INJECTIVITY AND SURJECTIVITY OF THE STIELTJES
MOMENT MAPPING IN GELFAND-SHILOV SPACES

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Abstract. The Stieltjes moment problem is studied in the framework of general
Gelfand-Shilov spaces defined via weight sequences. We characterize the injectivity
and surjectivity of the Stieltjes moment mapping, sending a function to its sequence
of moments, in terms of growth conditions for the defining weight sequence. Finally,
a related moment problem at the origin is studied.

1. Introduction

The moment problem, with its many variations and generalizations, has a long and
rich tradition that goes back to the seminal work of Stieltjes [20]. In 1939, Boas [1] and
Pólya [18] independently showed that, for every sequence \((c_p)_{p=0}^{\infty}\) of complex numbers,
there is a function \(F\) of bounded variation such that
\[
\int_0^{\infty} x^p dF(x) = c_p, \quad p \in \mathbb{N} = \{0, 1, 2, \ldots\}.
\]
This result was greatly improved by A. J. Durán [5] in 1989 who proved, in a construc-
tive way, that, for every sequence \((c_p)_{p \in \mathbb{N}}\) of complex numbers, the infinite system of
linear equations
\[
(1.1) \quad \int_0^{\infty} x^p \varphi(x) dx = c_p, \quad p \in \mathbb{N},
\]
adopts a solution \(\varphi \in \mathcal{S}(0, \infty)\), that is, \(\varphi\) belongs to the Schwartz space \(\mathcal{S}(\mathbb{R})\) of rapidly
decreasing smooth functions and \(\text{supp} \varphi \subseteq [0, \infty)\). We would like to point out that this
result also follows from a short non-constructive argument via Eidelheit’s theorem [17,
Thm. 26.27].

In this article, we study the (unrestricted) Stieltjes moment problem \((1.1)\) in the
context of Gelfand-Shilov spaces defined via weight sequences [7]; see [4, 13, 14] for
earlier works in this direction. Namely, given two sequences of positive real numbers
\(M = (M_p)_{p \in \mathbb{N}}\) and \(A = (A_p)_{p \in \mathbb{N}}\), we consider the spaces \(\mathcal{S}_M^A(0, \infty)\) and \(\mathcal{S}_M(0, \infty)\)
consisting of all \(\varphi \in \mathcal{S}(0, \infty)\) such that there exists \(h > 0\) with
\[
\sup_{p,q \in \mathbb{N}} \sup_{x \in \mathbb{R}} \frac{|x^p \varphi^{(q)}(x)|}{h^{p+q} p! M_p q! A_q} < \infty
\]
and
\[
\sup_{p \in \mathbb{N}} \sup_{x \in \mathbb{R}} \frac{|x^p \varphi^{(q)}(x)|}{h^p p! M_p} < \infty, \quad \forall q \in \mathbb{N},
\]
respectively. Obviously, \( S^A_M(0, \infty) \subset S^A_M(0, \infty) \). Now suppose that \( M \) is derivation closed, that is, \( M_{p+1} \leq C_0 H^{p+1} M_p, p \in \mathbb{N} \), for some \( C_0, H \geq 1 \). Clearly, for every \( \varphi \in S^M(0, \infty) \), its sequence of moments \( (\mu_p(\varphi))_{p \in \mathbb{N}} = (\int_0^\infty x^p \varphi(x) dx)_{p \in \mathbb{N}} \) belongs to the sequence space \( \Lambda^M = \{(c_p)_{p \in \mathbb{N}} \mid \forall h > 0 : \sup_{p \in \mathbb{N}} \frac{|c_p|}{h^{p+1} M_p} < \infty \} \). It is then natural to ask about the surjectivity of the Stieltjes moment mapping \( M \), given by \( M(\varphi) = (\mu_p(\varphi))_{p \in \mathbb{N}} \), when defined on either \( S^A_M(0, \infty) \) or \( S^M(0, \infty) \) and with range \( \Lambda^M \). As a first result, following a technique of A. L. Durán and R. Estrada \([6]\) that combines the Fourier transform with the Borel-Ritt theorem from asymptotic analysis, S.-Y. Chung, D. Kim and Y. Yeom \([4, \text{Thm. 3.1}]\) proved the surjectivity of \( M \) for \( S^M(0, \infty) \) with \( M = (p^{\alpha-1})_{p \in \mathbb{N}} \) (the Gevrey sequence) whenever \( \alpha > 2 \). Subsequently, A. Lastra and the third author \([13]\) refined this result by obtaining linear continuous right inverses for the Stieltjes moment mapping between suitable Fréchet subspaces of \( S^{pla-1}(0, \infty) \) and extended this result \([14]\) to \( S^M(0, \infty) \) for general strongly regular sequences \( M \), that is, sequences \( M \) that are log-convex, of moderate growth and strongly nonquasianalytic, whose growth index \( \gamma(M) \) is strictly greater than 1 (see Section 2 for the definition of these conditions and \( \gamma(M) \)). Conversely, it was proven in \([14]\) that if \( M \) is strongly regular, \( M : S^M(0, \infty) \to \Lambda^M \) is surjective and
\[
(1.2) \quad \sum_{p=0}^{\infty} \left( \frac{M_p}{M_{p+1}} \right)^{1/\gamma(M)} = \infty,
\]
then \( \gamma(M) > 1 \).

Our aim is to improve and complete these results by including the spaces \( S^A_M(0, \infty) \) in our considerations, by dropping some hypotheses on \( M \), specially moderate growth and \([12]\), and by also studying the injectivity of the Stieltjes moment mapping. Our key tools are: a better understanding of the meaning of the different growth conditions usually imposed on the sequence \( M \) and their expression in terms of indices of \( O\)-regular variation, as developed in \([10]\); the use of the Fourier transform in order to translate our problems into the corresponding ones for the asymptotic Borel mapping in certain ultraholomorphic classes on the upper half-plane; the enhanced information obtained in \([11]\) about the injectivity and surjectivity of the asymptotic Borel mapping for sequences \( M \) subject to minimal conditions.

The paper is organized as follows. In the preliminary Section 2 we gather the main facts needed regarding sequences, ultraholomorphic classes, the asymptotic Borel mapping \( B \) and growth indices related to the injectivity and surjectivity of \( B \), Gelfand-Shilov spaces and the Laplace transform. Section 3 contains the main results. Firstly, in Theorem 3.4 we characterize the injectivity of the Stieltjes moment mapping \( M \) (defined on either \( S^A_M(0, \infty) \) or \( S^M(0, \infty) \)) by an easy condition on the sequence \( M \), under minimal conditions on both \( M \) and \( A \). In Theorem 3.5 the surjectivity of \( M \) is characterized by the condition \( \gamma(M) > 1 \), for \( M \) log-convex and of moderate growth and \( A \) weakly log-convex and non-quasianalytic. In particular, this result significantly
improves those in [14] and, moreover, extends to a general situation the statement of surjectivity of $\mathcal{M}$ in the case of the space $S_{[0,\infty)}^{p[\alpha-1]}(0,\infty)$, with $\alpha > 2$ and $\beta > 1$, that appeared (without proof) in [4, Thm. 3.3] and which is, up to the best of our knowledge, the only known result dealing with spaces of the type $S_{\mathcal{M}}^\lambda(0,\infty)$. If moderate growth for $\mathcal{M}$ is substituted by the weaker condition of derivation closedness, we are only able to prove that $\gamma(\mathcal{M}) > 1$ is necessary for the surjectivity of $\mathcal{M}$. We conclude this section by showing that the Stieltjes moment mapping is never bijective and that there exist strongly regular sequences for which $\mathcal{M}$ is neither injective nor surjective.

The final Section 4 is devoted to the study of a related moment problem “at the origin”. More precisely, we consider the space $D_{\mathcal{M}}^\lambda(0,1)$ consisting of all $\varphi \in C^\infty(\mathbb{R})$ with $\operatorname{supp} \varphi \subset [0,1]$ such that there exists $h > 0$ with

$$\sup_{p \in \mathbb{N}} \sup_{x \in [0,1]} \frac{\left| \varphi^{(p)}(x) \right|}{h^p p! M_p} < \infty$$

and we define

$$\mu_0^p(\varphi) = \int_0^1 \frac{\varphi(x)}{x^p} dx, \quad p \in \mathbb{N}.$$ 

The study of the injectivity and surjectivity of the mapping $\mathcal{M}^0 : D_{\mathcal{M}}^\lambda(0,1) \to \Lambda_{\widehat{\mathcal{M}}} : \varphi \mapsto (\mu^0_p(\varphi))_p$, where $\widehat{\mathcal{M}} = (M_p/p!)_{p \in \mathbb{N}}$, is reduced to the one of $\mathcal{M}$ via a suitable use of the Fourier transform.

2. PRELIMINARIES

2.1. Weight sequences. We set $\mathbb{N} = \{0,1,2,\ldots\}$. Throughout this article $\mathcal{M} = (M_p)_{p \in \mathbb{N}}$ will stand for a sequence of positive real numbers with $M_0 = 1$. We define $m_p = M_{p+1}/M_p$, $p \in \mathbb{N}$. The sequence $\mathcal{M}$ is said to be a weight sequence if $m_p \to \infty$ as $p \to \infty$. Furthermore, we set $\widehat{\mathcal{M}} = (p! M_p)_{p \in \mathbb{N}}$ and $\widetilde{\mathcal{M}} = (M_p/p!)_{p \in \mathbb{N}}$. We shall use the following conditions on sequences $\mathcal{M}$:

- (lc) $\mathcal{M}$ is log-convex if $M_{p+1}^2 \leq M_p M_{p+1}$, $p \in \mathbb{Z}^+ = \{1, 2, \ldots\}$.
- (wlc) $\mathcal{M}$ is weakly log-convex if $\widehat{\mathcal{M}}$ satisfies (lc).
- (dc) $\mathcal{M}$ is derivation closed if $M_{p+1} \leq C_0 H P M_p$, $p \in \mathbb{N}$, for some $C_0, H \geq 1$.
- (mg) $\mathcal{M}$ has moderate growth if $M_{p+q} \leq C_0 H P M_p M_q$, $p, q \in \mathbb{N}$, for some $C_0, H \geq 1$.
- (nq) $\mathcal{M}$ is non-quasianalytic if $\sum_{p=0}^\infty \frac{1}{(p+1) m_p} < \infty$.
- (snq) $\mathcal{M}$ is strongly non-quasianalytic if $\sum_{q=p}^\infty \frac{1}{(q+1) m_q} \leq \frac{C}{m_p}$, $p \in \mathbb{N}$, for some $C > 0$.

Remark 2.1. All these properties are preserved when passing from $\mathcal{M}$ to $\widehat{\mathcal{M}}$. In particular, a sequence satisfying (lc) is also (wlc). However, only (dc) and (mg) are generally kept when going from $\mathcal{M}$ to $\widetilde{\mathcal{M}}$. 
A sequence $M$ satisfying (wlc) and (nq) is easily proved to be a weight sequence. A sequence $M$ is said to be strongly regular if it satisfies (lc), (mg) and (snq) (so, $M$ is a weight sequence). The Gevrey sequence $(p^\alpha)_p$ ($\alpha > 0$) is strongly regular.

In the classical work of H. Komatsu [12], the properties (lc), (dc) and (mg) are denoted by (M.1), (M.2)' and (M.2), respectively, while (nq) and (snq) for $M$ are the same as properties (M.3)' and (M.3) for $\hat{M}$, respectively.

For later use, we recall that (lc) (together with $M_0 = 1$) implies that
\begin{equation}
M_j M_p \leq M_{j+p}, \quad j, p \in \mathbb{N}.
\end{equation}

Following Komatsu [12], the relation $M \subset N$ between two sequences means that there are $C, h > 0$ such that $M_p \leq C h^p N_p$ for all $p \in \mathbb{N}$. The associated function of $M$ is defined as
\[ \omega_M(t) := \sup_{p \in \mathbb{N}} \log \frac{t^p}{M_p}, \quad t > 0, \]
and $\omega_M(0) := 0$. The following technical lemma shall be used later on.

**Lemma 2.2.** Let $M$ be a (weight) sequence satisfying (wlc) and (nq). Then, there is a (weight) sequence $N$ with $N \subset M$ satisfying (wlc), (dc) and (nq).

**Proof.** Define $a_p = \min\{2^{p+1}, (p+1)m_p\}, p \in \mathbb{N}$, and $N_0 = 1; N_p = \frac{1}{p!} \prod_{j=0}^{p-1} a_j, p \in \mathbb{Z}^+$. It is straightforward to check that $N = (N_p)_{p \in \mathbb{N}}$ satisfies all the requirements. \qed

### 2.2. Ultraholomorphic classes on the upper half-plane and the asymptotic Borel mapping

We write $\mathbb{H}$ for the open upper half-plane of the complex plane $\mathbb{C}$ and, given an open set $\Omega \subseteq \mathbb{C}$, we denote by $O(\Omega)$ the space of holomorphic functions in $\Omega$. Let $M$ be a weight sequence. For $h > 0$ we define $A_{M,h}(\mathbb{H})$ as the space consisting of all $f \in O(\mathbb{H})$ such that
\[ \sup_{p \in \mathbb{N}} \sup_{z \in \mathbb{H}} \frac{|f^{(p)}(z)|}{h^p p! M_p} < \infty. \]
We set $A_{M}(\mathbb{H}) = \bigcup_{h > 0} A_{M,h}(\mathbb{H})$. Next, for $h > 0$ we define $\mathcal{E}^{\infty}_{M,h}(\mathbb{R})$ as the space consisting of all $f \in C^\infty(\mathbb{R})$ such that
\[ \sup_{p \in \mathbb{N}} \sup_{x \in \mathbb{R}} \frac{|f^{(p)}(x)|}{h^p p! M_p} < \infty. \]
We set $\mathcal{E}^{\infty}_{\mathbb{R}} = \bigcup_{h > 0} \mathcal{E}^{\infty}_{M,h}(\mathbb{R})$.

The following result is standard; it follows from the fact that the elements of $A_{M,h}(\mathbb{H})$ together with all their derivatives are Lipschitz on $\mathbb{H}$.

**Lemma 2.3.** Let $M$ be a weight sequence and let $f \in A_{M,h}(\mathbb{H})$ for some $h > 0$. Then, $f_p(x) = \lim_{z \to x, \text{ z \in \mathbb{H}}} f^{(p)}(z) \in \mathbb{C}$ exists for all $x \in \mathbb{R}$ and $p \in \mathbb{N}$. Moreover, $f_0 \in C^\infty(\mathbb{R})$ and $(f_0)^{(p)} = f_p$ for all $p \in \mathbb{N}$. Consequently, $f_0 \in \mathcal{E}^{\infty}_{\mathbb{R}}$. 
Remark 2.4. Let $M$ be a weight sequence and let $f \in \mathcal{A}_M(\mathbb{H})$. In the sequel, we shall simply write

$$f(x) = \lim_{z \in \mathbb{H}, z \to x} f(z), \quad x \in \mathbb{R}.$$ 

Lemma 2.3 states that $f$ is continuous on $\mathbb{H}$, $f|_\mathbb{R} \in \mathcal{C}_\infty^M(\mathbb{R})$ and that

$$f^{(p)}(x) = \lim_{z \in \mathbb{H}, z \to x} f^{(p)}(z)$$

for all $x \in \mathbb{R}$ and $p \in \mathbb{N}$.

Let $M$ be a weight sequence. For $h > 0$ we define $\Lambda_{M,h}$ as the space consisting of all sequences $(c_p)_p \in \mathbb{C}^\mathbb{N}$ such that

$$\sup_{p \in \mathbb{N}} \frac{|c_p|}{h^p p! M_p} < \infty.$$ 

We set $\Lambda_M = \bigcup_{h > 0} \Lambda_{M,h}$. The asymptotic Borel mapping is defined as

$$\mathcal{B} : \mathcal{A}_M(\mathbb{H}) \to \Lambda_M : f \to (f^{(p)}(0))_p,$$

which is well-defined by Lemma 2.3 (see also Remark 2.4). For a fairly complete account on the injectivity and surjectivity of the asymptotic Borel mapping on various ultraholomorphic classes defined on arbitrary sectors we refer to [11]. There, two indices $\gamma(M)$ and $\omega(M)$, associated to the sequence $M$, play a prominent role. In [8, Ch. 2] and [10, Sect. 3], the connections between these indices, the growth properties usually imposed on sequences, and the theory of O-regular variation, have been thoroughly studied. We summarize here some facts. The first index, introduced by V. Thilliez [21, Sect. 1.3] for strongly regular sequences, may be defined for any weight sequence $M$ satisfying (lc) as

$$\gamma(M) := \sup\{ \mu > 0 \mid (m_p/(p+1)^\mu)_p \text{ is almost increasing} \} \in [0, \infty].$$

(a sequence $(c_p)_p$ is almost increasing if there exists $a > 0$ such that $c_p \leq ac_q$ for all $q \geq p$). On the other hand, for $\beta > 0$ we say that $M$ satisfies $(\gamma_\beta)$ if there is $C > 0$ such that

$$(\gamma_\beta) \quad \sum_{q=p}^\infty \frac{1}{(m_q)^{1/\beta}} \leq \frac{C(p+1)}{(m_p)^{1/\beta}}, \quad p \in \mathbb{N}.$$ 

Then one has that

$$\gamma(M) = \sup\{ \beta > 0 \mid M \text{ satisfies (}\gamma_\beta\text{)}\}.$$ 

Moreover, the following statements hold:

(i) $\gamma(M) > 0$ if and only if $M$ satisfies $(\text{snq})$.

(ii) $\gamma(M) > \beta$ if and only if $M$ satisfies $(\gamma_\beta)$.

The surjectivity of the asymptotic Borel mapping can be characterized as follows.

Theorem 2.5. ([11, Thm. 4.17]) Let $M$ be a strongly regular weight sequence. Then, $\mathcal{B} : \mathcal{A}_M(\mathbb{H}) \to \Lambda_M$ is surjective if and only if

$$\sup_{p \in \mathbb{N}} \frac{m_p}{p+1} \sum_{q=p}^\infty \frac{1}{m_q} < \infty.$$
or, equivalently, $\gamma(M) > 1$.

The second index $\omega(M)$ is given by

$$\omega(M) := \liminf_{p \to \infty} \frac{\log(m_p)}{\log(p)} \in [0, \infty],$$

and it turns out that

$$\omega(M) = \sup\{\mu > 0 \mid \sum_{p=0}^{\infty} \frac{1}{(m_p)^{1/\mu}} < \infty\} = \sup\{\mu > 0 \mid \sum_{p=0}^{\infty} \frac{1}{((p+1)m_p)^{1/(\mu+1)}} < \infty\}.$$

Concerning the injectivity of the asymptotic Borel mapping, we have the next result.

**Theorem 2.6.** ([19 Thm. 12], [11 Thm. 3.4]) Let $M$ be a weight sequence satisfying (lc). Then, $B : A_M(\mathbb{H}) \to \Lambda_M$ is injective if and only if

$$\sum_{p=0}^{\infty} \frac{1}{((p+1)m_p)^{1/2}} = \infty,$$

which in turn implies that $\omega(M) \leq 1$.

Finally, we mention that if $M$ is a weight sequence satisfying (lc), the asymptotic Borel mapping $B : A_M(\mathbb{H}) \to \Lambda_M$ is not bijective [11 Thm. 3.17].

### 2.3. Gelfand-Shilov spaces

Let $M$ and $A$ be weight sequences. For $h > 0$ we define $S^{A,h}_{M}(\mathbb{R})$ as the space consisting of all $\varphi \in C^\infty(\mathbb{R})$ such that

$$\sup_{p,q \in \mathbb{N}} \sup_{x \in \mathbb{R}} \frac{|x^p \varphi(q)(x)|}{hpq!M_pq!A_q} < \infty.$$

Notice that $\varphi \in C^\infty(\mathbb{R})$ belongs to $S^{A,h}_{M}(\mathbb{R})$ if and only if

$$\sup_{q \in \mathbb{N}} \sup_{x \in \mathbb{R}} \frac{|\varphi(q)(x)|e^{\omega(q)|x|/h}}{hq!A_q} < \infty.$$

We set $S^{A}_{M}(\mathbb{R}) = \bigcup_{h>0} S^{A,h}_{M}(\mathbb{R})$. Analogously, we define $S_{M,h}(\mathbb{R})$, $h > 0$, as the space consisting of all $\varphi \in C^\infty(\mathbb{R})$ such that, for all $q \in \mathbb{N},$

$$\sup_{p \in \mathbb{N}} \sup_{x \in \mathbb{R}} \frac{|x^p \varphi(q)(x)|}{hp!M_p} < \infty$$

and set $S_{M}(\mathbb{R}) = \bigcup_{h>0} S_{M,h}(\mathbb{R})$. As in the introduction, we define

$$S^{A}_{M}(0, \infty) := \{\varphi \in S^{A}_{M}(\mathbb{R}) \mid \text{supp } \varphi \subseteq [0, \infty)\}$$

and

$$S_{M}(0, \infty) := \{\varphi \in S_{M}(\mathbb{R}) \mid \text{supp } \varphi \subseteq [0, \infty)\}.$$

Recall that $S^{A}_{M}(0, \infty) \subset S_{M}(0, \infty)$. Suppose that $A$ satisfies (wlc), then $S^{A}_{M}(0, \infty)$ is non-trivial if and only if $A$ satisfies (nq), as follows from the Denjoy-Carleman theorem.
Proposition 2.7. Let \( M \) and \( A \) be weight sequences satisfying (wlc) and (dc). Then, the Fourier transform is an isomorphism from \( S_M^A(\mathbb{R}) \) onto \( S_M^A(\mathbb{R}) \).

Proof. Since the Fourier transform is an isomorphism on the Schwartz space \( S(\mathbb{R}) \) and \( \mathcal{F}^{-1}(\varphi)(\xi) = (2\pi)^{-1} \mathcal{F}(\varphi)(-\xi) \) for all \( \varphi \in S(\mathbb{R}) \), it suffices to show that \( \mathcal{F}(S_M^A(\mathbb{R})) \subseteq S_M^A(\mathbb{R}) \). Let \( h \geq 1 \) and \( \varphi \in S_M^{A,h}(\mathbb{R}) \) be arbitrary. Choose \( C > 0 \) such that

\[
\sup_{x \in \mathbb{R}} |x^p \varphi^{(q)}(x)| \leq C h^{p+q} p! M_p q! A_q, \quad p, q \in \mathbb{N}.
\]

Since \( M \) and \( A \) are weight sequences they are both increasing from some term on, and so there exists \( D \geq 1 \) such that \( M_j \leq D M_p \) and \( A_j \leq D A_p \) for all \( j \leq p \). Hence,

\[
\sup_{x \in \mathbb{R}} (1 + |x|)^p |\varphi^{(q)}(x)| \leq \sum_{j=0}^{p} \binom{p}{j} \sup_{x \in \mathbb{R}} |x^j \varphi^{(q)}(x)|
\]

\[
\leq C \sum_{j=0}^{p} \binom{p}{j} h^{j+q} j! M_j q! A_q
\]

\[
\leq C D (2h)^{p+q} p! M_p q! A_q
\]

for all \( p, q \in \mathbb{N} \). Therefore,

\[
\sup_{\xi \in \mathbb{R}} |\xi^q \hat{\varphi}^{(p)}(\xi)| \leq \sum_{j=0}^{\min\{p,q\}} \binom{q}{j} \frac{p!}{(p-j)!} \int_{-\infty}^{\infty} |x^{p-j} \varphi^{(q-j)}(x)| dx
\]

\[
\leq \sum_{j=0}^{\min\{p,q\}} \binom{q}{j} \binom{p}{j} j! \int_{-\infty}^{\infty} \frac{(1 + |x|)^{p+2-j} |\varphi^{(q-j)}(x)|}{(1 + |x|)^2} dx
\]

\[
\leq 8 C D h^2 \sum_{j=0}^{\min\{p,q\}} \binom{q}{j} \binom{p}{j} j! (2h)^{p+q-2j} (p + 2 - j)! M_{p+2-j} (q - j)! A_{q-j}
\]

\[
\leq 8 C h^2 D^3 (4h)^{p+q} (p + 2)! M_{p+2} q! A_q
\]

\[
\leq 32 C_0^2 H^3 C h^2 D^3 (8H^2 h)^{p+q} p! M_p q! A_q
\]

for all \( p, q \in \mathbb{N} \). □

In view of Proposition 2.7, the next result can be shown in a similar way as [3, Prop. 2.1].

Proposition 2.8. Let \( M \) and \( A \) be weight sequences satisfying (wlc) and (dc). Let \( \psi : \mathbb{R} \to \mathbb{C} \). Then, \( \psi \in \mathcal{F}(S_M^{0,\infty}) \) if and only if \( \psi \in S_M^A(\mathbb{R}) \) and there is \( \Psi : \mathbb{H} \to \mathbb{C} \) satisfying the following conditions:
\( (i) \) \( \Psi \big|_{\mathbb{R}} = \psi \).
\( (ii) \) \( \Psi \) is continuous on \( \mathbb{H} \) and analytic on \( \mathbb{H} \).
\( (iii) \) \( \lim_{\zeta \to \infty, \zeta \in \mathbb{H}} \Psi(\zeta) = 0 \).

2.4. The Laplace transform. Let \( M \) be a weight sequence. We define \( C_{M,h}[0, \infty) \) as the space consisting of all \( \varphi \in C([0, \infty)) \) such that

\[
\sup_{p \in \mathbb{N}} \sup_{x \in [0, \infty)} \frac{x^p|\varphi(x)|}{h^p p! M_p} < \infty
\]

or, in other words, such that

\[
\sup_{x \in [0, \infty)} |\varphi(x)| e^{\omega \tilde{\gamma}(|x|/h)} < \infty.
\]

We set \( C_M[0, \infty) = \bigcup_{h > 0} C_{M,h}[0, \infty) \). The **Laplace transform** of \( \varphi \in C_M[0, \infty) \) is defined as

\[
\mathcal{L}(\varphi)(\zeta) = \int_0^{\infty} \varphi(x)e^{ix\zeta} dx, \quad \zeta \in \mathbb{H}.
\]

**Remark 2.9.** Let \( M \) and \( A \) be weight sequences. We may view \( S^A_M(0, \infty) \) and \( S_M(0, \infty) \) as subspaces of \( C_M[0, \infty) \). Notice that \( L(\varphi)|_{\mathbb{R}} = \tilde{\varphi} \) for all \( \varphi \in S^A_M(0, \infty) \).

**Lemma 2.10.** Let \( M \) be a weight sequence satisfying \( (dc) \). Then, the mapping \( L : C_M[0, \infty) \to \Lambda_M(\mathbb{H}) \) is well-defined and injective.

**Proof.** The fact that \( L \) is well-defined follows along the same lines as the proof of Proposition 2.7. We now show that \( L \) is injective. Let \( \varphi \in C_M[0, \infty) \) be such that \( L(\varphi) \equiv 0 \) on \( \mathbb{H} \). Since \( L(\varphi) \) is continuous on \( \mathbb{H} \), we also have that \( L(\varphi) \equiv 0 \) on \( \mathbb{R} \). Define

\[
\tilde{\varphi}(x) = \begin{cases} 
\varphi(x), & x \geq 0, \\
0, & x < 0.
\end{cases}
\]

Then, \( \tilde{\varphi} \in L^1(\mathbb{R}) \) and \( \mathcal{F}(\tilde{\varphi}) = L(\varphi)|_{\mathbb{R}} \equiv 0 \). Since \( \mathcal{F} \) is injective on \( L^1(\mathbb{R}) \), \( \tilde{\varphi} = 0 \) almost everywhere. As \( \varphi \) is continuous on \( [0, \infty) \), we may conclude that \( \varphi \equiv 0 \) on \( [0, \infty) \). \( \square \)

3. The Stieltjes moment problem in Gelfand-Shilov spaces

Let \( M \) be a weight sequence. The \( p \)-th moment, \( p \in \mathbb{N} \), of an element \( \varphi \in C_M[0, \infty) \) is defined as

\[
\mu_p(\varphi) := \int_0^{\infty} x^p \varphi(x) dx.
\]

If \( M \) satisfies \( (dc) \), then the **Stieltjes moment mapping**

\[
\mathcal{M} : C_M[0, \infty) \to \Lambda_M : \varphi \mapsto (\mu_p(\varphi))_p
\]

is well-defined. The goal of this section is to characterize the injectivity and surjectivity of the Stieltjes moment mapping on \( C_M[0, \infty) \) and its subspaces of type \( S^A_M(0, \infty) \) and \( S_M(0, \infty) \) in terms of the defining weight sequence \( M \). We employ the same idea as in [6], which was later also used in [3, 13, 14]. Namely, we shall reduce these problems to
Proof of Lemma 3.1. Set \( H \) is harmonic and positive on their counterparts for the asymptotic Borel mapping (Theorems [2.6] and [2.5]) via the Laplace transform. In this regard, the following formula is fundamental
\[
\mathcal{L}(\varphi)^{(p)}(0) = i^p \mu_p(\varphi), \quad \varphi \in C_M[0, \infty), p \in \mathbb{N}.
\]
In the next lemma we construct an auxiliary function that shall be frequently used throughout this section (compare with the function \( G \) from [6]). We set \( \mathbb{H}_- = \{ z \in \mathbb{C} \mid \Im z > -1 \} \).

**Lemma 3.1.** Let \( A \) be a weight sequence satisfying (wlc) and (nq). Then, there is \( G \in \mathcal{O}(\mathbb{H}_-) \) satisfying the following conditions:

(i) \( G \) does not vanish on \( \mathbb{H}_- \).

(ii) \( \sup_{z \in \mathbb{H}_-} |G(z)| e^{\omega_A(|z|)} < \infty \).

(iii) \( \sup_{p \in \mathbb{N}} \sup_{x \in \mathbb{R}} \frac{|G^{(p)}(x)| e^{\omega_A(|x|/2)}}{2^{pp!}} < \infty \).

The construction of the function \( G \) from Lemma 3.1 is based on the following result.

**Lemma 3.2.** ([2] Lemma 2.2) Let \( \omega : [0, \infty) \to [0, \infty) \) be an increasing continuous function such that
\[
\int_0^\infty \frac{\omega(t)}{1 + t^2} dt < \infty
\]
and extend \( \omega \) as an even function to the whole real line. Then, the Poisson transform of \( \omega \) on \( \mathbb{H} \) given by
\[
P_\omega(z) = \frac{y}{\pi} \int_{-\infty}^{\infty} \frac{\omega(t)}{(t-x)^2 + y^2} dt, \quad z = x + iy \in \mathbb{H},
\]
is harmonic and positive on \( \mathbb{H} \) and satisfies
\[
P_\omega(z) \geq \frac{1}{4} \omega(|z|), \quad z \in \mathbb{H}.
\]

**Proof of Lemma 3.1** Set \( \omega = \omega_A(2 \cdot) \). Since \( A \) satisfies (wlc) and (nq), we have that [12] Lemma 4.1]
\[
\int_0^\infty \frac{\omega(t)}{1 + t^2} dt < \infty.
\]
Write \( U = 4P_\omega \) (cf. Lemma 3.2) and let \( V \) be the harmonic conjugate of \( U \) on \( \mathbb{H} \). Define \( G = e^{-(U(\cdot + i) + iV(\cdot + i))} \). It is clear that \( G \in \mathcal{O}(\mathbb{H}_-) \) and that (i) is satisfied. We now show (ii) and (iii).

(ii): For \( z \in \mathbb{H}_- \) with \( |z| \geq 2 \) we have that \( 2|z + i| \geq |z| \) and, thus,
\[
|G(z)| = e^{-U(z+i)} \leq e^{-\omega_A(2|z+i|)} \leq e^{-\omega_A(|z|)}.
\]
For \( z \in \mathbb{H}_- \) with \( |z| \leq 2 \) we have that
\[
|G(z)| \leq e^{\omega_A(2)} e^{-\omega_A(|z|)}.
\]
(iii): By the Cauchy estimates and (ii) there is \( C > 0 \) such that
\[
|G^{(p)}(x)| \leq 2^p p! \max_{|z-x| \leq 1/2} |G(z)| \leq C 2^p p! \max_{|z-x| \leq 1/2} e^{-\omega_A(|z|)}
\]
for all \( x \in \mathbb{R} \) and \( p \in \mathbb{N} \). For \( x \in \mathbb{R} \) with \( |x| \geq 1 \) we have that \( |z| \geq |x|/2 \) for all \( z \in \mathbb{C} \) with \( |z - x| \leq 1/2 \). Hence,
\[
|G^{(p)}(x)| \leq C2^p p! e^{-\omega \lambda (|x|/2)}, \quad p \in \mathbb{N}.
\]
For \( x \in \mathbb{R} \) with \( |x| \leq 1 \) we have that
\[
|G^{(p)}(x)| \leq C e^{\omega \lambda (1/2)} 2^p p! e^{-\omega \lambda (|x|/2)}, \quad p \in \mathbb{N}.
\]

\[\Box\]

Proposition 2.8 and Lemma 3.1 imply the following important lemma.

**Lemma 3.3.** Let \( M \) be a weight sequence satisfying \((\text{wlc})\) and \((\text{dc})\) and let \( A \) be a weight sequence satisfying \((\text{wlc})\), \((\text{dc})\) and \((\text{nq})\). Consider the function \( G \) from Lemma 3.1. Then, \( fG \in \mathcal{F}(S^A_M(0, \infty)) \) for all \( f \in A_M(\mathbb{H}) \).

We are ready to study the injectivity and surjectivity of the Stieltjes moment mapping.

**Theorem 3.4.** Let \( M \) be a weight sequence satisfying \((\text{lc})\) and \((\text{dc})\) and let \( A \) be a weight sequence satisfying \((\text{wlc})\) and \((\text{nq})\). Then, the following statements are equivalent:

1. \( \sum_{p=0}^{\infty} \frac{1}{((p+1)m_p)^{1/2}} = \infty \).
2. \( B : A_M(\mathbb{H}) \to \Lambda_M \) is injective.
3. \( M : C_M[0, \infty) \to \Lambda_M \) is injective.
4. \( M : S_M(0, \infty) \to \Lambda_M \) is injective.
5. \( M : S^A_M(0, \infty) \to \Lambda_M \) is injective.

**Proof.** (i) \( \Rightarrow \) (ii): By Theorem 2.6.

(ii) \( \Rightarrow \) (iii): Let \( \varphi \in C_M[0, \infty) \) be such that \( \mu_p(\varphi) = 0 \) for all \( p \in \mathbb{N} \). By Lemma 2.10 we have that \( \mathcal{L}(\varphi) \in A_M(\mathbb{H}) \). Moreover, \( \mathcal{L}(\varphi)(0) = p\mu_p(\varphi) = 0 \) for all \( p \in \mathbb{N} \) and, thus, \( \mathcal{L}(\varphi) \equiv 0 \). Since \( \mathcal{L} \) is injective (Lemma 2.10), we obtain that \( \varphi \equiv 0 \).

(iii) \( \Rightarrow \) (iv) \( \Rightarrow \) (v): Obvious.

(v) \( \Rightarrow \) (i): By Lemma 2.2 we may assume that \( A \) satisfies \((\text{dc})\). In view of Theorem 2.6 it suffices to show that \( B : A_M(\mathbb{H}) \to \Lambda_M \) is injective. Let \( f \in A_M(\mathbb{H}) \) be such that \( f^{(p)}(0) = 0 \) for all \( p \in \mathbb{N} \). Consider the function \( G \) from Lemma 3.1. By Lemma 3.3 we have that \( fG = \hat{\varphi} \) for some \( \varphi \in S^A_M(0, \infty) \). Observe that
\[
\mu_p(\varphi) = (-i)^p \hat{\varphi}^{(p)}(0) = (-i)^p (fG)^{(p)}(0) = (-i)^p \sum_{j=0}^{p} \binom{p}{j} f^{(j)}(0) G^{(p-j)}(0) = 0, \quad p \in \mathbb{N}.
\]
Hence, \( \varphi \equiv 0 \) and, thus, \( fG \equiv 0 \). Since \( G \) does not vanish (Lemma 3.1(i)), we obtain that \( f \equiv 0 \).

\[\Box\]

**Theorem 3.5.** Let \( M \) be a weight sequence satisfying \((\text{lc})\) and \((\text{dc})\) and let \( A \) be a weight sequence satisfying \((\text{wlc})\) and \((\text{nq})\). Then, the following statements are equivalent:

1. \( M : S^A_M(0, \infty) \to \Lambda_M \) is surjective.
2. \( M : C_M[0, \infty) \to \Lambda_M \) is surjective.
3. \( M : S_M(0, \infty) \to \Lambda_M \) is surjective.
4. \( M : S^A_M(0, \infty) \to \Lambda_M \) is surjective.

**Proof.** (i) \( \Rightarrow \) (ii): By Theorem 2.6.

(ii) \( \Rightarrow \) (iii): Let \( \varphi \in C_M[0, \infty) \) be such that \( \mu_p(\varphi) = 0 \) for all \( p \in \mathbb{N} \). By Lemma 2.10 we have that \( \mathcal{L}(\varphi) \in A_M(\mathbb{H}) \). Moreover, \( \mathcal{L}(\varphi)(0) = p\mu_p(\varphi) = 0 \) for all \( p \in \mathbb{N} \) and, thus, \( \mathcal{L}(\varphi) \equiv 0 \). Since \( \mathcal{L} \) is injective (Lemma 2.10), we obtain that \( \varphi \equiv 0 \).

(iii) \( \Rightarrow \) (iv) \( \Rightarrow \) (v): Obvious.

(v) \( \Rightarrow \) (i): By Lemma 2.2 we may assume that \( A \) satisfies \((\text{dc})\). In view of Theorem 2.6 it suffices to show that \( B : A_M(\mathbb{H}) \to \Lambda_M \) is surjective. Let \( f \in A_M(\mathbb{H}) \) be such that \( f^{(p)}(0) = 0 \) for all \( p \in \mathbb{N} \). Consider the function \( G \) from Lemma 3.1. By Lemma 3.3 we have that \( fG = \hat{\varphi} \) for some \( \varphi \in S^A_M(0, \infty) \). Observe that
\[
\mu_p(\varphi) = (-i)^p \hat{\varphi}^{(p)}(0) = (-i)^p (fG)^{(p)}(0) = (-i)^p \sum_{j=0}^{p} \binom{p}{j} f^{(j)}(0) G^{(p-j)}(0) = 0, \quad p \in \mathbb{N}.
\]
Hence, \( \varphi \equiv 0 \) and, thus, \( fG \equiv 0 \). Since \( G \) does not vanish (Lemma 3.1(i)), we obtain that \( f \equiv 0 \).

\[\Box\]
(ii) \( \mathcal{M} : \mathcal{S}_M(0, \infty) \to \Lambda_M \) is surjective.

(iii) \( \mathcal{M} : C_M[0, \infty) \to \Lambda_M \) is surjective.

(iv) \( \mathcal{B} : \mathcal{A}_M(\mathbb{H}) \to \Lambda_M \) is surjective.

Each of the previous statements implies the next one:

(vi) \( \sup_{\mathcal{p} \in \mathbb{N}} \frac{m_{\mathcal{p}}}{\mathcal{p} + 1} \sum_{q=\mathcal{p}}^{\infty} \frac{1}{m_{q}} < \infty \) or, equivalently, \( \gamma(\mathcal{M}) > 1 \).

If, in addition, \( \mathcal{M} \) satisfies (mg), then all the previous statements are equivalent.

In the proof of Theorem 3.5, we shall use the following lemma (cf. [6]).

**Lemma 3.6.** Let \( (c_p)_p \in \mathbb{C}^\mathbb{N} \) and let \( G \in C^\infty((-\delta, \delta)) \), for some \( \delta > 0 \), such that \( G(0) \neq 0 \). Set

\[
b_p = \sum_{j=0}^{\mathcal{p}} \binom{\mathcal{p}}{j} c_j \left( \frac{1}{G} \right)^{(\mathcal{p}-j)} (0), \quad \mathcal{p} \in \mathbb{N}.
\]

Then,

\[
\sum_{j=0}^{\mathcal{p}} \binom{\mathcal{p}}{j} b_j g^{(\mathcal{p}-j)}(0) = c_p, \quad \mathcal{p} \in \mathbb{N}.
\]

**Proof.** Choose \( 0 < \delta_1 \leq \delta \) such that \( G \) does not vanish on \((-\delta_1, \delta_1)\). By E. Borel’s theorem there is \( f \in C^\infty((-\delta_1, \delta_1)) \) such that \( f^{(\mathcal{p})}(0) = c_p \) for all \( \mathcal{p} \in \mathbb{N} \). Set \( g = f/G \in C^\infty((-\delta_1, \delta_1)) \). Then,

\[
g^{(\mathcal{p})}(0) = \sum_{j=0}^{\mathcal{p}} \binom{\mathcal{p}}{j} f^{(j)}(0) \left( \frac{1}{G} \right)^{(\mathcal{p}-j)} (0) = b_p, \quad \mathcal{p} \in \mathbb{N}.
\]

Hence,

\[
c_p = f^{(\mathcal{p})}(0) = (gG)^{(\mathcal{p})}(0) = \sum_{j=0}^{\mathcal{p}} \binom{\mathcal{p}}{j} g^{(j)}(0)G^{(\mathcal{p}-j)}(0) = \sum_{j=0}^{\mathcal{p}} \binom{\mathcal{p}}{j} b_j G^{(\mathcal{p}-j)}(0), \quad \mathcal{p} \in \mathbb{N}.
\]

**Proof of Theorem 3.5.** We first prove the equivalence of the statements (i) to (iv).

(i) \( \Rightarrow \) (ii) \( \Rightarrow \) (iii): Obvious.

(iii) \( \Rightarrow \) (iv): Let \((c_p)_p \in \Lambda_M\) be arbitrary. Pick \( \varphi \in C_M[0, \infty) \) such that \( \mu_p(\varphi) = (-i)^p c_p \) for all \( \mathcal{p} \in \mathbb{N} \). Then, \( f = \mathcal{L}(\varphi) \in \mathcal{A}_M(\mathbb{H}) \) (Lemma 2.10) and \( f^{(\mathcal{p})}(0) = i^p \mu_p(\varphi) = c_p \) for all \( \mathcal{p} \in \mathbb{N} \).

(iv) \( \Rightarrow \) (i): By Lemma 2.2, we may assume that \( \mathcal{A} \) satisfies (dc). Let \((c_p)_p \in \Lambda_M\) be arbitrary. Consider the function \( G \) from Lemma 3.1. Set

\[
b_p = \sum_{j=0}^{\mathcal{p}} \binom{\mathcal{p}}{j} i^j c_j \left( \frac{1}{G} \right)^{(\mathcal{p}-j)} (0), \quad \mathcal{p} \in \mathbb{N}.
\]

We claim that \((b_p)_p \in \Lambda_M\) (cf. [14, Prop. 6.4]). Indeed, choose \( C, h > 0 \) such that \( |c_p| \leq Ch^p p! M_p \) for all \( \mathcal{p} \in \mathbb{N} \). Next, since \( 1/G \) is holomorphic on a neighbourhood of the disk
with center the origin and radius 1/2, there is $C' > 0$ such that $|(1/G)^{(p)}(0)| \leq C' p!$ for all $p \in \mathbb{N}$. Hence,

$$|b_p| \leq CC'' \sum_{j=0}^{p} \left(\frac{p}{j}\right) h^j j! M_j 2^{p-j} (p-j)! \leq CC'' D (h+2)^p p! M_p, \quad p \in \mathbb{N},$$

where $D \geq 1$ is chosen so that $M_j \leq DM_p$ for all $j \leq p$. By assumption there is $f \in \mathcal{A}_M(\mathbb{H})$ such that $f^{(p)}(0) = b_p$ for all $p \in \mathbb{N}$. We have that $fG = \hat{\varphi}$ for some $\varphi \in \mathcal{S}^\infty_M(0, \infty)$ by Lemma 3.3. Finally, Lemma 3.6 implies that

$$\mu_p(\varphi) = (-i)^p \hat{\varphi}^{(p)}(0) = (-i)^p (fG)^{(p)}(0) = (-i)^p \sum_{j=0}^{p} \left(\frac{p}{j}\right) b_j G^{(p-j)}(0) = c_p, \quad p \in \mathbb{N}.$$

We now prove the statements related to (v). The implication (iv) $\Rightarrow$ (v) follows directly from [11] Thm. 4.14(i)]. If, in addition, $\mathcal{M}$ satisfies (mg), condition (v) implies that $\mathcal{M}$ satisfies (snq) as well (see Subsection 2.2), and so $\mathcal{M}$ is strongly regular. Then, Theorem 2.5 guarantees that (iv) holds. □

**Corollary 3.7.** Let $\mathcal{M}$ be a weight sequence satisfying (lc) and (dc) and let $\mathcal{A}$ be a weight sequence satisfying (wlc) and (nq). Then, $\mathcal{M} : C_M[0, \infty) \to \Lambda_M$, $\mathcal{M} : \mathcal{S}_M(0, \infty) \to \Lambda_M$ and $\mathcal{M} : \mathcal{S}^\Lambda_M(0, \infty) \to \Lambda_M$ are never bijective.

**Proof.** If any of the moment mappings were injective, we would have that $\sum_p ((p + 1) m_p)^{-1/2} = \infty$ by Theorem 3.4. From Subsection 2.2 we deduce that $\omega(\mathcal{M}) \leq 1$, which in turn implies that $\gamma(\mathcal{M}) \leq 1$ because $\gamma(\mathcal{N}) \leq \omega(\mathcal{N})$ for any weight sequence $\mathcal{N}$ satisfying (lc). Hence, (v) from Theorem 3.5 is violated and therefore none of (i) - (iii) from Theorem 3.5 can be satisfied. □

**Example 3.8.** There exist strongly regular sequences for which the Stieltjes moment mapping is neither injective nor surjective. E.g., in [9] Example 4.18, Remark 4.19 (see also [8], Example 2.2.26) the sequence $\mathcal{M}$ is defined via its sequence of quotients, $M_p = \prod_{j=0}^{p-1} m_j$, where

$$m_0 = 1; \quad m_p = e^{\delta_p / p} m_{p-1} = \exp \left(\sum_{k=1}^{p} \frac{\delta_k}{k}\right), \quad p \in \mathbb{Z}_+,$$

and the sequence $(\delta_k)_{k \in \mathbb{Z}_+}$ still has to be determined. Consider the sequences

$$k_j := 2^{3^j} < q_j := k_j^2 = 2^{3^j/2} < k_{j+1} = 2^{3^{j+1}}, \quad j \in \mathbb{N},$$

and choose $(\delta_k)_k$ as follows:

$$\delta_1 = \delta_2 = 2,$$
$$\delta_k = 3, \quad \text{if} \quad k \in \{k_j + 1, \ldots, q_j\}, j \in \mathbb{N},$$
$$\delta_k = 2, \quad \text{if} \quad k \in \{q_j + 1, \ldots, k_{j+1}\}, j \in \mathbb{N}.$$

One can prove that $\mathcal{M}$ is strongly regular and that $\gamma(\mathcal{M}) = 2 < \omega(\mathcal{M}) = 5/2$. Then, the sequence $\mathcal{M}^{1/2} := (M_p^{1/2})_{p \in \mathbb{N}}$ is again strongly regular and $\gamma(\mathcal{M}^{1/2}) = 1 < 5/4 =
\(\omega(M^{1/2})\). Hence, both the injectivity and surjectivity of the Stieltjes moment mapping are discarded.

Subsequently, in [8] (see also [10]), a general procedure has been designed to obtain strongly regular sequences with preassigned positive values of \(\gamma(M)\) and \(\omega(M)\). In particular, one can choose strongly regular sequences \(M\) with \(\gamma(M) \leq 1 < \omega(M)\) and thereby exclude both injectivity and surjectivity.

4. A MOMENT PROBLEM AT THE ORIGIN

Let \(M\) be a weight sequence. For \(h > 0\) we define \(D^{M,h}(0,1)\) as the space consisting of all \(\varphi \in C^\infty(\mathbb{R})\) with \(\text{supp } \varphi \subseteq [0,1]\) such that

\[
\|\varphi\|_{M,h} := \sup_{p \in \mathbb{N}} \sup_{x \in [0,1]} \left|\varphi^{(p)}(x)\right| h^p p! M_p < \infty.
\]

We set \(D^M(0,1) = \bigcup_{h>0} D^{M,h}(0,1)\). Suppose that \(M\) satisfies (wlc), then \(D^M(0,1)\) is non-trivial if and only if \(M\) satisfies (nq), as follows from the Denjoy-Carleman theorem. Notice that \(D^M(0,1) \subset \mathcal{S}^M(0,\infty)\) for all weight sequences \(A\).

**Lemma 4.1.** Let \(M\) be a weight sequence and let \(\varphi \in D^{M,h}(0,1)\) for some \(h > 0\). Then,

\[
|\varphi(x)| \leq \|\varphi\|_{M,h} h^p M_p x^p
\]

for all \(x \in [0,1]\) and \(p \in \mathbb{N}\).

**Proof.** Since \(\varphi^{(j)}(0) = 0\) for all \(j \in \mathbb{N}\), Taylor’s theorem implies that

\[
|\varphi(x)| \leq \frac{1}{p!} \sup_{t \in [0,x]} |\varphi^{(p)}(t)| x^p \leq \|\varphi\|_{M,h} h^p M_p x^p
\]

for all \(x \in [0,1]\) and \(p \in \mathbb{N}\). \(\square\)

Let \(M\) be a weight sequence. For \(\varphi \in D^M(0,1)\) we define

\[
\mu_p^0(\varphi) = \int_0^1 \frac{\varphi(x)}{x^p} \, dx, \quad p \in \mathbb{N}.
\]

The mapping

\[
\mathcal{M}^0 : D^M(0,1) \rightarrow \Lambda^\infty_n : \varphi \mapsto (\mu_p^0(\varphi))_p
\]

is well-defined by Lemma 4.1. The goal of this section is to characterize the injectivity and surjectivity of the mapping \(\mathcal{M}^0\) in terms of the defining weight sequence \(M\). We shall reduce these problems to their counterparts for the Stieltjes moment mapping (Theorems 3.4 and 3.5) via the following lemma.

**Lemma 4.2.** Let \(M\) be a weight sequence and let \(\varphi \in D^M(0,1)\). Then,

\[
\mu_p(\tilde{\varphi}) = i^{p+1} p! \mu_{p+1}^0(\varphi), \quad p \in \mathbb{N}.
\]
Proof. For all $p \in \mathbb{N}$ we have that
\[
\mu_p(\hat{\varphi}) = \int_0^\infty \xi^p \hat{\varphi}(\xi) d\xi = i^p \int_0^\infty \mathcal{F}(\varphi^{(p)})(\xi) d\xi = -i^{p+1} \int_0^\infty (\mathcal{F}(\varphi^{(p)}(x)/x))'(\xi) d\xi
\]
\[
= i^{p+1} \mathcal{F}(\varphi^{(p)}(x)/x)(0) = i^{p+1} \int_0^1 \frac{\varphi^{(p)}(x)}{x} dx = i^{p+1} p! \int_0^1 \frac{\varphi(x)}{xp+1} dx = i^{p+1} p! \mu_{p+1}^0(\varphi).
\]

We are ready to characterize the injectivity and surjectivity of the mapping $\mathcal{M}^0$.

**Theorem 4.3.** Let $\mathcal{M}$ be a weight sequence satisfying (lc), (dc) and (nq). Then, the following statements are equivalent:

(i) $\sum_{p=0}^\infty \frac{1}{((p+1)m_p)^{1/2}} = \infty$.

(ii) $\mathcal{M} : \mathcal{D}^\mathcal{M}(0,1) \to \Lambda^\mathcal{M}$ is injective.

(iii) $\mathcal{M}^0 : \mathcal{D}^\mathcal{M}(0,1) \to \Lambda^\mathcal{M}$ is injective.

**Remark 4.4.** If we assume that $\mathcal{M}$ satisfies (lc) and it does not satisfy (nq), then $\mathcal{D}^\mathcal{M}(0,1)$ is trivial and $\sum_{p=0}^\infty ((p+1)m_p)^{-1} = \infty$, and so the three previous statements clearly hold true. This justifies the hypothesis (nq) in Theorem 4.3 while condition (dc) is needed in order to apply our previous results about the moment mapping $\mathcal{M}$.

The proof of Theorem 4.3 is based on the next result.

**Proposition 4.5.** Let $\varphi \in L^1(\mathbb{R})$ with $\text{supp} \varphi \subseteq [0, \infty)$. If $\hat{\varphi}$ vanishes on a subset of $\mathbb{R}$ with positive Lebesgue measure, then $\varphi = 0$ almost everywhere.

Proposition 4.5 follows directly from the Lusin-Privalov theorem, which we include here for the reader’s convenience.

**Theorem 4.6** (of Lusin and Privalov [15]). Let $F \in \mathcal{O}(\mathbb{H})$ and suppose that there exists a set $A \subset \mathbb{R}$ with positive Lebesgue measure such that, for every $x \in A$ and $0 < \delta < 1$, it holds that
\[
\lim_{z \to x} F(z) = 0,
\]
where $S_{x,\delta} \subset \mathbb{H}$ is the sector with vertex at $x$, vertical bisecting direction and opening $\pi \delta$. Then, $F \equiv 0$ on $\mathbb{H}$.

**Proof of Proposition 4.3** (i) $\Rightarrow$ (ii): Follows directly from Theorem 3.4 since it is clear that $\mathcal{F}(\mathcal{D}^\mathcal{M}(0,1)) \subset C^\mathcal{M}[0, \infty)$.

(ii) $\Rightarrow$ (i): By Theorem 3.4 it suffices to show that $\mathcal{M} : C^\mathcal{M}[0, \infty) \to \Lambda^\mathcal{M}$ is injective. Let $\varphi \in C^\mathcal{M}[0, \infty)$ be such that $\mu_p(\varphi) = 0$ for all $p \in \mathbb{N}$. Set
\[
\tilde{\varphi}(x) = \begin{cases} 
\varphi(x), & x \geq 0, \\
0, & x < 0.
\end{cases}
\]
Next, by condition (nq), we can choose \( \psi \in \mathcal{D}^M(0,1) \) such that \( \psi(x) \neq 0 \) for all \( x \in (0,1) \). Then,

\[
\tilde{\varphi} \ast \tilde{\psi} = \mathcal{F}(\mathcal{L}(\varphi)_{\mathbb{R}}(\cdot) \cdot \psi) \in \mathcal{F}(\mathcal{D}^M(0,1)),
\]

as follows from Lemmas \ref{lem:2} and \ref{lem:4}. Moreover, we have that

\[
\mu_p(\tilde{\varphi} \ast \tilde{\psi}) = \sum_{j=0}^{p} \binom{p}{j} \mu_j(\varphi)\mu_{p-j}(\psi) = 0, \quad p \in \mathbb{N}.
\]

Hence, \( \tilde{\varphi} \ast \tilde{\psi} \equiv 0 \) and, thus, also \( \mathcal{F}^{-1}(\tilde{\varphi} \ast \tilde{\psi}) = \mathcal{L}(\varphi)_{\mathbb{R}}(\cdot) \cdot \psi \equiv 0 \). Since \( \psi(x) \neq 0 \) for all \( x \in (0,1) \), we obtain that \( \mathcal{F}(\tilde{\varphi})(\xi) = \mathcal{L}(\varphi)_{\mathbb{R}}(\xi) = 0 \) for all \( \xi \in (-1,0) \). Proposition \ref{prop:4.5} yields that \( \tilde{\varphi} = 0 \) almost everywhere. As \( \varphi \) is continuous on \( [0,\infty) \), we may conclude that \( \varphi \equiv 0 \) on \( [0,\infty) \).

\( (ii) \Rightarrow (iii) \): Follows directly from Lemma \ref{lem:4.2}

\( (iii) \Rightarrow (ii) \): Let \( \varphi \in \mathcal{D}^M(0,1) \) be such that \( \mu_p(\tilde{\varphi}) = 0 \) for all \( p \in \mathbb{N} \). By Lemma \ref{lem:4.2} we have that \( \mu_0^p(\varphi) = 0 \) for all \( p \in \mathbb{Z}_+ \). Condition (dc) implies that \( \varphi' \in \mathcal{D}^M(0,1) \). Moreover, we have that

\[(4.1) \quad \mu_p^0(\varphi') = \int_0^1 \varphi'(x)x^pdx = p \int_0^1 \varphi(x)x^{p+1}dx = p\mu_{p+1}^0(\varphi) = 0, \quad p \in \mathbb{Z}_+.\]

Hence, \( \varphi' \equiv 0 \) and, since \( \varphi \) is compactly supported, we obtain that \( \varphi \equiv 0 \). \( \square \)

**Theorem 4.7.** Let \( M \) be a weight sequence satisfying (lc) and (dc). Then, the following statements are equivalent:

\( (i) \) \( \mathcal{M}^0 : \mathcal{D}^M(0,1) \to \Lambda^M_\mathcal{M} \) is surjective.

\( (ii) \) \( \mathcal{M} : \mathcal{F}(\mathcal{D}^M(0,1)) \to \Lambda^M_\mathcal{M} \) is surjective.

Each of the previous statements implies the next one:

\( (iii) \) \( \sup_{p \in \mathbb{N}} \frac{m_p}{p+1} \sum_{q=p}^{\infty} \frac{1}{m_q} < \infty \) or, equivalently, \( \gamma(M) > 1 \).

If, in addition, \( M \) satisfies (mg), then all the previous statements are equivalent.

We need a lemma in preparation of the proof of Theorem 4.7.

**Lemma 4.8.** Let \( M \) be a weight sequence and let \( \chi \in C_M[0,\infty) \) be such that \( \mu_0(\chi) \neq 0 \). Define \( G = \mathcal{L}(\chi)_{\mathbb{R}} \) and notice that \( G(0) = \mu_0(\chi) \neq 0 \). For \( (c_p)_p \in \mathbb{C}^\mathbb{N} \) we set

\[
b_p = (-i)^p \sum_{j=0}^{p} \binom{p}{j} \varphi_j c_j \left( \frac{1}{G} \right)^{(p-j)}(0), \quad p \in \mathbb{N}.
\]

Then,

\[
\sum_{j=0}^{p} \binom{p}{j} b_j \mu_{p-j}(\chi) = c_p, \quad p \in \mathbb{N}.
\]
Proof. Lemma 3.6 implies that
\[ \sum_{j=0}^{p} \binom{p}{j} b_j \mu_{p-j}(\chi) = \sum_{j=0}^{p} \binom{p}{j} b_j (-i)^{p-j} G^{(p-j)}(0) = c_p, \quad p \in \mathbb{N}. \]

\[ \square \]

Proof of Theorem 4.7. We first prove the equivalence of (i) and (ii).

(i) \(\Rightarrow\) (ii): Follows directly from Lemma 4.2

(ii) \(\Rightarrow\) (i): Let \((c_p)_p \in \Lambda^M\) be arbitrary. Set \(b_p = i^p(p - 1)!c_p\), \(p \in \mathbb{N}\). Then, \((b_p)_p \in \Lambda^M\). Choose \(\varphi \in \mathcal{D}^M(0, 1)\) such that \(\mu_p(\hat{\varphi}) = b_p\) for all \(p \in \mathbb{N}\). Consider the function \(\varphi'\), which belongs to \(\mathcal{D}^M(0, 1)\) because of (dc). Then, the computation in (4.1) and Lemma 4.2 imply that
\[ \mu_p' (\varphi') = \frac{p \mu_p (\varphi)}{i^p p!} = \frac{p b_p}{i^p p!} = c_p, \quad p \in \mathbb{N}. \]

For the second part of the theorem, observe that, since \(\mathcal{F}(\mathcal{D}^M(0, 1)) \subset C^M_M[0, \infty)\), the implication (iii) \(\Rightarrow\) (iii) follows directly from (ii) \(\Rightarrow\) (v) in Theorem 3.5.

Finally, if \(M\) additionally satisfies (mg) and we depart from (iii), as before we first deduce that \(M\) is strongly regular and, thus, satisfies (nq). Let \((c_p)_p \in \Lambda^M\) be arbitrary. Choose \(\psi \in \mathcal{D}^M(0, 1)\) such that \(\mu_p(\hat{\psi}) \neq 0\). Define \(\chi = \hat{\psi}|_{[0, \infty)} \in C^M_M[0, \infty)\) and \(G = \mathcal{L}(\chi)|_{\mathbb{R}}\). Notice that, by Lemma 4.2, we have that \(G(0) = \mu_0(\chi) = i \mu_0(\hat{\psi}) \neq 0\).

Set \(b_p = (-i)^p \sum_{j=0}^{p} \binom{p}{j} i^j c_j \left(\frac{1}{G}\right)^{(p-j)}(0), \quad p \in \mathbb{N}.\)

We claim that \((b_p)_p \in \Lambda^M\). Indeed, choose \(C, h > 0\) such that \(|c_p| \leq Ch^p p!M_p\) for all \(p \in \mathbb{N}\). Next, since \(G \in \mathcal{E}^M_M(\mathbb{R})\) (Lemmas 2.3 and 2.10), \(G(0) \neq 0\) and \(M\) satisfies (lc), a classical result of Malliavin on the inverse-closedness of algebras of ultradifferentiable functions [16], p. 185, 4.1 implies that there are \(C', k > 0\) such that \(|(1/G)^{(p)}(0)| \leq C' k^p p! M_p\) for all \(p \in \mathbb{N}\). Hence,
\[ |b_p| \leq C C' \sum_{j=0}^{p} \binom{p}{j} h^j j! M_j k^{p-j}(p-j)! M_{p-j} \leq C' C'(h + k)^p p! M_p, \quad p \in \mathbb{N}, \]
where we have used (2.1). By Theorem 3.5 part (v) \(\Rightarrow\) (iii), there is \(\varphi \in C^\infty_M[0, \infty)\) such that \(\mu_p(\varphi) = b_p\) for all \(p \in \mathbb{N}\). Set
\[ \tilde{\varphi}(x) = \begin{cases} \varphi(x), & x \geq 0, \\ 0, & x < 0. \end{cases} \]

Then, \(\tilde{\varphi} \ast \hat{\psi} \in \mathcal{F}(\mathcal{D}^M(0, 1))\) (cf. the proof of Proposition 4.3). Finally, Lemma 4.8 implies that
\[ \mu_p(\tilde{\varphi} \ast \hat{\psi}) = \sum_{j=0}^{p} \binom{p}{j} b_j \mu_{p-j}(\hat{\psi}) = \sum_{j=0}^{p} \binom{p}{j} b_j \mu_{p-j}(\chi) = c_p, \quad p \in \mathbb{N}, \]
and so (ii) holds.

**Corollary 4.9.** Let \( M \) be a weight sequence satisfying (lc) and (dc). Then, \( \mathcal{M} : \mathcal{F}(\mathcal{D}^M(0,1)) \to \Lambda^M_\mathcal{M} \) and \( \mathcal{M}^0 : \mathcal{D}^M(0,1) \to \Lambda^M_{\mathcal{M}} \) are never bijective.

**Proof.** In view of Theorems 4.3 and 4.7, this can be shown in exactly the same way as Corollary 3.7.

**Remark 4.10.** A careful inspection of the proofs of Theorems 3.5 and 4.7 shows that, as long as \( M \) is a weight sequence satisfying (lc) and (dc), the surjectivity of the Borel mapping \( B : \mathcal{A}_M^\mathcal{M}(\mathbb{H}) \to \Lambda^M_\mathcal{M} \) implies, not only \( \gamma(M) > 1 \), but also the surjectivity of all the moment mappings considered in both statements. Although the condition (mg), combined with (lc) and \( \gamma(M) > 1 \), allows one to prove the surjectivity of \( B \) (see Theorem 2.5), in some cases where (mg) fails one can still show that \( B \) is surjective.

A very classical example is that of the so-called \( q \)-Gevrey sequences, \( M_q = (q^{p^2})_{p \in \mathbb{N}} \), where \( q > 1 \). These sequences satisfy (lc), (dc) and \( \gamma(M_q) > 1 \) (indeed, \( \gamma(M_q) = \infty \)) but not (mg). One can prove (see [21, Subsect. 3.3] for some hints and references) that \( B : \mathcal{A}_M_{M_q}(\mathbb{H}) \to \Lambda_{M_q}^M \) is surjective and so the previous considerations apply to this case.

**Acknowledgements:** The first author is supported by FWO-Vlaanderen, via the postdoctoral grant 12T0519N. The last two authors are partially supported by the Spanish Ministry of Economy, Industry and Competitiveness under the project MTM2016-77642-C2-1-P.

The authors wish to express their gratitude to Prof. Manuel Núñez, from the Universidad de Valladolid (Spain), for making them aware of Theorem 4.6.

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