Motivic classes of Nakajima quiver varieties

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Abstract

We prove, that Hausel’s formula for the number of rational points of a Nakajima quiver variety over a finite field also holds in a suitable localization of the Grothendieck ring of varieties. In order to generalize the arithmetic harmonic analysis in his proof we use Grothendieck rings with exponentials as introduced by Cluckers-Loeser and Hrushovski-Kazhdan.

1 Introduction

Let \( \Gamma = (I, E, s, t) \) be a quiver, that is a finite vertex set \( I \), a set of arrows \( E \subset I \times I \) and maps \( s, t : E \to I \) sending an arrow to its source and target. In \([14][15]\) Nakajima associates to \( \Gamma \) and two dimension vectors \( v, w \in \mathbb{N}^I \) a smooth algebraic variety \( M(v, w) \) called Nakajima quiver variety. A combinatorial formula for the Betti numbers of those varieties is proven in \([7]\) using arithmetic methods. More precisely Hausel counts the number of rational points of these varieties over finite fields of large enough characteristic and then deduces their Betti numbers by a theorem of Katz \([8, \text{Theorem 6.1.2.3}]\).

The main result of this article is Theorem 1.1, where we compute the class of \( M(v, w) \) in a suitable localization \( \mathcal{M} \) of the Grothendieck ring of varieties.

Let \( \lambda, \lambda' \in \mathcal{P} \) where \( \mathcal{P} \) is the set of partitions. For \( \lambda \in \mathcal{P} \) we write \( |\lambda| \) for its size and \( m_k(\lambda) \) for the multiplicity of \( k \in \mathbb{N} \) in \( \lambda \). Given any two partitions \( \lambda, \lambda' \in \mathcal{P} \) we define their inner product as \( \langle \lambda, \lambda' \rangle = \sum_{i, j \in \mathbb{N}} \min(i, j) m_i(\lambda)m_j(\lambda') \). Then we prove

**Theorem 1.1.** For a fixed dimension vector \( w \in \mathbb{N}^I \) the motivic classes of the Nakajima quiver varieties \( M_{\alpha, \chi}(v, w) \) in \( \mathcal{M} \) are given by the generating function

\[
\sum_{v \in \mathbb{N}^I} [M(v, w)] L^{d_{v, w}} T^v = \frac{\sum_{\lambda \in \mathcal{P}^I} \prod_{e \in E} L^{\langle \lambda_{s(e)}, \lambda_{t(e)} \rangle} \prod_{i \in I} L^{\langle 1, \lambda_i \rangle} \prod_{k \in \mathbb{N}} L^{k(\lambda_i)} (1-L^{-j})^{|\lambda|}}{\sum_{\lambda \in \mathcal{P}^I} \prod_{e \in E} L^{\langle \lambda_{s(e)}, \lambda_{t(e)} \rangle} \prod_{i \in I} L^{\langle \lambda_i, \lambda_i \rangle} \prod_{k \in \mathbb{N}} L^{k(\lambda_i)} (1-L^{-j})^{|\lambda|}} T^{|\lambda|},
\]

where \( d_{v, w} \) denotes half the dimension of \( M(v, w) \) and \( L \) the class of the affine line in \( \mathcal{M} \).
This implies in particular, that \([\mathcal{M}(v, w)]\) is given by a polynomial in \(L\).

The formula is the expected generalization of the count i.e. the cardinality \(q\) of the finite field is simply replaced by \(L\). However this generalization is not straightforward as we have to find a motivic analogue of the arithmetic harmonic analysis approach of [7]. We use the idea of [4] to 'add exponentials' to \(\mathcal{M}\) in order to define a naive motivic Fourier transform.

The author was informed by Ben Davison and Sergey Mozgovoy, that they both can prove formula (1) using different methods.

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\textit{Conventions:} Throughout the whole article we will work over an algebraically closed field \(k\) of characteristic 0. By a variety we mean a separated reduced scheme of finite type over \(k\).

\section{Grothendieck rings with exponentials and a naive Fourier transform}

In this section we start by introducing various Grothendieck rings with exponentials following closely [3]. This allows us to define a naive Fourier transform and prove a Fourier inversion formula for motivic functions. We should mention that nothing in this section is new, but rather a special case of the theory developed in [4].

The \textit{Grothendieck ring of varieties}, denoted by \(K\text{Var}\), is the quotient of the free abelian group generated by varieties modulo the relations

\[X - Y\]

if \(X\) and \(Y\) are isomorphic and

\[X - Z - U,\]

for \(Z \subset X\) a closed subvariety and \(U = X \setminus Z\). The multiplication is given by \([X] \cdot [Y] = [X \times Y]\), where we write \([X]\) for the class of a variety \(X\) in \(K\text{Var}\).

The \textit{Grothendieck ring with exponentials} \(K\text{ExpVar}\) is defined similarly. The generators are pairs \((X, f)\), where \(X\) is a variety and \(f : X \to \mathbb{A}^1 = \text{Spec}(k[T])\) a morphism. We impose three kinds of relations on the free abelian group generated by those pairs.

(i) For two varieties \(X, Y\), a morphism \(f : X \to \mathbb{A}^1\) and an isomorphism \(u : Y \to X\) the relation

\[(X, f) - (Y, u \circ f).\]
(ii) For a variety $X$, a morphism $f : X \to \mathbb{A}^1$, a closed subvariety $Z \subset X$ and $U = X \setminus Z$ the relation

$$(X, f) - (Z, f|_Z) - (U, f|_U).$$

(iii) For a variety $X$ and $pr_{\mathbb{A}^1} : X \times \mathbb{A}^1 \to \mathbb{A}^1$ the projection the relation

$$(X \times \mathbb{A}^1, pr_{\mathbb{A}^1}).$$

The class of $(X, f)$ in $\text{KExpVar}$ will be denoted by $[X, f]$. We define the product of two generators $[X, f]$ and $[Y, g]$ as

$$[X, f] \cdot [Y, g] = [X \times Y, f \circ pr_X + g \circ pr_Y],$$

where $f \circ pr_X + g \circ pr_Y : X \times Y \to \mathbb{A}^1$ is the morphism sending $(x, y)$ to $f(x) + g(y)$. This gives $\text{KExpVar}$ the structure of a commutative ring.

Denote by $L$ the class of $\mathbb{A}^1 \times S$ resp. $(\mathbb{A}^1 \times S, 0)$ in $\text{KVar}$ resp. $\text{KExpVar}$. The localizations of $\text{KVar}$ and $\text{KExpVar}$ with respect to the the multiplicative subset generated by $L$ and $L^n - 1$, where $n \geq 1$ are denoted by $\mathcal{M}$ and $\mathcal{E}xp\mathcal{M}$. For a variety $S$ there is a straightforward generalization of the above construction to obtain the relative Grothendieck rings $\text{KVar}_S$, $\text{KExpVar}_S$, $\mathcal{M}_S$ and $\mathcal{E}xp\mathcal{M}_S$. For example generators of $\text{KExpVar}_S$ are pairs $(X, f)$ where $X$ is a $S$-variety (i.e. a variety with a morphism $X \to S$) and $f : X \to \mathbb{A}^1$ a morphism. The class of $(X, f)$ in $\text{KExpVar}_S$ will be denoted by $[X, f]|_S$ or simply $[X, f]$ if the base variety $S$ is clear from the context.

There is a natural map

$$\text{KVar}_S \to \text{KExpVar}_S$$

$$[X] \mapsto [X, 0]$$

and similarly $\mathcal{M}_S \to \mathcal{E}xp\mathcal{M}_S$, which are both injective ring homomorphisms by $[3$, Lemma 1.1.3$]$. Hence we don’t need to distinguish between $[X]$ and $[X, 0]$ for a $S$-variety $X$.

For a morphism of varieties $u : S \to T$ we have induced maps

$$u : \text{KExpVar}_S \to \text{KExpVar}_T,$$

$$u^* : \text{KExpVar}_T \to \text{KExpVar}_S,$$

$$[X, f]|_S \mapsto [X, f]|_T$$

$$[X, f]|_T \mapsto [X \times T, S, f \circ pr_X]|_S.$$ 

In general $u^*$ is a morphism of rings and $u_!$ a morphism of additive groups. However it is straightforward to check that for any $u : S \to T$ and any $\varphi \in \text{KExpVar}_S$ we have

$$u_!(L \cdot \varphi) = L \cdot u_!(\varphi),$$

where $L$ denotes the class of $\mathbb{A}^1 \times S$ and $\mathbb{A}^1 \times T$ in $\text{KExpVar}_S$ and $\text{KExpVar}_T$ respectively.

Elements of $\text{KExpVar}_S$ can be thought of as motivic functions on $S$. The
evaluation of $\varphi \in \text{KExpVar}_S$ at a point $s : \text{Spec}(k) \to S$ is simply $s^*(\varphi) \in \text{KExpVar}_{\text{Spec}(k)} = \text{KExpVar}$. Computations with these motivic functions can sometimes replace finite field computations. More precisely let $\mathbb{F}_q$ be a finite field and fix a non-trivial additive character $\psi : \mathbb{F}_q \to \mathbb{C}^\times$. Assume that $S, X \to S$ and $f : X \to \mathbb{A}^1$ are also defined over $\mathbb{F}_q$. Then the class of $(X, f) \in \text{KExpVar}_S$ corresponds to the function

\[ S(\mathbb{F}_q) \to \mathbb{C}, \quad s \mapsto \sum_{x \in X_s(\mathbb{F}_q)} \psi(f(x)). \]

Furthermore for a morphism $u : S \to T$ the operations $u_!$ and $u^*$ correspond to summation over the fibres of $u$ and composition with $u$ respectively.

An important identity for computing character sums over finite fields is

\[ \sum_{v \in V} \psi(f(v)) = \begin{cases} q^{\dim(V)} & \text{if } f = 0 \\ 0 & \text{else,} \end{cases} \]

where $V$ is a $\mathbb{F}_q$ vector space and $f \in V^*$ a linear form.

To establish a similar identity in the motivic setting we let $V$ be a finite dimensional vector space over $k$ and $S$ a variety. We replace the linear form above with a family of affine linear forms i.e. a morphism $g = (g_1, g_2) : X \to V^* \times k$, where $X$ is a $S$-variety. Then we define $f$ to be the morphism

\[ f : X \times V \to k \\
(x, v) \mapsto (g_1(x), v) + g_2(x). \]

Finally we put $Z = g_1^{-1}(0)$.

**Lemma 2.1.** With the notation above we have the relation

\[ [X \times V, f] = \mathbb{L}^{\dim V}[Z, g_2|Z] \]

in $\text{KExpVar}_S$.

**Proof.** By using \[2\] we may assume $S = X$. Now because of \[3\], Lemma 1.1.8 it is enough to check for each point $x \in X$ the identity

\[ x^*(\langle X \times V, f \rangle) = x^*(\mathbb{L}^{\dim V}[Z, g_2|Z]) \]

and this is exactly Lemma 1.1.11 of loc. cit. $\square$

Now we’re ready to define a *naive motivic Fourier transform* for functions on a finite dimensional $k$-vector space $V$ and prove an inversion formula. All of this is a special case of \[4\], Section 7.1.

**Definition 2.2.** Let $p_V : V \times V^* \to V$ and $p_{V^*} : V \times V^* \to V^*$ be the obvious projections. *The naive Fourier transformation* $\mathcal{F}_V$ is defined as

\[ \mathcal{F}_V : \text{KExpVar}_V \to \text{KExpVar}_{V^*} \\
\varphi \mapsto p_{V^*!}(p_V^*\varphi : [V \times V^*, \langle , \rangle]). \]

Here $\langle , \rangle : V \times V^* \to k$ denotes the natural pairing.
We will often write $F$ instead of $F_V$ when there’s no ambiguity. Notice that $F$ is a homomorphism of groups and thus it is worth spelling out the definition in the case when $\varphi = [X, f]$ is the class of a generator in $\text{KExpVar}_V$. Letting $u : X \to V$ be the structure morphism we simply have

$$F([X, f]) = [X \times V^*, f \circ pr_X + \langle u \circ pr_X, pr_V \cdot \rangle].$$

Now we’re ready to prove an inversion formula for the naive Fourier transform.

**Proposition 2.3.** For every $\varphi \in \text{KExpVar}_V$ we have the identity

$$F(F(\varphi)) = L^{\dim(V)} \cdot i^*(\varphi),$$

where $i : V \to V$ is multiplication by $-1$.

**Proof.** Since $F$ is a group homomorphism it is enough to prove the lemma for $\varphi = [X, f]$ with $X \to V$. Iterating (3) we get

$$F(F([X, f])) = [X \times V \times V^*, f \circ pr_X + \langle u \circ pr_X + pr_V, pr_V \cdot \rangle].$$

Now we can apply Lemma 2.1 with $Z = \{(x, v) \in X \times V \mid u(x) + v = 0\}$ to obtain

$$[X \times V \times V^*, f \circ pr_X + \langle u \circ pr_X + pr_V, pr_V \cdot \rangle] = L^{\dim V^*} [Z, f \circ pr_X].$$

Notice that $Z$ is a $V$-variety via projection onto the second factor and hence the projection onto the first factor induces a $V$-isomorphism $Z \cong (X \overset{i}{\to} V)$, which gives the desired result.

### 3 Motives of moment map equations

The naive Fourier transform enables us to perform computations arising from the arithmetic harmonic analysis approach introduced in [6] in the motivic setting. In this section we prove a motivic version of the crucial Proposition 1 of loc. cit. on the number of points of certain moment map fibers.

Let $G$ be a reductive algebraic group over $k$ with Lie algebra $g$ and $\rho : G \to \text{GL}(V)$ a representation. The derivative of $\rho$ is the Lie algebra representation $\varrho : g \to \mathfrak{gl}_n$. We define the moment map

$$\mu : V \times V^* \to g^*$$

for $(v, w) \in V \times V^*$ and $X \in g$ by the formula

$$\langle \mu(v, w), X \rangle = \langle \varrho(X)(v), w \rangle.$$ 

Our goal is now to compute for $\xi \in g^*$ the motive of $\mu^{-1}(\xi)$ in $\text{ExpM}$. 


Remark 3.1. Notice that \( \rho \) induces an action of \( G \) on the symplectic vector space \( V \times V^* \) by the formula

\[
g \cdot (v, w) = (\rho(g)v, \rho(g^{-1})^*w)
\]

and if \( k = \mathbb{C} \) one can check that \( \mu \) is indeed a moment map for this action.

We define

\[
a_\varrho = \{(v,X) \in V \times g \mid \varrho(X)v = 0\},
\]

which is a \( g \)-variety via the projection onto the second factor \( \pi : a_\varrho \to g \). Analogous to \([6, Proposition 1]\) we have

**Proposition 3.2.** For any \( \xi \in g^* \) the identity

\[
[\mu^{-1}(\xi)] = L_{\dim V - \dim g} [a_\varrho, \langle -\pi, \xi \rangle]
\]

holds in \( \exp \mathcal{M} \).

Proof. We consider \( V \times V^* \) as a \( g^* \)-variety via the moment map \( \mu \). Then by (3) the naive Fourier transform of its class in \( K \text{ExpVar}_{g^*} \) is

\[
\mathcal{F}([V \times V^*]) = [V \times V^* \times g, \langle \mu \circ pr_{V \times V^*}, pr_g \rangle].
\]

Now by the definition (4) of \( \mu \) we have

\[
[V \times V^* \times g, \langle \mu \circ pr_{V \times V^*}, pr_g \rangle] = [V \times V^* \times g, \langle (\varrho \circ pr_g)pr_{V^*}, pr_{V^*} \rangle].
\]

Thus lemma 2.1 with \( X = V \times g \) and \( Z = a_\varrho \) gives

\[
\mathcal{F}([V \times V^*]) = L_{\dim V}[a_\varrho].
\]

Next we apply \( \mathcal{F} \) again and use the inversion lemma 2.3 to get

\[
L_{\dim g}[V \times V^*] = \mathcal{F}(L_{\dim V}[a_\varrho]) = L_{\dim V}[a_\varrho \times g^*, \langle \pi \circ pr_{a_\varrho}, pr_{g^*} \rangle].
\]

Finally passing to \( \exp \mathcal{M}_{g^*} \) to invert \( L_{\dim g} \) and using \( (i^*)^2 = \text{Id}_{g^*} \) gives

\[
[V \times V^*] = L_{\dim V}[a_\varrho \times g^*, \langle -\pi \circ pr_{a_\varrho}, pr_{g^*} \rangle].
\]

The result now follows from pulling back both sides along \( \xi : \text{Spec}(k) \to g^* \).

4 Nakajima Quiver varieties

In this section we recall the definition of Nakajima quiver varieties. Almost everything can be found in more detail in \([7]\) or in the original sources \([14, 15]\).

Let \( \Gamma = (I, E) \) be a quiver with \( I = \{1, 2, \ldots, n\} \) the set of vertices and \( E \) the set of arrows. We denote by \( s(e) \) and \( t(e) \) the source and target vertex of an arrow \( e \in E \). For each \( i \in I \) we fix finite dimensional vector spaces \( V_i, W_i \) and...
write \( v = (\dim V_i)_{i \in I}, w = (\dim W_i)_{i \in I} \in \mathbb{N}^I \) for their dimension vectors. From this data we construct the vector space

\[ \mathbb{V}_{v,w} = \bigoplus_{e \in E} \text{Hom}(V_{e(e)}, V_{t(e)}) \oplus \bigoplus_{i \in I} \text{Hom}(W_i, V_i), \]

the algebraic group

\[ G_v = \prod_{i \in I} \text{GL}(V_i) \]

and its Lie algebra

\[ \mathfrak{g}_v = \bigoplus_{i \in I} \mathfrak{gl}(V_i). \]

We have a natural representation

\[ \rho_{v,w} : G_v \to \text{GL}(\mathbb{V}_{v,w}) \]

and its derivative

\[ \theta_{v,w} : \mathfrak{g}_v \to \mathfrak{gl}(\mathbb{V}_{v,w}). \]

For \( g = (g_i)_{i \in I}, X = (X_i)_{i \in I} \) and \( \varphi = (\varphi_e, \varphi_i)_{e \in E, i \in I} \in \mathbb{V}_{v,w} \) they are given by the formulas

\[ \begin{align*}
\rho_{v,w}(g) \varphi &= (g_{t(e)} \varphi_e g_{s(e)}^{-1}, g_i \varphi_i)_{e \in E, i \in I} \\
\theta_{v,w}(X) \varphi &= (X_{t(e)} \varphi_e - \varphi_e X_{s(e)}, X_i \varphi_i)_{e \in E, i \in I}.
\end{align*} \]

Now we’re exactly in the situation of section 3 i.e. \( G_v \) acts on the vector space \( \mathbb{V}_{v,w} \times \mathbb{V}^*_{v,w} \) with moment map

\[ \mu_{v,w} : \mathbb{V}_{v,w} \oplus \mathbb{V}^*_{v,w} \to \mathfrak{g}_v^*, \]

given by (4).

Next we fix \( \alpha \in k \) and define the affine variety \( \mathcal{V}_\alpha(v, w) = \mu_{v,w}^{-1}(\alpha 1_v) \), where \( 1_v \in \mathfrak{g}_v^* \) is defined by \( 1_v(X) = \sum_{i \in I} \text{tr} X_i \) for \( X \in \mathfrak{g}_v \). Following [15] we set \( \chi \) to be the character of \( G_v \) given by \( \chi(g) = \prod_{i \in I} \det(g_i)^{-1} \). Furthermore we put

\[ k[\mathcal{V}_\alpha(v, w)]^{G_v \chi^m} = \{ f \in k[\mathcal{V}_\alpha(v, w)] \mid f(g(x)) = \chi(g)^m f(x) \quad \forall x \in \mathcal{V}_\alpha(v, w) \}. \]

Then \( \bigoplus_{m \geq 0} k[\mathcal{V}_\alpha(v, w)]^{G_v \chi^m} \) is a graded algebra and we define the Nakajima quiver variety as

\[ \mathcal{M}_{\alpha, \chi}(v, w) = \text{Proj} \left( \bigoplus_{m \geq 0} k[\mathcal{V}_\alpha(v, w)]^{G_v \chi^m} \right). \]

(5)

In this article we will mostly be concerned with the affine version given by

\[ \mathcal{M}_\alpha(v, w) = \text{Spec}(k[\mathcal{V}_\alpha(v, w)]^{G_v}). \]

A practical reason for this is the following fact.
Lemma 4.1. The quotient $V_\alpha(v, w) \rightarrow M_\alpha(v, w)$ is a Zariski locally trivial $G_v$-principal bundle.

Proof. Notice that any $G_v$-principal bundle is Zariski locally trivial, see [17, Lemma 5 and 6]. Hence it is enough to prove, that $V_\alpha(v, w) \rightarrow M_\alpha(v, w)$ is a $G_v$-principal bundle. In view of Proposition 0.9 and Amplification 1.3 of [13] it is enough to show, that $G_v$ acts scheme-theoretically freely on $V_\alpha(v, w)$ i.e. the natural map

$$G_v \times V_\alpha(v, w) \rightarrow V_\alpha(v, w) \times V_\alpha(v, w)$$

is a closed immersion, which can be done similarly to [16, Lemma 6.5].

Finally we notice, that motivically little is lost by restricting ourselves to $M_\alpha(v, w)$. Indeed, for $\alpha \in k\times$ the affinization map $M_\alpha,\chi(v, w) \rightarrow M_\alpha(v, w)$ (6)

is an isomorphism (cf. [3, Lemma 7]) and for $\alpha = 0$ we have

Proposition 4.2. The classes of $M_{0,\chi}(v, w)$ and $M_1(v, w)$ agree in $\mathcal{M}$.

Proof. The argument is similar to [7, Theorem 8]. Let $\mu : V_{v,w} \oplus V_{v,w} \oplus k \rightarrow g_v^*$ be the map given by $\mu(\varphi, \psi, z) = \mu_{v,w}(\varphi, \psi) - z1_v$. Letting $G_v$ act trivially on $k$, the fiber $V = \mu^{-1}(0)$ is $G_v$-invariant and we define the GIT quotient as in [5] by

$$N = \text{Proj} \left( \bigoplus_{m \geq 0} k[V]^{G_v \times k^m} \right).$$

We have a natural map $f : N \rightarrow k$ induced by the projection $V \rightarrow k$. Analogous to [12, Corollary 3.12] we deduce that $N$ is non-singular and furthermore $f^{-1}(\alpha) \cong M_{\alpha,\chi}(v, w)$ for every $\alpha \in k$ since $\text{Proj}$ is compatible with base change (cf. [3, Remark 13.27]).

Now the $k^\times$-action on $V$ given by

$$\lambda \cdot (\varphi, \psi, z) = (\lambda \varphi, \lambda \psi, \lambda^2 z),$$

descends to an action on $N$ and we have an equality of fixpoint sets $N^{k^\times} = f^{-1}(0)^{k^\times} = M_{0,\chi}(v, w)^{k^\times}$. Hence the Bialynicki-Birula Theorem [1, Theorem 4.1] (notice that the existence of a $k^\times$-invariant quasi-affine open covering is automatic by [18, Corollary 2]) implies $[N] = [L[M_{0,\chi}(v, w)]]$.

On the other hand using that $\mu_{v,w}$ is bilinear we obtain a trivialization $f^{-1}(k^\times) \cong M_{1,\chi}(v, w) \times k^\times$ and hence

$$[N] = [M_{0,\chi}(v, w)] + [f^{-1}(k^\times)] = [M_{0,\chi}(v, w)] + ([L] - 1)M_1(v, w),$$

where we also used the isomorphism (6). Comparing the two expressions for $[N]$ implies the result. 

5 The main computation

In this section we prove our main Theorem 1.1, a combinatorial formula for the motive of a Nakajima quiver variety \(\mathcal{M}_{\alpha,\chi}(v, w)\). By (the proof of) Proposition 4.2 it is enough to consider the following generating series

\[
\Phi(w) = \sum_{v \in \mathbb{N}^{i}} [\mathcal{M}_1(v, w)][U_{v,w}T^v] \in \mathcal{M}[[T_1, \ldots, T_n]],
\]

where we put \(d_{v,w} = \dim(g_v) - \dim(V_{v,w})\). Having proposition 3.2 available, we can argue along the lines of the finite field computations in [7], with one difference. Namely, given a fibration \(f : X \to Y\) with fiber \(F\) we cannot deduce in general

\[
[X] = [F][Y]
\]

in \(\text{KVar}\) or \(\mathcal{M}\), whereas a similar relation clearly holds over a finite field. However (8) holds if the fibration is Zariski-locally trivial i.e. \(Y\) admits an open covering \(Y = U_jU_j\) such that \(f^{-1}(U_j) \cong F \times U_j\). Indeed, in this case we have

\[
[X] = \sum_{j} [f^{-1}(U_j)] - \sum_{j_1 < j_2} [f^{-1}(U_{j_1} \cap U_{j_2})] + \ldots = [F][Y].
\]

Combining this with Lemma 4.1 and Proposition 3.2 we get

\[
\Phi(w) = \sum_{v \in \mathbb{N}^{i}} [V(v, w)] [U_{v,w}T^v] = \sum_{v \in \mathbb{N}^{i}} [a_{v,w} : (-\pi, 1_v)] [U_{v,w}T^v],
\]

with the notations

\[
a_{v,w} = \{(\varphi, X) \in V_{v,w} \times g_v | g_v(X)v = 0\}
\]

and \(\pi : a_{v,w} \to g_v\) the natural projection.

Next we use some basic linear algebra to split up the above generating series into a regular and a nilpotent part. Given a finite dimensional vector space \(V\) of dimension \(n\) and an endomorphism \(X\) of \(V\), we can write \(V = N(X) \oplus R(X)\), where \(N(X) = \ker(X^n)\) and \(R(X) = \text{Im}(X^n)\). With respect to this decomposition we have \(X = X^{\text{nil}} \oplus X^{\text{reg}}\) with \(X^{\text{nil}} = X|_{N(X)}\) nilpotent and \(X^{\text{reg}} = X|_{R(X)}\) regular.

Now let \(v' = (v'_i)_{i \in I}\) with \(v' \leq v\) (i.e the inequality holds for every entry). We define the three varieties

\[
a_{v,w}^{v'} = \{(\varphi, X) \in a_{v,w} | \dim(N(X_i)) = v'_i \text{ for } i \in I\}
\]

\[
a_{v,w}^{\text{nil}} = \{(\varphi, X) \in a_{v,w} | X\text{ nilpotent}\},
\]

\[
a_{v,w}^{\text{reg}} = \{(\varphi, X) \in a_{v,w} | X\text{ regular}\}.
\]

**Lemma 5.1.** For every \(v' \leq v\) we have the following relation in \(\text{Exp} \mathcal{M}\)

\[
[a_{v',w} : (-\pi, 1_v)] [G_{v'}] = [a_{v,w}^{\text{nil}} | a_{v,w}^{\text{reg}}, (-\pi, 1_{v-v'})][G_{v-v'}].
\]
We will prove that the morphism induces inclusions

$$V_{v',w} \oplus V_{v-v',0} \hookrightarrow V_{v,w} \quad \text{and} \quad g_{v'} \oplus g_{v-v'} \hookrightarrow g_v.$$ We will prove that the morphism

$$\Delta : a_{\delta_{v'},w}^{\text{nil}} \times a_{\delta_{v-v'},0}^{\text{reg}} \times G_v \to a_{\delta_{v},w}^{\varphi'}$$

$$(\varphi', X', \varphi'', X'', g) \mapsto (\rho_{\delta_{v},w}(g)(\varphi' \oplus \varphi''), \text{Ad}_g(X' \oplus X''))$$

is a Zariski-locally trivial $G_{\varphi'} \times G_{\varphi''}$-fibration. Since for every $(\varphi', X') \in a_{\delta_{v'},w}^{\text{nil}}$ we have

$$\langle -\pi, 1_{\varphi'} \rangle(\varphi', X') = \sum_{i \in I} \text{tr}X_i' = 0,$$

this will imply the lemma using (8).

First notice that $\Delta$ is well defined because

$$\rho_{\delta_{v},w} \circ \text{Ad}_g = \text{Ad}_{\rho_{\delta_{v},w}(g)} \circ \rho_{\delta_{v},w}.$$ The $G_{\varphi'} \times G_{\varphi''}$-action on the domain of $\Delta$ is given as follows. For $h = (h', h'') \in G_{\varphi'} \times G_{\varphi''}$ and $(\varphi', X', \varphi'', X'', g) \in a_{\delta_{v'},w}^{\text{nil}} \times a_{\delta_{v-v'},0}^{\text{reg}} \times G_v$ we set

$$h \cdot (\varphi', X', \varphi'', X'', g) = (\rho_{\delta_{v'},w}(h')\varphi', \text{Ad}_{h'}X', \rho_{\delta_{v-v'},0}(h'')\varphi'', \text{Ad}_{h''}X'', gh^{-1}),$$

where $gh^{-1}$ is understood via the inclusion $G_{\varphi'} \times G_{\varphi''} \hookrightarrow G_v$. One checks directly that $\Delta$ is invariant under this action and hence each fiber of $\Delta$ carries a free $G_{\varphi'} \times G_{\varphi''}$-action.

On the other hand, assume $\Delta(\varphi'_1, X'_1, \varphi''_1, X''_1, g_1) = \Delta(\varphi'_2, X'_2, \varphi''_2, X''_2, g_2)$. This implies

$$\text{Ad}_{g_2^{-1}g_1}(X'_1 \oplus X''_1) = X'_2 \oplus X''_2.$$

Since $X'_j$ is nilpotent and $X''_j$ regular for $j = 1, 2$, the decomposition $V_i = V'_i \oplus V''_i$ is preserved by $g_2^{-1}g_1 \in G_{\varphi'} \times G_{\varphi''}$, which shows that each fiber of $\Delta$ is isomorphic to $G_{\varphi'} \times G_{\varphi''}$.

Finally to trivialize $\Delta$ locally we notice, that there is an open covering $a_{\delta_{v'},w}^{\varphi'} = \bigcup_j U_j$ and algebraic morphisms $t_j : U_j \to G_v$ such that for $X \in U_j$ and $i \in I$ the columns of the matrix $t_j(X)_i$ form a basis of $N(X_i)$ and $R(X_i)$. \hfill \square

Now we use the stratification $a_{\delta_{v'},w} = \coprod_{v' \leq v} a_{\delta_{v'},w}$ together with lemma [5.1] to get
\[
\Phi(w) = \sum_{v \in \mathbb{N}} \frac{[a_{\varphi_v,w}, \langle -\pi, 1_v \rangle]}{[GL_v]} T^v \\
= \sum_{v \in \mathbb{N}} \sum_{v' \leq v} \frac{[a_{\varphi'_v,w}, \langle -\pi, 1_v \rangle]}{[GL_v]} T^v \\
\sum_{v \in \mathbb{N}} \sum_{v' \leq v} \frac{[a_{\varphi'_v,w}, \langle -\pi, 1_v \rangle]}{[GL_v]} T^v = \Phi_{\text{nil}}(w) \Phi_{\text{reg}},
\]
where we used the notations
\[
\Phi_{\text{nil}}(w) = \sum_{v \in \mathbb{N}} \frac{[a_{\varphi_v,w}]}{[G_v]} T^v
\]
and
\[
\Phi_{\text{reg}} = \sum_{v \in \mathbb{N}} \frac{[a_{\varphi_v,w}, \langle -\pi, 1_v \rangle]}{[G_v]} T^v.
\]
Now \cite[Lemma 3]{7} implies \(\Phi(0) = 1\) and therefore
\[
\Phi(w) = \frac{\Phi_{\text{nil}}(w)}{\Phi_{\text{nil}}(0)},
\]
which leaves us with computing \(\Phi_{\text{nil}}(w)\).

We denote by \(\mathcal{P}\) the set of all partitions \(\lambda = (\lambda^1, \lambda^2, \ldots)\), where \(\lambda^1 \geq \lambda^2 \geq \ldots\). The size of \(\lambda\) is \(|\lambda| = \lambda^1 + \lambda^2 + \ldots\). \(\mathcal{P}_n\) denotes the set of partitions of size \(n\). For \(\lambda \in \mathcal{P}_n\) we write \(C(\lambda)\) for the nilpotent conjugacy class, whose Jordan normal form is given by \(\lambda\). For \(\lambda = (\lambda_i) \in \mathcal{P}_I\) with \(\lambda_i \in \mathcal{P}_{v_i}\) we set
\[
a_{\varphi_v,w}^{\text{nil}}(\lambda) = \{(\varphi, X) \in a_{\varphi_v,w}^{\text{nil}} | X_i \in C(\lambda_i)\},
\]
which gives the stratification
\[
a_{\varphi_v,w}^{\text{nil}} = \prod_{\lambda_i \in \mathcal{P}_I, \lambda_i \in \mathcal{P}_{v_i}} a_{\varphi_v,w}^{\text{nil}}(\lambda).
\]
To compute \([a_{\varphi_v,w}^{\text{nil}}(\lambda)]\) we look at the projection
\[
\pi : a_{\varphi_v,w}^{\text{nil}}(\lambda) \to C(\lambda) = \prod_{i \in I} C(\lambda_i).
\]
The fiber of \(\pi\) over \(X \in C(\lambda)\) is simply \(\ker(\varphi_v,w(X))\). Because of \(\varphi_v,w \circ \text{Ad}_g = \text{Ad}_{\varphi_v,w(g)} \circ \varphi_v,w\) the dimensions of those kernels are constant and hence \(\pi\) is a vector bundle of rank, say, \(\kappa_{v,w}(\lambda)\).
Lemma 5.2. Denote by $Z(\lambda) \subset G_\lambda$ the centralizer of (some element in) $C(\lambda)$. We have the following relation in $\mathcal{M}$.

$$\frac{[\alpha_{\psi w}^{nil}(\lambda)]}{[G_\lambda]} = L^{\kappa_{\psi w}(\lambda)}_{[Z(\lambda)]}$$  \hspace{1cm} (15)

Proof. The formula (17) below shows in particular that $[Z(\lambda)]$ is invertible in $\mathcal{M}$. Since the projection (14) is a vector bundle, we’re left with proving $[G_\lambda]/[Z(\lambda)] = [C(\lambda)]$. Since $C(\lambda)$ is isomorphic to $G_\lambda/Z(\lambda)$, see for example [2, Chapter 3.9.1], it is enough to prove that the $Z(\lambda)$-principal bundle $G_\lambda \to G_\lambda/Z(\lambda)$ is Zariski locally trivial by (8). In fact, this is true for every $Z(\lambda)$-principal bundle, which follows from combining Propositions 3.13 and 3.16 of [12].

To compute $\kappa_{\psi w}(\lambda)$ and $[Z(\lambda)]$, denote by $m_k(\lambda)$ the multiplicity of $k \in \mathbb{N}$ in a partition $\lambda \in \mathcal{P}$. Then given any two partitions $\lambda, \lambda' \in \mathcal{P}$ we define their inner product to be

$$\langle \lambda, \lambda' \rangle = \sum_{i, j \in \mathbb{N}} \min(i, j)m_i(\lambda)m_j(\lambda').$$

Lemma 3.3 in [10] implies now

$$\kappa_{\psi w}(\lambda) = \sum_{c \in E} \langle \lambda_{s(c)}, \lambda_{t(c)} \rangle + \sum_{i \in I} \langle 1^w_i, \lambda_i \rangle,$$  \hspace{1cm} (16)

where $1^w_i \in \mathcal{P}_{w_i}$ denotes the partition $(1, 1, \ldots, 1)$.

For $[Z(\lambda)]$ we can use the formula (1.6) from [11, Chapter 2.1]. There the formula is worked out over a finite field but Lemma 1.7 of loc. cit. holds over any field. In our notation this gives (see [10, Chapter 3] for details)

$$[Z(\lambda)] = \prod_{i \in I} \prod_{k \in \mathbb{N}} \prod_{j=1}^{m_k(\lambda_i)} (1 - L^{-j}).$$  \hspace{1cm} (17)

Finally combining (12), (13), (15), (16) and (17) we obtain our main theorem.

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