MEAN-FIELD DOUBLY REFLECTED BACKWARD STOCHASTIC DIFFERENTIAL EQUATIONS

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ABSTRACT. We study mean-field doubly reflected BSDEs. First, using the fixed point method, we show existence and uniqueness of the solution when the data which define the BSDE are \( p \)-integrable with \( p = 1 \) or \( p > 1 \). The two cases are treated separately. Next by penalization we show also the existence of the solution. The two methods do not cover the same set of assumptions.

Keywords: Mean-field; Reflected backward SDEs; Dynkin game; Penalization.
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1. INTRODUCTION

In this paper we are concerned with the problem of existence and uniqueness of a solution of the doubly reflected BSDE of the following type: For any \( \xi \) depend on

\[
\begin{align*}
Y_t = \xi + \int_t^T f(s, Y_s, \mathbb{E}[Y_s], Z_s)ds + K_T^+ - K_T^- + \int_t^T Z_s dB_s; \\
h(t, \omega, Y_t, \mathbb{E}[Y_t]) \leq Y_t \leq g(t, \omega, Y_t, \mathbb{E}[Y_t]); \\
\int_0^T (Y_s - h(s, Y_s, \mathbb{E}[Y_s]))dK_s^+ = \int_0^T (Y_s - g(s, Y_s, \mathbb{E}[Y_s]))dK_s^- = 0 (K^\pm \text{ are increasing processes}).
\end{align*}
\]

It is said associated with the quadruple \( (f, \xi, h, g) \). Those BSDEs are of mean-field type because the generator \( f \) and the barriers \( h \) and \( g \) depend on the law of \( Y_t \) through its expectation. For simplicity reasons we stick to this framework, however it can be somewhat generalized (see Remark 3.6).

Since the introduction by Lasry and Lions [18] of the general mathematical modeling approach for high-dimensional systems of evolution equations corresponding to a large number of “agents” (the mean-field model), the interest to the mean-field models grows steadily in connection with several applications. Later standard mean-field BSDEs have been introduced in [3]. Since then, there have been several papers on mean-field BSDEs including (2, 4, 5, 6, 11, 23, 24, 25, etc) in relation with several fields and motivations in mathematics and economics, such as stochastic control, games, mathematical finance, utility of an agent inside an economy, PDEs, actuary, etc.

Mean-field one barrier reflected BSDEs have been considered first in the paper [23]. This latter generalizes the work in [1] to the reflected framework. Later Briand et al. [2] have considered another type of one barrier mean-field reflected BSDEs. Actually in [2], the reflection of the component \( Y \) of the solution holds in expectation. They show existence and uniqueness of the solution when the increasing process, which makes the constraint on \( Y \) satisfied, is deterministic. Otherwise the solution is not necessarily unique. The main motivation is the assessment of the risk of a position in a financial market.

In [11], Djehiche et al. consider the above problem (1.1) when there is only one reflecting barrier (e.g. take \( g \equiv +\infty \)). The authors show existence and uniqueness of the solution in several contexts of integrability of the data \( (f, \xi, h) \). The methods are the usual ones: Fixed point and penalization. Those methods do not allow for the same set of assumptions. For example, the fixed point method does not allow generators which depend on \( z \) while the penalization does, at the price of some additional regularity properties which are not required by the use of the first method. The main motivation for considering such a problem comes from actuary and namely the assessment of the prospective reserve of a representative contract in life-insurance (see e.g. [11] for more details). Later, there have been several works on this subject including (9, 10, 16).

In this paper we consider the extension of the framework of [11] to the case of two reflecting barriers. We first show existence and uniqueness of a solution of (1.1) by the fixed point method. We deal with the case when the data of the problem are only integrable or \( p \)-integrable with \( p > 1 \). Those cases are treated separately because one cannot deduce one of them from the other one. The generator \( f \) does not depend on \( z \) while the main requirement on \( h \) and \( g \) is only to be Lipschitz continuous with small enough Lipschitz constants (see condition (3.6)).
Later on, we use the penalization method to show the existence of a solution for (1.1) under an adaptation to our framework, of the well-known Mokobodski condition (see e.g. [7, 14], etc.) which plays an important role. Within this method, $f$ may depend on $z$ while the Lipschitz property of $h$ and $g$ is replaced with a monotonicity one. As a by-product, we provide a procedure to approximate the solution of (1.1) by a sequence of solutions of standard mean-field BSDEs.

The paper is organized as follows: In Section 2, we fix the notations and the frameworks. In Section 3, we deal with the case when $p > 1$ and finally with the case $p = 1$. Section 4 is devoted to the study of the penalization scheme which we show it is convergent and its limit provides a solution for (1.1). The adapted Mokobodski condition plays an important role since it makes that the approximations of the processes $K^\pm$ have mild increments and do not explode. As a by-product when the solution of (1.1) is unique, this scheme provides a way to approximate this solution.

2. Notations and formulation of the problems

2.1. Notations. Let $T$ be a fixed positive constant. Let $(\Omega, \mathcal{F}, \mathbb{P})$ denote a complete probability space with $B = (B_t)_{t \in [0,T]}$ a $d$-dimensional Brownian motion whose natural filtration is $(\mathcal{F}_t^0 := \sigma\{B_s, s \leq t\})_{0 \leq t \leq T}$. We denote by $\mathcal{F} = (\mathcal{F}_t)_{0 \leq t \leq T}$ the completed filtration of $(\mathcal{F}_t^0)_{0 \leq t \leq T}$ with the $\mathbb{P}$-null sets of $\mathcal{F}$, then it satisfies the usual conditions. On the other hand, let $\mathcal{P}$ be the $\sigma$-algebra on $[0, T] \times \Omega$ of the $\mathbb{F}$-progressively measurable sets.

For $p \geq 1$ and $0 \leq s_0 < t_0 \leq T$, we define the following spaces:

- $L^p := \{ \xi : F_T - \text{measurable random variable s.t. } \mathbb{E}[|\xi|^p] < \infty \}$;
- $\mathcal{H}^p_{\text{loc}} := \{(z_t)_{t \in [0,T]} : \mathcal{P} - \text{measurable process and } \mathbb{R}^d - \text{valued s.t. } \mathbb{E}\left(\int_0^T |z_t(\omega)|^2 ds\right)^{\frac{p}{2}} < \infty \}$;
- $\mathcal{H}^\infty_{\text{loc}} := \{(z_t)_{t \in [0,T]} : \mathcal{P} - \text{measurable process and } \mathbb{R}^m - \text{valued s.t. } \mathbb{P} - \text{a.s. } \mathbb{E}\left(\int_0^T |z_t(\omega)|^2 ds\right) < \infty \}$;
- $S^p := \{(y_t)_{t \in [0,T]} : \text{continuous and } \mathcal{P} - \text{measurable process s.t. } \mathbb{E}[\sup_{t \in [0,T]} |y_t|^p] < \infty \}; S^p(\{s_0, t_0\})$ is the space $S^p$ reduced to the interval $[s_0, t_0]$. If $y \in S^p(\{s_0, t_0\})$, we denote by $\|y\|_{S^p(\{s_0, t_0\})} := \mathbb{E}[\sup_{s \leq u \leq t_0} |y_u|^p]^{\frac{1}{p}}$;
- $\mathcal{A} := \{(k_t)_{t \in [0,T]} : \text{continuous, } \mathcal{P} - \text{measurable and non-decreasing process s.t. } k_0 = 0 \}; \mathcal{A}(\{s_0, t_0\})$ is the space $\mathcal{A}$ reduced to the interval $[s_0, t_0]$ (with $k_{s_0} = 0$);
- $S^2 := \{(y_t)_{t \in [0,T]} : \text{continuous and } \mathcal{P} - \text{measurable and non-decreasing process s.t. } \mathbb{E}[\sup_{t \in [0,T]} |y_t|^2] < \infty \}$;
- $T := \{\tau, \mathbb{F} - \text{stopping time s.t. } \mathbb{P} - \text{a.s. } t \leq \tau \leq T\};$
- $D := \{(\phi)_t \in [0,T] : \mathbb{F} - \text{adapted, } \mathbb{R} - \text{valued continuous process s.t. } ||\phi||_1 = \sup_{\tau \in [0,T]} \mathbb{E}[|y_\tau|] < \infty \}$. Note that the normed space $(D, ||\cdot||_1)$ is complete (e.g. [8], pp.90). We denote by $(D(\{s_0, t_0\}), ||\cdot||_1)$, the restriction of $D$ to the time interval $[s_0, t_0]$. It is a complete metric space when endowed with the norm $||\cdot||_1$ on $[s_0, t_0]$, i.e.,

$$||X||_{1, [s_0, t_0]} := \sup_{\tau \in [s_0, t_0]} \mathbb{E}[|X_\tau|] < \infty.$$ 

2.2. The class of doubly reflected BSDEs. In this paper we aim at finding $\mathcal{P}$-measurable processes $(Y, Z, K^+, K^-)$ which solves the doubly reflected BSDE of mean-field type associated with the generator $f(t, \omega, y, y')$, the terminal condition $\xi$, the lower barrier $h(y, y')$, and the upper barrier $g(y, y')$ when the data are $p$-integrable, in the cases $p > 1$ and $p = 1$ respectively. The two cases should be considered separately since one cannot deduce one case from another one. To begin with let us make precise definitions:

Definition 2.1. We say that the quaternary of $\mathcal{P}$-measurable processes $(Y_t, Z_t, K^+_t, K^-_t)_{t \leq T}$ is a solution of the mean-field reflected BSDE associated with $(f, \xi, h, g)$ if:

- $Y_T = \xi$;
- $Z_T = h(Y_T, Y'_T)$;
- $K^+_t = \sup_{s \leq t} \mathbb{E}[|Y_s|] < \infty$;
- $K^-_t = \sup_{s \leq t} \mathbb{E}[|Y_s|] < \infty$;
- $f(t, \omega, Y_t, Y'_t) = \mathbb{E}[f(t, \omega, Y'_t, Y'_t) | \mathcal{F}_t]$; 
- $g(Y_t, Y'_t) = \mathbb{E}[g(Y'_t, Y'_t) | \mathcal{F}_t]$; 
- $h(Y_t, Y'_t) = \mathbb{E}[h(Y'_t, Y'_t) | \mathcal{F}_t]$; 
- $|f(t, \omega, Y_t, Y'_t)|, |g(Y_t, Y'_t)|, |h(Y_t, Y'_t)|$ are $\mathcal{F}_t$-measurable.

Note that the terminal condition $\xi$ is a $\mathcal{P}$-measurable random variable, and $K^+_t$ and $K^-_t$ are non-negative $\mathcal{F}_t$-measurable processes, and $Y_t$ and $Y'_t$ are $\mathcal{F}_t$-measurable processes.
Case: $p > 1$

\[
\begin{aligned}
 & Y \in \mathcal{S}_p^p, \quad Z \in \mathcal{H}_{i.o.c}^d \quad \text{and} \quad K^+, K^- \in \mathcal{A}; \\
 & Y_t = \xi + \int_0^T f(s, Y_s, \mathbb{E}[Y_s]) ds + K^+_T - K^-_T - K^+_t - K^-_t - \int_t^T Z_s dB_s, \quad 0 \leq t \leq T; \\
 & h(t, Y_t, \mathbb{E}[Y_t]) \leq Y_t \leq g(t, Y_t, \mathbb{E}[Y_t]), \quad \forall t \in [0, T]; \\
 & \int_0^T (Y_s - h(s, Y_s, \mathbb{E}[Y_s])) dK^+_s = \int_0^T (Y_s - g(s, Y_s, \mathbb{E}[Y_s])) dK^-_s = 0.
\end{aligned}
\]

Case: $p = 1$

\[
\begin{aligned}
 & Y \in \mathcal{D}, \quad Z \in \mathcal{H}_{i.o.c}^d \quad \text{and} \quad K^+, K^- \in \mathcal{A}; \\
 & Y_t = \xi + \int_0^T f(s, Y_s, \mathbb{E}[Y_s]) ds + K^+_T - K^-_T - K^+_t - K^-_t - \int_t^T Z_s dB_s, \quad 0 \leq t \leq T; \\
 & h(t, Y_t, \mathbb{E}[Y_t]) \leq Y_t \leq g(t, Y_t, \mathbb{E}[Y_t]), \quad \forall t \in [0, T]; \\
 & \int_0^T (Y_s - h(s, Y_s, \mathbb{E}[Y_s])) dK^+_s = \int_0^T (Y_s - g(s, Y_s, \mathbb{E}[Y_s])) dK^-_s = 0.
\end{aligned}
\]

2.3. Assumptions on $(f, \xi, h, g)$. We now make precise requirements on the items $(f, \xi, h, g)$ which define the doubly reflected backward stochastic differential equation of mean-field type.

**Assumption (A1):**

(i) The coefficients $f, h, g$ and $\xi$ satisfy:

(a) $f$ does not depend on $z$ and the process $(f(t, 0, 0))_{t \leq T}$ is $\mathbb{P}$-measurable and such that $\int_0^T |f(t, 0, 0)| dt \in L^p$;

(b) $f$ is Lipschitz w.r.t $(y, y')$ uniformly in $(t, \omega)$, i.e., there exists a positive constant $C_f$ such that $\mathbb{P}$-a.s. for all $t \in [0, T], y_1, y_1', y_2$ and $y_2'$ elements of $\mathbb{R}$,

\[
|f(t, \omega, y_1, y_1') - f(t, \omega, y_2, y_2')| \leq C_f(|y_1 - y_1'| + |y_2 - y_2'|).
\]

(ii) $h$ and $g$ are mappings from $[0, T] \times \Omega \times \mathbb{R}^2$ into $\mathbb{R}$ which satisfy:

(a) $h$ and $g$ are Lipschitz w.r.t $(y, y')$, i.e., there exist pairs of positive constants $(\gamma_1, \gamma_2), (\beta_1, \beta_2)$ such that for any $t, x, x', y$ and $y' \in \mathbb{R}$,

\[
|h(t, \omega, x, x') - h(t, \omega, y, y')| \leq \gamma_1 |x - y| + \gamma_2 |x' - y'|,
\]

\[
|g(t, \omega, x, x') - g(t, \omega, y, y')| \leq \beta_1 |x - y| + \beta_2 |x' - y'|;
\]

(b) $\mathbb{P}$-a.s., $h(t, \omega, x, x') < g(t, \omega, x, x')$, for any $t, x, x' \in \mathbb{R}$;

(c) the processes $(h(t, \omega, 0, 0))_{t \leq T}$ and $(g(t, \omega, 0, 0))_{t \leq T}$ belong to $\mathcal{S}_p^p$ (when $p > 1$) and are continuous of class $\mathcal{D}$ (when $p = 1$).

(iii) $\xi$ is an $\mathcal{F}_T$-measurable, $\mathbb{R}$-valued r.v., $\mathbb{E}[|\xi|^p] < \infty$ and $\mathbb{P}$-a.s., $h(T, \xi, \mathbb{E}[\xi]) \leq \xi \leq g(T, \xi, \mathbb{E}[\xi])$.

3. Existence and Uniqueness of a Solution of the Doubly Reflected BSDE of Mean-Field Type

Let $Y = (Y_t)_{t \leq T}$ be an $\mathbb{R}$-valued, $\mathbb{P}$-measurable process and $\Phi$ the mapping that associates to $Y$ the following process $(\Phi(Y)_t)_{t \leq T}: \forall t \leq T$,

\[
\Phi(Y)_t := \text{ess sup} \sup \text{ess inf} \mathbb{E} \left\{ \int_t^{\omega \wedge T} f(s, Y_s, \mathbb{E}[Y_s]) ds + g(\sigma, Y_\sigma, \mathbb{E}[Y_\sigma]|=\sigma) \mathbb{1}_{\{\sigma < T\}} + h(\tau, Y_\tau, \mathbb{E}[Y_\tau]|=\tau) \mathbb{1}_{\{\tau \leq \sigma, \tau < T\}} + \xi \mathbb{1}_{\{\tau = T\}} \right\} |\mathcal{F}_t|.
\]

For the well-posedness of $\Phi(Y)$ one can see e.g. [19], Theorem 7.

The following result is related to some properties of $\Phi(Y)$. 
Lemma 3.1. Assume that assumptions (A1) are satisfied for \( p = 1 \) and \( Y \in \mathcal{D} \). Then the process \( \Phi(Y) \) belongs to \( \mathcal{D} \). Moreover there exist processes \( (\bar{Z}_t)_{t \leq T} \) and \( (\Delta_t^\pm)_{t \leq T} \) such that:

\[
\begin{align*}
\bar{Z} \in \mathcal{H}_{loc}^m, \Delta^\pm \in \mathcal{A}; \\
\Phi(Y)_t = \xi + \int_0^T f(s, Y_s, E[Y_s])ds + \Delta^+ - \Delta^- - \Delta^+ - \Delta^- - \int_0^T \bar{Z}_s dB_s, \ t \leq T; \\
\phi(t, Y_t, E[Y_t]) \leq \Phi(Y)_t \leq g(t, Y_t, E[Y_t]), \ t \leq T; \\
\int_0^T (\Phi(Y)_t - h(t, Y_t, E[Y_t]))d\Delta^+ = \int_0^T (\Phi(Y)_t - g(t, Y_t, E[Y_t]))d\Delta^- = 0.
\end{align*}
\]  

(3.1)

Proof. First note that since \( Y \in \mathcal{D} \) and \( g, h \) are Lipschitz then the processes \( (h(t, Y_t, E[Y_t]))_{t \leq T} \) and \( (g(t, Y_t, E[Y_t]))_{t \leq T} \) belong also to \( \mathcal{D} \). Next as \( h < g \) then, using a result by [15], Theorem 4.1 or [27], Theorem 3.1, there exist \( \mathcal{P} \)-measurable processes \( (Y_t)_{t \leq T}, (\bar{Z}_t)_{t \leq T} \) and \( (\Delta_t^\pm)_{t \leq T} \) such that:

\[
\begin{align*}
\mathcal{X} \in \mathcal{D}; \bar{Z} \in \mathcal{H}_{loc}^m, \Delta^\pm \in \mathcal{A}; \\
\Phi(Y)_t = \xi + \int_0^T f(s, Y_s, E[Y_s])ds + \Delta^+ - \Delta^- - \Delta^+ + \Delta^- - \int_0^T \bar{Z}_s dB_s, \ t \leq T; \\
\phi(t, Y_t, E[Y_t]) \leq \Phi(Y)_t \leq g(t, Y_t, E[Y_t]), \ t \leq T; \\
\int_0^T (\Phi(Y)_t - h(t, Y_t, E[Y_t]))d\Delta^+ = \int_0^T (\Phi(Y)_t - g(t, Y_t, E[Y_t]))d\Delta^- = 0.
\end{align*}
\]

Let us point out that in [27], Theorem 3.1, the result is obtained in the discontinuous framework, namely the obstacles are right continuous with left limits processes. However since in the case the processes \( (h(t, Y_t, E[Y_t]))_{t \leq T} \) and \( (g(t, Y_t, E[Y_t]))_{t \leq T} \) continuous then \( \mathcal{Y} \) and \( \Delta^\pm \) are continuous, and the frameworks of [15] and [27] are the same (one can see e.g. [21], pp.60). Finally the process \( \mathcal{Y} \) has the following representation as the value of a zero-sum Dynkin game: \( \forall t \leq T, \)

\[
\mathcal{Y}_t := \text{ess sup} \sup_{\sigma \geq t} \inf_{\tau \geq t} \mathcal{E}\{ \int_t^{\sigma \wedge \tau} f(s, Y_s, E[Y_s])ds + g(\sigma, Y_\sigma, E[Y_\sigma]_{t=\sigma})1_{(\sigma < T)} + h(\tau, Y_\tau, E[Y_\tau]_{t=\tau})1_{(\tau \leq \sigma < T)} + \xi 1_{(T = \sigma = T)} | F_t \}. \quad (3.2)
\]

Therefore \( \mathcal{Y} = \Phi(Y) \) and the claim is proved.

Remark 3.2. Note that we have also for any \( t \leq T, \)

\[
\begin{align*}
\mathcal{Y}_t := \text{ess inf} \inf_{\sigma \geq t} \sup_{\tau \geq t} \mathcal{E}\{ \int_t^{\sigma \wedge \tau} f(s, Y_s, E[Y_s])ds + g(\sigma, Y_\sigma, E[Y_\sigma]_{t=\sigma})1_{(\sigma < T)} + h(\tau, Y_\tau, E[Y_\tau]_{t=\tau})1_{(\tau \leq \sigma < T)} + \xi 1_{(T = \sigma = T)} | F_t \}. \quad (3.3)
\end{align*}
\]

3.1. The case \( p > 1 \). We will first show that \( \Phi \) is well defined from \( \mathcal{S}^p_Y \) to \( \mathcal{S}^p_Y \).

Lemma 3.3. Let \( f, h, g \) and \( \xi \) satisfy Assumption (A1) for some \( p > 1 \). If \( Y \in \mathcal{S}^p_Y \) then \( \Phi(Y) \in \mathcal{S}^p_Y \).

Proof. Let \( Y \in \mathcal{S}^p_Y \). For \( \sigma \) and \( \tau \) two stopping times, let us define:

\[
\begin{align*}
L(\tau, \sigma) = \int_0^{\tau \wedge \sigma} f(r, Y_r, E[Y_r])dr + g(\sigma, Y_\sigma, E[Y_\sigma]_{t=\sigma})1_{(\sigma < T)} + h(\tau, Y_\tau, E[Y_\tau]_{t=\tau})1_{(\tau \leq \sigma, T < \tau)} + \xi 1_{(T = \tau = \sigma)}.
\end{align*}
\]

Then for any \( t \leq T, \)

\[
\Phi(Y)_t + \int_0^t f(s, Y_s, E[Y_s])ds = \text{ess sup} \sup_{\sigma \geq t} \inf_{\tau \geq t} \mathcal{E}\{L(\tau, \sigma) | F_t\} = \text{ess inf} \inf_{\sigma \geq t} \sup_{\tau \geq t} \mathcal{E}\{L(\tau, \sigma) | F_t\}. \quad (3.4)
\]

As pointed out previously when \( Y \) belongs to \( \mathcal{S}^p_Y \) with \( p > 1 \), then it belongs to \( \mathcal{D} \). Therefore, under assumptions (A1), the process \( \Phi(Y) \) is continuous. On the other hand, the second equality in (3.4) is valid since by (A1)-(ii), (a)-(c), \( h < g \) and the processes \( (h(s, Y_s, E[Y_s]))_{s \leq T} \) and \( (g(s, Y_s, E[Y_s]))_{s \leq T} \) belongs to \( \mathcal{S}^p_Y \) since \( Y \) belongs to \( \mathcal{S}^p_Y \) (see e.g. [12] for more details).
Next let us define the martingale \( M := (M_t)_{0 \leq t \leq T} \) as follows:

\[
M_t := E \left\{ \int_0^T \left[ |f(s,0,0)| + C_f(|Y_s| + E[Y_s]) \right] ds + \sup_{s \leq t} |g(s,0,0)| + \beta_1 \sup_{s \leq T} |Y_s| + \beta_2 \sup_{s \leq T} E|Y_s| \\
+ \sup_{s \leq T} |h(s,0,0)| + \gamma_1 \sup_{s \leq T} |Y_s| + \gamma_2 \sup_{s \leq T} E|Y_s| + |\xi| |\mathcal{F}_t| \right\}.
\] (3.5)

As \( Y \) belongs to \( S^p \) and by Assumptions (A1), the term inside the conditional expectation belongs to \( L^p(d\mathcal{P}) \). As the filtration \( \mathcal{F} \) is Brownian then \( M \) is continuous and by Doob’s inequality with \( p > 1 \) (25, pp.54) one deduces that \( M \) belongs also to \( S^p \). Next as \( f, g \) and \( h \) are Lipschitz, then by a linearization procedure of those functions one deduces that:

\[
|E[\mathcal{L}(\tau,\sigma)|\mathcal{F}_t]| \leq M_t
\]

for any \( t \leq T \) and any stopping times \( \sigma, \tau \in \mathcal{T}_t \). Then we obtain

\[
\forall t \leq T, \ |\Phi(Y)_t + \int_0^t f(s,Y_s,E[Y_s])ds| \leq M_t.
\]

Therefore,

\[
E\{\sup_{t \leq T} |\Phi(Y)_t|^p \} \leq C_p \left\{ E \left( \int_0^T |f(s,Y_s,E[Y_s])| ds \right)^p + E\{\sup_{t \leq T} |M_t|^p \} \right\},
\]

where \( C_p \) is a positive constant that only depends on \( p \) and \( T \). It holds that \( \Phi(Y) \in S^p_\mathcal{P} \) since \( Y \in S^p_\mathcal{P} \) and \( f \) is Lipschitz. \( \square \)

Next we have the following result.

**Proposition 3.4.** Let Assumption (A1) holds for some \( p > 1 \). If \( \gamma_1, \gamma_2, \beta_1 \) and \( \beta_2 \) satisify

\[
(\gamma_1 + \gamma_2 + \beta_1 + \beta_2)^{p+1 \over p} \left[ \left( {p \over p-1} \right)^p (\gamma_1 + \beta_1) + (\gamma_2 + \beta_2) \right]^{1 \over p} < 1,
\] (3.6)

then there exists \( \delta > 0 \) depending only on \( p, C_f, \gamma_1, \gamma_2, \beta_1 \) and \( \beta_2 \) such that \( \Phi \) is a contraction on the time interval \([T - \delta, T]\).

**Proof.** Let \( Y, Y' \in S^p_\mathcal{P} \). Then, for any \( t \leq T \), we have,

\[
|\Phi(Y)_t - \Phi(Y')_t| \leq \text{ess sup} \sup_{\tau \geq t} \text{ess inf}_{\sigma \geq t} \left\{ \int_t^{\tau \wedge T} |f(s,Y_s,E[Y_s]) - f(s,Y'_s,E[Y'_s])| ds + |g(\sigma,Y_s,E[Y_t]=\sigma)| \right\} + \text{ess sup} \sup_{\tau \geq t} \text{ess inf}_{\sigma \geq t} \left\{ \int_t^{\tau \wedge T} |h(\sigma,Y'_s,E[Y'_t]=\sigma)| ds \right\} \leq \text{ess sup} \sup_{\tau \geq t} \text{ess inf}_{\sigma \geq t} \int_t^{\tau \wedge T} |f(s,Y_s,E[Y_s]) - f(s,Y'_s,E[Y'_s])| ds + |g(\sigma,Y_s,E[Y_t]=\sigma)| \right\} + \text{ess sup} \sup_{\tau \geq t} \text{ess inf}_{\sigma \geq t} \left\{ \int_t^{\tau \wedge T} |h(\sigma,Y'_s,E[Y'_t]=\sigma)| ds \right\} \leq E \left\{ \int_t^T |f(s,Y_s,E[Y_s]) - f(s,Y'_s,E[Y'_s])| ds + (\beta_1 + \gamma_1) \sup_{t \leq s \leq T} |Y_s - Y'_s| |\mathcal{F}_t| \right\} + \beta_2 \sup_{t \leq s \leq T} E[|Y_s - Y'_s|].
\] (3.7)
Fix now $\delta > 0$ and let $t \in [T - \delta, T]$. By the Lipschitz condition of $f$, (3.7) implies that

$$|\Phi(Y)_t - \Phi(Y')_t|$$

$$\leq E \left[ \sup_{T - \delta \leq s \leq T} |Y_s - Y'_s| + \sup_{T - \delta \leq s \leq T} E[|Y_s - Y'_s|] \right] + (\beta_1 + \gamma_1) \sup_{T - \delta \leq s \leq T} |Y_s - Y'_s|$$

$$+ (\beta_2 + \gamma_2) \sup_{T - \delta \leq s \leq T} E[|Y_s - Y'_s|]$$

$$= (\delta C_f + \gamma_1 + \beta_1) E \left[ \sup_{T - \delta \leq s \leq T} |Y_s - Y'_s| \right] + (\delta C_f + \gamma_2 + \beta_2) \sup_{T - \delta \leq s \leq T} E[|Y_s - Y'_s|].$$

(3.8)

As $p > 1$, thanks to the convexity inequality $(ax_1 + bx_2)^p \leq (a + b)^{p-1}(ax_1^p + bx_2^p)$ holding for any non-negative real constants $a, b, x_1$ and $x_2$, (3.8) yields

$$|\Phi(Y)_t - \Phi(Y')_t|^p \leq (2\delta C_f + \gamma_1 + \gamma_2 + \beta_1 + \beta_2)^{p-1} \left\{ (\delta C_f + \gamma_1 + \beta_1) \right\}^p$$

$$\left( E[\sup_{T - \delta \leq s \leq T} |Y_s - Y'_s|] \right)^p + (\delta C_f + \gamma_2 + \beta_2) \left( E[\sup_{T - \delta \leq s \leq T} |Y_s - Y'_s|] \right)^p \right\}. \right.$$ (3.9)

Next, by taking expectation of the supremum over $t \in [T - \delta, T]$ on the both hand-sides of (3.9), we have

$$E \left[ \sup_{T - \delta \leq s \leq T} \left| \Phi(Y)_s - \Phi(Y')_s \right|^p \right]$$

$$\leq (2\delta C_f + \gamma_1 + \gamma_2 + \beta_1 + \beta_2)^{p-1} \left\{ (\delta C_f + \gamma_1 + \beta_1) \right\}^p$$

$$\left( E[\sup_{T - \delta \leq s \leq T} |Y_s - Y'_s|] \right)^p + (\delta C_f + \gamma_2 + \beta_2) \left( E[\sup_{T - \delta \leq s \leq T} |Y_s - Y'_s|] \right)^p \right\}. \right.$$ (3.10)

By applying Doob’s inequality we have:

$$E \left[ \left( \sup_{T - \delta \leq s \leq T} \left| \Phi(Y)_s - \Phi(Y')_s \right| \right)^p \right] \leq \left( \frac{p}{p - 1} \right)^p E \left[ \sup_{T - \delta \leq s \leq T} |Y_s - Y'_s|^p \right]$$

(3.11)

and by Jensen’s one we have also

$$\left( E[\sup_{T - \delta \leq s \leq T} |Y_s - Y'_s|] \right)^p \leq E[\sup_{T - \delta \leq s \leq T} |Y_s - Y'_s|^p].$$

(3.12)

Plug now (3.11) and (3.12) in (3.10) to obtain:

$$\|\Phi(Y) - \Phi(Y')\|_{S^p_T([-\delta,T])} \leq \Lambda(C_f, p, \gamma_1, \gamma_2, \beta_1, \beta_2) (\delta) \|Y - Y'\|_{S^p_T([-\delta,T])}$$

where

$$\Lambda(C_f, p, \gamma_1, \gamma_2, \beta_1, \beta_2)(\delta) = (2\delta C_f + \gamma_1 + \gamma_2 + \beta_1 + \beta_2)^{p-1} \left( \frac{p}{p - 1} \right)^p (\delta C_f + \gamma_1 + \beta_1)$$

$$+ (\delta C_f + \gamma_2 + \beta_2)^{p-1} \left( \frac{p}{p - 1} \right)^p (\delta C_f + \gamma_1 + \beta_1)$$

Note that (3.6) is just $\Lambda(C_f, p, \gamma_1, \gamma_2, \beta_1, \beta_2)(0) < 1$. Now as

$$\lim_{\delta \to 0} \Lambda(C_f, p, \gamma_1, \gamma_2, \beta_1, \beta_2)(\delta) = \Lambda(C_f, p, \gamma_1, \gamma_2, \beta_1, \beta_2)(0) < 1,$$

then there exists $\delta$ small enough which depends only on $C_f, p, \gamma_1, \gamma_2, \beta_1, \beta_2$ and not on $\xi$ nor $T$ such that $\Lambda(C_f, p, \gamma_1, \gamma_2, \beta_1, \beta_2)(\delta) < 1$. It implies that $\Phi$ is a contraction on $S^p_T([-\delta,T])$. Then there exists a process which belongs to $S^p_T([-\delta,T])$ such that

$$Y_t = \Phi(Y)_t, \forall t \in [T - \delta, T].$$

□

We now show that the mean-field reflected BSDE (2.1) has a unique solution.
Theorem 3.5. Assume that Assumption (A1) holds for some \( p \) > 1. If \( \gamma_1 \) and \( \gamma_2 \) satisfy
\[
(\gamma_1 + \gamma_2 + \beta_1 + \beta_2)^{p+1} \left( \frac{p}{p-1} \right)^p (\gamma_1 + \beta_1) (\gamma_2 + \beta_2) \leq 1,
\]
then the mean-field doubly reflected BSDE (2.1) has a unique solution \((Y, Z, K^+, K^-)\).

Proof. Let \( \delta \) be as in Proposition 3.4. Then there exists a process \( Y \in S^p([T - \delta, T]) \), which is the fixed point of \( \Phi \) in this latter space and verifies: For any \( t \in [T - \delta, T] \),
\[
Y_t = \text{ess sup}_{\tau \geq t} \text{ess inf}_{c \geq t} \mathbb{E} \left\{ \int_t^{\sigma \wedge \tau} f(s, Y_s, \mathbb{E}[Y_s]) ds + g(\sigma, Y_\sigma, \mathbb{E}[Y_\sigma]) \mathbb{1}_{\{\sigma < \tau\}} \right\}
\]

Next since \( \zeta \in L^p(d\mathbb{P}), \mathbb{E}\left[\left(\int_0^T |f(s, \omega, Y_s, \mathbb{E}[Y_s])| ds\right)^p\right] < \infty \), the processes \((h(t, Y_t, \mathbb{E}[Y_t]))_{T-\delta \leq t \leq T}\) and \((g(t, Y_t, \mathbb{E}[Y_t]))_{T-\delta \leq t \leq T}\) belong to \( S^p([T - \delta, T]) \) since \( Y \) is so, and finally since \( h < g \), then there exist processes \( \bar{Y} \in S^p([T - \delta, T]), \bar{Z} \in \mathcal{H}^p_{loc}([T - \delta, T]) \) and \( \bar{K}^\pm \in \mathcal{A}([T - \delta, T]) \) (see e.g. [12] for more details) such that for any \( t \in [T - \delta, T] \), it holds:
\[
\begin{align*}
\bar{Y}_t &= \zeta_t + \int_{t}^{T} f(s, Y_s, \mathbb{E}[Y_s]) ds + \bar{K}^+_t - \bar{K}^-_t - \int_{t}^{T} \bar{Z}_s dB_s; \\
(h(t, Y_t, \mathbb{E}[Y_t])) &\leq \bar{Y}_t \leq g(t, Y_t, \mathbb{E}[Y_t]); \\
\int_{T-\delta}^{T} (\bar{Y}_s - h(s, Y_s, \mathbb{E}[Y_s])) d\bar{K}^+_s &= 0, \\
\int_{T-\delta}^{T} (\bar{Y}_s - g(s, Y_s, \mathbb{E}[Y_s])) d\bar{K}^-_s &= 0.
\end{align*}
\]

Therefore the process \( \bar{Y} \) has the following representation: \( \forall t \in [T - \delta, T], Y_t = \bar{Y}_t \). Thus \((Y, Z, K^\pm)\) verifies (2.1) and (3.14) on \([T - \delta, T]\), i.e., for \( t \in [T - \delta, T] \),
\[
\begin{align*}
Y_t &= \bar{Y}_t = \zeta_t + \int_{t}^{T} f(s, Y_s, \mathbb{E}[Y_s]) ds + \bar{K}^+_t - \bar{K}^-_t - \int_{t}^{T} \bar{Z}_s dB_s; \\
(h(t, Y_t, \mathbb{E}[Y_t])) &\leq Y_t \leq g(t, Y_t, \mathbb{E}[Y_t]); \\
\int_{T-\delta}^{T} (Y_s - h(s, Y_s, \mathbb{E}[Y_s])) d\bar{K}^+_s &= 0, \\
\int_{T-\delta}^{T} (Y_s - g(s, Y_s, \mathbb{E}[Y_s])) d\bar{K}^-_s &= 0.
\end{align*}
\]

But \( \delta \) of Proposition 3.4 does not depend on the terminal condition \( \zeta \) nor on \( T \), therefore there exists another process \( Y^1 \) which is a fixed point of \( \Phi \) in \( S^p([T - 2\delta, T - \delta]) \) with terminal condition \( Y_{T-\delta} \), i.e., for any \( t \in [T - 2\delta, T - \delta] \),
\[
Y^1_t = \text{ess sup}_{\tau \in [T - \delta, T-\delta]} \text{ess inf}_{c \in [T - \delta, T]} \mathbb{E} \left\{ \int_t^{\sigma \wedge \tau} f(s, Y^1_s, \mathbb{E}[Y^1_s]) ds + g(\sigma, Y^1_\sigma, \mathbb{E}[Y^1_\sigma]) \mathbb{1}_{\{\sigma < \tau\}} \right\}
\]

Then as previously, there exist processes \((\bar{Z}^1, \bar{K}^{1,\pm})\) \( (\bar{K}^{1,\pm} \in \mathcal{A}([T - 2\delta, T - \delta])) \) such that \((Y^1, \bar{Z}^1, \bar{K}^{1,\pm})\) verify: For any \( t \in [T - 2\delta, T - \delta] \),
\[
\begin{align*}
Y^1_t &= Y_{T - \delta} + \int_{T-\delta}^{t} f(s, Y^1_s, \mathbb{E}[Y^1_s]) ds + \bar{K}^{1,+}_t - \bar{K}^{1,-}_t - \int_{t}^{T-\delta} \bar{Z}^1_s dB_s; \\
(h(t, Y^1_t, \mathbb{E}[Y^1_t])) &\leq Y^1_t \leq g(t, Y^1_t, \mathbb{E}[Y^1_t]); \\
\int_{T-2\delta}^{T-\delta} (Y^1_s - h(Y^1_s, \mathbb{E}[Y^1_s])) d\bar{K}^{1,+}_s &= 0, \\
\int_{T-2\delta}^{T-\delta} (Y^1_s - g(Y^1_s, \mathbb{E}[Y^1_s])) d\bar{K}^{1,-}_s &= 0.
\end{align*}
\]

Concatenating now the solutions \((Y, \bar{Z}, \bar{K}^{\pm})\) and \((Y^1, \bar{Z}^1, \bar{K}^{1,\pm})\) we obtain a solution of (2.1) on \([T - 2\delta, T]\). Actually for \( t \in [T - 2\delta, T] \), let us set:
\[
\bar{Y}_t = Y^1_{T - \delta} + Y^1_{T - 2\delta}(T - \delta)(t) + Y^1_{T - 2\delta}(T - 2\delta)(t),
\]
\[
\dot{Z}_t = \dot{Z}_11_{[T-\delta,T]}(t) + \dot{Z}_1 1_{[T-2\delta,T-\delta]}(t), \\
\int_{T-2\delta}^T d\bar{K}^{1,\pm}_t = \int_{T-2\delta}^T \{1_{[T-\delta,T]}(s)d\bar{K}^{0,\pm}_s + 1_{[T-2\delta,T-\delta]}(s)d\bar{K}^{1,\pm}_s\}.
\]

Then \(\bar{Y} \in S^p_t([T-2\delta,T], \bar{Z} \in \mathcal{H}^4_{loc}([T-2\delta,T])\) and \(\bar{K}^{\pm} \in \mathcal{A}([T-2\delta,T])\) and they verify: For any \(t \in [T-2\delta,T]\),

\[
\begin{aligned}
\bar{Y}_t &= \zeta + \int_t^T f(s, \bar{Y}_s, \mathbb{E}[\bar{Y}_s])ds + \bar{K}^{+}_t - \bar{K}^{-}_t - \int_t^T \dot{Z}_s dB_s, \\
h(t, \bar{Y}_t, \mathbb{E}[\bar{Y}_t]) &\leq \bar{Y}_t \leq g(t, \bar{Y}_s, \mathbb{E}[\bar{Y}_s]), \\
\int_{T-2\delta}^T (\bar{Y}_s - h(s, \bar{Y}_s, \mathbb{E}[\bar{Y}_s]))d\bar{K}^{+}_s &= 0 \quad \text{and} \quad \int_{T-2\delta}^T (\bar{Y}_s - g(s, \bar{Y}_s, \mathbb{E}[\bar{Y}_s]))d\bar{K}^{-}_s = 0.
\end{aligned}
\]

But we can do the same on \([T-3\delta, T-2\delta], [T-4\delta, T-3\delta]\), etc. and at the end, by concatenation of those solutions, we obtain a solution \((Y, Z, K^{\pm})\) which satisfies (2.1).

Let us now focus on uniqueness. Assume there is another solution \((\mathbf{Y}, \mathbf{Z}, \mathbf{K}^{\pm})\) of (2.1). It means that \(Y\) is a fixed point of \(\Phi\) on \(S^p_{\mathbf{Y}}([T-\delta, T])\), therefore for any \(t \in [T-\delta, T]\), \(Y_t = \mathbf{Y}_t\). Next writing equation (2.1) for \(Y\) and \(\mathbf{Y}\) on \([T-2\delta, T-\delta]\), using the link with zeros-sum Dynkin games (see Lemma 3.1) and finally the uniqueness of the fixed point of \(\Phi\) on \(S^p_{\mathbf{Y}}([T-2\delta, T-\delta])\) to obtain that for any \(t \in [T-2\delta, T-\delta]\), \(Y_t = \mathbf{Y}_t\). By continuing this procedure on \([T-3\delta, T-2\delta], [T-4\delta, T-3\delta]\), etc. we obtain that \(Y = \mathbf{Y}\). The equality between the stochastic integrals implies that \(Z = \mathbf{Z}\). Finally as \(h < g\) and since \(Y = \mathbf{Y}\), then \(K^{+} = K^{+}\) and \(K^{-} = K^{-}\) (see e.g. (12)) for more details. Thus the solution is unique. The proof is complete. \(\square\)

**Remark 3.6.** There is no specific difficulty to consider the following more general framework of equations (2.1) and (2.2).

\[
\begin{aligned}
Y_t &= \zeta + \int_t^T f(s, Y_s, \mathbb{P}_{Y_s})ds + K^{+}_t - K^{-}_t - \int_t^T \dot{Z}_s dB_s, \quad 0 \leq t \leq T; \\
h(t, Y_t, \mathbb{P}_{Y_t}) &\leq Y_t \leq g(t, Y_s, \mathbb{P}_{Y_s}), \quad \forall t \in [0, T]; \\
\text{and} \quad \int_0^T (Y_s - h(s, Y_s, \mathbb{P}_{Y_s}))dK^{+}_s = 0, \quad \int_0^T (Y_s - g(s, Y_s, \mathbb{P}_{Y_s}))dK^{-}_s = 0
\end{aligned}
\]

where the Lipschitz property of \(f, h\) and \(g\) w.r.t. \(\mathbb{P}_{Y_s}\) should be read as: for \(\Psi \in \{f, g, h\}\) for any \(v, v'\) probabilities

\[|\Psi(v) - \Psi(v')| \leq Cd_p(v, v')\]

where \(d_p(\cdot, \cdot)\) is the \(p\)-Wasserstein distance on the subset \(\mathcal{P}_p(\mathbb{R})\) of probability measures with finite \(p\)-th moment, formulated in terms of a coupling between two random variables \(X\) and \(Y\) defined on the same probability space:

\[d_p(\mu, v) := \inf \left\{ (\mathbb{E} [\|X - Y\|^p])^{1/p}, \text{law}(X) = \mu, \text{law}(Y) = v \right\}. \square\]

### 3.2. The case \(p=1\).

We proceed as we did in the case when \(p > 1\). We have the following result.

**Proposition 3.7.** Let Assumptions (A1) hold for some \(p = 1\). If \(\gamma_1, \gamma_2, \beta_1, \beta_2\) satisfy

\[\gamma_1 + \gamma_2 + \beta_1 + \beta_2 < 1,\]

then there exists \(\delta > 0\) depending only on \(C_f, \gamma_1, \gamma_2, \beta_1, \beta_2\) such that \(\Phi\) is a contraction on the space \(\mathcal{D}([T-\delta,T])\).
Proof. Let $\delta$ be a positive constant and $\theta$ a stopping time which belongs to $[T - \delta, T]$. Therefore

$$|\Phi(Y)_\theta - \Phi(Y')_\theta| = \left| \text{ess sup}_{\sigma \geq \theta} \left| \mathbb{E} \left[ \int_{\theta}^{\sigma \wedge T} f(s, Y_s, \mathbb{E}[Y_s]) ds + g(\sigma, Y_\sigma, \mathbb{E}[Y_\sigma]) \mathbb{1}_{\{\sigma < T\}} \right] + h(\sigma, Y_\sigma, \mathbb{E}[Y_\sigma]|_{\sigma = T}) \mathbb{1}_{\{\sigma = T\}} \right| \right|_{F_\theta}$$

$$\leq \text{ess sup}_{\sigma \geq \theta} \text{ess sup}_{\tau \geq \theta} \mathbb{E} \left[ \int_{\sigma \wedge T}^{\tau \wedge T} |f(s, Y_s, \mathbb{E}[Y_s])| ds + |g(\sigma, Y_\sigma, \mathbb{E}[Y_\sigma])| \right]$$

Taking now expectation on both hand-sides to obtain:

$$\mathbb{E}[|\Phi(Y)_\theta - \Phi(Y')_\theta|] \leq 2\delta C_f \sup_{\tau \in [T - \delta, T]} \mathbb{E}[|Y_T - Y'_T|] + \text{ess sup}_{\sigma \geq \theta} \mathbb{E}\left[ |g(\sigma, Y_\sigma, \mathbb{E}[Y_\sigma]|_{\sigma = T}) - g(\sigma, Y'_\sigma, \mathbb{E}[Y'_\sigma]|_{\sigma = T})| \right]$$

Then for any stopping time $\theta$ valued in $[T - \delta, T]$, we have:

$$\mathbb{E}[|\Phi(Y)_\theta - \Phi(Y')_\theta|] \leq (2\delta C_f + \beta_1 + \beta_2 + \gamma_1 + \gamma_2) \sup_{\tau \in [T - \delta, T]} \mathbb{E}[|Y_T - Y'_T|].$$

Next since $\beta_1 + \beta_2 + \gamma_1 + \gamma_2 < 1$, then for $\delta$ small enough we have $\Sigma(\delta) < 1$ ($\delta$ depends neither on $\xi$ nor on $T$) and $\Phi$ is a contraction on the space $D([T - \delta, T])$. Therefore it has a fixed point $Y$, which then verifies:

$$Y \in D([T - \delta, T]) \quad \text{and} \quad \forall t \in [T - \delta, T],$$

$$Y_t = \text{ess sup}_{\sigma \geq t} \text{ess inf}_{\tau \geq t} \mathbb{E}\left\{ \int_{\sigma \wedge T}^{\tau \wedge T} f(s, Y_s, \mathbb{E}[Y_s]) ds + g(\sigma, Y_\sigma, \mathbb{E}[Y_\sigma]|_{\sigma = T}) \mathbb{1}_{\{\sigma < T\}} \right\} + h(\sigma, Y_\sigma, \mathbb{E}[Y_\sigma]|_{\sigma = T}) \mathbb{1}_{\{\sigma = T\}}$$

As a by-product we have the following result which stems from the link between the value of a zero-sum Dynkin game and doubly reflected BSDE with given in (3.1).

**Corollary 3.8.** Let Assumption (A1) hold for some $p = 1$. If $\gamma_1, \gamma_2, \beta_1$ and $\beta_2$ satisfy (3.20) then there exists $\delta > 0$, depending only on $C_f, \gamma_1, \gamma_2, \beta_1, \beta_2$, and $\mathbb{P}$-measurable processes $Z^0$, $K^0, \pm$ such that:

$$P - a.s., \int_{T - \delta}^{T} |Z^0|^2 ds < \infty; \quad K^0, \pm \in \mathcal{A} \quad \text{and} \quad K^0, \pm_{T - \delta} = 0;$$

$$Y_t = \xi + \int_{t}^{T} f(s, Y_s, \mathbb{E}[Y_s]) ds + K^0, + - K^0, - + K^0, + - K^0, - - \int_{t}^{T} Z^0 dB_s, \quad T - \delta \leq t \leq T;$$

$$h(t, Y_t, \mathbb{E}[Y_t]) \leq Y_t \leq g(t, Y_t, \mathbb{E}[Y_t]), \quad T - \delta \leq t \leq T;$$

$$\int_{T - \delta}^{T} (Y_t - h(t, Y_t, \mathbb{E}[Y_t])) dB^0_{t} = \int_{T - \delta}^{T} (Y_t - g(t, Y_t, \mathbb{E}[Y_t])) dB^0_{t} = 0.$$
Theorem 3.9. Let $f, h, g$ and $\xi$ satisfying Assumption (A1) for $p = 1$. Suppose that
\[ \gamma_1 + \gamma_2 + \beta_1 + \beta_2 < 1. \] (3.23)
Then, there exist $\mathcal{P}$-mesurable processes $(Y, Z, K^\pm)$ unique solution of the mean-field reflected BSDE (2.2), i.e.,
\[
\begin{cases}
Y_t = \xi + \int_t^T f(s, Y_s, E[Y_s]) ds + K_T^+ + K_T^- - K_t^+ + K_t^- - \int_t^T Z_s dB_s, & 0 \leq t \leq T; \\
h(t, Y_t, E[Y_t]) \leq Y_t \leq g(t, Y_t, E[Y_t]), & \forall t \in [0, T]; \\
\int_0^T (Y_s - h(s, Y_s, E[Y_s])) dK_s^+ = 0, & \int_0^T (Y_s - g(s, Y_s, E[Y_s])) dK_s^- = 0.
\end{cases}
\] (3.24)

Proof. Let $\delta$ be as in Proposition 3.7 and $Y$ the fixed point of $\Phi$ on $D([T - \delta, T])$ which exists since (3.20) is satisfied. Next let $Y^1$ be the fixed point of $\Phi$ on $D([T - 2\delta, T])$ with terminal condition $Y_{T-\delta}$, i.e., for any $t \in [T - 2\delta, T - \delta]$,
\[
Y_t^1 = \text{ess sup}_{\tau \in [t, T-\delta]} \text{ess inf}_{\sigma \in [t, T-\delta]} \mathbb{E} \left[ \int_t^{\sigma \wedge \tau} f(s, Y_s^1, E[Y_s^1]) ds + g(\sigma, Y_s^1, E[Y_s^1]) I_{\{\sigma < \tau\}} \right] + h(\tau, Y_t^1, E[Y_t^1]) I_{\{\tau \leq \sigma, \tau < T-\delta\}} + Y_{T-\delta} I_{\{\tau = \sigma = T-\delta\}} | \mathcal{F}_t \right].
\] (3.25)
The process $Y^1$ exists since condition (3.20) is satisfied and $\delta$ depends neither on $T$ nor on the terminal condition. Once more the link between reflected BSDEs and zero-sum Dynkin games (see Lemma 3.1) implies the existence of $\mathcal{P}$-mesurable processes $Z^1, K^{1, \pm}$ such that
\[
\begin{cases}
\mathbb{P} - \text{a.s.} \int_{T-2\delta}^{T-\delta} |Z|_\mathcal{F}_s^2 ds < \infty; & K^{1, \pm} \in \mathcal{A} \text{ and } K^{1, \pm}_{T-2\delta} = 0; \\
Y_t^1 = Y_{T-\delta} + \int_t^{T-\delta} f(s, Y_s^1, E[Y_s^1]) ds + K^{1, +}_{T-\delta} - K^{1, -}_{T-\delta} + K^{1, +}_t - K^{1, -}_t - \int_t^{T-\delta} Z_s dB_s, & t \in [T - 2\delta, T - \delta]; \\
h(t, Y_t^1, E[Y_t^1]) \leq Y_t^1 \leq g(t, Y_t^1, E[Y_t^1]), & t \in [T - 2\delta, T - \delta]; \\
\int_{T-2\delta}^{T-\delta} (Y_t^1 - h(t, Y_t^1, E[Y_t^1])) dK^{1, +}_t = \int_{T-2\delta}^{T-\delta} (Y_t^1 - g(t, Y_t^1, E[Y_t^1])) dK^{1, -}_t = 0.
\end{cases}
\] (3.26)
Concatenating now the solutions $(Y, Z^0, K^0)$ of (3.22) and $(Y^1, Z^1, K^{1, \pm})$ we obtain a solution of (2.2) on $[T - 2\delta, T]$. Actually for $t \in [T - 2\delta, T]$, let us set:
\[
\begin{align*}
\tilde{Y}_t &= Y_t 1_{[T-\delta, T]}(t) + Y_t^1 1_{[T-2\delta, T-\delta]}(t), \\
\tilde{Z}_t &= Z_t 1_{[T-\delta, T]}(t) + Z_t^1 1_{[T-2\delta, T-\delta]}(t), \\
\int_{T-2\delta}^t dK^{1, \pm}_s &= \int_{T-2\delta}^t 1_{[T-\delta, T]}(s) dK^0_{s, \pm} + 1_{[T-2\delta, T-\delta]}(s) dK^{1, \pm}_s.
\end{align*}
\] Then $\tilde{Y} \in D([T - 2\delta, T], \tilde{Z} \in \mathcal{H}^{d, \text{loc}}([T - 2\delta, T])$ and $K^\pm \in \mathcal{A}([T - 2\delta, T])$ and they verify: For any $t \in [T - 2\delta, T]$,
\[
\begin{cases}
\tilde{Y}_t = \xi + \int_t^T f(s, \tilde{Y}_s, E[\tilde{Y}_s]) ds + K^+_T - K^-_T + K^+_t - K^-_t - \int_t^T Z_s dB_s; \\
h(t, \tilde{Y}_t, E[\tilde{Y}_t]) \leq \tilde{Y}_t \leq g(t, \tilde{Y}_t, E[\tilde{Y}_t]); \\
\int_{T-2\delta}^T (\tilde{Y}_s - h(s, \tilde{Y}_s, E[\tilde{Y}_s])) dK^+_s = 0 \text{ and } \int_{T-2\delta}^T (\tilde{Y}_s - g(s, \tilde{Y}_s, E[\tilde{Y}_s])) dK^-_s = 0.
\end{cases}
\] (3.27)
But we can do the same on $[T - 3\delta, T - 2\delta], [T - 4\delta, T - 3\delta], \text{ etc.}$ and at the end, by concatenation of those solutions, we obtain a solution $(Y, Z, K^\pm)$ which satisfies (2.1).

Let us now focus on uniqueness. Assume there is another solution $(\tilde{Y}, \tilde{Z}, K^{\tilde{\pm}})$ of (2.1). It means that $Y$ is a fixed point of $\Phi$ on $D([T - \delta, T])$, therefore for any $t \in [T - \delta, T], Y_t = \tilde{Y}_t$. Next writing equation (2.1) for $Y$ and $\tilde{Y}$ on $[T - 2\delta, T - \delta]$, using the link with zero-sum Dynkin games (Lemma 3.1), the uniqueness of the fixed point of $\Phi$ on $D([T - 2\delta, T - \delta])$ implies that for any $t \in [T - 2\delta, T - \delta], Y_t = \tilde{Y}_t$. By continuing this procedure on $[T - 3\delta, T - 2\delta], [T - 4\delta, T - 3\delta], \text{ etc.}$ we obtain that $Y = \tilde{Y}$. The equality between the
stochastic integrals implies that \( Z = Z \). Finally as \( h < g \) and since \( Y = Y \), then \( K^+ = K^+ \) and \( K^- = K^- \) (see e.g. \[12\] for more details). Thus the solution is unique. The proof is complete.

Finally let us notice that the same Remark \[16\] is valid for this case \( p = 1 \). □

4. PENALTY METHOD FOR THE MEAN-FIELD DOUBLY REFLECTED BSDE

In this section, we study doubly reflected BSDE with mean-field term involved in the coefficients and obstacles by using the penalty method. Actually we consider the following mean-field reflected BSDE:

\[
\begin{align*}
Y_t & = \xi + \int_t^T f(s, Y_s, E[Y_s], Z_s) ds + K^+_T - K^-_T - (K^+_T - K^-_T) - \int_t^T Z_s dB_s, \ t \leq T; \\
h(t, Y_t, E[Y_t]) & \leq Y_t \leq g(t, Y_t, E[Y_t]), \ t \leq T; \\
\int_0^T (Y_t - h(t, Y_t, E[Y_t])) dK^+_t & = \int_0^T (g(t, Y_t, E[Y_t]) - Y_t) dK^-_t = 0.
\end{align*}
\]

(4.1)

For the ease of exposition, we assume that \( p = 2 \) in Assumption (A1). On the other hand, now \( f \) may depend on \( z \). Precisely the assumptions in this section are the following ones.

**Assumption (A2)**

(i) \( f(t, y, y', z) \) is Lipschitz with respect to \( (y, y', z) \) uniformly in \( (t, \omega) \), i.e. there exists a constant \( C_0 \) so that \( \mathbb{P} \)-a.s. for all \( t \in [0, T] \) and \( y_1, y'_1, y_2, y'_2 \in \mathbb{R}^d \),

\[
|f(t, y_1, y'_1, z) - f(t, y_2, y'_2, z)| \leq C(|y_1 - y_2| + |y'_1 - y'_2| + |z - z|),
\]

and \( (f(t, 0, 0, 0))_{t \leq T} \in \mathcal{H}^{d,1} \).

(ii) \( y' \mapsto f(t, y, y', z) \) is non-decreasing for fixed \( t, y, z \).

(iii) \( y \mapsto f(t, y, y', z) \) is non-decreasing for fixed \( t, y, z \).

(iv) \( \mathbb{P} \)-a.s. for any \( t \leq T \), \( h(\omega, t, y, y') \) and \( g(\omega, t, y, y') \) are non-decreasing w.r.t \( y \) and \( y' \).

(v) The processes \( (h(\omega, t, 0, 0))_{t \leq T} \) and \( (g(\omega, t, 0, 0))_{t \leq T} \) belong to \( \mathcal{S}_c^d \).

(vi) \( \mathbb{P} \)-a.s. \( |g(t, \omega, y, y') - g(t, \omega, 0, 0)| + |h(t, \omega, y, y') - h(t, \omega, 0, 0)| \leq C(1 + |y| + |y'|) \) for some constant \( C \) and any \( t, y, y' \).

(vii) adapted Mokobodski’s condition: There exists a process \( (X_t)_{t \leq T} \) satisfying

\[
\forall t \leq T, \ X_t = X_0 + \int_0^t J_s dB_s + V^+_t - V^-_t
\]

with \( J \in \mathcal{H}^{d,1} \) and \( V^+, V^- \in \mathcal{S}_c^d \) such that \( \mathbb{P} \)-a.s. for any \( t, y, y' \),

\[
h(\omega, t, y, y') \leq X_t \leq g(\omega, t, y, y');
\]

(viii) \( \mathbb{P} \)-a.s. \( h(t, \omega, y, y') < g(t, \omega, y, y') \), for any \( t, y, y' \in \mathbb{R} \).

We denote that, the constant \( C \) changes line by line in this paper.

**Remark 4.1.**

(a) Assumption (A2)-(i),(c) is not stringent since if \( f \) does not satisfy this property one can modify the mean-field doubly reflected BSDE \[4.1\] by using an exponential transform in such a way to fall in the case where \( f \) satisfies (A2)-(i),(c) (one can see e.g. \[13\], Corollary 4.1 for more details).

(b) In this penalization method, contrarily to the previous one, we cannot replace in \( f(\cdot) \) the expectation \( \mathbb{E}[Y_t] \) with the law \( \mathbb{P}_Y \), because of lack of comparison in this latter case. For the same reason \( f(\cdot) \) cannot depend on \( \mathbb{E}[Z_\omega] \) or more generally on \( \mathbb{P}_Z \) or the law of \( Z_\omega \).

(c) On the adapted Mokobodski’s condition: We use Assumption (A2)-(ii),(d) to let \( h(t, Y_t, E[Y_t]) \leq Y_t \leq g(t, Y_t, E[Y_t]), t \leq T, \) possible. When we do not have Assumption (A2)-(ii),(d), a counter-example is considered below:

For simplicity, we set \( g(t, y, y') = \bar{C} \) where \( \bar{C} \) is a large enough constant in such a way that \( 4.1 \) becomes a RBSDE, i.e.

\[
\begin{align*}
Y_t & = \xi + \int_t^T f(s, Y_s, E[Y_s], Z_s) ds + K_T - K_t - \int_t^T Z_s dB_s, \ t \leq T; \\
Y_t & \geq h(t, Y_t, E[Y_t]), \ t \leq T; \int_0^T (Y_t - h(t, Y_t, E[Y_t])) dK_t = 0.
\end{align*}
\]

(4.2)
Then, let $Y^K_t$ satisfy
\[ Y^K_t = \xi + \int_t^T f(s, Y^K_s, \mathbb{E}[Y^K_s], Z^K_s) ds + K_T - K_t - \int_t^T Z^K_s dB_s, \quad t \leq T, \]  
for a given increasing process $K$. If there exists a solution $Y$ of (4.2), then there exists an increasing process $K$ such that $Y_t \geq h(t, Y_t, \mathbb{E}[Y_t])$. Now if we do not have Assumption (A2)-(ii),(d), then, for some $h$ (see a explicit example below) and a fixed $t$, we obtain that
\[ Y^K_t - h(t, Y^K_t, \mathbb{E}[Y^K_t]) < 0, \quad \text{for any increasing processes } K, \]
which means (4.2) has no solution.

More precisely, when we take $h(t, y, y') = y + y' + 1$, $\xi = -T$ and $f = 1$, for $T = 1$, it follows that
\[ Y^K_t = -t + K_1 - K_t - \int_1^t Z^K_s dB_s, \quad \text{for an increasing process } K. \]

Then for $t \in [0, 1)$, we have
\[ \mathbb{E}[Y^K_t] = -t + \mathbb{E}[K_1 - K_t] \geq -t > -1, \]
which does not satisfy the inequality $\mathbb{E}[Y^K_t] \leq -1$. Therefore, corresponding Mean-field RBSDE, i.e.,
\[ \begin{cases} Y_t = -t + K_1 - K_t - \int_1^t Z_s dB_s, \\ Y_t \geq Y_t + \mathbb{E}[Y_t] + 1, \quad \text{and } \int_0^1 (\mathbb{E}[Y_t] + 1) dK_t = 0, \end{cases} \]
has no solution, because for $t \in [0, 1)$, $Y_t$ does not satisfy the obstacle constraint for any increasing process $K$.

However, if we have Assumption (A2)-(ii),(d), i.e., for any $t \in [0, T)$,
\[ \sup_{y, y'} h(t, y, y') \leq X_t, \]
then, for any $h$ and $t \in [0, T)$, there exists an increasing $K$ such that $Y^K_t$ satisfying
\[ Y^K_t \geq Y_t \geq \sup_{y, y'} h(t, y, y') \geq h(t, Y^K_t, \mathbb{E}[Y^K_t]). \]

Thus $Y$ satisfying $Y_t \geq h(t, Y_t, \mathbb{E}[Y_t])$, $t \leq T$, is possible in (4.2) for any $h$ satisfying Assumption (A2)-(ii),(d) in this case. \qed

Now we are going to show that (4.11) has a solution by penalization. First, we define the process $Y^{n,0}$ by the solution of a standard Mean-field BSDE, i.e.
\[ Y^{n,0}_t = \xi + \int_t^T f(s, Y^{n,0}_s, \mathbb{E}[Y^{n,0}_s], Z^{n,0}_s) ds - \int_t^T Z^{n,0}_s dB_s, \quad t \leq T, \]
whose existence and uniqueness is obtained thanks to [22] Theorem 3.1. By assuming that
\[ Y^{n-1} = Y^{n,0}, \quad Y^{-1,m} = Y^{0,m}, \quad \text{and } Y^{-1,-1} = Y^{0,0}, \quad \text{for any } n \geq 0, m \geq 0, \]
we present a series of mean-field BSDEs, for $n, m \geq 0$, as follows:
\[ \begin{cases} Y^{n,m}_t \in \mathcal{S}^2_{\mathcal{F}^t}, \quad Z^{n,m}_t \in \mathcal{H}^{2,\mathcal{F}}; \\ Y^{n,m}_t = \xi + \int_t^T f(s, Y^{n,m}_s, \mathbb{E}[Y^{n-1,m-1}_s], Z^{n,m}_s) ds + K^{n,m,+}_t - K^{n,m,+}_t - (K^{n,m,-}_t - K^{n,m,-}_t) - \int_t^T Z^{n,m}_s dB_s, \quad t \leq T, \end{cases} \]
where for any $t \leq T$,
\[ \begin{align*} K^{n,m,+}_t &:= m \int_0^t \left( Y^{n,m}_s - h(s, Y^{n-1,m-1}_s, \mathbb{E}[Y^{n-1,m-1}_s]) \right)^- ds \\ K^{n,m,-}_t &:= n \int_0^t \left( Y^{n,m}_s - g(s, Y^{n-1,m-1}_s, \mathbb{E}[Y^{n-1,m-1}_s]) \right)^+ ds. \end{align*} \]

By comparison theorem of mean-field BSDE (see [22] Theorem 2.2), we have the following inequality for any $n \geq 0, m \geq 0, t \in [0, T]$,
\[ Y^{m+1,m}_t \leq Y^{n,m}_t \leq Y^{m,m+1}_t. \]
It is obtained by induction on \( k \) such that \( n + m \leq k \). For \( k = 0 \) the property holds true since by comparison we have: For any \( t \leq T \),

\[
Y_{t}^{1,0} \leq Y_{t}^{0,0} \leq Y_{t}^{0,1}.
\] (4.12)

Next assume that it is satisfied for some \( k \) and let us show that it is also satisfied for \( n + m \leq k + 1 \). So it remains to show that it is satisfied for \( n + m = k + 1 \), i.e., to show that

\[
Y_{n+1,k+1-n} \leq Y_{n,k+2-n} \leq Y_{n,k+1-n}, \quad \text{for any } n = 0, 1, 2, \ldots, k + 1.
\] (4.13)

Here, we will only show \( Y_{n+1,k+1-n} \leq Y_{n,k+2-n} \) and the remainder is similar. By assumption, we have \( Y_{n,k-n} \leq Y_{n-1,k-n} \) for \( n = 0, 1, 2, \ldots, k + 1 \), where we use the notation \((4.8)\) for \( n = 0 \) and \( k = 1 \). Since \( f(t, y, y', z) \) is non-decreasing w.r.t \( y' \) and \( g(t, y, y') \) is non-decreasing w.r.t \( y, y' \), then for any \( n = 0, 1, \ldots, k + 1 \), we have

\[
-f(s, y, \mathbb{E}[Y_{s}^{n,k-n}], z) \leq f(s, y, \mathbb{E}[Y_{s}^{n-1,k-n}], z) \quad \text{and}
\]

\[
-(n + 1) \left( y - g(s, Y_{s}^{n,k-n}, \mathbb{E}[Y_{s}^{n,k-n}]) \right) \leq -n \left( y - g(s, Y_{s}^{n-1,k-n}, \mathbb{E}[Y_{s}^{n-1,k-n}]) \right).
\] (4.14)

Taking now into account of the monotonicity property of \( h \) (see (A2)-(ii),(a)) we deduce, by comparison, that

\[
Y_{n+1,k+1-n} \leq Y_{n+1,k+2-n} \leq Y_{n+1,k+1-n},
\]

Then the induction is completed.

Next for simplicity, we write

\[
\forall n, m \geq 0 \text{ and } s \leq T, \quad L_{s}^{n,m} := h(s, Y_{s}^{n,m}, \mathbb{E}[Y_{s}^{n,m}]), \quad U_{s}^{n,m} := g(s, Y_{s}^{n,m}, \mathbb{E}[Y_{s}^{n,m}]).
\]

Then we have the following estimates.

**Lemma 4.2.** There exists a constant \( C \geq 0 \) such that:

i) For any \( n, m \geq 0 \),

\[
\sup_{0 \leq t \leq T} \mathbb{E}[(Y_{t}^{n,m})^2] + \mathbb{E} \left[ \int_{0}^{T} |Z_{t}^{n,m}|^2 dt \right] \leq C.
\] (4.15)

ii) For any \( n, m \geq 0 \),

\[
\mathbb{E} \left[ m^2 \left( \int_{0}^{T} (y_{s,m} - L_{s}^{n-1,m-1})^2 dt \right)^2 + n^2 \left( \int_{0}^{T} (Y_{s,m} - U_{s}^{n-1,m-1})^+ dt \right)^2 \right] \leq C.
\] (4.16)

**Proof.** Let \( \eta \) and \( \tau \) be stopping times such that \( \eta \leq \tau \) \( \mathbb{P} \) - a.s.. We define two sequences of stopping times \( \{T_{k}\}_{k \geq 1} \) and \( \{S_{k}\}_{k \geq 0} \) by

\[
S_{0} = \eta, \quad T_{k} = \inf \{S_{k-1} \leq r \leq T : Y_{r}^{n,m} = U_{r}^{n-1,m-1} \} \land \tau, \quad k \geq 1,
\]

\[
S_{k} = \inf \{T_{k} \leq r \leq T : Y_{r}^{n,m} = L_{r}^{n-1,m-1} \} \land \tau, \quad k \geq 1.
\]

As \( h < g \) then when \( k \) tends to \( +\infty \), we have \( T_{k} \not\nearrow \tau \) and \( S_{k} \not\nearrow \tau \). Since \( Y_{n,m} \geq L^{n-1,m-1} \) on \( [T_{k}, S_{k}] \cap \{T_{k} < S_{k}\} \), we have

\[
Y_{T_{k}}^{n,m} = Y_{S_{k}}^{n,m} + \int_{T_{k}}^{S_{k}} f(s, Y_{s}^{n,m}, \mathbb{E}[Y_{s}^{n-1,m-1}], Z_{s}^{n,m}) ds - \int_{T_{k}}^{S_{k}} Z_{s}^{n,m} dB_{s} - n \int_{T_{k}}^{S_{k}} \left( Y_{s}^{n,m} - U_{s}^{n-1,m-1} \right)^+ ds,
\] (4.17)

which implies, for all \( k \geq 1 \),

\[
n \int_{T_{k}}^{S_{k}} \left( Y_{s}^{n,m} - U_{s}^{n-1,m-1} \right)^+ ds \leq X_{S_{k}} - X_{T_{k}} + \int_{T_{k}}^{S_{k}} f(s, Y_{s}^{n,m}, \mathbb{E}[Y_{s}^{n-1,m-1}], Z_{s}^{n,m}) ds - \int_{T_{k}}^{S_{k}} Z_{s}^{n,m} dB_{s}
\]

\[
\leq - \int_{T_{k}}^{S_{k}} J_{s} dB_{s} + Z_{s}^{n,m} - V_{s}^{+} - V_{s}^{-} + V_{s}^{+} - \int_{T_{k}}^{S_{k}} Z_{s}^{n,m} dB_{s} + \int_{T_{k}}^{S_{k}} |f(s, Y_{s}^{n,m}, \mathbb{E}[Y_{s}^{n-1,m-1}], Z_{s}^{n,m})| ds
\]

\[
\leq - \int_{T_{k}}^{S_{k}} J_{s} dB_{s} + V_{s}^{+} - V_{s}^{-} - \int_{T_{k}}^{S_{k}} Z_{s}^{n,m} dB_{s} + \int_{T_{k}}^{S_{k}} |f(s, Y_{s}^{n,m}, \mathbb{E}[Y_{s}^{n-1,m-1}], Z_{s}^{n,m})| ds.
\] (4.18)

Summing in \( k \), we get

\[
n \int_{\eta}^{\tau} \left( Y_{s}^{n,m} - U_{s}^{n-1,m-1} \right)^+ ds \leq - \int_{\eta}^{\tau} (J_{s} + Z_{s}^{n,m}) \sum_{k \geq 1} \{1_{[T_{k}, S_{k}]}(s)\} dB_{s} + (V_{s}^{+} - V_{s}^{-}) + \int_{\eta}^{\tau} |f(s, Y_{s}^{n,m}, \mathbb{E}[Y_{s}^{n-1,m-1}], Z_{s}^{n,m})| ds.
\] (4.19)
Take now $\eta = t$ and $\tau = T$, then squaring and taking expectation on the both sides of (4.19), we get

$$\begin{align*}
n^2 \mathbb{E} \left( \int_t^T (Y_{s}^{n,m} - U_{s}^{n-1,m-1})^2 ds \right)^2 \\
\leq \mathbb{C} \left( \int_t^T |J_s + Z^{n,m}_s|^2 ds + \mathbb{E} \left[ (V_{t}^{n,m})^2 \right] + \int_t^T |f(s, Y_{s}^{n,m}, \mathbb{E}[Y_{s}^{n-1,m-1}], Z^{n,m}_s)|^2 ds \right) \\
\leq \mathbb{C} \left( 1 + \int_t^T |Z^{n,m}_s|^2 ds + \int_t^T (Y_{s}^{n,m})^2 ds + \int_t^T (Y_{s}^{n-1,m-1})^2 ds \right),
\end{align*}$$

(4.20)

where we have used the Lipschitz continuity of $f$ and (A2)-(i),(a). Similarly considering (4.9) between $S_{k-1}$ and $T_k$, we obtain

$$m \int_\eta^T (Y_{s}^{n,m} - L_{s}^{n-1,m-1})^{-} ds \leq \int_\eta^T (-J_s + Z^{n,m}_s) \sum_{k \geq 1} \{ 1[ (S_{k-1}, T_k) ](s) \} dB_s + (V_{T}^{n,m} - V_{\eta}^{n,m})$$

$$- \int_\eta^T f(s, Y_{s}^{n,m}, \mathbb{E}[Y_{s}^{n-1,m-1}], Z^{n,m}_s) ds,$$

(4.21)

and then, as previously, we have:

$$m^2 \mathbb{E} \left( \int_t^T (Y_{s}^{n,m} - L_{s}^{n-1,m-1})^{-} ds \right)^2 \leq \mathbb{C} \left( 1 + \int_t^T |Z^{n,m}_s|^2 ds + \int_t^T (Y_{s}^{n,m})^2 ds + \int_t^T (Y_{s}^{n-1,m-1})^2 ds \right).$$

(4.22)

Next, applying Itô’s formula to $(Y_{t}^{n,m})^2$, for all constants $\alpha, \beta > 0$, we have

$$\mathbb{E}[Y_{t}^{n,m}]^2 + \mathbb{E} \left( \int_t^T Z^{n,m}_s ds \right)$$

$$= \mathbb{E}[1^2] + 2\mathbb{E} \left( \int_t^T Y_{s}^{n,m} f(s, Y_{s}^{n,m}, \mathbb{E}[Y_{s}^{n-1,m-1}], Z^{n,m}_s) ds \right) + \alpha \mathbb{E} \left[ \int_t^T Y_{s}^{n,m} (Y_{s}^{n,m} - L_{s}^{n-1,m-1})^- ds \right]$$

$$- 2\alpha \mathbb{E} \left[ \int_t^T Y_{s}^{n,m} \left( Y_{s}^{n,m} - U_{s}^{n-1,m-1} \right)^+ ds \right]$$

$$\leq \mathbb{C} \left( 1 + (2 + \frac{1}{\alpha}) \mathbb{E} \left( \int_t^T (Y_{s}^{n,m})^2 ds \right) + \alpha \mathbb{E} \left( \int_t^T (Y_{s}^{n-1,m-1})^2 ds \right) + \frac{1}{\beta} \mathbb{E} \sup_{s \in [t,T]} \left( (L_{s}^{n-1,m-1})^+ \right)^2$$

$$+ \frac{1}{\beta} \mathbb{E} \sup_{s \in [t,T]} \left( (U_{s}^{n-1,m-1})^- \right)^2 + \beta n^2 \mathbb{E} \left\{ \left( \int_t^T (Y_{s}^{n,m} - L_{s}^{n-1,m-1})^- ds \right)^2 \right\} + \beta n^2 \mathbb{E} \left\{ \left( \int_t^T (Y_{s}^{n,m} - U_{s}^{n-1,m-1})^- ds \right)^2 \right\},$$

(4.23)

where, according to Assumption (A2)-(ii),(d) we have

$$\mathbb{E} \left\{ \sup_{s \in [t,T]} \left( (L_{s}^{n-1,m-1})^+ \right)^2 + \sup_{s \in [t,T]} \left( (U_{s}^{n-1,m-1})^- \right)^2 \right\} \leq 2 \mathbb{E} \left\{ \sup_{s \in [t,T]} |X_s|^2 \right\} < +\infty.$$

(4.24)

Then, substituting (4.20), (4.22) and (4.24) to (4.23), we get

$$\mathbb{E}[Y_{t}^{n,m}]^2 + \mathbb{E} \left( \int_t^T |Z^{n,m}_s|^2 ds \right)$$

$$\leq \mathbb{C} \left( 1 + (2 + \frac{1}{\alpha}) \mathbb{E} \left( \int_t^T (Y_{s}^{n,m})^2 ds \right) + \alpha \mathbb{E} \left( \int_t^T (Y_{s}^{n-1,m-1})^2 ds \right) + \frac{1}{\beta} \mathbb{E} \left( \int_t^T |Z^{n,m}_s|^2 ds \right) \right)$$

$$+ C \beta \mathbb{E} \left( 1 + \mathbb{E} \left( \int_t^T (Y_{s}^{n,m})^2 ds \right) + \int_t^T (Y_{s}^{n-1,m-1})^2 ds + \int_t^T |Z^{n,m}_s|^2 ds \right).$$

(4.25)

By choosing $\alpha$ and $\beta$ satisfying $(\alpha + \beta)C = \frac{1}{4}$ in the previous inequality, we obtain

$$\mathbb{E}[Y_{t}^{n,m}]^2 \leq \bar{\mathbb{C}} \mathbb{E} \left( 1 + \mathbb{E} \left( \int_t^T (Y_{s}^{n,m})^2 ds \right) + \int_t^T (Y_{s}^{n-1,m-1})^2 ds \right),$$

(4.26)

where $\bar{\mathbb{C}}$ is a constant independent of $n$ and $m$.

Then, letting $n = m = 0$ and recalling $Y^{-1,-1} = Y^{0,0}$, it follows that

$$\mathbb{E}[Y_{t}^{0,0}]^2 \leq \bar{\mathbb{C}} \mathbb{E} \left( 1 + 2 \int_t^T (Y_{s}^{0,0})^2 ds \right).$$
Using Gronwall’s inequality, we have for all $t \in [0, T]$
\[ \mathbb{E}[(Y_t^{0,0})^2] \leq \gamma(t), \] (4.27)
where the function $\gamma : [0, T] \to \mathbb{R}$ is defined by
\[ \gamma(t) := \tilde{C}e^{\tilde{C}(T-t)}. \]

Next, we assume that for $n = k$, the following inequality is guaranteed for all $t \in [0, T]$.
\[ \mathbb{E}[(Y_t^{n,0})^2] \leq \gamma(t). \] (4.28)

Noticing the fact $\gamma(t) = \tilde{C}(1 + 2 \int_t^T \gamma(s)ds)$, we have
\[ \mathbb{E}[(Y_t^{k+1,0})^2 - \gamma(t)] \leq \tilde{C}\mathbb{E} \left( \int_t^T (Y_s^{k+1,0})^2 - \gamma(s) \right) ds + \tilde{C}\mathbb{E} \left( \int_t^T (Y_s^{k,0})^2 - \gamma(s) \right) ds \]
\[ \leq \tilde{C}\mathbb{E} \left( \int_t^T (Y_s^{k+1,0})^2 - \gamma(s) \right) ds. \]

Applying Gronwall’s inequality, we obtain
\[ \mathbb{E}[(Y_t^{k+1,0})^2] \leq \gamma(t) \] for any $t \in [0, T],$
which means (4.28) is also guaranteed for $n = k + 1$. Thus, (4.28) is correct for all $n \geq 0$.

Similarly, for a fixed $n$ and all $m \geq 0$, by using mathematical induction and Gronwall’s inequality, we can obtain the following inequality for any $t \in [0, T],$
\[ \mathbb{E}[(Y_t^{n,m})^2] \leq \gamma(t). \] (4.29)

Since $n$ is arbitrary, (4.29) is true for any $n, m \geq 0$. Noticing $\gamma(t)$ is bounded, we have
\[ \sup_{0 \leq t \leq T} \mathbb{E}[(Y_t^{n,m})^2] \leq C, \text{ uniformly in } (n, m). \] (4.30)

Then by (4.25), we can check
\[ \mathbb{E} \int_0^T |Z_t^{n,m}|^2 dt \leq C, \text{ uniformly in } (n, m). \] (4.31)

If we transfer (4.30) and (4.31) into (4.20) and (4.22) respectively, we get that
\[ n^2\mathbb{E} \left( \int_0^T (Y_t^{n,m} - U_t^{n,1,m-1})^2 ds \right) \leq C \text{ and } m^2\mathbb{E} \left( \int_0^T (Y_t^{m,n} - L_t^{n-1,m-1})^2 ds \right) \leq C. \] (4.32)

The result follows immediately. \[ \square \]

**Proposition 4.3.** For any $n \geq 0$, there exist processes $(Y^n, Z^n, K^n)$ that satisfy the following one barrier reflected BSDE:
\[
\begin{aligned}
Y_t^n &\in \mathcal{S}^2, \quad K_t^{n,+} \in \mathcal{S}^2 \text{ non-decreasing (}K_0^{n,+} = 0\text{) and } Z^n \text{ belongs to } \mathcal{H}^{2d};
Y_t^n &= \xi + \int_t^T f(s, Y_s^n, Y_t^n - U_t^{n-1}) + K_t^{n,+} - K_t^{n,-} \int_t^T (Y_s^n - U_s^{n-1})^+ ds - \int_t^T Z_s^n dB_s, t \leq T; \\
Y_t^n &\geq L_t^{n-1}, \quad t \leq T, \text{ and } \int_0^T (Y_t^n - L_t^{n-1})dK_t^{n,+} = 0,
\end{aligned}
\] (4.33)

where for any $n \geq 1$ and $t \leq T$, $L_t^{n-1} = h(t, Y_t^n, E[Y_t^{n-1}])$ and $U_t^{n-1} = g(t, Y_t^n, E[Y_t^{n-1}])$. Moreover the following estimates hold true:
\[ \mathbb{E}[\sup_{0 \leq t \leq T} (Y_t^n)^2] + \mathbb{E} \left[ \int_0^T |Z_t^n|^2 dt \right] + \mathbb{E}[\left( K_T^{n,+} \right)^2] + \mathbb{E}[n^2 \left( \int_0^T (Y_t^n - U_t^{n-1})^+ dt \right)^2] \leq C, \] (4.34)

where $C$ is a constant which does not depend on $n$. 

Proof. Recall that for any \(n,m \geq 0\), \((Y^{n,m}_t, Z^{n,m}_t)\) verify (4.39), i.e., for any \(t \leq T\),
\[
\begin{aligned}
Y^{n,m}_t &= \xi + \int_0^T f(s, Y^{n,m}_s, \mathbb{E}[Y^{n-1,m-1}_s])ds - n \int_0^T \left(Y^{n,m}_s - g(s, Y^{n-1,m-1}_s, \mathbb{E}[Y^{n-1,m-1}_s])\right) ds \\
&\quad + K^{n,m,+}_t - K^{n,m,-}_t - \int_0^T Z^{n,m}_s dB_s.
\end{aligned}
\]
(4.35)

On the other hand, we know that for any \(t \leq T\) and for any \(n, m \geq 0\), \(Y^{n,m}_t \leq Y^{n,m+1}_t\). Therefore for any \(n \geq 0\) and \(t \leq T\), let us set
\[
Y^m_t = \lim_{m \to \infty} Y^{n,m}_t.
\]
(4.36)

As \(Y^{n,m}\) is continuous adapted, therefore \(Y^n\) is a predictable process which is moreover l.s.c. On the other hand by Fatou’s Lemma and (4.15) we deduce that:
\[
\sup_{0 \leq t \leq T} \mathbb{E}[ (Y^n_t)^2 ] \leq C
\]
(4.37)

Next the proof will be divided into three steps.

Step 1: There exists a constant \(C\) such that for any \(n \geq 0\),
\[
\mathbb{E}[ \sup_{0 \leq t \leq T} (Y^n_t)^2 ] \leq C.
\]
(4.38)

For \(m \geq 0\), let \((\bar{Y}^m_n, \bar{Z}^m_n)\) be the solution of the following BSDE:
\[
\begin{aligned}
\bar{Y}^m_n &\in \mathcal{S}^2_t, \bar{Z}^m_n \in \mathcal{H}^{2,d}, \\
\bar{Y}^m_n(t) &= |\xi| + X_T + \int_0^T f(s, Y^n_s, \mathbb{E}[Y^{n-1}_s]) \bar{Z}^m_n(s)ds + m \int_0^T (\bar{Y}^m_n(s) - X_s\bar{Z}^m_n(s))ds - \int_0^T \bar{Z}^m_n(s)dB_s, \quad t \leq T.
\end{aligned}
\]
(4.39)

Now let \((\bar{Y}_n, \bar{Z}_n, \bar{K}_n)\) be the solution of the standard lower obstacle reflected BSDE associated with \((f(s, Y^n_s, \mathbb{E}[Y^{n-1}_s]), |\xi| + X_T, (X_t)_{t \leq T})\), i.e.,
\[
\begin{aligned}
\bar{Y}_n &\in \mathcal{S}^2_t, \bar{Z}_n \in \mathcal{H}^{2,d}, (\bar{K}_n \in \mathcal{S}^2_t, \text{non-decreasing and } \bar{K}_n(0) = 0); \\
\bar{Y}_n(t) &= |\xi| + X_T + \int_0^T f(s, Y^n_s, \mathbb{E}[Y^{n-1}_s]) \bar{Z}_n(s)ds + \bar{K}_n(T) - \bar{K}_n(t) - \int_0^T \bar{Z}_n(s)dB_s, \quad t \leq T. \\
\bar{Y}_n(t) &\geq X_t, \quad \forall t \leq T \quad \text{and} \quad \int_0^T (\bar{Y}_n(t) - X_t) d\bar{K}_n(t) = 0.
\end{aligned}
\]
(4.40)

Then there exists a constants \(C \geq 0\) such that
\[
\mathbb{E}[ \sup_{0 \leq t \leq T} (\bar{Y}^m_t)^2 ] \leq C \mathbb{E}[ (|\xi| + |X_T|)^2 ] + \sup_{t \leq T} |X_t|^2 + \int_0^T |f(s, Y^n_s, \mathbb{E}[Y^{n-1}_s], 0)|^2 ds \leq C.
\]
(4.41)

The second inequality stems from the facts that \(X \in \mathcal{S}^2, \xi \in L^2(d\mathbb{P})\) and finally in taking into account (4.37) and since \(f\) is Lipschitz and \((f(t, \omega, 0, 0, 0))_{t \leq T} \in \mathcal{H}^{2,1}\). Then by standard comparison results we have:
\[
Y^{n,0} \leq Y^{n,m} \leq Y^{n,m}_n \leq \bar{Y}_n,
\]
(4.42)

since \(h(t, y, y') \leq X_t\) for any \(t, y, y'\) (to infer the third inequality).

Next let \((\bar{Y}_n, \bar{X}_n, \bar{K}_n)\) be the solution of the following upper barrier reflected BSDE:
\[
\begin{aligned}
\bar{Y}_n &\in \mathcal{S}^2_t, \bar{Z}_n \in \mathcal{H}^{2,d}, (\bar{K}_n \in \mathcal{S}^2_t, \text{non-decreasing and } \bar{K}_n(0) = 0); \\
\bar{Y}_n(t) &= -|\xi| - |X_T| - \int_0^T f(s, \bar{Y}_n(s), \mathbb{E}[\bar{Y}_{n-1}(s)]) \bar{Z}_n(s)ds - \bar{K}_n(T) + \bar{K}_n(t) - \int_0^T \bar{Z}_n(s)dB_s, \quad t \leq T; \\
\bar{Y}_n(t) &\leq X_t, \quad \forall t \leq T \quad \text{and} \quad \int_0^T (\bar{Y}_n(t) - X_t) d\bar{K}_n(t) = 0.
\end{aligned}
\]
(4.43)

As previously in (4.41) and using the same inequality, there exists a constant \(C \geq 0\) which does not depend on \(n\) such that:
\[
\mathbb{E}[ \sup_{0 \leq t \leq T} (\bar{Y}_n(t))^2 ] \leq C \mathbb{E}[ (|\xi| + |X_T|)^2 ] + \sup_{t \leq T} |X_t|^2 + \int_0^T |f(s, \bar{Y}_n(s), \mathbb{E}[\bar{Y}_{n-1}(s)], 0)|^2 ds \leq C.
\]
(4.44)
On the other hand thanks to \( g(t, y, y') \geq X_t \) and the monotonicity property of \( f \) w.r.t. \( y \) and \( y' \), by using a standard comparison argument we deduce that \( \mathbb{P} - a.s., \) for any \( t \leq T, \) \( \bar{Y}_n(t) \leq Y^n_{t,0} \). Going back now to (4.42), in using the estimates obtained in (4.41) and (4.44), one deduces the existence of a constant \( C \) which does not depend on \( n \) such that the estimate (4.38) is satisfied. Finally let us notice that we have also

\[
\mathbb{E}[n^2 \left( \int_0^T (Y^n_t \mathbb{1}_{-U^n_{t-1}} + dt) \right)^2] \leq C,
\]

which is a consequence of the convergence of \((Y^{n,m})_{m \geq 0}\) to \( Y^n \) and the estimate 4.16.

Step 2: For any \( n \geq 0 \), \( Y^n \) is continuous.

Recall that \((Y^{n,m}, Z^{n,m})\) verify (4.35) and let \( n \) be fixed. For any \( m \geq 0 \), by (4.16)

\[
\mathbb{E}[\{m \int_0^T (Y^{n,m}_t - L^{n-1,m-1}_t)^2 \}^2] \leq C,
\]

where \( C \) is a constant independent of \( m \) (and also \( n \)). Therefore using a result by A.Uppman (28), Théorème 1, pp. 289), there exist a subsequence \((m_l)_{l \geq 0}\) and an adapted non-decreasing process \( K^{n,+}_\tau \) such that for any stopping time \( \tau \),

\[
K^{n,m_l,+}_\tau \rightarrow K^{n,+}_\tau \text{ weakly in the sense of } \sigma(L^1, L^\infty)-\text{topology, as } l \rightarrow \infty.
\]

But the uniform estimate (4.46) implies that

\[
K^{n,m_l,+}_\tau \rightarrow K^{n,+}_\tau \text{ weakly in } L^2(d\mathbb{P}) \text{ as } l \rightarrow \infty.
\]

Next using estimate (4.45) and since \( g \) is of linear growth at most, one deduces the existence of a constant \( C_n \) which may depend on \( n \) such that

\[
\mathbb{E}\left\{ \int_0^T \left| f(s, Y^{n,m}_s, \mathbb{E}[Y^{n-1,m-1}_s], Z^{n,m}_s) \right|^2 + |n| \left( Y^{n,m}_s - g(s, Y^{n-1,m-1}_s, \mathbb{E}[Y^{n-1,m-1}_s]) \right)^2 + |Z^{n,m}_s|^2 ds \right\} \leq C_n.
\]

Therefore there exists a subsequence which we still denote by \((m_l)_{l \geq 0}\) such that

\[
(f(s, Y^{n,m_l}_s, \mathbb{E}[Y^{n-1,m-1}_s], Z^{n,m_l}_s))_{s \leq T} \rightarrow (\Phi_n(s))_{s \leq T} \text{ weakly in } \mathcal{H}^{2,1} \text{ as } l \rightarrow \infty,
\]

\[
(n(Y^{n,m_l}_s - g(s, Y^{n-1,m-1}_s, \mathbb{E}[Y^{n-1,m-1}_s]))_{s \leq T} \rightarrow (\Sigma_n(s))_{s \leq T} \text{ weakly in } \mathcal{H}^{2,1} \text{ as } l \rightarrow \infty.
\]

and

\[
(Z^{n,m_l}_s)_{s \leq T} \rightarrow (Z_n(s))_{s \leq T} \text{ weakly in } \mathcal{H}^{2,1} \text{ as } l \rightarrow \infty.
\]

Going back now to (4.33) and considering the equation between 0 and \( \tau \) with \( m_l \) and finally sending \( l \) to infinity, we deduce that for any stopping time \( \tau \) it holds:

\[
\mathbb{P} - a.s., \quad Y^n_T = Y^n_0 - \int_0^T \Phi_n(s) ds + \int_0^T \Sigma_n(s) ds - K^{n,+}_\tau + \int_0^T Z_n(s) dB_s.
\]

Next let us consider \((\tau_k)_{k \geq 0}\) a decreasing sequence of stopping times which converges to some \( \theta \). As the filtration \((\mathcal{F}_t)_{t \leq T}\) satisfies the usual conditions (and then it is right continuous) then \( \theta \) is also an \((\mathcal{F}_t)_{t \leq T}\)-stopping time valued in \([0, T]\). On the other hand for any \( k \geq 0 \),

\[
\mathbb{P} - a.s., \quad Y^n_{\tau_k} - Y^n_\theta = -\int_\theta^{\tau_k} \Phi_n(s) ds + \int_\theta^{\tau_k} \Sigma_n(s) ds - (K^{n,+}_{\tau_k} - K^{n,+}_\theta) + \int_\theta^{\tau_k} Z_n(s) dB_s
\]

which, in taking expectation, yields:

\[
\mathbb{E}[Y^n_{\tau_k} - Y^n_\theta] = \mathbb{E}[-\int_\theta^{\tau_k} \Phi_n(s) ds + \int_\theta^{\tau_k} \Sigma_n(s) ds - \mathbb{E}[(K^{n,+}_{\tau_k} - K^{n,+}_\theta)].
\]

But the first term in the right-hand side converges to 0 as \( k \rightarrow \infty \). On the other hand by the weak limit and (4.21) we have:

\[
\mathbb{E}[(K^{n,+}_{\tau_k} - K^{n,+}_\theta)] = \lim_{k \rightarrow \infty} \mathbb{E}[(K^{n,m_l,+}_{\tau_k} - K^{n,m_l,+}_\theta)] \leq \limsup_{k \rightarrow \infty} \mathbb{E}[(V^{n,+}_{\tau_k} - V^{n,+}_\theta)] + \int_\theta^{\tau_k} |f(s, Y^{n,m_l}_s, \mathbb{E}[Y^{n-1,m-1}_s], Z^{n,m_l}_s)| ds
\]

\[
\leq \mathbb{E}[(V^{n,+}_{\tau_k} - V^{n,+}_\theta)] + \limsup_{k \rightarrow \infty} \mathbb{E}[(f(s, Y^{n,m_l}_s, \mathbb{E}[Y^{n-1,m-1}_s], Z^{n,m_l}_s)| ds.
\]

(4.47)
As the process \(|f(s, Y_s^n, m, Z_s^n, m)|\) belongs uniformly to \(\mathcal{H}^{2,1}\), then
\[
\lim_{k \to \infty} \mathbb{E}[(K_{s_k}^{n,+} - K_{s_k}^{n,-})] = 0
\]
and thus \(\lim_{k \to \infty} \mathbb{E}[Y_{s_k}^n - Y_{s_k}^n] = 0\). Now as \(Y\) is a predictable (and then optional) process, \(\tau\) and \(\tau_k\) are arbitrary, then, by a result by Dellacherie-Meyer [8], pp.120, Théorème 48) the process \(Y^n\) is right continuous. In the same way if \((\tau_k)_{k>0}\) is an increasing sequence of stopping times which converges to a predictable stopping time then \(\mathbb{E}[Y^n_{\tau} - Y^n_{\tau}] = 0\). As \(Y^n\) is predictable then \(Y^n\) is left continuous. Consequently \(Y^n\) is continuous (see [8], pp.120, Théorème 48).

Step 3: There exist processes \((Y^n, Z^n, K^n)\) that satisfy \(4.33\) and \(4.34\).

Actually as \(Y^{n,m} \not\subset Y^n\) and those processes are continuous then thanks to Dini’s theorem the convergence is uniform \(\omega\) by \(\omega\) \(\mathbb{P} - a.s.\). Finally by the Lebesgue dominated converge theorem and \(4.38\) we have
\[
\lim_{m \to n} \mathbb{E}[(Y^{n,m}_t - Y^n_t)^2] = 0.
\]

Next going back to \(4.35\) and using Itô’s formula with \((Y^{n,m} - Y^{n,q})^2\) and taking into account of \(4.16\), we obtain:
\[
\mathbb{E} \left[ \frac{\int_0^T (Z_s^{n,m} - Z_s^{n,q})^2 ds}{t} \right] \to 0 \quad \text{as} \quad m, q \to \infty.
\]

Thus let us denote by \(Z^n\) the limit in \(\mathcal{H}^{2,2}\) of the sequence \((Z^{n,m})_{m \geq 0}\) which exists since it is of Cauchy type in this normed complete linear space. Then we have also the convergence of \((f(s, Y_s^{n,m}, Z_s^{n,m})|s \leq T\) and \(n(Y_s^{n,m} - g(s, Y_s^{n,m-1}, Z_s^{n,m-1})), s \leq T)\) in \(\mathcal{H}^{2,1}\) toward \((f(s, Y_s^n, Z_s^n), s \leq T)\) respectively.

Next from \(4.2\), in taking the limit w.r.t \(m\), one deduces that
\[
\mathbb{E} \left[ \frac{\int_0^T (Y_s^n - h(s, Y_s^n, Z_s^n))^- ds}{t} \right] = 0
\]
which implies, by continuity, that for any \(s \leq T, Y_s \geq h(s, Y_s^n, Z_s^n)\). Finally for any \(t \leq T\), let us set
\[
K_{s_k}^{n,+} = Y_{s_k}^n - Y_{s_k}^n - \int_0^t f(s, Y_s^n, Z_s^n) ds + n \int_0^t (Y_s^n - U_s^{n-1})^+ ds + \int_0^T Z_s^n dB_s.
\]

Therefore \(K^{n,+}\) is continuous and is nothing but the limit w.r.t \(m\) in \(S^2\) of \(K^{n,m,+}\). As \(K^{n,m,+}\) is non-decreasing then \(K^{n,+}\) is also non-decreasing. Finally let us notice that for any \(m \geq 0\), we have
\[
\forall s \leq T, Y_s^{n,m} \wedge L_s^{n-1,m-1} \leq Y_s^{n,m}
\]
and
\[
\int_0^T (Y_s^n - Y_s^{n,m} \wedge L_s^{n-1,m-1}) dB_s^{K^{n,m,+}} = 0.
\]

As \((Y_s^{n,m} - Y_s^{n,m} \wedge L_s^{n-1,m-1}), s \leq T\) and \((K^{n,m,+}), s \leq T\) converge in \(S^2\) toward \((Y_s - L_s^{n-1}), s \leq T\) and \(K^{n,+}\) respectively (as \(m \to \infty\)) then by Helly’s Theorem ([17], pp.370) we have:
\[
\int_0^T (Y_s^n - h(s, Y_s^n, Z_s^n))^- ds = 0.
\]

It follows that \((Y^n, Z^n, K^{n,+})\) verify the reflected BSDE \(4.33\). The remaining estimates of \(4.34\) stem from: (i) the convergence of the sequence \((Z^{n,m})_{m \geq 0}\) in \(\mathcal{H}^{2,2}\) and \(4.31\); (ii) the above definition of \(K^{n,+}\) in combination with \(4.45\) mainly.

To resume, from \(4.11\), we deduce that for any \(n \geq 0\), \(Y^{n+1} \leq Y^n\). Then let us set
\[
\mathbb{P} - a.s., \forall t \leq T, Y_t = \lim_{n \to \infty} Y_t^n.
\]

Note that by \(4.38\), for any stopping time \(\tau\), \((Y^n_{\tau})_{n \geq 0}\) converges toward \(Y_{\tau}\) in \(L^1(d\mathbb{P})\). We are now ready to give the main result of this part.

**Theorem 4.4.** There exist processes \(Z\) and \(K^\pm\) such that \((Y, Z, K^\pm)\) verify:

\[
\begin{align*}
Y \in \mathcal{S}^2_{\mathcal{C}}, Z \in \mathcal{H}^{2,2}, K^\pm \in \mathcal{S}^2_{\mathcal{C}},
Y_t &= \xi + \int_0^T f(s, Y_s, E[Y_s], Z_s) ds + K_T^+ - K_T^- - (K_T^+ - K_T^-) - \int_0^T Z_s dB_s, t \leq T;

h(t, Y_t, E[Y_t]) \leq Y_t \leq g(t, Y_t, E[Y_t]), t \leq T;

\int_0^T (Y_t - h(t, Y_t, E[Y_t])) dK_t^+ = \int_0^T (g(t, Y_t, E[Y_t]) - Y_t) dK_t^- = 0.
\end{align*}
\]
Proof. : For any \( n \geq 0 \), \((Y^n, Z^n, K^{n+})\) verify (4.33), i.e.,

\[
\begin{aligned}
Y^n_t &= \xi + \int_t^T f(s, Y^n_s, \mathbb{E}[Y^n_s^{-1}], Z^n_s)ds + K^{n+}_t - K^{n+}_T - n \int_t^T (Y^n_s - U^{n-1}_s)^+ ds - \int_t^T Z^n_s dB_s, \quad t \leq T; \\
Y^n_T \geq L^{n-1}_t, \quad t \leq T, \quad \text{and} \quad \int_0^T (Y^n_t - L^{n-1}_t)dK^{n+}_t = 0.
\end{aligned}
\] (4.51)

First note that by (4.34) we have

\[
\mathbb{E}\left[(K^{n+}_t)^2\right] + \mathbb{E}\left[n^2 \left(\int_t^T (Y^n_s - U^{n-1}_s)^+ dt\right)^2\right] \leq C
\] (4.52)

where \( C \) is constant independent of \( n \). On the other hand by (4.19) and (4.21) for any stopping times \( \eta \leq \tau \) we have:

\[
\mathbb{E}[n \int_\eta^\tau (Y^n_s - g(s, Y^n_s^{-1}, \mathbb{E}[Y^n_s^{-1}]))^+ ds] \leq \mathbb{E}[(V^+ - V^+) + \mathbb{E}[\int_\eta^\tau |f(s, Y^n_s, \mathbb{E}[Y^n_s^{-1}], Z^n_s)|ds].
\] (4.53)

and

\[
\mathbb{E}[K^{n+}_\tau - K^{n+}_\eta] \leq \mathbb{E}[(V^+ - V^+) + \mathbb{E}[\int_\eta^\tau |f(s, Y^n_s, \mathbb{E}[Y^n_s^{-1}], Z^n_s)|ds].
\] (4.54)

Therefore once more by Uppman’s result ([28], Théorème 1, pp. 289) and (4.52), there exists a subsequence \((\eta_i)_{i \geq 0}\) such that for any stopping \( \tau \),

\[
K^{n_i+}_\tau \to K^+_\tau \text{ weakly in the sense of } \sigma(L^1, L^\infty)-\text{topology as } l \to \infty
\]

and

\[
K^{n_i-}_\tau := n \int_0^\tau (Y^n_s - g(s, Y^n_s^{-1}, \mathbb{E}[Y^n_s^{-1}]))^+ ds \to K^-_\tau \text{ weakly in the sense of } \sigma(L^1, L^\infty)-\text{topology as } l \to \infty,
\]

where \( K^\pm \) are adapted non-decreasing processes. Next using estimate (4.52) one deduces that the previous convergences hold also in \( L^2(d\mathbb{P}) \) weakly and not only for the \( \sigma(L^1, L^\infty)-\text{topology} \). Next since by (4.34), there exists a constant \( C \) independent of \( \eta_i \) such that

\[
\mathbb{E}[\int_0^\tau \{|f(s, Y^n_s, \mathbb{E}[Y^n_s^{-1}], Z^n_s)|^2 + |Z^n_s|^2\}ds] \leq C,
\]

then there exists a subsequence which we still denote by \((\eta_i)_{i \geq 0}\) such that \((f(s, Y^n_{s_i}, \mathbb{E}[Y^n_{s_i-1}], Z^n_{s_i}))_{s_i \leq T})_{i \geq 0}\) (resp. \((f(s, Y^n_{s_i}, \mathbb{E}[Y^n_{s_i-1}], Z^n_{s_i}))_{s_i \leq T})_{i \geq 0}\) resp. \((Z^n_{s_i})_{i \geq 0}\)) converges weakly in \( \mathcal{H}^{2,1} \) (resp. \( \mathcal{H}^{2,1} \), resp. \( \mathcal{H}^{2,1} \)) to some process \( \mathcal{F}(s)_{s \leq T}\) (resp. \( \mathbb{E}\), resp. \( Z\)) as \( l \to \infty \). Therefore from (4.53) and (4.54) and by Fatou’s Lemma we deduce that: For any stopping times \( \eta \leq \tau \),

\[
\mathbb{E}[K^+_\tau - K^-_\eta] \leq \mathbb{E}[(V^+ - V^+) + \mathbb{E}[\int_\eta^\tau \mathbb{E}(s)|ds]
\] (4.55)

and

\[
\mathbb{E}[K^-_\tau - K^-_\eta] \leq \mathbb{E}[(V^+ - V^+) + \mathbb{E}[\int_\eta^\tau \mathbb{E}(s)|ds].
\] (4.56)

Next from the equation (4.51) written forwardly, we deduce that for any stopping time \( \tau \) it holds:

\[
Y_\tau = Y_0 - \int_0^\tau \Phi(s)ds - K^+_\tau + K^-_\tau + \int_0^\tau Z_s dB_s.
\] (4.57)

Now as the process \( Y \) is predictable then if \((\tau_k)_{k \geq 0}\) is a decreasing sequence of stopping times that converges to \( \theta \) then

\[
\lim_{n \to \infty} \mathbb{E}[Y_{\tau_k} - Y_\theta] = 0
\]

since \( K^+ \) and \( K^- \) satisfy the inequalities in (4.55) and (4.56) respectively. Thus \( Y \) is right continuous ([8], pp.120, Théorème 48).

In the same way if \((\tau_k)_{k \geq 0}\) is an increasing sequence of predictable stopping times that converges to a predictable stopping time \( \theta \) then

\[
\lim_{n \to \infty} \mathbb{E}[Y_{\tau_k} - Y_\theta] = 0.
\]

Therefore, similarly, the predictable process \( Y \) is left continuous and then continuous.

Now we resume as we did in the proof of Proposition 4.3 Thanks to Dini’s Theorem and dominated convergence Theorem, the convergence of the sequence \((Y^n_{s})_{n \geq 0}\) to \( Y \) holds in \( \mathcal{S}^2 \) and by Itô’s formula
applied with \((Y^n - Y^g)^2\) we obtain that the sequence of processes \((Z^n)_{n \geq 0}\) is of Cauchy type in \(\mathcal{H}^{2,d}\) and then converges to a process \(Z\) which belongs to \(\mathcal{H}^{2,d}\). Now from the inequality (4.34) we deduce that

\[
\mathbb{E}\left[\int_0^T (Y^n_s - g(s, Y^n_{s-1}, \mathbb{E}[Y^n_{s-1}]))^+ ds \right] \leq Cn^{-1}.
\]

Sending \(n \to +\infty\) and using the uniform convergence of \((Y^n)_{n}\) and continuity of \(Y\) and \((g(t, Y_t, \mathbb{E}[Y_t]))_{t \leq T}\) to deduce that \(Y_t \leq g(t, Y_t, \mathbb{E}[Y_t])\) for any \(t \leq T\). Thus for any \(t \leq T\),

\[
h(t, Y_t, \mathbb{E}[Y_t]) \leq Y_t \leq g(t, Y_t, \mathbb{E}[Y_t])
\]

since \(Y^n \geq L^{n-1}\). Next the following properties hold true:

\[
\forall s \leq T, Y^n_s \geq U^{n-1}_s \geq Y^n_s \text{ and } \int_0^T (Y^n_s \vee U^{n-1}_s - Y^n_s) dK^n_s = 0. \tag{4.58}
\]

Combining this with the backward equation (4.51) that \(Y^n\) verifies and based on already known results on double barrier reflected BSDEs and zero-sum Dynkin games (see [2][14] for more details), one deduces that for any \(t \leq T\),

\[
Y^n_t = \text{ess sup}_{\tau \geq t} \max \mathbb{E}\left[ \int_t^\tau \left[ f(s, Y^n_s, \mathbb{E}[Y^n_{s-1}], Z^n_s) ds + \sup_{\sigma \geq t} \sup_{s \leq \tau} g(s, Y^n_s, \mathbb{E}[Y^n_s]) 1_{\sigma < \tau} + (L^{n-1}_s - Y^n_s) 1_{\sigma = \tau} \right] dB_s \right]. \tag{4.59}
\]

So let now \(\tilde{Y}\) be the process defined as follows: For any \(t \leq T\),

\[
\tilde{Y}_t = \text{ess sup}_{\tau \geq t} \max \mathbb{E}\left[ \int_t^\tau \left[ f(s, Y_s, \mathbb{E}[Y_s], Z_s) ds + \sup_{\sigma \geq t} \sup_{s \leq \tau} g(s, Y_s, \mathbb{E}[Y_s]) 1_{\sigma < \tau} + h(s, Y_s, \mathbb{E}[Y_s]) 1_{\sigma = \tau} \right] dB_s \right]. \tag{4.60}
\]

The process \(\tilde{Y}\) is related to double barrier reflected BSDEs in the following way: There exist processes \(\tilde{Z} \in \mathcal{H}^{2,d}\) and non-decreasing continuous processes \(\tilde{K}^\pm \in \mathcal{S}^2 \ (\tilde{K}^- = 0)\) such that

\[
\begin{align*}
\tilde{Y}_t &= \tilde{\xi} + \int_t^T f(s, Y_s, \mathbb{E}[Y_s], Z_s) ds + \tilde{K}^+_t - \tilde{K}^-_t - (\tilde{K}^+_t - \tilde{K}^-_t) - \int_t^T \tilde{Z}_s dB_s, \quad t \leq T; \\
h(t, Y_t, \mathbb{E}[Y_t]) &\leq \tilde{Y}_t = g(t, Y_t, \mathbb{E}[Y_t]), \quad t \leq T; \\
\int_0^T (\tilde{Y}_t - h(t, Y_t, \mathbb{E}[Y_t])) d\tilde{K}^+_t &= \int_0^T (g(t, Y_t, \mathbb{E}[Y_t]) - \tilde{Y}_t) d\tilde{K}^-_t = 0.
\end{align*}
\]

This is mainly due to the facts that Mokobodski’s condition is satisfied, i.e., for any \(t \leq T, h(t, Y_t, \mathbb{E}[Y_t]) \leq X_t \leq g(t, Y_t, \mathbb{E}[Y_t])\) and all those processes belong to \(\mathcal{S}^2\) (one can see e.g. [17][14]). Next since

\[
|\text{ess sup}_{\tau \geq t} \text{ess inf}_{\sigma \leq \tau} \Sigma_{1,\tau,\sigma} - \text{ess sup}_{\tau \geq t} \text{ess inf}_{\sigma \leq \tau} \Sigma_{2,\tau,\sigma}| \leq \text{ess sup}_{\tau \geq t} \text{ess sup}_{\sigma \geq t} |\Sigma_{1,\tau,\sigma} - \Sigma_{2,\tau,\sigma}|,
\]

then for any \(t \leq T\),

\[
|Y^n_t - \tilde{Y}_t| \leq \mathbb{E}\left[ \int_0^T |f(s, Y_s, \mathbb{E}[Y_s], Z_s) - f(s, Y^n_s, \mathbb{E}[Y^n_s], Z^n_s)| ds + \sup_{s \leq T} |Y^n_s \vee g(s, Y^n_{s-1}, \mathbb{E}[Y^n_{s-1}]) - g(s, Y_s, \mathbb{E}[Y_s])| + \sup_{s \leq T} |h(s, Y_s, \mathbb{E}[Y_s]) - h(s, Y^n_{s-1}, \mathbb{E}[Y^n_{s-1}])| \right].
\]

Then by Doob’s inequality ([26], pp.54), there exists a constant \(C \geq 0\) such that:

\[
\mathbb{E}\left[ \sup_{t \leq T} |Y^n_t - \tilde{Y}_t|^2 \right] \leq C \left\{ \mathbb{E}\left[ \sup_{s \leq T} |Y^n_s \vee g(s, Y^n_{s-1}, \mathbb{E}[Y^n_{s-1}]) - g(s, Y_s, \mathbb{E}[Y_s])|^2 \right] + \mathbb{E}\left[ \sup_{s \leq T} |h(s, Y_s, \mathbb{E}[Y_s]) - h(s, Y^n_{s-1}, \mathbb{E}[Y^n_{s-1}])|^2 \right] \right\} = A^n_2 + A^n_3 + A^n_4.
\]

But \(\lim_{n \to \infty} A^n_3 = 0\) since \((Y^n)_{n}\) converges to \(Y\) in \(\mathcal{S}^2\) and \((Z^n)_{n}\) converges to \(Z\) in \(\mathcal{H}^{2,d}\) and \(f\) is uniformly Lipschitz w.r.t \((y, y', z)\). Next, \(\lim_{n \to \infty} A^n_4 = 0\) by Dini’s Theorem and Lebesgue dominated convergence one
since \( h \) is monotone and \( h(t, Y_t, \mathbb{E}[Y_t]) \) is continuous w.r.t. \( t \). Finally using the same arguments one deduces that \( \lim_{n \to \infty} A^n_t = 0 \) as for any \( t \leq T, \ Y_t \leq \xi(t, Y_t, \mathbb{E}[Y_t]) \). Thus

\[
\lim_{n \to \infty} \mathbb{E}[\sup_{t \leq T} |Y^n_t - \bar{Y}_t|^2] = 0
\]

and then \( Y = \bar{Y} \). Next by (4.57) and (4.61) one deduces that \( Z = \bar{Z} \). Therefore \( (Y, Z, \bar{K}^\pm) \) verify (4.50), i.e., it is a solution of the mean-field reflected BSDE associated with \((f, \xi, h, g)\).

As a by-product we obtain the following result related to the approximation of the solution of (4.1). Its proof is based on uniqueness of the solution of that equation.

**Corollary 4.5.** Assume that the mean-field RBSDE (4.1) has a unique solution \((Y, Z, K^\pm)\). Then

\[
\lim_{n \to \infty} \mathbb{E}[\sup_{t \leq T} |Y^n_t - Y_t|^2] = 0.
\]

where for any \( n \geq 0, Y^n \) is the limit in \( S^2_t \) w.r.t of \( Y^{n,m} \) solution of (4.9) and which satisfies also (4.33).

**Remark 4.6.** The solution of (4.9) is unique when:

(i) \( g \) and \( h \) verify moreover (2.4) and (3.13) with \( p = 2 \).

(ii) \( f \) does not depend on \( z \) or \( f((t, \omega, y, y^0, z) = \Phi(t, \omega, y, y^0) + a_t z \) where \( (a_t)_{t \leq T}\) is, e.g., an adapted bounded process.

In this latter case of \( f \), uniqueness is obtained after a change of probability by using Girsanov’s theorem and arguing mainly as in Section 3.1.

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