Covariance of the One-Dimensional Mass Power Spectrum

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ABSTRACT

We analyse the covariance of the one-dimensional mass power spectrum along lines of sight. The covariance reveals the correlation between different modes of fluctuations in the cosmic density field and gives the sample variance error for measurements of the mass power spectrum. For Gaussian random fields, the covariance matrix is diagonal. As expected, the variance of the measured one-dimensional mass power spectrum is inversely proportional to the number of lines of sight that are sampled from each random field. The correlation between lines of sight in a single field may alter the covariance. However, lines of sight that are sampled far apart are only weakly correlated, so that they can be treated as independent samples. Using \( N \)-body simulations, we find that the covariance matrix of the one-dimensional mass power spectrum is not diagonal for the cosmic density field due to the non-Gaussianity and that the variance is much higher than that of Gaussian random fields. From the covariance, one will be able to determine the cosmic variance in the measured one-dimensional mass power spectrum as well as to estimate how many lines of sight are needed to achieve a target precision.

Key words: cosmology: theory – large-scale structure of universe

1 INTRODUCTION

The one-dimensional mass power spectrum (PS) and its relation to the three-dimensional mass PS have been frequently utilized to recover the linear mass PS from the Ly\(\alpha\) forest [Croft et al. 1998, 1999, 2002; Gnedin & Hamilton 2002; Kim et al. 2004]. This opens a great window for studying the large-scale structure of the universe over a wide range of redshift. As one starts to attempt precision cosmology using the Ly\(\alpha\) forest [Croft et al. 2002; Mandelbaum et al. 2003; Spergel et al. 2003], it becomes necessary to quantify systematic uncertainties of the PS analysis including the covariance.

For an ensemble of isotropic fields, the one-dimensional mass PS \( P_{1D}(k) \) is a simple integral of the three-dimensional mass PS \( P_{3D}(k) \) (Lumsden, Heavens & Peacock 1989),

\[
P_{1D}(k) = \frac{1}{2\pi} \int_{-\infty}^{\infty} P_{3D}(k') k' dk',
\]

where \( k \) is the line-of-sight wavenumber. However, there is only one observable universe. One has to replace the ensemble average with a spatial average. For instance, one may sample multiple line-of-sight densities from the three-dimensional cosmic density field and use the average PS of the one-dimensional densities in place of the ensemble-average quantity.

The covariance of the spatial-average PS may differ from that of the ensemble-average PS for at least two reasons. First, lines of sight (LOS’s) are no longer independent of each other. Correlations between close LOS’s could increase the covariance of the one-dimensional mass PS, although they may also be used to extract cosmological parameters from the Ly\(\alpha\) forest [Hui, Stebbins & Burles 1999; McDonald & Miralda-Escude 1999; Viel et al. 2002; McDonald 2003; Rollinde et al. 2003]. Second, the length of each LOS is always much less than the size of the universe, so that false correlations between different modes are introduced in the covariance (for the three-dimensional case, see Feldman, Kaiser & Peacock 1994). Because the cosmic density field is highly non-Gaussian on scales of interest, \( N \)-body simulations are helpful for quantifying the covariance.

The rest of the paper is organized as follows. Section 2 briefly describes the notation and the convention of Fourier transforms, the
three-dimensional mass PS, and its covariance. The spatial-average one-dimensional mass PS and its covariance are derived in Sections 3 and 4, respectively. The effects of the line-of-sight length is discussed in Section 5. Section 6 presents numerical results of the covariance from N-body simulations with different box sizes. The conclusions and discussions are given in Section 7. Unless specified otherwise, the PS refers to the mass PS.

2 THREE-DIMENSIONAL POWER SPECTRUM

We assume the following convention of Fourier transforms for a cubic density field of volume $V = B^3$:

$$\hat{\delta}(\mathbf{n}) = \int_V \delta(\mathbf{x}) e^{-2\pi i \mathbf{n} \cdot \mathbf{x}/B} \, d\mathbf{x}, \quad \delta(\mathbf{x}) = \frac{1}{V} \sum_{\mathbf{n} = -\infty}^{\infty} \hat{\delta}(\mathbf{n}) e^{2\pi i \mathbf{n} \cdot \mathbf{x}/B},$$

(2)

where $\delta(\mathbf{x})$ and $\hat{\delta}(\mathbf{n})$ are the overdensity and its Fourier counterpart, respectively, the summation $\sum_{\mathbf{n} = -\infty}^{\infty}$ is an abbreviation for $\sum_{n_1, n_2, n_3 = -\infty}^{\infty}$ and the wavevector $\mathbf{k} = 2\pi \mathbf{n}/B$. With the understanding that Fourier modes exist only at discrete wavenumbers, i.e. $n_1$, $n_2$, and $n_3$ are integers, one may use $\mathbf{k}$ and $\mathbf{n}$ interchangeably for convenience. To be complete, the orthonormality relations are

$$\frac{1}{V} \int_V e^{-2\pi i (\mathbf{n} - \mathbf{n'} \cdot \mathbf{x})/B} \, d\mathbf{x} = \delta_{\mathbf{n} \cdot \mathbf{n'}}^{\mathbf{K}}, \quad \frac{1}{V} \sum_{\mathbf{n} = -\infty}^{\infty} e^{2\pi i (\mathbf{x} - \mathbf{x'})/B} = \delta^D(\mathbf{x} - \mathbf{x'}),$$

(3)

where $\delta_{\mathbf{n} \cdot \mathbf{n'}}^{\mathbf{K}}$ is the three-dimensional Kronecker delta function, and $\delta^D(\mathbf{x} - \mathbf{x'})$ the Dirac delta function.

The three-dimensional PS of the universe is defined through

$$\langle \hat{\delta}(\mathbf{k}) \hat{\delta}^*(\mathbf{k'}) \rangle = P_{3D}(\mathbf{k}) \delta_{\mathbf{n} \cdot \mathbf{n'}}^{\mathbf{K}},$$

(4)

where $\langle \ldots \rangle$ stands for an ensemble average. One may define an observed PS, $P_{3D}(\mathbf{k}) = |\hat{\delta}(\mathbf{k})|^2/V$, so that $P_{3D}(\mathbf{k}) = \langle P_{3D}(\mathbf{k}) \rangle$. Note that a shot-noise term should be included if the PS is measured from discrete objects, and it is inversely proportional to the mean number density of the objects. We neglect the shot noise in this paper. For an isotropic universe, $P_{3D}(\mathbf{k})$ is a function of the length of $\mathbf{k}$ only, i.e. $P_{3D}(\mathbf{k}) \equiv P_{3D}(k)$. The four-point function of $\hat{\delta}(\mathbf{k})$ is

$$\langle \hat{\delta}(\mathbf{k}) \hat{\delta}^*(\mathbf{k'}) \hat{\delta}^*(\mathbf{k''}) \hat{\delta}^*(\mathbf{k''''}) \rangle = V^2 [P_{3D}(\mathbf{k}) P_{3D}(\mathbf{k'}) \delta_{\mathbf{n} \cdot \mathbf{n'}}^{\mathbf{K}} \delta_{\mathbf{n''} \cdot \mathbf{n'''}}^{\mathbf{K}} + P_{3D}(\mathbf{k}) P_{3D}(\mathbf{k''}) \delta_{\mathbf{n} \cdot \mathbf{n''}}^{\mathbf{K}} \delta_{\mathbf{n'''} \cdot \mathbf{n''''}}^{\mathbf{K}} + V T(\mathbf{k}, -\mathbf{k'}, -\mathbf{k''}, -\mathbf{k''''})],$$

(5)

where $T$ is the trispectrum, and we have restricted the wavevectors to be in the same hemisphere so that the term $P_{3D}(\mathbf{k}) P_{3D}(\mathbf{k'}) \delta_{\mathbf{n} \cdot \mathbf{n''}}^{\mathbf{K}} \delta_{\mathbf{n'''} \cdot \mathbf{n''''}}^{\mathbf{K}}$ does not appear. There is a redundancy in the variables of the trispectrum, which has only six degrees of freedom arising from relative coordinates of the four points under the constraint of homogeneity [Peebles 1980]. It is evident from the four-point function that the covariance of the three-dimensional PS is (see also [Meiksin & White 1999, Cooray & Hu 2001])

$$\sigma^2(\mathbf{k}, \mathbf{k'}) = |\langle P_{3D}(\mathbf{k}) - P_{3D}(\mathbf{k'}) \rangle|^2 = P_{3D}(\mathbf{k}) \delta_{\mathbf{n} \cdot \mathbf{n'}}^{\mathbf{K}} + V^{-1} T(\mathbf{k}, -\mathbf{k'}, -\mathbf{k''}, -\mathbf{k'''}),$$

(6)

For Gaussian random fields (GRFs), the trispectrum vanishes, and $\sigma^2(\mathbf{k}, \mathbf{k'}) = P_{3D}(\mathbf{k}) \delta_{\mathbf{n} \cdot \mathbf{n'}}^{\mathbf{K}}$. In reality, the survey volume is always smaller than the observable universe, and the survey geometry is more complex than the simple case we have assumed. These lead to a modification of the covariance for GRFs [Feldman, Kaiser & Peacock 1994].

$$\sigma^2(\mathbf{k}, \mathbf{k'}) \simeq P_{3D}^2(\mathbf{k}) \left[ \int \frac{u^2(\mathbf{x}) e^{-i(\mathbf{k} - \mathbf{k'}) \cdot \mathbf{x}} \, d\mathbf{x}}{\int u(\mathbf{x}) \, d\mathbf{x}} \right]^2,$$

(7)

where $u(\mathbf{x})$ is a weight function that depends on the survey volume and geometry, and the shot noise is neglected. For the low-redshift cosmic density field, the non-Gaussianity can dramatically boost the elements of the covariance matrix through the trispectrum.

3 ONE-DIMENSIONAL POWER SPECTRUM

For a line-of-sight density that is along the $x_3$-axis and sampled at $(x_1, x_2)$, the one-dimensional Fourier transform gives

$$\hat{\delta}(x_1, n_3) = \int_0^B \delta(x_1) e^{-2\pi i n_3 x_3/B} \, dx_3 = \frac{1}{B^2} \sum_{n_\perp = -\infty}^{\infty} \hat{\delta}(n_\perp, n_3) e^{2\pi i n_\perp x_1/B},$$

(8)

where the subscript $\perp$ signifies the first two components of a vector, i.e. $\mathbf{x}_\perp = (x_1, x_2)$. Similar to the three-dimensional PS, the one-dimensional PS is expected to follow

$$\langle \hat{\delta}(x_1, n_3) \hat{\delta}^*(x_1, n_3) \rangle = P_{1D}(n_3) B \delta_{n_\perp, n_3}^{\mathbf{K}}.$$

(9)

Substituting equation 8 in equation 7 and making use of equation 9, one finds the relation between the one-dimensional PS and the three-dimensional PS,
where the equality, the cosmic density field is not band-limited, aliasing can distort statistics of the field. The distortion on the three-dimensional PS cannot be added to the principal mode if they are not properly filtered out before sampling (for details, see Hockney & Eastwood 1981). Since

\[ P_{1D}(n_3) = \frac{1}{B^2} \sum_{n_\perp = -\infty}^{\infty} P_{3D}(n_\perp, n_3), \]  

(10)

which is a discrete analog of equation 1. Practically, one measures PS’s of LOS densities sampled at some locations, e.g. \( P_{1D}(x_\perp, k_3) = |\hat{\delta}(x_\perp, k_3)|^2 / B \). A simple estimator of the one-dimensional PS may be constructed by a spatial average over many LOS’s, i.e. \( P_{1D}(k_3) = \langle P_{1D}(x_\perp, k_3) \rangle_{x_\perp} \), where \( \langle \ldots \rangle_{x_\perp} \) runs over all LOS’s sampled in a single universe. To assess the performance of the estimator, two questions need to be addressed: (1) How does \( P_{1D}(k_3) \) relate itself to \( P_{3D}(k) \); and (2) What is the covariance of \( P_{1D}(k_3) \) with respect to \( P_{1D}(k_3) \)? The rest of this section answers the former, and Section 4 the latter.

For simplicity, we assume that LOS’s are sampled regularly in transverse directions at an interval of \( b = B / m \), where \( m \) is an integer. Each LOS has a length of \( B \), and \( x_\perp = l_1 b \) with \( l_1, l_2, \ldots, m - 1 \). The estimated one-dimensional PS is then

\[ P_{1D}(n_3) = \frac{1}{m^2 B^2} \sum_{l_1 = 0}^{m-1} \left| \sum_{n_\perp = -\infty}^{\infty} \hat{\delta}(n_\perp, n_3) e^{2\pi i n_\perp l_1 / m} \right|^2 = \frac{1}{B^2} \sum_{n_\perp, m j_\perp = -\infty}^{\infty} \hat{\delta}(n_\perp, n_3) \hat{\delta}^*(n_\perp + m j_\perp, n_3), \]  

(11)

where the equality,

\[ \frac{1}{m} \sum_{l=0}^{m-1} e^{2\pi i (n-n') l / m} = \delta^K_{n, n'+mj} \quad j = 0, \pm 1, \ldots, \pm \infty, \]  

(12)

has been used to obtain the last line in equation 11. It is easy to show with equation 11 that

\[ P_{1D}(k_3) = \langle P_{1D}(k_3) \rangle = \langle P_{1D}(x_\perp, k_3) \rangle. \]  

(13)

Note that there is no spatial average on the far right side of equation 11. It seems that only in the limit \( m \to \infty \) do \( P_{1D}(k_3) \) and \( P_{3D}(k) \) follow the same relation as equations 11 and 10, i.e. \( P_{1D}(k_3) = \sum_{n_\perp = -\infty}^{\infty} P_{3D}(k_3, k_3) / B^2 \), where we have assumed \( \delta^K(\infty) = 0 \) so that only \( j_\perp = (0, 0) \) terms contribute. In other words, to reduce the uncertainties of the recovered \( P_{3D}(k) \) and \( P_{1D}(k) \), one has to increase the sampling rate in the transverse direction.

Without losing generality, one may choose \( m \) to be even, so that equation 11 can be re-arranged into

\[ P_{1D}(n_3) = \frac{1}{B^2} \sum_{n_\perp = -m/2}^{m/2} P_{3D}(n_\perp, n_3) = \frac{1}{B^2} \sum_{n_\perp = -\infty}^{\infty} P_{3D}(n_\perp, n_3) + A(n_3), \]  

(14)

where

\[ P_{3D}(n_\perp, n_3) = \frac{1}{V} \left| \sum_{j_\perp = -\infty}^{\infty} \hat{\delta}(n_\perp + m j_\perp, n_3) \right|^2, \]  

(15)

and

\[ A(n_3) = \frac{1}{B^2} V \sum_{j_\perp = -\infty}^{\infty} \sum_{m j_\perp = -m/2}^{m/2} \sum_{j_\perp' \neq 0} \delta(n_\perp + m j_\perp, n_3) \delta^*(n_\perp + m(j_\perp + j_\perp'), n_3). \]  

(16)

Equation 14 provides two equivalent views of the sampling effect. On one hand, the one-dimensional PS is a sum of the aliased three-dimensional PS, \( P_{3D}(k) \), that is sampled by a grid of \( m \times m \) LOS’s. On the other hand, it is a complete sum of the underlying three-dimensional PS with an extra term \( A(k_3) \) that is determined by the sampling rate and properties of the density field.

The discrete Fourier transform cannot distinguish a principal mode at \( |k| \ll k_{Nyq} \) from its aliases at \( k = \pm 2k_{Nyq}, k = \pm 4k_{Nyq}, \ldots \), where \( k_{Nyq} \) is the sampling Nyquist wavenumber. For example, \( k_{Nyq} = \pi / b \) if one samples the field at an equal spacing of \( b \). The alias modes will be added to the principal mode if they are not properly filtered out before sampling (for details, see Hockney & Eastwood [1981]). Since the cosmic density field is not band-limited, aliasing can distort statistics of the field. The distortion on the three-dimensional PS cannot be quantified a priori, because it depends on relative phases between the principal mode and its aliases. The alias effect is less pronounced, if amplitudes of the alias modes are much smaller than that of the principal mode. Since the three-dimensional PS decreases toward small scales, a high sampling rate (or \( k_{Nyq} \)) can suppress aliasing for modes with \( k \ll k_{Nyq} \).

If the Fourier modes of the density field are uncorrelated, the term \( A(k_3) \) may be neglected even for a finite number of LOS’s and, therefore, validate equation 11. Strictly speaking, \( A(k_3) \) vanishes only as an ensemble average over many GRFs, but since there are so many independent modes in a shell of radius around \( k \) – especially at large wavenumbers – the summation in \( A(k_3) \) will tend to vanish even for a single GRF. The cosmic density field is more Gaussian at higher redshift, so equation 11 may be a good approximation then. At low redshift, however, the non-Gaussianity alters the statistics of one-dimensional fields (see e.g. Amendt [1994]), such that one might only be able to recover a heavily aliased three-dimensional PS from a sparse sample of LOS’s.

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4 \hspace{1cm} \textbf{COVARIANCE OF THE ONE-DIMENSIONAL POWER SPECTRUM}

The covariance of the PS is of interest because it tells us how likely an estimated PS is to represent the true PS and how much the modes on different scales are correlated. For the one-dimensional PS, the covariance is defined as

$$\sigma^2_{1D}(k_3, k_3') = \langle [P_{1D}(k_3) - P_{1D}(k_3)] [P_{1D}(k_3') - P_{1D}(k_3')] \rangle.$$  \hspace{1cm} (17)

The covariance of the mean PS of $N$ LOS’s that are sampled in a single field can be expanded into a sum of pair-wise covariances between two LOS’s separated by $s_{ij}^k = x_{ij}^k - x_{ij}^\perp$, e.g.

$$\sigma^2_{1D}(k_3, k_3') = \frac{1}{N^2} \sum_{s_{ij}^k} \sigma^2_{1D}(k_3, k_3'; s_{ij}^k),$$  \hspace{1cm} (18)

where

$$\sigma^2_{1D}(k_3, k_3'; s_{ij}^k) = \langle [P_{1D}(x_{ij}^k, k_3) - P_{1D}(k_3)] [P_{1D}(x_{ij}^k, k_3') - P_{1D}(k_3')] \rangle,$$

and $x_{ij}^k$ is the location of the $j \text{th}$ LOS in the $x_1$-$x_2$ plane.

For GRFs, the four-point function [equation 8] helps reduce the pair-wise covariance to

$$\sigma^2_{1D}(n_3, n_3'; s) = \frac{1}{B^2} \sum_{n_{\perp}, n_{\perp}', n_{\parallel}, n_{\parallel}'} \langle \delta(n_{\perp}, n_3) \delta^*(n_{\perp}', n_3) \delta(n_{\parallel}, n_3') \delta^*(n_{\parallel}', n_3') \rangle$$

$$\times e^{2\pi i (n_{\perp} - n_{\perp}') x_{\perp}^k + (n_{\parallel} - n_{\parallel}') x_{\parallel}^k / B} - P_{1D}(n_3) P_{1D}(n_3') \delta_{n_{\perp}, n_{\perp}'} \equiv \langle \xi(s, n_3) \rangle^2 \delta_{n_{\perp}, n_{\perp}'}^K,$$

where the subscript and superscript are dropped for $s$. Fourier transforms of $\xi(s, k_3)$ will give the three-dimensional PS and correlation function of the density field, and $P_{1D}(k_3) = \xi(0, k_3)$ (see also Viel et al. 2002). Because of isotropy, the pair-wise covariance $\sigma^2_{1D}(k_3, k_3'; s)$ and $\xi(s, k_3)$ depend only on the magnitude of the separation, i.e. $\sigma^2_{1D}(k_3, k_3'; s) = \sigma^2_{1D}(k_3, k_3; s)$ and $\xi(s, k_3) = \xi(s, k_3)$.

If one LOS is sampled from each GRF, the variance of the measured one-dimensional PS $P_{1D}(k_3)$ is $\sigma^2_{1D}(k_3, k_3) = \sigma^2_{1D}((k_3, k_3'; s)) = P_{1D}(k_3)$, analogous to the three-dimensional case. If $N = m^3$ LOS’s are sampled in each GRF on a regular grid as in Section 3, e.g. $s = (I - I') / B$, the covariance becomes

$$\sigma^2_{1D}(n_3, n_3') = \frac{1}{N^2} \sum_{I_1 = 0}^{m-1} \sigma^2_{1D}[n_3, n_3'; (I_1 - I_1') B] = \frac{1}{N^2 B^4} \sum_{n_{\perp}, n_{\perp}'} \sum_{I_1 = 0}^{\infty} P_{1D}(n_3, n_3) P_{1D}(n_3', n_3) \left| \sum_{I_1 = 0}^{m-1} e^{2\pi i (n_{\perp} - n_{\perp}') I_1 / B} \right|^2 \delta_{n_{\perp}, n_{\perp}'}^K,$$

where we have used equation 12 to reach the last line. This result can be easily obtained using equations 8 and 11 as well. It is seen that the summation in the last line of equation 20 runs over only one mode out of every $N$ modes in the Fourier space. If the three-dimensional PS were constant, the variance of the mean PS of the $N$ LOS’s would be $N$ times smaller than the variance of the PS of a single LOS. This is coincident with the theory of the variance of the mean. Since the cosmic density field is not a GRF in general, equation 20 is not expected to give an accurate estimate.

If the $N$ LOS’s are sampled randomly in each GRF, the summation over LOS’s in the second line of equation 20 should be re-cast to read

$$\sigma^2_{1D}(n_3, n_3') = \frac{1}{N^2 B^4} \sum_{n_{\perp}, n_{\perp}'} P_{1D}(n_3, n_3) P_{1D}(n_3', n_3) \sum_{j, l = 1}^{N} e^{2\pi i (n_{\perp} - n_{\perp}') (x_{\perp}^j - x_{\perp}^l) / B} \delta_{n_{\perp}, n_{\perp}'}^K,$$

where $x_{\perp}^j$ is randomly distributed, the second sum in equation 21 tends to vanish for a large number of LOS’s except when $j = l$. Thus, one obtains

$$\sigma^2_{1D}(n_3, n_3') \sim \frac{1}{N^2 B^4} \sum_{n_{\perp}, n_{\perp}'} P_{1D}(n_3, n_3) P_{1D}(n_3', n_3) \delta_{n_{\perp}, n_{\perp}'}^K = \frac{1}{N} P_{1D}^2(n_3) \delta_{n_{\perp}, n_{\perp}'}^K.$$

Again, it shows that the variance of $P_{1D}(k_3)$ is inversely proportional to the number of LOS’s, but this is valid only for a large number of LOS’s randomly sampled in a GRF.

Through the Gaussian case one can find the lowest bound of the uncertainty for a measured one-dimensional PS and the minimum number of LOS’s needed for a target precision. For example, to measure the one-dimensional PS of GRFs accurate to 5% on every mode,
The ensemble-averaged one-dimensional PS is

\[ P_{1D}(n_\perp, n_3)P_{1D}(n_\perp + mj_\perp, n_3) \delta^{K}_{n_\perp, n_3} + \frac{1}{B} T_{1D}(n_\perp, n_3'), \]

for GRFs, the covariance of one needs at least 400 LOS’s. However, the cosmic density field has a far more complex covariance due to its non-vanishing trispectrum, and it can have a much larger variance of the measured one-dimensional PS.

The trispectrum introduces an extra term to the covariance of the one-dimensional PS in addition to the Gaussian piece [equation (20)], i.e.

\[ \sigma_{1D}^2(n_\perp, n_3') = \frac{1}{B^3} \sum_{n_\perp, j_\perp = -\infty}^\infty P_{3D}(n_\perp, n_3)P_{3D}(n_\perp + mj_\perp, n_3) \delta^{K}_{n_\perp, n_3} + \frac{1}{B} T_{1D}(n_\perp, n_3'), \]

where

\[ T_{1D}(n_\perp, n_3') = \frac{1}{B^3} \sum_{n_\perp, j_\perp, n_\perp', j_\perp' = -\infty}^\infty T(n, -n - mj, n', -n' - mj'), \]

are the transverse components of \( n, j, n', \) and \( j', \) respectively, and \( j_3 = j_3' = 0. \) In deriving equation (23), we have made use of equations (5) and (11).

\section{5 Line-of-Sight Length}

Observationally, the length of a LOS is always much less than the size of the observable universe. For example, in the case of a pencil-beam survey, the length is limited by the depth of the survey. For the Ly\( \alpha \) forest, it is often determined by the distance between the starting redshift of Ly\( \alpha \) and Ly\( \beta \) lines. Damped Ly\( \alpha \) systems and other astrophysical or instrumental factors can break the spectrum into yet shorter chunks. Even if a very long LOS were obtained, one might still wish to measure the PS on smaller segments to avoid evolutionary effects. To account for the length of LOS’s, the one-dimensional PS and its covariance must be re-formulated.

For a LOS from \((x_\perp, 0)\) to \((x_\perp, L)\), its Fourier transform is

\[ \hat{\delta}(x_\perp, \tilde{n}_3) = \int_0^L \delta(x_\perp, x_3) e^{-2\pi i n_3 x_3 / L} dx_3 = \frac{L}{B^3} \sum_{n = -\infty}^\infty \hat{\delta}(n) e^{2\pi i n_3 x_3 / B} e^{2\pi i (n_3 L / B - \tilde{n}_3)} - 1, \]

so that

\[ \langle \hat{\delta}(x_\perp, \tilde{n}_3) \hat{\delta}^*(x_\perp, \tilde{n}_3') \rangle = \frac{L^2}{B^3} \sum_{n = -\infty}^\infty P_{3D}(n) w(n_3, \tilde{n}_3) w(n_3', \tilde{n}_3'), \]

where

\[ w(n_3, \tilde{n}_3) = (-1)^{n_3} \sin[(n_3 L / B - \tilde{n}_3) \pi] / (n_3 L / B - \tilde{n}_3) \pi. \]

The function \( w(n_3, \tilde{n}_3) \) is a window function that mixes Fourier modes of the density field along \( n_3 \) direction into the one-dimensional Fourier mode at \( \tilde{n}_3. \) Because \( w(n_3, \tilde{n}_3) \) and \( w(n_3', \tilde{n}_3') \) are not orthogonal to each other, the term \( \langle \hat{\delta}(x_\perp, \tilde{n}_3) \hat{\delta}^*(x_\perp, \tilde{n}_3') \rangle \) is no longer diagonal with respect to \( \tilde{n}_3 \) and \( \tilde{n}_3'. \) However, it remains diagonal-dominant.

One may define the observed one-dimensional PS at \( x_\perp \) as

\[ \tilde{P}_{1D}(x_\perp, \tilde{n}_3) = \langle \hat{\delta}(x_\perp, \tilde{n}_3) \hat{\delta}^*(x_\perp, \tilde{n}_3') \rangle / L. \]

The ensemble-averaged one-dimensional PS is

\[ \bar{P}_{1D}(\tilde{n}_3) = \langle \tilde{P}_{1D}(x_\perp, \tilde{n}_3) \rangle = \frac{L}{B^3} \sum_{n = -\infty}^\infty P_{3D}(n) w^2(n_3, \tilde{n}_3). \]

For GRFs, the covariance of \( \bar{P}_{1D}(\tilde{n}_3) \) is

\[ \sigma_{1D}^2(\tilde{n}_3, \tilde{n}_3') = \langle (\tilde{P}_{1D}(x_\perp, \tilde{n}_3) - \bar{P}_{1D}(\tilde{n}_3))(\tilde{P}_{1D}(x_\perp, \tilde{n}_3') - \bar{P}_{1D}(\tilde{n}_3')) \rangle = \frac{L^2}{B^3} \sum_{n_3, n_3' = -\infty}^\infty P_{3D}(n) P_{3D}(n') w(n_3, \tilde{n}_3) w(n_3', \tilde{n}_3') w(n_3', \tilde{n}_3) w(n_3, \tilde{n}_3'). \]

As expected, the covariance is not diagonal because of the window function \( w(n_3, \tilde{n}_3). \)

\section{N-Body Tests}

Even at moderately high redshift, the cosmic density field is already quite non-Gaussian on scales below 10 \( h^{-1} \) Mpc. When projected in one dimension, the small-scale non-Gaussianity will obviously affect the measured one-dimensional PS on much larger scales. The non-vanishing
trispectrum introduces an extra term, \( T_{1D} (k_3, k'_3) \), to the covariance of the one-dimensional PS [see equation (23)]. Although it is possible to derive an approximate trispectrum based on the halo model (e.g. Cooray & Hu 2001), the one-dimensional projection, unfortunately, obscures the contribution of the trispectrum to the one-dimensional PS. Therefore, numerical simulations are necessary for the study.

Three \( N \)-body simulations of 256\(^3\) cold dark matter (CDM) particles are used to quantify the covariance of the one-dimensional PS. The model parameters are largely consistent with WMAP results (Spergel et al. 2003), e.g. \((\Omega, \Omega_b, \Omega_\Lambda, h, \sigma_8, n) = (0.27, 0.04, 0.73, 0.71, 0.85, 1)\), where \( \Omega \) is the cosmic matter density parameter, \( \Omega_b \) the baryon density parameter, \( \Omega_\Lambda \) the energy density parameter associated with the cosmological constant, \( \sigma_8 \) the rms density fluctuation within a radius of 8 \( h^{-1} \) Mpc, and \( n \) is the power spectral index. The box sizes of the simulations are 128 \( h^{-1} \) Mpc (labelled as B128), 256 \( h^{-1} \) Mpc (B256), and 512 \( h^{-1} \) Mpc (B512). The baryon density parameter \( \Omega_b \) is used only for the purpose of calculating the transfer function using LINGER (Ma & Bertschinger 1995), which is then read by GRAFIC2 (Bertschinger 2001) to generate the initial condition. The CDM particles are evolved from \( z = 44.5 \) to the present using GADGET (Springel, Yoshida & White 2001). Additionally, GRFs are generated with the same box sizes as those of \( N \)-body simulations for comparison and are labelled as R128, R256, and R512, respectively.

The simulations produce snapshots at \( z = 3 \) and 0. For each snapshot, the particles are assigned to a density grid of 512\(^3\) nodes using the triangular-shaped-cloud scheme (Hockney & Eastwood 1981). LOS’s are then sampled on the \( x_1-x_2 \) plane, and each LOS is extracted with full resolution along the \( x_3 \)-axis. Because the density grid has a finite resolution, both the coordinates and the wavenumbers are discrete and finite. The equations in Sections 2–5 need to be modified accordingly, so that they do not sum over non-existing modes. For example, equation (10) becomes

\[
P_{1D}(n_3) = \frac{1}{B^2} \sum_{n_\perp = -M/2}^{M/2} P_{1D}(n_\perp, n_3),
\]

where \( M = 512 \) is the number of nodes of the density grid in transverse directions.

For brevity and the purpose of comparing the covariance, we introduce the following three covariance matrices: (1) The normalized pair-wise covariance between two LOS’s,

\[
C(k_3, k'_3; s) = \sigma_{1D}^2(k_3, k'_3; s) [P_{1D}(k_3) P_{1D}(k'_3)]^{-1},
\]

where \( s \) is the transverse separation between the two LOS’s. (2) The normalized covariance of the estimated one-dimensional PS,

\[
C(k_3, k'_3) = \sigma_{1D}^2(k_3, k'_3) [P_{1D}(k_3) P_{1D}(k'_3)]^{-1},
\]

where \( \sigma_{1D}^2(k_3, k'_3) \) is the covariance of the mean PS of \( N \) LOS’s, e.g. equation (11), and we have omitted its dependence on the number of LOS’s. (3) The reduced covariance

\[
\sigma_{1D}^2(k_3, k'_3; s) [P_{1D}(k_3) P_{1D}(k'_3)]^{-1},
\]
\[ \hat{C}(k_3, k'_3; s) = C(k_3, k'_3)[C(k_3, k_3)C(k'_3, k'_3)]^{-1/2}. \]

(34)

For GRFs, all these covariances are diagonal in matrix representation, if the length of LOS’s is the same as the size of the simulation box. In addition, we have \( C(k_3, k_3; 0) = 1 \) and \( C(k_3, k_3) = N^{-1} \), where \( N \) is the number of LOS’s that are sampled for estimating the one-dimensional PS. The advantage of \( \hat{C}(k_3, k'_3) \) is that it is 1 for all fields, so that they can be compared with each other in a single (grey) scale.

The normalized pair-wise covariance \( C(k_3, k'_3; s) \) is shown in Fig. 4 for GRFs R512 and the simulation B512 at \( z = 3 \). The covariances for GRFs are averaged over an ensemble of 2000 random realizations, while those for the simulation are averaged over 2000 pairs of LOS’s from a single field. The behaviour of the covariances is consistent with the expectation. Namely, \( C(k_3, k'_3; 0) \sim \delta_{k_3, k'_3} \) and \( C(k_3, k'_3; s) \) decreases as the separation \( s \) increases. The simulation deviates from GRFs because of the non-Gaussianity, which increases the variance \( C(k_3, k'_3; s) \).

Fig. 4 compares the diagonal elements \( C(k_3, k_3; s) \) with those calculated using equation (19). We note in passing that the expected values of \( C(k_3, k'_3; 3, \delta h^{-1} \text{Mpc}) \) are so close to 0 that they are even below the rms value of the off-diagonal elements of \( C(k_3, k'_3; 8 \delta h^{-1} \text{Mpc}) \) for 2000 GRFs. Hence, it is practically difficult to recover three-dimensional statistics from \( C(k_3, k'_3; s) \) if the LOS’s are too far apart. In theory, the modes of two LOS’s are always correlated as \( k_3 \rightarrow 0 \), regardless of their separation. However, large-scale LOS modes are greatly affected by small-scale three-dimensional perturbations, which are nearly-uncorrelated for large separations. As such, the correlation for \( s \gtrsim 8 \delta h^{-1} \text{Mpc} \) is so weak that it will not be easily detected against statistical uncertainties even for a sizable sample of 2000 pairs of LOS’s. Hence, Figs. 4 and 5 suggest that LOS’s sampled in a single cosmic density field are practically independent of each other as long as they are well separated. This condition is easily met. For example, McDonald et al. (2004) use 3035 \( z > 2.3 \) quasar spectra from the Sloan Digital Sky Survey (SDSS, over 2627 square degrees) to determine the \( \text{Ly}_\alpha \) flux PS. Assuming WMAP parameters, one will find that the mean separation between LOS’s is \( \sim 60 \) comoving \( \delta h^{-1} \text{Mpc} \) such that the covariance arisen from the correlation between LOS’s can be safely neglected.

Fig. 5 shows the one-dimensional PS measured by averaging over 64 and 1024 LOS’s from the B128 simulation. Although the mean PS of all the 512 planar LOS’s agrees with the result of a direct summation of the three-dimensional PS using equation (33), the deviation of the PS for any particular group of 64 or 1024 LOS’s is substantial, especially at \( z = 0 \). The variance is smaller at \( z = 3 \) than at \( z = 0 \) because the cosmic density field is more Gaussian earlier on, and it is roughly inversely proportional to the number of LOS’s. This can be seen better by comparing the lower right panel of Fig. 4 with that of Fig. 5. However, even at \( z = 3 \) the variance of the one-dimensional PS is still much higher than the variance for GRFs, \( N^{-1} P^2_{1D}(k_3) \), which indicates a heavy contribution from the trispectrum. The formulae in Sections 3.2.2 often assume that LOS’s are sampled on a grid with fixed spacing, which may not be applicable to realistic data such as inverted densities from the Ly\( \alpha \) forest (Nusser & Haehnelt 1999; Zhan 2003). Therefore, we sample the LOS’s in two ways in Fig. 5: grid sampling and random sampling. Since no significant difference is observed, random sampling can be safely applied in the rest of this paper.

The normalized covariances \( \hat{C}(k_3, k'_3) \) and \( C(k_3, k'_3) \) are quantified in Figs. 6 (\( N = 64 \)) and 6 (\( N = 1024 \)). The left column in each figure is the covariances of the spatially averaged one-dimensional PS from a single GRF that has the same box size and three-dimensional PS as its corresponding simulation. Clearly, the covariances based on spatial average are nearly diagonal with unity diagonal elements. This is in agreement with the expectations based on ensemble average for GRFs and is consistent with the ergodicity argument. The middle column is similar to the left column except that the density fields are from simulations at \( z = 3 \). The modes in the simulated density fields are strongly correlated, so that the covariances are no longer diagonal. In other words, the trispectrum is non-vanishing for the cosmic density field, as it is the only term that contributes to off-diagonal elements in the covariance. The cosmic density field becomes so non-Gaussian at
The non-Gaussianity is reflected not only in the correlations between different modes but also in the variance of the one-dimensional PS, i.e. the diagonal elements of the normalized covariance $C(k_3, k_3')$. The lower right panels of Figs. 4 and 5 compare $C(k_3, k_3')$ for five different density fields. The variance from the simulation B128 is orders of magnitude higher than the Gaussian value of $N^{-1} \hat{P}_{1D}(k_3)$, and it grows as the non-Gaussianity becomes stronger toward $z = 0$. As a result, the sample variance error estimated for GRFs is much lower than what can be actually measured from the cosmic density field. According to equation (12), both aliasing and the trispectrum contribute to the variance. Since the GRFs have the same three-dimensional PS and are sampled in the same way as the simulations, their near-Gaussian variances of spatially averaged one-dimensional PS’s suggest that the contribution of the aliasing effect is negligible. Comparisons of $C(k_3, k_3')$ for the same density field but with different sizes of sample ($N = 64$ and 1024) confirm the observation in Fig. 4 that the variance of the one-dimensional PS scales roughly as $N^{-1}$. This means that even though the non-Gaussianity drives up the sample variance error of the measured one-dimensional PS, one can still reduce the error by sampling a large number of LOS’s.

The statistics of a large sample may sometimes be dominated by a small fraction of extreme cases. It is inevitable that some of the LOS’s in our sample go through high-density halo structures, which may have significant contributions to the covariance matrices. To determine if a LOS falls in this category one has to assess the density profile intercepted by the LOS. We follow a much simpler approach by imposing a threshold to the mass column densities $\rho_{\text{col}}$ of the LOS’s. The one-point distribution function of column density contrast $\rho_{\text{col}}/\bar{\rho}_{\text{col}}$ from the simulation B128 (upper left panel of Fig. 6) peaks at $\rho_{\text{col}}/\bar{\rho}_{\text{col}} = 1$ as expected, and there clearly is a long-tail distribution (as compared to Gaussian or log-normal distributions) of $\rho_{\text{col}}/\bar{\rho}_{\text{col}} \gtrsim$ a few. We re-calculate the covariance $C(k_3, k_3')$ with a selection criterion that the column density of each LOS $\rho_{\text{col}}/\bar{\rho}_{\text{col}} < 3$. This criterion excludes 318 LOS’s from the selection of LOS’s for the statistics. As a total of 2000 × 64 LOS’s are randomly selected from 512$^3$ candidates, only half of the 318 LOS’s, i.e. 0.1% of the sample, would be selected without the criterion. The result is shown in the upper middle panel of Fig. 6 which is visually similar to the B128 $z = 3$ panel in Fig. 4. However, the cross sections of $\hat{C}(k_3, k_3')$ with the selection criterion (lower middle panel of Fig. 6) do demonstrate a several-percent reduction of the off-diagonal elements in contrast to those without the selection criterion (lower left panel). In other words, the 0.1% $\rho_{\text{col}}/\bar{\rho}_{\text{col}} \geq 3$ LOS’s contribute at least 10 times as much to the covariance per LOS as $\rho_{\text{col}}/\bar{\rho}_{\text{col}} < 3$ LOS’s do. It is also useful to have a comparison of the normalized covariance $C(k_3, k_3')$ because, unlike the reduced covariance $\hat{C}(k_3, k_3')$ that only tells the relative strength between the diagonal and off-diagonal elements, $C(k_3, k_3')$ reflects absolute values of the covariance up to a factor of the product of PS’s at $k_3$ and $k_3'$. Fig. 7
Figure 4. Reduced covariances $\hat{C}(k_3, k'_3)$ [see equation (34)] averaged over 2000 groups, each of which consists of 64 LOS’s ($N = 64$). For each panel in the left, the LOS’s are randomly drawn from a single GRF that has a box size of 128 $h^{-1}$Mpc (R128), or 256 $h^{-1}$Mpc (R256), or 512 $h^{-1}$Mpc (R512). The GRFs have the same three-dimensional mass PS as their corresponding simulations at $z = 0$, but note that $\hat{C}(k_3, k'_3)$ is independent of redshift for GRFs. Similarly, the middle column is for simulations at $z = 3$. The covariances are shown in a linear grey scale with black being 1.2 and white less than or equal to 0. At $z = 0$, the normalized covariances become much less diagonally dominant than the B128 $z = 3$ panel in the same grey scale, so only 4 cross sections along $Q = (k_3 + k'_3)/(\pi/4 \text{ h Mpc}^{-1})$ are plotted for B128 and B256. Diagonal elements $C(k_3, k_3)$ [see equation (33)] of R128 (squares), B128 at $z = 0$ (circles), B128 at $z = 3$ (diamonds), R512 (crosses), and B512 at $z = 3$ (pentagons) are shown with a multiplication factor $N$ in the lower right panel. For GRFs, $NC(k_3, k_3) = 1$.  

shows the diagonal elements of the normalized covariance for B128. It is seen that the variance $C(k_3, k_3)$ of the one-dimensional PS is indeed reduced by roughly a factor of 2 on all scales when the $0.1\% \rho_{\text{col}}/\bar{\rho}_{\text{col}} \geq 3$ LOS’s are excluded.

For a fixed number of particles, the simulation box sets a cut-off scale, below which fluctuations cannot be properly represented in low-density regions such as the Lyα forest. In other words, the number of particles and the size of the simulation determine the highest-wavenumber modes that are included in calculations of the one-dimensional PS and its covariance. Since the non-Gaussianity is stronger at smaller scales, a larger simulation box cuts off more small-scale fluctuations and may cause the correlation to appear weaker in Figs. 4 and 5. To test this, we assign the density field of the simulation B128 at $z = 3$ on a grid of $128^3$ nodes. The spatial resolution is the same as the simulation B512 on a grid of $512^3$ nodes. The covariance $\hat{C}(k_3, k'_3)$ is calculated in the same way as those in Fig. 4 but with fewer groups of LOS’s. Each group still has 64 LOS’s. The results are shown in the right column of Fig. 4 and in Fig. 5. Evidently, there is a significant reduction of the correlations between different modes as well as the variance of the one-dimensional PS. Indeed, the variance from the low-resolution calculation is close to that from the B512 simulation. Thus, the apparent resemblance between B512 and the GRF R512 in Figs. 4 and 5 is mostly due to the low resolution of the large-box simulation.
It is expected from equation (30) that the covariance matrix will not be diagonal if the length of LOS’s is less than the size of the simulation box. Fig. 8 shows the covariances that are calculated in the same way as those in Fig. 4 for the GRF R512 and the simulation B512, except that each LOS is only $128 \, h^{-1} \text{Mpc}$ long. The effect of the length is not visible for the GRF, but it does increase the correlation between different modes and doubles the variance of the one-dimensional PS for the simulation B512. For real observations, the LOS length is always much less than the size of the observable universe, so that the window function $w(k_3, \tilde{k}_3)$ in the LOS direction will cause stronger mixing of modes and more pronounced increase of correlation and variance.

7 CONCLUSIONS

We have investigated the covariance of the one-dimensional mass PS for both GRFs and simulated density fields. The non-linear evolution and the non-Gaussianity of the cosmic density field on small scales (see also Zhan, Jamkhedkar & Fang 2001; Smith et al. 2003; Zaldarriaga, Scoccimarro & Hu 2003; Zhan 2003) have caused the correlation between the fluctuations on different scales and increased the cosmic variance of the one-dimensional PS. Because of this, large number of LOS’s are needed to accurately measure the one-dimensional PS and recover the three-dimensional PS. The length of LOS’s introduces a window function in the line-of-sight direction, which mixes neighboring Fourier modes in the cosmic density field. Fig. 8 has demonstrated for simulations that the length of LOS’s (a quarter of the simulation box size) does affect the covariance...
Figure 6. The effect of the non-Gaussianity and resolution on the covariance. The left column shows the distribution function of column density \( \rho_{\text{col}} / \bar{\rho}_{\text{col}} \) (upper panel) and cross sections of the reduced covariance \( \hat{C}(k_3, k'_3) \) along \( Q = (k_3 + k'_3) / (\pi / 4 \, \text{h Mpc}^{-1}) \) (lower panel) for B128 at \( z = 3 \). The middle column shows \( \hat{C}(k_3, k'_3) \) (upper panel) and its cross sections (lower panel) for the same simulation output but with an exclusion of 318 LOS's that have \( \rho_{\text{col}} / \bar{\rho}_{\text{col}} > 3 \). The reduced covariances in the first two columns are calculated in the same way as in Fig. 4, i.e. with 2000 groups and \( N = 64 \). Thus, about half of the 318 LOS's would be selected without the criterion \( \rho_{\text{col}} / \bar{\rho}_{\text{col}} < 3 \). The right column is similar to the middle column, but the density is assigned on a grid of \( 128^3 \) nodes. All the \( 128^2 \) LOS's are selected and divided into 256 groups with \( N = 64 \). The grey scale is the same as Fig. 4.

Figure 7. The effect of the non-Gaussianity and resolution on the variance, i.e. diagonal elements of the covariance \( \sigma_{1D}^2(k_3, k'_3) \). Diamonds are the normalized variance \( \hat{C}(k_3, k_3) \) (multiplied by \( N \)) for B128 at \( z = 3 \). Circles correspond to the middle column of Fig. 6 which imposes the selection criterion \( \rho_{\text{col}} / \bar{\rho}_{\text{col}} < 3 \). Squares are from the low-resolution calculation in the right column of Fig. 6.
One may reduce the cosmic variance by binning independent Fourier modes. However, since the modes of the cosmic density field are strongly correlated, binning will be less effective in reducing the sample variance error. On the other hand, the non-Gaussian behaviour of the covariance provides important information of the field such as the trispectrum.

Our analysis shows that even with 64 LOS’s at \( z = 0 \), the cosmic variance on the PS can be several times of the PS itself (see Fig. 4). Thus, the extra power on the scale of \( 128 h^{-1} \text{Mpc} \) (\( \Omega = 1, \Omega_\Lambda = 0 \)) measured by Broadhurst et al. (1990) from 4 samples of pencil-beam galaxy surveys can still be consistent with standard cosmogonies (with considerations of effects such as redshift distortion and non-Gaussianity, see e.g. Kaiser & Peacock 1991; Park & God 1991; van de Weygaert 1991; Amendola 1994). One may also infer the one-dimensional density field and measure the PS from the Ly\( \alpha \) forest. Since the Ly\( \alpha \) flux is the direct observable, many works have been based on flux statistics, which are then used along with simulations to constrain cosmology. Using \( N \)-body simulations and the pseudo-hydro technique, Zhan (2003) find that the mapping between the nearly-Gaussian Ly\( \alpha \) flux PS (Zhan 2004) and the underlying one-dimensional mass PS (in redshift space) has a much larger scatter than that between the flux PS and the one-dimensional linear mass PS. This is expected because the one-dimensional PS of the cosmic density field has a variance much higher than the Gaussian variance. Therefore, one can measure the Ly\( \alpha \) flux PS precisely with a small sample of LOS’s, but to recover the mass PS to the same precision more LOS’s are needed.

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