Note on the luminosity distance

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We re-derive a formula relating the areal and luminosity distances, entirely in the framework of the classical Maxwell theory, assuming a geometric-optics type condition.

I. INTRODUCTION

It is accepted since Robertson [1] that the luminosity distance relates to the areal distance [2] as follows, in Friedman-Lemaitre-Robertson-Walker (FLWR henceforth) spacetimes,

\[ D = (1 + z)R. \]  

Formula (1) is usually derived in a way that mixes quantum mechanical and classical concepts [3]. Robertson has shown this result within the framework of the classical Maxwell theory using conservation laws. His approach suffers from a number of minor erroneous statements and gaps in the reasoning. One of that concerns the form of the electromagnetic energy-momentum tensor, another the energy flux. A comment on the latter. The Maxwell equations in FLRW spacetimes possess a mathematical strictly conserved energy which is different – as it is quite common in curved spacetimes – from the 'proper energy', the total energy detected by a comoving observer. The latter is directly related to the luminosity, but is not conserved; in fact it is equal to the former energy divided by the conformal factor (see below). Robertson assumes that the corresponding energy fluxes scale in the same way as the energies themselves; that can be true only approximately (but including all cases of astrophysical interest), as will be shown.

These shortcomings do not influence the validity of the main conclusion of [1]. The aim of this paper is to provide a derivation for the cosmological distance formula which is consistently classical and fills some gaps in the proof of Robertson [1]. The order of the rest of this paper is following. Sec. 2 brings fundamental definitions and equations. Sec. 3 is dedicated to the derivation of an explicit solution of the Maxwell equations. Next section shows how one incorporates initial data into this closed form solution. The first part of Sec. 5 contains a proof that the electromagnetic energy momentum tensor can satisfy assumptions made by Robertson. Sec 5B accomplishes the proof of (1).

II. EQUATIONS

The space-time geometry is defined by the FLWR line element,

\[ ds^2 = a(\eta)^2(-d\eta^2 + dr^2 + \sigma^2 d\Omega^2), \]

where \( \eta \) is a (conformal) time coordinate, \( r \) is a radial coordinate that coincides with the areal radius, \( d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2 \) is the line element on the unit sphere with \( 0 \leq \phi < 2\pi \) and \( 0 \leq \theta \leq \pi \). \( a(\eta) \) is a conformal factor and \( \sigma \) is given by \( \sinh (r) \) (for negative curvature \( k = -1 \)), \( r \) (for flat spacetime \( k = 0 \)) and \( \sin (r) \) (for positive curvature \( k = 1 \)). Throughout this paper \( c \) and \( G \), the velocity of light and the gravitational coupling constant respectively, are put equal to 1. The standard cosmological time is related to \( \eta \) by \( a d\eta = dt \).

The Maxwell equations read

\[ \nabla_{\mu} F^{\mu\nu} = 0, \]

where \( F_{\mu\nu} = \partial_{\mu} A_{\nu} - \partial_{\nu} A_{\mu} \) and \( A_{\mu} \) is the electromagnetic potential. It is convenient to assume \( A_0 = 0 \) and the Coulomb gauge condition \( \nabla_i A^i = 0 \). In such a case there are two independent degrees of freedom, represented by the magnetic or electric modes.

The magnetic modes are defined as follows
\[ A = \sum_{l=1}^{\infty} \sum_{m=-l}^{l} \frac{f_{lm}(\eta, r)}{\sqrt{l(l+1)}} \left( -m \frac{Y_{lm}}{\sin \theta} \, d\theta - i \sin \theta \partial_{\theta} Y_{lm} \, d\phi \right). \] (4)

Here \( Y_{lm} \) are the spherical harmonics. The multipole expansion coefficients \( f_{lm} \) depend on the conformal time \( \eta \) and the radial coordinate \( r \). Straightforward calculation shows that the equations of motion reduce to a system of hyperbolic equations

\[ (-\partial_{\eta}^2 + \partial_{r}^2) f_{lm} = \frac{l(l + 1)}{\sigma^2} f_{lm} \] (5)

for the multipoles \( f_{lm}(\eta, r) \).

The electric potential read

\[ A = \sum_{l=1}^{\infty} \sum_{m=-l}^{l} \frac{1}{\sqrt{l(l+1)}} \left( \sqrt{l(l+1)} h_{lm}(\eta, r) Y_{lm} \, dr + k_{lm}(\eta, r) \partial_{\theta} Y_{lm} \, d\theta + k_{lm}(\eta, r) \sin Y_{lm} \, d\phi \right). \] (6)

Here \( h_{lm}(\eta, r) \) and \( k_{lm}(\eta, r) \) are multipoles. From the Coulomb gauge condition \( \nabla A^{i} = 0 \) we have the relation between multipoles \( k_{lm}(\eta, r) \) and \( h_{lm}(\eta, r) \):

\[ \sqrt{l(l + 1)} k_{lm} = \partial_{r}(\tilde{h}_{lm}) \] (7)

where \( \tilde{h}_{lm} = \sigma^2 h_{lm} \). One can show that the first Maxwell equation \( (\nabla_{\mu} F^{\mu 0} = 0) \) is identity. The second one, \( (\nabla_{\mu} F^{\mu r} = 0) \), reduces to the same hyperbolic equations as in Eq.(5):

\[ (-\partial_{\eta}^2 + \partial_{r}^2) \tilde{h}_{lm} = \frac{l(l + 1)}{\sigma^2} \tilde{h}_{lm}. \] (8)

The two remaining Maxwell equations \( (\nabla_{\mu} F^{nk} = 0 \text{ where } k = \theta, \phi) \) yield

\[ (-\partial_{\eta}^2 + \partial_{r}^2) k_{lm} = \frac{l(l + 1)}{\sigma^2} k_{lm} - 2 \sqrt{l(l + 1)} (\partial_{\sigma} \sigma) k_{lm}. \] (9)

It’s easy to show that (8) is equivalent to (9). In order to see this, differentiate both sides of (8) with respect \( r \) and use the Coulomb gauge condition (7); a quick calculation gives (9).

### III. COMPACT FORM OF SOLUTIONS

Equations (5) and 8) have identical form; they can be written compactly as

\[ (-\partial_{\eta}^2 + \partial_{r}^2) \phi_{lm} = \frac{l(l + 1)}{\sigma^2} \phi_{lm}, \] (10)

where \( \phi = f, \tilde{h} \). Below we will drop the irrelevant index \( m \). The forthcoming constructions apply equally well to magnetic and electric degrees of freedom. The form of Eq. (10) suggests that each of the three cosmological models should be discussed separately, but we show later, that there exists a unique approach that works in all these cases. In this section we construct a solution of Eq.(10).

Define a set of generating functions (as we will sometimes refer to them in the forthcoming text) of the form

\[ f(\eta, r) = f(r - \eta) \] (11)

and also
Lemma. For any $f$ and $g$ the function

$$\phi_l(r, \eta) = \sigma^l \frac{\partial}{\partial \sigma} \frac{1}{\sigma} \frac{1}{\sigma} \cdots \frac{1}{\sigma} \frac{f + g}{\sigma}$$

solves Eq. (10).

Proof of Lemma.

The proof uses the method of mathematical induction. One can write the right hand side of (13) as

$$\sigma^l \frac{\partial}{\partial \sigma} \left( \left( \frac{\sigma^l - 1}{\sigma} \frac{1}{\sigma} \cdots \frac{1}{\sigma} \frac{f + g}{\sigma} \right) \right)$$

(herein we simply put $\frac{\sigma^l - 1}{\sigma}$ after the first sign of differentiation). Thence one obtains a recursive formula

$$\phi_l(r, \eta) = \frac{\partial}{\partial r} \phi_{l-1}(r, \eta) - \frac{l}{\sigma} \phi_{l-1}(r, \eta)$$

where

$$\phi_{l-1}(r, \eta) = \sigma^{l-1} \frac{\partial}{\partial \sigma} \frac{1}{\sigma} \frac{1}{\sigma} \cdots \frac{1}{\sigma} \frac{f + g}{\sigma}.$$  

For $l = 1$ we can write the right hand side of (13) as

$$\phi_1 = \sigma^0 \frac{f_1 + g_1}{\sigma}.$$  

This satisfies our equation (10), as can be easily checked. Now let $\phi_l$ be a solution of:

$$(-\partial_\eta^2 + \partial_r^2)\phi_l = \frac{l(l + 1)}{\sigma^2} \phi_l.$$  

We claim that $\phi_{l+1}$ defined as in (14) in terms of $\phi_l$ satisfies the equation (17) with $l + 1$ being put in place of $l$. That is easily proved by a direct calculation using the relation (14)), which shows that indeed $(-\partial_\eta^2 + \partial_r^2)\phi_{l+1} = \frac{(l+1)(l+2)}{\sigma^2} \phi_{l+1}$. Let us remark that explicit solutions of Eq. (10) are known in the literature [4]. Eq. (10) has the same form as in the Minkowski spacetime (strictly saying, it is so when $k = 0$. If $k = 1, -1$, Eq. (10) can be cast into the Minkowskian form by a suitable change of variables (see for instance [5], [6] and [7]). The specific form of the solution, presented in Eq. (13), can be new.

We would like to point out that the pair $(f, g)$ should be specified independently for each multipole $\phi_l$; the related pair of functions will be labelled by the multipole number $l$. The initial data for the wave equation (10) are $\phi_l, \phi_l$. In the next section we show that those initial data suffice to determine the pair $(f_l, g_l)$, which in turn determine solutions of Eq. (10) in the whole spacetime.

IV. GENERATING FUNCTIONS AND INITIAL DATA

Below we show how, having initial data one can construct the generating functions $f$ and $g$ that appear in the formula (13).

Let initial conditions be

$$\phi_l(r, \eta)|_{\eta=0} = \phi_l(r, 0) \quad \partial_\eta \phi_l(r, \eta)|_{\eta=0} = \phi_l(r, 0)$$

Assume that a support of initial data is compact and contained within $(a, b)$. Integrating $l$-times both sides of (13), one arrives at

$$(g_l + f_l)|_{\eta=0} = H_l(r) + M_l(r),$$
where quantities $H_l$ and $M_l$ are linear combinations of multiple integrals

$$M_l = \hat{M}_0 \sigma(r) + \sum_{s=1}^{l-1} \hat{M}_s \sigma(r) \int_a^r dr_1 \sigma(r_1)$$

$$\int_a^{r_1} dr_2 \sigma(r_2) \cdots \int_a^{r_{s-1}} dr_s \sigma(r_s),$$

$$H_l(r) = \sigma(r) \int_a^r dr_1 \sigma(r_1) \int_a^{r_1} dr_2 \sigma(r_2)$$

$$\cdots \int_a^{r_{(l-1)}} \phi_l(r, 0) \sigma(r_l) dr_l.$$  

(20)

Here $\hat{M}$’s are constants of integration.

From (13), taking into account (2) we have

$$\dot{\phi}_l(r, 0) = \sigma \frac{\partial}{\partial \sigma} \frac{1}{\sigma} \partial_{r_1} \sigma \partial_{r_2} \frac{1}{\sigma} \cdots \partial_{r_{(l-1)}} \frac{1}{\sigma} \partial_{r_l} (g_l - f_l).$$  

(21)

Therefore performing an adequate number of integrations, one arrives at

$$(f_l - g_l)|_{\eta=0} = -I_l - N_l,$$  

(22)

where

$$I_l(r) = \int_a^r dr_1 \sigma(r_1) \int_a^{r_1} dr_2 \sigma(r_2)$$

$$\cdots \int_a^{r_{(l-1)}} dr_l \sigma(r_l) \int_a^{r_l} \phi_l(r, 0) \sigma(r_{(l+1)}) dr_{(l+1)}.$$  

$$N_l = \hat{N}_0 + \sum_{s=1}^{l} \hat{N}_s \int_a^r dr_1 \sigma(r_1)$$

$$\int_a^{r_1} dr_2 \sigma(r_2) \cdots \int_a^{r_{(l-1)}} dr_l \sigma(r_l)$$  

(23)

and $\hat{N}$’s are the integration constants. The substraction of (22) from (19) yields

$$(g_l)|_{\eta=0} = \frac{1}{2} (H_l(r) + I_l(r)) + \frac{1}{2} M_l(r) + \frac{1}{2} N_l(r)$$  

(24)

Similary, adding (22) to (19) gives

$$(f_l)|_{\eta=0} = \frac{1}{2} (H_l(r, 0) - I_l(r, 0)) + \frac{1}{2} M_l - \frac{1}{2} N_l.$$  

(25)

Only quantities $H_l$ and $I_l$ carry information depending directly on initial data. In contrast, let us stress out that the constants $M_l$’s and $\hat{N}$’s are free – hence $M_l$ and $N_l$ are not determined by initial data. Therefore the generating functions are not specified uniquely by initial data. On the other hand, inverting this argument one can say, that this nonuniqueness does not influence initial data. Since a solution $\phi_l$ of Eq. (10) is uniquely determined by its initial data, one infers that properties of the evolving waves do not depend on specific values of the constants. Therefore we are free to adopt such values that are convenient to us, and this fact justifies the choice that will be made at the end of this section.

The quantities $M_l$ and $N_l$ are linear combinations of multiple integrals. Using the additivity of the integrals and splitting $\int_a^r = \int_a^b + \int_b^r$, one obtains
\[ M_l = \dot{M}_0 \sigma(r) + \sum_{s=1}^{l-1} \dot{M}_s \sigma(r) \int_b^r dr_1 \sigma(r_1) \int_r^{r_1} dr_2 \sigma(r_2) \]
\[ \cdots \int_b^{r_{s-1}} dr_s \sigma(r_s), \]
\[ N_l = \dot{N}_0 + \sum_{s=1}^{l} \dot{N}_s \int_b^r dr_1 \sigma(r_1) \int_r^{r_1} dr_2 \sigma(r_2) \]
\[ \cdots \int_b^{r_{s-1}} dr_s \sigma(r_s), \] (26)

with some constants \( \dot{M} \)'s and \( \dot{N} \)'s. In a similar token (but under additional condition \( r > b \)) one obtains

\[ H_l(r, 0) = h_0 \sigma + \sum_{s=1}^{l-1} h_s \sigma(r) \int_b^r dr_1 \sigma(r_1) \int_r^{r_1} dr_2 \sigma(r_2) \]
\[ \cdots \int_b^{r_{(s-1)}} \sigma(r_2) \]
\[ I_l(r, 0) = i_0 + \sum_{s=1}^{l} i_s \int_b^r dr_1 \sigma(r_1) \int_r^{r_1} dr_2 \sigma(r_2) \]
\[ \cdots \int_b^{r_{(s-1)}} \sigma(r_2). \] (27)

Here \( h \)'s and \( i \)'s are fixed constants, that depend on initial data, but whose specific form is not relevant. We would like to stress, that here the condition \( r > b \) is essential.

We choose the integration constants \( M \)'s, \( N \)'s such that the right hand sides of Eqs (24) and (25) become nullity for \( r > b \). Therefore the generating functions \( f_l, g_l \) vanish in the interval \((b, \infty)\) and become linear combinations \((\dot{M} \sigma^2 \dot{N}_0)/2 \) in \((0, a)\).

**V. THE LUMINOSITY FORMULA**

A lesson that can be drawn from the preceding section is this: an outgoing wave pulse that is initially comprised within \((r_E - \Delta, r_E)\) (as we assume from now on) will remain inside \((r_E + \eta - \Delta, r_E + \eta)\). A similar statement can be formulated about ingoing waves. This provides an explicit proof of the Huygens principle [8] in Friedman cosmological models [9].

In what follows we will restrict our attention only to the outgoing pulse of radiation having a width \( \Delta \), and located initially within \((r_E - \Delta, r_E)\). The stress-energy tensor of the electromagnetic field reads \( T_{\mu\nu} = F_{\mu\gamma}F^{\nu\gamma} - (1/4)g_{\mu\gamma}F_{\gamma\delta}F^{\gamma\delta} \)
and the time-like normal to a Cauchy hypersurface is \( (n_\mu) = (-a, 0, 0, 0) \). Define

\[ m(r, t) = -\int_{V(r)} dV aT_{00}^0 = \]
\[ 2\pi \int_0^r dr \Sigma_{l=1}^{\infty} \left( (\partial_\eta \phi_l)^2 + (\partial_r \phi_l)^2 + \frac{l(l+1)}{\sigma^2} \phi_l^2 \right). \] (28)

where \( dV \) is the proper volume element and the volume \( V(r) \) extends over the support of a radiation pulse that is enclosed inside the coordinate sphere \( S(r) \). \( t \) is the cosmic time, related to the conformal time by \( dt = ad\eta \). One finds that

\[ \partial_t m(r, t) = \frac{4\pi}{\alpha} \Sigma_{l=1}^{\infty} \partial_\eta \phi_l \partial_r \phi_l. \] (29)

The right hand side of Eq.(29) can be interpreted as the "energy flux" through the comoving coordinate sphere \( S(r) \). Assume that \( \Delta < \sigma(r_E) \) (we will refer later to this condition as to the geometric optics approximation). Then it follows from expression (13) that one can approximate \( \phi_l \) by its leading term, \( \partial_\mu f \), roughly speaking (see Appendix for a sketch of the precise argument). Therefore one has approximate equality valid almost everywhere.
\[
\partial_t m(r,t) = -\frac{4\pi}{a} \sum_{l=1}^{\infty} (\partial_r^{l+1} f_l)^2 .
\]  

The total energy \( m \equiv \lim_{r \to \infty} m(r,t) \) is strictly conserved, because the energy flux vanishes outside a coordinate sphere \( S(r_0 + \eta(t)) \). There is an interesting observation that can be made about the outgoing electromagnetic radiation in the geometric optics limit. If one adopts our point of view in which the emitter is not a point but rather a big sphere (say, a surface or an envelope of a galaxy of a radius \( r_E - \Delta \)), then one can show that \( |T_{\mu\nu}|, T_{00} \) and \( T_{rr} \) are much bigger that the remaining components of the energy-momentum tensor. In order to shorten the mathematical formulae, we will operate with quantities integrated over the coordinate sphere \( r_E \). (In the pointwise treatment one would have to deal with expressions of the type \( \int d^3 x Y_0 Y_{l0} A_{l\nu} \), where the symmetric tensor \( A_{l\nu} \) is given by \( A_{l\nu} = (1/2)(\partial_r \phi \partial_{\nu} \phi + \partial_{\nu} \phi \partial_r \phi) \) or \( A_{l\nu} = (1/2)(\partial_r \phi \partial_{\nu} \phi - \partial_{\nu} \phi \partial_r \phi) \).) The dominant parts of \( \int d^3 x S \hat{T}_{\mu\nu} |T_{\mu\nu}|, \int d^3 x d^3 y T_{00} \) and \( \int d^3 x d^3 y T_{rr} \), behave like the tangent pressures \( T_{\theta\theta} \) and \( T_{\phi\phi} \) in the limit of geometric optics, that is \( \Delta << r_E \), by \( 1 + \Delta/\sigma^{-1} \). In contrast with that, the surface integrals of the remaining components of the energy-momentum tensor are dominated by \( \sum_l \left( (\partial_r \phi l)^2 - (\partial_r \phi l)^2 \right) \). This expression can be approximated, in the same limit, by \( 0(\Delta/\sigma^{-1}) \sum_l (\partial_r^{l+1} f_l)^2 \).

Thus under the above conditions the electromagnetic radiation undergoes the process of partial isotropisation and behaves like a gas with nonisotropic pressure (the tangent pressures \( T_{\phi\phi} \) and \( T_{\theta\theta} \) are negligible). (That is an interesting fact in itself, which in principle opens a way for the explanation of the long standing riddle concerning the apparent mass over-abundance of spiral galaxies \([10]\).)

It was claimed in \([1]\) that all components of the energy-momentum tensor different from \( T^0_\nu, T^0_0 \) and \( T^r_r \) must vanish due to the isotropy of the discussed cosmological models. Robertson’s statement is obviously wrong, but the conclusion that \( T^0_\nu, T^0_0 \) and \( T^r_r \) are the only components that matter (in the limit of geometric optics, defined earlier), is correct. Therefore the remaining part of the reasoning of \([1]\) can be left intact, modulo another mistake mentioned in the introduction (see also below). Let us add here, for the convenience of an inspector reader, that the quantity \( \sum_{l=0}^{\infty} (\partial_r^{l+1} f_l)^2 \) is equal (modulo a constant factor) to the quantity \( V \) in the paper of Robertson (see formula (8) in \([1]\)).

A. The proper energy

The comoving observer will detect the local electromagnetic energy density \(-T^\mu_\nu n^\mu n^\nu = -T^0_0\). The total energy of a wave pulse,

\[
E(r,t) = -\int_{V(r)} dV T^0_0,
\]

is related to the quantity \( m \) by

\[
E(r,t) = \frac{m(r,t)}{a(t)} .
\]

The volume \( V(r) \) extends over an annulus \((r_E + \eta - \Delta, r_E + \eta)\), with the areal radial extension \( a\Delta \) being of the order of the longest wavelength present in the pulse. The energy flux through the comoving coordinate sphere \( S(r) \) reads

\[
\partial_t E(r,t) = -\frac{4\pi}{a^2} (\partial_r^{l+1} f_l)^2 - HE(r,t) .
\]

One finds, assuming (as in the preceding section) that \( \Delta << \sigma(r_E) \), that the energy term \( E(r,t) \) is approximated by \( 4\pi \int_{r_E + \eta - \Delta} r \sum_{l=1}^{\infty} (\partial_r^{l+1} f_l)^2 /a \approx (\Delta a) \sum_{l=1}^{\infty} (\partial_r^{l+1} f_l)^2 /a^2 \). It appears that \( H \approx 10^{-15}/km \) (remember that \( c = G = 1 \)) — that value is prohibitively small, so that for any realistic extension \( \Delta \) of the radiation pulse the quantity \( \Delta a H << 1 \). Therefore the second term in (33) is always negligible and the energy flux is well approximated by

\[
\partial_t E(r,t) = -\frac{4\pi}{a^2} \left( \partial_r^{l+1} f_l \right)^2 .
\]

Notice that the factor \((\partial_r f_0)^2\) is constant along the trajectory of the outgoing wave pulse. In order to compare this with Robertson, let us observe that \( \partial_t E(r,t) \) is equal (neglecting the \( H \)-related term) to \( 4\pi r^2 a^2 U \) where \( U \) is given by formula (2.10) in \([1]\).
B. The luminosity distance

We define \((r_E, t_E)\) and \((r_O, t_O)\) as coordinates of the emitter and the observer, respectively. One easily finds, using (32) and the constancy of \((\partial_t f_0)^2\), that

\[
\frac{\partial_t E(r, t)|_{t_O}}{\partial_t E(r, t)|_{t_E}} = \frac{a^2(t_E)}{a^2(t_O)} \frac{\partial_t E(r, t)|_{t_E}}{\partial_t E(r, t)|_{t_O}} = \frac{1}{(1+z)^2} \frac{\partial_t E(r, t)|_{t_E}}{\partial_t E(r, t)|_{t_O}}.
\]

(35)

Notice that \(\partial_t E(r, t)|_{t_E}\) is the luminosity of the emitted radiation. The apparent luminosity is given by the ratio \(\partial_t E(r, t)|_{t_O}/(4\pi R_O^2)\). From the definition of the luminosity distance follows now immediately

\[
D = (1+z)R_0,
\]

(36)
in accordance with the formula (1).

VI. APPENDIX

Let \(f \in C^\infty\) have a compact support \((a, b)\) such that \(\Delta = b - a << a\). It is easy to see that \(||\partial^l f/\sigma||_{L_2(a,r)} << ||\partial^{l+1} f||_{L_2(a,r)}\), for any interval \((a, r) \in (a, b)\). Therefore the amount \(m_{(a,r)}\) of radiation within the annulus \((a, r)\) is well approximated by \(||\partial^{l+1} f||_{L_2(a,r)}^2\) and the pointwise equality of Eq. (30) holds true almost everywhere.

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