Integral representations of $q$-analogues of the Hurwitz zeta function

Masato Wakayama and Yoshinori Yamasaki

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Abstract

Two integral representations of $q$-analogues of the Hurwitz zeta function are established. Each integral representation allows us to obtain an analytic continuation including also a full description of poles and special values at non-positive integers of the $q$-analogue of the Hurwitz zeta function, and to study the classical limit of this $q$-analogue. All the discussion developed here is entirely different from the previous work in [4].

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1 Introduction

By the integral expression of the gamma function $\Gamma(s)$, Hurwitz’s zeta function $\zeta(s, z) := \sum_{n=0}^{\infty} (n + z)^{-s}$ is obtained by the Mellin transform of the generating function $G(t, z)$ of Bernoulli polynomials $B_m(z)$ (see, e.g. [12]):

$$\zeta(s, z) = \frac{1}{\Gamma(s)} \int_0^{\infty} t^{s-1} G(t, z) \frac{dt}{t} \quad (\text{Re} (s) > 1),$$

(1.1)

$$G(t, z) := \frac{t e^{(1-z)t}}{e^t - 1} = \sum_{m=0}^{\infty} (-1)^m B_m(z) \frac{t^m}{m!} \quad (|t| < 2\pi).$$

(1.2)

For any $0 < a \leq +\infty$ and $0 < \varepsilon < \min\{a, 2\pi\}$, $\zeta(s, z)$ is represented also as

$$\zeta(s, z) = \frac{\Gamma(1-s)}{2\pi\sqrt{-1}} \int_{C(\varepsilon,a)} (-t)^{s-1} e^{(1-z)t} \frac{dt}{t} + \frac{1}{\Gamma(s)} \int_a^{\infty} t^{s-1} G(t, z) \frac{dt}{t},$$

(1.3)

where $C(\varepsilon,a)$ is a contour along the real axis from $a$ to $\varepsilon$, counterclockwise around the circle of radius $\varepsilon$ with center at the origin, and then along the real axis from $\varepsilon$ to $a$. This integral representation is straightforward from the idea due to Riemann in 1859. We need this kind of segmentation at $t = a$ for handling delicate relations presented among several $q$-series in the discussion. Since the contour integral defines an entire function, (1.3) provides a meromorphic
continuation of $\zeta(s, z)$. Moreover, by the residue theorem, one shows that $\zeta(s, z)$ has a simple pole at $s = 1$ with residue $B_0(z) = 1$ and $\zeta(1 - m, z) = -B_m(z)/m \ (m \in \mathbb{N})$. Furthermore, if we take $a = +\infty$, by the residue theorem again, the contour integral (1.3) yields the functional equation (= the symmetric invariance for $s \leftrightarrow 1 - s$) of the Riemann zeta function $\zeta(s)$ [8].

Let $0 < q < 1$ and $[z]_q := (1-q^z)/(1-q)$ for $z \in \mathbb{C}$. The following (Dirichlet-type) $q$-series has been studied in [4].

$$\zeta_q(s, t, z) := \sum_{n=0}^{\infty} \frac{q^{(n+z)t}}{(n+z)q^s} \quad (\text{Re} \ (t) > 0).$$

We put $\zeta_q^{(\nu)}(s, z) := \zeta_q(s, s - \nu, z)$ for $\nu \in \mathbb{N}$. The meromorphic continuation of $\zeta_q^{(\nu)}(s, z)$ was obtained in two ways; one is by the binomial expansion, while the other is by the Euler-Maclaurin summation formula. Though the expression obtained by the binomial expansion has much advantage for describing the location of poles and special values at non-positive integers, it is difficult to determine whether $\zeta_q^{(\nu)}(s, z)$ can give a proper $q$-analogue of $\zeta(s, z)$, i.e., if $\lim_{q \uparrow 1} \zeta_q^{(\nu)}(s, z) = \zeta(s, z)$ holds for any $s \in \mathbb{C}$. Actually, the proof of the main assertion in [4] which characterizes such proper $q$-analogues among the family of the functions $\zeta_q(s, \varphi(s), z)$, $\varphi(s)$ being a meromorphic function, could be achieved when employing the Euler-Maclaurin formula together with a careful piece of analysis.

The aim of the present paper is then, in contrast to the previous work, to study integral representations of $\zeta_q^{(\nu)}(s, z)$ which are considered respectively as analogues of (1.1) and (1.3). Each integral representation allows us not only to see the aforementioned facts for $\zeta_q^{(\nu)}(s, z)$ concerning the poles and special values but also to prove that $\zeta_q^{(\nu)}(s, z)$ realizes a proper $q$-analogue of $\zeta(s, z)$. During the course of a discussion for obtaining such integral representation, the Poisson summation formula plays an important role.

We remark that the works in [10, 11] have treated an integral representation of another $q$-analogue of Hurwitz’s zeta function (Mellin’s transform). The present study is, however, different from those in [10, 11] in the sense that our $q$-analogue of the zeta function is given by exactly a Dirichlet-type $q$-series, whereas the $q$-analogue in [10, 11] has some extra term. Precisely, see Corollary 2.4 [4].

The plan of the paper is as follows: In Section 2, we introduce functions $F^{(\nu)}_{q, \varepsilon}(t, z)$ for $\varepsilon \in \{+, -, 0\}$ and show that $\zeta_q^{(\nu)}(s, z)$ is expressed as the Mellin transform of $F^{(\nu)}_{q, +}(t, z)$ (Proposition 2.3). In Section 3, introducing a $q$-analogue $B^{(\nu)}_m(z; q)$ of the Bernoulli polynomial $B_m(z)$, we study relations between $F^{(\nu)}_{q, -}(t, z)$ and the generating function $G^{(\nu)}_q(t, z)$ of $B^{(\nu)}_m(z; q)$. Moreover, we show that $B^{(\nu)}_m(z; q) \rightarrow B_m(z)$ as $q \uparrow 1$ for each $\nu$ (Theorem 3.1). We then establish two integral representations of $\zeta_q^{(\nu)}(s, z)$ (Theorem 3.6), and as a corollary, we show $\lim_{q \uparrow 1} \zeta_q^{(\nu)}(s, z) = \zeta(s, z) \ (s \in \mathbb{C})$. The proof of these theorems and corollary are based on the result for $F^{(\nu)}_{q, 0}(t, z)$ (Proposition 2.4) obtained by the Poisson summation formula. In the final section, we introduce some two-variable function $Z_q(s, t)$ by an infinite product. If $s$ equals a positive integer $m$, $Z_q(m, t)$ coincides with the inverse of Appell’s $\mathcal{O}$-function (= the multiple elliptic gamma function). We show that the logarithmic derivative of $Z_q(s, t)$ gives $\zeta_q(s, t) := \zeta_q(s, t, 1)$ and obtain recurrence equations among $\zeta_q^{(\nu)}(s) := \zeta_q^{(\nu)}(s, 1)$.
Throughout the paper, we assume $0 < q < 1$. The number $\nu$ always represents a positive integer.

## 2 An integral expression of $\zeta_q^{(\nu)}(s, z)$

To obtain an integral expression of $\zeta_q^{(\nu)}(s, z)$ as \(13\) for $\zeta(s, z)$, we study functions $F_{q, \pm}(t, z)$ defined as

\[
F_{q, +}(t, z) := t^\nu \sum_{n=0}^{\infty} q^{-\nu(n+z)} \exp(-tq^{-(n+z)}[n + z]_q)
= t^\nu \sum_{n=0}^{\infty} q^{-\nu(n+z)} \exp(t[-(n + z)]_q), \tag{2.1}
\]

\[
F_{q, -}(t, z) := t^\nu \sum_{n=-\infty}^{1} q^{-\nu(n+z)} \exp(-tq^{-(n+z)}[n + z]_q)
= t^\nu \sum_{n=1}^{\infty} q^{\nu(n-z)} \exp(t[n - z]_q), \tag{2.2}
\]

when $z \in D_q := \{ z \in \mathbb{C} \mid |\text{Im}(z)| < \frac{\pi}{2} \frac{1}{\log q} \}$. We first note the

\textbf{Lemma 2.1.} (i) Let $z \in D_q$. Put $R_q(z) := \{ t \in \mathbb{C} \mid |\text{arg}(t) - (\text{Im}(z)) \log q| < \pi/2 \}$. Here we assume $-\pi \leq \text{arg}(t) < \pi$. Then the function $F_{q, +}(t, z)$ is holomorphic in $R_q(z)$.

(ii) The function $F_{q, -}(t, z)$ is entire.

(iii) The functions $F_{q, \pm}(t, z)$ satisfy respectively

\[
F_{q, \pm}(qt, z) = e^{-t}(F_{q, \pm}(t, z) \pm t^\nu q^{\nu(1-z)}e^{t[1-z]_q}). \tag{2.3}
\]

\textbf{Proof.} Since the condition $t \in R_q(z)$ implies $\text{Re} \left( t \exp(-\sqrt{-1} \text{Im}(z) \log q) \right) > 0$, it is easy to see that the series $F_{q, +}(t, z)$ converges absolutely whenever $t \in R_q(z)$. Hence we have (i). Since the function $[n-z]_q$ is bounded for $n \geq 0$, the series $F_{q, -}(t, z)$ converges absolutely for all $t \in \mathbb{C}$, whence the assertion (ii) follows. The functional equations (2.3) are straightforward. \hfill \Box

Note that $R_q(z) \supset \mathbb{R}_{>0}$, the positive real axis, for any $z \in D_q$.

\textbf{Lemma 2.2.} (i) For $a > 0$ and $w > 0$, $w^a e^{-w} \leq (ae^{-1})^a$ holds.

(ii) Put $z = x + \sqrt{-1} y \in D_q$ ($x, y \in \mathbb{R}$) and $\beta_y := \cos(y \log q)$. For $t > 0$, we have

\[
|F_{q, +}^{(\nu)}(t, z)| \leq \exp(-t \frac{\beta_y q^{-x} - 1}{1 - q}) \left( t^\nu q^{\nu x} + (\frac{qe^{-1}}{\beta_y})^\nu \frac{1}{1 - \exp(-t \beta_y q^{-x})} \right). \tag{2.4}
\]

Further, suppose $z \in J_q := \{ z = x + \sqrt{-1} y \in D_q \mid x > 0, \ q^{-x} \cos(y \log q) > 1 \}$. Then the function $t^\alpha F_{q, +}^{(\nu)}(t, z)$ is an integrable function on $[0, \infty)$ provided $\text{Re}(\alpha) > 0$. 

Proof. The inequality (i) is obvious. Using the relation \([n + z]_q = 1 + q + \cdots + q^{n-1} + q^n [z]_q\), we have

\[
F_{q,+}^{(\nu)}(t, z) = \exp(-tq^{-z}[z]_q) \left\{ t\nu q^{-\nu z} + \sum_{n=1}^{\infty} (tq^{-n+z})^\nu \exp(-tq^{-(n+z)} \prod_{j=1}^{n-1} \exp(-tq^{-z-j})) \right\}.
\]

Note that \(\text{Re} \ (q^{-z}) = \beta y q^{-x}\). Since \(q^{-x-j} \geq q^{-x} (1 \leq j \leq n-1)\), by (i) with \(a = \nu\) and \(w = t\beta y q^{-(n+x)}\), we get

\[
|F_{q,+}^{(\nu)}(t, z)| \leq \exp\left(-t \frac{\beta y q^{-x} - 1}{1 - q}\right) \left\{ t\nu q^{-\nu x} + \left(\frac{\nu e^{-1}}{\beta y}\right) \sum_{n=1}^{\infty} \exp(-t\beta y q^{-x}(n-1)) \right\}
\]

This shows (2.4). The rest of the assertion follows immediately from (2.4). \(\square\)

Estimate (2.4) verifies the expression of \(\zeta_q^{(\nu)}(s, z)\) by the Mellin transform of \(F_{q,+}^{(\nu)}(t, z)\).

**Lemma 2.3.** Retain the notation in Lemma 2.2. Suppose \(z \in J_q\). Then

\[
\zeta_q^{(\nu)}(s, z) = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-\nu} F_{q,+}^{(\nu)}(t, z) \frac{dt}{t} \quad (\text{Re} \ (s) > \nu + 1). \tag{2.5}
\]

Proof. Since \(\text{Re} \ (s) > \nu + 1\), \(t^{s-\nu-1} F_{q,+}^{(\nu)}(t, z)\) is integrable on \([0, \infty)\) by Lemma 2.2. Hence we have

\[
\int_0^\infty t^{s-\nu} F_{q,+}^{(\nu)}(t, z) \frac{dt}{t} = \int_0^\infty t^{s} \sum_{n=0}^{\infty} q^{-\nu(n+z)} \exp(-tq^{-(n+z)}[n + z]_q) \frac{dt}{t}
\]

\[
= \sum_{n=0}^{\infty} \frac{q^{-\nu(n+z)}}{(q^{-(n+z)}[n + z]_q)^s} \int_0^\infty w^s e^{-w} \frac{dw}{w} \quad (w = tq^{-(n+z)}[n + z]_q)
\]

\[
= \Gamma(s) \zeta_q^{(\nu)}(s, z).
\]

The change of order of the integration and summation is legitimate by the Lebesgue convergence theorem. Hence the lemma follows. \(\square\)

Define the function \(F_{q,0}^{(\nu)}(t, z)\) by

\[
F_{q,0}^{(\nu)}(t, z) = F_{q,+}^{(\nu)}(t, z) + F_{q,-}^{(\nu)}(t, z)
\]

\[
= t\nu \sum_{n \in \mathbb{Z}} q^{-\nu(n+z)} \exp(-tq^{-(n+z)}[n + z]_q) \quad (t \in R_q(z)). \tag{2.6}
\]

It is clear that \(F_{q,0}^{(\nu)}(t, z)\) is periodic, that is, \(F_{q,0}^{(\nu)}(t, z) = F_{q,0}^{(\nu)}(t, z+1)\). The following proposition is the key of our analysis and gives just the Fourier expansion of \(F_{q,0}^{(\nu)}(t, z)\).
Proposition 2.4. Let \( z \in D_q \). For \( t \in R_q(z) \), we have
\[
F^{(\nu)}_{q,0}(t, z) = -\frac{(1 - q)^{\nu}}{\log q} e^{\frac{t}{1-q}} \sum_{m \in \mathbb{Z}} \left( \frac{1 - q}{t} \right)^{m \delta_q} \Gamma(\nu + m \delta_q) e^{2\pi \sqrt{-1} mz},
\]
(2.7)
where \( \delta_q := 2\pi \sqrt{-1}/\log q \).

Proof. Let \( f^{(\nu)}_q(\eta, z) := q^{-(\nu+\eta z)} \exp(-tq^{-(\nu+\eta z)}[\eta + z]) \). Put \( a = t/(1 - q) \). Then the Fourier transform \( \hat{f}^{(\nu)}_q(\xi, z) \) of \( f^{(\nu)}_q(\eta, z) \) is calculated as
\[
\hat{f}^{(\nu)}_q(\xi, z) = \int_{-\infty}^{\infty} f^{(\nu)}_q(\eta, z) e^{-2\pi \sqrt{-1} \eta \xi} d\eta
\]
\[
= -\frac{e^{a + 2\pi \sqrt{-1} \xi z}}{\log q} \int_{-\infty}^{\infty} e^{-w} e^{-ae^{-w}} e^{-\delta_q \xi w} dw \quad (w = (\eta + z) \log q)
\]
\[
= -\frac{e^{a + 2\pi \sqrt{-1} \xi z}}{\log q} \int_{0}^{\infty} \alpha^\nu e^{-a\alpha} \alpha^{\delta_q \xi} d\alpha \quad (\alpha = e^{-w})
\]
\[
= -\frac{e^{a + 2\pi \sqrt{-1} \xi z}}{\log q} \left( \frac{1 - q}{t} \right)^{\nu + \delta_q \xi} \Gamma(\nu + \delta_q \xi).
\]
By the Poisson summation formula, we have
\[
F^{(\nu)}_{q,0}(t, z) = t^\nu \sum_{n \in \mathbb{Z}} f^{(\nu)}_q(n, z) = t^\nu \sum_{m \in \mathbb{Z}} \hat{f}^{(\nu)}_q(m, z).
\]
Hence the desired formula follows. \( \square \)

Let \( 0 < a < +\infty \) and \( m \in \mathbb{Z} \). To obtain an analytic continuation of \( \zeta^{(\nu)}_q(s, z) \) as a function of \( s \) via (2.5), it is useful to define the function \( \varphi^{(\nu)}_m(s; a, q) \) as
\[
\varphi^{(\nu)}_m(s; a, q) := \int_{0}^{a} t^{s-\nu - 1 - m \delta_q} e^{\frac{t}{1-q}} dt.
\]
(2.8)
Noting \( |\varphi^{(\nu)}_m(s; a, q)| \leq \varphi^{(\nu)}_0(\text{Re}(s); a, q) \), we see that \( \varphi^{(\nu)}_m(s; a, q) \) is holomorphic in \( \text{Re}(s) > \nu \).

Proposition 2.5. The function \( \varphi^{(\nu)}_m(s; a, q) \) admits a meromorphic continuation to the whole plane. It has simple poles at \( s = n + m \delta_q \) \( (n \in \mathbb{Z} \leq \nu) \) with
\[
\text{Res}_{s=n+m \delta_q} \varphi^{(\nu)}_m(s; a, q) = \frac{1}{(\nu - n)! (1 - q)^{-n + \nu}}.
\]
(2.9)

Proof. Assume \( \text{Re}(s) > \nu \). For \( l = -n \in \mathbb{Z}_{\geq -\nu} \), integration by parts yields
\[
\varphi^{(\nu)}_m(s; a, q) = \sum_{j=1}^{l+\nu+1} \frac{1}{(s - \nu - m \delta_q)^j} \frac{(-1)^j}{(1 - q)^{j-1}} a^{s-\nu - m \delta_q + j - 1} e^{a \frac{t}{1-q}}
\]
\[
+ \frac{1}{(s - \nu - m \delta_q)^{l+\nu+1}} \frac{(-1)^{l+\nu+1}}{(1 - q)^{l+\nu+1}} \varphi^{(\nu)}_m(s + l + \nu + 1; a, q).
\]
Here \( (s)_k := s(s+1) \cdots (s+k-1) \) for \( k \in \mathbb{N} \). This gives an analytic continuation of \( \varphi^{(\nu)}_m(s; a, q) \) to the region \( \text{Re}(s) > -l - 1 \). Moreover, since
\[
\varphi^{(\nu)}_m(s + l + \nu + 1; a, q) \bigg|_{s = -l + m \delta_q} = \int_{0}^{a} e^{\frac{t}{1-q}} dt = (1 - q) (e^{\frac{a}{1-q}} - 1),
\]
we have (2.9). \( \square \)
3 q-Analogue of Bernoulli polynomials and the main theorem

First we study a q-analogue $B^{(\nu)}_m(z; q)$ of the Bernoulli polynomial $B_m(z)$. Consider the q-difference equation

\[
\begin{cases}
G_q^{(\nu)}(0, z) = -\frac{(1-q)^\nu}{\log q} \nu! , \\
G_q^{(\nu)}(qt, z) = e^{-t}(G_q^{(\nu)}(t, z) + t^\nu q^{\nu(1-z)} e^{t(1-z)q}).
\end{cases}
\tag{3.1}
\]

Lemma 3.1. A solution of (3.1) which is continuous at $t = 0$ is (uniquely) given by

\[
G_q^{(\nu)}(t, z) = -\frac{(1-q)^\nu}{\log q} (\nu - 1)! e^{t(1-q)} - F_{q,-}^{(\nu)}(t, z).
\tag{3.2}
\]

In particular, $G_q^{(\nu)}(t, z)$ is holomorphic at $t = 0$.

Proof. Put $H_q^{(\nu)}(t, z) := e^{-t/1-q} G_q^{(\nu)}(t, z)$. Then $H_q^{(\nu)}(t, z)$ satisfies

\[
\begin{cases}
H_q^{(\nu)}(0, z) = -\frac{(1-q)^\nu}{\log q} \nu! , \\
H_q^{(\nu)}(qt, z) = H_q^{(\nu)}(t, z) + t^\nu q^{\nu(1-z)} \exp \left(-\frac{q^{1-z}}{1-q} t\right).
\end{cases}
\tag{3.3}
\]

It follows that

\[
H_q^{(\nu)}(q^m t, z) - H_q^{(\nu)}(t, z) = t^\nu \sum_{n=1}^m q^{\nu(n-z)} \exp \left(-\frac{q^{n-z}}{1-q} t\right) \quad (m \in \mathbb{N}).
\]

Since the function $H_q^{(\nu)}(t, z)$ is continuous at $t = 0$, letting $m \to \infty$, we have

\[
H_q^{(\nu)}(t, z) = -\frac{(1-q)^\nu}{\log q} \nu! - t^\nu \sum_{n=1}^\infty q^{\nu(n-z)} \exp \left(-\frac{q^{n-z}}{1-q} t\right).
\tag{3.4}
\]

This proves (3.2). \hfill \Box

Remark 3.2. We give another proof of (3.2) using the functional equations (2.3) and (3.1). Let $\psi_q^{(\nu)}(t, z) := G_q^{(\nu)}(t, z) + F_{q,-}^{(\nu)}(t, z)$. Since $F_{q,-}^{(\nu)}(0, z) = 0$, we have $\psi_q^{(\nu)}(0, z) = G_q^{(\nu)}(0, z)$.

Also, since $\psi_q^{(\nu)}(qt, z) = e^{-t} \psi_q^{(\nu)}(t, z)$, we see that $\psi_q^{(\nu)}(q^m t, z) = e^{-|m|q^t} \psi_q^{(\nu)}(t, z)$ for $m \in \mathbb{N}$. Hence by the continuity of $\psi_q^{(\nu)}(t, z)$ at $t = 0$, we have $\psi_q^{(\nu)}(t, z) = \lim_{m \to \infty} e^{-|m|q^t} \psi_q^{(\nu)}(q^m t, z) = e^{-t} \psi_q^{(\nu)}(0, z)$. This shows (3.2).

By the Taylor expansions of the exponential functions in (3.2) and (3.4), we may write

\[
G_q^{(\nu)}(t, z) = \sum_{m=0}^\infty (-1)^m \tilde{B}^{(\nu)}_m(z; q) \frac{t^m}{m!} \quad (t \in \mathbb{C}),
\tag{3.5}
\]
i.e., the series expansion has infinite radius of convergence at \( t = 0 \) when \( 0 < q < 1 \). Then we see from (3.1) that the function \( \tilde{B}_m^{(\nu)}(z; q) \) satisfies the recursion formula:

\[
\begin{aligned}
\tilde{B}_0^{(\nu)}(z; q) &= -\frac{(1-q)^\nu}{\log q} (\nu - 1)!, \\
\sum_{m=0}^{n} (-1)^m \left( \frac{n}{m} \right) q^m \tilde{B}_m^{(\nu)}(z; q) &= (-1)^n \tilde{B}_n^{(\nu)}(z; q) + \nu! \left( \frac{n}{\nu} \right) q^{\nu(1-z)} [1 - z]_q^{n-\nu}.
\end{aligned}
\]

Note that, in particular,

\[
\tilde{B}_n^{(\nu)}(z; q) = (-1)^{n+1} \frac{(1-q)^{\nu-n}}{\log q} (\nu - 1)! \quad (0 \leq n \leq \nu - 1).
\]

Define also the functions \( \{B_m^{(\nu)}(z; q)\}_{m \geq 0} \) by

\[
B_m^{(\nu)}(z; q) := (-1)^{\nu-1} \frac{m!}{(m+\nu-1)!} \tilde{B}_{m+\nu-1}^{(\nu)}(z; q).
\]

It is clear that \( B_m^{(1)}(z; q) = \tilde{B}_m^{(1)}(z; q) \). The first few are given by

\[
\begin{aligned}
B_0^{(\nu)}(z; q) &= \frac{q - 1}{\log q}, \quad B_1^{(\nu)}(z; q) = -\frac{q^{-\nu z}}{1-q^{-\nu}} + \frac{1}{\nu \log q}, \\
B_2^{(\nu)}(z; q) &= \frac{2}{q - 1} \left( -\frac{q^{-\nu z}}{1-q^{-\nu}} + \frac{q^{z(-\nu-1)}}{1-q^{-\nu-1}} + \frac{1}{\nu(\nu+1) \log q} \right), \quad \ldots.
\end{aligned}
\]

From Lemma (3.1) and (3.8), we have a closed expression of \( B_m^{(\nu)}(z; q) \).

**Proposition 3.3.** For \( m \in \mathbb{Z}_{\geq 0} \),

\[
B_m^{(\nu)}(z; q) = (q - 1)^{-m+1} \sum_{l=1}^{m} (-1)^l \left( \frac{m}{l} \right) l q^{z(-l+\nu+1)} \frac{1}{1-q^{-l+\nu+1}} + \left( \frac{m + \nu - 1}{\nu - 1} \right) \frac{1}{\log q} \in \mathbb{C}[q^{-z}].
\]

The following result shows the classical limit of \( B_m^{(\nu)}(z; q) \) reproduces the Bernoulli polynomial.

**Theorem 3.4.** For \( 0 < t < 2\pi \), we have

\[
\lim_{q \uparrow 1} G_q^{(\nu)}(t, z) = t^{\nu-1} G(t, z).
\]

In particular, \( \lim_{q \uparrow 1} B_m^{(\nu)}(z; q) = B_m(z) \) for all \( m \in \mathbb{Z}_{\geq 0} \).

**Proof.** For any \( t \in R_q(z) \), it is obvious that

\[
\lim_{q \uparrow 1} F_q^{(\nu)}(t, z) = t^\nu \sum_{n=0}^{\infty} e^{-t(n+z)} = t^{\nu-1} \frac{te^{(1-z)t}}{e^t - 1} = t^{\nu-1} G(t, z).
\]
Also, from (2.7) and (3.2), we have

\[ C_q^{(\nu)}(t, z) = \frac{(1 - q)^\nu}{\log q} \sum_{m \in \mathbb{Z} \setminus \{0\}} \left( \frac{1 - q}{t} \right)^{m\delta_q} \Gamma(\nu + m\delta_q) e^{2\pi \sqrt{-1} m z} + F_{q,+}^{(\nu)}(t, z). \]  

(3.12)

Then the series on the right hand side of (3.12) disappears when \( q \uparrow 1 \) by the Stirling formula (see [1]);

\[ |\Gamma(\nu + m\delta_q)| \sim \frac{(2\pi)^\nu |m|^{\nu - \frac{1}{2}}}{\log q^{\nu + \frac{1}{2}}} e^{-\frac{x^2|m|}{2 \log q}} \quad (q \uparrow 1). \]

(3.13)

In fact, since \( 0 < t < 2\pi < \frac{3}{4} \pi^2 \) and \( 1/\log q = -1/(1 - q) + 1/2 + O(1 - q) \), it follows that

\[ \frac{(1 - q)^\nu}{\log q} \left| e^{t - \tau} \Gamma(\nu + m\delta_q) \right| \sim (1 - q)^{-\nu - \frac{1}{2}}(2\pi)^\nu |m|^{\nu - \frac{1}{2}} \left( \frac{1 - q}{\log q} \right)^{\nu + \frac{1}{2}} e^{-\frac{1}{4} \log q - \tau} \left( 1 - q \right)^{\frac{1}{2}} e^{\frac{t}{1 - q} - \frac{1}{4} \log q} \]

\[ = O \left( e^{\frac{1}{4} \log q} \right) \cdot \exp \left( -\frac{\pi^2}{2} q + O(1) \right) \to 0 \quad (m \neq 0) \]

when \( q \uparrow 1 \). Hence, letting \( q \uparrow 1 \) in (3.12), we obtain (3.10) from (3.11). This completes the proof.

**Remark 3.5.** The function \( B_m^{(1)}(z; q) \) appeared essentially in [10, 11] (see also [5] and [9]). Actually, one can show that \( B_m^{(1)}(z) := (-1)^m \tilde{\beta}_m(1 - z, q) \), where \( \tilde{\beta}_m(z, q) \) is the one in [11]. Also, in [3], by the explicit expression (8.3) for \( \theta(z) \), it was shown in a different way that \( \lim_{q \uparrow 1} B_m^{(1)}(1; q) = B_m \quad (m \in \mathbb{Z}_{\geq 0}) \), where \( B_m = B_m^{(1)} \) is the Bernoulli number.

The following theorem is the main result.

**Theorem 3.6.** Let \( z \in J_q = \{ z = x + \sqrt{-1} y \in \mathbb{C} \mid x > 0, \ |y| < \frac{\pi}{2 \log q}, \ q^{-\varepsilon} \cos(y \log q) > 1 \} \). Assume \( \text{Re}(s) > \nu + 1 \). Then

1. For any \( 0 < a < +\infty \) and \( 0 < \varepsilon \leq a \), we have

\[ \zeta_q^{(\nu)}(s, z) = \frac{(-1)^\nu \Gamma(1 - s)}{2\pi \sqrt{-1}} \int_{C(\varepsilon, a)} (-t)^{s-\nu} G_q^{(\nu)}(t, z) \frac{dt}{t} + \frac{1}{\Gamma(s)} \int_a^{\infty} t^{s-\nu} F_{q,+}^{(\nu)}(t, z) \frac{dt}{t} \]

\[ - \frac{1}{\Gamma(s)} \cdot \frac{(1 - q)^\nu}{\log q} \sum_{m \in \mathbb{Z} \setminus \{0\}} (1 - q)^{m\delta_q} \Gamma(\nu + m\delta_q) e^{2\pi \sqrt{-1} m z} \varphi_m^{(\nu)}(s; a, q), \]

(3.14)

where \( C(\varepsilon, a) \) is the same contour as the one in (3.3).

2. For any \( N \in \mathbb{N} \), we have

\[ \zeta_q^{(\nu)}(s, z) = \frac{1}{\Gamma(s)} \left\{ \int_0^1 t^{s-\nu-1} \left( G_q^{(\nu)}(t, z) - \sum_{k=0}^{N+\nu-1} (-1)^k \tilde{B}_k^{(\nu)}(z; q) \frac{t^k}{k!} \right) dt \right. \]

\[ + \sum_{k=0}^{N+\nu-1} (-1)^k \tilde{B}_k^{(\nu)}(z; q) \frac{1}{k!} \frac{1}{s - \nu + k} + \int_1^{\infty} t^{s-\nu} F_{q,+}^{(\nu)}(t, z) \frac{dt}{t} \]

\[ - \frac{(1 - q)^\nu}{\log q} \sum_{m \in \mathbb{Z} \setminus \{0\}} (1 - q)^{m\delta_q} \Gamma(\nu + m\delta_q) e^{2\pi \sqrt{-1} m z} \varphi_m^{(\nu)}(s; q) \right\}. \]

(3.15)
The points in
analogues of the Hurwitz zeta function

where \( \varphi_m^{(\nu)} (s; q) := \varphi_m^{(\nu)} (s; 1, q) \).

Each integral representation shows that \( \zeta_q^{(\nu)} (s, z) \) is meromorphic in \( \mathbb{C} \) and has simple poles at
the points in \( n + \delta_q \mathbb{Z} \) (1 \( \leq n \leq \nu \)) and \( \mathbb{Z}_{\leq 0} + \delta_q \mathbb{Z} \{0\} \) with

\[
\text{Res}_{s = n + m \delta_q} \zeta_q^{(\nu)} (s, z) = - \left( \nu - 1 + m \delta_q \right) \left( 1 - q \right)^{n + m \delta_q} \log q e^{2\pi \sqrt{-1}} \text{Im} \quad (n \in \mathbb{Z}_{\leq \nu}, \ m \in \mathbb{Z}).
\] (3.16)

These exhaust all poles of \( \zeta_q^{(\nu)} (s, z) \). Further,

\[
\zeta_q^{(\nu)} (1 - m, z) = - \frac{B_m^{(\nu)} (z; q)}{m} \quad (m \in \mathbb{Z}_{\geq 0}).
\] (3.17)

Proof. For any \( a > 0 \), from (2.5), we have

\[
\Gamma(s) \zeta_q^{(\nu)} (s, z) = \int_0^a t^{s-\nu} F_{q, +}^{(\nu)} (t, z) \frac{dt}{t} + \int_a^\infty t^{s-\nu} F_{q, +}^{(\nu)} (t, z) \frac{dt}{t}.
\] (3.18)

The second integral obviously defines an entire function by Lemma 2.2. From (3.17) and (3.18) again, the first one turns to be

\[
\int_0^a t^{s-\nu} G_q^{(\nu)} (t, z) \frac{dt}{t} - \frac{1 - q}{} \sum_{m \in \mathbb{Z}_{\neq 0}} (1 - q)^{m \delta_q} \Gamma(\nu + m \delta_q) e^{2\pi \sqrt{-1} \text{Im}} \varphi_m^{(\nu)} (s; a, q).
\] (3.19)

1. Define the function \( I_q^{(\nu)} (s, z; a) \) by

\[
I_q^{(\nu)} (s, z; a) := \int_{C(\varepsilon, a)} (-t)^{s-\nu} G_q^{(\nu)} (t, z) \frac{dt}{t} \quad (0 < \varepsilon \leq a).
\]

The integral converges absolutely and uniformly with respect to \( s \) and does not depend on \( \varepsilon \) by the Cauchy integral theorem. Hence \( I_q^{(\nu)} (s, z; a) \) is entire as a function of \( s \). Moreover, we have

\[
\int_0^a t^{s-\nu} G_q^{(\nu)} (t, z) \frac{dt}{t} = \frac{(-1)^\nu \Gamma(s) (1 - s)}{2\pi \sqrt{-1}} I_q^{(\nu)} (s, z; a) \quad (\text{Re} (s) > \nu + 1).
\] (3.20)

In fact, it is easy to see that

\[
I_q^{(\nu)} (s, z; a) = \left( \int_0^\varepsilon + \int_{|t|=\varepsilon} + \int_\varepsilon^a \right) (-t)^{s-\nu} G_q^{(\nu)} (t, z) \frac{dt}{t}
\]

\[
= (-1)^\nu (e^{\pi s \sqrt{-1}} - e^{-\pi s \sqrt{-1}}) \int_\varepsilon^a t^{s-\nu} G_q^{(\nu)} (t, z) \frac{dt}{t}
\]

\[
+ \int_{|t|=\varepsilon} (-t)^{s-\nu} G_q^{(\nu)} (t, z) \frac{dt}{t}.
\] (3.21)

Since \( G_q^{(\nu)} (t, z) \) is bounded for \( t = |\varepsilon| \), the last integral in (3.21) disappears when \( \varepsilon \to 0 \)
provided \( \text{Re} (s) > \nu + 1 \). Therefore, letting \( \varepsilon \to 0 \) in (3.21) and using the relation
\( \Gamma(s) (1 - s) = \pi / \sin(\pi s) \), we obtain (3.20), whence (3.14). Since \( I_q^{(\nu)} (s, z; a) \) is entire,
\textbf{3.14} gives a meromorphic continuation of $\zeta_q^{(\nu)}(s, z)$ to the whole plane $\mathbb{C}$. By the residue theorem, it follows from \textbf{3.15} that

$$I_q^{(\nu)}(n, z; a) = \begin{cases} \frac{2\pi \sqrt{-1}}{(\nu - n)!} \tilde{B}_\nu^{(\nu)}(z; q) & \text{if } n = 1, 2, \ldots, \nu, \\ 0 & \text{if } n \in \mathbb{Z}_{\geq \nu+1}. \end{cases} \tag{3.22}$$

Hence, by Proposition \textbf{2.5}, we see that all poles of $\zeta_q^{(\nu)}(s, z)$ are simple and given by $n + \delta_q \mathbb{Z} \ (1 \leq n \leq \nu)$ and $\mathbb{Z}_{\leq 0} + \delta_q \mathbb{Z}\setminus\{0\}$. Further, if $s = n$ for $1 \leq n \leq \nu$, we have from \textbf{3.16} and \textbf{3.22} that

$$\text{Res}_{s=n} c_q^{(\nu)}(s, z) = (-1)^{\nu-n} \tilde{B}_{\nu-n}^{(\nu)}(z; q) \frac{1}{(n-1)!(\nu-n)!} = -\frac{(\nu - 1)(1-q)^n}{\log q}.$$

If $s = n + m\delta_q$ with $m \neq 0$, we have from \textbf{2.9}

$$\text{Res}_{s=n+m\delta_q} c_q^{(\nu)}(s, z) = -\frac{(1-q)^{n+m\delta_q} e^{2\pi \sqrt{-1} mz}}{\log q} \frac{\Gamma(\nu + m\delta_q)}{\Gamma(n + m\delta_q)(\nu-n)!} \frac{1}{e^{2\pi \sqrt{-1} mz}}.$$

Hence \textbf{3.16} follows. From \textbf{3.20} again, it follows that

$$\zeta_q^{(\nu)}(1-m, z) = \frac{(-1)^{\nu-1} \Gamma(m)}{2\pi \sqrt{-1}} I_q^{(\nu)}(1-m, z; a)$$

$$= (-1)^{\nu-1} \frac{(m-1)!}{(m+\nu-1)!} \tilde{B}_{m+\nu-1}^{(\nu)}(z; q) = -\frac{B_m^{(\nu)}(z; q)}{m}.$$

2. Since the first integral in \textbf{3.15} converges absolutely for $\text{Re}(s) > 1 - N$, it suffices to show \textbf{3.15}. This is easy because the integral $\int_0^1 t^{s-\nu-1} G_q^{(\nu)}(t, z) dt$ in \textbf{3.19} can be written as

$$\int_0^1 t^{s-\nu-1} \left( G_q^{(\nu)}(t, z) - \sum_{k=0}^{N+\nu-1} (-1)^k \tilde{B}_k^{(\nu)}(z; q) \frac{t^k}{k!} \right) dt + \int_0^1 t^{s-\nu-1} \sum_{k=0}^{N+\nu-1} (-1)^k \tilde{B}_k^{(\nu)}(z; q) \frac{t^k}{k!} dt$$

$$= \int_0^1 t^{s-\nu-1} \left( G_q^{(\nu)}(t, z) - \sum_{k=0}^{N+\nu-1} (-1)^k \tilde{B}_k^{(\nu)}(z; q) \frac{t^k}{k!} \right) dt + \sum_{k=0}^{N+\nu-1} (-1)^k \frac{\tilde{B}_k^{(\nu)}(z; q)}{k!} \frac{1}{s-\nu+k}.$$

This completes the proof of the theorem.

\textbf{Remark 3.7.} The binomial theorem yields the following series expression of $\zeta_q^{(\nu)}(s, z)$ (see [4], also [3]), which shows that $\zeta_q^{(\nu)}(s, z)$ is meromorphic in $\mathbb{C}$:

$$\zeta_q^{(\nu)}(s, z) = (1 - q)^s \sum_{l=0}^{\infty} \binom{s+l-1}{l} \frac{q^{z(s-\nu+l)}}{1-q^{s-\nu+l}}.$$

Using the expression, we can get the information about poles and special values.
By (3.10), we note that
\[
\lim_{q \uparrow 1} \text{Res}_{s=n} \zeta_q^{(\nu)}(s, z) = \begin{cases} 
1 & \text{if } n = 1, \\
0 & \text{if } 2 \leq n \leq \nu.
\end{cases}
\]
This shows the poles of \(\zeta_q^{(\nu)}(s, z)\) at \(s = 2, 3, \ldots, \nu\) disappear when \(q \uparrow 1\). We now see the classical limit of \(\zeta_q^{(\nu)}(s, z)\). In fact,

**Corollary 3.8.** We have
\[
\lim_{q \uparrow 1} \zeta_q^{(\nu)}(s, z) = \zeta(s, z) \quad (s \in \mathbb{C}, \ s \neq 1, 2, \ldots, \nu).
\] (3.23)

**Proof.** By comparing formulas (1.3) and (3.14) when \(a = 1\) (one may take any \(a\) satisfying \(0 < a < 2\pi\)), in view of Theorem 3.4, it is sufficient to show
\[
\lim_{q \uparrow 1} (1 - q)\nu \log q \sum_{m \in \mathbb{Z} \setminus \{0\}} (1 - q)^{m\delta} \Gamma(\nu + m\delta)e^{2\pi\sqrt{-1}mz} \varphi_m^{(\nu)}(s; q) = 0.
\] (3.24)

By the mean-value theorem, there exists a number \(0 < \tau < 1\) such that
\[
|\varphi_m^{(\nu)}(s; q)| \leq \int_0^1 e^{Re(s)-\nu-1}e^{\tau t}dt = \tau^{Re(s)-\nu-1}e^{\tau s}.
\]
Using the Stirling formula (3.13) again, we obtain (3.24). \(\square\)

### 4 Concluding remarks

Let \(\zeta_q(s, t) := \zeta_q(s, t, 1) = \sum_{n=1}^{\infty} q^n t[n]^{-s} \) and \(\zeta_q^{(\nu)}(s) := \zeta_q(s, s - \nu)\) for \(\nu \in \mathbb{N}\). As a final remark, we show the \(q\)-analogue \(\zeta_q^{(\nu)}(s)\) of \(\zeta(s)\) can be obtained from the function \(Z_q(s, t)\) defined by
\[
Z_q(s, t) := \prod_{l=0}^{\infty} (1 - q^{l+1})^{-\left(\frac{s+l-1}{l}\right)}.
\] (4.1)

Assume \(\text{Re}\ (t) > 0\). Then
\[
\log Z_q(s, t) = -\sum_{l=0}^{\infty} \left(\frac{s+l-1}{l}\right) \log (1 - q^{l+1}) = \sum_{l=0}^{\infty} \left(\frac{s+l-1}{l}\right) \sum_{n=1}^{\infty} \frac{q^{n(l+1)}}{n}
\]
\[
= \sum_{n=1}^{\infty} \frac{q^{nt}}{n} \sum_{l=0}^{\infty} \left(\frac{s+l-1}{l}\right) q^{nt} = \sum_{n=1}^{\infty} \frac{q^{nt}}{n}(1 - q^n)^{-s}.
\] (4.2)

Since the most right hand side converges absolutely in \(\text{Re}\ (t) > 0\), the infinite product (4.1) converges absolutely in this region. Though it is hard to expect any Euler product, the next proposition claims that \(\zeta_q^{(\nu)}(s)\) can be gotten by a specialization of the logarithmic derivative of \(Z_q(s, t)\).
Proposition 4.1. It holds that
\[ \zeta_q^{(\nu)}(s) = \frac{(1 - q)^s}{\log q} \frac{\partial}{\partial t} \log Z_q(s, t) \bigg|_{t=s-\nu}. \] (4.3)

Proof. From (4.2), we have
\[ \frac{\partial}{\partial t} \log Z_q(s, t) = (\log q) \sum_{n=1}^{\infty} q^{nt}(1 - q^n)^{-s} = \frac{\log q}{1 - q^s} \zeta_q(s, t). \]

Hence the claim follows. \[\square\]

Remark 4.2. Let \( p \) be a prime number and \( q = p^{-m} \ (m \in \mathbb{N}) \). Let \( \mathbb{F}_{q^{-n}} \) be the field of \( q^{-n} \) elements. By (4.2), \( Z_q(s, t) \) can be written also as
\[ Z_q(s, t) = \exp \left( \sum_{n=1}^{\infty} (q^n - 1)^{-s} u^n \right) = \exp \left( \sum_{n=1}^{\infty} (\# \mathbb{F}_{q^{-n}})^{-s} u^n \right), \]
where \( u = q^{t-s} \).

To see basic properties of the function \( Z_q(s, t) \), we recall Appell’s \( O \)-function \( O_q(t; \omega) \) defined by
\[ O_q(t; \omega) := \prod_{l_1, \ldots, l_m \geq 0} (1 - q^{l_1 + \cdots + l_m + t} \omega), \]
where \( \omega := (\omega_1, \ldots, \omega_m) \in \mathbb{C}^m \) with \( \text{Re} (\omega_i) > 0 \ (1 \leq i \leq m) \) (see [2], also [6]). Similarly to the discussion in [7], we have the

Proposition 4.3. (i) We have \( Z_q(0, t) = (1 - q^t)^{-1} \) and
\[ Z_q(m, t) = O_q(t; 1_m)^{-1}, \quad Z_q(-m, t) = \prod_{l=0}^{m} (1 - q^{t+l})^{(-1)^{l+1}(\gamma)} \ (m \in \mathbb{N}), \] (4.4)
where \( 1_m = (1, \ldots, 1) \).

(ii) For \( m \in \mathbb{N} \), we have
\[ Z_q(s + m, t) = \prod_{l=0}^{m} Z_q(s, t + l)^{m+l-i-1}, \quad Z_q(s - m, t) = \prod_{l=0}^{m} Z_q(s, t + l)^{(-1)^l(i)} \] (4.5)
\[ Z_q(s, t + m) = \prod_{l=0}^{m} Z_q(s - l, t)^{(-1)^l(i)}, \quad Z_q(s, t - m) = \prod_{l=0}^{m} Z_q(s - l, t)^{(m+l-i-1)}. \]

In particular,
\[ Z_q(s, t) = Z_q(s - 1, t)Z_q(s, t + 1). \] (4.6)

Proof. Equations (4.4) are obvious from the definition. One can check the ladder relations in (4.5) directly. \[\square\]
Formulas (4.5) together with Proposition 4.1 yield the following relation for $\zeta^{(\nu)}_q(s)$.

**Corollary 4.4.** For $m \in \mathbb{Z}$ satisfying $1 \leq m \leq \nu - 1$, we have

$$\zeta^{(\nu-m)}_q(s) = \sum_{l=0}^{m} (-1)^l \binom{m}{l} (1-q)^l \zeta^{(\nu-l)}_q(s-l). \quad (4.7)$$

In particular,

$$\zeta^{(\nu)}_q(s) = \zeta^{(\nu-1)}_q(s) + (1-q)\zeta^{(\nu-1)}_q(s-1). \quad (4.8)$$

**Proof.** The equation (4.7) follows from (4.3) and (4.5). \qed

**Remark 4.5.** The equation (4.8) was found in [13]. One can show it directly from the definition. More generally, we have

$$\zeta^{(\nu)}_q(s, z) = \zeta^{(\nu-1)}_q(s, z) + (1-q)\zeta^{(\nu-1)}_q(s-1, z).$$

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Masato Wakayama
Faculty of Mathematics, Kyushu University, Hakozaki Fukuoka 812-8581, Japan.
e-mail: wakayama@math.kyushu-u.ac.jp

Yoshinori Yamasaki
Graduate School of Mathematics, Kyushu University, Hakozaki Fukuoka 812-8581, Japan.
e-mail: ma203032@math.kyushu-u.ac.jp