Abstract. We show that one-loop quantum corrections to the potential energy density in supersymmetric hybrid inflation, outside the inflationary valley, cannot be neglected. A method is presented to calculate these one-loop corrections and they are applied to the case of D-term hybrid inflation, where a significant amount of inflation is shown to occur after spontaneous symmetry breaking. Taking this into account improves the agreement with WMAP measurements. A gauge coupling of up to 0.3 is still consistent with the CMB density perturbation. The spectral index is predicted in between 0.98 and 1.00 and the cosmic string contribution to the CMB anisotropy is sufficiently reduced.

Keywords: inflation, physics of the early universe

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1. Introduction

It is widely believed that the early universe goes through a period of accelerated expansion called ‘inflation’ [1]. This belief has been successfully confirmed by WMAP measurements of the cosmic microwave background radiation, which validate inflation as a mechanism to produce the initial irregularities of our universe [2]. Furthermore inflation has been
shown to solve a number of cosmological problems, such as the ‘horizon problem’, the ‘flatness problem’ and the ‘monopole problem’. There are many different inflationary models [3]. Experimentally these can be distinguished by their predictions for the main observable quantities: the density perturbation at decoupling, the spectral index of the CMB anisotropy and the cosmic string contribution to the CMB anisotropy [4].

These observables are derived from the precise shape of the effective scalar potential of a particular inflation model. Only potentials that contain a nearly flat direction give rise to inflation. Supersymmetric models contain these nearly flat directions naturally (without fine-tuning parameters), because of large cancellations of bosonic and fermionic one-loop quantum contributions to the effective potential, which give a slight tilt to classically flat directions. In supersymmetric hybrid inflation models, inflation occurs while the scalar fields are on the so-called ‘inflationary valley’ and it ends abruptly with a period of spontaneous symmetry breaking. On the inflationary valley the 'Coleman–Weinberg' formula [5] can be applied to get the one-loop quantum correction to the effective potential [6]. This gives the shape of the potential from which the aforementioned observables can be derived. For this derivation to be valid, it is crucial that inflation ends abruptly at the end of the inflationary valley, when spontaneous symmetry breaking occurs. This is usually assumed.

To make precise statements about this assumption, one needs to know the one-loop corrections outside the inflationary valley, during the spontaneous symmetry breaking phase. We will show that one cannot simply generalize the ‘Coleman–Weinberg’ formula to this case, because this introduces a non-renormalizable cutoff dependence and moreover it gives a one-loop correction on the inflationary valley at odds with earlier calculations. The cause of these problems is that the ‘Goldstone’ boson which enters the Higgs mechanism is not massless anymore for field-space points outside the inflationary valley. If we include this ‘Goldstone’ mass in the calculation of the one-loop corrections, the aforementioned problems disappear.

Using these off-valley one-loop corrections to the $D$-term hybrid inflation potential, we show that it is, in general, not allowed to neglect inflation after the critical point. Bearing this in mind, a new analysis is made of all relevant observables, which are in good agreement with the WMAP data for a large portion of the $D$-term parameter space.

This paper is organized as follows. Section 2 explains the basic ideas behind $D$-term hybrid inflation. The body of the paper is in section 3. It contains a way to calculate the one-loop quantum corrections to the full $D$-term potential, which is essential for a better understanding of its implications for WMAP observables. Section 3.1 introduces the ‘Coleman–Weinberg formula’ for the one-loop corrections. Then section 3.2 continues with the calculation of the field-dependent masses, needed for the Coleman–Weinberg formula. Section 3.3 explains a variety of subtleties in applying this formula and stumbles upon fundamental problems in calculating the one-loop corrections. Section 3.3.4 proposes an attempt to resolve these problems and finally in section 3.4 the result for the one-loop corrections is given. Section 4 explores the usefulness of these one-loop corrections in extending the inflationary phase. In section 5 we apply the one-loop corrections to get new cosmological constraints on the $D$-term hybrid inflation model. If the reader is only interested in new parameter bounds for $D$-term inflation, he can directly consult section 5. This paper will end with a small conclusion section 6.
2. $D$-term inflation

The simplest possible superpotential that can generate $D$-term inflation is [7, 8]

$$ W = \lambda S \Phi_+ \Phi_- . $$

(1)

The superfields $\Phi_+$ and $\Phi_-$ have charge 1 and $-1$ under a $U(1)_{FI}$ gauge symmetry. $S$ is uncharged and $\lambda$ is a coupling parameter. The corresponding tree-level effective scalar potential is

$$ V_{\text{tree}} = \lambda^2 |S|^2 (|\Phi_+|^2 + |\Phi_-|^2) + \lambda^2 |\Phi_+|^2 |\Phi_-|^2 + \frac{g^2}{2} (|\Phi_+|^2 - |\Phi_-|^2 + \xi)^2 , $$

(2)

where $S$ is called the inflaton field, $\xi$ is the Fayet–Illiopoulos gauge term (taken to be positive) and $g$ is the $U(1)_{FI}$ gauge coupling. $\Phi_+ = \Phi_- = 0$ represents a flat valley of local minima, because when both $\Phi_+$ and $\Phi_-$ vanish, the tree-level potential does not depend on $S$. In this case the potential energy density equals

$$ V = \frac{g^2 \xi^2}{2} . $$

(3)

This is the false vacuum energy density, which drives $D$-term inflation. This inflationary solution ($\Phi_+ = \Phi_- = 0$) can be either stable or unstable, depending on the value of the $S$ field.

For $|S| > S_c$ the $\Phi_-$-direction corresponds to a (stable) local minimum of the potential, but for $|S| < S_c$ the $\Phi_-$-direction corresponds to an (unstable) local maximum, which will cause spontaneous symmetry breaking. The fields will settle down in the true global minimum ($V = 0$, $S = 0$, $\Phi_+ = 0$, $|\Phi_-| = \sqrt{\xi}$) and inflation will stop. The $\Phi_+$ direction cannot cause a similar effect, because it always corresponds to a (stable) local minimum. The critical value of the inflaton field is given by

$$ S_c \equiv \frac{g \sqrt{\xi}}{\lambda} . $$

(4)

At tree level (classically) there is no term in the potential which drives the fields toward the true global minimum. Thus, assuming that the fields start out with random values after the Planck era, they will either settle down in the global minimum ($V = 0$, $S = 0$, $\Phi_+ = 0$, $|\Phi_-| = \sqrt{\xi}$) with no appreciable inflation, or in the local minimum ($V = g^2 \xi^2/2$, $|S| > S_c$, $\Phi_+ = 0$, $\Phi_- = 0$), in which case inflation goes on forever.

However, as shown in [6], (one-loop) quantum corrections to the tree-level potential will add a slight slope to the $D$-flat direction. Now the typical evolution of the fields after the Planck era will consist of three stages. (I) First the $\Phi_+$ and $\Phi_-$ fields will quickly roll down to the $D$-flat valley ($|S| > S_c$, $\Phi_+ = 0$, $\Phi_- = 0$). (II) Then a period of slow-roll inflation starts. The $S$ field rolls down until $S_c$ is reached. (III) Finally the spontaneous symmetry breaking puts a natural halt to the inflationary phase and the fields settle in the true global minimum, see figure 1.

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3 Quantum mechanically the effective potential can be interpreted as the energy density of the quantum state which minimizes this energy density, subject to the condition that the field expectation values (in this case $\langle \Phi_{\pm}, |S\rangle$) are as given. So for every point in field-space one can define an effective potential value. There are, however, subtleties in the interpretation of the effective potential in the case of a spontaneous symmetry breaking. These will be discussed in appendix B.

4 Loop corrections only contribute if supersymmetry is broken. If supersymmetry is not broken, the bosonic and fermionic contributions to the loop correction will cancel, so that the total loop correction vanishes. For this model supersymmetry is broken during inflation and then it is restored at the end of inflation in the global minimum.
Figure 1. This figure depicts the different stages for a typical time evolution of the fields in $D$-term inflation. In region I the $\Phi$ fields quickly roll down to zero. Then slow-roll inflation starts in region II (also called the inflationary valley). Finally in region III spontaneous symmetry breaking occurs and the fields roll down to the true global minimum, whereby $\Phi_-$ attains a vacuum expectation value.

For inflation model builders, the most important part of the potential is the valley ($|S| > S_c, \Phi_+ = 0, \Phi_- = 0$), which causes inflation (region II). Nevertheless the other parts of the potential can also be of interest. Region I is of interest if one wants to study the initial conditions after the Planck era [10,11]. In fact, not all homogeneous initial conditions give inflation, but most will, because the tree-level potential is very steep in the direction towards the inflationary valley. Region III is of interest if you want to study reheating. The initial value problem will probably not lead to observable consequences, because everything happening more than about 60 e-folds before the end of inflation will not have entered our observable universe today (it will have left its imprint on scales larger than our observable universe). On the other hand, all processes occurring in region III will be of observational interest, because they involve length scales that are within our past light-cone.

The study of quantum corrections to the effective potential in these two regimes, however, is still far from finished. Actually the only part for which we know the one-loop corrections for certain is the inflationary valley ($|S| > S_c, \Phi_+ = 0, \Phi_- = 0$) (see [6]). Work has been done, however, in the preheating region, [12,13]. In the following we will try to generalize their approach and to point out possible errors.

3. One-loop corrections

3.1. Coleman–Weinberg formula for calculating one-loop corrections to the effective potential

To compute the one-loop correction to the effective potential people normally use the Coleman–Weinberg formula. It is the total contribution to the effective potential of all Feynman diagrams, which have truncated external scalar propagators with zero momentum and one internal loop with any possible particle going around (in this case...
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one of the scalar particles, a fermionic superpartner, a gauge field or a gaugino). For the
derivation in a general (non-SUSY) context see [5]. It is given by

\[ \Delta V = \frac{1}{64\pi^2} \sum_i (-1)^F m_i^4 \ln \left( \frac{m_i^2}{\Lambda_m^2} \right), \]

(5)

with \( m_i \) being the mass of a given particle, the sum goes over all particles, \( F \) can be taken
as 0 for bosons and 1 for fermions and \( \Lambda_m \) is the renormalization mass. But this formula
is really a simplification of the more general [14]

\[ \Delta V = \frac{1}{64\pi^2} \text{Str} (M^0) \Lambda_c^4 \ln \left( \frac{\Lambda_c^2}{\mu^2} \right) + \frac{1}{32\pi^2} \text{Str} (M^2) \Lambda_c^2 + \frac{1}{64\pi^2} \text{Str} \left( M^4 \ln \left( \frac{M^2}{\Lambda_c^2} \right) \right) + \cdots, \]

(6)

with \( \Lambda_c \) being a momentum cutoff and \( \mu \) the scale parameter. The dots stand for \( \Lambda_c \)-independent contributions. Str is called the supertrace and is defined as follows:

\[ \text{Str} (M^a) = \sum_i (-1)^{2J_i} (2J_i + 1) m_i^a, \]

(7)

with \( m_i \) being the field-dependent mass eigenvalues and \( J_i \) being the spin of the
responding particles. The supertrace weights the number of degrees of freedom and
gives fermions the opposite sign of bosons.

It is important to emphasize that the two \( \Lambda \)'s are not the same. They are sometimes
confused because they appear in the same form (that is why there is an added subscript). The
renormalization mass \( \Lambda_m \) is an arbitrary mass scale the theory should not be
dependent on, which is guaranteed via the renormalization group equations. The
momentum cutoff \( \Lambda_c \), on the other hand, will go to infinity after renormalization, so
the potential had better not depend on this after renormalization. Both \( \Lambda \)'s appear in a
similar way in the formulae. In fact, equation (5) comes from the last term in equation (6)
after renormalizing (which makes \( \Lambda_c \) dependence of this term disappear, but introduces a
‘nicer’ \( \Lambda_m \) dependence, which happens to be of the same form).

Under what circumstances can we use equation (5) rather than equation (6)? The
first term in (6) will drop out in any supersymmetric theory, because \( \text{Str} (M^0) \) equals the
number of bosonic degrees of freedom minus the number of fermionic degrees of freedom,
which will be zero in any supersymmetric theory. What about the second term? In [15] it
is derived that, under suitable requirements (meaning no anomalies) for supersymmetric
theories, this term also vanishes. If, however, we look at the derivation of this cancellation
we see that explicit use has been made of the fact that all first-order partial derivatives of
the tree-level effective potential to the fields vanish. This shrinks the region of applicability
of the simple Coleman–Weinberg formula. Only at extremum (or saddle) points are we
justified to throw away the second term of equation (6). For the case at hand this means
that we can only use equation (5) in the global minimum and along the inflationary valley,
because here all derivatives in the tree-level potential vanish.

We stress this point, because these formulae are not always used with the appropriate
care. In [13] the one-loop corrections to the reheating part of the potential are calculated,
using the simplified formula, without checking whether the other part really drops out.
Let us now look at this case in full detail.
3.2. Calculation of the field-dependent masses in $D$-term inflation

3.2.1. Introduction. To calculate the one-loop correction we first have to find the masses of the fundamental particles in the theory. These will be dependent on the field values and so in the different regimes we may have different masses. Mass terms are all terms in the Lagrangian that are of second order in the fields, if we expand around any given point in field space. The most appropriate expansion is as follows:

$$S = \frac{1}{\sqrt{2}}(S_1 + s_1 + i(S_2 + s_2)), \quad (8)$$

$$\Phi_+ = \frac{1}{\sqrt{2}}(\Phi_{+1} + \phi_{+1} + i(\Phi_{+2} + \phi_{+2})), \quad (9)$$

$$\Phi_- = \frac{1}{\sqrt{2}}(\Phi + \phi)e^{i(\Theta + \theta)}. \quad (10)$$

All capital letters denote the fixed values of the fields around which we make an expansion. The small letters can be interpreted as the field excitations with a certain mass. Of course other choices can be made, for example $\Phi_+$ and $\Phi_-$ are treated differently.

The well-known expansion around the inflationary valley consists of setting $\Phi_{+1}$, $\Phi_{+2}$, $\Phi$ and $\Theta$ to zero and taking $|S| > S_c$. If we want to know the loop corrections off-valley in principle all fixed values are nonzero. We will comment later on this possibility, but for the moment it suffices to say that this will give very tedious calculations, so we should be more restrictive in our choice of fixed field values. In fact, we can set $\Phi_{+1}$ and $\Phi_{+2}$ to zero, if we restrict our attention to regions II and III, because the spontaneous symmetry breaking part for the $D$-term potential will be in the $\Phi_-$ sector. As discussed before the $\Phi_+$ will always be in a stable local maximum $\Phi_+ = 0$. One could also argue that nothing should depend on the phases of the $S_-$ and $\Phi_-$ fields, since the tree-level potential is also not dependent on these phases\(^5\). So one could take $S_-$ and $\Phi_-$ fields that are real. Here we will take them complex from the start, but we will indeed see in the end that the one-loop correction is independent of these phases.

3.2.2. The scalar masses. Plugging the expansion from equations (8)–(10) into the effective potential, equation (2), we can immediately read off the mass of the $\phi_{+1}$ and $\phi_{+2}$ fields. For the $s_1$, $s_2$ and $\phi$ fields we first have to diagonalize the second-order part of the Lagrangian in order to find the physical particles (which should have simple propagators, so no cross-terms):

$$\frac{1}{2} \begin{pmatrix} s_1 & s_2 & \phi \end{pmatrix} \begin{pmatrix} \frac{1}{2} \lambda^2 \Phi^2 & 0 & \lambda^2 \Phi S_1 \\ 0 & \frac{1}{2} \lambda^2 \Phi^2 & \lambda^2 \Phi S_2 \\ \lambda^2 \Phi S_1 & \lambda^2 \Phi S_2 & \frac{1}{2} \lambda^2 (S_1^2 + S_2^2) - g^2 \xi + \frac{3}{2} g^2 \Phi^2 \end{pmatrix} \begin{pmatrix} s_1 \\ s_2 \\ \phi \end{pmatrix}. \quad (11)$$

\(^5\)Unlike in the $F$-term inflation case, where the tree-level potential does depend on the phases of the $\Phi_+$- and $\Phi_-$ fields.
Now we can give a complete list of all scalar masses in the theory, see table 2.

Table 2. Diagonalization of the \((s_1, s_2, \phi)\) sector.

| (mass)\(^2\)-eigenvalue | (non-normalized) eigenstate |
|-------------------------|-----------------------------|
| \(\frac{1}{2} \lambda^2 \phi^2\) | \(\frac{1}{\sqrt{s_1^2 + s_2^2}} \begin{pmatrix} -s_2 \\ s_1 \\ 0 \end{pmatrix}\) |
| \(\frac{1}{2} (B - \sqrt{B^2 + 4C})\) | \(\begin{pmatrix} 2 \lambda^2 s_1 \\ 2 \lambda^2 s_2 \\ D - \sqrt{B^2 + 4C} \end{pmatrix}\) |
| \(\frac{1}{2} (B + \sqrt{B^2 + 4C})\) | \(\begin{pmatrix} 2 \lambda^2 s_1 \\ 2 \lambda^2 s_2 \\ D + \sqrt{B^2 + 4C} \end{pmatrix}\) |

Table 2. Mass and degrees of freedom of the scalar fields.

| Field | d.o.f. | (mass)\(^2\) |
|-------|--------|-------------|
| \(\phi_{+1}\) | 1 | \(\frac{1}{2} \lambda^2 (s_1^2 + s_2^2 + \phi^2) - \frac{3}{2} g^2 \phi^2 + g^2 \xi\) |
| \(\phi_{+2}\) | 1 | \(\ldots \) same \ldots |
| Lin. comb. of \(s_1, s_2\) | 1 | \(\frac{1}{2} \lambda^2 \phi^2\) |
| Lin. comb. of \(s_1, s_2, \phi\) | 1 | \(\frac{1}{2} (B - \sqrt{B^2 + 4C})\) |
| Lin. comb. of \(s_1, s_2, \phi\) | 1 | \(\frac{1}{2} (B + \sqrt{B^2 + 4C})\) |

The corresponding (mass)\(^2\) eigenvalues and (non-normalized) eigenstates are given in table 1, where we used the following definitions:

\[
B = \frac{1}{2} \lambda^2 (s_1^2 + s_2^2 + \phi^2) + \frac{3}{2} g^2 \phi^2 - g^2 \xi, \tag{12}
\]

\[
D = \frac{1}{2} \lambda^2 (s_1^2 + s_2^2 - \phi^2) + \frac{3}{2} g^2 \phi^2 - g^2 \xi, \tag{13}
\]

\[
C = \frac{1}{2} \lambda^4 \phi^2 (s_1^2 + s_2^2) + \frac{1}{2} \lambda^2 g^2 \xi \phi^2 - \frac{3}{4} \lambda^2 g^2 \phi^4. \tag{14}
\]

Now we can give a complete list of all scalar masses in the theory, see table 2.

3.2.3. The gauge boson mass. In the previous section we did not get a mass for the \(\theta\) field. Actually this is because we have chosen to write the complex field \(\Phi_-\) in (10) in terms of a modulus \(\phi\) and an angular part \(\theta\). This \(\theta\) field has a nontrivial kinetic term, which does not allow us to interpret the second-order part as a mass, like we did for the other fields. However, the reason for writing things out like this is that the interaction of the \(\theta\) field with the massless gauge boson \(A^\mu\) will cancel all \(\theta\) dependence of the Lagrangian in favour of a gauge boson \(B^\mu\), which acquires mass. This is the so-called Higgs mechanism.

\(\Phi_-\) has a charge \(-1\) under the local gauge transformation\(^6\). Hence the gauge-invariant kinetic term for \(\Phi_-\) is

\[
\mathcal{L}_{\text{kin, } \Phi_-} = (D_\mu \Phi_-)\dagger D^\mu \Phi_- = (\partial_\mu - ig A_\mu) \Phi_+^\dagger (\partial^\mu + ig A^\mu) \Phi_- = \frac{i}{2} \partial_\mu \phi \partial^\mu \phi + \frac{i}{2} (\phi + \phi^*)^2 \left( \partial_\mu \partial^\nu \theta + g^2 A_\mu A^\mu + 2 g \partial_\mu \partial^\nu \theta A^\mu \right). \tag{15}
\]

\(^6\) Under the gauge transformation \(\Phi \rightarrow e^{i q \xi} \Phi\) and \(A_\mu \rightarrow A_\mu + \frac{1}{g} \partial_\mu \xi\), with \(q\) the charge of the fields, so \(q = \pm 1, 0\) for \(\Phi_+\) and \(S\).
There is a non-canonical kinetic term for $\theta$ and an unwanted coupling of $\partial_\mu \theta$ to $A^\mu$. This quadratic coupling means we have not identified the physical particles correctly (it represents a mixed propagator). What we should do is write everything in terms of the gauge-invariant combination $B_\mu = A_\mu + \frac{1}{g} \partial_\mu \theta$. In doing so, equation (15) gets a simple form:

$$\mathcal{L}_{kin} = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi + \frac{1}{2} g^2 (\Phi + \phi)^2 B_\mu B^\mu.$$  

(16)

All $\theta$ dependence has cancelled and the gauge field has acquired a mass (a mass term for a gauge boson looks like $\frac{1}{2} m^2 A_{\mu} A^\mu$). The massive $B_\mu$ field now has three degrees of freedom instead of two for the massless $A_\mu$ field. The $\theta$ field behaves as the longitudinal component of $B_\mu$. The kinetic term for the gauge field remains unchanged under the redefinition, because the partial derivatives in $F_{\mu\nu} = (\partial_\mu A_\nu - \partial_\nu A_\mu)$ arising from the shift commute. The mass of the gauge field is included in table 3.

In principle the $\Phi$ field should be treated in the same way, but in this case it gives just simple kinetic and mass terms, because we expanded around the point $\Phi = 0$. If you expand around another point, you will get a more complicated interaction, but it will still be possible to define a Higgs mechanism which works for all field-space points.

Note that this derivation is done at any field value $\Phi$, so not only in the global minimum, as is done usually. The striking thing is that the $\theta$ dependence really drops out everywhere. A possible exception is the origin $\Phi = 0$, because here there is a coordinate singularity, which makes $B_\mu$ ill-defined.

### 3.2.4. The fermion masses.

In any supersymmetric theory all scalar bosons have fermionic superpartners and all gauge fields have corresponding fermionic gaugino fields, see for example [16]. The terms quadratic in the fermionic fields are summarized in the following matrix, which must again be diagonalized in order to find the field-dependent fermionic masses of the theory:

$$\begin{pmatrix}
  \begin{pmatrix}
    \psi_A & \psi_- & \psi_+ & \psi_S
  \end{pmatrix} &
  \begin{pmatrix}
    0 & g \Phi e^{i\Theta} & 0 & 0 \\
    g \Phi e^{-i\Theta} & 0 & 0 & \frac{\lambda}{\sqrt{2}} (S_1 + i S_2) \\
    0 & \frac{\lambda}{\sqrt{2}} (S_1 - i S_2) & 0 & \frac{\lambda}{\sqrt{2}} \Phi e^{i\Theta} \\
    0 & 0 & \frac{\lambda}{\sqrt{2}} \Phi e^{-i\Theta} & 0
  \end{pmatrix}
\end{pmatrix},$$  

(17)

where $\psi_-$, $\psi_+$ and $\psi_S$ are the Weyl fermions belonging to the chiral multiplet and $\psi_A$ is the gaugino. The corresponding mass eigenvalues are

$$\pm \frac{1}{2} \sqrt{E \pm \sqrt{E^2 - 8 \lambda^2 g^2 \Phi^4}},$$  

(18)

with

$$E = \lambda^2 (S_1^2 + S_2^2 + \Phi^2) + 2 g^2 \Phi^2.$$  

(19)

This completes the calculation of the fermionic masses. The results are summarized in table 4.
Table 4. Mass and degrees of freedom of the fermionic fields.

| Field | d.o.f. | (mass)² |
|-------|--------|---------|
| Lin. comb. of $\psi_A, \psi_-, \psi_+, \psi_S$ | 4 | $\frac{1}{4}(E + \sqrt{E^2 - 8\lambda^2 g^2 \Phi^4})$ |
| Lin. comb. of $\psi_A, \psi_-, \psi_+, \psi_S$ | 4 | $\frac{1}{4}(E - \sqrt{E^2 - 8\lambda^2 g^2 \Phi^4})$ |

3.3. Calculating the one-loop corrections to the $D$-term potential

Insertion of all masses in equation (5) in order to get a formula for the total one-loop correction is in principle straightforward. Some details can be found in the corresponding appendices. First of all, massless particles do not contribute (appendix A). For tachyonic masses we should only take the real part of the corresponding one-loop correction (appendix B) and imaginary mass-squared values do not occur (appendix C).

3.3.1. Cutoff dependence. Since we have obtained well-defined masses for all particles as a function of $S_1, S_2$ and $\Phi$, we can explicitly calculate the second term in equation (6):

$$\frac{1}{32\pi^2} \text{Str} \left( M^2 \right) \Lambda_c^2.$$ (20)

The supertrace should be equal to zero for the theory to make sense. Otherwise, when $\Lambda_c$ goes to infinity (or to a very large value like the Planck mass), the whole term blows up. Indeed, in calculating the supertrace most contributions cancel out, but not all. We are left with

$$\text{Str}(M^2) = -\frac{1}{2} \lambda^2 (S_1^2 + S_2^2) - \frac{1}{2} g^2 \Phi^2 + g^2 \xi.$$ (21)

There is no renormalization counterterm to cancel this field-dependent divergence. So we conclude that the effective potential is really ultraviolet-divergent, except possibly for field values where equation (21) vanishes. This happens exactly along the ‘spontaneous symmetry breaking valley’, by which we mean the valley determined by minimizing $\Phi$ for a given $S_1$ and $S_2$:

$$\frac{\partial V_{\text{tree}}}{\partial \Phi} = 0 \implies \frac{1}{2} \lambda^2 (S_1^2 + S_2^2) - \frac{1}{2} g^2 \Phi^2 + g^2 \xi = 0 \quad \text{or} \quad \Phi = 0.$$ (22)

3.3.2. Limit to the inflationary valley. A good way to check that the method in the enlarged parameter space makes sense is to compute whether it gives the same results as before on the inflationary valley ($\Phi = 0$). Mathematically one could argue that there is a problem in taking the limit $\Phi \to 0$, because $B_\mu$ is not well-defined for $\Phi = 0$. However, one can simply take a very small value for $\Phi$, then $B_\mu$ will be well defined and the Lagrangian will have no dependence on $\theta$. The particle masses we get along the valley are summarized in table 5.

The one-loop corrections on the inflationary valley are well known and easy to calculate if we just set $\Phi = 0$ from the very beginning. The particle masses that contribute are given in table 6.

Comparison of these two tables points out that the two methods really give different results. In the standard calculation there is an extra scalar particle of (mass)² equal to $\frac{1}{2} \lambda^2 (S_1^2 + S_2^2) - g^2 \xi$. As a result the one-loop correction derived from both methods will also differ. This makes the enlarged parameter space method not trustworthy.
Table 5. Mass and degrees of freedom of all fields along the inflationary valley in the limit $\Phi \to 0$ of the enlarged parameter space method. The gauge field has two degrees of freedom if $\Phi$ is really zero and three degrees of freedom if $\Phi$ is very small. For the computation of the one-loop corrections it makes no difference $(2 \times 0 = 3 \times 0 = 0)$.

| Field d.o.f. | (mass)$^2$ |
|-------------|-----------|
| $\phi_+^1$  | 1         | $\frac{1}{2} \lambda^2 (S_1^2 + S_2^2) + g^2 \xi$ |
| $\phi_+^2$  | 1         | ... same ... |
| $\sqrt{S_1^2 + S_2^2} (S_2 s_1 + S_1 s_2)$ | 1 | 0 |
| $\phi$      | 1         | $\frac{1}{2} \lambda^2 (S_1^2 + S_2^2) - g^2 \xi$ |
| $\phi_{-1}$ | 2 or 3    | 0 |
| Lin. comb. of $\psi_A$, $\psi_-$, $\psi_+^1$, $\psi_+^2$ | 4 | $\frac{1}{2} \lambda^2 (S_1^2 + S_2^2)$ |
| Lin. comb. of $\psi_A$, $\psi_-$, $\psi_+^1$, $\psi_+^2$ | 4 | 0 |

Table 6. Mass and degrees of freedom of all massive fields along the inflationary valley following the standard calculation.

| Field d.o.f. | (mass)$^2$ |
|-------------|-----------|
| $\phi_+^1$  | 1         | $\frac{1}{2} \lambda^2 (S_1^2 + S_2^2) + g^2 \xi$ |
| $\phi_+^2$  | 1         | ... same ... |
| $\phi_{-1}$ | 1         | $\frac{1}{2} \lambda^2 (S_1^2 + S_2^2) - g^2 \xi$ |
| $\phi_{-2}$ | 1         | ... same ... |
| Lin. comb. of $\psi_-$, $\psi_+$ | 4 | $\frac{1}{2} \lambda^2 (S_1^2 + S_2^2)$ |

3.3.3. The cutoff dependence and the limit problem. Obviously the described method similar to the one in [13] cannot be correct. Results are unphysical because of the cutoff dependence and just wrong because of the contradiction with the previous method along the inflationary valley. In the following we will describe a possible method which does not suffer from these two problems.

The origin of the limit problem seems to be the Higgs mechanism. The Higgs mechanism dictates that we write the $\Phi_-$ field in polar coordinates. In doing so the angular $\theta$ field becomes massless. This could well be the mass we are missing with respect to the usual case. Furthermore the cutoff-dependent supertrace contribution in equation (21) also amounts to the same (mass)$^2$ value of $\frac{1}{2} \lambda^2 (S_1^2 + S_2^2) - g^2 \xi$ in the limit $\Phi$ goes to zero. So maybe we can solve both problems at a time by evaluating the loop contributions with the masses given by the theory before applying the Higgs mechanism. First we will show that this indeed solves the aforementioned problems. Then we will discuss the physical validity of this method.

3.3.4. A possible solution. Since we suspect that the problems come from the massless $\theta$ field we will use the following expansion:

$$S = \frac{1}{\sqrt{2}} (S_1 + s_1 + i(S_2 + s_2)),$$

(23)
Comparing this extra mass with equation (21) reveals that using this method the Λ scalar masses (see section 3.2.2) we get the following mixing between the (mass)² difference between both methods lies in the inclusion of Feynman graphs with the mixed

Also we will set

→

become identical, resulting in a correct smooth limit to the known one-loop corrections

Diagonalizing this (mass)

σ

Diagonalizing this (mass)

The field spectrum is the same as in table 2 with an extra mass for the

Also we will set S₂ and Φ₋₂ to zero, because otherwise the calculations become unnecessarily bothersome and we have seen before that these phases really make no difference in D-term inflation. Going through the same steps as before in calculating the scalar masses (see section 3.2.2) we get the following mixing between the (s₁, s₂, φ₋₁, φ₋₂) fields:

\[
\frac{1}{2} \begin{pmatrix}
\phi₋₁ & \phi₋₂ & s₁ & s₂ \\
\frac{1}{2} \lambda² S₁ + \frac{3}{2} g² \phi₋₁² - g² \xi & 0 & \frac{1}{2} \lambda² S₁ & 0 \\
0 & \frac{1}{2} \lambda² S₁ & 0 & \frac{1}{2} \lambda² \phi₋₁² \\
\lambda² S₁ \phi₋₁ & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{1}{2} \lambda² \phi₋₁²
\end{pmatrix}
\begin{pmatrix}
\phi₋₁ \\
\phi₋₂ \\
s₁ \\
s₂
\end{pmatrix}
\]

Diagonalizing this (mass²) matrix and going back to the original notation (replacing S₁ by (S₁ + S₂) and Φ₋₁ by Φ) gives the following field-dependent particle masses, see table 7.

The mass spectrum is the same as in table 2 with an extra mass for the φ₋₂ field. Comparing this extra mass with equation (21) reveals that using this method the Λ² dependence really drops out. Also in the limit Φ → 0 the scalar masses in tables 7 and 6 become identical, resulting in a correct smooth limit to the known one-loop corrections along the inflationary valley. To be precise, in table 7 φ₋₂ and one of the linear combinations of (s₁, φ₋₁) get the same mass and the other linear combination becomes massless.

This new method works because the masses of the gauge field and the fermionic fields remain unaffected by this different choice of coordinates, so all other masses remain the same as before. For the fermionic part this is trivial, because the fermionic masses only depend on the fixed values of the scalar fields. For the gauge field this is less trivial, but comparison of equations (15) and (16) reveals that the mass of the gauge field is unaffected by the Higgs mechanism. In both cases the (mass)² value equals g²Φ². In principle the difference between both methods lies in the inclusion of Feynman graphs with the mixed
`Goldstone`-gauge propagator. In the next section we will present another way of viewing this situation.

3.3.5. Validity of the method. Of course we cannot just change the calculational method, because there is a physical background behind this. The reason for trying this out anyway is that there is no clear-cut answer in the literature which relates the Higgs mechanism to the Coleman–Weinberg formula and which states explicitly what are the masses to be used in applying the Coleman–Weinberg formula in this setting. This is because normally if one only looks at the global minimum it really makes no difference, since here the Goldstone mode is massless anyway and, as is shown in appendix A, massless fields do not contribute to the one-loop correction.

The Coleman–Weinberg formula, as used before, includes all one-loop Feynman graph contributions to the effective potential, where the external truncated scalar legs have zero momentum (since the vacuum has no momentum). The particle in the internal loop can be any particle in the theory that couples to the scalar external legs. In the cutoff-dependent $\text{Str}(M^2)$ part of the Coleman–Weinberg formula it really makes no difference whether we take the (mass)$^2$ values of the physical particles, after diagonalizing the (mass)$^2$ matrix, or we take the `unphysical` particle masses (without diagonalizing this matrix). This is because the trace of a matrix is invariant under a change of basis\(^7\). Somehow in mixing the scalar field with the gauge field via the Higgs mechanism, it seems that it does make a difference which basis we use.

In appendix D we describe a Higgs mechanism for a simple $U(1)$-invariant potential, using both polar and Cartesian field definitions. Surprisingly for the global $U(1)$ case the `Goldstone` boson has a mass, if it is outside the global minimum. For the local case one usually uses polar fields, because of the Higgs mechanism. However, it is also possible to define the equivalent Higgs mechanism with Cartesian fields. Then the mixed propagator term vanishes, as does the kinetic term for the `Goldstone` field, but the `Goldstone` mass term does not vanish\(^8\).

Of course the discussion in appendix D holds equally well for the Mexican hat type $U(1)$ symmetry we’re interested in. In this case the (mass)$^2$ value of the global Goldstone mode will be a smooth function of $\Phi$, which goes from a negative value at the top of the Mexican hat smoothly to zero at the global minimum circle to positive values outside this circle, see figure 2. Surprisingly the (mass)$^2$ of this Goldstone mode $\frac{1}{2}A^2(S_1^2 + S_2^2) + \frac{1}{2}g^2\Phi^2 - g^2\xi$ is exactly what we need to solve the cutoff dependence and the limit problem. So using second derivatives of the potential for the classical masses in the Coleman–Weinberg formula gives a well-defined one-loop correction even outside the global minimum and using the tree-level propagator masses does not.

There are now two ways of viewing the situation.

1. We should not include the Goldstone masses in a calculation of the one-loop corrections to the effective $D$-term potential outside the global minimum. As a result these one-loop corrections are ill defined for almost all field values.

\(^7\) Use $\text{Tr}(AB) = \text{Tr}(BA)$. If matrix $M$ is diagonalized as follows $M = CDC^{-1}$, then $\text{Tr}(M) = \text{Tr}(CDC^{-1}) = \text{Tr}(CC^{-1}D) = \text{Tr}(D)$. Also $\text{Tr}(M^n) = \text{Tr}(D^n)$.

\(^8\) Maybe `mass term` is not the right phrase to use here, because for the local $U(1)$ case it no longer corresponds to a pole in the propagator. It is the term quadratic in the corresponding `Goldstone` field.
Figure 2. The solid line gives the (mass)$^2$ value of the global ‘Goldstone’ mode as a smooth function of the Φ field for $S = 0$. The dashed line depicts the shape of the tree-level effective potential (the vertical axis does not refer to this). Note that the Goldstone mode is indeed massless at the global minimum of the potential.

(2) We should include the Goldstone masses in the calculation of the one-loop corrections to the effective $D$-term potential outside the global minimum. The resulting one-loop corrections agree with previous results along the inflationary valley and have no $\Lambda^2_r$ dependence. This is the viewpoint we will take.

There are some field-space points for which both methods agree, because the Goldstone boson is really massless there. These points lie exactly on the ‘spontaneous symmetry breaking valley’ defined by equation (22). Note that both methods disagree on the inflationary valley.

3.4. Graphs of the one-loop correction

This section is merely meant to give the reader a feeling of what the one-loop corrections look like in comparison with the tree-level potential and what are the possible implications. A quantitative analysis of the different aspects will be given later on. The results are shown with the aid of graphs, because the actual algebraic expression is too complicated.

There are four adjustable parameters: $\lambda$, $g$, $\xi$ and $\Lambda_m$, although the dependence on the last one will be small. The output, being a potential energy density, will have mass dimension 4 and will always be given in units of $(M_P^4)$. Some very small numbers appear due to the fourth power. First we will look at the case where all these parameters are 1, although this will not be the typical case, because normally $\xi$ is much less than one square Planck mass.

Figure 3(a) gives the classical tree-level potential in the region $0 < S < 2 \cdot S_c$. The $\Phi$ axis reaches to 1.2 times the global minimum value, so we can see the entire classical

In fact, the slightly broken supersymmetric nature of the one-loop correction makes it of the form: near-infinity minus near-infinity. Huge bosonic and fermionic contributions nearly cancel and the difference is the correction we are interested in. Many of these cancellations can be resolved analytically, by tricks of the form: $A^2 \ln(A) - B^2 \ln(B) = A^2 \ln(A/B) + (A^2 - B^2) \ln(B)$, which can make things much simpler if $A$ and $B$ are large, nearly equal quantities. Unfortunately we did not manage to get rid of all (near-infinity minus near-infinity) terms in this way, so the final formula for the total one-loop correction is still too complicated to analyse by hand. The near-infinity minus near-infinity nature can also lead to numerical problems, but these we have solved.
path during spontaneous symmetry breaking. Part (b) depicts the one-loop correction on the same domain. It is around a factor of 20 smaller than the tree-level potential and has an opposite shape. Of course we rely on the one-loop corrections to be (very) small in comparison to the tree-level potential, so that we are hopefully justified to ignore all two-loop and higher-order corrections. Part (c) depicts the total (tree + one-loop) potential. As mentioned in section 2 the quantum correction indeed adds a little slope to the inflationary valley ($\Phi = 0$). In slowly rolling down from $2 \cdot S_c$ to $S_c$, this universe will inflate by around 2900 e-folds.

The one-loop corrections in this case are small as compared to the tree-level potential. In most cases they will be even smaller, but this does not mean that they are unimportant, because sometimes it is the variation that matters. In figure 4, we have zoomed in on the tree, one-loop and total values of the potential in a region centred around the critical point ($S_c$). This part of the inflationary valley corresponds to around 52 e-folds. Although the tree-level potential is a factor of 60 larger, the one-loop corrections still dominate the behaviour around the critical point. The slope generated by the one-loop corrections can be easily seen. The assumption that inflation ends when the fields leave the inflationary valley, which is usually made, is probably not valid here: since the upper half of the picture corresponds to 52 e-folds, we can already see that the total potential does not change fast enough on the relevant timescales to end inflation abruptly (note that the scales on both axes are the same). The dependence on the renormalization mass scale ($\Lambda_m$) is very small. If we change the value of $\Lambda_m$ from 1 $M_{Pl}$ to 0.01 $M_{Pl}$, the slope changes only to give around 58 e-folds instead of 52 on the same domain.

Figures 5 and 6 represent the general shape of the one-loop corrections, which one obtains for almost all physically relevant parameter combinations. For these figures $g = 0.1$, $\lambda = 0.0001$, $\sqrt{\xi} = 0.00025M_{Pl}$ and $\Lambda_m = 0.35M_{Pl}$ (which is equal to the value of $S_c$). These values are compatible with WMAP observations, which will be explained in section 5. Anyway, this selection does not change the overall picture.
Figure 4. The tree-level potential, one-loop correction and the sum of both, for the same parameters as in figure 3 on a domain centred around the critical point ($S_c$). Both field axes have the same scale. The slope along the inflationary valley, arising from the one-loop corrections, is clearly visible now. It will give rise to 52 e-folds of inflation on the upper half of the picture, which therefore represents the cosmologically interesting region. It is apparent that the variations in the one-loop corrections dominate the tree-level variations in this region.

Figure 5. The tree-level potential, one-loop correction and the sum of both, for $g = 0.1$, $\lambda = 1.0 \times 10^{-4}$, $\sqrt{\xi} = 2.46 \times 10^{-4} M_{\text{Pl}}$ and $\Lambda_m = 0.35 M_{\text{Pl}}$ on the domain $0 < S < 2 \cdot S_c$, $0 < \Phi < 1.2 \cdot \Phi_{\text{min}}$. The one-loop correction is around a factor of 500 smaller than the tree-level potential. For $S < S_c$ and small $\Phi$, the one-loop corrections vary with $\Phi$ in the opposite direction with respect to the tree-level potential. The tilt produced by the one-loop corrections corresponds to around 18 000 e-folds in the upper half of the picture.

The main differences with the previous set of figures are that the one-loop corrections are even smaller as compared to the tree-level potential (around a factor of 500) and that, for $S < S_c$ and near $\Phi = 0$, the one-loop corrections are of opposite slope in the $\Phi$ direction as compared to the tree-level potential. For large $\Phi$ the one-loop corrections mimic the tree-level potential. So, only in the neighbourhood of the inflationary valley can we expect the corrections to be of interest.

Another thing, which is shared by all possible parameter choices, is that in the global minimum both the tree-level potential and the one-loop corrections vanish, giving a true, totally supersymmetric, non-inflating, ground state.
In the \((S = 0)\)-plane the one-loop corrections always counteract the tree-level potential, although normally the effect is too small to have important consequences.

Close to the inflationary valley the one-loop corrections behave more like, say, a parabola and the tree-level potential more like a fourth-order function. So, close to the inflationary valley, the one-loop corrections dominate the potential dependence on \(\Phi\).

Again, if we zoom in on the relevant region around the critical point we see that the one-loop corrections here will definitely have important consequences. This opens an interesting window. It seems that the, up to now neglected, off-valley quantum corrections will play a major role on all scales which are of cosmological interest. Most surely they will have consequences for the explanation of WMAP\(^{10}\) observations.

For a restricted region in parameter space it is possible that the negative one-loop corrections at the \((S, \Phi) = (0, 0)\) point become relatively large, say of the order of 10\% of the tree-level potential. This less common situation is depicted in figures 7 and 8. The lowering of the top of the Mexican hat would probably lead to less energetic cosmic strings. Of course, if the one-loop corrections become too big, we can no longer neglect the second-and higher-order loop contributions. Again we show a parameter combination which will be compatible with the WMAP density perturbation.

4. Inflation below the critical inflaton value

Usually it is assumed that no significant inflation occurs after the inflaton field \(S\) reaches the critical value \(S_c\). In this section we will show that even classically (without one-loop corrections) this assumption may be false. Then we will calculate explicitly the number of e-folds below \(S_c\) for the \(D\)-term parameter combinations which best fit WMAP measurements, according to [20]. We will see that indeed inflation below \(S_c\) cannot be neglected. A whole new analysis is needed in order to compare the predictions of the \(D\)-term model with WMAP measurements. This is done in section 5.

\(^{10}\) The Wilkinson Microwave Anisotropy Probe is a satellite which looks in great detail at the cosmic microwave background radiation, see http://map.gsfc.nasa.gov. Its predecessor was the COBE satellite.
Figure 7. The tree-level potential, one-loop correction and the sum of both, for $g = 1.0$, $\lambda = 0.01$, $\sqrt{\xi} = 6.8 \times 10^{-4} M_{Pl}$ and $\Lambda_m = 0.096 M_{Pl}$ ($= S_c$) on the domain $0 < S < S_c$, $0 < \Phi < 1.2 \cdot \Phi_{g,\text{min}}$. The one-loop corrections have lowered the top of the Mexican hat (origin in this picture) considerably and will thus slightly lower the resulting cosmic string tension.

Figure 8. The tree-level potential, one-loop correction and the sum of both, for the same parameters as in figure 7 on the domain $0 < S < 2 \cdot S_c$, small $\Phi$. This shows how the height of the Mexican hat goes down as the inflaton field $S$ goes to zero. The one-loop corrections will slightly decrease the tension of the resulting cosmic strings.

Substantial inflation after $S_c$ is interesting for two reasons. First of all this offers the possibility to dilute the cosmic strings which are produced in the symmetry breaking process. This is a good thing, since WMAP has put stringent bounds on the string contribution to the cosmic background anisotropy. It cannot be more than about $10\%$ \cite{17,18}.

Secondly, the cosmologically interesting region, which is around 50–60 e-folds before the end of inflation, will shift when substantial inflation occurs after $S_c$. This region, for example, determines the primordial density perturbation at scales observable in the CMB (which should be around $2 \times 10^{-5}$ in order to be compatible with WMAP measurements). So the density perturbation at recombination is dependent on the possible inflation below $S_c$.

\textsuperscript{11} The cosmic microwave background anisotropies have a fluctuating nature if they arise from quantum fluctuations that are enlarged by inflation. This is because of acoustic oscillations in the period between inflation and recombination. Anisotropies coming from cosmic strings are scale-free. The fluctuations observed by WMAP look very much like the ‘quantum fluctuation’ anisotropies, leaving at most a 10% contribution to cosmic strings.
Inflation takes place whenever the potential energy density of the scalar fields dominates the kinetic energy density [3]. This is possible only if the slope of the potential is very small. In this case the equation of motion becomes dominated by the friction term \((H\partial_t \phi)\), the fields are in slow-roll and the number of e-folds before the end of inflation is effectively determined by how long the fields manage to stay in slow-roll. A good approximation to the number of e-folds is [3]

\[
N = -8\pi \int \frac{V}{V'} \, d\phi.
\]

The integral is over the path in field space, units are in Planck masses and the approximation is valid as long as the slow-roll conditions are satisfied.

One normally assumes that inflation ends either before reaching \(S_c\), whenever the slow-roll parameters become too big (order 1), or at \(S_c\), because then spontaneous symmetry breaking starts and will speed up the fields very quickly. In the previous section we have seen that this is probably not always true, because the potential does not always become very steep in a region around \(S_c\) corresponding to about 50–60 e-folds. Partly this is because, close to the inflationary valley, the one-loop corrections are just opposite to the classical potential. But also even classically this assumption may be false.

Figure 9 gives the general shape of the tree-level potential in terms of three parameters \(a, b, c\) which correspond respectively to the width and height of the Mexican hat and to the value of \(S_c\). The relation with the \(D\)-term parameters is

\[
a = \sqrt{2\xi} \quad \xi = \frac{1}{2} a^2
\]

\[
b = \frac{a^2 \sqrt{2} \lambda}{2} \quad \text{or} \quad g = \frac{2\sqrt{2} b}{a^2}
\]

\[
c = \frac{a \sqrt{2} \lambda}{\sqrt{2}} \quad \lambda = \frac{2\sqrt{2} b}{ac}
\]

The conditions that favour inflation during spontaneous symmetry breaking are a large \(a\), small \(b\) and large \(c\), because these make the potential flat. Since the dependence on the
physical parameters is fairly simple this translates to a large $\xi$, small $g$ and small $\lambda$. We can even get a simple bound on when inflation happens approximately all the way down to the global minimum, assuming that the fields remain homogeneous and that cosmic string production does not bring inflation to a halt. Let us assume that $a$ is much smaller than $c$ and the field follows a straight line in going from $S_c$ to the global minimum, then we have the following average values:

$$\langle V \rangle = \frac{g^2 \xi^2}{4},$$

$$\langle V' \rangle = \frac{g^2 \xi^2}{2\sqrt{a^2 + c^2}} \approx \frac{g^2 \xi^2}{2c},$$

$$\langle \epsilon \rangle = \frac{1}{16\pi} \left( \frac{\langle V' \rangle}{\langle V \rangle} \right)^2 \approx \frac{1}{4\pi c^2},$$

with $\epsilon$ being one of the slow-roll parameters \[3\]. The condition for inflation to persist approximately all the way down becomes very simple:

$$\epsilon < 1, \quad S_c > \frac{1}{\sqrt{4\pi}} \approx 0.28M_{Pl}. \quad (32)$$

Inflation will happen during a large part of the spontaneous symmetry breaking if $S_c$ is bigger than $0.28M_{Pl}$. Otherwise it is still possible that the area around $S_c$ is flat enough to give substantial inflation in the beginning of the symmetry breaking, but certainly not during this whole stage. This will prove to be much more common.

We will now look at the parameter space which give the right magnitude of density perturbation according to \[20\]. Figure 2 in \[20\] gives various parameter combinations that lead to the right density perturbation\[13\]. Moreover combinations that lie under the horizontal line give a cosmic string contribution less than 10%.

Using their parameters as input, we are able to see if there is any inflation after $S_c$ (which is not included in their computation). We calculate numerically the steepest path down, using of course the total (tree-level + one-loop) potential and, with the aid of equation (27), we obtain the number of e-folds after $S_c$. Figure 10 gives the results for several parameter values with $g = 0.01$. Inflation stops when either $\epsilon$ or $\eta$ comes close to 1. One can see that the result does not depend much on the criterion used. In general $\eta$ will become large just before $\epsilon$ does so and even if there is a substantial difference in the field values at which respectively the $\eta$ and $\epsilon$ criterions fail, this will not give a substantial difference in e-folds, because when $\eta$ becomes large the slope is already too steep to have a considerable contribution from equation (27). However, $\epsilon$ is a more reliable numerical quantity than $\eta$, since it is dependent on the first derivative of $V$ instead of the second derivative. It is important to note that in all simulations the difference between both criteria never exceeds about 3 e-folds. Since the $\eta$ criterion can sometimes be off due to an enhanced numerical error, we will stick to the $\epsilon$ criterion from now on.

Figure 11 gives the results for all sampled parameter combinations. The remarkable thing is that the number of e-folds after $S_c$ can be as high as 100 000, so we are definitely

12 This is quite natural to assume, because large values of $\xi$ will give cosmic strings with a too large tension \[18\].

13 Note that $\kappa$ in \[20\] is $\lambda$ in this paper.
not always justified to ignore this. It is hard to extract further physical results from this, because as soon as the number of e-folds after $S_c$ become substantial, it could well be that this messes up the WMAP agreement in the density perturbation, which is dependent on the region about 60 e-folds before the end of inflation. In other words, the relation between $\lambda$, $g$ and $\xi$ used in [20] could be wrong. In the next section we will try to find the corrected relation between these parameters, that is compatible with WMAP.

5. New bounds on SUSY $D$-term inflation parameters

We have found a way to compute one-loop quantum corrections to the $D$-term effective potential in section 3. In section 3.4 we have seen that these one-loop corrections might be more important than thought previously. Especially in section 4 we saw that, using these one-loop corrections and looking at data from [20], there can be a huge amount of inflation after the critical $S$ value (where inflation is assumed to stop). The purpose of this section is to re-examine the agreement with the WMAP data [2], given the new insight of inflation after spontaneous symmetry breaking.

5.1. Matching with WMAP data

There are three free parameters $g$, $\lambda$ and $\xi$. $g$ is the dimensionless gauge coupling. Its most natural value would be somewhat smaller than unity. $\lambda$ is a dimensionless coupling and $\xi$ is the Fayet–Illiopoulos gauge term in units of $(M_{Pl})^2$. Its square root gives the field value of the $\Phi_-$ field after spontaneous symmetry breaking, see equation (28), which directly determines the tension of the resulting cosmic strings, see equation (39). The energy scale at which spontaneous symmetry breaking occurs is equal to $\sqrt{\xi}$ up to a factor $g/\lambda$, see equation (4).
Figure 11. This figure depicts the number of e-folds after \( S_c \) using datapoints from [20] spanning the whole range of parameters. As you can see in approximately half of the cases the number of e-folds exceeds 10, which will certainly give detectable consequences. In some cases the number of e-folds even exceeds 60, meaning that the whole region of inflation relevant to cosmology happens after \( S_c \).

The formulae for the amplitude of the (scalar) density perturbations \( A_S \) and gravitational waves (or tensor perturbations) \( A_T \) are

\[
A_S = \sqrt{\frac{512\pi}{75}} \frac{V^{3/2}}{V'}, \quad (33)
\]

\[
A_T = \sqrt{\frac{32}{75}} V^{1/2}. \quad (34)
\]

The input is again in Planck masses and both should be evaluated at 60 e-folds before the end of inflation (or later if one wishes to know the perturbations on smaller scales). The value of the potential energy density \( V \) and its derivative in the direction of steepest descent \( V' \) are coming from the total (tree-level plus one-loop) potential. As long as \( A_T \) is small in comparison with \( A_S \) it can be neglected and \( A_S \) should be approximately \( 2 \times 10^{-5} \) according to CMB measurements. In all simulations, the amplitude of the gravitational waves is very small and thus can be neglected.

We proceed as follows: we fix a given combination of \( \lambda \) and \( g \), set \( \Lambda_m \) equal to \( S_c \) so that we can compare results with [20], then we guess what could be a plausible value of \( \xi \). We do the whole simulation with this value of \( \xi \) and see whether we get the wanted

\footnote{Normally the one-loop corrections are very small in comparison to the tree-level potential. That is why people do not include the one-loop corrections in the value of \( V \) plugged into equations like (33) and (34). The nice thing is that derivatives of \( V \) are independent of \( \Lambda_m \). Neglecting the one-loop corrections in \( V \) is practically the same as setting \( \Lambda_m \) equal to the energy scale of interest, since this makes the one-loop corrections very small in the direct neighbourhood of this energy scale. As we saw before, the \( \Lambda_m \) dependence is very small anyway, so different choices of \( \Lambda_m \) do not give dramatically different results.}
Figure 12. This graph shows the parameter combinations that are compatible with the WMAP density perturbation. Points below the 10% cosmic string contribution line are also compatible with the WMAP measurements of the spectrum of the CMB anisotropies. There seems to be almost no dependence on the gauge parameter $g$. Datapoints for different $g$ overlap perfectly. The formula: $\sqrt{\xi} = 6.42 \times 10^{16} \lambda^{1/3} \text{ (GeV)}$ gives a perfect fit for $\lambda < 5 \times 10^{-4}$.

value of $2 \times 10^{-5}$ for the density perturbation, equation (33). If not, an algorithm makes an improved guess, and so on, until the density perturbation converges to the wanted limit.

The main differences between our analysis and the analysis in [20] are:

1. Their simulation stops at $S_c$, because they only have the one-loop corrections along the inflationary valley. This new simulation takes into account the number of e-folds after $S_c$ and the off-valley one-loop corrections.

2. Because of the aforementioned numerical problems of the kind ‘near-infinity minus near-infinity’, see footnote 9, in [20] they use an approximate formula for the one-loop corrections. We solved these numerical difficulties and use the real formula.

3. The simulation in [20] includes the cosmic string contribution in the determination of $\xi$. Our simulation does not. This is justified since we cannot assume right away that cosmic strings contribute, due to inflation after spontaneous symmetry breaking, see section 5.4. If the cosmic strings do contribute this will give a difference of at most 10% for datapoints consistent with WMAP.

This now happens to be one of the rare occasions in which increasing the complexity of the computation actually makes the results more simple, as can be seen by comparing figure 12 with figure 2 in [20]. It depicts the whole range of values of the three parameters $\lambda$, $g$ and $\xi$ that give the right density perturbation of $2 \times 10^{-5}$. The parameter range is

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15 There is really no hope to do such a thing analytically, because the one-loop corrections are way too complicated and also the trajectory is not a straight line for $S$ smaller than $S_c$. 

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larger than in [20]. There is an extra allowed region in parameter space for $g = 0.3$ and $g = 1.0$. Although a lot of computer work is put in to getting all these datapoints, the result is astonishingly simple. In [20] there is a large dependence on the gauge parameter $g$. This dependence has totally disappeared. Only for $g = 1.0$ can we detect a slightly different path. For $\lambda$ values up to about $5 \times 10^{-4}$ the remaining $(\lambda, \xi)$ dependence is perfectly described by

$$\sqrt{\xi} = 6.42 \times 10^{16} \lambda^{1/3} \text{ (GeV)}.$$  \hfill (35)

We cannot explain why a very involved simulation puts out such a nice precise number as $1/3$.

The value of the gauge coupling $g$ has no influence on relation (35), but it does determine at which point the number of off-valley e-folds becomes important. This can be seen in figure 13. For each value of $g$ the simulation stops when the number of off-valley e-folds comes close to 60. The value of $\lambda$ at which this occurs becomes smaller if $g$ becomes smaller. The simulation stops at this point, because when the 60th e-fold is off-valley the determination of the density perturbation becomes numerically very difficult and, more importantly, our algorithm to find the desired $\xi$ value consistent with WMAP does not converge any more. We do not exclude the possibility that there are still parameter combinations with smaller $\lambda$ values compatible with WMAP, but they do not lie in the neighbourhood of the other points. In mentioning the number of off-valley e-folds we really mean the number of e-folds after the $\Phi$ field leaves the valley ($\Phi = 0$). In general the one-loop corrections first extend the steepest descent path a bit along the $S$ axis and then spontaneous symmetry breaking start quite abruptly, see, for example, figure 6(c), but note that the scaling is different on both axes. This is the starting point for what we call 'off-valley inflation'.

The reason why the $g = 1$ plot does not perfectly overlap with the other $g$ values is probably that for $g = 1.0$ already off-valley inflation becomes important around $\lambda = 5 \times 10^{-4}$, before the curve bends around the corner.

The resulting parameter regime that is compatible with WMAP is thus much larger than previously estimated. For small $\lambda$ values all datapoints remain at low $\xi$ values, which puts them safely below the 10% cosmic string contribution line at $\sqrt{\xi} = 2.3 \times 10^{15}$ GeV. But even more importantly, beforehand the maximum value of $g$ which still had some datapoints below the 10% limit was $g = 0.03$ and this has gone up to about $g = 0.3$. In principle gauge couplings close to unity are more natural and they could arise from string theory. Even for gauge couplings $g$ bigger than 0.3, there might be combinations compatible with WMAP, because for all $g$‘s it is possible to get substantial off-valley inflation which might dilute the cosmic strings enough to suppress their contribution to the density perturbation observed in the CMB (an effect which is not included in the 10% bound in figure 12). If, for example, as a first guess we assume that there is one cosmic string produced on average per Hubble volume at the moment of symmetry breaking, then 60 e-folds of off-valley inflation would mean that there is, on average, about one cosmic string left in the whole of our observable universe. The contribution to the density perturbation would then be way less than 10%.
Figure 13. The same dataset as in figure 12 for different values of the gauge coupling $g$ separately. The cutoff at the lower $\lambda$ value corresponds to about 60 off-valley e-folds. Taking smaller $g$ values shifts this cutoff to consecutively smaller $\lambda$ values.
5.2. Number of e-folds

The reason why the results in section 5.1 have simplified so dramatically is because of the inclusion of a descent calculation of the amount of inflation below the critical inflaton field value. This shifts the cosmologically interesting region, which is 60 e-folds before the end of inflation and therefore changes all observables, like the density perturbation and the spectral index, see section 5.3. The effect will prove to be the largest for big values of $g$ and small values of $\lambda$. Let us now look at the number of e-foldings in more detail.

As mentioned before in section 4 the one-loop corrections will lengthen the path travelled on the inflationary valley. There are now two variables to look at: the number of e-folds after $S_c$, something which was neglected before, but more interesting even is the number of e-folds after the symmetry breaking, because this opens the possibility to dilute the cosmic strings after they are produced, which will be discussed in section 5.4.

In order to compute the number of e-folds we use the integral in equation (27). Figures 14–16 show the number of e-folds as a function of $g$ and $\lambda$ and the fitted $\xi$. Again the results are very simple. The following formulae give a good fit, except for $g = 1.0$:

$$N_{Sc} = 7.4 \times 10^{-3} g^{1.94} \lambda^{-1.40},$$

$$N_{val} = 7.4 \times 10^{-4} g^{1.91} \lambda^{-1.40},$$

where $N_{Sc}$ and $N_{val}$ give the number of e-folds respectively after $S_c$ and after leaving the $(\Phi = 0)$ valley. It is quite surprising that the fit works well in both the $g$ and $\lambda$ direction and moreover both functions are very similar, with about a factor of 10 difference. So the extension of the inflationary phase takes place for 90% on the extended inflationary valley and only for 10% after the new spontaneous symmetry breaking point.\footnote{We should note that equation (36) is very firm, but equation (37) should be looked at more skeptically, because it is dependent on the precise dynamics during spontaneous symmetry breaking. We took a homogeneous field slow-rolling down the steepest path in field space, but the true dynamics, including cosmic string formation, is bound to be more complicated and could well interfere with inflation.}

Figure 14. This graph shows the amount of inflation both after $S_c$ and after leaving the $(\Phi = 0)$ valley for all datapoints.
Figure 15. This graph shows the amount of inflation after $S_c$ for all datapoints. $N_{S_c} = 7.4 \times 10^{-3} g^{1.94} \lambda^{-1.40}$ is a good fit to all datapoints except $g = 1.0$.

Figure 16. This graph shows the amount of inflation after leaving the $(\Phi = 0)$ valley for all datapoints. $N_{S_c} = 7.4 \times 10^{-4} g^{1.91} \lambda^{-1.40}$ is a good fit to all datapoints except $g = 1.0$. The occasional point that is off the line is probably due to a slight mistake in the automatic recognition of the point where $\Phi$ really leaves the neighbourhood of $(\Phi = 0)$, which can be tricky.

In section 4 we argued that, even classically, inflation can be extended on a large portion of the trajectory during spontaneous symmetry breaking, if $S_c$ becomes bigger than about $0.28 M_{Pl}$, see equation (32). Figure 17 shows that this is indeed the case. Right around $S_c \approx 0.28 M_{Pl}$, the portion of the off-valley trajectory on which inflation still takes place becomes substantial. This does not depend much on any of the other parameters.
Figure 17. The $S/S_c$ field value at the end of inflation is plotted as a function of $S_c$. At around the predicted value of $0.28\text{M}_{\text{Pl}}$, this line goes steeply down, but also on the left of the picture there seems to be a possibility of a very flat potential up to around halfway down the $S$ axis. Most datapoints, however, lie very crowded on one curve.

5.3. Spectral index

The spectral index denotes how the amplitude of the tiny fluctuations depend on the scale. A spectral index of $n = 1$ would correspond to a scale-invariant perturbation. WMAP measurements reveal that this spectral index is very close to one. The 1-year results gave a value of $n = 0.99 \pm 0.04$. The accuracy improved with the 3-year results (which include CMB polarization measurements), giving a value of $n = 0.961 \pm 0.017$ (when the contribution of tensor modes is assumed to be negligible) [2]. So the spectral index is very close to one and probably a bit smaller ($n = 1$ is $2\sigma$ away from the best-fit value for the three-year results). Bayesian statistics give a somewhat larger interval as compared to the error estimate above [21].

During slow-roll there is a simple equation for the spectral index [3]:

$$n = 1 - 6\epsilon + 2\eta.$$  \hfill (38)

In our computer program this is evaluated between 58 and 60 e-folds before the end of inflation.

The results emerging from our simulations are very simple. Again there is almost no dependence on the gauge parameter $g$. The spectral index equals 1.0 for all $\lambda$ smaller than $10^{-4}$ and then goes to a value of $n = 0.983$ and remains constant for $\lambda > 0.01$, see figure 18, compare with reference [20], figure 2. The slight deviation for $g = 1$, from all other $g$ values, is exactly in the same $\lambda$ region that was also slightly off in figure 12.

The nice thing is that the model gives very precise predictions, so it can be falsified. In [20], for example, the spectral index is equal to 1.0 on most of the domain and then suddenly goes down very fast, meaning that one can get any small value of $n$ by fine-tuning the parameters, which is not desirable. Our value of $n = 0.983$ is at a $1.3\sigma$ deviation from
This figure depicts the values of the spectral index for all datapoints. They all lie on the same curve, with a dependence only on $\lambda$, except for a small portion of the ($g = 1$) dataset. So most points really consist of 6 datapoints right on top of each other. In [20] there was a large dependence on the gauge parameter $g$, which now has totally disappeared.

Figure 18.

WMAP measurements and $n = 1.0$ is at a 2.2σ deviation. So the values of $\lambda$ that are consistent with WMAP are approximately $\lambda > 0.001$, although smaller values are not excluded.

The region $\lambda > 10^{-3}$, which gives the best spectral index according to the 3-year WMAP data, does not overlap with the region consistent with the 10% cosmic string contribution bound, which is approximately $\lambda < 0.5 \times 10^{-4}$ or $\lambda < 10^{-3}$ if cosmic strings can be effectively diluted, see section 5.4. Of course, this can still be due to the statistical error in the WMAP-measurement (Bayesian statistics for WMAP do not exclude a spectral index equal to 1). It might also be possible to get a spectral index smaller than 1, by adding fluctuations to the path. Now we took the steepest path down, which is a sensible thing to do because of the large friction term in the equations of motion, but since the path on the inflationary valley is not necessarily long, it could be that small deviations from the steepest path have not damped out and these could give a spectral index different from 1 (since the spectral index has a large dependence on the second derivative of the path). Another possible, but maybe far-fetched solution, will be presented in section 5.5.

5.4. Diluting cosmic strings

In section 5.2 we have seen that a large amount of off-valley inflation is possible for any value of the gauge coupling $g$. Cosmic strings will form at the end of inflation, because of the spontaneous breaking of a $U(1)$ symmetry [9]. In this section we will see how off-valley inflation affects the cosmic string density.

We assume that the cosmic strings are formed at the moment when the $\Phi$ field leaves the inflationary valley. This happens if the process is relatively fast, in which case there will be a string production of approximately one length of string per Hubble volume.
Figure 19. This figure denotes the maximum angle with respect to the $S$ axis of the steepest descent path, for all datapoints. Generally the fields leave the $S$ axis quite abruptly and after that the angle gradually becomes smaller.

Figure 19 shows at which angle the $\Phi$ field will move from the $S$ axis. This angle always goes up very fast at the moment the fields leave the valley and then gradually goes down again. Since the leaving angle is generally quite big, it is reasonable to assume that the strings are produced at the moment the fields leave the valley, but it is possible that the quantum fluctuations are still big enough to pull the field over the top of the Mexican hat for some time [9], which would mean that the strings are effectively produced at some later instant, when the quantum fluctuations become small as compared to the height of the Mexican hat. It would be interesting to do a precise analysis or simulation of this process. In fact, this is also needed in order to investigate whether quantum fluctuations interfere with inflation.

Normally one assumes that one length of cosmic string is produced per Hubble volume at $S_c$. If there is inflation after the formation of strings the energy density stored in these cosmic strings will go down with the second power of the amount of inflation\(^{17}\). So, for example 30 off-valley e-folds, would reduce the energy density of the strings by a factor of $e^{60} \approx 10^{26}$.

This certainly seems enough to get the string contribution below the 10% bound, but this is not quite true. What matters is the string density at the moment of recombination. Between inflation and recombination the strings will interact and the string density will decrease. If the energy lost in this interaction goes into gravitational waves, then it does not contribute significantly to the WMAP anisotropy. As shown in [22] after a relevant cosmic string scale enters the Hubble radius again, it will go into a scaling regime, which makes the density of cosmic strings on a scale smaller than the Hubble radius approximately constant with respect to the Hubble radius. This means that the inter-string distance grows with the Hubble radius. This is why the 10% string contribution bound only depends on $\xi$ [17]: the density at recombination is always the same, but $\xi$

\(^{17}\) Not the third power, because a cosmic string is a one-dimensional object with a constant tension, which gets stretched by the expanding universe.
D-term inflation after spontaneous symmetry breaking
determines the tension of the string and therefore it determines the total energy density
stored in cosmic strings at recombination. The formula for the tension of a $D$-term string
is very simple because it is a ‘critical coupling’ or BPS string:

$$\mu = 2\pi \xi.$$  \hspace{1cm} (39)

Now if the number of off-valley e-folds exceeds 60, there will be no string contribution
for sure, because there simply are no strings. Up to about 5 e-folds less, the string
contribution will still be very small, because the inter-string distance will not have
entered the Hubble horizon at decoupling. For even smaller off-valley e-folds the string
contribution will go to the value as calculated before, due to the scaling. How fast the
string contribution will go to this value depends on how fast the scaling will take place.

5.5. Pushing the top of the Mexican hat down

As shown in section 3.4, for a restricted set of parameters it is possible to lower the top
of the Mexican hat of the fields in the true vacuum state. This could lead to strings with
less tension and since the effect seems to be the largest for the parameters which give the
preferred spectral index of 0.983, it is worth looking at.

Since in the origin ($S = 0, \Phi = 0$) many of the particles are massless, we can
actually do the analysis analytically. Evaluating all the masses, by looking carefully at
tables 3, 4 and 7, we see that only four scalar masses contribute: $M^2(\phi_{(1,2)}) = g^2 \xi$ and
$M^2(\phi_{-(1,2)}) = -g^2 \xi$. The rest of the fields are massless. Inserting this in the Coleman–
Weinberg formula gives

$$\Delta V = \frac{1}{64\pi^2} \left( 2(g^2 \xi)^2 \ln \left( \frac{g^2 \xi}{\Lambda^2_m} \right) + 2(-g^2 \xi)^2 \ln \left( \frac{-g^2 \xi}{\Lambda^2_m} \right) \right) = \frac{g^4 \xi^2}{16\pi^2} \ln \left( \frac{g^2 \xi}{\Lambda^2_m} \right).$$  \hspace{1cm} (40)

We should compare this with the tree-level potential in the origin, see equation (3). This
gives a relative one-loop correction in the origin of

$$\frac{\Delta V}{V} = \frac{g^2}{8\pi^2} \ln \left( \frac{g^2 \xi}{\Lambda^2_m} \right).$$  \hspace{1cm} (41)

For our datapoints we used $\Lambda_m = S_c$. If we insert this into equation (41) we get

$$\frac{\Delta V}{V} = \frac{g^2}{8\pi^2} \ln \left( \frac{\lambda^2}{2} \right).$$  \hspace{1cm} (42)

This will give a negative one-loop contribution for $\lambda < \sqrt{2} M_{Pl}$, so all datapoints will have
a one-loop correction in the origin which counteracts the tree-level potential. Figure 20
shows the relative size of these one-loop corrections.

One can see that the one-loop contribution in the origin is very small indeed for
almost all parameter combinations, but it can go up to about a maximum of 20% for
values of the gauge parameter $g$ very close to one. Of course we do not know what the
two-loop contribution is going to do, so we cannot make a definite judgement to what
extent this effect is important. The question remains open, whether the effect of this can
be big enough to lower the string contribution below the 10% bound for the datapoints
at large $\lambda$. In any case, the best thing to do in order to get an expression for the tension
of the strings, including this one-loop correction, is to make a numerical model, find the
string solution and integrate the energy density to get a corrected tension. Of course it
will come out smaller than in equation (39), but probably not small enough to get below
the 10% string contribution bound.
Figure 20. Relative size of the negative one-loop corrections in the origin $(S = 0, \Phi = 0)$ for all datapoints (a), zoomed in on the relevant region (b). For almost all choices of $g$ we get a negligible one-loop contribution. Only values of $g$ close to 1 can give a significant contribution.

6. Conclusion

In this paper we have studied the $D$-term hybrid inflation model in detail. The key ingredient we have added is a description of the fields after reaching the critical inflaton value and during spontaneous symmetry breaking, when the fields leave the inflationary valley. To analyse this we need a formula for the one-loop quantum corrections to the potential energy density, outside the inflationary valley. In calculating this, problems arise related to the non-equilibrium Higgs mechanism. We presented a good way out of these problems, but there still is room for a thorough theoretical analysis of this phenomenon.
Using our method, the parameter bounds of the $D$-term hybrid inflation model, compatible with WMAP measurements, improve a lot. A nice and simple relation between $\lambda$ and $\xi$ and independent of $g$ is presented that will give the correct CMB density perturbation. All parameter combinations with $\lambda < 4.5 \times 10^{-5}$ give a cosmic string contribution less than 10% and also for values of $\lambda$ approximately smaller than 0.001 there is the possibility of diluting the strings after formation, which could also give a string contribution less than 10%. Moreover, where beforehand only $g < 0.03$ was compatible, now all values of $g$ are possible. Results for the spectral index have no dependence on $g$. It lies between $n = 0.983$ and 1.0. The lower values are attained for approximately $\lambda$ bigger than 0.001. So the regions best compatible with the spectral index and with the cosmic string bound are exclusive.

For $D$-term inflation to be compatible with WMAP, either $n = 1.0$ must prove to be the right value (it is currently at a 2$\sigma$ deviation), or neglected dynamical effects must give a contribution to the spectral index, making it smaller than 1 for $\lambda < 0.001$, or one-loop effects which lower the top of the Mexican hat potential in the true vacuum must lower the string tension below the 10% bound.

In the future, work should be done related to the precise length scale of string formation in this model. The effect of quantum fluctuations on inflation after spontaneous symmetry breaking could be investigated. The scaling behaviour can be examined, in cases where there are a lot of off-valley e-foldings. Also the influence of insertion of the true equations of motion, instead of the steepest descent path, on the spectral index might be interesting. Of course one could try to calculate the tension of the resulting strings, which have a lowered top of the Mexican hat potential. We saw that we could not neglect one-loop corrections, so maybe even two- or higher-loop corrections may prove to be significant. But the first thing to do is to extend the method used to $F$-term hybrid inflation.

So there are still a lot of improvements that can be made, but the model as a whole looks quite promising, which is something that could not be said before.

**Appendix A. Contribution from massless particles**

For some field values, there will be massless particles contributing to the one-loop correction in (5). For example, on the inflationary valley the fields $B_\mu$ and $(S_1 S_2 - S_2 S_1)/|S|$ will be massless. For these massless loops there is an indeterminate expression in (5), but taking a simple limit reveals that these massless particles really do not contribute to the loop correction:

$$\lim_{M \to 0} \left( M^2 \ln(M/\Lambda_m^2) \right) = 0,$$

where $M$ is the mass-squared.

**Appendix B. Contribution from imaginary masses**

We have identified all field-dependent masses of the physical particles in the region ($\Phi_+ = 0, \Phi_-$ and $S$ arbitrary). In plugging them into equation (5) there is still a problem. As can be seen in table 2 the (mass)$^2$ values are not always positive for all
field values. Negative \((\text{mass})^2\) values plugged into the natural logarithm will give an imaginary contribution to the effective potential. This problem is resolved in [19]:

Quantum mechanically the effective potential can be interpreted as the energy density of the quantum state that minimizes this energy density, subject to the condition that the field expectation values (in this case \(\langle \Phi_\pm \rangle, \langle S \rangle \)) are as given. So for every point in field space you can define an effective potential value. For classical potentials that are convex everywhere, these quantum states will be homogeneous, meaning concentrated around a point. But for partially concave classical potentials (which can give a spontaneous symmetry breaking) these quantum states will be mixed states (because, if \(V''\) is negative, the mixed state will certainly have a lower energy than a homogeneous state).

For our purposes we do not want to know the energy density of these mixed states, but rather we want to know the energy density of a homogeneous state centred around the same concave point in field space. Weinberg and Wu show in [19] that the encountered imaginary one-loop corrections to the effective potential correspond to these concave points (for mixed states). The nice thing is that the real part of the one-loop correction can be interpreted as the one-loop correction to the energy density of the homogeneous state\(^{18}\). For our calculations we should therefore only take the real part of the one-loop effective potential.

### Appendix C. Contribution from imaginary mass-squared values

As can be seen from tables 2 and 4 there is a possibility that some bosonic and fermionic \((\text{mass})^2\) values are imaginary. This happens when the value under the square-root sign becomes negative. Therefore we would better hope that this never happens.

For the fermionic contribution it is easy to see that, indeed, \(E^2 - 8\lambda^2 g^2 \Phi^4\) is strictly positive for all field values, since it can be written as a sum of squares:

\[
E^2 - 8\lambda^2 g^2 \Phi^4 = \lambda^4 S^4 + 2\lambda^2 (\lambda^2 + 2g^2) S^2 \Phi^2 + (\lambda^2 - 2g^2)^2 \Phi^4 \geq 0,
\]

where \(S\) is shorthand notation for \(S^1_1 + S^2_2\).

For the scalar contribution this is less trivial, because it is not clear how to write \(B^2 + 4C\) as a sum of squares:

\[
B^2 + 4C = \left(\frac{1}{2} \lambda^2 S^2 - g^2 \xi\right)^2 + \left(\frac{1}{2} (\lambda^2 - 3g^2) \Phi^2 + g^2 \xi\right)^2 + \left(\frac{7}{2} \lambda^4 + \frac{3}{2} \lambda^2 g^2\right) S^2 \Phi^2 - g^4 \xi^2.
\]

However, it is still possible that this function is strictly positive, because the three squares cannot all be zero at the same time (fixing \(S\) and \(\Phi\) can make two squares vanish, but then the third will not vanish). Calculating the minimum of this function in the \((S, \Phi)\) plane for a given \(\lambda, g\) and \(\xi\), this indeed turns out to be strictly positive\(^{19}\):

\[
(B^2 + 4C)_{\text{min}} = \frac{4}{3} g^4 \xi^2 \left(\frac{\lambda^2}{\lambda^2 + g^2}\right) \geq 0.
\]

We conclude that there will never be imaginary mass-squared values in our computation, so we need not worry about this.

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\(^{18}\) The imaginary part of the one-loop corrections can be interpreted as a decay rate.

\(^{19}\) The minimum is attained for: \(\Phi^2 = \frac{1}{2} (g^2 / (\lambda^2 + g^2)) \xi\) and \(S^2 = (g^2 / \lambda^2 - 1/3)(g^2 / (\lambda^2 + g^2)) \xi\).
Appendix D. Higgs mechanism for polar and Cartesian fields

Let us take a closer look at the definition of field-dependent particle masses in the case of an $U(1)$ symmetry. Take the simplest possible global $U(1)$ invariant Lagrangian density:

$$\mathcal{L} = (\partial_\mu \phi)^\dagger \partial^\mu \phi - m^2 |\phi|^2. \quad (D.1)$$

We will look at it from two different coordinate systems at exactly the same point in field space $\phi = 1/\sqrt{2}\Phi$. The Cartesian coordinates

$$\phi = \frac{1}{\sqrt{2}}(\Phi + \phi_1 + i\phi_2) \quad (D.2)$$

give the Lagrangian density

$$\mathcal{L} = \frac{1}{2} \partial_\mu \phi_1 \partial^\mu \phi_1 + \frac{1}{2} \partial_\mu \phi_2 \partial^\mu \phi_2 - \frac{1}{2} m^2 (\phi_1^2 + \phi_2^2 + 2\phi_1 \Phi + \Phi^2). \quad (D.3)$$

Both scalar fields have normal kinetic terms and both have a classical mass $m$, because the tree-level propagator (being the inverse of the quadratic part of the Lagrangian) in momentum space has a pole at $k^2 = m^2$. If we look at the same Lagrangian in polar coordinates

$$\phi = \frac{1}{\sqrt{2}}(\Phi + \rho) e^{i\theta} \quad (D.4)$$

we get

$$\mathcal{L} = \frac{1}{2} \partial_\mu \rho \partial^\mu \rho + \frac{1}{2} (\Phi + \rho) \partial_\mu \theta \partial^\mu \theta - \frac{1}{2} m^2 (\rho^2 + 2\rho \Phi + \Phi^2). \quad (D.5)$$

The $\rho$ field enters in exactly the same way as the $\phi_1$ field before, but we cannot read off the $\theta$ mass directly, because of the non-standard kinetic term, so this looks like a bad choice of coordinates. There is no term proportional to $\theta^2$, because the second derivative of the potential along the circle trajectory corresponding to the $\theta$ field is zero. The real second derivative in the straight $\theta$ direction corresponds to the $\phi_2$ mass and is certainly not zero, see figure D.1. The trace of the (mass)$^2$ matrix is just equal to the Laplacian, which is of course coordinate-independent, as long as the coordinate trajectories are straight lines. This all seems obvious, and people will not interpret the $\theta$ field as a massless field in this case, because of the non-standard kinetic term.

However, if we make the global $U(1)$ symmetry local, the gauge field will enter. Its coupling with the $\theta$ field will cancel all $\theta$ dependence and the masses can be easily evaluated again. Indeed using $B_\mu = A_\mu + \frac{1}{4} \partial_\mu \theta$ the Lagrangian density becomes

$$\mathcal{L} = \frac{1}{2} \partial_\mu \rho \partial^\mu \rho + \frac{1}{2} g^2 (\Phi + \rho)^2 B_\mu B^\mu - \frac{1}{2} m^2 (\Phi + \rho)^2 - \frac{1}{4} F_{\mu \nu} F^{\mu \nu}. \quad (D.6)$$

The $\theta$ mass has disappeared.

If instead we stay in the Cartesian basis, in which the masses were easy to read off before making the global $U(1)$ invariance local, then the second-order term for the `Goldstone' boson does not disappear. We can either take exactly the same definition

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20 The Laplacian is the sum of all second derivatives in perpendicular directions.

21 Maybe it is not right to call this a Goldstone boson, because this terminology is normally used in the global minimum (of, for example, the Mexican hat potential). In this minimum the Goldstone boson will always be massless, even for the global case.
of $B_\mu$ as before, then the Lagrangian density becomes

$$\mathcal{L} = \frac{1}{2} \frac{\partial_\mu \phi_1 (\Phi + \phi_1) + \partial_\mu \phi_2 \phi_1}{(\Phi + \phi_1)^2 + \phi_2^2} - \frac{\Phi}{2} \partial_\mu \phi_2 - \frac{\phi_2}{2} B_\mu B^\mu - \frac{1}{2} m^2 ((\Phi + \phi_1)^2 + \phi_2^2) - \frac{1}{4} F_{\mu\nu} F^{\mu\nu}. \quad (D.7)$$

Or we can use a locally equivalent definition of the Higgs mixing $A_\mu = C_\mu - (1/g)(1/\Phi) \partial_\mu \phi_2$, yielding

$$\mathcal{L} = \frac{1}{2} \partial_\mu \phi_1 \partial_\nu \phi_1 + \frac{1}{2} g^2 \Phi^2 C_\mu C^\mu - \frac{1}{2} m^2 ((\Phi + \phi_1)^2 + \phi_2^2) - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \cdots, \quad (D.8)$$

where the dots stand for third- and higher-order terms in the fields. In both cases the second-order term for the Goldstone boson does not disappear in applying the Higgs mechanism. However we cannot associate a pole in the propagator with this term, because there is no kinetic term for the $\phi_2$ field where, in the global case, one could either define the classical mass to be used in the Coleman–Weinberg equation (6) by the pole in the propagator or by the second derivative in the diagonal field direction. For the local case, including the Higgs mixing, both definitions no longer agree. In fact, the second derivative in the ‘Goldstone’ field direction is the same as in the global case, but the ‘Goldstone’ mass coming from poles in the propagator arising from equation (D.6) has vanished.

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