STRING THEORY AND INTEGRABLE SYSTEMS

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Abstract  This is mainly a brief review of some key achievements in a “hot” area of theoretical and mathematical physics. The principal aim is to outline the basic structures underlying integrable quantum field theory models with infinite-dimensional symmetry groups which display a radically new type of quantum group symmetries. Certain particular aspects are elaborated upon with some detail: integrable systems of Kadomtsev-Petviashvili type and their reductions appearing in matrix models of strings; Hamiltonian approach to Lie-Poisson symmetries; quantum field theory approach to two-dimensional relativistic integrable models with dynamically broken conformal invariance. All field-theoretic models in question are of primary relevance to diverse branches of physics ranging from nonlinear hydrodynamics to string theory of fundamental particle interactions at ultra-high energies.

1. Introduction

One of the prevailing views in modern theoretical physics is that fundamental laws of Nature can be derived and understood in terms of field-theoretic models in a lower dimensional space-time possessing infinite-dimensional symmetry groups and, thus, as a rule being integrable.

These models in their various facets and disguises are encompassed in the extremely rich and rapidly developing branch of string theory [1]. It is widely believed that string theory is the most viable candidate for a unified theory of all fundamental interactions at ultra-short distances which, in particular, will provide a consistent reconciliation between General Relativity and Quantum Mechanics - one of the major challenges of this century’s Physics.

We have in mind two large classes of integrable models: conformal field theory (CFT) [2, 3] and massive completely integrable models [4, 5, 6] in $D = 2$ space-time

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dimensions. Typical examples of CFT are the rational CFT’s, of which the most extensively studied are the Wess-Zumino-Novikov-Witten (WZNW) models for various Lie groups G and models, obtained from them by gauging of different subgroups H of G [7]. Thoroughly studied models in the second class are: Sine-Gordon, (nonabelian) massive Thirring models, Toda models for various groups, the Korteweg-de Vries (KdV) and Kadomtsev-Petviashvili (KP) integrable soliton evolution equations and their hierarchies. The feature of integrability, common for both classes of models, stems from the infinite-dimensional Lie-algebraic structure they share: in the first class this being the Noether symmetry algebra, and in the second, the Hamiltonian structure. The underlying infinite-dimensional symmetries, manifesting themselves through the Virasoro (conformal) and affine Kac-Moody algebras [8], as well as through various (infinite-dimensional) generalizations thereof—e.g. the W-algebras [9], play a crucial role in solving and interpreting integrable models.

The interrelation between CFT and completely integrable models became recently explicit through the appearance of KdV and KP integrable hierarchies in the matrix model description [10] of (sub)critical strings (i.e., 2-dimensional gravity interacting with conformal “matter” fields). Thus, it is precisely the integrable field theories which provide the proper framework for incorporation of the huge symmetries of string theory models.

The property of integrability is studied in two aspects: (1) classical and (2) quantum. The principal questions, which one should answer, are the following: (1a) classification; (1b) action-angle type of variables; (1c) exact integration of the equations of motion—in the classical case, and (2a) classification; (2b) exact scattering amplitudes; (2c) exact correlation functions of local fields off the mass shell—in the quantum case.

The concepts and tools to approach the above topics invoke, besides the theory of infinite-dimensional Lie algebras and groups, another outstanding branch of mathematical physics—symplectic geometry or, equivalently, Hamiltonian mechanics. The aim here is to uncover the geometrical foundations of the relevant field theories. The use of geometric techniques offers powerful means for unifying various physical theories and obtaining new insights. It is just in the realm of string theory and the theory of completely integrable systems where the intertwining of Hamiltonian and Lie-group structures in field-theoretic models attains new immensely important qualities. Their quintessence is manifested in the development of the principal methods to solve the quantization problems in integrable models: quantum inverse scattering method [11, 12], representation theory of infinite-dimensional Lie algebras [8], Quantum Groups (non-commutative and non-cocommutative Hopf algebras) [13, 12].

The generic integrable models are massive field theories which in a sense can be regarded as integrable perturbations of conformal field theories [14]. Such models have the advantage of being relativistic invariant and classifiable by the conformal models of which they are perturbations. The latter describe the renormalization group fixed points of these massive integrable models. Their most essential feature, explicitly expos-
ing the intimate connection to conformal models, is the existence of multi-Hamiltonian structures, *i.e.* the existence of at least a second Hamiltonian structure which is compatible with the canonical $R$-matrix Kirillov-Kostant structure [12]. The corresponding fundamental Poisson brackets (linear $R$-matrix brackets and quadratic (Sklyanin) $R$-matrix brackets) naturally arise and are exhaustively understood within the classical "inverse scattering" method [8]. Also, they can be deduced in the semiclassical limit from the basic algebraic structures:

1. Yang-Baxter equation for the quantum version of the $R$-matrix;
2. Fundamental commutation relations for the quantum transfer matrix, involving the quantum $R$-matrix as "structure constants","n which enter the quantum "inverse scattering" method [13] - the first systematic method for quantization of completely integrable models.

In a related development, Drinfeld [16, 13] was the first to realize the deeper algebraic and geometric nature built-in into the theory of classical and quantum completely integrable models. Namely, he showed that the algebraic structures (1) and (2), listed above, constitute the basic structural relations of non-commutative and non-cocommutative Hopf algebras which ultimately received the name "Quantum Groups" and evolved into one of the "hottest" topics in mathematics and theoretical physics. Furthermore, it was realized that a quantum group is a deformation of a classical Lie group much in the same way quantum mechanics is a deformation of classical Hamiltonian (symplectic) mechanics [17]. Most importantly, in the "semiclassical" limit the basic quantum group algebraic structures (1) and (2), formulated above, transliterate into a special distinguished Hamiltonian structure on the classical Lie group $G$, called Lie-Poisson structure, which is compatible with the group multiplication. This is precisely the class of Hamiltonian structures given by the quadratic fundamental $R$-matrix Poisson brackets mentioned above in the context of classical completely integrable models.

The concept of quantum group symmetries in integrable quantum field and statistical mechanics' models lead in the recent years to numerous fruitful and exciting developments in theoretical physics: from generalization (*i.e.*, $q$-deformation) of the fundamental notions of internal and space-time symmetries in quantum field theory and spin-statistics connection to quantum magnetic chains and critical dynamics [18]. Furthermore, the concepts of integrability and perturbations around exactly solvable theories find their place in (close to) realistic models of elementary particles, e.g. in quantum chromodynamics [19].

In what follows, few particular topics among those mentioned above are discussed in some detail.

2. Matrix Models of Non-Perturbative Strings and Integrability

2.1 Conventional Perturbative String Theory
The standard geometric formulation of *perturbative* string theory \[20\] provides the following prescription for calculating physical observables (fermionic degrees of freedom are discarded for simplicity): to construct scattering amplitudes one considers functional integrals over (Euclidean) string world-sheets $\Sigma_{A,G}$ — smooth Riemann surfaces embedded in $D$-dimensional (Euclidean) space-time $\mathbb{R}^D$ of genus $G$ and area $A$:

$$Z_{\text{string}} = \sum_{G=0}^{\infty} g^{2G} \int dA e^{-\Lambda A} Z_{A,G}$$

$$Z_{A,G} = \int [Dh] DX \exp \left\{ -S_{\text{string}}[X, h] \right\} \prod_{i=1}^{n} \int d^2 \sigma_i \sqrt{h} V(X, h; k_i)$$

$$S_{\text{string}} = \frac{1}{2} \int d^2 \sigma \left[ \sqrt{h} \left( h_{ab} \partial_a X^\mu \partial_b X^\nu G_{\mu\nu}(X) + \Phi(X) R^{(2)}(h) \right) + e^{ab} \partial_a X^\mu \partial_b X^\nu B_{\mu\nu}(X) \right]$$

Here the following notations are used: $g$ denotes the string “coupling” constant, $\Lambda$ is the “cosmological” constant, the string action $S_{\text{string}}$ represents a typical $D=2$ conformally invariant field theory model describing $D=2$ gravity (given by the world-sheet metric $h_{ab}(\sigma)$) coupled to world-sheet “matter” fields $X^\mu(\sigma)$ (describing the embedding of $\Sigma_{A,G}$ in $\mathbb{R}^D$). The functionals $G_{\mu\nu}, \Phi, B_{\mu\nu}$ represent the space-time dilaton-gravity multiplet. String interactions are given in (1),(2) by geometrical splitting and joining of individual string world-sheets (thus creating handles on the total world-sheet), whereas asymptotic in-coming and out-going string states are given by vertex operators $V(X, h; k)$ ($k$ indicating the momentum of in/out-state).

Henceforth, for simplicity, we shall suppress the vertex operators $V$’s in (2), i.e., we shall concentrate on the string partition function.

The most difficult part in calculating (2) is the accurate treatment of the functional measure $[Dh]$ over the space of all *gauge-inequivalent* classes of metrics $h_{ab}$ on $\Sigma_{A,G}$ w.r.t. reparametrization and Weyl conformal invariance. In case of conformal gauge $h_{ab} = e^{\phi} \tilde{h}_{ab}(\tau)$ where $\phi$ is the Weyl conformal factor and $\tilde{h}_{ab}(\tau)$ is a reference metric with constant curvature $R^{(2)}(\tilde{h}_{ab})$ which depends in general on the moduli $\{\tau\}$ of the corresponding Riemann surface $\Sigma_{A,G}$, the standard Faddeev-Popov gauge-fixing procedure yields \[21, 22\] : $[Dh] = \delta \left( h_{ab} - e^{\phi} \tilde{h}_{ab}(\tau) \right) \Delta_{\phi} D\phi \left( d\tau \right)$, with the Faddeev-Popov determinant $\Delta_{\phi}$ giving rise to the well-known conformal anomaly.

An important result about the entropy of (random) surfaces with fixed area $A$, first obtained by Zamolodchikov \[22\] in the semi-classical approximation and subsequently strengthened in \[23\], states that for large $A$:

$$Z_{A,G} \simeq_{A \to \infty} \text{const}_G e^{\Lambda_c A} A^{-\chi(1-\gamma_0/2)-1}$$

where $\Lambda_c$ denotes a “critical” value of the “cosmological” constant $\Lambda$, $\chi = 2(1-G)$ is the Euler characteristics of the surfaces, and $\gamma_0$ denotes a critical exponent depending on the world-sheet “matter” fields. Relation (4) implies for the string partition function
which shows that one can obtain complete nonperturbative result for \( Z_{\text{string}} \) by taking the double scaling limit:

\[
\Lambda \rightarrow \Lambda_c \quad , \quad g^2 \rightarrow 0 \quad \text{such that} \quad g^2_{\text{ren}} \equiv \left( \frac{g^2}{(\Lambda - \Lambda_c)^{2-\gamma_0}} \right) = \text{fixed}
\]

### 2.2 Lattice Regularization of String Theory; Matrix Model Formulation

In a series of pioneering papers [24] it was found that statistical mechanical models of random matrices (“matrix models” for short) provide an adequate apparatus for nonperturbative description of lattice-regularized string theory based on the method of random triangulation (and, more generally, random polygonization) of the (Euclidean) string world-sheet. A decisive breakthrough occurred further in refs. [10] which proposed ways for correct implementation of the continuum limit as double scaling limit (6) allowing for exact solutions in string theory.

Let us note that, whereas matrix model formulation of random surfaces is adequate for solving integrable lattice models of planar statistical mechanics, its application to genuine string theory is limited so far to the case of \( D \leq 2 \) dimensional embedding space. Nonetheless, the exact solvability of matrix models provides an important testing ground and qualitative hints for the nonpertubative string theory solution in the realistic cases (for extensive reviews, see [26]).

Since our primary goal here is to elucidate the emergence of integrability structures in the matrix models of string theory, we shall consider for illustrative purpose the simplest one-matrix model whose partition function is given by:

\[
Z = \int d^N M \ e^{-N V(M)} \quad , \quad V(M) = \sum_{k \geq 0} t_k \left( \frac{N}{\beta} \right)^{k/2-1} \text{Tr} M^k
\]

with \( M = \| M_{ij} \| \) being a \( N \times N \) hermitian matrix. In ordinary perturbation theory defined in terms of a free “propagator” \( \langle M_{ij} M_{kl} \rangle_{(0)} \sim N^{-1} \delta_{ik} \delta_{jl} \) and \( k \)-leg vertices with weights \( t_k N \left( \frac{N}{\beta} \right)^{k/2-1} \), each diagram \( \Gamma \) gives contribution of the form:

\[
\prod_{k \geq 3} \left( t_k N \left( \frac{N}{\beta} \right)^{k/2-1} \right)^{V_k(\Gamma)} \cdot N^{-P(\Gamma)} \cdot N^{L(\Gamma)} = \mathcal{W}_{\Gamma} \left[ \{ t \} \right] \cdot N^{\chi(\Gamma)} \left( \frac{\beta}{N} \right)^{-L(\Gamma)}
\]

On the l.h.s. of (8) \( V_k(\Gamma) \), \( P(\Gamma) \), \( L(\Gamma) \) denote number of \( k \)-leg vertices, propagators (links) and closed loops (faces) of \( \Gamma \), whereas on the r.h.s. \( \chi(\Gamma) = L(\Gamma) - P(\Gamma) + \sum_{k \geq 3} V_k(\Gamma) \) denotes the Euler characteristics of the two-dimensional polygonized surface spanned by \( \Gamma \), and \( \mathcal{W}_{\Gamma} \left[ \{ t \} \right] \) indicates the product of the vertex weights. Clearly,
$L(\Gamma) \equiv A(\Gamma)$ can be understood as area of $\Gamma$. Thus, the partition function (7) can be written as:

$$Z = \exp \left\{ \sum_{\text{conn. surfaces } \Gamma} N \chi(\Gamma) e^{-\left(\ln \beta/N\right) A(\Gamma) + \ln W_\Gamma}\left\{t\right\}} \right\}$$

(9)

i.e., the free energy $\ln Z$ of the matrix model (9) represents the discretized regularized partition function of random surfaces (more precisely, “pure” $D = 2$ gravity with action $A(\Gamma)$ interacting with “matter” with action $\ln W_\Gamma$) upon making the following identifications (comparing with (1)-(2), (4)):

$1/N \approx g$ (“bare” string coupling constant), $N/\beta \approx e^{-(\Lambda - \Lambda_c)}$ ($\Lambda$ - the “cosmological” constant) and the double scaling limit (6) takes the form of a special continuum limit:

$$\frac{N}{\beta} \rightarrow 1 \quad N \rightarrow \infty,$$

such that $g_{\text{string}}^2 \equiv \beta^2 \left(\frac{\beta}{N} - 1\right)^{2-\gamma_0} = \text{fixed}$

(10)

Explicit solution for the partition function (7) is derived by the method of orthogonal polynomials [25]. Diagonalizing the hermitian matrix $M = U \text{diag}(\lambda_1, \ldots, \lambda_N) U^{-1}$ and integrating over angle variables in $U$, one gets (rescaling $M \rightarrow \left(\beta/N\right)^2 M$):

$$Z = \int \prod_{i=1}^{N} d\lambda_i \Delta(\lambda) \exp \left\{ -\beta \sum_{i=1}^{N} V(\lambda_i) \right\} \Delta(\lambda) ; \quad V(\lambda_i) = \sum_{k \geq 0} t_k \lambda_i^k, \quad \Delta(\lambda) = \prod_{i<j} (\lambda_i - \lambda_j)$$

(11)

Introducing a complete set of orthogonal polynomials $P_n(\lambda) = \lambda^n + \text{lower order terms}$:

$$\int d\lambda P_n(\lambda)e^{-\beta V(\lambda)}P_m(\lambda) = h_n \delta_{nm}$$

(12)

and re-expressing the van der Mond determinant as $\Delta(\lambda) = \text{det}\|P_i(\lambda_j)\|$, the integrals in (11) factorize and yield upon using (12):

$$Z \left[\left\{t\right\}\right] = \prod_{n=0}^{N} h_n \left(\left\{t\right\}\right)$$

(13)

where the dependence on the parameters of the random matrix potential is indicated. Explicit solutions for $h_n \left(\left\{t\right\}\right)$ can be found through solving the flow equations w.r.t. $t_k$ which correspond to integrable lattice hierarchies, as briefly discussed in the next subsection.

2.3 Differential Integrable Hierarchies from Matrix Models

The appearance of integrable hierarchies in the continuum (double scaling) limit (10) are extensively discussed in numerous papers (see [29] and refs. therein). It turns out, however, that flow equations inherent to integrable hierarchies can be extracted directly from discrete matrix models even before taking the continuum limit [27, 28], which reveals their close connection with topological field theories [29].
The above result is achieved most easily by employing again the method of orthogonal polynomials \((12)\). Namely, on the Hilbert space spanned by \(\{P_n(\lambda)\}_{n\geq 0}\) one introduces two conjugate operators \(Q, P\) with matrix elements defined as:

\[
\begin{align*}
h_n Q_{mn} &= \int d\lambda P_n(\lambda)e^{-\beta V(\lambda)}\lambda P_m(\lambda) \\
h_n P_{mn} &= \int d\lambda P_n(\lambda)e^{-\beta V(\lambda)} \frac{d}{d\lambda} P_m(\lambda)
\end{align*}
\]

From \((14), (15)\) one easily gets the matrix model “string” equation (the second one below):

\[
P = \beta \left( V'(Q) \right)_{(-)} \rightarrow \left[ \beta \left( V'(Q) \right)_{(-)} , Q \right] = 1
\]

where the subscript \((-)\) denotes taking strictly lower-diagonal part of the corresponding matrix. The “string” equation yields recursion relations for the matrix elements \((14)\) of \(Q\).

It is straightforward to deduce from \((14)\) and \((12)\) the following flow equations:

\[
\begin{align*}
\frac{\partial P_n(\lambda)}{\partial t_r} &= Q^r_{(-) nm} P_m(\lambda) , \quad Q_{nm} P_m(\lambda) = \lambda P_n(\lambda) \\
\frac{\partial Q}{\partial t_r} &= \left[ Q^r_{(-)} , Q \right]
\end{align*}
\]

Eq.\((18)\) is the integrability condition for eqs.\((17)\) and it is compatible with the “string” equation \((16)\). One can easily identify \((18)\) with the Lax form of the flow equations of the integrable Toda lattice hierarchy by inserting into \((18)\) the explicit form of the \(Q\) matrix elements:

\[
Q_{n,n+1} = 1 , \quad Q_{n,n} \equiv S_{n-1} , \quad Q_{n+1,n} = \frac{h_{n+1}}{h_n} \equiv R_n \equiv e^{\phi_n - \phi_{n-1}}
\]

the rest being zero as a consequence of the recurrence relations for orthogonal polynomials. Eqs.\((18)\) are now Hamiltonian, and the lowest one (with \(r = 1\)) is generated by the well-known Toda lattice Hamiltonian:

\[
H_{Toda} = \frac{1}{2} \sum_n S_n^2 + \sum_n \left( e^{\phi_{n+1} - \phi_n} - 1 \right) , \quad \{ S_n , \phi_m \} = \delta_{nm}
\]

Following \([23]\), it is possible to replace the discrete lattice integrable hierarchy \((18)\) by a differential hierarchy at each fixed lattice site \(n\) where the continuum variable is \(x = t_1\). Indeed, from the first \((r = 1)\) flow eqs. \((17)\) and \((18)\) yielding

\[
\begin{align*}
\frac{\partial P_{n+1}}{\partial t_1} &= R_n P_n , \quad \frac{\partial R_n}{\partial t_1} = R_n \left( S_{n-1} - S_n \right) , \quad \frac{\partial S_n}{\partial t_1} = R_n - R_{n+1}
\end{align*}
\]

one obtains (upto gauge transformation \(P_n \rightarrow \psi_n = \exp \{ \int dt_1 S_{n-1} \} P_n = h_n^{-1} P_n\), and similarly \(Q_{nm} \rightarrow \tilde{Q}_{nm} = h_n^{-1} Q_{nm} h_m\)):

\[
\lambda \psi_n = \tilde{Q}_{nm} \psi_m = h_n^{-1} ( P_{n+1} + S_{n-1} P_n + R_{n-1} P_{n-1} ) = \left[ \partial + R_n (\partial - S_n)^{-1} \right] \psi_n
\]
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where $\partial \equiv \frac{\partial}{\partial t_1}$. Once again using (21), one can rewrite the discrete evolution eqs. (17), (18) in a differential Lax form for a fixed lattice site $n$:

$$\frac{\partial \psi}{\partial t_r} = (L_r)^+ \psi, \quad \frac{\partial L}{\partial t_r} = \left[ (L_r)^+ , L \right]$$

where $r \geq 2$, $\psi \equiv \psi_n(t_1)$ and the subscript $+$ indicates taking the purely differential part of the corresponding pseudo-differential Lax operator (cf. last equality in (22)):

$$L = \partial + A(\partial - B)^{-1}, \quad A = R_n(t_1), \quad B = S_n(t_1)$$

Eqs. (23), (24) are immediately recognized as the well-known 2-boson reduction of KP integrable hierarchy.

Using generalization of the method of orthogonal polynomials, it is possible to derive flow equations for integrable hierarchies also in the general case of multi-matrix models (describing random surfaces interacting with $q$ different types of “matter”):

$$Z = \int \prod_{i=1}^q d^N M_i \exp \left\{ - \sum_{i=1}^q \left( \text{Tr} M_i M_{i+1} + \sum_{k \geq 0} t_{i,k} \text{Tr} M_i^k \right) \right\}$$

without passing to the continuum limit [28]. The appropriate generalization of (24) now reads:

$$L_q = \partial + \sum_{l=1}^q A_l (\partial - B_l)^{-1} (\partial - B_{l+1})^{-1} \cdots (\partial - B_q)^{-1}$$

which is a $2q$-boson reduction of KP integrable hierarchy (see subsection 3.3 for more details).

Finally, returning to the string partition function (13), one can show that $Z \{\{t\}\} = \tau_N(t)$ – the Toda lattice $\tau$-function [27] subject to the so called Virasoro constraints:

$$\mathcal{L}_s Z \{\{t\}\} = 0 \quad , \quad s \geq -1 \quad ; \quad [\mathcal{L}_r, \mathcal{L}_s] = (r - s) \mathcal{L}_{r+s}$$

which are equivalent to the constraints on the pertinent integrable hierarchies imposed by the “string” equation (16) and are in fact Ward identities due to the symmetry $\delta_s M = \varepsilon_s M^{s+1}$, $s \geq -1$ of (7). Similar relations hold for multi-matrix models. Moreover, in the continuum (double scaling) limit $Z \{\{t\}\}$ can be identified with $\tau$-functions of reduced KP hierarchies subject to the so-called $W$-constraints (generalizations of (27) which span $W$-algebras; cf. (49)-(50) below).

3. Integrable Systems in Classical Physics : Geometric Formulation
Now we pass to a brief review of some principal structures and properties of integrable systems in classical and quantum mechanics (or field theory). We first recall the notion of integrability.

**Complete integrability:** Consider a Hamiltonian system with \( n \) degrees of freedom possessing standard Hamiltonian structure with a Hamiltonian \( H(p,q) \) and Poisson bracket \( \{\cdot,\cdot\} \). A Hamiltonian system is called *completely* (or Liouville) *integrable* if it has \( n \) independent conserved quantities (integrals of motion) \( I_k(p,q), k = 1, \ldots, n \), which are in involution: \( \{I_i,I_j\} = 0 \). For such a system we can find the *action-angle* canonical variables and write explicitly the general solution to the equations of motion.

**Lax formulation:** For infinite-dimensional (field theory) integrable Hamiltonian systems, there exists the convenient Lax (or “zero-curvature”) formulation \[5\]. In the Lax formulation, the phase space of the Hamiltonian system is parametrized by elements \( L \) taking values in some Lie algebra \( G \) and the dynamical equations of motion can be written in terms of a Lax pair \( L, P \), the latter similarly taking values in \( G \), as the Lax-type equation

\[
\frac{dL}{dt} = [L, P] \tag{28}
\]

The Lax formulation leads straightforwardly to the construction of the involutive integrals of motion. Namely, for any \( \text{Ad} \)-invariant function \( I \) on \( G \), \( I(L) \) is a constant of motion. In fact, it can be shown that any completely integrable Hamiltonian system admits a Lax representation (at least locally) \[30\].

### 3.1 Adler-Kostant-Symes/Reyman-Semenov-Tyan-Shansky Scheme

A very wide class of integrable models can be constructed through the application of the AKS-RS approach \[31, 15\] having roots in the group coadjoint orbit method \[32\].

Let \( G \) denote a Lie group and \( \mathcal{G} \) be its Lie algebra. \( G \) acts on \( \mathcal{G} \) by the adjoint action: \( \text{Ad}(g) X = gXg^{-1} \), with \( g \in G \) and \( X \in \mathcal{G} \). Let \( \mathcal{G}^* \) be the dual space of \( \mathcal{G} \) relative to a non-degenerate bilinear form \( \langle \cdot | \cdot \rangle \) on \( \mathcal{G} \). The corresponding coadjoint action of \( G \) on \( \mathcal{G}^* \) is obtained from the duality of \( \langle \cdot | \cdot \rangle \): \( \langle \text{Ad}^*(g)U | X \rangle = \langle U | \text{Ad}(g^{-1})X \rangle \). We will denote the infinitesimal versions of adjoint and coadjoint transformations (for \( g = \exp Y \)) by \( \text{ad}(Y)X = [Y, X] \) and \( \langle \text{ad}^*(Y)U | X \rangle = -\langle U | [Y, X] \rangle \), respectively. In particular, when \( \mathcal{G} \) is endowed with an \( \text{ad} \)-invariant bilinear (Killing) form \( \langle \cdot, \cdot \rangle \) allowing to identify \( \mathcal{G}^* \) with \( \mathcal{G} \), we have \( \text{ad}^*(Y)U = [Y, U] \).

There exists a natural Poisson structure on the space \( C^\infty(\mathcal{G}^*, \mathbb{R}) \) of smooth, real-valued functions on \( \mathcal{G}^* \) (sometimes called Kirillov-Kostant (KK) bracket), given by:

\[
\{F, H\}(U) = -\left\langle U \left| \left[ \nabla F(U), \nabla H(U) \right] \right. \right\rangle \tag{29}
\]

where \( F, H \in C^\infty(\mathcal{G}^*, \mathbb{R}) \), the gradient \( \nabla F : \mathcal{G}^* \rightarrow \mathcal{G} \) is defined by the standard formula \( \frac{d}{dt}F(U + tV)|_{t=0} = \left\langle V \left| \nabla F(U) \right. \right\rangle \) and \( \langle \cdot, \cdot \rangle \) is the standard Lie commutator on \( \mathcal{G} \). On each orbit of \( G \) in \( \mathcal{G}^* \) the Poisson bracket \[23\] gives rise to a non-degenerate
symplectic structure. Moreover, for any Hamiltonian function \( H \) on such an orbit we have a Hamiltonian equation of motion \( dU/dt = \text{ad}^*(\nabla H(U))U \) (\( = [\nabla H(U), U] \) when \( \mathcal{G} \) has Killing form).

Ref.[15] introduced the \( R \)-operator (generalized \( R \)-matrix) as a linear map from a Lie algebra \( \mathcal{G} \) to itself such that the bracket:

\[
[X,Y]_R \equiv \frac{1}{2}[RX,Y] + \frac{1}{2}[X,RY]
\]

defines a second Lie-bracket structure on \( \mathcal{G} \), or equivalently, defines a second Lie algebra \( \mathcal{G}_R \) isomorphic to \( \mathcal{G} \) as a vector space. The Jacobi identity for the \( R \)-commutator (30) implies that the modified Yang-Baxter equation (YBE) for the \( R \)-matrix must hold (for arbitrary \( X_{1,2,3} \in \mathcal{G} \)):

\[
\sum_{\text{cyclic}(1,2,3)} \left[ X_1, \left[ RX_2, RX_3 \right] - R \left( \left[ RX_2, X_3 \right] + \left[ X_2, RX_3 \right] \right) \right] = 0 \quad (31)
\]

A sufficient condition (occurring in all interesting cases) for the fulfillment of (31) is:

\[
\left[ RX, RY \right] - R \left( \left[ RX, Y \right] + \left[ X, RY \right] \right) = -\alpha \left[ X, Y \right] \quad (32)
\]

(\( \alpha \) being arbitrary constant), which usually is written in terms of the “ordinary” \( r \)-matrix \( r \in \mathcal{G} \otimes \mathcal{G} \) canonically isomorphic (upto a factor) to \( R \in \mathcal{G} \otimes \mathcal{G}^* \) via the Killing form \( \mathcal{R} \):

\[
\left( 12 \right) r \equiv r_{ij}T^i \otimes T^j \otimes \mathbb{1} \in \mathcal{U}(\mathcal{G}) \otimes \mathcal{U}(\mathcal{G}) \otimes \mathcal{U}(\mathcal{G}) \quad \text{etc.}
\]

where \( \mathcal{U}(\mathcal{G}) \) is the universal enveloping algebra. Classification of solutions of (33) (depending on a spectral parameter) for simple \( \mathcal{G} \) is given in refs.[33].

With the help of (30) one can furthermore introduce on \( \mathcal{G}_R^* \simeq \mathcal{G}^* \) a new KK-type Poisson bracket \( \{ \cdot, \cdot \}_R \) called \( R \)-bracket by substituting the usual Lie commutator \( [\cdot,\cdot] \) for the \( R \)-Lie commutator \( [\cdot,\cdot]_R \) in (29):

\[
\{ F, H \}_R(U) = -\left\langle U \left| \left[ \nabla F(U), \nabla H(U) \right] \right\rangle \right._R \quad (34)
\]

A function \( H \) on \( \mathcal{G}_R^* \) is called \( \text{Ad}^* \)-invariant (Casimir) if \( H[\text{Ad}^*(g)U] = H[U] \) or, infinitesimally, \( \text{ad}^*(\nabla H(U))(U) = 0 \) for each \( U \in \mathcal{G}_R^* \). Then one can show [15] that:

1. the \( \text{ad}^* \)-invariant functions are in involution with respect to both brackets (29) and (14);
2. the Hamiltonian equation on \( \mathcal{G}_R^* \simeq \mathcal{G}_R^* \) takes the following (generalized Lax) form:

\[
dU/dt = \frac{1}{2}\text{ad}^* \left( R(\nabla H(U)) \right) U \quad \left( = \left[ \frac{1}{2} R(\nabla H(U)), U \right] \right) \quad \text{for} \ \mathcal{G} \ \text{with Killing form} \quad (35)
\]

\( 3_r = r_{ij}T^i \otimes T^j \) and \( \frac{1}{2}RX = T^i r_{ij} (T^j, X) \), where \{\( T^i \)\} denotes a basis in \( \mathcal{G} \).
Eq.(35) can be obtained from variational principle with the following geometric action:

\[ W[U] = -\int \langle U|\mathcal{Y}_R(U)\rangle - \int dt H[U] \] (36)

\[ dU = ad^*_R(\mathcal{Y}_R(U))U \quad \rightarrow \quad d\mathcal{Y}_R = \frac{1}{2}[\mathcal{Y}_R, \mathcal{Y}_R]_R \] (37)

where the integrals in (36) are along arbitrary smooth curve on the phase space \( G^* \); \( H[U] \) is any Casimir on \( G^* \); \( \mathcal{Y}_R(U) \) is Maurer-Cartan one-form on \( G^R \) and function of \( U \in G^* \) determined by the first eq.(37), with the \( R \)-coadjoint action :

\[ ad^*_R(X)U = \frac{1}{2}(ad^*(RX)U + R^*(ad^*(X)U)) \]

\[ = \frac{1}{2}[RX, U] - \frac{1}{2}R([X, U]) \quad \text{for } G \text{ with Killing form} \] (38)

Hence the above AKS-RS technique leads to a direct construction of completely integrable systems where the set of independent Casimir functions on \( G^* \) is precisely the complete set of integrals of motion in involution. The basic realization of this technique arises when the Lie algebra \( G \) decomposes as a vector space into two subalgebras \( G^+ \) and \( G^- \), i.e. \( G = G^+ \oplus G^- \). Let \( P_\pm \) be the corresponding projections on \( G_\pm \). Then \( R = P_+ - P_- \) satisfies the modified YBE (31) and provides a specific realization for the above scheme. In this case eqs.(30), (34), (35) and (38) take the following simple form:

\[ [X, Y]_R = [X_+, Y_+] - [X_-, Y_-] \quad , \quad (ad^*_R(X)U)_\pm = \mp[X_\pm, U_\pm]_\pm \] (39)

\[ \left\{ \langle U_\pm|X_\pm\rangle, U_\pm \right\} = \pm\left[X_\pm, U_\pm\right]_\pm \quad , \quad \frac{dU_\pm}{dt} + \left(\frac{\delta H}{\delta U}\right)_\pm, U_\pm = 0 \] (40)

where \( X_\pm = P_\pm X \in G_\pm \), \( U_\pm = P_\pm^*U \in (G^\pm)^* \), \( [X, U]_\pm = P_\pm^*([X, U]) \in (G^\pm)^* \).

### 3.2 Algebra of Pseudo-Differential Operators and Integrable Hierarchies of Kadomtsev-Petviashvili type

Here we will illustrate the AKS-RS construction on \( G = \Psi DO \) - the algebra of pseudo-differential operators on a circle. Recall that an arbitrary pseudo-differential operator \( X(x, D_x) = \sum_{k \geq -\infty} X_k(x)D_x^k \) is conveniently represented by its symbol \( \langle 34 \rangle \) - a Laurent series in the variable \( \xi : X(\xi, x) = \sum_{k \geq -\infty} X_k(x)\xi^k \), and the operator multiplication corresponds to the following symbol multiplication:

\[ X(\xi, x) \circ Y(\xi, x) = \sum_{N \geq 0} \frac{1}{N!} \frac{\partial^N X}{\partial \xi^N} \frac{\partial^N Y}{\partial x^N} \] (41)

which determines a Lie algebra structure given by : \([X, Y] \equiv (X \circ Y - Y \circ X)\). On \( \Psi DO \) one introduces an invariant, non-degenerate bilinear form:

\[ \langle L|X\rangle \equiv \text{Tr}_A(LX) = \int dx \text{Res}_\xi \left(L(\xi, x) \circ X(\xi, x)\right) \] (42)
which allows identification of the dual space $\Psi \mathcal{D} \mathcal{O}^*$ with $\Psi \mathcal{D} \mathcal{O}$.

There exist three natural decompositions of $\mathcal{G} = \Psi \mathcal{D} \mathcal{O}$ into a linear sum of two subalgebras $\mathcal{G} = \mathcal{G}_+^\ell \oplus \mathcal{G}_-^\ell$ labelled by the index $\ell$ taking values $\ell = 0, 1, 2$:

$$\mathcal{G}_+^\ell = \left\{ X_+ \equiv X_{\geq \ell} = \sum_{i=\ell}^\infty X_i(x) D^i \right\}; \quad \mathcal{G}_-^\ell = \left\{ X_- \equiv X_{<\ell} = \sum_{i=-\ell+1}^\infty X_{-i}(x) D^{-i} \right\},$$

(43)

Correspondingly the dual spaces to subalgebras $\mathcal{G}_+^\ell$ are given by:

$$\mathcal{G}_+^{\ell*} = \left\{ L_- \equiv L_{<\ell} = \sum_{i=\ell+1}^\infty D^{-i} \circ u_{-i}(x) \right\}; \quad \mathcal{G}_-^{\ell*} = \left\{ L_+ \equiv L_{\geq \ell} = \sum_{i=-\ell}^\infty D^i \circ u_i(x) \right\}.$$

(44)

Note that in (44) the differential operators are put to the left. Henceforth, we shall skip the sign $\circ$ in symbol products for brevity.

Defining $R_\ell = P_+ - P_-$ for each of the three cases, eqs.(43) take the form:

$$[X, Y]_{R_\ell} = [X_{\geq \ell}, Y_{\geq \ell}] - [X_{<\ell}, Y_{<\ell}] \quad \text{and} \quad \text{ad}_{R_\ell}^\ast(X)L = [X_{\geq \ell}, L_{<\ell}]_{<\ell} - [X_{<\ell}, L_{\geq \ell}]_{\geq \ell}$$

(45)

Now, choosing an infinite set of independent Casimir functions:

$$H_{m+1} = \frac{1}{m+1} \int dx \operatorname{Res} L^{m+1} \quad m = 0, 1, 2, \ldots$$

(46)

the three decompositions (44) of $\Psi \mathcal{D} \mathcal{O}$ labelled by $\ell = 0, 1, 2$ yield, according to the AKS-RS scheme, three different integrable hierarchies – the standard KP hierarchy ($\ell = 0$) and the first and second modified KP hierarchies ($\ell = 1, 2$):

$$\frac{\partial L}{\partial t_m} + \left[ (L^m)_{\geq \ell}, L \right] = 0$$

(47)

One can show that all three KP hierarchies are related through symplectic gauge transformations [33]. Here we shall concentrate on various Poisson reductions [34] of the standard KP hierarchy which appear, in particular, in the context of matrix models of strings.

The phase space of the standard KP integrable system is:

$$\mathcal{M}_{KP} = \left\{ L = D + \sum_{k=1}^\infty u_k(x) D^{-k} \right\}$$

(48)

\[\text{Let us recall the general notions [30]. Let } (\mathcal{M}, P) \text{ be a smooth Poisson manifold with Poisson structure } P : T^*(\mathcal{M}) \to T(\mathcal{M}) \text{ and let } S \text{ be a smooth submanifold of } \mathcal{M} \text{ with embedding } \mu : S \to \mathcal{M}. \text{ Now, a Poisson structure } P' : T^*(S) \to T(S) \text{ on } S \subset \mathcal{M} \text{ is called Poisson reduction of } P \text{ if for arbitrary functions on } \mathcal{M} \text{ the following property is satisfied: } \mu^\ast \{f_1, f_2\}_P = \{\mu^\ast f_1, \mu^\ast f_2\}_P. \text{ In other words, restriction of the Poisson brackets w.r.t. } P \text{ of arbitrary functions on } \mathcal{M} \text{ to the submanifold } S \text{ is equivalent to computing the Poisson brackets w.r.t. } P' \text{ of the restrictions on } S \text{ of these same functions.}\]
even more important, one can express all eqs. (23) indeed correspond to Hamiltonian integrable systems. Furthermore, what is called canonical pairs of coordinates and are recognized as the (centerless) \( \mathfrak{w}_{1+\infty} \) algebra (see [3]), which is isomorphic to \( D\mathcal{O}(S^1) \subset \Psi\mathcal{O} \) (algebra of purely differential operators) [37].

Now, returning to the flow eqs. in (multi)matrix models [23] with Lax operator [26], one can show [38] that the space of 2q-boson Lax operators:

\[
\mathcal{M}_q = \left\{ L_q = D + \sum_{l=1}^{q} A_l (D - B_l)^{-1} (D - B_{l+1})^{-1} \ldots (D - B_q)^{-1} \right\}
\]

is a legitimate Poisson reduction of the full KP hierarchy given by [18], i.e., the flow eqs. [23] indeed correspond to Hamiltonian integrable systems. Furthermore, what is even more important, one can express all 2q fields parametrizing (51) in terms of Darboux canonical pairs of coordinates \((a_r(x), b_r(x))_{r=1}^q\): \(B_l = b_l + b_{l+1} + \cdots b_q\), \(A_q = a_q\)

\[
A_{q-r}(a,b) = \sum_{n_r=r}^{q-1} \sum_{n_2=2}^{n_3-1} \sum_{n_1=1}^{n_2-1} \left( \partial + b_{n_r} + \cdots + b_{n_r-r+1} \right) \cdots \left( \partial + b_{n_1} \right) a_{n_1} \)
\]

\[
\{a_r(x), b_s(y)\}_{Pr} = -\delta_{rs} \delta(x-y)
\]

Expressing \(L_q\) in (51) as power series in \(D^{-1}\):

\[
L_q = D + \sum_{k=1}^{\infty} U_k[(a,b)](x) D^{-k}
\]

\[
U_k[(a,b)](x) = a_q P^{(1)}_{k-1}(b_q) + \sum_{r=1}^{q-r} A_{q-r}(a,b) P^{(r+1)}_{k-1-r}(b_q, b_q + b_{q-1}, \ldots, \sum_{l=q-r} b_l)
\]

where \(A_{q-r}(a,b)\) are the same as in [33], and \(P^{(N)}_n\) denote the (multiple) Faà di Bruno polynomials:

\[
P^{(N)}_n(B_N, B_{N-1}, \ldots, B_1) = \sum_{m_1 + \cdots + m_N = n} (-\partial + B_1)^{m_1} \cdots (-\partial + B_N)^{m_N} \cdot 1
\]

one obtains a series of explicit (Poisson bracket) realizations of \( \mathfrak{w}_{1+\infty} \) algebra in terms of finite number of 2q bosons \((a_r, b_r)_{r=1}^q\) for any \(q = 1, 2, 3, \ldots\) (cf. [19]-[30]):

\[
\{U_k[(a,b)](x), U_l[(a,b)](y)\} = \Omega_{k-1,l-1}(U[(a,b)]) \delta(x-y)
\]

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Similar “Darbouxization” of $L_q$ holds w.r.t. the second KP Hamiltonian structure (see next subsection).

3.3 Lie-Poisson Groups and Lie Bi-Algebras: Hamiltonian approach

Completely integrable systems possess another notorious feature – that of multi-Hamiltonian structure, i.e., they always possess at least two independent compatible Poisson structures. In the case of KP integrable hierarchies (and their reductions) it turns out that the “second” Poisson structure can be understood in terms of the Lie-Poisson structure on the Lie group $G$ corresponding to the Lie algebra $\mathcal{G}$ entering the AKS-RS construction, whereas the “first” Poisson structure is just the KK structure (34) on $G^\ast \otimes \mathbb{R}$.

The notions of a Lie-Poisson group and its “infinitesimal” version – Lie Bi-algebra were first introduced by Drinfeld [16]. A Lie group $G$ is called Lie-Poisson if on the algebra of smooth functions on it $\text{Fun}(G)$ there exists Poisson structure which is compatible with the group multiplication, i.e., $\Delta \left( \{ F_1, F_2 \} \right) = \{ \Delta (F_1), \Delta (F_2) \}$ where $\Delta$ denotes the coproduct in $\text{Fun}(G)$ ($\Delta (F(g)) = F(gh)$).

More precisely, the Lie-Poisson structure is given by:

$$\{ F_1(g), F_2(g) \} = \left\langle \nabla_L F_1(g) \otimes \nabla_L F_2(g) \right| r(g) \right\rangle$$

$$\left( = - \left\langle \nabla_R F_1(g) \otimes \nabla_R F_2(g) \right| r(g^{-1}) \right\rangle \right)$$

$$\left\langle \nabla_L F(g) \right| X \right\rangle = \left. \frac{d}{dt} \right|_{t=0} F(e^{tX}g) , \quad \left\langle \nabla_R F(g) \right| X \right\rangle = \left. \frac{d}{dt} \right|_{t=0} F(ge^{tX})$$

where $\nabla_{LR}$ denote left/right Lie-derivatives, and where $r(g)$ is a cocycle on $G$ with values in $\mathcal{G} \otimes \mathcal{G}$:

$$r(gh) = r(g) + \text{Ad}(g) \otimes \text{Ad}(g) r(h)$$

In the case when $r(g)$ is a coboundary, i.e.

$$r(g) = \text{Ad}(g) \otimes \text{Ad}(g) r_0 - r_0$$

with $r_0 \in \mathcal{G} \otimes \mathcal{G}$ being a constant element, one easily finds that the Jacobi identity for (59) reduces precisely to the YBE (33) for $r_0$ as a classical $r$-matrix.

Let us note that for matrix groups eq.(59) can be written in a simpler form:

$$\{ g \circ g \} = r(g) g \otimes g \quad \left( \{ g \circ g \} = - [ r_0 , g \otimes g ] \quad \text{in case of (62)} \right)$$

Eq.(61) implies the following exterior derivative equation:

$$\left. \frac{d}{dt} \right|_{t=0} r(e^{tX})$$

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Eq.(61) implies the following exterior derivative equation:

$$\left. \frac{d}{dt} \right|_{t=0} r(e^{tX})$$
where \( Y(g) = dg g^{-1} \) denotes the Maurer-Cartan one-form on \( \mathcal{G} \), and \( \phi(\cdot) \) defined in (65) is a \( \mathcal{G} \otimes \mathcal{G} \)-valued cocycle on \( \mathcal{G} \):

\[
\phi([X, Y]) = [X \otimes \mathbb{1} + \mathbb{1} \otimes X, \phi(Y)] + [\phi(X), Y \otimes \mathbb{1} + \mathbb{1} \otimes Y]
\]  

(66)

The solution of (64) is a coboundary \( r(g) \) (62) if and only if \( \phi(\cdot) \) is coboundary:

\[
\phi(X) = [r_0, X \otimes \mathbb{1} + \mathbb{1} \otimes X]
\]  

(67)

The cocycle \( \phi(\cdot) \) allows to introduce a Lie-commutator \([\cdot, \cdot]\) on the dual space \( \mathcal{G}^* \) and, correspondingly, coadjoint action \( ad^*_v(\cdot) \) of \( \mathcal{G}^* \) on \( \mathcal{G} \) as follows:

\[
\left\langle [U, V], X \right\rangle \equiv - \left\langle V, ad^*_v(U)X \right\rangle = \left\langle U \otimes V, \phi(X) \right\rangle \quad \forall U, V \in \mathcal{G}^*, \forall X \in \mathcal{G}
\]  

(68)

whereas the mixed commutator between elements of \( \mathcal{G} \) and \( \mathcal{G}^* \) is defined as:

\[
[X, U] = ad^*(X)U - ad^*_v(U)X
\]  

(69)

According to an important theorem by Drinfeld [16], the Lie algebra \( \mathcal{G} \) of each Lie-Poisson group \( G \) possesses a Lie-bialgebraic structure and vice versa. Namely, by means of (68) and (69) the direct sum (as vector space) \( \mathcal{D} = \mathcal{G} \oplus \mathcal{G}^* \) becomes itself a Lie algebra called “the double”, such that \( \mathcal{G} \) and \( \mathcal{G}^* \) are isotropic subalgebras of \( \mathcal{D} \) w.r.t. the Killing form on \( \mathcal{D} \):

\[
\left( (X_1, U_1), (X_2, U_2) \right) = (X_1 \otimes U_1) + (U_2 \otimes X_1) \quad \forall (X_{1,2}, U_{1,2}) \in \mathcal{D}
\]  

(70)

The triple \( (\mathcal{D}, \mathcal{G}, \mathcal{G}^*) \) is also called Manin triple. Furthermore, the double group \( \widetilde{\mathcal{D}} \simeq G \times G^* \), corresponding to \( \mathcal{D} \), is direct product (as a manifold) of its subgroups \( G \) and \( G^* \) corresponding to \( \mathcal{G} \) and \( \mathcal{G}^* \), respectively. Let us stress the symmetry of the above construction under the exchanges \( \mathcal{G} \leftrightarrow \mathcal{G}^* \), \( G \leftrightarrow G^* \).

Now it is easy to write down the explicit solution for the Lie-Poisson group cocycle \( r(g) \) (61):

\[
\langle U \otimes V | r(g) \rangle = \left\langle \left( g^{-1}Vg \right)_-, \left( g^{-1}Ug \right)_+ \right\rangle \quad \forall U, V \in \mathcal{G}^*
\]  

(71)

where the subscripts \((\pm)\) indicate projections in the double algebra \( \mathcal{D} \) along \( \mathcal{G}, \mathcal{G}^* \), respectively.

As in the AKS-RS Lie-algebraic scheme, one can construct integrable Hamiltonian systems on Lie-Poisson groups. Namely, from (61) or (62) one can easily verify that all \( Ad(\cdot) \)-invariant functions on \( G \), \( H[hgh^{-1}] = H[g] \), are in involution w.r.t. the Lie-Poisson structure (63) : \{ \( H_k[g] \), \( H_l[g] \) \} = 0 . Similarly, the analogues of the Hamiltonian eqs. of motion (63) and the associated geometric action (60) take now the form:

\[
\frac{\partial g}{\partial t_k} g^{-1} = -\hat{r}_g \left( \nabla_L H_k[g] \right)
\]  

(72)

\[
\mathcal{W}[g] = -\frac{1}{2} \int d^{-1} \left( \left\langle \hat{s}_g(Y(g)) \right| Y(g) \right\rangle \right) - \int dt H_k[g]
\]  

(73)
where the action of the operator \( \hat{r}_g : \mathcal{G}^* \to \mathcal{G} \) is defined by \( \langle U | \hat{r}_g (V) \rangle = \langle U \otimes V | r(g) \rangle \), cf. (71), \( \hat{s}_g = \hat{r}_g^{-1} \), and \( d^{-1} \) denotes inverse operator of the exterior derivative acting on the corresponding closed 2-form \( \Omega \) (recall \( Y(g) = dgg^{-1} \)).

The principal example is provided by the extended Volterra algebra of purely pseudo-differential operators (\( c \) is arbitrary constant):

\[
\mathcal{G} \equiv (\Psi \overline{DO})_\sim = \left\{ \sum_{k \geq 1} u_k(x)D^{-k} + c \ln D \right\}
\]

(74)

whose Lie-double is the extended algebra of all pseudo-differential operators:

\[
\mathcal{D} \equiv \Psi \overline{DO} = (\Psi \overline{DO})_\sim \oplus \mathcal{D}\mathcal{P} \quad ; \quad \mathcal{G}^* \equiv D\mathcal{P} \simeq W_{1+\infty} = \left\{ \sum_{l \geq 0} D^l v_l(x) + \alpha \hat{E} \right\}
\]

(75)

Here \( \hat{E} \) indicates the central element of \( W_{1+\infty} \), as well as of the whole \( \Psi \overline{DO} \), which is dual to \( \ln D \), cf. [37, 39]. The corresponding extended Volterra group (exponentiation of (74)):

\[
G \equiv (\Psi \overline{DO})_\sim = \left\{ g \equiv L = \left( 1 + \sum_{k \geq 1} \hat{u}_k(x)D^{-k} \right) \circ D^c \right\}
\]

(76)

can be viewed as a set of spaces (for each \( c \) fixed) of Lax operators of generalized KP hierarchies w.r.t. the second KP Poisson structure [40] (eq. (77) below). The latter is given precisely by the Lie-Poisson structure (63) with the cocycle \( r(g) \) (71) for \( G = (\Psi \overline{DO})_\sim \) (76) (see [39]).

Let us go back to the example of 2q-boson KP Lax operators appearing in the multi-matrix string models [11]. The second Hamiltonian structure for general Lax operators [18] has the form:

\[
\{ \langle L | X \rangle , \langle L | Y \rangle \} = \text{Tr}_A \left( (LX)_+ + LY - (XL)_+ YL \right) + \int dx \text{Res} \left( [L , X] \right) \partial^{-1} \text{Res} \left( [L , Y] \right)
\]

(77)

Here \( \text{Tr}_A \) denotes the Adler trace [12] and the subscript + indicates taking the purely differential part. For the coefficient fields \( u_k(x) \) of \( L \) [18] the second KP Poisson algebra (77) yields the nonlinear (i.e., non-Lie) \( \hat{W}_\infty \) algebra [12]. The latter appears as a unique (modulo certain homogeneity assumptions) nonlinear deformation of \( W_{1+\infty} \) algebra.

In analogy with eqs. (72)–(74) one can express [41] the coefficient fields \( (A_l , B_l)_{l=1}^q \) of \( L_q \) (11) in terms of Darboux canonical pairs of fields \( (c_r , \epsilon_r)_{r=1}^q \) w.r.t. the second KP
Hamiltonian structure \( (77) \):

\[
B_k = e_k + \sum_{l=k}^{q} c_l, \quad 1 \leq k \leq q \quad \text{;} \quad A_q = \sum_{r=1}^{q} (\partial + c_r) e_r
\]  
\[ (78) \]

\[
A_k = \sum_{n_k=1}^{k} \left( \partial + e_{n_k} - e_{n_k+q-k} + \sum_{l_k=n_k}^{n_k+q-k} c_{l_k} \right)
\times \sum_{n_{k-1}=1}^{n_k} \left( \partial + e_{n_{k-1}} - e_{n_{k-1}+q-1-k} + \sum_{l_{k-1}=n_{k-1}}^{n_{k-1}+q-1-k} c_{l_{k-1}} \right) \times \cdot \cdot \cdot
\]
\[ (79) \]

\[
\sum_{n_2=1}^{n_3} (\partial + e_{n_2} - e_{n_2+1} + c_{n_2} + c_{n_2+1}) \sum_{n_1=1}^{n_2} (\partial + c_{n_1}) e_{n_1}, \quad 1 \leq k \leq q - 1
\]
\[ (80) \]

These equations are equivalent to the following “dressing” form for the \( 2q \)-boson KP Lax operator:

\[
L_q = D + \sum_{l=1}^{q} A_l (D - B_l)^{-1} (D - B_{l+1})^{-1} \cdots (D - B_q)^{-1} = U_q \cdots U_1 D V_1^{-1} \cdots V_q^{-1}
\]  
\[ (81) \]

\[
U_k \equiv (D - e_k) e^\int c_k \quad \text{;} \quad V_k \equiv e^\int c_k (D - e_k), \quad k = 1, \ldots, q
\]  
\[ (82) \]

Eqs. (78)-(80) or, equivalently, eqs. (81)-(82) can be viewed as generalized Miura transformation for the \( 2q \)-boson KP hierarchy \( 8 \).

The Miura-transformed form of \( L_q \) (81) reads explicitly:

\[
L_q = D + \sum_{k=1}^{\infty} U_k[(c, e)](x) D^{-k}
\]  
\[ (83) \]

\[
U_k[(c, e)](x) = P_{k-1}^{(1)} (e_q + c_q) \sum_{l=1}^{q} (\partial + c_l) e_l + \sum_{r=1}^{\min(q-1,k-1)} A_{q-r}(c, e) P_{k-1-r}^{(r+1)} (e_q + c_q, c_q-1 + c_q, \ldots, e_{q-r} + c_{q-r} + \sum_{l=q-r}^{q} c_l)
\]  
\[ (84) \]

where \( A_{q-r}(c, e) \) are the same as in (79), and \( P_{n}^{(N)} \) denote the (multiple) Faá di Bruno polynomials \( 7 \).

Now, in complete analogy with eqs. (73)-(78), which yield a series of realizations of the linear \( W_{1+\infty} \) algebra in terms of \( 2q \) bosons, we obtain, upon substitution of (83)-(84) into (77), a series of explicit (Poisson bracket) realizations of the nonlinear \( W_{\infty} \) algebra in terms of \( 2q \) bosonic fields for any \( q = 1, 2, \ldots \). This algebra plays an important rôle as a “hidden” symmetry algebra in string-theory-inspired models with black hole solutions \( 12 \).

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8 For discussion of the generalized Miura transformation and the associated Kuperschmidt-Wilson theorem, we refer to \( 40 \).
Concluding this section, let us note that in the general case Lie-Poisson groups provide natural geometric description of the dressing symmetries in completely integrable models \([13, 14]\). Another outstanding rôle played by Lie-Poisson structures \([13]\) is their appearance in the context of the classical inverse scattering method \([3]\) as fundamental Sklyanin brackets for the monodromy matrix \(g \simeq T(\lambda)\) of the auxiliary linear spectral problem.

4. Quantum Integrable Models

4.1 Quantization of Lie-Poisson Groups: Quantum Groups

Historically, quantization of completely integrable models, whose Hamiltonian structure is based on group Lie-Poisson structures, laid for the first time to explicit construction, in the context of the quantum version of the inverse scattering method \([11]\), of quantum groups which were subsequently identified with quasi-triangular Hopf algebras \([13]\).

Among the various ways to introduce quantum groups there exists an approach \([12]\), whose conceptual point of view underscores both the quantum mechanical as well as the Hopf algebraic aspects in quantization of Lie-Poisson groups. Namely, on one hand \(Fun(G)\) can be viewed as Abelian associative algebra of “observables” of a classical Hamiltonian system \((\mathcal{M} = G, \mathcal{P})\) with a phase space \(\mathcal{M} = G\) and Poisson structure \(\mathcal{P} = \mathcal{P}_{LP}\) given by (85):

\[
\mathcal{P}_{LP}(F_1, F_2) \equiv \{ F_1, F_2 \} = \langle \nabla_L F_1 \otimes \nabla_L F_2 - \nabla_R F_1 \otimes \nabla_R F_2 | r_0 \rangle
\]

where the classical \(r\)-matrix \(r_0\) satisfies the classical YBE \([33]\) (from now on we shall consider only coboundary Lie-Poisson groups \([62]\)). On the other hand, one can easily check that \(Fun(G)\) is endowed with a structure of a commutative, but non-cocommutative, Hopf algebra \(A_0(m, \Delta, S, \varepsilon) \equiv Fun(G)\) with a product \(m(F_1, F_2) = F_1(g)F_2(g)\), coproduct \(\Delta(F)(g_1, g_2) = F(g_1g_2)\), antipode \(SF(g) = F(g^{-1})\) and counit \(\varepsilon(F) = F(e)\), and this Hopf structure is compatible with the Poisson structure \([85]\), i.e., \(\Delta \circ \mathcal{P} = \mathcal{P} \circ \Delta\).

Thus, quantization of a Lie-Poisson group \(G\) may be viewed as a generalization of Weyl quantization \(Fun(G) \longrightarrow Fun_h(G)\) of a classical Hamiltonian system defined by \((\mathcal{M} = G, \mathcal{P}_{LP})\), i.e., non-commutative deformation of the product \(m(\cdot, \cdot) \longrightarrow m_h(\cdot, \cdot)\) with a deformation parameter \(h\), which satisfies the additional condition that the deformed algebra \(Fun_h(G) \equiv A_h(m_h, \Delta, S, \varepsilon)\) is again (non-commutative and non-cocommutative) Hopf algebra which is a deformation of \(A_0(m, \Delta, S, \varepsilon) \equiv Fun(G)\).

Let us recall \([17]\), that in ordinary Hamiltonian mechanics on a Poisson manifold \((\mathcal{M}, \mathcal{P})\) with local coordinates \((x^i)\) and constant Poisson tensor \(\{ x^i, x^j \} = P^{ij}\), Weyl quantization is given by the associative and non-commutative Moyal product:

\[
m_h(F_1, F_2) \equiv F_1 *_h F_2 = m \circ e^{\frac{h}{2}P}(F_1, F_2) = F_1 \cdot F_2 + \frac{h}{2} \{ F_1, F_2 \} + O(h^2)
\]

\[
\mathcal{P} = P^{ij} \frac{\partial}{\partial x^i} \otimes \frac{\partial}{\partial x^j} : Fun(\mathcal{M}) \otimes Fun(\mathcal{M}) \longrightarrow Fun(\mathcal{M}) \otimes Fun(\mathcal{M})
\]
where $\mathcal{P}$ is the Poisson bi-vector field. Recall also, that the form of the first order term in the $h$-expansion (last eq.(86)) is dictated by the semiclassical correspondence principle.

In the case of Lie-Poisson groups $(G, \mathcal{P}_{LP})$, the deformed product $m_h(\cdot, \cdot)$ preserving the Hopf algebra structure and satisfying the semiclassical condition, can be constructed as follows [12]. Let us choose a basis $\{X^i\}$ in $\mathcal{U}(G)$ – the universal enveloping algebra of the Lie algebra $G$ of $G$, and let $\pi_{L,R}$ denote the representations of $\mathcal{U}(G)$ in terms of left/right Lie derivatives : $\pi_{L,R}(X^i) = \nabla^i_{L,R}$ (see eq.(64)). Then :

$$m_h(\cdot, \cdot) = m \circ \tilde{\Lambda} \; , \; \tilde{\Lambda} = (\pi_L \otimes \pi_L)(\Lambda) \circ (\pi_R \otimes \pi_R)(\Lambda^{-1}) \quad (88)$$

$$\Lambda(X, Y) = \sum_{\{\alpha\}, \{\beta\}} c_{\{\alpha\}, \{\beta\}}(h) \prod_{i=1}^{\dim G} (X^i)^{\alpha_i} \prod_{j=1}^{\dim G} (Y^j)^{\beta_j} = \mathbb{1} + \frac{h}{2} \epsilon_{ij} X^i Y^j + O(h^2) \quad (89)$$

with the following notations. The coefficients in $\Lambda(\cdot, \cdot) : \mathcal{U}(G) \otimes \mathcal{U}(G) \rightarrow \mathcal{U}(G) \otimes \mathcal{U}(G) [[h]]$ are power series in $h$ ; $\{X^i\}$ and $\{Y^j\}$ are generator basises in the first and second copy of $\mathcal{U}(G)$, respectively; $\|r_{ij}\| = r_0$ is just the classical $r$-matrix satisfying (83). Moreover, the associativity condition for $m_h(\cdot, \cdot)$ (88) implies the following basic quadratic equation on $\mathcal{U}(G) \otimes \mathcal{U}(G) \otimes \mathcal{U}(G) [[h]]$ (X, Y, Z below correspond to the first, second and third factor $\mathcal{U}(G)$ in the tensor product) :

$$\Lambda(X + Y, Z)\Lambda(X, Y) = \Lambda(X, Y + Z)\Lambda(Y, Z) \; ; \; \Lambda(X, 0) = \Lambda(0, Y) = \mathbb{1} \quad (90)$$

Defining $\tilde{R}(X, Y) = \Lambda^{-1}(Y, X)\Lambda(X, Y) \in \mathcal{U}(G) \otimes \mathcal{U}(G) [[h]]$, one obtains from (83) :

$$\tilde{R}(X, Y)\tilde{R}(X, Z)\tilde{R}(Y, Z) = \tilde{R}(Y, Z)\tilde{R}(X, Z)\tilde{R}(X, Y) \quad (91)$$

$$\tilde{R}(X, Y)\tilde{R}(Y, X) = \mathbb{1} \; ; \; \tilde{R}(X, Y) = \mathbb{1} + h \epsilon_{ij} X^i Y^j + O(h^2)$$

$\tilde{R}$ is called universal quantum $R$-matrix associated with the classical $r$-matrix $\|r_{ij}\| = r_0$. If $\rho : G \rightarrow \text{End}(\mathcal{V})$ is some representation of $G$ in a (finite-dimensional) vector space $\mathcal{V}$, then the matrix $R = (\rho \otimes \rho)(\tilde{R}) \in \text{End}(\mathcal{V} \otimes \mathcal{V})$ satisfies the famous quantum Yang-Baxter equation (QYBE) (plus the “unitarity” condition, $P \in \text{End}(\mathcal{V} \otimes \mathcal{V})$ being the permutation operator) :

$$\begin{pmatrix} 12 & 13 \end{pmatrix} (23) \begin{pmatrix} 12 & 13 \end{pmatrix} (23) \begin{pmatrix} 12 & 13 \end{pmatrix} \begin{pmatrix} R & R & R & R \end{pmatrix} = \begin{pmatrix} R & R & R & R \end{pmatrix} \begin{pmatrix} R & R & R & R \end{pmatrix} = \mathbb{1} \; ; \; RPRP = \mathbb{1} \; ; \; R = \mathbb{1} + h r_0 + O(h^2) \quad (92)$$

The indices (12), (13), (23) indicate the various embeddings of $R \in \text{End}(\mathcal{V} \otimes \mathcal{V})$ in $\text{End}(\mathcal{V} \otimes \mathcal{V} \otimes \mathcal{V})$.

In particular, from the deformed product (88) $m_h(F_1, F_2) = m \circ \bar{\Lambda}(F_1, F_2)$ where the functions $F_1(g) = g_{ab}, F_2(g) = g_{cd}$ are just matrix elements of the group element $g \in G$ (for matrix groups), one obtains using matrix tensor notations :

$$R(g \otimes \mathbb{1})(\mathbb{1} \otimes g) = (\mathbb{1} \otimes g)(g \otimes \mathbb{1})R \quad (93)$$
The semiclassical limit of Eq. (93) is precisely given by the Lie-Poisson bracket (63). Eq. (93) is nothing but the famous fundamental commutation relations for the matrix elements of the quantum monodromy matrix (with spectral parameter dependence suppressed) in the quantum inverse scattering method [11].

For various treatments and numerous applications of QYBE, see ref. [15]. For parallel developments in the abstract Hopf algebraic context, we refer to [46].

4.2 “Soliton” Scattering in Completely Integrable Models

Let us now consider integrable \( D = 2 \) relativistic field theories whose Hamiltonian dynamics is given by actions \( S[\phi] = \int d^2x \mathcal{L}(\phi, \partial \phi) \) – local functionals of the fundamental fields (collectively denoted by \( \phi(x^+, x^-) \)) and their derivatives. As usual, one uses the light-cone form of the space-time coordinates: \( x^\pm = \frac{1}{2}(x^1 \pm x^0) \). Complete integrability in this context implies the existence of an infinite number of independent integrals of motion in involution \( Q^{(s)} \), whose densities are local (as functionals of \( \phi \) and its derivatives) conserved currents:

\[
Q^{(s)} = \oint (T^{(s+1)} dx^- + \Theta^{(s-1)} dx^+) , \quad s = 1, 2, \ldots ; \quad \partial_\mp T^{(s+1)} = \mp \partial_\Theta^{(s-1)} \tag{94}
\]

(here \( s \) indicates the \( D = 2 \) Lorentz weight). Thus, quantization of completely integrable field theories means fulfilment of the quantum renormalized Ward identities for the renormalized quantum conserved currents \( \{ T^{(s+1)}, \Theta^{(s-1)} \} \):

\[
\partial_+ \langle \tilde{T}^{(s+1)}(x) \phi(x_1) \ldots \phi(x_n) \rangle - \partial_- \langle \tilde{\Theta}^{(s-1)}(x) \phi(x_1) \ldots \phi(x_n) \rangle = \delta - \text{function terms (95)}
\]

(as usual, \( \langle \ldots \rangle \) here denote time-ordered correlation functions). The infinite set of Ward identities lead to severe restrictions on the particle (“soliton”) scattering processes – conservation of all (odd) powers of momenta of incoming and outgoing particles:

\[
\sum_{l=1}^{N_{\text{in}}} p_{l(in)}^{2n+1} = \sum_{l=1}^{N_{\text{out}}} p_{l(out)}^{2n+1} \tag{96}
\]

which, in turn, implies [17]: (a) no multi-particle production, i.e., \( N_{\text{in}} = N_{\text{out}} \); and (b) factorization of multi-particle scattering amplitudes. The latter property is of tremendous importance, as it leads to the remarkable Zamolodchikov’s factorization eqs. for the 3-particle amplitudes [14], meaning that any 3-particle scattering process is accomplished as a sequence of 2-particle scatterings only and, moreover, the amplitude does not depend on the order in which these sequential 2-particle scatterings occur:

\[
S^{k_{12}}_{11} (\theta_{12}) \ S^{k_{13}}_{k_{1}k_{3}} (\theta_{13}) \ S^{k_{23}}_{k_{2}k_{3}} (\theta_{23}) = S^{k_{1}k_{2}}_{12} (\theta_{23}) \ S^{k_{1}k_{3}}_{13} (\theta_{13}) \ S^{k_{1}k_{2}}_{k_{1}k_{2}} (\theta_{12}) \tag{97}
\]

with \( \theta_{ab} \equiv \theta_a - \theta_b , \ a, b = 1, 2, 3 \). In [14] the following notations are used: \( S^{ij}_{kl} (\theta_{12}) \) denotes 2-particle scattering amplitude of incoming particles of “type” labelled by the indices \( i \) and \( j \) and with (on-mass-shell) momenta \( p_{1,2} = m_{1,2} (\cosh \theta_{1,2}, \sinh \theta_{1,2}) \), respectively (\( \theta_{1,2} \) are the relativistic “rapidities”).
Now, denoting by $V$ the vector space of internal particle symmetry (particle “types”), one can regard the matrix of the 2-particle amplitude as:

$$S(\theta_{12}) = \parallel S_{ij}^k (\theta_{12}) \parallel \in Mat(V) \otimes Mat(V) \quad \text{(for fixed } \theta_{12})$$

and, accordingly, for $S(\theta_{13})$ and $S(\theta_{23})$. Then it is straightforward to identify (97) with the QYBE (92) in the quantum group framework.

A closely related natural appearance of quantum group structure in “soliton” scattering is provided by the notion of asymptotic states’ symmetry [49]:

$$| (\theta, i) \rangle \rightarrow T_{ij}(\theta) | (\theta, j) \rangle; \quad | (\theta, i) \rangle \in V, \parallel T_{ij}(\theta) \parallel \in Mat(V),$$
as a result of the integrability. Namely, the 2-particle $S$-matrix:

$$| (\theta_1, i_1), (\theta_2, i_2) \rangle^{in} = S^{j_1 j_2}_{i_1 i_2} (\theta_{12}) | (\theta_2, j_2), (\theta_1, j_1) \rangle^{out}$$
can be viewed as a mapping (for fixed “rapidities”)

$$S^{(12)} : V \otimes V \rightarrow V \otimes V$$

and, therefore, its invariance under the asymptotic states’ symmetry:

$$S^{(12)} (\theta_{12}) T^{(1)} (\theta_1) T^{(2)} (\theta_2) = T^{(2)} (\theta_2) T^{(1)} (\theta_1) S^{(12)} (\theta_{12})$$
is straightforwardly identified with the structural relations (93) for quantum groups.

Let us point out that quantum group relations of exactly the same form as (97) and (100) do appear in exactly solvable lattice models of planar statistical mechanics, but in this case – with purely imaginary “rapidities” ($\theta = i\alpha$, $\alpha$ being angles characterizing the rectangular lattices), $S_{ij}^k (\alpha)$ being the matrix of Boltzmann weights at each lattice vertex, and $T_{ij}(\alpha)$ denoting the row transfer matrix [50].

4.3 Quantum Field Theory Approach to Integrable Models with Dynamically Broken Conformal Invariance

Finally, let us briefly discuss construction of higher local quantum conserved currents fulfilling the Ward identities (95), which is the heart of the quantum field theory approach to quantization of completely integrable models. Recently, Zamolodchikov [14] proposed powerful general formalism based on treating integrable models as mass perturbations of conformal field theories: $S[\phi] = S_{conf}[\phi] + \sum_i m_i \int d^2x B_i(\phi, \partial \phi)$, where the “coupling” constants $m_i$ have positive mass dimensions and $B_i(\phi, \partial \phi)$ are composite fields with conformal dimensions less than 2. Since in general it is not possible to find explicit expressions for $S_{conf}[\phi]$ and $B_i(\phi, \partial \phi)$ as local functionals of local fundamental fields $\{\phi\}$, Zamolodchikov’s approach is purely algebraic using results from representation theory of Virasoro algebra, in particular, information about the spectrum of conformal field dimensions.
There exist, however, interesting classes of $D = 2$ integrable field theories, i.e., $O(N)$ nonlinear sigma-models and their supersymmetric generalizations with Lagrangians:

$$\mathcal{L}_{NL\sigma} = \frac{1}{2} \partial_+ n^a \partial_- n_a \quad , \quad \vec{n}^2 = N/g \quad ; \quad \vec{n} = (n^1, \ldots, n^N) \quad (101)$$

$$\mathcal{L}_{susy-NL\sigma} = \frac{1}{2} \partial_+ n^a \partial_- n_a + i \bar{\psi}^a \gamma^\mu \partial_\mu \psi_a - \frac{g}{N} (\bar{\psi}^a \psi_a)^2 \quad , \quad \vec{n}^2 = N/g \quad , \quad n^a \psi_a = 0 \quad (102)$$

which are conformally invariant on the classical level but, upon quantization, they undergo dynamical dimensional transmutation, manifested through dynamical mass generation, leading to anomalous conformal symmetry breakdown. This clearly precludes the use of conformal perturbation approach to (101), (102). Fortunately, there exist long ago alternative nonperturbative\footnote{The term “nonperturbative” refers to expansions different from (or, e.g., partial resummations of) the ordinary perturbation theory w.r.t. the coupling constant $g$ in (101) and (102) which is plagued by infrared divergences in $D = 2$.} treatment of quantum field theory models with $O(N)$ or $SU(N)$ internal symmetry – the $1/N$ expansion\footnote{Bogoliubov-Parasiuk-Hepp-Zimmermann\cite{55}. As shown in\cite{54}, the (supersymmetric) nonlinear sigma-models (101) and (102) are renormalizable within the $1/N$ expansion also in $D = 3$ space-time dimensions in spite of their naive nonrenormalizability w.r.t. the ordinary coupling constant perturbation theory (note e.g. the presence of the four-fermion term in (102)).}.

Let us briefly illustrate the construction\footnote{\cite{52}} of higher local quantum conserved currents for (101)\footnote{\cite{53}} within the $1/N$ expansion framework (the same techniques applies to other $1/N$-expandable integrable models as well). The $1/N$ expansion is obtained from the generating functional of time-ordered correlation functions :

$$Z[J] = \int D\vec{n} \prod_x \delta \left( \vec{n}^2 - N/g \right) \exp \left\{ i \int d^2x \left[ \frac{1}{2} (\partial \vec{n})^2 + (\vec{J}, \vec{n}) \right] \right\} = \int D\sigma \exp \left\{ -\frac{N}{2} S_1[\sigma] + \frac{i}{g} \int d^2x d^2y \left( \vec{J}(x), (-\partial^2 + \sigma)^{-1} \vec{J}(y) \right) \right\} \quad (103)$$

$$S_1[\sigma] \equiv \text{Tr} \ln(-\partial^2 + \sigma) + \frac{i}{g} \int d^2x \sigma \quad (104)$$

by expanding the effective $\sigma$-field action (104) around its stationary point $\sigma_c \equiv m^2 = \mu^2 e^{-4\pi/g}$ (dynamically generated mass of the “Goldstone” field $\vec{n}$, $\mu$ being the renormalization scale), i.e., $\sigma(x) = m^2 + \frac{1}{\sqrt{N}} \tilde{\sigma}(x)$. As a result, one arrives at the $1/N$ diagram technique with (free) propagators in momentum space :

$$\langle n^a n^b \rangle_{(0)} = -i \left( m^2 + p^2 \right)^{-1} \delta^{ab} \quad ; \quad \langle \tilde{\sigma} \tilde{\sigma} \rangle_{(0)} = \left( \Sigma(p^2) \right)^{-1} \quad (105)$$

and ordinary tri-linear $\tilde{\sigma}nn$-vertices.

It turns out\footnote{\cite{54}}, that the $1/N$ expansion can be renormalized by adapting the well-known BPHZ renormalization technique. Another remarkable property of the $1/N$
expansion for (101) is that, in spite of the manifest linear \( O(N) \) symmetry of (105), the nonlinearity of the “Goldstone” field \( \vec{n}(x) \) is preserved on the quantum level as an identity on the correlation functions:

\[
\langle N[\vec{n}^2 P(\vec{n}, \partial \vec{n})](x) \ldots \rangle = \text{const} \langle N[P(\vec{n}, \partial \vec{n})](x) \ldots \rangle
\]

(106)

where \( P(\vec{n}, \partial \vec{n}) \) is arbitrary local polynomial of the fundamental fields and their derivatives, and \( N[\ldots] \) indicates renormalized normal product of the corresponding composite fields.

Armed with the above machinery, the first higher quantum conserved current (for \( s = 3 \) in the notations of (94), (95)) takes the following form:

\[
\tilde{T}^{(4)} = N\left( (\partial^2 \vec{n})^2 \right) + a_1 N\left( (\partial^{-\vec{n}})^2 \sigma \right), \quad \tilde{\Theta}^{(2)} = \left( \frac{1}{2} + a_2 \right) N\left( (\partial^{-\vec{n}})^2 \sigma \right) + a_3 \partial^2 \sigma
\]

(107)

where all coefficients \( a_{1,2,3} = O(1/N) \) are expressed in terms of one-particle irreducible correlation functions and their derivatives in momentum space at zero external momenta. Their explicit form can be found order by order in \( 1/N \) from the renormalized \( 1/N \)-diagram technique described above [52].

Let us stress, that the higher quantum conserved currents (107) and those for \( s = 5, 7, \ldots \) do not have analogues in the classical conformally invariant theory [56].

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