_{123}$-complex/linear operators. On the contrary, by working with a $σ_{21}$-complex geometry we have only one permitted right action, that is

$$1 \mid σ_{21} ,$$

which represents a $σ_{21}$-complex/linear operator. Why this counting of parameters? It is simple. In $Cl_{3,0}$ we work with 8 real parameters, but the most general linear transformation which can be performed on an element of $Cl_{3,0}$, adopting a $σ_{123}$-complex geometry, is

$$A + B \mid σ_{21} + C \mid σ_{23} + D \mid σ_{31} \quad A, B, C, D \in Cl_{3,0} ,$$

which contains 32 real parameters, the same number of $M_{4}(C)$. This explains the possibility of a direct translation between $4 \times 4$ complex matrices and the Pauli algebra with $σ_{123}$-complex geometry

$$\begin{pmatrix}
    ψ_1 \\
    ψ_2 \\
    ψ_3 \\
    ψ_4 \\
\end{pmatrix} \leftrightarrow \Psi = ψ_1 + σ_{21}ψ_2 + σ_{23}ψ_3 + σ_{31}ψ_4$$

$$M_{4}(C) \leftrightarrow A + B \mid σ_{21} + C \mid σ_{23} + D \mid σ_{31} .$$

### A. $σ_{123}$-complex geometry and Dirac equation

We have now all the tools to reproduce the Dirac equation within the algebra $Cl_{3,0}$. It is sufficient to translate the standard equation

$$i\Gamma^\mu \partial_\mu \Psi = m\Psi ,$$

by using the identification of $i = \sqrt{-1}$ by $σ_{123}$ and finding a representation of the Dirac matrices, $Γ^\mu$, by elements of the Pauli algebra. We observe that the $Γ^\mu$’s can be rewritten in terms of elements of $Cl_{3,0}$, by adopting pseudoscalar and left/right action of bivectors. To reproduce the right anticommutation relation which characterize the Dirac algebra, we perform the following identification

$$\vec{Γ} \sim (σ_{23}, σ_{31}, σ_{12}) .$$

To satisfy the anticommutation relation between $Γ^0$ and $\vec{Γ}$, we introduce right actions

$$Γ^0 \sim 1 \mid σ_{32} \quad \text{and} \quad Γ^{1,2,3} \sim 1 \mid σ_{31} .$$

Finally, the hermiticity conditions give
The Dirac equation reads
\[ \partial_t \Psi_{\sigma_{23}} + \sigma_{23} \partial_x \Psi_{\sigma_{13}} + \sigma_{31} \partial_y \Psi_{\sigma_{13}} + \sigma_{12} \partial_z \Psi_{\sigma_{13}} = m \Psi_{\sigma_{23}}. \] (3)

Let us multiply the previous equation by the barred operator \( \sigma_{123} | \sigma_{23} \),
\[ \sigma_{123} \partial_t \Psi_{\sigma_{23}} + \sigma_{123} \sigma_{23} \partial_x \Psi_{\sigma_{13}} + \sigma_{123} \sigma_{31} \partial_y \Psi_{\sigma_{13}} + \sigma_{123} \sigma_{12} \partial_z \Psi_{\sigma_{13}} = m \sigma_{123} \Psi_{\sigma_{23}}. \]

By observing that
\[ \sigma_{2}^{2} = -1, \quad \sigma_{13} \sigma_{23} = \sigma_{21}, \quad \sigma_{123} (\sigma_{23}, \sigma_{13}, \sigma_{12}) = - (\sigma_{1}, \sigma_{2}, \sigma_{3}), \]
we find
\[ \sigma_{123} \partial_t \Psi + \sigma_{1} \partial_x \Psi_{\sigma_{21}} + \sigma_{2} \partial_y \Psi_{\sigma_{21}} + \sigma_{3} \partial_z \Psi_{\sigma_{21}} = m \Psi_{\sigma_{1}}, \] (4)

which represents the Dirac equation in the Pauli algebra with \( \sigma_{123} \)-complex geometry. This equation is obtained by simple translation, so it reproduces the standard physical contents. We are now ready to perform the desired translation rules:
\[
\Psi = \begin{pmatrix}
\varphi_1 + i \eta_1 \\
\varphi_1 + i \eta_2 \\
\varphi_1 + i \eta_3 \\
\varphi_1 + i \eta_4
\end{pmatrix} \quad \leftrightarrow \quad (\varphi_1 + \sigma_{123} \eta_1) + \sigma_{21} (\varphi_2 + \sigma_{123} \eta_2) + \sigma_{23} (\varphi_3 + \sigma_{123} \eta_3) + \sigma_{31} (\varphi_4 + \sigma_{123} \eta_4),
\]
\[ \Phi \dagger \Psi \quad \leftrightarrow \quad (\Phi \dagger \Psi)_{(1, \sigma_{123})}. \]

To give the correspondence rules between 4×4 complex matrices and barred operators, we need to list only the matrix representations for the following barred operators
\[ 1, \quad \sigma_{21}, \quad \sigma_{23}, \quad \sigma_{123}, \quad 1|\sigma_{12}, \quad 1|\sigma_{23}, \]
all the other operators can be quickly obtained by suitable multiplications of the previous ones. The translation of 1 and \( \sigma_{123} \) is very simple:
\[ 1 \leftrightarrow I_{4 \times 4} \quad \text{and} \quad \sigma_{123} \leftrightarrow i I_{4 \times 4}. \]

The remaining four operators are represented by
\[
\sigma_{21} \leftrightarrow \begin{pmatrix}
0 & -1 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 \\
0 & 0 & 1 & 0
\end{pmatrix}, \quad 1|\sigma_{21} \leftrightarrow \begin{pmatrix}
0 & -1 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & -1 & 0
\end{pmatrix},
\]
\[
\sigma_{23} \leftrightarrow \begin{pmatrix}
0 & 0 & -1 & 0 \\
0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 \\
0 & -1 & 0 & 0
\end{pmatrix}, \quad 1|\sigma_{23} \leftrightarrow \begin{pmatrix}
0 & 0 & -1 & 0 \\
0 & 0 & 0 & -1 \\
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0
\end{pmatrix}.
\]

---

**B. \( \sigma_{21} \)-complex geometry and Dirac equation**

Let us now discuss the possibility to write down the Dirac equation in the Pauli algebra with a \( \sigma_{21} \)-complex geometry. At first glance a problem appears. We have not the needed parameters in the barred operators to perform a translation. In fact, the most general \( \sigma_{21} \)-complex/linear operator is
and consequently we count only 16 real parameters. We have no hope to settle down the 32 real parameters characterizing a generic \(4 \times 4\) complex matrix. Nevertheless, we must observe the possibility to perform the grade involution, which represents a \(\sigma_{21}\)-complex linear operation:

\[
[\Psi (\alpha + \sigma_{21}\beta)]^* = \Psi^\ast (\alpha + \sigma_{21}\beta) \quad \alpha, \beta \in \mathcal{R}.
\]

Thanks to this involution we double our real parameters. Let us show the desired translation rules

\[
\Psi \equiv \begin{pmatrix}
\varphi_1 + i\eta_1 \\
\varphi_1 + i\eta_2 \\
\varphi_1 + i\eta_3 \\
\varphi_1 + i\eta_4
\end{pmatrix} \leftrightarrow \begin{pmatrix}
(\varphi_1 + \sigma_{21}\eta_1) + \sigma_{23}(\varphi_2 + \sigma_{21}\eta_2) + \sigma_{123}(\varphi_3 + \sigma_{21}\eta_3) + \sigma_{123}\sigma_{23}(\varphi_4 + \sigma_{21}\eta_4)
\end{pmatrix},
\]

\[
\Phi^\dag \Psi \leftrightarrow (\Phi^\ast \Psi)^{(1,\sigma_{21})}.
\]

To give the correspondence rules between \(4 \times 4\) complex matrices and barred operators, we need to list only the matrix representations for the following barred operators

\[
1, \; \sigma_{21}, \; \sigma_{23}, \; \sigma_{123}, \; 1 | \sigma_{21},
\]

and give the matrix version of the grade involution. All the other operators can be quickly obtained by suitable combinations of the previous operations. The translation of \(1\) and \(1 | \sigma_{21}\) is soon obtained:

\[
1 \leftrightarrow \mathbb{1}_{4 \times 4} \quad \text{and} \quad 1 | \sigma_{21} \leftrightarrow i\mathbb{1}_{4 \times 4}.
\]

The remaining rules are

\[
\sigma_{21} \leftrightarrow i \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & -1
\end{pmatrix}, \quad \sigma_{23} \leftrightarrow \begin{pmatrix}
0 & -1 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 \\
0 & 0 & 1 & 0
\end{pmatrix}, \quad \sigma_{123} \leftrightarrow \begin{pmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
-1 & 0 & 0 & 0 \\
0 & -1 & 0 & 0
\end{pmatrix},
\]

and finally the grade involution is represented by the following matrix

\[
\ast\text{-involution} \leftrightarrow \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & -1 & 0 \\
0 & 0 & 0 & -1
\end{pmatrix}.
\]

Let us examine how to translate the Dirac equation

\[
i\Gamma^\mu \partial_\mu \Psi = m\Psi,
\]

by working with a \(\sigma_{21}\)-complex geometry. Firstly, we modify the previous equation by multiplying it by \(\Gamma^0\) on the left

\[
i\partial_\mu \Psi + i\Gamma^0 \Gamma^\mu \cdot \bar{\Psi} = m\Gamma^0 \Psi.
\]

We observe that (by using the standard representation \([8][9]\) for the Dirac matrices)

\[
\Gamma^0 \Psi \equiv \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & -1 & 0 \\
0 & 0 & 0 & -1
\end{pmatrix} \begin{pmatrix}
\varphi_1 + i\eta_1 \\
\varphi_2 + i\eta_2 \\
\varphi_3 + i\eta_3 \\
\varphi_4 + i\eta_4
\end{pmatrix}
\leftrightarrow (\varphi_1 + \sigma_{21}\eta_1) + \sigma_{23}(\varphi_2 + \sigma_{21}\eta_2) - \sigma_{123}(\varphi_3 + \sigma_{21}\eta_3) - \sigma_{123}\sigma_{23}(\varphi_4 + \sigma_{21}\eta_4)
\leftrightarrow (\varphi_1 + \sigma_{21}\eta_1) + \sigma_{23}(\varphi_2 + \sigma_{21}\eta_2) + \sigma_{123}(\varphi_3 + \sigma_{21}\eta_3) + \sigma_{123}\sigma_{23}(\varphi_4 + \sigma_{21}\eta_4)^\ast,
\]

and

\[
\Gamma^0 \Gamma^\mu \leftrightarrow (\sigma_1, \sigma_2, \sigma_3), \quad i\mathbb{1}_{4 \times 4} \leftrightarrow 1 | \sigma_{21}.
\]

Thus, the translated Dirac equation reads:

\[
\partial_\mu \Psi \sigma_{21} + \sigma_1 \partial_\mu \Psi \sigma_{21} + \sigma_2 \partial_\mu \Psi \sigma_{21} + \sigma_3 \partial_\mu \Psi \sigma_{21} = m\Psi^\ast.
\]
IV. COMPLEX GEOMETRIES EQUIVALENCE

In the previous sections, we have performed two translated versions of the Dirac equation. Explicitly,
\[
\sigma_{123}\text{-complex geometry} \quad (\sigma_{123}\partial_t + \nabla \cdot \sigma_{21}) \Psi = m\Psi \sigma_1 ,
\]
and
\[
\sigma_{21}\text{-complex geometry} \quad (\partial_t + \nabla) \Psi \sigma_{21} = m\Psi^* ,
\]
where
\[
\nabla \equiv \sigma_1 \partial_x + \sigma_2 \partial_y + \sigma_3 \partial_z .
\]
We discuss in this section the possibility to relate the two equations obtained by imposing different geometries. Let us start by taking the \( \bullet \)-involution of Eq. (6)
\[
\sigma_{123}\partial_t \Psi \bullet + \nabla \Psi \sigma_{21} = m\Psi^* \sigma_1 .
\]
By working with Eqs. (6,8) we can reobtain Eq. (7). To do it, we introduce the idempotents
\[
e_\pm = \frac{1}{2} (1 \pm \sigma_3),
\]
and give some relations which will be useful in the following
\[
[e_\pm, \sigma_{21}] = 0, \quad \sigma_1 e_\pm = e_\mp \sigma_1 ,
\]
and
\[
\sigma_{123} e_- = e_- \sigma_{21} , \quad \sigma_{123} e_+ = -e_+ \sigma_{21} .
\]
Let us multiply Eqs. (6) and (8) from the right respectively by \( e_- \) and \( \sigma_1 e_+ \),
\[
\sigma_{123}\partial_t \Psi e_- + \nabla \Psi e_- \sigma_{21} = m\Psi e_+ \sigma_1 ,
\]
\[
\sigma_{123}\partial_t \Psi \sigma_1 e_+ - \nabla \Psi \sigma_1 e_+ \sigma_{21} = m\Psi^* e_+ .
\]
By using the relations in Eq. (9), we can rewrite the previous equations as follows
\[
(\partial_t + \nabla) \Psi e_- \sigma_{21} = m\Psi e_+ \sigma_1 ,
\]
and
\[
(\partial_t + \nabla) \Psi \sigma_1 e_+ \sigma_{21} = -m\Psi^* e_+ .
\]
By taking the “difference” between these last two equations, we have
\[
(\partial_t + \nabla) [\Psi e_- - \Psi^* \sigma_1 e_+] \sigma_{21} = m [\Psi e_+ \sigma_1 + \Psi^* e_+] .
\]
By redefining
\[
\Phi \equiv \Psi e_- - \Psi^* \sigma_1 e_+ ,
\]
and noting that
\[
\Phi^* = \Psi^* e_+ + \Psi \sigma_1 e_- = \Psi^* e_+ + \Psi e_+ \sigma_1 ,
\]
we find
\[
(\partial_t + \nabla) \Phi \sigma_{21} = m\Phi^* ,
\]
as anticipated.
We conclude this section by discussing the phase transformations characterizing our equations. It is immediate to show that the phase transformation

$$\Psi \rightarrow \Psi e^{\sigma_{123}\alpha} \quad \alpha \in \mathcal{R},$$

implies the following transformation on $\Phi$

$$\Phi \rightarrow \Phi e^{\sigma_{21}\alpha}.$$ 

In fact,

$$\Phi' = \Psi e^{\sigma_{123}\alpha} e_\perp - \Psi* e^{-\sigma_{123}\alpha} \sigma_1 e_+$$

$$= \Psi e_\perp e^{\sigma_{21}\alpha} - \Psi* \sigma_1 e_+ e^{\sigma_{21}\alpha}$$

$$= \Phi e^{\sigma_{21}\alpha}.$$

At this stage, there is no difference in using a $\sigma_{123}$ or $\sigma_{21}$ complex geometry. So, we have an equivalence between $\sigma_{123}$ and $\sigma_{21}$ complex geometry within the Pauli algebra.

V. CONCLUSION

The possibility of using Clifford algebra to describe standard quantum mechanics receives a major thrust with the adoption of a complex scalar product (complex geometry). A second important step in this objective of translation is achieved with the introduction of the so-called barred operators, which permit to write down few translation rules which allow to quickly reproduce in the $\text{Cl}_{3,0}$ formalism the standard results of the Dirac theory. All the relations can be manipulated without introducing a matrix representation, greatly simplifying the algebra involved.

In this paper we worked with the Pauli algebra but we wish to remark that our considerations can be immediately generalized to the spacetime algebra, which represents the natural language for relativistic quantum mechanics.

In the standard literature, the unit scalar imaginary of quantum mechanics is replaced by a bivector. We showed that another possibility is also available, namely the identification of the unit scalar imaginary $i = \sqrt{-1}$ by the pseudoscalar $\gamma_{0123}$ of the spacetime algebra ($\sigma_{123}$ in the Pauli algebra). These two geometric interpretations reflect the two possible choices in defining a complex geometry within the multivector formalism. At the free-particle level, there is an equivalence in using these two complex scalar products.

We conclude by observing that a possible difference between the $\sigma_{21}$ and $\sigma_{123}$ complex geometries could appear in the formulation of the Salam-Weinberg model, where the electromagnetic group is obtained by symmetry breaking from the Glashow group $SU(2) \times U(1)$. It appears natural to use

$$\sigma_{21}, \quad \sigma_{23}, \quad \sigma_{31} \quad \text{and} \quad 1 \mid \sigma_{21},$$

as generators of the electroweak group. In this case the right choice should be the adoption of a $\sigma_{21}$ complex geometry. After symmetry breaking the remaining electromagnetic group will be identified by the left/right action of the generator $\sigma_{21}$. A complete discussion of the Salam-Weinberg model within the multivector formalism will be given in a forthcoming paper [20].

ACKNOWLEDGMENTS

One of the authors (SdL) enjoyed the help of many friends and colleagues: Capí, Dermevalle, Marcelo, Angela, Evelize, Lilian, Paula e Vera. In particular, the author would like to thank Mario e Dora for the hospitality during the stay in Brasil, and Luis for his genuine friendship. For financial support, SdL is indebted to the IMECC-UNICAMP.

[1] D. Hestenes, Space-Time Algebra (Gordon & Breach, New York, 1966).
[2] D. Hestenes, Am. J. Phys. 39, 1013 (1971).
[3] D. Hestenes and G. Sobczyk, Clifford Algebra to Geometric Calculus (D. Riedel Publishing Company, Dordrecht, 1984).
[4] P. Lounesto, *Clifford Algebras and Spinors* (Cambridge UP, Cambridge, 1997).
[5] D. Hestenes and A. Weingarthofer, *The Electron, New Theory and Experiment* (Kluwer Academic Publishers, Dordrecht, 1991).
[6] J. R. Zeni, in P. Letelier and W. A. Rodrigues (eds.), *Gravitation: The Space-Time Structure* (World Scientific, Singapore, 1994), p. 544.
[7] D. Hestenes, J. Math. Phys. 8, 798 (1967).
[8] J. Rembieliński, J. Phys. A 11, 2323 (1978).
[9] L. P. Horwitz and L. C. Biedenharn, Ann. Phys. 157, 432 (1984).
[10] S. De Leo and W. A. Rodrigues, *Quaternionic Quantum Mechanics: From Complex to Complexified Quaternions*, Int. J. Theor. Phys. 36, 2725 (1997).
[11] S. De Leo and W. A. Rodrigues, *Quaternionic Electron Theory: I-Dirac’s Equation and II-Geometry, Algebra and Dirac’s Spinors*, Int. J. Theor. Phys. (submitted).
[12] D. Finkelstein, J. M. Jauch, S. Schiminovich and D. Speiser, J. Math. Phys. 3, 207 (1962); 4, 788 (1963).
D. Finkelstein, J. M. Jauch and D. Speiser, J. Math. Phys. 4, 136 (1963).
[13] S. L. Adler, *Quaternion Quantum Mechanics and Quantum Field* (Oxford UP, New York, 1995).
[14] D. Hestenes, J. Math. Phys. 16, 556 (1975); Phys. Teach. 17, 235 (1979); Found. Phys. 20, 1213 (1990).
[15] P. Lounesto, in P. Letelier and W. A. Rodrigues (eds.), *Gravitation: The Space-Time Structure* (World Scientific, Singapore, 1994), p. 50; Found. Phys. 16, 967 (1986); 23, 1203 (1993).
[16] J. Keller, Adv. in Appl. Cliff. Alg. 3, 147 (1993).
[17] S. Gull, A. Lasenby and C. Doran, Found. Phys. 23, 1175 (1993); *ibidem*, 1239 (1993).
[18] C. Itzykson and J. B. Zuber, *Quantum Field Theory* (McGraw-Hill, New York, 1985).
[19] J. D. Bjorken and S. D. Drell, *Relativistic Quantum Mechanics* (McGraw-Hill, New York, 1964).
[20] S. De Leo, W. A. Rodrigues and J. Vaz, *Space-Time Algebra and Electroweak Theory* (work in progress).