Research Article

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Some new parameterized inequalities for co-ordinated convex functions involving generalized fractional integrals

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Abstract: In this study, we first obtain a new identity for generalized fractional integrals which contains some parameters. Then by this equality, we establish some new parameterized inequalities for co-ordinated convex functions involving generalized fractional integrals. Moreover, we show that the results proved in the main section reduce to several Simpson-, trapezoid- and midpoint-type inequalities for various values of parameters.

Keywords: Simpson’s 1/3 formula, integral inequalities, fractional calculus, co-ordinated convex functions

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1 Introduction

The inequalities discovered by C. Hermite and J. Hadamard for convex functions are considered significant in the literature. These inequalities state that if $F : I \to \mathbb{R}$ is a convex function on the interval $I$ of real numbers and $x_1, x_2 \in I$ with $x_1 < x_2$, then

$$F \left( \frac{x_1 + x_2}{2} \right) \leq \frac{1}{x_2 - x_1} \int_{x_1}^{x_2} F(x) \, dx \leq \frac{F(x_1) + F(x_2)}{2}. \quad (1.1)$$

Both inequalities hold in the reversed direction if $F$ is concave.

Over the last 20 years, numerous studies have focused on obtaining trapezoid- and midpoint-type inequalities which give bounds for the right-hand side and left-hand side of the inequality (1.1), respectively. For example, Dragomir and Agarwal first obtained trapezoid inequalities for convex functions in [1], whereas Kirmaci first established midpoint inequalities for convex functions in [2]. In [3], Sarikaya et al. generalized the inequalities (1.1) for fractional integrals and the authors also proved some corresponding trapezoid-type inequalities. Iqbal et al. presented some fractional midpoint-type inequalities for convex functions in [4]. On the other hand, Dragomir proved Hermite-Hadamard inequalities for co-ordinated convex mappings in [5]. In [6] and [7], the authors proved midpoint- and trapezoid-type inequalities for...
co-ordinated convex functions, respectively. Moreover, Sarikaya obtained fractional Hermite-Hadamard inequalities and fractional trapezoid for functions with two variables in [8]. Tunç et al. presented some fractional midpoint-type inequalities for co-ordinated convex functions in [9]. In [10], Sarikaya and Ertuğral first introduced new fractional integrals which are called generalized fractional integrals, and then, they proved Hermite-Hadamard inequalities and several trapezoid- and midpoint-type inequalities for generalized fractional integrals. In addition, Turkay et al. defined the generalized fractional integrals for functions with two variables and they presented Hermite-Hadamard- and trapezoid-type inequalities for this kind of fractional integrals in [11]. For the other similar inequalities, please refer to [12–19].

On the other hand, the following inequality is well known in the literature as Simpson’s inequality.

**Theorem 1.** Suppose that \( F : [\kappa_1, \kappa_2] \rightarrow \mathbb{R} \) is a four times continuously differentiable mapping on \((\kappa_1, \kappa_2)\), and let \( \|F^{(4)}\|_{\infty} = \sup_{t \in [\kappa_1, \kappa_2]} |F^{(4)}(t)| < \infty \). Then, one has the inequality

\[
\left| \frac{1}{3} \left( F(\kappa_1) + F(\kappa_2) \right) + 2F \left( \frac{\kappa_1 + \kappa_2}{2} \right) \right| - \frac{1}{\kappa_2 - \kappa_1} \int_{\kappa_1}^{\kappa_2} F(\tau) \, d\tau \leq \frac{1}{2880} \|F^{(4)}\|_{\infty} (\kappa_2 - \kappa_1)^4.
\]

In recent years, many authors have focused on Simpson-type inequalities for various classes of functions. For example, Dragomir et al. presented new Simpson-type results and their applications to quadrature formulas in numerical integration in [20]. In addition, some inequalities of Simpson-type for \( \eta \)-convex functions are deduced by Alomari et al. in [21]. Afterward, Sarikaya et al. observed the variants of Simpson-type inequalities based on convexity in [22]. Moreover, some papers are devoted to Simpson inequalities for co-ordinated convex functions [23–26]. On the other hand, some authors proved several Simpson-type inequalities for fractional integrals in [27–30]. In addition, in [31], Ertugral and Sarikaya obtained some inequalities of Simpson-type for generalized fractional integrals. For more recent developments, one can refer to [32–38].

The aim of this paper is to obtain some parameterized inequalities for co-ordinated convex functions via generalized fractional integrals. These inequalities reduce to Simpson, trapezoid and midpoint inequalities in the case of special choice of parameters. The overall structure of the study takes the form of six sections including an introduction. The remaining part of the paper proceeds as follows: In Section 2, the generalized fractional integral operators is summarized, along with some related theorems. In Section 3, an identity involving generalized fractional integrals is presented for partial differentiable functions. Then we prove several parameterized inequalities for functions whose partial derivatives in absolute value are co-ordinated convex in Section 4. Moreover, some special cases of the results in Section 4 are presented in Section 5. Finally, some conclusions and further directions of research are discussed in Section 6.

A formal definition for co-ordinated convex function may be stated as follows:

**Definition 1.** A function \( F : \Delta = [\kappa_1, \kappa_1] \times [\kappa_2, \kappa_2] \rightarrow \mathbb{R} \) is called co-ordinated convex on \( \Delta \), for all \((x, u), (y, v) \in \Delta \) and \( t, s \in [0, 1] \), if it satisfies the following inequality:

\[
F(tx + (1 - t)y, su + (1 - s)v) \leq tsF(x, u) + t(1 - s)F(x, v) + s(1 - t)F(y, u) + (1 - t)(1 - s)F(y, v).
\] (1.2)

The mapping \( F \) is a co-ordinated concave on \( \Delta \) if the inequality (1.2) holds in the reversed direction for all \( t, s \in [0, 1] \) and \((x, u), (y, v) \in \Delta \).

## 2 Generalized fractional integrals

Fractional calculus and applications have application areas in many different fields such as physics, chemistry and engineering as well as mathematics. The application of arithmetic carried out in classical analysis in fractional analysis is very important in terms of obtaining more realistic results in the solution of many problems. Many real dynamical systems are better characterized by using non-integer order dynamic
models based on fractional computation. While integer orders are a model that is not suitable for nature in classical analysis, fractional computation in which arbitrary orders are examined enables us to obtain more realistic approaches.

In this section, we summarize the generalized fractional integrals defined by Sarikaya and Ertuğral in [10].

**Definition 2.** Let \( F : [k_1, k_2] \to \mathbb{R} \) be an integrable function. The left-sided and right-sided generalized fractional integral operators are given by

\[
\kappa_1 I_{\varphi}^\tau \Gamma(\eta) = \int_{\kappa_1}^{\tau} \frac{\varphi(\eta - \xi)}{\xi} F(\xi) d\xi, \quad \eta > \kappa_1, \tag{2.1}
\]

and

\[
\kappa_2 I_{\varphi}^\tau \Gamma(\eta) = \int_{\kappa_2}^{\tau} \frac{\varphi(\xi - \eta)}{\xi} F(\xi) d\xi, \quad \eta < \kappa_2, \tag{2.2}
\]

respectively. Here, the function \( \varphi : [0, \infty) \to [0, \infty) \) satisfies the condition

\[
\int_{0}^{\infty} \frac{\varphi(\xi)}{\xi} d\xi < \infty.
\]

The most important feature of generalized fractional integrals is that they generalize some types of fractional integrals such as Riemann-Liouville fractional integral, \( k \)-Riemann-Liouville fractional integral, Katugampola fractional integrals, conformable fractional integral, Hadamard fractional integrals, etc. These important special cases of the integral operators (2.1) and (2.2) are mentioned below.

(i) If we take \( \varphi(\xi) = \xi \), the operators (2.1) and (2.2) reduce to the Riemann integral as follows:

\[
I_{\kappa_1}^\tau F(\eta) = \int_{\kappa_1}^{\tau} F(\xi) d\xi, \quad \eta > \kappa_1,
\]

\[
I_{\kappa_2}^\tau F(\eta) = \int_{\eta}^{\kappa_2} F(\xi) d\xi, \quad \eta < \kappa_2.
\]

(ii) If we take \( \varphi(\xi) = \frac{\xi}{(\alpha)} \), the operators (2.1) and (2.2) reduce to the Riemann-Liouville fractional integral as follows:

\[
J_{\alpha}^{\kappa_1} F(\eta) = \frac{1}{\Gamma(\alpha)} \int_{\kappa_1}^{\tau} (\eta - \xi)^{\alpha-1} F(\xi) d\xi, \quad \eta > \kappa_1,
\]

\[
J_{\alpha}^{\kappa_2} F(\eta) = \frac{1}{\Gamma(\alpha)} \int_{\eta}^{\kappa_2} (\xi - \eta)^{\alpha-1} F(\xi) d\xi, \quad \eta < \kappa_2.
\]

(iii) If we take \( \varphi(\xi) = \frac{1}{k_{\alpha}(\alpha)} \xi^{\alpha} \), the operators (2.1) and (2.2) reduce to the \( k \)-Riemann-Liouville fractional integral as follows:

\[
J_{\alpha}^{\kappa_1, k} F(\eta) = \frac{1}{k_{\alpha}(\alpha)} \int_{\kappa_1}^{\tau} (\eta - \xi)^{\alpha-1} F(\xi) d\xi, \quad \eta > \kappa_1,
\]

\[
J_{\alpha}^{\kappa_2, k} F(\eta) = \frac{1}{k_{\alpha}(\alpha)} \int_{\eta}^{\kappa_2} (\xi - \eta)^{\alpha-1} F(\xi) d\xi, \quad \eta < \kappa_2,
\]
where
\[
\Gamma_0(a) = \int_0^\infty x^{a-1}e^{-x}dx, \quad \Re(a) > 0
\]
and
\[
\Gamma_k(a) = k^{a-1}\Gamma\left(\frac{a}{k}\right), \quad \Re(a) > 0; \quad k > 0
\]
are given by Mubeen and Habibullah in [39].

In the literature, there are several papers on inequalities for generalized fractional integrals. Some of them please refer to [31, 40–47].

Inspired by this definition, Turkay et al. [11] give the following definitions:

**Definition 3.** Let \( F \in L_1([\kappa_1, \kappa_2] \times [\kappa_3, \kappa_4]) \). The generalized Riemann-Liouville fractional integrals
\[
I_{\varphi \psi}^{\kappa_1, \kappa_2} F(\tau_1, \tau_2),
\]
and
\[
I_{\varphi \psi}^{\kappa_3, \kappa_4} F(\tau_1, \tau_2)
\]
are defined by

\[
I_{\varphi \psi}^{\kappa_1, \kappa_2} F(\tau_1, \tau_2) = \int_{\kappa_1}^{\tau_1} \int_{\kappa_2}^{\tau_2} \frac{\varphi(\tau_1 - \tau)}{\tau_1 - \xi} \frac{\psi(\tau_2 - \eta)}{\tau_2 - \eta} F(\xi, \eta)d\eta d\xi, \quad \tau_1 > \kappa_1, \tau_2 > \kappa_2,
\]
(2.3)

\[
I_{\varphi \psi}^{\kappa_3, \kappa_4} F(\tau_1, \tau_2) = \int_{\kappa_3}^{\tau_1} \int_{\kappa_4}^{\tau_2} \frac{\varphi(\xi - \tau)}{\xi - \eta} \frac{\psi(\eta - \eta)}{\eta - \tau_2} F(\xi, \eta)d\eta d\xi, \quad \tau_1 > \kappa_3, \tau_2 < \kappa_4,
\]
(2.4)

\[
I_{\varphi \psi}^{\kappa_2, \kappa_4} F(\tau_1, \tau_2) = \int_{\kappa_2}^{\tau_1} \int_{\kappa_4}^{\tau_2} \frac{\varphi(\xi - \tau)}{\xi - \eta} \frac{\psi(\eta - \eta)}{\eta - \tau_2} F(\xi, \eta)d\eta d\xi, \quad \tau_1 > \kappa_2, \tau_2 > \kappa_3
\]
(2.5)

and

\[
I_{\varphi \psi}^{\kappa_3, \kappa_4} F(\tau_1, \tau_2) = \int_{\kappa_3}^{\tau_1} \int_{\kappa_4}^{\tau_2} \frac{\varphi(\xi - \tau)}{\xi - \eta} \frac{\psi(\eta - \eta)}{\eta - \tau_2} F(\xi, \eta)d\eta d\xi, \quad \tau_1 < \kappa_2, \tau_2 > \kappa_3
\]
(2.6)

where \( \varphi, \psi : [0, \infty) \rightarrow [0, \infty) \) are functions which satisfy
\[
\int_0^\infty \frac{\varphi(\xi)}{\xi} d\xi < \infty \quad \text{and} \quad \int_0^\infty \frac{\psi(\eta)}{\eta} d\eta < \infty, \quad \text{respectively.}
\]

In this definition, known fractional integrals can be obtained by some special choices. For example, (i) If we take \( \varphi(\xi) = \xi \) and \( \psi(\eta) = \eta \), then the operators (2.3), (2.4), (2.5) and (2.6) transform into the Riemann integrals on coordinates, respectively, as follows:

\[
I_{\varphi \psi}^{\kappa_1, \kappa_2} F(\tau_1, \tau_2) = \int_{\kappa_1}^{\tau_1} \int_{\kappa_2}^{\tau_2} F(\xi, \eta)d\eta d\xi, \quad \tau_1 > \kappa_1, \tau_2 > \kappa_2,
\]

\[
I_{\varphi \psi}^{\kappa_1, \kappa_2} F(\tau_1, \tau_2) = \int_{\kappa_1}^{\tau_1} \int_{\kappa_2}^{\tau_2} F(\xi, \eta)d\eta d\xi, \quad \tau_1 > \kappa_1, \tau_2 < \kappa_2,
\]

\[
I_{\varphi \psi}^{\kappa_1, \kappa_2} F(\tau_1, \tau_2) = \int_{\kappa_1}^{\tau_1} \int_{\kappa_2}^{\tau_2} F(\xi, \eta)d\eta d\xi, \quad \tau_1 < \kappa_1, \tau_2 > \kappa_2
\]

and

\[
I_{\varphi \psi}^{\kappa_1, \kappa_2} F(\tau_1, \tau_2) = \int_{\kappa_1}^{\tau_1} \int_{\kappa_2}^{\tau_2} F(\xi, \eta)d\eta d\xi, \quad \tau_1 < \kappa_2, \tau_2 < \kappa_2.
\]
(ii) If we take $\varphi(\xi) = \frac{\xi^\alpha}{\Gamma(\alpha)}$, $\psi(\eta) = \frac{\eta^\beta}{\Gamma(\beta)}$, then for $\alpha, \beta > 0$ the operators (2.3), (2.4), (2.5) and (2.6) transform into the Riemann-Liouville fractional integrals on coordinates [8], respectively, as follows:

$$J^{a,b}_{\alpha\beta}(\tau_1, \tau_2) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_{\tau_1}^{\tau_2} (\tau - \xi)^{\alpha-1}(\eta - \eta)^{\beta-1} F(\xi, \eta) d\eta d\xi, \quad \tau_1 > k_1, \tau_2 > k_3,$$

$$J^{a,b}_{\alpha\beta}(\tau_1, \tau_2) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_{\tau_1}^{\tau_2} (\tau - \xi)^{\alpha-1}(\eta - \eta)^{\beta-1} F(\xi, \eta) d\eta d\xi, \quad \tau_1 > k_1, \tau_2 < k_3,$$

$$J^{a,b}_{\alpha\beta}(\tau_1, \tau_2) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_{\tau_1}^{\tau_2} (\tau - \xi)^{\alpha-1}(\eta - \eta)^{\beta-1} F(\xi, \eta) d\eta d\xi, \quad \tau_1 < k_2, \tau_2 > k_3,$$

and

$$J^{a,b}_{\alpha\beta}(\tau_1, \tau_2) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_{\tau_1}^{\tau_2} (\tau - \xi)^{\alpha-1}(\eta - \eta)^{\beta-1} F(\xi, \eta) d\eta d\xi, \quad \tau_1 < k_2, \tau_2 < k_3,$$

where $\Gamma$ is the gamma function.

(iii) If we take $\varphi(\xi) = \frac{\xi^k}{\Gamma(k)}$ and $\psi(\eta) = \frac{\eta^k}{\Gamma(k)}$, for $\alpha, \beta, k > 0$, then the operators (2.3), (2.4), (2.5) and (2.6) transform into the Riemann-Liouville k-fractional integrals on coordinates [48], respectively, as follows:

$$J^{a,b,k}_{\alpha\beta}(\tau_1, \tau_2) = \frac{1}{k^2 \Gamma_k(\alpha)\Gamma_k(\beta)} \int_{\tau_1}^{\tau_2} (\tau - \xi)^{\alpha-1}(\eta - \eta)^{\beta-1} F(\xi, \eta) d\eta d\xi, \quad \tau_1 > k_1, \tau_2 > k_3,$$

$$J^{a,b,k}_{\alpha\beta}(\tau_1, \tau_2) = \frac{1}{k^2 \Gamma_k(\alpha)\Gamma_k(\beta)} \int_{\tau_1}^{\tau_2} (\tau - \xi)^{\alpha-1}(\eta - \eta)^{\beta-1} F(\xi, \eta) d\eta d\xi, \quad \tau_1 > k_1, \tau_2 < k_3,$$

$$J^{a,b,k}_{\alpha\beta}(\tau_1, \tau_2) = \frac{1}{k^2 \Gamma_k(\alpha)\Gamma_k(\beta)} \int_{\tau_1}^{\tau_2} (\tau - \xi)^{\alpha-1}(\eta - \eta)^{\beta-1} F(\xi, \eta) d\eta d\xi, \quad \tau_1 < k_2, \tau_2 > k_3,$$

and

$$J^{a,b,k}_{\alpha\beta}(\tau_1, \tau_2) = \frac{1}{k^2 \Gamma_k(\alpha)\Gamma_k(\beta)} \int_{\tau_1}^{\tau_2} (\tau - \xi)^{\alpha-1}(\eta - \eta)^{\beta-1} F(\xi, \eta) d\eta d\xi, \quad \tau_1 < k_2, \tau_2 < k_3,$$

where $\Gamma_k$ is the k-gamma function.

### 3 An Identity

Throughout this study, for brevity, we define

$$\Delta(\xi) = \int_0^\xi \varphi((k_2 - k_1)u) du, \quad \Lambda(\eta) = \int_0^\eta \psi((k_2 - k_1)u) du.$$  \quad (3.1)

Now we give an identity for generalized fractional integrals.
Lemma 1. Let $F : \Delta \to \mathbb{R}$ be a twice partially differentiable mapping on $\Delta$. If $\frac{\partial^2 F}{\partial \xi \partial \eta} \in L(\Delta)$, then we have the following equality:

$$
\Theta(\kappa_1, \kappa_2; \kappa_3, \kappa_4) = (\kappa_2 - \kappa_1)(\kappa_4 - \kappa_3) \int_0^1 \int_0^1 w(\xi, \eta) \frac{\partial^2 F}{\partial \xi \partial \eta} (\xi, \eta) \, d\xi \, d\eta
$$

for all $\lambda_i, \mu_i \geq 0$, $i = 1, 2$, where the mapping $w : [0, 1] \times [0, 1] \to \mathbb{R}$ is defined by

$$
w(\xi, \eta) = 
\begin{cases}
\Big((\Delta(\xi) - \Delta(1)\lambda_1)(\Delta(\eta) - \Delta(1)\lambda_2), & 0 \leq \xi \leq 1, 0 \leq \eta \leq \frac{1}{2}, \\
(\Delta(\xi) - \Delta(1)\lambda_1)(\Delta(\eta) - \Delta(1)\lambda_2), & 0 \leq \xi \leq \frac{1}{2}, \frac{1}{2} \leq \eta \leq 1, \\
(\Delta(\xi) - \Delta(1)\mu_1)(\Delta(\eta) - \Delta(1)\mu_2), & \frac{1}{2} \leq \xi \leq 1, 0 \leq \eta \leq \frac{1}{2}, \\
(\Delta(\xi) - \Delta(1)\mu_1)(\Delta(\eta) - \Delta(1)\mu_2), & \frac{1}{2} \leq \xi \leq 1, \frac{1}{2} \leq \eta \leq 1
\end{cases}
$$

and

$$
\Theta(\kappa_1, \kappa_2; \kappa_3, \kappa_4) = 
\Delta(1)(\mu_2 - \lambda_2)\Delta(1)(\mu_2 - \lambda_2) F \left( \frac{\kappa_2 + \kappa_3}{2}, \frac{\kappa_2 + \kappa_4}{2} \right) + \Delta(1)(1 - \mu_2)\Delta(1)(\mu_2 - \lambda_2) F \left( \frac{\kappa_2 + \kappa_3}{2}, \frac{\kappa_2 + \kappa_3}{2}, \kappa_3 \right)
$$

Proof. From the definition of the mapping $w(\xi, \eta)$, we have

$$
(\kappa_2 - \kappa_1)(\kappa_4 - \kappa_3) \int_0^1 \int_0^1 w(\xi, \eta) \frac{\partial^2 F}{\partial \xi \partial \eta} (\xi, \eta) \, d\xi \, d\eta
$$

$$
= (\kappa_2 - \kappa_1)(\kappa_4 - \kappa_3) \int_0^1 \int_0^1 (\Delta(\xi) - \Delta(1)\lambda_1)(\Delta(\eta) - \Delta(1)\lambda_2) \frac{\partial^2 F}{\partial \xi \partial \eta} (\xi, \eta) \, d\xi \, d\eta
$$

$$
+ \int_0^1 \int_0^1 (\Delta(\xi) - \Delta(1)\lambda_1)(\Delta(\eta) - \Delta(1)\mu_2) \frac{\partial^2 F}{\partial \xi \partial \eta} (\xi, \eta) \, d\xi \, d\eta
$$
Integrating by parts, one can easily obtain

\[ J_1 = \int_0^1 \int_0^1 \left[ \left( \Delta \left( \frac{1}{2} \right) - \Delta(1) \right) \Lambda(1) \right] F \left( \frac{k_1^2 + k_2^2}{2}, \frac{k_3 + k_4}{2} \right) \]

Hence, we obtain

\[ J_1 = \int_0^1 \int_0^1 \left[ \left( \Delta \left( \frac{1}{2} \right) - \Delta(1) \right) \Lambda(1) \right] F \left( \frac{k_1^2 + k_2^2}{2}, \frac{k_3 + k_4}{2} \right) \]
Similarly, we get

$$J_2 = \frac{1}{2} \int_0^1 (\Delta(\xi) - \Delta(1)\lambda_1)(\Lambda(\eta) - \Lambda(1)\mu_2) \frac{\partial^2 F}{\partial \xi \partial \eta}(\xi_2 + (1 - \xi)\nu_1, \eta\nu_2 + (1 - \eta)\nu_3) \, d\xi \, d\eta$$

$$= \frac{1}{(k_2 - k_3)(k_4 - k_3)} \left[ (\Delta(1) - \Delta(1)\lambda_1)(\Lambda(1) - \Lambda(1)\mu_2)F\left(\frac{k_1 + k_2}{2}, \frac{k_1 + k_4}{2}\right)
+ \Delta(1)\lambda_1(\Lambda(1) - \Lambda(1)\mu_2)F(k_1, k_3)
- \Delta(1)\mu_2(\Lambda(1) - \Lambda(1)\mu_2)F\left(\frac{k_1 + k_2}{2}, \frac{k_1 + k_4}{2}\right)ight]$$

$$= \frac{1}{(k_2 - k_3)(k_4 - k_3)} \left[ \left(\Delta(1) - \Delta(1)\lambda_1\right)\left(\Lambda(1) - \Lambda(1)\mu_2\right)F\left(\frac{k_1 + k_2}{2}, \frac{k_1 + k_4}{2}\right)
- \Delta(1)\lambda_1\left(\Lambda(1) - \Lambda(1)\mu_2\right)F(k_1, k_3)
- \Delta(1)\mu_2\left(\Lambda(1) - \Lambda(1)\mu_2\right)F\left(\frac{k_1 + k_2}{2}, \frac{k_1 + k_4}{2}\right)ight)$$

and

$$J_3 = \frac{1}{2} \int_0^1 (\Delta(\xi) - \Delta(1)\mu_2)(\Lambda(\eta) - \Lambda(1)\lambda_2) \frac{\partial^2 F}{\partial \xi \partial \eta}(\xi_2 + (1 - \xi)\nu_1, \eta\nu_2 + (1 - \eta)\nu_3) \, d\xi \, d\eta$$

$$= \frac{1}{(k_2 - k_3)(k_4 - k_3)} \left[ (\Delta(1) - \Delta(1)\mu_2)(\Lambda(1) - \Lambda(1)\lambda_2)F(k_2, k_3)
- \Delta(1)\lambda_2(\Lambda(1) - \Lambda(1)\mu_2)F\left(\frac{k_1 + k_2}{2}, \frac{k_1 + k_4}{2}\right)
- \Delta(1)\mu_1(\Lambda(1) - \Lambda(1)\mu_2)F\left(\frac{k_1 + k_2}{2}, \frac{k_1 + k_4}{2}\right)ight]$$

$$= \frac{1}{(k_2 - k_3)(k_4 - k_3)} \left[ \left(\Delta(1) - \Delta(1)\mu_2\right)\left(\Lambda(1) - \Lambda(1)\lambda_2\right)F(k_2, k_3)
- \Delta(1)\lambda_2\left(\Lambda(1) - \Lambda(1)\mu_2\right)F\left(\frac{k_1 + k_2}{2}, \frac{k_1 + k_4}{2}\right)
- \Delta(1)\mu_1\left(\Lambda(1) - \Lambda(1)\mu_2\right)F\left(\frac{k_1 + k_2}{2}, \frac{k_1 + k_4}{2}\right)ight]$$

and

$$J_3 = \frac{1}{2} \int_0^1 (\Delta(\xi) - \Delta(1)\mu_2)(\Lambda(\eta) - \Lambda(1)\lambda_2) \frac{\partial^2 F}{\partial \xi \partial \eta}(\xi_2 + (1 - \xi)\nu_1, \eta\nu_2 + (1 - \eta)\nu_3) \, d\xi \, d\eta$$

$$= \frac{1}{(k_2 - k_3)(k_4 - k_3)} \left[ (\Delta(1) - \Delta(1)\mu_2)(\Lambda(1) - \Lambda(1)\lambda_2)F(k_2, k_3)
- \Delta(1)\lambda_2(\Lambda(1) - \Lambda(1)\mu_2)F\left(\frac{k_1 + k_2}{2}, \frac{k_1 + k_4}{2}\right)
- \Delta(1)\mu_1(\Lambda(1) - \Lambda(1)\mu_2)F\left(\frac{k_1 + k_2}{2}, \frac{k_1 + k_4}{2}\right)ight]$$

$$= \frac{1}{(k_2 - k_3)(k_4 - k_3)} \left[ \left(\Delta(1) - \Delta(1)\mu_2\right)\left(\Lambda(1) - \Lambda(1)\lambda_2\right)F(k_2, k_3)
- \Delta(1)\lambda_2\left(\Lambda(1) - \Lambda(1)\mu_2\right)F\left(\frac{k_1 + k_2}{2}, \frac{k_1 + k_4}{2}\right)
- \Delta(1)\mu_1\left(\Lambda(1) - \Lambda(1)\mu_2\right)F\left(\frac{k_1 + k_2}{2}, \frac{k_1 + k_4}{2}\right)ight]$$
By the equalities (3.2)–(3.5), we have

\[
(\kappa_2 - \kappa_1)(\kappa_4 - \kappa_3)[I_1 + I_2 + J_3 + J_4]
\]
\[
= (\Delta(1)\mu_1 - \Delta(1)\lambda_1)(\Delta(1)\mu_2 - \Delta(1)\lambda_2)F\left(\frac{\kappa_1 + \kappa_2}{2}, \frac{\kappa_3 + \kappa_4}{2}\right) + \Delta(1)\lambda_2(\Delta(1)\mu_2 - \Delta(1)\lambda_2)F\left(\frac{\kappa_1 + \kappa_2}{2}, \frac{\kappa_3 + \kappa_4}{2}\right)
\]
\[
+ (\Delta(1) - \Delta(1)\mu_1)(\Delta(1)\mu_2 - \Delta(1)\lambda_2)F\left(\frac{\kappa_2}{2}, \frac{\kappa_3 + \kappa_4}{2}\right) + (\Delta(1) - \Delta(1)\lambda_2)(\Delta(1)\lambda_2)F\left(\frac{\kappa_2}{2}, \frac{\kappa_3 + \kappa_4}{2}\right)
\]
\[
+ (\Delta(1)\mu_1 - \Delta(1)\lambda_1)(\Delta(1)\mu_2 - \Delta(1)\lambda_2)F\left(\frac{\kappa_1 + \kappa_2}{2}, \kappa_3\right) + (\Delta(1)\lambda_2 - \Delta(1)\mu_2)(\Delta(1)\lambda_2)F\left(\kappa_2, \kappa_3\right)
\]
\[
- (\Delta(1)\mu_2 - \Delta(1)\lambda_2)\left[I_\phi F\left(\frac{\kappa_3 + \kappa_4}{2}, \frac{\kappa_3 + \kappa_4}{2}\right) + \frac{1}{2}I_\psi F\left(\kappa_3, \frac{\kappa_3 + \kappa_4}{2}\right)\right]
\]
\[
- (\Delta(1)\mu_1 - \Delta(1)\lambda_1)\left[I_\phi F\left(\frac{\kappa_2}{2}, \frac{\kappa_3 + \kappa_4}{2}\right) + \frac{1}{2}I_\psi F\left(\kappa_2, \frac{\kappa_3 + \kappa_4}{2}\right)\right]
\]
\[
- (\Delta(1)\mu_1)\left[I_\phi F\left(\kappa_1, \kappa_3\right) + \frac{1}{2}I_\psi F\left(\kappa_1, \kappa_3\right)\right]
\]
\[
+ (\Delta(1) - \Delta(1)\mu_1)(\Delta(1)\mu_2)F\left(\kappa_1, \kappa_3\right) + (\Delta(1) - \Delta(1)\mu_1)(\Delta(1)\mu_2)F\left(\kappa_2, \kappa_3\right)
\]
\[
- (\Delta(1)\mu_2 - \Delta(1)\lambda_2)\left[I_\phi F\left(\frac{\kappa_3 + \kappa_4}{2}, \frac{\kappa_3 + \kappa_4}{2}\right) + \frac{1}{2}I_\psi F\left(\kappa_3, \frac{\kappa_3 + \kappa_4}{2}\right)\right]
\]
\[
\]
which completes the proof.

**Remark 1.** If we assume \(\phi(\xi) = \xi\) and \(\psi(\eta) = \eta\) in Lemma 1, we obtain,

\[
\Phi(\kappa_1, \kappa_2; \kappa_3, \kappa_4) = (\kappa_2 - \kappa_1)(\kappa_4 - \kappa_3)\int_0^{1/2} \int_0^{1/2} (\xi - \lambda_1)(\eta - \lambda_2) \frac{\partial^2 F}{\partial \xi \partial \eta} (\xi \kappa_2 + (1 - \xi) \kappa_3, \eta \kappa_4 + (1 - \eta) \kappa_3) d\xi d\eta
\]
\[
+ (\kappa_1 - \kappa_2)(\kappa_3 - \kappa_4)\int_0^{1/2} \int_0^{1/2} (\xi - \lambda_1)(\eta - \mu_2) \frac{\partial^2 F}{\partial \xi \partial \eta} (\xi \kappa_2 + (1 - \xi) \kappa_3, \eta \kappa_4 + (1 - \eta) \kappa_3) d\xi d\eta
\]
\[
+ (\kappa_2 - \kappa_1)(\kappa_4 - \kappa_3)\int_0^{1/2} \int_0^{1/2} (\xi - \mu_1)(\eta - \lambda_2) \frac{\partial^2 F}{\partial \xi \partial \eta} (\xi \kappa_2 + (1 - \xi) \kappa_3, \eta \kappa_4 + (1 - \eta) \kappa_3) d\xi d\eta
\]
\[
+ (\kappa_1 - \kappa_2)(\kappa_3 - \kappa_4)\int_0^{1/2} \int_0^{1/2} (\xi - \mu_1)(\eta - \mu_2) \frac{\partial^2 F}{\partial \xi \partial \eta} (\xi \kappa_2 + (1 - \xi) \kappa_3, \eta \kappa_4 + (1 - \eta) \kappa_3) d\xi d\eta,
\]

where

\[
\Phi(\kappa_1, \kappa_2; \kappa_3, \kappa_4) = (\mu_1 - \lambda_1)(\mu_2 - \lambda_2)F\left(\frac{\kappa_1 + \kappa_2}{2}, \frac{\kappa_3 + \kappa_4}{2}\right) + \lambda_2(\mu_2 - \lambda_2)F\left(\frac{\kappa_1 + \kappa_2}{2}, \frac{\kappa_3 + \kappa_4}{2}\right)
\]
\[
+ (\mu_2 - \lambda_2)\left[I_\phi F\left(\frac{\kappa_2}{2}, \frac{\kappa_3 + \kappa_4}{2}\right) + \frac{1}{2}I_\psi F\left(\kappa_2, \frac{\kappa_3 + \kappa_4}{2}\right)\right]
\]
\[
+ (\mu_1 - \lambda_1)(\mu_2 - \lambda_2)F\left(\frac{\kappa_1 + \kappa_2}{2}, \kappa_3\right) + (\mu_1 - \lambda_1)(\mu_2 - \lambda_2)F\left(\kappa_2, \kappa_3\right)
\]
\[
+ \lambda_2(\mu_2 - \lambda_2)F\left(\kappa_2, \kappa_3\right) + (\mu_1 - \lambda_1)(\mu_2 - \lambda_2)F\left(\kappa_1, \kappa_3\right) + (\mu_1 - \lambda_1)(\mu_2 - \lambda_2)F\left(\kappa_2, \kappa_3\right)
\]
which is given by Budak and Ali in [49].

**Corollary 1.** In Lemma 1, if we set \( \varphi(x) = \frac{\xi^a}{\Gamma(a)} \) and \( \psi(y) = \frac{\xi^b}{\Gamma(b)} \), then we obtain the following new Riemann-Liouville fractional integral identity:

\[
\Omega(k_1, k_2; k_3, k_4) = (k_2 - k_1)(k_3 - k_1) \int_0^1 \int_0^1 \int_0^1 \int_0^1 (\xi^a - \lambda_1)(\eta^b - \lambda_2) \frac{\partial^2 F}{\partial \xi \partial \eta} \left( \xi k_2 + (1 - \xi) k_1, \eta k_2 + (1 - \eta) k_1 \right) d\xi d\eta
\]

where

\[
\Omega(k_1, k_2; k_3, k_4) = (\mu_1 - \lambda_1)(\mu_2 - \lambda_2) F \left( \frac{k_1 + k_2}{2} , \frac{k_3 + k_4}{2} \right) + \lambda_1 (\mu_2 - \lambda_2) F \left( k_1, \frac{k_3 + k_4}{2} \right) + (1 - \mu_1)(\mu_2 - \lambda_2) F \left( k_2, \frac{k_3 + k_4}{2} \right) + \lambda_1 (1 - \mu_2) F \left( k_1, k_2 \right) + \lambda_2 (1 - \mu_2) F \left( k_3, \frac{k_4}{2} \right) + \lambda_2 (1 - \mu_2) F \left( k_1, \frac{k_3 + k_4}{2} \right) - \lambda_1 (1 - \mu_2) F \left( k_1, k_2 \right) - \lambda_2 (1 - \mu_2) F \left( k_1, k_2 \right) - \lambda_2 (1 - \mu_2) F \left( k_1, k_2 \right) - \lambda_2 (1 - \mu_2) F \left( k_1, k_2 \right)
\]

where \( \Omega(k_1, k_2; k_3, k_4) \) is defined as in Corollary 1.
Corollary 2. In Lemma 1, if we set \( \varphi(\xi) = \frac{\xi^\alpha}{\Gamma(\alpha)} \) and \( \psi(\eta) = \frac{\eta^\beta}{\Gamma(\beta)} \), then we obtain the following new \( k \)-Riemann-Liouville fractional integral identity:

\[
\mathcal{J}(k_1, k_2; k_3, k_4) = \frac{1}{2} \left( \xi - \lambda_1 \right) \left( \eta^\beta - \lambda_2 \right) \frac{\partial^2 \mathcal{F}}{\partial \xi \partial \eta} (\xi k_2 + (1 - \xi) k_3, \eta k_4 + (1 - \eta) k_5) d\xi d\eta \\
+ (k_2 - k_1)(k_4 - k_3) \int_0^1 \left( \xi^\alpha - \lambda_1 \right) \left( \eta^\beta - \lambda_2 \right) \frac{\partial^2 \mathcal{F}}{\partial \xi \partial \eta} (\xi k_2 + (1 - \xi) k_3, \eta k_4 + (1 - \eta) k_5) d\xi d\eta \\
+ (k_2 - k_1)(k_4 - k_3) \int_0^1 \left( \xi^\alpha - \lambda_1 \right) \left( \eta^\beta - \lambda_2 \right) \frac{\partial^2 \mathcal{F}}{\partial \xi \partial \eta} (\xi k_2 + (1 - \xi) k_3, \eta k_4 + (1 - \eta) k_5) d\xi d\eta \\
+ (k_2 - k_1)(k_4 - k_3) \int_0^1 \left( \xi^\alpha - \lambda_1 \right) \left( \eta^\beta - \lambda_2 \right) \frac{\partial^2 \mathcal{F}}{\partial \xi \partial \eta} (\xi k_2 + (1 - \xi) k_3, \eta k_4 + (1 - \eta) k_5) d\xi d\eta,
\]

where

\[
\mathcal{J}(k_1, k_2; k_3, k_4) = (\mu_1 - \lambda_1)(\mu_2 - \lambda_2) \mathcal{F} \left( \frac{k_1 + k_2}{2}, \frac{k_3 + k_4}{2} \right) + \lambda(\mu_2 - \lambda_2) \mathcal{F} \left( k_1, \frac{k_3 + k_4}{2} \right) \\
+ (1 - \mu_1)(\mu_2 - \lambda_2) \mathcal{F} \left( k_2, \frac{k_1 + k_4}{2} \right) + (\mu_1 - \lambda_1)(1 - \mu_2) \mathcal{F} \left( k_1 + k_2, \frac{k_3 + k_4}{2} \right) + (1 - \mu_1) \mathcal{F} \left( k_1 + k_2, \frac{k_3 + k_4}{2} \right)
\]

where \( \mathcal{J}(k_1, k_2; k_3, k_4) \) is defined as in Corollary 2.

4 Some new inequalities for generalized fractional integrals

In this section, we establish some new Simpson-type inequalities for differentiable co-ordinated convex functions via generalized fractional integrals.
Theorem 2. We assume that the conditions of Lemma 1 hold. If the mapping \( \frac{\partial^\mathcal{F}}{\partial \xi \partial \eta} \) is co-ordinated convex on \([k_1, k_2]\), then the following inequality holds for generalized fractional integrals:

\[
|\Theta(k_1, k_2; \kappa, \kappa)| \leq (k_2 - k_1)(k_2 - k_3) \times \left[ \int\int \left( \frac{\partial^\mathcal{F}}{\partial \xi \partial \eta}(k_2, k_3) \right) \left( \Pi^\mathcal{F}_1(\lambda_1)\nabla^\mathcal{F}_1(\lambda_2) + \Pi^\mathcal{F}_2(\mu_1)\nabla^\mathcal{F}_2(\mu_2) + \Pi^\mathcal{F}_3(\mu_3)\nabla^\mathcal{F}_3(\mu_4) + \Pi^\mathcal{F}_4(\mu_4)\nabla^\mathcal{F}_4(\mu_4) \right) \right] + \frac{\partial^\mathcal{F}}{\partial \xi \partial \eta}(k_2, k_3) \left( \Pi^\mathcal{F}_1(\lambda_1)\nabla^\mathcal{F}_1(\lambda_2) + \Pi^\mathcal{F}_2(\mu_1)\nabla^\mathcal{F}_2(\mu_2) + \Pi^\mathcal{F}_3(\mu_3)\nabla^\mathcal{F}_3(\mu_4) + \Pi^\mathcal{F}_4(\mu_4)\nabla^\mathcal{F}_4(\mu_4) \right) + \frac{\partial^\mathcal{F}}{\partial \xi \partial \eta}(k_2, k_3) \left( \Pi^\mathcal{F}_1(\lambda_1)\nabla^\mathcal{F}_1(\lambda_2) + \Pi^\mathcal{F}_2(\mu_1)\nabla^\mathcal{F}_2(\mu_2) + \Pi^\mathcal{F}_3(\mu_3)\nabla^\mathcal{F}_3(\mu_4) + \Pi^\mathcal{F}_4(\mu_4)\nabla^\mathcal{F}_4(\mu_4) \right) + \frac{\partial^\mathcal{F}}{\partial \xi \partial \eta}(k_2, k_3) \left( \Pi^\mathcal{F}_1(\lambda_1)\nabla^\mathcal{F}_1(\lambda_2) + \Pi^\mathcal{F}_2(\mu_1)\nabla^\mathcal{F}_2(\mu_2) + \Pi^\mathcal{F}_3(\mu_3)\nabla^\mathcal{F}_3(\mu_4) + \Pi^\mathcal{F}_4(\mu_4)\nabla^\mathcal{F}_4(\mu_4) \right) \right],
\]

where

\[
\Pi^\mathcal{F}_1(\lambda_1) = \int_0^1 (1 - \xi)|\Delta(\xi) - \Delta(1)|d\xi, \quad \Pi^\mathcal{F}_2(\lambda_1) = \int_0^1 (1 - \xi)|\Delta(\xi) - \Delta(1)|d\xi,
\]

\[
\Pi^\mathcal{F}_3(\lambda_3) = \int_0^1 (1 - \eta)|\Delta(\eta) - \Delta(1)|d\eta, \quad \Pi^\mathcal{F}_4(\lambda_4) = \int_0^1 (1 - \eta)|\Delta(\eta) - \Delta(1)|d\eta,
\]

\[
\nabla^\mathcal{F}_1(\mu_1) = \int_0^1 (1 - \xi)|\nabla(\xi) - \nabla(1)|d\xi, \quad \nabla^\mathcal{F}_2(\mu_2) = \int_0^1 (1 - \xi)|\nabla(\xi) - \nabla(1)|d\xi,
\]

\[
\nabla^\mathcal{F}_3(\mu_3) = \int_0^1 (1 - \eta)|\nabla(\eta) - \nabla(1)|d\eta, \quad \nabla^\mathcal{F}_4(\mu_4) = \int_0^1 (1 - \eta)|\nabla(\eta) - \nabla(1)|d\eta.
\]

Proof. By taking the modulus in Lemma 1 and using the properties of the modulus, we obtain that

\[
|\Theta(k_1, k_2; \kappa, \kappa)| \leq (k_2 - k_1)(k_2 - k_3) \times \left[ \int\int \left( \frac{\partial^\mathcal{F}}{\partial \xi \partial \eta}(k_2, k_3) \right) \left( \Pi^\mathcal{F}_1(\lambda_1)\nabla^\mathcal{F}_1(\lambda_2) + \Pi^\mathcal{F}_2(\mu_1)\nabla^\mathcal{F}_2(\mu_2) + \Pi^\mathcal{F}_3(\mu_3)\nabla^\mathcal{F}_3(\mu_4) + \Pi^\mathcal{F}_4(\mu_4)\nabla^\mathcal{F}_4(\mu_4) \right) \right] + \frac{\partial^\mathcal{F}}{\partial \xi \partial \eta}(k_2, k_3) \left( \Pi^\mathcal{F}_1(\lambda_1)\nabla^\mathcal{F}_1(\lambda_2) + \Pi^\mathcal{F}_2(\mu_1)\nabla^\mathcal{F}_2(\mu_2) + \Pi^\mathcal{F}_3(\mu_3)\nabla^\mathcal{F}_3(\mu_4) + \Pi^\mathcal{F}_4(\mu_4)\nabla^\mathcal{F}_4(\mu_4) \right) + \frac{\partial^\mathcal{F}}{\partial \xi \partial \eta}(k_2, k_3) \left( \Pi^\mathcal{F}_1(\lambda_1)\nabla^\mathcal{F}_1(\lambda_2) + \Pi^\mathcal{F}_2(\mu_1)\nabla^\mathcal{F}_2(\mu_2) + \Pi^\mathcal{F}_3(\mu_3)\nabla^\mathcal{F}_3(\mu_4) + \Pi^\mathcal{F}_4(\mu_4)\nabla^\mathcal{F}_4(\mu_4) \right) + \frac{\partial^\mathcal{F}}{\partial \xi \partial \eta}(k_2, k_3) \left( \Pi^\mathcal{F}_1(\lambda_1)\nabla^\mathcal{F}_1(\lambda_2) + \Pi^\mathcal{F}_2(\mu_1)\nabla^\mathcal{F}_2(\mu_2) + \Pi^\mathcal{F}_3(\mu_3)\nabla^\mathcal{F}_3(\mu_4) + \Pi^\mathcal{F}_4(\mu_4)\nabla^\mathcal{F}_4(\mu_4) \right) \right],
\]

(4.3)
Since the mapping $\left| \frac{\partial F}{\partial \eta} \right|$ is co-ordinated convex on $\Delta$, therefore, we have

$$
\begin{align*}
\int_{\frac{1}{2}}^{\frac{1}{2}} \int_{0}^{1} |\Delta (\xi) - \Delta (1) \lambda | \Delta (\eta) - \Delta (1) \lambda | \frac{\partial F}{\partial \xi \eta} (\xi_2 + (1 - \xi) \kappa, \eta \kappa + (1 - \eta) \kappa) \, d\xi \, d\eta \\
&\leq \int_{\frac{1}{2}}^{\frac{1}{2}} \int_{0}^{1} |\Delta (\xi) - \Delta (1) \lambda | \Delta (\eta) - \Delta (1) \lambda | \left( \xi \frac{\partial F}{\partial \xi \eta} (\kappa \kappa, \kappa \kappa) + (1 - \xi) \frac{\partial F}{\partial \xi \eta} (\kappa \kappa, \kappa \kappa) \right) \\
&+ (1 - \xi) \eta \frac{\partial F}{\partial \xi \eta} (\kappa \kappa, \kappa \kappa) \right) d\xi \, d\eta \\
&= \Pi (\lambda_1) \Pi (\lambda_2) \left. \frac{\partial F}{\partial \xi \eta} (\kappa \kappa, \kappa \kappa) \right) + \Pi (\lambda_1) \Pi (\lambda_2) \left. \frac{\partial F}{\partial \xi \eta} (\kappa \kappa, \kappa \kappa) \right) + \Pi (\lambda_1) \Pi (\lambda_2) \left. \frac{\partial F}{\partial \xi \eta} (\kappa \kappa, \kappa \kappa) \right).
\end{align*}
$$

(4.4)

Similarly, we obtain

$$
\begin{align*}
\int_{\frac{1}{2}}^{\frac{1}{2}} \int_{0}^{1} |\Delta (\xi) - \Delta (1) \mu | \Delta (\eta) - \Delta (1) \mu | \frac{\partial F}{\partial \xi \eta} (\xi_2 + (1 - \xi) \kappa, \eta \kappa + (1 - \eta) \kappa) \, d\xi \, d\eta \\
&= \Pi (\mu_1) \Pi (\mu_2) \left. \frac{\partial F}{\partial \xi \eta} (\kappa \kappa, \kappa \kappa) \right) + \Pi (\mu_1) \Pi (\mu_2) \left. \frac{\partial F}{\partial \xi \eta} (\kappa \kappa, \kappa \kappa) \right) + \Pi (\mu_1) \Pi (\mu_2) \left. \frac{\partial F}{\partial \xi \eta} (\kappa \kappa, \kappa \kappa) \right).
\end{align*}
$$

(4.5)

By the inequalities (4.4)–(4.7), the proof is completed. □

**Remark 2.** In Theorem 2, if we take $\phi (\xi) = \xi$ and $\psi (\eta) = \eta$, we obtain the inequality

$$
|\Phi (\kappa _1, \kappa _2; \kappa _1, \kappa _2) | \leq (\kappa _2 - \kappa _1) (\kappa _4 - \kappa _3) \times \left[ \Psi _1 \left. \frac{\partial F}{\partial \xi \eta} (\kappa _2, \kappa _4) \right) + \Psi _2 \left. \frac{\partial F}{\partial \xi \eta} (\kappa _3, \kappa _2) \right) + \Psi _3 \left. \frac{\partial F}{\partial \xi \eta} (\kappa _3, \kappa _3) \right) + \Psi _4 \left. \frac{\partial F}{\partial \xi \eta} (\kappa _4, \kappa _3) \right).
$$

Some new parameterized inequalities
where $\Phi(\kappa_1, \kappa_2; \kappa_3, \kappa_4)$ is defined as in Remark 1 and

$$
\psi_1 = \left[ \frac{\lambda_1^3 + \mu_1^3}{3} - \frac{\lambda_1}{8} - 5\mu_1 \frac{5}{8} + \frac{5}{12} \right] \left[ \frac{\lambda_2^3 + \mu_2^3}{3} - \frac{\lambda_2}{8} - 5\mu_2 + \frac{5}{12} \right],
$$

$$
\psi_2 = \left[ \frac{\lambda_1^3 + \mu_1^3}{3} - \frac{\lambda_1}{8} - 5\mu_1 \frac{5}{8} + \frac{5}{12} \right] \left[ \frac{\lambda_2^3 + \mu_2^3}{3} + \lambda_2^2 + \mu_2^2 - 7\mu_2 + 3\lambda_2 + \frac{1}{3} \right],
$$

$$
\psi_3 = \left[ - \frac{\lambda_1^3 + \mu_1^3}{3} + \lambda_1^2 + \mu_1^2 - 7\mu_1 + 3\lambda_1 \frac{1}{8} + \frac{1}{3} \right] \left[ \frac{\lambda_2^3 + \mu_2^3}{3} - \frac{\lambda_2}{8} - 5\mu_2 + \frac{5}{12} \right],
$$

and

$$
\psi_4 = \left[ - \frac{\lambda_1^3 + \mu_1^3}{3} + \lambda_1^2 + \mu_1^2 - 7\mu_1 + 3\lambda_1 \frac{1}{8} + \frac{1}{3} \right] \left[ \frac{\lambda_2^3 + \mu_2^3}{3} + \lambda_2^2 + \mu_2^2 - 7\mu_2 + 3\lambda_2 + \frac{1}{3} \right],
$$

which is given by Budak and Ali in [49].

**Corollary 3.** In Theorem 2, if we use $\varphi(x) = \frac{e^x}{x^3}$ and $\psi(y) = \frac{e^y}{y^3}$, then we obtain the following parameterized Simpson-type inequality for Riemann-Liouville fractional integrals:

$$
|\Omega(\kappa_1, \kappa_2; \kappa_3, \kappa_4)| \leq (\kappa_3 - \kappa_1)(\kappa_4 - \kappa_2)
$$

\[
\times \left[ \frac{\partial F}{\partial \xi_1}(\kappa_1, \kappa_2) \left( \Pi_1^{\alpha}(\lambda_1) \Pi_1^{\beta}(\lambda_2) + \Pi_2^{\alpha}(\mu_1) \Pi_2^{\beta}(\mu_2) + \Pi_3^{\alpha}(\lambda_1) \Pi_3^{\beta}(\lambda_2) + \Pi_4^{\alpha}(\mu_1) \Pi_4^{\beta}(\mu_2) \right) \right.
\]

\[
+ \left. \frac{\partial F}{\partial \xi_2}(\kappa_1, \kappa_2) \left( \Pi_1^{\alpha}(\lambda_1) \Pi_2^{\beta}(\lambda_2) + \Pi_2^{\alpha}(\mu_1) \Pi_2^{\beta}(\mu_2) + \Pi_3^{\alpha}(\lambda_1) \Pi_3^{\beta}(\lambda_2) + \Pi_4^{\alpha}(\mu_1) \Pi_4^{\beta}(\mu_2) \right) \right]
\]

\[
+ \frac{\partial F}{\partial \xi_3}(\kappa_1, \kappa_2) \left( \Pi_2^{\alpha}(\lambda_1) \Pi_1^{\beta}(\lambda_2) + \Pi_2^{\alpha}(\mu_1) \Pi_2^{\beta}(\mu_2) + \Pi_3^{\alpha}(\lambda_1) \Pi_3^{\beta}(\lambda_2) + \Pi_4^{\alpha}(\mu_1) \Pi_4^{\beta}(\mu_2) \right)
\]

\[
+ \frac{\partial F}{\partial \xi_4}(\kappa_1, \kappa_2) \left( \Pi_3^{\alpha}(\lambda_1) \Pi_2^{\beta}(\lambda_2) + \Pi_3^{\alpha}(\mu_1) \Pi_3^{\beta}(\mu_2) + \Pi_4^{\alpha}(\lambda_1) \Pi_4^{\beta}(\lambda_2) + \Pi_4^{\alpha}(\mu_1) \Pi_4^{\beta}(\mu_2) \right)
\]

where $\Omega(\kappa_1, \kappa_2; \kappa_3, \kappa_4)$ is defined as in Corollary 1 and

$$
\Pi_1^{\alpha}(\lambda_1) = \frac{\alpha}{a + 2} \lambda_1^{a+1} - \frac{\lambda_1}{8} + \frac{1}{2^{a+1}(a + 2)},
$$

$$
\Pi_2^{\alpha}(\lambda_1) = \frac{2\alpha}{a + 1} \lambda_1^{a+1} - \frac{\lambda_1}{2} + \frac{1}{2^{a+1}(a + 1)} - \Pi_1^{\alpha}(\lambda_1),
$$

$$
\Pi_3^{\alpha}(\mu_1) = \frac{\alpha}{a + 2} \mu_1^{a+1} - \frac{5}{8} \mu_1 + \frac{2a+1}{2^{a+1}(a + 2)},
$$

$$
\Pi_4^{\alpha}(\mu_1) = \frac{2\alpha}{a + 1} \mu_1^{a+1} - \frac{3}{2} \mu_1 + \frac{2a+1}{2^{a+1}(a + 1)} - \Pi_3^{\alpha}(\mu_1),
$$

and

$$
\nabla_1^0(\lambda_1) = \frac{1}{\beta + 2} \lambda_1^\beta - \frac{\lambda_1}{8} + \frac{1}{2^{\beta+1}(\beta + 2)},
$$

$$
\nabla_2^0(\lambda_1) = \frac{2\beta}{\beta + 1} \lambda_1^\beta - \frac{\lambda_1}{2} + \frac{1}{2^{\beta+1}(\beta + 1)} - \nabla_1^0(\lambda_1),
$$

$$
\nabla_3^0(\mu_1) = \frac{\beta}{\beta + 2} \mu_1^\beta - \frac{5}{8} \mu_1 + \frac{2\beta+1}{2^{\beta+1}(\beta + 2)},
$$

$$
\nabla_4^0(\mu_1) = \frac{2\beta}{\beta + 1} \mu_1^\beta - \frac{3}{2} \mu_1 + \frac{2\beta+1}{2^{\beta+1}(\beta + 1)} - \nabla_3^0(\mu_1).
Corollary 4. In Theorem 2, if we use \( \varphi (\xi) = \frac{\delta^a}{\Gamma(a+\alpha)} \) and \( \psi(\eta) = \frac{\delta^b}{\Gamma(b+\beta)} \), then we obtain the following parameterized Simpson-type inequality for \( k \)-Riemann-Liouville fractional integrals:

\[
|\mathcal{I}(\kappa_1, \kappa_2, \kappa_3, \kappa_4)| \leq (\kappa_2 - \kappa_1)(\kappa_4 - \kappa_3)
\times \left[ \frac{a}{\Pi^{(a+k)}_{\beta}} (\lambda_2) \left( \alpha + k \right) ^{\alpha+2k} \right] ^{\frac{a}{\alpha+k}} - \frac{\lambda_2}{8} + \frac{1}{2^{\alpha+k}(a+k)}
\geq \left[ \frac{a}{\Pi^{(a+k)}_{\beta}} (\lambda_2) \left( \alpha + k \right) ^{\alpha+2k} \right] ^{\frac{a}{\alpha+k}} - \frac{\lambda_2}{8} + \frac{1}{2^{\alpha+k}(a+k)}
\]

where \( \mathcal{I}(\kappa_1, \kappa_2, \kappa_3, \kappa_4) \) is defined as in Corollary 2 and

\[
\begin{align*}
\frac{a}{\Pi^{(a+k)}_{\beta}} (\lambda_2) &= \frac{a}{a + 2k} \frac{\lambda_2}{2} \left( \alpha + k \right) ^{\alpha+2k} - \frac{\lambda_2}{8} + \frac{1}{2^{\alpha+k}(a+k)}, \\
\frac{a}{\Pi^{(a+k)}_{\beta}} (\mu_2) &= \frac{a}{a + k} \frac{\mu_2}{2} \left( \alpha + k \right) ^{\alpha+2k} - \frac{\mu_2}{8} + \frac{1}{2^{\alpha+k}(a+k)}.
\end{align*}
\]

and

\[
\begin{align*}
\frac{a}{\Pi^{(a+k)}_{\beta}} (\lambda_2) &= \frac{a}{\beta + 2k} \frac{\lambda_2}{2} \left( \beta + k \right) ^{\beta+2k} - \frac{\lambda_2}{8} + \frac{1}{2^{\beta+k}(a+k)}, \\
\frac{a}{\Pi^{(a+k)}_{\beta}} (\mu_2) &= \frac{a}{\beta + k} \frac{\mu_2}{2} \left( \beta + k \right) ^{\beta+2k} - \frac{\mu_2}{8} + \frac{1}{2^{\beta+k}(a+k)}.
\end{align*}
\]

Theorem 3. We assume that the conditions of Lemma 1 hold. If the mapping \( \frac{\partial^\gamma \phi}{\partial \zeta^\delta} \) is co-ordinated convex on \( \Delta, q \geq 1 \), then we have the following inequality:

\[
|\Theta(\kappa_1, \kappa_2, \kappa_3, \kappa_4)| \geq (\kappa_2 - \kappa_1)(\kappa_4 - \kappa_3)
\times \left[ \frac{a}{\Pi^{(a+k)}_{\beta}} (\lambda_2) \left( \alpha + k \right) ^{\alpha+2k} \right] ^{\frac{a}{\alpha+k}} - \frac{\lambda_2}{8} + \frac{1}{2^{\alpha+k}(a+k)}
\geq \left[ \frac{a}{\Pi^{(a+k)}_{\beta}} (\lambda_2) \left( \alpha + k \right) ^{\alpha+2k} \right] ^{\frac{a}{\alpha+k}} - \frac{\lambda_2}{8} + \frac{1}{2^{\alpha+k}(a+k)}
\]

(4.8)
\[
\begin{align*}
&+ \Pi^\phi_2(\lambda_2) \nabla^\psi_2(\lambda_2) \left| \frac{\partial^2 F}{\partial \xi \partial \eta}(\kappa_1, \kappa_2) \right|^\varpi + \Pi^\phi_2(\lambda_3) \nabla^\psi_3(\lambda_3) \left| \frac{\partial^2 F}{\partial \xi \partial \eta}(\kappa_1, \kappa_3) \right|^\varpi \\
&+ \left( \int_0^{1/2} \left| \Lambda(\eta) - \Lambda(1) \mu_2 \right| d\eta \right)^{1-\frac{1}{\varpi}} \left( \int_0^{1/2} \left| \Lambda(\xi) - \Lambda(1) \lambda_2 \right| d\xi \right)^{1-\frac{1}{\varpi}} \\
&\times \left( \Pi^\phi_2(\lambda_2) \nabla^\psi_2(\mu_2) \left| \frac{\partial^2 F}{\partial \xi \partial \eta}(\kappa_2, \kappa_3) \right|^\varpi + \Pi^\phi_3(\lambda_3) \nabla^\psi_3(\mu_3) \left| \frac{\partial^2 F}{\partial \xi \partial \eta}(\kappa_2, \kappa_3) \right|^\varpi \right) \\
&+ \Pi^\phi_3(\mu_3) \nabla^\psi_3(\lambda_3) \left| \frac{\partial^2 F}{\partial \xi \partial \eta}(\kappa_1, \kappa_3) \right|^\varpi + \Pi^\phi_3(\mu_4) \nabla^\psi_4(\lambda_4) \left| \frac{\partial^2 F}{\partial \xi \partial \eta}(\kappa_1, \kappa_4) \right|^\varpi \\
&+ \left( \int_0^{1/2} \left| \Lambda(\eta) - \Lambda(1) \mu_2 \right| d\eta \right)^{1-\frac{1}{\varpi}} \left( \int_0^{1/2} \left| \Lambda(\xi) - \Lambda(1) \mu_1 \right| d\xi \right)^{1-\frac{1}{\varpi}} \\
&\times \left( \Pi^\phi_3(\mu_3) \nabla^\psi_3(\mu_2) \left| \frac{\partial^2 F}{\partial \xi \partial \eta}(\kappa_2, \kappa_3) \right|^\varpi + \Pi^\phi_4(\mu_4) \nabla^\psi_4(\mu_2) \left| \frac{\partial^2 F}{\partial \xi \partial \eta}(\kappa_2, \kappa_3) \right|^\varpi \right) \\
&+ \Pi^\phi_4(\mu_4) \nabla^\psi_4(\mu_3) \left| \frac{\partial^2 F}{\partial \xi \partial \eta}(\kappa_1, \kappa_3) \right|^\varpi + \Pi^\phi_4(\mu_5) \nabla^\psi_5(\mu_2) \left| \frac{\partial^2 F}{\partial \xi \partial \eta}(\kappa_1, \kappa_4) \right|^\varpi \right)^{1-\frac{1}{\varpi}}. \\
\end{align*}
\]

where \( \Pi^\phi_i \) and \( \nabla^\psi_i \), \( i = 1, 2, 3, 4 \) are defined as in Theorem 2.

**Proof.** By using power mean inequality in (4.3) and co-ordinated convexity of \( \left| \frac{\partial^2 F}{\partial \xi \partial \eta} \right| \), we have

\[
\begin{align*}
&\int_0^{1/2} \left| \Lambda(\xi) - \Lambda(1) \lambda_1 \right| \left| \Lambda(\eta) - \Lambda(1) \lambda_2 \right| \left| \frac{\partial^2 F}{\partial \xi \partial \eta}(\xi \kappa_2 + (1 - \xi) \kappa_1, \eta \kappa_4 + (1 - \eta) \kappa_3) \right| d\xi d\eta \\
&\leq \left( \int_0^{1/2} \left| \Lambda(\xi) - \Lambda(1) \lambda_1 \right| d\xi \right)^{1-\frac{1}{\varpi}} \left( \int_0^{1/2} \left| \Lambda(\eta) - \Lambda(1) \lambda_2 \right| d\eta \right)^{1-\frac{1}{\varpi}} \\
&\times \left( \int_0^{1/2} \left| \Lambda(\xi) - \Lambda(1) \lambda_1 \right| \left| \Lambda(\eta) - \Lambda(1) \lambda_2 \right| \left| \frac{\partial^2 F}{\partial \xi \partial \eta}(\xi \kappa_2 + (1 - \xi) \kappa_1, \eta \kappa_4 + (1 - \eta) \kappa_3) \right|^\varpi d\xi d\eta \right)^{1-\frac{1}{\varpi}} \\
&\leq \left( \int_0^{1/2} \left| \Lambda(\xi) - \Lambda(1) \lambda_1 \right| d\xi \right)^{1-\frac{1}{\varpi}} \left( \int_0^{1/2} \left| \Lambda(\eta) - \Lambda(1) \lambda_2 \right| d\eta \right)^{1-\frac{1}{\varpi}}.
\end{align*}
\]

(4.8)
\[
\int_0^1 \left[ (\Delta(\xi) - \Delta(1)\lambda \xi - \Lambda(\eta) - \Lambda(1)\lambda) \left( \frac{\partial^2 F}{\partial \xi^2} (\kappa_{\xi}, \kappa_{\eta}) \right) + \xi(1 - \eta) \left( \frac{\partial^2 F}{\partial \xi^2} (\kappa_{\xi}, \kappa_{\eta}) \right) \right] \, d\xi 
\]

\[
+ (1 - \xi) \eta \left( \frac{\partial^2 F}{\partial \xi^2} (\kappa_{\xi}, \kappa_{\eta}) \right) + (1 - \xi)(1 - \eta) \left( \frac{\partial^2 F}{\partial \xi^2} (\kappa_{\xi}, \kappa_{\eta}) \right) \right] \, d\xi = \frac{1}{2} \left( \int_0^1 (\Lambda(\eta) - \Lambda(1)\lambda) \, d\eta \right)^{1/2} \left( \int_0^1 (\Lambda(\xi) - \Lambda(1)\lambda) \, d\xi \right)^{1/2}
\]

\[
+ \left( \int_0^1 \left( \frac{\partial^2 F}{\partial \xi^2} (\kappa_{\xi}, \kappa_{\eta}) \right) \right) \, d\xi = \frac{1}{2} \int_0^1 \left( \frac{\partial^2 F}{\partial \xi^2} (\kappa_{\xi}, \kappa_{\eta}) \right) \, d\xi
\]

Similarly, we obtain

\[
\int_0^1 \left[ (\Delta(\xi) - \Delta(1)\lambda \xi - \Lambda(\eta) - \Lambda(1)\lambda) \left( \frac{\partial^2 F}{\partial \xi^2} (\kappa_{\xi}, \kappa_{\eta}) \right) + \xi(1 - \eta) \left( \frac{\partial^2 F}{\partial \xi^2} (\kappa_{\xi}, \kappa_{\eta}) \right) \right] \, d\xi = \frac{1}{2} \left( \int_0^1 (\Lambda(\eta) - \Lambda(1)\lambda) \, d\eta \right)^{1/2} \left( \int_0^1 (\Lambda(\xi) - \Lambda(1)\lambda) \, d\xi \right)^{1/2}
\]

\[
+ \left( \int_0^1 \left( \frac{\partial^2 F}{\partial \xi^2} (\kappa_{\xi}, \kappa_{\eta}) \right) \right) \, d\xi = \frac{1}{2} \int_0^1 \left( \frac{\partial^2 F}{\partial \xi^2} (\kappa_{\xi}, \kappa_{\eta}) \right) \, d\xi
\]

By the inequalities (4.9)–(4.12), the proof is completed.
Corollary 5. In Theorem 3, if we take \( \varphi(\xi) = \xi \) and \( \psi(\eta) = \eta \), we have

\[
\left| \Phi(\kappa_1, \kappa_2, \kappa_3, \kappa_4) \right| \leq (\kappa_2 - \kappa_1)(\kappa_3 - \kappa_4)
\left( \Pi_1(\lambda_1) + \Pi_2(\lambda_2) \right)^{1/2} (v_1(\lambda_2) + v_2(\lambda_2))^{1/2}
\times
\left( \Pi_1(\lambda_1) V_1(\lambda_2) \left[ \frac{\partial \Phi}{\partial \xi} (\kappa_2, \kappa_3) \right] + \Pi_2(\lambda_2) V_2(\lambda_2) \left[ \frac{\partial \Phi}{\partial \eta} (\kappa_2, \kappa_3) \right] \right)^{1/2} + \left( v_1(\mu_2) + v_2(\mu_2) \right)^{1/2} \left( v_1(\lambda_2) + v_2(\lambda_2) \right)^{1/2}
\times \left( \Pi_1(\mu_1) V_1(\mu_2) \left[ \frac{\partial \Phi}{\partial \xi} (\kappa_2, \kappa_3) \right] + \Pi_2(\mu_2) V_2(\lambda_2) \left[ \frac{\partial \Phi}{\partial \eta} (\kappa_2, \kappa_3) \right] \right)^{1/2} + \left( v_1(\mu_2) + v_2(\mu_2) \right)^{1/2} \left( v_1(\mu_2) + v_2(\mu_2) \right)^{1/2}
\times \left( \Pi_1(\mu_1) V_1(\mu_2) \left[ \frac{\partial \Phi}{\partial \xi} (\kappa_2, \kappa_3) \right] + \Pi_2(\mu_2) V_2(\lambda_2) \left[ \frac{\partial \Phi}{\partial \eta} (\kappa_2, \kappa_3) \right] \right)^{1/2} + \left( v_1(\mu_2) + v_2(\mu_2) \right)^{1/2} \left( v_1(\mu_2) + v_2(\mu_2) \right)^{1/2}
\times \left( \Pi_1(\mu_1) V_1(\mu_2) \left[ \frac{\partial \Phi}{\partial \xi} (\kappa_2, \kappa_3) \right] + \Pi_2(\mu_2) V_2(\lambda_2) \left[ \frac{\partial \Phi}{\partial \eta} (\kappa_2, \kappa_3) \right] \right)^{1/2} + \left( v_1(\mu_2) + v_2(\mu_2) \right)^{1/2} \left( v_1(\mu_2) + v_2(\mu_2) \right)^{1/2}
\times \left( \Pi_1(\mu_1) V_1(\mu_2) \left[ \frac{\partial \Phi}{\partial \xi} (\kappa_2, \kappa_3) \right] + \Pi_2(\mu_2) V_2(\lambda_2) \left[ \frac{\partial \Phi}{\partial \eta} (\kappa_2, \kappa_3) \right] \right)^{1/2} + \left( v_1(\mu_2) + v_2(\mu_2) \right)^{1/2} \left( v_1(\mu_2) + v_2(\mu_2) \right)^{1/2},
\]

where \( \Phi(\kappa_1, \kappa_2, \kappa_3, \kappa_4) \) is defined as in Remark 1,

\[
\Pi_1(\lambda_1) = \frac{1}{3} \lambda_3^3 - \frac{\lambda_1}{8} + \frac{1}{24},
\]

\[
\Pi_2(\lambda_2) = \frac{1}{3} \lambda_2^3 + \lambda_2^2 - \frac{3\lambda_1}{8} + \frac{1}{12},
\]

\[
\Pi_3(\mu_3) = \frac{1}{3} \mu_3^3 - \frac{5}{8} \mu_1^2 + \frac{9}{24},
\]

\[
\Pi_4(\mu_4) = \frac{1}{3} \mu_1^3 + \mu_2^2 - \frac{7}{8} \mu_1 + \frac{1}{4}
\]

and

\[
v_1(\lambda_2) = \frac{1}{3} \lambda_3^3 - \frac{\lambda_2}{8} + \frac{1}{24},
\]

\[
v_2(\lambda_2) = \frac{1}{3} \lambda_2^3 + \lambda_2^2 - \frac{3\lambda_1}{8} + \frac{1}{12},
\]

\[
v_3(\mu_3) = \frac{1}{3} \mu_3^3 - \frac{5}{8} \mu_2^2 + \frac{9}{24},
\]

\[
v_4(\mu_4) = \frac{1}{3} \mu_2^3 + \mu_2^2 - \frac{7}{8} \mu_2 + \frac{1}{4}.\]
Corollary 6. In Theorem 3, if we use \( \varphi(\xi) = \frac{\xi^n}{\Gamma(a)} \) and \( \psi(\eta) = \frac{\eta^\beta}{\Gamma(\beta)} \), then we obtain the following parameterized Simpson-type inequality for Riemann-Liouville fractional integrals:

\[
|\Omega(\kappa_1, \kappa_2; \kappa_3, \kappa_4)| \leq (\kappa_2 - \kappa_1)(\kappa_4 - \kappa_3) \left[ (\Pi_1^\alpha(\lambda_1) + \Pi_2^\alpha(\lambda_2))^{1-\frac{1}{\nu}} (\nu F^\alpha(\lambda_2) + \nu F^\alpha(\lambda_2))^{1-\frac{1}{\nu}} \right] \\
\times \left[ \Pi_1^\alpha(\lambda_1) \frac{\partial^\alpha}{\partial \xi^\alpha} (\kappa_1, \kappa_2) \left[ \Pi_1^\alpha(\lambda_1) + \Pi_2^\alpha(\lambda_2) \right]^{1-\frac{1}{\nu}} + \Pi_1^\alpha(\lambda_1) \frac{\partial^\alpha}{\partial \xi^\alpha} (\kappa_1, \kappa_2) \right]^{\frac{1}{\nu}} \\
+ \Pi_2^\alpha(\lambda_2) \frac{\partial^\alpha}{\partial \xi^\alpha} (\kappa_1, \kappa_2) \left[ \Pi_1^\alpha(\lambda_1) + \Pi_2^\alpha(\lambda_2) \right]^{1-\frac{1}{\nu}} + \Pi_2^\alpha(\lambda_2) \frac{\partial^\alpha}{\partial \xi^\alpha} (\kappa_1, \kappa_2) \right]^{\frac{1}{\nu}}
\]

where \( \Omega(\kappa_1, \kappa_2; \kappa_3, \kappa_4) \) is defined as in Corollary 1 and \( \Pi_1^\alpha, \Pi_2^\alpha, i = 1, 2, 3, 4 \) are defined in Corollary 3.

Corollary 7. In Theorem 3, if we use \( \varphi(\xi) = \frac{\xi^n}{\Gamma(a)} \) and \( \psi(\eta) = \frac{\eta^\beta}{\Gamma(\beta)} \), then we obtain the following parameterized Simpson-type inequality for k-Riemann-Liouville fractional integrals:

\[
|\mathcal{C}(\kappa_1, \kappa_2; \kappa_3, \kappa_4)| \leq (\kappa_2 - \kappa_1)(\kappa_4 - \kappa_3) \left[ (\Pi_1^\alpha(\lambda_1) + \Pi_2^\alpha(\lambda_2))^{1-\frac{1}{\nu}} (\nu F^\alpha(\lambda_2) + \nu F^\alpha(\lambda_2))^{1-\frac{1}{\nu}} \right] \\
\times \left[ \Pi_1^\alpha(\lambda_1) \frac{\partial^\alpha}{\partial \xi^\alpha} (\kappa_1, \kappa_2) \left[ \Pi_1^\alpha(\lambda_1) + \Pi_2^\alpha(\lambda_2) \right]^{1-\frac{1}{\nu}} + \Pi_1^\alpha(\lambda_1) \frac{\partial^\alpha}{\partial \xi^\alpha} (\kappa_1, \kappa_2) \right]^{\frac{1}{\nu}} \\
+ \Pi_2^\alpha(\lambda_2) \frac{\partial^\alpha}{\partial \xi^\alpha} (\kappa_1, \kappa_2) \left[ \Pi_1^\alpha(\lambda_1) + \Pi_2^\alpha(\lambda_2) \right]^{1-\frac{1}{\nu}} + \Pi_2^\alpha(\lambda_2) \frac{\partial^\alpha}{\partial \xi^\alpha} (\kappa_1, \kappa_2) \right]^{\frac{1}{\nu}}
\]
Theorem 4. Where $\mathcal{I}(\kappa_1, \kappa_2; \kappa_3, \kappa_4)$ is defined as in Corollary 2 and $\nabla^\beta \frac{\partial F}{\partial \xi^\eta}, \beta = 1, 2, 3, 4$ are defined in Corollary 4.

**Theorem 4.** We assume that the conditions of Lemma 1 hold. If the mapping $\frac{\partial F}{\partial \xi^\eta}$ is co-ordinated convex on $\Delta$, then we have the following inequality:

$$
|\Theta(\kappa_1, \kappa_2; \kappa_3, \kappa_4)| \leq (\kappa_2 - \kappa_1)(\kappa_4 - \kappa_3) \left[ \frac{1}{2} \left( \int_0^1 |\Delta(\xi) - \Delta(1)\lambda|d\xi \right)^\gamma \right] \left[ \frac{1}{2} \left( \int_0^1 |\Delta(\eta) - \Delta(1)\lambda|d\eta \right)^\gamma \right] \\
\times \left[ \frac{1}{64} \left( \frac{3}{2} \frac{\partial F}{\partial \xi^\eta}(\kappa_2, \kappa_4)^\beta + 3 \frac{\partial F}{\partial \xi^\eta}(\kappa_2, \kappa_3)^\beta + 9 \frac{\partial F}{\partial \xi^\eta}(\kappa_1, \kappa_4)^\beta \right) + \frac{1}{64} \left( \frac{3}{2} \frac{\partial F}{\partial \xi^\eta}(\kappa_2, \kappa_3)^\beta + 3 \frac{\partial F}{\partial \xi^\eta}(\kappa_1, \kappa_4)^\beta + 9 \frac{\partial F}{\partial \xi^\eta}(\kappa_1, \kappa_3)^\beta \right) \right] \\
+ \left( \frac{1}{2} \left( \int_0^1 |\Delta(\eta) - \Delta(1)\lambda|d\eta \right)^\gamma \right) \left( \frac{1}{2} \left( \int_0^1 |\Delta(\xi) - \Delta(1)\lambda|d\xi \right)^\gamma \right) \\
\times \left[ \frac{1}{64} \left( \frac{3}{2} \frac{\partial F}{\partial \xi^\eta}(\kappa_2, \kappa_4)^\beta + 3 \frac{\partial F}{\partial \xi^\eta}(\kappa_2, \kappa_3)^\beta + 9 \frac{\partial F}{\partial \xi^\eta}(\kappa_1, \kappa_4)^\beta \right) + \frac{1}{64} \left( \frac{3}{2} \frac{\partial F}{\partial \xi^\eta}(\kappa_2, \kappa_3)^\beta + 3 \frac{\partial F}{\partial \xi^\eta}(\kappa_1, \kappa_4)^\beta + 9 \frac{\partial F}{\partial \xi^\eta}(\kappa_1, \kappa_3)^\beta \right) \right].
$$
where $\frac{1}{p} + \frac{1}{q} = 1$.

**Proof.** By using the Hölder inequality in (4.3) and co-ordinated convexity of $\frac{\partial^2 F}{\partial x \partial y}$, we have

$$
\int_0^1 \int_0^1 |\Delta(\xi) - \Delta(1)\lambda_j| |\Lambda(\eta) - \Lambda(1)\lambda_j| \left| \frac{\partial^2 F}{\partial x \partial y} \right| \left( \xi \kappa_2 + (1 - \xi)\kappa_3, \eta \kappa_4 + (1 - \eta)\kappa_5 \right) \, d\xi \, d\eta
\leq \left( \int_0^1 |\Delta(\xi) - \Delta(1)\lambda_j|^p \, d\xi \right)^{\frac{1}{p}} \left( \int_0^1 |\Lambda(\eta) - \Lambda(1)\lambda_j|^p \, d\eta \right)^{\frac{1}{p}}
\times \left( \int_0^1 \int_0^1 \left( \xi \frac{\partial^2 F}{\partial x \partial y}(\kappa_2, \kappa_3) + (1 - \xi)(1 - \eta) \frac{\partial^2 F}{\partial x \partial y}(\kappa_2, \kappa_3) \right)^{\frac{p}{2}} \, d\xi \, d\eta \right)^\frac{1}{p}
\leq \left( \int_0^1 |\Delta(\xi) - \Delta(1)\lambda_j|^p \, d\xi \right)^{\frac{1}{p}} \left( \int_0^1 |\Lambda(\eta) - \Lambda(1)\lambda_j|^p \, d\eta \right)^{\frac{1}{p}}
\times \left( \frac{\frac{\partial^2 F}{\partial x \partial y}(\kappa_2, \kappa_3)}{64} + 3 \frac{\partial^2 F}{\partial x \partial y}(\kappa_2, \kappa_3) + 3 \frac{\partial^2 F}{\partial x \partial y}(\kappa_2, \kappa_3) + 9 \frac{\partial^2 F}{\partial x \partial y}(\kappa_2, \kappa_3) \right)^{\frac{1}{p}}
\left( \frac{1}{p} \right)^{\frac{1}{p}}
$$

(4.13)

Similarly, we obtain

$$
\int_0^1 \int_0^1 |\Delta(\xi) - \Delta(1)\lambda_j| |\Lambda(\eta) - \Lambda(1)\mu_j| \left| \frac{\partial^2 F}{\partial x \partial y} \right| \left( \xi \kappa_2 + (1 - \xi)\kappa_3, \eta \kappa_4 + (1 - \eta)\kappa_5 \right) \, d\xi \, d\eta
\leq \left( \int_0^1 |\Lambda(\eta) - \Lambda(1)\mu_j|^p \, d\eta \right)^{\frac{1}{p}} \left( \int_0^1 |\Delta(\xi) - \Delta(1)\lambda_j|^p \, d\xi \right)^{\frac{1}{p}}
\times \left( \frac{\frac{\partial^2 F}{\partial x \partial y}(\kappa_2, \kappa_3)}{64} + 3 \frac{\partial^2 F}{\partial x \partial y}(\kappa_2, \kappa_3) + 3 \frac{\partial^2 F}{\partial x \partial y}(\kappa_2, \kappa_3) + 9 \frac{\partial^2 F}{\partial x \partial y}(\kappa_2, \kappa_3) \right)^{\frac{1}{p}}
\left( \frac{1}{p} \right)^{\frac{1}{p}}
$$

(4.14)
\[
\int_0^1 \left| \Delta(\xi) - \Delta(1)\mu_k |\Delta(\eta) - \Delta(1)\lambda_k \right| \frac{\partial^2 \mathcal{F}}{\partial \xi \partial \eta} \left( \xi_2 + (1 - \xi)\kappa + (1 - \eta)\kappa \right) \, d\xi \, d\eta \\
\leq \left( \int_0^1 \left| \Delta(\eta) - \Delta(1)\lambda_k \right| d\eta \right) \left( \int \left| \Delta(\xi) - \Delta(1)\mu_k \right| d\xi \right) \left( \frac{1}{2} \right)^{\frac{5}{2}} \cdot \left( \frac{3}{4} \right)^{\frac{5}{4}} \cdot \left( \frac{9}{16} \right)^{\frac{5}{8}} \cdot \left( \frac{1}{4} \right)^{\frac{5}{8}}}
\]

and
\[
\int_0^1 \left| \Delta(\xi) - \Delta(1)\mu_k |\Delta(\eta) - \Delta(1)\lambda_k \right| \frac{\partial^2 \mathcal{F}}{\partial \xi \partial \eta} \left( \xi_2 + (1 - \xi)\kappa + (1 - \eta)\kappa \right) \, d\xi \, d\eta \\
\leq \left( \int_0^1 \left| \Delta(\eta) - \Delta(1)\lambda_k \right| d\eta \right) \left( \int \left| \Delta(\xi) - \Delta(1)\mu_k \right| d\xi \right) \left( \frac{1}{2} \right)^{\frac{5}{2}} \cdot \left( \frac{3}{4} \right)^{\frac{5}{4}} \cdot \left( \frac{9}{16} \right)^{\frac{5}{8}} \cdot \left( \frac{1}{4} \right)^{\frac{5}{8}}}
\]

If we substitute the inequalities (4.13)–(4.16) in (4.3), then we obtain the required result. \[\Box\]

**Remark 3.** In Theorem 4, if we take \( \phi(\xi) = \xi \) and \( \psi(\eta) = \eta \), we obtain
\[
\left| \Phi(\kappa_1, \kappa_2; \kappa_3, \kappa_4) \right| \leq (\kappa_2 - \kappa_1)(\kappa_4 - \kappa_3) \left[ \int_0^1 |\xi - \lambda_k|^p \, d\xi \right] \left[ \int_0^1 |\eta - \lambda_k|^p \, d\eta \right] \left( \frac{1}{2} \right)^{\frac{5}{2}} \cdot \left( \frac{3}{4} \right)^{\frac{5}{4}} \cdot \left( \frac{9}{16} \right)^{\frac{5}{8}} \cdot \left( \frac{1}{4} \right)^{\frac{5}{8}} \]

\[
\times \left( \frac{3}{4} \right)^{\frac{5}{4}} \cdot \left( \frac{9}{16} \right)^{\frac{5}{8}} \cdot \left( \frac{1}{4} \right)^{\frac{5}{8}}
\]

\[
\times \left( \frac{3}{4} \right)^{\frac{5}{4}} \cdot \left( \frac{9}{16} \right)^{\frac{5}{8}} \cdot \left( \frac{1}{4} \right)^{\frac{5}{8}}
\]

\[
+ \left( \int_0^1 |\eta - \mu_k|^p \, d\eta \right) \left( \int_0^1 |\xi - \lambda_k|^p \, d\xi \right) \left( \frac{1}{2} \right)^{\frac{5}{2}} \cdot \left( \frac{3}{4} \right)^{\frac{5}{4}} \cdot \left( \frac{9}{16} \right)^{\frac{5}{8}} \cdot \left( \frac{1}{4} \right)^{\frac{5}{8}}
\]

\[
\times \left( \frac{3}{4} \right)^{\frac{5}{4}} \cdot \left( \frac{9}{16} \right)^{\frac{5}{8}} \cdot \left( \frac{1}{4} \right)^{\frac{5}{8}}
\]

\[
+ \left( \int_0^1 |\eta - \lambda_k|^p \, d\eta \right) \left( \int_0^1 |\xi - \mu_k|^p \, d\xi \right) \left( \frac{1}{2} \right)^{\frac{5}{2}} \cdot \left( \frac{3}{4} \right)^{\frac{5}{4}} \cdot \left( \frac{9}{16} \right)^{\frac{5}{8}} \cdot \left( \frac{1}{4} \right)^{\frac{5}{8}}
\]

\[
\times \left( \frac{3}{4} \right)^{\frac{5}{4}} \cdot \left( \frac{9}{16} \right)^{\frac{5}{8}} \cdot \left( \frac{1}{4} \right)^{\frac{5}{8}}
\]
Some new parameterized inequalities

**Corollary 8.** In Theorem 4, if we use $\varphi(\xi) = \frac{\xi^p}{\Gamma(\alpha)}$ and $\psi(\eta) = \frac{\eta^p}{\Gamma(\beta)}$, then we obtain the following parameterized Simpson-type inequality for Riemann-Liouville fractional integrals:

$$
|\Omega(\kappa_1, \kappa_2; \kappa_3, \kappa_4)| \leq (\kappa_2 - \kappa_1)(\kappa_4 - \kappa_3)
\left(\frac{1}{2}\int_0^{\frac{1}{2}} |\xi - \lambda_1|^p d\xi\right)\left(\frac{1}{2}\int_0^{\frac{1}{2}} |\eta - \lambda_2|^p d\eta\right)
\left(\frac{1}{2}\int_0^{\frac{1}{2}} |\eta^\beta - \lambda_3|^p d\eta\right)
\left(\frac{1}{2}\int_0^{\frac{1}{2}} |\xi^\alpha - \lambda_4|^p d\xi\right)
\times
\left(\frac{9}{64} \frac{\partial^\beta}{\partial \eta^\beta} (\kappa_2, \kappa_4) + 3 \frac{\partial^\beta}{\partial \eta^\beta} (\kappa_2, \kappa_3) + 3 \frac{\partial^\beta}{\partial \xi^\alpha} (\kappa_1, \kappa_4) + 9 \frac{\partial^\beta}{\partial \xi^\alpha} (\kappa_1, \kappa_3)\right)^{\frac{1}{7}}
\times
\left(\frac{1}{2}\int_0^{\frac{1}{2}} |\eta - \mu_2|^p d\eta\right)^{\frac{1}{7}}
\left(\frac{1}{2}\int_0^{\frac{1}{2}} |\xi - \mu_1|^p d\xi\right)^{\frac{1}{7}}
\left(\frac{1}{2}\int_0^{\frac{1}{2}} |\eta^\beta - \mu_3|^p d\eta\right)^{\frac{1}{7}}
\left(\frac{1}{2}\int_0^{\frac{1}{2}} |\xi^\alpha - \mu_1|^p d\xi\right)^{\frac{1}{7}}
\times
\left(\frac{3}{64} \frac{\partial^\beta}{\partial \eta^\beta} (\kappa_2, \kappa_4) + 3 \frac{\partial^\beta}{\partial \eta^\beta} (\kappa_2, \kappa_3) + 3 \frac{\partial^\beta}{\partial \xi^\alpha} (\kappa_1, \kappa_4) + 9 \frac{\partial^\beta}{\partial \xi^\alpha} (\kappa_1, \kappa_3)\right)^{\frac{1}{7}}
\times
\left(\frac{1}{2}\int_0^{\frac{1}{2}} |\eta - \lambda_2|^p d\eta\right)^{\frac{1}{7}}
\left(\frac{1}{2}\int_0^{\frac{1}{2}} |\xi - \lambda_1|^p d\xi\right)^{\frac{1}{7}}
\left(\frac{1}{2}\int_0^{\frac{1}{2}} |\eta^\beta - \lambda_3|^p d\eta\right)^{\frac{1}{7}}
\left(\frac{1}{2}\int_0^{\frac{1}{2}} |\xi^\alpha - \lambda_4|^p d\xi\right)^{\frac{1}{7}}
\times
\left(\frac{9}{64} \frac{\partial^\beta}{\partial \eta^\beta} (\kappa_2, \kappa_4) + 3 \frac{\partial^\beta}{\partial \eta^\beta} (\kappa_2, \kappa_3) + 3 \frac{\partial^\beta}{\partial \xi^\alpha} (\kappa_1, \kappa_4) + 9 \frac{\partial^\beta}{\partial \xi^\alpha} (\kappa_1, \kappa_3)\right)^{\frac{1}{7}},
$$

where $\Omega(\kappa_1, \kappa_2; \kappa_3, \kappa_4)$ is defined as in Corollary 1.
Corollary 9. In Theorem 4, if we use \( \varphi(\xi) = \frac{\xi^q}{k_1(\alpha)} \) and \( \psi(\eta) = \frac{\eta^p}{k_1(\beta)} \), then we obtain the following parameterized Simpson-type inequality for \( k \)-Riemann-Liouville fractional integrals:

\[
|\mathcal{J}(k_1, k_2; k_3, k_4)| \leq (k_2 - k_1)(k_4 - k_3) \left( \int_0^1 \left( k_2^q - \lambda_1 \right) \left( k_4^p - \lambda_2 \right) \, d\xi \right)^{\frac{1}{p}} \left( \int_0^1 \left( k_2^q - \lambda_1 \right) \left( k_4^p - \lambda_2 \right) \, d\eta \right)^{\frac{1}{q}}
\]

\[
\times \left( \int_0^1 k_2^q \left( k_2^q - \lambda_1 \right) \, d\eta \right)^{\frac{1}{p}} \left( \int_0^1 k_4^p \left( k_4^p - \lambda_2 \right) \, d\xi \right)^{\frac{1}{q}}
\]

\[
\times \left( \int_0^1 \left( k_2^q - \lambda_1 \right) \, d\eta \right)^{\frac{1}{p}} \left( \int_0^1 \left( k_4^p - \lambda_2 \right) \, d\xi \right)^{\frac{1}{q}}
\]

\[
\times \left( \int_0^1 \left( k_2^q - \lambda_1 \right) \, d\eta \right)^{\frac{1}{p}} \left( \int_0^1 \left( k_4^p - \lambda_2 \right) \, d\xi \right)^{\frac{1}{q}}
\]

\[
\times \left( \int_0^1 k_2^q \left( k_2^q - \lambda_1 \right) \, d\eta \right)^{\frac{1}{p}} \left( \int_0^1 k_4^p \left( k_4^p - \lambda_2 \right) \, d\xi \right)^{\frac{1}{q}}
\]

where \( \mathcal{J}(k_1, k_2; k_3, k_4) \) is defined as in Corollary 2.

Theorem 5. We assume that the conditions of Lemma 1 hold. If the mapping \( \frac{d^q}{d\xi^q} \) is co-ordinated convex on \( \Delta \), then we have the following inequality:

\[
|\Theta(k_1, k_2; k_3, k_4)| \leq (k_2 - k_1)(k_4 - k_3) \left( \int_0^1 |w(\xi, \eta)|^p \, d\eta \, d\xi \right)^{\frac{1}{p}} \left( \int_0^1 |w(\xi, \eta)|^q \, d\eta \, d\xi \right)^{\frac{1}{q}}
\]

\[
\times \left( \int_0^1 |w(\xi, \eta)|^p \, d\eta \, d\xi \right)^{\frac{1}{p}} \left( \int_0^1 |w(\xi, \eta)|^q \, d\eta \, d\xi \right)^{\frac{1}{q}}
\]

where \( \frac{1}{p} + \frac{1}{q} = 1 \).
**Proof.** Taking the modulus in Lemma 1 and using the Hölder inequality,

\[
|\Theta(\kappa_1, \kappa_2; \kappa_3, \kappa_4)| = \left| (\kappa_2 - \kappa_1)(\kappa_4 - \kappa_3) \int_0^1 \int_0^1 w(\xi, \eta) \frac{\partial^2 F}{\partial \xi \partial \eta} \left( \xi \kappa_2 + (1 - \xi) \kappa_1, \eta \kappa_4 + (1 - \eta) \kappa_3 \right) d\eta d\xi \right|
\]

\[
\leq (\kappa_2 - \kappa_1)(\kappa_4 - \kappa_3) \left( \int_0^1 \int_0^1 |w(\xi, \eta)| \frac{\partial^2 F}{\partial \xi \partial \eta} \left( \xi \kappa_2 + (1 - \xi) \kappa_1, \eta \kappa_4 + (1 - \eta) \kappa_3 \right) d\eta d\xi \right]^{\frac{1}{p}}
\]

\[
\times \left( \int_0^1 \int_0^1 \left| \frac{\partial^2 F}{\partial \xi \partial \eta} \left( \xi \kappa_2 + (1 - \xi) \kappa_1, \eta \kappa_4 + (1 - \eta) \kappa_3 \right) \right|^q d\eta d\xi \right]^{\frac{1}{q}}
\]

\[
\leq (\kappa_2 - \kappa_1)(\kappa_4 - \kappa_3) \left( \int_0^1 \int_0^1 |w(\xi, \eta)|^p d\eta d\xi \right]^{\frac{1}{p}}
\]

\[
\times \left( \int_0^1 \int_0^1 \left| \frac{\partial^2 F}{\partial \xi \partial \eta} \left( \xi \kappa_2 + (1 - \xi) \kappa_1, \eta \kappa_4 + (1 - \eta) \kappa_3 \right) \right|^q d\eta d\xi \right]^{\frac{1}{q}}
\]

Since \( \left| \frac{\partial^2 F}{\partial \xi \partial \eta} \right|^q \), \( q > 1 \), is a co-ordinated convex function on \( \Delta \), we obtain

\[
\int_0^1 \int_0^1 \left| \frac{\partial^2 F}{\partial \xi \partial \eta} \left( \xi \kappa_2 + (1 - \xi) \kappa_1, \eta \kappa_4 + (1 - \eta) \kappa_3 \right) \right|^q d\eta d\xi \leq \frac{\left| \frac{\partial^2 F}{\partial \xi \partial \eta} \left( \kappa_1, \kappa_3 \right) \right|^q + \left| \frac{\partial^2 F}{\partial \xi \partial \eta} \left( \kappa_2, \kappa_3 \right) \right|^q + \left| \frac{\partial^2 F}{\partial \xi \partial \eta} \left( \kappa_1, \kappa_4 \right) \right|^q + \left| \frac{\partial^2 F}{\partial \xi \partial \eta} \left( \kappa_2, \kappa_4 \right) \right|^q}{4}. \tag{4.17}
\]

\[\square\]

**Remark 4.** In Theorem 5, if we take \( \varphi(\xi) = \xi \) and \( \psi(\eta) = \eta \), we have

\[
|\Theta(\kappa_1, \kappa_2; \kappa_3, \kappa_4)| \leq (\kappa_2 - \kappa_1)(\kappa_4 - \kappa_3)(A_p(\lambda, \mu))^\frac{1}{p}
\]

\[
\times \left( \frac{\left| \frac{\partial^2 F}{\partial \xi \partial \eta} \left( \kappa_1, \kappa_3 \right) \right|^q + \left| \frac{\partial^2 F}{\partial \xi \partial \eta} \left( \kappa_2, \kappa_3 \right) \right|^q + \left| \frac{\partial^2 F}{\partial \xi \partial \eta} \left( \kappa_1, \kappa_4 \right) \right|^q + \left| \frac{\partial^2 F}{\partial \xi \partial \eta} \left( \kappa_2, \kappa_4 \right) \right|^q}{4} \right)^\frac{1}{q},
\]

where

\[
A_p(\lambda, \mu) = \frac{1}{(p + 1)^2} \left[ \lambda_{p+1}^2 + \left( \frac{1}{2} - \lambda_1 \right)^{p+1} + \left( \mu_1 - \frac{1}{2} \right)^{p+1} + (1 - \mu_1)^{p+1} \right]
\]

\[
\times \left[ \mu_{p+1}^2 + \left( \frac{1}{2} - \mu_2 \right)^{p+1} + \left( \lambda_2 - \frac{1}{2} \right)^{p+1} + (1 - \lambda_2)^{p+1} \right],
\]

given by Budak and Ali in [49].
Corollary 10. In Theorem 5, if we use \( \varphi(\xi) = \frac{\xi^p}{\Gamma(a)} \) and \( \psi(\eta) = \frac{\eta^p}{\Gamma(b)} \), then we obtain
\[
|\Theta(k_1, k_2; k_3, k_4)| \leq (k_2 - k_1)(k_4 - k_3) \left( \int_0^1 \int_0^1 |w_1(\xi, \eta)|^p \, d\eta \, d\xi \right)^{1/p} \times \left( \frac{3\varphi(\xi, k_1) + 3\varphi(\xi, k_2) + 3\varphi(\xi, k_3) + 3\varphi(\xi, k_4)}{4} \right)^{1/2},
\]
where the mapping \( w_1 : [0, 1] \times [0, 1] \to \mathbb{R} \) is defined by
\[
w_1(\xi, \eta) = \begin{cases} (\xi^\alpha - \lambda_1)(\eta^\beta - \lambda_2), & 0 \leq \xi \leq \frac{1}{2}, 0 \leq \eta \leq \frac{1}{2}, \\ (\xi^\alpha - \lambda_2)(\eta^\beta - \mu_2), & 0 \leq \xi \leq \frac{1}{2}, \frac{1}{2} \leq \eta \leq 1, \\ (\xi^\alpha - \mu_1)(\eta^\beta - \lambda_2), & \frac{1}{2} \leq \xi \leq 1, 0 \leq \eta \leq \frac{1}{2}, \\ (\xi^\alpha - \mu_2)(\eta^\beta - \mu_2), & \frac{1}{2} \leq \xi \leq 1, \frac{1}{2} \leq \eta \leq 1. \end{cases}
\]

Corollary 11. In Theorem 5, if we use \( \varphi(\xi) = \frac{\xi^p}{\Gamma(a)} \) and \( \psi(\eta) = \frac{\eta^p}{\Gamma(b)} \), then we obtain
\[
|\Theta(k_1, k_2; k_3, k_4)| \leq (k_2 - k_1)(k_4 - k_3) \left( \int_0^1 \int_0^1 |w_2(\xi, \eta)|^p \, d\eta \, d\xi \right)^{1/p} \times \left( \frac{3\varphi(\xi, k_1) + 3\varphi(\xi, k_2) + 3\varphi(\xi, k_3) + 3\varphi(\xi, k_4)}{4} \right)^{1/2},
\]
where the mapping \( w_2 : [0, 1] \times [0, 1] \to \mathbb{R} \) is defined by
\[
w_2(\xi, \eta) = \begin{cases} (\xi^\alpha - \lambda_1)(\eta^\beta - \lambda_2), & 0 \leq \xi \leq \frac{1}{2}, 0 \leq \eta \leq \frac{1}{2}, \\ (\xi^\alpha - \lambda_2)(\eta^\beta - \mu_2), & 0 \leq \xi \leq \frac{1}{2}, \frac{1}{2} \leq \eta \leq 1, \\ (\xi^\alpha - \mu_1)(\eta^\beta - \lambda_2), & \frac{1}{2} \leq \xi \leq 1, 0 \leq \eta \leq \frac{1}{2}, \\ (\xi^\alpha - \mu_2)(\eta^\beta - \mu_2), & \frac{1}{2} \leq \xi \leq 1, \frac{1}{2} \leq \eta \leq 1. \end{cases}
\]

5 Special cases of main results

In this section, we present some special cases of the results proved in the previous section.

Corollary 12. Under assumptions of Theorem 2 with \( \lambda_1 = \lambda_2 = \frac{1}{6} \) and \( \mu_1 = \mu_2 = \frac{5}{6} \), we have the following inequality:
\[
\Delta(1)\Delta(1) \left[ F\left( k_1, \frac{k_1 + k_4}{2} \right) + F\left( k_2, \frac{k_1 + k_4}{2} \right) + 4F\left( \frac{k_1 + k_2}{2}, \frac{k_1 + k_4}{2} \right) + F\left( \frac{k_1 + k_2}{2}, k_3 \right) + F\left( \frac{k_1 + k_2}{2}, k_4 \right) \right]
\]
\[ + \frac{F(K_1, K_3) + F(K_2, K_3) + F(K_1, K_4) + F(K_2, K_4)}{36} \]
\[ - \frac{2}{3} \left[ \Delta(I) \left( \frac{k_1 + k_3}{2}, i \psi \Phi \left( \frac{k_1 + k_3}{2}, k_3 \right) + i \psi \Phi \left( \frac{k_1 + k_3}{2}, k_4 \right) \right) \right] \]
\[ + \frac{\Delta(I)}{i \psi} \left( \frac{k_1 + k_2}{2}, i \psi \Phi \left( \frac{k_1 + k_2}{2}, k_3 \right) + i \psi \Phi \left( \frac{k_1 + k_2}{2}, k_4 \right) \right) \]
\[ - \frac{1}{6} \left[ \Delta(I) \left( \frac{k_1 + k_2}{2}, i \psi \Phi \left( \frac{k_1 + k_3}{2}, k_3 \right) + i \psi \Phi \left( \frac{k_1 + k_3}{2}, k_4 \right) \right) \right] \]
\[ + \frac{\Delta(I)}{i \psi} \left( \frac{k_1 + k_2}{2}, i \psi \Phi \left( \frac{k_1 + k_3}{2}, k_3 \right) + i \psi \Phi \left( \frac{k_1 + k_3}{2}, k_4 \right) \right) \]
\[ - \frac{1}{6} \left[ \Delta(I) \left( \frac{k_1 + k_2}{2}, i \psi \Phi \left( \frac{k_1 + k_3}{2}, k_3 \right) + i \psi \Phi \left( \frac{k_1 + k_3}{2}, k_4 \right) \right) \right] \]
\[ + \frac{\Delta(I)}{i \psi} \left( \frac{k_1 + k_2}{2}, i \psi \Phi \left( \frac{k_1 + k_2}{2}, k_2 \right) + i \psi \Phi \left( \frac{k_1 + k_2}{2}, k_3 \right) + i \psi \Phi \left( \frac{k_1 + k_2}{2}, k_4 \right) \right) \]
\[ + \frac{\Delta(I)}{i \psi} \left( \frac{k_1 + k_2}{2}, i \psi \Phi \left( \frac{k_1 + k_2}{2}, k_2 \right) + i \psi \Phi \left( \frac{k_1 + k_2}{2}, k_3 \right) + i \psi \Phi \left( \frac{k_1 + k_2}{2}, k_4 \right) \right) \]
\[ \leq (k_2 - k_1)(k_3 - k_2) \left| \frac{\partial^2 F}{\partial \xi^2 \partial \eta^2} (k_1, k_2) \right| \left( \Pi_1' \left( \frac{5}{6} \right) + \Pi_2' \left( \frac{5}{6} \right) \right) \left( \Pi_3' \left( \frac{5}{6} \right) + \Pi_4' \left( \frac{5}{6} \right) \right) \]
\[ + \frac{\partial^2 F}{\partial \xi^2 \partial \eta^2} (k_2, k_3) \left( \Pi_1' \left( \frac{5}{6} \right) + \Pi_2' \left( \frac{5}{6} \right) \right) \left( \Pi_3' \left( \frac{5}{6} \right) + \Pi_4' \left( \frac{5}{6} \right) \right) \]
\[ + \frac{\partial^2 F}{\partial \xi^2 \partial \eta^2} (k_3, k_4) \left( \Pi_1' \left( \frac{5}{6} \right) + \Pi_2' \left( \frac{5}{6} \right) \right) \left( \Pi_3' \left( \frac{5}{6} \right) + \Pi_4' \left( \frac{5}{6} \right) \right) \]
\[ + \frac{\partial^2 F}{\partial \xi^2 \partial \eta^2} (k_4, k_5) \left( \Pi_1' \left( \frac{5}{6} \right) + \Pi_2' \left( \frac{5}{6} \right) \right) \left( \Pi_3' \left( \frac{5}{6} \right) + \Pi_4' \left( \frac{5}{6} \right) \right) \]

where \( \Pi_i' \) and \( \psi_i', i = 1, 2, 3, 4 \) are defined as in (4.2).

**Remark 5.** In Corollary 12, if we take \( \varphi(\xi) = \xi \) and \( \psi(\eta) = \eta \), then Corollary 12 reduces to [24, Theorem 3].

**Remark 6.** In Corollary 12, if we take \( \varphi(\xi) = \frac{\alpha}{\Gamma(\alpha)} \) and \( \psi(\eta) = \frac{\beta}{\Gamma(\beta)} \), then we obtain
\[
\left( \begin{array}{c}
(k_2 - k_1)^\alpha (k_3 - k_2)^\beta \\
(\Gamma(\alpha + 1) / \Gamma(\beta + 1))
\end{array} \right) \left[ \frac{F(k_1, k_1 + k_2)}{36} + F(k_2, k_2 + k_3) + 4 \frac{F(k_3, k_3 + k_4)}{9} + F(k_4, k_4 + k_5)
\right]
\]
Remark 7. In Corollary 12, if we take \( \varphi(\xi) = \frac{\xi}{\xi_{1(\alpha)}} \) and \( \psi(\eta) = \frac{\eta}{\eta_{1(\beta)}} \), then we obtain

\[
(\kappa_2 - \kappa_1)^{\alpha} (\kappa_4 - \kappa_1)^{\beta} \left[ \frac{\Gamma(\alpha + 1) \Gamma(\beta + 1)}{\Gamma(\alpha + k) \Gamma(\beta + k)} \left[ \frac{\partial^2 \varphi}{\partial \xi \partial \eta} (\kappa_2, \kappa_1) \left( \Pi_{\alpha}^a \left( \frac{1}{6} \right) + \Pi_{\beta}^a \left( \frac{5}{6} \right) \right) + \frac{\partial^2 \psi}{\partial \xi \partial \eta} (\kappa_2, \kappa_1) \left( \Pi_{\alpha}^a \left( \frac{1}{6} \right) + \Pi_{\beta}^a \left( \frac{5}{6} \right) \right) \right] \right]
\]

\[
+ \frac{\partial^2 \varphi}{\partial \xi \partial \eta} (\kappa_2, \kappa_1) \left( \Pi_{\alpha}^a \left( \frac{1}{6} \right) + \Pi_{\beta}^a \left( \frac{5}{6} \right) \right) \right]
\]

\[
\leq (\kappa_2 - \kappa_1)(\kappa_4 - \kappa_1) \left[ \frac{\partial^2 \varphi}{\partial \xi \partial \eta} (\kappa_2, \kappa_1) \left( \Pi_{\alpha}^a \left( \frac{1}{6} \right) + \Pi_{\beta}^a \left( \frac{5}{6} \right) \right) + \frac{\partial^2 \psi}{\partial \xi \partial \eta} (\kappa_2, \kappa_1) \left( \Pi_{\alpha}^a \left( \frac{1}{6} \right) + \Pi_{\beta}^a \left( \frac{5}{6} \right) \right) \right]
\]

\[
+ \frac{\partial^2 \varphi}{\partial \xi \partial \eta} (\kappa_2, \kappa_1) \left( \Pi_{\alpha}^a \left( \frac{1}{6} \right) + \Pi_{\beta}^a \left( \frac{5}{6} \right) \right) \right]
\]
Corollary 13. Under assumptions of Theorem 2 with \( \lambda_1 = \lambda_2 = \mu_1 = \mu_2 = \frac{1}{2} \), we have the following inequality:

\[
\frac{(k_2 - k_1)^d (k_4 - k_3)^e}{\Gamma(\alpha + 1) \Gamma(\beta + 1)} \left| F(k_1, k_2) + F(k_2, k_3) + F(k_3, k_4) + F(k_4, k_1) - \frac{\Delta(1)}{2} \left[ k_{\lambda + k_3} \frac{I_{\mu + k_3}}{2} I_{\eta + k_3} \frac{F(k_1, k_3)}{2} + k_{\lambda + k_4} \frac{I_{\mu + k_4}}{2} I_{\eta + k_4} \frac{F(k_2, k_4)}{2} \right] \right| \leq (k_2 - k_1)(k_4 - k_3) \left[ \frac{\partial^2 F}{\partial \xi \partial \eta}(k_2, k_3) \left( \Pi_1^f \left( \frac{1}{2} \right) + \Pi_2^f \left( \frac{1}{2} \right) \right) \left( \psi^f \left( \frac{1}{2} \right) + \psi^f \left( \frac{1}{2} \right) \right) \right]
\]

where \( \Pi_i^f \) and \( \psi_i^f \), \( i = 1, 2, 3, 4 \) are defined as in (4.2).

Remark 8. In Corollary 13, if we take \( \phi(\xi) = \xi \) and \( \psi(\eta) = \eta \), then Corollary 13 reduces to [7, Theorem 2].

Remark 9. In Corollary 13, if we take \( \phi(\xi) = \frac{\xi^a}{\Gamma(\alpha)} \) and \( \psi(\eta) = \frac{\eta^b}{\Gamma(\beta)} \), then we obtain

\[
\frac{\left| F(k_1, k_2) + F(k_2, k_3) + F(k_3, k_4) + F(k_4, k_1) - \frac{\Delta(1)}{2} \left[ k_{\lambda + k_3} \frac{I_{\mu + k_3}}{2} I_{\eta + k_3} \frac{F(k_1, k_3)}{2} + k_{\lambda + k_4} \frac{I_{\mu + k_4}}{2} I_{\eta + k_4} \frac{F(k_2, k_4)}{2} \right] \right|}{2} \leq (k_2 - k_1)(k_4 - k_3) \left[ \frac{\partial^2 F}{\partial \xi \partial \eta}(k_2, k_3) \left( \Pi_1^f \left( \frac{1}{2} \right) + \Pi_2^f \left( \frac{1}{2} \right) \right) \left( \psi^f \left( \frac{1}{2} \right) + \psi^f \left( \frac{1}{2} \right) \right) \right]
\]
Remark 10. In Corollary 13, if we take $\varphi(\xi) = \frac{\xi^2}{k^{(a)}}$ and $\psi(\eta) = \frac{\eta^2}{k^{(b)}}$, then we obtain

\[
\left[\frac{F(k_1, k_2) + F(k_3, k_4) + F(k_5, k_6) + F(k_7, k_8)}{4}\right] 
- \frac{\Gamma(a + k)}{2(k_4 - k_3)^2} \left[ \int_{k_3}^{k_4} F(k_3, k_4) + \int_{k_4}^{k_5} F(k_4, k_5) + \int_{k_5}^{k_6} F(k_5, k_6) + \int_{k_6}^{k_7} F(k_6, k_7) \right] 
\]

\[
- \frac{\Gamma(a + k)}{2(k_2 - k_1)^2} \left[ \int_{k_1}^{k_2} F(k_1, k_2) + \int_{k_2}^{k_3} F(k_2, k_3) + \int_{k_3}^{k_4} F(k_3, k_4) + \int_{k_4}^{k_5} F(k_4, k_5) \right] 
\]

\[
+ \frac{\Gamma(\beta + k)}{2(k_2 - k_1)^2} \left[ \int_{k_1}^{k_2} F(k_1, k_2) + \int_{k_2}^{k_3} F(k_2, k_3) + \int_{k_3}^{k_4} F(k_3, k_4) + \int_{k_4}^{k_5} F(k_4, k_5) \right] 
\]

\[
\leq (k_2 - k_1)(k_4 - k_3) \left[ \frac{\partial F}{\partial \xi \partial \eta}(k_2, k_3) \left( \frac{\Pi_1^2(1/2)}{k_2} + \frac{\Pi_3^2(1/2)}{k_3} \right) \left( \frac{\partial F}{\partial \xi}(1/2) + \frac{\partial F}{\partial \eta}(1/2) \right) \right] 
\]

\[
+ \frac{\partial F}{\partial \xi \partial \eta}(k_3, k_4) \left( \frac{\Pi_1^2(1/2)}{k_3} + \frac{\Pi_3^2(1/2)}{k_4} \right) \left( \frac{\partial F}{\partial \xi}(1/2) + \frac{\partial F}{\partial \eta}(1/2) \right) 
\]

\[
+ \frac{\partial F}{\partial \xi \partial \eta}(k_4, k_5) \left( \frac{\Pi_1^2(1/2)}{k_4} + \frac{\Pi_3^2(1/2)}{k_5} \right) \left( \frac{\partial F}{\partial \xi}(1/2) + \frac{\partial F}{\partial \eta}(1/2) \right) 
\]

\[
+ \frac{\partial F}{\partial \xi \partial \eta}(k_5, k_6) \left( \frac{\Pi_1^2(1/2)}{k_5} + \frac{\Pi_3^2(1/2)}{k_6} \right) \left( \frac{\partial F}{\partial \xi}(1/2) + \frac{\partial F}{\partial \eta}(1/2) \right) 
\].

Corollary 14. Under assumptions of Theorem 2 with $\lambda_1 = \lambda_2 = 0$ and $\mu_1 = \mu_2 = 1$, we have the following inequality:

\[
\left| \Delta(1) \Lambda(1) F \left( \frac{k_1 + k_2}{2}, \frac{k_3 + k_4}{2} \right) - \Lambda(1) \left[ I_{\mu} F \left( \frac{k_1 + k_2}{2}, k_3 \right) + I_{\lambda} F \left( \frac{k_1 + k_2}{2}, k_4 \right) \right] \right| 
\]

\[
- \Lambda(1) \left[ I_{\mu} F \left( \frac{k_1 + k_2}{2}, k_3 \right) + I_{\lambda} F \left( \frac{k_1 + k_2}{2}, k_4 \right) \right] 
\]

\[
+ \frac{\partial F}{\partial \xi \partial \eta}(k_3, k_4) \left( \Pi_1^2(1/2) + \Pi_3^2(1/2) \right) \left( \Pi_1^2(1/2) + \Pi_3^2(1/2) \right) 
\]

\[
\leq (k_2 - k_1)(k_4 - k_3) \left[ \frac{\partial F}{\partial \xi \partial \eta}(k_2, k_3) \left( \Pi_1^2(1/2) + \Pi_3^2(1/2) \right) \left( \Pi_1^2(1/2) + \Pi_3^2(1/2) \right) \right] 
\]

\[
+ \frac{\partial F}{\partial \xi \partial \eta}(k_3, k_4) \left( \Pi_1^2(1/2) + \Pi_3^2(1/2) \right) \left( \Pi_1^2(1/2) + \Pi_3^2(1/2) \right) 
\]

\[
+ \frac{\partial F}{\partial \xi \partial \eta}(k_4, k_5) \left( \Pi_1^2(1/2) + \Pi_3^2(1/2) \right) \left( \Pi_1^2(1/2) + \Pi_3^2(1/2) \right) 
\]

\[
+ \frac{\partial F}{\partial \xi \partial \eta}(k_5, k_6) \left( \Pi_1^2(1/2) + \Pi_3^2(1/2) \right) \left( \Pi_1^2(1/2) + \Pi_3^2(1/2) \right) 
\],

where $\Pi_i^\mu$ and $\Pi_i^\psi$, $i = 1, 2, 3, 4$ are defined as in (4.2).
Remark 11. In Corollary 14, if we take $\phi(\xi) = \xi$ and $\psi(\eta) = \eta$, then Corollary 13 reduces to [6, Theorem 2].

Remark 12. In Corollary 14, if we take $\phi(\xi) = \frac{\xi^a}{\Gamma(a)}$ and $\psi(\eta) = \frac{\eta^\beta}{\Gamma(\beta)}$, then we obtain

$$\left| F\left(\frac{K_1 + K_2}{2}, \frac{K_3 + K_4}{2}\right) - \frac{1}{(K_2 - K_1)^\beta} \left(\frac{\Gamma(a + 1)}{K_1 + \frac{K_2}{2} - k} F\left(\frac{K_1 + K_2}{2}\right) + \frac{\Gamma(a + 1)}{K_2 + \frac{K_1}{2} - k} F\left(\frac{K_2 + K_1}{2}\right)\right) \right|$$

$$\leq (K_2 - K_1)(K_4 - K_3) \left[ \frac{\partial^2 F}{\partial \xi \partial \eta}(K_2, K_3) \left( \Pi^a(0) + \Pi^a(1) \right) + \frac{\partial^2 F}{\partial \xi \partial \eta}(K_3, K_4) \left( \Pi^\beta(0) + \Pi^\beta(1) \right) \right].$$

Remark 13. In Corollary 13, if we take $\phi(\xi) = \frac{\xi^a}{\Gamma(a)}$ and $\psi(\eta) = \frac{\eta^\beta}{\Gamma(\beta)}$, then we obtain

$$\left| F\left(\frac{K_1 + K_2}{2}, \frac{K_3 + K_4}{2}\right) - \frac{\Gamma(a + k)}{(K_2 - K_1)^\beta} \left(\frac{1}{K_1 + \frac{K_2}{2} - k} F\left(\frac{K_1 + K_2}{2}\right) + \frac{1}{K_2 + \frac{K_1}{2} - k} F\left(\frac{K_2 + K_1}{2}\right)\right) \right|$$

$$\leq (K_2 - K_1)(K_4 - K_3) \left[ \frac{\partial^2 F}{\partial \xi \partial \eta}(K_2, K_3) \left( \Pi^a(0) + \Pi^a(1) \right) + \frac{\partial^2 F}{\partial \xi \partial \eta}(K_3, K_4) \left( \Pi^\beta(0) + \Pi^\beta(1) \right) \right].$$

Remark 14. By special choices of $\lambda_1$, $\lambda_2$, $\mu_1$ and $\mu_2$ in Theorems 3, 4 and 5, one can obtain several new Simpson-, trapezoid- and midpoint-type inequalities. Writing these situations is left to the reader as it will make the article too long.
6 Conclusion

In this paper, we present several generalized inequalities for co-ordinated convex functions via generalized fractional integrals. It is also shown that the results given here are the strong generalization of some already published ones. It is an interesting and new problem that the forthcoming researchers can use the techniques of this study and obtain similar inequalities for different kinds of co-ordinated convexity in their next works.

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