ERGODICITY OF THE 2D NAVIER-STOKES EQUATIONS WITH DEGENERATE MULTIPLICATIVE NOISE

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ABSTRACT. Consider the two-dimensional, incompressible Navier-Stokes equations on the torus \( T^2 = [-\pi, \pi]^2 \) driven by a degenerate multiplicative noise

\[ dw_t = \nu \Delta w_t dt + B(Kw_t, w_t) dt + Q(w_t) dB_t. \]  

(0.1)

We use the Malliavin calculus to prove that the semigroup \( \{P_t\}_{t \geq 0} \) generated by the solutions to (0.1) is asymptotically strong Feller. Moreover, we use the coupling method to prove the semigroup \( \{P_t\}_{t \geq 0} \) is exponentially ergodic in some sense. Our result is stronger than that in [13].

Keywords: stochastic Navier-Stokes equation; asymptotically strong Feller property; ergodicity.

MSC 2000: 60H15; 60H07

1. INTRODUCTION, PRELIMINARIES AND PROPERTIES FOR SOLUTION

1.1. Introduction and Main Results. This work is motivated by paper [6], in which, Martin Hairer and Jonathan C. Mattingly considered the following two-dimensional, incompressible Navier-Stokes equations on the torus \( T^2 = [-\pi, \pi]^2 \) driven by an additive degenerate noise

\[ dw_t = \nu \Delta w_t dt + B(Kw_t, w_t) dt + QdB(t). \]  

(1.1)

With the asymptotically strong Feller property that they discovered, they proved the uniqueness and existence of the invariant measure for the semigroup generated by the solution to (1.1). Also, in [7] they proved the solution has a spectral gap in Wasserstein distance. In this article, we consider the same questions for the following two-dimensional, incompressible stochastic Navier-Stokes equations with degenerate multiplicative noise

\[ dw_t = \nu \Delta w_t dt + B(Kw_t, w_t) dt + Q(w_t) dB(t). \]  

(1.2)
We assume \( B_t \) is a cylindrical Wiener process of a Hilbert space \( U \), that is there exist an orthonormal basis \((\beta_n)\) of \( U \), and a family \((B_n)\) of independent brownian motions such that

\[
B_t = \sum_{n=1}^{\infty} B_n(t)\beta_n.
\]

Denote \( H = L_0^2 \), the space of real-valued square-integrable functions on the torus with vanishing mean and \( \| \cdot \| \) denotes the \( L^2 \)-norm on the Hilbert space \( H \). We make the following Hypotheses in this article on \( Q \):

**H1:** The function \( Q : H \to \mathcal{L}_2(U; H) \) is bounded. Assume

\[
B_0 = \sup_{u \in H} \|Q(u)\|_{\mathcal{L}_2(U; H)} < \infty,
\]

**H2:** The function \( Q : H \to \mathcal{L}_2(U; H) \) is Lipschitz and denote \( L_Q \) be the Lipschitz constant of \( Q \).

**H3:** There exist \( N \in \mathbb{N}^* \) and a bounded measurable map \( g : H \to L(H; U) \) such that for any \( u \in H \)

\[
Q(u)g(u) = P_N,
\]

here the meaning of \( P_N \) can see subsection 1.2. In the following, we set

\[
\|Q(u)\| = \|Q(u)\|_{\mathcal{L}_2(U; H)}.
\]

Denote by \( a \vee b := \max\{a, b\}, a \wedge b := \min\{a, b\} \). Let \( C(d) \) be some positive constant depending on \( d, B_0, L_Q \) and \( \nu \), and let \( C \) be some positive constant depending on \( B_0, L_Q \) and \( \nu \). The constant \( C \) or \( C(d) \) may changes from line to line.

Define \( \mathcal{V}_{\eta}(x) = e^{\|\eta\|_2^2}, \forall \eta > 0 \), and for any \( r \in (0, 1] \) and \( \eta > 0 \), we introduce a family of distances \( \rho^\eta_r \) on \( H \)

\[
\rho^\eta_r(x, y) := \inf_{\gamma} \int_0^1 V_{\eta}(\gamma(t)) \|\gamma'(t)\|dt,
\]

where the infimum runs over all paths \( \gamma \) such that \( \gamma(0) = x \) and \( \gamma(1) = y \). For simplify of writing, we denote \( \rho_r(x, y) := \rho^\eta_r(x, y) \). Let \( P_t \) be the transition semigroups of \( w_t \), that is

\[
P_t f(w_0) := \mathbb{E}_{w_0} f(w_t), \ f \in \mathcal{B}(H).
\]

Our main results in this article are the following three Theorems.

**Theorem 1.1** (Weak Form of Irreducibility). There exists some \( N_0 \) such that if \( H1,H2 \) hold and \( H3 \) holds \( N \geq N_0 \), then the solutions to (1.2) have the weak form of irreducibility. That is given any \( \eta \in (0, \frac{1}{8B_0}) \), \( C > 0 \), \( r \in (0, 1) \) and \( \delta > 0 \), there exists a \( T_0 \) such that for any \( T \geq T_0 \) there exists an \( a > 0 \) such that

\[
\inf_{\|\gamma\|} \sup_{\pi \in \Gamma(P^\eta_r, P^\eta_s, \delta)} \mathcal{P} \{(x', y') \in H \times H, \rho_r(x', y') \leq \delta \} > a.
\]
Theorem 1.2 (Gradient Estimate). There exists $\eta_0 > 0$, such that for any $\eta < \eta_0$, there exist some constant $N_0 := N_0(\eta) > 0$, such that if Hypotheses $H1,H2$ hold and $H3$ holds with $N \geq N_0$, then for any $f$

$$|\nabla P_t f(w_0)| \leq C(N)\exp\left(\left(\frac{4\eta}{\nu} + \eta\right)\|w_0\|^2\right)\sqrt{P_t|\varphi|^2(w_0)} + Ce^{-\nu \|w_0\|^2}e^{-\nu \varphi^2} \sqrt{P_t\|D\varphi\|^2(x)}.$$ 

We in introduce the following family of norms

$$\|\phi\|_{V^r_\eta} = \sup_{\varphi \in H^r} \frac{|\phi(x)| + \|D\phi(x)\|_{\nu(x)}}{V^r_\eta(x)}$$

When we take $r = 1$, we simply write $\|\phi\|_{V^1_\eta}$.

Theorem 1.3 (Exponential Mixing). For any $\eta \in (0, \frac{\nu}{16b_0})$, there exists some $N_0 := N_0(\eta)$ such that if $H1,H2$ hold and $H3$ holds $N \geq N_0$, then there exists a unique invariant probability measure $\mu^*$ for $P_t$ and positive constants $\theta$ and $C$ such that for any $\phi \in B$,

$$\|P_t\phi - \mu^*\phi\|_{V^1_\eta} \leq Ce^{-\theta t}\|\phi - \mu^*\phi\|_{V^1_\eta}, \quad \forall t > 0. \quad (1.3)$$

Remark 1.1. Odasso [13] Theorem 3.4] used coupling method to establish exponential mixing of the solutions of stochastic Navier-Stokes equation under hypotheses $H1,H2$ and $H3$, but he didn’t give the proof of asymptotically strong Feller property and the weak form of irreducibility. The results of theorem [13] is stronger than that in [13].

1.2. Preliminaries. Recall that the Navier-Stokes equations are given by

$$\partial_t u + (u \cdot \nabla) u = \nu \Delta u - \nabla p + \xi, \quad \text{div } u = 0.$$ 

where $\xi(x,t)$ is the external force field acting on the fluid.

The vorticity $w$ is defined by $w = \nabla \wedge u = \partial_2 u_1 - \partial_1 u_2$. $B(u, w) = -(u \cdot \nabla) w$. For $k = (k_1, k_2) \in \mathbb{Z}^2 \setminus \{(0,0)\}$, $k^\perp = (k_2, -k_1)$, $w_k = \langle w, (2\pi)^{-1} e^{ik \cdot x} \rangle_H$. The operator $K$ is defined in Fourier space by

$$(Kw)_k = \langle Kw, (2\pi)^{-1} e^{ik \cdot x} \rangle_H = -iw_k k^\perp / ||k||.$$ 

We write $\mathbb{Z}^2 \setminus \{(0,0)\} = \mathbb{Z}^2_+ \cup \mathbb{Z}^2_-$, where

$$\mathbb{Z}^2_+ = \{(k_1, k_2) \in \mathbb{Z}^2 : k_2 > 0\} \cup \{(k_1, 0) \in \mathbb{Z}^2 : k_1 > 0\},$$ 

$$\mathbb{Z}^2_- = \{(k_1, k_2) \in \mathbb{Z}^2 : -(k_1, k_2) \in \mathbb{Z}^2_+\},$$

and set, for $k \in \mathbb{Z}^2 \setminus \{(0,0)\}$,

$$e_k(x) = \begin{cases} 
\sin(k \cdot x) & k \in \mathbb{Z}^2_+, \\
\cos(k \cdot x) & k \in \mathbb{Z}^2_-.
\end{cases}$$

Then, $\{e_k, k \in \mathbb{Z}^2 \setminus \{(0,0)\}\}$ an orthonormal basis of $H$. For $\alpha \in \mathbb{R}$ and a smooth function $w$ on $[-\pi, \pi]^2$ with mean 0, denote $\|w\|_{\alpha}$ by

$$\|w\|_{\alpha}^2 = \sum_{k \in \mathbb{Z}^2 \setminus \{(0,0)\}} |k|^{2\alpha} |w_k|^2, \quad w_k = \langle w, (2\pi)^{-1} e^{ik \cdot x} \rangle_H,$$
and \( \|w\| := \|w\|_0 \). Denote \( H^{2\alpha} \) be the closure of smooth function with respect to the norm \( \|\cdot\|_{\alpha} \) in \( H \). We denote \( P_N \) and \( Q_N \) the orthogonal projection in \( H \) onto the space \( \text{Span}\{e_k, 1 \leq |k| \leq N\} \) and onto its complementary. Denote \( H_N = P_N H \).

Actually, (1.2) defines a stochastic flow on \( H \). That means a family of continuous map \( \Phi_t : W \times H \to H \) such that \( w_t = \Phi_t(B, w_0) \) is the solution to (1.2) with initial condition \( w_0 \) and noise \( B \).

Given a \( v \in L^2_{\text{loc}}(\mathbb{R}^+, U) \), the Malliavin derivative of the \( H \)-valued random variable \( w_t \) in the direction \( v \), denoted by \( \mathcal{D}^v w_t \), is defined by

\[
\mathcal{D}^v w_t = \lim_{\varepsilon \to 0} \frac{\Phi_t(B + \varepsilon V, w_0) - \Phi_t(B, w_0)}{\varepsilon},
\]

where the limit holds almost surely with respect to Wiener measure and \( V(t) = \int_0^t v(s) ds \). Let \( J_{s,t} \) be the derivative flow between times \( s \) and \( t \), i.e. for every \( \xi \in H \), \( J_{s,t} \xi \) is the solution of

\[
\begin{aligned}
&dJ_{s,t} \xi = v \Delta J_{s,t} \xi dt + \hat{B}(w_t, J_{s,t} \xi) dt + DQ(w_t) J_{s,t} \xi dB_t, \\
&J_{s,s} \xi = \xi,
\end{aligned}
\]

where \( \hat{B}(w, u) = B(Kw, u) + B(Ku, w) \). \( J_{s,t} \xi \) is the effect on \( w_t \) of an infinitesimal perturbation of the initial condition in the direction \( \xi \). \( DQ \) is Fréchet derivation of \( Q \). Observe that \( \mathcal{D}^v w_t = A_{0,t} v \), where \( A_{s,t} : L^2([s, t], U) \to H \)

\[
A_{s,t} v = \int_s^t J_{r,t} Q(w_r) v(r) dr.
\]

If \( B(u, v) = (u \cdot \nabla) v, S = \{ (s_1, s_2, s_3) \in \mathbb{R}_+^3 : \sum s_i \geq 1, s \neq (1, 0, 0), (0, 1, 0), (0, 0, 1) \} \). Then the following relations are useful. Its proof can be seen in \([3, 6]\).

\[
\langle B(u, v), w \rangle = -\langle B(u, w), v \rangle, \quad \text{if } \nabla \cdot u = 0,
\]

\[
|\langle B(u, v), w \rangle| \leq C \|u\|_{s_1} \|v\|_{1+s_2} \|w\|_{s_3}, \quad (s_1, s_2, s_3) \in S,
\]

\[
\|Kw\|_{s_0} = \|w\|_{s_0-c},
\]

\[
\|w\|_{s_0}^2 \leq \|w\|_{s_1} \|w\|_{s_2}.
\]

1.3. Properties For Solution. In this subsection, we will give some Lemmas and Propositions which will be used in section 2 and section 3.

**Lemma 1.1.** (\([10]\), Lemma A.1) Let \( M(s) \) be a continuous martingale with quadratic variation \( [M, M](s) \) such that \( \mathbb{E}[M, M] < \infty \). Define the semi-martingale \( N(s) = -\frac{\alpha}{2} [M, M](s) + M(s) \) for any \( \alpha > 0 \). If \( \gamma \geq 0 \), then for any \( \beta > 0 \) and \( T > \frac{1}{\beta} \)

\[
\mathbb{P}\left\{ \sup_{t \in [T-\frac{1}{\beta}, T]} \int_0^t e^{-\gamma(t-s)} dN(s) > \frac{e^{-\gamma}}{\alpha} K \right\} < e^{-K}.
\]
Specially,
\[ \mathbb{P} \left\{ \sup_{t>0} N(t) > \frac{1}{\alpha} K \right\} < e^{-K}. \]

**Lemma 1.2.** Assume **H1.** For every \( \eta \leq \frac{\nu}{2B_0} \), there exists a constant \( C = C(\nu, B_0) \) such that
\[ \mathbb{E} \exp \left\{ \sup_{t>0} (\eta \|w_t\|^2) \right\} \leq C e^{\eta \|w_0\|^2}, \]
and
\[ \mathbb{E} \left[ e^{\eta \|w_t\|^2} \right] \leq C e^{\eta \|w_0\|^2}. \]

**Proof.** By Itô’s formula,
\[ d\eta \|w_t\|^2 + 2\nu \|w_t\|^2 dt = 2\eta \langle w_t, Q(w_t) dB_t \rangle + \eta \|Q(w_t)\|^2 dt. \tag{1.10} \]
Using the fact that \( \|w_t\| \leq \|w_0\| \) and \( \|Q(w_t)\|^2 \leq B_0 \),
\[ d\eta \|w_t\|^2 + \nu \eta \|w_t\|^2 dt \leq 2\eta \langle w_t, Q(w_t) dB_t \rangle + \eta B_0 dt - \nu \|w_t\|^2 dt, \]
that is,
\[ \eta d(\|w_t\|^2 e^{\nu t}) \leq 2\eta e^{\nu t} \langle w_t, Q(w_t) dB_t \rangle + \eta B_0 e^{\nu t} dt - \nu \eta \|w_t\|^2 dt. \]
So,
\[ \eta \|w_t\|^2 - \eta e^{-\nu t} \|w_0\|^2 - \frac{\eta B_0}{\nu} \leq 2\eta \int_0^t e^{-\nu(t-s)} \langle w_s, Q(w_s) dB_s \rangle - \eta \nu \int_0^t e^{-\nu(t-s)} \|w_s\|^2 ds. \]
By Lemma 1.1, when \( \eta \leq \frac{\nu}{2B_0} \), one arrives at
\[ \mathbb{E} \exp \left\{ \sup_{t>0} \left( \eta \|w_t\|^2 - \eta e^{-\nu t} \|w_0\|^2 - \frac{\eta B_0}{\nu} \right) \right\} \leq 2, \tag{1.11} \]
here we use the fact that if a random variable \( X \) satisfies \( \mathbb{P}(X \geq C) \leq \frac{X}{C^2} \) for all \( C > 0 \), then \( \mathbb{E}X \leq 2 \). Then this lemma follows by (1.11). \( \square \)

**Lemma 1.3.** Assume **H1.** For every \( \eta \leq \frac{\nu}{2B_0} \), there exists an absolute constant \( C \) such that
\[ \mathbb{E} \exp \left( \eta \sup_{t>0} \left( \|w_t\|^2 + \nu \int_0^t \|w_r\|^2 dr - B_0 t \right) \right) \leq C \exp (\eta \|w_0\|^2). \]

**Proof.** Combining (1.10) with \( \|w_t\| \leq \|w_0\| \),
\[ \eta \|w_t\|^2 + \eta \nu \int_0^t \|w_r\|^2 dr - \eta B_0 t \leq 2\eta \int_0^t \langle w_r, Q(w_r) dB_r \rangle - \eta \nu \int_0^t \|w_r\|^2 dr. \]
Therefore
\[ \eta \|w_t\|^2 + \eta \nu \int_0^t \|w_r\|^2 dr - \eta B_0 t - \eta \|w_0\|^2 \leq 2\eta \int_0^t \langle w_r, Q(w_r) dB_r \rangle - \eta \nu \int_0^t \|w_r\|^2 dr. \]
Due to Lemma 1 when $\eta \leq \frac{\nu}{2B_0}$, for some absolutely constant $C$,

$$\Exp \exp \left( \eta \sup_{t \geq 0} (\|w_t\|^2 + \nu \int_0^t \|w_r\|^2 dr - B_0 t - \|w_0\|^2) \right) \leq C,$$

from which this lemma follows. \qed

2. Proof of Weak Form of Irreducibility

Let

$$F(w) = \nu \Delta w + B(Kw, w).$$

and $w = w(t, B, w_0^1)$ be the solution to following equation

$$\begin{cases}
    dw_t = \nu \Delta w_t dt + B(Kw_t, w_t)dt + Q(w_t)dB(t), \\
    w(0) = w_0^1.
\end{cases} \tag{2.1}$$

Let $\tilde{w} = \tilde{w}(t, B, \tilde{w}_0^2)$ be the solution to the following equation,

$$\begin{cases}
    d\tilde{w} = F(\tilde{w})dt + KP_\nu(\tilde{w} - w(t, B, w_0^1))dt + Q(\tilde{w})dB, \\
    \tilde{w}(0) = \tilde{w}_0^2.
\end{cases} \tag{2.2}$$

Therefore

$$\tilde{w}(t, B, \tilde{w}_0^2) = w(t, B + \int_0^t h_s ds, w_0^1),$$

here $h : H \times H \to U$ is given by

$$h_s := h(s, B) := -Kg(\tilde{w}_s)P_\nu(\tilde{w}_s - w(s, B, w_0^1)).$$

Denote

$$\rho'_r(x, y) = \int_0^1 e^{\nu \|x - y\|^2 (1 - \tau)} d\tau.$$ 

Lemma 2.1. There exists $C_1$ and $\kappa > 0$, such that

$$\Exp (\|w(t, B, w_0^1)\|^2) \leq e^{-\kappa t} |w_0|^2 + C_1$$

Lemma 2.2. There exists $C > 0$ such that for any $w_0^1, w_0^2$ in $H$ satisfying

$$\|w_0^1\|^2 + \|w_0^2\|^2 \leq 2C_1 \tag{2.3}$$

and there exists $\gamma_1, \gamma_2 > 0$, such that for any $t \geq 0$, we have

$$\Exp (\rho'_r(w(t, B_2, w_0^2), w(t, B_1, w_0^1)) \geq C e^{-\gamma_1 t},$$

$$\tilde{w}(\cdot, B_1, w_0^2) = w(\cdot, B_2, w_0^2) \text{ on } [0, t] \leq C e^{-\gamma_2 t}.$$
Proof. Denote \( x = \tilde{w}(t, B_1, w_0^2) \) and \( y = w(t, B_1, w_0^1) \)

\[
\mathbb{P}\left( \rho'_t(w(t, B_2, w_0^2), w(t, B_1, w_0^1)) \geq Ce^{-\gamma t} \right)
\leq \mathbb{P}\left( \rho'_t(\tilde{w}(t, B_1, w_0^2), w(t, B_1, w_0^1)) \geq Ce^{-\gamma t} \right)
\leq \mathbb{P}\left( e^{2r\|x\|^2+2r\|y\|^2} \|x - y\| \geq Ce^{-\gamma t} \right)
\leq \frac{1}{C}e^{\gamma t}\mathbb{E}\left[ \|x - y\|^2 \right]^{1/2} \mathbb{E}\left[ e^{2r\|x\|^2+4r\|y\|^2} \right]^{1/2},
\]

and therefore this Lemma follows by Lemma 1.3 and Lemma A.1. \( \square \)

**Lemma 2.3.** There exists \( p_1 > 0 \) such that for any \( w_0^1, w_0^2 \in H \) satisfying (2.3), we have

\[
\mathbb{P}\left( \int_0^\infty |h(t, B)|^2 dt \leq C \right) \geq p_1.
\]

Define

\[
\tau(B) = \inf \left\{ t > 0, \int_0^t |h(t, B)|^2 dt > 2C \right\}.
\]

Apply [13, corollary 1.5] to

\[
\left( B, B, B + \int_0^{\tau(B)} h(t, B) dt \right)
\]

we obtain \( B_1, B_2 \) cylindrical Wiener processes such that

\[
\left( B_2, B_1 + \int_0^{\tau(B)} h(t, B_1) dt \right)
\]

is a maximal coupling of \( \left( D(B), D + \int_0^{\tau(B)} h(t, B) dt \right) \) on \([0, \infty)\)

**Lemma 2.4.**

\[
\mathbb{P}\left( \tilde{w}(\cdot, B_1, w_0^2) = w(\cdot, B_2, w_0^2) \right) \geq p_1 \frac{1}{4e^{2C}}.
\]

**Proof.** Let us set

\[
\begin{align*}
A &= \{ B : \tau(B) = \infty \} \\
\Lambda_1 &= D(B), \\
\Lambda_2 &= D(B + \int_0^{\tau(B)} h(t, B) dt),
\end{align*}
\]

Novikov condition is obviously verified. So, Girsanov Transform gives

\[
\frac{d\Lambda_1}{d\Lambda_2}(B) = \exp \left( - \int_0^{\tau(B)} h(t, B) dB_t - \frac{1}{2} \int_0^{\tau(B)} |h(t, B)|^2 dt \right),
\]

7
which yields
\[
\int_{\Lambda} \left( \frac{d\Lambda_1}{d\Lambda_2} \right)^2 d\Lambda_1 \leq E \left[ e^{\int_0^T |\rho(t,B)|^2 dt} \right] \leq e^{2C}.
\]

By Lemma 2.3
\[
\Lambda_1(A) \geq p_1.
\]

By [13] Lemma 1.3
\[
\Lambda_1 \land \Lambda_2(A) \geq \frac{p_1}{4e^{2C}},
\]

combine it with [13] Lemma 1.2] yields the result of this Lemma.

Now we are in the position of the proof of Theorem 1.1

Proof.
\[
\sup_{\pi \in \Gamma(P_0^\delta, P_0^\delta)} \pi \{ (x, y) \in H \times H, \rho_\pi(x, y) \leq \delta \}
\geq \mathbb{P}\left( \rho'_\pi(w(t, B_2, w_0^2), w(t, B_1, w_0^1)) \leq \delta \right)
\geq \mathbb{P}\left( \rho'_\pi(w(t, B_2, w_0^2), w(t, B_1, w_0^1)) \leq Ce^{-\gamma_1 t} \right)
\geq \mathbb{P}\left( \tilde{w}(\cdot, B_1, w_0^2) = w(\cdot, B_2, w_0^2) \right)
\geq \mathbb{P}\left( \rho'_\pi(w(t, B_2, w_0^2) - w(t, B_1, w_0^1)) \geq Ce^{-\gamma_1 t}, \right.
\left. \tilde{w}(\cdot, B_1, w_0^2) = w(\cdot, B_2, w_0^2) \text{ on } [0, t] \right)
\]

Combing it with Lemma 2.2 and Lemma 2.4 yields
\[
\sup_{\pi \in \Gamma(P_0^\delta, P_0^\delta)} \pi \{ (x', y') \in H \times H, \rho_\pi(x', y') \leq \delta \} \geq \frac{p_1}{4e^{2C}} - Ce^{-\gamma_1 t},
\]

which finishes the proof of Theorem 1.1

3. Proof of Gradient Estimate

For any \( v \in L^2_{loc}(\mathbb{R}^+, U) \) and \( \xi \in H \) with \( ||\xi|| = 1 \), denote \( \rho_t = J_{0,t}^x \xi - D^tv_t = J_{0,t}^x \xi - A_{0,t}v \).

Then \( \rho_t \) satisfies the following equation
\[
d\rho_t = v \Delta \rho _t dt + \tilde{B}(w_t, \rho_t) dt + DQ(w_t) \rho_t dB_t - Q(w_t)v_t dt. \tag{3.1}
\]

Let \( \zeta_t \) be the solution to the following equation
\[
\begin{aligned}
d\zeta_t &= DQ(w_t) \zeta_t dB_t + B_1 \Delta \zeta_t dt \\
&\quad + \pi_t \tilde{B}(w_t, \zeta_t) dt + v \Delta \zeta_t^0 dt, \tag{3.2}
\end{aligned}
\]

\( \zeta_0 = \xi, \)

\( \)
here $B_1$ is a constant bigger than $vN^2$ and $\zeta_t^h = P_N\zeta_t$, $\zeta_t^h = Q_N\zeta_t$, $\pi_h \tilde{B}(w_t, \zeta_t) = Q_N \tilde{B}(w_t, \zeta_t)$. We set the infinitesimal perturbation $v$ by

$$
\begin{align*}
\nu_t &= g(w_t)F_t, \\
F_t &= \pi_t \tilde{B}(w_t, \zeta_t) - B_1 \Delta \zeta_t^l + v \Delta \zeta_t^l,
\end{align*}
$$

here $\pi_t \tilde{B}(w_t, \zeta_t) = P_N \tilde{B}(w_t, \zeta_t)$. By Hypothesis H3, $g$ depends on $N$. $\zeta_t$ also depends on $N$.

### 3.1. The estimate of $\zeta_t$

**Lemma 3.1.** For any $\eta \leq \frac{v^2}{24b_0}$, there exist some $N_0 := N_0(B_0, \eta, L_Q, v) > 0$, such that if $N \geq N_0$, then

$$
\mathbb{E}[|\|\zeta_t^h|^2 e^{v\nu N^2} e^{t\|\zeta_t^h\|^2} d\nu]| \leq 1, \forall t > 0.
$$

Furthermore, there exists an absolutely positive constant $C$ such that

$$
\mathbb{E}|\zeta_t^h| \leq Ce^{-\frac{1}{2}v\nu N^2} e^{\frac{t}{2}||\zeta_t^h||^2}, \forall t > 0.
$$

**Proof.** By Itô’s formula,

$$
d|\|\zeta_t^h|^2| \leq 2B_1 \langle \zeta_t^h, \Delta \zeta_t^l \rangle dt + 2v \langle \zeta_t^h, \Delta \zeta_t^l \rangle dt + 2\langle \zeta_t^h, \tilde{B}(w_t, \zeta_t) \rangle dt + h_d dB_t + L_Q|\zeta_t^h|^2 dt. \tag{3.3}
$$

Here we need to estimate $\langle \zeta_t^h, \tilde{B}(w_t, \zeta_t) \rangle$, observe that

$$
\langle \zeta_t^h, \tilde{B}(w_t, \zeta_t) \rangle = \langle \zeta_t^h, B(K\zeta_t, w_t) \rangle + \langle \zeta_t^h, B(Kw_t, \zeta_t) \rangle,
$$

and (1.7), for some $\hat{C}$,

$$
\langle \zeta_t^h, B(K\zeta_t, w_t) \rangle \leq \hat{C}|\zeta_t^h| ||w_t||_{L^2} ||\zeta_t^h||
\leq \eta||w_t||_{L^2}^2 ||\zeta_t^h||^2 + \frac{\hat{C}^2}{4\eta} ||\zeta_t^h||^2
\leq \eta||w_t||_{L^2}^2 ||\zeta_t^h||^2 + \frac{\hat{C}^2}{4\eta} ||\zeta_t^h|| ||\zeta_t^h||
\leq \eta||w_t||_{L^2}^2 ||\zeta_t^h||^2 + \frac{\hat{C}^2}{8\eta^2} ||\zeta_t^h||^2, \tag{3.4}
$$

and,

$$
\langle \zeta_t^h, B(Kw_t, \zeta_t) \rangle = \langle \zeta_t^h, B(Kw_t, \zeta_t) \rangle
\leq \hat{C}||w_t||_{L^2} ||\zeta_t^h|| ||\zeta_t^h||
\leq \eta||\zeta_t^h||^2 ||w_t||_{L^2}^2 + \frac{\hat{C}^2}{4\eta} ||\zeta_t^h||^2.
$$
Therefore, by (3.3),

\[
\begin{align*}
\frac{d}{dt} \|\zeta_t\|^2 &\leq -2B_1 \|\zeta_t^h\|^2 dt - 2\nu \|\zeta_t^h\|^2 dt + 4\eta \|\zeta_t^i\|^2 \|w_t\|^2 dt \\
&\quad + \frac{C^2}{2\eta} \|\zeta_t^i\|^2 + \frac{\nu}{3} \|\zeta_t^i\|^2 + \left[ -\frac{C^4}{4\eta^2 \nu} + L_Q \right] \|\zeta_t^i\|^2 \\
&\quad + h_t dB_t.
\end{align*}
\]

Hence, if \( B_1 > \nu N^2 > \frac{C^2}{2\eta} + \frac{\nu}{3} + \left[ -\frac{C^4}{4\eta^2 \nu} + L_Q \right] \) and \( \frac{2}{\nu} \nu N^2 > \left[ -\frac{C^4}{4\eta^2 \nu} + L_Q \right] \),

\[
\begin{align*}
\frac{d}{dt} \|\zeta_t\|^2 &\leq -B_1 \|\zeta_t^h\|^2 dt - \nu N^2 \|\zeta_t^h\|^2 dt + 4\eta \|\zeta_t^i\|^2 \|w_t\|^2 dt + h_t dB_t \\
&\leq -\nu N^2 \|\zeta_t\|^2 dt + 4\eta \|\zeta_t^i\|^2 \|w_t\|^2 dt + h_t dB_t,
\end{align*}
\]

from which,

\[
\mathbb{E}[\|\zeta_t\|^2 e^{\nu N^2 t - 4\eta \int_0^t \|w_t\|^2 dt}] \leq 1.
\] (3.6)

Therefore, by Hölder inequality, (3.6) and Lemma 3.3, if \( \eta \leq \frac{\nu^2}{8B_0 \nu} \) and \( \frac{\sqrt{8B_0 \eta \nu}}{\nu} \leq N \),

\[
\begin{align*}
\mathbb{E}[\|\zeta_t\|] &= \mathbb{E}\left[ \|\zeta_t\| e^{-\frac{1}{2} \nu N^2 t^2 + 2\eta \int_0^t \|w_t\|^2 dt} \cdot e^{\frac{1}{2} \nu N^2 t^2 - 2\eta \int_0^t \|w_t\|^2 dt} \right] \\
&\leq \left( \mathbb{E}[e^{-\nu N^2 t^2 + 2\eta \int_0^t \|w_t\|^2 dt}] \cdot \mathbb{E}[e^{\frac{1}{2} \nu N^2 t^2 - 2\eta \int_0^t \|w_t\|^2 dt}] \right)^{\frac{1}{2}} \\
&\leq Ce^{-\nu N^2 t} \cdot \left( \mathbb{E}[e^{-\nu N^2 t^2 + 2\eta \int_0^t \|w_t\|^2 dt}] \right)^{\frac{1}{2}} \\
&\leq Ce^{-\nu N^2 t} \cdot e^{\frac{2\nu}{\nu} \|w_t\|^2}.
\end{align*}
\]

\( \square \)

**Lemma 3.2.** For any \( n \in \mathbb{N} \) and \( \eta \leq \frac{\nu^2}{8B_0 \nu} \), there exists \( N_0 = N_0(n, \eta, B_0, \nu, L_Q) \), such that if \( N \geq N_0 \),

\[
\mathbb{E}[\|\zeta_t\|^{2n} e^{\nu N^2 t - n(\nu - 1)L_Q t + 4\eta \int_0^t \|w_t\|^2 dt}] \leq 1.
\]

and furthermore for some absolute constant \( C \),

\[
\mathbb{E}[\|\zeta_t\|^n] \leq Ce^{\frac{2\nu n}{\nu} \|w_t\|^2} e^{\nu N^2 t / 2}.
\]

**Proof.** By Itô’s formula,

\[
\begin{align*}
\frac{d}{dt} \|\zeta_t\|^2 &\leq 2B_1 \langle \zeta_t, \Delta \zeta_t^i \rangle dt + 2\nu \langle \zeta_t, \Delta \zeta_t^h \rangle dt + 2\zeta_t^h \langle B(w_t, \zeta_t) \rangle dt \\
&\quad + \langle \zeta_t, DQ(w_t) \zeta_t dB_t \rangle + \|DQ(w_t) \zeta_t\|^2 dt.
\end{align*}
\]

and

\[
\begin{align*}
\frac{d}{dt} \|\zeta_t\|^{2n} &= n\|\zeta_t\|^{2n-2} \frac{d}{dt} \|\zeta_t\|^2 + \frac{n(n - 1)}{2} \|\zeta_t\|^{2n-4} h_t^2 dt.
\end{align*}
\]
Hence, by H"older inequality, Lemma 1.2 and Lemma 3.2, for any
\[
\frac{n(n - 1)L_Q}{2} \|\zeta_i\|^{2n} dr.
\]
Therefore,
\[
\mathbb{E}\left[\|\zeta_i\|^{2n} e^{\left(vnN^2 - (n-1)L_Q/2\right)t - \int_0^t 4\eta \|\omega_t\|^2 dr}\right] \leq 1.
\]
By H"older inequality and the above inequality,
\[
\mathbb{E}\left[\|\zeta_i\|^n\right] \leq \mathbb{E}\left[e^{-\int_0^t 4\eta \|\omega_t\|^2 dr}\right]^{\frac{n}{2}},
\]
here we use the notation, for any random variable $X$, $\mathbb{E}[X]^p := \left[\mathbb{E}|X|^p\right]$. Hence, by Lemma 1.3, there exists $N_0 = N_0(n, \eta, B_0, \nu, L_Q)$, such that if $N \geq N_0$, then for some absolute constant $C$,
\[
\mathbb{E}\left[\|\zeta_i\|^n\right] \leq \mathbb{E}\left[e^{-\int_0^t 4\eta \|\omega_t\|^2 dr}\right]^{\frac{n}{2}} \leq Ce^{\frac{2n}{\nu} \|\omega_0\|^2} e^{-vnN^2t/2}.
\]
\[
\square
\]
3.2. The estimate of $\mathbb{E}\left[\left|\int_0^t v(s) dB_s(t)\right|^2\right]$.
\[
\left(\mathbb{E}\left|\int_0^t v(s) dW(s)\right|^2\right) = \int_0^t \mathbb{E}|v(s)|^2 ds \leq C \int_0^t \mathbb{E}|F_s|^2 ds.
\] (3.7)
Remaking that, by the definition of $F_t$ and $\|\pi_t \tilde{B}(u, w)\| \leq C(N) \cdot \|u\| \cdot \|w\|$ for some constant $C = C(N)$ (see [5] Lemma A.4),
\[
\mathbb{E}|F_s|^2 \leq C \cdot \left(\mathbb{E}\|\zeta_s\|^2 + \mathbb{E}(\|\omega_s\|^2 \|\zeta_s\|^2)\right) \leq C \cdot \left(\mathbb{E}\|\zeta_s\|^2 + \left[\mathbb{E}\|\omega_s\|^4\right]^{1/2} \left[\mathbb{E}\|\zeta_s\|^4\right]^{1/2}\right)
\]
Hence, by H"older inequality, Lemma 1.2 and Lemma 3.2, for any $\eta \leq \frac{2}{32b_0}$, there exist some $N_0 := N_0(B_0, \eta, L_Q, \nu) > 0$, such that if $N \geq N_0$, then there exists
\[
\mathbb{E}\left|\int_0^\infty v(s) dB(s)\right|^2 \leq C(N) \exp\left(\frac{4\eta}{\nu} \|\omega_0\|^2\right).
\] (3.8)
3.3. The proof of Theorem 1.2

Proof. For any $\xi$ with $\|\xi\| = 1$, for some constant $C = C(\nu, B_0, N, \eta)$,

$$\langle \nabla P_t \varphi(w_0), \xi \rangle$$

$$= \mathbb{E}_{w_0}(\langle \nabla \varphi(w_t), \xi \rangle) = \mathbb{E}_{w_0}(\langle \nabla \varphi(w_t) - \nabla \varphi(w_0), \xi \rangle)$$

$$= \mathbb{E}_{w_0}(\langle \nabla \varphi(w_t) \epsilon dB_s \rangle) + \mathbb{E}_{w_0}(\langle \nabla \varphi(w_t) + \rho_t, \xi \rangle)$$

$$= \mathbb{E}_{w_0}(\langle \nabla \varphi(w_t), \xi \rangle) + \mathbb{E}_{w_0}(\langle \nabla \varphi(w_t) \rho_t, \xi \rangle)$$

$$\leq \left[ \mathbb{E}_{w_0}(\varphi(w_t)^2) \right]^\frac{1}{2} \left[ \mathbb{E}_{w_0}(\int_0^t \epsilon dB_s)^2 \right]^\frac{1}{2} + \left[ \mathbb{E}_{w_0}(\varphi(w_t)^2) \right]^\frac{1}{2} \left[ \mathbb{E}_{w_0}(\rho_t^2) \right]^\frac{1}{2}$$

$$\leq C(N) \exp \left( \frac{4\eta}{\nu} \|w_0\|^2 \right) \sqrt{P_t \|\varphi\|^2(x_0)} + C e^{\frac{\|w_0\|^2}{\nu^2}} e^{-\nu N^2} \sqrt{P_t \|D\varphi\|^2(x)}$$

\[\Box\]

4. Proof of Exponential Mixing

For getting the exponential convergence, we use the methods in [7]. In the Assumption 4.1, 4.2, 4.3 and Theorem 4.1 below, we assume that we are given a random flow $\Phi_t$ on a Banach space $H$. We will assume that the map $x \mapsto \Phi_t(\omega, x)$ is $C^1$ for almost every element $\omega$ of the underlying probability space. We will denote by $D\Phi_t$ the Fréchet derivative of $\Phi_t(\omega, x)$ with respect to $x$.

Let $C(\mu_1, \mu_2)$ for the set of all measures $\Gamma$ on $H \times H$ such that $\Gamma(A \times H) = \mu_1(A)$ and $\Gamma(H \times A) = \mu_2(A)$ for every Borel set $A \subset H$. The following three assumptions are from [7].

Assumption 4.1. There exists a function $V : H \rightarrow [1, \infty)$ with the following properties:

(1) There exists two strictly increasing continuous functions $V^+$ and $V_+$ from $[0, \infty) \rightarrow [1, \infty)$ such that

$$V_+(\|x\|) \leq V(x) \leq V^+(\|x\|)$$

for all $x \in H$ and such that $\lim_{a \to \infty} V_+(a) = \infty$.

(2) There exists constants $C$ and $\kappa \geq 1$ such that

$$aV^+(a) \leq CV_+(a)$$

for every $a > 0$.

(3) There exists a positive constants $C$, $r_0 < 1$, a decreasing function $\zeta : [0, 1] \rightarrow [0, 1]$ with $\zeta(1) < 1$ such that for every $h \in H$ with $\|h\| = 1$

$$\mathbb{E} V^r(\Phi_t(x))(1 + \|D\Phi_t(x)h\|) \leq CV^r \zeta(h)(x),$$

for every $x \in H$, every $r \in [r_0, \kappa]$, and every $t \in [0, 1]$. 

If Assumption 4.1 is satisfied, then for every Fréchet differentiable function \( \varphi : H \to \mathbb{R} \), we introduce the following norm
\[
\| \varphi \|_V = \sup_{x \in H} \frac{|\varphi(x)| + \|D\varphi(x)\|}{V(x)},
\]
and for \( r \in (0, 1] \), a family of distance \( \rho_r \) on \( H \) is defined by
\[
\rho_r = \inf_{\gamma} \int_0^1 V'(\gamma(t))|\dot{\gamma}(t)|dt,
\]
where the infimum runs over all paths \( \gamma \) such that \( \gamma(0) = x \) and \( \gamma(1) = y \). For simple, we will write \( \rho \) for \( \rho_1 \).

**Assumption 4.2.** There exists a \( C_1 > 0 \) and \( p \in [0, 1) \) so that for every \( \alpha \in (0, 1) \) there exists positive \( T(\alpha) \) and \( C(\alpha) \) with
\[
\|D_P\varphi(x)\| \leq C_1 V^p(x) \left( C(\alpha) \sqrt{\langle P_t \varphi \rangle}(x) + \alpha \sqrt{\langle P_t \|D\varphi\|^2 \rangle}(x) \right), \tag{4.4}
\]
for every \( x \in H \) and \( t \geq T(\alpha) \).

**Assumption 4.3.** Given any \( C > 0 \), \( r \in (0, 1) \) and \( \delta > 0 \), there exists a \( T_0 \) so that for any \( T \geq T_0 \) there exists an \( a > 0 \) so that
\[
\inf_{|t|, |s| \leq C} \sup_{t \in G(x, y, \delta)} \Gamma \{ (x', y') \in H \times H : \rho_s(x', y') < \delta \} \geq a. \tag{4.5}
\]

If the setting of the semigroup \( P_t \) possesses an invariant measure \( \mu_* \), we define
\[
\| \varphi \|_\rho = \sup_{x \neq y} \left| \frac{\varphi(x) - \varphi(y)}{\rho(x, y)} \right| + \left| \int_H \varphi(x) \mu_*(dx) \right|.
\]

The next Theorem comes from Theorem 3.6, Corollary 3.5 and Theorem 4.5 in [7].

**Theorem 4.1.** Let \( \Phi_t \) be a stochastic flow on a Banach space \( H \) which is almost surely \( C^1 \) and satisfy Assumption 4.1. Denote by \( P_t \) the corresponding Markov semigroup and assume that it satisfies Assumption 4.2 and 4.3. Then there exists a unique invariant probability measure \( \mu \), for \( P_t \) and exists constants \( \gamma > 0 \) and \( C > 0 \) such that
\[
\| P_t \varphi - \mu_* \varphi \|_\rho \leq Ce^{-\gamma t}\| \varphi - \mu_* \varphi \|_\rho,
\]
\[
\| P_t \varphi - \mu_* \varphi \|_V \leq Ce^{-\gamma t}\| \varphi - \mu_* \varphi \|_V,
\]
for every Fréchet differentiable function \( \varphi : H \to \mathbb{R} \) and every \( t > 0 \).

The next Lemma comes from Lemma 5.1 in [7].

**Lemma 4.1.** Let \( U \) be a real-valued semi-martingale
\[
dU(t, \omega) = F(t, \omega)dt + G(t, \omega)dB(t, \omega),
\]
where $B$ is a standard Brownian motion. Assume that there exists a process $Z$ and positive constants $b_1, b_2, b_3$, with $b_2 > b_3$, such that $F(t, \omega) \leq b_1 - b_2 Z(t, \omega), U(t, \omega) \leq Z(t, \omega)$, and $G(t, \omega)^2 \leq b_3 Z(t, \omega)$ almost surely. Then the bound

$$
\mathbb{E} \exp \left( U(t) + \frac{b_2 e^{-b_2 \eta^2/2}}{2} \int_0^t Z(s) ds \right) \leq \frac{b_2 \exp(b_1 \eta^2)}{2} \exp \left( U(0) e^{-b_2 \eta^2 t} \right),
$$

holds for every $t \geq 0$.

**Proposition 4.1.** Let $H = L^2_0$ be the space of real-valued square-integrable functions on the torus $[-\pi, \pi]^2$ with vanishing mean, and the random flow $\Phi_t$ on the space $H$ is given by the solution to (1.2). Under the conditions of Theorem 4.1 Assumption 4.1 holds with

$$
V(x) = V_{\eta_0}(x) = e^{\eta_0 ||x||^2}, \quad \eta_0 = \frac{\nu}{16B_0}.
$$

**Proof.** By Itô formula,

$$
d\eta ||w_t||^2 + 2\eta \nu ||w_t||^2 dt = 2\eta (w_t, Q(w_t) dB_t) + \eta ||Q(w_t)||^2 dt.
$$

From Lemma 4.1,

$$
\mathbb{E} \exp \left( U(t) + \frac{b_2 e^{-b_2 \eta^2/2}}{4} \int_0^t Z(s) ds \right) \leq \frac{b_2 \exp(b_1 \eta^2)}{2} \exp \left( U(0) e^{-b_2 \eta^2 t} \right),
$$

where $U(t) = \eta ||w_t||^2$, $Z(t) = \eta ||w_t||_1^2$, $b_1 = \eta B_0$, $b_2 = 2\nu$, $b_3 = 4\eta B_0$. Therefore when $\eta \leq \frac{1}{4B_0}$,

$$
\mathbb{E} \exp \left( \eta ||w_t||^2 + \frac{2\nu e^{-2\nu \eta^2/2}}{4} \int_0^t \eta ||w_s||_1^2 ds \right) \leq 2 \exp\left( \frac{\eta B_0}{\nu} \right) \exp \left( \eta ||w_0||^2 e^{-2\nu t} \right).
$$

(4.6)

For $||x|| = 1$, denote $\xi_t = J_t \xi_t = D w_t \xi_t$, where $x$ is the initial value and $D$ is the differential operator with $x$. So $\xi_t$ satisfies the following equation

$$
d\xi_t = \nu \Delta \xi_t dt + \bar{B}(w_t, \xi_t) dt + DQ(w_t) \xi_t dB_t,
$$

(4.7)

and thus

$$
d||\xi_t||^2 \leq -2\nu ||\xi_t||^4 dt + 2 \langle B(K\xi_t, w_t), \xi_t \rangle dt + L_0 ||\xi_t||^2 dt + h_t dW_t.
$$

By the the similar step to get (3.4), one arrives at

$$
2 \langle B(K\xi_t, w_t), \xi_t \rangle \leq \eta ||w_t||^2 ||\xi_t||^2 + \nu ||\xi_t||^2 + \frac{16\hat{C}^4}{\eta^2 \nu} ||\xi_t||^2,
$$

and

$$
d||\xi_t||^2 \leq \eta ||w_t||^2 ||\xi_t||^2 dt + \left( \frac{16\hat{C}^4}{\eta^2 \nu} + L_0 \right) ||\xi_t||^2 dt + h_t dW_t.
$$

Define the function $h(\eta) = \left( \frac{16\hat{C}^4}{\eta^2 \nu} + L_0 \right)$, from the above inequality one arrives at

$$
\mathbb{E} \left| ||\xi_t||^2 \exp \left( -h(\eta) t + \int_0^t \eta ||w_s||_1^2 ds \right) \right| \leq 1, \quad \forall \eta > 0.
$$

(4.8)
Set $b = e^{-\frac{\nu}{2}}$, $\eta \leq \frac{\nu}{8B_0}$, $t \in [0, 1]$. From (4.6) and (4.8),

$$\mathbb{E}\{ \exp (\eta \|w_t\|^2) \|\xi_t\| \} = \mathbb{E}\left[ \exp (\eta \|w_t\|^2) \exp \left( \frac{b \nu}{2} \int_0^t \|w_s\|^2 ds \right) \|\xi_t\| \exp \left( -\frac{b \nu}{2} \int_0^t \|w_s\|^2 ds \right) \right]
$$

$$\leq \left( \mathbb{E} \exp (2\eta \|w_t\|^2 + b \nu \int_0^t \|w_s\|^2 ds) \mathbb{E}\|\xi_t\|^2 \exp (-b \nu \int_0^t \|w_s\|^2 ds) \right)^{\frac{3}{2}}
$$

$$\leq \left( 2 \exp \left( \frac{2\eta B_0}{\nu} \right) \exp \left( 2\eta \|w_0\|^2 e^{-\frac{b \nu}{2}} \right) \right)^{\frac{3}{2}} \exp \left( \frac{h(b \nu)}{2} \right)
$$

$$= C(\eta, B_0, \nu) \exp \left( \eta \|w_0\|^2 e^{-\nu t} \right) \exp \left( \frac{h(b \nu)}{2} \right).
$$

Set $\eta_0 = \frac{\nu}{16B_0}$, we know that the above inequality is satisfied for all $\eta \in [0, 2\eta_0]$. So $w_t$ satisfies Assumption 4.1 for $V(x) = e^{\alpha \|x\|^2}$, $\kappa = 2$, $r_0 = \frac{1}{2}$, $V_r(a) = V_r^*(a) = e^{\eta_0 a^2}$ and $\zeta_r = e^{-\frac{b}{2} r^2}$.

\section*{Appendix A.}

Let $r = \tilde{w}(t, B, w_0^2) - w(t, B, w_0^2)$.

**Proposition A.1.** Assume H1 and H2 hold. There exists $\gamma_1, \gamma_2 > 0$, $K_0 > 0$ and $N_0 = N_0(B_0, \nu, L_Q)$ such that for any $K \geq K_0$ and $N \geq N_0$ and any $(t, w_0^1, w_0^2) \in (0, \infty) \times H \times H$,

$$\mathbb{E}\|r\|^2 \leq 2e^{\nu \|w_0\|^2} \|r(0)\|^2 e^{-\gamma_2 t}.
$$

Here $r(t) = \tilde{w}(t, B, w_0^2) - w(t, B, w_0^2) = \tilde{w} - w$.

**Proof.** For any function $f$, denote $\delta f = f(\tilde{w}) - f(w)$. Taking the difference between (2.1) and (2.2), we obtain

$$dr = \nu \Delta r dt - K P_N r dt + \delta Q(w) dB_t + \left[ B(K\tilde{w}, \tilde{w}) - B(Kw, w) \right] dt
$$

$$= \nu \Delta r dt - K P_N r dt + \delta Q(w) dB_t + \left[ B(K\tilde{w}, r) + B(Kr, w) \right] dt.
$$

Apply Itô’s formula to $\|r\|^2$, we have

$$d\|r\|^2 = -2\nu \|r\|^2 dt - 2K\|P_N r\|^2 dt + 2\langle B(K\tilde{w}, r) \rangle dt + 2\langle r, \delta Q(w) dB_t \rangle + \|\delta Q(r)\|^2_{L^2(U, H)}.
$$

Remarking that for any $\eta > 0$,

$$\langle B(Kr, w), r \rangle \leq \|w\|_1 \cdot \|r\| \cdot \|r\|_{1/2}
$$

$$\leq \eta\|w\|^2_1 \|r\|^2 + C(\eta)\|r\|^2_{1/2}
$$

$$\leq \eta\|w\|^2_1 \|r\|^2 + \frac{\nu}{6} \|r\|^2 + C(\eta)\|r\|^2
$$

then

$$d\|r\|^2 \leq -\frac{3\nu}{2} \|r\|^2 dt - 2K\|P_N r\|^2 dt + C\|w\|^4_1 \|r\|^2 dt + 2\langle r, \delta Q(w) dB_t \rangle + L_0 \|r\|^2
$$

$$\leq \left[ -\frac{3}{2} \nu N^2 + \eta\|w\|^2_1 \|r\|^2 \right] dt + 2\langle r, \delta Q(w) dB_t \rangle,
$$

15
Integrating this formula and taking the expectation, if follows
\[
\mathbb{E}[||r||^2 \exp(\nu N^2 t - \eta \int_0^t ||w_s||^2_1 ds)] \leq ||r(0)||^2.
\]
Apply Itô’s formula to \([||r||^2]^2\), with a similar step, one arrives at
\[
d||r||^4 \leq [-\nu N^2 + \eta ||w||^2_1] ||r||^4 dt + 2\langle r, \delta Q(w) dB_t \rangle,
\]
and
\[
\mathbb{E}[||r||^4 \exp(\nu N^2 t - \eta \int_0^t ||w_s||^2_1 ds)] \leq ||r(0)||^4.
\]
It follows from the above inequality and Lemma [1.3] that for some \(\gamma_1, \gamma_2 > 0\),
\[
\mathbb{E}[||r||^2] \leq 2C e^{\gamma_1 ||w||^2_1} ||r(0)||^2 e^{-\gamma_2 t}.
\]

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