Conformal Superspace $\sigma$-Models

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σ-models in a nutshell

World-sheet

- 2D surface
  - (w/wo boundaries or handles)

Target space

- (Pseudo-)Riemannian manifold
  - (extra structure: gauge fields, ...)

σ-models = (quantum) field theories

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Conformal Superspace σ-Models
A simple example: The circle

The moduli space of circle theories

\[ R^2 / R \]

\[ R_0 \]

\[ S^1 \cong S^3 \]

Two lessons

- There is an equivalence: \( R \leftrightarrow R^2_0 / R \) (“T-duality”)
- In the quantum regime geometry starts to loose its meaning
A simple example: The circle

The moduli space of circle theories

Quantum regime

Classical regime

\( R^2 / R \)

\( R_0 \)

\( S^1 \cong S^3 \)

An open string partition function

\[
Z(q, z | R) = \text{tr} \left[ z^P q^{\text{Energy}(R)} \right] = \frac{1}{\eta(q)} \sum_{w \in \mathbb{Z}} z^w q^{w^2 / 2R^2}
\]
Adding curvature and supersymmetry...

Appearances of superspace $\sigma$-models

- **String theory**
  - Quantization of strings in flux backgrounds
  - String theory / gauge theory correspondence
  - Moduli stabilization in string phenomenology

- **Disordered systems**
  - Quantum Hall systems
  - Self avoiding random walks, polymer physics, ...
  - Efetov’s supersymmetry trick

Conformal invariance

- **String theory**: Diffeomorphism + Weyl invariance
- **Statistical physics**: Critical points / $2^{nd}$ order phase transitions
### Ingredients
- **Superspace $\sigma$-model encoding geometry and fluxes**
- Pure spinors: Curved ghost system
- BRST procedure

### Features
- Manifest target space supersymmetry
- Manifest world-sheet conformal symmetry
- Action quantizable, but quantization hard in practice
Strings on $\text{AdS}_5 \times S^5$ and $\text{AdS}_4 \times \mathbb{CP}^3$

**Spectrum accessible because of integrability**
- Factorizable S-matrix
- Structure fixed (up to a phase) by $\text{SU}(2|2) \rtimes \mathbb{R}^3$-symmetry
- Bethe ansatz, Y-systems, ...

**Open issues**
- String scattering amplitudes?
- 2D Lorentz invariant formulation?
- Other backgrounds? $\rightarrow$ Conifold, nil-manifolds, ...
### Overview

| Gauge theory | String theory |
|--------------|---------------|
| $\mathcal{N} = 4$ Super Yang-Mills | $\text{AdS}_5 \times S^5$ |
| $\mathcal{N} = 6$ Chern-Simons | $\text{AdS}_4 \times \mathbb{CP}^3$ |
| S-matrix, spectrum, ... | $\leftrightarrow$ S-matrix, spectrum, ... |
| t’Hooft coupling $\lambda$, ... | Radius $R$, ... |

### Problem

From this perspective, both sides need to be solved **separately**.
Proposal: Two step procedure...

Weakly coupled 4D gauge theory

Feynman diagram expansion

Weakly coupled 2D theory

"Well-established machinery"

Strongly curved 2D $\sigma$-model

(Topological $\sigma$-model)

[Berkovits] [Berkovits,Vafa] [Berkovits]
String theory in 10D
($\sigma$-model with constraints)

Gauge theory

Strong curvature
Weak curvature

Strong coupling
Weak coupling

$1/R$
$\lambda$
String theory in 10D
\((\sigma\text{-model with constraints})\)

\[ \sigma \]

“Some dual 2D theory”

Gauge theory

\[ 1/R \]

\[ g \]

\[ \lambda \]

Strong curvature

Weak curvature

Strong coupling

Weak coupling
The structure of this talk

Outline

1. Supercoset $\sigma$-models
   - Occurrence in string theory and condensed matter theory
   - Ricci flatness and conformal invariance

2. Particular examples
   - Superspheres
   - Projective superspaces

3. Quasi-abelian perturbation theory
   - Exact open string spectra
   - World-sheet duality for supersphere $\sigma$-models
## Appearance of supercosets

### String backgrounds as supercosets...

| Minkowski | AdS$_5 \times S^5$ | AdS$_4 \times \mathbb{CP}^3$ | AdS$_2 \times S^2$ |
|-----------|---------------------|-----------------------------|--------------------|
| super-Poincaré | PSU(2,2|4) SO(1,4)$\times$SO(5) | OSP(6|2,2) U(3)$\times$SO(1,3) | PSU(1,1|2) U(1)$\times$U(1) |
| Lorentz    |                     |                             |                    |

[Metsaev, Tseytlin] [Berkovits, Bershadsky, Hauer, Zhukov, Zwiebach] [Arutyunov, Frolov]

### Supercosets in statistical physics...

| IQHE   | Dense polymers | Dense polymers |
|--------|----------------|----------------|
| (non-conformal) | $S^{2S+1|2S}$ | $\mathbb{CP}^{S−1|S}$ |
| U(1,1|2) U(1|1)$\times$U(1|1) | OSP(2S+2|2S) U(S|S) | U(1)$\times$U(S−1|S) |

[IQHE] Dense polymers | [Weidenmüller] [Zirnbauer] [Read, Saleur] [Candu, Jacobsen, Read, Saleur]
A unifying construction

Definition of the cosets

\[ G/H : \quad gh \sim g \]

Some additional requirements for conformal invariance

- \( H \subset G \) is invariant subgroup under an automorphism
- Ricci flatness ("super Calabi-Yau") \( \iff \) vanishing Killing form

Examples: Cosets of \( \text{PSU}(N|N) \), \( \text{OSP}(2S + 2|2S) \), \( \text{D}(2, 1; \alpha) \).
Properties of conformal supercoset models

The moduli space of generic supercoset theories

Quantum regime \hspace{3cm} Classical regime

Radius

Properties at a glance

- Supersymmetry $G$: $g \mapsto kg$ (realized geometrically) [Kagan, Young] [Babichenko]
- Conformal invariance
- Integrability [Pohlmeyer] [Lüscher] ... [Bena, Polchinski, Roiban] [Young]
Properties of conformal supercoset models

The moduli space of generic supercoset theories

The general open string partition function

\[ Z(q, z|R) = \text{tr} \left[ z^{\text{Cartan}} q^{\text{Energy}(R)} \right] = \sum_{\Lambda} \psi_{\Lambda}(q, R) \chi_{\Lambda}^G(z) \]

Dynamics

Symmetry
Sketch of conformal invariance

The $\beta$-function vanishes identically...

$$\beta = \sum_{\text{certain } G\text{-invariants}} \kappa_{\mu\nu} = 0$$

**Ingredients:**
- Invariant form: $\kappa_{\mu\nu}$
- Structure constants: $f^{\mu\nu\lambda}$
The $\beta$-function vanishes identically...

$$f \bullet f = 0$$

**Ingredients:**

- Invariant form: $\kappa^{\mu\nu}$
- Structure constants: $f_{\mu\nu\lambda}$
The $\beta$-function vanishes identically...

There is a unique invariant rank 3 tensor!

[Bershadsky, Zhukov, Vaintrob’99] [Babichenko’06]
Sketch of conformal invariance

The $\beta$-function vanishes identically...

$\beta(f) \sim 0$

There is a unique invariant rank 3 tensor!

[Bershadsky, Zhukov, Vaintrob'99] [Babichenko'06]
Supersphere $\sigma$-models
The supersphere $S^{3|2}$

Realization of $S^{3|2}$ as a submanifold of flat superspace $\mathbb{R}^{4|2}$

$$\vec{X} = \begin{pmatrix} \vec{x} \\ \eta_1 \\ \eta_2 \end{pmatrix} \quad \text{with} \quad \vec{X}^2 = \vec{x}^2 + 2\eta_1\eta_2 = R^2$$

Symmetry

$$O(4) \times SP(2) \xrightarrow{\text{super-symmetrization}} OSP(4|2)$$

Realization as a supercoset

$$S^{3|2} = \frac{OSP(4|2)}{OSP(3|2)}$$
The supersphere $\sigma$-model

**Action functional**

\[ S_\sigma = \int \partial_\mu \vec{X} \cdot \partial^\mu \vec{X} \quad \text{with} \quad \vec{X}^2 = R^2 \]

**The space of states for freely moving open strings**

\[ \prod X^{a_i} \prod \partial_t X^{b_j} \prod \partial_t^2 X^{c_k} \ldots \quad \text{and} \quad \vec{X}^2 = R^2 \]

$\Rightarrow$ Products of coordinate fields and their derivatives

**Large volume partition function**

- “Single particle energies” add up $\rightarrow \#$ derivatives
- Partition function is pure combinatorics  
  [Candu, Saleur] [Mitev, TQ, Schomerus]
The large volume limit

Keeping track of quantum numbers...

- Symmetry
  
  $$\text{OSP}(4|2) \rightarrow \text{SP}(2) \times \text{SO}(4) \cong \text{SU}(2) \times \text{SU}(2) \times \text{SU}(2)$$

- Classify states according to the bosonic symmetry:
  
  $$\vec{X} = (\vec{x}, \eta_1, \eta_2) : \quad V = \begin{pmatrix} 0, \frac{1}{2}, \frac{1}{2} \end{pmatrix} \oplus \begin{pmatrix} \frac{1}{2}, 0, 0 \end{pmatrix}$$

  - bosons
  - fermions

- Other quantum numbers:
  - Energy $$q^E$$
  - Polynomial grade $$t^n$$ (broken by $$\vec{X}^2 = R^2$$)

- Use this to characterize all monomials
  
  $$\prod X^{a_i} \prod \partial_t X^{b_j} \prod \partial_t^2 X^{c_k} \ldots \quad \text{with} \quad \vec{X}^2 = R^2$$
Constituents of the partition function

A useful dictionary

| Field theoretic quantity | Contribution | Representation |
|--------------------------|--------------|----------------|
| 2 Fermionic coordinates  | $t z_1^{\pm 1}$ | $\frac{1}{2}$ |
| 4 Bosonic coordinates    | $t z_2^{\pm 1} z_3^{\pm 1}, t z_2^{\pm 1} z_3^{\mp 1}$ | $(\frac{1}{2}, \frac{1}{2})$ |
| Derivative $\partial$    | $q$          |                |
| Constraint $\vec{X}^2 = R^2$ | $1 - t^2$ |                |
| Constraint $\partial^n \vec{X}^2 = 0$ | $1 - t^2 q^n$ |                |

$t \leftrightarrow$ polynomial grade

$z_1, z_2, z_3 \leftrightarrow$ SU(2) quantum numbers
The full $\sigma$-model partition function

Summing up all contributions...

$$Z_\sigma(R_\infty) = \lim_{t \to 1} \left[ q^{-\frac{1}{24}} \prod_{n=0}^{\infty} (1 - t^2 q^n) \times \right.$$  

$$\left. \prod_{n=0}^{\infty} \frac{(1 + z_1 t q^n)(1 + z_1^{-1} t q^n)}{(1 - z_2 z_3 t q^n)(1 - z_2^{-1} z_3 t q^n)(1 - z_2^{-1} z_3^{-1} t q^n)} \right]$$

The problem...

Organize this into representations of OSP(4|2)!
Since the model is symmetric under $\text{OSP}(4|2)$ the partition function may be decomposed into characters of $\text{OSP}(4|2)$:

$$Z_{\sigma}(R_\infty) = \sum_{[j_1,j_2,j_3]} \psi_{\sigma}[j_1,j_2,j_3](q) \chi_{[j_1,j_2,j_3]}(z)$$

All the non-trivial information is encoded in

$$\psi_{\sigma}[j_1,j_2,j_3](q) = \frac{q^{-c[j_1,j_2,j_3]/2}}{\eta(q)^4} \sum_{n,m=0}^{\infty} (-1)^{m+n} q^{\frac{m}{2}(m+4j_1+2n+1)+\frac{n}{2}+j_1-\frac{1}{8}}$$

$$\times \left( q^{j_2-\frac{n}{2}^2} - q^{j_2+\frac{n}{2}+1} \right) \left( q^{j_3-\frac{n}{2}^2} - q^{j_3+\frac{n}{2}+1} \right)$$
Sketch of the large volume partition function

\[ \partial_t X^i \prod X^{a_j} \rightarrow 1 \]

\[ \prod X^{a_j} \rightarrow 0 \]

\( E \)

\( \infty \) many representations

Spherical harmonics on \( S^{3|2} \)

Weak curvature \((R = \infty)\)
Sketch of the large volume partition function

Quantum regime ($R = 1$)

Weak curvature ($R = \infty$)
A world-sheet duality
for supersphere $\sigma$-models
A world-sheet duality for superspheres?

Supersphere $\sigma$-model

Quantum regime

Large volume

$R^2 = 1 + g^2$

Weak coupling

Strong coupling

geometric

non-geometric (potential)

$Z_\sigma(q, z|R)$

$Z_{GN}(q, z|g^2)$

$\text{OSP}(4|2)$ Gross-Neveu model

[Candu, Saleur]$^2$ [Mitev, TQ, Schomerus]
In the two extreme limits the spectrum has the form...

Quantum regime ($R = 1$)  Weak curvature ($R = \infty$)
Evidence for the duality

Weak coupling

Free ghosts / WZW model
Affine symmetry

Lattice formulation

Strong coupling

Free theory
Combinatorics

Goal:
\[ Z_{\text{GN}}(q, z | g^2) = \sum \Lambda \psi_\sigma \Lambda(q, g^2) \chi_\Lambda(z) \]

[Candu, Saleur]² [Mitev, TQ, Schomerus]
Evidence for the duality

\[ Z_{GN}(q, z | g^2) = \sum_\Lambda \psi_\Lambda^\sigma(q, g^2) \chi_\Lambda(z) \]

[Candu, Saleur] [Mitev, TQ, Schomerus]
OSP(4|2) Gross-Neveu model
The OSP(4|2) Gross-Neveu model

**Field content**
- Fundamental OSP(4|2)-multiplet ($\psi_1, \psi_2, \psi_3, \psi_4, \beta, \gamma$)
- All these fields have scaling dimension 1/2

**Formulation as a Gross-Neveu model**

\[
S_{GN} = S_{\text{free}} + g^2 S_{\text{int}} \quad \begin{cases} 
S_{\text{free}} &= \int [\bar{\psi} \partial \psi + 2 \bar{\beta} \partial \gamma + \text{h.c.}] \\
S_{\text{int}} &= \int [\bar{\psi} \psi + \bar{\beta} \gamma - \bar{\gamma} \beta]^2 \end{cases}
\]

**Graphs**
- Weak coupling
- Strong coupling

**Goal:** $Z_{GN}(q, z | g^2)$
An open string spectrum

Formulation as a deformed \( \text{OSP}(4|2) \) WZW model

\[ S_{GN} = S_{WZW} + g^2 S_{\text{def}} \quad \text{with} \quad S_{\text{def}} = \int \text{str}(J\bar{J}) \]

Solution at \( g = 0 \)
- At \( g = 0 \) there is an \( \text{OSP}(4|2) \) Kac-Moody algebra symmetry
- Partition functions can be constructed using combinatorics

An open string partition function for \( g = 0 \)

\[ Z_{GN}(g^2 = 0) = \sum_{\Lambda} \psi_{\Lambda}^{\text{WZW}}(q) \chi_{\Lambda}(z) \]

- energy levels
- \( \text{OSP}(4|2) \) content
An open string spectrum

Formulation as a deformed OSP(4|2) WZW model

\[ S_{GN} = S_{WZW} + g^2 S_{\text{def}} \quad \text{with} \quad S_{\text{def}} = \int \text{str}(J\bar{J}) \]

Solution at \( g = 0 \)
- At \( g = 0 \) there is an OSP(4|2) Kac-Moody algebra symmetry
- Partition functions can be constructed using combinatorics

An open string partition function for all \( g \)

\[ Z_{GN}(g^2) = \sum_{\Lambda} q^{-\frac{1}{2}} \frac{g^2}{1+g^2} C_{\Lambda} \quad \psi_{\Lambda}^{WZW}(q) \quad \chi_{\Lambda}(z) \]
- anomalous dimension
- energy levels
- OSP(4|2) content
A specific D-brane in the OSP(4|2) WZW model...

The spectrum of a “twisted D-brane” is

\[ Z_{GN}(g^2 = 0) = \chi_0(q, z) + \chi_{1/2}(q, z) \]

The problem (yet again...)

Organize this into representations of OSP(4|2)!
Decomposition into representations of $\text{OSP}(4|2)$

Plugging in concrete expressions, one obtains

$$Z_{\text{GN}}(g^2 = 0) = \frac{\eta(q)}{\theta_4(z_1)} \left[ \frac{\theta_2(q^2, z_2^2)\theta_2(q^2, z_3^2)}{\eta(q)} + \frac{\theta_3(q^2, z_2^2)\theta_3(q^2, z_3^2)}{\eta(q)} \right]$$

$$= \sum \psi_{[j_1.j_2.j_3]}^{\text{WZW}}(q) \chi_{[j_1.j_2.j_3]}(z)$$
Plugging in concrete expressions, one obtains

\[
Z_{GN}(g^2 = 0) = \frac{\eta(q)}{\theta_4(z_1)} \left[ \frac{\theta_2(q^2, z_2^2)\theta_2(q^2, z_3^2)}{\eta(q)^2} + \frac{\theta_3(q^2, z_2^2)\theta_3(q^2, z_3^2)}{\eta(q)^2} \right]
\]

\[
= \sum \psi^{WZW}_{[j_1,j_2,j_3]}(q) \chi_{[j_1,j_2,j_3]}(z)
\]

\[
\psi^{WZW}_{[j_1,j_2,j_3]}(q) = \frac{1}{\eta(q)^4} \sum_{n,m=0}^{\infty} (-1)^{n+m} q^{\frac{m}{2}(m+4j_1+2n+1)+j_1+n\frac{1}{2}}
\]

\[
\times \left( q^{(j_2-n\frac{1}{2})^2} - q^{(j_2+n\frac{1}{2}+1)^2} \right) \left( q^{(j_3-n\frac{1}{2})^2} - q^{(j_3+n\frac{1}{2}+1)^2} \right)
\]
What did we achieve now?

WZW model ($g = 0$)

Affine \{0\}  

Affine \{1/2\}

$E$

1

$1/2$

Trivial

Adjoint

Fundamental $\otimes$ Adjoint

Fundamental

WZW model ($g = 0$)
Interpolation of the spectrum

\[ Z_{GN}(g^2) = \sum_{\Lambda} q^{-\frac{1}{2}} \frac{g^2}{1+g^2} c_{\Lambda} \psi_{\Lambda}(q) \chi_{\Lambda}(z) \]

anomalous dimension  
energy levels  
OSp(4\mid2) content

![Diagram showing energy levels and representations in WZW model and strong deformation](image)

**WZW model (g = 0)**  
**Strong deformation (g = \infty)**  
**Supersphere σ-model at R → \infty**
Quasi-abelian deformations
Consider a deformation...

\[ R = R_0 \sqrt{1 + \gamma} \]

Freely moving open strings on a circle of radius \( R \)...

\[
Z(q, z|R) = \frac{1}{\eta(q)} \sum_{w \in \mathbb{Z}} z^w q^{\frac{w^2}{2R^2}} = \frac{1}{\eta(q)} \sum_{w \in \mathbb{Z}} q^{\frac{2R^2_w (1+\gamma)}{2R_0^2}} \chi_w(z)
\]

Anomalous dimensions

\[
\delta_\gamma E_w = \frac{w^2}{2R_0^2} \left[ \frac{1}{1 + \gamma} - 1 \right] = -\frac{\gamma}{1 + \gamma} \frac{w^2}{2R_0^2} = -\frac{\gamma}{1 + \gamma} C_2(w)
\]
The effective deformation for conformal dimensions

The combinatorics of the perturbation series is determined by the current algebra

\[ J^\mu(z) J^\nu(w) = \frac{k \delta^{\mu \nu}}{(z - w)^2} + \frac{i f^{\mu \nu \lambda} J^\lambda(w)}{z - w} \approx \frac{k \delta^{\mu \nu}}{(z - w)^2} \]

Vanishing Killing form \( \Rightarrow \) the perturbation is quasi-abelian (for the purposes of calculating anomalous dimensions)

[Bershadsky, Zhukov, Vaintrob] [TQ, Schomerus, Creutzig]

In the OSP(4|2) WZW model a representation \( \Lambda \) shifts by

\[ \delta E^{\Lambda}(g^2) = -\frac{1}{2} \frac{g^2 C_\Lambda}{1 + g^2} = -\frac{1}{2} \left( 1 - \frac{1}{R^2} \right) C_\Lambda \]
Projective Superspaces
New features

- Family contains supertwistor space $\mathbb{CP}^3|4$ $\rightarrow$ [Witten]
- Non-trivial topology $\Rightarrow$ Monopoles and $\theta$-term
- Symplectic fermions as a subsector $[\text{Candu, Creutzig, Mitev, Schomerus}]
- $\theta$-term $\Rightarrow$ twists
- $\sigma$-model brane spectrum can be argued to be

$$Z_{R,\theta}(q, z) = q^{-\frac{1}{2} \lambda(R, \theta) \left[1 - \lambda(R, \theta)\right]} \sum_{\Lambda} q^{f(R, \theta) C_{\Lambda}} \psi_{\Lambda}^{\infty}(q) \chi_{\Lambda}(z)$$

result for $R \rightarrow \infty$ $[\text{Candu, Mitev, TQ, Saleur, Schomerus}]

- Currently no free field theory point is known...
Conclusions
Conclusions

- Using supersymmetry we determined the full spectrum of anomalous dimensions for certain open string spectra in various models as a function of the moduli.
- Our results provided strong evidence for a duality between supersphere $\sigma$-models and Gross-Neveu models.

Outlook

- Conformal invariance $\leftrightarrow$ Integrability
- Closed string spectra?
- Application to more stringy backgrounds...