Fractional Besov Trace/Extension-Type Inequalities via the Caffarelli–Silvestre Extension

Pengtao Li¹ · Rui Hu² · Zhichun Zhai²

Received: 21 December 2021 / Accepted: 14 June 2022 / Published online: 12 July 2022
© Mathematica Josephina, Inc. 2022

Abstract
Let \( u(\cdot, \cdot) \) be the Caffarelli–Silvestre extension of \( f \). The first goal of this article is to establish the fractional trace-type inequalities involving the Caffarelli–Silvestre extension \( u(\cdot, \cdot) \) of \( f \). In doing so, firstly, we establish the fractional Sobolev/logarithmic Sobolev/Hardy trace inequalities in terms of \( \nabla_{(x,t)} u(x, t) \). Then, we prove the fractional anisotropic Sobolev/logarithmic Sobolev/Hardy trace inequalities in terms of \( \partial_t u(x, t) \) or \( (-\Delta)^{-\gamma/2} u(x, t) \) only. Moreover, based on an estimate of the Fourier transform of the Caffarelli–Silvestre extension kernel and the sharp affine weighted \( L^p \) Sobolev inequality, we prove that the \( \dot{H}^{-\beta/2}(\mathbb{R}^n) \) norm of \( f(x) \) can be controlled by the product of the weighted \( L^p \)-affine energy and the weighted \( L^p \)-norm of \( \partial_t u(x, t) \).

The second goal of this article is to characterize non-negative measures \( \mu \) on \( \mathbb{R}^{n+1}_+ \) such that the embeddings

\[
\| u(\cdot, \cdot) \|_{\dot{H}^{q_0,p_0}(\mathbb{R}^{n+1})} \lesssim \| f \|_{\dot{H}^{r,q}(\mathbb{R}^{n+1})}
\]

hold for some \( p_0 \) and \( q_0 \) depending on \( p \) and \( q \) which are classified in three different cases: (1) \( p = q \in (n/(n + \beta), 1] \); (2) \( (p, q) \in (1, n/\beta) \times (1, \infty) \); (3) \( (p, q) \in (1, n/\beta) \times \{ \infty \} \). For case (1), the embeddings can be characterized in terms of an analytic condition of the variational capacity minimizing function, the iso-capacitary inequality of open balls, and other weak-type inequalities. For cases (2) and (3), the embeddings are characterized by the iso-capacitary inequality for fractional Besov capacity of open sets.

Zhichun Zhai
zhai2@macewan.ca

Pengtao Li
ptli@qdu.edu.cn

Rui Hu
hur3@macewan.ca

1 School of Mathematics and Statistics, Qingdao University, Qingdao, Shandong 266071, China
2 Department of Mathematics and Statistics, MacEwan University, Edmonton, AB T5J2P2, Canada
Keywords Sobolev inequality · Sobolev logarithmic inequality · Hardy inequality · Affine energy · Carleson embedding · Fractional Laplacian

Mathematics Subject Classification Primary 31 · 42 · 26D10 · 46E35 · 30H25

Contents

1 Introduction ............................................. 2
2 Preliminaries and Basic Lemmas .................................. 7
3 Fractional Trace Inequalities via the Caffarelli–Silvestre Extension ................ 14
  3.1 Fractional Trace Inequalities Involving ∇(x,t)u(x,t) ...................... 14
  3.2 Fractional Anisotropic Trace Inequalities Involving ∂tu(x,t) ............. 16
  3.3 Fractional Anisotropic Trace Inequalities Involving (−Δ)γ/2u(x,t) ......... 17
  3.4 Affine Fractional Poisson Trace Inequality ........................... 18
  3.5 Remarks on the General Case p ≥ 1 ................................ 19
4 Carleson Embeddings of Besov Spaces via the Caffarelli–Silvestre Extension ........... 21
  4.1 Case (1): p = q ∈ (n/(n + β), 1] .................................... 21
  4.2 Case (2): (p, q) ∈ (1, n/β) × (1, ∞) ................................ 27
  4.3 Case (3): (p, q) ∈ (1, n/β) × {∞} .................................. 28
References ................................................ 29

1 Introduction

Sobolev-type inequalities are extremely important in analysis and geometry, and have been applied in studying of partial differential equations. In this article, we will establish factional Besov trace and extension-type inequalities involving fractional Laplacian and the Caffarelli–Silvestre extension.

For $s \in (0, 2)$, the fractional Laplace operator $(-\Delta)^{s/2}$ in $\mathbb{R}^n$ is defined on the Schwartz class through the Fourier transform as

$$[(−\Delta)^{s/2} f](\xi) = |\xi|^s \hat{f}(\xi),$$

or via the Riesz potential as

$$(-\Delta)^{s/2} f(x) = \frac{s^s \Gamma((n + s)/2)}{2 \Gamma(1 - s/2) \pi^{n/2}} \text{P.V.} \int_{\mathbb{R}^n} \frac{f(x) - f(y)}{|x - y|^{n+s}} \, dy,$$

where $\hat{f}(\xi) = \int_{\mathbb{R}^n} e^{-ix \cdot \xi} f(x) \, dx$ is the Fourier transform of $f$, P.V. denotes the Cauchy principle value integral. The case $s = 2$ becomes the standard local Laplacian. The general case can be written in a local way by the so-called Caffarelli–Silvestre extension introduced by Caffarelli and Silvestre [8].

Let $f$ be a regular function in $\mathbb{R}^n$. We say that $u = u(\cdot, \cdot)$ is the Caffarelli–Silvestre extension of $f$ to the upper half-space $\mathbb{R}^{n+1}_+ := \mathbb{R}^n \times (0, \infty)$, if $u$ is a solution to the
problem

\[
\begin{cases}
\text{div}(t^{1-s}\nabla u) = 0, & \text{in } \mathbb{R}^{n+1}_+; \\
u = f, & \text{on } \mathbb{R}^n \times \{t = 0\}.
\end{cases}
\]  

(1.1)

The Caffarelli–Silvestre extension is well defined for smooth functions through the fractional Poisson kernel:

\[
p_t^s(x) := \frac{c(n, s)t^s}{|x|^2 + t^2(n+s)/2}, \quad c(n, s) = \frac{\Gamma((n + s)/2)}{\pi^{n/2} \Gamma(s/2)}
\]

as follows

\[
u(x, t) = p_t^s * f(x) = c(n, s)t^s \int_{\mathbb{R}^n} \frac{f(y)}{|x - y|^2 + t^2(n+s)/2} dy,
\]

where \(f * g\) means the convolution of \(f\) and \(g\). Here, the constant \(c(n, s)\) is the normalized constant such that \(\int_{\mathbb{R}^n} p_t^s(x)dx = 1\). Caffarelli and Silvestre [8] proved that, with \(c_s = \frac{\Gamma(s/2)}{2^{1-s}\Gamma(1-s/2)}\),

\[
(-\Delta)^{s/2} f(x) = -c_s \lim_{t \to 0^+} t^{1-s} \frac{\partial u}{\partial t}(x, t).
\]  

(1.2)

In the proof of (1.2), the key is the following identity

\[
\frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} |\xi|^{s} |\hat{f}(\xi)|^2 d\xi = \frac{2^{1-s} \Gamma(1 - s/2)}{\Gamma(s/2)} \int_{\mathbb{R}^{n+1}_+} |\nabla(x, t)u(x, t)|^2 t^{1-s} dx dt,
\]  

(1.3)

which means

\[
\|f\|_{\dot{B}^{s/2}_\infty} \approx \|\nabla(x, t)u(\cdot, \cdot)\|_{L^2(\mathbb{R}^{n+1}_+, t^{(1-s)/2} dx dt)}.
\]

To break this equivalence, in this article, our main purpose is to identify the space \(X\) of \(f\) on \(\mathbb{R}^n\) such that

\[
\|f\|_{X} \lesssim \|\nabla(x, t)u(\cdot, \cdot)\|_{L^2(\mathbb{R}^{n+1}_+, t^{(1-s)/2} dx dt)},
\]

and to identify the space \(Y\) of \(u(\cdot, \cdot)\) on \(\mathbb{R}^{n+1}_+\) such that

\[
\|u\|_{Y} \lesssim \|f\|_{\dot{A}^{p,q}_{\beta}(\mathbb{R}^n)},
\]

where \(\| \cdot \|\) denotes the norms of homogeneous Besov spaces which are defined in Definition 1.1 below. Here and henceforth, \(A \lesssim B\) means that \(A \leq CB\) for a constant.
C. $A \approx B$ means that $A \lesssim B$ and $B \lesssim A$. Similarly, one writes $V \gtrsim U$ for $V \geq cU$ with $c$ being a constant.

Equality (1.3) allows us to treat the fractional Sobolev/logarithmic Sobolev/Hardy inequalities as the fractional Sobolev/logarithmic Sobolev/Hardy trace inequalities. For any $f \in \dot{H}^{s/2}(\mathbb{R}^n)$ and its Caffarelli–Silvestre extension $u(x, t) = p_s^t * f(x) \forall (x, t) \in \mathbb{R}^{n+1}_+$, the fractional Sobolev trace inequality holds

$$
\left( \int_{\mathbb{R}^n} |f(x)|^{2n/(n-s)} \, dx \right)^{(n-s)/n} \lesssim \int_{\mathbb{R}^{n+1}_+} |\nabla_{(x,t)} u(x,t)|^2 \, t^{1-s} \, dx \, dt. \tag{1.4}
$$

Moreover, if $\int_{\mathbb{R}^n} |f(x)|^2 \, dx = 1$, the following fractional logarithmic Sobolev inequality holds:

$$
\exp \left( \frac{s}{n} \int_{\mathbb{R}^n} |f(x)|^2 \ln(|f(x)|^2) \, dx \right) \lesssim \int_{\mathbb{R}^{n+1}_+} |\nabla_{(x,t)} u(x,t)|^2 \, t^{1-s} \, dx \, dt. \tag{1.5}
$$

There also holds the fractional Hardy inequality (or the Kato inequality)

$$
\int_{\mathbb{R}^n} \frac{|f(x)|^2}{|x|^s} \, dx \lesssim \int_{\mathbb{R}^{n+1}_+} |\nabla_{(x,t)} u(x,t)|^2 \, t^{1-s} \, dx \, dt. \tag{1.6}
$$

Xiao [30] proved (1.4)–(1.5) for the Poisson extension. For the Caffarelli–Silvestre extension, Brändle et al. [6] proved (1.4). Inequality (1.5) was proved by Nguyen [27]. Inequalities similar to (1.6) have been studied in [5, 13, 17, 19, 27, 33, 35].

In (1.3), the order of the fractional derivative on left hand side is the half the order $s$ of the Caffarelli–Silvestre extension. In this article, we firstly observe that

$$
\int_{\mathbb{R}^{n+1}_+} |\nabla_{(x,t)} u(x,t)|^2 \, t^{1-\beta} \, dx \, dt \approx \int_{\mathbb{R}^n} |\xi|^{\beta} |\hat{f}(\xi)|^2 \, d\xi, \quad \beta \in (0, 2s), \tag{1.7}
$$

which allows us to establish inequalities similar to (1.4), (1.5), and (1.6) for $f \in \dot{H}^{\beta/2}(\mathbb{R}^n)$ with $\beta \in (0, 2s)$. Moreover, (1.7) is true when $\nabla_{(x,t)} u(x,t)$ is replaced by either $\partial_x u(x,t)$ or $\nabla_x u(x,t)$ (more general $(-\Delta)^{-\gamma/2}$). Based on these observations, the first goal of this article is to show that $\int_{\mathbb{R}^{n+1}_+} |\nabla_{(x,t)} u(x,t)|^2 \, t^{1-s} \, dx \, dt$ in (1.4), (1.5), and (1.6) can be replaced by

$$
\int_{\mathbb{R}^{n+1}_+} |\nabla_{(x,t)} u(x,t)|^2 \, t^{1-\beta} \, dx \, dt, \quad \int_{\mathbb{R}^{n+1}_+} \left| \frac{\partial u(x,t)}{\partial t} \right|^2 \, t^{1-\beta} \, dx \, dt,
$$

or

$$
\int_{\mathbb{R}^{n+1}_+} |(-\Delta)^{\gamma/2} u(x,t)|^2 \, t^{2\gamma-\beta-1} \, dx \, dt.
$$
Let \( \omega : \mathbb{R}_{+}^{n+1} \rightarrow \mathbb{R}_{+} \) be a positive measurable function. Denote by \( L^p(\mathbb{R}_{+}^{n+1}, \omega) \) the weighted Lebesgue space of all measurable functions \( g : \mathbb{R}_{+}^{n+1} \rightarrow \mathbb{R} \) with
\[
\|g\|_{L^p(\mathbb{R}_{+}^{n+1}, \omega)} := \left( \int_{\mathbb{R}_{+}^{n+1}} |g(x, t)|^p \omega(x, t) \, dx \, dt \right)^{1/p} < \infty.
\]

In [23], Lombardi and Xiao established the affine non-sharp Poisson trace inequality for \( \dot{H}^{-\beta/2}(\mathbb{R}^n) \) by using the weighted \( L^p \)-affine energy
\[
\mathcal{E}(g, \sigma) = c_{n, p} \left( \int_{\mathbb{S}^{n-1}} \|\nabla \xi g\|_{L^p(\mathbb{R}_{+}^{n+1}, \sigma)}^{-n} \, d\xi \right)^{-1/n},
\]
where \( c_{n, p} = (n \omega_n)^{1/n} \left( \frac{n \omega_n \nu_{n-1}}{2 \omega_n + n - 2} \right)^{1/p} \). We will extend the affine non-sharp Poisson trace inequality for \( \dot{H}^{-\beta/2}(\mathbb{R}^n) \) to the Caffarelli–Silvestre extension. The \( L^p \) affine energy has been applied to establish the affine Sobolev inequalities (cf. [24, 37, 38]) and the affine version of the Pólya–Szegő principle (cf. [9, 15]).

Let \( C_0^\infty(\mathbb{R}^n) \) stand for the set of all infinitely differentiable function with compact support. The homogeneous Besov space \( \dot{A}^p,q(\mathbb{R}^n) \) is defined as the completion of all \( C_0^\infty(\mathbb{R}^n) \) functions with \( \|f\|_{\dot{A}^p,q(\mathbb{R}^n)} < \infty \), where the norm \( \|f\|_{\dot{A}^p,q(\mathbb{R}^n)} \) is defined as follows.

**Definition 1.1** (i) If \( (\beta, p, q) \in (0, \infty) \times (0, \infty) \times (0, \infty) \), then
\[
\|f\|_{\dot{A}^p,q(\mathbb{R}^n)} := \left( \int_{\mathbb{R}^n} \|\Delta_h^k f\|_{L^p(\mathbb{R}^n)} \|h|^{-(n+\beta)q} \, dh \right)^{1/q}.
\]

(ii) If \( (\beta, p, q) \in (0, \infty) \times (0, \infty) \times \{\infty\} \), then
\[
\|f\|_{\dot{A}^p,q(\mathbb{R}^n)} = \sup_{\Delta_h, f \in \dot{A}^p,q(\mathbb{R}^n)} \|\Delta_h^k f\|^q_{L^p(\mathbb{R}^n)} \|h|^{-\beta}.
\]

Here, \( k = 1 + \lfloor \beta \rfloor \), \( \beta = [\beta] + \{\beta\} \) with \( [\beta] \in \mathbb{Z}_+ \), \( \{\beta\} \in (0, 1) \) and
\[
\Delta_h^k f = \Delta_h^k \Delta_h^{k-1} f, \quad \Delta_h^1 = f(x + h) - f(x) \quad \forall x \in \mathbb{R}^n.
\]

The second goal of this article is to study the Carleson-type embedding for \( \dot{A}^p,q(\mathbb{R}^n) \) via the Caffarelli–Silvestre extension. We will characterize the following embedding:
\[
\|u\|_{L^p_{\mu q}(\mathbb{R}_{+}^{n+1})} \lesssim \|f\|_{\dot{A}^p,q(\mathbb{R}^n)} \tag{1.8}
\]
for a non-negative Borel measure/outer measure by using the Besov capacities \( C^p,q(\cdot) \), when \( \beta \in (0, 1) \) or \( \beta \in (0, n) \), \( p \) and \( q \) satisfy one of the following three conditions:
\[
p = q \in (n/(n + \beta), 1] \quad (p, q) \in (1, n/\beta) \times (1, \infty) \quad (p, q) \in (1, n/\beta) \times \{\infty\}.
\]
For $0 < p, q < \infty$ and a non-negative Radon measure $\mu$ on $\mathbb{R}^{n+1}_+$, $L^{q,p}_\mu(\mathbb{R}^{n+1}_+)$ and $L^q_\mu(\mathbb{R}^{n+1}_+)$ denote the Lorentz space and the Lebesgue space of all functions $g(\cdot, \cdot)$ on $\mathbb{R}^{n+1}_+$, respectively. Moreover, $L^{q,\infty}_\mu(\mathbb{R}^{n+1}_+)$ denotes the set of all $\mu$-measurable functions $g(\cdot, \cdot)$ on $\mathbb{R}^{n+1}_+$ with

$$\|g(\cdot, \cdot)\|_{L^{q,\infty}_\mu(\mathbb{R}^{n+1}_+)} := \sup_{s > 0} \mu\left(\left\{(x, t) \in \mathbb{R}^{n+1}_+ : |g(x, t)| > s\right\}\right)^{1/q} < \infty,$$

respectively. Moreover, $L^{q,\infty}_\mu(\mathbb{R}^{n+1}_+)$ denotes the set of all $\mu$-measurable functions $g(\cdot, \cdot)$ on $\mathbb{R}^{n+1}_+$ with

$$\|g(\cdot, \cdot)\|_{L^{q,\infty}_\mu(\mathbb{R}^{n+1}_+)} := \sup_{s > 0} \mu\left(\left\{(x, t) \in \mathbb{R}^{n+1}_+ : |g(x, t)| > s\right\}\right)^{1/q} < \infty.$$

Let $C(\mathbb{R}^n)$ be the class of all continuous functions on $\mathbb{R}^n$. The Besov capacities $C^{p,q}_\beta(\cdot)$ have been studied in [1, 3] for $q < \infty$ and in [26] for $q = \infty$.

**Definition 1.2** (i) Let $(\beta, p, q) \in (0, \infty) \times (0, \infty) \times (0, \infty)$. For a compact set $K \subset \mathbb{R}^n$, the Besov capacity $C^{p,q}_\beta(E)$ is defined as

$$C^{p,q}_\beta(K) := \inf \left\{ \| f \|_{L^{p,q}_\beta(\mathbb{R}^n)} : f \in C^\infty_0(\mathbb{R}^n) \text{ and } f \geq 1_K \right\}$$

and for any set $E \subset \mathbb{R}^n$, one defines

$$C^{p,q}_\beta(E) := \inf_{\text{open } O \supseteq E} \sup_{\text{compact } K \subseteq O} \left\{ C^{p,q}_\beta(K) \right\},$$

where $1_E$ denotes the characteristic function of $E$.

(ii) Let $(\beta, p) \in (0, 1) \times (1, \infty)$. For a set $E \subset \mathbb{R}^n$, the $\infty$-Besov capacity $C^{p,\infty}_\beta(E)$ is defined as

$$C^{p,\infty}_\beta(E) := \inf \left\{ \| f \|_{L^{p,\infty}_\beta(\mathbb{R}^n)} : f \in L^{p,\infty}_\beta(\mathbb{R}^n) \cap C(\mathbb{R}^n) \text{ and } f \geq 1_N \text{ on a neighborhood } N \text{ of } E \right\}.$$

For any open set $O \subset \mathbb{R}^n$, the tent space $T(O)$ based on $O$ is defined as

$$T(O) = \left\{ (x, t) \in \mathbb{R}^{n+1}_+ : B(x, t) \subset O \right\}.$$
The variational capacity minimizing function associated with both $C^p_{\beta,q}$ and a non-negative measure $\mu$ on $\mathbb{R}^{n+1}_+$ is defined by

$$c^\beta_{p,q}(\mu; \lambda) := \inf \left\{ C^p_{\beta,q}(O) : \text{open } O \subseteq \mathbb{R}^n, \mu(T(O)) > \lambda \right\}$$

for $\lambda > 0$.

The study of (1.8) is mainly motivated by the work on similar embeddings via the classical Poisson kernel and the classical/fractional heat kernels. Adams and Xiao [3] proved embeddings similar to (1.8) via the classical Poisson kernel. Xiao [31, 32] studied the embeddings of $\dot{W}^1,p(R^n)$ and $\dot{A}^1,1(R^n)$ into the Lebesgue space $L^q_{\mu}(\mathbb{R}^{n+1}_+)$, under $(p, q) \in (1, \infty) \times (1, \infty)$, via the heat kernel. For fractional diffusion equations, motivated by Xiao [31], Zhai [36] explored the embeddings of the homogeneous Sobolev space $\dot{W}^{\beta,p}(R^n)$ into the Lebesgue space $L^q_{\mu}(\mathbb{R}^{n+1}_+)$. Xiao and Zhai [33] explored the embeddings for the homogeneous Besov space $\dot{A}^{p,p}(R^n)$ with $p < 1$ via fractional diffusion equations. Li et al. [22] studied the embedding of Sobolev spaces $\dot{W}^{\beta,p}(R^n)$ into Lebesgue spaces $L^q_{\mu}(\mathbb{R}^{n+1}_+)$ via the Caffarelli–Silvestre extension.

This article will be organized as follows. In Sect. 2, we will provide some preliminaries and basic results. Section 3 is devoted to establish the fractional anisotropic Sobolev trace inequalities and the affine trace inequality for the Caffarelli–Silvestre extension. In Sect. 4, we will characterize the non-negative Random measure $\mu$ on $\mathbb{R}^{n+1}_+$ such that $\|u(\cdot, \cdot)\|_{L^{q,p}_{\mu}(\mathbb{R}^{n+1}_+)} \lesssim \|f\|_{\dot{A}^{p,q}_\beta(R^n)}$ in three different cases.

2 Preliminaries and Basic Lemmas

At first, we want to find the Fourier transform of the fractional Poisson kernel $p^s_t(\cdot)$.

Proposition 2.1 The fractional Poisson kernel $p^s_t(\cdot)$ satisfies the following properties.

(i) The Fourier transform $\widehat{p^s_t}$ can be represented as

$$\widehat{p^s_t}(\xi) = C_{n,s} G_s(t|\xi|),$$

with a positive constant $C_{n,s}$ and

$$G_s(t) := \int_0^\infty \lambda^{s/2-1} e^{-\lambda - t^2/(4\lambda)} d\lambda, \quad t \in (0, \infty),$$

which satisfies

$$\begin{cases}
    \int_0^\infty G_s(t)^2 t^a dt < \infty, & a > -(s+1); \\
    \int_0^\infty G'_s(t)^2 t^a dt < \infty, & a > 1 - 2s.
\end{cases}$$ (2.2)
(ii) If \( a > -(s + 1) \), there exists a constant \( C(n, s, a) \) such that
\[
\int_0^\infty |\hat{p}_t^s(\xi)|^2 t^a \, dt = C(n, s, a) |\xi|^{-(a+1)}. \tag{2.3}
\]

**Proof** For (i), obviously, \( p_t^s(xt) = c(n, s)t^{-n}(|1 + |x|^2|^{-(n+s)/2}. \) Denote \( \tau = n + s. \)
For \( A > 0 \), it follows from the Gamma-function identity
\[
A^{-\tau/2} = \frac{1}{\Gamma(\tau/2)} \int_0^\infty e^{-\lambda A^{-\tau/2-1}} d\lambda
\]
that
\[
(1 + |x|^2)^{-\tau/2} = \frac{1}{\Gamma(\tau/2)} \int_0^\infty e^{-\lambda - \lambda |x|^2/\lambda^{\tau/2-1}} d\lambda.
\]
Thus, we obtain

\[
\hat{p}_t^s(\xi) = t^{-n} \hat{p}_1^s(\xi)
\]
\[
= \delta'(\hat{p}_1^s(\xi))
\]
\[
= \frac{c(n, s)}{\Gamma(\tau/2)} \delta'(e^{-\lambda |x|^2/4\lambda}) \int_0^\infty e^{-\lambda (e^{-\lambda |x|^2}) (\xi) \lambda^{\tau/2-1}} d\lambda
\]
\[
= C_{n, s} \delta'(e^{-\lambda |x|^2/4\lambda}) \lambda^{\tau/2-1} \int_0^\infty e^{-\lambda |t|^{2/4\lambda} \lambda^{s/2-1}} d\lambda
\]
\[
= C_{n, s} G_s(t|\xi|),
\]
which gives us (2.1). See also [18, Proposition 7.6] for a similar result for Bessel potentials.

It follows from [28, p. 132, (30)] and [18, Proposition 7.6] that there holds
\[
G_s(t) \lesssim \begin{cases} 
1 + t^s, & t \to 0; \\
e^{-ct}, & t \to \infty.
\end{cases} \tag{2.4}
\]

Now we verify (2.2). Since \( s + a > -1 \) and \( s \in (0, 2), \) (2.4) implies
\[
\int_0^\infty |G_s(t)|^2 t^a \, dt \lesssim \int_0^1 \left( 1 + t^{s+a} + t^{2s+a} \right) dt + \int_1^\infty e^{-2ct} t^a \, dt < \infty.
\]
Below we estimate the derivative \( G_s'(\cdot). \) Notice that
\[
G_s'(t) = -\frac{t}{2} \int_0^\infty \lambda^{s/2-2} e^{-\lambda - t^2/(4\lambda)} d\lambda.
\]
Then, for \( t \) near 0, letting \( 4\lambda = 1/w \), we obtain

\[
|t^{1-s} G'_s(t)| = \left| -\frac{t^{2-s}}{2} \int_0^\infty \lambda^{s/2-2} e^{-\lambda - t^2/(4\lambda)} d\lambda \right| \lesssim t^{2-s} \int_0^\infty w^{-s/2} e^{-t^2/w} dw,
\]

which, together with the change of variable: \( t^2 w = v \), gives

\[
|t^{1-s} G'_s(t)| \lesssim \int_0^\infty v^{-s/2} e^{-v} dv < \infty,
\]

where we have used the fact that \( s \in (0, 2) \).

Then, we assume that \( t \gg 1 \). For \( N > 1 \), taking \( 2l = N + s - 1 \), we can apply the change of variables \( 4\lambda = 1/w \) and \( t^2 w = v \) again to get

\[
|t^N G'_s(t)| \lesssim \int_0^\infty v^{l-s/2} e^{-v} dv < \infty.
\]

Therefore, we have proved that

\[
G'_s(t) \lesssim \begin{cases} 
   t^{s-1}, & t \to 0; \\
   t^{-N}, & t \to \infty \text{ and } \forall N > 1,
\end{cases}
\]

which indicates

\[
\int_0^\infty |G'_s(t)|^2 t^a dt < \infty, \quad a > 1 - 2s.
\]

(ii) For \( s + a > -1 \), using (2.2), we have

\[
\int_0^\infty \left( \hat{p}^x(t) \right)^2 t^a dt = C_{n,s} \int_0^\infty (G_s(t|\xi|))^{2} (t|\xi|)^a d(t|\xi|)|\xi|^{-(a+1)}
\]

\[
= C(n, s, a)|\xi|^{-(a+1)}.
\]

Thus (2.3) holds with \( C(n, s, a) = C_{n,s} \int_0^\infty G_s(t)^2 t^a dt \).
Denote by $M_f$ the Hardy–Littlewood maximal operator, i.e.,

$$M_f(x) = \sup_{r > 0} r^{-n} \int_{B(x,r)} |f(y)| dy, \quad x \in \mathbb{R}^n.$$ 

**Lemma 2.2** There exists a constant $C$ depending only on $n$ and $s$ such that

$$|p_t^s * f(x - y)| \leq C (1 + |y|^2/t) M_f(x) \quad \forall x, y \in \mathbb{R}^n \quad \forall f \in C^\infty_c (\mathbb{R}^n).$$

**Proof** For any ball $Q \subset \mathbb{R}^n$, denote by $x_Q$ the center of $Q$ and by $r_Q$ the radius of $Q$. We can see that if $x/t \in Q$ then $|x - tx_Q| < tr_Q$. This implies that

$$M_f(t \cdot)(x/t) \leq \sup_{x \in tQ} \frac{1}{|tQ|} \int_{tQ} |f(u)| du \leq M_f(x).$$

We only need to prove

$$|p_t^s * f(x - y)| \lesssim (1 + |y|)^n M_f(x). \quad (2.5)$$

In fact, if (2.5) holds, then by the change of variable $z/t = u$, we obtain

$$|p_t^s * f(x - y)| \approx \left| \int_{\mathbb{R}^n} \frac{t^n}{(t^n + |x - y - z|^2)^{(n+s)/2}} f(z) dz \right| \approx \left| \int_{\mathbb{R}^n} \frac{1}{t^n (1 + |x - y|/t)^{(n+s)/2}} \frac{1}{(1 + |u|^2 - |x - y - z|^2)^{(n+s)/2}} f(u) t^n du \right| \lesssim (1 + |y|/t)^n M_f(t \cdot)(x/t) \lesssim (1 + |y|/t)^n M_f(x).$$

Below we prove (2.5). We first estimate the kernel $p_t^s(x - y)$. If $|x| < |y|$, it is obvious that $p_t^s(x - y) \leq 1$. On the other hand, for $|x| \geq |y|$, noting that $||x| - |y|| \leq |x - y|$, it holds

$$p_t^s(x - y) \lesssim \frac{1}{(1 + (|x| - |y|)^2)^{(n+s)/2}}.$$ 

Set the decreasing radial majorant function of $p_t^s(x - y)$ as

$$\varphi_y^s(x) := \begin{cases} \frac{1}{(1 + (|x| - |y|)^2)^{(n+s)/2}}, & |x| \geq |y|; \\ 1, & |x| < |y|. \end{cases}$$

With a slight abuse of notation, let us write $\varphi_y^s(x) = \varphi_y^s(r)$ if $|x| = r$. We can get
\[ |p^\gamma_s \ast f(x - y)| \lesssim \int_{\mathbb{R}^n} \frac{1}{(1 + |x - y - z|^2)^{(n+s)/2}} |f(z)| \, dz \]
\[
\lesssim \int_{\mathbb{R}^n} \varphi^\gamma_s(x - z) |f(z)| \, dz \\
\approx \sum_{k=-\infty}^{\infty} \int_{2^k < |x - z| \leq 2^{k+1}} \varphi^\gamma_s(x - z) |f(z)| \, dz \\
\lesssim \sum_{k=-\infty}^{\infty} \varphi^\gamma_s(2^k) \int_{|x - z| < 2^{k+1}} |f(z)| \, dz \\
\lesssim \mathcal{M}f(x) \sum_{k=-\infty}^{\infty} \varphi^\gamma_s(2^k) 2^{n(k+1)} \\
\lesssim \mathcal{M}f(x) \sum_{k=-\infty}^{\infty} \varphi^\gamma_s(2^k) \int_{2^{k-1}}^{2^k} r^{n-1} \, dr \\
\lesssim \mathcal{M}f(x) \|\varphi^\gamma_s\|_{L^1(\mathbb{R}^n)}.
\]

By a direct computation, we obtain
\[
\|\varphi^\gamma_s\|_1 \approx \int_{|x| < |y|} 1 \, dx + \int_{|x| \geq |y|} \frac{1}{(1 + (|x| - |y|)^2)^{(n+s)/2}} \, dx \\
\lesssim |y|^n + \int_{|x| \geq |y|} \frac{1}{(1 + |x|^2 + |y|^2 - 2|x||y|)^{(n+s)/2}} \, dx \\
\lesssim |y|^n + \int_0^{\infty} \frac{(r + |y|)^{n-1} \, dr}{(1 + r^2)^{(n+s)/2}} \\
\lesssim |y|^n + \sum_{k=0}^{n-1} C_{n-1}^k |y|^{n-1-k} \int_0^{\infty} \frac{r^k}{(1 + r^2)^{(n+s)/2}} \, dr \\
\lesssim |y|^n + \sum_{k=0}^{n-1} C_{n-1}^k |y|^{n-1-k} \\
\leq C(1 + |y|)^n.
\]

In [30], for the heat kernel and variational capacities \(\text{cap}_\rho(\cdot)\), Xiao established some basic properties concerning tents, non-tangential maximal functions, and \(\text{cap}_\rho(\cdot)\). Following the idea of [30, Lemma 2.4], we can deduce the following properties for the fractional Poisson kernel \(p^\gamma_s(\cdot)\) and the related Besov capacities \(C^{\rho,q}_\beta(\cdot)\). Parts of the following result were proved for Sobolev functions in [22].

**Lemma 2.3** Let \(s \in (0, 2)\) and \(\beta \in (0, n)\). Given \(f \in C^\infty_0(\mathbb{R}^n)\), denote the Caffarelli–Silvestre extension of \(f\) by \(u(x, t) = p^\gamma_s * f(x)\). For \(\lambda > 0\) and a non-negative measure \(\mu\) on \(\mathbb{R}^{n+1}_+\), let

\[
E_{\lambda,s}(f) = \left\{ (x, t) \in \mathbb{R}^{n+1}_+ : |u(x, t)| > \lambda \right\}
\]

\(\square\) Springer
and

\[ O_{\lambda,s}(f) = \left\{ x \in \mathbb{R}^n : \sup_{|y-x| < t} |u(y, t)| > \lambda \right\}. \]

Then, the following four statements are true.

(i) \( \mu(E_{\lambda,s}(f)) \leq \mu(T(O_{\lambda,s}(f))) \).

(ii) For any natural number \( k \),

\[ \mu \left( E_{\lambda,s}(f) \cap T(B(0, k)) \right) \leq \mu \left( T \left( O_{\lambda,s}(f) \cap B(0, k) \right) \right). \]

(iii) Let \( c^{\beta}_{p,q} \) be the variational capacity minimizing function defined by (1.9). For any \( p, q \in (0, \infty) \) and any natural number \( k \),

\[ C^{p,q}_{\beta} \left( O_{\lambda,s}(f) \cap B(0, k) \right) \geq c^{\beta}_{p,q}(\mu; \mu \left( T \left( O_{\lambda,s}(f) \cap B(0, k) \right) \right)). \]

(iv) There exists a constant \( \theta_{n,s} > 0 \) such that

\[ \sup_{|y-x|<t} |u(y, t)| \leq \theta_{n,s} \mathcal{M}f(x), \quad x \in \mathbb{R}^n. \]

(v) There exists a constant \( \eta_{n,s} > 0 \) such that if \( (x, t) \in T(O) \), then

\[ (p_t^s * |f|)(x) \geq \eta_{n,s}, \]

where \( O \) is a bounded open set contained in \( \text{Int} \{ x \in \mathbb{R}^n : f(x) \geq 1 \} \).

**Proof** (i) and (ii). Let \( \Gamma(x) = \left\{ (y, t) \in \mathbb{R}^n+1_+ : |y - x| < t \right\} \). The non-tangential maximal function related with \( p_t^s \) is defined as

\[ N(f)(x) := \sup_{|y-x|<t} |p_t^s * f(y)|. \]

Then, \( O_{\lambda,s} = \left\{ x \in \mathbb{R}^n : N(f)(x) > \lambda \right\} \). For any \( x_0 \in O_{\lambda,s} \), \( N(f)(x_0) > \lambda \), i.e., there exists a \( y_0 \) such that \( |y_0 - x_0| < t \) and \( |p_t^s * f|(y_0) > \lambda \). For arbitrary \( x \in B(y_0, t) \), it holds

\[ N(f)(x) := \sup_{(z,t) \in \Gamma(x)} |p_t^s * f(z)| \geq |p_t^s * f(y_0)| > \lambda, \]

which indicates that \( B(y_0, t) \subseteq O_{\lambda,s} \) and \( N(f) \) is lower semi-continuous. We can see that \( O_{\lambda,s} \) is an open set in \( \mathbb{R}^n \). So, we get

\[
\begin{cases}
E_{\lambda,s}(f) \subseteq T(O_{\lambda,s}(f)); \\
\mu(E_{\lambda,s}(f)) \leq \mu(T(O_{\lambda,s}(f))),
\end{cases}
\]
and

\[ \mu(E_{\lambda,s}(f) \cap T(B(0,k))) \leq \mu(T(O_{\lambda,s}(f)) \cap T(B(0,k))) = \mu(T(O_{\lambda,s}(f) \cap B(0,k))). \]

(iii) It follows from the definition of \( c_{p,q}^\beta(\mu; t) \).

(iv) By Lemma 2.2, for \( y \) satisfying \( |x - y| < t \),

\[ p^x_t * f(x - y) \lesssim (1 + |y|/t)^n \mathcal{M}f(x), \]

which gives

\[ N(f)(x) = \sup_{|y| < t} |p^x_t * f(x - y)| \lesssim \mathcal{M}f(x) \sup_{|y| < t} (1 + |y|/t)^n \lesssim 2^n \mathcal{M}f(x). \]

(v) For any \((x, t) \in T(O)\), we have

\[ B(x, t) \subseteq O \subset \text{Int}(\{x \in \mathbb{R}^n : f(x) \geq 1\}). \]

We can see that for \( |x| \leq \sigma t \),

\[ p^x_t(x) \asymp \frac{t^s}{(t^2 + |x|^2)^{(n+s)/2}} \gtrsim \frac{t^s}{(t^2 + (\sigma t)^2)^{(n+s)/2}} \gtrsim \frac{1}{t^n}. \]

Then,

\[ p^x_t * |f|(x) = \int_{\mathbb{R}^n} p^x_t(x - y)|f|(y)dy \gtrsim \frac{1}{t^n} \int_{B(x, \sigma t) \cap \text{Int}(\{x \in \mathbb{R}^n : f(x) \geq 1\})} |f|(y)dy \gtrsim \frac{1}{t^n} |B(x, \sigma t) \cap \text{Int}(\{x \in \mathbb{R}^n : f(x) \geq 1\})|. \]

If \( \sigma > 1 \), then

\[ B(x, t) = B(x, t) \cap \text{Int}(\{x \in \mathbb{R}^n : f(x) \geq 1\}) \subset B(x, \sigma t) \cap \text{Int}(\{x \in \mathbb{R}^n : f(x) \geq 1\}). \]

If \( \sigma \leq 1 \), then

\[ B(x, \sigma t) \cap \text{Int}(\{x \in \mathbb{R}^n : f(x) \geq 1\}) = B(x, \sigma t). \]

Hence, \( p^x_t * |f|(x) \geq \eta_{n,s} \) for a constant \( \eta_{n,s} \). \hfill \Box

We need the following weak-/strong-type estimates for Besov capacities \( C_{p,q}^\beta (\cdot) \). Denote

\[ p \lor q = \max\{p, q\} \text{ and } p \land q = \min\{p, q\}. \]
Lemma 2.4  The following results hold.

(i)  When $\beta \in (0, 1) \& p = q \in (n/(n + \beta), 1)$, or $(\beta, p, q) \in (0, n) \times (1, n/\beta) \times (1, \infty)$,

\[
\int_0^\infty \left( C_\beta^{p,q} \left( \{ x \in \mathbb{R}^n : |\mathcal{M}_f(x)| > \lambda \} \right) \right)^{1/(q/p)} d\lambda \leq \| f \|_{\dot{A}^{p,q}_{\beta} (\mathbb{R}^n)} \forall f \in C_0^\infty (\mathbb{R}^n).
\]

(ii) When $(\beta, p, q) \in (0, 1) \times (1, n/\beta) \times \{\infty\}$,

\[
\sup_{\lambda \in (0, \infty)} \lambda^p C_\beta^{p,q} \left( \{ (x, t) \in \mathbb{R}^n : |\mathcal{M}_f(x)| > \lambda \} \right) \leq \| f \|_{\dot{A}^{p,q}_{\beta} (\mathbb{R}^n)} \forall f \in \dot{A}^{p,q}_{\beta} (\mathbb{R}^n) \cap C(\mathbb{R}^n).
\]

Proof  Statement (i) is due to Maz’ya [25] when $p = q > 1$. When $1 \leq p \leq q < \infty$ and $0 < \beta < 1$, Wu [39] proved (i). Adams and Xiao [3] established (i) when $0 < \beta < \infty$ and $(p, q) \in (1, n/\beta) \times (1, \infty)$. Xiao and Zhai [34] showed that (i) holds when $0 < \beta < 1$ and $n/(n + \beta) < p = q < 1$.

For (ii), it follows from [26, Proposition 2.8] that for $\lambda > 0$ and $f \in \dot{A}^{p,q}_{\beta} (\mathbb{R}^n) \cap C(\mathbb{R}^n)$,

\[
\left( \lambda^p C_\beta^{p,q} \left( \{ (x, t) \in \mathbb{R}^n : |\mathcal{M}_f(x)| > \lambda \} \right) \right)^{1/p} \leq \| \mathcal{M}_f \|_{\dot{A}^{p,q}_{\beta} (\mathbb{R}^n)} \leq \| f \|_{\dot{A}^{p,q}_{\beta} (\mathbb{R}^n)},
\]

where in the last inequality we have used the fact that the maximal function is bounded on $L^p(\mathbb{R}^n)$.

\[
\square
\]

3 Fractional Trace Inequalities via the Caffarelli–Silvestre Extension

3.1 Fractional Trace Inequalities Involving $\nabla_{(x,t)} u(x, t)$

Theorem 3.1  Let $f \in \dot{H}^{\beta/2} (\mathbb{R}^n)$ with $\beta \in (0, \min\{n, 2s\})$. Denote by $u(x, t) = p_t^s \ast f(x)$ the Caffarelli–Silvestre extension of $f$.

(i) There holds

\[
\left( \int_{\mathbb{R}^n} |f(x)|^{2(n/(n-\beta))} dx \right)^{1-\beta/n} \lesssim \int_{\mathbb{R}^{n+1}} |\nabla_{(x,t)} u(x, t)|^2 t^{1-\beta} dx dt. \tag{3.1}
\]

(ii) If $\| f \|_{L^2(\mathbb{R}^n)} = 1$, there holds

\[
\exp \left( \frac{\beta}{n} \int_{\mathbb{R}^n} |f(x)|^2 \ln(|f(x)|^2) dx \right) \lesssim \int_{\mathbb{R}^{n+1}} |\nabla_{(x,t)} u(x, t)|^2 t^{1-\beta} dx dt. \tag{3.2}
\]

(iii) There holds

\[
\int_{\mathbb{R}^n} |f(x)|^2 \frac{dx}{|x|^{\beta}} \lesssim \int_{\mathbb{R}^{n+1}} |\nabla_{(x,t)} u(x, t)|^2 t^{1-\beta} dx dt. \tag{3.3}
\]
Proof In order to prove Theorem 3.1, we need to establish the following result. For $\beta \in (0, 2s)$, there holds
\[
\int_{\mathbb{R}^{n+1}_+} |\nabla_{(x,t)} u(x, t)|^2 t^{1-\beta} \, dx \, dt \approx \int_{\mathbb{R}^n} |\xi|^\beta (\widehat{f}(\xi))^2 \, d\xi. \tag{3.4}
\]

In fact, since $u(x, t) = p_t^* \ast f(x)$, (2.2) of Proposition 2.1 implies
\[
\int_{\mathbb{R}^{n+1}_+} |\nabla_{(x,t)} u(x, t)|^2 t^{1-\beta} \, dx \, dt
= \int_{\mathbb{R}^{n+1}_+} |\nabla x u(x, t)|^2 t^{1-\beta} \, dx \, dt + \int_{\mathbb{R}^{n+1}_+} \left| \frac{\partial u(x, t)}{\partial t} \right|^2 t^{1-\beta} \, dx \, dt
= \int_0^\infty \int_{\mathbb{R}^n} |\xi|^2 |\widehat{u}(\xi, t)|^2 \, d\xi \, t^{1-\beta} \, dt + \int_0^\infty \int_{\mathbb{R}^n} \left| \frac{\partial \widehat{u}(\xi, t)}{\partial t} \right|^2 \, d\xi \, t^{1-\beta} \, dt
\approx \int_0^\infty \int_{\mathbb{R}^n} |\xi|^2 (\widehat{G_s}(t|\xi|))^2 (\widehat{f}(\xi))^2 \, d\xi \, t^{1-\beta} \, dt + \int_0^\infty \int_{\mathbb{R}^n} \left( \frac{\partial (G_s(t|\xi|))}{\partial t} \right)^2 (\widehat{f}(\xi))^2 \, d\xi \, t^{1-\beta} \, dt
\approx \int_{\mathbb{R}^n} \left( \int_0^\infty (G_s(t|\xi|))^2 u t^{1-\beta} \, dt + \int_0^\infty (G_s'(t|\xi|))^2 u t^{1-\beta} \, dt \right) |\xi|^\beta (\widehat{f}(\xi))^2 \, d\xi
\approx \int_{\mathbb{R}^n} |\xi|^\beta (\widehat{f}(\xi))^2 \, d\xi.
\]

In the last equality, we have used the requirement $s + 1 - \beta > -1$ and $1 - \beta > 1 - 2s$ which means that $\beta < 2 + s$ and $\beta < 2s$. Since $s \in (0, 2), 2s < 2 + s$ and $\beta \in (0, 2s)$.

It follows from (3.4) and the well-known fractional Sobolev inequality
\[
\| f \|_{L^{2n/(n-\beta)}(\mathbb{R}^n)} \leq B(n, \beta) \| (-\Delta)^{\beta/4} f \|_{L^2(\mathbb{R}^n)}
\]
for $\beta \in (0, n)$ and some constant $B(n, \beta)$ that (3.1) holds.

Now, we want to prove (3.2). Let $p = n(r - 2)/\beta$, $2 < r < 2n/(n - \beta)$ and $\beta \in (0, \min\{n, 2s\})$. Then, the Hölder inequality implies
\[
\| f \|_{L^r(\mathbb{R}^n)} = \int_{\mathbb{R}^n} |f(x)|^p |f(x)|^{-p} \, dx \leq \| f \|_{L^{2n/(n-\beta)}(\mathbb{R}^n)}^p \left( \int_{\mathbb{R}^n} |f(x)|^2 \, dx \right)^{1-p(n-\beta)/(2n)}.
\]

If $\| f \|_{L^2(\mathbb{R}^n)} = 1$, then
\[
\left( \int_{\mathbb{R}^n} |f(x)|^{r-2} |f(x)|^2 \, dx \right)^{1/(r-2)} = \left( \int_{\mathbb{R}^n} |f(x)|^r \, dx \right)^{1/(r-2)} \leq \| f \|_{L^{2n/(n-\beta)}(\mathbb{R}^n)}^{n/\beta}.
\]
So, the inequality (3.1) implies that for a positive constant $A(n, s, \beta)$,

\[
\left( \int_{\mathbb{R}^n} |f(x)|^{r-2}|f(x)|^2 \, dx \right)^{1/(r-2)} \leq \left( A(n, s, \beta) \int_{\mathbb{R}^n} |\nabla_{x,t} u(x, t)|^2 t^{1-\beta} \, dx \, dt \right)^{n/(2\beta)}.
\]

Since $\|f\|_{L^2(\mathbb{R}^n)} = 1$, $(f(x))^2 \, dx$ can be treated as a probability measure on $\mathbb{R}^n$. Thus, (3.2) can be obtained by letting $r \to 2$ in the previous inequality.

Inequality (3.3) follows from (3.4) and the fractional Hardy inequality

\[
\| f(\cdot) \|_{L^2(\mathbb{R}^n)} \leq H \| (-\Delta)^{\beta/4} f \|_{L^2(\mathbb{R}^n)}
\]

which is a special case of [33, (3.1) in Theorem 3.1].

### 3.2 Fractional Anisotropic Trace Inequalities Involving $\partial_t u(x, t)$

**Theorem 3.2** Let $f \in \dot{H}^{\beta/2}(\mathbb{R}^n)$ with $\beta \in (0, \min\{n, 2s\})$. Denote by $u(x, t) = p_t^s * f(x)$ the Caffarelli–Silvestre extension of $f$.

(i) There holds

\[
\left( \int_{\mathbb{R}^n} |f(x)|^{2n/(n-\beta)} \, dx \right)^{1-\beta/n} \lesssim \int_{\mathbb{R}^{n+1}_+} \left| \frac{\partial u(x, t)}{\partial t} \right|^2 t^{1-\beta} \, dx \, dt.
\]

(ii) If $\|f\|_{L^2(\mathbb{R}^n)} = 1$, there holds

\[
\exp \left( \frac{\beta}{n} \int_{\mathbb{R}^n} |f(x)|^2 \ln(|f(x)|^2) \, dx \right) \lesssim \int_{\mathbb{R}^{n+1}_+} \left| \frac{\partial u(x, t)}{\partial t} \right|^2 t^{1-\beta} \, dx \, dt.
\]

(iii) There holds

\[
\int_{\mathbb{R}^n} \frac{|f(x)|^2}{|x|^\beta} \, dx \lesssim \int_{\mathbb{R}^{n+1}_+} \left| \frac{\partial u(x, t)}{\partial t} \right|^2 t^{1-\beta} \, dx \, dt.
\]

**Proof** In order to prove Theorem 3.2, we need to establish the following result. For $\beta \in (0, \min\{n, 2s\})$, there exists $a(n, s, \beta)$ such that

\[
\int_{\mathbb{R}^{n+1}_+} \left| \frac{\partial u(x, t)}{\partial t} \right|^2 t^{1-\beta} \, dx \, dt = a(n, s, \beta) \int_{\mathbb{R}^n} |\xi|^\beta \left( \hat{f}(\xi) \right)^2 \, d\xi.
\]
In fact, noting that \( u(x, t) = p_t^s * f(x) \), we can apply Proposition 2.1 to deduce that

\[
\int_{\mathbb{R}_+^{n+1}} \left| \frac{\partial u(x, t)}{\partial t} \right|^2 t^{1-\beta} \, dx \, dt = \int_{\mathbb{R}_+^{n+1}} \left| \frac{\partial u(x, t)}{\partial t} \right|^2 t^{1-\beta} \, dx \, dt
\]

\[
= \int_0^\infty \int_{\mathbb{R}^n} \left| \frac{\partial \hat{u}(\xi, t)}{\partial t} \right|^2 d\xi \, t^{1-\beta} \, dt
\]

\[
= \int_0^\infty \int_{\mathbb{R}^n} \left( \frac{\partial \hat{p}_t^s(\xi)}{\partial t} \right)^2 (\hat{f}(\xi))^2 d\xi \, t^{1-\beta} \, dt
\]

\[
= \int_{\mathbb{R}^n} \left( \int_0^\infty (G_t'(u))^{2u-1} \, du \right) |\xi|^\beta (\hat{f}(\xi))^2 d\xi
\]

\[
= a(n, s, \beta) \int_{\mathbb{R}^n} |\xi|^\beta (\hat{f}(\xi))^2 d\xi.
\]

Then, Theorem 3.2 can be proved in a way similar to that of Theorem 3.1. \( \square \)

### 3.3 Fractional Anisotropic Trace Inequalities Involving \(-\Delta)^{\gamma/2}u(x, t)\)

**Theorem 3.3** For \( f \in \dot{H}^{\beta/2}(\mathbb{R}^n) \) with \( \beta \in (0, n) \), \( \gamma > \max \{(\beta - s)/2, 0\} \). Denote by \( u(x, t) = p_t^s * f(x) \) the Caffarelli–Silvestre extension of \( f \).

(i) There holds

\[
\left( \int_{\mathbb{R}^n} |f(x)|^{2n/(n-\beta)} \, dx \right)^{1-\beta/n} \lesssim \int_{\mathbb{R}_{+}^{n+1}} \left| (-\Delta)^{\gamma/2} u(x, t) \right|^{2t^{2\gamma-\beta-1}} \, dx \, dt.
\]

(3.9)

(ii) If \( \|f\|_{L^2(\mathbb{R}^n)} = 1 \), there holds

\[
\exp \left( \frac{\beta}{n} \int_{\mathbb{R}^n} |f(x)|^2 \ln(|f(x)|^2) \, dx \right) \lesssim \int_{\mathbb{R}_{+}^{n+1}} \left| (-\Delta)^{\gamma/2} u(x, t) \right|^{2t^{2\gamma-\beta-1}} \, dx \, dt.
\]

(3.10)

(iii) There holds

\[
\int_{\mathbb{R}^n} \frac{|f(x)|^2}{|x|^\beta} \, dx \lesssim \int_{\mathbb{R}_{+}^{n+1}} \left| (-\Delta)^{\gamma/2} u(x, t) \right|^{2t^{2\gamma-\beta-1}} \, dx \, dt.
\]

(3.11)

**Proof** We first establish the following result: for \( \beta \in (0, n) \), and \( \gamma > \max \{(\beta - s)/2, 0\} \), there holds

\[
\int_{\mathbb{R}_{+}^{n+1}} \left| (-\Delta)^{\gamma/2} u(x, t) \right|^{2t^{2\gamma-\beta-1}} \, dx \, dt \approx \int_{\mathbb{R}^n} |\xi|^\beta (\hat{f}(\xi))^2 \, d\xi.
\]

(3.12)
In fact, similar to the proof of (3.8), (i) of Proposition 2.1 implies

$$\int_{\mathbb{R}^{n+1}} \left| (-\Delta)^{\gamma/2} u(x,t) \right|^2 t^{2\gamma-\beta-1} \, dx \, dt \approx \int_0^\infty \int_{\mathbb{R}^n} |\xi|^{2\gamma} \left| \hat{u}(\xi, t) \right|^2 \, d\xi \, t^{2\gamma-\beta-1} \, dt$$

$$\approx \int_0^\infty \int_{\mathbb{R}^n} |\xi|^{2\gamma} (\hat{P}(\xi))^2 (\hat{f}(\xi))^2 \, d\xi \, t^{2\gamma-\beta-1} \, dt$$

$$\approx \int_{\mathbb{R}^n} \left( \int_0^\infty (G_s(t|\xi|))(2\gamma-\beta-1) \, d(t|\xi|) \right) \left| \xi \right|^{\beta} (\hat{f}(\xi))^2 \, d\xi$$

$$\approx \int_{\mathbb{R}^n} \left| \xi \right|^{\beta} (\hat{f}(\xi))^2 \, d\xi.$$  

Similar to the proof of Theorem 3.1, we can derive Theorem 3.3 by using (3.12).  

**3.4 Affine Fractional Poisson Trace Inequality**

In this section, we will prove an affine fractional Poisson trace inequality by applying the sharp affine weighted $L^p$ Sobolev inequality established in [16]. Let $\alpha > 0$. Define a function $\sigma$ on $\mathbb{R}^{n+1}$ as $\sigma(x, t) := t^\alpha \forall (x, t) \in \mathbb{R}^{n+1}$. For $1 \leq p < n + \alpha + 1$, denote by $L^p(\mathbb{R}^{n+1}, \sigma)$ the weighted Lebesgue space corresponding to the weight function $\sigma$.

**Theorem 3.4** [16, Theorem 1.1] Let $\alpha \geq 0$, $1 \leq p < n + \alpha + 1$ and $p^* = p(n + \alpha + 1) / (n + \alpha + 1 - p)$. There exists a sharp constant $J(n, p, \alpha)$ such that

$$\|g(\cdot, \cdot)\|_{L^{p^*}(\mathbb{R}^{n+1}, \sigma)} \leq J(n, p, \alpha)(\mathcal{E}_p(g, \sigma))^{n/(n+\alpha+1)} \left\| \frac{\partial g}{\partial t}(\cdot, \cdot) \right\|_{L^p(\mathbb{R}^{n+1}, \sigma)}^{(\alpha+1)/(n+\alpha+1)}.$$  

The equality in the above inequality holds if

$$g(x, t) = \begin{cases} 
\frac{c}{(1 + |\gamma t|^{1+1/p} + |B(x - x_0)|^{1+1/p})^{(1+n+\alpha-p)/p}}, & p > 1; \\
\frac{c}{1_{\mathbb{R}^{n+1}}(\gamma t, B(x - x_0))}, & p = 1;
\end{cases}$$

for some quadruple

$$(c, |\gamma|, x_0, B) \in \mathbb{R} \times \mathbb{R}_+ \times \mathbb{R}^n \times GL_n,$$

where $1_{\mathbb{R}^{n+1}}$ is the characteristic function of the unit ball in $\mathbb{R}^{n+1}$ and $GL_n$ denotes the set of all invertible real $n \times n$-matrices.
Theorem 3.5 Let \( p = 2(n + \beta)/(n + \beta + 2) \) and \( \beta \geq 1 \). For any \( f \in C_0^\infty(\mathbb{R}^n) \) and its Caffarelli–Silvestre extension \( u(x, t) = p_t^\ast f(x) \), there holds,

\[
\| f \|_{\dot{H}^{-\beta/2}(\mathbb{R}^n)} \lesssim (E_p(u, t^{\beta-1}))^{n/(n+\beta)} \left\| \frac{\partial u}{\partial t}(\cdot, \cdot) \right\|_{L^p(\mathbb{R}^{n+1}_+, t^{\beta-1})}^{\beta/(n+\beta)}.
\]

(3.13)

Proof Equality (2.3) implies

\[
\| u(\cdot, \cdot) \|_{L^2(\mathbb{R}^{n+1}_+, t^a)}^2 \approx \int_{\mathbb{R}^n} |\xi|^{-(a+1)} |\hat{f}(|\xi|)|^2 d\xi
\]

for \( a > -(s+1) \). In the following, taking \( a = \beta - 1 \geq 0 \) in Theorem 3.4, we can get \( \sigma := t^a = t^{\beta-1} \) and

\[
\| f \|_{\dot{H}^{-\beta/2}(\mathbb{R}^n)} = \| (-\Delta)^{-\beta/4} f \|_{L^2(\mathbb{R}^n)}
\approx \| u(\cdot, \cdot) \|_{L^2(\mathbb{R}^{n+1}_+, \sigma)}
\lesssim (E_p(u, \sigma))^{n/(n+\beta)} \left\| \frac{\partial u}{\partial t}(\cdot, \cdot) \right\|_{L^p(\mathbb{R}^{n+1}_+, \sigma)}^{\beta/(n+\beta)}.
\]

\[
\square
\]

3.5 Remarks on the General Case \( p \geq 1 \)

In Theorems 3.1, 3.2 and 3.3, the scope of \( (p, \beta) \) is \( p = 2 \) and \( 0 < \beta < \min\{n, 2s\} \). We can generalize the fractional trace-type inequalities Theorems 3.1, 3.2 and 3.3 to the general index \( p \in [1, \infty) \), \( \beta \in (0, 2n/p) \). Moreover, a general affine fractional Poisson trace inequality similar to Theorem 3.5 holds for \( f \in \dot{A}^{p_\ast}_{-\beta/2}(\mathbb{R}^n) \), where \( p_\ast \) satisfies \( p_\ast \geq \max\{2/\beta, 1\} \) and \( 1/p = 1/p_\ast + 2/(2n + p_\ast \beta) \).

Denote by \( u(x, t) = p_t^\ast f(x) \) the Caffarelli–Silvestre extension of \( f \). Applying [7, Theorems 1.1 and 1.3], Lenzmann and Schikorra [21, Proposition 10.8] obtained the following characterizations of Besov spaces \( \dot{A}^{p, p}_\beta(\mathbb{R}^n) \) with \( p \in (1, \infty) \),

\[
\left( \int_{\mathbb{R}^{n+1}_+} |\nabla_x u(x, t)|^{p_t p - p\beta/2} dx dt \right)^{1/p} \approx \| f \|_{\dot{A}^{p, p}_\beta(\mathbb{R}^n)},
\]
when \( \beta \in (0, 2) \);

\[
\left( \int_{\mathbb{R}^{n+1}_+} |\partial_t u(x, t)|^{p_t p - p\beta/2} dx dt \right)^{1/p} \approx \| f \|_{\dot{A}^{p, p}_\beta(\mathbb{R}^n)},
\]
when \( \beta \in (0, 2s) \);
and
\[
\left( \int_{\mathbb{R}^n_+} |(-\Delta)^{\gamma/2} u(x, t)|^{p} t^{(\gamma-\beta/2)-1} \, dx \, dt \right)^{1/p} \approx \| f \|_{\dot{A}_{\beta/2}^{p,p}(\mathbb{R}^n)},
\]
with \( \gamma > \beta/2 \).

(3.16)

The characterizations (3.14) and (3.15) indicate that
\[
\left( \int_{\mathbb{R}^n_+} |\nabla(x, t)u(x, t)|^{p} t^{p-1-p\beta/2} \, dx \, dt \right)^{1/p} \approx \| f \|_{\dot{A}_{\beta/2}^{p,p}(\mathbb{R}^n)},
\]
when \( \beta \in (0, \min\{2, 2s\}) \).

(3.17)

Equations (3.15)–(3.16) and (3.17), together with the fractional Sobolev inequality (cf. [4, Theorem 7.34] and [12, Theorem 6.5]):
\[
\left( \int_{\mathbb{R}^n} |f(x)|^{np/(n-p\beta/2)} \, dx \right)^{(n-p\beta/2)/np} \lesssim \| f \|_{\dot{A}_{\beta/2}^{p,p}(\mathbb{R}^n)}, \quad 1 \leq p < 2n/\beta,
\]
imply the following fractional Sobolev trace-type inequalities for \( f \in \dot{A}_{\beta/2}^{p,p}(\mathbb{R}^n) \) with \( 1 \leq p < 2n/\beta \),

\[
\| f \|_{L^{np/(n-p\beta/2)}(\mathbb{R}^n)} \lesssim \left\{ \begin{array}{l}
\left( \int_{\mathbb{R}^n_+} |\nabla(x, t)u(x, t)|^{p} t^{p-1-p\beta/2} \, dx \, dt \right)^{1/p}, \\
0 < \beta < \min\{2, 2n/p, 2s\};
\end{array} \right.
\]

\[
\left( \int_{\mathbb{R}^n_+} |\partial_t u(x, t)|^{p} t^{p-1-p\beta/2} \, dx \, dt \right)^{1/p},
\]

\[
0 < \beta < \min\{2n/p, 2s\};
\]

\[
\left( \int_{\mathbb{R}^n_+} |(-\Delta)^{\gamma/2} u(x, t)|^{p} t^{(\gamma-\beta/2)-1} \, dx \, dt \right)^{1/p},
\]

\[
\gamma > \beta/2 \& \beta \in (0, 2n/p).
\]

(3.18)

Moreover, for \( f \in \dot{A}_{\beta/2}^{p,p}(\mathbb{R}^n) \), there holds the logarithmic-type Sobolev inequality (cf. [17, Corollary 2.6]):
\[
\exp \left( \frac{\beta}{2n} \int_{\mathbb{R}^n} |f(x)|^p \ln |f(x)|^p \, dx \right) \lesssim \| f \|_{\dot{A}_{\beta/2}^{p,p}(\mathbb{R}^n)},
\]

\[
\| f \|_{L^p(\mathbb{R}^n)} = 1, \quad \beta \in (0, 2n/p) \quad \& \quad p \in (1, 2n/\beta),
\]

\( \dot{\text{Springer}} \)
and the fractional Hardy inequality (cf. [14, Theorem 1.1]):

\[
\left( \int_{\mathbb{R}^n} \frac{|f(x)|^p}{|x|^{p\beta/2}} \, dx \right)^{1/p} \lesssim \|f\|_{\dot{A}^{p,p}_{\beta/2}(\mathbb{R}^n)}, \quad \beta \in (0, 2) \quad \& \quad p \in [1, 2n/\beta). \tag{3.19}
\]

Similar to (3.18), we can establish the logarithmic and Hardy trace-type inequalities for general \( p \) with \( \beta \) in a similar range. We omit the details. Thus, Theorems 3.1, 3.2 and 3.3 can be generalized to \( p \geq 1 \).

Note that \( \|f\|_{\dot{H}^{-\beta/2}(\mathbb{R}^n)} \approx \|u(\cdot, \cdot)\|_{L^2(\mathbb{R}_+^{n+1}, \sigma)} \) is the key to prove Theorem 3.5. For \( f \in \dot{A}^{p,p}_{\beta/2}(\mathbb{R}^n) \) with \( \beta \in (0, 2n) \) and \( p \in (0, \infty) \), Ingmanns proved in [20, Corollary 5.2.3] that

\[
\left( \int_{\mathbb{R}^n+1} |u(x, t)|^{p^*_p \beta/2 - 1} \, dx \, dt \right)^{1/p} \approx \|f\|_{\dot{A}^{p,p}_{\beta/2}(\mathbb{R}^n)}, \tag{3.20}
\]

Thus, (3.20) and (3.4) imply that there holds

\[
\|f\|_{\dot{A}^{p\star, p\star}_{\beta/2}(\mathbb{R}^n)} \lesssim \left( \mathcal{E}_p \left( u, t^{p\star \beta/2 - 1} \right) \right)^{2n/(2n + p\star \beta)} \left\| \frac{\partial u}{\partial t} \right\|_{L^p(\mathbb{R}_+^{n+1}, t^{p\star \beta/2 - 1})}^{p\star \beta/(2n + p\star \beta)}
\]

with \( \beta \in (2/p\star, 2n) \), \( 1 \leq p < n + p\star \beta/2 \) with \( p\star \) satisfying \( p\star \geq \max\{2/\beta, 1\} \) and \( 1/p = 1/p\star + 2/(2n + p\star \beta) \).

Thus, an inequality similar to (3.13) in Theorem 3.5 holds for \( f \in \dot{A}^{p\star, p\star}_{\beta/2}(\mathbb{R}^n) \).

**4 Carleson Embeddings of Besov Spaces via the Caffarelli–Silvestre Extension**

In the following, we denote the Caffarelli–Silvestre extension of \( f \) by \( u(\cdot, \cdot) \). In this section, we study the Carleson embeddings via the Caffarelli–Silvestre extension \( u(\cdot, \cdot) \).

**4.1 Case (1): \( p = q \in (n/(n + \beta), 1) \)**

**Theorem 4.1** Let \( \beta \in (0, n) \) when \( p = 1 \), or \( \beta \in (0, 1) \) when \( p \in (n/(n + \beta), 1) \). Let \( q_0 \in [p, \infty) \) and \( \mu \) be a non-negative Radon measure on \( \mathbb{R}_+^{n+1} \). Then, the following statements are equivalent.

(i) There holds

\[
\|u(\cdot, \cdot)\|_{L^q(\mathbb{R}_+^{n+1})} \lesssim \|f\|_{\dot{A}^{p,p}_{\beta}(\mathbb{R}^n)} \quad \forall \ f \in C_0^\infty(\mathbb{R}^n).
\]
(ii) There holds
\[ \| u(\cdot, \cdot) \|_{L^p_{\mu}(\mathbb{R}^{n+1}_{+})} \leq \| f \|_{\dot{A}^p_{\mu}(\mathbb{R}^{n})} \quad \forall f \in C^\infty_0(\mathbb{R}^{n}). \]

(iii) There holds
\[ \| u(\cdot, \cdot) \|_{L^q_{\mu}(\mathbb{R}^{n+1}_{+})} \leq \| f \|_{\dot{A}^p_{\mu}(\mathbb{R}^{n})} \quad \forall f \in C^\infty_0(\mathbb{R}^{n}). \]

(iv) There holds
\[ \sup_{t>0} \frac{t^{p/q_0}}{c_{p, p}(\mu; t)} < \infty. \]

(v) There holds
\[ \sup \left\{ \frac{(\mu(T(O)))^{p/q_0}}{C^p_{\beta}(O)} : \text{open set } O \subset \mathbb{R}^{n} \right\} < \infty. \]

(vi) There holds
\[ \sup_{(x, r) \in \mathbb{R}^{n+1}_{+}} \frac{(\mu(T(B(x, r)))^{p/q_0}}{r^{n-p}} < \infty. \]

**Proof** We will prove the equivalence by showing (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (v) $\Rightarrow$ (i), (iii) $\Rightarrow$ (iv) $\Rightarrow$ (i), and (v) $\Leftrightarrow$ (vi).

**Part 1:** (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (v) $\Rightarrow$ (i). Firstly, note that the Lorentz spaces $L^p_{\mu}(\mathbb{R}^{n+1})$ are increasing with respect to $q$. We can get, as $p \leq q_0 \leq \infty$,
\[ \| u(\cdot, \cdot) \|_{L^{q_0}_{\mu}} \leq \| u(\cdot, \cdot) \|_{L^{q_0}_{\mu}(\mathbb{R}^{n+1}_{+}, \mu)} \leq \| u(\cdot, \cdot) \|_{L^\infty_{\mu}(\mathbb{R}^{n+1}_{+}, \mu)}. \]

This gives us (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii).

In the following, we show (iii) $\Rightarrow$ (v) $\Rightarrow$ (i). We first show (iii) $\Rightarrow$ (v). Assume that (iii) holds. Let $O \subset \mathbb{R}^{n}$ be an open set. Let $K$ be a compact subset of $O$. If a non-negative function $f \in C^\infty_0(\mathbb{R}^{n})$ satisfies $f \geq 1$ on $K$, noting that $B(x, t) \subset K$ for $(x, t) \in T(K)$, we can get for $|x - y| < t$,
\[ \frac{c(n, s)}{(t^2 + t^2)(n+s)/2) \geq \frac{c(n, s)}{2(n+s)/2 t^n}. \]

Hence, there exists a constant $c_{n, s}$ depending only on $n$ and $s$ such that
\[ |p_t^x f(x)| = \int_{\mathbb{R}^{n}} p_t^x (x - y) f(y) dy \geq \int_{K} p_t^x (x - y) dy. \]
\[
\geq c(n, s) \int_{B(x, t)} \frac{t^s}{(t^2 + |x - y|^2)^{(n+s)/2}} \, dy
\]
\[
\geq c_{n, s},
\]
which indicates that
\[
T(K) \subset \left\{(t, x) \in \mathbb{R}^{n+1} : |p_t^s \ast f(x)| > c_{n, s}\right\}.
\]
Hence, by (iii),
\[
\mu(T(K)) \leq \mu\left(\left\{(t, x) \in \mathbb{R}^{n+1} : |p_t^s \ast f(x)| > c_{n, s}\right\}\right)
\]
\[
\lesssim \|f\|_{A_p^{p, p}(\mathbb{R}^n)}^{q_0/p}.
\]
Taking the supermum on \( f \) gives
\[
\mu(T(K)) \leq (C_{p}^{p, p}(K))^{q_0/p}.
\]
By the definition of \( C_{p}^{p, p}(\cdot) \), we obtain
\[
\mu(T(O)) = \sup_{K \subset O} \mu(T(K))
\]
\[
\leq \sup_{K \subset O} (C_{p}^{p, p}(K))^{q_0/p}
\]
\[
\leq (C_{p}^{p, p}(O))^{q_0/p}.
\]
This proves (v).

Below we consider (v) \(\Rightarrow\) (i). Given \( f \in C_0^\infty(\mathbb{R}^n) \). It follows from (i)–(iv) and (v) of Lemma 2.3 that
\[
\mu(E_{\lambda, s}) \leq \mu\left(\left\{x \in \mathbb{R}^n : \mathcal{M} f(x) > \lambda/C\right\}\right)
\]
with a constant \( C \).

If (v) holds for every open set \( O \subset \mathbb{R}^n \),
\[
\mu(E_{\lambda, s}(f)) \lesssim \left(C_{p}^{p, p}\left(\left\{x \in \mathbb{R}^n : \mathcal{M} f(x) > \lambda/C\right\}\right)\right)^{q_0/p}.
\]
Then, (i) of Lemma 2.4 implies
\[
\int_0^\infty \left(\mu(E_{\lambda, s}(f))\right)^{p/q_0} d\lambda^p \lesssim \int_0^\infty C_{p}^{p, p}\left(\left\{x \in \mathbb{R}^n : \mathcal{M} f(x) > \lambda/C\right\}\right) d\lambda^p
\]
\[
\lesssim \|f\|_{A_p^{p, p}(\mathbb{R}^n)}^{p/p},
\]
i.e., (i) holds.
Part 2: (iii) \implies (iv) \implies (i). If (iii) holds, for any bounded open set \( O \subset \mathbb{R}^n \), let \( f \in C_0^\infty (\mathbb{R}^n) \) with \( f \geq 1 \) on \( O \). For \((x, t) \in T(O)\), then \( p_t^s \cdot |f(x)| > \eta_{n,s} \). Then,

\[
(\mu(T(O)))^{1/q_0} \leq \left( \mu(\{ (x, t) \in \mathbb{R}^{n+1}_+ : p_t^s \cdot f(x) > \eta_{n,s} \}) \right)^{1/q_0} \\
\leq K_p(\mu) \| f \|_{\mathcal{A}^{p,p}_\beta (\mathbb{R}^n)}.
\]

For any \( 0 < t < \mu(T(O)) \), we have \( t \leq (K_p(\mu) \| f \|_{\mathcal{A}^{p,p}_\beta (\mathbb{R}^n)})^{q_0} \). Taking the infimum on \( \| f \|_{\mathcal{A}^{p,p}_\beta (\mathbb{R}^n)} \) reaches \( t \leq (K_p(\mu))^{q_0}(C_{\beta}^{p,p}(O))^{q_0/p} \). This gives

\[
\sup_{t>0} \frac{t^{p/q_0}}{c_{p,p}^{\beta}(\mu; t)} \leq (K_p(\mu))^p.
\]

On the other hand, if (iv) holds, then we can get

\[
\int_0^\infty \mu(E_{\lambda,s}(f) \cap T(B(0,k)))^{p/q_0} d\lambda^p \\
\leq \int_0^\infty \left\{ \frac{(\mu(E_{\lambda,s}(f) \cap T(B(0,k))))^{p/q_0}}{c_{p,p}^\beta(\mu; \mu(E_{\lambda,s}(f) \cap T(B(0,k))))} \right\} \times C_{\beta}^{p,p}(O_{\lambda,s}(f) \cap B(0,k)) d\lambda^p \\
\leq \left( \sup_{t>0} \frac{t^{p/q_0}}{c_{p,p}^\beta(\mu; t)} \right) \int_0^\infty C_{\beta}^{p,p}(\{ x \in \mathbb{R}^n : \theta_{n,s}M f(x) > \lambda \} \cap B(0,k)) d\lambda^p \\
\leq \left( \sup_{t>0} \frac{t^{p/q_0}}{c_{p,p}^\beta(\mu; t)} \right) \| f \|_{\mathcal{A}^{p,p}_\beta (\mathbb{R}^n)}.
\]

Letting \( k \to \infty \) gives us (i).

Part 3: (v) \iff (vi). We first assume that (v) holds. Then, let \( O = B(x, r) \). By (v),

\[
(\mu(T(B(x,r))))^{p/q_0} \lesssim C_{\beta}^{p,p}(B(x,r)).
\]

For \( d \in (0, \infty) \), the classical Hausdorff capacity \( H_d^{(\infty)} \) is determined via

\[
H_d^{(\infty)}(K) := \inf \left\{ \sum_{i=0}^\infty m_i 2^{-di} \right\},
\]

where the infimum is taken over all countable coverings of \( K \subset \mathbb{R}^n \) by balls whose radii \( r_j < \infty \), while \( m_i \) is the number of balls from this covering whose radii \( r_j \) belong to the interval \((2^{-i-1}, 2^{-i}]\), \( i = 0, 1, 2, \ldots \). According to [1, Theorem 3.6, Theorem 3.8] and [2, p. 32], we have \( C_{\beta}^{p,p}(\cdot) \approx H_d^{(\infty)}(\cdot) \). Since \( C_{\beta}^{p,p}(B(x,r)) \approx \bullet \) Springer
\[ H_{n-p^\beta}(B(x,r)) \approx r^{n-p^\beta}, \]

\[
\sup_{(x,r) \in \mathbb{R}^{n+1}} \frac{(\mu(T(B(x,r))))^{p/q_0}}{r^{n-p^\beta}} \approx \sup_{(x,r) \in \mathbb{R}^{n+1}} \frac{(\mu(T(B(x,r))))^{p/q_0}}{C_{\beta}^{p,p}(B(x,r))} \]

\[
\lesssim \sup \left\{ \frac{(\mu(T(O)))^{p/q_0}}{C_{\beta}^{p,p}(O)} : \text{open set } O \subset \mathbb{R}^n \right\} < \infty.
\]

This gives (vi).

Conversely, we assume that (vi) holds. Then, we have

\[ T_{q_0,n,\beta,p}(\mu) := \sup_{(x,r) \in \mathbb{R}^{n+1}} \frac{\mu(T(B(x,r)))}{r^{q_0(n-p^\beta)/p}} < \infty. \]

Given an open set \( O \subset \mathbb{R}^n \) and let \( \{B(x_j, r_j)\}_{j \in \mathbb{Z}_+} \) be a sequence of covering balls of \( O \), i.e., \( O \subseteq \sqcup B(x_j, r_j) \). Below we only need to verify that (vi) holds for bounded sets. In fact, if (v) holds for bounded sets, for any open set \( O \subset \mathbb{R}^n \), let \( E_0 = B(0,1) \) and \( E_k = B(0,2^k) \setminus B(0,2^{k-1}) \), \( k \geq 1 \). Then, \( \mathbb{R}^n = \bigcup_{k=0}^{\infty} E_k \) and

\[ O = O \cap \left( \bigcup_{k=0}^{\infty} E_k \right) = \bigcup_{k=1}^{\infty} \left( O \cap E_k \right). \]

So, we can get

\[ (\mu(T(O)))^{p/q_0} \leq (\mu(\bigcup_{k=1}^{\infty} T((O \cap E_k))))^{p/q_0} \]

\[ \leq \left( \sum_{k=0}^{\infty} \mu(T(O \cap E_k)) \right)^{p/q_0} \]

\[ \lesssim \left( \sum_{k=0}^{\infty} C_{\beta}^{p,p}(O \cap E_k)^{q_0/p} \right)^{p/q_0} \]

\[ \lesssim \sum_{k=0}^{\infty} C_{\beta}^{p,p}(O \cap E_k) \]

\[ \lesssim C_{\beta}^{p,p}(\bigcup_{k=0}^{\infty}(O \cap E_k)) \]

\[ \lesssim C_{\beta}^{p,p}(O). \]

In the following, we assume that \( O \) is a bounded open subset of \( \mathbb{R}^n \). Then, there exists a dyadic cube sequence \( \{I_j^{(1)}\} \) in \( \mathbb{R}^n \) such that \( O \subseteq \bigcup_j I_j^{(1)} \) and \( |B(x_j, r_j)| \approx |I_j^{(1)}| \). On the other hand, according to Dafni and Xiao [10, Lemma 4.1], there exists
a sequence of dyadic cubes \( \{ I_j^{(2)} \} \) and denote by \( \{ I_j^{(3)} \} \) its \( 5\sqrt{n} \) expansion such that

\[
\begin{align*}
  \cup_j I_j^{(2)} &= \cup_j I_j^{(1)}; \\
  \sum_j |I_j^{(2)}|^{1-p\beta/n} &\leq \sum_j |I_j^{(1)}|^{1-p\beta/n}; \\
  T(O) &\subseteq \cup_j T(I_j^{(3)}).
\end{align*}
\]

Since \( |B(x_j, r_j)| \approx |I_j^{(1)}|, r_j \approx \ell(I_j^{(1)}) \approx \ell(I_j^{(3)}) \). Let \( x_I \) be the center of \( I_j^{(1)} \). Then, \( I_j^{(3)} \subseteq B(x_I, \ell(I_j^{(3)})) \). We can get

\[
\mu(T(O)) \leq \sum_j \mu(T(I_j^{(3)}))
\leq \sum_j \mu\left(T\left(B(x_I, \ell(I_j^{(3)}))\right)\right)
\leq \sum_j \left(\ell(I_j^{(3)})\right)^{q_0(\beta-n)/p} \mu\left(T\left(B(x_I, \ell(I_j^{(3)}))\right)\right) \left(\ell(I_j^{(3)})\right)^{q_0(n-\beta)/p}
\leq T_{q_0, n, \beta, p}(\mu) \sum_j \left(\ell(I_j^{(3)})\right)^{q_0(n-\beta)/p}
\leq T_{q_0, n, \beta, p}(\mu) \sum_j |I_j^{(3)}|^{q_0(\beta-n)/p}.
\] (4.1)

On the other hand, we have

\[
\sum_j |I_j^{(3)}|^{q_0(\beta-n)/p} \leq \sum_j |I_j^{(2)}|^{q_0(\beta-n)/p} \leq \left(\sum_j |I_j^{(1)}|^{1-p\beta/n}\right)^{q_0/p}
\leq \left(\sum_j |B(x_j, r_j)|^{1-p\beta/n}\right)^{q_0/p},
\]

which implies

\[
\sum_j |I_j^{(3)}|^{q_0(\beta-n)/p} \approx \left(H_n^{(\infty)}(O)^{q_0/p} \approx \left(C_{\beta}^{p, p}(O)^{q_0/p}ight.ight.
\] (4.2)

according to the definition of \( H_n^{(\infty)} \) and the equivalence of \( C_{\beta}^{p, p}(\cdot) \approx H_n^{(\infty)}(\cdot) \).

Thus, by (4.1) and (4.2), we get \( \mu(T(O)) \lesssim T_{q_0, n, \beta, p}(\mu) \left(C_{\beta}^{p, p}(O)^{q_0/p}\right) \) which implies (v).

\[\square\]
4.2 Case (2): $(p, q) \in (1, n/\beta) \times (1, \infty)$

Denote $p \vee q = \max \{p, q\}$ and $p \wedge q = \min \{p, q\}$.

**Theorem 4.2** Let $\mu$ be a non-negative Borel measure on $\mathbb{R}^{n+1}_+$. If $\beta \in (0, n)$ and $(p, q) \in (1, n/\beta) \times (1, \infty)$, then

$$\|u(\cdot, t)\|_{L_p^{\wedge q}, p^{\vee q}(\mathbb{R}^{n+1})} \lesssim \|f\|_{\dot{A}_\beta^{p, q}} \quad \forall f \in C_0^\infty(\mathbb{R}^n)$$

(4.3)

if and only if

$$\mu(T(O)) \lesssim (C_{\beta}^{p, q}(O))^{(p\wedge q)/p} \quad \forall \text{ open set } O \subset \mathbb{R}^n.$$  

(4.4)

**Proof** We first assume that (4.4) holds. Recall that

$$O_{\lambda, s}(f) = \left\{ x \in \mathbb{R}^n : \sup_{|y-x|<t} |u(y, t)| > \lambda \right\},$$

$$E_{\lambda, s}(f) = \left\{ (x, t) \in \mathbb{R}^{n+1}_+ : |u(x, t)| > \lambda \right\}.$$

It follows from (i) of Lemma 2.3 that

$$E_{\lambda, s}(f) \subset T(O_{\lambda, s}), \quad \mu(E_{\lambda, s}(f)) \leq \mu(T(O_{\lambda, s}(f))).$$

So, (4.4) implies

$$\mu((E_{\lambda, s}(f))) \leq \mu(T(O_{\lambda, s}(f))) \lesssim (C_{\beta}^{p, q}(O_{\lambda, s}(f)))^{(p\wedge q)/p},$$

which gives

$$\int_0^\infty (\mu(E_{\lambda, s}(f)))^{(p\vee q)/(p\wedge q)} d\lambda^{p\wedge q} \leq \int_0^\infty (\mu(T(O_{\lambda, s}(f))))^{(p\vee q)/(p\wedge q)} d\lambda^{p\wedge q} \lesssim \int_0^\infty C_{\beta}^{p, q}(O_{\lambda, s}(f))^{1/(q/p)} d\lambda^{p\wedge q}.$$ 

It follows from (iv) and (v) of Lemma 2.3 that $O_{\lambda, s} \subset \left\{ x \in \mathbb{R}^n : \mathcal{M}f(x) > \lambda/C \right\}$ with a constant $C$. So, we get

$$C_{\beta}^{p, q}(O_{\lambda, s}) \leq C_{\beta}^{p, q}(\left\{ x \in \mathbb{R}^n : \mathcal{M}f(x) > \lambda/C \right\}).$$
which implies that
\[
\int_0^\infty (\mu(E_{\lambda,s}(f)))^{(p/q)/(p\wedge q)} d\lambda^{p/q} \lesssim \int_0^\infty C_\beta^{p,q}((x \in \mathbb{R}^n : \mathcal{M}f(x) > \lambda/C))^{1/(q/p)} d\lambda^{p/q} \\
\lesssim \|f\|_{\dot{A}_\beta^{p,q}(\mathbb{R}^n)}^{p/q}
\]
according to (i) of Lemma 2.4. Hence, (4.3) holds.

4.3 Case (3): \((p, q) \in (1, n/\beta) \times \{\infty\}

Theorem 4.3 Let \(\beta \in (0, 1)\) and \(\mu\) be a non-negative outer measure on \(\mathbb{R}_{+}^{n+1}\). If \((\beta, p, q_0) \in (0, 1) \times (1, n/\beta) \times (0, \infty)\), then
\[
\|\mu(\cdot, \cdot)\|_{L_{\mu}^{q_0,\infty}(\mathbb{R}_{+}^{n+1})} \lesssim \|f\|_{\dot{A}_\beta^{p,\infty}(\mathbb{R}^n)} \ \ \forall \ f \in C_0^\infty(\mathbb{R}^n) \quad (4.5)
\]
if and only if
\[
\mu(T(O)) \lesssim (C_\beta^{p,\infty}(O))^{q_0/p} \ \ \forall \ \text{open set} \ O \subset \mathbb{R}^n. \quad (4.6)
\]

\textbf{Proof} Let \((\beta, p, q_0) \in (0, 1) \times (1, n/\beta) \times (0, \infty)\). Assume (4.5) holds. Then,
\[
K := \sup_{0 < \|f\|_{\dot{A}_\beta^{p,\infty}(\mathbb{R}^n)} < \infty, f \in C_0^\infty(\mathbb{R}^n)} \frac{\sup_{\lambda \in (0, \infty)} \left(\lambda^{q_0} \mu(E_{\lambda,s}(f))\right)^{1/q_0}}{\|f\|_{\dot{A}_\beta^{p,\infty}(\mathbb{R}^n)}} < \infty.
\]
For any open set \(O \subset \mathbb{R}^n\), we can take \(f \in C_0^\infty(\mathbb{R}^n)\) such that \(O \subset \text{Int}(\{x \in \mathbb{R}^n : f(x) \geq 1\})\). By (iii) of Lemma 2.3,
\[
(\mu(T(O)))^{1/q_0} \lesssim K \|f\|_{\dot{A}_\beta^{p,\infty}(\mathbb{R}^n)}.
\]
Then, the definition of \(C_\beta^{p,\infty}(\cdot)\) implies
\[
\mu(T(O)) \lesssim (C_\beta^{p,\infty}(O))^{q_0/p}.
\]
Thus, (4.6) holds.

On the other hand, assume that (4.6) holds. Then, for any \(f \in C_0^\infty(\mathbb{R}^n)\), (i)–(ii) of Lemma 2.3 imply
\[
\mu(E_{\lambda,s}(f)) \lesssim \mu(T(\{x \in \mathbb{R}^n : \mathcal{M}f(x) > \theta_{n,s}\lambda\})) \\
\lesssim (C_\beta^{p,\infty}(T(\{x \in \mathbb{R}^n : \mathcal{M}f(x) > \theta_{n,s}\lambda\})))^{q_0/p}.
\]
So, (ii) of Lemma 2.4 indicates that
\[
(\lambda^{q_0} \mu(E_{\lambda,s}(f)))^{1/q_0} \lesssim (\lambda^p C_\beta^{p,\infty}(T(\{x \in \mathbb{R}^n : \mathcal{M}f(x) > \theta_{n,s}\lambda\})))^{1/p} \lesssim \|f\|_{\dot{A}_\beta^{p,\infty}(\mathbb{R}^n)}.
\]
We have
\[
\sup_{\lambda > 0} \left( \lambda^{q_0} \mu \left( E_{\lambda, s} (f) \right) \right)^{1/q_0} \lesssim \| f \|^{-1}_{L^p_{\mu} (\mathbb{R}^n)},
\]
which gives (4.5). \hfill \Box

Acknowledgements The authors are very grateful to the anonymous reviewers for their numerous useful suggestions which lead to a substantial improvement of this manuscript. This work was in part supported by National Natural Science Foundation of China (Nos. 11871293, 12071272) and Shandong Natural Science Foundation of China (No. ZR2020MA004).

References

1. Adams, D.: The classification problem for capacities associated with the Besov and Triebel-Lizorkin spaces. Banach Center Publ. 22, 9–24 (1989)
2. Adams, D., Choquet integrals in potential theory. Publ. Mat. 42, 3–66 (1998)
3. Adams, D., Xiao, J.: Strong type estimates for homogeneous Besov capacities. Math. Ann. 325, 695–709 (2003)
4. Adams, R. A., Fournier J. J. F.: Sobolev Spaces, second ed., Academic Press, New York, (2003)
5. Beckner, W.: Pitt’s inequality and the uncertainty principle. Proc. Amer. Math. Soc. 123, 1897–1905 (1995)
6. Brändle, C., Colorado, E., Pablo, A., Sánchez, U.: A concaveconvex elliptic problem involving the fractional Laplacian. Proc. R. Soc. Edinb. A 143, 39–71 (2013)
7. Bui, H.Q., Candy, T.: A characterization of the Besov-Lipschitz and Triebel-Lizorkin spaces using Poisson like kernels, in: Functional analysis, harmonic analysis, and image processing: a collection of papers in honor of Björn Jawerth, in: Contemp. Math., vol. 693, Amer. Math. Soc., Providence, RI, pp. 109–141 (2017)
8. Caffarelli, L., Silvestre, L.: An extension problem related to the fractional Laplacian. Comm. Partial Differ. Equ. 32, 1245–1260 (2007)
9. Cianchi, A., Lutwak, E., Yang, D., Zhang, G.: Affine Moser-Trudinger and MorreySobolev inequalities. Calc. Var. Partial Differ. Equ. 36, 419–436 (2009)
10. Dafni, G., Xiao, J.: Some new tent spaces and duality theorems for fractional Carleson measures and Q(\mathbb{R}^n). J. Funct. Anal. 208, 377–422 (2004)
11. DeNápoli, P., Haddad, J., Jiménez, C., Montenegro, M.: The sharp affine L^2 Sobolev trace inequality and variants. Math. Ann. 370, 287–308 (2018)
12. Di Nezza, E., Palatucci, G., Valdinoci, E.: Hitchhiker’s guide to the fractional Sobolev spaces. Bulletin des sciences mathématiques 136(5), 521–573 (2012)
13. Eilertsen, S.: On weighted fractional integral inequalities. J. Funct. Anal. 185, 342–366 (2001)
14. Frank, R., Seiringer, R.: Non-linear ground state representations and sharp Hardy inequalities. J. Funct. Anal. 255, 3407–3430 (2008)
15. Haberl, C., Schuster, F., Xiao, J.: An asymmetric affine Polya-Szegö principle. Math. Ann. 352, 517–542 (2012)
16. Haddad, J., Jiménez, C., Montenegro, M.: Sharp affine weighted L^p Sobolev type inequalities. Trans. Amer. Math. Soc. 372, 2753–2776 (2019)
17. Hajajieh, H., Yu, X., Zhai, Z.: Fractional Gagliardo-Nirenberg and Hardy inequalities under Lorentz norms. J. Math. Anal. Appl. 396, 569–577 (2012)
18. Hao, C.: Introduction to harmonic analysis, lecture notes, available online
19. Herbst, I.: Spectral theory of the operator (\mu^2 + m^2)1/2 − Ze^2/r. Comm. Math. Phys. 53, 285–294 (1977)
20. Ingmanns, J.: Estimates for commutators of fractional differential operators via harmonic extension. arXiv preprint arXiv:2012.12072 (2020)
21. Lenzmann, E., Schikorra, A.: Sharp commutator estimates via harmonic extensions. Nonlinear Anal. 193, 111375 (2020)
22. Li, P., Shi, S., Hu, R., Zhai, Z.: Embeddings of Function Spaces via the Caffarelli-Silvestre Extension, capacities and Wolff potentials. Nonlinear Anal. 217, 112758 (2022)

23. Lombardi, N., Xiao, J.: Affine Poisson & non-Poisson trace inequalitys for $H^{-1-s}$, $s \leq \frac{1}{2}$ ($\mathbb{R}^{n-1} \geq 2$). Nonlinear Analysis, 111688 (2020)

24. Lutwak, E., Yang, D., Zhang, G.: Sharp affine $L^p$ Sobolev inequalities. J. Differential Geom. 62, 17–38 (2002)

25. Maz'ya, V.: On capacitary strong type estimates for fractional norms (Russian). Zup. Nauch. Sem. Leningrad otel. Math. Inst. Steklov (LOMI) 70, 161-168 (1977)

26. Milman, M., Xiao, J.: The $\infty$-Besov capacities problem. Math. Nachr. 290, 2961–2976 (2017)

27. Nguyen, V.: Some trace Hardy type inequalities and trace Hardy-Sobolev-Maz’ya type inequalities. J. Funct. Anal. 270, 4117–4151 (2016)

28. Stein, E.M.: Singular Integrals and Differentiability Properties of Functions. Princeton Univ. Press, Princeton, NY (1970)

29. Stein, E.M.: Harmonic Analysis: Real-variable Methods, Orthogonality, and Oscillatory Integrals. Princeton University Press, Princeton, New Jersey (1993)

30. Xiao, J.: A sharp Sobolev trace inequality for the fractional-order derivatives. Bull. Sci. Math. 130, 87–96 (2006)

31. Xiao, J.: Carleson embeddings for Sobolev spaces via heat equation. J. Differ. Equations 224, 277–295 (2006)

32. Xiao, J.: Homogeneous endpoint Besov space embeddings by Hausdorff capacity and heat equation. Adv. Math. 207, 828–846 (2006)

33. Xiao, J., Zhai, Z.: Fractional Sobolev, Moser-Trudinger, Morrey-Sobolev inequalities under Lorentz norms. J. Math. Science 166, 357–376 (2010)

34. Xiao, J., Zhai, Z.: C.S.I. for Besov Spaces $\dot{A}_p^{\alpha, q} (\mathbb{R}^n)$ with $(\alpha, (p, q)) \in (0, 1) \times (0, 1] \times (0, 1] (1, 1)$. Advanced Lectures in Mathematics Volume 34 Some Topics in Harmonic Analysis and Applications 34, 407-420 (2016)

35. Yafaev, D.: Sharp constants in the Hardy-Rellich inequalities. J. Funct. Anal. 168, 121–144 (1999)

36. Zhai, Z.: Carleson measure problems for parabolic Bergman spaces and homogeneous Sobolev spaces. Nonlinear Anal. TMA. 73, 2611–2630 (2010)

37. Zhai, Z.: Note on affine Gagliardo-Nirenberg inequalities. Potential Anal. 34, 1–12 (2011)

38. Zhang, G.: The affine Sobolev inequality. J. Differential Geom. 53, 183–202 (1999)

39. Wu, Z.: Strong type estimate and Carleson measures for Lipschitz spaces. Proc. Amer. Math. Soc. 127, 3243–3249 (1999)

Publisher’s Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.