Study of the Nonequilibrium Quenching and Annealing Dynamics for the Long-Range Ising Model

D. E. Rodríguez, M. A. Bab, and E. V. Albano

1Computational Physics Group, Instituto de Investigaciones Físicoquímicas Teóricas y Aplicadas (INIFTA), Facultad de Ciencias Exactas, Universidad Nacional de La Plata, CCT-La Plata CONICET; Suc. 4, CC16 (1900) La Plata, Argentina.

Abstract

Extensive Monte Carlo simulations are employed in order to study the dynamic critical behavior of the one-dimensional Ising magnet, with algebraically decaying long-range interactions of the form $\frac{1}{r^{d+\sigma}}$, with $\sigma = 0.75$. The critical temperature, as well as the critical exponents, is evaluated from the power-law behavior of suitable physical observables when the system is quenched from uncorrelated states, corresponding to infinite temperature, to the critical point. These results are compared with those obtained from the dynamic evolution of the system when it is suddenly annealed at the critical point from the ordered state. Also, the critical temperature in the infinite interaction limit is obtained by means of a finite-range scaling analysis of data measured with different cutoffs of the interaction range. All the estimated static critical exponents ($\gamma/\nu$, $\beta/\nu$, and $1/\nu$) are in good agreement with Renormalization Group (RG) predictions and previously reported numerical data obtained under equilibrium conditions. It is found that the dynamic exponent $z$ is different for quenching and annealing experiments, most likely due to the influence of the Kosterlitz-Thouless transition occurring at relatively similar algebraic decay of the interactions with $\sigma = 1$. However, for annealing experiments the measured exponent $z$ is close to the RG predictions. On the other hand, the relevant exponents of the dynamic behavior ($z$ and $\theta$) are slightly different than the RG predictions, most likely due to the fact that they may depend on the specific dynamics used (Metropolis in the present paper).

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I. INTRODUCTION

The study of the critical behavior of systems with long-range (LR) interactions is still a challenging topic in the field of statistical physics [1, 2, 3]. Furthermore, the understanding of the dynamic evolution of these systems, from far-from-equilibrium initial states towards a final equilibrium regime, poses an additional difficulty due to the fast relaxation of relevant physical observables due to the presence of LR interactions. For these reasons, the study of relaxational processes in simple Ising and Potts models with LR interactions plays an important role for the understanding of the dynamics of second-order phase transitions.

Within this context, the study of the short-time dynamics (STD) of critical systems has attracted great attention during the last two decades [1, 4, 5, 6]. The pioneering theoretical study of the STD, which was formulated in the context of the dynamic renormalization group [7], predicts the existence of a new exponent related to the initial increase of the order parameter. This prediction has subsequently been validated by a large body of numerical evidence obtained in a variety of models [4, 8, 9, 10, 11, 12]. However, only few studies have been performed in order to generalize these concepts to systems with LR interactions. In fact, the field-theoretical calculations of Janssen et al. [7] have been extended to the case of LR interactions decaying according to a power law for the case of the continuous $n$–vector model [1], the random Ising model [13], and the kinetic spherical model [14, 15]. On the other hand, theoretical studies of the relaxational dynamics of discrete models are still lacking, and only few preliminary numerical results on the STD of the Potts model have recently been reported [6].

In order to contribute to the understanding of the dynamics of phase transitions in discrete systems, the aim of this paper is to report and discuss extensive numerical simulations of the Ising model, in one dimension, with LR interactions decaying with the distance as a power law. For this purpose, we performed studies of both the STD of initially disordered states (i.e., quenching experiments) and the relaxation dynamics of initially ordered states (i.e., annealing experiments). Results obtained by applying these methods allow us to determine not only the critical temperature, but also the complete set of static and dynamic critical exponents (for the methodology used, see e.g. [16]). In this way, we can compare our results with theoretical Renormalization Group (RG) predictions [1, 17] and with independent numerical determinations of the static exponents performed under equilibrium conditions.
The paper is organized as follows: in Section II a brief description of the model and the simulation method is presented, Section III is devoted to a brief discussion of the theoretical background subsequently applied to the analysis of the results that are discussed in Section IV. Finally, our conclusions are stated in Section V.

II. THE ISING MODEL WITH LR INTERACTIONS AND THE SIMULATION METHOD

In this paper we present and discuss simulations of the Ising Model in $d = 1$ dimensions, whose Hamiltonian $H$ is given by

$$H = -J \sum_{\langle i,j \rangle} S_i S_j / r_{ij}^{d+\sigma},$$

where $J > 0$ is the (ferromagnetic) coupling constant, $S_i$ is the spin variable at the site of coordinates $i$, which can assume two values, namely, $S_i = \pm 1$, the summation is extended to all pairs of spins placed at distances $r_{i,j} = |r_i - r_j|$, and $\sigma$ is a parameter that controls the decay of LR interactions.

Simulations are performed by using samples of length $L = 10^5$ and taking periodic boundary conditions. The LR interactions described by the Hamiltonian of equation (1) are evaluated up to a distance $|r_i - r_j| = L/2$. Also, simulations with LR interactions truncated at the $N$th neighbor, i.e., $J = 0$ for $r > N$, have been performed in order to apply Finite Range Scaling (FRS) analysis [24] and the results will be briefly discussed. Spin update is performed by using the standard Metropolis dynamics. Also, during a Monte Carlo time step (mcs) all the spins of the sample are updated once, on average.

In order to carry out the calculations we chose $\sigma = 0.75$, because for this value of the parameter the critical exponents of the Ising model are expected to be sufficiently different from mean-field values to allow a meaningful comparison with RG predictions [18, 19, 20]. Furthermore, one also likes to be as far as possible from $\sigma = 1.00$, where strong Kosterlitz-Thouless behavior is known to occur [21].

During the simulations we recorded the time dependence of the following observables: (i) The order parameter or average magnetization $M(t, \tau)$ given by
\[ M(t, \tau) = \frac{1}{L} \left( \sum_{i=1}^{L} S_i(t, \tau) \right) \]  
\[ \text{(2)} \]

where \( \tau = \frac{T - T_c}{T_c} \) is the reduced temperature and \( T_c \) is the critical temperature.

(ii) The susceptibility \( (\chi(t, \tau)) \) evaluated as the fluctuations of the order parameter, namely

\[ \chi(t, \tau) = \frac{1}{LT} (M^2(t, \tau) - M(t, \tau)^2), \]  
\[ \text{(3)} \]

where \( M^2(t, \tau) = \frac{1}{L^2} \langle (\sum_{i=1}^{L} S_i(t, \tau))^2 \rangle \).

(iii) The autocorrelation of the spin variable

\[ A(t, \tau) = \frac{1}{L} \left( \sum_{i=1}^{L} S_i(t, \tau)S_i(0, \tau) \right). \]  
\[ \text{(4)} \]

(iv) The correlation of the order parameter at the critical point, when the initial condition corresponds to uncorrelated states, given by

\[ Q(t) = \frac{1}{L^2} \langle \sum_{i=1}^{L} S_i(t) \sum_{i=1}^{L} S_i(0) \rangle, \]  
\[ \text{(5)} \]

(v) The second-order Binder cumulant \( (U(t)) \), when the initial condition corresponds to the ground state, namely,

\[ U(t, \tau) = \frac{M^2(t, \tau)}{M(t, \tau)^2} - 1, \]  
\[ \text{(6)} \]

where in all cases the brackets indicate configurational averages performed with different samples started from equivalent (but different in the case of \( T = \infty \)) initial conditions.

### III. BRIEF THEORETICAL BACKGROUND

**Short-time dynamics (STD):** Let us now analyze the expected short-time dynamic behavior when the system starts from a disordered (uncorrelated) configuration, but with a small initial magnetization. According to the argument of Janssen et al. [7], the general scaling approach of the order parameter for the nonconservative dynamics of model A (according to the classification of Hohenberg and Halperin [22]), is given by

\[ M(t, \tau, L, M_0) = b^{-\beta/\nu} M \left( t/b^z, b^{1/\nu} \tau, L/b, b^{\tau_0} M_0 \right), \]  
\[ \text{(7)} \]
where $b$ is a scaling parameter, and $\beta$ and $\nu$ are the order parameter and correlation length (static) critical exponents, respectively. Also, $z$ is the dynamical exponent. Furthermore, $x_0$ is a new exponent, introduced by Janssen et al.\cite{Janssen1984}, which accounts for the scaling dimension of the initial magnetization $M_0$, in the $M_0 \to 0$ limit.

For sufficiently large lattices, at the critical point ($\tau \equiv 0$), and by setting $b = t^{1/z}$, equation (7) becomes

$$M(t, M_0) = t^{-\beta/\nu z} M(t^{x_0} M_0),$$

which holds for a time short enough such that the correlation length ($\xi(t) \propto t^{1/z}$) is not so large ($\xi \ll L$). Furthermore, for times even shorter than the crossover time ($t_x \approx M_0^{-z/x_0}$), but larger than the microscopic time that is set when the correlation length is of the order of a single lattice spacing, one has that equation (8) becomes

$$M(t) \propto M_0 t^\theta,$$

which describes the (power-law) initial increase of the magnetization with exponent $\theta = x_0/z - \beta/\nu z$.

In the absence of an initial magnetization ($M_0 \equiv 0$), and at criticality, the scaling behavior of the susceptibility is given by

$$\chi(t) \propto t^{\gamma/\nu z},$$

where $\gamma$ is the susceptibility exponent. Also, under these conditions ($\tau = 0$ and $M_0 = 0$), the time autocorrelation function is expected to follow a power law with time according to

$$A(t) \propto t^{-\lambda},$$

where the critical exponent is given by $\lambda = d/z - \theta$, i.e., even in the absence of an initial magnetization $\lambda$ depends on the exponent $\theta$ that describes the initial increase of the order parameter according to equation (9).

On the other hand, by starting with randomly generated configurations, the correlation function of the total magnetization is also expected to follow a power law with time according to
\[ Q(t) \propto t^\theta, \]  

i.e., a relationship that allows us to obtain the initial increase exponent avoiding the numerical extrapolation \( M_0 \to 0 \) \[5\].

**Standard relaxation dynamics (SRD).** STD measurements can be further reinforced by independent measurements of the SRD, which are started from a fully ordered or ground state configuration and are performed at criticality. In this way, one could be able not only to test the validity of some exponents evaluated by means of the STD method, as well as the critical temperature, but also obtain additional exponents and test the validity of relationships between them, e.g. the hyperscaling relationship \[4\]. In fact, by starting from a ground state configuration with all spins pointing in the same direction \((T = 0)\), upon annealing to criticality, the SRD scaling approach is given by (see also equation \[8\])

\[
M(t, \tau, L) = b^{-\beta/\nu} M(t/b^\gamma, b^{1/\nu} \tau, L/b).
\]  

(13)

For large lattices and by setting \( b = t^{1/z} \) this dynamic scaling form leads to

\[
M(t, \tau) \propto t^{-\beta/\nu z} M(t^{1/\nu z} \tau).
\]  

(14)

It is well known that this power-law decay of the order parameter is valid within the long-time regime, but several numerical results indicate that it also holds in the short-time regime.

On the other hand, by taking the logarithmic derivative of equation \[14\] with respect to the reduced temperature, evaluated at the critical point, one gets

\[
\frac{\partial \log M(t, \tau)}{\partial \tau} \bigg|_{\tau=0} \propto t^{1/\nu z},
\]  

(15)

which allows us to evaluate the exponent \( 1/\nu z \), by performing measurements at and slightly away from the critical point. Furthermore, just at the critical point the second-order Binder cumulant is expected to behave according to

\[
U(t) \propto t^{d/z}.
\]  

(16)
IV. RESULTS AND DISCUSSION.

A. Standard relaxation dynamics

FIG. 1: Log-log plots of the time evolution of the magnetization $M(t)$ obtained after annealing from $T = 0$ (ground state) to the indicated temperatures. The solid line shows the fit of the curve obtained for $T_c = 2.630$, according to equation (14). More details in the text.

Focusing our attention first on the relaxation dynamic behavior, at criticality, figure 1 shows the time evolution of the magnetization at different temperatures. The critical temperature $T_c = 2.630(3)$ was determined by searching the smallest standard deviation from the power law (equation (14)), and the error bar was assessed by considering closest temperatures that present noticeable but small deviations. Also, from the fit of the data the critical exponent $\beta/\nu z = 0.091(2)$ was determined. Figure 2 (a) shows the time evolution of the second-order Binder cumulant that can be fitted with a power law with exponent $d/z = 1.05(1)$. This value yields $z = 0.95(1)$ for the dynamic exponent that is significantly
larger than the RG predictions, which place $z$ between 0.7174 (for $\sigma = 0.70$) and 0.834 (for $\sigma = 0.80$) \[1, 6\]. In principle one could expect that this result may be most likely due to the fact that $z$ depends on the specific dynamics used, however, a more detailed discussion will be offered below. Finally, by using measurements of the magnetization performed at two adjacent temperature points, namely, $T = 2.620$ and $T = 2.640$, the logarithmic derivative of the magnetization with respect to the reduced temperature was obtained. Figure 2(b) shows that this observable also exhibits a power-law behavior with exponent $1/\nu z = 0.50(2)$. Furthermore, by replacing the obtained value of $z$ in the exponent corresponding to the logarithmic derivative, one gets $1/\nu = 0.47(2)$, in agreement with both the RG prediction, namely, $1/\nu = 0.477$ \[1, 17\], and with Monte Carlo simulations performed at equilibrium, $1/\nu = 0.469$ \[17\].

FIG. 2: Time evolutions obtained after annealing at $T_c = 2.630$ from $T = 0$ (ground state) of (a) the second-order Binder cumulant ($U(t)$), and (b) the logarithmic derivative of the magnetization with respect to the reduced temperature ($\frac{\partial \log M(t)}{\partial \tau}$). The solid lines indicate the fits with equations (15) and (16), respectively.
B. Short-time Dynamics

Now we turn our attention to the short-time dynamic measurements. The autocorrelation function (figure 3(a)) exhibits a power-law behavior with time, and the exponent $\lambda = 0.94(1)$ was obtained by means of a fit with equation (11). Furthermore, the susceptibility (figure 3(b)) can also be fitted by means of a power law yielding $\gamma/\nu z = 0.869(6)$. In contrast with these measurements performed by setting $M_0 \equiv 0$, the initial increase of the magnetization has to be measured for vanishingly small values of $M_0$, as shown in figure 4. The extrapolation of the obtained results for $M_0 \to 0$ yields $\theta = 0.211(7)$ (see equation (9)), namely a figure that is close to the RG prediction given by $\theta = 0.228$ [1]. Now, by using the relationship $\lambda = d/z - \theta$ and replacing the determined exponents one gets $z = (0.94(1) + 0.211(7))^{-1} = 0.868(9)$. This value of the dynamic exponent $z$ is closer to the RG prediction than the previous measurement obtained from the relaxation dynamics given by $z = 0.95(1)$. Also, it interpolates between previously published STD results corresponding to a system of size $L = 3000$, which are given by $z = 0.81(1)$ and $0.96(4)$, for $\sigma = 0.70$ and $0.80$, respectively [6]. So, this result strongly suggests that, for this model, both dynamic processes, the short-time and the relaxation one, are governed by different dynamic exponents. We expect that this finding may be an effect of the Kosterlitz-Thouless (KT) transition observed at $\sigma = 1.00$, which even for our choice of the interaction range ($\sigma = 0.75$) may influence the dynamic behavior of the system. KT effects seem to be also present in the evaluation of the exponent $1/\nu$, which for up to $\sigma \geq 0.75$ exhibits a rapid decay toward zero, in agreement with the absence of a power-law temperature dependence of the correlation length at $\sigma = 1.00$ [17]. On the other hand, it is well known that systems undergoing a KT transition exhibit different types of dynamic behavior upon quenching and annealing [25].

Additionally, one can use the values of both $\gamma/\nu z$ and $z$ in order to estimate $\gamma/\nu = 0.754(11)$, in excellent agreement with the RG prediction given by $\gamma/\nu = \sigma = 0.75$ [17, 23]. On the other hand, by assuming that the hyperscaling relationship $(d/z - 2\beta/\nu z = \gamma/\nu z)$ holds, one can obtain the short-time dynamic estimation of $\beta/\nu = 0.122(11)$, in excellent agreement with the RG prediction given by $\beta/\nu = \frac{d-\sigma}{2} = 0.125$ [17, 23].

Furthermore, just by starting with random configurations and measuring the correlation function of the total magnetization given by equation (12), one can also obtain the initial
FIG. 3: Time evolution measured after quenching from uncorrelated (disordered) states to $T_c = 2.630$ of (a) the autocorrelation $A(t)$, and (b) susceptibility $\chi(t)$. The solid lines indicate the fits with equations (3) and (4), respectively.

increase exponent $\theta = 0.207(2)$, as shown in figure 5. This value for the exponent is in good agreement with the previous measurement obtained by using the numerical extrapolation $M_0 \to 0$, namely $\theta = 0.211(7)$. Moreover, by using this independent estimation of $\theta$ and applying the previously described procedure, the following exponents: $z = 0.872(8)$, $\gamma/\nu = 0.758(10)$, and $\beta/\nu = 0.121(10)$ can be obtained, which of course, are in good agreement with our previous estimations.

C. Finite-Range Scaling (FRS) Analysis

A FRS analysis has been developed by analogy with the finite-size scaling[24]. Here, the basic idea is to truncate the range of the interaction at a certain prefixed distance and to obtain information on the critical behavior by using scaling properties. The truncated-interaction range $N$ is defined as the number of neighbors on each side of the central spin considered in order to evaluate the Hamiltonian given by equation (1). In this way, the dependence of the critical temperature on $N$ was studied. Note that due to the periodic boundary conditions used, and the largest system size studied in the present work, $L = 10^4$, the maximal interaction range considered is $N = 5000$. Figure 6 shows the behavior of the
FIG. 4: Log-log plot of $M(t)$ versus time results showing the initial increase of the magnetization after quenching from uncorrelated (desordered) states, with a small magnetization $M_0$, to $T_c = 2.630$. Data corresponding, from top to bottom, to $M_0 = 0.1, 0.05, 0.02, 0.015, 0.010 \text{ and } 0.006$, respectively. The solid lines show the best fits of the data obtained according to equation (9). The inset shows the linear extrapolation of the values of the exponent $\theta$ to $M_0 \to 0$.

magnetization, when the system is annealed from the ground state up to the ”critical” point ($T_c(N)$) for different values of $N$. Here one observes the power-law dependence expected from equation (14) for slightly different ($N$-dependent) ”critical” temperatures. Therefore, the inset shows the dependence of $T_c(N)$ on $N^{-1}$, which was fitted with the following scaling form

$$T_c(N) = T_c(\infty) + A/N^{x_T},$$

where $T_c(\infty)$ is the critical temperature for the infinite interaction range, $x_T$ is the convergence exponent, and $A$ is a constant. The fitting was performed by imposing $x_T = 1$, because this restriction does not have significant effect on the final results due to the small variation of $T_c(N)$ with $N$, and yields $T_c(\infty) = 2.646(3)$. The extrapolated critical tempera-
FIG. 5: Log-log plot of the time evolution of the correlation function of the total magnetization after quenching randomly generated configurations to $T_c = 2.630$. The exponent $\theta$ was determined by fitting the data with the aid of equation (12) and the solid line shows the obtained fit.

The critical temperature $T_c(\infty)$ interpolates between the previously reported values for $\sigma = 0.70$ ($T_c(\infty) = 2.929$ [24] and $T_c(\infty) = 2.9269$ [26]), and $\sigma = 0.80$ ($T_c(\infty) = 2.431$ [24] and $T_c(\infty) = 2.4299$ [26]), which were obtained by means of exact calculations with the transfer matrix method and FRS analysis.

V. CONCLUSIONS

In this paper we present and discuss the results of extensive simulations of the dynamic behavior of the LR Ising magnet with interactions decaying as $r^{-(d+\sigma)}$, in $d = 1$ dimensions and with $\sigma = 0.75$. Power-law behavior of the relevant observables is found at the critical temperature $T_c = 2.630(3)$ for both the relaxation and the short-time regimes. Also, the difference between the dynamic exponents $z$ determined for these regimes indicates that even for $\sigma = 0.75$ and the sample size used ($L = 10^4$), the effects of the Kosterlitz-Thouless transition ($\sigma = 1.00$) cannot be neglected. These findings lead us to conclude that both
FIG. 6: Log-log plot of the time evolution of the magnetization obtained after annealing from the ground state to the "critical" point $T_c(N)$ as measured for different values of $N$. The solid lines show the best fits of the data performed with equation (9). The inset shows convergence of the "critical" temperatures $T_c(N)$ to $T_c(\infty)$, and the solid line corresponds to the best fit performed with the aid of equation (17).

Types of dynamic measurements provide relevant information on the critical behavior. On the other hand, all the estimated static critical exponents ($\gamma/\nu$, $\beta/\nu$, and $1/\nu$) are in good agreement with RG predictions. However, the relevant exponents for the dynamic behavior ($z$ and $\sigma$) are slightly different from the RG predictions, most likely because they may depend on the specific Monte Carlo dynamics used (Metropolis in the present paper).

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