Planar matrices and arrays of Feynman diagrams

Freddy Cachazo¹, Alfredo Guevara¹,²,³,⁴, Bruno Umbert¹,⁵ and Yong Zhang¹,⁶,⁷,*

¹ Perimeter Institute for Theoretical Physics, Waterloo, ON N2L 2Y5, Canada
² Department of Physics and Astronomy, University of Waterloo, Waterloo, ON N2L 3G1, Canada
³ CECs Valdivia y Departamento de Física, Universidad de Concepción, Casilla 160-C, Concepción, Chile
⁴ Center for the Fundamental Laws of Nature, Society of Fellows and Black Hole Initiative, Harvard University, Cambridge, MA 02138, United States of America
⁵ Department of Applied Mathematics, Western University, London, ON N6A 5B7, Canada
⁶ CAS Key Laboratory of Theoretical Physics, Institute of Theoretical Physics, Chinese Academy of Sciences, Beijing 100190, China
⁷ School of Physical Sciences, University of Chinese Academy of Sciences, No. 19A Yuquan Road, Beijing 100049, China

E-mail: fcachazo@pitp.ca, aguevaragonzalez@fas.harvard.edu, bgimenez@uwo.ca and yzhang@perimeterinstitute.ca

Received 25 August 2023, revised 20 November 2023
Accepted for publication 28 November 2023
Published 29 February 2024

Abstract

Recently, planar collections of Feynman diagrams were proposed by Borges and one of the authors as the natural generalization of Feynman diagrams for the computation of $k = 3$ biadjoint amplitudes. Planar collections are one-dimensional arrays of metric trees satisfying an induced planarity and compatibility condition. In this work, we introduce planar matrices of Feynman diagrams as the objects that compute $k = 4$ biadjoint amplitudes. These are symmetric matrices of metric trees satisfying compatibility conditions. We introduce two notions of combinatorial bootstrap techniques for finding collections from Feynman diagrams and matrices from collections. As applications of the first, we find all 693, 13612 and 346710 collections for $(k, n) = (3, 7), (3, 8)$ and $(3, 9)$, respectively. As applications of the second kind, we find all 90608 and 30659424 planar matrices that compute $(k, n) = (4, 8)$ and $(4, 9)$ biadjoint amplitudes, respectively. As an example of the evaluation of matrices of Feynman diagrams, we present the complete form of the $(4, 8)$ and $(4, 9)$ biadjoint amplitudes. We also start a study of higher-dimensional arrays of Feynman diagrams, including the combinatorial version of the duality between $(k, n)$ and $(n - k, n)$ objects.

Supplementary material for this article is available online

Keywords: Feynman diagrams, biadjoint amplitudes, tropical Grassmannian

(Some figures may appear in colour only in the online journal)

1. Introduction

Tree-level scattering amplitudes in a cubic scalar theory with flavor group $U(N) \times U(N)$ admit a Cachazo–He–Yuan (CHY) formulation based on an integration over the configuration space of $n$ points on $\mathbb{C}P^{n-1}$ and the scattering equations [1–5]. [6],

* Author to whom any correspondence should be addressed.

Early, Mizera and two of the authors introduced generalizations to higher-dimensional projective spaces $\mathbb{C}P^{k-1}$. These higher $k$ biadjoint amplitudes’ were also shown to have deep connections to tropical Grassmannians. This led to the proposal that Feynman diagrams could be identified with facets of the corresponding Trop $G(k, n)$ [6–8]. Additionally, the exploration of generalized amplitudes and their attributes, such as generalized soft/hard theorems, alongside the pertinent scattering...
equations and their solution characterizations, has been expanded in several works [9–12].

Motivated by the connection between Trop $G(2, n)$ with metric trees, which can be identified as Feynman diagrams, and Trop $G(3, n)$ with metric arrangements of trees [13], Borges and one of the authors introduced a generalization to $k = 3$ called planar collections of Feynman diagrams as the objects that compute $k = 3$ biadjoint amplitudes [14].

The computation of a $k = 3$ biadjoint amplitude is completely analogous to that of the standard $k = 2$ amplitude but defined as a sum over planar collections of Feynman diagrams

$$m_n^{(3)}(\alpha, \beta) = \sum_{\mathcal{C} \in \Omega(\alpha) \cap \Omega(\beta)} \mathcal{R}(\mathcal{C}),$$

with $\Omega(\alpha)$ the set of all collections of Feynman diagrams that are planar with respect to the $\alpha$-ordering [14]. More explicitly, the $i$th tree in a collection is a tree with $n - 1$ leaves $\{1, 2, \ldots, n\} \setminus i$ which is planar with respect to the ordering induced by deleting $i$ from $\alpha$. This is why the collection is called planar but not the individual trees.

The value $\mathcal{R}(\mathcal{C})$ of a planar collection $\mathcal{C}$ is obtained from the following function

$$\mathcal{F}(\mathcal{C}) = \sum_{i,j,k} \tilde{\eta}_{ijk} s_{ijk},$$

defined in terms of the metrics of the trees in the collection $d_{ijk}^\beta$ which satisfy a compatibility condition $d_{ijk}^{(j)} = d_{ijk}^{(k)}$, thus defining a completely symmetric rank-three tensor $\tilde{\eta}_{ijk} := d_{ijk}^{(i)}$ [13]. Here $s_{ijk}$ is the $k = 3$ generalization of Mandelstam invariants, defined as completely symmetric rank-three tensors satisfying [6]

$$s_{ij} = 0, \quad s_{ijk} = 0, \quad \forall i \in \{1, 2, \ldots, n\}.\quad (1.3)$$

The explicit value is then computed as

$$\mathcal{R}(\mathcal{C}) = \int_\Delta d^{2(n-4)}f_j \exp \mathcal{F}(\mathcal{C}),$$

where the domain $\Delta$ is defined by the requirement that all internal lengths of all Feynman diagrams in the collection be positive [14].

In this work we continue the study of planar collections of Feynman diagrams by exploiting an algorithm proposed in [14] for determining all collections for $k = 3$ and $n$ points by a ‘combinatorial bootstrap’ starting from $k = 2$ and $n$-point planar Feynman diagrams. We review in detail the algorithm in section 2 and use it to construct all 693, 13612 and 346 710 collections for $(k, n) = (3, 7)$, $(3, 8)$ and $(3, 9)$, respectively. The 693 collections for $(k, n) = (3, 7)$ were already obtained in [14] by imposing a planarity condition on the metric tree arrangements presented by Herrmann, Jensen, Joswig and Sturmfels in their study of the tropical Grassmannian Trop $G(3, 7)$ in [13]. Also, there are deep connections between positive tropical Grassmannians and cluster algebras as explained by Speyer and Williams [15] and explored by Drummond, Foster, Gürdogan and Kalousios [16]. In the latter work, it was found that Trop $^+ G(3, 8)$ can be described in terms of 25 080 clusters. Here we show that our 13 612 planar collections for $(3, 8)$ encode exactly the same information as their 25 080 clusters. The cluster algebra analysis of Trop $^+ G(3, 9)$ has not appeared in the literature but it should be possible to obtain them from our 346 710 collections.

We also start to explore the next layer of generalizations of Feynman diagrams in section 3 and propose that $k = 4$ biadjoint amplitudes are computed using planar matrices of Feynman diagrams. In a nutshell, an $n$-point planar matrix of Feynman diagrams $\mathcal{M}$ is an $n \times n$ matrix with Feynman diagrams as entries. The $M_{ij}$ entry is a Feynman diagram with $n - 2$ leaves $\{1, 2, \ldots, n\} \setminus \{i, j\}$. Each tree has a metric defined by the minimum distance between leaves, $d_{ij}^{(i)}$. Here we use superscripts to denote the entry in the matrix of trees and subscripts for the two leaves whose distance is given. Planar matrices of Feynman diagrams must satisfy a compatibility condition on the metrics

$$d_{ij}^{(i)} = d_{ji}^{(i)} = d_{ij}^{(k)} = d_{jk}^{(i)} = d_{ik}^{(j)}.$$

This means that the collection of all metrics defines a completely symmetric rank-four tensor $\tilde{\eta}_{ijkl} := d_{ijkl}^{(i)}$.

Using this we generalize the prescription for computing the value $\mathcal{R}(\mathcal{T})$ and $\mathcal{R}(\mathcal{C})$ of $k = 2$ and $k = 3$ ‘diagrams’ to $\mathcal{R}(\mathcal{M})$ for $k = 4$ and therefore their contribution to generalized $k = 4$ amplitudes.

Moreover, we find that a second class of combinatorial bootstrap approach can be efficiently used to simplify the search for matrices of diagrams that satisfy the compatibility conditions (5). The idea is that any column of a planar matrix of Feynman diagrams must also be a planar collection of Feynman diagrams but with one less particle. In the first of our two main examples, any matrix for $(k, n) = (4, 8)$ must have columns taken from the set of 693 $(k, n) = (3, 7)$ planar collections. Using that the matrix must be symmetric, one can easily find 91 496 matrices of trees satisfying this purely combinatorial condition. Therefore the set of all valid planar matrices for $(k, n) = (4, 8)$ must be contained in the set of those 91 496 matrices. Surprisingly, we find that only 888 such matrices do not admit a generic metric satisfying (5). This means that there are exactly 90 608 planar matrices of Feynman diagrams for $(k, n) = (4, 8)$. We also find efficient ways of computing their contribution to $m_n^{(4)}(1, 1)$.

As the second main example of the technique, we use the $(3, 8)$ planar collections to construct candidate matrices in $(4, 9)$. We find 33 182 763 such symmetric objects. Computing their metrics we find that 2 523 339 of them are degenerate, and therefore the total number of planar matrices of Feynman diagrams for $(4, 9)$ is 30 659 424. We present all results, including the amplitudes, in the supplementary data and explain the results in section 4.

In section 5 we explain how to use efficient techniques to evaluate the contribution of a given planar array of Feynman diagrams to an amplitude by showing that the integration over the space of metrics is equivalent to the triangulations of certain polytopes and then show how software such as PolyMake can be used to carry out the computations.

After identifying collections with $(3, n)$ amplitudes and matrices with $(4, n)$, it is natural to introduce planar $(k - 2)$-dimensional arrays of Feynman diagrams as the objects relevant for the computation of $(k, n)$ biadjoint amplitudes. In
In this section, we give a short review of the definition and properties of planar collections of Feynman diagrams [14]. Emphasis is placed on a technique for constructing \(n\)-particle planar collections starting from special ones obtained by ‘pruning’ \(n\)-point planar Feynman diagrams and then applying a ‘mutation’ process. Here we borrow the terminology mutation from the cluster algebra literature [17–19]. The reason for this becomes clear below.

This pruning–mutating technique is the first combinatorial bootstrap approach we use in this work. The second kind is introduced in section 3 as a way of constructing planar matrices of Feynman diagrams from planar collections.

Without loss of generality, from now on we only consider the canonical ordering \(\mathcal{I} = (1, 2, \ldots, n)\), and every time an object is said to be planar it means with respect to \(\mathcal{I}\).

Recall that for \(k = 2\) the objects of interest are \(n\)-particle planar Feynman diagrams in a \(\phi^3\) scalar theory. There are exactly \(C_{n-2}\) such diagrams. When Feynman diagrams are thought of as metric trees, a length is associated with each edge and if any of the \(n - 3\) internal lengths becomes zero we say that the tree degenerates. Here is where the power of restricting to planar objects comes into play, once a given planar tree degenerates, there is exactly one more planar tree that shares the same degeneration. These two planar Feynman diagrams only differ by a single pole and we say that they are related by a mutation.

Starting from any planar Feynman diagram, one can get all other planar Feynman diagrams by repeating mutations. If no new Feynman diagrams are generated, we are sure we have obtained all the Feynman diagrams of certain ordering.

Here the terminology mutation precisely coincides with the one used in cluster algebras since planar Feynman diagrams are known to be in bijection with clusters of an \(A\)-type cluster algebra and mutations connect clusters in exactly the same way as degenerations connect planar metric trees. This precise connection between objects connected via degenerations and cluster mutations does not hold for higher \(k\), and therefore we hope the abuse of terminology will not cause confusion [14].

For the computation of \(k = 3\) biadjoint amplitudes, planar \(n\)-point Feynman diagrams are replaced by planar collections of \((n - 1)\)-point Feynman diagrams. Each collection is made out of \(n\) Feynman diagrams with the \(i\)th tree defined on the set \(\{1, 2, \ldots, n\}\) and planar with respect to the ordering \((1, 2, \ldots, i - 1, i + 1, \ldots, n)\). Each tree has its own metric defined as the matrix of minimal lengths from one leaf to another. The metric for the \(i\)th tree is denoted as \(d^{(i)}\) with \(j, k \in \{1, 2, \ldots, n\} \setminus \{i\}\). Moreover, the metrics have to satisfy a compatibility condition \(d^{(i)}_{jk} = d^{(k)}_{ij} = d^{(j)}_{ik}\).

A necessary condition for two planar collections of Feynman diagrams to be related is that their individual elements, i.e. the \((n - 1)\)-point Feynman diagrams, are either related by a mutation or are the same. Of course, in order to prove that the collections are actually related it is necessary to study the space of metrics and show that the two share a common degeneration.

The key idea is that we can get all planar collections of Feynman diagrams by repeated mutations, starting at any single collection. What is more, we can tell whether we have obtained all of the collections when there are no new collections produced by mutation.

A more efficient variant of the mutation procedure described above is obtained by introducing multiple initial collections. In fact, there is a canonical set of planar collections that are easily obtained from \(n\)-point planar Feynman diagrams.

Let us define the initial planar collections as those obtained via the following procedure. Consider any \(n\)-point planar Feynman diagram \(T\) and denote the tree obtained by pruning (or removing) the \(i\)th leaf by \(T_i\). Then the set \(\{T_1, T_2, \ldots, T_n\}\) is a planar collection of Feynman diagrams.

Let us illustrate this with a simple example shown in figure 1.

Using all such \(C_{n-2}\) collections as starting points one can then apply mutations to each and start filling out the space of planar collections in \(n\) points. When the method is applied to \((k, n) = (3, 5)\) we obtain all planar collections without the need for any mutations since every single planar collection in this case is dual to a \((2, 5)\) Feynman diagram. Next, we apply the technique to reproduce the known results for \((3, 6)\) starting from the \(C_4 = 14\) initial collections.

We find that after only three layers of mutations we get all planar collections. Repeating the procedure for \((3, 7)\) we find all 693 planar collections starting from the initial \(C_5 = 42\) collections after four layers of mutations.

Our first new results in this work are the computation of all 13 612 planar collections in \((3, 8)\) and all 346 710 in \((3, 9)\). Details on the results and the supplementary data where the collections are presented are provided in section 4.

All results are summarized in table 1. We classify the planar collections according to their number of mutations and count the number of collections for each kind as well. The precise definition of metrics and degenerations of planar collections of Feynman diagrams was given in [14].

\(\text{C}_n\) is the \(n\)th Catalan number.

[8] Here we assume that the set of all of the planar collections is connected. We have checked this to be the case up to \(n = 9\).
3. Planar matrices of Feynman diagrams

In the previous section, we introduced an efficient algorithm for finding all planar collections of Feynman diagrams based on a pruning–mutation procedure. Such collections compute $k = 3$ biadjoint amplitudes.

The next natural question is what replaces planar collections for $k = 4$ biadjoint amplitudes. Inspired by the way a single planar Feynman diagram defines a collection by pruning one leaf at a time, we start with a matrix of Feynman diagrams where the $i,j$ element is obtained by pruning the $i$th and $j$th leaves of an $n$-point planar Feynman diagram as the relevant objects for $k = 4$,

\[
\mathcal{M} = \begin{bmatrix}
\varnothing & T^{(1,2)} & \cdots & T^{(1,n-1)} & T^{(1,n)} \\
T^{(2,1)} & \varnothing & \cdots & T^{(2,n-1)} & T^{(2,n)} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
T^{(n-1,1)} & T^{(n-1,2)} & \cdots & \varnothing & T^{(n-1,n)} \\
T^{(n,1)} & T^{(n,2)} & \cdots & T^{(n,n-1)} & \varnothing
\end{bmatrix},
\]

as first proposed in [14]. We denote the Feynman diagram in the $i$th row and $j$th column, where labels $i$ and $j$ are absent, by $T^{(i,j)}$.

We add a metric to every Feynman diagram $T^{(i,j)}$ in the matrix, and denote the lengths of internal and external edges as $f^{(i,j)}$ and $e^{(i,j)}$, respectively. Correspondingly, we can use $d^{(i,j)}$ to denote the minimal distance between two leaves $k$ and $l$. Up to this point, the edge lengths and hence distances $d^{(i,j)}$ of different Feynman diagrams in the matrix have no relations. We can relate them by imposing compatibility conditions analogous to those for collections of Feynman diagrams. This leads to the following definition.

Definition 3.1. A planar matrix of Feynman diagrams is an $n \times n$ matrix $\mathcal{M}$ with component $\mathcal{M}_{ij}$ given by a metric tree with leaves $\{1, 2, \ldots, n\} \setminus \{i, j\}$ and planar with respect to the ordering $(1, 2, \ldots, j, \ldots, n)$ satisfying the following conditions:

- diagonal entries are the empty tree $\mathcal{M}_{ii} = \varnothing$
- compatibility (5)

\[
d^{(i,j)} = d^{(j,i)} = d^{(i,j)} = d^{(i,j)}.
\]

Note that the compatibility condition has several important consequences. The first is that since a given metric is symmetric in its labels, i.e. $d^{(i,j)} = d^{(j,i)}$, which is obvious from its definition as the minimum distance from $k$ to $l$, one finds that the matrix $\mathcal{M}$ must be symmetric as stated in the following lemma.
Lemma 3.2. Planar matrices of Feynman diagrams are symmetric.

Proof. The symmetry of the matrix follows from realizing that the compatibility condition requires that $d^{(ij)}_u = d^{(jk)}_u$, and therefore the symmetry of the metric on the left-hand side in the leave labels $k$ and $l$ implies that of the right-hand side is symmetric in the matrix labels $k$ and $l$. In order to complete the proof, it is enough to note that a binary metric tree is uniquely determined by its metric, as we show in appendix A.

Planar collections of Feynman diagrams have $(n - 4)n$ internal edges; $n - 4$ for each of the $n$ trees in the collection. However, only $2(n - 4)$ are independent once the compatibility condition is imposed on the metrics, as reviewed in [14]. In the case of planar matrices of Feynman diagrams there are $\binom{n}{2}(n - 4)$ internal lengths $f^{(ij)}_l$ with $1 \leq i < j \leq n$, $1 \leq l \leq n - 5$ while the compatibility conditions (5) reduce the number down to $3(n - 5)$ independent ones. This means that a planar matrix has at least $3(n - 5)$ possible degenerations. The precise number depends on the structure of the trees in the matrix.

In analogy with planar collections, we say that two planar matrices are related via a mutation if they share a codimension one degeneration.

Recall that an initial planar collection is obtained by pruning a leaf of the same $n$-point planar Feynman diagram to produce $n$ different $(n - 1)$-point trees. We can also get an initial planar matrix by pruning two different leaves at a time from the same $n$-point planar Feynman diagram (see figure 2 for an example).

Using all such $C_{n-2}$ matrices as starting points one can then apply mutations to each and start filling out the space of planar matrices in $n$ points.

The contribution to the amplitudes of every planar matrix can be calculated individually. Consider the function of a planar matrix of Feynman diagrams $M$,

$$\mathcal{F}(M) = \sum_{1 \leq j < k, l < n} \pi_{ijk} s_{ijkl},$$

(3.2)

with $\pi_{ijk} := d^{(ij)}_l$. Here $s_{ijkl}$ are the generalized symmetric Mandelstam invariants introduced in [6]. These satisfy the conditions

$$s_{ijk} = 0, \quad \sum_{j,k,l=1}^n s_{ijkl} = 0 \quad \forall i.$$  

At this point it is obvious, but these conditions make it possible to write $\mathcal{F}(M)$ in a form free of any length of leaves $s^{(ij)}_l$. In section 6 we explain this phenomenon in more generality for any value of $k$.

An integral of $\mathcal{F}(M)$ over independent internal lengths $\{f_1, f_2, \ldots, f_{3(n-5)}\}$ gives the contribution to $k = 4$ biadjoint amplitudes

$$\mathcal{R}(M) = \int_\Delta q^{3(n-5)} f_1 \exp \mathcal{F}(M),$$

(3.4)

where the domain $\Delta$ is defined by the condition that all $\binom{n}{2}(n - 4)$ internal lengths are positive and not only the $3(n - 5)$ independent ones. For future use, we comment that it is possible to consider (4) also for degenerate matrices, and in such cases it integrates to zero as its domain is a set of measure zero.

Another important observation is that, in the $j$th column or row of a planar matrix, all Feynman diagrams are free of particle $j$ and the compatibility condition (5) requires

$$d^{(ij)}_u = d^{(ik)}_u = d^{(il)}_u,$$

(3.5)

for every three different particles $i, k, l$ of the remaining $n - 1$ particles. This means that the $j$th column or row is nothing but a planar collection of Feynman diagrams. Each column of a planar matrix is therefore made out of planar collections of $(3, n - 1)$. Besides, once several columns have been fixed, the remaining columns have much fewer choices because of the symmetry requirement of the matrix. This simple but powerful observation leads to the second kind of combinatorial bootstrap, which we describe next.

3.1. Second combinatorial bootstrap

Suppose we have obtained all the $N$ planar collections for the ordering $(1, 2, \ldots, n - 1)$. Let us denote the set of all such collections as $E_{3,n-1} = \{C_1, C_2, \ldots, C_N\}$. The last column $\{T^{(1,a)}, T^{(2,a)}, \ldots, T^{(n-1,a)}\}$ (here we have omitted the trivial empty tree $\emptyset$) of any planar matrix $M$, where by definition particles $1, 2, \ldots, n - 1$ are deleted, respectively, and the common missing particle $n$ must be an element of $E_{3,n-1}$.

Now we consider a cyclic permutation with respect to the order $(1, 2, \ldots, n - 1, n)$ of particle labels of the set $E_{3,n-1}$,

$$E^{(a)}_{3,n-1} = \{C^{(a)}_1, C^{(a)}_2, \ldots, C^{(a)}_N\} = E_{3,n-1}^{(a)},$$

(3.6)

Clearly, particle labels are to be understood modulo $n$. One can see that $E^{(n)}_{3,n-1}$ is the set of all planar collections for the ordering $(1, 2, \ldots, a - 1, a + 1, \ldots, n)$ with particle $a$ absent. By definition, we have $E^{(a)}_{3,n-1} \equiv E^{(n)}_{3,n-1} \equiv E_{3,n-1}$. The $a$th column $\{T^{(1,a)}, T^{(2,a)}, \ldots, T^{(a-1,a)}, T^{(a+1,a)}, \ldots, T^{(n-1,a)}\}$ (here we have once again omitted the trivial tree $\emptyset$) of a planar matrix $M$ must belong to the set $E^{(a)}_{3,n-1}$. Thus any planar matrix of Feynman diagrams must take the form

$$M = \{C^{(a)}_{i_1}, C^{(a)}_{i_2}, \ldots, C^{(a)}_{i_n}\}, \quad 1 \leq i_1, \ldots, i_n \leq N.$$  

(3.7)

Naively, we have $N$ choices for each column and hence $N^n$ candidate planar matrices. In principle, one could take this set of $N^n$ matrices and impose the compatibility condition on the metrics thus reducing the set to that of all planar matrices of Feynman diagrams. However, this procedure is already impractical for $n = 7$ where $N = 693$.

Luckily, according to lemma 3.2, the symmetry requirement of a planar matrix reduces this number dramatically. It is much more efficient to find possible planar matrices from all the symmetric matrices of the form (7).

Using this method we have obtained all planar matrices up to $n = 9$. Table 2 gives a summary of our results.

More explicitly, when this method is applied to $(k, n) = (4, 6)$, we obtain exactly 14 planar matrices, which
are dual to the $C_4 = 14$ planar Feynman diagrams of $(2, 6)$. As there are 693 planar collections in $(3, 7)$, the duality between $(3, 7)$ and $(4, 7)$ implies that there should be 693 planar matrices in $(4,7)$ as well. In fact, our combinatorial bootstrap procedure results in exactly that number! Moreover, in section 6 we explain how the 693 planar matrices of Feynman diagrams map one-to-one onto the 693 planar collections via the duality.

Our second set of new results corresponds to the more interesting cases of $(4, 8)$ and $(4, 9)$, where our procedure leads to 91,416 and 33,182,763 symmetric matrices, respectively.

Table 2. Number of planar matrices of Feynman diagrams and number of degenerate matrices for different values of $n$. |
| (4, 6) | (4, 7) | (4, 8) | (4, 9) |
|---|---|---|---|
| Planar matrices | 14 | 693 | 90,608 | 30,659,424 |
| Degenerate matrices | 0 | 0 | 888 | 2,523,339 |

Having the set of all possible candidate matrices, we further determined that 90,608 and 30,659,424 of them, respectively, satisfy the compatibility conditions (5) while not becoming degenerate and thus get these numbers of planar matrices of Feynman diagrams. We see that in both cases the combinatorial bootstrap came very close to the correct answer. We comment that the extra 888 and 2,523,339 ‘offending’ symmetric matrices are actually degenerate planar matrices. This means that if we were to use all matrices obtained from the bootstrap in the formula for the amplitude we would still get the correct answer since the extra matrices integrate to zero under the formula (4). So we can just use all of the symmetric matrices to calculate the biadjoint amplitudes for $k = 4$ as well.

Below we show two explicit examples for $n = 6, 7$ in order to illustrate the procedure. These examples show why this is an efficient technique for getting planar matrices from collections of $(3, n - 1)$. Details on the results for $(4, 8)$ and $(4, 9)$ and the supplementary data where the collections are presented are provided in section 4.

Figure 2. An example of a six-point initial planar matrix. Above we show a six-point Feynman diagram. Below there is a symmetric matrix of four-point Feynman diagrams obtained by pruning two leaves at a time from the set $1, 2, \cdots, 6$ of the above Feynman diagram. The Feynman diagram from the $i$th column and $j$th row has the $i$th and $j$th leaves pruned.
3.2. A simple example: from (3,5) to (4,6)

Now we proceed to show an explicit example of how to obtain planar matrices of Feynman diagrams for (4, 6). In this example, given the duality (4, 6) ~ (2, 6), we could obtain the planar matrices by picking \( n = 6 \) Feynman diagrams in \( k = 2 \) and removing two leaves in a systematic way as shown in figure 2. Here, however, we introduce an algorithm to get the matrices using a second bootstrap approach constrained by the consistency conditions explained above, thus obtaining planar matrices from planar collections of Feynman diagrams of (3, 5).

This algorithm works for general \( n \), i.e. it obtains planar matrices of (4, \( n \)) from planar collections of (3, \( n - 1 \)), and is going to be particularly useful for larger \( n \), where the number of matrices is considerably large.

Before going through the algorithm, let us review the planar collections of Feynman diagrams \( E_{3,5} = \{ C_1, C_2, C_3, C_4, C_5 \} \) [14]. These can come from the caterpillar tree in \( n = 5 \) and its four cyclic permutations, shown in figure 3.

In what follows, we will adopt the notation \( T[ab|cd] \) for a four-point Feynman diagram with \( a, b \) and \( c, d \) sharing a vertex, i.e.

\[
T[ab|cd] := \begin{array}{c}
\begin{array}{c}
\text{b} \\
\text{a}
\end{array}
\end{array} \quad \begin{array}{c}
\begin{array}{c}
\text{c} \\
\text{d}
\end{array}
\end{array}
\]

(3.8)

By applying cyclic permutations (6) on \( E_{3,5} \) we get the set \( E_{3,5}^{(1)}, E_{3,5}^{(2)}, \ldots, E_{3,5}^{(6)} \) with \( E_{3,5}^{(6)} = E_{3,5} \). In the more compact notation defined above we have, for instance\(^{10}\)

\[
E_{3,5}^{(1)} = \{ C_1^{(1)}, C_2^{(1)}, C_3^{(1)}, C_4^{(1)}, C_5^{(1)} \}
\]

\[
\begin{align*}
T[45|63] & \quad T[34|56] & \quad T[45|63] & \quad T[34|56] & \quad T[34|56] \\
T[45|62] & \quad T[24|56] & \quad T[45|62] & \quad T[24|56] & \quad T[45|62] \\
= & \quad T[23|56] & \quad T[23|56] & \quad T[35|62] & \quad T[23|56] & \quad T[35|62] \\
T[23|46] & \quad T[34|56] & \quad T[23|46] & \quad T[23|46] & \quad T[34|56] \\
T[23|45] & \quad T[34|52] & \quad T[23|45] & \quad T[23|45] & \quad T[34|52]
\end{align*}
\]

(3.9)

\[
E_{3,5}^{(2)} = \{ C_1^{(2)}, C_2^{(2)}, C_3^{(2)}, C_4^{(2)}, C_5^{(2)} \}
\]

\[
\begin{align*}
T[34|56] & \quad T[45|63] & \quad T[34|56] & \quad T[34|56] & \quad T[45|63] \\
T[14|56] & \quad T[45|61] & \quad T[14|56] & \quad T[45|61] & \quad T[45|61] \\
= & \quad T[13|56] & \quad T[35|61] & \quad T[13|56] & \quad T[35|61] & \quad T[13|56] \\
T[34|61] & \quad T[34|61] & \quad T[13|46] & \quad T[34|61] & \quad T[13|46] \\
T[34|51] & \quad T[13|45] & \quad T[13|45] & \quad T[34|51] & \quad T[13|45]
\end{align*}
\]

(3.10)

The idea of the second bootstrap is that each column of a planar matrix is a planar collection. In other words, a planar matrix must take the form

\(^{10}\) Here, for example, one can see the cyclic permutation \( \{ 1 \rightarrow 3, 2 \rightarrow 4, 3 \rightarrow 5, 4 \rightarrow 6, 5 \rightarrow 1, 6 \rightarrow 2 \} \) of \( C_1 \) as \( [T[45|63], T[45|62], T[23|56], T[23|46], T[23|45]] \) with the leaves 3, 4, 5, 6 and 1 pruned, respectively, in addition to the common missing leaf 2. We rotate the list from the right by 1 to get a planar collection \( C_1^{(2)} \) with the leaves 1, 3, 4, 5 and 6 pruned, respectively.
\begin{align*}
\mathcal{M} &= \{c^{(1)}_1, c^{(2)}_1, \ldots, c^{(6)}_6\}, \\
\text{with } 1 \leq i_1, \ldots, i_6 \leq 5, & \quad (3.11)
\end{align*}

where each element in \( \mathcal{M} \) corresponds to a column, thus the \( \text{ith} \) column belongs to the set \( E^{(i)}_{3,6} \) subject to the \( \text{ith} \) permutation. There are five choices for the first column since there are five collections in (3.5). However, once one of the collections is chosen, the choices for the remaining five columns become substantially reduced.

For example, let us choose the first column of the matrix to be the first collection \( C^{(1)}_1 \). The symmetry of the matrix implies \( T^{(1,2)} = T^{(2,1)} \), thus the first tree of the second column \( C^{(2)}_2 \) must be the first tree of the first column, i.e. \( T^{[45][63]} \). By looking at (9) and (10) we find that only \( C^{(2)}_2 \) and \( C^{(3)}_2 \) satisfy this requirement. Similarly, we select candidates from \( E^{(1)}_{1,3} \) by again imposing the symmetry condition \( T^{(i,1)} = T^{(1,i)} \) now for \( i = 3, 4, 5, 6 \) (see figure 4 for a sketch).

With this approach, the number of choices for the remaining five columns has been reduced from the naive 5^5 = 3 125 to 2 × 2 × 3 × 3 × 3 = 108.

Therefore, we can now forget about the first column and focus on the possible 108 choices for the remaining five columns. Let us for instance choose \( C^{(2)}_2 = \{T^{[45][63]}, T^{[45][61]}, T^{[35][61]}, T^{[34][61]}, T^{[13][45]}\} \) for the second column of the matrix. Because of the symmetry condition \( T^{(2,1)} = T^{(1,2)} \), which happens to be \( \mathcal{M}_1 \) in table 3 and is also the example shown in figure 2.

Had we chosen \( C^{(2)}_2 \) for the second column instead of \( C^{(2)}_2 \), we would have found another two planar matrices using the same procedure, which correspond to \( \mathcal{M}_2 \) and \( \mathcal{M}_3 \) in table 3. Hence, we find a total of three planar matrices for the initial choice \( C^{(1)}_1 \).

Likewise, one finds three, two, four and two planar matrices for the initial choices \( c^{(1)}_1, c^{(1)}_1, c^{(1)}_1 \) and \( c^{(1)}_3 \), respectively, thus giving 14 planar matrices in total. One can check that all these 14 matrices satisfy the compatibility conditions (5). Therefore, all of them contribute to the biadjoint amplitude in \( k = 3 \).

In table 3 we present all 14 planar matrices of Feynman diagrams in (4, 6), explicitly showing the corresponding collections in each column.

### 3.3. A more interesting example: from (3, 6) to (4, 7)

Now we comment on another example, in this case on how to obtain planar matrices of Feynman diagrams for (4, 7) using the second bootstrap again. The starting point is the 48 planar collections of (3, 6), i.e. \( E_{1,6} = \{C_1, C_2, \ldots, C_{48}\} \), which can be obtained from the first bootstrap. The cyclic permutations (6)
can be inferred from the position of Table 3. Commun. Theor. Phys.

For the readers choices for the remaining columns allowed by the candidates at each step, we get 693 symmetric matrices in total, which are much more than the 42 initial planar matrices for (4, 7) used in the pruning–mutation procedure of section 2. There are 693 planar collections in (3, 7) as well and how they are dual to 693 planar matrices is explained in section 6.

The ordering of collections in $E_{3,6}$ is not relevant as long as its cyclic permutations $E_{3,6}^{1}, \ldots, E_{3,6}^{6}$ change covariantly. For the readers’ convenience, we borrow table 1 from [14] containing all 48 collections and place it as table 4 in appendix B. We adopt the same ordering notation as in [14] so that we can present more details of the second bootstrap.

A collection in table 4 is given by six trees characterized by six numbers. For example, the first collection $C_{1}$ expressed by $[4, 4, 4, 3, 3, 3]$ means the collection given in figure 1, where the ‘middle leaves’ are 4, 4, 4, 3, 3 and 3, respectively. Its cyclic permutations give $C_{1}^{(1)}, C_{1}^{(2)}, \ldots, C_{1}^{(6)}$ with $C_{1}^{(6)} = C_{1}$, which act as the first element of $E_{3,6}^{1}, E_{3,6}^{2}, \ldots, E_{3,6}^{6}$, respectively.

If we choose $C_{1}^{(1)}$ as the first column, it happens that from each $E_{3,6}^{1}, \ldots, E_{3,6}^{6}$ there are 14 collections satisfying the symmetry requirement $T^{(1, 0)} = T^{(0, 1)}$. For example, for the second and third columns, their 14 possible choices of collections are

| Matrix | Collections | Matrix | Collections |
|--------|-------------|--------|-------------|
| $A_{1}$ | $[C_{4}, C_{2}, C_{3}, C_{2}, C_{3}, C_{2}]$ | $A_{3}$ | $[C_{3}, C_{2}, C_{3}, C_{2}, C_{5}, C_{2}]$ |
| $A_{2}$ | $[C_{4}, C_{3}, C_{2}, C_{3}, C_{5}, C_{2}]$ | $A_{4}$ | $[C_{3}, C_{4}, C_{2}, C_{3}, C_{5}, C_{2}]$ |
| $A_{4}$ | $[C_{3}, C_{4}, C_{2}, C_{3}, C_{5}, C_{2}]$ | $A_{2}$ | $[C_{3}, C_{4}, C_{2}, C_{3}, C_{5}, C_{2}]$ |
| $A_{6}$ | $[C_{3}, C_{2}, C_{3}, C_{2}, C_{3}, C_{5}]$ | $A_{6}$ | $[C_{3}, C_{2}, C_{3}, C_{2}, C_{3}, C_{5}]$ |
| $A_{7}$ | $[C_{3}, C_{2}, C_{3}, C_{2}, C_{3}, C_{5}]$ | $A_{7}$ | $[C_{3}, C_{2}, C_{3}, C_{2}, C_{3}, C_{5}]$ |

Table 3. Planar matrices of Feynman diagrams in (4, 6). Here we abbreviate $[C_{1}^{(1)}, C_{1}^{(2)}, \ldots, C_{1}^{(6)}]$ as $[C_{1}, C_{2}, \ldots, C_{6}]$ since the superscripts can be inferred from the position of $C_{i}$ in the brackets.

We see that the naive number of choices for the remaining six columns reduces from $48^{6} \sim 1 \times 10^{10}$ down to $14^{6} \sim 8 \times 10^{6}$.

Now we can forget the first column and focus on the $14^{6}$ candidates for the remaining six columns. If we choose $C_{1}^{(1)}$ from (14) as the second column, we find only one collection $C_{8}^{(3)}$ from (14) that satisfies the requirement $T^{(2, 0)} = T^{(0, 2)}$. Similarly, we find that there is only one collection from 14 candidates for the remaining four columns satisfying the requirement $T^{(2, 0)} = T^{(0, 2)}$ as well for $i = 4, 5, 6, 7$. This time we see that the naive number of choices for the remaining five columns dramatically reduces from $14^{5} \sim 5 \times 10^{3}$ to 1. Hence the only choice that makes up a planar matrix of the form (13) is

$$[C_{1}^{(1)}, C_{1}^{(2)}, C_{1}^{(3)}] = [C_{48}, C_{41}, C_{27}, C_{18}, C_{8}].$$

Table 4. All 48 planar collections of trees for $n = 6$ in a compact notation tailored to this case and explained in the text.

Planar collections of trees in $k = 3$ and $n = 6$

We see that the naive number of choices for the remaining six columns reduces from $48^{6} \sim 1 \times 10^{10}$ down to $14^{6} \sim 8 \times 10^{6}$.

Now we can forget the first column and focus on the $14^{6}$

candidates for the remaining six columns. If we choose $C_{1}^{(1)}$

from (14) as the second column, we find only one collection $C_{8}^{(3)}$

from (14) that satisfies the requirement $T^{(2, 0)} = T^{(0, 2)}$. Similarly, we find that there is only one collection from 14 candidates for the remaining four columns satisfying the requirement $T^{(2, 0)} = T^{(0, 2)}$ as well for $i = 4, 5, 6, 7$. This time we see that the naive number of choices for the remaining five columns dramatically reduces from $14^{5} \sim 5 \times 10^{3}$ to 1. Hence the only choice that makes up a planar matrix of the form (13) is

$$[C_{1}^{(1)}, C_{1}^{(2)}, C_{1}^{(3)}] = [C_{48}, C_{41}, C_{27}, C_{18}, C_{8}].$$

Had we chosen the remaining collections $C_{2}^{(1)}, \ldots, C_{47}^{(1)}$ or $C_{48}^{(2)}$ in (14) as the second column instead of $C_{1}^{(1)}$, we would have found 1, 2, 1, 2, 1, 2, 1, 2, 1, 2, 3, and 9 planar matrices, respectively. Thus there are 32 planar matrices in total with $C_{1}^{(1)}$ as the first column.

Similarly, we can get all of the planar matrices with $C_{2}^{(1)}, \ldots, C_{48}^{(1)}$ or $C_{48}^{(2)}$ as the first column of the matrix. By adding them up, including the 32 for $C_{1}^{(1)}$, we obtain all the 693 planar matrices in (4, 7).
4. Main results

The main applications of the techniques introduced in this work are the computation of all the planar collections of Feynman diagrams for the cases (3, 6), (3, 7), (3, 8) and (3, 9). This is done using the first kind of combinatorial bootstrap. We have also computed all the planar matrices of Feynman diagrams for (4, 7), (4, 8) and (4, 9). In this section, we try to give a self-contained presentation of these results in the form of \texttt{.m} files and a MATHEMATICA notebook named \texttt{Arrays\_of\_FDs.nb} to read them.

Note that \texttt{.m} files can be opened not only by MATHEMATICA but also by any TEXTEDIT. When there is a file with a name in the form \texttt{*c.m}, it is compressed to reduce its size and can be uncompressed by using the command \texttt{Uncompress@Import\@"c.m"} in any MATHEMATICA notebook with the correct directory.

4.1. Planar collections of feynman diagrams

We have placed the results of all planar collections of Feynman diagrams for (3, 6), (3, 7), (3, 8), and (3, 9) as \texttt{.m} files. The files \texttt{col36.m} and \texttt{col37.m} are human readable and \texttt{col38c.m} and \texttt{col39c.m} will become readable after being uncompressed by MATHEMATICA.

Let us illustrate the content of the files. For example, in the file \texttt{col36.m}, there are all 14 planar collections of Feynman diagrams for (3, 6). Each of the 14 planar collections is a set of six five-point Feynman diagrams. The way we choose to store the information is better explained with an example. The first collection presented in the file reads

\[
\text{c[FD}[s[2, 3], s[2, 3, 4], 1], \text{FD}[s[1, 3], \\
\text{s[1, 3, 4, 2], \text{FD}[s[1, 2], s[1, 2, 4], 3, \\
\text{FD}[s[1, 2], s[1, 2, 3], 4, \text{FD}[s[1, 2], \\
\text{s[1, 2, 3], 5], \text{FD}[s[1, 2], s[1, 2, 3], 6].] \text{(4.1)}
\]

Here we have made use of the fact that a five-point Feynman diagram can be completely characterized by its two poles. Note that we have to assume that each Feynman diagram in the collection comes endowed with its valid kinematic data, i.e. for the \texttt{ith} tree one uses \texttt{s[i]} with \texttt{j, k} \in \{1, 2, 3, 4, 5, 6\} \texttt{\forall} \text{and satisfying momentum conservation in only the five particles present. In section 6 we show that these n copies of kinematic spaces are more than just a convenience and that each tree is indeed a fully fledged Feynman diagram.}

Let us continue with the example in (1). As mentioned above, the six five-point Feynman diagrams have leaves 1, 2, \ldots, 6 pruned respectively. Thus \text{FD}[s[2, 3], s[2, 3, 4], 1] means a Feynman diagram with the poles \texttt{s[2,3], s[2,3,4] = s[3,6]} and with leaf 1 pruned. This collection happens to be the example shown in figure 1.

The contribution of a given planar collection of Feynman diagrams to a \(k = 3\) biadjoint amplitude can be computed by integrating over the space of compatible metrics, for example using the formula (4). For the cases (3, 6) and (3, 7) this is easily done and the results are in agreement with previous computations \[6, 9, 16\]. However, we find that a straightforward application of such a method to (3, 8) and (3, 9) is not practical with modest computing resources. This is why we developed much more efficient but equivalent algorithms for computing such contributions to the amplitudes. In this section, we simply present the data and postpone the explanation of the algorithm to the next section where we discuss evaluations as computing volumes.

For convenience, we created a MATHEMATICA notebook named \texttt{Arrays\_of\_FDs.nb} that can read all the \texttt{.m} files results automatically. Let us illustrate the use of \texttt{Arrays\_of\_FDs.nb} with an example. The output of the command \texttt{col[3,8]} gives all 13 612 planar collections for (3, 8) taken from \texttt{col38c.m}.

We postpone the presentation of the integrated amplitude \(k = 3\) biadjoint amplitude \(m_0^{(3)}(l_1, l_2)\), using a more efficient method, to the next section.

4.2. Planar matrices of feynman diagrams

We express the planar matrices for (4, 7), (4, 8) and (4, 9) as a series of collections we already save and their cyclic permutations. More explicitly, recall that the second kind of bootstrap is based on the fact that each column of a planar matrix for \(n\) points must be one of the planar collections for \((n - 1)\)-points with the labels chosen appropriately.

For example, in the file \texttt{mat47.m} there are 693 sets for (4, 7). The first one reads

\[
\{1, 1, 48, 41, 27, 18, 8\}. \text{(4.2)}
\]

This notation might seem cryptic at first but it is actually both very efficient and simple. In order to gain familiarity with the notation note that this is exactly the matrix presented in (15) but it is given with a slightly less compact notation.

In practice, this means that we can get the planar matrix by picking out the 1st, 17th, \ldots, 3rd collections in \texttt{col36.m} and shifting their particle labels in a cyclic ordering (1, 2, 3, 4, 5, 6, 7) by 1, 2, 3, 4, 5, 6, 7 respectively.

For the user’s convenience, we introduced a command \texttt{matrix[k_, n_\_\_set_\_]} in the MATHEMATICA notebook \texttt{Arrays\_of\_FDs.nb} to expand the compact notation and produce the explicit planar matrices. In addition, the command \texttt{mat[k_, n_\_\_]} returns the complete set of all planar matrices automatically. For example, the output of \texttt{matrix[4,7][\{1, 1, 48, 41, 27, 18, 8\}\]} gives the explicit expression of the planar matrix shown in (15).

All planar matrices for (4, 8) and (4, 9) collected in \texttt{mat48c.m} and \texttt{mat49seedc.m}, respectively, which can be read by MATHEMATICA. So one can use \texttt{mat[4,9]} to produce all 30 659 424 (4,9) planar matrices. The output of \texttt{matrix[4, 9][\{98, 280, 5154, 7773, 9509, 11 334, 10 639, 9515, 5082\}] \text{(4.3)}

This gives an explicit expression of a planar matrix for (4, 9), whose columns are from the 98th, 280th, \ldots and 5082nd collections of (3, 8) presented in the file \texttt{col38c.m} respectively.

With all planar collections of Feynman diagrams for (3,8) and all planar matrices for (4,8) and (4,9), one can in principle...
produce any (3, 8), (4, 8), (4, 9) partial amplitudes with two arbitrary orderings according to (4) and (4). With advanced techniques explained in the next section, their amplitudes can be computed much faster. We provide a notebook file named 38_48_amplitudes.nb where one can set any kinematics satisfying (3) or (3) as input to produce numeric amplitudes $m^{(3)}_{	ext{fi}}$ and $m^{(4)}_{	ext{fi}}$ in reasonable time.\footnote{This has recently been applied in [21] for the context of string integrals.} We also provide another notebook file named 49_amplitudes.nb whose computations have been finished and which contains several numeric diagonal amplitudes $m^{(3)}_{	ext{fi}}$ and several analytic off-diagonal amplitudes $m^{(4)}_{	ext{fi}}$. More details will be explained in the next section.

\section{Evaluation: computing volumes from planar collections and matrices}

In this section, we introduce a geometric characterization of each collection as a facet of the full polytope. Motivated by the picture provided in [16], we then realize the amplitude (4) as the volume of the corresponding geometry. The characterization speeds up the evaluation, especially for the cases (3, 8), (4, 8) and (4, 9), as it can be implemented and automatized via the software PolyMake. The full amplitudes are provided in the supplementary data.

Let us describe the procedure for the collection of figure 5, where the compatibility equations fix

\begin{align}
  u &= x + w - z, \\
  v &= y + w - z.
\end{align}

As explained in [6, 14] this corresponds to a bipyramid facet in the tropical Grassmannian (3, 6). To make this identification precise, here we note that the six constraints on the collection metric, namely $x, y, z, w, u, v > 0$, can be written as

$$F \cdot Z_i > 0,$$

where we define $F := (x, y, z, w) \in \mathbb{R}^4$ and

$$Z_i = (1, 0, 0, 0), Z_2 = (0, 1, 0, 0), Z_3 = (0, 0, 1, 0), Z_4 = (0, 0, 0, 1),$$

$$Z_5 = (1, 0, -1, 1), Z_6 = (0, -1, 0, 1).$$

Each $Z_i$ corresponds to a plane that passes through the origin. As will be further explored in [20], such planes correspond to degenerated collections and hence to boundaries of our geometry. Indeed, the inequalities define a cone in $\mathbb{R}^4$

$$\Delta = \{F \in \mathbb{R}^4 \mid F \cdot Z_i > 0, i = 1, \ldots, 6\}.$$

A bounded three-dimensional region is obtained by intersecting $\Delta$ with an affine plane, for example $F \cdot (1, 1, 1, 1) = x + y + z + w = 1$. This allows us to draw a three-dimensional bipyramid as in figure 6, where we label the faces by the corresponding $Z_i$. For our purposes, there is a preferred affine plane $T$ given by

$$f(C^{\text{bip}}) = F \cdot (R, t_{1234}, t_{3456}, t_{5612} - R) = 1.$$  

We now show that the volume of the corresponding three-dimensional bipyramid is precisely the amplitude associated with such collection, as defined by the Laplace integral formula (4)

$$\int_{\Delta} d^4 F e^{-f(C^{\text{bip}})}.$$  

The statement is a particular case of a more general fact known as the Duistermaat–Heckman formula\footnote{This has recently been applied in [21] for the context of string integrals.}. To derive this, we note that the bipyramid $\Delta$ (projected into $F \cdot T = 1$) can be triangulated by two simplices, $\Delta = \Delta_1 \cup \Delta_2$, by inserting the auxiliary plane, also depicted in figure 6

$$F \cdot Z^4 := y - z = 0$$

corresponding to the base of the bipyramid. In practice, $Z = (0, 1, -1, 0)$ can be easily found from the input (3) using the TRIGON function of POLYMAKE. The two simplices are defined by the cones

$$\Delta_1 = \{F \in \mathbb{R}^4 \mid F \cdot Z_1 > 0, F \cdot Z_2 > 0\},$$

$$\Delta_2 = \{F \in \mathbb{R}^4 \mid F \cdot Z_3 > 0, F \cdot Z_4 > 0\}.$$
Figure 7. An illustration of the bipyramid as a cone in \( \mathbb{R}^3 \), bound by planes \( Z_i \). The vertices \( V_i \) correspond to rays.

\[
\Delta_1 = \{ F \in \mathbb{R}_+^4 \mid F \cdot Z_1 > 0, F \cdot Z_3 > 0, F \cdot Z_5 > 0, F \cdot Z^s > 0 \},
\]

\[
\Delta_2 = \{ F \in \mathbb{R}_+^4 \mid F \cdot Z_2 > 0, F \cdot Z_4 > 0, F \cdot Z_6 > 0, F \cdot Z^s < 0 \},
\]

after their projection to three dimensions via \( F \cdot T = 1 \) (see figure 7). Now,

\[
\int_{\Delta} d^4 F e^{-F(\alpha)} = \int_{\Delta} d^4 F e^{-F T} + \int_{\Delta} d^4 F e^{-F T},
\]

and it suffices to show that each integral computes the volume of the corresponding simplex. For this, we introduce the set of vertices for each simplex. For \( \Delta_1 \) we have the vertices (rays) \( \{V_1, V_2, V_3, V_4\} \) defined by

\[
V_1 \cdot Z_1 = V_1 \cdot Z_2 = V_1 \cdot Z_3 = 0, \quad (5.5)
\]

\[
V_2 \cdot Z_2 = V_2 \cdot Z_3 = V_2 \cdot Z^s = 0, \quad (5.6)
\]

\[
V_3 \cdot Z_3 = V_3 \cdot Z_2 = V_3 \cdot Z^s = 0, \quad (5.7)
\]

\[
V_4 \cdot Z_4 = V_4 \cdot Z_3 = V_4 \cdot Z^s = 0. \quad (5.8)
\]

We then have

\[
F \in \Delta_1 \iff F = \sum_{i=1}^{4} \beta V_i, \quad \beta^i > 0,
\]

and simple algebra leads to

\[
\int_{\Delta_1} d^4 F e^{-F T} = \frac{1}{(V_1 \cdot T)(V_2 \cdot T)(V_3 \cdot T)(V_4 \cdot T)} \left| (V_1, V_2, V_3, V_4) \right|. \quad (5.9)
\]

where \( (V_1, V_2, V_3, V_4) = e^{ijkl}V_{ij}V_{3k}V_{4l} \). The absolute value is conventional as it corresponds to a choice of orientation. This is precisely the volume spanned by the vectors \( \vec{V}_i = \frac{V_i}{V_i \cdot T} \) that also belong to the plane \( F \cdot T = 1 \), i.e. they satisfy \( \vec{V}_i \cdot T = 1 \). This thus proves that the amplitude is given by the volume of the cone \( \Delta \) projected into \( F \cdot T = 1 \).

As will be further explored in [20], we note that the vertices \( V_i \) are nothing but the poles emerging in the associated amplitude. This realizes the geometrical intuition provided in [9] that the bipyramid is formed by the poles \( \{ R, \tilde{R}, t_{1234}, t_{3456}, t_{5612} \} \), where \( R, \tilde{R} \) correspond to the apices. Indeed, recalling the definitions

\[
\begin{align*}
S_{abcd} &= s_{abc} + s_{ab} + s_{cd} + s_{cd}, \\
R_{ab,cd,ef} &= t_{abcd} + s_{cde} + s_{def},
\end{align*}
\]

with \( R = t_{1234}, \tilde{R} = t_{3456}, t_{5612} \), we find

\[
\frac{\left| (V_1, V_2, V_3, V_4) \right|}{(V_1 \cdot T)(V_2 \cdot T)(V_3 \cdot T)(V_4 \cdot T)} = \frac{1}{t_{1234}t_{3456}t_{5612}} \tilde{R},
\]

which corresponds to vertices of the upper half of the bipyramid in figure 6. Finally, one can check that the vertices of \( \Delta_2 \) are given by \( \{ V_2, V_4, V_5, V_6 \} \) where the new ray is defined by \( V_5 \cdot Z_4 = V_5 \cdot Z_5 = V_5 \cdot Z_6 = 0 \). This means that

\[
\int_{\Delta_2} d^4 F e^{-F T} = \frac{1}{(V_1 \cdot T)(V_2 \cdot T)(V_3 \cdot T)(V_4 \cdot T)} \left| (V_1, V_2, V_3, V_4) \right|.
\]

in agreement with, for example, [9]. We put the details on how to realize these calculations in POLYMAKE in appendix C.

For the cases (3, 8) and (4, 8), this method turns out to be necessary. We have used POLYMAKE to triangulate the cone \( \Delta \) for every one of their planar collections or matrices and stored the results in the supplementary data. The remaining procedure to produce the full numeric rational integrated amplitudes \( m_{ij}^{abc} \) for any given kinematics is implemented in the MATHEMATICA notebook \( 38\_48\_amplitudes.nb \). The (4, 8) amplitude \( m_{ij}^{abc} \) obtained this way was later found to coincide with that of [22] using Arkani–Hamed–Bai–He–Yan construction [23] in the context of Grassmannian stringy integrals [24], which is a strong consistency check for both sides.

We continued to apply this method for the case (4, 9), which has more than \( 3 \times 10^7 \) planar matrices. The number of simplices obtained by the TRIANGULATION function of POLYMAKE is even much bigger: 4 797 131 092. The files containing information on vertices; facets as well as replacement rules are too large to attach, so we just include four numeric results of the full amplitudes \( m_{ij}^{abc} \) for four sets of given kinematics data and five analytic results of off-diagonal amplitudes \( m_{ij}^{abc} \) in an auxiliary MATHEMATICA notebook \( 49\_amplitudes.nb \).

We close this section by emphasizing that the vertices \( V_i \) are dual to the planes \( Z_i \), which establishes a connection with the dual polytope. Explicitly, from the incidence relations (5) we can use \( V_i = e^{ijkl}Z_{ij}Z_{ik}Z_{jk} \), and analogously for all vertices to rewrite expression (9) as

This is the canonical form of the dual simplex [23], spanned by the rays \( \{Z_1, Z_2, Z_3, Z_4\} \). This perspective has been explored in detail in [25].
6. Higher \( k \) or planar arrays of Feynman diagrams and duality

Planar collections can be thought of as one-dimensional arrays with planar matrices as two-dimensional arrays of Feynman diagrams satisfying certain conditions. It is natural to propose that the computation of generalized biadjoint amplitudes for any \((k, n)\) can be done using \(k - 2\) dimensional arrays of Feynman diagrams.

**Definition 6.1.** A planar array of Feynman diagrams is a \((k - 2)\)-dimensional array \(\mathcal{A}\) with dimensions of size \(n\). The array has as component \(\mathcal{A}_{i_1 i_2 \ldots i_{k-2}}\), a metric tree with leaves in the set \(\{1, 2, \ldots, n\} \setminus \{i_1, i_2, \ldots, i_{k-2}\}\) and which is planar with respect to the ordering \((1, 2, \ldots, i_1, \ldots, i_{k-2}, i_{k-1}, \ldots, n)\) satisfying the following conditions:

- diagonal entries are the empty tree \(\mathcal{A}_{i_1 i_2 \ldots i_n} = \emptyset\)
- compatibility: \(d_{i_1 i_2 \ldots i_k}^{(i_1 \ldots i_k)}\) is completely symmetric in all \(k\) indices.

A point that has not been explained so far is why each element in a collection, matrix or in general an array is called a Feynman diagram. We now turn to this point. The contribution to an amplitude of a given planar array of Feynman diagrams is computed using the function

\[
\mathcal{F}(\mathcal{A}) = \sum_{i_1 i_2 \ldots i_k} s_{i_1 i_2 \ldots i_k} d_{i_1 i_2 \ldots i_k}^{(i_1 \ldots i_k)}.
\]  

(6.1)

For \(k = 2\) it is easy to show that this function is independent of the lengths of the external edges by writing \(d_{ij} = e_i + e_j + d_{ij}\) internal and using momentum conservation. For \(k = 3\) it was noted in [14] that the function \(\mathcal{F}(\mathcal{A})\) can also be written in such a way that it is also independent of the external edges. However, the proof is not as straightforward. In order to easily see this property all we have to do is to treat each tree in the array as a true Feynman diagram with its own kinematics.

The element in the array \(\mathcal{A}_{i_1 i_2 \ldots i_{k-2}}\) is an \((n - k + 2)\)-particle Feynman diagram with particle labels \(\{1, 2, \ldots, n\} \setminus \{i_1, i_2, \ldots, i_{k-2}\}\). As such, one has to associate the proper kinematic invariants satisfying momentum conservation. Let us introduce the notation \(\mathcal{I} = \{i_1, i_2 \ldots i_{k-2}\}\) and \(\overline{\mathcal{I}}\) for its complement. Then we have

\[
s^{(\mathcal{I})}_{i} = 0, \quad \sum_{j \in \overline{\mathcal{I}}} s^{(\mathcal{I})}_{j} = 0 \quad \forall i \in \mathcal{I}.
\]  

(6.2)

Using these kinematic invariants one can parametrize the \((k, n)\) invariants as

\[
s_{i_1 i_2 \ldots i_k} := \mathcal{F}(\mathcal{A}_{i_1 i_2 \ldots i_k}),
\]  

where the sum is over all possible ways of decomposing \(\{i_1, i_2, \ldots, i_k\}\) into two sets of \(k - 2\) and \(2\) elements respectively. To illustrate the notation consider \(k = 3\) where

\[
s_{ijk} = s^{(i)}_{ik} + s^{(j)}_{ij} + s^{(k)}_{jk}.
\]  

(6.3)

This parametrization is very redundant but, as any good redundancy, it makes at least one property of the relevant object manifest. In this case, it is the independence of the external edges of \(\mathcal{F}(\mathcal{A})\). Let us continue with the \(k = 3\) case in order not to clutter the notations but the general \(k\) version is clear.

Using (4) one can write

\[
\mathcal{F}(\mathcal{C}) = \sum_{i,j,k} s_{ijk} d^{(i)}_{jk},
\]

as

\[
\mathcal{F}(\mathcal{C}) = \frac{1}{3} \sum_{i,j,k} s^{(i)}_{jk} d^{(i)}_{jk}.
\]  

(6.6)

Here we used the symmetry property of \(d^{(i)}_{jk}\) to identify all three terms coming from using (4). The new form is nothing but a sum over the functions \(F(T)\) for each of the trees in the collection and therefore it is clearly independent of the external edges as expected.

Let us now discuss how the two kinds of combinatorial bootstraps work for general planar arrays of Feynman diagrams.

The first kind of combinatorial bootstrap, which we called pruning–mutating in section 2, is simply the process of producing \(C_{n-2}\) initial arrays of Feynman diagrams by starting with any given \(n\)-point planar Feynman diagram and pruning \(k - 2\) of its leaves in all possible ways to end up with an array of \((n - k + 2)\)-point Feynman diagrams. Starting from these initial planar arrays, one computes the corresponding metrics and finds all their possible degenerations. Approaching each degeneration one at a time one can produce a new planar array by resolving the degeneration only in the other possible planar way. Repeating the mutation procedure on all new arrays generated until no new array is found leads to the full set of planar arrays of Feynman diagrams.

The second kind of combinatorial bootstrap, as described in section 3 for planar matrices, is the idea that the compatibility conditions on the metrics of the trees making the array force it to be completely symmetric. This simple observation together with the fact that any subarray where some indices are fixed must in itself be a valid planar array of Feynman diagrams for some smaller values of \(k\) and \(n\) gives strong constraints on the objects.
As should be clear from the examples presented in section 3, the second bootstrap approach is more efficient than the first one if all planar arrays in \((k-1, n-1)\) are known. This means that one could start with \((3, 6)\) and produce the following sequence:

\[
(3, 6) \rightarrow (4, 7) \rightarrow (5, 8) \rightarrow (6, 9) \rightarrow (7, 10) \ldots
\]  

(6.7)

The reason to consider this sequence is that after obtaining all its elements, one can construct all \((3, n)\) planar collections via duality. Of course, in order to do that efficiently one has to find a combinatorial way of performing the duality directly at the level of the graphs.

6.1. Combinatorial duality

Let us start by defining some notation that will be used in this section. We will denote \(T_n\) as a planar tree in \((2, n)\), \(C_n\) as a planar collection in \((3, n)\) and \(M_n\) as a planar matrix in \((4, n)\). In general, a planar array \(A_n\) will correspond to a \((k-2)\)-dimensional array with dimensions of size \(n\). In order to understand how the combinatorial duality works, we also introduce the concept of combinatorial soft limit. The combinatorial soft limit for particle \(i\) applied to \(A_n\) is defined by removing the \(i\)th \((k-3)\)-dimensional array from \(A_n\), as well as removing the \(i\)th label to the remaining \((k-3)\)-dimensional arrays. Therefore, the combinatorial soft limit takes us from \((k, n)\) to \((k, n - 1)\).

It is useful to introduce a superscript \(A_n^{(i)}\) to refer to an array obtained from a combinatorial soft limit for particle \(i\). Notice that this notation slightly differs from the one used in earlier sections.

With this in hand, we can define the particular duality \((2, n) \sim (n - 2, n)\) as taking the tree \(T_n\) of \((2, n)\) and applying the combinatorial soft limit to particles \(i_1, \ldots, i_{n-2}\) in order to remove \(n - 2\) leaves to obtain the corresponding dual \(A_n^{(i_1, \ldots, i_{n-2})}\) of \((n - 2, n)\).

For general \((k, n)\) with \(k < n - 2\) the duality works as follows. Consider a \((k-2)\)-dimensional planar array \(A_n\) of \((k, n)\). By taking the combinatorial soft limit for particle \(i\), we end up with \(A_n^{(i)}\) of \((k, n - 1)\). Apply this step \(n\) times for all \(n\) particles. Now dualize each of the \(n\) objects to directly obtain the corresponding \((n - k - 2)\)-dimensional array \(A_n\) of \((n - k, n)\), hence the duality. The combinatorial duality can be simply summarized as

\[
\begin{align*}
(k, n) & \xrightarrow{\text{soft limit}} n \\
\times (k, n - 1) & \xrightarrow{\text{dualize}} (n - k, n).
\end{align*}
\]  

(6.8)

6.1.1. Illustrative example: \((3, 7) \sim (4, 7)\). Now we proceed to show the explicit example for \((3, 7) \sim (4, 7)\). The combinatorial soft limit for particle \(i\) applied to a planar collection \(C_n\) corresponds to removing the \(i\)th tree in \(C_n\) as well as removing the \(i\)th label in all the rest of the trees in \(C_n\). Therefore, it implies \(C_n \rightarrow C_n^{(i)}\). Similarly, the combinatorial soft limit for particle \(i\) applied to a planar matrix \(M_n\) corresponds to removing the \(i\)th column and row in \(M_n\) as well as removing the \(i\)th label in all the rest of the trees in \(M_n\). Therefore, it implies \(M_n \rightarrow M_n^{(i)}\).

Before studying \((3, 7) \sim (4, 7)\) let us consider \((3, 6) \sim (3, 6)\) as this will be useful below. Using (8) we can see

\[
(3, 6) \xrightarrow{\text{soft limit}} 6 \times (3, 5) \xrightarrow{\text{dualize}} (3, 6),
\]  

(6.9)

where the duality \((2, 5) \sim (3, 5)\) is one of the most basic ones which was used as a motivation for introducing planar collections in [14].

Now consider one planar collection \(C_{n=7}\) of \((3, 7)\). By taking the combinatorial soft limit for particle \(i\), we end up with a collection \(C_{n=6}^{(i)}\) in \((3, 6)\). Given that \((3, 6) \sim (3, 6)\), this collection is dual to another collection \(C_{n=6}^{(i)}\), which corresponds to the \(i\)th column of a planar matrix \(M_{n=7}\) in \((4, 7)\). This means that if we now take the combinatorial soft limit for the other particles in \(C_{n=7}\) we end up with the full matrix \(M_{n=7}\). Hence, the objects \(C_{n=7}\) and \(M_{n=7}\) are dual.

We can also see this by following an equivalent path. Consider one planar matrix \(M_{n=7}\) of \((4, 7)\). By taking the combinatorial soft limit for particle \(i\), we end up with a planar matrix \(M_{n=6}^{(i)}\) of \((4, 6)\). Notice that this matrix is dual to the planar tree \(T_{n=6,0}\) of \((2, 6)\) which is an element of \(C_{n=7}\), so by repeating the soft limit for all the remaining particles we end up with the full \(C_{n=7}\) of \((3, 7)\).

7. Future directions

Generalized biadjoint amplitudes as defined by a CHY integral over the configuration space of \(n\) points in \(\mathbb{CP}^{k-2}\) with \(k > 2\) provide a very natural step beyond standard quantum field theory [6]. An equally natural generalization of quantum field theory amplitudes is obtained by first identifying standard Feynman diagrams with metric trees and their connection to Trop \(G(2, n)\). In [13], arrangements of metric trees were introduced as objects corresponding to Trop \(G(3, n)\). A special class of such arrangement, called planar collections of Feynman diagrams, was then proposed as the simplest generalization of Feynman diagrams in [14]. In this work, we introduced \((k-2)\)-dimensional planar arrays of Feynman diagrams as the all \(k\) generalization. One of the most exciting phenomena is that these \((k-2)\)-dimensional arrays define generalized biadjoint amplitudes.

The fact that both definitions of generalized amplitudes, either as a CHY integral or as a sum over arrays, coincide is non-trivial. In fact, a rigorous proof of this connection, perhaps along the lines of the proof for \(k = 2\) given by Dolan and Goddard [5, 26], is a pressing problem. One possible direction is hinted by the observations made in section 6, where each Feynman diagram in an array was given its own kinematics along with its own metric. Of course, what makes the planar array interesting is the compatibility conditions for the metrics of the various trees in the array. Understanding the physical meaning of such conditions is also an important problem. However, this already gives a hint as to what to do...
with the CHY integral. Borrowing the \( k = 3 \) example in section 6, the kinematics is parametrized as \( s_{ijk} = s_{i}^{(k)} + s_{j}^{(l)} + s_{l}^{(i)} \). Recall that in the CHY formulation on \( \mathbb{CP}^2 \) introduced in [6] one starts with a potential function
\[
S_n^{(3)} = \sum_{i,j,k} s_{ijk} \log |ijk|, \tag{7.1}
\]
with \(|ijk|\) Plücker coordinates in \( G(3, n) \). Even though the object is antisymmetric in all its indices, only its absolute value is relevant in \( S_n^{(3)} \) since the way it enters in the CHY formula is only via the equations needed for the computation of its critical points. This means that \(|ijk|\) can be used to define ‘effective’ \( k = 2 \) Plücker coordinates of the form \(|jk|^{(l)} := |ijk|\). In other words, once a label is selected, say \( i \), then all other points in \( \mathbb{CP}^2 \) can be projected onto a \( \mathbb{CP}^1 \) using the \( i \)th point. This means that the potential \( S_n^{(3)} \) can be written as a sum over \( n \), \( k = 2 \) potentials in a way completely analogous to \( F(C) \) in (6), i.e.
\[
S_n^{(3)} = \frac{1}{3} \sum_{i,j,k} s_{ijk}^{(i)} |jk|^{(l)}. \tag{7.2}
\]
One can then write a \( k = 3 \) CHY formula as a product over \( n \), \( k = 2 \) CHY integrals linked by the ‘compatibility constraints’ imposing that the absolute value of \(|jk|^{(l)}|, |ij|^{(l)}, |ik|^{(l)}\) all be equal. Owing to the techniques developed in [11, 27–30], many non-trivial \( k = 3 \) CHY formulas have been verified to match the partial amplitudes obtained by using planar collections or matrices of Feynman diagrams, but general analysis such as we present here is still needed to understand the most general cases.

As explored in [20], by forcing a given planar array of Feynman diagrams to explore its degenerations of highest codimension one finds planar arrays of degenerate Feynman diagrams which encode the information of the poles of the contributions of this planar array to the amplitudes. Initial investigations into the residues of amplitudes obtained by using planar collections or matrices of Feynman diagrams, but general analysis such as we present here is still needed to understand the most general cases.

Note Added:

While the first version of this manuscript was being prepared for submission, the works [23, 47–49] appeared which have some overlap with our results, especially in (4, 8).

While the third version of this paper was being prepared, some new results on local planarity appeared [50, 51]. In this paper, we have studied an array of Feynman diagrams consistent with a global notion of planarity, which is closely related to the positive part of the tropical Grassmannian, Trop \( G'(k, n) \). The notion of generalized Feynman diagrams was first introduced in [6] and refined in [50, 51] to formalize the notion of local planarity or generalized color ordering. Generalized Feynman diagrams are expected to relate to the whole the tropical Grassmannian, Trop \( G(k, n) \), and it would be interesting to see how many properties of planar arrays of Feynman diagrams still hold there.

Acknowledgments

We would like to thank Nick Early and Song He for their useful discussions. Research at Perimeter Institute is supported in part by the Government of Canada through the Department of Innovation, Science and Economic Development Canada and by the Province of Ontario through the Ministry of Economic Development, Job Creation and Trade.

Appendix A. Proof of one-to-one map of a binary tree and its metric

Lemma A.1. Given that two cubic trees \( T_A \) and \( T_B \) have the same valid non-degenerate metric \( d_{ip} \) then \( T_A = T_B \).

Proof. We are going to provide a proof by induction. First, consider the base case where \( T_A \) and \( T_B \) are three-point trees. It is clear that there exists a unique solution to \( d_{12} = e_1 + e_2 \), \( d_{13} = e_1 + e_3 \) and \( d_{23} = e_2 + e_3 \). Since \( T_A \) and \( T_B \) have the same non-degenerate metric, the lengths \( e_i^{(A)} = e_i^{(B)} \) must be identical, thus \( T_A = T_B \).

Now let us assume that the lemma is true for all \((n - 1)\)-point cubic metric trees and consider two \( n \)-point cubic trees \( T_A \) and \( T_B \) that have the same non-degenerate metric \( d_{ip} \). Next let us find leaves \( i \) and \( j \) such that \( d_{il} - d_{lj} \) is \( l \) independent. Such a pair must exist because the condition is true for any pair of leaves that belong to the same ‘cherry’ as shown in the diagrams in figure 8. Moreover, only leaves in cherries satisfy this condition in a cubic non-degenerate tree.

Removing the cherries from both trees and introducing a new leaf \( \alpha \) one can define a metric for the \((n - 1)\)-point cubic trees in figure 9, whose leaves are given by \( \{(1, 2, \ldots, n) \setminus \{i, j\}\} \cup \{\alpha\} \).

Such a metric is defined in terms of the metric of the parent trees as follows: \( d_{ik}^{(A)} = d_{ik} \) if \( k, l \neq \alpha \) and \( d_{ik}^{(A)} = d_{ik} - e_i^{(A)} \). Likewise \( d_{ik}^{(B)} = d_{ik} \) if \( k, l \neq \alpha \) and \( d_{ik}^{(B)} = d_{ik} - e_i^{(B)} \). It is easy to see from the figure that the two metrics are identical, i.e. \( d_{ik}^{(A)} = d_{ik}^{(B)} \).

Using the induction hypothesis, the two metric trees in figure 9 must be the same. In order to complete the proof all we need is to show that \( e_i^{(A)} = e_i^{(B)} \). The fact that \( d_{il} = e_i^{(A)} + d_{il}^{(A)} = e_i^{(B)} + d_{il}^{(B)} \) immediately implies \( e_i^{(A)} = e_i^{(B)} \), hence \( T_A = T_B \). □
Appendix B. All planar collections of Feynman diagrams for (3, 6)

Here we reproduce for the reader’s convenience table 1 of [14] which contains all 48 planar collections of Feynman diagrams for (3, 6). The notation in this case is very compact and requires some explanation. Each collection for (3, 6) is made out of five-point trees. The tree in the $i$th position must be planar with respect to the ordering (1, 2, ..., $n$). There is a single topology of five-point trees, i.e. a caterpillar tree with two cherries and one leg. Therefore it is possible to specify it by giving the label of the leaf attached to the leg. Using this, each collection becomes a one-dimensional array of six numbers.

Appendix C. Triangulation functions in POLYMAKE

In this appendix we demonstrate how to use the TRIANGULATION function of POLYMAKE needed in section 5. In practice, the computation of the vertices $\{V_1, V_2, V_3, V_4, V_5\}$ as well as the triangulation that assigns them to two simplices $\Delta_1$ and $\Delta_2$ is automated by the software POLYMAKE. The only input needed is the set of faces (3). We now provide the corresponding script to automate the process, given by the following command lines:

```perl
open(INPUT, "<", "boundaries.txt");
$smttr=new Array((INPUT));$t1=time;
for(my$i=0;$i<scalar(@smttr);$i++)
{print "$i/n":@s1=split("\n",@s1->>[$i]);
 @arm=();
 for(my$j=0;$j<scalar(@s1);$j++)
 {@dst=split("\n",@s1[$j]);

(Continued.)
```

In order to implement the POLYMAKE script we need to create a file boundaries.txt, in which each row corresponds to the planes $Z_n$ of a collection. In a row, different vectors are separated by the character $\times$, and vector components are separated by commas. For instance, for the bipyramid case (3), the row reads

```
1,0,0,0,0,0,1,0,0,0,0,0,0,1,0,0,1,0,0,1,0,1,0,1,0,1,0,1,0,1
```

An arbitrary number of collections (for instance to obtain the full amplitude) can be processed simply by adding rows into the .txt file. The script will display a counter to indicate the row being processed, as well as the total time of the computation (in seconds) once it is completed.

The output of the script is two text files facets.txt and vertices.txt. For the previous example vertices.txt contains

```
0 0 0 1
0 0 1
1 0 0
1 1 0
0 1 0
0 0 1
```

This is just a list of vertices $V_i$ (note the unusual indexing starting from $i = 0$):

```
V_0 = (0, 0, 1, 1), V_1 = (1, 0, 0, 0), V_2 = (1, 1, 1, 0)
```

(C1)
meaning that the full object can be triangulated by two simplices, with vertices \( \{V_0, V_1, V_2, V_3 \} \) and \( \{V_0, V_1, V_3, V_4 \} \) respectively (these are relabelling of the previous simplices \( \Delta_1 \) and \( \Delta_2 \)). The corresponding contribution to the amplitude is then given by the volume formula (9).

**References**

[1] Fairlie D and Roberts D 1972 Durham University Preprint (PRINT-72-2440) Dual models without tachyons—a new approach https://inspirehep.net/literature/1116351

[2] Fairlie D B and Cauden A 2009 A coding of real null four-momenta into world-sheet coordinates Adv. Math. Phys. 2009 284689

[3] Cachazo F, He S and Yuan E Y 2014 Scattering equations and Kawai–Lewellen–tye orthogonality Phys. Rev. D 90 065001

[4] Cachazo F, He S and Yuan E Y 2014 Scattering of massless particles in arbitrary dimensions Phys. Rev. Lett. 113 171601

[5] Dolan L and Goddard P 2014 Proof of the formula of Cachazo, He and Yuan for Yang–Mills tree amplitudes in arbitrary dimension J. High Energy Phys. JHEP05(2014)010

[6] Cachazo F, Early N, Guevara A and Mizera S 2019 Scattering equations: from projective spaces to tropical Grassmannians J. High Energy Phys. JHEP06(2019)039

[7] Cachazo F and Sturmfelds B 2004 The tropical Grassmannian Adv. Geometry 4 389–411

[8] Cachazo F and Williams L 2005 The tropical totally positive Grassmannian J. Algebr. Comb. 22 189–210

[9] Cachazo F and Rojas J M 2020 Notes on biadjoint amplitudes, Trop Δ(3, 7) and \( \chi(3, 7) \) scattering equations J. High Energy Phys. JHEP04(2020)0176

[10] García Sepúlveda D and Guevara A 2019 A soft theorem for the tropical Grassmannian arXiv:1909.05291

[11] Cachazo F, Umbert B and Zhang Y 2020 Singular solutions in soft limits J. High Energy Phys. JHEP05(2020)048

[12] Abhishek M, Hegde S, Jatkar D P and Saha A P 2021 Double soft theorem for generalised biadjoint scalar amplitudes SciPost Phys. 10 036

[13] Herrmann S, Jensen A, Joswig M and Sturmfelds B 2009 How to draw tropical planes Electronic J. Combinatorics 16 6

[14] Borges F and Cachazo F 2020 Generalized planar feynman diagrams: collections J. High Energy Phys. JHEP11 (2020)164

[15] Cachazo F and Williams L K 2003 The tropical totally positive Grassmannian arXiv:math/012297v1

[16] Drummond J, Foster J, Gürdogan O and Kalousios C 2020 Tropical fans, scattering equations and amplitudes J. High Energy Phys. JHEP11(2020)071

[17] Gates S J, Hazel Mak S N, Spradlin M and Volovich A 2021 Cluster superalgebras and stringy integrals arXiv:2111.08186

[18] Cachazo F and Rojas J M 2020 Planar kinematic invariants, matroid subdivisions Adv. Math. Phys. JHEP11(2020)153

[19] He S, Li Z and Yang Q 2021 Truncated cluster algebras and Feynman integrals with algebraic letters J. High Energy Phys. JHEP12(2021)0110

[20] Ren L, Spradlin M and Volovich A 2021 Symbol alphabets from tensor diagrams J. High Energy Phys. JHEP12 (2021)079

[21] Cachazo F and Early N 2021 Minimal kinematics: an all k and n peek into Trop glued associahedra from residues of generalized amplitudes J. High Energy Phys. JHEP10(2023)015

[22] Cachazo F and Early N 2022 Planar kinematics: cyclic fixed points, mirror superpotential, k-dimensional catalan numbers, and root polytopes arXiv:2010.09708

[23] Cachazo F and Early N 2023 Biadjoint scalars and associahedra from residues of generalized amplitudes J. High Energy Phys. JHEP08(2022)252

[24] Early N 2023 Factorization for generalized biadjoint scalar amplitudes via matroid subdivisions arXiv:2211.16623

[25] Arkani-Hamed N, He S and Lam T 2021 2D scattering amplitudes of sparse finite type SIGMA 17 092

[26] Cachazo F, Early N and Giménez Ubert B 2022 Smoothly splitting amplitudes and semi-locality J. High Energy Phys. JHEP08(2022)252

[27] Cachazo F, Guevara A, Kawai K, Kido N and Tanihata T 2018 Causal diamonds, cluster polytopes and scattering amplitudes J. High Energy Phys. JHEP11(2018)153

[28] Arkani-Hamed N, He S, Li Z and Yang Q 2022 Truncated cluster algebras and Feynman integrals with algebraic letters J. High Energy Phys. JHEP12(2021)0110

[29] Ren L, Spradlin M and Volovich A 2021 Symbol alphabets from tensor diagrams J. High Energy Phys. JHEP12 (2021)079

[30] Lekkerkerker C, Sturmfelds B and Volovich A 2020 Stringy canonical forms and binary geometries from associahedra, cyclohedra and generalised permutohedra J. High Energy Phys. JHEP10 (2020)054

[31] Arkani-Hamed N and Papatheoasias G 2020 How tropical are seven- and eight-particle amplitudes? J. High Energy Phys. JHEP10 (2020)054
[48] Arkani-Hamed N, Lam T and Spradlin M 2021 Non-perturbative geometries for planar $\mathcal{N} = 4$ SYM amplitudes J. High Energy Phys. JHEP03(2021)065

[49] Drummond J, Foster J, Gürdogan O and Kalousios C 2021 Algebraic singularities of scattering amplitudes from tropical geometry J. High Energy Phys. JHEP04(2021)002

[50] Cachazo F, Early N and Zhang Y 2023 Color-dressed generalized biadjoint scalar amplitudes: local planarity arXiv:2212.11243

[51] Cachazo F, Early N and Zhang Y 2023 Generalized color orderings: CEGM integrands and decoupling identities arXiv:2304.07351