SECOND VARIATION FOR L-MINIMAL LEGENDRIAN
SUBMANIFOLDS IN PSEUDO-SASAKIAN MANIFOLDS

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Abstract. In this paper we provide the second variation formula for L-minimal
Lagrangian submanifolds in a pseudo-Sasakian manifold. We apply it to the
case of Lorentzian-Sasakian manifolds and relate the L-stability of L-minimal
Legendrian submanifolds in a Sasakian manifold $M$ to their L-stability in an
associated Lorentzian-Sasakian structure on $M$.

Introduction

Let $(M, g)$ be a Riemannian manifold. If $f : L \to M$ is a Riemannian submanifold,
then it is called minimal if $t = 0$ is a critical point of the volume functional
for all deformations $f_t : L \to M$ with $t \in (-\epsilon, \epsilon)$ and $f_0 = f$. Equivalently, $L$
is minimal if and only if its mean curvature vector vanishes. The submanifold is
called stable if $t = 0$ is actually a minimum, that is if the second derivative of the
volume functional at $t = 0$ is nonnegative.

The explicit expressions of the first and second derivatives of the volume are
standard and can be found, for instance, in [22].

When $(M, \omega)$ is Kähler of real dimension $2n$, it is natural to study the above
problem restricted to minimal Lagrangian submanifolds, namely $n$-dimensional sub-
manifolds that are minimal in the Riemannian geometric sense and where $\omega$
vanishes.

Let us restrict ourselves to deformations that keep $L$ Lagrangian, namely such
that $f^*_t \omega = 0$. Infinitesimally this can be seen in the fact that $\mathcal{L}_X \omega = d(i_X \omega) = 0$,
where $X$ is the normal component of the derivative of $f_t$. These deformations are
called Lagrangian.

In [14], Oh has introduced the notion of Hamiltonian stability. A minimal Lagrangian submanifold is Hamiltonian stable (H-stable) if its volume is a minimum among all infinitesimal Hamiltonian deformations, namely given infinitesimally by
normal fields $X$ such that $i_X \omega$ is exact, i.e. Hamiltonian vector fields.

He then computes the Jacobi operator of a minimal Lagrangian submanifold and
applies his second variation formula to provide a stability criterion for a subman-
ifold $L$ in a Kähler-Einstein ambient in terms of the first eigenvalue $\lambda_1(L)$ of the
Laplacian on $L$. Namely $L$ is H-stable if, and only if, $\lambda_1(L)$ is greater or equal than
the Einstein constant of $M$.

There are several examples of minimal H-stable submanifolds of $\mathbb{C}P^n$ or other
Hermitian symmetric spaces that are not stable in the usual sense. A survey of

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results and techniques, mostly for the homogeneous case, can be found in Ohnita’s paper [17].

A slight generalization of minimal Lagrangian submanifolds are $H$-minimal ones, namely Lagrangians that extremize the volume under all Hamiltonian variations or, equivalently, if the mean curvature vector is $L^2$-orthogonal to all Hamiltonian vector fields, see [15].

The odd dimensional counterpart of Kähler geometry is Sasakian geometry, that merges together Riemannian, contact and CR structures. A natural contact geometric object analogous to Lagrangian submanifolds are Legendrian submanifolds.

A submanifold $f : L^n \to (M^{2n+1}, \eta)$ of a contact manifold is Legendrian if $f^* \eta = 0$ and a deformation $f_t$ that preserves the Legendre condition is called Legendrian. Infinitesimally, it translates into having a variation field that is a contactomorphism.

Oh’s notion of $H$-stability is here replaced by Legendrian stability ($L$-stability), namely when the second derivative of the volume functional is nonnegative for all contact vector fields.

The computation of the second variation of minimal Legendrian submanifolds in Sasakian manifolds has been provided by Ono [19], along as a stability criterion – for a $\eta$-Sasaki-Einstein ambient – in terms of the spectrum of the Laplacian. Namely if the ambient Ricci tensor satisfies $\text{Ric} = A g + (2n - A) \eta \otimes \eta$, then $L$ is $L$-stable if, and only if, $\lambda_1 (L) \geq A + 2$, the Kähler-Einstein constant of the transverse metric.

Using Ono’s expression of the second variation and the known properties of the Jacobi operator, Calamai and the first author [10] were able to construct eigenfunctions of the Laplacian with eigenvalue $A + 2$, under the assumption of the presence of nontrivial Sasaki ambient automorphisms.

Ono’s work has been generalized by Kajigaya [13], who has introduced the notion of $L$-minimal Legendrian submanifolds, namely the ones that are stationary points of the volume under Legendrian deformations and computed their second variation. In this case, a criterion involving the spectrum of the Laplacian cannot be provided in dimension (of the ambient manifold) greater than three.

The minimality condition extends of course to the pseudo-Riemannian setting and is treated in Anciaux’s monograph [2]. The compatible combination of a pseudo-Riemannian metric and a complex structure leads to the notion of pseudo-Kähler structures that, being symplectic, allow us to speak about Lagrangian submanifolds. A, up to a certain point, similar structure of symplectic pseudo-Riemannian manifold is given by para-Kähler ones, for which we refer to [1], [11].

The study of the Hamiltonian stability of minimal Lagrangian in pseudo- and para-Kähler manifolds has been done by Anciaux and Georgiou [3], where they compute the second variation of such submanifolds and give a stability criterion analogous to Oh’s in case these are space-like.

In this paper we treat the analogous problem for pseudo-Sasakian manifolds. These structures have been introduced by Takahashi in [24] and consist in normal almost contact structures endowed with compatible pseudo-Riemannian metrics.

Our main result is the following (see Section 2).

**Theorem 2.5.** Let $L$ be a $L$-minimal Legendrian submanifold, possibly with boundary $\partial L$, of a pseudo-Sasakian manifold $(M, \eta, \xi, g, \varphi, \varepsilon)$ with $\varepsilon = |\xi|^2 = \pm 1$. 


Then, in the normal Legendrian direction \( V = f \xi + \frac{1}{2} \varphi \nabla f \) vanishing on \( \partial L \), the second variation of the volume is

\[
\frac{d^2}{dt^2} \bigg|_{t=0} \text{vol}(L_t) = \frac{1}{4} \int_L \left\{ (\Delta f)^2 - 2\varepsilon |\nabla f|^2 - \text{Ric}(\varphi \nabla f, \varphi \nabla f) \\
- 2g(H, h(\nabla f, \nabla f)) + g(H, \varphi \nabla f)^2 \right\} dv_0
\]

where \( H \) is the mean curvature vector, \( \text{Ric} \) is the Ricci tensor of \((M, g)\) and \( dv_0 \) is the volume form of \((L, g)\).

In the special case when \( L \) is minimal (\( H = 0 \)) and \( g \) is \( \eta \)-Einstein (\( \text{Ric} = Ag + (2n + \varepsilon A) \eta \otimes \eta \)), the formula above simplifies to

\[
\frac{d^2}{dt^2} \bigg|_{t=0} \text{vol}(L_t) = \frac{1}{4} \int_L \left\{ (\Delta f)^2 - (A + 2\varepsilon) |\nabla f|^2 \right\} dv_0
\]

and we are able to give the following stability criterion in case \( L \) is space-like.

**Proposition 2.7** The minimal space-like Legendrian \( L \) in the pseudo-Sasaki \( \eta \)-Einstein manifold \( M \) is Legendrian stable if and only if its first eigenvalue of the Laplacian on functions \( \lambda_1(L) \) satisfies

\[
(1) \quad \lambda_1(L) \geq A + 2\varepsilon.
\]

For \( \varepsilon = 1 \) we reobtain Ono’s formula and stability criterion.

Concerning usual stability in the pseudo-Riemannian case, it is known that every minimal submanifold is always unstable if the ambient metric is indefinite on its tangent or normal bundle, see [2, Thm. 37]. In contrast, we have the following.

**Corollary.** If \( A + 2\varepsilon \leq 0 \), then every minimal Legendrian submanifold is Legendrian stable. In particular this holds in the pseudo-Sasaki-Einstein case (with \( A = -2n \)).

Then we focus on Lorentzian-Sasakian manifolds, namely when the signature is \((2n, 1)\) and \( \varepsilon = -1 \). They appeared in [4], [6] in the study of twistor and Killing spinors on Lorentzian manifolds. Later their study has been proposed in Sasakian geometry, see [8] or [7, Sect. 11.8.1]. In particular it is proved in [8] that every negative Sasakian manifold admits a Lorentzian-Sasaki-Einstein metric and conversely.

In Subsection 3.1 we consider these deformations that map every Sasakian structure to a Lorentzian-Sasakian one. They generalize the well-known \( D \)-homothetic deformations of Tanno [24].

We then prove that for every minimal Lagrangian submanifold \( L \) in a Sasakian manifold \( M \) is Legendrian stable if, and only if, it is in the associated Lorentzian-Sasakian structure on \( M \).

1. **Pseudo-Sasakian manifolds**

In this section we recall the definition and main properties of pseudo-Sasakian structures, following [23].

Let \( M^{2n+1} \) be a differentiable manifold and let \( \xi \) be a vector field on \( M \), \( \eta \) a 1-form and \( \varphi \) a section of \( \text{End}(TM) \).

Then the triple \( (\xi, \eta, \varphi) \) is an almost contact structure if \( \eta(\xi) = 1 \) and \( \varphi^2 = -\text{id} + \eta \otimes \xi \), see e.g. [5]. If \( g \) is a pseudo-Riemannian metric, then we have the following.
Definition 1.1. The tuple $(\xi, \eta, \varphi, g, \varepsilon)$ is an almost contact metric structure if $(\xi, \eta, \varphi)$ is an almost contact structure and the following compatibility relations hold

1. $g(\xi, \xi) = \varepsilon \in \{\pm 1\}$;
2. $\eta(X) = \varepsilon g(\xi, X)$;
3. $g(\varphi X, \varphi Y) = g(X, Y) - \varepsilon \eta(X)\eta(Y)$.

A tuple as above is a contact metric structure if \[ d\eta = 2g(\varphi, \cdot). \]

Definition 1.2. A contact metric structure is normal or Sasakian if \[ (\nabla_X \varphi)Y = \varepsilon \eta(Y)X - g(X, Y)\xi \]
where $\nabla$ is the Levi-Civita connection of $g$.

For an almost contact metric structure we have the following.

Proposition 1.3. If the identity (2) holds, then $\nabla \xi = \varepsilon \varphi$, the field $\xi$ is Killing and the structure is contact metric.

In this paper we focus on a special kind of pseudo-Sasakian manifolds, namely Lorentzian Sasakian. They are characterized by their signature $(2n, 1)$ and $\varepsilon = -1$, see e.g. [4], [6].

Before giving some properties of pseudo-Sasakian manifolds we need to fix a sign convention for the curvature tensor $R$ of a connection $D$ on a vector bundle $E \to M$

\[ R(X, Y)\sigma = D_XD_Y\sigma - D_YD_X\sigma - D_{[X,Y]}\sigma \text{ for } X, Y \in \Gamma(TM) \text{ and } \sigma \in \Gamma(E). \]

We have the following properties, some of them proved in [20] for the Lorentzian case.

Lemma 1.4. Let $(M, g, \xi, \eta, \varepsilon)$ be a pseudo-Sasakian manifold, then for $X, Y, Z \in TM$ one has

1. $\varphi^2 X = -X + \eta(X)\xi,$
2. $(\nabla_X \varphi)Y = -g(X, Y)\xi + \varepsilon\eta(Y)X,$
3. $g(\varphi X, Y) = -g(X, \varphi Y),$
4. $\omega(X, Y) = (\nabla_X \eta)Y = g(\varphi X, Y),$
5. $(\nabla_X \omega)(Y, Z) = \varepsilon g(X, Z)\eta(Y) - \varepsilon g(X, Y)\eta(Z),$
6. $\text{Rm}(X, Y)\xi = \eta(X)Y - \eta(Y)X,$
7. $\text{Rm}(X, \xi, \eta, \xi, Y) = g(X, Y) - \varepsilon \eta(X)\eta(Y),$
8. $\text{Ric}(\xi, \xi) = 2n,$
9. $\text{Rm}(X, Y)\varphi Z = \varphi \text{Rm}(X, Y)Z + \varepsilon \Big(-g(\varphi Y, Z)X + g(\varphi X, Z)Y - g(Y, Z)\varphi X + g(X, Z)\varphi Y\Big)$

where $\text{Rm}$ is the Riemann curvature tensor of $(M, g)$ and $\text{Ric}$ is its Ricci tensor.

\[ \text{d}\eta(X, Y) = X\eta(Y) - Y\eta(X) - \eta([X, Y]). \]
1.1. Legendrian submanifolds. A submanifold \( f : L \to M \) of a contact manifold \( (M^{2n+1}, \eta) \) is called horizontal if it satisfies \( f^* \eta = 0 \). In particular, it follows \( f^* \omega = 0 \). A Legendrian submanifold is a maximally isotropic submanifold \( L^n \), i.e. a horizontal submanifold with \( \dim L = n \).

Let us consider a smooth Riemannian immersion \( f : L \to (M, g) \) into a Lorentzian manifold \( (M, g) \), i.e. \( f^* g \) defines a positive definite metric on \( L \). Then the second fundamental form \( h \in \Gamma(T^* L \otimes T^* L \otimes NL) \), where \( NL \) denotes the normal bundle of \( f : L \to (M, g) \), is given by

\[
h(X, Y) = \nabla_X (df) Y = (f^* \nabla)_X (df) Y - df(\nabla_X Y),
\]

where \( \nabla \) and \( \nabla \) are the Levi-Civita connections of \( g \) and \( f^* g \), respectively. Further, for a section \( \nu \) of \( NL \) we define the normal connection \( \nabla^\perp \) as the normal part and the shape operator \( A_\nu \) as the tangential part of \( f^* \nabla_X \nu \), i.e. via

\[
f^* \nabla_X \nu = \nabla^\perp_X \nu - \nu(X) A_\nu, X \in NL \oplus f_*(TL),
\]

where \( A \in \Gamma(N^* L \otimes T^* L \otimes TL) \). The mean curvature is defined as

\[
H := \text{tr}_g^L h.
\]

Later we need Gauss’ formula for pseudo-Riemannian submanifolds (see e.g. [18], p. 100)](3).

**Lemma 1.5.** The Riemann curvature tensor \( \text{Rm} \) of a pseudo-Riemannian submanifold \( L \) in the pseudo-Riemannian manifold \( (M, g) \) is related to the ambient curvature tensor \( \text{Rm} \) and to the second fundamental form \( h \) of the immersion by

\[
\text{Rm}(A, B, C, D) = \text{Rm}(A, B, C, D) - g(h(B, C), h(A, D)) + g(h(A, C), h(B, D))
\]

for vectors \( A, B, C, D \) tangent to \( L \).

Following [19], we give the following definition.

**Definition 1.6.** Let \( (M, g, \xi, \eta, \varphi, \varepsilon) \) be a pseudo-Sasakian manifold and \( f : L \to M \) a Legendrian immersion. A smooth family of immersions \( \{ f_t \}_{t \in (-\delta, \delta)} \) is called Legendrian deformation of \( L \), if \( f_t \) is Legendrian for all \( t \in (-\delta, \delta) \) and it is \( f_0 = f \).

By the curvature properties of pseudo-Sasakian metrics, we have the following.

**Lemma 1.7.** For a Legendrian submanifold \( L \) in a pseudo-Sasakian manifold \( (M, g, \xi, \eta, \varphi, \varepsilon) \) and in a normal orthonormal frame \( e_1, \ldots, e_n \) with \( \varepsilon_i = g(e_i, e_i) \), along the Legendrian submanifold \( L \) one has

1. \( \sum_{i=1}^n \varepsilon_i \text{Rm}(\varphi e_i, \xi, \varphi e_i) = n, \)
2. \( \varepsilon_i \text{Rm}(\varphi e_i, \xi, \varphi e_i, V_H) = 0, \) for \( V_H \in \mathcal{D}. \)

**Proof.** The first is a consequence of [8], i.e. \( \text{Rm}(\varphi e_i, \xi, \varphi e_i) = g(\varphi e_i, \varphi e_i) \). The second follows from [8] and \( \eta(\varphi e_i) = 0. \)

We state a property of the second fundamental form of a Legendrian submanifold in a pseudo-Sasakian manifold, whose proof is basically the same as for its Riemannian counterpart; see [19] Prop. 3.4.

**Lemma 1.8.** The second fundamental form \( h \) of a Legendrian submanifold \( L \) in a pseudo-Sasakian manifold \( (M, g, \eta, \xi, \varphi, \varepsilon) \) satisfies the following properties.

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2To keep notation short, we later write \( \nabla \) for \( f^* \nabla \) and \( g \) for \( f^* g \).

3Beware that O’Neill uses the opposite convention than ours for Riemannian curvature.
2. L-minimal Legendrian submanifolds of pseudo-Sasakian manifolds

2.1. The second variation along Legendrian deformations.

Definition 2.1. Let \( f : L \to M \) be a Legendrian immersion into a pseudo-Sasakian manifold \((M, g, \xi, \eta, \varphi, \varepsilon)\). Then \( f \) is called Legendrian minimal or L-minimal if

\[
\frac{d}{dt} \ \bigg|_{t=0} \ \text{vol}(L_t) = 0
\]

for all Legendrian deformations of \( L_t \).

Taking the normal field \( V = \left( \frac{\partial}{\partial t} \bigg|_{t=0} f_t(\cdot) \right)^1 \), then \( f_t \) is Legendrian if, and only if, \( L_V \eta = 0 \).

From the known expression of the first variation (see e.g. \cite{2}), namely

\[
\frac{d}{dt} \ \bigg|_{t=0} \ \text{vol}(L_t) = - \int_L g(X, H)dv_0,
\]

we see that L-minimality is equivalent to requiring \( H \) to be \( L^2 \)-orthogonal to all Legendrian vector fields.

For a normal field \( V \) we write \( V = f \xi + V_H \). It then is \( \iota_V d\eta = 2(\varphi V_H)^\flat \), implying, since \( L_V \eta = 0 \), that \( V_H = \frac{1}{2} \varphi \nabla f \). For positive signature the following is due to \cite{12}.

Proposition 2.2. The immersion \( f : L \to M \) of a manifold \( L \) is L-minimal (with respect to variations fixing the boundary) if and only if it is

\[
\delta \alpha_H = 0 \quad \text{or equivalently} \quad \text{div}(\varphi H) = 0
\]

where \( \alpha_H = d\eta(H, \cdot) \).

Proof. In fact, the well-known formula for the first variation along the normal direction \( X \) yields for variations with \( d\alpha_X = du \) for some function \( u \) on \( L \)

\[
\frac{d}{dt} \ \bigg|_{t=0} \ \text{vol}(L_t) = - \int_L g(X, H)dv_0 = - \int_L g(\alpha_X, \alpha_H)dv_0
\]

\[= - \int_L g(du, \alpha_H)dv_0 = - \int_L u\delta \alpha_H dv_0,\]

where we used that \( X \) vanishes on the boundary. Since this vanishes for arbitrary Legendrian variations we conclude \cite{13}. \( \square \)

Proposition 2.3. Let \((M, \eta, \xi, g, \varphi, \varepsilon)\) be a pseudo-Sasakian manifold and let \( L \) be an L-minimal Legendrian submanifold, possibly with boundary. Then the second variation of the volume of \( L \) under the normal direction \( V = f \xi + \frac{1}{2} \varphi \nabla f \), vanishing on \( \partial L \), is

\[
\frac{d^2}{dt^2} \ \bigg|_{t=0} \ \text{vol}(L_t) = \int_L \left\{ \text{tr}_{\text{II}}(\nabla^2 V, \nabla^2 V) + \text{Rm}(\cdot, V, \cdot, V) \right\} - g(A_V, A_V)
\]

\[- \frac{1}{4} g(h(\nabla f, \nabla f), H) + g(H, V)^2 \right\} dv_0\]
where \( L_t, \ t \in (\delta, \delta), \) is a family of submanifolds with variational vector field \( V \) and \( dv_t \) is the volume form of the induced metric at \( t = 0. \)

**Proof.** For this proof in positive signature we refer to [21]. We fix a local \((t\text{-dependent})\) frame \( e_i, i = 1, \ldots, n, \) for \( L. \) One starts with the well-known formula for the first variation along the direction \( X \)

\[
\frac{d}{dt} \text{vol}(L_t) = -\int_L g(X, H)dv_t
\]

where \( H \) is the mean curvature vector of the family \( L_t \) with variational vector field \( X \) and derives this expression at \( t = 0 \)

\[
\frac{d^2}{dt^2} \bigg|_{t=0} \text{vol}(L_t) = -\int_L \frac{d}{dt} g(H, X)|_{t=0} dv_0 + \int_L g(X, H)^2 dv_0
\]

where we have used the well-known fact that \( \frac{d}{dt} (dv_t) = -g(X, H)dv_t. \)

Here we write \( g \) for the induced metric on \( L_t, \) too. Further we also write \( \nabla \)

for the pull-back of the Levi-Civita connection along the immersion \((-\varepsilon, \varepsilon) \times L \ni (t, p) \mapsto f_t(p) \in M.\) The first two terms are

\[
\frac{d}{dt} g_{ij} = X \cdot g(e_i, e_j) = g(\nabla_X e_i, e_j) + g(e_i, \nabla_X e_j) = g(\nabla e_i, X, e_j) + g(e_i, \nabla e_j, X)
\]

where we recall

\[
\frac{d}{dt} g_{ij}(0) = -g^{ab} g_{ab} g_{ij}|_{t=0} = -\varepsilon_i \varepsilon_j g(h(e_i, e_j), X).
\]

We have

\[
-\frac{d}{dt} \bigg|_{t=0} g(H, X) = -2\varepsilon_i \varepsilon_j g(h(e_i, e_j), X)g(\nabla e_i, X, e_j) + \delta_{ij} \varepsilon_i X \cdot g(\nabla e_i, X, e_j)
\]

\[
= -2\varepsilon_i \varepsilon_j g(h(e_i, e_j), X)^2 + \varepsilon_i \left[ g(\nabla_X \nabla e_i, X, e_i) + g(\nabla e_i, X, \nabla_X e_i) \right]
\]

\[
= -2\varepsilon_i \varepsilon_j g(h(e_i, e_j), X)^2 + \varepsilon_i \left[ Rm(X, e_i, X, e_i) + g(\nabla e_i, \nabla_X X, e_i) + |\nabla e_i, X|^2 \right].
\]

On the other hand \( A_X e_i = g(A_X e_i, e_j) \varepsilon_j e_j, \) so \( \varepsilon_i g(A_X e_i, A_X e_i) = g(h(e_i, e_j), X)^2 \varepsilon_j. \)

So we have

\[
\frac{d^2}{dt^2} \bigg|_{t=0} \text{vol}(L_t) = \int_L \left\{ \varepsilon_i |\nabla^2 e_i X|^2 + \varepsilon_i Rm(X, e_i, X, e_i) - \varepsilon_i |A_X e_i|^2 \\
+ \text{div}(\nabla_X X)^T - g(\nabla_X X, H) + g(X, H)^2 \right\} dv_0
\]

The divergence term is \( \int_L \text{div}(\nabla_X X)^T dv = \int_{\partial L} g(\nabla_X X, \nu) = 0 \) since \( X \) vanishes on \( \partial L. \)
Finally we compute

\[
g(\nabla_X X, H) - \frac{1}{4} g(h(\nabla f, \nabla f), H) = g(\varphi \nabla_X X, \varphi H) - \frac{1}{4} g(\varphi h(\nabla f, \varphi H), \nabla f)
\]

\[
= g(\nabla_X \varphi X, \varphi H) - \frac{1}{4} g(\varphi h(\nabla f, \varphi H), \nabla f)
\]

\[
= X \cdot g(\varphi X, \varphi H) - g(\varphi X, \nabla_X \varphi H) - \frac{1}{4} g(\varphi h(\nabla f, \varphi H), \nabla f)
\]

\[
= (\nabla_X (\varphi X) \nabla_H (\varphi f) - \frac{1}{4} g(\varphi \nabla_H \varphi f, \nabla f)
\]

\[
= (\nabla_X (\varphi X) \nabla_H (\varphi f) + \frac{1}{2} g(\nabla_H \varphi X, \nabla f)
\]

\[
= (\nabla_X (\varphi X) \nabla_H (\varphi f) + \frac{1}{2} df(\nabla_H \varphi X)
\]

\[
= \frac{1}{2}(X df(\varphi H) - df(\nabla_X \varphi H) + df(\nabla_H \varphi X))
\]

\[
= \varphi H \cdot X \cdot f
\]

\[
= 0
\]

since \(X\) is normal to \(L\). So we can conclude (14). \(\square\)

Let us now compute the first term of (14). We have

\[
\nabla_{e_i}^\perp V = \left(\nabla_{e_i}(f \xi + \frac{1}{2} \varphi \nabla f)\right) \perp
\]

\[
= f(\nabla_{e_i} \xi) \perp + e_i \cdot f \xi + \frac{1}{2}(\nabla_{e_i} \varphi \nabla f) \perp
\]

\[
= \epsilon f \varphi e_i + e_i \cdot f \xi + \frac{1}{2}(\varphi \nabla_{e_i} \nabla f - g(e_i, \nabla f) \xi)
\]

\[
= \epsilon f \varphi e_i + \frac{1}{2} e_i \cdot f \xi + \frac{1}{2} \varphi \nabla_{e_i} \nabla f
\]

so we get summing over \(i\)

\[
\epsilon_i g(\nabla_{e_i}^\perp V, \nabla_{e_i}^\perp V) = f^2 \epsilon_i |\varphi e_i|^2 + \frac{1}{4}(e_i f)^2 \epsilon_i \epsilon + \frac{1}{4} |\nabla^2 f|^2 + \epsilon f g(e_i, \nabla_{e_i} \nabla f) \epsilon_i
\]

\[
= nf^2 + \frac{1}{4} \epsilon f \nabla f|^2 + \frac{1}{4} |\nabla^2 f|^2 - \epsilon f \Delta f
\]

where \(|\nabla^2 f|^2 = \epsilon_i |\nabla_{e_i} \nabla f|^2\) is the norm of the Hessian of \(f\).

Applying (12) we obtain the following, that says we have the symmetries analogous to the ones of the Kähler curvature tensor.

**Lemma 2.4.** We have \(\sum_i \epsilon_i \text{Rm}(\varphi e_i, V_H, \varphi e_i, V_H) = \sum_i \epsilon_i \text{Rm}(e_i, \varphi V_H, e_i, \varphi V_H)\).
Proof. Applying the identity \([12]\) we have

\[
\varepsilon_i \text{Rm}(\varphi e_i, V, \varphi e_i, V) = \varepsilon_i g(\varphi \text{Rm}(\varphi e_i, V_H) e_i, V_H) + \varepsilon_i |g(\varphi V_H, e_i) g(\varphi e_i, V_H) - g(e_i, e_i) g(V_H, V_H)|
\]

\[
= -\varepsilon_i \text{Rm}(\varphi e_i, V_H, e_i, V_H) + \varepsilon_i |V_H|^2 - n|V_H|^2
\]

\[
= -\varepsilon_i \text{Rm}(\varphi V_H, \varphi e_i, V_H) + \varepsilon(1 - n)|V_H|^2
\]

So the second term in our second variation is

\[
\varepsilon_i \text{Rm}(\varphi e_i, V, e_i, V) = -\varepsilon_i \text{Rm}(\varphi e_i, V, e_i, V) - \varepsilon_i \text{Rm}(\xi, V, \xi, V)
\]

\[
= -\text{Ric}(V, V) - \varepsilon_i f^2 \text{Rm}(\varphi e_i, \xi, \varphi e_i, \xi) - 2\varepsilon_i f \text{Rm}(\varphi e_i, V_H, \varphi e_i, \xi)
\]

\[
= -\text{Ric}(V, V) + n f^2 - \varepsilon_i \text{Rm}(\varphi e_i, V_H, \varphi e_i, V_H) + \varepsilon |V|^2 - \varepsilon f^2
\]

\[
= -\text{Ric}(V, V) + n f^2 - \varepsilon_i \text{Rm}(\varphi e_i, V_H, \varphi e_i, V_H) + \varepsilon |V_H|^2
\]

\[
= -\text{Ric}(V, V_H) - 2n f^2 + n f^2 + \varepsilon |V_H|^2
\]

\[
- \varepsilon_i \left( \text{Rm}(e_i, \varphi V_H, e_i, \varphi V_H) - g(h(\varphi V_H, e_i), h(\varphi V_H, e_i)) + g(h(e_i, e_i), h(\varphi V_H, \varphi V_H)) \right)
\]

where in the last equality we have used Gauss’ formula in Lemma \([13]\) and the fact that, from the definition of the shape operator, we have \(A_V e_i = -(\nabla e_i, V)^T\) and \(A_V = A_{V_H}\). We then compute that \(A_{V_H} e_i = \varphi h(e_i, \varphi V_H)\) and hence

\[
g(A_V, A_V) = \varepsilon_i g(h(e_i, \varphi V_H), h(e_i, \varphi V_H)).
\]
So (14) becomes
\[ \frac{d^2}{dt^2} \left|_{t=0} \right. \text{vol}(L_t) = \int_L \left\{ nf^2 + \frac{1}{4} \varepsilon |\nabla f|^2 + \frac{1}{4} \nabla^2 f^2 - \varepsilon f \Delta f 
\right. \\
- n f^2 - \frac{1}{4} \text{Ric(} \varphi \nabla f, \varphi \nabla f) + \frac{1}{4} \varepsilon |\nabla f|^2 + \frac{1}{4} \text{Ric(} \nabla f, \nabla f) \\
- g(H, h(\varphi V_H, \varphi V_H)) \\
+ \frac{1}{4} g(h(\nabla f, \nabla f), H) + \frac{1}{4} g(H, \varphi \nabla f)^2 \right\} dv_0 \\
= \frac{1}{4} \int_L \left\{ -2 \varepsilon |\nabla f|^2 - \text{Ric(} \varphi \nabla f, \varphi \nabla f) + |\nabla^2 f|^2 + \text{Ric(} \nabla f, \nabla f) \\
- 2 g(H, h(\nabla f, \nabla f) + g(H, \varphi \nabla f)^2 \right\} dv_0 \]

We can group the Hessian term and the Ricci term by means of the pseudo-Riemannian Bochner formula in the Appendix of [3]..

The formula on p. 609 of [3] differs by a sign as they define \( \Delta = \text{div(} \nabla \cdot) \).

---

Theorem 2.5. Let \( L \) be a \( L \)-minimal Legendrian submanifold, possibly with boundary \( \partial L \), of a pseudo-Sasakian manifold \((M, \eta, \xi, g, \phi, \varepsilon)\).

Then, in the normal Legendrian direction \( V = f \xi + \frac{1}{2} \phi \nabla f \) vanishing on \( \partial L \), the second variation of the volume is
\[ \frac{d^2}{dt^2} \left|_{t=0} \right. \text{vol}(L_t) = \int_L \left\{ (\Delta f)^2 - 2 \varepsilon |\nabla f|^2 - \text{Ric(} \varphi \nabla f, \varphi \nabla f) \\
- 2 g(H, h(\nabla f, \nabla f) + g(H, \varphi \nabla f)^2 \right\} dv_0 \]

where \( H \) is the mean curvature vector and \( dv_0 \) is the volume form of \((L, g)\).

2.2. The minimal case. Let us now consider the more special case where \( L \) is minimal and Riemannian (i.e. \( H = 0 \)) and \( M \) is \( \eta \)-Sasaki-Einstein, i.e. for some \( A, B \in \mathbb{R} \), it holds
\[ \overline{\text{Ric}} = Ag + B \eta \otimes \eta \]
where it must be \( B = 2n - \varepsilon A \).

In this case our second variation formula reads
\[ \frac{d^2}{dt^2} \left|_{t=0} \right. \text{vol}(L_t) = \int_L \left\{ |\Delta f|^2 - (A + 2 \varepsilon) |\nabla f|^2 \right\} dv_0 \]
and we recall that \( |df|^2 \geq 0 \) being \( L \) Riemannian.
Note that for the Riemannian case $\varepsilon = 1$ we have reobtained the formula of [19] (see also [16]).

In this case we have a sufficient condition for Legendrian stability coming from the positivity of the second term in the last expression.

**Proposition 2.6.** A minimal Legendrian $n$-submanifold in a pseudo-Sasakian $\eta$-Einstein manifold with constant $A$ is Legendrian stable if

$$A + 2\varepsilon \leq 0.$$  

In particular, if $M$ is Lorentzian-Sasaki-Einstein we have $A = -2n$ and $\varepsilon = -1$ so $L$ is always Legendrian stable.

With the same argument used in the Sasakian and Kähler case, using that $L$ is a Riemannian manifold so the space $C^\infty (L)$ admits a $L^2$-orthogonal decomposition given by the eigenspaces of the Laplacian, we can prove the following.

**Proposition 2.7.** The minimal space-like Legendrian $L$ in the pseudo-Sasakian $\eta$-Einstein manifold $M$ is Legendrian stable if and only if its first eigenvalue of the Laplacian on functions $\lambda_1(L)$ satisfies

$$\lambda_1(L) \geq A + 2\varepsilon.$$  

### 3. Lorentzian Sasakian manifolds

#### 3.1. Tanno deformations.** The following is a generalization of the well-known Tanno deformations [24]. Starting with a Sasakian manifold $(M, g, \eta, \xi, \varphi)$ one defines for fixed $\alpha \in \mathbb{R}_+$ and $\beta := \alpha + \alpha^2$

$$\tilde{g} := \tilde{g}_\alpha := \alpha g - \beta \eta \otimes \eta.$$  

This is a Lorentzian metric, since it is

$$\tilde{g}(\xi, \xi) = \alpha g(\xi, \xi) - (\alpha^2 + \alpha) = -\alpha^2.$$  

For the proof of this Proposition we need the next Lemma.

**Lemma 3.1.** If $\tilde{\nabla}$ is the Levi-Civita connection of $\tilde{g}_\alpha$ and $\nabla$ is the one of $g$, then we have

$$\tilde{\nabla}_X Y = \nabla_X Y - \alpha^{-1} \beta (\eta(X)\varphi Y + \eta(Y)\varphi X).$$  

For a proof in the case $\alpha = 1$ and $\beta = 2$ using Koszul’s formula we refer to Proposition 3.3 of Brunetti and Pastore [9]. The Riemannian case is due to Tanno [24], see also [7, Chap. 7]. We remark a sign difference with [9] due to the opposite convention in the definition of the fundamental 2-form.

**Proof.** Define $\tilde{\nabla}$ by

$$\tilde{\nabla}_X Y = \nabla_X Y + S_X Y,$$

where

$$S_X Y := -\alpha^{-1} \beta \left[ \eta(X)\varphi Y + \eta(Y)\varphi X \right].$$

The tensor field $S$ is symmetric, hence $\tilde{\nabla}$ is a torsion-free connection.
We compute
\[
(\tilde{\nabla}_X \tilde{g})(Y, Z) = \nabla_X \tilde{g}(Y, Z) - \tilde{g}(S_X Y, Z) - \tilde{g}(Y, S_X Z) \\
= -\beta \left( (\nabla_X \eta)(Y)\eta(Z) + (\nabla_X \eta)(Z)\eta(Y) \right) \\
+ \alpha^{-1}\beta \tilde{g} \left( [\eta(X)\varphi Y + \eta(Y)\varphi X], Z \right) \\
+ \alpha^{-1}\beta \tilde{g} \left( Y, [\eta(X)\varphi Z + \eta(Z)\varphi X] \right) \\
= 0,
\]
where we have used that \( \nabla_X \eta(Y) = g(\varphi X, Y) \), as a consequence of Proposition 3.2.

Hence, \( \tilde{\nabla} \) is metric for \( \tilde{g} \) and has no torsion, so it coincides with the Levi-Civita connection of \( \tilde{g} \). \( \square \)

The behavior (19) of the Levi-Civita connection of \( \tilde{g}_\alpha \) allows us to prove the following.

**Proposition 3.2.** Let \((\eta, \xi, \varphi, g)\) be a Sasakian structure. Then for \( \alpha > 0 \) the new structure \((\alpha \eta, \alpha^{-1}\xi, \varphi, \tilde{g}_\alpha)\) is Lorentzian Sasakian, where \( \tilde{g}_\alpha \) is defined in (18).

**Proof.** For completeness sake we prove this Proposition. Let \( \tilde{\xi} = \alpha^{-1}\xi \). First we observe, that \( Z \in \{\xi, \tilde{\xi}\} \) satisfies \( L_Z g = 0 \) and \( L_Z \eta = 0 \) and as a result \( L_Z \tilde{g} = 0 \). Hence \( \xi \) is a Killing vector field of length \(-1\).

Moreover, for the second term of \( \varphi^2 \) using \( \tilde{g}(\xi, \xi) = -\alpha^2 \) one has \( g(X, \xi)\xi = -\tilde{g}(X, \tilde{\xi})\tilde{\xi} \), which shows, that the relation (19) is satisfied.

Let us note, that it is \( \tilde{\nabla}_X Y = \nabla_X Y \) for \( X, Y \in D \) and \( \tilde{\nabla}_\xi \xi = \nabla_\xi \xi = 0 \). In order to check (2) we observe
\[
-g(X, Y)\xi + g(Y, \xi)X = -g(X, Y)\xi = -\tilde{g}(X, Y)\tilde{\xi}, \text{ for } X, Y \in D
\]
and \( (\tilde{\nabla}_\xi \varphi)\xi = 0 = (\tilde{\nabla}_\varphi \xi)\tilde{\xi} \). It remains to compute the expression for \( X \in D \)
\[
(\tilde{\nabla}_X \varphi)\xi = -\varphi(\tilde{\nabla}_X \xi) = -\varphi \left( \tilde{\nabla}_X \xi - \alpha^{-1}\beta(\varphi X) \right) = (\tilde{\nabla}_X \varphi)\xi - \alpha^{-1}\beta X
\]
and \( -g(X, \xi)\xi + g(\xi, \xi)X = -\tilde{g}(\xi, \xi)X \).

This shows
\[
(\tilde{\nabla}_X \varphi)\tilde{\xi} = \alpha^{-2}\beta X - \alpha^{-1}\tilde{g}(\xi, \xi)X = \frac{\alpha + \alpha^2 - \alpha \tilde{g}(\xi, \xi)X}{\alpha^2} = -\tilde{g}(X, \xi)\xi + \tilde{g}(\xi, \xi)X
\]
and finishes the proof of Proposition 3.2 since the converse statement goes along the same lines. \( \square \)

We can compute how the Ricci tensor behaves under these deformations.

**Proposition 3.3.** Let \((M, g, \eta, \xi, \varphi)\) be a Sasakian \( \eta \)-Einstein manifold with \( \text{Ric} = Ag + (2n + A)\eta \otimes \eta \) and let \( \tilde{g}_\alpha \) as above. Then \( \tilde{g}_\alpha \) is Lorentzian Sasakian \( \eta \)-Einstein with \( \text{Ric} = A_\alpha \tilde{g}_\alpha + (2n + A_\alpha)\eta \otimes \eta \) for \( A_\alpha = \frac{4\alpha^2}{\alpha} + 2 \).

From Lemma 3.1 we obtain the following about the curvature tensor.
Lemma 3.4. The curvature tensors $\tilde{R}_m$ of $\tilde{g}_\alpha$ and $R_m$ of $g$ are related by

$$\tilde{R}_m(X, Y)Z = R_m(X, Y)Z + \alpha^{-1} \beta \left( g(\varphi Y, Z)\varphi X - g(\varphi X, Z)\varphi Y - 2g(\varphi X, Y)\varphi Z \right)$$

for $X, Y, Z$ in $\mathcal{D}$.

Proof. We compute for $X, Y, Z$ in $\mathcal{D}$

$$\tilde{R}_m(X, Y)Z = \tilde{\nabla}_X \nabla_Y Z - \tilde{\nabla}_Y \nabla_X Z - \tilde{\nabla}_{[X,Y]} Z$$

$$= \nabla_X \nabla_Y Z - \alpha^{-1} \beta \eta(\nabla_Y Z)\varphi X - \nabla_Y \nabla_X Z$$

$$+ \alpha^{-1} \beta \eta(\nabla_X Z)\varphi Y - \nabla_{[X,Y]} Z + \alpha^{-1} \beta \eta([X,Y])\varphi Z$$

$$= R_m(X, Y)Z - \alpha^{-1} \beta \eta(\nabla_Y Z)\varphi X$$

$$+ \alpha^{-1} \beta \eta(\nabla_X Z)\varphi Y + \alpha^{-1} \beta \eta([X,Y])\varphi Z.$$

We have that

$$\eta(\nabla_Y Z) = Y \eta(Z) - (\nabla_Y \eta)(Z) = -g(\varphi Y, Z)$$

and similarly $\eta(\nabla_X Z) = -g(\varphi X, Z)$. Moreover, one has

$$\eta([X,Y]) = -d\eta(X, Y) = -2g(\varphi X, Y).$$

So we have

$$\tilde{R}_m(X, Y)Z = R_m(X, Y)Z + \alpha^{-1} \beta g(\varphi Y, Z)\varphi X$$

$$- \alpha^{-1} \beta g(\varphi X, Z)\varphi Y - 2\alpha^{-1} \beta g(\varphi X, Y)\varphi Z.$$

□

Proof of Proposition 3.3. Let $E_i$ be an orthonormal frame with respect to $g$ of $\mathcal{D}$ and let $\tilde{E}_i = \frac{1}{\sqrt{\alpha}} E_i$. We want to compute, for $X, Y$ in $\mathcal{D}$

$$\tilde{\text{Ric}}(X, Y) = \tilde{R}_m(X, \tilde{E}_i, \tilde{E}_i, Y) - \tilde{R}_m(X, \xi, \tilde{\xi}, Y)$$

$$= \tilde{g}(\tilde{R}_m(X, \tilde{E}_i)\tilde{E}_i, Y) - \tilde{g}(X, Y)$$

$$= g(R_m(X, E_i)E_i, Y) - \tilde{g}(X, Y)$$

$$= R_m(X, E_i, E_i, Y) + \beta \left( -g(\varphi X, E_i)g(\varphi E_i, Y) - 2g(\varphi X, E_i)g(\varphi E_i, Y) \right) - \tilde{g}(X, Y)$$

$$= R_m(X, E_i, E_i, Y) + 3\beta \frac{g(\varphi X, \varphi Y)}{\alpha} - \tilde{g}(X, Y)$$

$$= \text{Ric}(X, Y) - R_m(X, \xi, \xi, Y) + 3\beta \frac{g(\varphi X, \varphi Y)}{\alpha} - \tilde{g}(X, Y)$$

$$= Ag(X, Y) - g(X, Y) + 3\beta \frac{g(\varphi X, \varphi Y)}{\alpha} - \tilde{g}(X, Y)$$

$$= \left( \frac{A}{\alpha} - \frac{1}{\alpha^2} + 3\beta \frac{1}{\alpha^2} - 1 \right) \tilde{g}(X, Y)$$

$$= \left( \frac{A + 2}{\alpha} \right) \tilde{g}(X, Y).$$

Thus we have $A_\alpha = \frac{4A + 2}{\alpha} + 2$. □
3.2. Tanno deformations and Legendrian instability. Through the transformation of Proposition 3.2 it is easy to see the following.

**Proposition 3.5.** Let \( L \subset M \) be an \( n \)-dimensional submanifold. Then it is minimal Legendrian with respect to \((M, g, \eta, \xi, \varphi)\) if and only if it is also so with respect to \((M, \tilde{g}_\alpha, \tilde{n}, \tilde{\xi}, \varphi)\).

**Proof.** The contact structure does not change. As for minimality, from Lemma 3.1 we can write down the difference of the mean curvature vectors which turns out to be zero, since the restrictions of \( \nabla \) and \( \nabla \) to \( L \) coincide. \( \square \)

We emphasize the following observation.

**Remark 3.6.** The induced metrics \( g|_L \) and \( \tilde{g}_\alpha|_L = \alpha g|_L \) on \( L \) are homothetic. In particular, their Hodge-de-Rham Laplacians \( \Delta_L \) are related by \( \tilde{\Delta}_L = \alpha^{-1} \Delta_L \) and their first eigenvalues via \( \tilde{\lambda}_1(L) = \alpha^{-1} \lambda_1(L) \).

From Lemma 3.3, Remark 3.6 and Proposition 2.7 we can infer the following.

**Proposition 3.7.** Let \((M, g)\) be an \( \eta \)-Sasaki-Einstein manifold with constant \( A \). Then a \( L \)-minimal Legendrian submanifold \( L \) is Legendrian stable in \( g \) if, and only if, it is Legendrian stable in the associated Lorentzian-Sasakian metric \( \tilde{g}_\alpha \), for all \( \alpha > 0 \).

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