Nonlinear fluid dynamics of warped discs

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Abstract. A new description of the dynamics of warped accretion discs is presented. A theory of fully nonlinear, slowly varying bending waves is developed, involving a proper treatment of viscous fluid dynamics but neglecting self-gravitation. The special case of resonant wave propagation found in inviscid Keplerian discs is not considered. The resulting description goes beyond existing linear and nonlinear theories, and is suitable for numerical implementation.

1. Introduction

Non-planar accretion discs are believed to exist in many astrophysical situations. In some cases, such as the nuclear disc of the galaxy NGC 4258 or the circumstellar disc of β Pictoris, a warped profile is observed more or less directly. In others, including the X-ray binaries Her X-1 and SS 433, tilting of the disc is proposed in order to explain certain observational characteristics. A more complete discussion is given by Pringle (this volume).

In accretion discs, unlike the case of galactic warps, the effects of self-gravitation may often be neglected. The first attempts to describe the time-dependence of a warped disc within the framework of viscous fluid dynamics (Petterson 1978; Hatchett, Begelman and Sarazin 1981) indicated that the warp would simply diffuse away on a viscous time scale. However, Papaloizou and Pringle (1983) showed that these theories were oversimplified and failed to conserve angular momentum as a result of serious internal inconsistencies.

Special complications arise because of the existence of a resonance in thin discs that are both Keplerian (or very nearly so) and inviscid (or very nearly so). A distinction must therefore be made between the generic (non-resonant) case and the resonant case which occurs when the two conditions

\[
\left| \frac{\Omega^2 - \kappa^2}{\Omega^2} \right| \lesssim \frac{H}{r} \quad \text{and} \quad \alpha \lesssim \frac{H}{r}
\]

are both satisfied. Here \( \Omega \) is the angular velocity, \( \kappa \) is the epicyclic frequency, \( H/r \) is the ratio of the semi-thickness of the disc to the radius, and \( \alpha \) is the dimensionless viscosity parameter (Shakura and Sunyaev 1973). Whether the resonant case occurs in realistic astrophysical situations is a matter of some uncertainty. Current estimates suggest that the resonant case is not relevant to discs in X-ray binaries or active galactic nuclei, but may apply in protostellar
discs. However, the resonance is delicate and might be destroyed by the effects of self-gravitation, magnetic fields, or turbulence.

The non-resonant case was examined by Papaloizou and Pringle (1983), who considered a Keplerian disc with a significant viscosity. They treated the warp as a slowly varying disturbance with azimuthal wave number \( m = 1 \), using linear Eulerian perturbation theory. The resulting equation for the warp is a complex linear diffusion equation, which has been used in applications (e.g. Kumar and Pringle 1985).

Other cases were treated by Papaloizou and Lin (1994, 1995), who considered an inviscid, but not necessarily Keplerian, disc. Again, linear Eulerian perturbation theory was used, and the authors assumed the warp to be a slowly varying normal mode of the disc. In the non-Keplerian (non-resonant) case the warp obeys a dispersive linear wave equation, while in the Keplerian (resonant) case the warp obeys a non-dispersive linear wave equation. By considering the effect of a small viscosity on the inviscid modes, Papaloizou and Lin connected this theory with that of Papaloizou and Pringle (1983), showing how the transition occurs between wave-like and diffusive behaviour in Keplerian discs when \( \alpha \approx H/r \). This theory has also been used in applications (e.g. Papaloizou and Terquem 1995).

A difficulty arises with these theories because the Eulerian method is formally valid only when the tilt angle \( \beta(r,t) \) of the warp satisfies \( |\beta| \ll H/r \), a condition violated – almost by definition – by any interesting, observable warp. Of course, a linear theory may be derived by any convenient method, and violation of the above condition does not necessarily imply that the linearized dynamics ceases to apply, only that the Eulerian method cannot be used to justify it. Indeed, it is clear the Eulerian method is basically not well suited to the description of a warp or even a rigid tilt, and that some kind of Lagrangian method ought to be used to describe warps of significant amplitude and to discuss nonlinear effects. The Eulerian method offers little information about nonlinear effects, although it does predict that, in the resonant case, an amplitude \( |\partial \beta/\partial \ln r| \approx H/r \) would result in horizontal shearing motions in the disc comparable to the sound speed, which are expected to be unstable (Kumar and Coleman 1993; Gammie, Goodman and Ogilvie, in preparation).

Meanwhile, Pringle (1992) developed a different approach in which the forms of the equations governing a warped viscous disc are derived simply by requiring mass and angular momentum to be conserved, but without reference to the detailed internal fluid dynamics of the disc. In this scheme, neighbouring rings in the disc exchange angular momentum by means of viscous torques which are of two kinds. One kind of torque (associated with a kinematic viscosity coefficient \( \nu_1 \)) acts on the differential rotation in the plane of the disc, and leads to accretion; the other kind (associated with \( \nu_2 \)) acts to flatten the disc. The equations derived (originally by Papaloizou and Pringle 1983) are

\[
\frac{\partial \Sigma}{\partial t} + \frac{1}{r} \frac{\partial}{\partial r} (r \Sigma \bar{v}_r) = 0 \tag{2}
\]

for the surface density \( \Sigma(r,t) \), and

\[
\frac{\partial}{\partial t} (\Sigma r^2 \Omega \ell) + \frac{1}{r} \frac{\partial}{\partial r} (\Sigma \bar{v}_r r^3 \Omega \ell) = \frac{1}{r} \frac{\partial}{\partial r} \left( \nu_1 \Sigma r^3 \frac{\partial \Omega}{\partial r} \ell \right) + \frac{1}{r} \frac{\partial}{\partial r} \left( \frac{1}{2} \nu_2 \Sigma r^3 \frac{\partial \ell}{\partial r} \right) \tag{3}
\]
for the angular momentum, in the absence of external torques. Here $\bar{v}_r(r,t)$ is the mean radial velocity and $\ell(r,t)$ is the tilt vector, which is a unit vector parallel to the orbital angular momentum of the ring. These equations constitute a straightforward generalization of the equations used for a flat disc (e.g. Pringle 1981).

This approach is ostensibly valid for warps of large amplitude and would therefore represent an advance on the linear theory. It has been used in several applications, in particular to identify the radiation-driven instability (Pringle 1996) and to explore its linear theory (e.g. Maloney, Begelman and Pringle 1996) and nonlinear evolution (e.g. Pringle 1997). However, because the equations of Pringle (1992) are derived somewhat heuristically, without reference to the detailed internal fluid dynamics of the disc, some doubts remain over the validity of this method. For example, has any internal degree of freedom of the rings been neglected? Is the interaction between neighbouring rings purely of the assumed form of viscous torques? Are there any nonlinear fluid dynamical effects that might limit the amplitude of the warp? How are the viscosity coefficients $\nu_1$ and $\nu_2$ related? It is therefore important to investigate whether the equations of Pringle (1992) can be derived ab initio from the three-dimensional fluid dynamical equations, and to understand how they connect to the previously established linear theory.

An entirely different approach to warps of large amplitude, based on threedimensional numerical simulations, is described by Nelson (this volume).

2. Outline of the method

The basis of the approach is to develop a hydrodynamic theory of unforced bending waves of large amplitude in a thin disc. This is most naturally done in the case of a spherically symmetric external potential, in which a flat disc has no preferred plane of orientation. Continuous with the zero-frequency rigid-tilt mode, there exist bending waves – with azimuthal wave number $m = 1$ in linear theory – that are slowly varying in time and space. In contrast, most of the modes in a thin disc vary on a time-scale comparable to $\Omega^{-1}$ and on a length-scale comparable to $H$ (e.g. Lubow and Pringle 1993). Therefore these bending waves are special and deserve a special analysis.

A full account of the analysis has been given elsewhere (Ogilvie 1998). It will suffice here to explain the various steps involved with an emphasis on physical interpretation.

2.1. Warped coordinates

The limitations of an Eulerian approach to the fluid dynamics of a warped disc have already been noted. However, a fully Lagrangian approach is not suitable for dealing with a differentially rotating flow. Therefore a ‘semi-Lagrangian’ method is used, in which the equations are derived in Eulerian form but referred to a coordinate system that follows the principal warping motion of the disc. Warped spherical polar coordinates $(r, \theta, \phi)$ are defined as follows (Figure 1). $r$ is the spherical radial coordinate. On each sphere $r = \text{constant}$ the usual angular coordinates $(\theta, \phi)$ are defined, but with respect to an axis that is tilted to point in the direction of the unit vector $\ell(r,t)$. The intention is that the
disc matter on each sphere will lie close to $\theta = \pi/2$. This coordinate system is similar to that introduced by Petterson (1978) but has certain advantages. It is valid for any amplitude of warp, whereas Petterson’s system, which is based on tilted cylinders, breaks down for large amplitudes or for insufficiently thin discs. The present system also has the same Jacobian determinant as ordinary spherical polar coordinates, which simplifies the equations to some extent.

The equations of fluid dynamics can be readily derived in these coordinates using methods of elementary vector calculus. It is assumed that the fluid obeys the compressible Navier-Stokes equation with isotropic viscosity, but viscous heating is neglected. The principal features of the equations are (i) the appearance of a number of fictitious forces and (ii) the introduction of a ‘modified’ radial derivative operator. The effect of the latter feature is that in a warped disc the vertical pressure gradient gives rise to a radial force which drives horizontal motions (Figure 2).

2.2. Thin-disc asymptotics

The assumption that the disc is thin allows progress to be made by introducing scaled variables and asymptotic expansions. Although informal thin-disc approximations are common in accretion-disc theory, the rigorous procedures and wide applicability of asymptotic methods are not always appreciated. In the present problem, the basic small parameter $\epsilon$ is a characteristic value of $H/r$. The problem is simplified by (i) separating the ‘fast’ orbital time-scale from the ‘slow’ time-scale of viscous and dynamical evolution of the warp, (ii) separating the ‘fast’ orbital velocity, the ‘intermediate’ oscillatory velocities induced by the warp and the ‘slow’ accretion velocity, and (iii) allowing thin-disc geometrical simplifications. For example, the relative velocity components have
the expansions

\begin{align}
  v_r(r, \theta, \phi, t) &= \epsilon v_{r1}(r, \phi, \zeta, T) + \epsilon^2 v_{r2}(r, \phi, \zeta, T) + O(\epsilon^3), \\
  v_\theta(r, \theta, \phi, t) &= \epsilon v_{\theta1}(r, \phi, \zeta, T) + \epsilon^2 v_{\theta2}(r, \phi, \zeta, T) + O(\epsilon^3), \\
  v_\phi(r, \theta, \phi, t) &= r \Omega(r) \sin \theta + \epsilon v_{\phi1}(r, \phi, \zeta, T) + \epsilon^2 v_{\phi2}(r, \phi, \zeta, T) + O(\epsilon^3),
\end{align}

where \( \zeta = \epsilon^{-1}(\pi/2 - \theta) \) is a scaled dimensionless vertical coordinate in the disc, and \( T = \epsilon^2 t \) is the slow time coordinate. Note that all variables are non-axisymmetric except for the orbital angular velocity \( \Omega(r) \).

### 2.3. Formal structure of the problem

After substituting the asymptotic expansions into the dynamical equations, a number of equations are extracted at different orders. One of these determines the orbital angular velocity \( \Omega(r) \) as that of a free particle in a circular orbit in the given potential. The remaining equations may be divided into two sets. Physically, Set A determines the ‘intermediate’ velocities, while Set B determines the ‘slow’ velocities. Mathematically, Set A consists of coupled nonlinear partial differential equations (PDEs) in two dimensions \( (\phi, \zeta) \), while Set B consists of coupled linear PDEs whose coefficients depend on the solution of Set A.

It turns out that, while Set A must be solved in full, all the information required from Set B can be extracted by integration. Indeed, Set B can be manipulated to derive the basic conservation equations,

\begin{align}
  \frac{\partial \Sigma}{\partial t} + \frac{1}{r} \frac{\partial}{\partial r} (r \Sigma \bar{v}_r) &= 0, \\
  \frac{\partial}{\partial t} (\Sigma r^2 \Omega \ell) + \frac{1}{r} \frac{\partial}{\partial r} (\Sigma \bar{v}_r r^3 \Omega \ell) &= \frac{1}{r} \frac{\partial}{\partial r} \left( Q_1 I r^2 \Omega^2 \ell \right) + \frac{1}{r} \frac{\partial}{\partial r} \left( Q_2 I r^3 \Omega^2 \frac{\partial \ell}{\partial r} \right) \\
  &\quad + \frac{1}{r} \frac{\partial}{\partial r} \left( Q_3 I r^3 \Omega^2 \ell \times \frac{\partial \ell}{\partial r} \right).
\end{align}
for mass and angular momentum. Note that this angular momentum equation is similar to, but more general than, equation (3). Here \( I(r,t) \) is the azimuthally averaged second vertical moment of the density, which in cylindrical polar coordinates would be written

\[
I = \frac{1}{2\pi} \int_0^{2\pi} \left( \int_{-\infty}^{\infty} \rho z^2 \, dz \right) \, d\phi,
\]

and is an important dynamical quantity in the theory of bending waves. The quantities \( Q_i \) are dimensionless coefficients which can be calculated from the solution of Set A, and which depend on the amplitude of the warp as well as the rotation law and details of the thermodynamics and viscosity. The angular momentum equation implies that each ring in the disc experiences torques of three kinds from its neighbours: coefficient \( Q_1 \) represents a torque tending to spin up (or down) the ring. This would be the usual viscous torque proportional to \( d\Omega / dr \) in a flat disc, but in a warped disc there is an additional contribution, not proportional to \( d\Omega / dr \), due to a correlation between the radial and azimuthal velocities induced by the warp; this vanishes in an inviscid disc because the radial and azimuthal velocities are perfectly out of phase. Coefficient \( Q_2 \) represents a torque tending to align the ring with its neighbours, which acts to flatten the disc; this also vanishes in an inviscid disc. Coefficient \( Q_3 \) represents a torque tending to make the ring precess if it is misaligned with its neighbours; this leads to the dispersive wave-like propagation of the warp.

2.4. Separation of variables

The further assumptions that (i) the disc is polytropic (or isothermal) locally in radius and (ii) the viscosity coefficients are locally proportional to the pressure allow Set A to be solved by separation of variables, since the vertical dependence of the solution can be determined by inspection. The problem is then reduced to solving a set of dimensionless nonlinear ordinary differential equations (ODEs) in azimuth, subject to periodic boundary conditions, and extracting the three dimensionless coefficients \( Q_1, Q_2 \) and \( Q_3 \) from the solution. The system is of fourth order and involves five dimensionless parameters: the dimensionless amplitude of the warp,

\[
|\psi| = \left| \frac{\partial \ell}{\partial \ln r} \right|,
\]

the dimensionless squared epicyclic frequency,

\[
\tilde{\kappa}^2 = \frac{d\ln(r^4\Omega^2)}{d\ln r},
\]

the polytropic coefficient \( \Gamma \) and the dimensionless viscosity coefficients \( \alpha \) and \( \alpha_b \) (for shear and bulk viscosities, respectively).

2.5. Evaluation of the coefficients

Examples of the numerical evaluation of the coefficients are given by Ogilvie (1998). Of particular importance is the diffusion coefficient \( Q_2 \); this is shown in Figure 3 for the case of an isothermal, Keplerian disc without bulk viscosity \((\Gamma = 1; \tilde{\kappa}^2 = 1; \alpha_b = 0)\). It can be seen that the resonant behaviour associated with the limit \( \alpha \to 0 \) is diminished as the amplitude of the warp increases.
Figure 3. Contour plot of the coefficient $Q_2$ for an isothermal, Keplerian disc without bulk viscosity. The horizontal and vertical coordinates are the dimensionless amplitude of the warp and the dimensionless viscosity parameter, respectively.
2.6. Nature of the nonlinearity

A weakly nonlinear theory can be developed analytically (Ogilvie 1998) which provides truncated Taylor series of the form \( a + b|\psi|^2 + O(|\psi|^4) \) for the three coefficients. The weakly nonlinear theory therefore contains a cubic nonlinearity which arises through a three-mode coupling. The modes involved (Figure 4) are (i) the ‘tilt’ mode or f mode \((m = 1, \text{ of odd symmetry})\), consisting locally of a uniform vertical translation of the disc, (ii) an inertial or r mode \((m = 1, \text{ of odd symmetry})\), consisting of a horizontal epicyclic motion proportional to \(z\), and (iii) an acoustic or p mode \((m = 2, \text{ of even symmetry})\), consisting of a vertical motion proportional to \(z\). The compressive nature of the third mode explains why the nonlinear behaviour depends on the thermodynamics and, in principle, on the bulk viscosity.

3. Discussion

It has been shown that the nonlinear fluid dynamics of a warped accretion disc is described by conservation equations (7) and (8) for mass and angular momentum, respectively. The two basic assumptions are that (i) the disc is thin and (ii) the resonant case [equation (1)] does not occur. Equation (8) may be decomposed into an equation for the component of angular momentum parallel to \(\ell\),

\[
\Sigma \ddot{v}_r \frac{d(r^2 \Omega)}{dr} = \frac{1}{r} \frac{\partial}{\partial r} \left( Q_1 I r^2 \Omega^2 \right) - Q_2 I r^2 \Omega^2 \left| \frac{\partial \ell}{\partial r} \right|^2,
\]

and an equation for the tilt vector,

\[
\Sigma r^2 \Omega \left( \frac{\partial \ell}{\partial t} + \ddot{v}_r \frac{\partial \ell}{\partial r} \right) = Q_1 I r \Omega^2 \frac{\partial \ell}{\partial r} + \frac{1}{r} \frac{\partial}{\partial r} \left( Q_2 I r^3 \Omega^2 \frac{\partial \ell}{\partial r} \right)
+ Q_2 I r^2 \Omega^2 \left| \frac{\partial \ell}{\partial r} \right|^2 \ell + \frac{1}{r} \frac{\partial}{\partial r} \left( Q_3 I r^3 \Omega^2 \ell \times \frac{\partial \ell}{\partial r} \right).
\]

The warp therefore satisfies a kind of wave equation of parabolic type, featuring advection, diffusion and dispersion, each with a nonlinear dependence on the amplitude of the warp. This form of equation arises for the following reason. According to Set A, the velocities induced by the warp are determined
instantaneously and locally in radius. These velocities are in a ‘geostrophic’ state in which inertial forces balance the pressure gradients and viscous forces; time-derivatives of the velocities in the inertial frame do not appear because of the assumed separation of time-scales. This means that the velocities do not constitute additional dynamical degrees of freedom. In the resonant case this approximation breaks down and the time-derivatives must be restored; the resulting equations for the warp are hyperbolic rather than parabolic.

Since equation (5) is the angular momentum equation it is straightforward to add to the right-hand side any additional torque $T$ due to tidal forcing, Lense-Thirring precession, radiation forces, self-gravitation, etc.

Under the further assumptions of Section 2.4 concerning the thermodynamics and viscosity, the coefficients $Q_i$ can be consistently evaluated. For small-amplitude warps there is agreement with linear theory, but for larger amplitudes a numerical solution of the ODEs is required.

The dispersion coefficient $Q_3$ is generally smaller than the diffusion coefficient $Q_2$, and, to the extent that it can be neglected, the angular momentum equation (5) given by Pringle (1992) is valid. However, in general, both $\nu_1$ and $\nu_2$ depend on the amplitude of the warp are neither is equal to the usual vertically averaged viscosity $\bar{\nu}$. The most important result from linear theory for a Keplerian disc (Papaloizou and Pringle 1983) was that, although $\nu_1 \approx \bar{\nu}$,

$$\frac{\nu_2}{\nu_1} \approx \frac{2(1 + 7\alpha^2)}{\alpha^2(4 + \alpha^2)} \approx \frac{1}{2\alpha^2} \quad \text{for } \alpha \ll 1. \quad (14)$$

Therefore, for typical estimates in the range $0.01 \lesssim \alpha \lesssim 0.1$, a Keplerian disc is much more resistant to warping than would be estimated on the basis of the viscosity $\bar{\nu}$. The nonlinear analysis shows that this resonant behaviour is diminished as the amplitude of the warp increases: the ratio $\nu_2/\nu_1$ decreases with increasing amplitude of the warp, although it remains significantly larger than unity. The consequences of this and other complex properties of the system ought to be investigated using a numerical implementation of the governing equations incorporating a consistent determination of the three coefficients.

Acknowledgments. I thank Jim Pringle for many helpful discussions. I acknowledge the hospitality of the Isaac Newton Institute during the programme Dynamics of Astrophysical Discs, where I benefited from discussions with many participants, and where much of this work was completed.

References

Hatchett S. P., Begelman M. C. & Sarazin C. L. (1981). *Astrophys. J.* **247**, 677
Kumar S. & Coleman C. S. (1993). *Mon. Not. R. Astr. Soc.* **260**, 323
Kumar S. & Pringle J. E. (1985). *Mon. Not. R. Astr. Soc.* **213**, 435
Lubow S. H. & Pringle J. E. (1993). *Astrophys. J.* **409**, 360
Maloney P. R., Begelman M. C. & Pringle J. E. (1996). *Astrophys. J.* **472**, 582
Ogilvie G. I. (1998). *Mon. Not. R. Astr. Soc.,* submitted
Papaloizou J. C. B. & Lin D. N. C. (1994). In *Theory of Accretion Disks – 2*, eds W. J. Duschl et al. Kluwer, Dordrecht
Papaloizou J. C. B. & Lin D. N. C. (1995). *Astrophys. J.* **438**, 841
Papaloizou J. C. B. & Pringle J. E. (1983). *Mon. Not. R. Astr. Soc.* **202**, 1181
Papaloizou J. C. B. & Terquem C. (1995). *Mon. Not. R. Astr. Soc.* **274**, 987
Petterson J. A. (1978). *Astrophys. J.* **226**, 253
Pringle J. E. (1981). *Annu. Rev. Astron. Astrophys.* **19**, 137
Pringle J. E. (1992). *Mon. Not. R. Astr. Soc.* **258**, 811
Pringle J. E. (1996). *Mon. Not. R. Astr. Soc.* **281**, 357
Pringle J. E. (1997). *Mon. Not. R. Astr. Soc.* **292**, 136
Shakura N. I. & Sunyaev R. A. (1973). *Astron. Astrophys.* **24**, 337