Asymptotic Spectral Distribution of Crosscorrelation Matrix in Asynchronous CDMA

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Abstract

Asymptotic spectral distribution (ASD) of the crosscorrelation matrix is investigated for a large-system asynchronous direct sequence-code division multiple access (DS-CDMA) with random spreading sequences. The crosscorrelation matrix is formed by an infinite input symbol length. Two levels of asynchronism are considered. One is symbol-asynchronous but chip-synchronous (called chip-synchronous for short), and the other is chip-asynchronous. The results are applicable to arbitrary chip waveforms.

The existence of a nonrandom ASD is proven by moment convergence theorem, where the focus is on the existence of asymptotic eigenvalue moments (AEM) of the crosscorrelation matrix of all orders. A combinatorics approach based on noncrossing partition of set partition theory is adopted for AEM computation. It is shown that, in chip-synchronous CDMA, AEM are irrelevant to realizations of asynchronous delays and the adopted chip waveform, and AEM are equal to those of a symbol-synchronous CDMA system. The ASD converges almost surely to Marčenko-Pastur law. In chip-asynchronous CDMA, AEM are relevant to the shape of chip waveform. Whether AEM are relevant to realizations of asynchronous delays depends on chip waveform bandwidth. The empirical spectral distribution converges almost surely to an ASD, provided that a general constraint is satisfied by the chip waveform.

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I. INTRODUCTION

Direct sequence-code division multiple access (DS-CDMA) is one of the most flexible and commonly proposed multiple access techniques for wireless communication systems. To gain deeper insights into the performance of receivers in a DS-CDMA system, much work has been devoted to the analysis of random spreading DS-CDMA in the large-system regime, i.e. both the processing gain $N$ and the user number $K$ approach infinity with their ratio $K/N$ kept as a positive constant $\beta$ [1]–[3]. Such asymptotic analysis of random spreading CDMA enables random matrix theory to enter communication and information theory. In the last few years, a considerable amount of CDMA research has made substantial use of results in random matrix theory (see e.g. [4] and references therein).

Consider the linear vector memoryless channels of the form $y = Hx + w$, where $x$, $y$ and $w$ are the input vector, output vector and additive white Gaussian noise (AWGN), respectively, and $H$ denotes the random channel matrix independent of $w$. This linear model encompasses a variety of applications in communications such as multiuser channels, multi-antenna channels, multipath channels, etc., with $x$, $y$ and $H$ taking different meanings in each case. It is shown in [5] that $r = H^Ty = Rx + n$, with $R = H^TH$ and $n = H^Tw$, is a sufficient statistic for the estimation of $x$. Concerned with the linear model $r = Rx + n$, it is of particular interest to investigate the limiting distribution of eigenvalue $\lambda$’s of the random matrix $R$, called asymptotic spectral distribution (ASD), when the size of $R$ tends to infinity. Since ASD is deterministic and irrelevant to realizations of random parameters, it is convenient to use the asymptotic limit as an approximation for finite-size system design and analysis. Moreover, it is quite often that ASD provides us with much more insights than an empirical spectral distribution (ESD) does. Even though ASD is obtained with the large-system assumption, in practice, the system enjoys large-system properties for a moderate size of $R$. To show that an ESD tends to a nonrandom ASD $F(x)$, usually the moment convergence theory is employed, i.e. verifying the $n$-th order empirical eigenvalue moment of $\lambda$ converging to $m_n = \int x^n dF(x)$, called asymptotic eigenvalue moment (AEM), in some sense, and the Carleman’s condition [6] $\sum_{n=1}^{\infty} m_n^{-1/(2n)} = \infty$ is used to ensure the uniqueness of the limiting distribution [7].

Some applications of ASD in communication and information theory are exemplified below. It is known that a number of the system performance metrics, e.g. capacity and the minimum
mean-square-error (MMSE) achievable by a linear receiver, is determined by the ESD of $R$. The asymptotic capacity and MMSE obtained by using ASD as the approximation of ESD can be given in closed-form expressions [2], [5]. It is also shown in [8], [9] that empirical eigenvalue moments (or, more conveniently, AEM) of $R$ are relevant to finding the optimal weights of the reduced-rank and polynomial receivers and the output signal-to-interference-plus-noise ratio (SINR) in a large system. Moreover, a functional related to AEM is defined as the free expectation of random matrices in the free probability theory [10], which has been recently applied to the asymptotic random matrix analysis.

For synchronous DS-CDMA systems, it is well known that the ASD of the crosscorrelation matrix follows Marčenko–Pastur law. Also, explicit expressions for the AEM of $R$ under the environments of unfaded, frequency-flat fading, frequency-flat fading with antenna diversity, and frequency-selective fading are derived in [11], [12]. Actually, most of the research results on random spreading DS-CDMA making use of random matrix theory are applicable only for systems with synchronous transmission. Just a few of them investigate asynchronous systems [13]–[17]. The goal of this work is to find out the ASD of crosscorrelation matrix in asynchronous DS-CDMA systems given a set of asynchronous delays and an arbitrary chip waveform. As the uplink of a DS-CDMA system is asynchronous, this work is motivated by the needs to study the problem of asynchronous transmission that is important but much less explored by researchers in the area of random matrix theory.

Two levels of asynchronism are considered in this paper, i.e. symbol-asynchronous but chip-synchronous, and chip-asynchronous. In the sequel, chip-synchronous is used for short to denote the former, and symbol-synchronous represents an ideal synchronous system. To be more specific, the asynchronous delays are integer multiples of the chip duration in chip-synchronous CDMA, while they are any real numbers in chip-asynchronous CDMA.

In [13], it has been shown that, for both the linear MMSE receiver and decorrelator, the effective bandwidth of chip-synchronous CDMA is equal to that of symbol-synchronous CDMA if the observation window width of the former system is infinite. As both the MMSE receiver and decorrelator can be arbitrarily closely approximated by polynomial receivers [18], the result implies that the AEM of crosscorrelation matrices in symbol-synchronous and chip-synchronous CDMA are the same for an infinite-width observation window. The system model of [13] splits each interferer into two virtual users, which leads to a crosscorrelation matrix with neither
independent nor identically distributed entries. Results of [13] are obtained by employing results
about Stieltjes transform of ASD for random matrices of that type.

In this paper, the system model is constructed without the approximation of user splitting. We
consider an infinite input symbol length. The formulas for AEM are derived using a combinatorics
approach. In specific, we use noncrossing partition in set partition theory as the solving tool to
exploit all nonvanishing terms in the expression of AEM. Combinatorics approaches have been
adopted in [11], [19]–[23] to compute AEM of a random matrix corresponding to synchronous
systems. All of them, either explicitly or implicitly, make use of graphs to signify noncrossing
partitions. In this work, a notation of $K$-graph which is able to simultaneously represent two
noncrossing partitions is adopted. These two noncrossing partitions reside in the vertex and edge
sets of a $K$-graph, and they are Kreweras complementation map [24] of each other. The latter
property facilitates the employment of free probability theory.

Results of this paper are summarized below. For chip-synchronous CDMA systems, the
explicit expressions for AEM of the crosscorrelation matrix are given under the setting that
the asynchronous delays are deterministic constants. It is shown that the AEM formulas are
identical to those of symbol-synchronous CDMA, which is consistent with the implication of
[13]. We observe that AEM are irrelevant to the realizations of asynchronous delays and the
shape of chip waveform. Moreover, as the AEM satisfy the Carleman’s criterion [21] and an
almost-sure convergence test, it is concluded that the ASD in chip-synchronous CDMA converges
almost surely to Marčenko-Pastur law with ratio index $\beta$.

For chip-asynchronous CDMA systems, the convergence of ESD to a nonrandom ASD in an
almost-sure sense is proven. It is shown that, unlike chip-synchronous CDMA, the AEM are
related to chip waveform $\psi(t)$ through a set of $\int_{-\infty}^{\infty} \Psi^{2m}(\Omega) d\Omega/2\pi T_c^{m-1}$, where $\Psi(\Omega)$ is the
Fourier transform of $\psi(t)$, and $m$ is a natural number. On the other hand, the relationship of
AEM and asynchronous delays depends on the bandwidth of the chip waveform. Specifically,
when $\psi(t)$ has a bandwidth narrower than $1/(2T_c)$, with $T_c$ the chip duration, AEM are irrelevant
to the realizations of asynchronous delays $\tau_k$’s. On the contrary, if the bandwidth of $\psi(t)$ is wider
than $1/(2T_c)$, AEM are related to $\tau_k$’s. Nonetheless, if $\tau_k$’s are modeled as uniformly distributed
random variables in $[0, rT_c]$, with $r \in \mathbb{N}$, the averaged AEM over all realizations of $\tau_k$’s are
identical to those when $\psi(t)$ has bandwidth smaller than the threshold. As long as $r \in \mathbb{N}$, the
averaged AEM are not affected by the actual value of $r$, although choosing $r = N$ naturally fits
the scenario of asynchronous transmission. Related results are obtained in [17], where a symbol quasi-synchronous but chip-asynchronous system is considered with the asynchronous delays assumed to be uniformly distributed in $[0, T_c]$. Our result proves true the conjecture given there that asynchronous delay ranges $[0, T_c]$ and $[0, NT_c]$ yield the same performance. When the ideal sinc chip waveform is adopted, the AEM of chip-asynchronous CDMA can be shown to equal to those of chip-synchronous CDMA, which explains the equivalence result of [15] that the output SINR of the linear MMSE receiver converges to that of an equivalent chip-synchronous system. It also confirms the conjecture given in [15] that the equivalence result holds for the family of receivers considered in [18].

With the help of free probability theory, free cumulants of crosscorrelation matrices are also derived for both chip-synchronous and chip-asynchronous systems. It is also proven that the crosscorrelation matrix is asymptotically free with a random diagonal matrix having a general constraint. Based on the asymptotic freeness property, AEM for sum and product of crosscorrelation matrix and a random diagonal matrix are derived accordingly.

The organization of this paper is as follows. In Section II, the crosscorrelation matrices are given for chip-synchronous and chip-asynchronous CDMA systems. Explicit expressions for AEM are derived in Sections III and IV, respectively, for chip-synchronous and chip-asynchronous systems. In Section V, free probability theory is employed to obtain the spectra of sum and product of crosscorrelation matrix and a random diagonal matrix. Alternative expressions of crosscorrelation matrices are given in Section VI for new observations. Finally, this paper is concluded in Section VII.

II. CROSSCORRELATION MATRIX OF ASYNCHRONOUS CDMA

Consider asynchronous direct sequence-code division multiple access (DS-CDMA) systems where each user’s spreading sequence is chosen randomly and independently. To conduct large-system analysis, we assume both the user number $K$ and the spreading gain $N$ approach infinity with the ratio $K/N$ converging to a non-negative constant $\beta$. We focus on the uplink of the system and assume the receiver knows the spreading sequences and asynchronous delays of all users. Systems with two levels of asynchronism are considered, i.e. symbol-asynchronous but chip-synchronous, and chip-asynchronous. In the sequel, chip-synchronous is used for short to denote the former, and symbol-synchronous refers to an ideal synchronous system. To differentiate
notations of chip-synchronous and chip-asynchronous systems, subscripts in text form of "cs" and "ca" are used for notations in the former and the latter systems, respectively.

A. Chip-Synchronous CDMA

Denote the asynchronous delay of user $k$ as $\tau_k$. For convenience, users are labelled chronologically by their arrival time, and $\{\tau_k\}_{k=1}^K$ satisfy

$$0 = \tau_1 \leq \tau_2 \leq \cdots \leq \tau_K < NT_c,$$

where $T_c$ is the chip duration, and all $\tau_k$’s are integer multiples of $T_c$. Suppose that each user sends a sequence of symbols with indices from $-M$ to $M$, and binary phase-shift keying (BPSK) is adopted. In the complex baseband notation, the contribution of user $l$ to the received signal in a frequency-flat fading channel is

$$x_l(t) = \sum_{n=-M}^{M} A_l(n)b_l(n) \sum_{q=nN}^{(n+1)N-1} c_l^{(q)}(t-qT_c-\tau_l),$$

where $b_l(n)$ is the $n$-th symbol of user $l$ equiprobably in $\{-1, +1\}$, $A_l(n)$ is the complex amplitude at the time $b_l(n)$ is received, $c_l^{(q)}$ is the $(q \mod N)$-th chip at the $\lfloor q/N \rfloor$-th symbol of user $l$’s spreading sequence, and $\psi(t)$ is the normalized chip waveform having the zero inter-chip interference condition of

$$\int_{-\infty}^{\infty} \psi(t)\psi(t-rT_c)dt = \begin{cases} 
1, & r = 0, \\
0, & r \in \mathbb{Z} \setminus \{0\}.
\end{cases}$$

It is assumed that $b_l(n)$’s are independent and identically distributed (i.i.d.), and so do $c_l^{(q)}$’s. In particular, $c_l^{(q)}$’s are zero-mean random variables with variance equal to $1/N$ and the fourth order moment $O(1/N^2)$. The result of this paper does not depend on a particular distribution of $c_l^{(q)}$. The complex baseband received signal is given by

$$r(t) = \sum_{l=1}^{K} x_l(t) + w(t),$$

where $w(t)$ is baseband complex Gaussian ambient noise with independent real and imaginary components. The symbol matched filter output of user $k$’s symbol $m$, denoted as $y_k(m)$, is
obtained by correlating \( r(t) \) with the signature waveform of user \( k \)'s symbol \( m \)

\[
y_k(m) = \int_{-\infty}^{\infty} r(t) \left( \sum_{p=mN}^{(m+1)N-1} c_k^{(p)}(t-pT_c-\tau_k) \right) dt
\]

\[
= \sum_{l=1}^{K} \sum_{n=-M}^{M} A_l(n) b_l(n) \rho_{cs}(m, n; k, l) + v_k(m),
\]

where \( v_k(m) \) results from the ambient noise \( w(t) \), and \( \rho_{cs}(m, n; k, l) \) is the crosscorrelation of spreading sequences at user \( k \)'s \( m \)-th symbol and user \( l \)'s \( n \)-th symbol, given as

\[
\sum_{q=nN}^{(n+1)N-1} \sum_{p=mN}^{(m+1)N-1} c_l^{(q)} c_k^{(p)} \int_{-\infty}^{\infty} \psi(t-qT_c-\tau_l) \psi(t-pT_c-\tau_k) dt.
\]

Due to the zero inter-chip interference condition of (2), the integration in (6) is nonzero and equal to one only if \( pT_c + \tau_k = qT_c + \tau_l \). Thus, we obtain

\[
\rho_{cs}(m, n; k, l) = \sum_{p=mN}^{(m+1)N-1} \sum_{q=nN}^{(n+1)N-1} c_k^{(p)} c_l^{(q)} \delta(pT_c + \tau_k, qT_c + \tau_l),
\]

with \( \delta(i, j) \) the Kronecker delta function. Since \( 0 \leq \tau_k, \tau_l \leq (N-1)T_c \), for a specific symbol index \( m \), the \( \delta \) function in (7) is equal to zero if \( n \notin \{m-1, m, m+1\} \). It follows that the summation variable \( n \) of (5) can be revised to belong to \( \{m-1, m, m+1\} \). To be more specific, we rewrite (5) as

\[
y_k(m) = \sum_{l=1}^{K} \sum_{n=\max\{m-1, -M\}}^{\min\{m+1, M\}} A_l(n) b_l(n) \rho_{cs}(m, n; k, l) + v_k(m), \quad -M \leq m \leq M.
\]

Define the symbol matched filter output vector at the \( m \)-th symbol as

\[
\underline{y}(m) = [y_1(m), y_2(m), \cdots, y_K(m)]^T,
\]

and the transmitted symbol vector \( \underline{b}(m) \) and the noise component vector \( \underline{v}(m) \) have the same structures as that of \( \underline{y}(m) \). Moreover, we define a block matrix \( R_{cs} \) whose \((k, l)\)-th element of the \((m, n)\)-th block, with \(-M \leq m, n \leq M, 1 \leq k, l \leq K\), is equal to \( \rho_{cs}(m, n; k, l) \) of (7). The square bracket \([\cdot]\) is used to indicate a specific element of a matrix. For block matrices, two sets of indices are used, with the first and second representing the block and element locations, respectively. For instance, \([R_{cs}]_{mn,kl}\) represents the \((k, l)\)-th entry of the \((m, n)\)-th block of the block matrix \( R_{cs} \). When we just want to point out a specific block, only the first set of indices is used, i.e. \([R_{cs}]_{mn}\).
Using the notations defined above, we can show from (8) that
\[ y(m) = \min_{n=\max\{m-1,-M\}} \sum_{n} \left[ R_{cs} \right]_{mn} A(n) b(n) + v(m), \]
where \( A(n) = \text{diag}\{A_1(n), A_2(n), \ldots, A_K(n)\} \). Stacking up \( y(m) \)'s to yield the symbol matched filter output of the whole transmission period as
\[ y = [y^T(-M), y^T(-M+1), \ldots, y^T(M)]^T, \]
we obtain the discrete-time signal model
\[ y = R_{cs} A b + v, \tag{9} \]
where \( b \) and \( v \) have the same structures as \( y \), \( A = \text{diag}\{A(-M), A(-M+1) \ldots, A(M)\} \), and \( R_{cs} \) is a block matrix with a tri-diagonal block structure of
\[
\begin{bmatrix}
\cdot & \cdot & \cdots & \cdot \\
[\text{\scriptsize R}_{cs}]^{-1} & [\text{\scriptsize R}_{cs}]^{-1} & 0 & 0 \\
0 & [\text{\scriptsize R}_{cs}] & [\text{\scriptsize R}_{cs}] & 0 \\
0 & 0 & [\text{\scriptsize R}_{cs}] & [\text{\scriptsize R}_{cs}] \\
\end{bmatrix}.
\tag{10}
\]

Since \( \tau_k \leq \tau_l \) for \( k < l \), \( [R_{cs}]_{m} m-1 \) and \( [R_{cs}]_{m} m+1 \) are strict (zero diagonal) upper- and lower-triangular matrices, respectively. From the signal model given in (9), \( R_{cs} \) can be viewed as the crosscorrelation matrix of chip-synchronous CDMA.

**B. Chip-Asynchronous CDMA**

In chip-asynchronous CDMA, the assumption of \( \tau_k \)'s being integer multiples of \( T_c \) no longer exists. Similarly to (5), the symbol matched filter output \( y_k(m) \) can be expressed as
\[ y_k(m) = \sum_{l=1}^{K} \sum_{n=-M}^{M} A_l(n) b_l(n) \rho_{ca}(m, n; k, l) + v_k(m), \tag{11} \]
where \( \rho_{ca}(m, n; k, l) \) is different from \( \rho_{cs}(m, n; k, l) \) in (5) since the zero inter-chip interference condition does not hold when the time difference of chip waveforms is not integer multiples of \( T_c \). At this moment, the crosscorrelation \( \rho_{ca}(m, n; k, l) \) is
\[
\rho_{ca}(m, n; k, l) = \sum_{p=mN}^{(m+1)N-1} \sum_{q=nN}^{(n+1)N-1} c_{k}^{(p)} c_{l}^{(q)} R_{\psi}((p - q)T_c + \tau_k - \tau_l), \tag{12}
\]
where

\[ R_\psi(x) = \int_{-\infty}^{\infty} \psi(t)\psi(t-x)dt \]  

(13)
is the autocorrelation function of the chip waveform \( \psi(t) \). We define the block matrix \( R_{ca} \) whose

the \((k,l)\)-th component of the \((m,n)\)-th block, with \(-M \leq m,n \leq M\) and \(1 \leq k,l \leq K\), is equal to \( \rho_{ca}(m,n;k,l) \) given in (12). It can be shown that

\[ y(m) = \sum_{n=-M}^{M} [R_{ca}]_{mn} A(n)b(n) + \nu(m), \]  

(14)

and we obtain the discrete-time signal model

\[ y = R_{ca}Ab + \nu, \]

where \( R_{ca} \) is thus seen as the crosscorrelation matrix of a chip-asynchronous CDMA system.

Note that, unlike the tri-diagonal block structure of \( R_{cs} \) shown in (10), \( R_{ca} \) generally does not possess such block diagonal structure except when the autocorrelation function \( R_\psi(x) \) has a finite span.

C. Asymptotic Spectral Distribution of Crosscorrelation Matrix

Suppose that \( R_x, x \in \{cs, ca\} \), have eigenvalues \( \nu_1 \leq \nu_2 \leq \cdots \leq \nu_{(2M+1)K} \). Since \( R_x \) is symmetric, all \( \nu_i \)'s are real. A cumulative distribution function \( F^{(K)}(x) \) is defined as

\[ F^{(K)}(x) = \lim_{M \to \infty} (2M+1)^{-1}K^{-1}\#\{i : \nu_i \leq x\}, \]  

(15)

where \( \#\{\cdots\} \) denotes the number of elements in the indicated set. We take the limit \( M \to \infty \) in (15) to investigate the system behavior when the size of observation window tends to infinity. Let the crosscorrelation matrix be labeled as \( R_x^{(K)} \) when the user size is \( K \). The function \( F^{(K)}(x) \) is called the empirical spectral distribution (ESD) of \( R_x^{(K)} \). If \( F^{(K)}(x) \) tends to a nonrandom distribution function \( F(x) \) as \( K \to \infty \), then we say that the sequence \( \{R_x^{(K)} : K = 1,2,\cdots\} \) has an asymptotic spectral distribution (ASD) \( F(x) \).

Let \( \lambda_x^{(K)} \) denote the random variable governing the eigenvalues of \( R_x^{(K)} \). That is, the cumulative distribution function of \( \lambda_x^{(K)} \) is \( F^{(K)}(x) \). When the number of transmitted symbols \( 2M+1 \) tends
to infinity, we have

\[
\lim_{K,N,M \to \infty} E \left\{ (\lambda_x^{(K)})^n \right\} = \lim_{K,N,M \to \infty} (2M + 1)^{-1} K^{-1} E \left\{ \sum_{l=1}^{(2M+1)K} \nu_l^n \right\} \tag{16}
\]

\[
= \lim_{K,N,M \to \infty} (2M + 1)^{-1} K^{-1} E \left\{ \text{tr}((R_x^{(K)})^n) \right\}, \tag{17}
\]

where \( \text{tr}(\cdot) \) is the trace operator. Define \( \mu(\cdot) \) as a functional of the expected normalized trace

\[
\mu(B) = \lim_{K,N \to \infty} B^{-1} E\{\text{tr}(B)\} \tag{18}
\]

for a \( B \times B \) symmetric (or Hermitian) matrix \( B \). It is seen that the \( n \)-th order asymptotic eigenvalue moment (AEM) of \( R_x^{(K)} \) can be written as \( \mu(R_x^n) \), with both parameters in matrix size \( M, K \to \infty \). The relation between \( \mu(R_x^n) \) and the ASD \( F(x) \) is

\[
\mu(R_x^n) = \int x^n dF(x). \]

For our convenience, we will use

\[
\mu(R_x^n) = \lim_{K,N \to \infty} K^{-1} \left[ \lim_{M \to \infty} E \left\{ \text{tr}([R_x^{(K)}]^n_{00}) \right\} \right] \tag{19}
\]

for the computation of the \( n \)-th order AEM of \( R_x^{(K)} \). That is, the expectation under \( M \to \infty \) inside the square bracket is computed first, and then we take the limits of \( K \) and \( N \).

To show the existence of the ASD \( F(x) \) of \( \{R_x^{(K)} : K = 1, 2, \cdots \} \), we employ moment convergence theorem. That is, it is required to prove

1) \( \mu(R_x^n) \) exists for all \( n \in \mathbb{N} \), and

2) \( \sum_{n=1}^{\infty} \mu(R_x^{2n})^{-1/(2n)} = \infty \),

where condition 2) is due to Carleman’s criterion [6] for the uniqueness of a distribution given a moment sequence. In particular, to show that \( F^{(K)}(x) \) converges to \( F(x) \) almost surely, we need to prove

\[
\int x^n dF^{(K)}(x) \to \mu(R_x^n), \quad \text{almost surely.}
\]

Specifically, define

\[
v_K = \lim_{M \to \infty} (2M + 1)^{-1} K^{-1} \left\{ \text{tr}([R_x^{(K)}]^n) - E\{\text{tr}([R_x^{(K)}]^n)\} \right\}.
\]
By Borell-Cantelli lemma \([6]\), if
\[
\sum_{K=1}^{\infty} \text{Prob}(|v_K| > \epsilon) < \infty, \quad \forall \epsilon > 0,
\]
then \(v_K \to 0\) almost surely. Using Markov inequality that
\[
\text{Prob}(|v_K| > \epsilon) = \text{Prob}(v_K^2 > \epsilon^2) \leq \mathbb{E}\{v_K^2\}/\epsilon^2, \quad \forall \epsilon > 0,
\]
we can show the almost sure convergence of \(v_K\) to 0 by proving

3) \(\sum_{K=1}^{\infty} \mathbb{E}\{v_K^2\} < \infty\).

The almost-sure convergence of \(v_K\) to 0 is equivalent to saying that
\[
\lim_{M \to \infty} (2M + 1)^{-1} K^{-1} \text{tr}((R_x^{(K)})^n) = \int x^n dF^{(K)}(x)
\]
converges to
\[
\lim_{K,N \to \infty} \left[ \lim_{M \to \infty} (2M + 1)^{-1} K^{-1} \mathbb{E}\{\text{tr}((R_x^{(K)})^n)\} \right] = \mu(R_x^n)
\]
almost surely.

In Sections III and IV, criterions 1) – 3) listed above will be shown for \(R_{cs}\) and \(R_{ca}\), respectively. For simplicity, in the sequel, the superscript \((K)\) of \(R_x^{(K)}\) is omitted when no ambiguity occurs.

III. ASYMPTOTIC SPECTRAL DISTRIBUTION OF CHIP-SYNCHRONOUS CDMA

The AEM of \(R_{cs}\) is to be derived according to (19), where the asynchronous delays \(\tau_k\)'s are treated as known deterministic constants. By expanding matrix multiplications and letting
\[
[R_{cs}]; m_r, m_{r+1}, k_r, k_{r+1} = \rho_{cs}(m_r, m_{r+1}; k_r, k_{r+1})
\]
\[
= \sum_{p_r = m_r}^{(m_{r+1})N - 1} \sum_{q_{r+1} = m_{r+1}N}^{(m_{r+1} + 1)N - 1} c_{k_r}^{(p_r)} c_{k_{r+1}}^{(q_{r+1})} \delta(p_r T_c + \tau_k, q_{r+1} T_c + \tau_{k_{r+1}}),
\]
for \(1 \leq r \leq n\) with \(m_{n+1} = m_1\) and \(k_{n+1} = k_1\), we have

\[
\lim_{M \to \infty} \mathbb{E}\{\text{tr}([R_{cs}^n])_{00}\} = \lim_{M \to \infty} \sum_{K \in \mathcal{K}} \sum_{M' \in \mathcal{M}} \sum_{\mathbb{P} \in \mathcal{Z}_1} \cdots \sum_{\mathbb{P} \in \mathcal{Z}_n} \mathbb{E}\left\{\left(\begin{array}{c} c_{k_1}^{(p_1)} c_{k_2}^{(q_2)} \\ c_{k_2}^{(p_2)} c_{k_3}^{(q_3)} \\ \vdots \\ c_{k_n}^{(p_n)} c_{k_1}^{(q_1)} \end{array}\right) \times \delta(p_1 T_c + \tau_{k_1}, q_2 T_c + \tau_{k_2}) \delta(p_2 T_c + \tau_{k_2}, q_3 T_c + \tau_{k_3}) \cdots \delta(p_n T_c + \tau_{k_n}, q_1 T_c + \tau_{k_1})\right\}_{m_1 = 0},
\]
where \( K = \{k_1, \cdots, k_n\} \), \( X = [1, K] \times \cdots \times [1, K] = [1, K]^n \), \( M = \{m_2, \cdots, m_n\} \), \( Y = [-M, M]^{n-1} \), and \( \mathcal{P}_r = \{p_r, q_r\} \), \( Z_r = [m_rN, (m_r + 1)N - 1]^2 \) for \( 1 \leq r \leq n \). Since \( m_1 = 0 \), we have \( Z_1 = [0, N - 1]^2 \). Moreover, owing to the tri-diagonal block structure of \( R_{cs} \) shown in (10), there are constraints \( |m_r - m_{r+1}| \leq 1 \) for \( 1 \leq r \leq n \). The \( n \)-th order AEM is obtained by multiplying (20) with \( K^{-1} \) and taking limit of \( N, K = \beta N \to \infty \).

Computation of (20) can be done via considering the equality patterns of \( \{k_1, k_2, \cdots, k_n\} \). As equivalence relation and partition are essentially equivalent, the computation of (20) can be carried out with the aid of set partition theory, where if \( k_r \) and \( k_s \) take the same value in \( [1, K] \), they are partitioned in the same group. Set partition theory has been employed in [11,19] for AEM computation of a symbol-synchronous CDMA system. Some preliminaries are given in the following subsection.

A. Noncrossing Partition

Given a linearly ordered set of \( n \) elements \( K = \{k_1, k_2, \cdots, k_n\} \), where \( a \prec b \) denotes \( a \) precedes \( b \). A partition is a family of nonempty, pairwise disjoint sets, called \( \textit{classes} \), whose union is the \( n \)-element set. A partition is noncrossing if no two classes cross each other. That is, when \( a \) and \( b \) belong to one class and \( x \) and \( y \) to another, they are not arranged in the order \( a \prec x \prec b \prec y \). The number of noncrossing partitions that partition \( n \) elements into \( j \) classes is the Narayana number \( N_{n,j} \), given by

\[
N_{n,j} = \frac{1}{n} \binom{n}{j} \binom{n}{j-1}.
\]  

Moreover, if the \( j \) classes have sizes \( c_1, c_2, \cdots, c_j \) with \( c_1 \geq c_2 \geq \cdots \geq c_j \geq 1 \) (but not specifying which class gets which size), the number of noncrossing partitions is [24]

\[
\frac{n(n-1) \cdots (n-j+2)}{f(c_1, c_2, \cdots, c_j)},
\]  

where

\[
f(c_1, c_2, \cdots, c_j) = \prod_{k \geq 1} h_k !
\]

with \( h_k \) being the number of elements in \( (c_1, c_2, \cdots, c_j) \) that are equal to \( k \). It is clear that

\[
\sum_{c_1+c_2+\cdots+c_j=n \atop c_1 \geq c_2 \geq \cdots \geq c_j \geq 1} \frac{n(n-1) \cdots (n-j+2)}{f(c_1, c_2, \cdots, c_j)} = \frac{1}{n} \binom{n}{j} \binom{n}{j-1}.
\]
Fig. 1. (a) The partition \( \{ \{ k_1, k_4, k_6 \} \{ k_2, k_3, k_7, k_8 \} \{ k_5 \} \} \), (b) the partition \( \{ \{ k_1 \} \{ k_2, k_3, k_7, k_8 \} \{ k_4, k_5 \} \{ k_6 \} \} \), (c) the \( K \)-graph for the partition represented in (a), and (d) the \( K \)-graph for the partition represented in (b).

where the ordering constraint of \( \{ c_r \}_{r=1}^j \) is used to avoid duplications.

A partition can be represented graphically. For example, Figs. 1(a) and 1(b) show two partitions of \( \{ k_1, k_2, \cdots, k_8 \} \), where elements in the same class are joined successively by chords. A noncrossing partition is such that the chords intersect only at elements \( k_1, \cdots, k_n \). For instance, Fig. 1(b) is a noncrossing partition, while Fig. 1(a) is not. We define a representation of \( K \)-graph below.

**Definition 1:** Denote the \( K \)-graph of a \( j \)-class partition of \( \mathcal{K} = \{ k_1, k_2, \cdots, k_n \} \) by \( G = (\mathcal{V}, \mathcal{E}) \). The vertex set is \( \mathcal{V} = \{ v_1, v_2, \cdots, v_j \} \), and the edge set is \( \mathcal{E} = \{ e_1, e_2, \cdots, e_n \} \), where \( e_r, 1 \leq r \leq n \), connects vertices \( v_s \) and \( v_t \) with \( k_r \) and \( k_{r+1} \) (with \( k_{n+1} = k_1 \)) being partitioned into the \( s \)-th and \( t \)-th classes, respectively.

\[ \blacksquare \]

**Remark:** A \( K \)-graph can be interpreted in a more visually convenient way as follows. Let elements of \( \mathcal{K} = \{ k_1, k_2, \cdots, k_n \} \) be arranged orderly (either clockwise or counter-clockwise) as vertices of an \( n \)-vertex cycle. The \( K \)-graph of a partition of \( \mathcal{K} \) is obtained by merging vertices of the \( n \)-vertex cycle that are in the same class into one.

Each vertex of a \( K \)-graph represents one class of the associated partition of \( \mathcal{K} \). Figs. 1(c) and
1(d) present the $K$-graphs for the partitions of Figs. 1(a) and 1(b), respectively. It is seen that, if and only if the associated partition is noncrossing, the $K$-graph consists of cycles with any two of them connected by at most one vertex. Moreover, when the noncrossing partition has $j$ classes, there are $n - j + 1$ cycles in the $K$-graph. For example, Fig. 1(d) is composed of $8 - 4 + 1 = 5$ cycles. Any pair of these five cycles are connected by at most one of the two vertices labelled with \{ $k_2, k_3, k_7, k_8$ \} and \{ $k_4$, $k_5$ \}.

The cycles of a $K$-graph yield a noncrossing partition of the edge set. Specifically, let edges in the same cycle be partitioned into the same class. Then, these classes form a noncrossing partition of $\mathcal{E} = \{e_1, e_2, \cdots, e_n\}$. For example, Fig. 1(d) corresponds to \{ \{ $e_1, e_8$ \}, \{ $e_2$ \}, \{ $e_3, e_5, e_6$ \}, \{ $e_4$ \}, \{ $e_7$ \} \}, which is a noncrossing partition of \{ $e_1, e_2, \cdots, e_8$ \}. Clearly, there is a one-to-one onto mapping between the set of noncrossing partitions of $\mathcal{K}$ into $j$ classes and the set of noncrossing partitions of $\mathcal{E}$ into $n - j + 1$ classes.

A $K$-graph is a representation that is convenient in utilizing results of noncrossing partitions. Given a noncrossing partition $\varpi$ of an ordered set $\mathcal{K}$ and its corresponding $K$-graph $G = (V, E)$. We have the following properties.

1) There is a bijective correspondence between the class set of $\varpi$ and the vertex set of $G$.

2) The partition $\varsigma$ of $\mathcal{E}$, where edges belonging to the same cycle are in the same class, is noncrossing as well.

3) $\varpi$ and $\varsigma$ are the Kreweras complementation map [24] of each other.

The following fact is useful. The number of noncrossing partitions $\varpi$ of an $n$-element set meeting conditions of

1) $\varpi$ has $j$ classes with sizes $(c_1, c_2, \cdots, c_j)$, and

2) sizes of classes in the Kreweras complementation map of $\varpi$, denoted by $KC(\varpi)$, are $(b_1, b_2, \cdots, b_{n-j+1})$,

is [25,12]

$$n(n-j)!(j-1)! f(b_1, b_2, \cdots, b_{n-j+1}) f(c_1, c_2, \cdots, c_j). \tag{24}$$

In the following, the summation $\sum_{\mathcal{K} \in \mathcal{X}}$ in (20) is decomposed into several ones using properties of set partition theory given above. Let $\mathcal{X}_j$ denote the subset of $\mathcal{X} = \{1, K\}^n$ such that each element $x_j = (x_j(1), \cdots, x_j(n))$ of $\mathcal{X}_j$ corresponds to a $j$-class noncrossing partition $\varpi$ of $\mathcal{K}$. We mean $x_j$ corresponds to $\varpi$ by that $x_j(s) = x_j(t)$ if and only if $k_s$ and $k_t$ are partitioned
in the same class in \( \varpi \). Moreover, we let \( \mathcal{X}_j(b_1, b_2, \cdots, b_{n-j+1}) \) represent the union of \( x_j \)'s whose correspondent noncrossing partitions have Kreweras complementation maps having class sizes \((b_1, b_2, \cdots, b_{n-j+1})\). Since the Kreweras complementation map of a noncrossing partition is noncrossing as well, by (22), the member number of \( \mathcal{X}_j(b_1, b_2, \cdots, b_{n-j+1}) \) is given by

\[
\# \mathcal{X}_j(b_1, b_2, \cdots, b_{n-j+1}) = \frac{n(n-1) \cdots (j+1)}{f(b_1, b_2, \cdots, b_{n-j+1})} \cdot K(K-1) \cdots (K-j+1).
\]

The above equation says that the number of \( K \)-graphs associated with \( \mathcal{X}_j(b_1, b_2, \cdots, b_{n-j+1}) \) is \( n(n-1) \cdots (j+1)/f(b_1, b_2, \cdots, b_{n-j+1}) \), and each of these \( K \)-graphs has \( j \) vertices with each specified by a distinct integer from \([1, K]\).

We also denote \( \mathcal{X}_{cro} \) by the subset of \( \mathcal{X} = [1, K]^n \) such that each of its element is associated with a crossing partition of \( K \). With these settings, the summation \( \sum_{K \in \mathcal{X}} \) in (20) can be decomposed via the equivalence relation of

\[
\sum_{K \in \mathcal{X}} \equiv \sum_{j=1}^{n} \sum_{b_1+b_2+\cdots+b_{n-j+1}=n} \sum_{b_1 \geq b_2 \geq \cdots \geq b_{n-j+1} \geq 1} \sum_{K \in \mathcal{X}_j(b_1, b_2, \cdots, b_{n-j+1})} + \sum_{K \in \mathcal{X}_{cro}}. \tag{25}
\]

It will be shown later that every element of the set \( \mathcal{X}_j(b_1, b_2, \cdots, b_{n-j+1}) \) contributes the same amount to (20). Thus, \( \sum_{K \in \mathcal{X}_j(b_1, b_2, \cdots, b_{n-j+1})} \) can be simplified by calculating the contribution to (20) of any singly member of \( \mathcal{X}_j(b_1, b_2, \cdots, b_{n-j+1}) \), and then multiplying the result by the member number \( \# \mathcal{X}_j(b_1, b_2, \cdots, b_{n-j+1}) \).

**B. Computation of AEM**

In following Definition 1 to form a \( K \)-graph, let two ends of edge \( e_r, 1 \leq r \leq n \), in the \( K \)-graph be labelled with \( p_r \) and \( q_{r+1} \) (with \( q_{n+1} = q_1 \)) with the former and latter touching the vertices to which \( k_r \) and \( k_{r+1} \) belong, respectively. Or, equivalently, using the visually convenient interpretation stated in the remark of Definition 1, in the \( n \)-vertex cycle composed of vertices \( k_1, k_2, \cdots, k_n \) (e.g. Figs. 1(a) and 1(b)), two ends of the edge connecting \( k_r \) and \( k_{r+1} \) are labelled with \( p_r \) and \( q_{r+1} \), with the former and latter touching \( k_r \) and \( k_{r+1} \), respectively. We call these \( p_r \)'s and \( q_r \)'s as edge variables. Fig. 2 shows two \( K \)-graphs with edge variables labelled. Such labelling carries information of

\[1\] A number of these \( K \)-graphs are isomorphic.
Fig. 2. The $K$-graphs of (a) $n$-class partition $\{\{k_1\}, \{k_2\}, \ldots , \{k_n\}\}$, and (b) $(n - 1)$-class partition $\{\{k_1\}, \{k_2\}, \ldots , \{k_s, k_t\}, \ldots , \{k_n\}\}$ with $s < t$. 

1) the product of $c_{k_r}^{(p_r)}, c_{k_r}^{(q_r)}$ for each $k_r$, and 

2) the product of $\delta(p_rT_c + \tau_{k_r}, q_{r+1}T_c + \tau_{k_{r+1}})$ for each pair of $k_r$ and $k_{r+1}$ in (20), where $p_r$'s and $q_r$'s therein are summation variables. Another set of summation variables $m_r$'s are embedded in $p_r$'s and $q_r$'s by that the ranges of $p_r$ and $q_r$ are both $[m_rN, (m_r + 1)N - 1]$. Edge variables $p_r$'s and $q_r$'s are summation variables having degrees of freedom. We would like to inspect under what equivalence relation of $p_r$'s and $q_r$'s will the contribution to AEM nonvanishing in the large-system regime.

We start the computation of (20). Given natural numbers $(b_1, b_2, \ldots , b_{n-j+1})$ such that $b_1 + b_2 + \cdots + b_{n-j+1} = n$. The contribution of an element $x_j \in X_j(b_1, b_2, \ldots , b_{n-j+1})$ to the expression of (20) is evaluated.

First, we consider $j = n$. There is only one $K$-graph, shown in Fig. 2(a). Since all $k_r$'s are distinct, the expectation of spreading sequences in (20) is nonzero and equal to $N^{-n}$ if and only if $p_r = q_r$ for $1 \leq r \leq n$. The contribution of a singly element $x_j \in X_n(n)$ to (20) becomes

$$
N^{-n} \lim_{M \to \infty} \sum_{M \in \mathcal{Y}} \sum_{p_1 \in [0, N-1]} \sum_{p_2 \in Z'_n} \cdots \sum_{p_n \in Z'_n} \delta(p_1T_c + \tau_{k_1}, p_2T_c + \tau_{k_2}) \times \delta(p_2T_c + \tau_{k_2}, p_3T_c + \tau_{k_3}) \cdots \delta(p_nT_c + \tau_{k_n}, p_1T_c + \tau_{k_1}),
$$

where $Z'_n = [m_r, (m_r + 1)N - 1]$ and $|m_r - m_{r+1}| \leq 1$. The product of $\delta$ functions is nonzero and equal to one if and only if all $p_rT_c + \tau_{k_r}, 1 \leq r \leq n$, are equal. As $p_1 \in [0, N-1]$ and
0 ≤ τ_k ≤ (N−1)T_c for 1 ≤ k ≤ K, it is not difficult to see the term to the right-hand-side of \( N^{-n} \) in (26) is equal to \( N \). Consequently, (26) is equal to \( N^{-n} \cdot N = N^{-n+1} \).

Next, we consider \( j = n-1 \). Any \( K \)-graph associated with \( \mathcal{X}_{n-1} \) can be obtained from the \( K \)-graph of \( \mathcal{X}_n(n) \), denoted by \( G_n \), by merging two vertices into one. Suppose that vertices \( k_s \) and \( k_t \) of \( G_n \) are merged (with \( s < t \)), meaning that \( k_s = k_t \) in (20). Thus, the original \( n \)-vertex cycle is decomposed into two cycles with edge numbers \( e \triangleq t - s \) and \( n - e \) as shown in Fig. 2(b). We are going to demonstrate that, concerned with the four edge variables of the vertex merged from \( k_s, k_t \) in Fig. 2(b), it is sufficient to consider the condition that \( p_t = q_s \) and \( p_s = q_t \).

Since \( k_s = k_t \), to yield a nonzero expectation of \( \prod_{r=1}^{n} c_{kr}^{(q_r)} c_{kr}^{(p_r)} \) in (20), it is required that \( p_r = q_r \) for \( r \in \{1, 2, \cdots, n\} \setminus \{s, t\} \) and the four edge variables touching the vertex merged
from \( k_s, k_t \) are in pairs, i.e.

1) \( q_s = p_s \) and \( q_t = p_t \),
2) \( q_s = q_t \) and \( p_t = p_s \), or
3) \( q_s = p_t \) and \( q_t = p_s \).

Cases 1) – 3) are represented by the Figs. 3(a) – (c), respectively. For instance, in Fig. 3(a), except for the vertex labelled with \( k_s, k_t \), two edge variables touching the same vertex are identical. Besides, \( q_s \) and \( q_t \) in Fig. 2(b) are replaced with \( p_s \) and \( p_t \), respectively. In each of Figs. 3(a) – (c), the product of \( \delta \) functions in (20) are expressed in the form of cycle(s), indicated by thick lines traversing edges. The thick line passing through an edge labelled with variables \( p_\gamma \) and \( p_\epsilon \) represents \( \delta(p_\gamma T_c + \tau_{k_\gamma}, p_\epsilon T_c + \tau_{k_\epsilon}) \).

- Case 1): When \( p_s \neq p_t \), the expectation of \( \prod_{r=1}^{n} c_{k_r}(q_r)(p_r) \) in (20) is \( N^{-n} \), which makes (20) become (26). Since the thick line in Fig. 3(a) forms a single cycle, i.e.

\[
\prod_{(\gamma,\epsilon)\in \mathcal{I}} \delta(p_\gamma T_c + \tau_{k_\gamma}, p_\epsilon T_c + \tau_{k_\epsilon}),
\]

with \( \mathcal{I} = \{(1,2), (2,3), \cdots, (n-1,n), (n,1)\} \), the infinite sum of products of \( \delta \) functions is equal to \( N \). Hence, in this case, a singly element \( x_j \in \mathcal{X}_{n-1}(e, n-e) \) contributes to (20) by an amount of \( N^{-n} \cdot N = N^{-n+1} \). When \( p_s = p_t \), as the fourth order moment of \( c_{k_r}(p_r) \) is \( O(N^{-2}) \), the expectation of \( \prod_{r=1}^{n} c_{k_r}(q_r)(p_r) \) is \( O(N^{-n}) \). Moreover, since \( p_s = p_t \) forces \( m_s = m_t \), the correspondent infinite sum of products of \( \delta \) functions is no larger than \( N \). Thus, the contribution is \( O(N^{-n+1}) \).

- Case 2): Since \( q_s = q_t = p_t = p_s \) has already been considered in the above case, here we just look at \( q_s = q_t \neq p_t = p_s \). It is seen from Fig. 3(b) that, similarly to case 1), the product of \( \delta \) functions forms a single cycle of (27) with

\[
\mathcal{I} = \{(1,2), (2,3), \cdots, (s-1,s), (s,t-1), (t-1,t-2), \cdots, (s+2,s+1), (s+1,t), (t,t+1), \cdots, (n,1)\}
\]

and \( p_s \) therein replaced by \( q_s \). The different thing is that \( q_s = q_t \) and \( p_t = p_s \) in this case constrain \( m_s = m_t \). It follows that the infinite sum of products of \( \delta \) functions is no larger than \( N \). Thus, the contribution of this case to (20) is \( O(N^{-n+1}) \).
• Case 3): We consider \( q_s = p_t \neq q_t = p_s \), which implies \( m_s = m_t \). The product of \( \delta \) functions can be represented by concatenation of two cycles (see Fig. 3(c)), i.e.

\[
\prod_{(\gamma, \epsilon) \in \mathcal{I}} \delta(p_{\gamma}T_c + \tau_{k_{\gamma}}, p_sT_c + \tau_{k_s}) \prod_{(\eta, \zeta) \in \mathcal{J}} \delta(p_{\eta}T_c + \tau_{k_{\eta}}, p_cT_c + \tau_{k_c}),
\]

with \( \mathcal{I} = \{(1, 2), (2, 3), \ldots, (s - 2, s - 1), (s - 1, t), (t, t + 1), \ldots, (n, 1)\} \) and \( \mathcal{J} = \{(s, s + 1), (s + 1, s + 2), \ldots, (t - 2, t - 1), (t - 1, s)\} \). The contribution to (20) becomes

\[
N^{-n} \lim_{M \to \infty} \sum_{M \in \mathcal{Y}} \sum_{p_1, \ldots, p_{s-1}, p_s, p_{t+1}, \ldots, p_n} \prod_{(\gamma, \epsilon) \in \mathcal{I}} \delta(p_{\gamma}T_c + \tau_{k_{\gamma}}, p_sT_c + \tau_{k_s})
\]

\[
\times \sum_{p_s, p_{s+1}, \ldots, p_{t-1}} \prod_{(\eta, \zeta) \in \mathcal{J}} \delta(p_{\eta}T_c + \tau_{k_{\eta}}, p_cT_c + \tau_{k_c}) \bigg|_{m_s = m_t}.
\]  

(28)

Note that the constraint of \( m_s = m_t \) results in only \( N \) choices for \( p_s \in [m_tN, (m_t + 1)N - 1] \) in the second line of (28), given a specified \( m_t \). It follows that (28) is equal to \( N^{-n+2} \).

To sum up, when \( j = n - 1 \), the contributions of an element \( x_j \in \mathcal{X}_j(b_1, b_2, \ldots, b_{n-j+1}) \) to (20) are \( O(N^{-n+1}) \), \( O(N^{-n+1}) \), and \( N^{-n+2} \) for cases 1) – 3), respectively. It is sufficient to consider case 3), which results in a highest order of \( N \). That is, within each of the two cycles in \( K \)-graph, two edge variables touching the same vertex take the same value.

Any \( K \)-graph resulting from a noncrossing partition of \( K \) can be obtained by successively merging two vertices of a cycle. Specifically, there are \( n - j + 1 \) cycles in a \( j \)-vertex \( K \)-graph. This \( K \)-graph is obtained through \( n - j \) iterations of vertex mergence. At the \( r \)-th iteration (\( 1 \leq r \leq n - j \)), two vertices of any of \( r \) cycles are merged to yield in total \( r + 1 \) cycles. Within each of the two newly formed cycles, the edge variables touching the same vertex are assigned with the same value. The above observation leads to the following lemma.

**Lemma 1:** Given natural numbers \( (b_1, b_2, \ldots, b_{n-j+1}) \) such that \( b_1 + b_2 + \cdots + b_{n-j+1} = n \). In considering the contribution of an element \( x_j \in \mathcal{X}_j(b_1, b_2, \ldots, b_{n-j+1}) \) to the expression of (20), it is sufficient to consider the condition that, within each of the \( n - j + 1 \) cycles of \( x_j \)'s associated \( K \)-graph, two edge variables touching the same vertex take the same value. The contribution of \( x_j \) to (20) is given by

\[
N^{-n} \prod_{r=1}^{n-j+1} N = N^{-j+1},
\]

(29)

where \( \prod_{r=1}^{n-j+1} N \) means that the contribution of each cycle is \( N \), and the total contribution of the \( n - j + 1 \) cycles is obtained by multiplying the contribution of every singly cycle together.
From (29), each \( x_j \in \mathcal{X}_j(b_1, b_2, \ldots, b_{n-j+1}) \) results in the same contribution. Thus, the total contribution of \( \mathcal{X}_j(b_1, b_2, \ldots, b_{n-j+1}) \) to (20) is given by

\[
\#\mathcal{X}_j(b_1, b_2, \ldots, b_{n-j+1}) \cdot N^{-j+1}.
\]

**Theorem 1:** When \( M, K, N \to \infty \) with \( K/N \to \beta \), the \( n \)-th order AEM of \( R_{cs} \) is given by

\[
\mu(R_{cs}^n) = \frac{1}{n^\beta} \sum_{j=1}^{n} \binom{n}{j} \binom{n}{j-1} \beta^{j-1}.
\]

**Proof:**

We use the equivalence relation of (25). As shown in Lemma 1, \( \sum_{K \in \mathcal{X}_j(b_1, b_2, \ldots, b_{n-j+1})} \) leads to a contribution of \( \#\mathcal{X}_j(b_1, b_2, \ldots, b_{n-j+1}) \cdot N^{-j+1} \), which is equal to

\[
\frac{n(n-1) \cdots (j+1)}{f(b_1, b_2, \ldots, b_{n-j+1})} \prod_{r=0}^{j-1} (K-r) \cdot N^{-j+1}.
\]

To consider the contribution to \( \mu(R_{cs}^n) \) of noncrossing partitions of \( \mathcal{K} \), we take the other two summations \( \sum_{j=1}^{n} \sum_{b_1+b_2+\cdots+b_{n-j+1}=n, b_1 \geq b_2 \geq \cdots \geq b_{n-j+1} \geq 1} \) of (25) and \( \lim_{K, N \to \infty} K^{-1} of (19) \) into account. We obtain

\[
\lim_{K, N \to \infty} K^{-1} \sum_{j=1}^{n} \sum_{b_1+b_2+\cdots+b_{n-j+1}=n, b_1 \geq b_2 \geq \cdots \geq b_{n-j+1} \geq 1} \frac{n(n-1) \cdots (j+1)}{f(b_1, b_2, \ldots, b_{n-j+1})} \prod_{r=0}^{j-1} (K-r) \cdot N^{-j+1}
\]

\[
= \frac{1}{n} \sum_{j=1}^{n} \binom{n}{j} \binom{n}{j-1} \beta^{j-1},
\]

where the equality is due to

\[
\sum_{b_1+b_2+\cdots+b_{n-j+1}=n, b_1 \geq b_2 \geq \cdots \geq b_{n-j+1} \geq 1} \frac{n(n-1) \cdots (j+1)}{f(b_1, b_2, \ldots, b_{n-j+1})} = \frac{1}{n} \binom{n}{j} \binom{n}{j-1}.
\]

Note that the right-hand-side of (32) is the Narayana number denoting the number of noncrossing partitions of \( n \) elements into \( j \) (or \( n-j+1 \)) classes, while the summand in the left-hand-side is the number of noncrossing partitions of \( n \) elements into \( n-j+1 \) classes with sizes \( b_1, b_2, \ldots, b_{n-j+1} \).

Next, we consider the contribution of crossing partitions of \( \mathcal{K} \), i.e. \( \sum_{K \in \mathcal{X}_{cv}} \) of (25). Let \( G_j \) be a \( K \)-graph resulting from a crossing partition of \( \mathcal{K} \) into \( j \) classes. It can be seen that the number of cycles that \( G_j \) can be maximally decomposed into is less than \( n-j+1 \), i.e. at most \( n-j \). For example, the maximum number of cycles that Fig. 1(c) can be decomposed is
Thus, we have proven \(\lim_{K,N \to \infty, K/N \to \beta} K^{-1} \) of (19), we see that the contribution is zero in the large-system regime. Thus, we have proven \(\mu(R_{cs}^n)\) is equal to (31).

It is seen that (30) is the \(n\)-th order moment of Marčenko-Pastur distribution. As it has been shown in [21] that the moment sequence of (30) satisfies the Carleman’s criterion [6] such that a distribution can be uniquely determined, Theorem 1 establishes that, as \(K, N, M \to \infty\) with \(K/N \to \beta\), the spectral distribution of \(R_{cs}\) converges to Marčenko-Pastur law with ratio index \(\beta\), which is identical to that of symbol-synchronous CDMA.

**Corollary 1:** When \(M, K, N \to \infty\) with \(K/N \to \beta\), the \(n\)-th order AEM of \(A^\dagger R_{cs} A\), with \(\dagger\) denoting Hermitian transpose, is given by

\[
\mu((A^\dagger R_{cs} A)^n) = \sum_{j=1}^{n} \beta^{j-1} \sum_{c_1+c_2+\cdots+c_j=n \atop c_1 \geq c_2 \geq \cdots \geq c_j \geq 1} \frac{n(n-1)\cdots(n-j+2)}{f(c_1, c_2, \ldots, c_j)} \prod_{r=1}^{j} F^{(c_r)},
\]

where \(F^{(r)}\) is the \(r\)-th order moment of the random variable whose distribution is the empirical distribution of the squared magnitude of the complex amplitude \(|A_k(m)|^2\).

**Proof:** Let \(F = AA^\dagger\) and \(F_k(m) = |A_k(m)|^2\). We have

\[
\text{tr}\{[(A^\dagger R_{cs} A)^n]_{00}\} = \text{tr}\{[(R_{cs} F)^n]_{00}\}.
\]

Since \(F\) is diagonal, we have

\[
\lim_{M \to \infty} E\{\text{tr}\{[(R_{cs} F)^n]_{00}\}\} = \lim_{M \to \infty} \sum_{j=1}^{n} \sum_{b_1+b_2+\cdots+b_{n-j+1}=n} \sum_{K \in \mathcal{X}_j(b_1, b_2, \ldots, b_{n-j+1})} \sum_{M \in \mathcal{Y} \atop b_1 \geq b_2 \geq \cdots \geq b_{n-j+1} \geq 1} \sum_{M \in \mathcal{Y} \atop b_1 \geq b_2 \geq \cdots \geq b_{n-j+1} \geq 1} \sum_{P_1 \in \mathcal{Z}_1} \sum_{P_n \in \mathcal{Z}_n} \cdots \sum_{P_n \in \mathcal{Z}_n} \mathbb{E}\{c_{k_1}^{(p_1)}(q_1) c_{k_2}^{(p_2)}(q_3) \cdots c_{k_n}^{(p_n)}(q_1)\} \mathbb{E}\{F_{k_1}(m_1)F_{k_2}(m_2)\cdots F_{k_n}(m_n)\} \times \delta(p_1T_c + \tau_{k_1}, q_2T_c + \tau_{k_2}) \delta(p_2T_c + \tau_{k_2}, q_3T_c + \tau_{k_3}) \cdots \delta(p_nT_c + \tau_{k_n}, q_1T_c + \tau_{k_1})|_{m_1=0},
\]

with \(|m_j - m_{j+1}| \leq 1\).

We consider the contribution of an element \(x_j \in \mathcal{X}_j(b_1, b_2, \ldots, b_{n-j+1})\), which has a \(K\)-graph composed of \(n-j+1\) concatenated cycles with edge numbers \(b_1, b_2, \ldots, b_{n-j+1}\). Suppose that this \(K\)-graph is yielded by a \(j\)-class noncrossing partition of \(\mathcal{K}\) with sizes of classes \((c_1, c_2, \ldots, c_j)\).
Then, for this $x_j$, $\mathbb{E}\{F_{k_1}(m_1)F_{k_2}(m_2)\cdots F_{k_n}(m_n)\}$ in (34) becomes $\prod_{r=1}^{j} F^{(c_r)}$, where note that if $k_s$ and $k_t$ are partitioned in the same class, $m_s$ is equal to $m_t$.

As shown in (24), there are

$$\frac{n(n-j)!(j-1)!}{f(b_1, b_2, \cdots, b_{n-j+1})f(c_1, c_2, \cdots, c_j)}$$

numbers of such $x_j$. Thus, (34) can be expressed as

$$N^{-n} \sum_{j=1}^{n} \sum_{b_1+b_2+\cdots+b_{n-j+1}=n} \sum_{b_1\geq b_2\geq\cdots\geq b_{n-j+1}\geq 1} \sum_{c_1+c_2+\cdots+c_j=n} \frac{n(n-j)!(j-1)!}{f(b_1, b_2, \cdots, b_{n-j+1})f(c_1, c_2, \cdots, c_j)} \times N^{-n-j+1} \prod_{s=0}^{j-1} (K-s) \prod_{r=1}^{j} F^{(c_r)}$$

(35)

Multiplying (35) by $K^{-1}$ and taking the large-system limit, we obtain

$$\sum_{j=1}^{n} \beta^{j-1} \sum_{c_1+c_2+\cdots+c_j=n} \frac{n(n-1)\cdots(n-j+2)}{f(c_1, c_2, \cdots, c_j)} \prod_{r=1}^{j} F^{(c_r)}$$

where we make use of the equality

$$\sum_{b_1+b_2+\cdots+b_{n-j+1}=n} \frac{n(n-j)!(j-1)!}{f(b_1, b_2, \cdots, b_{n-j+1})} = n(n-1)\cdots(n-j+2).$$

Theorem 2: The empirical spectral distributions $\{F^{(K)}(x) : K = 1, 2, \cdots\}$ converge to a nonrandom limit almost surely when $N, K \to \infty$ with $K/N \to \beta$.

Proof: To prove this theorem, we require the three criterions listed in Subsection II-C. Criterion 1) has been established in Theorem 1, and criterion 2) is also ready. What we need to do here is to prove criterion 3). The proof is given in Appendix I.

IV. ASYMPTOTIC SPECTRAL DISTRIBUTION OF CHIP-ASYNCHRONOUS CDMA

In computing AEM of $R_{ca}$, the asynchronous delays $\tau_k$’s are regarded as either deterministic constants or random variables depending on the bandwidth of chip waveform $\psi(t)$. In specific,
when $ψ(t)$ has bandwidth smaller than $1/(2T_c)$, $τ_k$’s are treated as deterministic. When it is not, $τ_k$’s are taken as i.i.d. random variables uniformly distributed in $[0, NT_c)$. The reason of doing so is due to the property of $ψ(t)$ that will be presented in Lemmas 2 and 3 below. By similar procedures of reaching (20), we have

$$
\lim_{M \to \infty} E \{ \text{tr}(R^n_{\text{ca}}) \}
= \lim_{M \to \infty} \sum_{K \in X} \sum_{M \in Y} \sum_{n} \sum_{k} \sum_{j} \sum_{p} R_{\psi}(p_1 - q_1)T_c + (τ_{k_1} - τ_{k_2})R_{\psi}(p_2 - q_2)T_c + (τ_{k_2} - τ_{k_3})\cdots \times R_{\psi}(p_{m} - q_1)T_c + (τ_{k_m} - τ_{k_1}) \}
$$

where the expectation of the product of $R_{\psi}(\cdot)$’s is with respect to $τ_k$’s. When the bandwidth of $ψ(t)$ is smaller than $1/(2T_c)$, $τ_k$’s are regarded as deterministic, and the operator of expectation can be taken away.

A. Chip Waveform Property

Some properties about the chip waveform $ψ(t)$ are presented before proceeding.

**Lemma 2:** Denote the Fourier transform of $ψ(t)$ by $Ψ(Ω)$. Suppose that $Ψ(Ω) = 0$ for $Ω > π/T_c$. For any $m \in \mathbb{N}$, $n_0 \in \mathbb{Z}$ and all $\{η_j\}_{j=0}^{m-1} \in \mathbb{R}^m$, we have

$$
\sum_{n_1=-\infty}^{\infty} \sum_{n_2=-\infty}^{\infty} \cdots \sum_{n_{m-1}=-\infty}^{\infty} R_{\psi}(n_0 - n_1)T_c + η_0 - η_1)R_{\psi}(n_1 - n_2)T_c + η_1 - η_2)\cdots \times R_{\psi}(n_{m-1} - n_0)T_c + η_{m-1} - η_0)
= \frac{1}{2πT_c^{m-1}} \int_{-π/T_c}^{π/T_c} Ψ^{2m}(Ω) \, dΩ \triangleq \mathcal{W}_{ψ}^{(m)}.
$$

**Proof:** See Appendix II.

**Lemma 3:** Suppose that the bandwidth of $ψ(t)$ is larger than $1/(2T_c)$, i.e. $Ψ(Ω) \neq 0$ for some $Ω > π/T_c$. For any $m \in \mathbb{N}$, $n_0 \in \mathbb{Z}$ and all i.i.d. random variables $\{η_j\}_{j=0}^{m-1}$ uniformly distributed
in \((0, rT_c)\) with \(r \in \mathbb{N}\), we have
\[
\begin{align*}
\sum_{n_1=-\infty}^{\infty} \sum_{n_2=-\infty}^{\infty} \cdots \sum_{n_m=-\infty}^{\infty} & \mathbb{E} \left\{ R_\psi((n_0 - n_1)T_c + \eta_0 - \eta_1) R_\psi((n_1 - n_2)T_c + \eta_1 - \eta_2) \cdots \\ & \times R_\psi((n_{m-1} - n_0)T_c + \eta_{m-1} - \eta_0) \right\} \\
= & \frac{1}{2\pi T_c^{m-1}} \int_{-\infty}^{\infty} \Psi^{2m} (\Omega) \, d\Omega \triangleq \mathcal{W}_\psi^{(m)}.
\end{align*}
\]
(39)

**Proof:** See Appendix III.

The above two lemmas say that if the arguments \(\alpha_j\)’s of the chip waveform autocorrelation function \(R_\psi\)’s forms a cycle as \((\alpha_0 - \alpha_1), (\alpha_1 - \alpha_2), \cdots, (\alpha_{m-1} - \alpha_0)\), the \((m-1)\)-dimensional infinite sum of products of \(R_\psi\)’s is equal to a constant which is a function of \(\psi(t)\)’s Fourier transform.

**B. Computation of AEM**

**Lemma 4:** Given natural numbers \((b_1, b_2, \cdots, b_{n-j+1})\) such that \(b_1 + b_2 + \cdots + b_{n-j+1} = n\). In considering the contribution of an element \(x_j \in \mathcal{X}_j(b_1, b_2, \cdots, b_{n-j+1})\) to the expression of (36), it is sufficient to consider the condition that, within each of the \(n-j+1\) cycles of \(x_j\)’s associated \(K\)-graph, two edge variables touching the same vertex take the same value. The contribution of \(x_j\) to (36) is given by
\[
N^{-n} \prod_{r=1}^{n-j+1} \mathcal{N}_\psi^{(b_r)} = N^{-j+1} \prod_{r=1}^{n-j+1} \mathcal{W}_\psi^{(b_r)},
\]
(41)

where \(\prod_{r=1}^{n-j+1} \mathcal{N}_\psi^{(b_r)}\) means that the contribution of a \(b_r\)-edge cycle is \(\mathcal{N}_\psi^{(b_r)}\), and the total contribution of the \(n-j+1\) cycles is obtained by multiplying the contribution of each cycle together. From (41), each \(x_j \in \mathcal{X}_j(b_1, b_2, \cdots, b_{n-j+1})\) has the same contribution. Thus, the contribution of \(\mathcal{X}_j(b_1, b_2, \cdots, b_{n-j+1})\) to (36) is given by
\[
\#\mathcal{X}_j(b_1, b_2, \cdots, b_{n-j+1}) \cdot N^{-j+1} \prod_{r=1}^{n-j+1} \mathcal{W}_\psi^{(b_r)}.
\]
(42)

**Proof:** See Appendix IV.

**Theorem 3:** When \(M, K, N \to \infty\) with \(K/N \to \beta\), the \(n\)-th order AEM of \(R_{ca}\) is given by
\[
\mu (R_{ca}^n) = \sum_{j=1}^{n} \beta^{j-1} \sum_{b_1 + b_2 + \cdots + b_{n-j+1} = n} \frac{n(n-1) \cdots (j+1)}{f(b_1, b_2, \cdots, b_{n-j+1})} \prod_{r=1}^{n-j+1} \mathcal{W}_\psi^{(b_r)}.
\]
(43)
Proof:

By Lemma 4, $\mu(R_{ca}^n)$ is given by

$$\lim_{K,N \to \infty} K^{-1} \sum_{K/N \to \beta} \sum_{j=1}^{n} \sum_{b_1+b_2+\cdots+b_{n-j+1}=n} \#X_j(b_1, b_2, \cdots, b_{n-j+1}) \cdot N^{-j+1} \prod_{r=1}^{n-j+1} W_{\psi}^{(b_r)},$$

which is equal to (43).

When $\psi(t)$ has bandwidth larger than $1/(2T_c)$, the AEM formulas are obtained via the chip waveform property presented in Lemma 3. Since Lemma 3 holds when $\eta_j$’s are uniformly distributed in $(0, rT_c)$ for any $r \in \mathbb{N}$, the AEM formulas are applicable to not only chip-asynchronous CDMA. For example, the symbol quasi-synchronous but chip-asynchronous system discussed in [17] has the same AEM.

Corollary 2: When the ideal Nyquist sinc chip waveform

$$\psi^*(t) = \frac{1}{\sqrt{T_c}} \text{sinc} \left( \frac{t}{T_c} \right)$$

is used, $\mu(R_{ca}^n) = \mu(R_{cs}^n)$ for all $n \in \mathbb{N}$.

Proof: The Fourier transform of $\psi^*(t)$ is

$$\Psi^*(\Omega) = \sqrt{T_c} \text{rect} \left( \frac{T_c\Omega}{2\pi} \right).$$

By Lemma 2, $W_{\psi^*}^{(m)} = 1$ for all $m \in \mathbb{N}$. Thus, by the equality of (32), $\mu(R_{ca}^n)$ in (43) becomes $\mu(R_{cs}^n)$ in (30).

It is demonstrated in [15] that, if the ideal sinc chip waveform is used in a chip-asynchronous system and the observation window size tends to infinity, the performance of the linear MMSE receiver is the equivalent to those of infinite-window-size chip-synchronous CDMA and one-shot symbol-synchronous CDMA. Corollary 2 gives an explanation for the above equivalence result. It is shown in [18] that a linear MMSE receiver belongs to the family of linear receivers that can be arbitrarily well approximated by polynomials receivers formed by a crosscorrelation matrix\(^3\). As AEM are equivalent in systems of three synchronism levels under the indicated circumstances, both the coefficients of the three polynomial receivers approximating linear MMSE receivers

---

\(^3\)Although it is presented for symbol-synchronous CDMA, the proof (Lemma 5 of [18]) can be extended to asynchronous systems straightforwardly.
and their output signal-to-interference-plus-noise ratios (SINR) are identical. It is readily seen as well that the equivalence result is true not only for a linear MMSE receiver but also for the family of linear receivers defined in [18].

**Corollary 3:** When \( M, K, N \to \infty \) with \( K/N \to \beta \), the \( n \)-th order AEM of \( \text{\textit{A}} \text{\textdagger} R_{\text{ca}} \text{\textit{A}} \) is

\[
\mu((\text{\textit{A}} \text{\textdagger} R_{\text{ca}} \text{\textit{A}})^n) = \sum_{j=1}^{n} \beta^{j-1} \sum_{\begin{subarray}{c} b_1+b_2+\cdots+b_{n-j+1}=n \\ c_1+c_2+\cdots+c_j=n \\ b_1 \geq b_2 \geq \cdots \geq b_{n-j+1} \\ c_1 \geq c_2 \geq \cdots \geq c_j \end{subarray}} \frac{n(n-j)!(j-1)!}{f(b_1, b_2, \cdots, b_{n-j+1})f(c_1, c_2, \cdots, c_j)} \prod_{t=1}^{n-j+1} W_{\psi}^{(b_t)} \prod_{r=1}^{j} \mathcal{F}(c_r). \tag{44}
\]

**Proof:**

This proof basically follows the line of that of Corollary 1 with (35) being revised as

\[
N^{-n} \sum_{j=1}^{n} \frac{n(n-j)!(j-1)!}{f(b_1, b_2, \cdots, b_{n-j+1})f(c_1, c_2, \cdots, c_j)} \prod_{t=1}^{n-j+1} N W_{\psi}^{(b_t)} \prod_{s=0}^{j-1} (K-s) \prod_{r=1}^{j} \mathcal{F}(c_r),
\]

which is equal to (44) when we multiply it by \( K^{-1} \) and take the large-system limit.

**Theorem 4:** The empirical spectral distributions \( \{ \mathcal{F}(K)(x) : K = 1, 2, \cdots \} \) converge to a nonrandom limit almost surely when \( N, K \to \infty \) with \( K/N \to \beta \), if \( \sum_{n=1}^{\infty} (W_{\psi}^{2n})^{-1/2n} = \infty \).

**Proof:** We have proven in Theorem 3 that \( \mu(R_{\text{ca}}^{n}) \) exists for all \( n \in \mathbb{N} \). Also, as shown in [22], to show Carleman’s condition that \( \sum_{n=1}^{\infty} \mu(R_{\text{ca}}^{2n})^{-1/2n} = \infty \), it is sufficient to demonstrate \( \sum_{n=1}^{\infty} (W_{\psi}^{2n})^{-1/2n} = \infty \). At last, the proof of criterion 3) in Subsection II-C can be simply revised from Appendix I. Thus, we have completed the proof.

**V. ASYMPTOTIC FREEDOM**

**Definition 2:** [26] The Hermitian random matrices \( \textit{B} \) and \( \text{\textit{C}} \) are asymptotically free if, for all polynomials \( p_j(\cdot) \) and \( q_j(\cdot) \), \( 1 \leq j \leq n \), such that

\[
\mu(p_j(\textit{B})) = \mu(q_j(\text{\textit{C}})) = 0,
\]

we have

\[
\mu(p_1(\textit{B})q_1(\text{\textit{C}}) \cdots p_n(\textit{B})q_n(\text{\textit{C}})) = 0.
\]
Asymptotic freeness defined by Voiculescu [26] is related to the spectra of algebra of random matrices $B$ and $C$ when their sizes tend to infinity. However, in our context, the random matrices $R_{cs}$ and $R_{ca}$ have column and row sizes $(2M + 1)K$ controlled by two parameters $M$ and $K$. We let both $M$ and $K$ approach infinity.

**Theorem 5:** Suppose that $D(m) = \text{diag}\{d_1(m), d_2(m), \cdots, d_K(m)\}$ and $D = \text{diag}\{D(-M), D(-M + 1), \cdots, D(M)\}$, where $d_k(m)$’s are random variables having bounded moments, and $d_k(m_1)$ and $d_l(m_2)$ are independent if $k \neq l$. Also, $R_x, x \in \{cs, ca\}$, and $D$ are independent. Then $R_x$ and $D$ are asymptotically free as $M, K, N \to \infty$ with $K/N \to \beta$.

**Proof:** See Appendix V.

Before we proceed, some results of free probability about random matrices (see, for example, [27]) are summarized in the following theorem.

**Theorem 6:** [27] Let $B$ and $C$ be asymptotically free random matrices. The $n$-th order AEM of the sum $B + C$ and product $BC$ are given by

$$
\mu((B + C)^n) = \sum_{\omega} \prod_{V \in \omega} (c_{|V|}(B) + c_{|V|}(C)),
$$

and

$$
\mu((BC)^n) = \sum_{\omega} \prod_{V \in \omega} c_{|V|}(B) \prod_{U \in KC(\omega)} \mu(C^{[U]}),
$$

where each summation is over all noncrossing partitions $\omega$ of an $n$-element set, $V \in \omega$ means $V$ is a class of $\omega$, $|V|$ denotes the cardinality of $V$, $c_k(B)$ is the $k$-th order free cumulant of $B$, and $KC(\omega)$ is the Kreweras complementation map of $\omega$. Moreover, the relations between the moment and free cumulant sequences are

$$
\mu(B^n) = \sum_{\omega} \prod_{V \in \omega} c_{|V|}(B),
$$

$$
c_n(B) = \sum_{\omega} \prod_{V \in \omega} \mu(B^{[V]}) \prod_{U \in KC(\omega)} S_{|U|},
$$

where $S_k$ is given by

$$
S_k = (-1)^{k-1} \frac{1}{k} \binom{2k-2}{k-1}.
$$

With the aid of Theorem 6, we consider free cumulants of $R_{cs}$ and $R_{ca}$. Rewrite (43) as

$$
\mu(R_{ca}^n) = \sum_{j=1}^{n} \sum_{b_1 + b_2 + \cdots + b_j = n} \frac{n(n-1) \cdots (n - j + 2)}{f(b_1, b_2, \cdots, b_j)} \prod_{r=1}^{j} \mathcal{W}_\psi^{(b_r)} \beta_{b_r-1}. \tag{47}
$$
Let us interpret the summation variable $j$ in (47) as the class number of a noncrossing partition $\varpi$ of an $n$-element set, and $b_r$ as the size of the $r$-th class of $\varpi$. From (45), it is readily seen that the $k$-th order free cumulant of $R_{ca}$ is

$$c_k(R_{ca}) = \mathcal{W}_\psi^{(k)} \beta^{k-1}.$$ 

Similarly, we obtain the $k$-th order free cumulant of $R_{cs}$ as

$$c_k(R_{cs}) = \beta^{k-1}.$$ 

On the other hand, the $k$-th order free cumulant of the diagonal matrix $D$ defined in Theorem 5 can be acquired from (46) as

$$c_k(D) = \sum_{j=1}^{k} \sum_{b_1+b_2+\cdots+b_j=n} \sum_{b_1 \geq b_2 \geq \cdots \geq b_j \geq 1} \frac{k(k-j)! (j-1)!}{f(b_1, b_2, \ldots, b_j) f(c_1, c_2, \ldots, c_{k-j+1})} \prod_{r=1}^{j} \mu(D^{b_r}) \prod_{t=1}^{k-j+1} \mathcal{S}_{c_t}.$$ 

Thus, employing Theorem 6, we have the $n$-th order AEM of $R_{ca} + D$ and $R_{ca}D$ given as

$$\mu((R_{ca} + D)^n) = \sum_{j=1}^{n} \sum_{b_1+b_2+\cdots+b_j=n} \frac{n(n-1) \cdots (n-j+2)}{f(b_1, b_2, \ldots, b_j) f(c_1, c_2, \ldots, c_{n-j+1})} \prod_{r=1}^{j} \left( \mathcal{W}_\psi^{(b_r)} \beta^{b_r-1} + c_{b_r}(D) \right),$$

and

$$\mu((R_{ca}D)^n) = \sum_{j=1}^{n} \sum_{b_1+b_2+\cdots+b_j=n} \frac{n(n-1)! (j-1)!}{f(b_1, b_2, \ldots, b_j) f(c_1, c_2, \ldots, c_{n-j+1})} \prod_{r=1}^{j} \mathcal{W}_\psi^{(b_r)} \beta^{b_r-1} \prod_{t=1}^{n-j+1} \mu(D^{c_t}).$$

By setting the diagonal random matrix $D$ as $AA^\dagger$, we have $\mu(D^k) = \mathcal{F}^{(k)}$. In this way, (50) becomes (44).

Similarly, $\mu((R_{cs} + D)^n)$ and $\mu((R_{cs}D)^n)$ can be obtained from (49) and (50), respectively, by setting all $\mathcal{W}_\psi^{(k)}$’s equal to one. In this way, $\mu((R_{cs}D)^n)$ has a simpler form of

$$\mu((R_{cs}D)^n) = \sum_{j=1}^{n} \beta^{n-j} \sum_{c_1+c_2+\cdots+c_{n-j+1}=n} \frac{n(n-1) \cdots (j+1)}{f(c_1, c_2, \ldots, c_{n-j+1})} \prod_{r=1}^{n-j+1} \mu(D^{c_r}),$$

which is equal to (33) by letting $\mu(D^{c_r})$ equal to $\mathcal{F}^{(c_r)}$. 

October 14, 2008
VI. OTHER EXPRESSIONS OF CROSSCORRELATION MATRICES

The crosscorrelation matrices $R_{cs}$ and $R_{ca}$ can be decomposed into matrices products. Some remarks are made by means of the new expressions. To decompose $R_{cs}$ and $R_{ca}$, we have the following notation definitions:

$$C_k(m) = [c_k^{(mN)}, c_k^{(mN+1)}, \ldots, c_k^{((m+1)N-1)}]^T, \quad (51)$$

$$\mathcal{S}(m) = \text{diag}\{c_1(m), c_2(m), \ldots, c_K(m)\}, \quad (52)$$

$$\mathcal{S} = \text{diag}\{\mathcal{S}(-M), \mathcal{S}(-M+1), \ldots, \mathcal{S}(M)\}. \quad (53)$$

Moreover, define a $(2M+1)KN \times (2M+1)KN$ matrix $\Delta$ with the $(mKN + kN + p, nKN + lN + q)$-th element, $-M \leq m, n \leq M, 0 \leq k, l \leq K - 1, 0 \leq p, q \leq N - 1$, equal to

$$\delta((mN + p)T_c + \tau_{k+1}, (nN + q)T_c + \tau_{l+1}).$$

By means of (7), it can be shown that $R_{cs} = S^T \Delta S$.

Similarly, we define a $(2M+1)KN \times (2M+1)KN$ matrix $\Xi$ whose $(mKN + kN + p, nKN + lN + q)$-th element, $-M \leq m, n \leq M, 0 \leq k, l \leq K - 1, 0 \leq p, q \leq N - 1$, is equal to

$$R_{\psi}([[m-n]N + (p-q)]T_c + \tau_{k+1} - \tau_{l+1}).$$

With notations defined in (51) – (53), $R_{ca}$ can be decomposed as $R_{ca} = S^T \Xi S$.

Our AEM formulas for asynchronous CDMA resemble results of [22] which find application in symbol-synchronous CDMA. Consider a synchronous CDMA system. Define $S = [\xi_1, \xi_2, \ldots, \xi_K]$ with $\xi_k$ being the $N \times 1$ random spreading sequence vector of user $k$ at a specific symbol. Matrix $S$ can be seen as the counterpart of $S$ in a synchronous system. Let $T$ be a $N \times N$ symmetric random matrix independent of $S$ with compactly supported asymptotic averaged empirical eigenvalue distribution. The $n$-th order AEM of $SS^TT$ is computed in [22], given by

$$\mu((SS^TT)^n) = \sum_{j=1}^{n} \beta^j \sum_{b_1 + b_2 + \cdots + b_{n-j+1} = n} \frac{n(n-1) \cdots (j+1)}{f(b_1, b_2, \ldots, b_{n-j+1})} \prod_{r=1}^{n-j+1} \mu(T^{b_r}). \quad (54)$$

To build the relationship for results of this paper and [22], we note that $\text{tr}\{R_{ca}^n\} = \text{tr}\{(SS^T\Delta)^n\}$ and $\text{tr}\{R_{cs}^n\} = \text{tr}\{(SS^T\Xi)^n\}$. The formula of (54) shows remarkable similarity to $\mu(R_{ca}^n)$ given in (43), and $\mu(R_{cs}^n)$ can be obtained by setting $W^{(k)}_{\psi} = 1$ in (43) for all $k \in \mathbb{N}$. 
Another expression of $R_{cs}$ is given below. Define a $(2M + 2)N$-dimensional vector $\vec{c}_k(m)$ as

$$\vec{c}_k(m) = \left[ \Omega_{(M+m)+\tau_k/T_c}, \mathbf{L}^T(m), \Omega_{(M+1-m)+\tau_k/T_c} \right]^T,$$

where $\Omega_j = 0, 0, \ldots, 0$. Also, a $(2M + 2)N \times K$ matrix $\tilde{S}(m)$, given by

$$\tilde{S}(m) = [\tilde{c}_1(m), \tilde{c}_2(m), \ldots, \tilde{c}_K(m)],$$

and

$$\tilde{S} = [\tilde{S}(-M), \tilde{S}(-M+1), \ldots, \tilde{S}(M)]. \quad (55)$$

With these settings, we have $R_{cs} = \tilde{S}^T \tilde{S}$ and $[R_{cs}]_{mn} = \tilde{S}^T(m) \tilde{S}(n)$.

It is known that the ASD of the crosscorrelation matrix in symbol-synchronous CDMA converges almost surely to a Marčenko-Pastur law. Let us particularly use $\check{S}$ to denote the matrix $\tilde{S}$ in (55) when $\tau_k = 0$ for all $k$’s. Clearly, $\check{S}_*$ is a block diagonal matrix, and the ASD of $\check{S}_*^T \check{S}_*$ converges to the Marčenko-Pastur law with ratio index $\beta$ as $K, N \to \infty$ and $K/N \to \beta$. The equivalence of $\mu((\check{S}^T \check{S})^n) = \mu(R_{cs}^n)$ reveals that, if we let $M \to \infty$, for any arbitrary $\{\tau_k\}_{k=1}^K \in \{\gamma T_c : 0 \leq \gamma \leq N - 1\}^K$, $\check{S}^T \check{S}$ converges to the same ASD. Moreover, $\mu((\check{S}A^A^\dagger \check{S})^n) = \beta \cdot \mu((A^\dagger R_{cs}A)^n)$ can be written as

$$\sum_{j=1}^{n} \sum_{c_1+c_2+\ldots+c_j=n} \frac{n(n-1)\ldots(n-j+2)}{f(c_1, c_2, \ldots, c_j)} \prod_{r=1}^{j} F(c_r) \beta.$$

By the moment-cumulant formula of (45), the $k$-th order free cumulant of $\check{S}AA^A^\dagger \check{S}^T$ is $F^{(k)} \beta$, which is the same as that of its counterpart in symbol-synchronous CDMA, i.e. $\check{S}_* AA^A_0^\dagger \check{S}_*^T$, derived in [11].

VII. CONCLUSION

In this paper, the ASD of crosscorrelation matrices in chip-synchronous and chip-asynchronous CDMA systems are investigated with a particular emphasis on the derivation of AEM. Our results are obtained based on the setting of an infinite input symbol length, random spreading sequences and asynchronous delays known to the receiver, and an arbitrary chip waveform satisfying a general constraint of zero inter-chip interference. Since AEM of a random matrix are related the expected matrix trace, AEM are expressed as a sum of products of functions parameterized by spreading sequences, chip waveform and asynchronous delays. Noncrossing
partition of set partition theory is used to identify the nonvanishing terms in the sum of products in the large-system regime. A $K$-graph is defined to facilitate computing the contribution of each nonvanishing term to AEM. A convergence test built on Borell-Cantelli lemma is also employed to show that ESD converges almost surely to a nonrandom ASD in the large-system limit.

In chip-synchronous CDMA, it is shown that AEM are not relevant to chip waveform and any particular realization of asynchronous delays, and they have the same deterministic values as those of a symbol-synchronous CDMA system. Moreover, the ASD of the crosscorrelation matrix in chip-synchronous CDMA follows the Marčenko-Pastur law with ratio index $\beta$, which is identical to that of a symbol-synchronous CDMA system.

For chip-asynchronous CDMA, it is shown that, unlike chip-synchronous CDMA, the AEM are relevant to the chip waveform $\psi(t)$ through $\{W_\psi^{(m)} : 1 \leq m \leq n\}$ with $n$ equal to the moment order. When the bandwidth of $\psi(t)$ is wider (or narrower) than $1/(2T_c)$, the AEM are relevant (or irrelevant) to realizations of asynchronous delays $\tau_k$'s. In the relevance case, when AEM are averaged over all realizations of $\tau_k$’s with uniform distributions, the averaged AEM yield the same values as those of the irrelevance case. In particular, if the ideal Nyquist sinc function is employed as the chip waveform, AEM of chip-asynchronous CDMA are equal to those of chip-synchronous and symbol-synchronous CDMA, which enlightens the equivalence result presented by [15] that the output SINR of the linear MMSE receiver converges to that of an equivalent chip-synchronous and symbol-synchronous system under an infinite observation window width and confirms its conjecture that the equivalence result holds for the family of receivers considered in [18].

Asymptotic freeness relation of asynchronous crosscorrelation matrices and a random diagonal matrix is also established. Based on this property, spectra of sum and product of crosscorrelation matrices and a random diagonal matrix are derived accordingly. At last, alternative expressions of crosscorrelation matrices are given to allow more remarks.

**APPENDIX I**

**PART OF PROOF OF THEOREM 2: CRITERION 3) IN SUBSECTION II-C**

Define a normalized trace operator

$$\text{Tr}(R_{\text{cs}}^n) = \lim_{M \to \infty} (2M + 1)^{-1} K^{-1} \text{tr}((R_{\text{cs}}^{(K)})^n) = \lim_{M \to \infty} K^{-1} \text{tr}([([R_{\text{cs}}^{(K)})^n]_{00})$$
and its expectation \( \varphi(\mathbf{R}_c^n) = \mathbb{E}\{\text{Tr}(\mathbf{R}_c^n)\} \). We are going to show that

\[
\sum_{K=1}^{\infty} \mathbb{E}\{[\text{Tr}(\mathbf{R}_c^n) - \varphi(\mathbf{R}_c^n)]^2\} < \infty
\]

(56)

for all \( n \in \mathbb{N} \).

We have

\[
\mathbb{E}\{[\text{Tr}(\mathbf{R}_c^n) - \varphi(\mathbf{R}_c^n)]^2\} = \mathbb{E}\{[\text{Tr}(\mathbf{R}_c^n)]^2\} - [\varphi(\mathbf{R}_c^n)]^2
\]

\[
= \lim_{M \to \infty} K^{-2} \sum_{Q} Q(m_2, \ldots, m_n, m_{n+2}, \ldots, m_{2n}; k_1, \ldots, k_n, k_{n+1}, \ldots, k_{2n}),
\]

(57)

where

\[
Q(m_2, \ldots, m_n, m_{n+2}, \ldots, m_{2n}; k_1, \ldots, k_n, k_{n+1}, \ldots, k_{2n})
\]

\[
= \mathbb{E}\{[\mathbf{R}_{c1}]_{m_1m_2k_1k_2}[\mathbf{R}_{c2}]_{m_2m_3k_2k_3} \cdots [\mathbf{R}_{cK}]_{m_{2n}m_{2n+1}k_{2n}k_1} \}
\]

\[
\times [\mathbf{R}_{c1}]_{m_1+1m_{n+2}k_{n+3}}[\mathbf{R}_{c2}]_{m_{n+2}m_{n+3}k_{n+4}}[\mathbf{R}_{c3}]_{m_{n+3}m_{n+4}k_{n+5}} \cdots [\mathbf{R}_{cK}]_{m_{2n+1}m_{2n+2}k_{2n}} \}
\]

\[
- \mathbb{E}\{[\mathbf{R}_{c1}]_{m_1m_2k_1k_2}[\mathbf{R}_{c2}]_{m_2m_3k_2k_3} \cdots [\mathbf{R}_{cK}]_{m_{2n}m_{2n+1}k_{2n}k_1} \}
\]

\[
\times [\mathbf{R}_{c1}]_{m_1+1m_{n+2}k_{n+3}}[\mathbf{R}_{c2}]_{m_{n+2}m_{n+3}k_{n+4}}[\mathbf{R}_{c3}]_{m_{n+3}m_{n+4}k_{n+5}} \cdots [\mathbf{R}_{cK}]_{m_{2n+1}m_{2n+2}k_{2n}k_1} \}
\]

\[
|_{m_1=0} \]

(58)

and the summation is over all \(-M \leq m_2, \ldots, m_n, m_{n+2}, \ldots, m_{2n} \leq M\) and \( 1 \leq k_1, \ldots, k_n, k_{n+1}, \ldots, k_{2n} \leq K \). Moreover, we have \(|m_j - m_{j+1}| \leq 1\) for \( \{m_j\}_j=1^n \), and so do \( \{m_j\}_j=1^{n+1} \).

We consider two \( n \)-element noncrossing partitions of \( \{k_1, \ldots, k_n\} \) and \( \{k_{n+1}, \ldots, k_{2n}\} \). Suppose that there are \( j \) and \( l \) classes in the former and latter sets, respectively. Assume \( j \) classes of noncrossing partitions of \( \{k_1, \ldots, k_n\} \) take distinct values \( \{u_1, \ldots, u_j\} \) in \([1, K]\), and they have sizes \( \{a_1, \ldots, a_j\} \), respectively. Similarly, \( \{k_{n+1}, \ldots, k_{2n}\} \) take values \( \{v_1, \ldots, v_l\} \), which have sizes \( \{b_1, \ldots, b_l\} \), respectively.

First, consider the case that \( \{u_1, \ldots, u_j\} \) and \( \{v_1, \ldots, v_l\} \) have no common element, i.e. \( u_1, \ldots, u_j, v_1, \ldots, v_l \) are all distinct. Due to independence of spreading codes \( c_k^{(p)} \)'s, the first term of (59) (expectation of product of \( 2n \) elements) is equal to the second term. Thus, \( Q(m_2, \ldots, m_n, m_{n+2}, \ldots, m_{2n}; k_1, \ldots, k_n, k_{n+1}, \ldots, k_{2n}) \) is equal to zero. Thus, (56) follows trivially.

Secondly, consider the situation that \( \{u_1, \ldots, u_j\} \) and \( \{v_1, \ldots, v_l\} \) have only one element in
common. Without loss of generality, we let \( u_1 = v_1 = w \). In this case, (59) is equal to
\[
\sum \left\{ E \left( \prod_{\alpha=1}^{a_1} c_w^{(p_{1,\alpha})} c_w^{(q_1,\alpha)} \prod_{\gamma=1}^{b_1} c_w^{(r_{1,\gamma})} c_w^{(s_{1,\gamma})} \right) - E \left( \prod_{\alpha=1}^{a_1} c_w^{(p_{1,\alpha})} c_w^{(q_1,\alpha)} \right) E \left( \prod_{\gamma=1}^{b_1} c_w^{(r_{1,\gamma})} c_w^{(s_{1,\gamma})} \right) \right\}
\times \prod_{i=2}^{j} E \left( \prod_{\alpha=1}^{a_1} c_w^{(p_{i,\alpha})} c_w^{(q_{i,\alpha})} \right) \cdot \prod_{i=2}^{l} E \left( \prod_{\gamma=1}^{b_1} c_w^{(r_{i,\gamma})} c_w^{(s_{i,\gamma})} \right) \times \text{product of } \delta \text{ functions}, \tag{60}
\]
where the summation is over all \( p_{i,\alpha(i)}, q_{i,\alpha(i)}, i = 1, \ldots, j \), \( \alpha(i) = 1, \ldots, a_i \) and \( r_{i,\gamma(i)}, s_{i,\gamma(i)}, i = 1, \ldots, l, \gamma(i) = 1, \ldots, b_i \). Moreover, for given \( i \) and \( \alpha(i) \), \( p_{i,\alpha(i)} \) and \( q_{i,\alpha(i)} \) are edge variables touching the same vertex within a cycle of the \( K \)-graph. So do \( r_{i,\gamma(i)} \) and \( s_{i,\gamma(i)} \). Equation (60) is nonzero only if the following conditions are met:

1) for each \( 2 \leq i \leq j \), \( p_{i,1}, \ldots, p_{i,a_i}, q_{i,1}, \ldots, q_{i,a_i} \) are in pairs,
2) for each \( 2 \leq i \leq l \), \( r_{i,1}, \ldots, r_{i,b_i}, s_{i,1}, \ldots, s_{i,b_i} \) are in pairs,
3) \( p_{1,1}, \ldots, p_{1,a_1}, q_{1,1}, \ldots, q_{1,a_1}, r_{1,1}, \ldots, r_{1,b_1}, s_{1,1}, \ldots, s_{1,b_1} \) are in pair, and some of \( p_{1,1}, \ldots, p_{1,a_1}, q_{1,1}, \ldots, q_{1,a_1} \) are in pair with elements of \( r_{1,1}, \ldots, r_{1,b_1}, s_{1,1}, \ldots, s_{1,b_1} \),

where we say members of a set are in pair, if each member of the set can find odd number of other members that take the same value. In this case, due to Hölder inequality,
\[
\left\{ E \left( \prod_{\alpha=1}^{a_1} c_w^{(p_{1,\alpha})} c_w^{(q_{1,\alpha})} \prod_{\gamma=1}^{b_1} c_w^{(r_{1,\gamma})} c_w^{(s_{1,\gamma})} \right) - E \left( \prod_{\alpha=1}^{a_1} c_w^{(p_{1,\alpha})} c_w^{(q_{1,\alpha})} \right) E \left( \prod_{\gamma=1}^{b_1} c_w^{(r_{1,\gamma})} c_w^{(s_{1,\gamma})} \right) \right\}
\times \prod_{i=2}^{j} E \left( \prod_{\alpha=1}^{a_1} c_w^{(p_{i,\alpha})} c_w^{(q_{i,\alpha})} \right) \cdot \prod_{i=2}^{l} E \left( \prod_{\gamma=1}^{b_1} c_w^{(r_{i,\gamma})} c_w^{(s_{i,\gamma})} \right)
\]
in (60) is \( O(N^{-2n}) \). Moreover, note that \( p_{i,\alpha(i)} \)'s, \( q_{i,\alpha(i)} \)'s, \( r_{i,\gamma(i)} \)'s and \( s_{i,\gamma(i)} \)'s are related to \( m_1, \ldots, m_n, m_{n+1}, \ldots, m_{2n} \). In specific, for given \( i \) and \( \alpha \), \( p_{i,\alpha} \) and \( q_{i,\alpha} \) are associated with one same variable in \( m_1, \ldots, m_n \). Similarly, \( r_{i,\gamma} \) and \( s_{i,\gamma} \) are associated with the same variable in \( m_{n+1}, \ldots, m_{2n} \). To have the largest cardinality of summation variables \( m_2, \ldots, m_n, m_{n+2}, \ldots, m_{2n}, p_{i,\alpha(i)} \)'s, \( q_{i,\alpha(i)} \)'s, \( r_{i,\gamma(i)} \)'s and \( s_{i,\gamma(i)} \)'s, the pairing constraints listed in conditions 1) – 3) above should be as least as possible, which yields

4) for each \( 2 \leq i \leq j \) and \( 1 \leq \alpha(i) \leq a_i \), \( p_{i,\alpha(i)} \) is only paired with \( q_{i,\alpha(i)} \),
5) for each \( 2 \leq i \leq l \) and \( 1 \leq \gamma(i) \leq b_i \), \( r_{i,\gamma(i)} \) is only paired with \( s_{i,\gamma(i)} \),
6) there is exactly one \((\theta, \nu)\) such that \( \{p_{1,\theta}, r_{1,\nu}\} \) and \( \{q_{1,\nu}, s_{1,\nu}\} \) (or \( \{p_{1,\theta}, s_{1,\nu}\} \) and \( \{q_{1,\theta}, r_{1,\nu}\} \)) are in pair individually,
7) for \( p_{1,\alpha} \)'s and \( q_{1,\alpha} \)'s with \( 1 \leq \alpha \leq a_1 \) but \( \alpha \neq \theta \), \( p_{1,\alpha} \) is only paired with \( q_{1,\alpha} \),

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8) for \( r_{1,\gamma} 's \) and \( s_{1,\gamma} 's \) with \( 1 \leq \gamma \leq b_i \) but \( \gamma \neq \nu \), \( r_{1,\gamma} \) is only paired with \( s_{1,\gamma} \).

Under these circumstances, the product of \( \delta \) functions in (60) summed over variables \( m_2, \ldots, m_n, m_{n+2}, \ldots, m_{2n}, p_{i,\alpha(i)} 's, q_{i,\alpha(i)} 's, r_{i,\gamma(i)} 's \) and \( s_{i,\gamma(i)} 's \) is

\[
O(N^{n-j+1} \cdot N^{n-l+1} \cdot N^{-1}) = O(N^{2n-j-l+1}),
\]

where \( N^{n-j+1} \) and \( N^{n-l+1} \) are due to \( \{k_1, \ldots, k_n\} \) and \( \{k_{n+1}, \ldots, k_{2n}\} \) forming \( j \)-class and \( l \)-class noncrossing partitions, respectively, and \( N^{-1} \) is because of conditions 6) – 8) causing the cardinality reduced by one\(^4\). Thus, the contribution of (57) is

\[
K^{-2} \cdot O(N^{-2n}) \cdot O(K^{j+l-1}) \cdot O(N^{2n-j-l+1}) = O(K^{-2}),
\]

(61)

where \( O(K^{j+l-1}) \) is due to \( u_1, \ldots, u_j, v_1, \ldots, v_l \) are all distinct except for \( u_1 = v_1 \). Consider the infinite sum over \( K \) of (56). It is finite when (61) is summed over \( K \) from 1 to \( \infty \).

Next, we consider the situation that \( \{u_1, \ldots, u_j\} \) and \( \{v_1, \ldots, v_l\} \) have \( t \) elements, where \( t \geq 2 \), in common. Without loss of generality, we assume \( u_i = v_i = w_i, 1 \leq i \leq t \). In this case, (59) is equal to

\[
\sum \left\{ \prod_{i=1}^{t} \left( \prod_{\alpha=1}^{a_i} c_{w_i}^{(p_{i,\alpha})} c_{w_i}^{(q_{i,\alpha})} \right) \prod_{\gamma=1}^{b_i} c_{w_i}^{(r_{i,\gamma})} c_{w_i}^{(s_{i,\gamma})} \right\} - \prod_{i=1}^{t} \left( \prod_{\alpha=1}^{a_i} c_{w_i}^{(p_{i,\alpha})} c_{w_i}^{(q_{i,\alpha})} \right) \prod_{\gamma=1}^{b_i} c_{w_i}^{(r_{i,\gamma})} c_{w_i}^{(s_{i,\gamma})} \right\}
\]

\[
\times \prod_{i=t+1}^{j} \left( \prod_{\alpha=1}^{a_i} c_{u_i}^{(p_{i,\alpha})} c_{u_i}^{(q_{i,\alpha})} \right) \prod_{i=t+1}^{l} \left( \prod_{\gamma=1}^{b_i} c_{v_i}^{(r_{i,\gamma})} c_{v_i}^{(s_{i,\gamma})} \right) \times \text{product of } \delta \text{ functions}.
\]

(62)

The pairing constraints listed in 4) – 8) is one of the situations that yield (62) nonzero and have the largest cardinality of summation variables. Thus, the contribution of the sum of products of \( \delta \) functions is the same as that of when \( \{u_1, \ldots, u_j\} \) and \( \{v_1, \ldots, v_l\} \) have only one common element. That is, it is equal to \( O(N^{2n-j-l+1}) \). It is not difficult to see, when \( \{u_1, \ldots, u_j\} \) and \( \{v_1, \ldots, v_l\} \) have \( t \) common elements, the contribution to (57) is

\[
K^{-2} \cdot O(N^{-2n}) \cdot O(K^{j+l-t}) \cdot O(N^{2n-j-l+1}) = O(K^{-t-1}).
\]

(63)

When (63) is summed over \( K \) from 1 to \( \infty \), it is finite.

\(^4\)This is the same as cases 1) and 2) of Section III-B, where we compute the contribution of an element \( x_{n-1} \in \mathcal{X}(e, n-e) \) to (20). In these two cases, edge variables touching the same vertex in a cycle of a \( K \)-graph do not always take the same value.
Thus, we have proven
\[
\sum_{k=1}^{\infty} E \{ [\text{Tr}(R^n_k) - \varphi(R^n_k)]^2 \} < \infty
\]
for all natural numbers \( n \), which was our goal.

**Appendix II**

**Proof of Lemma 2**

For given \( \{n_0, \eta_0, \eta_1\} \), if we define \( \theta = n_0T_c + \eta_0 - \eta_1 \) and \( \tilde{R}_\psi(t) = R_\psi(t - \theta) \), then \( R_\psi((n_1 - n_0)T_c + \eta_1 - \eta_0) \) is the sample of \( \tilde{R}_\psi(t) \) at \( t = n_1T_c \). The discrete-time signal \( \tilde{R}_\psi(n) \) obtained from the continuous-time one \( \tilde{R}_\psi(t) \) by a sampling period of \( T_c \) is denoted by the same notation, but with the parameter being an integer.

Define \( \theta_0 = n_0T_c + \eta_0 - \eta_1, \theta_2 = n_2T_c + \eta_2 - \eta_1, \tilde{R}_{\psi,0}(t) = R_\psi(t - \theta_0) \), and \( \tilde{R}_{\psi,2}(t) = R_\psi(t - \theta_2) \).

By Parseval’s theorem, in (37), the summation with respect to \( n_1 \) is given by
\[
\sum_{n_1 = -\infty}^{\infty} R_\psi((n_0 - n_1)T_c + \eta_0 - \eta_1)R_\psi((n_1 - n_2)T_c + \eta_1 - \eta_2) = \sum_{n_1 = -\infty}^{\infty} \tilde{R}_{\psi,0}(n_1)\tilde{R}_{\psi,2}(n_1)
\]
(64)

\[
= \frac{1}{2\pi} \int_{-\pi}^{\pi} \text{DTFT}\{\tilde{R}_{\psi,0}(n_1)\} \text{DTFT}^*\{\tilde{R}_{\psi,2}(n_1)\} d\omega_1,
\]
(65)

where \( \text{DTFT}\{\cdot\} \) is the operator of discrete-time Fourier transform (DTFT) with
\[
\text{DTFT}\{x(n)\} = \sum_{n=-\infty}^{\infty} x(n)e^{-j\omega n}.
\]

As \( \tilde{R}_{\psi,0}(n_1) \) is the sample of \( R_\psi(t - \theta_0) \) at time \( t = n_1T_c \), and the Fourier transform of \( R_\psi(t) \) is \( \Psi^2(\Omega) \), we have
\[
\text{DTFT}\{\tilde{R}_{\psi,0}(n_1)\} = \frac{1}{T_c} \sum_{k=-\infty}^{\infty} e^{-j\frac{\omega - 2\pi k}{T_c} \theta_0} \Psi^2 \left( \frac{\omega - 2\pi k}{T_c} \right),
\]
where \( \omega = \Omega T_c \). Consequently, (65) is equal to
\[
\frac{1}{2\pi T_c^2} \sum_{k,l=-\infty}^{\infty} \int_{-\pi}^{\pi} e^{-j\frac{\omega - 2\pi k}{T_c} \theta_0 + j\frac{\omega_1 - 2\pi l}{T_c} \theta_2} \Psi^2 \left( \frac{\omega_1 - 2\pi k}{T_c} \right) \Psi^2 \left( \frac{\omega_1 - 2\pi l}{T_c} \right) d\omega_1,
\]
(66)

where, since \( \Psi(\Omega) \) is bandlimited to \( \pi/T_c \), only \( k = l = 0 \) has nonzero integration. Thus, (64) is equal to
\[
\frac{1}{2\pi T_c^2} \int_{-\pi}^{\pi} e^{-j\frac{\omega}{T_c} ((n_0-n_2)T_c+n_0-n_2)} \Psi^4 \left( \frac{\omega_1}{T_c} \right) d\omega_1.
\]
(67)
We consider the summation with respect to \( n_2 \) in (37). Define \( \theta_3 = n_3 T_c + \eta_3 - \eta_2 \) and \( \tilde{R}_{\psi,3}(t) = R_{\psi}(t - \theta_3) \). We have

\[
\sum_{n_2 = -\infty}^{\infty} \left[ \frac{1}{2\pi T_c^2} \int_{-\pi}^{\pi} e^{-j\frac{\omega_1}{T_c} (n_0 - n_2) T_c + \eta_0 - \eta_2} \Psi^4 \left( \frac{\omega_1}{T_c} \right) d\omega_1 \right] R_{\psi}(\theta_3) \right]
\]

where the summation inside the square brackets is the complex conjugate of the DTFT of \( \tilde{R}_{\psi,3}(n_2) \), given by

\[
\frac{1}{T_c} \sum_{k = -\infty}^{\infty} e^{j\frac{2\pi}{T_c} (n_0 T_c + \eta_0 - \eta_2)} \Psi^2 \left( \frac{\omega_1 - 2\pi k}{T_c} \right).
\]

Plugging (69) back to (68), we can see that the integration is nonzero only when \( k = 0 \), which results in

\[
\frac{1}{2\pi T_c^2} \int_{-\pi}^{\pi} e^{-j\frac{\omega_1}{T_c} (n_0 - n_3) T_c + \eta_0 - \eta_3} \Psi^4 \left( \frac{\omega_1}{T_c} \right) d\omega_1.
\]

In consequence, when the summations with respect to \( n_1 \) and \( n_2 \) are taken into account, the result is given in (70). Continuing this process, we obtain the final result as

\[
\frac{1}{2\pi T_c^2} \int_{-\pi}^{\pi} \Psi^2 \left( \frac{\omega_1}{T_c} \right) d\omega_1,
\]

which is equal to (38) by setting \( \Omega = \omega_1/T_c \).

APPENDIX III

PROOF OF LEMMA 3

Suppose that the bandwidth of \( \psi(t) \) is within \([\alpha/2T_c, (\alpha + 1)/2T_c]\) for \( \alpha \in \mathbb{N} \). Using the equality \( \mathbb{E}_{u,v}\{g(u,v)\} = \mathbb{E}_v\{\mathbb{E}_u\{g(u,v)|v\}\} \) with \( \mathbb{E}_u\{\cdot\} \) denoting the conditional expectation with respect to \( u \), we can see that

\[
\sum_{n_1 = -\infty}^{\infty} \mathbb{E}_{\eta_1}\{R_{\psi}((n_0 - n_1)T_c + \eta_0 - \eta_1) R_{\psi}((n_1 - n_2)T_c + \eta_1 - \eta_2)|\eta_0, \eta_2\}
\]

is nested in the multi-dimensional summation of (39). By employing the same procedures of getting (66), (71) becomes

\[
\frac{1}{2\pi T_c^2} \sum_{k,l = -\infty}^{\infty} \int_{-\pi}^{\pi} \mathbb{E}_{\eta_1}\left\{ e^{-j\frac{\omega_1}{T_c} (2\pi k \theta_0 + 2\pi l \theta_2)} |\eta_0, \eta_2\right\} \Psi^2 \left( \frac{\omega_1 - 2\pi k}{T_c} \right) \Psi^2 \left( \frac{\omega_1 - 2\pi l}{T_c} \right) d\omega_1.
\]
Since $E_{\eta_1}\left\{e^{j2\pi\eta_1/T_c}\right\} = 0$ for any nonzero integer $\gamma$, it is readily seen that we only need to consider $k = l$ in the above equation, which is given by

$$
\frac{1}{2\pi T_c^2} \sum_{k=-[\alpha/2]}^{[\alpha/2]} \int_{-\pi}^{\pi} e^{-j\omega_1 - j\omega_1} \frac{n_0 T_c + \eta_0 - \eta_2}{T_c} \Psi^4 \left(\frac{\omega_1 - 2\pi k}{T_c}\right) d\omega_1.
$$

(72)

Consider one more summation in (39), i.e. with respect to $n_2$, which yields

$$
\frac{1}{2\pi T_c^2} \sum_{k=-[\alpha/2]}^{[\alpha/2]} \int_{-\pi}^{\pi} \Psi^4 \left(\frac{\omega_1 - 2\pi k}{T_c}\right) \left[ \sum_{n_2=-\infty}^{\infty} E_{\eta_2} \left\{ e^{-j\omega_1 n_2} \tilde{R}_{\psi,3}(n_2) \right\} \eta_0, \eta_3 \right] d\omega_1,
$$

(73)

where the term inside the square bracket can be written as

$$
E_{\eta_2} \left\{ e^{-j\omega_1 - j\omega_1} (n_0 T_c + \eta_0 - \eta_2) \sum_{n_2=-\infty}^{\infty} e^{j\omega_1 n_2} \tilde{R}_{\psi,3}(n_2) \eta_0, \eta_3 \right\}
$$

$$
= \frac{1}{T_c} \sum_{l=-\infty}^{\infty} E_{\eta_2} \left\{ e^{-j\omega_1 n_2} (n_0 T_c + \eta_0 - \eta_2) e^{j\omega_1 n_2} (n_3 T_c + \eta_3 - \eta_2) \eta_0, \eta_3 \right\} \Psi^2 \left(\frac{\omega_1 - 2\pi l}{T_c}\right)
$$

(74)

with the expectation in (74) being nonzero only when $l = k$. Thus, the summations with respect to $n_1$ and $n_2$ of (39), i.e. (73), become

Continuing this process, we obtain the final result as

$$
\frac{1}{2\pi T_c^n} \sum_{k=-[\alpha/2]}^{[\alpha/2]} \int_{-\pi}^{\pi} \Psi^{2m} \left(\frac{\omega_1 - 2\pi k}{T_c}\right) d\omega_1 = \frac{1}{2\pi T_c^n} \int_{-\pi}^{\pi} \Psi^{2m} \left(\frac{\omega_1}{T_c}\right) d\omega_1,
$$

which is equal to (40) by changing variable from $\omega_1$ to $\Omega = \omega_1/T_c$.

APPENDIX IV

PROOF OF LEMMA 4

Consider $j = n$. The $K$-graph is shown in Fig. 2(a). The expectation of spreading codes in (36) is nonzero and equal to $N^{-n}$ if and only if $p_r = q_r$ for $1 \leq r \leq n$, and (36) becomes

$$
\frac{1}{2\pi T_c^n} \sum_{K \in \mathcal{X}_n} \sum_{M \in \mathcal{Y}} \sum_{p_1 \in [0,N-1]} \sum_{p_2 \in \mathcal{Z}_2^n} \cdots \sum_{p_n \in \mathcal{Z}_n^n} E \left\{ R_{\psi}((p_1 - p_2)T_c + \tau_{k_1} - \tau_{k_2}) \right\}
$$

$$
\times R_{\psi}((p_2 - p_3)T_c + \tau_{k_2} - \tau_{k_3}) \cdots R_{\psi}((p_n - p_1)T_c + \tau_{k_n} - \tau_{k_1}) \right\}
$$

$$
= \frac{1}{2\pi T_c^n} \int_{-\pi}^{\pi} \Psi^{2m} \left(\frac{\omega_1}{T_c}\right) d\omega_1,
$$

(75)
By Lemmas 2 and 3, the term to the right-hand-side of $\sum_{p_1 \in [0, N-1]}$ in (75) is equal to $\mathcal{W}_{\psi}^{(n)}$. Consequently, (75) is equal to

$$N^{-n} \cdot \# \mathcal{X}_n(n) \cdot N \cdot \mathcal{W}_{\psi}^{(n)} = \# \mathcal{X}_n(n) \cdot N^{-n+1} \mathcal{W}_{\psi}^{(n)}.$$ 

Thus, the statement is true for $j = n$.

Next, we consider $j = n - 1$. The $\mathcal{K}$-graph under consideration is composed of two cycles with edge numbers $e$ and $n - e$ (with $e \geq n - e$) shown in Fig. 2(b). We consider three cases of 1) $q_s = p_s$ and $q_t = p_t$, 2) $q_s = q_t$ and $p_t = p_s$, and 3) $q_s = p_t$ and $q_t = p_s$ for edge variables in Fig. 2(b). In each of Figs. 3(a) – (c), the thick line passing through the edge labelled with variables $p_\gamma$ and $p_\epsilon$ represents $R_\psi((p_\gamma - p_\epsilon)T_c + \tau_{k_\gamma} - \tau_{k_\epsilon})$.

We look at case 3) first. For case 3), due to $m_s = m_t$ implied by $q_s = p_t$ (as well as $q_t = p_s$), we have $p_s \in [\alpha N, (\alpha + 1)N - 1] = \mathcal{Z}_s'$. The contribution to (36) is given by

$$N^{-n} \sum_{\mathcal{K} \in \mathcal{X}_{n-1}(e, n-e)} \sum_{p_1 \cdot \cdots \cdot p_{n-t+1}} \mathbb{E}\left\{ \prod_{(\gamma, \epsilon) \in \mathcal{I}} R_\psi((p_\gamma - p_\epsilon)T_c + \tau_{k_\gamma} - \tau_{k_\epsilon}) \right\} \times \sum_{p_s \cdot p_{s+1} \cdot \cdots \cdot p_{t-1}} \mathbb{E}\left\{ \prod_{(\eta, \zeta) \in \mathcal{J}} R_\psi((p_\eta - p_\zeta)T_c + \tau_{k_\eta} - \tau_{k_\zeta}) \right\},$$ 

(76)

where $\mathcal{I} = \{(1, 2), \cdots, (s-2, s-1), (s-1, t), (t, t+1), \cdots, (n, 1)\}$, $\mathcal{J} = \{(s, s+1), \cdots, (t-2, t-1), (t-1, s)\}$, $p_1 \in [0, N-1]$, $p_s \in \mathcal{Z}_s''$ and all other $p_r$’s belong to $(-\infty, \infty)$. Consequently, (76) is equal to

$$N^{-n} \cdot \# \mathcal{X}_{n-1}(e, n-e) \cdot N \mathcal{W}_{\psi}^{(n-e)} \cdot N \mathcal{W}_{\psi}^{(e)} = \# \mathcal{X}_{n-1}(e, n-e) \cdot N^{-n+2} \mathcal{W}_{\psi}^{(n-e)} \mathcal{W}_{\psi}^{(e)}.$$ 

(77)

It can also be shown that both cases 1) and 2) result in a contribution of $O(N^{-n+1})$. Thus, when $N$ is large, the contributions of these two cases vanish compared with that of case 3). In consequence, the contribution of $\mathcal{X}_{n-1}(e, n-e)$ is given by (77), which confirms the statement of this theorem when $j = n - 1$.

The above procedure can be generalized to show, for a $\mathcal{K}$-graph of $\mathcal{X}_j(b_1, b_2, \cdots, b_{n-j+1})$, the $b_r$-edge cycle yields a value of $N \mathcal{W}_{\psi}^{(b_r)}$. Thus, the contribution of $\mathcal{X}_j(b_1, b_2, \cdots, b_{n-j+1})$ to (36) is equal to

$$N^{-n} \cdot \# \mathcal{X}_j(b_1, b_2, \cdots, b_{n-j+1}) \prod_{r=1}^{n-j+1} N \mathcal{W}_{\psi}^{(b_r)}.$$
APPENDIX V
PROOF OF THEOREM 5

Two lemmas are necessary to prove Theorem 5.

Lemma 5: Suppose that \( \varpi \) is a noncrossing partition of an \( n \)-element set without singletons. There exist classes of \( \varpi \) that contain adjacent elements in the set, where the adjacency is cyclic ordering, i.e. the first and last elements are adjacent.

Proof: It is shown in Section III-A that there is a bijective correspondence between the class set of \( \varpi \) and the vertex set of \( \varpi \)’s associated \( K \)-graph, denoted by \( G(\varpi) \). We have the following three observations. First, the size of the \( r \)-th class of \( \varpi \) is equal to \( d(v_r)/2 \), where \( v_r \) is the vertex in \( G(\varpi) \) that corresponds to class \( r \), and \( d(v_r) \) is the degree\(^5\) of \( v_r \). Secondly, a self-loop is introduced in \( G(\varpi) \) if two adjacent elements in the \( n \)-element set are partitioned into the same class. Thirdly, we cannot find any \( G(\varpi) \) (having the property that it is composed of cycles with any two of them connected by at most one vertex) that have no self-loops and all of the vertices have degrees equal to or larger than four. Based on the first two observations, the third one can be interpreted as the statement of the lemma. Thus, we have completed the proof.

Lemma 6: With \( x \in \{cs,ca\} \),

\[
\lim_{K,N,M \to \infty} \frac{1}{K/N} = \beta \sum_{K \in \mathcal{X}} \sum_{M \in \mathcal{Y}} \mathbb{E}\left\{ [R_x]_{m_1 m_2 k_1 k_2} [R_x]_{m_2 m_3 k_2 k_3} \cdots [R_x]_{m_n m_1 k_n k_1} \right\} |m_1 = 0 \tag{78}
\]

\[
= \mu(R_x^{n+s-t}) \cdot \mu(R_x^{t-s}).
\]

Proof: Only \( x = cs \) is proven here, and it is straightforward to extend to \( x = ca \). We consider noncrossing partition of \( K \) with \( k_s \) and \( k_t \) in the same class. Denote the cycles at the left- and right-hand-side of Fig. 2(b) as \( L \)-cycle and \( R \)-cycle, respectively. It is seen that \( L \)- and \( R \)-cycles have edge number \( n_l \overset{\Delta}{=} n + s - t \) and \( n_r \overset{\Delta}{=} t - s \), respectively. It is sufficient to consider those \( K \)-graphs obtained from performing vertex mergence on the two cycles individually. In this way, the resultant \( K \)-graphs consist of cycles with any two of them being connected by at most one vertex, which give non-vanishing contributions to (78) in the large-system regime.

\(^5\)The degree of a vertex is the number of edges that connect to that vertex. The singly vertex in a self-loop has a degree equal to two.
Consider the \( K \)-graph whose \( R \)- and \( L \)-cycles are decomposed into \( n_r - j_r + 1 \) and \( n_l - j_l + 1 \) cycles with edge sizes \( (b_{r,1}, b_{r,2}, \cdots, b_{r,n_r-j_r+1}) \) and \( (b_{l,1}, b_{l,2}, \cdots, b_{l,n_l-j_l+1}) \), respectively. Its contribution to (78) is given by

\[
K^{-1}N^{-n}\#X_{j_r}(b_{r,1}, b_{r,2}, \cdots, b_{r,n_r-j_r+1})\#X_{j_l}(b_{l,1}, b_{l,2}, \cdots, b_{l,n_l-j_l+1})K^{-1}N^{n_r-j_r+1+(n_l-j_l+1)},
\]

where the second \( K^{-1} \) is due to \( R \)- and \( L \)-cycles having one common vertex. The above term is equal to

\[
T(\{b_{r,i}\}_{i=1}^{n_r-j_r+1}, \{b_{l,i}\}_{i=1}^{n_l-j_l+1}) \triangleq \frac{n_r(n_r-1)\cdots(j_r+1)n_l(n_l-1)\cdots(j_l+1)}{f(b_{r,1}, b_{r,2}, \cdots, b_{r,n_r-j_r+1})f(b_{l,1}, b_{l,2}, \cdots, b_{l,n_l-j_l+1})^\beta}.
\]

in the large-system limit.

Summing up all \( K \)-graphs satisfying constraints, we have

\[
\sum_{j_r=1}^{n_r} \sum_{j_l=1}^{n_l} \sum_{\substack{b_{r,1}+b_{r,2}+\cdots+b_{r,n_r-j_r+1}=n_r \\ b_{l,1}+b_{l,2}+\cdots+b_{l,n_l-j_l+1}=n_l}} T(\{b_{r,i}\}_{i=1}^{n_r-j_r+1}, \{b_{l,i}\}_{i=1}^{n_l-j_l+1})
\]

\[
= \frac{1}{n_r n_l} \sum_{j_r=1}^{n_r} \binom{n_r}{j_r} \binom{n_r}{j_r} \beta^{j_r-1} \sum_{j_l=1}^{n_l} \binom{n_l}{j_l} \binom{n_l}{j_l} \beta^{j_l-1}
\]

\[
= \mu(R_{cS}^{n_r}) \cdot \mu(R_{cS}^{n_l}).
\]

**Proof:** [Theorem 5]

Suppose that, for \( 1 \leq j \leq n \), polynomials

\[
p_j(x) = \sum_{r_j \geq 0} a_{j,r_j} x^{r_j} \quad \text{and} \quad q_j(x) = \sum_{s_j \geq 0} b_{j,s_j} x^{s_j}
\]

give

\[
\sum_{r_j \geq 0} a_{j,r_j} \mu(R_{cS}^{r_j}) = \sum_{s_j \geq 0} b_{j,s_j} \mu(D^{s_j}) = 0.
\]

We have

\[
\mu(p_1(R_x)q_1(D) \cdots p_n(R_x)q_n(D))
\]

\[
= \sum_{\substack{\tau_1, \cdots, \tau_n \\ s_1, \cdots, s_n}} a_{1,\tau_1} b_{1,s_1} \cdots a_{n,\tau_n} b_{n,s_n} \lim_{M,N,K \to \infty} (2M + 1)^{-1} K^{-1} \beta \{ \text{tr} \{ R_{cS}^{s_1} \cdots R_{cS}^{s_n} D^{s_n} \} \}.
\]

(79)
where

\[
E \{ \text{tr} \{ R_x^{r_1} D^{s_1} \cdots R_x^{r_n} D^{s_n} \} \}
\]

\[
= \sum_{m_1, \ldots, m_n, k_1, \ldots, k_n} E \{ [R_x^{r_1}]_{m_1, m_2, k_1, k_2} [D^{s_1}]_{m_2, m_3, k_2, k_3} \cdots [R_x^{r_n}]_{m_n, k_n, k_1} [D^{s_n}]_{m_1, k_1, k_2} \}
\]

\[
= \sum_{m_1, \ldots, m_n, k_1, \ldots, k_n} E \{ (d_{k_2}(m_2))^{s_1} (d_{k_3}(m_3))^{s_2} \cdots (d_{k_1}(m_1))^{s_n} \}
\]

\[
\times E \{ [R_x^{r_1}]_{m_1, m_2, k_1, k_2} [R_x^{r_2}]_{m_2, m_3, k_2, k_3} \cdots [R_x^{r_n}]_{m_n, k_n, k_1} \}
\]

\[
= \sum_{1 \leq u_{j,(j)} \leq K, \sum_{v_{j,(j)}} v_{j,(j)} \leq M, 1 \leq j, l \leq n, 1 \leq l(j) \leq r_j} \times E \{ [R_x^{u_1}]_{v_1, v_2, u_1, u_2, 1, 2, 1} [R_x^{u_2}]_{v_1, v_2, u_3, u_1, 2, 1, 3} \cdots [R_x^{u_n}]_{v_1, v_2, u_n, u_1, 2, 1, n} \}
\]

\[
\times E \{ [R_x^{r_1}]_{v_1, v_2, u_1, u_2, 1, 2} [R_x^{r_2}]_{v_1, v_2, u_3, u_2, 2, 1, 3} \cdots [R_x^{r_n}]_{v_1, v_2, u_n, u_n, u_1, 1} \},
\] (80)

where, in the third equality, \(v_{j,1} \triangleq m_j\) and \(u_{j,1} \triangleq k_j\) for \(1 \leq j \leq n\).

Consider noncrossing partitions of \(\{u_{j,(j)}: 1 \leq j \leq n, 1 \leq l(j) \leq r_j\}\). We divide all of these noncrossing partitions into two groups as follows.

Group 1: At least one of \(\{u_{j,1}\}_{j=1}^n\) are singletons.

Group 2: None of \(\{u_{j,1}\}_{j=1}^n\) are singletons.

For Group 1, without loss of generality, we suppose that \(u_{1,1}\) is a singleton. Then, the expectation of \(d_{u_{j,1}}(v_{j,1})\)'s in (80) can be written as

\[
E \{(d_{u_{1,1}}(v_{1,1}))^{s_n}\} E \{(d_{u_{2,1}}(v_{2,1}))^{s_1} (d_{u_{3,1}}(v_{3,1}))^{s_2} \cdots (d_{u_{n,1}}(v_{n,1}))^{s_{n-1}}\},
\] (81)

since \(d_k(m_1)\) and \(d_l(m_2)\) are independent if \(k \neq l\). Due to Hölder inequality, for the second term of (81), we have

\[
E \{(d_{u_{2,1}}(v_{2,1}))^{s_1} (d_{u_{3,1}}(v_{3,1}))^{s_2} \cdots (d_{u_{n,1}}(v_{n,1}))^{s_{n-1}}\}
\]

\[
\leq E \{(d_{u_{2,1}}(v_{2,1}))^{s_1(n-1)}\}^{1/(n-1)} \cdots E \{(d_{u_{n,1}}(v_{n,1}))^{s_{n-1}(n-1)}\}^{1/(n-1)}
\]

\[
= O(1).
\] (83)
Thus, (80) can be written as

\[
C_1 \sum_{u_{1,1}, v_{1,1}} E\{(d_{u_{1,1}}(v_{1,1}))^{s_n}\} \\
\times \sum_{\{u_{j,l(j)}, v_{j,l(j)}\} : 1 \leq j \leq n, 1 \leq l(j) \leq r_j} \{u_{1,1}, v_{1,1}\} \times \ldots \times \left[ R_x \right]_{v_{n,1} v_{n,2} n, u_{n,2}} \left[ R_x \right]_{v_{n,2} v_{n,3} n, u_{n,3}} \ldots \left[ R_x \right]_{v_{n_r} v_{n_r} n, u_{n_r, u_{n,1}}} \right) \left[ R_k^1 \right]_{m_1 m_2 k_1 k_2} \right]_{m_n m_1 k_n k_1}
\]

(84)

where $C_1$ is a constant that is equal to the expectation of (82). It is seen that the second summation in (84) is equal to $\mu(R^2_x)$, with $\gamma = \sum_{j=1}^n r_j$, when $M, N, K = \beta N \to \infty$. It follows that

\[
\mu(p_1(R_x)q_1(D) \ldots p_n(R_x)q_n(D)) = \\
= \lim_{M,N,K \to \infty} (2M + 1)^{-1} K^{-1} \sum_{s_n} b_{n,s_n} \sum_{u_{1,1}, v_{1,1}} E\{(d_{u_{1,1}}(v_{1,1}))^{s_n}\} \\
\times \sum_{r_1, \ldots, r_n} a_{1,r_1} b_{1,s_1} \ldots a_{n,r_n} b_{n-1,s_{n-1}} C_1 \mu(R^2_x) \\
= \sum_{s_n} b_{n,s_n} \mu(D^{s_n}) \sum_{r_1, \ldots, r_n} a_{1,r_1} b_{1,s_1} \ldots a_{n,r_n} b_{n-1,s_{n-1}} C_1 \mu(R^2_x) \\
= 0.
\]

For Group 2, by Lemma 5, suppose that $u_{t,1}$ and $u_{t+1,1}$ are in the same class. Using Hölder inequality again in the first expectation of (80), we have

\[
E\{(d_{u_{2,1}}(v_{2,1}))^{s_1}(d_{u_{3,1}}(v_{3,1}))^{s_2} \ldots (d_{u_{1,1}}(v_{1,1}))^{s_n}\} = O(1).
\]

Suppose that it is equal to a constant $C_2$, which can be factored out in (80). Thus, (79) can be written as

\[
\lim_{M,N,K \to \infty} \frac{(2M + 1)^{-1}}{K^{-1}} \sum_{s_n} b_{n,s_n} \mu(D^{s_n}) \sum_{r_1, \ldots, r_n} a_{1,r_1} b_{1,s_1} \ldots a_{n,r_n} b_{n-1,s_{n-1}} C_1 \mu(R^2_x) \\
= C_2 \lim_{M,N,K \to \infty} \frac{(2M + 1)^{-1}}{K^{-1}} \sum_{s_n} b_{n,s_n} \mu(D^{s_n}) \sum_{r_1, \ldots, r_n} a_{1,r_1} b_{1,s_1} \ldots a_{n,r_n} b_{n-1,s_{n-1}} C_1 \mu(R^2_x) \\
= C_2 \lim_{M,N,K \to \infty} \frac{(2M + 1)^{-1}}{K^{-1}} \sum_{s_n} b_{n,s_n} \mu(D^{s_n}) \sum_{r_1, \ldots, r_n} a_{1,r_1} b_{1,s_1} \ldots a_{n,r_n} b_{n-1,s_{n-1}} C_1 \mu(R^2_x).
\]

(85)
For notational simplicity, the summation variables $u_{j,l}$'s and $v_{j,l}$'s are changed to $x_1, \ldots, x_\gamma$ and $y_1, \ldots, y_\gamma$, respectively. It is assumed that $u_{t,1} = x_\alpha$ and $v_{t,1} = y_\alpha$ so that $u_{t+1,1} = x_{\alpha + r_t}$ and $v_{t+1,1} = y_{\alpha + r_t}$. It follows that (85) yields

$$C_2 \lim_{M,N,K \to \infty} (2M + 1)^{-1} K^{-1} \sum_{x_1, \ldots, x_\gamma} \sum_{y_1, \ldots, y_\gamma} E\{[R_x]_{y_1y_2x_1x_2}[R_x]_{y_2y_3x_2x_3} \cdots [R_x]_{y_\gamma y_1x_1}\}$$

$$= C_2 \cdot \mu(R_x^{r_t}) \cdot \mu(R_x^{\gamma-r_t}),$$

where the equality is due to Lemma 6. We have

$$\mu(p_1(R_x)q_1(D) \cdots p_n(R_x)q_n(D))$$

$$= \sum_{r_t} a_{t,r_t} \mu(R_x^{r_t}) C_2 \sum_{r_1, \ldots, r_{\gamma-1}, r_{\gamma+1}, \ldots, r_n} \sum_{s_1, \ldots, s_n} \underbrace{a_{1,r_1}b_{1,s_1} \cdots a_{n,r_n}b_{n,s_n}}_{\text{without } a_{t,r_t}} \mu(R_x^{\gamma-r_t}),$$

$$= 0.$$

Thus, we have proven $R_x$ and $D$ are asymptotically free. ■

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