STRUCTURE OF THE ELECTRIC FIELD IN THE SKIN EFFECT PROBLEM

Yu.F. Alabina

Moscow State Regional University

105005, Moscow, Radio st., 10 a

e-mail: yf.alabina@gmail.com

Abstract

The structure of the electric field in a plasma has been elucidated for the skin effect problem. An expression for the distribution function in the half-space and the electric field profile have been obtained in the explicit form. The absolute value, the real part, and the imaginary part of the electric filed have been analyzed in the case of the anomalous skin effect near to a plasma resonance. It has been demonstrated that the electric field in the skin effect problem is predominantly determined by the discrete spectrum, i.e., the oscillation frequency of external field is the value of plasma frequency.

1. Introduction. Statement of problem.

The skin effect is associated with the response of an electron gas (in a metal or in a gas plasma) to an external alternating electromagnetic field that is tangential to the surface [1, 2]. This classical problem has been studied by many authors [3] – [6] and, up to now, has remained the subject of investigation. The main attention has been focused on the calculation
of the impedance. The distribution function of electrons and the electric field in plasma almost have not been investigated previously.

It has been demonstrated that the electric field is the sum of the integral term and two (or one) exponentially decreasing particular solutions to the initial system and that one particular solution disappears depending on the anomaly parameter.

Let’s Maxwell plasma fills the half-space \( x > 0 \). Here \( x \) is the orthogonal coordinate to the plasma boundary. Let’s the external electric field has only \( y \) component. Then the self-consistent electric field inside in plasma also has only \( y \) component \( E_y(x, t) = E(x)e^{-i\omega t} \). We consider the kinetic equation for the electron distribution function:

\[
\frac{\partial f}{\partial t} + v_x \frac{\partial f}{\partial x} + eE(x)e^{-i\omega t} \frac{\partial f}{\partial p_y} = \nu(f_0 - f(t, x, v)).
\]  

(1)

In (1) \( \nu \) is the frequency of electron collisions with ions, \( e \) is the charge of electron, \( f_0(v) \) is the equilibrium Maxwell distribution function, \( p = mv \) is the momentum of electron,

\[
f_0(v) = n \left( \frac{\beta}{\pi} \right)^{3/2} \exp(-\beta^2v^2), \quad \beta = \frac{m}{2k_B T}.
\]

Here \( m \) is the mass of electron, \( k_B \) is the Boltzmann constant, \( T \) is the temperature of plasma, \( v \) is the modulus of the velocity of the electron, \( n \) is the concentration of electrons (number density), \( c \) is the speed of light.
The electric field $E(x)$ satisfies to the equation:

$$E''(x) = -\frac{4\pi i e^{i\omega t}}{c^2} \int v_y f(t, x, v) \, d^3 v. \quad (2)$$

We assume that intensity of an electric field is such that linear approximation is valid. Then distribution function can be presented in the form:

$$f = f_0 \left(1 + C_y \exp(-i\omega t) h(x, \mu)\right),$$

where $C = \sqrt{\beta} v$ is the dimensionless velocity of electron, $\mu = C_x$. Let $l = v_T \tau$ is the mean free path of electrons, $v_T = 1/\sqrt{\beta}$, $v_T$ is the thermal electron velocity, $\tau = 1/\nu$. We introduce the dimensionless parameters and the electric field:

$$t_1 = \nu t, \quad x_1 = \frac{x}{l}, \quad e(x_1) = \frac{\sqrt{2} e}{\nu \sqrt{mk_BT}} E(x_1).$$

Later we substitute $x_1$ for $x$. The substitution produces the following form of the kinetic equation (1) and the equation on a field with the displacement current (2):

$$\mu \frac{\partial h}{\partial x} + z_0 h(x, \mu) = e(x) \quad (3)$$

$$e''(x) = -i \frac{\alpha}{\sqrt{\pi}} \int_{-\infty}^{\infty} \exp(-\mu'^2) h(x, \mu') \, d\mu'. \quad (4)$$

Here,

$$\alpha = \frac{2l^2}{\delta^2}, \quad z_0 = 1 - i\Omega, \quad \Omega = \omega \tau = \frac{\omega}{\nu}, \quad \delta = \frac{c^2}{2\pi \omega \sigma_0},$$

$$\mu = C_x.$$
δ is the classical depth of the skin layer \([\Pi]\), \(\sigma_0 = \frac{e^2 n}{m\nu}\), \(\sigma_0\) is the electric conductance, \(\alpha\) is the anomaly parameter.

The boundary conditions at the plasma surface for the distribution function of electrons in the case of specular reflection of electrons from the boundary can be written as follows \([\Pi]\):

\[
h(0, \mu) = h(0, -\mu), \quad 0 < \mu < \infty. \tag{5}
\]

The distribution function will be sought in the form of a decaying function far from the boundary; that is,

\[
h(+\infty, \mu) = 0, \quad -\infty < \mu < 0. \tag{6}
\]

The electric field deep in the plasma far from the surface decays. Taking into account this circumstance, the boundary conditions for the electric field are written in the form

\[
e(0) = 1, \tag{7}
\]

\[
e(+\infty) = 0. \tag{8}
\]

2. Decomposition on eigenfunctions

The separation of variables in (3) and (4) within several steps leads to the exponentially decreasing solutions

\[
h_\eta(x, \mu) = \exp\left(-\frac{z_0 x}{\eta}\right)\Phi(\eta, \mu), \quad e_\eta(x) = \exp\left(-\frac{z_0 x}{\eta}\right)E(\eta), \tag{9}
\]
where the separation parameter (also termed the spectral parameter) $\eta$ continuously fills the interval $(0, \infty)$, which, therefore, is called the continuous spectrum of the problem.

Substitution of relationships (9) into the initial system of equations (3) and (4) leads to the characteristic system of equations

$$(\eta - \mu)\Phi(\eta, \mu) = \frac{\eta}{z_0}E(\eta),$$

$$\frac{z_0^2}{\eta^2}E(\eta) = -i\frac{\alpha}{\sqrt{\pi}}\int_{-\infty}^{\infty}\exp(-\mu^2)\Phi(\eta, \mu)\,d\mu.$$  

The functions $\Phi(\eta, \mu)$ and $E(\eta)$, which are referred to as the eigenfunctions of the characteristic system and correspond to the eigenvalue (or characteristic value) of the parameter $\eta$, are defined by the expressions

$$\Phi(\eta, \mu) = \frac{a}{\sqrt{\pi}}\eta^3P\frac{1}{\eta - \mu} + \lambda(\eta)\exp(\eta^2)\delta(\eta - \mu), \quad (10a)$$

$$E(\eta) = \frac{az_0}{\sqrt{\pi}}\eta^2, \quad (10b)$$

where the dispersion function $\lambda(z)$ (see, for example, [3]) is given by the formula

$$\lambda(z) = 1 + \frac{az^3}{\sqrt{\pi}}\int_{-\infty}^{\infty}\frac{\exp(-\mu^2)}{\mu - z}d\mu, \quad a = -i\frac{\alpha}{z_0^3}. \quad (11)$$

With the use of the argument principle, it is possible to show that, in the $(\alpha, \Omega)$ plane, there exists a domain $D^+$ (Fig. 1a) so that, if the point $(\alpha, \Omega) \in D^+$, the dispersion function has four zeros $\pm \eta_0$ and $\pm \eta_1$, and
if \((\alpha, \Omega) \in D^-\) (where \(D^-\) is exterior of the domain \(D^+\)), the dispersion function has two zeros \(\pm \eta_0\). The designations \(\eta_0\) and \(\eta_1\) correspond to the zeros with the positive real parts: \(\text{Re} \ \eta_0 > 0\) and \(\text{Re} \ \eta_1 > 0\). The boundary of the domain \(D^+\) is found from the equation \(\omega^\pm(\mu) = 0\) and, in the parametric form, is determined by the equations \(1 - 3\Omega^2 \pm \alpha q(\mu) = 0, \ 3\Omega - \Omega^3 - \alpha p(\mu) = 0, \ -\infty < \mu < +\infty\).

It should be noted that the parameters \(\alpha\) and \(\Omega\) are proportional to the electric field frequency; i.e., they are not independent. In this respect, it seems quite natural to introduce the dimensionless independent frequencies

\[
\omega_1 = \frac{\omega}{\omega_p v_c}, \quad \nu_1 = \frac{\nu}{\omega_p v_c},
\]

and to construct the corresponding domains \(D^+_1\) and \(D^-_1\) (Fig. 1b) in their plane. Here, \(\alpha = \omega_1/(\nu_1^2)\), \(\Omega = \omega_1/\nu_1\), \(\omega_p\) is the plasma frequency and \(n\) is the electron concentration.

The zeros \(\eta_0\) and \(\eta_1\) correspond to the following eigenfunctions of the characteristic equation that are associated with the discrete spectrum:

\[
\Phi(\eta_k, \mu) = \frac{a \eta_k^3}{\sqrt{\pi} (\eta_k - \mu)}, \quad E(\eta_k) = \frac{az_0 \eta_k^2}{\sqrt{\pi}}, \quad k = 0, 1.
\]

The zeros of the dispersion function can be calculated in the explicit form with the use of the formulas for its factorization. In the case of two
zeros $\pm \eta_0$, the dispersion function (see [3]) can be represented in the form

$$\lambda(z) = a(\eta_0^2 - z^2)X(z)X(-z),$$

where

$$X(z) = \exp V(z),$$

$$V(z) = \frac{1}{2\pi i} \int_0^\infty \ln G(\tau) d\tau$$

In the case of four zeros $\pm \eta_0$ and $\pm \eta_1$, the dispersion function can be written as follows:

$$\lambda(z) = a(\eta_0^2 - z^2)(\eta_1^2 - z^2)X_1(z)X_1(-z),$$

where the function $X_1(z)$ is expressed through the function $X(z)$: $X_1(z) = X(z)/(z - 1)$.

By calculating the left- and right-hand sides of the former formula for the factorization of the dispersion function (for, example, at the point $z = 0$), after some transformations, we obtain the relationship for its zeros

$$\pm \eta_0 = \frac{1}{\sqrt{aX(z)X(-z)}} + 1.$$

In the skin effect theory, the normal and anomalous skin effects are recognized [7]. In the case of the normal skin effect, the mean free path of electrons is considerably smaller than the skin depth; i.e., the anomaly parameter satisfies the inequality $\alpha \ll 1$. The anomalous skin effect cor-
responds to the case where the mean free path of electrons is considerably larger than the characteristic skin depth: $\alpha \gg 1$.

Let us construct the general solution to the initial system of equations in the form of the expansion in eigenfunctions of the discrete and continuous spectra. Since the discrete spectra for zero and unit indices are different and the continuous spectrum does not depend on the index, the expansions of the solution in both cases differ only in the nonintegral terms corresponding to the discrete spectrum.

In [5], it was demonstrated that the distribution function of electrons and the electric field, which are the solution to the problem described by expressions (3)–(8), have the following expansions:

$$h(x, \mu) = \frac{a}{\sqrt{\pi}} \sum_{k=0}^{1} A_k \eta_k^3 \exp \left(-\frac{z_0 x}{\eta_k}\right) \phi \left(\frac{z_0 x}{\eta_k}\right) +$$

$$+ \int_{0}^{\infty} \exp \left(-\frac{z_0 x}{\eta}\right) A(\eta) \Phi(\eta, \mu) d\eta,$$

$$e(x) = \frac{az_0}{\sqrt{\pi}} \sum_{k=0}^{1} A_k \eta_k^2 \exp \left(-\frac{z_0 x}{\eta_k}\right) +$$

$$+ \frac{az_0}{\sqrt{\pi}} \int_{0}^{\infty} \exp \left(-\frac{z_0 x}{\eta}\right) \eta^2 A(\eta) d\eta.$$

Here, $\text{Re} \eta_k > 0$, $A_k$ ($k = 0, 1$) are unknown constant coefficients of expansions (12) and (13) (the so-called coefficients of the discrete spec-
trum), and $A(\eta)$ is an unknown function (the so-called coefficient of the continuous spectrum).

It should be noted that, in the case of two zeros of the dispersion function, it is necessary to set $k = 0$ in relationships (12) and (13). Therefore, the structure of the electric field depends on the domain $D^\pm$ that contains the point with the parameters $(\alpha, \Omega)$.

In [8], it was shown that the coefficient $A(\eta)$ of the continuous spectrum is represented in the form

$$A(\eta) = -\frac{\eta \exp(-\eta^2)}{z_0 I \lambda^+(\eta)\lambda^-(\eta)},$$

where

$$I = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{d\tau}{\lambda(i\tau)}.$$

The coefficients of the discrete spectrum are written in the following form:

$$A_k = -\frac{\sqrt{\pi}}{az_0 I \eta_k^2 \lambda'(\eta_k)}, \quad k = 0, 1.$$

The impedance is given by the formula [1]

$$Z = \frac{4\pi i\omega}{c^2} \cdot \frac{e(0)}{e'(0)}.$$

According to the boundary conditions for the field, we have $e(0) = 1$.

Therefore, the following expression holds true for the impedance:

$$Z = \frac{4\pi i\omega}{c^2 e'(0)} = \frac{8\pi i\omega l}{c^2 z_0} \left[ \frac{1}{\pi} \int_0^\infty \frac{d\tau}{\lambda(i\tau)} \right].$$
3. Distribution function and the electric field

With the use of the determined coefficients of the continuous and discrete spectra, the electric field profile in the half-space can be represented in the explicit form

\[ e(x) = \frac{1}{v\lambda'(\eta_0)} \exp \left( - \frac{z_0 x}{\eta_0} \right) - \frac{1}{v\lambda'(\eta_1)} \exp \left( - \frac{z_0 x}{\eta_1} \right) - \]

\[ - \frac{a}{\sqrt{\pi}} \int_0^\infty \exp \left( - \frac{z_0 x}{\eta} \right) \frac{\eta^3 \exp(-\eta^2)}{\lambda^+(\eta)\lambda^-(\eta)} d\eta. \]  

Formula (14) will be subsequently used for analyzing the behavior of the electric field in the half-space.

Now, we consider the profile of the distribution function of electrons in the half-space in the explicit form. The distribution function is represented in the form of two terms:

\[ h(x, \mu) = h_d(x, \mu) + h_c(x, \mu), \]

where the terms \( h_d(x, \mu) \) and \( h_c(x, \mu) \) correspond to the discrete and continuous spectra, respectively. With the use of the equality for the coefficients of the discrete and continuous spectra, these terms are written as follows:

\[ h_d(x, \mu) = -\frac{1}{z_0 v} \sum_{k=0}^1 \frac{\eta_k}{(\eta_k - \mu)\lambda'(\eta_k)} \exp \left( - \frac{z_0 x}{\eta_k} \right), \]

\[ h_c(x, \mu) = -\frac{1}{z_0 v} \int_0^\infty \exp \left( - \frac{z_0 x}{\eta} \right) \frac{\eta \exp(-\eta^2)}{\lambda^+(\eta)\lambda^-(\eta)} \Phi(\eta, \mu) d\eta. \]
At the plasma boundary, i.e., at $x = 0$, the last relationship can be calculated in the explicit form. As a result, we have

$$h_c(0, \mu) = \frac{1}{2\pi iz_0 I} \int_0^\infty \left( \frac{1}{\lambda^+(\eta)} - \frac{1}{\lambda^-}(\eta) \right) \frac{\eta d\eta}{\eta - \mu} +$$

$$+ \frac{\lambda(\mu)\theta(\mu) \exp(\mu^2)}{2iz_0 I a \sqrt{\pi} \mu^2} \left( \frac{1}{\lambda^+(\mu)} - \frac{1}{\lambda^-}(\mu) \right), \quad \mu \in (-\infty, +\infty).$$

Here, $\theta(\mu) = 1$ at $0 < \mu < \infty$ and $\theta(\mu) = 0$ at $-\infty < \mu < 0$.

By using contour integration methods [5], for the first term we find that

$$h_c(0, \mu) = \frac{1}{z_0 I} \left[ \mu \lambda(\mu)(1 - \theta(\mu)) \frac{\eta \eta_0}{(\eta_0 - \mu)\lambda'(\eta_0)} + \right.$$

$$+ \frac{\eta_1}{(\eta_1 - \mu)\lambda'(\eta_1)} + \left. \frac{1}{\pi} \int_0^\infty \tau^2 d\tau \frac{\lambda(i\tau)(\tau^2 + \mu^2)}{\lambda(i\tau)} \right].$$

By summing up the terms corresponding to the discrete and continuous spectra in the distribution function, we finally obtain

$$h(0, \mu) = \frac{1}{z_0 I} \left[ \mu \lambda(\mu)(1 - \theta(\mu)) \frac{1}{\lambda^+(\mu)\lambda^-(\mu)} + \frac{1}{\pi} \int_0^\infty \frac{\tau^2 d\tau}{\lambda(i\tau)(\tau^2 + \mu^2)} \right].$$

From this expression, for the distribution function of electrons moving at the metal boundary (i.e., in the case $-\infty < \mu < 0$), we have

$$h(0, \mu) = \frac{1}{z_0 I} \left[ \mu \lambda(\mu) \frac{\lambda^+(\mu)}{\lambda^-(\mu)} + \frac{1}{\pi} \int_0^\infty \frac{\tau^2 d\tau}{\lambda(i\tau)(\tau^2 + \mu^2)} \right], \quad -1 < \mu < 0,$$

For electrons specularly reflected from the metal boundary, we derive

$$h(0, \mu) = \frac{1}{\pi z_0 I} \int_0^\infty \frac{\tau^2 d\tau}{\lambda(i\tau)(\tau^2 + \mu^2)}, \quad 0 < \mu < 1.$$
These functions satisfy the specular boundary condition $h(0, \mu) = h(0, -\mu)$.

For all subsequent figures, we consider the typical case with the ratio $v_F/c = 0.003$.

The behavior of the real and imaginary parts of the distribution function at the boundary is illustrated in Fig. 2. In view of the specular boundary condition, the distribution functions of electrons reflected from the boundary $(0 < \mu < \infty)$ and electrons moving to the boundary $(-\infty < \mu < 0)$ are symmetric with respect to the point $\mu = 0$. The functions are constructed for the parameters $\alpha = 1, \Omega = 333$. Figure 2a depicts the real part of the electric field, fig. 2b depicts the imaginary part of the electric field. Let us compare the imaginary part of the electric field $\text{Im} \ e_c(x)$ with the real part of the field $\text{Re} \ e_c(x)$. The electric-field amplitude $|\text{Im} \ (h(0, \mu))| > 6 \cdot 10^3$ is considerably bigger for the imaginary part. It should be noted that the distribution function rapidly decreases with an increase in the quantity $\mu$. This circumstance is a manifestation of the ineffectiveness concept [2], according to which only electrons moving almost parallel to the surface, i.e., for which the quantity $\mu$ is considerably smaller than unity, are significant in the case of the anomalous skin effect.

The real and imaginary parts of the electric field in the vicinity of the boundary are presented in Figs. 3. The curves depicted in Fig. 3 cor-
respond to the following parameters: $\varepsilon = 10^{-4}$ ($\alpha = 900$, $\Omega = 1000$) for curves 1, $\varepsilon = 3 \cdot 10^{-4}$ ($\alpha = 100$, $\Omega = 333$) curves 2, and $\varepsilon = 9 \cdot 10^{-4}$ ($\alpha = 11$, $\Omega = 111$). This is an anomalous case. All the curves are considered near plasma resonance, i.e. the value $\gamma = 1$ and $\omega = \omega_p$.

Figure 3a shows the real part of the electric field $\text{Re} \ e_d(x)$, which corresponds to the discrete spectrum. An increase in the anomaly parameters leads to a drastic decrease in the depth of penetration of the electric field deep into the electron plasma.

Figure 3b depicts the real part of the electric field $\text{Re} \ e_c(x)$, which corresponds to the continuous spectrum. In this case, the electric-field amplitude — $|\text{Re} \ e_c(x)| < 1 \cdot 10^{-9}$ is considerably smaller that that for the real part due to the discrete spectrum.

The real part of the electric field $\text{Re} \ e_c(x)$, which is associated with the continuous spectrum, is nine orders of magnitude smaller than the real part of the field $\text{Re} \ e_d(x)$ corresponding to the discrete spectrum.

Therefore, the real part of the electric field in the vicinity of the plasma boundary is actually determined by the discrete spectrum.

The imaginary part of the electric field $\text{Im} \ e_d(x)$, which is associated with the discrete spectrum, is shown in Fig. 3c. As the anomaly parameter increases, the depth of penetration of the imaginary part of the electric
field deep into the plasma decreases slowly in contrast to the depth of penetration of the real part.

Figure 3d presents the imaginary part of the electric field $\text{Im} \ e_c(x)$ which corresponds to the continuous spectrum. Let us compare the imaginary part of the electric field $\text{Im} \ e_c(x)$ with the real part of the field $\text{Re} \ e_c(x)$. It can be seen that the amplitude has the same order of magnitude: $|\text{Im} \ e_c(x)| < 1, 2 \cdot 10^{-9}$. Therefore, the imaginary part of the electric field at the aforementioned values of the parameter, in actual fact, is also determined by the discrete spectrum. However, the imaginary part of the electric field corresponding to the discrete spectrum is nine orders of magnitude bigger than the real part of the electric field corresponding to the discrete spectrum.

As can be seen from the plots presented in Fig. 3, the contribution of the discrete spectrum at the aforementioned values of the parameter to the electric field is considerably larger than the contribution of the continuous spectrum. Thus, the above analysis of the electric field strength has demonstrated that, in the case of the anomalous skin effect, the electric field strength is determined in the vicinity of the boundary by the discrete spectrum.

Figure 4 the modulus of the electric field in the case of the anomalous
skin effect. On the $X$ axis is taken logarithmic scale. The curves depicted in these figures correspond to the following parameters: $\varepsilon = 3 \cdot 10^{-3}$ and $\gamma = 5$ ($\alpha = 5, \Omega = 1666$) for curve 1, $\varepsilon = 3 \cdot 10^{-4}$ and $\gamma = 1$ ($\alpha = 100, \Omega = 333$) for curve 2. It can be seen from Fig. 4 that with anomaly parameter increases in 20 times the modulus of the electric field in the anomalous case decreases one order of magnitude more rapidly.

**Conclusions**

Thus, in this paper, we have demonstrated that the electric field and the distribution function of electrons for the skin-effect problem are determined by their particular solutions. These solutions are the sums of the solutions corresponding to the discrete spectrum (decreasing particular solutions) and the continuous spectrum (solutions of the integral type) and that one particular solution disappears depending on the anomaly parameter.

It has been established that the zeros of the dispersion functions are necessary for the analytical solution of the problem and, in particular, for deriving the electric field and the distribution function of electrons in the explicit form in the half-space.

The analysis performed in this work has demonstrated that, in the case of the anomalous skin effect, the electric field in the skin effect problem is predominantly determined by the discrete spectrum. The real part of the
electric field which corresponds to the continuous spectrum is eight orders of magnitude smaller than the real part of the electric field corresponding to the discrete spectrum.

The imaginary part of the electric field at any anomaly parameters is considerably bigger than the real part. For $\varepsilon = 10^{-4}$ the amplitude of imaginary part is four orders of magnitude bigger than the amplitude of real part.

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REFERENCES

1. V.P. Silin and A.A. Rukhadze  Electromagnetic properties of plasma and mediums like plasma. Atomizdat, Moscow, 1961. P. 244.

2. L.D. Landau and E.M. Lifshitz  Electrodynamics of Continuous Media, Nauka, Moscow, 1992. P. 532.

3. I.D. Kaganovich, O.V. Polomarov and C.E. Theodosiou  Resisting the anomalous rf field penetration into a warm plasma// ArXiv: physics/0506135.
4. M. Opher, G.J. Morales and J.N. Leboeuf Krook collisional models of the kinetic susceptibility of plasmas// Phys. Rev. E. 2002 66(1), 016407, pp. 66 – 75.

5. A.V. Latyshev and A.A. Yushkanov Analytical solutions in the skin effect theory. Monography. Moscow State Regional University, Moscow, 2008. P. 285.

6. N.A. Zimbovskay ArXiv: physics/cond-mat/0506269.

7. A.F. Alexandrov, I.S. Bogdankevich and A.A. Rukhadze Principles of Plasma Electrodynamics. Springer–Verlag, New York, 1984.

8. Y.F. Alabina, A.V. Latyshev and A.A. Yushkanov The exact solution of the problem of skin effect in the gas plasma using the method of decomposition by eigenfunction. Proceedings of the Institute of Systems Analysis, Russian Academy of Sciences. 10(2) Moscow, 2006, pp. 66 – 72.
Fig. 1 a. Domains $D^+$ in the $(\alpha, \Omega)$ plane.

Fig. 1 b. Domains $D^+$ in the $(\omega_1, \nu_1)$ plane.
Fig. 2a. The real part of the distribution function

Fig. 2b. The imaginary part of the distribution function
Fig. 3a. The real part of the electric field for discrete spectra.

Fig. 3b. The real part of the electric field for continuous spectra.
Fig. 3c. The imaginary part of the electric field for the discrete spectra.

Fig. 3d. The imaginary part of the electric field continuous spectra.
Fig 4. The modulus of the electric field
