On General Solutions of Einstein Equations

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Abstract

We show how the Einstein equations with cosmological constant (and/or various types of matter field sources) can be integrated in a very general form following the anholonomic deformation method for constructing exact solutions in four and five dimensional gravity (S. Vacaru, IJGMMP 4 (2007) 1285). In this letter, we prove that such a geometric method can be used for constructing general non–Killing solutions. The key idea is to introduce an auxiliary linear connection which is also metric compatible and completely defined by the metric structure but contains some torsion terms induced nonholonomically by generic off–diagonal coefficients of metric. There are some classes of nonholonomic frames with respect to which the Einstein equations (for such an auxiliary connection) split into an integrable system of partial differential equations. We have to impose additional constraints on generating and integration functions in order to transform the auxiliary connection into the Levi–Civita one. This way, we extract general exact solutions (parametrized by generic off–diagonal metrics and depending on all coordinates) in Einstein gravity and five dimensional extensions.

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To construct the most general classes of metrics solving the gravitational field equations in Einstein gravity and extra dimension generalizations is of considerable importance in modern gravity, cosmology and astrophysics. This is a very difficult mathematical task because of high complexity of such systems of nonlinear partial differential equations. Various types of numerical and analytic approaches have not attempted to solve the problem in a general form but oriented to some particular types of exact or approximate solutions which seem to be of physical interest (black holes, cosmological solutions, nonlinear gravitational waves etc). Surprisingly, there were elaborated certain geometric methods which allows us to represent the field equations for various types of gravitational field theories in some convenient (for further integration) forms. Following this approach, to generate exact solutions with generic off–diagonal metrics, nonholonomic frame\textsuperscript{1} and various types of linear connections became a question of frame transforms and constraining integral varieties for corresponding systems of partial differential equations which can be solved in very general forms.

In the present paper, we prove that the Einstein equations with certain type of general sources (in particular, with nonzero, or vanishing, cosmological constants) can be solved following the anholonomic deformation method, see original results and reviews in Refs. \cite{1, 2, 3}. In our approach, we use the nonlinear connection formalism originally developed in Finsler and Lagrange geometry but recently modified for applications in general relativity and some ‘standard’ models of quantum gravity, noncommutative Ricci flow theory and string/brane gravity models on nonholonomic (pseudo) Riemannian and Riemann–Cartan manifolds. Such constructions were elaborated, for instance, in Refs. \cite{4, 5, 6} following geometric ideas originally considered for vector and tangent bundles \cite{7, 8}.

\textsuperscript{1}the word nonholonomic, equivalently, anholonomic means that our geometric constructions will be adapted with respect to certain classes of nonholonomic/nonintegrable frames

\textsuperscript{2}we use anholonomic deformations of frame, metric and connection structures which makes our approach more general than the Cartan’s moving frame method when the same fundamental geometric objects are equivalently re–defined with respect to certain convenient systems of reference; our idea is to solve the problem for a more general connection, also defined by the same metric structure in a unique metric compatible form, and than to constrain the solutions to generate Levi–Civita configurations
We emphasize that in this work the metrics and connections do not de-
pend on "velocities", i.e. we do not work with geometric objects on tangent
bundles, even a number of analogies with constructions in Lagrange–Finsler
geometry can be found. All results can be stated for four dimensional, 4–d,
(pseudo) Riemannian manifolds. Extensions to 5–d Einstein manifolds, with
c conventional 3 + 2 splitting of dimensions, and nonholonomic reductions to
2+2, will be used only because they simplify proofs of results and show
explicitly how the anholonomic deformation method of constructing exact
solutions can be generalized.

Let us consider a (pseudo) Riemannian 5–d manifold \( V \) endowed
with a metric \( g = g_{\alpha\beta}(u) du^\alpha \otimes du^\beta \) of arbitrary signature \( \epsilon_\alpha = (\epsilon_1 = \pm 1, \ldots, \epsilon_5 = \pm 1) \). The local coordinates on \( V \) are parametrized in the
form \( u^\alpha = (u^1, u^\alpha) \), where \( u^1 = (x^1, x^\hat{1}) \) and \( u^\alpha = (v, y) \), i. e. \( y^4 = v, y^5 = y \). Indices \( i, j, k, \ldots = 1, 2, 3 \); \( \hat{i}, \hat{j}, \hat{k}, \ldots = 2, 3 \) and \( a, b, c, \ldots = 4, 5 \) are used for a
conventional (3 + 2)–splitting of dimension and general abstract/coordinate
indices when \( \alpha, \beta, \ldots \) run values 1, 2, ..., 5. In 4–d constructions, we can
write \( V \) and \( u^{\hat{\alpha}} = (x^\hat{1}, y^\alpha) \), when the coordinate \( x^1 \) and values for indices
like \( \alpha, i, \ldots = 1 \) are not considered. In brief, we shall denote some partial
derivatives \( \partial_\alpha = \partial/\partial u^\alpha \) in the form \( s^* = \partial s/\partial x^2, s' = \partial s/\partial x^3, s^* = \partial s/\partial y^4 \).

We write \( \nabla = \{\Gamma^\alpha_{\beta\gamma}\} \) for the Levi–Civita connection\(^4\) with coefficients
stated with respect to an arbitrary local frame basis \( e_\alpha = (e_i, e_a) \) and its
dual basis \( e^\beta = (e^j, e^b) \). Using the Riemannian curvature tensor \( R = \{R^\alpha_{\beta\gamma\delta}\} \)
defined by \( \nabla \), one constructs the Ricci tensor, \( \mathcal{Ric} = \{R_{\beta\delta} \div R^\alpha_{\beta\alpha\delta}\} \), and
scalar curvature \( \mathcal{R} = g^{\beta\delta} R_{\beta\delta} \), where \( g^{\beta\delta} \) is inverse to \( g_{\alpha\beta} \). The Einstein
equations on \( V \), for an energy–momentum source \( T_{\alpha\beta} \), are written in the
form
\[
R_{\beta\delta} - \frac{1}{2} g_{\beta\delta} R = \kappa T_{\beta\delta},
\]
where \( \kappa = \text{const.} \) For the Einstein spaces defined by a cosmological constant
\( \lambda \), such gravitational field equations can be represented as \( R^\alpha_{\beta} = \lambda \delta^\alpha_{\beta} \), where
\( \delta^\alpha_{\beta} \) is the Kronecher symbol. The vacuum solutions are obtained for \( \lambda = 0 \).

The goal of this paper is to formulate and sketch the proof of (Main
\(^3\)In our works, we follow conventions from \[1, 2, 3\] when left up/l ow indices are used as
labels for geometric spaces and objects.

\(^4\)which is uniquely defined by a given tensor \( g \) to be metric compatible, \( \nabla g = 0 \), and
with zero torsion; we summarize on "up-low" repeating indices if the contrary is not stated

Result):
Theorem 0.1 If the gravitational field equations in Einstein gravity and its 5-d extension \([1]\) can be represented via frame transforms in the form

\[
R^\alpha_\beta = \Upsilon^\alpha_\beta \tag{2}
\]

for a given \(\Upsilon^\alpha_\beta = \text{diag}[\Upsilon_\gamma]\) with

\[
\Upsilon_1 = \Upsilon_2 + \Upsilon_4, \quad \Upsilon_2 \quad \Upsilon_3 = \Upsilon_2(x^k, v), \quad \Upsilon_4 \quad \Upsilon_5 = \Upsilon_4(x^k),
\]

for \(y^4 = v\), such equations can be solved in general form by metrics of type

\[
\begin{align*}
5g &= \epsilon_1 dx^1 \otimes dx^1 + g_1(x^k)dx^7 \otimes dx^7 + \omega^2(x^i, y^b)h_a(x^k, v)e^a \otimes e^a, \\
e^4 &= dy^4 + w_i(x^k, v)dx^i, \quad e^5 = dy^5 + n_i(x^k, v)dx^i,
\end{align*}
\]

where coefficients are determined by generating functions \(f(x^i, v), f^* \neq 0\), and \(\omega(x^i, y^b) \neq 0\) and integration functions \(0f(x^i), \quad 0h(x^i), \quad 1n_k(x^i)\) and \(2n_k(x^i)\), following formulas

\[
\begin{align*}
g_i &= \epsilon_1 \psi(x^k), \quad \text{for} \quad \epsilon_2 \psi^{**} + \epsilon_3 \psi'' = \Upsilon_4; \\
h_4 &= \epsilon_4 \quad 0h(x^i) [f^*(x^i, v)]^2 \varsigma(x^i, v) \quad \text{and} \quad h_5 = \epsilon_5 [f(x^i, v) - 0f(x^i)]^2; \\
w_i &= -\partial \varsigma(x^i, v)/\varsigma^*(x^i, v) \quad \text{and} \\
n_k &= 1n_k(x^i) + 2n_k(x^i) \int dv \varsigma(x^i, v)[f^*(x^i, v)]^2/[f(x^i, v) - 0f(x^i)]^3, \\
\end{align*}
\]

for \(\varsigma = \quad 0\varsigma(x^i) - \frac{\epsilon_4}{8} \quad 0h(x^i) \int dv \Upsilon_2(x^k, v) f^*(x^i, v)[f(x^i, v) - 0f(x^i)]; \\
\]

\[
\begin{align*}
e_k \omega &= \partial_k \omega + w_k \omega^* + n_k \partial \omega/\partial y^5 = 0, \tag{6}
\end{align*}
\]

when the so-called Levi–Civita integral varieties are selected by additional constraints

\[
w_i^* = e_i \ln |h_4|, \quad e_k w_i = e_i w_k, \quad n_i^* = 0, \quad \partial_i n_k = \partial_k n_i. \tag{7}
\]

In order to construct some explicit classes of exact solutions of Einstein equations \([2]\), we have to state certain boundary/ symmetry/ topology conditions which would allow to define the integration functions and systems of first order partial differential equations of type \([7]\). Perhaps all classes of exact solutions outlined in Refs. \([9, 10, 11, 2, 3]\) can be found as certain particular cases of metrics \([1]\) or equivalently redefined in such a form.
Remark 0.1

1. Analogs of Theorem 0.1 were proven in our previous works \[1, 2, 3\] for certain cases with \(\omega = 1\), and other types generalizations, which allowed us to generate various classes of generic off–diagonal exact solutions with one Killing vector (in such a case, metrics \(4\) do not depend on variable \(y^5\)). The key new result of this work is that we can consider any generating function \(\omega(x^j, y^b)\) depending on coordinate \(y^5\) but subjected to the condition \(5\). This allows us to construct very general classes of "non–Killing" exact solutions.

2. It should be emphasized that any (pseudo) Riemannian metric \(g = \{g_{\alpha\beta}(u^\alpha')\}\) depending in general on all five local coordinates on \(\mathbb{R}^5\) can be parametrized in a form \(g_{\alpha\beta}(4)\), \(g_{\alpha\beta} = e^{-u^\alpha}_\alpha e^{-u^\beta}_\beta g_{\alpha'\beta'}\), using frame transforms of type \(e_a = e^{-u^\alpha}_\alpha e^{\alpha}_\alpha\). So, the metrics constructed above define general solutions of Einstein equations for any type of sources \(\kappa T_{\beta\delta}\) which can be parametrized in a formally diagonalized form \(\tilde{g}\), with respect to a nonholonomic frame of reference \(\mathcal{F}\).

Let us provide the key points for a proof of Theorem 0.1 following Steps 1–6 of the anholonomic deformation/ frame method (proposed in Refs. \[11, 12, 13\], see recent reviews and generalizations in Refs. \[1, 2, 3, 4\]):

**Step 1: Ansatz for metrics and N–adapted frames**

We can consider a nonholonomic \((3 + 2)\)–splitting of a spacetime \(\mathbb{R}^5\) by introducing a non–integrable distribution stated by certain coefficients \(N = \{N_i^a\}\), when \(N = N_i^a(u^a)dx^i \otimes \frac{\partial}{\partial y^a}\). This defines a class of so–called N–adapted frames, (respectively) dual frames

\[
e^{-u^\alpha}_\alpha = \frac{\partial}{\partial x^i} - N_i^a \frac{\partial}{\partial y^a}, e_a = \frac{\partial}{\partial y^a}\),
\[
e^{-u^\alpha}_\alpha = (dx^i, e^a = dy^a + N_i^a dx^i) .
\]

The vielbeins \(9\) satisfy the nonholonomy relations

\[
[e_\alpha, e_\beta] = w_{\alpha\beta} e_\gamma
\]

\(5\) which can not be diagonalized by coordinate transform

\(6\) We have to solve certain systems of quadratic algebraic equations and define some \(e^{\alpha'}(u^a)\), choosing a convenient system of coordinates \(u^\alpha' = u^\alpha'(u^a)\).

\(7\) using chains of frame transforms, such parametrizations can be defined for 'almost' all physically important energy–momentum tensors
with (antisymmetric) nontrivial anholonomy coefficients \( w^b_{ia} = \partial_a N^b_i \) and 
\( w^a_{ji} = \Omega^a_{ij} \), where

\[
\Omega^a_{ij} = e_j (N^a_i) - e_i (N^a_j)
\]  
(11)

are the coefficients of N–connection curvature. The particular holonomic/integrable case is selected by the integrability conditions \( w^\gamma_{\alpha\beta} = 0 \).

Any (pseudo) Riemannian metric \( g \) on \( ^5V \) can be written in the form

\[
g = g_{ij}(u^a)e^i \otimes e^j + h_{ab}(u^a)e^a \otimes e^b,
\]  
(12)

for some N–adapted coefficients \([g_{ij}, h_{ab}]\) and \( N^a_i \). For instance, we get the metric (4) with \( \omega = 1 \), from (12) if we choose (omitting, for simplicity, priming of indices)

\[
g_{ij} = \text{diag}[\epsilon_1, g^i_k(x^k)], h_{ab} = \text{diag}[h_a(x^i, v)], N^4_k = w_k(x^i, v), N^5_k = n_k(x^i, v).
\]  
(13)

Such a metric has a Killing vector, \( e_5 = \partial/\partial y^5 \), symmetry because its coefficients do not depend on \( y^5 \). Introducing a nontrivial \( \omega^2(u^a) \) depending also on \( y^5 \), as a multiple before \( h_a \), we get a \((3 + 2)\) N–adapted parametrization, up to certain frame/coordinate transforms, for all metrics on \( ^5V \).

**Step 2: Metric compatible deformations of the Levi–Civita connection**

It is a cumbersome task to prove using the Levi–Civita connection \( \nabla \) (a unique one in general relativity being metric compatible, with zero torsion, and completely defined by the metric structure) that the Einstein equations (2) are solved by metrics of type (4). Our "main trick" is not only to adapt our constructions to N–adapted frames of type \( e_\alpha \) and \( e^\mu \) but also to use as an auxiliary tool (we emphasize, in Einstein gravity and generalizations) a new type of linear connection \( \hat{D} = \{\hat{\Gamma}^\alpha_{\beta\gamma}\} \), also uniquely defined by the metric structure. It can be defined as a 1–form \( \hat{\Gamma}^\alpha_{\beta\gamma} e^\gamma \) with coefficients \( \hat{\Gamma}^\gamma_{\alpha\beta} = (\hat{L}^i_{jk}, \hat{L}^a_{bk}, \hat{C}^i_{jc}, \hat{C}^a_{bc}) \) adapted to a \((3 + 2)\)–splitting. Such a linear connection is also metric compatible, \( \hat{D}g = 0 \), defined by any

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8In Lagrange–Finsler geometry, for \( ^4V = TM \), where \( TM \) is the total space of a tangent bundle on a manifold \( M \), such a \( N \) defines a nonlinear connection (N–connection) structure [7,8]; nevertheless, N–connections can be considered on nonholonomic manifolds, i.e. manifolds enabled with nonholonomic distributions, even in general relativity, see discussions in [1,3,4].

9We use boldface symbols for spaces (and geometric objects on such spaces) enabled with a structure of N–coefficients.
data \( g = \{g_{ij}, h_{ab}, N^a_i\} \) and contains an induced torsion (by the same metric coefficients)

\[
\hat{T}^\alpha = \hat{T}^\alpha_{\beta\gamma} e^\beta \wedge e^\gamma \hat{D}e^\alpha = de^\alpha + \hat{T}^\alpha_{\beta} \wedge e^\beta,
\]

with coefficients

\[
\begin{align*}
\hat{T}^i_{jk} &= \hat{T}^i_{jk} - \hat{T}^i_{kj}, \quad \hat{T}^i_{ja} = -\hat{T}^i_{aj} = \hat{C}^i_{ja}, \quad T^a_{ji} = \Omega^a_{ji}, \\
\hat{T}^a_{bi} &= -\hat{T}^a_{ib} = \frac{\partial N^a_i}{\partial y^b} - \hat{L}^a_{bi}, \quad \hat{T}^a_{bc} = \hat{C}^a_{bc} - \hat{C}^a_{cb}.
\end{align*}
\]

(15)

By straightforward computations, we shall prove that the (nonholonomically modified) Einstein equations in 5–d gravity can be solved in general form for the connection \( \hat{D} \).

Then imposing certain constraints when \( \hat{D} \to \nabla \), we shall construct the most general classes of solutions of gravitational field equations (2) which can be considered also in general relativity.

**Definition 0.1** A distinguished connection \( D \) (in brief, d–connection) on \( 5V \) is a linear connection preserving under parallelism a conventional horizontal and vertical splitting (in brief, h– and v–splitting) induced by a nonholonomic distribution \( N = \{N^a_i\} \) on tangent bundle

\[
T^5V = h^5V \oplus v^5V.
\]

(16)

We emphasize that the Levi–Civita connection \( \nabla \), for which \( \nabla g = 0 \) and \( T^\alpha \hat{\nabla} e^\alpha = 0 \), is not a d–connection because, in general, it is not adapted to a N–splitting defined by a Whitney sum (16).

**Theorem 0.2** There is a unique canonical d–connection \( \hat{D} \) satisfying the condition \( \hat{D}g = 0 \) and with vanishing ”pure” horizontal and vertical torsion coefficients, i.e. \( \hat{T}^i_{jk} = 0 \) and \( \hat{T}^a_{bc} = 0 \), see formulas (15).

**Proof.** It follows by a straightforward verification that

\[
\begin{align*}
\hat{D}_jg_{kl} = 0, \quad \hat{D}_ag_{kl} = 0, \quad \hat{D}_jh_{ab} = 0, \quad \hat{D}_ah_{bc} = 0,
\end{align*}
\]

(17)

i.e. \( \hat{D}g = 0 \), and computing N–adapted coefficients of torsion (15), by using the N–adapted coefficients

\[
\begin{align*}
\hat{L}^i_{jk} &= \frac{1}{2} g^{ir} (e_k g_{jr} + e_j g_{kr} - e_r g_{jk}), \\
\hat{L}^a_{bk} &= e_b (N^a_k) + \frac{1}{2} h^{ac} (e_k h_{bc} - h_{dc} e_b N^d_k - h_{db} e_c N^d_k), \\
\hat{C}^i_{jc} &= \frac{1}{2} g^{jk} e_c g_{jk}, \quad \hat{C}^a_{bc} = \frac{1}{2} h^{ad} (e_c h_{bd} + e_c h_{cd} - e_d h_{bc}).
\end{align*}
\]

(18)
Corollary 0.1 Any geometric construction for the canonical d–connection \( \hat{D} = \{ \hat{\Gamma}^\gamma_{\alpha\beta} \} \) can be re–defined equivalently into a similar one with the Levi–Civita connection \( \nabla = \{ \Gamma^\gamma_{\alpha\beta} \} \) following formulas

\[
\Gamma^\gamma_{\alpha\beta} = \hat{\Gamma}^\gamma_{\alpha\beta} + Z^\gamma_{\alpha\beta},
\]

where \( N \)–adapted coefficients of connections, \( \Gamma^\gamma_{\alpha\beta} \) and \( \hat{\Gamma}^\gamma_{\alpha\beta} \), and the distortion tensor \( Z^\gamma_{\alpha\beta} \) are determined in unique forms by the coefficients of a metric \( g^\alpha_\beta \).

Proof. It is similar to that presented for vector bundles in Refs. \([7, 8]\) but in our case adapted for (pseudo) Riemannian nonholonomic manifolds, see details in \([1, 3, 4]\). Here we write down the \( N \)–adapted components of the distortion tensor \( Z^\gamma_{\alpha\beta} \) computed as

\[
Z^a_{jk} = -\hat{C}^i_{jk} h^a_{ib}, \quad Z^i_{bk} = \frac{1}{2} \Omega^c_{jk} h^a_{cb} g^i_{ij} - \Xi^i_{jk} \hat{C}^i_{bb},
\]

\[
Z^a_{bk} = z^{ab}_{cd} \hat{T}^c_{kb}, \quad Z^i_{bk} = \frac{1}{2} \Omega^a_{jk} h^a_{cb} g^i_{ij} + \Xi^i_{jk} \hat{C}^i_{bb}, \quad Z^i_{jk} = 0,
\]

\[
Z^a_{jb} = -z^{ad}_{bc} \hat{T}^c_{jd}, \quad Z^a_{bc} = 0, \quad Z^i_{ab} = -\frac{g^i_{ij}}{2} \left[ T^c_{ja} h^b_{cb} + T^c_{ja} h^a_{cb} \right],
\]

for \( \Xi^i_{jk} = \frac{1}{2} (\delta^i_j \delta^k_h + g^i_{jk} g^k_h), \quad z^{ab}_{cd} = \frac{1}{2} (\delta^a_c \delta^b_d + h^a_{cd} h^b_{cd}) \) and \( \hat{T}^c_{ja} = \hat{L}^c_{ja} - \epsilon_a (N^c_j) \).

In 4–d, the Einstein gravity can be equivalently formulated in the so–called almost Kähler and Lagrange–Finsler variables, as we considered in Refs. \([4, 6, 14, 15]\). Such types of linear connections, like \( \hat{\Gamma}^\gamma_{\alpha\beta} \) and its nonholonomic deformations, are convenient not only for elaborating various
models of brane and deformation quantization of gravity and nonsymmetric generalizations but also in constructing general solutions of the Einstein equations for the Levi–Civita connection $\nabla$.

**Step 3: Nonholonomic deformations of Einstein equations**

In this and next steps, we shall work with the canonical $d$–connection. We can compute the nontrivial N–adapted components of curvature of $\hat{D}$ following formulas

$$\hat{\mathcal{R}}_{\alpha\beta} = d\hat{\Gamma}_{\alpha\beta} - \hat{\Gamma}_{\gamma\beta} \wedge \hat{\Gamma}^{\alpha}_{\gamma} = R_{\alpha\beta}^{\gamma\delta} e^\gamma \wedge e^\delta,$$  \hspace{1cm} (21)

The explicit formulas for the so–called N–adapted coefficients of curvature $\hat{\mathcal{R}}_{\alpha\beta\gamma\delta}$ of (pseudo) Riemannian spaces are provided, for instance, in Refs. [1, 3, 4].

Contracting respectively the N–adapted coefficients of $\hat{\mathcal{R}}_{\alpha\beta\gamma\delta}$ (21), one proves that the Ricci tensor $\hat{\mathcal{R}}_{\alpha\beta} = \hat{\mathcal{R}}^{\gamma}_{\alpha\beta\gamma}$ is characterized by $h$–$v$–components, i.e. the Ricci tensor $\hat{\mathcal{R}}_{\alpha\beta} = \{\hat{R}_{ij}, \hat{R}_{ia}, \hat{R}_{ai}, \hat{R}_{ab}\}$,  \hspace{1cm} (22)

The scalar curvature of $\hat{D}$ is defined

$$s\hat{R} = g^{\alpha\beta} \hat{\mathcal{R}}_{\alpha\beta} = g^{ij} \hat{R}_{ij} + h^{ab} \hat{R}_{ab}.$$  \hspace{1cm} (23)

The Einstein tensor of $\hat{D}$ is (by definition)

$$\hat{E}_{\alpha\beta} = \hat{R}_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} s\hat{R}.$$  \hspace{1cm} (24)

Here, one should be emphasized that tensors $\hat{R}^{a}_{\beta\gamma\delta}$, $\hat{R}_{\alpha\beta}$ and $\hat{E}_{\alpha\beta}$ (being constructed for the connection $\hat{D} \neq \nabla$) defer by corresponding distortion tensors from similar tensors $R^{a}_{\beta\gamma\delta}$, $R_{\alpha\beta}$ and $E_{\alpha\beta}$, derived for $\nabla$, even both classes of such tensors are completely defined by a same metric structure $g_{\alpha\beta}$. So, the nonholonomically modified gravitational field equations

$$\hat{E}_{\alpha\beta} = \kappa T_{\alpha\beta},$$  \hspace{1cm} (25)

are not equivalent, in general, to usual Einstein equations for the Levi–Civita connection $\nabla$. \hspace{1cm} (10)

\hspace{1cm} (10) In our previous works [4, 6, 14, 15], we noted that an equivalence of both types of filed equations would be possible, for instance, if we introduce a generalized source $\hat{T}_{\beta\delta}$ containing contributions of the distortion tensor [20].
Nevertheless, it is convenient to use a variant of equations (25),
\[ \hat{R}^{\alpha}_{\beta} = \Upsilon^{\alpha}_{\beta}, \] (26)
with a general source parametrize in the form (3), \( \Upsilon^{\alpha}_{\beta} = \text{diag}[\Upsilon_{\gamma}] \), because such equations can be integrated in general form and, for instance, play an important role in Finsler–Lagrange theories of gravity derived in low energy limits of string/brane gravity and noncommutative generalizations [2]. In Step 5, see below, we shall impose additional constraints on coefficients of solutions for \( \hat{D} \) when \( \hat{\Gamma}^{\gamma}_{\alpha\beta} \) will be the same as \( \Gamma^{\gamma}_{\alpha\beta} \), with respect to a chosen \( N \)-adapted frame (even, in general, \( \hat{D} \neq \nabla \))[11]. As a result, we shall select classes of solutions for equations (2) with the Ricci tensor \( R^{\alpha\beta} \).
Step 4: Solutions with Killing symmetry for nonholonomic gravitational fields

The system of equations used in Theorem 0.3 can be integrated in very general forms for any given $\Upsilon_2$ and $\Upsilon_4$. Here we note that the equation (27) relates an unknown function $g_2(x^2, x^3)$ to a prescribed $g_3(x^2, x^3)$, or inversely. The equation (28) contains only derivatives on $y^4 = v$ and allows us to define $h_4(x^i, v)$ for a given $h_5(x^i, v)$, or inversely, for $h_{4,5}^* \neq 0$; having defined $h_4$ and $h_5$, we can compute the coefficients (31), which allows us to find $w_i$ from algebraic equations (29) and to compute $n_i$ by integrating two times on $v$ as follow from equations (30). This way, we prove:

**Proposition 0.1** The general class of solutions of nonholonomic gravitational equations (26) with one Killing symmetry on $e_5 = \partial/\partial y^5$ is defined by an ansatz (4) with $\omega^2 = 1$ and coefficients $g^\hat{i}, h_a, w_k, n_k$ computed following formulas (5).

We note that such classes of solutions are very general ones and contain as particular cases all possible exact solutions for (non) holonomic Einstein spaces with Killing symmetry. They also can be generalized to include arbitrary finite sets of parameters as we considered in Ref. [1].

Step 5: Constraints generating solutions in Einstein gravity

Nevertheless, the solutions constructed following Proposition 0.1 are for the canonical d–connection, $\hat{D}$, and not for the Levi–Civita one, $\nabla$. We can see that both the torsion $\hat{T}^\alpha_{\beta\gamma}$ (15) and distortion tensor $Z^\gamma_{\alpha\beta}$ (20) became zero if and only if

$$\hat{C}_{ib} = 0, \Omega^a_{ji} = 0, \hat{T}^c_{ja} = 0,$$

with respect to a N–adapted basis (in general, such a basis is anholonomic because $w^{\beta}_a = \partial_a N^\beta_i$ is not obligatory zero, see formulas (10)). In such a case, the distortion relations (19) transform into $\Gamma^\gamma_{\alpha\beta} = \hat{\Gamma}^\gamma_{\alpha\beta}$.

**Corollary 0.2** An ansatz (4) with $\omega^2 = 1$ and coefficients $g^\hat{i}, h_a, w_k, n_k$ computed following formulas (5) defines solutions with one Killing symmetry on $e_5 = \partial/\partial y^5$ of the Einstein equations (2) for the Levi–Civita connection $\Gamma^\gamma_{\alpha\beta}$, all formulas being considered with respect to N–adapted frames, if the coefficients of metric are subjected additionally to the conditions (7).

**Proof.** By straightforward computations for ansatz defined by metrics (12) with coefficients (13), we get that the conditions (32) resulting in $\Gamma^\gamma_{\alpha\beta} = \hat{\Gamma}^\gamma_{\alpha\beta}$ are just those written as (7).
Steps 1–5 considered above result in:

**Conclusion:** In order to generate exact solutions with Killing symmetry in Einstein gravity and its 5–d extensions, we should consider N–adapted frames and nonholonomic deformations of the Levi–Civita connection to an auxiliary metric compatible d–connection (for instance, to the canonical d–connection, $\hat{D}$), when the corresponding system of nonholonomic gravitational field equations (27)–(30) can be integrated in general form. Subjecting the integral variety of such solutions to additional constraints of type (32), i.e. imposing the conditions (7) to the coefficients of metrics, we may construct new classes of exact solutions of Einstein equations for the Levi–Civita connection $\nabla$.

**Step 6: General solutions in Einstein gravity**

The last step which allows us to consider the most general classes of solutions of the nonholonomic gravitational field equations (26), and (for more particular cases), of Einstein equations (2), is to extend the anholonomic deformation method to the case of metrics depending on all coordinates $u^\alpha = (x^i, y^a)$, i.e. to solutions without any prescribed Killing symmetry.

Let us introduce a nontrivial multiple $\omega^2(x^i, y^a)$ before coefficients $h_a$ in a metric (12), when the rest of coefficients are parametrized in the form (13). We get an ansatz of type

$$\omega g = \epsilon_1 e^1 \otimes e^1 + g_j(x^k)e^j \otimes e^j + \omega^2(x^i, y^a) h_a(x^i, v)e^a \otimes e^a,$$

$$e^4 = dy^4 + w_i(x^k, v)dx^i, e^5 = dy^5 + n_i(x^k, v)dx^i. \quad (33)$$

Introducing coefficients of $\omega g$ into formulas (18), we compute $\omega \hat{\Gamma}^\gamma_{\alpha\beta}$, which allows us to define, see (22), $\omega \hat{R}_{\alpha\beta} = \{\hat{R}_{ij}, \hat{R}_{ia}, \omega \hat{R}_{ai}, \omega \hat{R}_{ab}\}$.

**Lemma 0.1** For a generalized ansatz (33), which for $\omega^2 = 1$ is a solution of nonholonomic gravitational equations (26) with Killing symmetry on $e_5 = \partial/\partial y$, we obtain

$$\omega \hat{R}_{ab} = \hat{R}_{ab} + \omega \hat{Z}_{ab} \quad \text{and} \quad \omega \hat{R}_{ai} = \hat{R}_{ai} = 0, \quad (34)$$

with $\omega \hat{Z}_{ab} = \text{diag}[\omega \hat{Z}_c(x^i, y^a)]$ determined for any $\omega^2(x^i, y^a)$ subjected to conditions $e^k \omega = 0$ (6) and $\hat{T}^c_{ja} = 0$.

**Proof.** By straightforward computations for ansatz defined by metrics (33), we see that the v–part containing coefficients $\omega^2 h_a$ results in certain
two dimensional conformal transforms of $\widehat{R}_{ab}$ to $\omega \widehat{R}_{ab}$ (we can consider in this case any fixed values $x^i$ but arbitrary coordinates $y^4$ and $y^5$) and certain additional terms to $\widehat{R}_{ai}$ giving a nonzero $\omega \widehat{R}_{ai}$. Nevertheless, we can satisfy the equations (34) with $\widehat{T}^c_{ja} = 0$ for any nontrivial factor $\omega$ for which $e_k \omega = 0$. Of course, for such nontrivial $\omega \widehat{Z}_c(x^i, y^a)$, we should redefine the sources (3), via frame/coordinate transform, which would allow us to solve equations of type (28), when $\widehat{R}^4_4 = \nabla^4 \Upsilon(x^i, v)$ and $\widehat{R}^5_5 = \nabla^5 \Upsilon(x^i, v)$, with contributions to the vertical conformal transforms, are equivalent to certain $\widehat{R}^4_4 = \widehat{R}^5_5 = -\Upsilon_2(x^i, v)$. For constraints of type $\widehat{T}^c_{ja} = 0$ and $e_k \omega = 0$, and dimensional vertical subspaces, one holds $\widehat{R}^4_4 = \widehat{R}^5_5 = \omega \widehat{R}^4_4 = \omega \widehat{R}^5_5$.

Summarizing the results of Theorems 0.2 and 0.3, Proposition 0.1, Corollary 0.2, and Lemma 0.1, we prove the Main Result stated in Theorem 0.1.

As a matter of principle, any exact solution in gravity theories (Einstein gravity and string/brane/gauge/Kaluza–Klein, Lagrange–Finsler, supersymmetric and/or noncommutative generalizations etc) can be represented in a form (34) or certain nonholonomic frame transforms/ deformations with extra–dimension coordinates and various types of commutative and noncommutative parameters, see more general/alternative constructions in Refs. [2, 3, 16, 17]. Perhaps, the anholonomic deformation method allows us to construct general solutions of gravitational equations in the form (2), for arbitrary dimension and source (3), when the Ricci tensor is determined by any generalized linear and nonlinear connections$^{13}$. The length of this paper does not allow us to speculate on symmetries and properties of such solutions and possible physical implications (for instance, how to consider black hole and cosmological solutions with singularities and horizons, and their nonholonomic deformations); for details and discussions, we send the reader to Refs. [1, 2, 3, 4].

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Remarks on submissions and publications This preprint version is almost identical to a published letter variant (see: S. Vacaru, IJGMMP 8 (2011) 9-21; submitted to arXiv.org on September 22, 2009). It is also related to another already published article [16] (further variants 2-4, extending the version v1, were put as arXiv: 0909.3949 [gr-qc] beginning October 1, 2009) containing detailed proofs and generalizations of results on

$^{13}$of course, the term "general solution" should be used in a quite approximate form because it may be not clear how to define a "general unique solution" in a rigorous mathematical form for some nonlinear systems of equations with possible singularities of coefficients and/or generalized topological and group symmetries etc.
exact solutions of Einstein equations for (pseudo) Riemannian spaces of arbitrary finite dimension \( n + m > 5 \). Following a discussion and suggestion of arXiv’s Moderator (from October 2009), we submitted two variants of electronic preprints because two different manuscripts were published in different journals (with different titles and lengths and rather different contents) and the letter variant may have certain priorities for readers interested in exact solutions in general relativity but not in extra dimension generalizations. On June 20, 2011, moderators of arXiv.org decided to provide a different number to the ”short” variant of paper as a submission to physics.gen-ph.

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