Testing Hypotheses about Covariance Matrices in General MANOVA Designs

Paavo Sattler*, Arne C. Bathke** and Markus Pauly*

Abstract

We introduce a unified approach to testing a variety of rather general null hypotheses that can be formulated in terms of covariances matrices. These include as special cases, for example, testing for equal variances, equal traces, or for elements of the covariance matrix taking certain values. The proposed method only requires very few assumptions and thus promises to be of broad practical use. Two test statistics are defined, and their asymptotic or approximate sampling distributions are derived. In order to improve particularly the small-sample behavior of the resulting tests, two bootstrap-based methods are developed and theoretically justified. Several simulations shed light on the performance of the proposed tests. The analysis of a real data set illustrates the application of the procedures.

Keywords: Multivariate Data, bootstrap, non-parametric tests, covariance matrix.

* TU Dortmund University, Faculty of Statistics, Germany
** Department for Mathematics, University of Salzburg, Austria
Motivation and Introduction

It is of substantial interest to have valid statistical methods for inference on covariance matrices available, for at least two major reasons. The first one is that a treatment effect may indeed best be described by a particular configuration of scale or covariance parameters – not by a mean difference. The second reason corresponds to a more indirect purpose, namely that the main interest of the investigation may be described by a location change under alternative, but some of the available inference methods for location effects rely on assumptions regarding variances or covariances that need to be assessed reliably. In either situation, a statistical test about hypotheses that are formulated in terms of covariance matrices is necessary. From a methodological point of view, such a test shall not make too many restrictive assumptions itself, for example regarding underlying distributions. Furthermore, it shall perform well for moderate sample sizes, where clearly the term moderate will have to be seen in connection with the number of parameters effectively being tested.

Considering the central importance and the widespread need for hypothesis tests on covariance matrices, it may come as a surprise that a general and unifying approach to this task has not been developed thus far. There are several tests for specialized situations, such as testing equality of variances or even covariance matrices. Many of these approaches will be mentioned below. However, they typically only address one particular question, and they often rely on restrictive distributional assumptions, such as normality (e.g. in Box (1953) and Anderson (1984)), elliptical distributions (e.g. in Muirhead (1982) and Fang and Zhang (1990)), or conditions on the characteristic functions (e.g. in Gupta and Xu (2006)). One exception is the test of Zhang and Boos (1993) which theoretically allows for testing a multitude of hypotheses without restrictive distributional conditions. Unfortunately, the small and medium sample performance of this procedure is comparatively poor, in particular regarding the power. Their technique to improve the performance requires a more restrictive null hypothesis that additionally postulates equality of certain moments. This makes it somewhat difficult to use this approach in practice as a rejection does not mean that the covariances are unequal.

The goal of the present article is to introduce a very general approach to statistical hypothesis testing where the hypotheses are formulated in terms of covariance matrices. This includes as special cases, for example, hypotheses formulated using their traces, hypotheses of equality of variances or of covariance matrices, and hypotheses in which a covariance matrix is assumed to have particular entries. The test procedures are based on a resampling approach whose asymptotic validity is shown theoretically, while the actual finite sample performance has been investigated by means of extensive simulation studies. Analysis of a real data
example illustrates the application of the proposed methods. In the following section, the statistical model and (examples for) different null hypotheses that can be investigated using the proposed approach will be introduced. Thereafter, the asymptotic distributions of the proposed test statistics are derived (Section 3) and proven to be regained by two different resampling strategies (Section 4). The simulation results regarding type-I-error control and power are discussed in Section 5 while an illustrative data analysis of EEG-data is conducted in Section 6. All proofs are deferred to a technical Appendix.

2 Statistical Model and Hypotheses

We consider a general semiparametric model given by independent \( d \)-dimensional random vectors

\[
X_{ik} = \mu_i + \epsilon_{ik}. \tag{1}
\]

Here, the index \( i = 1, \ldots, a \) refers to the treatment group and \( k = 1, \ldots, n_i \) to the individual, on which \( d \)-dimensional observations are measured. In this setting, \( E(X_{ik}) = \mu_i = (\mu_{i1}, \ldots, \mu_{id})^\top \in \mathbb{R}^d \) denotes the \( i \)-th group mean while the residuals \( \epsilon_{i1}, \ldots, \epsilon_{in_i} \) are assumed to be centered \( E(\epsilon_{i1}) = 0 \) and i.i.d. within each group, with finite fourth moment \( E(||\epsilon_{i1}||^4) < \infty \). Beyond this, no other distributional assumptions are presumed. In particular, the covariance matrices \( \text{Cov}(\epsilon_{i1}) = V_i \geq 0 \) may be arbitrary and do not even have to be positive definite. For convenience, we aggregate the individual vectors into \( X = (X_{11}^\top, \ldots, X_{an_a}^\top)^\top \) as well as \( \mu = (\mu_1^\top, \ldots, \mu_a^\top)^\top \). Stacking the covariance matrices \( V_i = (v_{irs})_{r,s} \) into the \( p := d(d+1)/2 \)-dimensional vector

\[
v_i = \text{vech}(V_i) = (v_{i11}, v_{i12}, \ldots, v_{i1d}, v_{i2d}, v_{i22}, \ldots, v_{idd})^\top \quad (i = 1, \ldots, a)
\]

containing the upper triangular entries of \( V_i \) we formulate hypotheses in terms of the pooled covariance vector \( v = (v_1^\top, \ldots, v_a^\top)^\top \) as

\[
\mathcal{H}_0^v : Cv = \zeta. \tag{2}
\]

Here, \( C \) denotes a suitable hypothesis matrix of interest, and \( \zeta \) is a fixed vector. It should be noted that we neither assume that \( C \) is a contrast matrix, not to mention a projection matrix. This is different to the frequently used hypotheses formulation about mean vectors in MANOVA designs (Konietschke et al. (2015), Friedrich et al. (2017), Bathke et al. (2018)), where one can usually work with a unique projection matrix. However, working with simpler contrast matrices (as we do) can help to save considerable computation time, see Remark 2.1 below.
In order to discuss some particular hypotheses included within the general setup (2), we fix the following notation:

Let $I_d$ be the $d$-dimensional unit matrix, $1_d = (1, \ldots, 1)^\top$ the $d$-dimensional column vector of 1’s and $J_d = 1_d 1_d^\top$ the $d$-dimensional matrix of 1’s. Furthermore, $P_a = I_a - \frac{1}{a} J_a$ denotes the $a$-dimensional centering matrix, while $\oplus$ and $\otimes$ denote direct sum and Kronecker product, respectively. Then the following null hypotheses of interest are covered:

(a) **Testing equality of variances**: For a univariate outcome with $d = 1$, testing the null hypothesis

$$H_{0v}^v : \nu_{111} = \nu_{211} = \cdots = \nu_{a11}$$

of equal variances is included within (2) by setting $C = P_a$ and $\zeta = 0$. Hypotheses of this type have been studied by Bartlett and Rajalakshman (1953) as well as Boos and Brownie (2004), Gupta and Xu (2006), and Pauly (2011), among others. In the special case of a two-armed design with $a = 2$, this is also the null hypothesis inferred by the popular F-ratio test which, however, is known to be sensitive to deviations from normality Box (1953).

(b) **Testing for a given covariance matrix**: Let $\Sigma$ be a given covariance matrix. It may represent, for example, an autoregressive or compound symmetry covariance structure. For $a = 1$, our general formulation also covers testing the null hypothesis

$$H_{0v}^v : V_1 = \Sigma$$

by setting $C = I_p$ and $\zeta = \text{vech}(\Sigma)$. Hypotheses of this kind have been studied by Gupta and Xu (2006).

(c) **Testing homogeneity of covariance matrices**: More general than in (a), let $C = P_a \otimes I_p$ and $\zeta = 0$ for arbitrary $d \in \mathbb{N}$. Then (2) describes the null hypothesis

$$H_{0v}^v : V_1 = \cdots = V_a.$$

For multivariate normally distributed random variables, this is the testing problem of Box’s-M-test Box (1953), for which extensions have been studied in Lawley (1963), Browne and Shapiro (1986), Zhu et al. (2002), and Yang and DeGruttola (2012). Moreover, Zhang and Boos (1992, 1993) proposed Bartlett-type tests with bootstrap approximations in a general model similar to ours. However, the pooled bootstrap method of Zhang and Boos (1992) requires equality of some special kind of fourth moments across groups while the separate bootstrap approximation proposed in Zhang and Boos (1993) exhibited unsatisfactory small sample behaviour in terms of size control or power.
Beyond the above choices, $H^\gamma_0$ in (2) even contains hypotheses about linear functions of matrices. To this end, set $h_d := (1, 0_{d-1}^\top, 1, 0_{d-2}^\top, \ldots, 1, 0, 1)^\top$ and consider the following examples:

(d) **Traces as effect measures:** Suppose we are interested in the total variance $\sum_{\ell=1}^d Var(X_{i_\ell \ell}) = \text{tr}(V_i)$ of all components as a univariate effect measure for each group. This may be an advantageous approach in terms of power, as illustrated in the data example analysis below. Then, their equality $H^\gamma_0: \text{tr}(V_1) = \cdots = \text{tr}(V_a)$ can be tested by choosing $C = P_a \otimes [h_d \cdot h_d^\top]/d$, and $\zeta = 0$.

(e) **Testing for a given trace:** Consider the situation of Example (d) with just one group $a = 1$. We then may be interested in testing for a given value $\gamma \in \mathbb{R}$ of the trace, i.e.

$$H^\gamma_0: \text{tr}(V_1) = \gamma.$$ 

Thereto we chose $C = h_d \cdot h_d^\top/d$ and $\zeta = \gamma/d \cdot h_d$.

(f) **Higher Way Layouts:** Moreover, we can even infer hypotheses about variances, covariance matrices, or traces in arbitrarily crossed multivariate layouts by splitting up indices. For example, consider a two-way cross-classified design with fixed factors $A$ and $B$ whose levels are $i_1 = 1, \ldots, \alpha$ and $i_2 = 1, \ldots, \beta$, respectively. Assume that the interest lies in measuring, for example, their effect on the total variance, that is, the trace (a similar approach works for variances and covariances). Setting $a = \alpha\beta$ we observe $n_{i_1i_2} > 0$ subjects for each factor level combination $(i_1, i_2)$. To formulate hypotheses of no main trace effects for each factor, as well as hypotheses of no interaction trace effects we write $\text{tr}(V_{i_1i_2}) = t + \alpha i_1 + \beta i_2 + (\alpha\beta)_{i_1i_2}$ with the usual side conditions $\sum_{i_1} \alpha_{i_1} = \sum_{i_2} \beta_{i_2} = \sum_{i_1}(\alpha\beta)_{i_1} = \sum_{i_2}(\alpha\beta)_{i_2} = 0$. Here, for example, $\alpha_{i_1}$ can be interpreted as the part of the total variance under factor level $i_1$ by factor $A$. Then, the choice $C = (P_a \otimes J_{b/b}) \otimes (h_d \cdot h_d^\top/d)$ and $\zeta = 0$ leads to a test for no main effect of factor $A$ (measured in the above trace effects),

$$H^\gamma_0: \alpha_1 = \cdots = \alpha_\alpha = 0,$$

while $C = (P_a \otimes P_b) \otimes (h_d \cdot h_d^\top/d)$ and $\zeta = 0$ result in the hypothesis of no interaction (again measured in trace effects) between the factors $A$ and $B$,

$$H^\gamma_0: \alpha\beta_{ij} \equiv 0 \text{ for all } i, j.$$
Remark 2.1:
Although in each of the used scenarios it is possible to find an idempotent symmetric hypothesis matrix $C$, the option $\zeta \neq 0_d$ allows for matrices which are neither symmetric nor idempotent. From a theoretical point of view, this does not really matter. However, from a practical point of view, the choice of the hypothesis matrix may actually have a great effect with regard to saving computation time. To this aim we allow $C \in \mathbb{R}^{m \times d}$ with $m \leq d$. For example $H_{\nu 0}^\dagger : \text{tr}(V_1) = \gamma$ could also be formulated by $h_d^\top \cdot \nu = \gamma$. It is clear that with this approach all our results can be used. In our simulations it was thereby possible to save $17 - 24\%$ of the computing time, depending on the hypothesis and the setting. For dimension $d = 10$, the time savings would amount to $72 - 79\%$ which shows the value of this approach.

In the subsequent sections we develop testing procedures for $H_{\nu 0}^\dagger$ in (2) and thus for all given examples (a)–(f) above. The basic idea is to use a quadratic form in the vector $C\hat{v} - \zeta$ of estimated and centered effects. For ease of presentation and its widespread use in our setting (with $\mathbb{E}((\|\epsilon_{11}\|^4) < \infty$), we thereby focus on empirical covariance matrices

$$\hat{V}_i = \frac{1}{n_i - 1} \sum_{k=1}^{n_i} (X_{ik} - \bar{X}_i) (X_{ik} - \bar{X}_i)^\top, \quad \hat{v}_i = \text{vech}(\hat{V}_i),$$

as estimators for $V_i$, $i = 1, \ldots, a$, where $\bar{X}_i = \frac{1}{n_i} \sum_{k=1}^{n_i} X_{ik}$. Other choices, as, for example, surveyed in Duembgen et al. (2013), may be part of future research. Thereby, inverting the resulting test procedures will lead to confidence regions about the effect measures of interest. For example, in case (e), we may obtain confidence intervals for the unknown trace $\text{tr}(V_1)$.

## 3 The Test Statistics and their Asymptotics

In order to obtain the mentioned inference procedures which are formulated using quadratic forms, we first have to study the asymptotic behaviour of the normalized $a \cdot p$ dimensional vector $\sqrt{N}(C\hat{v} - \zeta)$, where $\hat{v} = (\hat{v}_1^\top, \ldots, \hat{v}_a^\top)^\top$ is the pooled empirical covariance estimator of $v$. For convenience, we thereby assume throughout that the usual asymptotic sample size condition holds, namely, as $\min(n_1, \ldots, n_a) \to \infty$:

$$(A1) \quad \frac{n_i}{N} \to \kappa_i \in (0, 1], \quad i = 1, \ldots, a \quad \text{for} \quad N = \sum_{i=1}^{a} n_i.$$

As $\kappa_i > 0$ holds for all $i$, we have $\kappa_1 = 1$ if and only if $a = 1$. Under this framework, we obtain the first preliminary result towards the construction of proper test procedures.
Theorem 3.1:
Suppose Assumption (A1) holds. Then, as \( N \to \infty \), we have convergence in distribution
\[
\sqrt{N}C(\hat{v} - v) \overset{D}{\longrightarrow} N_{a \cdot p} \left( 0_{a \cdot p}, C \Sigma C^\top \right),
\]
where \( \Sigma = \bigoplus_{i=1}^{a} k_i^{-1} \cdot \Sigma_i \) and \( \Sigma_i = \text{Cov}(\text{vech}(\epsilon_{i1} \epsilon_{i1}^\top)) \) for \( i = 1, \ldots, a \).

Together with a consistent estimator for (all or certain parts of) \( \Sigma \), this result allows us to develop asymptotic tests for the null hypothesis (2). Thereby, potential test statistics may lean on well-known quadratic forms used for mean-based MANOVA-analyses in heteroscedastic designs Konietschke et al. (2015), Bathke et al. (2018). In particular, two natural candidates are given by test statistics of ANOVA- or Wald-type:

\[
\text{ATS}_v := A_N = N [C\hat{v} - \zeta]^\top [C\hat{v} - \zeta] / \text{tr} \left( C \hat{\Sigma} C^\top \right), \quad (3)
\]
\[
\text{WTS}_v := W_N = N [C\hat{v} - \zeta]^\top (C \hat{\Sigma} C^\top)^+ [C\hat{v} - \zeta]. \quad (4)
\]

Here, \( A^+ \) denotes the Moore-Penrose-inverse of a matrix \( A \), and \( \hat{\Sigma} = \bigoplus_{i=1}^{a} \frac{N}{n_i} \hat{\Sigma}_i \) with
\[
\hat{\Sigma}_i = \frac{1}{n_i - 1} \sum_{k=1}^{n_i} \text{vech} \left( \tilde{X}_{ik} \tilde{X}_{ik}^\top - \frac{1}{n_i} \sum_{\ell=1}^{n_i} \tilde{X}_{i\ell} \tilde{X}_{i\ell}^\top \right) \text{vech} \left( \tilde{X}_{ik} \tilde{X}_{ik}^\top - \frac{1}{n_i} \sum_{\ell=1}^{n_i} \tilde{X}_{i\ell} \tilde{X}_{i\ell}^\top \right)^\top,
\]
is the empirical estimator for \( \Sigma \). Moreover, \( \tilde{X}_{ik} := X_{ik} - \bar{X}_i \) denotes the centered version of observation \( k \) in group \( i \).

Another possibility would be to transfer the modified ANOVA-type statistic from Friedrich and Pauly (2017) to the present setup which has also been recommended for heteroscedastic MANOVA. However, in our setting, the modified ANOVA-type statistic did not show better finite sample properties in simulations, see the supplement for details.

Now, in order to define asymptotically correct tests for the null hypothesis \( H_0^v \) based on the proposed test statistics (3) and (4), we first have to analyze their asymptotic properties. To this end, we first prove consistency of the involved matrix \( \hat{\Sigma} \).

Lemma 3.2:
The estimator \( \hat{\Sigma}_i \) converges almost sure to \( \Sigma_i \) as \( N \to \infty \), as well as \( \hat{\Sigma} \) to \( \Sigma \).
Combining Lemma 3.2 with Theorem 3.1, Slutzky’s Lemma, and the continuous mapping theorem yields the asymptotic null distributions of both test statistics $A_N$ and $W_N$.

Theorem 3.3:
Under Assumption (A1) and the null hypothesis $\mathcal{H}_0^\flat : \mathbf{C}v = \zeta$, the following results hold.
(a) The statistic $A_N$ defined in (3) has asymptotically a “weighted $\chi^2$-distribution”, that is, as $N \to \infty$ we have

$$ATS_v \xrightarrow{D} \sum_{\ell=1}^{ap} \lambda_\ell B_\ell / \text{tr}(\mathbf{C}\mathbf{\Sigma}\mathbf{C}^\top),$$

where $B_\ell \sim \chi^2_1$ and $\lambda_\ell, \ell = 1, \ldots, ap$, are the eigenvalues of $(\Sigma^{1/2}\mathbf{C}^\top\mathbf{C}\Sigma^{1/2})$.
(b) Additionally assume that $\Sigma_i > 0$ for all $i = 1, \ldots, a$. Then the statistic $W_N$ defined in (4) has asymptotically a $\chi^2_f$-distribution with $f = \text{rank}(\mathbf{C})$ degrees of freedom, i.e. $WTS_v \xrightarrow{D} \chi^2_f$ as $N \to \infty$.

This result allows the definition of a natural test procedure in the WTS given by

$$\varphi_{WTS} = \mathbb{1}[WTS_v \notin (\chi^2_{f,\alpha/2}, \chi^2_{f,1-\alpha/2})].$$

However, the additional condition $\Sigma_i > 0$, ensuring asymptotic correctness of $\varphi_{WTS}$, may not always be satisfied in practice. Since this condition is not needed in the first part of the theorem, we merely focus on the ANOVA-type statistic $A_N = ATS_v$ in what follows. As its limit distribution depends on unknown quantities, we cannot calculate critical values from (5) directly. To this end, we employ resampling techniques for calculating proper critical values. We thereby focus on two resampling procedures: a parametric and a wild bootstrap as both methods have shown favorable finite sample properties in multivariate mean-based MANOVA (Konietschke et al. (2015), Friedrich et al. (2016), Friedrich and Pauly (2017), and Zimmermann et al. (2019)). That these procedures also lead to valid testing procedures in the current setting is proven in the subsequent section.

4 Resampling Procedures

To derive critical values for the non-pivotal $ATS_v$, we consider two common kinds of bootstrap techniques: a parametric and a wild bootstrap as applied for heteroscedastic MANOVA. Since we deal with covariances instead of expectations, some adjustments have to be made, in order to prove their asymptotic correctness.
4.1 Parametric Bootstrap

To motivate our first resampling strategy, note that
\[ \sqrt{N}(\hat{v}_i - v_i) = \sqrt{N} \text{vech} \left( \frac{1}{n_i - 1} \sum_{k=1}^{n_i} [e_{ik} e_{ik}^\top - V_i] \right) + O_p(1) \xrightarrow{D} N_p \left( 0_p, \Sigma_i \right) \] (6)
follows from the proof of Theorem 3.1.

Thus, to mimic its limit and afterwards the structure of the test statistic, we generate bootstrap vectors \( Y_{i1}^*, \ldots, Y_{in_i}^* \), for given realizations \( X_{i1}, \ldots, X_{in_i} \) with estimators \( \hat{\Sigma}_i \). We then calculate \( \hat{\Sigma}_i^* = \hat{\Sigma}_i(Y_{i1}^*, \ldots, Y_{in_i}^*) \) and \( \hat{\Sigma}^* := \bigoplus_{i=1}^{n_i} \frac{N}{n_i} \hat{\Sigma}_i^* \), the parametric bootstrap versions of the estimators \( \hat{\Sigma}_i \) and \( \hat{\Sigma} \), respectively. The next theorem ensures the asymptotic correctness of this approach.

Theorem 4.1:
Under Assumption (A1), the following results hold:
(a) For \( i = 1, \ldots, a \), the conditional distribution of \( \sqrt{N}Y_i^* \), given the data, converges weakly to \( N_p \left( 0_p, \kappa_i^{-1} \cdot \Sigma_i \right) \) in probability. Moreover we have \( \hat{\Sigma}_i^* \to \Sigma_i \) in probability.
(b) The conditional distribution of \( \sqrt{N}Y^* \), given the data, converges weakly to \( N_{a \cdot p} \left( 0_{a \cdot p}, \bigoplus_{i=1}^{n_i} \kappa_i^{-1} \cdot \Sigma_i \right) \) in probability. Moreover we have \( \hat{\Sigma}^* \to \Sigma \) in probability.

As a consequence, it is reasonable to calculate the bootstrap version of the ATS as
\[ \text{ATS}_v^* = N \left[ C Y^* \right] ^\top \left[ C Y^* \right] / \text{tr} \left( C \hat{\Sigma}^* C^\top \right). \]

In fact, its conditional distribution weakly approximates the null distribution of the ATS, in probability, as stated below.

Corollary 4.2:
For each parameter \( v \in \mathbb{R}^{a \cdot p} \) and \( v_0 \) with \( C v_0 = \zeta \), we have under Assumption (A1) that
\[ \sup_{x \in \mathbb{R}} |P_v(\text{ATS}_v^* \leq x|X) - P_{v_0}(\text{ATS}_{v_0} \leq x)| \xrightarrow{P} 0, \]
where \( P_v \) denotes the (un)conditional distribution of the test statistic when \( v \) is the true underlying vector.

Denoting with \( c_{\text{ATS}^*,1-\alpha} \) the \( (1 - \alpha) \)-quantile of the conditional distribution of \( \text{ATS}_v^* \) given the data, we obtain \( \varphi_{\text{ATS}^*} = 1 \{ \text{ATS}_v^* \notin (c_{\text{ATS}^*,\alpha/2}, c_{\text{ATS}^*,1-\alpha/2}) \} \) as
Beyond being necessary to carry out an asymptotic level $\alpha$ test in the ATS$_v$, re-sampling can also be used to enhance the finite sample properties of the WTS$_v$. In fact, utilizing Theorem 4.1 shows that a parametric bootstrap version of the WTS$_v$, say WTS$_v^*$, is also asymptotically $\chi^2_{\text{rank}(C)}$-distributed, under the assumption given in Theorem 3.1. Thus, it leads to a valid parametric bootstrap WTS$_v$-test as long as $\Sigma_i > 0$ for all $i = 1, \ldots, a$.

4.2 Wild Bootstrap

As a second resampling approach, we consider the Wild Bootstrap. In contrast to its application in the mean-based analysis where the realizations are multiplied with convenient wild bootstrap multipliers, we have to multiply them with $p$ dimensional random vectors of the kind $\text{vech}(\tilde{X}_{ik}\tilde{X}_{ik}^\top)$, to ensure asymptotic correctness due to (6). Specifically, generate i.i.d. random weights $W_{i1}, \ldots, W_{in_i}, i = 1, \ldots, a$, independent of the data, with $E(W_{i1}) = 0$ and $\text{Var}(W_{i1}) = 1$. Common choices are for example standard distributed random variables or random signs. Afterwards the wild bootstrap sample is defined as $Y_{ik}^* = W_{ik} \cdot \left[ \text{vech}(\tilde{X}_{ik}\tilde{X}_{ik}^\top) - \frac{1}{n_i} \sum_{\ell=1}^{n_i} \text{vech}(\tilde{X}_{i\ell}\tilde{X}_{i\ell}^\top) \right]$, where again centering is needed to capture the correct limit structure.

Setting $\hat{\Sigma}_i^* = \hat{\Sigma}(Y_{i1}^*, \ldots, Y_{in_i}^*)$ and $\hat{\Sigma}^* = \bigoplus_{i=1}^{a} \hat{\Sigma}_i^*$, we obtain the following theorem.

Theorem 4.3:

Under Assumption (A1), the following results hold:

(a) For $i = 1, \ldots, a$, the conditional distribution of $\sqrt{N} Y_{i}^*$, given the data converges weakly to $N_p \left( 0_p, \kappa_i^{-1} \cdot \Sigma_i \right)$ in probability. Moreover it holds that $\hat{\Sigma}_i^* \to \Sigma_i$ in probability.

(b) The conditional distribution of $\sqrt{N} Y^*$, given the data converges weakly to $N_{a \cdot p} \left( 0_{a \cdot p}, \bigoplus_{i=1}^{a} \kappa_i^{-1} \cdot \Sigma_i \right)$ in probability. Moreover we have $\hat{\Sigma}^* \to \Sigma$ in probability.

The result gives rise to define a wild bootstrap ATS$_v^*$ counterpart given by

$$ATS_v^* = N \left[ C\bar{Y}^* \right]^\top \left[ C\bar{Y}^* \right] / \text{tr} \left( C\hat{\Sigma}^* C^\top \right).$$

Similar to the parametric bootstrap, the next theorem guarantees the approximation of the original test statistic by its bootstrap version.
Corollary 4.4:
Under the assumptions of Corollary 4.2, we have convergence
\[
\sup_{x \in \mathbb{R}} \left| P_{\nu} (\text{ATS}^*_\nu \leq x | \mathbf{X} ) - P_{\nu_0} (\text{ATS}_\nu \leq x ) \right| \overset{p}{\to} 0.
\]

Therefore, analogous to \( \varphi^*_\text{ATS} \), we define \( \varphi^*_\text{ATS} := \mathbb{1} \{ \text{ATS}_\nu \not\in \left( c_{\text{ATS}^*, \alpha/2}, c_{\text{ATS}^*, 1-\alpha/2} \right) \} \) as asymptotic level \( \alpha \) test, with \( c_{\text{ATS}^*, 1-\alpha} \) denoting the \( (1-\alpha) \) quantile of the conditional distribution of \( \text{ATS}^*_\nu \) given the data.

Similar wild bootstrap versions of the \( \text{WTS}_\nu \) or comparable statistics can again be defined and used to calculate critical values if \( \Sigma > 0 \) is fulfilled, see Section 5 below for the WTS and the supplement for another, less favorable, possibility.

5 Simulations

The above results are valid for large sample sizes. For an evaluation of the finite sample behavior of all methods introduced above, we have conducted extensive simulations regarding

(i) their ability in keeping the nominal significance level and

(ii) their power to detect certain alternatives in various scenarios.

In particular, we studied three different kinds of hypotheses:

A) Equal Covariance Matrices: \( \mathcal{H}^*_0 : \mathbf{V}_1 = \mathbf{V}_2 \) with \( a = 2 \) groups.

B) Equal Diagonal Elements: \( \mathcal{H}^*_0 : \mathbf{V}_{111} = \ldots = \mathbf{V}_{1dd} \) in the one sample case.

C) Trace Test: \( \mathcal{H}^*_0 : \text{tr}(\mathbf{V}_1) = \gamma \) for a given \( \gamma \) and \( a = 1 \).

Each of these hypotheses can be formulated with a proper projection matrix \( \mathbf{C} \) although \( \zeta \neq 0_p \) in the last case. While \( \mathbf{C}(\mathbf{A}) = \mathbf{I}_d \) and \( \mathbf{C}(\mathbf{C}) = \mathbf{h}_d \cdot \mathbf{h}^\top_d / d \) follows directly from section 2, \( \mathbf{C}(\mathbf{B}) = \text{diag}(\mathbf{h}_d) - \mathbf{h}_d \cdot \mathbf{h}^\top_d / d \) is an adaptation of \( \mathbf{P}_d \).

For every hypothesis, we have simulated the two bootstrap methods based on the ANOVA-type statistic \( \varphi^*_\text{ATS} \) and \( \varphi^*_\text{ATS} \), as well as the Wald-type statistic \( \varphi^*_\text{WTS} \), \( \varphi^*_\text{WTS} \). The latter ones are based on the parametric bootstrap version of the WTS, given by

\[
\text{WTS}^* := \mathbb{N} \left[ \mathbf{cY}^* \right]^\top \left( \mathbf{cE}^* \mathbf{C}^\top \right)^+ \left[ \mathbf{cY}^* \right] \tag{7}
\]
and the wild bootstrap version given by

\[ WTS^* := N \left[ C \tilde{Y}^\top \left( C \tilde{\Sigma}^* C^\top \right)^+ \left[ C \tilde{Y}^\top \right] \right]. \tag{8} \]

Moreover, the asymptotic version \( \varphi_{WTS} \) based upon the \( \chi^2_{\text{rank}(C)} \)-approximation serves as another competitor.

In the special case of scenario A) we have also considered the tests from Zhang and Boos (1992, 1993) based on Bartlett’s test statistic, along with a so-called separate bootstrap as well as a pooled bootstrap to calculate critical values. We denote these tests by \( \varphi_{B-S} \) and \( \varphi_{B-P} \). While the first is asymptotically valid under the same conditions as our tests, the pooled bootstrap procedure additionally requires equality

\[ \mathbb{E} \left( \text{vech} \left( \epsilon_1 \epsilon_1^\top \right) \text{vech} \left( \epsilon_1 \epsilon_1^\top \right)^\top \right) = \mathbb{E} \left( \text{vech} \left( \epsilon_2 \epsilon_2^\top \right) \text{vech} \left( \epsilon_2 \epsilon_2^\top \right)^\top \right). \]

To examine the impact of deviations from this condition, we have also considered a setting where individual groups have different distributions and this assumption, therefore, is violated. Additionally, we have simulated Box’s M-test as it is the most popular test for scenario A), although it requires normally distributed data. There are two common ways to determine critical values for this test, Box (1949): Utilizing a \( \chi^2_f \)-approximation with \( f = \text{rank}(C) \) degrees of freedom or an \( F \)-approximation with estimated degrees of freedom. For ease of completeness, we decided to simulate both.

On an abstract level, the hypotheses considered thus far also fall into the framework presented by Zhang and Boos (1993). However, they do not provide concrete test statistics that we could use for comparison purposes. Other existing tests such as the one by Gupta and Xu (2006) rely on rather different model assumptions which also makes a comparative evaluation difficult. All simulations were conducted by means of the R-computing environment version 3.6.1 R Core Team (2019) with \( N_{\text{sim}} = 10^4 \) runs, 1000 bootstrap runs and \( \alpha = 5\% \).

Data generation

We considered 5-dimensional observations generated independently according to the model \( X_{ik} = \mu_i + \Sigma^{1/2} Z_{ik}, i = 1, \ldots, a, k = 1, \ldots, n_i \) with \( \mu_1 = (1^2, 2^2, 3^2, 4^2, 5^2)/4 \) and \( \mu_2 = 0_5 \). Here, the marginals of \( Z_{ik} = (Z_{ikj})_{j=1}^5 \) were either simulated independent from

- a centered exponential distribution with parameter 1, i.e. \( (Z_{ikj} + 1) \sim \text{Exp}(1) \),
- a standardized student t-distribution with 9 degrees of freedom, i.e. \( (3/\sqrt{7} \cdot Z_{ikj}) \sim t_9 \) or
- a standard normal distribution, i.e. \( Z_{ikj} \sim \mathcal{N}(0,1) \).
Table 1: Simulated type-I error rates ($\alpha = 5\%$) in scenario $A$ ($H_0^\prime : V_1 = V_2$) for ATS, WTS, MATS, Bartlett’s test and Box’s M-test, always with the same relation between group samples size by $n_1 := 0.6 \cdot N$ resp. $n_2 := 0.4 \cdot N$.

For the covariance matrix, a compound symmetry structure $\Sigma = I_5 + J_5$ was chosen. Results from further simulations with different covariance matrices can be found in the supplement.

Note that the chosen dimension of $d = 5$ leads to an effective dimension of $p = 15$ of the unknown parameter (i.e. covariance matrix) in each group. Hence in scenario $A$), the vector $\nu$ defining the null hypothesis (2) actually consists of 30 unknown parameters. To address this quite large dimension, we considered three different small to moderate total sample sizes of $N \in \{50, 100, 200\}$. Moreover, in scenario $A$) these were divided into two groups by setting $n_1 = 0.6 \cdot N$ and $n_2 = 0.4 \cdot N$. Thus, we had between 20 and 120 independent observations to estimate the unknown covariance matrix in each group.

### 5.1 Type-I error

The following tables display the simulated type-I error rates for all these settings.

In almost all simulation settings, the parametric bootstrap led to more conservative results, whereas the wild bootstrap was always more liberal. Apart from that, the quality of the bootstrap was rather comparable with a slight advantage of the parametric bootstrap.

Overall, the results of the ATS were preferable to the WTS. This matches the con-
Table 2: Simulated type-I error rates ($\alpha = 5\%$) in scenario A ($H_0^A: V_1 = V_2$) for ATS, WTS, MATS, Bartlett's test and Box’s M-test, always with the same relation between group samples size by $n_1 := 0.6 \cdot N$ resp. $n_2 := 0.4 \cdot N$.

|         | Exp(1) & t$_9$ | t$_9$ & Normal | Normal & Exp(1) |
|---------|---------------|---------------|---------------|
| N       | 50 | 100 | 200 | 50 | 100 | 200 | 50 | 100 | 200 |
| ATS-Para | .0240 | .0244 | .0359 | .0325 | .0381 | .0427 | .0456 | .0402 | .0425 |
| ATS-Wild | .0656 | .0543 | .0578 | .0599 | .0541 | .0530 | .0820 | .0775 | .0771 |
| WTS-Para | .0359 | .0333 | .0420 | .0345 | .0392 | .0436 | .1059 | .1504 | .1436 |
| WTS-Wild | .0679 | .0654 | .0649 | .0613 | .0577 | .0567 | .1773 | .2140 | .1924 |
| WTS-$\chi_{15}^2$ | .4350 | .1518 | .0741 | .3885 | .1394 | .0709 | .5941 | .3526 | .2235 |
| Bartlett-S | .0159 | .0338 | .0482 | .0125 | .0338 | .0418 | .0162 | .0420 | .0500 |
| Bartlett-P | .0184 | .0231 | .0331 | .0175 | .0298 | .0370 | .0423 | .0677 | .0766 |
| Box’s $M_{\chi_{15}^2}$ | .1988 | .2192 | .2463 | .0649 | .0616 | .0616 | .2316 | .2637 | .2727 |
| Box’s M-F | .1904 | .2167 | .2458 | .0622 | .0609 | .0615 | .2228 | .2618 | .2720 |

Table 2: Simulated type-I error rates ($\alpha = 5\%$) in scenario A ($H_0^A: V_1 = V_2$) for ATS, WTS, MATS, Bartlett's test and Box’s M-test, always with the same relation between group samples size by $n_1 := 0.6 \cdot N$ resp. $n_2 := 0.4 \cdot N$.

Conventional wisdom that the WTS generally exhibits a rather liberal behavior and requires large sample sizes to perform well. Moreover, the WTS requires the condition on the rank of $\Sigma$ which is difficult to check in practice, because of the special structure of $\Sigma$. In contrast, the ATS is capable to handle all these scenarios, including random variables originating from different distribution families, which is typically one of the most challenging settings when testing homogeneity. Therefore, it remains to compare these tests with the statistics based on Bartlett’s statistic.

First, we consider the situation with equal distributions, which can be seen in table 1 to have an arrangement were the conditions of all procedures are fulfilled. For bigger sample sizes, the type I-error rate of the separated bootstrap seemed comparable to the ATS with the parametric bootstrap, while the pooled bootstrap was a bit more conservative. But as already discussed in Zhang and Boos (1993) for small sample sizes, the $\alpha$-level of $\varphi_{B-S}$ and $\varphi_{B-P}$ was generally far too low. Here, $\varphi_{B-P}$ seemed to perform slightly better, but at the expense of a smaller null hypothesis. Moreover, for larger sample sizes this benefit in performance disappeared. Except for the exponential distribution, the ATS with the parametric bootstrap had a comparable approximation with the pooled bootstrap for significant smaller sample size and moreover does not require the additional conditions on moments. The popular Box’s M-test worked quite well under normality but showed poor results (type-I error rates of more than 20%) when this condition
### Table 3: Simulated type-I error rates ($\alpha = 5\%$) in scenario B ($H_0^*: V_{111} = ... = V_{155}$) for ATS and WTS.

|       | Exp(1) |       |       |       |       |       |       |
|-------|--------|-------|-------|-------|-------|-------|-------|
| N     | 25     | 50    | 100   | 25    | 50    | 100   | 25    |
|       |        |       |       | 25    | 50    | 100   | 25    |
|       |        |       |       |       |       |       |       |
| ATS-Para | .0150  | .0155 | .0237 | .0205 | .0285 | .0328 | .0320 |
| ATS-Wild | .0568  | .0575 | .0562 | .0482 | .0504 | .0501 | .0546 |
| WTS-Para | .1257  | .0988 | .0746 | .0599 | .0559 | .0566 | .0584 |
| WTS-Wild | .2348  | .1728 | .1244 | .0948 | .0798 | .0719 | .0799 |
| WTS-$\chi^2_l$ | .2376  | .1362 | .0874 | .1422 | .0897 | .0655 | .1308 |

### Table 4: Simulated type-I error rates ($\alpha = 5\%$) in scenario C ($H_0^*: \text{tr}(V_1) = 10$) for ATS and WTS.

|       | Exp(1) |       |       |       |       |       |       |
|-------|--------|-------|-------|-------|-------|-------|-------|
| N     | 25     | 50    | 100   | 25    | 50    | 100   | 25    |
|       |        |       |       | 25    | 50    | 100   | 25    |
|       |        |       |       |       |       |       |       |
| ATS-Para | .1504  | .1141 | .0930 | .1026 | .0798 | .0768 | .0963 |
| ATS-Wild | .1956  | .1374 | .1053 | .1164 | .0863 | .0805 | .1072 |
| WTS-Para | .1504  | .1141 | .0930 | .1026 | .0798 | .0768 | .0963 |
| WTS-Wild | .1956  | .1374 | .1053 | .1164 | .0863 | .0805 | .1072 |
| WTS-$\chi^2_l$ | .1576  | .1196 | .0947 | .1116 | .0823 | .0783 | .1031 |
was violated. This sensitivity to the moment assumption (or its violation) may have the consequence in practice that small p-values could be untrustworthy, independent of whether χ² or F distribution was used. Despite these well-known difficulties, Box’s M-test is still often used by practitioners.

Table 2 shows the performance in the more challenging case where the underlying distribution in the groups was from different families, and therefore the assumption \( E\left(\text{vech}\left(\epsilon_1\epsilon_1^\top\right)\right) = E\left(\text{vech}\left(\epsilon_2\epsilon_2^\top\right)\right) \) was violated. For all of our test statistics, the use of mixed distributions had nearly no systematic influence at all: some values got better, and other values got worse. This is not surprising because mixed distributions naturally make the situation more complicated, even for tests which allow for such circumstances. Similarly, there were variations in the type-I error rates of \( \varphi_{B-S} \), but no clear tendencies.

While for the first two mixed distributions also \( \varphi_{B-P} \) seemed quite robust against the violation of the additional condition, for the case of a normal distribution in one group and an exponential distribution in the other group the quality clearly decreased. Hereby the error rate for increasing samples sizes differed more and more from our \( \alpha \) level. Despite the unexpectedly good performance for the first two mixed distributions the behavior in the last setting indicates that this procedure has an essential disadvantage. The last row shows results for Box’s M test which had again error rates making it almost useless in this case.

The resampling procedure used in [Zhang and Boos (1993)] occasionally encountered covariance matrices without full rank, especially for smaller sample sizes. This creates issues in the algorithm because the determinant of these matrices is zero and the logarithm at this point is not defined. Regretfully this situation wasn’t discussed in the original paper, so we just excluded these values. Certainly, this would constitute a drastic user intervention in applying the bootstrap and also influencing the conditional distribution. Nevertheless, it was necessary to use this adaptation in all our simulations containing these tests. This effect can also occur in Box’s M-test, but comparatively rarely because there is no bootstrap involved.

All in all in scenario A) the ATS* and ATS* exhibited the best performance when we considered all distributions and in particular small sample sizes. Because of the more liberal results of \( \varphi_{ATS^*} \) and the more conservative result of \( \varphi_{ATS^*} \), it would be obvious to combine them to get a test which holds the level \( \alpha \) even better. But this would require the computation time of both tests together.

This combination would also be an option for scenario B) because again the ATS
with parametric bootstrap was a bit too conservative and the ATS with wild bootstrap was slightly too liberal. But in this case, the wild bootstrap had the best results, even for small sample sizes. Again the type-I error rates of the WTS were too high while also the wild bootstrap did not yield sufficient improvement for smaller samples sizes. Here, only the parametric bootstrap version showed considerable improvements regarding the $\chi^2$ distribution for the WTS.

For scenario C all tests considered had the same liberal behavior. Because of the fact that $\text{rank}(C) = 1$ even $\phi_{ATS}^\star$ and $\phi_{WTS}^\star$ take exactly the same values as well as $\phi_{ATS}^\ast$ and $\phi_{WTS}^\ast$. For better comparability in the simulation, both approaches WTS and ATS used the same observations as well as identical bootstrap samples. The presumable reason for the required larger sample size is also the small rank of the hypothesis matrix. Calculating the trace of the covariance matrix, just a really small part of the matrix is used and therefore a low proportion of the information contained in the data can be utilized.

The effect of other types of covariance matrices, which is considered in the appendix, was not significant and not systematic. Therein, we also investigated testing for a given covariance matrix. Here, only the type-I error rate of the ANOVA-type statistic with critical values obtained from the parametric bootstrap showed satisfying results.

To sum up, we only recommend the use of $\phi_{ATS}^\star$ and $\phi_{ATS}^\ast$ as both led to good results for comparably small sample sizes and are (asymptotically) valid without additional requirements on $\Sigma$. The additional simulations in the appendix also confirmed this. The fact that for some hypotheses, such as testing for homogeneity, $\phi_{ATS}^\star$ was a bit conservative while $\phi_{ATS}^\ast$ was a bit liberal, allows some freedom to choose a test according to personal preferences.

### 5.2 Power

For a power simulation, it is unfortunately not possible to merely shift the observations by a proper vector to control the distance from the null hypothesis. Thereto we have multiplied the observation vectors $X$ with a proper diagonal matrix, given by $\Delta = I_d + \text{diag}(1, 0, ..., 0) \cdot \delta$ for $\delta \in [0, 2.5]$. This was associated with a one-point-alternative that is known from testing expectation vectors to be challenging, namely a deviation in just one component, which is usually difficult to detect. In this way $C \text{vech} (\Delta V \Delta^\top) - \zeta \neq 0$, were $n_1 + n_2 = 50$ was used to investigate small size behavior, and $n_1 + n_2 = 100$ for moderate samples sizes, while the dimension was again $d = 5$, leading to $p = 15$. Due to computational reasons and because of the performance under null hypotheses described in the last section, we have only investigated the power of $\phi_{ATS}^\ast$ and $\phi_{ATS}^\star$, as well as
\( \varphi_{B-P} \) and \( \varphi_{B-S} \) from [Zhang and Boos (1993)](#).
Figure 2: Simulated power in scenario A ($H_0^v : V_1 = V_2$) for ATS with wild and parametric bootstrap as well as the test based on Bartlett’s statistic with separate and pooled bootstrap. The 5-dimensional vectors were based on the $t_9$ distribution, with covariance matrix $\Sigma = I_5 + J_5$ and $n_1 = 60$, $n_2 = 40$.

Illustrative Data Analysis

To demonstrate the use of the proposed methods, we have re-analyzed neurological data on cognitive impairments. In Bathke et al. (2018) the question was examined whether EEG- or SPECT-features are preferable to differentiate between three different diagnoses of impairments - subjective cognitive complaints (SCC), mild cognitive impairment (MCI), and Alzheimer disease (AD). The corresponding trial was conducted at the University Clinic of Salzburg, Department of Neurology. Here 160 patients were diagnosed with either AD, MCI, or SCC, based on neuropsychological diagnostics, as well as a neurological examination. This data set has been included in the R-package manova.rm by Sarah Friedrich (2019). The following table contains the number of patients divided by sex and diagnosis.

For each patient, $d = 6$ different kinds of EEG variables were investigated which leads to $p = 21$ variance and covariance parameters. As the male AD and SCC group only contain 12 and 20 observations, respectively, an application of the WTS
In Bathke et al. (2018) the authors descriptively checked the empirical covariances matrices to judge that the assumption of equal covariance matrices between the different groups is rather unlikely. However, this presumption has not been inferred statistically. To close this gap, we first test the null hypothesis of equal covariance matrices between the six different groups using the newly proposed methods. Applying the ATS with parametric resp. wild bootstrap led to p-values of 0.0275 and 0.0008.

In comparison, the Bartlett-S test of Zhang and Boos (1993) led to a p-value of 0.3484, potentially reflecting its bad power observed in Section 5 and also by the authors. Moreover, their Bartlett-P test for the smaller null hypothesis (additionally postulating equality of vectorized moments) shows a small p-value of 0.00019998.

As a next step, we take the underlying factorial structure of the data into account and test, for illustrational purposes, the following hypotheses:

a) Homogeneity of covariance matrices between different diagnoses,

b) Homogeneity of covariance matrices between different sexes,

c) Equality of total variance between different diagnosis,

d) Equality of total variance between different sexes.

For the first both hypotheses we calculated the ATS with wild and parametric bootstrap as well as Bartlett’s test statistic with separate and pooled bootstrap. Considering the trace just the first two tests are applicable, and in all cases, the one-sided tests are used based on 10,000 bootstrap runs. The results are presented in table 6 and table 7.

It is noticeable that both tests based on the ATS clearly reject the null hypothesis of equal covariances for AD and SCC for both sexes at level 5%, while the p-values of both Bartlett’s tests are not significant. An explanation for this combination with less samples may be given by the good small sample performance of the ATS observed in section 5 and the rather low power of Bartlett’s test statistic which was already mentioned in Zhang and Boos (1993). Moreover, the only cases where both Bartlett’s test-statistics have smaller p-values are for the combination with

|         | AD | MCI | SCC |
|---------|----|-----|-----|
| male    | 12 | 27  | 20  |
| female  | 24 | 30  | 47  |

Table 5: Number of observations for the different factor level combinations of sex and diagnosis.
Table 6: P-values of ATS with wild resp. parametric bootstrap and Bartlett’s test statistic with separate resp. pooled bootstrap for testing equality of covariance matrices.

| H₀ :          | ATS-Para p-value | ATS-Wild p-value | Bartlett-S p-value | Bartlett-P p-value |
|---------------|------------------|------------------|--------------------|--------------------|
| male AD vs. MCI | 0.1000           | **0.0282**       | 0.1744             | **0.0184**         |
| male AD vs. SCC | >0.0001          | **>0.0001**      | 0.0545             | 0.0634             |
| male MCI vs. SCC | 0.8767           | 0.9801           | 0.1388             | **0.0078**         |
| female AD vs. MCI | 0.0613           | 0.0559           | 0.1050             | 0.1480             |
| female AD vs. SCC | **0.0128**       | **0.0095**       | **0.0138**         | **0.0183**         |
| female MCI vs. SCC | 0.5656           | 0.6004           | 0.8964             | 0.8988             |
| AD male vs. female | 0.1008           | **0.0274**       | 0.2482             | 0.0542             |
| MCI male vs. female | 0.2455           | 0.2417           | 0.3695             | 0.4003             |
| SCC male vs. female | 0.2066           | 0.1914           | 0.2656             | 0.1648             |

The null hypothesis of equal total covariance resp. equal traces could be rejected significantly (at level 5%) by both bootstrap tests in three cases. Perhaps surprising at first is that the null hypothesis of equal covariance matrices between the female AD and MCI groups could not be rejected, but the joint univariate null hypothesis of equal traces could now be rejected at level 5%.

Although the hypothesis of equal covariance matrices couldn’t be rejected in each case, it shows that sex and diagnosis are likely to have an effect on the covariance matrix. This illustrative analysis underpins that the approach of Bathke et al. (2018), which can deal with covariance heterogeneity, was very reasonable.

7 Conclusion & Outlook

In the present paper, we have introduced and evaluated a unified approach to testing a variety of rather general null hypotheses formulated in terms of covariance matrices. The proposed method is valid under a comparatively small number of
Table 7: P-values of ATS with wild resp. parametric bootstrap for testing equality of traces containing covariance matrices.

| Hypothesis | Description | ATS-Para p-value | ATS-Wild p-value |
|------------|-------------|------------------|------------------|
| $H_0^c$:   | male AD vs. MCI | 0.0733           | 0.0635           |
| $H_0^c$:   | male AD vs. SCC | >0.0001          | >0.0001          |
| $H_0^c$:   | male MCI vs. SCC | 0.6146           | 0.6297           |
| $H_0^c$:   | female AD vs. MCI | 0.0074           | 0.0091           |
| $H_0^c$:   | female AD vs. SCC | 0.0006           | 0.0012           |
| $H_0^c$:   | female MCI vs. SCC | 0.3687           | 0.3811           |
| $H_0^d$:   | AD male vs. female | 0.0881           | 0.0834           |
| $H_0^d$:   | MCI male vs. female | 0.1582           | 0.1592           |
| $H_0^d$:   | SCC male vs. female | 0.3423           | 0.3744           |

Requirements which are verifiable in practice. Previously existing procedures for the situation addressed here had suffered from low power to detect alternatives, were limited to only a few specific null hypotheses, or needed various requirements in particular regarding the data generating distribution. Under weak conditions, we have proved the asymptotic normality of the difference between the vectorized covariance matrices and its corresponding vectorized empirical version. We considered two-test statistics which are based upon the vectorized empirical covariance matrix and an estimator of its own covariance: a Wald type statistic (WTS) as well as an ANOVA type statistic (ATS). These exhibit the usual advantages and disadvantages that are already well-known from the literature on mean-based inference. In order to take care of some of these difficulties, namely the critical value for the ATS being unknown and the WTS requiring a rather large sample size, two kinds of bootstrap were used. On this occasion, specific adaptions were needed to take account of the special situation where inference is not on the expectation vectors, but on the covariance matrices. To investigate the properties of the newly constructed tests, an extensive simulation study was done. For this purpose, several different hypotheses were considered and the type one error control, as well as the power to detect deviations from the null hypothesis, were compared to existing test procedures. The ATS showed a quite accurate error control in each of the hypotheses, in particular in comparison with competing procedures. Note that for most hypotheses, no appropriate competing test is available. The simulated power of the proposed tests was fine, even for moderately small sample sizes ($n_1 = 30, n_2 = 20$). This is a ma-
jor advantage when comparing with existing procedures for testing homogeneity of covariances, even considering that they usually require further assumptions.

In future research, we would like to investigate in more detail the large number of possible null hypotheses which are included in our model as special cases. For example, tests for given covariance structure (such as compound symmetry or autoregressive) with unknown parameters are of great interest. Moreover, our results allow for a variety of new tests for hypotheses that can derived from our model, for example testing the equality of determinants of covariances matrices.

8 Acknowledgment

Paavo Sattler and Markus Pauly like to thank the German Research Foundation for the support received within project PA 2409/4-1. Moreover, Arne Bathke was supported by Austrian Science Fund (FWF) I 2697-N31.
9 Appendix

The asymptotic distribution, discussed in Theorem 3.1 is well known, but based on the importance for the techniques present in this paper we will prove it shortly. Moreover this allows to get the idea of our bootstrap approaches later on.

Proof of Theorem 3.1: First we consider the difference between the vector $v_i$ and his estimated version $\hat{v}_i$, multiplied with $\sqrt{N}$

$$\sqrt{N} (\hat{v}_i - v_i) = \sqrt{N} \text{vech} \left( \frac{1}{n_{i-1}} \sum_{k=1}^{n_i} [e_{ik} e_{ik}^T - V_i] + \frac{1}{n_{i-1}} V_i - \frac{1}{n_{i-1}} \left( \sqrt{n_i} \epsilon_i \right) (\sqrt{n_i} \epsilon_i)^T \right).$$

Due to Slutzky and the multivariat Central limit theorem the second and third term tend to zero in probability. Thus, it is sufficient to consider the first term. But this converges to $N_d(0_d, \Sigma_i)$ in distribution again by the multivariate central limit theorem, which gives us the result by central mapping theorem. \hfill \Box

This convergence would also follow from Zhang and Boos (1993) but the bootstrap approach is based on this proof, so it is helpful to outline it again. To use this result, a consistent estimator for the covariance matrix $\Sigma$ is needed.

Proof of Lemma 3.2: We know that

$$\tilde{\Sigma}_i = \frac{1}{n_i - 1} \sum_{k=1}^{n_i} \left[ \text{vech}(e_{ik} e_{ik}^T) - \frac{1}{n_i} \text{vech}(e_{i} e_{i}^T) \right] \left[ \text{vech}(e_{ik} e_{ik}^T) - \frac{1}{n_i} \text{vech}(e_{i} e_{i}^T) \right]^T,$$

is a consistent estimator for $\Sigma_i$, since $\text{vech}(e_{ik} e_{ik}^T)$ are i.i.d. vectors. Thus, it is sufficient to prove that $\tilde{\Sigma}_i - \hat{\Sigma}_i$ converge almost sure to 0. This leads to

$$\hat{\Sigma}_i - \tilde{\Sigma}_i = \frac{4n_i}{n_i - 1} \left( \text{vech}(X_i \mu_i^T) \text{vech}(X_i \mu_i^T)^T - \text{vech}(X_i X_i^T) \text{vech}(X_i X_i^T)^T \right)$$

$$+ \frac{4}{n_i - 1} \sum_{k=1}^{n_i} \left[ \text{vech}(X_{ik} X_i^T) \text{vech}(X_{ik} X_i^T)^T - \text{vech}(X_{ik} \mu_i^T) \text{vech}(X_{ik} \mu_i^T)^T \right]$$

$$+ \frac{4}{n_i - 1} \sum_{k=1}^{n_i} \left[ \text{vech}(X_{ik} X_i^T) \text{vech}(X_{ik} X_i^T)^T - \text{vech}(X_{ik} X_i^T) \text{vech}(X_{ik} X_i^T)^T \right]$$

$$+ \frac{4}{n_i - 1} \sum_{k=1}^{n_i} \left[ \text{vech}(X_{ik} X_i^T) \text{vech}(X_{ik} X_i^T) \text{vech}(X_{ik} X_i^T)^T - \text{vech}(X_{ik} X_i^T) \text{vech}(X_{ik} X_i^T)^T \right]$$

$$= \frac{4n_i}{n_i - 1} \left( \text{vech}(X_i (\mu_i - X_i)^T) \text{vech}(X_i (\mu_i - X_i)^T)^T \right)$$

$$+ \frac{4}{n_i - 1} \sum_{k=1}^{n_i} \left[ \text{vech}(X_{ik} (X_i - \mu_i)^T) \text{vech}(X_{ik} (X_i - \mu_i)^T + 2X_{ik} \mu_i^T)^T \right]$$

$$+ \frac{4}{n_i - 1} \sum_{k=1}^{n_i} \left[ \text{vech}(X_{ik} X_{ik}^T) \text{vech}(X_{ik} (\mu_i - X_i)^T)^T \right]$$

$$+ \frac{4}{n_i - 1} \sum_{k=1}^{n_i} \left[ \text{vech}(X_{ik} X_{ik}^T) \text{vech}(X_{ik} (\mu_i - X_i)^T)^T \right].$$

24
It is enough to show that each component of this difference converges almost sure to zero. So with $|X|$ denoting the absolute value of each component we get for arbitrary $h, j \in \{1, \ldots, p\}$ that

\[
|(|\hat{\Sigma}_i - \Sigma_i|)_{h,j}| \leq \frac{4n_t}{n_{t-1}} |\text{vech}(\bar{X}_i(\mu_i - \bar{X}_i)\top)_h| |\text{vech}(\bar{X}_i\mu_i\top + \bar{X}_i\bar{X}_i\top)_j| + \frac{4}{n_{t-1}} \sum_{k=1}^{n_t} |\text{vech}(\bar{X}_{ik}(\bar{X}_i - \mu_i)\top)_h| |\text{vech}(\bar{X}_{ik}(\bar{X}_i - \mu_i)\top + 2\bar{X}_{ik}\mu_i\top)_j| \\
+ \frac{4}{n_{t-1}} \sum_{k=1}^{n_t} |\text{vech}(\bar{X}_{ik}\bar{X}_{ik}\top)_j| |\text{vech}(\bar{X}_i(\mu_i - \bar{X}_i)\top)_h| + \frac{4}{n_{t-1}} \sum_{k=1}^{n_t} |\text{vech}(\bar{X}_{ik}\bar{X}_{ik}\top)_j| |\text{vech}(\bar{X}_{ik}(\mu_i - \bar{X}_i)\top)_h| \\
\leq \max_{\ell = 1,\ldots,p} |(\mu_i)_\ell - (\bar{X}_i)_\ell| \frac{4n_t}{n_{t-1}} \cdot |\text{vech}(\bar{X}_i|1\top)_h| \cdot |\text{vech}(\bar{X}_i\mu_i\top + \bar{X}_i\bar{X}_i\top)_j| \\
+ \left( \max_{\ell = 1,\ldots,p} |(\mu_i)_\ell - (\bar{X}_i)_\ell| \right)^2 \frac{4}{n_{t-1}} \sum_{k=1}^{n_t} \text{vech}(\bar{X}_{ik}|1\top)_h \cdot \text{vech}(\bar{X}_{ik}|1\top)_j \\
+ \max_{\ell = 1,\ldots,p} |(\mu_i)_\ell - (\bar{X}_i)_\ell| \frac{4}{n_{t-1}} \sum_{k=1}^{n_t} \text{vech}(\bar{X}_{ik}\bar{X}_{ik}\top)_j \cdot \text{vech}(\bar{X}_i|1\top)_h \\
+ \max_{\ell = 1,\ldots,p} |(\mu_i)_\ell - (\bar{X}_i)_\ell| \frac{4}{n_{t-1}} \sum_{k=1}^{n_t} \text{vech}(\bar{X}_{ik}\bar{X}_{ik}\top)_j \cdot \text{vech}(\bar{X}_{ik}|1\top)_h.
\]

Here we used that the maximum doesn’t depend on the index of the sum, so this factor can be pulled out of the vech and the sum, which are both linear functions. Because of the strong law of large numbers we know $(\mu_i - \bar{X}_i) \overset{a.s.}{\rightarrow} 0_d$ which means that every component goes to zero almost sure and therefore also the maximum.

The general assumption (4) , which ensures that all occurring terms have finite expectation values together with another application of the SLLN leads to:

\[
\text{vech}(\bar{X}_i|1\top)_h \cdot \text{vech}(\bar{X}_i\mu_i\top + \bar{X}_i\bar{X}_i\top)_j \overset{a.s.}{\rightarrow} \text{vech}(\mu_i|1\top)_h \cdot \text{vech}(2\mu_i\mu_i\top)_j,
\]

\[
\frac{4}{n_{t-1}} \sum_{k=1}^{n_t} \text{vech}(\bar{X}_{ik}|1\top)_h \cdot \text{vech}(\bar{X}_{ik}|1\top)_j \overset{a.s.}{\rightarrow} 4 \cdot \mathbb{E} \left( \text{vech}(\bar{X}_{ii}|1\top)_h \cdot \text{vech}(\bar{X}_{ii}|1\top)_j \right)
\]

and equivalent for the other sums. So we have in all this cases the products goes almost sure to zero and therefore $(|\hat{\Sigma}_i - \Sigma_i|)_{h,j} \overset{a.s.}{\rightarrow} 0$.

With Slutzky and the continuous mapping theorem, we also get the results for $\hat{\Sigma}$ and $\hat{\Sigma}_0$.

With these results the asymptotic distribution of the applied test statistics can be proved. But first we introduce one more test statistic from Friedrich and Pauly (2017) which was simulated in our extended simulation study.
With $A_0$ as the notation for a matrix just containing the diagonal elements of $A$ we get

\[
\text{MATS}_v := M_N = N [C\hat{v} - \zeta]^T \left( C\hat{\Sigma}_0 C^T \right)^+ [C\hat{v} - \zeta].
\] (9)

**Proof of Theorem 3.3** All results are known, but usually, idempotent symmetric hypothesis matrices are considered, so we will repeat them for general matrices $C$. From Theorem 3.1 it follows that all these quadratic forms can be written as the sum of a quadratic form with normal distributed random vectors and vectors which converge in distribution to zero.

Therefore with $\sqrt{N}(\hat{v} - v) \xrightarrow{d} Z \sim N_{a \cdot p} (0_{a \cdot p}, \Sigma)$ we get

\[
\begin{align*}
A_N \cdot \text{tr}(C\Sigma C^T) &= N [C\hat{v} - \zeta]^T [C\hat{v} - \zeta] \\
&\xrightarrow{d_0} N \cdot (\hat{v} - v)^T C^T C(\hat{v} - v) \\
&\xrightarrow{d} Z^T C^T CZ \\
&= (\Sigma^{-1/2} Z)^T \Sigma^{1/2} C^T C \Sigma^{1/2} (\Sigma^{-1/2} Z) \\
&\xrightarrow{d} \sum_{\ell=1}^{a \cdot p} \lambda_\ell B_\ell,
\end{align*}
\]

with $\lambda_\ell, \ell = 1, \ldots, a \cdot p$ eigenvalues of $\Sigma^{1/2} C^T C \Sigma^{1/2}$ and $B_\ell \xrightarrow{i.i.d.} \chi^2_{1 \cdot \text{rank}(C)}$. Here it is used that $\Sigma^{1/2} C^T C \Sigma^{1/2}$ is symmetric and therefore has a spectral representation. The rest follows from the fact that the multivariate standard normal distribution is invariante under orthogonal transformations. With the fact that $\hat{\Sigma}$ is a consistent estimator and Slutzky’s theorem dividing the trace lead to the result.

For the WTS similar holds

\[
\begin{align*}
\text{WTS}_v &= N [C\hat{v} - \zeta]^T \left( C\hat{\Sigma} C^T \right)^+ [C\hat{v} - \zeta] \\
&\xrightarrow{d_0} N \cdot (\hat{v} - v)^T C^T (C\Sigma C^T)^+ C(\hat{v} - v) \\
&\xrightarrow{d} Z^T C^T (C\Sigma C^T)^+ CZ \\
&\xrightarrow{d} \sum_{\ell=1}^{\text{rank}(C)} 1 \cdot B_\ell \\
&\xrightarrow{d} \chi^2_{\text{rank}(C)}.
\end{align*}
\]

If $\Sigma > 0$ is fulfilled, the unknown matrix can be replaced by the estimated counterpart $\hat{\Sigma}$.  

26
Accordingly, for our last test statistic,

\[
\text{MATS}_\nu = N \left[ C\hat{\nu} - \zeta \right]^T \left( C\hat{\Sigma}_0 C^T \right)^+ \left[ C\hat{\nu} - \zeta \right]
\]

\[
\overset{\mathcal{L}_0}{=} N \cdot (\hat{\nu} - \nu)^T C^T (C\Sigma_0 C^T)^+ C(\hat{\nu} - \nu)
\]

\[
\overset{\mathcal{D}}{=} Z^T C^T (C\Sigma_0 C^T)^+ CZ
\]

\[
\overset{\mathcal{D}}{=} \sum_{\ell=1}^{p} \lambda'_\ell B_\ell
\]

with \( \lambda'_\ell, \ell = 1, ..., p \) eigenvalues of \( \Sigma^{1/2} C^T (C\Sigma_0 C^T)^+ C \Sigma^{1/2} \) and \( B_\ell \overset{i.i.d.}{\sim} \chi^2_1 \). Again \( \hat{\Sigma}_0 \) is a consistent estimator of \( \Sigma_0 \) due to the continuous mapping theorem and if \( \Sigma_0 > 0 \) holds, the result also can be used if the unknown diagonal matrix is replaced by his estimator. \( \square \)

The additional requirements which ensure that there is no jump in ranks with the result that also the general inverses converge are difficult to verify.

**Proof of Theorem 4.1**: It is sufficient to prove the part for the single groups because the second part is just the combination of all groups. This result follows from a part-wise application (given the data) of the multivariate Lindeberg-Feller-Theorem. So it remains to show that all conditions are fulfilled, for which we use the fact that \( Y^* \) under \( X \) is \( p \) dimensional normal distributed with expectation \( 0_p \) and variance \( \hat{\Sigma}_i \):

1. \[
\sum_{k=1}^{n_i} \frac{1}{n_i} \mathbb{E} \left( \frac{\sqrt{N}}{n_i} Y^*_{ik} \bigg| X \right) = \sum_{k=1}^{n_i} \frac{\sqrt{N}}{n_i} \cdot \mathbb{E} \left( Y^*_{ik} \bigg| X \right) = 0
\]

2. \[
\sum_{k=1}^{n_i} \text{Cov} \left( \frac{\sqrt{N}}{n_i} Y^*_{ik} \bigg| X \right) = \frac{1}{n_i} \sum_{k=1}^{n_i} \frac{N}{n_i} \hat{\Sigma}_i \overset{\mathcal{D}}{\to} \frac{1}{\kappa_i} \Sigma_i
\]

3. \[
\lim_{N \to \infty} \sum_{k=1}^{n_i} \mathbb{E} \left( \left| \frac{\sqrt{N}}{n_i} Y^*_{ik} \right|^2 \cdot \mathbb{I} \left| \frac{\sqrt{N}}{n_i} Y^*_{ik} \right| > \delta \bigg| X \right)
\]

\[
= \lim_{N \to \infty} \sum_{k=1}^{n_i} \mathbb{E} \left( \left| Y^*_{ik} \right|^2 \cdot \mathbb{I} \left| Y^*_{ik} \right| > \delta \frac{n_i}{\sqrt{N}} \bigg| X \right)
\]

\[
= \frac{1}{\kappa_i} \cdot \lim_{N \to \infty} \mathbb{E} \left( \left| Y^*_{ik} \right|^2 \cdot \mathbb{I} \left| Y^*_{ik} \right| > \delta \frac{n_i}{\sqrt{N}} \bigg| X \right)
\]

\[
\leq \frac{1}{\kappa_i} \cdot \lim_{N \to \infty} \sqrt{\mathbb{E} \left( \left| Y^*_{ik} \right|^2 \bigg| X \right) \cdot \mathbb{E} \left( \mathbb{I} \left| Y^*_{ik} \right| > \delta \frac{n_i}{\sqrt{N}} \bigg| X \right)} = 0
\]

Here we used the Cauchy-Bunjakowski-Schwarz-Inequality and the fact that given the date \( X \) we have a multivariate normal distributed random vector \( Y^* \), so we know \( \mathbb{E} \left( \left| Y^*_{ik} \right|^2 \bigg| X \right) \). Moreover with the condition \( n_i/N \to \kappa_i \) and therefore \( \delta \cdot n_i/\sqrt{N} \to \infty \) it holds \( \mathbb{P} \left( \left| Y^*_{ik} \right| > \delta \cdot n_i/\sqrt{N} \right) \to 0 \), which leads to the result.
Therefore given the data $X$ it follows that $\sqrt{N} \cdot \tilde{Y}_i^*$ converges in distribution to $N_p \left( 0_p, 1/k_i \cdot \Sigma_i \right)$ and with Slutsky also $\sqrt{N} \cdot Y^*$ converges in distribution to $N_{\alpha-p} \left( 0_{\alpha-p}, \bigoplus_{i=1}^{\alpha} 1/k_i \cdot \Sigma_i \right)$.

**Proof of Theorem 4.3**: Again we have to show the conditions of the Lindeberg-Feller Theorem part-wise, given the data $X = (X_{11}, \ldots, X_{an_a})^T$:

1. \[ \sum_{k=1}^{n_i} \mathbb{E} \left( \frac{\sqrt{N}}{n_i} Y_{ik}^* \bigg| X \right) = \sum_{k=1}^{n_i} \frac{\sqrt{N}}{n_i} \mathbb{E}(W_{ik}) \cdot \left[ \text{vech}(\tilde{X}_{ik} \tilde{X}_{ik}^T) - \sum_{i=1}^{n_i} \frac{\text{vech}(\tilde{X}_{ik} \tilde{X}_{ik}^T)}{n_i} \right] = 0 \]
2. \[ \sum_{k=1}^{n_i} \text{Cov} \left( \frac{\sqrt{N}}{n_i} Y_{ik}^* \bigg| X \right) = \sum_{k=1}^{n_i} \frac{N}{n_i} \mathbb{E}(W_{ik}^2) \cdot (n_i - 1) \cdot \hat{\Sigma}_i \]
\[ = \frac{n_i - 1}{n_i} \frac{N}{n_i} \sum_{i} \frac{\mathbb{E}(Y_{i1}^* \cdot 1_{\|Y_{i1}^*\| > \sqrt{\delta} \cdot \frac{n_i}{\sqrt{N}}})}{1/k_i \Sigma_i} \rightarrow 0 \]

For the last part we use that given the data $\|Y_{i1}^*\|^2 \cdot 1_{\|Y_{i1}^*\| > \sqrt{\delta} \cdot \frac{n_i}{\sqrt{N}}} \leq \|Y_{i1}^*\|^2$ has a finite expectation value. Moreover, Lebesgue’s dominated convergence theorem with $n_i/\sqrt{N} \to \infty$ and $\mathbb{P} \left( \|Y_{i1}^*\| > \delta \cdot n_i/\sqrt{N} \right) \to 0$, leads to the result.

3. \[ \lim_{n_i \to \infty} \sum_{k=1}^{n_i} \mathbb{E} \left( \left\| \frac{\sqrt{N}}{n_i} Y_{ik}^* \right\|^2 \cdot 1_{\|Y_{i1}^*\| > \sqrt{\delta} \cdot \frac{n_i}{\sqrt{N}}} \bigg| X \right) \]
\[ = \lim_{N \to \infty} \frac{N}{(n_i)^2} \sum_{k=1}^{n_i} \mathbb{E} \left( \left\| Y_{i1}^* \right\|^2 \cdot 1_{\|Y_{i1}^*\| > \sqrt{\delta} \cdot \frac{n_i}{\sqrt{N}}} \bigg| X \right) \]
\[ = \frac{1}{k_i} \cdot \lim_{N \to \infty} \mathbb{E} \left( \left\| Y_{i1}^* \right\|^2 \cdot 1_{\|Y_{i1}^*\| > \sqrt{\delta} \cdot \frac{n_i}{\sqrt{N}}} \bigg| X \right) \]
\[ = \frac{1}{k_i} \cdot \mathbb{E} \left( \lim_{N \to \infty} \left\| Y_{i1}^* \right\|^2 \cdot 1_{\|Y_{i1}^*\| > \sqrt{\delta} \cdot \frac{n_i}{\sqrt{N}}} \bigg| X \right) = 0 \]

Hence, given the data we have convergence in distribution of $\sqrt{N} \cdot \tilde{Y}_i$ and $\sqrt{N} \cdot Y^*$ to $N_p \left( 0_p, 1/k_i \cdot \Sigma_i \right)$ resp. $N_{\alpha-p} \left( 0_{\alpha-p}, \bigoplus_{i=1}^{\alpha} 1/k_i \cdot \Sigma_i \right)$.

**Proof of Corollary 4.2 and Corollary 4.4**: First it’s obvious that from $\Sigma > 0$ follows $\Sigma_0 > 0$.

Therefore we know $(C \Sigma C^T)^+ \xrightarrow{p} (C \Sigma C^T)^+$ and $(C \Sigma_0 C^T)^+ \xrightarrow{p} (C \Sigma_0 C^T)^+$ and the same for the wild bootstrap. So the result follows direct from Theorem 4.3 resp. Theorem 4.3. Therefore it the result especially allows the application of the parametric bootstrap version of the MATS defined in (9), given by

\[ \text{MATS}^* := N \left[ C \tilde{Y}^* \right]^T \left( C \Sigma C^T \right)^+ \left[ C \tilde{Y}^* \right] \] (10)

28
and the wild bootstrap version given by

$$\text{MATS}^* := N \left[ CY^* \right]^T (CS_0^* C^T)^+ \left[ CY^* \right]. \quad (11)$$

Further Simulations

In this more extensive simulation we consider more null hypotheses and additionally simulate bootstrap versions of the MATS statistic defined in (10) and (11). To investigate the influence of the covariance matrix, the distributional setting uses an additional covariance matrix. Thereto we define $\Sigma_2$ as an autoregressive covariance matrix with parameter 0.6.

For this random vectors we use we consider an additional scenario:

D) $a = 1 \quad J_{0}^* : V_1 = V$ for given $V$,

where also scenario D) can be formulated with an idempotent symmetric matrix $C (D) = I_{15}$. For scenario A) we consider $n_1 = 0.6 \cdot N$ and $n_2 = 0.4 \cdot N$ with $N = (50, 100, 200)$ and for all other scenarios $n_1 = (25, 50, 100)$. We should keep in mind that in this case $p$ is 15 which makes this sample sizes comparatively rather small or not more than medium. Although in practice it is quite difficult or even impossible to check the necessary conditions to use WTS resp. MATS we additionally calculated their type-I error rate. It seems natural that the requirements of the WTS are violated in case D and probably the less restrictive conditions of the MATS too.

It can be seen in all tables that the wild bootstrap leads to more liberal results and the parametric bootstrap has less liberal or even conservative test results. Except for table 10 the ATS provides the best type-I error control, while MATS is in this scenarios too liberal especially for the wild bootstrap. In table 8 for the wild bootstrap the ATS has the best error rates, but the test is too conservative wherefore MATS and WTS are slightly advantageous. Surprisingly in situation C) WTS and ATS has the same rates. Presumably, this is because of the special structure of the hypothesis matrix $C$ which has rank 1 in this situation. This seems also to be a reason for the comparably slow convergence in this scenario.

So all in all $\phi_{ATS}^*$ or $\phi_{ATS}^*$ leads to rather good results, even for these sample sizes. Especially in table 9 and table 11 were equality of covariances is tested between random vectors with different distributions, the results are realy convinc-
|                | Exp(1) |          |          |          |           |           |           |           |           |           |
|----------------|--------|----------|----------|----------|-----------|-----------|-----------|-----------|-----------|-----------|
|                | N      | 50       | 100      | 200      | t₉        | 50        | 100       | 200       | Normal    |           |
| ATS-Para       |        | .0211    | .0235    | .0306    | .0299     | .0339     | .0428     | .0356     | .0403     | .0454     |
| ATS-Wild       |        | .0772    | .0670    | .0604    | .0589     | .0555     | .0565     | .0593     | .0568     | .0526     |
| WTS-Para       |        | .0448    | .0423    | .0418    | .0310     | .0411     | .0449     | .0393     | .0468     | .0503     |
| WTS-Wild       |        | .0926    | .0880    | .0740    | .0573     | .0646     | .0614     | .0645     | .0680     | .0648     |
| WTS-χ²₁₅       |        | .4649    | .1683    | .0777    | .3995     | .1398     | .0803     | .4053     | .1545     | .0821     |
| MATS-Para      |        | .0380    | .0380    | .0403    | .0385     | .0380     | .0459     | .0401     | .0443     | .0471     |
| MATS-Wild      |        | .0735    | .0708    | .0624    | .0606     | .0558     | .0589     | .0628     | .0576     | .0533     |
| Bartlett-S     |        | .0179    | .0481    | .0528    | .0130     | .0285     | .0426     | .0131     | .0335     | .0418     |
| Bartlett-P     |        | .0204    | .0282    | .0343    | .0193     | .0274     | .0368     | .0196     | .0350     | .0428     |
| Box’s M-χ₁₅    |        | .3705    | .4310    | .4699    | .0866     | .0863     | .0969     | .0523     | .0517     | .0534     |
| Box’s M-F      |        | .3600    | .4280    | .4694    | .0816     | .0852     | .0965     | .0497     | .0515     | .0534     |

Table 8: Simulated type-I error rates ($\alpha = 5\%$) in scenario A ($\mathcal{H}_0: \mathbf{V}_1 = \mathbf{V}_2$) for ATS, WTS, MATS, Bartletts test and Box’s M-test. The observation vectors have dimension 5, covariance matrix $(\Sigma)_{ij} = I_5 + J_5$ and there is always the same relation between group samples size with $n_1 := 0.6 \cdot N$ resp. $n_2 := 0.4 \cdot N$. 
|          | Exp(1) & $t_9$ | $t_9$ & Normal | Normal & Exp(1) |
|----------|---------------|---------------|----------------|
|          | 50            | 100           | 200            | 50            | 100           | 200            | 50            | 100           | 200            |
| ATS-Para | .0240         | .0244         | .0359          | .0325         | .0381         | .0427          | .0456         | .0402         | .0425          |
| ATS-Wild | .0656         | .0543         | .0578          | .0599         | .0541         | .0530          | .0820         | .0775         | .0771          |
| WTS-Para | .0359         | .0333         | .0420          | .0345         | .0392         | .0436          | .1059         | .1504         | .1436          |
| WTS-Wild | .0679         | .0654         | .0649          | .0613         | .0577         | .0567          | .1773         | .2140         | .1924          |
| WTS-$\chi_{15}^2$ | .4350 | .1518 | .0741 | .3885 | .1394 | .0709 | .5941 | .3526 | .2235 |
| MATS-Para | .0337         | .0296         | .0412          | .0369         | .0401         | .0433          | .0666         | .0634         | .0662          |
| MATS-Wild | .0609         | .0536         | .0561          | .0573         | .0534         | .0529          | .0976         | .0878         | .0860          |
| Bartlett-S | .0159         | .0338         | .0482          | .0125         | .0338         | .0418          | .0162         | .0420         | .0500          |
| Bartlett-P | .0184         | .0231         | .0331          | .0175         | .0298         | .0370          | .0423         | .0677         | .0766          |
| Box’s M-$\chi_{15}^2$ | .1988 | .2192 | .2463 | .0649 | .0616 | .0616 | .2316 | .2637 | .2727 |
| Box’s M-F | .1904         | .2167         | .2458          | .0622         | .0609         | .0615          | .2228         | .2618         | .2720          |

Table 9: Simulated type-I error rates ($\alpha = 5\%$) in scenario A ($H_0: V_1 = V_2$) for ATS, WTS, MATS, Bartlett's test and Box's M-test. The observation vectors have dimension 5, covariance matrix $\{\Sigma\}_{ij} = I_5 + J_5$ and there is always the same relation between group samples size with $n_1 := 0.6 \cdot N$ resp. $n_2 := 0.4 \cdot N$. 

31
The advantage of WTS and MATS to be invariant under some transformations is handy but not enough.

|                  | Exp(1) |             |         | Normal |
|------------------|--------|-------------|---------|--------|
|                  | N=50   | N=100       | N=200   | N=50   | N=100  | N=200  |
| **ATS-Para**     | .0197  | .0207       | .0293   | .0285  | .0325  | .0392  | .0339  | .0388  | .0458  |
| **ATS-Wild**     | .0792  | .0658       | .0582   | .0594  | .0539  | .0562  | .0607  | .0556  | .0548  |
| **WTS-Para**     | .0433  | .0438       | .0424   | .0301  | .0413  | .0452  | .0379  | .0468  | .0516  |
| **WTS-Wild**     | .0926  | .0880       | .0740   | .0573  | .0646  | .0614  | .0645  | .0680  | .0648  |
| **WTS-χ₁₅**     | .4649  | .1683       | .0777   | .3995  | .1398  | .0803  | .4053  | .1545  | .0821  |
| **MATS-Para**    | .0327  | .0329       | .0379   | .0361  | .0368  | .0443  | .0387  | .0422  | .0470  |
| **MATS-Wild**    | .0758  | .0710       | .0626   | .0640  | .0570  | .0561  | .0620  | .0593  | .0556  |
| **Bartlett-S**   | .0181  | .0490       | .0531   | .0132  | .0291  | .0424  | .0130  | .0336  | .0415  |
| **Bartlett-P**   | .0204  | .0282       | .0343   | .0193  | .0274  | .0368  | .0196  | .0350  | .0428  |
| **Box’s M-χ₁₅** | .3705  | .4310       | .4699   | .0866  | .0863  | .0969  | .0523  | .0517  | .0534  |
| **Box’s M-F**    | .3600  | .4280       | .4694   | .0816  | .0852  | .0965  | .0497  | .0515  | .0534  |

Table 10: Simulated type-I error rates (α = 5%) in scenario A (\(V_0^y : V_1 = V_2\)) for ATS, WTS, MATS, Bartlett’s test and Box’s M-test. The observation vectors have dimension 5, covariance matrix \(\Sigma_{ij} = 0.6|i-j|\) and there is always the same relation between group samples size with \(n_1 := 0.6 \cdot N\) resp. \(n_2 := 0.4 \cdot N\).

For the power simulation, we again consider on one hand the ATS with a parametric bootstrap and a wild bootstrap and on the other hand bartlett’s test statistic with separate and pooled bootstrap and small as well as moderate sample size. Additional to the one-point-alternative we use \(\Delta = I_5 + diag(1/5, ..., 5/5) \cdot \delta\) to formulate a trend alternative and consider more different distributions.

The results from the one-point-alternative for normal-distribution and exponential distribution confirms the conclusions from the \(t_9\) distribution. Although for the trend-alternative the ATS with the wild bootstrap has a clearly higher power, especially for smaller delta. For moderate sample size except for \(\text{Exp}(1)\) distributed error term, the power of the \(\phi^*_\text{ATS}\) for the trend-alternative is higher or at least comparable to the Bartlett statistic. It turns out, that considering small sample sizes, the power rate of \(\phi^*_\text{ATS}\) and \(\phi^*\text{ATS}\) is always higher than from Bartlett’s statistic unattached from the chosen bootstrap technique.

Overall the ATS in particular with the wild bootstrap has quite good power curves and seems to merits over Bartlett’s test, especially for smaller numbers of observation, where it has often twice as much power.
|                  | Exp(1) & t₉  | t₉ & Normal | Normal & Exp(1) |
|------------------|--------------|-------------|-----------------|
| **N**            | 50    | 100 | 200 | 50    | 100 | 200 | 50    | 100 | 200 |
| ATS-Para         | .0215 | .0231 | .0378 | .0333 | .0348 | .0430 | .0427 | .0384 | .0415 |
| ATS-Wild         | .0658 | .0563 | .0612 | .0618 | .0554 | .0543 | .0845 | .0789 | .0764 |
| WTS-Para         | .0358 | .0327 | .0422 | .0339 | .0390 | .0434 | .1045 | .1499 | .1448 |
| WTS-Wild         | .0679 | .0654 | .0649 | .0613 | .0577 | .0567 | .1773 | .2140 | .1924 |
| WTS-χ¹⁵         | .4350 | .1518 | .0741 | .3885 | .1394 | .0709 | .5941 | .3526 | .2235 |
| MATS-Para        | .0283 | .0300 | .0393 | .0358 | .0372 | .0454 | .0692 | .0672 | .0701 |
| MATS-Wild        | .0632 | .0541 | .0562 | .0606 | .0556 | .0551 | .1105 | .0965 | .0961 |
| Bartlett-S       | .0164 | .0340 | .0484 | .0125 | .0331 | .0418 | .0163 | .0421 | .0502 |
| Bartlett-P       | .0184 | .0231 | .0331 | .0175 | .0298 | .0370 | .0423 | .0677 | .0766 |
| Box’s M-χ²¹      | .1988 | .2192 | .2463 | .0649 | .0616 | .0616 | .2316 | .2637 | .2727 |
| Box’s M-F        | .1904 | .2167 | .2458 | .0622 | .0609 | .0615 | .2228 | .2618 | .2720 |

Table 11: Simulated type-I error rates ($\alpha = 5\%$) in scenario A ($H_0^\text{A} : V_1 = V_2$) for ATS, WTS, MATS, Bartletts test and Box’s M-test. The observation vectors have dimension 5, covariance matrix $[\Sigma]_{ij} = 0.6^{|i-j|}$ and there is always the same relation between group samples size with $n_1 := 0.6 \cdot N$ resp. $n_2 := 0.4 \cdot N$.

|                  | Exp(1)  | t₉       | Normal |
|------------------|---------|----------|--------|
| **N**            | 25      | 50       | 100    |
| ATS-Para         | .0150   | .0155 | .0237 | .0205 | .0285 | .0328 | .0320 | .0342 | .0442 |
| ATS-Wild         | .0568   | .0575 | .0562 | .0482 | .0504 | .0501 | .0546 | .0488 | .0536 |
| WTS-Para         | .1257   | .0988 | .0746 | .0599 | .0559 | .0566 | .0584 | .0542 | .0540 |
| WTS-Wild         | .2348   | .1728 | .1244 | .0948 | .0798 | .0719 | .0799 | .0670 | .0616 |
| WTS-χ₂¹      | .2376   | .1362 | .0874 | .1422 | .0897 | .0655 | .1308 | .0877 | .0656 |
| MATS-Para        | .0634   | .0634 | .0585 | .0522 | .0527 | .0555 | .0516 | .0495 | .0561 |
| MATS-Wild        | .1847   | .1446 | .1118 | .1055 | .0816 | .0754 | .0921 | .0706 | .0682 |

Table 12: Simulated type-I error rates ($\alpha = 5\%$) in scenario B ($H_0^\text{B} : V_{111} = V_{211} = ... = V_{155}$) for ATS, WTS and MATS with 5-dimensional vectors and $([\Sigma])_{ij} = I_5 + J_5$. 

33
|                | Exp(1) | t₉   | Normal |
|----------------|--------|------|--------|
|                | 25     | 50   | 100    |
| N              | 25     | 50   | 100    |
| ATS-Para       | .0168  | .0183| .0249  |
| ATS-Wild       | .0583  | .0614| .0574  |
| WTS-Para       | .1166  | .0925| .0724  |
| WTS-Wild       | .2170  | .1674| .1216  |
| WTS-χ²        | .2497  | .1402| .0879  |
| MATS-Para      | .1166  | .0925| .0249  |
| MATS-Wild      | .2170  | .1674| .1216  |
| WTS-χ²        | .2497  | .1402| .0879  |
| MATS-Para      | .1166  | .0925| .0249  |
| MATS-Wild      | .2170  | .1674| .1216  |
| WTS-χ²        | .2497  | .1402| .0879  |

Table 13: Simulated type-I error rates ($\alpha = 5\%$) in scenario B ($H_0^\prime : V_{111} = V_{122} = ... = V_{155}$) for ATS, WTS and MATS with 5-dimensional vectors and $(\Sigma)_{ij} = 0.6^{\left|i-j\right|}$.

|                | Exp(1) | t₉   | Normal |
|----------------|--------|------|--------|
|                | 25     | 50   | 100    |
| N              | 25     | 50   | 100    |
| ATS-Para       | .1504  | .1141| .0930  |
| ATS-Wild       | .1956  | .1374| .1053  |
| WTS-Para       | .1504  | .1141| .0930  |
| WTS-Wild       | .1956  | .1374| .1053  |
| WTS-χ²        | .1576  | .1196| .0947  |
| MATS-Para      | .1576  | .1196| .0947  |
| MATS-Wild      | .2090  | .1443| .1089  |
| WTS-χ²        | .1576  | .1196| .0947  |
| MATS-Para      | .1576  | .1196| .0947  |
| MATS-Wild      | .2090  | .1443| .1089  |
| WTS-χ²        | .1576  | .1196| .0947  |

Table 14: Simulated type-I error rates ($\alpha = 5\%$) in scenario C ($H_0^\prime : \text{tr}(V_1) = 10$) for ATS, WTS and MATS with 5-dimensional vectors and $\Sigma = I_5 + J_5$. 34
Table 15: Simulated type-I error rates ($\alpha = 5\%$) in scenario C ($\mathcal{H}_0^C : \text{tr}(V_1) = 5$) for ATS, WTS and MATS with 5-dimensional vectors and $\Sigma_{ij} = 0.6^{|i-j|}$.

|          | $\text{Exp}(1)$ | $t_9$ | Normal |
|----------|----------------|-------|--------|
|          | 25 | 50 | 100 | 25 | 50 | 100 | 25 | 50 | 100 |
| ATS-Para | .1469 | .1102 | .0920 | .0928 | .0771 | .0696 | .0845 | .0719 | .0629 |
| ATS-Wild | .1917 | .1318 | .1057 | .1035 | .0831 | .0746 | .0921 | .0763 | .0664 |
| WTS-Para | .1469 | .1102 | .0920 | .0928 | .0771 | .0696 | .0845 | .0719 | .0629 |
| WTS-Wild | .1917 | .1318 | .1057 | .1035 | .0831 | .0746 | .0921 | .0763 | .0664 |
| WTS-$\chi^2_1$ | .1623 | .1180 | .0934 | .1137 | .0831 | .0777 | .1062 | .0810 | .0684 |
| MATS-Para | .1551 | .1139 | .0934 | .1008 | .0818 | .0717 | .0924 | .0748 | .0652 |
| MATS-Wild | .2053 | .1378 | .1093 | .1148 | .0883 | .0772 | .1040 | .0818 | .0681 |

Table 16: Simulated type-I error rates ($\alpha = 5\%$) in scenario D ($\mathcal{H}_0^D : V_1 = \Sigma$) for ATS, WTS and MATS with 5-dimensional vectors and $\Sigma_{ij} = I_5 + J_5$.

|          | $\text{Exp}(1)$ | $t_9$ | Normal |
|----------|----------------|-------|--------|
|          | 25 | 50 | 100 | 25 | 50 | 100 | 25 | 50 | 100 |
| ATS-Para | .0897 | .0698 | .0612 | .0681 | .0586 | .0555 | .0699 | .0589 | .0569 |
| ATS-Wild | .1800 | .1489 | .1214 | .1195 | .0945 | .0772 | .1160 | .0868 | .0693 |
| WTS-Para | .8421 | .7665 | .5965 | .6212 | .4940 | .3162 | .5192 | .3889 | .2383 |
| WTS-Wild | .8739 | .8171 | .6560 | .6723 | .5518 | .3619 | .5717 | .4417 | .2711 |
| WTS-$\chi^2_4$ | .9930 | .9156 | .7166 | .9733 | .7611 | .4666 | .9567 | .6843 | .3760 |
| MATS-Para | .3371 | .2418 | .1752 | .1756 | .1171 | .0893 | .1452 | .1013 | .0741 |
| MATS-Wild | .4084 | .3045 | .2246 | .2307 | .1506 | .1108 | .1908 | .1288 | .0860 |
Figure 3: Simulated power for an one-point-alternative in scenario A ($H_0^y : V_1 = V_2$) for ATS with wild and parametric bootstrap as well as the test based on Bartlett’s statistic with separate and pooled bootstrap. The error terms are based on the Exp(1) distribution with $\Sigma = I_5 + J_5$ and $n_1 = 30, n_2 = 20$ at the top resp. $n_1 = 60, n_2 = 40$ at the bottom.
Figure 4: Simulated power for an one-point-alternative in scenario A ($H_0' : V_1 = V_2$) for ATS with wild and parametric bootstrap as well as the test based on Bartlett’s statistic with separate and pooled bootstrap. The error terms are based on the $t_9$ distribution with $\Sigma = I_5 + J_5$ and $n_1 = 30, n_2 = 20$ at the top resp. $n_1 = 60, n_2 = 40$ at the bottom.
Figure 5: Simulated power for an one-point-alternative in scenario A ($H_0^V : V_1 = V_2$) for ATS with wild and parametric bootstrap as well as the test based on Bartlett’s statistic with separate and pooled bootstrap. The error terms are based on the $N(0,1)$ distribution with $\Sigma = I_5 + J_5$ and $n_1 = 30, n_2 = 20$ at the top resp. $n_1 = 60, n_2 = 40$ at the bottom.
Figure 6: Simulated power for a trend-alternative in scenario A ($H_0^\gamma : V_1 = V_2$) for ATS with wild and parametric bootstrap as well as the test based on Bartlett’s statistic with separate and pooled bootstrap. The error terms are based on the $\text{Exp}(1)$ distribution with $\Sigma = I_5 + J_5$ and $n_1 = 30, n_2 = 20$ at the top resp. $n_1 = 60, n_2 = 40$ at the bottom.
Figure 7: Simulated power for an one-point-alternative in scenario A \( (\mathcal{H}_0^Y : V_1 = V_2) \) for ATS with wild and parametric bootstrap as well as the test based on Bartlett’s statistic with separate and pooled bootstrap. The error terms are based on the \( t_9 \) distribution with \( \Sigma = I_5 + J_5 \) and \( n_1 = 30, n_2 = 20 \) at the top resp. \( n_1 = 60, n_2 = 40 \) at the bottom.
It is important to mention the fact, that multiplication with the diagonal matrix changes not even $V$ but also $\Sigma$. So for each $\Delta$ the eigenvalues of $\Sigma^{1/2}C^T\Sigma^{1/2}$ changes, which consequently changes the limit distribution of the ATS. Regrettably changing $V$ without changing $\Sigma$ is nearly impossible, and there exists no good approach to check the power in situations like this so far. Because of this for example in Zhang and Boos (1993) just one matrix is used for calculation of power instead of a whole sequence. Therefore our approach is pretty advanced and kind of intuitive.
Figure 8: Simulated power for an one-point-alternative in scenario A ($H_0^i : V_1 = V_2$) for ATS with wild and parametric bootstrap as well as the test based on Bartlett’s statistic with separate and pooled bootstrap. The error terms are based on the $\mathcal{N}(0, 1)$ distribution with $\Sigma = I_5 + J_5$ and $n_1 = 30, n_2 = 20$ at the top resp. $n_1 = 60, n_2 = 40$ at the bottom.
References

Anderson, T. W. (1984). *An Introduction to Multivariate Statistical Analysis*. 2nd ed. Wiley, New York, USA.

Bartlett, M. S. and Rajalakshman, D. V. (1953). Goodness of fit tests for simultaneous autoregressive series. *Journal of the Royal Statistical Society. Series B (Methodological)*, 15(1):107–124.

Bathke, A., Friedrich, S., Pauly, M., Konietschke, F., Staffen, W., Strobl, N., and Höller, Y. (2018). Testing mean differences among groups: Multivariate and repeated measures analysis with minimal assumptions. *Multivariate Behavioral Research*, 53.

Boos, D. D. and Brownie, C. (2004). Comparing variances and other measures of dispersion. *Statist. Sci.*, 19(4):571–578.

Box, G. E. P. (1949). A general distribution theory for a class of likelihood criteria. *Biometrika*, 36(3/4):317–346.

Box, G. E. P. (1953). Non-normality and tests on variances. *Biometrika*, 40(3-4):318–335.

Browne, M. and Shapiro, A. (1986). The asymptotic covariance matrix of sample correlation coefficients under general conditions. *Linear Algebra and its Applications*, 82:169 – 176.

Duembgen, L., Pauly, M., and Schweizer, T. (2013). A survey of m-functionals of multivariate location and scatter. *Statistics Surveys*, 9.

Fang, K. and Zhang, Y.-t. (1990). *Generalized multivariate analysis*. Springer-Verlag.

Friedrich, S., Brunner, E., and Pauly, M. (2017). Permuting longitudinal data in spite of the dependencies. *Journal of Multivariate Analysis*, 153:255–265.

Friedrich, S., Konietschke, F., and Pauly, M. (2016). A wild bootstrap approach for nonparametric repeated measurements. *Computational Statistics & Data Analysis*.

Friedrich, S. and Pauly, M. (2017). Mats: Inference for potentially singular and heteroscedastic manova. *Journal of Multivariate Analysis*, 165.

Gupta, A. K. and Xu, J. (2006). On some tests of the covariance matrix under general conditions. *Annals of the Institute of Statistical Mathematics*, 58(1):101–114.
Konietschke, F., Bathke, A., Harrar, S., and Pauly, M. (2015). Parametric and non-parametric bootstrap methods for general MANOVA. *Journal of Multivariate Analysis*, 140:291–301.

Lawley, D. N. (1963). On testing a set of correlation coefficients for equality. *Ann. Math. Statist.*, 34(1):149–151.

Muirhead, R. (1982). *Aspects of Multivariate Statistical Theory*. Wiley, New York.

Pauly, M. (2011). Weighted resampling of martingale difference arrays with applications. *Electronic Journal of Statistics*, 5:41–52.

R Core Team (2019). *R: A Language and Environment for Statistical Computing*. R Foundation for Statistical Computing, Vienna, Austria.

Sarah Friedrich, Frank Konietschke, M. P. (2019). MANOVA.RM: Analysis of Multivariate Data and Repeated Measures Designs. R package version 0.3.2.

Yang, Y. and DeGruttola, V. (2012). Resampling-based methods in single and multiple testing for equality of covariance correlation matrices. *The international journal of biostatistics*, 8:Article 13.

Zhang, J. and Boos, D. (1993). Testing hypotheses about covariance matrices using bootstrap methods. *Communications in Statistics - Theory and Methods*, 22:723–739.

Zhang, J. and Boos, D. D. (1992). Bootstrap critical values for testing homogeneity of covariance matrices. *Journal of the American Statistical Association*, 87(418):425–429.

Zhu, L.-X., Ng, K., and Jing, P. (2002). Resampling methods for homogeneity tests of covariance matrices. *Statistica Sinica*, 12.

Zimmermann, G., Pauly, M., and Bathke, A. C. (2019). Multivariate analysis of covariance when standard assumptions are violated.