A second-order characteristic line scheme for solving a juvenile–adult model of amphibians

Keng Deng* and Yi Wang

Department of Mathematics, University of Louisiana at Lafayette, Lafayette, LA 70504-1010, USA

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In this paper, we develop a second-order characteristic line scheme for a nonlinear hierarchical juvenile–adult population model of amphibians. The idea of the scheme is not to follow the characteristics from the initial data, but for each time step to find the origins of the grid nodes at the previous time level. Numerical examples are presented to demonstrate the accuracy of the scheme and its capability to handle solutions with singularity.

Keywords: age-structured juveniles; size-structured adults; characteristic lines; numerical scheme

1. Introduction

In this paper, we consider the following initial–boundary value problem which models the evolution of an amphibian juvenile–adult population:

\[ J_t + J_a + v(a, t, P(a, t))J = 0, \quad (a, t) \in (0, a_{\text{max}}) \times (0, T), \]
\[ A_t + (g(x, t, Q(x, t))A)_x + \mu(x, t, Q(x, t))A = 0, \quad (x, t) \in (x_{\text{min}}, x_{\text{max}}) \times (0, T), \]
\[ J(0, t) = \int_{x_{\text{min}}}^{x_{\text{max}}} \beta(x, t, Q(x, t))A(x, t) \, dx, \quad t \in (0, T), \]
\[ g(x_{\text{min}}, t, Q(x_{\text{min}}, t))A(x_{\text{min}}, t) = J(a_{\text{max}}, t), \quad t \in (0, T), \]
\[ J(a, 0) = J^0(a), \quad a \in [0, a_{\text{max}}], \]
\[ A(x, 0) = A^0(x), \quad x \in [x_{\text{min}}, x_{\text{max}}]. \]

Here \( J(a, t) \) is the density of juveniles of age \( a \) at time \( t \) and \( A(x, t) \) is the density of adults of size \( x \) at time \( t \). \( a_{\text{max}} \) is the maximum age of a juvenile, and \( x_{\text{min}} \) and \( x_{\text{max}} \) are the minimum size and

*Corresponding author. Email: deng@louisiana.edu
Author Email: amiami0911@gmail.com

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the maximum size of an adult, respectively. The functions \(P(a,t)\) and \(Q(x,t)\) are defined by

\[
P(a,t) = \alpha^J \int_0^a w^J(\xi) J(\xi,t) \, d\xi + \int_{a_0}^{a_{\text{max}}} w^J(\xi) J(\xi,t) \, d\xi, \quad 0 \leq \alpha^J \leq 1,
\]

and

\[
Q(x,t) = \alpha^A \int_{x_{\text{min}}}^x w^A(\xi) A(\xi,t) \, d\xi + \int_{x_{\text{min}}}^{x_{\text{max}}} w^A(\xi) A(\xi,t) \, d\xi, \quad 0 \leq \alpha^A \leq 1.
\]

\(P(a,t)\) is referred to as the environment for juveniles, and \(Q(x,t)\) is referred to as the environment for adults [4,8]. The coefficient \(\alpha^J\) is related to the level of hierarchy within juveniles, and \(\alpha^A\) is related to the level of hierarchy within adults. The functions \(w^J(a)\) and \(w^A(x)\) are weight functions for juveniles of age \(a\) and for adults of size \(x\), respectively. For example, when \(\alpha^J = \alpha^A = 1\), \(w^J(a) = w^A(x) = 1\), \(P(a,t) = \int_0^{a_{\text{max}}} J(\xi,t) \, d\xi\) is the total number of juveniles at time \(t\), and \(Q(x,t) = \int_{x_{\text{min}}}^{x_{\text{max}}} A(\xi,t) \, d\xi\) is the total number of adults at time \(t\). The function \(v\) represents the mortality rate of a juvenile of age \(a\) at time \(t\) with juvenile environment \(P\). The functions \(g\), \(\beta\), and \(\mu\) represent the growth rate, the reproduction rate, and the mortality rate of an adult of size \(x\) at time \(t\) with adult environment \(Q\), respectively.

The problem (1) is an extension of those developed in [1,5] wherein only the case \(\alpha^J = \alpha^A = 1\), \(w^J(a) = w^A(x) = 1\) was considered. In [3], the existence–uniqueness of the weak solution of Equation (1) was established via a finite difference approximation, and it was shown that when \(0 \leq \alpha^J, \alpha^A < 1\) the solution can become singular (measure valued) in finite time if \(g_Q > 0\).

Our main objective in this paper is to develop a numerical scheme for solving problem (1). Over the past few years, several numerical schemes have been developed for hierarchically structured population models that are composed of a single partial differential equation. In [4], a first-order upwind finite difference scheme was constructed and convergence of the scheme was proved. In [11], a second-order high-resolution scheme was developed, and stability and convergence of the scheme were shown. Later, in [12] a fifth-order accurate explicit finite difference weighted essentially non-oscillatory scheme was proposed. On the other hand, in [7] two second-order characteristic line methods for a fully nonlinear model were introduced and analysed. The first one, called ‘aggregation grid nodes method’, simply follows the characteristic lines, and at each time step one additional grid point in spatial direction is added; while the second one, called ‘selection grid nodes method’, selects some of the grid nodes at each time step. Recently, in [10] two other characteristic line schemes were developed. The idea is not to follow the characteristics, but at each time step to find the origins of the grid nodes at the previous time level.

For problem (1) in the case \(\alpha^J = \alpha^A = 1\), \(w^J(a) = w^A(x) = 1\), an explicit first-order finite difference scheme was developed in [5], which requires very fine mesh sizes to obtain accurate approximations. However, such a requirement results in time-consuming simulations, and thus the scheme is difficult to use especially when we tried to fit model (1) to field data using a least-squares approach which requires solving the model numerous times to achieve an optimal fit [6]. To overcome this difficulty, very recently in [3], an explicit second-order finite difference method was developed for Equation (1). It is a high-resolution non-oscillatory scheme even in the presence of solution singularities. In this paper, motivated by Marinov and Deng [10], we will develop a second-order characteristic line scheme for solving Equation (1), which has the advantages that it is very practically efficient and non-oscillatory even when the solution undergoes a singularity.

In order to carry out our program, letting \(\omega_1\) be a sufficiently large positive constant, as in [3] we impose the following regularity conditions on the parameters in the model (1):

1. \(v(a,t,P)\) is non-negative and continuously differentiable with respect to \(a\), \(t\), and \(P\). Furthermore,

\[
\sup_{(a,t,P) \in [0,a_{\text{max}}] \times [0,T] \times [0,\infty)} v(a,t,P) \leq \omega_1.
\]
We begin with the representation formulas for the solution juveniles in the model (1) given by

\[ J(H6) \beta(H4) g(H2) \]

rewrite the partial differential equations (1)\(^1\) and (1)\(^2\) as follows:

\[ \text{2. Second-order characteristic line scheme} \]

smooth problems, and its capability of application to solutions with singularity.

Furthermore, the characteristic line for juveniles, and for adults, the characteristic system is given by

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\[ \text{The paper is organized as follows. In Section 2, the second-order characteristic line scheme is constructed to approximate the solution of problem (1). In Section 3, two numerical examples are presented to demonstrate the convergence of the scheme, the order of its accuracy for solving smooth problems, and its capability of application to solutions with singularity.} \]

\[ \text{2. Second-order characteristic line scheme} \]

We begin with the representation formulas for the solution \((J, A)\) of problem (1). To this end, we rewrite the partial differential equations (1)\(^1\) and (1)\(^2\) as follows:

\[ J_t + J_a = -v(a, t, P(a, t)J), \quad (a, t) \in (0, a_{\text{max}}) \times (0, T), \]

\[ A_t + g(x, t, Q(x, t))A_x = -\tilde{\mu}(x, t, Q(x, t))A, \quad (x, t) \in (x_{\text{min}}, x_{\text{max}}) \times (0, T), \]

where

\[ \tilde{\mu}(x, t, Q(x, t)) = \mu(x, t, Q(x, t)) + (g(x, t, Q(x, t)))x. \]

If the vital rates \(v\) for juveniles, and \(g\) and \(\mu\) for adults are smooth, the characteristic system for juveniles in the model (1) is given by

\[ \frac{da}{dt} = 1, \]

\[ \frac{dJ}{dt} = -v(a, t, P(a, t)J), \]

and for adults, the characteristic system is given by

\[ \frac{dx}{dt} = g(x, t, Q(x, t)), \]

\[ \frac{dA}{dt} = -\tilde{\mu}(x, t, Q(x, t))A. \]

In view of Equation (5)\(^1\), the characteristic line for \(J\) that takes the value \(\hat{a}\) at time \(\hat{t}\) is given by

\[ a(t; \hat{a}, \hat{t}) = \hat{a} + (t - \hat{t}). \]
On the other hand, if we denote by \( X(t; \hat{x}, \hat{t}) \) the characteristic line for \( A \) passing through \((\hat{x}, \hat{t})\), then \( X \) is the solution of the following initial value problem:

\[
\frac{d}{dt} X(t; \hat{x}, \hat{t}) = g(X(t; \hat{x}, \hat{t}), t, Q(X(t; \hat{x}, \hat{t}), t)),
\]

\[X(\hat{t}; \hat{x}, \hat{t}) = \hat{x}.\]  

Thus, by Equations (5) and (7), \( J \) can be represented by

\[J(a, t; \hat{a}, \hat{t}) = J(\hat{a}, \hat{t}) \exp\left(-\int_{\hat{t}}^{t} \nu(a(\tau; \hat{a}, \hat{t}), \tau, P(a(\tau; \hat{a}, \hat{t}), \tau)) \, d\tau\right),\]

and by Equations (6) and (8), \( A \) can be represented by

\[A(x, t; \hat{x}, \hat{t}) = A(\hat{x}, \hat{t}) \exp\left(-\int_{\hat{t}}^{t} \bar{\mu}(X(\tau; \hat{x}, \hat{t}), \tau, Q(X(\tau; \hat{x}, \hat{t}), \tau)) \, d\tau\right).\]

Making use of the initial data and coupled boundary conditions in Equation (1), we then construct numerical schemes to approximate Equations (9) and (10). For this purpose, we introduce the following notation throughout the paper. Let \( I, L, \) and \( N \) be positive integers. We divide the intervals \([0, a_{\max}], [x_{\min}, x_{\max}], \) and \([0, T] \) into \( I, L, \) and \( N \) subintervals, respectively. We define the age, spatial, and time mesh lengths by \( s = a_{\max}/I, \) \( h = (x_{\max} - x_{\min})/L, \) and \( k = T/N, \) respectively. Then, the age and size mesh points are given by \( a_{j} = is, i = 0, 1, \ldots, I, x_{j} = x_{\min} + jh, j = 0, 1, \ldots, L, \) and the discrete time levels are \( t_{n} = nk, n = 0, 1, \ldots, N. \)

We now introduce the mesh functions

\[J^{n}_{i} = J(a_{i}, t_{n}), \quad A^{n}_{j} = A(x_{j}, t_{n}),\]

\[w^{j}_{i} = w^{j}(a_{i}), \quad w^{A}_{j} = w^{A}(x_{j}),\]

\[P^{n}_{i} = P(a_{i}, t_{n}), \quad Q^{n}_{j} = Q(x_{j}, t_{n}).\]

At the initial mesh points \( a_{i} (0 \leq i \leq I) \) and \( x_{j} (0 \leq j \leq L), \) \( J^{0}_{j} \) and \( A^{0}_{j} \) are computed by the initial data (1)\(_{5}\) and (1)\(_{6}\), respectively. Then by means of the trapezoidal rule and Equation (2), \( P^{0}_{i} (i = 0, 1, \ldots, I) \) is approximated as

\[P^{0}_{i} = \alpha^{j} \left( \sum_{m=1}^{I-1} w^{j}_{m} J^{0}_{m} + \frac{w^{j}_{0} J^{0}_{0} + w^{j}_{I} J^{0}_{I}}{2} \right) s + O(s^{3}). \]

and by means of the trapezoidal rule and Equation (3), \( Q^{0}_{j} (j = 0, 1, \ldots, L) \) is approximated as

\[Q^{0}_{j} = \alpha^{A} \left( \sum_{m=1}^{L-1} w^{A}_{m} A^{0}_{m} + \frac{w^{A}_{0} A^{0}_{0} + w^{A}_{L} A^{0}_{L}}{2} \right) h + O(h^{3}). \]

Next, we obtain approximations of \( J^{1}_{i} \) and \( A^{1}_{j} (1 \leq i \leq I, 1 \leq j \leq L) \) by the following equations:

\[J^{1}_{i} = J(\eta^{1}_{i}, t_{0}) \exp(-\nu(\eta^{1}_{i}, t_{0}, P(\eta^{1}_{i}, t_{0}))k) + O(k^{2}),\]

\[A^{1}_{j} = A(\xi^{1}_{j}, t_{0}) \exp(-\bar{\mu}(\xi^{1}_{j}, t_{0}, Q(\xi^{1}_{j}, t_{0}))k) + O(k^{2}).\]
We take a simple iteration procedure to solve the above system. Starting with an initial approximation $a_i - k$, $\xi_j = x_{j-1} + h \frac{h - g(x_{j-1}, t_0, Q_{j-1}^0)k}{h + (g(x_j, t_0, Q_j^0) - g(x_{j-1}, t_0, Q_{j-1}^0))k} + O(h^2 + k^2)$, and $J(\eta_j^0, t_0), P(\eta_j^0, t_0), A(\xi_j^0, t_0)$, and $Q(\xi_j^0, t_0)$ are approximated via the cubic spline interpolation.

We then find approximations of $A_0^1$ and $Q_0^1$. In view of the boundary condition (1), $A_0^1$ and $Q_0^1$ satisfy the following system:

$$A_0^1 = \frac{J_1^1}{g(x_0, t_1, Q_0^1)},$$

$$Q_0^1 = \left( \sum_{m=1}^{L-1} w_mA_m^1 + \frac{w_0^A A_0^1 + w_L^A A_L^1}{2} \right) h.$$

We then find approximations of $Q_j^1$ and $J_0^1$. Then, with the newly obtained value of $Q_j^1$, we calculate the new value of $A_0^1$ by Equation (15). The iteration procedure stops when the errors between the new and old values of $A_0^1$ and $Q_0^1$ are within $O(h^3)$. Then, with the newly obtained value of $Q_j^1$, we compute $J_0^1$ exactly by Equation (15). We then obtain approximations of $Q_j^1$ ($1 \leq j \leq L$) by the following equation:

$$Q_j^1 = \alpha^A \left( \sum_{m=1}^{L-1} w_mA_m^1 + \frac{w_0^A A_0^1 + w_L^A A_L^1}{2} \right) h + \left( \sum_{m=j+1}^{L-1} w_mA_m^1 + \frac{w_0^A A_0^1 + w_L^A A_L^1}{2} \right) h + O(h^3).$$

By means of the trapezoidal rule again, we now approximate $J_0^1$ as

$$J_0^1 = \sum_{m=1}^{L-1} \beta(x_m, t_1, Q_m^1)A_m^1 h + \frac{\beta(x_0, t_1, Q_0^1) A_0^1 + \beta(x_L, t_1, Q_L^1) A_L^1}{2} h + O(h^3).$$

Furthermore, we obtain approximations of $P_i^1$ ($0 \leq i \leq I$) by the following equation:

$$P_i^1 = \alpha^I \left( \sum_{m=1}^{I-1} w_m^I J_m^1 + \frac{w_0^I J_0^1 + w_I^I J_I^1}{2} \right) s + \left( \sum_{m=I+1}^{I-1} w_m^I J_m^1 + \frac{w_0^I J_0^1 + w_I^I J_I^1}{2} \right) s + O(s^3).$$

Since our idea is to utilize the characteristic lines, we need the solution at two consecutive time levels. We may take $J_i^n$ as an example and $A_i^j$ can be approximated in a similar manner. Tracking back the characteristic line of $J(a, t)$ from a given grid node $(a_i, t_n)$, we first compute the intersection of the characteristic line $a(t; a_i, t_n)$, which is given by Equation (7), at the previous time level $t_{n-1}$, and then compute its intersection at the time level $t_{n-2}$. Thus, $J(a_i, t_n)$ can be approximated by solving Equation (9).

Consider the general time step $n$ for $2 \leq n \leq N$. Suppose that the numerical approximations of $J$ and $A$ at the previous time levels $t_{n-2}$ and $t_{n-1}$ are computed. We then approximate $J_i^n$. To this end, we denote by $(\eta_i^{n-1}, t_{n-1})$ the intersection point of the characteristic line through the mesh point $(a_i, t_n)$ at the time level $t_{n-1}$. From the characteristic equation (7), we have (Figure 1)

$$\eta_i^{n-1} = a_i - k, \quad 1 \leq i \leq I.$$

Similarly, the intersection point $(\tilde{\eta}_i^{n-2}, t_{n-2})$ at the time level $t = t_{n-2}$ is computed exactly by

$$\tilde{\eta}_i^{n-2} = a_i - 2k, \quad 1 \leq i \leq I.$$
Proposition 2.1 Suppose that \( k/s < 0.5 \), then \( a_{i-1} < \eta_i^{n-1} < a_i \) and \( a_{i-1} < \bar{\eta}_i^{n-2} < a_i \) for \( 2 \leq n \leq N, 1 \leq i \leq I \).

Proof Since \( k < 0.5s \), we find
\[
a_{i-1} < a_i - 0.5s < \eta_i^{n-1} = a_i - k < a_i,
\]
\[
a_{i-1} = a_i - s < \bar{\eta}_i^{n-2} = a_i - 2k < a_i.
\]

We then interpolate the values of \( J(\bar{\eta}_i^{n-2}, t_{n-2}), J(\eta_i^{n-1}, t_{n-1}) \), and \( P(\eta_i^{n-1}, t_{n-1}), i = 1, 2, \ldots, I \), by means of the cubic spline interpolation.

By virtue of Equation (9), an exact formula for \( J(a_i, t_n) \) is given by
\[
J(a_i, t_n) = J(\bar{\eta}_i^{n-2}, t_{n-2}) \exp \left( - \int_{t_{n-2}}^{t_n} v(a(\tau; a_i, t_n), \tau, P(a(\tau; a_i, t_n), \tau)) \, d\tau \right). \tag{21}
\]

Thus, the midpoint rule can be used to discretize Equation (21) to approximate \( J^n_i \):
\[
J^n_i = J(\bar{\eta}_i^{n-2}, t_{n-2}) \exp(-2v(\eta_i^{n-1}, t_{n-1}, P(\eta_i^{n-1}, t_{n-1})))k + O(k^3) \tag{22}
\]
for \( 1 \leq i \leq I \).

Next, we consider the approximations of \( A(x_j, t_n) \) (\( 1 \leq j \leq L \)). Letting \( (\xi_j^{n-1}, t_{n-1}) \) be the point on the characteristic line passing through \( (x_j, t_n) \), we solve backwards Equation (8) to obtain the value of \( \xi_j^{n-1} \). Via the Taylor expansion, \( \xi_j^{n-1} \) is approximated implicitly as
\[
\xi_j^{n-1} = x_j + g(x_j^{n-1}, t_{n-1}, Q(x_j^{n-1}, t_{n-1})))k + O(k^2), \tag{23}
\]
and then the false position method is sufficient for us to find an explicit solution of the above equation as (Figure 2)
\[
\xi_j^{n-1} = x_j + h \frac{h - g(x_j-1, t_{n-1}, Q_j^{n-1})k}{h + (g(x_j, t_{n-1}, Q_j^{n-1}) - g(x_j-1, t_{n-1}, Q_j^{n-1})))k} + O(h^2 + k^2) \tag{24}
\]
for $1 \leq j \leq L - 1$. At the end mesh point, by hypothesis (H2), $g(x_{\text{max}}, t, Q(x_{\text{max}}, t)) = 0$. So the characteristic line at the point $(x_L, t)$ is $x = x_{\text{max}}$ for all time levels. Hence,

$$\xi_{L-1}^n = x_{\text{max}}. \quad (25)$$

We continue tracking the characteristic line $X(t; x_j, t_n)$ backwards to the time step $t_{n-2}$. We then find an intersection point $(\xi_{j-2}^{n-2}, t_{n-2})$ at this time level. Using the Taylor expansion again, $\xi_{j-2}^{n-2}$ is evaluated by

$$\xi_{j-2}^{n-2} = \xi_{j-1}^{n-1} - 2g(\xi_{j-1}^{n-1}, t_{n-1}, Q(\xi_{j-1}^{n-1}, t_{n-1}))k + O(k^2), \quad (26)$$

which can be combined with Equation (23) to derive the following approximation of $\xi_{j-2}^{n-2}$:

$$\xi_{j-2}^{n-2} = x_j - 2g(\xi_{j-1}^{n-1}, t_{n-1}, Q(\xi_{j-1}^{n-1}, t_{n-1}))k + O(k^2). \quad (27)$$

By solving Equations (23) and (24), $g(\xi_{j-1}^{n-1}, t_{n-1}, Q(\xi_{j-1}^{n-1}, t_{n-1}))k$ is expressed by

$$g(\xi_{j-1}^{n-1}, t_{n-1}, Q(\xi_{j-1}^{n-1}, t_{n-1}))k = \frac{h - g(x_{j-1}, t_{n-1}, Q_{j-1}^{n-1})k}{h + (g(x_j, t_{n-1}, Q_{j-1}^{n-1}) - g(x_{j-1}, t_{n-1}, Q_{j-1}^{n-1}))k} + O(h^2 + k^2). \quad (28)$$

Hence, Equations (27) and (28) enable us to write an explicit approximation for $\xi_{j-2}^{n-2}$ ($1 \leq j \leq L - 1$) as follows:

$$\xi_{j-2}^{n-2} = x_{j-1} - h \left(1 - 2\frac{h - g(x_{j-1}, t_{n-1}, Q_{j-1}^{n-1})k}{h + (g(x_j, t_{n-1}, Q_{j-1}^{n-1}) - g(x_{j-1}, t_{n-1}, Q_{j-1}^{n-1}))k} \right) + O(h^2 + k^2). \quad (29)$$

Because the characteristic line at the point $x_L$ is $x = x_{\text{max}}$ at all time levels under the condition $g(x_{\text{max}}, Q(x_{\text{max}}, t)) = 0$, we have

$$\xi_{L-2}^{n-2} = x_{\text{max}}. \quad (30)$$

**Proposition 2.2** Suppose that for each time step $t = t_n$,

$$\frac{k}{h} \max_{0 \leq j \leq L-1} g(x_{j-1}, t_n, Q_{j-1}^n) < 0.5,$$

then $x_{j-1} < \xi_{j-1}^{n-1} < x_j$ and $x_{j-1} < \xi_{j-2}^{n-2} < x_j$ for $1 \leq j \leq L - 1$.

**Proof** We divide Equation (23) by $h$, which implies

$$0 < \frac{x_j - \xi_{j-1}^{n-1}}{h} = g(\xi_{j-1}^{n-1}, t_{n-1}, Q(\xi_{j-1}^{n-1}, t_{n-1})) \frac{k}{h} < 0.5.$$

So $0 < x_j - \xi_{j-1}^{n-1} < 0.5h$, and thus $x_{j-1} < \xi_{j-1}^{n-1} < x_j$.

Similarly, when we divide Equation (27) by $h$, it follows that

$$0 < \frac{x_j - \xi_{j-2}^{n-2}}{h} = 2g(\xi_{j-1}^{n-1}, t_{n-1}, Q(\xi_{j-1}^{n-1}, t_{n-1})) \frac{k}{h} < 1.$$

So $0 < x_j - \xi_{j-2}^{n-2} < h$, and hence $x_{j-1} < \xi_{j-2}^{n-2} < x_j$. \qed
Using the cubic spline interpolation, we then obtain the approximations of $A(\bar{\xi}^{n-2}_j, t_{n-2})$, $A(\bar{\xi}^{n-1}_j, t_{n-1})$, and $Q(\bar{\xi}^{n-1}_j, t_{n-1})$ at the points $(\bar{\xi}^{n-2}_j, t_{n-2})$ and $(\bar{\xi}^{n-1}_j, t_{n-1})$ for $j = 1, 2, \ldots, L - 1$.

By Equation (9), $A(x_j, t_n)$ can be expressed as

$$A(x_j, t_n) = A(\bar{\xi}^{n-2}_j, t_{n-2}) \exp \left( - \int_{t_{n-2}}^{t_n} \tilde{\mu}(X(\tau; x_j, t_n), \tau) \frac{Q(X(\tau; x_j, t_n), \tau))}{d\tau} \right). \quad (31)$$

Then by means of the midpoint rule and Equation (31), $A^n_j (1 \leq j \leq L)$ is approximated numerically as follows:

$$A^n_j = A(\bar{\xi}^{n-2}_j, t_{n-2}) \exp(-2\tilde{\mu}(\bar{\xi}^{n-1}_j, t_{n-1}, Q(\xi^{n-1}_j, t_{n-1}))k) + O(k^2). \quad (32)$$

As for $A^0_0$ and $Q^0_0$, they form a system as

$$
\begin{align*}
A^n_0 &= \frac{J^n_0}{g(x_0, t_n, Q^0_0)}, \\
Q^n_0 &= \left( \sum_{m=1}^{L-1} w^n_m A^n_m + \frac{w^n_0 A^n_0 + w^n_L A^n_L}{2} \right) h.
\end{align*}
\quad (33)
$$

As before, this system can be solved by iteration. We then obtain the approximations of $Q^n_j (1 \leq j \leq L)$ by the following equation:

$$Q^n_j = A^n_j \left( \sum_{m=1}^{L-1} w^n_m A^n_m + \frac{w^n_0 A^n_0 + w^n_L A^n_L}{2} \right) h + \left( \sum_{m=1}^{L-1} w^n_m A^n_m + \frac{w^n_0 A^n_0 + w^n_L A^n_L}{2} \right) h + O(h^2). \quad (34)$$

Taking the boundary condition (1)$_4$ into consideration, we now approximate $J^n_0$ as

$$J^n_0 = \sum_{m=1}^{L-1} \beta(x_m, t_n, Q^n_m) A^n_m h + \frac{\beta(x_0, t_n, Q^n_0) A^n_0 + \beta(x_L, t_n, Q^n_L) A^n_L}{2} h + O(h^2). \quad (35)$$

Finally, we obtain the approximations of $P^n_i (0 \leq i \leq J)$ by the following equation:

$$P^n_i = \alpha^j \left( \sum_{m=1}^{L-1} w^n_m J^n_m + \frac{w^n_0 J^n_0 + w^n_L J^n_L}{2} \right) s + \left( \sum_{m=1}^{L-1} w^n_m J^n_m + \frac{w^n_0 J^n_0 + w^n_L J^n_L}{2} \right) s + O(s^2). \quad (36)$$

### 3. Numerical results

In this section, we carry out numerical experiments with the scheme developed in Section 2. Our interest is twofold. First, we want to confirm that the convergence rate of the scheme is second order in practice. Second, we want to demonstrate that the scheme also works well when the solution undergoes singularity. For this purpose, we consider two problems, one has an exact smooth solution and the other has a singular solution.
3.1. Approximation of a smooth solution

We use the same example of [3]. Thus, we let \( T = 0.5, a_{\text{max}} = 1, x_{\text{min}} = 0, x_{\text{max}} = 5, \alpha^J = 1, \alpha^A = 1, w^J(a) = 1, w^A(x) = 1 \). And we choose the following set of parameters:

\[
\begin{align*}
\nu(a, t, P) &= 1, \\
g(x, t, Q) &= 2 - 2 \exp(x - 5), \\
\beta(x, t, Q) &= 2 \exp(2), \\
\mu(x, t, Q) &= 1, \\
J^0(a) &= 2 \exp(2)(\exp(-2a) - \exp(-2a - 5)), \\
A^0(x) &= \exp(-x).
\end{align*}
\] (37)

It is easy to verify that with the above set of parameters, the boundary condition and the initial data are compatible at the origin, and the functions

\[
\begin{align*}
J(a, t) &= 2 \exp(2)(\exp(t - 2a) - \exp(t - 2a - 5)), \\
A(x, t) &= \exp(t - x)
\end{align*}
\] (38)

are the exact smooth solution components of the problem (1).

We use the second-order characteristic line scheme presented in Section 2 to compute the solution of Equation (1). The three-dimensional dynamics of the density of juveniles \( J(a, t) \) and adults \( A(x, t) \) are given, respectively, in Figures 3 and 4 with \( I = 640, L = 1280, \) and \( N = 12,800 \). The pointwise distribution of the numerical error for \( J(a, t) \) and \( A(x, t) \) at time \( t = 0.5 \) is given in Figure 5 with four different sets of \( I, L, \) and \( N \).

The \( L^1 \) norm of the error is calculated as follows:

\[
\|\text{err}\|_1 = \sum_{n=1}^{N} \sum_{i=1}^{I} |V^n_i - J(a_i, t_n)| sk + \sum_{n=1}^{N} \sum_{j=1}^{L} |U^n_j - A(x_j, t_n)| hk,
\]

and the rate of convergence is computed as

\[
\text{rate} = \log_2 \left| \frac{\text{err}_{2h}}{\text{err}_h} \right|
\]
Figure 4. The three-dimensional dynamics of the density of adults $A(x,t)$ for Example 1.

Figure 5. The differences between the numerical and exact values of $J$ and $A$ at $t = 0.5$ with four different sets of values of $I$, $L$, and $N$.

Table 1. $L^1$ errors and the order of accuracy for the characteristic line scheme.

| I   | L   | N   | $L^1$ error  | Order |
|-----|-----|-----|--------------|-------|
| 5   | 10  | 100 | 0.0714       |       |
| 10  | 20  | 200 | 0.0132       | 2.4379|
| 20  | 40  | 400 | 0.0028       | 2.2226|
| 40  | 80  | 800 | $6.5437 \times 10^{-4}$ | 2.1090|
| 80  | 160 | 1600| $1.5778 \times 10^{-4}$ | 2.0522|
| 160 | 320 | 3200| $3.8760 \times 10^{-5}$ | 2.0253|
| 320 | 640 | 6400| $9.6070 \times 10^{-6}$ | 2.0124|
| 640 | 1280| 12,800| $2.3915 \times 10^{-6}$ | 2.0062|

The $L^1$ norms of the differences between the numerical and exact values of the solution to the model (1) with eight different steps $s$, $h$, and $k$ are given in Table 1. It clearly indicates that our characteristic line scheme has a second-order accuracy for this smooth solution of the model.
3.2. **Approximation of a singular solution**

It is worth to point out that the assumption $g_Q(x, t, Q(x, t)) \leq 0$ for $0 \leq \alpha < 1$ in (H2) is crucial to ensure the existence of a global bounded variation weak solution of problem (1), because in [2,9] it was shown that without such an assumption the solution can become singular (measure valued) in finite time. However, the next example demonstrates that even if a singularity occurs, our scheme is capable of capturing the evolving Dirac measure $A(x, t)$ and the discontinuity in $Q(x, t)$.

Let $T = 0.2$, $a_{\text{max}} = 1$, $x_{\text{min}} = 0$, $x_{\text{max}} = 5$, $\alpha^J = 0.5$, $\alpha^A = 0.5$, $w^J(a) = 1$, $w^A(x) = 1$. And we select the following set of parameters:

\[
\begin{align*}
\nu(a, t, P) &= \frac{P}{2 + P}, \\
g(x, t, Q) &= 87.68(5 - x) \frac{Q}{16 + Q^2}, \\
\beta(x, t, Q) &= 12x^2 \frac{1}{1 + Q^2},
\end{align*}
\]
Figure 8. The two-dimensional distributions of $A(x, t)$ at $t = 0.2$ for different sets of values of $I$, $L$, and $N$.

Figure 9. The two-dimensional distributions of $Q(x, 0)$ and $Q(x, 0.2)$.

\[
\mu(x, t, Q) = \exp(x - 0.5)^2 \frac{Q}{1 + Q},
\]

\[
J^0(a) = \exp(2a),
\]

\[
A^0(x) = \exp(-8(x - 0.3)^2).
\]
Again it can be verified that with the above set of parameters, the boundary condition and the initial data are compatible at the origin. For the growth rate $g$ in Equation (39), we have

$$g_Q = 175.36(5 - x) \frac{8 - Q^3}{(16 + Q^3)^2}.$$ 

Thus, $g_Q < 0$ for $Q \in (2, \infty)$ and $g_Q \geq 0$ for $Q \in [0, 2]$.

In Figures 6 and 7, the three-dimensional dynamics of the density of juveniles and the density of adults are plotted, respectively, using $I = 640$, $L = 3200$, and $N = 12,800$. Figure 8 provides the density of adults at time $t = 0.2$ for several values of $I$, $L$, and $N$. We observe that a Dirac delta measure is forming at $x \approx 2.65$ and $t = 0.2$. Consequently, the Dirac delta measure $A$ results in a discontinuity in $Q$. In Figure 9, $Q(x, 0.2)$ is discontinuous at $x \approx 2.65$, while $Q(x, 0)$ is continuous on $[0, 5]$. Furthermore, one can note that $\max_{[0,5] \times [0,0.2]} Q(x,t) < 2$.

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