Memory effects in a Markov chain dephasing channel

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We study a dephasing channel with memory, modelled by a Markov chain. We show that even weak memory effects have a detrimental impact on the performance of quantum error correcting schemes designed for uncorrelated errors. We also discuss an alternative scheme that takes advantage of memory effects to protect quantum information.

Keywords: Quantum channels; non-Markovian noise; quantum error correction

I. INTRODUCTION

Quantum mechanics offers new and attractive perspectives for information processing and transmission. A large scale quantum computer, if constructed, would advance computing power much beyond the capabilities of classical computation, while quantum cryptography permits a provable secure data exchange. However, due to the unavoidable coupling of any quantum system to its environment, decoherence effects appear. This introduces noise, thus disturbing the programmed quantum coherent evolution. The decoherence problem is conveniently formulated in terms of quantum operations. Given the initial state $\rho$ of a quantum system $Q$ and an overall unitary evolution $U$ of system plus environment, the final system’s state $\rho'$ is obtained after tracing over the environment degrees of freedom: $\rho' = \mathcal{E}(\rho) = \text{Tr}_E[U(\rho \otimes w_0)U^\dagger]$, where $w_0$ is the initial state of the environment (we assume that initially the system and the environment are not entangled) and map $\mathcal{E}$ is known as a quantum operation or a superoperator. It is interesting to consider $\mathcal{E}$ as a quantum channel. This approach encompasses both noisy propagation in time and in space. In the first case, map $\mathcal{E}$ describes the evolution from time $t_i$ to time $t_f$ of some piece of quantum hardware, $\rho$ an $\rho'$ being the system’s states at $t_i$ and $t_f$, respectively. In the latter, the quantum system $Q$ plays the role of information carrier in a two-party communication scenario: $\rho$ is the quantum state at the entrance of the communication channel and $\rho'$ the output state, corrupted by noise effects described by the quantum operation $\mathcal{E}$. The quantum capacity of the channel $\mathcal{E}$ is the maximum amount of quantum information that can be reliably transmitted per channel use, in the asymptotic limit of a number $N \to \infty$ of channel uses.

Usually quantum channels are assumed to be memoryless, that is, the effect of the environment on each information carrier is always the same and described by map $\mathcal{E}$. In other words, there is no memory in the interaction between carriers and environment: the quantum operation for $N$ channel uses is given by $\mathcal{E}_N = \mathcal{E} \otimes \mathcal{E} \otimes \cdots \otimes \mathcal{E}$. However, in several physically relevant situations this is not a realistic assumption. Memory effects appear when the characteristic time scales for the environment dynamics are longer than the time between consecutive channel uses. For instance, solid state implementations, which are the most promising for their scalability and integrability, suffer from low frequency noise. In optical fibers, memory effects may appear due to slow birefringence fluctuations. This introduces correlation among channel uses, i.e., the effect of the environment on one carrier depends on the past interactions between the environment itself and the other carriers. This kind of channels are known as memory channels.

A very interesting question, raised for the first time in Ref. [9], is whether memory can enhance the transmission capacity of a quantum channel. This issue is relevant also for the performance of Quantum Error-Correcting Codes (QECCs). Since quantum capacity is the maximum rate of reliable quantum information transmission, it puts an upper bound to the asymptotic rate achievable by any QECC. On the other hand, realistic QECCs necessarily work on a finite number of channel uses. Moreover, present day experimental implementations are based on very few channel uses. Previous studies have investigated the impact of correlations on the performance of QECCs. Depending on the chosen model, correlations may have positive or negative impact on QECCs.

In this paper, we consider a Markov chain dephasing channel. In the system-environment Hamiltonian (written in the interaction picture with respect to the system), the $l$-th carrier (qubit) interacts with the environment by means
A single use dephasing channel has a very simple representation:

\[ \rho' = \mathcal{E}(\rho) = \sum_{m \in \{0, z\}} p_m B_m \rho B_m^\dagger, \quad B_m = \sigma_m, \]  

where \( \sigma_0 = \mathbb{1} \). Channel \( \mathcal{E} \) has an intuitive meaning: it leaves the qubit unchanged with probability \( p_0 \), while it introduces a phase-flip error with probability \( p_z = 1 - p_0 \). The generalization to \( N \) uses is straightforward:

\[ \rho'_N = \mathcal{E}_N(\rho_N) = \sum_{i_1, \ldots, i_N} p_{i_1 \ldots i_N} B_{i_1 \ldots i_N} \rho_N B_{i_1 \ldots i_N}^\dagger, \quad i_k = 0, z, \]

where \( \rho_N (\rho'_N) \) describes a \( N \)-qubit input (output) state, and the operators \( B_{i_1 \ldots i_N} \) are defined in terms of the Pauli operators \( \sigma_0 = \mathbb{1} \) and \( \sigma_z \):

\[ B_{i_1 \ldots i_N} = \sigma^{(1)}_{i_1} \otimes \cdots \otimes \sigma^{(N)}_{i_N}, \]

with \( \sum_{i_k} p_{i_1 \ldots i_N} = 1 \) and \( \sigma_{i_k}^{(k)} \) acting on the \( k \)-th qubit. The quantity \( p_{i_1 \ldots i_N} \) can be interpreted as the probability that the ordered sequence \( \sigma^{(1)}_{i_1} \ldots \sigma^{(N)}_{i_N} \) of Pauli operators is applied to the \( N \) qubits crossing the channel. We suppose that probability \( p_{i_1 \ldots i_N} \) is stationary: \( p_{i_q} = \sum_{i_k \neq q} p_{i_1 \ldots i_N} \), \( \{ p_{i_q} \} = \{ p_0, 1 - p_0 \} \) for all \( q = 1, \ldots, N \). Memory is introduced by assuming that the joint probability \( p_{i_1 \ldots i_N} \) cannot be factorized: \( p_{i_1 \ldots i_N} \neq p_{i_1} p_{i_2} \cdots p_{i_N} \). To describe the joint probabilities in \( \mathcal{E} \) we choose a Markov chain:

\[ p_{i_1 \ldots i_N} = p_{i_1} p_{i_2|i_1} \cdots p_{i_N|i_N-1}, \quad \text{where} \quad p_{i_k|i_{k-1}} = (1 - \mu) p_{i_k} + \mu \delta_{i_k, i_{k-1}}. \]

Here \( \mu \in [0, 1] \) measures the partial memory of the channel: it is the probability that the same operator (either \( \mathbb{1} \) or \( \sigma_z \)) is applied for two consecutive uses of the channel, whereas \( 1 - \mu \) is the probability that the two operators are uncorrelated. The limiting cases \( \mu = 0 \) and \( \mu = 1 \) correspond to memoryless channels and channels with perfect memory, respectively. In this noise model \( \mu \) might depend on the time interval between two consecutive channel uses. If the two qubits are sent at a time interval \( \tau \ll \tau_c \), where \( \tau_c \) denotes the characteristic memory time scale for the environment, then the same operator is applied to both qubits (\( \mu = 1 \)), while the opposite limit corresponds to the memoryless case (\( \mu = 0 \)). For this model correlations among different uses decay exponentially, and it can be proved \cite{5, 17} that the channel is forgetful \cite{5, 17}. This property allows us to use the quantum noisy channel coding theorem \cite{4, 18} to compute the quantum capacity of this channel \cite{15}:

\[ Q = 1 - p_0 H(q_0) - p_z H(q_z), \]

where \( q_{0,z} \equiv (1 - \mu) p_{0,z} + \mu \) are the conditional probabilities that the channel acts on two subsequent qubits via the same Pauli operator, and \( H(q_0), H(q_z) \) are binary Shannon entropies, defined by \( H(q) = -q \log_2 q - (1-q) \log_2(1-q) \). It is interesting to point out that \( Q \) increases for increasing degree of memory of the channel.
also for realistic QECCs acting on a small number of channel uses. Hereafter we compare the behaviour of two simple coding/decoding schemes for the Markovian dephasing channel \( \{2\} \), the three-qubit code \( \{1, 2\} \), designed for memoryless channels and from now on called code 1 or c1, and a two-qubit code \( \{16\} \), from now on called code 2 or c2, that exploits memory effects.

A proper way to measure reliability of quantum information transmission is the \textit{entanglement fidelity} \( F_e \). To define this quantity we look at the system \( Q \) as a part of a larger quantum system \( RQ \), initially in a pure entangled state \( |\psi^{RQ}\rangle \). The initial density operator of the system \( Q \) is then obtained from that of \( RQ \) by a partial trace over the reference system \( R \): \( \rho^{RQ} = \text{Tr}_R[|\psi^{RQ}\rangle\langle\psi^{RQ}|] \). The system’s state \( \rho^Q \) is then encoded by using an ancillary system \( A \) that consists of two qubits for code 1 (see Fig. 1 in Ref. \[10\]) and a single qubit for code 2 (see Fig. 6 in Ref. \[16\]).

The system and the ancillary qubits are then transmitted in a memoryless case (\( \mu = 0 \)), this happens with probability \( F_e^{(1)} \equiv (1 - P_e^{(1)}) \) (see Refs. \[1, 2\]). Memory changes this probability, and the entanglement fidelity reads

\[
F_e^{(1,m)} = P_0 q_0^2 + P_0 q_0 r_0 + P_0 r_0 p_z + P_z r_z q_0 = F_e^{(1)} - \mu(2 - \mu)(F_e^{(1)} - P_0),
\]

where \( r_{0,z} = 1 - q_{0,z} \) are the conditional probabilities that the channel acts on two subsequent qubits via a different Pauli operator. The first term in the right hand side of (8) is the probability that no errors occur during the transmission, while the other terms correspond to a single phase error during the first, the second, or the third qubit transmission. \( F_e^{(1,m)} \) is a monotonously decreasing function of \( \mu \), that is, memory degrades the performance of the three-qubit code (see Fig. 1). It is worth noticing that for any \( \mu \) the entanglement fidelity of the three-qubit code is better than for the simple transmission of the system qubit (in this latter case \( F_e = P_0 \)). It is interesting to consider the case of a small error probability \( \epsilon \equiv 1 - P_e \ll 1 \). Here the memoryless three-qubit code makes the transmission error probability \( P_e = 1 - F_e \propto \epsilon^2 \), while memory restores the \( \epsilon \)-dependence:

\[
P_e^{(1)} \approx 3 \epsilon^2 \quad \Rightarrow \quad P_e^{(1,m)} \approx \mu(2 - \mu)\epsilon.
\]

To grasp the implications of this remark, let us assume that we have a dephasing channel characterized by an error probability \( \epsilon = 10^{-3} \). At \( \mu = 0 \) the three-qubit code drastically lowers this error to \( P_e^{(1)} \approx 3 \cdot 10^{-6} \). However, a weak memory is sufficient to significantly degrade the performance of the code. For instance, \( P_e^{(1,m)} \approx 2 \cdot 10^{-4} \) at \( \mu = 0.1 \).

Now we turn to the simple coding-decoding scheme discussed in Ref. \[16\]. This two-qubit code is designed to take advantage of correlations: memory enhances the probability that the same operator (\( I \) or \( \sigma_z \)) acts on two qubits subsequently transmitted down the channel. Therefore, c2 encodes a qubit in the subspace spanned by \( \{01, 10\} \), which is noiseless with respect to the application of \( I \otimes I \) or \( \sigma_x \otimes \sigma_z \). The entanglement fidelity in presence of memory is given by

\[
F_e^{(2,m)} = P_0 q_0 + (1 - P_0) q_z = F_e^{(2)} + \mu(1 - F_e^{(2)}),
\]

where \( F_e^{(2)} \) is the entanglement fidelity of c2 at \( \mu = 0 \). A first question is whether this code improves the entanglement fidelity with respect to the simple transmission of the system qubit. This is the case provided that \( \mu > (2P_0 - 1)/2P_0 \). This condition is fulfilled for any \( P_0 \) when \( \mu > 0.5 \) and \( F_e^{(2,m)} \rightarrow 1 \) when \( \mu \rightarrow 1 \) (see again Fig. 1). For small dephasing probabilities \( \epsilon \) we obtain

\[
P_e^{(2,m)} \approx 2(1 - \mu)\epsilon.
\]

Note that \( P_e^{(2,m)} < P_e^{(1,m)} \) when \( \mu > 2 - \sqrt{2} \approx 0.6 \).
FIG. 1: Plot of the transmission error probability $P_e = 1 - F_e$ as a function of the memory factor $\mu$, for the transmission of a single qubit ($P_e = 1 - p_0 = 10^{-3}$, gray line), the three-qubit code ($P_e^{(c1,m)}$, full curve) and the two-qubit code ($P_e^{(c2,m)}$, dashed curve).

IV. FINAL REMARKS

In the case of small dephasing probabilities, $\epsilon \ll 1$, the quantum capacity \[^5\] is weakly affected by memory. On the other hand, in this regime small values of the memory factor $\mu$ are sufficient to have a significantly detrimental impact on the three-qubit code. The two-qubit code solves the problem only in the case of strong memory effects. It would be interesting to investigate codes working on a small number of qubits and suitable for the regime of weak noise and memory, $\epsilon \ll 1$, $\mu \ll 1$.

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