On 4n-dimensional neither pointed nor semisimple Hopf algebras and the associated weak Hopf algebras

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Abstract. For a class of neither pointed nor semisimple Hopf algebras $H_{4n}$ of dimension $4n$, it is shown that they are quasi-triangular, whose universal $R$-matrices are described. The corresponding weak Hopf algebras $\mathfrak{w}H_{4n}$ and their representations are constructed. Finally, their duality and their Green rings are established by generators and relations explicitly. It turns out that the Green rings of the associated weak Hopf algebras are not commutative even if the Green rings of $H_{4n}$ are commutative.

Keywords: quasi-triangularity, Green ring, weak Hopf algebra, representation.

Mathematics Subject Classification: 16G10, 16D70, 16T99.

Introduction

Let $k$ be an algebraically closed field of characteristic zero. In this paper, we will study representations of the class of neither pointed nor semisimple Hopf algebra $H_{4n}$ of dimension $4n$ (see Definition 1.1) and the associated weak Hopf algebras.

The class of Hopf algebras $H_{4n}$ plays an important role in constructing new Nichols algebras, new Hopf algebras and classifying Hopf algebras. Note that if $n = 2, 3$ and $a \neq 0$, it is just the unique neither pointed nor-semisimple 8-dimensional Hopf algebra $(A''_{C_4})^*$ (see [29]), or the 12-dimensional Hopf algebra $A_1^+$ (see [19]) up to isomorphism respectively. In [9], the authors determined all finite-dimensional Hopf algebras over $k$ whose coradical generated a Hopf subalgebra isomorphic to $H_8$. They also obtained new Nichols algebras of dimension 8 and new Hopf algebras of dimension 64. Based on this, [22] determined all finite-dimensional Nichols algebras over the semisimple objects in $H_8^H YD$ and obtained some new Nichols algebras of non-diagonal type and new Hopf algebras without the dual Chevalley property. By the equivalence $M D(H_{12}) \simeq H_{12} YD$, the authors ([11], [33]) obtained some new Nichols algebras which were not of diagonal type and some families of new Hopf algebras of dimension 216.

As is well known, the classification of finite dimensional Hopf algebras over $k$ is an important open problem. Since Kaplansky’s conjectures posed in 1975, several results on them have been obtained (see [36, 24, 3, 21, 22, 10, 7, 4]). In [3], the authors proved that there were exactly $4(q - 1)$ isomorphism classes of non-semisimple pointed Hopf algebras of dimension $pq^2$, of which, those Radford’s Hopf algebras (see [23]) occupied 1/4. It is remarked that the dual of $H_{4n}$ is just the Radford’s Hopf algebra in [23].

Given a Hopf algebra $H$, the decomposition problem of tensor products of indecomposable modules has attracted numerous attentions. In [8], Cibils classified the indecomposable modules over $k\mathbb{Z}_n(q)/I_d$, and gave the decomposition formulas of the tensor product of two indecomposable $k\mathbb{Z}_n(q)/I_d$-modules. Yang determined the representation type of a class of pointed Hopf algebras, classified all indecomposable modules of simple pointed Hopf algebra $R(q, a)$. The decomposition

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formulas of the tensor product of two indecomposable $R(q,a)$-modules is given (see [34]). It is noted that some results of $R(q,a)$ were recently extended to more general case of pointed Hopf algebras of rank one by Wang et al. (see [31]). Li and Hu described the Green rings of the 2-rank Taft algebra (at $q = -1$) (see [14]). Chen, Van Oystaeyen and Zhang gave the Green rings of the Taft algebra $H_n(q)$ (see [5]). Li and Zhang extended the results of [5], computed the Green rings of the Generalized Taft Hopf algebras $H_{n,d}$ by generators and generating relations, and determined all nilpotent elements in $r(H_{n,d})$ (see [15]). Su and Yang (see [25]) characterized the representation ring of small quantum group $\bar{U}_q(sl_2)$ by generators and relations. It turns out that the representation ring of $\bar{U}_q(sl_2)$ is generated by infinitely many generators subject to a family of generating relations.

The concept of weak Hopf algebra in the sense of Li was introduced by [13] in 1998 as a generalization of Hopf algebra. Since then, many weak Hopf algebras or weak quantum groups were constructed, for example, Aizawa and Isaac ([1]) constructed weak Hopf algebras corresponding to $U_q(sl_n)$ and Yang ([25]) constructed weak Hopf algebras $w_q^d(g)$ corresponding to quantized enveloping algebras $U_q(g)$ of a finite dimensional semisimple Lie algebra $g$. In [26], Su and Yang constructed the weak Hopf algebra $\tilde{H}_8$ corresponding to the non-commutative and non-cocommutative semisimple Hopf algebra $H_8$ of dimension 8. They described the representation ring of $\tilde{H}_8$ and studied the automorphism group of $r(\tilde{H}_8)$. In [27], Su and Yang studied the Green ring of the weak Generalized Taft Hopf algebra $r(w^*(H_{n,d}))$, showing that the Green ring of the weak Generalized Taft Hopf algebra was much more complicated than its Grothendieck ring.

In the present paper, it is shown that $H_{4n}$ is quasi-triangular, which universal $R$-matrices are described. the weak Hopf algebras $wH_{4n}$ and $wH^*_{4n}$ corresponding to the Hopf algebra $H_{4n}$ and its dual $H^*_{4n}$ are constructed. Then their representations and Green rings are explicitly described. It turns out that the Green rings of the associated weak Hopf algebras are not commutative even if the Green rings of $H_{4n}$ are commutative.

The paper is organized as follows. In Section 1, the definition of $H_{4n}$ by generators and relations is described first, then we prove that $H_{4n}$ is quasitriangular and describe all universal $R$-matrices $R$ explicitly. In Section 2, we compute the Green ring $r(H_{4n})$. In Section 3, we construct the weak Hopf algebra $wH_{4n}$ associated to $H_{4n}$. In Section 4, we study the representation ring $r(wH_{4n})$ of $wH_{4n}$ by generators and relations explicitly. In Section 5, we consider the dual Hopf algebra $H^*_{4n}$ and its weak Hopf algebra $wH^*_{4n}$, we also describe the representation rings $r(H^*_{4n})$ and $r(wH^*_{4n})$.

Throughout this paper, we work over an algebraically closed field $k$ of characteristic zero. For the theory of Hopf algebras and quantum groups, we refer to [13, 28, 12, 17].

1. The non-semisimple non-pointed Hopf algebra $H_{4n}$

First of all, let us give the definition of the Hopf algebra $H_{4n}$.

**Definition 1.1.** Let $n \geq 1$ and $q$ be a primitive $2n$-th root of unity. The Hopf algebra $H_{4n}$ is defined as follows. As an algebra it generated by $z, x$ with relations

$$z^{2n} = 1, \quad zx = qxz, \quad x^2 = 0$$

for any $a \in k$. 
The coalgebra structure is
\[ \Delta(z) = z \otimes z + a(1 - q^{-2})z^{n+1}x \otimes zx, \quad \Delta(x) = x \otimes 1 + z^n \otimes x; \]
\[ \epsilon(z) = 1, \quad \epsilon(x) = 0, \]
\[ S(z) = z^{-1}, \quad S(x) = -z^n x. \]

One sees that \( H_4 \) is just the 4-dimensional Sweedler’s Hopf algebra when \( n = 1 \). It is well known that \( H_4 \) is quasi-triangular with universal \( R \)-matrix
\[ R = \frac{1}{2} (1 \otimes 1 + 1 \otimes z + z \otimes 1 + z \otimes z) + \alpha (x \otimes x - x \otimes zg + zx \otimes x + zx \otimes zx) \]
for any \( \alpha \in k \).

In the sequel, we always assume that \( n > 1 \). Therefore \( q^2 \neq 1 \) and
\[ \Delta(z^i) = z^i \otimes z^i + a(1 - q^{-2})(1 + q^{-2} + \cdots + q^{-2(i-1)})z^{n+i}x \otimes z^i x. \]

Let \( C_i \) be the \( k \)-space spanned by \( z^i, z^{n+i}x, z^i x, z^{n+i}(1 \leq i \leq n-1) \) and
\[ T_4 = k1 \oplus kz^n \oplus kx^n x \oplus kx. \]

**Lemma 1.2.** If \( a \neq 0 \), then \( C_i \) is a simple subcoalgebra and as coalgebras
\[ H_{4n} = \bigoplus_{i=1}^{n-1} C_i \oplus T_4 \]
and \( T_4 \cong H_4 \) as Hopf algebras.

**Proof.** It is straightforward. \( \square \)

It follows that if \( a \neq 0 \), the Hopf algebra \( H_{4n}(n \geq 2) \) is not pointed.

**Example 1.3.** If \( q \) is 4-th primitive root of unity and \( a = 2 \), then \( H_8 \) is just the unique neither pointed nor semisimple 8-dimensional Hopf algebra \( (A''_4)^* \) (see [29]) up to isomorphism.

**Example 1.4.** If \( q \) is 6-th primitive root of unity and \( a \neq 0 \), then \( H_{12} \) is just the unique neither pointed nor semisimple 12-dimensional Hopf algebra \( A''_4^* \) (see [19]).

By [34, Lemma 2.2, Theorem 2.1], \( H_{4n} \) is a Nakayama algebra with \( 2n \) cyclic orientation and cyclic relations of length 2. In particular, it is of finite representation type.

For every integer \( j \), we set
\[ E_j = \frac{1}{2n} \sum_{i=0}^{2n-1} q^{-ij} z^i. \]

It is easy to see that \( E_0, \ldots, E_{2n-1} \) list the distinct \( E_j \)'s. Moreover, for \( 0 \leq j, k < 2n \), we have
\[ E_j z^k = \frac{1}{n} \sum_{i=0}^{2n-1} q^{-ij} z^{i+k} = q^{jk} \left( \frac{1}{n} \sum_{i=0}^{2n-1} q^{-(i+k)j} z^{i+k} \right) = q^{jk} E_j, \]
and
\[ xE_i = E_{i+1} x. \]

**Lemma 1.5.** \( \{E_0, \ldots, E_{2n-1}\} \) is a complete set of orthogonal idempotents of \( H_{4n} \).
Proof. Since $q^{-j}$ is also an $2n$-th root of unity different from 1 if $j \neq 0$, we get
\[
\sum_{i=0}^{n-1} E_i = \frac{1}{2n} \sum_{i=0}^{2n-1} \sum_{j=0}^{2n-1} q^{-ij} z^j = \frac{1}{2n} \sum_{j=0}^{2n-1} \left( \sum_{i=0}^{n-1} (q^{-j})^i \right) z^j = 1,
\]
Also, using (1.1), for $0 \leq l, j < 2n$:
\[
E_j E_l = \frac{1}{2n} \sum_{k=0}^{2n-1} q^{-lk} E_j z^k = \frac{1}{2n} \sum_{k=0}^{2n-1} q^{-lk+jk} E_j = \frac{1}{2n} \sum_{k=0}^{2n-1} (q^{j-l})^k E_j = \left\{ \begin{array}{ll} E_j & \text{if } l = j \\ 0 & \text{if } l \neq j \end{array} \right.
\]
Hence, \{\![E_0, \ldots, E_{2n-1}]\!\} is a complete set of orthogonal idempotents of $H_{4n}$. \hfill \qed

Quasi-triangular Hopf algebras play an important role in the theory of Hopf algebras and quantum groups, since they provide solutions to quantum Yang-Baxter equations. People try to construct quasi-triangular Hopf algebras and get a lot of results (see [23, 20, 6, 30, 16]). In this section, we shall show that $H_{4n}$ is quasitriangular and give all universal $R$-matrices explicitly. First, we recall the definition of quasi-triangular Hopf algebra.

Let $H$ be a finite dimensional Hopf algebra and $R \in H \otimes H$ an invertible element. The pair $(H, R)$ is said to be a quasi-triangular Hopf algebra and $R$ is said to be a universal $R$-matrix of $H$, if the following three conditions are satisfied.

(i) $\Delta'(h) = R\Delta(h)R^{-1}$, for all $h \in H$;

(ii) $(\Delta \otimes \text{id})(R) = R_{13}R_{23}$;

(iii) $(\text{id} \otimes \Delta)(R) = R_{13}R_{12}$.

Here $\Delta' = T \circ \Delta : H \otimes H \rightarrow H \otimes H, T(a \otimes b) = b \otimes a$, and $R_{ij} \in H \otimes H \otimes H$ is given by $R_{12} = R \otimes 1$, $R_{23} = 1 \otimes R$, $R_{13} = (T \otimes \text{id})(R_{23})$.

**Theorem 1.6.** $H_{4n}$ is a quasi-triangular Hopf algebra with universal $R$-matrix
\[
R = \sum_{i,j=0}^{2n-1} (-1)^{ij} E_i \otimes E_j + 2a \sum_{i,j=0}^{2n-1} (-1)^{j+1} E_ix \otimes E_jx.
\]

**Proof.** Note that here $n > 1$ and $q^2 \neq 1$.

Let $R \in H_{4n} \otimes H_{4n}$ be a universal $R$-matrix, and $T = k\{z|z^{2n} = 1\}$. First of all, we claim that $R \in T \otimes T + (T \otimes T)(x \otimes x)$.

Indeed, we assume that
\[
R = \sum_{h \in T} h \otimes X_h + \sum_{h \in T} hx \otimes Y_h, X_h, Y_h \in H_{4n}.
\]
Note that $\Delta(z^n) = z^n \otimes z^n$ and $\Delta^{\text{cop}}(z^n)R = R\Delta(z^n)$, The relations $zx = qxz$ implies that $xz^n = -z^n x$. From this relation, it follows that $X_h \in T$ and $Y_h \in T x$. Hence $R$ can be written as $R = R' + \hat{R}$ where $R' \in T \otimes T$ and $\hat{R} \in (T \otimes T)(x \otimes x)$. Let
\[
R' = \sum_{i,j=0}^{2n-1} a_{ij} E_i \otimes E_j \in T \otimes T.
\]
Note that $(\epsilon \otimes \text{id})(\hat{R}) = 0$ and $(\epsilon \otimes \text{id})(R) = 1$, therefore $(\epsilon \otimes \text{id})(R') = 1$. Thus we have
\[
a_{i0} = a_{0j} = 1 \quad \text{for all} \quad i, j = 0, 1, \ldots, 2n-1.
\]
Moreover, since $\Delta^\text{cop}(x)\hat{R} = R\Delta(x)$, and $\Delta^\text{cop}(x)\hat{\Delta} = 0 = \hat{R}\Delta(x)$, we see that $\Delta^\text{cop}(x)R' = R'\Delta(x)$,

$$
\sum_{i,j=0}^{2n-1} a_{ij} E_i \otimes x E_j + \sum_{i,j=0}^{2n-1} a_{ij} x E_i \otimes z^n E_j = \sum_{i,j=0}^{2n-1} a_{ij} E_i x \otimes E_j + \sum_{i,j=0}^{2n-1} a_{ij} E_i z^n \otimes E_j x
$$

Hence we get

$$
\sum_{i,j=0}^{2n-1} a_{ij} E_i \otimes x E_{j+1} x + \sum_{i,j=0}^{2n-1} (-1) i_{ij} E_{i+1} x \otimes E_j = \sum_{i,j=0}^{2n-1} (-1) i_{ij} E_i x \otimes E_j x + \sum_{i,j=0}^{2n-1} a_{ij} E_i x \otimes E_j.
$$

This implies that

$$
a_{i,j-1} = (-1)^i a_{ij}, \text{ and } a_{i-1,j} = (-1)^j a_{ij}
$$

and we have $a_{ij} = (-1)^{ij}$. Then any universal $R$-matrix $R$ of $H_{4n}$ can be expressed by

$$
R = \sum_{i,j=0}^{2n-1} (-1)^{ij} E_i \otimes E_j + \hat{R},
$$

where $\hat{R}$ can be written as

$$
\hat{R} = \sum_{i,j=0}^{2n-1} b_{ij} E_i x \otimes E_j x, \quad b_{ij} \in k.
$$

It is noted that

$$
\Delta(z) = (1 \otimes 1 + a(q^2 - 1)z^n x \otimes x)(z \otimes z).
$$

Compute both side of the equation

$$
\Delta^\text{cop}(z)R = R\Delta(z),
$$

then it is straightforward to see that the left hand side is

$$
\sum_{i,j=0}^{2n-1} (-1)^{ij} q^{i+j} E_i \otimes E_j + \sum_{i,j=0}^{2n-1} \left[a(q^2 - 1)(-1)^{(i-1)(j-1)+j} q^{i+j-2} + b_{ij} q^{i+j}\right] E_i x \otimes E_j x,
$$

and the right hand side is

$$
\sum_{i,j=0}^{2n-1} (-1)^{ij} q^{i+j} E_i \otimes E_j + \sum_{i,j=0}^{2n-1} \left[a(q^2 - 1)(-1)^{i+j+2} q^{i+j-2} + b_{ij} q^{i+j-2}\right] E_i x \otimes E_j x.
$$

Comparing the two-hand side of the above equation, we have

$$
a(q^2 - 1)(-1)^{(i-1)(j-1)+j} q^{i+j-2} + b_{ij} q^{i+j} = a(q^2 - 1)(-1)^{i+j+2} q^{i+j-2} + b_{ij} q^{i+j-2},
$$

and we get

$$
b_{ij} = 2a(-1)^{ij+i}.
$$

Hence, if $R$ is a universal $R$-matrix of $H_{4n}$, then $R$ must be equal to

$$
R = \sum_{i,j=0}^{2n-1} (-1)^{ij} E_i \otimes E_j + 2a \sum_{i,j=0}^{2n-1} (-1)^{(i+1)} E_i x \otimes E_j x.
$$

By direct computations we see that $(\Delta \otimes id)(R) = R_{13} R_{23}$ and $(id \otimes \Delta)(R) = R_{13} R_{12}$. Hence $R$ is a universal $R$-matrix of $H_{4n}$.

\[\Box\]
2. Indecomposable representations of $H_{4n}$

From this section, we always assume that $a \neq 0$ in Definition 1.1. The situation for $a = 0$ can be considered similarly. Let $H = H_{4n}$ and $M_i$ be the 2-dimensional cyclic $H$-module with bases \{\(v_i^1, v_i^2\}\), where $i \in \mathbb{Z}_{2n}$. The multiplication of $x$ and $z$ in $H$ provides the actions on $M_i$, that is

\[
x(v_i^1, v_i^2) = (v_i^1, v_i^2) \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},
\]

\[
z(v_i^1, v_i^2) = (v_i^1, v_i^2) \begin{pmatrix} q^i & 0 \\ 0 & q^{i+1} \end{pmatrix}.
\]

For any $i \in \mathbb{Z}_{2n}$, let $S_i$ be the 1-dimensional cyclic $H$-module with base \(\{v_i\}\), with the action $x \cdot v_i = 0$, $z \cdot v_i = q^i v_i$. Up to isomorphism, \(\{M_i | i \in \mathbb{Z}_{2n}\}\) provides the complete list of isomorphism classes of indecomposable $H$-modules with two dimension. Then we have the following decomposition formulas of the tensor product of two indecomposable $H$-modules.

**Theorem 2.1.** Let $i, j \in \mathbb{Z}_{2n}$, then as $H$-modules, we have

1. $S_i \otimes S_j \cong S_{i+j(mod2n)}$.
2. $S_i \otimes M_j \cong M_{i+j(mod2n)}$.
3. $M_i \otimes M_j \cong M_{i+j(mod2n)} \oplus M_{i+j+1(mod2n)}$.

**Proof.** Recall that $\Delta(z) = z \otimes z + a(1 - q^{-2})z^{n+1}x \otimes zx$, and $\Delta(x) = x \otimes 1 + z^n \otimes x$, for $i \in \mathbb{Z}_{2n}$, let $\sigma(i) = (-1)^i$, we have

1. $x \cdot (v_i \otimes v_j) = 0$, $z \cdot (v_i \otimes v_j) = q^{i+j}v_i \otimes v_j$, therefore $S_i \otimes S_j \cong S_{i+j(mod2n)}$.

2. For $j, k \in \{1, 2\}$ and $i \in \mathbb{Z}_{2n},$

\[
x \cdot (v_i \otimes v_k^j) = \begin{cases} 
\sigma(i)v_i \otimes v_2^j, & k = 1, \\
0, & k = 2.
\end{cases}
\]

\[
z \cdot (v_i \otimes v_k^j) = \begin{cases} 
q^{i+j}v_i \otimes v_2^j, & k = 1, \\
q^{i+j+1}v_i \otimes v_2^j, & k = 2.
\end{cases}
\]

so we have $S_i \otimes M_j \cong M_{i+j(mod2n)}$.

3. For $k, l \in \{1, 2\}$ and $i \in \mathbb{Z}_{2n}$, note that

\[
x \cdot (v_i^1 \otimes v_k^1) = v_i^1 \otimes v_k^1 + \sigma(i)v_i^1 \otimes v_2^2,
\]

\[
x \cdot (v_i^1 \otimes v_k^2) = v_i^2 \otimes v_k^1,
\]

\[
x \cdot (v_i^2 \otimes v_k^1) = \sigma(i+1)v_i^2 \otimes v_k^2,
\]

\[
x \cdot (v_i^2 \otimes v_k^2) = 0,
\]

\[
z \cdot (v_k^i \otimes v_l^i) = \begin{cases} 
q^{i+j}(v_i^1 \otimes v_l^1 + a(q^2 - 1)\sigma(i+1)v_i^2 \otimes v_l^2), & k + l = 2 \\
q^{i+j+1}v_i^1 \otimes v_l^1, & k + l = 3; \\
q^{i+j+2}v_i^2 \otimes v_l^2, & k + l = 4.
\end{cases}
\]

Let $w_1 = v_i^1 \otimes v_2^2$, $w_2 = v_i^2 \otimes v_2^1$, and

\[
w_3 = v_i^1 \otimes v_i^1 - a\sigma(i+1)v_i^2 \otimes v_2^2,
\]

\[
w_4 = v_i^2 \otimes v_i^1 + \sigma(i)v_i^1 \otimes v_2^2.
\]
then we have
\[ x(w_1, w_2) = (w_1, w_2) \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \]
\[ z(w_1, w_2) = (w_1, w_2) \begin{pmatrix} q^{i+j+1} & 0 \\ 0 & q^{i+j+2} \end{pmatrix}, \]
and
\[ x(w_3, w_4) = (w_3, w_4) \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \]
\[ z(w_3, w_4) = (w_3, w_4) \begin{pmatrix} q^{i+j} & 0 \\ 0 & q^{i+j+1} \end{pmatrix}. \]
Therefore, \( M_i \otimes M_j \cong M_{i+j(\text{mod}2n)} \otimes M_{i+j+1(\text{mod}2n)}. \)

Let \( H \) be a finite dimensional Hopf algebra and \( M \) and \( N \) be two finite dimensional \( H \)-modules. Recall that the Green ring or the representation ring \( r(H) \) of \( H \) can be defined as follows. As a group \( r(H) \) is the free Abelian group generated by the isomorphism classes of the finite dimensional \( H \)-modules \( M \), modulo the relations \([M \oplus N] = [M] + [N]\). The multiplication of \( r(H) \) is given by the tensor product of \( H \)-modules, that is, \([M][N] = [M \otimes N]\). Then \( r(H) \) is an associative ring with identity given by \([k_z]\), the trivial 1-dimensional \( H \)-module. Note that \( r(H) \) is a free abelian group with a \( \mathbb{Z} \)-basis \([\{[M] | M \in \text{ind}(H)\} \), where \( \text{ind}(H) \) denotes the set of finite dimensional indecomposable \( H \)-modules.

Denote \([S_1] = b, [M_0] = c\).

**Corollary 2.2.** The Green ring \( r(H_{4n}) \) is a commutative ring generated by \( b \) and \( c \). The set \( \{b^k | 0 \leq k \leq 2n-1\} \cup \{bc | 1 \leq i \leq 2n-1\} \) forms a \( \mathbb{Z} \)-basis for \( r(H_{4n}) \).

**Proof.** Firstly, \( r(H_{4n}) \) is a commutative ring since \( H_{4n} \) is a quasitriangular Hopf algebra. By Theorem 2.1, \( b^{2n} = 1 \) and there is a one to one correspondence between the set \( \{b^i | 0 \leq i \leq 2n-1\} \) and the set of one-dimensional simple \( H_{4n} \) module \( \{[S_i] | 0 \leq i \leq 2n-1\} \). Besides, for all \( 0 \leq i \leq 2n-1, [S_i]c = [M_i] \), hence \([M_i] = b^ic \) and all the two-dimensional simple \( H_{4n} \) modules \( \{[M_i] | 0 \leq i < j \leq n-1\} \) are obtained.

**Theorem 2.3.** The Green ring \( r(H_{4n}) \) is isomorphic to the quotient ring of the ring \( \mathbb{Z}[x_1, x_2] \) modulo the ideal \( I \) generated by the following elements
\[ x_1^{2n} - 1, \quad x_2^2 - x_1 x_2 - x_2 \]

**Proof.** By Corollary 2.2, \( r(H_{4n}) \) is generated by \( b \) and \( c \). Hence there is a unique ring epimorphism
\[ \Phi : \mathbb{Z}[x_1, x_2] \to r(H_{4n}) \]
such that
\[ \Phi(x_1) = b = [S_1], \quad \Phi(x_2) = c = [M_0]. \]
Since
\[ b^{2n} = 1, \quad c^2 = bc + c, \quad bc = cb, \]
we have
\[ \Phi(x_1^{2n} - 1) = 0, \quad \Phi(x_2^2 - x_1 x_2 - x_2) = 0, \quad \Phi(x_1 x_2 - x_2 x_1) = 0. \]
It follows that $\Phi(I) = 0$, and $\Phi$ induces a ring epimorphism

$$\overline{\Phi} : \mathbb{Z}[x_1, x_2]/I \to r(H_{4n}),$$

such that $\overline{\Phi}(v) = \Phi(v)$ for all $v \in \mathbb{Z}[x_1, x_2]$, where $\overline{\Phi} = \pi(v)$ (natural epimorphism $\pi : \mathbb{Z}[x_1, x_2] \to \mathbb{Z}[x_1, x_2]/I$). As $r(H_{4n})$ is a free $\mathbb{Z}$-module of rank $4n$, with a $\mathbb{Z}$-basis $\{b^i | 0 \leq i \leq 2n-1\} \cup \{b^i c^j | 0 \leq j \leq 2n-1\}$, we can define a $\mathbb{Z}$-module homomorphism:

$$\Psi : r(H_{4n}) \to \mathbb{Z}[x_1, x_2]/I,$$

$$b^i c^j \mapsto \pi(x_1^i x_2^j) = \pi(x_1^i) \pi(x_2^j), \quad 1 \leq i, j \leq 2n-1.$$

Observe that as a free $\mathbb{Z}$-module, $\mathbb{Z}[x_1, x_2]/I$ is generated by elements $x_1^i x_2^j, 0 \leq i, j \leq 2n-1$, we have

$$\Psi \Phi(x_1^i x_2^j) = \Psi \Phi(x_1^i) \Phi(x_2^j),$$

for all $0 \leq i, j \leq 2n-1$. Hence $\Psi \Phi = id$, and $\Phi$ is injective. Thus, $\overline{\Phi}$ is a ring isomorphism. \qed

3. Weak Hopf algebras corresponding to $H_{4n}$

Firstly, we recall the concept of weak Hopf algebra given by Li(see [13]). By definition, a weak Hopf algebra is $k$-bialgebra $H$ with a map $T \in \text{hom}(H, H)$ such that $T \ast id \ast T = T$ and $id \ast T \ast id = id$, where $\ast$ is the convolution map in $\text{hom}(H, H)$.

Let $\mathfrak{w}H_{4n}$ be the algebra generated by $Z, X$ with relations

$$Z^{2n+1} = Z, \quad ZX = qXZ, \quad X^2 = 0.$$

**Theorem 3.1.** $\mathfrak{w}H_{4n}$ is a noncommutative and noncocommutative weak Hopf algebra with comultiplication, counit and the weak antipode $T$ as follows

$$\Delta(Z) = Z \otimes Z + a(1 - q^{-2})Z^{n+1}X \otimes ZX, \quad \Delta(X) = X \otimes 1 + Z^n \otimes X;$$

$$\epsilon(Z) = 1, \quad \epsilon(X) = 0,$$

$$T(Z) = Z^{2n-1}, \quad T(X) = -Z^n X.$$

**Proof.** Firstly, it can be shown by direct calculations that the following relations hold:

$$\Delta(Z^{2n+1}) = \Delta(Z), \quad \Delta(Z)\Delta(X) = q\Delta(X)\Delta(Z), \quad \Delta(X)^2 = 0,$$

$$\epsilon(Z)^{2n+1} = \epsilon(Z), \quad \epsilon(Z)\epsilon(X) = q\epsilon(X)\epsilon(Z), \quad \epsilon(X)^2 = 0,$$

Therefore, $\Delta$ and $\epsilon$ can be extended to algebra morphism from $\mathfrak{w}H_{4n}$ to $\mathfrak{w}H_{4n} \otimes \mathfrak{w}H_{4n}$ and from $\mathfrak{w}H_{4n}$ to $k$ respectively. We also have

$$(\Delta \otimes id)\Delta(Y) = (id \otimes \Delta)\Delta(Y),$$

$$(\epsilon \otimes id)\epsilon(Y) = (id \otimes \epsilon)\epsilon(Y) = Y$$

for $Y = X, Z$. It follows that $\mathfrak{w}H_{4n}$ is a bialgebra.

Secondly, we prove that in the bialgebra $\mathfrak{w}H_{4n}$, the map $T$ can define a weak antipode in the natural way. To see this, note that the map $T : \mathfrak{w}H_{4n} \to \mathfrak{w}H_{4n}^{op}$ keeps the defining relations:

$$(T(Z))^{2n+1} = ((Z)^{2n-1})^{2n+1} = Z^{2n-1} = T(Z),$$

$$(T(X))^2 = (-Z^n X)^2 = 0.$$
\[ T(X)T(Z) = (-Z^nX)(Z)^{2n-1} = q^{1-2n}(Z)^{2n-1}(-Z^n) = qT(Z)T(X). \]

It follows that the map \( T \) can be extended to an anti-algebra homomorphism \( T : \mathfrak{m}H_{4n} \to \mathfrak{m}H_{4n} \).

Besides, it is easy to see that in \( \mathfrak{m}H_{4n} \),

\[
(id * T * id)(Z) = ZT(Z)Z = Z^{2n+1} = Z = id(Z),
\]

\[
(T * id * T)(Z) = T(Z)ZT(Z) = Z^{2n-1} = T(Z).
\]

and

\[
(id * T * id)(X) = \mu(id \otimes T \otimes id)(X \otimes 1 \otimes 1 + Z^n \otimes X \otimes 1 + Z^n \otimes Z^n \otimes X) = X + z^nT(X) + Z^nT(z)X = X - z^{2n}X + Z^nX = id(X),
\]

\[
(T * id * T)(X) = \mu(T \otimes id \otimes T)(X \otimes 1 \otimes 1 + Z^n \otimes X \otimes 1 + Z^n \otimes Z^n \otimes X) = T(X) + T(Z^n)X + T(Z^n)Z^nT(X) = -Z^nX + Z^nX - Z^{3n}X = -Z^nX = T(X).
\]

On the other hand, we have

\[
T * id(X) = X + Z^nT(X) = X - Z^{2n}X = X(1 - Z^{2n}),
\]

\[
T * id(X) = T(X) + T(Z)^nX = -Z^nX + Z^nX = 0.
\]

and \( T * T(Z) = ZT(Z) + a(1-q^{-2})Z^{n+1}XT(ZX) = Z^{2n} + a(1-q^{-2})Z^{n+1}(-Z^nX)Z^{2n-1} = Z^{2n} = T(Z). \)

These arguments show that for any \( h \in \mathfrak{m}H_{4n} \) we have \( id * T(h) \) and \( T * id(h) \) are in the center of \( \mathfrak{m}H_{4n} \). Now, if \( a, b \in \mathfrak{m}H_{4n} \) and

\[
T * id * T(a) = T(a), \ T * id * T(b) = T(b),
\]

\[
id * T * id(a) = a, \ id * T * id(b) = b,
\]

one can check that

\[
T * id * T(ab) = T(ab), \ id * T * id(ab) = ab.
\]

Hence \( T \) is indeed define a weak antipode of \( \mathfrak{m}H_{4n} \) and \( \mathfrak{m}H_{4n} \) is a weak Hopf algebra, which is non-commutative and non-cocommutative. \( \square \)

Let \( J = Z^{2n} \), it is easy to see that \( J \) and \( 1 - J \) are a pair of orthogonal central idempotents in \( \mathfrak{m}H_{4n} \). Let \( \mathfrak{w}_1 = \mathfrak{m}H_{4n}J, \mathfrak{w}_2 = \mathfrak{m}H_{4n}(1 - J) \).

**Proposition 3.2.** We have \( \mathfrak{w}H_{4n} = \mathfrak{w}_1 \oplus \mathfrak{w}_2 \) as two-sided ideals. Moreover, \( \mathfrak{w}_1 \cong H_{4n} \) as Hopf algebras and \( \mathfrak{w}_2 \cong k[y]/(y^2) \) as algebras.

**Proof.** The first statement is easy to see. Let us prove the second one.

Note that \( \mathfrak{w}_1 \) is generated by \( Z, \ XJ \) and with \( J \) as the identity and the relations

\[
JZ = ZJ = Z, \quad (XJ)^2 = 0, \quad Z(XJ) = q(XJ)Z.
\]

Let \( \rho : H_{4n} \to \mathfrak{w}_1 \) be the map defined by

\[
\rho(1) = J, \quad \rho(z) = Z, \quad \rho(z^{-1}) = Z^{2n-1} \quad \rho(x) = XJ.
\]
It is straightforward to see that \( \rho \) is well defined surjective algebraic homomorphism. Let \( \phi : \mathfrak{w}H_{4n} \to H_{4n} \) be the map given by

\[
\phi(1) = 1, \quad \phi(X) = x, \quad \phi(Z) = z.
\]

It is obvious that \( \phi \) is a well defined algebra homomorphism. If we consider the restricted homomorphism \( \phi|_{\mathfrak{w}1} \), then we have \( \phi|_{\mathfrak{w}1} \circ \rho = id_{H_{4n}} \). Hence, \( \rho \) is injective and \( \mathfrak{w}1 \cong H_{4n} \) as algebras. Furthermore, \( \mathfrak{w}2 \) is a Hopf algebra with comultiplication, counit and the antipode \( S \) as follows

\[
\Delta(Z) = Z \otimes Z + (1 - q^{-2})Z^{n+1}XJ \otimes ZXJ, \quad \Delta(XJ) = XJ \otimes 1 + Z^n \otimes XJ;
\]

\[
\epsilon(Z) = 1, \quad \epsilon(XJ) = 0,
\]

\[
S(Z) = Z^{2n-1}, \quad S(XJ) = -Z^nXJ.
\]

It is clear that \( \rho \) is a Hopf algebra isomorphism. Now we prove that \( \mathfrak{w}2 \cong k[y]/(y^2) \). We first claim that \( X(1 - J) \neq 0 \). Let \( N \) be the \( \mathfrak{w}H_{4n} \)-module with the basis \( \{w_1, w_2\} \). The action of \( \mathfrak{w}H_{4n} \) on \( N \) is given by

\[
Z \cdot w_i = 0, \quad i = 1, 2,
\]

\[
X \cdot w_i = \begin{cases} 
  w_2, & i = 1, \\
  0, & i = 2.
\end{cases}
\]

It follows that \( Jw_i = 0 \) for \( i = 1, 2 \) and \( [X(1 - J)]w_1 = w_2 \). Therefore, we have \( X(1 - J) \neq 0 \) and \( [X(1 - J)]^2 = 0 \).

Let \( \phi_1 : k[y]/(y^2) \to \mathfrak{w}2 \) be the map defined by

\[
\phi_1(y) = X(1 - J), \quad \phi_1(1) = 1 - J.
\]

It is easy to show that \( \phi_1 \) is an algebraic isomorphism, and we have \( \mathfrak{w}2 \cong k[y]/(y^2) \). \( \square \)

4. INDECOMPOSABLE REPRESENTATIONS OF \( \mathfrak{w}H_{4n} \)

By Proposition 3.2, \( \mathfrak{w}H_{4n} = H_{4n} \oplus k[y]/(y^2) \). Hence the indecomposable modules of \( H_{4n} \) and \( k[y]/(y^2) \) constitute all the indecomposable \( \mathfrak{w}H_{4n} \)-modules up to isomorphism.

For any \( i \in \mathbb{Z}_{2n} \), let \( S_i \) be the 1-dimensional cyclic \( \mathfrak{w}H_{4n} \)-module with base \( \{v_i\} \), with the action \( X \cdot v_i = 0, Z \cdot v_i = q^iv_i \), and \( M_i \) be the 2-dimensional cyclic \( \mathfrak{w}H_{4n} \)-module with bases \( \{v_i^1, v_i^2\} \). The module structures are as follows:

\[
X(v_i^1, v_i^2) = (v_i^1, v_i^2) \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},
\]

\[
Z(v_i^1, v_i^2) = (v_i^1, v_i^2) \begin{pmatrix} q^i & 0 \\ 0 & q^{i+1} \end{pmatrix}.
\]

In fact, \( S_i \) and \( M_i \) are just indecomposable \( \mathfrak{w}H_{4n} \)-modules corresponding to those of \( H_{4n} \)-modules.

Let \( N_0 \) be the \( k \)-vector space with a basis \( w_0 \), the actions of \( \mathfrak{w}H_{4n} \) on \( N_0 \) are defined by \( Z \cdot w_0 = 0, \quad X \cdot w_0 = 0 \). Let \( N_1 \) be the 2-dimensional \( \mathfrak{w}H_{4n} \)-module with bases \( \{w_1, w_2\} \). The
module structures are as follows:

\[ X(w_1, w_2) = (w_1, w_2) \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \]

\[ Z(w_1, w_2) = (w_1, w_2) \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}. \]

It is noted that \( N_0 \) and \( N_1 \) are just indecomposable \( \mathfrak{m}H_{4n} \)-modules corresponding to those of \( \mathbb{K}[y]/(y^2) \)-modules. Therefore, we have

**Proposition 4.1.** The set

\[ \{ S_i, M_i \mid i \in \mathbb{Z}_{2n} \} \cup \{ N_j \mid j = 0, 1 \} \]

forms a complete list of non-isomorphic indecomposable \( \mathfrak{m}H_{4n} \)-modules.

Now we establish the decomposition formulas of the tensor product of two indecomposable \( \mathfrak{m}H_{4n} \)-modules.

**Theorem 4.2.** Let \( i, j \in \mathbb{Z}_{2n} \), then as \( \mathfrak{m}H_{4n} \)-modules, we have

1. \( S_i \otimes S_j \cong S_{i+j(\text{mod}2n)} \cong S_j \otimes S_i \).
2. \( S_i \otimes M_j \cong M_{i+j(\text{mod}2n)} \cong M_j \otimes S_i \).
3. \( M_i \otimes M_j \cong M_{i+j(\text{mod}2n)} \cong M_i \otimes M_j \).
4. \( N_0 \otimes N_0 \cong N_0 \cong N_0 \otimes S_i \cong S_i \otimes N_0 \).
5. \( N_0 \otimes N_1 \cong N_0 \otimes N_0 \cong N_0 \otimes M_i \).
6. \( N_1 \otimes N_0 \cong N_1 \cong N_i \otimes N_0 \cong N_1 \otimes S_i \cong S_i \otimes N_1 \).
7. \( N_1 \otimes N_1 \cong N_1 \otimes N_1 \cong N_1 \otimes M_i \cong M_i \otimes N_1 \).

**Proof.** Recall that \( \Delta(X) = X \otimes 1 + Z^n \otimes X, \Delta(Z) = Z \otimes Z + a(1 - q^{-2})Z^{n+1} \otimes ZX \). For \( i \in \mathbb{Z}_{2n} \), let \( v_i \) be the basis of \( S_i \), \( \{ v_1, v_2 \} \) be the basis of \( M_i \), \( \{ w_0 \} \) be the basis of \( N_0 \) and \( \{ w_1, w_2 \} \) be the basis of \( N_1 \). Note that (1)-(3) can be obtained as \( \text{2.1} \).

4. It is clear since for \( i \in \mathbb{Z}_{2n} \), we have \( X \cdot w_0 \otimes w_0 = 0 = X \cdot w_0 \otimes v_i = X \cdot v_i \otimes w_0 \) and \( X \cdot w_0 \otimes w_0 = 0 = X \cdot w_0 \otimes v_i = X \cdot v_i \otimes w_0 \).

5. Note that for \( j, k \in \{ 1, 2 \} \) and \( i \in \mathbb{Z}_{2n} \), \( X \cdot w_0 \otimes w_j = 0 = X \cdot w_0 \otimes v_i, \) and \( Z \cdot w_0 \otimes w_j = 0 = X \cdot w_0 \otimes v_i, \) so we have \( N_0 \otimes N_1 \cong N_0 \otimes N_0 \cong N_0 \otimes M_i \).

6. Since for \( j \in \{ 1, 2 \} \) and \( i \in \mathbb{Z}_{2n} \),

\[ X \cdot w_1 \otimes w_0 = w_2 \otimes w_0, \]
\[ X \cdot v_1 \otimes w_1 = v_2 \otimes w_0, \]
\[ X \cdot v_1 \otimes v_i = w_2 \otimes v_i, \]

it follows that \( N_1 \otimes N_0 \cong N_1 \cong M_i \otimes N_0 \cong N_1 \otimes S_i \cong S_i \otimes N_1 \).
(7). For \(j, k \in \{1, 2\}\) and \(i \in \mathbb{Z}_{2n}\), let \(\sigma(i) = (-1)^i\).

\[
X \cdot v^i_j \otimes w_j = \begin{cases} 
  v^j_2 \otimes w_1 + \sigma(i) v^j_1 \otimes w_2, & j = 1; \\
  v^j_2 \otimes w_2, & j = 2.
\end{cases}
\]

\[
X \cdot v^j_2 \otimes w_j = \begin{cases} 
  \sigma(i + 1) v^j_2 \otimes w_2, & j = 1; \\
  0, & j = 2.
\end{cases}
\]

\[
Z \cdot v^j_k \otimes w_j = 0.
\]

Therefore, if we set \(w_1 = v^1_i \otimes w_1, w_2 = v^2_i \otimes w_1 + \sigma(i) v^1_i \otimes w_2, w_3 = v^1_i \otimes w_1, w_4 = \sigma(i + 1) v^2_i \otimes w_2\). Then we have

\[
X \cdot w_1 = w_2, \quad X \cdot w_2 = 0, \quad X \cdot w_3 = w_4, \quad X \cdot w_4 = 0, \quad Z \cdot w_l = 0 (l = 1, 2, 3, 4),
\]

and we obtain \(M_i \otimes N_1 \cong N_1 \oplus N_1\). Besides, note that

\[
X \cdot w_j \otimes v^i_k = \begin{cases} 
  w_2 \otimes v^i_1, & j = 1; \\
  0, & j = 2.
\end{cases}
\]

\[
X \cdot w_j \otimes v^2_k = \begin{cases} 
  w_2 \otimes v^2_1, & j = 1; \\
  0, & j = 2.
\end{cases}
\]

\[
Z \cdot v^i_k \otimes w_j = 0.
\]

\[Z \cdot w_j \otimes v^i_k = 0.
\]

\[Z \cdot w_j \otimes v^2_k = 0.
\]

\[w_2 \otimes v^2_k, & j = 1; \\
  0, & j = 2.
\]

\[w_2 \otimes 0, & j = 1; \\
  0, & j = 2.
\]

\[
Z \cdot w_j \otimes 0 = 0.
\]

Without confusion, we denote \([S_1] = b, [M_0] = c\), and \([N_0] = d\).

**Corollary 4.3.** The Green ring \(r(\mathfrak{w}H_{4n})\) is a ring generated by \(b, c\) and \(d\). The set \(\{b^i c^j \mid 0 \leq i \leq 2n - 1, j = 0, 1\} \cup \{c^kd \mid k = 0, 1\}\) forms a \(\mathbb{Z}\)-basis for \(r(\mathfrak{w}H_{4n})\).

**Proof.** By Theorem [1.2] \(b^{2n} = 1\) and \(\{b^i \mid [S_i] \mid 0 \leq i \leq 2n - 1\}\). Besides, for all \(0 \leq i \leq 2n - 1, [S_i]c = [M_i]\), hence \([M_i] = b^i c\) and all the two-dimensional simple \(H_{4n}\) module \(\{M_i \mid 0 \leq i \leq 2n - 1\}\) are obtained. Note that \([N_0] = d\) and \(N_1 \cong M_0 \otimes N_0\), we have \([N_1] = cd\). The result is obtained. \(\square\)

**Theorem 4.4.** The Green ring \(r(\mathfrak{w}H_{4n})\) is isomorphic to the quotient ring of the ring \(\mathbb{Z}\langle x_1, x_2, x_3 \rangle\) module the ideal \(I\) generated by the following elements

\[
x^{2n - 1}_1, \quad x^{2}_2 - x_1 x_2 - x_2, \quad x_1 x_2 - x_2 x_1, \quad x^{2}_3 - x_3, \quad x_1 x_3 - x_3, \quad x_3 x_1 - x_3, \quad x_3 x_2 - 2x_3.
\]

**Proof.** By Corollary [2.2] \(r(\mathfrak{w}H_{4n})\) is generated by \(b, c\) and \(d\). Hence there is a unique ring epimorphism

\[
\Phi : \mathbb{Z}\langle x_1, x_2, x_3 \rangle \rightarrow r(\mathfrak{w}H_{4n})
\]

such that

\[
\Phi(x_1) = b = [S_1], \quad \Phi(x_2) = c = [M_0], \quad \Phi(x_3) = d = [N_0].
\]
By Theorem [1.2]

\[ b^{2n} = 1, \quad c^2 = bc + c, \quad bc = cb, \]
\[ d^2 = d, \quad ad = da = d, \quad dc = 2d. \]

Thus we have

\[ \Phi(x_1^{2n} - 1) = 0, \quad \Phi(x_2^2 - x_1x_2 - x_2) = 0, \quad \Phi(x_1x_2 - x_2x_1) = 0, \]
\[ \Phi(x_3^2 - x_3) = 0, \quad \Phi(x_1x_3 - x_3) = 0, \quad \Phi(x_3x_1 - x_3) = 0, \quad \Phi(x_3x_2 - 2x_3). \]

It follows that \( \Phi(I) = 0 \), and \( \Phi \) induces a ring epimorphism

\[ \overline{\Phi} : \mathbb{Z}(x_1, x_2, x_3)/I \to r(\mathfrak{w}_H). \]

Comparing the rank of \( \mathbb{Z}(x_1, x_2, x_3)/I \) and \( r(\mathfrak{w}_H) \), it is easy to see that \( \overline{\Phi} \) is a ring isomorphism. \( \Box \)

5. The Dual \( H^*_4 \) of \( H_{4n} \) and \( \mathfrak{w}H^*_4 \)

In this section, we consider the dual Hopf algebra \( H^*_4 \) of \( H_{4n} \) and its weak Hopf algebra \( \mathfrak{w}H^*_4 \), we also describe the representation ring \( r(\mathfrak{w}_H^*_4) \) of \( \mathfrak{w}_H^*_4 \).

Let \( \alpha \) and \( \eta \) be the linear forms on \( H_{4n} \) defined on the basis \( \{z^ix^j\}_{0 \leq i < 2n, j = 0, 1} \) by

\[ \langle \alpha, z^ix^j \rangle = \delta_{i,0}q^i \text{ and } \langle \eta, z^ix^j \rangle = \delta_{j,1}q^i. \]

It is easy to determine that \( H^*_4 \) is generated by \( \alpha \) and \( \eta \) with the following relations

\[ \alpha^{2n} = 1, \quad \eta^2 = a(1 - \alpha^2), \quad \alpha\eta = -\eta\alpha, \]
\[ \Delta(\alpha) = \alpha \otimes \alpha, \quad \Delta(\eta) = \eta \otimes 1 + 1 \otimes \eta; \]
\[ \epsilon(\alpha) = 1, \quad \epsilon(\eta) = 0, \]
\[ S(\alpha) = \alpha^{-1}, \quad S(\eta) = -\alpha^{-1}\eta. \]

Without lost of generality, we take \( a = 1 \) and we get \( \eta^2 = 1 - \alpha^2 \). The representations of \( H^*_4 \) and their tensor products decompositions have been described in [34], and the corresponding representation ring are obtained in [31]. By Theorem 8.2 [31], the Green ring of \( H^*_4 \) is a commutative ring generated by \( Y, Z, X_1, \ldots, X_{n-1} \) with the relations

\[ Y^2 = 1, \quad Z^2 = Z + YZ, \quad YX_1 = X_1, \quad ZX_1 = 2X_1, \]
\[ X_1^{2j} = 2^{j-1}X_j \text{ for } 1 \leq j \leq n-1, \quad X_1^n = 2^{n-2}Z^2. \]

Let \( \mathfrak{w}H^*_4 \) be the algebra generated by \( G, X \) with relations

\[ G^{2n+1} = G, \quad GX = -XG, \quad X^2 = 1 - G^2. \]

Then \( \mathfrak{w}H^*_4 \) is a noncommutative and noncozommunative weak Hopf algebra with comultiplication, counit and the weak antipode \( T \) as follows

\[ \Delta(G) = G \otimes G, \quad \Delta(X) = X \otimes 1 + 1 \otimes X; \]
\[ \epsilon(G) = 1, \quad \epsilon(X) = 0, \]
\[ T(G) = G^{2n-1}, \quad T(X) = -G^{2n-1}X. \]

Let \( J = G^{2n} \), it is easy to see that \( J \) and \( 1 - J \) are a pair of orthogonal central idempotents in \( \mathfrak{w}H^*_4. \) Let \( \mathfrak{w}_1 = \mathfrak{w}H^*_4J, \mathfrak{w}_2 = \mathfrak{w}H^*_4(1 - J). \)
Proposition 5.1. We have \( \mathfrak{w}H_{4n}^* = \mathfrak{w}_1 \oplus \mathfrak{w}_2 \) as two-sided ideals. Moreover, \( \mathfrak{w}_1 \cong H_{4n}^* \) as Hopf algebras and \( \mathfrak{w}_2 \cong k[y]/(y^2 - 1) \) as algebras.

The proof is similar to Proposition 5.1 and we omit it here. By Proposition 5.1, \( \mathfrak{w}H_{4n}^* = H_{4n}^* \oplus k[y]/(y^2 - 1) \). Hence up to isomorphism, the indecomposable \( H_{4n}^* \) modules and \( k[y]/(y^2 - 1) \)-modules constitute all the indecomposable \( \mathfrak{w}H_{4n}^* \)-modules.

For \( s = 0 \) or \( n \), let \( M[1, s] \) be the 1-dimensional cyclic \( \mathfrak{w}H_{4n}^* \)-module with the base \( \{v_s\} \) defined by \( X \cdot v_s = 0 \), \( G \cdot v_s = (-1)^s v_s \). Let \( M[2, s] \) be the 2-dimensional cyclic \( \mathfrak{w}H_{4n}^* \)-module with bases \( \{v_1^s, v_2^s\} \) defined as follows

\[
X(v_1^s, v_2^s) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},
\]

\[
G(v_1^s, v_2^s) = \begin{pmatrix} (-1)^s & 0 \\ 0 & (-1)^s + 1 \end{pmatrix}.
\]

For \( 1 \leq j \leq n - 1 \), let \( P_j \) be the 2-dimensional \( \mathfrak{w}H_{4n}^* \)-module with bases \( \{p_1^j, p_2^j\} \) and module structures as follows:

\[
X(p_1^j, p_2^j) = \begin{pmatrix} 0 & 1 - q^2j \\ 1 & 0 \end{pmatrix},
\]

\[
G(p_1^j, p_2^j) = \begin{pmatrix} q^j & 0 \\ 0 & -q^j \end{pmatrix}.
\]

In fact, \( M[k, s], k = 1, 2; s = 0, n \) and \( P_j, 1 \leq j \leq n - 1 \) are just indecomposable \( \mathfrak{w}H_{4n}^* \)-modules corresponding to those of \( H_{4n}^* \)-modules.

Let \( N_i (i = 0, 1) \) be the \( k \)-vector space with a basis \( w_i \), the actions of \( \mathfrak{w}H_{4n}^* \) on \( N_i \) are defined by \( X \cdot w_i = (-1)^i w_i \), \( G \cdot w_i = 0 \). It is noted that \( N_0, N_1 \) are just indecomposable \( \mathfrak{w}H_{4n}^* \)-modules corresponding to those of \( k[y]/(y^2 - 1) \)-modules. Therefore, we have

Proposition 5.2. The set

\[
\{M[k, s], P_j \mid k = 1, 2; s = 0, n; 1 \leq j \leq n - 1\} \cup \{N_i \mid i = 0, 1\}
\]

forms a complete list of non-isomorphic indecomposable \( \mathfrak{w}H_{4n}^* \)-modules.

Now we establish the decomposition formulas of the tensor product of two indecomposable \( \mathfrak{w}H_{4n}^* \)-modules.

Theorem 5.3. As \( \mathfrak{w}H_{4n}^* \)-modules, we have

1. For \( 1 \leq i, j \leq n - 1 \), \( P_i \otimes P_j \cong \begin{cases} M[2, 0] \oplus M[2, n], & n \mid i + j; \\ 2P_{i+j}, & n \mid i + j. \end{cases} \)
2. For \( k \in \{1, 2\}, s \in \{0, n\}, 1 \leq j \leq n - 1 \), \( M[k, s] \otimes P_j \cong kP_j \cong P_j \otimes M[k, s] \).
3. For \( k, l \in \{1, 2\}, s, j \in \{0, n\} \),

\[
M[k, s] \otimes M[l, j] \cong \begin{cases} M[2, 0] \oplus M[2, n], & k + l = 4; \\ M[k + l - 1, s + j \mod{2n}], & k + l < 4. \end{cases}
\]
4. For \( i, j \in \{0, 1\} \), \( N_i \otimes N_j \cong N_i \).
5. For \( k \in \{1, 2\}, s \in \{0, n\}, j \in \{0, 1\} \), \( M[k, s] \otimes N_j \cong \begin{cases} N_j + 1, & k = 1; \\ N_0 \oplus N_1, & k = 2. \end{cases} \)
(6) For \( k \in \{1, 2\}, s \in \{0, n\}, j \in \{0, 1\}, N_j \otimes M[k, s] \cong kN_j. \)

(7) For \( i \in \{0, 1\}, 1 \leq j \leq n - 1, N_i \otimes P_j \cong 2N_i, P_j \otimes N_i \cong N_0 \oplus N_1. \)

**Proof.** Recall that \( \Delta(G) = G \otimes G \) and \( \Delta(X) = X \otimes 1 + G \otimes X \). (1)-(3) can be proved similarly as in \cite{[31],[32].}

(4). Note that \( G \cdot w_i = 0, \quad X \cdot w_i = (-1)^iw_i, \) therefore \( N_i \otimes N_j \cong N_i \), for \( i, j \in \{0, 1\}. \)

(5) and (6). Let \( k \in \{1, 2\}, s \in \{0, n\}, j \in \{0, 1\} \) and \( v_s \) be the basis of \( M[1, s] \), then \( X \cdot v_s = 0 \) and \( G \cdot v_s = (-1)^jv_s \), so we have

\[
G \cdot (v_s \otimes w_j) = 0, \quad X \cdot (v_s \otimes w_j) = (-1)^j(v_s \otimes w_j),
\]

\[
G \cdot (w_j \otimes v_s) = 0, \quad X \cdot (w_j \otimes v_s) = (-1)^j(v_s \otimes w_j),
\]

hence \( M[1, s] \otimes N_j \cong N_{j+1} \) and \( N_j \otimes M[1, s] \cong N_j. \)

Let \( \{v_s^1, v_s^2\} \) be the basis of \( M[2, s] \), then

\[
X \cdot (v_s^1 \otimes w_j + v_s^2 \otimes w_j + (-1)^j(v_s^1 \otimes w_j) = v_s^2 \otimes w_j + (-1)^j(v_s^1 \otimes w_j + v_s^1 \otimes w_j),
\]

\[
G \cdot (v_s^1 \otimes w_j + v_s^2 \otimes w_j + (-1)^j(v_s^1 \otimes w_j) = 0,
\]

\[
X \cdot (v_s^1 \otimes w_j - v_s^2 \otimes w_j - (-1)^j(v_s^1 \otimes w_j) = v_s^2 \otimes w_j + (-1)^j(v_s^1 \otimes w_j - v_s^1 \otimes w_j),
\]

\[
G \cdot (v_s^2 \otimes w_j + (-1)^j(v_s^1 \otimes w_j) = 0,
\]

therefore \( M[2, s] \otimes N_j \cong N_0 \oplus N_1. \) Besides,

\[
X \cdot (w_j \otimes v_s^1) = (-1)^j(w_s^1 \otimes v_s^2), \quad X \cdot (w_j \otimes v_s^2) = (-1)^j(w_s^2 \otimes v_s^2), \quad G \cdot (w_j \otimes v_s^k) = 0,
\]

therefore \( N_j \otimes M[2, s] \cong 2N_j. \)

(7). Note that

\[
X \cdot (p_t^1 \otimes w_i) = p_t^2 \otimes w_i + (-1)^jq^kq^i \otimes w_i,
\]

\[
X \cdot (p_t^2 \otimes w_i) = (1 - q^{2k})p_t^1 \otimes w_i - (-1)^jq^kq^i \otimes w_i,
\]

Let \( \omega_1 = (1 + (-1)^jq^k)p_t^1 \otimes w_i + p_t^2 \otimes w_i, \omega_2 = (1 - (-1)^jq^k)p_t^1 \otimes w_i - p_t^2 \otimes w_i, \) then we have \( X \cdot \omega_1 = \omega_1, X \cdot \omega_2 = -\omega_2 \) and \( G \cdot \omega_k = 0, k = 0, 1 \). Therefore \( P_j \otimes N_i \cong N_0 \oplus N_1. \) Besides,

\[
X \cdot (w_i \otimes p_t^k) = (-1)^jw_i \otimes p_t^k \) and \( G \cdot (w_i \otimes p_t^k) = 0 \) for \( k = 1, 2, \) therefore \( N_i \otimes P_j \cong 2N_j. \)

Denote \( M[1, n] = b, M[2, 0] = c, P_j = a_j, j \in \{1, 2, \cdots n - 1\}, \) and \( N_0 = d, \) then we have

**Corollary 5.4.** The Green ring \( r(\mathfrak{m}H^n_{1n}) \) is a ring generated by \( b, c, d \) and \( a_j \). The set \( \{a_j, b^ie^k | 1 \leq j \leq n - 1, i, k = 0, 1\} \cup \{b^id | i = 0, 1\} \) forms a \( Z \)-basis for \( r(\mathfrak{m}H^n_{1n}) \).

**Proof.** By Theorem \(5.5\) \( b^2 = 1, bc = cb = M[2, n] \) and \( c^2 = c + bc. \) Therefore, the set \( \{a_j, b^ie^k | 1 \leq j \leq n - 1, i, k = 0, 1\} \) has a one to one correspondence with the modules \( \{M[k, s], P_j\} \). Besides, note that \( d^2 = d, \) and \( [N_1] = bd, \) the result is obtained.

**Theorem 5.5.** The Green ring \( r(\mathfrak{m}H^n_{4n}) \) is isomorphic to the quotient ring of the ring \( \mathbb{Z}(Y, Z, X_j, W) \) module the ideal \( I \) generated by the following elements

\[
(5.1) Y^2 - 1, \quad Z^2 - Z - YZ, \quad YX_1 - X_1, \quad ZX_1 - 2X_1, \quad YZ - ZY, YX_1 - X_1Y, X_1Z - ZX_1;
\]

\[
(5.2) X_j^2 - 2^{-1}X_j(1 \leq j \leq n - 1), \quad X_j^n - 2^{n-2}Z^2;
\]

\[
(5.3) W^2 - W, \quad WY - W, \quad WZ - 2W, \quad ZW - W - YW, \quad WX_1 - 2W, \quad X_1W - W - YW.
\]
Proof. By Corollary 5.4, \( r(\mathfrak{w}H_{4n})^* \) is generated by \( b, c, d \) and \( a_j \). Hence there is a unique ring epimorphism

\[
\Phi : \mathbb{Z}(Y, Z, X_j, W) \to r(\mathfrak{w}H_{4n}^*)
\]

such that

\[
\Phi(Y) = b, \quad \Phi(Z) = c, \quad \Phi(X_j) = a_j (1 \leq j \leq n-1), \quad \Phi(W) = d.
\]

By Theorem 5.3, it is easy to see that \( \Phi \) vanishes at the generators of the ideal \( I \) given by (5.1)-(5.3). It follows that \( \Phi \) induces a ring epimorphism

\[
\overline{\Phi} : \mathbb{Z}(Y, Z, X_j, W)/I \to r(\mathfrak{w}H_{4n}^*).
\]

Comparing the rank of \( \mathbb{Z}(Y, Z, X_j, W)/I \) and \( r(\mathfrak{w}H_{4n}^*) \), it is easy to see that \( \overline{\Phi} \) is a ring isomorphism.

\[\square\]

Declarations

Ethical Approval. This declaration is not applicable.
Competing interests. The authors declare that they have no conflict of interest.
Authors’ contributions. S. Yang developed the idea for the study, all the authors did the analyses, and J. Chen and S. Yang wrote the paper.
Funding. This work was Supported by the National Natural Science Foundation of China (Grant Nos.11701019, 11671024, 11871301), the Beijing Natural Science Foundation (Grant No.1162002), and the Science and Technology Project of Beijing Municipal Education Commission (Grant No. KM202110005012).
Availability of data and materials. The data that support the findings of this study are available on request from the corresponding author upon reasonable request.

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