Two and three-dimensional spin systems
with gonihedric action

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Abstract

We perform numerical simulations of the two and three-dimensional spin systems with competing interaction. They describe the model of random surfaces with linear-gonihedric action. The degeneracy of the vacuum state of this spin system is equal to \(d \cdot 2^N\) for the lattice of the size \(N^d\). We observe the second order phase transition of the three-dimensional system, at temperature \(\beta_c \simeq 0.439322\) which almost coincides with \(\beta_c\) of the 2D Ising model. This confirms the earlier analytical result for the case when self-interaction coupling constant \(k\) is equal to zero. We suggest the full set of order parameters which characterize the structure of the vacuum states and of the phase transition.

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1.

It is well known that the partition function of three-dimensional Ising ferromagnet is equivalent to the summation over closed surfaces with an action which is proportional to the surface area. In two and three dimensions the Ising model undergoes a second order phase transition and at the critical point, in two-dimensions, corresponds to free fermions and probably to interacting string theory in three dimensions.

The aim of this article is to investigate the phase structure of the two- and three-dimensional spin systems introduced in [1]. In three dimensions the partition function of this spin system can be represented as a sum over closed self-intersecting random surfaces with an action which is proportional to the linear size of the surfaces.

We observe the second order phase transition of the system with linear-gonihedric action in three dimensions. This numerical result encourages the hope that the system may describe the propagation of an almost free fermionic string. This hope is based on the fact that the the 3D gonihedric model can be rewritten as a model of propagating loops with the interaction term which is proportional to the overlapping length, while 3D Ising model can be rewritten in the same way, but with the interaction term which is proportional to overlapping area, and therefore in the latter case the interaction is much stronger.

In the next sections we describe the two- and three-dimensional models with gonihedric action. We stress that the vacuum is strongly degenerated and that the rate of degeneracy depends on the self-intersection coupling constant $k$. If $k \neq 0$, the degeneracy of the vacuum state is equal to $d \cdot 2^N$ for the lattice of the size $N^d$ and is equal to $2^{dN}$ when $k = 0$. The last case is a sort of supersymmetric point in the space of gonihedric Hamiltonians. The rate of degeneracy is between Ising model $Z_2$ and Wegner gauge spin systems $Z_{2N^d}$.

In two dimensions the system is in completely disordered regime because of the tunneling phenomenon between the vacuum states through the instantons. The instantons remove the degeneracy of the ground state and the symmetry restoreds.

In three dimensions the vacua are well separated and we have different symmetry breaking phases. We observed the second order phase transition at $\beta_c \simeq 0.439322$. We suggest the full set of order parameters which characterize the structure of the vacuum states and the phase transition. At the critical point the correlation length became infinite and the theory may have a continuum limit.

2.
The Hamiltonian of the system in two dimensions has the form \[ H^{2d}_{\text{gonimetric}} = -k \sum \sigma_\vec{r} \sigma_{\vec{r}+\vec{a}} + \frac{k}{2} \sum \sigma_\vec{r} \sigma_{\vec{r}+\vec{a}+\vec{\zeta}} - \frac{1}{2} \sum \sigma_\vec{r} \sigma_{\vec{r}+\vec{a}} \sigma_{\vec{r}+\vec{a}+\vec{\zeta}} \sigma_{\vec{r}+\vec{\zeta}} \] (1)

The low temperature expansion of the partition function

\[ Z(\beta) = \sum_{\{\sigma\}} \exp(-\beta H^{2d}_{\text{gonimetric}}) \] (2)

can be represented as a sum over random walks or paths with the energy of the path which is proportional to the number of right corners of the path. In the self-intersection points the interaction energy is equal to \(4k\) and therefore the constant \(k\) is called an intersection coupling constant \([13, 14]\). The energy of a given path \(P\) is equal to

\[ E_{\text{path}} \equiv k(P) = \text{right}(P) + 4k \cdot \text{inter}(P) \] (3)

where \(\text{right}(P)\) is the number of right angles of the path and \(\text{inter}(P)\) is the number of its self-intersection points. \(E_{\text{path}}\) is a scale invariant quantity: if we enlarge the contour let’s say \(\lambda\) times, then \(E_{\text{path}}\) does not change.

Comparing this system with 2\(d\) Ising ferromagnet

\[ H^{2d}_{\text{Ising}} = -\sum \sigma_\vec{r} \sigma_{\vec{r}+\vec{\zeta}} \] (4)

one can see that the energy of paths in the low-temperature expansions of the partition function is proportional to the length of the path

\[ E_{\text{path}} \equiv l(P) = \text{links}(P) \] (5)

where \(\text{links}(P)\) is the number of the links of the closed path.

The intersection coupling constant \(k\) defines the intensity of self-intersections of the paths and at the same time defines the rate of degeneracy of the vacuum states. When \(k \neq 0\), then the vacuum is degenerated by the factor \(d \cdot 2^N\) for the lattice of the size \(N^d\), where \(d\) is the dimension of the lattice. The degeneracy increases when \(k = 0\) and is equal to \(2^{dN}\). This means that the point \(k = 0\) in the “space” of Hamiltonians corresponds to a system with a higher symmetry. The rate of degeneracy of the vacuum is between the Ising ferromagnet and Gauge-spin models which is \(Z_2\) and \(Z_2^{N^d}\) correspondingly.
In the case when the intersection coupling constant is equal to zero ($k = 0$) the Hamiltonian (1) and the partition function reduce to the form

$$H_{gonimetric}^{2d} = -\frac{1}{2} \sum_{\vec{r}, \vec{\alpha}, \vec{\beta}} \sigma_{\vec{r}} \sigma_{\vec{r} + \vec{\alpha}} \sigma_{\vec{r} + \vec{\beta}} \sigma_{\vec{r} + \vec{\alpha} + \vec{\beta}}$$  \hspace{1cm} (6)

$$Z(\beta) = \sum_{\{\sigma\}} \exp(-\beta H_{gonimetric}^{2d}) = \sum_{\{P\}} e^{-\beta k(P)},$$  \hspace{1cm} (7)

where $k(P)$ is the total curvature of the path (3). In this particular case $k = 0$ the model can be solved exactly [2]. Indeed the high temperature expansion of the partition function is:

$$Z(\beta) = \sum_{\{\sigma\}} \exp(-\beta H_{gonimetric}^{2d}) = (2 \cosh \beta)^N (1 + O(N) + \ldots)$$  \hspace{1cm} (8)

and the free energy is equal to

$$-\beta f = \ln(2 \cosh \beta)$$  \hspace{1cm} (9)

The system is in a completely disordered regime like the one-dimensional Ising model with energy density and the specific heat equal to:

$$u = \frac{<k(P)>}{N^2} = \frac{1 - th\beta}{2}, \quad C = \beta^2(1 - th^2\beta).$$  \hspace{1cm} (10)

The reason why the system is in the disordered regime is connected with the fact that the potential barrier between these vacua is finite like in the 1D Ising model and the tunneling phenomenon through the instantons removes the degeneracy of the ground state and the symmetry restores.

The question to which we would like to address here is the nature of the phase transition in the case $k \neq 0$. The model with $k \neq 0$ can be formulated as an eight vertex model with the weights [14]

$$\omega_1 = 1, \omega_2 = \omega^{4k}, \omega_3 = 1, \omega_5 = \omega_6 = \omega_7 = \omega_8 = \omega$$  \hspace{1cm} (11)

where $\omega = \exp(-\beta)$, and the partition function can be rewritten as a fermionic partition function with quadratic and quartic field operators. Perturbation expansion around quadratic operators ($k = 0$) shows that the system is in the disordered regime for small values of intersection coupling constant $k$.

To confirm this universal behavior of the system for different values of the self-intersection coupling constant $k$ one can use the Monte-Carlo simulation of the spin system (1). Let us consider the case $k = 1$ [4].
\[ H_{gonimetric}^{2d} = -\sum_{\vec{r},\vec{\alpha}} \sigma_{\vec{r}} \sigma_{\vec{r} + \vec{\alpha}} + \frac{1}{2} \sum_{\vec{r},\vec{\alpha},\vec{\beta}} \sigma_{\vec{r}} \sigma_{\vec{r} + \vec{\alpha} + \vec{\beta}} \]  \hspace{1cm} (12)

We performed a simulation of the system on a $100^2$ lattice with periodic boundary conditions. We measured the magnetization $M = \langle \sigma \rangle_{\beta}$, the spin-spin correlation function and the average energy density. We saw that in two dimensions there is no phase transition for any positive temperature in accordance with the arguments above.

In order to measure magnetization we start from a high value of $\beta$ with a completely magnetized state $M = 1$. While we heat up the system, magnetization drops and the remaining fluctuations are around $M = 0$. As we cool down again, we see that the system does become magnetized spontaneously. This is the first similarity encountered between the 2D gonimetric model and 1D Ising.

The second similarity comes from the measurements of the spin-spin correlation function $G(\vec{r}) = \langle \sigma(0), \sigma(\vec{r}) \rangle$ which was seen to drop very fast with distance in the large temperature interval. Our measurements of the critical exponent $\nu$ for the correlation length $\xi$ at $\beta_c = \infty$ indicate that $\nu$ is close to one. The third similarity observed is that typical configurations for high $\beta$ have a simple structure: the islands formed by spins are rectangular blocks, the system becomes more and more ordered as $\beta$ becomes higher. At high temperature the islands have curly boundaries, but still the intersections are very disfavoured. This is a consequence of the high energy cost associated with them.

Again one can explain this disordered phase by analyzing the vacuum structure. As we already mentioned in the case of Hamiltonian (1) with $k \neq 0$ the vacuum is $2 \cdot 2^N$ times degenerated, because flat layers of spins with opposite directions have zero interface energy. The vacuum state of the 2D Ising model is twice degenerated and the potential barrier between those vacua is infinite, while in the gonimetric model (1) the barriers between the vacuum states are finite and are equal to four units of energy. This is the reason why the system is always in a disordered regime. The kinklike excitations which connect those vacua are shown on Fig. 1.

In summary for the two-dimensional system with gonimetric action we have the following properties: (i) The system has strong degeneracy of the vacuum states, similar to gauge or spin glass systems, which are separated by finite potential barriers. There is a tunneling phenomenon between those vacua through instantons. (ii) Consequently, the instantons remove the degeneracy of the ground state and the symmetry restoreds. The system is in the disordered regime for all
non-zero temperatures and for large values of the intersection coupling constant $k$. (iii) The typical configurations of interface are such that the paths of interface are self-avoiding.

3.

In three dimensions the corresponding Hamiltonian is \[ H_{gonihedric}^{3d} = -2k \sum_{\vec{r},\vec{\alpha}} \sigma_{\vec{r}+\vec{\alpha}} + \frac{k}{2} \sum_{\vec{r},\vec{\alpha},\vec{\beta}} \sigma_{\vec{r}+\vec{\alpha}} \sigma_{\vec{r}+\vec{\alpha}+\vec{\beta}} - \frac{1}{2} \sum_{\vec{r},\vec{\alpha},\vec{\beta}} \sigma_{\vec{r}+\vec{\alpha}} \sigma_{\vec{r}+\vec{\alpha}+\vec{\beta}} \]

and the low temperature expansion of the partition function

\[ Z(\beta) = \sum_{\{\sigma\}} \exp(-\beta H_{gonihedric}^{3d}) \]

can be expressed as a sum over surfaces of interface with the amplitude which is proportional to linear size of the surface \[ Z(\beta) = \sum_{\{M\}} e^{-\beta A(M)} \quad A(M) = \sum_{<i,j>} \lambda_{ij} \cdot |\pi - \alpha_{i,j}| \]

where $\lambda_{ij}$ is the length of the surface edges (on the lattice $\lambda$ is equal to lattice size $a$), and $\alpha_{ij}$ is the dihedral angle equal to 0, $\pi/2$ or $\pi$.

The constant $k$ again plays the role of intersection coupling constant and defines the rate of degeneracy of the vacuum. If we take periodic boundary conditions and $k = 0$, then the vacuum is degenerated by the factor $3 \cdot 2^N$ and consists of spin layers of different width in the x, y, and z direction. In the same way as in two dimensions the system simplifies in the symmetric point where the self-intersection coupling constant is zero $k = 0$ \[ H_{gonihedric}^{3d} = -\frac{1}{2} \sum_{\vec{r},\vec{\alpha},\vec{\beta}} \sigma_{\vec{r}+\vec{\alpha}} \sigma_{\vec{r}+\vec{\alpha}+\vec{\beta}} \]

The vacuum is strongly degenerate $2^N \cdot 2^N \cdot 2^N$ times, because now all flat layers in all directions form different vacuum states. The system (17) is highly symmetric and even allows to construct the dual Hamiltonian \[ H_{gonihedric}^{3d} = -\frac{1}{2} \sum_{\vec{r},\vec{\alpha},\vec{\beta}} \sigma_{\vec{r}+\vec{\alpha}} \sigma_{\vec{r}+\vec{\alpha}+\vec{\beta}} \]

Little is known about phase transition point of the three-dimensional system (14) even in supersymmetric point $k = 0$ (17). Curvature representation of the linear action (16) \[ \text{Curvature representation of the linear action (16)} \]

allows to find equivalent representations of the partition function for the system (17) in terms of propagation of the polygon loops or strings $P$ in a given direction with the transition amplitude \[ \exp \left( -k(P) - 2 \cdot l(P) \right) \]
where \( l(P) \) is the length and \( k(P) \) is the absolute curvature of \( P \). The interaction is proportional to the overlapping length of the loops

\[
A_{\text{int}} = l(P_1 \cap P_2).
\]

In the first approximation ignoring the interaction term one can solve the model and see that the system undergoes a second order phase transition at \( \beta_c \) which is similar to 2D Ising ferromagnet and that it describes the propagation of an almost free string of 2D Ising fermions [2].

One can understand the nature of this second order phase transition analyzing the structure of the vacuum. Indeed the potential barrier between those vacua is infinite like in 2D Ising model and is of order \( N \) for the lattice of the size \( N^3 \). Therefore one can have well separated phases of the system at low temperature, but rigorous prove of there existence is still missing. The main problem is to count properly the entropy of the given configurations.

This symmetry breaking phenomenon cannot be uniquely characterized by an order parameter like magnetization, because the number of vacuum states with zero magnetization is equal to

\[
\frac{N!}{((\frac{N}{2})!)^2} \approx \frac{2^N}{\sqrt{N}}
\]

and the probability to have zero magnetization dominates. The degeneracy is less for the vacuum states with nonzero magnetization. The magnetization does not uniquely characterize the system and only partly play the role of an order parameter of the system. One can introduce the full set of order parameters to characterize this phases. Let as denote the vacuum spin configurations of the system by \( \sigma_\mu^{\text{vac}}(\vec{r}) \) where \( \mu = 1, 2, ..., 2^{3N} \), then the generalized magnetization \( M^\mu \) is equal to

\[
M^\mu = < \sum_{\vec{r}} \sigma_\mu^{\text{vac}}(\vec{r}) \cdot \sigma_\vec{r} >
\]

and uniquely characterize the vacuum states and the phase transition. The internal energy, which is equal to the number of corners can also be used as an order parameter.

It is interesting therefore to have Monte-Carlo simulations of the system in three dimensions and to study the nature and the dependence of the phase transition as a function of the self-intersection coupling constant \( k \) in (14). We will consider here the case \( k = 1 \) [4], when

\[
H_{\text{gonihedric}}^{3d} = -2 \sum_{\vec{r}, \vec{\alpha}} \sigma_\vec{r} \sigma_{\vec{r}+\vec{\alpha}} + \frac{1}{2} \sum_{\vec{r}, \vec{\alpha}, \vec{\beta}} \sigma_\vec{r} \sigma_{\vec{r}+\vec{\alpha}+\vec{\beta}}
\]
and the spin system includes only the competing ferromagnetic and antiferromagnetic interaction.

We consider the 3D lattice of size \( N^3 = 24^3 \) and \( 32^3 \). Measuring the energy density

\[
u(\beta) = \frac{\mathcal{A}(M)}{N^3} = \frac{\partial}{\partial \beta}(\beta f(\beta))
\]

we observe the sharp behavior at temperature \( \beta_c \simeq 0.439322 \), see Fig. 2. The two point correlation function drastically increase at critical temperature, see Tab.1. As we approach the transition point from higher temperatures the correlation function blows-up uniformly in all three directions. After critical point, at low temperatures, the correlation function is unisotropic and we observe symmetry breaking phenomena. All energy histograms obtained near \( \beta_c \) have only one pick, see Fig. 3. The typical spin configurations, while we set near critical point, describe large surfaces without self-intersections.

We came to the conclusion that the system in three dimensions undergoes a second order phase transition and therefore may have well defined continuum limit. For a rigorous proof of this picture one should have a more detailed study of the phenomena near the critical point. We still don’t know the critical indices of the model to distinguish them from the ones in 3D Ising model.

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Figure caption

Fig. 1. (a) one of the vacuum states, (b) excitation - the kink of energy four, (c) new vacuum after transition.

Fig. 2. Total number of corners as a function of $\beta$ in $32^3$ 3d lattice, at $k = 1$.

Tab. 1 Correlation function at $\beta = 0.439320$.

Fig. 3. Energy histogram at $\beta = 0.439322$
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