ANALYSIS AND ENTROPY STABILITY OF THE LINE-BASED DISCONTINUOUS
GALERKIN METHOD

WILL PAZNER AND PER-OLOF PERSSON

Abstract. We develop a discretely entropy-stable line-based discontinuous Galerkin method for hyperbolic conservation laws based on a flux differencing technique. By using standard entropy-stable and entropy-conservative numerical flux functions, this method guarantees that the discrete integral of the entropy is non-increasing. This nonlinear entropy stability property is important for the robustness of the method, in particular when applied to problems with discontinuous solutions or when the mesh is under-resolved. This line-based method is significantly less computationally expensive than a standard DG method. Numerical results are shown demonstrating the effectiveness of the method on a variety of test cases, including Burgers’ equation and the Euler equations, in one, two, and three spatial dimensions.

1. Introduction

High-order numerical methods for the solution of partial differential equations have seen success in a wide range of application areas [39]. In particular, the discontinuous Galerkin (DG) method [8, 32], an arbitrary-order finite element method suitable for use on unstructured geometries, possesses many desirable properties, making it well-suited for a large number of applications. Variants of the DG method, such as the discontinuous Galerkin spectral element method (DG-SEM) [1, 30, 19] and the line-based discontinuous Galerkin method (Line-DG) [26, 27] have been introduced in order to retain the attractive properties of the DG method while reducing its computational cost.

Of particular interest are the stability properties of these methods. It has been shown that the standard discontinuous Galerkin method satisfies a cell entropy inequality in the scalar and symmetric system case, leading to $L^2$ stability [18, 15]. However, these results do not immediately translate to general systems of conservation laws. Additionally, these results rely on the use of an exactly integrated DG methods, which may be impractical or prohibitively expensive due to nonlinearities in the fluxes. To maintain stability in the general nonlinear case, a variety of methods have been proposed, including limiters [42, 9] and artificial viscosity [28, 43], however these methods can result in reduced order of accuracy, and often require parameter tuning. Recently, discretely entropy-stable DG and DG-SEM methods have been developed based on a technique known as flux differencing [4, 7, 3, 11, 13, 12]. These methods are based on the entropy-conservative and entropy-stable fluxes developed by Tadmor [36, 35, 22], which have been used in the context of finite volume methods.

In this work, we extend the flux differencing methodology to the line-based DG (Line-DG) methods developed in [26, 27]. These methods are closely related both to standard DG methods and to DG-SEM methods, and are based on solving a sequence of one-dimensional Galerkin problems along lines of nodes within a tensor-product element. We modify the Line-DG method by introducing entropy-stable and entropy-conservative flux functions, combined with appropriate projection operators required to ensure discrete entropy stability. Then we show that this modified method satisfies an entropy inequality consistent with the quadrature chosen for the scheme. This discrete entropy stability property is shown to be important for the robustness of the scheme. For instance, for Burgers’ equation, entropy stability implies $L^2$ stability. For the Euler equations, we additionally must require that the density and pressure remain positive. We remark that although the method remains stable, the numerical solution may still develop strong oscillations in the vicinity of a discontinuity, suggesting the utility of other shock-capturing techniques for problems with strong shocks. However, this increased robustness could prove to be particularly important for under-resolved turbulent flows, for which standard methods have been observed to be unstable [40, 20]. The structure of the paper is as follows. In Section 2 we describe the governing equations and define the Line-DG method. We then modify the Line-DG scheme to achieve discrete entropy stability, and analyze the accuracy of the resulting method. In Section 3 we discuss implementation details and computational efficiency. In Section 4 we provide a range of numerical experiments, including Burgers’ equation and the Euler equations of gas dynamics, in one, two, and three spatial dimensions. We end with concluding remarks in Section 5.

2. Discretization and equations

2.1. Governing equations and entropy analysis. We consider a system of hyperbolic conservation laws in $d$ dimensions in a spatial domain $\Omega \subseteq \mathbb{R}^d$,

\begin{equation}
\frac{\partial u}{\partial t} + \nabla \cdot f = 0.
\end{equation}

2.2. Numerical flux. For the Line-DG method, we use a numerical flux $\mathbf{f}(\mathbf{u}_i, \mathbf{u}_j)$ that is both entropy-conservative and entropy-stable. The entropy-conservative flux is given by

\begin{equation}
\mathbf{f}(\mathbf{u}_i, \mathbf{u}_j) = \mathbf{u}_i + \mathbf{f}(\mathbf{u}_i) - \mathbf{f}(\mathbf{u}_j).
\end{equation}

The entropy-stable flux is defined as

\begin{equation}
\mathbf{f}_{\text{stab}}(\mathbf{u}_i, \mathbf{u}_j) = \mathbf{f}(\mathbf{u}_i) - \mathbf{f}_e(\mathbf{u}_i) - \mathbf{f}_e(\mathbf{u}_j) + \mathbf{f}_e(\mathbf{u}_i) - \mathbf{f}_e(\mathbf{u}_j) + \mathbf{f}_e(\mathbf{u}_j).
\end{equation}

where $\mathbf{f}_e(\mathbf{u})$ is the entropy flux.

2.3. Quadrature and projection. The quadrature used in the Line-DG method is based on a tensor-product of Gauss-Legendre quadrature in one dimension. The projection operators used to enforce the discrete entropy inequality are given by

\begin{equation}
P_{\text{stab}}(\mathbf{u}) = \mathbf{u} - \mathbf{f}_e(\mathbf{u})/\mathbf{f}''(\mathbf{u}),
\end{equation}

where $\mathbf{f}''(\mathbf{u})$ is the second derivative of the flux.

2.4. Numerical results. We demonstrate the effectiveness of the Line-DG method on a variety of test cases, including Burgers’ equation and the Euler equations, in one, two, and three spatial dimensions. The numerical results show that the method is stable and accurate, with reduced computational cost compared to standard DG methods.

3. Implementation details and computational efficiency

3.1. Algorithmic details. The Line-DG method is implemented using a parallel computational framework, which allows for efficient parallelization of the computational cost.

3.2. Performance analysis. We analyze the computational efficiency of the Line-DG method, showing that it is significantly more computationally efficient than standard DG methods.

4. Numerical experiments

4.1. Burgers’ equation. We demonstrate the effectiveness of the Line-DG method on Burgers’ equation, showing that it is stable and accurate in one, two, and three spatial dimensions.

4.2. Euler equations. We demonstrate the effectiveness of the Line-DG method on the Euler equations, showing that it is stable and accurate in one, two, and three spatial dimensions.

5. Concluding remarks

We conclude with a few remarks on the future development of the Line-DG method, including potential extensions to more complex systems of conservation laws.

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The number of solution components is denoted \( n_c \), and so the solution \( u \) is a function \( u(x, t) : \mathbb{R}^d \times \mathbb{R} \rightarrow \mathbb{R}^{n_c} \), and the flux function is written \( f(u) : \mathbb{R}^{n_c} \rightarrow \mathbb{R}^{n_c \times d} \).

A convex function \( U(u) : \mathbb{R}^{n_c} \rightarrow \mathbb{R} \) is called an \textit{entropy function} if there exist \textit{entropy fluxes} \( F_j : \mathbb{R}^{n_c} \rightarrow \mathbb{R} \), \( 1 \leq j \leq d \), such that
\[
U'(u)f'_j(u) = F'_j(u),
\]
where the derivatives are taken with respect to the state variables \( u \). In regions where \( u \) is smooth, the entropy satisfies the related equation
\[
\frac{\partial U}{\partial t} + \nabla \cdot F = 0.
\]
However, hyperbolic conservation laws admit solutions with discontinuities, for which the physically-relevant solutions must dissipate entropy. Thus, the \textit{entropy solution} \( u \) to equation (1) satisfies
\[
\frac{\partial U}{\partial t} + \nabla \cdot F \leq 0
\]
for all entropy functions \( U \). Assuming periodic or compactly supported boundary conditions, and integrating (1) over \( \Omega \), we obtain
\[
\frac{d}{dt} \int_{\Omega} U \, dx \leq 0,
\]
and thus conclude that the total entropy is monotonically non-increasing in time. The inequality (5) can be seen as a non-linear analogue to standard \( L^2 \) stability. Furthermore, if the entropy function \( U \) is uniformly convex, then entropy stability can be used to guarantee \( L^2 \) stability, thus motivating the development of numerical schemes that satisfy a discrete entropy stability property.

### 2.1.1. Entropy variables.
We define the \textit{entropy variables} by
\[
v = U'(u).
\]
If \( U \) is uniformly convex, then the mapping \( u \mapsto v \) is invertible, and is considered as a change of variables. Defining \( g(v) = f(u(v)) \), we obtain a system of hyperbolic conservation laws equivalent to (1),
\[
\hat{u}'(v) \frac{\partial w}{\partial t} + \nabla \cdot g = 0.
\]
Convexity of the entropy function \( U \) implies symmetry of \( g'(v) \), and so there exist functions \( \psi_j(v) \), called \textit{flux potentials}, satisfying
\[
\psi'_j(v) = g'_j(v).
\]
One can verify that \( \psi_j \) is given by
\[
\psi_j = g_j(v)^T v - F_j(u(v)).
\]

### 2.1.2. Two-point numerical fluxes.
In order to obtain discrete entropy stability, we introduce \textit{entropy-conservative} and \textit{entropy-stable} numerical flux functions. An \textit{entropy-conservative} two-point numerical flux is a function \( f_{i,S} \), \( 1 \leq i \leq d \) that satisfies the following properties:
1. Consistency: \( f_{i,S}(u, u) = f_i(u) \).
2. Symmetry: \( f_{i,S}(u_L, u_R) = f_{i,S}(u_R, u_L) \).
3. Entropy conservation: \( (v_R - v_L)^T f_{i,S}(u_L, u_R) = \psi_{i,R} - \psi_{i,L} \).

Additionally, we introduce an \textit{entropy-stable} two-point numerical flux function \( \hat{f}_S \) that satisfies:
1. Consistency: \( \hat{f}_S(u, u) = f(u) \).
2. Symmetry: \( \hat{f}_S(u_L, u_R) = \hat{f}_S(u_R, u_L) \).
3. Entropy stability: \( (v_R - v_L)^T \hat{f}_S(u_L, u_R) \cdot n \leq (\psi_{i,R} - \psi_{i,L}) \cdot n \).

We point out that in the entropy-stable case, the equality in the third property has been replaced by an inequality.
2.2. Euler equations and entropy variables. As a particularly important example of governing equations, we consider the Euler equations of gas dynamics in $d$ spatial dimensions, written in conservative form

\[ \frac{\partial u}{\partial t} + \nabla \cdot f(u) = 0, \]

where $u = (\rho, \rho w, \rho E)$ is the vector of conserved variables (density, momentum, and energy, respectively). The flux function is given by

\[ f = (u) \begin{pmatrix} \rho w \\ \rho w \otimes w^T + pI \end{pmatrix}. \]

Here $I$ is the $d \times d$ identity matrix, $p$ is the pressure, and $H = E + p/\rho$ is the stagnation enthalpy. The pressure is defined through the equation of state

\[ p = (\gamma - 1)\rho(E - \|w\|^2/2), \]

where the constant $\gamma$ is the ratio of specific heats, taken in this work as $\gamma = 7/5$.

We wish to introduce an entropy pair that simultaneously symmetrizes the Euler equations as well as the viscous term in the compressible Navier-Stokes equations. In this case, the entropy pair is unique \[16\], and is given by

\[ U = -\rho s, \quad F = -\rho ws, \]

where $s = \log (pp^-)$. Given this entropy pair, the entropy variables can be written in terms of the conservative variables as

\[ v = \begin{pmatrix} -pE(\gamma - 1)/p + (\gamma + 1 - s) \\ \rho w(\gamma - 1)/p \\ -\rho(\gamma - 1)/p \end{pmatrix}. \]

Likewise, we can transform from the entropy variables to conservative variables by

\[ u = \begin{pmatrix} -pv_{d+2}/(\gamma - 1) \\ p_{j}/(\gamma - 1) \\ p \left(1 - \frac{1}{2} \sum_{i=2}^{d+1} v_i^2/v_{d+2}\right) /(\gamma - 1) \end{pmatrix}, \quad j = 2, \ldots, d + 1, \]

where, in terms of the entropy variables, we have

\[ s = \gamma - v_1 + \frac{1}{2} \sum_{i=2}^{d+1} v_i^2/v_{d+2}, \quad p/(\gamma - 1) = \left(\frac{\gamma - 1}{-v_{d+2}}\right)^{1/(\gamma - 1)} \exp \left(\frac{-s}{\gamma - 1}\right). \]

The entropy potential flux is given by

\[ \psi = (\gamma - 1)\rho w. \]

2.2.1. Entropy-conservative and entropy-stable numerical fluxes. There has been much recent interest in the development of entropy-conservative and entropy-stable numerical flux functions for the Euler equations \[17, 29, 5, 31\]. In this work, for the volume fluxes, we will use the two-point entropy-conservative flux of Chandrashekar \[5\]. For $d = 2$, the flux is defined as follows. We introduce the convenient notation for the arithmetic and logarithmic means

\[ \{\phi\} = \frac{1}{2} (\phi_L + \phi_R), \quad \{\phi\}_\text{log} = \frac{\phi_R - \phi_L}{\log(\phi_R) - \log(\phi_L)}. \]
A numerically stable procedure for evaluating \( \{ \phi \}_\log \) was given by Ismail and Roe [17]. Chandrashekar’s entropy-conservative numerical flux is given by

\[
\begin{align*}
\bf{j}^{(1)}_{1,S} &= \{ \rho \}_\log \{ w_1 \}, \\
\bf{j}^{(2)}_{1,S} &= \{ w_1 \} \bf{j}^{(1)}_{1,S} + \{ \rho \}_\log \{ \beta \} \\
\bf{j}^{(3)}_{1,S} &= \{ w_2 \} \bf{j}^{(1)}_{1,S} \\
\bf{j}^{(4)}_{1,S} &= \left( \frac{1}{2(2\gamma - 1) \{ \beta \}_\log} - \frac{1}{4} \right) \bf{j}^{(1)}_{1,S} + \{ w_1 \} \bf{j}^{(2)}_{1,S} + \{ w_2 \} \bf{j}^{(3)}_{1,S} \\
\bf{j}^{(1)}_{2,S} &= \{ \rho \}_\log \{ w_2 \} \\
\bf{j}^{(2)}_{2,S} &= \{ w_1 \} \bf{j}^{(1)}_{2,S} \\
\bf{j}^{(3)}_{2,S} &= \{ w_2 \} \bf{j}^{(1)}_{2,S} + \{ \rho \}_\log \{ \beta \} \\
\bf{j}^{(4)}_{2,S} &= \left( \frac{1}{2(2\gamma - 1) \{ \beta \}_\log} - \frac{1}{4} \right) \bf{j}^{(1)}_{2,S} + \{ w_1 \} \bf{j}^{(2)}_{2,S} + \{ w_2 \} \bf{j}^{(3)}_{2,S}
\end{align*}
\]  

(19)

where \( \beta \) is the inverse temperature, defined by

\[
\beta = \frac{1}{2RT} = \frac{\rho}{2p}.
\]  

(20)

At element interfaces, we must introduce an entropy-stable numerical flux function. It is shown in [7] that exactly solving the Riemann problem at element interfaces results in an entropy-stable numerical flux. However, this process can be computationally expensive, requiring the solution of a system of nonlinear equations for every evaluation. For this reason, we opt to use a simple local Lax-Friedrichs (LLF) flux function, defined by

\[
\tilde{f}(\bf{u}_L, \bf{u}_R) = \frac{1}{2} \left( \bf{f}(\bf{u}_L) + \bf{f}(\bf{u}_R) \right) + \frac{\lambda}{2} (\bf{u}_L - \bf{u}_R),
\]  

(21)

where \( \lambda \) is chosen to bound the leftmost and rightmost wave speeds in the corresponding Riemann problem. This numerical flux function is a special case of the Harten-Lax-Van Leer (HLL) approximate Riemann solver [14], which can be shown to be entropy-stable [7].

2.3. Line-based DG discretization. The line-based discontinuous Galerkin discretization (Line-DG) is constructed by modifying the standard nodal DG discretization on tensor-product elements, so that a sequence of one-dimensional Galerkin problems are solved along each coordinate dimension. To be more specific, the spatial domain \( \Omega \) is partitioned into a conforming mesh \( \mathcal{T}_h = \{ K_j : 1 \leq j \leq n_e \} \), such that \( \bigcup_{j=1}^{n_e} K_j = \Omega \). Each element \( K_j \in \mathcal{T}_h \) is taken to be the image of the reference element \( \mathcal{R} = [0,1]^d \) under a transformation map \( T_j : \mathbb{R}^d \rightarrow \mathbb{R}^d \).

We now focus on a single element, \( K \in \mathcal{T}_h \), with corresponding transformation map \( T \). We use the convention that \( \bf{x} = T(\bf{\xi}) \), and refer to \( \bf{x} \) as physical coordinates, and \( \bf{\xi} \) as reference coordinates. We wish to perform a change of variables to rewrite the conservation law [1] in the reference domain \( \mathcal{R} \). Let \( J \) denote the Jacobian of \( T \) (referred to as the deformation gradient),

\[
J = \left( \frac{\partial x_i}{\partial \xi_j} \right),
\]  

(22)

and let \( g = \det(J) \). Following a standard procedure to change spatial variables, we define

\[
\tilde{u}(\bf{\xi}, t) = g u(T(\bf{\xi}), t),
\]  

(23)

and

\[
\tilde{f}(\tilde{u}) = gJ^{-1} f(\tilde{u}/g).
\]  

(24)

Then, \( \tilde{u} \) evolves according to the transformed hyperbolic conservation law

\[
\frac{\partial \tilde{u}(\bf{\xi}, t)}{\partial t} + \nabla \cdot \tilde{f}(\tilde{u}(\bf{\xi}, t)) = 0,
\]  

(25)

where, in this case, the divergence is understood to be taken with respect to the reference coordinates \( \bf{\xi} \). In order to introduce the Line-DG method, we discretize equation (25) directly.

For simplicity of presentation, we describe the method for \( d = 2 \). The extension to three or more spatial dimensions is straightforward. The reference coordinates are written \( \bf{\xi} = (\xi, \eta) \). We fix a polynomial degree \( p \geq 1 \), and introduce nodal interpolation points \( \bf{\xi}_{ij} = (\xi_i, \eta_j) \in \mathcal{R} \) according to a tensor-product Gauss-Lobatto rule, for \( 0 \leq i, j \leq p \).
We approximate the solution using time-dependent nodal values \( \tilde{u}_{ij}(t) \approx \tilde{u}(\xi_{ij}, t) \). The standard Line-DG semi-discretization in space reads:

\[
\frac{\partial \tilde{u}_{ij}}{\partial t} + q_{ij} + r_{ij} = 0,
\]

where \( q_{ij} \) and \( r_{ij} \) are discretizations of the derivatives

\[
q_{ij} \approx \frac{\partial \tilde{f}_1}{\partial \xi}(\xi, \xi), \quad r_{ij} \approx \frac{\partial \tilde{f}_2}{\partial \eta}(\xi, \xi),
\]

which are both obtained through a sequence of one-dimensional Galerkin problems, described as follows.

We begin by defining \( q_{ij} \), which approximates the \( \xi \)-derivative of \( \tilde{f}_1 \). \( r_{ij} \) is defined through an analogous procedure. We fix an index \( j \), \( 0 \leq j \leq p \). Then, we consider all the points \( q_{ij} \) that lie along this line. We view these nodal values as defining an interpolating polynomial, which we write as

\[
\tilde{q}(\xi) = \sum_{i=0}^{p} a_{ij} \phi_i(\xi),
\]

where \( \phi_i(\xi) = \delta_{ij} \) is the Lagrange interpolating polynomial defined at the one-dimensional Gauss-Lobatto points \( \xi_k \). Similarly, we define the polynomial \( \tilde{u}_j(\xi) = \sum_{i=0}^{p} \tilde{u}_{ij} \phi_i(\xi) \). We choose the coefficients \( q_{ij} \) such that they satisfy the Galerkin problem

\[
\int_0^1 q_{ij}(\xi) \cdot w(\xi) d\xi = - \int_0^1 \tilde{f}_1(\tilde{u}_j(\xi)) \cdot w(\xi) d\xi + \tilde{f}_1 \cdot w|_0^1,
\]

for all test functions \( w \in [P^p([0, 1])]^n \) (the space of vector-valued polynomials of degree \( p \)), where \( \tilde{f}_1 \) is an appropriately-defined numerical flux function. Analogously, for fixed \( i \), \( r_{ij} \) is defined to satisfy

\[
\int_0^1 r_{ij}(\eta) \cdot w(\eta) d\eta = - \int_0^1 \tilde{f}_2(\tilde{u}_i(\eta)) \cdot w(\eta) d\eta + \tilde{f}_2 \cdot w|_0^1.
\]

Our discretization is complete once we specify the numerical flux functions \( \tilde{f}_1 \) and \( \tilde{f}_2 \). We note that given a vector \( N \) normal to the reference element \( \mathcal{R} \), we obtain a transformed vector \( n = gJ^{-T}N \), normal to the physical element \( K \). We then note that

\[
n^T f = (gJ^{-T}N)^T f = N^T gJ^{-1}f = N^T \tilde{f}.
\]

Thus, given a numerical flux function \( \tilde{f}(u^-, u^+) \) from any standard discontinuous Galerkin discretization, we can define

\[
N^T \tilde{f}(\tilde{u}^-, \tilde{u}^+) = n^T \tilde{f}(u^-, u^+),
\]

allowing us to reuse standard DG numerical flux functions for the purposes of our Line-DG discretization.

2.4. Discrete entropy stability. In order to solve the one-dimensional problems (28) and (29), we discretize the integrals using a quadrature rule. For this purpose, we use a Gauss-Lobatto rule with \( \mu \geq p + 1 \) points, which is exact for polynomials of degree \( 2\mu - 3 \). Here we emphasize that if \( \mu = p + 1 \), then the Line-DG method is exactly equivalent to the Gauss-Lobatto DG spectral element method (DG-SEM) method. However, if \( \mu > p + 1 \), then the method is distinct from DG-SEM, and possesses different properties.

Using techniques similar to those developed in [11, 3, 17, 4], we modify the Line-DG discretization as follows in order to obtain discrete entropy stability. Let \( \xi_\alpha \), \( 1 \leq \alpha \leq \mu \) denote the Gauss-Lobatto quadrature points, and let \( w_\alpha \) denote the quadrature weights. We define the rectangular \( \mu \times (p+1) \) quadrature evaluation matrix \( G \) by

\[
G_{\alpha i} = \phi_i(\xi_\alpha).
\]

Similarly, we consider the \( \mu \times (p+1) \) differentiation matrix

\[
D_{\alpha i} = \phi_i'(\xi_\alpha).
\]

We also define the \( \mu \times \mu \) diagonal quadrature weight matrix by

\[
W = \text{diag}(w_1, w_2, \ldots, w_\mu).
\]

Additionally, we define the differentiation matrix at quadrature points,

\[
\tilde{D}_{\alpha \beta} = \tilde{\phi}_{\beta}'(\xi_\alpha),
\]

where \( \tilde{\phi} \) are the Lagrange interpolating polynomials defined using the quadrature points \( \xi_\alpha \). Finally, we define the \( \mu \times \mu \) boundary evaluation matrix \( B \) by

\[
B = \text{diag}(-1, 0, \ldots, 0, 1).
\]

**Proposition 1.** We briefly summarize some of the properties of the above matrices.

(i) \( D = \tilde{D} \)

(ii) \( W \tilde{D} + \tilde{D}^T W = B \) (summation-by-parts property)
(iii) \( \sum_{\beta=1}^{p} \tilde{D}_{\alpha,\beta} = 0 \) (derivative of a constant is zero)

(iv) \( \sum_{\alpha=1}^{n} (W \tilde{D})_{\alpha,\beta} = \begin{cases} -1, & \beta = 1, \\ 1, & \beta = \mu, \\ 0, & \text{otherwise}. \end{cases} \)

Proof. (i). Let \( a \in \mathbb{R}^{p+1} \). Define the polynomial \( a(\xi) = \sum_{i=0}^{p} a_i \phi_i(\xi) \). Then, \((Da)_\alpha = a'(\xi_\alpha)\) for \( 1 \leq \alpha \leq \mu \) because \( D \) exactly differentiates polynomials of degree \( p \). Similarly, \( \tilde{D} \) exactly differentiates polynomials of degree \( \mu - 1 \geq p \) (given in terms of their values at quadrature points), and thus \((\tilde{D}a)_\alpha = a'(\xi_\alpha)\).

(ii). Consider two polynomials, \( a(\xi) = \sum_{\alpha=1}^{\mu} a_\alpha \phi_\alpha(\xi) \) and \( b(\xi) = \sum_{\alpha=1}^{\mu} b_\alpha \phi_\alpha(\xi) \). Then, \( \int_{0}^{1} a'(\xi) b(\xi) \, d\xi = b^T W \tilde{D} a \), since the quadrature rule is exact for polynomials of degree \( 2\mu - 3 \). Integrating by parts, we have

\[
\int_{0}^{1} a'(\xi) b(\xi) \, d\xi = - \int_{0}^{1} b'(\xi) a(\xi) \, d\xi + ab[1]_{0}^{1} = b^T (-\tilde{D}^T W + B) a,
\]

and since \( a \) and \( b \) were arbitrary, we conclude \( WD^T + \tilde{D}^T W = B \).

(iii). This is immediate since \( \tilde{D} \) is exact for polynomials of degree \( \mu - 1 \).

(iv). This follows from properties (ii) and (iii), since

\[
\sum_{\alpha=1}^{n} (W \tilde{D})_{\alpha,\beta} = 1^T W \tilde{D} = 1^T W \tilde{D} + 1^T \tilde{D}^T W = 1^T B.
\]

We define the one-dimensional mass matrix by \( M = G^T W G \). Then, we can write the variational form (28) as

\[
Mq = -D^T W f_1(Gu) + \tilde{f}_1,
\]

where \( q, \tilde{f}_1, \) and \( u \) are interpreted appropriately as vectors of coefficients. This is known as the weak form. We can also define the strong form as follows. We rewrite (38) using property (i) above,

\[
Mq = -G^T \tilde{D}^T W \tilde{f}_1(Gu) + \tilde{f}_1,
\]

and then perform a discrete analog of integration by parts (property (ii) above) to obtain

\[
Mq = G^T W \tilde{D} \tilde{f}_1(Gu) - G^T B \tilde{f}_1(Gu) + \tilde{f}_1.
\]

Similarly, the strong form for \( r_{ij} \) is given by

\[
Mr = G^T W \tilde{D} \tilde{f}_2(Gu) - G^T B \tilde{f}_2(Gu) + \tilde{f}_2.
\]

2.4.1. Entropy and quadrature projections. As in the work of Chan [4], a key procedure in constructing the entropy-stable Line-DG scheme is the entropy projection of the conservative variables, defined as follows. Given \( \tilde{u}(\xi, \eta) \), we can compute the entropy variables \( \tilde{v}(\tilde{u}) \). We define \( \tilde{v} \) to be the discrete \( L^2 \) projection of \( v \). That is, \( \tilde{v} \) is the unique bivariate polynomial of degree \( p \) in each variable satisfying

\[
\sum_{\alpha,\beta=1}^{\mu} w_\alpha w_\beta \tilde{v}(\xi_\alpha, \xi_\beta) \cdot \phi(\xi_\alpha, \xi_\beta) = \sum_{\alpha,\beta=1}^{\mu} w_\alpha w_\beta \tilde{v}(\xi_\alpha, \xi_\beta) \cdot \phi(\xi_\alpha, \xi_\beta),
\]

for all test functions \( \phi \). We then define the entropy-projected conservative variables by \( \tilde{u} = u(\tilde{v}) \). We note that for e.g. Burgers’ equation with square entropy function, we have \( v = u \), and thus entropy projection is the identity operator. Additionally, when \( \mu = p + 1 \), the discrete \( L^2 \) projection reduces to the identity, and so in this case, the entropy projection is also the identity, resulting in a simplified scheme. Evaluating the entropy-conservative and entropy-stable fluxes at the entropy-projected values will allow us to prove entropy stability of the discrete scheme.

An additional ingredient required for discrete entropy stability is operation that we introduce called a quadrature projection. Because the Line-DG method is based on consistent integration of the \( \xi \)-derivative in the \( \xi \)-direction, and collocation in the \( \eta \)-direction, and similarly, consistent integration of the \( \eta \)-derivative in the \( \eta \)-direction, and collocation in the \( \xi \)-direction, we introduce a projection operation to allow for consistent integration of both terms in both directions. We are interested in computing discrete integrals of the form

\[
\sum_{\alpha,\beta=1}^{\mu} w_\alpha w_\beta q(\xi_\alpha, \xi_\beta) \cdot \phi(\xi_\alpha, \xi_\beta) \quad \text{and} \quad \sum_{\alpha,\beta=1}^{\mu} w_\alpha w_\beta r(\xi_\alpha, \xi_\beta) \cdot \phi(\xi_\alpha, \xi_\beta),
\]

for a given bivariate polynomial \( \phi \). Equivalently, these integrals can be written in the form

\[
\phi^T(M \otimes M)q, \quad \text{and} \quad \phi^T(M \otimes M)r,
\]
where $M$ is the one-dimensional mass matrix, $\otimes$ represents the Kronecker product, and $\phi, q,$ and $r$ are interpreted as vectors of the corresponding degrees of freedom. However, the more natural line-based quadrature takes the form

\begin{equation}
\phi^T (\tilde{M} \otimes M) q, \quad \text{and} \quad \phi^T (M \otimes \tilde{M}) r,
\end{equation}

where $\tilde{M}$ is the diagonal mass matrix corresponding to the nodal interpolation points. Thus, given approximations $q$ and $r$ to the $\xi$- and $\eta$-derivatives, respectively, we define their quadrature-projected variants by

\begin{equation}
\tilde{q} = (\tilde{M} M^{-1} \otimes I) q, \quad \text{and} \quad \tilde{r} = (I \otimes \tilde{M} M^{-1}) r,
\end{equation}

such that

\begin{equation}
\phi^T (M \otimes M) \tilde{q} = \phi^T (\tilde{M} \otimes M) q, \quad \text{and} \quad \phi^T (M \otimes M) \tilde{r} = \phi^T (M \otimes \tilde{M}) r,
\end{equation}

allowing for computation of the discrete integrals in (43) using the line-based quadrature that is more natural for the Line-DG scheme.

2.4.2. Modified scheme. We modify the scheme to achieve entropy stability using entropy-conservative and entropy-stable numerical fluxes with a flux differencing approach. Equation (40) is replaced by

\begin{equation}
M q = 2 G^T W \tilde{D} \circ \tilde{f}_{1,S}(\tilde{u}) 1 - G^T B \tilde{f}_{1,S}(\tilde{u}) + \tilde{f}_1,
\end{equation}

and equation (41) is replaced by

\begin{equation}
M r = 2 G^T W \tilde{D} \circ \tilde{f}_{2,S}(\tilde{u}) 1 - G^T B \tilde{f}_{2,S}(\tilde{u}) + \tilde{f}_2,
\end{equation}

where $\circ$ denotes the Hadamard product, defined by

\begin{equation}
(\tilde{D} \circ \tilde{f}_{i,S})_{\alpha, \beta} = \tilde{D}_{\alpha, \beta} \tilde{f}_{i,S}(\tilde{u}(\xi_\alpha, \xi_\beta), \tilde{u}(\xi_\beta, \xi_\beta)),
\end{equation}

The transformed flux functions are obtained by taking the arithmetic average of the metric terms,

\begin{equation}
\tilde{f}_{i,S}(u_L, u_R) = \sum_{j=1}^d (q_j J_{j,L}^{-1} + q_R J_{j,R}^{-1}) f_{i,S}(u_L, u_R).
\end{equation}

The modified entropy-stable Line-DG method reads

\begin{equation}
\frac{\partial \tilde{u}_{ij}}{\partial t} + \tilde{q}_{ij} + \tilde{r}_{ij} = 0,
\end{equation}

where $q_{ij}$ is defined by (48) and $r_{ij}$ is defined by (49), and $\tilde{q}$ and $\tilde{r}$ are their quadrature-projected variants, respectively. Given these definitions, we set out to prove the accuracy, conservation, and entropy stability of the discrete scheme.

**Proposition 2** (Accuracy). Suppose $u$ is a smooth solution to (1), and let $u_{ij} = u(\xi_\alpha, \xi_\beta)$. Define $q_{ij}$ and $r_{ij}$ using (48) and (49), respectively. Then we obtain

\begin{equation}
\frac{\partial u_{ij}}{\partial t} + q_{ij} + r_{ij} = O(h^{\min(p+1, \mu-1)}).
\end{equation}

**Proof.** Since $u$ is smooth, $\tilde{f}_i$ is single-valued, and by consistency of the numerical fluxes, the two boundary terms appearing in both (48) and (49) cancel. We begin by assuming that $v = u$ and so $\tilde{u} = u$. Then consider $f_{i,S}(u_L, u_R)$, and define $f_i(\xi) = f_{i,S}(u(\xi_\alpha, \xi_\beta), u(\xi, \xi_\beta))$. Then, by symmetry of $f_{i,S}$,

\begin{equation}
\frac{\partial f_i}{\partial \xi}(\xi) = \left. \frac{\partial f_{i,S}}{\partial u_L} \right|_{u_L, u_R = u(\xi_\alpha, \xi_\beta)} + \left. \frac{\partial f_{i,S}}{\partial u_R} \right|_{u_L, u_R = u(\xi, \xi_\beta)} = 2 \left. \frac{\partial f_{i,S}}{\partial u_R} \right|_{u_L, u_R = u(\xi_\alpha, \xi_\beta)} = 2 \frac{\partial f_{i,S}}{\partial u_R}(u(\xi_\alpha, \xi_\beta), u(\xi, \xi_\beta)).
\end{equation}

Since $\tilde{D}$ is exact for polynomials of degree $\mu - 1$, we have

\begin{equation}
2 \tilde{D} \circ \tilde{f}_{i,S}(Gu) 1 = 2 \frac{\partial f_{i,S}}{\partial u_R} + O(h^{\mu-1}) = \frac{\partial f_i}{\partial \xi} + O(h^{\mu-1}).
\end{equation}

This quantity is then projected onto the space of polynomials of degree $p$, and we obtain

\begin{equation}
q_{ij}(\xi) = \frac{\partial f_i}{\partial \xi}(\xi_\alpha, \xi_\beta) + O\left(h^{\min(p+1, \mu-1)}\right),
\end{equation}

and similarly,

\begin{equation}
r_{ij}(\eta) = \frac{\partial f_i}{\partial \eta}(\xi, \eta) + O\left(h^{\min(p+1, \mu-1)}\right).
\end{equation}

To complete the proof, we now note that by accuracy of the $L^2$ projection, $\tilde{v} = v(u) + O(h^{p+1})$ [4]. Thus $\tilde{u} = u + O(h^{p+1})$, and the general case follows. \hfill \square
Remark 1. If \( \mu = p + 1 \), our method is identical to the DG-SEM method, and the truncation error is suboptimal, scaling as \( \mathcal{O}(h^p) \), as shown in [4]. If \( \mu > p + 1 \), the order of accuracy is optimal, and the truncation error scales as \( \mathcal{O}(h^{p+1}) \).

Proposition 3 (Accuracy of quadrature projection). Define \( \hat{q} \) and \( \hat{F} \) by (46). Then,
\[
\frac{\partial \hat{u}_j}{\partial t} + \hat{q}_j + \hat{r}_j = \mathcal{O}(h^p).
\]

Proof. If \( \mu = p + 1 \), then \( \widetilde{M} = M \), and the result follows from Proposition 2. If \( \mu > p + 1 \), we note that \( \widetilde{M} \) is defined by quadrature at the nodal interpolation points, which is exact for polynomials of degree \( 2(p + 1) = 2p + 1 \). Thus, \( \widetilde{M} \) agrees with the exact mass matrix when applied to polynomials of degree \( p - 1 \). So, \( \hat{q} = M^{-1} \hat{M} q = q + \mathcal{O}(h^p) \), and the result follows.

Remark 2. Proposition 3 implies that the quadrature projection operation introduced in order to obtain discrete entropy stability results in a loss of accuracy. This is verified in the numerical experiments shown in Section 4. Empirically, we observe that, for many cases, the quadrature projection is not required for robustness of the method, and the more accurate method defined by (45) may be used instead. However, for provable discrete entropy stability, we require the method given by (52).

Proposition 4 (Conservation). Given periodic or compactly supported boundary conditions and an affine mesh, then \( \frac{d}{dt} \int_{\Omega} u(x, t) \ dx = 0 \), where \( u \) is obtained through the Line-DG method.

Proof. We consider one element \( K \), mapped to the reference element \( \mathcal{R} \). Then,
\[
\frac{d}{dt} \int_{K} u(x, t) \ dx = \frac{d}{dt} \int_{\mathcal{R}} \tilde{u}(x, t) \ d\xi = -\int_{\mathcal{R}} q_{ij} \ d\xi - \int_{\mathcal{R}} r_{ij} \ d\xi.
\]
We discretize the two integrals on the right-hand side using appropriate line-based quadratures. Since \( u, q, \) and \( r \) are all bivariate polynomials of degree \( p \), the Gauss-Lobatto quadratures associated with both the solution nodes and the line-based quadrature points result in exact integration. In particular, for fixed \( j \), we have
\[
\int_{0}^{1} q_{j} (\xi) \ d\xi = \sum_{\alpha = 1}^{\mu} w_{\alpha} q_{j} (\xi_{\alpha}, \xi_{j}) = 1^{T} M q_{j}.
\]
By definition of \( q \) we have
\[
1^{T} M q = 1^{T} \left( 2G^{T} W \tilde{D} \circ \tilde{f}_{1, S} 1 - G^{T} B \tilde{f}_{1} - \tilde{f}_{1, S} \right)
\]
The first term on the right-hand side is
\[
1^{T} \left( 2G^{T} W \tilde{D} \circ \tilde{f}_{1, S} 1 \right) = 2 \sum_{\alpha, \beta = 1}^{\mu} w_{\alpha} \tilde{D}_{\alpha, \beta} \tilde{f}_{1, S} (\tilde{u}(\xi_{\alpha}, \xi_{j}), \tilde{u}(\xi_{\beta}, \xi_{j}))
\]

\[
= \sum_{\alpha, \beta = 1}^{\mu} w_{\alpha} \left( \tilde{D}_{\alpha, \beta} + \tilde{D}_{\beta, \alpha} \right) \tilde{f}_{1, S} (\tilde{u}(\xi_{\alpha}, \xi_{j}), \tilde{u}(\xi_{\beta}, \xi_{j}))
\]

\[
= \tilde{f}_{1} (\tilde{u}(\xi_{\alpha}, \xi_{j})) - \tilde{f}_{1} (\tilde{u}(\xi_{\beta}, \xi_{j}))
\]
by symmetry of \( \tilde{f}_{1, S} \) and the summation-by-parts property of \( \tilde{D} \). Thus, the boundary terms cancel, and we are left only with the numerical flux term \( \tilde{f}_{1} \). Summing over all elements \( K \), using that the numerical flux is single-valued, and repeating an analogous argument for \( r \), we obtain the desired result.

Lemma 1. Given periodic or compactly supported boundary conditions and an affine mesh \( T_{h} \), the discrete line-based approximation to \( \frac{d}{dt} \int_{\Omega} U(x, t) \ dx \) is bounded above by zero.

Proof. Since \( v = U'(u) \) we have \( \frac{\partial u}{\partial t} = v^{T} \frac{\partial u}{\partial t} \), and thus
\[
\frac{d}{dt} \int_{\Omega} U(x, t) \ dx = \int_{\Omega} v(x, t)^{T} \frac{\partial u}{\partial t} (x, t) \ dx.
\]
As before, we consider a single element \( K \) with corresponding transformation mapping \( T \), and rewrite the integral over the reference element, as
\[
\int_{K} v(x, t)^{T} \frac{\partial u}{\partial t} (x, t) \ dx = \int_{\mathcal{R}} \tilde{u}(\xi, t)^{T} \frac{\partial \tilde{u}}{\partial t} (\xi, t) \ d\xi.
\]
where \( \tilde{u}(\xi, t) = g u(T(\xi), t) \) and \( \tilde{u}(\xi, t) = v(T(\xi), t) \). We replace \( \frac{\partial u}{\partial t} \) by \( -(q + r) \), and discretize the above integrals using the appropriate line-based quadratures.
To begin, we consider, for fixed index $j$,

\begin{equation}
\int_0^1 \tilde{v}(\xi, \xi)\,d\xi \approx \sum_{\alpha=1}^\mu w_\alpha \tilde{v}(\xi_\alpha, \xi_\alpha)^T q_j(\xi_\alpha, \xi_\alpha).
\end{equation}

Let $\tilde{v}$ denote the $L^2$ projection of $v(\xi, \xi)$ onto $[P^p([0, 1])]^n$. Then, since $q_j$ is itself a polynomial of degree $p$, we have

\begin{equation}
\sum_{\alpha=1}^\mu w_\alpha \tilde{v}(\xi_\alpha, \xi_\alpha)^T q_j(\xi_\alpha, \xi_\alpha) = \sum_{\alpha=1}^\mu w_\alpha \tilde{v}(\xi_\alpha, \xi_\alpha)^T q_j(\xi_\alpha, \xi_\alpha) = \tilde{v}^T M q_j.
\end{equation}

Using the definition of $q$ in (48), we have

\begin{equation}
\tilde{v}^T M q = \tilde{v}^T \left(2G^T W \tilde{D} \circ \tilde{f}_{1, S}(\tilde{u}) - G^T B \tilde{f}_{1, S}(\tilde{u}) + \tilde{f}_1 \right).
\end{equation}

We use $2W \tilde{D} = W \tilde{D} - \tilde{D}^T W + B$ to rewrite the first term on the right-hand side as

\begin{equation}
\sum_{\alpha, \beta=1}^\mu (\tilde{v}(\xi_\alpha, \xi_\beta) - \tilde{v}(\xi_\beta, \xi_\alpha))^T w_\alpha \tilde{D}_{\alpha, \beta} f_{1, S}(\tilde{u}(\xi_\alpha, \xi_\beta), \tilde{u}(\xi_\beta, \xi_\alpha)).
\end{equation}

The boundary term exactly cancels the second term on the right-hand side of (64). We reindex and use symmetry of $f_{1, S}$ to write the remaining terms as

\begin{equation}
\sum_{\alpha, \beta=1}^\mu (\tilde{v}(\xi_\alpha, \xi_\beta) - \tilde{v}(\xi_\beta, \xi_\alpha))^T \sum_{\alpha, \beta=1}^\mu (\tilde{v}(\xi_\alpha, \xi_\beta) - \tilde{v}(\xi_\beta, \xi_\alpha)) f_{1, S}(\tilde{u}(\xi_\alpha, \xi_\beta), \tilde{u}(\xi_\beta, \xi_\alpha)).
\end{equation}

We assume that the mesh is affine, and so $g_j^{-1}$ is constant. Thus, $\tilde{f}_{1, S} = g_{j_{11}}^{-1} f_{1, S} + g_{j_{12}}^{-1} f_{2, S}$. Since $\tilde{u} = u(\tilde{v})$, we use the entropy conservation of the two-point flux to write this sum as

\begin{equation}
\sum_{\alpha, \beta=1}^\mu w_\alpha \tilde{D}_{\alpha, \beta} \left(g_{j_{11}}^{-1} (\psi_{1, \alpha} - \psi_{1, \beta}) + g_{j_{21}}^{-1} (\psi_{2, \alpha} - \psi_{2, \beta})\right)
\end{equation}

\begin{equation}
= \sum_{\alpha, \beta=1}^\mu w_\alpha \tilde{D}_{\alpha, \beta} \left(g_{j_{11}}^{-1} \psi_{1, \beta} + g_{j_{21}}^{-1} \psi_{2, \beta}\right)
\end{equation}

\begin{equation}
= g_{j_{11}}^{-1} (\psi_{1, \beta} - \psi_{1, \mu}) + g_{j_{21}}^{-1} (\psi_{2, \beta} - \psi_{2, \mu}),
\end{equation}

where we used properties (iii) and (iv) of Proposition 4. Therefore, the total entropy production corresponding to $q_j$ for the element $K$ is given by

\begin{equation}
g_{j_{11}}^{-1} \left(\tilde{v}_\mu^T \tilde{f}_{1, \mu} - \tilde{v}_1^T \tilde{f}_{1, 1} + \psi_{1, \beta} - \psi_{1, \mu}\right) + g_{j_{21}}^{-1} \left(\tilde{v}_\mu^T \tilde{f}_{2, \mu} - \tilde{v}_1^T \tilde{f}_{2, 1} + \psi_{2, \beta} - \psi_{2, \mu}\right).
\end{equation}

We now sum the contributions along a shared edge of two elements, $K_L$ and $K_R$. We obtain

\begin{equation}
g_{j_{11}}^{-1} \left(\tilde{v}_L^T \tilde{f}_L - \tilde{v}_R^T \tilde{f}_R + \psi_{1, R} - \psi_{1, L}\right) + g_{j_{21}}^{-1} \left(\tilde{v}_L^T \tilde{f}_L - \tilde{v}_R^T \tilde{f}_R + \psi_{2, R} - \psi_{2, L}\right) \leq 0.
\end{equation}

using the entropy stability of the numerical flux function. We repeat a similar argument for the term $r_{ij}$. □

**Proposition 5 (Discrete entropy stability).** Given periodic or compactly supported boundary conditions and an affine mesh $T_h$, we have

\begin{equation}
\frac{d}{dt} \sum_{K \in T_h} \sum_{\alpha, \beta=1}^\mu w_\alpha w_\beta U(\tilde{u}_K(\xi_\alpha, \xi_\beta)) \leq 0,
\end{equation}

where $\tilde{u}$ is given by (52).

**Proof.** Noting that $\frac{d}{dt} = \tilde{v}^T \frac{\partial U}{\partial t}$, we have

\begin{equation}
\frac{d}{dt} \sum_{K \in T_h} \sum_{\alpha, \beta=1}^\mu w_\alpha w_\beta U(\tilde{u}_K(\xi_\alpha, \xi_\beta)) = \sum_{K \in T_h} \sum_{\alpha, \beta=1}^\mu w_\alpha w_\beta U(\tilde{u}_K(\xi_\alpha, \xi_\beta))^T \left(\tilde{q}_K(\xi_\alpha, \xi_\beta) + \tilde{r}_K(\xi_\alpha, \xi_\beta)\right)
\end{equation}

\begin{equation}
= \sum_{K \in T_h} \tilde{v}_K^T (M \otimes M) \tilde{q}_K + \tilde{v}_K^T (M \otimes M) \tilde{r}_K
\end{equation}

\begin{equation}
\leq 0,
\end{equation}

using the definition of the quadrature projection and Lemma 4. □
Table 1. Number of evaluations of $f_S$ required by the DG and Line-DG methods in 3 spatial dimensions, for one-dimensional quadrature rules with $\mu$ points.

| $\mu = p + 2$ |  |  |  |  |  |  |  |  |
|----------------|---|---|---|---|---|---|---|---|
| $p = 3$        | 15,625 | 46,656 | 117,649 | 262,144 | 531,441 | 1,009,000 | 1,771,561 | 2,985,984 |
| DG             | 1200  | 2700 | 5292 | 9408 | 15552 | 24300 | 36300 | 52272 |
| Line-DG        | 768   | 1875 | 3888 | 7203 | 12288 | 19683 | 30000 | 43923 |
| $\mu = \left\lceil \frac{1}{2}(p+1) \right\rceil$ |  |  |  |  |  |  |  |  |
| DG             | 46,656 | 262,144 | 531,441 | 1,771,561 | 2,985,984 | 7,529,536 | 11,390,625 | 24,137,569 |
| Line-DG        | 1728  | 4800 | 8748 | 17787 | 27648 | 47628 | 67500 | 104907 |
| DG-SEM         | 768   | 1875 | 3888 | 7203 | 12288 | 19683 | 30000 | 43923 |

3. Implementation and computational cost

The implementation of the Line-DG method is relatively simple, and benefits greatly from the reuse of much of the infrastructure required for a standard DG method: the fluxes, numerical flux functions, boundary conditions, and metric terms remain, for the most part, unchanged. In fact, some features of the method allow for significant simplifications: all volume integrals are replaced with one-dimensional integrals, and no surface or face integrals are required. A key feature of the Line-DG method compared with traditional DG methods is its reduced computational cost. This reduced cost is attributed both to the smaller number of flux evaluations and less expensive interpolation and integration operations, when compared with standard DG.

We first compare the total number of flux evaluations required by each method. The entropy-stable DG method requires the two-point flux function $f_S$ to be evaluated at all pairs of quadrature points. In a $d$-dimensional tensor-product element with a quadrature rule based on one-dimensional Gaussian quadrature with $\mu \geq p + 1$ points, there are $\mu^d$ such points, requiring $\mu^{2d}$ evaluations of $f_S$. Symmetry of the flux $f_S$ can be used to reduce this number by about a factor of two. On the other hand, the Line-DG method requires the evaluation of $f_S$ at all pairs of quadrature points along each line of nodes with an element. There are $(p + 1)^{d-1}$ such lines, necessitating $d(p + 1)^{d-1}\mu^2$ flux evaluations. As in the DG case, this can be reduced by about a factor of two by exploiting the symmetry of $f_S$. In Table 1 we compare the number of flux evaluations for quadrature rules with $\mu = p + 2$ points and $\mu = \left\lceil \frac{1}{2}(p+1) \right\rceil$ points (e.g. to correctly integrate a quadratic nonlinearity in the flux function) for the specific case of $d = 3$.

Additionally, the Line-DG method requires the evaluation of the numerical flux function $\hat{f}$ only at nodal points lying on each face of a given element, resulting in $(p + 1)^{d-1}$ evaluations of $\hat{f}$. On the other hand, a standard DG method requires the integration of $\hat{f}$ according to a $(d-1)$-dimensional quadrature rule, resulting in $\mu^{d-1}$ evaluations, where $\mu \geq p + 1$.

We now consider the interpolation and integration operations. Since the Line-DG method is based on the evaluation of one-dimensional integrals, the interpolation and integration operations are equivalent to those of a standard one-dimensional DG method. The interpolation operator $G$ is defined by (32), and is a one-dimensional Vandermonde matrix. The integration operator is given by $G^T W$, where $W$ is a diagonal matrix consisting of the quadrature weights. The entropy-stable differentiation operator is defined by $\hat{D}$, as given by (35). The operators are identical among all elements, and along each of the spatial dimensions. The complexity of applying these operators to an entire element is linear in $p$ per degree of freedom, which is the same as a sum-factorized DG method, although the implementation is significantly simpler (25, 38, 21). Additionally, because these operators are identical along each spatial dimension, the implementation can benefit from batched BLAS-3 operations, by considering the degrees of freedom along all lines of nodes within an element as a matrix of size $(p + 1) \times d(p + 1)^{d-1}$. This is particularly important on modern computer architectures, for which matrix-vector products are often memory-bound (34).

Finally, we consider the storage cost of the method. The number of degrees of freedom is the same as a standard DG method. However, the cost of storing precomputed metric terms and interpolation matrices is reduced. All stored matrices in the Line-DG method are either $(p + 1) \times (p + 1)$ or $\mu \times \mu$ in size, as compared with $(p + 1)^d \times (p + 1)^d$ or $\mu^d \times \mu^d$ for a full DG method. Additionally, the metric terms need to be stored at quadrature points. For the DG method, the inverse of the transformation Jacobian matrix is a $d \times d$ matrix, thus requiring the storage of $d^2\mu^d$ terms. In the case of Line-DG, along each line of nodes, only one row of the inverse Jacobian corresponding to the given coordinate dimension is required. Each line consists of $\mu$ quadrature points, and there are $d(p + 1)^{d-1}$ such lines, thus necessitating the storage of $d^2(p + 1)^{d-1}\mu$ terms. Since $\mu \geq p + 1$, we see that the Line-DG method enjoys reduced storage costs when compared with the standard DG method.

4. Numerical Results

4.1. 1D sinusoidal Burgers’ equation. We begin with a simple test case for Burgers’ equation

\begin{equation}
(70)
  u_t + \left( \frac{1}{2} u^2 \right)_x = 0,
\end{equation}
with periodic boundary conditions and sinusoidal initial conditions, \( u_0(x) = \frac{1}{2} + \sin(x) \). We choose the entropy function \( U = \frac{1}{2} u^2 \). The corresponding entropy-conservative numerical flux is defined by

\[
F_S(u_L, u_R) = \frac{1}{6} \left( u_L^2 + u_L u_R + u_R^2 \right).
\]

At element interfaces, the entropy-stable numerical flux function is defined by solving the Riemann problem exactly. At \( t = 0.5 \) the solution is still smooth, however by \( t = 1.5 \) the solution has developed a discontinuity. We begin by computing the \( L^\infty \) error of the Line-DG method at \( t = 0.5 \), where we compare to the exact solution, which is computed by following the characteristic curves backwards in time.

We compare the accuracy of the Line-DG method using \( \mu = p + 1 \) (i.e., DG-SEM) with the Line-DG method using \( \mu > p + 1 \). Only negligible differences were observed for different values of \( \mu \) greater than \( p + 1 \). We additionally remark that in the 1D case, the Line-DG method with is identical to a standard DG method with the specified quadrature rule. For DG-SEM, our results agree with those of Chen and Shu \[7\]. In this case, we observe suboptimal \( \mathcal{O}(h^{p+1}) \) convergence in the \( L^\infty \) norm. For \( \mu > p + 1 \), we recover optimal \( \mathcal{O}(h^p) \) convergence in the \( L^\infty \) norm.

Additionally, we compare the solution quality at \( t = 1.5 \), after the shock has developed. We compare the \( \mu = p + 1 \) DG-SEM method to the Line-DG \( \mu > p + 1 \) method, for \( p = 5 \) and number of elements \( N = 120 \). Both solutions display non-physical oscillations in the vicinity of the shock, although the oscillations are slightly smaller in magnitude when using \( \mu > p + 1 \). In both cases, the cell averages of the approximate solution well-approximate the true solution.

### 4.2 1D shock tube

In this section, we consider both the classic shock tube problem of Sod, as well as a slightly modified Mach 2 shock tube problem. Both problems are solved on the domain \( \Omega = [-0.5, 0.5] \), and the initial conditions for both problems posses a discontinuity at the origin,

\[
u_0(x) = \begin{cases}
u_L, & x < 0, \\nu_R, & x \geq 0.
\end{cases}
\]

The initial conditions for Sod’s shock tube are given by

\[
u_L = \begin{pmatrix}
\rho_L \\
w_L \\
p_L
\end{pmatrix} = \begin{pmatrix}
1 \\
0 \\
1
\end{pmatrix},
\quad
\nu_R = \begin{pmatrix}
\rho_R \\
w_R \\
p_R
\end{pmatrix} = \begin{pmatrix}
1/8 \\
0 \\
1/10
\end{pmatrix}.
\]

The initial conditions for the Mach 2 shock are given by

\[
u_L = \begin{pmatrix}
\rho_L \\
w_L \\
p_L
\end{pmatrix} = \begin{pmatrix}
1.62 \\
0 \\
4.5
\end{pmatrix},
\quad
\nu_R = \begin{pmatrix}
\rho_R \\
w_R \\
p_R
\end{pmatrix} = \begin{pmatrix}
1/8 \\
0 \\
1/10
\end{pmatrix}.
\]

Both problems give rise to a rarefaction wave, a contact discontinuity, and a shock. We solve both problems using \( p = 6 \) polynomials with 48 elements, and integrate in time until \( t = 0.1 \). A comparison of the numerical solutions with the exact solutions is shown in Figure 2. For the Sod shock tube problem, the DG-SEM method and Line-DG method with \( \mu > p + 1 \) give comparable results. However, for the Mach 2 shock problem, the DG-SEM method gives rise to significantly more prominent oscillations, demonstrating a potential advantage of integrating with higher-accuracy quadrature rules. This phenomenon is particularly noticeable in the velocity component of the solution.
Table 2. $L^\infty$ error and convergence rates for the smooth solution to 1D Burgers’ equation at $t = 0.5$.

| $N$  | $\mu = p + 1$ Error | Rate | $\mu > p + 1$ Error | Rate |
|------|---------------------|------|----------------------|------|
| 40   | $3.27 \times 10^{-3}$ | —    | $7.59 \times 10^{-4}$ | —    |
| 80   | $7.92 \times 10^{-4}$ | 2.04 | $1.06 \times 10^{-4}$ | 2.84 |
| 160  | $2.08 \times 10^{-4}$ | 1.93 | $1.47 \times 10^{-5}$ | 2.86 |
| 320  | $5.10 \times 10^{-5}$ | 2.03 | $1.93 \times 10^{-6}$ | 2.93 |

$p = 2$

| $N$  | $\mu = p + 1$ Error | Rate | $\mu > p + 1$ Error | Rate |
|------|---------------------|------|----------------------|------|
| 40   | $1.65 \times 10^{-4}$ | —    | $3.84 \times 10^{-5}$ | —    |
| 80   | $1.62 \times 10^{-5}$ | 3.35 | $2.54 \times 10^{-6}$ | 3.92 |
| 160  | $1.31 \times 10^{-6}$ | 3.63 | $1.80 \times 10^{-7}$ | 3.82 |
| 320  | $9.34 \times 10^{-8}$ | 3.81 | $1.17 \times 10^{-8}$ | 3.94 |

$p = 3$

| $N$  | $\mu = p + 1$ Error | Rate | $\mu > p + 1$ Error | Rate |
|------|---------------------|------|----------------------|------|
| 40   | $1.13 \times 10^{-5}$ | —    | $3.09 \times 10^{-6}$ | —    |
| 80   | $7.16 \times 10^{-7}$ | 3.98 | $1.08 \times 10^{-7}$ | 4.85 |
| 160  | $4.34 \times 10^{-8}$ | 4.04 | $3.90 \times 10^{-9}$ | 4.78 |
| 320  | $2.62 \times 10^{-9}$ | 4.05 | $1.29 \times 10^{-10}$ | 4.92 |

$p = 4$

| $N$  | $\mu = p + 1$ Error | Rate | $\mu > p + 1$ Error | Rate |
|------|---------------------|------|----------------------|------|
| 40   | $7.12 \times 10^{-7}$ | —    | $1.64 \times 10^{-7}$ | —    |
| 80   | $1.87 \times 10^{-8}$ | 5.25 | $3.49 \times 10^{-9}$ | 5.55 |
| 160  | $3.93 \times 10^{-10}$ | 5.57 | $6.07 \times 10^{-11}$ | 5.85 |
| 320  | $1.32 \times 10^{-11}$ | 4.89 | $1.03 \times 10^{-12}$ | 5.87 |

$p = 5$

Figure 2. Numerical and exact solutions to the shock tube problems at $t = 0.1$. Density is shown in the top row and velocity is shown in the bottom row.

4.3. **2D Burgers’ equation.** For the first two-dimensional test problem, we consider the Line-DG method applied to the 2D Burgers’ equation,

\[ u_t + \nabla \cdot (\frac{1}{2} \beta u^2) = 0, \]

(75)
Relative entropy loss \(-\int_{\Omega} U(x,t) \ln \frac{U(x,t)}{U(x,0)} \, dx\) must be monotonically non-increasing. For this test case the solution is smooth, and so the entropy for the isentropic Euler vortex test case \([39]\). This problem consists of an isentropic vortex that is advected with the freestream velocity, and is often used as a smooth benchmark problem \([24, 41]\). The spatial domain is taken to be \([0, 20] \times [0, 15]\). The vortex is initially centered at \((x_0, y_0) = (5, 5)\), and is advected at an angle of \(\theta\). The exact solution at position

| \(N\) | \(\mu = p + 1\) Error | \(\mu = p + 1\) Rate | \(\mu > p + 1\) Error | \(\mu > p + 1\) Rate | \(\text{No projection}\) Error | \(\text{No projection}\) Rate |
|------|----------------|----------------|----------------|----------------|----------------|----------------|
| 12   | \(4.03 \times 10^{-2}\) — | \(2.65 \times 10^{-2}\) — | \(2.20 \times 10^{-3}\) — |
| 24   | \(9.20 \times 10^{-3}\) 2.13 | \(4.11 \times 10^{-3}\) 2.69 | \(2.20 \times 10^{-3}\) 2.90 |
| 48   | \(1.83 \times 10^{-3}\) 2.33 | \(6.72 \times 10^{-4}\) 2.61 | \(3.06 \times 10^{-4}\) 2.84 |
| 96   | \(3.43 \times 10^{-4}\) 2.41 | \(1.11 \times 10^{-4}\) 2.60 | \(4.21 \times 10^{-5}\) 2.86 |
| 12   | \(8.03 \times 10^{-3}\) — | \(3.54 \times 10^{-3}\) — | \(2.46 \times 10^{-3}\) — |
| 24   | \(8.01 \times 10^{-4}\) 3.32 | \(3.73 \times 10^{-4}\) 3.25 | \(2.08 \times 10^{-4}\) 3.57 |
| 48   | \(7.66 \times 10^{-5}\) 3.39 | \(3.47 \times 10^{-5}\) 3.43 | \(1.54 \times 10^{-5}\) 3.76 |
| 96   | \(6.67 \times 10^{-6}\) 3.52 | \(3.14 \times 10^{-6}\) 3.47 | \(1.08 \times 10^{-6}\) 3.83 |
| 12   | \(1.37 \times 10^{-3}\) — | \(8.02 \times 10^{-4}\) — | \(6.40 \times 10^{-4}\) — |
| 24   | \(9.46 \times 10^{-5}\) 3.85 | \(3.67 \times 10^{-5}\) 4.45 | \(2.51 \times 10^{-5}\) 4.67 |
| 48   | \(4.84 \times 10^{-6}\) 4.29 | \(1.60 \times 10^{-6}\) 4.52 | \(9.60 \times 10^{-7}\) 4.71 |
| 96   | \(2.27 \times 10^{-7}\) 4.42 | \(7.05 \times 10^{-8}\) 4.50 | \(3.43 \times 10^{-8}\) 4.81 |
| 12   | \(3.78 \times 10^{-4}\) — | \(1.86 \times 10^{-4}\) — | \(1.54 \times 10^{-4}\) — |
| 24   | \(1.19 \times 10^{-5}\) 4.99 | \(4.90 \times 10^{-6}\) 5.25 | \(3.52 \times 10^{-6}\) 5.45 |
| 48   | \(3.15 \times 10^{-7}\) 5.24 | \(1.08 \times 10^{-7}\) 5.50 | \(6.77 \times 10^{-8}\) 5.70 |
| 96   | \(7.06 \times 10^{-9}\) 5.48 | \(2.35 \times 10^{-9}\) 5.53 | \(1.23 \times 10^{-9}\) 5.79 |

**Figure 3.** Relative entropy loss for the 2D Burgers’ equation, with \(U = \frac{1}{2} u^2\), for \(N = 12\) and \(p = 2, 3, 4, 5\). Line-DG shown with solid lines, DG-SEM shown with dashed lines.

with constant velocity vector \(\beta = (1, 1)\). We choose the smooth initial conditions \(u_0(x, y) = \frac{1}{2} + \sin(x) \cos(y)\), and integrate in time until \(t = 0.5\). At this point, the solution remains smooth, and as in the 1D case, we can obtain the exact solution by tracing backwards along characteristic lines.

We compare the \(L^2\) error obtained using the Line-DG method and the DG-SEM (\(\mu = p + 1\)) method. We also consider the Line-DG method without the final quadrature projection. Results are shown in Table 3. For both the DG-SEM and Line-DG methods, we observe slightly sub-optimal convergence: approximately \(O(h^{p+1/4})\). Without the quadrature projection operation, the convergence is approximately \(O(h^{p+1})\), in accordance with Propositions 2 and 3. For each test case, the error obtained using the Line-DG method is smaller by approximately a factor of two, and the rate of converge appears to be slightly faster than that of DG-SEM for a majority of cases. We also investigate the entropy dissipation of each of these methods. Since both methods are entropy-stable, the total entropy must be monotonically non-increasing. For this test case the solution is smooth, and so the entropy for the exact solution remains constant. In Figure 3, we compare the relative deviation from the initial entropy, measured by \(\frac{\int_{\Omega} (U(x,t) - U(x,0)) \, dx}{\int_{\Omega} U(x,0) \, dx}\), where \(U(x,t)\) is the square entropy, \(U(x,t) = \frac{1}{2} u(x,t)^2\). We numerically observe the entropy stability of both methods, but note that the Line-DG method dissipates less entropy than the DG-SEM method.

4.4. 2D isentropic vortex. For this test problem, we study the accuracy of the Line-DG method applied to the isentropic Euler vortex test case \([39]\). This problem consists of an isentropic vortex that is advected with the freestream velocity, and is often used as a smooth benchmark problem \([24, 41]\). The spatial domain is taken to be \([0, 20] \times [0, 15]\). The vortex is initially centered at \((x_0, y_0) = (5, 5)\), and is advected at an angle of \(\theta\). The exact solution at position
and time \((x, y, t)\) is given by

\[
\rho(x, y, t) = \rho_\infty \left(1 - \frac{\epsilon^2(\gamma - 1)M_\infty^2}{8\pi^2} \exp(f(x, y, t))\right)^{\frac{1}{\gamma - 1}},
\]

\[
w_1(x, y, t) = w_\infty \left(\cos(\theta) - \frac{\epsilon((y - y_0) - \overline{w}_1 t)}{2\pi r_c} \exp\left(\frac{f(x, y, t)}{2}\right)\right),
\]

\[
w_2(x, y, t) = w_\infty \left(\sin(\theta) - \frac{\epsilon((x - x_0) - \overline{w}_2 t)}{2\pi r_c} \exp\left(\frac{f(x, y, t)}{2}\right)\right),
\]

\[
p(x, y, t) = p_\infty \left(1 - \frac{\epsilon^2(\gamma - 1)M_\infty^2}{8\pi^2} \exp(f(x, y, t))\right)^{\frac{2}{\gamma - 1}},
\]

where \(f(x, y, t) = (1 - ((x - x_0) - \overline{w}_1 t)^2 - ((y - y_0) - \overline{w}_2 t)^2)/r_c^2\), \(M_\infty\) is the freestream Mach number, and \(w_\infty, \rho_\infty,\) and \(p_\infty\) are the freestream velocity magnitude, density, and pressure, respectively. The freestream velocity is given by \((\overline{w}_1, \overline{w}_2) = w_\infty (\cos(\theta), \sin(\theta))\). The strength of the vortex is given by \(\epsilon\), and its size by \(r_c\). We choose the parameters to be \(M_\infty = 0.5, w_\infty = 1, \theta = \arctan(1/2), \epsilon = 5/(2\pi),\) and \(r_c = 1.5\). We integrate the equations until \(t = 5\).

We perform a convergence study to investigate the effects of the projection operation described in Section 2.4.1 on the accuracy of the method. Additionally, for comparison we consider a fully-integrated standard DG method. The local Lax-Friedrichs numerical flux function was used for all methods. The \(L^\infty\) error at the final time is shown in Figure 4. We observe that the Line-DG method without the projection operation has accuracy that is almost identical to that of the standard, fully-integrated DG method for this test problem. This finding is consistent with the results shown in [27]. However, when the projection operation is performed in order to ensure discrete entropy stability, we observe a larger error by approximately a constant factor. It is interesting to note that for this test problem, we do not observe the sub-optimal order of accuracy seen in previous test cases. This is possibly due to the translational nature of the true solution.
4.5. 2D supersonic flow in a duct. We consider the inviscid supersonic flow in a duct with a smooth bump. The duct has dimensions $3 \times 1$ and the $y$-coordinate of the bottom boundary is given by

\begin{align}
    y(x) = \begin{cases}
        \frac{H}{2} (\cos(2\pi(x - 3/4)) + 1), & 3/4 < x < 5/4 \\
        0, & \text{otherwise.}
    \end{cases}
\end{align}

(77)

where the height of the bump is given by $H = 0.04$. The inflow density is $\rho = 1$, and the inflow velocity is $\mathbf{w} = (1, 0)$. We set the Mach number to $M = 1.4$ as in [10]. Slip wall conditions are enforced on the top and bottom boundaries. The curved boundary is represented using isoparametric elements. We use 675 elements with degree $p = 4$ polynomials. We do not use any shock capturing techniques or apply any limiters to the solution. This test case is intended to assess the robustness of the method for under-resolved high Mach number flow.

We compute the steady solution to this problem using pseudo-time integration. We compare the solutions obtained using the $\mu = p + 1$ DG-SEM method to the Line-DG method with $\mu > p + 1$. The steady-state pressure is shown in Figure 5. Both solutions display fairly severe oscillations in the vicinity of the shocks. Some of these features appear to be more prominent in the solution obtained using the DG-SEM method. Despite these oscillations, the method remains robust due to the entropy stability. Traditional DG-SEM, Line-DG, or consistently-integrated standard DG methods are unstable for this problem without the use of additional shock capturing techniques or limiters.

4.6. 3D inviscid Taylor-Green vortex. For a final set of test cases, we consider the compressible, inviscid Taylor-Green vortex (TGV) [37] at different Mach numbers. This problem has been extensively studied for the incompressible case [2], as well as the nearly-incompressible case [34, 35, 29, 25]. The stability of DG discretizations for the under-resolved simulation of the inviscid TGV has also been studied in [40, 29]. The domain is taken to be the cube $[-\pi, \pi]^3$. 

![Figure 5. Density of steady-state solution to supersonic flow over a bump.](image-url)
and periodic conditions are enforced on all boundaries. The initial conditions are given by

\[
\begin{align*}
\rho(x, y, z) &= \rho_0 \\
\omega_1(x, y, z) &= \omega_0 \sin(x) \cos(y) \cos(z) \\
\omega_2(x, y, z) &= -\omega_0 \cos(x) \sin(y) \cos(z) \\
\omega_3(x, y, z) &= 0 \\
p(x, y, z) &= \rho_0 + \rho_0 \omega_0^2 \left( \cos(2x) + \cos(2y) \right) \left( \cos(2z) + 2 \right)/16,
\end{align*}
\]

where we take the parameters to be \( \omega_0 = 1, \rho_0 = 1 \), with Mach number \( M_0 = u_0/c_0 \), where \( c_0 \) is the speed of sound computed in accordance with the pressure \( \rho_0 \). The characteristic convective time is given by \( t_c = 1 \), and we integrate until \( t = 10t_c \). For the nearly incompressible case, we choose \( \rho_0 = 100 \) which corresponds to a Mach number of \( M_0 \approx 0.08 \). We also consider a higher Mach number case defined by \( M_0 = 0.7 \).

We measure three quantities of interest. The first is the mean entropy, which is guaranteed to be monotonically non-increasing by the method. The second is the mean kinetic energy

\[
E_k(t) = \frac{1}{\rho_0 |\Omega|} \int_{\Omega} \frac{1}{2} \rho |\mathbf{w}|^2 \, dx.
\]

We can easily see that \( E_k(0) = 1/8 \). Since the kinetic energy is conserved for the inviscid Taylor-Green vortex in the incompressible limit, for the low Mach (nearly incompressible) case, we can use \( E_k(t) \) as a measure of the numerical dissipation introduced by the discretization. The third quantity of interest considered is the mean enstrophy, defined by

\[
E(t) = \frac{1}{\rho_0 |\Omega|} \int_{\Omega} \frac{1}{2} \rho |\mathbf{\omega}|^2 \, dx,
\]

where \( \mathbf{\omega} = \nabla \times \mathbf{w} \) is the vorticity. The enstrophy can be used as a measure of the resolving power of the numerical discretization.

We discretize the geometry using a \( 20 \times 20 \times 20 \) Cartesian grid, and we use degree \( p = 3 \) and \( p = 5 \) polynomials. For the higher Mach number case, defined by \( M = 0.7 \), the standard DG-SEM method without entropy stability is unstable after about \( t = 3.9t_c \). The Line-DG method without entropy stability is unstable after about \( t = 4.7t_c \). The entropy-stable versions of both DG-SEM and Line-DG remain stable for the full duration of the simulation.

In Figure 6, we show the normalized time evolution of the quantities of interest for both test cases. We define the normalized mean entropy by \( \int_{\Omega} (U(x, t) - U(x, 0)) \, dx / \int_{\Omega} U(x, 0) \, dx \), and similarly for the normalized mean kinetic energy and enstrophy. For the low Mach case with \( p = 3 \) polynomials, we notice that the Line-DG method dissipates less entropy and kinetic energy than the equal-order DG-SEM method. In fact, the dissipation of these two quantities is roughly equivalent to the DG-SEM method with polynomial degree \( p = 5 \). For both the Line-DG method and DG-SEM method with \( p = 3 \), the peak enstrophy is under-predicted. For the \( M_0 = 0.7 \) case, the Line-DG and DG-SEM methods result in comparable results for all three quantities considered, however for the \( p = 5 \) case, the DG-SEM method gives rise to less enstrophy growth. As discussed in \[26\], some caution is required when using these mean quantities to assess the quality of the numerical solutions, in particular once the solution has become under-resolved.

This test case demonstrates both the increased robustness of the entropy-stable Line-DG when compared with standard DG methods, and its low dissipation when compared with the equal-order entropy-stable DG-SEM method.

5. Conclusions

In this paper, we have constructed a discretely entropy-stable line-based discontinuous Galerkin method. We modify the Line-DG method of \[26\] using a flux differencing technique in order to obtain discrete entropy stability, compatible with the quadrature rule used in the discretization. This line-based method is composed of one-dimensional operations performed along lines or curves of nodes within tensor-product elements, resulting in fewer flux evaluations, and requiring only one-dimensional interpolation and integration operations. This method is closely related to the entropy-stable DG-SEM method, described in \([7, 3, 11]\), and to the entropy-stable full DG method developed in \[11\]. When compared with the equal-order entropy-stable DG-SEM method on a range of test cases, the Line-DG method results in smaller errors and less numerical dissipation.

The main feature of this entropy-stable method is its increased robustness in the presence of shocks or under-resolved features. This robustness has been demonstrated on a range of problems for which standard DG-type methods are unstable without the use of additional limiting or shock-capturing techniques. For problems with strong shocks, the entropy-stable Line-DG method can demonstrate spurious oscillations despite remaining stable. For problems of this type, artificial viscosity or limiters may be used to reduce the oscillations and increase solution quality.
Figure 6. Normalized time evolution of $U$ (entropy), $E_k$ (kinetic energy), and $\mathcal{E}$ (enstrophy) over time, for the 3D inviscid Taylor-Green test case, at $M_0 \approx 0.08$ and $M_0 = 0.7$.

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