Small Ball and Discrepancy Inequalities

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Dedicated to the Memory of Walter Philipp, Teacher and Steadfast Friend
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Preface

We discuss an inequality for three dimensional Haar functions motivated by questions in a range of areas. These are

• Irregularity of Distributions of points in the unit cube, relative to boxes in the standard coordinate basis.
• Chung’s Law for the Brownian Sheet, or equivalently, sharp estimates for the probability that the Brownian Sheet has a small sup norm in the unit cube.
• Lower bounds on the number of $L_\infty$ balls of small radius needed to cover certain compact classes of functions with bounded mixed derivative in three dimensions.

Of these three questions, the first admits the easiest description, and has the longest history, beginning with van Aardenne-Ehrenfest [2, 3], with significant contributions by a variety of authors over many years. See the first chapter of Beck and Chen [5]. Our methods are influenced by many of these contributions; the reader will find references to them in the pages below. Indeed, these notes are our effort to understand the famous contribution of József Beck [4] to the irregularities of distribution in three dimensions, and its connection with other questions in analysis. Along the way, we will simplify and extend his argument, in a manner that raises hopes that one could resolve the issue in three dimensions.

The latter two problems listed above have a more sophisticated description, indeed one that admits an abstract formulation. The relationship between them is rather precise, and well known, [22, 23].

These topics are unified by their methods of proofs. In its simplest manifestation, this is a particular inequality about Haar functions in three dimensions, a question which can be viewed as just beyond the reach of Littlewood Paley theory. We take this question as our main focus, as doing so will permit us to develop the necessary analytical tools with some efficiency. We establish a partial result in the direction of the main conjecture in the subject, Theorem 1.1.7. Afterwards, we discuss the other subjects above.

In the subject of Irregularities of Distribution, the principal new result is an extension of the result of Beck already cited, namely Theorem 2.1.9. The entire subject is also of interest in two dimensions; we include this theory in our notes, as it is the foundation from which one must generalize. The two dimensional case is substantially easier, and all important elements of that theory have been developed see [20, 28, 34, 35] among other references listed in the paper below.

The central methods of this paper are those of Harmonic Analysis: Riesz products; Littlewood Paley inequalities; conditional expectation arguments; and product theory.
These notes are written with a focus on these issues. (This is the area of expertise of the author.) We have written a separate chapter recalling some of these basic issues in a separate chapter, see Chapter 3. As our subject touches a range of issues, we have also included background material on Irregularities of Distributions, Approximation Theory, and Probability Theory. These are offered for the convenience of the reader, with the caveat that the author is not an expert in these subjects.

Notation. The language and notation of probability and expectation is used throughout. Thus,

$$\mathbb{E}f = \int_{[0,1]^d} f(x) \, dx$$

and \(\mathbb{P}(A) = \mathbb{E}1_A\). This serves to keep formulas simpler. As well, certain conditional expectation arguments are essential to us. We use the notation

$$\mathbb{P}(B \mid A) = \mathbb{P}(A)^{-1}\mathbb{P}(A \cap B).$$

For a sigma field \(\mathcal{F}\),

$$\mathbb{E}(f \mid \mathcal{F})$$

is the conditional expectation of \(f\) given \(\mathcal{F}\). In all instances, \(\mathcal{F}\) will be generated by a finite collection of atoms \(\mathcal{F}_{\text{atoms}}\), in which case

$$\mathbb{E}(f \mid \mathcal{F}) = \sum_{A \in \mathcal{F}_{\text{atoms}}} \mathbb{P}(A)^{-1}\mathbb{E}(f 1_A) \cdot 1_A.$$

We suppress many constants which do not affect the arguments in essential ways.

\(A \lesssim B\) means that there is an absolute constant so that \(A \leq KB\). Thus \(A \lesssim 1\) means that \(A\) is bounded by an absolute constant. And \(A \simeq B\) means \(A \lesssim B \lesssim A\).

Acknowledgment. Walter Philipp, my thesis advisor who passed away unexpectedly in the summer of 2006, introduced me to this topic while I was in graduate school. Vladimir Temlyakov lead me through the theory that had been developed since graduate school days. I report on joint work with Dmitry Bilyk. We have benefited from several conversations with Mihalis Kolountzakis and Vladimir Temlyakov on this subject. A substantial part of this manuscript was written while in residence at the University of Crete.
CHAPTER 1

The Small Ball Problem

1.1. The Principal Conjecture

In one dimension, the class of dyadic intervals are
$$D := \{ [j2^k, (j+1)2^k) : j, k \in \mathbb{Z} \}.$$ Each dyadic interval has a left and right half, indicated below, which are also dyadic. Define
$$h_l := -1_{\text{left}} + 1_{\text{right}}$$
Note that this is an $L^\infty$ normalization of these functions, which we will keep throughout these notes. This will cause some formulas to look a little odd to readers accustomed to an $L^2$ normalization for Haar functions.

In dimension $d$, a dyadic rectangle is a product of dyadic intervals, thus an element of $D^d$. A Haar function associated to $R$ we take to be the product of the Haar functions associated with each side of $R$, namely
$$h_{R_1 \times \cdots \times R_d}(x_1, \ldots, x_d) := \prod_{j=1}^d h_{R_j}(x_j).$$
This is the usual ‘tensor’ definition.¹

We will concentrate on rectangle with a fixed volume, and consider a local problem. This is the ‘hyperbolic’ assumption, that pervades the subject. Our concern is the following Theorem and Conjecture concerning a lower bound on the $L^\infty$ norm of sums of hyperbolic Haar functions:

1.1.1. Talagrand’s Theorem. For dimensions $d \geq 2$, we have
\begin{equation}
2^{-n} \sum_{|R|=2^{-n}} |\alpha(R)| \lesssim n^{\frac{1}{2}(d-2)} \left\| \sum_{|R| \geq 2^{-n}} \alpha(R)h_R \right\|_{L^\infty}
\end{equation}
Here, the sum on the right is taken over all rectangles with area at least $2^{-n}$.

1.1.3. Small Ball Conjecture. For dimension $d \geq 3$ we have the inequality
\begin{equation}
2^{-n} \sum_{|R|=2^{-n}} |\alpha(R)| \lesssim n^{\frac{1}{2}(d-2)} \left\| \sum_{|R| \geq 2^{-n}} \alpha(R)h_R \right\|_{L^\infty}
\end{equation}

¹Note that we are not claiming that these functions form a basis.

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This conjecture is, by one square root, better than the trivial estimate available from Cauchy Schwartz, see § 1.2. As well, see that section for an explanation as to why the conjecture is sharp. The motivations for the conjecture are indirect, a subject we return to in the discussion of functions with $L^2$ mixed partials below, § 4.1. Nevertheless, we have begun with this conjecture as it provides the quickest path to the essential technical aspects behind the various conjectures of these notes.

The result in the case of $d = 2$ is that of Talagrand [34]. We will give the easier proof of Temlyakov [35], which proof resonates with the ideas of Roth [25], Schmidt [28], and Halász [20]. Compare § 1.3 and § 2.4.

For many applications of interest, one can restrict attention to this version of the conjecture

1.1.5. Restricted Small Ball Conjecture. We have the inequality (1.1.2), or (1.1.4), in the case where the coefficients $\alpha(R) \in \{0, \pm 1\}$, for $|R| = 2^{-n}$ and about half of the $\alpha(R) \neq 0$. Namely, under these assumptions on the coefficients $\alpha(R)$ we have the inequality

\[
\left\| \sum_{|R| = 2^{-n}} \alpha(R) h_R \right\|_{\infty} \gtrsim n^{d/2}.
\]

It is possible that the proof would simplify considerably—and be of interest—if one in addition assumed that $|\alpha(R)| \equiv 1$. But some of the applications may not be available in this case.

The principal point of these notes is to expound on the three dimensional case, providing a partial resolution of this case. We extend and simplify an approach of J. Beck [4], establishing this result.

1.1.7. Theorem. In dimension $d = 3$, there is a small positive $\epsilon > 0$ for which we have the estimate

\[
2^{-n} \sum_{|R| = 2^{-n}} |\alpha_R| \lesssim n^{1-\epsilon} \left\| \sum_{|R| = 2^{-n}} \alpha_R h_R \right\|_{\infty}.
\]

This result is due to Bilyk and Lacey [1]. Beck [4] established this inequality with $n^{-\epsilon}$ replaced by a term logarithmic in $n^2$.

The organization of the proof, at the highest level, and outlined in § 1.6, is that of József Beck [4]. At the same time, both the exact construction and subsequent details are in many respects easier than in Beck’s paper. In particular, the construction in that section is a Riesz product construction, following the lines of § 1.3. But, the product, with our current understanding, must be taken to be ‘short,’ a dictation to us from the third dimension: The ‘product rule’ 1.3.1 does not hold in dimension three. This unfortunate, and critical fact, forces the definition of ‘strongly distinct’ on us. See Definition 1.5.5.

Critically, József Beck observed that in the case of that the ‘strongly distinct’ does not hold, there is a gain over naive estimates. See Lemma 1.7.2 and Theorem 1.9.1. We will

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2. Beck did not state the result this way, as the principal concern of that paper is on the question of irregularities of distribution. See § 2.1.
refer to any instance of this phenomena as the Beck Gain. The simplest instance of this is discussed in detail in § 1.7. Here, we obtain a better range of results, and a larger gain, than Beck.

Beck’s insight is that this gain permits one to carry out a proof, provided the Riesz product is sufficiently short, so short that the combinatorial explosion generated by the expansion of the Riesz product does not overwhelm the gain.

Beck’s gain has other surprising implications, namely in § 1.8 we see that hyperbolic sums of Haar functions obey a range of sub-gaussian estimates, not predicted by the general theory in § 1.4. This section employs a conditional expectation argument to permit an effective application of the Beck gain.

Concerning the value of $\epsilon$ for which our Theorem holds, it is computable, but we do not carry out this step, as the particular $\epsilon$ we would obtain is certainly not optimal. Instead, the point of this proof is that the methods pioneered by József Beck are more powerful than originally suspected. We expect more efficient organizations of the proof will yield quantifiable and substantive improvements to the results of this paper.

1.2. The Trivial Bounds

The inequality (1.1.2) with an extra square root of $n$ is easy to prove.

1.2.1. Lemma. It is the case that

$$\sum_{|R|=2^{-n}} |\alpha(R)| \cdot |R| \leq n^{\frac{1}{2}(d-1)} \left\| \sum_{|R|=2^{-n}} \alpha(R) h_R \right\|_\infty.$$ 

Proof. Each point $x \in [0, 1]^d$, is in at most $n^{d-1}$ possible rectangles. This is the essential point dictated by the hyperbolic nature of the problem. Using this, and the Cauchy–Schwartz inequality, we have

$$\sum_{|R|=2^{-n}} |\alpha_R| \cdot |R| = \left\| \sum_{|R|=2^{-n}} |\alpha_R| 1_R \right\|_1 \leq n^{\frac{1}{2}(d-1)} \left\| \left( \sum_{|R|=2^{-n}} |\alpha_R|^2 1_R \right)^\frac{1}{2} \right\|_1 \leq n^{\frac{1}{2}(d-1)} \left\| \sum_{|R|=2^{-n}} \alpha(R) h_R \right\|_2 \leq n^{\frac{1}{2}(d-1)} \left\| \sum_{|R|=2^{-n}} \alpha(R) h_R \right\|_\infty \square

3This observation is not essential to our main theorem.
Let us also see that the Small Ball Conjecture is sharp. Indeed, we take the $\alpha(R)$ to be random choices of signs. It is immediate that

$$2^{-n} \sum_{|R|=2^{-n}} |\alpha_R| \approx n^{d-1}.$$ 

We now turn to properties of Rademachers outlined in Chapter 3. On the other hand, for fixed $x \in [0,1]^d$ we have

$$\mathbb{E} \left| \sum_{|R|=2^{-n}} \alpha(R) h_R(x) \right| \approx n^{\frac{1}{2}(d-1)}.$$ 

It is also well known that sums of Rademacher random variables obey a sub–Gaussian distributional estimate. The supremum of such sums admit easily estimated upper bounds. In particular, it is enough to test the $L^\infty$ norm of the sum at a grid of $2^{nd}$ points in the unit cube, hence we have

$$\mathbb{E} \left\| \sum_{|R|=2^{-n}} \alpha(R) h_R \right\|_\infty \lesssim \sqrt{\log 2^{nd}} \cdot \sup_x \mathbb{E} \left| \sum_{|R|=2^{-n}} \alpha(R) h_R(x) \right| \lesssim n^{d/2}.$$ 

Comparing these two estimates shows that the Small Ball Conjecture is sharp.

The Small Ball Conjecture could be substantially resolved if one could directly show that in the random case that this estimate is sharp.

### 1.3. Proof of Talagrand’s Theorem

We follow the approach of V. Temlyakov [35] to the stronger inequality (1.1.4) in the case of $d = 2$, and invite the reader to compare this argument to the proof of Schmidt’s Theorem in § 2.4.

The decisive point in two dimensions is that one has a ‘product rule.’ Let us formalize it as this proposition, and leave the proof to the reader.

#### 1.3.1. Product Rule in Dimension 2.

Let $R, R'$ be two dyadic rectangles of the same area. Then,

$$h_R \cdot h_{R'} \in \{0, 1_R, h_{R \cap R'}\}.$$ 

More generally, let $R_1, R_2, \ldots, R_k$ be dyadic rectangles of equal area and distinct lengths in e.g. their first coordinates. Then

$$\prod_{j=1}^k h_{R_j} \in \{0, \pm h_{R_1 \cap \cdots \cap R_k}\}.$$ 

The proof of (1.1.4) is by duality. Fix

$$H = \sum_{|R|\geq 2^{-n}} \alpha(R) h_R.$$
We will construct a function $\Psi$ with $L^1$ norm at most 1, for which the inner product

$$\langle H, \Psi \rangle = 2^{-n-1} \sum_{|R|=2^{-n}} |\alpha(R)|.$$ 

This clearly implies the Theorem. Moreover, the function $\Psi$ is defined as a Riesz product. Our Riesz product is

$$\Psi := \prod_{s=1}^{n} (1 + \frac{1}{2} \psi_s),$$

$$\psi_s = \sum_{R: |R_1|=2^{-s}, |R_2|=2^{-n-s}} \text{sgn}(\alpha(R)) h_R$$

Of course $\Psi$ is non-negative. Moreover, it has $L^1$ norm one: Expanding the product, the leading term is 1. All products of $\psi_s$ are, by Proposition 1.3.1, a sum of Haar functions, hence have mean zero.

The Proposition also implies that

$$\langle H, \Psi \rangle = \sum_{s=1}^{n} \langle H, \psi_s^n \rangle = 2^{-n-1} \sum_{|R|=2^{-n}} |\alpha(R)|.$$ 

The proof is complete.

Remark. If one considers the case of $\alpha(R) \equiv 1$, it is clear that the $L^\infty$ norm is achieved—or nearly achieved—on a set of measure approximately $2^{-cn}$. That is, the supremum is achieved on a very thin set. Experience shows that Riesz products are very useful in such situations.

Remark. Traditionally, a Riesz product is of the form

$$\prod_{k=1}^{\infty} (1 + \cos 4^k x).$$

By a well known heuristic, the functions $\cos 4^k x$ behave as independent random variables, so we don’t make a distinction between the classical Riesz product and the Riesz products of our proofs. Using Riesz products as above has a long history in the subject of irregularities of distributions.

1.4. Exponential Moments

We state a distributional estimate for sums of hyperbolic Haars which shapes the potential forms of approach to the Small Ball Conjecture. However, while the estimates we describe here are in general sharp, they admit certain improvements, for small $p$; see § 1.8.

Background on these issues are developed on Chapter 3.
1.4.1. **Theorem.** In dimension $d \geq 2$ we have the estimate below, phrased in terms of the exponential Orlicz Lebesgue spaces.

\[
\left\| \sum_{|R|=2^{-n}} \alpha(R) h_R \right\|_{\exp(L^{2/(d-1)})} \leq \left\| \sum_{|R|=2^{-n}} \alpha(R)^2 1_R \right\|^{1/2}_{\infty}.
\]

**Remark.** The estimates above, specialized to hyperbolic sums in dimension 3 or higher, are better than those that appear in the literature associated to the Discrepancy function.

Here we are using a typical definition of the exponential integrability classes, as given in §3.1. This definition could be for instance

\[
\|X\|_{\exp(L^\alpha)} \approx \sup_{p \geq 1} p^{-1/\alpha} \|X\|_p
\]

The equivalence holding on any probability space.

Of principal relevance to us is the three dimensional case, where the estimate above asserts that the hyperbolic sums are exponentially integrable.

**Proof.** The tool is the vector valued Littlewood Paley inequality, with sharp rate of growth in the constants as $p \to \infty$. As such the proof is a standard one, see [19,24].

Applying the one dimensional Littlewood Paley inequality in the coordinate $x_1$ we see that

\[
\left\| \sum_{|R|=2^{-n}} \alpha(R) h_R \right\|_p \leq \sqrt{p} \left\| \left[ \sum_{r_1=1}^n \sum_{|R|=2^{-r}} \alpha(R) h_R \right]^2 \right\|_p^{1/2}
\]

If we are in dimension 2, note that

\[
\left| \sum_{|R|=2^{-n}} \alpha(R) h_R \right|^2 = \sum_{|R|=2^{-n}} \left| \alpha(R) \right|^2 1_R
\]

so our proof is complete in this case.

In the higher dimensional case, the key point is to observe that the last term can be viewed as an $\ell^2$ space valued function. Then, the Hilbert space analog of the Littlewood Paley inequalities applies to the second coordinate, to give us

\[
\left\| \sum_{|R|=2^{-n}} \alpha(R) h_R \right\|_p \leq p \left\| \sum_{r_1=1}^n \sum_{r_2=1}^n \sum_{|R|=2^{-r}} \alpha(R) h_R \right\|_p^{1/2}
\]

Observe that we have a full power of $p$, due to the two applications of the Littlewood Paley inequalities. And if $d = 3$, then analog of (1.4.4) holds, completing the proof in this case.

In the case of dimension $d \geq 4$ note that we can continue applying the Littlewood Paley inequalities inductively. They need only be used $d - 1$ times due to the hyperbolic
assumption. Thus, we have the inequality
\[ \left\| \sum_{|R|=2^{-m}} \alpha(R) h_R \right\|_p \lesssim p^{(d-1)/2} \left\| \sum_{|R|=2^{-m}} \alpha(R)^2 1_R \right\|_p^{1/2}, \quad 2 \leq p < \infty. \]

The implied constant depends upon dimension; the main point we are interested in is the rate of growth of the $L^p$ norms. Assuming that the Square Function of the sum is bounded in $L^\infty$, the $L^p$ norms can only grow at the rate of $p^{(d-1)/2}$, which completes the proof. \qed

Remark. It is a thesis of A. Zygmund that when one is concerned with product domain questions, the relevant estimates are governed by the effective number of parameters involved. This thesis in the hyperbolic setting, says that relevant estimates should be those of $d-1$ parameters in dimension $d$. We have just seen one instance of this. While it is known that this thesis does not hold in full generality, the hyperbolic setting is simple enough that it should hold for most, if not all, questions of interest.

1.5. Definitions and Initial Lemmas for Dimension Three

The principal difficulty in three and higher dimensions is that the product of Haar functions is not necessarily a Haar function. On this point, we have the following proposition which does not admit any essential extension.

1.5.1. Proposition. Suppose that $R_1, \ldots, R_k$ are rectangles such that there is no choice of $1 \leq j < j' \leq k$ and no choice of coordinate $1 \leq t \leq d$ for which we have $R_{j,t} = R_{j',t}$. Then, for a choice of sign $\epsilon \in \{\pm 1\}$ we have
\[ \prod_{j=1}^k h_{R_j} = \epsilon h_S, \quad S = \bigcap_{j=1}^k R_j. \]

Proof. Expand the product as
\[ \prod_{m=1}^\ell h_{R_m}(x_1, \ldots, x_d) = \prod_{m=1}^\ell \prod_{t=1}^d h_{R_m,t}(x_t) \]

Here $\epsilon_m \in \{\pm 1\}$. Our assumption is that for each $t$, there is exactly one choice of $1 \leq m_0 \leq \ell$ such that $R_{m_0,t} = S_t$. And moreover, since the minimum value of $|R_{m,t}|$ is obtained exactly once, for $m \neq m_0$, we have that $h_{R_{m,t}}$ is constant on $S_t$. Thus, in the $t$ coordinate, the product is
\[ \epsilon_{m_0} h_{S_t}(x_t) \prod_{1 \leq m \neq m_0 \leq \ell} \epsilon_m h_{R_{m,t}}(S_t). \]

This proves our Lemma. \qed

Let $\vec{r} \in \mathbb{N}^d$ be a partition of $n$, thus $\vec{r} = (r_1, r_2, r_3)$, where the $r_j$ are non negative integers and $|\vec{r}| := \sum_j r_j = n$. Denote all such vectors at $\mathbb{H}_n$. (‘$\mathbb{H}$’ for ‘hyperbolic.’) For vector $\vec{r}$ let $\mathcal{R}_{\vec{r}}$ be all dyadic rectangles $R$ such that for each coordinate $k$, $|R_k| = 2^{-r_k}$. 

1.5.3. Definition. We call a function $f$ an $r$ function with parameter $\vec{r}$ if

$$f = \sum_{R \in R_r} \varepsilon_R h_R, \quad \varepsilon_R \in \{\pm 1\}.$$  

We will use $f_{\vec{r}}$ to denote a generic $r$ function. A fact used without further comment is that $f_{\vec{r}}^2 \equiv 1$.

1.5.5. Definition. For vectors $\vec{r}_j \in \mathbb{N}^3$, say that $\vec{r}_1, \ldots, \vec{r}_J$ are strongly distinct iff for coordinates $1 \leq t \leq 3$ the integers $\{r_{jt} : 1 \leq j \leq J\}$ are distinct. The product of strongly distinct $r$ functions is also an $r$ function.

The $r$ functions we are interested in are:

$$f_{\vec{r}} := \sum_{R \in R_r} \text{sgn}(a(R)) h_R$$

### 1.6. József Beck’s Short Riesz Product

Let us define relevant parameters by

$$q = an^\varepsilon, \quad b = \frac{1}{4}$$

$$\tilde{\rho} = aq^b n^{-1}, \quad \rho = \sqrt{q} n^{-1}.$$ 

Here, $a$ are small positive constants, we use the notation of $b = 1/4$ throughout, so as not to obscure those aspects of the argument that that dictate this choice of $b$. $\tilde{\rho}$ is a ‘false’ $L^2$ normalization for the sums we consider, while the larger term $\rho$ is the ‘true’ $L^2$ normalization. Our ‘gain over the trivial estimate’ in the Small Ball Conjecture is $q^b = n^{\varepsilon/4}$. $0 < \varepsilon < 1$ is a small constant. It certainly can’t be more than $1/6$ in view of (1.8.3) though there are other more severe restrictions on the size of $\varepsilon$; the exact determination of what we could take $\varepsilon$ equal to in this proof doesn’t seem to be worth calculating.

In Beck’s paper, the value of $q = q_{\text{Beck}} = \frac{\log n}{\log \log n}$ was much smaller than our value of $q$. The point of this choice is that $q_{\text{Beck}}^{\text{Beck}} \approx n$, with the term $q^a$ controlling many of the combinatorial issues concerning the expansion of the Riesz product.\footnote{Specifically, $q^{2q}$ is a naive bound for the number of admissible graphs, as defined in § 1.9.} With our substantially larger value of $q$, we need to introduce additional tools to control the combinatorics. These tools are

- A Riesz product that will permit us to implement various conditional expectation arguments.
- Attention to $L^p$ estimates of various sums, and their growth rates in $p$.
- Systematic use of the Littlewood Paley inequalities, with the sharp exponents in $p$.\footnote{Specifically, $q^{2q}$ is a naive bound for the number of admissible graphs, as defined in § 1.9.}
Divide the integers \( \{1, 2, \ldots, n\} \) into \( q \) disjoint intervals \( I_1, \ldots, I_q \), and let \( A_i := \{ \vec{r} \in \mathbb{H}_n : r_1 \in I_i \} \). Let 
\[
F_i = \sum_{\vec{r} \in A_i} f_{\vec{r}}.
\]
The Riesz product is now a ‘short product.’

\[
\Psi := q \prod_{t=1}^{q} (1 + \bar{\rho} F_t).
\]

Note the subtle way that the false \( L^2 \) normalization enters into the product. It means that the product is, with high probability, positive. And of course, for a positive function \( F \), we have \( \mathbb{E} F = \|F\|_1 \), with expectations being typically easier to estimate. This heuristic is made precise below.

We need to decompose the product \( \Psi \) into
\[
\Psi = 1 + \Psi^{sd} + \Psi^{-},
\]
where the two pieces are the ‘strongly distinct’ and ‘not strongly distinct’ pieces. To be specific, for integers \( 1 \leq u \leq q \), let
\[
\Psi^{sd}_k := \bar{\rho}^k \sum_{1 \leq v_1 < \cdots < v_k \leq q} \sum_{\vec{r} \in A_{v_t}} \prod_{t=1}^{u} f_{\vec{r}_t}
\]
where \( \sum^{sd} \) is taken to be over all \( k \) tuples of vectors \( \{(\vec{r}_1, \ldots, \vec{r}_k) \in \prod_{t=1}^{k} A_{v_t} \} \) such that:
\[
\text{the vectors } \{\vec{r}_t : 1 \leq t \leq k \} \text{ are strongly distinct.}
\]

Then define
\[
\Psi^{sd} := \sum_{u=1}^{q} \Psi^{sd}_u
\]

With this definition, it is clear that we have
\[
\langle H_n, \Psi^{sd} \rangle = \langle H_n, \Psi^{sd}_1 \rangle \geq q^b \cdot n^1,
\]
so that \( q^b \) is our ‘gain over the trivial estimate.’

The bulk of the proof is taken up with the proof of the technical estimates below. The main point of the Lemma is the last estimate, (1.6.14), which with (1.6.7) above proves Theorem 1.1.7.

1.6.8. Lemma. We have these estimates:
\[
\begin{align*}
\mathbb{P}(\Psi < 0) &\leq \exp(-Aq^{1-2b}); \\
\|\Psi\|_2 &\leq \exp(a'q^{2b}); \\
\mathbb{E}\Psi &= 1;
\end{align*}
\]
Here, 0 < \( a' < 1 \), in (1.6.10), is a small constant, decreasing to zero as \( a \) in (1.6.1) goes to zero; and \( A > 1 \), in (1.6.9) is a large constant, tending to infinity as \( a \) in (1.6.1) goes to zero.

Proof. We give the proof of the Lemma, assuming our main inequalities proved in the subsequent sections. In particular, the first two estimates of our Lemma are substantial, as they reflect the influence of the non trivial sub–gaussian estimates of § 1.8.

Proof of (1.6.9). The main tool is the distributional estimate (1.8.3). Observe that

\[
\mathbb{P}(\Psi < 0) \leq \sum_{t=1}^{q} \mathbb{P}(\tilde{\rho}F_t < -1)
\]

\[
= \sum_{t=1}^{q} \mathbb{P}(\rho F_t < -a^{-1}q^{1/2-b})
\]

\[
\lesssim q \exp(ca^{-1}q^{1-2b}).
\]

Proof of (1.6.10). The proof of this is detailed enough that we postpone it to Lemma 1.8.5 below.

Proof of (1.6.11). Expand the product in the definition of \( \Psi \). The leading term is one. Every other term is a product

\[
\prod_{k \in V} \tilde{\rho} F_k,
\]

where \( V \) is a non-empty subset of \( \{1, \ldots, q\} \). This product is in turn a product of \( r \) functions. Among this product, the maximum in the first coordinate is unique. This fact tells us that the expectation of this product of \( r \) functions is zero. So the expectation of the product above is zero. The proof is complete.

Proof of (1.6.12). We use the first two estimates of our Lemma. Observe that

\[
||\Psi||_1 = \mathbb{E}\Psi - 2\mathbb{E}\Psi 1_{\Psi < 0}
\]

\[
\leq 1 + 2\mathbb{P}(\Psi < 0)^{1/2}||\Psi||_2
\]

\[
\leq 1 + \exp(-Aq^{1-2b}/2 + a'q^{2b}) .
\]

We have taken \( b = 1/4 \) so that \( 1 - 2b = 2b \). For sufficiently small \( a \) in (1.6.1), we will have \( A \geq a' \). We see that (1.6.12) holds.\(^5\)

Indeed, Lemma 1.8.5 proves a uniform estimate, namely

\[
\sup_{\nu \leq 1} \mathbb{E} \prod_{t \in \nu} (1 + \tilde{\rho}F_t)^2 \leq \exp(a'q^{2b}) .
\]

\(^5\)Here of course we are strongly using the fact \( \Psi \) is positive with high probability.
Hence, the argument above proves

\[ \sup_{V \subset \{1, \ldots, q\}} \left\| \prod_{v \in V} (1 + \tilde{\rho} F_t) \right\|_1 \lesssim 1. \]

**Proof of (1.6.13).** The primary facts are (1.6.15) and Theorem 1.9.1; we use the notation devised for that Theorem.

Note that the Inclusion Exclusion principle gives us the identity

\[ \Psi^- = \sum_{V \subset \{1, \ldots, q\} \atop |V| \geq 2} (-1)^{|V| + 1} \text{Prod}(\text{NSD}(V)) \cdot \prod_{t \in \{1, \ldots, q\} \setminus V} (1 + \tilde{\rho} F_t). \]

We use the triangle inequality, the estimates of Lemma 1.8.5, Hölder’s inequality, with indices \(1 + 1/q^{2b}\) and \(q^{2b}\), and the estimate of (1.9.2) in the calculation below. Notice that we have

\[ \sup_{V \subset \{1, \ldots, q\}} \left\| \prod_{v \in V} (1 + \tilde{\rho} F_t) \right\|_{1 + q^{2b}} \leq \sup_{V \subset \{1, \ldots, q\}} \left\| \prod_{v \in V} (1 + \tilde{\rho} F_t) \right\|_1^{(1 + q^{-2b})/(1 - q^{-2b})} \]
\[ \times \left\| \prod_{v \in V} (1 + \tilde{\rho} F_t) \right\|_{2 + q^{-2b}} \]
\[ \lesssim 1. \]

And recall that \(q^{2b} = q^{1/2}\) is a small power of \(n\). So the \(L^p\) norms that we need on terms arising from NSD(V) below are for moderate values of \(p\), namely we only need \(p \leq q^{2b}\). This is a key reason why we can control the combinatorial explosion associated with our short Riesz product.

We estimate

\[ \|\Psi^-\|_1 \leq \sum_{V \subset \{1, \ldots, q\} \atop |V| \geq 2} \left\| \text{Prod}(\text{NSD}(V)) \cdot \prod_{t \in \{1, \ldots, q\} \setminus V} (1 + \tilde{\rho} F_t) \right\|_1 \]
\[ \leq \sum_{V \subset \{1, \ldots, q\} \atop |V| \geq 2} \|\text{Prod}(\text{NSD}(V))\|_{q^{2b}} \cdot \left\| \prod_{t \in \{1, \ldots, q\} \setminus V} (1 + \tilde{\rho} F_t) \right\|_{1 + q^{-2b}} \]
\[ \lesssim \sum_{v=2}^q [q^c n^{1/6}]^v \]
\[ \lesssim q^c n^{1/6} \]
\[ \lesssim n^{-\epsilon'} \lesssim 1. \]

**Proof of (1.6.14).** This follows from (1.6.13) and (1.6.12) and the identity \(\Psi = 1 + \Psi^\text{sd} + \Psi^-\) and the triangle inequality.

\[ \square \]
1.7. The Beck Gain in the Simplest Instance

Beck considered sums of products of $r$ functions that are not strongly distinct, and observed that the $L^2$ norm of the same are smaller than one would naively expect. This is what we call the Beck Gain. A product of $r$ functions will not be strongly distinct if the product involves two or more vectors which agree in one or more coordinates. In this section, we study the sums of products of two $r$ functions which are not strongly distinct. A later section, §1.9, will study the general case.

In this section, and again in §1.9, we will use this notation. For a subset $C \subset \mathbb{H}_n^k$, let

\[
\text{Prod}(C) := \sum_{(\vec{r}_1, \ldots, \vec{r}_k) \in C} \prod_{j=1}^k f_{\vec{r}_j}.
\]

In this section, we are exclusively interested in $k = 2$.

Let $C(2) \subset \mathbb{H}_n^2$ consist of all pairs of distinct $r$ vectors $\{\vec{r}_1, \vec{r}_2\}$ for which $r_{1,2} = r_{2,2}$. J. Beck calls such terms ‘coincidences’ and we will continue to use that term. We need norm estimates on the sums of products of such $r$ vectors.

1.7.2. Lemma. [The Simplest Instance of the Beck Gain.] We have these estimates for arbitrary subsets $C \subset C(2)$

\[
||\text{Prod}(C)||_p \lesssim p^{5/4} n^{7/4}.
\]

Moreover, if we have $C = C(2) \cap A_s \times A_t$ for some $0 \leq s, t \leq q$ we have

\[
||\text{Prod}(C)||_p \lesssim p^{3/2} n^{3/2} q^{-1/2}.
\]

Finally, we have the estimate

\[
\left\| \sum_{\vec{r} \neq \vec{s} \in A_s \atop r_1 = s_1} f_\vec{r} \cdot f_\vec{s} \right\|_p \lesssim p^{3/2} n^{3/2} q^{-1}.
\]

We will use the second estimate of the Lemma, which we do not claim for arbitrary subsets of $C(2)$. This estimate appears to be sharp, in that the collection $C(2)$ has three free parameters, and the estimates is in terms of $n^{3/2}$. Note that for $p \approx n$ we have

\[
||\text{Prod}(C_2)||_n \approx ||\text{Prod}(C_2)||_\infty.
\]

And the latter term can be as big as $n^3$, which matches the bound above.

The proof of the Lemma requires we pass through an intermediary collection of four tuples of $r$ vectors. Let $B(4) \subset \mathbb{H}_n^4$ be four tuples of distinct vectors $(\vec{r}', \vec{s}', \vec{t}', \vec{u}')$ for which (i) $r_1 = s_1$ and $t_1 = u_1$; and (ii) in the second and third coordinate two of the vectors agree.

Proof. The method of proof is probably best explained by considering first the case of $p = 2$. Observe that

\[
||\text{Prod}(C)||_2^2 = \mathbb{E} \text{Prod}(B),
\]
where \( \mathcal{B} = \mathbb{C} \times \mathbb{C} \cap \mathcal{B}(4) \). Indeed, the main point is that in order for
\[
\mathbb{E} f_{\hat{r}_1} \cdot f_{\hat{r}_2} \cdot f_{\hat{r}_3} \cdot f_{\hat{r}_4} \neq 0
\]
there is a coincidence among the four vectors in each coordinate. But this is the definition of \( \mathcal{B}(4) \). Thus the case \( p = 2 \) follows immediately from Lemma 1.7.7.

Now, let us consider \( 4 \leq p \leq n \), as the inequalities we prove are trivial for \( p > n \). Let \( K_{3/2} \) be the best constant in the inequality
\[
N(p) := \sup \| \text{Prod}(\mathbb{C}) \|_p \leq K p^{3/2} n^{3/2}.
\]
Here the supremum is over all choices of \( n \) and \( \mathbb{r} \) functions. We give an *a priori* estimate of \( K_{3/2} \). We define \( K_{7/4} \) similarly.

Each pair \((\hat{r}, \hat{s}) \in \mathbb{C}\) must be distinct in the first and third coordinates. Therefore, we can apply the Littlewood Paley inequalities in these coordinates to estimate
\[
N(p) := \| \text{Prod}(\mathbb{C}) \|_p \leq p \left( \sum_{a,b} \left( \sum_{(\hat{r}, \hat{s}) \in \mathbb{C}} \prod_{\text{max}_{\{1,2\}} = a \atop \text{max}_{\{3,4\}} = b} f_{\hat{r}} \cdot f_{\hat{s}} \right)^2 \right)^{1/2}.
\]
Here, we have a full power of \( p \), as we apply the Littlewood Paley inequalities twice. Observe that
\[
\sum_{a,b} \left( \sum_{(\hat{r}, \hat{s}) \in \mathbb{C}} \prod_{\text{max}_{\{1,2\}} = a \atop \text{max}_{\{3,4\}} = b} f_{\hat{r}} \cdot f_{\hat{s}} \right)^2 = \# \mathbb{C} + \sum_{i \neq j \in \{1,2,3,4\}} \text{Prod}(\mathbb{C}_{i,j}) + \text{Prod}(\mathbb{B}_{\text{max}}).
\]
The term \( \# \mathbb{C} \) arises from the diagonal of the square. The terms \( \mathbb{C}_{i,j} \) are
\[
\mathbb{C}_{i,j} := \{(\hat{r}_1, \hat{r}_2, \hat{r}_3, \hat{r}_4) \in \mathbb{C} \times \mathbb{C} : \hat{r}_i = \hat{r}_j, \text{ and the other two vectors are distinct}\}
\]
Note that by definition, \( \mathbb{C}_{1,2} = \mathbb{C}_{3,4} = \emptyset \). The term \( \mathbb{B}_{\text{max}} \) is
\[
\mathbb{B}_{\text{max}} := \{(\hat{r}_1, \hat{r}_2, \hat{r}_3, \hat{r}_4) \in \mathbb{C} \times \mathbb{C} : \text{the maximum in the first and third coordinates occur twice}\}
\]
Then, we can estimate by the triangle inequality, and the sub-additivity of \( x \mapsto \sqrt{x} \),
\[
\| \text{Prod}(\mathbb{C}) \|_p \leq p (\# \mathbb{C})^{1/2} + p \sum_{i \neq j \in \{1,2,3,4\}} \| \text{Prod}(\mathbb{C}_{i,j}) \|^{1/2}_p + p \| \text{Prod}(\mathbb{B}_{\text{max}}) \|_p
\]
\[
\leq p^{3/2} + pn^{1/2}N(p/2)^{1/2} + p \| \text{Prod}(\mathbb{B}_{\text{max}}) \|_p.
\]
Using the estimate (1.7.8), we see that
\[
\| \text{Prod}(\mathbb{C}) \|_p \leq pn^{3/2} + pn^{1/2}N(p/2)^{1/2} + p^{5/4}n^{7/4}
\]
\[
\leq p^{5/4}n^{7/4} + pn^{1/2}N(p/2)^{1/2}
\]
This implies that
\[
K_{7/4} \leq 1 + \sup_{2 \leq p \leq n} \left( p^{-5/4} n^{7/4} \right) \left( K p^{5/4} n^{7/4} \right)^{1/2} \\
\leq 1 + K_{7/4}^{1/2}.
\]
Clearly, this implies \( K_{7/4} \leq 1 \).

Using the estimate (1.7.12), the proof that \( K_{3/2} \leq 1 \) is entirely similar.

\[\square\]

Recall that \( \mathbb{B}(4) \subset \mathbb{H}_p^4 \) be four tuples of distinct vectors \((\vec{r}, \vec{s}, \vec{t}, \vec{u})\) for which (i) \( r_1 = s_1 \) and \( t_1 = u_1 \); and (ii) in the second and third coordinate two of the vectors agree.

1.7.7. Lemma. For any subset \( \mathbb{B} \subset \mathbb{B}(4) \)

(1.7.8) \[\| \text{Prod}(\mathbb{B}) \|_p \leq \sqrt{p} n^{7/2}.\]

Moreover, for \( \mathbb{B} \subset \mathbb{B}(4) \cap (A_s \times A_t)^2 \), for any choice of \( 0 \leq s \neq t \leq q \), we have

(1.7.9) \[\| \text{Prod}(\mathbb{B}) \|_p \leq c \sqrt{p} n^{7/2} q^{-2}.\]

If we do not consider arbitrary subsets, the estimates improve. We have the estimates

(1.7.10) \[\| \text{Prod}(\mathbb{B}(4)) \|_p \leq p n^3,\]

(1.7.11) \[\| \text{Prod}(\mathbb{B}(4) \cap (A_s \times A_t)^2) \|_p \leq pn^3.\]

Finally, define

\[\mathbb{B}_{\text{max}} := \{ (\vec{r}, \vec{s}, \vec{t}, \vec{u}) \in \mathbb{B}(4) \cap (A_s \times A_t)^2 : \text{the maximum in second and third coordinates occur twice}\}\]

Then, we have the estimate

(1.7.12) \[\| \text{Prod}(\mathbb{B}_{\text{max}}) \|_p \leq pn^3.\]

This Lemma, with exponents on \( n \) being \( n^{7/2} \) appears in Beck’s paper [4], in the case of \( p = 2 \). The \( L^p \) variants, following from consequences of Littlewood Paley inequalities, are important for us.

The first group of estimates are recorded, as it is interesting that they apply to arbitrary subsets of \( \mathbb{B}(4) \). We will rely upon the second group of estimates. Pointed out to us by Mihalis Kolountzakis, these estimates are better for all ranges of \( p \leq n \).

Proof. We discuss (1.7.8) explicitly, and note as we go the improvements needed to get the estimate (1.7.9).

The proof is a case analysis, depending upon the number of \( \{\vec{r}, \vec{s}, \vec{t}, \vec{u}\} \) at which the maximums occur in the second and third coordinates. We proceed immediately to the cases.

Let \( \mathbb{B}_1 \subset \mathbb{B} \) consist of those four–tuples \( \{\vec{r}, \vec{s}, \vec{t}, \vec{u}\} \) for which

\[ r_2 = t_2 = \max\{r_2, s_2, t_2, u_2\}, \quad r_3 = t_3 = \max\{r_3, s_3, t_3, u_3\}. \]
This collection is empty, for necessarily we must have \( r_1 = s_1 = t_1 = u_1 \), but then \( \vec{r} = \vec{s} \), as the parameters of all vectors is \( n \). This violates the definition of \( \mathbb{B} \).

Let \( \mathbb{B}_3 \subset \mathbb{B} \) consist of those four–tuples \( \{\vec{r}, \vec{s}, \vec{t}, \vec{u}\} \) for which

\[
r_2 = t_2 = \max\{r_2, s_2, t_2, u_2\}, \quad r_3 = u_3 = \max\{r_3, s_3, t_3, u_3\}.
\]

That is, the maximal values involve three distinct vectors. These four vectors can be depicted as

\[
\vec{r} = \begin{pmatrix} r_1 \\ r_2(\Box) \\ r_3 \end{pmatrix}, \quad \vec{s} = \begin{pmatrix} r_1 \\ * \\ \Box \end{pmatrix}, \quad \vec{t} = \begin{pmatrix} t_1 \\ t_2 \\ \Box \end{pmatrix}, \quad \vec{u} = \begin{pmatrix} t_1 \\ \Box \\ r_3 \end{pmatrix}
\]

A \( \Box \) denotes a parameter which is determined by other choices. It is essential to note that choices of \( r_1 \) and \( r_3 \) determine the value of \( r_2 \) (hence the \( \Box \) in the middle coordinate for \( \vec{r} \)), and so the vector \( \vec{r} \). The only free parameters are (say) \( s_2 \), denoted by an \( * \) above.

But, note that we must then have \(|s| = s_1 + s_2 + s_3 < n\). Therefore this case is empty.

Let \( \mathbb{B}_4 \) be those four tuples four tuples \( \{\vec{r}, \vec{s}, \vec{t}, \vec{u}\} \in \mathbb{B} \) such that \( s_2 = t_2 \) and \( r_3 = u_3 \). That is there are four vectors involved in the maximums of the second and third coordinates. These four vectors can be represented as

\[
(1.7.13) \quad \vec{r} = \begin{pmatrix} r_1 \\ \Box \\ r_3 \end{pmatrix}, \quad \vec{s} = \begin{pmatrix} r_1 \\ s_2 \\ \Box \end{pmatrix}, \quad \vec{t} = \begin{pmatrix} t_1 \\ s_2 \\ \Box \end{pmatrix}, \quad \vec{u} = \begin{pmatrix} t_1 \\ \Box \\ r_3 \end{pmatrix}
\]

The next argument proves \((1.7.8)\). Let \( \mathbb{B}_4(a, a', b) \) be those four tuples \( \{\vec{r}, \vec{s}, \vec{t}, \vec{u}\} \in \mathbb{B} \) such that

\[
r_1 = s_1 = a, \quad t_1 = u_1 = a', \quad s_2, t_2 = b.
\]

The point to observe is that

\[
||\text{Prod}(\mathbb{B}_4(a, a', b))||_p \leq C \sqrt{n} \sqrt{n}.
\]

As there at most \( n^3 \) choices for \( a, a' \) this proves the Lemma. (And, in the case of \((1.7.9)\), there are at most \( n(n/q)^2 \) choices for these three parameters.)

Indeed, we have not specified \( r_3 = u_3 \). Since all vectors are distinct, \( a \neq a' \), and in considering the norm above, we ignore \( \vec{s} \) and \( \vec{t} \), as they are completely specified by the datum \( (a, a', b) \). The product \( f_{\vec{r}} \cdot f_{\vec{u}} \), in the second coordinate, is equal in distribution to a Rademacher function. And then the estimate above follows. The proof of \((1.7.8)\) and \((1.7.9)\) are finished.

We turn to the proof of \((1.7.10)\) and \((1.7.11)\), arguing similarly. Let \( \mathbb{B}_4(a, a') \) be those four tuples \( \{\vec{r}, \vec{s}, \vec{t}, \vec{u}\} \in \mathbb{B} \) such that

\[
r_1 = s_1 = a, \quad t_1 = u_1 = a'.
\]

The point to observe is that

\[
||\text{Prod}(\mathbb{B}_4(a, a'))||_p \leq Cp n.
\]
As there at most $\leq n^2$ choices for $a, a'$ this proves the Lemma. (And, in the case of (1.7.11), there are at most $(n/q)^2$ choices for these two parameters.)

The point is that $\text{Prod}(\mathcal{B}_4(a, a'))$ splits into a product. Namely,

$$
\text{Prod}(\mathcal{B}_4(a, a')) = \text{Prod}(\mathcal{B}_{4,1}(a, a')) \cdot \text{Prod}(\mathcal{B}_{4,2}(a, a'))
$$

$$
\text{Prod}(\mathcal{B}_{4,1}(a, a')) := \{ \{ \vec{r}, \vec{u} \} : \text{there exists } \{ \vec{s}, \vec{t} \} \in \mathcal{B}_2^2 \text{ with } \{ \vec{r}, \vec{s}, \vec{t}, \vec{u} \} \in \mathcal{B}_4(a, a') \},
$$

$$
\text{Prod}(\mathcal{B}_{4,2}(a, a')) := \{ \{ \vec{s}, \vec{t} \} : \text{there exists } \{ \vec{r}, \vec{r} \} \in \mathcal{B}_2^2 \text{ with } \{ \vec{r}, \vec{s}, \vec{t}, \vec{u} \} \in \mathcal{B}_4(a, a') \}.
$$

We estimate

$$
||\text{Prod}(\mathcal{B}_4(a, a'))||_p \leq ||\text{Prod}(\mathcal{B}_{4,1}(a, a'))||_{2p} \cdot ||\text{Prod}(\mathcal{B}_{4,2}(a, a'))||_{2p}.
$$

Both of the last two norms are at most $\leq \sqrt{p} \cdot n^{1/2}$, which will finish the proof.

That is the estimate is

$$
(1.7.14) \quad ||\text{Prod}(\mathcal{B}_{4,1}(a, a'))||_{2p} \leq \sqrt{p} \cdot n^{1/2}.
$$

We may assume without loss of generality that $a > a'$. The pairs in $\text{Prod}(\mathcal{B}_{4,1}(a, a'))$ consist of the two vectors $\vec{r}$ and $\vec{u}$ in (1.7.13). These two vectors are parameterized by $u_2$, say. Since $a = r_1 < a' = u_1$, and $r_3 = u_3$, the hyperbolic assumption implies $u_2$ is the maximal coordinate. Therefore, the Littlewood Paley inequality applies.

The proof of (1.7.11) is exactly the same, just noting that $a, a'$ can only take $(n/q)^2$ values in that case.

We turn to the proof of the estimate (1.7.12). Here, it suffices to prove that

$$
(1.7.15) \quad ||\text{Prod}(\mathcal{B}_4) \cap (\mathcal{A}_s \times \mathcal{A}_t)^2 - \mathcal{B}_{\max}||_p \leq pn^3.
$$

This last collection of four tuples of vectors can be further subdivided into finite number of collections, $\mathcal{B}_j'$, for $1 \leq j \leq 6$. Take $\mathcal{B}_1'$ to be a subset of four tuples $(\vec{r}, \vec{s}, \vec{t}, \vec{u}) \in \mathcal{B}_4 \cap (\mathcal{A}_s \times \mathcal{A}_t)^2 - \mathcal{B}_{\max}$ with

$$
\vec{r} = \begin{pmatrix} r_1 \\ r_2 \\ r_3 \end{pmatrix}, \quad \vec{s} = \begin{pmatrix} s_1 \\ r_2 \\ s_3 \end{pmatrix}, \quad \vec{t} = \begin{pmatrix} t_1 \\ t_2 \\ s_3 \end{pmatrix}, \quad \vec{u} = \begin{pmatrix} s_1 \\ t_2 \\ t_3 \end{pmatrix}
$$

Here we assume that $r_1$ is the unique maximal integer among $\{r_1, s_1, t_1, u_1\}$. Note that $\vec{r}$ and $\vec{s}$ have a coincidence in the second coordinate; $\vec{s}, \vec{t}$ have a coincidence in the first coordinate; and $\vec{t}, \vec{u}$ have a coincidence in the third coordinate. The other collections $\mathcal{B}_j'$ differ in the location of the maximums in either the first and third coordinates, and the particular patterns of coincidences.

It is important to observe that we necessarily have $r_1 > t_1 > s_2 = u_2$. And we will apply the Littlewood Paley inequality in the $r_1$ and $t_1$ variables. Clearly, we can apply the Littlewood Paley inequality in $r_1$ to get the estimate

$$
\|\text{Prod}(\mathcal{B}_1')\|_p \leq \sqrt{p} \left\| \sum_a \text{Prod}(\mathcal{B}_1(a))^2 \right\|^{1/2}_p
$$
Here, $\mathcal{B}_1(a)$ is the collection of all four tuples $\{\vec{r}, \vec{s}, \vec{t}, \vec{u}\} \in \mathcal{B}_1'$ with $r_1 = a$.

Next, we use the triangle inequality in the values of $r_2$ and $s_3$. Note that with $r_1, r_2, s_3$ specified, the values of $r_3$ and $s_1$ are then forced. Let $\mathcal{B}_1'(a, b, c)$ be the pairs of vectors $\{\vec{r}, \vec{s}\}$ for which there are vectors $\{\vec{r}, \vec{s}, \vec{t}, \vec{u}\} \in \mathcal{B}_1'$, with in addition

$$r_1 = a, \quad r_2 = b, \quad s_3 = b.$$ 

By the triangle inequality, we can estimate

$$\|\text{Prod}(\mathcal{B}_1')\|_p \lesssim \sqrt{p} \cdot n^{5/2} \sup_{a, b, c} \|\text{Prod}(\mathcal{B}_1'(a, b, c))\|_p.$$ 

Now, among the pairs of vectors in $\mathcal{B}_1'(a, b, c)$ have only one free parameter, which can be taken to be the maximum in the first coordinate. Thus, by the Littlewood Paley inequality we see that

$$\|\text{Prod}(\mathcal{B}_1')\|_p \lesssim p n^3.$$ 

The analysis of the other possible forms of the collections $\mathcal{B}_j'$ proceeds along similar lines. We omit the details.

\[\Box\]

There is another corollary to the proof above required at a later stage of the proof. For an integer $a$, let $\mathcal{B}_a(4) \subset \mathbb{H}^4$ be four tuples of distinct vectors $\{\vec{r}, \vec{s}, \vec{t}, \vec{u}\}$ for which (i) $r_1 = s_1$ and $t_1 = u_1$; and (ii) in the second coordinate we have $r_2 = t_2 = a$; and (iii) two of the four vectors agree in the third coordinate.

1.7.16. L**E**MMA. For any integer $a$, and subset $\mathcal{B} \subset \mathcal{B}_a(4)$ we have

(1.7.17) $$\|\text{Prod}(\mathcal{B})\|_p \lesssim p n^{5/2}.$$ 

Moreover, for $\mathcal{B}_a \subset \mathcal{B}(4) \cap (A_s \times A_t)^2$, for any choice of $0 \leq s \neq t \leq q$, we have

(1.7.18) $$\|\text{Prod}(\mathcal{B})\|_p \leq c p n^{5/2} q^{-2}.$$ 

The point of this estimate is that we reduce the number of parameters of $\mathcal{B}(4)$ by one, and gain a full power of $n$ in the size of the $L^p$ norm, as compared to the estimate in (1.7.9).

**Proof.** In the proof of Lemma 1.7.7, in the analysis of the terms $\mathcal{B}_3$ and $\mathcal{B}_4$ we used the triangle inequality over the term $b = r_2 = t_2$. Treating this coordinate as fixed, we gain a term $n^{-1}$ in the previous proof, hence proving the Lemma above. The additional powers of $q$ are obtained by using the fact that the first coordinates can only vary over a set of size $\approx n/q$.

\[\Box\]

A further sub-case of the inequality (1.7.3) demands attention. Using the notation of Lemma 1.7.2, let

(1.7.19) $$\mathcal{C}_{2, b} := \{(\vec{r}_1, \vec{r}_2) \in \mathcal{C}_2 : r_{1,1} = b\}, \quad 1 \leq a \leq n.$$
Thus, this collection consists of pairs of distinct vectors, with a coincidence in the second coordinate, and the first coordinate of \( r'_1 \) is fixed. Note that these collections of variables have two free parameters. At \( L^2 \) we find a 1/4 gain over the ‘naive’ estimate.

1.7.20. **Lemma.** For any \( b \) and any subset \( C \subset C_{2,b} \) we have the estimates

\[
\|\text{Prod}(C)\|_p \leq p \cdot n^{5/4}, \quad 2 \leq p < \infty .
\]

Moreover, if \( C \subset A_s \times A_t \), for any choice of \( 0 \leq s \neq t \leq q \), we have

\[
\|\text{Prod}(C)\|_p \leq p \cdot n^{5/4} q^{-1/2}, \quad 2 \leq p < \infty .
\]

**Proof.** As in the proof of Lemma 1.7.2, we begin with the case \( p = 2 \). Observe that

\[
\|\text{Prod}(C)\|_2^2 = \mathbb{E} \text{Prod}(B),
\]

where \( B = C_{2,b} \times C_{2,b} \cap B_b(4) \), with the last collection defined in Lemma 1.7.16. Therefore, the Lemma in this case follows from that Lemma.

More generally, no pair of vectors in \( C_{2,b}(2) \) can have a coincidence in the third coordinate, so we can use the Littlewood Paley inequalities in that coordinate to estimate

\[
\|\text{Prod}(C)\|_p \leq \sqrt{p} \left[ \sum_c \left| \sum_{(r'_1, r'_2) \in C} f_{r'_1} \cdot f_{r'_2} \right|^2 \right]^{1/2}.
\]

Observe that

\[
\sum_c \left| \sum_{(r'_1, r'_2) \in C} f_{r'_1} \cdot f_{r'_2} \right|^2 = |C| + \sum_{i<j \in \{1,2,3,4\}} \text{Prod}(C_{i,j}) + \text{Prod}(B).
\]

Similar to before, we define the collections \( C_{i,j} \) as follows.

\[
C_{i,j} := \{(r'_1, r'_2, r'_3, r'_4) \in C \times C : r'_i = r'_j \}, \text{ and the other two vectors are distinct}\]

In this case, observe that four of these collections are empty, namely

\[
C_{1,2} = C_{2,3} = C_{1,4} = C_{2,4} = 0 .
\]

The only non-empty collection is \( C_{1,3} \). Yet, in \( C_{1,3} \), the vectors \( r'_2 \) and \( r'_3 \) have a coincidence in the second coordinate. Thus, Lemma 1.7.2 applies to \( C_{1,3} \), so that we have the estimate

\[
\|\text{Prod}(C_{1,3})\|_p \leq p^{5/4} n^{7/4} .
\]

Let us prove (1.7.21). Combining these observations with (1.7.23) and Lemma 1.7.16 we see that

\[
p^{-1/2}\|\text{Prod}(C)\|_p \leq n + \|\text{Prod}(C_{1,3})\|_p^{1/2} + \|\text{Prod}(B)\|_p^{1/2}
\]

\[
\leq n + p^{5/8} n^{7/8} + p^{1/2} n^{5/4} .
\]
Concerning the right hand side, note that for $2 < p < n^3$, we have $p^{5/8}n^{7/8} < p^{1/2}n^{5/4}$. Hence we have proved

$$\|\text{Prod}(C)\|_p \leq pn^{5/4}, \quad 1 < p < n^3.$$ 

Yet, for $p \geq n$ the $L^p$ norm above is comparable to the $L^\infty$ norm, so we have finished the proof of (1.7.21).

The case of (1.7.22) is left to the reader. 

\[\Box\]

### 1.8. Norm Estimates Particular to the Hyperbolic Assumption

The result of Theorem 1.4.1 admits an improvement, which we state in the a form adapted to our Riesz product. These improvements are subtle consequences of the detailed information we have about the Beck Gain.

#### 1.8.1. Theorem

Using the notation of (1.6.2) and (1.6.3), we have this estimate, valid for all $1 \leq t \leq q$.

\[(1.8.2) \quad \|\rho F_t\|_p \leq \sqrt{p}, \quad 1 \leq p \leq cn^{1/3}.\]

As a consequence, we have the distributional estimate

\[(1.8.3) \quad \mathbb{P}(\rho G_t > x) \leq \exp(-cx^2), \quad x < cn^{1/6} .\]

Here $0 < c < 1$ is an absolute constant.

**Remark.** It is perhaps worth emphasizing that we do not need this Theorem to deduce our main result, Theorem 1.1.7 on the Small Ball Conjecture in three dimensions.\(^6\) Nevertheless, we will use the result above. And we find the proof to be a compelling application of the Beck Gain.

**Remark.** There are limits to validity to these kinds of inequalities: Recall that one has $\ell^\infty \simeq \ell^{\log N}$. Thus, for appropriate $F_t$ we would have

$$\|\rho F_t\|_\infty \simeq \|\rho F_t\|_{3n} \simeq n/\sqrt{q}.$$ 

Hence, the sub–gaussian bound above can’t hold for this range of $p$, unless $q \simeq n$, but then the sub–gaussian estimate is immediate.

**Proof.** Recall that

$$F_t = \sum_{\vec{r} \in A_t} f_{\vec{r}}.$$ 

where $A_t := \{\vec{r} \in \mathbb{H}_n : r_1 \in I_t\}$, and $I_t$ in an interval of integers of length $n/q$, so that $\#A_t \simeq \rho^2$, with $\rho$ defined in (1.6.2).

---

\(^6\)If one does not use the result above, a smaller value of $b = \frac{1}{3}$ is required.
Apply the Littlewood Paley inequality in the first coordinate. This results in the estimate
\[
\| \rho F_t \|_p \lesssim \sqrt{p} \left[ \sum_{s \in I_j} \left| \sum_{r: r_1 = s} f_r \right|^2 \right]^{1/2} \]
\[
\lesssim \sqrt{p} \left( 1 + \| \Gamma_t \|_{p/2} \right) \]
\[
\lesssim \sqrt{p} \left( 1 + \| \Gamma_t \|_{p/2} \right)
\]
\[
\Gamma_t := \rho^2 \sum_{\substack{r \in A_t \atop r_1 \neq s_1}} f_r \cdot f_s.
\]

Of course the terms $\Gamma_t$ are controlled by the estimate in (1.7.4). In particular, we have
\[
\| \Gamma_t \|_p \leq C p^{3/2} n^{-1/2}.
\]
Hence (1.8.2) follows.

The second distributional inequality is a well known consequence of the norm inequality. Namely, one has the inequality below, valid for all $x$:
\[
\mathbb{P}(\rho F_t > x) \leq C p^{p/2} x^{-p}, \quad 1 \leq p \leq cn^{1/3}.
\]
If $x$ is as in (1.8.3), we can take $p \approx x^2$ to prove the claimed exponential squared bound. \ □

Remark. The proof above does permit better than ‘naive’ estimates for $\| \rho F_t \|_p$ for a range of $p > n^{1/3}$. The estimate we have is
\[
\| \rho F_t \|_p \leq \min\{p, \sqrt{p(1 + p^{3/2} n^{-1/2})}\}.
\]
The first estimate is from Theorem 1.4.1 while the second estimate is from the proof above. The minimum will be the second estimate provided $p \leq n^{1/2}$. Thus, for $n^{1/3} < x < n^{1/2}$ one can achieve an estimate that is better than from that of Theorem 1.4.1.

We now prove a central estimate of the proof.

1.8.5. Lemma. The estimate (1.6.10) holds. Moreover, we have
\[
\sup_{V \subset \{1, \ldots, q\}} \mathbb{E} \prod_{v \in V} (1 + \widetilde{\rho} F_t)^2 \lesssim \exp(a' q^{2b}).
\]
Here, $\widetilde{\rho}$ is as in (1.6.2), and $a'$ is a fixed constant times $0 < a < 1$, the small constant that enters into the definition of $\tilde{\rho}$.

Remark. A conditional expectation argument is essential to this proof. This Lemma is also proved in Beck’s paper. Yet, due to a more complicated Riesz product, the use of our line of reasoning was not available to him.
Proof. The supremum over \( V \) will be an immediate consequence of the proof below, and so we don’t address it specifically.

Let us give the initial, essential observation. We expand

\[
E \prod_{v=1}^q (1 + \tilde{\rho} F_i)^2 = E \prod_{v=1}^q (1 + 2\tilde{\rho} F_i + (\tilde{\rho} F_i)^2).
\]

Hold the \( x_2 \) and \( x_3 \) coordinates fixed, and let \( \mathcal{F} \) be the sigma field generated by \( F_1, \ldots, F_{q-1} \). We have

\[
E (1 + 2\tilde{\rho} F_q + (\tilde{\rho} F_q)^2 \mid \mathcal{F}) = 1 + E((\tilde{\rho} F_q)^2 \mid \mathcal{F})
\]

\[
= 1 + a^2 q^{2h-1} + \tilde{\rho}^2 \Gamma_q,
\]

(1.8.7)

where \( \Gamma_t := \sum_{\vec{r} \neq \vec{s} \in A_t} f_{\vec{r}} \cdot f_{\vec{s}} \).

Then, we see that

\[
E \prod_{v=1}^q (1 + 2\tilde{\rho} F_i + (\tilde{\rho} F_i)^2) = E \left\{ \prod_{v=1}^{q-1} (1 + 2\tilde{\rho} F_i + (\tilde{\rho} F_i)^2) \times E(1 + 2\tilde{\rho} F_i + (\tilde{\rho} F_i)^2 \mid \mathcal{F}) \right\}
\]

(1.8.8)

\[
\leq (1 + a^2 q^{2h-1}) E \prod_{v=1}^q (1 + 2\tilde{\rho} F_i + (\tilde{\rho} F_i)^2)
\]

(1.8.9)

\[
+ \left| E \prod_{v=1}^q (1 + 2\tilde{\rho} F_i + (\tilde{\rho} F_i)^2) \cdot \tilde{\rho}^2 \Gamma_q \right|
\]

This is the main observation: one should induct on (1.8.8), while treating the term in (1.8.9) as an error, as the ‘Beck Gain’ estimate (1.7.4) applies to it.

Let us set up notation to implement this line of approach. Set

\[
N(V; r) := \left\| \prod_{v=1}^{V} (1 + \tilde{\rho} F_i) \right\|_r, \quad V = 1, \ldots, q.
\]

We will use the trivial inequality available from the exponential moments

\[
N(V; 4) \leq \prod_{v=1}^{V} \|1 + \tilde{\rho} F_i\|_4 V
\]

\[
\leq (1 + Cq^{b-1/2} V)^V
\]

\[
\leq (Cq)^{Cq}.
\]

This of course is a terrible estimate, but we now use interpolation, noting that

\[
N(V; 2(1 - 1/q)^{-1}) \leq N(V; 2)^{1-1/q} \cdot N(V; 4)^{1/q}.
\]

(1.8.10)
We see that (1.8.8), (1.8.9) and (1.8.10) give us the inequality
\[ N(V + 1; 2) \leq (1 + a^2q^{2b-1})^{1/2}N(V; 2) + C \cdot N(V; 2(1 - 1/q)^{-1}) \cdot \|\tilde{\rho}^2\Gamma_q\|_q \]
\[ \leq (1 + a^2q^{2b-1})^{1/2}N(V; 2) + CN(V; 2)^{1-1/q} \cdot N(V; 4)^{1/q} \|\tilde{\rho}^2\Gamma_q\|_q \]
\[ \leq (1 + a^2q^{2b-1})^{1/2}N(V; 2) + Cq^Cn^{-1/2}N(V; 2)^{1-1/q} \cdot \|\tilde{\rho}^2\Gamma_q\|_q \]
(1.8.11)
In the last line we have used the inequality (1.7.4).

Of course we only apply this as long as \(N(V; 2) \geq 1\). Assuming this is true for all \(V \geq 1\), we see that
\[ \prod_{j \in V} r_j \leq (1 + a^2q^{2b-1} + Cq^Cn^{-1/2})q \]
\[ \leq e^{a'q^{2b}}. \]
Here of course we need \(Cq^Cn^{-1/2} \leq aq^{2b-1}\), which we certainly have for large \(n\). □

1.9. The Beck Gain

Let us state the main result of this section. Given \(V \subset \{1, \ldots, q\}\) let
\[ \text{NSD}(V) := \{(r_j' : j \in V) \in \times_{j \in V} A_j \mid \text{for each } j \in V, \text{there is a choice of } j' \in V - \{j\} \text{ and } \ell = 2, 3 \text{ so that } r_{j,\ell} = r_{j',\ell}' \}. \]
That is, we take tuples of \(r\) vectors, indexed by \(V\), requiring that each \(r_j\) be in a coincidence. Such sums admit a favorable estimate on their \(L^2\) norms.

1.9.1. Theorem. [The Beck Gain.] There are positive constants \(C_0, C_1, C_2, C_3, \eta\) for which we have the estimate
\[ p^{\|\prod(\text{NSD}(V))\|_p} \leq [C_0|V|^{C_1p^{C_2}q^{C_3}n^{-\eta}}]^{V}, \quad V \subset \{1, \ldots, q\}. \]

Remark. The novelty in this estimate is that we find that (a) the gain is proportional to the number of vertices, and (b) the gain also holds in \(L^p\) norms. In application, \(p \leq q^{2b} = \sqrt{q} \approx n^{\epsilon'}, \) so the polynomial growth in \(p\) and in \(q\) is acceptable to us. Beck [4] found a gain in \(L^2\) norm of order \(n^{-1/4}\), for all \(V\). Such a small gain of course forces a much shorter Riesz product.

Remark. It is disappointing that we cannot identify a reasonable value of \(\eta > 0\), which is in large measure, the amount of the gain. Yet, the goal of this proof is to have a relatively simple method of proof. Obviously, a finer understanding of this estimate, among other issues, will be central to future progress on the range of questions discussed in these notes.

The proof of this Theorem requires a careful analysis of the variety of ways that a product can fail to be strongly distinct. That is, we need to understand the variety of ways that coincidences can arise, and how coincidences can contribute to a smaller norm.
It is important at the outset to recognize that patterns of coincidences can be quite complex, a point best illustrated by a few examples of such patterns. Consider the specific product

\[(1.9.3) \prod_{j=1}^{7} \sum_{\vec{r}_j \in A_j} f_{\vec{r}_j},\]

and the ways that summands in such a product could fail to be strongly distinct. One could consider those terms in which the first three choices of \(\vec{r}_j\) agree in the second coordinate:

\[r_{1,2} = r_{2,2} = r_{3,2},\]

while imposing no restriction on the remaining four vectors \(\vec{r}_4, \vec{r}_5, \vec{r}_6, \vec{r}_7\). Note that

\[(1.9.4) \prod_{j=1}^{7} \sum_{\vec{r}_j \in A_j} f_{\vec{r}_j} = \left(\prod_{j=1}^{3} \sum_{\vec{r}_j \in A_j, r_{1,2}=r_{2,2}=r_{3,2}} f_{\vec{r}_j}\right) \cdot \left(\prod_{j=4}^{7} \sum_{\vec{r}_j \in A_j, r_{1,2}=r_{2,2}=r_{3,2}} f_{\vec{r}_j}\right)\]

That is, we have a product of terms, with a ‘simple’ coincidence in the first term, and no restriction on the sum in the second. In this instance, we would take \(V = \{1, 2, 3\}\).

Similarly, a pattern of coincidences could be

\[r_{1,2} = r_{2,2} = r_{3,2}, \quad r_{4,3} = r_{5,3} = r_{6,3} = r_{7,3}.\]

As in the first case, the corresponding sum would break into a product. And the \(L^1\) norm would be substantially smaller, due to the presence of two sets of ‘simple’ coincidences.

Yet, one could have a more complicated set of coincidences, such as

\[r_{1,2} = r_{2,2} = r_{3,2}, \quad r_{1,3} = r_{4,3} = r_{5,3}, \quad r_{2,3} = r_{6,3} = r_{7,3}.\]

Here, the first and second vectors are both involved in two distinct sets of coincidences. This case, as it turns out, are also substantially smaller in \(L^1\) norm than the first case, due to the ‘overlapping’ coincidences.

Following Beck, we will use the language of Graph Theory to describe these general patterns of coincidences.

Before passing to the general description of these results, the reader should keep forefront in their minds these points:

- Coincidences can only occur in the second and third coordinates, due to the specific way we form our products.
- Our graphs will have as vertices the integers \(j \in \{1, 2, \ldots, q\}\), the index of the product in (1.9.3).
- Edges in the graph represent a coincidence between two vectors. Edges come in two different types, or colors, associated to coincidence in the second or third coordinates.
- Equality is transitive, so the edges in e.g. the second coordinate will naturally decompose into cliques.
As we work in three dimensions, a clique in the second coordinate, and a clique in the third coordinate can contain at most one common vertex, as two common vertices would imply that our product contains two equal vectors. This case is specifically excluded from our consideration.

The presence of an edge will mean that we enforce a coincidence of that type in the products we consider. The absence of an edge will mean that no such condition is assumed—not that equality is forbidden. This will permit product formulas such as (1.9.4) above hold.

A graph is naturally associated to sums of products of \( r \) functions. We seek effective \( L^p \) norms on these sums. Larger cliques, and more overlapping cliques serve to reduce the number of parameters, and give smaller norms.

**Graph Theory Nomenclature.** We adopt familiar nomenclature from Graph Theory.\(^7\) The class of graphs that we are interested satisfy particular properties. A graph \( G \) is the triple of \((V(G), E_2, E_3)\), of the vertex set \( V(G) \subset \{1, \ldots, q\} \), and edge sets \( E_2 \) and \( E_3 \), of color \( 2 \) and \( 3 \) respectively. Edge sets are are subsets of

\[
E_j \subset V(G) \times V(G) - \{(k, k) \mid k \in V(G)\}.
\]

Edges are symmetric, thus if \((v, v') \in E_j\) then necessarily \((v', v) \in E_j\).

A clique of color \( j \) is a maximal subset \( Q \subset V(G) \) such that for all \( v \neq v' \in Q \) we have \((v, v') \in E_j\). By maximality, we mean that no strictly larger set of vertices \( Q' \supset Q \) satisfies this condition.

Call a graph \( G \) admissible iff

- The edges sets, in both colors, decompose into a union of cliques.
- Any two cliques \( Q_2 \) in color \( 2 \) and clique \( Q_3 \) in color \( 3 \) can contain at most one common vertex.
- Every vertex is in at least one clique.

A graph \( G \) is connected iff for any two vertices in the graph, there is a path that connects them. A path in the graph \( G \) is a sequence of vertices \( v_1, \ldots, v_k \) with an edge of either color, spanning adjacent vertices, that is \((v_j, v_{j+1}) \in E_2 \cup E_3\).

**Reduction to Admissible Graphs.** Given admissible graph \( G \) on vertices \( V \), we set \( X(G) \) to be those tuples of \( r \) vectors

\[
[r_v : v \in V] \in \prod_{v \in V} A_v
\]

so that if \((v, v')\) is an edge of color \( j \) in \( G \), then \( r_{v,j} = r_{v',j} \).

We will prove the Lemma below in the following two sections.

---

\(^7\)There is no graph theoretical fact that we need, rather the use of this language is just a convenient way to do some bookkeeping.
1.9.5. **Lemma.** For an admissible graph $G$ on vertices $V$ we have the estimate below for positive, finite constants $C_0, C_1, C_2, C_3$:

\[
\rho^{|V|}||\text{Prod}(X(G))||_1 \leq [C_0 |V|^{C_1 p + C_2 q - C_3} n^{-\eta}]^{|V|}, \quad 2 < p < \infty.
\]

Let us give the proof of Theorem 1.9.1 assuming this Lemma. Our tool is the Inclusion Exclusion Principle, but to apply it we need additional concepts.

Given two admissible graphs $G_1, G_2$ on the same vertex set $V$, let $G_1 \wedge G_2$ be the smallest admissible graph which contains all the edges in $G_1$ and in $G_2$. By smallest, we mean the graph with the fewest number of edges; and such a graph may not be defined, in which case we take $G_1 \wedge G_2$ to be undefined. We recursively define $G_1 \wedge \cdots \wedge G_k := (G_1 \wedge \cdots \wedge G_{k-1}) \wedge G_k$.

This wedge product is associative.

Let $G_0$ be the set admissible graphs on $V$ which are not of the form $G_1 \wedge G_2$ for admissible $G_1, G_2$. These are the ‘prime’ graphs. (If $V$ is of cardinality 2 or 3, every graph is prime.) For instance, in the case of $V = \{v_1, v_2, v_3, v_4\}$ the two graphs below are prime.

\[
\begin{array}{cccc}
\square & \square & \square & \square \\
\bullet & = & \bullet & = \quad \text{and} \quad \bullet & = & \bullet & = & \bullet
\end{array}
\]

The only difference between the two is the ordering of the vertices in the top row. There are no coincidences in the third row, and the first row, with the $\square$s, never has a coincidence. These two graphs are distinct, and clearly members of $G_0$. Note that their wedge product is

\[
\begin{array}{cccc}
\square & \square & \square & \square \\
\bullet & = & \bullet & = & \bullet & = \end{array}
\]

Now define $G_k$ to be those graphs which are equal to a wedge product $G_1 \wedge \cdots \wedge G_k$, with $G_j \in G_0$, and moreover, $k$ is the smallest integer for which this is true. Clearly, we only need to consider $k \leq q$.

Then, by the inclusion exclusion principle,

\[
(1.9.7) \quad \text{Prod}(\text{NSD}(V)) = \sum_{k=0}^{q} (-1)^k \sum_{G \in G_k} \text{Prod}(X(G)).
\]

The number of admissible graphs on a set of vertices $V$ is at most $2^{|V|!} |V|! < 2^{|V|} |V|^{|V|}$. So that using (1.9.6) clearly implies Theorem 1.9.1.

**Norm Estimates for Admissible Graphs.** We begin this section with a further reduction to connected admissible graphs. Let us write $G \in \text{BG}(C_0, C_1, C_2, C_3, \eta)$ if the estimates (1.9.6) holds. ('BG' for 'Beck Gain.') We need to see that all admissible graphs are in $\text{BG}(C_0, C_1, C_2, C_3, \eta)$ for non-negative, finite choices of the relevant constants.
1.9.8. Lemma. Let $C_0, C_1, C_2, C_3, \eta$ be non-negative constants. Suppose that $G$ is an admissible graph, and that it can be written as a union subgraphs $G_1, \ldots, G_k$ where all $G_j \in \text{BG}(C_0, C_1, C_2, C_3, \eta)$. Then,

$$G \in \text{BG}(C_0, C_1, C_2, C_2 + C_3, \eta).$$

With this Lemma, we will identify a small class of graphs for which we can verify the property (1.9.6) directly, and then appeal to this Lemma to deduce Theorem 1.9.1. Accordingly, we modify our notation. If $\mathcal{G}$ is a class of graphs, we write $\mathcal{G} \subset \text{BG}(\eta)$ if there are constants $C_0, C_1, C_2, C_3$ such that $\mathcal{G} \subset \text{BG}(C_0, C_1, C_2, C_3, \eta)$.

**Proof.** We then have by Proposition 1.9.9

$$\text{Prod}(X(G)) = \prod_{j=1}^{k} \text{Prod}(X(G_j)).$$

Using Hölder’s inequality, we can estimate

$$||\text{Prod}(X(G))||_p \leq \prod_{j=1}^{k} ||\text{Prod}(X(G_j))||_{kp}$$

$$\leq \prod_{j=1}^{k} [C_0(kp)^{C_1} q^{C_2} n^{-\eta}]^{|V_j|}$$

$$\leq [C_0 p^{C_1} q^{C_2 + C_3} n^{-\eta}]^{|V|}.$$ 

Here, we use the fact that since the graphs are non-empty, we necessarily have $k \leq q$. 

**1.9.9. Proposition.** Let $G_1, \ldots, G_p$ be admissible graphs on pairwise disjoint vertex sets $V_1, \ldots, V_p$. Extend these graphs in the natural way to a graph $G$ on the vertex set $V = \bigcup V_i$. Then, we have

$$\text{Prod}(X(G)) = \prod_{t=1}^{p} \text{Prod}(X(G_i)).$$

**Connected Graphs Have the Beck Gain.** Let $\mathcal{G}_{\text{connected}}$ be the collection of of all admissible connected graphs on $V \subset \{1, \ldots, q\}$.

1.9.10. Lemma. We have $\mathcal{G}_{\text{connected}} \subset \text{BG}(\frac{1}{13})$.

One can depict small examples of these graphs as follows.

\[ \square \square \square \square \square \square \square \square \square \square \] 
\[ \bullet = \bullet = \bullet = \bullet \] 
\[ \bullet = \bullet = \bullet = \bullet = \bullet \]

These are graphs on 2, 3 and 4 vertices respectively. We will have to pay special attention to the case of 2 and 3 vertices, as these cases are not amenable to the general procedure
we invoke below. It is important to observe that the first coordinates, represented by □
above, are necessarily distinct, and have the partial order inherited from the vertex set \( V \).
Namely, the vertex set \( V \subset \{1, \ldots, q\} \), and \( V \) inherits the order from the integers. By the
construction of our Riesz product, the first coordinates inherit this same order.

Unfortunately, even working with this class of admissible graphs, our proof is of an
ad hoc nature, and we won’t actually specify a value of \( \eta > 0 \) for which the Lemma above
holds.

General Remarks on Littlewood Paley Inequality. These remarks are essential to our analy-
isis of this lemma, and the Theorem we are proving. The vertex set \( V \) is a subset of \( \{1, \ldots, q\} \)
and it inherits an order from that set. Moreover, the tuples of \( r \) vectors do as well. Namely,
writing
\[
V = \{v_1 < \cdots < v_{\ell}\},
\]
for \( \{r_1, \ldots, r_\ell\} \in X(G) \), we have, by construction, \( r_1 < \cdots < r_{\ell,1} \). This since \( r_{m,1} \in I_{m} \), where
\( I_m \) is the increases sequence of intervals of length equal to \( n/q \) that partition \( \{1, \ldots, n\} \).

Continuing this line of thought, we see that there is a natural way to apply the Little-
wood Paley inequalities. For integer \( b_\ell \in I_\ell \), let \( X(G; b_\ell) \) be the tuple of \( r \) vectors \( \{\tilde{r}_1, \ldots, \tilde{r}_\ell\} \)
such that \( \tilde{r}_{\ell,1} = b_\ell \). We have
\[
\|\text{Prod}(X(G))\|_p \leq \sqrt{\sum_{b_\ell \in I_\ell} \left| \text{Prod}(X(G; b_\ell)) \right|^2}^{1/2}.
\]
It is tempting to continue this procedure, by applying the Littlewood Paley inequality
again to the vertex \( v_{\ell-1} \). Yet—and this in an important point—due to the nature of \( r \)
functions, this option is blocked to us. The vertex \( v_\ell \) is in at least one clique \( Q \) of, say, color 2. We could choose a value \( c_Q \) for that clique, thereby specifying all coordinates of the vector \( \tilde{r}_\ell \). Set \( X(G; b_\ell; c_Q) \) be the tuple of \( r \) vectors \( \{\tilde{r}_1, \ldots, \tilde{r}_{\ell-1}\} \) such that
\[
\{\tilde{r}_1, \ldots, \tilde{r}_{\ell-1}, (a_\ell, b_\ell, n - a_\ell - b_\ell)\} \in X(G; a_\ell).
\]
Here, \( X(G; b_\ell; c_Q) \) consists of tuples of length \( \ell - 1 \), since the vector \( \tilde{r}_\ell \) is completely specified. Thus, we see that
\[
\|\text{Prod}(X(G))\|_p \leq n \sup_{a_\ell, c_Q} \left\| \sum_{b_\ell} \text{Prod}(X(G; b_\ell; c_Q))^2 \right\|^{1/2}_p.
\]
At this point, the (Hilbert space) Littlewood Paley inequalities will again apply.

We will refer to the notation above. Keep in mind that \( \hat{b} \) is for the coordinates specified
by a Littlewood Paley inequality; \( \hat{c} \) are for the coordinates in a coincidence that we use the
triangle inequality on. We shall return to these themes momentarily.

Proof of Lemma 1.9.10. We begin the proof with a discussion of the case of two and
three vertices, which will not be susceptible to the general methods related to the Little-
wood Paley inequality outlined above.
1. THE SMALL BALL PROBLEM

The Case of Two Vertices. Notice that if \( G \) consists of only two vertices, the relevant estimate is (1.7.4). Namely, we have

\[
\|\text{Prod}(X(G))\|_p \leq C p^{3/2} n^{3/2} q^{-1}.
\]

Equivalently, \( G \in \text{BG}(C_0, 3/4, 0, 1/4) \).

The Case of Three Vertices. The case of \( G \in \mathcal{G}_2 \) having three vertices depends critically on the same phenomena behind the Beck Gain for graphs on two vertices. We will deduce this case as a corollary to the case of two vertices.

There are two distinct sub-cases. The more delicate of the two cases is as follows. The graph is depicted as

\[
\begin{array}{ccc}
\bullet & = & \bullet \\
\square & & \square \\
\end{array}
\]

where \( v_1 < v_2 < v_3 \). (The case of \( v_2 < v_1 < v_3 \) is entirely the same, and we don’t discuss it directly.)

By our general remarks on the Littlewood Paley inequality, this inequality applies in the first coordinate, to the vertex \( v_3 \). Using the notation in (1.9.11), we have

\[
\|\text{Prod}(X(G))\|_p \leq \sqrt{p} \left[ \sum_{b_3 \in \mathcal{I}_{v_3}} \|\text{Prod}(X(G; b_3))\|_p^2 \right]^{1/2}.
\]

The vectors \( v_2 \) and \( v_3 \) have a coincidence in the third coordinate. Therefore, we specify the value of the coincidence to be \( c_3 \) and estimate

\[
\|\text{Prod}(X(G))\|_p \leq \sqrt{p} \cdot n^{3/2} \cdot \sup_{b_3, c_3} \|\text{Prod}(X(G; b_3; c_3))\|_p.
\]

Recall that \( X(G; a_3; b_3) \) consists only of pairs of vectors. This graph can be depicted as

\[
\begin{array}{cc}
\bullet & = \\
\square & \\
\end{array}
\]

But this is the case considered in (1.7.22). From that inequality, we see that we have the estimate

\[
\|\text{Prod}(X(G; b_3; c_3))\|_p \leq \sqrt{p} n^{3/4} q^{-1/2}.
\]

Therefore, from (1.9.14), we see that

\[
\|\text{Prod}(X(G))\|_p \leq p^{3/2} n^{9/4} q^{-3/2}.
\]

Recall that the point of comparison is to \( n^3 q^{-3/2} \), and the estimate above is smaller by \( n^{-1/4} \). Thus the class of graphs given by (1.9.13) are contained in \( \text{BG}(\frac{1}{12} - \epsilon) \).
1.9. THE BECK GAIN

The other case is when the graph can be depicted by

\[ \begin{array}{ccc}
& v_1 & v_3 \\
v_2 & \square & \square \\
\bullet & \bullet & = \\
\end{array} \]

where \(v_3\), the maximal index is in both cliques. This case is much easier, as one application of the Littlewood Paley inequality, and the triangle inequality will determine the value of both cliques. It is very easy to see that this class of graphs is in \(BG(1/6)\), and the details are omitted. Hence the discussion graphs on three vertices, with all cliques of size 2 is complete.

A General Estimate. We now present a general recursive estimate for the \(L^p\) norm of \(\text{Prod}(X(G))\), assuming that \(G\) is a connected graph on at least four vertices. Write \(V\) as

\[ V = \{v_1 < \cdots < v_\ell\} . \]

The estimate is obtained recursively. Along the way we will construct two disjoint subsets \(V_{3/2}, V_{1/2} \subset V\). \(V_{3/2}\) will be the vertices to which we apply both the Littlewood Paley and triangle inequalities, thus these vertices contribute \(n^{3/2} \sqrt{q^{1/2}}\) to our estimate. \(V_{1/2}\) will be the vertices to which we apply only the Littlewood Paley inequality, thus these vertices contribute \((n/q)^{1/2}\) to our estimate. Those vertices not in \(V_{3/2} \cup V_{1/2}\) will be those which are determined by earlier steps in the procedure. They contribute nothing to our estimate. In estimating an \(L^p\) norm, the power of \(p\) is one-half of the number of applications of the Littlewood Paley inequality, namely \(\frac{1}{2\ell}(V_{3/2} \cup V_{1/2})\).

The purpose of these considerations is to prove the estimate

\[ \|\text{Prod}(X(G))\|_p \leq (C \sqrt{p})^{|V_{3/2}|+|V_{1/2}|} (n/q)^{|V_{3/2}|} n^{|V_{3/2}|} . \tag{1.9.16} \]

Initialize

\[ V_{3/2} \leftarrow \emptyset, \quad V_{1/2} \leftarrow \emptyset, \quad Q_{\text{fixed}} \leftarrow \emptyset . \]

The last collection consists of those cliques which are specified by earlier stages of the argument.

At each stage, we will have an estimate for the form

\[ \|\text{Prod}(X(G))\|_p \leq (C \sqrt{p})^{|V_{3/2}|+|V_{1/2}|} n^{|V_{3/2}|} \]

\[ \times \sup_{\vec{c} \in [1, \ldots, n]^{Q_{\text{fixed}}}} \left\| \sum_{\vec{b} \in [1, \ldots, n]^{V_{3/2} \cup V_{1/2}}} \text{Prod}(X(G; \vec{b}; \vec{c}))^2 \right\|^{1/2}_p . \tag{1.9.17} \]

Here, \(X(G; \vec{b}; \vec{c})\) denotes those tuples \(\{\vec{r}_v : v \in V\}\) such that if \(v \in V_{3/2} \cup V_{1/2}\) then, \(r_{v,1} = b_v\). And if \(v\) is in a clique \(Q \in Q_{\text{fixed}}\) of color \(t\), then \(r_{v,t} = c_Q\).

Base Case of the Recursion. We update \(V_{3/2} \leftarrow \{v_\ell\}\), since it is the maximal element. We update \(Q_{\text{fixed}}\) to those cliques which contain \(v_\ell\). Then (1.9.17) is a consequence of (1.9.12).

Recursive Case. At this point, we have the datum \(V_{3/2}, V_{1/2}\), and \(Q_{\text{fixed}}\). We also have datum \(\vec{b} \in [1, \ldots, n]^{V_{3/2} \cup V_{1/2}}, \text{ and } \vec{c} \in [1, \ldots, n]^{Q_{\text{fixed}}}\). Notice that this datum can completely
specify the \( r \) vectors associated to vertices not in \( V_{3/2} \cup V_{1/2} \)—think of a vertex that is in two cliques in \( Q_{\text{fixed}} \). The recursion stops if every vertex \( v_k \) is determined by this datum. Otherwise, let \( k \) to be the largest integer such that \( \vec{r}_{v_k} \) is not determined by this datum. If no clique in \( Q_{\text{fixed}} \) contains \( v_k \), update

\[
V_{3/2} \leftarrow V_{3/2} \cup \{v_k\},
\]

and update \( Q_{\text{fixed}} \) to include those cliques which contain \( v_k \). By application of the Littlewood Paley inequality and the triangle inequality, the estimate (1.9.17) continues to hold for these updated values.

If some clique in \( Q_{\text{fixed}} \) contains \( v_k \), then there can be exactly one clique \( Q_{v_k} \) which does, for otherwise \( \vec{r}_{v_k} \) would be completely specified by these two cliques. Update

\[
V_{1/2} \leftarrow V_{1/2} \cup \{v_k\},
\]

and update \( Q_{\text{fixed}} \) to include all cliques which contain \( v_k \). By application of the Littlewood Paley inequality and the triangle inequality, the estimate (1.9.17) continues to hold for these updated values.

Once the recursion stops the inequality (1.9.17) holds. But note that we necessarily have

\[
\text{Prod}(X(G; \vec{b}; \vec{c}))^2 \equiv 1,
\]
as all \( r \) vectors are completely determined by \( \vec{b} \) and \( \vec{c} \). Therefore, we have proven (1.9.16).

The Conclusion of the Proof. Since \( V_{3/2} \) and \( V_{1/2} \) are disjoint subsets of \( V \), we have proven the inequality

(1.9.18)

\[
\rho^{\text{Vol}[\text{Prod}(X(G))]_p} \leq (C \sqrt{p} n^{2|V_{3/2}|+\frac{1}{2}|V_{1/2}|} - |V|).
\]

And the remaining analysis concerns the exponent on \( n \) above, namely we should see that

(1.9.19)

\[
|V|^{-1}\left[\frac{3}{2}|V_{3/2}| + \frac{1}{2}|V_{1/2}| - |V|\right] \leq -\eta,
\]

for a fixed positive choice of \( \eta \), and all connected graphs \( G \) on at least four vertices. We would conclude that this collection of graphs is in \( BG(\eta) \).

It would be helpful to consider a couple of simple cases. Consider the graph on five vertices

\[
v_1 \quad v_4 \quad v_2 \quad v_5 \quad v_3
\]

(1.9.20)

\[
\bullet = \bullet \quad \bullet = \bullet \quad \bullet = \bullet
\]

Note that we specify a particular order on the vertices in the top row, and indicate the membership of each vertex in \( V_{3/2}, V_{1/2} \), and in \( V_0 := V - V_{3/2} - V_{1/2} \). Note that the zeros
at $v_4$ and $v_5$ are forced. Consider the graph on six vertices

$$\begin{array}{cccccc}
v_1 & v_6 & v_2 & v_5 & v_3 & v_4 \\
\frac{3}{2} & 0 & \frac{3}{2} & 0 & \frac{3}{2} & \frac{1}{2}
\end{array}$$

(1.9.21)

Here, there is one vertex in $V_{1/2}$, but of course all vertices in $V_{1/2}$ contribute to the Beck Gain. But the reader should keep in mind that the graphs can in general have a much more complicated structure than these two linear examples.

The extremal cases in the estimate (1.9.19) are those cases in which $V_{3/2}$ is as large as possible. To continue, we note another formula. Let $E(G)$ be the total number of edges in the graph $G$, and let $E(v)$ be the number of edges in $G$ with one endpoint of the edge being $v$.

For $v \in V_{3/2} \cup V_{1/2}$, let $F(v)$ be the number of edges which are specified upon the selection of that vertex in our recursive procedure. It is clear that we have $E(v) = F(v)$ if $v \in V_{3/2}$. But also,

$$\sum_{v \in V_{3/2} \cup V_{1/2}} F(v) = E(G).$$

It follows that to maximize the cardinality of $V_{3/2}$, those vertices must be in small cliques. There are two different classes of graphs which are extremal with respect to these criteria.

The first extremal class consists of graphs $G$ with all cliques being of size 2, and the number of cliques is $|V| + 1$, that is the graphs are like in (1.9.20) and (1.9.21). For such graphs, $|V_{3/2}| \leq \left\lfloor \frac{1}{2} |V| \right\rfloor$, and if the value is maximal then $V_{1/2}$ is either 0 if $|V|$ is odd, and 1 if $|V|$ is even. It is straightforward to see that the maximum of (1.9.19) occurs at $|V| = 5$, and is $-\frac{1}{10}$. Here, it is vital that we have already discussed the case of two and three vertices!

The second class are graphs on an even number of vertices, with half the vertices in a clique $Q$, and each vertex $v \in Q$ is in one other clique of size 2. One can depict the graph as

$$\begin{array}{ccccccc}
v_1 & v_2 & v_3 & v_4 & v_5 & v_6 \\
* & * & * & a & b & c
\end{array}$$

The vertices are written in increasing order: $v_1 < v_2 < v_3 < v_4 < v_5 < v_6$. Note that $v_1, v_2, v_3$ form a single clique of color 2. There are three additional cliques of size 2, all of color 3. They are $\{v_j, v_{j+3}\}$ for $j = 1, 2, 3$. For such a graph, it is clear that $|V_{3/2}| = \frac{1}{2} |V|$, and $|V_{1/2}| = 1$.\footnote{If for example the maximal vertex $v_6$ where in the clique of size 3, our algorithm then predicts a smaller estimate for the such a graph.}

The term (1.9.19) behaves exactly like the first class of extremal graphs on an even number of vertices. Our proof is complete. \hfill $\square$
CHAPTER 2

Irregularities of Distributions

2.1. Discrepancy

We outline the Discrepancy Theory, highlighting its relevance to the Small Ball Problem. In $d$ dimensions, one takes $\mathcal{A}_N$ to be $N$ points in the unit cube, and considers the function

$$D_N(x) = \# \mathcal{A}_N \cap [\vec{0}, \vec{x}) - N |[\vec{0}, \vec{x})|$$

Here, $[\vec{0}, \vec{x}) = \prod_{j=1}^d[0, x_j)$, that is a rectangle with antipodal corners being $\vec{0}$ and $\vec{x}$. We will typically suppress the dependence upon the selection of points $\mathcal{A}_N$. A set of points will be well distributed if this function is small in some appropriate function space. Thus, it of interest to understand the ‘min–max’ function

$$\inf_{\mathcal{A}_N} \|D_N\|_{L^p([0,1]^d)}, \quad 0 < p \leq \infty.$$ 

For the purposes of this note, we will primarily be concerned with lower bounds for this quantity, with $1 \leq p \leq \infty$. Dimension will be held fixed, with $N$ large. Many variants of this question are interesting; interested readers is encouraged to consult one of the excellent references in this area.

It turns out that relevant norms of this function must tend to infinity, in dimensions 2 and higher. Using the basic facts of the next section, we can prove the Theorem below, which concatenates results of Roth [25] in the case of $p = 2$. Indeed the proof we give below is the ‘hyperbolic orthogonal function’ method he initiated; and Schmidt [29] for other values of $p$. The end point estimate below is a consequence of the method, and don’t seem to be as well known.

2.1.2. Theorem. For any collection of points $\mathcal{A}_N \subset [0,1]^d$, we have the estimates

$$\|D_N\|_p \gtrsim (\log N)^{(d-1)/2}$$

More particularly, we have the endpoint estimate

$$\|D_N\|_{L((\log L)^{(d-1)/2})} \gtrsim (\log N)^{(d-1)/2}$$

Proof. As is usual, the proof is by duality, following Roth [25], and we use the Haar function approach of Schmidt [28].

We stick to the hyperbolic setting, with the rationale that extremal point distributions, whatever they might be, must have about one point in any rectangle of volume about $2^{-n}$. 

For each \( \vec{r} \in \mathbb{H}_n \) construct the \( \vec{r} \) function \( f_{\vec{r}} \) as in Proposition 2.3.1, and set

\[
F := \sum_{\vec{r} \in \mathbb{H}_n} f_{\vec{r}}.
\]

By construction we have

\[
h^{d-1} \lesssim \langle D_N, F \rangle \leq \|D_N\|_2 \|F\|_2 \leq \|D_N\|_2 n^{(d-1)/2}.
\]

This prove (2.1.3) in the case of \( p = 2 \), and by extension to all \( p \geq 2 \). To finish the proof, recall that \( L(\log L)^{(d-1)/2} \) and \( \exp(L^2/(d-1)) \) are dual spaces, see § 3.1. Thus, we we should observe that

\[
\|F\|_{\exp(L^2/(d-1))} \lesssim n^{(d-1)/2}.
\]

But, the square function of \( F \)

\[
S(F) := \left[ \sum_{\vec{r} \in \mathbb{H}_n} |f_{\vec{r}}|^2 \right]^{1/2} \lesssim n^{(d-1)/2}.
\]

The last estimate is an \( L^\infty \) estimate. Therefore, by Theorem 1.4.1, we conclude that \( \|F\|_{\exp(L^2/(d-1))} \lesssim n^{(d-1)/2} \). This implies the \( L(\log L)^{(d-1)/2} \) endpoint estimate for \( D_N \) in (2.1.4).

While this last Theorem is quite adequate for \( L^p \), the endpoint cases of \( L^\infty \) and \( L^1 \) are not amenable to the same techniques, and the relevant fact is that the \( L^\infty \) bound should be larger. In dimension 2, the end point estimates are known. At \( L^\infty \), it is the Theorem of Schmidt [28].

2.1.5. Schmidt’s Theorem. We have the estimates below, valid for all collections \( \mathcal{A}_N \subset [0,1]^2 \):

\[
\|D_N\|_\infty \gtrsim \log N.
\]

We shall see that this is a rather precise analog of Talagrand’s theorem; the proof we give will share a great deal of similarity with the proof of Temlyakov we have described in § 1.3.

Let us comment that there is an interpolant between the result of Schmidt and the \( L^p \) results, provided one uses the scale of exponential Orlicz classes.

2.1.7. Theorem. We have the estimates below, valid for all collections \( \mathcal{A}_N \subset [0,1]^2 \):

\[
\|D_N\|_{\exp(L^p)} \gtrsim (\log N)^{1-1/p}, \quad 2 < p < \infty.
\]

In dimensions 3 and higher, there is the following improvement on J. Beck’s result [4], due to Lacey and Bilyk [1].

2.1.9. Theorem. There is a choice of 0 < \( \eta < \frac{1}{2} \) for which the following estimate holds for all collections \( \mathcal{A}_N \subset [0,1]^3 \):

\[
\|D_N\|_\infty \gtrsim (\log N)^{1+\eta}.
\]
Beck’s result is as above, with \((\log N)^\eta\) replaced by a doubly logarithmic term. There is no further result known about the Small Ball Problem, nor the Discrepancy Function in higher dimensions.\(^1\)

Halász established the \(L^1\) endpoint estimate for the Discrepancy function in two dimensions. Namely

**2.1.11. Halász’ Theorem.** For any collection of points \(A_N \subset [0, 1]^2\) of cardinality \(N\) we have

\[
\|D_N\|_1 \gtrsim \sqrt{\log N}.
\]

While the \(L^\infty\) case is in close analogy to the Small Ball Conjecture, this analogy breaks down in this case. We will give Halász’ proof of this result, as well as a new one, which is again a duality method, but the construction of the dual function is not by way of a Riesz product. See §2.6.

In the reverse direction, concerning point distributions with small Discrepancy function, the following is known.

**2.1.13. Theorem.** In dimension \(d\), there are point distributions \(A_N\) with

\[
\|D_N\|_p \lesssim (\log N)^{(d-1)/2}, \quad 0 < p < \infty.
\]

These constructions are delicate, and the product of significant effort over a period of decades. See especially Davenport [11], Roth [26, 27], and Chen [14]. These earlier constructions were random in nature; recently Chen and Skriganov [12, 13] found subtle deterministic constructions.

On the other hand, Schmidt’s result is sharp, for Halton [21] has constructed point sets with Discrepancy function of \(L^\infty\) norm that matches his lower bound.

**2.1.14. Halton’s Theorem.** For dimension \(d \geq 2\) there are point sets \(A_N\) with

\[
\|D_N\|_\infty \gtrsim (\log N)^{(d-1)/2}.
\]

The \(L^\infty\) Conjectures. In light of the close connection between the proof of the lower bounds in the \(L^\infty\) case and the Small Ball Conjecture, one suspects that an extra square root of \(n \approx \log N\) is all that should be obtainable at the end point estimate at \(L^\infty\) for the discrepancy function.

**2.2. Conjectures for Discrepancy**

**2.2.1. Hyperbolic \(\sup\) Norm Conjecture.** For all choices of \(N\) points \(A_N\) we have

\[
\|D_N\|_\infty \gtrsim (\log N)^{d/2}.
\]

\(^1\)The student of the literature will find an article published some years ago that claims an extension of Beck’s result to higher dimensions. While this paper can serve as a useful summary of Beck’s argument, an early critical Lemma in that paper is in error; a technique to repair the error is unknown to me.
What should be clear, in light of the sharpness of the Small Ball Conjecture, is that those who hold the conviction that this last conjecture falls short of the truth will necessarily seek a proof other than the hyperbolic one.

2.2.3. Sharpness of the Hyperbolic Sup Norm Conjecture. We have the estimate

\[ \inf_{\mathcal{R}_N} \|D_N\|_{\infty} \lesssim (\log N)^{d/2}. \]

In this paper, we emphasize the similarity in proof techniques in the Small Ball Problem and the Discrepancy problems. It would be of interest to establish some formal connection between these two problems.

The reader can consult the survey article by Temlyakov [37] for a discussion of the connection between the Discrepancy function in \( L^\infty \) and cubature formulas.

One suspects that Theorem 2.1.7 is sharp. (Compare to [16].)

2.2.5. Conjecture. In dimension 2, one has

\[ \min_{\mathcal{R}_N} \|D_N\|_{\exp(\log N)} \simeq (\log N)^{1-1/\alpha}, \quad 2 \leq \alpha < \infty. \]

The \( L^1 \) Conjecture. The other outstanding conjecture concerns the \( L^1 \) norm endpoint.

2.2.6. \( L^1 \) Norm Conjecture. In any dimension \( d \) one has the estimate

\[ \|D_N\|_1 \gtrsim (\log N)^{(d-1)/2}. \]

It appears that any improvement in the estimate (2.1.4), by e.g. replacing the logarithmic Orlicz space by one closer to \( L^1 \), will generate an interesting new proof technique.

The \( L^p \) Conjecture, for \( 0 < p < 1 \). One can ask about the size of the Discrepancy Function in \( L^p \), for \( 0 < p < 1 \). The absence of duality methods has prevented any progress towards this conjecture.

2.2.7. Conjecture. We have the estimate below, for all \( 0 < p < 1 \).

\[ \|D_N\|_p \gtrsim (\log N)^{(d-1)/2}. \]

Here, we indicate a result in this direction.

2.2.8. Theorem. For \( 0 < p < 1 \), and dimension \( d \geq 2 \) we have the estimate

\[ \|M D_N\|_p \gtrsim (\log N)^{(d-1)/2}. \]

Here, \( M \) denotes the strong maximal function in \( d \) dimensions, thus

\[ M f(x) = \sup_{R \text{ dyadic}} 1_R(x) \mathbb{E}(f \mid R). \]

Proof. We are uncertain as to how interesting this is, so our proof is somewhat abbreviated. The only real observation to make is that the theory of multi-parameter Hardy space is relevant. See [8,9]. In particular, letting \( H^p \) denote Hardy space, one has

\[ \|f\|_{H^p} \simeq \|M f\|_p \simeq \|S(f)\|_p, \quad 0 < p \leq 1. \]
We apply this to $D_N$. Let $\mathcal{G}$ be the class of good rectangles, as defined in Proposition 2.3.1. We then have

$$\|M D_N\|_p \simeq \|S(D_N)\|_p \gtrsim \left\| \sum_{R \in \mathcal{G}} 1_R \right\|_p^{1/2}$$

It is an elementary exercise to see that the last term is $\gtrsim (\log N)^{(d-1)/2}$.

2.3. Elementary Propositions

Throughout, we will specify $n$ by $2N \leq 2^n < 4N$, so that $n \approx \log N$. The value of $n$ plays the same role in this section as it does in our discussion of the Small Ball Conjecture. In this section, we use the notation and definitions of §1.5.

Recall that $f$ an $r$ function if it is equal to

$$f = \sum_{R \in R_{\vec{r}}} \varepsilon_R h_R$$

where $\varepsilon_R \in \{-1, 0, 1\}$. Recall that $R_{\vec{r}}$ consists of all dyadic rectangles $R$ with $|R_j| = 2^{-r_j}$ for all coordinates $j$.

2.3.1. Proposition. For each $\vec{r} \in \mathbb{H}_n$, there is an $r$ function $f_{\vec{r}}$ with

$$\langle D_N, f_{\vec{r}} \rangle \geq c_d.$$  

Here $c_d$ is a dimensional constant.

Proof. There is a very elementary one dimensional fact: For all dyadic intervals $I$,

$$(2.3.2) \quad \mathbb{E}x 1_I(x) h_I(x) = \frac{1}{4} |I|^2.$$  

This immediately implies that in any dimension

$$\mathbb{E}h_R(\vec{x})[0,\vec{x}] = 4^{-d}|R|^2.$$  

We shall rely upon the construction of the this function $f_{\vec{r}}$ below. Recall that $\mathcal{A}_N$, the distribution of $N$ points in the unit cube, is fixed. Call a cube $R \in R_{\vec{r}}$ good if $R$ does not intersect $\mathcal{A}_N$, otherwise call it bad. Set

$$(2.3.3) \quad f_{\vec{r}} := \sum_{R \in R_{\vec{r}}} h_R + \sum_{R \in R_{\vec{r}}} \text{sgn}(\langle D_N, h_R \rangle) h_R.$$  

Each bad rectangle contains at least one point in $\mathcal{A}_N$, and $2^n \geq 2N$, so there are at least $N$ good rectangles. Moreover, since the counting function $\#\mathcal{A}_N \cap [0,\vec{x}]$ is constant over each good rectangle, we have

$$\langle D_N, h_R \rangle = N \prod_{j=1}^d \langle x_j, h_{R_j} \rangle = N 2^{-2n-2d} \gtrsim 2^{-n}.$$
Hence, we can estimate

\[
\langle D_N, f_R \rangle \geq \sum_{R \in R, \text{R is good}} \langle D_N, h_R \rangle \geq 2^{-n} \# \{ R \in R : R \text{ is good} \} \geq 1.
\]

And so our proof is complete. \(\square\)

Another proposition of a similar flavor is this.

2.3.4. Proposition. Let \( f_\vec{x} \) be any \( r \) function with \( |\vec{s}| > n \). We have

\[
|\langle D_N, f_\vec{x} \rangle| \lesssim N 2^{-|\vec{s}|}.
\]

Proof. This is a brute force proof. Consider the linear part of the Discrepancy function. By (2.3.2), we have

\[
|\langle N \prod_{j=1}^{d} x_j, f_\vec{x} \rangle| \lesssim N 2^{|\vec{s}|},
\]
as claimed.

Consider the part of the Discrepancy function that arises from the point set. Observe that for any point \( \vec{x}_0 \) in the point set, we have

\[
|\langle 1_{[\vec{s}, \vec{x}]}(\vec{x}_0), f_\vec{x} \rangle| \lesssim 2^{-|\vec{s}|}.
\]

Indeed, of the different Haar functions that contribute to \( f_\vec{x} \) there is at most one with non zero inner product with the function \( 1_{[\vec{s}, \vec{x}]}(\vec{x}_0) \) as a function of \( \vec{x} \). It could only be the one rectangle which contains \( x_0 \) in its interior. Thus the inequality above follows. Summing it over the \( N \) points in the point set finish the proof of the Proposition. \( \square \)

A final, general proposition is relevant.

2.3.5. Proposition. Fix a collection of \( r \) functions \( \{ f_\vec{r} : \vec{r} \in \mathcal{H}_n \} \). Fix \( \vec{s} \) with \( |\vec{s}| > n \), and let \( 3 \leq k \leq |\vec{s}| - n + 1 \). Let \( \text{Count}(\vec{s}; k) \) be the number of ways to choose strongly distinct \( r_1, \ldots, r_k \in \mathcal{H}_n \) so that \( \prod_{j=1}^{k} f_\vec{r} \) is an \( \vec{s} \) function. We have

\[
\text{Count}(\vec{s}; k) \lesssim (|\vec{s}| - n)^2 \cdot k^3 \cdot \left( (|\vec{s}| - n)^{(d-1)} / k - 3 \right).
\]

For \( k = 2 \) we have

\[
\text{Count}(\vec{s}; 2) \lesssim |\vec{s}| - n.
\]

Proof. This estimate is only of interest for \( |\vec{s}| < dn \), and is very crude. Fix \( \vec{s} \). We want to choose strongly distinct \( r_1, \ldots, r_k \in \mathcal{H}_n \) so that for all coordinate \( 1 \leq t \leq d \) we have

\[
\max \{ r_{1,t}, \ldots, r_{k,t} \} = s_k.
\]
(Of course if the $\vec{r}_k$ are not strongly distinct, the product need not be a $r$ function.) Observe that for given $\vec{s}$, there are at most $\leq (|s| - n)^{d-1}$ vectors $\vec{r} \in H_n$ with $r_t \leq s_t$ for all coordinates $t$.

Since the product is to be an $r$ function with parameter $\vec{s}$, we must have either two or three of the chosen $\vec{r}$ functions whose parameters are maximal, and equal to $\vec{s}$. There are at most $k^3$ ways to select these $r$ functions among the $k$ terms were are forming the product over. And having selected them, there are at most $(|s| - k)^2$ ways to select these $r$ functions. The remaining $k - 3$ $r$ functions can be selected freely. This gives (2.3.6).

The second estimate (2.3.7) is easier.

\[\square\]

2.4. Proof of Schmidt’s Theorem

We prove the Theorem of Schmidt; this section should be compared to § 1.3. With the $r$ functions as constructed in the the proof of Proposition 2.3.1, we set

$$\Psi := \prod_{\vec{r} \in P_n} (1 + \alpha f_{\vec{r}}).$$

Here, $0 < \alpha < \frac{1}{2}$, and to be specific, we can choose $\alpha = 2^{-6}$. Clearly, this is a non negative function, with $\int \Psi \, dx = 1$. And so we should argue that

$$\langle D_N, \Psi \rangle \geq n.$$

Write the function $\Psi$ as

$$\Psi = \sum_{k=0}^{n} \psi_k$$

$$\psi_k = \sum_{W \subset P_n} \alpha^{\#W} \prod_{\vec{r} \in W} f_{\vec{r}}$$

where we understand that $\psi_0 = 1_{[0,1]^n}$.

Clearly, $\langle D_N, \psi_0 \rangle = 0$. By Proposition 2.3.1, we have

(2.4.1) \[\sum_{\vec{r} \in P_n} \langle D_N, \alpha f_{\vec{r}} \rangle \geq an\]

For this, recall that we are specializing to the case of dimension 2.

We provide an upper bound on the remaining inner products $\langle D_N, \psi_k \rangle$ for $k \geq 2$. For a subset $W \subset P_n$ of cardinality at least 2. Then, the product

$$\prod_{\vec{r} \in W} f_{\vec{r}}$$

\[\text{Note that in the small ball problem, this set is not needed!}\]
is again a sum of Haar functions, by the Product Rule! See Theorem 1.3.1. By Proposition 2.3.4,

$$\left| \left\langle D_N, \prod_{x \in W} f_x \right\rangle \right| \lesssim N2^{-|\omega|}.$$  

Now, for a fixed $k$ and $\vec{w}$ with $n + k \leq |\vec{w}| \leq 2n$, we count the number of distinct ways of choosing $W$ so that $\prod_{x \in W} f_x$ is a $\vec{w}$ function. The first coordinates of the vectors $\vec{r}$ must be $k$ distinct integers in the range $n - w_2 \leq r_1 \leq w_1$. Moreover, there must be choices of $\vec{r} \in W$ whose first coordinates are equal to either endpoint. There are clearly at most

$$\binom{|\vec{w}| - n - 2}{k - 2}$$

choices of $W$.

For an integer $n \leq \omega \leq 2n$, there are at most $2n$ vectors $\vec{w}$ with $|\vec{w}| = \omega$. Therefore,

$$\left| \langle D_N, \psi_k \rangle \right| \leq 2n\alpha^kN \sum_{\omega=n+k}^{2n} \binom{\omega - n - 2}{k - 2} 2^{-\omega} = n\alpha^kN 2^{-n-k+1} \sum_{\omega=0}^{n-k} \binom{\omega + k - 2}{k - 2} 2^{-\omega}$$

This must be summed over $2 \leq k \leq n$. This sum is treated by two changes of variables. (One is $v = \omega + k$.)

$$\sum_{k=2}^{n} \sum_{\omega=0}^{n-k} n\alpha^kN 2^{-n-k+1} \binom{\omega + k - 2}{k - 2} 2^{-\omega} = \alpha^2 2^{-n-1} N \sum_{k=0}^{n-1} \sum_{\omega=0}^{n-k} \alpha^k 2^{-k-\omega} \binom{\omega + k}{k} \leq n\alpha^2 2^{-n-1} N \sum_{v=0}^{n} \sum_{k=0}^{n-v} \alpha^k 2^{-v} \binom{v}{k} \leq n\alpha^2 2^{-n-1} N \sum_{v=0}^{n} 2^{-v}(1 + \alpha)^v \leq 4n\alpha^2$$

For $\alpha$ sufficiently small, we see that this estimate is much smaller than the lower bound in (2.4.1), so that our proof is complete.

The proof of Theorem 2.1.7 is a simple corollary to the proof above. Since $\|\Psi\|_\infty \leq 2^n$, it is clear that we have

$$\|\Psi\|_{L^1(\log L)^\alpha} \lesssim (\log N)^\alpha.$$
Therefore, we can estimate for $2 < p < \infty$
\[
\log N \leq \langle D_N, \Psi \rangle \leq \|D_N\|_{\exp(L^p)} \cdot \|\Psi\|_{L^1(\log L)^{1/p}} \leq (\log N)^{1/p} \|D_N\|_{\exp(L^p)}.
\]

## 2.5. Proof of Theorem 2.1.9

We rely upon §1.6. We see that for $\Psi^{\text{sd}}$ as defined in (1.6.6), that we have $\|\Psi^{\text{sd}}\|_1 \leq 1$. Moreover, we have
\[
\langle D_N, \Psi^{\text{sd}}_1 \rangle \simeq a q b n \approx an^{1+c/4}
\] (2.5.1)

Here, $q$ is defined as in (1.6.1), and $0 < a < 1$ is a small constant. Again, $q^b \simeq n^{c/4}$ is the ‘gain over the trivial estimate.’

Consider the terms arising from $\Psi^{\text{sd}}_k$. These are products of strongly distinct $r$ vectors. Hence, we combine the estimates from Proposition 2.3.4 and Proposition 2.3.5 as follows.

For $k = 2$ we have
\[
|\langle D_N, \Psi^{\text{sd}}_k \rangle| \leq \sum_{h=1}^{3n} \sum_{\|s\|=n+h} \left[ \frac{q^b}{n} \right]^2 \cdot N 2^{-n-h} \cdot \text{Count}(s^2; 2)
\]
\[
\leq \left[ \frac{q^b}{n} \right]^2 \sum_{h=1}^{3n} (n+2)^{2h} \cdot 2^{-h}
\]
\[
\leq q^{2b} = n^{c/2}.
\]

This is much smaller than the main term (2.5.1).

We treat the terms arising from $\Psi_k$ for $k \geq 3$ as follows.
\[
\sum_{k=3}^{q} |\langle D_N, \Psi^{\text{sd}}_k \rangle| \leq \sum_{k=2}^{q} \sum_{h=k}^{2n} \sum_{\|s\|=n+h} N 2^{-n-h} \left[ \frac{aq^b}{n} \right]^k \cdot \text{Count}(s^2; k)
\]
\[
\leq \sum_{k=3}^{q} \sum_{h=k}^{2n} h^2 2^{-h} \sum_{j=0}^{h^2} \left[ \frac{aq^b}{n} \right]^j \binom{h^2}{j}
\]
\[
\leq q^3 \left[ \frac{aq^b}{n} \right]^3 \sum_{h=3}^{2n} h^2 2^{-h} \sum_{j=0}^{h^2} \left[ \frac{aq^b}{n} \right]^j \binom{h^2}{j}
\]

We have crudely estimated a term or two, and reversed the order of summation. Observe that $q = n^c$ is much smaller than $n$, so that we can estimate
\[
\sum_{j=0}^{h^2} \left[ \frac{aq^b}{n} \right]^j \binom{h^2}{j} \leq \sum_{j=0}^{h^2} \left[ \frac{aq^b}{n} \right]^j \cdot \left[ 1 - \frac{aq^b}{n} \right]^{h^2-j} \binom{h^2}{j} \leq 1.
\]
It follows that
\[ \sum_{k=3}^{q} |\langle D_N, \Psi_k \rangle| \leq q^3 \left( \frac{aqb}{n} \right)^3 \sum_{h=3}^{2n} h^2 2^{-h} \lesssim q^6 \cdot n^{-3} \]
which is again much smaller than the main term (2.5.1). Our proof is complete.

2.6. The \( L^1 \) bound in dimension 2

We will indicate two proofs of Halász’ Theorem 2.1.11. The first is the proof of Halász. Let \( f_r \) be the \( r \) functions has in Proposition 2.3.1. Consider the Riesz product
\[ \Psi := \prod_{i=1}^{n} (1 + i \frac{a}{\sqrt{n}} f_i) . \]
Here, \( 0 < a < 1 \) is a small constant to be chosen. Because of the imposition of the imaginary number, it is evident that this \( \Psi \) is a bounded complex valued function. But one can argue that
\[ \langle D_N, \text{Im}(\Psi) \rangle \gtrsim \sqrt{n} . \]
much as the lines of the argument used to prove Schmidt’s theorem. We omit the details.

The second proof, as far as the author knows, is new; as with Haász’ proof, it does not admit a straightforward extension to higher dimensions. We offer it as a technically interesting object, as the function we use is not a Riesz product, rather it is
\[ (2.6.1) \Phi := \sin \left( \frac{a}{\sqrt{n}} \sum_{|\vec{r}|=n} f_{\vec{r}} \right) . \]
As usual, \( 0 < a < 1 \) is a sufficiently small constant. And we argue that \( \langle D_N, \Phi \rangle \gtrsim \sqrt{n} . \)

Recall that the argument of the sine function above has \( \exp(L^2) \) norm bound independent of \( n \). Thus, as one may directly check, the Taylor expansion of \( \Phi \) is convergent in all \( L^p \). That is, we may expand
\[ (2.6.2) \Phi = \sum_{k=0}^{\infty} \frac{(-1)^{2k-1}}{(2k+1)! \cdot n^{(2k+1)/2}} \left( \sum_{|\vec{r}|=n} f_{\vec{r}} \right)^{2k+1} . \]
and the sum is convergent in all \( L^p, 1 < p < \infty \). A remarkable fact is that this infinite expansion is in fact a finite sum. To see this, let us observe the odd powers above have a simple closed form.

2.6.3. Lemma. For integers \( k \)
\[ n^{-(2k+1)/2} \left( \sum_{|\vec{r}|=n} f_{\vec{r}} \right)^{2k+1} = \sum_{v=1}^{\min(n,2k+1)} \frac{(2k+1)!}{2^v} n^{-v} G_v \]
where \( G_v := \sum_{\vec{r}_1, \ldots, \vec{r}_v \text{ distinct}} \prod_{w=1}^{v} f_{\vec{r}_w} \). The last sum is over all distinct \( v \) tuples of \( r \) vectors with \(|\vec{r}| = n\).

**Proof.** Only odd products of \( \vec{r} \) functions can occur in the expanded product. Fix \( v \) odd, and distinct \( r \) vectors \( \vec{r}_1, \ldots, \vec{r}_v \). It suffices to count the number of ways this product can arise from the expanded product. But this is

\[
\binom{2k + 1}{v} \cdot v! \cdot \frac{(2k + 1 - v)}{2^v} n^{(2k+1-v)/2}
\]

Indeed from the terms

\[
\left( \sum_{|\vec{r}| = n} f_{\vec{r}} \right)^{2k+1}
\]

we choose \( v \) terms from which we take one of the pre-specified \( r \) functions \( f_{\vec{r}_1}, \ldots, f_{\vec{r}_v} \). These products can be specified in one of \( v! \) ways.

In the remaining \( 2k + 1 - v \) terms, we divide them into groups of two. And select one of \( n \) \( r \) functions for each pair. This proves the Lemma. \( \square \)

Expanding the Taylor series we see that

\[
\Phi = \sum_{k \text{ odd}} \frac{(-1)^{(k+1)/2}}{k!} n^{-k/2} 2^{-k} \left[ \sum_{\vec{r} \in H_n} f_{\vec{r}} \right]^k
\]

\[
= \sum_{k \text{ odd}} (-1)^{(k+1)/2} 2^{-3k/2} \sum_{v=1}^{n} 2^{v/2} n^{-v/2} G_v
\]

\[
= c \sum_{v=3}^{n} (-1)^{(v+1)/2} 2^{-v} n^{-v/2} G_v.
\]

(2.6.4)

Here, \( c = (1 - 4^{-3/2})^{-1} \).

We turn our attention to the terms in (2.6.4). Now, by construction, we have

\[
\langle D_N, n^{-1/2} G_1 \rangle \gtrsim n^{-1/2} \sum_{\vec{r} \in H_n} \langle D_N, f_{\vec{r}} \rangle \gtrsim n^{1/2} \approx \sqrt{\log N}.
\]

As for the terms \( 3 \leq v \leq n \), note that by Proposition 2.3.4, (2.4.2) and the definition of \( G_v \), we have

\[
|\langle D_N, G_v \rangle| \lesssim N \sum_{s=v+1}^{2n} 2^{-s} \binom{s - n - 1}{v - 2}.
\]
And so we estimate as follows. Here is convenient that the sum is only over odd $v \geq 3$.

$$\sum_{\substack{v=3 \\ v \text{ odd}}}^{n} 2^{-v} n^{-v/2} \langle D_{N}, G_{v} \rangle \lesssim N \sum_{\substack{v=3 \\ v \text{ odd}}}^{n} \sum_{s=n+v-1}^{2n} 2^{-s-v} n^{-v/2} \binom{s-n-1}{v-2}$$

$$\lesssim N n^{-1} \sum_{s=n+3}^{2n} \sum_{v=0}^{s-n-1} 2^{-s-v} n^{-v/2} \binom{s-n-1}{v}$$

$$\lesssim N n^{-1} \sum_{s=n}^{2n} 2^{-s} \left(1 + 1/\sqrt{n}\right)^{s-n+1}$$

$$\lesssim n^{-1}.$$ 

Since this estimate tends to zero with $n$, this proves our Theorem for sufficiently large $N$. 
CHAPTER 3

Some Aspects of Harmonic Analysis

3.1. Exponential Orlicz Classes

Let $\psi : \mathbb{R} \rightarrow \mathbb{R}$ be a symmetric convex function with $\psi(x) = 0$ iff $x = 0$. Define the Orlicz norm

$$\|f\|_{\psi} := \inf\{C > 0 : \mathbb{E}\psi(f/C) \leq 1\}. \quad (3.1.1)$$

We take the infimum of the empty set to be $+\infty$, and denote by $L^\psi$ to be the collection of functions for which $\|f\|_{\psi} < \infty$.

It is straightforward to see that $\|\cdot\|_{\psi}$ is in fact a norm, with the triangle inequality following from Jensen’s inequality. If $\psi(x) = x^p$, then $\|\cdot\|_{\psi}$ is the usual $L^p$ norm.

We are especially interested in the class of $\psi$ given by

$$\psi_\alpha (x) = e^{|x|^\alpha}, \quad |x| \geq 1.$$ 

Here, we insist upon equality for $|x|$ sufficiently large, depending upon $x$. We will write $L^{\psi_\alpha} = \exp(L^\alpha)$. These are the exponential Orlicz classes.

Especially important is the case of $\alpha = 2$, which is the class $\exp(L^2)$, of exponentially square integrable functions, of which the Gaussian random variables are a canonical example. A function $f \in \exp(L^2)$ is said to be sub-gaussian.

Using Stirling’s formula, and the Taylor expansion for $e^x$, one can check that

3.1.2. Proposition. We have the equivalence of norms

$$\|f\|_{\exp(L^\alpha)} \simeq \sup_{p \geq 1} p^{-1/\alpha} \|f\|_p \quad \simeq \sup_{\lambda > 0} \lambda^{-\alpha} \log \mathbb{P}(|f| > \lambda).$$

One also has a familiar Lemma for the maximum of random variables.

3.1.3. Lemma. Let $X_1, \ldots, X_N$ be random variables in $L^\psi$ of norm at most one. Then, we have

$$\mathbb{E} \sup_{n \leq N} |X_N| \lesssim \psi^{-1}(N).$$

So for $X_1, \ldots, X_N \in \exp(L^2)$ of norm one, we have

$$\mathbb{E} \sup_{n \leq N} |X_N| \lesssim \sqrt{\log N + 1}. \quad (3.1.4)$$

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Indeed, we will leave to the reader to verify that under the assumptions above

\[(3.1.5) \quad \|\sup_{n \leq N} |X_n|\|_{\exp(L^2)} \leq \sqrt{\log N + 1}.\]

**Proof.** By Jensen’s inequality

\[\psi(\mathbb{E} \sup_{n \leq N} |X_n|) \leq \mathbb{E} \sup_{n \leq N} \psi(|X_n|) \leq \sum_{n=1}^{N} \mathbb{E} \psi(|X_n|) \leq N.\]

The proof is complete. \(\square\)

Another class of relevant spaces are given by the convex functions

\[\varphi_{\beta}(x) := |x| (\log 2 + |x|).\]

We denote \(L^{\varphi_{\beta}} = L(\log L)^{\beta}\). The connection with the exponential Orlicz classes is by way of duality.

\[(3.1.6) \quad [\exp(L^\alpha)]^* = L(\log L)^{1/\alpha}.\]

These spaces are closely associated with the *extrapolation* principle.

3.1.7. **Proposition.** Let \(T\) be a linear operator with

\[(3.1.8) \quad \|T\|_{L^p(\{0,1\}^d) \to L^p(\{0,1\}^d)} \leq (p - 1)\alpha, \quad 1 < p \leq 2, \; 0 < \alpha < 1.\]

We then have the inequality

\[(3.1.9) \quad \|Tf\|_{L^1} \leq \|f\|_{L(\log L)^{1/\alpha}}.\]

More generally,

\[(3.1.10) \quad \|Tf\|_{L^1(\log L)^\beta} \leq \|f\|_{L(\log L)^{1+\beta}}, \quad 0 < \beta < \infty.\]

**Proof.** Let us consider (3.1.9). This inequality is dual to

\[\|T^* f\|_{\exp(L^{1/\alpha})} \leq \|f\|_{\infty} \cdot \]

But, taking \(f \in L^\infty\), with \(\|f\|_{\infty} = 1\), and using (3.1.8), we have for \(2 < p < \infty\),

\[\|T^* f\|_p \leq p^\alpha\]

and so the dual estimate follows Proposition 3.1.2. The inequality (3.1.10) is entirely similar. \(\square\)
3.2. Khintchine Inequalities

The utility of the exponential Orlicz classes is that they allow a concise expression of a range of inequalities. This is especially relevant to the classical Khintchine Inequalities. In other instances we shall see, that Orlicz spaces express sharp inequalities forms of different inequalities.

Let \( \{r_k : k \geq 1\} \) be independent, identically distributed random variables, with \( P(r_1 = 1) = P(r_1 = -1) = \frac{1}{2} \). Such random variables are referred to as Rademacher random variables. They admit different realizations, of which the most direct is

\[
r_k = \text{sgn}(\sin(2^k \pi x)), \quad 0 \leq x \leq 1.
\]

Such random variables are in particular orthogonal, so that we have

\[
\left\| \sum_k a_k r_k \right\|_2 = \left[ \sum_k a_k^2 \right]^{1/2}.
\]

This holds for all finite sequences of constants \( \{a_k\} \).

The Khintchine Inequality says that these sums, in all \( L^p \), are controlled by the \( L^2 \) norms. In its sharp form, this inequality states

3.2.1. Khintchine Inequalities. For all finite sequences of constants \( \{a_k\} \)

\[
\left\| \sum_k a_k r_k \right\|_{L^p} \lesssim \left[ \sum_k a_k^2 \right]^{1/2}. \tag{3.2.2}
\]

Proof. The classical proof of this is quite elementary, passing through the Moment Generating Function. We can restrict attention to the case where

\[
\left[ \sum_k a_k^2 \right]^{1/2} = 1.
\]

Consider the moment generating function, given by

\[
\varphi(\lambda) = \mathbb{E} e^{\lambda \sum_k a_k r_k}, \quad \lambda > 0
\]

\[
= \prod_k \mathbb{E} e^{\lambda a_k r_k}
\]

\[
= \prod_k \frac{1}{2}(e^{-\lambda a_k} + e^{\lambda a_k})
\]

\[
\leq \prod_k e^{\lambda^2 a_k^2}
\]

\[
\leq e^{\lambda^2}
\]

Here, we have relied statistical independence of the random variables. In particular, if \( X, Y \) are independent random variables, then

\[
\mathbb{E} X \cdot Y = \mathbb{E} X \cdot \mathbb{E} Y.
\]
We have also used the elementary inequality

\[(3.2.3) \quad \frac{1}{2}(e^{-\mu} + e^{\mu}) = \sum_{j=1}^{\infty} \frac{\mu^{2j}}{(2j)!} \leq e^{\mu^2}, \quad \mu \in \mathbb{R}.
\]

Now estimate

\[\mathbb{P}\left(\sum_{k} a_k r_k > t\right) \leq \varphi(\lambda) e^{-\lambda t} \leq e^{\lambda^2 - \lambda t}, \quad \lambda > 0.
\]

The minimum over \(\lambda > 0\) of the right hand side occurs at \(\lambda = t/2\), giving us the estimate

\[\mathbb{P}\left(\sum_{k} a_k r_k > t\right) \leq e^{-t^2/4}.
\]

In view of the symmetry of the Rademacher random variables and Proposition 3.1.2, this proves the Theorem.

\[\square\]

### 3.3. Maximal Function Estimates

While our primary interest is in the Littlewood Paley Theory, the maximal function and its relevant estimates are essential to the subject.

Define

\[(3.3.1) Mf(x) = \sup_{x \in I} \mathbb{E}(f \mid I).
\]

The principal properties of the Maximal function are

#### 3.3.2. Theorem

We have the estimates

\[(3.3.3) \sup_{\lambda} \lambda \mathbb{P}(Mf > \lambda) \leq \|f\|_1
\]

\[\|Mf\|_p \leq (1 + 1/(p - 1))\|f\|_p, \quad 1 < p \leq \infty.
\]

The left hand side of the first inequality is referred to as the weak \(L^1\) norm, and we write it as \(\|Mf\|_{1,\infty}\). More generally, we define

\[(3.3.4) \|f\|_{p,\infty} = \sup_{\lambda \geq 0} \lambda^{-1}\mathbb{P}(|f| > \lambda)^{1/p}.
\]

As with the Orlicz norms, in certain instances these norms define sharp inequalities.

**Proof of Theorem 3.3.2.** This is especially easy as we are working with the dyadic maximal function, this is especially easy. We begin with the weak type inequality.

Fix \(\lambda > 0\), and let \(\Lambda\) be the collection of maximal dyadic intervals with \(\mathbb{E}(f \mid I) \geq \lambda\). By maximality these intervals are disjoint, so

\[\lambda \mathbb{P}(Mf > \lambda) = \sum_{I \in \Lambda} \mathbb{E} f 1_I \leq \mathbb{E} f \leq \|f\|_1.
\]
For the proof of the remaining inequalities, one interpolates with the obvious $L^\infty$ bound, as is described in Stein and Weiss [33].

The norm estimate we give above, as $p \downarrow 1$ is sharp, which extrapolates to this estimate

3.3.5. Theorem. We have the estimate

\[(3.3.6) \quad \|Mf\|_1 \lesssim \|f\|_{L \log L}.\]

Proof. This nearly follows from Proposition 3.1.7, but $M$ is not a linear operator. Yet, the bound for the maximal operator in Theorem 3.3.2 is equivalent to the same bound for the family of linear operators

\[Tf := \sum_{I \in \mathcal{D}} 1_{E(I)}(f|_I),\]

where $\{E(I) : I \in \mathcal{D}\}$ is a family of pairwise disjoint sets with $E(I) \subset I$ for all $I$. (For a given $f$, one takes $E(I)$ to be the set of $x \in I$ for which the supremum in the definition of $M$ is achieved at $I$.)

These operators, being linear, satisfy the estimate (3.3.6), by Proposition 3.1.7. Therefore the Lemma follows.

There is a striking converse to this last Theorem,

3.3.7. Theorem. [E. M. Stein] If $M|f| \in L$, then we have $f \in L \log L$.

Proof. We can assume that $f \geq 0$. Let us first show that

\[(3.3.8) \quad \lambda^{-1} \mathbb{E} f 1_{\{f \geq \lambda\}} \leq \mathbb{P}(Mf > \lambda), \quad \lambda > \mathbb{E} f.\]

Indeed, let $I$ be the collection of maximal dyadic intervals with $\{Mf > \lambda\} = \bigcup_{I \in I} I$. Then, if $x \notin \{Mf > \lambda\}$, we must have $f(x) \leq \lambda$ by the Martingale Convergence Theorem. In addition, $\lambda > \mathbb{E} f$, so no $I \in I$ can be maximal. That implies that $\mathbb{E}(f|I) \leq 2\lambda$. But then,

\[
\mathbb{E} f 1_{\{f \geq \lambda\}} \leq \sum_{I \in I} \mathbb{E} f 1_I \\
\leq 2\lambda \sum_{I \in I} |I| \\
= 2\lambda \mathbb{P}(Mf > \lambda).
\]

Hence, we can estimate

\[
\int_{\mathbb{E} f} \lambda^{-1} \mathbb{E} f 1_{\{f \geq \lambda\}} d\lambda \leq \int_{\mathbb{E} f} \mathbb{P}(Mf > \lambda) d\lambda,
\]

and our conclusion follows easily from this.
3.4. Littlewood Paley Theory

We consider the Haar basis on $[0,1]$, given by \{1_{[0,1]} \cup \{h_I : I \in \mathcal{D}\}$, where we remind the reader that $\mathcal{D}$ consists of the dyadic intervals in $[0,1]$. We also remind the reader that the Haar functions are normalized to have $L^\infty$ norm one, so that our formulas are different from most of our references.

It is important to our applications that we consider the Haar basis as one for vector valued functions. The vector space should be a Hilbert space $H$, and by $L^p_H$ we mean the class of measurable functions $f : [0,1] \rightarrow H$ such that
\[
\mathbb{E}|f|^p_H < \infty.
\]

The Haar Square Function is
\[
S(f) := \left[|\mathbb{E}|f|^2 + \sum_{I \in \mathcal{D}} \frac{|\langle f, h_I \rangle|^2}{|I|} 1_I\right]^{1/2}.
\]

Here, we are taking the Hilbert space norm of those terms that involve $f$. Of course we have $\|f\|_2 = \|S(f)\|_2$ just by the fact that the Haar basis is an orthogonal basis.

The Littlewood Paley Inequalities are a profound extension of this equality, to an approximate version that holds on all $L^p$, $1 < p < \infty$.

3.4.1. Littlewood Paley Inequalities. For $1 < p < \infty$ there are absolute constants $0 < A_p < B_p < \infty$ so that
\[
\|f\|_p \leq B_p \|S(f)\|_p, \quad 1 < p < \infty
\]
\[
B_p \leq 1 + \sqrt{p}.
\]

In the reverse direction, we have
\[
A_p \|S(f)\|_p \leq \|f\|_p, \quad 1 < p < \infty
\]
\[
A_p \approx 1 + 1/\sqrt{p-1}.
\]

We stress that these results are delicate. Burkholder [6] has shown that the best constants in the inequality above for general martingales are $A_p^{-1} = B_p = \max\{p, q\} - 1$. However, a Haar series is not a general martingale; it is dyadic, which forces conditional symmetry. See [38].

The constants above are sharp. To see that $B_p \approx \sqrt{p}$ is sharp for $p$ large, just use the Central Limit Theorem for Rademacher random variables, or the sharpness of the Khintchine Inequality. A duality argument shows that one can take $A_p = B_p^{-1}$, where $p'$ is the conjugate index to $p$.

The inequality (3.4.2) holds for $0 < p < 2$, but we do not need that case, so don’t discuss it.
Duality Principle. With the Littlewood Paley Inequalities, there is an important duality principle which permits us to pass from one inequality to another. Let us see that we can take

\[ B_p = A_q^{-1}, \quad \frac{1}{p} + \frac{1}{q} = 1. \]  

Assume the inequality \( A_q \|S(g)\|_q \leq \|g\|_q \). Fix \( f \in L^p \), and choose \( g \in L^q \) of norm one so that we have

\[
\|f\|_p = \langle f, g \rangle = \mathbb{E} f \cdot \mathbb{E} g + \sum_{I \in \mathcal{D}} \langle f, h_I \rangle \cdot \mathbb{E} \langle g, h_I \rangle \\
\leq \|S(f)\|_p \cdot \|S(g)\|_q \\
\leq A_q^{-1} \|S(f)\|_p
\]

So (3.4.4) holds.

The Chang Wilson Wolff Inequality. A key step in the proof of this inequality is to first prove the Chang Wilson Wolff inequality, [10].

3.4.5. Chang Wilson Wolff Inequality. We have the estimate below for Hilbert space valued \( f \).

\[
\|f\|_{\text{exp}(L^2)} \lesssim \|S(f)\|_\infty.
\]  

Proof. It is immediately clear that if we knew \( \|f\|_p \lesssim \sqrt{p}\|S(f)\|_p \) for \( p \geq 2 \), in the Hilbert space valued case, then the inequality (3.4.6) would follow.

Our strategy is to first prove the inequality (3.4.6) in the case that the function \( f \) is real valued. From this, we will deduce a quadratic inequality, which will prove the Littlewood Paley inequalities for large \( p \), in the Hilbert space valued case. This will complete the proof of the Chang Wilson Wolff inequality as we have stated it.

We give the proof of Chang Wilson and Wolff, in the real valued case, which they learned from Herman Rubin. Indeed, this proof can be regarded as the conditional version of the proof we have already given of the Khintchine inequalities.

Let us recall that a sequence of functions \( g_1, \ldots, \) form a martingale iff for all sequences

\[
\mathbb{E}(g_{n+1} \mid g_1, \ldots, g_n) = g_n.
\]

Here, we are taking the conditional expectation of \( g_{n+1} \) with respect to the sigma field generated by \( g_1, \ldots, g_n \).

Let \( \mathcal{F}_n \) be the sigma field generated by the dyadic intervals of length \( 2^{-n} \), so that

\[
f_n := \mathbb{E}(f \mid \mathcal{F}_n) = \mathbb{E} f + \sum_{|I| \geq 2^{-n}} \frac{\langle f, h_I \rangle}{|I|} h_I
\]

is a dyadic martingale. We assume that \( \mathbb{E} f = 0 \).
For $t > 0$ we define a new martingale by the formula
\[ q_n := e^{tf_n} \prod_{j=1}^{n-1} \mathbb{E}(e^{t(f_{j+1}-f_j)} | \mathcal{F}_j)^{-1}. \]

Of course, it is hardly obvious that $q_n$ is a martingale, and so we check this now. Clearly, $q_n$ is $\mathcal{F}_n$ measurable. We should then check that
\[ \mathbb{E}(q_n + 1 | \mathcal{F}_n) = q_n. \]

And therefore, $\mathbb{E}q_n = 1$ for all $n$.

The fact that we work with a dyadic martingale enters. For we can appeal to (3.2.3) to see that
\[ \prod_{j=1}^{n-1} \mathbb{E}(e^{t(f_{j+1}-f_j)} | \mathcal{F}_j) \leq \prod_{j=1}^{n-1} \mathbb{E}(e^{2(f_{j+1}-f_j)} | \mathcal{F}_j) = \prod_{j=1}^{n-1} e^{2(f_{j+1}-f_j)^2} = e^{2S(f)^2}. \]

Therefore, under the assumption that $||S(f)||_\infty \leq 1$, we see that
\[ \mathbb{E}e^{tf_n - t^2} \leq \mathbb{E}q_n = 1. \]
As this holds for all $n$, we can take $n \to \infty$. Therefore, we have for $\lambda > 0$,
\[ \mathbb{P}(f > \lambda) \leq e^{-t\lambda} \mathbb{E}e^{t} \leq e^{-t\lambda + t^2}. \]
Taking $t = \lambda/2$ proves the Chang Wilson Wolff inequality in the case that $f$ is real valued. \hfill \square

**Proof of the Littlewood Paley Inequalities.** The first step is to derive a ‘Good $\lambda$ Inequality,’ as below. This exotic looking inequality, first devised in [7], has proven to be a very powerful technique.

**3.4.7. Good $\lambda$ Inequality.** For $\lambda > 0$ we have the inequality
\[ (3.4.8) \quad \mathbb{P}(M f > 2\lambda; S(f) < \epsilon\lambda) \leq e^{-c\epsilon^2} \mathbb{P}(M f > \lambda), \quad 0 < \epsilon < \frac{1}{2}. \]
Here $M f$ is the dyadic maximal function, and $0 < c < 1$ is an absolute constant. The point of the estimate is that it holds for all $0 < \epsilon < \frac{1}{2}$, with the constant on the right tending to zero as $\epsilon \downarrow 0$. 


Proof. Define a stopping time by
\[ \tau = \min\{ n : \sum_{j=1}^{n} (f_j - f_{j-1})^2 \geq \epsilon \lambda \} . \]
As is usual, the minimum of the empty set will be taken to be \(+\infty\).

Let \( f_I = P(I)^{-1} \mathbb{E} f 1_I \) be the average value of \( f \) on \( I \).

Let \( Q \) be the maximal dyadic intervals with \( f_I \geq \lambda P(I) \), so that \( \{ M f > \lambda \} = \bigcup_{I \in Q} I \).

On each \( I \) the event \( E_I := I \cap \{ M f > 2\lambda ; S(f) < \epsilon \lambda \} \). This is the main point: If \( E_I \) is non-empty then \( \mathbb{E} f 1_I \leq (1 + \epsilon) \lambda P(I) \). Indeed, let \( I' \) denote the dyadic interval which contains it and is twice as long. So the average value of \( f \) on \( I' \) is less than \( \lambda \). If our claim is not true, then
\[ |\langle f, h_I' \rangle| \geq \epsilon \lambda P(I) , \]
contradicting \( E_I \) being non-empty.

Now observe that
\[ P(E_I) = P(M f > 2\lambda ; \tau = \infty) \leq P(M(f_\tau - f_I) > (1 - \epsilon) \lambda) . \]
Moreover, \( \| S(f) \|_\infty \leq \epsilon \lambda \). Therefore, by the Chang Wilson Wolff inequality applied to the renormalized martingale \( f_\tau - f_I \),
\[ P(E_I) \leq e^{-c \epsilon^{-2}} P(I) . \]
By summing over \( I \in Q \) we complete the proof. \( \Box \)

There is a standard way to pass from the Good \( \lambda \) Inequalities to norm inequalities, illustrated by this computation. Since \( |f| \leq M f \), it suffices to prove the estimate \( \| M f \|_p \leq B_p \| S(f) \|_p \). First observe that
\[ P(M f > 2\lambda) \leq P(S(f) \leq \epsilon \lambda) + P(M f > 2\lambda ; S(f) < \epsilon \lambda) \]
\[ \leq P(S(f) \leq \epsilon \lambda) + C e^{-c \epsilon^{-2}} P(M f > \lambda) . \]
Then, we can compute
\[ \| M f \|_p^p = p 2^p \int_0^\infty \lambda^{p-1} P(M f > 2\lambda) d\lambda \]
\[ \leq p 2^p \int_0^\infty P(S(f) \leq \epsilon \lambda) d\lambda + p 2^p C e^{-c \epsilon^{-2}} \int_0^\infty \lambda^{p-1} P(M f > \lambda) d\lambda \]
\[ \leq (2/\epsilon)^p \| S(f) \|_p^p + p 2^p C e^{-c \epsilon^{-2}} \| M f \|_p^p . \]
Observe that if we take \( \epsilon \approx p^{-1/2} \), we can conclude
\[ \| M f \|_p \leq (C \sqrt{p})^p \| S(f) \|_p^p \]
which proves the desired inequality.

To recap, we have proved (3.4.2) in the range $1 < p < \infty$ for real valued functions $f$. By
the duality principle, this proves (3.4.3) in the same range.

To deduce the stronger result, for Hilbert space valued functions $f$, we need a different formulation of the Chang Wilson Wolff inequality. Fefferman and Pipher [19] have devised an elegant proof, inspired by the work of Wilson [39]. Also see [24]

3.4.9. Definition. For $1 < p < \infty$, a function $w \geq 0$ on $[0, 1]$, say that it is in dyadic $A_p$ if

$$\|w\|_{A_p} := \sup_{I \in D} |I|^{-1} \|w1_I\|_p \cdot \|w^{-1}1_I\|_{p/(p−1)} < \infty.$$  (3.4.10)

We are especially interested in the endpoint cases. To be explicit, these are

$$\|w\|_{A_1} := \sup_{I \in D} |I|^{-1} \|w1_I\|_1 \cdot [\inf_{x \in I} w(x)]^{-1} < \infty$$

$$\|w\|_{A_\infty} := \sup_{I \in D, x \in I} w(x) \cdot |I|^{-1} \|w^{-1}1_I\|_1 < \infty$$

The functions $w \geq 0$ are ‘weights’ that we use to construct $L^p(w)$ spaces, with norm $\|f\|_{L^p(w)} = \mathbb{E}f^p \cdot w$. By an abuse of notation, we will write this last expectation as

$$\mathbb{E}_w f := \mathbb{E} f \cdot w.$$  

Likewise $\mathbb{P}_w(A) = \mathbb{E}_w 1_A$. The result we are interested in is:

3.4.11. Theorem. We have the inequality

$$\|f\|_{L^2(w)} \lesssim \|w\|_{A_1}^{1/2} \|S(f)\|_{L^2(w)}$$  (3.4.12)

This holds for all Hilbert space valued $f$.

There are two key observations about this Theorem. First, the estimate is quadratic in nature, a key reason for passing to this level of generality. In particular, in order to establish this Hilbert space valued $f$, it suffices to establish it for real valued $f$. Indeed, if $f$ takes values in a Hilbert space, then we can assume that the Hilbert space is $\ell^2$, and write $f = (f_k : k \in \mathbb{N})$. Assuming the real valued version, we can just sum on $k$.

$$\|f\|_{L^2(w)}^2 = \sum_{k \in \mathbb{N}} \|f_k\|_{L^2(w)}^2 \leq \sum_{k \in \mathbb{N}} \|w\|_{A_1} \|S(f_k)\|_{L^2(w)}^2.$$  

So the Hilbert space case is immediate.

Second, the dependence in terms of the $A^1$ constant is sharp, which permits the deduction of the sharp growth rate in $L^p$ constants, for $p > 2$. This is a standard argument, following Rubio de Francia. For $p > 2$, write

$$\|f\|_p^2 \leq \mathbb{E}f^2 \cdot \varphi$$
for some non-negative $\varphi$ with $\|\varphi\|_{(p/2)'} = 1$. We dominate $\varphi$ by an $A^1$ weight, which is given as follows.

\[(3.4.13) \quad v := \sum_{k=0}^{\infty} (2\mu((p/2)'))^k M^k \varphi .\]

In this display, $M$ denotes the dyadic maximal function, and $M^k$ denotes the $k$th power of $M$. We interpret the 0th power to be the identity. The constant $\mu(q)$ is the norm of the Maximal Function on $L^q$. The relevant fact for us here is that $\mu(q) \sim (q - 1)^{-1}$ as $q \downarrow 1$. In particular, $\mu((p/2)') \approx p$ as $p \to \infty$. It is clear that $\|v\|_{(p/2)'} \lesssim 1$.

Now $v$ satisfies $\|v\|_{A^1} \lesssim p$, since for any dyadic interval $I$

$$E(v \mid I) = \sum_{k=0}^{\infty} (2\mu((p/2)'))^k \cdot |I|^{-1} E_1 M^k \varphi$$

$$\leq 2\mu((p/2)') \sum_{k=1}^{\infty} (2\mu((p/2)'))^k \inf_{x \in I} M^k \varphi$$

$$\lesssim p \inf_{x \in I} v(x).$$

But then, we have

$$\|f\|_p^2 \leq E_v f^2$$

$$\lesssim \|v\|_{A^1} E_v S(f)^2$$

$$\lesssim p E_v S(f)^2$$

$$\leq p \|S(f)\|_p^2 .$$

So the Littlewood Paley estimates holds for all $p > 2$, in the Hilbert space valued case.

**Proof of Theorem 3.4.11.** We need an additional result on the way in which $A^1$ weights embed in $A^\infty$ weights.

**3.4.14. Lemma.** [Lemma 3.6, [19].] Given $0 < \eta < 1$, there is a $C > 0$ so that for all $w \in A^1$ and sets $E \subset I$ where $I$ is dyadic, we have

$$P(E \mid I) < e^{-C \|w\|_{A^1}} \quad \text{implies} \quad P_w(E \mid I) < \eta$$

**Proof.** As we work on a probability space, we have the Hölder inequality

$$E|f| \leq \|f\|_p , \quad 1 < p < \infty ,$$

as well as the Orlicz variants, $E|f| \leq \|f\|_{L^1(\log L)\mid |f|)}$. It is a key attribute of the weighted theory that one can reverse some of these inequalities for weights $w \in A_p$. In the case of $A_1$ the reverse Hölder inequality is

$$\|w\|_{L^1(\log L)(t|dx|/|I|)} \lesssim \|w\|_{A^1} \|w\|_{L^1(t|dx|/|I|)} .$$

This follows immediately from Theorem 3.3.7 and the definition of $A^1$. 
But then we can estimate

\[ E(w_1 | I) \leq \| w \|_{L^1((I; dx/I))} \| 1_E \|_{\exp(L(I; dx/I))} \]
\[ \leq \| w \|_{A^1} \| w \|_{L^1(I; dx/I)} \log \| P(E | I)^{-1} \]
\[ \leq \eta \| w \|_{L^1(I; dx/I)} \]

This proves our Lemma. \( \square \)

Recall the Chang Wilson Wolff good \( \lambda \) inequality

\[ P(Mf > 2\lambda ; S(f) < \epsilon \lambda) \leq e^{-c\epsilon^{-2}} P(Mf > \lambda), \quad 0 < \epsilon < \frac{1}{2}. \]

Taking \( \epsilon \approx \| w \|_{A^1}^{-1/2} \), we can deduce the weighted good \( \lambda \) inequality

\[ P_w(Mf > 2\lambda ; S(f) < \epsilon \lambda) \leq \eta P(Mf > \lambda). \]

And the standard way to prove the \( L^2 \) estimate from the good \( \lambda \) inequality gives us the inequality

\[ \| f \|_{L^2(w)} \leq c\epsilon^{-2} \| S(f) \|_2 \approx \| 2\|_{A^1} \| S(f) \|_2, \]

and so the proof is done.

**Weak \( L^1 \) Estimate.** At \( L^1 \), the equivalence \( \| f \|_1 \approx \| S(f) \|_1 \) fails. Nevertheless, there is an endpoint estimate of interest to us. It is

3.4.15. **Weak \( L^1 \) Bound for the Square Function.** We have the inequality

\[ \sup_{\lambda > 0} \lambda P(S(f) > \lambda) \leq \| f \|_1. \]

We stress that this inequality holds for Hilbert space valued functions \( f \).

**Remark.** Traditional approaches to these issues treat the weak \( L^1 \) estimate first, and then interpolate to \( L^p \). We are interested in the sharp constants for the square function, which are not available by way of the weak \( L^1 \) norm.

Central to the proof of this estimate is the *Calderón Zygmund Decomposition*.

3.4.17. **Calderón Zygmund Decomposition.** For \( f \in L^1_M \) of norm one, and let \( \lambda > 0 \). Then, we can write \( f = g_1 + g_2 \) so that \( \| g_1 \|_\infty \leq \lambda \), and \( g_2 \) is supported on disjoint dyadic intervals \( \{ I_j : j \geq 1 \} \), with

\[ \bigcup_{j \geq 1} I_j \leq \lambda^{-1}, \quad \| g_2 \|_I = 0. \]

\[ ^1\text{Instead, one has } \| Mf \| \approx \| S(f) \|_1 \text{ where } M \text{ is Maximal function. The theory of Hardy space } H^1 \text{ depends critically on this equivalence.} \]
Proof. This is a stopping time argument, but as we work on the dyadic grid, the details simplify considerably. Take \(|I_j|\) to be the maximal dyadic intervals such that
\[
E(|f| | I_j) \geq \lambda.
\]
Maximality assures us that these intervals are disjoint. Since \(|f|_1 = 1\), we have
\[
\sum_j |I_j| \leq \lambda \sum_j E(|f| 1_{I_j}) \leq 1.
\]
Set
\[
g_1(x) = \begin{cases} E(|f| | I_j) & x \in I_j, \ j \geq 1, \\ f(x) & \text{otherwise.} \end{cases}
\]
By the Lebesgue Differentiation Theorem (or Martingale Convergence Theorem), \(\|g_1\|_\infty \leq \lambda\).

It is then clear that we have
\[
g_21_{I_j} = f1_{I_j} - E(|f| | I_j).
\]
Thus, \(g_2\) satisfies all its desired properties. \(\square\)

Proof of (3.4.16). Fix \(f \in L^1_H\) of norm one and \(\lambda > 0\). As we work on a probability space, we can further restrict attention to \(\lambda > 1\). Apply the Calderón Zygmund Decomposition, writing \(f = g_1 + g_2\).

Note that we have
\[
P(S(f) > 2\lambda) \leq P(S(g_1) > \lambda) + P(S(g_2) > \lambda),
\]
so that is suffices to analyze the two terms on the right separately.

For \(g_1\), we use the \(L^2\) bound for the Square Function so that
\[
\lambda^2 P(S(g_1) > \lambda) \leq \|S(g_1)\|_2^2 \\
\leq \|g_1\|_2^2 \\
= 2 \int_0^\lambda u P(|g_1| > u) \, du \\
\leq 2\lambda.
\]
The matches the required bound from (3.4.16).

The case of \(g_2\) is simpler. The function \(g_2\) is supported on the dyadic intervals \(I_j\), and has mean zero on each dyadic interval. Thus, if \(J\) is any dyadic interval that strictly contains an \(I_j\), we must have \(\langle g_2, h_J \rangle = 0\). It follows that the square function of \(g_2\) is supported on the \(I_j\), so that
\[
P(S(g_2) > 0) \leq \sum_j P(I_j) \leq \lambda^{-1}.
\]
Our proof is complete. \(\square\)
We need further extensions of the Chang Wilson Wolff inequality, namely these extensions, which are essentially known.

3.4.19. **Theorem.** For $\beta \geq 0$ we have

$$\|S(f)\|_{L^1(\log L)^\beta} \lesssim \|f\|_{L^1(\log L)^{\beta+1/2}}$$

Again, this holds for Hilbert space valued functions $f$.

**Proof.** A variant of the duality principle is useful to us. We can choose a function $g$ so that $S(g)$ has $\exp(L^{1/\beta})$ norm one for which

$$\|S(f)\|_{L^1(\log L)^\beta} = \langle f, g \rangle \leq \|f\|_{L^1(\log L)^{\beta+1/2}} \cdot \|g\|_{\exp(L^{(\beta+1/2)-1})}$$

Now, by Proposition 3.1.2, and the sharp Littlewood Paley inequalities,

$$\|g\|_{\exp(L^{(\beta+1/2)-1})} \simeq \sup_{r>2} r^{-(\beta+1/2)} \|g\|_r \leq \sup_{r>2} r^{-\beta} \|S(g)\|_r \lesssim 1.$$

Our proof is complete. $\square$

### 3.5. Product Theory

The product theory is a branch of Harmonic Analysis devoted to a range of issues that are effectively analyzed with tensor products of Haar bases.\(^2\)

To describe this, again due to the local nature of the questions, we need to slightly modify the dyadic intervals. Before, we used $\mathcal{D}$ to denote the dyadic intervals contained in $[0, 1]$. Let us set $\mathcal{D}_s$ to be these dyadic intervals together with the interval $[0, 2]$. Let us define the Haar function associated with $[0, 2]$ to be the constant function.

$$h_{[0,2]} = 1_{[0,1]}.$$

(We could have taken these steps earlier, but it would have been confusing to do so.) Then, $\{h_I : I \in \mathcal{D}_s\}$ is an orthogonal basis for $L^2([0, 1])$.

Let us construct the tensor product basis for $L^2([0, 1]^d)$. The basis elements are indexed by $\mathcal{R}_d := \mathcal{D}_d^d$, and for $R = R_1 \times \cdots \times R_d \in \mathcal{R}_d$, set

$$h_R(x_1, \ldots, x_d) = \prod_{s=1}^d h_{R_s}(x_s).$$

---

\(^2\)A more typical description involves questions that are invariant under a family of dilations of two or more dilations. Dilations don’t appear in these notes due to the local nature of the questions studied.
This is an orthogonal basis for $L^2([0, 1]^d)$. As in the one parameter setting, we are interested in the vector valued version of this space.

The Haar Square Function in this setting is

$$(3.5.1) \quad S(f) := \left[ \sum_{R \in \mathbb{R}^d} \frac{|\langle f, h_R \rangle|^2}{|R|^2} \mathbf{1}_R \right]^{1/2}.$$ 

As in the one parameter setting, it is clear that $\|f\|_2 = \|S(f)\|_2$, and there is a deep extension of this equivalence to all $L^p$. Again, we are interested in the version of this result which has the sharp dependence in $p$.

3.5.2. Theorem. We have the inequalities below, valid on $[0, 1]^d$.

$$(3.5.3) \quad \|f\|_p \lesssim p^{d/2} \|S(f)\|_p, \quad 1 < p < \infty$$

$$(3.5.4) \quad \|S(f)\|_p \lesssim (p - 1)^{-d/2} \|f\|_p, \quad 1 < p < \infty.$$ 

Proof. The Duality Principle is still in effect, and so it suffices to prove one of the set of inequalities above. We refer to prove the first inequalities.

The method of proof is a standard iteration of the one parameter inequalities, in the vector valued setting, a common technique in the subject, see for instance [31, 32].

Observe that the product Square Function is the composition of Square Functions applied in each coordinate. These Square Functions are then applied to Hilbert space valued functions. In particular, let $S_j$ be the one parameter square function applied in the coordinate $x_j$. Then,

$$S = S_1 \circ \cdots \circ S_d.$$ 

Note that in applying $S_1, \ldots, S_{d-1}$, one should interpret it as applied to a Hilbert space valued functions. Namely in two dimensions, we interpret

$$S_1 f(x_1, x_2) := \left\{ \frac{\langle f(x_1, x_2), h_{I_1}(x_1) \rangle}{\sqrt{|I_1|}} \mathbf{1}_{I_1}(x_1) : I_1 \in \mathcal{D}_+ \right\},$$

and one computes the $\ell^2(\mathcal{D}_+)$ norm of this quantity. Then,

$$S_2 \circ S_1 f(x_1, x_2) := \left\{ \frac{\langle f(x_1, x_2), h_{I_1}(x_1) h_{I_2}(x_2) \rangle}{\sqrt{|I_1| \cdot |I_2|}} \mathbf{1}_{I_1}(x_1) \mathbf{1}_{I_2}(x_2) : I_1, I_2 \in \mathcal{D}_+ \right\},$$

And one computes the $\ell^2 \mathcal{D}_+ \times \mathcal{D}_+$ norm of the right hand side.

It is clear that the Theorem then follows from the one parameter Littlewood Paley inequalities.

Remark. Alternately, one can use the weighted inequality in Theorem 3.4.11, applied $d$ times. Details are left to the reader.

We briefly mention some other relevant inequalities. The weak type estimate is replaced by
3.5.5. Theorem. We have the inequality below on $[0, 1]^d$.

\[(3.5.6) \quad \|S(f)\|_{1,\infty} \lesssim \|f\|_{L^1(\log L)^{d-1}} \]

The Maximal Function is

$$M f(x) = \sup_{R \in \mathcal{D}} 1_R(x) \mathbb{E}(f \mid R).$$

The principal inequalities are below, and in general are sharp.

3.5.7. Theorem. We have the inequalities

\[
\|M f\|_{L^{1,\infty}} \lesssim \|f\|_{L^1(\log L)^{d-1}},
\]

\[
\|M f\|_p \lesssim [1 + (p - 1)^{-1}]^{-d} \|f\|_p.
\]

As we don’t use this estimate, we do not prove it.
CHAPTER 4

Other Applications: Approximation Theory and Probability Theory

4.1. Mixed Derivatives

We will take an abbreviated view of the subject of this chapter, referring the reader to references, especially [36] for more information. In \( d \) dimensions, consider the map

\[
\text{Int}_d f(x_1, \cdots, x_d) := \int_0^{x_1} \cdots \int_0^{x_d} f(y_1, \cdots, y_d) \, dy \cdots dy_d
\]

We consider this as a map from \( L^p([0,1]^d) \) into \( C([0,1]^d) \). Clearly, the image of \( \text{Int}_d \) consists of functions with \( L^p \) integrable mixed partial derivatives. Let us set

\[
\text{Ball}(\text{MW}^p([0,1]^d)) := \text{Int}_d(\{ f \in L^p([0,1]^d) : \|f\|_p \leq 1 \}).
\]

That is, this is the image of the unit ball of \( L^p \). This is the unit ball of the space of functions with \( L^p \) integrable mixed partial derivatives. Let us set

\[
\text{Ball}(\text{MW}^p([0,1]^d)) := \text{Int}_d(\{ f \in L^p([0,1]^d) : \|f\|_p \leq 1 \}).
\]

That is, this is the image of the unit ball of \( L^p \). This is the unit ball of the space of functions with \( L^p \) integrable mixed partial derivatives. Our main theorem Theorem 1.1.7 has consequences for the case of \( p = 1 \), but in this discussion we concentrate of the case of \( p = 2 \), for which we have no new results.

These sets are compact in \( C([0,1]^d) \), and it is of relevance to quantify the compactness. The traditional way to do this is through entropy numbers. For \( 0 < \epsilon < 1 \), set \( N(\epsilon) \) to be the least number \( N \) of points \( x_1, \cdots, x_N \in C([0,1]^d) \) so that

\[
\text{Ball}(\text{MW}^2([0,1]^d)) \subset \bigcup_{n=1}^{N} x_n + \epsilon B_{\infty}.
\]

Here, \( B_{\infty} \) is the unit ball of \( C([0,1]^d) \). An upper bound on these numbers is known,

\[
\log N(\epsilon) \lesssim \epsilon^{-1} (- \log \epsilon)^{d-1/2}
\]

And the task at hand is to prove that this estimate is sharp. The case of \( d = 2 \) below follows from Talagrand [34].

4.1.2. Conjecture. For \( d \geq 2 \) one has the estimate

\[
\log N(\epsilon) \approx \epsilon^{-1} (\log 1/\epsilon)^{d-1/2}, \quad \epsilon \downarrow 0.
\]

How does Small Ball Conjecture enter in? We should use a ‘smooth’ version of the Small Ball Conjecture. That is, in the Small Ball Conjecture, (1.1.3), one should replace the
‘rough’ Haar functions by smooth variants. There is no canonical way to do this,\footnote{One can replace the splines below by tensor products of wavelets, or by appropriate hyperbolic trigonometric polynomials.} and so we simply choose the ‘spline variant’ of Talagrand [34]. For dyadic interval $R \in \mathcal{D}$ in dimension $d$, set $u_R = \text{Int}_d h_R$.

4.1.3. Smooth Small Ball Conjecture. For all sequences $\alpha(R)$, we have the estimate below valid for all integers $n$.

\[
2^{-2n} \sum_{|R|=2^{-n}} |\alpha(R)| \leq n^{(d-2)/2} \left\| \sum_{|R|=2^{-n}} \alpha(R) u_R \right\|_\infty
\]  

(4.1.4)

The power $2^{-2n}$ is explained by the fact that the functions $u_R$ have $L^\infty$ norm comparable to $2^{-n}$. This inequality is true, and proved by Talagrand [34], but the methods of § 1.3 will provide simple proofs of related facts.

Let us explain how this conjecture provides lower bounds for entropy numbers. Given a choice of signs $\sigma : \{R : |R| = 2^{-n}\} \rightarrow \{\pm 1\}$, we consider the functions

\[
F_\sigma := n^{(1-d)/2} \sum_{|R|=2^{-n}} \sigma(R) h_R.
\]

Then, the mixed derivative of $F_\sigma$ has norm about 1. The point of view is to let $\sigma$ vary to construct sets of points in $\text{Int}_d(\mathcal{B}_2)$ that are widely separated.

Suppose that for two different choices of $\sigma$ and $\sigma'$, we have

\[
\sum_{|R|=2^{-n}} |\sigma(R) - \sigma'(R)| \geq n^{d-1} 2^n.
\]  

(4.1.5)

Then, Conjecture 4.1.3 enters in the following way:

\[
\|\text{Int}_d(F_\sigma - F_{\sigma'})\|_\infty = \left\| \sum_{|R|=2^{-n}} (\sigma(R) - \sigma'(R)) u_R \right\|_\infty
\]

\[
\geq n^{d+3/2} 2^{-2n} \sum_{|R|=2^{-n}} |\sigma(R) - \sigma'(R)|
\]

\[
\geq n^{1/2} 2^{-n}
\]  

(4.1.6)

Thus, a collection of $F_\sigma$ satisfying (4.1.5) are uniformly separated in $L^\infty$ norm.

Notice that we have reduced the problem to one of finding many proportional subsets of $\mathcal{R}_n$ that are essentially disjoint from each other. This is addressed in a general fashion by this proposition.

4.1.7. Proposition. There is a constant $c > 0$ so that for all integers $m$, there is a collection of subsets $\mathcal{A}$ of $\{1, \ldots, m\}$ so that

\[
\text{card}(A \triangle A') \geq cm, \quad A \neq A' \in \mathcal{A},
\]  

(4.1.8)

\[
\text{card}(\mathcal{A}) \geq \exp(cm).
\]  

(4.1.9)
Apply this proposition the collection of dyadic rectangles \( \{ R : |R| = 2^{-n} \} \). Let \( \mathcal{A} \) be the corresponding subsets of this collection, thus for \( A, A' \in \mathcal{A} \) we have \( |A \Delta A'| \geq n^{d-1}2^n \). Let \( A \) also stand for the function

\[
A(R) := \begin{cases} 
1 & \text{if } R \in A \\
-1 & \text{if } R \not\in A
\end{cases}.
\]

Consider the collection \( \{ F_A : A \in \mathcal{A} \} \). Any two distinct functions in this collection obey the estimate (4.1.6), hence it follows that

\[
\log N(n^{1/2}2^{-n}) \geq \log(\#\mathcal{A}) \geq n^{d-1}2^n.
\]

Setting \( \epsilon \approx n^{1/2}2^{-n} \), we see that we have

(4.1.10) \[
\log N(\epsilon) \geq \delta^{-1}(\log 1/\epsilon)^{d-1/2}, \quad \epsilon \downarrow 0.
\]

This would match the known upper bound, (4.1.1). Again, this inequality is known, and a consequence of Talagrand’s work, in dimension \( d = 2 \).

**A Coding Theory Result.** A useful observation is that Proposition 4.1.7 is concerned with the central issues of coding theory. Namely, each subset of \( \{1, \ldots, m\} \) is identified with a word of length \( m \), in an alphabet of two colors. The condition (4.1.8) implies that the words differ in a constant times \( m \) slots—that is that their Hamming distance is proportionally as large as possible. And the condition (4.1.8) assures us that the code has a large capacity. Fortunately, we can appeal to a well known result from Coding Theory to address this proposition.

4.1.11. **Theorem.** [Varshamov–Gilbert Bound]. We view \( \{0, 1\}^n \) as a linear vector space mod 2. It contains \( V \), a linear subspace mod 2 with

\[
|x - y|_A \geq d \quad x, y \in V
\]

iff the inequality below holds.

(4.1.12) \[
\binom{n-1}{n-1} + \cdots + \binom{n-1}{d-2} < 2^{n-k}
\]

To prove Proposition 4.1.7, in this Theorem, we take \( d = k = \alpha n \), for a small constant \( \alpha \) to be chosen. Note that the left hand side of (4.1.12) is at most

\[
d\binom{n-1}{d-2} \leq \alpha n \binom{n}{\alpha n}
\]

\[
\sim \frac{n^n e^{-n}}{\alpha^n (1-\alpha)^n n^n e^{-n}} \sqrt{\alpha(1-\alpha)}
\]

\[
= \alpha \sqrt{\alpha(1-\alpha)} n(\alpha^\alpha (1-\alpha)^{1-\alpha} - n
\]

\[
< 2^{(1-\alpha)n}.
\]
Here, we are using Stirling’s formula $m! \sim m^{m-1/2}e^{-m}$, meaning that the ratio of these two terms approaches a non zero constant. Observe that $\alpha^\alpha \to 0$ as $\alpha \to 0$, so that we can make a choice of $\alpha$ for which this inequality will be true for all large $n$.

### 4.2. The Brownian Sheet

**General Gaussian Processes.** A Gaussian process is a random map $X_t : T \to \mathbb{R}$ where $T$ is some index set, so that for all finite $S \subset T$ and reals $a_s$,

$$
\sum_{s \in S} a_s X_s
$$

is a random variable with a Gaussian distribution. It is a fundamental property that a mean zero Gaussian process is characterized by the covariances

$$
\rho(s, t) := \mathbb{E} X_s \cdot X_t.
$$

Throughout, we will be concerned with processes which are almost surely have bounded sample paths, namely

$$
\mathbb{P}(\sup_{t \in T} |X_t| < \infty) = 1.
$$

The Small Ball Problem concerns estimates for the probability

$$
\mathbb{P}(\sup_{t \in T} |X_t| < \epsilon), \quad \downarrow 0.
$$

See [22] for a survey on these types of questions.

If one is given a subset $K$ of a Hilbert space $\mathcal{H}$, then one can define an associated mean zero Gaussian process $X_s$ for $s \in K$ by defining

$$
\mathbb{E} X_s \cdot X_t := \langle s, t \rangle_{\mathcal{H}}
$$

where the last inner product is the one associated with $\mathcal{H}$. This is a canonical relationship with profound consequences: Most Gaussian processes of interest can be described in this manner, and the Hilbert space has function theoretic description which in turn reflects the structure of the Gaussian process.

For instance, assume that associated with $\{X_t : t \in T\}$ are covariance kernel functions $K_t$ and measure $\mu$ on $T$ so that $\{K_t : t \in T\} \subset L^2(T, d\mu)$ and

$$
\mathbb{E} X_s \cdot X_t = \int_T K_s \cdot K_t \, d\mu.
$$

Let $\mathcal{H}_X$ be the $L^2(\mu)$ completion of the set of functions $\{K_t : t \in T\}$. This spaces is called the Reproducing Kernel Hilbert Space associated with the Gaussian process $X_t$.

Following on the work Talagrand, Kuelbs and Li [22] uncovered a close connection between the the Small Ball Probabilities and the covering numbers associated with the unit ball of $\mathcal{H}_X$ in the $L^\infty(T)$ metric. We will recall this result in the particular instances of the Brownian sheet below.
4.2. THE BROWNIAN SHEET

4.2.1. The Brownian Sheet. The Brownian sheet is a canonical Gaussian process indexed by points \( s \in [0,1]^d \). Calling the process \( B(s) \), it is characterized by requiring it to be a mean zero process with covariance structure

\[
\mathbb{E}B(s)B(t) = \prod_{j=1}^{d} \min(s_j, t_j)
\]

Note that this covariance functional is given by

\[
\mathbb{E}B(s)B(t) = \int_{[0,1]^d} \mathbf{1}_{[0,s)} \cdot \mathbf{1}_{[0,t)} \, dx.
\]

The Reproducing Kernel Hilbert Space associated with the Brownian sheet is \( WM^2_d \), the Sobolev space of functions with square integrable mixed derivatives in dimension \( d \). A particular case of the result of Kuelbs and Li [23] states that

4.2.1. Theorem. As \( \epsilon \downarrow 0 \) we have

\[
\log P(\|B\|_{C([0,1]^d)} < \epsilon) \approx \epsilon^{-2} (\log 1/\epsilon)^\beta \quad \text{iff} \quad \log N(\epsilon) \approx \epsilon^{-1} (\log 1/\epsilon)^{\beta/2}.
\]

Thus, the Conjecture (4.1.3) gives a result on these processes. And the form of the relevant conjecture here is as follows.

4.2.3. Small Ball Problem for the Brownian Sheet. For dimension \( d \geq 2 \), we have

\[
\log P(\|B\|_{C([0,1]^d)} < \epsilon) \approx \epsilon^{-2} (\log 1/\epsilon)^{2d-1}, \quad \epsilon \downarrow 0.
\]

This is known for \( d = 2 \). For all \( d \geq 3 \), the upper bound on the Small Ball probabilities is known; the issue is to obtain the appropriate lower bound. In dimension \( d \geq 3 \), the best known lower bounds miss the conjecture above by a single power of \( \log 1/\epsilon \).
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