Polynomial bounds for chromatic number.

IV. A near-polynomial bound for excluding the five-vertex path

Alex Scott\textsuperscript{1}
Mathematical Institute, University of Oxford, Oxford OX2 6GG, UK

Paul Seymour\textsuperscript{2}
Princeton University, Princeton, NJ 08544

Sophie Spirkl\textsuperscript{3}
University of Waterloo, Waterloo, Ontario N2L3G1, Canada

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Abstract

A graph $G$ is $H$-free if it has no induced subgraph isomorphic to $H$. We prove that a $P_5$-free graph with clique number $\omega \geq 3$ has chromatic number at most $\omega \log_2(\omega)$. The best previous result was an exponential upper bound $(5/27)^{3\omega}$, due to Esperet, Lemoine, Maffray, and Morel. A polynomial bound would imply that the celebrated Erdős-Hajnal conjecture holds for $P_5$, which is the smallest open case. Thus, there is great interest in whether there is a polynomial bound for $P_5$-free graphs, and our result is an attempt to approach that.
1 Introduction

If $G, H$ are graphs, we say $G$ is $H$-free if no induced subgraph of $G$ is isomorphic to $H$; and for a graph $G$, we denote the number of vertices, the chromatic number, the size of the largest clique, and the size of the largest stable set by $|G|$, $\chi(G)$, $\omega(G)$, $\alpha(G)$ respectively.

The $k$-vertex path is denoted by $P_k$, and $P_4$-free graphs are well-understood; every $P_4$-free graph $G$ with more than one vertex is either disconnected or disconnected in the complement [24], which implies that $\chi(G) = \omega(G)$. Here we study how $\chi(G)$ depends on $\omega(G)$ for $P_5$-free graphs $G$.

The Gyárfás-Sumner conjecture [10, 25] says:

1.1 Conjecture: For every forest $H$ there is a function $f$ such that $\chi(G) \leq f(\omega(G))$ for every $H$-free graph $G$.

This is open in general, but has been proved [10] when $H$ is a path, and for several other simple types of tree ([3, 11, 12, 13, 14, 17, 19]; see [18] for a survey). The result is also known if all induced subdivisions of a tree are excluded [17].

A class of graphs is hereditary if the class is closed under taking induced subgraphs and under isomorphism, and a hereditary class is said to be $\chi$-bounded if there is a function $f$ such that $\chi(G) \leq f(\omega(G))$ for every graph $G$ in the class (thus, the Gyárfás-Sumner conjecture says that, for every forest $H$, the class of $H$-free graphs is $\chi$-bounded). Louis Esperet [8] made the following conjecture:

1.2 (False) Conjecture: Let $G$ be a $\chi$-bounded class. Then there is a polynomial function $f$ such that $\chi(G) \leq f(\omega(G))$ for every $G \in G$.

Esperet’s conjecture was recently shown to be false by Briański, Davies and Walczak [2]. However, this raises the further question: which $\chi$-bounded classes are polynomially $\chi$-bounded? In particular, the two conjectures 1.1 and 1.2 would together imply the following, which is still open:

1.3 Conjecture: For every forest $H$, there exists $c > 0$ such that $\chi(G) \leq \omega(G)^c$ for every $H$-free graph $G$.

This is a beautiful conjecture. In most cases where the Gyárfás-Sumner conjecture has been proved, the current bounds are very far from polynomial, and 1.3 has been only been proved for a much smaller collection of forests (see [15, 20, 22, 23, 21, 5, 16]). In [23] we proved it for any $P_5$-free tree $H$, but it has not been settled for any tree $H$ that contains $P_5$. In this paper we focus on the case $H = P_5$.

The best previously-known bound on the chromatic number of $P_5$-free graphs in terms of their clique number, due to Esperet, Lemoine, Maffray, and Morel [9], was exponential:

1.4 If $G$ is $P_5$-free and $\omega(G) \geq 3$ then $\chi(G) \leq (5/27)3^{\omega(G)}$.

Here we make a significant improvement, showing a “near-polynomial” bound:

1.5 If $G$ is $P_5$-free and $\omega(G) \geq 3$ then $\chi(G) \leq \omega(G)^{\log_2(\omega(G))}$.

(The cycle of length five shows that we need to assume $\omega(G) \geq 3$. Sumner [25] showed that $\chi(G) \leq 3$ when $\omega(G) = 2$.) Conjecture 1.3 when $H = P_5$ is of great interest, because of a famous conjecture due to Erdős and Hajnal [6, 7], that:
1.6 Conjecture: For every graph $H$ there exists $c > 0$ such that $\alpha(G)\omega(G) \geq |G|^c$ for every $H$-free graph $G$.

This is open in general, despite a great deal of effort; and in view of [4], the smallest graph $H$ for which 1.6 is undecided is the graph $P_5$. Every forest $H$ satisfying 1.3 also satisfies the Erdős-Hajnal conjecture, and so showing that $H = P_5$ satisfies 1.3 would be a significant result. (See [1] for some other recent progress on this question.)

We use standard notation throughout. When $X \subseteq V(G)$, $G[X]$ denotes the subgraph induced on $X$. We write $\chi(X)$ for $\chi(G[X])$ when there is no ambiguity.

2 The main proof

We denote the set of nonnegative real numbers by $\mathbb{R}_+$, and the set of nonnegative integers by $\mathbb{Z}_+$. Let $f : \mathbb{Z}_+ \rightarrow \mathbb{R}_+$ be a function. We say

- $f$ is non-decreasing if $f(y) \geq f(x)$ for all integers $x, y \geq 0$ with $y > x \geq 0$;
- $f$ is a binding function for a graph $G$ if it is non-decreasing and $\chi(H) \leq f(\omega(H))$ for every induced subgraph $H$ of $G$; and
- $f$ is a near-binding function for $G$ if $f$ is non-decreasing and $\chi(H) \leq f(\omega(H))$ for every induced subgraph $H$ of $G$ different from $G$.

In this section we show that if a function $f$ satisfies a certain inequality, then it is a binding function for all $P_5$-free graphs. Then at the end we will give a function that satisfies the inequality, and deduce 1.5.

A cutset in a graph $G$ is a set $X$ such that $G \setminus X$ is disconnected. A vertex $v \in V(G)$ is mixed on a set $A \subseteq V(G)$ or a subgraph $A$ of a graph $G$ if $v$ is not in $A$ and has a neighbour and a non-neighbour in $A$. It is complete to $A$ if it is adjacent to every vertex of $A$. We begin with the following:

2.1 Let $G$ be $P_5$-free, and let $f$ be a near-binding function for $G$. Let $G$ be connected, and let $X$ be a cutset of $G$. Then

$$\chi(G \setminus X) \leq f(\omega(G) - 1) + \omega(G)f(\lfloor \omega(G)/2 \rfloor).$$

Proof. We may assume (by replacing $X$ by a subset if necessary) that $X$ is a minimal cutset of $G$; and so $G \setminus X$ has at least two components, and every vertex in $X$ has a neighbour in $V(B)$, for every component $B$ of $G \setminus X$. Let $B$ be one such component; we will prove that $\chi(B) \leq f(\omega(G) - 1) + \omega(G)f(\lfloor \omega(G)/2 \rfloor)$, from which the result follows.

Choose $v \in X$ (this is possible since $G$ is connected), and let $N$ be the set of vertices in $B$ adjacent to $v$. Let the components of $B \setminus N$ be $R_1, \ldots, R_k, S_1, \ldots, S_{\ell}$, where $R_1, \ldots, R_k$ each have chromatic number more than $f(\lfloor \omega(G)/2 \rfloor)$, and $S_1, \ldots, S_{\ell}$ each have chromatic number at most $f(\lfloor \omega(G)/2 \rfloor)$. Let $S$ be the union of the graphs $S_1, \ldots, S_{\ell}$; thus, $\chi(S) \leq f(\lfloor \omega(G)/2 \rfloor)$. For $1 \leq i \leq k$, let $Y_i$ be the set of vertices in $N$ with a neighbour in $V(R_i)$, and let $Y = Y_1 \cup \cdots \cup Y_k$.

(1) For $1 \leq i \leq k$, every vertex in $Y_i$ is complete to $R_i$. 

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Let \( y \in Y_i \). Thus, \( y \) has a neighbour in \( V(R_i) \); suppose that \( y \) is mixed on \( R_i \). Since \( R_i \) is connected, there is an edge \( ab \) of \( R_i \) such that \( y \) is adjacent to \( a \) and not to \( b \). Now \( v \) has a neighbour in each component of \( G \setminus X \), and since there are at least two such components, there is a vertex \( u \in V(G) \setminus (X \cup V(B)) \) adjacent to \( v \). But then \( u-v-y-a-b \) is an induced copy of \( P_5 \), a contradiction. This proves (1).

(2) \( \chi(Y) \leq (\omega(G) - 1)f(\lfloor \omega(G)/2 \rfloor) \).

Let \( 1 \leq i \leq k \). Since \( f(\lfloor \omega(G)/2 \rfloor) < \chi(R_i) \leq f(\omega(R_i)) \), and \( f \) is non-decreasing, it follows that \( \omega(R_i) > \omega(G)/2 \). By (1), \( \omega(G[Y_i]) + \omega(R_i) \leq \omega(G) \), and so \( \omega(G[Y_i]) < \omega(G)/2 \). Consequently \( \chi(Y_i) \leq f(\lfloor \omega(G)/2 \rfloor) \), for \( 1 \leq i \leq k \). Choose \( I \subseteq \{1, \ldots, k\} \) minimal such that \( \bigcup_{i \in I} Y_i = Y \). From the minimality of \( I \), for each \( i \in I \) there exists \( y_i \in Y_i \) such that for each \( j \in I \setminus \{i\} \) we have that \( y_i \notin Y_j \); and so the vertices \( y_i \ (i \in I) \) are all distinct. For each \( i \in I \) choose \( r_i \in V(R_i) \). For all distinct \( i, j \in I \), if \( y_i, y_j \) are nonadjacent, then \( r_i-y_i-v-y_j-r_j \) is isomorphic to \( P_5 \), a contradiction. Hence the vertices \( y_i \ (i \in I) \) are all pairwise adjacent, and adjacent to \( v \); and so \( |I| \leq \omega(G) - 1 \). Thus, \( \chi(Y) = \chi(\bigcup_{i \in I} Y_i) \leq (\omega(G) - 1)f(\lfloor \omega(G)/2 \rfloor) \). This proves (2).

All the vertices in \( N \setminus Y \) are adjacent to \( v \), and so \( \omega(G[N \setminus Y]) \leq \omega(G) - 1 \). Moreover, for \( 1 \leq i \leq k \), each vertex of \( R_i \) is adjacent to each vertex in \( Y_i \), and \( Y_i \neq \emptyset \) since \( B \) is connected, and so \( \omega(R_i) \leq \omega(G) - 1 \). Since there are no edges between any two of the graphs \( G[N \setminus Y], R_1, \ldots, R_k \), their union (\( Z \) say) has clique number at most \( \omega(G) - 1 \) and so has chromatic number at most \( f(\omega(G) - 1) \). But \( V(B) \) is the union of \( Y, V(S) \) and \( V(Z) \); and so

\[
\chi(B) \leq f(\omega(G) - 1) + (\omega(G) - 1)f(\lfloor \omega(G)/2 \rfloor) + f(\lfloor \omega(G)/2 \rfloor).
\]

This proves 2.1.

2.2 Let \( \Omega \geq 1 \), and let \( f : \mathbb{Z}_+ \to \mathbb{R}_+ \) be non-decreasing, satisfying the following:

- \( f \) is a binding function for every \( P_5 \)-free graph \( H \) with \( \omega(H) \leq \Omega \); and
- \( f(w - 1) + (w + 2)f(\lfloor w/2 \rfloor) \leq f(w) \) for each integer \( w > \Omega \).

Then \( f \) is a binding function for every \( P_5 \)-free graph \( G \).

**Proof.** We prove by induction on \(|G|\) that if \( G \) is \( P_5 \)-free then \( f \) is a binding function for \( G \). Thus, we may assume that \( G \) is \( P_5 \)-free and \( f \) is near-binding for \( G \). If \( G \) is not connected, or \( \omega(G) \leq \Omega \), it follows that \( f \) is binding for \( G \), so we assume that \( G \) is connected and \( \omega(G) > \Omega \). Let us write \( w = \omega(G) \) and \( m = \lfloor w/2 \rfloor \). If \( \chi(G) \leq f(w) \) then \( f \) is a binding function for \( G \), so we assume, for a contradiction, that:

(1) \( \chi(G) > f(w - 1) + (w + 2)f(m) \).

We deduce that:

(2) Every cutset \( X \) of \( G \) satisfies \( \chi(X) > 2f(m) \).
If some cutset $X$ satisfies $\chi(X) \leq 2f(m)$, then since $\chi(G \setminus X) \leq f(w - 1) + wf(m)$ by 2.1, it follows that $\chi(G) \leq f(w - 1) + (w + 2)f(m)$, contrary to (1). This proves (2).

(3) If $P, Q$ are cliques of $G$, both of cardinality at least $w/2$, then $G[P \cup Q]$ is connected.

Suppose not; then there is a minimal subset $X \subseteq V(G) \setminus (P \cup Q)$ such that $P, Q$ are subsets of different components ($A, B$ say) of $G \setminus X$. From the minimality of $X$, every vertex $x \in X$ has a neighbour in $V(A)$ and a neighbour in $V(B)$. If $x$ is mixed on $A$ and mixed on $B$, then since $A$ is connected, there is an edge $a_1a_2$ of $A$ such that $x$ is adjacent to $a_1$ and not to $a_2$; and similarly there is an edge $b_1b_2$ of $B$ with $x$ adjacent to $b_1$ and not to $b_2$. But then $a_2-a_1-x-b_1-b_2$ is an induced copy of $P_5$, a contradiction; so every $x \in X$ is complete to at least one of $A, B$. The set of vertices in $X$ complete to $A$ is also complete to $P$, and hence has clique number at most $m$, and hence has chromatic number at most $f(m)$; and the same for $B$. Thus, $\chi(X) \leq 2f(m)$, contrary to (2). This proves (3).

If $v \in V(G)$, we denote its set of neighbours by $N(v)$, or $N_G(v)$. Let $a \in V(G)$, and let $B$ be a component of $G \setminus (N(a) \cup \{a\})$; we will show that $\chi(B) \leq (w - m + 2)f(m)$.

A subset $Y$ of $V(B)$ is a joint of $B$ if there is a component $C$ of $B \setminus Y$ such that $\chi(C) > f(m)$ and $Y$ is complete to $C$. If $\emptyset$ is not a joint of $B$ then $\chi(B) < f(m)$ and the claim holds, so we may assume that $\emptyset$ is a joint of $B$; let $Y$ be a joint of $B$ chosen with $Y$ maximal, and let $C$ be a component of $B \setminus Y$ such that $\chi(C) > f(m)$ and $Y$ is complete to $C$.

(4) If $v \in N(a)$ has a neighbour in $V(C)$, then $\chi(V(C) \setminus N(v)) \leq f(m)$.

Let $N_C(v)$ be the set of neighbours of $v$ in $V(C)$, and $M = V(C) \setminus N_C(v)$; and suppose that $\chi(M) > f(m)$. Let $C'$ be a component of $G[M]$ with $\chi(C') > f(m)$, and let $Z$ be the set of vertices in $N_C(v)$ that have a neighbour in $V(C')$. Thus, $Z \neq \emptyset$, since $N_C(v), V(C') \neq \emptyset$ and $C$ is connected. If some $z \in Z$ is mixed on $C'$, let $p_1p_2$ be an edge of $C'$ such that $z$ is adjacent to $p_1$ and not to $p_2$; then $a-v-z-p_1-p_2$ is an induced copy of $P_5$, a contradiction. So every vertex in $Z$ is complete to $V(C')$; but also every vertex in $Y$ is complete to $V(C)$ and hence to $V(C')$, and so $Y \cup Z$ is a joint of $B$, contrary to the maximality of $Y$. This proves (4).

(5) $\chi(Y) \leq f(m)$ and $\chi(C) \leq (w - m + 1)f(m)$.

Let $X$ be the set of vertices in $N(a)$ that have a neighbour in $V(C)$. Since $C$ is a component of $B \setminus Y$ and hence a component of $G \setminus (X \cup Y)$, and $a$ belongs to a different component of $G \setminus (X \cup Y)$, it follows that $X \cup Y$ is a cutset of $G$. By (2), $\chi(X \cup Y) > 2f(m)$. Since $\omega(C) \geq m + 1$ (because $\chi(C) > f(m)$, and $f$ is near-binding for $G$) and every vertex in $Y$ is complete to $V(C)$, it follows that $\omega(G[Y]) \leq w - m - 1 \leq m$, and so has chromatic number at most $f(m)$ as claimed; and so $\chi(X) > f(m)$. Consequently there is a clique $P \subseteq X$ with cardinality $w - m$. The subgraph induced on the set of vertices of $C$ complete to $P$ has clique number at most $m$, and so has chromatic number at most $f(m)$; and for each $v \in P$, the set of vertices of $C$ nonadjacent to $v$ has chromatic number at most $f(m)$ by (4). Thus, $\chi(C) \leq (|P| + 1)f(m) = (w - m + 1)f(m)$. This proves (5).
(6) $\chi(B) \leq (w - m + 2)f(m)$.

By (3), every clique contained in $V(B) \setminus (V(C) \cup Y)$ has cardinality less than $w/2$ (because it is anticomplete to the largest clique of $C$) and so

$$\chi(B \setminus (V(C) \cup Y)) \leq f(m);$$

and hence $\chi(B \setminus Y) \leq (w - m + 1)f(m)$ by (5), since there are no edges between $C$ and $V(B) \setminus (V(C) \cup Y)$. But $\chi(Y) \leq f(m)$ by (5), and so $\chi(B) \leq (w - m + 2)f(m)$. This proves (6).

By (6), $G \setminus N(a)$ has chromatic number at most $(w - m + 2)f(m)$. But $G[N(a)]$ has clique number at most $w - 1$ and so chromatic number at most $f(w - 1)$; and so $\chi(G) \leq f(w - 1) + (w - m + 2)f(m)$, contrary to (1). This proves 2.2.

Now we deduce 1.5, which we restate:

2.3 If $G$ is $P_5$-free and $\omega(G) \geq 3$ then $\chi(G) \leq \omega(G)^{\log_2(\omega(G))}$.

**Proof.** Define $f(0) = 0$, $f(1) = 1$, $f(2) = 3$, and $f(x) = x^{\log_2(x)}$ for every real number $x \geq 3$. Let $G$ be $P_5$-free. If $\omega(G) \leq 2$ then $\chi(G) \leq 3 = f(2)$, by a result of Sumner [25]; if $\omega(G) = 3$ then $\chi(G) \leq 5 \leq f(3)$, by an application of the result 1.4 of Esperet, Lemoine, Maffray, and Morel [9]; and if $\omega(G) = 4$ then $\chi(G) \leq 15 \leq f(4)$, by another application of 1.4. Consequently every $P_5$-free graph $G$ with clique number at most four has chromatic number at most $f(\omega(G))$.

We claim that

$$f(x - 1) + (x + 2)f(\lfloor x/2 \rfloor) \leq f(x)$$

for each integer $x > 4$. If that is true, then by 2.2 with $\Omega = 4$, we deduce that $\chi(G) \leq f(\omega(G))$ for every $P_5$-free graph $G$, and so 1.5 holds. Thus, it remains to show that

$$f(x - 1) + (x + 2)f(\lfloor x/2 \rfloor) \leq f(x)$$

for each integer $x > 4$. This can be verified by direct calculation when $x = 5$, so we may assume that $x \geq 6$.

The derivative of $f(x)/x^4$ is

$$(2 \log_2(x) - 4)x^{\log_2(x)-5},$$

and so is nonnegative for $x \geq 4$. Consequently

$$\frac{f(x - 1)}{(x - 1)^4} \leq \frac{f(x)}{x^4}$$

for $x \geq 5$. Since $x^2(x^2 - 2x - 4) \geq (x - 1)^4$ when $x \geq 5$, it follows that

$$\frac{f(x - 1)}{x^2 - 2x - 4} \leq \frac{f(x)}{x^2},$$

that is,

$$f(x - 1) + \frac{2x + 4}{x^2}f(x) \leq f(x),$$

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when \( x \geq 5 \). But when \( x \geq 6 \) (so that \( f(x/2) \) is defined and the first equality below holds), we have

\[
f([x/2]) \leq f(x/2) = (x/2)^{\log_2(x/2)} = (x/2)^{\log_2(x)} - 1 = (2/x)(x/2)^{\log_2(x)} = (2/x^2)f(x),
\]

and so

\[
f(x - 1) + (x + 2)f([x/2]) \leq f(x)
\]

when \( x \geq 6 \). This proves 2.3.

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