Analytical benchmark for the non-equilibrium Marshak diffusion problem in a planar slab of finite thickness

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Abstract

An analytical solution to the non-equilibrium Marshak diffusion problem in a planar slab of finite thickness is presented. Analytic expressions for the radiation and material energy densities as a function of space and time are derived using the Laplace transform method by summing over the first few residues at the poles of the transcendental equation. Integrated energy densities and leakage currents are also obtained in analytical form. Results for a planar slab of any finite thickness can be generated using the analytic expressions of these quantities unlike the previous works wherein numerical results were generated to a specified degree of accuracy for a semi-infinite medium with semi analytical solutions. The benchmark results obtained in this work can be used to validate and verify non equilibrium radiation diffusion computer codes.

Keywords: Non-equilibrium Marshak diffusion; analytic solution; planar slab of finite thickness; Laplace transform method
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1 Introduction

The time dependent non-equilibrium radiation transport equation is non linearly coupled to the material energy equation. Also the material properties have complex dependence on the independent variables. As a result, the time dependent thermal radiation transport problems are commonly solved numerically. Benchmark results for test problems are necessary to validate and verify the numerical codes. Analytical solutions producing explicit expressions for the radiation and material energy density, integrated densities, leakage fluxes, etc. are the most desirable.

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In the past, considerable amount of efforts have been applied for solving the Radiation Transport problem analytically. All the available results are semi-analytical and the desired quantities are tabulated to some fixed accuracy level for a particular value of material parameters. The first semi-analytical solution was obtained by Marshak in which he considered a semi infinite planar slab with radiation incident upon the surface. Assuming that the radiation and material fields are in equilibrium, similarity solutions were applied and the problem reduced to a second order ordinary differential equation which was solved numerically. Later, Pomraning considered the non equilibrium radiation diffusion problem with the assumption that the material specific heat is proportional to the cube of the temperature so that the equations become linear in the radiative intensity and the fourth power of the material temperature. Using Laplace transform method, Pomraning derived the semi analytical solutions for the surface quantities, integral quantities, and the distributions of radiative energy and material temperature as functions of space and time for the case when the speed of light is treated as infinity. Subsequently, the semi analytical results for finite speed of light was obtained by Su and Olson. Semi analytical results were also obtained for the transport problem with a point source applied for a finite time. The non-grey benchmark results for the two temperature non equilibrium radiative transfer were also generated. The normal-mode expansion technique and spherical harmonics have been applied for solving the steady state radiation transport equation in a finite slab. Numerical results for reflectivity and transmittivity in a finite planar slab having specularly reflecting boundaries has been obtained using the travelling wave transformation.

In this paper, we solve the non equilibrium Marshak diffusion problem for a plane slab of finite thickness with radiation incident upon one of its surfaces. The difference from the semi infinite slab is that radiation will leak from the other surface due to its finite thickness and vacuum boundary condition. Non equilibrium diffusion codes can be more easily validated and verified against these benchmark results as there is no need to take a slab of very large size for avoiding boundary effects. As all real life systems are finite in size, these results are also more practical and experimental results are more easily modelled. For the semi infinite slab, because of the multiple valuedness of the functions obtained by Laplace transform, inverting them using the inverse Laplace transform required evaluation of contributions from all the branch cuts. This resulted in integrals which had to be computed numerically. The oscillations in the integrand resulted in difficulty in their convergence. The advantage of solving the finite problem is that because of the single valuedness of the Laplace transformed functions, the inversion using inverse Laplace transform is very simple. The sum of the residues at the singularities (poles) give the required solution. Thus analytical expressions for all the quantities of interest can be obtained for any slab width and parameter value. These results will be very useful for benchmarking and checking the sensitivity of non equilibrium radiation diffusion codes.

The remainder of the paper is organised as follows. In Section 2 the ana-
2 Analytical solution

We consider a planar slab of finite thickness which is purely absorbing and homogeneous occupying $0 \leq z \leq l$. The medium is at zero temperature initially. At time $t=0$, a time independent radiative flux ($F_{inc}$) is incident on the surface at $z=0$ as shown in Fig. 1. Neglecting hydrodynamic motion and heat conduction, the one group radiative transfer equation (RTE) in the diffusion approximation and the material energy balance equation (ME) are

\[
\frac{\partial E(z,t)}{\partial t} - \frac{\partial}{\partial z} \left[ c \frac{\partial E(z,t)}{\partial z} \right] = c \kappa(T) [aT^4(z,t) - E(z,t)] \tag{1}
\]

\[
C_v(T) \frac{\partial T(z,t)}{\partial t} = c \kappa(T) [E(z,t) - aT^4(z,t)] \tag{2}
\]

where $E(z,t)$ is the radiation energy density, $T(z,t)$ is the material temperature, $\kappa(T)$ is the opacity (absorption cross section), $c$ is the speed of light, $a$ is the radiation constant, and $C_v(T)$ is the specific heat of the material.

The Marshak boundary condition on the surface at $x=0$ is given by

\[
E(0,t) - \left( \frac{2}{3\kappa[T(0,t)]} \right) \frac{\partial E(0,t)}{\partial z} = \frac{4}{c} F_{inc} \tag{3}
\]

where $F_{inc}$ is the flux incident upon the surface $z=0$.

And that at $z = l$ is

\[
E(l,t) + \left( \frac{2}{3\kappa[T(l,t)]} \right) \frac{\partial E(l,t)}{\partial z} = 0 \tag{4}
\]

The initial conditions on these two equations are

\[
E(z,0) = T(z,0) = 0 \tag{5}
\]
To remove the nonlinearity in the RTE (Eq. 1) and ME (Eq. 2), opacity \( \kappa \) is assumed to be independent of temperature and specific heat \( C_v \) is assumed to be proportional to the cube of the temperature. i.e., \( C_v = \alpha T^3 \). The RTE and the ME are recast into the dimensionless form by introducing the dimensionless independent variables given by

\[
x \equiv \sqrt{3} \kappa z, \tau \equiv \left( \frac{4 \alpha c \kappa}{\alpha} \right) t
\]

and new dependent variables given by

\[
u(x, \tau) \equiv \left( \frac{c}{4} \right) \left[ \frac{E(z, t)}{F_{inc}} \right], u(x, \tau) \equiv \left( \frac{c}{4} \right) \left[ \frac{\alpha T^4(z, t)}{F_{inc}} \right]
\]

With these new variables, the RTE and ME take the dimensionless form

\[
\varepsilon \frac{\partial u(x, \tau)}{\partial \tau} = \frac{\partial^2 u(x, \tau)}{\partial x^2} + v(x, \tau) - u(x, \tau)
\]

\[
\frac{\partial v(x, \tau)}{\partial \tau} = u(x, \tau) - v(x, \tau)
\]

with the initial conditions

\[
u(x, 0) = 0
\]

\[
u(x, 0) = 0
\]

And the boundary conditions on the surfaces are

\[
u(0, \tau) - \frac{2}{\sqrt{3}} \frac{\partial \nu(0, \tau)}{\partial x} = 1
\]

\[
u(b, \tau) + \frac{2}{\sqrt{3}} \frac{\partial \nu(b, \tau)}{\partial x} = 0
\]

where \( b = \sqrt{3} \kappa l \) and the parameter \( \varepsilon \) is defined as

\[
\varepsilon = \frac{16 \sigma}{c \alpha} = \frac{4a}{\alpha}
\]

To solve Eqs. 8 - 13, we introduce the Laplace transform according to

\[
\tilde{f}(s) = \int_0^\infty d\tau e^{-s \tau} f(\tau)
\]

to obtain

\[
\varepsilon s \tilde{u}(x, s) - \frac{\partial^2 \tilde{u}(x, s)}{\partial x^2} = \tilde{v}(x, s) - \tilde{u}(x, s)
\]

\[
s \tilde{v}(x, s) = \tilde{u}(x, s) - \tilde{v}(x, s)
\]

\[
\tilde{u}(0, s) - \frac{2}{\sqrt{3}} \frac{\partial \tilde{u}(0, s)}{\partial x} = \frac{1}{s}
\]

\[
\tilde{u}(b, s) + \frac{2}{\sqrt{3}} \frac{\partial \tilde{u}(b, s)}{\partial x} = 0
\]
The solutions of Eqs. (16)-(19) in s space are obtained as

\[
\bar{u} = \frac{3\sin[\beta(s)(b-x)] + 2\sqrt{3}\beta(s)\cos[\beta(s)(b-x)]}{s[3\sin(\beta(s)b) + 4\sqrt{3}\beta(s)\cos(\beta(s)b) - 4\beta(s)^2\sin(\beta(s)b)]} + 2\sqrt{3} \beta(s)[3\sin(\beta(s)b) + 4\sqrt{3}\beta(s)\cos(\beta(s)b) - 4\beta(s)^2\sin(\beta(s)b)]
\]

\[
\bar{v} = \frac{3\sin[\beta(s)(b-x)] + 2\sqrt{3}\beta(s)\cos[\beta(s)(b-x)]}{s(s + 1)[3\sin(\beta(s)b) + 4\sqrt{3}\beta(s)\cos(\beta(s)b) - 4\beta(s)^2\sin(\beta(s)b)]}
\]

where \(\beta(s)\) is given by

\[
\beta^2(s) = -\frac{s}{s + 1}[1 + \varepsilon(s + 1)]
\]

Before solving for the radiation and material energy densities by inverting \(\bar{u}\) and \(\bar{v}\), we first obtain the small and large \(\tau\) limits of \(u(x,\tau)\) and \(v(x,\tau)\) from the large and small s limits of Eqs. (20) and (21) respectively. Using the theorems

\[
\lim_{s \to \infty}[s\bar{f}(s)] = \lim_{\tau \to 0}[f(\tau)]
\]

\[
\lim_{s \to 0}[s\bar{f}(s)] = \lim_{\tau \to \infty}[f(\tau)]
\]

we have

\[
u(x,0) = v(x,0) = 0
\]

\[
u(x,\tau \to \infty) \to v(x,\tau \to \infty) \to \frac{3b + 2\sqrt{3} - 3x}{3b + 4\sqrt{3}}
\]

Thus according to Eq. (25), at the initial instant, both the material and radiation energy densities are zero inside the slab. Eq. (26) asserts that at infinite time the radiation and material energy density equilibrate among themselves. However, because of the finite thickness of the slab, flux leaks out of the right edge so that the energy densities vary linearly along the length of the slab.

The solutions for \(u(x,\tau)\) and \(v(x,\tau)\) follow from \(\bar{u}(x,s)\) and \(\bar{v}(x,s)\) by inverting them using the Laplace inversion theorem

\[
f(\tau) = \frac{1}{2\pi i} \int_{C} dse^{\tau s} \bar{f}(s)
\]

where the integration contour is a line parallel to the imaginary s axis to the right of all the singularities of \(\bar{f}(s)\). Both \(\bar{u}\) and \(\bar{v}\) are single valued functions and hence there are no branch points. However, there are an infinite number of poles obtained from the roots of the transcendental equation

\[
3\sin(\beta(s)b) + 4\sqrt{3}\beta(s)\cos(\beta(s)b) - 4\beta^2(s)\sin(\beta(s)b) = 0
\]

or, \(\tan(\beta(s)b) = \frac{4\sqrt{3}\beta(s)}{4\beta^2(s) - 3}\)

The roots of the transcendental equation can by obtained by graphical plotting (as shown in Fig. 2 for \(b=1\).
Figure 2: Finding the roots of the transcendental equation \( \tan(\beta(s)) = f(\beta) = \frac{4\sqrt{3}\beta(s)}{4\beta^2(s) - 3} \)

The contour is closed in the left half plane so that the large semi circle gives a zero contribution.

For ease of generating results, in this work, MATHEMATICA has been used to obtain the first 30 roots namely, 0, 1.22826, 3.61221, 6.54624, 9.60463, 12.7025, 15.8174, 18.9409, 22.0696, 25.2014, 28.3354, 31.4709, 34.6076, 37.745, 40.8831, 44.0216, 47.1606, 50.2999, 53.4395, etc [12].

Corresponding to each root of \( \beta(s) \), there exists two values of \( s \), i.e., two simple poles. The poles are obtained from solution of Eq. [22]. The root \( \beta(s) = 0 \) gives poles at \( s = 0 \) and \( s = -11 \). The residue at \( s = -11 \) is zero and that at \( s = 0 \) is redundant as we already have a simple pole at \( s = 0 \) as seen in the denominator of Eq. [20]. According to the residue theorem, \( \int_C dse^{\tau} \hat{f}(s) = 2\pi i \times \text{(sum of the residues at the singularities)} \). Hence the residue at \( s = 0 \) is

\[
\lim_{s \to 0} e^{\tau}(s - 0) \frac{3\sin[\beta(s)(b - x)] + 2\sqrt{3}\beta(s)\cos[\beta(s)(b - x)]}{s[3\sin(\beta(s)b) + 4\sqrt{3}\beta(s)\cos(\beta(s)b) - 4\beta^2(s)\sin(\beta(s)b)]} = \frac{3b + 2\sqrt{3} - 3x}{3b + 4\sqrt{3}}
\]

which has a dependence only on position \( x \) and no dependence on time \( \tau \). Thus the asymptotic (steady state) solution for the radiation energy density is

\[
u(x, \infty) = \frac{3b + 2\sqrt{3} - 3x}{3b + 4\sqrt{3}}
\]

which is also obtained by equating \( \frac{\partial u(x, \tau)}{\partial \tau} \) and \( \frac{\partial v(x, \tau)}{\partial \tau} \) in Eqs. [23] and [24] to zero, solving \( \frac{\partial^2 u(x, \tau)}{\partial x^2} = 0 \) and obtaining the values of the constants from the
BC given by Eqs. (12) and (13). The contribution to the time dependent part will come from the other poles. Adding all these contributions will give us the complete space and time dependence of the radiation energy density. Similarly, the residue at s=0 for the material energy density \( \bar{\nu} \) is

\[
\lim_{s \to 0} e^{\tau s}(s - 0) \frac{3\sin[\beta(s)(b - x)] + 2\sqrt{3}\beta(s)\cos[\beta(s)(b - x)]}{s(s + 1)[3\sin(\beta(s)b) + 4\sqrt{3}\beta(s)\cos(\beta(s)b) - 4\beta^2(s)\sin(\beta(s)b)]} = \frac{3b + 2\sqrt{3} - 3x}{3b + 4\sqrt{3}}
\]

(31)

and that at s=-1 is 0. Hence, the asymptotic (steady state) solution for the material energy density is

\[
\nu(x, \infty) = \frac{3b + 2\sqrt{3} - 3x}{3b + 4\sqrt{3}}
\]

(32)

which is the same as the radiation energy density. This shows that the radiation and material energy density attain equilibrium after sufficient time. The poles for the second root of the transcendental equation \( \beta = 1.22826 \) are s=-25.49448516 and -0.59174484. Now, the residue at a simple pole \( s = s_n \) is

\[
\lim_{s \to s_n} e^{\tau s}(s - s_n) \frac{3\sin[\beta(s)(b - x)] + 2\sqrt{3}\beta(s)\cos[\beta(s)(b - x)]}{s[3\sin(\beta(s)b) + 4\sqrt{3}\beta(s)\cos(\beta(s)b) - 4\beta^2(s)\sin(\beta(s)b)]}
\]

As \( s = s_n \) is a pole of \( \bar{\nu} \), both the numerator and the denominator tend to 0. Applying L’Hospital’s rule to both the numerator and the denominator, the residue at a pole \( s = s_n \) is

\[
e^{\tau s_n}[3\sin(\beta(s_n)(b - x)) + 2\sqrt{3}\beta(s_n)\cos(\beta(s_n)(b - x))] / s_n[(3b + 4\sqrt{3} - 4\beta(s_n)^2)b\cos(\beta(s_n)b) - (4\sqrt{3}\beta(s_n)b + 8\beta(s_n))\sin(\beta(s_n)b)]d\beta(s_n)/ds
\]

where

\[
\frac{d\beta(s_n)}{ds} = \frac{\epsilon(s_n + 1)^2 + 1}{2(s_n + 1)^2[\sqrt{s_n(1 + 1/(s_n+1))}]^{1/2}}
\]

(33)

For each value of \( \beta(s) \), there are two poles (i.e., s values), namely

\[
\pm(\epsilon + \beta(s) + 1) \sqrt{\epsilon + \beta(s) + 1} \pm \sqrt{\epsilon + \beta(s) + 1} \pm \sqrt{\epsilon + \beta(s) + 1}
\]

The solution is obtained by summing over the residues from the first few poles.

\[
u(x, \tau) = \frac{3b + 2\sqrt{3} - 3x}{3b + 4\sqrt{3}} + \sum_n \frac{e^{\tau s_n}[3\sin(\beta(s_n)(b - x)) + 2\sqrt{3}\beta(s_n)\cos(\beta(s_n)(b - x))]}{s_n[(3b + 4\sqrt{3} - 4\beta(s_n)^2)b\cos(\beta(s_n)b) - (4\sqrt{3}\beta(s_n)b + 8\beta(s_n))\sin(\beta(s_n)b)]d\beta(s_n)/ds}
\]

(34)
Similarly, the solution for the material density is

\[
v(x, \tau) = \frac{3b + 2\sqrt{3} - 3x}{3b + 4\sqrt{3}} + \sum_n \frac{e^{s_n \tau} [3\sin(\beta(s_n)(b - x)) + 2\sqrt{3}\beta(s_n)\cos(\beta(s_n)(b - x))]}{s_n(s_n + 1)}\left[(3b + 4\sqrt{3} - 4\beta(s_n)^2b)\cos(\beta(s_n)b) - (4\sqrt{3}\beta(s_n)b + 8\beta(s_n))\sin(\beta(s_n)b)\right] \frac{d\beta(s_n)}{ds}
\]

(35)

3 Results

3.1 Solution for \( \epsilon = 0 \)

We first consider the \( \epsilon = 0 \) case which arises when the speed of light is taken to be infinite so that radiation is not retarded initially. At infinite time, the radiation and material energy densities assume the same spatial dependence as for \( \epsilon \neq 0 \) case.

\[
u(x, \tau \to \infty) \to v(x, \tau \to \infty) \to \frac{3b + 2\sqrt{3} - 3x}{3b + 4\sqrt{3}}
\]

(36)

However, for \( \tau = 0 \), as \( s \to \infty \) for \( \epsilon = 0 \), we obtain \( \beta = i \) where \( i = \sqrt{-1} \). Thus,

\[
u(x, 0) = \frac{3\sinh(b - x) + 2\sqrt{3}\cosh(b - x)}{7\sinh(b) + 4\sqrt{3}\cosh(b)}
\]

(37)

\[
v(x, 0) = 0
\]

(38)

Thus the material energy density is zero at \( \tau = 0 \) as predicted by the initial condition. However, because of the absence of retardation effects, the radiation energy density attains a finite value consistent with the incoming flux of radiation. This behaviour is in agreement with that obtained in the case of a semi infinite planar slab for the no retardation case.

We now obtain the solution \( u(x, \tau) \) and \( v(x, \tau) \) by inverting the Eqs. (20) and (21) using inverse Laplace transform as in the general case with \( \epsilon = 0 \). The difference from the \( \epsilon \neq 0 \) case is that only one pole is obtained corresponding to a value of beta i.e., \( s = -\frac{\beta^2(s)}{\beta^2(s) + 1} \). In all the results generated in this paper, sum of the residues at the first 30 roots have been considered. Also, the value of \( b \) is chosen to be 1.

The material energy density increases with time from a zero initial value whereas the radiation energy density attains a finite value even at very early times due to the absence of retardation effects as seen from Figs. 3 and 4. Both the energy densities attain the steady state value at \( \tau = 10 \) and maintains that value at a later time \( \tau = 100 \).
Figure 3: Radiation energy density vs position in the slab at different times for $\epsilon = 0$

Figure 4: Material energy density vs position in the slab at different times for $\epsilon = 0$
Figure 5: Radiation energy density vs position in the slab at different times for $\epsilon = 0.1$

3.2 Solution for $\epsilon \neq 0$

For a finite value of $\epsilon$, both the radiation and material energy density increase gradually from zero as shown in Figs. 5 and 6 (for $\epsilon = 0.1$).

The first derivatives w.r.t. position of the radiation and material energy density are obtained as

$$\frac{\partial u(x, \tau)}{\partial x} = \frac{-3}{3b + 4\sqrt{3}}$$

$$+ \sum_n \frac{e^{s_n \tau}[-3\beta(s_n)\cos(\beta(s_n)(b-x)) + 2\sqrt{3}\beta^2(s_n)\sin(\beta(s_n)(b-x))]}{s_n[(3b + 4\sqrt{3} - 4\beta^2(s_n)b)\cos(\beta(s_n)b) - (4\sqrt{3}\beta(s_n)b + 8\beta(s_n))\sin(\beta(s_n)b)]}\frac{d\beta(s_n)}{ds}$$

(39)

and

$$\frac{\partial v(x, \tau)}{\partial x} = \frac{-3}{3b + 4\sqrt{3}}$$

$$+ \sum_n \frac{e^{s_n \tau}[-3\beta(s_n)\cos(\beta(s_n)(b-x)) + 2\sqrt{3}\beta^2(s_n)\sin(\beta(s_n)(b-x))]}{s_n(s_n + 1)\beta(s_n)[(3b + 4\sqrt{3} - 4\beta^2(s_n)b)\cos(\beta(s_n)b) - (4\sqrt{3}\beta(s_n)b + 8\beta(s_n))\sin(\beta(s_n)b)]}\frac{d\beta(s_n)}{ds}$$

(40)

and plotted in Figs. 7 and 8. The gradient of both radiation and material energy densities obtain a constant value of $\frac{-3}{3 + 4\sqrt{3}} = -0.30217$ after infinite
Figure 6: Material energy density vs position in the slab at different times for $\epsilon = 0.1$

time showing that there is a constant leakage of flux from the right surface due to the finite thickness. This result is different from the semi-infinite slab result where at infinite time, the entire halfspace is at a constant temperature with a uniform radiation field and hence there is no gradient and no flux [4].

The current of radiation leaking out from the left and right surfaces of the slab are

$$J_- = u(0, \tau) + \frac{2}{\sqrt{3}} \frac{\partial u(0, \tau)}{\partial x}$$ (41)

$$J_+ = u(b, \tau) - \frac{2}{\sqrt{3}} \frac{\partial u(b, \tau)}{\partial x}$$ (42)

The leakage currents are plotted as a function of time in Fig. 6. At later times, the values are seen to converge to the asymptotic values of 0.30217 and 0.6978 respectively from the left and right surfaces.

The averaged or integrated radiation energy density is given by

$$\psi_r(\tau) = \int_0^b u(x, \tau) dx = \int_0^b \frac{3b + 2\sqrt{3} - 3x}{3b + 4\sqrt{3}} dx$$

$$+ \sum_n s_n [(3b + 4\sqrt{3} - 4\beta^2 s_n)b\cos(\beta(s_n)b) - (4\sqrt{3}\beta(s_n)b + 8\beta(s_n))\sin(\beta(s_n)b)] \frac{d\beta(s_n)}{dx}$$

$$\times \int_0^b [3\sin(\beta(s_n)(b-x)) + 2\sqrt{3}\beta(s_n)\cos(\beta(s_n)(b-x))] dx$$
Figure 7: Space derivative of radiation energy density vs position in the slab at different times for $\epsilon = 0.1$

Figure 8: Space derivative of material energy density vs position in the slab at different times for $\epsilon = 0.1$
Figure 9: Leakage currents from the two surfaces of the slab

\[
\psi_m(\tau) = \int_0^b v(x, \tau) dx
\]

\[= \frac{3b + 2\sqrt{3}b}{3b + 4\sqrt{3}} - \frac{3}{3b + 4\sqrt{3}} \frac{b^2}{2}\]

\[+ \sum_n e^{s_n \tau} s_n[(3b + 4\sqrt{3} - 4\beta^2(s_n)b)\cos(\beta(s_n)b) - (4\sqrt{3}\beta(s_n)b + 8\beta(s_n))\sin(\beta(s_n)b)] \frac{d\beta(s_n)}{ds}\]

\[\times \left[ \frac{3}{\beta(s_n)}(1 - \cos(\beta(s_n)b)) + 2\sqrt{3}\sin(\beta(s_n)b) \right] \quad (43)\]

Similarly the averaged or integrated material energy density is

\[
\psi_m(\tau) = \int_0^b v(x, \tau) dx
\]

\[= \frac{3b + 2\sqrt{3}b}{3b + 4\sqrt{3}} - \frac{3}{3b + 4\sqrt{3}} \frac{b^2}{2}\]

\[+ \sum_n e^{s_n \tau} s_n(s_n + 1)[(3b + 4\sqrt{3} - 4\beta^2(s_n)b)\cos(\beta(s_n)b) - (4\sqrt{3}\beta(s_n)b + 8\beta(s_n))\sin(\beta(s_n)b)] \frac{d\beta(s_n)}{ds}\]

\[\times \left[ \frac{3}{\beta(s_n)}(1 - \cos(\beta(s_n)b)) + 2\sqrt{3}\sin(\beta(s_n)b) \right] \quad (44)\]

For \(b=1\), the steady state integrated value is 0.5 as seen from Fig.10.
To check the consistency of the final results, we add Eqs. (8) and (9) and integrate over $x$ from 0 to $b$, yielding
\[
\int_{0}^{b} \left( \epsilon \frac{\partial u(x, \tau)}{\partial \tau} + \frac{\partial v(x, \tau)}{\partial \tau} \right) dx = \int_{0}^{b} \frac{\partial^2 u(x, \tau)}{\partial x^2} dx = \frac{\partial u(b, \tau)}{\partial x} - \frac{\partial u(0, \tau)}{\partial x}
\]
i.e.,
\[
\epsilon \frac{\partial \psi_r(\tau)}{\partial \tau} + \frac{\partial \psi_m(\tau)}{\partial \tau} = \frac{\partial u(b, \tau)}{\partial x} - \frac{\partial u(0, \tau)}{\partial x} \tag{45}
\]
Using the expressions for the energy densities, their first derivatives in space and the integrated quantities, we find that both the left and right hand sides reduce to
\[
\sum_n \frac{e^{i \pi n}}{|3 \beta(s_n) b + 4 \sqrt{3} \beta(s_n) b + 8 \beta(s_n) \sin(\beta(s_n) b)|} \times \frac{3}{8 \beta(s_n)} (1 - \cos(\beta(s_n) b)) + 2 \sqrt{3} \sin(\beta(s_n) b)| \epsilon + \frac{1}{s_n + 1} \]
proving the consistency of the obtained solutions.

As there are infinite number of residues, the exact solution is obtained only on adding all of them. However, the contribution from the poles decrease very sharply. To study convergence, we plot percentage error as a function of number of roots of the transcendental equation considered. As seen from Fig. 11, 2.1% error in the value of $u(0, 2.5)$ is observed on considering only the first two roots i.e., the steady state result and residue for the two non zero poles. The errors arising due to non inclusion of higher order terms is more initially as the higher order poles contribute only at very small times because of the exponential term. The error falls sharply to a negligible value (0.005%) on considering the contribution from the first 6 roots i.e., first 11 poles. More accurate results can be obtained by adding residues from higher order poles.
Figure 11: Percentage error in the radiation energy density as a function of number of roots considered (N)

4 Conclusions

In this paper we have derived completely analytical expressions for the space and time dependent radiation and material energy densities inside a planar slab of finite thickness under the diffusion approximation. All other quantities of importance like the integrated values and the leakage currents are obtained directly from these analytical expressions for any parameter value like slab thickness, specific heat, etc. The series solutions are found to converge quickly and depending on the required degree of accuracy, the number of poles to be considered is decided. These results can serve as benchmark for time dependent non equilibrium one dimensional radiation diffusion codes.

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