Exact and approximate solutions of Schrödinger’s equation with hyperbolic double-well potentials

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Analytic and approximate solutions for the energy eigenvalues generated by the hyperbolic potentials
\[ V_m(x) = -U_0 \sinh^{2m}(x/d) / \cosh^{2m+2}(x/d), \quad m = 0, 1, 2, \ldots \]
are constructed. A byproduct of this work is the construction of polynomial solutions for the confluent Heun equation along with necessary and sufficient conditions for the existence of such solutions based on the evaluation of a three-term recurrence relation. Very accurate approximate solutions for the general problem with arbitrary potential parameters are found by use of the asymptotic iteration method.

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I. INTRODUCTION

We study the one-dimensional Schrödinger equation
\[ -\frac{\hbar}{2\mu} \frac{d^2\psi}{dx^2} + V(x)\psi = E\psi, \quad V(x) = -U_0 \frac{\sinh^4(x/d)}{\cosh^6(x/d)}, \quad \psi(\pm\infty) = 0, \]  
(1)

with a double-well potential that has two physical parameters, \( U_0 \) and \( d \), representing the potential’s depth and width. This problem has been the subject of several recent studies \[1-5\]. Besides being a useful model for a wide variety of applications, from heterostructure physics to the trapping of Bose-Einstein condensates, it becomes an algebraically solvable system when certain constraints on the potential parameters \( U_0 \) and \( d \) are satisfied. Under suitable transformations of the dependent and independent variables, the equation itself transforms into the poorly understood confluent Heun-type equation. The interesting spectral problem studied in this paper illuminates the contribution of the confluent Heun equation to mathematical physics, and bridges an elusive physics problem to mathematical analysis. In the present work, we introduce a concrete approach to find both analytic and approximate solutions for a class of hyperbolic potentials given by:
\[ -\frac{\hbar}{2\mu} \frac{d^2\psi}{dx^2} + V_m(x; U_0, d)\psi = E\psi, \quad V_m(x; U_0, d) = -U_0 \frac{\sinh^{2m}(x/d)}{\cosh^{2m+2}(x/d)}, \quad \psi(\pm\infty) = 0, \quad m = 0, 1, 2, \ldots \]  
(2)

This potential family includes, for \( m = 0 \), the classical modified Pöschl-Teller potential \( V(x) = -U_0 / \cosh^2(x) \), one of the few exactly solvable potentials in quantum mechanics \[3, 7\].

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II. A CLASS OF POTENTIALS

Although the potential family $V(x; U_0, d)$ characterized by the two parameters $U_0$ and $d$, a simple change of variable $z = x/d$ transforms the equation into the following one-parameter Schrödinger equation

$$\left[-\frac{d^2}{dz^2} + V_m(z; v)\right] \psi = \varepsilon \psi, \quad V_m(z; v) = -v \frac{\sinh^{2m}(z)}{\cosh^{2m+2}(z)}, \quad v > 0, \quad -\infty < z < \infty, \quad \psi(\pm\infty) = 0,$$

where $v = 2\mu U_0 d^2/\hbar^2$ and $\varepsilon = 2\mu E d^2/\hbar^2$. The graph of the potential $V_m(z; 1)$ for different values $m = 0, 1, 2$ is displayed in Fig. 1. Clearly, $V_{m+1}(z) > V_m(z)$ for all $m$.

![Potential Graph](image)

FIG. 1: The potential $V_m(z; 1) = -\sinh^{2m}(z)/\cosh^{2m+2}(z)$ for $m = 0, 1, 2$.

The minimum of the potential occurs at $z = \pm[(\cosh(1+2m))/2]/\cosh$ with a minimum value of $V_{\text{min}}(z; v) = -m^m/(1+m)^{1+m}v$. We observe that $\lim_{m \to \infty} V_{\text{min}}(z, v) = 0$. Since

$$\int_{-\infty}^{\infty} V(z)dz = -v \int_{-\infty}^{\infty} \tanh^{2m}(z) \text{sech}^2(z)dz = -\frac{2v}{1+2m} < 0, \quad m = 0, 1, 2, \ldots.$$

the potential $V_m(z; v)$ has at least one negative eigenvalue [3] for any positive value of $v$ with eigenvalues $\varepsilon$ satisfying $V_{\text{min}}(z, v) < \varepsilon < 0$. For each $m \geq 0$, the hyperbolic potential [3] has a finite number of bound-state $\mathfrak{N}$ with degeneracy one, and an upper bound on $\mathfrak{N}$ given by:

$$\mathfrak{N} < 1 + \sqrt{2 \left[ \int_{-\infty}^{\infty} z^2 V(z)dz \int_{-\infty}^{\infty} V(z)dz \right]^{1/4}}.$$  

(5)

However, for the hyperbolic potential [2],

$$\int_{-\infty}^{\infty} z^2 V(z)dz = -\frac{4v}{2m+1} \left( \frac{z^2}{24} + \sum_{j=0}^{m-1} \frac{1}{4(m-j)} (\log(4) + H_{(2m-j)-1}/2) \right)$$

where $H_m$ is the $m^{th}$ Harmonic number $H_m = \sum_{k=1}^{m} 1/k$. Thus, for each $m \geq 0$, the number of bound state energies is bounded above by

$$\mathfrak{N} < 1 + \sqrt{\frac{v}{2m+1} \left( \frac{4\pi^2}{3} + 8 \sum_{j=0}^{m-1} \frac{\log(4) + H_{(2m-2j)-1}/2}{(m-j)} \right)^{1/4}}, \quad m = 0, 1, 2, \ldots.$$

(6)

For example, for the modified Pöschl-Teller potential, the number of the bound states is bounded above by $\mathfrak{N} < 1 + \sqrt{2\pi v}/3^{1/4}$. 
III. GENERAL DIFFERENTIAL EQUATION

The change of variable $\eta = 1/\cosh^2(z)$ maps the infinite interval $-\infty < z < \infty$ into $0 < \eta \leq 1$. Since the potential $V(z)$ is an even function, the energy eigenfunctions may be classified as even $\psi_+(z)$ or odd $\psi_-(z)$ functions of $z$. If we write $\psi(z) = \phi(\eta)$, the boundary condition requirement $\psi(\pm \infty) = 0$ is equivalent to the condition $\phi(0) = 0$. Thus, the mapping has the feature that the change of variable $z \rightarrow \eta$ covers the interval $(0, 1)$ twice and vanishes at the end point $\eta = 1$ only once corresponding to $z = 0$. Thus, for both even and odd cases, for each zero of the wave function $\phi(\eta)$, where $\eta \in (0, 1)$, there are two zeros of the wave function $\psi(z)$ for $z \in (\infty, \infty)$; whereas, in the odd case, $\psi(z)$ has one extra zero $\psi_-(0) = \phi(1) = 0$. This change of variable reduces equation (3) to

$$4\eta^2(1-\eta)\frac{d^2\phi(\eta)}{d\eta^2} + [4\eta - 6\eta^2] \frac{d\phi(\eta)}{d\eta} + (\varepsilon + \nu (1-\eta)^m) \phi(\eta) = 0,$$

(7)

with boundary condition(s) $\phi(0) = 0$ and $\phi(1) \neq 0$ or $\phi(0) = 0$ and $\phi(1) = 0$. For each $m \geq 0$, the differential equation (7) has two regular singular points at $\eta = 0$ with exponents $\{\pm \sqrt{-\varepsilon} / 2\}$ where $\varepsilon < 0$ and at the singular point $\eta = 1$ with exponents $\{0, 1/2\}$. For the singular point at $\eta = \infty$, the transformation $\xi = 1/\eta$ is used and the resulting equation is examined for the regularity at $\xi = 0$. It is not difficult to deduce that the resulting equation has a regular singular point at $\xi = 0$ with exponents $\{(1 \pm \sqrt{1+8\nu})/4\}$ if $m = 0$. On other hand, the transformed equation has an irregular singular point at $\xi = 0$ for all $m \geq 1$. Consequently, the general solution of (7) may assume the form

$$\phi(\eta) = \eta^\alpha (1-\eta)^\beta e^{-\gamma \eta} f(\eta), \quad \eta \in (0, 1)$$

(8)

where $\alpha = \sqrt{-\varepsilon}/2$ and $\beta$ takes either the value of $\beta = 0$ or $\beta = 1/2$. The parameter $\gamma = 0$ for $m = 0$ is used to sustain the regularity at infinity, and $\gamma \geq 0$ for $m \geq 1$. Again, because of the two possible values of the parameter $\beta$, we have to distinguished between two cases: For $\beta = 0$, the wave function (8) vanishes at $\eta = 0$ and since $\eta \neq 1$, the boundary condition $\phi(0) = 0$ equivalent to $\psi(\pm \infty) = 0$ and the resulting wave function (8) is even $\psi_+$. For $\beta = 1/2$, the wave function (8) vanishes at $\eta = 0$ in addition to $\eta = 1$, the boundary conditions $\phi(0) = \phi(1) = 0$, in this case, equivalent to $\psi(\pm \infty) = \psi(0) = 0$ and the resulting wave function $\psi_-(z)$ is odd with respect to $z$. On substituting (8) into (7), the unknown function $f(\eta)$ has the following differential equation

$$f''(\eta) + \left(\frac{2\alpha + 1}{\eta} + \frac{4\beta + 1}{2(\eta - 1)} - 2\gamma\right) f'(\eta) + \left(\frac{\beta(2\beta - 1)}{2(\eta - 1)^2} + \frac{4\alpha^2 + \varepsilon}{4\eta^2} + \gamma^2 + \frac{2\alpha + 4\beta + 8\alpha\beta - \varepsilon - 2\gamma - 8\beta\gamma}{4(\eta - 1)}\right) f(\eta) = 0.$$  

(9)

Since $\alpha = \sqrt{-\varepsilon}/2$ for either value of $\beta = 0$ or $\beta = 1/2$, the term $\beta(2\beta - 1) = 0$ and the equation (9) reduce to

$$f''(\eta) + \left(\frac{2\alpha + 1}{\eta} + \frac{4\beta + 1}{2(\eta - 1)} - 2\gamma\right) f'(\eta) + \left(\gamma^2 + \frac{2\alpha + 4\beta + 8\alpha\beta - \varepsilon - 2\gamma - 8\beta\gamma}{4(\eta - 1)}\right) f(\eta) = 0.$$  

(10)

This is the general differential equation that we attempt to solve, either analytically or approximately, for the non-negative integer $m = 0, 1, 2, \ldots$.

IV. THE MODIFIED PÖSCHL-TELLER POTENTIAL

For $m = 0$, the potential $V_{\nu=0}(z) = -\nu / \cosh^2(z)$ is the classical modified Pöschl-Teller potential often used as a realistic model for molecular potentials. The one-dimensional Schrödinger equation with this potential has been analyzed long ago [6, 11] and has been studied extensively ever since. Thus, we briefly outline our solution within the application of the general equation (10). For $m = 0$, $\gamma = 0$ and the equation (10) reduces to

$$f''(\eta) + \left(\frac{(3 + 4\alpha + 4\beta)\eta - 2 - 4\alpha}{2\eta(\eta - 1)}\right) f'(\eta) + \left(\frac{4\beta + 2\alpha(1 + 4\beta) - \varepsilon - \nu}{4\eta(\eta - 1)}\right) f(\eta) = 0, \quad 0 < \eta < 1.$$  

(11)

Equation (11) has three regular singular points at $\eta = 0, 1, \infty$, and according the general theory of the hypergeometric equation [11], the general solution is expressible in terms of Gauss’s hypergeometric function $2F_1(\alpha, \beta; \gamma; z) = \cdots$.
\[
\sum_{k=0}^{\infty} (\alpha)_{k}(\beta)_{k}/[(\gamma)_{k} k!] z^{k}, \text{ where } (z)_{k} \text{ is the Pochhammer symbol defined in terms of Gamma function by } (z)_{k} = z(z + 1) \ldots (z + k - 1) = \Gamma(z + k)/\Gamma(z). \text{ Indeed, it is not difficult to show that the differential equation has the solution, see also [12],} \\
f(\eta) = 2F_1 \left( \beta + \frac{1 + 2\sqrt{-\varepsilon} - \sqrt{1 + 4v}}{4}, \beta + \frac{1 + 2\sqrt{-\varepsilon} + \sqrt{1 + 4v}}{4}; 1 + 2\alpha; \eta \right),
\]
where \(\beta(2\beta - 1) = 0\) and \(4\alpha^{2} + \varepsilon = 0\) have been employed. The infinite series representation of (12) terminates to an \(n\)-degree polynomial if
\[
\beta + \frac{1 + 2\sqrt{-\varepsilon} - \sqrt{1 + 4v}}{4} = -n, \quad n = 0, 1, 2, \ldots,
\]
that yields \(\sqrt{-\varepsilon} = (-1 - 4n - 4\beta + \sqrt{1 + 4v})/2\). Since the left-hand side of this equation is positive, it is necessary that \(-1 - 4n - 4\beta + \sqrt{1 + 4v} > 0\) which bounds on the number of the eigenenergies given by the formula \(n < (-1 - 4\beta + \sqrt{1 + 4v})/4\). Thus, in summary, the exact solutions of Schrödinger’s equation
\[
\left[ \frac{d^2}{dz^2} - \frac{v}{\cosh^{2}(z)} \right] \psi_{n} = \varepsilon_{n} \psi_{n}, \quad -\infty < z < \infty, \quad \psi_{n}(\pm \infty) = 0,
\]
are given explicitly by
\[
\psi_{n}(z) = \begin{cases} 
\text{sech}^{\sqrt{-\varepsilon_{n}}} (z)_{2} F_{1} \left( -n, \frac{1}{2} + \sqrt{-\varepsilon_{n}} + n; 1 + \sqrt{-\varepsilon_{n}}; \text{sech}^{2}(z) \right), & \text{(Even states, } \beta = 0, \ n = 0, 1, \ldots), \\
\text{sech}^{\sqrt{-\varepsilon_{n}}} (z) \tanh(z)_{2} F_{1} \left( -n, \frac{1}{2} + \sqrt{-\varepsilon_{n}} + n; 1 + \sqrt{-\varepsilon_{n}}; \text{sech}^{2}(z) \right), & \text{(Odd states, } \beta = \frac{1}{2}, \ n = 0, 1, \ldots). 
\end{cases}
\]
where
\[
\varepsilon_{n} = -\frac{1}{4} \left(-1 - 4\beta - 4n + \sqrt{1 + 4v} \right)^2,
\]
for \(n < (-1 - 4\beta + \sqrt{1 + 4v})/4\) or \(v > 2(\beta + n)(1 + 2\beta + 2n)\) where \(\beta = 0, 1/2\). The exact number of the bound-states of the modified Pöschl-Teller potential, given \(v\) and \(\beta\), is precisely
\[
\mathcal{N} = 1 + \left[ \frac{-1 - 4\beta + \sqrt{1 + 4v}}{4} \right],
\]
where \([x]\) is the greatest integer less than or equal to \(x\).

V. THE ASYMPTOTIC ITERATION METHOD

The asymptotic iteration method (AIM) is an iterative algorithm originally introduced [13] to investigate the analytic and approximate solutions of a second-order linear differential equation
\[
y'' = \lambda_{0}(r)y' + s_{0}(r)y, \quad (^{'} = \frac{d}{dr})
\]
where \(\lambda \equiv \lambda_{0}(r)\) and \(s_{0} \equiv s_{0}(r)\) are \(C^{\infty}(a, b)\)—differentiable functions. AIM states [13]: **Given \(\lambda_{0}\) and \(s_{0}\) in \(C^{\infty}(a, b)\), the differential equation** [13] **has the general solution**
\[
y(r) = \exp \left( - \int_{c}^{r} \frac{s_{n-1}(t)}{\lambda_{n-1}(t)} \, dt \right) \left[ C_{2} + C_{1} \int_{c}^{r} \exp \left( \int_{\tau}^{t} \left[ \lambda_{0}(\tau) + \frac{2s_{n-1}(\tau)}{\lambda_{n-1}(\tau)} \right] \, d\tau \right) \, dt \right]
\]
where \(C_{1}\) and \(C_{2}\) are the integration constants, if for sufficiently large \(n > 0\)
\[
\delta_{n} = \lambda_{n}s_{n-1} - \lambda_{n-1}s_{n} = 0,
\]
(20)
The AIM sequences \( \lambda_n \) and \( s_n, n = 1, 2, \ldots, \) are computed recursively using
\[
\lambda_n = \lambda'_{n-1} + s_{n-1} + \lambda_0 \lambda_{n-1} \quad \text{and} \quad s_n = s'_{n-1} + s_0 \lambda_{n-1}.
\] (21)

Over the past decade, AIM has proved to be an efficient and effective algorithm for solving many eigenvalue problems that occur in relativistic and non-relativistic quantum mechanics. The first step in applying the AIM algorithm is to construct a product of an asymptotic solution to the given boundary-value problem with an unknown function (to be determine by AIM). Thus the original problem is transformed into the eigenvalue problem with the form \( \text{AIM} \). The second step is to evaluate the termination condition \( \text{AIM} \) by using the AIM sequences \( \{\lambda_n(r)\} \) and \( \{s_n(r)\} \) recursively, as given by (21). The resulting expressions for \( \delta_n \) are usually functions of the (unknown) eigenvalue \( E \) and the independent variable \( r \). A one-dimensional root-finding method is then employed to evaluate the roots of the equation \( \delta_n(E, r) = 0 \) for a suitable initial value \( r_0 \) of \( r \). If the eigenvalue problem is solvable, the number of wave function zeros is equal to the iteration number \( n \) and the roots of the termination condition Eq.\([19]\) are precisely the exact eigenvalues \( \varepsilon_n \) regardless of the given initial value \( r_0 \). In other cases, the iteration sequence is arbitrarily stopped and AIM is employed as an approximation method with the advantage of being a simple programmable algorithm.

The modified Pöschl-Teller potential serves as a perfect test example to examine the accuracy of the AIM algorithm and as a benchmark for the more difficult problems to be solved. In Table I, we check our computer program, written using Maple 16 running on a MacBook Pro 2.5 GHz Intel Core 17 with 8GB RAM against the exact values as given by formula (16). The number of the bound-states indicated by AIM is in complete agreement with the exact number as given by (4).

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| \( \beta \) | \( n \) | \( \varepsilon_n \) (Exact) | \( \varepsilon_n \) (AIM) |
|---|---|---|---|
| 0 | 0 | 0.381 966 001 250 105 151 79 | 0.381 966 001 250 105 151 79 | 3.96 (3.96) |
| 0 | 4 | 2.348 447 187 191 169 725 09 | 2.348 447 187 191 169 725 09 | 3.96 (4.05) |
| 0 | 9 | 6.458 618 734 850 890 155 50 | 6.458 618 734 850 890 155 50 | 3.96 (4.05) |
| 1 | 0 | 2.939 093 674 254 450 777 50 | 2.939 093 674 254 450 777 50 | 3.96 (4.05) |
| 1 | 10 | 12.468 871 128 850 725 178 82 | 12.468 871 128 850 725 178 82 | 3.96 (4.05) |
| 1 | 16 | 2.344 355 629 253 625 869 08 | 2.344 355 629 253 625 869 08 | 3.96 (4.05) |
| 1 | 25 | 20.475 062 189 439 554 864 89 | 20.475 062 189 439 554 864 89 | 3.96 (4.05) |
| 1 | 34 | 6.375 310 947 197 774 324 45 | 6.375 310 947 197 774 324 45 | 3.96 (4.05) |
| 2 | 0 | 2.755 595 704 955 993 784 01 | 2.755 595 704 955 993 784 01 | 4.05 (3.75) |

TABLE I: Exact eigenenergies \( \beta = 0 \) evaluated using formula (16) comparing with AIM results. Number of iterations along with the computational times in seconds, used by AIM, are given as subscripts of the column \( \varepsilon_n \) (AIM). In all of our computations, the iterative process started with the value \( r_0 = 1/2 \).

VI. A CLASS OF HYPERBOLIC DOUBLE-WELL POTENTIALS

In this section, the case \( m = 1 \) is analyzed, namely Schrödinger’s equation
\[
-\frac{d^2 \psi}{dz^2} - v \frac{\sinh^2(z)}{\cosh^4(z)} \psi = \varepsilon \psi, \quad -\infty < z < \infty, \quad \psi(\pm \infty) = 0.
\] (22)

For this equation, the assumed general solution \( [9] \) leads in general to the differential equation \( [19] \), that is to say
\[
\eta^2 (4 - 4 \eta) \frac{d^2 \phi(\eta)}{d\eta^2} + \eta [4 - 6 \eta] \frac{d\phi(\eta)}{d\eta} + (\varepsilon + v \eta - v \eta^2) \phi(\eta) = 0,
\] (23)
and we find in this case

\[ f''(\eta) + \left( \frac{2\alpha + 1}{\eta} + \frac{4\beta + 1}{2(\eta - 1)} - 2\gamma \right) f'(\eta) \\
+ \left( \gamma^2 + \frac{2\alpha + 4\beta + 8\alpha\beta - \varepsilon - 2\gamma - 8\beta\gamma}{4(\eta - 1)} + \frac{\varepsilon + v - 2\alpha - 4\beta - 8\alpha\beta - 4\gamma - 8\beta\gamma}{4\eta} \right) f(\eta) = 0, \tag{24} \]

where \(\beta(2\beta - 1) = 0\) and \(4\alpha^2 + \varepsilon = 0\) has been used. Equation (24) does not admit any polynomial solution \[16\], as the criterion for polynomial solutions is not satisfied. Further, since \(\eta = \infty\) is an irregular singular point for arbitrary value of \(\gamma \geq 0\), we set \(\gamma = 0\) and this reduces equation (24) to

\[ f''(\eta) = - \left( \frac{1 + 4\beta}{2(\eta - 1)} + \frac{1 + \sqrt{-\varepsilon}}{\eta} \right) f'(\eta) - \left( \frac{\varepsilon + v + 4\beta + 4\beta\sqrt{-\varepsilon} - \varepsilon - v}{4\eta(\eta - 1)} \right) f(\eta). \tag{25} \]

For this differential equation, the coefficients of the infinite series solution

\[ f(\eta) = \sum_{n=0}^{\infty} c_n \eta^n, \tag{26} \]

by Frobenius’s method obey the recurrence relation

\[ c_0 = 1, \quad c_1 = \frac{4\beta + 2\alpha(1 + 4\beta) - \varepsilon - v}{4 + 8\alpha}, \]

\[ c_n + \left( \frac{v + 4\beta + \varepsilon + \alpha(6 - 8\beta - 8n) + 6n - 8\beta n - 4n^2 - 2}{4n(2\alpha + n)} \right) c_{n-1} - v c_{n-2} = 0, \quad n \geq 2. \tag{27} \]

These coefficients have an interesting property that allows us to evaluate the series coefficients of (26) as

\[ f(\eta) = \sum_{n=0}^{\infty} \frac{P_n(\alpha)}{n! (1 + 2\alpha)_n} \left( \frac{\eta}{4} \right)^n, \tag{28} \]

where for \(\beta = 0\), the polynomials \(\{P_n(\alpha)\}_{n=0}^{\infty}\) satisfy the three-term recurrence relation

\[ P_{n+1}(\alpha) = (2n(2n + 1) + (8n + 2)\alpha + 4\alpha^2 - v)P_n(\alpha) + 4n(n + 2\alpha)P_{n-1}(\alpha), \quad P_{-1}(\alpha) = 0, P_0(\alpha) = 1, \tag{29} \]

while for \(\beta = 1/2\), the series solution (26) takes the form

\[ f(\eta) = \sum_{n=0}^{\infty} \frac{P_n(\alpha)}{n! (1 + 2\alpha)_n} \left( \frac{\eta}{4} \right)^n, \tag{30} \]

where now the polynomials \(\{P_n(\alpha)\}_{n=0}^{\infty}\) satisfy the recurrence relation

\[ P_{n+1}(\alpha) = (2n + 1)(2n + 1) + (8n + 6)\alpha + 4\alpha^2 - v)P_n(\alpha) + 4n(n + 2\alpha)P_{n-1}(\alpha), \quad P_{-1}(\alpha) = 0, P_0(\alpha) = 1. \tag{31} \]

The discrete spectrum of the Hamiltonian (21) evaluated using AIM initiated with

\[ \lambda_0 = - \left( \frac{1 + 4\beta}{2(\eta - 1)} + \frac{1 + \sqrt{-\varepsilon}}{\eta} \right), \quad \text{and} \quad s_0 = - \left( \frac{\varepsilon + v + 4\beta + 4\beta\sqrt{-\varepsilon} - \varepsilon - v}{4\eta(\eta - 1)} \right) \tag{32} \]

are reported in Tables (I) and (II). Starting with \(r_0 = 1/2 \in (0, 1)\), in Table (I) we report our finding of \(\varepsilon\) using the roots of the termination condition (19) determined accurately to the first 24 decimal places, along with the number of iteration \(N\) used by AIM. In Table (II) we also report the eigenvalues for higher values of the parameter \(v\). AIM converges fast as indicated by the low number of iterations for very high precision of the eigenvalues for the given potential strength \(v\). With this finding, the coefficients of the wave function are easily computed using Eqs. (29) and (31).
TABLE II: The eigenvalues for the potential \(-v\sinh^2(z)/\cosh^4(z)\) for very small values of the potential parameter \(v\). The computational times in seconds, used by AIM, are given as subscript values of the column \(\varepsilon_n(AIM)\).

| \(\beta\) | \(n\) | \(v\) | \(V_{\text{min}}\) | \(\varepsilon_n(AIM)\) | \(N_{\text{iteration}}\) |
|-------|-------|------|----------------|----------------|------------------|
| 0     | 0     | 0.00001 | 0.000 025 | 0.000 000 001 110 997 544 530 833(0.173) | 5 |
| 0     | 0     | 0.00004 | 0.000 100 | 0.000 000 017 770 512 138 752 699(0.767) | 6 |
| 0     | 0     | 0.00009 | 0.000 225 | 0.000 000 089 917 289 559 488 488(0.545) | 5 |
| 0     | 0     | 0.00016 | 0.000 400 | 0.000 000 283 980 113 514 486 522(0.687) | 6 |
| 0     | 0     | 0.00025 | 0.000 625 | 0.000 000 692 675 074 023 441 258(0.741) | 6 |

TABLE III: The eigenvalues for the potential \(-v\sinh^2(z)/\cosh^4(z)\) for higher values of the parameter \(v\). The computational times in seconds, used by AIM, are given as subscript values of the column \(\varepsilon_n(AIM)\).

| \(\beta\) | \(n\) | \(v\) | \(V_{\text{min}}\) | \(\varepsilon_n(AIM)\) | \(N_{\text{iteration}}\) |
|-------|-------|------|----------------|----------------|------------------|
| 0     | 0     | 5    | -1.25         | -0.547 952 205 065 460 959 101 243(1.558) | 14 |
| 0     | 0     | 10   | -2.5          | -1.284 258 416 184 695 724 376 712(2.848) | 16 |
| 0     | 0     | 20   | -5            | -0.625 853 590 393 309 267 849 407(2.072) | 14 |
| 0     | 0     | 30   | -7.5          | -4.674 864 616 067 671 875 486 969(6.830) | 19 |
| 0     | 0     | 50   | -12.5         | -8.462 774 605 628 490 576 718 186(12.437) | 21 |
| 0     | 0     | 100  | -25           | -18.764 147 649 169 376 439 874 972(24.872) | 24 |

VII. ANOTHER CLASS OF HYPERBOLIC DOUBLE-WELL POTENTIAL

In this section, the case \(m = 2\) is examined and both the quasi-exact and the approximate solutions for the entire discrete spectrum are evaluated for the Schrödinger equation

\[-\frac{d^2\psi}{dz^2} - v\frac{\sinh^4(z)}{\cosh^4(z)}\psi = \varepsilon \psi, \quad -\infty < z < \infty, \quad \psi(\pm \infty) = 0. \tag{33}\]

For this equation, the assumed solution \([8]\) of the differential equation \([10]\),

\[4\eta^2(1-\eta)\frac{d^2\phi(\eta)}{d\eta^2} + [4\eta - 6\eta^2] \frac{d\phi(\eta)}{d\eta} + (\varepsilon + v \eta (1-\eta)^2) \phi(\eta) = 0, \quad \phi(0) = \phi(1) = 0, \tag{34}\]

becomes explicitly

\[(2\eta^2 - 2\eta)f''(\eta) + (-4\eta\gamma^2 + (3 + 4\alpha + 4\beta + 4\gamma)\eta - 2(1 + 2\alpha))f'(\eta) + (\gamma(2\gamma - 3 - 4\alpha - 4\beta)\eta + \alpha + 2\alpha^2 + 2\beta + 4\alpha\beta + 2\gamma + 4\alpha\gamma - 2\gamma^2)f(\eta) = 0 \tag{35}\]

where we have used the relations \(2\beta^2 - \beta = 0, \ 4\alpha^2 + \varepsilon = 0, \) and \(4\gamma^2 - v = 0.\) As noticed earlier \([1]\), this is Heun’s confluent-type differential equation \([15]\). It has a solution around the regular singular point \(\eta = 0\) given in terms of the confluent Heun function \([1]\) that can be explicitly expressed using, for example, Maple computing software. However, we introduce in Theorem VIII.1 a slightly easier method for evaluating the exact solutions in terms of a recurrence relation instead of the correlation between polynomial equations and matrix determinants usually used \([15]\). We first give a general result valid for a class of differential equations.
Theorem VII.1. The necessary condition for the existence of $N$-degree polynomial solutions of the differential equation

$$\left(a_2 z^2 + a_1 z\right)f''(z) + \left(b_2 z^2 + b_1 z + b_0\right)f'(z) - (\tau_1 z + \tau_0)f(z) = 0$$

is

$$\tau_1 = Nb_2, \quad N = 0, 1, 2, \ldots.$$  

and the polynomial solutions are given explicitly by

$$f_N(z) = \sum_{k=0}^{N} \frac{P_k(\tau_0)}{k! a_k (b_0/a_1)_k} z^k,$$

where, for each $N$, the finite sequence of the polynomials $\{P_k(\tau_0)\}_{k=0}^{N}$ satisfies a three-term recurrence relation, for $0 \leq k \leq n+1$,  

$$P_{k+1}(\tau_0) = (\tau_0 - k(k-1)a_2 - kb_1)P_k(\tau_0) + kb_2(N-k+1)((k-1)a_1 + b_0)P_{k-1}(\tau_0),$$

initialized with $P_{-1}(\tau_0) = 0$, $P_0(\tau_0) = 1$.  

The proof of this theorem is given in the appendix along the explicit forms of the first few polynomial solutions. Direct comparison of equation (35) with (36) gives the necessary condition for polynomial solutions of (34) as

$$3 + 4\beta + 2\sqrt{\varepsilon} + 4N - \sqrt{v} = 0$$

from which we obtain the following formula for exact eigenvalues

$$\varepsilon_N = -\frac{1}{4} \left( \sqrt{v} - 3 - 4\beta - 4N \right)^2, \quad v > (3 + 4\beta + 4N)^2, \quad \beta = 0, 1/2.$$  

It should be clear that $N$ is the degree of polynomial solution, not necessary the number of nodes $n$ of the full wave function, as discussed earlier. For each $N$, the polynomial solution is given by

$$f_N(\eta) = \sum_{k=0}^{N} \frac{P_k}{k! (1 + \sqrt{\varepsilon})^k} \left( -\frac{\eta}{2} \right)^k,$$  

where the polynomial coefficients $\{P_k\}_{k=0}^{N}$ are evaluated in terms of $\varepsilon$ and $v$ using

$$P_{k+1} = \left( \frac{\varepsilon}{2} - \frac{\sqrt{-\varepsilon}}{2} (1 + 4\beta + 4k + 2\sqrt{v}) + \frac{v}{2} - (2\beta + k)(1 + 2k - (1 + 2k)\sqrt{v}) \right) P_k + 4k(N-k+1)\sqrt{v} (\sqrt{-\varepsilon} + k) P_{k-1}, \quad 0 \leq k \leq N+1,$$

initialized by $P_{-1} = 0$ and $P_0 = 1$. The sufficient condition for the polynomial solution is given explicitly by (43) for $k = N + 1$. In the next subsections, the polynomial solutions of degree $N = 0, 1, 2$, are discussed in detail. Higher order polynomial solutions may be constructed similarly.

A. Zero-degree polynomial solution

We note that although only $\psi(z)$ are even or odd functions, the corresponding $\phi(\eta)$ functions will be written with the same symmetry subscripts: thus $\psi_{\pm}(z) \leftrightarrow \phi_{\pm}(\eta)$.

In the case $N = 0$, the constant solution

$$f_0(\eta) = 1,$$

is subject to the following two conditions, relating $\varepsilon$ and $v$,

$$3 + 4\beta + 2\sqrt{-\varepsilon} - \sqrt{v} = 0, \quad \varepsilon - (4\beta + 2\sqrt{v} + 1)\sqrt{-\varepsilon} - 4\beta - 2\sqrt{v} + v = 0.$$  

The non-zero solutions of this system yields, for $\beta = 0$, $v = 29 + 8\sqrt{13}$, and $\varepsilon = -(7 + \sqrt{13})/2$ with wave function $\phi(\eta)$ given by

$$\phi_+(\eta) = \eta^2 e^{-\sqrt{v+8\sqrt{13}}\eta}, \quad \phi_+(0) = 0, \quad \phi_+(1) \neq 0,$$

while for $\beta = 1/2$, the polynomial solution (44) is subject to the constraints $v = 125 + 16\sqrt{61}$ and $\varepsilon = -(35 + 3\sqrt{61})/2$, with the odd wave function

$$\phi_-(\eta) = \eta^2 \sqrt{1 - \eta} e^{-\sqrt{125 + 16\sqrt{61}}\eta}, \quad \phi_-(0) = \phi_-(1) = 0.$$

The corresponding full wave functions of Schrödinger’s equation (32) are then

$$\psi_+(z) = \text{sech}^2(z), \quad \psi_-(z) = \text{tanh}(z) \text{sech}^2(z).$$

The graph of these exact bound-state wave functions are displayed in Fig. 2 along with the plot of the associated potential and the exact eigenvalue $E_0$ and the exact wave function $\psi_{exp}(z)$ (Inset). Right: Plot of the potential $V(z) = -(29 + 8\sqrt{13}) \sinh^2(z)/\cosh^2(z)$ along with the exact eigenvalue $E_0$ and the exact wave function $\psi_{exp}(z)$ (Inset).

The graph of these exact bound-state wave functions are displayed in Fig. 2 along with the plot of the associated potential and the exact eigenvalue. We note that the minimum of the potential $V(x) = -v \sinh^2(z)/\cosh^2(z)$ is $V_{\text{min}} = -4v/27$, if the potential strength is fixed at $v = 29 + 8\sqrt{13}$ and at $v = 125 + 16\sqrt{61}$, the minimum of the potential is, respectively, $V_{\text{min}} = -4(29 + 8\sqrt{13})/27 \sim -8.569$ and $V_{\text{min}} = -4(125 + 16\sqrt{61})/27 \sim -37.031$. It is natural to ask whether the potential supports the existence of other bound-states beside the exact $\psi_+(z)$ and $\psi_-(z)$. To find out, we rely on AIM to evaluate all the possible (discrete) eigenvalues, including the exact ones, as a test example. Writing Eq. (34) as $f''(\eta) = \lambda_0(\eta)f'(\eta) + s_0(\eta)f(\eta)$, where

$$\lambda_0(\eta) = -\left(1 + \frac{\sqrt{-\varepsilon}}{\eta} + \frac{4\beta + 1}{2(\eta - 1)} - \sqrt{v}\right),$$

$$s_0(\eta) = -\left(-4\beta\sqrt{v} + 4\beta\sqrt{\varepsilon} - \sqrt{v - \varepsilon} + \sqrt{\varepsilon - 2\sqrt{v} - 2\sqrt{\varepsilon} - 2\varepsilon + 4\beta + v}\right).$$

The eigenvalues evaluated using the roots of the termination condition (19) by means of the AIM sequences $\lambda_n(\eta)$ and $s_n(\eta)$, $n = 0, 1, 2, \ldots$, initiated with $\lambda_0$ and $s_0$ as given by equation (47) with $r_0$ is fixed at $r_0 = 1/2$, are reported in Table IV. In this table, $N$ refers to the state level not the number of possible nodes of the exact wave function.

**B. First-degree polynomial solutions**

In the case $N = 1$, the first-degree polynomial solution reads, see Theorem VIII.1

$$f_1(\eta) = 1 + \left(\frac{4\beta + 2\sqrt{v} + \sqrt{-\varepsilon}(1 + \sqrt{-\varepsilon}) - v}{4(1 + \sqrt{-\varepsilon})}\right)\eta.$$
subject to the following two constraints

\[ 7 + 4\beta + 2\sqrt{-\varepsilon} - \sqrt{v} = 0 \implies \varepsilon = -\frac{1}{4}(\sqrt{v} - 7 - 4\beta)^2, \quad \text{for} \quad v > (7 + 4\beta)^2, \quad (50) \]

and

\[
\begin{align*}
&v^2 - 4 \left(2 + \sqrt{-\varepsilon}\right) v^{3/2} - 2 \left(\varepsilon - 5\sqrt{-\varepsilon} + 4\beta \left(2 + \sqrt{-\varepsilon}\right) - 3\right) v \\
&+ 4 \left(7 + 4\beta \left(3 + 4\sqrt{-\varepsilon} - \varepsilon\right) + 11\sqrt{-\varepsilon} + (\varepsilon)^{3/2} - 5\varepsilon\right) \sqrt{v} \\
&+ 16\beta^2 \left(3 + 4\sqrt{-\varepsilon} - \varepsilon\right) + 6 \sqrt{-\varepsilon} + 6 \left(-\varepsilon\right)^{3/2} - 11\varepsilon + \varepsilon^2 - 8\beta \left(\varepsilon - 3 - 7\sqrt{-\varepsilon}\right) = 0.
\end{align*}
\]

The non-zero solutions of this constraint system, for \(\beta = 0\), are

\[ v = 149.574 256 933 312 63 \ldots, \quad \varepsilon = -6.838 370 069 149 139 \ldots, \]
\[ v = 595.838 654 872 035 2 \ldots, \quad \varepsilon = -75.775 340 860 142 68 \ldots, \]

with exact wave functions

\[ \psi_+(z) = e^{-6.115 027 737 739 \ldots \text{sech}^2(z) \text{sech}^{2.615 027 737 739 \ldots} (z)} (1 - 3.575 137 130 402 \ldots \times \text{sech}^2(z)), \quad (52) \]

and

\[ \psi_+(z) = e^{-12.204 903 265 409 \text{sech}^2(z) \text{sech}^{8.704 903 265 409 \ldots} (z)} (1 - 0.967 778 541 967 \ldots \times \text{sech}^2(z)), \quad (53) \]

respectively. For \(\beta = 1/2\), the solution of the constraint system, (51) and (52), gives

\[ v = 426.232 048 026 951 5 \ldots, \quad \varepsilon = -33.903 765 749 927 9850 \ldots, \]
\[ v = 1092.798 974 117 571 6 \ldots, \quad \varepsilon = -144.690 948 072 182 15 \ldots. \]

with wave functions, respectively, given by

\[ \psi_-(z) = \tanh(z) e^{-10.322 694 028 534 \ldots \text{sech}^2(z) \text{sech}^{5.822 694 028 534 631 \ldots} (z)} (1 - 3.339 804 928 329 \ldots \times \text{sech}^2(z)), \quad (54) \]

and

\[ \psi_-(z) = \tanh(z) e^{-16.528 755 050 803 \ldots \text{sech}^2(z) \text{sech}^{12.028 755 050 803 \ldots} (z)} (1 - 0.933 039 213 973 \ldots \times \text{sech}^2(z)). \quad (55) \]

The plot of the exact wave functions (52), (55) are displayed in Figures 3-4. For each exact case, the rest of the discrete spectrum can be evaluated by using AIM and some of the eigenvalues are displayed in Table IV. The accuracy of the eigenvalues to much higher number of decimal places can be obtain with some patience specially for higher iteration numbers for which the AIM computations may become tedious.
FIG. 3: Left: Plot of the potential \( V(z) = -(149.574 \ 256 \ 933 \ 312 \ldots) \sinh^4(z) / \cosh^6(z) \) along with the exact eigenvalue and the exact wave function (Inset). Right: Plot of the potential \( V(z) = -(595.838 \ 654 \ 872 \ 035 \ldots) \sinh^4(z) / \cosh^6(z) \) along with the exact eigenvalue and the exact wave function (Inset).

FIG. 4: Left: Plot of the potential \( V(z) = -(426.232 \ 048 \ldots) \sinh^4(z) / \cosh^6(z) \) along with the exact eigenvalue and the exact wave function (Inset). Right: Plot of the potential \( V(z) = -(1092.798 \ 974 \ldots) \sinh^4(z) / \cosh^6(z) \) along with the exact eigenvalue and the exact wave function (Inset).

| \( v = 149.574 \ 256 \ 933 \ 312 \ldots \) | | \( v = 595.838 \ 654 \ 872 \ 035 \ldots \) |
|---|---|---|
| \( \beta = 0 \) | \( \beta = 0 \) | \( \beta = 0 \) |
| \( \beta = \frac{1}{2} \) | \( \beta = \frac{1}{2} \) | \( \beta = \frac{1}{2} \) |
| \( \alpha \) | \( \alpha \) | \( \alpha \) |
| 0 | 0 | 0 |
| 1 | 1 | 1 |
| 2 | 2 | 2 |
| 3 | 3 | 3 |
| 4 | 4 | 4 |

TABLE V: The discrete spectra supported by the potentials \( V(z) = -(149.574 \ldots) \sinh^4(z) / \cosh^6(z) \), \( V(z) = -(595.838 \ldots) \sinh^4(z) / \cosh^6(z) \), \( V(z) = -(426.232 \ldots) \sinh^4(z) / \cosh^6(z) \), and \( V(z) = -(1092.798 \ldots) \sinh^4(z) / \cosh^6(z) \). The subscripts refer to the number of iterations and the computational times (in seconds) used by AIM.
C. Second-degree polynomial solution

In the case $N = 2$, the second-degree polynomial solutions for $\beta = 0$ is

$$f_2(\eta) = 1 + \frac{(2\sqrt{\varepsilon} + \sqrt{-\varepsilon})(1 + \sqrt{-\varepsilon}) - v}{4(1 + \sqrt{-\varepsilon})} \eta$$

$$+ \frac{v^2 - 4(2 + \sqrt{-\varepsilon})v^{3/2} + 2(5 + 7\sqrt{-\varepsilon} - \varepsilon) v - 4\sqrt{-\varepsilon}(\varepsilon - 3\sqrt{-\varepsilon} - 2)v^{1/2} - \sqrt{-\varepsilon}(6 - 11\sqrt{-\varepsilon} + \varepsilon\sqrt{-\varepsilon})}{32(1 + \sqrt{-\varepsilon})(2 + \sqrt{-\varepsilon})} \eta^2$$

subject to the exact values of $\varepsilon$ and $v$ given as:

$$(\varepsilon, v) = (-235.972 747 187 322 915 \ldots, 1740.792 785 280 901 009 \ldots),$$

$$(\varepsilon, v) = (-84.008 276 606 551 \ldots, 860.319 634 232 780 644 \ldots),$$

$$(\varepsilon, v) = (-8.143 960 834 197 \ldots, 279.141 396 634 685 463 \ldots).$$

For $\beta = 1/2$, the second-order polynomial solution reads

$$f_2(\eta) = 1 + \frac{(2 + 2\sqrt{\varepsilon} + \sqrt{-\varepsilon})(1 + \sqrt{-\varepsilon}) - v}{4(1 + \sqrt{-\varepsilon})} \eta$$

$$+ \frac{v^2 - 4(2 + \sqrt{-\varepsilon})v^{3/2} + 2(1 + 5\sqrt{-\varepsilon} - \varepsilon) v + 4(4 + \sqrt{-\varepsilon}(8 - 5\sqrt{-\varepsilon} - \varepsilon))v^{1/2} + 24 + 10\sqrt{-\varepsilon}(5 - \varepsilon) - \varepsilon(35 - \varepsilon)}{32(1 + \sqrt{-\varepsilon})(2 + \sqrt{-\varepsilon})} \eta^2$$

subject to the exact values of $\varepsilon$ and $v$ as

$$(\varepsilon, v) = (-349.620 385 570 634 540 \ldots, 2539.784 747 349 247 690 \ldots),$$

$$(\varepsilon, v) = (-156.512 009 095 311 555 \ldots, 1445.592 787 027 749 297 \ldots),$$

$$(\varepsilon, v) = (-8.143 960 834 197 \ldots, 642.496 980 045 734 503 \ldots).$$

Plots of the exact wave functions are displayed in Fig. 5 and in Fig. 6 along with potential. For each exact case, the rest of the discrete spectrum can be approximated by the use of AIM, and some are displayed in Table 6. The accuracy of the eigenvalues to a much higher number of decimal places can be obtained by using more iterations provided the numerical computing environment can support it.

| $v$ | $\varepsilon_n$ | $v$ | $\varepsilon_n$ | $v$ | $\varepsilon_n$ |
|-----|----------------|-----|----------------|-----|----------------|
| $\beta = 0$ | | | | | |
| 0 | -235.972 747 187 322 | $\beta = 0$ | | | |
| 1 | -333.052 738 536 688 | | | | |
| 2 | -18.591 111 216 762 | | | | |
| 3 | -8.143 960 834 198 | | | | |
| $\beta = 1/2$ | | | | | |
| 0 | -349.620 385 570 634 | $\beta = 1/2$ | | | |
| 1 | -82.177 087 688 263 | | | | |
| 2 | -58.252 330 895 293 | | | | |
| 3 | -38.115 338 099 145 | | | | |

TABLE VI: The discrete spectra supported by the potentials $V(z) = -149.574 \ldots \sinh^4(z)/\cosh^6(z), V(z) = -595.838 \ldots \sinh^2(z)/\cosh^6(z), V(z) = -426.232 \ldots \sinh^4(z)/\cosh^6(z), V(z) = -1092.798 \ldots \sinh^2(z)/\cosh^6(z).$ The subscript refer to the number of iterations and the computational times (in seconds) used by AIM.

VIII. CONCLUSION

In this work, the exact and approximate solutions of Schrödinger's equation with various hyperbolic potentials [3], $m = 0, 1, 2$ are discussed. For $m = 2$, the corresponding Schrödinger equation admits polynomial solutions provided
that certain constraints on the potential parameters are satisfied. A general existence theorem is devised that allows us to enumerate all such solutions of the confluent Heun equation: the proof of this theorem is presented in Appendix I. This theory has the advantage of easy implementation requiring only the computation of three-term recurrence relation. The theory is illustrated by the computation of the exact eigenvalues for the higher order polynomial solutions which are reported in Appendix II. For non-polynomial cases AIM is also used to provide very accurate numerical approximations.

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Appendix I: The proof of Theorem VIII.1

Theorem VIII.1. The necessary condition for the existence of $N$-degree polynomial solutions of the differential equation

$$
(a_2 z^2 + a_1 z) f''(z) + (b_2 z^2 + b_1 z + b_0) f'(z) - (\tau_1 z + \tau_0) f(z) = 0
$$

is

$$
\tau_1 = Nb_2, \quad N = 0, 1, 2, \ldots
$$
and the polynomial solutions are give explicitly by
\[ f_N(z) = \sum_{k=0}^{N} \frac{P_k(\tau_0)}{k!a_k^2(b_0/a_1)_k} z^k, \]
where, for each \( N \), the finite sequence of the polynomials \( \{P_k(\tau_0)\}_{k=0}^{N} \) satisfies a three-term recurrence relation, for \( 0 \leq k \leq n + 1 \),
\[ P_{k+1}(\tau_0) = (\tau_0 - k(k-1)a_2 - kb_1)P_k(\tau_0) + kb_2(N-k+1)((k-1)a_1 + b_0)P_{k-1}(\tau_0), \]
initialized with \( P_{-1}(\tau_0) = 0, \ P_0(\tau_0) = 1 \).

Proof

The differential equation
\[ (a_2z^2 + a_1z)f''(z) + (b_2z^2 + b_1z + b_0)f'(z) - (\tau_1 z + \tau_0)f(z) = 0, \]
with real constants \( a_j, j = 1, 2, b_k, k = 2, 1, 0 \) has two regular singular points, namely at \( z = 0 \) and \( z = -a_1/a_2 \), in addition to an irregular singular point at \( z = \infty \). The domain of definition is \( z \in (0, -a_1/a_2) \) if \( a_2a_1 < 0 \) or \( z \in (-a_1/a_2, 0) \) if \( a_2a_1 > 0 \). In the neighbourhood of the singular point \( z = 0 \), the formal series solution takes the form \( y(z) = \sum_{k=0}^{\infty} c_k z^k \) since the exponents of the regular point \( z = 0 \) are \( s = 0 \) and \( s = 1 - b_0/a_1 \). On substituting this expression for \( y(z) \) into the differential equation and employing all the necessary shifting of the summation indices, the recurrence relation for the coefficients \( c_k \) reads
\[ (k(k+1)a_1 + (k+1)b_0)c_{k+1} + (k(k-1)a_2 + kb_1 - \tau_0)c_k + ((k-1)b_2 - \tau_1)c_{k-1} = 0, \]
for \( k = 0, 1, 2, \ldots \), with the convention \( c_{-1} = 0, a_0 = 1 \) For an \( N \)th degree polynomial solution, this linear system can be expressed as
\[ \sum_{j=0}^{2} [(k-j)(k-j+1)a_{j+1} + (k-j+1)b_j - \tau_{j-1}] c_{k-j+1} = 0 \]
where \( a_3 = \tau_{-1} = 0 \) and \( k = 0, 1, \ldots, N + 1 \). The system of the \( N + 2 \) equations breaks down into three subclasses. For \( k = N + 1 \), the necessary condition for polynomial solutions is
\[ (N b_2 - \tau_1)c_N = 0. \]
The second subclass also consists of a single equation, namely \( k = N \), that yields the sufficient condition as
\[ (N(N-1)a_2 + N b_1 - \tau_0)c_N + ((N-1)b_2 - \tau_1)c_{N-1} = 0. \]
Using \( \tau_1 = N b_2 \), it easily follows that
\[ (N(N-1)a_2 + N b_1 - \tau_0)c_N - b_2c_{N-1} = 0. \]
The third subclass consists of \( N \) equations given by \( k = 0, 1, \ldots, \ N-1 \) and used to evaluate the polynomial coefficients \( c_N \), for example, by Cramer’s rule. Indeed, the \( N \)-equations can be written as
\[
\begin{pmatrix}
\beta_1 & 0 & 0 & 0 & 0 & \ldots & 0 & 0 & 0 & 0 \\
\gamma_2 & \beta_2 & 0 & 0 & 0 & \ldots & 0 & 0 & 0 & 0 \\
\eta_3 & \gamma_3 & \beta_3 & 0 & 0 & \ldots & 0 & 0 & 0 & 0 \\
0 & \eta_4 & \gamma_4 & \beta_4 & 0 & \ldots & 0 & 0 & 0 & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\
0 & 0 & 0 & 0 & 0 & \ldots & \eta_{N-1} & \gamma_{N-1} & \beta_{N-1} & 0 \\
0 & 0 & 0 & 0 & 0 & \ldots & \eta_N & \gamma_N & \beta_N & 0
\end{pmatrix}
\begin{pmatrix}
c_1 \\
c_2 \\
c_3 \\
c_4 \\
\vdots \\
c_N-1 \\
c_N
\end{pmatrix}
= \begin{pmatrix}
\tau_0 \\
\tau_1 \\
c_3 \\
c_4 \\
\vdots \\
c_{N-1} \\
c_N
\end{pmatrix},
\]
where
\[ \beta_j = j(j-1)a_1 + jb_0, \quad j = 1, 2, \ldots, N, \]
\[ \gamma_j = (j-2)(j-1)a_1 + (j-1)b_0 - \tau_0, \quad j = 2, 3, \ldots, N-1, \]
\[ \eta_j = (j-2)b_2 - \tau_1, \quad j = 3, 4, \ldots, N - 2. \]
The non-trivial solutions $c_k, k = 1, 2, \ldots, N$ require that $\prod_{i=1}^{N} \beta_j \neq 0$ which yields $N! a_0^N (b_0/a_1)_N \neq 0$. The application of Cramer’s rule allows us to express the polynomial solutions of the differential equation as

$$f_N(z) = \sum_{k=0}^{N} \frac{P_k(\tau_0)}{k! a_k^N (b_0/a_1)_k} z^k,$$

where, for each $N$, the finite sequence of the polynomials $\{P_k(\tau_0)\}_{k=0}^{N}$ satisfies a three-term recurrence relation, for $0 \leq k \leq n + 1$,

$$P_{k+1}(\tau_0) = (\tau_0 - k(k - 1)a_2 - kb_1)P_k(\tau_0) + kb_2(N - k + 1)((k - 1)a_1 + b_0)P_{k-1}(\tau_0),$$

utilized with $P_{-1}(\tau_0) = 0, P_0(\tau_0) = 1$. The first few degree-$N$ polynomial solutions of the differential equation are

$$(a_2 z^2 + a_1 z) f''(z) + (b_2 z^2 + b_1 z + b_0) f'(z) - (\tau_1 z + \tau_0) f(z) = 0$$

- $N = 0$: the constant solution is
  $$f_0(\eta) = 1,$$
  subject to $\tau_1 = 0$ and $\tau_0 = 0$.

- $N = 1$: the first degree solution is
  $$f_1(\eta) = 1 + \frac{\tau_0}{b_0} \eta,$$
  subject to $\eta_1 = b_2$ and $\tau_0^2 - b_1 \tau_0 + b_0 b_2 = 0$.

- $N = 2$: the second degree solution is
  $$f_2(\eta) = 1 + \frac{\tau_0}{b_0} \eta + \frac{\tau_0^2 + b_1 \tau_0 + 2b_0 b_2}{2b_0(a_1 + b_0)} \eta^2,$$
  subject to
  $$\tau_1 = 2b_2,$$
  $$\tau_0^3 - (2a_2 + 3b_1) \tau_0^2 + 2(b_1(a_2 + b_1) + (a_1 + 2b_0)b_2) \tau_0 - 4b_0(a_2 + b_1)b_2 = 0.$$

- $N = 3$: the third-degree solution is
  $$f_3(\eta) = 1 + \frac{\tau_0}{b_0} \eta + \frac{\tau_0^2 - b_1 \tau_0 + 3b_0 b_2}{2b_0(a_1 + b_0)} \eta^2 + \frac{\tau_0^3 - (2a_2 + 3b_1) \tau_0^2 - (2a_2b_1 + 2b_2^2 + 4a_1 b_2 + 7b_0 b_2) \tau_0 - 6(a_2 + b_1)b_0 b_2}{6b_0(a_1 + b_0)(2a_1 + b_0)} \eta^3,$$
  subject to
  $$\tau_1 = 3b_2,$$
  $$\tau_0^4 - 2(4a_2 + 3b_1) \tau_0^3 + (12a_2^2 + 26a_2 b_1 + 11b_1^2 + 10(a_1 + b_0)b_2) \tau_0^2$$
  $$- 6(b_1(a_2 + b_1)(2a_2 + b_1) + (a_1 a_2 + 8a_2 b_0 + 3a_1 b_1 + 5b_0 b_1)b_2) \tau_0 + 9b_0 b_2(2(a_2 + b_1)(2a_2 + b_1) + (2a_1 + b_0)b_2) = 0.$$

- $N = 4$: the fourth-degree solution is
  $$f_4(\eta) = 1 + \frac{\tau_0}{b_0} \eta + \frac{\tau_0^2 - b_1 \tau_0 + 4b_0 b_2}{2b_0(a_1 + b_0)} \eta^2 + \frac{\tau_0^3 - (2a_2 + 3b_1) \tau_0^2 - (2a_2b_1 + 2b_2^2 + 6a_1 b_2 + 10b_0 b_2) \tau_0 - 8(a_2 + b_1)b_0 b_2}{6b_0(a_1 + b_0)(2a_1 + b_0)} \eta^3$$
  $$+ \frac{\xi}{24b_0(a_1 + b_0)(2a_1 + b_0)(3a_1 + b_0)} \eta^4,$$
  where

$$\tau_0^4 - 2(4a_2 + 3b_1) \tau_0^3 + (12a_2^2 + 26a_2 b_1 + 11b_1^2 + 18a_1 b_2 + 16b_0 b_2) \tau_0^2$$
$$- 2(6a_2^2 b_1 + 9a_2 b_2^2 + 3b_1^3 + 18a_1 a_2 b_2 + 34a_2 b_0 b_2 + 15a_1 b_1 b_2 + 22b_0 b_1 b_2) \tau_0$$
$$+ 24b_0 b_2(2a_2^2 + 3a_2 b_1 + b_1^2 + 2a_1 b_2 + b_0 b_2) = 0.$$
subject to

\[ \tau_1 = 4b_2, \]
\[ \tau_0^5 - 10(2a_2 + b_1)\tau_0^4 + (108a_2^2 + 130a_2b_1 + 35b_1^2 + 30a_1b_2 + 20b_2b_1)\tau_0^3 \]
\[ - 2(72a_2^3 + 186a_2b_1 + 127a_2b_1^2 + 253b_1^2 + 138a_1a_2b_2 + 134a_2b_2b_1 + 69a_1b_1b_2 + 60b_2b_1b_2)\tau_0^2 \]
\[ + 8(18a_2b_1 + 33a_2b_1^2 + 18a_2b_1^3 + 3b_1^4 + 54a_1a_2b_2 + 108a_2b_2b_1 + 66a_1a_2b_1b_2 + 110a_2b_1b_2 + 18a_1b_2^2 + 26b_2b_2^2 + 9a_2b_2^2 + 24a_1b_2^2 + 8b_2^2)\tau_0 - 32b_0b_2(18a_2^2 + 33a_2b_1 + 18a_2b_1^2 + 3b_1^3 + 21a_1a_2b_2 + 10a_2b_2b_1 + 9a_1b_1b_2 + 4b_2b_1b_2) = 0. \]

**Appendix II: High-order polynomial solutions for \( m = 2 \)**

For the third-degree polynomial solution \( N = 3 \), in the symmetric case \( \beta = 0 \), we have the exact eigenvalue corresponding the potential strength as follows:

\[ (\varepsilon, \nu) = (-485.633 694 118 171 701 \ldots, 3489.760 664 445 471 178 \ldots), \]
\[ (\varepsilon, \nu) = (-251.403 743 430 891 744 \ldots, 2181.957 959 376 591 216 \ldots), \]
\[ (\varepsilon, \nu) = (-91.612 310 090 595 802 \ldots, 1165.735 158 983 190 844 \ldots), \]
\[ (\varepsilon, \nu) = (-9.314 641 090 299 513 \ldots, 445.377 945 981 708 794 \ldots). \]

while for the anti-symmetric case, \( \beta = 1/2 \), the exact pair of the eigenvalues and the potential strength are

\[ (\varepsilon, \nu) = (-644.012 578 551 987 881 \ldots, 4590.713 712 215 915 292), \]
\[ (\varepsilon, \nu) = (-368.672 229 400 112 178 \ldots, 3069.345 989 249 095 172 \ldots), \]
\[ (\varepsilon, \nu) = (-167.611 705 221 567 520 \ldots, 1839.808 408 499 296 398 \ldots), \]
\[ (\varepsilon, \nu) = (-41.996 677 058 480 376 \ldots, 897.659 642 186 544 995 \ldots). \]

For the fourth-degree polynomial solution \( N = 4 \), in the symmetric case \( \beta = 0 \), we have the exact eigenvalue corresponding the potential strength as follows:

\[ (\varepsilon, \nu) = (-824.756 984 779 279 158 \ldots, 5842.640 214 370 125 564 \ldots), \]
\[ (\varepsilon, \nu) = (-508.312 652 989 564 072 \ldots, 4107.730 662 518 437 769 \ldots), \]
\[ (\varepsilon, \nu) = (-266.051 362 321 527 012 \ldots, 2664.847 602 800 234 089 \ldots), \]
\[ (\varepsilon, \nu) = (-98.740 362 436 827 665 \ldots, 1511.159 657 792 798 942 \ldots), \]
\[ (\varepsilon, \nu) = (-10.392 841 434 278 337 \ldots, 647.579 635 239 168 471 \ldots). \]

while for the anti-symmetric case, \( \beta = 1/2 \), the exact pair of the eigenvalues and the potential strength are

\[ (\varepsilon, \nu) = (-1027.866 880 025 139 082 \ldots, 7245.538 017 777 396 086 \ldots), \]
\[ (\varepsilon, \nu) = (-670.322 583 106 494 693 \ldots, 5297.099 783 021 461 760 \ldots), \]
\[ (\varepsilon, \nu) = (-386.896 254 447 370 708 \ldots, 3640.838 016 388 864 321 \ldots), \]
\[ (\varepsilon, \nu) = (-178.134 857 500 181 954 \ldots, 2274.663 666 881 406 389 \ldots), \]
\[ (\varepsilon, \nu) = (-45.630 488 564 892 802 \ldots, 1190.944 836 521 014 392 \ldots). \]

For the fifth-degree polynomial solution \( N = 5 \), in the symmetric case \( \beta = 0 \), we have the exact eigenvalue corresponding the potential strength as follows:

\[ (\varepsilon, \nu) = (-1253.342 243 407 552 930 \ldots, 8799.405 778 488 506 227 \ldots), \]
\[ (\varepsilon, \nu) = (-854.700 662 342 345 719 \ldots, 6637.446 939 245 948 353 \ldots), \]
\[ (\varepsilon, \nu) = (-530.130 397 260 421 230 \ldots, 4767.781 177 102 649 014 \ldots), \]
\[ (\varepsilon, \nu) = (-280.049 454 485 697 568 \ldots, 3188.788 212 420 147 429 \ldots), \]
\[ (\varepsilon, \nu) = (-105.490 706 285 346 518 \ldots, 1895.882 577 289 245 947 \ldots), \]
\[ (\varepsilon, \nu) = (-11.402 086 879 730 237 \ldots, 885.264 529 414 657 142 \ldots). \]
while for the anti-symmetric case, $\beta = 1/2$, the exact pair of the eigenvalues and the potential strength are
\[
(\varepsilon, v) = (-1501.183 \, 061 \, 037 \, 531 \, 656 \ldots, 10504.242 \, 614 \, 555 \, 052 \, 587 \ldots),
\]
\[
(\varepsilon, v) = (-1061.446 \, 073 \, 598 \, 550 \, 701 \ldots, 8128.768 \, 446 \, 594 \, 073 \, 010 \ldots),
\]
\[
(\varepsilon, v) = (-695.745 \, 365 \, 800 \, 573 \, 168 \ldots, 6045.680 \, 016 \, 488 \, 841 \, 376 \ldots),
\]
\[
(\varepsilon, v) = (-404.415 \, 553 \, 322 \, 089 \, 445 \ldots, 4253.670 \, 799 \, 351 \, 802 \, 332 \ldots),
\]
\[
(\varepsilon, v) = (-188.182 \, 487 \, 408 \, 993 \, 217 \ldots, 2749.526 \, 171 \, 415 \, 990 \, 809 \ldots),
\]
\[
(\varepsilon, v) = (-49.069 \, 088 \, 680 \, 629 \, 925 \ldots, 1521.769 \, 671 \, 468 \, 867 \, 580 \ldots).
\]

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