A complete classification of the $(15_4 20_3)$-configurations with at least three $K_5$-graphs

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Abstract

The class of $(\binom{n+1}{2} \binom{n+1}{3})$-configurations which contain at least $n-2$ $K_n$-graphs coincides with the class of so called systems of triangle perspectives i.e. of configurations which contain a bundle of $n-2$ Pasch configurations with a common line. For $n = 5$ the class consists of all binomial partial Steiner triple systems on 15 points, that contain at least three $K_5$-graphs. In this case a complete classification of respective configurations is given and their automorphisms are determined.

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Introduction

It is a quite common research project to characterize (and classify) configurations (more generally: block designs) which contain (or not) subconfigurations in a definite class (comp. [7], [2], [3]). In the case of our paper these are: partial Steiner triple systems with complete graphs ‘inside’.

The minimal size of a partial Steiner triple system i.e. of a $(v, b_3)$-configuration which contains a complete graph $K_n$ is $v = \binom{n+1}{2}$, $b = \binom{n+1}{3}$, $r = n - 1$, so such a minimal configuration PSTS is a binomial configuration.

Generally, a binomial $(\binom{n+1}{2} \binom{n+1}{3})$-configuration may contain $0, 1, \ldots, n - 1, n+1$ graphs $K_n$. All the (minimal) configurations which contain $K_4$ were classified in [8]. These all are well known $10_3$-configurations (comp. [1], [9]).

Binomial (minimal) configurations with $K_5$ are $(15_4 20_3)$-configurations, so each of them may contain $0, 1, 2, 3, 4$ or $6 K_5$-graphs. In this paper we classify all these $(15_4 20_3)$-configurations which contain at least three copies of $K_5$. We prove that there are seventeen such configurations (four of them are so called multi-Veblen configurations MVC with 4 or 6 copies of $K_5$), we present each of these configurations, and we determine the automorphism group of each of them.

As a technical tool to achieve our result we use the representation of our $(15_4 20_3)$-configurations as so-called systems of triangle perspectives STP.
In accordance with the general theory every binomial \( \binom{n+1}{2} - \binom{n+1}{3} \)-configuration with at least \( n - 2 \) copies of \( K_n \) can be represented as a system of triangle perspectives STP (comp. [11]). The number of STP’s grows rapidly; for \( n = 4 \) there are three STP’s, two of them are MVC, for \( n = 5 \) there are seventeen STP’s, three of them are (simple) MVC, for \( n = 6 \) there are seven simple MVC’s (cf. [13]) and at least thirty other STP’s (the classification is not yet completed). The problem to classify all the minimal PST’s with at least \( n - 2 \) \( K_n \)-graphs in each is, theoretically, solved: it is equivalent to classification of all the edge-colorings of the complete digraph on \( n - 2 \) vertices by the elements of the group \( S_3 \). Practically, for \( n > 5 \) this classification cannot be achieved, we think, ‘by hand’ and computer methods must be used.

1 Definitions and the results

1.1 Some (basic) combinatorial configurations

Recall that the term combinatorial configuration (or simply a configuration) is usually (cf. [4], [13], [6]) used for a finite incidence point-line structure with constant line size and point rank provided that two different points are incident with at most one line. A \((v_r, b_k)\)-configuration is a combinatorial configuration with \( v \) points and \( b \) lines such that there are \( r \) lines through each point, and there are \( k \) points on each line. A binomial configuration is a combinatorial \((v, b, k)\)-configuration such that \( v = \binom{r+k-1}{r} \) and \( b = \binom{r+k-1}{k} \). In what follows we shall be primarily interested in binomial partial Steiner triple systems i.e. in the \( \binom{n+1}{2} - \binom{n+1}{3} \)-configurations with arbitrary integer \( n \geq 3 \).

Let \( k \) be a positive integer and \( X \) a set; we write \( \wp_k(X) \) for the set of all \( k \)-subsets of \( X \). A multiset with repetitions \((a \text{ multiset})\) of cardinality \( k \) with elements in the set \( X \) is a function \( f: X \to \mathbb{N} \) such that \( |f| := \sum_{x \in X} f(x) = k < \infty \). We write \( \eta_k(X) \) for the family of all such multisets. Clearly, if \( f \in \eta_k(X) \) then \( \text{supp}(f) = \{x \in X: f(x) \neq 0\} \) is finite. In particular, if \( X \) is finite, we can identify \( f \) with the (formal) polynomial \( f = \prod_{x \in \text{supp}(f)} x^{f(x)} = \prod_{x \in X} x^{f(x)} \) with variables in \( X \); we have \( |\eta_k(X)| = \binom{|X|+k-1}{k} \).

The incidence structure

\[ G_k(X) := (\wp_k(X), \wp_{k+1}(X), \subseteq) \]

will be called a combinatorial Grassmannian (cf. [13]).

The \((m)\text{-th}\) combinatorial Veronesian over \( X \) is the incidence structure

\[ V_m(X) := (\eta_m(X), \mathcal{L}^*, \subseteq) \]

where \( \mathcal{L}^* = \{eX^r : 1 \leq r \leq m, e \in \eta_{m-r}(X)\} \) and \( eX^r = \{ex^r : x \in X\} \) (cf. [12]).

Let \( X \) be a nonempty set and \( |X| = n \); we write \( V_m(n) = V_m(X) \) and \( G_k(n) = G_k(X) \). Let us quote three classical examples: \( G_2(4) \cong V_2(3) \) is the Veblen (Pasch) configuration, \( G_2(5) \) is the Desargues configuration \( \text{DES} \), and \( V_3(3) \) is the \( 10_3G \) Kantor configuration (see [12], [5]). The last one is sometimes called the Veronese configuration ([12]).

Let \( X \) be a non void set. Formally, a nondirected loopless graph defined on \( X \) is a structure of the form \((X, \mathcal{P})\) with \( \mathcal{P} \subset \wp_2(X) \) but in the sequel we shall frequently
identify it with $\mathcal{P}$ and we shall call $\mathcal{P}$ simply a graph. We write $K_X$ and $N_X$ for the complete graph with the vertices $X$ and for its complement, respectively.

Finally, let us recall one of the definitions of a multiveblen configuration (cf.\cite{bib:3}). Let $\mathcal{P}$ be a graph defined on $X$, $|X| = n$, and let $\mathcal{S} = (\wp X, \mathcal{L})$ be a partial Steiner triple system. The points of the multiveblen configuration $\mathfrak{M} := \wp(\wp X, \mathcal{L})$ are the following: $p, a_i, b_i$ with $i \in X$, $c_z$ with $z \in \wp X$. The lines of $\mathfrak{M}$ are the following sets:

- $\{a_i, b_j, c_{i,j}\}$ and $\{a_j, b_i, c_{i,j}\}$ for $\{i, j\} \in \wp X \setminus \mathcal{P}$;
- $\{a_i, a_j, c_{i,j}\}$ and $\{b_i, b_j, c_{i,j}\}$ for $\{i, j\} \in \mathcal{P}$;
- $\{p, a_i, b_i\}$, $i \in X$; $\{a_u, c_v, c_w\}$, where $\{u, v, w\}$ is a line of $\mathcal{L}$.

A multiveblen configuration is simple if $\mathcal{S} = G_2(X)$. The point $p$ is a center of $\mathfrak{M}$.

**Fact 1.1.** $V_m(k)$ and $G_{k-1}(n)$ are $\binom{n}{m}$-configurations, where $n = m + k - 1$.

If $\mathcal{S}$ is a $\binom{n}{2}$ configuration then $\wp(\wp X, \mathcal{L})$ is a $\binom{n+2}{3}$ configuration.

For every distinct $i, j \in X$ the substructure of $\wp(\wp X, \mathcal{L})$ with the points $p, a_i, a_j, b_i, b_j, c_{i,j}$ is the Veblen configuration.

If $|X| = n$ then $\wp(\wp X, G_2)$ contains $G_2(n)$.

Note that combinatorial Grassmannians $G_2(n)$, combinatorial Veronesians $V_k(3)$, and multiveblen configurations defined above are binomial partial Steiner triples.

We say that a configuration $\mathfrak{M} = (S, \mathcal{L})$ freely contains the complete graph $K_X$ if and only if $X \subset S$, for every edge $e \in \wp X$ there is a unique block $\tau \in \mathcal{L}$ that contains $e$, and any two edges $e_1, e_2 \in \wp X$ such that $\overline{e_1} \cap \overline{e_2} \neq \emptyset$ have a common vertex in $X$.

### 1.2 Systems of triangle perspectives

In the paper we shall consider also configurations defined by the following construction, which proposes another approach to “gluing” Veblen configurations, more general than the notion of a multiveblen configuration.

**Construction 1.2.** Let $I$ be an $n$-element set, $\mathcal{T} := \{a, b, c\}$, and $I \cap \mathcal{T} = \emptyset$.

Moreover, adopt the convention $X \in \{A, B, C\}$ ($X = A$ if $x = a$ etc., where $x \in \mathcal{T}$). Let $n$ Veblen configurations $\mathcal{V}$ labelled by the elements $i \in I$,

$$\mathcal{V} = \{q^a, q^b, q^c, a_i, b_i, c_i, \{L, A_i, B_i, C_i\}, i \in I\},$$

have a common line $L = \{q^a, q^b, q^c\}$, let $q^a \mapsto X_i, x_i \not\in X_i$ for $i \in I$, $x \in \mathcal{T}$.

Let perspectives $\xi^{i,j}: \mathcal{V}^i \rightarrow \mathcal{V}^j$ with centers $q^{i,j}$ be given for distinct $i, j \in I$; then the triples $\{q^{i,j}, x_i, y_j\}$ $(x, y \in \mathcal{T})$ for $y_j = \xi^{i,j}(x_i), i, j \in I$ are considered as “perspective rays”.

In other words, let $\xi$ be a map $\mathcal{I}: I \times I \rightarrow \mathcal{S}_\mathcal{T}$ such that $\xi(i, i) = \id$ and $\xi(j, i) = (\xi(i, j))^{-1}$ for $i, j \in I$; then $\xi^{i,j}(x_i) = (\xi(i, j)(x_i))_j$ and $q^{i,j}$ are abstract “new” points such that the perspective rays have form $\{q^{i,j}, x_i, \xi^{i,j}(x_i)\}$ for all distinct $i, j \in I$ and $x \in \mathcal{T}$. Finally, let $\mathcal{S} = (\wp(\mathcal{V}), L_0')$ be a $\binom{n}{2}$ configuration and let $L_0$ be the image of the family $L'_0$ under the map $\wp \mathcal{V}$.\footnote{The term “perspective” used here may be slightly misleading; $\xi^{i,j}$ is not any “real” perspective of $\mathcal{V}$ onto $\mathcal{V}^j$, as the latter should fix the points on $L$, which yields $\xi(i, j) = \id$. Therefore, in the sequel we prefer to use the term “perspective of triangles.”}
The union of the configurations \( V^i, i \in I \), the points \( q^{ij}, \{i, j\} \in \mathcal{S}_2(I) \), the perspective rays, and the lines \( L_0 \) will be denoted by \( \mathcal{P}_{\mathcal{F}_0} \) and will be called a system of triangle perspectives. The line \( L \) is called its basis.

A system of triangle perspectives is simple if \( \mathcal{F} = G_2(I) \).

In the sequel in most parts we shall adopt simply \( I = \{1, \ldots, n\} \) and we shall consider, mainly, simple STP’s.

Let \( \rho \in S_7 \) be a “rotation”: \( \rho(a) = b, \rho(b) = c, \rho(c) = a \). Let us write \( \sigma_x \) for the map \( \sigma_x \in S_7 \) such that \( \{x, y, \sigma_x(y)\} = \mathcal{F} \) for any distinct \( x, y \in \mathcal{F} \). Evidently, \( \xi(i, j) \in \{id, \rho, \rho^{-1}, \sigma_x : x \in \mathcal{F}\} \) for every \( i, j \in I \). It is clear that \( \mathcal{P}_{\mathcal{F}_0} \) is a \( \binom{n+3}{2} - \text{configuration, where } n = |I| \). Note that if \( n = 3 \) then \( \mathcal{F} \) is the line \( G_2(3) \) and thus corresponding STP’s are simple.

1.3 Main results

Recall a known classifying theorem:

**Theorem** (see [8]). Let \( \mathcal{M} \) be a \((10_3 10_3)\)-configuration (i.e. let it be a binomial \( \binom{n+1}{2} - \text{configuration with } n = 4 \)). Assume that \( \mathcal{M} \) freely contains at least 2 (= \( n - 2 \)) graphs \( K_n \). Then either \( \mathcal{M} \) is the Desargues Configuration \( G_2(5) \) with \( n + 1 = 5 \) subgraphs \( K_4 \), or it is the Kantor \( 10_3 \) configuration with \( n - 1 = 3 \) subgraphs \( K_4 \), or it is the six configuration with \( n - 2 \) subgraphs \( K_4 \). No \( 10_3 \)-configuration freely contains a \( K_5 \)-graph.

In our investigations the following theorem, which directly follows from a general theorem [11] Thm. 2.20, Cor. 2.21], is crucial

**Theorem.** Let \( \mathcal{M} \) be a \((15_4 20_3)\)-configuration (i.e. let it be a binomial \( \binom{n+1}{2} - \text{configuration with } n = 5 \)). Assume that \( \mathcal{M} \) freely contains at least 3 (= \( n - 2 \)) graphs \( K_n \). Then either \( \mathcal{M} \) is the combinatorial Grassmannian \( G_2(6) \) with \( 6 = n + 1 \) graphs \( K_5 \), or it is a simple multiveblen configuration on 15 points, with at least 4 = \( n - 1 \) graphs \( K_5 \), or it is a (simple) system of triangle perspectives on 15 points. No such a configuration freely contains a \( K_6 \)-graph.

On the other hand, the simple multiveblen configurations on 15 points are all known:

**Theorem** (see [13] Thm. 4]). Up to an isomorphism, there are exactly three simple multiveblen configurations on 15 points; these are the following:

\[ \mathcal{M}_{K_5}^J \cong G_2(6), \quad \mathcal{M}_{N_5}^J \cong \mathcal{P}_4 \] (cf. [13] for a definition of the latter structure), and \( \mathcal{M}_{L_5}^J \cong G_2(J) \),

where \( L_J \) is a linear graph on \( J = \{1, 2, 3, 4\} \).

The aim of this paper is to prove the following complete classification of all the \((15_4 20_3)\)-configurations which freely contain at least three graphs \( K_5 \):
Theorem (main). There are exactly seventeen binomial partial Steiner triple systems on 15 points that freely contain at least three graphs $K_5$ each. These are systems $\mathcal{P}_{15,20_3}G_2(I)$ with $I = \{1, 2, 3\}$ of triangle perspectives determined by the following triples $(\xi(1, 2), \xi(2, 3), \xi(1, 3))$:
\[
(\rho, \rho, \rho), (\rho, \rho, \rho), (\rho, \rho, \rho), (\rho, \rho, \rho), (\rho, \rho, \rho), (\rho, \rho, \rho), (\rho, \rho, \rho), (\rho, \rho, \rho), (\rho, \rho, \rho), (\rho, \rho, \rho), (\rho, \rho, \rho), (\rho, \rho, \rho), (\rho, \rho, \rho), (\rho, \rho, \rho), (\rho, \rho, \rho), (\rho, \rho, \rho), (\rho, \rho, \rho), (\rho, \rho, \rho), \]

$x, y, z$ are arbitrary, with $\{x, y, z\} = \{a, b, c\}$.

The last three triples determine a PSTS with (at least) four graphs $K_5$, and among them the last one contains six graphs $K_5$.

Some remarks on specific features of the geometry of corresponding configurations together with characterizations of their automorphism groups are given in the next section, where, in several steps we prove our main result.

2 Reasoning

Let us adopt the notation of Subsection 1.2. We write $p \sim p’$ when points $p, p’$ are collinear.

2.1 Relations between multiveblen configurations and systems of triangle perspectives

Systems of triangle perspectives generalize multiveblen configurations: every simple multiveblen configuration is a simple system of triangle perspectives. More precisely, we have the following

Proposition 2.1. Let $I = \{1, \ldots, n\}, p = \{n+1, n+2\}, \mathcal{P}$ be a graph defined on the set $I \cup \{0\} =: I’$, and $\mathcal{S}’$ be a $(\binom{n+1}{2}, \binom{n+4}{3})$-configuration defined on the point-set $\mathcal{P}_2(I’)$ such that $\{(0, i), (0, j), (i, j)\}$ is a line of $\mathcal{S}’$ for every $i, j \in \mathcal{P}_2(I)$. Let $\mathcal{S}_I$ be the subconfiguration of $\mathcal{S}_I’$ with the point set $\mathcal{P}_2(I)$. Then there is a map $\xi: I \times I \rightarrow S_I$ such that $\mathcal{N}_{\mathcal{P}_2(I’)}^{p’} \mathcal{S}_I’ \cong \mathcal{P}_{\mathcal{P}_2(I’)}\mathcal{S}_I’$.

Consequently, for every $\mathcal{P}$ as above, $\mathcal{N}_{\mathcal{P}_2(I’)}^{p’}G_2(I’) \cong \mathcal{P}_{\mathcal{P}_2(I’)}\mathcal{S}_I’$.

Proof. In the first step we note that there is a graph $\mathcal{P}’$ such that $\{0, i\} \in \mathcal{P}’$ for every $i \in I$ and $\mathcal{N}_{\mathcal{P}_2(I’)}^{p’} \mathcal{S}_I’ \cong \mathcal{N}_{\mathcal{P}_2(I’)}^{p} \mathcal{S}_I’$. For two graphs $\mathcal{P}_1, \mathcal{P}_2$ on $I’$ and $i_0 \in I’$ we write $\mathcal{P}_2 = \mu_{i_0}(\mathcal{P}_1)$ iff the two conditions hold: $\{i_0, j\} \in \mathcal{P}_2$ if $\{i_0, j\} \notin \mathcal{P}_1$ for $j \neq i_0$, and $\{i, j\} \in \mathcal{P}_2$ if $\{i, j\} \in \mathcal{P}_1$ for $j, i \neq i_0$. Let $\{i_1, \ldots, i_s\} = \{i \in I : \{0, i\} \notin \mathcal{P}\}$ and let $\mathcal{P}’ = \xi: I \times I \rightarrow S_I$ such that $\mathcal{N}_{\mathcal{P}_2(I’)}^{p’} \mathcal{S}_I’ \cong \mathcal{N}_{\mathcal{P}_2(I’)}^{p} \mathcal{S}_I’$. Define $\xi(i, j)(c) = c$ and $\xi(i, j)(a, b) = \begin{cases} (a, b) & \text{if } \{i, j\} \notin \mathcal{P}’ \\ (b, a) & \text{if } \{i, j\} \notin \mathcal{P}’ \end{cases}$ for all $\{i, j\} \in \mathcal{P}_2(I)$. It is seen that the following relabelling $q^f \mapsto p, q^b \mapsto a_0, q^a \mapsto b_0, a_i \mapsto a_i, b_i \mapsto b_i, c_i \mapsto c_{i(0, i)}$ for $i \in I$, and $q^{1, i} \mapsto c_{i(1, i)}$ for $\{i, j\} \in \mathcal{P}_2(I)$ establishes an isomorphism of $\mathcal{P}_{\mathcal{P}_2(I’)}\mathcal{S}_I’$ onto $\mathcal{N}_{\mathcal{P}_2(I’)}^{p’} \mathcal{S}_I’$. \boxed
Let us make some comments on the idea of the proof of [2.1]. Note that, in view of [1.2], the role of the points \( q^a, q^b, q^c \) on the line \( L \) is symmetric. Observing the proof we note that the point \( q^a \) was chosen in \( \mathcal{P}_{\xi}^p \mathcal{S} \) so that it appears to be the center of the given \( \mathcal{N}_{I^{P_P}, \mathcal{S}'} \). But this center can be chosen arbitrary on \( L \) and thus, reversing the reasoning of that proof we obtain

**Proposition 2.2.** Let \( I, I', p \) be as in [2.1]; let \( \mathcal{S} \) be a \( \left( \binom{n}{2}_{n-2}, \binom{n}{3}_{3}\right) \)-configuration on the pointset \( \mathcal{V}_2(I) \). Assume that \( \xi : I \times I \to \mathcal{S}_I \) fixes some \( x \in \mathcal{T} \) i.e. \( \xi_{i,j}(x) = x \) for all \( i, j \). Then there is a graph \( \mathcal{P} \) on \( I' \) and a \( \left( \binom{n+1}{2}_{n-1}, \binom{n+1}{3}_{3}\right) \)-configuration \( \mathcal{S}' \) extending \( \mathcal{S} \) such that \( \mathcal{P}_{I'}^p \mathcal{S} \cong \mathcal{N}_{I^{P_P}, \mathcal{S}'} \).

In particular, for every such \( \xi \) there is \( \mathcal{P} \) with \( \mathcal{P}_{I'}^p \mathcal{G}_2(I) \cong \mathcal{N}_{I^{P_P}, \mathcal{S}'} \).

However, there are systems of triangle perspectives that are not multivedeben configurations (see [2.9]–[2.10]).

### 2.2 Systems of triangle perspectives and their characteristic subconfigurations

We start with determining Desargues and Veronese subconfigurations contained in a given system of triangle perspectives. In essence, we shall determine such subconfigurations with one of their lines being the distinguished base line \( L \). Let us begin with some observations.

**Representation 2.3.** Let us consider a particular case \( I = \{1, 2\} \) and \( \mathcal{E} = \mathcal{P}_{I^{P_P}, \mathcal{S}} \), where \( \mathcal{S} = \mathcal{G}_2(I) \) is a trivial structure consisting of one single point. Clearly, \( \mathcal{E} \) is a \((20_3)\)-configuration. Then \( \xi = \xi(1, 2) \in \mathcal{S}_I \) is one of the following three types.

(i; \( \xi = \text{id} \)) Then \( \mathcal{E} \) is the classical Desargues configuration \( \text{DES} \).

(ii; \( \xi = \sigma_{x} \)) Then \( \mathcal{E} \) is the Veronese configuration \( \mathcal{V}_3(3) \).

(iii; \( \xi = \rho \)) Then \( \mathcal{E} \) is another cousin of the \( \text{DES} \)-configuration, visualized in Figure 3. We shall call it the \( \text{DES}'' \)-configuration (it is sometimes called the fez configuration, see [8]).

Note (comp. [8]) that through (i)–(iii) we have shown all the possible \((20_3)\)-configurations that can be presented as a perspective of two triangles.

Now, let \( I \) be arbitrary and \( \mathfrak{N} = \mathcal{P}_{I^{P_P}, \mathcal{S}} \). For any distinct \( i', i'' \in I \) the two Veblen subconfigurations \( \mathcal{V}_{i'}, \mathcal{V}_{i''} \) of \( \mathfrak{N} \) with the common line \( L \) completed by the point \( q^{i',i''} \) yield a \((20_3)\)-subconfiguration \( \mathfrak{N}_{i',i''} \) and \( \mathfrak{N}_{i',i''} \cong \mathcal{E} \), where \( \mathcal{E} \) is one of the three introduced through (i)–(iii) above; namely: \( \text{DES} \) for \( \xi(i', i'') = \text{id} \), \( \text{DES}' \) for \( \xi(i', i'') = \sigma_x, x \in \mathcal{T} \), and \( \text{DES}'' \) for \( \xi(i', i'') = \rho, \rho^{-1} \).

Finally, comparing the Pasch subconfigurations of \( \mathfrak{N} \) and \( \mathcal{E} \) we can justify that, conversely, if \( \mathcal{E} \) is a substructure of \( \mathfrak{N} \) such that \( L \) is a line of \( \mathcal{E} \) and \( \mathcal{E} \) is isomorphic to one of (i)–(iii) then \( \mathcal{E} \) is spanned by \( \mathcal{V}_{i'}, \mathcal{V}_{i''} \) for some \( i', i'' \in I \).

The following is immediate (it can be even stated without proof, we think), though it is quite central in our investigations.
Lemma 2.4. Let $I = \{1, \ldots, m\}$, $\xi : I \times I \rightarrow S_3$, and $\mathfrak{M} = \mathcal{P}_{P_{\xi}}G_2(I) = \langle S, \mathcal{L} \rangle$ be an STP. Then

(i) the set
$$K_i = \{a_i, b_i, c_i\} \cup \{q_{i,j} : j \in I \setminus \{i\}\}$$
is a $K_{m+2}$-graph freely contained in $\mathfrak{M}$ for each $i \in I$.

Recall that $\mathfrak{M}$ is a binomial $\binom{m+3}{2}_{m+1} \binom{m+3}{3}$ partial Steiner triple system, so the maximal size of a complete subgraph of $\mathfrak{M}$ is $n = m + 2$.

Assume, moreover, that $\mathfrak{M}$ freely contains exactly $n - 2 = m$ graphs $K_n$.

(ii) The family $\mathcal{K} = \{K_i : i \in I\}$ consists of the complete $K_n$-graphs freely contained in $\mathfrak{M}$.

(iii) $\Omega := \{q^{i,j} : \{i, j\} \in \mathcal{P}_2(I)\} = \{\mathcal{K}' \cap \mathcal{K}'' : \mathcal{K}', \mathcal{K}'' \in \mathcal{K}, \mathcal{K}' \neq \mathcal{K}''\}$, so $\Omega$

Points in a row are the points of a triangle of one of the Veblen figures with common line $L$ that are not on $L$; points in a column correspond each to other in that they are collinear with the same points on $L$. Lines indicate that the respective points (from distinct triangles) are collinear.

Table 1: The diagram of the line $L$ in the case (viii) of 2.8.
remains distinguishable in M as well.

(iv) \( L = \bigcap \{ S \setminus K : K \in \mathcal{K} \} \) and therefore \( L \) is distinguishable in \( M \).

(v) Set \( \mathcal{D}_x = \{ x_i : i \in I \} \) for \( x \in \mathcal{T} \). Then \( \mathcal{D}_x = \{ a \in S : a \sim q^x, a \notin Q \} \), so the family \( \mathcal{D} = \{ \mathcal{D}_x : x \in \mathcal{T} \} \) is distinguishable in terms of the structure \( M \).

Consequently, the set \( L \) and the families \( \mathcal{K}, \mathcal{Q}, \) and \( \mathcal{D} \) remain invariant under the automorphisms of \( M \).

**Corollary 2.5.** Assume that for every \( x \in \mathcal{T} \) there is \( \{ i, j \} \in \mathcal{Q}_2(I) \) such that \( \xi(i,j)(x) \neq x \) (cf. \( \mathcal{I}, \mathcal{K} \)). Then the structure \( \mathcal{P}_n \mathcal{G}_2(I) \) is not a simple multiveblen configuration.

**Proof.** Suppose that \( \mathfrak{M} := \mathcal{P}_n \mathcal{G}_2(I) \) contains one more complete \( K_n \)-graph \( Y \).

Here, we use some observations of \([11]\). For each \( i, j \) \( \mathcal{M} \)-automorphisms of \( \mathcal{D} \) remains distinguishable in \( \mathcal{T} \) perspectives \( \mathcal{I} \).

Let \( \mathcal{T} \subseteq \mathcal{I} \cap \mathcal{K}_i, Y \cap \mathcal{K}_j, \mathcal{K}_i \cap \mathcal{K}_j \) are on a line for any two distinct \( i, j \in I \). This means that \( \{ z_i, z_j, q^{ij} \} \) is a line of \( \mathfrak{N} \) i.e. \( \xi(i,j)(z) = z \) for all \( i, j \).

This contradicts the assumptions. \( \square \)

One more evident observation will be also useful.

**Lemma 2.6.** Let \( \alpha \in \mathcal{S}_I, \beta \in \mathcal{S}_T, \) and \( \xi \) be a matrix defining a system of triangle perspectives \( \mathcal{P}_n \mathcal{G}_2(I) \). Let \( \mathfrak{R} \) be the image of \( \mathfrak{S} \) under the map \( \{ i, j \} \mapsto \{ \alpha(i), \alpha(j) \} \) defined on \( \mathcal{Q}_2(I) \); finally, let a matrix \( \zeta \) be defined by \( \zeta(i,j) = \beta \circ \xi(\alpha^{-1}(i), \alpha^{-1}(j)) \circ \beta^{-1}, \) for all \( i, j \in I \).

Then the following map \( F \),

\[
F(x_i) = (\beta(x))_{\alpha(i)}, \quad F(q^x) = q^{\beta(x)}, \quad F(q^{ij}) = q^{\alpha(i), \alpha(j)}, \quad x \in \mathcal{T}, \; i, j \in I,
\]

is an isomorphism of \( \mathcal{P}_n \mathcal{G}_2(I) \) onto \( \mathcal{P}_n \mathcal{R} \). The map \( F \) will be denoted by \( \beta \times \alpha \).

As an immediate consequence we get

**Lemma 2.7.** Let \( \alpha \in \mathcal{S}_I \) and \( \beta \in \mathcal{S}_T \). If the map \( F_0: x_i \mapsto \beta(x)_{\alpha(i)} (x \in \mathcal{T}, i \in I) \) preserves the diagram of \( L \) (cf. Table \( \mathcal{I} \)), i.e. \( \xi(i,j)(x) = y \) iff \( \xi(\alpha(i), \alpha(j))(\beta(x)) = \beta(y) \) for all \( x, y \in \mathcal{T}, i, j \in I \) then \( \beta \times \alpha \) is a (unique) automorphism of \( \mathcal{P}_n \mathcal{G}_2(I) \) extending \( F_0 \).

Conversely, if \( F \in \mathbb{A}(\mathcal{P}_n \mathcal{G}_2(I)) \) preserves the line \( L \) then there are \( \alpha \in \mathcal{S}_I \) and \( \beta \in \mathcal{S}_T \) such that \( F = \beta \times \alpha \).

**Proof.** The first statement is evident in view of \( 2.6 \). Let \( F \in \mathbb{A}(\mathcal{P}_n \mathcal{G}_2(I)) \) preserve \( L \). So, \( F \) determines a permutation \( \alpha \) of the Veblen subconfigurations of \( \mathcal{P}_n \mathcal{G}_2(I) \) having \( L \) as one of its lines defined by the condition \( F(V^i) = V^{\alpha(i)} \). Then for every \( i \in I \) there is \( \beta_i \in \mathcal{S}_T \) with \( F(x_i) = \beta_i(x)_{\alpha(i)} \); this gives \( F(q^x) = q^{\beta_i(x)} \) for \( x \in \mathcal{T} \). Thus \( \beta_i = \beta_j =: \beta \) for all \( i, j \in I \) and, finally, \( F = \beta \times \alpha \). \( \square \)
2.3 Classifications

**Classification 2.8.** Let $I = \{1, 2, 3\}$ and $\mathcal{A} = \{a_i, b_i, c_i : i \in I\}$. Recall that $\rho(a,b,c) = (b,c,a)$. Consider the substructure of $\mathfrak{N} := \mathfrak{F}_{\rho_2}\mathbb{G}_2(I)$ determined by $\mathcal{A}$. In the following list we indicate types of possible triples $\xi(1,2), \xi(2,3), \xi(1,3)$.

(i): $\rho, \rho, \rho$ \quad Let $\xi(1,2) = \xi(2,3) = \xi(1,3) = \rho$. Then $\mathcal{A}$ is the 9-gon $(a_1, b_2, c_3, b_1, c_2, a_3, c_1, a_2, b_3)$.

(ii): $\rho, \rho, \text{id}$ \quad Set $\xi(1,2) = \rho = \xi(2,3)$ and $\xi(1,3) = \text{id}$. Then $\mathcal{A}$ is the 9-gon $(a_1, b_2, c_3, a_2, b_3, b_1, c_1, a_3)$.

(iii): $\rho, \text{id}, \text{id}$ \quad Let $\xi(1,2) = \rho$, $\xi(1,3) = \xi(2,3) = \text{id}$. Then $\mathcal{A}$ is the 9-gon $(a_1, b_2, b_3, c_1, a_2, c_3, a_3, c_1, a_2, b_3)$.

(iv): $\rho, \rho, \rho^{-1}$ \quad Set $\xi(1,2) = \rho = \xi(2,3) = \xi(3,1)$. Then $\mathcal{A}$ consists of three triangles $(a_1, b_2, c_3), (b_1, c_2, a_3), \text{and } (a_1, a_2, b_3)$.

(v): $\rho, \rho^{-1}, \text{id}$ \quad Set $\xi(1,2) = \rho = \xi(3,2)$ and $\xi(1,3) = \text{id}$. Then $\mathcal{A}$ consists of three triangles $(a_1, b_2, a_3), (b_1, c_2, b_3), \text{and } (a_1, a_2, c_3)$.

(vi): $\sigma_x, \sigma_y, \sigma_z$ \quad Let $\xi(1,2) = \sigma_a, \xi(1,3) = \sigma_c, \text{and } \xi(2,3) = \sigma_b$. Then $\mathcal{A}$ is the union of the hexagon $(a_2, c_3, b_1, b_3, a_1)$ and the triangle $(c_2, b_1, a_3)$.

(vii): $\sigma_x, \sigma_y, \text{id}$ \quad Set $\xi(1,2) = \sigma_a, \xi(2,3) = \sigma_c, \text{and } \xi(1,3) = \text{id}$. Then $\mathcal{A}$ is the 9-gon $(a_1, a_2, b_3, b_1, c_2, c_1, b_2, a_3)$.

(viii): $\sigma_x, \rho, \text{id}$ \quad Set $\xi(1,2) = \sigma_a, \xi(2,3) = \rho, \text{and } \xi(1,3) = \text{id}$. Then $\mathcal{A}$ is the union of the hexagon $(a_1, a_2, b_3, b_1, c_2, a_3)$ and the triangle $(c_1, b_1, a_3)$.

(ix): $\sigma_x, \sigma_y, \rho$ \quad Set $\xi(1,2) = \sigma_a, \xi(2,3) = \sigma_c, \text{and } \xi(1,3) = \rho$. Then $\mathcal{A}$ consists of three triangles $(a_1, a_2, b_3), (b_1, c_2, c_3), \text{and } (c_1, b_2, a_3)$.

(x): $\sigma_x, \sigma_y, \rho^{-1}$ \quad Set $\xi(1,2) = \sigma_a, \xi(2,3) = \sigma_c, \text{and } \xi(3,1) = \rho$. Then $\mathcal{A}$ is the 9-gon $(a_1, a_2, b_3, c_1, b_2, c_1, c_2, a_3)$.

(xi): $\sigma_x, \sigma_y, \sigma_z$ \quad Set $\xi(1,2) = \sigma_a = \xi(2,3) = \xi(1,3) = \sigma_a$. Then $\mathcal{A}$ consists of the hexagon $(b_1, a_2, b_3, c_1, c_2, c_3)$ and the triangle $(a_1, b_2, a_3)$.

(xii): $\sigma_x, \sigma_z, \rho$ \quad Set $\xi(1,2) = \sigma_a = \xi(2,3) = \xi(1,3) = \rho$. Then $\mathcal{A}$ is the 9-gon $(a_1, b_2, a_3, c_1, c_2, b_3, b_1, a_2, b_3)$.

(xiii): $\rho, \rho, \sigma_z$ \quad Set $\xi(1,2) = \rho = \xi(2,3) = \xi(1,3) = \sigma_c$. Then $\mathcal{A}$ consists of the hexagon $(a_1, b_2, c_3, c_1, a_2, b_3) \text{ and the triangle } (b_1, c_2, a_3)$.

(xiv): $\rho, \rho^{-1}, \sigma_z$ \quad Set $\xi(1,2) = \rho, \xi(2,3) = \rho^{-1}$ and $\xi(1,3) = \sigma_c$. Then $\mathcal{A}$ consists of the hexagon $(a_1, b_2, a_3, c_1, b_2, c_3)$ and the triangle $(c_1, a_2, c_3)$.

From [2,3] we get that in every of the above cases $\mathfrak{N}$ is not a simple multiveblen configuration. The numbers of DES, DES', and DES"-configurations containing $L$ and contained in respective STP’s, together with other important parameters are presented in Table 2. \hfill \Box

**Proposition 2.9.** Let $|I| = 3$. The list $[i] - [xv]$ of [2.8] exhausts all the possible systems $\xi$ (up to permutations of $I$ and of $I$) such that $\mathfrak{F}_{\rho_2}\mathbb{G}_2(I)$ is not a simple multiveblen configuration (cf. [2.2, 2.3]).

The structures in this list are pairwise nonisomorphic. Consequently, every $(15_4, 20_3)$-system of triangle perspectives that is not a simple multiveblen configuration is isomorphic to exactly one of the structures defined through $[i] - [xv]$ of [2.8].
Proof. Any system $\xi$ is determined by three maps: $\xi(1,2), \xi(2,3), \xi(1,3)$, each one in one of the following classes $P = \{\rho, \rho^{-1}\}$, $\Sigma = \{\sigma_a \sigma_b, \sigma_c\}$, and $\Delta = \{\text{id}\}$. Thus (cf. 2.6) the type of $\xi$ can be represented by a 3-multiset with elements in the above three classes. Let us take into account three observations. Firstly, the type $3 \times P$ represents both $2 \times \rho + \rho^{-1}$ and $3 \times \rho$. Secondly, $3 \times \Sigma$ represents $\sigma_x_1, \sigma_x_2, \sigma_x_3$ where either the $x_i$ are distinct or some of them coincide (similar remark concerns $2 \times P$ and $2 \times \Sigma$). Thirdly, $\sigma_x \sigma_y = \rho$ or $\sigma_x \sigma_y = \rho^{-1}$; the arising configurations are distinguishable by 2.4 (look at the diagram of $L$ in corresponding structures). Then canceling types which, in view of 2.2 produce a multiveblen configuration (e.g. $(\sigma_x, \sigma_x, \text{id})$, which fixes $x$) we obtain the list considered in 2.8.

We write $N_n$ for the structure defined in (n) of 2.8 with $n = 1, ..., 14$. To prove that these structures are pairwise nonisomorphic note, first, that if two of them would be isomorphic then the numbers of their DES, DES', and DES"-configurations having $L$ as one of its lines should agree. If these numbers agree, one compares the type of polygons $A$ (i.e. the diagrams of $L$). The above procedure does not distinguish structures in the following three pairs only: $((vi),(xi))$, $((xi),(xii))$, and $((xiii),(xiv))$.

Let us consider the pair $N_{\text{evin}}$, $N_{\text{eviv}}$. Suppose that $F$ is an isomorphism of $N_{\text{evin}}$ onto $N_{\text{eviv}}$. Then $F$ maps the line $L$ onto itself. Moreover, $F$ maps the triangle (unique in the diagram of $L$) $\Delta_1 = (c_1, a_2, c_3)$ of $N_{\text{evin}}$ onto the triangle (analogously the unique one) $\Delta_2 = (b_1, c_2, a_3)$ of $N_{\text{eviv}}$. The triangle $\Delta_2$ crosses each set in the family $D$ defined over $N_{\text{evin}}$ while $\Delta_1$ misses $D_b$ of $N_{\text{eviv}}$. The respective families $D$ should be preserved by isomorphisms, so no isomorphism may map $\Delta_1$ onto $\Delta_2$ and thus $N_{\text{evin}} \not\cong N_{\text{eviv}}$.

Let us consider the pair $N_{\text{evii}}$, $N_{\text{eviii}}$. Suppose that $F$ is an isomorphism of the respective structures. Then $F$ maps the triangle $\Delta_1 = (c_2, b_1, a_3)$ of $N_{\text{evii}}$ onto the triangle $\Delta_2 = (a_1, b_2, a_3)$ of $N_{\text{eviii}}$. The reasoning as above shows, that no isomorphism may map $\Delta_1$ onto $\Delta_2$ and, consequently, $N_{\text{evii}} \not\cong N_{\text{eviii}}$.

Finally, let us consider the pair $N_{\text{evix}}$, $N_{\text{evx}}$. The 9-gon $A$ of $N_{\text{evix}}$ contains three consecutive points in one element of the family $D$ (namely: the sequence $(c_1, c_2, c_3)$ in $D_c$), while no such subsequence can be found in the 9-gon around $L$ in $N_{\text{evx}}$. Therefore $N_{\text{evix}} \not\cong N_{\text{evx}}$.

\begin{table}[h]
\centering
\begin{tabular}{|c|ccccccccccccc|}
\hline
the number of & \text{I} & \text{ii} & \text{iii} & \text{iv} & \text{v} & \text{vi} & \text{vii} & \text{viii} & \text{ix} & \text{x} & \text{xi} & \text{xii} & \text{xiii} & \text{xiv} \\
\hline
\text{DES-subconfigurations} & 0 & 1 & 2 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\text{DES'-subconfigurations} & 0 & 0 & 0 & 0 & 3 & 2 & 1 & 2 & 2 & 3 & 2 & 1 & 1 & 1 \\
\text{DES"-subconfigurations} & 3 & 2 & 1 & 3 & 2 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 2 \\
\hline
\text{triangles around } L & 0 & 0 & 0 & 3 & 3 & 1 & 0 & 1 & 3 & 0 & 1 & 0 & 1 & 1 \\
\text{hexagons around } L & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 \\
\text{9-gons around } L & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\
\hline
\end{tabular}
\caption{Parameters of $(15_4 20_3)$-systems of triangle perspectives.}
\end{table}

\textbf{Remark 2.10.} Note that the three $(15_4 20_3)$ simple multiveblen configurations that are systems of triangle perspectives can be associated (as in 2.8) with the following...
triples \((\xi(1,2), \xi(2,3), (\xi(1,3))) (J = \{1,2,3,4\})\):

(i: id, id, id) \(\mathbb{W}^p_{p_{K_J}} G_2(J) \cong G_2(6)\),

(ii: \(\sigma_x, \sigma_x, \sigma_x\) \(\mathbb{W}^p_{p_{N_J}} G_2(J) \cong \mathbb{P}(4)\) (cf. [10] for a definition of the latter structure),

(iii: \(\sigma_x, \sigma_x, \text{id}, \text{id}, \text{id}, \sigma_x\) \(\mathbb{W}^p_{p_{L_J}} G_2(J)\), where \(L_J\) is a linear graph on \(J\).

**Remark 2.11.** Let \(I = \{1,2,3\}\), \(\xi(1,2) = \text{id}\), \(\xi(2,3) = \rho\), and \(\xi(1,3) = \rho\) (cf. [2,8]). Then \(\mathcal{P}_{\rho} G_2(I) \cong \mathbb{W}^{q_{1,2}, q_{2,3}}\mathfrak{J}\), where \(\mathfrak{J} = \mathcal{I} \cup \{0\}\) and the Veblen configuration \(\mathfrak{J}\) defined on \(\mathfrak{P}(\mathfrak{Z})\) has the lines \(\mathcal{P}_2(\mathcal{J})\), \(\{\{a, b\}, \{b, c\}\}\), \(\{\{0, b\}, \{0, c\}, \{a, c\}\}\), and \(\{\{0, c\}, \{0, a\}, \{a, b\}\}\).

Clearly, \(\mathcal{P}_{\rho} G_2(I)\) is not any simple multiveblen configuration.

**Proof.** Let us write in the definition of a multiveblen configuration \(j_a\) instead of \(a_j\), and \(j_b\) instead of \(b_j\) \((j\) is an element of the “indexing” set). Next, replace the symbols \(a, b\) used as indices by the symbols ‘1’ and ‘2’ resp. (in particular, this operation relabels \((a_a, b_a, a_b, b_b, a_c, b_c) \mapsto (a_1, a_2, b_1, b_2, c_1, c_2)\)). Finally, it suffices to denote \(p = q_{1,2}^3, 0_1 = q_{1,3}^3, 0_2 = q_{1,3}^5\), write down the points on lines through \(p\), and apply definitions of a multiveblen configuration and [12] Note two examples: the triples \((b_1, c_1, q^a)\) and \((b_2, c_2, q^a)\) are collinear and thus \(q^a = c_{(b,c)}\); the triples \((q_{1,3}^3, c_1, a_3)\) and \((q_{1,3}^5, c_2, a_3)\) are collinear and thus \(a_3 = c_{(0,c)}\); and so on.

**Remark 2.12.** Note that a multiveblen configuration has a center i.e. a point \(p\) such that any two lines through \(p\) yield a Veblen configuration. Considering the lists of 2.8 and 2.10 and taking into account 2.11, we get that exactly four \((15_420_3)\)-multiveblen configurations can be presented as a system of triangle perspectives: those quoted in 2.10 and the one defined in 2.5. Since there are more than four \((15_420_3)\)-multiveblen configurations (cf. [10]) we get that not every multiveblen configuration can be presented as a system of triangle perspectives.

**Remark 2.13.** It is known that every Veronese configuration \(V_k(3)\) freely contains exactly three \(K_{k+1}\)-graphs. In particular, \(V_4(3)\) is a \((15_420_3)\)-configuration with three \(K_5\)-graphs, so it is an STP. Indeed, \(V_4(3)\) is isomorphic to the structure defined in [20].

**Proof.** Let \(\mathfrak{Y} = V_4(\mathcal{J});\) set \(I = \{1,2,3\}\), \(\xi(1,2) = \sigma_a, \xi(2,3) = \sigma_c,\) and \(\xi(1,3) = \sigma_b\). The following relabelling establishes an isomorphism of \(\mathfrak{Y}\) and \(\mathcal{P}_{\rho} G_2(I)\): \(abc \mapsto L, a^2bc \mapsto q^a, ab^2c \mapsto q^b, abc^2 \mapsto q^c, ab^3 \mapsto a_1, a^3b \mapsto b_1, a^2b^3 \mapsto c_1, ac^3 \mapsto a_2, a^2c^2 \mapsto b_2, a^3c \mapsto c_2, b^2c^2 \mapsto a_3, bc^3 \mapsto b_3, b^3c \mapsto c_3, c^4 \mapsto q_{1,2}, c^4 \mapsto q_{1,3}, b^4 \mapsto q_{1,3}, T^4 \mapsto Q\).

**2.4 Automorphisms**

In view of 2.7 and 2.1, if the structure \(\mathcal{P}_{\rho} G_2(I)\) is not a simple multiveblen configuration to determine its automorphism group it suffices to determine automorphisms of the diagram of \(L\) (suitable permutations of its rows and columns).
Proposition 2.14. Let $\mathfrak{H} = \mathbb{P}_f \ast G_2(I)$ be one of the structures defined in 2.8. We write $P$ for the $C_3$-group generated by $\rho$. If $i \in I = \{1, 2, 3\}$ we write $\sigma_i$ for the transposition of the elements in $I \setminus \{i\}$ – similarly we write $\rho$ for the cycle $(2, 3, 1)$ and $R$ for the group generated by $\rho$. The group $\mathfrak{G}$ of automorphisms of $\mathfrak{H}$ consists of all the maps $\beta \times \alpha$ where $\alpha \in S_I$ and $\beta \in S_T$ such that:

- (i): $\alpha = \text{id}$, $\beta = P$ or $\alpha = \sigma_i$, $\beta = \sigma_x$, $x \in T$, where in the corresponding cases $i_0 = 2$, $i_0 = 2$, and $i_0 = 3$; then $\mathfrak{G} \cong C_2 \times C_3 = S_3$.
- (ii): $\alpha \in R$, $\beta \in P$ or $(\alpha, \beta) = (\sigma_i, \sigma_x)$ with $x \in T$, $i \in I$; then $\mathfrak{G} \cong C_2 \times (C_3 \oplus C_3)$.
- (iii): $\alpha \in \{\text{id}, \sigma_2\}$, $\beta \in P$; then $\mathfrak{G} \cong C_2 \oplus C_3$.
- (iv): $(\alpha, \beta) = (\rho^m, \rho^n)$ with $m = 0, 1, 2$ or $(\alpha, \beta) = (\sigma_i, \sigma_x)$ with $(x, i) \in \{(b, 1), (c, 2), (a, 3)\}$; then $\mathfrak{G} \cong S_T = S_3$ (cf. 2.13 and [13, Prop. 5]).
- (v): $(\alpha, \beta) = (\text{id}, \text{id})$ or $(\alpha, \beta) = (\sigma_2, \sigma_{x_0})$, where in the corresponding cases $x_0 = b$, $x_0 = b$, $x_0 = b$, $x_0 = c$, and $x_0 = c$; then $\mathfrak{G} \cong C_2$.
- (vi): $\alpha = \text{id}$, $\beta = \text{id}$.
- (vii)-(xiv): $\alpha \in \{\text{id}, \sigma_2\}$, $\beta = \text{id}$; then $\mathfrak{G} \cong C_2$.

Proof. It is easy to check that in every one of the cases (i)-(xiv) given maps yield automorphisms of the diagram of $L$. We must verify that $\mathfrak{H}$ has no other automorphisms.

We say that a set of points $\{a_1, \ldots, a_{n+1}\}$ is a path of length $n$ if points $a_i, a_{i+1}$ are collinear for all $i = 1, \ldots, n$ and no three consecutive points from this set are on a line.

Let $F = \beta \times \alpha \in \text{Aut}(\mathfrak{H})$. Recall that $F(\mathcal{V}^i) = \mathcal{V}^{\alpha(i)}$. Moreover, if $F(X_i) = Y_j$ then $F(x_i) = y_j$. Let us examine corresponding cases of 2.8.

(i): Clearly, $F$ preserves the 9-gon $A$ in the diagram of $L$; let $f$ be the induced automorphism of $A$. If $f$ is the rotation $\tau_1$ of $A$ on 1 item, then $f(a_1) = b_2$, $f(a_3) = c_1$, so, inconsistently, $c = \beta(a) = b$. If $f$ is the rotation $\tau_2$ on 2 items then $f(a_1) = c_3$, $f(a_3) = a_2$ and again a contradiction arises. Finally, only the rotation $\tau_3$ of $A$ on 3 items and, consequently, the rotation $\tau_6$ on 6 items can be written in the form $\beta \times \alpha$ and then $\alpha = \text{id}$, $\beta \in P$. Now, let $f$ be a reflection in a point $d$ of $A$. Analyzing $A$ we obtain $d = a_2$, $d = b_2$ or $d = c_2$, for $\beta = \sigma_a, \sigma_b, \sigma_c$ respectively, and $\alpha = \sigma_2$.

(ii): First, note that $\alpha(2) = 2$ since there is only one $\text{DES}$-configuration that contains $L$ and this one contains $\mathcal{V}^1$ and $\mathcal{V}^3$. Therefore $F$ leaves the set $H := \{a_2, b_2, c_2\}$ invariant. Consider the 9-gon $A$ in the diagram of $L$; an automorphism of $A$ that preserves simultaneously $H$ is either the rotation on 3 or 6 items ($b_2 \mapsto a_2$ or $b_2 \mapsto c_2$) or the reflection in one of the points of $H$. This gives the claim.

(iii): Note that $A_3, B_3, C_3$ are the unique lines distinct from $L$, which are contained in both of $\text{DES}$-subconfigurations of $\mathfrak{H}$. Therefore, $F$ preserves these lines, so $\alpha(3) = 3$. Moreover, $F$ yields an automorphism $f$ of the 9-gon $A$ i.e. its suitable rotation or symmetry; taking into account the fact that the set $\{a_3, b_3, c_3\}$ must be invariant under $f$ we get the claim.
(iv): It suffices to show that the maps \( f_1 = \sigma_c \times \text{id} \) and \( f_2 = \text{id} \times \sigma_2 \) do not preserve the diagram of \( L \). Indeed, \( b_1 \sim c_2 \) and \( f_1(b_1) = \sigma_c \circ c_2 \neq f_1(c_2) \), and \( b_1 \sim a_3 \) and \( f_2(b_1) = \beta_3 \neq a_1 = f_2(a_3) \).

(v): The unique DES\(^{-}\)-subconfiguration of \( \mathfrak{N} \) containing \( L \) contains \( \mathbb{V}^1 \) and \( \mathbb{V}^3 \) and thus \( F \) maps \( \mathbb{V}^2 \) onto itself; therefore \( \alpha(2) = 2 \) and \( \alpha = \text{id}, \sigma_2 \). Finally, note that for every \( x \in \mathbb{T} \) the diagram of \( L \) contains the triangle \( (x_1, x_3, \rho(x_2)) \) and no triangle \( (x_1, x_3, y_2) \) with \( y \neq \rho(x) \); consequently, \( \beta \neq \sigma_x \).

(vi): It suffices to show that \( F \) leaves the triangle \( (b_1, c_2, a_3) \), the unique one in the diagram of \( L \), invariant.

(vii): Note that the set \( D_x \cap \mathcal{A} \) is either a path of length 2 if \( x = a, c \), or a union of a path of length 1 and the set \( \{b_2\} \) if \( x = b \). Thus \( \beta(b) = b \), and also \( F(b_2) = b_2 \), which gives \( \alpha(2) = 2 \). The point \( a_1 \) is the center of the path \( D_a \cap \mathcal{A} \), and \( c_3 \) is the center of the path \( D_c \cap \mathcal{A} \). Hence, \( F \) fixes each of these two centers or interchanges them. Finally, we get that either \( F = \sigma_b \times \sigma_2 \) or \( F = \text{id} \times \text{id} \).

(viii): Considering substructures of \( \mathfrak{N} \) spanned by \( \mathbb{V}^1 \cup \mathbb{V}^3 \) (that are pair wise non isomorphic) we get that \( \alpha = \text{id} \). Then, since \( F \) leaves the triangle \( (c_1, b_2, c_3) \) invariant, we conclude with \( \beta = \text{id} \).

(ix): The unique DES\(^{-}\)-subconfiguration of \( \mathfrak{N} \) containing \( L \) contains \( \mathbb{V}^1 \) and \( \mathbb{V}^3 \) and thus \( \alpha(2) = 2 \). The unique triangle in the diagram of \( L \) which contains a point of each of \( D_x \) is \( (c_1, b_2, a_3) \), so \( F \) preserves it. Particularly, \( F(b_2) = b_2 \) and the set \( \{c_1, a_3\} \) remains invariant under \( F \). Finally we get that either \( F = \sigma_b \times \sigma_2 \) or \( F = \text{id} \times \text{id} \).

(x): As in (ix) we obtain \( \alpha(2) = 2 \). The set \( D_x \cap \mathcal{A} \) contains no path only for \( x = b \), so \( \beta(b) = b \). Since a path \( \Gamma \) is contained in a set \( D_a \cap \mathcal{A} \), \( x \in \mathbb{T} \) exactly when \( \Gamma = \{a_1, a_2\} \) or \( \Gamma = \{c_2, c_3\} \), either the map \( F \) preserves each of these paths and then \( F = \text{id} \times \text{id} \), or \( F \) interchanges them, that gives \( F = \sigma_b \times \sigma_2 \).

(xi): The set \( D_x \) has two, one, and none common elements with the unique triangle \( (a_1, b_2, a_3) \) in the diagram of \( L \) exactly when \( x = a, x = b, \) and \( x = c \), respectively. Therefore, we get \( \beta = \text{id} \). It means also that, in particular, \( F(b_2) = b_2 \), so \( \alpha = \text{id}, \sigma_2 \).

(xii): The set \( D_x \cap \mathcal{A} \) is a path of length 2 only for \( x = c \), thus \( \beta(c) = c \) and \( F(D_c) = D_c \). The point \( c_2 \) is the center of this path, so \( F(c_2) = c_2 \), and then \( \alpha(2) = 2 \). The unique points from \( \mathcal{A} \setminus D_c \) that are collinear with a point of \( D_c \) are \( a_3, b_1 \). Therefore the set \( \{a_3, b_1\} \) remains invariant under \( F \). Consequently, either \( F = \sigma_c \times \sigma_2 \) or \( F = \text{id} \times \text{id} \).

(xiii): The unique DES\(^{-}\)-subconfiguration of \( \mathfrak{N} \) having \( L \) as its line contains \( \mathbb{V}^1 \) and \( \mathbb{V}^3 \), so \( \alpha(2) = 2 \). Consider the unique triangle \( (b_1, c_2, a_3) \) in the diagram of \( L \); in view of the above, \( F(c_2) = c_2 \) which gives \( \beta(c) = c \). Thus either \( F \) fixes \( b_1, a_3 \) and then \( \beta = \text{id} \) and \( \alpha = \text{id} \), or \( F \) interchanges the points \( b_1, a_3 \), which gives \( \beta = \sigma_c \) and \( \alpha = \sigma_2 \).

(xiv): Consider the unique triangle \( \Delta = (c_1, a_2, c_3) \) in the diagram of \( L \); clearly, \( |\Delta \cap D_a| = 1, |\Delta \cap D_c| = 2 \). Since \( F \) preserves \( \Delta \), \( \beta(a) = a, \beta(c) = c \), so \( \beta = \text{id} \). Therefore, \( F(a_2) = a_2 \). Consequently, \( \alpha(2) = 2 \), so \( \alpha = \text{id}, \sigma_2 \).

In view of 2.7 and 2.4 this closes the proof. \( \square \)
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