Statistical Learning Aided List Decoding of
Semi-Random Block Oriented
Convolutional Codes

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Abstract

In this paper, we propose a statistical learning aided list decoding algorithm, which integrates a serial list Viterbi algorithm (SLVA) with a soft check instead of the conventional cyclic redundancy check (CRC), for semi-random block oriented convolutional codes (SRBO-CCs). The basic idea is that, compared with an erroneous candidate codeword, the correct candidate codeword for the first sub-frame has less effect on the output of Viterbi algorithm (VA) for the second sub-frame. The threshold for testing the correctness of the candidate codeword is then determined by learning the statistical behavior of the introduced empirical divergence function (EDF). With statistical learning aided list decoding, the performance-complexity tradeoff and the performance-delay tradeoff can be achieved by adjusting the statistical threshold and extending the decoding window, respectively. To analyze the performance, a closed-form upper bound and a simulated lower bound are derived. Simulation results verify our analysis and show that: 1) The statistical learning aided list decoding outperforms the sequential decoding in high signal-to-noise ratio (SNR) region; 2) under the constraint of equivalent decoding delay, the SRBO-CCs have comparable performance with the polar codes.

Index Terms

block Markov superposition transmission (BMST), convolutional code, list decoding, statistical learning, ultra-reliable and low latency communication (URLLC).

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I. Introduction

It has been pointed out by Shannon [1] that the error free transmission is possible with infinite coding length as long as the transmission rate is below the channel capacity. To approach the channel capacity, a number of powerful iteratively decodable channel codes with long block length have been proposed. For example, low-density parity check (LDPC) codes [2] and turbo codes [3] perform within a few hundredths of a decibel from the Shannon limits under iterative belief propagation (BP) decoding algorithm. However, long codes are not suitable for emerging applications that are sensitive to the delay, such as automated driving, smart grids, industrial automation and medical applications. Designing a good code with strict latency constraint is challenging since most constructions developed for long block length do not deliver good codes in the short block length regime. For this reason, more attention has been paid recently to the design of short and medium block length codes (e.g., a thousand or less information bits) [4].

One solution is to construct LDPC codes by progressive edge growth (PEG) algorithm [5], which can deliver better codes than randomly constructed LDPC codes in short block length regime. Polar codes [6], another promising solution for short packet transmission, have been adopted by 5G control channel. Many works on constructions, decoding algorithms and decoder implementations for short polar codes have been proposed [7]–[11]. Powerful classical short codes with near maximum likelihood decoding algorithm were also investigated for low latency communication. In [4], the extended Bose-Chaudhuri-Hocquenghem (BCH) codes were shown to perform near the normal approximation benchmark under ordered statistics decoding (OSD) [12]. As shown in [13], in the short block length regime, the tail-biting convolutional codes (TBCCs) with wrap-around Viterbi algorithm (WAVA) [14] outperform significantly state-of-the-art iterative coding schemes.

All the aforementioned codes are block codes with short coding length, whereas convolutional codes with limited decoding window can be alternative choices for the streaming services with strict latency constraint, such as real-time online games and video conferences. The comparison in [15] [16] between convolutional codes and PEG-LDPC codes showed that convolutional codes outperform LDPC codes for very short delay when the bit error rate is used as a performance metric.

A coding scheme called block Markov superposition transmission (BMST) was proposed in [17] to construct iteratively decodable convolutional codes with long constraint length from
simple basic codes. The construction of BMST codes is flexible, in the sense that it applies to any short basic codes. Hence, it delivers a wide range of code rates with Hadamard transform coset codes as the basic codes [18] and essentially all code rates of interest in the interval (0, 1) with basic codes consisting of repetition codes and single-parity-check codes [19]. The BMST codes are also capable of supporting a wide range of delays but with a small amount of extra implementation complexity [20]. The extrinsic information transfer (EXIT) chart analysis in [21] showed that BMST codes have near-capacity performance in the waterfall region and an error floor that can be controlled by the encoding memory. However, even with the sliding window decoding algorithm, the BMST codes still suffer from a large decoding delay and hence are not suitable for low latency communication. This is because the BP decoding algorithm performs far worse than the optimal decoding algorithm when the layers (sub-blocks) become short.

To solve this issue, the semi-random block oriented convolutional code (SRBO-CC) was proposed in [22] with a Cartesian product of short code as the basic code. The SRBO-CCs can be decoded by the sequential decoding, whose memory load is heavy due to the requirement of a large amount of stack memory. In [23], taking the truncated convolutional codes as the basic codes, we proposed a list decoding algorithm for SRBO-CCs. However, the frame error rate of short convolutional codes without termination is relatively high. In [24], the TBCCs were taken as the basic codes to improve the performance. As extension works of [22] [23] [24], we present in this paper more details on the SRBO-CCs. The SRBO-CCs are decoded by a sliding window decoding algorithm with successive cancellation, whose performance depends critically on the performance of the first sub-frame. To recover the first sub-frame reliably, list decoding is conducted and the transmitted codeword is identified from the list with the help of the empirical divergence function. With statistical learning aided list decoding, the SRBO-CCs, which are designed for low latency communication, preserve almost all the advantages of the BMST codes, which are designed to approach the capacity for long block length. In particular, the SRBO-CCs have the following three attractive features.

- The construction of SRBO-CCs is flexible, in the sense that any codes with fast encoding algorithms and efficient list decoding algorithms can be taken as the basic codes. This suggests that the SRBO-CCs can support a wide range of code rates by simply adapting the basic codes to the desired rate.
- Given the weight enumerating functions (WEFs) of the basic codes, the closed-form WEFs of the SRBO-CCs can be derived for analyzing the performance. Simulation results show
that, in high SNR region, the performance of the SRBO-CCs are well predicted by the upper bounds.

- The performance-complexity tradeoff and the performance-delay tradeoff can be achieved by adjusting the statistical threshold and extending the decoding window, respectively.

This paper is organized as follows. In Section II we present the encoding algorithm and the algebraic description of the SRBO-CC. In Section III the statistical learning aided list decoding is proposed. In Section IV by analyzing the performance and decoding complexity, the performance-complexity tradeoff and performance-delay tradeoff are discussed. Simulation results are presented in Section V. Finally, some concluding remarks are given in Section VI.

II. SEMI-RANDOM BLOCK ORIENTED CONVOLUTIONAL CODE

A. Encoding Algorithm

Let \( u = (u^{(0)}, u^{(1)}, \ldots, u^{(L-1)}) \) be the data to be transmitted, where \( u^{(t)} = (u_0^{(t)}, u_1^{(t)}, \ldots, u_{k-1}^{(t)}) \in \mathbb{F}_2^k \) for \( 0 \leq t \leq L-1 \). Taking a binary linear code \( C \) of dimension \( k \) and length \( n \) as the basic code, the encoding algorithm of the SRBO-CC is described in Algorithm 1 (see Fig. 1 for reference). The code rate of the SRBO-CC is \( R = k/n \times L/(L+1) \), which is slightly less than that of the basic code \( C \). However, the rate loss is negligible for large \( L \).

**Algorithm 1 Encoding of the SRBO-CC**

- **Initialization:** Let \( v^{(-1)} = 0 \in \mathbb{F}_2^n \).
- **Iteration:** For \( 0 \leq t \leq L-1 \),
  1. **Structured Encoding:** Encode \( u^{(t)} \) into \( v^{(t)} \in \mathbb{F}_2^n \) by the encoding algorithm of the basic code \( C \). Equivalently, \( v^{(t)} = u^{(t)} S \), where \( S \) is the generator matrix of the basic code \( C \).
  2. **Random Encoding:** Compute \( w^{(t)} = v^{(t-1)} R \in \mathbb{F}_2^n \), where \( R \) is a random matrix of order \( n \) whose elements are generated independently according to the Bernoulli distribution with success probability \( 1/2 \).
  3. **Superposition:** Compute \( c^{(t)} = v^{(t)} + w^{(t)} \in \mathbb{F}_2^n \), which will be taken as the sub-frame for transmission at time \( t \).
- **Termination:** The sub-frame at time \( L \) is set to \( c^{(L)} = v^{(L-1)} R \), which is equivalent to setting \( u^{(L)} = 0 \).

B. Algebraic Description

Generally speaking, a terminated (time-invariant) convolutional code can be treated as a linear block code, whose generator matrix has a banded diagonal form. More precisely, the generator
Fig. 1. Encoding structure of the SRBO-CC.

The generator matrix of a terminated convolutional code can be written as
\[
G_{cc} = \begin{pmatrix}
G_0 & G_1 & \cdots & G_m \\
G_0 & G_1 & \cdots & G_m \\
\vdots & \vdots & \ddots & \vdots \\
G_0 & G_1 & \cdots & G_m \\
\end{pmatrix},
\]
where \(G_i\) is a \(k \times n\) matrix for \(0 \leq i \leq m\). The SRBO-CC can be viewed as a convolutional code with memory one. The encoding memory is fixed to be one due to the following reasons. First, for a larger encoding memory, \(L\) should be set larger to reduce the rate loss caused by the termination, resulting in a performance degradation (as discussed in Subsection II-C). Second, to fulfill the latency constraint, we attempt to keep the decoding window (hence the complexity and the decoding delay) as small as possible. The generator matrix of the SRBO-CC can be written as
\[
G = \begin{pmatrix}
S & SR \\
S & SR \\
\vdots & \vdots \\
S & SR \\
\end{pmatrix},
\]
where \(S\) is a structured matrix of size \(k \times n\) and \(R\) is a randomly generated matrix of size \(n \times n\).

As a kind of convolutional code, the SRBO-CC has streaming properties. In other words, the encoded bits can be generated without waiting for the whole input block while the received signal can be decoded by a sliding window decoding algorithm with tunable delays. Different from block codes, in the SRBO-CC coded system, the latency constraint is fulfilled by the limited
decoding window instead of the short coding length.

Unlike commonly accepted classical convolutional codes with small $k$, the SRBO-CCs typically have large $k$ (hence large constraint length) induced by the block oriented encoding process, as is the same case for the convolutional LDPC codes. Due to the large constraint length, the SRBO-CCs are typically non-decodable by the Viterbi algorithm (VA). Another difference between the SRBO-CCs and classical convolutional codes is the randomness introduced by the random linear transformation. Good convolutional codes with short constraint length are usually constructed by computer search [25], while good SRBO-CCs can be constructed easily by generating $R$ randomly.

As a special class of block oriented convolutional codes, the SRBO-CCs are similar to but different from the BMST codes proposed in [17] in the following three aspects.

- The BMST codes are designed to approach the capacity for long block length and the length of the basic codes is relatively large (e.g., $k > 1000$). However, the SRBO-CCs, which are designed for low latency communication, take short codes as the basic codes (e.g., $k < 50$).
- In the BMST codes, the basic codes are required to have an efficient soft-in soft-out decoding algorithm for the iterative BP decoding algorithm. However, in the SRBO-CCs, the basic codes are required to have an efficient list decoding algorithm.
- In the BMST codes, the interleaved versions of the previous codewords are superimposed on a fresh codeword. That is, $G_i = G_0 \Pi_i$ for $1 \leq i \leq m$, where $\Pi_i$ is a randomly generated permutation matrix of order $n$. The reason of using an interleaver is that there exists no efficient decoding algorithm for a general transformation when the length of the basic codes is large. The use of interleaver also makes it convenient to exchange the soft messages between adjacent layers. In contrast, a random linear transformation is employed in the SRBO-CCs. This is necessary for enhancing the minimum distance in the case of encoding memory one.

C. Performance Metric

Suppose that $c^{(t)}$ is modulated with binary phase-shift keying (BPSK) signals and transmitted over additive white Gaussian noise (AWGN) channels, resulting in a noisy version $y^{(t)} \in \mathbb{R}^n$ at the receiver. We focus on a sliding window decoding algorithm with the decoding window $w$, which attempts to recover $u^{(t)}$ from $(y^{(t)}, \ldots, y^{(t+w-1)})$. In other words, the decoding delay is $wn$ in terms of bits.
Given a decoding algorithm, define \( f_{ER_t} \) for \( 0 \leq t \leq L-1 \) as the probability that the decoding result \( \hat{u}^{(t)} \) is not equal to the transmitted vector \( u^{(t)} \) and \( FER \) as the probability that the decoding result \( \hat{u} \) is not equal to \( u \). It is not difficult to verify that

\[
f_{ER_0} \leq \max_t f_{ER_t} \leq FER \leq \sum_{t=0}^{L-1} f_{ER_t}. \tag{3}
\]

We define

\[
f_{ER} = \frac{1}{L} \sum_{t=0}^{L-1} f_{ER_t}, \tag{4}
\]

which is used as the performance metric in this paper\(^1\) and can be evaluated in practice by

\[
f_{ER} = \frac{\text{number of erroneous decoded sub-frames}}{\text{total number of transmitted sub-frames}}. \tag{5}
\]

The event that the decoding result \( \hat{u}^{(0)} \) is not equal to the transmitted vector \( u^{(0)} \) is referred to as the first error event \( E_0 \). In general, we say that the first error event occurs at time \( t \), which is denoted by \( E_t \), if \( \hat{u}^{(i)} = u^{(i)} \) for all \( i < t \) but \( \hat{u}^{(t)} \neq u^{(t)} \). The probability that the first error event occurs at time \( t \) is given by

\[
\Pr\{E_t\} = f_{ER_0} \cdot (1 - f_{ER_0})^t, \tag{6}
\]

by noticing that \( \Pr\{\hat{u}^{(i)} \neq u^{(i)} | \hat{u}^{(i-1)} = u^{(i-1)}\} = f_{ER_0} \) for \( 1 \leq i \leq L - 1 \). In the worst case, the first error event at time \( t \) causes catastrophic error-propagation. That is, the event \( E_t = \{\hat{u}^{(t)} \neq u^{(t)}\} \) can cause \( \hat{u}^{(j)} \neq u^{(j)} \) for all \( j > t \). The \( f_{ER_t} \) can be bounded by

\[
f_{ER_t} = \sum_{i=0}^{t} \Pr\{E_i\} \Pr\{\hat{u}^{(t)} \neq u^{(t)} | E_i\} \leq \sum_{i=0}^{t} \Pr\{E_i\}. \tag{7}
\]

\(^1\)For conventional block codes, such as polar codes, we define \( f_{ER} \) as the probability that the decoding codeword is not equal to the transmitted codeword. That is, \( f_{ER} = FER \).
Therefore, the fER can be upper bounded by
\[
\text{fER} \leq \sum_{t=0}^{L-1} \frac{L-t}{L} \Pr\{E_t\} \quad (8)
\]
\[
= \sum_{t=0}^{L-1} \frac{L-t}{L} fER_0 \cdot (1 - fER_0)^t \quad (9)
\]
\[
\leq \sum_{t=0}^{L-1} \frac{L-t}{L} fER_0 \quad (10)
\]
\[
= \frac{L+1}{2} \cdot fER_0. \quad (11)
\]

III. SUCCESSIVE CANCELLATION DECODING

As a kind of convolutional codes with large constraint length, the SRBO-CCs are typically non-decodable by VA. In [22], the authors analyze the tree structure of the SRBO-CCs and employ the sub-optimal sequential decoding, whose memory load is heavy due to the requirement of a large amount of stack memory.

In this paper, we propose a sliding window algorithm with successive cancellation. The first and critical step is to recover reliably \( v(0) \), which is not interfered by any other sub-frames. By removing the effect of the first sub-frame, the second sub-frame is then decoded in the same way. This process will be continued until all sub-frames are decoded. In this section, we focus on the methods to estimate \( v(0) \) from \((y(0), y(1))\). The complete decoding algorithm is summarized in Algorithm 2 and the extension to the recovery of \( v(0) \) from \((y(0), y(1), y(2))\) will be discussed in Subsection IV-B.

For illustrating the basic idea, we first introduce the maximum a posteriori (MAP) decoding and the maximum likelihood (ML) decoding, although they seem to be less practical. The statistical learning aided list decoding is then proposed.
A. Maximum A Posteriori Decoding

The MAP decoding is optimal in the sense that the error probability of \( v^{(0)} \) (i.e., fER\( _0 \)) is minimized. The MAP decoder always outputs the codeword

\[
\hat{v}^{(0)} = \arg \max_{v^{(0)}} P(v^{(0)} | y^{(0)} y^{(1)})
\]

\[
= \arg \max_{v^{(0)}} \frac{P(v^{(0)})}{P(y^{(0)} y^{(1)})} P(y^{(0)} y^{(1)} | v^{(0)}),
\]

where \( P(\cdot) \) is the probability mass (or density) function. Since the channel is memoryless and \( v^{(0)} \) is independent with \( v^{(1)} \), we have

\[
P(y^{(0)} y^{(1)} | v^{(0)}) = P(y^{(0)} | v^{(0)}) P(y^{(1)} | v^{(0)})
\]

\[
= P(y^{(0)} | v^{(0)}) \sum_{v^{(1)}} P(v^{(1)}) P(y^{(1)} | v^{(0)} v^{(1)}).
\]

Therefore, we have

\[
P(v^{(0)} | y^{(0)} y^{(1)}) \propto P(y^{(0)} | v^{(0)}) \sum_{v^{(1)}} P(y^{(1)} | v^{(0)} v^{(1)}),
\]

by noticing that \( \frac{P(v^{(0)})}{P(y^{(0)} y^{(1)})} \) is constant for all \( v^{(0)} \).

To find such a codeword \( \hat{v}^{(0)} \), the MAP decoder explores all \( 2^{2nR} \) possible codewords \( (v^{(0)}, v^{(1)}) \), which implies that the complexity increases exponentially with the length of the basic code.

B. Maximum Likelihood Decoding

Different from the MAP decoder, the ML decoder minimizes the error probability of the codeword \( (v^{(0)}, v^{(1)}) \). The ML decoder selects \( \hat{v}^{(0)} \) as output such that

\[
(\hat{v}^{(0)}, \hat{v}^{(1)}) = \arg \max_{(v^{(0)}, v^{(1)})} P(y^{(0)} y^{(1)} | v^{(0)} v^{(1)}).
\]

Since the channel is memoryless, we have

\[
P(y^{(0)} y^{(1)} | v^{(0)} v^{(1)}) = P(y^{(0)} | v^{(0)}) P(y^{(1)} | v^{(0)} v^{(1)}).
\]

Without causing much ambiguity, we will use \( \hat{v}^{(0)} \), \( \hat{c}^{(0)} \) and \( \hat{u}^{(0)} \) interchangeably in the remainder of this paper.
Equivalently, the ML decoder outputs the codeword

$$\hat{v}^{(0)} = \arg\max_{v^{(0)}} P(y^{(0)}|v^{(0)}) \left[ \max_{v^{(1)}} P(y^{(1)}|v^{(0)}v^{(1)}) \right].$$

(20)

The ML decoding can also be viewed as an approximation to the MAP decoding, since \(\max_{v^{(1)}} P(y^{(1)}|v^{(0)}v^{(1)})\) is the dominant term in \(\sum_{v^{(1)}} P(y^{(1)}|v^{(0)}v^{(1)})\).

Given \(v^{(0)}\), the inner maximization over \(v^{(1)}\) in (20) can be achieved by performing VA, which is more efficient than exploring all possible \(v^{(1)}\). Unfortunately, no efficient algorithm to achieve the outer maximization, except exploring all \(2^{nR}\) possible \(v^{(0)}\), which implies that the complexity is lower than the MAP decoding but still increases exponentially with the length of the basic code.

C. Statistical Learning Aided Decoding

One obvious way to reduce the complexity of the ML decoding is to limit the search space for \(v^{(0)}\). Let \(\mathcal{L} \subset \mathcal{C}\) be a list of \(\ell_{\text{max}}\) codewords. The decoder outputs the codeword

$$\hat{v}^{(0)} = \arg\max_{v^{(0)} \in \mathcal{L}} P(y^{(0)}|v^{(0)}) \left[ \max_{v^{(1)}} P(y^{(1)}|v^{(0)}v^{(1)}) \right].$$

(21)

Obviously, if the transmitted codeword \(v^{(0)}\) is included in the list \(\mathcal{L}\), the decoder with reduced search space performs no worse than the ML decoder. In contrast, an error must occur if the transmitted codeword is not in the list. Therefore, we need to generate efficiently a list \(\mathcal{L}\) which contains the transmitted codeword with high probability.

We assume that the basic code \(\mathcal{C}\) can be efficiently decoded by outputting a list of candidate codewords. To avoid messy notation, we omit the superscript of \(v^{(0)}\) and assume that a codeword \(v \in \mathcal{C}\) is transmitted. Upon receiving its noisy version \(y = (y_0, y_1, \cdots, y_{n-1})\), the decoder serially outputs a list of candidate codewords \(\hat{v}_\ell, \ell = 1, 2, \cdots, \ell_{\text{max}},\) where \(\ell_{\text{max}}\) is a parameter to trade off the performance against the complexity. We will not focus on the detailed implementation in this paper but simply conduct the serial list Viterbi algorithm (SLVA) [26] over the trellis representation of the basic code. For ease of notation, we use SLVA\((y, \ell)\) to represent the \(\ell\)-th output of the SLVA. In particular, SLVA\((y, 1)\), simply denoted by VA\((y)\), is the output of the VA.

The list decoding is successful if the transmitted codeword occurs in the list. Obviously, the probability of the list decoding being successful can be as high as required by enlarging the list size \(\ell_{\text{max}}\). Example [1] shows the performance of a TBCC under list decoding.
Fig. 2. Performance of the TBCC under list decoding in Example 1. The 16-state \((2, 1, 4)\) TBCC defined by the polynomial generator matrix \([25]\) \(G(D) = [D^4 + D^2 + D + 1, D^4 + D^3 + 1]\) (denoted as \([27, 31]_8\) in octal form for short) with information length \(k = 32\) \((n = 64)\) is considered. Here the label “Failed Probability” for the ordinate stands for the probability that the transmitted codeword is not contained in the list.

**Example 1:** The 16-state \((2, 1, 4)\) TBCC defined by the polynomial generator matrix \([25]\) \(G(D) = [D^4 + D^2 + D + 1, D^4 + D^3 + 1]\) (denoted as \([27, 31]_8\) in octal form for short) with information length \(k = 32\) \((n = 64)\) is considered. The list decoding performance is shown in Fig. 2.

For a large list size (e.g., \(\ell_{\text{max}} = 64\)), the transmitted codeword is included in the list with high probability. However, the average list sizes required to contain the correct candidate codeword can be much smaller than \(\ell_{\text{max}} = 64\), as tabulated in Table I. This implies that, in many cases, the list size can be smaller than \(\ell_{\text{max}}\). A question arises: can we check the correctness of the candidate codeword and stop the SLVA earlier when the transmitted codeword is found? One solution is to invoke the cyclic redundancy check (CRC), as embedded in polar codes \([27]\). However, the overhead (rate loss) due to the CRC is intolerable especially for a short basic code. Motivated by the jointly typical set decoding, which is employed in \([28]\) Section 3.2.

| SNR | 2.0 | 2.5 | 3.0 | 3.5 | 4.0 |
|-----|-----|-----|-----|-----|-----|
| list size | 1.256 | 1.069 | 1.019 | 1.005 | 1.001 |
to prove the channel coding theorem, we consider checking the correctness of the candidate codeword by typicality. The list decoding process will stop if a candidate codeword is found to be “jointly typical” with the received signal.

To proceed, we need the following concept. For the received signal \( y = (y_0, y_1, \cdots, y_{n-1}) \in \mathbb{R}^n \), define an empirical divergence function (EDF) as

\[
D(x, y) = \frac{1}{n} \log_2 \frac{P(y|x)}{P(y)},
\]

(22)

for \( x \in \mathbb{F}_2^n \), where

\[
P(y) = \prod_{i=0}^{n-1} \left( \frac{1}{2} P(y_i|0) + \frac{1}{2} P(y_i|1) \right).
\]

(23)

Note that, in the above definition, \( P(y) \) is not equal to \( 2^{-k} \sum_{v \in \mathcal{C}} P(y|v) \) but to \( 2^{-n} \sum_{x \in \mathbb{F}_2^n} f(y|x) \). Also note that \( x \) is not necessarily a codeword of \( \mathcal{C} \). Especially, we are interested in the following cases.

1) If \( v \) is the transmitted one, we have \( D(v, y) \approx I(X; Y) > 0 \), where \( \approx \) is used to indicate that the EDF is around in probability its expectation for large \( n \). Here \( I(X; Y) \) is the mutual information between the channel output \( Y \) and the uniform binary input \( X \). To be precise, \( D(v, y) \approx I(X; Y) \) means that, for an arbitrary small positive number \( \epsilon \),

\[
\lim_{n \to \infty} P \left( \left| D(v, y) - I(X; Y) \right| \leq \epsilon \right) = 1,
\]

(24)

as guaranteed by the weak law of large numbers (WLLN).

2) If \( x \) is randomly generated (hence typically not equal to the transmitted one), from the WLLN, we have

\[
D(x, y) \approx \mathbb{E}_{Y|V} \left[ \frac{1}{2} \log_2 \frac{f(Y|0)}{f(Y)} + \frac{1}{2} \log_2 \frac{f(Y|1)}{f(Y)} \right],
\]

(25)

which is negative from the concavity of the function \( \log_2(\cdot) \).

3) What are the typical values of \( D(\hat{v}, y) \), where \( \hat{v} = \text{VA}(y) \)? Given \( y \), since \( D(\hat{v}, y) = \max_{v \in \mathcal{C}} D(v, y) \), we expect that \( D(\hat{v}, y) \geq D(v, y) \approx I(X; Y) > 0 \).

4) What about \( D(\tilde{v}, \tilde{y}) \)? Here \( \tilde{v} = \text{VA}(\tilde{y}) \) where \( \tilde{y} = x \odot y \) with \( x \) being a totally random bipolar vector and \( \odot \) stands for component-wise product. That is, we first randomly flip the received vector \( y \), and then execute the VA to find the first candidate codeword \( \tilde{v} \). We expect that \( D(\tilde{v}, \tilde{y}) \) is located between \( D(v, y) \) of the first case and \( D(x, y) \) of the
second case.

**Example 2:** We consider the TBCC in **Example 1** again and set SNR = 4 dB, at which the mutual information is \( I(X;Y) \approx 0.79 \). The histogram is shown in Fig. 3, from which we observed that \( D(v, y) \) is likely to be large with \( v \) being the transmitted one (or the output of the VA corresponding to \( y \)). Note that the statistical behavior of \( D(\tilde{v}, \tilde{y}) \) is different from that of \( D(x, y) \), since \( \tilde{v} \) is dependent on \( \tilde{y} \). The typical values of \( D(\tilde{v}, \tilde{y}) \) are greater than those of \( D(x, y) \) but less than those of \( D(v, y) \).

The statistical behavior of the EDF can be helpful in the decoding process of the SRBO-CCs. In the case when the decoding result of the first sub-frame \( \hat{v}^{(0)} \) equals to \( v^{(0)} \), \( y^{(1)} \circ \phi(\hat{v}^{(0)}R) \) is the Gaussian noisy version of \( v^{(1)} \), where \( \phi(v^{(0)}R) \) is the BPSK signal corresponding to the binary vector \( v^{(0)}R \). In contrast, in the case when \( \hat{v}^{(0)} \neq v^{(0)} \), \( y^{(1)} \circ \phi(\hat{v}^{(0)}R) \) is the randomly flipped Gaussian noisy version of \( v^{(1)} \). Since these two cases have different statistical impact on the EDF, we are able to distinguish with high probability whether \( y^{(1)} \circ \phi(\hat{v}^{(0)}R) \) is the randomly flipped Gaussian noisy version of \( v^{(1)} \) (equivalently, \( \hat{v}^{(0)} \) is erroneous) or not.

Given \( y^{(0)} \), the SLVA is implemented to deliver serially a list of candidate codewords \( \hat{v}_\ell^{(0)} \), \( 1 \leq \ell \leq 16 \).
Fig. 4. Statistical behavior of $M_2(\hat{v}_\ell^{(0)})$ in Example 3. The 16-state $(2, 1, 4)$ TBCC defined by the polynomial generator matrix $G(D) = [27, 31]_8$ with information length $k = 32$ ($n = 64$) is taken as the basic code. The setup of SNR = 3 dB and $\ell_{\text{max}} = 64$ is considered.

For each candidate codeword, we define a soft metric

$$M_2(\hat{v}_\ell^{(0)}) = D(\hat{v}_\ell^{(0)}, y^{(0)}) + D(\tilde{v}_\ell, y^{(1)} \odot \phi(\hat{v}_\ell^{(0)} R)),$$  

(26)

where $\tilde{v}_\ell$ is the output of the VA with $y^{(1)} \odot \phi(\hat{v}_\ell^{(0)} R)$ as the input. The first term in the right hand side of (26) specifies the EDF between the candidate codeword and the received vector $y^{(0)}$, while the second term is the EDF between $y^{(1)} \odot \phi(\hat{v}_\ell^{(0)} R)$ and its corresponding VA output $\tilde{v}_\ell$. Both of them are likely to be large in the case when the candidate codeword is the transmitted one. Heuristically, we will set a threshold on $M_2(\hat{v}_\ell^{(0)})$ to check the correctness of the candidate codeword, as illustrated in Example 3.

**Example 3**: The TBCC in Example 1 is taken as the basic code. We set SNR = 3 dB and $\ell_{\text{max}} = 64$. With the help of the histogram shown in Fig. 4, we set a threshold $T$ to distinguish the correct candidate codeword from the erroneous one. The candidate codeword $\hat{v}_\ell^{(0)}$ is treated to be correct only if $M_2(\hat{v}_\ell^{(0)}) \geq T$, where $T$ is usually set large (e.g., $T = 1.2$ in this example) to reduce the probability that an erroneous candidate is mistaken as the correct one. The threshold $T$, depending on SNRs and coding parameters, can be learned off-line and stored for use in the decoding algorithm.
The statistical learning aided list decoding algorithm, as summarized in Algorithm 2, is outlined as follows. The decoder employs the SL VA to compute the candidate codewords, which will be checked by (26) with a preset threshold, until finding a qualified one. If the list size reaches the maximum $\ell_{\text{max}}$ and no candidate codeword is qualified, the decoder delivers $\hat{v}^{(0)}_{\ell}$ with the maximum $M_2(\hat{v}^{(0)}_{\ell})$ as output.

Algorithm 2 Successive cancellation decoding for the SRBO-CC

- **Global initialization:** Set the threshold $T$. Assume that $y^{(0)}$ has been received and set $z^{(0)} = y^{(0)}$.
- **Sliding-window decoding:** For $0 \leq t \leq L - 1$, after receiving $y^{(t+1)}$,
  1) **Local initialization:** Set $M_{\text{max}} = -\infty$ and $\ell = 1$.
  2) **List:** While $M_{\text{max}} \leq T$ and $\ell \leq \ell_{\text{max}}$,
     a) Perform SL VA to find $\hat{v}^{(t)}_{\ell} = \text{SLVA}(z^{(0)}, \ell)$ and compute $D(\hat{v}^{(t)}_{\ell}, z^{(0)})$.
     b) Flip the received vector $y^{(t+1)}$, resulting in $z^{(1)} = y^{(t+1)} \odot \phi(\hat{v}^{(t)}_{\ell}R)$.
     c) Perform VA to find $\tilde{v}_{\ell} = \text{VA}(z^{(1)})$ and compute $D(\tilde{v}_{\ell}, z^{(1)})$.
     d) If $M_2(\hat{v}^{(t)}_{\ell}) = D(\hat{v}^{(t)}_{\ell}, z^{(0)}) + D(\tilde{v}_{\ell}, z^{(1)}) \geq M_{\text{max}}$, replace $M_{\text{max}}$ by $M_2(\hat{v}^{(t)}_{\ell})$ and replace $\hat{v}^{(t)}_{\ell}$ by $\tilde{v}_{\ell}$.
     e) Increment $\ell$ by one.
  3) **Decision:** Output $\hat{u}^{(t)}$, the corresponding information vector to $\hat{v}^{(t)}_{\text{max}}$, as the decoding result of the $t$-th sub-frame.
  4) **Cancellation:** Remove the effect of the $t$-th sub-frame on the $(t + 1)$-th sub-frame. That is, update $z^{(0)}$ by computing

$$z^{(0)} = y^{(t+1)} \odot \phi(\hat{v}^{(t)}_{\text{max}} R).$$

IV. PERFORMANCE AND COMPLEXITY ANALYSIS

A. Upper Bound

In this subsection, we derive an upper bound on $f_{\text{ER}_0}$ under the ML decoding. Because of the linearity of the code, we assume that all zero codeword is transmitted. The ML decoder selects $\hat{v}^{(0)}$ as output such that the codewords $(\hat{v}^{(0)}, \hat{v}^{(1)})$ maximize $P(y^{(0)}y^{(1)}|v^{(0)}v^{(1)})$. The ML decoding is successful if $\hat{v}^{(0)} = 0$ and an error occurs if $\hat{v}^{(0)} \neq 0$. Note that $\hat{v}^{(0)}$ can be
correct even if \( \mathbf{v}^{(1)} \neq 0 \). The \( \text{fER}_0 \) can be upper bounded by

\[
\text{fER}_0 = \Pr \left\{ \bigcup_{v^{(0)} \neq 0} (v^{(0)}, v^{(1)}) \text{ is the most likely} \right\} 
\]

\[
\leq \Pr \left\{ \bigcup_{v^{(0)} \neq 0} (v^{(0)}, v^{(1)}) \text{ is more likely than } (0, 0) \right\} \tag{28}
\]

\[
\leq \sum_{v^{(0)} \neq 0} \Pr \left\{ (v^{(0)}, v^{(1)}) \text{ is more likely than } (0, 0) \right\} . \tag{29}
\]

This bound is indeed the well-known union bound and can be calculated by deriving the weight distribution of the truncated code

\[
\mathcal{C}^{(0,1)} = \left\{ (c^{(0)}, c^{(1)}) \mid c = (c^{(0)}, \ldots, c^{(L)}) \text{ is a coded sequence with } c^{(0)} \neq 0 \right\} . \tag{30}
\]

Let \( A(X) \) be the WEF of the basic code \( \mathcal{C} \setminus 0 \) (all non-zero codewords). Then the ensemble WEF of the truncated code \( \mathcal{C}^{(0,1)} \) with \( R \) being totally random is given by

\[
B(X) = 2^{-n+k}(1+X)^n A(X) = \sum_{w=1}^{2n} B_w X^w . \tag{31}
\]

The upper bound on \( \text{fER}_0 \) under the ML decoding is given by

\[
\text{fER}_0 \leq \sum_{w=1}^{2n} B_w Q \left( \sqrt{\frac{w}{\sigma^2}} \right) , \tag{32}
\]

where \( \sigma^2 \) is the variance of the noise.

**B. Lower Bound and Extended Windowed Decoding**

Obviously, in the statistical learning aided list decoding, the first sub-frame can be decoded correctly only if the transmitted codeword is included in the list. Therefore, the \( \text{fER}_0 \) performance is not better than the list decoding performance of the basic code, which can be regarded as a lower bound and obtained by simulating the list decoding of the basic code. Example 4 is presented to illustrate the lower bound on \( \text{fER}_0 \).
Example 4: The basic code is the 16-state (2, 1, 4) convolutional code with information length \( k = 32 \) \((n = 64)\), which is truncated without termination and defined by the polynomial generator matrix \( G(D) = [27, 31]^8 \). The list size is \( \ell_{\text{max}} = 64 \) and the thresholds are set properly based on off-line statistical learning. The fER\(_0\) performance of the statistical learning aided list decoding are shown in Fig. 5, where “\( w = 2 \)” corresponds to Algorithm 2. The corresponding lower bound is also plotted. We see that Algorithm 2 performs about 0.5 dB away from the lower bound, implying that the statistical check is not always able to identify the transmitted codeword in the list. This gap can be narrowed, however, if the constraint on complexity and latency is relaxed. Indeed, we can extend Algorithm 2 which recovers \( \hat{v}^{(0)} \) from \( y^{(0)} \) and \( y^{(1)} \), to improve the performance by recovering \( \hat{v}^{(0)} \) from \( y^{(0)} \), \( y^{(1)} \) and \( y^{(2)} \), as shown by the curve “\( w = 3 \)” in Fig. 5. The details of such an extension is omitted here, while the basic idea is described below.

After receiving \( y^{(0)} \) and \( y^{(1)} \), the decoder first attempts to recover \( \hat{v}^{(0)} \) by Algorithm 2. In the case when the decision on \( \hat{v}^{(0)} \) is not that confident, we keep a list of candidates for further processing. For each candidate \( \hat{v}^{(0)} \), we perform Algorithm 2 to find \( \hat{v}^{(1)} \) and \( \hat{v}^{(2)} \) from \( y^{(1)} \) and \( y^{(2)} \). Finally, we select \( \hat{v}^{(0)} \) such that \( (\hat{v}^{(0)}, \hat{v}^{(1)}, \hat{v}^{(2)}) \) is the most likely candidate with respect to \( (y^{(0)}, y^{(1)}, y^{(2)}) \).
C. Decoding Complexity

In this subsection, taking the add-compare-select operation (the basic operation in both VA and SLVA) as an atomic operation, we analyze the complexity of the statistical learning aided list decoding. Assume that the basic code $\mathcal{C}[n,k]$ has a trellis representation with $s$ states. To find the best candidate codeword by SLVA (equivalently, by VA), $sn$ operations are needed. With the $(\ell - 1)$-th best candidate codeword known, only $n$ operations are needed to find the $\ell$-th best candidate codeword by SLVA.

Let $\bar{\ell} \leq \ell_{\text{max}}$ be the average list size. Then the SLVA requires on average $sn + (\bar{\ell} - 1)n$ operations. For each candidate, VA is employed to calculate the soft metric, which needs $\bar{\ell}sn$ operations. Hence the total operations for decoding each sub-frame is given by

$$\#\text{Operations} = (s + \bar{\ell} - 1 + \bar{\ell}s)n. \quad (33)$$

We see that the complexity is dominated by $\bar{\ell}sn$. For fixed $n$, we can reduce the complexity is to reduce the average list size $\bar{\ell}$ by tuning down the threshold.

V. SIMULATION RESULTS

In this section, all simulations are conducted by assuming BPSK modulation and AWGN channels. The SRBO-CCs are terminated every $L = 49$ blocks. All codes are decoded by Algorithm 2 with the maximum list size $\ell_{\text{max}} = 64$ and properly thresholds obtained by off-line statistical learning, unless otherwise specified. All upper bounds taken as the benchmarks are derived by combining (11) and (32).

A. Impact of Parameter on the Performance

Example 5: The basic code is the 16-state $(2,1,4)$ convolutional code, which is truncated without termination and defined by the polynomial generator matrix $G(D) = [27,31]_8$. We consider the sub-frame information lengths $k = 32$ and $k = 48$. The maximum list sizes are $\ell_{\text{max}} = 64$ and $\ell_{\text{max}} = 128$ for $k = 32$ and $k = 48$, respectively. The fER is shown in Fig. 6. We see that the performance can be improved by increasing $k$ (hence the decoding delay). It is worth pointing out that a larger $k$ usually requires a larger maximum list size $\ell_{\text{max}}$. 
The basic code is the 16-state (2, 1, 4) convolutional code, which is truncated without termination and defined by the polynomial generator matrix $G(D) = [27, 31]_8$. We consider the sub-frame information lengths $k = 32$ and $k = 48$. The maximum list sizes are $\ell_{\text{max}} = 64$ and $\ell_{\text{max}} = 128$ for $k = 32$ and $k = 48$, respectively.

**TABLE II**

| SNR | 2.0 | 2.5 | 3.0 | 3.5 | 4.0 |
|-----|-----|-----|-----|-----|-----|
| $T_A$ | 1.3 | 1.35 | 1.4 | 1.45 | 1.5 |
| $T_B$ | 0.95 | 1.0 | 1.05 | 1.1 | 1.15 |
| list size for $T_A$ | 38 | 30 | 23 | 18 | 14 |
| list size for $T_B$ | 25 | 8.2 | 2.6 | 1.3 | 1.1 |

**B. Tradeoff Between Performance and Complexity**

**Example 6**: The 16-state (2, 1, 4) TBCC defined by the polynomial generator matrix $G(D) = [27, 31]_8$ is taken as the basic code. The sub-frame information length is $k = 32$. We consider two sets of thresholds $T_A$ and $T_B$ specified in Table II. The fER is shown in Fig. 7, while the average list sizes needed for decoding a sub-frame are shown in Table II. We see that the complexity (average list size), at the cost of performance loss, can be reduced by tuning down the threshold. For example, at SNR = 4 dB, the computational complexity (average list size) can be reduced more than 10 times if a performance degradation (fER deterioration) is tolerated from $10^{-5}$ to $10^{-4}$. 
Fig. 7. Performance of the SRBO-CC in Example 6. The 16-state (2, 1, 4) TBCC defined by the polynomial generator matrix $G(D) = [27, 31]_8$ is taken as the basic code. The sub-frame information length is $k = 32$. We consider two sets of thresholds $T_A$ and $T_B$ specified in Table II.

C. Performance with Different Rate

Example 7: The 16-state (2, 1, 4) TBCC defined by the polynomial generator matrix $G(D) = [27, 31]_8$, the (3, 1, 4) TBCC defined by the polynomial generator matrix $G(D) = [25, 33, 37]_8$ and the (4, 1, 4) TBCC defined by the polynomial generator matrix $G(D) = [25, 27, 33, 37]_8$ are taken as the basic code. The sub-frame information lengths and the total rates are specified in the legends. The fER is shown in Fig. 8. We see that the SRBO-CCs can support a wide range of code rates by simply adapting the basic code to the desired rate.

D. Comparison with Sequential Decoding

Example 8: The Cartesian product of Reed-Muller code RM[8, 4] is taken as the basic code. The sub-frame information length is $k = 32$. For comparison, the same code is also decoded by the sequential decoding [22] with the same decoding window and a stack of size 20000. The fER is shown in Fig. 9. We see that the statistical learning aided list decoding algorithm outperforms the sequential decoding algorithm in high SNR region.
Fig. 8. Performance of the SRBO-CCs in Example 7. The 16-state \((2, 1, 4)\) TBCC defined by the polynomial generator matrix 
\(G(D) = [27, 31]_8\), the \((3, 1, 4)\) TBCC defined by the polynomial generator matrix 
\(G(D) = [25, 33, 37]_8\) and the \((4, 1, 4)\) TBCC defined by the polynomial generator matrix 
\(G(D) = [25, 27, 33, 37]_8\) are taken as the basic code. The sub-frame information lengths and the total rates are specified in the legends.

Fig. 9. Performance of the SRBO-CC in Example 8. The Cartesian product of Reed-Muller code \(RM[8, 4]^8\) is taken as the basic code. The sub-frame information length is \(k = 32\). For comparison, the same code is also decoded by the sequential decoding \([22]\) with the same decoding window and a stack of size 20000.
Fig. 10. Performance of the SRBO-CC in Example 9. The 16-state $(2, 1, 4)$ TBCC defined by the polynomial generator matrix $G(D) = [27, 31]_8$ is taken as the basic code. The sub-frame information length is $k = 32$. For comparison, we have also redrawn the performance curve of the polar code [9] with length 128 (the same decoding delay as the SRBO-CC). Note that “SCL(16)” represents the successive cancellation list algorithm [8] with list size 16.

E. Comparison with Other Codes

Example 9: The 16-state $(2, 1, 4)$ TBCC defined by the polynomial generator matrix $G(D) = [27, 31]_8$ is taken as the basic code. The sub-frame information length is $k = 32$. For comparison, we have also redrawn the performance curve of the polar code [9] with length 128 (the same decoding delay as the SRBO-CC). The fER is shown in Fig. 10 where “SCL(16)” represents the successive cancellation list algorithm [8] with list size 16. We see that the SRBO-CC with statistical learning aided list decoding is competitive with the polar code.

VI. CONCLUSION

In this paper, we have presented more details on the SRBO-CCs, which can be decoded by a statistical learning aided list decoding. The decoder outputs serially a list of decoding candidates and identifies the correct one by a statistical threshold, which can be designed by statistical learning. With statistical learning aided list decoding, the performance-complexity tradeoff and the performance-delay tradeoff can be achieved by adjusting the statistical threshold and extending the decoding window, respectively. A closed-form upper bound based on the weight enumerating function was derived to analyze the performance in high SNR region. Simulation
results showed that the statistical learning aided list decoding outperforms the sequential decoding in high SNR region and that under the constraint of equivalent decoding delay, the SRBO-CCs have comparable performance with the polar codes.

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