Vector-valued Poincaré inequality in infinite dimension with respect to a weighted Gaussian measure and applications

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Abstract

We consider the spaces $L^p(X, \nu; V)$, where $X$ is a separable Banach space, $\mu$ is a centred non-degenerate Gaussian measure, $\nu := Ke^{-U}\mu$ with normalizing factor $K$ and $V$ is a separable Hilbert space. In this paper we prove a vector-valued Poincaré inequality for functions $F \in W^{1,p}(X, \nu; V)$, which allows us to show that for any $p \in (1, +\infty)$ and any $k \in \mathbb{N}$ the norm in $L^p(X, \nu)$ is equivalent to the graph norm of $D^k H$ in $L^p(X, \nu)$. Further, we provide a logarithmic Sobolev inequality for vector-valued functions $F \in FC^1_b(X; V)$ and, as a consequence, we obtain that the vector-valued perturbed Ornstein-Uhlenbeck semigroup $(T^V(t))_{t \geq 0}$ is hypercontractive. To conclude, we show exponential decay estimates for $(T^V(t))_{t \geq 0}$ as $t \to +\infty$. Useful tools are the study of the asymptotic behaviour of the scalar perturbed Ornstein-Uhlenbeck $(T(t))_{t \geq 0}$, and pointwise estimates for $|D^k H T(t)f|_p$ by means both of $T(t)|D^k H f|_p$ and of $T(t)|f|_p$.

Keywords: Vector-valued Poincaré inequality; Sobolev spaces; vector-valued logarithmic Sobolev inequalities; abstract Wiener spaces; vector-valued perturbed Ornstein-Uhlenbeck semigroup

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1 Introduction

Let $X$ be a separable Banach space, and let $(X, \mu, H)$ be an abstract Wiener space, i.e., $\mu$ is a centred non-degenerate Gaussian measure on $X$ and $H$ is the Cameron-Martin space associated to $\mu$. Further, let $\nu := Ke^{-U}\mu$, where $U : X \to \mathbb{R}$ is a smooth enough convex function and $K = \|e^{-U}\|_{L^1(X, \mu)}^{-1}$ is the normalizing constant which gives $\nu(X) = 1$, and let $V$ be a separable Hilbert space. The aim of this paper is to generalize to the spaces $L^p(X, \nu)$ and $L^p(X, \nu; V)$ some important results which are already known in the $L^p(X, \mu)$ setting.

Abstract Wiener spaces have been introduced by Gross in [16] to study the properties of Gaussian measures on infinite-dimensional spaces. The fundamental idea in the Theory of Gaussian measures is that any centred Gaussian measure is the realization of the same "canonical" Gaussian measure: the countable product of the standard normal Gaussian distributions on the line. This fact, and the fact that in infinite dimension does not exists an analogous of Lebesgue measure, have increased the interest around Gaussian measures in infinite dimension, both from an analytic and a probabilistic point of view (see e.g. [7, 18, 21, 23]). In an abstract Wiener space, the Gaussian measure $\mu$ factors according to an orthogonal decomposition of $H$, and this implies that many results can be obtained arguing in finite dimension and then letting the dimension to infinity. The situation completely change

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when $\mu$ is replaced by the measure $\nu$; in this case the finite dimensional approximations do not always work, and even when it happens computations are delicate and much more complicated.

The central point of our investigation is a family of vector-valued Poincaré inequalities, and, as a consequence, the characterization of the Sobolev spaces $W^{k,p}(X,\nu)$, with $k \in \mathbb{N}$ and $p \in (1, +\infty)$. To be more precise, by means of the asymptotic behaviour of the vector-valued perturbed Ornstein-Uhlenbeck semigroup $(T^V(t))_{t \geq 0}$ we are able to show that, for any $p \in [1, +\infty)$, there exists a positive constant $k_p$ such that for any $F \in W^{1,p}(X,\nu;V)$ it holds that

$$
\|F - \nu(F)\|_{L^p(X,\nu;V)} \leq k_p \|D_H F\|_{L^p(X,\nu;H \otimes V)},
$$

where

$$
\nu(F) := \int_X F d\nu \in V.
$$

We stress that in the case of Gaussian measures, i.e., when $U = 0$, and when $p \in (1, +\infty)$, these inequalities for vector-valued functions have been obtained in [19, Proposition 3.1] in the setting of Malliavin Calculus and when $V$ is a UMD space. However, the authors use a vector-valued version of Meyer’s Multiplier Theorem, which does not work for $p = 1$ and which cannot be adapted to our situation since it strongly relies on the decomposition of the Gaussian measure $\mu$, which is no longer available for $\nu$. In the specific case when $V = \mathcal{H}_k(H)$, $f \in W^{k+1,p}(X,\nu)$ and $F := D_H^k f$, inequality (1.1) reads as

$$
\|D_H^k f - \nu(D_H^k f)\|_{L^p(X,\nu;\mathcal{H}_k(H))} \leq k_p \|D_H^{k+1} f\|_{L^p(X,\nu;\mathcal{H}_{k+1}(H))}. \tag{1.2}
$$

Formula (1.2) allows us to prove the characterization of the Sobolev spaces $W^{k,p}(X,\nu)$, with $k \in \mathbb{N}$ and $p \in (1, +\infty)$ by means of the graph norm. These spaces have been introduced in [13] and they are defined as the domain of the closure of the operator

$$(D_H, \ldots, D_H^k) : \mathcal{F}C^\infty_b(X) \to L^p(X,\nu;H) \times \ldots \times L^p(X,\nu;\mathcal{H}_k(H)),$$

in $L^p(X,\nu)$, with $k \in \mathbb{N}$ and $p \in (1, +\infty)$, endowed with the norm

$$
\|f\|_{k,p} := \left( \sum_{j=0}^{k} \|D_H^j f\|_{L^p(X,\nu;\mathcal{H}_j(H))}^p \right)^{1/p}, \quad f \in W^{k,p}(X,\nu).
$$

We prove that the question if $\| \cdot \|_{k,p}$ is equivalent to the graph norm

$$
\|f\|_{p,D_H^k} := \|f\|_{L^p(X,\nu)} + \|D_H^k f\|_{L^p(X,\nu;\mathcal{H}_k(H))}, \quad f \in \mathcal{F}C^\infty_b(X;V),
$$

has positive answer, and we extend the result of [19, Corollary 3.2] where this equivalence is shown for the Gaussian measure $\mu$.

We conclude the paper by providing a family of logarithmic Sobolev inequalities for vector-valued functions, which allows us to prove an hypercontractivity property for the vector-valued perturbed Ornstein-Uhlenbeck semigroup $(T^V(t))_{t \geq 0}$. Finally, we refine the asymptotic behaviour of $(T^V(t))_{t \geq 0}$ by showing that it satisfies an exponential decay. All these results seem to be new also in finite dimension, where the above issues are studied and proved only for scalar functions.

The paper is organized as follows.

In Section 2 we collect the background which we need in the sequel of the paper. In particular, we present the structure of abstract Wiener space, listing the main features of the Cameron-Martin space, of the Ornstein-Uhlenbeck semigroup and of the Sobolev spaces $W^{k,p}(X,\mu)$. Then, we give the definition and the principal properties of the Wiener chaos decomposition which plays a crucial role in the proof of Theorems 5.1 and 5.3. Later, we provide the assumptions on the function $U$, we introduce the Sobolev spaces $W^{k,p}(X,\nu)$ and $W^{1,p}(X,\nu;V)$, with $k \in \mathbb{N}$ and $p \in [1, +\infty)$, where $V$ is
a separable Hilbert space. Besides the results in [13], we show other properties of the elements of the functions belonging to these Sobolev spaces which will be used in the following. We also introduce the semigroup \( (T(t))_{t \geq 0} \) in \( L^2(X, \nu) \) associated with the symmetric bilinear form

\[
\mathcal{E}(u, v) := \int_X [D_H u, D_H v] d\nu, \quad u, v \in W^{1,2}(X, \nu),
\]

and its vector-valued extension \( (T^V(t))_{t \geq 0} \) in \( L^p(X, \nu; V) \), \( p \in [1, +\infty) \). Finally, inspired by [4, Section 2] we provide smooth approximations of the semigroup \( (T(t))_{t \geq 0} \).

Section 3 is devoted to the study of the asymptotic behaviour of \( (T(t))_{t \geq 0} \) in \( L^p(X, \nu; V) \) and to provide pointwise estimates of \( |D_H T(t)f|^{p'}_{p'} \) by means of both of \( T(t)|D_H f|^{p'}_{p'} \) and of \( T(t)|f|^p \). We also generalize these results to the vector-valued semigroup \( (T^V(t))_{t \geq 0} \), of which we investigate both the asymptotic behaviour in \( L^p(X, \nu; V) \) for \( p \in [1, +\infty) \) and pointwise estimates of \( |D_H T^V(t)f|^{p'}_{p'} \) when \( p \in [2, +\infty) \). We stress that, even if the gradient estimates follow from computations similar to those in the proofs of [4, Theorems 3.1 & 3.3], the asymptotic behaviour of \( (T(t))_{t \geq 0} \) is obtained with different techniques with respect to those in the quoted paper: here, we take advantage of the asymptotic estimate (3.2) for \( \|D_H T(t)f\|_{L^2(X, \nu; H)} \) as \( t \to +\infty \), which allows us to characterize the limit of \( T(t)f \) in \( L^p(X, \nu) \) as \( t \to +\infty \).

Sections 4 and 5 are the core of this paper. In the former we provide the vector-valued Poincaré inequalities (1.1) and (1.2), in the latter we give the equivalence of the norms \( \| \cdot \|_{k,p} \) and \( \| \cdot \|_{p, D^k_H} \). We remark that the proof of (1.1) relies on duality arguments: when \( p \geq 2 \) we use the fact that, for any \( F \in W^{1,p}(X, \nu) \), the function \( F^* := |F|^{p-2}F \in W^{1,p'}(X, \nu) \) and an explicit formula for \( D_H F^* \) is available (see Lemma 2.16), while when \( p \in [1, 2) \) we employ the duality between \( L^p(X, \nu; H \otimes V) \) and \( L^{p'}(X, \nu; H \otimes V) \). Regarding the equivalence of \( \| \cdot \|_{k,p} \) and \( \| \cdot \|_{p, D^k_H} \), we adapt to our situation the idea of [19, Corollary 3.2], which we explain when \( k = 2 \). In this case, it is enough to estimate \( \|D_H f\|_{L^p(X, \nu; H)} \) by means of \( \|f\|_{p, D^2_H} \). From (1.2) we have

\[
\|D_H f\|_{L^p(X, \nu; H)} \leq \|D_H f - \nu(D_H f)\|_{L^p(X, \nu; H)} + \|\nu(D_H f)\|_H \\
\leq k_p \| D^2_H f \|_{L^p(X, \nu; H^2_2(H))} + \|\nu(D_H f)\|_H.
\]

To conclude, it remains to prove that there exists a positive constant \( c \) such that \( \|\nu(D_H f)\|_H \leq c\|f\|_{L^p(X, \nu)} \). The proof of Theorem 5.1 shows that this estimate is a consequence of an integration by parts and of the Wiener chaos decomposition. However, the presence of the function \( e^{-U} \) heavily complicates the computations, since additional terms arise and the estimate of these terms is quite involving. The same arguments allows us to extend the equivalence for \( k \geq 3 \), provided further assumptions on \( U \). In this case we need to iterate the integration by parts \( k - 1 \) times, and the main effort consists in estimating the new terms which appear.

Finally, in Section 6 we get a family of logarithmic Sobolev inequalities for vector-valued functions. These inequalities are the counterpart of the Sobolev embeddings, which does not hold for the measure \( \nu \). Indeed, it is possible to prove that Sobolev embeddings holds true when the measure considered is doubling, and this is not the case of the Gaussian measure \( \mu \) and so neither of \( \nu \). By taking advantage of these Logarithmic Sobolev Inequalities, we are able to prove that the vector-valued semigroup \( (T^V(t))_{t \geq 0} \) is hypercontractive, i.e., given \( q \in (1, +\infty) \) and \( t > 0 \), for any \( F \in L^q(X, \nu; V) \) we have \( T^V(t)F \in L^p(X, \nu; V) \) for any \( p \leq p(t) := 1 + (q-1)e^{2t} \). The last result of this section is an improvement of the asymptotic behaviour of \( (T^V(t))_{t \geq 0} \), for which we show an exponential decay in \( L^p(X, \nu) \) as \( t \to +\infty \) for any \( p \in [1, +\infty) \).

1.1 Notations

Let \( X \) be a separable Banach space, let \( X^* \) be its topological dual and let us denote by \( \langle \cdot, \cdot \rangle \) and by \( |\cdot|_X \) its duality and its norm, respectively. For any \( k \in \mathbb{N} \cup \{\infty\} \) we denote by \( C^k(X) \) the set of \( k \)-times (infinitely many times if \( k = +\infty \) ) Fréchet differentiable functions \( f : X \to \mathbb{R} \). We denote by \( C^k_b(X) \)
the set of functions \( f \in C^k(X) \) which are bounded together with their derivatives up to order \( k \). We denote by \( \mathcal{C}^k_{b}(X) \) the set of functions \( f \in C^k(X) \) such that there exist \( n \in \mathbb{N} \), \( x_1, \ldots, x_n \in X \) and \( \varphi \in C^k_b(\mathbb{R}^n) \) such that \( f(x) = \varphi((x, x_1), \ldots, (x, x_n)) \) for any \( x \in X \). For any \( f \in C_b(X) \) we set \( \|f\|_\infty := \sup\{|f(x)| : x \in X\} \).

Let \( V \) be a separable Hilbert space with norm \( |\cdot|_V \) and inner product \( \langle \cdot, \cdot \rangle_V \). For any \( k \in \mathbb{N} \cup \{\infty\} \) we denote by \( \mathcal{F}C_k^b(X; V) \) the space of \( k \)-times (infinitely many times if \( k = +\infty \)) Fréchet differentiable functions \( f : X \to V \) such that there exist \( n \in \mathbb{N}, v_1, \ldots, v_n \in V \) and \( f_1, \ldots, f_n \in \mathcal{F}C_k^b(X) \) such that

\[
 f(x) = \sum_{j=1}^n f_j(x)v_j, \quad x \in X.
\]

For any \( f \in C_b(X; V) \) we set \( \|f\|_\infty := \sup\{|f(x)|_V : x \in X\} \), where \( C_b(X; V) \) denotes the space of functions \( f : X \to V \) which are bounded and continuous with respect to the strong topology.

We denote by \( B(X) \) the Borel \( \sigma \)-field of \( X \). Let \( \gamma \) be a Borel positive finite measure on \( X \). For any \( p \in [1, +\infty) \) we denote by \( L^p(X, \gamma; V) \) the space of the (equivalence classes of) functions \( F \) which are Bochner integrable endowed with the norm

\[
 \|F\|_{L^p(X, \gamma; V)} := \left( \int_X |F|^p d\gamma \right)^{1/p}, \quad F \in L^p(X, \gamma; V).
\]

Let \( K \) be a separable Hilbert space with norm \( |\cdot|_K \) and inner product \( \langle \cdot, \cdot \rangle_K \). We denote by \( V \times K \) the set \( \{(v, k) : v \in V, k \in K\} \) and by \( V \otimes K \) the tensor product of \( V \) and \( K \), i.e., the closure of \( \text{span}\{v \otimes k : v \in V, k \in K\} \) with respect to the inner product

\[
 \langle v_1 \otimes k_1, v_2 \otimes k_2 \rangle_{V \otimes K} := \langle v_1, v_2 \rangle_V \langle k_1, k_2 \rangle_K, \quad v_1, v_2 \in V, k_1, k_2 \in K.
\]

We set \( \mathcal{L}(V; K) \) the space of bounded linear operators from \( V \) to \( K \). If \( V = K \) we simply write \( \mathcal{L}(V) \). We can see \( V \otimes K \subset \mathcal{L}(V; K) \) or \( V \otimes K \subset \mathcal{L}(K; V) \) by setting \( (v \otimes k)w := [v, w]_V k \) for any \( v, w \in V \) and \( k \in K \), or \( (v \otimes k)h := [k, h]_K v \) for any \( v \in V \) and any \( k, h \in K \), respectively. For any \( k \in \mathbb{N} \) we set

\[
 V^{\otimes_k} := V \otimes \cdots \otimes V
\]

As above, we can see \( V^{\otimes_2} \) as a subset of \( \mathcal{L}(V) \) by setting \( (v \otimes w)(h) = [v, w]_V h \) for any \( v, w, h \in V \).

We say that a symmetric non-negative operator \( A \in \mathcal{L}(V) \) is a Trace class operator if there exists an orthonormal basis \( \{v_n : n \in \mathbb{N}\} \) of \( V \) such that

\[
 \text{Tr}[A]_V := \sum_{n \in \mathbb{N}} [Av_n, v_n]_V < +\infty.
\]

We denote by \( \mathcal{L}_{(1)}(V) \) the space of Trace class operators endowed with the norm

\[
 \|A\|_{\mathcal{L}_{(1)}(V)} := \text{Tr}[A]_V.
\]

For any \( k \in \mathbb{N} \) we denote by \( \mathcal{H}_k(V) \) the space of \( k \)-linear Hilbert-Schmidt operators on \( V \), i.e., the space of the operators \( A : V^k \to \mathbb{R} \) such that there exists an orthonormal basis \( \{v_n : n \in \mathbb{N}\} \) of \( V \) which gives

\[
 \|A\|_{\mathcal{H}_k(V)} := \sum_{i_1, \ldots, i_k = 1}^\infty |A(v_{i_1}, \ldots, v_{i_k})|^2 < +\infty.
\]

The space \( \mathcal{H}_k(V) \) with inner product

\[
 (A, B)_{\mathcal{H}_k(V)} := \sum_{i_1, \ldots, i_k = 1}^\infty A(v_{i_1}, \ldots, v_{i_k})B(v_{i_1}, \ldots, v_{i_k}),
\]
is a Hilbert space. Thanks to the Riesz representation Theorem it is possible to identify \( V \) with \( \mathcal{H}_1(V) \) by setting, for any \( h \in V \), \( h(v) := [h, v]_V \), for any \( v \in V \). For any \( A \in \mathcal{H}_2(V) \) we have
\[
\|A\|_{\mathcal{H}_2(V)} = \sum_{n \in \mathbb{N}} |[Av_n, Av_n]_V|,
\]
where \( \{v_n : n \in \mathbb{N}\} \) is an orthonormal basis of \( V \). Further, \( A^* A \) is a trace class operator and
\[
\text{Tr}[A^* A]_V = \|A\|^2_{\mathcal{H}_2(V)} = \sum_{n \in \mathbb{N}} |Av_n|_V^2, \quad \{v_n : n \in \mathbb{N}\} \text{ orthonormal basis of } V.
\]
We notice that for any \( f \in X \) the characteristic function of the set \( B \) is \( \mathcal{H}_k(V) \): indeed, for any orthonormal basis \( \{v_n : n \in \mathbb{N}\} \) of \( V \) we have
\[
\sum_{i_1, \ldots, i_k = 1}^{\infty} |(w_1 \otimes \cdots \otimes w_k)(v_{i_1}, \ldots, v_{i_k})|^2 = \sum_{i_1, \ldots, i_k = 1}^{\infty} |w_1, v_{i_1}|_V^2 |w_2, v_{i_2}|_V^2 \cdots |w_k, v_{i_k}|_V^2 = \prod_{j=1}^k |w_j|_V^2.
\]
We are now able to define the Cameron–Martin space \( H \) (see [7, Section 2.2]).

2 Preliminaries

2.1 The abstract Wiener space

Let \( X \) be a separable Banach space and let \( \mu \) be a centred non-degenerate Gaussian measure on \( X \). We follow [7, Chapter 2] to construct the Cameron–Martin space \( H \) associated to \( \mu \). This construction will give us the abstract Wiener space \( (X, \mu, H) \) which will be the primary setting of our studies. From the Fernique’s theorem [7, Theorem 2.8.5], it follows that \( X^* \subseteq L^2(X, \mu) \), and we denote by \( j : X^* \to L^2(X, \mu) \) the injection of \( X^* \) in \( L^2(X, \mu) \), namely for any \( x^* \in X^* \) we have
\[
(j(x^*))(x) := \langle x, x^* \rangle, \quad x \in X.
\]
We remark that by [7, Theorem 2.2.4] there exists a nonnegative symmetric linear bounded operator \( Q : X^* \to X \) such that for every \( x_1^*, x_2^* \in X^* \) we have
\[
\langle Qx_1^*, x_2^* \rangle = \int_X j(x_1^*)j(x_2^*)d\mu.
\]
We denote by \( X^*_\mu \) the closure of \( j(X^*) \) in \( L^2(X, \mu) \) and we define \( R : X^*_\mu \to (X^*)' \) by
\[
R(f)(x^*) := \int_X fj(x^*)d\mu, \quad f \in X^*_\mu, \quad x^* \in X^*.
\]
It is well-known that \( R(X^*_\mu) \subseteq X \). This means that for every \( f \in X^*_\mu \) there exists \( R(f) \in X \) such that for every \( x^* \in X^* \)
\[
\langle R(f), x^* \rangle = \int_X fj(x^*)d\mu.
\]
In particular the operator \( R : L^2(X, \mu) \to X \) is the adjoint of \( j \). Again from [7, Theorem 2.8.5], for any \( h \in X^*_\mu \) there exists a positive constant \( c \) such that
\[
\int_X e^{c(j(x))} \mu(dx) < +\infty.
\]
We are now able to define the Cameron–Martin space \( H \) (see [7, Section 2.2]).
Definition 2.1. The Cameron–Martin space associated to \( \mu \) is \( H := \{ h \in X : |h|_H < +\infty \} \), where \( |h|_H := \sup \{ \langle h, x^* \rangle \mid x^* \in X^* \text{ such that } \|j(x^*)\|_{L^2(X, \mu)} \leq 1 \} \).

From [7, Lemma 2.4.1] it follows that \( h \in H \) if and only if there exists \( \hat{h} \in X^*_\mu \) such that \( \mathcal{R}(\hat{h}) = h \). Furthermore \( H \) is a Hilbert space if endowed with the inner product
\[
[h, k]_H = \int_X \hat{h} k d\mu, \quad h, k \in H.
\]
(2.3)

We stress that for any \( x^* \in X^* \), from (2.1) and (2.2) we have \( Qx^* \in H \) and that \( \mathcal{R}(j(x^*)) = Qx^* \), i.e., \( \hat{Q}x^* = j(x^*) \). Further, from (2.3) we deduce that
\[
\langle Qx_1^*, x_2^* \rangle = \langle Qx_1^*, Qx_2^* \rangle_H, \quad x_1^*, x_2^* \in X^*.
\]
(2.4)

We provide the following characterization of \( H \).

Lemma 2.2. \( H \) is the RKHS associated to \( Q \) in \( X \), i.e., \( H = \overline{QX^*|\mu} \), where \( [Qx^*, Qy^*]_H := \langle Qx^*, y^* \rangle \).

Proof. The proof is quite simple but we provide it for the convenience of the reader. Let \( h \in H \). Then, there exists \( \hat{h} \in X^*_\mu \) such that \( \mathcal{R}(\hat{h}) = h \). In particular, there exists \( (x_n^*) \subseteq X^* \) such that the sequence \( (j(x_n^*)) \) converges to \( \hat{h} \) in \( L^2(X, \mu) \). We claim that the sequence \( (Qx_n^*) \) converges to \( h \) in \( H \). Indeed, by (2.3) and recalling that \( Qx_n^* = j(x_n^*) \) for any \( n \in \mathbb{N} \) and (2.3), it follows that
\[
\lim_{n \to +\infty} |Qx_n^* - h|_H^2 = \lim_{n \to +\infty} \int_X |j(x_n^*) - \hat{h}|^2 d\mu = 0.
\]
This means that \( H \subseteq \overline{QX^*|\mu} \). The converse inclusion follows by analogous arguments. \( \square \)

Let us denote by \( i \) the injection \( i : H \to X \), and let \( i^* : X^* \to H \) be the adjoint operator of \( i \) (here we have identified \( H^* \) with \( H \) by means of the Riesz representation theorem), then \( Q = i \circ i^* \). Indeed, for any \( x_1^*, x_2^* \in X^* \), by (2.1) and (2.3) we have
\[
\langle (i \circ i^*)x_1^*, x_2^* \rangle = \langle i^* x_1^*, i^* x_2^* \rangle_H = \int_X j(x_1^*)j(x_2^*)d\mu = \langle Qx_1^*, x_2^* \rangle_H,
\]
(2.5)
which gives \( Q = i \circ i^* \).

Finally we introduce the Ornstein-Uhlenbeck semigroup \( (P(t))_{t \geq 0} \): for any \( f \in C_b(X) \) we set
\[
P(t)f(x) = \int_X f(\sqrt{e^{-t}x + \sqrt{1 - e^{-2t}y}})d\mu(dy), \quad x \in X, \quad t \geq 0.
\]
\( (P(t))_{t \geq 0} \) extends to a positive strongly continuous semigroup of contractions on \( L^p(X, \mu) \) for any \( p \in [1, +\infty) \) (see [7, Theorem 2.9.1]). We suggest [7, Section 2.9] for an in-depth study of \( ((P(t))_{t \geq 0} \).

2.2 The \( H \)-gradient and the Sobolev spaces \( W^{k,p}(X, \mu) \)

In this subsection we define the Sobolev spaces \( W^{k,p}(X, \mu) \). For an in-depth study of these spaces we refer to [7, Chapter 5]. We consider the operator \( D_H \) defined on \( C^1(X) \) by
\[
D_H F(x) := i^* DF(x), \quad F \in C^1(X), \quad x \in X.
\]
On the space \( \mathcal{F}^{\infty} \) the operator \( D_H \) acts as follows:
\[
D_H f(x) = \sum_{j=1}^n \frac{\partial \phi}{\partial x_j}(x, x_1^*, \ldots, x_n^*)ix_j^*, \quad x \in X,
\]
(2.6)
where \( f(x) = \varphi((x, x_1), \ldots, (x, x_n)) \) for some \( n \in \mathbb{N} \), \( \varphi \in C^1(\mathbb{R}^n) \) and \( x_1, \ldots, x_n \in X^* \). We say that a function \( f \) is \( H \)-differentiable at \( x \in X \) if there exists \( k \in H \) such that
\[
f(x + h) = f(x) + [k, h]_H + o(h), \quad h \to 0.
\]
(2.7)

In this case we set \( D_H f(x) := k \). It is not hard to see that if \( f \in \mathcal{F}C^\infty_b(X) \) then the definitions of \( D_H \) given in (2.6) and (2.7) coincide.

We define the Sobolev spaces \( W^{1,p}(X, \mu) \) as the domain of the closure of the operator \( D_H : \mathcal{F}C^\infty_b(X) \to L^p(X, \mu; H) \) in \( L^p(X, \mu) \), for any \( p \in [1, +\infty) \), and we denote by \( D_H^\mu \) its closure. These spaces are Banach spaces if endowed with the norm
\[
\|f\|_{W^{1,p}(X, \mu)} := \left( \|f\|_{\mathcal{F}C^\infty_b(X)}^p + \|D_H^\mu f\|_{L^p(X, \mu; H)}^p \right)^{1/p}, \quad f \in W^{1,p}(X, \mu),
\]
and the space \( W^{1,2}(X, \mu) \) is a Hilbert space if endowed with the scalar product
\[
(f, g)_{W^{1,2}(X, \mu)} := (f, g)_{L^2(X, \mu)} + \langle D_H^\mu f, D_H^\mu g \rangle_{L^2(X, \mu; H)}, \quad f, g \in W^{1,2}(X, \mu).
\]

For any \( k \in \mathbb{N} \) we define the operator \( D_H^k \) which acts on \( \mathcal{F}C^k(X) \) as follows:
\[
D_H^k f(x) = \sum_{j_1, \ldots, j_k=1}^n \frac{\partial^{j_1} f}{\partial x_{j_1} \cdots \partial x_{j_k}}(\langle x, x_1^1 \rangle, \ldots, \langle x, x_n^* \rangle) x_{j_1}^i \cdots x_{j_k}^i, \quad x \in X,
\]
where \( f(x) = \varphi((x, x_1^1), \ldots, (x, x_n^*)) \) for some \( n \in \mathbb{N} \), \( \varphi \in C^k(\mathbb{R}^n) \) and \( x_1, \ldots, x_n \in X^* \). The Sobolev spaces \( W^{k,p}(X, \mu) \) are the domain of the closure of the operator \( D_H^k : \mathcal{F}C^\infty_b(X, \mu) \to L^p(X, \mu; H) \times \cdots \times L^p(X, \mu; H) \) in \( L^p(X, \mu) \), for any \( p \in [1, +\infty) \), and we denote by \( D_H^{\mu,k} \) its closure. These spaces are Banach spaces if endowed with the norm
\[
\|f\|_{W^{k,p}(X, \mu)} := \left( \|f\|_{\mathcal{F}C^\infty_b(X, \mu)}^p + \sum_{j=1}^k \|D_H^{\mu,j} f\|_{L^p(X, \mu; H_j)}^p \right)^{1/p}, \quad f \in W^{k,p}(X, \mu),
\]
and \( W^{k,2}(X, \mu) \) is a Hilbert space if endowed with the scalar product
\[
(f, g)_{W^{k,2}(X, \mu)} := (f, g)_{L^2(X, \mu)} + \sum_{j=1}^k \langle D_H^{\mu,j} f, D_H^{\mu,j} g \rangle_{L^2(X, \mu; H_j)}, \quad f, g \in W^{k,2}(X, \mu).
\]

### 2.3 Wiener chaos decomposition

In this subsection we briefly present the Wiener chaos decomposition. For a systematic study of the Wiener chaos decomposition in the setting of Malliavin calculus we refer to [18], while for the main results of the Wiener chaos decomposition in our setting we refer to [7, Section 2.9]. For any \( n \in \mathbb{N} \) let \( H_n \) be the \( n \)-th Hermite polynomial, which is defined by
\[
H_n(\xi) := \frac{(-1)^n}{n!} e^{\xi^2/2} \frac{d^n}{d\xi^n} \left( e^{-\xi^2/2} \right), \quad \xi \in \mathbb{R}.
\]
We notice that \( H_1(\xi) = \xi \) and \( H_2(\xi) = \frac{1}{2}(\xi^2 - 1) \). We denote by \( \Lambda \) the set of multiindices \( \alpha = (\alpha_1, \ldots, \alpha_n, \ldots) \) such that all the terms, except a finite number of them, vanish. For any \( \alpha \in \Lambda \) we set
\[
\alpha! := \prod_{i=1}^\infty \alpha_i, \quad |\alpha| := \sum_{i=1}^\infty \alpha_i.
\]
For any multiindex $\alpha$ we define the generalized Hermite polynomial $H_\alpha$ by

$$H_\alpha(\xi) := \prod_{i=1}^{\infty} H_{\alpha_i}(\xi_i), \quad \xi = (\xi_1, \ldots, \xi_n) \in \mathbb{R}^n, \quad n = |\alpha|.$$

Let us fix an orthonormal basis $\{h_n = i^*(x_n^*) : x_n^* \in X^*, \ n \in \mathbb{N}\}$ of $H$ (this basis exists since $j(X^*)$ is dense in $X^*_\mu$), and for any $\alpha \in \Lambda$ let us define

$$\Phi_\alpha(x) := \sqrt{\alpha!} \prod_{i=1}^{\infty} H_{\alpha_i}(\langle x, x_i^* \rangle), \quad x \in X.$$

It is possible to prove (see [18, Proposition 1.1.1]) that the set $\{\Phi_\alpha : \alpha \in \Lambda\}$ is a complete orthonormal system in $L^2(X, \mu)$. Further, if for any $n \in \mathbb{N} \cup \{0\}$ we set

$$E_n := \text{span}\{\Phi_\alpha : \alpha \in \Lambda, \ |\alpha| = n\},$$

then $E_n$ and $E_m$ are orthogonal subspaces of $L^2(X, \mu)$ when $n \neq m$ and

$$L^2(X, \mu) := \bigoplus_{n \in \mathbb{N} \cup \{0\}} E_n.$$

Let us denote by $I_n$ the projection on $E_n$, $n \in \mathbb{N} \cup \{0\}$. Then, for any $f \in L^2(X, \mu)$ we have

$$f = \sum_{n=0}^{\infty} I_n f, \quad I_n f = \sum_{\alpha \in \Lambda, |\alpha| = n} \left( \int_X f \Phi_\alpha d\mu \right) \Phi_\alpha.$$

**Remark 2.3.** The properties of $\Phi_\alpha$ and $E_n$ are independent of the choice of the orthonormal basis $\{h_n : n \in \mathbb{N}\}$ of $H$, i.e., if $\{k_n : k_n = i^*(y_n^*), \ y_n^* \in X^*, \ n \in \mathbb{N}\}$ is another orthonormal basis of $H$ and we set

$$\Phi'_\alpha(x) := \sqrt{\alpha!} \prod_{i=1}^{\infty} H_{\alpha_i}(\langle x, y_i^* \rangle), \quad x \in X, \ \alpha \in \Lambda, \ E'_n := \text{span}\{\Phi'_\alpha : \alpha \in \Lambda, \ |\alpha| = n\},$$

then $E'_n$ and $E'_m$ are orthogonal subspaces of $L^2(X, \mu)$ when $n \neq m$,

$$L^2(X, \mu) := \bigoplus_{n \in \mathbb{N} \cup \{0\}} E'_n,$$

and for any $f \in L^2(X, \mu)$ we have

$$f = \sum_{n=0}^{\infty} I_n f, \quad I_n f = \sum_{\alpha \in \Lambda, |\alpha| = n} \left( \int_X f \Phi'_\alpha d\mu \right) \Phi'_\alpha.$$

**Remark 2.4.** For any multiindex $\alpha$ with $|\alpha| = 1$ it follows that $\Phi_\alpha(x) = \tilde{h}_n(x)$ for any $x \in X$, where $\alpha_n = 1$, with $n \in \mathbb{N}$, is the unique component of $\alpha$ different from 0. Further, for any multiindex $\alpha$ with $|\alpha| = 2$, we have $\Phi_\alpha(x) = \frac{1}{\sqrt{2}} \left( (\tilde{h}_n(x))^2 - 1 \right)$ for any $x \in X$, if $\alpha_n = 2$, with $n \in \mathbb{N}$, is the unique component of $\alpha$ different from 0, while $\Phi_\alpha(x) = \tilde{h}_n(x) \tilde{h}_m(x)$ for any $x \in X$, if $\alpha_n = \alpha_m = 1$, with $n, m \in \mathbb{N}, n \neq m$, are the unique components of $\alpha$ which don’t vanish.

The Ornstein-Uhlenbeck semigroup $(P(t))_{t \geq 0}$ behaves good on $E_n$: indeed, (see [7, Theorem 2.9.2]) for any $t \geq 0$ and any $f \in L^2(X, \mu)$ we have

$$P(t) f = \sum_{n=0}^{\infty} e^{-nt} I_n f.$$
In particular, for any $f \in E_n$ it follows that $T(t)f = e^{-nt}f$, for any $n \in \mathbb{N}$ and any $t \geq 0$.

We prove the following useful lemma which is a consequence of Nelson’s hypercontractivity theorem for the Ornstein-Uhlenbeck semigroup $(P(t))_{t \geq 0}$ (see [7, Theorem 5.5.3]).

**Lemma 2.5.** Let $p \in (1, +\infty)$. The projection $I_k$ is a bounded linear operator from $L^p(X, \mu)$ onto $L^2(X, \mu)$, and

$$
\|I_k f\|_{L^2(X, \mu)} \leq \frac{1}{(p-1)^k} \|f\|_{L^p(X, \mu)}, \quad f \in L^p(X, \mu), \quad p \in (1, 2),
$$

$$
\|I_k f\|_{L^2(X, \mu)} \leq (p-1)^{k/2} \|f\|_{L^p(X, \mu)}, \quad f \in L^p(X, \mu), \quad p \in [2, +\infty),
$$

**Proof.** The case $p = 2$ follows from the fact that $I_k$ is a projection on $L^2(X, \mu)$. If $p \in (2, +\infty)$ the statement is easy to prove. Indeed, let $p \in (2, +\infty)$ and let $f \in L^p(X, \mu)$. Then, $f \in L^2(X, \mu)$ and from [23, Chapter IV, Theorem 1] we have

$$
\|I_k f\|_{L^p(X, \mu)} \leq (p-1)^{k/2} \|f\|_{L^p(X, \mu)}.
$$

Further,

$$
\|I_k f\|_{L^2(X, \mu)} \leq \|I_k f\|_{L^p(X, \mu)},
$$

and combining the above inequalities we have the thesis.

Let $p \in (1, 2)$, let $k \in \mathbb{N}$ and let $f \in L^p(X, \mu)$. We consider a sequence $(f_n) \subset C_b(X) \subset L^2(X, \mu)$ which converges to $f$ in $L^p(X, \mu)$ as $n \to +\infty$. We recall that for any element $g \in E_k$ we have $P(t)g = e^{-kt}g$. Then,

$$
\|I_k f_n\|_{L^2(X, \mu)} = e^{kt} \|P(t)I_k f_n\|_{L^2(X, \mu)}, \quad n \in \mathbb{N}.
$$

Further, the hypercontractivity property of $(P(t))_{t \geq 0}$ (see [7, Theorem 5.5.3]) gives

$$
\|P(t)I_k f_n\|_{L^2(X, \mu)} \leq \|I_k f_n\|_{L^p(X, \mu)}, \quad n \in \mathbb{N},
$$

with $t = -\ln(p-1)^{1/2}$. It follows that

$$
\|I_k f_n\|_{L^2(X, \mu)} \leq e^{kt} \|I_k f_n\|_{L^p(X, \mu)} = \frac{1}{(p-1)^{k/2}} \|I_k f_n\|_{L^p(X, \mu)}, \quad n \in \mathbb{N}.
$$

[23, Chapter IV, Theorem 1] implies that

$$
\|I_k f_n\|_{L^p(X, \mu)} \leq \frac{1}{(p-1)^{k/2}} \|f_n\|_{L^p(X, \mu)}.
$$

Therefore, $(I_k f_n)$ is a Cauchy sequence in $L^2(X, \mu)$. If we denote by $I_k f$ the limit of $(I_k f_n)$ in $L^2(X, \mu)$ as $n \to +\infty$, it follows that $I_k f$ is well defined, it belongs to $L^2(X, \mu)$ and

$$
\|I_k f\|_{L^2(X, \mu)} \leq \frac{1}{(p-1)^{k/2}} \|f\|_{L^p(X, \mu)}.
$$

\[\square\]

### 2.4 The Sobolev spaces $W^{1,p}(X, \nu)$ and the $H$-divergence operator

Let us provide the assumptions on the function $U$, which will be used to define the weighted Gaussian measure $\nu$.

**Hypothesis 2.6.** $U$ is a convex function which belongs to $C^2(X) \cap W^{1,p}(X, \mu)$ for any $p \in (1, +\infty)$. 


The convexity of $U$ implies that $U$ is bounded from below by a linear functional. From Fernique’s theorem (see [7, Theorem 2.8.5]) we infer that the function $e^{-U}$ belongs to $L^p(X,\mu)$ for any $p \in [1, +\infty)$. We introduce the probability measure

$$\nu := Ke^{-U}\mu, \quad K = \|e^{-U}\|_{L^1(X,\mu)},$$

(2.8)

where $K$ is the normalizing factor. Hypothesis 2.6 implies that $|D_H U|_H \in L^p(X,\nu)$ for any $p \in [1, +\infty)$. Indeed, for any $p \in [1, +\infty)$ we have

$$\int_X |D_H U|_H^p d\nu = \int_X |D_H U|^p e^{-U} d\mu \leq \|D_H U\|_{L^p(X,\mu)} \|e^{-U}\|_{L^p(X,\mu)} < +\infty,$$

for any $q \in (1, +\infty)$. Analogously, we can prove that $\hat{h} \in L^q(X,\nu)$ for any $q \in [1, +\infty)$. We define the Sobolev spaces $W^{1,p}(X,\nu)$ as in [13]. The integration by parts formula (see [13, Lemma 3.1])

$$\int_X [D_H f, h]_H d\nu = \int_X f(\hat{h} + [D_H U, h]_H) d\nu,$$

(2.9)

implies that the operator $D_H : \mathcal{F}C^\infty_b(X) \rightarrow L^p(X,\nu;H)$ is closable for $p \in [1, +\infty)$.

**Proposition 2.7.** The operator $D_H : \mathcal{F}C^\infty_b(X) \rightarrow L^p(X,\nu;H)$ is closable in $L^p(X,\nu)$ for any $p \in [1, +\infty)$. We denote by $D_H$ its closure and by $W^{1,p}(X,\nu)$ the domain of its closure. $W^{1,p}(X,\nu)$ is a Banach space if endowed with the norm

$$\|f\|_{1,p} := \left(\|f\|_{L^p(X,\nu)}^p + \|D_H f\|_{L^p(X,\nu;H)}^p\right)^{1/p}, \quad f \in W^{1,p}(X,\nu),$$

$W^{1,2}(X,\nu)$ is a Hilbert space if endowed with inner product

$$\langle f, g \rangle_{W^{1,2}(X,\nu)} = \langle f, g \rangle_{L^2(X,\nu)} + \langle D_H f, D_H g \rangle_{L^2(X,\nu;H)}, \quad f, g \in W^{1,2}(X,\nu).$$

**Remark 2.8.** We have introduced the notation $D_H^p$ to underline the difference between the closure of $D_H$ in $L^p(X,\mu)$ and in $L^p(X,\nu)$; the importance of this different notation will arise in the proof of Proposition 2.10, where we use a property of $D_H^p$. This means that the notation $D_H f$ represents both the action of the closure of $D_H$ in $L^p(X,\nu)$ for $f \in W^{1,p}(X,\nu)$, and the action of the operator $D_H$ defined in (2.6) on smooth functions $f \in C^1(X)$.

**Proof.** If $p > 1$ the statement has already proved in [13, Proposition 3.2]. Let us consider $p = 1$ and let $\{h_m : m \in \mathbb{N}\}$ be an orthonormal basis of $H$. In this case we cannot directly use the method applied in the proof of [13, Proposition 3.2], since $\hat{h} - [D_H U, h]_H$ does not belong to $L^\infty(X,\nu)$. We introduce the function $\theta \in C^0_b(\mathbb{R})$ such that $\theta(0) = 0$ and $\theta'(0) \neq 0$. Let $(f_n) \subset \mathcal{F}C^\infty_b(X)$ be such that $f_n \rightarrow 0$ in $L^1(X,\nu)$ and $D_H f_n \rightarrow G$ in $L^1(X,\nu;H)$ as $n \rightarrow +\infty$. Then, for any $\varphi \in C^\infty_b(X)$ and any $m \in \mathbb{N}$ we have

$$\int_X |\theta \circ f_n, h_m|_H \varphi d\nu = \int_X \langle \theta' \circ f_n \rangle |D_H f_n, h_m|_H \varphi d\nu.$$

(2.10)

Letting $n \rightarrow +\infty$, the right-hand side converges to $\int_X \theta'(0) |G, h_m|_H \varphi d\nu$. Indeed,

$$\left|\int_X \langle \theta' \circ f_n \rangle |D_H f_n, h_m|_H \varphi d\nu - \int_X \theta'(0) |G, h_m|_H \varphi d\nu\right|$$

$$\leq \int_X \left|\theta' \circ f_n|D_H f_n, h_m|_H \varphi d\nu - \theta'(0) |G, h_m|_H \varphi d\nu\right|$$

$$+ \int_X \left|\theta'(0) |G, h_m|_H \varphi d\nu - \theta'(0) |G, h_m|_H \varphi d\nu\right|$$

$$+ \int_X \left|\theta'(0) |G, h_m|_H \varphi d\nu - \theta'(0) |G, h_m|_H \varphi d\nu\right|$$
\[
\leq \int_X \| (\theta' \circ f_n) \varphi \| |D_H f_n, h_m|_H - |G, h_m|_H |d\nu + \int_X \| (\theta' \circ f_n) - \theta'(0) \| |G, h_m|_H \varphi |d\nu.
\]

Since \( \theta' \) and \( \varphi \) are bounded, the first integral vanishes as \( n \to +\infty \). Further, there exists a subsequence \((f_{k_n}) \subset (f_n)\) such that \( f_{k_n}(x) \to 0 \) for \( \nu \)-a.e. \( x \in X \). Moreover,
\[
\| (\theta' \circ f_n) - \theta'(0) \| |G, h_m|_H \varphi | \leq 2 \| \theta' \|_\infty \| \varphi \|_\infty |G|_H.
\]

By the dominated convergence theorem we infer that
\[
\int_X \| (\theta' \circ f_{k_n}) - \theta'(0) \| |G, h_m|_H \varphi |d\nu \to 0, \quad n \to +\infty. \tag{2.11}
\]

In particular, we have prove that any subsequence \((f_{k_n}) \subset (f_n)\) admits a subsequence \((f_{k_{n_k}}) \subset (f_{k_n})\) such that (2.11), with \((f_{k_{n_k}})\) replaced by \((f_{k_n})\) holds true. Hence,
\[
\int_X \| (\theta' \circ f_{k_n}) - \theta'(0) \| |G, h_m|_H \varphi |d\nu \to 0, \quad n \to +\infty.
\]

Let us apply (2.9) to the left-hand side of (2.10). We get
\[
\int_X |D_H(\theta \circ f_n, h_m|_H \varphi |d\nu = \int_X (\theta \circ f_n)(\hat{h}_m \varphi + |D_H U, h_m|_H \varphi - |D_H \varphi, h_m|_H) |d\nu.
\]

As above, let \((f_{k_n}) \subset (f_n)\) be such that \( f_{k_n}(x) \to 0 \) for \( \nu \)-a.e. \( x \in X \). Since
\[
\left| (\theta \circ f_n)(\hat{h}_m \varphi + |D_H U, h_m|_H \varphi - |D_H \varphi, h_m|_H) \right| \\
\leq \| \theta \|_\infty \| \hat{h}_m \| \| \varphi \|_\infty + |D_H U|_H \| \varphi \|_\infty + |D_H \varphi|_\infty \in L^1(X, \nu),
\]

by the dominated convergence theorem we infer that
\[
= \int_X (\theta \circ f_{k_n})(\hat{h}_m \varphi + |D_H U, h_m|_H \varphi - |D_H \varphi, h_m|_H) |d\nu \to 0, \quad n \to +\infty. \tag{2.12}
\]

In particular, we have prove that any subsequence \((f_{k_n}) \subset (f_n)\) admits a subsequence \((f_{k_{n_k}}) \subset (f_{k_n})\) such that (2.12) holds true with \((f_{k_{n_k}})\) replaced by \((f_{k_n})\). Hence,
\[
\int_X (\theta \circ f_{k_n})(\hat{h}_m \varphi + |D_H U, h_m|_H \varphi - |D_H \varphi, h_m|_H) |d\nu \to 0, \quad n \to +\infty.
\]

Therefore, we get
\[
\theta'(0) \int_X |G, h_m|_H \varphi |d\nu = 0,
\]

for any \( \varphi \in C^1_b(X) \). By recalling that \( \theta'(0) \neq 0 \), it follows that \( |G, h_m|_H = 0 \) for \( \nu \)-a.e. in \( X \) for any \( m \in \mathbb{N} \), which gives \( G = 0 \) for \( \nu \)-a.e. in \( X \).

The second part of the statement follows from standard arguments.

The following technical lemma will be used in the sequel to prove important results.

**Lemma 2.9.** Let \( p \in [1, +\infty) \) and let \( f \in W^{1,p}(X, \nu) \). Then, \( |f|, f^+, f^- \in W^{1,p}(X, \nu) \) and \( D_H|f| = \text{sgn}(f) D_H f, D_H f^+ = D_H f_{\chi_{\{f > 0\}}} \) and \( D_H f^- = D_H f_{\chi_{\{f < 0\}}} \). Further, \( D_H f = 0 \) for \( \nu \)-a.e. in \( f^{-1}(0) \). The same results hold true if we replace \( \nu \) with \( \mu \) and \( D_H \) with \( D^\mu_H \).

**Proof.** The proof of this fact can be obtain by repeating verbatim that of [11, Lemma 2.7].
Thanks to Lemma 2.9 we are able to prove that a bounded function \( g \in W^{1,p}(X, \nu) \) with \( D_H g = 0 \) is constant for \( \nu \text{-a.e.} \) in \( X \).

**Proposition 2.10.** Let \( g \in W^{1,p}(X, \nu) \), with \( p > 1 \), be a bounded function such that \( D_H^p g = 0 \). Then, \( g \) equals a constant for \( \nu \text{-a.e.} \) in \( X \).

**Proof.** We show that \( g \in W^{1,1}(X, \mu) \) and \( D_H^p g = 0 \). From [7, Example 5.4.16] and the equivalence of the measures \( \mu \) and \( \nu \) the thesis follows. Let \( (g_n) \subset \mathcal{F}_b^1(X) \) be such that \( g_n \to g \) in \( W^{1,p}(X, \nu) \) as \( n \to +\infty \). Since \( U \in C^2(X) \) we have

\[
D_H^\mu (g_n e^{-U}) = e^{-U} (D_H^\mu g_n) - g_n e^{-U} (D_H U) = e^{-U} (D_H g_n) - g_n e^{-U} (D_H U) \\
= -e^{-U} (D_H^\mu g) - ge^{-U} (D_H U) = -ge^{-U} (D_H U), \quad n \to +\infty,
\]

in \( L^q(X, \mu) \) for any \( 1 < q < p \). Hence, \( ge^{-U} \in W^{1,q}(X, \mu) \) and \( D_H^\mu (ge^{-U}) = -ge^{-U} (D_H U) \). Let us set \( U_n := U \land n \). For any \( n \in \mathbb{N} \) the function \( U_n \) is bounded, from Lemma 2.9 we infer that \( e^{U_n} \in W^{1,r}(X, \mu) \) for any \( r \in [1, +\infty) \) and \( D_H^\mu (e^{U_n}) = e^{U_n} (D_H U) \chi_{\{U \leq n\}} \). If we consider \( r = q' \) the conjugate exponent of \( q \), we have \( ge^{-U} e^{U_n} = g\chi_{\{U \leq n\}} + ge^{n-U} \chi_{\{U > n\}} \in W^{1,1}(X, \mu) \) and

\[
D_H^\mu (ge^{-U} e^{U_n}) = e^{U_n} (D_H^\mu (ge^{-U})) + ge^{-U} (D_H^\mu e^{U_n}) \\
= -ge^{U_n-U} (D_H U) + ge^{U_n-U} (D_H U) \chi_{\{U \leq n\}} \\
= -ge^{U_n-U} (D_H U) \chi_{\{U > n\}} \\
= -ge^{-U} (D_H U) \chi_{\{U > n\}}.
\]

We notice that \((ge^{-U} e^{U_n}) = (g\chi_{\{U \leq n\}} + ge^{n-U} \chi_{\{U > n\}})\) is a sequence of bounded functions which pointwise converges to \( g \) as \( n \to +\infty \), and \( ge^{n-U} (D_H U) \chi_{\{U > n\}} \to 0 \) in \( L^r(X, \mu; H) \) for any \( r \in [1, +\infty) \). Hence, \( g \in W^{1,1}(X, \mu) \) and \( D_H^\mu g = 0 \), which gives that \( g \) is constant \( \mu \text{-a.e.} \) in \( X \).

Now we define the \( \nu \text{-H-divergence operator} \) as the adjoint of \( D_H \) in \( L^2(X, \nu) \).

**Definition 2.11.** Let

\[
D(D_H^\ast) := \left\{ f \in L^2(X, \nu; H) : \exists g \in L^2(X, \nu), \int_X [f, D_H g]_{H} d\nu = -\int_X g d\nu, \forall v \in \mathcal{F}_b(X) \right\}, \\
D_H^\ast f := g.
\]

We say that the operator \( D_H^\ast : D(D_H^\ast) \subset L^2(X, \nu; H) \to L^2(X, \nu) \) is the \( \nu \text{-H-divergence operator} \). Since \( D(D_H) = W^{1,2}(X, \nu) \) is densely defined in \( L^2(X, \nu) \) and \( D_H \) is a closed operator in \( L^2(X, \nu) \), from [20, Theorem 13.12] it follows that \( D(D_H^\ast) \) is densely defined in \( L^2(X, \nu; H) \) and \((D_H^\ast)^\ast = D_H \).

### 2.5 The Sobolev spaces \( W^{k,p}(X, \nu) \) and \( W^{1,p}(X, \nu; V) \)

Let us consider the second-order \( H \text{-gradient} D_H^2 \) defined on functions \( f \in \mathcal{F}_b^\infty(X) \) by

\[
D_H^2 f(x) = \sum_{j,k=1}^n \frac{\partial^2 \varphi}{\partial x_j \partial x_k}((x, x_1^*)\ldots,(x, x_n^*)) i^* x_j^* \otimes i^* x_k^*, \quad x \in X,
\]

where \( f(x) = \varphi((x, x_1^*),\ldots,(x, x_n^*)) \) for some \( n \in \mathbb{N} \), \( \varphi \in C_b^\infty(\mathbb{R}^n) \) and \( x_1^*,\ldots,x_n^* \in X^* \). For any \( x \in X \) the operator \( D_H^2 f(x) \in \mathcal{L}(1(H) \cap H_2(H)) \). Indeed, let \( \{e_m : m \in \mathbb{N}\} \) be an orthonormal basis of \( H \). Then, we have

\[
\text{Tr}(D_H^2 f(x))_{H} = \sum_{j,k=1}^n \frac{\partial^2 \varphi}{\partial x_j \partial x_k} \sum_{m \in \mathbb{N}} ([i^* x_j^* \otimes i^* x_k^*] e_m, e_m)_{H}
\]
we get the following result.

Let

\[ \text{Tr}[ (D_H^2 f(x))^2 ]_H = \| D_H^2 f(x) \|_{L^2(H)}^2 \]

= \[ \sum_{j,k,t,s=1}^n \frac{\partial^2 \varphi}{\partial \xi_j \partial \xi_k} \frac{\partial^2 \varphi}{\partial \xi_t \partial \xi_s} \sum_{m \in \mathbb{N}} [(i^* x_j^* \otimes i^* x_k^*) e_m, (i^* x_t^* \otimes i^* x_s^*) e_m]_H \]

= \[ \sum_{j,k,t,s=1}^n \frac{\partial^2 \varphi}{\partial \xi_j \partial \xi_k} \frac{\partial^2 \varphi}{\partial \xi_t \partial \xi_s} [i^* x_j^*, i^* x_k^*]_H \sum_{m \in \mathbb{N}} [i^* x_j^*, e_m]_H [i^* x_t^*, e_m]_H \]

= \[ \sum_{j,k,t,s=1}^n \frac{\partial^2 \varphi}{\partial \xi_j \partial \xi_k} \frac{\partial^2 \varphi}{\partial \xi_t \partial \xi_s} [i^* x_j^*, i^* x_k^*]_H [i^* x_j^*, i^* x_t^*]_H. \]

and, by recalling (1.3),

\[ \text{Tr}[ (D_H^2 f(x))^2 ]_H = \| D_H^2 f(x) \|_{L^2(H)}^2 \]

Arguing as in Proposition 2.7 we get the following result.

**Proposition 2.12.** The operator \((D_H, D_H^2) : \mathcal{F}C^\infty_b(X) \to L^p(X, \nu; H) \times L^p(X, \nu; H_2(H))\) is closable in \(L^p(X, \nu)\) for any \(p \in [1, +\infty)\). We still denote by \((D_H, D_H^2)\) its closure and by \(W^{2,p}(X, \nu)\) the domain of its closure. \(W^{2,p}(X, \nu)\) is a Banach space if endowed with the norm

\[ \| f \|_{2,p} := \left( \| f \|_{1,p}^p + \| D_H f \|_{L^p(X, \nu; H_2(H))}^p \right)^{1/p}, \quad f \in W^{2,p}(X, \nu). \]

Finally, \(W^{2,2}(X, \nu)\) is a Hilbert space if endowed with inner product

\[ \langle f, g \rangle_{W^{2,2}(X, \nu)} = \langle f, g \rangle_{W^{1,2}(X, \nu)} + \langle D_H^2 f, D_H^2 g \rangle_{L^2(X, \nu; H_2(H))}, \quad f, g \in W^{2,2}(X, \nu). \]

For any \(k \geq 3, k \in \mathbb{N}\), we introduce the \(k\)-th order \(H\)-gradient \(D_H^k\) defined on functions \(f \in \mathcal{F}C^\infty_b(X)\) by

\[ D_H^k f(x) = \sum_{j_1, \ldots, j_k = 1}^n \frac{\partial^k \varphi}{\partial \xi_{j_1} \cdots \partial \xi_{j_k}} ((x, x_1^*), \ldots, (x, x_n^*)) i^* x_{j_1}^* \otimes \cdots \otimes i^* x_{j_k}^*, \quad x \in X, \quad (2.14) \]

where \(f(x) = \varphi((x, x_1^*), \ldots, (x, x_n^*))\) for some \(n \in \mathbb{N}, \varphi \in C^\infty_b(\mathbb{R}^n)\) and \(x_1^*, \ldots, x_n^* \in X^*\). Arguing as above and taking (1.4) into account, it is easy to prove that \(D_H^3 f \in \mathcal{H}_k(H)\). The next proposition states that the operator \((D_H, D_H^2, D_H^3)\) is closable in \(L^p(X, \nu)\).

**Proposition 2.13.** Let \(k \geq 3, k \in \mathbb{N}\). The operator \((D_H, \ldots, D_H^k) : \mathcal{F}C^\infty_b(X) \to L^p(X, \nu; H) \times \cdots \times L^p(X, \nu; H_k(H))\) is closable in \(L^p(X, \nu)\) for any \(p \in [1, +\infty)\). We denote by \((D_H, \ldots, D_H^k)\) its closure and by \(W^{k,p}(X, \nu)\) the domain of its closure. \(W^{k,p}(X, \nu)\) is a Banach space if endowed with the norm

\[ \| f \|_{k,p} := \left( \| f \|_{k-1,p}^p + \| D_H^k f \|_{L^p(X, \nu; H_k(H))}^p \right)^{1/p}, \quad f \in W^{k,p}(X, \nu). \]

Finally, \(W^{k,2}(X, \nu)\) is a Hilbert space if endowed with inner product

\[ \langle f, g \rangle_{W^{k,2}(X, \nu)} = \langle f, g \rangle_{W^{k-1,2}(X, \nu)} + \langle D_H^k f, D_H^k g \rangle_{L^2(X, \nu; H_k(H))}, \quad f, g \in W^{k,2}(X, \nu). \]

**Remark 2.14.** Let us apply (2.9) to the function \(D_H^k f\). Let \(f \in \mathcal{F}C^\infty_b(X)\) be as in (2.14), and let \(h_1, \ldots, h_k \in H\). We have

\[ [D_H (D_H^{k-1} f(h_1, \ldots, h_{k-1})) (x), h_k]_H = [D_H^k f(x)(h_1, \ldots, h_{k-1}), h_k]_H. \]
\[ D^k_H f(x)(h_1, \ldots, h_{k-1}) = \sum_{i_1, \ldots, i_k=1}^n \frac{\partial^k \varphi}{\partial \xi_{i_1} \cdots \partial \xi_{i_k}} ((x, x_{i_1}^1), \ldots, (x, x_{i_k}^n)) [i_{i_1}^* x_{i_1}^1, h_1] \cdots [i_{i_k}^* x_{i_k}^n, h_{k-1}] H f_{i_{i_k}^* x_{i_k}^n}, \quad x \in X, \]

where

\[ D^k_H f(x)(h_1, \ldots, h_{k-1}) = \sum_{i_1, \ldots, i_k=1}^n \frac{\partial^k \varphi}{\partial \xi_{i_1} \cdots \partial \xi_{i_k}} ((x, x_{i_1}^1), \ldots, (x, x_{i_k}^n)) [i_{i_1}^* x_{i_1}^1, h_1] \cdots [i_{i_k}^* x_{i_k}^n, h_{k-1}] H f_{i_{i_k}^* x_{i_k}^n}, \quad x \in X, \]

is seen as an element of \( H \). Formula (2.9) gives

\[
\begin{align*}
\int_X D^k_H f(x)(h_1, \ldots, h_k) d\nu &= \int_X [D_H (D^{k-1}_H f(h_1, \ldots, h_{k-1}))], h_k]_H d\nu \\
&= \int_X D^{k-1}_H f(h_1, \ldots, h_{k-1}) \langle \hat{h}_k + [D_H U, h_k]_H \rangle d\nu. \tag{2.15}
\end{align*}
\]

Now we define the \( H \)-gradient on vector-valued functions. Let \( V \) be a separable Hilbert space. We introduce the operator \( \overline{D}_H \) defined on smooth functions \( F \in \mathcal{F}^\infty_0(X; V) \) as follows:

\[
\overline{D}_H F(x) = \sum_{i=1}^m D_H f_i(x) \otimes v_i \in H \otimes V, \quad x \in X,
\]

where

\[
F(x) = \sum_{i=1}^m f_i(x)v_i, \quad f_i \in \mathcal{F}^\infty_0(X), \quad v_i \in V, \quad i = 1, \ldots, m.
\]

The arguments of Proposition 2.7, adapted to vector-valued functions, allow us to prove the closability of \( \overline{D}_H \) in \( L^p(X, \nu; V) \) with \( p \in [1, +\infty) \).

**Proposition 2.15.** For any \( p \in [1, +\infty) \) the operator \( \overline{D}_H : \mathcal{F}^\infty_0(X; V) \to L^p(X, \nu; H \otimes V) \) is closable in \( L^p(X, \nu; V) \). We still denote by \( \overline{D}_H \) its closure and by \( W^{1,p}(X, \nu; V) \) the domain of its closure. \( W^{1,p}(X, \nu; V) \) is a Banach space if endowed with the norm

\[
\| F \|_{1,p,V} := \left( \| F \|^p_{L^p(X, \nu; V)} + \| \overline{D}_H F \|^p_{L^p(X, \nu; H \otimes V)} \right)^{1/p}, \quad F \in W^{1,p}(X, \nu; V).
\]

Finally, \( W^{1,2}(X, \nu; V) \) is a Hilbert space if endowed with the inner product

\[
\langle F, G \rangle_{W^{1,2}(X, \nu; V)} = \langle F, G \rangle_{L^2(X, \nu; V)} + \langle \overline{D}_H F, \overline{D}_H G \rangle_{L^2(X, \nu; H \otimes V)}, \quad F, G \in W^{1,2}(X, \nu; V).
\]

We conclude this subsection by showing that for any \( F \in W^{1,p}(X, \nu; V) \) with \( p \in [2, +\infty) \), the dual function \( F^* := \| F \|_{V}^{p-2} F \in W^{1,p'}(X, \nu; V) \).

**Lemma 2.16.** Let \( F \in W^{1,p}(X, \nu; V) \) with \( p \in [2, +\infty) \). Then, the function \( F^* := \| F \|_{V}^{p-2} F \in W^{1,p'}(X, \nu; V) \), where \( p' = \frac{p}{p-1} \), and

\[
\overline{D}_H F^* = (p-2)\| F \|_{V}^{p-4} \overline{D}_H |F|_{V}^{p-2} F + |F|_{V}^{p-2} \overline{D}_H F. \tag{2.16}
\]

Here, \( \overline{D}_H |F|_{V}^{p-2} F \) is seen as an element of \( H \).

**Proof.** The fact that \( F^* \in L^p(X, \nu; V) \) is trivial. Let us consider a sequence \((F_n) \subset \mathcal{F}^\infty_0(X; V)\) such that \( F_n \to F \) in \( W^{1,p}(X, \nu; V) \) as \( n \to +\infty \). For any \( n \in \mathbb{N} \) we set \( F_n^* := \| F_n \|_{V}^{p-2} F_n \). Hence, there exists a subsequence \((F_{n_k}) \subset (F_n)\) such that

\[
F_{n_k}^*(x) \to F^*(x), \quad F_{n_k}(x) \to F(x), \quad \overline{D}_H F_{n_k}(x) \to \overline{D}_H F(x), \quad \nu\text{-a.e. } x \in X. \tag{2.17}
\]
Further, 
\[ D_H F_{k_n}^* = (p-2) |F_{k_n}|_{V}^{p-4} [D_H F_{k_n}, F_{k_n}]_V \otimes F_{k_n} + |F_{k_n}|_{V}^{p-2} D_H F_{k_n}, \quad n \in \mathbb{N}, \quad (2.18) \]
where the writing \([D_H F_{k_n}, F_{k_n}]_V\) is meant as an element of \(H\). From (2.17) it follows that 
\[ D_H F_{k_n}^*(x) \to (p-2) |F(x)|_{V}^{p-4} [D_H F(x), F(x)]_V \otimes F(x) + |F(x)|_{V}^{p-2} D_H F(x) =: \Phi(x), \quad \nu\text{-a.e.} \ x \in X, \]
as \(n \to +\infty\). We claim that \(F_{k_n}^* \to F^*\) in \(L^{p'}(X, \nu; V)\) and that \(D_H F_{k_n}^* \to \Phi\) in \(L^{p'}(X, \nu; H \otimes V)\). If the claim is true, the fact that \(D_H\) is a closed operator in \(L^{p'}(X, \nu; V)\) implies that \(F^* \in W^{1,p'}(X, \nu; V)\) and \(D_H F^* = \Phi\), which gives the thesis. Hence, it remain to prove the claim. Egoroff’s Theorem (see [8, Theorem 2.2.1], whose proof can be easily generalized to vector-valued functions) implies that for any \(\varepsilon > 0\) there exists a Borel set \(X_\varepsilon \subset X\) such that \(\nu(X_\varepsilon) < \varepsilon\) and \(F_{k_n}^* \to F^*\) uniformly on \(X \setminus X_\varepsilon\). Let \(\varepsilon > 0\). For any \(n \in \mathbb{N}\) we have
\[ \int_X |F_{k_n}^* - F^*|^p V \, d\nu = \int_{X \setminus X_\varepsilon} |F_{k_n}^* - F^*|^p V \, d\nu + \int_{X_\varepsilon} |F_{k_n}^* - F^*|^p V \, d\nu. \quad (2.19) \]
Since \(F_{k_n}^* \to F^*\) uniformly on \(X \setminus X_\varepsilon\), it follows that 
\[ \int_{X \setminus X_\varepsilon} |F_{k_n}^* - F^*|^p V \, d\nu \to 0, \quad n \to +\infty. \quad (2.20) \]
Further, from the definition of \(F_{k_n}^*\) and of \(F^*\) it follows that
\[ \int_{X_\varepsilon} |F_{k_n}^* - F^*|^p V \, d\nu = \|\chi_{X_\varepsilon}(F_{k_n}^* - F^*)\|_{L^{p'}(X, \nu; V)}^p \]
\[ \leq \left(\|\chi_{X_\varepsilon} F_{k_n}^*\|_{L^{p'}(X, \nu; V)} + \|\chi_{X_\varepsilon} F^*\|_{L^{p'}(X, \nu; V)}\right)^p \]
\[ = \left(\|\chi_{X_\varepsilon} F_{k_n}\|_{L^{p}(X, \nu; V)} + \|\chi_{X_\varepsilon} F\|_{L^{p}(X, \nu; V)}\right)^p. \quad (2.21) \]
From (2.19), (2.20) and (2.21), and recalling that \(F_{k_n} \to F\) in \(L^{p}(X, \nu; V)\) as \(n \to +\infty\), it follows that 
\[ \limsup_{n \to +\infty} \int_X |F_{k_n}^* - F^*|^p V \, d\nu \leq 2 \|\chi_{X_\varepsilon} F\|_{L^{p}(X, \nu; V)}^p, \]
for any \(\varepsilon > 0\). Since \(\nu(X_\varepsilon) < \varepsilon\) for any \(\varepsilon > 0\), we conclude that \(F_{k_n}^* \to F^*\) in \(L^{p'}(X, \nu; V)\) as \(n \to +\infty\).

Let us consider \(D_H F_{k_n}^*\). Again from Egoroff’s Theorem, for any \(\varepsilon > 0\) there exists a Borel set \(X_\varepsilon \subset X\) such that \(\nu(X_\varepsilon) < \varepsilon\) and \(D_H F_{k_n}^* \to \Phi\) uniformly on \(X \setminus X_\varepsilon\). Let us fix \(\varepsilon > 0\). For any \(n \in \mathbb{N}\) we have
\[ \int_X |D_H F_{k_n}^* - \Phi|^p_{H \otimes V} \, d\nu = \int_{X \setminus X_\varepsilon} |D_H F_{k_n}^* - \Phi|^p_{H \otimes V} \, d\nu + \int_{X_\varepsilon} |D_H F_{k_n}^* - \Phi|^p_{H \otimes V} \, d\nu =: I_1^n + I_2^n, \quad n \in \mathbb{N}. \]
Since \(D_H F_{k_n}^* \to \Phi\) uniformly on \(X \setminus X_\varepsilon\), it follows that \(I_1^n \to 0\) as \(n \to +\infty\). Let us estimate \(I_2^n\). We have
\[ I_2^n = \|\chi_{X_\varepsilon} (D_H F_{k_n}^* - \Phi)\|_{L^{p'}(X, \nu; H \otimes V)}^p \]
\[ \leq \left(\|\chi_{X_\varepsilon} D_H F_{k_n}^*\|_{L^{p'}(X, \nu; H \otimes V)} + \|\chi_{X_\varepsilon} \Phi\|_{L^{p'}(X, \nu; H \otimes V)}\right)^p \]
\[ =: (J_1^n + J_2^n)^p, \quad n \in \mathbb{N}. \quad (2.22) \]
Taking into account (2.18) we infer that
\[
(J^n)^p' \leq (p-1)^p' \int_X \chi_{X_n} |F_{k_n}|_{L^p(V)}^p |D_H F_{k_n}|_{H \otimes V}^p \, d\nu
\]
\[
\leq (p-1)^p' \left( \int_X \chi_{X_n} |F_{k_n}|_{L^p(V)}^p \, d\nu \right)^{\frac{p-2}{p-1}} \cdot \left( \int_X \chi_{X_n} |D_H F_{k_n}|_{L^p(X,\nu;H \otimes V)}^p \, d\nu \right)^{\frac{1}{p-1}}
\]
where in the second inequality we have applied the Hölder’s inequality with \(q = p-1\) and \(q' = \frac{p-1}{p-2}\), and we have used the fact that \(p' = \frac{p}{p-1}\). Recalling that \(F_n \to F\) in \(W^1,p(X,\nu;V)\) as \(n \to +\infty\), we deduce that
\[
\limsup_{n \to +\infty} J^n \leq (p-1)\|\chi_{X_n} F\|_{L^p(X,\nu;V)}^{p-2} \cdot \|\chi_{X_n} D_H F\|_{L^p(X,\nu;H \otimes V)}.
\]
As far as \(J_2\) is considered, arguing as in (2.23) with \(F_{k_n}\) replaced by \(F\) it follows that
\[
J_2 \leq (p-1)\|\chi_{X_n} F\|_{L^p(X,\nu;V)}^{p-2} \cdot \|\chi_{X_n} D_H F\|_{L^p(X,\nu;H \otimes V)}.
\]
From (2.22), (2.24) and (2.25), it follows that
\[
\limsup_{n \to +\infty} \int_X |D_H F_{k_n} - \Phi|_{H \otimes V}^p \, d\nu \leq (2p - 1)^p' \|\chi_{X_n} F\|_{L^p(X,\nu;V)}^{p(p-2)} \cdot \|\chi_{X_n} D_H F\|_{L^p(X,\nu;H \otimes V)}^p,
\]
for any \(\varepsilon > 0\). Since \(\nu(X_n) < \varepsilon\) for any \(\varepsilon > 0\), we get
\[
\limsup_{n \to +\infty} \int_X |D_H F_{k_n} - \Phi|_{H \otimes V}^p \, d\nu = 0,
\]
which proves the claim. \(\square\)

**Remark 2.17.** Hereafter, we fix an orthonormal basis \(\Phi : \{e_m = i^*(e^*_m) : m \in \mathbb{N}\} \) of \(H\) (the existence of this basis follows from the density of \(j(X^*)\) in \(L^2(X,\mu)\) and from the isometry between \(L^2(X,\mu)\) and \(H\)), and for any \(k \in \mathbb{N} \cup \{\infty\}\) let us define \(\mathcal{F} \mathcal{E}_{b,\Phi}(X) := \{ f \in \mathcal{F} \mathcal{E}_{b}(X) : f(x) = \varphi((x,e^*_1),\ldots,(x,e^*_m)) \}, m \in \mathbb{N}, \varphi \in C_b^k(\mathbb{R}^m)\). For any \(p \in [1, +\infty)\) the closure of the operator \(D_H : \mathcal{F} \mathcal{E}_{b,\Phi}^\infty(X) \to L^p(X,\nu;H)\) and the domain of its closure coincide with \(D_H\) and with \(W^{1,p}(X,\nu)\), respectively. Indeed, let \(p \in [1, +\infty)\) and let \(f \in \mathcal{F} \mathcal{E}_{b,\Phi}^\infty(X)\) be such that \(f(x) = \varphi((x,x^*_1),\ldots,(x,x^*_n))\) for some \(n \in \mathbb{N}\), \(\varphi \in C_b^\infty(\mathbb{R}^n)\) and \(x^*_1,\ldots,x^*_n \in X^*\). For any \(m \in \mathbb{N}\) let us denote by \(P_m : X \to H\) the projection
\[
P_m x := \sum_{i=1}^m (x,e^*_i) e_i, \quad x \in X,
\]
on \(\text{span}\{e_1,\ldots,e_m\}\), and let us set \(f_m(x) := f(P_m x)\) for any \(x \in X\). Clearly, \(f_m \in \mathcal{F} \mathcal{E}_{b,\Phi}^\infty(X)\) for any \(m \in \mathbb{N}\). Further, from [7, Corollary 3.5.8] and from the dominated convergence theorem we infer that \(f_m \to f\) in \(L^p(X,\nu)\) as \(n \to +\infty\). Let us compute \(D_H f_m\). We have
\[
D_H f_m(x) = P_m D_H f(P_m x), \quad x \in X, \ m \in \mathbb{N}.
\]
[7, Corollary 3.5.8] and the dominated convergence theorem give \(D_H f_m \to D_H f\) in \(L^p(X,\nu;H)\) as \(n \to +\infty\). As a byproduct, \(\mathcal{F} \mathcal{E}_{b,\Phi}^\infty(X)\) is dense in \(W^{1,p}(X,\nu)\) for any \(p \in [1, +\infty)\). Analogously, it is possible to prove that \(\mathcal{F} \mathcal{E}_{b,\Phi}^\infty(X)\) is dense in \(W^{k,p}(X,\nu)\) for any \(k \in \mathbb{N}\) and any \(p \in [1, +\infty)\). Finally, we set \(\Pi_n : X \to \mathbb{R}^n\) the projection defined by
\[
\Pi_n x := ((x,f^*_1),\ldots,(x,f^*_n)) \in \mathbb{R}^n, \quad x \in X,
\]
and for any \(n \in \mathbb{N}\) we define \(\Sigma_n : \mathbb{R}^n \to H\) as
\[
\Sigma_n \xi := \sum_{i=1}^n \xi_i e_i, \quad \xi = (\xi_1,\ldots,\xi_n) \in \mathbb{R}^n.
\]
2.6 The perturbed Ornstein-Uhlenbeck operator in $L^p(X, \nu)$

We introduce the symmetric bilinear form

$$\mathcal{E}(u, v) := \int_X [D_H u, D_H v] \nu, \quad u, v \in W^{1,2}(X, \nu),$$

with domain $\mathcal{D} = W^{1,2}(X, \nu)$. From Proposition 2.7 it follows that $\mathcal{E}$ is a symmetric bilinear form which satisfies the strong sector condition, hence it is closed and coercive. [17, Theorem 2.8 & Corollary 2.10] implies that the operator $(L_2, D(L_2))$ defined as

$$D(L_2) := \{ u \in W^{1,2}(X, \nu) : \exists g \in L^2(X, \nu), \mathcal{E}(u, v) = -\int_X g v \nu, \forall v \in \mathcal{F}\mathcal{C}^\infty_b(X) \},$$

$$L_2 u := g,$$

is the infinitesimal generator of an analytic symmetric strongly continuous semigroup of contractions $(T_2(t))_{t \geq 0}$ on $L^2(X, \nu)$. The integration by parts formula (2.9) allows us to provide an explicit expression to $L$ applied to smooth functions.

**Proposition 2.18.** $\mathcal{F}\mathcal{C}^3_b(X) \subset D(L_2)$ and for any $u \in \mathcal{F}\mathcal{C}^3_b(X)$ it follows that

$$(L_2 u)(x) = \text{Tr}[D_H^2 u(x)]_H - \langle x, Du(x) \rangle - [D_H U(x), D_H u(x)]_H, \quad x \in X. \tag{2.26}$$

Further, if $u \in \mathcal{F}\mathcal{C}^3_b(X)$ then $L_2 u \in \mathcal{F}\mathcal{C}^4_b(X)$ and for any $h \in H$ we have

$$[(D_H L_2 u)(x), h]_H = (L_2 [D_H u(-), h]_H)(x) - [D_H u(x), h]_H - [D_H^2 U(x) h, D_H u(x)]_H. \tag{2.27}$$

**Proof.** Formula (2.26) is well known and it is a direct consequence of integration by parts (2.9). Further, By differentiating (2.26), long but straightforward computations give (2.27). \qed

We conclude this subsection by providing some properties of $(T_2(t))_{t \geq 0}$ which arise from the theory of Dirichlet forms. For reader’s convenience, we recall the definition of Dirichlet forms, Dirichlet operators and sub-Markovian semigroups and their main properties (see e.g. [17, Chapter 1, Definitions 4.1 & 4.5, Proposition 4.3 & Theorem 4.4]).

**Definition 2.19.** Let $(E, B, \mu)$ be a measure space and let $\mathcal{H} := L^2(E, \mu)$ be a Hilbert space.

(i) A symmetric closed coercive form $(\mathcal{E}, D(\mathcal{E}))$ on $\mathcal{H}$ is called a Dirichlet form if for any $u \in D(\mathcal{E})$ one has $\mathcal{E}(u^+ \wedge 1, u^+ \wedge 1) \leq \mathcal{E}(u, u)$.

(ii) A semigroup $(S(t))_{t \geq 0}$ on $\mathcal{H}$ is called sub-Markovian if for any $t \geq 0$ and any $f \in \mathcal{H}$ with $0 \leq f \leq 1$ $\mu$-a.e. in $E$, we have $0 \leq S(t)f \leq 1$ $\mu$-a.e. in $E$.

(iii) A closed linear densely defined operator $A$ on $\mathcal{H}$ is called Dirichlet operator on $\mathcal{H}$ if

$$\int_E Au(u - 1)^+ d\mu \leq 0, \quad u \in D(A).$$

**Proposition 2.20.** Let $(\mathcal{E}, D(\mathcal{E}))$ be a symmetric closed coercive Dirichlet form on $L^2(E, \mu)$, let $A$ be the operator associated to $\mathcal{E}$ and let $(S(t))_{t \geq 0}$ be the strongly continuous contraction semigroup on $L^2(E, \mu)$ generated by $A$. Then, the following are equivalent:

(i) $(\mathcal{E}, D(\mathcal{E}))$ is a Dirichlet form on $L^2(E, \mu)$.

(ii) $(S(t))_{t \geq 0}$ is a sub-Markovian semigroup on $L^2(E, \mu)$.

(iii) $A$ is a Dirichlet operator on $L^2(E, \mu)$. 
Proposition 2.21. The bilinear form \((\mathcal{E}, W^{1,2}(X, \nu))\) is a Dirichlet form on \(L^2(X, \nu)\). Then:

(i) \((T_2(t))_{t \geq 0}\) is non-negative, i.e., for any \(t > 0\) we have \(T(t)f \geq 0\) \(\nu\)-a.e. in \(X\) for any \(f \in L^2(X, \nu)\) such that \(f \geq 0\) \(\nu\)-a.e. in \(X\). Further, if \(f \in L^2(X, \nu)\) and has positive infimum, i.e., there exists a positive constant \(c\) such that \(f \geq c\) \(\nu\)-a.e. in \(X\), then \(T_2(t)f \geq c\) \(\nu\)-a.e. in \(X\) for any \(t > 0\).

(ii) For any \(f \in C_b(X)\) we have \(|T_2(t)f| \leq T_2(t)|f|\) \(\nu\)-a.e. in \(X\) and \(\|T_2(t)f\|_\infty \leq \|f\|_\infty\) for any \(t \geq 0\).

(iii) for any \(1 \leq p \leq q \leq +\infty\) we have
\[
(T_2(t)(|f|^p))^{1/p} \leq (T_2(t)(|f|^q))^{1/q}, \quad \nu\text{-a.e. in } X, \quad f \in C_b(X), \quad t \geq 0.
\] (2.28)

(iv) For any \(p \in (1, +\infty)\) we have
\[
|T_2(t)(fg)| \leq (T_2(t)(|f|^p))^{1/p}(T_2(t)(|g|^p'))^{1/p'}, \quad \nu\text{-a.e. in } X, \quad f, g \in C_b(X), \quad t \geq 0,
\] (2.29)
where \(p'\) is the conjugate exponent of \(p\), i.e., \(p' = \frac{p}{p-1}\).

Proof. At first we show that \((\mathcal{E}, W^{1,2}(X, \nu))\) is a Dirichlet form. From Definition 2.19 it is enough to show that for any \(u \in W^{1,2}(X, \nu)\) we have \(u^+ \land 1 \in W^{1,2}(X, \nu)\) and
\[
\mathcal{E}(u^+ \land 1, u^+ \land 1) \leq \mathcal{E}(u, u), \quad u \in W^{1,2}(X, \nu).
\]
From Lemma 2.9 we infer that \(u^+ \land 1 \in W^{1,2}(X, \nu)\) for any \(u \in W^{1,2}(X, \nu)\) and \(D_H(u^+ \land 1) = D_Hux\{u \in (0,1)\}\). Then,
\[
\mathcal{E}(u^+ \land 1, u^+ \land 1) = \int_{\{u \in (0,1)\}} |D_Hu|^2 \, d\nu \leq \int_X |D_Hu|^2 \, d\nu = \mathcal{E}(u, u),
\]
which implies that \((\mathcal{E}, W^{1,2}(X, \nu))\) is a Dirichlet form. From Proposition 2.20 it follows that \((T_2(t))_{t \geq 0}\) is a sub-Markovian semigroup on \(L^2(X, \nu)\).

Let us prove (i). Let \(f \in L^\infty(X, \nu)\) such that \(f \geq 0\) \(\nu\)-a.e. in \(X\), and let \(t > 0\). Then, \(0 \leq f\|f\|_\infty^1 \leq 1\) \(\nu\)-a.e. in \(X\), which gives
\[
0 \leq T_2(t)(f\|f\|_\infty^1) \leq 1, \quad \nu\text{-a.e. in } X,
\]
and this implies that \(0 \leq T_2(t)f \leq \|f\|_\infty\) \(\nu\)-a.e. in \(X\). If \(f \in L^2(X, \nu)\) satisfies \(f \geq 0\) \(\nu\)-a.e. in \(X\), we consider the sequence \((f_n := f \land n) \subset L^\infty(X, \nu)\). Since \(f_n \geq 0\) \(\nu\)-a.e. in \(X\), it follows that \(T_2(t)f_n \geq 0\) \(\nu\)-a.e. in \(X\). Moreover, \((T_2(t)f_n)\) converges to \(T_2(t)f\) in \(L^2(X, \nu)\), and, up to a subsequence, pointwise \(\nu\)-a.e. in \(X\), hence, \(T_2(t)f \geq 0\) \(\nu\)-a.e. in \(X\). To prove the second part, we notice that if \(f \geq c > 0\) \(\nu\)-a.e. in \(X\), the function \(g := f - c \geq 0\) \(\nu\)-a.e. in \(X\). By recalling that \(T_2(t)c = a\) for any \(a \in \mathbb{R}\), it follows that
\[
T_2(t)f - c = T_2(t)f - T_2(t)c = T_2(t)(f - c) \geq 0,
\]
which gives the thesis.

Now we prove (ii). Let \(f \in C_b(X)\) and let us consider \(T_2(t)f^+ + T_2(t)f^-\). From (i) it follows that \(T_2(t)f^+ + T_2(t)f^- \geq 0\) \(\nu\)-a.e. in \(X\). Then,
\[
|T_2(t)f| = |T_2(t)(f^+ - f^-)| \leq |T_2(t)f^+| + |T_2(t)f^-| = T_2(t)(f^+ + f^-) = T_2(t)|f|.
\]
In the proof of (i) we have shown that \(0 \leq T_2(t)|f| \leq \|f\|_\infty\). Hence, we get \(|T_2(t)f| \leq \|f\|_\infty\), which gives the second part of (ii).

Finally, to prove (iii) and (iv) it is enough to repeat the computations in [22, Lemma 2.1].
Finally, arguing as in [1, Proposition 3.7] we infer that \((T_2(t))_{t \geq 0}\) extends to a positive strongly continuous semigroup of contractions \((T_p(t))_{t \geq 0}\) on \(L^p(X, \nu)\) for any \(p \in [1, +\infty)\). We state this result in the following proposition.

**Proposition 2.22.** \((T_2(t))_{t \geq 0}\) extends to a positive strongly continuous semigroups of contraction \((T_p(t))_{t \geq 0}\) on \(L^p(X, \nu)\) with infinitesimal generator \(L_p\) for any \(p \in [1, +\infty)\). These semigroups are consistent in the sense that if \(f \in L^p(X, \nu)\) for some \(p \in [1, +\infty)\), then \(T_p(t)f = T_q(t)f\) for any \(q \in [1, p]\) and any \(t \geq 0\).

**Remark 2.23.** Where there is no danger of confusion we omit the subscript \(p\) and we simply denote by \((T(t))_{t \geq 0}\) and by \(L\) the semigroup and its infinitesimal generator on \(L^p(X, \nu)\), respectively.

When \(p \in [2, +\infty)\) we can associate a bilinear form to \(L_p\).

**Corollary 2.24.** For any \(p \in [2, +\infty)\) we have
\[
\int_X L_p f g d\nu = -\int_X [D_H f, D_H g]_{H^1} d\nu,
\]
for any \(f \in D(L_p)\) and any \(g \in W^{1,p'}(X, \nu)\).

**Proof.** Since \(D(L_p) \subset D(L_2)\), the thesis follows from the definition of \(L_2\) and from the density of \(\mathcal{F}c_b^\infty(X)\) in \(W^{1,p'}(X, \nu)\).

\[\square\]

### 2.7 The perturbed vector-valued Ornstein-Uhlenbeck semigroup in \(L^p(X, \nu; V)\)

Let \(V\) be a separable Hilbert space. It is well-known that \(\mathcal{F}c_b^\infty(X; V)\) is dense in \(L^p(X, \nu; V)\). We consider the vector-valued semigroup \((T^V_p(t))_{t \geq 0}\) on \(L^p(X, \nu; V)\), extension of \((T_p(t))_{t \geq 0}\) (for vector-valued extensions of positive operators see [15, Subsection 4.5.3]), defined on \(F \in L^p(X, \nu; V)\) with finite range by
\[
T^V_p(t)F := \sum_{i=1}^n T_p(t)f_i v_i, \quad F := \sum_{i=1}^n f_i v_i, \quad f_i \in L^p(X, \nu), \quad v_i \in V, \quad i = 1, \ldots, n.
\]

It turns out that \((T^V_p(t))_{t \geq 0}\) is a strongly continuous semigroup of contractions on \(L^p(X, \nu; V)\). We denote by \(L^V_p\) its infinitesimal generator, which acts on functions \(F \in D(L^V_p)\) with finite range as
\[
L^V_p F = \sum_{i=1}^n L_p f_i v_i, \quad F := \sum_{i=1}^n f_i v_i, \quad f_i \in D(L_p), \quad v_i \in V, \quad i = 1, \ldots, n,
\]
where we have supposed, without loss of generality, than \(\{v_1, \ldots, v_n\}\) is an orthonormal system in \(V\).

Where there is no danger of confusion we omit the subscript \(p\) and we simply denote by \((T^V(t))_{t \geq 0}\) and by \(L^V\) the semigroup and its infinitesimal generator on \(L^p(X, \nu; V)\), respectively. We generalize some properties of the scalar semigroup and of its infinitesimal generator to \((T^V(t))_{t \geq 0}\) and of \(L^V\) in the following lemma.

**Lemma 2.25.** Let \((T^V(t))_{t \geq 0}\) and \(L^V\) be defined as above. Then:

(i) \((T^V(t))_{t \geq 0}\) is symmetric with respect to \(\nu\), i.e.,
\[
\int_X [T^V(t)F, G]_{V} d\nu = \int_X [F, T^V(t)G]_{V} d\nu, \quad F \in L^p(X, \nu; V), \quad G \in L^p(X, \nu; V).
\]

(ii) Let \(p \in [2, +\infty)\). For any \(F \in D(L^V_p)\) and \(G \in W^{1,p'}(X, \nu; V)\) we have
\[
\int_X [L^V_p F, G]_{V} d\nu = \int_X [D_H F, D_H G]_{H^1 \otimes V} d\nu.
\]

(2.30)
Proof. (i) By density we can limit ourselves to prove the statement for \( F, G \in \mathcal{F}C_{\infty}(X; V) \). Hence, we fix

\[
F := \sum_{i=1}^{n} f_i v_i, \quad G := \sum_{j=1}^{n} g_j w_j, \quad f_i, g_j \in \mathcal{F}C_{\infty}(X), \quad v_i, w_j \in V, \quad i, j = 1, \ldots, n.
\]

Then,

\[
\int_X [T^V(t)F, G]_V d\nu = \sum_{i,j=1}^{n} [v_i, w_j]_V \int_X T(t)f_i g_j d\nu = \sum_{i,j=1}^{n} [v_i, w_j]_V \int_X f_i T(t)g_j d\nu
\]

where we have used the property of symmetry of the scalar semigroup \((T(t))_{t \geq 0}\).

(ii) By approximation we can limit ourselves to prove the statement for functions with finite range. Let us fix

\[
F := \sum_{i=1}^{n} f_i v_i, \quad G := \sum_{j=1}^{n} g_j w_j, \quad f_i \in D(L^p), \quad g_j \in W^{1,p'}(X, \nu), \quad v_i, w_j \in V, \quad i, j = 1, \ldots, n,
\]

where we have supposed, without loss of generality, than \( \{v_1, \ldots, v_n\} \) and \( \{w_1, \ldots, w_n\} \) are orthonormal systems in \( V \). From Corollary 2.24 we infer that

\[
\int_X [L^p F, G]_V d\nu = \sum_{i,j=1}^{n} [v_i, w_j]_V \int_X L^p f_i g_j d\nu = -\sum_{i,j=1}^{n} [v_i, w_j]_V \int_X [D_H f_i, D_H g_j]_H d\nu
\]

\[= - \int_X [D_H F, D_H G]_{H \otimes V} d\nu.\]

\[\square\]

2.8 Smooth approximations

This section is devoted to the approximation procedure which will be crucial in the following of the paper. We recall the definition of \( H \)-Lipschitz function: let \( Y \) be a separable Banach space with norm \( \| \cdot \|_Y \), we say that \( F : X \to Y \) is a \( H \)-Lipschitz continuous function if there exists a positive constant \( M \) such that

\[
\| F(x + h) - F(x) \|_Y \leq M|h|_H, \quad \forall h \in H, \quad \mu\text{-a.e. } x \in X.
\]

(2.31)

We denote by \([F]_{H-Lip}\) the smaller constant \( M \) which realizes (2.31).

For any \( f \in L^1(X, \mu) \) we introduce the conditional expectation \( \mathbb{E}_n f \) as

\[
\mathbb{E}_n f(x) := \int_X f(P_n x + (I - P_n)y) \mu(dy), \quad \mu\text{-a.e. } x \in X,
\]

(2.32)

where \( P_n \) has been defined in Remark 2.17. We recall the following results (see [7, Corollary 3.5.2 & Proposition 5.4.5]).

**Proposition 2.26.** Let \( p \in [1, +\infty) \) and let \( f \in L^p(X, \mu) \). Then, \( \mathbb{E}_n f \to f \) in \( L^p(X, \mu) \) and for any \( n \in \mathbb{N} \) we have

\[
\| \mathbb{E}_n f \|_{L^p(X, \mu)} \leq \| f \|_{L^p(X, \mu)}.
\]
Further, if \( f \in W^{1,p}(X,\mu) \) then \( \mathbb{E}_n f \to f \) in \( W^{1,p}(X,\mu) \) and for any \( n \in \mathbb{N} \) we have

\[
[D_H(\mathbb{E}_n f), h_j]_H = \begin{cases} 
\mathbb{E}_n([D_H f, h_j]_H), & 1 \leq j \leq n, \\
0, & j \geq n + 1.
\end{cases}
\]

Finally, the same results, with the obvious modifications, are true for \( f \in W^{2,p}(X,\mu) \).

The final tool we need are the Moreau-Yosida approximants of \( U \) along \( H \). Below we state the main results we use in the following, and we refer to [6, Section 12.4] for the classical theory, and to [3, 4, 9] for the case here considered.

**Proposition 2.27.** Let \( f : X \to \mathbb{R} \cup \{+\infty\} \) be a proper convex and lower semicontinuous function and denote by \( \text{dom}(f) = \{ x \in X : f(x) < +\infty \} \). For \( \varepsilon > 0 \) and \( x \in X \), let us set

\[
f_\varepsilon(x) := \inf \left\{ f(x + h) + \frac{1}{2\varepsilon} |h|^2_H \mid h \in H \right\}.
\]

Then:

(i) \( f_\varepsilon(x) \leq f(x) \) for any \( \varepsilon > 0 \) and \( x \in X \). Moreover, \( f_\varepsilon(x) \) monotonically converges to \( f(x) \) as \( \varepsilon \to 0^+ \) for any \( x \in X \).

(ii) \( f_\varepsilon \) is \( H \)-differentiable and \( D_H f_\varepsilon \) is \( H \)-Lipschitz continuous in \( X \).

(iii) \( f_\varepsilon \in W^{2,p}(X,\mu) \) whenever \( f \in L^p(X,\mu) \) with \( p \in [1, +\infty) \).

(iv) If \( x \in \text{dom}(f) \) and \( f \in W^{1,p}(X,\mu) \) for some \( p \in [1, +\infty) \) then \( D_H f_\varepsilon(x) \) converges to \( D_H f(x) \) as \( \varepsilon \to 0^+ \).

(v) If \( f \in C^2(X) \cap W^{2,p}(X,\mu) \) for some \( p \in [1, +\infty) \), then \( D^2_H f_\varepsilon(x) \) exists and converges to \( D_H^2 f(x) \) as \( \varepsilon \to 0^+ \) for any \( x \in \text{dom}(f) \). Moreover, \( D^2_H f_\varepsilon \) is \( H \)-continuous in \( X \), i.e., for any \( x \in X \) we have

\[
\lim_{|h|_H \to 0} D^2_H f_\varepsilon(x + h) = D^2_H f(x).
\]

Let us introduce the smooth approximations of \( U \). For any \( \varepsilon > 0 \), let \( U_\varepsilon \) be the Moreau-Yosida approximants of \( U \). Further, we set

\[
\psi_{\varepsilon,n}(\xi) := \mathbb{E}_n(U_\varepsilon)(\Sigma_n \xi), \quad \psi_{\varepsilon,n,\eta}(\xi) := (\psi_{\varepsilon,n} \ast \theta_\eta)(\xi), \quad \xi \in \mathbb{R}^n,
\]

where the second term is the convolution of \( \psi_{\varepsilon,n} \) with the family of mollifiers \( \theta_\eta, \eta \in \mathbb{R}^+ \). Here, \( \theta_\eta(\xi) = \theta(\eta \xi) \) and \( \theta \in C^\infty_c(\mathbb{R}^n) \) is a positive function with support contained in the unit ball such that \( \int_{\mathbb{R}^n} \theta(\xi)d\xi = 1 \). For any \( \varepsilon > 0 \) we set

\[
\nu_\varepsilon := e^{-U_\varepsilon} \mu.
\]

Arguing as in [9, Proposition 5.12] it follows that there exists \( x^* \in X^* \) and \( r \in \mathbb{R} \) such that

\[
U_\varepsilon(x) \geq \langle x, x^* \rangle + r, \quad x \in X,
\]

for any \( \varepsilon \in (0,1] \). This fact and the Fernique Theorem (see [7, Theorem 2.8.5]) imply that for any \( \varepsilon \in (0,1] \) we have \( e^{-U_\varepsilon} \in L^p(X,\mu) \) for any \( p \in [1, +\infty) \), and so the measure \( \nu_\varepsilon \) is well defined. We also deduce the following useful lemma.

**Lemma 2.28.** Let \( U_\varepsilon \) be as above. Then, for any \( p \in [1, +\infty) \) and any decreasing and vanishing sequence \( (\varepsilon_n)_{n \in \mathbb{N}} \) we have

\[
\lim_{n \to +\infty} \| e^{-U_{\varepsilon_n}} - e^{-U} \|_{L^p(X,\mu)} = 0.
\]
Proof. Thank to 2.27(i), the thesis follows from the monotone convergence theorem.

**Proposition 2.29.** For any $\varepsilon \in (0, 1]$, $n \in \mathbb{N}$ and $\eta \in \mathbb{R}^+$, the function $\psi_{\varepsilon, n, \eta} \in C^\infty_c(\mathbb{R}^n)$. Further, if we set $U_{\varepsilon, n, \eta}(x) := \psi_{\varepsilon, n, \eta}(\Sigma_n \Pi_n x)$, $x \in X$, for any $\varepsilon \in (0, 1]$ there exists a decreasing vanishing sequence $(\eta_n) \subset \mathbb{R}^+$ such that

$$L^2(X, \nu_\varepsilon; H) - \lim_{n \to +\infty} D_H U_{\varepsilon, n, \eta_n} = D_H U_\varepsilon,$$  \hspace{1cm} (2.33)

$$L^2(X, \nu_\varepsilon; H_2(H)) - \lim_{n \to +\infty} D_H^2 U_{\varepsilon, n, \eta_n} = D_H^2 U_\varepsilon.$$  \hspace{1cm} (2.34)

Proof. The first part of the statement follows arguing as in [4, Lemma 2.3]. To show (2.33), at first we prove that for any vanishing sequence $(\eta_n)$ we have

$$L^2(X, \nu_\varepsilon; H) - \lim_{n \to +\infty} D_H U_{\varepsilon, n, \eta_n} = D_H U_\varepsilon.$$  \hspace{1cm}

Let $U_{\varepsilon, n} := \mathbb{E}_n(U_\varepsilon)$ for any $\varepsilon \in (0, 1]$ and any $n \in \mathbb{N}$. We have

$$\|D_H U_{\varepsilon, n, \eta_n} - D_H U_\varepsilon\|_{L^2(X, \nu_\varepsilon; H)}^2 \leq 2\|D_H U_{\varepsilon, n} - D_H U_\varepsilon\|^2_{L^2(X, \nu_\varepsilon; H)} + 2\|D_H U_{\varepsilon, n, \eta_n} - D_H U_{\varepsilon, n}\|^2_{L^2(X, \nu_\varepsilon; H)} =: I_1^n + I_2^n.$$  \hspace{1cm}

As far as $I_1^n$ is considered, for any $p \in (1, +\infty)$ from Proposition 2.26 we get

$$I_1^n \leq \left( \int_X e^{-p' U_\varepsilon} d\mu \right)^{1/p'} \|D_H U_{\varepsilon, n} - D_H U_\varepsilon\|_{L^{p'}(X, \mu; H)}^2 \to 0, \hspace{0.5cm} n \to +\infty,$$  \hspace{1cm}

where $p'$ is the conjugate exponent of $p$. Let us estimate $I_2^n$. From the definition of $\psi_{\varepsilon, n, U_{\varepsilon, n, \eta_n}}$ and $U_{\varepsilon, n, \eta_n}$, and from Proposition 2.27(ii) it follows that

$$I_2^n = \int_X \int_X \left( D_H U_{\varepsilon, n}(P_n x + (I - P_n) y) - \int_{\mathbb{R}^n} D_H U_\varepsilon((P_n x + (I - P_n) y + \eta_n(\Sigma_n \xi)) \theta(\xi) d\xi) d\nu_{\varepsilon}\right)^2 d\nu_{\varepsilon}$$

$$\leq \|D_H U_\varepsilon\|^2_{L^{p'}(X, \mu; H)} \int_{\mathbb{R}^n} |\xi|^2 \theta(\xi) d\xi \to 0, \hspace{0.5cm} n \to +\infty.$$  \hspace{1cm}

Let us consider the convergence of the second order derivatives. We claim that for any $\varepsilon \in (0, 1]$ and any $n \in \mathbb{N}$, $D_H U_{\varepsilon, n, \eta_n} \to D_H U_{\varepsilon, n}$ in $L^2(X, \nu_\varepsilon; H_2(H))$ as $\eta \to 0^+$. If the claim is true, a diagonal argument as in the proof of [4, Lemma 2.4] allows us to conclude. It remains to prove the claim. We have

$$\|D_H^2 U_{\varepsilon, n, \eta_n} - D_H^2 U_{\varepsilon, n}\|^2_{L^2(X, \nu_\varepsilon; H_2(H))}$$

$$\leq \int_X \left( \int_{\mathbb{R}^n} \left| D_H^2 U_{\varepsilon, n}(P_n x + (I - P_n) y) - D_H^2 U_{\varepsilon, n}(P_n x + (I - P_n) y - \eta_n(\Sigma_n \xi)) \right|^2 d\mu \right) \theta(\xi) d\xi \right) d\nu_{\varepsilon}.$$  \hspace{1cm} (2.35)

From Hypothesis 2.6 and Proposition 2.27(v) it follows that $D_H^2 U_{\varepsilon, n} \in H$-continuous. This implies that the integrand converges to 0 as $\eta \to 0^+$. Further, Proposition 2.27(ii) gives that $D_H^2 U_{\varepsilon, n} \in \mu$-essentially bounded in $X$. The dominated convergence theorem implies that the right-hand side in (2.35) vanishes as $\eta \to 0$, and the claim is so proved.

By means of the family of measures $\nu_\varepsilon$, $\varepsilon \in (0, 1]$, we introduce a family of operators $L_\varepsilon$. We set

$$D(L_\varepsilon) := \left\{ u \in W^{1, 2}(X, \nu_\varepsilon) : \exists \eta \in L^2(X, \nu_\varepsilon), \int_X [D_H u, D_H v] d\nu_\varepsilon = -\int_X g v d\nu_\varepsilon, \forall v \in \mathcal{C}_b^\infty(X) \right\},$$

$$L_\varepsilon u := g,$$  \hspace{1cm} (2.36)

where $W^{1, 2}(X, \nu_\varepsilon)$ is the domain of the closure of $D_H : \mathcal{C}_b^\infty(X) \to L^2(X, \nu_\varepsilon; H)$ in $L^2(X, \nu_\varepsilon)$. We denote by $(T_\varepsilon)_{\varepsilon \geq 0}$ the analytic symmetric strongly continuous semigroup of contractions generated by $L_\varepsilon$ on $L^2(X, \nu_\varepsilon)$. Arguing as in Proposition 2.18 we deduce the following result.
Proposition 2.30. We have $\mathcal{F}\mathcal{C}_{b,\Phi}^2(X) \subset D(L_2^2)$ and for any $u \in \mathcal{F}\mathcal{C}_{b,\Phi}^2(X)$ it follows that

$$(L_2^2 u)(x) = \text{Tr}[D_{H}^2 u(x) |_{H} - \langle x, Du(x) \rangle - [D_H U_{\varepsilon}(x), D_H u(x)]_H], \quad x \in X.$$  

(2.37)

Further, if $u \in \mathcal{F}\mathcal{C}_{b,\Phi}^2(X)$ then $L_2 u \in \mathcal{F}\mathcal{C}^1(X)$ and for any $h \in H$ we have

$$[(D_H L_2^2 u)(x), h]_H = (L_2^2 [D_H u(\cdot), h]_H)(x) - [D_H U_{\varepsilon}(x), h]_H - [D_H^2 U_{\varepsilon}(x) h, D_H u(x)]_H.$$  

(2.38)

Remark 2.31. Let $\varepsilon \in (0, 1]$. If we denote by $(T_{\varepsilon}(t))_{t \geq 0}$ the analytic $C_0$-semigroup generates from $L_2^2$ in $L^2(X, \nu_{c})$, then Proposition 2.21 can be extended to $(T_{\varepsilon}(t))_{t \geq 0}$.

Let $f \in \mathcal{F}\mathcal{C}_{b,\Phi}^\infty(X)$ and let $\lambda > 0$. From [9, for any $\varepsilon \in (0, 1]$ there exists a sequence $u_{\varepsilon,n} \in \mathcal{F}\mathcal{C}_{b,\Phi}^3(X)$ such that

$$\lambda u_{\varepsilon,n} - L_2^2 u_{\varepsilon,n} = f + [D_H U_{\varepsilon}, D_H U_{\varepsilon} - D_H U_{\varepsilon,n}]_H =: f_{\varepsilon,n}.  

(2.39)

For any $\lambda > 0$ and any $\varepsilon \in (0, 1]$, we denote by $R(\lambda, L_2^2)$ and by $R(\lambda, L_2)$ the resolvent of $L_2^2$ and of $L_2$, respectively.

Proposition 2.32. Let $\varepsilon \in (0, 1]$. For any $f \in L^2(X, \nu_{c})$ there exists a sequence $(f_{\varepsilon,n}) \subset W^{1,2}(X, \nu_{c})$ such that $f_{\varepsilon,n} \to f$ in $L^2(X, \nu_{c})$, the sequence $(R(\lambda, L_2^2) f_{\varepsilon,n}) \subset \mathcal{F}\mathcal{C}_{b,\Phi}^3(X)$ converges to $R(\lambda, L_2)f$ in $W^{2,2}(X, \nu_{c})$ as $n \to +\infty$, and

$$\|R(\lambda, L_2^2)f\|_{W^{2,2}(X, \nu_{c})} \leq \max\{\sqrt{2}, \lambda^{-1}, \lambda^{-1/2}\} \|f\|_{L^2(X, \nu_{c})}.  

(2.40)

If $f \in W^{1,2}(X, \nu_{c})$, then $D_H f_{\varepsilon,n} \to D_H f$ in $L^1(X, \nu_{c}; H)$ as $n \to +\infty$. Further, if $f \in \mathcal{F}\mathcal{C}_{b,\Phi}^\infty(X)$ the sequence $(f_{\varepsilon,n})$ is the sequence defined in (2.39).

Proof. The proof is contained in the proof of [9, Propositions 5.6 & 5.10]. The unique point which we have to show is that $D_H f_{\varepsilon,n} \to D_H f$ in $L^1(X, \nu_{c}; H)$ as $n \to +\infty$. As usual, by density it is enough to prove the statement for $f \in \mathcal{F}\mathcal{C}_{b,\Phi}^\infty(X)$. In this case, $f_{\varepsilon,n}$ is the function defined in (2.39). By differentiating (2.39) along $H$ in the direction of $e_i$, $i = 1, \ldots, n$, by multiplying by $[D_H u_{\varepsilon,n}, e_i]_H$ and by summing up i from 1 to n, taking (2.38) into account we get

$$(\lambda + 1)\|D_H u_{\varepsilon,n}\|^2_H - \sum_{i=1}^{n} (L_2^2[D_H u_{\varepsilon,n}(\cdot), e_i]_H)[D_H u_{\varepsilon,n}, e_i]_H + [D_H^2 U_{\varepsilon} D_H u_{\varepsilon,n}, D_H u_{\varepsilon,n}]_H  

= [D_H f, D_H u_{\varepsilon,n}]_H + [D_H^2 u_{\varepsilon,n} D_H U_{\varepsilon} - D_H U_{\varepsilon,n}]_H + [D_H u_{\varepsilon,n}, (D_H^2 U_{\varepsilon} - D_H^2 U_{\varepsilon,n}) D_H u_{\varepsilon,n}]_H.$$  

We recall that, from [9, Proposition 4.4], there exists a positive constant $K$, independent of $n$, such that

$$\|D_H u_{\varepsilon,n}\| \leq K\|D_H f\|, \quad n \in N.$$  

(2.41)

The convexity of $U_{\varepsilon}$, the definition of $L_2^2$ and (2.41) imply that

$$\int_X |D_H^2 u_{\varepsilon,n}|_{H^2(X)}^2 d\nu_{c} \leq C \left(\|D_H f\|^2 + \alpha \int_X |D_H^2 u_{\varepsilon,n}|_{H^2(X)}^2 d\nu_{c} + \frac{1}{\sigma} \int_X |D_H U_{\varepsilon} - D_H U_{\varepsilon,n}|^2_H d\nu_{c} + \int_X |D_H^2 U_{\varepsilon} - D_H^2 U_{\varepsilon,n}|_{H^2(H)}^2 d\nu_{c} \right),$$

for some positive constant $C$ independent on $n$, for any $\sigma > 0$. By choosing $\sigma = (2C)^{-1}$ and taking into account (2.33) and (2.34), it follows that there exists a positive constant $M$, independent of $n$, such that

$$\|D_H^2 u_{\varepsilon,n}\|_{L^2(X, \nu_{c}, H^2(H))} \leq M, \quad n \in N.$$  

(2.42)
We are ready to prove that $D_H f_{\varepsilon,n} \to D_H f$ in $L^1(X,\nu_\varepsilon;H)$ as $n \to +\infty$. Indeed, we have

$$
\|D_H f_{\varepsilon,n} - D_H f\|_{L^1(X,\nu_\varepsilon;H)} \leq \int_X \|D_H^2 u_{\varepsilon,n} D_H u_{\varepsilon,n} - D_H U_{\varepsilon,n}\|_H |dv_\varepsilon
+ \int_X \|D_H u_{\varepsilon,n} (D_H^2 U_{\varepsilon,n} - D_H^2 U_{\varepsilon,n}) D_H u_{\varepsilon,n}\|_H |dv_\varepsilon
\to 0, \quad n \to +\infty,
$$

from (2.33), (2.34), (2.41) and (2.42).

**Proposition 2.33.**

(i) Let $\varepsilon \in (0,1]$, let $t > 0$, let $f \in L^2(X,\nu_\varepsilon)$ and let $(f_{\varepsilon,n})$ be as in Proposition 2.32. The sequence $(T_\varepsilon(t) f_{\varepsilon,n}) \subset \mathcal{F} C^{\frac{3}{2}}_{c,0}(X)$ converges to $T_\varepsilon(t)f$ in $W^{2,2}(X,\nu_\varepsilon)$ as $n \to +\infty$.

(ii) For any $f \in C_b(X)$ and any $t > 0$, we have $T_\varepsilon(t)f \to T(t)f$ as $\varepsilon \to 0^+$ weakly in $W^{2,2}(X,\nu)$.

**Proof.** (i). The proof is identical to that of [4, Proposition 2.8(i)].

(ii). Let $f \in C_b(X)$, and let $\varepsilon \in (0,1]$. Since $U(x) \geq U_\varepsilon(x)$ for any $x \in X$, from (2.40) it follows that the family $\{R(\lambda, L^2_\varepsilon) : \varepsilon \in (0,1]\}$ is bounded in $W^{2,2}(X,\nu)$. Hence, there exists a vanishing sequence $(\varepsilon_n) \subset (0,1]$ such that $R(\lambda, L_{\varepsilon_n})f$ weakly converges to $g \in W^{2,2}(X,\nu)$ in $W^{2,2}(X,\nu)$. Same arguments as in the proof of [9, Theorem 1.2] imply that $g = R(\lambda, L_2)f$. Since any vanishing sequence $(\varepsilon_n) \subset (0,1]$ admits a subsequence $(\varepsilon_{k_n})$ such that $R(\lambda, L^2_{\varepsilon_{k_n}})f \to R(\lambda, L_2)f$ in $W^{2,2}(X,\nu)$ as $n \to +\infty$, it follows that $R(\lambda, L^2_\varepsilon)f \to R(\lambda, L_2)f$ weakly in $W^{2,2}(X,\nu)$ as $\varepsilon \to 0^+$. Since $(T_\varepsilon(t))_{t \geq 0}$ is an analytic semigroup for any $\varepsilon \in (0,1]$, it follows that

$$
T_\varepsilon(t)f = \frac{1}{2\pi i} \int_{\sigma} e^{\lambda t} R(\lambda, L^2_\varepsilon)f d\lambda, \quad t > 0, \quad T(t)f = \frac{1}{2\pi i} \int_{\sigma} e^{\lambda t} R(\lambda, L_2)f d\lambda, \quad t > 0,
$$

where $\sigma$ is an unbounded curve in $\mathbb{C}$ which leaves on the left a sector containing the spectrum of $L^2_\varepsilon$. We remark that it is possible to choose $\sigma$ independent of $\varepsilon$. Therefore, for any $g \in C_b(X)$ we have

$$
\int_X T_\varepsilon(t)f g d\nu = \frac{1}{2\pi i} \int_X \left( \int_{\sigma} e^{\lambda t} R(\lambda, L^2_\varepsilon)f d\lambda \right) g d\nu_\varepsilon = \frac{1}{2\pi i} \int_{\sigma} e^{\lambda t} \left( \int_X R(\lambda, L^2_\varepsilon)f g d\nu \right) d\lambda.
$$

By the dominated convergence theorem, letting $\varepsilon \to 0^+$ we get

$$
\lim_{\varepsilon \to 0^+} \int_X T_\varepsilon(t)f g d\nu = \int_X e^{\lambda t} \left( \int_X R(\lambda, L_2)f g d\nu \right) d\lambda = \int_X T(t)f g d\nu.
$$

This proves that $T_\varepsilon(t)f \to T(t)f$ as $\varepsilon \to 0^+$ weakly in $L^2(X,\nu)$. The proof of the convergence of $D_H T_\varepsilon(t)f$ and of $D_H^2 T_\varepsilon(t)f$ is analogous.

**3  Analysis of $(T(t))_{t \geq 0}$ and of $(T^V(t))_{t \geq 0}$**

**3.1  Asymptotic behaviour of $(T(t))_{t \geq 0}$**

In this subsection we show that for any $p \in [1, +\infty)$ the semigroup $(T(t))_{t \geq 0}$ is ergodic in $L^p(X,\nu)$, i.e., for any $f \in L^p(X,\nu)$ we have $T(t)f \to \nu(f)$ in $L^p(X,\nu)$ as $t \to +\infty$, where

$$
\nu(f) := \int_X f d\nu, \quad f \in L^p(X,\nu).
$$

To prove this fact we use the following intermediate result, whose proof is inspired by that in [10, Corollary 3.6 & Proposition 3.7] in finite dimension.
Lemma 3.1. Let \( f \in L^2(X, \nu) \). Then,
\[
\lim_{t \to +\infty} \| D_H T(t) f \|_{L^2(X, \nu; H)}^2 = 0. \tag{3.2}
\]

Proof. We claim that for any \( f \in D(L_2) \), the function \( t \mapsto \chi_f(t) := \| D_H T(t) f \|_{L^2(X, \nu; H)}^2 \in L^1(0, +\infty) \). To prove the claim, let us fix \( f \in D(L_2) \) and \( t > 0 \). We have
\[
\frac{d}{dt} \int_X |T(t)f|^2 d\nu = 2 \int_X (T(t)f)(L_2 T(t)f) d\nu = -2 \int_X |T(t)f|^2_H d\nu.
\]
Integrating between 0 and \( t \) we get
\[
\| T(t)f \|_{L^2(X, \nu)}^2 - \| f \|_{L^2(X, \nu)}^2 = -2 \int_0^t \| D_H T(s)f \|_{L^2(X, \nu; H)}^2 ds, \quad t \geq 0.
\]
This implies that
\[
\| T(t)f \|_{L^2(X, \nu)}^2 + 2 \int_0^t \| D_H T(s)f \|_{L^2(X, \nu; H)}^2 ds \leq \| f \|_{L^2(X, \nu)}^2, \quad t \geq 0,
\]
and the claim is so proved. Let us consider \( f \in D(L_2^2) \), where \( L_2^2 \) is the square power of the operator \( L_2 \). This means that both \( \chi_f \) and \( \chi_{L_f} \) belong to \( L^1(0, +\infty) \). Since
\[
\frac{d}{dt} \chi_f(t) = 2 \int_0^t |D_H T(s)L_2 f, D_H T(t)f|_{H^1} d\nu \leq \chi_f(t) + \chi_{L_f}(t),
\]

it follows that both \( \chi_f \) and \( \chi'_f \) belong to \( L^1(0, +\infty) \). This implies that \( \chi_f \in W^{1,1}(0, +\infty) \), and therefore
\[
\lim_{t \to +\infty} \chi_f(t) = 0, \quad f \in D(L_2^2).
\]
Since \( (T(t))_{t \geq 0} \) is an analytic semigroup in \( L^2(X, \nu) \), it follows that \( T(1)f \in D(L_2^2) \) for any \( n \in \mathbb{N} \). Then,
\[
\lim_{t \to +\infty} \| D_H T(t)f \|_{L^2(X, \nu; H)}^2 = \lim_{t \to +\infty} \| D_H T(t-1)T(1)f \|_{L^2(X, \nu; H)}^2 = 0, \quad f \in L^2(X, \nu).
\]

\[\square\]

Proposition 3.2. For any \( p \in [1, +\infty) \) and any \( f \in L^p(X, \nu) \) we have
\[
\lim_{t \to +\infty} \| T(t)f - \nu(f) \|_{L^p(X, \nu)} = 0, \tag{3.3}
\]
where \( \nu(f) \) has been defined in (3.1).

Proof. Let us split the proof into three steps. In the former we show that for any \( f \in C_b(X) \) the function \( T(t)f \) weakly converges to \( \nu(f) \) in \( L^2(X, \nu) \) as \( t \to +\infty \), in the second we prove (3.3) for \( f \in C_b(X) \), in the latter we conclude.

**STEP 1**. Let \( f \in C_b(X) \). Since \( (T(t))_{t \geq 0} \) is a semigroup of contractions in \( L^2(X, \nu) \), it follows that there exists a sequence \( (t_n) \) diverging to \( +\infty \) and a function \( g \in L^2(X, \nu) \) such that \( T(t_n)f \to g \) weakly in \( L^2(X, \nu) \) as \( n \to +\infty \). Further, \( g \) is bounded and \( \| g \|_\infty \leq \| f \|_\infty \). Indeed, for any positive, bounded and continuous function \( \nu \), from Proposition 2.21(ii) we have
\[
\left| \int_X T(t_n)f d\nu \right| \leq \| f \|_\infty \int_X \nu d\nu.
\]
Letting $n \to +\infty$ we get
\[
\left| \int_X g d\nu \right| \leq \|f\|_{\infty} \int_X v d\nu.
\]
The arbitrariness of $v$ implies that $-\|f\|_{\infty} \leq g \leq \|f\|_{\infty}$ for $\nu$-a.e. in $X$.

Let us consider $u \in D(D_H^*)$. We have
\[
\int_X g D_H^* u d\nu = \lim_{n \to +\infty} \int_X T(t_n) f D_H^* u d\nu = \lim_{n \to +\infty} \int_X [D_H T(t_n) f, u]_H d\nu = 0,
\]
where the last equality follows from Lemma 3.1. Therefore, $g \in D(D_H^*)^* = W^{1,2}(X,\nu)$ and $D_H g = 0$.

From Proposition 2.10 it follows that $g$ is constant $\nu$-a.e. in $X$. Finally, we have
\[
g = \int_X g d\nu = \lim_{n \to +\infty} \int_X T(t_n) f d\nu = \lim_{n \to +\infty} \int_X f d\nu = \nu(f),
\]
where the third equality follows from the fact that $\nu$ is an invariant measure for $(T(t))_{t \geq 0}$. In particular, the above arguments show that for any sequence $(t_n) \subset (0, +\infty)$ diverging to $+\infty$ as $n \to +\infty$ there exists a subsequence $(t_{k_n}) \subset (t_n)$ such that $(T(t_{k_n})) f \to \nu(f)$ weakly in $L^2(X,\nu)$ as $n \to +\infty$. Hence, $(T(t)) f \to \nu(f)$ weakly in $L^2(X, \nu)$ as $t \to +\infty$.

**STEP 2.** From Step 1 we know that $(T(t)) f \to \nu(f)$ weakly in $L^2(X,\nu)$ for any $f \in C_b(X)$. Then,
\[
\|T(t) f\|_{L^2(X,\nu)}^2 = \int_X (T(t) f)(T(t) f) d\nu = \int_X (T(2t) f)(f) d\nu \to \int_X \nu(f) f d\nu = \|\nu(f)\|_{L^2(X,\nu)}^2,
\]
as $t \to +\infty$. Here, we have used the symmetry of $(T(t))_{t \geq 0}$ with respect to $\nu$ and the semigroup property of $(T(t))_{t \geq 0}$. (3.4) implies that $(T(t)) f \to \nu(f)$ in $L^2(X,\nu)$ as $t \to +\infty$. Therefore, for any sequence $(t_n) \subset (0, +\infty)$ diverging to $+\infty$ as $n \to +\infty$ there exists a subsequence $(t_{k_n}) \subset (t_n)$ such that $(T(t_{k_n})) f(x) \to \nu(f)$ for $\nu$-a.e. $x \in X$. By the dominated convergence theorem it follows that $(T(t_{k_n})) f \to \nu(f)$ in $L^p(X,\nu)$ as $n \to +\infty$, for any $p \in [1, +\infty)$. This means that $(T(t)) f \to \nu(f)$ in $L^p(X,\nu)$ as $t \to +\infty$ for any $p \in [1, +\infty)$.

**STEP 3.** Let $f \in L^p(X,\nu)$ and let $(f_n) \subset C_b(X)$ be such that $f_n \to f$ in $L^p(X,\nu)$ as $n \to +\infty$. From Hölder’s inequality we get
\[
\|\nu(f_n) - \nu(f)\|_{L^p(X,\nu)} \leq \|f_n - f\|_{L^p(X,\nu)}, \quad n \in \mathbb{N}.
\]
Hence,
\[
\|T(t) f - \nu(f)\|_{L^p(X,\nu)} \leq \|T(t) f - T(t) f_n\|_{L^p(X,\nu)} + \|T(t) f_n - \nu(f_n)\|_{L^p(X,\nu)}
\leq 2\|f - f_n\|_{L^p(X,\nu)} + \|T(t) f_n - \nu(f_n)\|_{L^p(X,\nu)},
\]
where we have used the fact that $(T(t))_{t \geq 0}$ is a semigroup of contractions in $L^p(X,\nu)$. Let us fix $\varepsilon > 0$. There exists $\overline{\varepsilon} \in \mathbb{N}$ such that $\|f - f_n\|_{L^p(X,\nu)} \leq \varepsilon/4$ for any $n \geq \overline{\varepsilon}$. Further, from Step 2 there exists $\overline{t} > 0$ such that $\|T(t) f - \nu(f)\|_{L^p(X,\nu)} \leq \varepsilon/2$ for any $t > \overline{t}$. This means that, by choosing $n = \overline{\varepsilon}$ in (3.5), for any $\varepsilon > 0$ there exists $\overline{t} > 0$ such that $\|T(t) f - \nu(f)\|_{L^p(X,\nu)} \leq \varepsilon$ for any $t > \overline{t}$. This gives the thesis.

### 3.2 Pointwise gradient estimates to $(T(t))_{t \geq 0}$

We show two different estimates of the $D_H T(t)f$. In Proposition 3.3 we estimate $|D_H T(t)f|^p_H$ by means of $T(t)|D_H f|^p_H$, in Proposition 3.4 we estimate $|D_H T(t)f|^p_H$ by means of $(T(t)|f|^p_H$.
Proposition 3.3. For any $p \in [1, +\infty)$ and any $f \in W^{1,p}(X, \nu)$ we have
\[ |D_H T(t)f(x)|_H^p \leq e^{-pt}(T(t))|D_H f|_H^p(x), \quad t \geq 0, \ \nu\text{-a.e. } x \in X. \]  
(3.6)

Proof. We split the proof into three steps. In the former we prove that for any $\varepsilon \in (0, 1]$ and any $f \in \mathcal{F}_{E_{0}\phi}(X)$ there exists a $\nu_{\varepsilon}$-measurable set $N_{\varepsilon} \subset X$ such that $\nu_{\varepsilon}(N_{\varepsilon}) = 0$ and $|D_H T_{\varepsilon}(t)f(x)|_H \leq e^{-tT_{\varepsilon}(t))}|D_H f|_H(x)$ for any $x \in X \setminus N_{\varepsilon}$, in the second one we prove the statement for any $f \in \mathcal{F}_{E_{0}\phi}(X)$ and any $p \in [1, +\infty)$, in the latter we conclude.

**STEP 1.** Let $\varepsilon \in (0, 1]$, let $t > 0$, let $f \in \mathcal{F}_{E_{0}\phi}(X)$, and let $g$ be a positive, bounded and continuous function on $X$. For any $\sigma > 0$ we set $\eta_{\sigma} : [0, +\infty) \to [0, +\infty)$ defined by $\eta_{\sigma}(\xi) := \sqrt{\xi + \sigma - \sqrt{\sigma}}$, for any $\xi \in [0, +\infty)$. We notice that $\eta_{\sigma}$ satisfies the following:
\[ (i) \eta_{\sigma}(\xi) \leq \sqrt{\xi}, \quad (ii) \xi \eta'_{\sigma}(\xi) \geq \frac{1}{2} \eta_{\sigma}(\xi), \quad (iii) \eta''_{\sigma}(\xi) + 2\xi \eta''_{\sigma}(\xi) \geq 0, \quad \xi \in [0, +\infty). \]  
(3.7)

To lighten the notations we set $w_{\varepsilon}^n(s) := |D_H T_{\varepsilon}(s)f_{\varepsilon,n}|_H^2$, for any $n \in \mathbb{N}$ and any $s \geq 0$, where the sequence $(f_{\varepsilon,n})$ is as in Proposition 2.33. We introduce the function
\[ G(s) := \int_X \eta_{\sigma}(w_{\varepsilon}^n(t-s))T_{\varepsilon}(s)gd\nu_{\varepsilon}, \quad t \in [0, +\infty), s \in [0, t], \ n \in \mathbb{N}. \]

The smoothness of $T_{\varepsilon}(t)f_{\varepsilon,n}$ (see Proposition 2.33(i)) implies that
\[ \frac{d}{ds} \eta_{\sigma}(w_{\varepsilon}^n(t-s)) = \frac{d}{ds} |D_H T_{\varepsilon}(t-s)f_{\varepsilon,n}, D_H T_{\varepsilon}(t-s)f_{\varepsilon,n}|_H \]
\[ = -2 \eta'_{\sigma}(w_{\varepsilon}^n(t-s))|D_H L_2 T_{\varepsilon}(t-s)f_{\varepsilon,n}, D_H T_{\varepsilon}(t-s)f_{\varepsilon,n}|_H. \]

Then, we get
\[ G'(s) = -2 \int_X \eta'_{\sigma}(w_{\varepsilon}^n(t-s))|D_H L_2 T_{\varepsilon}(t-s)f_{\varepsilon,n}, D_H T_{\varepsilon}(t-s)f_{\varepsilon,n}|_H T_{\varepsilon}(s)gd\nu_{\varepsilon} \]
\[ + \int_X \eta_{\sigma}(w_{\varepsilon}^n(t-s))L_2 T_{\varepsilon}(s)gd\nu_{\varepsilon}. \]  
(3.8)

Let us take into account the second addend in the right-hand side (3.8). From the definition of $L_2$ we get
\[ \int_X \eta_{\sigma}(w_{\varepsilon}^n(t-s))L_2 T_{\varepsilon}(s)gd\nu_{\varepsilon} = - \int_X \eta'_{\sigma}(w_{\varepsilon}^n(t-s))[(D_H w_{\varepsilon}^n(t-s), D_H T_{\varepsilon}(s))_H]gd\nu_{\varepsilon} \]
\[ = - \int_X [D_H w_{\varepsilon}^n(t-s), D_H (\eta'_{\sigma}(w_{\varepsilon}^n(t-s))T_{\varepsilon}(s))_H]gd\nu_{\varepsilon} \]
\[ + \int_X \eta''_{\sigma}(w_{\varepsilon}^n(t-s))|D_H w_{\varepsilon}^n(t-s)|^2 T_{\varepsilon}(s)gd\nu_{\varepsilon} \]
\[ = \int_X \eta''_{\sigma}(w_{\varepsilon}^n(t-s))T_{\varepsilon}(s)gL_2 w_{\varepsilon}^n(t-s)gd\nu_{\varepsilon} \]
\[ + \int_X \eta''_{\sigma}(w_{\varepsilon}^n(t-s))|D_H w_{\varepsilon}^n(t-s)|^2 T_{\varepsilon}(s)gd\nu_{\varepsilon}. \]  
(3.9)

It follows that
\[ G'(s) = 2 \int_X \eta''_{\sigma}(w_{\varepsilon}^n(t-s))T_{\varepsilon}(s)g \left( \frac{1}{2} L_2 w_{\varepsilon}^n(t-s) - [D_H L_2 T_{\varepsilon}(t-s)f_{\varepsilon,n}, D_H T_{\varepsilon}(t-s)f_{\varepsilon,n}]_H \right) \]
\[ + \int_X \eta''_{\sigma}(w_{\varepsilon}^n(t-s))|D_H w_{\varepsilon}^n(t-s)|^2 T_{\varepsilon}(s)gd\nu_{\varepsilon}. \]  
(3.10)
Long but straightforward computations reveal that for any \( x \in X \) we have

\[
\frac{1}{2} L^2 w_n^\varepsilon(t-s)(x) = \text{Tr}[(D_H^2 T_\varepsilon(t-s) f_{\varepsilon,n})^2(x)]_H + L^2 [D_H T_\varepsilon(t-s) f_{\varepsilon,n}(-), T_\varepsilon(t-s) f_{\varepsilon,n}(x)]_H(x),
\]

where in the second addend of the right-hand side of (3.11), for any \( x \in X \), the term \( T_\varepsilon(t-s) f_{\varepsilon,n}(x) \) is seen as a fixed element of \( H \). By combining (2.38) and (3.11) we get

\[
\frac{1}{2} L^2 w_n^\varepsilon(t-s) - [D_H L^2 T_\varepsilon(t-s) f_{\varepsilon,n}, D_H T_\varepsilon(t-s) f_{\varepsilon,n}]_H \\
= \text{Tr}[(D_H^2 T_\varepsilon(t-s) f_{\varepsilon,n})^2]_H + [D_H T_\varepsilon(t-s) f_{\varepsilon,n}]_H^2 + [D_H^2 U_\varepsilon D_H T_\varepsilon(t-s) f_{\varepsilon,n}, D_H T_\varepsilon(t-s) f_{\varepsilon,n}]_H \\
\geq \|D_H^2 T_\varepsilon(t-s) f_{\varepsilon,n}\|_{\mathcal{H}_2(H)}^2 + [D_H T_\varepsilon(t-s) f_{\varepsilon,n}]_H^2,
\]

where in the last inequality we have used (1.3), the symmetry of \( D_H^2 T_\varepsilon(t-s) f_{\varepsilon,n} \) as operator from \( H \times H \) onto \( \mathbb{R} \) and the convexity of \( U_\varepsilon \). Further,

\[
[D_H w_n^\varepsilon(t-s)]_H^2 = 4[D_H^2 T_\varepsilon(t-s) f_{\varepsilon,n}, D_H T_\varepsilon(t-s) f_{\varepsilon,n}]_H \\
\leq 4\|D_H T_\varepsilon(t-s) f_{\varepsilon,n}\|_{\mathcal{H}_2(H)}^2 |D_H T_\varepsilon(t-s) f_{\varepsilon,n}|_H^2.
\]

By collecting (3.8)-(3.13) and recalling (3.7) we infer that

\[
G'(s) \geq 2 \int_X (\eta_{\alpha}(w_n^\varepsilon(t-s)) + 2 w_n^\varepsilon(t-s) \eta_{\alpha}'(w_n^\varepsilon(t-s))) T_\varepsilon(s) g \text{Tr}[(D_H^2 T_\varepsilon(t-s) f_{\varepsilon,n})^2]_H dv_\varepsilon \\
+ 2 \int_X \eta_{\alpha}'(w_n^\varepsilon(t-s)) w_n^\varepsilon(t-s) T_\varepsilon(s) g dv_\varepsilon \\
geq \int_X \eta_{\alpha}(w_n^\varepsilon(t-s)) T_\varepsilon(s) g dv_\varepsilon \\
= G(s).
\]

This gives

\[
G(s) \geq G(0)e^s, \quad s \in [0,t].
\]

In particular, if we choose \( s = t \) we get

\[
\int_X \left( \sqrt{\sigma + |D_H T_\varepsilon(t) f_{\varepsilon,n}|_H^2} - \sqrt{\sigma} \right) g dv_\varepsilon = G(0) \leq e^{-t} G(t) \\
= e^{-t} \int_X \left( \sqrt{\sigma + |D_H f_{\varepsilon,n}|^2} - \sqrt{\sigma} \right) T_\varepsilon(t) g dv_\varepsilon.
\]

By letting \( \sigma \to 0^+ \) and by applying the symmetry of \( T_\varepsilon(t) \) on \( L^2(X, \nu_\varepsilon) \) we infer that

\[
\int_X |D_H T_\varepsilon(t) f_{\varepsilon,n}|_H g dv_\varepsilon \leq e^{-t} \int_X T_\varepsilon(t) |D_H f_{\varepsilon,n}|_H g dv_\varepsilon.
\]

Letting \( n \to +\infty \) and recalling Proposition 2.33(i) we infer that

\[
\int_X |D_H T_\varepsilon(t) f|_H g dv_\varepsilon \leq e^{-t} \int_X T_\varepsilon(t) |D_H f|_H g dv_\varepsilon,
\]

for any \( \varepsilon \in (0,1] \). The arbitrariness of \( g \) implies that for any \( \varepsilon \in (0,1] \) there exists a \( \nu_\varepsilon \)-measurable set \( N_\varepsilon \subset X \) such that \( \nu_\varepsilon(N_\varepsilon) = 0 \) and \( |D_H T_\varepsilon(t) f(x)|_H \leq e^{-t} T_\varepsilon(t) |D_H f|_H(x) \) for any \( x \in X \setminus N_\varepsilon \).
STEP 2. Let us consider a decreasing and vanishing sequence \((\varepsilon_n) \subset (0, 1]\) and let us set \(N := \bigcup_{n \in \mathbb{N}} N_{\varepsilon_n}\), where \(N_{\varepsilon_n}\) has been defined in Step 1. Since \(\nu_\varepsilon\) and \(\nu\) are equivalent measures on \(X\) for any \(\varepsilon > 0\), we have \(\nu(N) = 0\) and
\[
|D_H T_{\varepsilon_n}(t) f(x)|_H \leq e^{-t} T_{\varepsilon_n}(t) |D_H f|_H(x), \quad \forall x \in X \setminus N, \; \forall n \in \mathbb{N}.
\] (3.14)
Let \(g\) be a positive, bounded and continuous function. By multiplying both the sides of (3.14) by \(g\) and integrating on \(X\) with respect to \(\nu\) we get
\[
\int_X |D_H T_{\varepsilon_n}(t) f|_H d\nu \leq e^{-t} \int_X T_{\varepsilon_n}(t) |D_H f|_H d\nu, \quad \forall n \in \mathbb{N}.
\] (3.15)
Let us consider the left-hand side of (3.15). Since \(g\) is positive, it follows that
\[
\int_X |D_H T_{\varepsilon_n}(t) f|_H d\nu = \int_X |gD_H T_{\varepsilon_n}(t) f|_H d\nu.
\] (3.16)
Let us set \(V_n := gD_H T_{\varepsilon_n}(t) f\). For any \(\Phi \in L^\infty(X, \nu; H)\), from Proposition 2.33(ii) we have
\[
\int_X [V_n, \Phi]_H d\nu = \int_X |D_H T_{\varepsilon_n}(t) f, g\Phi|_H d\nu \to \int_X |D_H T(t) f, g\Phi|_H d\nu = \int_X |gD_H T(t) f, \Phi|_H d\nu,
\] as \(n \to +\infty\). We recall that \(L^\infty(X, \nu; H) = (L^1(X, \nu; H))^\ast\) (see [12, Chapter IV, Theorem 1]). Therefore, \(V_n \to gD_H T(t) f\) weakly in \(L^1(X, \nu; H)\) as \(n \to +\infty\). This implies that
\[
\int_X |D_H T(t) f|_H d\nu = \int_X |gD_H T(t) f|_H d\nu \leq \lim inf_{n \to +\infty} \int_X |V_n|_H d\nu = \lim inf_{n \to +\infty} \int_X |D_H T_{\varepsilon_n}(t) f|_H g d\nu.
\] (3.17)
(3.15), (3.16) and (3.17) give
\[
\int_X |D_H T(t) f|_H d\nu \leq e^{-t} \lim inf_{n \to +\infty} \int_X T_{\varepsilon_n}(t) |D_H f|_H g d\nu.
\] (3.18)
From Proposition 2.33(ii) we infer that
\[
\lim_{n \to +\infty} \int_X T_{\varepsilon_n}(t) |D_H f|_H g d\nu = \int_X T(t) |D_H f|_H g d\nu.
\]
This formula and (3.18) give
\[
\int_X |D_H T(t) f|_H d\nu \leq e^{-t} \int_X T(t) |D_H f|_H g d\nu.
\]
The arbitrariness of \(g\) implies that
\[
|D_H T(t) f(x)|_H \leq e^{-t} (T(t) |D_H f|_H(x), \quad t \geq 0, \; \nu \text{ a.e. } x \in X,
\] (3.19)
which gives the thesis for \(f \in \mathcal{F} \Phi_{b, \Phi}^\infty(X)\) and \(p = 1\).
Let \(p > 1\). From (2.28) (with \(q = p\) and \(p = 1\)) and (3.19) we infer that
\[
|D_H T(t) f(x)|^p_H \leq e^{-pt} (T(t) |D_H f|_H(x))^p \leq e^{-pt} (T(t) |D_H f|_H^p(x), \quad t \geq 0, \; \nu\text{-a.e. } x \in X.
\]
STEP 3. The general case follows by approximation. Let \(p \in [1, +\infty)\), let \(f \in W^{1,p}(X, \nu)\) and let \((g_n) \subset \mathcal{F} \Phi_{b, \Phi}^\infty(X)\) be such that \(g_n \to f\) in \(W^{1,p}(X, \nu)\). We get
\[
\int_X |D_H T(t) (g_n - g_m)|^p_H d\nu \leq \int_X T(t) |D_H (g_n - g_m)|^p_H d\nu = \int_X |D_H g_n - D_H g_m|^p_H d\nu,
\]
29
for any \( n, m \in \mathbb{N} \), where in the last equality we have used the fact that \( \nu \) is an invariant measure for \( (T(t))_{t \geq 0} \). This implies that \( (D_H T(t)g_n) \) is a Cauchy sequence in \( L^p(X, \nu; H) \) for any \( t \geq 0 \), and we notice that \( T(t)g_n \to T(t)f \) in \( L^p(X, \nu) \) as \( n \to +\infty \). The fact that \( (D_H, W^{1,p}(X, \nu)) \) is a closed operator in \( L^p(X, \nu) \) implies that \( T(t)f \in W^{1,p}(X, \nu) \) and

\[
D_H T(t) f = L^p - \lim_{n \to +\infty} D_H T(t)g_n, \quad t \geq 0.
\]

Let us fix \( t \geq 0 \). From Step 2 we know that for any \( n \in \mathbb{N} \) there exists \( N_n \in B(X) \), with \( \nu(N_n) = 0 \), such that

\[
|D_H T(t) g_n(x)|_H^p \leq e^{-pt}(T(t)|D_H g_n|_H^p)(x), \quad x \in X \setminus N_n.
\]

We set \( N := \cup_n N_n \). Then, \( \nu(N) = 0 \) and

\[
|D_H T(t) g_n(x)|_H^p \leq e^{-pt}(T(t)|D_H g_n|_H^p)(x), \quad x \in X \setminus N, \quad n \in \mathbb{N}.
\] (3.20)

Finally, we recall that \( |D_H g_n|_H \to |D_H f|_H \) in \( L^1(X, \nu) \) as \( n \to +\infty \), and therefore for any \( t \geq 0 \) we have \( T(t)|D_H g_n|_H \to T(t)|D_H f|_H \) in \( L^1(X, \nu) \) as \( n \to +\infty \). We consider a subsequence \( (g_{n_k}) \) such that \( T(t)|D_H g_{n_k}|_H \to T(t)|D_H f|_H \) and \( D_H T(t)g_{n_k} \to D_H T(t)f \) pointwise \( \nu \)-a.e. in \( X \). Letting \( n \to +\infty \) in (3.20) with \( g_n \) replaced by \( g_{n_k} \) we get

\[
|D_H T(t) f|_H^p \leq e^{-pt}(T(t)|D_H f|_H^p), \quad \nu \text{-a.e. in } X.
\]

\( \square \)

**Proposition 3.4.** Let \( p \in (1, +\infty) \). Then, there exists a positive constant \( c_p \) which only depends on \( p \) and which equals \( 2^{-p/2} \) for \( 2, +\infty \), such that for any \( f \in L^p(X, \nu) \) we have

\[
|D_H T(t) f(x)|_H^p \leq c_p t^{-p/2}(T(t)|f|_H^p)(x), \quad t > 0, \quad \nu \text{-a.e. } x \in X.
\] (3.21)

**Proof.** For reader’s convenience we split the proof into three steps. In the former we prove that for any \( p \in (1, 2] \) and any \( f \in \mathcal{F}\mathcal{C}_{b,q}^\infty(X) \) we have

\[
|D_H T_c(t) f(x)|_H^p \leq c_p t^{-p/2}(T_c(t)|f|_H^p)(x), \quad t > 0, \quad \epsilon > 0, \quad x \in X.
\] (3.22)

in the second we show (3.21) for any \( p \in (1, +\infty) \) and any \( f \in \mathcal{F}\mathcal{C}_{b,\Phi}^\infty(X) \), in the latter we conclude.

**STEP 1.** Let \( f \in \mathcal{F}\mathcal{C}_{b,\Phi}^\infty(X) \), let \( (f_{\epsilon,n}) \) be the approximating sequence of \( f \) defined in Proposition 2.32, let \( p \in (1, 2] \) and let us fix \( t > 0 \). For any \( \delta > 0 \) and any \( n \in \mathbb{N} \) we set

\[
G_{\delta,n}(s) := T_c(t - s) \left( (T_c(s)f_{\epsilon,n})^2 + \delta \right)^{p/2} - \delta^{p/2}, \quad 0 < s < t.
\]

\( G_{\delta,n} \) is differentiable in \((0, t)\), and differentiating it we get

\[
(G_{\delta,n}'(s)) = -L_2^2 T_c(t - s) \left( (T_c(s)f_{\epsilon,n})^2 + \delta \right)^{p/2} - \delta^{p/2}
\]

\[
+ T_c(t - s) \left[ p \left( (T_c(s)f_{\epsilon,n})^2 + \delta \right)^{p/2 - 1} (T_c(s)f_{\epsilon,n})(L_2^2 T_c(s)f_{\epsilon,n}) \right]
\]

\[
= T_c(t - s) \left[ -L_2^2 \left( (T_c(s)f_{\epsilon,n})^2 + \delta \right)^{p/2} - \delta^{p/2} \right] + p \left( (T_c(s)f_{\epsilon,n})^2 + \delta \right)^{p/2} (T_c(s)f_{\epsilon,n})(L_2^2 T_c(s)f_{\epsilon,n}) \right],
\] (3.23)

where we have used the fact that \( (T_c(s)f_{\epsilon,n})^2 + \delta \) \( \in \mathcal{F}\mathcal{C}_{b,\Phi}^\infty(X) \subset D(L_2^2) \). Further,

\[
L_2^2 \left( (T_c(s)f_{\epsilon,n})^2 + \delta \right)^{p/2} = p \left( (T_c(s)f_{\epsilon,n})^2 + \delta \right)^{p/2 - 1} (T_c(s)f_{\epsilon,n})(L_2^2 T_c(s)f_{\epsilon,n})
\]
Now we estimate $|DH T_s(s)|^p_H$. Let $s \in (0, t)$. From the semigroup property of $(T_s(t))_{t \geq 0}$, Proposition 2.33(i) and (3.6) we infer that

$$|DH T_s(t) f_{\varepsilon,n}|^p_H = |DH T_s(t-s) T_s(s) f_{\varepsilon,n}|^p \leq e^{-p(t-s)} T_s(t-s) (|DH T_s(s) f_{\varepsilon,n}|^p_H).$$

We multiply and divide the argument of $T_s(t-s)$ by $|T_s(s) f_{\varepsilon,n}|^2 + \delta)^{p/2}$. By applying (2.29) with $q = \frac{1}{p}$ and $q' = \frac{1}{2 - p}$, we infer that

$$T_s(t-s) (|DH T_s(s) f_{\varepsilon,n}|^p_H)$$

$$= T_s(t-s) (|T_s(s) f_{\varepsilon,n}|^2 + \delta) - \frac{2^2 - p}{2} |DH T_s(s) f_{\varepsilon,n}|^p_H (|T_s(s) f_{\varepsilon,n}|^2 + \delta)^{p/2}$$

$$\leq (T_s(t-s) (|T_s(s) f_{\varepsilon,n}|^2 + \delta)^{p/2-1} |DH T_s(s) f_{\varepsilon,n}|^2_H)$$

$$\cdot (T_s(s) f_{\varepsilon,n}|^2 + \delta)^{p/2}$$

$$\leq \frac{p}{2} |DH T_s(s) f_{\varepsilon,n}|^p_H (|T_s(s) f_{\varepsilon,n}|^2 + \delta)^{p/2}$$

$$+ \frac{2 - p}{2} \eta^{2/(p-2)} T_s(t-s) (|T_s(s) f_{\varepsilon,n}|^2 + \delta)^{p/2},$$

for any $\eta > 0$, where in the last inequality we have applied the Young’s inequality. Further, the positivity of $(T_s(t))_{t \geq 0}$ and the fact that $p \in (1, 2]$ give

$$T_s(t-s) (|T_s(s) f_{\varepsilon,n}|^2 + \delta)^{p/2} \leq T_s(t-s) (|T_s(s) f_{\varepsilon,n}|^p + \delta^{p/2}) \leq T_s(t-s) (T_s(s)|f_{\varepsilon,n}|^p + \delta^{p/2})$$

$$\leq T_s(t)|f_{\varepsilon,n}|^p + \delta^{p/2}.$$
Putting together (3.26), (3.27) and (3.28) we infer that
\[
e^{pt} |D_H T_\varepsilon(t) f_{\varepsilon,n}|_H^p \leq \frac{p}{2} \eta^{2/p} T_\varepsilon(t) s \left( (|T_\varepsilon(s) f_{\varepsilon,n}|^2 + \delta)^{p/2 - 1} |D_H T_\varepsilon(s) f_{\varepsilon,n}|_H^2 \right) + \frac{2 - p}{2} \eta^{2/(p-2)} T_\varepsilon(t) |f_{\varepsilon,n}|^p + \delta^{p/2}.
\]
Integrating with respect to \( s \) between 0 and \( t \) and recalling (3.25) we infer that
\[
\frac{e^{pt} - 1}{p} |D_H T_\varepsilon(t) f_{\varepsilon,n}|_H^p \leq \frac{\eta^{2/p}}{2(p-1)} T_\varepsilon(t) |f_{\varepsilon,n}|^p + \frac{2 - p}{2} \eta^{2/(p-2)} t T_\varepsilon(t) |f_{\varepsilon,n}|^p + \frac{2(p-1)}{2} \eta^{2/(p-2)} t T_\varepsilon(t) |f_{\varepsilon,n}|^p,
\]
for any \( \eta > 0 \). Therefore,
\[
\frac{e^{pt} - 1}{p} |D_H T_\varepsilon(t) f_{\varepsilon,n}|_H^p \leq \min_{\eta > 0} \left\{ \frac{\eta^{2/p}}{2(p-1)} + \frac{2 - p}{2} \eta^{2/(p-2)} t \right\} T_\varepsilon(t) |f_{\varepsilon,n}|^p = c_p t^{p - \frac{p}{2} + 1} T_\varepsilon(t) |f_{\varepsilon,n}|^p,
\]
(3.29)
for some positive constant \( c_p \) only depending on \( p \). By dividing both the sides of (3.29) by \((e^{pt} - 1)p^{-1}\)
we infer that
\[
|D_H T_\varepsilon(t) f_{\varepsilon,n}|_H^p \leq \frac{p t}{e^{pt} - 1} c_p t^{-p/2} T_\varepsilon(t) |f_{\varepsilon,n}|^p \leq \tilde{c}_p t^{-p/2} T_\varepsilon(t) |f_{\varepsilon,n}|^p;
\]
Since the function \( t \mapsto pt(e^{pt} - 1)^{-1} \) is bounded in \((0, +\infty)\). We notice that if \( p = 2 \) computations
simplify and we get
\[
\frac{e^{2t} - 1}{2} |D_H T_\varepsilon(t) f_{\varepsilon,n}|_H^2 \leq \frac{1}{2} T_\varepsilon(t) |f_{\varepsilon,n}|^2.
\]
Hence,
\[
|D_H T_\varepsilon(t) f_{\varepsilon,n}|_H^2 \leq \frac{2t}{2(e^{2t} - 1)} t^{-1} T_\varepsilon(t) |f_{\varepsilon,n}|^2,
\]
which gives \( c_2 = \frac{1}{2} \). In both the cases, we get
\[
|D_H T_\varepsilon(t) f_{\varepsilon,n}|_H^p \leq c_p t^{-p/2} T_\varepsilon(t) |f_{\varepsilon,n}|^p, \quad t > 0, \quad p \in (1, 2].
\]
(3.30)
From Proposition 2.33(i), up to a subsequence, the left-hand side of (3.30) converges to \(|D_H T_\varepsilon(t) f(x)|_H^p\)
as \( n \to \infty \) for \( \nu_\varepsilon \)-a.e. \( x \in X \). Let us consider the right-hand side of (3.30). Since \(|f_{\varepsilon,n}|^p \to |f|^p\) in \( L^1(X, \nu_\varepsilon) \) it follows that
\[
T_\varepsilon(t) |f_{\varepsilon,n}|^p \to T_\varepsilon |f|^p, \quad n \to +\infty, \text{ in } L^1(X, \nu_\varepsilon).
\]
Hence, up to a subsequence, \( T_\varepsilon(t) |f_{\varepsilon,n}|^p(x) \to T_\varepsilon |f|^p(x) \) as \( n \to +\infty \) for \( \nu_\varepsilon \)-a.e. \( x \in X \). This gives (3.22) for \( p \in (1, 2] \).
STEP 2. Let \( f \in \mathcal{F}^{\infty}_{b, \Phi}(X) \) and let \( p \in (1, 2] \). Let us multiply both the sides of (3.22) by a positive, bounded and continuous function \( g \) and let us integrate on \( X \) with respect to \( \nu \). We get

\[
\int_X |D_H T_\varepsilon(t) f|^p_H g d\nu \leq p^{-p/2} \int_X T_\varepsilon(t) |f|^p g d\nu, \quad t > 0.
\]

Since \( D_H T_\varepsilon(t) f \to D_H T(t) f \) weakly in \( L^p(X, \nu) \) as \( \varepsilon \to 0^+ \) (see Proposition 2.33(ii)), arguing as in Step 2 in the proof of Proposition 3.3 it is possible to prove that \( g^{1/p} D_H T_\varepsilon(t) f \) weakly converges to \( g^{1/p} D_H T(t) f \) in \( L^p(X, \nu) \). Hence,

\[
\int_X |D_H T(t) f|^p_H g d\nu \leq \liminf_{\varepsilon \to 0^+} \int_X |D_H T_\varepsilon(t) f|^p_H g d\nu.
\]

From Proposition 2.33(ii) we deduce that

\[
\int_X T_\varepsilon(t) |f|^p g d\nu \to \int_X T(t) |f|^p g d\nu, \quad \varepsilon \to 0^+.
\]

It follows that

\[
\int_X |D_H T(t) f|^p_H g d\nu \leq c_p t^{-p/2} \int_X T(t) |f|^p g d\nu, \quad t > 0.
\]

The arbitrariness of \( g \) implies that

\[
|D_H T(t) f(x)|^p_H \leq c_p t^{-p/2} T(t) |f|^p(x), \quad \nu\text{-a.e. } x \in X. \tag{3.31}
\]

If \( p > 2 \), we apply (3.22) with \( p = 2 \) and (2.28) with \( p = 1 \) and \( q = p/2 \), and we get

\[
|D_H T(t) f(x)|^p_H = (|D_H T(t) f(x)|^2_H)^{p/2} \leq \left( \frac{1}{2} T^{-1} T(t) |f|^2(x) \right)^{p/2} \leq c_p t^{-p/2} T(t) |f|^p(x), \quad \nu\text{-a.e. } x \in X,
\]

with \( c_p = 2^{-p/2} \).

STEP 3. Let \( f \in L^p(X, \nu) \) and let \( (g_n) \subset \mathcal{F}^{\infty}_{b, \Phi}(X) \) converge to \( f \) in \( L^p(X, \nu) \) as \( n \to +\infty \). Replacing \( f \) with \( g_n - g_m \) in (3.31) and integrating on \( X \) with respect to \( \nu \) we get

\[
\int_X |D_H T(t) (g_n - g_m)|^p_H g d\nu \leq c_p t^{-p/2} \int_X T(t) |g_n - g_m|^p g d\nu = c_p t^{-p/2} \int_X |g_n - g_m|^p g d\nu, \tag{3.32}
\]

where in the last part we have used the fact that \( \nu \) is an invariant measure for \((T(t))_{t \geq 0}\). Conclusion follows by repeating the same computations as in Step 3 in the proof of Proposition 3.3.

\[\square\]

3.3 Asymptotic behaviour and gradient estimates to the vector-valued semigroup \( (T^V(t))_{t \geq 0} \)

Let \( V \) be a separable Hilbert space. In this subsection we show that the \( V \)-valued semigroup \((T^V(t))_{t \geq 0}\) inherits asymptotic behaviour and gradient estimates from the scalar semigroup \((T(t))_{t \geq 0}\).

**Proposition 3.5.** For any \( p \in [1, +\infty) \) we have

\[
\nu(F) := \int_X F d\nu = L^p - \lim_{t \to +\infty} T^V(t) F, \quad F \in L^p(X, \nu; V). \tag{3.33}
\]

**Proof.** Let \( p \in [1, +\infty) \). Let us prove the result for \( F \in \mathcal{F}^{\infty}_{b, \Phi}(X; V) \). Let

\[
T^V(t) F = \sum_{i=1}^n T(t) f_i v_i, \quad F = \sum_{i=1}^n f_i v_i, \quad f_i \in \mathcal{F}^{\infty}_{b, \Phi}(X), \quad v_i \in V, \quad i = 1, \ldots, n,
\]
Without loss of generality we can assume that \( \{v_1, \ldots, v_n\} \) are orthonormal vectors in \( V \). We have
\[
\nu(F) = \sum_{i=1}^{n} \left( \int_X f_i d\nu \right) v_i.
\]

From Proposition 3.2 we infer that there exists a sequence \( (t_m) \subset (0, +\infty) \) diverging to \(+\infty\) as \( m \to +\infty \) such that \( T^V(t_m)F \to \int_X F d\nu \) pointwise \( \nu\text{-a.e.} \) in \( X \). Further, from the last part of Proposition 2.21(ii) we infer that
\[
\left| T^V(t_m)F - \int_X F d\nu \right|_V^p = \left( \left| T^V(t_m)F - \int_X F d\nu \right|_V^2 \right)^{p/2} = \left( \sum_{i=1}^{n} \left| T(t_m)f_i - \int_X f_i d\nu \right|^2 \right)^{p/2}
\leq \left( \sum_{i=1}^{n} 2(\|T(t_m)f_i\|_\infty^2 + \|f_i\|_\infty^2) \right)^{p/2} \leq 2^p \left( \sum_{i=1}^{n} \|f_i\|_\infty^2 \right)^{p/2}.
\]

By the dominated convergence theorem we get \( T^V(t_m)F \to \int_X F d\nu \) in \( L^p(X, \nu; V) \) as \( m \to +\infty \). In particular, same arguments give that for any sequence \( (t_m) \subset (0, +\infty) \) there exists a subsequence \( (t_{m_k}) \) such that \( T^V(t_{m_k})F \to \int_X F d\nu \) in \( L^p(X, \nu; V) \) as \( m \to +\infty \). This means that \( T^V(t)F \to \int_X F d\nu \) in \( L^p(X, \nu; V) \) as \( t \to +\infty \).

Let us consider a function \( F \in L^p(X, \nu) \) and let \( (F_m) \subset \mathcal{F}_{\mathcal{C}}^\infty(X; V) \) be a sequence converging to \( F \) in \( L^p(X, \nu; V) \) as \( n \to +\infty \). From the properties of Bochner integral (see [12, Chapter II, Thereom 4(ii)]) and Hölder’s inequality we infer that
\[
\left\| \int_X F_m d\nu - \int_X F d\nu \right\|_{L^p(X, \nu; V)} \leq \|F - F_m\|_{L^p(X, \nu; V)}.
\]

Since \( (T^V(t))_{t \geq 0} \) is a semigroup of contractions in \( L^p(X, \nu; V) \) it follows that
\[
\left\| T^V(t)F - \int_X F d\nu \right\|_{L^p(X, \nu; V)} \leq \left\| T^V(t)F - T^V(t)F_m \right\|_{L^p(X, \nu; V)} + \left\| T^V(t)F_m - \int_X F_m d\nu \right\|_{L^p(X, \nu; V)} + \left\| \int_X F_m d\nu - \int_X F d\nu \right\|_{L^p(X, \nu; V)} \leq 2\|F - F_m\|_{L^p(X, \nu; V)} + \left\| T^V(t)F_m - \int_X F_m d\nu \right\|_{L^p(X, \nu; V)}.
\]

Let \( \varepsilon > 0 \). Then, there exists \( \overline{m} \in \mathbb{N} \) such that \( \|F - F_m\|_{L^p(X, \nu; V)} \leq \varepsilon/4 \) for any \( m \geq \overline{m} \). Further, there exists \( \overline{t} \) such that for any \( t > \overline{t} \) we have
\[
\left\| T^V(t)F_{\overline{m}} - \int_X F_{\overline{m}} d\nu \right\|_{L^p(X, \nu; V)} \leq \varepsilon/2.
\]

Therefore, from (3.34) with \( m \) replaced by \( \overline{m} \), for any \( \varepsilon > 0 \) there exists \( \overline{t} > 0 \) such that for any \( t > \overline{t} \) we have
\[
\left\| T^V(t)F - \int_X F d\nu \right\|_{L^p(X, \nu; V)} \leq \varepsilon.
\]

This gives the thesis. \( \Box \)

The next two propositions are the vector-valued version of Propositions 3.3 and 3.4, respectively.
Proposition 3.6. For any $p \in [2, +\infty)$ we have
\[
|\langle DH T^V(t) F \rangle_{H \otimes V}|_p \leq e^{-pt} T(t) \| DH F \|_{H \otimes V}^p, \quad t > 0, \ F \in L_p(X, \nu; V), \ \nu\text{-a.e. } x \in X. \quad (3.35)
\]
Further,
\[
\int_X |DH T^V(t) F|_{H \otimes V}^p \, d\nu \leq e^{-pt} \int_X |DH F|_{H \otimes V}^p \, d\nu, \quad t > 0, \ F \in L_p(X, \nu; V). \quad (3.36)
\]
Proof. Formula (3.36) follows by integrating (3.35) on $X$ with respect to $\nu$ and by recalling that $\nu$ is an invariant measure for $(T(t))_{t \geq 0}$. Let us prove the first part. By density we can limit ourselves to prove the result for $F \in \mathcal{F} \mathcal{E}_b^\infty(X; V)$.

Let $p \geq 2$, let $t > 0$ and let $F \in \mathcal{F} \mathcal{E}_b^\infty(X; V)$ of the form
\[
F = \sum_{i=1}^n f_i v_i, \quad f_i \in \mathcal{F} \mathcal{E}_b^\infty(X), \ v_i \in V, \ i = 1, \ldots, n.
\]
Without loss of generality we can assume that $\{v_1, \ldots, v_n\}$ are orthonormal vectors in $V$. From (3.6) with $p = 2$ we infer that
\[
|DH T^V(t) F|_{H \otimes V}^2 = \sum_{i=1}^n |DH T(t) f_i(x)|_H^2 \\
\leq e^{-2t} \sum_{i=1}^n |T(t)| |DH f_i(x)|_H^2 \\
= e^{-2t} T(t) \left( \sum_{i=1}^n |DH f_i(x)|_H^2 \right) \\
= e^{-2t} T(t) \left( |DH F|_{H \otimes V}^2 \right).
\]
This implies that
\[
|DH T^V(t) F|_{H \otimes V}^p \leq \left( |DH T^V(t) F|_{H \otimes V}^2 \right)^{p/2} \\
\leq e^{-2t} T(t) \left( |DH F|_{H \otimes V}^2 \right)^{p/2} \\
= e^{-pt} (T(t) \left( |DH F|_{H \otimes V}^2 \right))^{p/2}.
\]
By applying (2.28) with $q = p$ and $p = 1$ we get the thesis.

Proposition 3.7. For any $p \in [2, +\infty)$ we have
\[
|\langle DH T^V(t) F \rangle_{H \otimes V}|_p \leq \frac{1}{(2t)^{p/2}} T(t) \| F \|_{V}^p \ (x), \quad t > 0, \ F \in L^p(X, \nu; V), \ \nu\text{-a.e. } x \in X. \quad (3.37)
\]
Further,
\[
\int_X |DH T^V(t) F|_{H \otimes V}^p \, d\nu \leq \frac{1}{(2t)^{p/2}} \int_X |F|_V^p \, d\nu, \quad t > 0, \ F \in L^p(X, \nu; V). \quad (3.38)
\]
Proof. The further part follows by integrating (3.37) on $X$ with respect to $\nu$ and by recalling that $\nu$ is an invariant measure for $(T(t))_{t \geq 0}$. Let us prove the first part. By density we can limit ourselves to prove the result for $F \in \mathcal{F} \mathcal{E}_b^\infty(X; V)$. The thesis follows by arguing as in the proof of Proposition 3.6 and by applying (3.21) with $p = 2$.  

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Finally, we put together Propositions 3.6 and 3.7 to get the following result.

**Corollary 3.8.** Let $p \in [2, +\infty)$. Then,
\[
\|DH^{TV}(t)F\|_{L^p(X, \nu; H \otimes V)} \leq \frac{\max\{t^{-1/2}, 1\}}{\sqrt{2}} \min\{1, e^{-t+1}\} \|F\|_{L^p(X, \nu; V)}, \quad t > 0,
\] (3.39)
for any $F \in L^p(X, \nu; V)$.

**Proof.** By density, it is enough to show that (3.39) holds for $F \in \mathcal{F} \mathcal{C}^\infty_b(X; V)$. Let $p \in [2, +\infty)$, let $F \in \mathcal{F} \mathcal{C}^\infty_b(X; V)$ and let $t \in (0, 1]$. From (3.38) we get
\[
\int_X |DH^{TV}(t)F|_{H \otimes V}^p d\nu \leq \frac{1}{(2t)^{p/2}} \int_X |F|_V^p d\nu.
\] (3.40)
If $t \geq 1$, from the semigroup property of $(TV(t))_{t \geq 0}$ and by applying (3.36) with $t - 1$ and (3.38) with $t = 1$ we have
\[
\int_X |DH^{TV}(t)F|_{H \otimes V}^p d\nu = \int_X |DH^{TV}(t-1)TV(1)F|_{H \otimes V}^p d\nu
\leq e^{-p(t-1)} \int_X |DH^{TV}(1)F|_{H \otimes V}^p d\nu
\leq \frac{1}{2^{p/2}} e^{-p(t-1)} \int_X |F|_V^p d\nu.
\] (3.41)
By combining (3.40) and (3.41) we get the thesis. \hfill \Box

\section{Vector-valued Poincaré inequality}

Thank to Propositions 3.5 and 3.7 and to Corollary 3.8 we are able to prove a vector-valued Poincaré inequality.

**Theorem 4.1.** Let $V$ be a separable Hilbert space and let $p \in [1, +\infty)$. Then, there exists a positive constant $k_p$ defined by
\[
k_p = \begin{cases} \sqrt{p-1}, & p \in [2, +\infty), \\ \frac{3}{\sqrt{2}}, & p \in [1, 2), \end{cases}
\] (4.1)
such that for any $F \in W^{1,p}(X, \nu; V)$ we have
\[
\|F - \nu(F)\|_{L^p(X, \nu; V)} \leq k_p \|DH^F\|_{L^p(X, \nu; H \otimes V)},
\] (4.2)
where $\nu(F)$ has been defined in (3.33).

**Proof.** We split the proof into two parts. In the former we prove the thesis for $p \in [2, +\infty)$, in the latter we consider the remaining cases $p \in [1, 2)$. As usual, by density it is enough to prove the statement for $F \in \mathcal{F} \mathcal{C}^\infty_b(X; V)$.

**STEP 1.** Let $p \in [2, +\infty)$ and let $F \in \mathcal{F} \mathcal{C}^\infty_b(X; V)$ with
\[
F := \sum_{i=1}^m f_i v_i, \quad f_i \in \mathcal{F} \mathcal{C}^\infty_b(X), \quad v_i \in V, \quad i = 1, \ldots, m,
\]
and \( \{v_1, \ldots, v_m\} \) orthonormal vectors in \( V \). Then,
\[
T^V(t)F = \sum_{i=1}^{m} T(t) f_i v_i,
\]
belongs to \( D(L_p) \). We define \( G := F - \nu(F) \in \mathcal{F}\mathcal{C}^\infty_b(X; V) \). From Lemma 2.16 we infer that \( G^* := |G|^p G \) \( \in \mathcal{W}^{1,p'}(X, \nu; V) \) and
\[
\overline{D_HG}^* = (p - 2)|G|^{p-2}G |D_HG|_V \otimes G + |G|^{p-2}D_HG, \quad (4.3)
\]
where \( [D_HG, G]_V \) is meant as an element of \( H \). Formula (3.33) gives
\[
\int_X |G|^p_V d\nu = \int_X [G, G^*)_V d\nu = \int_X [F - \nu(F), G^*)_V d\nu = \lim_{t \to +\infty} \int_X [T^V(0)F - T^V(t)F, G^*)_V d\nu. \quad (4.4)
\]
Let us consider the argument of the limit. We get
\[
\int_X [T^V(0)F - T^V(t)F, G^*)_V d\nu = - \int_X \left( \int_0^t \left[ \frac{d}{ds} T^V(s)F, G^*_V \right] ds \right) d\nu = \int_X \left( \int_0^t \left[ L^V_p T^V(s)F, G^*_V \right] ds \right) d\nu = \int_0^t \left( \int_X [D_HT^V(s)F, D_HG^*_V]_{H \otimes V} d\nu \right) ds, \quad (4.5)
\]
where we have used the Fubini’s theorem and Lemma 2.25(ii). From (4.3) we have
\[
\left| \int_X [D_HT^V(s)F, D_HG^*]_{H \otimes V} d\nu \right| \leq \int_X [D_HT^V(s)F]_{H \otimes V} |D_HG^*|_{H \otimes V} d\nu \leq (p - 1) \int_X |D_HT^V(s)F|_{H \otimes V} |D_HG|_{H \otimes V} p - 2 d\nu, \quad (4.6)
\]
since \( \overline{D_HG} = \overline{D_HF} \). Let us consider \( p > 2 \). From the generalized Hölder’s inequality for three functions with exponents \( p, p \) and \( \frac{p}{p-2} \) and (3.36) it follows that
\[
\int_X [D_HT^V(s)F]_{H \otimes V} |D_HG|_{H \otimes V} p - 2 d\nu \\
\leq \left( \int_X [D_HT^V(s)F]_{H \otimes V} p d\nu \right)^{1/p} \left( \int_X |D_HG|_{H \otimes V} p d\nu \right)^{1-2/p} \left( \int_X [G]^p_V d\nu \right)^{1-2/p} e^{-s} \left( \int_X |G|^p_V d\nu \right)^{2/p} \left( \int_X [G]^p_V d\nu \right)^{1-2/p}. \quad (4.7)
\]
Putting together (4.4)-(4.7) we infer that
\[
\int_X |G|^p V d\nu \leq (p - 1) \left( \int_0^{\infty} e^{-s} ds \right) \left( \int_X [D_HF]_{H \otimes V} p d\nu \right)^{2/p} \left( \int_X |G|^p_V d\nu \right)^{1-2/p} = (p - 1) \left( \int_X [D_HF]_{H \otimes V} p d\nu \right)^{2/p} \left( \int_X [G]^p_V d\nu \right)^{1-2/p}. \quad (4.8)
\]
Dividing both the sides by \((\int_X |G|^p \, d\nu)^{1-2/p}\) we get
\[
\left( \int_X |G|^p \, d\nu \right)^{2/p} \leq (p-1) \left( \int_X |\overline{D}_H F|^p \, d\nu \right)^{2/p},
\]
which gives the thesis with \(k_p = (p-1)^{1/2}\). If \(p = 2\), then \(G^* = G\) and \(\overline{D}_H G^* = \overline{D}_H F\), and from (3.36) we get
\[
\left| \int_X [\overline{D}_H T^V(s)F, \overline{D}_H G^*]_{H \otimes V} \, d\nu \right| = \left| \int_X [\overline{D}_H T^V(s)F, \overline{D}_H F]_{H \otimes V} \, d\nu \right| \tag{4.8}
\]
notag \leq \left( \int_X |\overline{D}_H T^V(s)F|_{H \otimes V}^2 \, d\nu \right)^{1/2} \left( \int_X |\overline{D}_H F|_{H \otimes V}^2 \, d\nu \right)^{1/2}
\leq e^{-s} \left( \int_X |D_H F|_{H \otimes V}^p \, d\nu \right), \tag{4.9}
\]
for any \(s > 0\). By collecting (4.5), (4.5) and (4.9) we get the thesis with \(c_2 = 1 = (2-1)^{1/2}\).

**STEP 2.** Let \(p \in (1, 2)\) and let \(F \in \mathcal{F}'_{\infty}(X; V)\). Then, we have
\[
\|F - \nu(F)\|_{L_p'}(X, \nu; V) = \sup_{G \in \mathcal{F}'_{\infty}(X; V), \|G\|_{L_p'}(X, \nu; V) \leq 1} \int_X [F - \nu(F), G|_V \, d\nu. \tag{4.10}
\]
Let \(G \in \mathcal{F}'_{\infty}(X; V)\) with \(\|G\|_{L_p'}(X, \nu; V) \leq 1\). Then,
\[
\int_X [F - \nu(F), G|_V \, d\nu = \lim_{t \to +\infty} \int_X [T^V(0)F - T^V(t)F, G|_V \, d\nu = \lim_{t \to +\infty} \int_X [F, T^V(0)G - T^V(t)G|_V \, d\nu, \tag{4.11}
\]
where we have applied Lemma 2.25(i) and (3.33). Arguing as in Step 1 we infer that
\[
\int_X [F, T^V(0)G - T^V(t)G|_V \, d\nu = \int_0^t \left( \int_X |\overline{D}_H F, \overline{D}_H T^V(s)G|_{H \otimes V} \, d\nu \right) \, ds
\leq \int_0^t \left( \int_X |\overline{D}_H F|_{H \otimes V} |\overline{D}_H T^V(s)G|_{H \otimes V} \, d\nu \right) \, ds
\leq \left( \int_X |\overline{D}_H F|_{H \otimes V}^p \, d\nu \right)^{1/p} \int_0^t \left( \int_X |\overline{D}_H T^V(s)G|_{H \otimes V}^{p'} \, d\nu \right)^{1/p'} \, ds. \tag{4.12}
\]
By applying (3.39) to \(\int_X |\overline{D}_H T^V(s)G|_{H \otimes V}^{p'} \, d\nu\) and recalling that \(\|G\|_{L_p'}(X, \nu; V) \leq 1\) we infer that
\[
\left( \int_X |\overline{D}_H T^V(s)G|_{H \otimes V}^{p'} \, d\nu \right)^{1/p'} \leq \frac{\max\{s^{-1/2}, 1\}}{\sqrt{2}} \min\{1, e^{-s+1}\} \|G\|_{L_p'}(X, \nu; V)
\leq \frac{\max\{s^{-1/2}, 1\}}{\sqrt{2}} \min\{1, e^{-s+1}\}, \tag{4.13}
\]
which gives
\[
\int_0^t \left( \int_X |\overline{D}_H T^V(s)G|_{H \otimes V}^{p'} \, d\nu \right)^{1/p'} \, ds \leq \frac{1}{\sqrt{2}} \left( \int_0^1 s^{-1/2} \, ds + \int_1^t e^{-s+1} \, ds \right) = \frac{3}{\sqrt{2}} - e^{-t+1}, \tag{4.14}
\]
which gives
for any $t > 1$. Collecting (4.11)-(4.14) we get
\[
\int_X |F - \nu(F), G|_V \, d\nu \leq \frac{3}{\sqrt{2}} \|D_H F\|_{L^p(X, \nu; H \otimes V)},
\]
for any $G \in \mathcal{F}C_b^\infty (X; V)$ with $\|G\|_{L^p(X, \nu; V)} \leq 1$. From (4.10) we get
\[
\|F - \nu(F)\|_{L^p(X, \nu; V)} \leq \frac{3}{\sqrt{2}} \|D_H F\|_{L^p(X, \nu; H \otimes V)},
\]
for any $p \in (1, 2)$ and any $F \in \mathcal{F}C_b^\infty (X; V)$. Since the constant $k_p = \frac{3}{\sqrt{2}}$ does not depend on $p$, letting $p \to 1^+$ we get (4.15) also for $p = 1$. \hfill \Box

As a byproduct of Theorem 4.1 we get the following result.

**Corollary 4.2.** Let $p \in [1, +\infty)$ and let $f \in W^{k+1,p}(X, \nu)$ with $k \in \mathbb{N}$. Then,
\[
\|D_H^k f - \nu(D_H^k f)\|_{L^p(X, \nu; H_k(H))} \leq k_p \|D_H^{k+1} f\|_{L^p(X, \nu; H_{k+1}(H))},
\]
where $k_p$ is the positive constant in (4.1).

**Proof.** The thesis follows by applying Theorem 4.1 with $V = H_k(H)$ and $F = D_H^k f$. \hfill \Box

## 5 Equivalent definitions of Sobolev spaces $W^{k,p}(X, \nu)$

The combination of Wiener chaos decomposition (see Subsection 2.3) and of vector-valued Poincaré inequality (4.16) allows us to prove that the norm $\| \cdot \|_{k,p}$ is equivalent to the graph norm of $D_H^k$ in $L^p(X, \nu)$ for any $p \in (1, +\infty)$ and any $k \in \mathbb{N}$. We split this fact into two different theorems, since the case $k = 2$ can be proved if $U$ only satisfies Hypothesis 2.6, while for the case $k \geq 3$ we need additional assumptions on $U$.

**Theorem 5.1.** Let $p \in (1, +\infty)$. Then, $W^{2,p}(X, \nu)$ coincides with the space $\mathcal{W}^{2,p}(X, \nu)$ defined as the closure of $\mathcal{F}C_b^\infty (X)$ with respect to the graph norm $\| \cdot \|_{p, D_H^2}$ of $D_H^2$, i.e.,
\[
\|f\|_{p, D_H^2} := \|f\|_{L^p(X, \nu)} + \|D_H^2 f\|_{L^p(X, \nu; H_2(H))}, \quad f \in \mathcal{F}C_b^\infty (X).
\]

**Proof.** Since the norm $\| \cdot \|_{2, p}$ is stronger than $\| \cdot \|_{p, D_H^2}$, the continuous embedding $W^{2,p}(X, \nu) \subset \mathcal{W}^{2,p}(X, \nu)$ follows. To prove the converse inclusion, we show that for any $p \in (1, +\infty)$ there exist a positive constant $C_p$, which only depends on $p$ and $U$, such that for any $f \in \mathcal{F}C_b^\infty (X)$ we have
\[
\|D_H f\|_{L^p(X, \nu; H_2(H))} \leq k_p \|D_H^2 f\|_{L^p(X, \nu; H_2(H))} + C_p \|f\|_{L^p(X, \nu)},
\]
and $k_p$ is the constant in (4.16). Let $p \in (1, +\infty)$ and let $f \in \mathcal{F}C_b^\infty (X)$. Then,
\[
\|D_H f\|_{L^p(X, \nu; H)} \leq \|D_H f - \nu(D_H f)\|_{L^p(X, \nu; H)} + \|\nu(D_H f)\|_{H}
\leq k_p \|D_H^2 f\|_{L^p(X, \nu; H_2(H))} + \left( \sum_{n \in \mathbb{N}} \|D_H f, e_n\|_{H} \right)^2 \right)^{1/2},
\]
where in the last inequality we have taken advantage from (4.16). Let us estimate the second addend above. Integrating by parts we get
\[
\sum_{n \in \mathbb{N}} \left| \int_X [D_H f, e_n]_H d\nu \right|^2 = \sum_{n \in \mathbb{N}} \left| \int_X f(e_n + [D_H U, e_n]_H) d\nu \right|^2
\]

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\[ \leq 2 \sum_{n \in \mathbb{N}} \left| \int_X e^{-U} f c_n d\mu \right|^2 + 2 \sum_{n \in \mathbb{N}} \left| \int_X f [D_H U, e_n] H d\nu \right|^2. \]  

(5.2)

From Remarks 2.3 and 2.4 it follows that \( \{ c_n : n \in \mathbb{N} \} \) is an orthonormal basis the first Wiener chaos \( E_1 \) and that

\[ \sum_{n \in \mathbb{N}} \left| \int_X e^{-U} f c_n d\mu \right|^2 = \| I_1(e^{-U} f) \|^2_{L^2(X, \mu)}. \]

Let \( q = \frac{p+1}{2} \in (1, p) \). From Lemma 2.5 there exists a positive constant \( c_p \), which only depends on \( p \), such that

\[ \| I_1(e^{-U} f) \|^2_{L^2(X, \mu)} \leq C_p \| e^{-U} f \|^2_{L^2(X, \mu)}. \]

By applying the Hölder’s inequality with \( r = \frac{2}{q} \) we get

\[ \| e^{-U} f \|^2_{L^p(X, \mu)} = \left\| \left( e^{-\frac{U}{p}} |f|^q e^{-\frac{q}{p} U} \right)^{2/q} \right\|_{L^p(X, \mu)} \leq \| e^{-U} f \|^2_{L^p(X, \mu)} \left( \int_X e^{-r(q-\frac{q}{p})U} d\mu \right)^{2/(qr)} =: C_p \| f \|^2_{L^p(X, \mu)}, \]  

(5.3)

where \( C_p \) is a constant which only depends on \( p \) and \( U \). Further,

\[ \sum_{n \in \mathbb{N}} \left| \int_X f [D_H U, e_n] H d\nu \right|^2 = \left| \int_X f \sum_{n \in \mathbb{N}} [D_H U, e_n] H e_n d\nu \right|^2_H = \left| \int_X f D_H U d\nu \right|^2_H \leq \left( \int_X |f|^2 |D_H U| H d\nu \right)^2 \leq \| f \|^2_{L^p(X, \nu)} \| D_H U \|^2_{L^{p'}(X, \nu)}. \]  

(5.4)

Since both \( e^{-U} \) and \( |D_H U|_H \) belong to \( L^s(X, \mu) \) for any \( s \in (1, +\infty) \), we deduce that

\[ \| D_H U \|^2_{L^{p'}(X, \nu)} = \| D_H U \|_{H e^{-U/p'}}^2 < +\infty. \]  

(5.5)

Collecting together (5.2)-(5.5) we infer that

\[ \left( \sum_{n \in \mathbb{N}} \left| \int_X [D_H f, e_n] H d\nu \right|^2 \right)^{1/2} \leq \sqrt{2} \left( C_p + \| D_H U \|_{L^{p'}(X, \nu)} \right)^{1/2} \| f \|_{L^p(X, \nu)} =: \tilde{C}_p \| f \|_{L^p(X, \nu)}, \]

which gives the thesis. \( \Box \)

Without any additional assumption on \( U \) we are able to extend (5.1) to \( D_H^k \) for any \( k \in \mathbb{N}, k \geq 2 \).

**Proposition 5.2.** Let \( p \in (1, +\infty) \) and let \( k \in \mathbb{N}, k \geq 2 \). Then, there exist a positive constant \( \tilde{C}_p \), which only depends on \( p \) and \( U \), such that for any \( f \in W^{k+1,p}(X, \nu) \) we have

\[ \| D_H^k f \|_{L^p(X, \nu; H_k(H))} \leq k_p \| D_H^{k+1} f \|_{L^p(X, \nu; H_{k+1}(H))} + \tilde{C}_p \| D_H^{k-1} f \|_{L^p(X, \nu; H_{k-1}(H))}, \]  

(5.6)

where \( k_p \) is the constant in (4.16).

**Proof.** Let \( k \in \mathbb{N}, k \geq 2 \), and let \( f \in \mathcal{F} \mathcal{C}_h^{\infty}(X) \). From (4.16) we have

\[ \| D_H^k f \|_{L^p(X, \nu; H_k(H))} \leq \| D_H^k f - \nu(D_H f)^k \|_{L^p(X, \nu; H_k(H))} + \nu(D_H f)\|_{H_k(H)}, \]

(4.16)
We notice that

\[ D^k_H f(e_{i_1}, \ldots, e_{i_k}) = [D^k_H f(e_{i_1}, \ldots, e_{i_{k-1}}), e_{i_k}]_H, \quad i_1, \ldots, i_{k-1} \in \mathbb{N}, \]

where \( D^k_H f(e_{i_1}, \ldots, e_{i_{k-1}}) \) is seen as an element of \( H \). By applying formula (2.15) to the last addend in (5.7) we get

\[
\left\| D^k_H f(e_{i_1}, \ldots, e_{i_k}) \right\|_{L^2(X, \mu)}^2 = \sum_{i_1, \ldots, i_k \in \mathbb{N}} \left| \int_X D^k_H f(e_{i_1}, \ldots, e_{i_k}) \, d\mu \right|^2
\]

By applying Lemma 2.5 for any \( i_1, \ldots, i_{k-1} \in \mathbb{N} \) we get

\[
\left\| D^k_H f(e_{i_1}, \ldots, e_{i_{k-1}}) \right\|_{L^2(X, \mu)}^2 = \left\| I_1(D^k_H f(e_{i_1}, \ldots, e_{i_{k-1}}) e^{-U}) \right\|_{L^2(X, \mu)}^2 \leq c_p \left\| D^k_H f(e_{i_1}, \ldots, e_{i_{k-1}}) e^{-U} \right\|_{L^q(X, \mu)}^2
\]

where \( q = \min \left\{ \frac{p+1}{2}, \frac{3}{2} \right\} \) and \( c_p \) is a positive constant which only depends on \( p \). Summing up \( i_1, \ldots, i_{k-1} \) over \( \mathbb{N} \) we get

\[
\sum_{i_1, \ldots, i_{k-1} \in \mathbb{N}} \left| \int_X D^k_H f(e_{i_1}, \ldots, e_{i_{k-1}}) \, d\mu \right|^2 \leq c_p \sum_{i_1, \ldots, i_{k-1} \in \mathbb{N}} \left\| D^k_H f(e_{i_1}, \ldots, e_{i_{k-1}}) e^{-U} \right\|_{L^q(X, \mu)}^2
\]

Let us apply the Minkowski’s integral inequality with \( \mu_1 = \mu, \mu_2 \) being the product of \( k-1 \) counting measures on \( \mathbb{N} \) and \( p = 2/q > 1 \). It follows that

\[
\sum_{i_1, \ldots, i_{k-1} \in \mathbb{N}} \left( \int_X \left| D^k_H f(e_{i_1}, \ldots, e_{i_{k-1}}) e^{-U} \right| q \, d\mu \right)^{2/q} \leq \left( \int_X \left( \sum_{i_1, \ldots, i_{k-1} \in \mathbb{N}} \left| D^k_H f(e_{i_1}, \ldots, e_{i_{k-1}}) e^{-U} \right|^2 \right)^{q/2} \, d\mu \right)^{2/q}
\]

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Theorem 5.3. \( \text{Let} \) for any \( p \)

\[
\\|D^k_{H} f\|_{H_k-1(H)} e^{\nu} \leq C_p \|D^k_{H} f\|_{L^p(X,\nu;H_k-1(H))},
\]

for some positive constant \( C_p \) which only depends on \( p \) and \( U \). Moreover,

\[
\sum_{i_1,\ldots,i_k \in \mathbb{N}} \sum_{i \in \mathbb{N}} \left( \int_X D^k_{H} f(e_{i_1},\ldots,e_{i_k}) d\nu \right)^2 \leq \bar{C}_p \|D^k_{H} f\|_{L^p(X,\nu;H_k-1(H))}^{1/2}.
\]

As we said above, a generalization of Theorem 5.1 is available for \( k > 2 \) under additional assumptions on \( U \). Indeed, the idea is going on with integrations by parts of the last addend

\[
\sum_{i_1,\ldots,i_k \in \mathbb{N}} \int_X D^k_{H} f(e_{i_1},\ldots,e_{i_k}) d\nu^2
\]

in (5.7) in order to estimate this term by means of \( \|f\|_{L^p(X,\nu)} \). By applying this procedure the derivatives \( D^j_{H} U \) arise, with \( j = 1,\ldots,k - 1 \), and so we need that the function \( U \) belongs to \( W^{k-1,p}(X,\nu) \).

Theorem 5.3. \( \text{Let} \) \( p \in (1,\infty) \) and let \( k \in \mathbb{N} \), \( k \geq 3 \). \( \text{If} \) \( U \in W^{k-1,q}(X,\mu) \) \( \text{for any} \ q \in [1,\infty) \),

\( \text{then the space} \ W^{k,p}(X,\nu) \ \text{coincides with the space} \ \bar{W}^{k,p}(X,\nu) \ \text{defined as the closure of} \ \mathcal{F}\mathcal{E}_\infty^\mu(X) \ \text{with respect to the graph norm of} \ D^k_{H} \ \text{in} \ L^p(X,\nu), \ i.e.,} \)

\[
\|f\|_{p,D^k_{H}} := \|f\|_{L^p(X,\nu)} + \|D^k_{H} f\|_{L^p(X,\nu;H_k-1(H))}, \ f \in \mathcal{F}\mathcal{E}_\infty^\mu(X).
\]

Proof. We limit ourselves to prove the statement when \( k = 3 \), the other cases following by analogous computations. Let \( k = 3 \) and let \( p \in (1,\infty) \). Since the norm \( \|\cdot\|_{3,p} \) is stronger than \( \|\cdot\|_{p,D^3_{H}} \), the continuous embedding \( W^{3,p}(X,\nu) \subset W^{3,p}(X,\nu) \) follows.

To prove the converse inclusion, at first we notice that from Theorem 5.1 there exists a positive constant \( M_p \) such that

\[
\|f\|_{2,p} \leq M_p \left( \|f\|_{L^p(X,\nu)} + \|D^2_{H} f\|_{L^p(X,\nu;H_2(H))} \right), \ f \in \mathcal{F}\mathcal{E}_\infty^\mu(X).
\]

This gives

\[
\|f\|_{3,p} \leq \|f\|_{2,p} + \|D^3_{H} f\|_{L^p(X,\nu;H_3(H))}
\]
\[ \leq M_p \|f\|_{L^p(X,\nu)} + M_p \|D_H^2 f\|_{L^p(X,\nu;\mathcal{H}_2(H))} + \|D_H^3 f\|_{L^p(X,\nu;\mathcal{H}_3(H))}, \quad f \in \mathcal{C}^\infty_b(X). \]

Hence, it is enough to find a positive constant \( c \), independent of \( f \), such that
\[ \|D_H^2 f\|_{L^p(X,\nu;\mathcal{H}_2(H))} \leq c \left( \|f\|_{L^p(X,\nu)} + \|D_H^3 f\|_{L^p(X,\nu;\mathcal{H}_3(H))} \right), \quad f \in \mathcal{C}^\infty_b(X). \]

Arguing as in (5.7) and taking into account (4.16), we get
\[ \|D_H^2 f\|_{L^p(X,\nu;\mathcal{H}_2(H))} \leq k_p \|D_H^2 f\|_{L^p(X,\nu;\mathcal{H}_3(H))} + \left( \sum_{i,j \in \mathbb{N}} \left| \int_X [D_H^2 f e_i, e_j]_H d\nu \right|^2 \right)^{1/2}, \quad (5.12) \]
for any \( f \in \mathcal{C}^\infty_b(X) \). Two integrations by parts in the second addend of the right-hand side of (5.12) give
\[ \int_X [D_H^2 f e_i, e_j]_H d\nu = \int_X [D_H f, e_i]_H (\hat{e}_j + [D_H U, e_j]_H) d\nu \]
\[ = \int_X f((\hat{e}_i + [D_H U, e_i]_H)(\hat{e}_j + [D_H U, e_j]_H) - \delta_{ij} - [D_H^2 U e_i, e_j]_H) d\nu \]
\[ = \int_X f(\hat{e}_i \hat{e}_j - \delta_{ij}) d\nu - \int_X f[D_H^2 U e_i, e_j]_H d\nu + \int_X f[D_H U, e_i]_H [D_H U, e_j]_H d\nu \]
\[ + \int_X f(\hat{e}_i [D_H U, e_j]_H + \hat{e}_j [D_H U, e_i]_H) d\nu \]
\[ =: J_1^{i,j} + J_2^{i,j} + J_3^{i,j} + J_4^{i,j}, \quad (5.13) \]
for any \( i, j \in \mathbb{N} \), where \( \delta_{ij} \) is the Kronecker symbol and \([D_H \hat{e}_i, e_j]_H = \delta_{ij}\) follows from [7, Lemma 2.10.5]. Let us separately estimate the four terms in the last line of (5.13). From Remarks 2.3 and 2.4 it follows that \( J_1^{i,j} \) is related to the second Wiener chaos \( E_2 \); indeed,
\[ J_1^{i,j} = \int_X \Phi_{\alpha(i,j)} f e^{-U} d\mu, \quad i, j \in \mathbb{N}, \quad i \neq j, \]
where \( \alpha(i, j) \) is the multiindex which satisfies \(|\alpha| = 2\) and \( \alpha_i = \alpha_j = 1 \), and
\[ J_1^{i,i} = \sqrt{2} \int_X \Phi_{\alpha(i)} f e^{-U} d\mu, \quad i \in \mathbb{N}, \]
where \( \alpha(i) \) is the multiindex which satisfies \(|\alpha| = 2\) and \( \alpha_i = 2 \). Then,
\[ \sum_{i, j \in \mathbb{N}} \left| J_1^{i,j} \right|^2 = \sum_{i, j \in \mathbb{N}} \int_X f e^{-U}(\hat{e}_i \hat{e}_j - \delta_{i,j}) d\mu = 2 \| J_2(f e^{-U}) \|^2_{L^2(X,\mu)} \leq c_p \| f e^{-U} \|^2_{L^p(X,\mu)} \leq C_p \| f \|^2_{L^p(X,\nu)}, \quad (5.14) \]
where the last inequality can be obtained arguing as in (5.3) and \( C_p \) is a positive constant which only depends on \( p \) and \( U \).

As far as \( J_2^{i,j} \) is considered, we get
\[ \sum_{i, j \in \mathbb{N}} \left| J_2^{i,j} \right|^2 = \sum_{i, j \in \mathbb{N}} \left( \int_X f[D_H^2 U e_i, e_j]_H d\nu \right)^2 = \left( \int_X D_H^2 U f d\nu \right)^2_{\mathcal{H}_2(H)} \]
\[ \leq \left( \int_X \|D_H^2 U\|_{\mathcal{H}_2(H)} |f| d\nu \right)^2 \leq \left\| D_H^2 U \right\|_{\mathcal{H}_2(H)}^2 \left\| f \right\|^2_{L^p(X,\nu)}, \quad (5.15) \]
where in the last inequality we have applied the Hölder’s inequality with exponents $p$ and $p'$. From the assumptions on $U$, as for (5.5) we can prove that

$$\|\|D_H^2 U\|_{\mathcal{H}^1(U)}\|_{L^{p'}(X,\nu)} < +\infty,$$

(5.16)

since both $\|D_H^2 f\|_{\mathcal{H}^1(U)}$ and $e^{-U}$ belongs to $L^q(X,\mu)$ for any $q \in (1, +\infty)$. $J_{3,i}^j$ can be estimated as follows:

$$\sum_{i,j \in \mathbb{N}} \left| J_{3,i}^j \right|^2 = \sum_{i,j \in \mathbb{N}} \left| \int_X f(e_i[D_H U, e_j]_H + e_j[D_H U, e_i]_H) d\nu \right|^2 \leq 2 \sum_{i,j \in \mathbb{N}} \left| \int_X f e_i[D_H U, e_j]_H d\nu \right|^2 + 2 \sum_{i,j \in \mathbb{N}} \left| \int_X e_j[D_H U, e_i]_H d\nu \right|^2

= 4 \sum_{i \in \mathbb{N}} \sum_{j \in \mathbb{N}} \left| \int_X f e_i[D_H U, e_j] H^e^{-U} d\mu \right|^2. \quad (5.19)$$

Finally, we take into account $I_{4,i}^j$. We get

$$\sum_{i,j \in \mathbb{N}} \left| J_{4,i}^j \right|^2 = \sum_{i,j \in \mathbb{N}} \left| \int_X f e_i[D_H U, e_j]_H d\nu \right|^2 \leq 2 \sum_{i,j \in \mathbb{N}} \left| \int_X f e_i[D_H U, e_j]_H d\nu \right|^2 + 2 \sum_{i,j \in \mathbb{N}} \left| \int_X e_j[D_H U, e_i]_H d\nu \right|^2

= 4 \sum_{i \in \mathbb{N}} \sum_{j \in \mathbb{N}} \left| \int_X f e_i[D_H U, e_j] H^e^{-U} d\mu \right|^2. \quad (5.19)$$

We recall that for any $j \in \mathbb{N}$ the element

$$\int_X f e_i[D_H U, e_j] H^e^{-U} d\mu,$$

is the projection of $f(D_H U, e_j) H^e^{-U}$ on the subspace of $E_1$ generated by $e_i$, for any $i \in \mathbb{N}$. Hence, from Lemma 2.5 we have

$$\sum_{i \in \mathbb{N}} \left| \int_X f e_i[D_H U, e_j] H^e^{-U} d\mu \right|^2 = \| I_1(f[D_H U, e_j] H^e^{-U}) \|_{L^2(X,\nu)}^2 \leq C_p \| f(D_H U, e_j) H^e^{-U} \|_{L^q(X,\mu)}^2, \quad (5.20)$$

for any $j \in \mathbb{N}$, where $q = \min \left\{ \frac{1+p}{2}, \frac{q}{2} \right\}$ and $C_p$ is a positive constant which only depends on $p$. By repeating the same computations as in (5.9) with $k = 2$ we infer that

$$\sum_{j \in \mathbb{N}} \| f(D_H U, e_j) H^e^{-U} \|_{L^q(X,\mu)}^2 \leq \| f(D_H U) H^e^{-U} \|_{L^q(X,\mu)}^2. \quad (5.21)$$

We apply the Hölder’s inequality with $r = \frac{p}{q'}$ and we get

$$\| f(D_H U) H^e^{-U} \|_{L^q(X,\mu)}^2 \leq \| f \|_{L^p(X,\nu)}^2 \left( \int_X |D_H U|_{L^q(X,\mu)}^r e^{-r'((q'-\frac{2}{2})^U) d\mu} \right)^{2/r'}, \quad \leq C_p \| f \|_{L^p(X,\nu)}^2, \quad (5.22)$$

and the last inequality follows from the Hölder’s inequality with exponents $p$ and $p'$. Again, arguing as in (5.5) we infer that

$$\|\|D_H^2 U\|_{\mathcal{H}^1(U)}\|_{L^{p'}(X,\nu)} < +\infty.$$
where \( C_p \) is a positive constant which only depends on \( p \) and \( U \). By collecting (5.12)-(5.22) and noticing that
\[
\left( \sum_{i,j \in \mathbb{N}} \left| \int_X |D_H^2 f e_i, e_i|_H d\nu \right|^2 \right)^{1/2} \leq 2 \left( \sum_{i,j \in \mathbb{N}} \left( |J_1^{i,j}|^2 + |J_2^{i,j}|^2 + |J_3^{i,j}|^2 + |J_4^{i,j}|^2 \right) \right)^{1/2},
\]
we conclude that there exists a positive constant \( \tilde{C}_p \), which only depends on \( p \) and \( U \), such that
\[
\left( \sum_{i,j \in \mathbb{N}} \left| \int_X |D_H^2 f e_i, e_i|_H d\nu \right|^2 \right)^{1/2} \leq \tilde{C}_p \| f \|_{L^p(X,\nu)}.
\]
This gives the thesis for \( k = 3 \).

The cases \( k \geq 4 \) can be obtained arguing as for \( k = 3 \), simply iterating the integrations by parts as in (5.13) \( k - 1 \)-times and estimating the terms which arise as for \( k = 3 \). Computations are long but straightforward, and we left them to the reader.

6 Logarithmic Sobolev Inequality, hypercontractivity and exponential decay

In this section we provide other important results which involve vector-valued functions, and which have been already proved in the scalar case, both in finite dimension for more general operators (see [2, 5, 14]) and when \( X \) is a Hilbert space (see [4, Section 4]). We begin by proving a Logarithmic Sobolev Inequality for functions \( F \in C_b^1(X; V) \) which generalize the scalar case. Thanks to this result, we are able to show that the vector-valued semigroup \( \{ (T^V(t))_{t \geq 0} \} \) is hypercontractive. Finally, we show an exponential decay of \( (T^V(t))_{t \geq 0} \) in \( L^p(X,\nu; V) \) as \( t \to +\infty \), which refines the asymptotic behaviour (3.33).

Proposition 6.1. Let \( V \) be a separable Hilbert space. Then, for any \( p \in [1, +\infty) \) and any \( f \in \mathcal{S}'_b(X; V) \) we have
\[
\int_X |F|^p \log |F|^p \, d\nu \leq \| F \|_{L^p(X,\nu; V)}^p \log(\| F \|_{L^p(X,\nu; V)}) + \frac{p^2}{2} \int_X |F|^p \log(|F|_V^p) \, d\nu + \frac{p^2}{2} \int_X |D_H F|_V^2 \log(|F|_V^p) \, d\nu,
\]
where \( |D_H F|_V \) is seen as an element of \( H \).

Proof. The case \( V = \mathbb{R} \) can be proved repeating verbatim the proof of [4, Proposition 4.3]. Indeed, in this case we have \( |f|^{p-4} |D_H f|_H^2 = |f|^{p-2} |D_H f|_H^2 \). Hence, for any \( p \in [1, +\infty) \) and any \( f \in C_b^1(X) \) we get
\[
\int_X |f|^p \log |f|^p \, d\nu \leq \| f \|_{L^p(X,\nu; V)}^p \log(\| f \|_{L^p(X,\nu; V)}) + \frac{p^2}{2} \int_X |f|^{p-2} |D_H f|_V^2 \log(|f|_V^p) \, d\nu.
\]

Let \( V \) be a separable Hilbert space, and let
\[
F = \sum_{i=1}^n f_i v_i, \quad f_i \in \mathcal{S}'_b(X), \quad v_i \in V, \quad i = 1, \ldots, n,
\]
be such that \( \{v_1, \ldots, v_n\} \) are orthonormal vectors in \( V \). We assume that there exists a positive constant \( c < 1 \) such that \( c \leq |F|_V^2 \leq 1 \) on \( X \). This means that \( |F|_V \in C_b^1(X) \) and
\[
D_H |F|_V = |F|_V^{-1} |D_H F|_V,
\]
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where $|D_H F, F|_V$ is seen as an element of $H$. By applying (6.2) with $f$ replaced by $|F|_V$ we infer that

$$
\int_X |F|^p_V \log (|F|^p_V) \, d\nu \leq \|F\|^p_{L^p(X,\nu;V)} \log (\|F\|^p_{L^p(X,\nu;V)}) + \frac{p^2}{2} \int_X |F|^{p-4}_V \|D_H F, F\|^2_H \chi_{\{|F|_V \neq 0\}} \, d\nu,
$$

$$
= \|F\|^p_{L^p(X,\nu;V)} \log (\|F\|^p_{L^p(X,\nu;V)}) + \frac{p^2}{2} \int_X |F|^{p-4}_V \|D_H F, F\|^2_H \, d\nu.
$$

(6.4)

Hence, (6.1) is proved for $F \in \mathcal{F} \mathcal{C}^1_H(X; V)$ such that $|F|_V$ has positive lower bound. To prove the statement for an arbitrary $F \in \mathcal{F} \mathcal{C}^1_H(X; V)$ of the form (6.3), with $\{v_1, \ldots, v_n\}$ orthonormal vectors in $V$, it is enough to consider the sequence $(F_m) \subset \mathcal{F} \mathcal{C}^1_H(X; V)$ defined by

$$
F_m := \sum_{i=1}^n f_{i,m} v_i.
$$

Here, for any $m \in \mathbb{N}$ we have

$$
f_{i,m} := (n + \|F\|_\infty)^{-1} \sqrt{f_i^2 + m^{-1}}, \quad i = 1, \ldots, n.
$$

Clearly, we have

$$
\frac{\sqrt{n}}{\sqrt{m(n + \|F\|_\infty)}} \leq |F_m|_V \leq 1, \quad m \in \mathbb{N}.
$$

From (6.4) for any $m \in \mathbb{N}$ we get

$$
\int_X |F_m|^p_V \log (|F_m|^p_V) \, d\nu \leq \|F_m\|^p_{L^p(X,\nu;V)} \log (\|F_m\|^p_{L^p(X,\nu;V)}) + \frac{p^2}{2} \int_X |F_m|^{p-4}_V \|D_H F_m, F_m\|^2_H \, d\nu,
$$

(6.5)

for any $m \in \mathbb{N}$. As $m \to +\infty$, the left-hand side of (6.5) converges to

$$
(n + \|F\|_\infty)^{-p} \int_X |F|^p_V \log (|F|^p_V) \, d\nu,
$$

(6.6)

and the first addend in the right-hand side of (6.5) converges to

$$
(n + \|F\|_\infty)^{-p} \|F\|^p_{L^p(X,\nu;V)} \log \left( \frac{\|F\|^p_{L^p(X,\nu;V)}}{(n + \|F\|_\infty)^p} \right).
$$

(6.7)

Finally, we notice that

$$
D_H F_m = \frac{1}{n + \|F\|_\infty} \sum_{i=1}^n \frac{f_i}{\sqrt{f_i^2 + m^{-1}}} D_H f_i v_i, \quad m \in \mathbb{N}.
$$

Then,

$$
[D_H F_m, F_m]_V = \frac{1}{(n + \|F\|_\infty)^2} \sum_{i=1}^n f_i D_H f_i = \frac{1}{(n + \|F\|_\infty)^2} \|D_H F, F\|_V, \quad m \in \mathbb{N}.
$$

The monotone convergence theorem for $p \in [1, 2)$, and the dominated convergence theorem otherwise, imply that the second addend in the right-hand side of (6.5) converges to

$$
\frac{p^2}{2} (n + \|F\|_\infty)^{-p} \int_X |F|^{p-4}_V \|D_H F, F\|^2_H \chi_{\{|F|_V \neq 0\}} \, d\nu,
$$

(6.8)

as $m \to +\infty$. The thesis follows letting $m \to +\infty$ in (6.5) and taking into account (6.6)-(6.8). \qed
Proposition 6.2. Let $V$ be a separable Hilbert space, let $t > 0$ and let $q \in (1, +\infty)$. Then, for any $p \leq 1 + (q - 1)e^{2t}$ the operator $T^V(t)$ is a contraction from $L^p(X, \nu; V)$ into $L^p(X, \nu; V)$, i.e.,

$$||T^V(t)F||_{L^p(X, \nu; V)} \leq ||F||_{L^p(X, \nu; V)}, \quad F \in L^p(X, \nu; V).$$

Proof. At first we prove the thesis for $F \in \mathcal{F}C_b^\infty(X; V)$ with

$$F = \sum_{i=1}^n f_i v_i, \quad f_i \in \mathcal{F}C_b^\infty(X), \quad v_i \in V, \quad i = 1, \ldots, n,$$

such that $\{v_1, \ldots, v_n\}$ are orthonormal vectors in $V$ and $|F|_V$ has positive lower bound. Let $q \in (1, +\infty)$, let $t > 0$ and let us set $p(t) = 1 + (q - 1)e^{2t}$. For any $s \in [0, t]$ we set

$$G(s) := (K(s))^{1/p(s)}, \quad K(s) := \int_X |T^V(s)F|^p(s)/V d\nu.$$

We recall that

$$T^V(s)F = \sum_{i=1}^n T(s)f_i v_i,$$

which means that $T^V(s)F \in W^{1,2}(X, \nu; V) \cap L^\infty(V; \nu; V)$ and that

$$\frac{d}{ds}(T^V(s)F) = \frac{d}{ds} \sum_{i=1}^n T(s)f_i v_i = \sum_{i=1}^n LT(s)f_i v_i = L^V T^V(s)F, \quad \nu\text{-a.e. in } X.$$

By differentiating $K(s)$ with respect to $s$ we get

$$K'(s) = p'(s) \int_X |T^V(s)F|^p(s)/V \ln(|T^V(s)F|_V) d\nu + p(s) \int_X |T^V(s)F|^p(s)-2[T^V(s)F, L^V T^V(s)F]_V d\nu.$$

(6.9)

Let us integrate by parts the second addend in the right-hand side above. From (2.30) it follows that

$$\int_X |T^V(s)F|^p(s)-2[T^V(s)F, L^V T^V(s)F]_V d\nu \begin{align*}
= & \sum_{i=1}^n \int_X |T^V(s)F|^p(s)-2(LT(s)f_i)(T(s)f_i) d\nu \\
= & - (p(s) - 2) \sum_{i=1}^n \int_X |T^V(s)F|^p(s)-4 \left[ [D_H T^V(s)F, T^V(s)F]_V, (T(s)f_i)(D_H T(s)f_i) \right]_H d\nu \\
& \quad - \sum_{i=1}^n \int_X |T^V(s)F|^p(s)-2 |D_H T(s)f_i|_H^2 d\nu \\
= & - (p(s) - 2) \int_X |T^V(s)F|^p(s)-4 \left[ [D_H T^V(s)F, T^V(s)F]_V \right]^2_H d\nu \\
& \quad - \int_X |T^V(s)F|^p(s)-2 |D_H T^V(s)F|_H^2 \otimes V d\nu.
\end{align*}$$

(6.10)

Since $||D_H T^V(s)F, T^V(s)F|_V^2_H \leq |D_H T^V(s)F|_H^2 \otimes V |T^V(s)F|_V^2$, it follows that

$$- \int_X |T^V(s)F|^p(s)-2 |D_H T^V(s)F|_H^2 \otimes V d\nu = - \int_X |T^V(s)F|^p(s)-4 |D_H T^V(s)F|_H^2 \otimes V |T^V(s)F|_V^2 d\nu$$

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\[ \leq - \int_X |T^V(s)F|^{|p(s)|^{-4}} \| [\overline{D_H} T^V(s)F, T^V(s)F]_V \|^2_H d\nu. \] (6.11)

From (6.10) and (6.11) we infer that
\[ \int_X |T^V(s)F|^{p(s)-2}[T^V(s)F, L^V T^V(s)F]_V d\nu \leq -(p(s) - 1) \int_X |T^V(s)F|^{|p(s)|^{-4}} \| [\overline{D_H} T^V(s)F, T^V(s)F]_V \|^2_H d\nu. \] (6.12)

Differentiating \( G \) with respect to \( s \) and taking into account (6.9) and (6.12) we get
\[
G'(s) = G(s) \left( - \frac{p'(s)}{(p(s))^2} \ln(K(s)) + \frac{1}{p(s)} \frac{K'(s)}{K(s)} \right)
\leq G(s) \left( \frac{p'(s)}{(p(s))^2} K(s) \right) \left( - \ln(K(s)) + \int_X |T^V(s)F|^{|p(s)|^{-4}} \ln(|T^V(s)F|^{p(s)}) d\nu \right)
- \frac{p(s) - 1}{K(s)} \int_X |T^V(s)F|^{|p(s)|^{-4}} \| [\overline{D_H} T^V(s)F, T^V(s)F]_V \|^2_H d\nu.
\]

From (6.1) it follows that
\[
- \ln(K(s)) + \int_X |T^V(s)F|^{|p(s)|^{-4}} \ln(|T^V(s)F|^{p(s)}) d\nu \leq \frac{(p(s))^2}{2} \int_X |T^V(s)F|^{|p(s)|^{-4}} \| [\overline{D_H} T^V(s)F, T^V(s)F]_V \|^2_H d\nu.
\]

Therefore,
\[
G'(s) \leq G(s) \left( \frac{p'(s)}{2} - (p(s) - 1) \right) \int_X |T^V(s)F|^{|p(s)|^{-4}} \| [\overline{D_H} T^V(s)F, T^V(s)F]_V \|^2_H d\nu = 0,
\]

since \( p'(s) = 2(p(s) - 1) \). It follows that \( G \) is a decreasing function, which means that \( G(t) \leq G(0) \). This gives
\[
\| T^V(t)F \|_{L^p(X,\nu;V)} \leq \| F \|_{L^p(X,\nu;V)}.
\]

Let \( p \in (1, p(t)) \). Since \( \nu \) is a probability measure, it follows that
\[
\| T^V(t)F \|_{L^p(X,\nu;V)} \leq \| T^V(t)F \|_{L^{p(t)}(X,\nu;V)} \leq \| F \|_{L^p(X,\nu;V)},
\]

which gives the thesis for any \( F \in \mathcal{F}C_b^\infty(X;V) \) such that \( |F|_V \) has positive lower bound. For a general \( F \in \mathcal{F}C_b^\infty(X;V) \) we can argue as in the second part of the proof of Proposition 6.1 by approximating \( F \) by means of \( (F_m) \subset \mathcal{F}C_b^\infty(X;V) \), and from the density of \( \mathcal{F}C_b^\infty(X) \) in \( L^q(X,\nu;V) \) we conclude. \( \square \)

**Proposition 6.3.** Let \( V \) be a separable Hilbert space. Then:

(i) For any \( p \in [2, +\infty) \), any \( t > 1 \) and any \( F \in L^p(X,\nu;V) \), we have
\[
\| T^V(t)F - \nu(F) \|_{L^p(X,\nu;V)} \leq c_p e^{-t} \| F \|_{L^p(X,\nu;V)},
\]

where \( c_p \) is a positive constant given by
\[
c_p := e^{\sqrt{\frac{p-1}{2}}}. \] (6.13)
(ii) For any $p \in [1, 2)$, any $t > 2$ and any $F \in L^p(X, \nu; V)$ we have

$$\|T^V(t)F - \nu(F)\|_{L^p(X, \nu; V)} \leq \frac{c^2}{2} e^{-t} \|F\|_{L^p(X, \nu; V)}.$$  

Proof. Let $V$ be a separable Hilbert space. By density, we can limit ourselves to prove the statement for $F \in \mathcal{F}C^\infty_b(X; V)$.

(i). Let $p \in [2, +\infty)$ and let $t > 1$. Let $F \in \mathcal{F}C^\infty_b(X; V)$, and let us set $G := T^V(t)F - \nu(F) \in \mathcal{F}C^\infty_b(X)$. Then, from (4.3), (4.4) and (4.5) it follows that

$$\int_X |G|^p \nu \, dv = \int_t^\infty \left( \int_X |\overline{D_H T^V(s)}F, \overline{D_H G^*}|_{H \otimes V} \, dv \right) \, ds,$$

where $G^* := |G|^{p-2}G$. Since $T^V(t)F \in W^{1, p}(X, \nu; V)$, from (2.16) we infer that $G^* \in W^{1, \frac{p}{p-1}}(X, \nu)$ and

$$\overline{D_H G^*} = (p - 2)|G|^{\frac{p-4}{2}}|\overline{D_H T^V(t)}F, G|_{V} \otimes G + |G|^{\frac{p-4}{2}}|\overline{D_H T^V(t)}F, G - \nu(F)|_{V}.$$  

Arguing as in (6.4), from (6.14) and (6.15) we deduce that

$$\int_X |G|^p \nu \, dv \leq (p - 1) \int_t^\infty \left( \int_X |\overline{D_H T^V(s)}F|_{H \otimes V}|\overline{D_H T^V(t)}F|_{H \otimes V}|G|^{\frac{p-2}{p}} \, dv \right) \, ds.$$  

Let $p > 2$. We apply the generalized Hölder’s inequality with exponents $p, p$ and $\frac{p}{p-2}$, which gives

$$\int_X |G|^p \nu \, dv \leq (p - 1) \left( \int_t^\infty \|\overline{D_H T^V(s)}F\|_{L^p(X, \nu; H \otimes V)} \, ds \right) \left( \int_t^\infty \|\overline{D_H T^V(t)}F\|_{L^p(X, \nu; H \otimes V)} |G|^{\frac{p-2}{p}} \, dv \right).$$

Recalling (3.39) and that $t > 1$, it follows that

$$\int_X |G|^p \nu \, dv \leq \frac{(p - 1)e^{-t+1}}{2} \int_t^\infty e^{-s+1} ds \|F\|_{L^p(X, \nu; V)}^2 |G|^{\frac{p-2}{p}} \|G\|_{L^p(X, \nu; V)}^{-2}.$$  

Dividing both the sides of (6.16) by $\|G\|_{L^p(X, \nu; V)}^{\frac{p-2}{p}}$ we get

$$\|T(t)F - \nu(F)\|_{L^p(X, \nu; V)} \leq c_p e^{-t} \|F\|_{L^p(X, \nu; V)},$$

with $c_p$ is the constant in (6.13). The case $p = 2$ follows noticing that $G = G^*$ and $\overline{D_H G^*} = \overline{D_H F}$, applying the Hölder’s inequality with $p = 2$ in (6.14) and concluding as for $p > 2$ by means of (3.39).

(ii). Let $p \in (1, 2)$, let $t > 2$ and let $F \in \mathcal{F}C^\infty_b(X; V)$. We have

$$\|T^V(t)F - \nu(F)\|_{L^p(X, \nu; V)} = \sup_{G \in \mathcal{F}C^\infty_b(X; V), \|G\|_{L^p(X, \nu; V)} \leq 1} \int_X |F - \nu(F), G|_{V} \nu \, dv.$$  

Let $G \in \mathcal{F}C^\infty_b(X; V)$ with $\|G\|_{L^p(X, \nu; V)} \leq 1$. From the semigroup property of $(T^V(t))_{t \geq 0}$, Lemma 2.25(i) and (3.33) it follows that

$$\int_X |T^V(t)F - \nu(F), G|_{V} \nu \, dv = \lim_{r \to +\infty} \int_X |T^V(t)F - T^V(0)F, G|_{V} \nu \, dv$$

$$= \lim_{r \to +\infty} \int_X |T^V(t)F, T^V(t/2)G - T^V(r - t/2)G|_{V} \nu \, dv.$$  

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Arguing as in (4.5), for any \( r > t \) we have

\[
\int_X |T^V(t/2)F, T^V(t/2)G - T^V(r - t/2)G|_V d\nu = \int_{t/2}^{r-t/2} \left( \int_X |D_H T^V(t/2)F, D_H T^V(s)G|_{H@V} d\nu \right) ds \\
\leq \left( \int_X |D_H T^V(t/2)F|_{H@V}^p d\nu \right)^{1/p} \int_{t/2}^{r-t/2} \left( \int_X |D_H T^V(s)G|_{H@V}^{p'} d\nu \right)^{1/p'} ds.
\]

By applying (3.39) to \( \int_X |D_H T^V(s)G|_{H@V}^{p'} d\nu \) and recalling that \( \|G\|_{L^p(X,\nu;V)} \leq 1 \) and that \( s > t/2 > 1 \), we infer that

\[
\left( \int_X |D_H T^V(s)G|_{H@V}^{p'} d\nu \right)^{1/p'} \leq \frac{e^{-s+1}}{\sqrt{2}}, \quad s \in (t/2, r - t/2),
\]

which gives

\[
\int_{t/2}^{r-t/2} \left( \int_X |D_H T^V(s)G|_{H@V}^{p'} d\nu \right)^{1/p'} ds \leq \frac{1}{\sqrt{2}} \int_{t/2}^{r-t/2} e^{-s+1} ds = \frac{e}{\sqrt{2}} \left( e^{-t/2} - e^{-r+t/2} \right),
\]

for any \( r > t \). Further, since \( t > 2 \) estimate (3.39) gives

\[
\left( \int_X |D_H T^V(t/2)F|_{H@V}^p d\nu \right)^{1/p} \leq \frac{e^{-t/2+1}}{\sqrt{2}} ||F||_{L^p(X,\nu;V)}.
\]

Collecting (6.17)-(6.21) we get

\[
\|T^V(t)F - \nu(F)\|_{L^p(X,\nu;V)} \leq \frac{e^2}{2} e^{-t/2} ||F||_{L^p(X,\nu;V)},
\]

which gives the thesis for \( p \in (1, 2) \). Since the constant in (6.22) does not depend on \( p \), letting \( p \to 1^+ \) the thesis follows also for \( p = 1 \). 

\[
\square
\]

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