Evaluating Two Determinants

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Abstract

This article evaluates the determinants of two classes of special matrices, which are both from a number theory problem. Applications of the evaluated determinants can be found in [arXiv:math.NT/0509523, 2005].

Note that the two determinants are actually special cases of Theorems 20 and 23 in [arXiv:math.CO/9902004], respectively. Since this paper does not provide any new results, it will not be published anywhere.

1 The Determinants

Theorem 1 Assume $m \geq 1$. Given a $2m \times 2m$ matrix $A = \begin{bmatrix} A_1 \\ A_2 \end{bmatrix}$, where $A_1 = [X_i^{j-1}]_{1 \leq i \leq m}^{1 \leq j \leq 2m}$ and $A_2 = [jX_i^{j-1}]_{1 \leq i \leq m}^{1 \leq j \leq 2m}$, i.e.,

$$A = \begin{bmatrix}
1 & X_1 & \cdots & X_1^{m-1} & X_1^m & \cdots & X_1^{2m-1} \\
1 & X_2 & \cdots & X_2^{m-1} & X_2^m & \cdots & X_2^{2m-1} \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
1 & X_m & \cdots & X_m^{m-1} & X_m^m & \cdots & X_m^{2m-1} \\
1 & 2X_1 & \cdots & mX_1^{m-1} & (m+1)X_1^m & \cdots & 2mX_1^{2m-1} \\
1 & 2X_2 & \cdots & mX_2^{m-1} & (m+1)X_2^m & \cdots & 2mX_2^{2m-1} \\
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
1 & 2X_m & \cdots & mX_m^{m-1} & (m+1)X_m^m & \cdots & 2mX_m^{2m-1} \\
\end{bmatrix}.$$ 

Then, $|A| = (-1)^{m(m-1)/2} \prod_{i=1}^{m} X_i \prod_{1 \leq i < j \leq m} (X_j - X_i)^4$.

Corollary 1 Assume $m \geq 1$. Given a $2m \times 2m$ matrix $A = \begin{bmatrix} A_1 \\ A_2 \end{bmatrix}$, where $A_1 = [X_i^{j+1}]_{1 \leq i \leq m}^{1 \leq j \leq 2m}$ and $A_2 = [(j+1)X_i^{j-1}]_{1 \leq i \leq m}^{1 \leq j \leq 2m}$. Then, $|A| = (-1)^{m(m-1)/2} \prod_{i=1}^{m} X_i^4 \prod_{1 \leq i < j \leq m} (X_j - X_i)^4$.

Proof: Factoring out $X_i^2$ from each row of $A_1$ and factoring out $X_i$ from each row of $A_2$, one has $A_1^{(1)} = [X_i^{j-1}]_{1 \leq i \leq m}^{1 \leq j \leq 2m}$ and $A_2^{(1)} = [(j+1)X_i^{j-1}]_{1 \leq i \leq m}^{1 \leq j \leq 2m}$. Then, for $i = 1 \sim m$, subtracting row $i$ of $A_1^{(1)}$ from row $i$ of $A_2^{(1)}$, one has $A_2^{(2)} = [jX_i^{j-1}]_{1 \leq i \leq m}^{1 \leq j \leq 2m}$. From Theorem 1, one immediately gets

$$|A| = \prod_{i=1}^{m} X_i^3 \left((-1)^{m(m-1)/2} \prod_{i=1}^{m} X_i \prod_{1 \leq i < j \leq m} (X_j - X_i)^4\right) = (-1)^{m(m-1)/2} \prod_{i=1}^{m} X_i^4 \prod_{1 \leq i < j \leq m} (X_j - X_i)^4.$$ 

$\blacksquare$
Theorem 2 Assume $m \geq 1, n \geq l \geq 1$ and $A$ is a block-wise $ml \times ml$ matrix as follows:

$$A = \begin{bmatrix} A_1 \\ A_2 \\ \vdots \\ A_m \end{bmatrix},$$

where for $i = 1 \sim m$,

$$A_i = \begin{bmatrix} (n+j-1)X_i^{j-1} \\ \vdots \\ 0 \end{bmatrix}_{1 \leq j \leq ml}.$$

Then, $|A| = \prod_{i=1}^{m} X_i^{(ml-1)} \prod_{1 \leq i < j \leq m} (X_j - X_i)^2$.

2 The Proofs

2.1 Proof of Theorem 1

Proof: We use mathematical induction on $m$ to prove this theorem.

1) When $m = 1$, $A = \begin{bmatrix} 1 & X_1 \\ 1 & 2X_1 \end{bmatrix}$. Directly calculating the determinant, $|A| = X_1 = (-1)^{(m-1)/2} X_1$.

2) Assume this theorem is true for $m-1 \geq 1$, let us prove the case of $m \geq 2$.

For $j = 2 \sim 2m$, subtracting $X_1$ times of column $(j-1)$ from column $j$ of $A$, one gets

$$A_{i}^{(1)} = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 1 & 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & X_1 & X_2 & \cdots & X_{2m} \\ 1 & 1 & 1 & \cdots & 1 \end{bmatrix},$$

where $A_{i}^{(1)}$ and $A_{i}^{(2)}$ are both $(m-1) \times (2m-1)$ matrices: $A_{i}^{(1)} = [X_j^{j-1}X_{i+1} - X_1]_{1 \leq i \leq m-1}$ and $A_{i}^{(2)} = [jX_j^{j-1}X_{i+1} + X_j^{j}]_{1 \leq i \leq m-1}$. Apparently, $|A|$ is equal to the determinant of the following $(2m-1) \times (2m-1)$ matrix:

$$A^{(1)} = \begin{bmatrix} A_{1}^{(1)} \\ A_{2}^{(1)} \\ \vdots \\ X_1 \end{bmatrix}.$$

Moving the row matrix between $A_{i}^{(1)}$ and $A_{i}^{(2)}$ to the top of $A^{(1)}$, one get another matrix $A^{(2)}$ and has $|A| = (-1)^{m-1}|A^{(2)}|$. Factoring $(X_{i+1} - X_1)$ out from each row of $A_{i}^{(1)}$, one gets a new sub-matrix $A_{i}^{(3)} = [X_j^{j-1}X_{i+1}]_{1 \leq i \leq m-1}$ and

$$A^{(3)} = \begin{bmatrix} X_1 & X_2 & \cdots & X_{2m} \\ A_{1}^{(3)} \\ A_{2}^{(1)} \end{bmatrix}.$$
2.2 Two Proofs of Theorem 2

In this subsection, we give two inductive proofs of this theorem, one uses induction on \( m \) and another uses induction on \( n \). The two proofs are based on the same idea of reducing the matrix, though the first proof is simpler in organization and understanding.

2.2.1 The First Proof (Induction on \( m \))

We first prove a lemma to simplify the first proof of Theorem 2. This lemma is actually a special case of the theorem under study when \( m = 1 \) and \( X_1 = 1 \).

**Lemma 1** When \( 1 \leq m \leq n \), the determinant of the \( m \times m \) matrix \( A_{n,m} = \binom{n+j-1}{i-1} \binom{n+i-1}{j} \) is always equal to 1.

**Proof:** We use induction on \( m \) to prove this lemma.

1) When \( m = 1 \), \( A_{n,1} = \binom{n}{0} = [1] \). It is obvious that \( |A_{n,1}| = 1 \).

2) Suppose this lemma is true for \( m - 1 \geq 1 \), let us prove the case of \( m \geq 2 \). Write \( A_{n,m} \) as follows:

\[
\begin{bmatrix}
1 & 1 & \cdots & 1 \\
2 & 3 & \cdots & n+1 \\
\begin{array}{c}
\binom{n}{1} \\
\binom{n}{2} \\
\vdots \\
\binom{n}{m-1}
\end{array}
& \begin{array}{c}
\binom{n+1}{1} \\
\binom{n+1}{2} \\
\vdots \\
\binom{n+1}{m-1}
\end{array}
& \cdots & \begin{array}{c}
\binom{n+m-1}{1} \\
\binom{n+m-1}{2} \\
\vdots \\
\binom{n+m-1}{m-1}
\end{array}
\end{bmatrix}
\]

\[
(1)
\]
For $i = 2 \sim m$, subtract column $(i - 1)$ column from column $i$, one gets the following matrix:

$$
\begin{bmatrix}
1 & 0 & \cdots & 0 \\
n & 1 & \cdots & 1 \\
\binom{n}{2} & \binom{n+1}{2} & \cdots & \binom{n+m-1}{2} - \binom{n+m-2}{2} \\
\vdots & \vdots & \ddots & \vdots \\
\binom{n}{m-1} & \binom{n+1}{m-1} & \cdots & \binom{n+m-1}{m-1} - \binom{n+m-2}{m-1}
\end{bmatrix}.
$$

From the property of binomial coefficients [1], $\binom{i}{j} - \binom{i-1}{j} = \binom{i-1}{j-1}$, so the above matrix becomes

$$
\begin{bmatrix}
1 & 0 & \cdots & 0 \\
\binom{n}{1} & \binom{n+1}{1} & \cdots & \binom{n+m-1}{1} \\
\vdots & \vdots & \ddots & \vdots \\
\binom{n}{m-1} & \binom{n+1}{m-1} & \cdots & \binom{n+m-1}{m-1}
\end{bmatrix} = \begin{bmatrix} \binom{n}{2} \mathbf{A}_{n,m-1} \end{bmatrix}.
$$

Then, from the hypothesis, $|\mathbf{A}_{n,m}| = 1 \cdot |\mathbf{A}_{n,m-1}| = 1$. Thus this lemma is proved. ■

**The First Proof of Theorem 2:** In this proof, we use induction on $m$ to prove this theorem.

1) When $m = 1$ and $n \geq l \geq 1$, $\mathbf{A}$ is simplified into an $l \times l$ matrix as follows:

$$
\mathbf{A} = \begin{bmatrix}
\binom{n}{0} & \binom{n+1}{0} X_1 & \cdots & \binom{n+(l-1)}{0} X_1^{l-1} \\
\binom{n}{1} & \binom{n+1}{1} X_1 & \cdots & \binom{n+(l-1)}{1} X_1^{l-1} \\
\vdots & \vdots & \ddots & \vdots \\
\binom{n}{l-1} & \binom{n+1}{l-1} X_1 & \cdots & \binom{n+(l-1)}{l-1} X_1^{l-1}
\end{bmatrix}_{l \times l}.
$$

Factor the common terms in each column, the above matrix is reduced to be

$$
\tilde{\mathbf{A}} = \begin{bmatrix}
\binom{n}{0} & \binom{n+1}{0} & \cdots & \binom{n+(l-1)}{0} \\
\binom{n}{1} & \binom{n+1}{1} & \cdots & \binom{n+(l-1)}{1} \\
\vdots & \vdots & \ddots & \vdots \\
\binom{n}{l-1} & \binom{n+1}{l-1} & \cdots & \binom{n+(l-1)}{l-1}
\end{bmatrix}_{l \times l}.
$$

From Lemma 1, $|\tilde{\mathbf{A}}| = 1$, so $|\mathbf{A}| = X_1^{1+\cdots+(l-1)} |\tilde{\mathbf{A}}| = X_1^{\frac{1}{2}(l+1)}$, which is equal to $\prod_{i=1}^{l} X_i^{\frac{1}{2}(l+1)} \prod_{1 \leq i < j \leq l} (X_j - X_i)^2$ (the second term actually does not exist).

2) Suppose this theorem is true for $m - 1$ and $n \geq l \geq 1$, let us prove the case of $m \geq 2$ and $n \geq l \geq 1$.

Before starting this part, we give a brief introduction to the basic idea underlying the proof. The matrix $\mathbf{A}$ has a special feature after the following elementary matrix operations: for $j = 2 \sim ml$, subtracting column $j - 1$ multiplied by $X_1$ from column $j$, row 1 of $\mathbf{A}$ becomes $[1 \ 0 \ 0 \ \cdots \ 0]$. Then, one can remove row 1 and column 1 from $\mathbf{A}$ and reduce $\mathbf{A}$ in some way. Repeat this process for $n$ rounds, $\mathbf{A}_1$ can be completely removed from $\mathbf{A}$, which means that the value of $m$ decreases by one and the hypothesis can be applied to prove the result of $m \geq 2$ and $n \geq l \geq 1$.

In the following, let us see how to reduce the matrix in the first round of the process. Here, to achieve a clearer description of the process, we use bracketed superscripts with increased digits to denote the new matrices, each sub-matrices, and their elements after different matrix operations (including reductions of the size). For example, $\mathbf{A}^{(1)}$ denotes the matrix obtained after the above subtractions, and $\mathbf{A}_i^{(1)} = \begin{bmatrix} a_{ij,k}^{(1)} \end{bmatrix}_{1 \leq i \leq m, \ 1 \leq j \leq n}^{1 \leq k \leq m}$ denotes the $i$-th sub-matrix of $\mathbf{A}^{(1)}$. Specially the original matrix is always written as $\mathbf{A}$ (without any superscript) and its sub-matrix as $\mathbf{A}_i$. 


For $j = 2 \sim ml$, multiplying column $j - 1$ by $X_1$ and subtract it from column $j$, the element of $A_i^{(1)}$ at position $(j, 1)$ becomes $a_{i,j,1}^{(1)} = X_i^{j-1} - X_i^{j-2}X_1 = X_i^{j-2}(X_i - X_1)$ and the element at position $(j, k \geq 2)$ becomes:

$$a_{i,j,k}^{(1)} = \binom{n + j - 1}{k - 1} X_i^{j-1} - \binom{n + j - 2}{k - 1} X_i^{j-2}X_1$$

$$= \left( \binom{n + j - 2}{k - 1} + \binom{n + j - 2}{k - 2} \right) X_i^{j-1} - \binom{n + j - 2}{k - 1} X_i^{j-2}X_1$$

$$= \binom{n + j - 2}{k - 1} X_i^{j-2}(X_i - X_1) + \binom{n + j - 2}{k - 2} X_i^{j-1}.$$

When $i = 1$, the above elements become: $a_{1,j,1}^{(1)} = 0$ and $a_{1,j,k}^{(1)} = \binom{n + j - 2}{k - 2} X_1^{j-1} (k \geq 2)$. So, row 1 of $A_1$ become $\begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix}$. Then, $|A|$ is equal to the determinant of the following $(ml - 1) \times (ml - 1)$ matrix after removing row 1 and column 1 of $A^{(1)}$:

$$A^{(2)} = \begin{bmatrix} A_1^{(2)} \\ A_2^{(2)} \\ \vdots \\ A_m^{(2)} \end{bmatrix},$$

where $A_1^{(2)} = \left[ \binom{n + j - 1}{k - 1} X_1^j \right]_{1 \leq j \leq mn - 1}^{1 \leq k \leq n - 1}$ is an $(ml - 1) \times (l - 1)$ matrix, and for $2 \leq i \leq m$, $A_i^{(2)}$ is an $(ml - 1) \times l$ matrix as follows:

$$A_i^{(2)} = \left[ a_{i,j,k}^{(2)} \right]_{1 \leq j \leq ml - 1}^{1 \leq k \leq l} = \left[ a_{i,j+1,k}^{(2)} \right]_{1 \leq j \leq ml - 1}^{1 \leq k \leq l} = \begin{bmatrix} X_i^{j-1}(X_i - X_1) \\ \vdots \\ \binom{n + j - 1}{k - 1} X_i^{j-1}(X_i - X_1) + \binom{n + j - 1}{k - 2} X_i^{j-1} \\ k \geq 2 \end{bmatrix}.$$

Then, let us reduce $A^{(2)}$ to be of the same form as $A$. For $A_1^{(2)}$, we simply factor out $X_1$ from each row and get $A_1^{(3)} = \left[ \binom{n + j - 1}{k - 1} X_1^j \right]_{1 \leq j \leq ml - 1}^{1 \leq k \leq l}$. Then, consider $A_i^{(2)}$ for $i \geq 2$. Factoring out $(X_i - X_1)$ from row 1, one gets $a_{i,j,1}^{(3)} = X_i^{j-1}$. Then, multiplying row 1 by $X_i$ and subtracting it from row 2, one has $a_{i,j,2}^{(3)} = \binom{n + j - 1}{k - 1} X_i^{j-1}(X_i - X_1)$. Then, factoring out $(X_i - X_1)$ from row 2, one gets $a_{i,j,2}^{(3)} = \binom{n + j - 1}{k - 1} X_i^{j-1}$. Repeat the above procedure for other rows, one can finally get $A_i^{(3)} = \left[ \binom{n + j - 1}{k - 1} X_i^{j-1} \right]_{1 \leq j \leq ml - 1}^{1 \leq k \leq l}$ and $(X_i - X_1)^n$ is factored out. Combining the above results, we have

$$|A| = X_1^{l-1} \prod_{2 \leq i \leq m} (X_i - X_1)^l |A^{(3)}|.$$ 

Note that the above equation becomes $|A| = \prod_{2 \leq i \leq m} (X_i - X_1)^l |A^{(3)}|$ when $l = 1$. Observing the $m$ sub-matrices, one can see that each sub-matrix is of the same form as the original one in $A$, except that row $l$ and column $ml$ are removed from $A_1$ and column $ml$ is removed from $A_i$ ($i \geq 2$).

Next, repeat the above process on $A^{(3)}$, we can finally get the following $(ml - 2) \times (ml - 2)$ matrix:

$$A^{(4)} = \begin{bmatrix} A_1^{(4)} \\ A_2^{(4)} \\ \vdots \\ A_m^{(4)} \end{bmatrix},$$

where for $A_i^{(4)} = \left[ \binom{n + j - 1}{k - 1} X_i^{j-1} \right]_{1 \leq j \leq ml - 2}^{1 \leq k \leq l - 2}$ and for $2 \leq i \leq m$, $A_i^{(4)} = \left[ \binom{n + j - 1}{k - 1} X_i^{j-1} \right]_{1 \leq j \leq ml - 2}^{1 \leq k \leq l}$. In addition, we also have

$$|A^{(4)}| = X_1^{l-2} \prod_{2 \leq i \leq m} (X_i - X_1)^l |A^{(3)}|.$$
one has

\[ |A^{(j)}| = X_1^{l-j} \prod_{2 \leq i \leq m} (X_i - X_1)^j |A^{(j-1)}|, \]

where \( A^{(j)} \) denotes the reduced matrix of size \((ml - j) \times (ml - j)\) obtained after the \( j \)-th round of the above process finishes, specially, \( A^{(1)} = A^{(3)} \) and \( A^{(2)} = A^{(4)} \).

After total \( l \) rounds of the above process, one finally gets an \((ml - l) \times (ml - l)\) matrix

\[
A^{(n)} = \begin{bmatrix}
A_2^{(n)} \\
A_3^{(n)} \\
\vdots \\
A_m^{(n)}
\end{bmatrix},
\]

in which the first sub-matrix \( A_1 \) is completely removed and all other sub-matrices are untouched. Apparently, now \( A^{(l)} \) is a matrix of the same kind with parameter \( m - 1 \) and \( l \).

Combining the relation between \(|A|\) and \( A^{(1)} \), and the relationships between \(|A^{(j)}|\) and \(|A^{(j-1)}|\) \(2 \leq j \leq l\), one has

\[
|A| = X_1^{(l-1)\cdots+1} \prod_{2 \leq i \leq m} (X_i - X_1)^l |A^{(l)}| = X_1^{i(l-1)+\cdots+1} \prod_{2 \leq i \leq m} (X_i - X_1)^i |A^{(i)}|.
\]

Then, applying the hypothesis for \( A^{(l)} \), we finally have

\[
|A| = \left( \prod_{2 \leq i \leq m} (X_i - X_1)^i \right) \cdot \left( \prod_{i=2}^{m} X_i^{i(l-1)+\cdots+1} \prod_{2 \leq i < j \leq m} (X_j - X_i)^{i} \right)
\]

\[
= \prod_{i=1}^{m} X_i^{i(l-1)+\cdots+1} \prod_{1 \leq i < j \leq m} (X_j - X_i)^{i}.
\]

This proves the case of \( m \geq 2 \) and \( n \geq l \geq 1 \).

From the above two cases, this theorem is thus proved.

**2.2.2 The Second Proof (Induction on \( l \))**

*The Second Proof of Theorem 2:* In this proof, we use induction on \( l \) to prove this theorem.

1) When \( l = 1 \), \( m \geq 1 \) and \( n \geq l \), \( A \) is simplified into an \( m \times m \) matrix as follows:

\[
A = \begin{bmatrix}
1 & X_1 & \cdots & X_1 \\
1 & X_2 & \cdots & X_2^{m-1} \\
\vdots & \vdots & \ddots & \vdots \\
1 & X_m & \cdots & X_m^{m-1}
\end{bmatrix}_{m \times m}
\]

This is a Vandermonde matrix, so \(|A| = \prod_{1 \leq i < j \leq m} (X_j - X_i)\) for \( m \geq 1 \) \([2, \S 4.4]\), which is equal to

\[
\prod_{i=1}^{m} X_i^0 \prod_{1 \leq i < j \leq m} (X_j - X_i) = \prod_{i=1}^{m} X_i^{i(l-1)+\cdots+1} \prod_{1 \leq i < j \leq m} (X_j - X_i)^{i}.
\]
2) Suppose this theorem is true for \( l-1, n \geq l \) and \( m \geq 1 \), let us prove the case of \( l \geq 2, n \geq l \) and \( m \geq 1 \).

Before starting this part, we give a brief introduction to the basic idea underlying the proof. The matrix \( \mathbf{A} \) has a special feature after the following elementary matrix operations: for \( i = 1 \sim m \) and \( j = 2 \sim ml \), subtracting column \( j - 1 \) multiplied by \( X_i \) from column \( j \), row 1 of \( \mathbf{A}_i \) becomes \( \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \end{bmatrix} \). After removing row 1 of each sub-matrix and column 1 of \( \mathbf{A} \), the whole matrix is reduced to be of size \( m(l - 1) \times m(l - 1) \) and each sub-matrix is reduced to be of size \( m(l - 1) \times (l - 1) \). More importantly, after a series of matrix operations, the matrix can be finally reduced to be a matrix of the same form as the original one (with only different size). As a result, we can then use the hypothesis on the case of \( l - 1 \) and \( m, n \) to prove the result on \( l \) and \( m, n \).

As the first step, for \( j = 2 \sim ml \), multiplying column \( j - 1 \) by \( X_1 \) and subtract it from column \( j \), let us see how the matrix can be reduced. In the following proof, to achieve a clearer description of the process, we use bracketed superscripts with increased digits to denote the new matrices, each sub-matrices, and their elements after different matrix operations (including reductions of the size). For example, \( \mathbf{A}^{(1)} \) denotes the matrix obtained after the above subtractions, and \( \mathbf{A}^{(1)}_{i,j,k} = \left[ a_{i,j,k}^{(1)} \right]_{\substack{1 \leq i \leq m \\ 1 \leq j \leq l}} \) denotes the \( i \)-th sub-matrix of \( \mathbf{A}^{(1)} \). Specially the original matrix is always written as \( \mathbf{A} \) (without any superscript) and its sub-matrix as \( \mathbf{A}_i \).

After the above subtraction transformations, the element of \( \mathbf{A}^{(1)}_i \) at position \((j,1)\) becomes \( a_{i,j,1}^{(1)} = X_i^{j-1} - X_i^{j-2}X_1 = X_i^{j-2}(X_i - X_1) \) and the element at position \((j,k \geq 2)\) becomes:

\[
a_{i,j,k}^{(1)} = \begin{pmatrix} n+j-1 \\ k-1 \end{pmatrix} X_i^{j-1} - \begin{pmatrix} n+j-2 \\ k-1 \end{pmatrix} X_i^{j-2} X_1 \\
\begin{pmatrix} n+j-2 \\ k-2 \end{pmatrix} X_i^{j-1} - \begin{pmatrix} n+j-2 \\ k-1 \end{pmatrix} X_i^{j-2} X_1 \\
\begin{pmatrix} n+j-2 \\ k-1 \end{pmatrix} X_i^{j-2}(X_i - X_1) + \begin{pmatrix} n+j-2 \\ k-2 \end{pmatrix} X_i^{j-1}.
\]

When \( i = 1 \), the above elements become: \( a_{1,j,1}^{(1)} = 0 \) and \( a_{1,j,k}^{(1)} = \begin{pmatrix} n+j-2 \\ k-2 \end{pmatrix} X_1^{j-1} \) \((k \geq 2)\). So, row 1 of \( \mathbf{A}_1 \) become \([1 \ 0 \ 0 \ \cdots \ 0] \). Then, \( |\mathbf{A}| \) is equal to the determinant of the following \((ml - 1) \times (ml - 1)\) matrix after removing row 1 and column 1 of \( \mathbf{A}^{(1)} \):

\[
\mathbf{A}^{(2)} = \begin{bmatrix}
\mathbf{A}_1^{(2)} \\
\mathbf{A}_2^{(2)} \\
\vdots \\
\mathbf{A}_m^{(2)}
\end{bmatrix},
\]

where \( \mathbf{A}_1^{(2)} = \left[ \begin{pmatrix} n+j-1 \\ k-1 \end{pmatrix} X_1^{j-1} \right]_{1 \leq j \leq m-1, 1 \leq k \leq l-1} \) is an \((ml - 1) \times (l - 1)\) matrix, and for \( 2 \leq i \leq m, \) \( \mathbf{A}_i^{(2)} \) is an \((ml - 1) \times l\) matrix as follows:

\[
\mathbf{A}_i^{(2)} = \left[ a_{i,j,k}^{(2)} \right]_{\substack{1 \leq j \leq m-1 \\ 1 \leq k \leq l}} = \left[ a_{i,j+1,k}^{(1)} \right]_{\substack{1 \leq j \leq m-1 \\ 1 \leq k \leq l}} \begin{pmatrix} X_i^{j-1}(X_i - X_1), & k = 1 \\
\begin{pmatrix} n+j-1 \\ k-1 \end{pmatrix} X_i^{j-1}(X_i - X_1) + \begin{pmatrix} n+j-1 \\ k-2 \end{pmatrix} X_i^{j-1}, & k \geq 2 \end{pmatrix}_{\substack{1 \leq j \leq m-1 \\ 1 \leq k \leq l}}.
\]

Then, let us reduce \( \mathbf{A}^{(2)} \) to be of the same form as \( \mathbf{A} \). For \( \mathbf{A}_1^{(2)} \), we simply factor out \( X_1 \) from each row and get \( \mathbf{A}_1^{(3)} = \left[ \begin{pmatrix} n+j-1 \\ k-1 \end{pmatrix} X_1^{j-1} \right]_{1 \leq j \leq m-1, 1 \leq k \leq l-1} \). Then, consider \( \mathbf{A}_i^{(2)} \) for \( i \geq 2 \). Factoring out \((X_i - X_1)\) from row 1, one gets \( a_{i,j,1}^{(3)} = X_i^{j-1} \). Then, multiplying row 1 by \( X_i \) and subtracting it from row 2, one has \( a_{i,j,2}^{(3)} = \begin{pmatrix} n+j-1 \\ k-1 \end{pmatrix} X_i^{j-1}(X_i - X_1) \). Then, factoring out \((X_i - X_1)\) from row 2, one gets \( a_{i,j,2}^{(3)} = \begin{pmatrix} n+j-1 \\ k-1 \end{pmatrix} X_i^{j-1} \). Repeat the above procedure for other rows, one can finally get \( \mathbf{A}_i^{(3)} = \left[ \begin{pmatrix} n+j-1 \\ k-1 \end{pmatrix} X_i^{j-1} \right]_{1 \leq j \leq m-1, 1 \leq k \leq l-1} \) and \((X_i - X_1)^{j} \) is factored out. Combining the above results, we have

\[
|\mathbf{A}| = X_1^{l-1} \prod_{2 \leq i \leq m} (X_i - X_1)^{l} |\mathbf{A}^{(3)}|.
\]
Note that the above equation becomes $|A| = X_1^{l-1} |A^{(3)}|$ when $m = 1$. Observing the $m$ sub-matrices, one can see that each sub-matrix is of the same form as the original one in $A$, except that row $l$ and column $ml$ are removed from $A_1$ and column $ml$ is removed from $A_i (i \geq 2)$.

Next, repeat the above process on $A^{(3)}$ after replacing $X_1$ by $X_2$. Due to the similarity of the whole process, we omit the details and finally get the following $(ml - 2) \times (ml - 2)$ matrix:

$$A^{(4)} = \begin{pmatrix}
A_1^{(4)} \\
A_2^{(4)} \\
\vdots \\
A_m^{(4)}
\end{pmatrix},$$

where for $1 \leq i \leq 2$, $A_i^{(4)} = \left( \binom{n+j-1}{k-1} X_i^{j-1} \right)_{1 \leq j \leq m-2}$, and for $3 \leq i \leq m$, $A_i^{(4)} = \left( \binom{n+j-1}{k-1} X_i^{j-1} \right)_{1 \leq j \leq m-2}$. In addition, we have $|A^{(4)}| = (-1)^{j-1} X_2^{j-1} (X_1 - X_2)^{l-1} \prod_{3 \leq i \leq m} (X_i - X_2)^{l-1} |A^{(3)}|$, where $(-1)^{j-1}$ is induced by the fact that $b^{(3)}_{i,j}$ is at the position of $(l,1)$ in the full matrix $A^{(3)}$ (note that $A_1^{(3)}$ has only $l-1$ rows).

Repeat the above procedure for $j = 3 \sim m$, one can get

$$|A^{(j)}| = (-1)^{(j-1)(l-1)} X_j^{l-1} \prod_{1 \leq i < j \leq m} (X_i - X_j)^{l-1} \prod_{j+1 \leq i \leq m} (X_i - X_j)^{l-1} |A^{(j-1)}|,$$

where $A^{(j)}$ denotes the reduced matrix of size $(ml - j) \times (ml - j)$ obtained after $j$ rounds of the above process, specially, $A^{(1)} = A^{(3)}$ and $A^{(2)} = A^{(4)}$. After total $m$ rounds of the above process, one finally gets an $(ml - m) \times (ml - m)$ matrix $A^{(m)}$, in which each sub-matrix is an $(ml - m) \times (l - 1)$ matrix defined by $A_i^{(m)} = \left( \binom{n+j-1}{k-1} X_i^{j-1} \right)_{1 \leq j \leq m, 1 \leq k \leq l - 1}$. Apparently, $A^{(m)}$ is a matrix of the same kind with parameter $l - 1$ and $m, n$.

Combining the relation between $|A|$ and $A^{(1)}$, and the relationships between $|A^{(j)}|$ and $|A^{(j-1)}|$ ($2 \leq j \leq m$), one has

$$|A| = (-1)^{(l-1)+\ldots+(m-1)} \prod_{i=1}^{m} X_i^{l-1} \prod_{1 \leq i < j \leq m} (X_i - X_j)^{l-1} \prod_{1 \leq i \leq j \leq m} (X_j - X_i)^{l-1} |A^{(m)}|$$

$$= (-1)^{(l-1)+\ldots+(m-1)(l-1)} \prod_{i=1}^{m} X_i^{l-1} \cdot \left( (-1)^{(l-1)+\ldots+(m-1)(l-1)} \prod_{1 \leq i < j \leq m} (X_j - X_i)^{l-1} \prod_{1 \leq i < j \leq m} (X_j - X_i)^{l-1} |A^{(m)}| \right)$$

$$= \prod_{i=1}^{m} X_i^{l-1} \prod_{1 \leq i < j \leq m} (X_j - X_i)^{2l-1} |A^{(m)}|.$$

Then, applying the hypothesis on $A^{(m)}$, we finally have

$$|A| = \left( \prod_{i=1}^{m} X_i^{l-1} \prod_{1 \leq i < j \leq m} (X_j - X_i)^{2l-1} \right) \cdot \left( \prod_{i=1}^{m} X_i^{l-1(l-2)} \prod_{1 \leq i < j \leq m} (X_j - X_i)^{(l-1)^2} \right)$$

$$= \prod_{i=1}^{m} X_i^{\frac{l(l-1)}{2}} \prod_{1 \leq i < j \leq m} (X_j - X_i)^{l^2}.$$

This proves the case of $l \geq 2$, $n \geq l$ and $m \geq 1$.

From the above two cases, this theorem is thus proved. 
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