Inequivalent Goldstone Hierarchies for Spontaneously Broken Spacetime Symmetries

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Abstract

The coset construction is a powerful tool for building theories that non-linearly realize symmetries. We show that when the symmetry group is not semisimple and includes spacetime symmetries, different parametrizations of the coset space can prefer different Goldstones as essential or inessential, due to the group’s Levi decomposition. This leads to inequivalent physics. In particular, we construct a theory of a scalar and vector Goldstones living in de Sitter spacetime and non-linearly realizing the Poincaré group. Either Goldstone can be seen as inessential and removed in favor of the other, but the theory is only healthy when both are kept dynamical. The corresponding coset space is the same, up to reparametrization, as that of a Minkowski brane embedded in a Minkowski bulk, but the two theories are inequivalent.

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1 Introduction

The concept of spontaneous symmetry breaking is a powerful organizational principle for effective theories found throughout different branches of physics. Because objects transforming under such symmetry group $G$ are not in a representation of $G$, the group is often said to be non-linearly realized. The coset construction – so called because it relates the Goldstones bosons arising from the breaking pattern $G \to H$ to the coset space $G/H$ – is part and parcel of building theories that non-linearly realize symmetry groups, the machinery for which was first introduced more than 50 years ago in [1, 2].

While the coset construction applied to internal (compact and semisimple) groups is well understood, considerable effort has been made in the last years to study its application to spacetime symmetry groups. After all, the breaking of these groups is at the heart of many concepts of physics, such as cosmology and condensed matter [3, 4, 5, 6]. One of the most significant distinctions from internal groups is that a non-linear realization of spacetime symmetries can have fewer degrees of freedom than there are broken generators, the phenomenon of Goldstone inessentially [7, 8, 9].

Given spacetime coordinates $x$ and two kinds of Goldstones $\pi$ and $\xi$, we say $\pi$ is essential if (a) it transforms without reference to $\xi$, that is, $\pi \to \tilde{\pi}(\pi, x)$; and (b) $x$ and $\pi$ fully realize the symmetry, meaning their transformations depend on all group parameters. In this case, $\xi$ is unneeded to realize the group and can be discarded; it’s inessential.

Goldstone inessentiality can also be formulated in terms of inverse Higgs constraints (IHCs). These are relations built out of the invariants that connect $\pi$ and $\xi$. If this relation includes derivatives of $\pi$ while $\xi$ appears only algebraically, the constraint can be solved to eliminate $\xi$ in terms of $\pi$ and its derivatives.

This hierarchy between Goldstones is peculiar, and the question then arises: is such structure unique? That is, given any parametrization (choice of coordinates) for some coset space, can we uniquely determine the essentiality of a Goldstone boson? If true, it would mean the coset construction for spacetime symmetries is universal, meaning that different physicists arrive at the same theory regardless of how they choose to parametrize their coset spaces, assuming a common set of rules. On the other hand, the construction would be not unique if the two physicists end up eliminating different degrees of freedom from their theory and arrive at inequivalent actions.

We show the second case can happen. We provide a toy geometrical example in Section 2.1 and then a proper physical example in Section 4, where changing the parametrization changes the essential nature of the Goldstones, leading to different theories.

The reason is that the hierarchical property "]\pi transforms without reference to $\xi" is a kind of structure. Reparametrizing the coset space induces a field redefinition between all objects that is guaranteed to preserve the group product structure, but not necessarily any other structure, including this hierarchical property. Equivalently, each IHC will always be mapped to a new IHC, but this new constraint might be unusable if it can’t be solved. This opens the door for physicists to make different choices of which Goldstones to eliminate (or which IHC to use). We discuss this in more detail in Section 2.2.

In the literature, it’s customary to perform the coset construction by means of the distinguished Maurer-Cartan form (e.g. [10, 11]). This is convenient, since we can refrain from explicitly deriving the transformation laws for our objects. But because those transformations are precisely our focus here – in particular, which kinds of Goldstones transform without reference to the others – we introduce in Section 3 an alternative method based on [12] that requires computation of those transformations but dispenses use of the Maurer-Cartan form. Readers interested only in discussions and results are invited to skip it.

Conventions. We work in $(-+++\ldots)$ signature. All transformations here are treated under the passive viewpoint, meaning we don’t pullback the arguments of functions if the dependent variables transform, e.g., $x \to \tilde{x}(x)$ and $f(x) \to f(\tilde{x})$. Unless said otherwise, quantities written in capital letters (e.g., $X$ or $\Pi$) are invariant under the symmetry group under consideration.
2 Non-uniqueness of coset construction

First, we provide in Section 2.1 a toy geometrical example where the coset construction fails to deliver a unique result. For the interested reader, we discuss the more technical aspects of why this happens for non-semisimple groups in Section 2.2.

2.1 How to draw curves

Suppose two physicists, Rachel and Leo, are asked to build the action for plane curves from the coset ISO(2)/\{1\}, where \{1\} is the trivial group containing only the identity. The two agree on the following common rules:

- They will employ reparametrization invariance for their curves.
- The resulting action can contain only first or second derivatives.
- They should attempt to eliminate inessential degrees of freedom if possible.

Rachel’s theory. Rachel decides to parametrize the coset space as \( \ell_R = e^{xP_1}e^{yP_2}e^{\theta J} \), where \( P_i \) are translation generators and \( J \) the rotation one. All the objects \( \{x, y, \theta\} \) are functions of some diffeomorphism parameter \( \lambda \). She computes the transformation laws of these objects under a Euclidean group element \( (a^i, \phi) \) and finds:

\[
\begin{align*}
    x &\to x \cos(\phi) - y \sin(\phi) + a^1, \\
y &\to y \cos(\phi) + x \sin(\phi) + a^2, \\
\theta &\to \theta + \phi.
\end{align*}
\]

Rachel notices that just \( x \) and \( y \) are sufficient to fully realize the symmetry and that they transform without reference to \( \theta \); she keeps them as essential and discards \( \theta \) as inessential. Equivalently, she can find the constraint \( \tan(\theta) \partial_\lambda x = \partial_\lambda y \), which is algebraic in \( \theta \). She then computes the following action:

\[
S_R = \int d\lambda \sqrt{(x')^2 + (y')^2} P \left( \frac{-y'x'' + x'y''}{((x')^2 + (y')^2)^{3/2}} \right),
\]

where \( P \) is some arbitrary function and primes denote \( \partial_\lambda \). This is of course familiar from plane geometry; the object inside the \( P \) function is the extrinsic curvature \( \kappa \) of a curve embedded in Euclidean space.

It is useful to understand how this action represents a prescription for drawing curves on paper. Rachel slides a ruler against the paper in a fixed direction. With the other hand, she holds a pen next to the ruler, allowing the pen to be pushed by it. Reparametrization invariance arises because the speed of the ruler can be removed as a degree of freedom; Rachel’s actual degree of freedom is in moving the pen along the direction parallel to the ruler. For linear equations of motion obtained from \( P(\kappa) = 1 \), she doesn’t move the pen at all, only letting it be pushed by the ruler, and draws a straight line.

Leo’s theory. Leo then takes the parametrization Rachel used, but to be contrarian, flips the order of the exponentials, writing his as \( \ell_L = e^{\theta J}e^{\pi P_2}e^{\sigma P_1} \). His transformation laws are:

\[
\begin{align*}
    \theta &\to \theta + \varphi, \\
\pi &\to \pi + a^1 \cos(\theta) + a^2 \sin(\theta), \\
\sigma &\to \sigma + a^2 \cos(\theta) - a^1 \sin(\theta),
\end{align*}
\]
Leo now notices that $\theta$ and $\pi$ form an essential pair: together they fully realize the group, and neither one transforms with reference to the inessential $\sigma$. Leo then discards $\sigma$, which he could also do through the constraint $\sigma \partial_\lambda \theta = \partial_\lambda \pi$, and derives the following action:

$$ S_L = \int d\lambda \theta' F \left( \pi + \frac{-\pi' \theta'' + \theta' \pi''}{(\theta')^3} \right). \tag{8} $$

Leo’s action is more peculiar. The object inside the $F$ function is a notion of torsion $\tau$, which we’ll discuss shortly. In terms of drawing curves, it works as follows. He places a wheel together with a ruler on the paper. He then rotates the ruler without slipping around the wheel, which is kept fixed. Again, the ruler’s angular speed can be removed as a degree of freedom; the actual one is in moving the pen parallel to the ruler. For linear equations of motion obtained from $F(\tau) = \tau^2$, Leo doesn’t move the pen with respect to the ruler and draws an involute of the wheel. See Figure 1 for visualization.

The invariant $\tau$ can be called torsion because it is connected to the winding of the pen around the wheel and thus to the displacement of the pen from its original position after one rotation cycle. A curve that doesn’t close after one cycle must necessarily have nonzero torsion.

**Inequivalence between the two.** In Appendix A, we show that no invertible redefinition (either global or local) between Rachel’s $(x, y)$ and Leo’s $(\theta, \pi)$ exists; the two actions are inequivalent and represent two distinct ways of drawing curves. While not particularly relevant for physics, the point of this example was to show that the coset construction doesn’t necessarily produce unique results. Rachel and Leo started from the same coset space and employed the same prescription of removing inessential degrees of freedom, but arrived at inequivalent results.

In the following subsection, we discuss the reason why this can happen. In Section 4 we apply the same logic Leo did to produce a physical example in spacetime.

### 2.2 Why it’s not unique

We will first briefly review the connection of Goldstone bosons and homogeneous spaces in Section 2.2.1 and the fundamentals of the coset construction in Section 2.2.2. Then, in Section 2.2.3, we discuss inequivalencies that arise in non-semisimple groups.

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1. Alternatively, he could’ve eliminated $\pi$ in favor of $\sigma$ but the final result is the same.
2. Different from the torsion of spatial curves.
3. Readers familiar with children’s toys will recognize this as a spirograph drawing.
2.2.1 Goldstone bosons live in homogeneous spaces

Perhaps the fundamental property of Goldstone bosons is that a zero background value for one of these bosons can be transformed into a nonzero one under action of the symmetry group $G$:

$$\pi = 0 \rightarrow \pi \neq 0,$$  \hspace{1cm} (9)

so that the actual value of the Goldstone’s vacuum is irrelevant. This property goes by the name of transitive group action. Isometry groups of maximally symmetric manifolds act transitively as well: the translations of Minkowski spacetime can move the origin somewhere else, and so can the homotheties (translations plus dilations) of de Sitter spacetime.\(^4\) The points of a maximally symmetry manifold are indistinguishable from each other.

The symmetries that move a field’s background are said to be broken, while those that move the origin of spacetime are said to be inhomogeneous. If all inhomogeneous symmetries are unbroken, we can collect all spacetime coordinates $x$ and all Goldstones $\pi$ into a single space $Q = \{x, \pi\}$ and the action of $G$ on $Q$ remains transitive. This $Q$ is then called a homogeneous space under $G$. Put another way, a homogeneous space has a single orbit under $G$: the whole space itself.

The $\pi$’s in this context should be seen as just coordinates of the space $Q$, not yet as functions of spacetime. A specific solution of the equations of motion is then the subspace given by the embedding $\pi(x)$.\(^5\)

2.2.2 A homogeneous space is equivalent to a coset space . . .

The fact all coordinates and fields live in a homogeneous space turns out to be powerfully constraining for model building, because homogeneous spaces are mostly unique. To specify one such space under the group $G$, we simply need to know the stability subgroup $S$ of the origin of $Q$. That’s the subgroup of $G$ that leaves both the origin of spacetime and the backgrounds of all fields invariant (i.e., unbroken homogeneous symmetries). Then the orbit-stabilizer theorem establishes that $Q$ corresponds to a coset space:

$$Q \sim \frac{G}{S},$$  \hspace{1cm} (10)

where $\sim$ means equivalence in the sense that it preserves the group product (i.e., homeomorphism), but not necessarily any additional structure $Q$ might have. Concretely, this means that any $q \in Q$ can be written in terms of some group element $\ell s \in G$ acting on the origin of $Q$, where $\ell \in G$ is called a lift (or coset space representative) and $s \in S$ is an arbitrary stability element.

So for each element in $Q$ there’s a corresponding element in $G$, with some $S$-ambiguity, thus intuitively $Q \sim G/S$. But a left coset space has a canonical group action by left multiplication of $g \in G$:

$$\ell S \rightarrow \tilde{\ell} S \overset{\text{def}}{=} gS.$$  \hspace{1cm} (11)

Hence, specifying the full symmetry group $G$ and the stability subgroup $S$ automatically specifies the transformation laws of all coordinates and fields, which by extension fixes all invariants that can be used to build an action. This construction – mapping the physical entities in $Q$ to some coset space and deriving invariants – goes by the name of coset construction.

2.2.3 . . . up to additional structure

The orbit-stabilizer theorem guarantees any homogeneous space $Q$ with a $G$-action is equivalent, up to additional structure, to the coset space $G/S$, the key phrase here being "up to additional structure."

Perhaps the first such structure one may think of is topology. This is a valid argument. Nonetheless, in the context of an effective theory, we typically are interested only in expanding fields perturbatively around the vacuum, so that the topology of field space is of little interest. Spacetime

\(^4\)Recall the origin of de Sitter spacetime in conformal time is $-1$/Hubble, so not invariant under dilations, even though they act linearly.

\(^5\)As our focus is to build effective actions, we can always do so classically and then quantize the action afterwards. It would be interesting to extend the formalism discussed here to work with operators and Hilbert spaces from the get-go, though we won’t do it here.
itself could have nontrivial topology as well, but by the same token we would prefer to restrict ourselves to local measurements that can’t probe such exotica. So although topology might indeed lead to non-trivial physics beyond perturbation theory, we leave it aside in the following.

Typically in physics we have spacetime coordinates $x$ and internal space coordinates $\phi$. Crucially, spacetime is distinguished from internal space, because the $x$’s can only transform among themselves, i.e., $x \rightarrow \tilde{x}(x,g)$ under some $g \in G$. Another way of stating the same thing is that the isometries of a spacetime are intrinsic to the spacetime itself; they can’t depend on what you put inside. On the other hand, the $\phi$ are allowed to mix with $x$, i.e., $\phi \rightarrow \tilde{\phi}(\phi,x,g)$; this is simply a non-uniform symmetry for our fields.

Similarly, if $Q$ contains essential Goldstones $\pi$ and inessential ones $\xi$, then by definition the action of $G$ only mixes the $\pi$ among themselves (and possibly the coordinates $x$) without reference to $\xi$.

This kind of hierarchy where some objects transform without reference to others is a type of additional structure. Not all field redefinitions will preserve it. For example, replacing $\pi$ by $\bar{\pi} = \tilde{\pi}(\pi,\xi)$ will typically cause the transformation of $\bar{\pi}$ to also depend on $\xi$, and now the hierarchy is lost. Equivalently, a coset space parametrization in which an inverse Higgs constraint is algebraic can be mapped to one where the constraint is now differential and typically unsolvable.

This isn’t particularly surprising. The orbit-stabilizer theorem only guarantees reparametrizations of the coset space preserve the group product structure, not this kind of hierarchical structure. But since this hierarchy is precisely linked to the removal of physical degrees of freedom, if multiple hierarchies exist then different physics can arise.

Here’s one example where inequivalent hierarchies for some coset space are possible. Following the Levi decomposition of $G$, any group can be written as $G = R \rtimes L$, so that each symmetry can be classified as belonging to either the radical $R$ or the (semi)simple factor $L$. See Appendix B for details. Then when we parametrize the coset space, there are at least two orderings of the exponentials that endow the objects with different hierarchical structure; we leave the proof of this statement for Appendix B.1. The two Levi orderings are:

$$\ell_R \overset{\text{def}}{=} \text{radical symmetries } \times \text{simple symmetries} \quad (12)$$

$$\ell_L \overset{\text{def}}{=} \text{simple symmetries } \times \text{radical symmetries} \quad (13)$$

In $\ell_R$, the radical objects fully realize the group, without reference to the simple ones. In $\ell_L$, the simple ones together with a reduced number of the radicals might fully realize the group. By means of example (both in Sections 2.1 and later in 4), we know that these orderings can swap which Goldstones are essential or inessential, so they can potentially lead to different physics for any symmetry breakdown.

Of course, the above assumes $R$ even exists to begin with. If $G$ is a simple group, then the Levi decomposition is trivial and it’s not clear if such ambiguities can arise. That they don’t for internal groups is well established, but we don’t know if that’s the case for simple spacetime groups such as the conformal group $\text{SO}(2,D)$ (see also [13] for possible ambiguities in conformal group breakdown).

3 Normalization construction

We now describe the general method for constructing objects that realize some symmetry, linearly or non-linearly. The procedure is essentially based on [11, 12], and we direct the reader to those references for formal proofs of the method.

The reason why we use this technique rather than the usual one based on the Maurer-Cartan form is to highlight the importance of the transformation laws themselves, which dictate whether additional structure is present for the objects in our theory and, by extension, whether the resulting effective action is unique.

The basic idea is the notion that anything that can be transformed away by the symmetry group cannot, by definition, be an invariant. But the group is finite, so there’s only a finite amount of quantities it can transform away before its symmetries have been used up. Anything that remains afterwards is an invariant.

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6It’s a kind of fiber bundle structure, though somewhat different from how the concept is used in physics.
For example, consider a nonrelativistic particle in 3D Euclidean space with position \( \vec{x} \) under the Galilean group. We can spend all three translations moving \( \vec{x} \) to \( \vec{0} \). Then we can spend all three boosts shifting the velocity \( \dot{\vec{x}} \) to \( \vec{0} \). Now only rotations remain, but it’s impossible to eliminate the acceleration \( \ddot{\vec{x}} \) simply by rotating; at most, we can align it with some preferred axis. Thus \( |\ddot{\vec{x}}| \) is what remains; it’s the invariant of the Galilean group.

**Step 1.** As input, one must inform the physicist about the full symmetry group of the problem, the broken symmetries, and the inhomogeneous symmetries in spacetime. This determines \( G \) and its stability subgroup \( S \).

**Step 2.** Next, we must parametrize the lift of \( G/S \). As discussed in Section 2.2.3, two convenient choices for non-semisimple groups are given by:

\[
\ell_R \overset{\text{def}}{=} \text{radical symmetries} \times \text{simple symmetries},
\]

(14)

\[
\ell_L \overset{\text{def}}{=} \text{simple symmetries} \times \text{radical symmetries}.
\]

(15)

**Step 3.** We now derive the transformed lift \( \tilde{\ell} \) under a group element \( g \in G \), following the canonical group action on a coset via the group product: \( \ell S \to \tilde{\ell} S = g \ell S \). This gives the transformation laws \( x \to \tilde{x} \) and \( \pi \to \tilde{\pi} \).

**Step 4.** When deriving the transformation laws for the objects in the homogeneous space, we might observe that some transform without reference to the others, meaning the space might have some hierarchical structure. Suppose \( Q = \{q, p\} \) with action under \( g \in G \):

\[
q \to \tilde{q}(q; g),
\]

(16)

\[
p \to \tilde{p}(p, q; g),
\]

(17)

where \( \tilde{q} \) depends on all parameters of \( g \). Then we can construct a new homogeneous space \( \tilde{Q} \overset{\text{def}}{=} \{q\} \) deprived of the \( p \)'s, which still has a consistent action under \( G \). Also, because \( \tilde{q} \) depends on all parameters of \( g \), this reduced space still realizes the full group \( G \) (i.e., the action is faithful).

If this is possible then we can forget the \( p \)'s exist and perform the construction solely on the \( q \)'s, in which case the \( p \)'s are called inessential and the \( q \)'s essential. This is equivalent to imposing an inverse Higgs constraint after deriving the invariants, except here we do this from the very beginning.

Which objects are essential or inessential can depend on the ordering selected in step 2. This is because a reparametrization of the lift (i.e., a field redefinition) will not, in general, preserve the hierarchy between objects in the coset space.

**Step 5.** The final step is to derive the actual invariants. At this point, we have a (possibly reduced in the previous step) homogeneous space with coordinates and fields, \( Q = \{x, \pi\} \) and the transformation rules for \( \tilde{x} \) and \( \tilde{\pi} \) which follow from \( \tilde{\ell} \). We now try to use the group action to set to zero as many objects in \( Q \) as possible.

Obviously \( Q \) is an homogeneous space, so by definition everything in it can be eliminated. But we know how \( x \) and \( \pi \) transforms, so we know how \( \partial_x \pi \) does too, as well as all higher derivatives. Thus we take our original homogeneous space \( Q = \{x, \pi\} \) and extend it with a finite amount of derivatives \( \partial_x \pi, \partial^2_x \pi \), and so on.\(^7\) We then transform those quantities under some special \( g \in G \) to be determined later. These transformed objects are denoted with capital letters rather than tildes (e.g., \( X \) instead of \( \tilde{x} \)) due to their special status as putative invariants.

---

\(^7\)Formally, the homogeneous space is a fiber bundle, so it can be prolonged into a jet bundle [14, Ch. 4].
Transforming \((x, \pi)\) under the \(g_\ast\), we schematically have:

\[
x^{g_\ast} X, \\
\pi^{g_\ast} \Pi, \\
\frac{\partial \pi}{\partial x}^{g_\ast} \frac{d \Pi}{dX}, \\
\frac{\partial^2 \pi}{\partial x^2}^{g_\ast} \frac{d^2 \Pi}{dX^2}, \\
\vdots
\]

1. We start by normalizing \(X\) and \(\Pi\) to zero,\(^8\) which allows us to solve for some of the parameters of \(g_\ast\). If this completely fixes \(g_\ast\), then \(d \Pi / dX\) are the invariants of the theory,\(^9\) and we’re done.

2. If not, we then attempt to normalize as many of the \(d \Pi / dX\) to zero as possible, which lets us fix more of the \(g_\ast\). If \(g_\ast\) is completely fixed by now, then the remaining \(d \Pi / dX\) are the invariants we’re after. If none remain, then the \(d^2 \Pi / dX^2\) are the invariants.

3. If \(g_\ast\) still hasn’t been fixed, we repeat the procedure, setting as many of the \(d^2 \Pi / dX^2\) to zero as possible, and so on. In the end, when \(g_\ast\) is completely determined (which can always be done since the group is finite-dimensional, so a finite number of normalizations fixes all parameters), the lowest order in derivatives \(d^n \Pi / dX^n\) that survived the process are the invariants.

Invariant one-forms then follow by transforming the basis \(dx\) under the \(g_\ast\) found above. By extension we can build the invariant volume form \(dV\):

\[
dx^{g_\ast} dX, \\
dV \overset{\text{def}}{=} \frac{1}{D!} dX^0 \wedge dX^1 \wedge \ldots \wedge dX^D.
\]

An invariant derivative can also be constructed, by inverting the invariant one-form as usual. That is, if \(dX^\alpha = M^\alpha_\beta dx^\beta\), then the invariant derivative is \(d / dX^\alpha = (M^{-1})^\beta_\alpha \partial_\beta\). Such derivatives can act on the invariants we obtained to produce higher-order invariants, or act on additional matter fields that don’t transform under the group \(G\) and weren’t part of the construction.

This gives the complete toolbox needed to build the most general invariant action.

Notice that in this procedure, we must use normalization constraints to fix all parameters of the group element \(g_\ast\). However, in many cases, the objects we work with will transform linearly (i.e., in a representation) under some subgroup of \(G\), typically the unbroken subgroup or the stability subgroup. In this case, if we contract objects covariantly under this subgroup, the corresponding group parameters will naturally drop out anyway. So these parameters don’t need to be fixed, which saves us some time. But it’s not always guaranteed that a certain lift parametrization will automatically induce a linear transformation that let us exploit covariance of objects. We encounter such issue in Section 4.

Example: curvature of curves. Let’s look at the quintessential example of planar curves. The coset space is \(\text{ISO}(2)/\{1\}\) and we parametrize the lift as \(\ell_R = e^{xP_1} e^{yP_2} e^{\theta J}\). The three generators admit a matrix representation:

\[
P_1 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad P_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad J = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},
\]

so that the group product can be easily computed in terms of matrix products. This gives the transformation laws that Rachel found in (1, 2). Since \((x, y)\) fully realize the group, we discard

\(^8\)We can set it to any constant without affecting the result. For clarity of notation, we set it to zero.

\(^9\)Notice that setting \(\Pi = 0\) doesn’t imply \(d \Pi = 0\).
We have the following quantities by transforming $x, y$ and derivatives of $y$ with respect to $x$, under the special group element $g_* = (a_1^*, \varphi_*)$:

\begin{align*}
X &= x \cos(\varphi_*) - y \sin(\varphi_*) + a_1^*, \\
Y &= y \cos(\varphi_*) + x \sin(\varphi_*) + a_2^*, \\
\frac{dY}{dX} &= \frac{y' \cos(\varphi_*) + \sin(\varphi_*)}{\cos(\varphi_*) - y' \sin(\varphi_*)}, \\
\frac{d^2Y}{dX^2} &= \frac{y''}{(\cos(\varphi_*) - y' \sin(\varphi_*))^2}.
\end{align*}

Normalizing $X = Y = \frac{dY}{dX} = 0$ solves for the group element $g_*:
\begin{align*}
a_1^* &= \frac{-x - yy'}{\sqrt{1 + (y')^2}} \quad a_2^* = \frac{-y + xy'}{\sqrt{1 + (y')^2}} \quad \varphi_* = -\arctan(y'),
\end{align*}

leaving us with the invariant curvature and measure:
\begin{align*}
\frac{d^2Y}{dX^2} &= \frac{y''}{(1 + (y')^2)^{3/2}} \quad dX = dx \sqrt{1 + (y')^2}.
\end{align*}

### 3.1 Coordinate independence

We can also perform the construction in a coordinate-independent manner. We declare all objects in the homogeneous space, both $x$ and $\pi$, to be functions of $D$ external diffeomorphism parameters $\lambda$. While $\lambda$ transforms under $\text{Diff}(D)$, the basis forms $d\lambda$ transform under local $\text{GL}(D)$:
\begin{equation}
d\lambda^a \rightarrow J^a_b d\lambda^b,
\end{equation}
where $J$ is the Jacobian of the diffeomorphism. Following a similar logic as before, we can transform $d\lambda$ under some special Jacobian $J_*$ to be determined to produce the invariant one-forms:
\begin{equation}
d\lambda \xrightarrow{J_*} d\Lambda.
\end{equation}

Now, in addition to fixing the special group element $g_*$, we also need to fix the special Jacobian $J_*$. That’s $D^2$ extra parameters to fix! Luckily, because $\lambda$ is now our independent variable, we don’t work with the quantities $d\Pi/dX$, but rather $d\Pi/d\Lambda$ and $dX/d\Lambda$:
\begin{align*}
\frac{\partial x}{\partial \lambda} g_* \frac{dX}{d\Lambda}, \\
\frac{\partial \pi}{\partial \lambda} g_* \frac{d\Pi}{d\Lambda}.
\end{align*}

The $dX/d\Lambda$ now give precisely the extra quantities that can be normalized to convenient values in order to fix the Jacobian.

**Example: diffeomorphic curvature of curves.** As in the previous example, but now we impose diffeomorphism symmetry. Instead of taking derivatives with respect to the form $dx$, which gets transformed into $dX$ under the special group element $g_*$, we take derivatives with respect to the form $d\lambda$, which becomes $d\Lambda$ under the special Jacobian $J_*$. Importantly, $J$ is an element of *local* $\text{GL}(1)$, so while $dg = 0$, we have $dJ \neq 0$. Thus, our quantities are:
\[ X = x \cos(\varphi_\alpha) - y \sin(\varphi_\alpha) + a_1^\ast, \]
\[ Y = y \cos(\varphi_\alpha) + x \sin(\varphi_\alpha) + a_2^\ast, \]
\[ \frac{dX}{d\Lambda} = \frac{1}{J_\ast} \left[ x' \cos(\varphi_\alpha) - y' \sin(\varphi_\alpha) \right], \]
\[ \frac{dY}{d\Lambda} = \frac{1}{J_\ast} \left[ y' \cos(\varphi_\alpha) + x' \sin(\varphi_\alpha) \right], \]
\[ \frac{d^2X}{d\Lambda^2} = \frac{J_\ast'}{J_\ast^2} \left( \sin(\varphi_\alpha) y'' - \cos(\varphi_\alpha) x' \right) + J_\ast \left( \cos(\varphi_\alpha) x'' - \sin(\varphi_\alpha) y'' \right), \]
\[ \frac{d^2Y}{d\Lambda^2} = \frac{J_\ast'}{J_\ast^2} \left( \sin(\varphi_\alpha) x'' + \cos(\varphi_\alpha) y'' \right) - J_\ast' \left( \sin(\varphi_\alpha) x' + \cos(\varphi_\alpha) y' \right). \]

Setting \( X = Y = dY/d\Lambda = 0 \) and \( dX/d\Lambda = 1 \) fixes everything:

\[ a_1^\ast = \frac{-xx' - yy'}{\sqrt{(x')^2 + (y')^2}} \quad a_2^\ast = \frac{-yx' + xy'}{\sqrt{(x')^2 + (y')^2}} \quad \varphi_\ast = -\arctan(y'/x') \quad J_\ast = \sqrt{(x')^2 + (y')^2}, \]

so that the invariant curvature and measure are:

\[ \frac{d^2Y}{d\Lambda^2} = \frac{-y'x'' + x'y''}{\left[ (x')^2 + (y')^2 \right]^{3/2}} \quad d\Lambda = d\Lambda \sqrt{(x')^2 + (y')^2}. \]

Note that \( d^2X/d\Lambda^2 = 0 \) after imposing (42) so that the final number of invariant observables is the same as in the problem without diffeomorphism invariance. This is expected since coordinate independence is simply a redundancy in the description; the two problems are physically the same.

### 3.2 Quasi-invariants

The previous procedure concerns the construction of a strictly invariant action. Physics, however, isn’t that strict and can tolerate actions that change by a total derivative. Terms that do so are called quasi-invariants, or Wess-Zumino terms.

To find these in \( D \) spacetime dimensions, we must locate invariant \((D+1)\)-forms \( \beta \) that are exact, so \( \beta = d\alpha \), but with \( \alpha \) itself not being invariant. Then the invariance of \( \beta \) together with \( d^2 = 0 \) imply the quasi-invariance of \( \alpha \). And of course \( \alpha \) is a \( D \)-form, so \( \int \alpha \) will be a valid supplement to the action.

We have invariant one-forms given by:

\[ dx \xrightarrow{g_\ast} dX, \]
\[ d\pi \xrightarrow{g_\ast} d\Pi, \]

evaluated under the special group element \( g_\ast \) that we fixed before, and treating \( dx \) as an independent form, that is, we don’t write \( d\pi^\ast = \partial_\alpha \pi^a dx^\mu \). Higher forms can be constructed with sufficient applications of the wedge product between the \( dX \) and \( d\Pi \). The procedure is fairly standard, so we simply direct the reader to [15] for more detailed instructions.

### 4 Extended example: Poincaré to de Sitter

Let’s consider an extended example in spacetime and in higher dimensions that illustrates many of the ambiguities and inequivalencies that can arise when performing the coset construction for spacetime symmetry groups.

**Step 1.** Suppose we are given the symmetry breaking pattern \( \text{ISO}(1,D) \to \text{SO}(1,D) \). This covers the broken symmetries, but to fully determine the stability subgroup we need to know which symmetries are inhomogeneous in spacetime. There are two canonical options:
• Since ISO(1, D) is the isometry group of Minkowski spacetime \( M^{D+1} \), we could take the spacetime origin to be the origin of \( M^{D+1} \). Then the inhomogeneous transformations are the translations, so that the overall stability group is \( S = \text{SO}(1, D) \).

• Since SO(1, D) is the isometry group of de Sitter spacetime \( dS^D \), we could take the spacetime origin to be the origin of \( dS^D \). Then the inhomogeneous transformations are the homotheties, so that the overall stability group is \( S = \text{SO}(1, D - 1) \) (see the next step).

The choices are inequivalent; this is trivial to see since the first corresponds to the coset space \( \text{ISO}(1, D)/\text{SO}(1, D) \) while the second to \( \text{ISO}(1, D)/\text{SO}(1, D - 1) \). The first gives rise to the usual embedding of the de Sitter hyperboloid in an ambient Minkowski space; as it has already been explored in [16], we won’t focus on it here. We will thus pick the second option, the coset space \( \text{ISO}(1, D)/\text{SO}(1, D - 1) \).

**Step 2.** Let us now parametrize the lift that connects the elements in our theory to a coset in \( \text{ISO}(1, D)/\text{SO}(1, D - 1) \). Once again we are presented with inequivalent choices. One option would be to write:

\[
\ell_R = e^{x^\mu P^\mu} e^{\pi P^\mu} e^{\eta^\mu M^\mu}, \tag{46}
\]

where \( P^\mu \) are the usual translations of Minkowski space and \( M^\mu \) Lorentz transformations; the Greek indices \( \mu, \nu \) range from 0 to \( d = D - 1 \). But this lift parametrization gives the well known DBI action for a Minkowski brane embedded in Minkowski bulk [16]:

\[
S = \int d^D x \sqrt{1 + (\partial \pi)^2}. \tag{47}
\]

Let’s use the other Levi ordering, thus inverting the radical-then-simple order of (46). For clarity, we define a new basis for the simple generators:

\[
D^{\text{dS}} = M^M_0, \tag{48}
\]

\[
P_i^{\text{dS}} = M^M_0 - M^M_D, \tag{49}
\]

\[
M^{\mu
\nu} = M^M_{\mu\nu}. \tag{50}
\]

with Latin indices \( i, j \) ranging from 1 to \( d \). The \( D^{\text{dS}} \) and \( P_i^{\text{dS}} \) generators satisfy the homothety algebra, that is, they are spacetime dilation and space translations, respectively:

\[
[P_i^{\text{dS}}, P_j^{\text{dS}}] = 0 \quad [P_i^{\text{dS}}, D^{\text{dS}}] = P_i^{\text{dS}}, \tag{51}
\]

so that the stability subgroup (unbroken group minus homotheties) is indeed \( \text{SO}(1, d) \), as advertised above. We thus write the simple-then-radical lift as:

\[
\ell_L = e^{x^\mu P^\mu} e^{\xi D^\mu} e^{\pi P^\mu + \eta P^\mu}. \tag{52}
\]

**Step 3.** We now need the transformation laws, which follow from the group action on a coset element: \( tS \rightarrow g tS \). While straightforward, the computation itself can be tedious (it helps to switch to conformal time \( t = -\log(-\tau) \), with Hubble = 1). One concern we encounter is that then \( \xi^\mu \) isn’t a vector. For instance, it’s strictly invariant rather than covariant under a dilation. In principle this isn’t an issue; the procedure in Section 3 doesn’t require covariance under the unbroken subgroup. By inspection, though, we can see that a field redefinition \( \xi^\mu = A^\mu/\tau \) gives the proper covariant transformation for \( A^\mu \), so we will make use of this for simplicity. We stress, however, that this step is ad hoc; had we been unable to find this convenient field redefinition, we would have had to perform the full construction, without exploiting covariance.

Under the simple part of the group (unbroken \( \text{SO}(1, D) \)) we obtain that \( x^\mu = (\tau, x^i) \) transform as the (flat slicing) coordinates of de Sitter spacetime in conformal time (see Appendix C for explicit expressions), \( \tau \) as a scalar and \( A^\mu \) as a vector.
As for the radical part of the group (broken $\mathbb{R}^{D+1}$), we have:

\begin{align}
  x^\mu &\rightarrow x^\mu, \\
  \pi &\rightarrow \pi - \theta, \\
  A_\mu &\rightarrow A_\mu + \partial_\mu \theta, \\
  \theta &\overset{\text{def}}{=} \frac{1}{\tau} (c + b^i x^j \delta_{ij} + \frac{1}{2} a x^\mu x^\nu \eta_{\mu\nu}),
\end{align}

where $a, b^i, c$ are the parameters of the broken translations. Notice how $A^\mu$ transforms as if it were a gauge vector, with $\pi$ its longitudinal mode. However, we aren’t interested in imposing gauge invariance, that is, for any choice of $\theta$, but rather only for the specific $\theta$ given above.

For convenience, it is useful to note that $\theta$ satisfies:

\[ [\nabla_\mu \nabla_\nu + g_{\mu\nu}] \theta = 0, \quad (57) \]

for $\nabla_\mu$ and $g_{\mu\nu}$ the usual geometrical objects of de Sitter space (in this context, $\theta$ is a scalar).

Also note that if we restore the Hubble constant $H$, switch to physical time, then take the Minkowski $H \rightarrow 0$ limit, we obtain $\theta = 0$ and the symmetry is trivialized.

**Step 4.** Our bosons are antisocial: $A_\mu$ transforms without $\pi$ and $\pi$ without $A_\mu$, and any by itself still fully realizes the broken translations (and the $x^\mu$ realize the rest of the group). This means we could, in principle, remove either one. However, as we will see in Section 4.2, either choice leads to a sick theory. So we’ll keep both.

**Step 5.** To derive invariants, we first transform all objects under some special group element $g_*$, whose specific form will be fixed later:

\begin{align}
  x^\alpha &\overset{g_*}{\rightarrow} X^\alpha, \\
  \pi &\overset{g_*}{\rightarrow} \Pi, \\
  A_\beta &\overset{g_*}{\rightarrow} A_\beta.
\end{align}

We wish to shift those objects back to the origin of spacetime and field space, so we set $X^0 = -1$, $X^i = 0$, $\Pi = 0$ and $A_\mu = 0$. This solves for all the group parameters of the Minkowski translations $a, b^i$ and $c$, the dilation $\Lambda$, and the de Sitter translations $d^i$:

\begin{align}
  a_* &= A_0 + \pi, & [b_i]_* &= A_i, & c_* &= \frac{1}{2} (A_0 - \pi) & \Lambda_* &= -\frac{1}{\tau} & d^i_* &= -x^i.
\end{align}

The group parameters for the stability group $SO(1,d)$ remain. However, both $\Pi$ and $A_\alpha$, as well as derivatives $d/dX^\alpha$, transform covariantly under it, so we don’t need to fix those parameters as long as we perform manifestly invariant contractions of the $\alpha, \beta$ indices.

Since we have exhausted the zeroth order objects $\Pi$ and $A$, we extend it to their derivatives:

\begin{align}
  \partial_\alpha \pi &\overset{g_*}{\rightarrow} \frac{d\Pi}{dX^\alpha}, \\
  \partial_\alpha A_\beta &\overset{g_*}{\rightarrow} \frac{dA_\beta}{dX^\alpha},
\end{align}

which must be evaluated under the $g_*$ we found above. The covariant one-forms $dX$ are found in a similar manner, from transforming $dx$ under $g_*$. The result is:

\begin{align}
  dX^\alpha &= \frac{1}{\tau} \delta^\alpha_\mu dx^\mu, \\
  \frac{d\Pi}{dX^\alpha} &= \tau \delta^\mu_\alpha (A_\mu + \partial_\mu \pi), \\
  \frac{dA_\beta}{dX^\alpha} &= \tau^2 \delta^\mu_\alpha \delta^\nu_\beta (\nabla_\mu A_\nu - \pi g_{\mu\nu}).
\end{align}
These objects live in flat spacetime, so they must be contracted with \( \eta_{\alpha\beta} \) or \( \varepsilon_{\alpha_1\alpha_2...} \). For ease of notation, we can write the corresponding objects living in the curved de Sitter space together with the volume measure, via the tetrad property (\( \tau \delta^\mu_\nu)(\tau \delta^\nu_\mu)\eta^{\alpha\beta} = g^{\mu\nu} \):

\[
\begin{align*}
\text{d}V &= \frac{1}{D!} \varepsilon_{\alpha_1\alpha_2...} \text{d}X^{\alpha_1} \wedge \text{d}X^{\alpha_2} \wedge \ldots = \frac{\text{d}^Dx}{\tau^D} = \text{d}^Dx \sqrt{-g}, \\
V_\mu &= A_\mu + \partial_\mu \pi, \\
F_{\mu\nu} &= \partial(\mu A_\nu), \\
S_{\mu\nu} &= \nabla(\mu V_\nu) - \pi g_{\mu\nu},
\end{align*}
\]

where \( F_{\mu\nu} \) and \( S_{\mu\nu} \) come from splitting (66) into its antisymmetric and symmetric parts, and \( V_\mu \) is just (65) renamed. The \( \mu, \nu \) indices are to be contracted with \( g^{\mu\nu} \). Note that \( F_{\mu\nu} \) and \( V_\mu \) are \( U(1) \)-invariant, by accident as that wasn’t part of the original construction, but \( S_{\mu\nu} \) isn’t.

There’s another parametrization of these invariants that’s more useful. By symmetrizing \( \nabla_\mu V_\nu \), we can rewrite \( S_{\mu\nu} \) without explicit reference to \( A_\mu \):

\[ S_{\mu\nu} = \nabla(\mu V_\nu) - [\nabla(\mu \nabla_\nu) + g_{\mu\nu}]\pi. \tag{71} \]

But \( S_{\mu\nu} \) and \( V_\mu \) are covariant so the following operator must be covariant as well:

\[ H_{\mu\nu} \overset{\text{def}}{=} [\nabla(\mu \nabla_\nu) + g_{\mu\nu}]\pi. \tag{72} \]

In principle, the strictly invariant action (no Wess-Zumino terms yet) then is:

\[ S = \int \frac{\text{d}^Dx}{\tau^D} P(V_\mu, H_{\mu\nu}, \nabla_\mu), \tag{73} \]

where \( F_{\mu\nu} \) is implicitly included given \( V_\mu \) and \( \nabla_\mu \). Making sure the action is healthy, however, further constrains it:

- The scalar \( \pi \) appears only in \( H_{\mu\nu} \), with second derivatives. To avoid propagating these extra ghostly degrees of freedom, such scalar would need to be a curved spacetime Horndeski, e.g. [17]. But other than a tadpole \( \sim \pi \), no such Horndeski terms can be built out of only \( H_{\mu\nu} \) and \( g_{\mu\nu} \).
- Similarly, the vector \( V_\mu \) can have kinetic terms of the form \( \nabla_\mu V^\mu \) and \( \nabla(\mu V_\nu) \). Those, too, have to appear in a special combination that does not propagate a ghost, as a massive vector should have only three degrees of freedom. Such generalized Proca theories in curved spacetime have already been derived in [18]; we must simply specialize to the case of de Sitter. This fixes the \( V_\mu \) part of the action.

After these considerations, the final strictly invariant Lagrangian is simply the generalized Proca one:

\[ \mathcal{L}_{\text{gen.Proca}}(V_\mu; \nabla_\mu), \tag{74} \]

described in [18]. Since the decomposition of the invariant \( V_\mu \) in terms of the Goldstones is \( V_\mu = A_\mu + \partial_\mu \pi \), this theory begs to be rewritten following the usual Stückelberg procedure. Defining the generalized Stückelberg Lagrangian, \( \mathcal{L}_{\text{gen.Stück}}(A_\mu, \partial_\mu \pi) = \mathcal{L}_{\text{gen.Proca}}(A_\mu + \partial_\mu \pi) \), we write the action as:

\[ S = \int \text{d}^Dx \sqrt{-g} \mathcal{L}_{\text{gen.Stück}}(A_\mu, \partial_\mu \pi; \nabla_\mu). \tag{75} \]

It is not unusual that the strict invariants for \( \pi \) ended up higher order in derivatives. From the transformation (54), \( \pi \) appears to be a galileon. As discussed in [15], galileon invariants tend to be higher order in derivatives. But now the theory has too much symmetry. While it’s technically ISO(1, \( D \)) invariant, that group gets drowned in the infinite \( U(1) \) gauge group. In order to rescue it while preserving the theory’s health, we will use quasi-invariants that break \( U(1) \) but not ISO(1, \( D \)).
4.1  Adding quasi-invariants

In addition to strict invariants, we also have quasi-invariants, or Wess-Zumino terms, that change by a total derivative. For the sake of expediency, we just write down the first three, restoring the Hubble constant $H$:

\begin{align}
W_1 &= \pi, \\
W_2 &= (\partial \pi)^2 - DH^2 \pi^2, \\
W_3 &= (\Box \pi) \left[ (\partial \pi)^2 - (D - 1)H^2 \pi^2 \right] - \frac{2}{3} D(D - 1)H^4 \pi^3,
\end{align}

though we will require only $W_2$ for building a healthy theory. Notice these terms appear like the usual galileon operators, except with some corrections due to $H$. Indeed, in the limit $H \to 0$, we get the Minkowski galileons [15, 19] as expected from the group contraction.

Looking at $W_2$, the kinetic term has the wrong sign compared to the mass term. This is not a problem, because $(\partial \pi)^2$ also appears in the generalized Stückelberg Lagrangian, so we may hope to combine those two into something with the proper sign. Indeed, we can extract the mass term $-(m^2/2) (A_\mu + \partial_\mu \pi)^2$ from $L_{\text{gen. Stück}}$ and add it to $qm^2W_2$ where $q$ is some dimensionless constant.

Performing the canonical normalization $\pi_c \overset{\text{def}}{=} m\sqrt{1-q}\pi$ then gives the following action:

\begin{align}
S = \int d^Dx \sqrt{-g} \left[ -\frac{1}{4} F^2 - \frac{1}{2} m^2 A^2 - \frac{m}{\sqrt{1-q}} (A \cdot \partial) \pi_c + L_{\text{int. Stück}}^{\text{gen. Stück}} - \frac{1}{2} (\partial \pi_c)^2 - \frac{1}{2} \left( \frac{q}{1-q} \right) DH^2 \pi_c^2 + \text{other WZ terms} \right],
\end{align}

where $L_{\text{int. Stück}}^{\text{gen. Stück}}$ denotes all generalized Stückelberg interactions. The theory is healthy as long as $0 \leq q < 1$. Furthermore, the special case $q = 0$ together with setting all other Wess-Zumino terms to zero restores the $U(1)$ gauge symmetry.

To sum up, we have a healthy action (79) constructed from the same coset space as the action for a Minkowski brane embedded in Minkowski bulk (47), but the two theories have nothing to do with which other. This is a result of changing the coset space parametrization in a way that flipped the Levi ordering.

4.2  No inessential Goldstones

This problem is peculiar in that both Goldstones are essential and inessential: either $\pi$ or $A_\mu$ can be eliminated in favor of derivatives of the other. One way to see this is to return to the transformation laws (54) and (55) and recall that the $\pi$ doesn’t mix with the $A_\mu$ and vice-versa. We could’ve eliminated either and straightforwardly derived invariants using only one of them. It might also be instructive to look at this issue from the inverse Higgs constraint (IHC) point of view. Note that in this case the IHCs will not be equivalent to integrating out fields via their equations of motion.

To eliminate $A_\mu$, we perform the following covariant normalization:

\begin{align}
V_\mu = 0 \implies A_\mu = -\partial_\mu \pi.
\end{align}

Alternatively, $\pi$ can be eliminated through the invariant normalization of $g^{\mu\nu} S_{\mu\nu}$:

\begin{align}
S = 0 \implies \pi = \frac{1}{D} \nabla_\mu A^\mu.
\end{align}

In both cases, the resulting theory becomes sick:

**Pure $\pi$ theory.** If we set $V_\mu = 0$, then the kinetic term for $\pi$ is sick as we lack the $(m^2 A^2/2)V^2$ mass term. We discussed this in Section 4.1.
Pure $A_\mu$ theory. Setting $S = 0$ introduces second derivatives of $A_\mu$ in the action (unless the action is just generalized Maxwell, with only $F$), so it won’t be healthy. One way to see this is through a Stückelberg decomposition again. Even though we split $V_\mu = A_\mu + \partial_\mu \pi$, the inverse Higgs constraint itself breaks the gauge symmetry, so $A_\mu$ will not be a gauge boson. Decomposing $A_\mu$ will mean its longitudinal mode carries more than two derivatives, so it’s an unhealthy scalar.\footnote{As discussed in [20], scalar theories with more than two derivatives are unhealthy, but that’s not necessarily true for multi-field ones. Following that reference, we can take the decoupling limit of our action after removing $\pi$ and show that the resulting theory for the longitudinal mode still involves more than two derivatives, thus sick.}

The conclusion is that while the possibility of eliminating Goldstones exists from a purely group-theoretical point of view, once physics is considered, none can be removed. In a sense, this is reassuring: since either can be eliminated, the most democratic option is to keep both.

5 Discussion

In this paper we have returned to the first principles of the coset construction to investigate its universality when applied to spacetime symmetry groups. We discussed the natural hierarchical structure that Goldstones can acquire when the group $G$ is not semisimple, which dictates which Goldstones are essential and which can be eliminated. By direct example, we showed that an arbitrary reparametrization of the coset space might not preserve this structure, changing which inessential Goldstones can be conveniently removed.

In particular, by reparametrizing the coset space of a Minkowski brane in Minkowski bulk, we constructed a theory for a scalar and vector Goldstones living in de Sitter space and non-linearly realizing the Poincaré group. At first sight, one may wonder how this is possible, given that gauge Goldstones that non-linearly realize spacetime symmetries should not exist according to [21] (our vector boson isn’t gauge, but can of course be decomposed into one plus a scalar). Simply put, the assumptions of the no-go theorem aren’t satisfied: our unbroken subgroup is de Sitter, whereas that of [21] is Poincaré, and [21] assumes removal of the inessential Goldstones, which in our case can’t be executed. Also, as was discussed below (56), our action doesn’t realize any broken symmetry in the $H \to 0$ limit, so group contraction doesn’t provide a counter-example to the no-go theorem. Indeed, this is an example of a symmetry realization unique to de Sitter without analogue in Minkowski, a possibility brought up in [22].

That the universality of the coset construction isn’t protected under transformations that change the Goldstone hierarchy isn’t a new fact. In [23], the authors construct two inequivalent theories related by a map that mixes essential and inessential Goldstones, though they didn’t bring up the hierarchy issue. It’s not surprising these theories would then have different hierarchical structures.

A possible extension of this work would be to classify all possible hierarchical structures for a given coset space with inequivalent physics. We provided two candidates, based on the Levi ordering, but we don’t know if they are exhaustive. Furthermore, even if a reparametrization mixes essential and inessential Goldstones, it doesn’t necessarily mean the resulting theories will be inequivalent; an example is given in [13, Sec. 4.2].

Another avenue of further research is the study of simple spacetime symmetry groups. Non-linear realizations of such groups can still involve inessential bosons. The basic example is how the breaking of the conformal group down to Poincaré gives rise to an essential dilaton and an inessential special conformal boson [24]. Yet such groups have a trivial Levi decomposition, so wherever it is that their Goldstone hierarchy is coming from, it’s not coming from there. Further study to determine whether the coset construction is unique in this case is required.

A last question concerns the issue of UV completion, which we haven’t touched upon at all. Given two inequivalent theories derived from the same coset space, it would be interesting to see what the theories look like once the broken symmetries are restored and if they relate in any way.

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A Rachel and Leo don’t understand each other

Here we show that the two actions describing plane curves that Rachel and Leo found in Section 2.1 are inequivalent.

Suppose they are equivalent and Rachel’s language can be translated to Leo’s and vice-versa. Then, we have to find an invertible redefinition that maps Rachel’s variables to Leo’s. A naive first attempt would be to set $\ell_R = \ell_L$, then use the inverse Higgs constraints that Rachel and Leo found to eliminate the inessential fields in terms of derivatives of the essentials. However, this redefinition would mix fields with their own derivatives, and one straightforward shows they are not invertible. For example, mapping Rachel to Leo then back to Rachel doesn’t output the original input.

Let’s first select the $x' = 1$ gauge for Rachel. Her Lagrangian now is:

$$L_R = \sqrt{1 + (y')^2} P \left( \frac{y''}{1 + (y')^2} \right). \quad (82)$$

As for Leo, we keep his reparametrization invariance. Recall his action was:

$$L_L = \theta' F \left( \pi + \frac{-\pi' \theta'' + \theta' \pi''}{(\theta')^3} \right). \quad (83)$$

We need to find redefinitions for $\pi$ and $\theta$ that will put Leo’s Lagrangian in the form of (82). In fact, the definition of $\theta$ is unimportant since we still have the gauge freedom to set it to whatever, which we will use in the end. Notice that in Rachel’s Lagrangian her field $y$ only appears with derivatives, so the same needs to be true for Leo. If the action only depends on the derivatives of some field $\phi(\lambda)$, the field possesses a shift symmetry of the form $\delta \phi = c$, and its equation of motion is manifestly integrable:

$$Q \overset{\text{def}}{=} \frac{\partial L}{\partial \phi'} - \frac{d}{d\lambda} \frac{\partial L}{\partial \phi''} + \ldots, \quad (84)$$

$$\Rightarrow \frac{dQ}{d\lambda} = 0, \quad (85)$$

which gives $Q = \text{cst}$, precisely the conserved charge of the symmetry. Recalling the exact variation for $\pi$ under the Euclidean group,

$$\delta \pi = a^1 \cos \theta + a^2 \sin \theta, \quad (86)$$

we see $\pi$ doesn’t have a shift symmetry. Thus the objective is to redefine $\pi$ so one of the Euclidean symmetries becomes the desired shift, protecting it from appearing without derivatives in the action.\(^{11}\) The only candidate is the translation by $a^1$, as it’s the only one for which $\pi$ transforms inhomogeneously (i.e., can map $\theta = 0$ and $\pi = 0$ to something else), just like a shift. So we need to compute the charge of this translation, which is really the momentum generator $P_1$.

To do so, we perform the usual trick of allowing $a^1$ to be a local parameter and vary the action, putting it in the form $\delta S_L = \int d\lambda P_1 \partial_\lambda a^1$ through integration by parts. This gives:

$$P_1 = -\frac{d}{d\lambda} \left[ \frac{\cos \theta}{\theta'} F_r(\tau) \right] - \frac{2(\theta')^2 \sin \theta + \theta'' \cos \theta}{(\theta')^2} F_r(\tau) = \text{cst}, \quad (87)$$

where we called $\tau$ the argument of the $F$ function and $F_r = \partial_r F$. If $L_L = \theta' F(\tau)$ is to be a Lagrangian where its dependent variable (call it $\phi$) appears only with derivatives, then its equation of motion must be manifestly integrable:

$$\frac{d}{d\lambda} \left[ -\frac{d}{d\lambda} \frac{\partial L_L}{\partial \phi''} + \frac{\partial L_L}{\partial \phi'} \right] = 0, \quad (88)$$

$$\Rightarrow -\frac{d}{d\lambda} \left[ \theta' \frac{\partial_{\phi'} F_r(\tau)}{\partial \phi''} + \theta' \frac{\partial_{\phi'} F_r(\tau)}{\partial \phi'} \right] = \text{cst}. \quad (89)$$

\(^{11}\)This shift symmetry does need to come from the Euclidean group, otherwise Leo’s action will have more symmetries (Euclid plus this new shift) than Rachel’s.
By Noether’s theorem, charge conservation and equation of motion are equivalent, so we match Equations (87) and (89), giving the following differential relation between $\pi$ and $\phi$:

$$
\tau = \pi + \frac{-\pi'\theta'' + \theta'\pi''}{(\theta')^3},
$$

(90)

$$
= \frac{\cos \theta}{(\theta')^2} \phi'' - \frac{2(\theta')^2 \sin \theta + \theta' \cos \theta}{(\theta')^3} \phi'.
$$

(91)

It can be solved\textsuperscript{12} into an algebraic relation:

$$
\pi = c_1 \sin \theta + (c_2 + \phi) \cos \theta,
$$

(92)

where $c_1$ are $c_2$ and integration constants (one can check they don’t alter the conclusions, so we set them to 0). At this level we should start to worry, since the redefinition isn’t invertible whenever $\cos \theta = 0$, but at least it’s locally invertible around the origin $\theta = 0$. So maybe the two theories can match in some region of space. Regardless, the Lagrangian in this new variable $\phi$ is:

$$
L_L = \theta' F \left( \frac{\cos \theta}{(\theta')^2} \phi'' - \frac{2(\theta')^2 \sin \theta + \theta' \cos \theta}{(\theta')^3} \phi' \right),
$$

(93)

and $\phi$ varies infinitesimally as:

$$
\delta \phi = a^1 + a^2 \tan \theta + \varphi \phi \tan \theta + O(a^1 \varphi, a^2 \varphi),
$$

(94)

so $a^1$ is indeed shifty and protects $\phi$ from appearing without derivatives. Now we can attempt to pick some gauge choice for $\theta$, to have the action match Rachel’s form (82). To reproduce the square root factor that Rachel has, Leo would need $\theta' = \sqrt{1 + (\theta')^2}$. However, $\theta$ appears explicitly in Leo’s Lagrangian (93); this choice would then introduce non-local terms that depend on $\int d\lambda \sqrt{1 + (\theta')^2}$. But Rachel’s action is manifestly local. Therefore, the two are inequivalent.

### B Levi decomposition

Let us quickly recall the notion of the Levi decomposition. Any finite Lie group $G$ can be decomposed\textsuperscript{13} using a single semidirect product [25]:

$$
G = R \rtimes L,
$$

(95)

where:

- $L$ is the simple (or Levi) factor, a semisimple group;
- $R$ is the radical, a group whose algebra is maximally solvable.

A subalgebra $i$ of $g$ is solvable if it’s an ideal (so $[i, g] \subseteq i$) and if it telescopes to zero upon repeated application of the commutator, meaning:

$$
[i, i] = i_1 \subseteq i,
$$

(96)

$$
[i_1, i_1] = i_2 \subseteq i_1,
$$

(97)

$$
[i_2, i_2] = i_3 \subseteq i_2,
$$

(98)

$$
\vdots
$$

(99)

$$
[i_n, i_n] = 0,
$$

(100)

after a finite number $n$ of steps. The largest such ideal is then the algebra’s radical.

\textsuperscript{12}Pick the $\theta' = 1$ gauge so the differential equation is easily solvable, then restore gauge symmetry by replacing all explicit instances of $\lambda$ in the solution by $\theta'(\lambda)$.

\textsuperscript{13}To make the claim mathematically unimpeachable, we note the decomposition also admits discrete group factors and that the simple factor is only unique up to conjugation by the group’s nilradical. We ignore these technicalities here.
Some examples of Levi decomposition:

- Poincaré \(\sim\) translations \(\rtimes\) Lorentz,
- Galileo \(\sim\) (translations and boosts) \(\rtimes\) rotations,
- General affine \(\sim\) (translations and dilation) \(\rtimes\) special linear.

This classification could be further refined depending on the kind of radical in play (abelian, nilpotent, etc) though that’s not required for our purposes here.

**B.1 Levi ordering**

Now suppose we want to work with the coset space \(G/S\). We will assume that \(G\) is not semisimple, but \(S\) is. Ideally we would like to classify all possible hierarchical structures the homogeneous space could have, but we will limit ourselves to only showing at least two inequivalent ones exist.

Let \(P\) be the generators living in the radical of \(G\), and \(T\) those in the simple factor. Finally, if a simple generator is not in the stability group \(S\), denote it by \(A\); conversely, denote a simple generator in \(S\) by \(V\). Then at least two parametrizations of the lift give different structures:

\[
\ell_R \overset{\text{def}}{=} e^{zP}e^{\xi A} \quad (z \text{ transforms by itself}),
\]

\[
\ell_L \overset{\text{def}}{=} e^{\xi A}e^{zP} \quad (\xi \text{ transforms by itself}).
\]

To establish this, we act with the group \(G\) and check the form of each transformation law. We will make use of the following braiding identities:

\[
e^{aP}e^{uT} = e^{uT}e^{MaP}, \tag{103}
\]

\[
e^{aP}e^{vT} = e^{vT}e^{MaP}, \tag{104}
\]

\[
e^{aP}e^{\xi A} = e^{\xi A}e^{v\xi V}e^{M_t aP}, \tag{105}
\]

where \(M_u\) is a linear map. They follow from the Baker–Campbell–Hausdorff formula upon usage of the ideal property of the radical, \([P,T] \sim P\), together with closure of the \(T\)'s and \(V\)'s.

**Radical first.** We act with some \(g = e^{uT}e^{aP}\) on \(\ell_R S\) to derive transformation laws:

\[
\tilde{\ell}_R S = e^{uT}e^{aP}e^{zP}e^{\xi A S} = e^{uT}e^{\tilde{z}(z; a)P}e^{\xi A S} \tag{106}
\]

\[
= e^{uT}e^{\tilde{z}(z; a)P}e^{\xi A S} \tag{107}
\]

\[
= e^{M_a \tilde{z}(z; a) P}e^{uT}e^{\xi A S} \tag{108}
\]

\[
= e^{M_a \tilde{z}(z; a) P}e^{\tilde{\xi}(\xi; u) V}e^{M_t aP} \tag{109}
\]

where we used, in order, closure of \(R\), then braiding, then closure of \(L\); the underline denotes where we use each argument. To conclude, the objects in the homogeneous space transform as:

\[
z \rightarrow M_a \tilde{z}(z; a), \quad \xi \rightarrow \tilde{\xi}(\xi; u), \tag{110}
\]

so that \(z\) fully realizes the group: it transforms by itself and its transformation depends on all group parameters.

**Simple first.** Without loss of generality, flip the order of the group element, so now \(g = e^{aP}e^{uT}\) acting on \(\ell_L S\):

\[
\tilde{\ell}_L S = e^{aP}e^{uT}e^{\xi A}e^{zP}S = e^{aP}e^{\tilde{\xi}(\xi; u) A}e^{uT}e^{\tilde{z}(z; \xi; u) aP}S \tag{111}
\]

\[
= e^{aP}e^{\tilde{\xi}(\xi; u) A}e^{uT}e^{M_t aP}S \tag{112}
\]

\[
= e^{\tilde{\xi}(\xi; u) A}e^{e(u; \xi) V}e^{M_t aP}e^{M_t zP}S \tag{113}
\]

\[
= e^{\tilde{\xi}(\xi; u) A}e^{e(u; \xi) V}e^{M_t aP}e^{M_t zP}S \tag{114}
\]

\[
= e^{\tilde{\xi}(\xi; u) A}e^{\tilde{z}(z; \xi; u, a) P}S, \tag{115}
\]

18
using closure of $L$, then braiding, then braiding again, then closure of $P$ with $V$. Thus the transformations are:

$$\xi \rightarrow \tilde{\xi}(\xi; u), \quad z \rightarrow \tilde{z}(z; \xi; u, a),$$

(116)

so $\xi$ transforms by itself, but $z$ transforms with reference to $\xi$.

C Finite de Sitter isometries

The transformation laws for $x^\mu = (\tau, x^i)$ derived in Section 4 correspond to the de Sitter isometries. The straightforward ones are the translations $d^i$, dilation $\Lambda$ and rotations $\theta_{ij}$:

$$\tau \rightarrow \Lambda \tau,$$

(117)

$$x^i \rightarrow \Lambda R(\theta)^i_j(x^j + d^j).$$

(118)

Meanwhile, a boost with rapidity $\beta$ along the $x$-direction is given by:

$$\tau \rightarrow \frac{2\tau}{(1 - H^2 \eta_{\mu\nu} x^\mu x^\nu) + (1 + H^2 \eta_{\mu\nu} x^\mu x^\nu) \cosh(\beta) + 2 H x \sinh(\beta)},$$

(119)

$$x^i \rightarrow \frac{2 x \cosh(\beta) + \sinh(\beta)(1 + H^2 \eta_{\mu\nu} x^\mu x^\nu)/H}{(1 - H^2 \eta_{\mu\nu} x^\mu x^\nu) + (1 + H^2 \eta_{\mu\nu} x^\mu x^\nu) \cosh(\beta) + 2 H x \sinh(\beta)},$$

(120)

$$y^j \rightarrow \frac{2 y^j}{(1 - H^2 \eta_{\mu\nu} x^\mu x^\nu) + (1 + H^2 \eta_{\mu\nu} x^\mu x^\nu) \cosh(\beta) + 2 H x \sinh(\beta)},$$

(121)

for $j \neq 1$. Boosts along the other $y^j$-directions follow identically by rotational symmetry. Upon changing to physical time and taking the $H \rightarrow 0$ limit, we retrieve the Minkowski boost:

$$t \rightarrow \cosh(\beta)t - \sinh(\beta)x,$$

(122)

$$x \rightarrow \cosh(\beta)x - \sinh(\beta)t,$$

(123)

$$y^j \rightarrow y^j.$$  

(124)

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