Refinement of the Reactor Dynamics Mathematical Model

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Abstract. A mathematical model of reactor dynamics described by a system of integro-differential equations consisting of an unsteady-state multivelocity kinetic transfer equation and a delayed neutrons balance equation is considered. To refine some parameters of this system a number of inverse problems of reactor dynamics with a new definition of integral form are solved. Statements of the linear inverse problem of recovery of a part of the function of inner neutron sources, and non-linear inverse problems of recovery of a part of the absorption coefficient are presented. The sought for parts of the said parameters are functions depending on the velocity and time variables. Theorems of existence and uniqueness of generalized solutions have been proved.

1. Introduction
Currently, construction of more and more nuclear generating stations calls for reliable, simple to control and safe nuclear power plants. This is an additional driver for the development of the fairly young and continuously evolving mathematical theory of nuclear reactors. This is evidenced by a number of works [1-14] dedicated to these topics. Researchers, together with engineers, aim to create such reactor models the operation of which has least dependence on the human factor. The issues on the agenda are the mathematical modeling of processes in nuclear reactors. To develop mathematical models researchers should realize that the nuclear reactor is to be considered a complex conglomerate of interacting processes; the contribution of each process in this or that case can hardly be neglected. However, our task is not to construct a holistic mathematical model describing the nuclear reactor as an object on the whole. The model under study is the reactor dynamics system given in [5]. Our task is to refine several parameters of this system using additional measurements of occurring processes. A refined model is required: 1) to have a better insight into how a specific plant works, what its structure and properties are; 2) to learn to control this object, or the processes running inside this object, and determine the best methods of control within specified objectives and criteria; 3) to predict the behavior of the reactor and evaluate the consequences of different methods and forms of action on this object.

2. Statement of linear inverse problem
Let us first examine a mathematical model of reactor dynamics without considering temperature feedback from [4], described by the system of integro-differential equations:
1) by the non-stationary anisotropic multi-speed kinetic transfer equation

\[
\frac{\partial u}{\partial t}(x,v,t) + (v, \nabla_x) u(x,v,t) + \Sigma(x,v,t)u(x,v,t) = \sum_{k=1}^{N} z_k R_k(x,v,t) + \int_{V} J_k(x,v,v',t)u(x,v',t)dv' + F(x,v,t);
\]

(1)

2) by the delayed neutrons balance equation

\[
\frac{\partial R_k}{\partial t}(x,v,t) = -z_k R_k(x,v,t) + \int_{V} J_k(x,v,v',t)u(x,v',t)dv' \quad \forall \ k=1,N,
\]

(2)

where \((x,v,t) \in D = G \times V \times [0,T].\)

In this model the function \(u(x,v,t)\) describes the density distribution of neutrons flying through point \(x \in G\) with velocity \(v \in V\) at the time moment \(t \in [0,T].\) Functions \(\Sigma(x,v,t), \ J(x,v,v',t), \ F(x,v,t)\) specify properties of the medium in which the mass transfer process is occurring. \(\Sigma(x,v,t), \) at this, is the absorption coefficient, \(J(x,v,v',t)\) is the dispersion index, \(F(x,v,t)\) is the density of the radiation source. The kernels of the scattering integrals \(J_k(x,v,v',t)\) specify densities of distribution of secondary neutrons, and the function \(R_k(x,v,t)\) is the distribution density of the supports of the delayed neutrons of the \(k\)-th group. Strongly convex region \(G\) is the region of variation of space coordinates; the closed set \(V\) lying in the ball layer \(\left\{ 0 < v_0 \leq |v| \leq v_1 < \infty \right\}\) is the set of variation of velocities \(v\) of emitted particles.

Let us make a natural assumption that the process under consideration occurs in the absence of an external neutron source, i.e. the reactor walls to not transmit external radiation. Mathematically this is expressed by the following boundary condition:

\[
u(x,v,t) = 0, \quad (x,v,t) \in \partial \gamma \times [0,T],\]

(3)

where \(\gamma = \left\{ (x,v) \in \partial G \times V : (v,n_x) < 0 \right\}\), and \(n_x\) is the outer normal to the boundary \(\partial G\) of region \(G\) at the point \(x\).

In addition, let the initial conditions be specified:

\[
u(x,v,0) = \varphi(x,v), \quad (x,v) \in G \times V.
\]

(4)

\[
R_k(x,v,0) = R_{id}(x,v) \quad \forall \ k=1,N, \quad (x,v) \in G \times V.
\]

(5)

Let us now turn to the statement of linear the inverse problem stating the over-specifying condition in the form:

\[
\int_{G} \omega_{\tilde{x}'}(x)u(x,v,t)dx = \chi(v,t), \quad (v,t) \in V \times [0,T].
\]

(6)

From the physical viewpoint condition (6) means the internal measurement of the neutron population in a certain vicinity of the point \(x^\theta \in G\). The finite function \(\omega_{\tilde{x}'} \in L_{2}(G)\), called the «instrument function» and specifying parameters of the instrument used to make the measurements at the point \(x^\theta\) has a pronounced maximum. The support of this function, at this, \(\text{supp } \omega_{\tilde{x}'} \subset G\).

Assume that it is necessary to refine the function of internal sources \(F(x,v,t)\) in equation (1). On the basis of condition (6) look for this function in the following form:

\[
F(x,v,t) = f(v,t)g_j(x,v,t) + h_j(x,v,t),
\]

(7)

where \(f(v,t)\) is the sought for part of the source function, and \(g_j(x,v,t), h_j(x,v,t)\) are the a priori prescribed functions (correction function).
Inverse problem 1. To find the triple of functions \( \{ u(x, v, t), R_h(x, v, t), f(v, t) \} \), satisfying conditions (1)-(6), where \( F(x, v, t) \) has form (7).

3. Solution of inverse problem 1

From the viewpoint of inversion theory inverse problem 1 is linear.

For further studies consider the following classes of functions examined in [6]:
1) \( H_\infty(D) \) is the space of function \( u \), lying in the class \( L_\infty(D) \) together with its generalized derivatives \( u \), and \( (v, \nabla_x)u \) on \( D \) and possessing traces \( u|_{\Gamma_-} \) on \( \Gamma_- = \gamma_- \times (0, T) \) from \( L_\infty(\Gamma_-) \), i.e.

\[
H_\infty(D) = \left\{ u \in L_\infty(D) : \frac{\partial u}{\partial t}(v, \nabla_x)u \in L_\infty(D), u|_{\Gamma_-} \in L_\infty(\Gamma_-) \right\},
\]

which is Banach relative to the norm
\[
\| u \|_{H_\infty} = \left[ \| u \|_{L_\infty,D} + \| \frac{\partial u}{\partial t} \|_{L_\infty,D} + \| (v, \nabla_x)u \|_{L_\infty,D} + \| h|_{\Gamma_-} \|_{L_\infty,\Gamma_-} \right] < \infty.
\]

2) \( L'_\infty(D) \) is the space of functions \( R_h \), belonging to the class \( L_\infty(D) \) together with its generalized derivatives \( \frac{\partial R_h}{\partial t} \) on \( D \), i.e.

\[
L'_\infty(D) = \left\{ R_h \in L_\infty(D) : \frac{\partial R_h}{\partial t} \in L_\infty(D) \right\},
\]

which is Banach relative to the norm
\[
\| R_h \|_{L'_\infty} = \left[ \| R_h \|_{L_\infty,D} + \| \frac{\partial R_h}{\partial t} \|_{L_\infty,D} \right] < \infty.
\]

Here \( \| \cdot \|_{L_\infty,D} \) is the norm in the space \( L_\infty(D) \).

Definition 1. By generalized solution of inverse problem 1 we mean the triple of functions \( \{ u(x, v, t), R_h(x, v, t), f(v, t) \} \), almost everywhere satisfying equations (1), (2) and conditions (3)-(6) and such, that \( u \in H_\infty(D), R_h \in L'_\infty(D) \) and \( f \in L_\infty(V \times [0, T]) \).

Sufficient conditions of generalized solvability of the stated inverse problem 1 were obtained in the following theorem.

Theorem 1. Let \( \Sigma \in L'_\infty(D) ; J \in L'_\infty(D \times V) ; \varphi, (v, \nabla_x)\varphi \in L_\infty(G \times V) ; \varphi|_{\Gamma_-} \in L_\infty(\gamma_-) ;
\)

\[
g_{1}, (v, \nabla_x)g_{1} \in L_\infty(D), \quad \int_{G} \omega_{g}(x)g_{1}(x, v, t) \, dx \geq g_{1} > 0 ; \quad h_{1} \in L'_\infty(D) ; \quad J_{k} \in L'_\infty(D \times V) ;
\]

\[
R_{k} \in L_\infty(G \times V) ; \quad \forall \ k = 1, \overline{N} ; \quad \chi \in L'_\infty(V \times [0, T]) ; \quad \text{and the matching conditions are fulfilled:}
\]

\[
\varphi(x, v) = 0 \quad \text{at} \quad (x, v) \in \gamma_- , \quad \text{and} \quad \int_{G} \omega_{g}(x)\varphi(x, v) \, dx = \chi(v, 0) , \quad \text{at almost all} \quad v \in V . \quad (8)
\]

Then inverse problem 1 has a unique generalized solution in the sense of definition 1.

Proof. To prove the theorem we used the method proposed in [8-10] for a kinetic transfer equation. After integration by characteristics the system of equations (1)-(7) is reduced to the system of two integral equations:

\[
u(y + v\xi, \xi, t) = \Phi(y, v, \xi, t) + \int_{\eta} \left[ P_{u}(y + v\xi, v, \theta - \xi + i) + f(v, t) \xi_{y_{1}}(y + v\xi, v, \theta - \xi + i) + h_{1}(y + v\xi, v, \theta - \xi + i)d\theta \right] . \quad (9)
\]
\[ f(v,t) = \left[ \int_0^\infty \omega_x e^{\frac{1}{2} (y + v_\xi)) g_1(y + v_\xi, v, \tau) d\tau \right]^{-1} \left\{ \frac{\partial}{\partial t} \varphi (v, t) + \int_0^\infty \omega_x e^{\frac{1}{2} (y + v_\xi)) \left( v, \nabla \right) \varphi (y + v(x - t), v) \right\} - \left( P \varphi + h \right)(y + v(x - t), v, \theta) - \\
- \left[ \frac{\partial}{\partial t} \left( P \varphi + h \right)(y + v(x - t), v, \theta - \xi + 1) + \int_0^\infty f(v, \tau)(v, \nabla) g_1(v + v(\tau - \xi - t), v, \tau) d\tau \right] - \\
- \sum_{k=1}^N z_k \int R_k(y + v \xi, v) e^{-z_k \theta} d\theta 
\]

where \( y + v_\xi = x \) is a characteristic straight line with a directing vector \( v \), \( \xi \) is the parameter from segment \( \xi - , \xi + \), at this \( y + v_\xi \in \partial G \), and \( y + v_\xi \in \gamma - ; \)

\[ \Phi(y, v, \xi, t) = \varphi (y + v(x - t), v) + \sum_{k=1}^N z_k \int R_k(y + v \theta, v) e^{-z_k \theta} d\theta \] \( \eta = \xi - t \)
at \( 0 \leq t \leq \xi - \xi - , \xi \in [\xi - , \xi +] \)
and \( \Phi(y, v, \xi, t) = \sum_{k=1}^N z_k \int R_k(y + v \theta, v) e^{-z_k \theta} d\theta \) \( \eta = \xi - \) at \( \xi - \xi - < t \leq T, \xi \in [\xi - , \xi +] \)

\[ (P \varphi)(y + v_\xi, v, t) \equiv \sum_{k=1}^N z_k \int_0^\infty e^{-z_k (t - \tau)} J_k(y + v_\xi, v, \nu, \tau) u(y + v_\xi, \nu, \tau) d\nu d\tau + \int_0^\infty J(y + v_\xi, v, \nu, \tau) u(y + v_\xi, \nu, \tau) d\nu - \Sigma(y + v_\xi, v, \nu) u(y + v_\xi, \nu, \tau). \]

It is easy to prove that the solution of inverse problem 1 will be the solution of the system of integral equations (9)-(10) and, conversely, the solution of the system of integral equations (9)-(10) is the solution of inverse problem 1, i.e. fulfills conditions (1)-(6) in the above said classes. Hence, inverse problem 1 has a unique generalized solution \( u \in H_\infty(D) \), \( R_k \in L_\infty(D) \) \( f \in L_\infty(V \times[0,T]) \) when, and only when, the system of integral equations (9), (10) is unambiguously solvable in the specified class of functions. And this is possible if condition \[ \int_0^\infty \omega_x e^{\frac{1}{2} (y + v_\xi)) g_1(x, v, \tau) d\tau \geq g_{\theta 1} > 0 \), where \( g_{\theta 1} \) as a certain constant is satisfied.

Further on, go over to investigating the solvability of the system of integral equations (9), (10). For this write it down in the operator form:

\[ \{ u, f \} = \mathcal{A}_1 \{ u, f \} = \mathcal{A}_2 \{ u, f \} = \mathcal{A}_3 \{ u, f \} \]

where \( \mathcal{A}_1, \mathcal{A}_2 \) are the operators within the range of definition \( H_\infty(D) \times L_\infty(V \times[0,T]) \), acting on the pair of functions \( \{ u, f \} \) according to the rule defined by the right-hand sides of equations (9), (10) respectively. At this, the operator \( \mathcal{A} \) continuously reflects the Banach space \( H_\infty(D) \times L_\infty(V \times[0,T]) \)
into itself. As image \( \mathcal{A}_f \{ u, f \} \) has generalized derivatives belonging to the space \( L_\infty(D) \), and its trace is an element of space \( L_\infty(\Gamma_c) \). The image \( \mathcal{A}_f \{ u, f \} \) belongs to the space \( L_\infty(V \times [0, T]) \).

Investigating contractiveness of operator \( \mathcal{A} \) we have that its second degree will be the contraction operator, providing \( \left| \int_G \omega(x) g_j(x, v, t) \, dx \right| \geq g_{\theta_1} > 0 \).

Therefore, by virtue of the Banach principle about the fixed point there exists a unique solution \( \{ u, f \} \) of the system of integral equations (9)-(10) from the space \( H_\infty(D) \times L_\infty(V \times [0, T]) \); which means that the inverse problem 1 also has a unique generalized solution \( \{ u, f \} \) at the found restrictions on the initial data of the problem, i.e. sufficient conditions of the unique generalized solution of inverse problem 1 have been met.

So, theorem 1 is completely proved.

Hence, on the basis of theorem 1 we can refine the density of internal sources, and, consequently, the entire mathematical model of reactor dynamics represented by equations (1), (2).

**Remark 1.** In theorem 1 conditions \( \Sigma \in L'_\infty(D) \); \( J \in L'_\infty(D \times V) \), \( h_1 \in L'_\infty(D) \) and \( J_k \in L'_\infty(D \times V) \), \( \forall \ k = 1, N \) can be replaced with conditions \( \Sigma, (v, \nabla)v \Sigma \in L_\infty(D) \), \( J, (v, \nabla)J \in L_\infty(D \times V) \), \( h_1, (v, \nabla)h_1 \in L_\infty(D) \), \( J_k, (v, \nabla)J_k \in L_\infty(D \times V) \) \( \forall \ k = 1, N \), respectively. The results of theorem 1 will still stand, at this.

4. Nonlinear inverse problem

Now assume that we need to refine the absorption coefficient (the function of a complete macroscopic neutron cross-section) \( \Sigma(x, v, t) \) in equation (1). Using conditions (6), look for this function in the following form:

\[
\Sigma(x, v, t) = \sigma(v, t) g_2(x, v, t) + h_2(x, v, t), \tag{11}
\]

where \( \sigma(v, t) \) is the sought for part of function \( \Sigma(x, v, t) \), and \( g_2(x, v, t), h_2(x, v, t) \) are the a priori prescribed functions (correction functions).

**Inverse problem 2.** To find the triple of functions \( \{ u(x, v, t), R_k(x, v, t), \sigma(v, t) \} \), satisfying conditions (1)-(6), where \( \Sigma(x, v, t) \) has form (11).

From the view point of inverse theory inverse problem 2 is nonlinear.

Sufficient conditions of unambiguous solvability of inverse problem 2 have been obtained in the following theorem.

**Theorem 2.** Let \( h_2 \in L'_\infty(D) \); \( J \in L'_\infty(D \times V) \); \( F \in L'_\infty(D) \); \( \varphi, (v, \nabla)\varphi \in L_\infty(G \times V) \), \( \varphi_{|_{\gamma_+}} \in L_\infty(\gamma_+) \); \( g_2 \in L'_\infty(D) \), \( \left| \int_G \omega(x) g_2(x, v, t) \, dx \right| \geq g_{\theta_2} > 0 \); \( J_k \in L'_\infty(D \times V) \);

\( R_k \in L_\infty(G \times V) \); \( \forall \ k = 1, N \); \( \chi \in L'_\infty(V \times [0, T]) \), \( \left| \chi(v, t) \right| \geq \chi_0 > 0 \); and the matching conditions are fulfilled:

\( \varphi(x, v) = 0 \) at \( (x, v) \in \gamma_+ \), and \( \left| \int_G \omega(x) \varphi(x, v) \, dx \right| = \chi(v, 0) \), at almost all \( v \in V \).

Then inverse problem 2 has a unique generalized solution in the sense of definition 1 on a certain closed restricted set at a certain restriction on \( v_0 \) and on \( \text{diam}G \).
Sketch of the proof. Similar to the proof of theorem 1, reduce inverse problem 2, set by equations (1)-(6) under condition (11), to a system of integral equations. Providing $r_0^{-1}\text{diam}\, G \leq b$, where $b$ is a certain constant, the inverse problem 2 has a unique generalized equation when, and only when, the obtained system of integral equations of the second kind has a unique solution on the following set $U \times \Omega = \left\{ (u, \sigma) : u \in H_\infty(D), \sigma \in L_\infty(V \times [0, T]), \| u \|_{H_\infty} \leq k_1, \| \sigma \|_{L_\infty, D} \leq k_2 \right\}$, where the constants $k_1, k_2$ depend on the norms of initial data of the problem. Examining the obtained system of integral equation further in the operator form proves that the second degree of the right hand sides of these systems will be a contraction operator on the set $U \times \Omega$.

This means that the system of integral equation has a unique solution belonging to the set $U \times \Omega$, and inverse problem 2 has a unique generalized solution. So, the inverse problem makes it possible to refine the absorption coefficient $\mathcal{Z}(x, v, t)$ in equation (1).

Remark 2. The results of theorem 2 will remain the same, if the conditions $J \in L_\infty(D \times V), F \in L_\infty(D), h_2 \in L_\infty(D), J_k \in L_\infty(D \times V), \forall k = 1, N$ are replaced by $J, (v, \nabla)J \in L_\infty(D \times V), F, (v, \nabla)F \in L_\infty(D), h_2, (v, \nabla)h_2 \in L_\infty(D), J_k, (v, \nabla)J_k \in L_\infty(D \times V) \forall k = 1, N$ respectively.

So, the work considers linear and nonlinear inverse problems for the system of reactor dynamics described by equations (1), (2). The proposed methods allow, at this, not only to synthesize the sought for parts of investigated parameters, but also to define phase conditions of the object under the study corresponding to them.

It should also be noted that as the proofs of the above given theorems inherently comprise the method of successive approximations this makes it possible to constructively create solutions of the studied inverse problem 1. The proposed methods make it possible to construct numerical algorithms of recovery of the said functions.

The work has been done within the framework of the Competitiveness Enhancement Program of the National Research Nuclear University MEPhI (Moscow Engineering and Physics Institute); № 02.a03.21.0005, 27.08.2013.

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