On symplectic eigenvalues of positive definite matrices

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Abstract

If $A$ is a $2n \times 2n$ real positive definite matrix, then there exists a symplectic matrix $M$ such that $M^T AM = \begin{bmatrix} D & O \\ O & D \end{bmatrix}$ where $D = \text{diag}(d_1(A), \ldots, d_n(A))$ is a diagonal matrix with positive diagonal entries, which are called the symplectic eigenvalues of $A$. In this paper we derive several fundamental inequalities about these numbers. Among them are relations between the symplectic eigenvalues of $A$ and those of $A^t$, between the symplectic eigenvalues of $m$ matrices $A_1, \ldots, A_m$ and of their Riemannian mean, a perturbation theorem, some variational principles, and some inequalities between the symplectic and ordinary eigenvalues.

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1 Introduction

Let $\mathbb{M}(2n)$ be the space of $2n \times 2n$ real matrices, $\mathbb{P}(2n)$ the subset of $\mathbb{M}(2n)$ consisting of positive definite matrices, and $Sp(2n)$ the group of real symplectic matrices; i.e.,

$$Sp(2n) = \{ M \in \mathbb{M}(2n) : M^T J M = J \}.$$
Here \( J = \begin{bmatrix} O & I \\ -I & O \end{bmatrix} \), and \( J \) itself is a symplectic matrix.

If \( A \) is an element of \( \mathbb{P}(2n) \), then there exists a symplectic matrix \( M \) such that
\[
M^T A M = \begin{bmatrix} D & O \\ O & D \end{bmatrix},
\]
where \( D \) is a diagonal matrix with positive entries
\[
d_1(A) \leq d_2(A) \leq \cdots \leq d_n(A).
\]
This is often called Williamson’s theorem \([1], [7], [17]\). In \([11]\) it is pointed out that this was known to Weierstrass. The numbers \( d_j(A) \) are uniquely determined by \( A \) and characterise the orbits of \( \mathbb{P}(2n) \) under the action of the group \( \text{Sp}(2n) \). We call them the symplectic eigenvalues of \( A \). They play an important role in classical Hamiltonian dynamics \([1]\), in quantum mechanics \([2]\), in symplectic topology \([11]\), and in the more recent subject of quantum information; see e.g., \([8], [10], [13], [17]\).

The goal of this paper is to present some fundamental inequalities for symplectic eigenvalues.

It is clear from the definition that if the symplectic eigenvalues of \( A \) are enumerated as in \((2)\), then those of \( A^{-1} \) are
\[
\frac{1}{d_n(A)} \leq \frac{1}{d_{n-1}(A)} \leq \cdots \leq \frac{1}{d_1(A)}.
\]
No relation between the symplectic eigenvalues of \( A \) and those of \( A' \) is readily apparent. Our first theorem unveils such relationships.

Given \( x \in \mathbb{R}_+^m \), we denote by \( x^\downarrow = (x_1^\downarrow, \ldots, x_m^\downarrow) \) the vector whose coordinates are the coordinates of \( x \) rearranged in decreasing order \( x_1^\downarrow \geq \cdots \geq x_m^\downarrow \). If \( x \) and \( y \) are two \( m \)-vectors with positive coordinates, then we say that \( x \) is log majorised by \( y \), in symbols \( x \prec_{\log} y \), if
\[
\prod_{j=1}^k x_j^\downarrow \leq \prod_{j=1}^k y_j^\downarrow, \quad 1 \leq k \leq m
\]
and
\[
\prod_{j=1}^m x_j^\downarrow = \prod_{j=1}^m y_j^\downarrow.
\]
By classical theorems of Weyl and Polya, log majorisation implies the usual weak majorisation relation $x \prec_w y$ characterised by the inequalities

$$
\sum_{j=1}^{k} x_j^\downarrow \leq \sum_{j=1}^{k} y_j^\downarrow, \quad 1 \leq k \leq m.
$$

(6)

See Chapter II of [3].

It is convenient to introduce a $2n$-vector $\hat{d}(A)$ whose coordinates are

$$
\hat{d}_1(A) \geq \hat{d}_2(A) \geq \cdots \geq \hat{d}_{2n}(A),
$$

(7)

which are the symplectic eigenvalues of $A$, each counted twice and rearranged in decreasing order. (Thus $\hat{d}_1(A) = \hat{d}_2(A) = d_n(A)$ and $\hat{d}_{2n-1}(A) = \hat{d}_{2n}(A) = d_1(A)$.) With these notations we have the following.

**Theorem 1.** Let $A$ be any element of $\mathbb{P}(2n)$. Then

$$
\hat{d}(A^t) \prec_{\log} \hat{d}^t(A) \quad \text{for} \quad 0 \leq t \leq 1,
$$

(8)

and

$$
\hat{d}^t(A) \prec_{\log} \hat{d}(A^t) \quad \text{for} \quad 1 \leq t < \infty.
$$

(9)

**Corollary 2.** The symplectic eigenvalues of $A$ have the properties:

(i) If $0 \leq t \leq 1$, then for all $1 \leq k \leq n$

$$
\prod_{j=1}^{k} d_j(A^t) \geq \prod_{j=1}^{k} d_j^t(A).
$$

(10)

(ii) If $t \geq 1$, then for all $1 \leq k \leq n$

$$
\prod_{j=1}^{k} d_j(A^t) \leq \prod_{j=1}^{k} d_j^t(A).
$$

(11)

Given two $n \times n$ positive definite matrices $A$ and $B$, their geometric mean $G(A, B)$, also denoted as $A \# B$, is defined as

$$
G(A, B) = A \# B = A^{1/2}(A^{-1/2}BA^{-1/2})^{1/2}A^{1/2}.
$$

(12)
This was introduced by Pusz and Woronowicz [18], and has been much studied in connection with problems in physics, electrical networks, and matrix analysis. Recently there has been renewed interest in it because of its interpretation as the midpoint of the geodesic joining \( A \) and \( B \) in the Riemannian manifold \( \mathbb{P}(n) \). The Riemannian distance between \( A \) and \( B \) is defined as

\[
\delta(A, B) = \left( \sum_{i=1}^{n} \log^2 \lambda_i (A^{-1} B) \right)^{1/2},
\]

where \( \lambda_i(X), 1 \leq i \leq n \), are the eigenvalues of \( X \). With this metric \( \mathbb{P}(n) \) is a nonpositively curved space. Any two points \( A \) and \( B \) in \( \mathbb{P}(n) \) can be joined by a unique geodesic. A natural parametrisation for this geodesic is

\[
A \#_t B = A^{1/2} (A^{-1/2} B A^{-1/2})^t A^{1/2}, \quad 0 \leq t \leq 1.
\]

(14)

\( G(A, B) \) is evidently the midpoint of this geodesic. Our next theorem links the symplectic eigenvalues of \( A \#_t B \) with those of \( A \) and \( B \).

**Theorem 3.** Let \( A, B \) be any two elements of \( \mathbb{P}(2n) \). Then for \( 0 \leq t \leq 1 \),

\[
\hat{d}(A \#_t B) \prec_{\log} \hat{d}^{1-t}(A) \hat{d}^t(B).
\]

In particular

\[
\hat{d}(G(A, B)) \prec_{\log} \left( \hat{d}(A) \hat{d}(B) \right)^{1/2}.
\]

(15)

Next let \( A_1, A_2, \ldots, A_m \) be \( m \) points in \( \mathbb{P}(n) \). Their geometric mean, variously called the Riemannian mean, the Cartan mean, the Karcher mean, the Riemannian barycentre, is defined as

\[
G(A_1, \ldots, A_m) = \arg\min \sum_{j=1}^{m} \frac{1}{m} \delta^2(A_j, X).
\]

(17)

This object of classical differential geometry has received much attention from operator theorists and matrix analysts in the past ten years, and many new properties of it have been established. It has also found applications in diverse areas such as statistics, machine learning, image processing, brain-computer interface, etc. We refer the reader to [4] for a basic introduction to this area and to [5] for an update.
A little more generally, a weighted geometric mean of \( A_1, \ldots, A_m \) can be defined as follows. Given positive numbers \( w_1, \ldots, w_m \) with \( \sum w_j = 1 \), let

\[
G(w, A_1, \ldots, A_m) = \operatorname{argmin} \sum_{j=1}^{m} w_j \delta^2(A_j, X). \tag{18}
\]

The object in (17) is the special case when \( w_j = \frac{1}{m} \) for all \( j \). In case \( m = 2 \), we have \( (w_1, w_2) = (1-t, t) \) for some \( 0 \leq t \leq 1 \), and then \( G(w, A, B) \) reduces to the matrix in (14). With \( t = 1/2 \) it reduces further to (12).

Our next theorem is a several-variables version of Theorem 3.

**Theorem 4.** Let \( A_1, \ldots, A_m \) be elements of \( \mathbb{P}(2n) \) and let \( w = (w_1, \ldots, w_m) \) be a positive vector with \( \sum w_j = 1 \). Then

\[
\hat{d}(G(w, A_1, \ldots, A_m)) \prec \log \prod_{j=1}^{m} \hat{d}^{w_j}(A_j). \tag{19}
\]

In particular

\[
\hat{d}(G(A_1, \ldots, A_m)) \prec \log \left( \prod_{j=1}^{m} \hat{d}(A_j) \right)^{1/m} \tag{20}
\]

In the study of eigenvalues of Hermitian matrices, a very important role is played by variational principles, such as the Courant-Fischer-Weyl minmax principle, Cauchy’s interlacing theorem and Ky Fan’s theorems on extremal characterisations of sums and products of eigenvalues. It will be valuable to assemble a similar arsenal of techniques for symplectic eigenvalues. In Section 4 of this paper we give an exposition of some of these ideas. We provide an outline of proofs of a minmax principle and an interlacing theorem (both of which are known results). Then we use this to provide a unified simple proof of the following theorem. To emphasize the dependence on \( n \) we use the notation \( J_{2n} \) for the \( 2n \times 2n \) matrix \[
\begin{bmatrix}
O & I \\
-I & O
\end{bmatrix}.
\]
The minimum in Theorem 5 below is taken over \( 2n \times 2k \) matrices \( M \) satisfying \( M^T J_{2n} M = J_{2k} \).

**Theorem 5.** Let \( A \in \mathbb{P}(2n) \). Then for all \( 1 \leq k \leq n \)

\[
(i) \quad 2 \sum_{j=1}^{k} d_j(A) = \min_{M: M^T J_{2n} M = J_{2k}} \text{tr} M^T A M, \tag{21}
\]
\[(ii) \prod_{j=1}^{k} d^2_j(A) = \min_{M:M^T J_{2n} M = J_{2k}} \det M^T AM. \quad (22)\]

Part (i) of this theorem has been proved by Hiroshima \[10\], and was an inspiration for our work. Our proof might be simpler and more conceptual. An interesting property of symplectic matrices crops up as a byproduct of our analysis.

Every element \( M \) of \( Sp(2n) \) has a block decomposition

\[ M = \begin{bmatrix} A & B \\ C & G \end{bmatrix}, \quad (23)\]

in which \( A, B, C, G \) are \( n \times n \) matrices satisfying the conditions

\[ AG^T - BC^T = I, \quad AB^T - BA^T = 0, \quad CG^T - GC^T = 0. \quad (24)\]

We associate with \( M \) an \( n \times n \) matrix \( \tilde{M} \) whose entries are given by

\[ \tilde{m}_{ij} = \frac{1}{2} \left( a_{ij}^2 + b_{ij}^2 + c_{ij}^2 + g_{ij}^2 \right). \quad (25)\]

This matrix has some nice properties and can be put to good use in the study of symplectic matrices. In the course of our proof of Theorem \[5\] we will see that for every \( M \in Sp(2n) \) the matrix \( \tilde{M} \) has the properties

\[ \sum_{j=1}^{n} \tilde{m}_{ij} \geq 1, \quad 1 \leq i \leq n, \quad \text{and} \]

\[ \sum_{i=1}^{n} \tilde{m}_{ij} \geq 1, \quad 1 \leq j \leq n. \quad (26)\]

It turns out that more is true.

An \( n \times n \) matrix \( A \) is said to be \textit{doubly stochastic} if \( a_{ij} \geq 0 \) for all \( i, j, \)

\[ \sum_{j=1}^{n} a_{ij} = 1 \quad \text{for all} \quad 1 \leq i \leq n \]

and

\[ \sum_{i=1}^{n} a_{ij} = 1 \quad \text{for all} \quad 1 \leq j \leq n. \]

A matrix \( B \) with nonnegative entries is called \textit{doubly superstochastic} if there exists a doubly stochastic matrix \( A \) such that \( b_{ij} \geq a_{ij} \) for all \( i, j \). Our next theorem shows that \( \tilde{M} \) is a doubly superstochastic matrix.
Theorem 6. Let $M \in \text{Sp}(2n)$, and let $\tilde{M}$ be the $n \times n$ matrix associated with $M$ according to the rule (25). Then $\tilde{M}$ is doubly superstochastic. Further $\tilde{M}$ is doubly stochastic if and only if $M$ is orthogonal.

Doubly stochastic, superstochastic and substochastic matrices play an important role in the theory of inequalities; see the monograph [16]. Theorem 6 is thus likely to be very useful in deriving inequalities for symplectic matrices.

For the usual eigenvalues of Hermitian matrices there are several perturbation bounds available. See [3]. Our next theorem gives such inequalities for symplectic eigenvalues. The continuity implied by these bounds will be used in our proofs of Theorems 1, 3, 4. But they are of independent interest.

We use the symbol $\|\cdot\|$ to denote any unitarily invariant norm on the space of matrices [3]. Particular examples are the operator norm

$$\|A\| = \lambda_1(A^T A)^{1/2} = \sup_{\|x\|=1} \|Ax\|,$$

and the Frobenius norm

$$\|A\|_2 = (\text{tr} A^T A)^{1/2} = \left(\sum |a_{ij}|^2\right)^{1/2}.$$

(27)

Here $\lambda_1$ stands for the maximum eigenvalue.

Theorem 7. Let $A, B$ be two elements of $\mathbb{P}(2n)$, and let $\hat{D}(A), \hat{D}(B)$ be the diagonal matrices whose diagonals are $\hat{d}(A)$ and $\hat{d}(B)$. Then for every unitarily invariant norm we have

$$\|\hat{D}(A) - \hat{D}(B)\| \leq (\|A\|^{1/2} + \|B\|^{1/2}) \|A - B\|^{1/2}.$$  (29)

The special cases of the operator norm and the Frobenius norm give

$$\max_{1 \leq j \leq n} |d_j(A) - d_j(B)| \leq (\|A\|^{1/2} + \|B\|^{1/2}) \|A - B\|^{1/2},$$  (30)

$$\sqrt{2} \left(\sum_{j=1}^{n} |d_j(A) - d_j(B)|^2\right)^{1/2} \leq (\|A\|^{1/2} + \|B\|^{1/2}) (\text{tr} |A - B|)^{1/2}.\)  (31)

(Here $|X|$ denotes the matrix absolute value defined as $|X| = (X^T X)^{1/2}$.)
The rest of the paper is organised as follows. In Section 2 we give a proof of Theorem 7 and in Section 3 of Theorems 1, 3 and 4. In Section 4 we prove Theorem 5, and in Section 5 we prove Theorem 6. Some other results are proved along the way either as prerequisites or as supplements.

Let us recall here two facts about symplectic eigenvalues and associated pairs of eigenvectors. The imaginary numbers $\pm id_j(A)$, $1 \leq j \leq n$, constitute the set of eigenvalues of the skew-symmetric matrix $A^{1/2}JA^{1/2}$. To each $d_j(A)$ there corresponds a pair of vectors $u_j, v_j$ in $\mathbb{R}^{2n}$ such that

$$Au_j = d_j(A)Jv_j, \quad Av_j = -d_j(A)Ju_j.$$ 

We may normalize these vectors so that the Euclidean inner product $\langle u_j, Jv_j \rangle = 1$. Then we call $(u_j, v_j)$ a symplectic eigenvector pair corresponding to the symplectic eigenvalue $d_j$. Together, these $2n$ vectors constitute a symplectic eigenbasis for $\mathbb{R}^{2n}$; i.e.,

$$\langle u_i, Ju_j \rangle = \langle v_i, Jv_j \rangle = 0 \text{ for all } i, j,$$

and

$$\langle u_i, Jv_j \rangle = \delta_{ij} \text{ for all } i, j.$$

## 2 Proof of Theorem 7

A norm $\|\cdot\|$ on $\mathbb{M}(n)$ is called unitarily invariant if $\|UXV\| = \|X\|$, for all $X \in \mathbb{M}(n)$ and for all unitary matrices $U, V$. If $X, Y, Z$ are any three matrices, then $\|XYZ\| \leq \|X\|\|Y\|\|Z\|$. See Chapter IV of [3] for properties of such norms.

Let $A$ be a Hermitian matrix and $\text{Eig}^+(A)$ the diagonal matrix whose diagonal entries are the decreasingly ordered eigenvalues of $A$. By the famous Lidskii-Wielandt theorem (see (IV.62)) in [3]) we have

$$\|\text{Eig}^+(A) - \text{Eig}^+(B)\| \leq \|A - B\|.$$ 

Now let $A \in \mathbb{P}(2n)$. The symplectic eigenvalues $d_j(A)$ with their negatives are the eigenvalues of the Hermitian matrix $iA^{1/2}JA^{1/2}$. So, from the Lidskii-
Wielandt theorem we obtain, for any \( A, B \) in \( \mathbb{P}(2n) \)

\[
\| \hat{D}(A) - \hat{D}(B) \| \leq \| A^{1/2}JA^{1/2} - B^{1/2}JB^{1/2} \|
\]

\[
\leq \| A^{1/2}JA^{1/2} - A^{1/2}JB^{1/2} \| + \| A^{1/2}JB^{1/2} - B^{1/2}JB^{1/2} \|
\]

\[
= \| A^{1/2}J (A^{1/2} - B^{1/2}) \| + \| (A^{1/2} - B^{1/2}) JB^{1/2} \|
\]

\[
\leq \| A^{1/2}J \| \| A^{1/2} - B^{1/2} \| + \| A^{1/2} - B^{1/2} \| \| JB^{1/2} \|
\]

\[
= (\| A^{1/2} \| + \| B^{1/2} \|) \| A^{1/2} - B^{1/2} \|.
\]

By theorem X.1.3 in [3]

\[
\| A^{1/2} - B^{1/2} \| \leq \| A - B \|^{1/2}
\]

Combining these inequalities we obtain (29). Using the definitions of \( \| \cdot \| \)
and \( \| \cdot \|_2 \), we get (30) and (31) from this.

**Example** Let \( \gamma \) be a positive number, and let

\[
A = \begin{bmatrix} \gamma I & O \\ O & I \end{bmatrix}, \quad B = \begin{bmatrix} I & O \\ O & I \end{bmatrix}.
\]

Then

\[
\hat{D}(A) = \begin{bmatrix} \gamma^{1/2}I & O \\ O & \gamma^{1/2}I \end{bmatrix}, \quad \hat{D}(B) = \begin{bmatrix} I & O \\ O & I \end{bmatrix}
\]

So, if \( \gamma \geq 1 \), then \( \| \hat{D}(A) - \hat{D}(B) \| = \gamma^{1/2} - 1 \), and \( \| A - B \| = \gamma - 1 \). This shows that for large \( \gamma \), both \( \| \hat{D}(A) - \hat{D}(B) \| \) and \( \| A - B \|^{1/2} \) are close to \( \gamma^{1/2} \). Thus the bound given by Theorem 7 has the right order.

### 3 Proofs of Theorems 1, 3 and 4

We first prove the relation (19) in the special case \( t = 2 \). We use that to establish (15), and then derive (8) from it. From this we obtain (9) for all \( t \geq 1 \). Finally, we use the relation (15) to get Theorem 4.

We use two elementary properties of the operator norm \( \| \cdot \| \). For any matrix \( X \) we have \( \| X \|^2 = \| XX^T \| = \| X^T X \| \). If \( X \) and \( Y \) are any two matrices such that \( XY \) is normal, then \( \| XY \| \leq \| YX \| \). This is so because the norm of a normal matrix is equal to its spectral radius, the spectral radius
of $XY$ and $YX$ are equal, and in general the norm of $X$ is bigger than the spectral radius of $X$.

Now let $A \in \mathbb{P}(2n)$. Then $\hat{d}_1(A)$ is the maximum eigenvalue of $iA^{1/2}JA^{1/2}$. So, using the properties stated above, we get

$$\hat{d}_1^2(A) = \| A^{1/2}JA^{1/2} \|^2 = \| A^{1/2}JA^T A^{1/2} \|$$
$$\leq \| AJA^T \| = \| AJA \| = \hat{d}_1(A^2).$$

Apply these same considerations to the $k$th antisymmetric tensor power $\Lambda^k A$. This gives

$$\prod_{j=1}^k \hat{d}_j^2(A) = \| \Lambda^k (A^{1/2}JA^{1/2}) \|^2$$
$$\leq \| \Lambda^k A \Lambda^k J \Lambda^k A \Lambda^k J^T \|$$
$$= \| \Lambda^k(AJA) \| = \prod_{j=1}^k \hat{d}_j(A^2),$$

for all $1 \leq k \leq 2n$. When $k = 2n$, the two extreme sides of the last inequality are equal to $\det A^2$. This establishes (9) for the special case $t = 2$.

Now let $A, B$ be any two elements of $\mathbb{P}(2n)$ and put

$$U = \left( A^{-1/2}BA^{-1/2} \right)^{1/2} A^{1/2}B^{-1/2}.$$

Then $U^TU = I$, and so $U$ is orthogonal. From the formula (12) we see that

$$G(A, B) = A^{1/2}UB^{1/2} = B^{1/2}U^TA^{1/2},$$

for brevity put $G = G(A, B)$. By what we have already proved

$$\hat{d}_1^2(G) \leq \hat{d}_1(G^2) = \| GJG \|.$$

Using (32) we see that

$$\| GJG \| = \| A^{1/2}UB^{1/2}JB^{1/2}U^TA^{1/2} \|$$
$$\leq \| AU \| \| B^{1/2}JB^{1/2}U^T \|$$
$$\leq \| AU \| \| B^{1/2}JB^{1/2}U^T \|$$
$$= \| A \| \| B^{1/2}JB^{1/2} \| = \| A \| \hat{d}_1(B).$$
Thus, we have
\[ \hat{d}_1^2(G) \leq \|A\|\hat{d}_1(B). \] (33)

By the invariance of \(G(A, B)\) under congruence transformations, we have for every \(M \in GL(2n)\)
\[ M^TG(A, B)M = G(M^TAM, M^TBM). \]

If \(M\) is symplectic, then the symplectic eigenvalues of \(M^TG(A, B)M\) are the same as those of \(G(A, B)\). So
\[ \hat{d}_1(G(A, B)) = \hat{d}_1(G(M^TAM, M^TBM)). \] (34)

Choose \(M \in Sp(2n)\) so that \(M^TAM = \begin{bmatrix} D & O \\ O & D \end{bmatrix}\). Then \(\|M^TAM\| = \|D\| = \hat{d}_1(A)\). Using this fact we obtain from (33) and (34)
\[ \hat{d}_1(G(A, B)) \leq (\hat{d}_1(A)\hat{d}_1(B))^{1/2}. \] (35)

Once again, applying this to \(\Lambda^kA\) and \(\Lambda^kB\) we obtain the log majorisation (16).

The equation (14) gives a natural parametrisation of the geodesic joining \(A\) and \(B\). Hence
\[ A\#_{1/4}B = A\#_{1/2}(A\#_{1/2}B). \]

So, from (35) we obtain
\[
\hat{d}_1(A\#_{1/4}B) \leq \hat{d}_1^{1/2}(A) \hat{d}_1^{1/2}(A\#_{1/2}B) \\
\leq \hat{d}_1^{1/2}(A) \hat{d}_1^{1/4}(A) \hat{d}_1^{1/4}(B) \\
= \hat{d}_1^{3/4}(A) \hat{d}_1^{1/4}(B).
\]

This argument can be repeated to show that
\[ \hat{d}_1(A\#_tB) \leq \hat{d}_1^{1-t}(A) \hat{d}_1^t(B), \] (36)

for all dyadic rationals \(t\) in \([0, 1]\). By the continuity of symplectic eigenvalues, this is then true for all \(t\) in \([0, 1]\). Using antisymmetric tensor powers, we obtain (15) from (36). This completes the proof of Theorem 3.
The inequality (8) is a special case of (15), since $I#_t A = A^t$ for all $0 \leq t \leq 1$. If $t \geq 1$, let $s = 1/t$. Then from (5) we have $\hat{d}(A^s) \prec_{\log} \hat{d}^s(A)$. Replace $A^s$ by $A$ to obtain (9). This completes the proof of Theorem 1.

Now we turn to Theorem 4. It was shown by E. Cartan that the minimising problem in (18) has a unique solution, and this is also the unique positive definite solution of the equation

$$
\sum_{j=1}^{m} \log \left( X_{1/2}^{-1} A_j^{-1} X_{1/2}^{-1} \right) = 0.
$$

(37)

See e.g. [4], [5]. A direct description of $G$ suitable for some operator theoretic problems has been found recently. This describes $G(A_1, \ldots, A_m)$ as the limit of a “walk” in the Riemannian metric space $\mathbb{P}$. Consider the sequence $S_k$ defined as

$$
S_1 = A_1 \\
S_2 = S_1 #_{1/2} A_2 \\
\vdots \\
S_{k+1} = S_k #_{1/k+1} A_{k+1}, \quad \text{where} \quad \bar{k} = k \pmod{m}.
$$

Then it turns out that

$$
G(A_1, \ldots, A_m) = \lim_{k \to \infty} S_k.
$$

(38)

A stochastic version of this was proved in [14] and some simplifications made in [6]. The statement (38) was first proved in [12] and then a considerably simpler proof given in [15]. The effectiveness of this formula stems from the fact that it gives $G$ as a limit of the binary mean operation $#$ rather than the solution to an $m$-variable minimisation problem as in (17), or as the solution of an $m$-variable nonlinear matrix equation as in (37).

The majorisation relation (20) can be derived now from (15). First use it to get a majorisation for $\hat{d}(S_k)$ as in the proof of (15), and then take the limit as $k \to \infty$. The proof of the weighted version (19) is a modification of this idea. We can proceed either as in [6], first proving it for rational weights and then taking a limit, or as in [15] where the definition of $S_k$ is modified to include weights.

An element $A$ of $\mathbb{P}(2n)$ is called a Gaussian matrix (or, more precisely, the covariance matrix corresponding to a Gaussian state) if $A \pm \frac{1}{2} J$ is positive.
definite. Using (1) one can see that this condition is equivalent to saying that
\( d_1(A) \geq 1/2 \). Gaussian matrices are being intensely studied in the current
literature on quantum information. Theorems 1, 2, 4 have an interesting
corollary.

**Corollary 8.**

(i) Let \( A \) be a Gaussian matrix. Then for every \( 0 \leq t \leq 1 \), \( A^t \) is Gaussian.

(ii) Let \( A, B \) be Gaussian matrices. Then every point on the Riemannian
geodesic \( A^\#_t B, 0 \leq t \leq 1 \) is a Gaussian matrix. Thus the set of Gauss-
sian matrices is a geodesically convex set in the Riemannian metric
space \((\mathbb{P}(2n), \delta)\).

(iii) The geometric mean of any \( m \)-tuple of Gaussian matrices is Gaussian.

**Proof.** Imbedded in (10) is the inequality \( d_1(A^t) \geq d_1(A) \) for \( 0 \leq t \leq 1 \). So,
the statement (i) follows. Likewise (ii) and (iii) follow from Theorems 3 and
4. \( \square \)

### 4 Variational principles and a proof of Theorem 5

The Courant-Fischer-Weyl minmax principle is one of the most powerful
tools in the analysis of eigenvalues of Hermitian matrices. Such a principle
is known also for symplectic eigenvalues. We state it in a form suitable for
us and, for the convenience of the reader, indicate its proof. The idea is
borrowed from \[11\], p.39.

We denote the usual Euclidean inner product on \( \mathbb{R}^m \) or on \( \mathbb{C}^m \) by \( \langle \cdot, \cdot \rangle \).
In the latter case we assume that the inner product is conjugate linear in the
first variable. Given \( A \in \mathbb{P}(2n) \), introduce another inner product on \( \mathbb{C}^{2n} \) by putting
\[
(x, y) = \langle x, Ay \rangle.
\] (39)

Call the resulting inner product space \( \mathcal{H} \). Let \( A^\# = iA^{-1}J \). Then
\[
(x, A^\# y) = i\langle x, Jy \rangle = (A^\# x, y).
\]
So $A^\#$ is a Hermitian operator on $\mathcal{H}$. The symplectic eigenvalues of $A^{-1}$ arranged in decreasing order are $\frac{1}{d_1(A)} \geq \frac{1}{d_2(A)} \geq \cdots \geq \frac{1}{d_n(A)}$. The (usual) eigenvalues of $A^\#$ are $1_{d_1(A)} \geq 1_{d_2(A)} \geq \cdots \geq 1_{d_n(A)} \geq -1_{d_n(A)} \geq \cdots \geq -1_{d_1(A)}$.

So, from the usual minmax principle (Corollary III.1.2 in [3]) applied to $A^\#$ we get the following.

**The minmax principle for symplectic eigenvalues.**

Let $A \in \mathbb{P}(2n)$. Then for $1 \leq j \leq n$

$$\frac{1}{d_j(A)} = \max_{M \subset \mathbb{C}^{2n}} \min_{\dim M = j} \min_{x \in M, \langle x, Ax \rangle = 1} \langle x, iJx \rangle, \quad (40)$$

and also

$$\frac{1}{d_j(A)} = \min_{M \subset \mathbb{C}^{2n}} \max_{\dim M = 2n - j + 1} \max_{x \in M, \langle x, Ax \rangle = 1} \langle x, iJx \rangle, \quad (41)$$

One of the important corollaries of the minmax principle for Hermitian matrices is the interlacing principle for eigenvalues of $A$ and those of a principal submatrix. So it is for symplectic eigenvalues:

**The interlacing theorem for symplectic eigenvalues.**

Let $A \in \mathbb{P}(2n)$. Partition $A$ as $A = [A_{ij}]$ where each $A_{ij}$, $i, j = 1, 2$, is an $n \times n$ matrix. A matrix $B \in \mathbb{P}(2n - 2)$ is called an $s$-principal submatrix of $A$ if $B = [B_{ij}]$, and each $B_{ij}$ is an $(n - 1) \times (n - 1)$ principal submatrix of $A_{ij}$ occupying the same position in $A_{ij}$ for $i, j = 1, 2$. In other words, $B$ is obtained from $A$ by deleting, for some $1 \leq i \leq n$, the $i$th and $(n + i)$th rows and columns of $A$. Then

$$d_j(A) \leq d_j(B) \leq d_{j+2}(A), \quad 1 \leq j \leq n - 1,$$  

where we adopt the convention that $d_{n+1}(A) = \infty$.

The proof is similar to the one in the classical Hermitian case. See [3], p.59. This observation has been made in [13].

Now we come to the proof of Theorem 5. We begin with a proof of the inequalities (26). From the condition $AG^T - BC^T = I$ in (24), we have for
1 \leq i \leq n

\begin{align*}
1 &= \sum_{j=1}^{n} (a_{ij}g_{ij} - b_{ij}c_{ij}) \\
&\leq \sum_{j=1}^{n} \frac{1}{2} (a_{ij}^2 + g_{ij}^2) + \sum_{j=1}^{n} \frac{1}{2} (b_{ij}^2 + c_{ij}^2) \\
&= \sum_{j=1}^{n} \tilde{m}_{ij}.
\end{align*}

Applying the same argument to $M^T$ we see that the second inequality in (26) also holds. Now we can prove Part (i) of Theorem 5 in the special case $k = n$.

Without loss of generality, we may assume that $A = \tilde{D} = \begin{bmatrix} D & O \\ O & D \end{bmatrix}$. Let $M$ be any element of $Sp(2n)$ and decompose it as $M = \begin{bmatrix} P & Q \\ R & S \end{bmatrix}$ according to the rules (23) and (24). Then

\[ \text{tr} \ M^T \tilde{D} M = \text{tr} \ (P^TDP + Q^TDQ + R^TDR + S^TDS) \]

\[ = \sum_{i=1}^{n} d_i(A) \sum_{j=1}^{n} (p_{ij}^2 + q_{ij}^2 + r_{ij}^2 + s_{ij}^2) \]

\[ = \sum_{i=1}^{n} d_i(A) \sum_{j=1}^{n} 2\tilde{m}_{ij} \]

\[ \geq 2 \sum_{i=1}^{n} d_i(A), \]

using (26). When $M = I$, the two extreme sides of this equality are equal. Thus

\[ \min_{M \in Sp(2n)} \text{tr} M^T AM = 2 \sum_{j=1}^{n} d_j(A). \tag{43} \]

This is the special case of (21) when $k = n$.

Let $M$ be a $2n \times 2k$ matrix satisfying the condition $M^T J_{2n} M = J_{2k}$. Partition $M$ as $M = \begin{bmatrix} P' & Q' \\ R' & S' \end{bmatrix}$, where each block is an $n \times k$ matrix. Then
we can find a $2n \times 2n$ symplectic matrix $L = \begin{bmatrix} P & Q \\ R & S \end{bmatrix}$ in which each block is an $n \times n$ matrix and the first $k$ columns of $P', Q', R', S'$ are the columns of $P', Q', R', S'$, respectively. The matrix $M^TAM$ is then a $2k \times 2k$ s-principal submatrix of $L^TAL$.

The symplectic eigenvalues of $L^TAL$ are $d_1(A) \leq d_2(A) \leq \cdots \leq d_n(A)$. Let those of $M^TAM$ be $d'_1 \leq d'_2 \leq \cdots \leq d'_k$. By the interlacing principle $d'_j \geq d_j(A)$ for $1 \leq j \leq k$.

Now we can complete the proof of Theorem 5. First from the special case of (i) proved above we can see that

$$\text{tr } M^TAM \geq 2 \sum_{j=1}^{k} d'_j.$$  \hspace{1cm} (44)

Then from the interlacing principle we see that

$$\text{tr } M^TAM \geq 2 \sum_{j=1}^{k} d_j(A).$$  \hspace{1cm} (44)

By the same arguments we see that

$$\det M^TAM \geq \prod_{j=1}^{k} d'_j^2 \geq \prod_{j=1}^{k} d_j^2(A).$$  \hspace{1cm} (45)

There is equality in the inequalities (44) and (45) when $M$ is the matrix whose columns are the symplectic eigenvectors of $A$ corresponding to $d_1(A), \ldots, d_k(A)$. This proves Theorem 5. \hspace{1cm} ■

An immediate corollary of this theorem is that if $A, B \in \mathbb{P}(2n)$, then for all $1 \leq k \leq n$, we have

$$\sum_{j=1}^{k} d_j(A + B) \geq \sum_{j=1}^{k} d_j(A) + \sum_{j=1}^{k} d_j(B),$$  \hspace{1cm} (46)

$$\prod_{j=1}^{k} d_j^2(A + B) \geq \prod_{j=1}^{k} d_j^2(A) + \prod_{j=1}^{k} d_j^2(B).$$  \hspace{1cm} (47)
5 Proof of Theorem 6

We use a theorem of Elsner and Friedland [9]. This says that if $R$ is an $n \times n$ matrix with singular values $s_1(R) \geq \cdots \geq s_n(R)$, then there exist doubly stochastic matrices $P$ and $Q$ for which

$$s_n(R)^2 \ p_{ij} \leq |r_{ij}|^2 \leq s_1(R)^2 \ q_{ij}$$

(48)

for all $1 \leq i, j \leq n$.

The Euler decomposition theorem says that every symplectic matrix $M$ can be decomposed as

$$M = O_1 \left[ \begin{array}{cc} \Gamma & O \\ O & \Gamma^{-1} \end{array} \right] O_2^T,$$

(49)

where $O_1$ and $O_2$ are orthogonal and symplectic, and $\Gamma = \text{diag}(\gamma_1, \ldots, \gamma_n)$ with

$$\gamma_1 \geq \gamma_2 \geq \cdots \geq \gamma_n \geq 1.$$

(50)

There is a correspondence between $2n \times 2n$ real orthogonal symplectic matrices and $n \times n$ complex unitary matrices that tells us that we can find $n \times n$ unitary matrices $U$ and $V$ such that

$$U = X + iY, \quad V = Z + iW,$$

(51)

where $X, Y, Z, W$ are real, and

$$O_1 = \left[ \begin{array}{cc} X & -Y \\ Y & X \end{array} \right], \quad O_2 = \left[ \begin{array}{cc} Z & -W \\ W & Z \end{array} \right].$$

(52)

Both the theorems cited above may be found in [2] or [7].

Using (23), (49) and (52) we see that

$$A = X \Gamma Z^T + Y \Gamma^{-1} W^T, \quad B = X \Gamma W^T - Y \Gamma^{-1} Z^T,$$

$$C = Y \Gamma Z^T - X \Gamma^{-1} W^T, \quad G = Y \Gamma W^T + X \Gamma^{-1} Z^T.$$

(53)

From (51) we have

$$X = \frac{1}{2} (U + \overline{U}), \quad Y = \frac{1}{2i} (U - \overline{U}),$$

$$Z = \frac{1}{2} (V + \overline{V}), \quad W = \frac{1}{2i} (V - \overline{V}).$$

(54)
Here $\overline{U}$ stands for the entrywise complex conjugate of $U$. We will use the notation $U^*$ for $U^T$. Let
\begin{align*}
\Sigma &= \frac{1}{2}(\Gamma + \Gamma^{-1}), \quad \Delta = \frac{1}{2}(\Gamma - \Gamma^{-1}) \quad (55)
\end{align*}
Both are positive diagonal matrices. Let
\begin{align*}
\Sigma &= \text{diag}(\sigma_1, \ldots, \sigma_n), \quad \Delta = \text{diag}(\delta_1, \ldots, \delta_n). \quad (56)
\end{align*}

From the first equation in (53), and the equations (54) and (55) we see after a little calculation that
\begin{align*}
A &= \frac{1}{2} (U\Delta V^T + U\Sigma V^* + \overline{U}\Sigma V + \overline{U}\Delta V^*). \quad (57)
\end{align*}

Another calculation involving the entries of the matrices in (57) shows that
\begin{align*}
a_{ij} &= \sum_{k=1}^n \delta_k \Re(u_{ik}v_{jk}) + \sum_{k=1}^n \sigma_k \Re(u_{ik}\overline{v}_{jk}). \quad (58)
\end{align*}

Similar calculations with the other three equations in (53) show that
\begin{align*}
b_{ij} &= \sum_{k=1}^n \delta_k \Im(u_{ik}v_{jk}) + \sum_{k=1}^n \sigma_k \Im(u_{ik}\overline{v}_{jk}), \quad (59)
\end{align*}
\begin{align*}
c_{ij} &= \sum_{k=1}^n \delta_k \Im(u_{ik}v_{jk}) - \sum_{k=1}^n \sigma_k \Im(u_{ik}\overline{v}_{jk}), \quad (60)
\end{align*}
\begin{align*}
g_{ij} &= -\sum_{k=1}^n \delta_k \Re(u_{ik}v_{jk}) + \sum_{k=1}^n \sigma_k \Re(u_{ik}\overline{v}_{jk}). \quad (61)
\end{align*}

Squaring the equations (58)-(61), adding them and simplifying the resulting expression, we see that
\begin{align*}
\frac{1}{2} \left( a_{ij}^2 + b_{ij}^2 + c_{ij}^2 + g_{ij}^2 \right) &= \left| \sum_{k=1}^n \delta_k u_{ik}v_{jk} \right|^2 + \left| \sum_{k=1}^n \sigma_k u_{ik}\overline{v}_{jk} \right|^2. \quad (62)
\end{align*}

This shows that
\begin{align*}
\tilde{m}_{ij} &\geq \left| \sum_{k=1}^n \sigma_k u_{ik}\overline{v}_{jk} \right|^2. \quad (63)
\end{align*}
Now let $R = U\Sigma V^*$. Then the right-hand side of (63) is equal to $|r_{ij}|^2$. From (55) and (56) we see that the smallest singular value of $R$ is $\sigma_n = \frac{1}{2}(\gamma_n + \gamma_n^{-1})$. So, from (48) we see that there exists a doubly stochastic matrix $P$ such that

$$m_{ij} \geq |r_{ij}|^2 \geq \sigma_n^2 p_{ij}. \quad (64)$$

Since $\frac{1}{2}(x+x^{-1}) \geq 1$ for any positive number $x$, we have $\sigma_n \geq 1$. So, it follows from (64) that $\tilde{M}$ is doubly superstochastic. This proves the first statement of Theorem 6.

Now suppose $M$ is symplectic and orthogonal. We have noted earlier that then there exists a complex unitary matrix $U = X + iY$ such that

$$M = \begin{bmatrix} X & -Y \\ Y & X \end{bmatrix}.$$ 

It is clear from this that the matrix $\tilde{M}$ associated with this via (20) is doubly stochastic.

To prove the converse, return to the relation (62). We have already seen that if the second term on the right-hand side is equal to $|r_{ij}|^2$, then the matrix $R$ dominates entrywise a doubly stochastic matrix $P$. So, a necessary condition for $\tilde{M}$ to be doubly stochastic is that

$$\left| \sum_{k=1}^{n} \delta_k u_{ik} v_{jk} \right|^2 = 0 \quad \text{for all} \quad i, j.$$ 

Translated to matrices, this says that $U\Delta V^T = 0$. By the definition of $\Delta$ in (56), this is equivalent to the condition $\gamma_j - \gamma_j^{-1} = 0$ for $1 \leq j \leq n$; or in other words $\gamma_j = 1$ for $1 \leq j \leq n$. In turn, this means that $M$ is orthogonal. The proof of Theorem 6 is complete.

For the theory of majorisation and the role of doubly superstochastic matrices in it we refer the reader to the comprehensive treatise [16].

Let $x = (x_1, \ldots, x_n)$ be any element of $\mathbb{R}^n$ and let $x^\uparrow = (x_1^\uparrow, \ldots, x_n^\uparrow)$ be the vector obtained from $x$ by rearranging its coordinates in increasing order

$$x_1^\uparrow \leq x_2^\uparrow \leq \cdots \leq x_n^\uparrow.$$ 

We say $x$ is supermajorised by $y$, in symbols $x \prec^w y$, if for $1 \leq k \leq n$

$$\sum_{j=1}^{k} x_j^\uparrow \geq \sum_{j=1}^{k} y_j^\uparrow. \quad (65)$$
A fundamental theorem in the theory of majorisation says that the following two conditions are equivalent:

(i) An $n \times n$ matrix $A$ is doubly superstochastic.

(ii) $Ax \prec_w x$ for every positive $n$-vector $x$.

Inequalities like (46) express a supermajorisation. An alternative proof of Theorem 5(ii) can be obtained using Theorem 6.

6 Some remarks

Let $m_1, m_2, \ldots, m_k$ be positive integers, and let $n = m_1 + m_2 + \cdots + m_k$. If $A_j, 1 \leq j \leq k$, are $m_j \times m_j$ matrices, we write $\oplus A_j$ for their usual direct sum. This is the $n \times n$ block-diagonal matrix with entries $A_1, \ldots, A_k$ on its diagonal and zeros elsewhere. Given an $n \times n$ matrix $A$ partitioned into blocks as $A = [A_{ij}]$, where the diagonal blocks $A_{jj}$ are $m_j \times m_j$ in size, the pinching of $A$ is the block diagonal matrix $\oplus A_{jj}$. This is denoted by $C(A)$.

We introduce a version of direct sum and pinching adapted to the symplectic setting. Let

$$A_j = \begin{bmatrix} P_j & Q_j \\ R_j & S_j \end{bmatrix}, \ 1 \leq j \leq k$$

be $2m_j \times 2m_j$ matrices partitioned into blocks of size $m_j \times m_j$. The $s$-direct sum of $A_j$ is defined to be the $2n \times 2n$ matrix

$$\oplus^s A_j = \begin{bmatrix} \oplus P_j & \oplus Q_j \\ \oplus R_j & \oplus S_j \end{bmatrix}.$$ 

Then, one can see that $\oplus^s J_{2m_j} = J_{2m}$, the $s$-direct sum of symplectic matrices is symplectic, and the $s$-direct sum of positive definite matrices is positive definite. If $A$ is a $2n \times 2n$ and $B$ a $2m \times 2m$ positive definite matrix, then the symplectic eigenvalues of their $s$-direct sum are the symplectic eigenvalues of $A$ and $B$ put together. Let $C$ be a pinching on $n \times n$ matrices. Then we define the $s$-pinching of a $2n \times 2n$ matrix $A = \begin{bmatrix} P & Q \\ R & S \end{bmatrix}$ as

$$C^s(A) = \begin{bmatrix} C(P) & C(Q) \\ C(R) & C(S) \end{bmatrix}.$$ 

If $A$ is positive definite, then so is $C^s(A)$. Our next theorem gives a majorisation relation between the symplectic eigenvalues of $A$ and those of $C^s(A)$. 20
Theorem 9. Let $A$ be any element of $\mathbb{P}(2n)$ and let $C^s(A)$ be an $s$-pinching of $A$. Then
\[
\hat{d}(C^s(A)) \prec^w \hat{d}(A).
\]

Proof. It is enough to consider the case when $n = m_1 + m_2$ and $C$ is a pinching into two blocks; i.e.,
\[
T = \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix} \quad \text{and} \quad C(T) = \begin{bmatrix} T_{11} & O \\ O & T_{22} \end{bmatrix}.
\]
The general case can be derived by repeated applications of such p inchings. Partition the $2n \times 2n$ positive definite matrix $A$ as
\[
A = \begin{bmatrix} P & Q \\ Q^T & R \end{bmatrix} = \begin{bmatrix} P_{11} & P_{12} & Q_{11} & Q_{12} \\ P_{21} & P_{22} & Q_{21} & Q_{22} \\ Q_{11}^T & Q_{21}^T & R_{11} & R_{12} \\ Q_{12}^T & Q_{22}^T & R_{21} & R_{22} \end{bmatrix},
\]
where $P_{11}$ and $R_{11}$ are $m_1 \times m_1$, and $P_{22}$ and $R_{22}$ are $m_2 \times m_2$ matrices with $m_1 + m_2 = n$. Then
\[
C^s(A) = \begin{bmatrix} P_{11} & O & Q_{11} & O \\ O & P_{22} & O & Q_{22} \\ Q_{11}^T & O & R_{11} & O \\ O & Q_{22}^T & O & R_{22} \end{bmatrix}.
\]
Evidently, $C^s(A)$ is the $s$-direct sum of a $2m_1 \times 2m_1$ matrix $B$ and a $2m_2 \times 2m_2$ matrix $C$ defined as
\[
B = \begin{bmatrix} P_{11} \\ Q_{11}^T \\ R_{11} \end{bmatrix}, \quad C = \begin{bmatrix} P_{22} \\ Q_{22}^T \\ R_{22} \end{bmatrix}.
\]
The symplectic eigenvalues of $C^s(A)$ are the symplectic eigenvalues of $B$ and those of $C$ put together. So, given $1 \leq k \leq n$, there exist $k_1, k_2$ such that $1 \leq k_1 \leq m_1$, $1 \leq k_2 \leq m_2$, $k_1 + k_2 = k$ and
\[
\sum_{j=1}^{k} d_j(C^s(A)) = \sum_{j=1}^{k_1} d_j(B) + \sum_{j=1}^{k_2} d_j(C).
\]
Using (21) we can choose a $2m_1 \times 2k_1$ matrix $M_1$ and a $2m_2 \times 2k_2$ matrix $M_2$ such that
\[ M_1^T J_{2m_1} M_1 = J_{2k_1}, \quad M_2^T J_{2m_2} M_2 = J_{2k_2}, \]
and
\[ 2 \sum_{j=1}^{k_1} d_j(B) = \text{tr} M_1^T B M_1, \quad 2 \sum_{j=1}^{k_2} d_j(C) = \text{tr} M_2^T C M_2. \tag{68} \]
Let
\[ M_1 = \begin{bmatrix} P_1 & Q_1 \\ R_1 & S_1 \end{bmatrix}, \quad M_2 = \begin{bmatrix} P_2 & Q_2 \\ R_2 & S_2 \end{bmatrix}, \]
where $P_1, Q_1, R_1, S_1$ are $m_1 \times k_1$ matrices and $P_2, Q_2, R_2, S_2$ are $m_2 \times k_2$ matrices, and then let
\[ M = \begin{bmatrix} P_1 & O & Q_1 & O \\ O & P_2 & O & Q_2 \\ R_1 & O & S_1 & O \\ O & R_2 & O & S_2 \end{bmatrix}. \]
Using the relations (24) it can be seen that the $2n \times 2k$ matrix $M$ satisfies the equation
\[ M^T J_{2n} M = J_{2k}. \]
Further,
\[ \text{tr} M_1^T B M_1 + \text{tr} M_2^T C M_2 = \text{tr} M^T A M. \tag{69} \]
Combining (67), (68) and (69) we see that
\[ 2 \sum_{j=1}^{k} d_j(C^*(A)) = \text{tr} M^T A M. \]
It follows from (21) that
\[ 2 \sum_{j=1}^{k} d_j(C^*(A)) \geq 2 \sum_{j=1}^{k} d_j(A). \]
This proves (66).

Using standard arguments from the theory of majorisation one has the following consequence.
Corollary 10. Let $f : \mathbb{R}^n_+ \to \mathbb{R}$ be any function that is permutation invariant, concave and monotone increasing. Then

\[ f(d_1(C(A)), \ldots, d_n(C(A))) \geq f(d_1(A), \ldots, d_n(A)). \tag{70} \]

Among functions that satisfy the requirements of Corollary 10 are

\[ f(x_1, \ldots, x_n) = s_k(x_1, \ldots, x_n), \]

and

\[ f(x_1, \ldots, x_n) = \left( s_k(x_1, \ldots, x_n) \right)^{1/k}, \]

where $s_k$ are the elementary symmetric polynomials, $1 \leq k \leq n$. The functions

\[ f(x_1, \ldots, x_n) = \sum_{j=1}^{n} \frac{x_j}{1 + x_j}, \]

\[ f(x_1, \ldots, x_n) = \sum_{j=1}^{n} \log x_j, \]

\[ f(x_1, \ldots, x_n) = \left( \frac{1}{n} \sum_{j=1}^{n} x_j^r \right)^{1/r}, \ r < 1, \]

also satisfy the conditions in Corollary 10.

Finally, we present some inequalities between the symplectic eigenvalues and the usual eigenvalues of $A$.

Theorem 11. Let $A \in \mathbb{P}(2n)$. Let $d_j(A), 1 \leq j \leq n$ be the symplectic eigenvalues of $A$ counted as in (2) and $\hat{d}(A)$ the $2n$-tuple defined in (7). Let $\lambda_1(A), \lambda_2(A), \ldots, \lambda_{2n}(A)$ be the usual eigenvalues of $A$. Arranged in decreasing order they will be denoted by $\lambda^\downarrow_j(A)$ and in increasing order by $\lambda^\uparrow_j(A)$. Then

\[ (i) \quad \hat{d}(A) \prec \log \lambda(A) \tag{71} \]

\[ (ii) \quad \lambda^\downarrow_j(A) \leq d_j(A) \leq \lambda^\uparrow_{n+j}(A), \ 1 \leq j \leq n. \tag{72} \]

Proof. (i) By the arguments seen in Section 3

\[ \hat{d}_1(A) = \|A^{1/2}JA^{1/2}\| \leq \|AJ\| = \|A\| = \lambda^\downarrow_1(A). \]
Arguing as before with $\Lambda^k A$ we get

$$\prod_{j=1}^{k} \hat{d}_j(A) \leq \prod_{j=1}^{k} \lambda_j(A), \quad 1 \leq k \leq 2n.$$ 

When $k = 2n$ both sides are equal to $\det A$. This proves (71).

(ii) It follows from the inequality $iJ \leq I$ that $A^{1/2} i J A^{1/2} \leq A$. The eigenvalues of $A^{1/2} i J A^{1/2}$ arranged in increasing order are

$$-d_n(A) \leq \cdots \leq -d_1(A) \leq d_1(A) \leq \cdots \leq d_n(A),$$

and those of $A$ are

$$\lambda_1(A) \leq \cdots \leq \lambda_n(A) \leq \lambda_{n+1}(A) \leq \cdots \leq \lambda_{2n}(A).$$

By Weyl's monotonicity principle [3], p. 63

$$d_j(A) \leq \lambda_{n+j}(A) \quad \text{for} \quad 1 \leq j \leq n.$$ 

Replacing $A$ by $A^{-1}$ in this inequality we see that $\frac{1}{d_j(A)} \leq \frac{1}{\lambda_j(A)}$ for all $1 \leq j \leq n$. This proves (72).

Caveat. In this paper we have chosen $J_{2n} = \begin{bmatrix} O & I \\ -I & O \end{bmatrix}$. Some authors choose instead $J_{2n} = J_2 \oplus \cdots \oplus J_2$ (n copies). Then the class of symplectic matrices, as well as the symplectic eigenvalues change. All our theorems remain valid with these changes.

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