REGULARITY OF SOLUTIONS TO THE DIRICHLET PROBLEM FOR MONGE-AMPÈRE EQUATIONS

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ABSTRACT. We study Hölder continuity of solutions to the Dirichlet problem for measures having density in $L^p$, $p > 1$, with respect to Hausdorff-Riesz measures of order $2n - 2 + \epsilon$ for $0 < \epsilon \leq 2$, in a bounded strongly hyperconvex Lipschitz domain and the boundary data belongs to $C^{0,\alpha}(\partial\Omega)$, $0 < \alpha \leq 1$.

INTRODUCTION

Let $\mu$ be a finite Borel measure on a bounded domain $\Omega \subset \mathbb{C}^n$. Given $f \in L^p(\Omega, \mu)$ for $p > 1$, and $\varphi \in C(\partial\Omega)$, the Dirichlet problem for the complex Monge-Ampère equation asks for a function $u$ such that

$$
\text{Dir}(\Omega, \varphi, f d\mu) : \begin{cases}
    u \in PSH(\Omega) \cap C(\bar{\Omega}), & \\
    (dd^c u)^n = f d\mu & \text{in } \Omega, \\
    u = \varphi & \text{on } \partial\Omega,
\end{cases}
$$

where $PSH(\Omega)$ denotes the set of plurisubharmonic (psh) functions in $\Omega$ and $(dd^c)^n$ is the complex Monge-Ampère operator.

This problem has attracted attention for many years, we refer the reader to [BT76, Ce84, CP92, Bł96, Ko98], and references therein, for more details.

In the case when $\Omega$ is a strongly pseudoconvex domain and $\mu$ is the Lebesgue measure, Bedford and Taylor [BT76] proved the existence and uniqueness of the solution to $\text{Dir}(\Omega, \varphi, f dV_{2n})$ with $0 \leq f \in C(\Omega)$. Moreover, the solution is $\alpha$-Hölder continuous when $\varphi \in Lip_{2a}(\partial\Omega)$ and $f^{1/n} \in Lip_{\alpha}(\Omega)$ for $0 < \alpha \leq 1$. For more general domains, the existence and regularity of the solution had been studied in [Bł96, Ch15].

Kołodziej [Ko96, Ko98] demonstrated that this problem still admit a unique solution when $f \in L^p(\Omega)$, $p > 1$, and more generally when the right hand side of the complex Monge-Ampère equation is a measure satisfying some sufficient condition which is close to be best possible.

Hölder continuity of solutions to this problem for densities in $L^p$ with respect to the Lebesgue measure was studied in [GKZ08], [N14], [Ch15] and [BKPZ15]. On a compact Kähler manifold, the existence of solutions is due to [Ko98] and Hölder regularity of solutions to complex Monge-Ampère equations has been investigated by many authors, we refer to [Ko08, DDGP14] for more details.

Date: November 6, 2015.

2010 Mathematics Subject Classification. 32W20, 32U15.

Key words and phrases. complex Monge-Ampère equation, Hausdorff-Riesz measure, strongly hyperconvex Lipschitz domain, plurisubharmonic function.
In the case of singular measures with respect to the Lebesgue measure, H. H. Pham [Ph10] proved the Hölder continuity of the solution to the complex Monge-Ampère equation on a compact Kähler manifold. There is no study about the regularity in the local case in \( \mathbb{C}^n \).

Our purpose in this paper is to explore the Hölder continuity of the solution to the Dirichlet problem \( \text{Dir}(\Omega, \varphi, f d\mu) \) when \( \mu \) is a Hausdorff-Riesz measure of order \( 2n-2+\epsilon \), for \( 0 < \epsilon \leq 2 \) (see Definition 1.6). Precisely, we prove the following.

**Theorem 0.1.** Let \( \Omega \) be a bounded strongly hyperconvex Lipschitz domain in \( \mathbb{C}^n \) and \( \mu \) be a Hausdorff-Riesz measure of order \( 2n-2+\epsilon \) for \( 0 < \epsilon \leq 2 \). Suppose that \( \varphi \in C^{1,1}(\partial \Omega) \) and \( 0 \leq f \in L^p(\Omega, \mu) \) for some \( p > 1 \), then the unique solution to \( \text{Dir}(\Omega, \varphi, f d\mu) \) is Hölder continuous on \( \Omega \) of exponent \( \frac{\epsilon \gamma}{2} \) for any \( 0 < \gamma < 1/(nq+1) \) and \( 1/p + 1/q = 1 \).

This result generalizes the one proved in [GKZ08, Ch15] from which the main idea of our proof originates.

When the boundary data is merely Hölder continuous we state the regularity of the solution using the previous theorem.

**Theorem 0.2.** Let \( \Omega \) be a bounded strongly hyperconvex Lipschitz domain in \( \mathbb{C}^n \) and \( \mu \) be a Hausdorff-Riesz measure of order \( 2n-2+\epsilon \) for \( 0 < \epsilon \leq 2 \). Suppose that \( \varphi \in C^{0,\alpha}(\partial \Omega) \), \( 0 < \alpha \leq 1 \) and \( 0 \leq f \in L^p(\Omega, \mu) \) for some \( p > 1 \), then the unique solution to \( \text{Dir}(\Omega, \varphi, f d\mu) \) is Hölder continuous on \( \Omega \) of exponent \( \frac{\epsilon \gamma}{2} \min\{\alpha, \epsilon \gamma\} \) for any \( 0 < \gamma < 1/(nq+1) \) and \( 1/p + 1/q = 1 \).

Moreover, when \( \Omega \) is a smooth strongly pseudoconvex domain the Hölder exponent of the solution will be \( \frac{\epsilon \gamma}{2} \min\{\alpha, \epsilon \gamma\} \), for any \( 0 < \gamma < 1/(nq+1) \).

In the case of the Lebesgue measure, i.e. \( \epsilon = 2 \), in a smooth bounded strongly pseudoconvex domain we get the Hölder exponent \( \min\{\alpha/2, \gamma\} \) which is better than the one obtained in [BKPZ15].

Finally, a natural question is that if we have a Hölder continuous subsolution to the Dirichlet problem, can we get a Hölder continuous solution? This question is still open in the local case (see [DDGPKZ14] for a positive answer in the compact setting). However, we deal in Theorem 3.6 some particular case.

**Acknowledgements.** I am very grateful to my advisor, Professor Ahmed Zeriahi, for useful discussions and suggestions.

1. Existence of solutions to the Dirichlet problem

This section is devoted to explain briefly the existence of continuous solutions to the Dirichlet problem \( \text{Dir}(\Omega, \varphi, \mu) \) for measures \( \mu \) dominated by Bedford-Taylor’s capacity, as in (1.2) below, in a bounded strongly hyperconvex Lipschitz domain.

We begin by recalling (see [Ch15]) that a bounded domain \( \Omega \subset \mathbb{C}^n \) is called strongly hyperconvex Lipschitz domain if there exist an open neighborhood \( \Omega' \) of \( \Omega \) and a Lipschitz plurisubharmonic defining function \( \rho : \Omega' \to \mathbb{R} \) such that

1. \( \Omega = \{\rho < 0\} \) and \( \partial \Omega = \{\rho = 0\} \),
2. \( dd^c \rho \geq 2\beta \) in \( \Omega \) in the weak sense of currents, where \( \beta \) is the standard Kähler form on \( \mathbb{C}^n \).

**Example 1.1.**
(1) Any bounded strictly convex domain is a bounded strongly hyperconvex Lipschitz domain.

(2) The nonempty finite intersection of smooth strongly pseudoconvex bounded domains in $\mathbb{C}^n$ is a bounded strongly hyperconvex Lipschitz domain.

(3) The domain $\Omega = \{ z = (z_1, \cdots, z_n) \in \mathbb{C}^n; |z_1| + \cdots + |z_n| < 1 \} \ (n \geq 2)$ is a bounded strongly hyperconvex Lipschitz domain in $\mathbb{C}^n$ with non-smooth boundary.

We need in the sequel the following property of a bounded strongly hyperconvex Lipschitz domain.

**Lemma 1.2.** Let $\Omega$ be a bounded strongly hyperconvex Lipschitz domain. Then there exist a defining function $\tilde{\rho} \in \text{PSH}(\Omega) \cap C^0(\overline{\Omega})$ such that near $\partial \Omega$ we have

$$c \text{dist}(z, \partial \Omega) \geq -\tilde{\rho}(z) \geq \text{dist}(z, \partial \Omega)^2,$$

for some $c > 0$ depending on $\Omega$.

Moreover $dd^c \tilde{\rho} \geq \beta$ in the weak sense of currents on $\Omega$.

**Proof.** Since $\Omega$ is a strongly hyperconvex Lipschitz domain, there exists a defining function $\rho$ such that $dd^c \rho \geq 2\beta$ in the weak sense of currents on $\Omega$. Let us fix $\xi \in \partial \Omega$, then the function defined by $\tilde{\rho}_\xi(z) := \rho(z) - |z - \xi|^2$ is Lipschitz continuous in $\overline{\Omega}$ and satisfies $dd^c \tilde{\rho}_\xi \geq \beta$ in the weak sense of currents on $\Omega$. Hence $\tilde{\rho}_\xi \in \text{PSH}(\Omega) \cap C^0(\partial \Omega)$. Set

$$\tilde{\rho} := \sup \{ \tilde{\rho}_\xi; \xi \in \partial \Omega \}.$$

It is clear that $\tilde{\rho} \in C^0(\overline{\Omega}) \cap \text{PSH}(\Omega)$ and thus the first inequality in (1.1) holds. For any $\xi \in \partial \Omega$ we have $-\tilde{\rho}_\xi(z) \geq C|z - \xi|^2$, so we infer that

$$-\tilde{\rho}(z) \geq \text{dist}(z, \partial \Omega)^2,$$

for any $z$ near $\partial \Omega$.

The last statement follows from the fact that for any $\xi \in \partial \Omega$, $dd^c \tilde{\rho}_\xi \geq \beta$ in the weak sense of currents on $\Omega$. \qed

**Remark 1.3.** When $\Omega$ is a smooth strongly pseudoconvex domain, we know that the defining function $\rho$ satisfies near the boundary,

$$-\rho \approx \text{dist}(., \partial \Omega).$$

**Definition 1.4.** A finite Borel measure $\mu$ on $\Omega$ is said to satisfy Condition $\mathcal{H}(\tau)$ for some fixed $\tau > 0$ if there exists a positive constant $A$ such that

$$\mu(K) \leq A \text{Cap}(K, \Omega)^{1+\tau},$$

for any Borel subset $K$ of $\Omega$.

Kołodziej [Ko98] demonstrated the existence of a continuous solution to $\text{Dir}(\Omega, \varphi, \mu)$ when $\mu$ verifies (1.2) and some local extra condition in a bounded strongly pseudoconvex domain with smooth boundary. Furthermore, he disposed of the extra condition in [Ko99] using Cegrell’s result [Ce98] about the existence of a solution in the energy class $\mathcal{F}_1$.

Here, we only summarize the steps of the proof of the existence of continuous solutions to $\text{Dir}(\Omega, \varphi, \mu)$ in a bounded strongly hyperconvex Lipschitz domain following the lines of Kołodziej and Cegrell’s arguments in [Ko98, Ce98].
**Theorem 1.5.** Let $\mu$ be a measure satisfying Condition $\mathcal{H}(\tau)$, for some $\tau > 0$, on a bounded strongly hyperconvex Lipschitz domain $\Omega \subset \mathbb{C}^n$ and $\varphi \in C(\partial \Omega)$. Then there exists a unique continuous solution to $\text{Dir}(\Omega, \varphi, \mu)$.

**Proof.** Suppose first that $\mu$ has compact support in $\Omega$. Let us consider a subdivision $I^s_\chi$ of $\text{supp} \mu$ consisting of $3^{2ns}$ congruent semi-open cubes $I^s_j$ with side $d_s = d/3^s$, where $d := \text{diam}(\Omega)$ and $1 \leq j \leq 3^{2ns}$. Thanks to Proposition 5.3 in [Ch15], one can find $u_s \in \text{PSH}(\Omega) \cap C(\bar{\Omega})$ such that

$$u_s = \varphi \text{ on } \partial \Omega,$$

and

$$(dd^c u_s)^n = \mu_s := \sum_j \frac{\mu(I^s_j)}{d_s^{2n}} \chi_{I^s_j} dV_{2n} \text{ in } \Omega,$$

where $\chi_{I^s_j}$ is the characteristic function of $I^s_j$. We can ensure following [Ko98] that there exist $s_0 > 0$ and $B = B(n, \tau) > 0$ such that for all $s > s_0$ the measure $\mu_s$ satisfies

$$\mu_s(K) \leq B \text{Cap}(K, \Omega)^{1+\tau},$$

for all Borel subsets $K$ of $\Omega$. Then, we prove that the $L^\infty$-norm of $u_s$, for $s > s_0$, is bounded by an absolute constant depending only on $n$ and $\tau$.

We set $u := (\lim sup u_s)^*$ which is a candidate to be the solution to $\text{Dir}(\Omega, \varphi, \mu)$. The delicate point is then to show that $(dd^c u_s)^n$ converges to $(dd^c u)^n$ in the weak sense of measures as in [Ko98] (see also [GZ07]). For this purpose, we invoke Cegrell’s techniques [Ce98] to ensure that

$$\int_\Omega u_s \, d\mu \to \int_\Omega u \, d\mu \text{ and } \int_\Omega |u_s - u| \, d\mu_s \to 0, \text{ when } s \to +\infty.$$

For the general case, let $\chi_j$ is a nondecreasing sequence of smooth cut-off function, $\chi_j \nearrow 1$ in $\Omega$. We get solutions $u_j$ to the Dirichlet problem for the measures $\chi_j \mu$. Then, the solutions $u_j$ are uniformly bounded. We set $u := (\lim sup u_j)^* \in \text{PSH}(\Omega) \cap L^\infty(\bar{\Omega})$ and the last argument yields that $u$ is the required bounded solution to $\text{Dir}(\Omega, \varphi, \mu)$.

Finally, we assert the continuity of the solution in the spirit of [Ko98].

We introduce an important class of Borel measures on $\Omega$ containing Riesz measures and closely related to Hausdorff measures which play an important role in geometric measure theory [Ma95]. We call such measures Hausdorff-Riesz measures.

**Definition 1.6.** A finite Borel measure on $\Omega$ is called a Hausdorff-Riesz measure of order $2n - 2 + \epsilon$, for $0 < \epsilon \leq 2$ if it satisfies the following condition:

$$(1.3) \quad \mu(B(z, r) \cap \bar{\Omega}) \leq Cr^{2n-2+\epsilon}, \quad \forall z \in \bar{\Omega}, \ \forall 0 < r < 1,$$

for some positive constant $C$.

We give some interesting examples of Hausdorff-Riesz measures.

**Example 1.7.**

1. The Lebesgue measure $dV$ on $\Omega$, for $\epsilon = 2$.
2. The surface measure of a compact real hypersurface, for $\epsilon = 1$.
3. Measures of the type $dd^c v \wedge \beta^{n-1}$, where $v$ is a $\alpha$-Hölder continuous subharmonic function in a neighborhood of $\bar{\Omega}$, for $\epsilon = \alpha$. 
(4) The measure $1_E \mathcal{H}^{2n-2+\varepsilon}$, where $\mathcal{H}^{2n-2+\varepsilon}$ is the Hausdorff measure and $E$ is a Borel set such that $\mathcal{H}^{2n-2+\varepsilon}(E) < +\infty$.

(5) If $\mu$ is a Hausdorff-Riesz measure of order $2n-2+\varepsilon$, then $f d\mu$ is a Hausdorff-Riesz measure of order $(2n-2+\varepsilon)/q$, for any $f \in L^p(\Omega, \mu)$, $p > (2n-2+\varepsilon)/\varepsilon$.

The existence of continuous solutions to $\text{Dir}(\Omega, \varphi, f d\mu)$ for such measures follows immediately from Theorem 1.5 and the following lemma.

**Lemma 1.8.** Let $\Omega$ be a bounded strongly hyperconvex Lipschitz domain and $\mu$ be a Hausdorff-Riesz measure of order $2n-2+\varepsilon$, for $0 < \varepsilon \leq 2$. Assume that $0 \leq f \in L^p(\Omega, \mu)$ for $p > 1$, then for all $\tau > 1$ there exists $D > 0$ depending on $\tau, \varepsilon$ and $\text{diam}(\Omega)$ such that for any Borel set $K \subset \Omega$,

\begin{equation}
\int_K f d\mu \leq D \|f\|_{L^p(\Omega, \mu)} [\text{Cap}(K, \Omega)]^\tau. \tag{1.4}
\end{equation}

**Proof.** By the Hölder inequality we have

\[ \int_K f d\mu \leq \|f\|_{L^p(\Omega, \mu)} \mu(K)^{1/q}. \]

Let $z_0 \in \Omega$ be a fixed point and $R := 2 \text{diam}(\Omega)$. Hence, $\Omega \Subset B := B(z_0, R)$. For any Borel set $K \subset \Omega$ we get, by Corollary 5.2 in [Z04] and Alexander-Taylor’s inequality, that

\[ \mu(K) \leq C(T_R(K))^{r/2} \leq C \exp(-\varepsilon/2 \text{Cap}(K, B)^{1/n}) \leq C \exp(-\varepsilon/2\text{Cap}(K, \Omega)^{1/n}), \]

where $C > 0$ depends on $\varepsilon$ and $\text{diam}(\Omega)$.

Now, for any $\tau > 1$, we can find $D > 0$ depending on $\tau, \varepsilon$ and $\text{diam}(\Omega)$ such that

\[ \int_K f d\mu \leq D \|f\|_{L^p(\Omega, \mu)} [\text{Cap}(K, \Omega)]^\tau. \]

□

2. Preliminaries

We introduce in this section basic ingredients of proofs of our results.

We prove in the following proposition that the total mass of Laplacian of the solution is finite when the boundary data is $C^{1,1}$-smooth.

**Proposition 2.1.** Let $\mu$ be a finite Borel measure satisfying Condition $\mathcal{H}(\tau)$ on $\Omega$ and $\varphi \in C^{1,1}(\partial \Omega)$. Then the solution $u$ to $\text{Dir}(\Omega, \varphi, d\mu)$ has the property that

\[ \int_\Omega \Delta u \leq C, \]

where $C > 0$ depends on $n, \Omega$ and $\mu(\Omega)$.

**Proof.** Let $u_0$ be the solution to the Dirichlet problem $\text{Dir}(\Omega, 0, d\mu)$. We first claim that the total mass of $\Delta u_0$ is finite in $\Omega$. Indeed, let $\rho$ be the defining function of $\Omega$. Then by Corollary 5.6 in [Ce04] we get

\begin{equation}
\int_\Omega (dd^c \rho)^{n-1} \leq \left( \int_\Omega (dd^c u_0)^n \right)^{1/n} \left( \int_\Omega (dd^c \rho)^n \right)^{(n-1)/n}, \tag{2.1}
\end{equation}

\[ \leq \mu(\Omega)^{1/n} \left( \int_\Omega (dd^c \rho)^n \right)^{(n-1)/n}. \]
Since $\Omega$ is a bounded strongly hyperconvex Lipschitz domain, there exists a constant $c > 0$ such that $dd^c \rho \geq c \beta$ in $\Omega$. Hence, (2.1) yields
\[
\int_{\Omega} dd^c u_0 \wedge \beta^{n-1} \leq \frac{1}{c^{n-1}} \int_{\Omega} (dd^c \rho)^n \leq \frac{\mu(\Omega)}{c^{n-1}} \left( \frac{\int_{\Omega} (dd^c \rho)^n}{(n-1)/n} \right).
\]

Now we note that the total mass of complex Monge-Ampère measure of $\rho$ is finite in $\Omega$ by the Chern-Levine-Nirenberg inequality and since $\rho$ is psh and bounded in a neighborhood of $\Omega$. Therefore, the total mass of $\Delta u_0$ is finite in $\Omega$.

Let $\tilde{\varphi}$ be a $C^{1,1}$-extension of $\varphi$ to $\Omega$ such that $\|\tilde{\varphi}\|_{C^{1,1}(\Omega)} \leq C\|\varphi\|_{C^{1,1}(\partial\Omega)}$ for some $C > 0$. Now, let $v = A\rho + \tilde{\varphi} + u_0$ where $A \gg 1$ such that $A\rho + \tilde{\varphi} \in PSH(\Omega)$. By the comparison principle we see that $v \leq u$ in $\Omega$ and $v = u = \varphi$ on $\partial\Omega$. Since $\rho$ is psh in a neighborhood of $\Omega$ and $\|\Delta u_0\|_\Omega < +\infty$, we deduce that $\|\Delta v\|_\Omega < +\infty$. This completes the proof. \hfill $\square$

**Definition 2.2.** A finite Borel measure $\mu$ on $\Omega$ is said to satisfy Condition $\mathcal{H}(\infty)$ if for any $\tau > 0$ there exists a positive constant $A$ depending on $\tau$ such that
\[
\mu(K) \leq A\text{Cap}(K, \Omega)^{1+\tau},
\]
for any Borel subset $K$ of $\Omega$.

Let $\mu$ be a measure satisfying Condition $\mathcal{H}(\infty)$, $0 \leq f \in L^p(\Omega, \mu)$, $p > 1$ and $\varphi \in C(\partial\Omega)$. Let also $u$ be the continuous solution to $Div(\Omega, \varphi, fd\mu)$ and consider
\[
u_+ := \sup_{z \in \overline{\Omega}} u(z + \zeta), \quad z \in \Omega_\delta,
\]
where $\Omega_\delta := \{ z \in \Omega; \text{dist}(z, \partial\Omega) > \delta \}$.

To ensure the Hölder continuity of the solution in $\Omega$, we need to control the $L^\infty$-norm of $u_\delta - u$ in $\Omega_\delta$.

It is shown in [GKZ08] that the Hölder norm of the solution $u$ can be estimated by using either $\sup_{\Omega_\delta} (u_\delta - u)$ or $\sup_{\Omega_\delta} (\hat{u}_\delta - u)$, where
\[
\hat{u}_\delta(z) := \frac{1}{\tau_{2n}\delta^{2n}} \int_{|\zeta| \leq \delta} u(\zeta)dV_{2n}(\zeta), \quad z \in \Omega_\delta,
\]
and $\tau_{2n}$ is the volume of the unit ball in $\mathbb{C}^n$.

It is clear that $\hat{u}_\delta$ is defined in $\Omega_\delta$, so we extend it with a good control near the boundary $\partial\Omega$. To this end, we assume the existence of $\nu$-Hölder continuous function $v$ such that $v \leq u$ in $\Omega$ and $v = u$ on $\partial\Omega$. Then, we present later the construction of such a function.

**Lemma 2.3.** Let $\Omega$ be a bounded strongly hyperconvex Lipschitz domain and $\varphi \in C^{0,\alpha}(\partial\Omega)$, $0 < \alpha \leq 1$. Assume that there is a function $v \in C^{0,\nu}(\overline{\Omega})$ for $0 < \nu \leq 1$, such that $v \leq u$ in $\Omega$ and $v = \varphi$ on $\partial\Omega$. Then there exist $\delta_0 > 0$ small enough and $c_0 > 0$, depending on $\Omega$, $\|\varphi\|_{C^{0,\alpha}(\partial\Omega)}$ and $\|v\|_{C^{0,\nu}(\overline{\Omega})}$, such that for any $0 < \delta_1 \leq \delta < \delta_0$ the function
\[
\hat{u}_{\delta_1} = \begin{cases} 
\max\{\hat{u}_{\delta_1}, u + c_0\delta^{\nu_1}\} & \text{in } \Omega_\delta, \\
u_1 \in \Omega \setminus \Omega_\delta,
\end{cases}
\]
is plurisubharmonic in $\Omega$ and continuous on $\Omega$, where $\nu_1 = \min\{\nu, \alpha/2\}$. 

Proof. If we prove that \( \hat{u}_{\delta_1} \leq u + c_0\delta^n \) on \( \partial \Omega_\delta \), then the required result can be obtained by the standard gluing procedure. Thanks to Corollary 4.6 in [Ch15], we find a plurisuperharmonic function \( \tilde{v} \in C^{0,\alpha/2}(\bar{\Omega}) \) such that \( \tilde{v} = \varphi \) on \( \partial \Omega \) and

\[
\| \tilde{v} \|_{C^{0,\alpha/2}(\bar{\Omega})} \leq C\| \varphi \|_{C^{0,\alpha}(\partial \Omega)},
\]

where \( C \) depends on \( \Omega \). From the maximum principle we see that \( u \leq \tilde{v} \) in \( \Omega \) and \( \tilde{v} = u \) on \( \partial \Omega \).

Fix \( z \in \partial \Omega_\delta \), there exists \( \zeta \in \mathbb{C}^n \) with \( \| \zeta \| = \delta_1 \) such that \( \hat{u}_{\delta_1}(z) \leq u(z + \zeta) \). Hence we obtain

\[
\hat{u}_{\delta_1}(z) - u(z) \leq u(z + \zeta) - u(z) \leq \tilde{v}(z + \zeta) - v(z).
\]

We choose \( \zeta_0 \in \mathbb{C}^n \), with \( \| \zeta_0 \| = \delta \), such that \( z + \zeta_0 \in \partial \Omega \). Since \( \tilde{v}(z + \zeta_0) = v(z + \zeta_0) \), we get

\[
\tilde{v}(z + \zeta) - v(z) \leq [\tilde{v}(z + \zeta) - \tilde{v}(z + \zeta_0)] + [v(z + \zeta_0) - v(z)] \\
\leq 2\| \tilde{v} \|_{C^{0,\alpha/2}(\Omega)}\delta_1^{\alpha/2} + \| v \|_{C^{0,\alpha}(\Omega)}\delta^n \\
\leq c_0\delta^n,
\]

where \( c_0 := 2C\| \varphi \|_{C^{0,\alpha}(\partial \Omega)} + \| v \|_{C^{0,\alpha}(\Omega)} > 0 \).

Moreover, we can conclude from the last argument that

\[
|u(z_1) - u(z_2)| \leq 2c_0\delta^n,
\]

for all \( z_1, z_2 \in \Omega \setminus \Omega_\delta \) such that \( |z_1 - z_2| \leq \delta \).

Remark 2.4. When \( \varphi \in C^{1,1}(\partial \Omega) \), the last lemma holds for \( \nu_1 = \nu \). Indeed, let \( \hat{\varphi} \) be a \( C^{1,1} \)-extension of \( \varphi \) to \( \bar{\Omega} \). We define the plurisuperharmonic Lipschitz function \( \hat{v} := -A\rho + \varphi \), where \( A \gg 1 \) and \( \rho \) is the defining function of \( \Omega \). Hence, the constant \( c_0 \) in Lemma 2.3 will depend only on \( \partial \Omega \), \( \| \varphi \|_{C^{1,1}(\partial \Omega)} \), and \( \| v \|_{C^{0,\alpha}(\Omega)} \).

The following weak stability estimate, proved in [GKZ09] for the Lebesgue measure, plays an important role in our work. A similar, but weaker, estimate was established by Kołodziej [Ko02] and in the compact setting it was proved by Eyssidieux, Guedj and Zeriahi [EGZ09]. This estimate is still true for any finite Borel measure \( \mu \) satisfying Condition \( \mathcal{H}(\infty) \).

Theorem 2.5. Let \( \mu \) be a finite Borel measure on \( \Omega \) satisfying Condition \( \mathcal{H}(\infty) \) and \( 0 \leq f \in L^p(\Omega, \mu), \ p > 1 \). Suppose that \( v_1, v_2 \) are two bounded psh functions in \( \Omega \) such that \( \liminf_{z \to \partial \Omega}(v_1 - v_2)(z) \geq 0 \) and \( (dv^n_1)^n = fd\mu \). Fix \( r \geq 1 \) and \( 0 < \gamma < r/(nq + r), 1/p + 1/q = 1 \). Then there exists a constant \( c_1 = c_1(r, \gamma, n, q) > 0 \) such that

\[
\sup_{\Omega}(v_2 - v_1) \leq c_1(1 + \| f \|_{L^p(\Omega, \mu)}^\gamma)(v_2 - v_1) + \| f \|_{L^q(\Omega, \mu)}^\gamma,
\]

where \( (v_2 - v_1)_+ = \max(v_2 - v_1, 0) \) and \( \eta = \frac{1}{n} + \frac{\gamma q}{r - \gamma(r + nq)} \).
3. Proofs of main results

Theorem 3.1. Let $\Omega$ be a bounded strongly hyperconvex Lipschitz domain and let $\mu$ be a finite Borel measure on $\Omega$ satisfying Condition $\mathcal{H}(\infty)$. Suppose that $\varphi \in C^{0,\alpha}(\partial \Omega)$, $0 < \alpha \leq 1$, and $0 \leq f \in L^p(\Omega, \mu)$ for $p > 1$. Then the solution $u$ to $\text{Dir}(\Omega, \varphi, f \mu)$ is Hölder continuous on $\Omega$ of exponent $\frac{1}{\alpha} \min\{\nu, \alpha/2, \tau \gamma\}$, for any $\gamma < 1/(nq + 1)$ and $1/p + 1/q = 1$, if the two following conditions hold:

(i) there exists $c > 0$ such that $c_1 \leq c < \infty$,

(ii) and $\|\hat{u}_{\delta_i} - u\|_{L^1(\Omega_{\delta_i}, \mu)} \leq c \delta^\gamma$, where $c, \tau > 0$ and $0 < \delta_i = \delta^\gamma_\lambda$ for some $\lambda \geq 1$.

Moreover, if $\varphi \in C^{1,1}(\partial \Omega)$ then the Hölder exponent of $u$ will be $\frac{1}{\alpha} \min\{\nu, \tau \gamma\}$.

Proof. It follows from Lemma 2.3 that there exist $c_0 > 0$ and $\delta_0 > 0$ so that

$$\hat{u}_{\delta_i} = \begin{cases} \max\{\hat{u}_{\delta_i}, u + c_0 \delta^\gamma\} & \text{in } \Omega_{\delta_i}, \\ u + c_0 \delta^\gamma & \text{in } \Omega \setminus \Omega_{\delta_i}, \end{cases}$$

belongs to $P^SH(\Omega) \cap C(\bar{\Omega})$, for $0 < \delta_i \leq \delta < \delta_0$ and $\nu_1 = \min\{\nu, \alpha/2\}$.

By applying Theorem 2.5 with $v_1 := u + c_0 \delta^\gamma$ and $v_2 := \hat{u}_{\delta_i}$, we infer that

$$\sup_{\Omega_{\delta_i}} (\hat{u}_{\delta_i} - u - c_0 \delta^\gamma) \leq \sup_{\Omega} (\hat{u}_{\delta_i} - u - c_0 \delta^\gamma) \leq c_1 (1 + \|f\|_{L^p(\Omega, \mu)}^\eta) \|\hat{u}_{\delta_i} - u - c_0 \delta^\gamma\|_{L^1(\Omega, \mu)}^\eta,$$

where $\eta := 1/n + \gamma q/[1 - (\gamma(1 + nq)]$, $c_1 = c_1(n, q, \gamma)$ and $0 < \gamma < 1/(nq + 1)$ is fixed. Since $\hat{u}_{\delta_i} = u + c_0 \delta^\gamma$ in $\Omega \setminus \Omega_{\delta_i}$ and

$$\| (\hat{u}_{\delta_i} - u - c_0 \delta^\gamma)_{+} \|_{L^1(\Omega, \mu)} \leq \|\hat{u}_{\delta_i} - u\|_{L^1(\Omega_{\delta_i}, \mu)},$$

we conclude that

$$\sup_{\Omega_{\delta_i}} (\hat{u}_{\delta_i} - u) \leq c_0 \delta^\gamma + c_1 (1 + \|f\|_{L^p(\Omega, \mu)}^\eta) \|\hat{u}_{\delta_i} - u\|_{L^1(\Omega_{\delta_i}, \mu)}^\eta.$$

By hypotheses we have

$$\sup_{\Omega_{\delta_i}} (\hat{u}_{\delta_i} - u) \leq c_0 \delta^\gamma + c_1 \delta^\gamma (1 + \|f\|_{L^p(\Omega, \mu)}^\eta) \delta^\gamma \nu_1.$$

Let us set $c_2 := (c_0 + c_1)\delta^\gamma (1 + \|f\|_{L^p(\Omega, \mu)}^\eta)$. We derive from the last inequality that

$$\sup_{\Omega_{\delta_i}} (\hat{u}_{\delta_i} - u) \leq c_2 \delta^\gamma \min\{\nu_1, \tau \gamma\}.$$

This means that

$$\hat{u}_{\delta_i} - u \leq c_2 \delta^\gamma \min\{\nu_1, \tau \gamma\} \text{ in } \Omega_{\delta_i}.$$

Hence, by Lemma 4.2 in [GKZ08], there exists $c_3, \delta_0 > 0$ such that for all $0 < \delta < \delta_0$ we have

$$u_{\delta} - u \leq c_3 \delta^\gamma \min\{\nu_1, \tau \gamma\} \text{ in } \Omega_{\delta}.$$

Thus, (3.1) and (2.2) yield the Hölder continuity of $u$ on $\bar{\Omega}$ of exponent $\frac{1}{\alpha} \min\{\nu, \alpha/2, \tau \gamma\}$, for any $\gamma < 1/(nq + 1)$ and $1/p + 1/q = 1$.

Finally, if $\varphi \in C^{1,1}(\partial \Omega)$, we get that the Hölder exponent is $\frac{1}{\alpha} \min\{\nu, \tau \gamma\}$, since $\nu_1 = \nu$ (see Remark 2.4).

The first step in the proof of Theorem 0.1 is to estimate $\|\hat{u}_{\delta} - u\|_{L^1(\Omega, \mu)}$, so we present the following lemma.
Lemma 3.2. Let $\Omega \subset \mathbb{C}^n$ be a strongly hyperconvex Lipschitz domain and let $\mu$ be a Hausdorff-Riesz measure of order $2n - 2 + \epsilon$ on $\Omega$ for $0 < \epsilon \leq 2$. Suppose that $\varphi \in C^{1,1}(\partial \Omega)$ and $f \in L^p(\Omega, \mu)$, $p > 1$, then the solution $u$ to the Dirichlet problem $\text{Dir}(\Omega, \varphi; f \, d\mu)$ satisfies
\[
\int_{\Omega_\delta} [\hat{u}_\delta(z) - u(z)] d\mu(z) \leq C \delta^\epsilon,
\]
where $C$ is a positive constant depending on $n$, $\epsilon$, $\Omega$, $\|f\|_{L^p(\Omega, \mu)}$ and $\mu(\Omega)$.

Proof. Let us denote by $\sigma_{2n-1}$ the surface measure of the unit sphere. It follows from the Poisson-Jensen formula, for $z \in \Omega_\delta$ and $0 < r < \delta$, that
\[
\frac{1}{\sigma_{2n-1} r^{2n-1}} \int_{\partial B(z,r)} u(\xi) d\sigma(\xi) - u(z) = c_n \int_0^r t^{1-2n} \left( \int_{B(z,t)} \Delta u(\xi) \right) dt.
\]
Using polar coordinates we obtain for $z \in \Omega_\delta$,
\[
\hat{u}_\delta(z) - u(z) = c_n \frac{1}{\sigma_{2n-1} \delta^{2n}} \int_0^\delta r^{2n-1} dr \int_0^r t^{1-2n} dt \left( \int_{B(z,t)} \Delta u(\xi) \right).
\]
Now, we integrate on $\Omega_\delta$ with respect to $d\mu$ and use Fubini’s theorem
\[
\int_{\Omega_\delta} [\hat{u}_\delta(z) - u(z)] d\mu(z) = c_n \frac{1}{\sigma_{2n-1} \delta^{2n}} \int_0^\delta r^{2n-1} dr \int_0^r t^{1-2n} dt \left( \int_{\Omega} \Delta u(\xi) \right) d\mu(z)
\]
\[
= c_n \frac{1}{\sigma_{2n-1} \delta^{2n}} \int_0^\delta r^{2n-1} dr \int_0^r t^{1-2n} dt \int_{\Omega} \left( \int_{B(z,t)} \Delta u(\xi) \right) d\mu(z)
\]
\[
\leq c_n \frac{1}{\sigma_{2n-1} \delta^{2n}} \int_0^\delta r^{2n-1} dr \int_0^r t^{1-2n} dt \int_{\Omega} \Delta u(\xi) d\mu(z)
\]
\[
\leq \frac{c_n}{\epsilon (2n + \epsilon)} \left( \int_{\Omega} \Delta u \right) \delta^\epsilon.
\]
By Proposition 2.1, the total mass of $\Delta u$ is finite in $\Omega$ and this completes the proof. \hfill \Box

When $\varphi$ is not $C^{1,1}$-smooth, the measure $\Delta u$ may have infinite mass on $\Omega$. Fortunately, we can estimate $\|\hat{u}_\delta - u\|_{L^1(\Omega_\delta, \mu)}$ for some $\delta_1 < \delta \leq 1$.

Lemma 3.3. Let $\Omega \subset \mathbb{C}^n$ be a strongly hyperconvex Lipschitz domain and let $\mu$ be a Hausdorff-Riesz measure of order $2n - 2 + \epsilon$ on $\Omega$, for $0 < \epsilon \leq 2$. Suppose that $0 \leq f \in L^p(\Omega, \mu)$, $p > 1$ and $\varphi \in C^{0,\alpha}(\partial \Omega)$, $\alpha \leq 1$. Then for any small $\epsilon_1 > 0$, we have the following inequality
\[
\int_{\Omega_\delta} [\hat{u}_{\delta_1}(z) - u(z)] d\mu(z) \leq C \delta^{\epsilon_1/2 - \epsilon_1},
\]
where $\delta_1 = (1/2) \delta^{1/2 + 3/\epsilon}$ and $C$ is a positive constant depending on $n$, $\Omega$, $\epsilon$, $\epsilon_1$ and $\|u\|_{L^\infty(\bar{\Omega})}$.

Proof. One sees as in the proof of Lemma 3.2 that
\[
\hat{u}_{\delta_1}(z) - u(z) = c_n \frac{1}{\delta_1^{2n}} \int_0^{\delta_1} r^{2n-1} dr \int_0^r t^{1-2n} dt \left( \int_{B(z,t)} \Delta u(\xi) \right).
\]
Then, we integrate on $\Omega_\delta$ with respect to $\mu$ and use Fubini’s Theorem
\[
\int_{\Omega_3} [\hat{u}_{\delta_1}(z) - u(z)]d\mu(z) = \frac{c_n}{\delta_1^2} \int_0^{\delta_1} r^{2n-1}dr \int_0^r t^{1-2n}dt \int_{\Omega_3} \left( \int_{B(z,t)} \Delta u(\xi) \right) d\mu(z) \\
\leq \frac{c_n}{\delta_1^2} \int_0^{\delta_1} r^{2n-1}dr \int_0^r t^{1-2n}dt \int_{\Omega_3} d\mu(z) \Delta u(\xi) \\
\leq \frac{c_n}{\delta_1^2} \sup_{\Omega_3} \Delta u(\xi) \leq \frac{c_n}{\delta_1^2} \sup_{\Omega_3} \Delta u(\xi) \\
\leq \frac{c_n}{\delta_1^2} \sup_{\Omega_3} \Delta u(\xi) \leq \frac{c_n}{\delta_1^2} \sup_{\Omega_3} \Delta u(\xi) \\
\leq \frac{4c_n\delta^{-3-\epsilon}}{\epsilon(2n+\epsilon)} \Delta u(\xi) \\
\leq C_1 \delta^{3/2-\epsilon} \Delta u(\xi),
\]

where \( \hat{\rho} \in PSH(\Omega) \cap C^{0,1}(\overline{\Omega}) \) is as in Lemma 1.2 and \( C_1 \) is a positive constant depending on \( n \) and \( \epsilon \). To complete the proof we demonstrate that the mass \( \|(-\hat{\rho})^{(3+\epsilon)/2} \Delta u\|_{\Omega} \) is finite. The following idea is due to [BKPZ15] with some appropriate modifications. We set for simplification \( \theta := (3+\epsilon)/2 \). Let \( \rho_\eta \) be the standard regularizing kernels with \( \text{supp} \rho_\eta \subset B(0, \eta) \) and \( \int_{B(0, \eta)} \rho_\eta dV_{2n} = 1 \). Hence, \( u_\eta = u * \rho_\eta \in C^\infty \cap PSH(\Omega_\eta) \) decreases to \( u \) in \( \Omega \). It is clear that \( \|u_\eta\|_{L^\infty(\Omega_\eta)} \leq \|u\|_{L^\infty(\Omega)} \) and the first derivatives of \( u_\eta \) have \( L^\infty \)-norms less than \( \|u\|_{L^\infty(\Omega)}/\eta \). We denote by \( \chi_{\Omega_\eta} \) the characteristic function of \( \Omega_\eta \) and by \( \rho \) the defining function of \( \Omega \). Since \( u_\eta \searrow u \) in \( \Omega \), we have \( \chi_{\Omega_\eta}(-\hat{\rho}) \Delta u_\eta \) converges to \( (-\hat{\rho})^\theta \Delta u \) in the weak sense of measures.

It is sufficient to show that

\[
I := \int_{\Omega_\eta} (-\hat{\rho})^\theta \Delta u_\eta,
\]

is bounded by an absolute constant independent of \( \eta \). We have by Stokes' theorem

\[
I = \int_{\partial \Omega_\eta} (-\hat{\rho})^\theta d^c u_\eta \wedge \beta^{n-1} + \theta \int_{\Omega_\eta} (-\hat{\rho})^{\theta-1} d\hat{\rho} \wedge d^c u_\eta \wedge \beta^{n-1}.
\]

Note that

\[
\int_{\partial \Omega_\eta} (-\hat{\rho})^{\theta-1} u_\eta d^c \hat{\rho} \wedge \beta^{n-1} = \int_{\Omega_\eta} (-\hat{\rho})^{\theta-1} u_\eta d\hat{\rho} \wedge \beta^{n-1} + \\
+ \int_{\Omega_\eta} (-\hat{\rho})^{\theta-1} u_\eta d\hat{\rho} \wedge \beta^{n-1} \\
- (\theta - 1) \int_{\Omega_\eta} (-\hat{\rho})^{\theta-2} u_\eta d\hat{\rho} \wedge \beta^{n-1}.
\]
Hence, we get
\[
I = \int_{\partial \Omega} (\tilde{\rho})^\theta d^\nu u_\eta \wedge \beta^{n-1} + \theta \int_{\partial \Omega} (\tilde{\rho})^{\theta-1} u_\eta d^\nu \tilde{\rho} \wedge \beta^{n-1}
- \theta \int_{\Omega} (\tilde{\rho})^{\theta-1} u_\eta d^\nu \tilde{\rho} \wedge \beta^{n-1} + \theta(\theta - 1) \int_{\bar{\Omega}} (\tilde{\rho})^{\theta-2} u_\eta d\tilde{\rho} \wedge d^\nu \tilde{\rho} \wedge \beta^{n-1}
\leq C\|u\|_{L^\infty(\bar{\Omega})} \left(\int_{\partial \Omega} d\sigma + \int_{\Omega} d^\nu \tilde{\rho} \wedge \beta^{n-1} + \int_{\bar{\Omega}} (\tilde{\rho})^{\theta-2} \beta^n\right),
\leq C\|u\|_{L^\infty(\bar{\Omega})} \left(\int_{\partial \Omega} d\sigma + \int_{\Omega} d^\nu \tilde{\rho} \wedge \beta^{n-1} + \int_{\bar{\Omega}} (\tilde{\rho})^{\theta-2} \beta^n\right),
\]
where \(d\sigma = d^\nu \rho \wedge (d^\nu \rho)^{n-1}\) and \(\rho\) is the defining function of \(\Omega\). Since \(\rho\) is psh in a neighborhood of \(\bar{\Omega}\), the second integral in the last inequality is finite. Thanks to Lemma 1.2 we have \(-\tilde{\rho} \geq \text{dist}(., \partial \Omega)^2\) near \(\partial \Omega\) and so the third integral will be finite since \(\epsilon_1 > 0\) small enough. Consequently, we infer that \(I\) is bounded by a constant independent of \(\eta\) and then this proves our claim. \(\square\)

**Corollary 3.4.** When \(\Omega\) is a smooth strongly pseudoconvex domain, then Lemma 3.3 holds also for \(\delta_1 = (1/2)\delta^{1/2+1/\epsilon}\).

**Proof.** Let \(\rho\) be the defining function of \(\Omega\). In view of Remark 1.3 and the last argument, we can estimate \(\|(-\rho)^{1+\epsilon_1} \Delta u\|_{\Omega}\), for \(\epsilon_1 > 0\) small enough. So the proof of Lemma 3.3 is still true for \(\delta_1 := (1/2)\delta^{1/2+1/\epsilon}\). \(\square\)

We are now in a position to prove the main theorems. We begin to prove the Hölder continuity of the solution to \(\text{Dir}(\Omega, \varphi, f d\mu)\) where \(\mu\) is a Hausdorff-Riesz measure of order \(2n - 2 + \epsilon\) and \(\varphi \in C^{1,1}(\partial \Omega)\).

**Proof of Theorem 0.1.** We first assume that \(f\) equals to zero near the boundary \(\partial \Omega\), that is, there exists a compact \(K \subset \Omega\) such that \(f = 0\) on \(\Omega \setminus K\). Since \(\varphi \in C^{1,1}(\partial \Omega)\), we extend it to \(\tilde{\varphi} \in C^{1,1}(\bar{\Omega})\) such that \(\|\tilde{\varphi}\|_{C^{1,1}(\bar{\Omega})} \leq C\|\varphi\|_{C^{1,1}(\partial \Omega)}\) for some constant \(C\). Let \(\rho\) be the defining function of \(\Omega\) and let \(A \gg 1\) be so that \(v := A\rho + \tilde{\varphi} \in \text{PSH}(\Omega)\) and \(v \leq u\) in a neighborhood of \(K\). Moreover, by the comparison principle, we see that \(v \leq u\) in \(\Omega \setminus K\). Consequently, \(v \in \text{PSH}(\Omega) \cap C^{1,1}(\bar{\Omega})\) and satisfies \(v \leq u\) on \(\partial \Omega\) and \(v = u = \varphi\) on \(\partial \Omega\). It follows from Theorem 3.1 and Lemma 3.2 that \(u \in C^{0,\epsilon_1}(\bar{\Omega})\), for any \(0 < \gamma < 1/(nq + 1)\).

In the general case, fix a large ball \(B \subset \mathbb{C}^n\) containing \(\Omega\) and define a function \(\tilde{f} \in L^p(B, \mu)\) so that \(\tilde{f} := f\) in \(\Omega\) and \(\tilde{f} := 0\) in \(B \setminus \Omega\). Hence, the solution to the following Dirichlet problem

\[
\begin{cases}
  v_1 \in \text{PSH}(B) \cap C(B), \\
  (d^\nu v_1)^n = \tilde{f} d\mu & \text{in } B, \\
  v_1 = 0 & \text{on } \partial B,
\end{cases}
\]

belongs to \(C^{0,\gamma'}(B)\), with \(\gamma' = \epsilon \gamma\) for any \(\gamma < 1/(nq + 1)\).

Let \(h_{\varphi - v_1}\) be the continuous solution to \(\text{Dir}(\Omega, \varphi - v_1, 0)\). Then, Theorem A in [Ch15] implies that \(h_{\varphi - v_1}\) belongs to \(C^{0,\gamma'/2}(\bar{\Omega})\).

This enables us to construct a Hölder barrier for our problem. We take \(v_2 = v_1 + h_{\varphi - v_1}\). It is clear that \(v_2 \in \text{PSH}(\Omega) \cap C^{0,\gamma'/2}(\bar{\Omega})\) and \(v_2 \leq u\) on \(\bar{\Omega}\) by the comparison principle.
Hence, Theorem 3.1 and Lemma 3.2 imply that the solution $u$ to $Dir(\Omega, \varphi, fd\mu)$ is Hölder continuous on $\Omega$ of exponent $\epsilon \gamma /2$ for any $0 < \gamma < 1/(nq + 1)$. \hfill \Box

Proof of Theorem 0.2. Let $v_1$ be as in the proof of Theorem 0.1 and $h_{\varphi-v_1}$ be the solution to $Dir(\Omega, \varphi - v_1, 0)$. In order to apply Theorem 3.1 we set $v = v_1 + h_{\varphi-v_1}$. Hence, $v \in PSH(\Omega) \cap C(\Omega)$, $v = \varphi$ on $\partial \Omega$ and $(dd^c v)^n \geq fd\mu$ in $\Omega$. The comparison principle yields $v \leq u$ in $\Omega$. Moreover, By Theorem A in [Ch15], we have $h_{\varphi-v_1} \in C^{0, \gamma''}(\Omega)$ with $\gamma'' = 1/2 \min \{\alpha, \epsilon \gamma\}$. Hence, it stems from Theorem 3.1 and Lemma 3.3 that the solution $u$ is Hölder continuous on $\Omega$ of exponent $\frac{\epsilon \gamma}{\epsilon \gamma + 1} \min \{\alpha, \epsilon \gamma\}$, for any $0 < \gamma < 1/(nq + 1)$.

Moreover, when $\Omega$ is a strongly pseudoconvex domain, we get using Theorem 3.1 and Corollary 3.4 that the solution is Hölder continuous with better exponent $\frac{\epsilon \gamma}{\epsilon \gamma + 1} \min \{\alpha, \epsilon \gamma\}$, for any $0 < \gamma < 1/(nq + 1)$. \hfill \Box

Corollary 3.5. Let $\Omega$ be a finite intersection of smooth strongly pseudoconvex domains in $\mathbb{C}^n$. Assume that $\varphi \in C^{0, \alpha}(\partial \Omega)$, $0 < \alpha \leq 1$, and $0 \leq f \in L^p(\Omega)$ for some $p > 1$. Then the solution $u$ to the Dirichlet problem $Dir(\Omega, \varphi, fd\nu_{2n})$ belongs to $C^{0, \alpha'}(\Omega)$, with $\alpha' = \min \{\alpha/2, \gamma\}$, for any $\gamma < 1/(nq + 1)$ and $1/p + 1/q = 1$.

Moreover, if $\varphi \in C^{1, 1}(\partial \Omega)$, then the solution $u$ is $\gamma$-Hölder continuous on $\Omega$.

The first part of this corollary was proved in Theorem 1.2 in [BKPZ15] with the Hölder exponent $\min \{2\gamma, \alpha\} \gamma$ and the second part was proved in [GKZ08] and [Ch15] (see also [N14, Ch14] for the complex Hessian equation).

Our final purpose concerns how to get the Hölder continuity of the solution to the Dirichlet problem $Dir(\Omega, \varphi, fd\mu)$, by means of the Hölder continuity of subsolutions to $Dir(\Omega, \varphi, d\mu)$ for some special measure $\mu$ on $\Omega$.

Theorem 3.6. Let $\mu$ be a finite Borel measure on a bounded strongly hyperconvex Lipschitz domain $\Omega$. Let also $\varphi \in C^{0, \alpha}(\partial \Omega)$, $0 < \alpha \leq 1$ and $0 \leq f \in L^p(\Omega)$, $p > 1$. Assume that there exists a $\lambda$-Hölder continuous plurisubharmonic function $w$ in $\Omega$ such that $(dd^c w)^n \geq \mu$. If, near the boundary, $\mu$ is Hausdorff-Riesz of order $2n - 2 + \epsilon$ for some $0 < \epsilon \leq 2$, then the solution $u$ to $Dir(\Omega, \varphi, fd\mu)$ is Hölder continuous on $\Omega$.

Proof. Let $\Omega_1 \Subset \Omega$ be an open set such that $\mu$ is a Hausdorff-Riesz measure on $\Omega \setminus \Omega_1$. Let also $\tilde{\mu}$ be a Hausdorff-Riesz measure of order $2n - 2 + \epsilon$ so that $\tilde{\mu}$ equals $\mu$ in $\Omega \setminus \Omega_1$.

As $\varphi$ is not $C^{1, 1}$-smooth, we can not control $\|\tilde{u}_\delta - u\|_{L^1(\Omega_\delta, \mu)}$ as in the proof of Lemma 3.2. Thus, we will estimate $\|\tilde{u}_\delta - u\|_{L^1(\Omega_\delta, \mu)}$ with $\delta_1 := (1/2)\delta^{1/2 + 3/\epsilon}$.

We have

$$\int_{\Omega_\delta} (\tilde{u}_\delta - u) d\mu \leq \int_{\Omega_1} (\tilde{u}_\delta - u) d\mu + \int_{\Omega_\delta} (\tilde{u}_\delta - u) d\tilde{\mu}.$$ 

Fix $\epsilon_1 > 0$ small enough and let $U \Subset \Omega$ be a a neighborhood of $\Omega_1$. Then, Theorem 4.3 in [DDGPKZ14] and Lemma 3.3 yield that

$$\int_{\Omega_\delta} (\tilde{u}_\delta - u) d\mu \leq \int_{\Omega_1} (\tilde{u}_\delta - u)(dd^c w)^n + \int_{\Omega_\delta} (\tilde{u}_\delta - u) d\tilde{\mu}$$

$$\leq C_1 \|\Delta u\|_{U_{\delta_1}^{\frac{nq}{2}}} + C_2 \delta^{\epsilon/2 - \epsilon_1},$$
where $C_1 = C_1(\Omega, U)$ is a positive constant and $C_2$ depends on $n, \Omega, \epsilon, \epsilon_1$ and $\|u\|_{L^\infty(\Omega)}$. Since the mass of $\Delta u$ is locally bounded, there exists a constant $C_3 > 0$ such that

$$\int_{\Omega} (\bar{u} \delta_i - u) d\mu \leq C_3 \delta^\tau,$$

where $\tau = \min\{\frac{\gamma}{\epsilon + 6}, \frac{\lambda(\epsilon + 6)}{\epsilon(\lambda + 2\epsilon)}\}$.

The last requirement to apply Theorem 3.1 is to construct a function $v \in C^{0, \nu}(\overline{\Omega})$ for $0 < \nu \leq 1$ such that $v \leq u$ in $\Omega$ and $v = \varphi$ on $\partial \Omega$. Let us denote by $w_1$ the solution to $\text{Dir}(\Omega, 0, f d\mu)$ and $h_\varphi$ the solution to $\text{Dir}(\Omega, \varphi, 0)$. Now, set $v = w_1 + h_\varphi + A \rho$ with $A > 1$ so that $v \leq u$ in a neighborhood of $\Omega$. It is clear that $v \in \text{PSH}(\Omega) \cap C(\Omega)$, $v = \varphi$ on $\partial \Omega$ and $v \leq u$ in $\Omega$ by the comparison principle. Moreover, by Theorem A in [Ce98] and Theorem 0.1, we infer that $v \in C^{0, \nu}(\overline{\Omega})$, for $\nu = 1/2 \min\{\epsilon, \alpha\}$ and any $\gamma < 1/(\epsilon + 6 + 1)$. Finally, we get from Theorem 3.1 that $u$ is Hölder continuous on $\Omega$ of exponent $\min\{\alpha, \epsilon, 2\gamma/\epsilon(\lambda + 2\epsilon)\}$.

The following are nice applications of Theorem 3.6.

**Corollary 3.7.** Let $\Omega \subset \mathbb{C}^n$ be a bounded strongly hyperconvex Lipschitz domain and let $\mu$ be a finite Borel measure with compact support on $\Omega$. Let also $\varphi \in C^{0, \alpha}(\partial \Omega)$, $0 < \alpha \leq 1$ and $0 \leq f \in L^p(\Omega, \mu)$, $p > 1$. Assume that there exists a $\lambda$-Hölder continuous psh function $w$ in $\Omega$ such that $(dd^c w)^n \geq \mu$. Then the solution to the Dirichlet problem $\text{Dir}(\Omega, \varphi, f d\mu)$ is Hölder continuous on $\overline{\Omega}$ of exponent $\min\{\frac{\alpha}{2}, \frac{2\alpha}{1 + 2\alpha}\}$, for any $\gamma < 1/(\epsilon + 6 + 1)$ and $1/p + 1/q = 1$.

**Example 3.8.** Let $\mu$ be a finite Borel measure with compact support on a bounded strongly hyperconvex Lipschitz domain $\Omega$. Let also $\varphi \in C^{0, \alpha}(\partial \Omega)$, $0 < \alpha \leq 1$ and $0 \leq f \in L^p(\Omega, \mu)$, $p > 1$. Suppose that $\mu \leq dV_n$, where $dV_n$ is the Lebesgue measure of the totally real part $\mathbb{R}^n$ of $\mathbb{C}^n$, then the solution to the Dirichlet problem $\text{Dir}(\Omega, \varphi, f d\mu)$ is Hölder continuous on $\overline{\Omega}$ of exponent $\min\{\frac{\alpha}{2}, \frac{2\alpha}{1 + 2\alpha}\}$, for any $\gamma < 1/(\epsilon + 6 + 1)$ and $1/p + 1/q = 1$.

Indeed, since $\mathbb{R}^n = \{Imz_j = 0, j = 1, ..., n\}$, one can present the Lebesgue measure of the totally real part $\mathbb{R}^n$ of $\mathbb{C}^n$ in the form

$$\left(dd^c \sum_{j=1}^n (Imz_j)\right)^n.$$

Let us set $w = \sum_{j=1}^n (Imz_j)^+$. It is clear that $w \in \text{PSH}(\Omega) \cap C^{0, 1}(\overline{\Omega})$ and $\mu \leq (dd^c w)^n$ on $\Omega$. Corollary 3.7 yields that the solution $u$ belongs to $C^{0, \alpha'}(\overline{\Omega})$ with $\alpha' = \min\{\alpha/2, \frac{2\gamma}{1 + 2\alpha}\}$, for any $\gamma < 1/(\epsilon + 6 + 1)$.

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