Chiral Langevin theory for non-Abelian plasmas

Yukinao Akamatsu\(^1\) and Naoki Yamamoto\(^2\)

\(^1\)Kobayashi-Maskawa Institute for the Origin of Particles and the Universe, Nagoya University, Nagoya 464-8602, Japan
\(^2\)Maryland Center for Fundamental Physics, Department of Physics, University of Maryland, College Park, Maryland 20742-4111, USA

Charged plasmas with chirality imbalance are unstable and tend to reduce the imbalance. This chiral plasma instability is, however, not captured in (anomalous) hydrodynamics for high-temperature non-Abelian plasmas. We derive a Langevin-type classical effective theory with anomalous parity-violating effects for non-Abelian plasmas that describes the chiral plasma instability at the magnetic scale. We show that the time scale of the instability is of order \(\frac{g^4T\ln(1/g)}{1}\) at weak coupling.

I. INTRODUCTION

Matter with chirality imbalance is expected to appear in a wide range of physical systems, such as quark-gluon plasma created in relativistic heavy ion collision experiments at RHIC and LHC \(^{[1,2]}\), the electroweak plasma in the early Universe \(^{[3]}\), electron plasmas inside neutron stars \(^{[4]}\), among others. One of the direct consequences of the chirality imbalance is an unusual transport phenomena in the presence of a strong magnetic field \(B\) (which is also expected in most of these systems) called the chiral magnetic effect \(^{[5-9]}\):

\[
\mathbf{j} = \frac{e^2\mu_5}{2\pi^2} \mathbf{B} \equiv \kappa \mathbf{B}.
\]

Here \(\mathbf{j}\) is the electric current and \(\mu_5\) is the chiral chemical potential that characterizes the chirality imbalance. This current is unusual in that it flows in the direction of the magnetic field and it is dissipationless. Both facts are indeed related to the quantum anomalies \(^{[10,11]}\)—an intrinsic property of relativistic quantum field theories. The chiral magnetic effect may have been observed in heavy ion collision experiments at RHIC (for the current status, see Refs. \(^{[12,13]}\) and references therein) and may be observable in new materials called Weyl semimetals \(^{[14-16]}\).

Recently, it was argued based on the kinetic theory with Berry curvature corrections (or simply chiral kinetic theory) \(^{[17-20]}\) that charged plasmas with chirality imbalance have unstable infrared collective modes that tend to reduce the imbalance, and thus, quark-gluon plasmas exhibiting the chiral magnetic effect do not exist stably \(^{[21]}\). This is the chiral plasma instability.\(^1\) (See also Refs. \(^{[23,24]}\).) A related instability had also been pointed

\(^1\) One might wonder why we call it the plasma instability. Our terminology parallels the conventional Weibel
out previously for the electroweak theory at high density [26–30] and for the primordial magnetic field in the early Universe [3, 31], using different theoretical frameworks. (See also Ref. [25] for the similar instability based on the nonlocal hard dense loop effective action.) In this paper, we shall concentrate on high-temperature non-Abelian plasmas with chirality imbalance, whose dynamics will turn out to be richer than Abelian ones.

It would be useful to have a simple local effective theory (simpler than the kinetic theory) to describe the dynamical evolution of non-Abelian chiral plasmas. To this end, one might come up with hydrodynamics, which is an effective theory valid at long distance and long time scale compared with mean free path, $l_{\text{mfp}}$, and mean free time, $\tau_{\text{mfp}}$. Indeed, it has proven quite useful to describe a quark-gluon plasma after thermalization. In the hydrodynamic description for high-temperature non-Abelian plasmas, we need not care about non-Abelian gauge fields, because both chromoelectric and chromomagnetic fields are screened at the scales (inverse of the Debye mass $\sim gT$ and the magnetic scale $\sim g^2T$, respectively) much shorter than the mean free path, $l_{\text{mfp}} \sim (g^4T)^{-1}$, as we shall review in the following (see Sec. I A). This is the reason why the effective theory must be hydrodynamics rather than chromomagneto-hydrodynamics. However, as was found in Ref. [21] (and as we shall also show later), it is exactly this magnetic scale where the non-Abelian chiral plasma instability occurs in the presence of the chirality imbalance, $\mu_5 \sim T$. (Here and below we mean by “$\sim$” that they are parametrically of the same order in $g$.) This means that the dynamics of non-Abelian chiral plasma instability is completely missed in the hydrodynamic description. This is so even in the anomalous hydrodynamics [32] which was recently constructed after the finding in the gauge/gravity duality computations [33, 34]; see also Refs. [35–37] for other formulations and Ref. [38] for a numerical analysis of anomalous hydrodynamics.

The purpose of this paper is to construct an effective theory at the magnetic scale that describes the dynamical evolution of non-Abelian chiral plasmas. Our main result is given by the Langevin-type equation for the gauge field, Eq. (21) [which can be equivalently rewritten in the form of Eqs. (12) and (22)], supplemented by the evolution equation for the chiral charge, Eq. (27). This is a generalization of the Langevin-type equation without anomalous effects previously derived in Refs. [39–41]. When we are only interested in the physics at the magnetic scale, our effective theory is much simpler than the chiral kinetic theory that also includes the (semi-)hard degrees of freedom with momenta $k \gtrsim gT$. It is also cheaper in plasma instability [22] which occurs when the distribution function of particles is anisotropic in momentum space. The Weibel plasma instability is manifested in tachyonic collective modes of gauge fields that tend to make the distribution function isotropic. Similarly, the chiral plasma instability appears when fermions have chiral asymmetry, and it is manifested in tachyonic modes that tend to make right- and left-handed fermions equal [21]. In other words, the chiral plasma instability is a generalization of the conventional plasma instability to the case where the distribution function of fermions and the polarization tensor of gauge fields are parity non-invariant [21].

In contrast, Abelian plasmas do not have the magnetic screening, and magnetic fields persist even in the hydrodynamic regime; this is why the long-range effective theory for Abelian plasmas is magneto-hydrodynamics, in which case the Abelian chiral plasma instability can be captured.
performing practical numerical simulations than the chiral kinetic theory, since the former depends only on the coordinate \( x \), while the latter depends on both \( x \) and momentum \( p \). On the basis of this Langevin theory, we shall give parametric estimates of time scale of the chiral plasma instability and that of variation of the chiral charge.\(^3\)

Throughout this paper, we assume massless fermions, which should be a good approximation when the temperature \( T \) is sufficiently large compared with the fermion mass \( m \). We also assume \( \mu_5 \sim T \).

### A. Hierarchy of scales and effective theories

We pause here to recapitulate the hierarchy of scales in high-temperature non-Abelian plasmas and their low-energy effective theories. (For a pedagogical review, see Ref. [12].) We consider a sufficiently high temperature regime so that the coupling constant is small, \( g \ll 1 \), where there is a definite separation of momentum scales: \( g^4 T \ll g^2 T \ll gT \ll T \) (up to logarithmic corrections). In the following, we will refer to the scales, \( T \), \( gT \), and \( g^2 T \), the hard, semi-hard, and soft (or magnetic) scales, respectively. Among others, it is easy to see that that the gauge boson acquires a semi-hard Debye screening mass \( \sim gT \) in medium by perturbatively computing one-loop Feynman diagrams. However, the magnetic scale \( g^2 T \) emerges nonperturbatively and it cannot be understood in an analogous way.

To understand the origin of the magnetic scale, consider the regime where the amplitude of fluctuations of the gauge field \( A(k) \) with momentum \( k \) becomes nonperturbative. This amounts to the condition that the contributions of nonlinear interactions are comparable to those of linear ones, \( kA \sim gA^2 \). On the other hand, the law of equipartition of energy states that the magnetic field has the energy \( \sim T \), and hence, \( B^2 R^3 \sim T \), where \( R \sim 1/k \) is the typical spatial size. Combining these two conditions, the typical scales of the momentum and the gauge field read

\[
  k \sim g^2 T, \quad A \sim gT, \quad B \sim g^3 T^2. \tag{2}
\]

In this way, the magnetic (or the soft) scale \( k \sim g^2 T \) emerges.

Let us then explain that the mean free path is given by \( l_{\text{mfp}} \sim (g^4 T)^{-1} \) (up to logarithmic corrections). Remember first the textbook formula, \( l_{\text{mfp}} \sim (n\sigma v)^{-1} \), where \( n \sim T^3 \) is the density of scatterers, \( \sigma \) is the cross section, and \( v \sim 1 \) is the relative velocity. Because the scattering amplitude involves \( g^2 \), the cross section must be proportional to \( g^4 \). Note also that the cross section is proportional to \( l^2 \), where the length scale \( l \) is given by the inverse of the typical momentum exchange, \( l \sim T^{-1} \). Hence, we have \( \sigma \sim g^4 l^2 \sim g^4 / T^2 \). Inserting them into the formula above, we get \( l_{\text{mfp}} \sim (g^4 T)^{-1} \).

---

\(^3\) In a quark-gluon plasma with chirality imbalance, both photons and gluons that mediate the electromagnetic and strong forces become tachyonic, because quarks have both U(1) electric and SU(3) color charges: there are Abelian and non-Abelian chiral plasma instabilities at the same time in this case. Here we are interested in the non-Abelian chiral plasma instability that is not captured by hydrodynamics.
In summary, we have a hierarchy of length scales as depicted in Fig. 1. Due to the screening effects, there are no chromoelectric and chromomagnetic fields at $L \gg (gT)^{-1}$ and $L \gg (g^2T)^{-1}$, respectively. Kinetic theory describes the dynamics at $L \gtrsim (gT)^{-1}$, while hydrodynamics is only applicable at $L \gg l_{\text{mfp}} \sim (g^4T)^{-1}$. The Langevin theory that we shall derive in this paper describes the physics at $L \gtrsim (g^2T)^{-1}$, which has some overlapping regime with the kinetic theory, but is beyond the applicability of hydrodynamics. It is this scale where the non-Abelian chiral plasma instability occurs, as we are going to explain in a moment below.

B. Intuitive picture of the chiral plasma instability

We now give an intuitive argument of the chiral plasma instabilities.\textsuperscript{4} Here we consider an Abelian plasma for simplicity, but our argument is also applicable to non-Abelian plasmas as long as the amplitude of the gauge fields is sufficiently small such that the nonlinear gauge interactions are negligible (in which case the Yang-Mills equation reduces to the Maxwell equation). This argument is based only on the basic laws of electromagnetism (Maxwell equations) and the chiral magnetic effect in Eq. (1), assuming a homogeneously distributed chirality imbalance, $\mu_5 > 0$, in the initial state. For further simplicity, we ignore the effects of dissipation and do not consider the Ohmic current $j_{\text{ohm}} = \sigma E$. These effects will be discussed in detail in the following sections.\textsuperscript{5}

Suppose there is a fluctuation of a magnetic field $B_{\text{in}} = B_z e_z$ in some finite domain

\textsuperscript{4} The presence of the instability for an Abelian chiral plasma was shown in Ref. 3 using the anomalous Maxwell equations (Maxwell equations plus the chiral magnetic effect). However, we are not aware of any previous, physical argument, and we believe it worth while to provide it here explicitly.

\textsuperscript{5} For non-Abelian plasmas, the effects of dissipation are essential at the quantitative level: the typical frequency of the chiral plasma instability would be $\omega \sim g^4T$ without dissipation, while it is $\omega \sim g^4T \ln(1/g)$ with dissipation \cite{21}; see also below. The argument here provides, at least, qualitative understanding of the emergence of the chiral plasma instability.
along the $z$-axis with a typical radius $R_1$ (of order the wavelength of the magnetic field, $\lambda$), as depicted in Fig. 2. Here we use a cylindrical coordinate system characterized by three unit vectors, $(e_r, e_\theta, e_z)$. By the chiral magnetic effect, a current density $j_{\text{in}} = j_z e_z$ with $j_z = \kappa B_z$ is induced. Then, according to Ampère’s law, this current density leads to a magnetic field around the $z$-axis. At a distance $R$ larger than $R_1$, the magnetic field is given by $B(R) = B_\theta(R)e_\theta$ with $B_\theta(R) = \pi R_1^2 j_z / 2\pi R$. Again by the chiral magnetic effect, a current density $j(R) = j_\theta(R)e_\theta$ with $j_\theta(R) = \kappa B_\theta(R)$ is there. Ampère’s law suggests that the induced current $j_\theta(R)$ in the region $R_1 < R < R_2$ produces a magnetic field along the $z$-axis, $B'_z = B'_z e_z$, where

$$B'_z = \int_{R_1}^{R_2} j_\theta(R) dR = \frac{\kappa^2}{2} \frac{R_2^2}{R_1^2} B_z \ln \left( \frac{R_2}{R_1} \right). \quad (3)$$

Here the UV and IR cutoffs, $R_1$ and $R_2$, are comparable to $\lambda$, so the logarithmic factor is just some numerical factor.

Let us first consider the static case and ignore the time dependence of electromagnetic fields. In this case, the condition $B_z = B'_z$ must be satisfied. This amounts to the condition, $\nabla \times B = \kappa B$ (known as the Beltrami field), which is a force-free field because $j \times B = 0$. In the context of electroweak plasmas, such a static solution was given in Refs. [29, 30] and is called the Chern-Simons wave (see also Refs. [23, 43] for recent works). Then one finds that $\lambda_c \sim 1/\kappa$ is the critical value for the static solution.

If $\lambda > \lambda_c$ ($B'_z > B_z$), on the other hand, the static situation can no longer be sustained and the (electro)magnetic fields grow, and thus, they are unstable. This is the chiral plasma instability. In the presence of such an instability, one needs to take into account the time-dependence of electromagnetic fields. A growing magnetic field $B_z(t)$ as a func-
tion of time means that, according to Faraday’s law, an electric field is also induced at a distance \( R \): \( \mathbf{E}(R) = E_\theta(R) \mathbf{e}_\theta \) with \( E_\theta(R) < 0 \). Note that the direction of this electric field \( \mathbf{E}(R) \) is the opposite as that of the magnetic field \( \mathbf{B}(R) \). Now remember the anomaly relation, which connects the nonconservation of the chiral charge to the electromagnetic fields: \( \Delta N_5 = e^2/\left(2\pi^2\right) \int d^3x \mathbf{E} \cdot \mathbf{B} \). That the induced electric and magnetic fields are in the opposite directions means, according to the anomaly relation, that the chiral charge \( N_5 \) (and consequently, \( \mu_5 \)) must decrease as a function of time. In this way, the chirality imbalance tends to switch itself off, and thereby, the chiral plasma instability is weakened.

One may repeat a similar argument for non-Abelian plasmas. In this case, the critical value \( 1/\lambda_c \) above is the magnetic scale \( g^2 T \) for \( \mu_5 \sim T \), which is much shorter than the mean free path, \( l_{\text{mfp}} \sim (g^4 T)^{-1} \), as mentioned above. This is why hydrodynamics cannot describe the non-Abelian chiral plasma instability. Apparently, such dynamics should be important for the nonequilibrium evolution of a chiral plasma. For example, one would naively expect that the typical time scale of the color fluctuations is modified in the presence of the chiral plasma instability. In this paper, we describe these systems by a simple effective theory at the magnetic scale, and show that the typical time scale of the chiral plasma instability is of order \( [g^4 T \ln(1/g)]^{-1} \) (which is comparable to the mean free time for large angle scattering).

The paper is organized as follows. In Sec. II, after reviewing the conventional Langevin theory for the dynamics of soft gauge fields, we provide a physical derivation of the Langevin theory with anomalous effects (the chiral Langevin theory). In Sec. III, we derive the chiral Langevin theory from the chiral kinetic theory by integrating out the (semi-)hard degrees of freedom. In Sec. IV, we apply the chiral Langevin theory and compute the typical time scale of the chiral plasma instability. Section V is devoted to conclusions.

II. PHYSICAL ARGUMENTS

A. Langevin theory for soft gauge fields without anomalous effects

We first briefly review the Langevin-type effective theory without anomalous parity-violating effects that describes the nonperturbative physics at the soft scale \( g^2 T \). This effective theory was first constructed starting from the Boltzmann-Vlasov-Maxwell-type kinetic equation in Refs. [39], and later a more intuitive derivation was given in Refs. [40] (see also Refs. [41] for another derivation). We here explain the intuitive argument of Refs. [40].

We split the gauge fields into those at the soft scale \( k \sim g^2 T \) and (semi-)hard scale \( k \gg g^2 T \). Let us now remember that the dynamics of soft modes is classical. This is because, for the soft modes \( k \sim g^2 T \), the Bose-Einstein factor

\[
n(k) = \frac{1}{e^{k/T} - 1} \simeq \frac{T}{k} \gg 1,
\]

is so large that quantum effects are negligible. Thus, the soft modes are effectively described
by the classical Yang-Mills equation

\[ D \times B = D_t E + j_{\text{hard}}, \]

(5)

where \( B = D \times A \) and the covariant derivatives \( D \) are understood as only involving the soft modes, and \( j_{\text{hard}} \) is the color current consisting of hard modes. Here and below, we suppress the color indices of colored quantities, unless there is a possibility of confusion.

In the parity-invariant system (i.e., in the system without a chirality imbalance), the color current is written in the form of a non-Abelian analogue of the Ohmic law

\[ j_{\text{hard}} = \sigma_c E, \]

(6)

where \( \sigma_c \) is the color conductivity expressed by \[40, 44, 45\]

\[ \sigma_c = \frac{m_D^2}{3\gamma} \sim \frac{T}{\ln(1/g)}, \]

(7)

where \( m_D \) is the Debye mass and

\[ \gamma = N_c \alpha_g T \ln(1/g) \]

(8)

is the damping rate for hard thermal bosons with \( N_c \) being the number of colors and \( \alpha_g = g^2/(4\pi) \). As will be verified \textit{a posteriori}, the typical time scale \( \tau \) of Eq. (57) is much larger than \( \sigma^{-1} \)  [see Eq. (14)], and thus the \( D_t E \) term is negligible compared with the \( \sigma_c E \) term. In the \( A_0 = 0 \) gauge, the Yang-Mills equation above reduces to

\[ \sigma_c \partial_t A = -D \times B. \]

(9)

However, this equation does not account for the correct equilibrium fluctuations of the soft modes. This is because the soft modes are not only relaxed away from the equilibrium, but also excited from thermal noise due to the interactions with hard modes, the latter of which is missed here. To take into account the latter effect, we add the noise term \( \zeta \) to have

\[ \sigma_c \partial_t A = -D \times B + \zeta. \]

(10)

It is simple to obtain the equation for \( \zeta \) to satisfy; according to the fluctuation-dissipation theorem, which connects dissipation to thermal noise, \( \zeta \) must satisfy

\[ \langle \zeta_i(x) \zeta_j(x') \rangle = 2\sigma_c T \delta_{ij} \delta^{(4)}(x - x'), \]

(11)

where \( \sigma_c \) is the same conductivity as above.

Note that Eq. (10) may also be written in a more familiar form of the Langevin-type equation

\[ \sigma_c \partial_t A = -\nabla_A H_{\text{eff}}(A) + \zeta, \]

(12)
where $H_{\text{eff}}$ is the 3-dimensional effective Hamiltonian given by

$$H_{\text{eff}}(A) = \frac{1}{2} \int d^3x \, B^2.$$  \hfill (13)

It should be remarked that $H_{\text{eff}}/T$ is exactly the magnetostatic Yang-Mills action, which can be obtained by performing the dimensional reduction in the $T$-direction and by integrating out the electrostatic field $A_0$\cite{46}.

Once the effective theory (10) is obtained, one can read off the typical time scale of the dynamics as

$$\tau \sim \frac{\sigma_c}{k^2} \sim \frac{1}{g^4 T \ln(1/g)},$$  \hfill (14)

where Eq. (7) is used. Therefore, $\sigma_c$ in Eq. (7) is much larger than $gA \sim g^2 T$ and $\tau^{-1}$, and the approximation to obtain Eq. (9) is indeed justified. This result was first obtained in Refs. [39].

**B. Intuitive derivation of the chiral Langevin theory**

We now consider the system with a chirality imbalance parametrized by the chiral chemical potential $\mu_5 = (\mu_R - \mu_L)/2$. In this case, the color current consists of not only the usual Ohmic current, but also the anomalous current,

$$j_{\text{hard}} = \sigma_c E + \sigma_{\text{anom}} B.$$  \hfill (15)

Looking at this relation, one might think that the current proportional to $B$ must be prohibited due to the parity; while the (non-Abelian) Ohmic law is consistent with parity (as $j_{\text{hard}}$ is parity odd and $E$ is parity odd), the anomalous current is not (as $B$ is parity even). However, the anomalous current with $\sigma_{\text{anom}} \propto \mu_5$ is in fact consistent with parity as $\mu_5$ is also parity odd. As shown below, one finds that $\sigma_{\text{anom}}$ is given by

$$\sigma_{\text{anom}} = \frac{N_f g^2 \mu_5}{4\pi^2},$$  \hfill (16)

where $N_f$ is the number of flavors. Note that the usual conductivity $\sigma_c$ is dissipative (as $j_{\text{hard}}$ is time reversal odd and $E$ is time reversal odd), while the anomalous conductivity $\sigma_{\text{anom}}$ is dissipationless (as $B$ is time reversal odd). Note also that the prefactor $1/4\pi^2$ (per each flavor) is different from that of the Abelian chiral magnetic effect [5–9], $1/2\pi^2$, due to the normalization of the non-Abelian charges, $\text{tr}[t^at^b] = (1/2)\delta^{ab}$.

From Eq. (14), the magnitude of the electric field $E$ is

$$E \sim \tau^{-1} A \sim g^5 T^2 \ln(1/g),$$  \hfill (17)

and, combined with Eq. (7), we have

$$j_{\text{ohm}} \sim g^5 T^3.$$  \hfill (18)
On the other hand, the anomalous current can be evaluated as
\[ j_{\text{anom}} \sim g^5 \mu_5 T^2, \tag{19} \]
where Eq. (2) is used. Therefore, \( j_{\text{ohm}} \sim j_{\text{anom}} \) for \( \mu_5 \sim T \).

Let us now turn to the intuitive derivation of the Langevin theory for this system. Our argument is based on the special nature of the anomalous current that it is topological and has nothing to do with collisional effects (i.e., dissipations or fluctuations). This may also be understood from the viewpoint of symmetries: \( \sigma_{\text{anom}} \) is time reversal invariant, while collisional effects are not, and they are not related to each other. This property leads us to expect that the following replacement would be sufficient to obtain the desired Langevin equation:
\[ j_{\text{ohm}} \rightarrow j_{\text{ohm}} + j_{\text{anom}}, \tag{20} \]
with the rest parts of the Langevin equation concerning hard modes, which are dissipative, unchanged. It will turn out in Sec. III that this handwaving argument gives the correct answer. The resulting effective theory, which we shall call the chiral Langevin theory, is given by
\[ \sigma_c \partial_t A = -D \times B - \frac{N_f g^2 \mu_5}{4\pi^2} B + \zeta, \tag{21} \]
where \( \zeta \) again satisfies Eq. (11).

In terms of the effective Hamiltonian \( H_{\text{eff}} \) in Eq. (12), the above modification corresponds to the following modified Hamiltonian:
\[ H_{\text{eff}}(A) = \int d^3 x \left( \frac{1}{2} B^2 + 2N_f \mu_5 n_{\text{CS}} \right), \]
\[ n_{CS} = \frac{g^2}{32\pi^2} \epsilon_{ijk} \left( F_{ij}^a A_k^a - \frac{g}{3} f_{abc} A_i^a A_j^b A_k^c \right). \tag{22} \]
The second term in \( H_{\text{eff}} \) is the induced Chern-Simons term at finite \( \mu_5 \), which was derived by integrating out fermion degrees of freedom in Ref. [26] and was studied in the context of the electroweak theory in Refs. [29, 30]. Note that the Chern-Simons term should not receive any perturbative or nonperturbative corrections because it has a topological origin; as we will see below, it originates from the Berry curvature in the kinetic theory. In the presence of \( \mu_5 \), the magnetostatic Yang-Mills action in Ref. [46] must be thus modified to the form of Eq. (22). Note also that the modified Yang-Mills action \( H_{\text{eff}}/T \) with Eq. (22) has only one scale \( g^2 T \) for \( \mu_5 \sim T \) (for which \( j_{\text{ohm}} \sim j_{\text{anom}} \) as mentioned above).

In the electroweak theories considered in Refs. [26, 30], dissipative effects are not included. We stress that our Langevin-type equation in Eq. (12) together with Eq. (22) is a more (and the most) complete framework taking into account the effects of the dissipation and thermal fluctuations systematically. It is thus appropriate to be applied to study the dynamical evolution of non-Abelian chiral plasmas. Converting this chiral Langevin equation with
Gaussian white noise into a Fokker-Planck equation, one can check that it reproduces the correct “equilibrium” distribution \( e^{-H_{\text{eff}}/T} \) (see Appendix A).\(^6\)

In Sec. [11], we provide a more detailed derivation of the chiral Langevin theory, starting from the chiral kinetic theory and integrating out (semi-)hard degrees of freedom. As found in Refs. [17–20], anomalous parity-violating effects are taken into account by introducing Berry curvature corrections [47]—a notion widely applied in condensed matter physics [48, 49]. Because the kinetic theory with Berry curvature corrections can be derived from the underlying quantum field theories [19, 20], this will complete the derivation of the chiral Langevin theory from quantum field theories.

**C. Evolution equation for \( \mu_5 \)**

Here we comment on the chiral chemical potential \( \mu_5 \). One might suspect that \( \mu_5 \) cannot be introduced as the usual chemical potential associated with a conserved charge, because the chiral charge is not a conserved quantity due to the axial anomaly:

\[
\partial_{\mu} j^{\mu_5} = \frac{N_f g^2}{4\pi^2} \mathbf{E} \cdot \mathbf{B}.
\]

Here \( j^{\mu_5} = (n_5, j^5) \) is the (color and flavor singlet) axial current and \( N_f \) is the number of flavors. Nonetheless, one may introduce \( \mu_5 \) as an external parameter if time scale of its variation is slow enough compared with the typical time scale of the system.\(^7\) This will be actually justified \textit{a posteriori} for \( g \ll 1 \) in Sec. [IV]. In general, one needs to treat \( \mu_5 \) as a dynamical variable which also evolves as a function of time.

Let us derive the evolution equation for \( \mu_5(t, x) \). For this purpose, let us recall the Fick’s law of diffusion,

\[
j^5 = -D_f \nabla n_5.
\]

Here \( D_f \) is the diffusion constant which is computed in the high-temperature (weak-coupling) regime of SU(3) gauge theory in Ref. [50]. Substituting Eq. (24) into Eq. (23), we get

\[
(\partial_t - D_f \nabla^2) n_5 = \frac{N_f g^2}{4\pi^2} \mathbf{E} \cdot \mathbf{B},
\]

---

6 One might suspect that \( e^{-H_{\text{eff}}/T} \) is not the equilibrium distribution in the usual sense, because, as we argued, the system with \( \mu_5 \) has a chiral plasma instability which tends to reduce \( \mu_5 \) dynamically. However, it will be justified \textit{a posteriori} that \( \mu_5 \) changes very slowly compared with the typical time scale of the system for \( g \ll 1 \) (see Sec. [IV]), so \( e^{-H_{\text{eff}}/T} \) can be approximately regarded as the “equilibrium” distribution.

7 Remember that the baryon chemical potential is also introduced although baryon charge is \textit{not} strictly a conserved charge due to quantum anomalies in the standard model. The reason why it makes sense to do so is because the time scale of baryon-number-changing processes, such as the proton lifetime \( \tau_p \), is too large (much larger than the age of the Universe), and during the time scale much less than \( \tau_p \), the baryon charge is regarded as conserved.
In equilibrium, the axial number density \( n_5 \) is expressed by the chiral chemical potential \( \mu_5 \) via
\[
n_5 \equiv n_R - n_L = N_c N_f \left[ \frac{1}{3 \pi^2} (\mu_5^3 + 3 \mu_5 \mu^2) + \frac{1}{3} T^2 \mu_5 \right],
\tag{26}
\]
which we assume to remain valid, as the system is close to equilibrium.

For fixed \( T \) and \( \mu \), one obtains the equation in terms of \( \mu_5(t, x) \) by substituting Eq. \( \text{(26)} \) into Eq. \( \text{(25)} \),
\[
(\chi \partial_t - \sigma_f \nabla^2) \mu_5 = \frac{N_f g^2}{4 \pi^2} E \cdot B,
\tag{27}
\]
where
\[
\chi \equiv \frac{\partial n_5}{\partial \mu_5} = N_c N_f \left[ \frac{1}{\pi^2} (\mu_5^2 + \mu^2) + \frac{T^2}{3} \right]
\tag{28}
\]
is the susceptibility and \( \sigma_f \) is the electrical conductivity. Here we used the Einstein relation \( \sigma_f = \chi D_f \), assuming that the diffusion constants are the same for left-handed and right-handed (or vector and axial-vector) fermions. Note that \( \sigma_f \) here is the flavor (or electrical) conductivity, but \textit{not} the color conductivity \( \sigma_c \), as Eq. \( \text{(24)} \) is related to color-singlet momentum diffusion.

In summary, the dynamical variables in our chiral Langevin theory are \( A(t, x) \) and \( \mu_5(t, x) \), whose evolutions are described by Eqs. \( \text{(21)} \) and \( \text{(27)} \).

III. FROM CHIRAL KINETIC THEORY TO THE CHIRAL LANGEVIN THEORY

In this section, we derive the chiral Langevin theory from the chiral kinetic theory for a non-Abelian plasma.

A. Chiral kinetic theory

Let us first recall the (collisionless) Boltzmann equation for an Abelian plasma (for which we use the coupling constant \( e \) instead of \( g \)). It is formulated in terms of the single-particle phase space distribution function of hard particles, \( n(t, x, p) \). According to Liouville’s theorem, it follows that
\[
\frac{dn}{dt} = 0.
\tag{29}
\]

\[8\] This Einstein relation can be understood as follows: since \( \mu \) behaves as the time component of an external gauge field coupled to the particle number current, \( \mu \) corresponds to an “external electric field” \( E = -\nabla \mu \). This field induces the current \( j = \sigma_f E = -\sigma_f \chi^{-1} \nabla n \), leading to the relation \( D_f = \sigma_f \chi^{-1} \).
Noting that $n$ is a function of $t$, $\mathbf{x}$, and $\mathbf{p}$, the left hand side of the above equation can be expanded as

$$\frac{dn}{dt} = (\partial_t + \dot{x} \cdot \nabla_x + \dot{p} \cdot \nabla_p) n. \quad (30)$$

Usually, the classical equations of motion for hard particles are given by

$$\dot{x} = \mathbf{v}, \quad (31a)$$
$$\dot{p} = e(E + \dot{x} \times \mathbf{B}), \quad (31b)$$

where $\mathbf{v} = d\epsilon_p/d\mathbf{p} = \mathbf{p}/|\mathbf{p}| \equiv \dot{\mathbf{p}}$ is the group velocity of the plasma particles. (We here defined $p \equiv |\mathbf{p}|$.) Then one obtains

$$\partial_t n_p + \mathbf{v} \cdot \nabla_x n_p + e(E + \dot{x} \times \mathbf{B}) \cdot \nabla_p n_p = 0, \quad (32)$$

which is the conventional Boltzmann equation. The electric current is given by

$$j \equiv \int \frac{d^3p}{(2\pi)^3} \dot{x} n_p = \int \frac{d^3p}{(2\pi)^3} \mathbf{v} n_p. \quad (33)$$

For chiral fermions, the equations of motion are modified by the effects of a Berry curvature: the correct equations of motion are given by [51]

$$\dot{x} = \frac{\partial \epsilon_p}{\partial \mathbf{p}} + \dot{\mathbf{p}} \times \Omega_p, \quad (34a)$$
$$\dot{p} = e(E + \dot{x} \times \mathbf{B}) - \frac{\partial \epsilon_p}{\partial \mathbf{x}}. \quad (34b)$$

Here $\Omega_p = \pm \mathbf{p}/(2p^3)$ is the Berry curvature and the signs $\pm$ correspond to the right- and left-handed fermions [17, 19, 49]; $\epsilon_p$ is the energy of the fermion as a function of $\mathbf{p}$ and its form is determined from the Lorentz covariance as [19]

$$\epsilon_p = p(1 - e\mathbf{B} \cdot \Omega_p). \quad (35)$$

Physically, the second term in Eq. (35) stands for the magnetic moment of chiral fermions in a magnetic field (the Zeeman effect). Note that Eqs. (34) reduce to Eqs. (31) in the absence of Berry curvature corrections as it should.

---

9 This modification may be understood as follows. First, remember the Hamiltonian for a spin in a magnetic field, $H_{\text{spin}} = \mathbf{\sigma} \cdot \mathbf{B}$, which was considered in the original paper by Berry [17]. In this case, the Berry curvature in the $\mathbf{B}$ space is found to be $\Omega_B = \pm \frac{B}{2\pi|\mathbf{B}|^2}$ in the magnetic-field space ($B_x, B_y, B_z$), where the signs $\pm$ correspond to the spin polarizations along the direction of the magnetic field [17]. In our case, the Hamiltonian for a chiral fermion is $H_{\text{chiral}} = \mathbf{\sigma} \cdot \mathbf{p}$, which is equivalent to the Hamiltonian above if we replace $\mathbf{B}$ by $\mathbf{p}$; the corresponding Berry curvature in the momentum space is given by $\Omega_p = \pm \frac{P}{2|\mathbf{p}|^3}$, where the signs $\pm$ correspond to the chiralities of fermions.
By solving Eqs. (34) for $\dot{x}$ and $\dot{p}$, one finds

$$
(1 + eB \cdot \Omega_p) \dot{x} = \dot{v} + e\tilde{E} \times \Omega_p + (\dot{v} \cdot \Omega_p)eB
$$

$$
(1 + eB \cdot \Omega_p) \dot{p} = e\tilde{E} + \dot{v} \times eB + e^2(\tilde{E} \cdot B)\Omega_p,
$$

where $\dot{v} \equiv \partial \epsilon_p / \partial p$ and $\tilde{E} \equiv E - \nabla (pB \cdot \Omega_p)$. Then the modified Boltzmann equation with Berry curvature corrections is [19]

$$(1 + eB \cdot \Omega_p) \partial_t n_p + (\dot{v} + e\tilde{E} \times \Omega_p + (\dot{v} \cdot \Omega_p)eB) \cdot \nabla_x n_p
$$

$$
+ (e\tilde{E} + \dot{v} \times eB + e^2(\tilde{E} \cdot B)\Omega_p) \cdot \nabla_p n_p = 0.
$$

(38)

(See also Refs. [18, 52, 53] for the case with a homogeneous magnetic field, where $\dot{v}$ and $\tilde{E}$ reduce to $\mathbf{v}$ and $\mathbf{E}$, respectively.) The modification of the equations of motion means that the phase space measure is also modified from $\mathcal{d}\Gamma = dx dp$ to 

$$(39)
$$

As a result, the expression for the electric current is [17–19]

$$
\mathbf{j} = -e \int \frac{d^3p}{(2\pi)^3} 
\left[ \epsilon_p \nabla_p n_p + e (\Omega_p \cdot \nabla_p n_p) \epsilon_p \mathbf{B} + \epsilon_p \Omega_p \times \nabla_x n_p \right] + e^2 \mathbf{E} \times \mathbf{\sigma},
$$

(40)

where

$$
\mathbf{\sigma} = \int \frac{d^3p}{(2\pi)^3} \Omega_p n_p.
$$

(41)

**B. Linearized Boltzmann equation**

Let us now linearize the Boltzmann equation above. We define the “scalar potential” $W = W(x, \mathbf{v})$ such that $n(x, \mathbf{p}) = n^\text{eq}(\epsilon_p - eW)$. Then the distribution function $n(x, \mathbf{p})$ can be expanded up to linear terms in $W$. Using Eq. (35), it follows that

$$
n(x, \mathbf{p}) \simeq n_p^\text{eq} - \frac{dn_p^\text{eq}}{dp} e(W + pB \cdot \Omega_p),
$$

where $n_p^\text{eq}$ is the equilibrium distribution function, $n_p^\text{eq} = (e^{(\mathbf{p} - \mu)/T} + 1)^{-1}$. In terms of the deviation $\delta n$ from equilibrium (including the Zeeman effect in a magnetic field), it is also written as

$$
\delta n \equiv n(x, \mathbf{p}) - n^\text{eq}(\epsilon_p) \simeq -eW \frac{dn_p^\text{eq}}{dp}.
$$

(43)

The linearized Boltzmann equation takes the following form

$$
\mathbf{v} \cdot \partial_x W_{R,L}(x, \mathbf{v}) = \mathbf{v} \cdot \mathbf{E}(x) = \frac{\mathbf{v}}{2p} \cdot \partial_t \mathbf{B}(x),
$$

(44)
for right- and left-handed fermions, where \( v \cdot \partial_x \equiv \partial_t + v \cdot \nabla_x \). Similarly, the linearized Boltzmann equation for antiparticles is

\[
v \cdot \partial_x \overline{W}_{R,L}(x, v) = -v \cdot E(x) \mp \frac{v}{2p} \cdot \partial_t B(x),
\]

(45)

Note that antiparticles of right-(left-)handed fermions are left-(right-)handed. It is clear from Eqs. (44) and (45) that the electric field \( E \) couples to a scalar potential carrying electric charge, \( W_+ \equiv (W_R + W_L - \overline{W}_R - \overline{W}_L)/4 \), while the magnetic field \( B \) couples to that carrying magnetic moment, \( W_- \equiv (p/2)(W_R + \overline{W}_L - W_L - \overline{W}_R) \). The linearized Boltzmann equations for the scalar potentials \( W_+ \) and \( W_- \) are obtained as

\[
v \cdot \partial_x W_+(x, v) = v \cdot E(x),
\]

(46)

\[
v \cdot \partial_x W_-(x, v) = -v \cdot \partial_t B(x).
\]

(47)

Summing over \( W_{R,L} \) and \( \overline{W}_{R,L} \) in Eq. (40), taking \( W_+ = -\overline{W}_R \) and \( W_- = -\overline{W}_L \), and performing the integral over \( p \equiv |p| \), we have

\[
j = m_D^2 \int_v v W_+ + \frac{\mu_5 e^2}{2\pi^2} \int_v (v W_- + B - v \times \nabla W_+),
\]

(48)

where \( \int_v \equiv \int d\Omega/(4\pi) \) is the angular integral and

\[
m_D^2 \equiv -e^2 \int \frac{d^3p}{(2\pi)^3} \frac{dn^{eq}_p}{dp} = e^2 \left( \frac{T^2}{3} + \frac{\mu^2 + \mu_5^2}{\pi^2} \right).
\]

(49)

C. Linearized non-Abelian Boltzmann-Vlasov equation

We shall generalize this set of linearized Boltzmann equation to the case of a non-Abelian plasma with \( N_c \) colors and \( N_f \) flavors. Here we do not attempt to derive it from a first principles Wigner function approach or the Kadanoff-Baym formalism (see, e.g., Ref. [55] for a review), but rather we try to derive it on physical grounds. Such a microscopic derivation should be doable by generalizing the argument of Ref. [19] to non-Abelian gauge fields.

First, notice that the color current is carried by both fermions and bosons in non-Abelian plasmas (e.g., quarks and gluons in QCD) while the electric current is carried only by fermions in Abelian plasmas (e.g., electrons in QED). So one needs to introduce the single particle phase space distribution functions both for fermions and bosons, which will be denoted as \( n^a \) and \( N^a \). Similarly to the Abelian plasma, we parametrize

\[
C_F \delta n^a(x, p) = -g W_F^a \frac{dn^{eq}_F}{dp}, \quad C_B \delta N^a(x, p) = -g W_B^a \frac{dn^{eq}_B}{dp}.
\]

(50)

Here \( n^{eq}_{F,B} \) is the equilibrium Fermi/Bose distribution function (which is colorless), and \( W_B = W_B^a t^a \) and \( W_F = W_F^a t^a \), where \( t^a (a = 1, 2, \cdots, N_c^2 - 1) \) are the generators of fundamental representations of \( SU(N_c) \) with the normalization such that \( \text{tr}(t^a t^b) = (1/2)\delta^{ab} \). The factors
$C_{F,B}$ reflect the number of degrees of freedom of fermions/bosons, and are given by $C_F = 1/2$ and $C_B = N_c$.

Second, note that the motion of chiral fermions is affected by the Berry curvature, while that of gauge bosons is not. For gauge bosons, $W_B$ thus consists only of the parity-even part $W_{B+}$ and the linearized Boltzmann equation is known as [55]

$$[v \cdot D, W_{B+}(x, v)] = v \cdot E(x),$$  
(51)

where $v \cdot D = v^\mu D_\mu$ and $D_\mu = \partial_\mu - igA_\mu$ is the covariant derivative. Again, the color indices are suppressed for simplicity. The color current is

$$j^{\mu a}_B = m_D^2 \int_v v^\mu W^{a}_{B+}(x, v).$$  
(52)

For chiral fermions, $W_F$ consists of parity-even and parity-odd parts, $W_{F+}$ and $W_{F-}$, similarly to the Abelian case above, and the linearized Boltzmann equation is given by a non-Abelian generalization of Eq. (46):

$$[v \cdot D, W_{F+}(x, v)] = v \cdot E(x),$$  
(53)

$$[v \cdot D, W_{F-}(x, v)] = -v \cdot \partial_t B(x).$$  
(54)

The color current reads

$$n^{a}_F = m_D^2 \int_v W^{a}_{F+},$$  
(55)

$$j^{a}_F = m_D^2 \int_v vW^{a}_{F+} + \frac{N_f g^2 \mu_5}{4\pi^2} \int_v (vW^{a}_{F-} + B^a - v \times \nabla W^{a}_{F+}).$$  
(56)

They must be solved together with the Yang-Mills equation

$$[D_\nu, F^{\nu\mu}] = j^{\mu}, \quad j^{\mu a} = j^{\mu a}_B + j^{\mu a}_F,$$  
(57)

which describes the evolution of gauge fields.

These equations can be further simplified by considering the counting in $g$. Observe in Eq. (56) that, for the semi-hard and soft modes, the final term $\sim g^2\mu_5KW_{F+} \leq g^3T^2W_{F+}$ is much smaller than the first term $\sim m_D^2W_{F+} \sim g^2T^2W_{F+}$ and is negligible. Then, the parity-even terms for fermions and bosons are combined into $W_+ \equiv W_{B+} + W_{F+}$, so that (hereafter $W_{F-}$ will be denoted as $W_-$ with suppressing the subscript “F”)

$$[v \cdot D, W_+(x, v)] = v \cdot E(x),$$  
(58)

$$[v \cdot D, W_-(x, v)] = -v \cdot \partial_t B(x),$$  
(59)

$$[D_\nu, F^{\nu\mu}] = m_D^2 \int_v vW_+ + \frac{N_f g^2 \mu_5}{4\pi^2} \int_v vW_- + \frac{N_f g^2 \mu_5}{4\pi^2} B$$  
(60)

with

$$m_D^2 = (N_f + 2N_c)\frac{g^2 T^2}{6} + N_f \frac{g^2 (\mu^2 + \mu_5^2)}{2\pi^2}.$$  
(61)

This set of equations constitutes the effective Boltzmann-Vlasov equation with anomalous parity-violating effects.
D. Integrating out semi-hard degrees of freedom

We now integrate out semi-hard degrees of freedom with \( k \gg g^2 T \) in the Boltzmann-Vlasov equation to derive the chiral Langevin equation. The case without the anomalous parity-violating effects was previously worked out in Refs. [39] (see also Refs. [40]). We shall follow their procedure and extend it to the case with parity-violating effects below.

Let us first summarize the procedure:

1. We decompose the fields into semi-hard and soft modes, and obtain their equations of motion.

2. We explicitly solve the equation of motion for the soft modes in terms of the fields involving semi-hard modes (defined as \( h_\pm \) below).

3. We substitute the above solutions into the expressions for the fluctuations (defined as \( \xi_\pm \) below) and take the statistical average thereof.

4. By integrating out \( W_\pm \), we express the color current in terms of the gauge fields (\( E \) and \( B \)) and the fluctuation \( \zeta \) alone.

5. Combined with the Yang-Mills equation, we obtain the Langevin-type equation for soft fields (21). The fluctuation-dissipation theorem (11) also follows from the explicit expression for \( \zeta \).

From now on, we shall proceed to take the above steps one by one. We work in the \( A_0 = 0 \) gauge. We decompose the fields \( A, E, B, \) and \( W_\pm \) into

\[
A = \tilde{A} + a, \quad E = \tilde{E} + e, \quad B = \tilde{B} + b, \quad W_\pm = \tilde{W}_\pm + w_\pm,
\]

where \( \tilde{A}, \tilde{E}, \tilde{B}, \) and \( \tilde{W}_\pm \) are soft components (components with momenta \( k < \mu \), where \( g^2 T \ll \mu \ll gT \)) and \( a, e, b, \) and \( w \) are semi-hard components (components with \( k > \mu \)).

For simplicity, we shall rename \( \tilde{A}, \tilde{E}, \tilde{B}, \) and \( \tilde{W}_\pm \) to \( A, E, B, \) and \( W_\pm \) in the following. Substituting these expressions into Eqs. (58) and (59), the equations for the soft fields read

\[
[v \cdot D, W_+(x, v)] = v \cdot E(x) + \xi_+(x, v),
\]

\[
[v \cdot D, W_-(x, v)] = -v \cdot \partial_t B(x) + \xi_-(x, v),
\]

\[
[D_v, F^{\mu\nu}] = m_D^2 \int_v vW_+ + \frac{N_f g^2 \mu_5}{4\pi^2} \int_v vW_- + \frac{N_f g^2 \mu_5}{4\pi^2} B,
\]

where

\[
\xi_\pm (x, v) \equiv ig (v \cdot a(x), w_\pm (x, v))_{soft},
\]

The subscript “soft” stands for the soft components with \( k < \mu \). Note that we have ignored the interactions between semi-hard and soft modes in Eq. (65), because they correspond to non-hard thermal loop vertices and are negligible at the leading order in \( g \) [39].
We then write down equations of motion for the semi-hard modes. For semi-hard modes with momentum $k \sim g T$ (and with the amplitude $a \sim g^{1/2} T$), the linear term, $ka$, dominates over the nonlinear term, $ga^2$, so we can ignore the latter. Then the equations of motion read

$$v \cdot \partial w_+ = v \cdot e + h_+, \quad (67)$$

$$v \cdot \partial w_- = -v \cdot \partial b + h_-, \quad (68)$$

$$\partial^2 a - \nabla (\nabla \cdot a) = m_0^2\int_v v w_+ + \frac{N_f g^2 \mu_5}{4\pi^2} \int_v v w_- + \frac{N_f g^2 \mu_5}{4\pi^2} b. \quad (69)$$

where $h_\pm$ are the interaction terms involving the soft fields,

$$h_\pm = ig \left(\langle v \cdot a, W_\pm \rangle + [v \cdot A, w_\pm]\right). \quad (70)$$

Note that, for the semi-hard modes with $k \sim g T$, the final term on the right-hand side of Eq. (69) is suppressed compared with the left-hand side by a factor of $g$, and we will ignore it in what follows.

Below we will concentrate on the transverse part of the gauge field defined by $a^i_T(t, k) = P^i_T a^j(t, k)$, which we shall rename to $a(t, k)$ for simplicity. (Here $P^i_T = \delta^{ij} - \hat{k}^i \hat{k}^j$ is the transverse projector with $\hat{k}^i$ the unit vector $\hat{k}^i = k^i/|k|$.) In the $(t, k)$ space, the equations of motion for semi-hard modes above can be written as

$$\partial_t w_+(t, k, v) + iv \cdot kw_+(t, k, v) = -v \cdot \partial a(t, k) + h_+(t, k, v), \quad (71)$$

$$\partial_t w_-(t, k, v) + iv \cdot kw_-(t, k, v) = -i v \cdot (k \times \partial a(t, k)) + h_-(t, k, v), \quad (72)$$

$$\partial^2 a(t, k) + |k|^2 a(t, k) = m_0^2\int_v v_T w_+(t, k, v) + \frac{N_f g^2 \mu_5}{4\pi^2} \int_v v_T w_-(t, k, v), \quad (73)$$

where $v^i_T = P^i_T v^j$. These equations can be solved with the help of the one-sided Fourier transform. The solutions are given by (see Appendix B)

$$a^i(K) = a^i_0(K) + \int_v \Delta^i_{12}(K, v) \left[h_+(K, v) + \frac{N_f g^2 \mu_5}{4\pi^2 m_D^2} h_-(K, v)\right] = \Delta^i_{12}(K, v), \quad (74a)$$

$$w_+(K, v) = w_+^{(0)}(K, v) + \int_{v'} \left[\Delta_{22}(K, v, v') h_+(K, v') + \frac{N_f g^2 \mu_5}{4\pi^2 m_D^2} \Delta_{23}(K, v, v') h_-(K, v')\right] = \Delta_{22}(K, v), \quad (74b)$$

$$w_-(K, v) = w_-^{(0)}(K, v) + \frac{ih_-(K, v)}{v \cdot K} = \Delta_{23}(K, v), \quad (74c)$$

where $K^\mu = (k_0, k)$, and $a^i_0$ and $w^{(0)}_\pm$ are the solutions to the equations of motion for $h_\pm = 0$, and they can be expressed by the initial values at $t = 0$, $a_{in}$ and $w_{in}$, alone; e.g., $w_-^{(0)} = w_-^{(in)}$. Here we also defined the propagators

$$\Delta^i_{12}(K, v) = \frac{im_D^2}{v \cdot K} \Delta_T(K) v^i, \quad (75a)$$

$$\Delta_{22}(K, v, v') = \delta^{(3)}(v - v') - \frac{im_D^2 k_0 v_T \cdot v'_T}{(v \cdot K)(v' \cdot K)} \Delta_T(K), \quad (75b)$$

$$\Delta_{23}(K, v, v') = -\frac{im_D^2 k_0 v_T \cdot v'_T}{(v \cdot K)(v' \cdot K)} \Delta_T(K), \quad (75c)$$
where $\Delta_T(K)$ is the hard thermal loop resummed propagator,
\[
\Delta_T(K) = \frac{1}{-K^2 + \Pi_T(K)}, \quad \Pi_T(K) = \frac{1}{2} \frac{m_0^2}{k_0} \int \frac{v_T^2}{v \cdot K},
\]  
(76)
and $\delta^{(S^2)}$ is the delta function on the two-dimensional sphere $S^2$ such that
\[
\int_{v'} f(v') \delta^{(S^2)}(v - v') = f(v).
\]  
(77)

Now remember that $h_\pm$ themselves are expressed by $a$ and $w_\pm$ [see Eq. (70)]. This means one can solve these equations iteratively; more specifically, inserting $a^{(0)}$ and $w^{(0)}_\pm$ into Eq. (70) yields $h^{(1)}_\pm$, and inserting this $h^{(1)}_\pm$ into the above equations gives $a^{(1)}$ and $w^{(1)}_\pm$. Repeating this procedure, one has the expansions,
\[
a = a^{(0)} + a^{(1)} + a^{(2)} + \cdots, \quad (78)
\]
\[
w_\pm = w^{(0)}_\pm + w^{(1)}_\pm + w^{(2)}_\pm + \cdots, \quad (79)
\]
\[
h_\pm = h^{(1)}_\pm + h^{(2)}_\pm + h^{(3)}_\pm + \cdots, \quad (80)
\]
where the expansion parameter in these expansions is the amplitude of the soft field $W$ and/or $A$, and the indices "(n)" ($n = 0, 1, 2, \cdots$) represents the $n$-th order in soft fields. Note that the expansion parameter is not the coupling constant $g$ itself. Note also that the expansion of $h_\pm$ starts from the first order in soft fields, as the soft fields $W$ and $A$ enter in Eq. (70).

Substituting these expansions into $\xi_\pm$ in Eq. (66), one obtains the expansion
\[
\xi_\pm = \xi^{(0)}_\pm + \xi^{(1)}_\pm + \xi^{(2)}_\pm + \cdots, \quad (81)
\]
where, e.g.,
\[
\xi^{(0)}_\pm = ig [v \cdot a^{(0)}_\pm, w^{(0)}_\pm]_{\text{soft}}, \quad \xi^{(1)}_\pm = ig (v \cdot a^{(1)}_\pm, w^{(0)}_\pm)_{\text{soft}} + (v \cdot a^{(0)}_\pm, w^{(1)}_\pm)_{\text{soft}}. \quad (82)
\]

One might expect that $\xi_\pm \simeq \xi^{(0)}_\pm$ is a good approximation in the equations of motion for the soft fields, (63) and (64). However, when one takes the thermal average over initial conditions, one finds
\[
\langle \xi^{(0)}_\pm \rangle = -gf^{abc} \langle (v \cdot a^{(0)}_b w^{(0)}_{\pm,c})_{\text{soft}} \rangle \propto f^{abc} \delta_{bc} = 0, \quad (83)
\]
where we used $\langle a^{(0)}_b w^{(0)}_{\pm,c} \rangle \propto \delta_{bc}$ and the antisymmetric property of the structure constant $f^{abc}$. So one also needs to take into account the subleading term $\xi^{(1)}_{\pm}$ in the effective theory. Notice that though $\langle \xi^{(0)}_{\pm} \rangle$ is vanishing, the two point function $\langle \xi^{(0)}_{\pm}(x, v) \xi^{(0)}_{\pm}(x', v) \rangle$ is not necessarily vanishing. Actually, $\langle \xi^{(0)}_{\pm}(x, v) \xi^{(0)}_{\pm}(x', v) \rangle$ was previously computed in Ref. 39 to be
\[
\langle \xi^{(0)}_{+,a}(x, v) \xi^{(0)}_{+,b}(x', v) \rangle = \frac{2T}{3 \sigma_c} I_{+}(v, v') \delta_{ab} \delta^{(4)}(x - x'), \quad (84)
\]
One can further show that \( I_+(v, v') = \delta^{(2)}(v - v') - \frac{4}{\pi} \frac{(v \cdot v')^2}{\sqrt{1 - (v \cdot v')^2}} \).

(85)

On the other hand, \( \langle \xi^{(0)}_{\pm}(x, v) \xi^{(0)}_{\pm}(x', v) \rangle \) will be shown to be irrelevant to our final effective theory, and we will not try to compute its detailed form in this paper.

From the expression \([39, 40]\) and the solutions \((74)\), it is easy to find that \( \xi^{(1)}_{\pm} \sim g^2 T W_{\pm} \).

(86)

Taking into account the theory, and we will not try to compute its detailed form in this paper.

with some functions \( J_{\pm}(v, v') \). The explicit form of \( \delta C_{\pm}[W_{\pm}] \) was previously found to be \([39, 40]\)

\[
\delta C[W] = \gamma \int_{v'} I_+(v, v') W_+(x, v'),
\]

(87)

where \( \gamma \) is the damping rate given by Eq. \([8]\) and \( I_+(v, v') \) is given by Eq. \((85)\). Again, we will see below that \( J_{\pm}(v, v') \) is irrelevant to our effective theory of interest. Physically, \( \delta C_{\pm} \) correspond to the collision terms and the typical scale of \( \delta C_{\pm} \) is the scale of small-angle scattering that causes color diffusion, \( \gamma \sim g^2 T \ln(1/g) \) \([40]\).

Inserting \( \xi_{\pm} \approx \xi^{(0)}_{\pm} + \xi^{(1)}_{\pm} \) and Eq. \((86)\) into the equations of the soft fields, \((63)\) and \((64)\), we arrive at

\[
[v \cdot D, W_+(x, v)] = v \cdot E(x) + \xi^{(0)}_+(x, v) + \delta C_+[W_+],
\]

(88)

\[
[v \cdot D, W_-(x, v)] = -v \cdot \partial_t B(x) + \xi^{(0)}_-(x, v) + \delta C_-[W_-].
\]

(89)

E. Chiral Langevin theory

We can now show that \( W_- \) and \( \xi_- \) are irrelevant to our effective theory at the soft scale to the leading order in \( g \). As we have seen in Sec. \([I]\) and as we will show in Sec. \([IV]\) for chiral plasmas, the typical time scale is \( \tau \sim (g^4 T)^{-1} \) [up to a factor of \( \ln(1/g) \)]. Taking into account \( [v \cdot D, W_{\pm}] \sim g^2 T W_{\pm} \) and \( \delta C_{\pm}[W_{\pm}] \sim g^2 T W_{\pm}, \)

(86)

one can estimate the typical

\(10\) One can further show that \( \xi^{(n)}_{\pm} \geq 2 \) is suppressed compared with \( \xi^{(1)}_{\pm} \) by some factors of \( g \) as follows: since \( \xi^{(n)}_{\pm} \) is \( n \)-th order in the soft fields and since each iteration of the equation of motion for semi-hard modes comes with the factor \( g \), \( \xi^{(n)}_{\pm} \) must be proportional to \( (g W_{4n})^n \) \((n \geq 1)\). Considering the dimensionality and the fact that \( g T \) is the only other scale available in the equations of motion for the semi-hard modes, one has \( \xi^{(n)}_{+} \sim (g W_{4n})^n (g T)^{2-n} \) and \( \xi^{(n)}_{-} \sim (g W_{n})^n (g T)^{3-2n} \). As will be shown momentarily, \( W_+ \sim g^2 T \) and \( W_- \sim g^{4n+3} T^2 \); \( \xi^{(n)}_{\pm} \geq 2 \) is negligible compared with \( \xi^{(1)}_{\pm} \), respectively.

\(11\) A more careful analysis shows that the latter is larger than the former by a factor of \( \ln(1/g) \). This property will be used in the later discussion, but is irrelevant here.
amplitudes of $W_\pm$ as
\begin{align}
W_+ &\sim \frac{A}{r k} \sim g^3 T \\
W_- &\sim \frac{A}{r} \sim g^5 T,
\end{align}
up to factors of $\ln(1/g)$, where $A \sim gT$ is used (see Sec. 1). Hence, in the current (56), the first and the third terms are $\sim g^5 T^3$, while the second one is $\sim g^7 T^3$, so the contribution from $W_-$ is suppressed compared with others. As a result, the leading-order contributions to the current read
\begin{equation}
\mathbf{j}^a = m_D^2 \int v W_+^a + \frac{N_f \mu_5 g^2}{4\pi^2} \mathbf{B}^a.
\end{equation}

We still need to find the form of the first term in Eq. (92), but it was already worked out in Refs. [39, 40]; in Eq. (88), the left-hand side is much smaller than $\delta C[W_+]$ by a factor of $\ln(1/g) \gg 1$ and is negligible. The resulting equation is (hereafter $\xi_{(0)}$ will be denoted by $\xi$ for simplicity)
\begin{equation}
\mathbf{v} \cdot \mathbf{E} + \xi \approx \delta C[W_+],
\end{equation}
and its formal solution is given by
\begin{equation}
W_+ = (\delta C)^{-1} (\mathbf{v} \cdot \mathbf{E} + \xi),
\end{equation}
where $\delta C^{-1}$ is understood as an operator that acts on the space of functions of $v$. The first term in Eq. (92) then reduces to [40]
\begin{equation}
m_D^2 \int_v v W_+^a = m_D^2 \int_v \mathbf{v} [(\delta C)^{-1} (\mathbf{v} \cdot \mathbf{E} + \xi)] = \sigma_c \mathbf{E}^a + \zeta^a,
\end{equation}
where $\sigma_c = m_D^2 / (3\gamma)$ is the color conductivity and
\begin{equation}
\zeta^a = 3\sigma_c \int_v v \xi
\end{equation}
is the noise term. Using Eq. (84), one can show Eq. (11) we asserted on the basis of the fluctuation-dissipation theorem.

Collecting altogether, to the leading order in $g$, the current takes the form:
\begin{equation}
\mathbf{j} = \sigma_c \mathbf{E} + \frac{N_f \mu_5 g^2}{4\pi^2} \mathbf{B} + \zeta.
\end{equation}
Combining it to the spatial part of the Yang-Mills equation (57) (or Ampère’s law), one arrives at the chiral Langevin equation (21) postulated in Sec. II B.

\textsuperscript{12} Precisely speaking, this is not completely justified because the operator $\delta C$ has a zero mode. One can develop a more rigorous argument by taking it into account and can check that the result remains unchanged at the end [40].
IV. CHIRAL PLASMA INSTABILITIES REVISITED

In light of the chiral Langevin theory we have derived, Eqs. (21) and (27), we are now ready to study the dynamical evolutions of non-Abelian chiral plasmas. First, notice that the behavior of the mean value of the gauge field in Eq. (21) is governed by the anomalous Yang-Mills equation (Yang-Mills equation plus non-Abelian analogue of the chiral magnetic effect). From the argument in Sec. II B one finds that it exhibits a chiral plasma instability; gauge fields grow rapidly.

We can also estimate the typical time scale of the chiral plasma instability. From Eq. (21), it is easy to see that

\[ \tau_{\text{inst}} \sim \frac{\sigma_c}{k^2} \sim \frac{1}{g^4T \ln(1/g)}. \]  

This is the same time scale as Eq. (14) without anomalous effects, because in the effective theory there is only one length scale \( R \sim k^{-1}_{\text{soft}} \sim (g^2T)^{-1} \) and the scale of color diffusion \( \sigma_c \) for \( \mu_5 \sim T \). This result is also consistent with the time scale previously obtained in Ref. [21] based on the Boltzmann-Vlasov equation with Berry curvature corrections, where importance of the color diffusion was found (see the footnote 5 in this paper). Note that the analysis of Ref. [21] is based on the linear response theory and is justified only when the nonlinear gauge interactions are negligible, while the present result using the chiral Langevin theory is general in that it is valid even when the nonlinear interactions are comparable to the linear ones.

This time scale in turn provides the typical scale of the amplitude of the color electric field as in Eq. (17): \( E \sim g^5T^2 \ln(1/g) \). Combined it with the amplitude of the color magnetic field, \( B \sim g^3T^2 \), and from the equation (27) for \( \mu_5 \), we can estimate the typical time scale at which the chiral charge \( N_5 \) varies:

\[ \tau_{N_5} \sim \frac{T^3}{g^2EB} \sim \frac{1}{g^{10}T \ln(1/g)}. \]  

This is much larger than \( \tau_{\text{inst}} \), and hence, \( \mu_5 \) almost “freezes” during the time \( \tau_{\text{inst}} \), at least within the applicability of this effective theory. Therefore, it justifies the very first assumption that the chiral charge \( N_5 \) can be regarded as conserved during the typical time scale of the system and that the chiral chemical potential \( \mu_5 \) is well-defined.

There is an alternative way to see that \( \mu_5 \) is well-defined in the regime under consideration. Rewrite Eq. (23) as

\[ \partial_t (N_5 + N_{\text{CS}}) = 0, \]  

where

\[ N_5 \equiv \int d^3x \, j^{05}, \quad N_{\text{CS}} \equiv \int d^3x \, n_{\text{CS}}, \]
are global chiral charge and Chern-Simons number, respectively. Equation (100) suggests that the combination $N_5 + N_{CS}$ is a conserved charge, so one can safely introduce a chemical potential associated with it, which we denote $\mu'_5$. Then consider the saturation regime where the distribution of the gauge field is the equilibrium distribution with fixed $T$ and $\mu'_5$,

$$P_{eq}(A) \sim e^{-H_{eff}(A)/T}. \quad (102)$$

Here $H_{eff}$ is given in Eq. (22) with the replacement $\mu_5 \to \mu'_5$. From the condition of the equilibrium distribution, $H_{eff}|_{k \sim g^2 T} \sim T$ for $\mu'_5 \sim T$, one can easily estimate the amplitude of the gauge field as $A \sim g T$. (Here we assumed that saturation occurs at the magnetic scale, $k \sim g^2 T$.) The magnitude of the Chern-Simons number is then estimated as

$$N_{CS} \sim g^2 k A^2 \sim g^6 T^3. \quad (103)$$

On the other hand, the chiral charge carried by fermions is $N_5 \sim T^3$; only $O(g^6)$ fraction of the conserved charge $N'_5$ is carried by the gauge field. So, even if the system starts from a given initial condition, the saturation regime is characterized by $\mu_5 = [1 + O(g^6)] \mu'_5 \sim T$, indicating that $\mu_5$ is well-defined.

It should be remarked that the separations of scales, $\tau_{inst} \ll \tau_{N_5}$ and $N_{CS} \ll N_5$, we have shown for $g \ll 1$, do not necessarily exist for $g \sim 1$; if not, one may not define $\mu_5$ itself. Indeed, in the quark-gluon plasma created in real heavy ion collision experiments, the QCD coupling constant is no longer small, $g \sim 1$, and it is not clear at all whether one can define $\mu_5$. A similar discussion in a different context is given in Ref. [56].

Note also that our chiral Langevin theory is applicable all the way to saturation as long as the definite separation of scales characterized by the small coupling constant $g \ll 1$ exists. On the other hand, the presence of the chiral plasma instability means that the prefactor $\lambda$ defined by $A = \lambda g T$ grows very rapidly. In our paper, we have always assumed that $\lambda \sim 1$ (which could be a large factor not captured by the expansion in $g$) and that $\lambda g \ll 1$. Beyond the counting scheme of the present paper, one could imagine the situation that the large $\lambda$ overwhelms the small $g$ such that $\lambda g \gtrsim 1$. It would be an interesting question to consider a possible effective theory description in this regime where the naive expansion in $g$ breaks down. Such an effective theory might give a new insight to understand the physics of the chiral plasma instability towards the saturation regime.

V. CONCLUSION

In this paper, we have constructed a Langevin-type effective theory that describes the dynamics of non-Abelian plasmas with chirality imbalance at the magnetic scale $\sim g^2 T$. This chiral Langevin theory, in particular, describes the evolution of the chiral plasma instability towards saturation, which is completely missed in hydrodynamics for non-Abelian plasmas. Using this equation, the time scale of the chiral plasma instability is easily found to be

$$\tau_{inst} \sim [g^4 T \ln(1/g)]^{-1},$$

as is consistent with the estimate previously found in Ref. [21] based
on the Boltzmann-Vlasov equation with Berry curvature corrections. On the other hand, the time scale of the variation of the chiral charge is $\tau_{N_5} \sim \left(g^{10} T \ln(1/g)\right)^{-1}$ and is much longer than $\tau_{\text{inst}}$; the chiral charge is thus shown to be approximately conserved during the chiral plasma instability, which ensures that $\mu_5$ is well-defined for $g \ll 1$.

In this paper, we have derived the chiral Langevin equation both from a physical argument and from a microscopic analysis. For the latter, we started with the linearized non-Abelian Boltzmann-Vlasov equation with Berry curvature corrections and integrated out (semi-)hard degrees of freedom. Alternatively, one should also be able to arrive at the same Langevin-type equation starting from a different classical transport theory \cite{57} with Berry curvature corrections \cite{58} where the trajectories of a particle is specified by $x, p$, and non-Abelian charge $Q^a$ (known as the Wong equations \cite{59}). Such a derivation was done in the case without anomalous parity-violating effects in Refs. \cite{41}.

Detailed numerical analysis of the chiral Langevin theory should allow us to understand the numerical coefficient in Eq. (98), how the system approaches saturation after the chiral plasma instability, and what the configuration at the saturation stage looks like. These investigations are deferred to future work \cite{60}.

Acknowledgments

The authors thank D. Bödeker, C. Manuel, K. Rajagopal, A. Rothkopf, and D. T. Son for discussions. One of the authors (N.Y.) also thanks the hospitality of IFT UAM-CSIC, where a part of this work was carried out.

Appendix A: Fokker-Planck equation

For completeness, we review the derivation of the Fokker-Planck equation from the Langevin equation \cite{12}. The Fokker-Planck equation here describes how the probability function for the gauge field $A(t, x)$ relaxes to the thermal equilibrium distribution $e^{-H_{\text{eff}}/T}$ over a long time span. Our derivation essentially follows Ref. \cite{61}.

We first introduce the probability

$$ P [A(x), t | A_0(x), t_0] = \langle \delta [A(x) - A(t, x)] \rangle_{A_0, t_0}, \tag{A1} $$

that the gauge field has the configuration $A(x)$ at time $t$, given the initial gauge field configuration $A_0(x)$ at time $t_0$. Then, the probability function satisfies

$$ P [A(x), t + \Delta t | A_0(x), t_0] = \int D A' P [A(x), t + \Delta t | A'(x), t] P [A'(x), t | A_0(x), t_0], \tag{A2} $$

with infinitesimally small $\Delta t$, where

$$ P [A(x), t + \Delta t | A'(x), t] = \langle \delta [A(x) - A(t + \Delta t, x)] \rangle_{A', t}. \tag{A3} $$
Below we shall compute it explicitly.

Using the Langevin equation (12), we have
\[
A(t + \Delta t, x) = A'(x) - \frac{\Delta t}{\sigma} \delta H_{\text{eff}}(A') + \frac{1}{\sigma} \int_t^{t+\Delta t} dt' \zeta(t', x).
\] (A4)

Then we substitute it into Eq. (A3) and expand up to linear terms in $\Delta t$. In this process, note that the average of the third term on the right hand side of Eq. (A4) is vanishing, while the average of its square is
\[
\int_t^{t+\Delta t} dt_1 \int_t^{t+\Delta t} dt_2 \langle \zeta(t_1, x_1) \zeta_j(t_2, x_2) \rangle = 2\sigma T \delta_{ij} \Delta t \delta(x_1 - x_2).
\] (A5)

Note also that averages of higher order terms in \( \int dt \zeta(t, x) \) are higher order in $\Delta t$ and are negligible. This is because they can be expressed as products of \( \langle \zeta(t_1, x_1) \zeta_j(t_2, x_2) \rangle \sim \delta_{ij} \Delta t^{-1} \delta(x_1 - x_2) \) due to the fact that $\zeta$ is a Gaussian white noise. We thus have
\[
\langle \delta [A - A(t + \Delta t)] \rangle_{A', t} = \left[ 1 + \Delta t \frac{T}{\sigma} \int d^3x \left( \frac{1}{T} \frac{\delta H_{\text{eff}}(A')}{\delta A'} \cdot \frac{\delta}{\delta A} + \frac{\delta^2}{\delta A^2} \right) \right] \delta [A - A'],
\] (A6)

where the argument $x$ of $A$ is suppressed for notational simplicity.

Remembering the definition of $P$ in Eq. (A1), we arrive at
\[
\frac{\partial P}{\partial t} = \frac{T}{\sigma} \int d^3x \frac{\delta}{\delta A} \cdot \left[ \left( \frac{1}{T} \frac{\delta H_{\text{eff}}(A)}{\delta A} + \frac{\delta}{\delta A} \right) P \right]
\] (A7)

This is the Fokker-Planck equation. From the condition $\partial P/\partial t = 0$, we obtain the equilibrium distribution
\[
P_{eq} \sim e^{-H_{\text{eff}}/T}.
\] (A8)

We note that the derivation here does not depend on the details of $H_{\text{eff}}$ and is valid for both $H_{\text{eff}}$ given in Eqs. (13) and (22).

Appendix B: Solution to the equations of motion for semi-hard modes

In this appendix, we give a detailed procedure to solve Eqs. (71), (72), and (73) with the use of the one-sided Fourier transform. The one-sided Fourier transform is defined for a function $f(t)$ by
\[
f(k_0) = \int_0^{\infty} dt e^{ik_0 t} f(t).
\] (B1)

From the definition, it is easy to derive the relations
\[
\int_0^{\infty} dt e^{ik_0 t} \partial_t f(t) = -f_{in} - ik_0 f(k_0),
\] (B2)
\[
\int_0^{\infty} dt e^{ik_0 t} \partial_t^2 f(t) = -\partial_t f_{in} + ik_0 f_{in} - k_0^2 f(k_0),
\] (B3)
where the subscript “in” stands for the initial values at $t = 0$. Using these relations for Eqs. (71) and (72), we can express $w_{\pm}$ in terms of $a$ as

$$w_+(K, v) = \frac{1}{v \cdot K} [k_0 v \cdot a(K) + il_+(K, v)], \quad (B4)$$

$$w_-(K, v) = \frac{1}{v \cdot K} [ik_0 v \cdot (k \times a(K)) + il_-(K, v)], \quad (B5)$$

where $K^\mu = (k_0, k)$ and

$$l_+(K, v) \equiv w_{\text{in}}^+(k, v) + v \cdot a_{\text{in}}(k) + h_+(K, v), \quad (B6)$$

$$l_-(K, v) \equiv w_{\text{in}}^-(k, v) + iv \cdot (k \times a_{\text{in}}(k)) + h_-(K, v). \quad (B7)$$

Substituting these expressions into the one-sided Fourier transform of Eq. (73),

$$-K^2 a_T(K) - m_D^2 \int_v v_T w_+(K, v) - \frac{N_f g^2 \mu_5}{4 \pi^2} \int_v v_T w_-(K, v) = l(K), \quad (B8)$$

with

$$l(K) \equiv -2 T_{ij} (k) - ik_0 a_{\text{T}}^{\text{in}}(k), \quad (B9)$$

one finds that the first term in Eq. (B5) and the second term in Eq. (B7) in $w_-$ (both of which have the opposite contributions for right and left-handed fermions) are subleading in $g$ compared with the contributions from $w_+$. One can then solve this equation in terms of $a$, and as a result, one can write $w_{\pm}$ as

$$a_i(K) = \Delta_{11}^{ij}(K) l_j(K) + \int_v \Delta_{12}^{ij}(K, v) \left[ l_+(K, v) + \frac{N_f g^2 \mu_5}{4 \pi^2 m_D^2} l_-(K, v) \right], \quad (B10a)$$

$$w_+(K, v) = \Delta_{21}^{ij}(K, v) l_j(K) + \int_{v'} \left[ \Delta_{22}^{ij}(K, v, v') l_+(K, v') + \frac{N_f g^2 \mu_5}{4 \pi^2 m_D^2} \Delta_{23}^{ij}(K, v, v') l_-(K, v') \right], \quad (B10b)$$

$$w_-(K, v) = \frac{il_-(K, v)}{v \cdot K}, \quad (B10c)$$

where the propagators $\Delta_{11}^{ij}$ and $\Delta_{21}^{ij}$ are defined as

$$\Delta_{11}^{ij}(K) = P_{T}^{ij} \Delta_T(K), \quad (B11)$$

$$\Delta_{21}^{ij}(K, v) = -\frac{k_0}{v \cdot K} \Delta_T(K) v_T^j, \quad (B12)$$

and $\Delta_{12}^{ij}, \Delta_{22}^{ij}$, and $\Delta_{23}^{ij}$ are defined in Eqs. (75). Substituting the expressions for $l_{\pm}$ and $l$ into Eqs. (B10), one obtains Eqs. (74).

[1] D. Kharzeev and A. Zhitnitsky, Nucl. Phys. A 797, 67 (2007).
[2] D. E. Kharzeev, L. D. McLerran and H. J. Warringa, Nucl. Phys. A 803, 227 (2008).
[3] M. Joyce and M. E. Shaposhnikov, Phys. Rev. Lett. 79, 1193 (1997).
[4] J. Charbonneau and A. Zhiltzitsky, JCAP 1008, 010 (2010).
[5] A. Vilenkin, Phys. Rev. D 22, 3080 (1980).
[6] H. B. Nielsen and M. Ninomiya, Phys. Lett. B 130, 389 (1983).
[7] A. Y. Alekseev, V. V. Cheianov, and J. Frohlich, Phys. Rev. Lett. 81, 3503 (1998).
[8] D. T. Son and A. R. Zhiltzitsky, Phys. Rev. D 70, 074018 (2004).
[9] K. Fukushima, D. E. Kharzeev, and H. J. Warringa, Phys. Rev. D78, 074033 (2008).
[10] S. Adler, Phys. Rev. 177, 2426 (1969).
[11] J. S. Bell and R. Jackiw, Nuovo Cimento A 60, 47 (1969).
[12] D. E. Kharzeev, arXiv:1312.3348 [hep-ph].
[13] J. Liao, arXiv:1401.2500 [hep-ph].
[14] X. Wan, A. M. Turner, A. Vishwanath, and S. Y. Savrasov, Phys. Rev. B 83, 205101 (2011).
[15] A. A. Burkov and L. Balents, Phys. Rev. Lett. 107, 127205 (2011).
[16] G. Xu, H. Weng, Z. Wang, X. Dai, and Z. Fang, Phys. Rev. Lett. 107, 186806 (2011).
[17] D. T. Son and N. Yamamoto, Phys. Rev. Lett. 109, 181602 (2012).
[18] M. A. Stephanov and Y. Yin, Phys. Rev. Lett. 109, 162001 (2012).
[19] D. T. Son and N. Yamamoto, Phys. Rev. D 87, 085016 (2013).
[20] J. -W. Chen, S. Pu, Q. Wang and X. -N. Wang, Phys. Rev. Lett. 110, 262301 (2013).
[21] Y. Akamatsu and N. Yamamoto, Phys. Rev. Lett. 111, 052002 (2013).
[22] E. S. Weibel, Phys. Rev. Lett. 2, 83 (1959).
[23] Z. V. Khaidukov, V. P. Kirilin, A. V. Sadofyev and V. I. Zakharov, arXiv:1307.0138 [hep-th].
[24] K. Jensen, P. Kovtun and A. Ritz, arXiv:1307.3234 [hep-th].
[25] M. Laine, JHEP 0510, 056 (2005).
[26] A. N. Redlich and L. C. R. Wijewardhana, Phys. Rev. Lett. 54, 970 (1985).
[27] K. Tsokos, Phys. Lett. B 157, 413 (1985).
[28] A. J. Niemi and G. W. Semenoff, Phys. Rev. Lett. 54, 2166 (1985).
[29] V. A. Rubakov, Prog. Theor. Phys. 75, 366 (1986).
[30] V. A. Rubakov and A. N. Tavkhelidze, Phys. Lett. B 165, 109 (1985).
[31] A. Boyarsky, O. Ruchayskiy and M. Shaposhnikov, Phys. Rev. Lett. 109, 111602 (2012).
[32] D. T. Son and P. Surowka, Phys. Rev. Lett. 103, 191601 (2009).
[33] J. Erdmenger, M. Haack, M. Kaminski and A. Yarom, JHEP 0901, 055 (2009).
[34] N. Banerjee, J. Bhattacharya, S. Bhattacharyya, S. Dutta, R. Loganayagam and P. Surowka, JHEP 1101, 094 (2011).
[35] N. Banerjee, J. Bhattacharya, S. Bhattacharyya, S. Jain, S. Minwalla and T. Sharma, JHEP 1209, 046 (2012).
[36] K. Jensen, Phys. Rev. D 85, 125017 (2012).
[37] F. M. Haehl, R. Loganayagam and M. Rangamani, arXiv:1312.0610 [hep-th].
[38] M. Hongo, Y. Hirono and T. Hirano, arXiv:1309.2823 [nucl-th].
[39] D. Bodeker, Phys. Lett. B 426, 351 (1998); Nucl. Phys. B 559, 502 (1999).
[40] P. B. Arnold, D. T. Son and L. G. Yaffe, Phys. Rev. D 59, 105020 (1999); Phys. Rev. D 60, 025007 (1999).
[41] D. F. Litim and C. Manuel, Phys. Rev. Lett. 82, 4981 (1999); Nucl. Phys. B 562, 237 (1999).
[42] P. B. Arnold, Int. J. Mod. Phys. E 16, 2555 (2007).
[43] C. Manuel and J. M. Torres-Rincon, arXiv:1312.1158 [hep-ph].
[44] A. Selikhov and M. Gyulassy, Phys. Lett. B 316, 373 (1993).
[45] H. Heiselberg, Phys. Rev. Lett. 72, 3013 (1994).
[46] E. Braaten and A. Nieto, Phys. Rev. D 53, 3421 (1996).
[47] M. V. Berry, Proc. R. Soc. Lond. A 392, 45 (1984).
[48] D. Xiao, M.-C. Chang, and Q. Niu Rev. Mod. Phys. 82, 1959 (2010).
[49] G. E. Volovik, The Universe in a Helium Droplet, Clarendon Press, Oxford (2003).
[50] P. B. Arnold, G. D. Moore and L. G. Yaffe, JHEP 0011, 001 (2000).
[51] G. Sundaram and Q. Niu, Phys. Rev. B 59, 14915 (1999).
[52] C. Duval, Z. Horváth, P. A. Horváthy, L. Martina, and P. Stichel, Mod. Phys. Lett. B 20, 373 (2006).
[53] D. T. Son and B. Z. Spivak, Phys. Rev. B 88, 104412 (2013).
[54] D. Xiao, J. Shi, and Q. Niu, Phys. Rev. Lett. 95, 137204 (2005).
[55] J. -P. Blaizot and E. Iancu, Phys. Rept. 359, 355 (2002).
[56] G. D. Moore and M. Tassler, JHEP 1102, 105 (2011).
[57] P. F. Kelly, Q. Liu, C. Lucchesi and C. Manuel, Phys. Rev. Lett. 72, 3461 (1994); Phys. Rev. D 50, 4209 (1994).
[58] M. Stone and V. Dwivedi, Phys. Rev. D 88, 045012 (2013).
[59] S. K. Wong, Nuovo Cim. A 65, 689 (1970).
[60] Y. Akamatsu, A. Rothkopf, and N. Yamamoto, work in progress.
[61] P. M. Chaikin and T. C. Lubensky, Principles of Condensed Matter Physics, Cambridge Univ. Press (Cambridge, 1995), Chap. 7.5.5.