Modulational instability of two-component Bose-Einstein condensates in an optical lattice

Guang-Ri Jin\textsuperscript{1}, Chul Koo Kim\textsuperscript{1}, and Kyun Nahm\textsuperscript{2}

\textsuperscript{1} Institute of Physics and Applied Physics, Yonsei University, Seoul 120-749, Korea
\textsuperscript{2} Department of Physics, Yonsei University, Wonju 220-710, Korea

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We study modulational instability of two-component Bose-Einstein condensates in an optical lattice, which is modelled as a coupled discrete nonlinear Schrödinger equation. The excitation spectrum and the modulational instability condition of the total system are presented analytically. In the long-wavelength limit, our results agree with the homogeneous two-component Bose-Einstein condensates case. The discreteness effects result in the appearance of the modulational instability for the condensates in miscible region. The numerical calculations confirm our analytical results and show that the interspecies coupling can transfer the instability from one component to another. PACS numbers: 05.45.-a, 03.75.Lm

I. INTRODUCTION

The modulational instability (MI) is a general phenomenon in nonlinear wave equations and can occur in various physical systems, such as fluid dynamics, plasma physics, and nonlinear optics. Due to the interplay between nonlinearity and the dispersive effects, a weak perturbation on a plane wave may induce an exponential growth, and result in the carrier wave break up into a train of localized waves [1]. Recently, the MI of Bose-Einstein condensates (BEC) has attracted much interest, especially for the single-component BEC in an optical lattice [2,3]. In the superfluid regime, the BEC in an optical lattice can be modeled by a discrete nonlinear Schrödinger equation [4,5]. Following Ref. [6], the MI of a BEC in a deep optical lattice has been studied both theoretically [7–10] and experimentally [11,12].

Compared with the single-component BEC system, binary mixtures of BECs have much richer structures [13]. In fact, the excitation spectrum of a homogenous two-component BECs has been studied in Refs. [14–16]. It was shown that the MI depends strongly on the sign of $\Lambda_{12}^2 - \Lambda_1 \Lambda_2$, where $\Lambda_1$ and $\Lambda_2$ are the intraspecies scattering strengths, respectively. In Ref. [17], the MI of the two-component BECs in a deep optical lattice was studied with a Josephson-like coupling term. However, their works were limited on a fixed perturbation wave number (i.e., $q = \pi$). In addition, they did not compare their results with that of the homogenous two-component BECs case [14–16].

In this paper, we study the MI of two-component BECs in an optical lattice. The explicit expression of the excitation spectrum is presented analytically. In order to outline the effects of the discreteness, we compare the MI condition of the modulated plane waves with that of the homogeneous two-component BECs case, in which the MI can occur only for $\Lambda_{12}^2 > \Lambda_1 \Lambda_2$, i.e., the BECs in phase separation region [14–16]. Our results show that, the MI of the two component BECs in an optical lattice depends not only on the sign of $\Lambda_{12}^2 - \Lambda_1 \Lambda_2$, but also on the wave number $k$ of the carrier waves. For $\Lambda_{12}^2 < \Lambda_1 \Lambda_2$, the MI of the miscible condensates can also take place when the condensates have relatively large wave number $k > \pi/2$. The effect of the interspecies coupling is investigated by numerical calculations. Our results show that the coupling between the two condensates can transfer the instability from one component to another.

II. THEORETICAL MODEL

We consider a cloud of BEC atoms which have two internal states labeled by $|1\rangle$ and $|2\rangle$ in an optical lattice. When the heights of the interwell barriers are much higher than the chemical potentials, the condensate wave functions can be expressed as a sum of wave functions localized in each well of the periodic potential $\Psi_{\sigma}(r,t) = \sqrt{N} \sum_j \psi_{j,\sigma}(t) \phi_{\sigma}(r - r_j)$, where $N$ is the total number of condensate atoms, and $\phi_{\sigma}(r - r_j)$ is the spatial wave function localized in the $j$th site. With the help of the tight-binding approximation [4], the probability amplitudes $\psi_{j,\sigma}$ obey a set of coupled DNLS equations ($\hbar = 1$):

$$i \frac{\partial \psi_{j,\sigma}}{\partial t} = -K_{\sigma}(\psi_{j-1,\sigma} + \psi_{j+1,\sigma}) + \Lambda_{\sigma,\sigma'}|\psi_{j,\sigma}|^2 + \Lambda_{\sigma,\sigma'}|\psi_{j,\sigma'}|^2 \psi_{j,\sigma},$$

where $K_{\sigma}$ and $\Lambda_{\sigma,\sigma'}$ with $\sigma = 1, 2$ describe the nearest-neighbor hopping term and the on-site atomic collisions, respectively. The stationary solutions of the DNLS equations (1) are plane waves $\psi_{j,\sigma} = e^{i(k_j - \mu_\sigma t)}$ with the chemical potentials of the two-component condensed atoms

$$\mu_\sigma = -2K_\sigma \cos(k) + \Lambda_{\sigma,\sigma'}|\psi_{\sigma}^{(0)}|^2 + \Lambda_{\sigma,\sigma'}|\psi_{\sigma'}^{(0)}|^2.$$ (2)

Here, the chemical potentials $\mu_\sigma$ are to be determined by the normalization condition $\sum_{j,\sigma} |\psi_{j,\sigma}|^2 = 1$. The terms $-2K_\sigma \cos(k)$ in Eq. (2) are the kinetic energies aroused from tunnelling. It should be noted that, we
study the BECs system in the superfluid regime with \( (N/M)K_2 > \Lambda_{\sigma,\sigma} \) \( (M \) is the lattice number), and thus the dynamics can be well described by the DNLS equations.

In the above derivations, we have assumed \( \psi^{(0)} \) to be real, and taken \( K_1 = K_2 = K \), then \( \epsilon_{q,1} = \epsilon_{q,2} = \epsilon_q = 4K \cos(k) \sin(q/2) \) [18]. In the absence of the interspecies interactions \( (\Lambda_{12} = 0) \), \( \Delta_\sigma = 2\Lambda_1(\psi^{(0)}_1)^2 \) or \( 2\Lambda_2(\psi^{(0)}_2)^2 \). The two-component BECs are fully decoupled, and the spectra are the same with single-component case [9,18]. According to Refs. [6,8], the plane waves of the condensates are modulationally unstable when the eigenfrequencies becomes imaginary, i.e., \( \epsilon_q(\epsilon_q + \Delta_\sigma) < 0 \). In this case, the system undergoes exponential growth with the growth rate determined by the imaginary part of \( \omega^{(\pm)}_{q,\sigma} \). From Eq. (4), we know that: (i) if \( \Lambda_{12}^2 < \Lambda_1\Lambda_2 \), i.e., the case of the two components being miscible, both \( \Delta_1 \) and \( \Delta_2 \) are positive; (ii) if \( \Lambda_{12}^2 > \Lambda_1\Lambda_2 \), then \( \Delta_1 < 0 \) and \( \Delta_2 > 0 \). Within the second scenario, the condensates tend to separate spatially, namely the phenomena of phase separation [19,20]. The sign of \( (\Lambda_{12}^2 - \Lambda_1\Lambda_2) \) is of crucial important in discussing the stability of the eigenmodes.

III. MODULATIONAL INSTABILITY

In this section, we study the modulational instability by using Eq. (3). To understand the effect of the discreteness clearly, we first study Eq. (3) in the long-wavelength limit [6]. Eq. (3) in the long-wavelength limit, i.e., \( k \to 0 \) and \( q \to 0 \), is reduced to:

\[
\omega^{(\pm)}_{q,\sigma} = 2Kqk \pm \sqrt{Kq^2(Kq^2 + \Delta_\sigma)},
\]

which agrees with the usual expression obtained from continuous NLS equations [14–16] by replacing \( Kq^2 \) with \( q^2/2m \). It was shown that only immiscible BECs (with \( \Lambda_{12}^2 > \Lambda_1\Lambda_2 \)) exhibit the MI [21].

A. The MI condition for the miscible BECs

The MI condition for the two-component BECs are determined by the condition: \( \epsilon_q(\epsilon_q + \Delta_\sigma) < 0 \). For the miscible BECs with \( \Lambda_{12}^2 < \Lambda_1\Lambda_2 \), the MI occurs for phonon modes are taken in the traditional form \( \delta \psi_{j,\sigma} = u_{\sigma}e^{i(\omega - \omega t)} + v_{\sigma}e^{-i(\omega - \omega t)} \), with the constants \( u_{\sigma} \) and \( v_{\sigma} \) to be determined. Then we obtain the excitation spectra as

\[
\omega^{(\pm)}_{q,\sigma} = 2K \sin(k) \sin(q) \pm \sqrt{\epsilon_q(\epsilon_q + \Delta_\sigma)},
\]

where

\[
\Delta_\sigma = \Lambda_1(\psi^{(0)}_1)^2 + \Lambda_2(\psi^{(0)}_2)^2 + (1)^\sigma \sqrt{\left[\Lambda_1(\psi^{(0)}_1)^2 - \Lambda_2(\psi^{(0)}_2)^2\right]^2 + 4\Lambda_{12}^2(\psi^{(0)}_1\psi^{(0)}_2)^2}.
\]

Therefore, the miscible BECs may exhibits the MI when \( \pi/2 < k < 3\pi/2 \) and \( \psi_0^2 > -\epsilon_q/\Omega_\sigma \). More specially, both of the two-component with relative large wave number \( k > \pi/2 \) will be unstable for arbitrary \( q \) provided that \( \psi_0^2 > \psi_0^2_{\text{cr}} = 4K/\Omega_1 \). Below the critical amplitude, the MI region depends on the perturbation’s wave number \( q \). In Fig. 1 (a), we consider the case \( \psi_0^2 < \psi_0^2_{\text{cr}} \), where the atomic collisions are taken as \( a_1 : a_{12} : a_2 :: 1.007 : 1 : 1.01 \) [22]. We take \( \Lambda_1 = \Lambda \), then \( \Lambda_{12} = 0.993\Lambda \) and \( \Lambda_2 = 1.00298\Lambda \), which satisfy the miscible criterion. Our results show that there exist three different regions in the \( (q, k) \) plane. The dotted line in Fig. 1(a) separates the unstable regime into a fully unstable and a partially unstable, in which one component is stable and the second one is unstable.

B. The MI condition for the BECs in phase separation region

To study the MI of the immiscible BECs, we take the atomic collisions are taken as \( a_1 : a_{12} : a_2 :: 1.03 : 1 : 0.97 \) [23]. Thus we have \( \Lambda_{12} = 0.9709\Lambda \) and \( \Lambda_2 = 0.9417\Lambda \), which meet the condition \( \Lambda_{12}^2 > \Lambda_1\Lambda_2 \). Unlike the previous miscible case, the MI condition considered here depends on the wave number \( k \). We find that for \( \cos k > 0 \) (i.e., \( k < \pi/2 \) or \( k > 3\pi/2 \)), the MI condition is \( \psi_0^2 > \psi_0^2_{\text{cr}} = 4K/\Omega_1 \). Below the critical amplitude, the MI region in the \( (q, k) \) plane is shown in Fig. 1 (b). The MI can take place for any wave number \( k \).
tion condition. The wave numbers are chosen as \((l = 100)\), \((A_{amplitude as we choose the unit lattice constant. \alpha\) background amplitudes \(\alpha\) modulation amplitudes \(\alpha\) in accordance with our previous discussions. The modu-

\[ M\]

matic lattice with the total number of the sites \(K\) taken as \(1\), \(\Lambda = 100\), and \(\psi_0 = 1/(2M + 1) < \psi_{cr}\).

IV. NUMERICAL ANALYSIS

The linear-stability analysis can determine the onset of MI, however it does not yield any dynamical information beyond the instability point. Therefore, we perform numerical simulations of the DNLS equations. The initial conditions are two modulated plane waves

\[ \psi_{j,1}(0) = \psi_{j,2}(0) = [A + \alpha \cos(qj)]e^{ikj}, \]

in accordance with our previous discussions. The modulation amplitudes \(\alpha\) are assumed small compared with the background amplitudes \(\alpha = 0.05\). We consider the optical lattice with the total number of the sites \(M = 400\). From the periodic boundary condition, the wave numbers take the form of \(k = 2\pi l/M\) and \(q = 2\pi s/M\), where we choose the unit lattice constant.

We first consider the miscible BEC case. We take the amplitude as \(A = 1/\sqrt{2M + 1}\) to insure the normalization condition. The wave numbers are chosen as \((l = 50, s = 100)\), \((l = 150, s = 100)\), and \((l = 150, s = 50)\), which correspond to the points labeled by 1, 2, 3 in Fig. 1(a). The first two points are stable, and the last one is in an unstable regime with the growth rate being 0.1863. Density of the first component \(|\psi_{j,1}(t)|^2\) is plotted in Fig. 2. As expected from the analytical prediction, for the first point \((k = \pi/4, q = \pi/2)\), the modulated wave is stable. However, for the second point \((k = 3\pi/4, q = \pi/2)\) as shown in Fig. 2(b), the density increase sharply at the final stage. It is not predicted by the linear stability analysis. The reason for this phenomenon is the complex interactions between three fundamental eigenmodes of wave numbers \(k, k - q,\) and \(k + q\) contained in the initial state, and additional components of other wave numbers [6]. Fig. 2(c) shows the case that \(\omega_{q,1}\) is real and \(\omega_{q,2}\) is imaginary, which means that the component 1 is stable and the component 2 is unstable. However, the instability is transferred from one component to another due to the effects interspecies coupling, which in turn leads to the MI of total system. The appearance of spatially localized modes with large amplitude indicates the modulational instability of the miscible BECs.

![FIG. 1. Region of the MI in the \((q, k)\) plane for (a) miscible BEC \(\Lambda_1 = 0.993\) and \(\Lambda_2 = 1.00298\). (b) immiscible BEC \(\Lambda_1 = 0.9709\) and \(\Lambda_2 = 0.9417\). Other parameters are taken as \(K = 1\), \(\Lambda_1 = \Lambda = 100\), and \(\psi_0 = 1/(2M + 1) < \psi_{cr}\).](image1)

![FIG. 2. Density of component 1 for the miscible case \((\Lambda_{12}^2 < \Lambda_1 \Lambda_2)\). Component 2 is similar with component 1. (a) \(k = \pi/4, q = \pi/2\). (b) \(k = 3\pi/4, q = \pi/2\), and (c) \(k = 3\pi/4, q = \pi/4\). Other parameters are the same as Fig. 1(a).](image2)
analytical prediction. Other simulations in the unstable regime show that the increase of $q$ in the monotonic regime ($0 < q < \pi$) can enhance the growth rate, i.e., the instability growth time becomes more and more shorter with the increase of $q$.

For the smaller $q$ wave number species can be modulationally unstable for any given $q$ BECs with the phase separation can take place for any whole system. The discreteness effect leads to the appearance of the MI for the two-component BECs not only in the miscible region but also in the phase separation region. In summary, with the help of linear-stability analysis, we have studied the stability of the modulated plane waves, which is essential to predict the existence of nonlinear localized modes, such as dark solitons, in the discrete nonlinear Schrödinger equation [24,25].

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![Graph](image_url)

**FIG. 3.** Density of component 1 for the immiscible case ($\Lambda^2_2 > \Lambda_1 \Lambda_2$). Component 2 is similar with component 1. (a) $k = 3\pi/4$, $q = \pi/40$, (b) $k = 3\pi/4$, $q = \pi/20$. Other parameters are the same as Fig. 1(b).

### V. CONCLUSIONS

We have studied the modulational instability of two-component BEC in the superfluid regime. The explicit expression of the excitation spectrum is presented analytically. In the long-wavelength limit, our results can recover previous results of two-species BECs in homogeneous case. From the Bogoliubov dispersion relation we studied the MI condition of the modulated plane waves. The discreteness effect leads to the appearance of the MI for the two-component BECs not only in the miscible region but also in the phase separation region.

For the miscible condensates, the MI can occur for relatively large wave number of the condensates $k > \pi/2$. For the smaller $\psi_0^2$ compared with the critical density $\psi_{0,\text{cr}}^2$, the MI condition depends also on the perturbation wave number $q$. The MI conditions in the two-component BECs with the phase separation can take place for any $k$. However, unlike to the miscible case, only one of the species can be modulationally unstable for any given $q$ and $k$. To confirm the analytical results, we also performed numerical calculations. We find that the inter-species coupling can transfer the instabilities from one component to the other, which lead to the MI of the whole system.

\[ \text{In summary, with the help of linear-stability analysis, we have studied the stability of the modulated plane waves, which is essential to predict the existence of nonlinear localized modes, such as dark solitons, in the discrete nonlinear Schrödinger equation [24,25].} \]

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