Abstract

The classical multivariate extreme-value theory concerns the modeling of extremes in a multivariate random sample. The observations with large values in at least one component is an example of this. The theory suggests the use of max-stable distributions. In this work the classical theory is extended to the case where aggregated data, such as maxima of a random number of observations, are the actual interest. A new limit theorem concerning the domain of attraction for the distribution of the aggregated data is derived, which boils down to a new family of max-stable distributions. A practical implication of our result is, for instance, the derivation of an approximation of the joint upper tail probability for the aggregated data. The connection between the extremal dependence structure of classical max-stable distributions and that of our new family of max-stable distributions is established. By means of the so-called inverse problem, a semiparametric composite-estimator for the extremal dependence of the unobservable data is derived, starting from a preliminary estimator of the extremal dependence obtained with the aggregated data. The large-sample theory of the composite-estimator is developed and its finite-sample performance is illustrated by means of a simulation study.

Keywords: Extremal dependence; Extreme value copula; Inverse problem; Multivariate max-stable distribution; Nonparametric estimation; Pickands dependence function; Random number of maxima; Random scaling.

1 Introduction and background

The multivariate extreme-value theory aims to quantify the probability that the extremes of multiple dependent observations take place. Two commonly employed approaches for modelling extremes in high dimensions are: the componentwise maximum, where for each of the involved variables the partial maximum values are taken into account, e.g., yearly maxima, (e.g., Falk, Hüsler, and Reiss 2010, Ch. 4); the exceedance of high thresholds where only the observations that exceed a high
threshold, for at least one variable, are taken into account (e.g., Falk, Hüsler, and Reiss 2010, Ch. 5; Rootzén and Tajvidi 2006). Recent advances in these topics are discussed for instance in Dombry et al. (2017), Ho and Dombry (2017), Krupskii et al. (2018), see also the references therein. The utility of the first approach is twofold. By means of the so-called extreme-value copula and the tail-dependence function (which are indeed linked to each other), it helps to describe the stochastic behavior of component-wise maxima as well as that of raw observations (from which maxima could be derived) in the joint upper tail of their own distribution function. This makes it appealing for a wide range of applications. In this contribution we focus on the componentwise maxima approach providing new developments in the theory of the extremal dependence.

Basic foundations of the componentwise maxima approach are here briefly introduced. First, however, we specify the notation that we use throughout the paper. Given $X \subset \mathbb{R}^n$, $n \in \mathbb{N}$, let $\ell^\infty(X)$ denote the spaces of bounded real-valued functions on $X$. For $f : X \to \mathbb{R}$, let $\|f\|_\infty = \sup_{x \in X} |f(x)|$.

The arrows "$\Rightarrow$", "$\to\text{p}$", "$\Rightarrow$" denote convergence (outer) almost surely, convergence in (outer) probability and convergence in distribution of random vectors (see van der Vaart 1998, Ch. 2) or weak convergence of random functions in $\ell^\infty(X)$ (see van der Vaart 1998, Ch. 18–19), the distinction between the two will be clear from the context. For a non-decreasing function $f$, let $f^-$ denote the left-continuous inverse of $f$. The abbreviation $a \sim b$ stands for $a$ is approximately equal to $b$. Finally, the multiplication, division and maximum operation between vectors is intended componentwise.

Let $X = (X_1, \ldots, X_d)$ be a $d$-dimensional random vector with distribution $F_X$ and margins $F_X^j$, $j = 1, \ldots, d$ and $X_1, X_2, \ldots$ be independent and identically distributed (iid) copies of $X$. Assume that $F_X$ is in the maximum-domain of attraction (simply domain of attraction) of a multivariate extreme-value distribution $G$, in symbols $F_X \in \mathcal{D}(G)$. This means that there are sequences of constants $a_n > 0 = (0, \ldots, 0)$ and $b_n \in \mathbb{R}^d$ such that $(\max(X_1, \ldots, X_n) - b_n)/a_n \Rightarrow \eta$ as $n \to \infty$, where the distribution of $\eta$ is a multivariate extreme-value distribution (e.g., Falk, Hüsler, and Reiss 2010 Ch. 4) in the form

$$G(x) = C_G(G_1(x_1), \ldots, G_d(x_d)), \quad x \in \mathbb{R}^d.$$

Precisely, $G_j$’s are members of the generalized extreme-value distribution (GEV) (e.g., Falk, Hüsler, and Reiss 2010 p. 21), $C_G$ is an extreme-value copula, i.e.,

$$C_G(u) = \exp(-L ((-\ln u_1), \ldots, (-\ln u_d))), \quad u \in (0, 1]^d,$$

where $L : [0, \infty)^d \mapsto [0, \infty)$ is the so-called stable-tail dependence function (Huang 1992). $G$ is a max-stable distribution, i.e., for $k = 1, 2, \ldots$, there are norming sequences $a_k > 0$ and $b_k \in \mathbb{R}^d$ such that $G^k(a_k x + b_k) = G(x)$, for all $x \in \mathbb{R}^d$. The extreme-value copula expresses the dependence among extremes, while the stable-tail dependence function is useful because it provides a simple way to compute an approximate probability that observations fall in the upper tail region. Indeed,
for large $n$ we have

$$
\mathbb{P}(F_{X_1}(X_1) > 1 - z_1/n \text{ or } \ldots \text{ or } F_{X_d}(X_d) > 1 - z_d/n) \sim L(n^{-1}z), \quad z \in [0, \infty)^d,
$$

that is the probability that, at least one component among $X_1, \ldots, X_d$ exceeds a high percentile of its own distribution, is approximated by the stable-tail dependence function. Examples of parametric extreme-value copula models are: the Logistic or Gumbel copula (Gumbel 1960), the Hüsler-Reiss copula, (Hüsler and Reiss 1989) the extremal-$t$ copula (Nikoloulopoulos et al. 2009), just to name a few. An extensive list of additional models is available in Joe (2015, Ch. 4). Since $L$ is a homogeneous function of order 1, then the stable-tail dependence function is conveniently represented as

$$
L(z) = (z_1 + \cdots + z_d) A(t), \quad z \in [0, \infty)^d,
$$

(1.1)

where $t_j = z_j / (z_1 + \cdots + z_d)$ for $j = 2, \ldots, d$, $t_1 = 1 - t_2 - \cdots - t_d$. The function $A$, named Pickands (dependence) function, denotes the restriction of $L$ on the $d$-dimensional unit simplex

$$
S_d := \left\{ (v_1, \ldots, v_d) \in [0, 1]^d : v_1 + \cdots + v_d = 1 \right\}.
$$

It summarizes the extremal dependence among the components of $\eta$, specifically it holds that $1/d \leq \max(t_1, \ldots, t_d) \leq A(t) \leq 1$, where the lower and upper bounds represent the cases of complete dependence and independence. A synthesis of the extremal dependence is provided by the extremal coefficient, that is

$$
\theta(G) = dA(1/d, \ldots, 1/d) \in [1, d].
$$

It can be interpreted as the (fractional) number of independent variables with joint distribution $G$ and common margins. An alternative summary index that measure the dependence among observations that fall in the upper tail region is the coefficient of upper tail dependence (e.g., Joe 2015, Ch. 2.13). In the bivariate case it is equal to

$$
\lambda(F_X) = \lim_{u \uparrow 1} \mathbb{P}(X_1 > F_{X_1}^{-1}(u), X_2 > F_{X_2}^{-1}(u)) = \lim_{u \uparrow 1} \mathbb{P}(X_2 > F_{X_2}^{-1}(u), X_1 > F_{X_1}^{-1}(u)) \in [0, 1].
$$

It is said that $F_X$ exhibits independence or dependence in the upper tail whenever $\lambda(F_X) = 0$ or $\lambda(F_X) > 0$, respectively, with the case of complete dependence covered when $\lambda(F_X) = 1$. The coefficient $\lambda(F_X)$ is linked to the extremal coefficient by the relationship $\theta(G) = 2 - \lambda(F_X)$.

From this point, our new developments are described. Let $N$ be a discrete random variable taking values in $\mathbb{N}_+ = \mathbb{N} \setminus \{0\}$. Assume hereafter that $N$ with distribution $F_N$ is independent of $X_i$’s. In some applications the interest is to analyze aggregated data such as the total amounts or maximum amounts obtained on a random number of observations. The study of

$$
S_N = \left( \sum_{i=1}^N X_{i,1}, \ldots, \sum_{i=1}^N X_{i,d} \right), \quad M_N = \left( \max_{1 \leq i \leq N} X_{i,1}, \ldots, \max_{1 \leq i \leq N} X_{i,d} \right),
$$

(1.2)
may be of particular interest in insurance, finance and risk management, and in big-data problems such as the analysis of Internet traffic data. Since the number of measurements of Internet traffic is huge, the data processing is feasible only after suitable aggregation. For dimension $d = 1$ the tail’s behaviour of $S_N$ has been extensively studied in the literature (e.g., Embrechts et al. 1997; Robert and Segers 2008), while in the multivariate case there are no results and only few results are known on the joint upper tail probability relative to the random vector $M_N$ (Hashorva et al. 2017; Freitas, Häusler, and Temido 2012).

In this contribution, we derive the extreme-value copula and the stable-tail dependence function for the random vector $M_N$. To do so we characterize the joint extremal behaviour of $(M_N, N)$ by establishing a new limit theorem concerning the domain of attraction for their joint distribution. The interest is to pinpoint the conditions that guarantee asymptotic dependence and independence between $N$ and $M_N$. Although $M_N$ and $N$ are dependent, it is possible to have (under some restrictions) asymptotic independence. Roughly speaking this happens when $E(N) < \infty$. Consequently the extremal properties of $M_N$ and $N$ can be studied separately. In the literature there are no results which can point to how different the extremal behaviour of $F_{M_N}$ is with respect to $F_X$. In the special case that $P(N > y) = y^{-\alpha}L(y), \ y > 0, \ \alpha \in (0, 1)$, where $L$ is a slowly varying function at infinity, we have $E(N) = \infty$ and the condition $F_X \in D(G)$ implies $F_{M_N} \in D(G_\alpha)$, where $G_\alpha, \ \alpha \in (0, 1)$, is a new max-stable distribution with an extreme-value copula $C_{G_\alpha}$, given in (2.4), which differs from $C_G$, the extreme-value copula of $G$. The coefficient $\alpha \in (0, 1)$ influences the extremal dependence of the distribution function $G_\alpha$. We find that the extremal properties of $F_{M_N}$ can be recovered by knowing the extremal properties of $F_X$ and the tail behaviour of $N$. Practical implications of our findings are the following:

For large $n$, the joint upper tail probability of $M_N$ can be approximated as follows

$$
P(F_{M_N}^{X_1} > 1 - z_1/n \text{ or } \ldots \text{ or } F_{M_N}^{X_d} > 1 - z_d/n) \sim L^\alpha(n^{-1}z^{1/\alpha}), \quad z \in [0, \infty)^d,$$

where $M_N^{X_j} = \max_{1 \leq i \leq N} X_{j,i}$, $j = 1, \ldots, d$ and $L$ is the stable-tail dependence function relative to the extreme-value copula $C_G$. Furthermore, we have

$$
\lambda(F_X) = 2 - [2 - \lambda(F_{M_N})]^{1/\alpha}. \quad (1.3)
$$

This link among extreme properties gives rise to an interesting problem of inversion, that is, in applications where $M_N$ and $N$ are observable, it is of interest to determine the extremal properties of $F_X$. By means of the so-called inverse problem, from preliminary estimates of $\lambda(F_{M_N})$ and $\alpha$ we can reconstruct an estimate for $\lambda(F_X)$.
use a likelihood- and moments-based estimator for $\alpha$ and we use three nonparametric estimators for $A_\alpha$ (existing in the literature). Then, we appropriately combine the preliminary estimators together, obtaining a new semi-parametric composite-estimator for $A$. We develop the asymptotic theory for the proposed composite-estimator and by means of a Monte Carlo simulation study we show its finite-sample performances.

The rest of the paper is organized as follows. In Section 2, we derive our new limit theorem concerning the domain of attraction for the joint distribution of $(M_N, N)$. Different representations for the limit distribution $G_\alpha$ are described in Section 3. In Section 4, we describe an inversion method for estimating $A$ that relies on our proposed composite-estimator. We investigate the theoretical properties of the composite-estimator and by a simulation study we show its finite-sample performance. The paper concludes with a discussion in Section 5. The proofs are reported in the Appendix whereas some technical details and additional simulation results are included in the Supplementary Material.

2 Domains of Attraction

First, recall that members of the GEV distribution are: the $\alpha$-Fréchet (heavy-tailed distribution), Gumbel (light-tailed distribution) and Weibull (short-tailed distribution), in symbols, $\Phi_\alpha(x) = \exp(-x^{-\alpha})$, with $x > 0$ and $\alpha > 0$, $\Lambda(x) = \exp(-e^{-x})$ with $x \in \mathbb{R}$ and $\Psi_\alpha(x) = \exp((-x)^{-\alpha})$ with $x < 0$. We also recall that a random variable, say $S$, is positive (asymmetric) $\alpha$-stable with index parameter $0 < \alpha < 1$ if its Laplace transform is $L_S(s) = E(e^{-sS}) = e^{-s^{-\alpha}}$, $s > 0$.

Let $N$ be a random block size and $M_N$ be a vector of componentwise maxima obtained with a randomly sized block of iid random vectors $X_1, X_2, \ldots$ with common distribution $F_X$, defined in (1.2). Under the assumptions that $F_X \in \mathcal{D}(G)$ and $F_N \in \mathcal{D}(H)$, where either $H \equiv \Phi_\alpha$ or $H \equiv \Lambda$ since $N$ is positive integer-valued (Robert and Segers 2008), we establish a new limit result concerning the domain of attraction for the joint distribution of the random vector $(M_N, N)$.

**Theorem 2.1.** Assume that $F_X \in \mathcal{D}(G)$ and $F_N \in \mathcal{D}(H)$ with $H \equiv \Phi_\alpha$ or $H \equiv \Lambda$. Then, there exist norming constants $c_n > 0$, $d_n \in \mathbb{R}$, $d_n \in \mathbb{R}$ such that

$$
\lim_{n \to \infty} P_n \left( \frac{M_N - d_n}{c_n} \leq x, \frac{N - d_n}{c_n} \leq y \right) = Q(x, y),
$$

where $Q$ is a $(d+1)$-dimensional max-stable distribution. Precisely, when

1. $F_N \in \mathcal{D}(\Phi_\alpha)$, then

$$
-\ln Q(x, y) = \begin{cases} 
  y^{-\alpha}e^{-y\sigma(x, \alpha)} + \sigma^\alpha(x, \alpha) \gamma \{1 - \alpha, y \sigma(x, \alpha)\}, & \alpha \in (0, 1] \\
  -\ln G(x) + y^{-\alpha}, & \alpha > 1
\end{cases}
$$

(2.1)

for all $x \in \mathbb{R}^d$ and $y > 0$, where $\sigma(x, \alpha) = \{-\ln G(x)\}/\Gamma^{1/\alpha}(1 - \alpha)$ and $\Gamma$, $\gamma$ denote the Euler Gamma and Lower Incomplete Gamma functions with the convention $\Gamma(0) = 1$. 


2. \( F_N \in \mathcal{D}(\Lambda) \), then

\[
- \ln Q(x, y) = - \ln G(x) + e^{-y}, \quad x \in \mathbb{R}^d, \ y \in \mathbb{R}.
\] (2.2)

The margins of \( Q \) are

\[
\begin{align*}
G_\alpha(x) &:= \exp\{-\ln G(x)\}^\alpha, \quad \alpha \in (0, 1) \\
G(x), &\quad \alpha \geq 1
\end{align*}
\] (2.3)

and \( \Phi_\alpha(y) \), when \( F_N \in \mathcal{D}(\Phi_\alpha) \), \( \alpha > 0 \). While they are equal to \( G(x) \), \( x \in \mathbb{R}^d \), and \( \Lambda(y) \), \( y \in \mathbb{R} \), when \( F_N \in \mathcal{D}(\Lambda) \). Specifically, the distribution \( \Gamma_\alpha \), \( \alpha \in (0, 1) \), is a max-stable distribution with margins \( \Gamma_{\alpha,1}, \ldots, \Gamma_{\alpha,d} \), that are members of the GEV class, and extreme-value copula in the form

\[
C_{\Gamma_\alpha}(u) = \exp\{-L\alpha\{(-\ln u_1)^{1/\alpha}, \ldots, (-\ln u_d)^{1/\alpha}\}\}, \quad u \in (0, 1]^d, \ \alpha \in (0, 1),
\] (2.4)

where \( L \) is the stable-tail dependence function of the max-stable distribution \( G \).

A probabilistic interpretation of the problem addressed in Theorem 2.1 is as follows. Let \( N_1, N_2, \ldots \) be iid copies of \( N \) and set \( S_n = N_1 + \cdots + N_n \), \( M_n = \max(N_1, \ldots, N_n) \). Then,

\[
\mathbb{P}\left( \frac{M_{S_n} - d_n}{c_n} \leq x, \frac{M_n - d_n}{c_n} \leq y \right) = \mathbb{P}^n\left( \frac{M_N - d_n}{c_n} \leq x, \frac{N - d_n}{c_n} \leq y \right).
\]

Loosely speaking, the interest concerns the asymptotic distribution of the random vector \((M_{S_n}, M_n)\) appropriately normalized (a.n.). When \( F_N \in \mathcal{D}(\Lambda) \) (light-tailed) or \( F_N \in \mathcal{D}(\Phi_\alpha) \) (heavy-tailed), with \( \alpha > 1 \), then \( \mu = \mathbb{E}(N) < \infty \). Since \( n^{-1}S_n \) converges to \( \mu \), then the asymptotic distributions of \( M_{S_n} \) and \( M_{\lceil n\mu \rceil} \) a.n. are approximately the same. Furthermore, \( n^{-1/2}S_n \) and \( c_n^{-1}(M_n - d_n) \) are asymptotically independent (van der Vaart 1998, Lemma 21.19). Accordingly, \( M_N \) and \( N \) are asymptotically independent. When \( F_N \in \mathcal{D}(\Phi_\alpha) \) (heavy-tailed), with \( 0 < \alpha \leq 1 \), then \( \mathbb{E}(N) = \infty \) and \( S_n \) and \( M \) a.n. are asymptotically dependent, implying that \( M_N \) and \( N \) a.n. are asymptotically dependent too. In addition when \( \alpha \in (0, 1) \), \( c_n^{-1}S_n \) converges in distribution to a positive stable random variable \( S \), where \( c_n := c_n \Gamma^{1/\alpha}(1 - \alpha) \), implying that the asymptotic distribution of \( M_{S_n} \) and \( M_{\lceil c_nS \rceil} \) a.n. coincide and it is equal to \( \Gamma_\alpha \) in the first line of (2.3), see Corollary 2.2, which is a location-scale mixture of the max-stable distribution \( G \) obtained with a deterministic block size (see also Tawn 1990; Fougères et al. 2009 for related results).

**Corollary 2.2.** Let \( F_X \in \mathcal{D}(G) \), \( F_N \in \mathcal{D}(\Phi_\alpha) \), \( \alpha \in (0, 1) \) and \( c_n, d_n \) and \( c_n \) as in Theorem 2.1. Then \( \tilde{c}_n^{-1}S_n \sim S \), as \( n \to \infty \), where \( \tilde{c}_n = c_n \Gamma^{1/\alpha}(1 - \alpha) \) and

\[
\lim_{n \to \infty} \mathbb{P}(M_{\lceil \tilde{c}_nS \rceil} \leq c_nx + d_n) = G_\alpha(x).
\]

The dependence structure of \( Q \) defined in (2.1) and (2.2) can be synthesized by means of its extremal coefficient.
Corollary 2.3. When the expression of $Q$ is given in (2.1), then the extremal coefficient is

$$
\theta(Q) = \begin{cases} 
\exp \left( -\frac{\theta(G)}{\Gamma^{1/\alpha}(1-\alpha)} \right) + \frac{\theta(G)}{\Gamma^{1/\alpha}(1-\alpha)} \gamma \left( 1 - \alpha, \frac{\theta(G)}{\Gamma^{1/\alpha}(1-\alpha)} \right), & \alpha \in (0, 1) \\
\exp \{-\theta(G)\} + \theta(G)(\text{li}\exp \{-\theta(G)\} + 1), & \alpha = 1 \\
\theta(G) + 1, & \alpha > 1 
\end{cases}
$$

where $\text{li}$ denotes the Logarithmic Integral function. When the expression of $Q$ is given in (2.2), $\theta(Q) = \theta(G) + 1$.

The extremal coefficient for the distribution $G_\alpha$ with $\alpha \in (0, 1)$, $\theta(G_\alpha)$, is given in (3.5). In the bivariate case, using the relationship between the extremal coefficient and the coefficient of upper tail dependence, i.e., $\theta(G) = 2 - \lambda(F_X)$ then by (3.5) we obtain $\theta(G_\alpha) = (2 - \lambda(F_X))^{\alpha}$, which implies that

$$
\lambda(F_X) = 2 - (\theta(G_\alpha))^{1/\alpha}.
$$

At the same time it is also true that $\theta(G_\alpha) = 2 - \lambda(F_M)$). Hence plugging the right-hand side of the last expression into the above display we obtain the result (1.3).

### 3 Representations of the model $G_\alpha$

In this section we show that there are different representations that yield a max-stable distribution with the same copula $C_{G_\alpha}$ in (2.4) of the limit distribution $G_\alpha$. For the extreme-value copula $C_{G_\alpha}$ we derive the corresponding Pickands function.

Let $S$ be a positive $\alpha$-stable random variable with index parameter $0 < \alpha < 1$. Let $Z$ be a random vector with distribution $G_\ast$, that is a max-stable distribution with common unit-Fr´echet marginals. Assume $S$ and $Z$ to be independent. Define $R = (SZ_1, \ldots, SZ_d)$, then for every $y > 0$

$$
P(R \leq y) = E \left( G_\ast^S (y) \right) = L S \{ - \ln G_\ast (y) \} = \exp \left[ - \{ - \ln G_\ast (y) \}^\alpha \right] =: G_{\ast\alpha}(y).
$$

Distribution $G_{\ast\alpha}$ is max-stable with common $\alpha$-Fr´echet margins and extreme-value copula $C_{G_\alpha}$.

By (3.1), it follows easily that for any $\alpha_1, \alpha_2 \in (0, 1)$ $(G_{\ast\alpha_1\alpha_2})_{\alpha_1} = (G_{\ast\alpha_2})_{\alpha_1} = G_{\ast\alpha_1\alpha_2}$, i.e., its iteration does not produce a new multivariate max-stable distribution. In the special case that the components of $Z$ are independent, the copula of $G_{\ast\alpha}$ is

$$
C_{G_{\ast\alpha}}(u) = \exp \left[ -\left\{ (-\ln u_1)^{1/\alpha} + \cdots + (-\ln u_1)^{1/\alpha} \right\}^\alpha \right], \quad u \in (0, 1]^d, \quad \alpha \in (0, 1),
$$

which is the well-know Logistic or Gumbel copula (Gumbel 1960). Therefore, the elements of $Z$ are dependent for any $\alpha \in (0, 1)$, and they become nearly independent as $\alpha \to 1$ and almost completely dependent as $\alpha \to 0$.

The de Haan characterization of max-stable processes (de Haan 1984) establishes a Poisson point process construction of a random vector with any max-stable distribution $G_\ast$. A natural question
that arises here is: What is the spectral representation of a random vector \( R \) defined by the random scaling construction? The next result establishes that the representation derived in Robert (2013) indeed provides the spectral representation of \( R \).

**Proposition 3.1.** Let \( Z_1, Z_2, \ldots \) be iid copies of \( Z \), with distribution \( G_\ast \), independent of \( P_1, P_2, \ldots \) that are points of a Poisson process on \((0, \infty)\) with intensity measure \( \alpha r^{-\alpha-1} dr \), \( \alpha \in (0, 1) \). Define

\[
R = \frac{1}{\Gamma(1-\alpha)} \left( \max_{i \geq 1} P_i Z_{1i}, \ldots, \max_{i \geq 1} P_i Z_{id} \right).
\]

Then, the distribution of \( R \) is \( G_{\ast \alpha} \).

Here we provide an alternative, more general proof than that given in Robert (2013).

Next, we derive the domain of attraction relative to the generic random scaling and centering construction.

**Proposition 3.2.** Let \( X_1, \ldots, X_n \) be iid copies of the random vector \( X \), with distribution \( F_X \). Assume \( F_X \in D(G) \). Let \( S \) be a positive \( \alpha \)-stable random variable, \( \alpha \in (0, 1) \). Assume \( S \) is independent of \( X \). Define

\[
W_n := a \lfloor nS \rfloor, \quad V_n := b_n - b \lfloor nS \rfloor \frac{a_n}{a_{nS}},
\]

where \( a_n \) and \( b_n \) are the usual norming constants of \( F_X \). Then,

\[
a_n^{-1} \left( \max \{ W_n(X_1 - V_n), \ldots, W_n(X_n - V_n) \} - b_n \right) \rightsquigarrow G_\alpha, \quad n \to \infty.
\]

A direct consequence of Proposition 3.2 is the following. Transforming \( X \) into \( Y \), a random vector with common unit-Fréchet marginal distributions, and setting \( R = SY \), implies that \( F_R \in D(G_{\ast \alpha}) \). This means that the attractors of \( F_R \) and \( F_{MN} \), when \( F_N \in D(\Phi_\alpha) \), \( \alpha \in (0, 1) \), share the same extreme-value copula, \( C_{G_\alpha} \). Finally, we derive the explicit form of the Pickands function corresponding to the extreme-value copula \( C_{G_\alpha} \).

**Proposition 3.3.** The Pickands function corresponding to the \( C_{G_\alpha} \) in (2.4) is

\[
A_\alpha(t) = \|t\|_{1/\alpha} A_\alpha \left( \frac{t}{\|t\|_{1/\alpha}} \right)^{1/\alpha}, \quad t \in S_d, \ \alpha \in (0, 1),
\]

where

\[
\|t\|_{1/\alpha} = \left( 1 - \sum_{i=1}^{d-1} t_i^{1/\alpha} + \sum_{i=1}^{d-1} t_i^{1/\alpha} \right)^{\alpha}, \quad t \in S_d, \ \alpha \in (0, 1)
\]

is the Pickands function corresponding to the Logistic copula.

As a direct consequence of Proposition 3.3, the following facts follow. The smaller the parameter \( \alpha \), the more \( A_\alpha \) represents a stronger dependence level than \( A \). Since we have that \( \|1/d, \ldots, 1/d\|_{\alpha} = d^{\alpha-1} \), then by the definition of the extremal coefficient in Section 1, we obtain

\[
\theta(G_\alpha) = (\theta(G))^{\alpha}.
\]
By solving for $A$ in equation (3.3), we obtain the inverse relation between $A_\alpha$ and $A$, i.e.,

$$A^*(t) := A\left(\left(t / \|t\|_{1/\alpha}\right)^{1/\alpha}\right) = \left(A_\alpha(t) / \|t\|_{1/\alpha}\right)^{1/\alpha}, \quad t \in S_d,$$

and $A(t) = A^*(t^{\alpha/\|t^\alpha\|_1})$, providing the expression for the Pickands function in (1.1).

4 Inferring the Pickands function with the inverse method

4.1 A semiparametric estimator

In this section we introduce a new semiparametric procedure to estimate the Pickands function $A$ in (1.1). Several nonparametric estimators are already available for $A$, see e.g., Gudendorf and Segers (2012), Berghaus et al. (2013), Marcon et al. (2017) among others and see Vettori et al. (2018) for a review. Here we propose a novel approach for inferring $A$. Unlike the above references, we work with the maxima of aggregated data and we construct an estimator for $A$, exploiting an inversion method via (3.6). Observe that, since $t \mapsto (t / \|t\|_{1/\alpha})^{1/\alpha}$ is a bijective map, estimating $A^*$ is equivalent to estimating $A$.

Let $(\eta_1, \xi_1), (\eta_2, \xi_2), \ldots, \eta_n$ be iid random vectors with joint distribution in (2.1) with $\alpha \in (0, 1)$. Assume that a sample of $n$ observations from such a sequence is available. An estimate of $A^*$ is obtained by combining the results of a two-step procedure: we estimate $\alpha$ and $A_\alpha$, we plug the estimates in (3.6). Precisely, $\xi_1, \ldots, \xi_n$ follows a $\alpha$-Fréchet distribution. For estimating $\alpha$ we consider two well-know estimators: the Generalized Probability Weighted Moment (GPWM) (Guillou et al. 2014) and the Maximumu Likelihood (ML). In the first case the estimator is

$$\hat{\alpha}_{n_{GPWM}} := \left(k - 2 \frac{\hat{\mu}_{1,k}}{\hat{\mu}_{1,k-1}}\right)^{-1},$$

for $k \in \mathbb{N}_+$, where

$$\hat{\mu}_{a,b} = \int_0^1 H_{\alpha}^\left(-v\right)v^\alpha\left(-\ln v\right)^b dv, \quad a, b \in \mathbb{N}$$

and

$$H_{\alpha}(y) = \frac{1}{n} \sum_{i=1}^n \mathbb{I}(\xi_i \leq y), \quad y > 0. \quad (4.2)$$

In the second case the estimator is

$$\hat{\alpha}_{n_{ML}} := \arg\max_{\alpha \in (0, \infty)} \sum_{i=1}^n \ln \hat{\Phi}_{\alpha}(\xi_i),$$

where $\hat{\Phi}_{\alpha}(x) = \partial/\partial x \Phi_{\alpha}(x), \quad x > 0$.

The sequence $\eta_1, \ldots, \eta_n$ follows the distribution $G_\alpha$. For estimating $A_\alpha$ we consider three well-know estimators: Pickands (P) (Pickands 1981), Capéraà-Fougère-Genest (CFG) (Capéraà et al. 1997)
and Madogram (MD) (Marcon et al. 2017). In the first case the estimator is
\[
\hat{A}_{\alpha,n}(t) = \left( \frac{1}{n} \sum_{i=1}^{n} \hat{\varphi}_i(t) \right)^{-1},
\]
where
\[
\hat{\varphi}_i(t) := \min_{1 \leq j \leq d} \left\{ -\frac{1}{t_j} \ln \left( \frac{n}{n+1} G_{n,j}(\eta_{i,j}) \right) \right\}
\]
and for every \( x \in \mathbb{R} \) and \( j \in \{1, \ldots, d\} \)
\[
G_{n,j}(x) := \frac{1}{n} \sum_{i=1}^{n} \mathbb{1}(\eta_{i,j} \leq x).
\]
In the second case the estimator is
\[
\hat{A}^{\text{CFG}}_{\alpha,n}(t) = \exp \left( -\frac{1}{n} \sum_{i=1}^{n} \ln \hat{\varphi}_i(t) - \varsigma \right),
\]
where \( \varsigma \) is the Euler’s constant. Finally, in the third case the estimator is
\[
\hat{A}^{\text{MD}}_{\alpha,n}(t) := \hat{\nu}_n(t) + c(t),
\]
where
\[
c(t) = \frac{t_1}{1 + t_1} + \cdots + \frac{t_d}{1 + t_d}.
\]
For brevity we denote the estimators of \( \alpha \) and \( A_{\alpha} \) by \( \hat{\alpha}_n \) and \( \hat{A}_{\alpha,n} \), respectively, where the symbols “•” and “◦” are representative of the labels “GPWM”, “ML” and “P”, “CFG”, “MD”, respectively.

Then, plugging the estimators into equation (3.6) we obtain the following composite-estimator for \( A^* \)
\[
\hat{A}^{\text{•◦}}_n(t) := \left( \hat{A}^{\text{•}}_{\alpha,n}(t)/\|t\|_{1/\hat{\alpha}_n^*} \right)^{1/\hat{\alpha}_n^*}, \quad t \in \mathcal{S}_d.
\]
Next, we establish the asymptotic theory of the composite-estimator in (4.9) defined by all the combinations of the GPWM and ML estimators for \( \alpha \) with the P, CFG and MD estimators for \( A_{\alpha} \).
Our results rely on the following assumptions.

**Condition 4.1.** For \( j \in \{1, \ldots, d\} \), let \( \mathcal{U}_j = \{ u \in [0,1]^d : 0 < u_j < 1 \} \). Assume in the following that:

(i) for \( j \in \{1, \ldots, d\} \), the first-order partial derivative \( \hat{C}_{G_{\alpha}}(u) := \partial / \partial u_j C_{G_{\alpha}}(u) \) exists and is continuous in \( \mathcal{U}_j \);}
(ii) for \(i, j \in \{1, \ldots, d\}\), the second-order partial derivative \(\ddot{C}_{G_\alpha}(u) := \frac{\partial^2}{\partial u_i \partial u_j} C_{G_\alpha}(u)\) exists and is continuous in \(U_i \cap U_j\) and

\[
\sup_{u \in U_i \cap U_j} \max(u_i, u_j) |\ddot{C}_{G_\alpha}(u)| < \infty.
\]

**Theorem 4.1.** Assume Condition 4.1(i) holds true. Additionally, Condition 4.1(ii) is also assumed to hold true with the estimators \(\hat{A}_{n}^{P}, \hat{A}_{n}^{CFG}\) and condition \(\alpha > 1/(k-1)\) for any fixed \(k \in \mathbb{N}_+\) is also assumed to be valid with the estimators \(\hat{A}_{n}^{MD, GPWM}\). Then as \(n \to \infty\)

\[
\sqrt{n} \left( \hat{A}_{n}^{P} - A^* \right) \rightsquigarrow \phi_{\alpha, \bullet}(C_Q)
\]

in \(\ell^\infty(S_d)\), where the map \(\phi_{\alpha, \bullet}\) is given in Appendix A.6, \(C_Q\) is a zero-mean Gaussian process with covariance function

\[
\text{Cov}(C_Q(u), C_Q(v)) = C_Q(\min(u, v)) - C_Q(u)C_Q(v), \quad u, v \in [0,1]^{d+1},
\]

where

\[
C_Q(u, v) = Q(G_{\alpha, 1}^\leftarrow(u_1), \ldots, G_{\alpha, d}^\leftarrow(u_d), \Phi_{\alpha}^\leftarrow(v))
\]

and the minimum is taken componentwise. Moreover,

\[
\|\hat{A}_{n}^{P} - A^*\|_\infty \xrightarrow{p} 0, \quad \|\hat{A}_{n}^{MD, GPWM} - A^*\|_\infty \xrightarrow{a.s.} 0, \quad n \to \infty.
\]

**Remark 4.2.** In Gudendorf and Segers (2012) and Guillou et al. (2018) modified versions of the estimators \(P, CFG\) and \(MD\) for \(A_{\alpha}\) are proposed to guarantee that \(\hat{A}_{\alpha,n}(e_j) = 1\) for all \(n = 1, 2, \ldots\) and \(j = 1, \ldots, d\) where \(e_j = (0, \ldots, 0, 1, 0, \ldots, 0)\). The results in Theorem 4.1 are also valid when such adjusted estimators are considered in place of (4.4), (4.6) and (4.7), respectively. This outcome follows from asymptotic arguments already established in the works referenced here above.

### 4.2 Simulation study

We show the finite sample performance of the composite-estimator \(\hat{A}_{n}^{P} \circ \bullet\) through a simulation study consisting of two experiments. Hereafter we consider, for the \(P, CFG\) and \(MD\) estimators, the adjusted versions mentioned in Remark 4.2.

**First experiment:** For simplicity we sample from the limiting distribution \(G_{\alpha}\) in (3.1). Specifically, consider \(R = (SZ)\), where \(Z\) is a two-dimensional random vector with max-stable distribution with common unit-Fréchet margins and Symmetric Logistic copula with dependence parameter \(\psi \in (0,1]\) (e.g., Tawn 1988) and \(S\) is a positive \(\alpha\)-stable random variable with \(\alpha \in (0,1)\). The distribution of \(R\) is \(G_{\alpha}\) with common \(\alpha\)-Fréchet margins. Set \(\xi = \max(R_1, R_2)\) and \(\eta = R\). In this setup, although the joint distribution of \((\xi, \eta)\) is not exactly \(Q\) in the first line of (2.1), its marginal distributions are \(\alpha\)-Fréchet (with some scale parameter) and \(G_{\alpha}\), respectively.
Then, we simulate \( n \) independent replications of \((\xi, \eta)\) and we estimate \( \alpha \) by the GPWM estimator \( \hat{\alpha}_{n}^{\text{GPWM}} \) in equation (4.1), with \( k = 5 \), and the ML estimator \( \hat{\alpha}_{n}^{\text{ML}} \) in (4.3). Next, we estimate \( A_{\alpha} \) by the P estimator \( \hat{A}_{\alpha,n}^{P} \) in (4.4), CFG estimator \( \hat{A}_{\alpha,n}^{\text{CFG}} \) in (4.6) and MD estimator \( \hat{A}_{\alpha,n}^{\text{MD}} \) in (4.7). Finally, by the inverse method we estimate \( \hat{A}^{\star} \) using the composite-estimator \( \hat{A}_{n}^{\circ \circ} \) in equation (4.9). The asymptotic properties of \( \hat{A}_{n}^{\circ \circ} \) within the present setting are discussed in Section 4 of the supplementary material, where we show that they are almost the same as those described in Theorem 4.1, as expected.

We repeat the simulation and estimation steps for different values of the dependence parameters \( \alpha \) and \( \psi \) and different sample sizes. Precisely, we consider \( \alpha = 0.5, 0.633, 0.767, 0.9 \) and 15 equally spaced values in \([0.1, 1]\) for \( \psi \) and \( n = 50, 100 \). We repeat this experiment (considering different values of the parameters and sample sizes) 1000 times and we compute a Monte Carlo approximation of the Mean Integrated Squared Error (MISE), i.e.,

\[
\text{MISE}(\hat{A}_{n}^{\circ \circ}, A^{\star}) = E \left( \int_{S_d} \{ \hat{A}_{n}^{\circ \circ}(t) - A^{\star}(t) \}^2 \, dt \right)
\]

where the first and second terms in the second line are known as integrated squared bias (ISB) and integrated variance (IV) (Gentle 2009, Ch. 6.3).

The results for the sample size \( n = 50 \) are summarized in Figure 1. The MISE, ISB and IV (\( \times 1000 \)) of the GPWM-based estimators are reported from the first to the third row. The solid black, dashed green and dotted red lines report the results obtained estimating \( A_{\alpha} \) with P, CFG and MD estimators, respectively. The results for the different values of \( \alpha \) are reported along the columns. For each fixed value of \( \alpha \) we see that IV is close to zero at the strongest dependence level (\( \psi = 0.1 \)), then it increases with the decrease of the dependence level (\( \psi \) increases approaching one).

On the contrary, ISB takes the largest value at \( \psi = 0.1 \) and then it decreases with the decrease of the dependence level, for the cases \( \alpha = 0.5, 0.633 \). Overall, for the case \( \alpha = 0.5 \), MISE takes the largest value at \( \psi = 0.1 \) and then it decreases with the decrease of the dependence level. For the case \( \alpha = 0.633 \), MISE does not change much over the whole range of dependence levels, since ISB and IV compensate each other. While, for the cases \( \alpha = 0.767, 0.9 \), IV grows much more than ISB decreases, implying that MISE increases with the decrease of the dependence level. The smallest values of ISB and IV are obtained with the CFG-based and P-based estimator, respectively. Overall on the basis of the MISE, the best performance is obtained with the CFG-based estimator, although there is little difference with the P-based estimator. Finally, we show in the supplementary material that there is not much difference in the performance of the P-, CFG- and MD-based estimators already for the sample size \( n = 100 \).

In Figure 2 the comparison between the estimation results obtained with the GPWM- and ML-based
Figure 1: MISE, ISB and IV for 1000 samples of size 50 from the bivariate extreme-value copula in (2.4) with the logistic Pickands dependence model, for different values of the dependence parameters \( \psi \) and \( \alpha \). The Pickands function \( A^* \) is estimated by the composite-estimator \( \hat{A}_{GPWM}^* \) in formula (4.9).

Estimators are reported. Precisely, from the first to the third row, the ratio between the MISE, ISB and IV computed estimating the Pickands function \( A^* \) by the GPWM- and ML-based estimators are displayed. On the basis of the ISB, for the cases \( \alpha = 0.5, 0.633 \), the GPWM- and ML-based estimators perform very similarly. However, for \( \alpha = 0.633 \), the ML-based estimators outperform the GPWM-based estimators when \( \psi \) is close to 1, that is, at weak dependence levels. For the cases \( \alpha = 0.733, 0.9 \), the ML-based estimators clearly outperform the GPWM-based estimators. On the basis of the IV, the GPWM-based estimators outperform the ML-based estimators for all cases, except for the GPWM-MD-based estimator. Indeed, for the cases \( \alpha = 0.633, 0.767, 0.9 \), the GPWM-MD-based estimator outperforms the ML-MD-based estimator only for strong dependence levels, while the opposite is true for weak dependence levels. Overall on the basis of the MISE, the GPWM-based estimators outperform the ML-based estimators for the case \( \alpha = 0.5 \), while they
perform very similarly for the case $\alpha = 0.633$. On the contrary, for the cases $\alpha = 0.767, 0.9$, the ML-based estimators outperform the GPWM-based estimators, except for the GPWM-P-based and ML-P-based estimators that perform very similarly.

**Second experiment:** We study the performance of the composite-estimator $\widehat{A}_{n}^{\circ \ast}$ in (4.9) when it is used with data that are only approximately coming from the limit distribution $Q$. This is a more realistic scenario. Specifically, we set $N = \lceil N' \rceil$, where we assume that $N'$ follows a standard Pareto distribution with shape parameter $\alpha \in (0, 1)$. We simulate $N$ observations of a two-dimensional random vector $X$ with a standard bivariate Student-$t$ distribution with a fixed value of the correlation $\rho$ and the degrees of freedom $\upsilon$. We recall that a Student-$t$ distribution is in the domain of attraction of a multivariate extreme-value distribution with an extreme-value copula that is the so-called extremal-$t$ (e.g., Nikoloulopoulos et al. 2009). In the bivariate case, the extremal coefficient of the extremal-$t$ copula is $\theta = 2T_{\upsilon+1}[(v+1)(1-\rho)/(1+\rho)]^{1/2}$, where $T_{\upsilon+1}$ is a univariate standard Student-$t$ distribution with $\upsilon + 1$ degrees of freedom. Next, with the simulated
Figure 3: MISE, ISB and IV for 1000 samples of size 50 from an approximated distribution $Q$, obtained on the basis of the standard Pareto distribution for $N$ and the bivariate Student-$t$ distribution for $X$, for different values of the parameters $\alpha$ and $\rho$, $\upsilon$. The parameter $\theta$ is the extremal coefficient related to the corresponding extreme-value copula extremal-$t$.

data we compute the observed value of the componentwise maxima $M_N$ in (1.2). We repeat this simulation steps $n' = 500$ times generating $n'$ independent observations from the pair $(N, M_N)$ with which we compute an observation from the random variable $\xi = \max(N_1, \ldots, N_{n'})$ and vector $\eta = \max(M_{N_1}, \ldots, M_{N_{n'}})$, where the later maximum is meant componentwise. We repeat these simulation steps $n$ times generating a data sample approximately drawn from the distribution $Q$, whose expression is given in the first line (2.1) and where the expression of $G$ can been deduced in Nikoloulopoulos et al. (2009).

Then, we estimate $\alpha$ using the observations generated from the sequence $\xi_1, \ldots, \xi_n$ with the GPWM and ML estimators in (4.1) and (4.3). We estimate $A_\alpha$ using the observations generated from the sequence $\eta_1, \ldots, \eta_n$ with the P, CFG and MD estimators in (4.4)-(4.7). Finally, we estimate $A^*$ by the composite-estimator $\hat{A}^*_n$. 

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The simulation and estimation steps are repeated 1000 times and an approximation of the MISE is computed. We repeat the experiment for different values of the model parameters and different sample sizes. In particular, for the Student-$t$ distribution we consider, for the degrees of freedom $\nu = 1$ and 15, equally spaced values of $\rho$ in $[-0.99, 0.99]$. With these parameters’ values the extremal coefficient $\theta$ (related to the extremal-$t$ copula) takes values $[1, 2]$, where the lower and upper bounds represent the cases of complete dependence and independence. We also consider the same values of $\alpha$ as in the previous experiment and the sample sizes $n = 50, 100$.

Figure 3 displays the results obtained with the GPWM-based estimators for the sample size $n = 50$. Although in this experiment we consider synthetic data from a more complicated model than that of the previous experiment, and notwithstanding that the data only come approximately from the distribution $Q$, the results summarised by ISB, IV and MISE are very similar in both experiments. Due to this, here we do not comment on the results for each individual estimator. This finding is a good outcome indicating that our proposed composite-estimator provides a good performance.

Figure 4: Ratio between MISE, ISB and IV computed estimating the Pickands function $A^*$ by the estimator $\hat{A}^*_{GPWM, n}$ and $\hat{A}^*_{ML, n}$ in formula (4.9). The same setting as Figure 3 is considered.
overall. In particular, our composite-estimator displays a moderate distortion, despite that the uniform consistency guarantee of Theorem 4.1 does not directly extend to the present setting. Similar conclusions are obtained with sample size $n = 100$ and in this case the results are reported in the supplementary material.

Figure 4 displays the comparison between the estimates obtained with the GPWM- and ML-based estimators. Concerning the configurations with $\alpha > 0.5$ the ML-based estimators considerably outperform GPWM-based estimators in terms of MISE. These conclusions are valid for all three P-, CFG- and MD-based estimators. More specifically, IV is smaller for the ML-based estimators and better performances of them are obtained for weaker dependence structures (when $\theta$ approaches 2). The ML-based estimators are much less biased than then GPWM-based estimators and the difference is much more pronounced, increasing the values of $\alpha$ and for weaker dependence structures (when $\theta$ approaches 2), although for the P- and CFG-based estimators such a difference diminishes when $\theta$ is close to 2.

5 Discussion

The analysis of aggregated data is an important topic in statistics. In insurance, finance, risk management and big-data problems examples of aggregated data that are of particular interest are the total amounts or maximum amounts, computed on a random number of observations. These can be described through the random vectors $S_N$ and $M_N$ in (1.2). Apart from the univariate case, where several results on the extreme behaviour of $S_N$ are available, a multivariate extreme-value theory that characterizes the extreme behaviour of $S_N$ and $M_N$ is not yet available. This contribution makes a first step in establishing such a theory. Although $S_N$ is usable on a wider range of applications than $M_N$, we studied the extreme behaviour of the latter as it is considerably simpler with respect to the former. This was our stating point.

Nevertheless, the study on the extreme behaviour of $M_N$ is anything but simple and there are still some open problems. For instance, it still remains to be established an exact algorithm in order to draw samples from the limit distribution $Q$ in the first line of (2.1). The modelling of the joint upper tail associated to the distribution function of $M_N$ would benefit from the derivation of new nonparametric estimators defined on the basis of threshold exceedances (for at least one component). This would make our theory useful to all applications where there are few block-maxima are available, but many threshold exceedances are at hand.

Concluding, an important step forward on the multivariate extreme-value theory of aggregated data would be fulfilled by extending our results (probabilistic and inferential) to the case of random vector $S_N$, that as already mentioned, has a broader practical applicability.
A Appendix: Proofs

A.1 Proof of Theorem 2.1

Let $U_j(t) := F_{X_j}(1 - 1/t)$, $t > 1$, $D_j(x) = G_{j}^{-1}(e^{-1/x})$, $x \in \text{supp}(G_j)$, for $j = 1, \ldots, d$, and

$$F_\ast(x) = F_X\{U_1(\cdot), \ldots, U_d(\cdot)\}, \quad G_\ast(y) = G\{D_1(y_1), \ldots, D_d(y_d)\}, \quad y \in (0, \infty)^d.$$ 

Since $F_X \in \mathcal{D}(G)$, then

$$\lim_{n \to \infty} \frac{1 - F_\ast(ny)}{1 - F_\ast(n1)} = -\frac{\ln G_\ast(y)}{\theta(G)}, \quad y \in (0, \infty)^d,$$

see e.g., Resnick (2007, Ch. 5).

The proof is organized in three parts: the derivation of the norming constants, a preliminary result and the main body of the proof.

A.1.1 Norming constants

We first derive $c_n$ and $d_n$. When $F_N \in \mathcal{D}(\Phi_\alpha)$, $\alpha \in (0, 1]$, then $\bar{F}_N(y) = 1 - F_N(y)$, $y > 0$, satisfies

$$\bar{F}_N(y) \sim \mathcal{L}(y) y^{-\alpha}, \quad y \to \infty. \quad (A.2)$$

Set $z_n = F_N^{-1}(1 - \{\theta(G)\}^\alpha/(n\Gamma(1 - \alpha)))$, where $\Gamma(0) := 1$ by convention, then

$$\mathcal{L}(z_n) z_n^{-\alpha} \sim \{\theta(G)\}^\alpha/(n\Gamma(1 - \alpha)), \quad n \to \infty$$

and

$$\Gamma(1 - \alpha)\{\bar{F}_\ast(m_n)/\theta(G)\}^\alpha \mathcal{L} \{1/\bar{F}_\ast(m_n)\} \sim n^{-1}, \quad n \to \infty, \quad (A.3)$$

where $m_n = F_N^{-1}(1 - 1/z_n)$ with $F_\ast(y) = F_\ast(y1)$, $y > 0$, and $z_n$ satisfies $z_n \sim 1/\bar{F}_\ast(m_n)$ as $n \to \infty$. We highlight that the symbols $F_\ast(x)$ and $F_\ast(x)$ are used to denote two different functions, a multivariate and a univariate function, respectively. Which of the two we are referring to will be clear from the context. Hence, in this case we set $d_n = 0$ and

$$c_n = \theta(G)/(\bar{F}_\ast(m_n)\Gamma^{1/\alpha}(1 - \alpha)).$$

Instead, when $F_N \in \mathcal{D}(\Phi_\alpha)$, $\alpha > 1$, or $F_N \in \mathcal{D}(\Lambda)$, since $E(N) \in (0, \infty)$ we denote $m_n = nE(N)$ and define $c_n$ and $d_n$ in the standard way, as described for instance in Resnick (2007, pp. 48-54).

We now derive $c_n$ and $d_n$. When $F_N \in \mathcal{D}(\Phi_\alpha)$, $\alpha > 0$,

(i) if $F_{X_j} \in \mathcal{D}(\Phi_{\beta_j})$, then we set $d_{n,j} = 0$ and

$$c_{n,j} = \begin{cases} U_j(m_n), & \alpha \in (0, 1] \\ U_j(n\{E(N)\})^{1/\beta_j}, & \alpha > 1, \end{cases}$$

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When \( F \in D(\Lambda) \), then we set
\[
\begin{align*}
c_{n,j} &= \begin{cases} 
\omega_j(d_{n,j}), & \alpha \in (0, 1) \\
\omega_j\{\Upsilon_j^{-}(1 - 1/\delta_j n)\}, & \alpha > 1 
\end{cases} \\
d_{n,j} &= \begin{cases} 
\Upsilon_j^{-}(1 - 1/\delta_j n), & \alpha \in (0, 1) \\
c_{n,j}\ln E(N) + \Upsilon_j^{-}(1 - 1/\delta_j n), & \alpha > 1,
\end{cases}
\end{align*}
\]
where \( \Upsilon_j \) is the Von Mises function associated to \( \bar{F}_{X_j} \), \( \omega_j \) its auxiliary function (e.g., Resnick 2007, pp. 40-43) and \( \delta_j = \lim_{x \to \infty} F_{X_j}(x)/(1 - \Upsilon_j(x)) \);

(iii) if \( F_{X_j} \in D(\Psi_{\beta_j}) \), then we set \( d_{n,j} = x_{0,j} \), where \( x_{0,j} = \sup\{x : F_{X_j}(x) < 1\} \), and
\[
\begin{align*}
c_{n,j} &= \begin{cases} 
\{\bar{F}_{X_j}^{-}(1 - 1/m_n)\}^{-1}, & \alpha \in (0, 1) \\
\{\bar{F}_{X_j}^{-}(1 - 1/n)E(N)^{1/\beta_j}\}^{-1}, & \alpha > 1,
\end{cases}
\end{align*}
\]
where \( \bar{F}_{X_j}(x) = F_{X_j}(x_{0,j} - 1/x) \).

When \( F_N \in D(\Lambda) \), then we set \( c_n \) and \( d_n \) equal to the case \( F_N \in D(\Phi_\alpha) \), with \( \alpha > 1 \).

With these norming constants, we obtain the following approximations as \( n \) goes to infinity
\[
U_j^+(c_{n,j}x_j + d_{n,j}) \sim m_n D_j^+(x_j) = \begin{cases} 
m_n x_j^{\beta_j}, & F_{X_j} \in D(\Phi_{\beta_j}) \\
m_n e^{x_j}, & F_{X_j} \in D(\Lambda) \} \\
m_n (-x_j)^{-\beta_j}, & F_{X_j} \in D(\Psi_{\beta_j}) 
\end{cases}
\]

A.1.2 Preliminary result

Lemma A.1. Let \( p_n(x) = F_X(c_n x + d_n) \). If \( F_N \in D(\Phi_\alpha), \alpha \in (0, 1) \), then for all \( x \in \mathbb{R}^d \) we have
\[
\lim_{n \to \infty} n\{1 - E\{p_n^N(x)\}\} = \{-\ln G(x)\}^\alpha.
\]

If \( F_N \in D(\Lambda) \) or \( F_N \in D(\Phi_\alpha), \alpha > 1 \), then we have
\[
\lim_{n \to \infty} n\{1 - E\{p_n^N(x)\}\} = -\ln G(x).
\]

Proof. When \( F_N \in D(\Phi_\alpha), \alpha \in (0, 1) \), then by Corollary 8.17 in Bingham et al. (1989) we have
\[
1 - L_N(s) \sim \Gamma(1 - \alpha)s^{\alpha}, \quad s \downarrow 0,
\]
with \( L_N \) the Laplace transform of \( N \). Therefore, as \( n \to \infty \)
\[
1 - E\{p_n^N(x)\} = 1 - L_N\{-\ln p_n(x)\} \\
\sim \Gamma(1 - \alpha)s\{-1/\ln p_n(x)\}\{-\ln p_n(x)\}^\alpha \\
\sim \Gamma(1 - \alpha)s[1/(1 - p_n(x))]{\{1 - p_n(x)\}}^\alpha.
\]
By (A.1) and (A.4) we obtain
\[ 1 - p_n(x) \sim 1 - F_s \{ m_n D_1^- (x_1), \ldots, m_n D_d^- (x_d) \} \]
\[ \sim \bar{F}_s (m_n) \{ - \ln G(x) \} / \theta(G) \] (A.5)
as \( n \to \infty \). Using this last approximation and (A.3), we derive the following asymptotic equivalence
\[ n \{ 1 - E (p_n^N (x)) \} \sim \Gamma (1 - \alpha) \{ \} / \theta(G) \]
\[ \sim \Gamma (1 - \alpha) \{ - \ln G(x) \} / \theta(G) \]
as \( n \to \infty \). When \( F_N \in D (\Phi^\alpha) \), \( \alpha > 1 \), or \( F_N \in D (\Lambda) \), it holds that \( E (N) < \infty \). Consequently,
\[ 1 - L_N (s) \sim s E (N), \quad s \downarrow 0. \]

Therefore, the following asymptotic equivalence holds true
\[ 1 - E (p_n^N (x)) = 1 - L_N \{ - \ln p_n (x) \} \]
\[ \sim - \ln p_n (x) E (N) \sim \{ 1 - p_n (x) \} E (N), \quad n \to \infty. \]

We recall that in these cases \( m_n = n E (N) \in (0, \infty) \), thus by (A.1) and (A.4) we have that
\[ 1 - p_n (x) \sim 1 - F_s \{ n E (N) D_1^- (x_1), \ldots, n E (N) D_d^- (x_d) \} \]
\[ \sim \bar{F}_s \{ n E (N) \} \{ - \ln G(x) \} / \theta(G) \]
\[ \sim - \ln G(x) / \{ n E (N) \}, \quad n \to \infty. \] (A.6)

Finally, we obtain the equality
\[ \lim_{n \to \infty} n \{ 1 - E (p_n^N (x)) \} = - \ln G(x). \]
and thus the proof is complete. \( \square \)

A.1.3 Main body of the proof

As \( n \to \infty \), we have
\[ -n \ln \mathbb{P} (M_N \leq c_n x + d_n, N \leq c_n y + d_n) \sim n \{ 1 - \mathbb{P} (M_N \leq c_n x + d_n, N \leq c_n y + d_n) \} \]
\[ = n \{ 1 - E (p_n^N (x) \mathbb{1} \{ N \leq u_n (y) \}) \} \]
\[ = n \{ 1 - E (p_n^N (x)) \} + n E (p_n^N (x) \mathbb{1} \{ N > u_n (y) \}) \]
\[ \equiv T_{1,n} + T_{2,n}, \]

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where \( u_n(y) = c_n y + d_n \). The limiting behavior of \( T_{1,n} \) has been established in Lemma A.1. Note that for every \( v \in [0, 1] \) we have
\[
\mathbb{P} \left( p_n^N(x) \mathbb{I} \{ N > u_n(y) \} \leq v \right) = F_N \{ u_n(y) \} + \mathbb{P} \{ N \geq \ln v / \ln p_n(x), N > u_n(y) \}
\]
\[
= \begin{cases} 
F_N \{ u_n(y) \}, & v = 0 \\
F_N \{ u_n(y) \} + \bar{F}_N \{ \ln v / \ln p_n(x) \}, & 0 < v < p_n^{u_n(y)}(x) \\
1, & v \geq p_n^{u_n(y)}(x). 
\end{cases}
\]
Therefore,
\[
n \mathbb{E} \left( p_n^N(x) \mathbb{I} \{ N > u_n(y) \} \right) = n \int_0^{p_n^{u_n(y)}(x)} \mathbb{P} (p_n^N(x) \mathbb{I} \{ N > u_n(y) \} > v) dv \\
= n p_n^{u_n(y)}(x) \bar{F}_N \{ u_n(y) \} - n \int_0^{p_n^{u_n(y)}(x)} \bar{F}_N \left( \frac{\ln v}{\ln p_n(x)} \right) dv \\
= n I_{n,1} + n I_{n,2}.
\]
When \( F_N \in \mathcal{D}(\Phi_\alpha) \) with \( \alpha \in (0, 1) \), from (A.2), (A.3), (A.5) and the definition of \( c_n \) it follows that
\[
n I_{n,1} \sim n \mathcal{L}(c_n y) (c_n y)^{-\alpha} \exp[-y c_n \{ 1 - p_n(x) \}]
\]
\[
\sim n \Gamma(1 - \alpha) \mathcal{L} \left( \frac{\theta(G)y}{F_s(m_n) \Gamma^{1/\alpha}(1 - \alpha)} \right) \left( \frac{\theta(G)}{F_s(m_n)} \right)^{-\alpha} y^{-\alpha} \exp[-y \{ - \ln G(x) \} / \Gamma^{1/\alpha}(1 - \alpha)]
\]
\[
\sim y^{-\alpha} \exp[-y \{ - \ln G(x) \} / \Gamma^{1/\alpha}(1 - \alpha)] =: y^{-\alpha} \pi(x, y), \quad n \to \infty,
\]
where we have used the fact that
\[
p_n^{u_n(y)}(x) \to \pi(x, y)
\]
as \( n \to \infty \). Furthermore, by uniform convergence (Resnick 2007, Proposition 0.5) we also obtain
\[
n I_{n,2} \sim -n \int_0^{\pi(x, y)} \bar{F}_N \left( \frac{\ln v}{\ln p_n(x)} \right) dv
\]
\[
\sim -n \bar{F}_N \left( \frac{1}{- \ln p_n(x)} \right) \int_0^{\pi(x, y)} (- \ln v)^{-\alpha} dv
\]
\[
\sim -\frac{(- \ln G(x))^{\alpha}}{\Gamma(1 - \alpha)} \int_{-\ln \{ \pi(x, y) \}}^{\infty} t^{-\alpha} \exp(-t) dt
\]
\[
= \frac{(- \ln G(x))^{\alpha}}{\Gamma(1 - \alpha)} \gamma \left( 1 - \alpha, \frac{y \{ - \ln G(x) \}}{\Gamma^{1/\alpha}(1 - \alpha)} \right) - \{ - \ln G(x) \}^{\alpha},
\]
as \( n \to \infty \) and the first part of (2.1) follows.
When \( F_N \in \mathcal{D}(\Phi_\alpha) \) with \( \alpha > 1 \), then by (A.6) we have that
\[
\lim_{n \to \infty} p_n^{u_n(y)}(x) = 1
\]
and hence \( n I_{n,1} \to y^{-\alpha} \) as \( n \to \infty \). Recall that \( u_n(y) = c_n y \) in this case, with \( c_n/n = o(1) \).

Furthermore, by Karamata's theorem (Resnick 2007, p. 17) and (A.6) we also obtain

\[
 n|I_{n,2}| = -n \ln p_n(x) \int_{u_n(y)}^{\infty} \bar{F}_N(t) \exp(-t(-\ln p_n(x))) dt \\
\leq -n \ln p_n(x) \int_{u_n(y)}^{\infty} \bar{F}_N(t) dt \\
\sim -n \ln p_n(x) u_n(y) \bar{F}_N(u_n(y)), \quad n \to \infty \\
\sim \left[\{-\ln G(x)/E(N)\}u_n(y) \bar{F}_N\{u_n(y)\}\right], \quad n \to \infty.
\]

Since \( u_n(y) \bar{F}_N\{u_n(y)\} = c_n y \bar{F}_N(c_n y) \), then by item (v) in (Resnick 2007, p. 23) we have

\[
c_n y \bar{F}_N(c_n y) \sim y^{1-\alpha} c_n/n \sim y^{1-\alpha} n^{1/\alpha - 1} \to 0
\]
as \( n \to \infty \). As a consequence \( \lim_{n \to \infty} n|I_{n,2}| = 0 \) and thus the second part of (2.1) follows.

When \( F_N \in \mathcal{D}(\Lambda) \), using Propositions 0.10 and 0.16 and Lemma 1.2 in Resnick (2007) it can be shown that \( u_n(y)/n \to 0 \) as \( n \to \infty \) (see Section 2 in the Supplementary Material). Consequently, by (A.6) it holds that

\[
nI_{n,1} \sim \exp(-y) \exp[u_n\{1 - p_n(x)\}] \\
\sim \exp(-y) \exp[u_n\{-\ln G(x)/E(N)\}/n], \quad n \to \infty \\
\sim \exp(-y), \quad n \to \infty.
\]

Furthermore, by (A.7)

\[
n|I_{n,2}| \leq -n \ln p_n(x) \int_{u_n(y)}^{\infty} \bar{F}_N(t) dt \\
\sim \left[\{-\ln G(x)/E(N)\}\int_{u_n(y)}^{\infty} \bar{F}_N(t) dt\right], \quad n \to \infty.
\]

The term in the second line of (A.9) goes to zero as \( n \to \infty \) implying that \( n|I_{n,2}| \to 0 \) (see Section 2 in the Supplementary Material). Consequently, equation (2.2) follows and the proof is now complete.

A.2 Proof of Corollary 2.2

Let \( c_n, d_n, c_n \) be the norming sequences defined in Subsection A.1 for the case \( F_N \in \mathcal{D}(\Phi_\alpha) \), \( \alpha \in (0,1) \). In particular \( d_n = 0 \), \( nF_N(c_n) \sim 1 \) as \( n \to \infty \) and

\[
\tilde{c}_n \sim F_N^+\left(1 - \frac{1}{n\Gamma(1 - \alpha)}\right), \quad n \to \infty.
\]

Therefore the first result \( \tilde{c}_n^{-1} S_n \sim S \) as \( n \to \infty \), now follows from Theorem 5.4.2 in Uchaikin and Zolotarev (2011).
Let $p_n(x) = F_X(c_n x + d_n)$, in the previous subsection it has been established that

$$p_n^{c_n}(x) \sim \exp \left( - \frac{-\ln G(x)}{\Gamma^{1/\alpha}(1-\alpha)} \right), \quad n \to \infty.$$ 

Consequently, by dominated convergence

$$\lim_{n \to \infty} \mathbb{P}(M_{[\tilde{c}_n S]} \leq c_n x + d_n) = \lim_{n \to \infty} \int_0^\infty p_n^{\tilde{c}_n s}(x) dF_S(s)$$

$$= \lim_{n \to \infty} \int_0^{\infty} \{p_n^{c_n}(x)\}^{\tilde{c}_n s / c_n} dF_S(s)$$

$$= \int_0^{\infty} \left( e^{\frac{-\ln G(x)}{\Gamma^{1/\alpha}(1-\alpha)}} \right) s^{\Gamma^{1/\alpha}(1-\alpha)} dF_S(s)$$

$$= L_S(-\ln G(x))$$

$$= G_\alpha(x)$$

establishing the proof.

### A.3 Proof of Proposition 3.1

Observe that $E(Z_1^\alpha) = \Gamma(1-\alpha)$ for every $0 < \alpha < 1$. Set $K_\alpha = 1/\Gamma(1-\alpha) > 0$. With this notation we have

$$\frac{1}{K_\alpha} = E(Z_1^\alpha) = \int_0^\infty \mathbb{P}(Z_1 > t^{1/\alpha}) dt$$

$$= \int_0^\infty \left( 1 - e^{-t^{1/\alpha}} \right) dt = \int_0^\infty \left( 1 - e^{-v^{1/\alpha}} \right) v^{-2} dv. \quad (A.10)$$

For any positive $y_1, \ldots, y_d$ we have

$$-\ln \mathbb{P}(R_1 \leq y_1, \ldots, R_d \leq y_d) = K_\alpha E\left( \max_{1 \leq j \leq d} Z_j^\alpha / y_j^\alpha \right)$$

$$= K_\alpha \int_0^\infty \mathbb{P} \left( \exists j \in \{1, \ldots, d\} : Z_j^\alpha > y_j^\alpha \right) dv$$

$$= \int_0^\infty \mathbb{P} \left( \exists j \in \{1, \ldots, d\} : v K_\alpha Z_j^\alpha > y_j^\alpha \right) v^{-2} dv$$

$$= \int_0^\infty \mathbb{P} \left( \exists j \in \{1, \ldots, d\} : Z_j > y_j(K_\alpha v)^{-1/\alpha} \right) v^{-2} dv$$

$$= \int_0^\infty \left\{ 1 - \mathbb{P} \left( \forall j \in \{1, \ldots, d\} : Z_j \leq y_j(K_\alpha v)^{-1/\alpha} \right) \right\} v^{-2} dv.$$ 

Now, we have

$$\mathbb{P} \left( \forall j \in \{1, \ldots, d\} : Z_j \leq y_j(K_\alpha v)^{-1/\alpha} \right) = \exp \left( -L(1/y_1, \ldots, 1/y_d) K_\alpha^{1/\alpha} v^{1/\alpha} \right),$$

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where $L(z)$, with $z = 1/y$, is the stable-tail dependence function. Consequently, by (A.10) we obtain the final result

\[-\ln P(R_1 \leq y_1, \ldots, R_d \leq y_d) = \int_0^\infty \left[ 1 - e^{-L(1/y_1, \ldots, 1/y_d) v^{1/\alpha}} \right] v^{-2} dv = (L(1/y_1, \ldots, 1/y_d))^{\alpha} K_\alpha \int_0^\infty \left[ 1 - e^{-v^{1/\alpha}} \right] v^{-2} dv = L^{\alpha}(1/y_1, \ldots, 1/y_d).\]

A.4 Proof of Proposition 3.2

Let $M_n := \max\{X_1, \ldots, X_n\}$. By the max-stability of $G$ and equation (5.18) in (Resnick 2007), for every $s > 0$ we have

\[
\frac{a_{[ns]}}{a_n} \left( \frac{M_n - b_n}{a_n} - \frac{b_n - b_{[ns]}}{a_{[ns]}} \right) \rightsquigarrow G^s, \ n \to \infty.
\]

Consequently, by the equality $G^{\alpha}(x) = E(G^S(x))$, $x \in \mathbb{R}^d$, and the dominated convergence theorem

\[
G^{\alpha}(x) = \int_0^\infty G^s(x)dF_S(s) = \lim_{n \to \infty} \int_0^\infty \Pr \left\{ a_{[ns]} \left( \frac{M_n - b_n}{a_n} - \frac{b_n - b_{[ns]}}{a_{[ns]}} \right) \leq x \right\} dF_S(s) = \lim_{n \to \infty} \Pr \left( a_n^{-1} \left\{ \max\{W_n(X_1 - V_n), \ldots, W_n(X_n - V_n)\} - b_n \right\} \leq x \right).
\]

The proof is now complete.

A.5 Proof of Proposition 3.3

Let $A^{\alpha}$ be the Pickands function of $G^{\alpha}$, $\alpha \in (0, 1)$. By the definition of the stable-tail dependence function we have

\[
L^{\alpha}(z_1, \ldots, z_d) = (z_1 + \cdots + z_d)A^{\alpha}(t), \quad t \in S_d.
\]

On the other hand, using the representation of $G^{\alpha}$ by the distribution $G^s$ and the Pickands function $A$ of $G^s$ we have

\[
L^{\alpha}(z_1, \ldots, z_d) = \left( \sum_{j=1}^d z_j^{1/\alpha} \right)^\alpha \left\{ A \left( \frac{t^{1/\alpha}}{\sum_{j=1}^{d-1} t_j^{1/\alpha} + (1 - \sum_{j=1}^{d-1} t_j^{1/\alpha})} \right) \right\}^\alpha.
\]

Choosing $(z_1, \ldots, z_d)$ such that $\sum_{j=1}^d z_j = 1$ we obtain further

\[
A^{\alpha}(t) = \left( \sum_{j=1}^{d-1} t_j^{1/\alpha} + (1 - \sum_{j=1}^{d-1} t_j)^{1/\alpha} \right)^\alpha \left( \frac{t^{1/\alpha}}{\sum_{j=1}^{d-1} t_j^{1/\alpha} + (1 - \sum_{j=1}^{d-1} t_j)^{1/\alpha}} \right),
\]

hence the result in (3.3) follows.
A.6 Proof of Theorem 4.10

The proof is organized in four parts: notation, some preliminary results, the main body of the proof and auxiliary results.

A.6.1 Notation

Recall that \((\eta_1, \xi_1), \ldots, (\eta_n, \xi_n)\), \(n = 1, 2, \ldots\), are iid random vectors with distribution \(Q\) in (2.1) with \(\alpha \in (0, 1)\). For \(i = 1, 2, \ldots\), and \(j \in \{1, \ldots, d\}\), let

\[
V_i := \Phi_\alpha(\xi_i), \quad U_{i,j} := G_{\alpha,j}(\eta_{i,j}), \quad \hat{U}_{i,j} := G_{n,j}(\eta_{i,j}),
\]

where \(\Phi_\alpha\) is the \(\alpha\)-Fréchet distribution, \(G_{\alpha,j}\) is the \(j\)-th margin of the distributions in the first line of (2.3) and \(G_{n,j}\) is as in (4.5). Set \(U_i = (U_{i,1}, \ldots, U_{i,d})\) and \(\hat{U}_i = (\hat{U}_{i,1}, \ldots, \hat{U}_{i,d})\). In the sequel when the index \(i\) is omitted we refer to a single observation. For every \(u \in [0, 1]^d\) and \(v \in [0, 1]\), define the copula functions

\[
C_{Q,n}(u, v) := \frac{1}{n} \sum_{i=1}^{n} 1(U_i \leq u, V_i \leq v), \quad C_{G_{\alpha,n}}(u) = C_{Q,n}(u, 1), \quad B_n(v) = C_{Q,n}(1, v),
\]

where \(1 = (1, \ldots, 1)\), and the copula processes

\[
C_{Q,n}(u, v) := \sqrt{n}(C_{Q,n}(u, v) - C_Q(u, v)), \quad C_{G_{\alpha,n}}(u) = C_{Q,n}(u, 1), \quad B_n = C_{Q,n}(1, v).
\]

Let \(C_{G_{\alpha}}(u) := C_{Q}(u, 1), \quad u \in [0, 1]^d\). The covariance function of \(C_{G_{\alpha}}\) is as in (4.11), with \(C_Q\) replaced by \(C_{G_{\alpha}}\). Furthermore, for every \(u \in [0, 1]^d\), define the empirical copula function and process

\[
\hat{C}_{G_{\alpha,n}}(u) = \frac{1}{n} \sum_{j=1}^{n} 1(\hat{U}_j \leq u), \quad \hat{C}_{G_{\alpha,n}} = \sqrt{n}(\hat{C}_{G_{\alpha,n}} - C_{G_{\alpha}}).
\]

For every \(f \in \ell([0, 1]^{d+1})\), let \(g_{\epsilon} : \ell([0, 1]^{d+1}) \mapsto \ell([0, 1]^{d+1})\) be the weighting map defined by

\[
(g_{\epsilon}(f))(z) = \begin{cases} w_{\epsilon}^{-1}(z)f(z), & z \in (0, 1]^{d+1} \setminus \{1, \ldots, 1\} \\ 0, & \text{otherwise} \end{cases}
\]

where, for any fixed \(\epsilon \in [0, 1/2)\), \(w_{\epsilon}\) denotes the weighting function

\[
w_{\epsilon} : [0, 1]^{d+1} \mapsto [0, 1] : z \mapsto \min_{1 \leq j \leq d+1} z_j^\epsilon \left(1 - \min_{1 \leq j \leq d+1} z_j\right)^{\epsilon}.
\]

To keep a light notation, when \(f(z)\) is computed at \(z = (u, 1)\) we still write \(g_{\epsilon}(f)\). The difference in meaning will be clear from the context.

By Genest and Segers (2009) and Gudendorf and Segers (2012) we have that as \(n \to \infty\)

\[
g_{\epsilon}(C_{Q,n}) \Rightarrow g_{\epsilon}(C_Q), \quad g_{\epsilon}(C_{G_{\alpha,n}}) \Rightarrow g_{\epsilon}(C_{G_{\alpha}}),
\]

(A.16)
in $\ell([0,1])^{d+1}$ and $\ell([0,1])^d$, respectively. For every $f \in \ell([0,1])^{d+1}$, let $\pi_1, \ldots, d : \ell([0,1])^{d+1} \mapsto \ell([0,1])^d$ and $\pi_{d+1} : \ell([0,1])^{d+1} \mapsto \ell([0,1])$ be the selection maps defined by

\[(\pi_1, \ldots, d(f))(u) := f(u_1, \ldots, u_d, 1), \quad (\pi_{d+1}(f)) := f(1, \ldots, 1, v).\] (A.17)

For every $\alpha \in (0, 1)$, $\epsilon \in [0, 1/2)$ and $u \in (0, 1]^d \setminus \{1, \ldots, 1\}$, let $\omega_{\epsilon,u} : [0,1]^d \mapsto \mathbb{R}$ be the weighted function defined by

\[\omega_{\epsilon,u}(v) := \frac{1}{(\pi_1, \ldots, d(w_{\epsilon}))(u)}(v \leq u - CG_{\alpha}(u)).\] (A.18)

For every $u, v \in [0,1]^d$, set

\[\omega_{\epsilon,u}'(v) = \begin{cases} 
\omega_{\epsilon,u}(v), & u \in (0, 1]^d \setminus \{1, \ldots, 1\} \\
0, & \text{otherwise.}
\end{cases}\] (A.19)

Finally, for any given a signed measure $M$ on the measurable space $(\mathcal{X}, \mathcal{X})$ and a measurable function $f$, denote $Mf := \int f dM$ - see e.g. van der Vaart and Wellner (1996, p.80).

**A.6.2 Preliminary results**

The results in this section rely on the following conditions.

**Condition A.1.** Let $\phi : \ell^\infty([0,1]^d) \mapsto \ell^\infty(S_d)$ be a continuous linear map and $\hat{A}_{\alpha,n}$ be an estimator of $A_{\alpha}$ that allows the representation

\[\sqrt{n}(\ln \hat{A}_{\alpha,n} - \ln A_{\alpha}) = \phi_{g_{\epsilon}}(CG_{\alpha,n}) + o_p(1)\]

for some $\epsilon \in (0, 1/2)$, where $\phi_{g_{\epsilon}} = \phi \circ g_{\epsilon}$, $g_{\epsilon}$ is as in (A.15) and $CG_{\alpha,n}$ is as in (A.13).

**Condition A.2.** Let $\hat{\alpha}_n$ be an estimator of $\alpha$ satisfying one of the following properties:

(i) there is a continuous linear map $\tau : \ell^\infty([0,1]) \mapsto \mathbb{R}$ such that

\[\sqrt{n}(\hat{\alpha}_n - \alpha) = \tau(B_n) + o_p(1),\]

where $B_n$ is as in (A.13);

(ii) there is a measurable function $\zeta : (0, +\infty) \mapsto \mathbb{R}$ with $\Phi_{\alpha}\zeta = 0$ and $\Phi_{\alpha}\zeta^2 < \infty$, such that

\[\sqrt{n}(\hat{\alpha}_n - \alpha) = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \zeta(\xi_i) + o_p(1).\]

The next two propositions establish weak convergence results for the composite-estimator $\hat{A}^*_n$ in (4.9) obtained combining an estimator of $A_{\alpha}$ that satisfies Condition A.1 with an estimator of $\alpha$ that satisfies Condition A.2(i) and A.2(ii), respectively.
Proposition A.2. Let \( \hat{A}_n^* \) be the estimator in (4.9) obtained by the composition of two estimators \( \hat{A}_{\alpha,n} \) and \( \tilde{\alpha}_n \) satisfying Condition A.1 and Condition A.2(i), respectively. Then, in \( \ell^\infty([0,1]^d) \)
\[
\sqrt{n}(\hat{A}_n^* - A^*) \sim A \left\{ \phi'_{g_\epsilon}(\mathbb{C}_Q) + K_\alpha \tau_{g_\epsilon}(\mathbb{C}_Q) \right\}, \quad n \to \infty. \tag{A.20}
\]
Specifically, \( \phi'_{g_\epsilon} = \alpha^{-1} \phi \circ \pi_1 \circ \ldots \circ \pi_d \circ g_\epsilon, \tau_{g_\epsilon} = \tau \circ \pi_{d+1} \circ (w_\epsilon, g_\epsilon), \) where \( \pi_1, \ldots, \pi_d \) and \( \pi_{d+1} \) are as in (A.17), \( w_\epsilon \) as in (A.15), and for any \( t \in S_d \)
\[
K_\alpha(t) = \alpha^{-2} \left\{ \|t\|_{1/\alpha}^{-1/\alpha} \sum_{1 \leq j < d; t_j > 0} t_j^{1/\alpha} \ln t_j - \ln A_\alpha(t) \right\}. \tag{A.21}
\]

Proof. The result in (A.20) relies on the following result.

Lemma A.3. For \( \|t\|_{1/\alpha} \) in (3.4), the map
\[
g : (0, \infty) \mapsto \ell^\infty(S_d) : h \mapsto \left( \ln \|t\|_{1/\alpha}^{1/h} \right)_{t \in S_d}
\]
is Hadamard differentiable at \( \alpha \) with derivative
\[
\{(g_\alpha(h)) (t) \}_{t \in S_d} = \left\{ -h \alpha^{-2} \|t\|_{1/\alpha}^{-1/\alpha} \sum_{1 \leq j < d; t_j > 0} t_j^{1/\alpha} \ln t_j \right\}, \quad 0 < h < \infty.
\]
For the proof see Section 3 of the supplementary material. For simplicity we focus on \( \ln \hat{A}_n^* \) and \( \ln \hat{A}_n \). Then, we obtain
\[
\sqrt{n}(\ln \hat{A}_n^* - \ln A^*) = \sqrt{n} \left( \frac{1}{\hat{\alpha}_n} \ln \hat{A}_{\alpha,n} - \frac{1}{\alpha} \log A_\alpha \right) - \sqrt{n} \left( \ln \|\cdot\|_{1/\hat{\alpha}_n} - \ln \|\cdot\|_{1/\alpha} \right)
= : T_{1,n} + T_{2,n}.
\]

By Condition A.2(i), the functional version of Slutsky’s lemma (e.g., van der Vaart and Wellner 1996, page 32) and the delta method (e.g., van der Vaart 1998, Theorem 3.1) it follows that
\[
T_{1,n} = \frac{1}{\alpha} \sqrt{n}(\ln \hat{A}_{\alpha,n} - \ln A_\alpha) - \alpha^{-2} \ln A_\alpha \sqrt{n}(\hat{\alpha}_n - \alpha) + o_p(1).
\]
By Lemma A.3 and the functional delta method (van der Vaart 1998, Ch. 20) it follows that
\[
T_{2,n} = (K_\alpha + \alpha^{-2} \log A_\alpha) \sqrt{n}(\hat{\alpha}_n - \alpha) + o_p(1),
\]
where \( K_\alpha \) is as in (A.21). Then, by Condition A.1 and A.2(i) we obtain
\[
T_{1,n} + T_{2,n} = \frac{1}{\alpha} \phi_{g_\epsilon}(\mathbb{C}_\alpha,_{n}) + K_\alpha \sqrt{n}(\hat{\alpha}_n - \alpha) + o_p(1)
= \frac{1}{\alpha} \phi_{g_\epsilon}(\mathbb{C}_\alpha,_{n}) + K_\alpha \tau_{g_\epsilon}(\mathbb{R}_n) + o_p(1)
= \phi'_{g_\epsilon}(\mathbb{C}_Q,_{n}) + K_\alpha \tau_{g_\epsilon}'(\mathbb{C}_Q,_{n}) + o_p(1). \tag{A.22}
\]
Now, the result in (A.20) follows by applying the continuous mapping theorem and the functional delta method in the last line of (A.22). \( \square \)
Proposition A.4. Let \( \hat{A}_n^* \) be the estimator in (4.9) obtained by the composition of two estimators \( \hat{A}_{\alpha,n} \) and \( \hat{\alpha}_n \) satisfying Condition A.1 and Condition A.2(ii), respectively, with:

\[
(\phi(f))(y) = \sum_{i=0}^{m} \int_{a}^{b} f(\beta_{i,1}(z;t_1), \ldots, \beta_{i,d}(z;t_d)) K_i(z;t) dz,
\]

where \( z \mapsto \beta_{i,j}(z,t_j) \) is a bijective and continuous function, with \( i = 1, \ldots, m \), for \( m = 1, 2, \ldots \), \( j \in \{1, \ldots, d\} \) and \( t \in S_d \), and where \( K_i \) is a function satisfying

\[
\sup_{t \in S_d} \max_{0 \leq i \leq m} |K_i(z;t)| \leq K(z), \quad z \in (a, b),
\]

for \( -\infty \leq a < b \leq \infty \) and some integrable function \( K \); and \( \varphi = \zeta \circ \Phi_\alpha^\perp \) satisfying

\[
-\infty < \mathbb{E}[\omega_{\epsilon,u}(U)\varphi(V)] < \infty
\]

with \( \omega_{\epsilon,u} \) as in (A.19). Then, in \( \ell^\infty(S_d) \) as \( n \to \infty \)

\[
\sqrt{n} \left( \hat{A}_n^* - A^* \right) \sim_A \phi_{\alpha,\varphi}^\perp(C_Q).
\]

Specifically, \( \phi_{\alpha,\varphi}^\perp = \phi'' \circ \phi_{\alpha,\varphi}^\prime \), where \( \phi' : \ell^\infty([0,1]^d) \to \ell^\infty(S_d) : f \mapsto \alpha^{-1} \phi(f) + K_{m+1} f(1, \ldots, 1), \)

\[
K_{m+1}(t) = K_\alpha(t) - \frac{1}{\alpha} \sum_{i=0}^{m} \int_{a}^{b} K_i(z;t) dz, \quad t \in S_d
\]

and \( \phi_{\alpha,\varphi}^\prime(C_Q) \) is a zero-mean Gaussian process with covariance function defined in (A.25).

Proof. For any \( \{u_1, \ldots, u_k\} \subset [0,1]^d \), the random vectors \( (\omega_{\epsilon,u_1}(U_1), \ldots, \omega_{\epsilon,u_k}(U_k), \varphi(V_i)) \), \( i = 1, \ldots, n \), are i.i.d. with zero-mean and finite pairwise covariances, by arguments in Genest and Segers (2009), Gudendorf and Segers (2012), Condition A.2(ii) and (A.24). Let

\[
g_{\epsilon}^\prime(C_Q,n)(u) = n^{-1/2} \sum_{i=1}^{n} \omega_{\epsilon,u}(U_i), \quad \varphi_n(u) = \delta(u) \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \varphi(V_i), \quad u \in [0,1]^d,
\]

where \( \delta(u) = 1 \) for all \( u \in [0,1]^d \). Note that \( g_{\epsilon}^\prime(C_Q,n)(u) = g_{\epsilon}(C_{Q,n})(u) \), with \( u \in [0,1]^d \). Then, \( g_{\epsilon}^\prime(C_Q,n) \) and \( \varphi_n \) are asymptotically tight (e.g., van der Vaart and Wellner 1996, Definition 1.3.7) and by the central limit theorem we have that \( (g_{\epsilon}^\prime(C_Q,n)(u_1), \ldots, g_{\epsilon}^\prime(C_Q,n)(u_k), \varphi_n(u_1), \ldots, \varphi_n(u_k)) \sim N(0, \Sigma) \) as \( n \to \infty \). By arguments in van der Vaart and Wellner (1996, p. 42 point 3), these facts are sufficient to claim that the class of functions \( Y_{\epsilon,\varphi} := \{ (a, b) \mapsto \omega_{\epsilon,u}(a) + \delta(u)\varphi(b) : u \in [0,1]^d \} \)

is \( C_Q \)-Donsker (van der Vaart and Wellner 1996, pp. 80-82). Indeed, introducing the map

\[
g_{\epsilon,\varphi} : M \mapsto \{ Mf : f \in Y_{\epsilon,\varphi} \}
\]

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defined on the space of signed measure \(M\) on \([0, 1]^{d+1}\), for all \(u \in [0, 1]^d\), we have that \(g'_{\epsilon, \varphi}(C_{Q, n})(\omega_{\epsilon, u} + \delta(u)\varphi) = g'_{\epsilon}(C_{Q, n})(u) + \tilde{\varphi}_n(u)\). Then, as \(n \to \infty\), \(g'_{\epsilon, \varphi}(C_{Q, n}) \sim g'_{\epsilon, \varphi}(C_Q)\) in \(\ell^\infty(\mathcal{Y}_{\epsilon, \varphi})\), where \(g'_{\epsilon, \varphi}(C_Q)\) is a zero-mean Gaussian process with covariance function

\[
\text{Cov}\left\{g'_{\epsilon, \varphi}(C_Q)(\omega_{\epsilon, u} + \delta(u)\varphi), g'_{\epsilon, \varphi}(C_Q)(\omega_{\epsilon, v} + \delta(v)\varphi)\right\} = \begin{cases} 
E\{\omega_{\epsilon, u}(U) + \varphi(V)\{\omega_{\epsilon, v}(U) + \varphi(V)\}, & u, v \in \mathcal{V} \\
E\{\omega_{\epsilon, u}(U) + \varphi(V)\} \varphi(V), & u \in \mathcal{V}, v \in \mathcal{V}^c \\
E(\varphi^2(V)), & u, v \in \mathcal{V}^c
\end{cases}
\]  
(A.25)

and where \(\mathcal{V} = (0, 1]^d \setminus \{1, \ldots, 1\}\). Since each element of \(\mathcal{Y}_{\epsilon, \varphi}\) corresponds to a unique \(u \in [0, 1]^d\), we can consider the processes \(g'_{\epsilon, \varphi}(C_{Q, n}), g'_{\epsilon, \varphi}(C_Q)\) as indexed on the latter set. By (A.24), Condition A.2(ii) and the first line of (A.22) it follows that

\[
\sqrt{n}(\hat{A}_{n, \epsilon}^* - A^*) = \phi'' \circ g'_{\epsilon, \varphi}(C_{Q, n}) + o_p(1).
\]

The final results follow by applying the continuous mapping theorem and the functional delta method to the above expression.

\[\square\]

A.6.3 Main body of the proof

We start analyzing the case when \(\alpha\) is estimated with the ML estimator in (4.3) and \(A_\alpha\) with the MD estimator in (4.7). We recall that the estimator in (4.3) is the unique solution of log-likelihood equation \(n^{-1} \sum_{i=1}^n \hat{L}_\alpha(\xi_i) = 0\), where

\[
\hat{L}_\alpha(x) = \partial/\partial \hat{\alpha} \ln \Phi_{\hat{\alpha}}(x) = 1/\hat{\alpha} + \ln x(x^{-\hat{\alpha}} - 1), \quad x > 0.
\]

Noting that \(\xi^{-1}\) is a Weibull random variable, then by using similar arguments to van der Vaart (1998, Theorem 5.41 and 5.42) it follows that \(\hat{\alpha}_n^\text{ML} \overset{p}{\to} \alpha\) as \(n \to \infty\) and

\[
\sqrt{n}(\hat{\alpha}_n^\text{ML} - \alpha) = \frac{1}{i_\alpha} \frac{1}{\sqrt{n}} \sum_{i=1}^n \hat{L}_\alpha(\xi_i) + o_p(1),
\]  
(A.26)

where \(i_\alpha = \alpha^{-2}\{(1 - \zeta^2) + \pi^2/6\}\) is the Fisher information and \(\zeta\) is Euler’s constant.

Assuming that Condition 4.1(i) holds true, then \(\hat{A}_{\alpha,n}^\text{MD}\) satisfies Condition A.1 by Lemma A.5 with \(\phi = \phi_{\text{MD}}\) where \(\phi_{\text{MD}}\) is as in (A.29). Furthermore, \(\hat{\alpha}_n^\text{ML}\) satisfies Condition A.2(i) by (A.26) with \(\zeta = \zeta_{\text{ML}} = i_\alpha^{-1} \hat{L}_\alpha\). Define

\[
\varphi_{\text{ML}}(v) := \zeta_{\text{ML}} \circ \Phi_{\alpha}^{-1}(v) = i_\alpha^{-1} \alpha^{-1}\{1 + (1 + \ln v)\ln(-\ln v)\}, \quad v \in (0, 1),
\]  
(A.27)

then (A.24) is satisfied with \(\varphi = \varphi_{\text{ML}}\), by Lemma A.6. Therefore, from Proposition A.4 it follows that in \(\ell^\infty([0, 1]^d)\),

\[
\sqrt{n}(\hat{A}_{n, \epsilon}^{\text{MD, ML}} - A^*) \sim \phi_{\text{MD, ML}}(C_Q), \quad n \to \infty.
\]
Specifically, \( \phi_{\text{MD,ML}} = A_\alpha \phi' \circ g_{0,\varphi_{\text{ML}}} \), with \((\phi'(f))(t) = \alpha^{-1}(\phi_{\text{MD}}(f))(t) + K_{M_{d+1}}(t)f(1,\ldots,1)\) for every \( f \in \ell^\infty([0,1]^d) \) and \( t \in S_d \), where \( \phi_{\text{MD}} \) is given in (A.29), \( K_{M_{d+1}} \) is defined via

\[
K_{M_{d+1}}(t) = K_\alpha(t) - \frac{(1 + A_\alpha(t))^2}{\alpha A_\alpha(t)} \left( \sum_{j=1}^d \int_0^1 \tilde{C}_{G_{\alpha,j}}(v^t,\ldots,v^t) dv - 1 \right)
\]

and \( K_\alpha \) is as in (A.21). Moreover, \( g'_{0,\varphi_{\text{ML}}}(C_Q) \) is a zero-mean Gaussian process with covariance function

\[
\text{Cov}\{g'_{0,\varphi_{\text{ML}}}(C_Q)(u), g'_{0,\varphi_{\text{ML}}}(C_Q)(v)\} = C_{G_\alpha}(\min(u,v)) - C_{G_\alpha}(u)C_{G_\alpha}(v) + T_\alpha(u) + T_\alpha(v) + 1,
\]

for every \( u, v \in [0,1]^d \), with

\[
T_\alpha(\cdot) = \frac{1}{i_\alpha \alpha} \left( C_{G_\alpha}(\cdot) - \int_0^1 \frac{\partial}{\partial v} C_Q(u,v)(1 + \ln v) \ln(-\ln v) dv \right).
\]

Finally, from the weak convergence result it follows by the functional version of Slutsky’s lemma that \( \|\hat{A}_n^{\text{MD,ML}} - A^*\|_\infty \xrightarrow{p} 0 \) as \( n \to \infty \). Next, we study the case when \( \alpha \) is estimated with the GPWM estimator in (4.1) and \( A_\alpha \) with the MD estimator in (4.7). Here, we additionally assume that \( \alpha > 1/(k - 1) \). By Lemma A.8, the estimator \( \hat{\alpha}_n^{\text{GPWM}} \) satisfies Condition A.2(i) with \( \tau = \tau_{\text{GPWM}} \) given in (A.31). Then, by Proposition (A.2) it follows that in \( \ell^\infty([0,1]^d) \),

\[
\sqrt{n} \left( \hat{A}_n^{\text{MD,GPWM}} - A^* \right) \xrightarrow{\text{as}} \phi_{\text{MD,GPWM}}(C_Q), \quad n \to \infty,
\]

where \( \phi_{\text{MD,GPWM}}(f) = A_\alpha \{ \phi_{g_0}'(f) + K_\alpha \tau_{g_0}'(f) \} \), with \( \phi = \phi_{\text{MD}} \) given in (A.29), \( K_\alpha \) in (A.21) and \( \tau = \tau_{\text{GPWM}} \) in (A.31). Furthermore, given the result in Lemma A.8 and since \( \|n^{-1/2} E_n\|_\infty \xrightarrow{\text{as}} 0 \) then we have that \( \hat{\alpha}_n^{\text{GPWM}} \xrightarrow{\text{as}} \alpha \) as \( n \to \infty \). Therefore, by Lemma A.7 it follows that

\[
\left\| \hat{A}_n^{\text{MD,GPWM}} - A^* \right\|_\infty \xrightarrow{\text{as}} 0, \quad n \to \infty.
\]

Now, we study the case when \( \alpha \) is estimated with the ML estimator in (4.3) and \( A_\alpha \) with the P and CFG estimator in (4.4) and (4.6), respectively. Assuming that Conditions 4.1(i) and 4.1(ii) hold true, then by Segers (2012, Proposition 3.1 and 4.2) we have

\[
\hat{C}_{G_{\alpha,n}}(u) = C_{G_{\alpha,n}} - \sum_{j=1}^d \hat{C}_{\alpha,j} C_{G_{\alpha,n}}(1,\ldots,1,u_j,1,\ldots,1) + R_n,
\]

where \( \hat{C}_{G_{\alpha,n}} \) is as in (A.14) and almost surely

\[
R_n = O(n^{-1/4}(\ln n)^{1/2}(\ln \ln n)^{1/4}), \quad n \to \infty.
\]
Then, using similar arguments to those in Gudendorf and Segers (2012, pp. 3082–3083) (and the functional delta method for the P estimator) we have that $\hat{A}_{\alpha,n}^\phi$ and $\hat{A}_{\alpha,n}^{CFG}$ satisfy Condition A.1 with $\phi = \phi_P$ and $\phi = \phi_{CFG}$. Specifically, for any fixed $\epsilon \in (0, 1/2)$

$$\sqrt{n}(\ln \hat{A}_{\alpha,n} - \ln A_{\alpha}) = \phi_{g,\alpha}(C_{G_{\alpha,n}}) + o_p(1),$$

where $\phi_{g,\alpha} = \phi_g \circ g$ and $\phi_g : \ell^\infty([0, 1]^d) \to \ell^\infty(S_d)$ is defined via

$$(\phi_g(f))(t) = \int_0^\infty \int_0^v f(e^{-vt}, \ldots, e^{-vt^d})e^{-\epsilon \gamma(t)}dv,$$

where for every $f \in \ell^\infty([0, 1]^d)$ and $v \in [0, 1)$ we have $h_{\alpha}(t) = 0$ for $t \in S_d$

The map $\phi_g$ satisfies the representation in (A.23). By Lemma A.6, the expectation in (A.24) is finite for $\varphi = \varphi_{ML}$ and any given $\epsilon \in (0, 1/2)$. Then, by Proposition (A.4) we have that in $\ell^\infty([0, 1]^d)$

$$\sqrt{n}(\hat{A}_{\alpha,n}^{\phi_{ML}} - A^*) \rightsquigarrow \phi_{g,\alpha}(C_Q), \quad n \to \infty.$$

Precisely, $\phi_{g,\alpha} = \alpha^{-1}(\phi_{g}(f))(t) + K_{d+1}(t)f(1, \ldots, 1)$, with $\phi_g$ as in (A.28) and

$$K_{d+1}(t) = K_{\alpha}(t) - \int_0^\infty \beta_\epsilon \left( e^{-\epsilon \gamma(t)} \right) \frac{h_{\alpha}(t; v)}{\alpha} dv + \sum_{j=1}^d \int_0^\infty \check{C}_{G_{\alpha,j}}(v^j, \ldots, v^d) \beta_\epsilon \left( e^{-\epsilon \gamma(t)} \right) \frac{h_{\alpha}(t; v)}{\alpha} dv.$$

Moreover, $\check{g}_{\alpha,\varphi_{ML}}(C_Q)$ is a zero-mean Gaussian process with covariance function (A.25). Finally, from this result and the functional version of Slutsky’s lemma, it follows that $\|\hat{A}_{\alpha,n}^{P,ML} - A^*\|_\infty \overset{p}{\to} 0$ and $\|\hat{A}_{\alpha,n}^{CFG,ML} - A^*\|_\infty \overset{p}{\to} 0$ as $n \to \infty$.

Concluding, we study the case when $\alpha$ is estimated with the GPWM estimator in (4.1) and $A_{\alpha}$ with the P and CFG estimators in (4.4) and (4.6). Assuming in addition to the previous case that $\alpha > 1/(k-1)$, then by Proposition A.2 we have that in $\ell^\infty([0, 1]^d)$

$$\sqrt{n}(\hat{A}_{\alpha,n}^{\phi_{GPWM}} - A^*) \rightsquigarrow \phi_{g,\alpha}(C_Q), \quad n \to \infty,$$

where for every $f \in \ell^\infty([0, 1]^{d+1})$, $\epsilon \in (0, 1/2)$ we have $\phi_{g,\alpha}(f) = A_{\alpha}\{\phi_{g}'(f) + K_{\alpha}\tau_{\epsilon}(f)\}$, with $\phi_{g}'$ and $\tau_{\epsilon}$ now obtained via $\phi = \phi_g$ in (A.28) and $\tau = \tau_{GPWM}$ in (A.31), respectively, and $K_{\alpha}$ as in (A.21). Ultimately, from this result and the functional version of Slutsky’s lemma, it follows that

$$\|\hat{A}_{\alpha,n}^{P,GPWM} - A^*\|_\infty \overset{p}{\to} 0, \quad \|\hat{A}_{\alpha,n}^{CFG,GPWM} - A^*\|_\infty \overset{p}{\to} 0$$

as $n \to \infty$.  

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A.6.4 Auxiliary results

Lemma A.5. Under Condition 4.1(i) we have
\[ \sqrt{n} \left( \ln \hat{A}_{\alpha,n}^{MD} - \ln A_\alpha \right) = \phi_{MD}(C_{G_{\alpha,n}}) + o_p(1), \]
where \( \phi_{MD} : \ell^\infty([0,1]^d) \to \ell^\infty(S_d) \) is defined, for every \( t \in S_d \), by
\[ (\phi_{MD}(f))(t) = \frac{(1 + A_\alpha(t))^2}{A_\alpha(t)} \left( \sum_{j=1}^d \int_0^1 \hat{C}_{\alpha,j}(v^{t_1}, \ldots, v^{t_d}) f(v^{t_1}, \ldots, v^{t_d}, 1, \ldots, 1) dv \right) \]
\[ - \int_0^1 f(v^{t_1}, \ldots, v^{t_d}) dv \].

(A.29)

Lemma A.6. Let \((U,V)\) be defined as in (A.11). Let \( \omega_{\epsilon,u} \) be the function defined in (A.19) and \( \varphi_{ML} \) in (A.27). Then, for every \( \epsilon \in (0,1/2) \) and \( u \in (0,1)^d \), we have \( E\{\omega_{\epsilon,u}(U)\varphi_{ML}(V)\} \in \mathbb{R} \).

Lemma A.7. Let \( \hat{\alpha}_n \) be an estimator of \( \alpha \) satisfying \( \hat{\alpha}_n \overset{a.s.}{\to} \alpha \) as \( n \to \infty \) and \( \hat{A}_{\alpha,n}^{MD} \) be the estimator of \( A_\alpha \) given in (4.7). Let
\[ \hat{A}_{\alpha,n}^*(t) = \left( \frac{\hat{A}_{\alpha,n}^{MD}(t)/\|t\|_{1/\hat{\alpha}_n}}{1/\hat{\alpha}_n} \right)^{1/\hat{\alpha}_n}, \quad t \in S_d. \]

Then,
\[ \left\| \hat{A}_{\alpha,n}^* - A^* \right\|_\infty \overset{a.s.}{\to} 0, \quad n \to \infty. \]

(A.30)

Lemma A.8. Assume that \( \alpha > 1/(k - 1) \). Then, almost surely as \( n \to \infty \)
\[ \sqrt{n}(\hat{\alpha}_n^{GPWM} - \alpha) = \tau_{GPWM}(B_n) + o(1), \]
where \( \tau_{GPWM} : \ell^\infty([0,1]) \to \mathbb{R} \) is defined as
\[ \tau_{GPWM}(f) = -2 \int_0^1 f(v) \left\{ v(-\ln v)^k \mu_{1,k-1} - \mu_{1,k}/(-\ln v) \right\} \Phi_{\alpha}(\Phi_{\alpha}^{-1}(v)) (k\mu_{1,k-1} - 2\mu_{1,k})^2 dv, \]
\[ \mu_{a,b} = \int_0^1 \Phi_{\alpha}^{-1}(v)v^a(-\ln v)^b dv, \quad a, b \in \mathbb{N} \]
\[ \Phi_{\alpha}(v) = \partial/\partial v \Phi_{\alpha}(v), \quad v \in (0,1). \]

For the proofs see Section 3 of the supplementary material.

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