Locating domination in bipartite graphs and their complements

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Abstract

A set $S$ of vertices of a graph $G$ is distinguishing if the sets of neighbors in $S$ for every pair of vertices not in $S$ are distinct. A locating-dominating set of $G$ is a dominating distinguishing set. The location-domination number of $G$, $\lambda(G)$, is the minimum cardinality of a locating-dominating set. In this work we study relationships between $\lambda(G)$ and $\lambda(\overline{G})$ for bipartite graphs. The main result is the characterization of all connected bipartite graphs $G$ satisfying $\lambda(\overline{G}) = \lambda(G) + 1$. To this aim, we define an edge-labeled graph $G^S$ associated with a distinguishing set $S$ that turns out to be very helpful.

Keywords: domination; location; distinguishing set; locating domination; complement graph; bipartite graph.

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1 Introduction

Let $G = (V, E)$ be a simple, finite graph. The neighborhood of a vertex $u \in V$ is $N_G(u) = \{v : uv \in E\}$. We write $N(u)$ or $d(v, w)$ if the graph $G$ is clear from the context. For any $S \subseteq V$, $N(S) = \cup_{u \in S} N(u)$. A set $S \subseteq V$ is dominating if $V = S \cup N(S)$ (see [7]). For further notation and terminology, we refer the reader to [4].

A set $S \subseteq V$ is distinguishing if $N(u) \cap S \neq N(v) \cap S$ for every pair of different vertices $u, v \in V \setminus S$. In general, if $N(u) \cap S \neq N(v) \cap S$, we say that $S$ distinguishes the pair $u$ and $v$. A locating-dominating set, LD-set for short, is a distinguishing set that is also dominating. Observe that there is at most one vertex not dominated by a distinguishing set. The location-domination number of $G$, denoted by $\lambda(G)$, is the minimum cardinality of a locating-dominating set. A locating-dominating set of cardinality $\lambda(G)$ is called an LD-code [12, 13]. Certainly, every LD-set of a non-connected graph $G$ is the union of LD-sets of its connected components and the location-domination number is the sum of the location-domination number of its connected components. Both, LD-codes and the location-domination parameter have been intensively studied during the last decade; see [1, 2, 3, 5, 6, 8, 9, 10]. A complete and regularly updated list of papers on locating-dominating codes is to be found in [11].

The complement of $G$, denoted by $\overline{G}$, has the same set of vertices of $G$ and two vertices are adjacent in $\overline{G}$ if and only if they are not adjacent in $G$. This work is devoted to approach the relationship between $\lambda(G)$ and $\lambda(\overline{G})$ for connected bipartite graphs.

It follows immediately from the definitions that a set $S \subseteq V$ is distinguishing in $G$ if and only if it is distinguishing in $\overline{G}$. A straightforward consequence of this fact are the following results.

Proposition 1 ([9]). Let $S \subseteq V$ be an LD-set of a graph $G = (V, E)$. Then, $S$ is an LD-set of $\overline{G}$ if and only if $S$ is a dominating set of $\overline{G}$;

Proposition 2 ([8]). Let $S \subseteq V$ be an LD-set of a graph $G = (V, E)$. Then, the following properties hold.

(a) There is at most one vertex $u \in V \setminus S$ such that $N(u) \cap S = S$, and in the case it exists, $S \cup \{u\}$ is an LD-set of $\overline{G}$.

(b) $S$ is an LD-set of $\overline{G}$ if and only if there is no vertex in $V \setminus S$ such that $N(u) \cap S = S$.

Theorem 1 ([8]). For every graph $G$, $|\lambda(G) - \lambda(\overline{G})| \leq 1$. 
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According to the preceding inequality, \( \lambda(G) \in \{ \lambda(G) - 1, \lambda(G), \lambda(G) + 1 \} \) for every graph \( G \), all cases being feasible for some connected graph \( G \). We intend to determine graphs such that \( \lambda(G) > \lambda(G) \), that is, we want to solve the equation \( \lambda(G) = \lambda(G) + 1 \). This problem was completely solved in \cite{9} for the family of block-cactus.

In this work, we carry out a similar study for bipartite graphs. For this purpose, we first introduce in Section 2 the graph associated with a distinguishing set. This graph turns out to be very helpful to derive some properties related to LD-sets and the location-domination number of \( G \), and will be used to get the main results in Section 3.

In Table 1, the location-domination number of some families of bipartite graphs are displayed, along with the location-domination number of its complement graphs. Concretely, we consider the path \( P_n \) of order \( n \geq 4 \); the cycle \( C_n \) of (even) order \( n \geq 4 \); the star \( K_{1,n-1} \) of order \( n \geq 4 \), obtained by joining a new vertex to \( n - 1 \) isolated vertices; the complete bipartite graph \( K_{r,n-r} \) of order \( n \geq 4 \), with \( 2 \leq r \leq n - r \) and stable sets of order \( r \) and \( n - r \), respectively; and finally, the bi-star \( K_2(r,s) \) of order \( n \geq 6 \) with \( 3 \leq r \leq s = n - r \), obtained by joining the central vertices of two stars \( K_{1,r-1} \) and \( K_{1,s-1} \) respectively.

**Proposition 3** (\cite{9}). Let \( G \) be a graph of order \( n \geq 4 \). If \( G \) is a graph belonging to one of the following classes: \( P_n, C_n, K_{1,n-1}, K_{r,n-r}, K_2(r,s) \), then the values of \( \lambda(G) \) and \( \lambda(G) \) are known and they are displayed in Table 1.

| \( G \) | \( P_n \) | \( P_n \) | \( C_n \) | \( C_n \) |
|-------|--------|--------|--------|--------|
| \( n \) | \( 4 \leq n \leq 6 \) | \( n \geq 7 \) | \( 4 \leq n \leq 6 \) | \( n \geq 7 \) |
| \( \lambda(G) \) | \( \lfloor \frac{2n}{5} \rfloor \) | \( \lfloor \frac{2n}{5} \rfloor \) | \( \lfloor \frac{2n}{5} \rfloor \) | \( \lfloor \frac{2n}{5} \rfloor \) |
| \( \lambda(G) \) | \( \lfloor \frac{2n}{5} \rfloor \) | \( \lfloor \frac{2n-2}{5} \rfloor \) | \( \lfloor \frac{2n}{5} \rfloor \) | \( \lfloor \frac{2n-2}{5} \rfloor \) |

| \( G \) | \( K_{1,n-1} \) | \( K_{r,n-r} \) | \( K_2(r,s) \) |
|-------|--------|--------|--------|
| \( n \) | \( n \geq 4 \) | \( 2 \leq r \leq n - r \) | \( 3 \leq r \leq s \) |
| \( \lambda(G) \) | \( n - 1 \) | \( n - 2 \) | \( n - 2 \) |
| \( \lambda(G) \) | \( n - 1 \) | \( n - 2 \) | \( n - 3 \) |

Table 1: The values of \( \lambda(G) \) and \( \lambda(G) \) for some families of bipartite graphs.

Notice that in all cases considered in Proposition 3, we have \( \lambda(G) \leq \lambda(G) \). Moreover, observe also that, for every pair of integers \((r,s)\) with \( 3 \leq r \leq s \), we have examples of bipartite graphs with stable sets of order \( r \) and \( s \) respectively, such that \( \lambda(G) = \lambda(G) \) and such that \( \lambda(G) = \lambda(G) - 1 \).
2 The graph associated with a distinguishing set

Let $S$ be a distinguishing set of a graph $G$. We introduce in this section a labeled graph associated with $S$ and study some general properties. Since LD-sets are distinguishing sets that are also dominating, this graph allows us to derive some properties related to LD-sets and the location-domination number of $G$.

**Definition 1.** Let $S$ be a distinguishing set of cardinality $k$ of a graph $G = (V, E)$ of order $n$. The so-called $S$-associated graph, denoted by $G^S$, is the edge-labeled graph defined as follows.

i) $V(G^S) = V \setminus S$;

ii) If $x, y \in V(G^S)$, then $xy \in E(G^S)$ if and only if the sets of neighbors of $x$ and $y$ in $S$ differ in exactly one vertex $u(x, y) \in S$;

iii) The label $\ell(xy)$ of edge $xy \in E(G^S)$ is the only vertex $u(x, y) \in S$ described in the preceding item.

![Figure 1: A graph $G$ (left) and the graph $G^S$ associated with the distinguishing set $S = \{1, 2, 3, 4, 5\}$ (right). The neighbors in $S$ of each vertex are those enclosed in brackets.](image)

Notice that if $xy \in E(G^S)$, $\ell(xy) = u \in S$ and $|N(x) \cap S| > |N(y) \cap S|$, then $N(x) \cap S = (N(y) \cap S) \cup \{u\}$. Therefore, we can represent the graph $G^S$ with the vertices lying on $|S| + 1 = k + 1$ levels, from bottom (level 0) to top (level $k$), in such a way that vertices with exactly $j$ neighbors in $S$ are at level $j$. For any $j \in \{0, 1, \ldots, k\}$ there are at most $\binom{k}{j}$ vertices at level $j$. So, there is at most one vertex at level $k$ and, if it is so, this vertex is adjacent to all vertices of $S$. There is at most one vertex at level 0 and, if it is so, this vertex has no neighbors in $S$. Notice that $S$ is an LD-set if and only if there is no vertex at level 0.
The vertices at level 1 are those with exactly one neighbor in $S$. See Figure 1 for an example of an LD-set-associated graph.

Next, we state some basic properties of the graph associated with a distinguishing set that will be used later.

**Proposition 4.** Let $S$ be a distinguishing set of $G = (V, E)$, $x, y \in V \setminus S$ and $u \in S$. Then,

1. $S$ is a distinguishing set of $\overline{G}$.
2. The associated graphs $G^S$ and $\overline{G}^S$ are equal.
3. The representation by levels of $\overline{G}^S$ is obtained by reversing bottom-top the representation of $G^S$.
4. $xy \in E(G^S)$ and $\ell(xy) = u$ if and only if $x$ and $y$ have the same neighborhood in $S \setminus \{u\}$ and (thus) they are not distinguished by $S \setminus \{u\}$.
5. If $xy \in E(G^S)$ and $\ell(xy) = u$, then $S \setminus \{u\}$ is not a distinguishing set.

![Figure 2](image_url)

Figure 2: $S = \{1, 2, 3\}$ is distinguishing, $S' = \{1, 2\}$ is not distinguishing and $G^S$ has no edges.

The converse of Proposition 4 (5) is not necessarily true. For example, consider the graph $G$ of order 6 displayed in Figure 2. By construction, $S = \{1, 2, 3\}$ is a distinguishing set. However, $S' = S \setminus \{3\} = \{1, 2\}$ is not a distinguishing set, because $N(3) \cap S' = N([12]) \cap S' = \{1, 2\}$, and the $S$-associated graph $G^S$ has no edge with label 3 (in fact, $G^S$ has no edges since the neighborhoods in $S$ of all vertices not in $S$ have the same size).

As a straight consequence of Proposition 4 (5), the following result is derived.

**Corollary 1.** Let $S$ be a distinguishing set of $G$ and let $S' \subseteq S$. Consider the subgraph $H^S_{S'}$ of $G^S$ induced by the edges with a label from $S'$. Then, all the vertices belonging to the same connected component in $H^S_{S'}$ have the same neighborhood in $S \setminus S'$, concretely, it is the neighborhood in $S$ of a vertex lying on the lowest level.
For example, consider the graph shown in Figure 1. If \( S' = \{1, 2\} \), then vertices of the same connected component in \( H_{S'} \) have the same neighborhood in \( S \setminus S' \). Concretely, the neighborhood of vertices \([1234], [234], [134] \) and \([34]\) in \( S \setminus \{1, 2\} \) is \( \{3, 4\} \); the neighborhood of vertices \([13]\) and \([3]\) in \( S \setminus \{1, 2\} \) is \( \{3\} \); and the neighborhood of vertices \([1245], [245]\) in \( S \setminus \{1, 2\} \) is \( \{4, 5\} \) (see Figure 3).

![Figure 3](image)

**Figure 3:** If \( S' = \{1, 2\} \), then \( H_{S'} \cong C_4 + 2K_2 \) has three components. Vertices of the same component in \( H_{S'} \) have the same neighborhood in \( S \setminus S' \).

**Proposition 5.** Let \( S \) be a distinguishing set of cardinality \( k \) of a connected graph \( G \) of order \( n \). Let \( G^S \) be its associated graph. Then, the following conditions hold.

1. \( |V(G^S)| = n - k \).
2. \( G^S \) is bipartite.
3. Incident edges of \( G^S \) have different labels.
4. Every cycle of \( G^S \) contains an even number of edges labeled \( v \), for all \( v \in S \).
5. Let \( \rho \) be a walk with no repeated edges in \( G^S \). If \( \rho \) contains an even number of edges labeled \( v \) for every \( v \in S \), then \( \rho \) is a closed walk.
6. If \( \rho = x_i x_{i+1} \ldots x_{i+h} \) is a path satisfying that vertex \( x_{i+h} \) lies at level \( i + h \), for any \( h \in \{0, 1, \ldots, h\} \), then

   a. the edges of \( \rho \) have different labels;
   b. for all \( j \in \{i+1, i+2, \ldots, i+h\} \), \( N(x_{j}) \cap S \) contains the vertex \( \ell(x_k x_{k+1}) \), for any \( k \in \{i, i+1, \ldots, j-1\} \).

**Proof.** (1) It is a direct consequence from the definition of \( G^S \).
(2) Take \( V_1 = \{ x \in V(G^S) : |N(x) \cap S| \text{ is odd} \} \) and \( V_2 = \{ x \in V(G^S) : |N(x) \cap S| \text{ is even} \} \). Then, \( V(G^S) = V_1 \cup V_2 \) and \( V_1 \cap V_2 = \emptyset \). Since \( |N(x) \cap S| - |N(y) \cap S| = 1 \) for any \( xy \in E(G^S) \), it is clear that the vertices \( x, y \) are not in the same subset \( V_i, i = 1, 2 \).

(3) Suppose that edges \( e_1 = xy \) and \( e_2 = yz \) have the same label \( l(e_1) = l(e_2) = v \). This means that \( N(x) \cap S \) and \( N(y) \cap S \) differ only in vertex \( v \), and \( N(y) \cap S \) and \( N(z) \cap S \) differ only in vertex \( v \). It is only possible if \( N(x) \cap S = N(z) \cap S \), implying that \( x = z \).

(4) Let \( \rho \) be a cycle such that \( E(\rho) = \{ x_0x_1, x_1x_2, \ldots, x_hx_0 \} \). The set of neighbors in \( S \) of two consecutive vertices differ exactly in one vertex. If we begin with \( N(x_0) \cap S \), then each time we add (remove) the vertex of the label of the corresponding edge, we have to remove (add) it later in order to obtain finally the same neighborhood, \( N(x_0) \cap S \). Therefore, \( \rho \) contains an even number of edges with label \( v \).

(5) Consider the vertices \( x_0, x_1, x_2, x_3, \ldots, x_{2k} \) of \( \rho \). In this case, \( N(x_{2k}) \cap S \) is obtained from \( N(x_0) \cap S \) by either adding or removing the labels of all the edges of the walk. As every label appears an even number of times, for each element \( v \in S \) we can match its appearances in pairs, and each pair means that we add and remove (or remove and add) it from the neighborhood in \( S \). Therefore, \( N(x_{2k}) \cap S = N(x_0) \cap S \), and hence \( x_0 = x_{2k} \).

(6) It straightly follows from the fact that \( N(x_j) \cap S = (N(x_{j-1}) \cap S) \cup \{ \ell(x_{j-1}x_j) \} \), for any \( j \in \{ i + 1, \ldots, i + h \} \).

In the study of distinguishing sets and LD-sets using its associated graph, a family of graphs is particularly useful, the cactus graph family. A block of a graph is a maximal connected subgraph with no cut vertices. A connected graph \( G \) is a cactus if all its blocks are either cycles or edges. Cactus are characterized as those connected graphs with no edge shared by two cycles.

**Lemma 1.** Let \( S \) be a distinguishing set of a graph \( G \) and \( \emptyset \neq S' \subseteq S \). Consider a subgraph \( H \) of \( G^S \) induced by a set of edges containing exactly two edges with label \( u \), for each \( u \in S' \subseteq S \). Then, all the connected components of \( H \) are cactus.

**Proof.** We prove that there is no edge lying on two different cycles of \( H \). Suppose, on the contrary, that there is an edge \( e_1 \) contained in two different cycles \( C_1 \) and \( C_2 \) of \( H \). Note that \( C_1 \) and \( C_2 \) are cycles of \( G^S \), since \( S' \subseteq S \). Hence, if the label of \( e_1 \) is \( u \in S' \subseteq S \), then by Proposition 5 both cycles \( C_1 \) and \( C_2 \) contain the other edge \( e_2 \) of \( H \) with label \( u \). Suppose that \( e_1 = x_1y_1 \) and \( e_2 = x_2y_2 \) and assume without loss of generality that there exist \( x_1 - x_2 \) and \( y_1 - y_2 \) paths in \( C_1 \) not containing edges \( e_1, e_2 \). Let \( P_1 \) and \( P'_1 \) denote respectively those paths (see Figure 4 a).

We have two possibilities for \( C_2 \): (i) there are \( x_1 - x_2 \) and \( y_1 - y_2 \) paths in \( C_2 \) not containing neither \( e_1 \) nor \( e_2 \). Let \( P_2 \) denote the \( x_1 - x_2 \) path in \( C_2 \) in that case (see Figure 4 b).
(ii) there are $x_1 - y_2$ and $y_1 - x_2$ paths in $C_2$ not containing neither $e_1$ nor $e_2$ (see Figure 4c).

In case (ii), the closed walk formed with the path $P_1$, $e_1$ and the $y_1 - x_2$ path in $C_2$ would contain a cycle with exactly one edge labeled with $u$, a contradiction (see Figure 4d).

In case (i), at least one the following cases hold: either the $x_1 - x_2$ paths in $C_1$ and in $C_2$, $P_1$ and $P_2$, are different, or the $y_1 - y_2$ paths in $C_1$ and in $C_2$ are different (otherwise, $C_1 = C_2$).

Assume that $P_1$ and $P_2$ are different. Let $z_1$ be the last vertex shared by $P_1$ and $P_2$ advancing from $x_1$ and let $z_2$ be the first vertex shared by $P_1$ and $P_2$ advancing from $z_1$ in $P_2$. Notice that $z_1 \neq z_2$. Take the cycle $C_3$ formed with the $z_1 - z_2$ paths in $P_1$ and $P_2$. Let $P_1^*$ and $P_2^*$ be respectively the $z_1 - z_2$ subpaths of $P_1$ and $P_2$ (see Figure 4c). We claim that the internal vertices of $P_2^*$ do not lie in $P_1^*$. Otherwise, consider the first vertex $t$ of $P_1^*$ lying also in $P_2^*$. The cycle beginning in $x_1$, formed by the edge $e_1$, the $y_1 - t$ path contained in $P_1^*$, the $t - z_1$ path contained in $P_2^*$ and the $z_1 - x_1$ path contained in $P_1$ has exactly one appearance of an edge with label $u$, which is a contradiction (see Figure 4f).

By Proposition 3 the labels of edges belonging to $P_1^*$ appear an even number of times in cycle $C_3$, but they also appear an even number of times in cycle $C_1$. But this is only possible if they appear exactly two times in $P_1^*$, since $H$ contains exactly two edges with the same label. By Proposition 3 $P_1^*$ must be a closed path, which is a contradiction.

Next, we establish a relation between some parameters of bipartite graphs having cactus as connected components. We denote by $cc(G)$ the number of connected components of a graph $G$.

**Lemma 2.** Let $H$ be a bipartite graph of order at least 4 such that all its connected components are cactus. Then, $|V(H)| \geq \frac{3}{4}|E(H)| + cc(H) \geq \frac{3}{4}|E(H)| + 1$. 
Proof. Let $\text{cy}(H)$ denote the number of cycles of $H$. Since $H$ is a planar graph with $\text{cy}(H) + 1$ faces and $\text{cc}(H)$ components, the equality follows from the generalization of Euler's Formula:

$$(\text{cy}(H) + 1) + |V(H)| = |E(H)| + (\text{cc}(H) + 1).$$

Let $\text{ex}(H) = |E(H)| - 4 \text{cy}(H)$. Then,

$$|V(H)| = |E(H)| - \text{cy}(H) + \text{cc}(H) = |E(H)| - \frac{1}{4}(|E(H) - \text{ex}(H)| + \text{cc}(H))$$

$$= \frac{3}{4}|E(H)| + \frac{1}{4}\text{ex}(H) + \text{cc}(H).$$

But $\text{cc}(H) \geq 1$, and $\text{ex}(H) \geq 0$ as all cycles of a bipartite graph have at least 4 edges. Thus,

$$|V(H)| = \frac{3}{4}|E(H)| + \frac{1}{4}\text{ex}(H) + \text{cc}(H) \geq \frac{3}{4}|E(H)| + \text{cc}(H) \geq \frac{3}{4}|E(H)| + 1.$$

The preceding result allows us to give a lower bound of the order of some subgraphs of the graph associated with a distinguishing set.

**Corollary 2.** Let $S$ be a distinguishing set of a graph $G$ and $\emptyset \neq S' \subseteq S$. Consider a subgraph $H$ of $G^S$ induced by a set of edges containing exactly two edges with label $u$ for each $u \in S' \subseteq S$. Let $r' = |S'|$. Then, $|V(H)| \geq \frac{3}{2}r' + 1$.

**Proof.** Since two edges of $G^S$ with the same label have no common endpoints, we have $|V(H)| \geq 4$, and the result follows by applying Lemmas 1 and 2 to $H$. $\square$

Next lemma states a property about the difference of the order and the number of connected components of a subgraph and will be used to prove the main result of this work.

**Lemma 3.** If $H$ is a subgraph of $G$, then $|V(G)| - \text{cc}(G) \geq |V(H)| - \text{cc}(H)$.

**Proof.** Since every subgraph of $G$ can be obtained by successively removing vertices and edges from $G$, it is enough to prove that the inequality holds whenever a vertex or an edge is removed from $G$.

Let $u \in V(G)$. If $u$ is an isolated vertex in $G$, then the order and the number of components decrease in exactly one unit when removing $u$ from $G$, so that the given inequality holds. If $u$ is a non-isolated vertex, then the order decreases in one unit while the number of components does not decrease when removing $u$ from $G$. Thus, the given inequality holds.

Now let $e \in E(G)$. Notice that the order does not change when removing an edge from $G$. If $e$ belongs to a cycle, then the number of components does not change when removing $e$ from $G$, and the given inequality holds. If $e$ does not belong to a cycle, then the number of components increases in exactly one unit when removing $e$ from $G$, and the given inequality holds. $\square$
3 The bipartite case

This section is devoted to solve the equation $\lambda(G) = \lambda(G) + 1$ when we restrict ourselves to bipartite graphs. In the sequel, $G = (V, E)$ stands for a bipartite connected graph of order $n = r + s \geq 4$, such that $V = U \cup W$, where $U$ and $W$ are the stable sets and

$$1 \leq |U| = r \leq s = |W|.$$ 

In the study of LD-sets, vertices with the same neighborhood play an important role, since at least one of them must be in an LD-set. We say that two vertices $u$ and $v$ are twins if either $N(u) = N(v)$ or $N(u) \cup \{u\} = N(v) \cup \{v\}$.

**Lemma 4.** Let $S$ be an LD-code of $G$. Then, $\lambda(G) \leq \lambda(G)$ if any of the following conditions hold.

i) $S \cap U \neq \emptyset$ and $S \cap W \neq \emptyset$.

ii) $r < s$ and $S = W$.

iii) $2^r \leq s$.

**Proof.** If $S$ satisfies item i), then the LD-code of $G$ is a distinguishing set of $G$ and it is dominating in $G$ because there is no vertex in $G$ with neighbors in both stable sets. Thus, $\lambda(G) \leq \lambda(G)$.

Next, assume that $r < s$ and $S = W$. In this case, $\lambda(G) = |W| > |U|$ and thus $U$ is not an LD-set, but it is a dominating set since $G$ is connected. Therefore, there exists a pair of vertices $w_1, w_2 \in W$ such that $N(w_1) = N(w_2)$. Hence, $W - \{w_1\}$ is an LD-set of $G - w_1$. Let $u \in U$ be a vertex adjacent to $w_1$ (it exists since $G$ is connected), and notice that $(W \setminus \{w_1\}) \cup \{u\}$ is an LD-code of $G$ with vertices in both stable sets, which, by the preceding item, means that $\lambda(G) \leq \lambda(G)$.

Finally, if $2^r \leq s$ then $S \neq U$, which means that $S$ satisfies either item i) or item ii). ☐

**Proposition 6.** If $G$ has order at least 4 and $1 \leq r \leq 2$, then $\lambda(G) \leq \lambda(G)$.

**Proof.** If $r = 1$, then $G$ is the star $K_{1,n-1}$ and $\lambda(G) = \lambda(G) = n - 1$.

Suppose that $r = 2$. We distinguish cases (see Figure 5).

- If $s \geq 2^2 = 4$ then, by Lemma 4, $\lambda(G) \leq \lambda(G)$.
- If $s = 2$, then $G$ is either $P_4$ and $\lambda(P_4) = \lambda(P_4) = 2$, or $G$ is $C_4$ and $\lambda(C_4) = \lambda(C_4) = 2$. 

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\[ \lambda(G) = \lambda(G) \]

\[ r = 1 \]

\[ K_{1,n-1} \]

\[ r = 2 \]

\[ s = 2 \]

\[ P_4 \]

\[ C_4 \]

\[ r = 2 \]

\[ s = 3 \]

\[ P_5 \]

\[ K_{2,3} \]

\[ K_{2}(2,3) \]

\[ P_5 \]

\[ P_4 \]

\[ C_4 \]

\[ K_{1,n-1} \]

\[ K_{2,3} \]

\[ P \]

Figure 5: Some bipartite graphs with \( 1 \leq r \leq 2 \).

- If \( s = 3 \), then \( G \) is either \( P_5, K_{2,3}, K_2(2,3) \), or the banner \( P \), and \( \lambda(P_5) = \lambda(P) = 2 \), \( \lambda(K_{2,3}) = \lambda(K_{2,3}) = 3 \), \( 2 = \lambda(K_2(2,3)) < \lambda(K_2(2,3)) = 3 \), and \( 2 = \lambda(P) < \lambda(P) = 3 \).

Since \( \lambda(K_2) = 1 \), \( \lambda(K_2) = 2 \), and \( \lambda(P) = \lambda(P) = 2 \), by Proposition 6 we have that \( K_2 \) is the only bipartite graph \( G \) satisfying \( \lambda(G) = \lambda(G) + 1 \), whenever \( r \in \{1, 2\} \). From now on, we assume that \( r \geq 3 \).

**Proposition 7.** If \( r = s \), then \( \lambda(G) \leq \lambda(G) \).

*Proof.* If \( G \) has an LD-code with vertices at both stable sets, then \( \lambda(G) \leq \lambda(G) \) by Lemma 4. In any other case, \( G \) has at most two LD-codes, \( U \) and \( W \).

If both \( U \) and \( W \) are LD-codes, then we distinguish the following cases.

- If there is no vertex \( u \in U \) such that \( N(u) = W \), then \( W \) is an LD-set of \( G \), and consequently, \( \lambda(G) \leq \lambda(G) \).

- Analogously, if there is no vertex \( w \in W \) such that \( N(w) = U \), then we derive \( \lambda(G) \leq \lambda(G) \).

- If there exist vertices \( u \in U \) and \( w \in W \) such that \( N(u) = W \) and \( N(w) = U \), then \( (U - \{u\}) \cup \{w\} \) would be an LD-set of \( G \), and thus \( \lambda(G) \leq \lambda(G) \).

Next, assume that \( U \) is an LD-code and \( W \) is not an LD-code of \( G \). If there is no vertex \( w \in W \) such that \( N(w) = U \), then \( U \) is an LD-set of \( G \), and so \( \lambda(G) \leq \lambda(G) \). Finally, suppose that there is a vertex \( w \in W \) such that \( N(w) = U \). Note that \( W \) is not a distinguishing set of \( G \) (otherwise, it would be an LD-code because \( W \) is a dominating set of size \( r \)). Therefore, there exist vertices \( x, y \in U \) such that \( N(x) = N(y) \). In such a case, \( (U \setminus \{x\}) \cup \{w\} \) is an LD-set of \( G \), and thus \( \lambda(G) \leq \lambda(G) \). \( \square \)
From Lemma 4 and Propositions 6 and 7 we derive the following result.

**Corollary 3.** If \( \lambda(\overline{G}) = \lambda(G) + 1 \), then \( 3 \leq r < s \leq 2^r - 1 \) and \( U \) is the only LD-code of \( G \).

Next theorem characterizes connected bipartite graphs satisfying the equation \( \lambda(\overline{G}) = \lambda(G) + 1 \) in terms of the graph associated with a distinguishing set.

**Theorem 2.** Let \( 3 \leq r < s \). Then, \( \lambda(\overline{G}) = \lambda(G) + 1 \) if and only if the following conditions hold:

1) \( W \) has no twins.

2) There exists a vertex \( w \in W \) such that \( N(w) = U \).

3) For every vertex \( u \in U \), the graph \( G^U \) has at least two edges with label \( u \).

**Proof.** \( \Leftarrow \) Condition i) implies that \( U \) is a distinguishing set. Moreover, \( U \) is an LD-set of \( G \), because \( G \) is connected. Hence, \( \lambda(G) \leq r \). Let \( S \) be an LD-code of \( \overline{G} \). We next prove that \( S \) has at least \( r + 1 \) vertices. Condition ii) implies that \( U \) is not a dominating set in \( \overline{G} \), thus \( S \neq U \). If \( U \subseteq S \), then \( |S| \geq |U| + 1 = r + 1 \) and we are done. If \( U \setminus S \neq \emptyset \), consider the graph \( G^U \) associated with \( U \). Let \( H_{U \setminus S} \) be the subgraph of \( G^U \) induced by the set of edges with a label in \( U \setminus S \neq \emptyset \). By Corollary 1, the vertices of a same connected component in \( H_{U \setminus S} \) have the same neighborhood in \( U \cap S \). Besides, \( W \) induces a complete graph in \( \overline{G} \). Hence, \( S \cap W \) must contain at least all but one vertex from every connected component of \( H_{U \setminus S} \), otherwise \( \overline{G} \) would contain vertices with the same neighborhood in \( S \). Therefore, \( S \cap W \) has at least \( |V(H_{U \setminus S})| - \text{cc}(H_{U \setminus S}) \) vertices.

Condition iii) implies that there are at least two edges with label \( u \), for every \( u \in U \setminus S \). Let \( H \) be a subgraph of \( H_{U \setminus S} \) induced by a set containing exactly two edges with label \( u \) for every \( u \in U \setminus S \). Since \( U \setminus S \neq \emptyset \), the subgraph \( H \) has at least two edges. By Proposition 6 (3), edges with the same label in \( G^U \) have no common endpoint, thus we have \( |V(H)| \geq 4 \). Hence, by applying Lemmas 1 and 2 we derive

\[
|V(H)| - \text{cc}(H) \geq \frac{3}{4} |E(H)| = \frac{3}{2} |U \setminus S|.
\]

Since \( H \) is a subgraph of \( H_{U \setminus S} \), Lemma 3 applies. Therefore

\[
|S| = |S \cap U| + |S \cap W| \\
\quad \geq |S \cap U| + |V(H_{U \setminus S})| - \text{cc}(H_{U \setminus S}) \\
\quad \geq |S \cap U| + |V(H)| - \text{cc}(H) \\
\quad \geq |S \cap U| + \frac{3}{2} |U \setminus S| \\
\quad = |U| + \frac{1}{2} |U \setminus S| > |U| = r.
\]
\( \Rightarrow \) By Corollary 3, \( U \) is the only LD-code of \( G \) and hence, \( U \) is not an LD-set of \( \overline{G} \). Therefore, \( W \) has no twins and \( N(w_0) = U \) for some \( w_0 \in W \). It only remains to prove that condition iii) holds. Suppose on the contrary that there is at most one edge in \( G^U \) with label \( u \) for some \( u \in U \). We consider two cases.

If there is no edge with label \( u \), then by Proposition 4, \( U \setminus \{u\} \) distinguishes all pairs of vertices of \( W \) in \( \overline{G} \). Let \( S = (U \setminus \{u\}) \cup \{w_0\} \). We claim that \( S \) is an LD-set of \( \overline{G} \). Indeed, \( S \) is a dominating set in \( \overline{G} \), because in this graph \( u \) is adjacent to any vertex in \( U \setminus \{u\} \) and vertices in \( W \setminus \{w_0\} \) are adjacent to \( w_0 \). It only remains to prove that \( S \) distinguishes the pairs of vertices of the form \( u \) and \( v \), when \( v \in W \setminus \{w_0\} \). But \( w_0 \in N_{\overline{G}}(v) \cap S \) and \( w_0 \notin N_{\overline{G}}(u) \cap S \). Thus, \( S \) is an LD-set of \( \overline{G} \), implying that \( \lambda(\overline{G}) \leq |S| = |U| = \lambda(G) \), a contradiction.

If there is exactly one edge \( xy \) with label \( u \), then only one of the vertices \( x \) or \( y \) is adjacent to \( u \) in \( G \). Assume that \( ux \in E(G) \). Recall that \( x, y \in W \). By Proposition 4, \( U \setminus \{u\} \) distinguishes all pairs of vertices of \( W \), except the pair \( x \) and \( y \), in \( \overline{G} \). Let \( S = (U \setminus \{u\}) \cup \{x\} \). We claim that \( S \) is an LD-set of \( \overline{G} \). Indeed, \( S \) is a dominating set in \( \overline{G} \), because \( u \) is adjacent to any vertex in \( U \setminus \{u\} \) and vertices in \( W \setminus \{x\} \) are adjacent to \( x \). It only remains to prove that \( S \) distinguishes the pairs of vertices of the form \( u \) and \( v \), when \( v \in W \setminus \{x\} \). But \( x \in N_{\overline{G}}(v) \cap S \) and \( x \notin N_{\overline{G}}(u) \cap S \). Thus, \( S \) is an LD-set of \( \overline{G} \), implying that \( \lambda(\overline{G}) \leq |S| = |U| = \lambda(G) \), a contradiction. \( \square \)

Observe that condition iii) of Theorem 2 is equivalent to the existence of at least two pairs of twins in \( G - u \), for every vertex \( u \in U \). Therefore, it can be stated as follows.

**Theorem 3.** Let \( 3 \leq r < s \). Then, \( \lambda(\overline{G}) = \lambda(G) + 1 \) if and only if the following conditions hold:

i) \( W \) has no twins.

ii) There exists a vertex \( w \in W \) such that \( N(w) = U \).

iii) For every vertex \( u \in U \), the graph \( G - u \) has at least two pairs of twins in \( W \).

We already know that it is not possible to have \( \lambda(\overline{G}) = \lambda(G) + 1 \) when \( s \geq 2^r \). However, the condition \( s \leq 2^r - 1 \) is not sufficient to ensure the existence of bipartite graphs satisfying \( \lambda(\overline{G}) = \lambda(G) + 1 \). We next show that there are graphs satisfying this equation if and only if \( \frac{3r}{2} + 1 \leq s \leq 2^r - 1 \).

**Proposition 8.** If \( r \geq 3 \) and \( \lambda(\overline{G}) = \lambda(G) + 1 \), then \( \frac{3r}{2} + 1 \leq s \leq 2^r - 1 \).

**Proof.** If \( r \geq 3 \) and \( \lambda(\overline{G}) = \lambda(G) + 1 \), then by Corollary 3, we have that \( s \leq 2^r - 1 \) and \( U \) is the only LD-code of \( G \). Moreover, since \( G \) satisfies Condition iii) of Theorem 2
the $U$-associated graph $G^U$ contains a subgraph $H$ with exactly two edges labeled with $u$, for every $u \in U$. Recall that $V(H) \subseteq V(G) \setminus U = W$. Hence, by Corollary 2 we have $s = |W| \geq |V(H)| \geq \frac{3r}{2} + 1$.

**Proposition 9.** For every pair of integers $r$ and $s$ such that $3 \leq r$ and $\frac{3r}{2} + 1 \leq s \leq 2^r - 1$, there exists a bipartite graph $G(r, s)$ such that $\lambda(G) = \lambda(G) + 1$.

**Proof.** Let $s = \left\lceil \frac{3r}{2} + 1 \right\rceil$. Let $[r] = \{1, 2, \ldots, r\}$ and let $\mathcal{P}([r])$ denote the power set of $[r]$. Take the bipartite graph $G(r, \left\lceil \frac{3r}{2} + 1 \right\rceil)$ such that $V = U \cup W$, $U = [r]$, and $W \subseteq \mathcal{P}([r]) \setminus \{\emptyset\}$ is defined as follows (see Figure 6). For $r = 2k$ even:

$$W = \left\{ [r] \right\} \cup \left\{ [r] \setminus \{i\} : i \in [r] \right\} \cup \left\{ [r] \setminus \{2i - 1, 2i\} : 1 \leq i \leq k \right\}$$

and for $r = 2k + 1$ odd:

$$W = \left\{ [r] \right\} \cup \left\{ [r] \setminus \{i\} : i \in [r] \right\} \cup \left\{ [r] \setminus \{2i - 1, 2i\} : 1 \leq i \leq k \right\} \cup \left\{ [r] \setminus \{r - 1, r\} \right\}.$$

![Figure 6: The labeled graph $G^U$, for $G = G(r, \left\lceil \frac{3r}{2} + 1 \right\rceil)$ and $U = \{1, \ldots, r\}$.](image)

The edges of $G(r, \left\lceil \frac{3r}{2} + 1 \right\rceil)$ are defined as follows. If $u \in U = [r]$ and $w \in W$, then $u$ and $w$ are adjacent if and only if $u \in w$.

By construction, $W$ has no twins, there is a vertex $w$ such that $N(w) = U$ and the $U$-associated graph, $G^U$, has at least two edges with label $u$ for every $u \in U$. Hence, $\lambda(G^U) = \lambda(G) + 1$ by Theorem 2.

Finally, if $\left\lceil \frac{3r}{2} + 1 \right\rceil < s \leq 2^r - 1 = |\mathcal{P}([r]) \setminus \{\emptyset\}|$, consider a set $W'$ obtained by adding $s - \left\lceil \frac{3r}{2} + 1 \right\rceil$ different subsets from $\mathcal{P}([r]) \setminus (W \cup \{\emptyset\})$ to the set $W$. Take the bipartite graph $G'$ having $U \cup W'$ as set of vertices and edges defined as before, i.e., for every $u \in U$ and $w \in W'$, $uw \in E(G')$ if and only if $u \in w$. Then, $|W'| = s$ and, by construction, $W'$ has no twins. Moreover, the vertex $[r] \in W \subseteq W'$ satisfies $N([r]) = U$ and, since $W \subseteq W'$, the $U$-associated graph $(G')^U$ has at least two edges with label $u$ for every $u \in U$. By Theorem 2, $\lambda(G') = \lambda(G') + 1$, and the proof is completed. □
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