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G-Frobenius manifolds

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By  Byeongho Lee

Entitled
G-FROBENIUS MANIFOLDS

For the degree of  Doctor of Philosophy

Is approved by the final examining committee:

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Approved by Major Professor(s):  Ralph M. Kaufmann

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Head of the Departmental Graduate Program  Date
G-FROBENIUS MANIFOLDS

A Dissertation
Submitted to the Faculty
of
Purdue University
by
Byeongho Lee

In Partial Fulfillment of the
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of
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May 2015
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West Lafayette, Indiana
To Sejin, Dylan, and Claire.
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ABSTRACT

Lee, Byeongho PhD, Purdue University, May 2015. G-Frobenius Manifolds. Major Professor: Ralph M. Kaufmann.

The goal of this dissertation is to introduce the notion of $G$-Frobenius manifolds for any finite group $G$. This work is motivated by the fact that any $G$-Frobenius algebra yields an ordinary Frobenius algebra by taking its $G$-invariants. We generalize this on the level of Frobenius manifolds.

To define a $G$-Frobenius manifold as a braided-commutative generalization of the ordinary commutative Frobenius manifold, we develop the theory of $G$-braided spaces. These are defined as $G$-graded $G$-modules with certain braided-commutative “rings of functions”, generalizing the commutative rings of power series on ordinary vector spaces.

As the genus zero part of any ordinary cohomological field theory of Kontsevich-Manin contains a Frobenius manifold, we show that any $G$-cohomological field theory defined by Jarvis-Kaufmann-Kimura contains a $G$-Frobenius manifold up to a rescaling of its metric.

Finally, we specialize to the case of $G = \mathbb{Z}/2\mathbb{Z}$ and prove the structure theorem for (pre-)$\mathbb{Z}/2\mathbb{Z}$-Frobenius manifolds. We also construct an example of a $\mathbb{Z}/2\mathbb{Z}$-Frobenius manifold using this theorem, that arises in singularity theory in the hypothetical context of orbifolding.
1. Introduction

1.1 Orbifolding Frobenius Algebras

Although there is no universal definition of the procedure of orbifolding, we do have several examples, and we can describe common features of them. Given a Frobenius algebra $H_e$ with a finite group $G$ acting on it, using the help of some external data, we first construct a $G$-Frobenius algebra\(^1\) $H$ that contains the original Frobenius algebra as a subalgebra, called the untwisted sector, corresponding to the identity $e \in G$. The extended factors corresponding to nontrivial elements of $G$ are called twisted sectors. Since the resulting $G$-Frobenius algebra also has a $G$-action extending the original one, one can take its subalgebra of $G$-invariants $H^G$, and this is a Frobenius algebra. Note that this is graded by conjugacy classes of $G$, that is inherited from the $G$-grading of $H$. It also contains $(H_e)^G$, and again, this is called the untwisted sector of $H^G$.

Two main classes of examples of Frobenius algebras are cohomology rings and Milnor rings\(^2\) of isolated singularities of complex valued functions. We review examples of orbifolding in these classes.

In [8], Fantechi and Göttsche defined what is later called a stringy cohomology ring, that is an example of a $G$-Frobenius algebra. Given a complex manifold $Y$ and a finite group $G$ acting on it, they defined the stringy multiplication on the vector space $H^*(Y, G) := \bigoplus_{g \in G} H^*(Y^g, \mathbb{Q})$ using top Chern classes of certain vector bundles on some submanifolds of $Y$. After taking $G$-invariants, this ring yields the Chen-Ruan orbifold cohomology of the orbifold $[Y/G]$. Note that Fantechi-Göttsche’s construction greatly simplified the original definition of orbifold cohomology.

\(^1\)See Section 3.2.
\(^2\)Also known as local rings, or Jacobian rings.
In [12], Kaufmann showed that one can construct a $G$-Frobenius algebra given a type of Frobenius algebra and a finite group $G$ acting on it, using group-cohomological data of $G$. He used this to give a mathematical treatment of the orbifolding of the $A_{2n-3}$ to $D_n$ singularities on the level of their Milnor rings. After constructing a $\mathbb{Z}/2\mathbb{Z}$-Frobenius algebra for $A_{2n-3}$, taking $\mathbb{Z}/2\mathbb{Z}$ invariants of it yields the Milnor ring for $D_n$. A physics argument of the corresponding procedure can be found in [5] on the level of Frobenius manifolds.

1.2 Orbifolding Frobenius Manifolds

Then, a natural question that one can ask is if we could extend these procedures to the level of Frobenius manifolds mathematically. In analogy with the case of Frobenius algebras, one may want to construct something that could be called a $G$-Frobenius manifold out of an ordinary Frobenius manifold with $G$-action, that would yield another Frobenius manifold upon taking a suitable notion of $G$-invariants.

There is another motivation to want a mathematical theory of orbifolding Frobenius manifolds. Let $X$ be a smooth projective variety. Genus 0 Gromov-Witten invariants of $X$ can be encoded as a Frobenius manifold structure on its cohomology ring. In [4], Costello showed that genus $g$ Gromov-Witten invariants of $X$ can be written in terms of genus 0 ones on its symmetric products, $[X^{g+1}/S_{g+1}]$. These will give rise to the structures of Frobenius manifolds on the Chen-Ruan orbifold cohomology rings of the symmetric products. The procedure of orbifolding should give us a way to produce Frobenius manifolds of symmetric products out of those of ordinary products, which can be obtained as in [17].

The key ingredients missing in this procedure are the notion of $G$-Frobenius manifolds and the way to take their $G$-invariants. The goal of this dissertation is to provide possible answers to these questions.
1.3 Summary of Results

It is known that a formal Frobenius manifold is equivalent to the genus 0 part of a cohomological field theory [16]. Since $G$-cohomological field theory is a natural generalization of it that contains an ordinary cohomological field theory and a $G$-Frobenius algebra [10], we are led to construct a $G$-Frobenius manifold so that it would be contained in the genus 0 part of the $G$-cohomological field theory.\(^3\)

Looking at the proof of the statement that the genus 0 part of a cohomological field theory gives rise to a formal Frobenius manifold, we realize that the key point is to interpret degree $n$ polynomials as symmetric $n$-linear forms.\(^4\) Since the state space of a $G$-cohomological field theory is an object\(^5\) of $D(k[G])\text{-Mod}$, the category of modules over the quasitriangular Hopf algebra of the Drinfeld’s double of the group algebra of $G$, that is a braided monoidal category, the analog of the symmetric $n$-linear forms are braided $n$-linear forms, that are invariant under the action of the braid group $B_n$.\(^6\) Since the ring of power series can be taken as the ring of functions on vector spaces, we can generalize this to our situation by interpreting it as the ring of symmetric multilinear forms, considering the ring of braided multilinear forms. We view $G$-graded $G$-modules as a braided space with this ring of functions.\(^7\)

The important step to interpret degree $n$ polynomials as symmetric $n$-linear forms is the use of symmetrization map, that is the averaging map for the $S_n$-orbits. There is a problem if we want to do an analogous step for braid groups: $B_n$ is infinite for $n \geq 2$. But fortunately for us, the $D(k[G])\text{-Mod}$ has some finiteness with regard to $B_n$-orbits. This is realized as a finite groupoid structure on the set $G^n$, and the $n$th tensor power of any $G$-graded $G$-module give rise to a representation of this groupoid. We define braidization in $D(k[G])\text{-Mod}$ using this representation.\(^8\)

---

\(^3\)The converse is a hard problem since the topology of the moduli space $\overline{M}_{0,n}^G$ of pointed admissible $G$-covers is not very well understood compared to that of $\overline{M}_{0,n}$.
\(^4\)See Sections 2.1 and 3.1.
\(^5\)called a $G$-graded $G$-module
\(^6\)See Section 2.2.
\(^7\)See Section 2.4.
\(^8\)See Sections 2.3 and 2.4.
In fact, we develop the theory of braided tensors and braidization for general braided monoidal categories that satisfy certain conditions. We show that \( D(k[G])\)-\textbf{Mod} satisfies these conditions and proved that we can define the unital associative ring of braided multilinear forms for any finite dimensional \( G \)-graded \( G \)-modules.

Any \( G \)-graded \( G \)-module contains two distinguished subspaces: the \( G \)-degree \( e \) part and the \( G \)-invariants. To define the notion of \( G \)-Frobenius manifolds, we use the fact that the braided multilinear forms restrict to the ordinary symmetric multilinear forms on these two spaces. Since we want our \( G \)-Frobenius manifold to contain a Frobenius manifold for its untwisted sector and another one on the subspace of \( G \)-invariant vectors, we define a pre-\( G \)-Frobenius manifold as a finite dimensional \( G \)-graded \( G \)-module with a braided multilinear form that restricts to two power series on the two distinguished subspaces that gives them the structure of formal Frobenius manifolds. Since \( G \)-Frobenius algebras can be defined in terms of braided bi- and tri-linear forms on \( G \)-graded \( G \)-modules, we further define \( G \)-Frobenius manifolds as pre-\( G \)-Frobenius manifolds with the degree 2 and 3 terms of their braided multilinear forms defining \( G \)-Frobenius algebra structures on the underlying \( G \)-graded \( G \)-modules. Here we see that taking \( G \)-invariants for \( G \)-Frobenius manifolds is the same thing as taking the \( G \)-invariants for the underlying \( G \)-graded \( G \)-module and restricting its braided multilinear form to it.

As noted in the first paragraph of this section, we prove that any \( G \)-cohomological field theory contains a \( G \)-Frobenius manifold. In fact, one can produce two types of correlation functions out of a \( G \)-cohomological field theory: the symmetric one and the braided one. Using Jarvis-Kaufmann-Kimura’s result that any \( G \)-cohomological field theory gives rise to an ordinary cohomological field theory on the \( G \)-invariants of the original state space, it is straightforward to show that the symmetric correlation

\[ \text{See Section 2.2.} \]
\[ \text{See Theorem 2.4.13.} \]
\[ \text{See Section 2.5.} \]
\[ \text{See Definition 3.3.1.} \]
\[ \text{See Remark 4.4.3.} \]
functions restrict to the correlation functions on the ordinary Frobenius manifold on the \( G \)-invariants. On the other hand, it is the braided correlation functions that we want to have this property. It turns out that the restrictions of these two types of correlation functions coincide on the distinguished subspaces.\(^{14}\)

Lastly, we specialize to \( G = \mathbb{Z}/2\mathbb{Z} \) case and discover a series of \( \mathbb{Z}/2\mathbb{Z} \)-Frobenius manifolds that arise in singularity theory. After proving the structure theorem for pre-\( \mathbb{Z}/2\mathbb{Z} \)-Frobenius manifolds,\(^{15}\) we show that the Frobenius manifolds of simple singularities \( A_{2n-3} \) and \( D_n \) together give rise to the \( G \)-Frobenius manifold structure on the \( \mathbb{Z}/2\mathbb{Z} \)-Frobenius algebras of \( A_{2n-3} \).\(^{16}\) Here we use already known facts about these singularities, but the important question of orbifolding still remains unsolved. The fact that the data of \( D_n \) is already contained in \( A_{2n-3} \)\(^{17}\) makes the question more interesting in this setting.

### 1.4 Notation

Throughout this dissertation, \( G \) is a finite group, \( k \) a field of characteristic zero. We use the Einstein convention, namely, we sum over the repeated upper and lower indices. The identity in \( G \) is denoted by \( e \). We do not consider the trace axiom of \( G \)-Frobenius algebras in this paper. We use boldface letters to denote elements of \( G^n \). For example, \( m \) means \( (m_1, \ldots, m_n) \in G^n \). The identity in \( G \) is denoted by \( e \).

\(^{14}\)See Section 4.3.  
\(^{15}\)See Theorem 5.1.3.  
\(^{16}\)See Sections 5.3, 5.4, and 5.5.  
\(^{17}\)See Section 5.3.
2. G-braided Spaces

2.1 Functions on Vector Spaces

We review some well-known facts about ordinary vector spaces. The goal of this section is to fix notations and emphasize the viewpoint that will be useful for the generalization we have in mind.

Let $H$ be a finite dimensional vector space over $k$. We take the ring of power series $k[[H^*]]$ to be its ring of functions. This is required for the definition of Frobenius manifold. See Section 3.1.

It is well known that $k[[H^*]]_n$, the homogeneous polynomials of degree $n$, can be identified with $\text{Sym}^n H^* := [(H^*)^\otimes n]^{S_n}$, the subspace of invariants under the natural $S_n$ action, also known as the symmetric $n$-linear forms. Hence we have isomorphisms of vector spaces

$$k[[H^*]] \cong \prod_{n=0}^{\infty} \text{Sym}^n H^* := \text{Sym}[H^*].$$

Also, observe that there is another characterization of $\text{Sym}^n H^*$, considering the $S_n$ action on $H^\otimes n$. Namely,

$$\text{Sym}^n H^* = \{ x \in (H^*)^\otimes n | x(v) = x(\sigma \cdot v) \text{ for any } \sigma \in S_n \text{ and } v \in H^\otimes n \}.$$

Note that $k[[H^*]]$ also has the structure of a commutative ring. We transfer this ring structure to $\text{Sym}[H^*]$. Note also that $\text{Sym}[H^*]$ is a vector subspace of the complete tensor algebra $\prod_{n=0}^{\infty} (H^*)^\otimes n := T[H^*]$. The latter also has the natural ring structure of juxtaposition, but $\text{Sym}[H^*]$ is not a subring of $T[H^*]$.

If we consider how these two multiplication rules relate to each other and view the ring structure of $\text{Sym}[H^*]$ intrinsically, we realize that the following map is in action.
Definition 2.1.1 For any $n \geq 0$, the $n$th symmetrization on $H$ is the map

$$S_n : (H^*)^\otimes n \to \text{Sym}^n H^*, \quad x \mapsto \frac{1}{|S_n|} \sum_{\sigma \in S_n} \sigma x.$$  

Also, the symmetrization on $H$ is the map

$$S := \prod_{n=0}^{\infty} S_n : \mathcal{T}[H^*] \to \text{Sym}[H^*].$$

We observe that the ring structure on $\text{Sym}[H^*]$ is given as follows. Let $\mu$ be the multiplication map in $\mathcal{T}[H^*]$, namely, the juxtaposition.

Proposition 2.1.2 The composition of the following maps

$$\text{Sym}[H^*] \otimes \text{Sym}[H^*] \to \mathcal{T}[H^*] \otimes \mathcal{T}[H^*] \overset{\mu}{\to} \mathcal{T}[H^*] \overset{\delta}{\to} \text{Sym}[H^*]$$

is the multiplication map of $\text{Sym}[H^*]$.

Remark 2.1.3 In view of the identification of $k[H^*]$ with $\text{Sym}[H^*]$, we regard $k[H^*]$ as the ring of symmetric multilinear forms.

The following functorial property is also well known.

Proposition 2.1.4 Let $\phi : H \to K$ be a linear map of finite dimensional vector spaces. Then we have the induced map of algebra with unity

$$\phi^* : \text{Sym}[K^*] \to \text{Sym}[H^*].$$

2.2 Braided Tensors

In this section, we introduce the notion of braided tensors in certain subcategories of $\textbf{Vect}_k$, that are also braided monoidal. We also define ring structures for them. Note that we work in the covariant setting. The dual setting will be considered when we specialize to the category of $G$-graded $G$-modules in Section 2.4.
We work with a subcategory $\mathbf{C}$ of $\mathbf{Vect}_k$, and fix a braided monoidal structure $(\mathbf{C}, \otimes, 1 = k, \Phi)$. Here $\otimes$ is the usual tensor product of vector spaces, but $\Phi$ may be different from the usual braided (actually, symmetric) monoidal structure of $\mathbf{Vect}_k$. We may simply use $\mathbf{C}$ understanding its braided monoidal structure.

We first generalize the notion of symmetric tensors on vector spaces. For any $H \in \mathbf{C}$, note that the braid group $B_n$ acts on $H^{\otimes n}$ canonically. Let $\mathbf{C}_{\text{fin}}$ be the full subcategory of finite dimensional objects.

**Definition 2.2.1** For each $H \in \mathbf{C}_{\text{fin}}$, set

$$\mathcal{Br}^n H := (H^{\otimes n})^{B_n},$$

the subspace of $B_n$-invariant $n$-tensors. Its elements are called the braided $n$-tensors on $H$. Also, set

$$\mathcal{Br}[H] := \prod_{n=0}^{\infty} \mathcal{Br}^n H \subset T[H].$$

Its elements are called the braided tensors on $H$.

To prove that braided tensors are functorial, we need the following lemma.

**Lemma 2.2.2** Let $\phi : H \to K$ be a morphism in $\mathbf{C}_{\text{fin}}$. Then for any $b \in B_n$, we have the following commutative diagram.

$$\begin{array}{ccc}
H^{\otimes n} & \xrightarrow{\phi^{\otimes n}} & K^{\otimes n} \\
\downarrow b & & \downarrow b \\
H^{\otimes n} & \xrightarrow{\phi^{\otimes n}} & K^{\otimes n}
\end{array}$$

**Proof** This follows from the functoriality of the braiding in $\mathbf{C}$. \hfill \blacksquare

**Remark 2.2.3** We regard that $\phi^{\otimes n} = id_k$ when $n = 0$.

**Proposition 2.2.4** $\mathcal{Br}^n(\cdot)$ and $\mathcal{Br}[\cdot]$ are functors $\mathbf{C}_{\text{fin}} \to \mathbf{Vect}_C$. 
Proof Let $\phi : H \to K$ be a morphism in $\mathfrak{C}_{\text{fin}}$. Consider the restriction of the morphism $\phi^\otimes_n : H^\otimes_n \to K^\otimes_n$ on $\mathfrak{Br}^n H$. Let $v \in \mathfrak{Br}^n H$. Then by Lemma 2.2.2, we have

$$b \cdot \phi^\otimes_n(v) = \phi^\otimes_n(b \cdot v) = \phi^\otimes_n(v).$$

showing that $\phi^\otimes_n(v) \in \mathfrak{Br}^n K$.

Set $\phi_* := \prod_{n=0}^{\infty} \phi^\otimes_n$. Then $\phi_*$ maps $\mathfrak{Br}[H]$ to $\mathfrak{Br}[K]$.

If $\psi$ is another morphism in $\mathfrak{C}_{\text{fin}}$, the equations $(\psi \circ \phi)^\otimes_n = \psi^\otimes_n \circ \phi^\otimes_n$ and $(\psi \circ \phi)_* = \psi_* \circ \phi_*$ are inherited from the ones before restricting to braided tensors.

We also observe that $id^\otimes_n = id$ and $id_* = id$. 

Note that $\mathfrak{Br}^n H$ and $\mathfrak{Br}[H]$ are vector spaces, but they need not be objects of $\mathfrak{C}$. We want them to be objects in $\mathfrak{C}$.

Definition 2.2.5 $\mathfrak{C}$ is regular if, for each $H \in \mathfrak{C}_{\text{fin}}$, $\mathfrak{Br}^n H$ and $\mathfrak{Br}[H]$ are objects of $\mathfrak{C}$, and their braidings are natural in the following sense: given another object $W$ of $\mathfrak{C}$, the braiding map

$$\Phi_{\mathfrak{Br}^n H,W} : \mathfrak{Br}^n H \otimes W \to W \otimes \mathfrak{Br}^n H$$

is given by the restriction of the braiding map

$$\Phi_{H^\otimes_n,W} : H^\otimes_n \otimes W \to W \otimes H^\otimes_n,$$

and the braiding map

$$\Phi_{\mathfrak{Br}[H],W} : \mathfrak{Br}[H] \otimes W \to W \otimes \mathfrak{Br}[H]$$

comes from $\Phi_{\mathfrak{Br}^n H,W}$ applied degree by degree. Similarly for the braiding maps $\Phi_{W,\mathfrak{Br}^n H}$ and $\Phi_{W,\mathfrak{Br}[H]}$.

Note that $\mathfrak{Br}[H]$ need not be a subalgebra of $T[H]$. Instead, we define a different multiplication on it. To this end, we generalize the notion of symmetrization in Definition 2.1.1 in the covariant setting.
Definition 2.2.6 A braidization in $\mathbf{C}$ is the following data.

- (nth braidization) For each $H \in \mathbf{C}_{\text{fin}}$ and each $n \geq 0$, a surjective linear map 
  
  $\mathcal{B}_n : H^{\otimes n} \to \mathbf{Br}^n H$

  that satisfies

  (i) $\mathcal{B}_n^2 = \mathcal{B}_n$,

  (ii) $\mathcal{B}_n(v) = \mathcal{B}_n(b \cdot v)$ for any $v \in H^{\otimes n}$ and $b \in B_n$.

These maps should satisfy the following two conditions.

- (Functoriality) For each morphism $\phi : H \to K$ in $\mathbf{C}_{\text{fin}}$, the following diagram is commutative for each $n$.

  $\begin{array}{ccc}
  H^{\otimes n} & \xrightarrow{\mathcal{B}_n} & \mathbf{Br}^n H \\
  \downarrow{\phi^{\otimes n}} & & \downarrow{\phi^{\otimes n}} \\
  K^{\otimes n} & \xrightarrow{\mathcal{B}_n} & \mathbf{Br}^n K
  \end{array}$

- (Associativity) For each $H \in \mathbf{C}_{\text{fin}}$, $v \in H^{\otimes n}$, $w \in H^{\otimes m}$ and $z \in H^{\otimes k}$,

  $\mathcal{B}_{n+m+k}(\mathcal{B}_{n+m}(v \otimes w) \otimes z) = \mathcal{B}_{n+m+k}(v \otimes \mathcal{B}_{m+k}(w \otimes z))$.

$\mathbf{C}$ is braidizable if $\mathbf{C}$ admits a braidization. After fixing a braidization, $\mathbf{C}$ is braidized. If $\mathbf{C}$ is braidized, for each $H \in \mathbf{C}_{\text{fin}}$, set the total braidization

$\mathcal{B} := \prod_{n=0}^{\infty} \mathcal{B}_n : \mathcal{T}[H] \to \mathbf{Br}[H]$.

Remark 2.2.7 Note that the surjectivity and condition (i) of $\mathcal{B}_n$ implies that it is the identity on $\mathbf{Br}^n H$.

Now we are ready to generalize Proposition 2.1.2 in the covariant setting.
Proposition 2.2.8  Suppose that $C$ is regular and braidized. Then for each $H \in C_{\text{fin}}$, the composition of the following maps

$$\circ : \mathcal{B}r[H] \otimes \mathcal{B}r[H] \hookrightarrow \mathcal{T}[H] \otimes \mathcal{T}[H] \xrightarrow{\mu} \mathcal{T}[H] \xrightarrow{\mathcal{B}} \mathcal{B}r[H]$$

gives $\mathcal{B}r[H]$ the structure of a braided-commutative algebra with unity. Moreover, for any morphism $\phi : H \to K$ in $C_{\text{fin}}$, the induced linear map

$$\phi_* : \mathcal{B}r[H] \to \mathcal{B}r[K]$$

is a morphism of algebra with unity.

Proof  Braided-commutativity follows from the following equations. Let $\sum_{m=0}^{\infty} v_m$, $\sum_{n=0}^{\infty} w_n \in \mathcal{B}r[H]$ with $v_m \in \mathcal{B}r^m H$ and $w_n \in \mathcal{B}r^n H$. Using the regularity of $C$, we have

$$\mathcal{B}(\mu(\sum_{m=0}^{\infty} v_m \otimes \sum_{n=0}^{\infty} w_n)) = \mathcal{B}(\sum_{m,n=0}^{\infty} v_m \otimes w_n),$$

$$= \mathcal{B}(\sum_{m,n=0}^{\infty} \Phi(v_m \otimes w_n)) \text{ by (ii) of Definition 2.2.6},$$

$$= \mathcal{B}(\mu \Phi(\sum_{m=0}^{\infty} v_m \otimes \sum_{n=0}^{\infty} w_n)).$$

Associativity of $\circ$ follows from that of the braidization. Unity exists by Remark 2.2.7.

We have the following commutative diagram by the functoriality of the braidization in $C$.

$$\begin{array}{ccc}
\mathcal{B}r^n H \otimes \mathcal{B}r^m H & \xrightarrow{\mu} & H^{\otimes (n+m)} \\
\downarrow_{\phi^{\otimes (n+m)}} & & \downarrow_{\phi^{\otimes (n+m)}} \\
\mathcal{B}r^n K \otimes \mathcal{B}r^m K & \xrightarrow{\mu} & K^{\otimes (n+m)} \\
\end{array}$$

Recall that $\mu$ in the left box denotes the juxtaposition. This diagram shows that $\phi_*$ preserves the multiplication. Unity is preserved since $\phi^{\otimes n} = id_k$ on $\mathcal{B}r^n H = \mathcal{B}r^n K = k$ when $n = 0$. \qed
2.3 A Groupoid Structure on \( G^n \)

Before specializing to the case of \( G \)-graded \( G \)-modules, we study the \( B_n \)-action on the \( G^n \)-degrees on \( H^\otimes n \) for any \( G \)-graded \( G \)-module \( H \).

\( G^n \rtimes S_n \) acts on \( G^n \) by taking the componentwise adjoint action of \( G^n \) first, and then switching positions. This induces a natural \( B_n \) action on \( G^n \) in the following way. Let \( b_1 \) be one of the standard generators of \( B_n \) that braids the first two strands. Then

\[
b_1(\gamma_1, \ldots, \gamma_n) := (\gamma_1 \gamma_2 \gamma_1^{-1}, \gamma_1, \gamma_3, \ldots, \gamma_n),
\]

and similarly for the other generators. This shows that given any \( b \in B_n \) and \( \gamma \in G^n \), we have an element \( b_\gamma \in G^n \rtimes S_n \) that really acts on \( \gamma \). For example, we see that \( b_{1, \gamma} = (e, \gamma_1, e, \ldots, e) \times (1, 2) \). Note that this does not define a homomorphism from \( B_n \) to \( G^n \rtimes S_n \) since \( b_\gamma \) depends on \( \gamma \). We have the following structure instead.

**Definition 2.3.1** The \( B_n \)-groupoid \( G^n \) is defined in the following way:

\[
\text{Obj}(G^n) = G^n,
\]

\[
\text{Hom}(\gamma_1, \gamma_2) = \{b_\gamma \in G^n \rtimes S_n | b_\gamma = \gamma_2 \text{ for some } b \in B_n\}.
\]

**Remark 2.3.2** A related structure is the action groupoid \( B_n \rtimes G^n \). See [21] for its definition. This is different from \( G^n \) in that different elements of \( B_n \) can give rise to the same arrow in \( G^n \). In particular, \( G^n \) is a finite groupoid, whereas \( B_n \rtimes G^n \) is not.

**Remark 2.3.3** \( \text{Obj}(G^n) \) has the natural \( G \)-grading that is defined as \( \prod \gamma_i \) for \( \gamma \in \text{Obj}(G^n) \). Since it is invariant under the action of \( B_n \), each component of \( G^n \) has the constant \( G \)-degree.

**Remark 2.3.4** Consider the diagonal adjoint \( G \)-action on \( \text{Obj}(G^n) \). We observe that this commutes with the action of \( B_n \). This implies that \( G \) acts on the set of components of \( G^n \). If a component \( C \) has the \( G \)-degree \( \gamma \), the \( G \)-degree of \( g \cdot C \) is equal to \( g \gamma g^{-1} \).
We want to introduce the concept of reflection map on $B_n$, that will be useful when we consider braided tensors.

**Definition 2.3.5** The reflection $r$ on $B_n$ is an antihomomorphism that sends $b_i$ to $b_{n-i}$.

**Remark 2.3.6** This definition makes sense since the defining relations on the generators of $B_n$ are also satisfied among the images of them. $r$ being an antihomomorphism does not matter because of the form of the relations.

We want to see how this map works in the categories $G^n$. Let $\gamma := (\gamma_1, \ldots, \gamma_n) \in G^n$. Let

$$r : G^n \to G^n, \quad (\gamma_1, \ldots, \gamma_n) \mapsto (\gamma_n^{-1}, \ldots, \gamma_1^{-1})$$

be a set map. Then we have the following commutative diagram

$$
\begin{array}{ccc}
(\gamma_1, \ldots, \gamma_n) & \xrightarrow{r} & (\gamma_n^{-1}, \ldots, \gamma_1^{-1}) \\
\downarrow b_1 & & \uparrow b_{n-1} \\
(\gamma_1 \gamma_2 \gamma_1^{-1}, \gamma_1, \gamma_3, \ldots, \gamma_n) & \xrightarrow{r} & (\gamma_n^{-1}, \ldots, \gamma_1^{-1}, \gamma_1 \gamma_2^{-1} \gamma_1^{-1})
\end{array}
$$

and similarly for the other generators of $B_n$. This allows us to make the following definition.

**Definition 2.3.7** The reflection map on $G^n$ is the functor $r : (G^n)^{\circ} \to G^n$ that maps $\gamma$ to $r(\gamma)$ in the sense above, and sends $b_{i,\gamma}$ to $b_{n-i, r(b_i, \gamma)}$.

**Remark 2.3.8** Given a morphism in $G^n$, note that we should choose an element in $B_n$ that acts as the given one to know the image of it under $r$. One sees that this does not depend on the choice of the representative by looking at the diagram above. Namely, suppose we return to the original $\gamma$ after several application of $b_i$’s and its inverses on the left side of the above diagram. Then the same thing should happen to the right side since the right side is the exact “reflection” of the left side.

This observation also implies the following proposition.

**Proposition 2.3.9** $r$ is fully faithful.
2.4 \textit{G-braided Spaces}

In this section, we consider the category of $G$-graded $G$-modules, and interpret its finite dimensional objects as braided-commutative spaces.

\textbf{Definition 2.4.1} A $G$-graded $G$-module $(H, \rho)$ is a vector space $H$ with the decomposition

$$H = \bigoplus_{m \in G} H_m$$

and a $G$-action $\rho$ respecting the $G$-grading, in the sense that

$$\rho(\gamma)H_m = H_{\gamma m \gamma^{-1}}$$

for any $\gamma, m \in G$.

\textbf{Remark 2.4.2} We may suppress $\rho$ for the $G$-action and use $\cdot$ instead, when no confusion should arise.

\textbf{Remark 2.4.3} $G$-graded $G$-modules form the category $D(k[G])$-$\textbf{Mod}$, the category of $D(k[G])$-modules $[14, 19]$. The latter is the Drinfel’d double of the group algebra $k[G]$. It is a quasitriangular Hopf algebra, hence $D(k[G])$-$\textbf{Mod}$ is a braided monoidal category. The braiding is given by

$$\Phi : V_g \otimes W_h \to W_{ghg^{-1}} \otimes V_g, \quad w \otimes v \mapsto g \cdot v \otimes w$$

and extending linearly. Their morphisms are $G$-equivariant linear maps that preserve the $G$-gradings. See [19] for details.

\textbf{Remark 2.4.4} A (linear) representation of a groupoid is a functor from the groupoid to the category of $k$-vector spaces. Given a $G$-graded $G$-module $H$, $H^{\otimes n}$ gives rise to a $G^n$-representation. For $\gamma := (\gamma_1, \ldots, \gamma_n)$, let $H_\gamma := H_{\gamma_1} \otimes \cdots \otimes H_{\gamma_n}$. Then we have $H^{\otimes n} = \bigoplus H_\gamma$. The representation assigns $\gamma$ to $H_\gamma$ and $b_\gamma$ to $b : H_\gamma \to H_{b \gamma}$. This is well defined since if $b_\gamma = b'_\gamma$, the maps $b$ and $b' : H_\gamma \to H_{b \gamma} = H_{b' \gamma}$ are the same.
Remark 2.4.5 When $G$ is trivial, $D(k[G])$-$\text{Mod}$ is nothing but the symmetric monoidal category of vector spaces. Hence its braided multilinear forms on finite dimensional objects are the same as the symmetric multilinear forms.

Note that $\mathcal{Br}^n H$ is graded by components of $G^n$. That is, let $C$ be a component of $G^n$. Then $H_C := \bigoplus_{\gamma \in C} H_\gamma$ is an invariant subspace of the map

$$b : H^{\otimes n} \to H^{\otimes n},$$

for any $b \in B_n$. Let $C(G^n)$ denote the set of components in $G^n$ and define $\mathcal{Br}^n_C H := \mathcal{Br}^n H \cap H_C$. Then we have

$$\mathcal{Br}^n H = \bigoplus_{C \in C(G^n)} \mathcal{Br}^n_C H,$$

where $B_n$ acts on $\mathcal{Br}^n_C H$ trivially. This follows from the equation $b \cdot v = v$ for any $v \in \mathcal{Br}^n H$ and $b \in B_n$. We give a characterization of vectors in $\mathcal{Br}^n_C H$.

**Proposition 2.4.6** Let $C \in C(G^n)$ and $v \in H_C$. Write

$$v = \sum_{\gamma \in C} v_\gamma,$$

where $v_\gamma$ denotes the $G^n$-homogeneous component of $v$. Then $v \in \mathcal{Br}^n_C H$ if and only if

$$b \cdot v_\gamma = v_{b \cdot \gamma}$$

for any $b \in B_n$.

**Proof** Suppose $v \in \mathcal{Br}^n_C H$. Then the equation $b \cdot v = v$ implies

$$\sum_{\gamma \in C} b \cdot v_\gamma = \sum_{\gamma \in C} v_\gamma.$$

Comparing the homogeneous components yields the desired equation. Conversely, suppose $b \cdot v_\gamma = v_{b \cdot \gamma}$ for $b \in B_n$. We compute

$$b \cdot v = \sum_{\gamma \in C} b \cdot v_\gamma = \sum_{\gamma \in C} v_{b \cdot \gamma} = \sum_{\gamma \in C} v_\gamma = v,$$

since the map $b : C \to C$, defined by acting on elements in $C$, is bijective. \qed
Now we are ready to prove that \( D(k[G])\text{-Mod} \) is regular and braidizable.

**Proposition 2.4.7** \( D(k[G])\text{-Mod} \) is regular.

**Proof** Let \( H \) be a finite dimensional \( G \)-graded \( G \)-module. To see the \( G \)-graded \( G \)-module structure of \( \text{Br}^n H \), note that \( \text{Br}^n C \) has the natural \( G \)-degree equal to that of \( C \) as defined in Remark 2.3.3. This defines a \( G \)-grading on \( \text{Br}^n H \).

To prove the \( G \)-module structure, consider the diagonal \( G \)-action on \( H \otimes G \). We want to show that \( \text{Br}^n H \) is invariant under this action. Suppose that

\[
v = \sum_{\gamma \in C} v_\gamma \in \text{Br}^n C H
\]

for some \( C \in C(G^n) \). Then we have

\[
w := g \cdot v = \sum_{\gamma \in C} g \cdot v_\gamma,
\]

and set \( w_{g \cdot \gamma} := g \cdot v_\gamma \), so that

\[
w = \sum_{\gamma \in g \cdot C} w_\gamma.
\]

We use Proposition 2.4.6 to show that \( w \in \text{Br}^n_{g \cdot C} H \). Recall from Remark 2.3.4 that the \( B_n \) and \( G \)-actions on \( \text{Obj}(G^n) \) commutes with each other. Moreover, we observe that the two actions on \( H \otimes G \) also commute with each other. Hence for any \( b \in B_n \),

\[
b \cdot w_\gamma = b \cdot (g \cdot v_{g^{-1} \cdot \gamma}) = g \cdot (b \cdot v_{g^{-1} \cdot \gamma})
\]

\[
= g \cdot v_{b \cdot (g^{-1} \cdot \gamma)} = g \cdot v_{g^{-1} \cdot (b \cdot \gamma)} = w_{b \cdot \gamma}.
\]

The compatibility of its \( G \)-grading and \( G \)-action follows by observing that

\[
\text{deg}_G(g \cdot C) = g \cdot \text{deg}_G C \cdot g^{-1}
\]

where \( \text{deg}_G \) denotes the \( G \)-degree.

Next, we prove that \( \text{Br}[H] \) is a \( G \)-graded \( G \)-module. Using the \( G \)-grading on \( \text{Br}^n H \), write

\[
\text{Br}^n H = \bigoplus_{g \in G} (\text{Br}^n H)_g
\]
where the subscript denotes the $G$-degree. Using the finiteness of $G$, we have

$$\mathcal{B}r[H] = \prod_{n=0}^{\infty} \mathcal{B}r^n H = \prod_{n=0}^{\infty} \bigoplus_{g \in G} (\mathcal{B}r^n H)_g$$

$$= \bigoplus_{g \in G} \prod_{n=0}^{\infty} (\mathcal{B}r^n H)_g = \bigoplus_{g \in G} (\mathcal{B}r[H])_g,$$

by setting that $(\mathcal{B}r[H])_g := \prod_{n=0}^{\infty} (\mathcal{B}r^n H)_g$. $G$-action on $\mathcal{B}r[H]$ is defined term by term diagonally. The $G$-invariance of $\mathcal{B}r^n H$ guarantees that of $\mathcal{B}r[H]$. The compatibility of its $G$-grading and $G$-action also follows from that of $\mathcal{B}r^n H$.

The naturality of their braidings follows from the fact that the $G$-grading on $\mathcal{B}r^n H$ is inherited from $H^{\otimes n}$.

**Proposition 2.4.8** $D(k[G])$-Mod is braidizable.

**Proof** (*nth braidization*) Let $H$ be a finite dimensional $G$-graded $G$-module. We use the $G^n$-representation structure of Remark 2.4.4 on $H^{\otimes n}$. For $\gamma \in \text{Obj}(G^n)$, let $A_\gamma$ be the set of all the morphisms in the component of $\gamma$ with the source $\gamma$. That is,

$$A_\gamma := \prod_{\gamma' = b\gamma \text{ for some } b \in B_n} \text{Hom}_{G^n}(\gamma, \gamma').$$

For $v \in H_\gamma$, define

$$\mathcal{B}_n(v) := \frac{1}{|A_\gamma|} \sum_{b_\gamma \in A_\gamma} b_\gamma \cdot v,$$

and extend linearly.

We show that

$$b_i \cdot \mathcal{B}_n(v) = \mathcal{B}_n(v)$$

for any generator $b_i$ of $B_n$. It is still enough to assume that $v \in H_\gamma$. Write

$$b_i \cdot \mathcal{B}_n(v) = \frac{1}{|A_\gamma|} \sum_{b_\gamma \in A_\gamma} b_i \cdot (b_\gamma \cdot v)$$
and observe that the map

\[ b_i : A_\gamma \to A_\gamma \]

defined by postcomposing the element \( b_{i,\gamma'} \in G^n \rtimes S_n \) is a bijection. Here \( \gamma' \) denotes the target of each arrows in \( A_\gamma \). Hence

\[ b_i : \mathcal{B}_n(v) = \frac{1}{|A_\gamma|} \sum_{b'_{\gamma} \in A_\gamma} b'_{\gamma} \cdot v = \mathcal{B}_n(v), \]

where \( b'_{\gamma} \) denotes each element in the image of the map \( b_i \) above.

To show the surjectivity and condition (i) of Definition 2.2.6, it is enough to show that

\[ \mathcal{B}_n(v) = v \]

for any \( v \in \mathcal{B}r^a H \). Considering the \( C(\mathcal{G})^n \)-grading of \( \mathcal{B}r^a H \), it is enough to consider that \( v \in \mathcal{B}r^a C \) for \( C \in C(\mathcal{G}) \) and put

\[ v = \sum_{\gamma \in C} v_{\gamma}. \]

Observe that \( |A_\gamma| \) is constant for \( \gamma \in C \). Write

\[ n_C := |A_\gamma| \]

for any choice of \( \gamma \in C \). Also, \( |\text{Hom}_{\mathcal{G}}(\gamma, \gamma')| \) is constant for any two objects \( \gamma \) and \( \gamma' \) in \( C \). Define

\[ m_C := |\text{Hom}_{\mathcal{G}}(\gamma, \gamma')| \]

for any choice of two objects \( \gamma \) and \( \gamma' \) in \( C \). Then we have

\[ n_C = |C| \cdot m_C. \tag{2.1} \]

By the definition of \( \mathcal{B}_n \), we have

\[ \mathcal{B}_n(v) = \frac{1}{n_C} \sum_{\gamma \in C} b_\gamma \cdot v_\gamma. \]

Proposition 2.4.6 implies

\[ \sum_{b_\gamma \in A_\gamma} b_\gamma \cdot v_\gamma = m_C \cdot v \]
for any fixed $\gamma \in C$. Thus
\[
\mathcal{B}_n(v) = \frac{1}{n_C} \sum_{\gamma \in C} m_C \cdot v = \frac{1}{n_C} \cdot |C| \cdot m_C \cdot v = v,
\]
by Equation (2.1).

To prove condition (ii), it is enough to consider $v \in H_{\gamma}$. Then $b \cdot v \in H_{b \gamma}$. Hence
\[
\mathcal{B}_n(b \cdot v) = \frac{1}{|A_{b \gamma}|} \sum_{b_{b \gamma} \in A_{b \gamma}} b_{b \gamma} \cdot (b \cdot v).
\]
Since the map
\[
b : A_{b \gamma} \to A_{\gamma}
\]
defined by precomposing $b_{\gamma}$ is a bijection, and using the fact that $|A_{b \gamma}| = |A_{\gamma}|$, we have
\[
\mathcal{B}_n(b \cdot v) = \frac{1}{|A_{\gamma}|} \sum_{b'_{\gamma} \in A_{\gamma}} b'_{\gamma} \cdot v,
\]
where $b'_{\gamma}$ denotes each element in the image of the map $b$ above.

*(Functoriality)* It is enough to consider $G^n$-homogeneous vectors. Hence, suppose $v \in H_{\gamma}$. Then by the definition of $n$th braidization, we have
\[
\phi^{\otimes n}(\mathcal{B}_n(v)) = \frac{1}{|A_{\gamma}|} \sum_{b_{\gamma} \in A_{\gamma}} \phi^{\otimes n}(b_{\gamma} \cdot v).
\]
For each $b_{\gamma} \in A_{\gamma}$, choose $b \in B_n$ such that $b$ acts as $b_{\gamma}$ on $H_{\gamma}$. Let $A'$ be the finite subset of $B_n$ of such $b$’s. Then we have
\[
\phi^{\otimes n}(\mathcal{B}_n(v)) = \frac{1}{|A_{\gamma}|} \sum_{b \in A'} \phi^{\otimes n}(b \cdot v) = \frac{1}{|A_{\gamma}|} \sum_{b \in A'} b \cdot \phi^{\otimes n}(v)
\]
by Lemma 2.2.2. Since $\phi^{\otimes n}$ preserves the $G^n$-grading, we have
\[
\phi^{\otimes n}(\mathcal{B}_n(v)) = \frac{1}{|A_{\gamma}|} \sum_{b_{\gamma} \in A_{\gamma}} b_{\gamma} \cdot \phi^{\otimes n}(v) = \mathcal{B}_n(\phi^{\otimes n}(v)).
\]

*(Associativity)* Suppose $v, w$ and $z$ in $T[H]$ are homogeneous of degree $n, m$ and $k$, respectively. By linearity, it is enough to assume that they are also monomials of homogeneous $G^n, G^m$ and $G^k$-degrees, respectively. We observe that
\[
\mathcal{B}_{n+m+k}(\mathcal{B}_{n+m}(v \otimes w) \otimes z) = \mathcal{B}_{n+m+k}(v \otimes w \otimes z)
\]
\[
= \mathcal{B}_{n+m+k}(v \otimes \mathcal{B}_{m+k}(w \otimes z))
\]
since the $G^{n+m+k}$-orbits of monomials of $\mathcal{B}_{n+m}(v \otimes w) \otimes z$ and $v \otimes \mathcal{B}_{m+k}(w \otimes z)$ coincide
with that of $v \otimes w \otimes z$, with correct averaging constants. □

**Remark 2.4.9** We assume that $D(k[G])$-$\text{Mod}$ is braided in this way hereafter.

**Remark 2.4.10** Note that we recover the usual symmetrization when $G$ is trivial.

The following proposition enables us to use the results of the previous section in
the dual setting.

**Proposition 2.4.11** In $D(k[G])$-$\text{Mod}$, taking the dual space is a contravariant functor on its full subcategory of finite dimensional $G$-graded $G$-modules.

**Proof** Given a finite dimensional $G$-graded $G$-module $H$, its induced $G$-action on $H^*$ is given by the following formula.

$$(\gamma \cdot x)(v) = x(\gamma^{-1} \cdot v). \quad (2.2)$$

One consistent choice of $G$-grading on $H^*$ can be obtained by requiring that the trace map

$$\text{tr} : H \otimes H^* \to \mathbb{C}, \ v \otimes x \mapsto x(v)$$

are $G$-degree preserving. Namely, the $G$-grading on $H^*$ can be given as

$$(H^*)_m = (H_{m-1})^*.$$  

We observe that Equation (2.2) implies that

$$\gamma \cdot (H^*)_m = (H^*)_{m\gamma^{-1}}.$$  

Let $\phi : H \to K$ be a morphism between finite dimensional objects. Then $\phi^* : K^* \to H^*$ is a linear map. To prove that $\phi^*$ preserves the $G$-degree, suppose that $x \in (K^*)_m, x \neq 0$. If $\phi^* x = 0$ for any $x$, then $\phi^*$ preserves degree $m$. If not, there exist a homogeneous vector $v \in H$ such that $\phi^* x(v) \neq 0$. But $\phi^* x(v) = x(\phi(v))$ and $\phi$ is degree-preserving, it follows that $\deg_G v = m^{-1}$. We conclude that $\phi^* x$ is
homogeneous of $G$-degree $m$. To show the $G$-equivariance, let $x \in K^*$ and $v \in H$. Using the $G$-equivariance of $\phi$, we compute

\[
\phi^*(g \cdot x)(v) = (g \cdot x)(\phi(v)) = x(g^{-1} \cdot \phi(v))
\]

\[
= x(\phi(g^{-1} \cdot v)) = \phi^* x(g^{-1} \cdot v) = g \cdot (\phi^* x)(v).
\]

The conditions on composition and identity follows from that of linear maps. □

**Remark 2.4.12** We fix this $G$-graded $G$-module structure for dual spaces hereafter.

The following theorem summarizes what we have been proving.

**Theorem 2.4.13** Let $H$ be a finite dimensional $G$-graded $G$-module, and $\circ$ be defined as in Proposition 2.2.8. Then $(\mathcal{B}r[H^*], \circ)$ is a braided-commutative algebra with unity. Moreover, for each morphism $\phi : H \rightarrow K$ of finite dimensional $G$-graded $G$-modules, the induced linear map

\[
\phi^* : \mathcal{B}r[K^*] \rightarrow \mathcal{B}r[H^*]
\]

is a morphism of algebra with unity.

**Proof** This follows from Propositions 2.2.8, 2.4.7, 2.4.8, and 2.4.11. □

**Remark 2.4.14** We call the elements of $\mathcal{B}r^n H^*$ and $\mathcal{B}r[H^*]$ the braided $n$-linear forms and braided multilinear forms, respectively.

**Remark 2.4.15** We adopt the viewpoint that any finite dimensional $G$-graded $G$-module is a braided-commutative space with its ring of braided multilinear forms.

Lastly, we compare the $B_n$ actions on $H^\otimes n$ and $(H^*)^\otimes n$. We use the notation of Definition 2.3.5 for the following proposition.

**Proposition 2.4.16** For any $x \in (H^*)^\otimes n$ and $v \in H^\otimes n$, the following equations hold.

\[
(b \cdot x)(v) = x(r(b) \cdot v).
\]
Recall that the trace map is given by

$$\text{tr} : (H^*)^n \otimes H^\otimes n \rightarrow \mathbb{C},$$

$$x_1 \otimes \cdots \otimes x_n \otimes v_1 \otimes \cdots \otimes v_n \mapsto x_n(v_1) \cdot x_{n-1}(v_2) \cdots x_1(v_n)$$

so that there is no need of braiding.

To prove the equation, it is enough to prove the following equations

$$(b_i \cdot x)(v) = x(b_{n-i} \cdot v),$$

$$(b_i^{-1} \cdot x)(v) = x(b_{n-i}^{-1} \cdot v).$$

By linearity, we assume that $x$ is a $G^n$-homogeneous monomial. For simplicity, we only consider $i = 1$. Suppose that $x \in H^*_1$ in the notation of Remark 2.4.4, and write

$$x = x_{\gamma_1} \otimes \cdots \otimes x_{\gamma_n}.$$  

Then we have

$$b_1 \cdot x = \gamma_1 \cdot x_{\gamma_2} \otimes x_{\gamma_1} \otimes x_{\gamma_3} \otimes \cdots \otimes x_{\gamma_n}.$$

It is enough to consider $G^n$-homogeneous $v$ of matching $G^n$-degrees, otherwise both sides are equal to zero. Hence, write

$$v = v_{\gamma_{n-1}}^{-1} \otimes \cdots \otimes v_{\gamma_3}^{-1} \otimes v_{\gamma_1}^{-1} \otimes v_{\gamma_1 \gamma_2 \gamma_1^{-1} \gamma_1^{-1}}.$$

Then we have

$$b_{n-1} \cdot v = v_{\gamma_{n-1}}^{-1} \otimes \cdots \otimes v_{\gamma_3}^{-1} \otimes \gamma_1^{-1} \cdot v_{\gamma_1 \gamma_2 \gamma_1^{-1} \gamma_1^{-1}} \otimes v_{\gamma_1}^{-1}.$$  

We compute

$$(b_1 \cdot x)(v) = x_{\gamma_n}(v_{\gamma_{n-1}}^{-1}) \cdots x_{\gamma_3}(v_{\gamma_3}^{-1}) \cdot x_{\gamma_1}(v_{\gamma_1}^{-1}) \cdot (\gamma_1 \cdot x_{\gamma_2})(v_{\gamma_1 \gamma_2 \gamma_1^{-1} \gamma_1^{-1}})$$

$$= x_{\gamma_n}(v_{\gamma_{n-1}}^{-1}) \cdots x_{\gamma_3}(v_{\gamma_3}^{-1}) \cdot x_{\gamma_1}(v_{\gamma_1}^{-1}) \cdot x_{\gamma_2}(\gamma_1^{-1} \cdot v_{\gamma_1 \gamma_2 \gamma_1^{-1} \gamma_1^{-1}})$$

$$= x(b_{n-1} \cdot v)$$

by Equation (2.2). The second equation can be proved in a similar fashion.
This leads to the following characterization of braided multilinear forms on $H$.

**Corollary 2.4.17** For any finite dimensional $G$-graded $G$-module $H$, we have

$$\mathcal{Br}^n H^* = \{ x \in (H^*)^\otimes_n | x(v) = x(b \cdot v) \text{ for any } b \in B_n \text{ and } v \in H^\otimes_n \},$$

the subspace of $B_n$-invariant multilinear forms.

**Proof**

$$x \in \mathcal{Br}^n H^*$$

$$\iff x \in [(H^*)^\otimes_n]^{B_n}$$

$$\iff x = b \cdot x, \ \forall b \in B_n$$

$$\iff x(v) = (b_i \cdot x)(v) = x(b_{n-i} \cdot v), \ \{b_i\} \text{ standard generators of } B_n,$$

$$\forall v \in H^\otimes_n$$

$$\iff x(v) = x(b \cdot v), \ \forall b \in B_n, \ \forall v \in H^\otimes_n.$$  

The previous proposition also allows us to relate the $n$th braidization on $\mathcal{Br}^n H$ and $\mathcal{Br}^n H^*$.

**Corollary 2.4.18** For any $x \in (H^*)^\otimes_n$ and $v \in H^\otimes_n$, we have

$$[\mathcal{B}_n(x)](v) = x(\mathcal{B}_n(v)).$$

**Proof** We use the notation of Definition 2.3.7.

Let $x \in H^*_\gamma$. Suppose that $C$ is the component of $\gamma \in G^n$ and choose $\gamma' \in C$. We assume that $v \in H_{r(\gamma')}$. Otherwise both sides are zero. Then we have

$$[\mathcal{B}_n(x)](v) = \frac{1}{|A_\gamma|} \left[ \sum_{b_\gamma \in A_\gamma} b_\gamma \cdot x \right](v) = \frac{1}{|A_\gamma|} \sum_{b_\gamma \in \text{Hom}(\gamma, \gamma')} [(b_\gamma \cdot x)(v)].$$
Choose $b \in B_n$ for each $b_{\gamma} \in \text{Hom}_{G^0}(\gamma, \gamma')$ so that $b$ acts as $b_{\gamma}$ on $H_{\gamma}^*$. Let $A'$ be the subset of $B_n$ of such $b$'s. Using Proposition 2.4.16 we have

$\left[\mathcal{B}_n(x)\right](v) = \frac{1}{|A_{\gamma}|} \sum_{b \in A'} [(b \cdot x)(v)]$

$= \frac{1}{|A_{\gamma}|} \sum_{b \in A'} x(r(b) \cdot v) = x \left( \frac{1}{|A_{\gamma}|} \sum_{b \in A'} r(b) \cdot v \right)$.

By Proposition 2.3.9, we see that the action of $\{r(b)\mid b \in A'\}$ is the same as that of $\text{Hom}_{G^0}(r(\gamma'), r(\gamma))$. Since $x \in H_{\gamma}^*$ and $|A_{\gamma}| = |A_{r(\gamma')}|$ by Proposition 2.3.9, we conclude

$\left[\mathcal{B}_n(x)\right](v) = x(\mathcal{B}_n(v)).$

\[\Box\]

### 2.5 Distinguished Subspaces

For any finite dimensional $G$-graded $G$-module $H$, we have two distinguished subspaces $H_e$ and $H^G$. $H_e$ denotes the $G$-homogeneous subspace of $G$-degree $e$. $H^G$ is the subspace of $G$-invariants.

Let us first consider $H_e$. Since $H_e$ is a $G$-graded $G$-module concentrated at $G$-degree $e$, we can talk about its braided multilinear forms. But in a similar way as Remark 2.4.5, notice that the braided multilinear forms are the same as symmetric multilinear forms in this case. Remark 2.4.15 shows that this identification is as rings. Let $i_e : H_e \hookrightarrow H$ be the inclusion map. Note that this is a morphism of $G$-graded $G$-modules. Applying Theorem 2.4.13, we have the following proposition.

**Proposition 2.5.1** We have the following induced morphism of algebras with unity.

$(i_e)^* : \text{Br}[H^*] \rightarrow \text{Br}[H_e^*] = \text{Sym}[H_e^*]$

Next, we consider the subspace of $G$-invariants, $H^G$. Let $\overline{g} \in \overline{G}$, the conjugacy class of $g$. Since $\bigoplus_{h \in g} H_h$ is an invariant space for any $g \in G$, we see that $H^G$ has a $\overline{G}$-grading. Let $H^G_{\overline{g}}$ denote the homogeneous part of $\overline{G}$-degree $\overline{g}$. 
In particular, $H^G$ is not a $G$-graded $G$-module since $H_h$ is not an invariant subspace of the map $g : H \to H$ in general. Hence we cannot talk about braided multilinear forms on $H^G$, but we can consider the restrictions of those of $H$ on $H^G$.

**Proposition 2.5.2** Let $i^G : H^G \hookrightarrow H$ be the inclusion. Then we have the linear map

$$(i^G)^* : \mathcal{Br}[H^*] \to \text{Sym}[[H^G]^*], \quad \sum_{n=0}^{\infty} x_n \mapsto \sum_{n=0}^{\infty} x_n \mathcal{B}r_{H^G}^n,$$

where $x_n \in \mathcal{Br}^n H^*$.

**Proof** Let $x \in \mathcal{Br}^n H^*$ and $v \in (H^G)^{\otimes n} \subset H^{\otimes n}$. Let $s : B_n \to S_n, b_i \mapsto (i, i + 1)$ be the standard surjective homomorphism. Then for any $b \in B_n$, we have $b \cdot v = s(b) \cdot v$. To see this, suppose $v = v_1 \otimes \cdots \otimes v_n$ with $v_i \in H^G$. Suppose further that $v_1 = \sum_{g \in G} v_g$, where $v_g \in H_g$. Note that each $v_g$ need not be in $H^G$. We consider $b_1 \cdot v$. The other generators are similar. We have

$$b_1 \cdot v = b_1 \cdot \left[ \left( \sum_{g \in G} v_g \right) \otimes v_2 \otimes \cdots \otimes v_n \right] = \left( \sum_{g \in G} g \cdot v_2 \otimes v_g \right) \otimes v_3 \otimes \cdots \otimes v_n = \sum_{g \in G} v_2 \otimes v_g \otimes v_3 \otimes \cdots \otimes v_n = v_2 \otimes \left( \sum_{g \in G} v_g \right) \otimes v_3 \otimes \cdots \otimes v_n = v_2 \otimes v_1 \otimes v_3 \otimes \cdots \otimes v_n = v_2 \otimes v_1 \otimes v_3 \otimes \cdots \otimes v_n.$$

Given $\sigma \in S_n$ choose $b \in B_n$ such that $s(b) = \sigma$. Using Corollary 2.4.17, we have

$$x(\sigma \cdot v) = x(s(b) \cdot v) = x(b \cdot v) = x(v),$$

showing that $(i^G)^*(x) \in \text{Sym}^n(H^G)^*$.

**Remark 2.5.3** $(i^G)^*$ need not preserve the multiplication since $\mathcal{B}_n$ on $(H^*)^{\otimes n}$ and $S_n$ on $[(H^G)^*]^{\otimes n}$ need not be compatible.

We mention the following functorial property.
Proposition 2.5.4 Suppose $\phi : H \to K$ is a morphism of finite dimensional $G$-graded $G$-modules. Then we have the following morphisms of algebras with unity

$$\phi^* : \text{Sym}[^*\!_K] \to \text{Sym}[^*\!_H], \quad \phi^* : \text{Sym}[^*\!_{(K^G)^*}] \to \text{Sym}[^*\!_{(H^G)^*}].$$

Proof Since $\phi$ preserves the $G$-grading, it maps $H_e$ into $K_e$. Note that this restriction is also a morphism of $G$-graded $G$-modules. Hence the existence of the first map follows from Theorem 2.4.13.

Since $\phi$ is $G$-equivariant, it maps $H^G$ into $K^G$. Applying Proposition 2.1.4, we have the second map.

$\blacksquare$
3. G-Frobenius Manifolds

3.1 Ordinary Frobenius Manifolds

We review the concept of formal Frobenius manifold in this section. See [16], [20] for details.

Let $H$ be a finite dimensional vector space, and fix a basis $\{\partial_a\}$. Let $\{x^a\}$ be its dual basis. Recall from Section 2.1 that we have the identification $\text{Sym}[H^*] = k[H^*]$, which we interpret as functions on $H$. We identify the vectors in $H$ with differential operators on $\text{Sym}[H^*]$. Then we can regard $\mathfrak{X}H := \text{Sym}[H^*] \otimes H$ to be global vector fields on $H$. Suppose also that we have a symmetric nondegenerate bilinear form $g$ on $H$.

Given a function $Y \in \text{Sym}[H^*]$, we define a multiplication $\circ_Y$ on $\mathfrak{X}H$ in the following way.

$$\partial_a \circ_Y \partial_b := \partial_a \partial_b \partial_k Y_{g}^{kl} \partial_l,$$

and extending to $\mathfrak{X}H$ using the symmetric monoidal structure of vector spaces and commutative ring structure of $\text{Sym}[H^*]$.

Since the order of differentiation does not matter, we see that $\circ_Y$ is commutative. But $\circ_Y$ need not be associative. In fact, its associativity is equivalent to the following system of partial differential equations known as WDVV-equations. We use the notation $Y_a := \partial_a Y$, etc.

$$Y_{abk} g^{kl} Y_{lcd} = Y_{bck} g^{kl} Y_{lad}.$$

**Definition 3.1.1** $(H, g, Y)$ is a formal Frobenius manifold if $Y$ satisfies the WDVV-equations.

It follows that a formal Frobenius manifold yields a commutative algebra structure on $\mathfrak{X}H$. 
A closely related concept is that of cohomological field theory (CohFT) of Kontsevich-Manin [16]. For a definition of it, see Section 4.2. They proved the following theorem.

**Theorem 3.1.2** A formal Frobenius manifold is equivalent to the genus zero part of a CohFT.

We outline the proof of one direction of this theorem. Namely, we review how one obtains a formal Frobenius manifold out of a CohFT.

Suppose that we are given a CohFT \( \Lambda := (H, g, \{ \Lambda_n \}) \). For each \( n \geq 3 \), the \( n \)th correlation function \( Y_n \) of \( \Lambda \) is a symmetric \( n \)-linear form on \( H \) defined as

\[
Y_n : H^\otimes n \to k, \quad v \mapsto \int_{\overline{M}_{0,n}} \Lambda_n(v),
\]

where \( \overline{M}_{0,n} \) denotes the moduli space of rational stable curves with \( n \) marked points. Now, set

\[
Y := \sum_{n \geq 3} \frac{1}{n!} Y_n \in \text{Sym}[[H^*]].
\]

Then the axioms of CohFT and the topology of \( \overline{M}_{0,n} \) together imply that \( Y \) satisfies the WDVV-equations.

### 3.2 G-Frobenius Algebras

We recall the definition of \( G \)-Frobenius algebras (also known as Turaev algebras, or crossed \( G \)-algebras) in this section. See [10], [12], [15], [22] and [26] for details.

These are certain algebra structures on \( G \)-graded \( G \)-modules. First, we put the following compatibility condition on \( G \)-action with the \( G \)-grading. Let \((H, \rho)\) be a \( G \)-graded \( G \)-module.

**Definition 3.2.1** \((H, \rho)\) is self-invariant if \( \gamma \) acts on \( H_\gamma \) trivially, for any \( \gamma \in G \).

We consider a bilinear form on \((H, \rho)\) that is compatible with the \( G \)-action.
**Definition 3.2.2** A symmetric bilinear form $\eta$ on $(H, \rho)$ is $G$-invariant if

$$\eta(\rho(\gamma)v_1, \rho(\gamma)v_2) = \eta(v_1, v_2)$$

for any $\gamma \in G$ and $v_1, v_2 \in H$.

Our bilinear form should also be compatible with the $G$-grading in the following sense. Let $v_g, v_h \in H$ be $G$-homogeneous vectors of $G$-degree $g$ and $h$.

**Definition 3.2.3** A bilinear form $\eta$ on $H$ preserves the $G$-grading if $\eta(v_g, v_h)$ equals zero unless $gh = e$.

**Remark 3.2.4** $\eta$ preserves the $G$-grading if and only if $\eta \in [(H^*)^\otimes 2]_e$, the $G$-degree $e$ part.

**Remark 3.2.5** We note that a symmetric bilinear form on a self-invariant $G$-graded $G$-module that preserves the $G$-grading is actually a braided bilinear form in the sense of Remark 2.4.14.

We observe the following characterization about nondegenerate bilinear forms on $H$.

**Lemma 3.2.6** Suppose $\eta \in [(H^*)^\otimes 2]_e$. Then $\eta$ is nondegenerate if and only if the following linear map

$$H_g \to [H^*]_g = (H_{g^{-1}})^*, \quad v_g \mapsto \eta(v_g, \cdot)$$

is an isomorphism for each $g \in G$.

The following lemma plays an important role in the theory of $G$-Frobenius algebras.

**Lemma 3.2.7** Let $\eta$ be a $G$-invariant nondegenerate symmetric bilinear form that preserves the $G$-grading on $H$. Then $(i^G)^*\eta$ is nondegenerate on $H^G$.
Our main definition in this section follows. We denote the $G$-degrees of homogeneous elements as subscripts.

**Definition 3.2.8** A $G$-Frobenius algebra $((H, \rho), \eta, \cdot, 1)$ is a unital associative algebra on a finite dimensional self-invariant $G$-graded $G$-module $(H, \rho)$ with a $G$-invariant nondegenerate symmetric bilinear form $\eta$ that preserves the $G$-grading, and satisfying the following compatibility conditions:

1. $G$-equivariance of multiplication. $\rho(\gamma)v_1 \cdot \rho(\gamma)v_2 = \rho(\gamma)(v_1 \cdot v_2)$ for any $\gamma \in G$ and $v_1, v_2 \in H$.

2. $G$-graded multiplication. $v_{m_1} \cdot v_{m_2} \in H_{m_1 m_2}$.

3. Braided commutativity. $v_{m_1} \cdot v_{m_2} = \rho(m_1^{-1})v_{m_2} \cdot v_{m_1}$.

4. Invariance of the metric. $\eta(v_1 \cdot v_2, v_3) = \eta(v_1, v_2 \cdot v_3)$ for any $v_1, v_2, v_3 \in H$.

5. $G$-invariant identity. $\rho(\gamma)1 = 1$.

**Remark 3.2.9** The standard definition of a $G$-Frobenius algebra requires that it satisfies the trace axiom. We do not consider it in this paper. See [10, Definition 4.13].

**Remark 3.2.10** [12] shows that a $G$-Frobenius algebra can also be defined in terms of certain $G$-braided bi- and tri-linear forms.

**Remark 3.2.11** In [14,15] the authors show that a $G$-Frobenius algebra is a monoid object in $D(k[G])$-Mod, satisfying further conditions.

**Remark 3.2.12** Note that the identity should be $G$-homogeneous with $G$-degree equal to the identity in $G$.

**Remark 3.2.13** When $G$ is trivial, we recover the notion of Frobenius algebra. In particular, $H_e$ of any $G$-Frobenius algebra $H$ has the induced structure of a Frobenius algebra. Moreover, [12, Proposition 2.1] shows that the restriction of the algebra structure and the bilinear form on $H^G$ also yields a Frobenius algebra structure on it.
3.3 G-Frobenius Manifolds

Let \((H, \rho)\) be a self-invariant \(G\)-graded \(G\)-module of finite dimension, \(\Phi\) the braiding as in Remark 3.2.11, \(\eta\) a \(G\)-invariant nondegenerate symmetric bilinear form on it that preserves the \(G\)-grading. Let \(Y^n \in [\mathcal{Br}^H H^*]_e\) for each \(n \geq 3\) and set \(Y := \sum Y^n \in [\mathcal{Br}[H^*]]_e\). Write \(\eta_e := (i_e)^* \eta, \eta^G := (i_G)^* \eta, Y_e := (i_e)^* Y,\) and \(Y^G := (i^G)^* Y\).

**Definition 3.3.1** \(((H, \rho), \eta, Y)\) is a pre-\(G\)-Frobenius manifold if both \(Y_e\) and \(Y^G\) satisfies the WDVV-equations. Moreover, if there is a vector \(1 \in H_e\) such that \(((H, \rho), \eta, Y^3, 1)\) is a \(G\)-Frobenius algebra, then \(((H, \rho), \eta, Y, 1)\) is a \(G\)-Frobenius manifold.

**Remark 3.3.2** Here we are using Remark 3.2.10.

**Remark 3.3.3** \((H_e, \eta_e, Y_e)\) and \((H^G, \eta^G, Y^G)\) become ordinary formal Frobenius manifolds. The nondegeneracies of \(\eta_e\) and \(\eta^G\) are guaranteed by Lemmas 3.2.6 and 3.2.7.

**Remark 3.3.4** Kaufmann envisioned such a structure as an ingredient for generalizing the results of [12] to the level of Frobenius manifolds. I realized it by interpreting \(G\)-graded \(G\)-modules as \(G\)-braided spaces.
4. Correlation Functions of $G$-Cohomological Field Theories

4.1 Moduli Spaces

We review the theory of moduli spaces of pointed admissible $G$-covers from [10]. We start with (unpointed) admissible $G$-covers from [1] and [10].

Let $(C \to T, p_1, \ldots, p_n)$ be a stable curve over $T$ of genus zero with $n$ marked points $p_1, \ldots, p_n$.

**Definition 4.1.1** A finite morphism $\pi : E \to C$ is an admissible $G$-cover if it satisfies the following conditions.

(i) $E/T$ is a nodal curve, possibly disconnected.

(ii) Nodes of $E$ maps to nodes of $C$.

(iii) $G$ acts on $E$, respecting the fibers.

(iv) $\pi$ is a principal $G$-bundle outside of the special points (nodes or marked points).

(v) Over the nodes of $C$, the structure of the maps $E \to C \to T$ is locally analytically isomorphic to the following:

$$\text{Spec } A[z, w]/(zw - t) \to \text{Spec } A[x, y]/(xy - tr) \to \text{Spec } A,$$

where $t \in A$, $x = z^r$, and $y = w^r$ for some positive integer $r$.

(vi) Over the marked points of $C$, the structure of the maps $E \to C \to T$ is locally analytically isomorphic to the following:

$$\text{Spec } A[z] \to \text{Spec } A[x] \to \text{Spec } A,$$

where $x = z^s$ for some positive integer $s$. 
(vii) At each node \( q \) of \( E \), the eigenvalues of the action of the stabilizer \( G_q \) on the two tangent spaces should be the multiplicative inverses of each other.

**Remark 4.1.2** [1] shows that the stack of admissible \( G \)-covers is isomorphic to the stack \( \overline{M}_{0,n}(BG) \) of balanced twisted stable maps into the classifying stack of \( G \), and is a smooth Deligne-Mumford stack, flat, proper, and quasi-finite over \( \overline{M}_{0,n} \).

As in the case of principal \( G \)-bundles, admissible \( G \)-covers are classified by their holonomies up to the adjoint action of \( G \). Let \( C_{\text{gen}} \) and \( E_{\text{gen}} \) be the points that are neither nodes nor marked points in \( C \) and \( E \). Choose a point \( p_0 \in C_{\text{gen}} \).

**Proposition 4.1.3** If \( C_{\text{gen}} \) is connected, then there is a one to one correspondence between the set of homomorphisms \( \pi_1(C_{\text{gen}}, p_0) \to G^n \) and the set of isomorphism classes of admissible \( G \)-covers over \( C \) together with a point \( \tilde{p}_0 \) in the fiber of \( p_0 \).

Let \( \mathcal{G} \) be the set of conjugacy classes of \( G \), and \( \bar{\gamma} \in \mathcal{G}^n \) be the \( n \)-tuple of conjugacy classes determined by \( \gamma \in G^n \). [11] shows that we have the following decomposition:

\[
\overline{M}_{0,n}(BG) = \bigsqcup_{\bar{\gamma} \in \mathcal{G}^n} \overline{M}_{0,n}(BG; \bar{\gamma}),
\]

where \( \overline{M}_{0,n}(BG; \bar{\gamma}) \) denote the substack with holonomies that can be labeled by \( \bar{\gamma} \). Note that \( \overline{M}_{0,n}(BG; \bar{\gamma}) \) consists of multiple components.

We now come to the pointed version of the previous moduli space.

**Definition 4.1.4** For a \( g = 0 \) stable curve with \( n \geq 3 \) marked points \( (C \to T, p_1, \ldots, p_n) \), an \( n \)-pointed admissible \( G \)-cover is an admissible \( G \)-cover \( \pi : E \to C \) with \( n \) marked points \( \tilde{p}_i \in E \) such that \( \pi(\tilde{p}_i) = p_i \). The morphisms in the stack of admissible \( G \)-covers \( \overline{M}_{0,n}^G \) are the \( G \)-equivariant fibered diagrams preserving the points \( \tilde{p}_i \).

**Remark 4.1.5** We have forgetful morphisms

\[
\overline{M}_{0,n}^G \to \overline{M}_{0,n}(BG) \to \overline{M}_{0,n}.
\]

Let \( \text{st} := \tilde{\text{st}} \circ \tilde{\text{st}} \). [10] shows that \( \overline{M}_{0,n}^G \) are smooth Deligne-Mumford stacks, flat, proper, and quasi-finite over \( \overline{M}_{0,n} \).
Given a pointed admissible $G$-cover, let $m := (m_1, \ldots, m_n)$ be the monodromy around the marked points on the cover, and $\overline{M}_{0,n}^G(m)$ denote the substack with the monodromy $m$. Then we have the decomposition

$$\overline{M}_{0,n}^G = \coprod_{m \in G^n} \overline{M}_{0,n}^G(m).$$

Given a pointed admissible $G$-cover, the structure of the fiber over a marked point on the underlying curve with monodromy $m$ is the same as $G/\langle m \rangle$ as a $G$-set. This gives rise to the $G^n$-action on $\overline{M}_{0,n}^G$. Also, we have the $S_n$-action on it given by switching the labels of the marked points on underlying curves.

We now describe two morphisms that are relevant in $g = 0$: the \textit{forgetting tails} morphism $\tau$ and the \textit{gluing morphism} $gl$.

$\tau$ is defined when the monodromy of the last point is trivial. Suppose $m \in G^n$. Then

$$\tau : \overline{M}_{0,n+1}^G(m, e) \to \overline{M}_{0,n}^G(m)$$

is given simply by forgetting the last marks on the underlying stable curve and the point over it. If we get an unstable curve as the result, we stabilize it and [10] shows that we have a way to define a suitable pointed admissible $G$-cover on the stabilized curve.

We can glue two $G$-covers at marked points when their monodromies are inverse to each other. Let $\mu \in G$ and $n = n_1 + n_2$. Then

$$gl : \overline{M}_{0,n_1+1}^G(m, \mu) \times \overline{M}_{0,n_2+1}^G(\mu^{-1}, m') \to \overline{M}_{0,n}^G(m, m')$$

is given by gluing the underlying curve at the last marked point of the first curve and the first marked point of the second curve, and gluing the $G$-cover equivariantly at the points over the marked points just glued. The result is already a pointed admissible $G$-cover. More generally, the gluing morphism can be defined for any partition $I \sqcup J$ of $\{1, \ldots, n\}$ as

$$gl : \overline{M}_{0,n_1+1}^G(m_I) \times \overline{M}_{0,n_2+1}^G(m_J) \to \overline{M}_{0,n}^G(m)$$
making use of the symmetric group action. Here \( m_I \) and \( m_J \) are obtained by adjoining \( \mu \) and \( \mu^{-1} \) at some position to the corresponding components of \( m \).

Next, we define distinguished components of \( \overline{M}^G_{0,n} \) following [10]. Suppose that we have \( n \) marked points at \( P := \{ e^{2\pi i k/n} : 1 \leq k \leq n \} \) on \( \mathbb{P}^1 = \mathbb{C} \cup \{ \infty \} \). We label the points from 1 to \( n \) counterclockwise starting from the point \( 1 \in \mathbb{C} \). Draw line segments from 0 to each marked point. Then this determines a set of generators of the fundamental group of \( \mathbb{P}^1 - P \) at 0. Let \( m \in G^n \) with \( \prod_i m_i = e \). Then \( m \) determines a holonomy, and Proposition 4.1.3 gives us an admissible \( G \)-cover and a point \( p \) over 0. Parallel transporting \( p \) along the line segments above yields \( n \) points over the marked points on the underlying \( \mathbb{P}^1 \). Note that the monodromy at these points are also given by \( m \). We denote \( [m] \) the component of \( \overline{M}^G_{0,n}(m) \) that contains the point representing this pointed admissible \( G \)-cover.

**Remark 4.1.6** The homology classes of \( \overline{M}^G_{0,n} \) in this paper denote those of its coarse moduli space. Namely, we do not divide by the orders of automorphism groups.

### 4.2 \( G \)-Cohomological Field Theory

In this section, we recall the notion of \( G \)-cohomological Field Theory(\( G \)-CohFT) from [10], and review how one obtains an ordinary cohomological field theory(\( \text{CohFT} \)) and a \( G \)-Frobenius algebra out of it. We consider only \( g = 0 \) case.

**Definition 4.2.1** A genus zero \( G \)-CohFT is a quadruple \( ((H, \rho), \eta, \{ \Lambda_n \}, 1) \) where

(i) \( (H, \rho) \) is a \( G \)-graded \( G \)-module,

(ii) \( \Lambda_n \in \bigoplus_m H^\bullet(\overline{M}_n^G(m)) \otimes H^*_m \) and \( G^n \rtimes S_n \)-invariant,

(iii) \( 1 \neq 0 \in H_e \) such that

   (a) \( 1 \) is \( G \)-invariant,

   (b) \( \Lambda_{n+1}(v_m, 1) = \tau^* \Lambda_n(v_m) \) under the forgetting tails morphism \( \tau \),
(iv) \( \eta \) is a nondegenerate symmetric bilinear form on \( H \) such that

\[
\eta(v_{m_1}, v_{m_2}) := \int_{[m_1, m_2, 1]} \Lambda_3(v_{m_1}, v_{m_2}, 1), \tag{4.1}
\]

(v) for any basis \( \{e_\alpha\} \) of \( H \),

\[
gl^* \Lambda_n(v_m) = \sum_{\alpha, \beta} \Lambda_{n_1+1}(v_{m_1}, e_\alpha) \eta^{\alpha \beta} \Lambda_{n_2+1}(e_\beta, v_{m_j})
\]

under the gluing morphism \( gl \), for any partition \( I \sqcup J \) of \( \{1, \ldots, n\} \).

**Remark 4.2.2** The notion of CohFT as defined in [16] does not involve the axioms regarding 1. Hence we recover the notion of the \( g = 0 \) part of the CohFT by considering trivial group \( G \) and removing the axiom (iii) above and Equation (4.1).

We now review how to obtain an ordinary CohFT from a \( G \)-CohFT. This is a two-step process that involves the three moduli spaces \( \overline{M}_{0,n}^G, M_{0,n}(BG) \) and \( \overline{M}_{0,n} \).

The first step involves the forgetful map

\[
\tilde{st}_m : \overline{M}_{0,n}^G(m) \rightarrow M_{0,n}(BG; m),
\]

where

\[
\overline{M}_{0,n}^G(m) := \coprod_{m \in \mathbb{m}} \overline{M}_{0,n}^G(m).
\]

We have the following proposition.

**Proposition 4.2.3** There are unique classes

\[
\hat{\Lambda}_n \in \bigoplus_{m \in \mathbb{m}} H^\bullet(\overline{M}_{0,n}(BG; \mathbb{m})) \otimes (H^G_\mathbb{m})^*
\]

such that

\[
\tilde{st}_m^* \hat{\Lambda}_n(v_m) = \Lambda_n(v_m)
\]

for all \( v_m \in H^G_\mathbb{m} \).

Once we obtain \( \hat{\Lambda}_n \), we define \( \overline{\Lambda}_n \) in the following way.
Definition 4.2.4

\[ \overline{\Lambda}_n := \hat{\text{st}}_n \Lambda_n \in H^\bullet(M_{0,n}) \otimes [(H^G)^*]^{\otimes n}. \]

Let \( \eta^G \) be the restriction of \( \eta \) to \( H^G \). Then we get the CohFT we wanted as follows.

Theorem 4.2.5 \( (H^G, \eta^G, \{\overline{\Lambda}_n\}) \) is a CohFT.

Remark 4.2.6 In [10], the authors use \( \overline{\eta} := (1/|G|)\eta^G \) instead of \( \eta^G \). This is because they include 1 as a data for the definition of a CohFT. Once we remove the conditions about 1, we can normalize \( \eta^G \) as needed, and we still get a CohFT.

Next, we consider the \( G \)-Frobenius algebra that is contained in any \( G \)-CohFT \( ((H, \rho), \eta, \Lambda_n, 1) \). Let \( g, h, \) and \( k \) be in \( G \) such that \( ghk = e \). Fix a basis \( \{e_\alpha\} \) for \( H_k \) and \( \{f_\beta\} \) for \( H_{k^{-1}} \). Let \( v \in H_g \) and \( w \in H_h \). Define

\[ v \cdot w := \int_{[g,h,k]} \Lambda_3(v, w, e_\alpha) \eta^{\alpha \beta} f_\beta. \quad (4.2) \]

Theorem 4.2.7 \( ((H, \rho), \eta, \cdot, 1) \) is a \( G \)-Frobenius algebra.

Remark 4.2.8 In particular, \( \rho \) is self-invariant, and \( \eta \) is \( G \)-invariant and preserves the \( G \)-grading.

Remark 4.2.9 Due to our convention in Remark 4.1.6 on the homology classes in \( M_{0,n}^G \), the homology class that is used for the definition of \( \cdot \) in [10] is the same class as we use here.

4.3 Correlation Functions

In this section, we define two types of correlation functions that we can get out of a \( G \)-CohFT \( ((H, \rho), \eta, \{\Lambda_n\}, 1) \), a symmetric one and a braided one, and prove that they coincide on \( H^G \).
Given \( \mathbf{m} = (m_1, \ldots, m_n) \in G^n \), let \( \tau \) be a component of \( \overline{M}_{0,n}(BG; \mathbf{m}) \). Then \( \tilde{\text{st}}_{\mathbf{m}}^{-1}(\tau) \) is the union of the components of \( \overline{M}_{0,n}^G(\mathbf{m}) \) that are mapped to \( \tau \). Given a component \( \kappa \) of \( \overline{M}_{0,n}^G(\mathbf{m}) \), let \( \tau(\kappa) \) denote the component of \( \overline{M}_{0,n}(BG; \mathbf{m}) \) that contains the image of \( \kappa \) under \( \tilde{\text{st}}_{\mathbf{m}} \). Set \( u(\kappa) := \tilde{\text{st}}_{\mathbf{m}}^{-1}(\tau(\kappa)) \) and define \( p(\kappa) \) to be the degree of the forgetful map

\[
\tilde{\text{st}}_{\mathbf{m}}|_{u(\kappa)} : u(\kappa) \to \tau(\kappa).
\]

In particular, we write \( p(m) := p([m]) \). Also, since \( \gamma : [m] \to [\gamma[m]] \) is an isomorphism that is compatible with the forgetful map, we have \( p(\gamma[m]) = p(m) \). Let \( \{\kappa_i(m)\} \) be the set of components of \( \overline{M}_{0,n}^G(m) \). This is a finite set since any component in \( \overline{M}_{0,n}^G \) can be written as \( \gamma[m'] \) for some \( \gamma \) and \( m' \) with \( \prod m'i = e \) in \( G^n \).

**Definition 4.3.1** Given a \( G \)-CohFT \( ((H, \rho), \eta, \{A_n\}, 1) \), its degree \( n \) symmetric correlation function \( Y^n_S \) is defined as

\[
Y^n_S(v) := \sum_i \frac{1}{p(\kappa_i(m))} \int_{\kappa_i(m)} \Lambda_n(v),
\]

for any \( v \in H_m \).

**Proposition 4.3.2** \( Y^n_S \) is a symmetric \( n \)-linear form.

**Proof** It is enough to consider homogeneous vectors. Suppose \( v \in H_m \) and \( \sigma \in S_n \).

\[
Y^n_S(\sigma v) = \sum_i \frac{1}{p(\kappa_i(\sigma m))} \int_{\kappa_i(\sigma m)} \Lambda_n(\sigma v)
= \sum_i \frac{1}{p(\kappa_i(\sigma m))} \int_{\kappa_i(\sigma m)} \sigma \Lambda_n(v) \quad \text{by the } G^n \rtimes S_n \text{ invariance},
= \sum_i \frac{1}{p(\kappa_i(\sigma m))} \int_{\sigma^{-1} \kappa_i(\sigma m)} \Lambda_n(v).
\]

Note that we have the following commutative diagram, where the two \( \sigma \)'s are isomorphisms.

\[
u(\kappa_j(m)) \quad \xrightarrow{\sigma} \quad u(\kappa_i(\sigma m)) \]

\[
\Rightarrow \quad \tilde{\text{st}}_{\mathbf{m}} \quad \xrightarrow{\sigma} \quad \tilde{\text{st}}_{\mathbf{m}} \]

\[
\tau(\kappa_j(m)) \quad \xrightarrow{\sigma} \quad \tau(\kappa_i(\sigma m))
\]
This implies that \( p(\kappa_i(\sigma \mathbf{m})) = p(\kappa_j(\mathbf{m})) \). Hence,

\[
Y^n_S(\sigma \mathbf{v}) = \sum_j \frac{1}{p(\kappa_j(\mathbf{m}))} \int_{\kappa_j(\mathbf{m})} \Lambda_n(\mathbf{v}) = Y^n_S(\mathbf{v}).
\]

Since the ways of representing components as \( \gamma[\mathbf{m}'] \) are not unique, let \( \nu(\kappa) \) be the number of ways that \( \kappa \) can be written in this way. In particular, let \( \nu(\mathbf{m}) := \nu([\mathbf{m}]) \).

Note that \( \nu \) is invariant under the \( B_n \)-action, and \( \nu(\gamma[\mathbf{m}]) = \nu(\mathbf{m}) \) for any \( \gamma \in G^n \) and \( \mathbf{m} \), since these actions are isomorphisms. Also, we see that the component \( \gamma[\mathbf{m}] \) has the constant monodromy \( md(\gamma[\mathbf{m}]) := (\gamma_1 \gamma_1^{-1}, \ldots, \gamma_n \gamma_n^{-1}) \).

**Definition 4.3.3** The degree \( n \) braided correlation function \( Y^n_B \) of a \( G \)-CohFT \( ((H, \rho), \eta, \{\Lambda_n\}, 1) \) is defined as

\[
Y^n_B(\mathbf{v}) := \sum_{\prod m = e} \frac{|G|^n}{p(\mathbf{m}) \nu(\mathbf{m})} \int_{|\mathbf{m}|} \Lambda(\mathbf{v}),
\]

for any \( \mathbf{v} \in H^\otimes n \).

**Remark 4.3.4** Note that \( Y^n_B \in (\text{Br}^n H^*)_e \) for each \( n \).

The analogue of Proposition 4.3.2 follows.

**Proposition 4.3.5** \( Y^n_B \) is a braided \( n \)-linear form.

**Proof** Note that the \( B_n \) action on \( \overline{M}_{0,n}(BG) \) amounts to choosing a new set of generators of the fundamental group on the underlying curve, to determine the holonomy of its admissible \( G \)-cover. Using the \( B_n \)-invariance of \( \nu \), and the fact that \( b[\mathbf{m}] = [b \mathbf{m}] \), the proof is completely analogous to that of Proposition 4.3.2.

Note that \( Y^n_B \) restricted to \( (H^G)^\otimes n \) are symmetric \( n \)-linear forms. Moreover, we have the following proposition.
Proposition 4.3.6 For $v \in (H^G)^{\oplus n}$,
\[ Y^n_S(v) = Y^n_B(v). \]

**Proof** We first prove Equation (4.3) below, that relates $Y^n_S$ and $Y^n_B$ for any homogeneous vector $w$. Hence, suppose $w \in H_m$. Then we have
\[
Y^n_S(w) = \sum_i \frac{1}{p(\kappa_i(m))} \int_{\kappa_i(m)} \Lambda(w) = \sum_{m' = e \atop md(\gamma[m']) = m} \frac{1}{\nu(m')} \frac{1}{p(\gamma[m'])} \int_{[m']} \Lambda(w).
\]

By the $G^n \rtimes S_n$-invariance of $\Lambda_n$ and $G^n$-invariance of $p$,
\[
Y^n_S(w) = \sum_{m' = e \atop md(\gamma[m']) = m} \frac{1}{\nu(m')} \frac{1}{p(\gamma[m'])} \int_{[m']} \Lambda(\gamma^{-1}w).
\]

By the definition of $Y^n_B$,
\[
Y^n_S(w) = \sum_{m' = e \atop md(\gamma[m']) = m} \frac{1}{|G|^n} Y^n_B(\gamma^{-1}w).
\]

Since $Y^n_B$ is zero on the vectors of total degree not equal to $e$,
\[
Y^n_S(w) = \frac{1}{|G|^n} Y^n_B(\sum_{\gamma \in G^n} \gamma w). \tag{4.3}
\]

Now we use this equation to prove the proposition. Let $c(\gamma)$ be the number of elements in the conjugacy class of $\gamma$. For $m \in G^n$, define $c(m) := \prod c(m_i)$. Suppose $v \in H^G_m$ and let
\[
v = \sum_{i=1}^{c(m)} w_i
\]
be the homogeneous decomposition of $v$. Then
\[
|G|^n v = \sum_{\gamma \in G^n} \gamma v = \sum_{\gamma \in G^n} \sum_{i=1}^{c(m)} \gamma w_i.
\]
Hence we conclude
\[
Y^n_S(v) = Y^n_S\left(\sum_{i=1}^{c(m)} w_i\right) = \sum_{i=1}^{c(m)} Y^n_S(w_i) \\
= \sum_{i=1}^{c(m)} \frac{1}{|G|^n} Y^n_B\left(\sum_{\gamma \in G^n} \gamma w_i\right) = Y^n_B(v).
\]

4.4 Pre-\(G\)-Frobenius Manifold Structure

In this section, we prove that any \(G\)-cohomological field theory gives rise to a pre-\(G\)-Frobenius manifold structure on its state space. We first make the following observation. Let \(H\) be a finite dimensional vector space.

Lemma 4.4.1 If \(Y := \sum Y^n \in \text{Sym}[H^*]\) satisfies the WDVV-equations, then \(Y' := \sum a^n Y^n\) for some \(a \in k\) also satisfies the WDVV-equations.

Proof This is true since WDVV-equations can be proved degree-by-degree.

Given a \(G\)-CohFT \(((H, \rho), \eta, \{\Lambda_n\}, 1)\), set
\[
Y_B := \sum_{n \geq 3} \frac{1}{n!} Y^n_B.
\]

Theorem 4.4.2 \(((H, \rho), \eta, Y_B)\) is a pre-\(G\)-Frobenius manifold.

Proof By Remark 4.2.8, \(\rho\) is self-invariant, and \(\eta\) is \(G\)-invariant and preserves the \(G\)-grading.

We first consider \((i^G)_*(Y_B)\). Let \((H^G, \eta^G, \bar{\Lambda}_n)\) be the CohFT obtained by taking the \(G\)-quotient of our \(G\)-CohFT. Recall that its correlation functions are defined as
\[
(Y^G)^n(v) := \int_{\overline{M}_{0,n}} \bar{\Lambda}(v)
\]
for any \(v \in (H^G)^{\otimes n}\). In view of the equivalence of CohFT's and Frobenius manifolds, and Proposition 4.3.6, it is enough to show
\[
Y^n_S(v) = (Y^G)^n(v)
\]
for any $v \in (H^G)^{\otimes n}$. Suppose $v$ is homogeneous of $G$-degree equal to $\overline{m}$, and $v = \sum_{m \in \overline{m}} w_m$ be its homogeneous decomposition. Since $\Lambda_n(w_m)$ is supported on $\overline{M}_{0,n}(m)$, we have

$$Y^n_S(v) = \sum_{m \in \overline{m}} \sum_i \frac{1}{p(\kappa_i(m))} \int_{\kappa_i(m)} \Lambda_n(w_m)$$

$$= \sum_{m \in \overline{m}} \sum_i \frac{1}{p(\kappa_i(m))} \int_{\kappa_i(m)} \Lambda_n(v) = \int \sum_{m \in \overline{m}} \sum_i \frac{1}{p(\kappa_i(m))} \kappa_i(m) \Lambda_n(v).$$

Using the projection formula, we have

$$Y^n_S(v) = \int_{\overline{M}_{BG;\overline{m}}} \widehat{\Lambda}_n(v).$$

Using the Gysin map, we conclude that

$$Y^n_S(v) = \int_{\overline{M}_{0,n}} \overline{\Lambda}(v) = (Y^G)^n(v).$$

Consider $(i_e)^*(Y_B)$ next. Let $e := (e, \ldots, e)$, and $v \in H_e$. Note that $\nu(e) = |G|$ since $[e]$ can be also written as $g[e]$ where $g := (g, \ldots, g)$, for any $g \in G$. Since $p(e) = 1$, we have

$$Y^n_B(v) = |G|^{n-1} \int_{[e]} \Lambda_n(v).$$

Observe that

$$Y' := \sum_{n} \frac{1}{|G|^{n-1}} \cdot \frac{1}{n!} Y^n_B$$

satisfies the WDVV-equations since $[e]$ is isomorphic to $\overline{M}_{0,n}$ and we are reduced to an ordinary CohFT. A simple modification of Lemma 4.4.1 implies that $(i_e)^*(Y_B)$ satisfies the WDVV-equations.

**Remark 4.4.3** In fact, we can say a little bit more. By comparing Equation (4.2) and Definition 4.3.3, we notice that the multiplication structure of the $G$-Frobenius algebra that is contained in the $G$-CohFT differs by constants from the one given by the degree 3 braided correlation functions, and these constants depend on the $G$-degrees of the vectors multiplied together. This means that we can identify the two multiplication rules by rescaling the metric on the underlying $G$-graded $G$-module. In
other words, a $G$-CohFT contains a $G$-Frobenius manifold up to a rescaling of its metric.

**Remark 4.4.4** One can ask if it is possible to get a $G$-CohFT starting from a pre-$G$-Frobenius manifold, in analogy with the case of ordinary CohFT’s and Frobenius manifolds. A difficulty answering this question is that the topology of $\overline{M}_{0,n}^G$ is not very well understood compared to that of $\overline{M}_{0,n}$. 
5. Examples of $\mathbb{Z}/2\mathbb{Z}$-Frobenius Manifolds

5.1 $G = \mathbb{Z}/2\mathbb{Z}$

In this section, we specialize to the case $G = \mathbb{Z}/2\mathbb{Z} = \{e, g\}$ and prove the structure theorem 5.1.3 for pre-$\mathbb{Z}/2\mathbb{Z}$-Frobenius manifolds.

We first state some general facts about $\mathbb{Z}/2\mathbb{Z}$-graded $\mathbb{Z}/2\mathbb{Z}$-modules.

**Proposition 5.1.1** Let $(H, \rho)$ be a $\mathbb{Z}/2\mathbb{Z}$-graded $\mathbb{Z}/2\mathbb{Z}$-module that is self-invariant. Then $H = H_e \oplus H_g$ has the following properties.

(i) The non-twisted sector $H_e$ has the following decomposition:

\[ H_e = H_i \bigoplus H_v, \]

where $H_i$ is invariant under $\mathbb{Z}/2\mathbb{Z}$ and $\rho(g)v = -v$ for any $v \in H_v$.

(ii) The $G$-invariants decomposes as

\[ H^G = H_i \bigoplus H_g. \]

**Proof** (i) follows from the theory of representations of finite groups, since $H_e$ is a $\mathbb{Z}/2\mathbb{Z}$-module. See, for example, [23]. For (ii), note that the conjugacy classes of $\mathbb{Z}/2\mathbb{Z}$ are $\mathbb{Z}/2\mathbb{Z}$ itself, and $H_g$ is $\mathbb{Z}/2\mathbb{Z}$-invariant. Also, $(H_e)^G = H_i$. 

Fix a homogeneous basis $\{x^p\}$ of $H^*$ under the grading

\[ H^* = H_i^* \bigoplus H_v^* \bigoplus H_g^*, \]

and denote their homogeneous $G$-degree as subscripts. We have the following description of the braided multilinear forms on $H$. 

Proposition 5.1.2  Given a basis of $H^*$ as above, we have the isomorphism of vector spaces

$$\mathcal{Br}[H^*] \cong k[[x^p]]/\mathcal{I},$$

where $\mathcal{I}$ is generated by $\{x_v^p, x_g^q\}$.

Proof Suppose $X \in \mathcal{Br}^n H^*$. Then it is symmetric on $H^G$ and $H_e$ by Propositions 2.5.1 and 2.5.2. In view of Proposition 5.1.1, it remains to consider the value of $X$ on vectors $w \in H^{\otimes n}$ that has at least one factor $w_g \in H_g$ and another $w_v \in H_v$. By linearity it is enough to assume that $w$ has only one term. Then $w$ can be written as one of the following two forms.

$w = \cdots \otimes w_v \otimes w_{i_1} \otimes \cdots \otimes w_{i_k} \otimes w_g \otimes \cdots$

or

$w = \cdots \otimes w_g \otimes w_{i_1} \otimes \cdots \otimes w_{i_l} \otimes w_v \otimes \cdots$

where $w_{i_j}$ denotes vectors in $H_i$.

We use Corollary 2.4.17 and consider only the first case. The other one is similar.

$$X(w) = X(\cdots \otimes w_v \otimes w_{i_1} \otimes \cdots \otimes w_{i_k} \otimes w_g \otimes \cdots)$$

$$= X(\cdots \otimes w_v \otimes w_g \otimes \cdots)$$

by $B_n$ invariance and the fact that $H_i$ is invariant under the action of $\mathbb{Z}/2\mathbb{Z}$. Since $w_v$ is in $H_e$, we have

$$X(w) = X(\cdots \otimes w_g \otimes w_v \otimes \cdots)$$

$$= -X(\cdots \otimes w_v \otimes w_g \cdots) = -X(w)$$

since $w_v$ is in the eigenspace of $g$ with eigenvalue $-1$. Hence we have

$$X(w) = 0$$

if $w$ has factors of $H_v$ and $H_e$ together. This implies that any element in $\mathcal{Br}^n H^*$ does not have any term that has both $x_v^p$ and $x_g^q$ as its factors. The proposition follows
by the identification of symmetric $n$-linear forms and homogeneous polynomials of degree $n$.

Once we have a pre-$\mathbb{Z}/2\mathbb{Z}$-Frobenius manifold structure on $H = H_i \oplus H_v \oplus H_g$, we have ordinary Frobenius manifold structures on $H_e = H_i \oplus H_v$ and $H^G = H_i \oplus H_g$ with induced potentials. On the other hand, we may be able to produce a pre-$\mathbb{Z}/2\mathbb{Z}$-Frobenius manifold starting from two ordinary Frobenius manifolds in the following way.

Let $(H_e, \eta_e, \rho_e)$ and $(H^G, \eta^G, \rho^G)$ be formal Frobenius manifolds. Suppose that there is a vector space $H_i$ and two linear injections $\iota_e : H_i \to H_e$ and $\iota^G : H_i \to H^G$ such that $(\iota_e)^* Y_e = (\iota^G)^* Y^G := Y_i$, and $(\iota_e)^* \eta_e = (\iota^G)^* \eta^G := \eta_i$. Fix decompositions $H_e = H_i \oplus H_v$ and $H^G = H_i \oplus H_g$. Write $Y_e = Y_i + Y_v$ and $Y^G = Y_i + Y_g$, and $\eta_e = \eta_i + \eta_v$ and $\eta^G = \eta_i + \eta_g$ where $(\iota_e)^* Y_v = (\iota^G)^* Y_g = 0$ and $(\iota_e)^* \eta_v = (\iota^G)^* \eta_g = 0$. Let $H := H_i \oplus H_v \oplus H_g$ be a $\mathbb{Z}/2\mathbb{Z}$-graded $\mathbb{Z}/2\mathbb{Z}$-module as in Proposition 5.1.1. Set $Y := Y_i + Y_v + Y_g$ and $\eta := \eta_i + \eta_v + \eta_g$, extending by zero.

Let $S := \{i, v, g\}$ be the set of the subscript in the decomposition of $H$ above. Then we can talk about the $S^2$-degrees of homogeneous terms of $\eta$.

**Theorem 5.1.3 (Structure of pre-$\mathbb{Z}/2\mathbb{Z}$-Frobenius manifolds)** Suppose that $Y \in [T[H^*]]_e$, and $\eta_i, \eta_v, \eta_g$ are $S^2$-homogeneous of degrees $(i, i)$, $(v, v)$, and $(g, g)$ respectively. Then $((H, \rho), \eta, Y)$ is a pre-$\mathbb{Z}/2\mathbb{Z}$-Frobenius manifold. Conversely, the $\eta$ and $Y$ of any pre-$\mathbb{Z}/2\mathbb{Z}$-Frobenius manifold $((H, \rho), \eta, Y)$ is of this form, and uniquely determined by the metrics and potentials of the two Frobenius manifolds it contains.

**Proof** We first prove that $((H, \rho), \eta, Y)$ constructed as above is a pre-$\mathbb{Z}/2\mathbb{Z}$-Frobenius manifold. By construction, $\rho$ is self-invariant, and $\eta$ is $\mathbb{Z}/2\mathbb{Z}$-invariant, symmetric, and preserves the $\mathbb{Z}/2\mathbb{Z}$-grading. Since the $S^2$-degree of $\eta_g$ is $(g, g)$, the condition of Lemma 3.2.6 is satisfied for $\mathbb{Z}/2\mathbb{Z}$-degree $e$ by the nondegeneracy of $\eta_e$. The $S^2$-degrees of $\eta_i$ and $\eta_g$, and the nondegeneracy of $\eta^G$ also implies that of $\eta_g$, satisfying the condition of the Lemma for $g$. It follows that $\eta$ is nondegenerate.
Proposition 5.1.2 implies that $Y$ is automatically in $\mathcal{Br}[H^*]$. It remains to show that $Y^e = (i_e)^*Y$ and $Y^G = (i^G)^*Y$. Since we are extending by zero and $(i^G)^*Y_g = 0$, we have $(i_e)^*Y_g = 0$. With the completely analogous reasoning, we also have $(i^G)^*Y_v = 0$.

To prove the converse, we first consider $\eta$. Writing the $S^2$-degrees as subscripts, we have the decomposition

$$\eta = \eta_i + \eta_{iv} + \eta_{ig} + \eta_{ve} + \eta_{vg} + \eta_{gg}.$$ 

Since $\eta$ preserves the $G$-degree, we have $\eta_{ig} = \eta_{vg} = 0$. To see that $\eta_{iv} = 0$, take $w_i \in H_i$ and $w_v \in H_v$. Using the $\mathbb{Z}/2\mathbb{Z}$-invariance of $\eta$, we have

$$\eta_{iv}(w_i, w_v) = \eta_{iv}(g \cdot w_i, g \cdot w_v) = \eta_{iv}(w_i, -w_v) = -\eta_{iv}(w_i, w_v).$$

For $Y$, let $Y_i$ be the sum of terms that only have factors of $S$-degree $i$, $Y_v$ the sum of terms that have at least a factor of $S$-degree $v$, and $Y_g$ the sum of terms that have at least a factor of $S$-degree $g$. Proposition 5.1.2 implies that every term of $Y$ is uniquely a term of either $Y_i$, $Y_v$ or $Y_g$. Hence we have the unique decomposition of $Y$ as

$$Y = Y_i + Y_v + Y_g.$$ 

Note that $(i_e)^*Y_g = (i^G)^*Y_v = 0$, and $(i_e)^*Y_v = (i^G)^*Y_g = 0$ by construction.

Uniqueness is obvious from the decompositions of $\eta$ and $Y$.

5.2 $A_n$ and $D_n$ Singularities and Their Milnor Rings

We review the concept of Milnor rings and their Frobenius algebra structures for certain singularities. See [7] for details.

We consider two series of polynomials called $A_n$ and $D_n$.

$$A_n : \frac{1}{n+1}z^{n+1}$$

$$D_n : \frac{1}{2}xy^2 + \frac{1}{2n-2}x^{n-1}$$
Note that these polynomials define complex valued functions on $\mathbb{C}$ or $\mathbb{C}^2$, and they have isolated singularities (critical points) at the origin.

Let $f$ be one of the polynomials. Then its Milnor ring (Jacobian ring or local ring) $A_f$ is given by the quotient ring of the polynomial ring by the Jacobian ideal $J_f$ of $f$. For example, the Milnor ring of $D_4$ is $\mathbb{C}[x,y]/(y^2 + x^2, xy)$. Note that the Milnor ring for $A_n$ is generated by $\{1, z, \ldots, z^{n-1}\}$ and for $D_n$ by $\{1, x, \ldots, x^{n-2}, y\}$ as vector spaces. Hence their dimensions are equal to $n$.

$A_f$ has the structure of a Frobenius algebra with counit $\epsilon$ as follows:

$$A_n: \epsilon(z^i) = \delta_{i,n-1} \text{ for } i = 0, \ldots, n - 1,$$
$$D_n: \epsilon(x^i) = \delta_{i,n-2} \text{ for } i = 0, \ldots, n - 2 \text{ and } \epsilon(y) = 0$$

5.3 Frobenius Manifold Structures on Milnor Rings

We review the Frobenius manifold structures associated with the universal unfoldings of the singularities $A_n$ and $D_n$. For details, see [5] and [20].

Any vector in the Milnor rings of $A_n$ and $D_n$ can be uniquely written as follows.

$$A_n: k_{n-1}z^{n-1} + \cdots + k_1 z + k_0$$
$$D_n: l_{n-2}x^{n-2} + \cdots + l_1 x + l_0 + l_* y$$

These identify their Milnor rings with $\mathbb{C}^n$, whose coordinates can be written as $(k_{n-1}, \ldots, k_0)$ for $A_n$ and $(l_{n-2}, \ldots, l_0, l_*)$ for $D_n$. It follows that we have the following identifications of vectors.

$$A_n: \partial_{k_i} \leftrightarrow z^i$$
$$D_n: \partial_{l_j} \leftrightarrow x^j \text{ and } \partial_* \leftrightarrow y$$
We also associate a deformation of the defining polynomials of $A_n$ and $D_n$ tautologically for each vector in the Milnor ring simply by addition.

\[
A_n : \frac{1}{n+1} z^{n+1} + k_{n-1}z^{n-1} + \cdots + k_1 z + k_0 := F_{A_n}
\]

\[
D_n : \frac{1}{2} xy^2 + \frac{1}{2n-2} x^{n-1} + l_{n-2}x^{n-2} + \cdots + l_1 x + l_0 + l_\ast y := F_{D_n}
\]

We define multiplication rules between vector fields in the way that each tangent space is isomorphic to the Jacobian ring of the deformed polynomial. To be more precise, the ring of formal vector fields are isomorphic to the following.

\[
A_n : \mathbb{C}\left[ k_0, \ldots, k_{n-1} \right][z]/JF_{A_n}
\]

\[
D_n : \mathbb{C}\left[ l_0, \ldots, l_{n-2}, l_\ast \right][x,y]/JF_{D_n}
\]

Here $JF_{A_n}$ and $JF_{D_n}$ denote the Jacobian ideal of $F_{A_n}$ and $F_{D_n}$ with respect to $z$ for $A_n$ and $x, y$ for $D_n$.

Next we describe the metrics. The metrics are given by the residues. Their multi-variable version can be found, for example, in [9]. We divide by 2 for $D_n$ for normalization purposes. Fix a tangent space, take $f \mapsto g$ in the Jacobian ring, and let $F$ be equal to either $F_{A_n}$ or $F_{D_n}$. $P$ runs through all the poles.

\[
A_n : \eta(f, g) := \sum_P \text{Res}_P \frac{fg}{F^\prime} dz
\]

\[
D_n : \eta(f, g) := \frac{1}{2} \sum_P \text{Res}_P \frac{fg}{\frac{\partial F}{\partial x} \frac{\partial F}{\partial y}} dx \, dy
\]

One can show that the metric at the origin is the same as the one in the previous section. Thus we can identify the undeformed Milnor ring as the tangent space at the origin.

Consider the odd dimensional ones in $A_n$ series. First we identify a series of Frobenius manifolds called $B_{n-1}$ as submanifolds in both $A_{2n-3}$ and $D_n$. For generalities about Frobenius submanifolds, see [25].

Consider the $(n - 1)$-dimensional submanifold with coordinates $a_i := k_{2i} = l_i, i = 0, \ldots, n - 2$, that is, we are considering hypersurfaces given by $k_{2i-1} = 0$ and $l_\ast = 0$. 
The algebra of tangent vectors at any point on the submanifold is generated by \( z^2 \) or \( x \). One can show that the ring homomorphism of tangent spaces given by \( z^2 \mapsto x \) at each point of the submanifold is an isomorphism that also preserves the metric. We will use \((a_0, a_1, \ldots, a_{2n-4})\) and \((a_0, a_2, \ldots, a_{2n-4}, a_*)\) for coordinates of \( A_{2n-3} \) and \( D_n \), respectively, so that \((a_0, a_2, \ldots, a_{2n-4})\) will be the coordinates for \( B_{n-1} \).

Before considering the potential, note that the coordinates we are using are not flat, meaning that the metric between the coordinate vector fields are not constants. We first describe how to get a set of flat coordinates for \( A_n \). If we set

\[
\frac{w^{n+1}}{n+1} = F_{A_n} = \frac{1}{n+1}z^{n+1} + k_{n-1}z^{n-1} + \cdots + k_1z + k_0,
\]

then we have the Laurent series expansion

\[
z = w + \frac{t_{n-1}}{w} + \frac{t_{n-2}}{w^2} + \cdots + \frac{t_0}{w^n} + O\left(\frac{1}{w^{n+1}}\right).
\]

It is known that \( a_i \) is a polynomial of \( t_i, t_{i+2}, \ldots, t_{n-1} \) for each \( i \), where the linear term is equal to \(-t_i \) and the higher order terms are functions of \( t_{i+2}, \ldots, t_{n-1} \) [5]. Thus this gives us a global diffeomorphism, and the inverse map can be found easily. It is also easy to see that the coordinate vector fields \( \partial_i := \partial/\partial t_i \) can be identified with \( \partial_i F_{A_n} \). Moreover, it can be shown that the metric \( \eta_{ij} := \eta(\partial_i, \partial_j) \) is constant by change of coordinates, hence its values can be obtained by looking at the origin as shown in the previous subsection.

The uniqueness of the series expansion implies that the submanifold \( B_{n-1} \) is also the hyperplane defined by the equations \( t_{2i-1} = 0 \) in the flat coordinates. Thus \( \{t_{2i}\} \) constitutes a flat coordinate system for \( B_{n-1} \). Moreover, it can be shown that \( \{t_{2i}, t_* := -a_*\} \) is also a flat coordinate system for \( D_n \).

Now we are ready to describe the potentials. It is known that the potential for \( A_n \) is a polynomial of degree \( \leq n + 2 \) in the flat coordinates. Its explicit form can be determined from the ring structure and the metric. The potential for \( B_{n-1} \) can be obtained by setting \( t_{2i-1} = 0 \) from the potential for \( A_{2n-3} \). Finally, it is known that the potential for \( D_n \) can be obtained by adding the term \((-1/2)a_0t_*^2\) to the \( A_{2n-3} \)
potential and setting $t_{2i-1} = 0$, where $a_0$ is the constant term in the usual coordinates, and is a function of the flat coordinates.

We list some explicit formulas for $n = 3$ and 4. Let $\Phi_{A_{2n-3}}$ denote the potential for $A_{2n-3}$, etc.

**Example 5.3.1** ($n = 3$) *Flat coordinates and their inverse:*

\[
\begin{align*}
t_2 &= -a_2 & a_2 &= -t_2 \\
t_1 &= -a_1 & a_2 &= -t_1 \\
t_0 &= -a_0 + \frac{1}{2}a_2^2 & a_0 &= -t_0 + \frac{1}{2}t_2^2 \\
\Phi_{A_3} &= -\frac{1}{2}t_0^2t_2 - \frac{1}{2}t_0^2t_1^2 - \frac{1}{4}t_1^2t_2 - \frac{1}{60}t_2^5 \\
\Phi_{D_3} &= -\frac{1}{2}t_0^2t_2 + \frac{1}{2}t_0^2t_1^2 - \frac{1}{4}t_2^2t_*^2 - \frac{1}{60}t_2^5
\end{align*}
\]

**Example 5.3.2** ($n = 4$) *Flat coordinates and their inverse:*

\[
\begin{align*}
t_4 &= -a_4 & a_4 &= -t_4 \\
t_3 &= -a_3 & a_3 &= -t_3 \\
t_2 &= -a_2 + \frac{3}{2}a_4^2 & a_2 &= -t_2 + \frac{3}{2}t_4^2 \\
t_1 &= -a_1 + 2a_3a_4 & a_1 &= -t_1 + 2t_3t_4 \\
t_0 &= -a_0 + \frac{1}{2}a_3^2 + a_2a_4 - \frac{7}{6}a_4^3 & a_0 &= -t_0 + \frac{1}{2}t_3^2 + t_2t_4 - \frac{1}{3}t_4^3 \\
\Phi_{A_5} &= -\frac{1}{2}t_0^2t_4 - t_0t_1t_3 - \frac{1}{2}t_0t_2^2 - \frac{1}{2}t_1^2t_2 - \frac{1}{4}t_1^2t_4 - t_1t_2t_3t_4 - \frac{1}{6}t_1t_3^3 \\
&- \frac{1}{6}t_2^3t_4 - \frac{1}{2}t_2t_3t_2 - \frac{1}{6}t_2t_4^2 - \frac{1}{2}t_2t_3t_4^2 - \frac{1}{6}t_3^4t_4 - \frac{1}{8}t_3^4t_4 - \frac{1}{210}t_4^7 \\
\Phi_{D_4} &= -\frac{1}{2}t_0^2t_4 - \frac{1}{2}t_0t_2^2 + \frac{1}{2}t_0t_2^2 - \frac{1}{6}t_2^3t_4 - \frac{1}{2}t_2t_4t_*^2 - \frac{1}{6}t_2^3t_*^2 + \frac{1}{6}t_*^2t_4^3 - \frac{1}{210}t_*^7
\end{align*}
\]
5.4 \( \mathbb{Z}/2\mathbb{Z}\)-Frobenius Algebra for \( A_{2n-3} \)

We review the \( \mathbb{Z}/2\mathbb{Z}\)-Frobenius algebra structure for \( A_{2n-3} \) as described in [12]. Call it \( H \). Let \( \mathcal{I} \) be the ideal of \( \mathbb{C}[z,y] \) generated by \( \{z^{2n-3}, yz, y^2 + z^{2n-4}\} \). Then its ring structure is

\[
H = \mathbb{C}[z,y]/\mathcal{I}.
\]

Note that this is a \( 2n - 2 \) dimensional vector space having \( \{1, z, \ldots, z^{2n-4}, y\} \) as a basis.

The \( \mathbb{Z}/2\mathbb{Z}\)-graded \( \mathbb{Z}/2\mathbb{Z}\)-module structure is the following.

\[
H_i = \text{Span} \{1, z^2, \ldots, z^{2n-4}\}
\]

\[
H_v = \text{Span} \{z, z^3, \ldots, z^{2n-5}\}
\]

\[
H_y = \text{Span} \{y\}
\]

If we fix an ordered basis for \( H \) as \( (1, z^2, \ldots, z^{2n-4}, z, z^3, \ldots, z^{2n-5}, y) \), then \( \eta \) is given by the following matrix.

\[
\begin{pmatrix}
0 & \cdots & 1 & 0 & 0 \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
1 & \cdots & 0 & 0 & 0 \\
0 & \cdots & 1 & \ddots & \vdots \\
0 & \cdots & 0 & \cdots & 0 & -1
\end{pmatrix}
\]

It is straightforward to check that \( H \) thus defined is a \( \mathbb{Z}/2\mathbb{Z}\)-Frobenius algebra.

Remark 5.4.1 \textit{Note that} \( \eta \) \textit{satisfies the conditions of Theorem 5.1.3, and the restriction of} \( \eta \) \textit{on} \( H_e \) \textit{coincides with the metric on the Frobenius algebra of} \( A_{2n-3} \).

Remark 5.4.2 \textit{We observe that} \( H^{\mathbb{Z}/2\mathbb{Z}} \) \textit{is isomorphic to the Frobenius algebra of} \( D_n \) \textit{via the map} \( z^2 \mapsto x \). \textit{Note that it also preserves the metric.}
5.5 $\mathbb{Z}/2\mathbb{Z}$-Frobenius Manifold for $A_{2n-3}$

We use the flat coordinates $\{t_0, \ldots, t_{2n-3}, t_*\}$ as defined in Section 5.3 on the underlying $\mathbb{Z}/2\mathbb{Z}$-graded $\mathbb{Z}/2\mathbb{Z}$-module of the $\mathbb{Z}/2\mathbb{Z}$-Frobenius algebra of $A_{2n-3}$ as described in the previous section. Theorem 5.1.3 gives us a pre-$\mathbb{Z}/2\mathbb{Z}$-Frobenius manifold structure on it. This is because we can obtain the potential for $B_{n-1}$ by restricting either that of $A_{2n-3}$ or that of $D_n$, and the condition on the metric is also met by Remark 5.4.1. Since $t_*$ always appear as $t_*^2$ in the terms of $\Phi_{D_n}$, the $G$-degree requirement for $Y$ is also satisfied. We identify $H_e$ as the Frobenius manifold of $A_{2n-3}$ and $H^G$ as that of $D_n$.

To see that this is a $\mathbb{Z}/2\mathbb{Z}$-Frobenius manifold, note that we have

$$Y^3 = \frac{1}{2} t_0 t_*^2$$

regardless of the value of $n$, from the discussions in Section 5.3. Considering the metric, we notice that this term implies all the multiplication rule that involves $\partial_* \leftrightarrow -y$ (at the origin) described in Section 5.4, up to the factor of $1/6$. Of course, $Y^3_i + Y^3_v$ is just the degree 3 term of $A_{2n-3}$. Thus these terms accounts for the multiplication rules for $\partial_i \leftrightarrow -z^i$ (at the origin), again up to the factor of $1/6$. Hence it contains the same $\mathbb{Z}/2\mathbb{Z}$-Frobenius algebra as in Section 5.4, up to the constant factor of $1/6$. 
6. Prospects

6.1 A Differential Geometry of $G$-braided Spaces

Although we concentrated on the formal aspects of Frobenius manifolds so far, their theory becomes much richer once we start considering their differential geometric aspects [6, 20]. One can then ask if we can extend this part of their theory to our case. This leads to the question of developing a differential calculus on the ring of braided multilinear forms.

The usual way of introducing a differential calculus on the ring of functions in noncommutative geometry is to define a derivation that satisfies the ordinary Leibniz rule on the ring. But in our case it is more natural to define a differential operator in terms of multilinear forms. Namely, the usual rule of differentiation on polynomials can be transferred into one on symmetric multilinear forms via their identifications. Once we extend the same rule to all multilinear forms, then it turns out that this rule restricted to braided multilinear forms yields braided multilinear forms.

The problem with this approach is in that this definition of differentiation rule need not satisfy the usual Leibniz rule. This is not surprising since the usual rule does not seem to take into account any braided structure. Namely, the differential operator and the functions freely change their positions, except possibly gaining the minus sign. It seems that the usual Leibniz rule lives in symmetric monoidal categories.

Hence it will be interesting to generalize the Leibniz rule so that it would work in braided monoidal categories. In fact, one such generalization can be found in [19, Chapter 10] in his version of braided geometry. We will need a version that would work for our purposes.
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