Nonlinear Realizations of Superconformal Groups and Spinning Particles

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The method of nonlinear realizations is applied for the conformally invariant description of the spinning particles in terms of geometrical quantities of the parameter spaces of the one dimensional $N$-extended superconformal groups. We develop the superspace approach to the cases of spin 0, $\frac{1}{2}$, 1 particles and describe the alternative component approach in the application to the spin-$\frac{1}{2}$ particle.

1. INTRODUCTION

The conformally invariant description of the relativistic (spinning) particle treats on the same footings the coordinates of the particle and (super)einbein, needed for the local reparametrization invariance of the action [1,2]. The space-time coordinates in this approach are not fundamental ones. They are ratios of some more basic variables, each transforming as the one-dimensional vielbein. In total, for the description of the particle in $D$ dimensions, one need $D+2$ such variables (einbeins) entering the starting action identically.

As was shown in [3–5], these einbeins (and their superpartners in the case of $N = 1$ spinning particle) can be naturally described as dilatons (with superpartners), parametrizing $D + 2$ different elements of the one-dimensional superconformal group. In this paper we generalize the consideration of [3,4] to the case of $N = 2$ spinning particle making use of the superspace approach. We also point out the problems arising when attempts to generalize the method to higher spins and describe the alternative component approach in the application to the spin-$\frac{1}{2}$ particle. The algebra of $N$-extended superconformal group is described in the Appendix.

2. RELATIVISTIC MASSLESS PARTICLE AND $N = 1$ SPINNING PARTICLE

2.1. The nonlinear realizations method

In this method the representation space of the group coincides with the whole space of the group parameters or with its subspace, which describes the coset space of the group over some of its subgroup. Let us consider the general group element and its transformation under the left multiplication by some fixed element $g$.

$$G(a_i) \rightarrow G' = g \cdot G(a_i) = G(a'_i).$$ (1)

So, the parameters $a_i$ of the group space realize the representation of the group with the transformation law: $a'_i = a'_i(a_k, g)$. If the group admits the parametrization in the form

$$G = K \cdot H = K(k_m) \cdot H(h_s),$$ (2)

then

$$a'_i = a'_i(k_m, h_s).$$ (3)
where $H$ is some subgroup of the group $G$ and $K$ parametrizes the corresponding coset $K = G/H$, the group action can be realized not only on the whole space of parameters $\{k_m, h_s\}$, but on the coset space $\{k_m\}$ as well. The transformation law in this case is

$$k'_m = k'_m(k_m, g), \quad h'_s = h'_s(k_m, h_t, g). \quad (3)$$

One can generalize the approach and consistently consider simultaneously more than one group elements carrying the external index $\mathcal{I}$

$$G_{\mathcal{I}} = K \cdot H_{\mathcal{I}} \quad (4)$$

with the same coset element $K$ and different elements $H_{\mathcal{I}}$ of the subgroup $H$. This property (the equality of the coset elements for all $G_{\mathcal{I}}$) is invariant with respect to the left multiplication

$$G_{\mathcal{I}} \to G'_{\mathcal{I}} = g \cdot G_{\mathcal{I}} \quad (5)$$

with any group element $g$.

The differential invariant Cartan’s $\Omega$ - forms can be constructed for each of these group elements

$$\Omega_{\mathcal{I}} = G_{\mathcal{I}}^{-1} dG_{\mathcal{I}}. \quad (6)$$

Moreover, the following group elements (strictly speaking they belong to the subgroup $H$)

$$G_{\mathcal{I}, \mathcal{J}} = G_{\mathcal{I}}^{-1} G_{\mathcal{J}} = H_{\mathcal{I}}^{-1} H_{\mathcal{J}} \quad (7)$$

are also invariant with respect to the left multiplication $\Omega_{\mathcal{I}}$. This fact gives the additional possibilities for construction of invariants of the group. Let us underline that in contrast to the $\Omega$ - forms $\Omega_{\mathcal{I}}$, which belong to the algebra, new invariants $G_{\mathcal{I}, \mathcal{J}}$ belong to the group itself.

In what follows we will consider as starting groups the superconfined infinite dimensional groups of the (super)spaces with one bosonic and $N$ grassmann coordinates $Z^M = (\tau, \theta_a)$. The corresponding (super)Virasoro algebras contain among their generators the translation $P = L_{-1}$ and $N$ supercharges $G^{a}_{-1/2}$. In this case it is convenient to parametrize the group element in the form

$$G = e^{i \tau L_{-1}} \cdot e^{\theta_a G^{a}_{-1/2}} \cdot \tilde{G} \quad (8)$$

and after that consider all parameters in $\tilde{G}$ as functions in the superspace $(\tau, \theta_a)$. The coordinates $(\tau, \theta_a)$ transform indeed as they should transform. Such consideration automatically will lead to superfield constructions. However, as we will see later, in some cases, especially when the number of supersymmetries $N$ grows, such superfield approach leads to difficulties. Instead, one can consider already $\theta_a$ as the functions $\theta_a(\tau)$ of a single bosonic coordinate $\tau$. This situation corresponds to the phase of the spontaneously broken supersymmetry and functions $\theta_a(\tau)$ play the role of the corresponding Goldstone fields. Such component approach is alternative to the superfield one and it, possibly, will help to overcome the mentioned difficulties.

### 2.2. Virasoro algebra and massless particle

One dimensional diffeomorphisms algebra is the simplest example of the $N$ - extended superconformal algebras described in the Appendix. It coincide with the Virasoro (centerless) algebra

$$[L_m, L_n] = -i(m-n) L_{m+n}. \quad (9)$$

If one limit themselves to the positive part of this algebra, which generate in the one dimensional space the transformations which are regular at the origin, the most natural is the following parametrization of the group element

$$G = e^{i \tau L_{-1}} e^{U^{(1)} L_1} e^{U^{(2)} L_2} e^{U^{(3)} L_3} \ldots e^{U^{(0)} L_0}, \quad (10)$$

in which all multipliers with the exception of $e^{i \tau L_{-1}}, U \equiv U^{(0)}$, are ordered by the dimensionality $dim L_n = n$ of the correspondent generators.

Such structure of the group element simplifies the evaluation of the variations $\delta U^{(m)}$ under the infinitesimal left action

$$G' = (1 + i \epsilon) G, \quad (11)$$

where $\epsilon = \sum_{m=0}^{\infty} \epsilon^{(m)} L_{m-1}$ belongs to the algebra of the diffeomorphisms group. In particular $\delta \tau = \sum_{m=0}^{\infty} \epsilon^{(m)} \tau^m \equiv \epsilon(\tau)$. All other $U^{(m)}$ transforms through $\tau$ and $U^{(m)}, \ m < n$.

At this stage it is natural to consider all parameters $U^{(n)}$ as the fields $U^{(n)}(\tau)$ in one dimensional space parametrized by the coordinate $\tau$. The field $U^{(0)}(\tau)$ transforms as a one dimensional dilaton.
Simultaneously $\mathcal{U}^{(1)}(\tau)$ transforms as one-dimensional Cristoffel symbol.

Having in mind that the generators $L_0$ and $L_1$ form the subalgebra, one can consider more than one group elements ($\mathcal{I} = 0, 1, \ldots, D + 1$)

$$G_\mathcal{I} = e^{i\nu L_{-1}} e^{iU^{(2)} L_2} e^{iU^{(3)} L_3} \ldots e^{iU^{(1)} L_1} e^{iU L_0},$$

which have identical values of parameters $\tau$ and $U^{(m)}(\tau)$, $m \geq 2$, and differ in the values of the parameters $U^{(1)}_I$ and $U^{(0)}_I \equiv U_\mathcal{I}$. This property is valid when all of these group elements are transformed with the same infinitesimal transformation parameter $\epsilon$ in (11).

Consider the Cartan’s differential form for each value of the index $\mathcal{I}$

$$\Omega_\mathcal{I} = G_\mathcal{I}^{-1} dG_\mathcal{I} = i\Omega^{(-1)}_\mathcal{I} L_{-1} + i\Omega^{(0)}_\mathcal{I} L_0 + \ldots (13)$$

All its components $(\Omega^{(-1)}_\mathcal{I}, \Omega^{(0)}_\mathcal{I}, \Omega^{(1)}_\mathcal{I}, \ldots)$ are invariant with respect to the left transformation (5) (or (11)). The explicit expressions for some components of the $\Omega$-form are:

$$\Omega^{(-1)}_\mathcal{I} = e^{-U_2} d\tau, \quad (14)$$

$$\Omega^{(0)}_\mathcal{I} = dU_\mathcal{I} - 2d\tau U^{(1)}_\mathcal{I}, \quad (15)$$

$$\Omega^{(1)}_\mathcal{I} = (dU^{(1)}_\mathcal{I} + d\tau(U^{(2)}_\mathcal{I} - 3d\tau U^{(2)})) e^{U_3}. \quad (16)$$

The first of these forms is the differential one-form einbein. The covariant derivatives (carrying the external index $\mathcal{I}$) calculated with its help are

$$D_\tau e^{U_2} = e^{U_2} \frac{d}{d\tau}. \quad (17)$$

The most interesting is the form $\Omega^{(1)}_\mathcal{I}$. Using it one can write the following invariant expression for the action

$$S = \frac{1}{2} \Sigma_\mathcal{I} \int \Omega^{(1)}_\mathcal{I} = \frac{1}{2} \Sigma_\mathcal{I} \int d\tau e^{U_2} (U^{(1)}_\mathcal{I} + U^{(1)}_\mathcal{I})^2 - 3d\tau U^{(2)}),$$

where $\Sigma_\mathcal{I}$ is the signature of $D + 2$-dimensional space-time (with two times)

$$\Sigma_\mathcal{I} = (- + + \ldots + + -). \quad (19)$$

and summation over external index $\mathcal{I}$ is understood.

After the elimination of $U^{(1)}_\mathcal{I}$ with the help of its equation of motion the action (18) in terms of new variables

$$x_\mathcal{I} = e^{U_2(\tau)/2}, \quad \lambda = -3U^{(2)}(\tau)$$

has the form

$$S = \int d\tau(\frac{1}{2} \dot{x}^2 - \frac{1}{2} \lambda x^2). \quad (21)$$

The relation of the action (21) with the usual $D$-dimensional action is established by solving the equation of motion for the Lagrange multiplier $\lambda$

$$x^2 = 0. \quad (22)$$

Note the triviality of dynamics implied by this equation in the absence of the external index $\mathcal{I}$ as well as in the case of positive definite signature $\Sigma_\mathcal{I}$.

In terms of new variables

$$\dot{x}_i = \frac{x_i}{x_+}, \quad e = \frac{1}{x_+}, \quad (x_- = \frac{-x^i x_i}{2x_+}), \quad (23)$$

the Lagrangian in (21) becomes the standard one

$$L = \frac{1}{2} \dot{x}^2. \quad (24)$$

The expressions (23) for coordinates $\dot{x}_i$ show that they are indeed scalars with respect to the transformations of the one-dimensional diffeomorphisms group. At the same time $e(\tau)$ transforms correctly as an einbein. All this is a result of the transition from $D + 2$ dimensional to $D$ dimensional consideration which is implied by the constraint $x^2 = 0$.

2.3. $N = 1$ spinning particle in the superspace approach

To generalize the approach on the spinning particles we consider the $N = 1$ SCA which is the simplest of the algebras described in the Appendix. Its generators are placed on the first two lines of the Picture 1 and have the following commutation relations in addition to (10)

$$[L_m, G_s] = -i(\frac{m}{2} - s)G_{m+s} \quad (25)$$

$$\{G_r, G_s\} = 2L_{r+s}. \quad (26)$$
Following the considerations of the previous subsections we write $D + 2$ group elements as $(\mathcal{I} = 0, 1, \ldots, D + 1)$

$$G_\mathcal{I} = e^{i\tau L_{-1}} \cdot e^{i\theta G_{-1/2}} \cdot e^{i\Theta^{(1/2)}G_{1/2}} \cdot e^{iU^{(2)}L_2} \cdot e^{i\theta G_{1/2}} \cdot e^{iU^{(1)}L_1} \cdot e^{iU^{(0)}L_0}. \quad (27)$$

Last three multipliers in this expression form the subgroup of the whole $N = 1$ superconformal group and they consistently can carry external index $\mathcal{I}$. All parameters (Grassmann $\Theta$-s and commuting $U$-s) are considered as superfunctions in the $(1, 1)$ superspace parametrized by $\tau$ and $\theta$. The variations of these superspace coordinates under the left action of infinitesimal superconformal transformation are given by the general expressions (A.4).

To calculate the invariant differential $\Omega$ - forms one should take into account that Grassmann parity of differential of any variable is opposite to its own Grassmann parity, i.e. $d\tau$ is odd and $d\theta$ is even. The general expression for Cartan’s $\Omega$ - form is now

$$\Omega_\mathcal{I} = G_\mathcal{I}^{-1}dG_\mathcal{I} = i\Omega_\mathcal{I}^{(1)}L_{-1} + i\Omega_\mathcal{I}^{(-1)/2}G_{-1/2} + i\Omega_\mathcal{I}^{(0)}L_0 + i\Omega_\mathcal{I}^{(1/2)}G_{1/2} + i\Omega_\mathcal{I}^{(1)}L_1 + \ldots. \quad (28)$$

Its two first components

$$\Omega_\mathcal{I}^{\tau} \equiv \Omega_\mathcal{I}^{(1)} = (d\tau - id\theta)e^{-U_\mathcal{I}} = dx^M E^A_{MI}, \quad (29)$$

$$\Omega_\mathcal{I}^{\theta} \equiv \Omega_\mathcal{I}^{(-1)/2} = \{d\theta - (d\tau - id\theta)\Theta\}_e^{-U_\mathcal{I}/2} = dx^M E^A_{MI} \quad (30)$$

define supervielbeins $(x^1 \equiv \tau, x^2 \equiv \theta)$:

$$E^A_{MI} = \begin{vmatrix} e^{-U_\mathcal{I}} & -\Theta_e^{-U_\mathcal{I}/2} \\ -ie^{-U_\mathcal{I}} \cdot \theta & e^{-U_\mathcal{I}/2}(1 + i\Theta_e \cdot \theta) \end{vmatrix}$$

The invariant integration measure is

$$dV_\mathcal{I} = d\tau d\theta Ber(E^A_{MI}), \quad (31)$$

where $d\theta$ is the Berezin differential and

$$Ber(E^A_{MI}) = e^{-U_\mathcal{I}/2}. \quad (32)$$

Note, that the integration measure, as well as the supervielbeins, depend on the external index $\mathcal{I}$.

The action for $N = 1$ spinning particle is constructed by using the coefficient $\Gamma_\mathcal{I}$ in the expression of the invariant component $\Omega_\mathcal{I}^{(1)}$ in terms of the full system of invariant differential forms $\Omega_\mathcal{I}$ and $\Omega_\mathcal{I}^{\theta}$:

$$\Omega_\mathcal{I}^{(1)} = \Omega_\mathcal{I}^{\theta} Y_\mathcal{I} + \Omega_\mathcal{I}^{\theta} \Gamma_\mathcal{I}. \quad (33)$$

This odd coefficient, as well as the even one $Y_\mathcal{I}$, is also invariant. The invariant action is

$$S = \frac{i}{2} \Sigma_\mathcal{I} \int dV_\mathcal{I} \Gamma_\mathcal{I} = \frac{i}{2} \Sigma_\mathcal{I} \int d\tau d\theta e^{U_\mathcal{I}}(D_\theta U^{(1)}_\mathcal{I} - iD_\theta \Theta_\mathcal{I} + 2i\Theta_\mathcal{I} U^{(1)}_\mathcal{I} - 2i(\Theta^{(3/2)})). \quad (34)$$

After the solution of the equations of motion for auxiliary fields $U^{(1)}_\mathcal{I}$, and introduction of new variables $X_\mathcal{I} = e^{U_\mathcal{I}/2} = \tau + i\theta \gamma_1(\tau)$ the action (34) becomes

$$S = -\frac{i}{2} \int d\tau d\theta (X_\mathcal{I} D_\theta X_\mathcal{I} + 2i\Theta^{3/2}X^2_\mathcal{I}). \quad (35)$$

The superfield $\Theta^{3/2}(\tau, \theta) = \frac{1}{2}(\rho - \theta \lambda)$ in this action plays the role of Lagrange multiplier leading to the constraint $X^2_\mathcal{I} = 0$ which is the supersymmetric generalization of the constraint (22). After the Berezin integration over the $\theta$ the action (35) coincides with the manifestly conformal component action for the $N = 1$ spinning particle

$$S = \frac{1}{2} \int d\tau(x^2 + i\gamma_1 x^2 - \lambda x^2 - 2i\rho \gamma_1 x_\mathcal{I}). \quad (36)$$

3. $N = 2$ SUPERCONFORMAL ALGEBRA AND SPIN-1 PARTICLE IN THE SUPERSPACE APPROACH

$N = 2$ Superconformal Algebra (SCA) in complex notations

$$G_s = G_s^1 + iG_s^2, \quad T_m = -\frac{i}{2} T_m^{12}, \quad (37)$$

(see Appendix) have the following form

$$[L_m, L_n] = -i(m - n)L_{m+n},$$

$$[L_m, T_n] = i\lambda T_{m+n},$$

$$[L_m, G_s] = -i\frac{m}{2} - s)G_{m+s},$$
\[ [L_m, \bar{G}_s] = -i \left( \frac{m}{2} - s \right) \bar{G}_{m+s} \]  
\[ [T_m, G_s] = -i \frac{s}{2} G_{m+s}, \]
\[ [T_m, \bar{G}_s] = -i \frac{s}{2} \bar{G}_{m+s}, \]
\[ \{ G_r, \bar{G}_s \} = 2L_{r+s} + 2T_{r+s}. \]

Following the previous considerations we consider simultaneously \( D + 2 \) elements of \( N = 2 \) superconformal group \( (I = 0, 1, \ldots, D + 1) \)

\[ G_I = e^{iL_{-1} + \delta G_{1/2} + \frac{\tau}{2} \bar{G}_{1/2}}e^{V^1 T_1}, \]
\[ e^{\bar{\Gamma}^{(1/2)} G_{1/2} + \bar{\theta} \bar{G}_{1/2} + e^{U^{(2)} I_2} L_2 \ldots} \]
\[ e^{\theta G_{1/2} + \theta \bar{G}_{1/2} + e^{V^1 I_1} L_1} e^{U^1 I_0} e^{V_2 T_0}. \]

Last four multipliers in this expression form the subgroup of the whole \( N = 2 \) superconformal group and they consistently can carry external index \( I \).

The line of calculations is the same. Firstly we find the expressions for Cartan's \( \Omega \) - form components

\[ \Omega_I = G_I^{-1} dG_I = i\Omega_I^{(-1)} L_{-1} + \bar{\Omega}_I^{(-1/2)} G_{1/2} + \Omega_I^{(-1/2)} G_{-1/2} + \frac{1}{i} \Omega_I^{(0)} L_0 + \bar{\Omega}_I^{(1/2)} G_{1/2} + \Omega_I^{(1/2)} \bar{G}_{1/2} + \ldots. \]

Its three first components

\[ \Omega_{\tau}^\tau \equiv \Omega_{\tau}^{(-1)} = d_\tau^M E_{\tau}^{M \bar{M}} \]
\[ = (d\tau - i\theta \bar{\theta} - i\bar{\theta} \theta)e^{-U_I}, \]
\[ \Omega_{\tau}^\theta \equiv \Omega_{\tau}^{(-1/2)} = d_\tau^M E_{\tau}^{M \bar{M}} \]
\[ = \{d\theta - (d\tau - i\theta \bar{\theta} - i\bar{\theta} \theta) \theta \bar{\theta} \}
\[ e^{-U_I/2 - iV_I/2}, \]
\[ \Omega_{\bar{\tau}}^\bar{\theta} \equiv \Omega_{\bar{\tau}}^{(-1/2)} = d_{\bar{\tau}}^M E_{\bar{\tau}}^{\bar{M} \bar{M}} \]
\[ = \{(d\bar{\theta} - (d\bar{\tau} - i\theta \bar{\theta} - i\bar{\theta} \theta) \bar{\theta} \bar{\theta} \}
\[ e^{-U_I/2 + iV_I/2}, \]

define supervielbein \( E_{\tau}^A \) in the notations: \( x^1 = \tau, x^2 = \theta, x^3 = \bar{\theta} \).

The invariant integration measure is simply

\[ dV_I = d\tau d\theta d\bar{\theta}. \]

because \( Ber(E_{\tau}^A) = 1 \) for the case of \( N = 2 \) SCA, as one can calculate using the expressions \([11]-[13]\). For construction of the action one need also the expressions

\[ \Omega_{I}^{\Omega_I^{(1/2)}} = \{d\theta I - (d\tau - (d\tau - i\theta \bar{\theta} - i\bar{\theta} \theta) \theta \bar{\theta} \}U_I^{(1)} \]
\[ -i\bar{\theta} \theta I - \frac{i}{2} dV_I^1 e^{U_I/2 - iV_I/2}, \]
\[ \bar{\Omega}_I^{\Omega_I^{(1/2)}} = \{d\bar{\theta} I - (d\bar{\tau} - (d\bar{\tau} - i\theta \bar{\theta} - i\bar{\theta} \theta) \bar{\theta} \bar{\theta} \}U_I^{(1)} \]
\[ +i\bar{\theta} \theta \bar{\theta} I + \frac{i}{2} dV_I^1 e^{U_I/2 - iV_I/2}, \]

Their expansion in terms of fundamental forms \([11]-[13]\) contains invariant coefficients

\[ \Gamma = \{D \theta I - U_I^1 - i\bar{\theta} \bar{\theta} \}
\[ \bar{\Gamma} = \{\bar{D} \bar{\theta} I - U_I^1 + i\bar{\theta} \bar{\theta} \}. \]

which can be used for the construction of the action

\[ S_{N=2} = \frac{i}{4} \int d\tau d\theta d\bar{\theta} (\Gamma - \bar{\Gamma}) = \int dV \mathcal{L} \]
\[ \mathcal{L} = \frac{i}{4} \{D \theta I - \bar{D} \bar{\theta} I - 2i\bar{\theta} \bar{\theta} I - iV_I^1 \}. \]

After the solution of the equations of motion for auxiliary fields \( \theta I, \bar{\theta} I \) this lagrangian becomes

\[ \mathcal{L} = \frac{1}{2} DX_I DX_I + \frac{1}{4} V_I X_I^2. \]

Here \( X_I = e^{U_I/2} = x_I(\tau) + i\bar{\theta} \gamma_I(\tau) + i\bar{\theta} \bar{\theta} \gamma_I(\tau) + \theta \phi \gamma_I(\tau) \) are the \( N = 2 \) superfield coordinates and \( D = \partial / \partial \theta + i\theta \bar{\theta} / \partial \tau, \bar{D} = \partial / \partial \bar{\theta} + i\bar{\theta} \theta / \partial \tau \) - the flat covariant derivatives. The \( N = 2 \) superfield \( V_I(\tau, \theta, \bar{\theta} \) in this lagrangian plays the role of Lagrange multiplier leading to the constraint \( X^2_I = 0 \) which is the \( N = 2 \) supersymmetric generalization of the constraint \([2],[4]\). The integration over the grassmann coordinates, normalized as \( \int d\theta d\bar{\theta} d\theta \bar{\theta} = -1 \) leads to the \( D + 2 \) - dimensional component lagrangian for \( N = 2 \) spinning particle received in \([3]\).

4. \( N = 1 \) SPINNING PARTICLE IN THE COMPONENT APPROACH

The analysis of the possible generalization of described scheme on higher \( N \) spinning particles reveals the following obstacle. In all considered examples the dimensionality of the action in the
units \( \dim \tau = +1, \dim \theta_e = +1/2 \) is \( \dim S = -1 \). The dimensionality of the integration volume for \( N \) extended supersymmetry is \( \dim dV = 1 - N/2 \) whereas the dimensionality of the components of Cartan’s \( \Omega \) - form is nonpositive integers or halfintegers. It means that starting with \( N = 4 \) there do not exist the appropriate invariant coefficients in the expansions of these \( \Omega \) - form components in terms of vielbeins which can be taken as the lagrangian (see the Picture 1).

One possible way out consists in the using as lagrangians the more complicated (nonlinear) functions of these invariant coefficients. The structure of these functions in the every case needs the additional analysis. However, there exists the universal approach to all cases of \( N \) - extended supersymmetry. This is the so called component approach, in which all parameters of the group are the functions of a parameter \( \tau \) (the proper time) only. In some sense this approach is more economic, because it reduces the number of the fields by \( 2^N \) times (the number of component fields in a superfield). The only price, as was described in the Sect.2, is the appearance of the grassmann Goldstone fields, which corresponds to the \( \tau \) dependent parameters at the supertranslations generators. In all the cases the action can have the form analogous to \( [18] \)

\[
S = -\frac{1}{2} \Sigma \int \Omega_I^{(1)} \tag{52}
\]

where \( \Omega_I^{(1)} \) is the component of the \( \Omega \) - form corresponding to the generator \( L_1 \). So defined action by construction is the supersymmetrization of the spinless particle action. The only thing one should to do - is to redefine the variables in such a way that the dependence of the action on the grassmann Goldstone fields disappears. Below we illustrate this approach for the simplest case of \( N = 1 \) supersymmetry.

Once more consider the parametrization of the \( D + 2 \) elements of the \( N = 1 \) SCA. This time we consider the spontaneously broken realization of the supersymmetry transformation, i.e. the corresponding parameter instead of being the Grassmann coordinate of the superspace \((1,1)\) is now the Goldstone field \( \theta(\tau) \) which depends on the only bosonic coordinate \( \tau \)

\[
G_\tau = e^{i \tau L_{-1}} e^{i \theta(\tau) G_{-1/2}} e^{i \theta^{(3/2)} G_{3/2}} e^{U/(2) L_2} \ldots e^{i \theta \tau G_{1/2}} e^{U/(1) L_1} e^{U(0) L_0} \tag{53}
\]

All other parameters are the functions of \( \tau \) as well.

The explicit expression for the \( \Omega_I^{(1)} \) component is the following

\[
\Omega_I^{(1)} = \{ dU_I^\tau + d\tau (U_I^{(1)})^2 - 3d\tau U^{(2)} - id\theta \Omega_I + 2id\theta \Omega_I U_I^{(1)} - id\theta^2 (U_I^{(1)})^2 + 3id\theta^2 U^{(2)} - 2id\theta U^{(3/2)} + 2id\theta U^{(3/2)} + 2id\theta \Omega_I (3/2) \} e^{U(z)}
\]

One can eliminate the auxiliary fields \( U_I^{(1)} \) and introduce new fields (the dot denote the \( \tau \)-derivative)

\[
x_I = e^{U_I^\tau/(2)} (1 - i \theta \Omega_I) \tag{55}
\]

\[
\gamma_I = -e^{U_I^\tau/(2)} (1 - i \theta \partial) \Omega_I + \frac{1}{2} \Omega_I e^{U_I^\tau/(2)} \partial, \tag{56}
\]

\[
\lambda = -3U^{(2)} (1 - i \theta \partial) - 4i \theta \partial (3/2) - 2i \theta \partial (3/2) \tag{57}
\]

\[
\rho = -2 \partial (3/2) (1 - 3i \theta \partial) - 6 \partial U^{(2)} \tag{58}
\]

In terms of these new variables the action \( [2] \) coincides with the manifestly conformal form of the action for \( N = 1 \) spinning particle \( (39) \) \( (42) \).

5. CONCLUSIONS

All considered examples illustrate the close connection between the physical systems and their symmetry groups, which consists in the possibility to describe the system in terms of the parameters of its symmetry group. One more well known example is the gravity which can be described in terms of the metric tensor \( [10] \) (see also \( [11] \) for supergravity) or vielbein \( [12] \) parametrizing the diffeomorphisms group of the space-time.

So, it would be interesting to apply the approach developed here to the cases when the bosonic part of the (super)space is not one dimensional. The simplest are the two dimensional spaces, which correspond to the (super)strings. In addition, the method can be applied to the nonlinearly realized W-algebras which are the symmetry groups for particles with rigidity and their supersymmetric generalizations.
Acknowledgement

I would like to thank the members of the Institut für Theoretische Physik, Universität Hannover, where considerable part of this work was finished, for their hospitality. Especially I would like to thank Prof. Olaf Lechtenfeld for his interest and enlightening discussions. The work was supported in part by the Russian Foundation of Fundamental Research, under the grant 99-02-18417 and the joint grant RFFR-DFG 99-02-04022, and by a grant of the Committee for Collaboration between Czech Republic and JINR.

6. APPENDIX. \(N\) EXTENDED SUPERCONFORMAL ALGEBRA

Having in mind the application of nonlinear realizations of \(N\) extended Superconformal Algebra to the description of \(N\) extended spinning particle we describe in this Appendix the \(N\) extended SCA as subalgebra of the diffeomorphism algebra of the \((1,N)\) superspace \((s, \eta_a), (a = 1, 2, \cdots N)\). The generators of the corresponding diffeomorphism group regular at the origin can be written in the coordinate representation as

\[
P^{(m)\alpha_1,\alpha_2,\cdots,\alpha_n}_0 = i \sum_{\alpha}^{m} \eta^{\alpha_1} \eta^{\alpha_2} \cdots \eta^{\alpha_n} \frac{\partial}{\partial s}, \quad (A.1)
\]

\[
P^{(m)\alpha_1,\alpha_2,\cdots,\alpha_n}_a = i \sum_{\alpha}^{m} \eta^{\alpha_1} \eta^{\alpha_2} \cdots \eta^{\alpha_n} \frac{\partial}{\partial \eta_a}, n \leq N.
\]

With the help of this representation one can easily calculate the corresponding algebra of diffeomorphisms. All of these generators can be naturally ordered in accordance with their dimensionality \((\text{dim } s = +1, \text{dim } \eta_a = +1/2)\):

\[
\text{dim } P_0 = -1, \text{dim } P_a = \text{dim } P_{a0} = -\frac{1}{2}, \quad (A.2)
\]

\[
\text{dim } P_{00} = \text{dim } P_{a1} = \text{dim } P_{a1a2} = 0, \text{ etc.}
\]

The \(N\) extended SCA in the \((s, \eta_a)\) superspace is characterized by \(N\) supercovariant derivatives

\[
D_a = \partial/\partial \eta^a + i \eta_a \partial/\partial s, \quad (A.3)
\]

which transform homogeneously under the transformations of the \(N\) extended SCA. One can show, \(\square\), that the corresponding infinitesimal transformations of the coordinates \((s, \eta_a)\) can be described in terms of an unconstrained scalar superfunction \(\Lambda(s, \eta_a)\):

\[
\delta s = \Lambda - \frac{1}{2} \eta_a D_a \Lambda, \quad \delta \eta_a = -\frac{i}{2} D_a \Lambda. \quad (A.4)
\]

The composition law for two transformations with parameters \(\Lambda_1\) and \(\Lambda_2\) \(([\delta_1, \delta_2] = \delta_3)\) is

\[
\Lambda_3 = \Lambda_1 \partial_s \Lambda_2 - \Lambda_2 \partial_s \Lambda_1 - \frac{i}{2} D_a \Lambda_1 D_a \Lambda_2. \quad (A.5)
\]

Each function \(\Lambda\) is in one to one correspondence with the generators of the SCA. Naturally all functions \(\Lambda\) are divided on classes by their dependence on the Grassmann coordinates \(\eta_a\). Their correspondence with the generators of the SCA is illustrated by the following table

\[
\begin{array}{l}
\Lambda^0_n = \epsilon_n s^n \Leftrightarrow -i \epsilon_n L_{n-1}, \\
\Lambda_{1/2}^1 = -2i \epsilon_n \eta_a s^n \Leftrightarrow -i \epsilon_n G^a_{n-1/2}, \\
\Lambda_1^n = 2i \epsilon_n \eta_a \eta_b s^n \Leftrightarrow -i \epsilon_n T_a^n B^{ab}, \\
\Lambda_{3/2}^n = -2i \epsilon_n \eta_a \eta_b \eta_c s^n \Leftrightarrow -i \epsilon_n R^{abc} F_{n-1/2}, \\
\vdots \\
\Lambda_{N/2}^n = 2(-i)^{N(N+1)/2} \epsilon_n a_1 \cdots a_N \eta_{a_1} \eta_{a_2} \cdots \eta_{a_N} s^n \\
\Leftrightarrow -i \epsilon_n a_1 \cdots a_N R^{a_1 \cdots a_N} F_{n-1/2}.
\end{array}
\]

All \(\Lambda\)s at the left are hermitian (with hermitian \(\epsilon\)s) and normalization is chosen to get a convenient definition of the generators at the right.

\[
\begin{array}{cccccc}
\text{Bose} & L_{-1} & L_0 & L_1 & L_2 & L_3 & L_4 \\
\text{Fermi} & G^a_{-1/2} & G^a_{1/2} & G^a_3/2 & G^a_5/2 & G^a_7/2 & \\
\text{Bose} & T^a_0 & T^a_1 & T^a_2 & T^a_3 & T^a_4 & \\
\text{Fermi} & F^{abc} & F^{abc} & F^{abc} & F^{abc} & F^{abc} & \\
\text{Bose} & \Lambda^{abc} & \Lambda^{abc} & \Lambda^{abc} & \Lambda^{abc} & \Lambda^{abc} & \\
\text{Fermi} & H^{abcde} & H^{abcde} & H^{abcde} & H^{abcde} & H^{abcde} & \\
\end{array}
\]

Picture 1. Encircled are generators whose \(\Omega\)-form components are used to construct the actions for \(N = 0, 1 - (L_1)\) and \(N = 2 - (G^a_{1/2})\) spinning particles in the superfield approach.

The first lines of the table and Picture 1. contain the generators \(L_n\) of the Virasoro algebra with \(n \geq -1\). The second line contains \(N\) series \(G^a_r\), \((r \geq -1/2)\), each starting from the
supercharge $G_{a}^{-1/2}$. The next line starts from the generators $T_{ab}^{c}$ of the SO($N$) algebra and corresponds to its Kac–Moody generalization. The generators $G_{a}^{c}$ form the vector representation of this algebra. All generators and parameters with even (odd) number of indices are bosonic (fermionic).

The algebra of some lower generators $L_{m}, G_{a}^{r}, T_{ab}^{c}, F_{abc}^{s}, H_{abcd}^{t}, \ldots$, $(m, n, k – \text{integer}, r, s, t – \text{halfinteger})$ is as follows:

$$
\begin{align*}
\{L_{m}, L_{n}\} &= -i(m - n)L_{m+n}, \\
\{L_{m}, G_{a}^{r}\} &= -i(m - r)G_{m+r}, \\
\{G_{a}^{r}, G_{b}^{s}\} &= 2\delta_{ab}L_{r+s} + (r-s)T_{ab}^{c}, \\
\{T_{ab}^{c}, G_{d}^{e}\} &= -2\delta_{cd}T_{r+s}^{ab} - 2\delta_{cd}T_{r+s}^{ab} - \delta_{cd}T_{r+s}^{ab} - \Lambda_{abcd}^{m+n}, \\
\{F_{abc}^{r}, G_{d}^{e}\} &= i\delta_{da}F_{r+s}^{ebc} + i\delta_{db}F_{r+s}^{ecb} + i\delta_{dc}F_{r+s}^{eba} - i\delta_{dc}F_{r+s}^{eba} - i\delta_{dc}F_{r+s}^{eba} - \delta_{dc}F_{r+s}^{eba}, \\
\{T_{ab}^{c}, T_{de}^{f}\} &= i\delta_{da}T_{r+s}^{ebc} + i\delta_{db}T_{r+s}^{ecb} + i\delta_{dc}T_{r+s}^{eba} - i\delta_{dc}T_{r+s}^{eba} - i\delta_{dc}T_{r+s}^{eba} - \delta_{dc}T_{r+s}^{eba}.
\end{align*}
$$

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