Dynamic characteristic of combined viscous and Coulomb damping under varying nature of material and coefficient of friction

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Abstract. Practically, most systems possess characteristics of both viscous damping and Coulomb damping. A model of combined viscous and Coulomb damping is analysed while taking into account the different nature of materials. Different material contact surfaces exhibit distinct coefficients of friction which has an effect on the displacement response of a vibratory system. The study involves analysis of a linear spring-mass damper system by varying its material parameters, taking frictional force into account. The response in terms of maximum displacement amplitude is examined while varying the values of coefficient of friction. Material parameters such as natural frequency (varied by mass and stiffness each), damping ratio and coefficient of friction are studied. The research is conducted using theory of combined viscous and dry friction damping and graphical observations are made to study the response in view of different surface contacts.

Keywords: Coefficient of friction; Coulomb damping; viscous damping; response; amplitude

1. Introduction

Nomenclature

| Symbol | Parameter                        | Unit  |
|--------|----------------------------------|-------|
| m      | Mass of the system               | kg    |
| k      | Spring Constant of the system    | N/m   |
| c      | Damping coefficient of the system| Ns/m  |
| t      | Time                             | s     |
| x₀     | Initial Displacement             | m     |
| x      | Displacement                     | m     |
| ẋ     | Velocity                         | m/s   |
| ẍ     | Acceleration                     | m/s²  |
| ẋ₀    | Initial Velocity                 | m/s   |
| µ      | Coefficient of Friction          |       |
| N      | Normal force                     | N     |
| Fᵣ     | Frictional force                 | N     |
| dx     | Frictional Decay                 | m     |
| α      | Viscous Decay                    | s⁻¹   |
| ω₀     | Natural frequency                | rad/s |
| ωₜ     | Damped frequency                 | rad/s |
| ξ      | Damping ratio                    |       |

Extensive research is done in the field of vibration, considering harmonic oscillator model or simple pendulum model. Damping becomes very important to analyze vibratory systems near resonance. The phenomenon of amplification in magnitude of vibration is called resonance. Resonance is a large
cause of failure in mechanical systems and one effective way of avoiding it is through damping. Damping is the phenomenon that gradually reduces the amplitude of vibrations, caused due to internal friction developed in the system. Damping provides a means of controlling amplitude of a body experiencing simple harmonic motion in terms of displacement, velocity, acceleration and energy. Damping ratio (\(\xi\)) is defined as the ratio of damping constant to the critical damping constant. Coulomb damping (dry Friction damping) is chiefly due to conversion of kinetic energy into heat dissipation in a body. Viscous damping considers these energy losses under the influence of a fluid lubrication system.

Hartog [1] developed an analytical solution for forced vibrations undergoing combined Coulomb and viscous damping. He found that in case the friction force is small in relation to the magnitude of the disturbing force, the ensuing motion would be constant, whereas a single cycle of motion may consist of regions of variable motion for large values of friction force. Peters et. al., [2] developed a model including the long-period physical pendulum inspired by the behavior of a simple harmonic oscillator subject to dry sliding friction. The new model predicted a logarithmic decrement of the motion which is quadratic in the period. In the presence of viscous and dry friction, Molina[3] compared the amplitude decay for a harmonic oscillator. Liang et. al.[4] obtained a method for estimating Coulomb and viscous damping parameters simultaneously in linear forced oscillators Fay [5] adopted an energy approach to investigate trajectories of mechanical vibrating systems consisting of second order differential equations for dry friction. It is applied to the harmonic oscillator example and a pendulum model. Cveticanin [6] investigated the properties of oscillatory system with non-polynomial damping.

Two numerical cases are compared; first where the damping force coefficient is small and second where the damping force is similar to the dry friction. Anuse [7] identified information of a system undergoing both viscous and coulomb damping where the damper properties are studied to avoid resonance condition. These dampers are used by dissipating resonant vibration energy to reduce resonant stress by providing sliding contact between points experiencing relative motion due to vibration. Vitorino et. al.[8] compared the amount of sliding friction and the amount of viscous damping that experimental models take. Wang et. al.[10] suggested an experimental approach to help accurately define the damping model parameters compared to the standard acceleration response sensitivity process. Hinrichsen et.al.[9] studied the acceleration, velocity and displacement of a glider using an accelerometer on an air track undergoing oscillatory motion due to viscous and dry frictional damping. A theory of oscillatory motion acceleration, assuming viscous damping proportional to velocity plus an independent Coulomb friction force, is presented and compared with the experiment.

1.1. Response of spring-mass damper system
Practically, most of the systems possess characteristics of both viscous and Coulomb damping. A model of combined viscous and Coulomb damping is considered with different combination of materials in contact. The response spectrum is a graphical representation showing the variation of the maximum response (displacement, velocity, acceleration or jerk) with the natural frequency (or natural time period) of a single-degree-of-freedom (SDOF) system to a specified force function.

The response of the spring-mass system with viscous damping is given by Rao[11] as

\[
x(t) = e^{-\omega_n t} \left\{ x_o \cos \omega_d t + \frac{(\dot{x}_o + \omega_n x_o)}{\omega_d} \sin \omega_d t \right\}
\]  (1.1)

The response of spring-mass system with Coulomb damping for the first half cycle, that is, for \(0 \leq t \leq \pi/\omega_n\) is given as

\[
x(t) = \left( x_o - \frac{\mu N}{k} \right) \cos \omega_n t + \frac{\mu N}{k}
\]  (1.2)

For the second half cycle, that is, for \(\pi/\omega_n \leq t \leq 2\pi/\omega_n\) its response is given by

\[
x(t) = \left( x_o - \frac{3\mu N}{k} \right) \cos \omega_n t - \frac{\mu N}{k}
\]  (1.3)

Similarly, its response for the next half cycle, that is, for \(2\pi/\omega_n \leq t \leq 3\pi/\omega_n\) is given by
\[ x(t) = \left( x_0 - \frac{5\mu N}{k} \right) \cos \omega_n t + \frac{\mu N}{k} \]

Hence the response \( x(t) \) of spring-mass system with Coulomb damping can be written in a generalized form for every \( n \)th half cycle, for \( (n - 1)\pi / \omega_n \leq t \leq (n + 1)\pi / \omega_n \) as
\[ x(t) = \left( x_0 - \frac{n(n + 1)\mu N}{2k} \right) \cos \omega_n t - (-1)^n \frac{\mu N}{k} \]

Differentiating equation (1.5), the standard form velocity equation for Coulomb damping is obtained
\[ \dot{x}(t) = -\omega_n \left( x_0 - \frac{n(n + 1)\mu N}{2k} \right) \sin \omega_n t \]

Similarly, on further differentiating equation (1.6), the standard form acceleration equation for Coulomb Damping in given as
\[ \ddot{x}(t) = \omega_n^2 \left( x_0 - \frac{n(n + 1)\mu N}{2k} \right) \cos \omega_n t \]

1.2. Response of spring-mass damper system combined viscous and Coulomb damping

In the present work, a spring mass damper system is considered with both viscous and Coulomb damping as seen in Figure 1 along with their free body diagram in Figure 2. The friction force \( F_f \) parallel to the contact surface is assumed to be independent of velocity or position and such as to oppose the motion. Then for a mass \( m \) attached to a spring with spring constant \( k \), subjected to both Coulomb friction \( F_f = \mu N \) and a viscous damping force \( F(\dot{x}) = -c\dot{x} \) proportional to velocity and the equation of motion is
\[ m\ddot{x} = -kx - c\dot{x} - F_f Sgn(\dot{x}) \]

\[ \ddot{x} + 2\alpha \dot{x} + \omega_n^2 (x + \Delta x Sgn(\dot{x})) = 0 \]

Here \( \alpha = c/2m \), \( Sgn(\dot{x}) = 1, 0, -1 \) for \( \dot{x} > 0, \dot{x} = 0, \dot{x} < 0 \) respectively and \( \Delta x = F_f / k \) taking the magnitude of \( x \) at the instant when the friction force is just equal to the spring force

Natural frequency of the system can be found as
\[ \omega_n = \sqrt{-\frac{\ddot{x} - 2\alpha \dot{x}}{x + \Delta x Sgn(\dot{x})}} \]

For the first half oscillation according to the theory of combined viscous and Coulomb damping by Hinrichsen response in terms of displacement is given by
\[ x(t) = \frac{\omega_n}{\omega} (x_o - \Delta x) e^{-\alpha t} \cos (t - t_o) + x \]

The displacement has a minimum after the first half of the oscillation, the velocity reaches zero and
the friction force reverses. On further differentiation response in terms of velocity and acceleration are obtained respectively
\[
\dot{x}(t) = -\frac{\omega_n^2}{\omega^3} (x_o - \Delta x) e^{-\alpha t} \sin \omega t \\
\ddot{x}(t) = -\frac{\omega_n}{\omega} (x_o - \Delta x) e^{-\alpha t} \cos \omega(t + t_a)
\] (1.12) (1.13)

2. Response analysis
In order to study the nature of a system under going combined coulomb and viscous damping, response graphs are plotted (figure 3 and figure 4) for the equations of viscous damping (1.1) and Coulomb damping (1.5). The Hinrichsen [9] experimental model using theory of combined viscous and Coulomb damping, considering acceleration plot is validated. The theoretical plot is described using equation (1.11) and compared with experimental Hinrichsen model (refer Figure 5). Values for the plot as seen in Table 1 are as per his air-glider experimental values [9]. These values are taken as a foundation for the rest of the parametric study. MATLAB is used for plotting and Plot Digitizer for plot validation.

| Table 1. Initializing parameters according to Hinrichsen’s experimental work |
|---------------------------------|------|-----|
| Parameters                      | Value | Unit |
| Mass of the system \((m)\)      | 10 kg |      |
| Spring constant of the system \((k)\) | 1000 N/m |      |
| Damping coefficient of the system \((c)\) | 20 Ns/m |      |
| Initial Displacement \((x_o)\)  | 0.25 m |      |
| Initial Velocity \((x_d)\)     | 0 m/s |      |
| Coefficient of Friction \((\mu)\) | 0.3  |      |
| Damping ratio \((\zeta)\)      | 0.1   |      |
| Frictional force \((F_f)\)     | 294.3 N |      |

**Figure 3.** Response for viscous damping  
**Figure 4.** Response for Coulomb friction damping
Figure 5. Acceleration response for combined viscous and Coulomb friction damping in terms of acceleration and validated Hinrichsen’s model

3. Parametric Study

The study is conducted considering displacement response varying the material parameters. The maximum displacement amplitude is found in each case, using MATLAB to understand the effects of different material properties on the oscillation.

3.1. Displacement response for varying $\omega_n$ keeping $k$, $\zeta$, $\mu$, $F_f$ and $t$ constant

A displacement response spectrum is analyzed taking a series of oscillators subjected to motion by a single, fixed force.

3.1.1. Varying mass ($m$)

By varying the mass of the system, the natural frequency is varied as in Table 2. For a displacement response plot of the same range it is observed that as the mass of the system increases, its natural frequency decreases and the time period for damping increases (as seen in Figure 6).

| Value of $m$ (kg) | Value of $\omega_n$ (rad/s) |
|------------------|----------------------------|
| 1                | 31.62                      |
| 5                | 14.14                      |
| 10               | 10                         |
| 20               | 7.07                       |

3.1.2. Varying stiffness ($k$)

Natural frequency is varied by varying only stiffness of system with a range as per Table 3. The displacement plot (Figure 7) for the same range shows that as the stiffness increases, the time period decreases.

| Value of $k$ (N/m) | Value of $\omega_n$ (rad/s) |
|--------------------|-----------------------------|
| 1000               | 10                          |
| 2500               | 15.81                       |
| 10000              | 31.62                       |
| 25000              | 50                          |
3.2. Displacement response for varying damping ratio ($\xi$) keeping $\omega_n$, $\mu$, $F_f$ and $t$ constant
Damping ratio has wide application in mechanics of systems and its practical application can be seen in automotive suspension systems employing shock absorbers. The displacement response varying the damping ratio ($\xi$) in the range of 0.01 to 0.5 is analyzed.
It is observed that as time period decreases, the damping ratio increases (as seen in Figure 8).

3.3. Displacement response for varying coefficient of friction ($\mu$) keeping $\omega_n$, $F_f$, $\xi$ and $t$ constant
Varying the coefficient of friction indicates variable material contact surfaces taken. This gives an insight of the friction acting between the two surfaces. Coefficients of friction ranging from 0.1 to close to 1 are considered in the displacement plot. Table 4 shows the coefficient of friction along with their contact surface.
From the displacement response plot (Figure 9), it is observed that on increasing the coefficient of friction, the time period decreases and the mean axis about which oscillation occurs shifts upwards, indicating increased displacement response.
### Table 4. Coefficient of friction for different contact surfaces

| Contact surface                  | Dynamic coefficient of friction |
|----------------------------------|---------------------------------|
| For polished oiled metal surfaces | Less than 0.1                   |
| For glass on glass               | 0.4                             |
| For rubber on tarmac             | Close to 1                       |

![Graphical representation of displacement response varying \( \mu \) as mentioned in Table 4](image1)

**Figure 9.** Graphical representation of displacement response varying \( \mu \) as mentioned in Table 4

![Exploded view of Figure 8. at the maximum first amplitude](image2)

**Figure 10.** Exploded view of Figure 8. at the maximum first amplitude

#### 4. Conclusion

A new formulation of the response of a Coulomb damping system in terms of displacement, velocity and acceleration is presented.

A study while varying parameters such as natural frequency (while changing mass and stiffness each), damping ratio and coefficient of friction is conducted. The response of the system in each of these cases in terms of displacement is analysed. The coefficient of friction is a material property and can be considered for different contact surfaces. In this way the response of different material surfaces is studied. This study can be used in studying the response in different surface contact applications.

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