ON THE CLASS OF 2D $q$-APPELL POLYNOMIALS

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Abstract. In this research, as the new results of our previously proposed definition for the new class of 2D $q$-Appell polynomials in [1], we derive some interesting relations including the recurrence relation and partial $q$-difference equation of the aforementioned family of $q$-polynomials. Next, as some famous examples of this new defined class of $q$-polynomials, we obtain the corresponding relations to the 2D $q$-Bernoulli polynomials, 2D $q$-Euler polynomials as well as 2D $q$-Genocchi polynomials.

1. Introduction

In [1], Eini and Mahmudov defined 2D $q$-Appell Polynomials by means of the following generating function

(1) $A_q(x, y; t) := A_q(t)e_q(tx)E_q(ty) = \sum_{n=0}^{\infty} A_{n,q}(x, y) \frac{t^n}{[n]_q!}$,

where

(2) $A_q(t) := \sum_{n=0}^{\infty} A_{n,q} \frac{t^n}{[n]_q!}, \quad A_q(t) \neq 0$,

is an analytic function at $t = 0$, and $A_{n,q} := A_{n,q}(0,0)$. Taking $q$-derivative of $A_q(x, y, t)$ with respect to the variable $x$, from one hand we obtain

$$D_{q,x}(A_q(x, y, t)) = D_{q,x}(A_q(t)e_q(tx)E_q(ty))$$

$$= tA_q(t)e_q(tx)E_q(ty) = \sum_{n=0}^{\infty} A_{n,q}(x, y) \frac{t^{n+1}}{[n]_q!}$$

(3) $$= \sum_{n=1}^{\infty} \frac{[n]_q}{[n]_q!} A_{n-1,q}(x, y) \frac{t^n}{[n]_q!}$$

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From another hand we can write

\[ D_{q,x}(A_q(x, y; t)) = D_{q,x}\left(\sum_{n=0}^{\infty} A_{n,q}(x, y) \frac{t^n}{[n]_q!}\right) \]
\[ = \sum_{n=0}^{\infty} D_{q,x}(A_{n,q}(x, y)) \frac{t^n}{[n]_q!}. \]  

Comparing the coefficients of \( \frac{t^n}{[n]_q!} \) in the relations (3) and (4), leads to obtain

\[ D_{q,x}(A_{n,q}(x, y)) = [n]_q A_{n-1,q}(x, y). \]  

Using a similar technique for taking \( q \)-derivative of \( A_q(x, y; t) \) with respect to the variable \( y \), we have

\[ D_{q,y}(A_{n,q}(x, y)) = [n]_q A_{n-1,q}(x, qy). \]  

Now, according to relations (5) and (6), we define the following lowering operators

\[ \Phi_{n,q_x} = \frac{1}{[n]_q} D_{q,x}, \quad \Phi_{n,q_y} = \frac{1}{[n]_q} D_{q,y}. \]  

Therefore, we may reexpress the relations (5) and (6) in the form of the following operational identities

\[ \Phi_{n,q_x} A_{n,q}(x, y) = A_{n-1,q}(x, y), \quad \text{and} \quad \Phi_{n,q_y} A_{n,q}(x, y) = A_{n-1,q}(x, qy), \]  

respectively. Eventually applying the above operators \( k \) times, leads to obtain

\[ A_{n-k,q}(x, y) = (\Phi_{n-k,q_x} \circ \ldots \circ \Phi_{n,q_x}) A_{n,q}(x, y) = \frac{[n-k]_q!}{[n]_q!} D_{q,x}^k A_{n,q}(x, y), \]  

and

\[ A_{n-k,q}(x, qk y) = (\Phi_{n-k,q_y} \circ \ldots \circ \Phi_{n,q_y}) A_{n,q}(x, y) = \frac{[n-k]_q!}{[n]_q!} D_{q,y}^k A_{n,q}(x, y), \]  

respectively.
2. Recurrence Relations and $q$-Difference Equations for the Class of 2D $q$-Appell Polynomials

In 2002, Bretti et. al. proposed a generating function for the family of 2D Appell polynomials, [2]. They, also, obtained the corresponding recurrence relations and differential equations to the aforementioned family by calculating raising and lowering operators. In [3], Mahmudov applied an innovative technique in order to derive the recurrence relations and difference equations of the polynomials in the class of $q$-Appell polynomials only by using only lowering operators that are $q$-derivatives. Inspired by his novel approach, in the following we will use a similar technique in order to derive the corresponding relations to the class of 2D $q$-Appell polynomials.

**Theorem 1.** The following linear homogeneous recurrence relation holds for the class of 2D $q$-Appell polynomials

\[
A_{n,q}(qx, y) = \frac{1}{[n]_q} \sum_{k=1}^{n} \binom{n}{k} q^{n-k} A_{n-k}(x, y) \left( (\alpha_k + \frac{\beta_{k-1}}{[k]_q} y) + qx A_{n-1,q}(x, y) \right), \quad n \geq 1,
\]

or equivalently,

\[
A_{n,q}(qx, y) = q^n (x + (\alpha_1 + \beta_0 y) q^{-1}) A_{n-1,q}(x, y) +
\]

\[
\frac{1}{[n]_q} \sum_{k=1}^{n-1} \binom{n}{k-1} q^{k-1} A_{k-1,q}(x, y) \left( \alpha_{n-k+1} + \frac{y}{[n-k+1]_q} \beta_{n-k} \right), \quad n \geq 1.
\]

**Proof.** Starting with taking the $q$-derivative of the generating function in relation (1) with respect to $t$, we have

\[
D_{q,t}(A_q(qx, y; t)) =
\]

\[
y A_q(t) e_q(qtx) E_q(qty) + D_{q,t}(A_q(t)) e_q(qtx) E_q(qty) + qx A_q(x, y; qt).
\]
Now, multiplying both sides of the identity (13) by $t$ and factorizing $A_q(x, y; qt)$ form its left hand side, we obtain

\[(14) \quad tD_{q,t}(A_q(qx, y; t)) = A_q(x, y; qt)\left[\frac{tD_{q,t}(A_q(t))}{A_q(qt)} + t qx + ty \frac{A_q(t)}{A_q(tq)}\right].\]

Suppose that $t \frac{D_{q,t}(A_q(t))}{A_q(qt)} = \sum_{n=0}^{\infty} \alpha_n t^n [n]_q!$, and $t \frac{A_q(t)}{A_q(qt)} = \sum_{n=0}^{\infty} \beta_n t^n [n]_q!$. Starting from taking $q$-derivative of the left hand side of relation (14) with respect to $t$ and also substituting the assumptions above in the right hand side of the same equation, we can continue as

\[(15) \quad \sum_{n=1}^{\infty} [n]_q A_{n,q}(qx, y) \frac{t^n}{[n]_q!} = A_q(x, y; qt)\left[\sum_{n=0}^{\infty} \alpha_n \frac{t^n}{[n]_q!} + \sum_{n=0}^{\infty} y \beta_n \frac{t^{n+1}}{[n]_q!} + t qx\right].\]

The last part of identity above can be written as

\[(16) \quad = \sum_{n=0}^{\infty} q^n A_{n,q}(x, y) \frac{t^n}{[n]_q!} \left[\sum_{n=0}^{\infty} \alpha_n \frac{t^n}{[n]_q!} + y \sum_{n=1}^{\infty} [n]_q \beta_n \frac{t^n}{[n]_q!} + t qx\right],\]

which is equivalent to

\[
\begin{align*}
&= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \binom{n}{k} q^{n-k} A_{n-k,q}(x, y) \alpha_k \frac{t^n}{[n]_q!} + \\
&y \sum_{n=0}^{\infty} \sum_{k=0}^{n} \binom{n}{k} q^{n-k} A_{n-k,q}(x, y) \beta_k \frac{t^{n+1}}{[n]_q!} + x \sum_{n=0}^{\infty} q^{n+1} A_{n,q}(x, y) \frac{t^{n+1}}{[n]_q!}.
\end{align*}
\]
This means that

\[
\sum_{n=1}^{\infty} [n]_q A_{n,q}(qx, y) \frac{t^n}{[n]_q} = A_{0,q}(x, y)\alpha_0 + \\
\sum_{n=1}^{\infty} \left( \sum_{k=0}^{n} \binom{n}{k}_q q^{-k} A_{n-k,q}(x, y)\alpha_k + y \sum_{k=0}^{n-1} \binom{n-1}{k}_q [n]_q q^{-k-1} A_{n-k-1,q}(x, y)\beta_k \right) \frac{t^n}{[n]_q}.
\]

(17)

\[
x[n]_q^n A_{n-1,q}(x, y) \frac{t^n}{[n]_q}.
\]

(18)

\[
= \sum_{n=0}^{\infty} \sum_{k=0}^{n} \binom{n}{k}_q q^{-k} A_{n-k}(x, y)(\alpha_k + \beta_k y) + [n]_q x q^n A_{n-1,q}(x, y) \frac{t^n}{[n]_q}.
\]

Comparing the coefficients of $\frac{t^n}{[n]_q}$ in both sides of relation (17) and noting to the fact that $\alpha_0 = 0$, lead to obtain the following identity for $n \geq 1$

\[
[n]_q A_{n,q}(qx, y) = \sum_{k=1}^{n} \binom{n}{k}_q q^{-k} A_{n-k,q}(x, y)\alpha_k + \\
y \sum_{k=1}^{n} \binom{n-1}{k-1}_q [n]_q q^{-k} A_{n-k,q}(x, y)\beta_{k-1} + x[n]_q q^n A_{n-1,q}(x, y),
\]

whence the result. \qed

**Theorem 2.** The following partial $q$-difference equations hold for the polynomials in the class of 2D $q$-Appell

\[
\sum_{k=1}^{n} \frac{q^{n-k}}{[k]_q} (\alpha_k + \beta_{k-1} y) D_{q,x}^k + x q^n D_{q,x} A_{n,q}(x, y) - [n]_q A_{n,q}(qx, y) = 0.
\]

(19)

\[
\sum_{k=1}^{n} \frac{q^{n-k}}{[k]_q} (\alpha_k + \beta_{k-1} y) D_{q,y}^k A_{n,q}(x, y) + x q^n D_{q,y} A_{n,q}(x, y) - [n]_q A_{n,q}(qx, y) = 0.
\]

(20)

Proof. The proof is the direct result of replacing relations (9) and (10) in the linear homogeneous recurrence relation (11) given in Theorem (1), respectively. \qed
3. q-Difference Equations For Various Members Of The Family Of q-Appell Polynomials

Choosing different functions as \( A_q(t) \) in Definition (1), leads to generate different members of 2D q-Appell polynomials. In the following we introduce some of the most famous 2D q-Appell polynomials and derive the corresponding recurrence relations and partial q-difference equations to them.

3.1. 2D q-Bernoulli polynomials. Taking \( A_q(t) = t e_q(t) \) in Definition (1), leads to obtain 2D q-Bernoulli polynomials, \( B_{n,q}(x, y) \), [4], [5].

\[
B_q(x, y; t) := \frac{t}{e_q(t) - 1} e_q(tx) E_q(ty) = \sum_{n=0}^{\infty} B_{n,q}(x, y) \frac{t^n}{[n]_q},
\]

Lemma 3. Suppose that

\[
t \frac{D_{q,t}(A_q(t))}{A_q(qt)} = t \frac{D_{q,t} \frac{t}{e_q(qt) - 1}}{\frac{t}{e_q(qt) - 1}} = \sum_{n=0}^{\infty} \alpha_n \frac{t^n}{[n]_q},
\]

and

\[
\frac{A_q(t)}{A_q(qt)} = \frac{t}{e_q(qt) - 1} = \sum_{n=0}^{\infty} \beta_n \frac{t^n}{[n]_q},
\]

then

\[
\alpha_n = -\frac{1}{q} b_{n,q}, \quad \alpha_1 = -\frac{1}{[2]_q},
\]

and

\[
\beta_n = \frac{q-1}{q} \sum_{k=0}^{n} \binom{n}{k} b_{k,q}, \quad \text{for } n \geq 1 \quad \text{and } \beta_0 = 1,
\]

where \( b_{n,q} = B_{n,q}(0, 0) \) is the n-th q-Bernoulli number and can be obtained from the generating function \( \frac{1}{e_q(t) - 1} = \sum_{n=0}^{\infty} b_{n,q} \frac{t^n}{[n]_q} \).
Theorem 4. The following linear homogeneous recurrence relation holds for the class of 2D $q$-Bernoulli polynomials for every $n \geq 1$

$$B_{n,q}(qx, y) = q^n(x + \left(\frac{-1}{[2]_q} + y\right)q^{-1})B_{n-1,q}(x, y) +$$

\[
\frac{1}{[n]_q} \sum_{k=1}^{n-1} \binom{n}{k-1}_q q^{k-2}B_{k-1,q}(x, y)(-b_{n-k+1,q} + \frac{(q-1)y}{[n-k+1]_q}) \sum_{l=0}^{n-k} \binom{n-k}{l}_q b_{l,q}.
\]

Theorem 5. The following partial $q$-difference equations hold for the polynomials in the class of 2D $q$-Bernoulli

\[
(xq^n + y - \frac{1}{[2]_q})D_{q,x} + \sum_{k=2}^{n} \frac{q^{n-k-1}}{[k]_q!}(-b_{k,q} + (q-1)y) \sum_{l=0}^{k-1} \binom{k-1}{l}_q b_{l,q} D_{q,x}^k
\]

\[
\times B_{n,q}(x, y) - [n]_q B_{n,q}(qx, y) = 0.
\]

\[
(xq^n + y - \frac{1}{[2]_q})D_{q,y} A_{n,q}(x, \frac{y}{q}) + \sum_{k=2}^{n} \frac{q^{n-k-1}}{[k]_q!}(-b_{k,q} + (q-1)y) \sum_{l=0}^{k-1} \binom{k-1}{l}_q b_{l,q} D_{q,y}^k
\]

\[
\times D_{q,y} B_{n,q}(x, \frac{y}{q^k}) - [n]_q A_{n,q}(qx, y) = 0.
\]

3.2. 2D $q$-Euler polynomials. Taking $A_q(t) = \frac{2}{e_q(t)+1}$ in Definition (1), leads to obtain 2D $q$-Euler polynomials, $E_{n,q}(x, y)$, as follows\[4, 5\].

\[
E_q(x, y; t) := \frac{2}{e_q(t)+1}e_q(tx)E_q(ty) = \sum_{n=0}^{\infty} E_{n,q}(x, y) \frac{t^n}{[n]_q!},
\]

Lemma 6. Suppose that

\[
\frac{t D_{q,t}(A_q(t))}{A_q(qt)} = \frac{D_{q,t} \frac{2}{e_q(t)+1}}{\frac{2}{e_q(qt)+1}} = \sum_{n=0}^{\infty} \alpha_n \frac{t^n}{[n]_q!},
\]

and

\[
\frac{A_q(t)}{A_q(qt)} = \frac{\frac{2}{e_q(t)+1}}{\frac{2}{e_q(qt)+1}} = \sum_{n=0}^{\infty} \beta_n \frac{t^n}{[n]_q!},
\]
then
\( \alpha_n = \frac{1}{2} E_{n-1,q}, \quad \alpha_1 = \frac{-1}{2}, \)

and
\( \beta_n = \frac{q-1}{2} \sum_{k=0}^{n} \binom{n}{k} \frac{E_{k,q}}{q}, \) for \( n \geq 1 \) and \( \beta_0 = \frac{q+1}{2}, \)

where \( E_{n,q} = E_{n,q}(0,0) \) is the \( n \)-th \( q \)-Euler number and can be obtained from the generating function \( \frac{2}{e_q(t)+1} = \sum_{n=0}^{\infty} E_{n,q} \frac{t^n}{[n]_q}. \)

**Theorem 7.** The following linear homogeneous recurrence relation holds for the class of \( 2D \) \( q \)-Euler polynomials for every \( n \geq 1 \)

\[
A_{n,q}(qx, y) = q^n (x + \frac{(q+1)y-1}{2q}) E_{n-1,q}(x, y) + \frac{1}{[n]_q} \sum_{k=1}^{n-1} \binom{n}{k-1} q^{k-1} E_{k-1,q}(x, y) \left( \frac{1}{2} E_{n-k,q} + \frac{(q-1)y}{q[n-k+1]_q} \sum_{l=0}^{n-k} \binom{n-k}{l} E_{l,q} \right).
\]

**Theorem 8.** The following partial \( q \)-difference equations hold for the polynomials in the class of \( 2D \) \( q \)-Euler

\[
\left( xq^n + \frac{(q+1)y}{2q[k]_q} - \frac{1}{2} \right) D_{q,x} + \sum_{k=2}^{n} q^{n-k} \frac{1}{k[k]_q!} \left( \frac{1}{2} E_{k-1,q} + \frac{(q-1)y}{2q[k]_q} \sum_{l=0}^{k-1} \binom{k-1}{l} E_{l,q} \right) D_{q,x}^k \\
\times E_{n,q}(x, y) - [n]_q E_{n,q}(qx, y) = 0.
\]

\[
\left( xq^n + \frac{(q+1)y}{2q[k]_q} - \frac{1}{2} \right) D_{q,y} E_{n,q}(x, y) + \sum_{k=2}^{n} \frac{q^{n-k}}{k[k]_q!} \left( \frac{1}{2} E_{k-1,q} + \frac{(q-1)y}{2q[k]_q} \sum_{l=0}^{k-1} \binom{k-1}{l} E_{l,q} \right) D_{q,y}^k \\
\times D_{q,y} E_{n,q}(x, y) - [n]_q E_{n,q}(qx, y) = 0.
\]

3.3. \( 2D \) \( q \)-Genocchi polynomials. Taking \( A_q(t) = \frac{2t}{e_q(t)+1} \) in Definition (1), leads to obtain \( 2D \) \( q \)-Genocchi polynomials, \( G_{n,q}(x, y) \), as follows

\[
G_q(x, y; t) := \frac{2t}{e_q(t)+1} e_q(tx) G_q(ty) = \sum_{n=0}^{\infty} G_{n,q}(x, y) \frac{t^n}{[n]_q},
\]
Lemma 9. Suppose that

\begin{equation}
\frac{D_{q,t}(A_q(t))}{A_q(qt)} = \frac{e_{q(t)+1}}{e_{q(t)+1}} \sum_{n=0}^{\infty} \alpha_n \frac{t^n}{[n]_q!},
\end{equation}

and

\begin{equation}
\frac{A_q(t)}{A_q(qt)} = \frac{e_{q(t)+1}}{e_{q(t)+1}} \sum_{n=0}^{\infty} \beta_n \frac{t^n}{[n]_q!},
\end{equation}

then

\begin{equation}
\alpha_n = \frac{1}{2q} G_{n,q}, \quad \text{for } n \geq 2, \text{ and } \alpha_0 = \frac{1}{q}, \alpha_1 = -\frac{1}{q},
\end{equation}

and

\begin{equation}
\beta_n = \frac{q-1}{2q} \sum_{k=0}^{n} \binom{n}{k}_q G_{k,q}, \quad \text{for } n \geq 1 \text{ and } \beta_0 = \frac{1}{q},
\end{equation}

where $G_{n,q} = G_{n,q}(0,0)$ is the $n$-th $q$-Genocchi number and can be obtained from the generating function $\frac{2t}{e_{q(t)+1}} = \sum_{n=0}^{\infty} G_{n,q} \frac{t^n}{[n]_q!}$.

Theorem 10. The following linear homogeneous recurrence relation holds for the class of $2D$ $q$-Genocchi polynomials for every $n \geq 1$

\begin{equation}
G_{n,q}(qx, y) = q^n(x + \frac{y-1}{q^2})G_{n-1,q}(x, y) +
\frac{1}{2[n]_q} \sum_{k=1}^{n-1} \binom{n}{k}_q q^{k-2} G_{k-1,q}(x, y)(G_{n-k+1,q} - \frac{(q-1)y}{[n-k+1]_q} \sum_{l=0}^{n-k} \binom{n-k}{l}_q G_{l,q}) G_{k,q}.
\end{equation}

Theorem 11. The following partial $q$-difference equations hold for the polynomials in the class of $2D$ $q$-Genocchi

\begin{equation}
\left[ (xq^n + \frac{y-1}{q}) D_{q,x} + \sum_{k=2}^{n} q^{n-k-1} \frac{2k}{[k]_q!} (G_{k,q} + \frac{(q-1)y}{[k]_q} \sum_{l=0}^{k-1} \binom{k-1}{l}_q G_{l,q}) D_{q,x} \right]
G_{n,q}(x, y) - [n]_q G_{n,q}(qx, y) = 0.
\end{equation}
\[(xq^n + \frac{y-q}{q^2})D_{q,y}G_{n,q}(x, \frac{y}{q}) + \sum_{k=2}^{n} \frac{q^{n-k-1}}{2[k]_q} (G_{k,q} + \frac{(q-1)y}{q^k[k]_q} \sum_{l=0}^{k-1} \left[ \frac{k-1}{l} \right]_q) \times D_{q,y}G_{n,q}(x, \frac{y}{q^k}) \]

\[\times D_{q,y}G_{n,q}(x, \frac{y}{q^k}) - [n]_q G_{n,q}(qx, y) = 0.\]

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