Stability of Poiseuille-type Flows for an MHD Model of an Incompressible Polymeric Fluid

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Received November 25, 2018; revised July 25, 2019; accepted July 31, 2019

Abstract—A new rheological model, an extension of the Pokrovskii—Vinogradov rheological model, describing the flows of melts and solutions of incompressible viscoelastic polymeric media in external uniform magnetic field in the presence of a temperature drop and conduction current is studied. An asymptotic representation of the linear problem spectrum resulting from the linearization of the initial boundary value problem in an infinite plane channel about a Poiseuille-type flow is obtained. For this Poiseuille-type flow the parameter domain of linear Lyapunov’s stability is determined.

Keywords: Pokrovskii—Vinogradov model, model of an incompressible viscoelastic polymeric fluid, temperature, magnetic field, conduction current, Poiseuille-type flow, spectrum of linearized mixed problem, linear Lyapunov stability

DOI: 10.1134/S0015462819080020

1. INTRODUCTION

We study an extension of the structural-phenomenological model of Pokrovskii and Vinogradov (PVM) describing the flows of solutions and melts of incompressible viscoelastic polymeric media to the nonisothermal case. Moreover, we additionally assume that the medium is subject to a magnetic field. In the PVM the polymeric medium is considered the suspension of macromolecules of polymer moving in the anisotropic fluid generated, for instance, by a solvent and other macromolecules. The action of environment on the macromolecule is approximated by the action upon the linear chain of Brownian particles, each of which is a sufficiently large part of the macromolecule. The Brownian particles (beads) are interrelated by elastic forces (springs). In the case of slowly varying motions, the macromolecule is modeled as the chain of two particles (dumbbell).

This physical representation of the flow of linear polymers leads to the PVM formulation [1–3]

\[
\frac{\partial}{\partial t} v_i + v_k \frac{\partial}{\partial x_k} v_i = \frac{\partial}{\partial x_k} \sigma_{ik}, \quad \frac{\partial v_i}{\partial x_j} = 0, \quad \sigma_{ik} = -p \delta_{ik} + 2\eta_0 \frac{\tau_0}{a_{ik}}, \\
\frac{d}{dt} a_{ik} - v_j a_{jk} - v_k a_{kj} + \frac{1}{\gamma_0} a_{ik} = 2 \gamma_0 \frac{3\beta}{\tau_0} a_{ik}, \\
I = a_{11} + a_{22} + a_{33}, \quad \gamma_0 = \frac{v_{ik} + v_{ki}}{2},
\]

where \( \rho \) is the density of the polymer, \( v_i \) is the \( i \)th component of the velocity, \( \sigma_{ik} \) is the stress tensor, \( p \) is the hydrostatic pressure, \( \eta_0 \) and \( \tau_0 \) are the initial values of the shear viscosity and the relaxation time for the viscoelastic part, \( a_{ik} \) is the second-rank symmetric tensor of additional stresses, \( v_{ij} \) is the velocity gradient tensor, \( I \) is the first invariant of the tensor of additional stresses, \( \gamma_0 \) is the symmetrized velocity gradient tensor, and \( k \) and \( \beta \) are the phenomenological parameters accounting for the dimensions and shape.
of the molecular coil in the equations of macromolecular dynamics. Here, Eq. (1.1) is the equation of motion and the condition of incompressibility, and (1.2) and (1.3) are the rheological relations linking the kinematic characteristics of the flow and its thermodynamic parameters.

Some generalizations of the model (1.1)–(1.3), for instance, in the case when the summand taking into account the shear viscosity is added in Eq. (1.2) and the parameter $\beta$ is additionally dependent on the first invariant of the anisotropy tensor give good results in computational experiments for viscometric flows [4]. Therefore, we may suppose that the PVM modifications will be fruitful in modeling the motion of polymers under complex deformation conditions, for instance, for the stationary and nonstationary flows in circle channels and in channels with abruptly varying cross-section area and for the flows with a free surface. The two- and three-dimensional character of these flows is their important feature.

In this paper, we consider one of such generalizations: we take into account the effects of heat and magnetic field on the motion of a polymeric fluid. We focus on the analog of the well-known shear flow in the infinite plane channel, the Poiseuille flow, which has several features. In particular, as the computations show, with certain parameter values the velocity profile is elongated along the direction perpendicular to the direction of the pressure forces (Section 3).

The main results are formulated in Section 4: firstly, we obtain the asymptotic representation for the spectrum of the linearized problem about the Poiseuille-type flow, and, secondly, we write the condition under which the main solution becomes asymptotically stable in the chosen class of perturbations periodic in the variable changing along the boundary of the infinite plane channel.

For the case of the viscous fluid, there is a well-known result of A. N. Krylov about the linear instability of the Poiseuille flow with sufficiently large Reynolds numbers [5] confirming the Heisenberg hypothesis [6] (an improvement of the Krylov result was obtained [7]).

2. NONLINEAR MODEL OF FLOW OF A POLYMERIC FLUID IN A PLANE CHANNEL UNDER AN EXTERNAL MAGNETIC FIELD

Using the well-known results [3, 8–12], we formulate the mathematical model of magnetohydrodynamic flows of an incompressible polymeric fluid. We consider the variant where the dissipative summands are introduced in the equation characterizing the variation in the internal energy (the equation of heat influx) of the fluid, similarly to the work of Shibata [13]. In the dimensionless form we obtain (in the following, we will use the notation accepted earlier [12])

$$\text{div } u = u_x + v_y = 0, \quad \text{div } H = L_x + M_y = 0,$$

$$\frac{du}{dt} + \nabla P = \text{div}(Z\Pi) + \sigma_m(H, \nabla)H + Gr(Z - 1)
\begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad P = p + \sigma_m \frac{L^2 + (1 + M)^2}{2},$$

$$\frac{da_{1}}{dt} - 2A_{1}u_{x} - 2a_{12}u_{y} + L_{11} = 0,$$

$$\frac{da_{2}}{dt} - 2A_{2}v_{y} - 2a_{12}v_{x} + L_{22} = 0,$$

$$\frac{dZ}{dt} = \frac{1}{Pr} \Delta_{x,y}Z + \frac{A_z}{Pr} Z D_{y} + \frac{A_m}{Pr} \sigma_m D_m, \quad \frac{dH}{dt} - (H, \nabla)u - b_m \Delta_{x,y}H = 0.$$

Here, $u = (u, v)$ and $H = (L, 1 + M)$ is the velocity vector and the magnetic field vector in the Cartesian system of coordinates $x, y$, $t$ is the time, $Z = \frac{T - T_0}{T_0}$, $T$ is the temperature, $T_0$ is some averaged value of the temperature (room temperature, we will further take $T_0 = 300$ K), $\Pi = \frac{1}{Re} (a_{ij})$, $i, j = 1, 2$, $Re = \frac{u_H l}{\eta_0}$ is the Reynolds number, $(\rho = \text{const})$ is the density of the medium, $u_H$ is the characteristic length, $l$ is the characteristic length, $\eta_0$ is the initial value of the shear viscosity at room temperature $T_0$, $a_{11}, a_{22}, a_{12}$ are the components of the second-rank symmetric tensor of anisotropy, $Gr = \frac{Ra}{Pr}$ is the Grashof number, $Ra =$.
is the Rayleigh number, }b\text{ is the thermal expansion coefficient of the polymeric fluid, }g\text{ is the gravity acceleration, }\sigma_m = \frac{\mu_0 H_0^2}{\rho u_H^2} \text{ is the magnetic pressure, }\mu\text{ is the magnetic permeability of the polymeric fluid, }\mu_0\text{ is the magnetic permeability of the vacuum, }H_0\text{ is the characteristic value of the magnetic field, }A_i = \text{We}^{-1} + a_i, i = 1, 2, \text{ We} = \frac{\tau_0 u_H}{l}\text{ is the Weissenberg number, }\tau_0^*\text{ is the initial value of the time relaxation at room temperature }T_0[3, 12], L_i = \frac{K_i a_i + (a_i^2 + a_i^2)}{o(Z)}\text{, }K_i = \text{We}^{-1} + \frac{k}{3} I, k = k_1, I = a_1 + a_2,\text{ is the first invariant of the anisotropy tensor; }k, \text{ and } (0 < \beta < 1)\text{ are the phenomenological parameters of the rheological model}[3], \tilde{K}_i = K_i + I, I = a_1 + a_2,\text{ is the first invariant of the anisotropy tensor, }\tau_0(Z) = \frac{1}{ZJ(Z)}, J(Z) = \exp\left[\frac{E_A Z - 1}{Z}\right], E_A = \frac{E_A}{T_0}\text{, }E_A\text{ is the energy of activation, }A_\alpha = \frac{\alpha_\mu^2 Pr}{T_0 c_v}, D = a_i u_i + (v_x + u_x) a_2 + a_2 v_y, A_m = \frac{\alpha_m u_H Pr}{T_0 c_v}, \alpha\text{ is the thermal equivalent of work}[14], \alpha_m\text{ is the magnetothermal equivalent of work, }D_m = L^2 u_x + (1 + M)(v_x + u_x) + (1 + M)^2 v_y, b_m = \frac{1}{Re_m}, \text{ Re}_m = \sigma_m \mu_0 u_H l\text{ is the magnetic Reynolds number, }\sigma\text{ is the electric conductance of the medium, }\frac{\partial}{\partial t} + u \frac{\partial}{\partial x} + v \frac{\partial}{\partial y}, \text{ and } \Delta_{x,y} = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}.

In the system of Eqs. (2.1)–(2.4) the variables and the complexes of variables }t, (x, y), (u, v), p, (a_1, a_2, a_3), L, \text{ and } M\text{ are the results of relating the corresponding dimensional variables to their following characteristic values: }l/u_H, l, u_H, \rho u_H^2, \text{ We}/3, \text{ and } H_0 (\text{Fig. 1}).

Note 1. The magnetohydrodynamic Eqs. (2.1)–(2.4) are derived by invoking the Maxwell equations[9, 10], and the magnetic induction vector is taken in the form

\[ \mathbf{B} = \mu_0 \mu H = (1 + \chi) \mu_0 H, \quad \chi = \frac{\chi_0}{Z}, \] (2.5)

where }\chi\text{ is the magnetic susceptibility and }\chi_0\text{ is the magnetic susceptibility at room temperature }T_0 (= 300 K)[15, 16]. In the following, we will assume that }\mu = 1 (\chi_0 = 0)\text{ for the polymeric fluid.

We will consider the problem of solving the system of Eqs. (2.1)–(2.4), describing the magnetohydrodynamic flow of an incompressible polymeric fluid in a plane channel with the thickness }l\text{ limited by the horizontal walls, the electrodes }C^+\text{ and }C^-\text{ along which the electric conduction current flows with the value }J^+\text{ and }J^-\text{. The channel is put under the external magnetic field with the components }L = 0\text{ and }M = 0 \text{ (Fig. 1).}

The domains }S_1^+\text{ and }S_1^-\text{ external with respect to the channel are the magnetic with the magnetic susceptibilities }\chi_1^+\text{ and }\chi_1^-;\text{ they are also subject to the above described magnetic field whose components have the corresponding indices. The no slip conditions are satisfied at the channel walls

\[ y = \pm \frac{1}{2}; \quad u = 0 \] (2.6)

along with the condition }T = T_0\text{ in the domain }S_1^+\text{ and at the electrode }C^+

\[ y = \frac{1}{2}; \quad Z = 1 \] (2.7)
Due to relation (2.5) and conditions (2.6)–(2.8), we obtain

\[
(2.9)
\]

Note 2. We will further assume that either \( k = \beta \) or \( k = 1.2\beta \). We note that, as was shown in work [3], at \( k = 1.2\beta \) the results for the melts of polymers agree in the best way with the experimental data.

Note 3. The boundaries \( C^+ \) and \( C^- \) are the interfaces between the two homogeneous isotropic magnetics. Therefore, the following boundary conditions are satisfied at them [15, 17]:

\[
y = \frac{1}{2}(C^+)\quad L = -J^+, \quad M = \frac{\chi_0^+}{1 + \theta}, \quad \theta = T - T_0, \quad \theta = T - T_0,
\]

(2.10)

We will consider the analog of the Poiseuille flow as the main flow regime of the incompressible fluid.

Note 4. We again note that, in contrast to, for instance, works [12, 18–20], the first one of Eqs. (2.4) describing the variation in internal energy (the equation of heat flux) contains the dissipative terms characterizing the heat flux appearing at nonzero gradients of velocity. The presence of dissipative terms leads, in particular, to the presence of nonisothermality and to \( \bar{\theta} = 0 \) (and it is correct, because the heat flux will occur due to the influence of the anisotropy tensor components \( \hat{a}_{11}, \hat{a}_{12}, \) and \( \hat{a}_{22} \)).

Conversely, in the model accepted earlier [12], at \( \bar{\theta} = 0 \) we have \( \hat{Z} = 1 \) and \( \tau_6(\hat{Z}) = 1 \), that is, the isothermal process appears (a hat denotes the main solution and the values computed at the main solution).
3. POISEUILLE-TYPE MAGNETOHYDRODYNAMIC FLOW

We denote \( U(t, x, y) = (u, v, a_{11}, a_{12}, a_{22}, Z, L, M)^T \). We will seek for the partial solution of the special kind to the system of Eqs. (2.1)–(2.4),

\[
U = \hat{U}(y), \\
p = \hat{p}(y) + \hat{p}_0 - \hat{A}x, \\
\hat{A} = \frac{\Delta p}{u_{ij} h}
\]  

(3.1)

which corresponds to the stationary flow of an incompressible polymeric fluid in a plane infinite channel (Fig. 1) under the action of a constant pressure drop along the channel axis \( y = 0 \). Here,

\[
\hat{U}(y) = (\hat{u}(y), 0, \hat{a}_{11}(y), \hat{a}_{12}(y), \hat{a}_{22}(y), \hat{Z}(y), \hat{L}(y), \hat{M}(y))^T
\]

and \( \hat{p}(y) (\hat{p}(0) = 0) \) is some function to be determined, \( \hat{p}_0 \) is the pressure value at the channel axis \( y = 0 \) at \( x = 0 \), \( \hat{A} \) is the dimensionless pressure drop at the segment \( h \) taken with the opposite sign, and \( \Delta p > 0 \) (Fig. 1).

To determine the values as functions of \( y \) from Eqs. (2.1)–(2.4), (2.6)–(2.8), and (2.19), we obtain the relations which allow formulating the iteration algorithm for computing \( \hat{Z}(y, \bar{C}), \hat{L}(y, \bar{C}), \hat{a}_{22}(y, \bar{C}) \), and the constant \( \bar{C} \); the results of the performed calculations were given in details earlier in [21].

Some solutions, or the components of solutions \( \hat{u}, \hat{Z}, \) and \( \hat{L} \), are presented in Figs. 2–5. In the first (main) case \( \hat{A} = 1, 1, m = 1, \) \( \text{Re} = 1, \) \( \text{We} = 1, = 0.5, \) \( A_{-} = 1, A_{+} = 1, b_{m} = 1, b_{m} = 1, E_{A} = 1, J_{+} = 2, \) and \( J_{-} = 1. \) And in the second, third, and fourth cases only one of parameters \( A, J_{+} \), and \( \theta \) is varied, respectively, whereas the remaining parameters remain unchanged.

Note the important qualitative features in the behavior of stationary modes of the flow of a polymeric fluid: in contrast to work [12], where the effect of the parameter \( E_{A} \) was discovered to be related to the energy of activation \( E_{A}; E_{A} = E_{A}/T_{0} \), it affects the shape of the velocity profile and makes it lose its sym-
metry typical for the parabolic velocity profile of the Poiseuille flow of the viscous fluid [11], the solutions of the model (2.1)–(2.10) possess a wider spectrum of interesting properties.

In particular, in the main case (Fig. 2) the velocity profile is elongated against the direction of the pressure forces (which is caused by the influence of the magnetic field!). In Fig. 3, the pressure drop is (by the absolute value) increased, so the velocity profile is turned to the right, and in Fig. 4 the absolute value of the pressure is again moderate; however, due to the fact that at the upper electrode the current direction is changed to the opposite one, the velocity profile is again turned to the right. Finally, strong cooling of the channel’s lower boundary leads to the fluid velocity in the channel’s lower part vanishing almost entirely (Fig. 5).

The physical realizability of the solution including, in particular, its Lyapunov stability, is the most important issue arising in view of the stationary solution to problem (2.1)–(2.10), the Poiseuille-type flow.

4. LINEARIZED MODEL OF FLOW OF A POLYMERIC FLUID IN A PLANE CHANNEL: FORMULATION OF THE MAIN RESULTS

We linearize the model (2.1)–(2.10) of the analog of the Poiseuille flow described in the previous section accounting for additional conditions (2.1) and derive a cumbersome system which may be written in the structural form as

\[
\begin{align*}
V_t + BV_x + CV_y + RV + \Gamma &= 0, \\
\Delta_{x,y} \Omega &= F, \\
Z_t + \hat{u}Z_x + \hat{\Omega} &= G, \\
L_t + \hat{u}L_x + \hat{\Omega}L - (1+\mu_y - \hat{u}M - b_m\Delta_{x,y}L &= 0, \\
M_t + \hat{u}M_x - \hat{\Omega}M - (1+\mu_y - b_m\Delta_{x,y}M &= 0, \quad -\frac{1}{2} < y < \frac{1}{2},
\end{align*}
\]

where

\[
V = (u, v, a_{11}, a_{12}, a_{22})^T, \quad B = \hat{U}\hat{I}_5 + B_0,
\]
where $B_0 = (b_{ij}^0)_{i,j=1}^5$, $C = (c_{ij}^0)_{i,j=1}^5$ are the sufficiently sparse matrices with the elements depending on $\hat{Z}$ and $\hat{y}$, $\Omega = P - \hat{Z}_{22} - Z_{22}$, $\alpha_y = \frac{\hat{y}}{\Re}$, the components of the vector $\Gamma$, and the functions $F$ and $G$ are the linear functions of the unknowns and their derivatives with respect to the variables $x, y$, and the coefficients of the system are determined by the main solution and by the parameters of the original system of Eqs. (2.1)–(2.4).

We supplement the system of Eqs. (4.1)–(4.5) with the boundary conditions

$$
y = \frac{1}{2}; \quad u = v = Z = L = M = 0, \quad \Omega_y = (1 + \bar{\theta})(\alpha_{12})_x,
$$

and with the initial data. Here, the initial data must satisfy Eq. (4.2) and additional conditions (2.1).

In the following, we will seek the partial solutions of a special kind to the problem formulated above

$$
(\Omega, Z, L, M)^T = \exp(t + ix)(\Omega^*(y), Z^*(y), L^*(y), M^*(y))^T, \quad \lambda = \eta + i\xi, \quad \xi, \omega \in R^+.
$$

We formulate the main results of the work.

**Theorem 1.** If problem (4.1)–(4.7) has the solution of form (4.8) (the parameter $\omega$ is fixed), then the following asymptotic representation is valid for $\lambda$

$$
\lambda_k = \frac{1}{2} \int_{-1/2}^{1/2} d\xi \left( \int_{-1/2}^{1/2} d\xi \left( -i \xi \omega + \frac{R_{44}}{\Re} \hat{\alpha}_{12} \right) \right) + O\left( \frac{1}{k} \right), \quad k \to \infty,
$$

where $\hat{\alpha}_2 = \hat{\alpha}_{22} + \frac{1}{R \Re \We}$, $R_{34} = -2\hat{\omega} + 2\beta \hat{\alpha}_{12} \hat{Z}_0^*$, $\hat{\alpha}_0 = \frac{1}{\Re}\hat{Z}_0^*$, $R_{44} = \hat{Z}_0^* \hat{K}_j$. ($R_{34}$ and $R_{44}$ are the components of the matrix $R$ in system (4.1)).

Representation (4.9) implies the necessary condition of asymptotic stability of the Poiseuille-type flow found in Section 3.

**Theorem 2.** For the asymptotic stability of the Poiseuille-type flow, the following inequality needs to be fulfilled

$$
\int_{-1/2}^{1/2} d\xi \left( \hat{\alpha}_{11} \left( \hat{K}_3 + \hat{\beta} \right) + \frac{1}{2} \Re \We \right) < 0.
$$

The proofs of the theorems are not given, because they are cumbersome. Note that in the proofs we use the ideas of works [22–24] related with the asymptotic representation of fundamental solutions to systems of ordinary differential equations in the case with increasing absolute value of the parameter.

The statements of the theorems generalize the results [18–20, 25–28] to the case of nonisothermal flows of polymers under a magnetic field.

**CONCLUSIONS**

We determined the formula describing the distribution of the points of the spectrum (when their absolute value increases) for the linear mixed problem obtained with linearization of the MHD model of an incompressible polymeric fluid with respect to an analog of the Poiseuille flow in an infinite plane channel in the class of periodic perturbations. We formulated the necessary condition of its asymptotic stability determining the domain of admissible parameters of the model for which this property is valid.

**FUNDING**

The work is supported by the Russian Foundation for Basic Research, projects nos. 17-01-00791a and 19-01-00261a.
ACKNOWLEDGMENTS

The authors thank A.V. Egitov for his help in preparing the manuscript.

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