DISTORTION REVERSAL IN APERIODIC TILINGS

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ABSTRACT. It is proved that homeomorphic images of certain two-dimensional aperiodic tilings, such as Ammann-A2 tilings, are recognizable, in both mathematical and practical senses. One implication of the results is that it is possible to search for distorted aperiodic structures in nature, where they may be hiding in plain sight.

1. INTRODUCTION

Herein we prove that homeomorphic images of certain two-dimensional self-similar aperiodic tilings are recognizable, in both mathematical and practical senses. In the practical sense our results say it is possible to search for and recognize distorted aperiodic structures, for example in natural and physical settings. We recall that the mathematical discovery of aperiodic tilings by Penrose in 1972 and Ammann in 1975 preceded the discovery by Shechtman in 1982 of quasicrystals, and the subsequent examples of quasicrystals in meteorites in 2009 [13].

In this paper we focus on Ammann-A2 tilings [2], but the ideas are more generally applicable. Our results illustrate that aperiodic tilings can be retrieved from their images distorted by unknown homeomorphisms. For example, we show how distorted Ammann-A2 tilings, such as the ones in Figure 1, are recognizable both practically and mathematically. In Figure 2, an algorithm (without knowledge of the distorting homeomorphism) is applied to the distorted tiling to successively retrieve a patch of the original Ammann-A2 tiling.

The situation is a bit strange: the unknown distortion may be such as to change the Hausdorff dimension of the boundaries of the tiles in a very rough way, or to transform an Ammann-A2 tiling to a tiling by triangles. Nevertheless, the key properties of the tiling may still be discernable.

There is a substantial literature on the occurrence of two-dimensional tilings in physics [14]. Many of these are distortions of tilings by hexagons. Many naturally occurring tilings have an average of six faces per tile, and typical tiles are hexagons. Ammann-A2 tiles are hexagonal and comprise among the simplest of self-similar tilings. Despite this, they do not seem to show up in nature. Are they hiding in plain sight? We provide some tools which may be applied to this question.

2. ORGANIZATION AND MAIN RESULTS

Let \( P \) be a finite aperiodic prototile set. This means that there exist tilings of the plane by copies of the tiles in \( P \), but every such tiling is non-periodic, i.e., has

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no translational symmetry. Such tilings have been objects of fascination, both in research and recreational mathematics, since their introduction in the 1960’s. The most well-known such prototile set consists of the two Penrose tiles. To develop the ideas in this paper we concentrate, because of their simplicity, on tilings based on the set of two Ammann-A2 tiles. There are uncountably many corresponding Ammann-A2 tilings, obtained either by matching rules or by substitution rules. Basic properties of such Ammann-A2 tilings are provided in Section 3.

Many non-periodic tilings have a hierarchical structure, meaning that if $T$ is such a tiling, then there exist a sequence of tilings $T = T_0, T_1, T_2, \ldots$, such that, for all $n \geq 1$, each tiling in $T_n$ is the non-overlapping union of tiles in $T_{n-1}$. For the Ammann-A2 tilings, the tiling $T_n$, appropriately scaled, is again an Ammann-A2 tiling. Specifically, there is a constant $s$ such that $T_n := s^n T_n$ is an Ammann-A2 tiling for all $n$. There is a well-defined procedure, which we call amalgamation, taking $T_n$ to $T_{n+1}$, and denoted $T_{n+1} = a(T_n)$. Amalgamation and hierarchy for the Ammann-A2 tilings is explored in Section 4.

Here is the question that is the subject of this paper. Suppose that we are given a distorted version $T'$ of an Ammann-A2 tiling $T$. Specifically, there is an unknown (not revealed to us) homeomorphism $h$ of the plane such that $T' = h(T)$. Is it possible to retrieve the undistorted Ammann-A2 tiling $T$ from its distorted image $T'$ without knowledge of $h$? As long as the distance between any point $x$ in the plane and its image $h(x)$ is bounded, we answer this question in the affirmative.
Suppose that amalgamation can be carried over to the distorted tiling $T'$. Then Section 5 contains the following somewhat surprising result: if $T'_n = a^n(T')$, $n \geq 0$, are the tilings obtained by successive application of the amalgamation operator $a$ applied to the distorted image $T'$, then the sequence $\{T_n\}$ of tilings converges, in the sense of Theorem 1 in Section 5 and Remark 1, to the original tiling $T$. This is illustrated in Figure 2.

The above distortion reversal result is predicated on being able to amalgamate tiles of the distorted tiling $T'$, a tiling in which metric and geometric properties of the original tiling are lost. Only the combinatorial properties of $T$ are carried over to $T'$. In Section 6 we obtain, completely combinatorially, the needed amalgamation operator $a$ applied to $T'$. This is done in Algorithm A in Section 6; see Theorem 2. Of course, the plane and tilings of it, are unbounded. We would like to perform the distortion reversal efficiently on a large patch of the tiling $T'$, say on a patch that contains the disk $D_R$ of radius $R$ centered at the origin. Algorithm B, a variation of Algorithm A, accomplishes this. Theorem 3 in Section 6 states that each application of the combinatorial amalgamation applied to such a patch can be performed in running time $O(R^2)$.

An alternative method for reversing the distortion in an Ammann-A2 tiling is provided in Section 7. It uses the combinatorial amalgamation operator applied to the distorted tiling $T''$ to generate an infinite binary string $c$, called the code of the tiling. The code is the input for an iterated function system based method to retrieve the original tiling $T$. This method, encapsulated in Theorem 4 of Section 7, has the advantage that there is no restriction on the homeomorphism $h$. It has the disadvantage that there is no sequence of increasingly accurate images converging.
to the original tiling \( T \); it simply produces \( T \) or, in finite time, an arbitrarily large patch of \( T \).

3. AMMANN-A2 TILINGS

In this paper a tile is defined as a set in the plane homeomorphic to a closed disk. A tiling of the plane is a set of non-overlapping tiles whose union is \( \mathbb{R}^2 \). By non-overlapping is meant that the intersection of any two distinct tiles has empty interior. For all tilings in this paper, the intersection of any two distinct tiles is connected (possibly empty). A patch of a tiling is a finite number of tiles whose union is a topological disk.

The classical Ammann-A2 hexagon \( G \), sometimes referred as the golden bee, is depicted in Figure 3. It is the only polygon, other than any right triangle and any parallelogram with side lengths in the ratio \( \sqrt{2} : 1 \), that is the non-overlapping union of two smaller similar copies of itself [12]. These two smaller copies are isometric to \( sG \) and \( s^2G \), where \( s = 1/\tau \) and \( \tau \) is the square root of the golden ratio, i.e., \( \tau \) is the positive real root of \( x^4 - x^2 - 1 = 0 \).

![Figure 3. Golden bee G](image)

An Ammann-A2 tiling is a tiling of the plane by non-overlapping isometric copies of \( sG \) and \( s^2G \), which we will refer to as the large and small tiles, respectively. The tiling must obey matching rules dictated by the elliptical markings on the upper tiles of the left panel in Figure 4. Other alternative but equivalent sets of markings are also possible, for example the markings shown in the right panel of Figure 4. Although the decorations on small and large tiles at the top of the left panel and the decorations on the small and large tiles of the right panel differ, matching according to either set of decorations define the Ammann-A2 tilings, as observed in [7, p. 551]; see also [1, 2, 4, 10].

Hereafter we will call an Ammann-A2 tiling simply an Ammann tiling. A portion of an Amman tiling is shown in Figure 5. Two tilings \( T \) and \( T' \) are said to be congruent if there is an isometry \( U \) such that \( T' = U(T) \). Much has been written on Ammann tilings both in mathematics journals, for example [4, 7] and in recreational sources, notably [8, 11]. There are uncountable many Ammann tilings, each being non-periodic. Every Ammann tiling is repetitive and every pair of Ammann tilings are locally isomorphic. A tiling \( T \) is repetitive, also called quasiperiodic, if, for every finite patch \( P \) of \( T \), there is a real number \( R \) such that every ball of radius \( R \) contains a patch congruent to \( P \). Two tilings are locally isomorphic if any patch in either tiling also appears in the other tiling.
The Ammann tilings may be defined, as an alternative to matching rules, by substitution rules, see [4, 6]. Another method of construction appears in [3] and is used in Section 7 of this paper.

Let $T$ be an Ammann tiling. For each small tile $s$ in $T$, there is a partner, defined as the unique large tile $b$ in $T$ such that the union of $s$ and $b$, called the amalgamation of $s$ and $b$, is a hexagon congruent to $G$. Any two distinct small tiles
have distinct partners. The tiling obtained from $T$ by amalgamating each small tile with its partner is denoted $A(T)$. The scaled tiling $a(T) := sA(T)$ is an Ammann tiling called the amalgamation of $T$.

**Proposition 4.1.** If $T$ is an Ammann tiling, then so is $a(T)$.

**Proof.** The upper left and right panels of Figure 4 show two different decorations on the small and large tiles, call them the $D1$ and the $D2$ decorations, respectively.

Each small tile in $a(T)$ (left tile in the right panel of Figure 4) is a (scaled by $s$) large tile in $T$ (right tile in the left panel); and each large tile in $a(T)$ (right tile in the right panel) is the scaled amalgamation of a small and large tile in $T$ (bottom of the left panel).

As previously noted, the matching rules dictated by the $D1$ and $D2$ decorations are equivalent in that they both define the Ammann tilings. Therefore if $T$ is an Ammann tiling, then so is $a(T)$. □

Related comments on matching rules for the Ammann tilings are given in [1, 2, 4]. In particular see [1] for a discussion of replacement of markings and the notion of ghost markings. Equivalence of different sets of markings is discussed in [10].

The sequence of tilings $H(T) := \{T, A(T), A^2(T), \ldots\}$, consisting of Ammann tilings at larger and larger scales, will be referred to as the hierarchy of $T$. In general, if $t$ is any tile at any level of the hierarchy, then $t$ is the non-overlapping union of tiles in $T$. Denote this tiling of $t$ by $T(t)$. The tiling $T(t)$ is sometimes referred to as a supertile in the tilings literature; see for example [5, 7, 9]. Any large tile $t$ in $A^n(T)$ is congruent to $\tau^{n-1}(G)$. For such a large tile $T$ in $A^n(T)$, denote $T(t)$ by $T_n$ which, by Proposition 4.2 below, is well-defined, independent of $t$. The tiling $T_6$ appear in Figure 6.

**Proposition 4.2.** If $p$ and $q$ are congruent tiles at any level of the hierarchy, then $T(p)$ and $T(q)$ are congruent.
Proof. It is true at level 1; $A(T)$ is the non-overlapping union of a small and a large Ammann tile in a unique way. Proceeding by induction on the level, assume that the statement of the proposition is true at level $n - 1$, and consider congruent tiles $p$ and $q$ of $A^n(T)$. Then either (1) $p$ and $q$ are small tiles of $A^n(T)$, in which case they are large tiles of $A^{n-1}(T)$, or (2) $p$ and $q$ are large tiles of $A^n(T)$, in which case they are the amalgamation of a large and a small tile of $A^{n-1}(T)$. In case (1) we have that $T(p)$ and $T(q)$ are congruent by the induction hypothesis. In case (2) the decomposition of $p$ into a non-overlapping union of a large and small tile of $A^{n-1}(T)$ is the same as for $q$. Moreover, by the induction hypothesis, the decompositions of large and small tiles of $A^{n-1}(T)$ is unique. Therefore we again have that $T(p)$ and $T(q)$ are congruent.

Lemma 1 below is needed to prove a main result in Section 6. We first introduce a useful combinatorial notion. Define a planar map $M$ to be a 2-cell embedding of a locally finite simple graph $\Gamma$ in the plane. By locally finite is meant that the degree of each vertex is finite, and by simple is meant no loops or multiple edges. The faces of $M$ are the closures of the connected components of $\mathbb{R}^2 \setminus \Gamma$. By 2-cell embedding is meant that each face of $M$ is homeomorphic to a closed disk. Two planar maps $M$ and $M'$ with underlying graphs $\Gamma$ and $\Gamma'$, respectively, are isomorphic if there is a graph isomorphism taking $\Gamma$ to $\Gamma'$ that preserves faces.

A tiling $T$ can be considered a planar map. A vertex of $T$ is the intersection point of three or more distinct tiles of $T$, assuming that the intersection is not empty, and an edge of $T$ is the intersection, if not empty or a single point, of two distinct tiles of $T$. The faces of $T$ are the tiles of $T$.

From an Ammann tiling $T$, a tiling $\hat{T}$ is constructed as follows. Note that the degree (number of incident edges) of any vertex of an Ammann tiling $T$ is either 3 or 4. Color red each edge of $T$ (and its two incident vertices) that joins two vertices of degree 4. For any red vertex lying on only one red edge, remove the color red from that vertex and remove the incident edge. Let $\hat{T}$ be the tiling induced by the red edges and vertices, i.e., by removing all edges and vertices not colored red. For any face $f$ of $\hat{T}$, it’s red boundary $\partial f$ together with all enclosed tiles of $T$ is a finite tiling which is denoted $T(f)$.

Lemma 1. With notation as above, if $T$ is an Ammann tiling, then $\hat{T}$ has the following properties.

1. If $f$ is a tile of $\hat{T}$, then $T(f) = T_4$ or $T(f) = T_5$.
2. $\hat{T} = A^5(T)$.

Proof. In $T_n$, any edge adjacent to exactly one tile is called a boundary edge. A boundary vertex is a vertex that lies on a boundary edge. In $T_n$, color edges and vertices red by the same rule as used to color $T$, but, in addition, color red any edge (and incident vertex) joining a vertex of degree 4 to a boundary vertex and color all boundary edges (and incident vertices) red. As previously, for any red vertex lying on only one red edge, remove the color red from that vertex and remove the incident edge. Denote by $\hat{T}_n$ the tiling induced by only the red edges and vertices. Figure 6 shows the tiling $T_6$ with red edges designated and the corresponding tiling $\hat{T}_6$. Note that, for any red edge $e$ in $T_6$ not on the boundary, one incident tile is the reflection of the other incident tile in the line that extends $e$. Also note that $\hat{T}_6$ has two faces, say $f$ and $f'$ such that $T(f) = T_4$ and $T(f') = T_5$. We claim that the
same is true for all $T_n$, $n \geq 6$, i.e., (1) for any red edge $e$ of $T_n$ not on the boundary, one incident tile is the reflection of the other in the line that extends $e$ (and this implies that any red vertex of $T_n$, not on the boundary, has degree 4), and (2) for any tile $f$ of $T_n$ it is the case that $T(f) = T_4$ or $T(f) = T_5$.

The claim will be proved by induction on $n$. Assume that assertions (1) and (2) above are true for $T_n$ and $T_n$. Obtain $T_{n+1}$ by first subdividing each large tile in $T_n$ into one large and one small tile, then enlarging the resulting tiling by a factor of $\tau$. Now color the edges of $T_{n+1}$ as previously prescribed. If $e$ is a red edge in $T_n$ and $f$ and $f'$ are the adjacent tiles, then either they are both small tiles in $T_n$, and hence not subdivided, or they are both large tiles in $T_n$, and hence subdivided in exactly the same way, so that the reflection property remains true in $T_{n+1}$. Therefore, each red edge in $T_n$ induces either one or two red edges in $T_{n+1}$. Let $E$ be this set of induced red edges and let $T_{n+1}'$ be $T_{n+1}$, but with only $E$ and the boundary edges (and incident vertices) colored red. Since each tile of $T_n'$ is either $T_4$ or $T_5$, each tile in $T_{n+1}'$ is either $T_5$ or $T_6$. But when additional red faces are added to $T_{n+1}'$ to obtain $T_{n+1}$, no additional red edges are added to each $T_5$ because $T_5$ has no degree 4 internal (not on the boundary) vertices, and the additional red edges added to each $T_6$, as shown in Figure 6, divides each $T_6$ into one $T_4$ and one $T_5$. Our claim has now been proved.

To extend the result from $T_n'$ to $\hat{T}$, let $f$ be any tile of $\hat{T}$. If $f$ lies in the interior (no edge of $f$ on the boundary) of a tile in $A^n(T)$ for some $n$, then by the paragraphs above, $T(f)$ is either $T_4$ or $T_5$. So assume that there is an edge $e$ of $f$ that lies the on the boundary of some tile $t_n$ of $A^n(T)$ for all $n$ sufficiently large, say $n \geq n_0$. Denote by $T_n$ the subset of the tiling $T$ that lies in $t_n$. Then $T_n \cup T_{n+1} \subseteq T_{n+2} \subseteq \cdots$. The nested union $\bigcup_{n \geq n_0} T_n$ is an Ammann tiling of a proper subset of the plane. But it is known [4] that this occurs only if it is a tiling of a half-space bounded by the line $L$ that extends $e$ or a quadrant of the plane bounded by two perpendicular rays $L_1$ and $L_2$. Moreover, if $T$ is an Ammann tiling of the half plane, then the only extension to an Ammann tiling of the entire plane is obtained by reflecting $T$ in the line $L$ of $T$. Similarly for a tiling obtained from a tiling $T$ of a quadrant by reflecting in the two perpendicular border lines $L_1$ and $L_2$. In the quadrant case, the edges and vertices of $T$ on $L$ will be colored red, and in the quadrant case, the edges and vertices of $T$ on $L_1$ and $L_2$ will be colored red. So again, $T(f)$ is either $T_4$ or $T_5$, and statement (1) of Lemma 1 is proved.

Concerning statement (2), the tiles in both $A^5(T)$ and $\hat{T}$ are congruent copies of $\tau_5(G)$ and $\tau_4(G)$. Let $t \in \hat{T}$. If $s$ is an small tile in $A^j(T)$, $j < 5$, that is contained in $t$, and $b$ is its partner, then the shapes dictate that $b$ is also contained in $t$. Therefore $\hat{T} = A^5(T)$. \hfill $\square$

5. Bounded Distortion Homeomorphism

**Definition 1.** A homeomorphism $h$ of the plane has **bounded distortion** if there is a constant $C$ such that

$$|x - h(x)| \leq C$$

for all $x \in \mathbb{R}^2$. In this case, $h$ is said to have **bounded distortion** $C$. 
**Lemma 2.** Let $h$ be a homeomorphism of the plane with bounded distortion $C$. If $\phi_r$ is a similarity transformation with scaling ratio $r$, then

$$|x - \phi_r^{-1} \circ h \circ \phi_r(x)| \leq \frac{1}{r} C$$

for all $x \in \mathbb{R}^2$.

**Proof.** Let $x \in \mathbb{R}^2$, $y = \phi_r(x)$, $z = h(y)$, $w = \phi_r^{-1}(z)$. Then

$$|x - \phi_r^{-1} \circ h \circ \phi_r(x)| = |x - w| = |\phi_r^{-1}(y) - \phi_r^{-1}(z)| = \frac{1}{r} |y - z|$$

$$= \frac{1}{r} |y - h(y)| \leq \frac{1}{r} C$$

for all $x \in \mathbb{R}^2$. □

Let $\mathcal{X}$ denote the set of tilings as defined in Section 3. For a tiling $T \in \mathcal{X}$, let $\partial T$ denote the union of the boundaries of the tiles of $T$. Define a distance function on $\mathcal{X}$ as follows.

$$d(T, T') = d_H(\partial T, \partial T').$$

where $d_H$ is the Hausdorff distance. Intuitively, the $d$-distance between tilings is small if the tilings are almost the same over the entire plane. The next corollary follows immediately from Lemma 2.

**Corollary 1.** If $h$ a homeomorphism of bounded distortion $C$ and $\phi_r$ is a similarity transformation with scaling ratio $r$, then

$$d(\phi_r^{-1} \circ h \circ \phi_r)(T), T) \leq \frac{1}{r} C$$

for all tilings $T \in \mathcal{X}$.

If $T$ is an Ammann tiling and $h$ is a homeomorphism of the plane, then $h$ acts on $T$. Let $T' = h(T)$ be the image of an Ammann tiling $T$ under a homeomorphism $h$. The small and large tiles in $T'$ are defined to be the images of the small and large tiles in $T$. If $s'$ is a small tile in $T'$ corresponding to a small tile $s$ in $T$, then the partner of $s'$ is the image under $h$ of the partner of $s$. Therefore the amalgamation $a(T')$ and the hierarchy $H(T')$ can be defined for $T'$ exactly as they are for $T$.

**Theorem 1.** Let $T$ be an Ammann tiling, $h$ a bounded distortion homeomorphism of the plane, and $T' = h(T)$. Then

$$\lim_{k \to \infty} d(a^k(T'), a^k(T)) = 0.$$
From this it follows that
\[
\lim_{k \to \infty} d(\alpha_k(T'), \alpha_k(T)) = 0. \quad \square
\]

Remark 1. Theorem 1 does not state that \(\alpha_k(T')\) becomes arbitrary close to \(T\) as \(k \to \infty\). It does imply, however, that an arbitrarily large finite patch of the tiling \(\alpha_k(T')\) becomes arbitrarily close to a patch \(P_k\) of \(T\) as \(k \to \infty\). This follows from Proposition 4.1 and the fact that any two Ammann tilings are locally isomorphic.

6. Combinatorial Amalgamation

Let \(T\) be the set of all images of Ammann tilings of the plane under homeomorphisms of the plane. Essential to Theorem 1 is being able to apply the amalgamation operator \(\alpha : T \to T\). Let \(T\) be an Ammann tiling and \(T' = h(T)\), where \(h\) is a homeomorphism of the plane. In this section it is proved that \(\alpha(T')\) can be determined combinatorially, without knowledge of the homeomorphism \(h\).

Theorem 2. Let \(T\) be an Ammann tiling, \(h\) a homeomorphism of the plane and \(T' = h(T)\). Then Algorithm A below computes \(\alpha(T')\), without knowledge of \(h\).

The proof of Theorem 2 appears after a few simple lemmas. With terminology from Section 4, the following lemma is clear.

Lemma 3. If \(T\) is a tiling of the plane and \(h\) is a homeomorphism of the plane, then as planar maps \(T\) and \(h(T)\) are isomorphic.

Two planar maps \(M_1\) (left) and \(M_2\) (right) are shown in Figure 7 (ignoring the dots and arrows). Recalling the definition of \(T_n\) from Section 4 the following two lemmas are apparent by inspection.

Lemma 4. The two planar maps \(M_1\) and \(M_2\) in Figure 7 are isomorphic to \(T_4\) and \(T_5\), respectively, considered as planar maps.

Lemma 5. Each of the two planar maps in Figure 7, ignoring the arrows, has trivial automorphism group.

Figure 7. Rule for locating the partner of a small tile.
Each arrow in Figure 7 points from a face bounded by a 4-cycle to an adjacent face. A consequence of Lemma 5 is that, if $M$ is any planar map isomorphic to $M_1$ or $M_2$ in Figure 7, then arrows can be uniquely assigned to $M$ such that there is an arrow from face $f_1$ to $f_2$ in $M$ if and only if there is an arrow from $\phi(f_1)$ to $\phi(f_2)$ in $M_1$ or $M_2$, where $\phi$ is the isomorphism.

**Algorithm A**

Input: A tiling $T'$ that is the image of an Amman tiling, distorted by an unknown homeomorphism.

Output: The amalgamated tiling $a(T')$.

Perform the following steps to obtain the tiling $a(T')$ from $T$.

1. Color red each edge (and its two incident vertices) of $T'$ that joins two vertices of degree 4. But, for any red vertex lying on only one red edge, remove the color red from that vertex and remove the incident edge.

2. Let $\hat{T}'$ be the tiling induced by just the red edges and vertices of $T'$, i.e., by removing all edges and vertices not colored red.

3. For every tile $f \in \hat{T}'$, let $T'(f)$ be the set of tiles of $T'$ contained in $f$. Viewed as a planar map, each $T'(f)$ is isomorphic to one of the planar maps in Figure 7. (This is shown to be the case in the proof below of Theorem 2.) For every $f \in \hat{T}'$ do:
   - For every tile $s$ of $T'(f)$ that is bounded by a 4-cycle, do:
     - Locate the partner $b$ of $s$ using the relevant arrow in Figure 7.
     - Replace the two tiles $s$ and $b$ in $T'$ by their union to form a single tile in $A(T')$, and hence in $a(T')$ after scaling by $s = 1/\tau$.

**Proof of Theorem 2.** The notation in this proof follows the notation in Section 4 just prior to Lemma 1. Let $s'$ be any small tile in $T'$. It must be shown that Algorithm A pairs $s'$ with its large partner tile in $T'$. The tile $s$ is contained in a unique tile $f \in A^5(T') = \hat{T}$, the last equality by statement (2) of Lemma 1. Therefore $s' := h(s)$ is contained in $f' := h(f) \in A^4(T')$. Since $f \in \hat{T}$, we know that $f' \in h(\hat{T})$.

By Lemma 3, $T$ and $h(T)$, as well as $\hat{T}$ and $h(\hat{T})$, are isomorphic as planar maps. Therefore, by statement (1) of Lemma 1, if $T(f')$ denotes the set of all tiles in $T'$ contained in $f'$, then $T(f')$ is isomorphic to $T_4$ or $T_5$. By Lemma 4, we know that $T(f')$ is isomorphic to one of the planar maps in Figure 7.

Note that the boundary of any face in $T'$ is a 4-cycle if the corresponding tile is small and a 5 or 6-cycle if the corresponding tile is large. Therefore, by Lemma 5 and the remarks following it, the partner of $s' \in f'$ is uniquely determined by the arrows in Figure 7. This completes the proof of Theorem 2.

Algorithm A acts on the tiling $T'$ of the entire plane and hence does not run in finite time. From a practical point of view, it may be asked whether it is possible to efficiently compute the amalgamation of the distorted tiling $T'$ on a finite subset of the plane, for instance on a patch containing the disk $D_R$ of radius $R$ centered at the origin. We tweak Algorithm A to obtain a combinatorial Algorithm B that
applies to a finite patch of a tiling. Algorithm B computes \( a(T') \) on \( D_R \) and runs in time quadratic in \( R \). This is the content of Theorem 3 below. Figure 8 illustrates Algorithm A above and Algorithm B below, showing the red edges of a distorted Ammann tiling before and after the red edges that do not belong to cycles are removed.

The notation \( T'\mid R \) and \( a(T'\mid R) \) in Theorem 3 are defined as follows. If \( T' \) is a tiling, \( T'\mid R \) the set of tiles of \( T' \) with non-empty intersection with \( D_R \). If \( T' \) is an Ammann tiling or a homeomorphic image of an Ammann tiling, the amalgamation \( a(T'\mid R) \) is obtained by replacing every small tile \( t \) in \( T'\mid R \) by the union of \( t \) and its partner - even if the partner of \( t \) has empty intersection with \( D_R \).

Lemma 6. Let \( R \) be a positive real number. If \( T \) is an Ammann tiling and \( h \) is a homeomorphism of bounded distortion \( C \), then

1. \( h(A^5(T))\mid R \subset D_{R+2C+\tau^6} \),
2. the number of tiles of \( h(T) \) contained in the disk \( D_R \) is less than \( \tau\pi(\frac{15}{4} - \sqrt{5})(R+C)^2 \).

Proof. Concerning statement (1), if \( x \) is any point in a tile \( t' \in h(A^5(T))\mid R \), and \( y \) is any point in \( t' \cap D_R \), then by the triangle inequality and the fact that \( h \) is of bounded distortion \( C \), we have

\[
|x| \leq |y| + |x-y| \leq R + \text{diam}(t') = R + \text{diam}(h(t)) \leq R + \text{diam}(t) + 2C \\
\leq R + \text{diam}(\tau^5G) + 2C = R + \tau^6 + 2C,
\]

where \( \text{diam} \) denotes the diameter of the set.

Concerning statement (2), because \( h \) is of bounded distortion \( C \), we have \( D_R \subset h(D_{R+C}) \). Each small tile in an Ammann tiling is congruent to \( s^2G \). Therefore there are less than \( \text{area}(D_{R+C})/\text{area}(s^2G) = \tau\pi(\frac{15}{4} - \sqrt{5})(R+C)^2 \) Ammann tiles in \( D_{R+C} \). \( \square \)

Algorithm B

Input: A tiling \( T' \) that is the image of an Amman tiling, distorted by an unknown homeomorphism of bounded distortion \( C \), and a positive real number \( R \).

Output: A subset \( T'' \) of the amalgamated tiling \( a(T') \) that contains \( a(T'\mid R) \).

(1) Set \( \hat{R} = R + 2C + \tau^6 \)

Perform the following steps to obtain the tiling \( T'' \) from the tiling \( T'|\hat{R} \):

(2) Color some edges and vertices of \( T'|\hat{R} \) red according to the following rules:

- Color red each edge (and its two incident vertices) of \( T'|\hat{R} \) that joins two vertices of degree 4.
- Remove the color red from any red vertex or edge not lying on a red cycle in \( T'|\hat{R} \).

(3) Let \( \hat{T}' \) be the tiling induced by the red edges and vertices of \( T'|\hat{R} \).
Theorem 3. Let \( T \) be an Ammann tiling, \( h \) a homeomorphism of the plane of bounded distortion, and \( T' = h(T) \). Then Algorithm B above computes \( a(T'|R) \) in time quadratic in \( R \), without knowledge of \( h \).

\begin{proof}
In addition to what was shown in the proof of Theorem 3, it must be verified that the finite tiling \( \hat{T}' \) constructed in Algorithm B is contained in \( D_{\hat{R}} \), which follows from statement (2) of Lemma 1 and statement (1) of Lemma 6.
\end{proof}
7. **Direct Retrieval of an Ammann tiling from a Distorted Image**

Let $T$ be an Ammann tiling and $T' = h(T)$, where $h$ is a homeomorphism of the plane. In this section we extract from $T'$, without using $h$, an infinite binary string $\theta = \theta_1 \theta_2 \theta_3 \cdots$. A tiling $T(\theta)$ will be combinatorially constructed from $\theta$ such that $T = T(\theta)$.

**Definition 2.** Let $S$ be the set of all pairs $(T', t')$, where $T' = h(T)$ and $t' = h(t)$ for some Ammann tiling $T$, large tile $t \in T$, and homeomorphism $h$ of the plane. Let $\{1, 2\}^\infty = \{c_1 c_2 c_3 \cdots : c_i \in \{1, 2\} \text{ for all } i\}$, i.e., the set of infinite binary strings. Also let $\{1, 2\}^*$ denote the set of finite binary strings. To define a map $g : S \rightarrow S$ and a function $c : S \rightarrow \{1, 2\}^\infty$, called the code of $(T', t')$, let $(T', t') \in T$. Define $(\tilde{T}, \tilde{t}) = g(T', t')$ as follows. There are two cases.

Case 1. If $t'$ has a partner in $T'$, then let $\tilde{T} = A(T')$, the amalgamation of $T'$, and let $\tilde{t}$ be union of $t'$ and its partner, a large tile in $\tilde{T}$. Define $g(T', t') = (\tilde{T}, \tilde{t})$ and $\bar{c}(T', t') = 1$.

Case 2. If $t'$ has no partner in $T'$, then let $\tilde{T} = A^2(T')$, the second amalgamation of $T'$. Then $t'$ is a small tile in $A(T')$ that has a partner. In $\tilde{T}$ let $\tilde{t}$ be the union of $t'$ and its partner in $A(T')$. Let $g(T', t') := (\tilde{T}, \tilde{t})$. Note that $\tilde{t}$ is a large tile in $\tilde{T}$. Define $\bar{c}(T', t') = 2$.

Define a sequence of tilings $T'_0 = T', T'_1, T'_2, \ldots$ and tiles $t'_0 = t', t'_1, t'_2, \ldots$, recursively by $(T'_{n+1}, t'_{n+1}) = g(T'_n, t'_n)$. Then define the code of $(T', t')$ to be the binary string

$$c(T', t') = \bar{c}(T'_0, t'_0) \bar{c}(T'_1, t'_1) \bar{c}(T'_2, t'_2) \cdots ,$$

and note that, as done in Section 6, the amalgamation used to determine the code $c(T', t')$ of $T'$ can be done combinatorially without knowing the homeomorphism.

**Theorem 4** below states that $c(T', t')$ completely determines the original Ammann tiling $T$, and can be used to generate $T$. An intuitive way to see this is to start with a patch $P_0$ consisting of a single large Ammann tile $t$. Let $c(T, t) = c_1 c_2 \cdots$. If $c_1 = 1$, embed $P_0$ in a patch $P_1$ as shown at the start of the bottom row in Figure 9; if $c_1 = 2$, embed $P_0$ in a patch $P_1$ as at the start of the top two rows in Figure 9. Continue in this way to get a nested sequence of patches as in Figure 9. The nested union is $T$.

**Remark 2.** It is well known that there are a couple of special cases where the nested union $\sqcup \{P_n : n \geq 0\}$ is a tiling that fills a half plane or a quadrant of the plane. Therefore, in this section such tilings will also be classified as Ammann tilings. If $T$ is an Ammann tiling of the half plane, then a tiling of the entire plane satisfying the matching rules can be obtained by reflecting $T$ in the border line of $T$. These tilings will not be classified as Ammann tilings in this section. Similarly for a tiling obtained from a tiling $T$ of a quadrant by reflecting in the two border rays.
A more rigorous formulation of the above method of retrieving the original tiling from the code uses the following combinatorial construction. Recall that $s = 1/\tau$, where $\tau$ is the square root of the golden ratio. The fact that the golden bee $G$ is the non-overlapping union of two small similar copies of itself can be stated more precisely as follows.

$$G = f_1(G) \cup f_2(G),$$

where $f_1(G)$ and $f_2(G)$ are non-overlapping and

$$f_1 \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 & -s \\ s & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} s \\ 0 \end{pmatrix}, \quad f_2 \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} s^2 & 0 \\ 0 & -s^2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$
Then winding through Definition 3, the code of this implies that \( t(\theta, k, \sigma) \) in Definition 3 below to be congruent to either a large or small Ammann tile.

**Definition 3.** For each \( \theta \in \{1, 2\}^\infty \), a tiling \( T(\theta) \) is constructed in three steps.

1. A single tile. For each integer \( k \geq 1 \) and each \( \sigma \in [N]^* \), construct a single tile \( t(\theta, k, \sigma) \) that is similar to \( G \):
   \[
   t(\theta, k, \sigma) := (f_{-\theta | K} \circ f_{\theta})(sG).
   \]
2. A patch of tiles. Form a patch \( T(\theta, k) \) of tiles given by:
   \[
   T(\theta, k) := \{ t(\theta, k, \sigma) : \sigma \in W(\theta, k) \}.
   \]
3. A tiling. The tiling \( T(\theta) \), depending only on \( \theta \in \{1, 2\}^\infty \), is the union of the patches \( T(\theta, k) \), which is known \[3\] to be a nested union:
   \[
   T(\theta) := \bigcup_{k \geq 1} T(\theta, k).
   \]

The set \( \{1, 2\}^\infty \) is called the **parameter set.** For each parameter \( \theta \), the tiling \( T(\theta) \) is called the **\( \theta \)-tiling.** Modulo Remark 2 a \( \theta \)-tiling is an Ammann tiling.

**Theorem 4.** Let \( T' = h(T) \) be the image of Ammann tiling \( T \) under a homeomorphism \( h \) of the plane. Let \( t' \) be any large tile in \( T' \), and let \( \theta := c(T', t') \) be the code of \( (T', t') \). Then \( T = T(\theta) \), independent of which large tile \( t' \in T' \) is chosen.

**Proof.** Let \( t = h^{-1}(t') \in T \), and note that \( c(T', t') = c(T, t) \). Let \( \theta := c(T, t) \). It must be shown that \( T(\theta) = T \) for any large tile \( t \in T \). Since an Ammann tiling is determined by its code, it suffices to show that, for any large tile \( t \in T \), there is a large tile \( p \in T(\theta) \) such that that \( c(T(\theta), p) = \theta \).

A tile \( t(\theta, k, \sigma) \in T(\theta) \) is a large tile congruent to \( sG \) in \( T(\theta) \) if \( c(\sigma) - e(\theta | K) = 0 \) and a small tile congruent to \( s^2G \) if \( c(\sigma) - e(\theta | K) = 1 \). From the definitions, this implies that \( t(\theta, k, \sigma) \in T(\theta) \) is a small tile if and only if \( \sigma_1 = 2 \), i.e., \( \sigma = \sigma_1 \sigma_2 \cdots \sigma_{j-1} 2 \), and the tile \( t(\theta, k, \omega) \), where \( \omega = \sigma_1 \sigma_2 \cdots \sigma_{j-1} 1 \), is the large partner of \( t(\theta, k, \sigma) \). Let \( p = t(\theta, k, \sigma) \), where \( \sigma = \sigma_1 \sigma_2 \cdots \sigma_j \), be an arbitrary tile in \( T(\theta) \). Then winding through Definition 3 the code of \( c(T(\theta), p) \) is given by \( c(T(\theta), p) = \sigma_j \sigma_{j-1} \cdots \sigma_1 \theta_{k+1} \theta_{k+2} \cdots \).

Consider two cases. If the original large tile \( t \in T \) has a partner, then \( \theta_1 = 1 \), i.e., \( c(T, t) = 1 \theta_2 \theta_3 \cdots \). In \( T(\theta) \), let \( p = t(\theta, 1, 1) \). Then the code of \( c(T(\theta), p) \) is given by \( c(T(\theta), p) = 1 \theta_2 \theta_3 \cdots \). If the original large tile \( t \in T \) has no partner, then \( \theta_1 = 2 \), i.e., \( c(T, t) = 2 \theta_2 \theta_3 \cdots \). In \( T(\theta) \), let \( p = t(\theta, 1, 2) \). Then the code of \( c(T(\theta), p) \) is given by \( c(T(\theta), p) = 2 \theta_2 \theta_3 \cdots \). In either case \( c(T(\theta), p) = \theta \).

**Remark 3.** An arbitrarily large patch of the original tiling \( T \) can be obtained in finite time and efficiently from the distorted tiling \( T' \) by first using the combinatorial
amalgamation Algorithm B to obtain a finite concatenation $\theta|k$ of a code $\theta$ for $T'$ and then by using the procedure in Definition 3 to construct the patch $T(\theta,k)$.

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