Abstract. In this paper we studied the Möbius disjointness for C*-algebras with its automorphisms and proved that Möbius function is linearly disjoint from canonical anticommutation relation algebra with Bogoliubov automorphism if and only if its Voiculescu-Brown entropy of the automorphism is zero. We also obtained some results of Möbius disjointness for irrational rotation algebras, finite von Neumann algebras etc.

Keywords: Möbius function, Singular spectrum, Bogoliubov automorphism, Irrational rotation.

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1. Introduction

Let $\mu$ be the Möbius function, that is $\mu(n)$ is 0 if $n$ is not square-free, and is $(-1)^t$ if $n$ is a product of $t$ distinct primes. We say $\mu$ is linearly disjoint from a flow $(X, T)$, where $X$ is compact topological space and $T : X \to X$ is a continuous map, if

$$\frac{1}{N} \sum_{n \leq N} \mu(n) f(T^n x) \to 0, \quad \text{as } N \to \infty$$

for any $f$ in $C(X)$ and $x$ in $X$. The Möbius disjointness conjecture posted by Peter Sarnak (which is also known as Sarnak’s conjecture) states that $\mu$ is linearly disjoint from every flow $(X, T)$ when the topological entropy of $(X, T)$ is zero. Recently, B. Green and T. Tao [10] showed that the Möbius function is strongly asymptotically orthogonal to any polynomial nilsequences; Liu and Sarnak [14] showed that the Möbius function is linearly disjoint from an analytic skew product on the 2-torus with additional conditions on the Fourier coefficients.

In the view of operator algebra, the algebra $C(X)$ of all continuous function on a compact Hausdorff space is a C*-algebra, any point $x$ gives a pure state $\rho_x$ on $C(X)$, and the continuous map $T : X \to X$ will induce an endomorphism $\alpha_T$ on $C(X)$. Then the condition (1) can be rephrased as

$$\frac{1}{N} \sum_{n \leq N} \mu(n) \rho_x(\alpha^n_T(f)) \to 0, \quad \text{as } N \to \infty$$

for any $f$ in $C(X)$ and $x$ in $X$. Hence it is reasonable to consider the Möbius disjointness for a C*-algebra with its endomorphism.

Let $\mathfrak{A}$ be a C*-algebra and $\alpha$ its endomorphism, The pair $(\mathfrak{A}, \alpha)$ is said to be a noncommutative flow. When $\mathfrak{A}$ is a von Neumann algebra, we need its endomorphism $\alpha$ to be weak-operator continuous. We then can define Möbius disjointness for $(\mathfrak{A}, \alpha)$. The Möbius function $\mu$ is linearly disjoint
from \((\mathfrak{A}, \alpha)\) if
\[
\frac{1}{N} \sum_{n \leq N} \mu(n) \rho(\alpha^n(A)) \to 0, \quad \text{as } N \to \infty
\]
for any state \(\rho\) on \(\mathfrak{A}\) and \(A\) in \(\mathfrak{A}\).

The Möbius disjointness conjecture also concerns the Kolmogorov-Sinai entropy of a flow \((X, T)\), i.e., the topological entropy for \((X, T)\). There are several ways to define an entropy for a \(C^*\)-algebra with an automorphism (or endomorphism). In [7], A. Connes and E. Størmer introduced entropy \(h_\tau(\alpha)\) of automorphism \(\alpha\) of II\(_1\) von Neumann algebra \(\mathcal{M}\) with a tracial state \(\tau\). Later, A. Connes, H. Narnhofer and W. Thirring [6] defined a dynamic entropy \(h_\varphi(\alpha)\) for any \(C^*\) algebra \(\mathfrak{A}\) (or von Neumann algebra) with its automorphism \(\alpha\) with respect to a \(\alpha\)-invariant state \(\varphi\) of \(\mathfrak{A}\). This entropy for \(C^*\) algebras is called CNT entropy. In 1995 D. Voiculescu [24] introduced topological entropy \(ht(\alpha)\) for a nuclear \(C^*\) algebra \(\mathfrak{A}\) with its automorphism \(\alpha\). N. Brown [3] showed that the entropy given by Voiculescu is good for exact \(C^*\) algebra and Dykema [9] shows that \(h_\varphi(\alpha) \leq ht(\alpha)\) for a exact \(C^*\) algebra \(\mathfrak{A}\) with its automorphism \(\alpha\), where \(\varphi\) is an \(\alpha\)-invariant state on \(\mathfrak{A}\). The entropy is called Voiculescu-Brown entropy. We will show that the Möbius function is linearly disjoint from any factor of type II\(_1\) with its automorphism in sense of weak-operator topology, but its CNT entropy might be greater zero. Hence we will pick Voiculescu-Brown entropy for exact \(C^*\)-algebras. As N. Brown pointed out in [3], the Voiculescu-Brown entropy is also well-defined for endomorphisms of exact \(C^*\) algebras.

With Möbius disjointness and Voiculescu-Brown entropy for a noncommutative flow \((\mathfrak{A}, \alpha)\), we formulate Möbius disjointness conjecture for exact \(C^*\) algebras.

**Conjecture 1.1.** The Möbius function \(\mu\) is linearly disjoint from a noncommutative flow \((\mathfrak{A}, \alpha)\) when the Voiculescu-Brown entropy of \((\mathfrak{A}, \alpha)\) is zero, where \(\mathfrak{A}\) is a unital exact \(C^*\)-algebra and \(\alpha\) is an endomorphism of \(\mathfrak{A}\).

It is clear that the Möbius disjointness conjecture for exact \(C^*\) algebra implies the Möbius disjointness conjecture.

In this paper, we will show that the Möbius disjointness conjecture for exact \(C^*\) algebra is true when

1. \((\mathfrak{A}, \alpha)\), \(\mathfrak{A}\) is finite-dimensional \(C^*\) algebra and \(\alpha\) is any automorphism of \(\mathfrak{A}\); \(\mathfrak{A}\) is AF-algebra and \(\alpha\) is inner automorphism; \(\mathfrak{A}\) is the algebra of all compact operators on a Hilbert space with its automorphism whose entropy is zero;
2. \((\text{CAR}(\mathcal{H}), \alpha_U)\), \(\text{CAR}(\mathcal{H})\) is the canonical anti commutation relation algebra with respect to a Hilbert space \(\mathcal{H}\) and \(\alpha\) is the Bogoliubov automorphism induced by a unitary operator on \(\mathcal{H}\);
3. \((A_\theta, \alpha)\), \(A_\theta\) is the irrational rotation algebra and \(\alpha\) is noncommutative toral automorphism on \(A_\theta\);
4. \((C_r^*(F_\mathbb{Z}), \alpha)\), \(C_r^*(F_\mathbb{Z})\) is the reduced \(C^*\) algebra of free group on \(\mathbb{Z}\) generators and \(\alpha\) is the shift on generators.

The rest of paper will contain six sections. Section two we will introduce some basic properties for Möbius disjointness for \(C^*\) algebras. Section three the Möbius disjointness for finite-dimensional \(C^*\) algebras is studied and we also show that the Möbius disjointness conjecture for exact \(C^*\) algebras holds for AF-algebras with its inner automorphisms. Section four we will show that the Möbius function is linearly disjoint from CAR algebras with Bogoliubov automorphisms if and only if its entropy is zero. Section five the Möbius disjointness for irrational rotation algebra is studied and
we obtain results parallel to the commutative torus. Section six we show that Möbius function is linearly disjoint from reduced free group C$^*$ algebra with its shift on generators. Section seven we explain why there is no Möbius disjointness conjecture for finite von Neumann algebras in the sense of weak operator topology.

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2. Preliminaries

We begin by recalling the definition of Voiculescu-Brown entropy.

Let $\mathfrak{A}$ be an exact (equivalently, nuclearly embeddable) C$^*$ algebra and $\alpha$ an automorphism of $\mathfrak{A}$. Let $\pi : \mathfrak{A} \to \mathbb{B}(\mathcal{H})$ be a faithful *-representation. For a finite set $\Omega \subset \mathfrak{A}$ and $\delta > 0$ we denote by $CPA(\pi, \Omega, \delta)$ the collection of triples $\varphi, \psi, \mathfrak{B}$ where $\mathfrak{B}$ is a finite dimensional C$^*$ algebra and $\varphi : \mathfrak{A} \to \mathfrak{B}$ and $\psi : \mathfrak{B} \to \mathbb{B}(\mathcal{H})$ are contractive completely positive maps such that $\| (\psi \circ \varphi)(a) - \pi(a) \| < \delta$ for all $a \in \Omega$. This collection is nonempty by nuclear embeddability. We define $rcp(\Omega, \delta)$ to be the infimum of rank $\mathfrak{B}$ over all $(\varphi, \psi, \mathfrak{B}) \in CPA(\pi, \Omega, \delta)$ with rank referring to the dimension of a maximal abelian C$^*$ subalgebra. We then set $ht(\alpha, \Omega, \delta) = \lim sup_{n \to \infty} n^{-1} \log rcp(\Omega \cup \alpha \Omega \cup \cdots \cup \alpha^{n-1}, \delta)$.

where the last supremum is taken over all finite sets $\Omega \subset \mathfrak{A}$. The quantity $ht(\alpha)$ is a C$^*$-dynamical invariant which is the Voiculescu-Brown entropy of $\alpha$.

For the definition of Connes-Narnhofer-Thirring entropy (briefly CNT entropy) for C$^*$ algebras, we skip it here and refer to [3]. Let $\mathfrak{A}$ be a unital nuclear C$^*$ algebra and $\alpha$ its automorphism. Brown [3] showed that $ht(\alpha) = h_\varphi(\alpha)$, where $\varphi$ is $\alpha$-invariant state on $\mathfrak{A}$ and $h_\varphi(\alpha)$ is the CNT entropy of $\alpha$ with respect to $\varphi$. In general $ht(\alpha) \geq h_\varphi(\alpha)$.

Next, we will introduce the Möbius disjointness for a C$^*$ algebra $\mathfrak{A}$ with an endomorphism $\alpha$. We begin with the definition of the Möbius function $\mu$.

The Möbius function $\mu(n)$, $n = 1, 2, 3, \ldots$ is defined by

$$
\mu(n) = \begin{cases} 
1 & \text{if } n = 1 \\
0 & \text{if } n \text{ is not square free} \\
(-1)^t & \text{if } n \text{ is a product of } t \text{ distinct primes}
\end{cases}
$$

Throughout the paper, the function $e(x)$ will denote $e^{2\pi ix}$ for real $x$.

Lemma 2.1. Let $p$ be positive integer and $0 \leq l < p$. For arbitrary $h > 0$,

$$
\frac{1}{N} \sum_{n \leq N, n \equiv l \pmod{p}} \mu(n) e(a_d n^d + a_{d-1} n^{d-1} + \cdots + a_1 n + a_0) = O((\log N)^{-h})
$$
where the implied constant may depend on $h$ and $p$, but is independent of any of coefficients $a_d, \cdots, a_0$. In particular,

$$\frac{1}{N} \sum_{n \leq N} \mu(n) e(\theta n) = O((\log N)^{-h})$$

uniformly in $\theta$.

This second equation in the lemma is proved by Davenport [8]. The general case is proved by Hua [12].

Let $A$ be a $C^\ast$ algebra and $\alpha$ an automorphism of $A$. The pair $(A, \alpha)$ is a noncommutative flow. The Möbius function $\mu$ is linearly disjoint from $(A, \alpha)$ if

$$\frac{1}{N} \sum_{n=1}^{N} \mu(n) \rho(\alpha^n(A)) \to 0 \quad \text{as } N \to \infty$$

for every $A$ in $A$ and any state $\rho$ on $A$.

Similarly, we can define Möbius disjointness for von Neumann algebras. Let $M$ be a von Neumann algebra and $\alpha$ is normal automorphism of $M$, we say $\mu$ is weakly linearly disjoint from $(M, \alpha)$ if

$$\frac{1}{N} \sum_{n=1}^{N} \mu(n) \rho(\alpha^n(A)) \to 0 \quad \text{as } N \to \infty$$

for every $A$ in $M$ and any normal state $\rho$ on $M$.

**Lemma 2.2.** Let $A$ be a $C^\ast$ algebra and $\alpha \in \text{Aut}(A)$. Suppose that $\alpha^q = \text{Id}$ for some $q$ in $\mathbb{N}$. Then $\mu$ is linearly disjoint from $(A, \alpha)$.

**Proof.** Lemma 2.1 can be applied here directly. □

Thus we will focus on aperiodic automorphisms of $C^\ast$ algebras.

**Lemma 2.3.** Let $A$ be a $C^\ast$ algebra and $\alpha \in \text{Aut}(A)$. Suppose that $\Omega$ is a subset of $A$ such that $A$ is the norm closure of linear span of $\Omega$. If

$$\frac{1}{N} \sum_{n=1}^{N} \mu(n) \rho(\alpha^n(A)) \to 0 \quad \text{as } N \to \infty$$

for any state $\rho$ on $A$ and $A$ in $\Omega$, then $\mu$ is linearly disjoint from $(A, \alpha)$.

**Proof.** For any $A$ in $\mathfrak{A}$, any $\epsilon > 0$, there exist $\lambda_1, \ldots, \lambda_k \in \mathbb{C}$ and $A_1, \ldots, A_k \in \Omega$ such that $\|A - \sum_{j=1}^{k} \lambda_j A_j\| < \epsilon/2$. Since $A_1, \ldots, A_k$ satisfies the property list in the lemma, there exists a $N_0 \in \mathbb{N}$ such that for any $N \geq N_0$, we have

$$\left| \frac{1}{N} \sum_{n=1}^{N} \mu(n) \rho(\alpha^n(A_j)) \right| < \frac{1}{2} \sum_{j=1}^{k} |\lambda_j| \epsilon, \quad j = 1, \ldots, k.$$
Then we have that
\[
\left| \frac{1}{N} \sum_{n=1}^{N} \mu(n)\rho(\alpha^n(A)) \right| \\
\leq \left| \frac{1}{N} \sum_{n=1}^{N} \mu(n)\rho(\alpha^n(A - \sum_{j=1}^{k} \lambda_j A_j)) \right| + \left| \frac{1}{N} \sum_{n=1}^{N} \mu(n) \sum_{j=1}^{k} \lambda_j \rho(\alpha^n(A_j)) \right| \\
\leq \varepsilon/2 + \sum_{j=1}^{k} |\lambda_j| \left| \frac{1}{N} \sum_{n=1}^{N} \mu(n)\rho(\alpha^n(A_j)) \right| \leq \varepsilon/2 + \varepsilon = \varepsilon.
\]

This shows that \(\mu\) is linearly disjoint from \((\mathfrak{A}, \alpha)\). \(\square\)

**Lemma 2.4.** Let \(\mathfrak{A}\) be a C*-algebra and \(\alpha \in \text{Aut}(\mathfrak{A})\). Suppose that \(\mathfrak{A}_0\) is a C*-subalgebra of \(\mathfrak{A}\) such that \(\alpha(\mathfrak{A}_0) = \mathfrak{A}_0\). If \(\mu\) is linearly disjoint from \((\mathfrak{A}, \alpha)\), then \(\mu\) is linearly disjoint from \((\mathfrak{A}_0, \alpha|_{\mathfrak{A}_0})\).

**Proof.** By Hahn-Banach Theorem, any state \(\rho\) of \(\mathfrak{A}_0\) can be extended to a state \(\rho'\) of \(\mathfrak{A}\) such that \(\rho'|_{\mathfrak{A}_0} = \rho\), we see the lemma holds. \(\square\)

**Lemma 2.5.** Let \(\mathfrak{A}\) be a C*-algebra and \(\alpha \in \text{Aut}(\mathfrak{A})\). Then \(\mu\) is linearly disjoint from \((\mathfrak{A}, \alpha)\) if and only if \(\mu\) is linearly disjoint from \((\mathfrak{A} \rtimes_{\alpha} \mathbb{Z}, \text{Ad}(U))\), where \(U \in \mathfrak{A} \rtimes_{\alpha} \mathbb{Z}\) is the unitary element implementing \(\alpha\).

**Proof.** By Lemma 2.4, we only have to show that if \(\mu\) is linearly disjoint from \((\mathfrak{A}, \alpha)\), then \(\mu\) is linearly disjoint from \((\mathfrak{A} \rtimes_{\alpha} \mathbb{Z}, \text{Ad}(U))\). So we assume that \(\mu\) is linearly disjoint from \((\mathfrak{A}, \alpha)\). Let \(\Omega = \{AU^k | A \in \mathfrak{A}, k \in \mathbb{Z}\}\). Then the linear span of \(\Omega\) is norm dense in \(\mathfrak{A}\). By Lemma 2.3, we only have to show
\[
\frac{1}{N} \sum_{n=1}^{N} \mu(n)\rho(U^n(\text{Ad}(U)^k)U^{*n}) \to 0 \quad \text{as } N \to \infty
\]
Since \(U^n(\text{Ad}(U)^k)U^{*n} = \alpha^n(A)U^k\) and \(\rho(U^k)\) gives a bounded linear functional on \(\mathfrak{A}\), we have the lemma proved. \(\square\)

**Corollary 2.6.** If the conjecture is true for any \((\mathfrak{A}, \text{Ad}(U))\), where \(\mathfrak{A}\) is an exact C*-algebra and \(\text{Ad}(U)\) is an inner automorphism of \(\mathfrak{A}\). Then the conjecture is true when automorphism is considered.

**Proof.** Let \(\mathfrak{B}\) be an exact C*-algebra and \(\beta \in \text{Aut}(\mathfrak{B})\). Suppose that \(ht_{\mathfrak{B}}(\beta) = 0\). By [3], we have that \(ht_{\mathfrak{B}}(\beta) = 0\), and where \(U \in \mathfrak{B} \rtimes_{\beta} \mathbb{Z}\) is the unitary element which implements \(\beta\). Then \(\mu\) is linearly disjoint from \((\mathfrak{B} \rtimes_{\beta} \mathbb{Z}, \text{Ad}(U))\). By Lemma 2.5, we obtain \(\mu\) is linearly disjoint from \((\mathfrak{B}, \beta)\). \(\square\)

**Lemma 2.7.** Let \(\mathfrak{A}\) be a C*-algebra and \(\alpha, \beta \in \text{Aut}(\mathfrak{A})\). Then \(\mu\) is linearly disjoint from \((\mathfrak{A}, \alpha)\) if and only if \(\mu\) is linearly disjoint from \((\mathfrak{A}, \beta\alpha\beta^{-1})\).

**Proof.** Suppose that \(\mu\) is linearly disjoint from \((\mathfrak{A}, \alpha)\). Then
\[
\frac{1}{N} \sum_{n=1}^{N} \mu(n)\rho(\alpha^n(A)) \to 0 \quad \text{as } N \to \infty
\]
for any $A$ in $\mathfrak{A}$ and state $\rho$ of $\mathfrak{A}$. Replace $\rho$ by $\rho \circ \beta$ and $A$ by $\beta^{-1}(A)$, we have that

$$\frac{1}{N} \sum_{n=1}^{N} \mu(n) \rho(\beta \alpha^n(\beta^{-1}(A))) \to 0 \quad \text{as} \quad N \to \infty$$

for any $A$ in $\mathfrak{A}$ and state $\rho$ of $\mathfrak{A}$. Since $\beta \alpha^n \beta^{-1} = (\beta \alpha \beta^{-1})^n$, the lemma holds. \qed

**Lemma 2.8.** Let $\mathfrak{A}$ be a $C^*$ algebra and $\alpha \in \text{Aut}(\mathfrak{A})$. Suppose $\mathcal{I}$ is an ideal of $\mathfrak{A}$ such that $\alpha(\mathcal{I}) = \mathcal{I}$. Let $\overline{\alpha}$ denote the induced automorphism on $\mathfrak{A}/\mathcal{I}$. If $\mu$ is linearly disjoint from $(\mathfrak{A}, \alpha)$, then $\mu$ is linearly disjoint from $(\mathfrak{A}/\mathcal{I}, \overline{\alpha})$.

**Proof.** Let $\Phi : \mathfrak{A} \to \mathfrak{A}/\mathcal{I}$ be the canonical quotient homomorphism. Then $\overline{\alpha} \circ \Phi = \Phi \circ \alpha$. For any $A$ in $\mathfrak{A}/\mathcal{I}$, there exists $A_0$ in $\mathfrak{A}$ such that $\Phi(A_0) = A$. Let $\rho$ be a state of $\mathfrak{A}/\mathcal{I}$. Then $\rho(\overline{\alpha}^n(A)) = \rho(\Phi(\alpha^n(A_0)))$. While $\rho \circ \Phi$ is a state on $\mathfrak{A}$, we see that the lemma holds. \qed

**Lemma 2.9.** Let $\mathfrak{A}$ be a $C^*$ algebra and $\alpha \in \text{Aut}(\mathfrak{A})$. Let $\{\rho_m\}_m$ be a sequence of bounded linear functionals on $\mathfrak{A}$ such that $\|\rho_m - \rho\| \to 0$ as $m \to \infty$ for some bounded linear functional $\rho$ on $\mathfrak{A}$. If for $A$ in the unit ball of $\mathfrak{A}$ and any $m$

$$\frac{1}{N} \sum_{n \leq N} \mu(n) \rho_m(\alpha^n(A)) \to 0 \quad \text{as} \quad N \to \infty$$

then

$$\frac{1}{N} \sum_{n \leq N} \mu(n) \rho(\alpha^n(A)) \to 0 \quad \text{as} \quad N \to \infty.$$

**Proof.** We will show that for any $\epsilon > 0$, there exists $N_0$ in $\mathbb{N}$ such that for $N > N_0$ we have $\left| \frac{1}{N} \sum_{n \leq N} \mu(n) \rho(\alpha^n(A)) \right| < \epsilon$. Since $\|\rho_m - \rho\| \to 0$ as $m \to \infty$, there exist $m_0$ in $\mathbb{N}$ such that $\|\rho_m - \rho\| < \epsilon/2$ for $m \geq m_0$. From the fact that $\frac{1}{N} \sum_{n \leq N} \mu(n) \rho_m(\alpha^n(A)) \to 0$ as $N \to \infty$, we have there exists $N_0$ in $\mathbb{N}$ such that $\left| \frac{1}{N} \sum_{n \leq N} \mu(n) \rho_m(\alpha^n(A)) \right| \leq \epsilon/2$ for $N \geq N_0$. Then for $N \geq N_0$

$$\left| \frac{1}{N} \sum_{n \leq N} \mu(n) \rho(\alpha^n(A)) \right|$$

$$= \left| \frac{1}{N} \sum_{n \leq N} \mu(n) (\rho - \rho_m)(\alpha^n(A)) \right| + \left| \frac{1}{N} \sum_{n \leq N} \mu(n) \rho_m(\alpha^n(A)) \right|$$

$$\leq \epsilon/2 + \epsilon/2 = \epsilon.$$

This proved the lemma. \qed

**Remark 2.1.** Let $\mathcal{A}$ be a $C^*$ algebra, $\alpha$ its automorphism, and $\varphi$ is an $\alpha$-invariant state on $\mathcal{A}$. Suppose that $\mu$ is linearly disjoint from $(\mathcal{A}, \alpha)$. Let $\mathcal{M} = \pi_\varphi(\mathcal{A})''$ where $\pi_\varphi$ is the GNS representation arising from $\varphi$ and $\hat{\alpha}$ be the automorphism of $\mathcal{M}$ extended by $\alpha$. Then $\mu$ is weakly disjoint from $(\mathcal{M}, \hat{\alpha})$.

### 3. Finite-dimensional $C^*$ Algebras

In this section, we will show that the Möbius disjointness conjecture for exact $C^*$ algebra holds for finite-dimensional $C^*$ algebras with its automorphisms.

First we would like to give the matrix version for Lemma 2.8.
Lemma 3.1. Let $p$ be a positive integer and $0 \leq l < p$. Let $U_1, \ldots, U_d$ be unitary matrices in $M_k(\mathbb{C})$, $A_1, \ldots, A_d$ matrices in $M_k(\mathbb{C})$ with $\|A_j\| \leq 1$, $j = 1, \ldots, d$, and $\phi_1, \ldots, \phi_d$ polynomials with real coefficients where $d$ is an positive integer. Then for arbitrary $h > 0$,

$$
\frac{1}{N} \sum_{n \equiv (l \mod p)} \mu(n) \text{tr}_k(U_1^{\phi_1(n)} A_1 U_2^{\phi_2(n)} A_2 \cdots U_d^{\phi_d(n)} A_d) = t^{d/2} O((\log N)^{-h})
$$

uniformly in unitary elements $U_1, \ldots, U_d$, $A_1, \ldots, A_d$ and independent in coefficients of $\phi_j$, $j = 1, \ldots, d$, where $\text{tr}_k$ is the normalized trace of $M_k(\mathbb{C})$, the implied constant only depends on the maximal degree of $\phi_j$s and $p$. If $d = 2$, we have

$$
\frac{1}{N} \sum_{n \equiv (l \mod p)} \mu(n) \text{tr}_k(U_1^{\phi_1(n)} A_1 U_2^{\phi_2(n)} A_2) = O((\log N)^{-h})
$$

which is independent of $k$.

Proof. We assume that $U_j = \text{diag}(e(\theta_1^{(j)}), \ldots, e(\theta_k^{(j)}))$ for $j = 1, \ldots, d$ and $A_j = (a_{ts}^{(j)})$ for $j = 1, \ldots, d$. Then

$$
\text{tr}_k(U_1^{\phi_1(n)} A_1 U_2^{\phi_2(n)} A_2 \cdots U_d^{\phi_d(n)} A_d) = \text{tr}_k((e(\theta_1^{(1)} \phi_1(n)) a_{t_1s_1}^{(1)} \cdots (e(\theta_d^{(d)} \phi_d(n)) a_{t_ds_d}^{(d)}))_{t,s}
$$

$$
= \frac{1}{k} \sum_{t_1, \ldots, t_d = 1}^k e(\theta_1^{(1)} \phi_1(n) + \cdots + \theta_d^{(d)} \phi_d(n)) a_{t_1t_2}^{(1)} \cdots a_{t_dt_d}^{(d)}
$$

Hence

$$
\frac{1}{N} \sum_{n \leq N} \mu(n) \text{tr}_k(U_1^{\phi_1(n)} A_1 U_2^{\phi_2(n)} A_2 \cdots U_d^{\phi_d(n)} A_d)
$$

$$
= \frac{1}{N} \sum_{n \leq N} \mu(n) \frac{1}{k} \sum_{t_1, \ldots, t_d = 1}^k e(\theta_1^{(1)} \phi_1(n) + \cdots + \theta_d^{(d)} \phi_d(n)) a_{t_1t_2}^{(1)} \cdots a_{t_dt_d}^{(d)}
$$

$$
= \frac{1}{k} \sum_{t_1, \ldots, t_d = 1}^k \left( \frac{1}{N} \sum_{n \leq N} \mu(n) e(\theta_1^{(1)} \phi_1(n) + \cdots + \theta_d^{(d)} \phi_d(n)) \right) a_{t_1t_2}^{(1)} \cdots a_{t_dt_d}^{(d)}
$$

By Lemma 2.1,

$$
\frac{1}{N} \sum_{n \leq N} \mu(n) e(\theta_1^{(1)} \phi_1(n) + \cdots + \theta_d^{(d)} \phi_d(n)) = O((\log N)^{-h})
$$

which the implied constant only depends on the maximal degree of $\phi_j$s and $p$. Note that $\frac{1}{k} \sum_{t_1, \ldots, t_d = 1}^k |a_{t_1t_2}^{(1)} \cdots a_{t_dt_d}^{(d)}|$ is the trace of $A_1' A_2' \cdots A_d'$, where $A_j' = (|a_{ts}^{(j)}|)_{t,s}$ and $\|A_j'\| \leq \sqrt{k}$. Therefore

$$
\frac{1}{k} \sum_{t_1, \ldots, t_d = 1}^k |a_{t_1t_2}^{(1)} \cdots a_{t_dt_d}^{(d)}| = \text{tr}_k(A_1' A_2' \cdots A_d') \leq \|A_1'\| \cdots \|A_d'\| \leq k^{d/2}
$$

This completes the proof for general $d$. If $d = 2$, we have

$$
\frac{1}{k} \sum_{t_1, t_2 = 1}^k |a_{t_1t_2}^{(1)} a_{t_1t_2}^{(2)}| \leq \frac{1}{k} \left( \sum_{t_1, t_2 = 1}^k |a_{t_1t_2}^{(1)}|^2 \right)^{1/2} \left( \sum_{t_1, t_2 = 1}^k |a_{t_1t_2}^{(2)}|^2 \right)^{1/2} = \|A_1\| \cdot \|A_2\| \leq 1.
$$
Hence we see that when $d = 2$, it is independent of $k$. \hfill \Box

**Theorem 3.2.** Suppose that $\mathfrak{A}$ be a finite dimensional $C^*$ algebra and $\alpha \in \text{Aut}(\mathfrak{A})$. Then $ht(\alpha) = 0$ and $\mu$ is linearly disjoint from $(\mathfrak{A}, \alpha)$.

**Proof.** Let $\psi$ be a faithful $\alpha$-invariant state on $\mathfrak{A}$. By GNS construction, we assume that $\mathfrak{A}$ acts on finite-dimensional Hilbert space $L^2(\mathfrak{A}, \psi)$ and $\alpha(A) = UAU^*$ for some unitary operator $U$ on $L^2(\mathfrak{A}, \psi)$. Let $\tau$ be the normalized trace. For any state $\rho$ on $\mathfrak{A}$, by Hahn-Banach theorem, there is a state $\rho'$ extending $\rho$. Then we have to show
\[
\frac{1}{N} \sum_{n \leq N} \mu(n)\tau(U^n A U^n B) \to 0 \quad \text{as } N \to \infty
\]

for any $A, B$ in $\mathcal{B}(L^2(\mathfrak{A}, \psi))$, but it is directly from Lemma 2.3. Hence $\mu$ is linearly disjoint from $(\mathfrak{A}, \alpha)$.

By the definition of Voiculescu-Brown entropy for exact $C^*$ algebras, we see that $ht(\alpha) = 0$. \hfill \Box

**Proposition 3.3.** Let $\mathcal{H}$ be a Hilbert space and $\alpha$ is an automorphism of $\mathcal{C}(\mathcal{H})$, the algebra of all compact operators on $\mathcal{H}$. Then $\mu$ is linearly disjoint from $(\mathcal{C}(\mathcal{H}), \alpha)$.

**Proof.** We will show that for any automorphism $\alpha$ of $\mathcal{C}(\mathcal{H})$ there exists a unitary operator $U$ on $\mathcal{H}$ such that $\alpha = \text{Ad}(U)$. Let $\{E_{jk}\}_{j,k}$ be a system of matrix units for $\mathcal{C}(\mathcal{H})$. Since $E_{11}, \alpha(E_{11})$ are equivalent in $\mathcal{B}(\mathcal{H})$, there exist a partial isometry $V$ such that $VV^* = E_{11}$ and $V^*V = \alpha(E_{11})$. Then a unitary operator $U$ implementing $\alpha$ can be given as $U = \sum_{j=1}^{\infty} E_{j1} V \alpha(E_{1j})$.

It is known that the dual of $\mathcal{C}(\mathcal{H})$ is the space of all normal linear functionals of $\mathcal{B}(\mathcal{H})$, i.e. the predual $\mathcal{B}(\mathcal{H})^*$ of $\mathcal{B}(\mathcal{H})$. Hence we can focus on vector state on $\mathcal{B}(\mathcal{H})$.

Since every operator is finite sum of positive operator and every positive compact operator can be approximated by finite linearly combination of rank one projections. By Lemma 2.3 it suffices to show that
\[
\frac{1}{N} \sum_{n \leq N} \mu(n)\langle U^n P \xi U^n \eta, \eta \rangle \to 0, \quad \text{as } N \to \infty,
\]

for any unit vector $\xi, \eta$ in $\mathcal{H}$.

Meanwhile
\[
\left| \frac{1}{N} \sum_{n \leq N} \mu(n)\langle U^n P \xi U^n \eta, \eta \rangle \right| = \left| \frac{1}{N} \sum_{n \leq N} \mu(n)\langle \langle U^n \eta, \xi \rangle U^n \eta \rangle \right|
\]
\[
= \left| \frac{1}{N} \sum_{n \leq N} \mu(n)\langle U^n \eta, \xi \rangle \langle U^n \eta, \eta \rangle \right|
\]
\[
= \left| \frac{1}{N} \sum_{n \leq N} \mu(n)\langle U^n \eta \otimes U^n \eta, \xi \otimes \eta \rangle \right|
\]
\[
\leq \max_{z \in \mathbb{T}} \left| \frac{1}{N} \sum_{n \leq N} \mu(n)z^n \right| = O((\log N)^{-h})
\]
for any fixed $h > 0$ by [3] or Lemma 2.1.

A $C^*$ algebra $\mathfrak{A}$ is approximately finite dimensional (or AF) if it is the closure of an increasing union of finite dimensional subalgebras $\mathfrak{A}_k$.

**Proposition 3.4.** Let $\mathfrak{A}$ be an AF-algebra and $U$ a unitary element in $\mathfrak{A}$. Then $ht(Ad(U)) = 0$ and $\mu$ is linearly disjoint from $(\mathfrak{A}, Ad(U))$.

**Proof.** For any $A$ in $\mathfrak{A}$ and $N$ in $\mathbb{N}$, there exists a unitary element $U_0$ and $A_0$ in a finite dimensional subalgebra $\mathfrak{B}_N$ such that

$$||U^n AU^{*-n} - U_0^n A_0 U_0^{*-n}|| \leq \frac{1}{N}, n = 1, \ldots, N.$$  

We may assume that $\mathfrak{B}_N \subset \mathfrak{B}_{N+1}$ and $\cup_N \mathfrak{B}_N$ is norm dense in $\mathfrak{A}$. Let $\tau$ be a trace on $\mathfrak{A}$. For any state $\rho$ on $\mathfrak{A}$, there exists $B_N$ in $\mathfrak{B}_N$ such that $\rho|_{\mathfrak{B}_N} = \tau(B_N)|_{\mathfrak{B}_N}$. We have $||\tau(B_N) - \rho|| \rightarrow 0$ as $N \rightarrow \infty$. By Lemma 2.9 we only have to check whether the following term converges to zero when $\rho$ is replaced by $\tau(B_m)$ for any $m$. Moreover, we will continue using the technique in the rest of paper without explanation.

$$\frac{1}{N} \sum_{n \leq N} \mu(n)\rho(U^n AU^{*-n}) \leq \frac{1}{N} \sum_{n \leq N} \mu(n)\rho(U^n AU^{*-n} - U_0^n A_0 U_0^{*-n}) + \frac{1}{N} \sum_{n \leq N} \mu(n)\rho(U_0^n A_0 U_0^{*-n}).$$

By Theorem [3,3] or Lemma [3,1] we see that it converges to zero as $N$ goes to infinity. Hence $\mu$ is linearly disjoint from $(\mathfrak{A}, Ad(U))$.

By [6] Corollary V.4., we have that $h_\phi(AdU) = 0$ for any $Ad(U)$-invariant state $\phi$ on $\mathfrak{A}$. Since $\mathfrak{A}$ is nuclear algebra, we obtain $ht(Ad(U)) = 0$.

**Corollary 3.5.** Let $\mathfrak{B}$ be a $C^*$ algebra and $U$ is a unitary element in $\mathfrak{B}$. Suppose that $\mathfrak{B}$ can be imbedded into an AF-algebra. Then $\mu$ is linearly disjoint from $(\mathfrak{B}, Ad(U))$ and $ht(Ad(U)) = 0$.

**Proof.** By Lemma 2.4 and Proposition 2.1 in [3].

**Corollary 3.6.** Let $\mathfrak{A}$ be a $C^*$ algebra and $\alpha \in Aut(\mathfrak{A})$. Suppose that $\mathfrak{A} \rtimes_\alpha \mathbb{Z}$ is AF embeddable. Then $\mu$ is linearly disjoint from $(\mathfrak{A}, \alpha)$.

**Proof.** By Corollary 3.5 $\mu$ is linearly disjoint from $(\mathfrak{A} \rtimes_\alpha \mathbb{Z}, Ad(U))$ where $U$ is the unitary in $\mathfrak{A}$ implementing $\alpha$. By Lemma 2.5 $\mu$ is linearly disjoint from $(\mathfrak{A}, \alpha)$.

**Remark 3.1.** This corollary says that if $\mu$ is not linearly disjoint from $(\mathfrak{A}, \alpha)$, then $\mathfrak{A} \rtimes_\alpha \mathbb{Z}$ is not AF-embeddable.

Let $\mathfrak{A}$ be an AF-algebra and $\alpha \in Aut(\mathfrak{A})$ then we denote by $H_\alpha$ the subgroup of $K_0(\mathfrak{A})$ given by all elements of the form $\alpha_*(x) - x$ for $x \in K_0(\mathfrak{A})$.

**Corollary 3.7.** If $H_\alpha \cap K^0(\mathfrak{A}) = \{0\}$, then $\mu$ is linearly disjoint from $(\mathfrak{A}, \alpha)$.

**Proof.** By result in [2] we see that $\mathfrak{A} \rtimes_\alpha \mathbb{Z}$ is AF embeddable.
4. CAR Algebras

In this section, we will investigate the Möbius disjointness for Canonical Anticommutation Relation algebras with Bogoliubov automorphisms.

Let $\mathcal{H}$ be a Hilbert space. The full Fock space $T(\mathcal{H})$ of $\mathcal{H}$ is given by

$$T(\mathcal{H}) = \bigoplus_{n=0}^{\infty} \otimes^n \mathcal{H},$$

where $\otimes^0 \mathcal{H}$ is $\mathbb{C}1$.

Define a linear map $P$ of $T(\mathcal{H})$ given by

$$P(\xi_1 \otimes \cdots \otimes \xi_n) = \frac{1}{n!} \sum_{\sigma \in S_n} (-1)^{\sigma} \xi_{\sigma(1)} \otimes \cdots \otimes \xi_{\sigma(n)},$$

where $S_n$ is the permutation group on $\{1, 2, \ldots, n\}$. One can check that $P$ is a projection on $T(\mathcal{H})$. We denote by $\Lambda^n \mathcal{H}$ the Hilbert subspace $P(\otimes^n \mathcal{H})$ and by $\Lambda \mathcal{H}$ the Hilbert space $P(T(\mathcal{H})) = \bigoplus_{n=0}^{\infty} \Lambda^n \mathcal{H}$.

For $f \in \mathcal{H}$ we define a linearly map $a(f)$ on $\Lambda \mathcal{H}$ given by $a(f) : \Lambda^n \mathcal{H} \to \Lambda^{n+1} \mathcal{H}$

$$a(f)(\xi_1 \wedge \cdots \wedge \xi_n) = f \wedge \xi_1 \wedge \cdots \wedge \xi_n$$

The operators $a(f), a(g)$ satisfy Canonical Anticommutation Relation (CAR):

\[
\begin{align*}
  a(f)a(g) + a(g)a(f) &= 0 \\
  a(f)a(g)^* + a(g)^*a(f) &= \langle f, g \rangle 1
\end{align*}
\]

Then the CAR algebra $\text{CAR}(\mathcal{H})$ the $*$-algebra generated by $a(f)$ represented on $\Lambda \mathcal{H}$. We have that $\text{CAR}(\mathcal{H})$ is a C$^*$ algebra when $\mathcal{H}$ is a Hilbert space. When $\dim \mathcal{H} = 1$, one see that $\text{CAR}(\mathcal{H})$ is isomorphic to $M_2(\mathbb{C})$. In this case $a(\xi)$ is a partial isometry from $\mathbb{C}1$ onto $\mathbb{C} \xi$ and $\Lambda \mathcal{H}$ is 2-dimensional Hilbert space spanned by $1, \xi$.

For any unitary operator $U$ on $\mathcal{H}$, $\alpha_U$ is an automorphism of $\text{CAR}(\mathcal{H})$ given by $\alpha_U(a(f)) = a(Uf)$ for any $f$ in $\mathcal{H}$. The automorphism $\alpha_U$ arising from a unitary operator $U$ on $\mathcal{H}$ is called Bogoliubov automorphism of $\text{CAR}(\mathcal{H})$.

Let $T \in B(\mathcal{H})$ and $0 \leq T \leq 1$. The quasi-free state $\varphi_T$ on $\text{CAR}(\mathcal{H})$ given by

$$\varphi_T(a(g_1)^* \cdots a(g_i)^* a(f_1) \cdots a(f_n)) = \delta_{m,n} \det(\langle Tf_i, g_j \rangle)_{i,j=1}^{n}$$

and $\varphi_T(1) = 1$

**Lemma 4.1.** Let $U$ be a unitary operator on a Hilbert space $\mathcal{H}$ whose spectrum has absolutely continuous part. Then $\mu$ is not (weakly) linearly disjoint from $(B(\mathcal{H}), \text{Ad}(U))$ and $\mu$ is not linearly disjoint from $(\text{CAR}(\mathcal{H}), \alpha_U)$.

**Proof.** Suppose that the absolutely continuous part $U_\alpha$ of $U$ acts on the subspace $\mathcal{H}_\alpha$ of $\mathcal{H}$.

We can write $\mathcal{H}_\alpha = \int_\mathbb{T} \mathcal{H}_z d\lambda(z)$ and $U_\alpha = \int_\mathbb{T} z d\lambda(z)$ where $\lambda$ is the Lebesgue measure on $\mathbb{T}$.

Let $\xi = \int_\mathbb{T} \xi_z d\lambda(z)$ be a unit vector in $\mathcal{H}_\alpha$. We then define $\xi_k = \int_\mathbb{T} z^k \xi_z d\lambda(z)$ for $k \in \mathbb{Z}$. By direct computation, $\{\xi_k\}_{k \in \mathbb{Z}}$ is orthogonal family of unit vectors in $\mathcal{H}_\alpha$.

Let $P_k$ be the projection of $\mathcal{H}$ onto $\mathbb{C}\xi_k$ and $T = \sum_{k=1}^{\infty} \mu(k) P_{-k}$. Then $T$ is bounded.

One can checks that $U_\alpha^n P_k U_\alpha^{-n} = P_{k+n}$ for any $k, n$ in $\mathbb{Z}$. Then we see that

$$\sum_{n \leq N} \mu(n) \langle U_\alpha^n T U_\alpha^{-n} \xi, \xi \rangle = \sum_{n \leq N} |\mu(n)|.$$
It is known that $\frac{1}{N}\sum_{n \leq N} |\mu(n)|$ does not converge to zero. Hence $\mu$ is not (weakly) linearly disjoint from $(B(\mathcal{H}), \text{Ad}(U))$.

For CAR algebra $\text{CAR}(\mathcal{H})$, we consider the quasi-free state $\varphi_{(T+I)/2}$. Then

$$\varphi_{(T+I)/2}(\alpha^*_U (a(\xi)^* a(\xi))) = \varphi_{(T+I)/2}(a(U^n \xi)^* a(U^n \xi))$$
$$= \left(\frac{T+I}{2} U^n \xi, U^n \xi\right)$$
$$= \frac{1}{2} (\mu(n) + 1),$$

and

$$\sum_{n \leq N} \mu(n) \varphi_{(T+I)/2}(\alpha^*_U (a(\xi)^* a(\xi))) = \frac{1}{2} \sum_{n \leq N} (|\mu(n)| + \mu(n)).$$

Hence $\mu$ is not linearly disjoint from $(\text{CAR}(\mathcal{H}), \alpha_U)$. $\square$

By Lemma 4.1 we will study unitary operator $U$ on a Hilbert space $\mathcal{H}$ whose spectrum measure is singular. The techniques here benefit a lot from Lemma 5.1 in [23].

**Lemma 4.2.** $\mu$ is weakly linearly disjoint from $(B(\mathcal{H}), \text{Ad}(U))$ if the spectrum of $U$ is singular.

**Proof.** It suffices to show that

$$\frac{1}{N} \sum_{n \leq N} \mu(n) \langle U^n T^2 U^n \xi, \xi \rangle \rightarrow 0, N \rightarrow \infty$$

for any unit vector $\xi$ in $\mathcal{H}$ and any positive element $T$ in the unit ball of $B(\mathcal{H})$.

Since the spectral measure of $U$ is singular, there is a set $\sigma \subset \mathbb{T}$ such that the following hold for given $\delta > 0$

(a) $\sigma = \sigma_1 \cup \sigma_2 \cup \cdots \cup \sigma_q$ where $\sigma_j (1 \leq j \leq q)$ are arcs, $\sigma_j = \{ e(\theta) : \alpha_j \leq \theta \}$

(b) $q \max_{1 \leq j \leq q} (\beta_j - \alpha_j) < \delta$

(c) If $E(\sigma)$ is the spectral projection of $U$ for set $\sigma$ then $\| (1 - E(\sigma)) \xi \| < \delta$

For a number $m > 0$ consider the $mq$ arcs $\sigma_{1,m}, \ldots, \sigma_{mq,m}$ obtained by subdividing each $\sigma_j$ into $m$ arcs of equal length. Let $E(\sigma_{j,m})$ be the corresponding spectral projections of $U$, which are pairwise
orthogonal. With \( \theta_j \in \sigma_{j,m} \),
\[
\| \sum_{j=1}^{mq} e(n\theta_j)E(\sigma_{j,m})\xi - U^n\xi \|^2 \\
\leq 2 \sum_{j=1}^{mq} \| e(\theta_j n)E(\sigma_{j,M})\xi - U^n E(\sigma_{j,M})\xi \|^2 + 2\| U^n E(\sigma)\xi - U^n \xi \|^2 \\
\leq 2 \sum_{j=1}^{mq} \| (e(\theta_j n) - U^n)E(\sigma_{j,m})\xi \|^2 \| E(\sigma_{j,m})f \|^2 + 2\delta^2 \\
\leq 2 \max_{1 \leq j \leq mq} \| (e(\theta_j n) - U^n)E(\sigma_{j,m})\xi \|^2 + 2\delta^2 \\
\leq 2 \max_{1 \leq j \leq mq} \sup_{z \in \sigma_{j,m}} |e(\theta_j n) - z^n|^2 + 2\delta^2 \\
\leq 2(2\pi (\text{length } \sigma_{j,m})n^2 + 2\delta^2 \\
\leq 2(\frac{2\pi n}{m} \max_{1 \leq j \leq MN} (\beta_j - \alpha_j))^2 + 2\delta^2 \\
< \frac{8\pi^2 n^2 \delta^2}{m^2 q^2} + 2\delta^2 < \frac{8\pi^2 N^2 \delta^2}{m^2 q^2} + 2\delta^2
\]

Let \( m = \left\lfloor \frac{\log N}{q} \right\rfloor + 1 \) and \( \delta < \frac{1}{81N} \). Then we have
\[
\frac{8\pi^2 n^2 \delta^2}{m^2 q^2} + 2\delta^2 < \frac{8\pi^2 N^2 \frac{1}{81 N^2}}{N} + 2\delta^2 \leq \frac{2\pi^2}{81 \log N} + \frac{2}{81 N^2} < \frac{1}{\log N}
\]

Therefore
\[
\| \sum_{j=1}^{mq} e(n\theta_j)E(\sigma_{j,m})\xi - U^n\xi \| < \log^{-1/2} N, \quad n = 1, \ldots, N
\]
\[
\begin{align*}
\left| \frac{1}{N} \sum_{n \leq N} \mu(n)(U^n T^2 U^n \xi, \xi) \right| & = \left| \frac{1}{N} \sum_{n \leq N} \mu(n)\| T U^n \xi \|^2 \right| \\
\leq \left| \frac{1}{N} \sum_{n \leq N} \mu(n)\| T U^n \xi - T \sum_{j=1}^{mq} e(n\theta_j)E(\sigma_{j,m})\xi \|^2 \right| + \left| \frac{1}{N} \sum_{n \leq N} \mu(n)\| T \sum_{j=1}^{mq} e(n\theta_j)E(\sigma_{j,m})\xi \|^2 \right| \\
\leq \log^{-1/2} N + \sum_{j,k=1}^{mq} \frac{1}{N} \sum_{n \leq N} \mu(n)e(n(\theta_j - \theta_k))(TE(\sigma_{j,m})\xi, E(\sigma_{k,m})\xi) \\
\leq O((\log N)^{-1}) \sum_{j,k=1}^{mq} |(TE(\sigma_{j,m})\xi, E(\sigma_{k,m})\xi)| \leq O((\log N)^{-1}) \sum_{j,k=1}^{mq} \| E(\sigma_{j,m})\xi \| \| E(\sigma_{k,m})\xi \| \\
\leq O((\log N)^{-1}) \sqrt{mq} \leq O((\log N)^{-1/2})
\end{align*}
\]
This completes the proof the lemma. \( \square \)

Now we need to find the relation between that Möbius disjointness for \( B(\mathcal{H}) \) with inner automorphism and that for \( CAR(\mathcal{H}) \) with Bogoliubov automorphism.
\textbf{Theorem 4.3.} Let $U$ be a unitary operator on a Hilbert space $H$ and $\tilde{U} = U \oplus I$ on Hilbert space $H \oplus \mathbb{C} \xi_0(= H)$ for some unit vector $\xi_0$. Then $\mu$ is linearly disjoint from $(\text{CAR}(H), \alpha_U)$ if $\mu$ is weakly disjoint from $(\mathcal{B}(\tilde{H}^\otimes m), \text{Ad}(\tilde{U}^* \otimes m))$ for any $m \geq 1$. In particular, when $U$ has pure point spectrum 1, then $\mu$ is linearly disjoint from $(\text{CAR}(H), \alpha_U)$ if $\mu$ is weakly disjoint from $(\mathcal{B}(\tilde{H}^\otimes m), \text{Ad}(U^* \otimes m))$ for any $m \geq 1$.

\textbf{Proof.} Suppose that $\mu$ is weakly linearly disjoint from $(\mathcal{B}(\tilde{H}^\otimes m), \text{Ad}(\tilde{U}^* \otimes m))$.

Since the linearly span of

$$a(\xi_1) \cdots a(\xi_m), m \geq 1, a(\eta_k)^* \cdots a(\eta_l)^*, k \geq 1$$

and

$$a(\eta_k)^* \cdots a(\eta_l)^* a(\xi_1) \cdots a(\xi_m), k, m \geq 1$$

is norm dense in $\text{CAR}(H)$, we will consider element $A$ in $\text{CAR}(H)$ in the three forms as above.

When $A = a(\xi_1) \cdots a(\xi_m)$, we have

$$\| \frac{1}{N} \sum_{n \leq N} \mu(n) a_U^n(A) \| = \| \frac{1}{N} \sum_{n \leq N} \mu(n) a(U^n \xi_1) \cdots a(U^n \xi_m) \|$$

$$= \| \frac{1}{N} \sum_{n \leq N} \mu(n) a(U^n \xi_1 \wedge \cdots \wedge U^n \xi_m) \|$$

$$= \| \frac{1}{N} \sum_{n \leq N} \mu(n) (U^n \otimes \cdots \otimes U^n) \xi_1 \wedge \cdots \wedge \xi_m \|$$

$$\leq \frac{1}{N} \sum_{n \leq N} \mu(n) (U^n \otimes \cdots \otimes U^n) \| \xi_1 \wedge \cdots \wedge \xi_m \|$$

$$\leq \frac{1}{N} \max_{z \in \text{sp}(U \otimes \cdots \otimes U)} | \sum_{n \leq N} \mu(n) z^n | \| \xi_1 \wedge \cdots \wedge \xi_m \| \to 0$$

The reason for the last step is in \[3\].

When $A = a(\eta_k)^* \cdots a(\eta_l)^*$, we consider its adjoint $A^*$. When $A = a(\eta_k)^* \cdots a(\eta_l)^* a(\xi_1) \cdots a(\xi_m)$, we assume that $k \leq m$. If $k > m$, we consider $A^*$. Let $\rho$ be a state on $\text{CAR}(H)$. Then

$$\rho(a_U^n(A)) = \rho(a_U^n(a(\eta_k)^* \cdots a(\eta_l)^* a(\xi_1) \cdots a(\xi_m)))$$

$$= \rho(a(U^n \eta_k)^* \cdots a(U^n \eta_l)^* a(U^n \xi_1) \cdots a(U^n \xi_m))$$

$$= \rho(a(U^n \eta_1 \wedge \cdots \wedge U^n \eta_k)^* a(U^n \xi_1 \wedge \cdots \wedge U^n \xi_m))$$

Let $P_n$ be the projection of $\otimes^n \tilde{H}$ onto $\otimes^n H$. Let $V$ be transformation from $\otimes^k \tilde{H}$ into $\otimes^m \tilde{H}$ given by $V \xi = \xi \otimes \xi_0 \cdots \otimes \xi_0$ for any $\xi$ in $\otimes^k \tilde{H}$.

For any $\xi$ in $\otimes^m \tilde{H}$ and $\eta$ in $\otimes^m \tilde{H}$, let $B(\eta, \xi) = \rho(a(P P_k V^* \eta)^* a(P P_m \xi))$. Then $B(\cdot, \cdot)$ is linearly in the first variable and conjugate-linearly in the second variable. Moreover $B(\cdot, \cdot)$ is bounded. Hence there exists a bounded element $T$ in $\mathcal{B}(\otimes^m \tilde{H})$ such that $\rho(a(P P_k V^* \eta)^* a(P P_m \xi)) = \langle T \xi, \eta \rangle$ for any $\xi, \eta$ in $\otimes^m \tilde{H}$.

Let $\eta = \eta_1 \wedge \cdots \wedge \eta_k \otimes \xi_0 \cdots \otimes \xi_0$ and $\xi = \xi_1 \wedge \cdots \wedge \xi_m$. Then $P P_m \tilde{U} \otimes \cdots \otimes \tilde{U} \eta = \tilde{U} \otimes \cdots \otimes \tilde{U} \xi$ and $P P_k V^* \tilde{U} \otimes \cdots \otimes \tilde{U} \eta = P P_k U^* \eta_1 \wedge \cdots \wedge U^* \eta_k = U^* \eta_1 \wedge \cdots \wedge U^* \eta_k$. On the other hand,

$$\rho(a_U^n(A)) = \langle \tilde{U}^n \otimes \cdots \otimes \tilde{U}^n T \tilde{U}^n \otimes \cdots \otimes \tilde{U}^n \xi, \eta \rangle$$
Hence by the assumption, we have that \( \frac{1}{n} \sum_{n \leq N} \mu(n)\rho(\alpha^0_n(A)) \to 0 \) for any element \( A \) in the linearly span of \( a(\xi_1) \cdots a(\xi_m), m \geq 1, a(\eta_1)^* \cdots a(\eta_k)^*, k \geq 1 \) and \( a(\eta_k)^* a(\xi_1) \cdots a(\xi_m), k, m \geq 1 \).

Therefore \( \mu \) is linearly disjoint from \( (CAR(\mathcal{H}), \alpha_U) \).

If \( U \) has pure point spectrum 1, we let \( \xi'_0 \) be its unit eigenvector. Then we define \( V' \) to be the transformation from \( \otimes^k \mathcal{H} \) into \( \otimes^m \mathcal{H} \) by \( V'\xi = \xi \otimes \xi'_0 \otimes \cdots \otimes \xi'_0 \). Similarly, we have

\[
\rho(\alpha^0_n(U)) = \langle U^n \otimes \cdots \otimes U^{*n}TU^n \otimes \cdots \otimes U^n, \eta \rangle.
\]

Then the conclusion follows by a similar argument above.

\[ \square \]

**Remark 4.1.** It is clear that if \( \mu \) is linearly disjoint from \( (CAR(\mathcal{H}), \alpha_U) \), then \( \mu \) is weakly linearly disjoint from \( (B(\mathcal{H}), Ad(U)) \). Later we will show that they are equivalent.

The CNT entropy of Bogoliubov automorphism was first computed by E. Størmer and D. Voiculescu [23]. Later in [15], Neshveyev give a complete description of the CNT entropy of \( \alpha_U \) with respect to a quasi-free state \( \varphi_T \) which is

\[
h_{\varphi_T}(\alpha_U) = \int_T \text{Tr}(\eta(T) + \eta(I - T))d\lambda(z),
\]

where \( \eta(x) = -x \log x, 0 \leq T \leq I \) on \( \mathcal{H} \) commuting with \( U \) and \( T|_{\mathcal{H}_a} = \int_T \frac{d\lambda(\cdot)}{\lambda(\cdot)} \), \( \lambda \) is the Lebesgue measure on \( \mathbb{T} \) and \( \lambda(\mathbb{T}) = 1 \), \( \mathcal{H}_a \) is the Hilbert subspace corresponding to absolutely continuous part of \( U \). The entropy \( h_{\varphi_T}(\alpha_U) = 0 \) if and only if \( T = 0 \) is a projection commuting with \( U \).

Hence if \( U \) has absolutely continuous part, the CNT entropy will be greater than zero with respect to a quasi-free state. Moreover if \( U \) has absolutely continuous part, then \( ht(\alpha_U) > 0 \). Since CAR algebra is nuclear, \( ht(\alpha_U) = h_{\varphi}(\alpha_U) = 0 \) if the spectrum of \( U \) is singular (see also [17]).

**Theorem 4.4.** The Möbius function \( \mu \) is linearly disjoint from \( (CAR(\mathcal{H}), \alpha_U) \) if and only if \( ht(\alpha_U) = 0 \).

**Proof.** If \( ht(\alpha_U) = 0 \), then the spectrum of \( U \) is singular and the spectrum of \( U^{\otimes n} \) and \( U^{\otimes n} \) are singular again. By Lemma 3.4 we have that \( \mu \) is weakly linearly disjoint from \( (B(\mathcal{H}^{\otimes n}), Ad(U^{\otimes n})) \). By Theorem 4.3 we have that \( \mu \) is linearly disjoint from \( (CAR(\mathcal{H}), \alpha_U) \).

If \( ht(\alpha_U) > 0 \), then the spectrum of \( U \) has absolutely continuous part and by Lemma 4.1 we have that \( \mu \) is not linearly disjoint from \( (CAR(\mathcal{H}), \alpha_U) \). \[ \square \]

**Corollary 4.5.** Let \( U \) be a unitary operator on a Hilbert space \( \mathcal{H} \). Then the spectrum of \( U \) is singular if and only if \( \mu \) is weakly linearly disjoint from \( (B(\mathcal{H}), Ad(U)) \) (or equivalently \( \mu \) is linearly disjoint from \( (CAR(\mathcal{H}), \alpha_U) \)).

**Remark 4.2.** This might give a new way to determine whether the spectrum of a unitary operator on a Hilbert space is singular. In [1], El Houcein el Abdalaoui, Mariusz Lemanczyk and Thierry de la Rue show that all sequences realized in the symbolic models associated to some rank-one constructions are orthogonal to the Möbius function. Furthermore the maximal spectral type of those maps are singular.

5. **Irrational Rotation Algebras**

In this section, we will study Möbius disjointness for the noncommutative torus automorphisms of irrational rotation algebras which is parallel to affine automorphisms of commutative torus.

Let \( \mathcal{A}_\theta \) be irrational rotation algebra, i.e. the \( C^* \) algebra generated by unitary elements \( U, V \) satisfying \( UV = e(\theta)VU \) with irrational number \( \theta \), where \( e(\theta) = e^{2\pi i \theta} \).
Let \( \alpha_b, b = (b_1, b_2) \in \mathbb{T}^2 \) be the automorphism of \( \mathcal{A}_\theta \) such that \( \alpha_b(U) = e(b_1)U \) and \( \alpha_b(V) = e(b_2)V \). Let \( T \) be in \( SL(2, \mathbb{Z}) \). If \( T = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \), then \( \alpha_T : \mathcal{A}_\theta \rightarrow \mathcal{A}_\theta \) given by \( \alpha_T(U) = U^{a_{21}}V^{a_{21}} \) and \( \alpha_T(V) = U^{a_{12}}V^{a_{22}} \) is an automorphism of \( \mathcal{A}_\theta \). We denote by \( \alpha_{T,b} \) the automorphism \( \alpha_T\alpha_b \) of \( \mathcal{A}_\theta \).

**Proposition 5.1.** The Möbius function \( \mu \) is linearly disjoint from \((\mathcal{A}_\theta, \alpha_{T,b})\) when \( ht(\alpha_{T,b}) = 0 \).

**Proof.** Let \( a_1, a_2 \) be the eigenvalues of \( T \) and \( \lambda_j = \max\{|a_j|, 1\} \). By the results of [24] and [13], we have that

\[
\frac{1}{2} \log |\lambda_1 \lambda_2| \leq ht(\alpha_{T,b}) \leq \log |\lambda_1 \lambda_2|.
\]

Hence \( ht(\alpha_{T,b}) = 0 \) if and only if \( |a_1| = |a_2| = 1 \). Since \( T \) is in \( SL(2, \mathbb{Z}) \), we have \( a_1a_2 = 1 \) and \( a_1 + a_2 \) must be an integer (or equivalently \( a_j \) is a root of \( x^2 + px + 1 = 0 \) for \( p = -2, -1, 0, 1, 2 \)). Then the Jordan standard form of \( T \) will be one of the following

\[
\begin{pmatrix}
1 & 0 \\
0 & 1 
\end{pmatrix}, \quad 
\begin{pmatrix}
1 & 1 \\
0 & 1 
\end{pmatrix}, \quad 
\begin{pmatrix}
-1 & 0 \\
0 & -1 
\end{pmatrix}, \quad 
\begin{pmatrix}
-1 & 1 \\
0 & -1 
\end{pmatrix}, \\
\begin{pmatrix}
0 & i \\
i & 0 
\end{pmatrix}, \quad 
\begin{pmatrix}
0 & -i \\
i & 0 
\end{pmatrix}, \quad 
\begin{pmatrix}
0 & \frac{1+i\sqrt{3}}{2} \\
-\frac{1-i\sqrt{3}}{2} & 0 
\end{pmatrix}, \quad 
\begin{pmatrix}
0 & \frac{1-i\sqrt{3}}{2} \\
\frac{1+i\sqrt{3}}{2} & 0 
\end{pmatrix}
\]

By noting that

\[
\begin{pmatrix}
1 & 1 \\
0 & 1 
\end{pmatrix}^n = \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix}, \quad 
\begin{pmatrix}
-1 & 1 \\
0 & -1 
\end{pmatrix}^n = \begin{pmatrix} (-1)^n & (-1)^{n+1}n \\ 0 & (-1)^n \end{pmatrix}
\]

and the rest Jordan form are periodic, we have that \( T \) has form

\[
\begin{pmatrix}
\phi_{11,n}(n) & \phi_{12,n}(n) \\
\phi_{21,n}(n) & \phi_{22,n}(n) 
\end{pmatrix}
\]

where \( \phi_{11,n}, \phi_{12,n}, \phi_{21,n}, \phi_{22,n} \) are polynomials with real coefficients and degree at most 1. If \( a_1 = a_2 = 1 \), \( \phi_{11,n} = \phi_{11,n+1}, \phi_{12,n} = \phi_{12,n+1}, \phi_{21,n} = \phi_{21,n+1}, \phi_{22,n} = \phi_{22,n+1} \) for any \( n \). If \( a_1 = a_2 = -1 \), \( \phi_{11,n} = \phi_{11,n+2}, \phi_{12,n} = \phi_{12,n+2}, \phi_{21,n} = \phi_{21,n+2}, \phi_{22,n} = \phi_{22,n+2} \) for any \( n \). If \( a_1 = i, a_2 = -i \), \( \phi_{11,n} = \phi_{11,n+4}, \phi_{12,n} = \phi_{12,n+4}, \phi_{21,n} = \phi_{21,n+4}, \phi_{22,n} = \phi_{22,n+4} \) for any \( n \). For

\[
\begin{pmatrix}
-\frac{1+i\sqrt{3}}{2} & 0 \\
0 & -\frac{1-i\sqrt{3}}{2} 
\end{pmatrix}, \quad 
\begin{pmatrix}
\frac{1+i\sqrt{3}}{2} & 0 \\
0 & \frac{1-i\sqrt{3}}{2} 
\end{pmatrix}
\]

we have \( \phi_{11,n} = \phi_{11,n+3}, \phi_{12,n} = \phi_{12,n+3}, \phi_{21,n} = \phi_{21,n+3}, \phi_{22,n} = \phi_{22,n+3} \) for any \( n \) and \( \phi_{11,n} = \phi_{11,n+6}, \phi_{12,n} = \phi_{12,n+6}, \phi_{21,n} = \phi_{21,n+6}, \phi_{22,n} = \phi_{22,n+6} \) for any \( n \) respectively.
Now we will compute $\alpha_{T,b}^{n}(U^{m}V^{k})$. Let $(m_{j}, k_{j}) = T^{j}(m, k)$, where $T(m, k) = (a_{11}m + a_{12}k, a_{21}m + a_{22}k)$. Then

$$\alpha_{T,b}^{n}(U^{m}V^{k}) = e(mb_{1} + kb_{2})(U^{a_{11}}V^{a_{21}})^{m}(U^{a_{12}}V^{a_{22}})^{k}$$

$$= e(mb_{1} + kb_{2})e(-\frac{1}{2}(m_{1}k_{1} - ma_{11}a_{21} - ka_{22}a_{12}))U^{m_{1}}V^{k_{1}}$$

$$\alpha_{T,b}^{n}(U^{m}V^{k}) = e(mb_{1} + kb_{2})\cdots e(m_{n-1}b_{1} + k_{n-1}b_{2}) \cdot$$

$$\cdot e(-\frac{1}{2}(m_{1}k_{1} - ma_{11}a_{21} - ka_{22}a_{12})) \cdots$$

$$\cdot e(-\frac{1}{2}(m_{n}k_{n} - m_{n-1}a_{11}a_{21} - k_{n-1}a_{22}a_{12}))U^{m_{n}}V^{k_{n}}$$

$$= e((\sum_{j=0}^{n-1} m_{j})(b_{1} + \frac{1}{2}a_{11}a_{21}) + (\sum_{j=0}^{n-1} k_{j})(b_{2} + \frac{1}{2}a_{12}a_{22}) \cdot$$

$$\cdot e(-\frac{1}{2}\sum_{j=1}^{n} m_{j}k_{j})U^{m_{n}}V^{k_{n}}.$$ 

Since $(m_{j}, k_{j}) = T^{j}(m, n) = (\phi_{11}(j)m + \phi_{12}(j)k, \phi_{21}(j)m + \phi_{22}(j)k)$, we obtain that $\sum_{j=0}^{n-1} m_{j}$ and $\sum_{j=0}^{n-1} k_{j}$ are polynomials in $n$ with degree at most 2 and $\sum_{j=1}^{n} m_{j}k_{j}$ is a polynomial with degree at most 3. Let

$$P_{n}(n) = (\sum_{j=0}^{n-1} m_{j})(b_{1} + \frac{1}{2}a_{11}a_{21}) + (\sum_{j=0}^{n-1} k_{j})(b_{2} + \frac{1}{2}a_{12}a_{22}) - \frac{1}{2}\sum_{j=1}^{n} m_{j}k_{j}.$$ 

Then if $a_{1} = a_{2} = 1$, $P_{n} = P_{n+1}$ for any $n$; if $a_{1} = a_{2} = -1$, $P_{n} = P_{n+2}$ for any $n$; if $a_{1} = i, a_{2} = -i$, $P_{n} = P_{n+4}$ for any $n$; if $a_{1} = -\frac{1+i\sqrt{3}}{2}, a_{2} = -\frac{1-i\sqrt{3}}{2}, P_{n+3} = P_{n}$ for any $n$; if $a_{1} = \frac{1+i\sqrt{3}}{2}, a_{2} = \frac{1-i\sqrt{3}}{2}, P_{n+6} = P_{n}$ for any $n$. To show $\mu$ is linearly disjoint from $(A_{\theta}, \alpha_{T,b})$, it suffices to show that

$$\frac{1}{N} \sum_{n \leq N} \mu(n)\rho(\alpha_{T,b}^{n}(U^{m}V^{k})) \rightarrow 0, \quad \text{as } N \rightarrow \infty$$

for any state $\rho$ on $A_{\theta}$.

By [13], there exists an AF-algebra $A$ such that $A_{\theta}$ is imbedded as a $C^{*}$ subalgebra of $A$. By Hahn-Banach theorem, $\rho$ can be extended to a state on $A$, denoted by $\rho$ again. Hence for $U, V$ in $A_{\theta}$ there exist unitary matrices $U_{0}$ and $V_{0}$ in a finite dimensional $C^{*}$ subalgebra $B$ such that

$$\|U^{m_{n}}V^{k_{n}} - U_{0}^{m_{n}}V_{0}^{k_{n}}\| \leq \frac{1}{N}, n = 1, \ldots, N.$$
Proposition 6.1. The M"obius entropy algebra. Let $\alpha$

Proof. Let $\hat{\tau}$ be the free group on generators $\ldots, g_{-1}, g_0, g_1, \ldots$ and $C^*_r(F_2)$ the reduced free group C$^*$

Then we have that

$$\frac{1}{N} \sum_{n \leq N} \mu(n)\rho(\alpha_T^p(U^mV^k)) = \frac{1}{N} \sum_{n \leq N} \mu(n)e(P_n(n))\rho(U^mV^k)$$

$$\leq \frac{1}{N} \sum_{n \leq N} \mu(n)e(P_n(n))\rho(U^mV^k - U^mV^k_0) + \frac{1}{N} \sum_{n \leq N} \mu(n)e(P_n(n))\rho(U^mV^k_0)$$

Hence we still have to estimate $\frac{1}{N} \sum_{n \leq N} \mu(n)e(P_n(n))\rho(U^mV^k_n)$. Note that the restriction of

$$\rho$$

to $B$ is a state again, we have $\rho(\cdot) = \text{tr}(\cdot B)$ for some $B \in B$. Let $q = \text{rank}B$, i.e. $U_0, V_0$ are

$q \times q$ matrices and $\text{diag}(e(\theta_0^{(1)}), \ldots, e(\theta_j^{(j)}))$ conjugate to $U_0, V_0$ for $j = 1, 2$ respectively. $(b_l^{(j)})$ the

matrices interpolated between the unitary diagonal matrices as mentioned before, then follow the computation in Lemma 3.1 we have

$$\frac{1}{N} \sum_{n \leq N} \mu(n)e(P_n(n))\rho(U^mV^k) = \frac{1}{N} \sum_{n \leq N} \mu(n)e(P_n(n))\text{tr}(U^mV^k B)$$

$$= \frac{1}{k} \left( \frac{1}{N} \sum_{n \leq N} \mu(n)e(P_n(n)) + m_n \theta_1^{(1)} + k_n \theta_2^{(2)} \right) \left| b_l^{(1)} b_l^{(2)} \right|$$

If $P_n, \phi_{11,n}, \phi_{12,n}, \phi_{21,n}, \phi_{22,n}$ have period $p$, by Lemma 2.1 we have that

$$\frac{1}{N} \sum_{n \leq N} \mu(n)\rho(U^mV^k) = O((\log N)^{-h})$$

for any fixed $h > 0$ and $0 \leq l < p$. Therefore $\mu$ is linearly disjoint from $(\mathcal{A}_\theta, \alpha_T, b)$. □

Remark 5.1. We do not know whether a similar result holds for noncommutative $d$-torus $(d_2, \varnothing)$, because in Lemma 3.1, the implied constant depends on the dimension of estimated matrices when $d > 2$.

6. Free Group C$^*$ Algebras

Let $F_2$ be the free group on generators $\ldots, g_{-1}, g_0, g_1, \ldots$ and $C^*_r(F_2)$ the reduced free group C$^*$

algebra. Let $\alpha$ be the automorphism of $C^*_r(F_2)$ such that $\alpha(g_i) = g_{i+1}$.

Proposition 6.1. The M"obius function $\mu$ is linearly disjoint from $(C^*_r(F_2), \alpha)$ whose Voiculescu-

Brown entropy $h(\alpha) = 0$.

Proof. Let $\bar{m} = (i_1, \ldots, i_k)$ when $k \geq 1$ and $\bar{m} = \emptyset$ when $k = 0$, where $i_1, \ldots, i_k \in \mathbb{Z}$. Denote by

$g_{\bar{m}} = g_{i_1} \cdots g_{i_k}$, $g_{0} = e$ and $|\bar{m}|$ the length of $\bar{m}$.

For any given $g_{\bar{m}}$, we let $l = \max\{|i_1|, \ldots, |i_k|\}$. 


Let
\[ B_{k,N} = \sum_{p=0}^{[N/(2l+1)]-1} \mu(p(2l+1) + k)\alpha^{p(2l+1)+k}(g_{\tilde{m}}) \]
for \( k = 1, \ldots, 2l+1 \).

It is clear that \( B_{k,N} \) is sum of free elements, i.e. \( B_{k,N} = D_{1,k,N} + \cdots + D_{q,k,N} \), where \( q \leq [N/(2l+1)] \) and \( D_{1,k,N}, \ldots, D_{q,k,N} \) are free. We see that \( \frac{1}{2}(D_{1,k,N} + D_{1,k,N}^*) \), \( \frac{1}{2}(D_{q,k,N} + D_{q,k,N}^*) \) are free. By the central limit Theorem in [25], we obtain that
\[ \frac{1}{\sqrt{q}} (\frac{1}{2}(D_{1,k,N} + D_{1,k,N}^*) + \cdots + \frac{1}{2}(D_{q,k,N} + D_{q,k,N}^*) ) \]
converges in distribution to a semicircle element. Then \( \| (2l+1)B_{k,N}/N \| \to 0 \) as \( N \to \infty \) and hence there exists \( N_0 \) in \( \mathbb{N} \) such that \( \| (2l+1)B_{k,N}/N \| < \epsilon \) for \( k = 1, \ldots, 2l+1 \) and \( N > N_0 \). Therefore
\[ \left| \frac{1}{N} \sum_{n \leq N} \mu(n)\rho(U^*nT\eta) \right| = \left| \frac{1}{N} \sum_{k=1}^{2l+1} \rho(B_{k,N}) \right| < \epsilon \]
for any \( \eta \) in the unit ball of \( \mathcal{M} \) and normal state \( \rho \). By Lemma 2.10 in [11] and Lemma 2.9, we may assume that \( \rho(\cdot) = \langle \cdot A\eta, A\eta \rangle \), where \( \eta \) is the canonical trace vector in \( L^2(\mathcal{M}, \tau) \) and \( A \) in the unit ball of \( \mathcal{M} \).

By spectral decomposition theorem, for any \( \epsilon > 0 \) there is a unitary element \( V = \sum_{k=1}^m \epsilon(\theta_k)P_k \) in \( \mathcal{M} \) such that \( \| U^m - V^m \| \leq \epsilon \), where \( P_k, k = 1, \ldots, m \) are orthogonal projections and \( n = 1, \ldots, N \).

7. Finite von Neumann Algebras

In the commutative case, let \( F = (X_F, T, \nu_F) \) such that \( T \) is measure-preserving, it is pointed out by P. Sarnak in [20] that such disjointness (orthogonality) is valid universally, i.e. for every \( F_\nu \). We will present a similar result in noncommutative case.

**Proposition 7.1.** Let \( \mathcal{M} \) be a finite von Neumann algebra with a faithful normal tracial state \( \tau \) and \( U \) a unitary element in \( \mathcal{M} \). Then \( \mu \) is weakly linearly disjoint from \( (\mathcal{M}, Ad(U)) \).

**Proof.** We assume that \( \mathcal{M} \) acts standardly on the Hilbert space \( L^2(\mathcal{M}, \tau) \). To show \( \mu \) is weakly linearly disjoint from \( (\mathcal{M}, Ad(U)) \), we have to estimate
\[ \frac{1}{N} \sum_{n \leq N} \mu(n)\rho(U^*nT\eta) \]
for any \( T \) in the unit ball of \( \mathcal{M} \) and normal state \( \rho \). By Lemma 2.10 in [11] and Lemma 2.9 we may assume that \( \rho(\cdot) = \langle \cdot A\eta, A\eta \rangle \), where \( \eta \) is the canonical trace vector in \( L^2(\mathcal{M}, \tau) \) and \( A \) in the unit ball of \( \mathcal{M} \).

By spectral decomposition theorem, for any \( \epsilon > 0 \) there is a unitary element \( V = \sum_{k=1}^m \epsilon(\theta_k)P_k \) in \( \mathcal{M} \) such that \( \| U^m - V^m \| \leq \epsilon \), where \( P_k, k = 1, \ldots, m \) are orthogonal projections and \( n = 1, \ldots, N \).
We have

\[
\frac{1}{N} \sum_{n=1}^{N} \mu(n) \langle TU^n \eta, U^n \eta \rangle \\
\leq \frac{1}{N} \sum_{n=1}^{N} \mu(n) \langle T(U^n - V^n) \eta, U^n \eta \rangle 
+ \frac{1}{N} \sum_{n=1}^{N} \mu(n) \langle TV^n \eta, (U^n - V^n) \eta \rangle \\
+ \frac{1}{N} \sum_{n=1}^{N} \mu(n) \langle TV^n \eta, V^n \eta \rangle \\
\leq 2\epsilon + \left| \sum_{l,k} \frac{1}{N} \sum_{n=1}^{N} \mu(n) e(n(\theta_l - \theta_k)) \langle TP_l \eta, P_k \eta \rangle \right| \\
\leq 2\epsilon + O((\log N)^{-h}) \sum_{l,k} \tau(P_k TP_l AA^* P_k) \\
\leq 2\epsilon + O((\log N)^{-h}) \sum_{l,k} \|P_k TP_l\|_2 \|P_k AA^* P_l\|_2 \\
\leq 2\epsilon + O((\log N)^{-h}) \left( \sum_{l,k} \|P_k TP_l\|^2_2 \right)^{1/2} \left( \sum_{l,k} \|P_k AA^* P_l\|^2_2 \right)^{1/2} \\
= 2\epsilon + O((\log N)^{-h}) \|T\|_2 \|AA^*\|_2 
\leq 2\epsilon + O((\log N)^{-h}) = O((\log N)^{-h})
\]

Hence \(\mu\) is weakly linearly disjoint from \((M, Ad(U))\).

\(\square\)

**Proposition 7.2.** Let \(M\) be a finite von Neumann algebra with a faithful normal tracial state \(\tau\). Suppose \(\alpha\) is a trace-preserving automorphism of \(M\). Then \(\mu\) is weakly linearly disjoint from \((M, \alpha)\).

**Proof.** We assume that \(\alpha^n \neq \text{id}\) for any nonzero \(n \in \mathbb{Z}\). By considering the crossed product \(M \rtimes \alpha \mathbb{Z}\) of \(M\) by \(\mathbb{Z}\), we have that \(M \rtimes \alpha \mathbb{Z}\) is a finite von Neumann algebra. Let \(U\) be the unitary element in \(M \rtimes \alpha \mathbb{Z}\) implementing \(\alpha\). Then \(AdU\) is an inner automorphism of \(M \rtimes \alpha \mathbb{Z}\) and \(\mu\) is weakly linearly disjoint from \((M \rtimes \alpha \mathbb{Z}, AdU)\). Hence \(\mu\) is weakly linearly disjoint from \((M, \alpha)\).

\(\square\)

**Remark 7.1.** In [7], Connes and Størmer showed that the entropy for finite von Neumann algebra of the shift automorphism of the hyperfinite factor of type II_1 is greater than zero. But the proposition shows that the Möbius function is weakly linearly disjoint from the shifts. So it is trivial to consider Möbius disjointness conjecture for von Neumann algebras in the sense of weak-operator topology. We actually characterize the Möbius disjointness for type I factors in section 4. Currently, we do not know too much about the Möbius disjointness for type III factors.

**References**

[1] El Houcein el Abdalaoui, Mariusz Lemanczyk and Thierry de la Rue On spectral disjointness of powers for rank-one transformations and Möbius orthogonality [arXiv:1301.0134v1, 2013]

[2] N. Brown AF embeddability of crossed products of AF algebras by integers J. Funct. Anal. 160, 1998, no.1 150-175.

[3] N. Brown Topological entropy in exact C* algebras Math. Ann. 314, 1999, 347-367.
[4] M. Choda Entropy for automorphisms of free groups Proceedings of the American Mathematical Society 134, 2006, no. 10, 2905-2911.
[5] A. Connes Classification of Injective Factors Case $II_1$, $II_\infty$, $III_\lambda$, $\lambda \neq 1$ The Annals of Mathematics 104, 1976, no. 1, 73-115.
[6] A. Connes, H. Narfhofer, W. Thirring Dynamical entropy of $C^*$-algebras and von Neumann algebras Commun. Math. Phys 112, 1987, 691-719.
[7] A. Connes, E. Stormer Entropy of automorphisms of $II_1$-von Neumann algebras Acta Math. 134, 1975, 289-306.
[8] H. Davenport On some infinite series involving arithmetical functions(I) Quarterly Journal of Mathematics 8, 1937, 313-320.
[9] K. Dykema Topological entropy of some automorphisms of reduced amalgamated free product $C^*$-algebras Ergod. Th. and Dynam. Sys. 21, 2001, 1683-1693.
[10] B. Green, T. Tao The Möbius function is strongly orthogonal to nilsequences Ann. Math.(2) 175, 2012, 541-566.
[11] U. Haagerup The standard form of von Neumann algebras Math. Scand. 37, 1975, 271-283.
[12] Jinsong Wu and Wei Yuan University of Science and Technology of China Hefei, Anhui, China 230026 wjsl@ustc.edu.cn
[13] D. Kerr, H. Li Positive Voiculescu-Brown entropy in non-commutative toral automorphisms. Ergodic Theory and Dynamical Systems 26, 2006, no.6, 1855-1866.
[14] J. Liu, P. Sarnak The Möbius function and distal flows arXiv:1303.4957v2
[15] V. Neshveyev Entropy of Bogoliubov automorphisms of CAR and CCR algebras with respect to quasi-free states. Reviews in Mathematical Physics 13, 2001, no.01, 29-50.
[16] S. Neshveyev, E. Stomer Entropy in type I algebras Pacific J. Math. 201, 2001, No.2, 421-428.
[17] S. Neshveyev, E. Stomer Dynamical Entropy in operator algebras A series of modern surveys in mathematics, vol 50, Springer, 2006.
[18] M. Pinsker and D. Voiculescu Imbedding the irrational rotation $C^*$-algebra into an AF-algebra Journal of Operator Theory 4, 1980, no. 2, 201-210.
[19] N. M. dos Santos and R. Urzua-Luz Minimal homomorphisms on low-dimensional tori Ergodic Theory Dynam. Systems 29, 2009, 1515-1528.
[20] P. Sarnak Three lectures on the Möbius function, randomness and dynamics IAS Lecture Notes, 2009;
[21] P. Sarnak Möbius randomness and dynamics Not. S. Afr. Math. Soc. 43, 2012, 89-97.
[22] E. Stomer Entropy of some inner automorphisms of the hyperfinite $II_1$ factor Int. J. Math. 04, 1993, 319-322
[23] E. Stomer, D. Voiculescu Entropy of Bogoliubov automorphisms of the Cannonical Anticommutation Relations Commun. Math. Phys. 133, 1990, 521-542.
[24] D. Voiculescu Dynamical approximation entropies and topological entropies in operator algebras Commun. Math. Phys. 170, 1995, 249-281.
[25] D. Voiculescu, K. Dykenma, A. Nica Free Random Variables No.1 American Mathematical Soc. 1992.

Jinsong Wu
University of Science and Technology of China
Hefei, Anhui, China 230026
wjsl@ustc.edu.cn

Wei Yuan
Academy of Mathematics and Systems Science, CAS
Beijing, China 100190
wyuan@math.ac.cn