Multiqubit entanglement witness

Lin Chen and Yi-Xin Chen
Zhejiang Institute of Modern Physics, Zhejiang University, Hangzhou 310027, China

We introduce a feasible method of constructing the entanglement witness that detects the genuine entanglement of a given pure multiqubit state. We illustrate our method in the scenario of constructing the witnesses for the multiqubit states that are broadly theoretically and experimentally investigated. It is shown that our method can construct the effective witnesses for experiments. We also investigate the entanglement detection of symmetric states and mixed states.

I. INTRODUCTION

Quantum nonlocality is one of the most remarkable features distinguishing the quantum and classical world [1]. The “true” nonlocality predicts the existence of entanglement, which has been proved an extensively useful quantum resource in quantum information theory (QIT) for a review see [2, 3]. Here we emphasize the word “true” in the sense that it is not a kind of classical correlation, and one cannot prepare it through classical simulation [4]. Quantum nonlocality has been verified in some recent experiments [5].

Motivated by the further understanding of quantum nonlocality and novel quantum-information processing such as quantum cryptography [6], we think it is a meaningful and important job to tell whether a quantum state is entangled or separable (classically correlated). Generally, this is difficult, even for pure states of multipartite system. Several useful theories have been founded in this context. The Peres-Horodecki criterion gives a necessary condition on which a bipartite state is separable, and it is also sufficient for the states in the $2 \times 2$ or $2 \times 3$ Hilbert spaces [7]. A sufficient condition verifying entanglement can be obtained via the violation of Bell inequality [8]. Although the above approaches as well as the further investigation of them [3, 10], can effectively detect entanglement of many bipartite states, there exist some cases in which we need new tools for entanglement detection [11]. In addition, one will face a more puzzling situation when applying these theories to multipartite states.

On the other hand, the entanglement witness (EW) was introduced as a sufficient condition on which one can learn a given state is entangled or not [12]. The EW is an Hermitian observable, so it could be used for the experimental demonstration of entanglement of particular system ( notice the partial transpose is not completely positive and it cannot be physically realizable [13] ). Unlike the Peres-Horodecki criterion and Bell inequality detecting entanglement in a regular way, one may construct different types of witnesses for a given state. Adopting the EW for entanglement detection is thus more flexible than using the former techniques. By choosing appropriate witness for a target state, verifying its entanglement via the present experimental techniques is likely.

In past years, many EWs have been constructed for different families of entangled states [12, 13, 14, 15, 16, 17, 18, 19]. In particular, much attentions have been paid to EWs for entangled multiqubit states due to two main reasons. First, it has been shown that the multiqubit entanglement lies at the very heart of quantum-information processing, such as quantum teleportation and dense coding [20], error correction [21], quantum telecloning [22] and quantum computation [23, 24]. Second, Many kinds of multiqubit entangled states have been realized in recent experiments, like the Greenberger-Horne-Zeilinger (GHZ) [25, 26, 27, 28], W [29, 30, 31], cluster [28, 32] and some special multipartite states [33, 34], by using of spontaneous parametric down-conversion (SPDC) [35] and tomography [36]. The theoretical and experimental progress indicate that the multiqubit state could be an applicable and promising quantum resource for the novel tasks in QIT. In this case, it is important to give a further investigation of EWs for multiqubit entanglement.

In this paper, we propose a feasible method of constructing the EW that detects a pure genuine entangled multiqubit state $|\psi\rangle_N$ of $N$ parties, i.e., the state whose reduced density operator of any subsystem has the rank larger than 1. We do it by showing that the state $|\psi\rangle_N$ can be converted into a state of standard form through some local invertible local operators (ILOs) $A_0 \otimes \cdots \otimes A_{N-1}$ with each nonsingular operator $A_k, k = 0, ..., N - 1$ acting on the corresponding subsystem of $|\psi\rangle_N$. Then, it is easy to construct the EW for the state of standard form, and we can obtain the witness for the original state $|\psi\rangle_N$ via the operators $A_0 \otimes \cdots \otimes A_{N-1}$. Our method is generally applicable to the multiqubit state of genuine entanglement, and it does not require the full knowledge of some states. Furthermore, the proposed EW here for any state $|\psi\rangle_N$ can be measured by at most $N^2 - N + 1$ local devices in experiment. Compare to the existing method requiring an exponentially increasing number of measuring devices with $N$ [17], our method essentially reduces the necessary experimental effort. We then respectively construct the EWs for the states which are so far theoretically and experimentally investigated, including the two-qubit, three-qubit, GHZ, W, the four-photon state $|\Psi^{(4)}\rangle$ [33], the four-photon cluster state [32] and the four-photon Dicke state [34]. All the EWs are applicable to experiment. We also make a study of symmetric multiqubit state on entanglement detection. Finally, we apply our method to detect mixed state entanglement.

Our paper is organized as follows. In Sec. II we develop the method of constructing the EW detecting a
pure genuine entangled multiqubit state. By using of this method and other techniques, we present the EWs for different states and show their effect when used in an experiment in Sec. III. In addition, we investigate the symmetric state and propose some application of our method to mixed states. We present our conclusion in Sec. IV.

II. CONSTRUCTION OF ENTANGLEMENT WITNESS FOR MULTIQUBIT STATE

Let us start by recalling the definition of the EW\[12\]. An entanglement witness operator $W$ is an Hermitian observable which has non-negative expectation values for all separable states, and thus the entanglement of a particular state is indicated through the negative expectation value. That is,

$$\text{Tr}(W\rho) \geq 0, \quad \rho \in S,$$

(1)

where $S$ denotes the set of separable states and

$$\text{Tr}(W\sigma) < 0$$

(2)

for some genuine entangled state $\sigma$. Since there are other forms of multipartite states, e.g., the biseparable state, the tri-separable state and so on \[17\], we use the word “genuine” to distinguish them from our target states. We only consider the pure genuine entangled multiqubit states in this paper.

A universal EW detecting genuine entanglement close to $|\psi\rangle$ has been constructed by \[17\],

$$W_c = cI - |\psi\rangle\langle\psi|, \quad c = \max_{|\phi\rangle \in B} |\langle\phi|\psi\rangle|^2,$$  

(3)

where $I$ denotes the identity operator and $B$ represents the set of biseparable states. For general $N$-partite states, determining the value of $c$ is difficult, since one has to find out the maximal square of the Schmidt coefficient over $2^{N-1} - 1$ possible bipartitions of the state $|\psi\rangle$ \[17\]. In addition, the witness $W_c$ often needs an exponentially increasing number of measuring devices \[18\], so more experimental effort is required. Taken in this sense, constructing the witness $W_c$ is a universal, but not always applicable method for entanglement detection.

The concept of ILOs has been firstly introduced to entanglement manipulation under the criterion of stochastic local operation and classical communication (SLOCC) \[37, 38\]. It is the essential property of the ILOs that an entangled or separable state will remain entangled or separable after being operated by some ILOs. The following lemma is thus easily derived from the property.

Lemma 1. Let $W$ be an EW for some $N$-partite genuine entangled state, and $A_0, \ldots, A_{N-1}$ ILOs. Then \(W' = \otimes_{i=0}^{[N-1]} A_iW \otimes \prod_{i=0}^{[N-1]} A_i^{-1}\) is also an EW detecting some state of $N$-partite genuine entanglement. Concretely, if the state $\rho$ is detected by $W$, then the state $\rho' = \otimes_{i=0}^{[N-1]} (A_i^{-1})^* \rho \otimes \prod_{i=0}^{[N-1]} A_i^{-1}$ is detected by $W'$.\[36\]

Such conclusion has been used for entanglement detection recently \[31, 34\]. Clearly, the state $\rho'$ is also completely entangled due to ILOs. Lemma 1 implies that if two states of genuine entanglement are equivalent under SLOCC, then one can derive the EW for one of them from the other. So the existing achievements in entanglement manipulation are helpful to construct the EW for genuine entanglement. Since what interests us is the construction of multiqubit witness, a direct way to do it is to find out all different kinds of multiqubit states under SLOCC. Despite of a few results in this context \[37, 39, 40, 41\], it is impossible to generally catalog the multiqubit states due to sophisticated mathematics. In fact, there is no necessity to find out the classification of multiqubit states, since we are only concerned about the genuine entanglement.

For the $N$-partite $W$ state

$$|W_N\rangle = \frac{1}{\sqrt{N}} \left(|0,0,\ldots,0\rangle + |0,1,\ldots,0\rangle + \cdots + |0,0,\ldots,1\rangle\right),$$  

(4)

its witness turns out to be \[17\]

$$W_{W_N} = \frac{N-1}{N}I - |W_N\rangle\langle W_N|.$$  

(5)

It was first pointed out by Dür et al \[37\] that the $W$ state has a kind of “robust” entanglement in the sense that the state remains entangled after losing some particles of the system. However, we are interested in the witness of more general entangled state. Let us consider the state

$$|\Phi_N\rangle \equiv a_1,0|1,0,\ldots,0\rangle + a_{1,1}|0,1,\ldots,0\rangle + \cdots + a_{1,N-1}|0,0,\ldots,0\rangle + \sum_{k=0}^{(N)} a_{2,k}P_k(|1,1,0,\ldots,0\rangle) + \sum_{k=0}^{(N)-1} a_{3,k}P_k(|1,1,1,0,\ldots,0\rangle) + \cdots + a_{N,0}|1,1,\ldots,1\rangle, a_{1,k} \neq 0, k \in [0, N-1],$$  

(6)

where $\{P_k\}$ denotes the set of all distinct permutations of the spins. This state contains another kind of “robust” entanglement in the sense that the state contains another kind of “robust” entanglement in the sense that the state contains another kind of “robust” entanglement in the sense that it is always fully entangled. The state $|\Phi_N\rangle$ plays the central role in our work and we call it the standard multiqubit (SMQ) state. The SMQ state actually represents a family of multiqubit states, since the constraint on it is to keep the coefficients $a_{1,k}, k = 0, \ldots, N-1$ nonvanishing. We have written the terms corresponding to the coefficients $a_{1,k}, k = 0, \ldots, N-1$, e.g., $a_{1,0}$’s term is $|1,0,\ldots,0\rangle$, $a_{1,1}$’s term is $|0,1,\ldots,0\rangle$, etc.

It is easy to show that the SMQ state must be fully entangled. We rewrite $|\Phi_N\rangle$ with respect to an arbitrary
The bipartition of the system,
\[ |\Phi_N\rangle = \left( a_{1,0}^{*} |1,0,0,\ldots,0\rangle + \cdots + a_{1,m-1}^{*} |0,\ldots,0,1\rangle \right) \otimes \left( |0,0,\ldots,0\rangle + (a_{1,m}^{*} |0,0,\ldots,0\rangle + |X_1\rangle) \otimes |1,0,0,\ldots,0\rangle + |X_2\rangle \otimes |0,1,0,\ldots,0\rangle + \cdots + |X_{2^{N-m}-1}\rangle \otimes |1,1,\ldots,1\rangle \right). \]

Here, the coefficients \( a_{i,k} \), \( k = 0, \ldots, m \) come from a permutation of the initial coefficients \( a_{1,k} \), \( k = 0, \ldots, m \), due to the bipartition. Notice the two states \( |X_0\rangle \) and \( |X_1\rangle \) do not contain the term \( |0,0,\ldots,0\rangle \), which implies the local rank of the bipartite system is not less than 2. So the SMQ state is always fully entangled.

It seems difficult to seek an EW for a generic SMQ state by using of the existing method in [17]. We have found a way to construct the EWs for SMQ states via a special adjustment of the witness \( W_{SMQ} \).

**Lemma 2.** Given an SMQ state \( |\Phi_N\rangle \), it can be detected by a family of witnesses \( W_{SMQ} = \otimes \prod_{k=0}^{N-1} \left( \begin{array}{cc} 1 & 0 \\ 0 & \frac{1}{\sqrt{2}} \end{array} \right) \). \( W_{SMQ} \) is a positive number satisfying the SMQ-inequality
\[ \left| a_{1,0}^{-1} a_{1,1}^{-1} \cdots a_{1,N-1}^{-1} \right|^{2} b^{2N-2} + \left( \left| a_{1,0}^{-1} a_{1,1}^{-1} \cdots a_{1,N-2}^{-1} \right|^2 \right)^{b^{2N-4}} + \cdots + \left| a_{1,0}^{-1} a_{1,1}^{-1} \cdots a_{1,N-1}^{-1} \right|^{2} b^{2N-4} \]
\[ + \left| a_{2,0}^{-1} a_{2,1}^{-1} a_{1,1}^{-1} \cdots a_{1,N-1}^{-1} \right|^{2} b^{2N-4} + \cdots + \left| a_{2,0}^{-1} a_{2,1}^{-1} a_{1,1}^{-1} \cdots a_{1,N-1}^{-1} \right|^{2} b^{2N-4} < \frac{N}{N-1}. \]

Here, \( |a_{m,n}|^2 \) appears in the term \( b^{2m-2} \) in the polynomial of the left hand side (l.h.s. of the inequality). The extra \( |a_{1,k}|^{-2} \) is determined by the places of \( |1\rangle \)'s in the term of the state. For example, \( a_{1,0}^{-1} \)'s term is \( [1,1,\ldots,1] \), thus the coefficient of \( b^{2N-2} \) is \( |a_{1,0}^{-1} a_{1,1}^{-1} \cdots a_{1,N-2}^{-1}|^2 \); \( a_{1,1}^{-1} \)'s term is \( [1,1,\ldots,1,0] \), thus the coefficient of \( b^{2N-4} \) is \( |a_{1,0}^{-1} a_{1,1}^{-1} \cdots a_{1,N-2}^{-1}|^2 \), etc.

**Proof.** First, the operator \( W_{SMQ} \) is an EW due to lemma 1. To show it indeed detects the entanglement of \( |\Phi_N\rangle \), it suffices to show the expectation value is negative, namely \( \langle \Phi_N | W_{SMQ} | \Phi_N \rangle < 0 \). This is equivalent to the SMQ-inequality by some simple calculation.

One can derive the upper bound \( b_{upp} \) of \( b \) from this inequality. An arbitrary choice of \( b \in (0, b_{upp}) \) leads to an EW for the given SMQ state. We provide a feasible method for solving the SMQ-inequality. Suppose the state \( |\Phi_N\rangle \) is normalized and it is always able to set \( b \leq 1 \) beforehand. Then the l.h.s. of the SMQ-inequality is not more than
\[ \left| a_{1,0}^{-1} a_{1,1}^{-1} \cdots a_{1,N-1}^{-1} \right|^{2} \sum_{j=2}^{N} \left| a_{i,j} \right|^2 b^{j-2} \]
\[ \left[ \prod_{i=0}^{N-1} |a_{i,i}|^{-2} (1 - \sum_{j=0}^{N-1} |a_{i,j}|^2) \right] b^{N-1} < \frac{N}{N-1}. \]

This inequality is easily solvable and the solution of \( b \) from it must make the SMQ-inequality hold. Specially when the SMQ state has merely the terms \( P_l((1,0,\ldots,0), l = 0,\ldots,N-1) \) and the permutation of spins means \( \sum_{i=0}^{N-1} a_{i}P_l((1,0,\ldots,0)) = |1,0,\ldots,0\rangle, P_l((1,0,\ldots,0)) = |0,1,\ldots,0\rangle, \ldots, P_{N-1}((1,0,\ldots,0)) = |0,0,\ldots,1\rangle \), the l.h.s. of the SMQ-inequality equals zero. Hence, any value \( b > 0 \) leads to a witness \( W_{SMQ} \) of this special state. In fact, it is easy to see that the SMQ state becomes a ‘pseudo’ W state \( \sum_{i=0}^{N-1} a_{i}P_l((1,0,\ldots,0)) \) when all other terms do not exist, so its witness must be \( W_{SMQ} \) regardless of the change of \( b \). It is also likely to find out other efficient ways to solve the SMQ-inequality, according to the specific form of given state.

There is another interesting issue we can refer to here. As we know, it usually requires the full knowledge of the state to be detected [42]. We regard it does hold in the following context for simplicity. Even so, the SMQ-inequality indicates we can analytically obtain the upper bound of \( b \), if we are merely aware of the content of coefficients \( a_{1,n}, n = 0,\ldots,N-1 \). So our method can construct the EWs for the situations in which a little information is provided for the target SMQ states, which is helpful to the experimental implementation of multiquit state by using of tomography [26, 30, 37].

Clearly, the set of SMQ states doesn’t include all genuine entangled multiqubit states, e.g., the GHZ state. As mentioned above, an SMQ state is always fully entangled. In what follows we consider the inverse question: is every genuine entangled multiqubit state can be transformed into the SMQ state by some ILOs? If so, it is then feasible to construct the EWs detecting genuine entanglement by means of lemma 1 and 2. Suppose the general multiqubit state is
\[ |\Psi_N\rangle = \sum_{i_0,i_1,\ldots,i_{N-1}} c_{i_0,i_1,\ldots,i_{N-1}} |i_0,i_1,\ldots,i_{N-1}\rangle \]
and the operator performed on it is
\[ V_N = \otimes \prod_{k=0}^{N-1} \left( \alpha_{k0} \alpha_{k1} \alpha_{k2} \right), \text{det}V_N \neq 0. \]

We have the following result.

**Theorem 1.** Any genuine entangled multiqubit state \( |\Psi_N\rangle \) can be converted into an SMQ state \( |\Phi_N\rangle \) by some ILOs \( V_N \). The EW detecting the state \( |\Psi_N\rangle \) is \( V_N^{-1}W_{SMQ}V_N \).

**Proof.** See appendix.

We have given a general method of constructing the EWs detecting genuine entanglement. If a given multiqubit state is an SMQ state, one construct its EW by virtue of lemma 2. On the other hand if a state \( |\Psi_N\rangle \) is not the SMQ state, one first transform it into an SMQ state by some ILOs \( V_N \), whose witness can be constructed by lemma 1 and lemma 2. In the appendix, we have provided a method of finding out the
ILOs $V_N$ in the first case in terms of $f_k$ and $g_k$, where we set $\alpha_{k,0} = x^{5k}$, $\alpha_{k,1} = x^{5k+N-1}$, $k = 1, \ldots, N-1$ (the coefficient of $P_l([1,0,\ldots,0])$ in $V_N[\Psi_N]$ can be written as $\alpha_{k,2}f_k + \alpha_{k,3}g_k$, $k = 0, 1, \ldots, N-1$, see appendix). One can thus set the variable $x = e^{i\theta}$, so that the large exponentials $5^k, k = 1,2,\ldots$ can be removed by the phase. The weakness of this method is that the finding of $\theta$ which makes the coefficients of $P_l([1,0,\ldots,0]), l = 0, \ldots, N-1$ nonvanishing, becomes difficult when the number of coefficients $c_{0,\ldots,i}N-1$ is large.

There seems no general method for this case since the distribution of coefficients is stochastic, so it is the concrete situation that uniquely determines how we create the ILOs $V_N$, e.g., when the coefficients are regularly disposed. Nevertheless, one can try a tentative method like this. Notice the similarity of coefficients of $[0,0,\ldots,0]$ and $P_l([1,0,\ldots,0]), l = 0, \ldots, N-1$ in the appendix, one can set $\alpha_{k,0} = 0, \alpha_{k,1} = 1, k = 1, \ldots, N-1$. The criterion of SMQ states then requires

$$c_{0,1,\ldots,1}0_0 + c_{1,1,\ldots,1}0_1 = 0,$$

$$c_{0,0,1,\ldots,1}0_0 + c_{1,0,1,\ldots,1}0_1 \neq 0, \ldots,$$

$$c_{0,1,\ldots,1}0_0 + c_{1,1,\ldots,1}0_1 \neq 0. \quad (12)$$

When the coefficients satisfy these relations, one can easily find out the ILOs $V_N$. If unfortunately, there is four proportional coefficients such as $c_{0,1,\ldots,1}c_{1,1,\ldots,1} = c_{0,0,1,\ldots,1}c_{1,0,1,\ldots,1}$, the relations cannot hold. Then one can set $\alpha_{k,0} = 0, \alpha_{k,1} = 1, k = 2,3,\ldots, N-1$, and carry out a similar procedure. In addition, one can set $\alpha_{k,0} = 0, \alpha_{k,1} = 1, k = 2,3,\ldots, N-1$, which increases the number of free variables. It reduces the possibility generating proportional coefficients. The character of this method is the necessary amount of calculations is small at the risk of failure. Generally, it is feasible to construct an EW for the genuine entangled multiqubit state via our methods.

### III. Entanglement Detection of Practical Multiqubit States

We have proposed a theoretical approach to construct EWs in the preceding section. Now, let us move to investigate its practical use for multiqubit states. Two main characters of a practical witness $W$ in experiments are that, how many measuring devices are necessary for its realization and how much it tolerates noise [12]. The basis $\sigma_{x,z}$ states then requires $5$ settings as shown above. However, such a decomposition is useful to detect the SMQ states.

The basis $\{|a_{\nu}^{(k,m)}\rangle\}$ are orthogonal vectors for a fixed $(k,m)$. $m$ is the ordinal number of party, which we will omit if unnecessary.

It has been shown that each observable $M_k$ can be measured with one local measuring device in experiments [12]. So when the number $K$ reaches its minimum, we say the decomposition of $W$ is optimal. On the other hand, suffering the noise from the decoherence coupling with the environment is always unavoidable when implementing quantum-information tasks. The present experimental techniques require that a superior EW should be considerably resistant against the noise.

The problem of optimally decomposing a given witness is technically difficult, and it has been addressed by some authors [12,15,16,17,18]. For example, the optimal decomposition of the witness $W_{W_3}$ has been found as follows [16].

$$W_{W_3} = \frac{2}{3} I - |W_3\rangle\langle W_3|$$

According to the definition of decomposition of the witness, 5 local measuring devices are required for the realization of this witness, namely $\sigma_{x,z}^3 (I + \sigma_{+} + \sigma_{z})^3$.

The superscripts $(i), i = 0,\ldots, N-1$ represent the parties. The number of measuring settings in this decomposition is $N^2 - N + 1$, namely $\sigma_{x,z}^i (I + \sigma_{+} + \sigma_{z})^3 \prod_{k=0, k \neq i, j}^{N-1} \prod_{k=0, k \neq i, j}^{N-1} (|0\rangle\langle 0|^k)$.

The universal decomposition of $W_{W_N}$ requires more settings than the optimal one, e.g., when $N = 3$, it requires 7, while the optimal one requires 5 settings as shown above. However, such a decomposition is useful to detect the SMQ states.
Recall the form of the witness \( W_{SMQ} \) in the lemma 2. It is easy to see that the entanglement close to an SMQ state \( |\Phi_N) \) can be detected by using of the universal decomposition, namely by \( N^2 - N + 1 \) measuring devices.

\[
\sigma_z^{\otimes N}, \left[ \otimes_{k=0}^{N-1} \begin{pmatrix} 1 & 0 \\ 0 & \alpha_{k} \end{pmatrix} \right] (k)^{\dagger} \cdot \left[ \otimes_{k=0}^{N-1} \begin{pmatrix} 1 & 0 \\ 0 & \alpha_{k} \end{pmatrix} \right] \cdot \left[ \otimes_{k=0}^{N-1} \begin{pmatrix} 1 & 0 \\ 0 & \alpha_{k} \end{pmatrix} \right] ^{(k)}
\]

\[
\left[ \otimes_{k=0}^{N-1} \begin{pmatrix} 1 & 0 \\ 0 & \alpha_{k} \end{pmatrix} \right] \cdot \left[ \otimes_{k=0}^{N-1} \begin{pmatrix} 1 & 0 \\ 0 & \alpha_{k} \end{pmatrix} \right] \cdot \left[ \otimes_{k=0}^{N-1} \begin{pmatrix} 1 & 0 \\ 0 & \alpha_{k} \end{pmatrix} \right] ^{(k)}
\]

\[
\left[ \otimes_{k=0}^{N-1} \begin{pmatrix} 1 & 0 \\ 0 & \alpha_{k} \end{pmatrix} \right] \cdot \left[ \otimes_{k=0}^{N-1} \begin{pmatrix} 1 & 0 \\ 0 & \alpha_{k} \end{pmatrix} \right] \cdot \left[ \otimes_{k=0}^{N-1} \begin{pmatrix} 1 & 0 \\ 0 & \alpha_{k} \end{pmatrix} \right] ^{(k)}
\]

\[
i > j = 0, ..., N-1.
\]

The parameter \( b \) is determined by the SMQ-inequality. As there may be better method of decomposing the witness \( W_{SMQ} \), we assert that one can detect the entanglement close to an SMQ state through at most \( N^2 - N + 1 \) measuring devices. In addition, this result also applies to the general multiqubit state \( |\Psi_N) \), which is shown to be converted into some SMQ state \( |\Phi_N) \) by local unitary operations. According to theorem 1, there exist some ILOs

\[
V_N = \otimes_{k=0}^{N-1} \begin{pmatrix} \alpha_{k0} \alpha_{k1} \\ \overline{\alpha_{k0}} \overline{\alpha_{k1}} \end{pmatrix} \] such that \( |\Phi_N) = V_N |\Psi_N) \).

The form of SMQ state is unchanged under the ILOs

\[
V_N' = \otimes_{k=0}^{N-1} \begin{pmatrix} x_k & 0 \\ y_k & 1 \end{pmatrix}, \forall x_k \neq 0.
\]

\[
x_k = \frac{\alpha_{k0} \alpha_{k2} - \alpha_{k0} \alpha_{k3}}{\sqrt{\alpha_{k0}^2 + \alpha_{k1}^2}}, \quad y_k = \frac{-\alpha_{k0} \alpha_{k2} - \alpha_{k3} \alpha_{k0}}{\sqrt{\alpha_{k0}^2 + \alpha_{k1}^2}}
\]

for \( k = 0, 1, ..., N-1 \). It is easy to check \( V_N' V_N \) is unitary after dividing a constant \( a \). We can thus transform \( |\Psi_N) \) into an SMQ state by the unitary operation \( V_N'^{\dagger} V_N / a \). Because the necessary number of devices measuring an EW is invariant under the local unitarity, the following result is hence derived from lemma 1.

**Theorem 2.** An arbitrary genuine entangled multiqubit state \( |\Psi_N) \) can be transformed into an SMQ state by local unitary transformation \( V_N \). The multiqubit entanglement close \( |\Psi_N) \) is detected by a witness \( V_N'^{\dagger} W_{SMQ} V_N \), which can be measured by at most \( N^2 - N + 1 \) local measuring devices \( V_N'^{\dagger} M_k V_N \) with the settings \( M_k, k = 1, ..., N^2 - N + 1 \) expressed in (16).

Theorem 2 asserts that we are able to detect the genuine entanglement of multiqubit state by not more than \( N^2 - N + 1 \) devices, which is a polynomial of the party number \( N \). Compare to the existing result in [17] which often requires an exponentially increasing number of devices, we have essentially reduced the necessary experimental effort. One can use the given method in Sec. II to find out the unitary transformation \( V_N \), or use other tricks based on the specific situation.

To give an example, we consider the experimentally realizable state \( \Psi(4) \):

\[
\Psi(4) = \frac{1}{\sqrt{3}} \left[ |0011) + |1100) - \frac{1}{2} (|0110) + |1001) + |0101) + |1010) \right]
\]

whose entanglement has been detected by 15 measuring devices [17]. However, our method shows that at most \( 4^2 - 4 + 1 = 13 \) devices are enough to measure the EW for this state. Finding the unitary transformation \( V_N \) is easy, like

\[
\left( \frac{1}{\sqrt{2}} \frac{1}{\sqrt{2}} \right) \left( \frac{1}{\sqrt{2}} \frac{1}{\sqrt{2}} \right) \left( \frac{\sqrt{2}}{\sqrt{5}} \frac{\sqrt{2}}{\sqrt{5}} \right) \left( \frac{3}{5} \frac{4}{5} \right) \left( \frac{4}{5} \frac{3}{5} \right) \left( \frac{1}{0} \frac{0}{1} \right).
\]

Of course one can choose other operation \( V_N \) transforming \( |\Psi(4)\rangle \) into an SMQ state, for the robustness of witness against the noise is also considered when the necessary number of devices is unchanged. In addition, we provide a simple way to reduce the number of measuring devices. Rewrite the witness \( W_{W_4} \) as follows

\[
W_{W_4} = \frac{3}{4} I - |W_4)\langle W_4|
\]

\[
= \frac{3}{4} I - \frac{1}{4} \left[ |W_3)\langle W_3| (0) + (0010) \langle 0001| \right.
+ (0001) \langle 0010| + (0100) \langle 0001| + (0001) \langle 0010| \right.
+ \left. (0100) \langle 0001| + (0001) \langle 0010| + (0010) \langle 0001| \right]
\]

We replace the projector \( |W_3)\langle W_3| \) by \( \frac{3}{4} I - W_{W_3}, \) which has the decomposition requiring 5 devices including \( \sigma_z^{\otimes 3} \) (see the last page). The operator in each bracket of equation (19) can be measured by 2 devices due to the identity \( |0,1)\langle 1,0| + |1,0)\langle 0,1| = \frac{1}{2} (\sigma_x \sigma_z + \sigma_y \sigma_y). \) So the witness \( W_{W_4} \) has a better decomposition containing only 11 correlated devices’ settings, and so does the 4-body witness \( W_{SMQ}. \) Since local unitary operations do not change the necessary number of devices, one can detect the entanglement of \( |\Psi(4)\rangle \) by 11 devices. Similarly, one can detect the entanglement of 4-qubit cluster state [32] by using of 11 devices. In what follows we give more examples to illustrate our techniques.

(i) The 2-qubit state \( |\Psi\rangle_{AB} \). It is a kind of state that has been intensively investigated, both theoretically and experimentally [13, 44]. One can always write the state as

\[
|\Psi\rangle_{AB} = U_A \otimes U_B (\cos \theta |00) + \sin \theta |11)\]

\[
= U_A (V_A)^{-1} \otimes U_B (V_B)^{-1} \cdot [V_A \otimes V_B (\cos \theta |00) + \sin \theta |11)]
\]

where unitary operators \( U_A \) and \( U_B \) are easily known and \( V_A = V_B = \left( \begin{array}{cc} 1 & i \sqrt{\cos \theta} \\ 0 & 1 \end{array} \right) \), so the state in the square bracket is an SMQ state. Then we can find out the unitary transformation \( V_N \) based on the ILOs \( U_A (V_A)^{-1} \).
and $U_B(V_B)^{-1}$. It implies the entanglement of every 2-qubit state can be detected via at most $2^2 - 2 + 1 = 3$ measuring devices. This reaches the same effect as the witness $(|\Psi\rangle_A|\Psi\rangle_B)^T_A$ in [13]. In addition, we investigate the robustness of our witness against white noise. Let $\theta \in [0, \pi/4]$. The analytical calculation shows our witness detects a state $p|1/4 + (1-p)|\Psi\rangle_A|\Psi\rangle_B$ with
\[ p < \frac{8\cos^2\theta \sin^2\theta}{2 - \cos4\theta + \sqrt{2 - \cos 4\theta - 2\sin 2\theta}}. \] (21)

This upper bound is less than $\frac{2\sin 2\theta}{1 + 2\sin 2\theta}$, which is the optimal value of noise tolerated by the witness $(|\Psi\rangle_A|\Psi\rangle_B)^T_A$. On the other hand, the upper bound is better than $\frac{1}{4} \sin^2 \theta$, which is the optimal value tolerated by the witness $W_\mathcal{C}$ in [15].

(ii.) The 3-qubit state $|\Psi\rangle_{ABC}$. It is more sophisticated than the 2-qubit state in configuration. There are two types of genuine entanglement here, namely, the GHZ state and W state. The ILOs converting the 3-qubit states into them can be found in [37, 45]. The W state is already an SMQ state, so it suffices to find out the ILOs converting the GHZ state into an SMQ state. This is easily done by, e.g., $\left( \begin{array}{ccc} 1 & -1 & 0 \\ 0 & 1 & 1 \end{array} \right)^{\otimes 3}$. By using of these ILOs, we can construct the unitary transformation $V_N$. As the witness $W_{SMQ}$ can be measured by 5 devices due to the optimal decomposition of $W_{W_3}$, we have

Lemma 3. The genuine entanglement of every 3-qubit state can be detected by 5 measuring devices, whose specific forms are easily obtained in terms of $W_{W_3}$ and the unitary operator $V_N$. ■

(iii.) The $N$-partite GHZ state. An optimal witness for this celebrated state has been constructed in a recent paper [18], which requires two local measurements. Here we propose another method of constructing a witness $W$ for the GHZ state. Let $V_N = \left( \begin{array}{c} \sqrt{2}/2 & -\sqrt{2}/2 & \cdots & -(1-N^{-1}) \end{array} \right)^{\otimes N}$, which transforms the GHZ state into an SMQ state. So at most $N^2 - N + 1$ measuring devices are enough to detect the witness $W_{SMQ}$ for the GHZ state. Specialy, the number of devices can be reduced to 5 when $N = 3$ in view of the optimal decomposition of $W_{W_3}$. This is better than another usual witness $W_{1/2} = \frac{1}{2}I - |\text{GHZ}\rangle\langle\text{GHZ}|$, which is decomposed into $2^N - 1$ devices [18]. The witness $W$ can detect a state $|\Psi\rangle_A|\Psi\rangle_B|\text{GHZ}\rangle$ with the white noise up to, e.g., $p \approx 0.3336$ when $N = 3$. The ability against the noise gradually declines with an increasing $N$, which is similar to the EWs $W_{1/2}$ and that in [18].

(iv.) The $N$-partite W state. It contains the robust entanglement [37], and there has been recently remarkable progress in experimental preparation [29, 30, 31]. The usual witness for the $N$-partite W state is $W_{W_N}$ given by equation (5). We have presented a set of devices measuring the witness $W_{W_N}$ (see (15) below), which detects a state $p|\Psi\rangle_A|\Psi\rangle_B|\text{W}\rangle\langle\text{W}|$ with white noise up to $p_W < N^{-1}(1 - 2^{-N})$. So the witness $W_{W_N}$ becomes weaker and weaker against the noise, as the number of parties in the system increases. To overcome this shortcoming, we provide another witness $W_{W_N}$ via $W_{SMQ}$, namely
\[ W_{W_N} = \otimes_{k=0}^{N-1} \begin{pmatrix} 1 & 0 \\ 0 & b\sqrt{N} \end{pmatrix} W_{W_N} \otimes \otimes_{k=0}^{N-1} \begin{pmatrix} 1 & 0 \\ 0 & b\sqrt{N} \end{pmatrix}. \] (22)

Then by optimizing $\text{Tr}[\rho(p)W_{W_N}]$ with $b = 1/\sqrt{N^2 - N}$, one can obtain the upper bound
\[ p_{W'} < \frac{2^N}{2N - N + (N - 1)^2 - N N^2}, \] (23)

which monotonically increases from $N = 4$. The upper bound $p_{W'} > p_W$, and it tends to 1 when $N$ becomes very large. So the new witness $W_{W_N}'$ is more robust against the noise when the party number increases, which is different from many existing EWs. On the other hand, the witness $W_{W_N}'$ can be measured by not more than $N^2 - N + 1$ devices presented in (16), with $a_{1,i} = N^{-1/2}, i = 0, ..., N - 1$. In particular, the necessary measuring devices for $W_{W_N}'$ seems not more than that for $W_{W_N}$, e.g., in the case of $N = 3$. So it is more efficient to use the witness $W_{W_N}'$ to detect the W state in experiments.

(v.) The Dicke state $|m, N\rangle$ [46]. It is a kind of symmetric state with the form
\[ |m, N\rangle = \left( \frac{N}{m} \right)^{-1/2} \sum_k P_k(|1, ..., 1, 0, ..., 0\rangle). \] (24)

The excitation number $m$ ranges from 1 to $N - 1$, and the state $|1, N\rangle$ or $|N - 1, N\rangle$ is just the W state. A theoretical method of detecting entanglement of the Dicke state around $m = N/2$ is proposed recently, by using of the witness $W_{W_N}'$ [47]. There is also the latest progress in experiment [34], which present the observation of $|2, 4\rangle$ with a fidelity nearly 0.844 by using of quantum tomography. The Dicke state ($m \in [2, N - 2]$) also has a kind of robust entanglement in the sense that it remains entangled if one of the qubit in the state is projected onto some space. It helps the Dicke state become the source of EPR singlet, as well as the quantum channel in telecloning and teleportation [11]. So the entanglement detection of the Dicke state is a valuable work.

One may construct the EW for $|2, 4\rangle$ by using of the method in [17] or in the present paper, and the latter will require a less amount of experimental effort. Nevertheless, we have found a more efficient way to accomplish the object. We propose the witness
\[ W_{(2,4)} = \otimes_{k=0}^{3} \begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ i/\sqrt{2} & i/\sqrt{2} \end{pmatrix} \cdot \left( 2I - \sigma_z^{\otimes 4} - \frac{1}{4} \prod_{k=1}^{3} (\sigma_z^{(k-1)}\sigma_z^{(k)}) + I \right) \otimes \otimes_{k=0}^{3} \begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ i/\sqrt{2} & i/\sqrt{2} \end{pmatrix}. \] (25)
This is indeed an EW since the operator in the square bracket is the witness \(W^{(\text{GHZ})}\) in \(13\), which can be measured with two measuring devices \(\sigma_x^{(4)}\) and \(\sigma_y^{(4)}\). So it suffices to measure the witness \(W_{(2,4)}\) by virtue of only two devices.

\[
\left[ \begin{array}{cc}
\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\
\frac{i}{\sqrt{2}} & \frac{i}{\sqrt{2}}
\end{array} \right] \begin{array}{cc}
\sigma_x & \\
\sigma_y
\end{array}^{(4)}
\]

and

\[
\left[ \begin{array}{cc}
\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\
\frac{i}{\sqrt{2}} & \frac{i}{\sqrt{2}}
\end{array} \right] \begin{array}{cc}
\sigma_x & \\
\sigma_y
\end{array}^{(4)}
\]

It is easy to check the witness \(W_{(2,4)}\) detects entanglement in the vicinity of state \([2,4]\), especially it detects a state \(p\mathbf{I}/16 + (1-p)[2,4]/2, 2,4\) with white noise up to \(p < 2/9\).

Furthermore, for the generally pure symmetric multiqubit (PSMQ) state, we investigate some properties of them on entanglement detection. A PSMQ state is invariant under the exchange of any two particles in the system \(48\). Based on this property, it is easy to derive that any PSMQ state has the form

\[
|\Psi_N\rangle_{\text{PSMQ}} = \sum_{m=0}^{N} c_m |m,N\rangle,
\]

where we denote the states \(|0,N\rangle = |0\rangle^\otimes N, |N,N\rangle = |1\rangle^\otimes N\). Evidently, the PSMQ state can be fully entangled such as the Dicke state, or fully separable such as the state \(|0,N\rangle\). Then there is a question that whether the PSMQ state can be partially entangled, namely \(|\Psi_N\rangle_{\text{PSMQ}} = \otimes \bigotimes \{|\phi_i\rangle\}^{\otimes N}\) with at least one state \(|\phi_i\rangle\) is fully entangled. To address it, we extract a part \(|\phi_k\rangle \otimes |\phi_i\rangle\) from the PSMQ state. Suppose there is a particle \(A\) in the system \(AC\) of \(|\phi_k\rangle\), and \(B\) in that of \(|\phi_i\rangle\). Then, exchanging \(A\) and \(B\) leads to the particles \(B,C\) are entangled due to the symmetry of the PSMQ state. It contradicts with the precondition, so the PSMQ state cannot be partially entangled.

**Lemma 4.** A PSMQ state is either fully entangled or fully separable.

This character helps construct the EWs for the PSMQ states. One can divide a PSMQ state into two parts and explore whether this bipartite state is entangled. Namely a two-body EW is enough to detect the multiqubit entanglement. On the other hand, since the PSMQ state is separable if and only if it has the form \((a|0\rangle + b|1\rangle)^\otimes N = a^N|0,0,...,0\rangle + a^{N-1}b \sum P_i(|1,0,...,0\rangle) + \cdots + b^N|1,1,...,1\rangle\), a more efficient method of detecting the symmetric state is to observe its coefficients \(c_m, m = 0,...,N\). This can be implemented by quantum tomography and only at most \(N+1\) coefficients are necessarily observed. It is also feasible to construct the witness by using of our method in the appendix, since there are at most \(N+1\) different coefficients in the PSMQ state. This greatly helps determine the operator \(V_N\) that transforms the PSMQ state into an SMQ state. However, it is not easy to create the witness \(W_{\text{PSMQ}}\) for general PSMQ state, though it is highly symmetric \([47]\).

It is interesting that one cannot generalize lemma 4 to the case of mixed symmetric multiqubit (MSMQ) state, whose definition resembles that of PSMQ state. For example, the 3-body MSMQ state \(\rho_1 = \frac{1}{4}(|000\rangle + |111\rangle)(|000\rangle + |111\rangle) + \frac{1}{2}|111\rangle|111\rangle\) is fully entangled since any bipartition of \(\rho_1\) is an entangled state, due to the Peres-Horodecki criterion. On the other hand, the MSMQ state \(\rho_2 = \frac{1}{7}(|000\rangle|000\rangle + \frac{1}{7}|111\rangle|111\rangle\) is fully separable \([4]\). However, we consider the entangled edge state in expression (14) of \([14]\).

\[
\rho_3 = \frac{2}{19} \left( (|000\rangle + |111\rangle)(|000\rangle + |111\rangle) + 2(|001\rangle|010\rangle + |010\rangle|010\rangle + |100\rangle|100\rangle) + \frac{1}{2}(|011\rangle|101\rangle + |101\rangle|101\rangle + |110\rangle|110\rangle) \right),
\]

which is biseparable with respect to any partition. Because \(\rho_3\) is symmetric, lemma 4 does not hold for the case of MSMQ states.

The study shows there is a more abundant content of entanglement detection for mixed states \([19]\), so finally we apply our method to detect the mixed state entanglement. If a given witness \(W\) detects several pure entangled states \(\sigma_i, i = 1,2,...\), then it also detects the mixed state \(\sum \rho_i \sigma_i\). By contrast, we often face a given state \(\sigma = \sum \rho_i \sigma_i\) to be detected, while it is difficult to find out an EW detecting each state \(\sigma_i\) simultaneously, and thus to detect the state \(\sigma\). Nevertheless, the method of detecting SMQ state sometimes works here, since one can provide several EWs for the same SMQ state due to lemma 2. For example, if the state \(\rho_1\) is detected via \(W_{\text{SMQ}}\) and \(V_{\text{SMQ}}V_N\), while the given state has the decomposition form \(\sigma' = \sigma_1 + V_N\sigma_1V_N^\dagger\), then one can detect the state \(\sigma'\) by the witness \(W_{\text{SMQ}}\).

It is feasible to apply this skill to detect more general state \(\rho' = \sum \rho_i \sigma_iV_iV_i^\dagger\), where the state \(\rho\) is detected by many EWs \(V_iV_N^\dagger\text{SMQ}V_i, i = 1,...\). The state \(\rho'\) can be the output of a quantum channel, or more general map taking \(\rho\) as the input state.

In summary, we have found out a decomposition of the witness \(W_{W_N}\). Based on this, it has been shown that not more than \(N^2 - N + 1\) measuring devices are enough to detect the entanglement of any \(N\)-partite SMQ state and furthermore any \(N\)-partite genuine entangled multiqubit state \(|\Psi_N\rangle\). We have explicitly presented the measuring devices for \(|\Psi_N\rangle\). By using these facts, we have constructed the EWs for the entanglement detection of several practical states, including the \(|\Psi_N\rangle\) state, the 2-qubit, 3-qubit, the GHZ, W and the Dicke state. We also discussed the entanglement detection of PSMQ and MSMQ states, as well as the mixed states. One can also apply our method to detect the entanglement of other typical multiqubit states, such as the GHZ-W-type states \([39]\).
IV. CONCLUSIONS

We have proposed a feasible method of constructing the entanglement witness detecting the genuine entanglement of pure multiqubit state. The method is efficient in the sense that it could reduces the theoretical and experimental effort in many cases. Our method can be used to construct the entanglement witnesses for the multiqubit states that are theoretically and experimentally investigated in literatures. It is a problem that how to find out the optimal witness for a given state by means of the proposed techniques, according to the specific requirement of experiment.

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APPENDIX: CONVERTING A MULTIQUBIT STATE INTO AN SMQ STATE

Before expanding our proof, we propose a useful proposition.

Proposition. Suppose \( a > 1 \) is a natural number, \( \{p_1, p_2, \ldots\} \) is a set of integers with \( \forall p_i \in [1, a-1] \) and similarly for the set \( \{q_1, q_2, \ldots\} \). Suppose two series of natural numbers, \( m_0 < m_1 < \cdots < m_k \), \( n_0 < n_1 < \cdots < n_{k-1} \). Then \( \sum_{i=0}^{k-1} p_i a^{m_i} = \sum_{i=0}^{k-1} q_i a^{n_i} \) if and only if \( k = l \) and \( m_i = n_l = m_i, i = 0, \ldots, k-1 \).

Proof. It suffices to verify the necessity. Let \( \sum_{i=0}^{k-1} p_i a^{m_i} = \sum_{i=0}^{k-1} q_i a^{n_i} \), and it is no loss of generality to suppose \( n_j \geq m_0 > n_{j-1} \). By dividing a factor \( a^{m_0} \) on both sides, we have

\[
\sum_{i=0}^{k-1} p_i a^{m_i-m_0} = \sum_{i=0}^{j-1} q_i a^{n_i-m_0} + \sum_{i=j}^{l-1} q_i a^{n_i-m_0}.
\]

Clearly, the l.h.s. of the equation is a natural number, and so is the second term of the r.h.s. of the equation. However, the first term of the r.h.s. is a proper fraction since \( a^{m_0} - \sum_{i=0}^{j-1} q_i a^{n_i} \geq a^{n_j} - \sum_{i=0}^{j-1} q_i a^{n_i} \geq \cdots \geq a^{n_k} - q_k a^{n_0} > 0 \), which contradicts with the equation. So it is only possible that \( m_0 = n_0 \), and hence \( p_0 = q_0 \). Then one obtains a new equation \( \sum_{i=1}^{k-1} p_i a^{m_i} = \sum_{i=1}^{j-1} q_i a^{n_i} \). Repeating the above procedure leads to \( m_1 = n_1 \), and hence \( p_1 = q_1 \). In the same vein one finally verifies the assertion in the proposition.

The proposition indeed asserts that the decomposition of a natural number with respect to some less natural number is unique. This is similar to the case of binary system, i.e., \( N = \sum_i a_i \cdot 2^i, a_i = 0 \) or 1 is of unique decomposition.

From now on we address the problem of converting a multiqubit state \( |\Psi_N\rangle \) into an SMQ state. The state equivalent to \( |\Psi_N\rangle \) under SLOCC is \( V_N \ |\Psi_N\rangle \). To find out its relationship with the SMQ state, we write out some coefficients of terms in \( V_N |\Psi_N\rangle \),

\[
|0, 0, \ldots, 0\rangle : \left( \begin{array}{c} c_{0,0,0,0,0} \\ c_{0,0,0,0,1} \\ \cdots \\ c_{1,1,1,1,1} \end{array} \right)^T \left[ \prod_{k=0}^{N-1} \left( \frac{\alpha_k,0}{\alpha_k,1} \right) \right],
\]

\[
P_k([1,0,\ldots,0]) : \left( \begin{array}{c} c_{0,0,0,0,0} \\ c_{0,0,0,0,1} \\ \cdots \\ c_{1,1,1,1,1} \end{array} \right)^T \left[ \prod_{k=0}^{l-1} \left( \frac{\alpha_k,0}{\alpha_k,1} \right) \prod_{k=l+1}^{N-1} \left( \frac{\alpha_{l},2}{\alpha_{l},3} \right) \right], l = 0, \ldots, N - 1.
\]

Because an SMQ state contains no the term \( |0, 0, \ldots, 0\rangle \), we set

\[
\alpha_{0,0} = - \left( \begin{array}{c} c_{1,0,0,0,0} \\ c_{1,0,0,0,1} \\ \cdots \\ c_{1,1,1,1,1} \end{array} \right)^T \left[ \prod_{k=0}^{N-1} \left( \frac{\alpha_k,0}{\alpha_k,1} \right) \right],
\]

\[
\alpha_{0,1} = \left( \begin{array}{c} c_{0,0,0,0,0} \\ c_{0,0,0,0,1} \\ \cdots \\ c_{0,0,1,1,1} \end{array} \right)^T \left[ \prod_{k=0}^{N-1} \left( \frac{\alpha_k,0}{\alpha_k,1} \right) \right].
\]

Hence, the coefficient of \( P_k([1,0,\ldots,0]) \) can be written as \( \alpha_{k,2} \beta_{k,3} f_k (\alpha_{k,0}, \alpha_{k,1}, \ldots, \alpha_{N-1,0}, \alpha_{N-1,1}) + \alpha_{k,3} g_k (\alpha_{k,0}, \alpha_{k,1}, \ldots, \alpha_{N-1,0}, \alpha_{N-1,1}), k = 0, 1, \ldots, N - 1 \). The equations \( f_k \) and \( g_k \) are the polynomials of the variables \( \alpha_{i,j}, i = 1, \ldots, N - 1, j = 0, 1. \) For example when \( N = 2 \), it holds that \( f_k = \sum_{i,j=0}^{2} \beta_{i,j} \alpha_{i,0} \alpha_{j,1} \) with some constant coefficient \( \beta_{i,j} \). The SMQ state requires that every coefficient of \( P_k([1,0,\ldots,0]) \) is nonvanishing. Because \( \alpha_{k,2}, \alpha_{k,3} \) can be freely determined, we analyze the situation in terms of \( f_k \) and \( g_k \).

For the first case, there is no pair of equations \( f_k \) and \( g_k \) simultaneously identical to zero, namely \( \prod_{k_0, k_1} f_{k_0} g_{k_1} \neq 0 \), \( k_0 \in S_0, k_1 \in S_1, S_0 \cup S_1 = \{ 0, 1, \ldots, N - 1 \} \). This implies not every constant coefficient \( b_{i_1, \ldots, i_{N-2}} \) equals zero. One can choose nonzero variables \( \alpha_{i,j}, i = 1, \ldots, N - 1, j = 0, 1 \) making the product \( \prod_{k_0, k_1} f_{k_0} g_{k_1} \neq 0 \). For example, let \( \alpha_{k,0} = x^{2^k}, \alpha_{k,1} = x^{2^k}, k = 1, \ldots, N - 1, x \) is a variable. Then the powers of \( x \), \( \prod_{k=0}^{N-1} \prod_{k=0}^{N-1} f_{k} g_{k}(x) = 0 \) has a finite number of solutions \( x \in S_2 \). Similar results are applicable to \( \alpha_{0,0} \) and \( \alpha_{0,1} \), which have finite solution sets \( S_3 \) and \( S_4 \), respectively. Then we choose \( \alpha \) \( \not\in S_2 \cup S_3 \cup S_4 \), and suitable \( \alpha_{2,3}, \alpha_{2,3}, k = 0, \ldots, N - 1 \) making every coefficient of \( P_k([1,0,\ldots,0]) \) nonvanishing, as well as the non-singularity of the operator \( V_N \). So we have transformed the initial state into an SMQ state.

For the second case, there is at least one pair of equations \( f_k \) and \( g_k \) simultaneously identical to zero no mat-
ter how the variables $\alpha_{k,0}, \alpha_{k,1}, k = 1, \ldots, N - 1$ change, which means at least one term $P_k(1,0,\ldots,0)$ is always removed. We show in this case that the state $|\Psi_N\rangle$ must be separable. For the case of $f_0 = g_0 \equiv 0$ namely $\alpha_{0,0} = \alpha_{0,1} \equiv 0$, choosing $\alpha_{k,0} = x^5^k, \alpha_{k,1} = x^{5k+N-1}, k = 1, \ldots, N - 1$ leads to that every coefficient $c_{i,0,\ldots,i,\ldots,i,N-1}$ equals zero, due to the proposition. So it is impossible that $f_0 = g_0 \equiv 0$.

On the other hand, there may be the case of $f_k = g_k \equiv 0, k \in [1, N - 1]$. It suffices to investigate the case of $f_1 = g_1 \equiv 0$, and other situations can be similarly dealt with. Taking into account the expressions of $f_1$ and $g_1$ and removing the parameters $\alpha_{0,0}, \alpha_{0,1}$, we have

$$
\begin{align*}
\tilde{c}_{1,0} = \left[ \bigotimes_{k=2}^{N-1} \left( \frac{\alpha_{k,0}}{\alpha_{k,1}} \right) \right] = \tilde{c}_{1,1} = \left[ \bigotimes_{k=2}^{N-1} \left( \frac{\alpha_{k,0}}{\alpha_{k,1}} \right) \right],
\end{align*}
$$

where the $2^{N-2} \times 1$ coefficient vector $\tilde{c}_{i,j}$ represents

$$
\tilde{c}_{i,j} = \begin{pmatrix}
\alpha_{i,0,0,0,0,1} \\
\alpha_{i,0,0,1,0,1} \\
\vdots \\
\alpha_{i,j,1,1,1,1}
\end{pmatrix}, i, j = 0, 1,
$$

and similarly for $\tilde{c}_{i,j,k}, i, j, k = 0, 1$. Let us analyze the condition making the equation (4) an identity. If some vector $\tilde{c}_{i,j} = 0$, then it holds that $\tilde{c}_{1-i,j} = 0$ or $\tilde{c}_{1,j-i} = 0$, which is derived by choosing $\alpha_{k,0} = x^{5k}, \alpha_{k,1} = x^{5k+N-1}, k = 1, \ldots, N - 1$ again. In this case, the state $|\Psi_N\rangle$ is separable. In what follows we suppose no vector $\tilde{c}_{i,j} = 0, i, j = 0, 1$.

For the case of $N = 3$, the equation (4) becomes

$$
\begin{align*}
\tilde{c}_{1,0,0,0} |\Psi_0\rangle + \tilde{c}_{1,0,1,0} |\Psi_1\rangle = \tilde{c}_{1,1,0,0} |\Psi_0\rangle + \tilde{c}_{1,1,1,0} |\Psi_1\rangle.
\end{align*}
$$

As the variables $\alpha_{2,0}$ and $\alpha_{2,1}$ arbitrarily change, simple algebra leads to $\tilde{c}_{0,0} = k_1 \tilde{c}_{0,0}, \tilde{c}_{1,1} = k_1 \tilde{c}_{1,0},$ or $\tilde{c}_{0,0} = k_2 \tilde{c}_{0,1}, \tilde{c}_{0,1} = k_2 \tilde{c}_{0,1},$ with $k_1, k_2$ two proportional constants. Either of them makes (4) an identity. Suppose the result applies to the case of $N = m$, namely if the identity (4) holds then it always holds $\tilde{c}_{0,0} = k_1 \tilde{c}_{0,0}, \tilde{c}_{1,1} = k_1 \tilde{c}_{1,0},$ or $\tilde{c}_{0,0} = k_2 \tilde{c}_{0,1}, \tilde{c}_{0,1} = k_2 \tilde{c}_{0,1}$. For the case of $N = m + 1$, we rewrite the equation (4) as

$$
\begin{align*}
\left(\tilde{c}_{1,0,0,0,0} |\Psi_0\rangle + \tilde{c}_{1,0,1,0,1} |\Psi_1\rangle \right) \bigotimes_{k=3}^{m} \left( \frac{\alpha_{k,0}}{\alpha_{k,1}} \right) = \\
\left(\tilde{c}_{1,0,0,0,0} |\Psi_0\rangle + \tilde{c}_{1,0,1,0,1} |\Psi_1\rangle \right) \bigotimes_{k=3}^{m} \left( \frac{\alpha_{k,0}}{\alpha_{k,1}} \right).
\end{align*}
$$

Applying the assumption that the case of $N = m$ holds to this expression, we have

$$
\begin{align*}
\tilde{c}_{1,0,0,0} + \tilde{c}_{1,0,1,0} = k(\tilde{c}_{0,0,0} + \tilde{c}_{1,0,1,0}), \quad (5.1) \\
\tilde{c}_{1,1,0,0} + \tilde{c}_{1,1,1,0} = k(\tilde{c}_{0,0,1} + \tilde{c}_{1,1,1,0}), \quad (5.2)
\end{align*}
$$

or

$$
\begin{align*}
\tilde{c}_{1,0,0,0} + \tilde{c}_{1,0,1,0} = k(\tilde{c}_{0,0,0} + \tilde{c}_{1,0,1,0}), \quad (5.3) \\
\tilde{c}_{0,0,0} + \tilde{c}_{1,1,1,0} = k(\tilde{c}_{0,0,0} + \tilde{c}_{1,1,1,0}), \quad (5.4)
\end{align*}
$$

We first analyze equations (5.1) and (5.2), in which the proportional number $k$ can be the function of $\alpha_{2,0}$ and $\alpha_{2,1}$. Let the constant $C_{i,j,k}^l$ be the $l$’th entry of the $2^{m-2} \times 1$ vector $\tilde{c}_{i,j,k}, i, j, k = 0, 1$. Similar to the case of $N = 3$, we have

$$
\begin{align*}
\frac{C_{1,0,0}^l}{C_{0,0,0}} = \frac{C_{1,0,1}^l}{C_{0,0,1}} = \frac{C_{1,1,0}^l}{C_{0,1,0}} = \frac{C_{1,1,1}^l}{C_{0,1,1}}, \quad (5.5)
\end{align*}
$$

or

$$
\begin{align*}
\frac{C_{1,0,0}}{C_{1,0,1}} = \frac{C_{0,0,0}}{C_{0,0,1}} = \frac{C_{1,1,0}}{C_{0,1,0}} = \frac{C_{1,1,1}}{C_{0,1,1}} \quad (5.6)
\end{align*}
$$

with $l_1 \in S_5, l_2 \in S_6, S_5 \cup S_6 = \{0,1,\ldots,2^{m-2} - 1\}$. If the equation (5.5) holds for some $l_1$, it is easy to verify $k$ is a constant and thus $\tilde{c}_{0,0} = k\tilde{c}_{0,0}, \tilde{c}_{0,1} = k\tilde{c}_{0,1}$. If no equation (5.5) holds, namely the equation (5.6) holds for any $l_2 \in [0, 2^{m-2} - 1]$, we have $\tilde{c}_{0,0} = k'\tilde{c}_{0,0}, \tilde{c}_{0,1} = k'\tilde{c}_{0,1}$. Analyzing the equations (5.3) and (5.4) leads to the same conclusion. So we have shown by induction that the equation (4) becomes an identity if and only if $\tilde{c}_{0,0} = k\tilde{c}_{0,0}, \tilde{c}_{0,1} = k\tilde{c}_{0,1}, \tilde{c}_{1,0} = k\tilde{c}_{1,0}$, or $\tilde{c}_{0,0} = k'\tilde{c}_{0,0}, \tilde{c}_{0,1} = k'\tilde{c}_{0,1}$. Either of them asserts the state $|\Psi_N\rangle$ is separable, concretely $|\Psi_N\rangle = |\psi_0\rangle \otimes |\psi_1,\ldots,N-1\rangle$ or $|\Psi_N\rangle = |\psi_0\rangle \otimes |\psi_1,\ldots,N-1\rangle$, respectively. This completes the proof for $f_1 = g_1 \equiv 0$.

For other cases $f_k = g_k \equiv 0, k = 2, \ldots, N - 1$, it holds that the state $|\Psi_N\rangle = |\psi_0\rangle \otimes |\psi_1,\ldots,N-1\rangle$ or $|\Psi_N\rangle = |\psi_0\rangle \otimes |\psi_1,\ldots,N-1\rangle$, which can be verified by following the technique similar to the proof for $f_1 = g_1 \equiv 0$. In conclusion, if a multiqubit state $|\Psi_N\rangle$ cannot be converted into an SMQ state via some ILOs, then it must be separable. It means the genuine entangled multiqubit state can always be converted into an SMQ state via some ILOs.
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