SQUARE INTEGRABILITY OF REPRESENTATIONS ON $p$-ADIC SYMMETRIC SPACES

SHIN-ICHI KATO AND KEIJI TAKANO

Abstract. A symmetric space analogue of Casselman’s criterion for square integrability of representations of a $p$-adic group is established. It is described in terms of exponents of Jacquet modules along parabolic subgroups associated to the symmetric space.

Introduction

Let $G$ be a connected reductive group defined over a non-archimedean local field $F$ equipped with an involutive $F$-automorphism $\sigma : G \to G$, and $H$ the subgroup of all $\sigma$-fixed points of $G$. The quotient space $G/H$ where $G = G(F)$ and $H = H(F)$ is called a $p$-adic symmetric space. We are interested in representations of $G$ which can be realized in the space of functions on $G/H$. Such representations are often said to be $H$-distinguished. In this paper, we are concerned especially with discrete series for $G/H$; roughly speaking, the representations which have realizations in the space of square integrable functions on $G/H$.

Let $(\pi, V)$ be a finitely generated admissible representation of $G$ which carries a non-zero $H$-invariant linear form $\lambda \in (V^*)^H$. We consider the functions $\varphi_{\lambda,v}$ on $G/H$ for $v \in V$ given by $\varphi_{\lambda,v}(g) = \langle \lambda, \pi(g^{-1})v \rangle$ $(g \in G)$.

Such functions are called $H$-matrix coefficients of $\pi$ defined by $\lambda$. Any non-trivial realization of $\pi$ in the space of functions on $G/H$ is formed by these $H$-matrix coefficients for some non-zero $\lambda \in (V^*)^H$. Note that $H$-matrix coefficients are not the matrix coefficients in the usual sense, but are generalized matrix coefficients, since $H$-invariant linear forms are not smooth in general. In a previous work [KT], we have studied representations whose $H$-matrix coefficients have compact support modulo $Z_G H$. Here $Z_G$ denotes the center of $G$. We have called such representations $(H, \lambda)$-relatively cuspidal, and established a criterion for $(H, \lambda)$-relative cuspidality of $\pi$ by using Jacquet modules ([KT, 6.2]). In the present work, we shall deal with...
a different class of representations. For simplicity, suppose that $\pi$ has a unitary central character. Then $|\varphi_{\lambda,v}(\cdot)|$ is regarded as a function on $G/ZG/H$. We say that $\pi$ is $H$-square integrable with respect to $\lambda$ if $|\varphi_{\lambda,v}(\cdot)|$ is square integrable on $G/ZG/H$ for all $v \in V$, namely, if

$$\int_{G/ZG/H} |\varphi_{\lambda,v}(g)|^2 dg < \infty$$

for all $v \in V$. We shall establish a criterion for $H$-square integrability of $\pi$ in this paper.

Before stating our main theorem, let us recall Casselman’s criterion for the usual square integrability. We say that $\pi$ is square integrable if all the usual matrix coefficients (defined by smooth linear forms) are square integrable on $G/Z$. For each parabolic subgroup $P$ of $G$ with the $F$-split component $A_P$, let $(\pi_P, V_P)$ be the normalized Jacquet module of $\pi$ along $P$ and $\mathcal{E}xp_{A_P}(\pi_P)$ the set of all quasi-characters $\chi$ of $A_P$ having non-zero generalized eigenvectors in $V_P$. Let $A^-_P$ and $A^+_P$ denote the dominant part of $A_P$ and the $O_F$-points of $A_P$ respectively.

**Casselman’s criterion.** ([C 4.4.6]) The representation $\pi$ is square integrable if and only if for every parabolic subgroup $P$, the condition $|\chi(a)| < 1$ holds for all $\chi \in \mathcal{E}xp_{A_P}(\pi_P)$ and $a \in A^-_P \setminus Z_G A^+_P$.

Also our criterion for $H$-square integrability is stated in terms of exponents of Jacquet modules. However we use only those along $\sigma$-split parabolic subgroups (see [L5]). In our previous work [KT] (and also in Lagier [L]), a canonical mapping $r_P : (V^*)^H \to (V_P^*)^{M \cap H}$ was introduced for each $\sigma$-split parabolic subgroup $P = MU$ (see [L2]). Now, for a given $\lambda \in (V^*)^H$ and a $\sigma$-split parabolic subgroup $P$ with the $(\sigma,F)$-split component $S_P$ (see [L5]), we put

$$\mathcal{E}xp_{S_P}(\pi_P, r_P(\lambda)) = \left\{ \chi \in \mathcal{E}xp_{S_P}(\pi_P) \mid r_P(\lambda) \neq 0 \text{ on the generalized } \chi\text{-eigenspace in } V_P \right\}.$$

The main theorem of this paper is the following (Theorem 4.7):

**Main Theorem.** The representation $\pi$ is $H$-square integrable with respect to $\lambda$ if and only if for every $\sigma$-split parabolic subgroup $P$, the condition $|\chi(s)| < 1$ holds for all $\chi \in \mathcal{E}xp_{S_P}(\pi_P, r_P(\lambda))$ and $s \in S^-_P \setminus Z_G S^+_P$.

This is an analogue, and even a generalization, of Casselman’s criterion. Actually, if one applies the above theorem to the symmetric space $G/H = (G_1 \times G_1)/G_1$ where the involution is the permutation of two factors (referred to as the group case), then one recovers Casselman’s criterion for the group $G_1$.

Let us summarize the contents of this paper. In Section 1 we prepare notation and several definitions used throughout. In Section 2, we recall the analogue of Cartan decomposition for $p$-adic symmetric spaces given by
Benéist-Oh [BO] and Delorme-Sécherre [DS]. After that, we give two ingredients for the proof of the main theorem; a disjointness assertion (Proposition 2.3) and some volume estimate (Proposition 2.6). Section 3 is essentially a recollection of [KT, §5] and [L, §2] on the asymptotic behavior of $H$-matrix coefficients described by the mapping $r_F$. Section 4 is devoted to the proof of the main theorem. We give simple examples of $H$-square integrable representations in Section 5.

1. Notation and definitions

1.1. Basic notation.

Let $F$ be a non-archimedean local field with the absolute value $|·|_F$. The valuation ring of $F$ is denoted by $O_F$ and the order of the residue field of $F$ by $q_F$. Throughout this paper, we assume that the residual characteristic of $F$ is not equal to 2. Let $\nu_F : F^\times \to \mathbb{Z}$ denote the additive valuation defined by

$$|x|_F = q_F^{-\nu_F(x)} \quad (x \in F^\times).$$

Let $G$ be a connected reductive group defined over $F$ and $\sigma$ an $F$-involution on $G$. The $F$-subgroup $\{h \in G \mid \sigma(h) = h\}$ consisting of all $\sigma$-fixed points of $G$ is denoted by $H$. Let $Z$ be the $F$-split component of $G$, that is, the largest $F$-split torus lying in the center of $G$. Note that $Z$ is $\sigma$-stable. We put $Z_0 = \left(\{z \in Z \mid \sigma(z) = z^{-1}\}\right)^0$, which we shall call the $(\sigma, F)$-split component of $G$. Here and henceforth, $(\cdot)^0$ stands for the identity component in the Zariski topology.

The group $G(F)$ consisting of all the $F$-rational points of $G$ is denoted by $G$. Similarly, for any $F$-subgroup $R$ of $G$, we shall write $R = R(F)$.

For a connected $F$-group $M$, let $X^*(M)$ (resp. $X^*(M)_F$) denote the free $\mathbb{Z}$-module of rational (resp. $F$-rational) characters of $M$. If $A$ is an $F$-split torus, one has $X^*(A) = X^*(A)_F$. Let $M$ be a connected reductive $F$-group with its $F$-split component $A$. We put $a_M = \text{Hom}(X^*(M)_F, \mathbb{R})$.

The natural homomorphism $X^*(M)_F \to X^*(A)$ defined by restriction induces an isomorphism

$$(1.1.1) \quad \text{Hom}(X^*(A), \mathbb{R}) \simeq a_M.$$

We define a homomorphism

$$\nu_M : M = M(F) \to a_M$$

by

$$\langle \alpha, \nu_M(m) \rangle = \nu_F(m^\alpha)$$

for all $m \in M$ and $\alpha \in X^*(M)_F$. We can define $\langle \alpha, \nu_M(m) \rangle$ also for $\alpha \in X^*(A)$ through the identification $(1.1.1)$. The kernel of $\nu_M$ is denoted by $M^1$. Note that $A^1 = A(O_F)$ for an $F$-split torus $A$. 

1.2. \textit{H}-matrix coefficients of representations.

Let $C^\infty(G/H)$ denote the space of all smooth $\mathbb{C}$-valued functions on $G/H$ on which $G$ acts by left translation. A smooth representation $(\pi, V)$ of $G$ is said to be \textit{H-distinguished} if $(V^*)^H \neq \{0\}$. Take a non-zero $\lambda \in (V^*)^H$ and consider the functions on $G$ given by

$$\varphi_{\lambda,v}(g) = \langle \lambda, \pi(g^{-1})v \rangle \quad (g \in G)$$

for $v \in V$. We call these functions \textit{H-matrix coefficients} of $\pi$ defined by $\lambda$. Let us identify right $H$-invariant functions on $G$ with functions on $G/H$. Then, H-matrix coefficients belong to $C^\infty(G/H)$ and the mapping $T_\lambda : V \to C^\infty(G/H), \quad T_\lambda(v) = \varphi_{\lambda,v}$ gives a $G$-morphism. Any realization of $V$ in $C^\infty(G/H)$ is determined by an $H$-invariant linear form in this way.

Let $\omega_0$ be a quasi-character of $Z_0 = \mathbb{Z}_0(F)$. A smooth representation $(\pi, V)$ is called an $\omega_0$-representation if $Z_0$ acts on $V$ by the the character $\omega_0$. If $(\pi, V)$ is an $H$-distinguished $\omega_0$-representation, then for any $\lambda \in (V^*)^H$, the image $T_\lambda(V)$ of $V$ is contained in the space $C^\infty_{\omega_0}(G/H)$ consisting of functions $\varphi \in C^\infty(G/H)$ which satisfy

$$\varphi(zgH) = \omega_0(z)^{-1}\varphi(gH) \quad (z \in Z_0, \; gH \in G/H).$$

1.3. \textit{H}-square integrability.

Both $G$ and $H$ are unimodular groups. Thus the quotient space $G/Z_0H$ carries a left $G$-invariant measure, which is denoted by $\int_{G/Z_0H} dg$. Let $\omega_0$ be a unitary character of $Z_0$. We define $L^2_{\omega_0}(G/H)$ to be the space of $(L^2$-classes of) all functions $\varphi \in C^\infty_{\omega_0}(G/H)$ satisfying

$$\int_{G/Z_0H} |\varphi(g)|^2 dg < \infty.$$

Take a non-zero $\lambda \in (V^*)^H$. We say that an $H$-distinguished $\omega_0$-representation $(\pi, V)$ is \textit{H-square integrable with respect to} $\lambda$ if the H-matrix coefficients $\varphi_{\lambda,v}$ defined by $\lambda$ are square integrable on $G/Z_0H$ for all $v \in V$, or equivalently, if $T_\lambda(V)$ is contained in $L^2_{\omega_0}(G/H)$. Note that this definition agrees with the one given in the Introduction, since $Z_0H/Z_0H$ is compact. Irreducible $H$-square integrable representations are said to be \textit{in the discrete series} for $G/H$.

In [KT], we have put the following definition: An $H$-distinguished admissible $\omega_0$-representation $(\pi, V)$ of $G$ is said to be \textit{(H, $\lambda$)-relatively cuspidal} if all the $H$-matrix coefficients of $\pi$ defined by $\lambda$ are compactly supported modulo $Z_0H$. We gave examples of such representations in [KT] §8. It is clear from the definition that (H, $\lambda$)-relatively cuspidal $\omega_0$-representations are $H$-square integrable with respect to $\lambda$ provided that $\omega_0$ is unitary.
1.4. Tori and roots associated to symmetric spaces.

We shall fix notation on tori, roots, and parabolic subgroups associated to the involution $\sigma$. For the reference, see [MII] and also [KT, §2].

A torus $S$ is said to be $(\sigma, F)$-split if it is $F$-split and $\sigma(s) = s^{-1}$ for all $s \in S$. We fix a maximal $(\sigma, F)$-split torus $S_0$ of $G$ and a maximal $F$-split torus $A_0$ containing $S_0$. Then $A_0$ is necessarily $\sigma$-stable, so $\sigma$ acts naturally on $X^*(A_0)$.

Let $\Phi \subset X^*(A_0)$ be the root system of $(G, A_0)$. It is $\sigma$-stable. We choose a $\sigma$-basis $\Delta$ of $\Phi$ that has the property

$$\alpha > 0, \sigma(\alpha) \neq -\alpha \implies \sigma(\alpha) < 0$$

in the corresponding order. The subset of all $\sigma$-fixed roots in $\Phi$ (resp. $\Delta$) is denoted by $\Phi_\sigma$ (resp. $\Delta_\sigma$).

Let $p : X^*(A_0) \to X^*(S_0)$ be the homomorphism defined by restriction to $S_0$. It is surjective and its kernel coincides with the submodule of all $\sigma$-fixed elements of $X^*(A_0)$. Let us put

$$\overline{\Phi} = p(\Phi) \setminus \{0\} = p(\Phi \setminus \Phi_\sigma).$$

It is well-known that $\overline{\Phi}$ is a root system in $X^*(S_0)$ with a basis

$$\overline{\Delta} = p(\Delta) \setminus \{0\} = p(\Delta \setminus \Delta_\sigma).$$

For each subset $\mathcal{T}$ of $\overline{\Delta}$, we consider the subset

$$[\mathcal{T}] := (p^{-1}(\mathcal{T}) \cap \Delta) \cup \Delta_\sigma$$

of $\Delta$. In this paper we say that a subset of $\Delta$ is $\sigma$-split if it is of the form $[\mathcal{T}]$ for some $\mathcal{T} \subset \overline{\Delta}$. This terminology agrees with that in [KT, 2.3]. The correspondence $\mathcal{T} \mapsto [\mathcal{T}]$ is an inclusion-preserving bijection between subsets of $\overline{\Delta}$ and $\sigma$-split subsets of $\Delta$. The inverse of this correspondence is given by $I \mapsto p(I \setminus \Delta_\sigma)$ for a $\sigma$-split subset $I$ of $\Delta$.

Note that maximal proper $\sigma$-split subsets of $\Delta$ are written in the form $[\overline{\Delta} \setminus \{\overline{\alpha}\}]$ for some $\overline{\alpha} \in \overline{\Delta}$.

1.5. Parabolic subgroups associated to symmetric spaces.

A parabolic $F$-subgroup $P$ of $G$ is said to be $\sigma$-split if $P$ and $\sigma(P)$ are opposite. In such a case, we always take $M = P \cap \sigma(P)$ for a ($\sigma$-stable) Levi subgroup of $P$.

Let $P_0$ be the minimal parabolic $F$-subgroup of $G$ corresponding to the choice of $\Delta$ as in [1.4]. The centralizer $Z_G(A_0)$ of $A_0$ in $G$ is denoted by $M_0$, which is a Levi subgroup of $P_0$. Put $M_0 = Z_G(S_0)$ and $P_0 = P_0M_0$. Then $P_0$ is a minimal $\sigma$-split parabolic $F$-subgroup of $G$ with a $\sigma$-stable Levi subgroup $M_0$. Let $U_0$ be the unipotent radical of $P_0$.

For each subset $I$ of $\Delta$, let $P_I$ denote the standard parabolic $F$-subgroup (that contains $P_0$) of $G$ corresponding to $I$. If $I$ is a $\sigma$-split subset of $\Delta$, then $P_I$ is $\sigma$-split. Standard $\sigma$-split parabolic $F$-subgroups of $G$ correspond to $\sigma$-split subsets of $\Delta$ in this way. Furthermore, it is remarked in [KT].
2.5] that any $\sigma$-split parabolic $F$-subgroup of $G$ is of the form $\gamma^{-1}P_{I}\gamma$ for some $\sigma$-split $I \subseteq \Delta$ and $\gamma \in (M_{0}H)(F)$. For each $\sigma$-split $I \subseteq \Delta$, we put $M_{I} = P_{I} \cap \sigma(P_{I})$. Note that $P_{\Delta_{\sigma}}$ and $M_{\Delta_{\sigma}}$ coincide with $P_{0}$ and $M_{0}$ respectively. Let $U_{I}$ be the unipotent radical of $P_{I}$, so that we have a Levi decomposition $P_{I} = M_{I}U_{I}$. The $F$-split component (resp. $(\sigma, F)$-split component) of $M_{I}$ is denoted by $A_{I}$ (resp. $S_{I}$). These are also called the $F$-split and $(\sigma, F)$-split component of $P_{I}$ respectively.

We can describe $\sigma$-split parabolic subgroups and their $(\sigma, F)$-split components also by using subsets of the restricted basis $\Delta$ through the bijective correspondence $I \mapsto [\bar{I}]$ in [1,4] We shall occasionally use notation based on restricted roots if it is convenient. For each subset $\mathcal{T}$ of $\Delta$, we put

$$S_{\mathcal{T}} = \left(\bigcap_{\pi \in \mathcal{T}} \ker(\pi: S_{0} \to G_{m})\right)^{0}.$$  

It is easy to see the following equalities:

(1.5.1)  
$$S_{0} = S_{\mathcal{T}} : S_{\Delta \setminus \mathcal{T}}, \quad S_{\mathcal{T}} \cap S_{\Delta \setminus \mathcal{T}} = S_{\Delta} = Z_{0}. $$

Observe that if $\mathcal{T} \subseteq \Delta$ corresponds to a $\sigma$-split subset $I \subseteq \Delta$, then $S_{\mathcal{T}}$ coincides with $S_{I}$.

For a positive real number $\varepsilon \leq 1$ and a $\sigma$-split subset $I$ of $\Delta$, we put

$$S_{I}^{\varepsilon}(\varepsilon) = \left\{ s \in S_{I} \big| |s^{\alpha}|_{F} \leq \varepsilon (\alpha \in \Delta \setminus I) \right\},$$

$$S_{0,I}^{\varepsilon}(\varepsilon) = \left\{ s \in S_{0} \big| |s^{\alpha}|_{F} \leq \varepsilon (\alpha \in \Delta \setminus I), \quad |s^{\alpha}|_{F} \leq 1 (\alpha \in I) \right\},$$

and

$$I_{I}S_{0}^{-}(\varepsilon) = \left\{ s \in S_{0} \big| |s^{\alpha}|_{F} \leq \varepsilon (\alpha \in \Delta \setminus I), \quad \varepsilon < |s^{\alpha}|_{F} \leq 1 (\alpha \in I) \right\}. $$

For example, $S_{I}^{-}(\varepsilon)$ can be written also as

(1.5.2)  
$$S_{I}^{-}(\varepsilon) = \left\{ s \in S_{I} = S_{\mathcal{T}} \big| |\pi^{\mathcal{T}}|_{F} \leq \varepsilon \ (\pi \in \Delta \setminus \mathcal{T}) \right\}.$$

if $I$ corresponds to $\mathcal{T} \subseteq \Delta$ as in [1,4] We abbreviate $S_{I}^{-}(1)$ as $S_{I}^{-}$ and write

$$S_{0}^{-} (= S_{\Delta_{\sigma}}^{-}(1)) = \left\{ s \in S_{0} \big| |s^{\alpha}|_{F} \leq 1 (\alpha \in \Delta) \right\}.$$

It is obvious from the definition that

$$S_{I}^{-}(\varepsilon) \subset I_{I}S_{0}^{-}(\varepsilon) \subset S_{0,I}^{-}(\varepsilon) \subset S_{0}^{-},$$

and that

$$\varepsilon \leq \varepsilon' \implies S_{0,I}^{-}(\varepsilon) \subset S_{0,I}^{-}(\varepsilon').$$

1.6. Lemma. Let $I$ be a $\sigma$-split subset of $\Delta$.

(1) For any positive real number $\varepsilon \leq 1$, there exists a positive real number $\varepsilon' \leq 1$ such that

$$S_{0,I}^{-}(\varepsilon') \subset S_{I}^{-}(\varepsilon) \cdot S_{0}^{-}. $$
(2) For any positive real number $\varepsilon \leq 1$, there exist a positive real number $\varepsilon'$ and finitely many elements $t_1, \ldots, t_k$ of $S_0^-$ such that

$$I S_0^-(\varepsilon) \subset \bigcup_i S_I^-(\varepsilon') t_i Z_0 S_0^1.$$ 

Moreover one can take $\varepsilon' \leq 1$ if $\varepsilon$ is sufficiently small.

(3) For any positive real number $\varepsilon \leq 1$, one has a decomposition

$$S_0^- = \bigcup_{I \subseteq \Delta; \sigma\text{-split}} I S_0^- (\varepsilon) \text{ (disjoint)}$$

where $I$ ranges over all $\sigma$-split subsets of $\Delta$.

Proof. The proof of (1) is exactly the same as that of [C, 4.3.1]. Regard the union in (3) as

$$\bigcup_{I \subseteq \Delta} I S_0^- (\varepsilon) \quad (I = \{I\})$$

where $\overline{\Delta}$ ranges over all subsets of $\Delta$ including the empty set. Then the assertion of (3) can be seen by the same way as in [C, remark preceding 4.3.4]. For (2), let $\overline{I}$ be the subset of $\overline{\Delta}$ corresponding to $I$. First observe that $S_I \cdot S_{\Delta \setminus I}$ is of finite index in $S_0$ by (1.5.1). We can take a finite set $\Gamma_I$ of representatives of $S_0 / (S_I \cdot S_{\Delta \setminus I})$ from $S_0^-$. We put

$$c = c_I = \min_{\gamma \in \Gamma_I, \overline{\alpha} \in \Delta \setminus \overline{I}} (|\gamma|_F)$$

and take $t_1, \ldots, t_k \in S_0^-$ so that $\bigcup_i t_i Z_0 S_0^1$ contains the subset

$$\left\{ s \in S_0^- \left| \begin{array}{l} \varepsilon < |s|_F, \\
\varepsilon |\overline{\alpha}|_F \leq 1 \ (\overline{\alpha} \in \overline{I}), \\
\varepsilon |\overline{\gamma}|_F \leq 1 \ (\overline{\gamma} \in \Delta \setminus \overline{I}) \end{array} \right. \right\}. $$

Now, let us write $s \in I S_0^- (\varepsilon)$ as $s = s_1 s_2 \gamma$ with $s_1 \in S_I = S_I^-$, $s_2 \in S_{\Delta \setminus \overline{I}}$, and $\gamma \in \Gamma_I$. We show that $s_1 \in S_I^-(\varepsilon')$ for some $\varepsilon'$ and that $s_2 \gamma \in \bigcup_i t_i Z_0 S_0^1$. For any $\overline{\alpha} \in \overline{\Delta} \setminus \overline{I}$, we have

$$|s_1|_F \cdot c \leq |(s_1 \gamma)|_F = |s|_F \leq \varepsilon.$$ 

Therefore $s_1$ belongs to $S_I^-(\varepsilon')$ for $\varepsilon' = \varepsilon c^{-1}$ (see (1.5.2)). Note that $\varepsilon' \leq 1$ if $\varepsilon \leq c = c_I$. On the other hand, we have

$$\varepsilon < |s|_F = |(s_2 \gamma)|_F \leq 1$$

for each $\overline{\alpha} \in \overline{I}$, while

$$c \leq |\gamma|_F = |(s_2 \gamma)|_F \leq 1$$

for each $\overline{\alpha} \in \overline{\Delta} \setminus \overline{I}$. This completes the proof. $\square$
2. Relative Cartan decomposition

In this section we recall the analogue of Cartan decomposition for \( p \)-adic symmetric spaces given by Benoist-Oh [BO] and Delorme-Sécherre [DS]. Concerning this decomposition, we shall give a disjointness result (Proposition 2.3) and some volume estimate (Proposition 2.6). These will be key ingredients for the proof of the main theorem in Section 4.

2.1. The analogue of Cartan decomposition.

Let us fix the data \((S_0, A_0, \Delta)\) as in 1.4. We shall write \(S_0^+ I(\varepsilon), S_0^- I(\varepsilon), I S_0^+ \) respectively. We recall the analogue of Cartan decomposition given in [BO] and [DS]. Let us state it in the following form as in [KT, 3.4]: There exists a compact subset \( \Omega \) of \( G \) and a finite subset \( \Gamma \) of \((M_0 H)(F)\) such that
\[
G = \Omega S_0^+ \Gamma H.
\]

2.2. Groups with Iwahori factorization.

Let \( K \) be a \( \sigma \)-stable open compact subgroup of \( G \) and \( P = MU \) a \( \sigma \)-split parabolic subgroup of \( G \) standard with respect to \( \Delta \). We put \( P^- = \sigma(P) \) and \( U^- = \sigma(U) \), so that \( P \cap P^- = M \) and \( P^- = MU^- \). We say that \( K \) has the Iwahori factorization with respect to \( P \) if the product map induces a bijection
\[
U_K^- \times M_K \times U_K \simeq K,
\]
where \( U_K^- = U^- \cap K \), \( M_K = M \cap K \), and \( U_K = U \cap K \).

Let \( K_{\text{max}} \) be an \( A_0 \)-good maximal compact subgroup of \( G \). We take a \( \sigma \)-stable open compact subgroup \( K_0 \), having the Iwahori factorization with respect to \( P_0 \), such that \( S_0^1 \subset K_0 \subset K_{\text{max}} \). For example, it suffices to take \( K_0 = B \cap \sigma(B) \) for an Iwahori subgroup \( B \) contained in \( K_{\text{max}} \). Choose a finite set \( \Xi \subset G \) such that \( \Omega \subset \bigcup_{\xi \in \Xi} \xi K_0 \). We may decompose \( G \) as
\[
G = \bigcup_{\xi \in \Xi} \bigcup_{\gamma \in \Gamma \backslash \xi K_0^+} \xi K_0^+ \gamma H,
\]
hence
\[
(2.2.1) \quad G/Z_0 H = \bigcup_{\xi \gamma \in S^+_0} \xi K_0^+ \gamma H/H.
\]
It is not known whether these are disjoint. However we can assert the following.

2.3. Proposition. For each \( \gamma \in (M_0 H)(F) \), the union
\[
\bigcup_{\xi \gamma \in S^+_0} K_0^+ \gamma H
\]
is disjoint.
Proof. We use the mapping $\tau : G \to G$ given by $\tau(g) = gs(g)^{-1}$ and the action

$$(g, x) \mapsto g * x = gxs(g)^{-1} \quad (g \in G, x \in \tau(G))$$

of $G$ on $\tau(G)$ to study cosets modulo $H$. Suppose that $K_0s_1\gamma H = K_0s_2\gamma H$ for $s_1, s_2 \in S_0^+$. Applying the Iwahori factorization, write

$$K_0 * (s_1^2m_\gamma) = K_0 * (s_2^2m_\gamma)$$

where $m_\gamma = \tau(\gamma) \in M_0$. There is an element $k \in K_0$ such that $k * (s_1^2m_\gamma) = s_2^2m_\gamma$. Thus

$$ks_1^2m_\gamma = s_2^2m_\gamma \sigma(k).$$

Using the Iwahori factorization, write $k$ and $\sigma(k) \in K_0$ as

$$k = u_1^-m_1u_1, \quad \sigma(k) = u_2^-m_2u_2$$

where $u_i^- \in (U^-_{0})_{K_0}$, $m_i \in (M_0)_{K_0}$, and $u_i \in (U_0)_{K_0}$. Then we have

$$u_1^-m_1u_1s_1^2m_\gamma = s_2^2m_\gamma u_2^-m_2u_2,$$

hence

$$u_1^- \cdot (m_1s_1^2m_\gamma) \cdot (m_\gamma^{-1}(s_1^2)^{-1}u_1s_1^2m_\gamma) = (m_\gamma s_2^2u_2 (s_2^2)^{-1}m_\gamma^{-1}) \cdot (m_\gamma s_2^2m_2) \cdot u_2.$$

By the uniqueness of expressions in $U^-_0M_0U_0$, we must have

$$(2.3.1) \quad m_1s_1^2m_\gamma = m_\gamma s_2^2m_2.$$

Now we use the usual Cartan decomposition (see e.g. [W, 1.1(4)]) for $M_0$ to write $m_\gamma \in M_0$ as

$$m_\gamma = k_1ak_2 \quad (k_1, k_2 \in M_0 \cap K_{\text{max}}, a \in \overline{M}_{0}^{\Delta_{\gamma}}).$$

Here $\overline{M}_{0}^{\Delta_{\gamma}}$ denotes the set of elements $m$ of $M_0$ satisfying

$$\langle \alpha, \nu_{M_0}(m) \rangle \leq 0$$

for all $\alpha \in \Delta_{\gamma}$. Note that $m_1, m_2 \in M_0 \cap K_0 \subset M_0 \cap K_{\text{max}}$, and $S_0^+ \subset \overline{M}_{0}^{\Delta_{\gamma}}$. Since $s_1^2$ and $s_2^2$ are central in $M_0$, we have

$$m_1k_1 \cdot (s_1^2a) \cdot k_2 = k_1 \cdot (s_2^2a) \cdot k_2m_2$$

by (2.3.1). From the uniqueness of Cartan decomposition, we may conclude that

$$s_1^2a \equiv s_2^2a \mod M_0^1.$$

Since $M_0^1 \cap S_0 = S_0^1$, we have $s_1^2 \equiv s_2^2 \mod S_0^1$, and in turn, $s_1 \equiv s_2 \mod S_0^1$. □
2.4. **Some volume computation.**

Let us fix a left $G$-invariant measure $\int_{G/H} dg$ on the quotient space $G/H$. For each open compact subgroup $K$ of $G$ and an element $a \in G$, the set $KaH/H$ is open and compact in $G/H$. Let $\int_{KaH/H} dg$ denote the restriction of $\int_{G/H} dg$ to the open subset $KaH/H$.

We have to evaluate the volume of $K_{0s\gamma}H/H$. For this purpose we first need to take a $\sigma$-stable open compact subgroup $K$ from an adapted family given in [KT, 4.3]: Besides the Iwahori factorization, we need a good filtration ([KT, 4.3 (2)]) inside $K$ (and its subgroups) to use the result [KT, 4.6]. Then we can compute the volume of $K_{sH/H}$ for such a $K$ and $s \in S_0^+$.

Let $\delta_P$ denote the modulus character of $P = P(F)$ for a parabolic $F$-subgroup $P$ of $G$.

2.5. **Lemma.** If $K$ is a $\sigma$-stable open compact subgroup as above, then

$$\text{vol}(KsH/H) = \delta_P(s) \cdot \text{vol}(KeH/H)$$

for all $s \in S_0^+$.

**Proof.** We use the mapping $\tau$ to identify $\tau(G)$ with $G/H$ and transport the left $G$-invariant measure on $G/H$ to the measure on $\tau(G)$ invariant under the $*$-action of $G$. The subset $KsH/H$ is identified with $(Ks)_*e$ and $\text{vol}((Ks)_*e) = \text{vol}(s*(s^{-1}Ks)_*e) = \text{vol}((s^{-1}Ks)_*e)$

by the invariance under $\ast$. Put $K_s = s^{-1}K_s$. The Iwahori factorization

$$K = U_{0,K}^- M_{0,K} U_{0,K}$$

of $K$ with respect to $P_0$ implies

$$K_s = (s^{-1}U_{0,K}^- s) \cdot M_{0,K} \cdot (s^{-1}U_{0,K}^- s)$$

for $s \in S_0^+$ with

$$s^{-1}U_{0,K}^- s \supset U_{0,K}^-; \quad s^{-1}U_{0,K}^- s \subset U_{0,K}.$$

Put $K'_s = K_s \cap \sigma(K_s) = s^{-1}Ks \cap sKs^{-1}$. It is $\sigma$-stable and has a factorization

$$K'_s = (sU_{0,K}^- s^{-1})M_{0,K}(s^{-1}U_{0,K}^- s).$$

We have $K'_s \subset K$. So we may apply [KT 4.6] to the group $K'_s$, which asserts that

$$s^{-1}U_{0,K}^- s \subset (sU_{0,K}^- s^{-1})M_{0,K}H.$$

As a result, we have

$$K_s * e = (s^{-1}U_{0,K}^- s)M_{0,K}(s^{-1}U_{0,K}^- s) * e$$

$$\subset (s^{-1}U_{0,K}^- s)M_{0,K}(sU_{0,K}^- s^{-1})M_{0,K} * e = (s^{-1}U_{0,K}^- s)M_{0,K} * e.$$
We also note that \((s^{-1}U_{0,K}s)M_{0,K} = P_0^− \cap K_s\). This shows that the orbit \((P_0^− \cap K_s) * e\) of the subgroup \(P_0^− \cap K_s\) of \(K_s\) coincides with the whole \(K_s\)-orbit \(K_s * e\). Now, fix Haar measures \(du^−\) and \(dm\) on \(s^{-1}U_{0,K}s\) and \(M_{0,K}\) respectively. The \((P_0^− \cap K_s)\)-invariant measure

\[
f \mapsto \int_{s^{-1}U_{0,K}s} \int_{M_{0,K}} f(u^− * (m * e)) \, dm \, du^−
\]
on the orbit \(K_s * e\) has to be \(K_s\)-invariant. Consequently, the volume of \(K_s * e\) is proportional to

\[
\text{vol}(s^{-1}U_{0,K}s) \cdot \text{vol}(M_{0,K}) = \text{vol}(U_{0,K}) \cdot \delta_{P_0}(s) \cdot \text{vol}(M_{0,K})
\]

which is a constant multiple of \(\delta_{P_0}(s)\). The constant turns out to be the volume of \(KeH/H\), if we consider \(s = e\).

\[\square\]

2.6. **Proposition.** Let \(K\) be an arbitrary open compact subgroup of \(G\). For each \(\gamma \in \Gamma\), there exist positive real constants \(c_1\) and \(c_2\) such that

\[
c_1 \cdot \delta_{P_0}(s) \leq \text{vol}(Ks\gamma H/H) \leq c_2 \cdot \delta_{P_0}(s)
\]

for all \(s \in S_0^+\).

**Proof.** We put \(K' = \gamma^{-1}K\gamma\), \(S_0' = \gamma^{-1}S_0\gamma\), and \(P_0' = \gamma^{-1}P_0\gamma\). Then \(P_0'\) is a \(\sigma\)-split parabolic subgroup with the \((\sigma,F)\)-split component \(S_0'\). We have

\[
\text{vol}(Ks\gamma H/H) = \text{vol}(\gamma K'\gamma^{-1}s\gamma H/H) = \text{vol}(K's'H/H)
\]

where \(s' = \gamma^{-1}s\gamma \in (S_0')^+ = \gamma^{-1}S_0\gamma\). Take a member \(K''\) from the adapted family (corresponding to the \(\gamma\)-conjugated data) such that \(K'' \subset K'\). Then we have

\[
\text{vol}(K''s'H/H) \leq \text{vol}(K's'H/H) \leq [K' : K''] \cdot \text{vol}(K''s'H/H).
\]

Thus the claim follows from Lemma [2.5].

\[\square\]

3. **Asymptotic behavior of \(H\)-matrix coefficients**

In this section we shall describe asymptotic behavior of \(H\)-matrix coefficients through the mapping \(r_P\) (see [3,2]). This section is essentially a recollection of [3,1], §5 and [3,2].

From now on, we shall briefly say that \(P\) is a \(\sigma\)-split parabolic subgroup of \(G\) if \(P\) is the group of \(F\)-points of a \(\sigma\)-split parabolic \(F\)-subgroup \(P = P_I\) of \(G\) etc, by abuse of terminology.

3.1. **Normalized Jacquet modules.**

For an admissible representation \((\pi, V)\) of \(G\) and a parabolic subgroup \(P = MU\) of \(G\), the normalized Jacquet module of \(\pi\) along \(P\) is denoted by \((\pi_P, V_P)\): The space \(V_P\) is defined as the quotient \(V/V(U)\) where \(V(U)\) is the subspace of \(V\) spanned by all the elements of the form \(\pi(u)v - v\) \((u \in U, v \in V)\). The action \(\pi_P\) of \(M\) on \(V_P\) is normalized so that

\[
\pi_P(m)j_P(v) = \delta_P^{-1/2}(m)j_P(\pi(m)v)
\]
3.2. The mapping $r_P$.

When $P = MU$ is a $\sigma$-split parabolic subgroup, we have defined in [KT] a linear mapping

$$r_P : (V^*)^H \to (V^*_P)^{M \cap H}$$

between the spaces of invariant linear forms. If $v \in V$ is a canonical lifting (in the sense of [C, §4]) of $\bar{v} \in V_P$ with respect to a suitable $\sigma$-stable open compact subgroup (a member in an adapted family in [KT] 4.3]), then $r_P(\lambda)$ for $\lambda \in (V^*)^H$ is defined by

$$\langle r_P(\lambda), \bar{v} \rangle = \langle \lambda, v \rangle$$

(see [KT] 5.3 (2) and 5.4). The same mapping was constructed independently by N. Lagier [L] in a different manner. P. Delorme extended the construction of such mappings to any smooth representation $s$ in [D].

In [KT] §5, we gave asymptotic behavior of $H$-matrix coefficients through the mapping $r_P$. The result [KT] Proposition 5.5 can be extended to the next proposition. This is a generalization of Casselman’s result [C, 4.3.3] to symmetric spaces, and was already proved essentially in [L] Théorème 2. The proof provided by [L] invokes Casselman’s result itself. We shall give a proof which do not rely on [C, loc.cit] (but follow the lines similar to [C] as we did in [KT] §5). It would yield [C, loc.cit] when we apply this to the group case.

3.3. Proposition. Let $I$ be a $\sigma$-split subset of $\Delta$ and $P = P_I$ the corresponding $\sigma$-split parabolic subgroup with the $(\sigma, F)$-split component $S = S_I$. Let $(\pi, V)$ be an admissible representation of $G$ and $V_1 \subset V$ a finite dimensional subspace. Then there exists a positive real number $\varepsilon = \varepsilon_I \leq 1$ such that

$$\langle \lambda, \pi(s)v \rangle = \delta_I^{1/2}(s)\langle r_P(\lambda), \pi_P(s)j_P(v) \rangle$$

for all $s \in S_{0,I}(\varepsilon)$, $v \in V_1$, and $\lambda \in (V^*)^H$.

Proof. For a compact subgroup $K$ of $G$, let $p_K : V \to V^K$ denote the projection defined by $p_K(v) = \int_K \pi(k)v \, dk$. For an open compact subgroup $K$ of $G$ (from the adapted family), there is a positive real number $\varepsilon \leq 1$ such that the space $p_K(\pi(s)V^K)$ does not depend on $s \in S_I(\varepsilon)$ and is isomorphic to $(V_P)^{MK}$ by the restriction of $j_P : V \to V_P$. The vectors in $p_K(\pi(s)V^K)$ are called canonical liftings over $(V_P)^{MK}$ with respect to $K$. First, using (1) of Lemma [L] we can even construct the space $p_K(\pi(s)V^K)$ of canonical liftings by taking $s$ from $S_{0,I}(\varepsilon)$ (replacing $\varepsilon$ suitably). This step is entirely the same as the derivation of [C, 4.3.2] from [C, 4.3.1]. Next, choose $K$ small enough so that $V_1 \subset V^K$. Then, $\pi(s)v \in V^{M_{0,K}U_{0,K}}$ for each $v \in V_1$ and $s \in S_{0,I}(\varepsilon)$. By [KT] 5.3 (1), we have

$$\langle \lambda, \pi(s)v \rangle = \langle \lambda, p_{U_{0,K}}(\pi(s)v) \rangle = \langle \lambda, p_K(\pi(s)v) \rangle.$$
Since \( p_K(\pi(s)v) \) is now a vector in the space of canonical liftings, this is further equal to

\[
\langle r_P(\lambda), j_P(p_K(\pi(s)v)) \rangle = \langle r_P(\lambda), j_P(p_{U_0,K}(\pi(s)v)) \rangle
\]

by the definition (3.2.1) of \( r_P(\lambda) \). We use the decomposition \( U_0 = U \cdot U' \)
where \( U' = M \cap U_0 \). This implies that \( U_{0,K} = U_K \cdot U_{K}', \) hence \( p_{U_0,K} = p_{U_K} \circ p_{U_K}' \). Since \( j_P \circ p_{U_K} = j_P \) and \( U_K' \subset M \), the right hand side of (3.3.1) is equal to

\[
\langle r_P(\lambda), j_P(p_{U_K'}(\pi(s)v)) \rangle = \langle r_P(\lambda), p_{U_K'}(j_P(\pi(s)v)) \rangle
\]

Finally, as a vector in the representation \( \pi_P \) of \( M \), \( j_P(v) \) is \( M \)-fixed and thus \( \pi_P(s)j_P(v) \in \langle V_p \rangle_{M_0,K(U_{0,K}^\cap M)} \). Applying [K, 5.3 (1)] for the \( M \cap H \)-invariant linear form \( r_P(\lambda) \), we have

\[
\langle r_P(\lambda), j_P(p_{U_K'}(\pi(s)v)) \rangle = \langle r_P(\lambda), \pi_P(s)j_P(v) \rangle.
\]

This completes the proof. \( \square \)

3.4. Remark. When we take a \( \sigma \)-split parabolic subgroup \( P \) without specifying the initial data \((S_0, A_\emptyset, \Delta)\), we may think of \( P \) as a standard one \( P_I \) for a suitable choice of \((S_0, A_\emptyset, \Delta)\) and \( I \subset \Delta \). However, sometimes we have to fix the data \((S_0, A_\emptyset, \Delta)\) and deal with an arbitrary (possibly non-standard) \( \sigma \)-split parabolic subgroup written in the form \( P = \gamma^{-1}P_I \gamma \) with \( \gamma \in (M_0H)(F) \). The relation as in (3.3) for such a \( P \) can be derived in the same way, and is presented as follows: There exists a positive real number \( \varepsilon = \varepsilon_{\gamma,I} \leq 1 \) such that

\[
\langle \lambda, \pi(\gamma^{-1}s\gamma)v \rangle = \delta_P^{1/2}(\gamma^{-1}s\gamma) \langle r_P(\lambda), \pi_P(\gamma^{-1}s\gamma)j_P(v) \rangle
\]

for all \( s \in S_{0,I}(\varepsilon) \), \( v \in V_1 \), and \( \lambda \in (V^*)^H \).

4. The main theorem

In this section we prepare notation on exponents, give a preliminary result (Proposition 4.3) on the relation between \( H \)-square integrability and exponents, and establish a criterion for \( H \)-square integrability (Theorem 4.17). We also give a non-trivial relation between \( H \)-square integrability and the usual square integrability (Proposition 4.10).

4.1. Exponents.

Let \( Z_1 \) be a closed subgroup of the center of \( G \) and \( \mathcal{X}(Z_1) \) be the set of all quasi-characters of \( Z_1 \). For a smooth representation \((\pi, V)\) of \( G \) and a quasi-character \( \omega \in \mathcal{X}(Z_1) \), we put

\[
V_{\omega, \infty} = \left\{ v \in V \left| \begin{array}{l}
\text{There exists a } d \in \mathbb{N} \text{ such that } \\
(\pi(z) - \omega(z))^dv = 0 \text{ for all } z \in Z_1
\end{array} \right. \right\}.
\]
This is a $G$-submodule of $V$. We put
\[ \mathcal{E}xp_{Z_1}(V) = \mathcal{E}xp_{Z_1}(\pi) = \{ \omega \in \mathcal{X}(Z_1) \mid V_{\omega, \infty} \neq 0 \}. \]
If $(\pi, V)$ is finitely generated and admissible, then the set $\mathcal{E}xp_{Z_1}(\pi)$ is finite and $V$ has a direct sum decomposition
\[ V = \bigoplus_{\omega \in \mathcal{E}xp_{Z_1}(\pi)} V_{\omega, \infty} \]
(see [C, 2.1.9]). Let $Z_1$ and $Z_2$ be closed subgroups of the center of $G$ such that $Z_1 \supset Z_2$. As is easily seen, the mapping
\[ \mathcal{E}xp_{Z_1}(\pi) \to \mathcal{E}xp_{Z_2}(\pi) \]
defined by restriction is surjective.

### 4.2. Exponents along parabolic subgroups.

Let $(\pi, V)$ be a finitely generated admissible representation of $G$. For each parabolic subgroup $P$ of $G$ with the $F$-split component $A$, we consider the finite set $\mathcal{E}xp_A(\pi_P)$. The elements of $\mathcal{E}xp_A(\pi_P)$ are called exponents of $\pi$ along $P$. If $P$ is a $\sigma$-split parabolic subgroup with the $(\sigma, F)$-split component $S$, we also consider the finite set $\mathcal{E}xp_S(\pi_P)$. The mapping
\[ (4.2.1) \quad \mathcal{E}xp_A(\pi_P) \to \mathcal{E}xp_S(\pi_P) \]
defined by restriction is surjective.

Let $P_1 = M_1 U_1$ and $P_2 = M_2 U_2$ be $\sigma$-split parabolic subgroups of $G$ with the $Q$-split components $S_1$ and $S_2$ respectively. If $P_1 \subset P_2$, then $M_1 \subset M_2$ and $S_1 \supset S_2$. The intersection $M_2 \cap P_1$ is a $\sigma$-split parabolic subgroup of $M_2$ having $M_1$ as a $\sigma$-stable Levi subgroup. As is well-known, $(\pi_P)_{M_2 \cap P_1}$ is naturally isomorphic to $\pi_P$ as an $M_1$-module. Through this isomorphism, it is easy to see that
\[ (4.2.2) \quad \chi \in \mathcal{E}xp_{S_1}(\pi_{P_1}) \Rightarrow \chi|_{S_2} \in \mathcal{E}xp_{S_2}(\pi_{P_2}). \]

In the course of the proofs in this section, we need to fix the initial data $(S_0, A_0, \Delta)$ as in [1.4]. In dealing with an arbitrary $\sigma$-split parabolic subgroup $P$, we have to write it as $P = \gamma^{-1} P_I \gamma$ by a $\sigma$-split subset $I \subset \Delta$ and an element $\gamma \in (M_0 H)(F)$. The $(\sigma, F)$-split component of $P$ is then written as $S = \gamma^{-1} S_I \gamma$. We also write $S^- = \gamma^{-1} S_I^{-1}(1) \gamma$ in such a case.

Now, for a given finitely generated admissible representation $(\pi, V)$ of $G$, let us consider the following condition on a $\sigma$-split parabolic subgroup $P$:
\[ (\sharp_P) \quad |\chi(s)| < 1 \text{ for all } \chi \in \mathcal{E}xp_S(\pi_P) \text{ and all } s \in S^- \setminus Z_0 S^1. \]

### 4.3. Proposition.

Let $\omega_0$ be a unitary character of $Z_0$ and $(\pi, V)$ a finitely generated $H$-distinguished admissible $\omega_0$-representation of $G$. If the condition $(\sharp_P)$ is satisfied for every $\sigma$-split parabolic subgroup $P$ of $G$, then $(\pi, V)$ is $H$-square integrable with respect to any $\lambda \in (V^*)^H$. 
Proof. Let $\Gamma$, $K_0$, $\Xi$ be as in 2.2 for the data $(S_0, A_0, \Delta)$. Take a non-zero vector $v_0 \in V$ and let $V_0$ be the finite dimensional subspace of $V$ generated by $\pi(k^{-1}\xi^{-1})v_0$ ($k \in K_0$, $\xi \in \Xi$). Further, let $V_1$ be the finite dimensional subspace of $V$ generated by $\pi(\gamma^{-1})v$ ($\gamma \in \Gamma$, $v \in V_0$). We take a positive real number $\varepsilon \leq 1$ satisfying

- $\varepsilon \leq \varepsilon_{\gamma,I}$ for all $\gamma \in \Gamma$ and all $\sigma$-split $I \subset \Delta$ where $\varepsilon_{\gamma,I}$ is such that (3.4.1) is valid for all $v \in V_1$, and
- $\varepsilon \leq c_I$ for all $\sigma$-split $I \subset \Delta$ where $c_I$ is the constant given in the proof of Lemma 1.6.

We use a disjoint decomposition

$$S_0^+ = \bigcup_{I \subset \Delta, \sigma\text{-split}} I S_0^+(\varepsilon)$$

obtained from Lemma 1.6 (3) for the number $\varepsilon$ as above. Let us put

$$G_{I,\gamma} = \bigcup_{\delta \in I S_0^+(\varepsilon)/Z_0 S_0^1} K_0 \delta s I,\gamma$$

for each $\sigma$-split subset $I \subset \Delta$ and $\gamma \in \Gamma$. Then, by (2.2.1) we have

$$G/Z_0 H = \bigcup_{\xi,\gamma,I} \xi G_{I,\gamma}/H.$$

Now we start to evaluate the $L^2$-norm of the $H$-matrix coefficient $\varphi_{\lambda,v_0}$. It is clear that

$$\int_{G/Z_0 H} \left| \varphi_{\lambda,v_0}(g) \right|^2 dg \leq \sum_{\xi,\gamma,I} \left( \int_{\xi G_{1,\gamma}/H} \left| \varphi_{\lambda,v_0}(g) \right|^2 dg \right).$$

It is enough to study the convergence of

$$\int_{\xi G_{1,\gamma}/H} \left| \varphi_{\lambda,v_0}(g) \right|^2 dg = \int_{G_{1,\gamma}/H} \left| \varphi_{\lambda,\pi(\xi^{-1})v_0}(g) \right|^2 dg$$

for each $\xi$, $\gamma$ and $I$. So the proof of the proposition is completed once the following claim is shown.

Claim. If (2P) is satisfied for $P = \gamma^{-1}P I,\gamma$, then

$$\int_{G_{1,\gamma}/H} \left| \varphi_{\lambda,v}(g) \right|^2 dg < \infty$$

for all $v \in V_0$.

Let us prove this. It is obvious that

$$\int_{G_{1,\gamma}/H} \left| \varphi_{\lambda,v}(g) \right|^2 dg \leq \sum_{\delta \in I S_0^+(\varepsilon)/Z_0 S_0^1} \int_{K_0 \delta s I,\gamma H/H} \left| \varphi_{\lambda,v}(g) \right|^2 dg.$$

There is an element $k_0 \in K_0$ such that

$$\left| \varphi_{\lambda,v}(k \delta \gamma h) \right|^2 \quad (k \in K_0)$$
attains the maximum at $k = k_0$. We have

\[
\int_{K_0 \delta \gamma H/H} |\varphi_{\lambda,v}(g)|^2 dg \leq \text{vol}(K_0 \delta \gamma H/H) \cdot |\varphi_{\lambda,v}(k_0 \delta \gamma h)|^2 \\
\leq C \cdot \delta P_0(\delta) \cdot |\varphi_{\lambda,v}(k_0 \delta \gamma h)|^2
\]

for some positive real constant $C$ which does not depend on $\delta$ by \ref{2.6}. We may replace $\pi(k_0^{-1})v \in V_0$ by $v$. Applying Proposition \ref{3.3} (or \ref{3.4}) along $P = \gamma^{-1}P_I\gamma$, we have

\[
\varphi_{\lambda,v}(\delta \gamma) = \langle \lambda, \pi(\gamma^{-1} \delta^{-1})v \rangle = \langle \lambda, \pi(\gamma^{-1} \delta^{-1})\pi(\gamma^{-1})v \rangle \\
= \delta P(\gamma^{-1} \delta^{-1})^{1/2} \langle r_P(\lambda), \pi_P(\gamma^{-1} \delta^{-1})\pi(\gamma^{-1})v \rangle
\]

since $\delta^{-1} \in I_{S_0^{-1}}(\varepsilon) \subset S_{0,I}(\varepsilon)$ and $\pi(\gamma^{-1})v \in V_1$. Next, we use Lemma \ref{1.6} (2) to write

\[
\delta^{-1} = \delta_I \cdot t_i, \quad \delta_I \in S_I^{-1}(\varepsilon')/Z_0 S_I^1,
\]

where $\varepsilon' \leq 1$ by our choice of $\varepsilon$. Putting $\pi_P(\gamma^{-1} t_i \gamma) \pi(\gamma^{-1})v = \overline{\nu}$ for simplicity, we have

\[
\varphi_{\lambda,v}(\delta \gamma) = \delta P_I(\delta_I t_i)^{-1/2} \langle r_P(\lambda), \pi_P(\gamma^{-1} \delta_I \gamma)\overline{\nu} \rangle.
\]

The function $\langle r_P(\lambda), \pi_P(\cdot)\overline{\nu} \rangle$ on $S = \gamma^{-1}S_I\gamma$ is $S$-finite. Thus it can be written as

\[
(4.3.3) \quad \langle r_P(\lambda), \pi_P(s')\overline{\nu} \rangle = \sum_{\chi \in \text{Exp}_S(\pi_P)} \chi(s') \mathcal{P}_\chi(\nu_S(s'))
\]

for all $s' \in S$ using suitable polynomials $\mathcal{P}_\chi$ on $a_S$ (see \cite[I.2]{W}). Let us write $s'_I = \gamma^{-1}s_I\gamma$. Then, returning to \eqref{4.3.2}, we have a bound for $\int_{K_0 \delta \gamma H/H} |\varphi_{\lambda,v}(g)|^2 dg$ by

\[
C \cdot \delta P_0(\delta_I t_i) \delta P_I(\delta_I t_i)^{-1} \sum_{\chi} \chi(s'_I) \mathcal{P}_\chi(\nu_S(s'_I)) \bigg| \bigg| \\
= C \cdot \delta P_0(t_i) \delta P_I(t_i)^{-1} \sum_{\chi} \chi(s'_I) \mathcal{P}_\chi(\nu_S(s'_I)) \bigg| \bigg| \\
\leq C \cdot \delta P_0 \delta P_I^{-1}(t_i) \sum_{\chi, \lambda'} \chi'(s'_I) \mathcal{P}_\chi(\nu_S(s'_I)) \mathcal{P}_{\lambda'}(\nu_S(s'_I)) \bigg| \bigg|.
\]

Here we used the fact that $\delta P_0 \equiv \delta P_I$ on $S_I$. Returning further to \eqref{4.3.1}, the integral $\int_{G_{I,\gamma}/H} |\varphi_{\lambda,v}(g)|^2 dg$ is bounded by

\[
\sum_{\delta_I \in S_I^{-1}(\varepsilon')/Z_0 S_I^1} \sum_{i} \delta P_0(\delta_I t_i) \delta P_I(t_i) \sum_{\chi, \lambda'} \chi'(s'_I) \mathcal{P}_\chi(\nu_S(s'_I)) \mathcal{P}_{\lambda'}(\nu_S(s'_I)) \bigg| \bigg|.
\]
Now, we may regard \( S^{-}(e')/Z_{0}S_{1}^{1} \) as a set of lattice points in a positive cone. Then the infinite sum
\[
\sum_{s_{1} \in S^{-}(e')/Z_{0}S_{1}^{1}} \left| \chi'(s_{1}') \mathcal{P}_{\chi}(\nu_{S}(s_{1}')) \mathcal{P}_{\chi}(\nu_{S}(s_{1}')) \right|
\]
is essentially a power series with polynomial coefficients. This series converges if \( |\chi'(s_{1}')| < 1 \) for all \( \chi \) and \( s_{1}' = \gamma^{-1}s_{1}\gamma' \) with \( s_{1}' \in S^{-}(e')/Z_{0}S_{1}^{1} \), except for those \( s_{1}' \in Z_{0}S_{1}^{1} \) which represent the origin in the lattice. Thus the condition \( \left( 3 \right) \) for \( P = \gamma^{-1}P_{1}\gamma' \) is sufficient for the convergence of \( \int_{G_{1}/H} |\varphi_{\lambda,v}(g)|^{2} \, dg \). \( \square \)

4.4. Exponents with respect to \( \lambda \).
Let \((\pi, V)\) be a smooth representation of \( G \) and take an \( H \)-invariant linear form \( \lambda \) on \( V \). We define
\[
\mathcal{E}\text{xp}_{Z_{0}}(\pi, \lambda) := \left\{ \omega \in \mathcal{E}\text{xp}_{Z_{0}}(\pi) \mid \lambda|_{V_{\omega, \infty}} \neq 0 \right\}.
\]
Assume that \((\pi, V)\) is finitely generated and admissible. For each \( \sigma \)-split parabolic subgroup \( P \) of \( G \) with the \( (\sigma, F) \)-split component \( S \), we consider the subset \( \mathcal{E}\text{xp}_{S}(\pi_{P}, r_{P}(\lambda)) \) of \( \mathcal{E}\text{xp}_{S}(\pi_{P}) \). As a relation similar to \( \left( 4.2.2 \right) \), we have the following.

4.5. Lemma. Let \( P_{1} \) and \( P_{2} \) be \( \sigma \)-split parabolic subgroups of \( G \) such that \( P_{1} \subset P_{2} \), with the \( (\sigma, F) \)-split components \( S_{1} \) and \( S_{2} \) respectively. Then,
\[
\chi \in \mathcal{E}\text{xp}_{S_{1}}(\pi_{P_{1}}, r_{P_{1}}(\lambda)) \implies \chi|_{S_{2}} \in \mathcal{E}\text{xp}_{S_{2}}(\pi_{P_{2}}, r_{P_{2}}(\lambda)).
\]

Proof. Suppose that \( \chi \in \mathcal{E}\text{xp}_{S_{1}}(\pi_{P_{1}}, r_{P_{1}}(\lambda)) \). For each \( \tilde{v}_{1} \in (V_{P_{1}})_{\chi, \infty} \), we can take a \( \tilde{v}_{2} \in (V_{P_{2}})_{|\chi|_{S_{2}, \infty}} \) such that \( j_{M_{2} \cap P_{1}}(\tilde{v}_{2}) = \tilde{v}_{1} \) (in the notation of \( \left( 1.2 \right) \) regarding \( \left( 4.2.2 \right) \)). Applying Proposition \( \left( 3.3 \right) \) to the \( M_{2} \cap H \)-matrix coefficients, we have
\[
\langle r_{P_{2}}(\lambda), \pi_{P_{2}}(s)\tilde{v}_{2} \rangle = \delta_{M_{2} \cap P_{1}}(s)^{1/2} \langle r_{M_{2} \cap P_{1}}(r_{P_{2}}(\lambda)), \pi_{P_{1}}(s)\tilde{v}_{1} \rangle
\]
at least for some \( s \in S_{1} \). The right hand side is written as
\[
\delta_{M_{2} \cap P_{1}}(s)^{1/2} \langle r_{P_{1}}(\lambda), \pi_{P_{1}}(s)\tilde{v}_{1} \rangle
\]
by the transitivity result \( r_{M_{2} \cap P_{1}} \circ r_{P_{2}} = r_{P_{1}} \) given in \( \left[ KT \right] \) Proposition 5.9 (and also \( \left[ L \right] \) Théorème 3)). Hence \( r_{P_{2}}(\lambda) \) cannot be identically zero on \( (V_{P_{2}})_{\chi|_{S_{2}, \infty}} \) provided that \( r_{P_{1}}(\lambda)|(V_{P_{1}})_{\chi, \infty} \neq 0 \). \( \square \)

Fix a non-zero \( \lambda \in (V^{*})^{H} \). Let us take up the following condition on a \( \sigma \)-split parabolic subgroup \( P \):
\[
(\sharp_{P, \lambda}) \quad |\chi(s)| < 1 \quad \text{for all} \quad \chi \in \mathcal{E}\text{xp}_{S}(\pi_{P}, r_{P}(\lambda)) \quad \text{and all} \quad s \in S^{-} \setminus Z_{0}S_{1}^{1}.
\]

4.6. Lemma. If the condition \( (\sharp_{P, \lambda}) \) holds for every maximal \( \sigma \)-split parabolic subgroup \( P \) of \( G \), then it holds for every \( \sigma \)-split parabolic subgroup \( P \) of \( G \).
Proof. It is enough to derive (\(E_{P,\lambda}\)) for a standard \(P = P_I\) assuming (\(E_{P,\lambda}\)) for all maximal standard \(P\). For a given \(\sigma\)-split subset \(I\) of \(\Delta\), let us write
\[
\Delta \setminus I = \{\varpi_1, \varpi_2, \ldots, \varpi_r\}.
\]

For each \(i\), let \(P_i\) be the maximal standard \(\sigma\)-split parabolic subgroup corresponding to \(\{\varpi \neq \varpi_i\}\) (see the remark at the end of 4.4). These are maximal \(\sigma\)-split parabolic subgroups containing \(P_I\). The \((\sigma, F)\)-split component \(S_i\) of \(P_i\) is given by
\[
S_i = S_{\Delta \setminus \{\varpi_i\}} = \left( \bigcap_{\varpi \neq \varpi_i} \ker(\varpi) \right)^0.
\]

The \((\sigma, F)\)-split component \(S = S_I\) of \(P = P_I\) can be decomposed as \(S = S_1 S_2 \ldots S_r\), so that \(S_1 S_2 \ldots S_r\) is of finite index in \(S\). Thus for any given \(s \in S^-\), there exists an integer \(m\) such that \(s^m \in S_1 S_2 \ldots S_r\). We may write \(s^m = s_1 s_2 \ldots s_r\) for each \(i\). Now, for any \(\chi \in \mathcal{E}xp_S(\pi_P, r_P(\lambda))\), we have \(\chi|_{S_i} \in \mathcal{E}xp_{S_i}(\pi_{P_i}, r_{P_i}(\lambda))\) by Lemma 4.5. Therefore, assumptions (\(E_{P,\lambda}\)) for all \(P = P_I\) imply that
\[
|\chi(s)|^m = |\chi(s^m)| = |\chi(s_1)| |\chi(s_2)| \ldots |\chi(s_r)| < 1.
\]
This completes the proof. \(\Box\)

Now we shall state the main theorem of this paper.

4.7. Theorem. Let \(\omega_0\) be a unitary character of \(Z_0\) and \((\pi, V)\) a finitely generated \(H\)-distinguished admissible \(\omega_0\)-representation of \(G\). Then, for a non-zero \(H\)-invariant linear form \(\lambda\) on \(V\), the representation \((\pi, V)\) is \(H\)-square integrable with respect to \(\lambda\) if and only if the condition (\(E_{P,\lambda}\)) is satisfied for every \(\sigma\)-split parabolic subgroup \(P\) of \(G\).

Proof. Note that only the exponents in \(\mathcal{E}xp_S(\pi_P, r_P(\lambda))\) contribute to the evaluation of \(\int_{G/Z_0H} |\varphi_{\lambda,v}(g)|^2 dg\) in the proof of Proposition 4.3 specifically at the expression (4.3.3). So the if part is already proved in 4.3. To prove the only if part, suppose that (\(E_{P,\lambda}\)) fails for some \(\sigma\)-split parabolic subgroup \(P\). By Lemma 4.6 we may suppose that \(P\) is a maximal one, say, \(P = \gamma^{-1} P_I \gamma\) with \(I = \{\Delta \setminus \{\varpi_i\}\}\) for some \(\varpi_i \in \Delta\). Then there exists an exponent \(\chi \in \mathcal{E}xp_S(\pi_P, r_P(\lambda))\) and an element \(s = \gamma^{-1} s_I \gamma \in S^- \setminus Z_0 S^1, s_I \in S_I^- \setminus Z_0 S^1\), such that \(|\chi(s)| \geq 1\). Take a vector \(\tilde{v} \in (V_P)_{\chi,\infty}\) with \(\langle r_P(\lambda), \tilde{v} \rangle \neq 0\) and let \(v \in V\) be such that \(j_P(\pi(\gamma^{-1}) v) = \tilde{v}\).

From Proposition 2.3 the union \(\bigcup_{n \geq 0} K_0 s_I^{-n} \gamma H\) is disjoint. So it is enough to see that
\[
\sum_{n \geq 0} \int_{K_0 s_I^{-n} \gamma H/H} |\varphi_{\lambda,v}(g)|^2 dg
\]
is divergent. Take an open compact subgroup $K \subset K_0$ which fixes $v$. For each $n$, the function $\varphi_{\lambda,v}$ is constant on $Ks^{-n}_I\gamma H$, hence
\[
\int_{Ks^{-n}_I\gamma H/H} |\varphi_{\lambda,v}(g)|^2 \, dg \geq \int_{Ks^{-n}_I\gamma H/H} |\varphi_{\lambda,v}(g)|^2 \, dg
\]
\[
= \text{vol}(Ks^{-n}_I\gamma H/H) |\langle \lambda, \pi(\gamma^{-1}s^n_I)\gamma v \rangle|^2
\]
\[
= C \cdot \delta_{P_\theta}(s^{-n}_I) |\langle \lambda, (\gamma^{-1}s^n_I)\gamma (\gamma^{-1})v \rangle|^2
\]
(4.7.1)

for some constant $C$ by Proposition 2.6. Now, let us take $\varepsilon \leq 1$ such that the relation (3.4.1) holds for all $s \in S_I^{-}(\varepsilon)$. Since $I = [\Delta \setminus \{\tau\}]$ here, the set $S_I^{-}(\varepsilon)$ is described as
\[
S_I^{-}(\varepsilon) = \{ s \in S_I \mid |s\pi|_F \leq \varepsilon \}.
\]
The element $s_I \in S_I^{-} \setminus Z_0S^I$ satisfies $|(s_I)\pi|_F < 1$. Thus we can take an integer $N$ such that $s^n_I \in S_I^{-}(\varepsilon)$ for all $n \geq N$. As a result, (4.7.1) continues as
\[
C \cdot \delta_{P_\theta}(s^{-n}_I) \delta_P(s^n_I) |\langle r_P(\lambda), \pi_P(\gamma^{-1}s^n_I)\gamma j_P(\gamma^{-1})v \rangle|^2
\]
\[
= C \cdot |\langle r_P(\lambda), \pi_P(s^n)v \rangle|^2
\]
whenever $n \geq N$. Since $\tilde{v} \in (V_P)_{\chi,\infty}$, the function $\langle r_P(\lambda), \pi_P(\cdot)v \rangle$ on $S$ is written just as $\chi(\cdot)P_\chi(\nu_S(\cdot))$ by a single polynomial $P_\chi$ on $as$. Finally, the series
\[
\sum_{n \geq N} \chi(s^n)P_\chi(\nu_S(s^n)) = \sum_{n \geq N} |\chi(s)|^{2n}|P_\chi(\nu_S(s^n))|^2
\]
is obviously divergent, hence the proof is completed. \hfill \Box

4.8. Remark. We have actually shown that the following three conditions are equivalent:

(i) $(\pi,V)$ is $H$-square integrable with respect to $\lambda$.

(ii) $\langle r_P, \lambda \rangle$ is satisfied for every $\sigma$-split parabolic subgroup $P$ of $G$.

(iii) $\langle r_P, \lambda \rangle$ is satisfied for every maximal $\sigma$-split parabolic subgroup $P$ of $G$.

See [W, III.1.1] for a similar statement on the usual square integrability.

4.9. $H$-square integrability and the usual square integrability.

Let $(\tilde{\pi},\tilde{V})$ be the contragredient of $(\pi,V)$. For $v \in V$ and $\tilde{v} \in \tilde{V}$, the usual matrix coefficient $c_{\tilde{v},v}$ is defined by
\[
c_{\tilde{v},v}(g) = \langle \tilde{v}, \pi(g^{-1})v \rangle.
\]
Let $\omega$ be a unitary character of the $F$-split component $Z$ of $G$. A smooth $\omega$-representation $(\pi,V)$ of $G$ is said to be square integrable if the functions $|c_{\tilde{v},v}(\cdot)|$ are square integrable on $G/Z$ for all $v$ and $\tilde{v}$. For the $F$-split component $A = A_I$ of a parabolic subgroup $P = P_I$ of $G$, the dominant part $A^-$ of $A$ is defined by
\[
A^- = \{ a \in A \mid |a^\alpha| \leq 1 (\alpha \in \Delta \setminus I) \}.
\]
There is a well-known criterion for square integrability due to Casselman ([C, 4.4.6]). In terms of normalized Jacquet modules, it is stated as follows.

**Casselman’s criterion.** A finitely generated admissible \(\omega\)-representation \((\pi, V)\) of \(G\) is square integrable if and only if for every parabolic subgroup \(P\), \(\left|\chi(a)\right| < 1\) holds for all \(\chi \in \mathcal{E}\pi_{\lambda}(\pi_P)\) and \(a \in A^- \setminus ZA^1\).

Although the usual square integrability and \(H\)-square integrability are different notions, we have the following relation.

**4.10. Proposition.** Let \(\omega\) be a unitary character of \(Z\) and \((\pi, V)\) a finitely generated admissible \(\omega\)-representation of \(G\). If \((\pi, V)\) is square integrable and is \(H\)-distinguished, then it is \(H\)-square integrable for all \(\lambda \in (V^*)^H\).

**Proof.** This is immediate from Casselman’s criterion and Proposition 4.3 in view of the surjectivity of (4.2.1). \(\Box\)

An example of square integrable \(H\)-distinguished representations can be found in [H, §7] for the symmetric space \(GL_2(E) / GL_2(F)\) where \(E/F\) is a quadratic extension.

### 5. Examples of \(H\)-square integrable representations

We shall give two simple examples of \(H\)-square integrable representations which are not square integrable. In both cases, there are constructions ([D] and [HR]) of an \(H\)-distinguished representation \(\pi(\rho)\) attached to a representation \(\rho\) of \(GL_2(F)\). We have observed in [KT, §8] that \(\pi(\rho)\) is \(H\)-relatively cuspidal if \(\rho\) is cuspidal. In what follows, we shall observe that \(\pi(\rho)\) is \(H\)-square integrable if \(\rho\) is square integrable.

#### 5.1. The symmetric space \(GL_3(F) / (GL_2(F) \times GL_1(F))\)

Let \(G\) be the group \(GL_3(F)\) and \(\sigma\) the inner involution \(\sigma = \text{Int}(\epsilon)\) defined by the anti-diagonal permutation matrix \(\epsilon = \left( \begin{array}{cc} 1 & 1 \\ 1 & 1 \end{array} \right)\). Then the \(\sigma\)-fixed point subgroup \(H\) is isomorphic to \(GL_2(F) \times GL_1(F)\). For this symmetric space, all the irreducible \(H\)-distinguished representations were determined by D. Prasad [P]. We shall use notation in [KT, 8.2] for the case \(n = 3\).

Let \(Q = LU_Q\) be the standard parabolic subgroup of \(G\) of type \((1, 2)\). Thus \(L \simeq GL_1(F) \times GL_2(F)\). Let \(\rho\) be an infinite dimensional irreducible admissible representation of \(GL_2(F)\) with trivial central character and form the normalized induction

\[
\pi(\rho) = \text{Ind}_Q^G(1_{GL_1(F)} \otimes \rho)
\]

where \(1 = 1_{GL_1(F)}\) denotes the trivial character of \(GL_1(F) = F^\times\). Then \(\pi(\rho)\) is irreducible and \(H\)-distinguished ([P, Theorem 2 (2)]).

We take a maximal \((\sigma, F)\)-split torus \(S_0\) as the one consisting of diagonal matrices of the form \(\text{diag}(s, 1, s^{-1})\) with \(s \in F^\times\), and a maximal \(F\)-split torus \(A_0\) of all diagonal matrices. The \((\sigma, F)\)-split component \(Z_0\) of \(G\) is trivial. As a minimal \(\sigma\)-split parabolic subgroup \(P_0\), we may take the Borel...
subgroup consisting of upper triangular matrices. In this case, \( P_0 \) is the only proper \( \sigma \)-split parabolic of \( G \) up to \( H \)-conjugacy. Let us determine the exponents of \( \pi(\rho) \) along \( P_0 \).

By the Geometric Lemma of [BZ 2.12], we have
\[
(\pi(\rho))^{s,s}_{P_0} = (\text{Ind}^G_B(1 \otimes \rho))^{s,s}_{P_0} \cong \bigoplus_{w \in [W_L\backslash W]} \mathcal{F}_w(1 \otimes \rho)^{s,s}
\]
where \( W \) and \( W_L \) denote the Weyl group of \( A_2 \) in \( G \) and \( L \) respectively, \( [W_L\backslash W] \) as in [C 1.1.3], \( \cdots \)^{s,s} denotes the semisimplified form, and
\[
\mathcal{F}_w(1 \otimes \rho) = \text{Ind}^M_B(1 \otimes \rho)(w \cdot (1_{\text{GL}_1(F)} \otimes \rho)_{L \cap w^{-1}P_0w}).
\]
The set \( [W_L\backslash W] \) consists of three elements. In forms of permutation matrices, those are
\[
e = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}.
\]
It is easy to compute the \( M_0 = A_2 \)-module \( \mathcal{F}_w(1 \otimes \rho) \) for each \( w \):
\[
\mathcal{F}_w(1 \otimes \rho) = w \cdot (1 \otimes \rho_{B_2})
\]
where \( B_2 \) denotes the Borel subgroup consisting of upper triangular matrices of \( \text{GL}_2(F) \).

Now take \( \rho \) to be the Steinberg representation \( \text{St}_2 \) of \( \text{GL}_2(F) \). We shall observe that the irreducible \( H \)-distinguished representation \( \pi(\text{St}_2) \) is \( H \)-square integrable. As is well-known, \( (\text{St}_2)_{B_2} \) is given by the one-dimensional character \( s^{1/2} \). So the set \( E \times p_{A_2}(\pi(\text{St}_2)_{P_0}) \) of exponents of \( \pi(\text{St}_2) \) along \( P_0 \) consists of three characters \( \chi_1, \chi_2, \chi_3 \) respectively given by
\[
\chi_1 \left( \begin{array}{cc} a_1 & a_2 \\ 0 & a_3 \end{array} \right) = |a_2|^F_a|a_3|_F^{-1}, \quad \chi_2 \left( \begin{array}{cc} a_1 & a_2 \\ 0 & a_3 \end{array} \right) = |a_1|^F_a|a_3|_F^{-1}, \quad \chi_3 \left( \begin{array}{cc} a_1 & a_2 \\ 0 & a_3 \end{array} \right) = |a_1|^F_a|a_2|_F^{-1}.
\]
Finally, since \( S_0^- \) (resp. \( S_1^+ \)) consists of \( \text{diag}(s, 1, s^{-1}) \) with \( |s|_F \leq 1 \) (resp. \( |s|_F = 1 \)), we may conclude that the restriction to \( S_0 \) of each exponent \( \chi_i \) satisfies \( |\chi_i(s)| < 1 \) for all \( s \in S_0^- \setminus S_1^+ \). Hence the claim follows from Proposition 4.3.

5.2. The symmetric space \( \text{GL}_4(F)/\text{Sp}_2(F) \).

Let \( G \) be the group \( \text{GL}_4(F) \) and \( \sigma \) the involution on \( G \) defined by
\[
\sigma(g) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \cdot g^{-1} \cdot \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}^{-1} \quad (g \in G).
\]
Then the \( \sigma \)-fixed point subgroup \( H \) is the symplectic group \( \text{Sp}_2(F) \). For this symmetric space, \( H \)-distinguished representations were studied by Heumos-Rallis [HR]. See also [KT 8.3].

We take
\[
S_0 = \{ \text{diag}(s_1, s_1, s_2, s_2) \mid s_1, s_2 \in F^\times \}\]
as a maximal \((\sigma, F)\)-split torus of \(G\). The \((\sigma, F)\)-split component \(Z_0\) of \(G\) consists of all the scalar matrices of \(G\) in this case. Let \(P_0 = M_0U_0\) be the standard parabolic subgroup of \(G\) of type \((2, 2)\). This is the only proper \(\sigma\)-split parabolic subgroup of \(G\) up to \(H\)-conjugacy. Note that

\[(5.2.1) \quad M_0 \cong \text{GL}_2(F) \times \text{GL}_2(F), \quad M_0 \cap H \cong \text{SL}_2(F) \times \text{SL}_2(F).\]

Let \(\rho\) be an irreducible admissible representation of \(G_2 := \text{GL}_2(F)\) and let us form the normalized induction

\[I(\rho) = \text{Ind}_{G_0}^G(\rho \cdot |\det(\cdot)|_{F}^{1/2} \otimes \rho \cdot |\det(\cdot)|_{F}^{-1/2}).\]

Then \(I(\rho)\) is \(H\)-distinguished (\([\text{HR} 11.1(a)]\)). If further \(\rho\) is square integrable, then \(I(\rho)\) has the unique irreducible quotient \(\pi(\rho)\) which also is \(H\)-distinguished (\([\text{HR} 11.1(b)]\)).

We shall investigate the \(M_0\)-module \((I(\rho))_{P_0}\) and \(M_0 \cap H\)-distinguished components therein. Note that irreducible \(M_0\cap H\)-distinguished \(M_0\)-modules have to be one dimensional according to (5.2.1).

Let \(W\) (resp. \(W_{M_0}\)) be the Weyl group of \((G, A_0)\) (resp. of \((M_0, A_0)\)). Following the definition of \([\text{C} 1.1.3]\), we can give the coset representatives \([W_{M_0}]W/W_{M_0}\) as

\[e, \left(\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array}\right), \text{ and } \left(\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array}\right).\]

We use the abbreviation \([\rho] = \rho \cdot |\det(\cdot)|_{F}^{1/2} \otimes \rho \cdot |\det(\cdot)|_{F}^{-1/2}\). The Geometric Lemma \([\text{BZ} 2.12]\) asserts that

\[(I(\rho))_{P_0}^{s} \simeq \bigoplus_{w \in [W_{M_0}]W/W_{M_0}} \mathcal{F}_w([\rho])^{s, s}\]

where

\[\mathcal{F}_w([\rho]) = \text{Ind}_{M_0 \cap wP_0w^{-1}}^{M_0}(w([\rho]_{M_0 \cap wP_0w^{-1}})).\]

It is easy to determine the three pieces \(\mathcal{F}_w([\rho])\) for each \(w\).

(i) \(w = e\): Nothing but

\[\mathcal{F}_w([\rho]) = [\rho],\]

which is not \(M_0 \cap H\)-distinguished unless \(\rho\) is one dimensional.

(ii) \(w = \left(\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array}\right)\): In this case, we have

\[M_0 \cap wP_0w^{-1} = M_0 \cap w^{-1}P_0w = M_0,\]

so that

\[\mathcal{F}_w([\rho]) = w[\rho] = \rho \cdot |\det(\cdot)|_{F}^{-1/2} \otimes \rho \cdot |\det(\cdot)|_{F}^{1/2}.\]

Again this cannot be \(M_0 \cap H\)-distinguished unless \(\rho\) is one dimensional.

(iii) \(w = \left(\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array}\right)\): We have

\[M_0 \cap wP_0w^{-1} = M_0 \cap w^{-1}P_0w = \left\{\left(\begin{array}{cc} * & 0 \\ 0 & 0 \end{array}\right) \right\} \simeq B_2 \times B_2\]
where $B_2$ is as in [5.1]. So we may write
\[
\mathcal{F}_w([\rho]) = \text{Ind}_{B_2 \times B_2}^{G_2 \times G_2}(w([\rho]_{B_2 \times B_2}))
\]
in this case.

Now, take $\rho$ to be the Steinberg representation $\text{St}_{2}$ of $G_2$. We claim that the irreducible $H$-distinguished representation $\pi(\text{St}_{2})$ is $H$-square integrable. It is enough to look at the exponents coming from $M_0 \cap H$-distinguished components in $\mathcal{I}(\rho)_{P_0}$. So we may look at only the case (iii) above. Put $T_2 = \{(s_0^0) \in B_2\}$. Under this notation, the $T_2 \times T_2$-module $[\rho]_{B_2 \times B_2}$ is given by
\[
\rho_{B_2} \big| \det(\cdot) \big|_{F}^{1/2} \otimes \rho_{B_2} \big| \det(\cdot) \big|_{F}^{-1/2} = \delta_{B_2}^{1/2} \big| \det(\cdot) \big|_{F}^{1/2} \otimes \delta_{B_2}^{-1/2} \big| \det(\cdot) \big|_{F}^{-1/2}.
\]
This is a character of $T_2 \times T_2$ written as
\[
((t_1, t_2), (t_3, t_4)) \mapsto |t_1|_F \cdot |t_4|_F^{-1}.
\]
Applying $w$ and inducing up to $G_2 \times G_2$, we have
\[
(5.2.2) \quad \mathcal{F}_w([\text{St}_{2}]) = \text{Ind}_{B_2}^{G_2}(1 \cdot | \cdot \otimes 1) \otimes \text{Ind}_{B_2}^{G_2}(1 \otimes | \cdot | F^{-1}).
\]
This is a reducible principal series having one dimensional quotient. So the only possible element of $\mathcal{E}xp_{S_0}(\pi(\text{St}_{2})_{P_0}, r_{P_0}(\lambda))$ (for any $\lambda \in (\pi(\rho)^H)$ is the restriction of the central character of (5.2.2), which is given by
\[
\chi(\text{diag}(s_1, s_2, s_2)) = |s_1|_F \cdot |s_2|_F^{-1}.
\]
We may conclude that $|\chi(s)| < 1$ for all $s \in S_0 \setminus Z_0 S_0$, hence the claim follows from Theorem 4.7.

References

[BO] Y. Benoist and H. Oh, Polar decomposition for $p$-adic symmetric spaces, Int. Math. Res. Not. 2007, no. 24, Art. ID rnm121, 20pp.

[BZ] I.N. Bernstein and A.V. Zelevinsky, Induced representations of $p$-adic groups.I, Ann. Sci. Éc. Norm. Sup. 10 (1977), 441-472.

[C] W. Casselman, Introduction to the theory of admissible representations of $p$-adic reductive groups, Unpublished manuscript, 1974, available at http://www.math.ubc.ca/~people/faculty/cass/research.html

[D] P. Delorme, Constant term of smooth $H_0$-spherical functions on a reductive $p$-adic group: Application to finiteness theorems, to appear in Trans. Amer. Math. Soc.

[DS] P. Delorme and V. Sécherre, An analogue of the Cartan decomposition for $p$-adic reductive symmetric spaces, preprint (2006), arXiv:math.RT/0612545.

[H] J. Hakim, Distinguished $p$-adic representations, Duke Math. J. 62(1) (1991), 1-22.

[HH] A.G. Helminck and G.F. Helminck, A class of parabolic $k$-subgroups associated with symmetric $k$-varieties, Trans. Amer. Math. Soc. 350 (1998), 4669-4691.

[HR] M. J. Heumos and S. Rallis, Symplectic-Whittaker models for $G\ell_n$, Pacific J. Math. 146 (1990), 247-279.

[KT] S. Kato and K. Takano, Subrepresentation theorem for $p$-adic symmetric spaces, Int. Math. Res. Not. 2008, no. 11, Art. ID run028, 40pp.

[L] N. Lagier, Termé constant de fonctions sur un espace symétrique réductif $p$-adique, J. Funct. Anal. 254(4) (2008), 1088-1145.
[P] D. Prasad, *On the decomposition of a representation of $GL(3)$ restricted to $GL(2)$ over a $p$-adic field*, Duke Math. J. 69(1) (1993), 167–177.

[W] J-L. Waldspurger, *La formule de Plancherel pour les groupes $p$-adiques, d’après Harish-Chandra*, J. Inst. Math. Jussieu 2(2) (2003), 235–333.