Well-posedness for hyperbolic equations whose coefficients lose regularity at one point

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Abstract
We prove some $C^\infty$ and Gevrey well-posedness results for hyperbolic equations whose coefficients lose regularity at one point.

Keywords
Gevrey space · Well-posedness · Strictly hyperbolic · Modulus of continuity

Mathematics Subject Classification
35L10 · 35A22 · 46F12

1 Introduction
In this paper we deal with the well-posedness of the Cauchy problem for a linear hyperbolic operator whose coefficients depend only on time. Namely, we consider the equation

$$u_{tt} - \sum_{i,j=1}^{n} a_{ij}(t)u_{x_i x_j} = 0 \quad (1.1)$$

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in \([0, T] \times \mathbb{R}^n\), with initial data

\[ u(0, x) = u_0(x), \quad u_t(0, x) = u_1(x) \tag{1.2} \]

in \(\mathbb{R}^n\). The matrix \((a_{ij})_{i,j}\) is supposed to be real and symmetric. Setting

\[ a(t, \xi) := \sum_{i,j=1}^n a_{ij}(t) \xi_i \xi_j / |\xi|^2, \quad (t, \xi) \in [0, T] \times (\mathbb{R}^n \setminus \{0\}), \tag{1.3} \]

we assume throughout that \(a(\cdot, \xi) \in L^\infty(0, T)\) for all \(\xi \in \mathbb{R}^n \setminus \{0\}\). Moreover, we suppose that the equation (1.1) is strictly hyperbolic, i.e.

\[ \Lambda_0 \geq a(t, \xi) \geq \lambda_0 > 0 \tag{1.4} \]

for all \((t, \xi) \in [0, T] \times (\mathbb{R}^n \setminus \{0\})\).

It is a classical result that if the coefficients \(a_{ij}(t)\)’s are real integrable functions, then the Cauchy problem (1.1), (1.2) is well posed in \(A'_{\mathbb{R}^n}\), the space of real analytic functionals; moreover, if the initial data vanish in a ball, then the solution vanishes in a cone, whose slope depends on the coefficients \(a_{ij}(t)\)’s (see [1, Theorems 1 and 3.a]). On this basis, various well-posedness results can be proved by mean of the Paley-Wiener theorem (in the version of [1, p. 517], to which we refer here and throughout) and some energy estimates. If the coefficients \(a_{ij}(t)\)’s are Lipschitz-continuous then the Cauchy problem (1.1), (1.2) is well posed in Sobolev spaces. Relaxing this regularity assumption, one has that if the \(a_{ij}(t)\)’s are Log-Lipschitz-continuous or Hölder-continuous of index \(\alpha\), then (1.1), (1.2) is well posed in \(C^\infty\) or in the Gevrey space \(\gamma(s)\) for \(s < 1/(1-\alpha)\) respectively (see [1, Theorem 3.b,c]). Suitable counterexamples show that in each case the regularity assumption on the \(a_{ij}(t)\)’s is sharp for the well posedness of (1.1), (1.2) in the corresponding function space.

It is a remarkable fact that in the above mentioned counterexamples the coefficients \(a_{ij}(t)\)’s are in fact \(C^\infty\) for \(t \neq 0\), and each time the specific regularity fails only at \(t = 0\). In [2] the authors showed that a control on the rate of the loss of Lipschitz regularity of the \(a_{ij}(t)\)’s as \(t \to 0\) allows to recover well-posedness of (1.1), (1.2) in suitable function spaces. To be more specific, if the \(a_{ij}(t)\)’s are of class \(C^1\) in \([0, T]\) and \(|a'_{ij}(t)| \leq Ct^{-p}\), then (1.1), (1.2) is well posed in \(C^\infty\) when \(p = 1\), and in the Gevrey space \(\gamma(s)\) for \(s < \frac{1}{p-1}\) when \(p > 1\). Concerning \(C^\infty\) well-posedness, it was proved in [3] that a control on the second derivative of the \(a_{ij}\)’s as \(t \to 0\) allows to relax slightly the growth assumption on the first derivative up to \(|a'_{ij}(t)| \leq Ct^{-1}|\log t|^p\). In [5] some of the above results were extended to the case in which the coefficients \(a_{ij}\)’s depend also on the \(x\) variable in \(C^\infty\) fashion.

In this paper we consider non Lipschitz coefficients whose regularity is ruled by a modulus of continuity \(\mu\), with a constant which blows up as \(t \to 0\). More precisely, we assume that

\[ |a_{ij}(t + \tau) - a_{ij}(t)| \leq \frac{C}{v(t)} \mu(\tau), \quad 0 \leq \tau \leq \tau_0, \quad t, t + \tau \in [0, T], \tag{1.5} \]
where $v(t)^{-1}$ is possibly non integrable at $t = 0$ and where $\mu$-continuity is possibly strictly weaker than Lipschitz continuity. We investigate how the interaction between $v$ and $\mu$ affects the well-posedness of (1.1), (1.2).

In Sect. 2 we prove a technical regularization result for the coefficients $a_{ij}$’s.

In Sect. 3 we consider locally Hölder continuous coefficients satisfying

$$|a_{ij}(t + \tau) - a_{ij}(t)| \leq \frac{C}{t^p} \tau^\alpha, \quad 0 \leq \tau, \quad t, t + \tau \in ]0, T] \quad (1.6)$$

with $0 < \alpha < 1$ and $p > 1$, and we obtain well-posedness in the Gevrey space $\gamma(\sigma)$ for $\sigma < \frac{p}{p-\alpha}$, a condition which fits perfectly with the ones of [1] and [2].

In Sect. 4 we consider the problem of $C^\infty$ well-posedness and we identify a precise relation between $\mu$ and $v$ which guarantees the latter. In particular we obtain well-posedness for coefficients satisfying

$$|a_{ij}(t + \tau) - a_{ij}(t)| \leq \frac{C}{t|\log t|} \frac{\tau|\log \tau|}{|\log|\log \tau|}, \quad 0 \leq \tau, \quad t, t + \tau \in ]0, T], \quad (1.7)$$

where one can easily see that $v(t)^{-1}$ is non integrable and $\mu$-continuity is strictly weaker than Lipschitz continuity. Also in this situation the results fits with the ones contained in [1] and [2] and contain them as particular cases.

### 2 Approximation

We begin by recalling the notion of *modulus of continuity*.

**Definition 1** Let $\tau_0 > 0$. A function $\mu : [0, \tau_0] \to [0, +\infty]$ is a modulus of continuity if it is continuous, concave, strictly increasing and $\mu(0) = 0$.

Let $\mu$ be a modulus of continuity and let $a : [0, T] \to \mathbb{R}$ be a bounded function. Without loss of generality we can assume that $\tau_0 \leq T$. We assume that

$$|a(t + \tau) - a(t)| \leq \frac{C}{v(t)} \mu(\tau), \quad 0 \leq \tau \leq \tau_0, \quad t, t + \tau \in ]0, T], \quad (2.1)$$

where $v : ]0, T] \to ]0, +\infty]$ is a non-decreasing continuous function such that, for some $\kappa > 0$,

$$v(t/2) \geq \kappa v(t), \quad t \in ]0, T]. \quad (2.2)$$

**Remark 1** Condition (2.2) is satisfied whenever $v$ is concave. Moreover, it is satisfied by $v(t) = t^p$ for every real exponent $p > 0$. On the other hand, it is not satisfied if $v(t)$ tends to 0 too fast as $t \to 0$, e.g. by $v(t) = e^{-1/t}$. 

Now let $0 < \epsilon \leq \tau_0 \leq T$ and define

$$
\tilde{a}_\epsilon(t) := \begin{cases} 
    a(\epsilon) & \text{for } t \leq \epsilon, \\
    a(t) & \text{for } \epsilon \leq t \leq T, \\
    a(T) & \text{for } T \leq t. 
\end{cases}
$$

(2.3)

Let $\rho \in C^\infty(\mathbb{R})$ with supp $\rho \subset [-1, 1]$, $\rho(s) \geq 0$, $\int_{\mathbb{R}} \rho(s) \, ds = 1$, set $\rho_\epsilon(s) := \frac{1}{\epsilon} \rho(\frac{s}{\epsilon})$, and define

$$
a_\epsilon(t) := \int_{-\epsilon}^{\epsilon} \rho_\epsilon(s) \tilde{a}_\epsilon(t - s) \, ds, \quad t \in \mathbb{R}.
$$

(2.4)

We have the following

**Proposition 2** Under the above hypotheses, there exist constants $C'$ and $C'' > 0$ such that, for $0 < \epsilon \leq \tau_0$,

$$
|a_\epsilon(t) - \tilde{a}_\epsilon(t)| \leq C' \min \left\{1, \frac{1}{v(t)} \mu(\epsilon) \right\}, \quad t \in ]0, T].
$$

(2.5)

and

$$
|a'_\epsilon(t)| \leq \frac{C''}{\epsilon} \min \left\{1, \frac{1}{v(t)} \mu(\epsilon) \right\}, \quad t \in ]0, T].
$$

(2.6)

The constants $C'$ and $C''$ depend only on $C$, $\rho$, $\kappa$ and $\|a\|_\infty$.

**Proof** We have

$$
|a_\epsilon(t) - \tilde{a}_\epsilon(t)| = \left| \int_{t-\epsilon}^{t+\epsilon} \rho_\epsilon(t-s)(\tilde{a}_\epsilon(s) - \tilde{a}_\epsilon(t)) \, ds \right|
\leq \int_{t-\epsilon}^{t+\epsilon} \rho_\epsilon(t-s) |\tilde{a}_\epsilon(s) - \tilde{a}_\epsilon(t)| \, ds.
$$

If $t \geq 2\epsilon$, then $t - \epsilon \geq t/2 \geq \epsilon$, so $v(t-\epsilon) \geq v(t/2) \geq \kappa v(t)$. Therefore, we have

$$
|a_\epsilon(t) - \tilde{a}_\epsilon(t)| \leq \int_{t-\epsilon}^{t+\epsilon} \rho_\epsilon(t-s) |\tilde{a}(s) - a(t)| \, ds
\leq \int_{t-\epsilon}^{t+\epsilon} \rho_\epsilon(t-s) \frac{C}{v(t-\epsilon)} \mu(|s-t|) \, ds
\leq \int_{t-\epsilon}^{t+\epsilon} \rho_\epsilon(t-s) \frac{C}{v(t/2)} \mu(|s-t|) \, ds
\leq \int_{t-\epsilon}^{t+\epsilon} \rho_\epsilon(t-s) \frac{C/\kappa}{v(t)} \mu(|s-t|) \, ds
\leq \frac{C/\kappa}{v(t)} \mu(\epsilon).
$$
If $0 < t \leq \epsilon$, then $\tilde{a}_\epsilon(s) = \tilde{a}_\epsilon(t) = a(\epsilon)$ for $s \leq \epsilon$, and therefore,

$$|a_\epsilon(t) - \tilde{a}_\epsilon(t)| \leq \int_{t-\epsilon}^{t+\epsilon} \rho_\epsilon(t-s)|a(s) - a(\epsilon)| \, ds$$

$$\leq \int_{t-\epsilon}^{t+\epsilon} \rho_\epsilon(t-s) \frac{C}{\nu(\epsilon)} \mu(|s-\epsilon|) \, ds$$

$$\leq \int_{\epsilon}^{t+\epsilon} \rho_\epsilon(t-s) \frac{C}{\nu(t)} \mu(\epsilon) \, ds \leq \frac{C}{\nu(\epsilon)} \mu(\epsilon).$$

If $\epsilon \leq t \leq 2\epsilon$, then $0 \leq t - \epsilon \leq \epsilon \leq t$. Therefore, we have

$$|a_\epsilon(t) - \tilde{a}_\epsilon(t)| \leq \int_{t-\epsilon}^{\epsilon} \rho_\epsilon(t-s)|a(\epsilon) - a(t)| \, ds + \int_{\epsilon}^{t+\epsilon} \rho_\epsilon(t-s)|a(s) - a(\epsilon)| \, ds$$

$$\leq \int_{t-\epsilon}^{\epsilon} \rho_\epsilon(t-s) \frac{C}{\nu(\epsilon)} \mu(t-\epsilon) \, ds + \int_{\epsilon}^{t+\epsilon} \rho_\epsilon(t-s) \frac{C}{\nu(t)} \mu(|t-s|) \, ds$$

$$\leq \frac{C}{\nu(\epsilon)} \mu(\epsilon) \leq \frac{C}{\nu(t/2)} \mu(\epsilon) \leq \frac{C}{\nu(t)} \mu(\epsilon).$$

The thesis follows setting $C' := \max\{C, C/\kappa, 2\|a\|_\infty\}$.

In order to estimate $a_\epsilon'$, we observe that

$$|a_\epsilon'(t)| = \left| \int_{t-\epsilon}^{t+\epsilon} \rho_\epsilon'(t-s) \tilde{a}_\epsilon(s) \, ds \right| = \left| \int_{t-\epsilon}^{t+\epsilon} \rho_\epsilon'(t-s) (\tilde{a}_\epsilon(s) - \tilde{a}_\epsilon(t)) \, ds \right|$$

$$\leq \int_{t-\epsilon}^{t+\epsilon} |\rho_\epsilon'(t-s)||\tilde{a}_\epsilon(s) - \tilde{a}_\epsilon(t)| \, ds.$$
ball and whose slope depends on the coefficients $a_{ij}$’s. Therefore, it will be sufficient to show that if $u_0$ and $u_1$ belong to a suitable Gevrey space $\gamma^{(\sigma)}$ and have compact support, then the corresponding solution $u$ is not only in $W^{2,1}([0, T], \mathcal{A}'(\mathbb{R}^n))$, but it belongs to the same Gevrey space in the $x$ variable for all $t \in [0, T]$. The result for initial data which do not have compact support follows by an exhaustion argument. Our main tools in the proof will be the Paley-Wiener theorem and energy estimates.

**Theorem 3** Let $p > 1$ and $0 < \alpha < 1$, and assume that there exists a constant $C > 0$ such that the function $a = a(t, \xi)$ defined by (1.3) satisfies

$$|a(t + \tau, \xi) - a(t, \xi)| \leq \frac{C}{t^p} \tau^\alpha, \quad 0 \leq \tau, \quad t, t + \tau \in [0, T]$$

for all $\xi \in \mathbb{R}^n \setminus \{0\}$. Then the Cauchy problem (1.1), (1.2) is $\gamma^{(\sigma)}$-well-posed for $1 \leq \sigma < \frac{p}{p - \alpha}$.

**Remark 2** For a fixed $p > 1$, passing to the limit as $\alpha \to 1$ we regain the result of [2]. In the same way, for a fixed $\alpha < 1$, passing to the limit as $p \to 1$ we extend to $p = 1$ the result of [1] which was valid only for $p < 1$. The case $\alpha = 1$, $p = 1$ was considered in [2] and will be reconsidered here in a more general context: in this case one has well posedness in $C^\infty$.

**Remark 3** The result in Theorem 3 can be considered sharp in the following sense. Let $p_0 > 1$ and $0 < \alpha_0 < 1$. It is possible to construct a positive function $a \in C^\infty([0, T]) \cap C([0, T])$ such that

$$|a(t + \tau) - a(t)| \leq \frac{C}{t^{p_0}} \tau^{\alpha_0}, \quad 0 \leq \tau, \quad t, t + \tau \in [0, T],$$

and it is possible to construct two functions $u_0, \ u_1 \in \gamma^{(s)}(\mathbb{R})$, for all $s > \frac{p_0}{p_0 - \alpha_0}$ such that the Cauchy problem

$$\begin{cases}
  u_{tt} - a(t)u_{xx} = 0 \\
  u(0, x) = u_0(x), \ u_t(0, x) = u_1(x)
\end{cases}$$

has no solution in $C^1([0, r[; \mathcal{D}'(\mathbb{R}))$, for all $s > \frac{p_0}{p_0 - \alpha_0}$ and for all $r > 0$ (here $\mathcal{D}'(\mathbb{R})$ denotes the set of Gevrey-ultradistributions of index $s$). The construction of such a counterexample is exactly the same as that contained in Theorem 5 in [2].

**Remark 4** A result analogous to that of Theorem 3 can be proved if the singularity of the $a_{ij}$’s is located at $t = T$, with only minor obvious changes in the proof. As a consequence, the result is still valid if the coefficients have a finite number of singularities, where the loss of regularity is controlled as in (3.1).

**Proof of Theorem 3** We take the Fourier transform of $u$ with respect to $x$, and we denote it by $\hat{u}$. Equation (1.1) then transforms to

$$\hat{u}_{tt}(t, \xi) + a(t, \xi)|\xi|^2 \hat{u}(t, \xi) = 0.$$  

(3.2)
Let $\epsilon$ be a positive parameter and for each $\epsilon$ let $a_\epsilon : [0, T] \times (\mathbb{R}^n \setminus \{0\}) \to \mathbb{R}$ be defined according to (2.3)–(2.4).

We define the approximate energy of $\hat{u}$ by

$$E_\epsilon(t, \xi) := a_\epsilon(t, \xi)|\xi|^2|\hat{u}(t, \xi)|^2 + |\hat{u}_t(t, \xi)|^2, \quad (t, \xi) \in [0, T] \times (\mathbb{R}^n \setminus \{0\}). \quad (3.3)$$

Differentiating $E_\epsilon$ with respect to $t$ and using (3.2) we get

$$E_\epsilon'(t, \xi) = a_\epsilon'(t, \xi)|\xi|^2|\hat{u}(t, \xi)|^2 + 2a_\epsilon(t, \xi)|\xi|^2\text{Re}(\hat{u}_t(t, \xi)\bar{\hat{u}}(t, \xi))$$

$$+ 2\text{Re}(\hat{u}_{tt}(t, \xi)\bar{\hat{u}}(t, \xi)) \leq \left( \frac{|a_\epsilon'(t, \xi)|}{a_\epsilon(t, \xi)} + \frac{|a_\epsilon(t, \xi) - a(t, \xi)|}{a_\epsilon(t, \xi)^{1/2}} |\xi| \right) E_\epsilon(t, \xi).$$

By Gronwall’s lemma we obtain

$$E_\epsilon(t, \xi) \leq E_\epsilon(0, \xi) \exp \left( \int_0^T \frac{|a_\epsilon'(t, \xi)|}{a_\epsilon(t, \xi)} dt + |\xi| \int_0^T \frac{|a_\epsilon(t, \xi) - a(t, \xi)|}{a_\epsilon(t, \xi)^{1/2}} dt \right) \quad (3.4)$$

for all $t \in [0, T]$ and for all $\xi \in \mathbb{R}^n$, $|\xi| \geq 1$.

By Proposition 2 with $\mu(\tau) = \tau^\alpha$ and $\nu(t) = t^p$ we have

$$\int_0^T \frac{|a_\epsilon'(t, \xi)|}{a_\epsilon(t, \xi)} dt + |\xi| \int_0^T \frac{|a_\epsilon(t, \xi) - a(t, \xi)|}{a_\epsilon(t, \xi)^{1/2}} dt$$

$$\leq \int_0^T \frac{|a_\epsilon'(t, \xi)|}{a_\epsilon(t, \xi)} dt + |\xi| \int_0^T \left( \frac{|a(t, \xi) - \tilde{a}_\epsilon(t, \xi)|}{a_\epsilon(t, \xi)^{1/2}} + \frac{|a_\epsilon(t, \xi) - \tilde{a}_\epsilon(t, \xi)|}{a_\epsilon(t, \xi)^{1/2}} \right) dt$$

$$\leq \int_0^{\epsilon^\alpha/p} \frac{C''}{\lambda_0^2} dt + \int_0^T \frac{C''}{\lambda_0^{1/2} t^{-p} \epsilon^\alpha} dt + \frac{2\Lambda_0}{\lambda_0^{1/2}} |\xi|\epsilon$$

$$+ |\xi| \left( \int_0^{\epsilon^\alpha/p} \frac{C'}{\lambda_0^{1/2}} dt + \int_0^T \frac{C'}{\lambda_0^{1/2} t^{-p} \epsilon^\alpha} dt \right)$$

$$\leq M|\xi|\epsilon + M \left( |\xi| + \frac{1}{\epsilon} \right) \left( \epsilon^\alpha/p + (\epsilon^\alpha/p)^{1-p} \epsilon^\alpha \right)$$

$$= M|\xi|\epsilon + 2M \left( |\xi| + \frac{1}{\epsilon} \right) \epsilon^\alpha/p,$$

where $M$ depends on $C', C'', \lambda_0, \Lambda_0, \alpha$ and $p$. Choosing $\epsilon = |\xi|^{-1}$ we get

$$\left[ \int_0^T \frac{|a_\epsilon'(t, \xi)|}{a_\epsilon(t, \xi)} dt + |\xi| \int_0^T \frac{|a_\epsilon(t, \xi) - a(t, \xi)|}{a_\epsilon(t, \xi)^{1/2}} dt \right]_{\epsilon = |\xi|^{-1}} \leq M + 4M|\xi|^{p-\alpha}. \quad (3.5)$$

Putting together (3.4) and (3.5) we get

$$E_{1/|\xi|}(t, \xi) \leq e^M e^{4M|\xi|^{p-\alpha}} E_{1/|\xi|}(0, \xi) \quad (3.6)$$
and, finally,
\[ |\dot{u}_t(t, \xi)|^2 + |\xi|^2|\dot{u}(t, \xi)|^2 \leq \frac{e^{M_0}}{\lambda_0} e^{M_0|\xi|} \left( |\dot{u}_t(0, \xi)|^2 + |\xi|^2|\dot{u}(0, \xi)|^2 \right). \] (3.7)

Now if \( u_0, u_1 \in \gamma^{(\sigma)} \cap C_0^\infty \), the Paley-Wiener theorem ensures that there exist \( K, \delta > 0 \) such that
\[ |\hat{u}(0, \xi)|^2 + |\hat{u}_t(0, \xi)|^2 \leq K \exp(-\delta|\xi|^{1/\sigma}) \] (3.8)
for all \( \xi \in \mathbb{R}^n, |\xi| \geq 1 \). It follows from (3.7) that if \( \sigma < p / (p - \alpha) \), then there exist \( K', \delta' > 0 \) such that
\[ |\hat{u}(t, \xi)|^2 + |\hat{u}_t(t, \xi)|^2 \leq K' \exp(-\delta'|\xi|^{1/\sigma}) \] (3.9)
for all \( t \in [0, T] \) and for all \( \xi \in \mathbb{R}^n, |\xi| \geq 1 \) and, therefore, \( u \in W^{2,1}([0, T], \gamma^{(\sigma)}) \).

The proof is complete. \( \square \)

4 Well posedness in \( C^\infty \)

Let \( \psi : [1, +\infty[ \to [0, +\infty[ \) be a strictly increasing continuous function, such that \( \psi' \) is non-increasing and \( e^r \psi'(r) \) is non-decreasing. Moreover, we assume that

1. \( \lim_{r \to +\infty} \psi(r) = \chi, 0 < \chi \leq +\infty; \)
2. \( \lim_{r \to +\infty} \psi'(r) = \eta, 0 \leq \eta < +\infty; \)

We set
\[ \nu(t) := \begin{cases} \frac{t}{\psi(|\log t|)} & \text{for } 0 < t \leq e^{-1}, \\ \frac{e^{-1}}{\psi(1)} & \text{for } e^{-1} \leq t. \end{cases} \] (4.1)

A direct computation shows that \( \nu \) is a non-decreasing continuous function and that \( \nu(t/2) \geq (1/2)\nu(t) \) for \( t \in [0, T] \). We define
\[ \mu(\tau) := \frac{\tau |\log \tau|}{\psi(|\log \tau|)} \] (4.2)

and we assume that \( \mu \) is strictly increasing and concave in \([0, \tau_0] \) for a suitable \( \tau_0 > 0 \), so it is a modulus of continuity.

**Theorem 4** Let \( \nu = \nu(t) \) and \( \mu = \mu(\tau) \) be as above, and assume that there exists a constant \( C > 0 \) such that the function \( a = a(t, \xi) \) defined by (1.3) satisfies
\[ |a(t + \tau, \xi) - a(t, \xi)| \leq \frac{C}{\nu(t)} \mu(\tau), \quad 0 \leq \tau \leq \tau_0, \quad t, t + \tau \in [0, T], \] (4.3)
for all $\xi \in \mathbb{R}^n \setminus \{0\}$. Then the Cauchy problem (1.1), (1.2) is well-posed in $C^\infty$.

**Remark 5** Examples of functions satisfying all the above properties are $\psi(r) = 1 - e^{-\alpha r}$ with $0 < \alpha \leq 1$, $\psi(r) = 1 + \log r$ and $\psi(r) = r^\beta$ with $0 < \beta \leq 1$. In particular, we have:

- if $\psi(r) = r$ we have $\eta = 1$ and $\chi = +\infty$ and we get $\mu(\tau) = \tau$ and $\nu(t) = t$, that is the situation considered in [2];
- if $\psi(r) = 1 - e^{-\alpha r}$ we have $\eta = 0$ and $\chi = 1$ and we get $\mu(\tau) = \tau |\log \tau|/(1 - \tau^\alpha)$, which is equivalent to $\mu(\tau) = \tau |\log \tau|$, and $\nu(t) = \alpha t^{1-\alpha}$, that is a situation covered by the result of [1], since $\nu(t)^{-1}$ is integrable;
- if $\psi(r) = 1 + \log r$ or $\psi(r) = r^\beta$ with $0 < \beta < 1$, we have $\eta = 0$ and $\chi = +\infty$, and we get $\mu(\tau) = \tau |\log \tau|/(1 + \log |\log \tau|)$ or $\mu(\tau) = \tau |\log \tau|^{1-\beta}$. In both cases $\mu$-continuity is weaker than Lipschitz continuity. Moreover we have $\nu(t) = t |\log t|$ or $\nu(t) = t |\log t|^{1-\beta}$, so in both cases $\nu(t)^{-1}$ is not integrable.

The case in which $\mu(\tau) = \tau |\log \tau|$ and $\nu(t)^{-1}$ is not integrable is not covered by Theorem 4, and we were not able to find a counterexample to $C^\infty$ well posedness either, so the question remains open. On the other hand, when $\mu(\tau) = \tau |\log \tau|$ and $\nu(t) = t$ by Theorem 3 we get automatically $\gamma^{(\infty)}$ well posedness.

**Remark 6** A result analogous to that of Theorem 4 can be proved if the singularity of the $a_{ij}$’s is located at $t = T$, with only minor obvious changes in the proof. As a consequence, the result is still valid if the coefficients have a finite number of singularities, where the loss of regularity is controlled as in (4.3).

**Proof of Theorem 4** Like in the proof of Theorem 3, we take the Fourier transform $\hat{u}$ of $u$. Equation (1.1) then transforms to

$$\hat{u}_{tt}(t, \xi) + a(t, \xi)|\xi|^2\hat{u}(t, \xi) = 0. \quad (4.4)$$

For $0 < \epsilon \leq \tau_1 := \min\{\tau_0, T, e^{-1}\}$ we define $a_\epsilon : [0, T] \times (\mathbb{R}^n \setminus \{0\}) \to \mathbb{R}$ according to (2.3)–(2.4). Again, we define an approximate energy of $\hat{u}$ by

$$E_\epsilon(t, \xi) := a_\epsilon(t, \xi)|\xi|^2|\hat{u}(t, \xi)|^2 + |\hat{u}_t(t, \xi)|^2, \quad (t, \xi) \in [0, T] \times (\mathbb{R}^n \setminus \{0\}). \quad (4.5)$$

Differentiating $E_\epsilon$ with respect to $t$ and using (4.4) we get

$$E'_\epsilon(t, \xi) = a'_\epsilon(t, \xi)|\xi|^2|\hat{u}(t, \xi)|^2 + 2a_\epsilon(t, \xi)|\xi|^2\text{Re}(\hat{u}_t(t, \xi)\tilde{\hat{u}}(t, \xi)) + 2\text{Re}(\hat{u}_{tt}(t, \xi)\tilde{\hat{u}}(t, \xi)) \leq \left(\frac{|a'_\epsilon(t, \xi)|}{a_\epsilon(t, \xi)} + \frac{|a_\epsilon(t, \xi) - a(t, \xi)|}{a_\epsilon(t, \xi)^{1/2}}|\xi|\right) E_\epsilon(t, \xi).$$

By Gronwall’s lemma we obtain

$$E_\epsilon(t, \xi) \leq E_\epsilon(0, \xi) \exp\left(\int_0^T \frac{|a'_\epsilon(t, \xi)|}{a_\epsilon(t, \xi)} \, dt + |\xi| \int_0^T \frac{|a_\epsilon(t, \xi) - a(t, \xi)|}{a_\epsilon(t, \xi)^{1/2}} \, dt\right) \quad (4.6)$$
for all $t \in [0, T]$ and for all $\xi \in \mathbb{R}^n$, $|\xi| \geq 1$. By Proposition 2 with $\mu(\tau)$ and $v(t)$ given by (4.2) and (4.1), we have

$$\int_0^T \frac{a_e'(t, \xi)}{a_e(t, \xi)} \, dt = \int_0^\epsilon \frac{a_e'(t, \xi)}{a_e(t, \xi)} \, dt + \int_\epsilon^{\epsilon-1} \frac{a_e'(t, \xi)}{a_e(t, \xi)} \, dt + \int_{\epsilon-1}^T \frac{a_e'(t, \xi)}{a_e(t, \xi)} \, dt \leq \frac{C''}{\lambda_0} \left( \epsilon + \int_\epsilon^{\epsilon-1} \frac{\psi'(|\log t|)}{t} \frac{\epsilon |\log \epsilon|}{\psi(|\log \epsilon|)} \, dt + \int_{\epsilon-1}^T \psi'(1) \frac{\epsilon |\log \epsilon|}{\psi(|\log \epsilon|)} \, dt \right).$$

Since

$$\frac{\psi'(|\log t|)}{t} = -\frac{d}{dt} \psi(|\log t|)$$

and $\psi(|\log \epsilon|) \geq \psi(|\log \tau_1|)$, we obtain

$$\int_0^T \frac{a_e'(t, \xi)}{a_e(t, \xi)} \, dt \leq M'' (1 + |\log \epsilon|). \quad (4.7)$$

On the other hand

$$\int_0^T \frac{a_e(t, \xi) - a(t, \xi)}{a_e(t, \xi)^{1/2}} \, dt = \int_0^T \left( \frac{|a(t, \xi) - \tilde{a}_e(t, \xi)|}{a_e(t, \xi)^{1/2}} + \frac{|a_e(t, \xi) - \tilde{a}_e(t, \xi)|}{a_e(t, \xi)^{1/2}} \right) \, dt \leq \frac{2\Lambda_0}{\lambda_0^{1/2}} \epsilon + \frac{C'}{\lambda_0^{1/2}} \left( \epsilon + \int_\epsilon^{\epsilon-1} \frac{\psi'(|\log t|)}{t} \frac{\epsilon |\log \epsilon|}{\psi(|\log \epsilon|)} \, dt \right).$$

Arguing as above, we get

$$\int_0^T \frac{|a_e(t, \xi) - a(t, \xi)|}{a_e(t, \xi)^{1/2}} \, dt \leq M' \epsilon (1 + |\log \epsilon|). \quad (4.8)$$

Choosing $\epsilon = |\xi|^{-1}$ we get

$$\left[ \int_0^T \frac{a_e'(t, \xi)}{a_e(t, \xi)} \, dt + |\xi| \int_0^T \frac{|a_e(t, \xi) - a(t, \xi)|}{a_e(t, \xi)^{1/2}} \, dt \right]_{\epsilon=|\xi|^{-1}} \leq M((1 + \log |\xi|)) \quad (4.9)$$

for $|\xi| \geq \tau_1^{-1}$.

Putting together (4.6) and (4.9) we get

$$E_{1/|\xi|}(t, \xi) \leq e^M |\xi|^M E_{1/|\xi|}(0, \xi) \quad (4.10)$$

and, finally,

$$|\hat{u}_t(t, \xi)|^2 + |\xi|^2 |\hat{u}(t, \xi)|^2 \leq \frac{e^M \Lambda_0}{\lambda_0} |\xi|^M \left( |\hat{u}_t(0, \xi)|^2 + |\xi|^2 |\hat{u}(0, \xi)|^2 \right). \quad (4.11)$$
Now if \( u_0, u_1 \in C^\infty_0 \), the Paley-Wiener theorem ensures that for all \( \zeta > 0 \) there exists \( K_\zeta > 0 \) such that

\[
|\hat{u}(0, \xi)|^2 + |\hat{u}_t(0, \xi)|^2 \leq K_\zeta |\xi|^{-\zeta}
\]

for all \( \xi \in \mathbb{R}^n, |\xi| \geq \tau_1^{-1} \). It follows from (4.11) that for all \( \theta > 0 \) there exist \( K'_\theta > 0 \) such that

\[
|\hat{u}(t, \xi)|^2 + |\hat{u}_t(t, \xi)|^2 \leq K'_\theta |\xi|^{-\theta}
\]

for all \( t \in [0, T] \) and for all \( \xi \in \mathbb{R}^n, |\xi| \geq \tau_1^{-1} \), and therefore, \( u \in W^{2,1}([0, T], C^\infty_0) \).

The proof is complete. \( \square \)

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