Information on some recent applications of umbral extensions to discrete mathematics

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Abstract

At the first part of our communicate we show how specific umbral extensions of the Stirling numbers of the second kind result in new type of Dobinski-like formulas. In the second part among others one recovers how and why Morgan Ward solution of uncountable family of ψ- difference calculus nonhomogeneous equations \( \Delta_\psi f = \varphi \) in the form

\[
f(x) = \sum_{n \geq 1} \frac{B_n}{n!} \varphi^{(n-1)}(x) + \int_\psi \varphi(x) + p(x)
\]

extends to ψ- Appell polynomials case automatically. Illustrative specifications to q-calculus case and Fibonomial calculus case are made explicit due to the usage of upside down notation for objects of Extended Finite Operator Calculus.

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INTRODUCTION

The upside down convenient notation for objects of Extended Finite Operator Calculus (EFOC) recently being developed and promoted by the present author [1, 2, 3, 4] is to be used throughout the whole exposition of both parts.

The EFOC is an implementation of operator approach of Gian Carlo Rota to various by now formulations of extended of umbral difference calculi. For abundant references - thousands of them - see [5] and [1, 2, 3, 4].

In [1, 2, 3, 4, 5] and here also ψ denotes a number or functions' sequence - sequence of functions of a parameter q. ψ constitutes the Morgan Ward [6] extension (see: after then Viskov [7]) of \( \frac{1}{1-q} \) \( n \geq 0 \) sequence to quite arbitrary one (the so called - "admissible" in [8] and after then in [1,2,3,4,5]). The specific choices are for example: Fibonomialy-extended sequence \( \frac{1}{1-q} \) \( n \geq 0 \) (\( \langle F_n \rangle \) - Fibonacci sequence) or just "the usual" ψ-sequence \( \frac{1}{1-q} \) \( n \geq 0 \) or the famous Gauss q-extended \( \frac{1}{1-q} \) \( n \geq 0 \) admissible sequence of extended umbral operator calculus, where \( n_q = \frac{1-q^n}{1-q} \) and \( n_q! = n_q(n-1)_q! \), 0\! = 1 - see more below. With such type extension we frequently may in a "ψ-mnemonic" way repeat reasoning and this what was done by Rota (see at first (FOC) of Rota in [9]).

The simplicity of the first steps to be done while identifying general properties of ψ-extended objects consists in writing objects of these extensions in mnemonic convenient upside down notation [1, 2, 3, 4], [10] which is here down introduced as follows:

\[
\frac{\psi(n-1)}{\psi_n} \equiv n_\psi, n_\psi! = n_\psi(n-1)_\psi!, n > 0, x_\psi \equiv \frac{\psi(x-1)}{\psi(x)}, \tag{1}
\]

\[
x_\psi^k = x_\psi(x-1)_\psi(x-2)_\psi... (x - k + 1)_\psi \tag{2}
\]

\[
x_\psi(x-1)_\psi... (x - k + 1)_\psi = \frac{\psi(x-1)\psi(x-2)\psi(x-k)}{\psi(x)\psi(x-1)\psi(x-k+1)}. \tag{3}
\]

If one writes the above in the form \( x_\psi \equiv \frac{\psi(x-1)}{\psi(x)} \equiv \Phi(x) \equiv \Phi_x \equiv x_\Phi \), one sees that the name upside down notation is legitimate.

You may consult [1, 2, 3, 4, 5] and [10] for further development and use of this notation.
In the first part of our communicate umbral extensions of the Stirling numbers of the both kinds are considered and the resulting new type of Dobinski-like formulas are discovered [10]. These extensions naturally encompass the well known $q$-extensions. The fact that $q$-extended Stirling numbers giving rise to the umbral $q$-extended Dobinski formula interpreted as the average of powers of random variable $X_q$ with the $q$-Poisson distribution and are equivalent by re-scaling with the other Comtet -like $q$-extended Stirling numbers [10] - singles out the $q$-extensions itself which appear to be a kind of bifurcation point in the domain of umbral extensions. The further consecutive umbral extensions of Carlitz-Gould $q$-Stirling numbers are therefore realized in [10] in a two-fold way.

In the second part of our communicate one displays [11] how and why Morgan Ward solution [6] of $\psi$- difference calculus nonhomogeneous equation $\Delta \psi f = \varphi$ in the form

$$f(x) = \sum_{n \geq 1} \frac{B_n}{n!} \varphi^{(n-1)}(x) + \int_{\psi} \varphi(x) + p(x)$$

recently proposed by the present author [12]- extends here now to $\psi$- Appell polynomials case - almost automatically. Illustrative specifications to $q$-calculus case and Fibonomial calculus case [2, 13, 14] were already made explicit in [12] exactly due to the of upside down notation for objects of the EFOC.

The First Part - Stirling numbers extensions and Dobinski-like formulae

In this part of our communicate we follow [10] and we refer Reader for further details to consult [10]. At the start let us recall that the two standard $[15] [16] [17]$ $q$-extensions Stirling numbers of the second kind might be defined as follows:

$$x_q^n = \sum_{k=0}^{n} \binom{n}{k}_q x_q^k,$$

where $x_q = \frac{1 - x}{1 - q}$ and $x_q^k = x_q(x - 1)_q... (x - k + 1)_q$, which corresponds to the $\psi$ sequence choice in the $q$-Gauss form $\langle \frac{1}{n_q} \rangle_{n \geq 0}$
and $q$\textsuperscript{\textasciitilde}-Stirling numbers

$$x^n = \sum_{k=0}^{n} \binom{n}{k}_q \chi_k(x)$$

where $\chi_k(x) = x(x - 1_q)(x - 2_q)\ldots(x - [k - 1]_q)$

For these two classical by now $q$-extensions of Stirling numbers of the second kind - the "$q$-standard" recurrences hold respectively:

$$\left\{ \binom{n+1}{k}_q \right\} = \sum_{l=0}^{n} \binom{n}{l}_q q^l \left\{ \binom{l}{k-1}_q \right\}; n \geq 0, k \geq 1,$$

$$\left\{ \binom{n+1}{k}_q \right\} \sim = \sum_{l=0}^{n} \binom{n}{l}_q q^{l-k+1} \left\{ \binom{l}{k-1}_q \right\} \sim; n \geq 0, k \geq 1.$$

From the above it follows immediately that corresponding $q$-extensions of $B_n$ Bell numbers satisfy respective recurrences:

$$B_q(n+1) = \sum_{l=0}^{n} \binom{n}{l}_q q^l B_q(l); n \geq 0,$$

$$B_q^\sim(n+1) = \sum_{l=0}^{n} \binom{n}{l}_q q^{l-k+1} \\overline{B_q}(l); n \geq 0$$

where

$$\overline{B_q}(l) = \sum_{k=0}^{l} q^k \binom{l}{k}_q \sim.$$  

**Note** that both definitions via (4) and (5) equations consequently correspond to different $q$-counting [16] .

For applications to coherent state physics see [17] and references therein.

With any other choice out of countless choices of the $\psi$ sequence the equation (5) becomes the definition of $\psi$\textsuperscript{\textasciitilde}- Stirling (vide "Fibonomial-Stirling") numbers of the second kind $\left\{ \binom{n}{k}_\psi \right\}$ and then $\psi$\textsuperscript{\textasciitilde}-Bell numbers $B_q^\sim(\psi)$ are defined as sums as usual - where now $\chi_k(x)$ in (5) is to be replaced by $\psi_k(x) = x(x - 1_{\psi})(x - 2_{\psi})\ldots(x - [k - 1]_{\psi})$. These $\psi$\textsuperscript{\textasciitilde}- Stirling numbers of the second kind recognized for $q$ case properly as Comtet numbers in Wagner's terminology [16] satisfy familiar recursion and are given by familiar
formulas to be presented soon.

The extension of definition (4) of the $q$-Stirling numbers of the second kind beyond this $q$-case i.e. beyond the $\psi = \left\{ \frac{1}{(nq)!} \right\}_{n \geq 0}$ choice is not that mnemonic and the ” behavior ” of the naturally expected recursion under extension comprises quite a surprise (see: appendix 2.2 in [10]).

Therefore further consecutive umbral extension of Carlitz-Gould $q$-Stirling numbers $\{n \atop k\}_q$ and $\{n \atop k\}_q \sim \psi$ is realized two-fold way [10].

The first ”easy way” consists in almost mnemonic sometimes replacement of $q$ subscript by $\psi$ because we learn from [16] that the equation (5) defines as a matter of fact the specific case of Comtet numbers [16] i.e. we define $\psi$-extended Comtet-Stirling numbers of the second kind as follows:

$$x^n = \sum_{k=0}^{n} \left\{ \begin{array}{c} n \\ k \end{array} \right\}_q \psi_k(x)$$

(6)

where $\psi_k(x) = x(x - 1_\psi)(x - 2_\psi)...(x - [k - 1]_\psi)$.

As a consequence we have ”for granted”:

$$\left\{ \begin{array}{c} n + 1 \\ k \end{array} \right\}_\psi = \left\{ \begin{array}{c} n \\ k \end{array} \right\}_\psi + k_\psi \left\{ \begin{array}{c} n \\ k - 1 \end{array} \right\}_\psi; \quad n \geq 0, k \geq 1;$$

(7)

where $\left\{ \begin{array}{c} n \\ 0 \end{array} \right\}_\psi = \delta_{n,0}$, $\left\{ \begin{array}{c} n \\ k \end{array} \right\}_\psi = 0, \quad k > n$; and the easy derivable recurrence relations for ordinary generating function now read

$$G_{k_\psi}(x) = \frac{x}{1 - k_\psi} G_{k_\psi - 1}(x), \quad k \geq 1$$

(8)

where

$$G_{k_\psi}(x) = \sum_{n \geq 0} \left\{ \begin{array}{c} n \\ k \end{array} \right\}_\psi x^n, \quad k \geq 1$$

from where as a consequence we arrive to what follows:

$$G_{k_\psi}(x) = \frac{x^k}{(1 - 1_\psi x)(1 - 2_\psi x)...(1 - k_\psi x)}, \quad k \geq 0$$

(9)
and adapting reasoning from [18] we derive the following explicit formula

$$\left\{ \frac{n}{k} \right\} = \frac{1}{k^{\psi}} \sum_{r=1}^{k} (-1)^{k-r} \binom{k}{r_{\psi}}; \quad n, k \geq 0. \quad (10)$$

Expanding the right hand side of (9) results in another explicit formula for these $\psi$-case Comtet numbers i.e. we have

$$\left\{ \frac{n}{k} \right\} = \sum_{1 \leq i_1 \leq i_2 \leq \ldots \leq i_{n-k} \leq k} (i_1)^{\psi} (i_2)^{\psi} \ldots (i_{n-k})^{\psi}; \quad n, k \geq 0. \quad (11)$$

or equivalently (compare with [16])

$$\left\{ \frac{n}{k} \right\} = \sum_{d_1 + d_2 + \ldots + d_k = n-k, \quad d_i \geq 0} 1^{d_1} 2^{d_2} \ldots k^{d_k}; \quad n, k \geq 0. \quad (12)$$

With help of $\psi$-Stirling numbers of the second kind being defined equivalently by (6) , (7), (11) or (12) we define now $\psi$-Bell numbers in a standard way

$$B_n^{\sim}(\psi) = \sum_{k=0}^{n} \left\{ \frac{n}{k} \right\}; \quad n \geq 0.$$

The recurrence for $B_n^{\sim}(\psi)$ is already quite involved and complicated for the $q$-extension case (see: the first section in [10])- and no acceptable readable form of recurrence for the $\psi$-extension case is known to us.

Nevertheless after adapting the the corresponding Wilf’s reasoning from [18] we derive for two variable ordinary generating function for $\left\{ \frac{n}{k} \right\}$ Stirling numbers of the second kind and the $\psi$-exponential generating function for $B_n^{\sim}(\psi)$ Bell numbers the following formulæ

$$C_{\psi}(x, y) = \sum_{n \geq 0} A_n^{\sim}(\psi, y)x^n, \quad (13)$$

where the $\psi$-exponential-like polynomials $A_n^{\sim}(\psi, y)$

$$A_n^{\sim}(\psi, y) = \sum_{k=0}^{n} \left\{ \frac{n}{k} \right\} y^k$$

do satisfy the recurrence
\[ A_n^\sim(\psi, y) = [y(1 + \partial_\psi)]A_{n-1}^\sim(\psi, y) \quad n \geq 1, \]

hence
\[ A_n^\sim(\psi, y) = [y(1 + \partial_\psi)]^n1, \quad n \geq 0, \]

where the linear operator \( \partial_\psi \) acting on the algebra of formal power series is being called (see: [1,2,3,4,5] and references therein) the "\( \psi \)-derivative" and \( \partial_\psi y^n = n_\psi y^{n-1} \).

The \( \psi \)-exponential generating function \( F_n^\sim(\psi)(x) = \sum_{n \geq 0} B_n^\sim(\psi) \frac{x^n}{n_\psi!} \)

for \( F_n^\sim(\psi) \) Bell numbers - again - after cautious adaptation of the method from the Wilf’s generatingfunctionology book [18] we get only a little bit involved formula
\[
B_n^\sim(\psi)(x) = \sum_{r \geq 0} \epsilon(\psi, r) \frac{e_\psi[r_\psi x]}{r_\psi!}, \tag{14}
\]

where (see: [6,7,1,2,3,4])
\[
e_\psi(x) = \sum_{n \geq 0} \frac{x^n}{n_\psi!}
\]

while
\[
\epsilon(\psi, r) = \sum_{k=r}^{\infty} \frac{(-1)^{k-r}}{(k_\psi - r_\psi)!}, \tag{15}
\]

and for \( \psi \)-extension the Dobinski like formula here now reads
\[
B_n^\sim(\psi) = \sum_{r \geq 0} \epsilon(\psi, r) \frac{r_\psi^n}{r_\psi!}, \tag{16}
\]

In the case of Gauss \( q \)-extended choice of \( \left(\frac{1}{n_\psi}\right)_{n \geq 0} \) admissible sequence of extended umbral operator calculus we have then
\[
\epsilon(q, r) = \sum_{k=r}^{\infty} \frac{(-1)^{k-r}}{(k - r)_q q^{-}(\xi)}q^{-}(\xi), \tag{17}
\]

and the new \( q^\sim \)-Dobinski formula is given by
\[
B_n^\sim(\psi) = \sum_{r \geq 0} \epsilon(\psi, r) \frac{r_\psi^n}{r_\psi!}. \tag{18}
\]
which for \( q = 1 \) becomes the Dobinski formula from 1887 [19].

As for the problem of how eventually one might interpret the \( \psi \)-Dobinski formulae (16) and (18) in the Rota-like way see: [10].

In [10] you find also details in support of our conviction that \( q \)-extensions seem appear as a kind of bifurcation point in the domain of umbral extensions.

For the discussion of the other way to arrive eventually to another type of Dobinski formula - via an attempt to \( \psi \)-extend the equation (4) - we again refer the Reader to [10] as this is quite a longer story with a surprise.

The parallel treatment of the Comtet \( \left[ \begin{array}{c} n \\ k \end{array} \right] \sim \psi \) Stirling numbers of the first kind is now not difficult.

Namely we define as in [10] the \( \psi \)-Stirling numbers of the first kind as the coefficients in the following expansion

\[
\psi_k(x) = \sum_{r=0}^{k} \left[ \begin{array}{c} k \\ r \end{array} \right] \sim \psi x^r
\]

where - recall \( \psi_k(x) = x(x-1_\psi)(x-2_\psi)...(x-[k-1_\psi]) \); From the above we then get

\[
\sum_{r=0}^{k} \left[ \begin{array}{c} k \\ r \end{array} \right] \sim \psi \left\{ \begin{array}{c} r \\ l \end{array} \right\} \sim \psi = \delta_{k,l}.
\]

Another (expected Whithney numbers of the first kind) \( \psi \)-Stirling numbers of the first kind [10] are defined as follows:

\[
\psi_k^c(x) = \sum_{r=0}^{k} \left[ \begin{array}{c} k \\ r \end{array} \right] \sim \psi c x^r
\]

where - now \( \psi_k^c(x) = x(x+1_\psi)(x+2_\psi)...(x+[k-1_\psi]) \); More on that - see [10].

The Second Part -\( \psi \)- Appell polynomials\( ^\dagger \) solutions of the \( Q(\partial_\psi) \)- difference nonhomogeneous equation

The second part is based on [11,12] to which we refer for more details.
At first recall [3] [1] the simple fact to be used in what follows.

**Proposition 0.1.** $Q(\partial_\psi)$ is a $\psi$- delta operator iff there exists invertible $S \in \Sigma_\psi$ such that $Q(\partial_\psi) = \partial_\psi S$.

Formally: "$S = Q/\partial_\psi$" or "$S^{-1} = \partial_\psi/Q$". In the sequel we use this abbreviation $Q(\partial_\psi) \equiv Q$.

$\psi$- Appell or generalized Appell polynomials $\{A_n(x)\}_{n \geq 0}$ are defined according to

$$\partial_\psi A_n(x) = n_\psi A_{n-1}(x)$$

and they naturally do satisfy the $\psi$ - Sheffer-Appell identity [3] [1]

$$A_n(x + \psi y) = \sum_{s=0}^{n} \binom{n}{s}_\psi A_s(y)x^{n-s}.$$ (23)

$\psi$-Appell or generalized Appell polynomials $\{A_n(x)\}_{n \geq 0}$ are equivalently characterized via their $\psi$- exponential generating function

$$\sum_{n \geq 0} z^n \frac{A_n(x)}{n_\psi!} = A(z) \exp_\psi \{xz\},$$ (24)

where $A(z)$ is a formal series with constant term different from zero - here normalized to one.

The $\psi$- exponential function of $\psi$-Appell-Ward numbers $A_n = A_n(0)$ is

$$\sum_{n \geq 0} z^n \frac{A_n}{n_\psi!} = A(z).$$ (25)

Naturally $\psi$- Appell $\{A_n(x)\}_{n \geq 0}$ satisfy the $\psi$- difference equation

$$QA_n(x) = n_\psi x^{n-1}; \quad n \geq 0,$$ (26)

because $QA_n(x) = QS^{-1}x^n = Q(\partial_\psi/Q)x^n = \partial_\psi x^n = n_\psi x^{n-1}; n \geq 0$. Therefore they play the same role in $Q(\partial_\psi)$- difference calculus as Bernoulli polynomials do in standard difference calculus or $\psi$-Bernoulli-Ward polynomials (see Theorem 16.1 in [1] and consult also [12]) in $\psi$-difference calculus due to the following: The central problem of the $Q(\partial_\psi)$ - difference calculus is:

$$Q(\partial_\psi)f = \varphi \quad \varphi =?,$$

where $f, \varphi$ - are for example formal series or polynomials.
The idea of finding solutions is the \( N \)-Finite Operator Calculus \[1, 3, 4, 2\] standard. As one knows \([1, 3]\)) any \( N \)-delta operator \( Q \) is of the form

\[
Q(\partial_N) = \partial_N S
\]

where \( S \in \Sigma_N \). Let \( Q(\partial_N) = \sum_{k \geq 0} \frac{q_k}{(k+1)!} \partial_N^k \), \( q_1 \neq 0 \). Consider then \( Q(\partial_N = \partial_N S) \) with \( S = \sum_{k \geq 0} \frac{s_k}{k!} \partial_N^k \); \( s_0 = q_1 \neq 0 \).

We thus have for \( S^{-1} \equiv \hat{A} \) - call it: \( N \)-Appell operator - the obvious expression

\[
\hat{A} \equiv S^{-1} = \frac{\partial_N}{Q_N} = \sum_{n \geq 0} \frac{A_n}{n!} \partial_N^n.
\]

Now multiply the equation \( Q(\partial_N) f = \varphi \) by \( \hat{A} \equiv \sum_{n \geq 0} \frac{A_n}{n!} \partial_N^n \) thus getting

\[
\partial_N f = \sum_{n \geq 0} \frac{A_n}{n!} \varphi^{(n)}(x) \varphi^{(n-1)} = \partial_N \varphi^{(n-1)}.
\]

The solution then reads:

\[
f(x) = \sum_{n \geq 1} \frac{A_n}{n!} \varphi^{(n-1)}(x) + \int_\psi \varphi(x) + p(x),
\]

where \( p \) is "\( Q(\partial_N)\)- periodic" i.e. \( Q(\partial_N)p = 0 \). Compare with \([12]\) for "\( N \)- periodic" i.e. \( p(x + \psi 1) = p(x) \) i.e. \( \Delta_N p = 0 \). Here the relevant \( N \)- integration \( \int_\psi \varphi(x) \) is defined as in \([1]\). We recall it in brief. Let us introduce the following representation for \( \partial_N\) "difference-ization"

\[
\partial_N = \hat{n}_N \partial_0; \quad \hat{n}_N x^{n-1} = n_N x^{n-1}; \quad n \geq 1,
\]

where \( \partial_0 x^n = x^{n-1} \) i.e. \( \partial_0 \) is the \( q = 0 \) Jackson derivative. \( \partial_0 \) is identical with divided difference operator. Then we define the linear mapping \( \int_\psi \) accordingly:

\[
\int_\psi x^n = \left( \frac{x}{n_N} \right) x^n = \frac{1}{(n + 1)_N} x^{n+1}; \quad n \geq 0
\]

where of course \( \partial_N \circ \int_\psi = id \).
1 Examples

(a) The case of $\psi$-Bernoulli-Ward polynomials and $\Delta_{\psi}$-difference calculus
was considered in detail in [12] following [6].

(b) Specification of (a) to the Gauss and Heine originating $q$-umbral calculus case [6, 4, 5, 2] was already presented in [12].

(c) Specification of (a) to the Lucas originating FFOC case was also presented in [12] (here: FFOC=Fibonomial Finite Operator Calculus), see example 2.1 in [2]). Recall: the Fibonomial coefficients -already known in 19-th century to Lucas [20]- are defined "binomially" as

\[
\binom{n}{k}_F = \frac{F_n!}{F_k! F_{n-k}!} = \binom{n}{n-k}_F,
\]

where ($F_n$-Fibonacci numbers in up-side down notation: $n_F \equiv F_n \neq 0$, $n_F! = n_F(n-1)_F(n-2)_F(n-3)_F \ldots 2_F 1_F$; $0_F! = 1$; $n^+_F = n_F(n-1)_F \ldots (n-k+1)_F$; $\binom{n}{k}_F \equiv \frac{n^+_k}{k_F!}$).

We shall call the corresponding linear difference operator $\partial_F$; $\partial_F x^n = n_F x^{n-1}$; $n \geq 0$ the $F$-derivative. Then in conformity with [6] and with notation as in [1-13, 14] one has:

\[
E^\alpha(\partial_F) = \sum_{n \geq 0} \frac{\alpha^n}{n_F!} \partial^n_F
\]

for the corresponding [3,1,4] generalized translation operator $E^\alpha(\partial_F)$. The $\psi$-integration for the moment is still not explored $F$-integration and we arrive at the $F$-Bernoulli polynomials unknown till now.

Note: recently a combinatorial interpretation of Fibonomial coefficient has been found [13] by the present author.

(d) The other examples of $Q(\partial_\psi)$-difference calculus - expected naturally to be of primary importance in applications are provided by the possible use of such $\psi$-Appell polynomials as:

- $\psi$-Hermite polynomials \{$H_{n,\psi}\}_{n \geq 0}$:
\[ H_{n,\psi}(x) = \left[ \sum_{k \geq 0} \left( -\frac{1}{2} \right)^k \frac{\partial^2 \psi^k}{\psi_k!} \right] x^n \quad n \geq 0; \]

- \( \psi \) - Laguerre polynomials \( \{L_{n,\psi}\}_{n \geq 0} \) [3]:

\[ L_{n,q}(x) = \frac{n_q}{n} \hat{x}_\psi \left[ \frac{1}{\partial \psi - 1} \right]^{-n} x^{n-1} = \frac{n_q}{n} \hat{x}_\psi (\partial \psi - 1)^n x^{n-1} = \]

\[ = \frac{n_q}{n} \hat{x}_\psi \sum_{k=1}^{n} (-1)^k \binom{n}{k} \partial^{n-k}_\psi x^{n-1} = \]

\[ = \frac{n_q}{n} \sum_{k=1}^{n} (-1)^k \binom{n}{k} (n-1)^{n-k}_\psi \frac{k}{k_q} x^k. \]

For \( q = 1 \) in \( q \)-extended case one recovers the known formula:

\[ L_{n,q=1}(x) = \sum_{k=1}^{n} (-1)^k \frac{n_q!}{k_q!} \binom{n-1}{k-1} x^k. \]

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