Symplectic Dynamical Symmetries in Algebraic Models of Nuclear Structure

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Abstract. Based on a generalized reduction scheme for boson representations of symplectic algebras of the type $Sp(4k, R)$, we consider the symplectic extension of a boson realization of compact unitary algebras for the $k = 1$, $k = 3$ and $k = 6$ cases, which have relevance in nuclear structure theory. First we review an application of the $k = 1$ case for the creation of a $Sp(4, R)$ classification scheme, which is used for obtaining a generalized phenomenological description of important nuclear characteristics in terms of the classification quantum numbers for large sets of nuclei. Then for the $k = 3$ and $k = 6$ cases we outline some of the new possibilities that appear in the symplectic extensions of the Interacting Vector Boson Model (IVBM) and the Interacting Boson Model (IBM-2), respectively. The examples presented are used to describe the collective modes of the nuclear spectra in individual nuclei as well as in sequences of nuclei.

1. Introduction

In terms of group theory – the language of symmetries, the concept of the dynamical symmetry of a nuclear system is represented by chains of group-subgroups that reduce the initial symmetry to the $SO(3)$ symmetry of the angular momentum. Each of these chains provide for a labeling of the basis states in terms of the representations of their constituents and for defining the respective Hamiltonians in terms of their Casimir invariants. This approach is commonly employed in two types of applications. The first is related to the classification of the many-body systems under consideration \cite{1}, and the second to the development of algebraic models for the description of the different collective modes in the individual nuclei or in sets of nuclei with specific properties. An important advantage of this approach is that exact analytic solutions of the associated eigenvalue problems can be given, which correspond to different limiting cases of physical phenomena.

Symplectic algebras have been applied extensively in the theory of nuclear structure. They are used generally to describe systems with changing number of particles or excitation quanta and in this way provide for larger representation spaces and richer subalgebraic structures. Based on a general reduction scheme of the boson representations of symplectic algebras of the type $Sp(4k, R)$ \cite{2}, we review the two types of applications in nuclear structure physics, starting with the $k = 1$ case where we present a classification scheme of the nuclei in the major nuclear shells, which leads to a generalized description of some of their properties.
Next we consider the $k = 3$ case illustrated by the Interacting Vector Boson Model (IVBM) [3] based on the embedding $Sp(12, R) \supset U(6)$, where the symplectic symmetry allows for a change in the number of bosons used to build basis states, which in turn leads to mixing of the collective modes in their description. Additional chains of subgroups that appear as a result of the symplectic generalization of the group of dynamical symmetry are also considered. This approach reveals new features of the collective spectra of the individual heavy even-even nuclei.

A new and different application of this general approach is presented by the $k = 6$ case, where the symplectic extension $Sp(24, R) \supset U(12)$ of the dynamical symmetry group of the Interacting Boson Model-2 (IBM-2) [4] is considered. In this case a change in the number of bosons is associated with a change of the number of valence protons and neutrons used to describe a nuclear system. New reduction chains that arise can be used for a classification of the even-even nuclei in the major nuclear shells, which results in a consistent group-theoretical investigation of the development of collectivity in their spectras and respectvelly in nuclear shapes and deformations, allowing these to be treated in a unified way.

2. Reductions of the boson representations of the $Sp(4k, R)$

The boson representation of the $Sp(4k, R)$ algebra is realized in terms of bilinear combinations of creation $a_{\alpha i}^\dagger$ and annihilation $a_{\alpha i}$ operators with two indexes, $\alpha = 1, 2; i = 1, 2, \ldots, k$, that satisfy Bose commutation relations: $[a_{\alpha i}, a_{\beta j}^\dagger] = \delta_{\alpha \beta} \delta_{ij}$ (all other commutators are zero). Specifically, by appending the operators

$$a_{\alpha i}^\dagger a_{\beta j}, \quad a_{\alpha i} a_{\beta j}, \quad \alpha, \beta = 1, 2; i, j = 1, \ldots, k$$

(1)

to the Weyl generators $a_{\alpha i}^\dagger a_{\beta j}$ of the $U(2k)$ group, the boson representation of the $Sp(4k, R)$ algebra is obtained [5]. It is a reducible one that decomposes into two irreducible representations. One of them acts in the space $H_+$, spanned over the vectors for which the number of bosons $n$ is even, and the other acts in $H_-$ defined by the condition $n$ – odd so that $H = H_+ + H_-$. By construction, each of the subspaces $H_+$ and $H_-$ spans a reducible representation of $U(2k)$ which decomposes into a direct sum of eigensubspaces of the first Casimir invariant $N = \sum a_{\alpha i}^\dagger a_{\alpha i}$ of $U(2k)$. In this way the totally symmetric irreducible unitary representation (IUR) of $U(2k)$, denoted by $[n]_{2k}$ is realized.

Therefore, the group $U(2k)$ appears as a maximal compact subgroup, which further contains the direct product $U(2) \otimes U(k)$ of the two mutually complementary subgroups generated by the operators $F_{\alpha \beta} = \sum a_{\alpha i}^\dagger a_{\beta i}$ and $A_{ij} = \sum a_{\alpha i}^\dagger a_{\alpha j}$, respectively. The operator $N = F_{\alpha \alpha} = A_{ii}$ is the first-order Casimir operator for the groups $U(2)$ as well as $U(k)$. Next, the IURs of the groups $SU(2)$, $SU(k)$, and $SU(2) \otimes SU(k)$ at $n$ = fixed, can be labeled by the eigenvalues $F(F + 1)$ of the operator $F^2$, where $F = n/2, n/2 - 1, \ldots, 0$ or $1/2$ for $n$ even or odd, respectively. Thus when $n$ is fixed and $T$ is fixed, $2F + 1$ equivalent representations of the group $SU(k)$ arise. Each of them is labeled by the eigenvalues of the operator $F_0$: $-F, -F + 1, \ldots, F$.

On the other hand, the so called ladder representation of the noncompact group $U(k, k)$ acts in the space of the boson representation of the $Sp(4k, R)$ algebra. There exists a connection between this ladder representation and the boson representation of $U(2k)$, which is realized through the third generator $F_0$ of the multiplier $SU(2)$ of the already mentioned direct product. This operator is also the first Casimir operator of the group $U(k, k)$. Different aspects of this relationship will be revealed in more details in the applications. As shown in [2], both reduction chains

$$Sp(4k, R) \supset U(2k) \supset U(2) \otimes U(k) \supset SU(k) \quad (2)$$

$$Sp(4k, R) \supset U(k, k) \supset U(k) \otimes U(k) \supset SU(k) \quad (3)$$

are equally convenient for the description of the representations of the final group $SU(k)$. 

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3. The $k = 1$ case and the $Sp(4,R)$ classification scheme with application

To illustrate the $k = 1$ case, we apply the above algebraic construction to a classification of the many-body nuclear systems. Similar methods are used for the classification of elementary particles. In this case we realize the generators in terms of two types of one-dimensional (scalar) creation ($\pi^+, \nu^+$) and annihilation ($\pi, \nu$) operators. The reduction of the boson representation of the classification group $Sp(4,R)$ to its compact $u(2)$ and non-compact $u(1,1)$ subalgebras [6],

\[
\begin{align*}
sp(4,R) & \twoheadrightarrow u(2) \quad F_0 \\
F_0 & \twoheadrightarrow u(1,1) \quad N_t
\end{align*}
\]

is the mathematical underpinning of the scheme. As illustrated by (4), the reduction is realized by means of the operator that counts the total number of particles, $N_t = (N_\pi + N_\nu)$ ($N_\pi = \pi^+ \pi, N_\nu = \nu^+ \nu$) which is the first order invariant of $u(2)$, and the operator of the third projection of the $F$-spin, $F_0 = \frac{1}{2}(N_\pi - N_\nu)$, which does not differ essentially from the first order Casimir of $u(1,1)$. $N_t$ reduces the space $\mathcal{H}_t$ in which the boson representation of $sp(4,R)$ acts, into a direct sum of a totally symmetric irreducible unitary representations /IUR/ of the classification group $Sp$ and $u$, respectively. The same is obtained by reducing the $u(1,1)$ ladders with the operator $N_t$.

The relation of the algebraic operators used in the classification scheme to the nuclear characteristics in the valence shell, is quite natural when $N_\pi = \frac{1}{2}(N_p - Z^{(1)})$ and $N_\nu = \frac{1}{2}(N_n - N^{(1)})$ are counted as the numbers of proton and neutron valence pairs of the nucleus from a given shell, in which $Z^{(1)}$ and $N^{(1)}$ are the numbers of protons and neutrons of the double magic nucleus at the beginning of the shell. Then $N_t$ and $F_0$ are exactly the operators reducing the $sp(4,R)$ spaces, and their interpretation corresponds to the one of the Interacting Boson Model - 2 (IBM-2) [4], as the total number of valence bosons and the third projection $F_0$.

| $N_t$ | 56Ni | 60Zn | 64Ni | 68Ge | 72Zn | 72Se | 76Kr | 76Ge | 80Sr | 80Kr | 84Zr | 84Sr | 84Kr | 88Mo | 88Zr | 88Sr | 92Mo |
|-------|------|------|------|------|------|------|------|------|------|------|------|------|------|------|------|------|
| 0     |      |      |      |      |      |      |      |      |      |      |      |      |      |      |      |      |
| 2     | 56Ni | 60Zn | 64Ni | 68Ge | 72Zn | 72Se | 76Kr | 76Ge | 80Sr | 80Kr | 84Zr | 84Sr | 84Kr | 88Mo | 88Zr | 88Sr | 92Mo |
| 4     |      |      |      |      |      |      |      |      |      |      |      |      |      |      |      |      |      |
| 6     |      |      |      |      |      |      |      |      |      |      |      |      |      |      |      |      |      |
| 8     |      |      |      |      |      |      |      |      |      |      |      |      |      |      |      |      |      |
| 10    |      |      |      |      |      |      |      |      |      |      |      |      |      |      |      |      |      |
| 12    |      |      |      |      |      |      |      |      |      |      |      |      |      |      |      |      |      |
| 14    |      |      |      |      |      |      |      |      |      |      |      |      |      |      |      |      |      |
| 16    |      |      |      |      |      |      |      |      |      |      |      |      |      |      |      |      |      |
| 18    |      |      |      |      |      |      |      |      |      |      |      |      |      |      |      |      |      |

Table 1. Nuclei from the (28, 28|50, 50|)\_ shell mapped on the $\mathcal{H}_-$ ($N_t$-odd) subspace of $Sp(4,R)$. Each nucleus is defined by the total number of bosons $N_t$ or boson holes $\bar{N}_t$, which label the rows from the left and from the right side respectively and the third projections $F_0$ ($\bar{F}_0$) of the $F-$ spin, which label the columns on the top and bottom, respectively.
of the $F$-spin. This is illustrated on Table 1 for the even-even nuclei from the major shell with $28 < Z = N < 50$, labeled as (28, 28[50], 50)).

In the application of this classification scheme presented below [7], a component of the nuclear mass, called the “semiempirical microscopic mass” ($SEM$, where $SEM = M_{\text{exp}} - M_{\text{macr}}$) is investigated empirically for all the nuclei from the shell (28, 28[50], 50). The $SEM$ values are defined as the differences between the experimental masses $M_{\text{exp}}$ [8] and the calculated [9] spherical macroscopic mass $M_{\text{macr}}$. The structure independent macroscopic part $M_{\text{macr}}$ is evaluated explicitly for spherical shapes through the dependence on the mass number $A$, neutron, $N$, or proton, $Z$, numbers of the nucleus considered. Obviously the residual microscopic part $SEM$ of the nuclear masses contains the structure and deformation dependence, which is best studied in the above mentioned systematics related to the valence nuclear shells. We start with an empirical investigation of the behavior of the $SEM$s as functions of $F_0$ for extended multiplets containing all of the even as well as odd $A$ nuclei from the major shell (28, 28[50], 50). The plots of $SEM$ versus $F_0$ (see Fig. 1) show a clearly expressed parabolic behavior in the isotopic chains. The parabolas form a dome with rather evenly spread points.

Figure 1. Experimental values of $SEM$s (shapes) in the isotopic chains (l) of the (28, 28[50], 50) shell plotted versus $F_0^{l;i}$ and their parabolic approximations (full lines) with function (5).

Next we obtain a general functional dependence of the $SEM_l$ on $F_0^{l;i}$ (Fig. 1) in the isotopic chains (enumerated by $l$) of the considered shell under consideration. Obviously the most convenient fitting function is a parabolic one:

$$SEM(F_0^{l;i}) = C_0^l + C_1^l F_0^{l;i} + C_2^l (F_0^{l;i})^2, \quad l = 0, 1, 2, ..., 16.$$  (5)

The values of the parameters $C_0^l$, $C_1^l$ and $C_2^l$ were determined by means of a fitting procedure for each of the isotopic chains with their respective uncertainties. The latter were further approximated with second order polynomials in the number of valence protons $N^l_n = Z_l - 28$, defining the isotopic chains:

$$C_i^l = D_0^i + D_1^i N^l_n + D_2^i N^l_n^2, \quad i = 0, 1, 2.$$  (6)
The isotopic chains with $N^i_z > 11$ were not included in the second step of the fitting procedure. In this way, the 9 coefficients $D^i_k$, $k = 0, 1, 2$; $i = 0, 1, 2$ (6) given in Table 2 with their respective uncertainties $\Delta D^i_k$, were determined.

As a result, the general expression (5) for the SEMs of 188 nuclei from the (28, 28)50, 50) shell, depending on 9 phenomenological parameters defined in (6) and with values given in Table 2 was obtained.

To test the accuracy and predictive power of the generalized formula (5), we compare the theoretical values of the nucleonic masses $M_{\text{predicted}} = \text{SEM}(F_0^{(i)}) + M_{\text{macro}}$ to the experimental $M_{\text{Exp}}$ ones [8]. In Fig. 2, the differences $|M_{\text{Exp}} - M_{\text{predicted}}|$ are plotted and compared with the differences $|M_{\text{Exp}} - M_{\text{Moller}Nix}|$, where $M_{\text{Moller}Nix}$ are the masses obtained by the rather elaborate evaluation of Moller and Nix [9]. The results obtained from the parabolic dependence of SEMs as a function of $F_0$ (Fig. 1) are closer to the experimental ones than those of Moller and Nix for about 75% of the nuclei considered.

In summary, all the nuclei in the $A = 80$ region with $N = Z = 28 - 50$ can be described quite well by one phenomenological formula with 9 parameters. And in turn, the formula can be used to predict unknown masses of the nuclei from this region.

4. The $k = 3$ case and the symplectic extension of the Interacting Vector Boson Model

The $k = 3$ case of general reductions of boson representations of the $Sp(4k, R)$ (2), (3) algebra will be illustrated through the example of the Interacting Vector Boson Model (IVBM) [3]. The main assumption of the model is that the nuclear dynamics can be described by means of two types of vector “quasiparticles”, which are also characterized by another quantum number – a “T - spin” (an analogue of the $F$-spin). The non-compact symplectic group $Sp(12, R)$ appears as the group of dynamical symmetry for the problem of two interacting vector bosons. The symplectic extension of the $U(6)$ algebra, which was the dynamical symmetry of the number preserving version of the model [10] allows the change in the number of phonons, needed to build the collective states, that results in larger model spaces, which can accommodate the more complex structural effects observed in the contemporary experiment.

Applications of the symplectic extension of the IVBM, are obtained by exploiting new chains in the reduction of $Sp(12, R)$ to the angular momentum subgroup $SO(3)$. The corresponding exactly solvable limiting cases are applied to achieve a description of complex nuclear collective spectra of even-even nuclei in the rare earth and actinide regions up to states of very high angular momentum. This generalized reduction scheme links the three dynamical symmetry “pathways” from $Sp(12, R)$ to $SO(3)$ and their respective physical interpretations. In mathematical terms, the relations in Fig. 3 are based on the appearance of the physically important $U(2)$ [11] group of the “T-spin” as the maximal compact subgroup of $Sp(4, R)$ [12] (4), as well as its non-compact $Sp(2, R)$ (or $SU(1, 1)$) counterpart [13].

The first reduction that we exploit is one that extends the rotational limit of the number preserving version of the model; namely, $Sp(12, R) \supset U(6) \supset U(2) \otimes U(3)$ [11]. Another limit of the symplectic IVBM, $Sp(12, R) \supset Sp(2, R) \otimes SO(6)$, contains in a natural way the 6-dimensional Davidson potential [13]. A common feature of these applications is the possibility of mixing with
Figure 3. Generalized reduction scheme of the symplectic extension of IVBM.

varying strengths the two main collective modes – vibrational and rotational, which results in an accurate description even of nuclei at the critical points of phase/shape transitions in the framework of these exactly solvable cases, which is illustrated in Fig. 4. It has been established [3] that the two reduction schemes, which describe the developments of collective bands in various types of nuclear spectra, one through $U(6)$ [11] and one through $SO(6)$, [13] although using different realizations of the basis states and the Hamiltonians, because of their connection through the content of the $Sp(4, R)$ (4), yield very similar applications (see Figs. 4 and 5) for a description of the ground bands and the excited positive and negative parity bands. This reveals the important role of the structure of the band-heads, in particular the number of

Figure 4. Comparison of the $U(6)$ limit theoretical and experimental energies of the states from several collective positive and negative parity bands of the $^{152}Sm$ nucleus at the critical point $X(5)$ symmetry.

Figure 5. Comparison of theoretical energies calculated in the $Sp(2,R) \otimes SO(6)$ limit of the IVBM with experimental energies for ground and excited bands of the $^{162}Dy$. 
bosons from which they are built, for the development of the excited bands, a feature that is due to the consideration of the symplectic extension of the model. The structure of band-head configurations, whose importance is established in the first two limits, is examined in the third reduction, $Sp(12, R) ⊃ Sp(4, R) \otimes SO(3)$ [12]. In this respect the energy distribution of states with fixed angular momenta, and in particular for low-lying ones with $L = 0^+_1, 2^+_1, 4^+_1$, most of which are band-head configurations, obtained in the connecting $Sp(4, R)$ [12], provide a tool to obtain their corresponding $N_L$-values upon which the excited bands can be build. From a physical point of view this structure also enables one to distinguish typical collective vibrational and rotational spectra. By means of this vertical structure (the second column in (3)) the dynamical symmetries describing the ground and excited bands are connected with the dynamical symmetry describing the sets of states with fixed angular momentum.

Because of the inclusion relations between the subgroups in the first and second rows of (3) and the relation between the second order Casimir operators of the mutually complementary groups, $SU(3)$ and $SU(2)$, we use the same Hamiltonian and basis as in the $U(6)$ limit [11], but in this case the Hamiltonian eigenvalues give the energies of the states with a fixed value of $L$ with respect to $N$. In the former the dependence of the energies of the collective states on the number of phonons (vector bosons) $N$ is parabolic. All the rest of the quantum numbers defining the states $T, T_0$ and $L$ are expressed in terms $N$ by means of the reduction procedure given in [12]. This result confirms the conclusions of the empirical investigation of the states with fixed angular momentum [14], that their energies are well described by the simple phenomenological formula $E_L(n) = an - bn^2$, where $a > 0$ and $b > 0$ are fitting parameters and $n = N_L/4$ is an integer number corresponding to each of the states with given $L_i$. The examples presented in Fig. 6 show that the procedure is accurate and appropriate for obtaining the values of the $N_i$ from which the excited $0^+$ band-heads can be obtained. As a demonstration of the predictive capability of this approach a second $0^+$ excited state with energy 0.6813 MeV was experimentally

![Figure 6](image_url)

**Figure 6.** Parabolic distribution of the excited $K^\pi = 0^+$ states in $^{180}W$, $^{152}Gd$, $^{176}Hf$ and $^{162}Dy$ [16].
confirmed in the $^{160}Dy$ nucleus [15].

While a major appeal of this approach is the simplicity of its application in that it only requires the determination of a relatively small number of model parameters that can be achieved through fits to experimental data, its real power lies in its ability to predict the energetics of excited bands. This predictive capacity is related to the symplectic extension, which allows it to be used to identify the boson number of other band-head configurations and to predict the energies of states that belong to these bands. In order to obtain even greater predicting power of the symplectic extension of the IVBM, we need a further systematic investigation of the behavior of the model parameters as functions of the specific nuclear characteristics [17], as well as of the energy distributions of the band-head states in sequences of nuclei, which is a future aim. So, the symplectic extension of the IVBM permits a fuller classification of states than its unitary version and can be applied to rather diverse nuclear spectra. Another important characteristic of the model it is ability to yield a correct description of experimental energies when interactions between the proton and neutron subsystems are prominent, while still retaining the exact analytic nature of solutions in each of the applications considered.

5. The $k = 6$ case and the generalized symplectic dynamical symmetry of the Interacting Boson Model – 2

The building blocks of the IBM model [18] are the boson creation and annihilation operators $s^i, s$ and $d_{m}^l, d_m$ ($m = 0, \pm 1, \pm 2$), representing pairs of valence nucleons coupled to $l = 0$ and $l = 2$ respectively. The algebra that describes these systems is the $U(6)$ algebra. The version of the model, denoted as IBM-2, that considers the proton and neutron nuclear subsystems has dynamical symmetry represented by the direct product of the two algebras $U(6) \times U(6)$, which is obviously contained in $U(12)$ [19]. Based on the simple classification properties of the $Sp(4, R)$ group, outlined in Section 3, and the interpretation of the reduction operators in terms of the IBM-2 [4], it is straight forward to realize the symplectic extension of the group of dynamical symmetry of the model $U(12) \supset Sp(24, R)$ [1]. In this extension of the model, and in analogy with the generalized reduction scheme of the IVBM, (Fig. 3) we can explore the reduction scheme of the symplectic extension of IBM-2, given in Fig. 7. The formal algebraic aspects of this construction are in a close analogy with the particle-hole version of the IBM and

![Diagram](image-url)
details on it can be found in [20].

From the $u(6)$ algebra down to the algebra of the angular momentum one can proceed with the reductions defining the three limiting cases of IBM-1 [18] through the $U(5)$, $U(3)$ and $O(6)$, which have the anharmonic vibrator, the axial rotor and the $\gamma$-unstable rotor as geometrical analogs. Actually the $O(6)$ multiplier of the algebra of $Sp(4, R)$ on the second order of (7) is exactly the one that will start the respective limit of the model. It was shown in [21] that using an infinite dimensional algebraic technique based on the relationship of the quantum numbers of representations and the second order Casimir invariants of the bases $U(5) \supset SO(5)$ and $SO(6) \supset SO(5)$ with the $SU^d(1, 1) \supset U(1)$ and the $SU^d(1, 1) \supset U(1)$ respectively, exact analytic solutions can be obtained for the $O(6) \leftrightarrow U(5)$ transitional cases.

In this reduction scheme (Fig. 7) we have, as for the $k = 3$ case of the IVBM, the vertical structure in the reduction of $Sp(4, R)$ (4), but in this case related to the classification of all the even-even nuclei from a given major nuclear shell. This follows from the physical interpretation of the reduction operators as the operators of the total number of valence bosons (proton and neutron pairs) $- N = (N_p + N_n)$, the valence isospin $- F = \frac{N_p - N_n}{2}$ and its third projection $F_0 = \frac{1}{2}(N_p - N_n)$. This construction involving a classification group in a larger dynamical group allows us to treat in a unified way the properties of sequences of nuclei. Furthermore, a way that is similar to the considerations in [21], it allows us to obtain analytic solutions for the energy spectrum and the transition operators of sequences of nuclei defined by the classification quantum numbers.

In this regard we are motivated by the empirical investigation of the experimental energies of the ground state bands of all even-even nuclei with $A > 20$ [6], based on the $Sp(4, R)$-classification scheme [1]. This reveals a smooth and periodic behavior of the energies as classified in fixed $F_0$ multiplets for changing values of $N_t$, which yields a generalized phenomenological description by means of the expression

$$E_L = \beta L (L + \Omega) = \beta L (L + 1) + \beta L(\Omega - 1),$$

where the inertial parameter $\beta$ is evaluated as a function of the classification quantum numbers and a parameter $\Omega$ in the geometrical part of the interaction is introduced to account for the mixing of rotational and vibrational degrees of freedom. In a more consistent application of the reduction scheme (Fig. 7), the interactions can be prescribed by the respective dynamical symmetry, as suggested in [6].

6. Conclusions

Based on general reduction schemes for boson representations of symplectic algebras of the type $Sp(4k, R)$ [2], we first considered applications of the simplest $k = 1$ case. Specifically, we showed that the $Sp(4, R)$ algebra is very convenient for classifying many-body nuclear systems within a major nuclear shell when we use an interpretation of the reduction operators in terms of IBM-2 bosons ([19]), representing pairs of valence protons and neutrons, the constituent particles of any nuclei. This interpretation was advanced for a generalized description [6], [1] of nuclear properties based on the quantum labels of the associated classification scheme. The scheme was illustrated through the example of a unified phenomenological formula with only 9 parameters for reproducing $SEMs$ within the major $28 < Z = N < 50$ shell as functions of the third projection $F_0$ of the $F$–spin. It was also shown to have predictive capability for yet-to-be observed structures like excited band-head configurations and states within these bands [7].

Next we considered the $k = 3$ case by introducing the symplectic extension of the dynamical symmetry $U(6) \supset Sp(12, R)$ of the IVBM, that is constructed by means of two types of vector bosons and used it to describe the collective modes and their interactions of heavy even-even nuclear systems [3]. This extension yields a rich subgroup structure of $Sp(12, R)$ within which
some new non-compact subgroup structures appear. The reduction through the direct product $Sp(4, R) \otimes SO(3)$ [12] is of particular importance as it is not only used to describe sequences of states with fixed angular momentum, but also to elucidate the connection of this chain with the other dynamical symmetries of the model through its $SU(2)$ and $SU(1, 1)$ subgroups that are also substantial parts of the two other dynamical symmetries considered. In this way it plays the role of a generalized group classification scheme, one that orders (distributes) the collective excitations in spectra of individual nuclei in terms of the collective (boson) structure of their band-head configurations.

And finally we explored a generalized reduction scheme for the symplectic extension ($k = 6$) of the proton-neutron version of the IBM-2 [19]. A point of interest in this case is that the $k = 1$ results reappear with the same interpretation of the $Sp(4, R)$ structure and reductions as discussed in [1]. This limit of the theory provides us with an opportunity to describe, within the framework of the $Sp(24, R)$ dynamical symmetry, the development of collectivity across an entire nuclear shell, allowing for an investigation of transitions between the different limiting cases, while at the same time retaining the strategic advantage of dynamical symmetries for obtaining exact analytic solutions. This approach can also provide for an algebraic evaluation of critical point features, such as phase/shape transitions, in terms of the nuclear characteristics employed in their classification.

In summary, we explored the richer possibilities that the symplectic extensions of unitary algebras provide, and made use of their classification properties in order to achieve generalized descriptions of the nuclear collective behavior.

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