SOME CHARACTERIZATIONS OF SPHERES AND ELLIPTIC PARABOLOIDS II

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Abstract. We show some characterizations of hyperspheres in the \((n+1)\)-dimensional Euclidean space \(\mathbb{E}^{n+1}\) with intrinsic and extrinsic properties such as the \(n\)-dimensional area of the sections cut off by hyperplanes, the \((n+1)\)-dimensional volume of regions between parallel hyperplanes, and the \(n\)-dimensional surface area of regions between parallel hyperplanes. We also establish two characterizations of elliptic paraboloids in the \((n+1)\)-dimensional Euclidean space \(\mathbb{E}^{n+1}\) with the \(n\)-dimensional area of the sections cut off by hyperplanes and the \((n+1)\)-dimensional volume of regions between parallel hyperplanes. For further study, we suggest a few open problems.

1. Introduction

Let \(S^n(a)\) be a hypersphere with radius \(a\) in the Euclidean space \(\mathbb{E}^{n+1}\). For a fixed point \(p \in S^n(a)\) and for a sufficiently small \(t > 0\), let’s denote by \(\Phi\) a hyperplane parallel to the tangent space \(\Psi\) of \(S^n(a)\) at \(p\) with distance \(t\) which intersects \(S^n(a)\).

We denote by \(A_p(t), V_p(t)\) and \(S_p(t)\) the \(n\)-dimensional area of the section in \(\Phi\) enclosed by \(\Phi \cap S^n(a)\), the \((n+1)\)-dimensional volume of the region bounded by the sphere and the plane \(\Phi\) and the \(n\)-dimensional surface area of the region of \(S^n(a)\) between the two planes \(\Phi\) and \(\Psi\), respectively.

Then, for a sufficiently small \(t > 0\), we can have the following properties of the sphere \(S^n(a)\).

(A): The \(n\)-dimensional area \(A_p(t)\) of the section is independent of the point \(p\).

(V): The \((n+1)\)-dimensional volume \(V_p(t)\) of the region is independent of the point \(p\).

(S): The \(n\)-dimensional surface area \(S_p(t)\) of the region is independent of the point \(p\).

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If \( n = 2 \), Archimedes proved that \( S_p(t) = 2\pi at \) holds for \( S^2(a) \) ([11], p.78). For a differential geometric proof, see Archimedes’ Theorem ([8], pp.116-118).

Conversely, it is natural to ask the following question:

**Question 1.** “Are there any other hypersurfaces in Euclidean space which satisfy the above properties?”

For the case of \( n = 2 \) about the property \((S)\), the authors answered negatively as follows ([5]) (See also [2] and [10]):

**Proposition 2.** Let \( M \) be a closed and convex surface in the 3-dimensional Euclidean space \( \mathbb{E}^3 \). Suppose that \( M \) satisfies the condition:

\( (S) \quad S_p(t) = \phi(t), \) which depends only on \( t \).

Then \( M \) is a round sphere.

In this article, first, we study convex hypersurfaces \( M \) in the \((n + 1)\)-dimensional Euclidean space \( \mathbb{E}^{n+1} \) which satisfy the above mentioned properties. For a point \( p \in M \subset \mathbb{E}^{n+1} \), \( A_p(t), V_p(t) \) and \( S_p(t) \) are defined as above.

In Section 3, as a result, we prove the following:

**Theorem 3.** Let \( M \) be a complete and convex hypersurface in the \((n + 1)\)-dimensional Euclidean space \( \mathbb{E}^{n+1} \). Suppose that \( M \) satisfies one of the following conditions.

\( (A) \): The \( n \)-dimensional area \( A_p(t) \) of the section is independent of the point \( p \in M \).
\( (V) \): The \((n + 1)\)-dimensional volume \( V_p(t) \) of the region is independent of the point \( p \in M \).
\( (S) \): The \( n \)-dimensional surface area \( S_p(t) \) of the region is independent of the point \( p \in M \).

Then the hypersurface \( M \) is a round hypersphere \( S^n(a) \).

Second, suppose that \( M \) is a smooth convex hypersurface in the \((n + 1)\)-dimensional Euclidean space \( \mathbb{E}^{n+1} \) defined by the graph of a convex function \( f : \mathbb{R}^n \to \mathbb{R} \). For a fixed point \( p = (x, f(x)) \in M \) and for a real number \( k > 0 \), consider a hyperplane \( \Phi \) through \( v = (x, f(x) + k) \) which is parallel to the tangent hyperplane \( \Psi \) of \( M \) at \( p \).

We denote by \( A^*_p(k), V^*_p(k) \) and \( S^*_p(k) \) the \( n \)-dimensional area of the section in \( \Phi \) enclosed by \( \Phi \cap M \), the \((n + 1)\)-dimensional volume of the region bounded by \( M \) and the hyperplane \( \Phi \), and the \( n \)-dimensional surface area of the region of \( M \) between the two hyperplanes \( \Phi \) and \( \Psi \), respectively.

For elliptic paraboloids, in [5] the authors proved the following.

**Proposition 4.** Let \( M \) be a smooth convex hypersurface in the \((n + 1)\)-dimensional Euclidean space \( \mathbb{E}^{n+1} \) defined by the graph of a convex function \( f : \mathbb{R}^n \to \mathbb{R} \). Then \( M \) is an elliptic paraboloid if and only if it satisfies the following condition:
(L): The 
(n + 1)-dimensional volume \( V_p^*(k) \) is \( ak^{(n+2)/2} \) for some constant \( a \) which depends only on the hypersurface \( M \).

In Section 3, we generalize Proposition 4 as follows.

**Theorem 5.** Let \( M \) be a smooth convex hypersurface in the \((n + 1)\)-dimensional Euclidean space \( \mathbb{E}^{n+1} \) defined by the graph of a convex function \( f: \mathbb{R}^n \to \mathbb{R} \). Then \( M \) is an elliptic paraboloid if and only if it satisfies one of the following conditions:

\( (V^*): \ V_p^*(k) \) is a nonnegative function \( \phi(k) \), which depends only on \( k \).

\( (A^*): \ A_p^*(k)/W(p) \) is a nonnegative function \( \psi(k) \), which depends only on \( k \).

Here, we denote \( p = (x, f(x)) \) and \( W(p) = \sqrt{1 + |\nabla f(x)|^2} \), where \( \nabla f \) is the gradient of \( f \).

In view of the conditions in Theorem 5, it is reasonable to ask the following question.

**Question 6.** Which hypersurfaces satisfy the following condition \((S^*)\)?

\( (S^*): \ S_p^*(k)/W(p) \) is a nonnegative function \( \eta(k) \), which depends only on \( k \).

Finally, using harmonic function theory \([\text{II}]\), we answer Question 6 negatively as follows.

**Theorem 7.** Let \( M \) be a smooth convex hypersurface in the \((n + 1)\)-dimensional Euclidean space \( \mathbb{E}^{n+1} \) defined by the graph of a convex function \( f: \mathbb{R}^n \to \mathbb{R} \). Then \( M \) does not satisfy condition \((S^*)\).

In this paper, in order to prove our theorems, we prove a lemma (Lemma 8), extending a lemma in \([\text{3}]\), about a new meaning of Gauss-Kronecker curvature \( K(p) \) of convex hypersurface \( M \) at a point \( p \in M \) in three ways.

We now state some questions for further study as follows.

**Question A.** Let \( M \) be a convex (not complete) hypersurface in the \((n+1)\)-dimensional Euclidean space \( \mathbb{E}^{n+1} \). Suppose that \( M \) satisfies one of the conditions in Theorem 3. Then, is it an open part of a round hypersphere \( S^n(a) \)?

An elliptic paraboloid satisfies the following conditions. For a proof, see the proof of Theorem 5, which is given in Section 3.

\( (V^{**}): \ V_p(t) = C(p)t^{(n+2)/2} \), where \( C(p) \) is a function of \( p \in M \).

\( (A^{**}): \ A_p(t) = D(p)t^{n/2} \), where \( D(p) \) is a function of \( p \in M \).

Due to \((2.5)\), the above two conditions are equivalent.

**Question B.** Let \( M \) be a complete and convex hypersurface in the \((n + 1)\)-dimensional Euclidean space \( \mathbb{E}^{n+1} \), which is not necessarily a graph of a function. Suppose that \( M \) satisfies the condition \((V^{**})\). Then, is it an elliptic paraboloid?
For \( n = 1 \), Question A is true because the plane curvature is a nonzero constant. In [6], the authors answered Question B for \( n = 1 \), affirmatively.

Throughout this article, all objects are smooth and connected, otherwise mentioned.

2. Preliminaries

Suppose that \( M \) is a smooth convex hypersurface in the \((n+1)\)-dimensional Euclidean space \( \mathbb{E}^{n+1} \). For a fixed point \( p \in M \) and for a sufficiently small \( t > 0 \), consider a hyperplane \( \Phi \) parallel to the tangent hyperplane \( \Psi \) of \( M \) at \( p \) with distance \( t \) which intersects \( M \).

We denote by \( A_p(t), V_p(t) \) and \( S_p(t) \) the \( n \)-dimensional area of the section in \( \Phi \) enclosed by \( \Phi \cap M \), the \((n + 1)\)-dimensional volume of the region bounded by the hypersurface and the hyperplane \( \Phi \) and the \( n \)-dimensional surface area of the region of \( M \) between the two hyperplanes \( \Phi \) and \( \Psi \), respectively.

Now, we may introduce a coordinate system \((x, z) = (x_1, x_2, \cdots, x_n, z)\) of \( \mathbb{E}^{n+1} \) with the origin \( p \), the tangent space of \( M \) at \( p \) is the hyperplane \( z = 0 \). Furthermore, we may assume that \( M \) is locally the graph of a non-negative convex function \( f : \mathbb{R}^n \rightarrow \mathbb{R} \).

Then, for a sufficiently small \( t > 0 \) we have

\[
A_p(t) = \iint_{f(x)<t} 1 \, dx, \quad (2.1)
\]

\[
V_p(t) = \iint_{f(x)<t} \{ t - f(x) \} \, dx \quad (2.2)
\]

and

\[
S_p(t) = \iint_{f(x)<t} \sqrt{1 + |\nabla f|^2} \, dx, \quad (2.3)
\]

where \( x = (x_1, x_2, \cdots, x_n), \, dx = dx_1 dx_2 \cdots dx_n \) and \( \nabla f \) denotes the gradient vector of the function \( f \).

Note that we also have

\[
V_p(t) = \iint_{f(x)<t} \{ t - f(x) \} \, dx \quad (2.4)
\]

\[
= \int_{z=0}^{t} \{ \iint_{f(x)<z} 1 \, dx \} \, dz.
\]

Hence, together with the fundamental theorem of calculus, (2.4) shows that

\[
V_p'(t) = \iint_{f(x)<t} 1 \, dx = A_p(t). \quad (2.5)
\]

First of all, we prove
Lemma 8. Suppose that the Gauss-Kronecker curvature $K(p)$ of $M$ at $p$ is positive with respect to the upward unit normal to $M$. Then we have the following:

1)  
\[ \lim_{t \to 0} \frac{1}{(\sqrt{t})^n} A_p(t) = \frac{(\sqrt{2})^n \omega_n}{\sqrt{K(p)}}, \]  
(2.6)

2)  
\[ \lim_{t \to 0} \frac{1}{(\sqrt{t})^{n+2}} V_p(t) = \frac{(\sqrt{2})^{n+2} \omega_n}{(n+2)\sqrt{K(p)}}, \]  
(2.7)

3)  
\[ \lim_{t \to 0} \frac{1}{(\sqrt{t})^n} S_p(t) = \frac{(\sqrt{2})^n \omega_n}{\sqrt{K(p)}}, \]  
(2.8)

where $\omega_n$ denotes the volume of the $n$-dimensional unit ball.

**Proof.** We denote by $x$ the column vector $(x_1, x_2, \ldots, x_n)^t$. Then for a symmetric $n \times n$ matrix $A$, we have from the Taylor’s formula of $f(x)$ as follows:

\[ f(x) = x^t Ax + f_3(x), \]  
(2.9)

where $f_3(x)$ is an $O(|x|^3)$ function. Then the Hessian matrix of $f$ at the origin is given by $D^2 f(0) = 2A$. Hence, for the upward unit normal to $M$ we have

\[ K(p) = \det D^2 f(0) = 2^n \det A. \]  
(2.10)

By the assumption, we see that every eigenvalue of $A$ is positive. Thus, there exists a nonsingular symmetric matrix $B$ satisfying

\[ A = B^t B, \]  
(2.11)

where $B^t$ denotes the transpose of $B$. Therefore, we obtain

\[ f(x) = |Bx|^2 + f_3(x). \]  
(2.12)

We consider the decomposition of $S_p(t)$ as follows:

\[ S_p(t) = A_p(t) + N_p(t), \]  
(2.13)

where

\[ A_p(t) = \int \int_{f(x) < t} 1 \, dx \]  
(2.1)

and

\[ N_p(t) = \int \int_{f(x) < t} (\sqrt{1 + |\nabla f|^2} - 1) \, dx. \]  
(2.14)

First, we show (2.6) as follows. We let $t = \epsilon^2$ and $x = \epsilon y$. Then (2.1) gives

\[ \frac{1}{(\sqrt{t})^n} A_p(t) = \frac{1}{(\sqrt{\epsilon})^n} \int \int_{f(x) < \epsilon^2} 1 \, dx = \int \int_{|By|^2 + \epsilon g_3(y) < 1} 1 \, dy, \]  
(2.15)
where \( g_3(y) \) is an \( O(|y|^3) \) function. As \( \epsilon \to 0 \), it follows from (2.15) that
\[
\lim_{t \to 0} \frac{1}{(\sqrt{t})^n} A_p(t) = \int\int_{|By|^2 < 1} 1 dy.
\] (2.16)

If we let \( w = By \), then from (2.16) we get
\[
\lim_{t \to 0} \frac{1}{(\sqrt{t})^n} A_p(t) = \frac{1}{|\det B|} \int\int_{|w| < 1} 1 dw = \frac{\omega_n}{|\det B|}.
\] (2.17)

Hence, it follows from (2.10) and (2.11) that (2.6) holds.

Together with (2.5) and (2.6), L'Hospital's rule implies (2.7).

In order to prove (2.8), it suffices to show that
\[
\lim_{t \to 0} \frac{1}{(\sqrt{t})^n} N_p(t) = 0.
\] (2.18)

Note that the following inequality holds
\[
N_p(t) \leq \frac{1}{2} \int\int_{f(x) < t} |\nabla f|^2 dx.
\] (2.19)

The function \( f \) satisfies
\[
|\nabla f(x)|^2 = 4|Ax|^2 + h_2(x),
\] (2.20)
where \( h_2(x) \) is an \( O(|x|^2) \) function. Thus, there exists a positive constant \( C \) satisfying in a neighborhood of the origin
\[
|\nabla f(x)|^2 \leq C|x|^2.
\] (2.21)

In the same argument as above, putting \( t = \epsilon^2 \) and \( x = \epsilon y \), it follows from (2.19) and (2.21) that
\[
\frac{1}{(\sqrt{t})^n} N_p(t) \leq \frac{C\epsilon^2}{2} \int\int_{|By|^2 + \epsilon g_3(y) < 1} |y|^2 dy.
\] (2.22)

Since the integral of the right side in (2.22) tends to a constant as \( \epsilon \to 0 \), by letting \( t \to 0 \) in (2.22), we get (2.18). Together with (2.6) and (2.13), (2.18) shows that (2.7) holds. This completes the proof. \( \square \)

3. PROOFS OF THEOREMS

In this section, first, we prove Theorem 3.

Let \( M \) be a complete and convex hypersurface in the \((n + 1)\)-dimensional Euclidean space \( \mathbb{E}^{n+1} \). Then the Gauss-Kronecker curvature \( K(p) \) of \( M \) with respect to the inward unit normal to \( M \) is nonnegative and positive somewhere.

Suppose that \( M \) satisfies one of the conditions in Theorem 3. Then Lemma 8 shows that the Gauss-Kronecker curvature \( K(p) \) is constant on the nonempty open set \( \Omega = \{ p \in M \mid K(p) > 0 \} \). Hence the continuity of \( K \) implies that \( \Omega = M \), that is, \( K(p) \) is
constant on \( M \). Thus, it follows from Theorem 7.1 in [9] or the main theorem in [3] that \( M \) is a round hypersphere.

The converse is obvious.

Second, we give a proof of Theorem 5.

Suppose that \( M \) is a smooth convex hypersurface in the \((n+1)\)-dimensional Euclidean space \( E^{n+1} \) defined by the graph of a convex function \( f : \mathbb{R}^n \to \mathbb{R} \). We fix a point \( p = (x, f(x)) \) in \( M \). For a positive constant \( k \), consider a hyperplane \( \Phi \) through \( v = (x, f(x) + k) \), which is parallel to the tangent hyperplane \( \Psi \) to \( M \) at \( p \). Let \( W(x) = \sqrt{1 + |\nabla f(x)|^2} \). Then for a constant \( t \) with \( k = tW \), we have \( V^*_p(k) = V_p(t) \), \( A^*_p(k) = A_p(t) \) and \( S^*_p(k) = S_p(t) \).

1) The condition \((V^*)\) shows that \( V_p(t) = \phi(k) \), which is independent of \( p \in M \). Hence we have

\[
\frac{V_p(t)}{(\sqrt{t})^{n+2}} = \frac{\phi(k)}{(\sqrt{k})^{n+2}} (\sqrt{W})^{n+2}. \tag{3.1}
\]

Thus, Lemma 8 implies that

\[
\lim_{k \to 0} \frac{\phi(k)}{(\sqrt{k})^{n+2}} = W^{-(n+2)/2} \lim_{t \to 0} \frac{V_p(t)}{(\sqrt{t})^{n+2}} = \frac{(\sqrt{2})^{n+2} \omega_n}{(n + 2) \sqrt{K(p)}} W^{-(n+2)/2}. \tag{3.2}
\]

If we denote by \( \alpha \) the limit of the left side of (3.2), which is independent of \( p \), then we have

\[
K(p) = \frac{2^{n+2} \omega_n^2}{\alpha^2(n + 2)^2 W(x)^{n+2}}. \tag{3.3}
\]

Since the Gauss-Kronecker curvature \( K(p) \) of \( M \) at \( p \) is given by ([12], p.93)

\[
K(p) = \frac{\det D^2 f(x)}{W^{n+2}}, \tag{3.4}
\]

it follows from (3.3) that the determinant \( \det D^2 f(x) \) of the Hessian of \( f(x) \) is a positive constant. The continuity of \( \det D^2 f(x) \) shows that it is a positive constant on the whole space \( \mathbb{R}^n \). Thus \( f(x) \) is a globally defined quadratic polynomial ([4, 7]), and hence \( M \) is an elliptic paraboloid.

2) The condition \((A^*)\) shows that \( A^*_p(k)/W(p) = \psi(k) \), which is independent of \( p \in M \). Hence we have

\[
\frac{A_p(t)}{(\sqrt{t})^n} = \frac{\psi(k)}{(\sqrt{k})^n} \frac{(\sqrt{W})^{n+2}}{(\sqrt{K(p)})}. \tag{3.5}
\]

Thus, Lemma 8 implies that

\[
\lim_{k \to 0} \frac{\psi(k)}{(\sqrt{k})^n} = W^{-(n+2)/2} \lim_{t \to 0} \frac{A_p(t)}{(\sqrt{t})^n} = \frac{(\sqrt{2})^{n+2} \omega_n}{\sqrt{K(p)}} W^{-(n+2)/2}. \tag{3.6}
\]
If we denote by $\beta$ the limit of the left side of (3.6), which is independent of $p$, then we have

$$K(p) = \frac{2^n \omega^n_2}{\beta^2 W(x)^{n+2}}.$$  \tag{3.7}$$

Hence, as in the proof of Case 1), we see that $M$ is an elliptic paraboloid.

This completes the proof of the if part of Theorem 5.

Conversely, consider an elliptic paraboloid $M : z = f(x) = \Sigma_{i=1}^n a_i^2 x_i^2, a_i > 0$, a tangent hyperplane $\Psi$ to $M$ at a fixed point $p = (x, z) \in M$, a hyperplane $\Phi$ through $v = (x, z + k), k > 0$ which is parallel to the tangent hyperplane $\Psi$ of $M$ at $p$. Then the proof of Theorem 5 of [5] shows that

$$V_p^*(k) = \alpha_n k^{(n+2)/2}, \quad \alpha_n = \frac{2\sigma_{n-1}}{n(n+2)a_1a_2\cdots a_n}, \tag{3.8}$$

where $\sigma_{n-1}$ denotes the surface area of the $(n-1)$-dimensional unit sphere. Since $V_p^*(k) = V_p(t)$ with $k = tW$, we get

$$V_p(t) = \alpha_n W(p)^{(n+2)/2} t^{(n+2)/2}. \tag{3.9}$$

Hence, it follows from (2.5) that

$$A_p(t) = \frac{n + 2}{2} \alpha_n W(p)^{(n+2)/2} t^{n/2} \tag{3.10}$$

and

$$A_p^*(k) = \frac{n + 2}{2} \alpha_n W(p)k^{n/2}. \tag{3.11}$$

Thus, (3.8) and (3.11) show that an elliptic paraboloid $M$ satisfies conditions ($V^*$) and ($A^*$) in Theorem 5, respectively.

This completes the proof of the only if part of Theorem 5.

Finally, we prove Theorem 7.

Suppose that a smooth convex hypersurface $M$ in the $(n + 1)$-dimensional Euclidean space $\mathbb{E}^{n+1}$ defined by the graph of a convex function $f : \mathbb{R}^n \to \mathbb{R}$ satisfies the condition ($S^*$). Then, as in the proof of Theorem 5, we can prove that $M$ is an elliptic paraboloid given by $z = f(x) = \Sigma_{i=1}^n a_i^2 x_i^2, a_i > 0$.

Consider a hyperplane $\Phi$ intersecting $M$, a point $p = (p_1, \cdots, p_n, \Sigma_{i=1}^n a_i^2 p_i^2) \in M$ where the tangent hyperplane $\Psi$ of $M$ is parallel to $\Phi$, and a point $v$ where the line through $p$ parallel to the $z$-axis meets $\Phi$ with $||p - v|| = k$. Then we have the following.

$$\Phi : z = 2a_1^2 p_1 x_1 + \cdots + 2a_n^2 p_n x_n - (a_1^2 p_1^2 + \cdots + a_n^2 p_n^2) + k,$$

$$S_p^*(k) = \int_{D_p(k)} W(x)dx, \tag{3.12}$$

$$D_p(k) : \Sigma_{i=1}^n a_i^2 (x_i - p_i)^2 < k,$$
where \( W(x) = \sqrt{1 + \sum_{i=1}^{n} 4a_i^2 x_i^2} \).

By the linear transformation \( y_i = a_ix_i, i = 1, 2, \ldots, n \), we obtain

\[
S^*_p(k) = \frac{1}{a_1a_2\cdots a_n} \int_{B_q(\sqrt{k})} V(y) dy,
\]

(3.13)

where \( V(y) = \sqrt{1 + 4\sum_{i=1}^{n} a_i^2 y_i^2} \) and \( q = (a_1p_1, \ldots, a_np_n) \).

Since \( M \) satisfies the condition \((S^*)\) with \( W(p) = V(q) \), by letting \( r = \sqrt{k} \), it follows from (3.13) that \( V(y) \) satisfies the following.

\[
\int_{B_q(r)} V(y) dy = V(q)g(r), \quad q \in \mathbb{R}^n, r \geq 0,
\]

(3.14)

where \( B_q(r) = \{ y||y-q|<r \} \) is the ball of radius \( r \) centered at \( q \) and \( g(r) \) is a function of \( r \).

For a function \( g = g(r), r \geq 0 \), we denote by \( C_g \) the set of all functions \( f : \mathbb{R}^n \to \mathbb{R} \) satisfying (3.14). Then, it is straightforward to show the following ([1]).

**Lemma 9.** The set \( C_g \) satisfies the following.

1) If \( g(r) \) is the volume \( \omega_n r^n \) of \( B_q(r) \) for sufficiently small \( r > 0 \), then \( C_g \) is the set of all harmonic functions on \( \mathbb{R}^n \).

2) If a positive function in \( C_g \) has a local maximum (local minimum, respectively), then \( g(r) \leq \omega_n r^n \) (\( g(r) \geq \omega_n r^n \), respectively) for sufficiently small \( r > 0 \).

3) If \( f \in C_g \), then every partial derivative of \( f \) also belongs to \( C_g \).

4) Every linear combination of functions in \( C_g \) also belongs to \( C_g \).

In order to complete the proof of Theorem 7, we use Lemma 9 as follows. By differentiating, we have

\[
V_{ii} = 4a_i^2 \frac{(V^2 - 4a_i^2 x_i^2)}{V^3}, \quad i = 1, 2, \ldots, n,
\]

(3.15)

where \( V_i \) means the \( i \)-th partial derivative of \( V \), etc.. Hence, we get

\[
U = \frac{1}{4} \sum_{i=1}^{n} \left( V_{ii} \frac{1}{a_i^2} \right) = \frac{(n-1)V^2 + 1}{V^3},
\]

(3.16)

which is again an element of \( C_g \).

Note that \( V \) has a strict minimum \( V(0) = 1 \) and \( U \) has a strict maximum \( U(0) = n \). Thus, we see that \( g(r) = \omega_n r^n \), and hence that \( V \) is harmonic. This is a contradiction by the maximum principle of harmonic functions, which completes the proof of Theorem 7.

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