KAZHDAN-LUSZTIG BASIS FOR GENERIC SPECHT MODULES

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Abstract. In this paper, we let $H$ be the Hecke algebra associated with a finite Coxeter group $W$ and with one-parameter, over the ring of scalars $A = \mathbb{Z}[q, q^{-1}]$. With an elementary method, we introduce a cellular basis of $H$ indexed by the sets $E_J(J \subseteq S)$ and obtain a general theory of "Specht modules". Our main purpose is to provide an algorithm for $W$-graphs for the "generic Specht module", which associates with the Kazhdan and Lusztig cell (or more generally, a union of cells of $W$) containing the longest element of a parabolic subgroup $W_J$ for appropriate $J \subseteq S$. As an example of applications, we show a construction of $W$-graphs for the Hecke algebra of type $A$.

Preliminaries

Let $W$ be a finite Coxeter group with $S$ the set of simple reflections, and let $H$ be the corresponding Hecke algebra. We use a variation of the definition given in [3], taking $H$ to be an algebra over $A = \mathbb{Z}[q, q^{-1}]$, the ring of Laurent polynomials with integer coefficients in the indeterminate $q$. Then $H$ is an algebra generated by $(T_s)_{s \in S}$ subject to

$$T_s^2 = 1 + (q - q^{-1})T_s$$

$$T_rT_s \cdots T_{m_{rs}} = T_sT_r \cdots$$

$m_{rs}$ factors

$m_{rs}$ factors

(for all $r, s \in S$).

Moreover, $H$ has $A$-basis $\{ T_w \mid w \in W \}$ where $T_w = T_{s_1}T_{s_2} \cdots T_{s_l}$ whenever $s_1s_2 \cdots s_l$ is a reduced expression for $w$, and

$$T_rT_w = \begin{cases} 
T_{sw} & \text{if } \ell(sw) > \ell(w) \\
T_{sw} + (q - q^{-1})T_w & \text{if } \ell(sw) < \ell(w),
\end{cases}$$

(1)

for all $w \in W$ and $s \in S$. We also define $A^+ = \mathbb{Z}[q]$, the ring of polynomials in $q$ with integer coefficients, and let $a \mapsto \overline{a}$ be the involutory automorphism of $A$ such that $\overline{q} = q^{-1}$. This involution on $A$ extends to an involution on $H$ satisfying $T_s = T_s^{-1} = T_s + (q^{-1} - q)$ for all $s \in S$. This gives $\overline{T_w} = T_w^{-1}$ for all $w \in W$. The map $\mathcal{H} \to \mathcal{H}$, $h \mapsto \overline{h}$ is a ring involution such that

$$\sum_{w \in W} a_wT_w = \sum_{w \in W} \overline{a_w}T_w^{-1}, a_w \in A.$$
0.1. Kazhdan-Lusztig basis. There are two types of Kazhdan-Lusztig bases of \( \mathcal{H} \), denoted by \( \{ C_w | w \in W \} \) and \( \{ C'_w | w \in W \} \) in the original article by Kazhdan-Lusztig \cite{3}. It will be technically more convenient to work with the \( C \)-basis. The reason can be seen, for example, in Lusztig \cite[chap.18]{5}. The basis element \( C_w \) is uniquely determined by the conditions that \( C_w = C_w \) and \( C_w \equiv T_w \mod \mathcal{H}_{>0} \), where \( \mathcal{H}_{>0} := \sum_{w \in W} q^A \tau(T_w) \), see \cite{5}. Or more clearly

\[
C_w = T_w + \sum_{y \in W, y < w} p_{y,w} T_y,
\]

where \( \leq \) denotes the Bruhat-Chevalley order on \( W \) and \( p_{y,w} \in q^A^+ \) for all \( y < w \) in \( W \). We write \( y < w \) if \( y \leq w \) and \( y \neq w \).

The polynomials \( p_{y,w} \) are related to the polynomials \( P_{y,w} \) of \cite{3} (the Kazhdan-Lusztig polynomials) by \( p_{y,w}(q) = (-q)^{\ell(w) - \ell(y)} P_{y,w}(q^2) \). That is, to get \( p_{y,w} \) from \( P_{y,w} \) replace \( q \) by \( q^2 \), apply the bar involution, and then multiply by \( (-q)^{\ell(w) - \ell(y)} \).

0.2. Multiplication rules for \( C \)-basis. For \( s \in S, w \in W \), we have

\[
T_s C_w = \begin{cases} 
-q^{-1}C_w, & \text{if } sw < w \\
qC_w + \sum_{y < w, sy < y} \mu(y,w) C_y, & \text{if } sw > w.
\end{cases}
\]

The quantity \( \mu(y,w) \), which is the coefficient of \( q^{(\ell(w) - \ell(y) - 1)} \) in \( P_{y,w} \), is the coefficient of \( q \) in \((-1)^{\ell(w) - \ell(y)} P_{y,w} \). However, since Kazhdan and Lusztig show that \( \mu(y,w) \) is nonzero only when \( \ell(w) - \ell(y) \) is odd, therefore \( \mu(y,w) \in \mathbb{Z} \) can also be described as the coefficient of \( q \) in \(-P_{y,w} \), as above.

The following notion of \( W \)-graph was introduced by Kazhdan and Lusztig in \cite{3}.

**Definition of \( W \)-graph.** Since we have slightly modified the definition of Hecke algebra used in \cite{3}, we are forced to also slightly alter the definition of \( W \)-graph.

We define a \( W \)-graph datum to be a triple \( ( \Gamma, I, \mu ) \) consisting of a set \( \Gamma \) (the vertices of the graph), a function

\[
I : \gamma \mapsto I_\gamma
\]

from \( \Gamma \) to the set of all subsets of \( S \), and a function

\[
\mu : \Gamma \times \Gamma \to \mathbb{Z}
\]

such that \( \mu(\delta, \gamma) \neq 0 \) if and only if \( \{ \delta, \gamma \} \) is an edge of the graph. These data are subject to the requirement that \( A\Gamma \), the free \( A \)-module on \( \Gamma \), has an \( \mathcal{H} \)-module structure satisfying

\[
T_s \gamma = \begin{cases} 
-q^{-1} \gamma, & \text{if } s \in I_\gamma \\
q^\gamma + \sum_{\delta \in \Gamma | s \in I_\delta} \mu(\delta, \gamma) \delta, & \text{if } s \notin I_\gamma,
\end{cases}
\]

for all \( s \in S \) and \( \gamma \in \Gamma \). If \( \tau_s \) is the \( A \)-endomorphism of \( A\Gamma \) such that \( \tau_s(\gamma) \) is the right-hand side of Eq. \( \mathbf{3} \) then this requirement is equivalent to the condition that for all \( s, t \in S \) such that \( st \) has finite order, we require that

\[
\frac{\tau_s \tau_t \tau_s \ldots}{m \text{ factors}} = \frac{\tau_t \tau_s \tau_t \ldots}{m \text{ factors}}.
\]

where \( m \) is the order of \( st \). (Note that the definition of \( \tau_s \) guarantees that \((\tau_s + q^{-1})(\tau_s - q) = 0 \) for all \( s \in S \).)
Lemma 1.1. It is also an easy consequence of [19, Prop. 5.9].

0.3. Cells in W-graphs. Following [3], given any W-graph \( \Gamma \) we define a preorder relation \( \leq \) on \( \Gamma \) as follows: for \( \gamma, \gamma' \in \Gamma \) we say that \( \gamma \leq \gamma' \) if there exists a sequence of vertices \( \gamma = \gamma_0, \gamma_1, \ldots, \gamma_n = \gamma' \) such that for each \( i \) (\( 1 \leq i \leq n \)), we have both \( \mu(\gamma_{i-1}, \gamma_i) \neq 0 \) and \( I_{\gamma_{i-1}} \not\subset I_{\gamma_i} \). We shall refer to \( \leq \) as the Kazhdan-Lusztig preorder on \( \Gamma \).

Let \( \sim \) be the equivalence relation on \( \Gamma \) associated to the Kazhdan-Lusztig preorder; thus \( \gamma \sim \gamma' \) means that \( \gamma \leq \gamma' \) and \( \gamma' \leq \gamma \). The corresponding equivalence classes are called the cells of \( \Gamma \).

In this paper, the preorder \( \leq \) is generated by Kazhdan-Lusztig left preorder [3]: \( x \leq_L y \) if \( C_x \) occurs with nonzero coefficient in the expression of \( T_s C_y \) in the \( C \)-basis, for some \( s \in S \). Their equivalence classes are called left cells, see [3, 5, 11] where right cells and two-sided cells are also defined.

0.4. Left cell module. Let \( \mathcal{C} \) be a left cell or, more generally, a union of left cells of \( W \). We define an \( \mathcal{H} \)-module by \( [\mathcal{C}]_A := \mathcal{J}_C/\mathcal{J}_C \) where

\( \mathcal{J}_C := \langle C_w | w \leq_L z \text{ for some } z \in \mathcal{C} \rangle_A \)

\( \mathcal{J}_C := \langle C_w | w \not\in \mathcal{C}, w \leq_L z \text{ for some } z \in \mathcal{C} \rangle_A \)

are the \( A \)-spanned modules.

This paper is organized as follows. In Sect. 1 we introduce the indexing sets \( D_J, D_J^+ \) for the basis of \( \mathcal{H} \)-module \( \mathcal{H}C_{w_J} \), and \( E_J \) for the so called general Specht module. In Sect. 2, we obtain a version of cellular basis for \( \mathcal{H} \) in general and set up the concept of general Specht module. In Sect. 3 we show the construction of W-graph basis by introducing a new family of \( E_J \)-Kazhdan-Lusztig polynomials \( p_{xyJ} \), and show an inductive procedure for computing \( p_{xyJ}^+ \). In Sect. 4 we consider an example of type A and discuss the applications of our results, we show the transition between Murphy basis and W-graph basis.

1. The indexing sets

For each \( J \subseteq S \), let \( \hat{J} = S \setminus J \) (the complement of \( J \)) and define \( W_J = \langle J \rangle \), the corresponding parabolic subgroup of \( W \) and let \( w_J \in W_J \) be the unique element of maximal length. Let \( \mathcal{H}_J \) be the Hecke algebra associated with \( W_J \). As is well known, \( \mathcal{H}_J \) can be identified with a subalgebra of \( \mathcal{H} \).

1.1. Sets \( D_J, D_J^+ \) and \( E_J \). Let \( D_J = \{ w \in W | \ell(ws) > \ell(w) \text{ for all } s \in J \} \), the set of minimal coset representatives of \( W/W_J \). The following lemma is well known, it is also an easy consequence of [19, Prop. 5.9].

Lemma 1.1 (Deodhar [2] Lemma 3.2]). Let \( J \subseteq S \) and \( s \in S \), and define

\( D_J^- = \{ d \in D_J | \ell(sd) < \ell(d) \} \),

\( D_J^+ = \{ d \in D_J | \ell(sd) > \ell(d) \} \) and \( sd \in D_J \),

\( D_J^0 = \{ d \in D_J | \ell(sd) > \ell(d) \} \) and \( sd \not\in D_J \),
so that \( D_J \) is the disjoint union \( D_{J,s}^- \cup D_{J,s}^+ \cup D^0_{J,s} \). Then \( sD^+_J = D_{J,s}^- \), and if \( d \in D^0_{J,s} \) then \( sd = dt \) for some \( t \in J \).

Define

\( E_J = \{ d \in W \mid \ell(ds) < \ell(d) \text{ for all } s \in J \text{ and } \ell(ds) > \ell(d) \text{ for all } s \notin J \} \)

(4) that is, \( E_J \) is the set of maximal coset representatives of \( W/W_J \) and the minimal ones of \( W/W_J \). Clearly \( \sharp E_J = \sharp E_J \), where \( E_J \) was introduced and written as \( Y_J \) in [7].

Let \( \leq \) denote the left weak Bruhat order on \( W \). That is, \( x \leq y \) if and only if \( y = wx \) for some \( w \in W \) such that \( \ell(y) = \ell(w) + \ell(x) \). McDonough-Pallikaras [15] also say that \( x \) is a prefix of \( y \) if \( x \leq y \). Given \( x, y \in W \) let \( [x, y]_\leq = \{ z \in w \mid x \leq z \leq y \} \) be the left interval they determine.

Let

\[ D_J = D_J w_J, \]

then

\[ D_J = \{ d \in W \mid \ell(ds) < \ell(d) \text{ for all } s \in J \} \]

is the set of longest coset representatives of \( W_J \) in \( W \). Thus,

\[ E_J = \overline{D_J} \cap D_J, \]

and directly from the definition,

\[ \overline{D_J} = \bigcup_{J \subseteq K \subseteq S} E_K, \]

where the union is disjoint.

**Proposition 1.2.** Let \( J \subseteq S \) and \( s \in S \), we define

\[ E^-_{J,s} = \{ d \in E_J \mid \ell(sd) < \ell(d) \text{ and } sd \in E_J \}, \]

\[ E^+_{J,s} = \{ d \in E_J \mid \ell(sd) > \ell(d) \text{ and } sd \in E_J \}, \]

\[ E^0_{J,s} = \{ d \in E_J \mid sd \notin E_J \}, \]

so that \( E_J \) is the disjoint union \( E^-_{J,s} \cup E^+_{J,s} \cup E^0_{J,s} \); let

\[ E^{0-}_{J,s} = \{ d \in E_J \mid \ell(sd) < \ell(d) \text{ and } sd \notin E_J \}, \]

\[ E^{0+}_{J,s} = \{ d \in E_J \mid \ell(sd) > \ell(d) \text{ and } sd \notin E_J \}, \]

then \( E^0_{J,s} = E^{0-}_{J,s} \cup E^{0+}_{J,s} \) (disjoint union); if \( d \in E^{0-}_{J,s} \) then \( sd = dt \) for some \( t \in J \), if \( d \in E^{0+}_{J,s} \) then \( sd = dt \) for some \( t \in J \).

**Proof.** For any \( d \in E_J \), we write \( d = d'w_J \), where \( d' \in D_J \) and \( w_J \) the longest element of \( W_J \). Given \( s \in S \), we have either \( sd < d \) or \( sd > d \).

Case(a): if \( sd < d \) then we have either \( sd \in E_J \) or \( sd \notin E_J \). If \( sd \in E_J \) then \( d \in E^-_{J,s} \).

We now consider the case \( sd \notin E_J \). Since \( d \in E_J \) that is, \( d \in \overline{D_J} \) and \( d \in D_J \), we have \( sd \in D_J \). Thus \( sd \notin \overline{D_J} \), that is \( sd' \notin D_J \), this is the case \( d' \in D^0_{J,s} \) in the statement of Lemma 1.1, so we have \( sd' > d' \) and \( sd' = dt' \) for some \( t' \in J \), and

\[ sd = s(d'w_J) = (sd')w_J = (d't)w_J = (d'w_J)t' = dt' \]

where \( t' = w_Jtw_J \in J \). This is the case \( d \in E^{0-}_{J,s} \).
Case(b): if $sd > d$ then again we have either $sd \in E_J$ or $sd \notin E_J$. If $sd \in E_J$ then $d \in E_{J,s}^+$, we consider the case $sd \notin E_J$.

Since $sd = s(d'w_J) = (sd')w_J$, where $d' \in D_{J,s}^+$ (according to the above discussion, the case $d' \in D_{J,s}^0$ can not happen), and clearly $d' \notin D_{J,s}^+$. So $sd \in \overline{D_J}$, and by the assumption $sd \notin E_J$, we have $sd \notin D_J$.

Applying Lemma 1.1 to the set $D_J$, we have $sd = dt$ for some $t \in \hat{J}$, which is the case $d \in E_{J,s}^0$.

For $w \in W$ we set $\mathcal{L}(w) = \{ s \in S; sw < w \}$, $\mathcal{R}(w) = \{ s \in S; ws < w \}$ and refer them to be the left and right descent set of $w$.

**Lemma 1.3.** \[ \text{[5] Prop.8.6}] \text{Let } w, w' \in W, \text{ then}

(a) if $w \leq_L w'$, then $\mathcal{R}(w') \subseteq \mathcal{R}(w)$. If $w \sim_L w'$, then $\mathcal{R}(w') = \mathcal{R}(w)$.

(b) if $w \leq_R w'$, then $\mathcal{L}(w') \subseteq \mathcal{L}(w)$. If $w \sim_R w'$, then $\mathcal{L}(w') = \mathcal{L}(w)$.

The linear map $\varepsilon_j : \mathcal{H}_j \to A$ defined by $\varepsilon_j(T_w) = \epsilon_wq^{-\ell(w)}$ for any $w \in W_J$ is an algebra homomorphism, called the sign representation. We denote by $\text{Ind}^S_j(\varepsilon_j)$, the $\mathcal{H}$-module obtained by induction from $\varepsilon_j$.

We now introduce the element $C_{w,j}$ in the Kazhdan-Lusztig $C$-basis of $\mathcal{H}$. By [5] Cor. 12.2], it has the expression

$$C_{w,j} = \epsilon_{w,j}q^{\ell(w)} \sum_{w \in W_J} \epsilon_wq^{-\ell(w)}T_w.$$  

**Lemma 1.4.** \[ \text{[5] Lemma 2.8}] \text{The followings hold}

(a) For any $w \in W_J$, we have $T_wC_{w,j} = \epsilon_wq^{-\ell(w)}C_{w,j}$.

(b) We have $C_{w,j}^2 = \epsilon_{w,j}q^{-\ell(w)}P_JC_{w,j}$, where $P_J = \sum_{w \in W_J} q^{2\ell(w)}$.

(c) The set $\overline{D_J} = D_Jw_J$ is a union of left cells in $W$, we have

$$\overline{D_J} = \{ w \in W \mid w \leq_L w_J \},$$

and $\overline{D_J}_A \cong \text{Ind}^S_j(\varepsilon_j) \cong \mathcal{H}C_{w,j}$ (isomorphisms as left $\mathcal{H}$-modules).

**Proposition 1.5.** For $J \subseteq S$, then

(1) $E_J$ is the left cell, or union of left cells with right descent set $J$.

(2) The Bruhat order $\leq$ for the elements of $E_J$ is exactly the weak order $\leq_{\mathcal{L}}$. If $x, y \in E_J$ and $x \leq y$, then $[x, y]_{\mathcal{L}} \subseteq E_J$.

**Proof.** (1) is directly from Lemma 1.3 and 1.4.

(2) is from Prop. 1.2. \[ \Box \]

**Remark** For convenience, in the following sections we still use the usual notations of Bruhat order $\leq$, $<$ for the weak Bruhat orders $\leq_{\mathcal{L}}, <_{\mathcal{L}}$ for the elements of $E_J$, unless indicated.

1.2. Some multiplication rules. For $J \subseteq S$, let $M^J = \mathcal{H}C_{w,J}$ be a $\mathcal{H}$-module, then

**Lemma 1.6.** (1) Let $J \subseteq S$, then $M^J$ is a free $A$-module with basis \[ \{ T_wC_{w,J} \mid w \in D_J \}, \text{ or alternatively } \{ T_wC_{w,J} \mid w \in \overline{D_J} \}. \]

the multiplication of $\mathcal{H}$ with respect to this basis:

$$T_s(T_wC_{w,J}) = \begin{cases} T_{sw}C_{w,J} + (q - q^{-1})T_wC_{w,J} & \text{if } w \in D_{J,s}^- \text{ or } w \in \overline{D}_{J,s}^- \\ T_{sw}C_{w,J} & \text{if } w \in D_{J,s}^+ \text{ or } w \in \overline{D}_{J,s}^+ \\ -q^{-1}T_wC_{w,J} & \text{if } w \in D_{J,s}^0 \text{ or } w \in \overline{D}_{J,s}^0. \end{cases}$$
for all $s \in S$.

(2) For $w \in E_J$, we have:

$$T_s(T_w C_{w,J}) = \begin{cases} T_{sw} C_{w,J} + (q - q^{-1}) T_w C_{w,J} & \text{if } w \in E_{J,s}^- \\ T_{sw} C_{w,J} & \text{if } w \in E_{J,s}^+ \\ -q^{-1} T_w C_{w,J} & \text{if } w \in E_{J,s}^0 \\ q T_w C_{w,J} + T_w C_{tw,J} & \text{if } w \in E_{J,s}^{0,+}, t = w^{-1} s w \in \hat{J} \end{cases}$$

Proof. (1) $M^J$ is spanned by the elements $T_w C_{w,J}$, where $w \in W$; however, if $w = dv$ for $d \in D_J$ and $v \in W_J$, then $T_w C_{w,J} = \varepsilon_v q^{-(\ell(v)-\ell(d))} T_d C_{w,J}$. It follows that $M^J$ is a free $A$-module with the basis shown and it remains to verify the multiplication formulæ.

According to Eq. (1) we immediately get the first two rules. By the multiplication formulæ for the $C$-basis elements (Eq. (2)), we have:

$$T_s C_{w,J} = \begin{cases} -q^{-1} C_{w,J} & \text{if } s \in J \\ q C_{w,J} + C_{sw,J} & \text{if } s \in \hat{J} \end{cases}$$

if $w \in D_{0,s}$, let $t = w^{-1} s w$ and $t \in J$ then $s w = w t < w$, we have

$$T_s(T_w C_{w,J}) = \left[ T_{sw} + (q - q^{-1}) T_w \right] C_{w,J}$$

$$= \left[ T_{sw} + (q - q^{-1}) T_w \right] C_{w,J}$$

$$= \left[ T_{sw} T_{t^{-1}} + (q - q^{-1}) T_w \right] C_{w,J}$$

$$= T_w T_{t^{-1}} C_{w,J} + (q - q^{-1}) T_w C_{w,J}$$

$$= T_w \left[ T_t + (q^{-1} - q) \right] C_{w,J} + (q - q^{-1}) T_w C_{w,J}$$

$$= -q^{-1} T_w C_{w,J}.$$  

(2) If $w \in E_{J,s}^{0,+}$ and $t = w^{-1} s w \in \hat{J}$, again by the multiplication rules for $C_{w,J}$

$$T_s(T_w C_{w,J}) = T_w (T_t C_{w,J}) = T_w (q C_{w,J} + C_{tw})$$

2. A cellular basis and generic Specht modules

The concept of "cellular algebras" was introduced by Graham-Lehrer [14]. It provides a systematic framework for studying the representation theory of non-semisimple algebras which are deformations of semisimple ones. The original definition was modeled on properties of the Kazhdan-Lusztig basis [3] in Hecke algebras of type $A$. There is now a significant literature on the subject, and many classes of algebras have been shown to admit a "cellular" structure, including Ariki-Koiki algebras, $q$-Schur algebras, Temperly-Lieb algebras, and a variety of other algebras with geometric connections.

As we discussed above, $H$ is the one-parameter Hecke algebra associated to finite Weyl group $W$. Furthermore, if $H$ is defined over a ground ring in which "bad" primes for $W$ are invertible, Geck [9] used deep properties of the Kazhdan-Lusztig basis and Lusztig’s $a$-function, he showed that $H$ has a natural cellular structure in the sense of Graham-Lehrer.
For the purpose of this paper, we show a new version of cellular basis of $\mathcal{H}$. Thus, we also obtain a general theory of ”Specht modules” for Hecke algebras of finite type.

We introduce an $A$-linear anti-involution: $*: \mathcal{H} \rightarrow \mathcal{H}$ by $T_w^* = T_{w^{-1}}$ for $w \in W$. Clearly, $C_{w_j}^* = C_{w_j}$; for any $J \subseteq S$ and let $x, y \in D_J$ (or $x, y \in \overline{D}_J$), we define $m_{xy} = T_z C_{w_J} T_y$. Then $m_{xy}^* = m_{yx}$. For convenience, we use the indexing set $\overline{D}_J$ in the following context.

**Remark** If $J = \emptyset$ then $D_J = W$, as an $A$-modules, $M^\emptyset = \mathcal{H}$ so the elements

$$\{m_{xy} \mid x, y \in \overline{D}_\emptyset\}$$

certainly span $\mathcal{H}$.

In order to show that $\mathcal{H}$ is cellular, we have to show that $m_{xy}$ with $x, y \in \overline{D}_J$, can be written as an $A$-linear combination of $\{m_{uv} \mid u, v \in E_K, J \subseteq K\}$.

**Lemma 2.1.** For any $x \in \overline{D}_J$, we have

$$T_x C_{w_J} = \sum_{x' \in E_J} r_{x'} T_{x'} C_{w_J} + \sum_{u \in E_K, J \subseteq K} r_u T_u C_{w_K},$$

where $r_{x'}, r_u \in A$.

**Proof.** As we have found $\overline{D}_J = \bigcup_{J \subseteq K \subseteq S} E_K$, where the union is disjoint. If $x \in E_J$ there is nothing to prove; suppose that $x \notin E_J$, then $x \in E_K$ where $K \supseteq J$. By Prop. 1.2 we have $x = w w_K$ and $w_K = g w_J$ where $w \in W$(or more exactly $w \in D_K$) and $g \in D^K_K = D_J \cap W_K$, with $\ell(x) = \ell(w) + \ell(w_K)$ and $\ell(w_K) = \ell(g) + \ell(w_J)$.

Since $T_g C_{w_J}$ is the sum of $C_{w_K} = C_{w_K}$ and a linear combination of terms $C_{h w_J}$ where $h \in D^K_K$ and $h < g$ (this is the special case of [III Prop.2.3]). On the other hand, $C_{h w_J}$ is the sum of $T_h C_{w_J}$ and an $A$-linear combination of terms $T_f C_{w_J}$, where $f < h, f \in D^K_K$. As a result, $T_g C_{w_J}$ is the sum of $C_{w_K}$ and an $A$-linear combination of these terms $T_f C_{w_J}$. Thus

$$T_x C_{w_J} = T_{w(g w_J)} C_{w_J} = \epsilon_{w, f} q^{-\ell(w_J)} T_w C_{w_J} = \epsilon_{w, f} q^{-\ell(w_J)} T_w (C_{w_K} + \sum_{f < g, f \in D^K_K} r_f T_f C_{w_J}) = r_u T_u C_{w_K} + \sum_{z \in D_J, z < w} r_z T_z C_{w_J},$$

where $r_u, r_f, r_z \in A$. By induction, each term $T_z C_{w_J}$ has also the required form. \hfill $\square$

**Lemma 2.2.** Let $J \subseteq S$ and suppose that $x, y \in \overline{D}_J$, then there exist $x', y, u v \in A$ such that

$$m_{xy} = \sum_{x' \in E_J} r_{x' y} m_{x' y} + \sum_{u \in E_K, v \in D_K, J \subseteq K} r_{u v} m_{u v}.$$
where we write
\[
\text{the}
\]
Hence the set
\[
\text{Theorem 2.4.}
\]
(1) The
\[
\text{for all}
\]
v \in \{M
\]
Consequently,
\[
\text{Proof.}
\]
By Lemma 2.1, we have
\[
\text{where}
\]
T
\[
\text{This is clear because}
\]
We first show that
\[
\text{Proof.}
\]
The Hecke algebra
\[
\text{Theorem 2.3.}
\]
\[
\text{and}
\]
\[
C_{wK}T_y^* = (T_yC_{wK})^*
\]
where
\[
T_yC_{wK} \in \mathcal{H}C_{wK},
\]
is an anti-isomorphism of
\[
\text{Proof.}
\]
Let \(\Omega^{\text{lex}} = \{J \mid J \subseteq S\}\) be a set ordered lexicographically.

**Theorem 2.3.** The Hecke algebra \(\mathcal{H}\) is free as an \(\mathcal{A}\)-module with basis
\[
\mathcal{M} = \{m_{uv} \mid u, v \in E_J \text{ for some } J \subseteq S\}.
\]

**Proof.** We first show that \(\mathcal{M}\) spans \(\mathcal{H}\) by showing that whenever \(x, y \in \overline{\mathcal{D}}\) then \(m_{xy}\) can be written as a \(\mathcal{A}\)-linear combination of terms \(m_{uv}\) in \(\mathcal{M}\). When \(J = S\) this is clear because \(\mathcal{H}C_{wJ} = AC_{wJ}\). If \(J \neq S\), by Lemma 2.2, we have
\[
m_{xy} = \sum_{x' \in E_J} r_{x'y}m_{x'y} + \sum_{(u,v), J \subseteq K} r_{uv}m_{uv},
\]
where \(r_{x'}, r_{uv} \in \mathcal{A}\), and the second sum is over the pairs \((u,v)\) where \(u \in E_K\), \(v \in \overline{\mathcal{D}}\). However, \(m_{xy}^* = m_{yx}\) so by induction on the elements of \(\Omega^{\text{lex}}\) again (start with \(J = S\), clearly \(C_{wJ} = C_{wJ}\)), \(m_{xy}\) can be written as an \(\mathcal{A}\)-linear combination of elements of \(\mathcal{M}\). Finally, let \(J = \emptyset\), then \(\mathcal{H} = \mathcal{H}C_{w\emptyset} = \mathcal{H}\).

Therefore \(\mathcal{M}\) spans \(\mathcal{H}\).

By Wedderburn’s theorem \(\dim(\mathcal{H}) = |W| = \sum_{J \subseteq S} |\mathcal{M}(J)|^2\), where
\[
\mathcal{M}(J) = \{m_{uv} \mid u, v \in E_J \text{ for a fixed } J, J \subseteq S\}.
\]

Hence the set \(\mathcal{M}\) has the correct cardinality. \(\square\)

Define \(\hat{\mathcal{H}}^J\) to be the \(\mathcal{A}\)-module with basis
\[
\{m_{uv} \mid w, v \in E_K \text{ for some } K \text{ such that } J \subset K \subseteq S\},
\]
where we write \(J \subset K\) when \(J \subseteq K\) and \(J \neq K\). Similarly, we define \(\mathcal{H}_J\) to be the \(\mathcal{H}\)-module with basis \(m_{uv}\) where \(u, v \in E_K\) with \(J \subseteq K \subseteq S\).

**Theorem 2.4.** (1) The \(\mathcal{A}\)-linear map determined by
\[
m_{uv} \mapsto m_{vu}
\]
for all \(m_{uv} \in \mathcal{M}\), is an anti-isomorphism of \(\mathcal{H}\).

(2) Suppose that \(h \in \mathcal{H}\) and that \(u \in E_J\), there exist \(r_u \in \mathcal{A}\) such that for all \(v \in E_J\)
\[
hm_{uv} \equiv \sum_{w \in E_J} r_w m_{wu} \mod \hat{\mathcal{H}}^J.
\]
Consequently, \(\{\mathcal{M}, \Omega^{\text{lex}}\}\) is a cellular basis of \(\mathcal{H}\).
further, if \( J \subseteq S \) for calculating these elements in \( \mathcal{H} \) coincide since \( m_{uv}^* = m_{vu} \) for all \( m_{uv} \in \mathcal{M} \). This proves (1) since \( * \) is an anti-

isomorphism of \( \mathcal{H} \).

(2) We argue by induction on \( J \in \Omega^{lex} \). By (1), if \( J = S \) then \( \mathcal{H}C_{w,J} = AC_{w,J} \), there is nothing to prove. Suppose that \( J \subseteq S \). First we consider \( v = w_J \). Since \( \mathcal{M} \) is a basis of \( \mathcal{H} \), for any \( h \in \mathcal{H} \) we may write

\[
hm_{u,w_J} = \sum_{x,y \in E_{K}, K \subseteq S} r_{xy}m_{xy}
\]

for some \( r_{xy} \in \mathcal{A} \). Now \( hm_{u,w_J} \) belongs to \( M^J \), clearly, if \( r_{xy} \neq 0 \) then \( J \subseteq K \); further, if \( J = K \) then we must also have \( v = w_J \). Hence,

\[
hm_{u,w_J} = \sum_{x \in E_J} r_xm_{x,w_J} \mod \hat{\mathcal{H}}^J
\]

where \( r_x = r_{x,w_J} \in \mathcal{A} \). This completes the proof of (2) when \( v = w_J \).

Now, if \( K \supseteq J \) and \( u, y \in E_K \) then \( m_{uy}T_v^* = (T_vm_{uy})^* \in \mathcal{H}K \subseteq \hat{\mathcal{H}}^J \) by induction on \( J \in \Omega^{lex} \). Therefore, we can multiply the Eq. (1) on the right by \( T_v^* \), to complete the proof.

So we can now introduce the following:

**Definition 2.5.** Let \( S^J = \langle T_uC_{w_J}, \hat{\mathcal{H}}^J \mid u \in E_J \rangle_\mathcal{A} \), then \( S^J \) is an \( \mathcal{H} \)-submodule of \( \mathcal{H}^J / \hat{\mathcal{H}}^J \). We call this the **generic Specht module** of \( \mathcal{H} \) associated with \( J \).

**The bar involution for \( S^J \).** For all \( x, y \in E_J \) we define elements \( R_{x,y} \in \mathcal{A} \) by the formula

\[
R_{x,y}(\mod \hat{\mathcal{H}}^J) = \begin{cases} 
R_{x,y} & \text{if } x \in E_{J,s}^-, \\
R_{x,y} + (q^{-1} - q)R_{x,y} & \text{if } x \in E_{J,s}^+, \\
-qR_{x,y} & \text{if } x \in E_{J,s}\neg, \\
q^{-1}R_{x,y} & \text{if } x \in E_{J,s}\neg.
\end{cases}
\]

We may use induction on \( \ell(y) \) to establish that \( R_{x,y} = 0 \) unless \( x \trianglelefteq y \) in the weak Bruhat partial order on \( E_J \); this follows from the fact that if \( sy \trianglelefteq y \) and \( x \trianglelefteq y \) then both \( x \trianglelefteq y \) and \( sx \trianglelefteq y \). It is also easily seen that \( R_{x,x} = 1 \).

3. \textbf{W-graphs for generic Specht modules}

Let \( C_{w,J} \) be a left cell, or more generally, a union of left cells containing \( w_J \), then the transition between the bases of the left cell module \( [C_{w,J}]_\mathcal{A} \) and the generic Specht module \( S^J \) is described as the following:

**Theorem 3.1.** The \( \mathcal{H} \)-module \( S^J \) has a unique basis \( \{ C_w \mid w \in E_J \} \) such that \( \overline{C_w} = C_w \) for all \( w \in E_J \), and

\[
C_w = \sum_{y \in E_J} P_{y,w}T_yC_{w_J} \mod \hat{\mathcal{H}}^J.
\]
for some elements \( P_{y,w} \in \mathcal{A}^+ \) with the following properties:

(i) \( P_{y,w} = 0 \) if \( y \notin w \);
(ii) \( P_{w,w} = 1 \);
(iii) \( P_{y,w} \) has zero constant term if \( y \neq w \).

Comparing with the original Kazhdan-Lusztig's polynomials in \([3]\), we called \( \{ P_{y,w} \mid y,w \in E_J \} \) the family of \( E_J \)-relative Kazhdan-Lusztig polynomials. We shall show that the basis \( \{ C_w \mid w \in E_J \} \) give \( S_J \) the structure of a \( W \)-graph. That is, there is a \( W \)-graph \( \Lambda \) with vertex elements \( \{ C_w \mid w \in E_J \} \). Before showing the proof of Theorem 3.1, we describe the edge weights and descent sets for \( \Lambda \).

Given \( y, w \in E_J \) with \( y \neq w \), we define an integer \( \mu(y, w) \) as follows. If \( y < w \) then \( \mu(y, w) \) is the coefficient of \( q \) in \( -P_{y,w} \).

We write \( y \prec w \) if \( y < w \) and \( \mu(y, w) \neq 0 \).

The (left) descent set associated with the vertex element \( C_w \) \( (w \in E_J) \) of \( \Lambda \) is

\[
I(w) = \{ s \in S \mid \ell(sw) < \ell(w) \} = \{ s \in S \mid w \in E_{J,s} \} \cup \{ s \mid w \in E_{0,s} \}
\]

In accordance with the notation introduced in Section 2, we define

\[
\Lambda_s^- = \{ w \in E_J \mid s \in I(w) \} = \{ w \mid w \in E_{J,s} \text{ or } w \in E_{0,s} \},
\]

and similarly \( \Lambda_s^+ = \{ w \in E_J \mid s \notin I(w) \} \). Our proof of Theorem 3.1 will also incorporate a proof of the following result, which will be an important component of the subsequent proof that \( \Lambda \) is a \( W \)-graph.

**Theorem 3.2.** Let \( v \in E_J \). Then for all \( s \in S \) such that \( \ell(sv) > \ell(v) \) and \( sv \in E_J \) we have

\[
T_s C_v = q C_v + C_{sv} + \sum_{z \in E_J} \mu(z, v) C_z,
\]

where the sum is over all \( z \in \Lambda_s^- \) such that \( z < v \).

The following is the proof of Theorem 4.1.

*Proof.* Uniqueness is proved similarly with that of \([3, \text{Theorem 1.1}]\), we omit the details.

Existence. We give a recursive procedure for constructing elements \( P_{x,w} \) satisfying the requirements of Theorem 3.1. We start with the definition

\[
P_{w_J,w_J} = 1
\]

so that \( C_w = C_w \) holds for \( w = w_J \), as do Conditions (i), (ii) and (iii).

Now assume that \( w \neq w_J \) and that for all \( v \in E_J \) with \( \ell(v) < \ell(w) \) the elements \( P_{x,v} \) have been defined (for all \( x \in E_J \)) so that the requirements of Theorem 3.1 are satisfied. Thus the elements \( C_v \) are known when \( \ell(v) < \ell(w) \). We may choose \( s \in S \) such that \( w = sv \) with \( \ell(w) = \ell(v) + 1 \); note that \( v \in E_J \) by Lemma 1.6. In accordance with the formula in Theorem 3.2 we define

\[
C_w = (T_s - q) C_v - \sum_{z \prec v \atop z \in \Lambda_s^-} \mu(z, v) C_z.
\]
Since \( T_s - q = T_s - q \), induction immediately gives \( C_w = C_w \). We define \( P'_{y,w} \) and \( P''_{y,w} \) by

\[
(T_s - q)C_v = \sum_{y \in E_J} P'_{y,w} T_y C_{w,j} \\
\sum_{z < v} \mu(z, v)C_z = \sum_{y \in E_J} P''_{y,w} T_y C_{w,j}
\]

and define \( P_{y,w} = P'_{y,w} - P''_{y,w} \).

If \( y \in E_J \) then

\[
(T_s - q)T_y = \begin{cases} 
T_{sy} - qT_y & \text{if } y \in E^+_{j,s} \\
T_{sy} - q^{-1}T_y & \text{if } y \in E^-_{j,s} \\
T_y(T_1 - q) & \text{if } y \in E^0_{j,s} \\
T_{sy} - qT_y & \text{if } y \in E^0_{j,s} 
\end{cases}
\]

where we have written \( t = y^{-1}sy \) in the case \( y \in E^0_{j,s} \). Thus we see that

\[
(T_s - q)C_v = \sum_{y \in E^+_{j,s}} P_{y,v}(T_{sy} - qT_y)C_{w,j} + \sum_{y \in E^-_{j,s}} P_{y,v}(T_{sy} - q^{-1}T_y)C_{w,j} \\
+ \sum_{y \in E^0_{j,s}} P_{y,v}T_y(T_1 - q)C_{w,j} + \sum_{y \in E^0_{j,s}} P_{y,v}(T_{sy} - qT_y)C_{w,j}
\]

\[
= \sum_{y \in E^+_{j,s}} (P_{sy,v} - q^{-1}P_{y,v})T_y C_{w,j} + \sum_{y \in E^-_{j,s}} (P_{sy,v} - qP_{y,v})T_y C_{w,j} \\
+ \sum_{y \in E^0_{j,s}} P_{y,v}(-q^{-1} - q)T_y C_{w,j} \\
+ \sum_{y \in E^0_{j,s}} P_{y,v}[(qT_yC_{w,j} + T_y C_{tw,j}) - qT_y C_{w,j}]
\]

Now comparing Eq. (8) with the expression for \( (T_s - q)C_v \) obtained above we obtain the following formulas for the cases \( y \in E^+_{j,s} \) (case (a)), \( y \in E^-_{j,s} \) (case (b)), \( y \in E^0_{j,s} \) (case (c)) and \( y \in E^{0,+}_{j,s} \) (case (d)):

\[
P'_{y,w} = \begin{cases} 
P_{sy,v} - qP_{y,v} & \text{(case (a))}, \\
P_{sy,v} - q^{-1}P_{y,v} & \text{(case (b))}, \\
(-q - q^{-1})P_{y,v} & \text{(case (c))}, \\
0 & \text{(case (d))} 
\end{cases}
\]

Since \( C_z = \sum_{y \in E_J} P_{y,z} T_y C_{w,j} \), we have

\[
\sum_{y \in E_J} \mu(z, v)C_z = \sum_{y \in E_J} \sum_{z < v, z \in \Lambda^-} \mu(z, v)P_{y,z} T_y C_{w,j}
\]

and by comparison with Eq. (9)

\[
P''_{y,w} = \sum_{z < v, z \in \Lambda^-} \mu(z, v)P_{y,z}.
\]
We omit the details here. □

We may check that with $P_{y,w}′$ and $P_{y,w}′′$ given by Eq’s (10) and (11), the elements $P_{y,w} = P_{y,w}′ - P_{y,w}′′$ lie in $A^+$ and satisfy Conditions (i), (ii) and (iii) of Theorem 3.1. We omit 3.1 may be written as

(12) $\tilde{T}_w = T_w C_{w, j}$.

Observe that the formula for $C_w$ in Theorem 3.1 may be written as

$$C_w = \tilde{T}_w + \sum_{y < w, y \in E_j} P_{y,w} \tilde{T}_y,$$

and inverting this gives

(12) $\tilde{T}_w = C_w + \sum_{y < w, y \in E_j} Q_{y,w} C_y$

where the elements $Q_{y,w}$ (defined whenever $y < w$) are given recursively by

$$Q_{y,w} = -P_{y,w} - \sum_{z | y < z < w} Q_{y,z} P_{z,w}.$$ 

In particular, $Q_{y,w}$ is in $A^+$, has zero constant term, and has coefficient of $q$ equal to $\mu(y, w)$.

We now state our main result.

**Theorem 3.3.** The basis $\{C_w | w \in E_j\}$ gives the generic Specht module $S^j$ the structure of a W-graph, as described above.

**Proof.** The proof is similar with [21, Theorem 2.6], modified appropriately. We start by using induction on $\ell(w)$ to prove that for all $s \in S$

(13) $T_s C_w = \begin{cases} 
-q^{-1}C_w & \text{if } w \in \Lambda^-_s, \\
q C_w + \sum_{z \in E_j, z \in \Lambda^-_s} \mu(z, w) C_z & \text{if } w \notin \Lambda^-_s.
\end{cases}$

or more exactly

(14)

$$T_s C_w \text{ (mod } \mathfrak{H}^j) = \begin{cases} 
-q^{-1}C_w & \text{if } w \in E_{j,s}^- \text{ or } w \in E_{j,s}^{0-}, \\
q C_w + C_{sw} + \sum_{z \in E_{j,s}^+, z < w} \mu(z, w) C_z & \text{if } w \in E_{j,s}^+, \\
q C_w + \sum_{z \in E_{j,s}^+, z < w} \mu(z, w) C_z & \text{if } w \in E_{j,s}^{0+}.
\end{cases}$$

If $w \in E_{j,s}^+$, then $w \notin \Lambda^-_s$, and Eq. (13) follows immediately from Theorem 3.2 (applied with $v$ replaced by $w$), since the only $z \in \Lambda^-_s$ with $\mu(z, w) \neq 0$ and $\ell(z) \geq \ell(w)$ is $z = sw$.

For the case $w \in E_{j,s}^{0+}$, the term $C_{sw}$ can not appear in the sum of Eq. (13).

If $w \in E_{j,s}^-$, which implies that $w \in \Lambda^-_s$, then writing $v = sw$ and applying Theorem 3.2 gives

$$C_w = (T_s - q) C_v - \sum \mu(z, v) C_z,$$

where $z < v$ and $z \in \Lambda^-_s$ for all terms in the sum. The inductive hypothesis thus gives $T_s C_z = -q^{-1}C_z$, and since we also have $T_s(T_s - q) = -q^{-1}(T_s - q)$ it follows that $T_s C_w = -q^{-1}C_w$, as required.
Now suppose that \( w \in E^0_{s^i} \), and as usual let us write \( sw = wt \). Suppose first that \( t = w^{-1}sw \in J \), so that \( w \in \Lambda^-_s \). By Eq. [12],

\[
C_w = \tilde{T}_w - \sum_{\{y|y < w, y \in E_J\}} Q_{y,w} C_y,
\]

and since \( T_s T_w C_{w,J} + q^{-1}T_w C_{w,J} = T_w (T_s C_{w,J} + q^{-1}C_{w,J}) = 0 \) we find that

\[
T_s C_w + q^{-1}C_w = -\sum_{\{y|y < w, y \in E_J\}} Q_{y,w} (T_s C_y + q^{-1}C_y).
\]

By the inductive hypothesis,

\[
T_s C_y + q^{-1}C_y = \begin{cases} 
0 & \text{if } y \in \Lambda^-_s \\
(q + q^{-1})C_y + \sum_{z \in \Lambda^-_s} \mu(z,y)C_z & \text{if } y \notin \Lambda^-_s,
\end{cases}
\]

and so Eq. (15) gives

\[
T_s C_w + q^{-1}C_w = -\sum_{y \notin \Lambda^-_s \atop y < w} Q_{y,w} (q + q^{-1})C_y + X
\]

for some \( X \) in the \( A \)-submodule spanned by the elements \( C_z \) for \( z \in \Lambda^-_s \). Now since \( T_s = T_s^{-1} + (q - q^{-1}) \) it follows that

\[
(T_s + q^{-1})C_w = (T_s + q^{-1})C_w
\]

\[
= -\sum_{y \notin \Lambda^-_s \atop y < w} Q_{y,w} (q^{-1} + q)C_y + X,
\]

and comparing with Eq. (16) shows that for all \( y \) with \( y < w (y \in E_J) \) and \( y \notin \Lambda^-_s \),

\[
Q_{y,w} = Q_{y,w}.
\]

Since \( Q_{y,w} \) is in \( A^+ \) and has zero constant term, Eq. (17) forces \( Q_{y,w} \) to be zero whenever \( y < w \) and \( y \notin \Lambda^-_s \). Therefore the right hand side of Eq. (15) is zero, since \( T_s C_y + C_y = 0 \) whenever \( y \in \Lambda^-_s \). So

\[
T_s C_w = -q^{-1}C_w,
\]

as required. \( \square \)

4. Applications to Type A

Throughout this section, we apply our results to the Hecke algebra of type \( A \).

Let \( W = \mathfrak{S}_n \) be the symmetric group acting on the left on \( \{1, 2, \cdots, n\} \). Another reference is the exposition by Mathas [1]. For \( i = 1, 2, \cdots, n - 1 \) let \( s_i \) be the basic transposition \( (i, i + 1) \) and let \( S = \{s_1, s_2, \cdots, s_{n-1}\} \), the generating set of \( \mathfrak{S}_n \).
4.1. Notations. Let $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_r)$ be a partition of $n$ with the notation $\lambda \vdash n$. A standard $\lambda$-tableau is a tableau whose entries are exactly $1, 2, \cdots, n$ and which has both increasing rows and increasing columns, the set is denoted $T(\lambda)$. Let $t^\lambda$ (resp. $t_\lambda$) be the $\lambda$-tableau in which the numbers $1, 2, \cdots$ appear in order from left to right (resp. top to bottom) and down along successive rows (resp. columns), then $t^\lambda, t_\lambda \in T(\lambda)$. For a Young tableau $t$, we put

$$I(t) = \{ i \mid 1 \leq i \leq n - 1, i + 1 \text{ is in a lower position than } i \text{ in } t \}$$

and call it the descent set of $t$. Let

$$I_0(t) = \{ i \in I(t) \mid i + 1 \text{ is in the left side of } i \text{ in } t \},$$

$$I_1(t) = \{ i \in I(t) \mid i + 1 \text{ is directly below } i \text{ in } t \}.$$

Lemma 4.1. [17] For a standard tableau $t$ of shape $\lambda \vdash n$,

(1) $I(t) = I_0(t) \cup I_1(t)$;

(2) $I(t) \cup I(t') = \{ 1, 2, \ldots, n - 1 \}$;

(3) $I_0(t) = \emptyset$ if and only if $t = t_\lambda$;

(4) $I_0(t') = \emptyset$ if and only if $t = t^\lambda$.

The Young subgroup $G = G_\lambda \times \cdots \times G_r$ of $G_n$ is the row stabilizer of $t^\lambda$. Let $D_\lambda$ be the set of distinguished left coset representatives of $G_\lambda$ in $G_n$, by Dipper-James [1] and Mathas [6], we have the following explicit description:

$$D_\lambda = \{ w \in G_n \mid wt^\lambda \text{ is row-standard} \}.$$

As in [1] [12] [6], if $t$ is a row-standard $\lambda$-tableau, the unique element $d \in D_\lambda$ such that $t = dt^\lambda$ will be denoted by $d(t)$. Let $w_{J(\lambda)}$ be the longest element of the Young subgroup $G_\lambda$, an element $w_\lambda$ is defined by $t_\lambda = w_\lambda t^\lambda$.

Given partitions $\mu = (\mu_1, \mu_2, \ldots)$ and $\lambda = (\lambda_1, \lambda_2, \ldots)$ of $n$, we say $\mu$ dominates $\lambda$, and write $\lambda \trianglelefteq \mu$, if

$$\lambda_1 \leq \mu_1, \lambda_1 + \lambda_2 \leq \mu_1 + \mu_2, \lambda_1 + \lambda_2 + \lambda_3 \leq \mu_1 + \mu_2 + \mu_3, \ldots$$

we write $\lambda \trianglelefteq \mu$ if $\lambda \leq \mu$ and $\mu \neq \lambda$. The partial order $\trianglelefteq$ on the set of partitions(or shapes) of $n$ will be referred to as the dominance order.

For a fixed $\lambda \vdash n$, $s, t \in T(\lambda)$. We write $s \trianglelefteq t$ if $\ell(d(s)) \leq \ell(d(t))$, and $s \triangleright t$ if $s \trianglelefteq t$ and $s \neq t$. We note that the notation here is different with [3][pp.31].

4.2. Cells. The cells of $W = G_n$ may be described in terms of the Robinson-Schensted correspondence. The correspondence is a bijection of $S_n$ to pairs of standard tableaux $(P, Q)$ of the same shape corresponding to partitions of $n$, so that if $w \mapsto (P(w), Q(w))$ then $Q(w) = P(w^{-1})$. In particular, the involutions are the elements $w \in W$ for which $Q(w) = P(w)$. If $\lambda \vdash n$, the pair of tableaux corresponding to $w_{J(\lambda)}$ has the form $(t_\lambda, t^\lambda)$. Hence, the tableaux corresponding to $w_{J(\lambda)}$ have shape $\lambda'$, where $\lambda'$ denotes the partition conjugate to $\lambda$.

If $R$ is a fixed standard tableau then the set $\{ w \in W : Q(w) = R \}$ is a left cell of $W$ and the set $\{ w \in W : P(w) = R \}$ is a right cell of $W$. See [3] and also [4] for an alternative proof of this result.

Lemma 4.2. Let $\lambda \vdash n$ and $t \in T(\lambda)$. The element of $G_n$, which corresponds to the pair of tableaux $(t^\lambda, t_\lambda)$ under the Robinson-Schensted correspondence, is $w_\lambda w_{J(\lambda)}$. 

The following is the corollaries of the discussion in Section 1, see also in [13 Lemma 3.3] and Du [16 Lemma 1.2].

**Lemma 4.3.** The followings hold (i) $w_\lambda w_{J(\lambda)} \in D_\lambda$, (ii) $dw_{J(\lambda)} \in D_\lambda$ for each prefix $d$ of $w_\lambda$, (iii) $dw_{J(\lambda)} \in D_\lambda$ is in the same left cell as $w_{J(\lambda)}$ for each prefix $d$ of $w_\lambda$.

As in Section 1, we write $E_{J(\lambda)} = \{ e \mid e = dw_{J(\lambda)} \text{ and } d \text{ is a prefix of } w_\lambda \}$, for any $s_i = (i, i+1) \in S$ we define

- $E_{J(\lambda), s_i} = \{ e \in E_{J(\lambda)} \mid \ell(s_i e) < \ell(e) \text{ and } s_i e \in E_{J(\lambda)} \}$,
- $E_{J(\lambda), s_i}^+ = \{ e \in E_{J(\lambda)} \mid \ell(s_i e) > \ell(e) \text{ and } s_i e \in E_{J(\lambda)} \}$,
- $E_{J(\lambda), s_i}^0 = \{ e \in E_{J(\lambda)} \mid s_i e \notin E_{J(\lambda)} \}$

so that $E_{J(\lambda)}$ is the disjoint union $E_{J(\lambda), s_i}^- \cup E_{J(\lambda), s_i}^+ \cup E_{J(\lambda), s_i}^0$, then

$$s_i E_{J(\lambda), s_i}^0 = E_{J(\lambda), s_i}^-;$$

let

- $E_{J(\lambda), s_i}^{-0} = \{ e \in E_{J(\lambda)} \mid \ell(s_i e) < \ell(e) \text{ and } s_i e \notin E_{J(\lambda)} \}$,
- $E_{J(\lambda), s_i}^{0+} = \{ e \in E_{J(\lambda)} \mid \ell(s_i e) > \ell(e) \text{ and } s_i e \notin E_{J(\lambda)} \}$,

then $E_{J(\lambda), s_i}^0 = E_{J(\lambda), s_i}^{-0} \cup E_{J(\lambda), s_i}^{0+}$ (disjoint union); if $e \in E_{J(\lambda), s_i}^{-0}$ then $s_i e = t$ for some $t \in J(\lambda)$, if $e \in E_{J(\lambda), s_i}^{0+}$ then $s_i e = t$ for some $t \in J(\lambda)$, where $J(\lambda) = S \backslash J(\lambda)$.

We have the following observation

- $E_{J(\lambda), s_i}^- = \{ d(t)w_{J(\lambda)} \mid t \in T(\lambda), i \in I_0(t') \}$,
- $E_{J(\lambda), s_i}^+ = \{ d(t)w_{J(\lambda)} \mid t \in T(\lambda), i \in I_0(t) \}$,
- $E_{J(\lambda), s_i}^{0-} = \{ d(t)w_{J(\lambda)} \mid t \in T(\lambda), i \in I_1(t') \}$,
- $E_{J(\lambda), s_i}^{0+} = \{ d(t)w_{J(\lambda)} \mid t \in T(\lambda), i \in I_1(t) \}$,

Let

$$C_{w_{J(\lambda)}} = \epsilon_{w_{J(\lambda)}} q^{\ell(w_{J(\lambda)})} \sum_{w \in S_\lambda} \epsilon_w q^{-\ell(w)} T_w.$$

then the following statement is a corollary of Lemma 2.1.

**Lemma 4.4.** Mathas2 Let $\lambda \vdash n$, then $\mathcal{H} C_{w_{J(\lambda)}}$ is a free $\mathcal{A}$-module with basis

$$\{ T_{d(t)} C_{w_{J(\lambda)}} \mid t \text{ a row standard } \lambda\text{-tableau} \}.$$

Moreover, if $t$ is row standard and $s = s_i t$ for some $1 \leq i \leq n - 1$, then

$$T_i T_{d(t)} C_{w_{J(\lambda)}} = \begin{cases} T_{d(s)} C_{w_{J(\lambda)}}, & \text{if } i \in I_0(t) \\ T_{d(s)} C_{w_{J(\lambda)}} + (q - q^{-1}) T_{d(t)} C_{w_{J(\lambda)}}, & \text{if } i \in I_0(t') \\ -q^{-1} T_{d(t)} C_{w_{J(\lambda)}}, & \text{if } i \in I_1(t') \end{cases}$$

where $T_i := T_{s_i}$. 
4.3. Murphy basis and W-graph basis. The following is a corollary of the main Theorems in Section 2.

**Theorem 4.5.** \cite{12, 13} For any \( \lambda \vdash n \) and \( s, t \in \mathbb{T}(\lambda) \), we define elements of \( \mathcal{H} \) by

\[
m_{st} = T_d(s)C_{w_{J(\lambda)}}T_d(t)^{-1}
\]

then the following hold (a) The set \( \{m_{st}|s,t \in \mathbb{T}(\lambda)\text{ for some } \lambda \vdash n\} \) is an \( \mathcal{A} \)-basis of \( \mathcal{H} \); (b) For any \( \lambda \vdash n \), let \( \mathcal{H}^\lambda \) be the \( \mathcal{A} \)-submodule of \( \mathcal{H} \) spanned by all elements \( m_{st} \) where \( s, t \in \mathbb{T}(\mu) \) for some \( \lambda \leq \mu \), then \( \mathcal{H}^\lambda \) is a two-sided ideals in \( \mathcal{H} \).

Note that the element that we denote by \( T_w \) corresponds to the element \( q^{t(w)}T_w \) in Murphy’s notation. Thus the element denoted by \( C_{w_{J(\lambda)}} \) in the above statement is exactly as in Murphy’s work, except the associated coefficient \( \epsilon_{w_{J(\lambda)}}q^{t(w_{J(\lambda)})} \). However, this does not affect the validity of (a) and (b) since \( q \) is invertible in \( \mathcal{A} \). The statement in (a) can be found in Murphy \cite[Th. 3.9]{12} or Murphy \cite[Th. 4.17]{13}. The statement(b) is proved in \cite[Th. 4.18]{13}.

Murphy also obtains the following result concerning the Specht modules of \( \mathcal{H} \). For any \( \lambda \vdash n \), let \( \hat{\mathcal{H}}^\lambda \) be the \( \mathcal{A} \)-submodule of \( \mathcal{H} \) spanned by all \( m_{st} \) where \( s, t \in \mathbb{T}(\mu) \) for some \( \mu \vdash n \) such that \( \lambda \leq \mu \). Thus, we have

\[
\hat{\mathcal{H}}^\lambda = \sum_{\mu} \mathcal{H}^\mu
\]

where the sum runs over all \( \mu \vdash n \) such that \( \lambda \leq \mu \). In particular, \( \hat{\mathcal{H}}^\lambda \) is a two-sided ideal and we have \( \mathcal{H}^\lambda = \mathcal{H}C_{w_{J(\lambda)}}\mathcal{H} + \hat{\mathcal{H}}^\lambda \).

**Definition 4.6.** \( \lambda \vdash n \), the Specht module \( S^\lambda \) is defined to be the left \( \mathcal{H} \)-module \( \hat{\mathcal{H}}^\lambda + C_{w_{J(\lambda)}}\mathcal{H} \).

Note that \( \hat{\mathcal{H}}^\lambda + C_{w_{J(\lambda)}} \) is an element of the \( \mathcal{H} \)-module \( \mathcal{H} / \hat{\mathcal{H}}^\lambda \) so that \( S^\lambda \) is a submodule of \( \mathcal{H} / \hat{\mathcal{H}}^\lambda \). As we defined it, the Specht module \( S^\lambda \) is isomorphic to the dual of the Specht module which Dipper and James \cite{1} indexed by \( \lambda' \).

For a standard \( \lambda \)-tableau \( t \) let \( m_t = m_{t\lambda} + \hat{\mathcal{H}}^\lambda = T_d(t)C_{w_{J(\lambda)}} + \hat{\mathcal{H}}^\lambda \), We have

**Theorem 4.7.** \( \mathcal{H} \) The Specht module \( S^\lambda \) is free as an \( \mathcal{H} \)-module with basis \( \{m_t|t \in \mathbb{T}(\lambda)\} \), and \( \mathcal{H} / \hat{\mathcal{H}}^\lambda \) is a direct sum of \( |\mathbb{T}(\lambda)| \) copies of \( S^\lambda \).

While

**Lemma 4.8.** \( \lambda \vdash n \), suppose \( t \in \mathbb{T}(\lambda) \) such that \( i \in I_1(t) \), then for all \( s \in \mathbb{T}(\lambda) \)

\[
T_im_s \equiv qm_s + \sum_{v \leq s} r_v m_v \quad \text{ mod } \hat{\mathcal{H}}^\lambda
\]

for some \( r_v \in \mathcal{A} \).

**Corollary 4.9.** Let \( t \in \mathbb{T}(\lambda) \) and \( s = s,t \) for some \( 1 \leq i \leq n - 1 \), then

\[
T_im_t = \begin{cases} 
m_s, & \text{if } i \in I_0(t) \\
m_s + (q - q^{-1})m_t, & \text{if } i \in I_0(t') \\
-q^{-1}m_s, & \text{if } i \in I_1(t') \\
qm_t + \sum_{v \leq t} r_v m_v & \text{ mod } \hat{\mathcal{H}}^\lambda, \text{if } i \in I_1(t).
\end{cases}
\]

where \( r_v \in \mathcal{A} \).
We apply with Theorem 4.1 and 4.3 to establish the transition between Murphy’s basis and W-graph basis of the Specht module. We also note that in the references, the authors related the Kazhdan-Lusztig cell module and the corresponding Specht module in the case of symmetry group, group algebra and Hecke algebra of type $A$. See Naruse [17], Garsia-MacLarman [18] and MacDonough and Pallicaros [17] ect.

**Theorem 4.10.** For a fixed $\lambda \vdash n$, we define the elements of the $C$-basis for $S^\lambda$

$$C_{d(s)w_J(\lambda)} = m_s - q \sum_{d(t) < d(s)} p_{t,s} m_t,$$

$$= T_{d(s)} C_{w_J(\lambda)} - q \sum_{t < s} p_{t,s} T_{d(t)} C_{w_J(\lambda)} \mod(\hat{H}).$$

where $s, t \in T(\lambda)$ and $p_{t,s} \in \mathbb{Z}(q)$ will be defined recursively by

$$T_i C_{d(t)w_J(\lambda)} = \begin{cases} 
-q^{-1} C_{d(t)w_J(\lambda)}, & \text{if } i \in I(t') \\
 q C_{d(t)w_J(\lambda)} + \sum_{i \in I(t'), u < t} \mu(u, t) C_{d(u)w_J(\lambda)}, & \text{if } i \in I_1(t) \\
 q C_{d(t)w_J(\lambda)} + C_{s_i d(t)w_J(\lambda)} + \sum_{i \in I(t'), u < t} \mu(u, t) C_{d(u)w_J(\lambda)}, & \text{if } i \in I_0(t)
\end{cases}$$

where $\mu(u, t)$ is the constant term of the polynomial $p_{u,t}$.

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