Quaternion based generalization of Chern-Simons theories in arbitrary dimensions

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Abstract

A generalization of Chern-Simons gauge theory is formulated in any dimension and arbitrary gauge group where gauge fields and gauge parameters are differential forms of any degree. The quaternion algebra structure of this formulation is shown to be equivalent to a three $\mathbb{Z}_2$-gradings structure, thus clarifying the quaternion role in the previous formulation.
1 Introduction

A formulation of gauge theory in terms of differential forms has the advantage that it automatically generates a general coordinate invariant formulation since an explicit metric dependence does not appear. One interesting approach along this line is the applications of Chern-Simons action to 3-dimensional gravity [1].

In the formulation of the standard gauge theory only 1-form gauge fields and 0-form gauge parameters play a role as differential forms. It is natural to ask if one can formulate gauge theories in terms of all the degrees of differential forms. A positive answer was given by one of authors (N.K.) and Watabiki many years back with a graded Lie algebra setting [2]. In this paper we focus on the generalization of the Chern-Simons action to arbitrary dimensions with arbitrary degrees of differential forms as gauge fields and parameters for Lie algebra setting and clarify the origin of the quaternion structure which was discovered in the original formulation [2]. In the present formulation the introduction of graded Lie algebra is not required.

2 The origin of the quaternion structure for a three grading gauge system

When we consider standard Abelian gauge theory with differential forms we identify gauge field as one-form and gauge parameter as zero-form. In this gauge system $\mathbb{Z}_2$-grading structure of even-form and odd-form is present. If we define $\Lambda_+$ as a set of even forms and $\Lambda_-$ as a set of odd forms, we have

$$\lambda_+ \wedge \lambda'_+ = \lambda'_+ \wedge \lambda_+ \in \Lambda_+, \quad \lambda_+ \wedge \lambda_- = \lambda_- \wedge \lambda_+ \in \Lambda_-, \quad \lambda_- \wedge \lambda'_- = -\lambda'_- \wedge \lambda_- \in \Lambda_+, \quad (1)$$

where $\lambda_+, \lambda'_+ \in \Lambda_+, \quad \lambda_-, \lambda'_- \in \Lambda_-$ and $\wedge$ is a wedge product. Fermionic and bosonic fields have similar $\mathbb{Z}_2$-grading structure with an obvious correspondence.

Let us consider two types of fields $\Phi_{(a,b,c)}$ and $F_{(a,b,c)}$ which have a three $\mathbb{Z}_2$-grading structure $(a, b, c)$ with $a, b, c = 0$ or $1$. For simplicity we assume that these fields have Abelian nature. Then we introduce two types of commuting structure with respect to the three gradings:

$$\Phi_{(a,b,c)} \Phi'_{(a',b',c')} = (-1)^{aa'+bb'+cc'} \Phi'_{(a',b',c')} \Phi_{(a,b,c)}, \quad (2)$$

$$F_{(a,b,c)} F'_{(a',b',c')} = (-1)^{(a+b+c)(a'+b'+c')} F'_{(a',b',c')} F_{(a,b,c)}. \quad (3)$$

In [2] the three gradings are independent whereas in [3] there is one global grading corresponding $a + b + c$. Let us introduce an object $q(a, b, c)$ satisfying the following commuting structure:

$$q(a, b, c) q(a', b', c') = (-1)^{aa'+bb'+cc'+(a+b+c)(a'+b'+c')} q(a', b', c') q(a, b, c). \quad (4)$$

Then we have the following commuting relation:

$$(\Phi_{(a,b,c)} q(a, b, c)) (\Phi'_{(a',b',c')} q(a', b', c')) = (-1)^{(a+b+c)(a'+b'+c')} (\Phi'_{(a',b',c')} q(a', b', c')) (\Phi_{(a,b,c)} q(a, b, c)), \quad (5)$$

where we assume $\Phi_{(a,b,c)}$ and $q(a, b, c)$ are not interacting and thus commuting:

$$\Phi_{(a,b,c)} q(a, b, c) = q(a, b, c) \Phi_{(a,b,c)}. \quad (6)$$

We recognize now that a field of the $F$-type, namely with a single global grading, can be written in terms of a field $\Phi_{(a,b,c)}$ with three distinct gradings as

$$F_{(a,b,c)} = \Phi_{(a,b,c)} q(a, b, c). \quad (7)$$
In order to define a product of the \( \mathcal{F} \)-fields, we have to define a product in the \( q(a, b, c) \) space that satisfies (1). This is given by

\[
q(a, b, c)q(a', b', c') = (-1)^{aa' + bb' + cc' + a'b + b'c + c'a} q(a + a', b + b', c + c'),
\]

where the sums are defined modulo 2. This product is associative. There are eight possible \( q(a, b, c) \)'s for \( a, b, c = 0, 1 \). However with respect to their commuting structure (1) and with respect to the product (8) there are only four independent \( q(a, b, c) \). In fact the sign factors in (1) and (8) are invariant for \( (a, b, c) \rightarrow (a + 1, b + 1, c + 1) \) and the same for \( (a', b', c') \). The units \( q(a, b, c) \) can then be identified in pairs and renamed as:

\[
q(1, 1, 1) = q(0, 0, 0) \equiv 1, \quad q(1, 0, 0) = q(0, 1, 1) \equiv i, \\
q(0, 1, 0) = q(1, 0, 1) \equiv j, \quad q(0, 0, 1) = q(1, 1, 0) \equiv k.
\]

It is easy to recognize now that \( 1, i, j, k \) satisfy the quaternion algebra

\[
i^2 = j^2 = k^2 = -1, \\
i j = -ji = k, \quad jk = -kj = i, \quad ki = -ik = j.
\]

It is important to notice that eq.(8) defines the quaternion algebra in an unconventional way, different from its standard mathematical introduction, and that it links in an unexpected way the quaternion algebra to the existence of three independent gradings.

We can introduce now two types of \( \mathcal{F} \)-fields, corresponding respectively to \( a + b + c \) odd and even:

\[
\mathcal{A} = q(1, 1, 1)\Phi_{(1,1,1)} + q(1, 0, 0)\Phi_{(1,0,0)} + q(0, 1, 0)\Phi_{(0,1,0)} + q(0, 0, 1)\Phi_{(0,0,1)} \\
= 1\Phi_{(1,1,1)} + i\Phi_{(1,0,0)} + j\Phi_{(0,1,0)} + k\Phi_{(0,0,1)}, \\
\mathcal{V} = q(0, 0, 0)\Phi_{(0,0,0)} + q(0, 1, 1)\Phi_{(0,1,1)} + q(1, 0, 1)\Phi_{(1,0,1)} + q(1, 1, 0)\Phi_{(1,1,0)} \\
= 1\Phi_{(0,0,0)} + i\Phi_{(0,1,1)} + j\Phi_{(1,0,1)} + k\Phi_{(1,1,0)}.
\]

Assuming that the fields \( \Phi_{(a,b,c)} \) are Abelian \( \mathcal{A} \) and \( \mathcal{V} \) are odd and even elements in a \( \mathbb{Z}_2 \) commutative algebra:

\[
\mathcal{A}\mathcal{A}' = -\mathcal{A}'\mathcal{A}, \quad \mathcal{A}\mathcal{V} = \mathcal{V}\mathcal{A}, \quad \mathcal{V}\mathcal{V}' = \mathcal{V}'\mathcal{V},
\]

where \( \mathcal{A}' \) and \( \mathcal{V}' \) are defined by \( \Phi'(a', b', c') \).

It also follows immediately from (11), (12) and the multiplication rules of quaternions that if we denote by \( \Lambda_- \) and \( \Lambda_+ \) the sets of fields respectively of the \( \mathcal{A} \)-type and \( \mathcal{V} \)-type then

\[
\mathcal{A}\mathcal{A}' = -\mathcal{A}'\mathcal{A} \in \Lambda_+, \quad \mathcal{A}\mathcal{V} = \mathcal{V}\mathcal{A} \in \Lambda_-, \quad \mathcal{V}\mathcal{V}' = \mathcal{V}'\mathcal{V} \in \Lambda_+.
\]

In conclusion \( \mathcal{A} \) and \( \mathcal{V} \) are anticommuting and commuting quaternionic fields whose component fields \( \Phi_{(a,b,c)} \) possess three independent \( \mathbb{Z}_2 \)-gradings whose physical meaning and interpretation may be different. We shall use them in what follows to formulate gauge theories with higher differential forms.

The importance of the different sign choice in (2) and (3) was noticed in [3] for two- and three-gradings formulations of supersymmetric gauge theories.

3 Higher form gauge fields and non-Abelian extension

We shall consider fields \( \Phi_{(a,b,c)}(x) \) with space-time dependence so it is natural to associate one of the gradings, -conventionally the second one labeled by \( b \), as representing the grading of even
and odd differential forms in space-time. We shall also take the first grading, labeled by $a$, as some sort of fermion-boson grading (whose exact nature will be briefly discussed at the end) while the third grading will be left unspecified and denoted by the suffix $c (= 0, 1)$.

We introduce then the following notations:

$$
\Phi_{(0,0,c)}(x) \equiv \hat{A}_c(x) = \text{direct sum of bosonic even forms},
$$

$$
\Phi_{(0,1,c)}(x) \equiv A_c(x) = \text{direct sum of bosonic odd forms},
$$

$$
\Phi_{(1,0,c)}(x) \equiv \hat{\psi}_c(x) = \text{direct sum of fermionic even forms},
$$

$$
\Phi_{(1,1,c)}(x) \equiv \psi_c(x) = \text{direct sum of fermionic odd forms}.
$$

Here and in the following we shall denote bosons and fermions, respectively, with Roman and Greek letters while even forms and odd forms are differentiated by hat and non-hat.

We introduce the exterior derivative operator $d = dx^\mu \partial_\mu$ as an $A$-type operator since it has grading; $(0, 1, 0)$:

$$
Q = q_{(0,1,0)}d = jd.
$$

A more general form of $Q$ will be briefly discussed at the end of the paper.

Extending from Abelian to non-Abelian gauge field theory, we identify the following $A$ and $V$ as generalized gauge field and gauge parameter respectively:

$$
A = A^BT^B = (1\psi_1^B + i\hat{\psi}_0^B + jA_0^B + k\hat{A}_1^B)T^B,
$$

$$
V = V^BT^B = (1\hat{\psi}_0^B + i\psi_1^B + jA_1^B + kA_0^B)T^B,
$$

where $T^B$’s are generators of Lie algebra. It is important to realize at this stage that $A^B$ and $V^B$ satisfy the same grading structure as $A$-type and $V$-type in (14). Hereafter all products such as $A^A A^B$, $A^A V^B$, $V^A V^B \cdots$ are understood as wedge products.

It is important to remark at this stage that $A$, $V$ and $Q$ have the same formal properties as respectively one form gauge fields, zero form gauge parameters and the differential operator $d$ in gauge theories. So any gauge theory that can be entirely written in terms of forms, without use of the Hodge operator, like Chern-Simons theory and Einstein gravity, admits a generalization where all fields and parameters are replaced by the quaternionic analogues $A$ and $V$ and gauge invariance in terms of the generalized parameters is automatically preserved. In this paper we shall concentrate on the generalized Chern-Simons theory, which was already studied in a similar framework in [2].

## 4 Generalized Chern-Simons actions in arbitrary dimensions

We can now construct a dimension independent formulation of generalized Chern-Simons action:

$$
S = \int \text{Tr } \left( \frac{1}{2}AQ + \frac{1}{3}A^3 \right) = \text{Tr } S^1_1 + iS^2_1 + jS^3_1 + kS^k_1,
$$

where the Tr is taken on a representation of the Lie algebra. Notice that $A$ contains now forms of arbitrary degree and so the integral is not restricted to be on a three dimensional space-time but can be formulated in any dimensions. This generalized Chern-Simons action is invariant under the following generalized gauge transformation:

$$
\delta A = QV + [A, V] = 1\delta \psi_1 + i\delta \psi_0 + j\delta A_0 + k\delta \hat{A}_1.
$$

The proof of the gauge invariance of the generalized Chern-Simons action can be derived from the following properties of the generalized gauge fields and parameters:
1. $Q^2 = 0,$

2. $\{\overline{Q}, \lambda_-\} = Q\lambda_-, \quad [\overline{Q}, \lambda_+] = Q\lambda_+,$

3. $\text{Tr}(\lambda_+\lambda'_+) = \text{Tr}(\lambda'_+\lambda_+), \quad \text{Tr}(\lambda_-\lambda'_-) = \text{Tr}(\lambda_-\lambda_-), \quad \text{Tr}(\lambda_-\lambda'_-) = -\text{Tr}(\lambda'_-\lambda_-),$

where $\lambda_-,$ $\lambda'_-$ and $\lambda_+,$ $\lambda'_+$ are, respectively, $A$-type and $V$-type fields and parameters in (17). Here $\{ , \}$ and $[ , ]$ are anti-commutator and commutator, respectively. $\overline{Q} = \mathbf{j} \overline{d}$ is an exterior derivative operating on the right.

In this quaternion valued formulation each sector of quaternion coefficients are equivalent in (18) and (19) since quaternions commute with fields and interact only among themselves. We can thus derive the following four types of generalized Chern-Simons actions:

$$S^1 = \int \text{Tr} \left[ -\psi_1(dA_0 + A_0^2 + \hat{A}_1^2 + \hat{\psi}_0^2) + \hat{A}_1(d\hat{\psi}_0 + [A_0, \hat{\psi}_0]) + \frac{1}{3}\hat{\psi}_0^3 \right],$$

$$S^2 = \int \text{Tr} \left[ -\hat{\psi}_0(dA_0 + A_0^2 + \hat{A}_1^2 - \psi_1^2) - \hat{A}_1(d\hat{\psi}_1 + \{A_0, \psi_1\}) - \frac{1}{3}\hat{\psi}_1^3 \right],$$

$$S^j = \int \text{Tr} \left[ -\frac{1}{2}A_0dA_0 - \frac{1}{3}A_0^3 + \frac{1}{2}\hat{A}_1(d\hat{A}_1 + [A_0, \hat{A}_1]) \right. 
+ \frac{1}{2}\hat{\psi}_0(d\hat{\psi}_0 + [A_0, \hat{\psi}_0]) + \frac{1}{2}\psi_1(d\psi_1 + \{A_0, \psi_1\}) - \hat{\psi}_0\{\psi_1, \hat{A}_1\} \bigg],$$

$$S^k = \int \text{Tr} \left[ -\hat{A}_1(dA_0 + A_0^2 + \hat{\psi}_0^2 - \psi_1^2) - \frac{1}{3}\hat{A}_1^3 - \psi_1(d\hat{\psi}_0 + [A_0, \hat{\psi}_0]) \right],$$

which are invariant under the following generalized gauge transformations:

$$\delta A_0 = d\alpha_0 + [A_0, \delta a_0] + \{\hat{A}_1, a_1\} + [\psi_1, \hat{\alpha}_1] - \{\hat{\psi}_0, \alpha_0\},$$

$$\delta \hat{A}_1 = -da_1 - \{A_0, a_1\} + [\hat{A}_1, \hat{\alpha}_0] + [\psi_1, \alpha_0] + \{\hat{\psi}_0, \hat{\alpha}_1\},$$

$$\delta \psi_1 = -d\alpha_1 - [A_0, \hat{\alpha}_1] - [\hat{A}_1, \alpha_0] + [\psi_1, \hat{\alpha}_0] - \{\hat{\psi}_0, a_1\},$$

$$\delta \hat{\psi}_0 = d\alpha_0 + \{A_0, \alpha_0\} - \{\hat{A}_1, \alpha_1\} + [\psi_1, a_1] + \{\hat{\psi}_0, \hat{\alpha}_0\}.$$  

In these generalized gauge transformations we find that commutators and anti-commutators are mixed. It is, however, important to realize that the generators of the Lie algebra appear only in commutators and thus are algebraically closed. For example $\{A_0, \alpha_0\} = A_0^B \alpha_0^C [T^B, T^C], \quad \{\hat{\psi}_0, \hat{\alpha}_1\} = \hat{\psi}_0^B \hat{\alpha}_1^C [T^B, T^C], \quad \{\hat{A}_1, a_1\} = \hat{A}_1^B a_1^C [T^B, T^C], \quad \cdots$ and so on. Notice that the standard 3-form Chern-Simons action and the standard gauge transformation are included respectively in the first two terms of $S^j$ and of $\delta A_0.$

It is important to recognize that the integrand of the generalized Chern-Simons action has the $A$-type nature given in (11) with respect to the quaternion structure. $S^j$ is thus bosonic odd-dimensional action and $S^k$ is bosonic even-dimensional action. $S^1$ and $S^j$ are, respectively, fermionic odd-dimensional and fermionic even-dimensional actions, whose physical interpretation is not yet clear at this moment. Notice that the structure of $S^j$ is different from the one of the other actions $S^1, S^i$ and $S^k$ which have instead a similar field structure. The origin of this difference and similarity comes respectively from the special choice of the exterior derivative operator as a $j$-component quaternion; $Q = \mathbf{j} \overline{d}$ and from the permutation invariance in the quaternion space.

5 Connection with graded Lie algebra formulation

The quaternion structure in formulating higher form gauge theory was discovered long time ago.
by one of authors (N.K.) and Watabiki [2]. In fact the generalized Chern-Simons actions (20) and
the gauge transformations (21) were already given in [2] in a graded Lie algebra framework. At
the time the origin of quaternions in formulating the generalized gauge theory was not clear. In
the present paper the origin of the quaternion is clarified and is based on the 3-grading structure.
Due to this clarification a Lie algebra formulation instead of graded Lie algebra formulation has
been successfully realized. Here we show how these two formulations are related in the current
context.

In order that a product of algebra valued fields and parameters of \( A \)-type and \( V \)-type again
belongs to \( A \)-type or \( V \)-type field or parameter, the products should be defined by the following
graded commutator:

\[
\mathcal{A} \cdot \mathcal{A}' \equiv \frac{1}{2} \{ \mathcal{A}, \mathcal{A}' \}, \quad \mathcal{A} \cdot \mathcal{V} \equiv \frac{1}{2} [ \mathcal{A}, \mathcal{V} ], \quad \mathcal{V} \cdot \mathcal{V}' \equiv \frac{1}{2} [ \mathcal{V}, \mathcal{V}' ],
\]

(22)

where \( \mathcal{A}, \mathcal{A}' \) and \( \mathcal{V}, \mathcal{V}' \) are (graded) Lie algebra valued \( \mathcal{A} \)- and \( \mathcal{V} \)-type fields and/or parameters,
respectively. In [2] the necessity of algebraic closure for the product was noticed but not clearly
stated while this point was stressed in [4].

Let us now consider the case where we do not introduce the third grading denoted by the
suffix \( c = (0,1) \), but introduce two types of generators \( T^B \) and \( \Sigma^\alpha \) which close as a graded Lie
algebra:

\[
[T^B, T^C] = f^{BC}_D T^D, \quad [T^B, \Sigma^\alpha] = g^\alpha_B \Sigma^\beta, \quad \{ \Sigma^\alpha, \Sigma^\beta \} = h^{\alpha\beta}_B T^B,
\]

(23)

where \( f^{BC}_D, g^\alpha_B \) and \( h^{\alpha\beta}_B \) are the structure constants of the graded Lie algebra.

If we define \( A \)-type and \( V \)-type valued fields and/or parameters as:

\[
\mathcal{A} = \mathbf{1}_{\psi^\alpha \Sigma^\alpha} + \mathbf{i} \hat{\psi}^B T^B + j \hat{A}^B T^B + k \hat{\Lambda}^\alpha \Sigma^\alpha \in \Lambda'_-, \quad \mathcal{V} = \mathbf{1}_{\hat{a}^B T^B + i a^\alpha \Sigma^\alpha + j \hat{a}^\alpha \Sigma^\alpha + k \alpha^B T^B} \in \Lambda'_+,
\]

(24)

where \( \Lambda'_- \) and \( \Lambda'_+ \) denote the set of \( \mathcal{A} \)-type and \( \mathcal{V} \)-type fields and/or parameters, respectively. We can then show

\[
\mathcal{A} \cdot \mathcal{A}' \in \Lambda'_+, \quad \mathcal{A} \cdot \mathcal{V} \in \Lambda'_-, \quad \mathcal{V} \cdot \mathcal{V}' \in \Lambda'_+,
\]

(25)

which are the extended version of the closure relation (14) for graded Lie algebra.

In order to obtain the gauge invariance of the generalized Chern-Simons action (18) for the
graded Lie algebra framework, it is necessary the item 3. in section 4 to be fulfilled for graded
Lie algebra counterparts:

\[
\text{Str} [ \mathcal{V}, \mathcal{V}' ] = \text{Str} [ \mathcal{A}, \mathcal{V} ] = \text{Str} \{ \mathcal{A}, \mathcal{A}' \} = 0.
\]

(26)

All these conditions can be satisfied if super trace (Str) satisfies the following relations for graded
generators:

\[
\text{Str} [ T^B, T^C ] = \text{Str} [ T^B, \Sigma^\alpha ] = \text{Str} \{ \Sigma^\alpha, \Sigma^\beta \} = 0.
\]

(27)

In particular the odd generators \( \Sigma^\alpha \)'s anti-commute within the super trace.

This result shows that the 3rd grading with Lie algebra formulation is equivalent to the
graded Lie algebra formulation without 3rd grading but with the super trace relations (27).

In the earlier generalized gauge theory formulation, the introduction of graded Lie algebra is
necessary to formulate odd dimensional generalized Chern-Simons actions including all degrees
of differential forms equally successful as for the even dimensional case [2]. In other words the
3rd grading of the current formulation is mandatory for a equal footing treatment of generalized
Chern-Simons actions for both even and odd dimensions. Thus introduction of three grading, equivalently the quaternion structure, is very important for the dimension independent treatment of the generalized gauge theory.

As an application of the above graded Lie algebra formulation, Clifford algebra formulation which satisfies loosened version of the condition [22]; a product of $T^B$ and $\Sigma^\alpha$ generators close within product, was investigated to formulate conformal gravity in two and four dimensions [5].

6 D-dimensional generalized Chern-Simons actions

It is important to realize that in the generalized Chern-Simons action the generalized gauge invariance is valid order by order in the form degree. In other words 0-form, 1-form, 2-form, 3-form, 4-form, · · · sectors of generalized Chern-Simons actions are separately invariant under the generalized gauge transformations of the gauge fields.

To derive explicit forms of the generalized Chern-Simons actions with third grading ($c = 0, 1$) for Lie algebra setting we introduce the following notations to clarify the differential form degrees:

$$
\mathcal{A} = 1\psi_1 + i\hat{\psi}_0 + jA_0 + k\hat{A}_1
$$

$$
\nu = 1\alpha_0 + i\alpha_1 + j\alpha_1 + k\alpha_0
$$

where $\phi_1^{(0)}, \omega_0^{(1)}, B_1^{(2)}, \Omega_0^{(3)}, H_1^{(4)}, · · ·$ are bosonic gauge fields of 0-, 1-, 2-, 3-, 4-form, · · · with the form degree in the parentheses. Similarly $v_0^{(0)}, u_1^{(1)}, b_0^{(2)}, U_1^{(3)}, h_0^{(4)}, · · ·$ are bosonic gauge parameters of 0-, 1-, 2-, 3-, 4-form, · · ·, respectively. Fermionic gauge fields and parameters are simply denoted by $\psi$ and $\alpha$ and the suffix “i” in (i) denotes differential form degree. Hereafter we omit the differential form degree assignments for bosonic gauge fields and parameters for simplicity.

Component expressions of the bosonic generalized Chern-Simons actions in 0-, 1-, 2-, 3-, 4-dimensions are given by

$$
S_0^k = \int \text{Tr} \left[ -\phi_1^{(0)}(\hat{\psi}_0^{(0)})^2 - \frac{1}{3}\phi_1^{(0)} \right]
$$

$$
S_1^j = \int \text{Tr} \left[ \frac{1}{2}\phi_1^{(0)}(d\phi_1 + [\omega_0, \phi_1]) - \hat{\psi}_0^{(0)}\{\psi_1^{(1)}, \phi_1\} + \frac{1}{2}\hat{\psi}_0^{(0)}(d\hat{\psi}_0^{(0)} + [\omega_0, \hat{\psi}_0^{(0)}]) \right]
$$

$$
S_2^k = \int \text{Tr} \left[ -\phi_1^{(0)}(d\omega_0 + \omega_0^2 + \{\hat{\psi}_0^{(0)}, \hat{\psi}_0^{(0)}\}) - (\psi_1^{(1)})^2 \right]
$$

$$
-\psi_1^{(1)}(d\hat{\psi}_0^{(0)} + [\omega_0, \hat{\psi}_0^{(0)}]) - \phi_1^{(0)}B_1 - (\hat{\psi}_0^{(0)})^2B_1
$$

$$
S_3^j = \int \text{Tr} \left[ -\frac{1}{2}\omega_0d\omega_0 - \frac{1}{3}\omega_0^3 + \hat{\psi}_0^{(0)}(d\hat{\psi}_0^{(0)} + [\omega_0, \hat{\psi}_0^{(0)}]) - \{\psi_1^{(1)}, B_1\} - \{\psi_1^{(0)}, \phi_1\} \right]
$$

$$
+\phi_1^{(0)}(dB_1 + [\omega_0, B_1]) - \Omega_0(\psi_1^{(1)})^2B_1 \right]
$$

$$
S_4^k = \int \text{Tr} \left[ -B_1(d\omega_0 + \omega_0^2 + \{\hat{\psi}_0^{(0)}, \hat{\psi}_0^{(0)}\}) - (\psi_1^{(1)})^2 - H_1((\hat{\psi}_0^{(0)})^2 + \phi_1^{(0)})
$$
\(-\phi_1(d\Omega_0 + \{\omega_0, \Omega_0\} + B_1^2 + \{\hat{\psi}_0^{(0)}, \psi_0^{(4)}\} - \{\psi_1^{(1)}, \psi_1^{(3)}\})
\)
\(-H_1((\hat{\psi}_0^{(0)})^2 + \phi_1^2) - \psi_1^{(1)}(d\hat{\psi}_0^{(2)} + [\omega_0, \hat{\psi}_0^{(2)}] + [\Omega_0, \hat{\psi}_0^{(2)}]) - \psi_1^{(3)}[\omega_0, \hat{\psi}_0^{(2)}])\]

where \(S_{(2k+1)}^j\) and \(S_{(2k)}^j\) for \(k = 0, 1, 2, \cdots\) are, respectively, \((2k + 1)\)-dimensional and \((2k)\)-dimensional actions. These generalized Chern-Simons actions are invariant under the following generalized gauge transformations for bosons:

\[
\begin{align*}
\delta \phi_1 &= [\phi_1, v_0] + \{\hat{\psi}_0^{(0)}, \alpha_1^{(0)}\}, \\
\delta \omega_0 &= dv_0 + [\omega_0, v_0] + \{\phi_1, u_1\} + [\hat{\psi}_0^{(1)}, \alpha_1^{(0)}] - \{\psi_0^{(0)}, \alpha_0^{(1)}\}, \\
\delta B_1 &= -du_1 - \{\omega_0, u_1\} + [\phi_1, b_0] + [B_1, v_0] + \{\hat{\psi}_0^{(0)}, \hat{\alpha}_1^{(2)}\} + [\psi_1^{(1)}, \alpha_0^{(1)}] + \{\hat{\psi}_0^{(2)}, \alpha_1^{(0)}\}, \\
\delta \Omega_0 &= db_0 + [\omega_0, b_0] + \{\phi_1, U_1\} + [B_1, u_1] + [\Omega_0, v_0] \\
&\quad - \{\hat{\psi}_0^{(0)}, \alpha_1^{(3)}\} + [\psi_1^{(0)}, \alpha_1^{(2)}] - \{\hat{\psi}_2^{(0)}, \alpha_1^{(1)}\} + [\psi_1^{(3)}, \alpha_1^{(0)}], \\
\delta H_1 &= -dU_1 - \{\omega_0, U_1\} + [\phi_1, h_0] + [B_1, b_0] - \{\Omega_0, u_1\} + [H_1, v_0] \\
&\quad + \{\hat{\psi}_0^{(0)}, \alpha_1^{(4)}\} + [\psi_1^{(0)}, \alpha_1^{(3)}] + \{\hat{\psi}_0^{(2)}, \hat{\alpha}_1^{(2)}\} + [\psi_1^{(3)}, \alpha_0^{(1)}] + \{\hat{\psi}_0^{(4)}, \alpha_1^{(0)}\},
\end{align*}
\]

and for fermions:

\[
\begin{align*}
\delta \hat{\psi}_0^{(0)} &= [\hat{\psi}_0^{(0)}, v_0] - \{\phi_1, \hat{\alpha}_1^{(0)}\}, \\
\delta \hat{\psi}_1^{(1)} &= -d\hat{\alpha}_1^{(0)} - [\omega_0, \alpha_1^{(0)}] - \{\phi_1, \alpha_0^{(1)}\} - \{\hat{\psi}_0^{(0)}, u_1\} + [\psi_1^{(1)}, v_0], \\
\delta \hat{\psi}_0^{(2)} &= d\alpha_0^{(1)} + \{\omega_0, \alpha_0^{(1)}\} - \{\phi_1, \hat{\alpha}_1^{(2)}\} - \{B_1, \alpha_1^{(0)}\} + [\hat{\psi}_0^{(0)}, b_0] + [\psi_1^{(1)}, u_1] + [\hat{\psi}_0^{(2)}, v_0], \\
\delta \hat{\psi}_1^{(3)} &= -d\hat{\alpha}_1^{(2)} - [\omega_0, \hat{\alpha}_1^{(2)}] - \{\phi_1, \alpha_0^{(3)}\} - \{B_1, \alpha_1^{(1)}\} - [\Omega_0, \hat{\alpha}_1^{(0)}] \\
&\quad - \{\hat{\psi}_0^{(0)}, U_1\} + [\psi_1^{(1)}, b_0] - [\hat{\psi}_2^{(0)}, u_1] + [\psi_1^{(3)}, v_0], \\
\delta \hat{\psi}_0^{(4)} &= d\alpha_0^{(3)} + \{\omega_0, \alpha_0^{(3)}\} - \{\phi_1, \alpha_1^{(4)}\} - \{B_1, \hat{\alpha}_1^{(2)}\} + [\Omega_0, \alpha_0^{(1)}] - \{H_1, \hat{\alpha}_1^{(0)}\} \\
&\quad + [\hat{\psi}_0^{(0)}, h_0] + [\psi_1^{(1)}, U_1] + [\hat{\psi}_2^{(0)}, b_0] + [\psi_1^{(3)}, u_1] + [\hat{\psi}_0^{(4)}, v_0].
\end{align*}
\]

It is possible to derive component expressions of fermionic generalized Chern-Simons actions in 0-, 1-, 2-, 3-, 4-dimensions. We do not have an interpretation of the actions corresponding to \(S^1\) and \(S^3\) in (20).

## 7 2-grading gauge system

We have given a formulation of higher form gauge theory in terms of the 3-grading structure with the quaternion algebra accommodating the generalized gauge system. We may wonder what happens if we consider only a 2-grading structure. It turns out that the quaternion structure is kept to classify \(A\)-type and \(V\)-type generalized gauge fields and parameters. In the current assignment of the 3-grading structure, a 2-grading formulation can be derived by simply setting fermionic gauge fields and parameters to be zero:

\[
\psi = 0, \quad \alpha = 0.
\]

In this setting a bosonic version of generalized Chern-Simons actions can be obtained with all the higher degrees of forms included. As one can see the leading terms of the actions in each dimensions have a typical \(BF\)-type structure except for 0 dimension.
Let us specifically consider here the 3- and 4-dimensional generalized Chern-Simons actions without fermions, which include the standard Chern-Simons action in 3 dimensions and $BF$ action in 4 dimensions, respectively:

\[ S_3^g(\psi = 0) = \int \text{Tr} \left[ -\frac{1}{2} \omega_0 d\omega_0 - \frac{1}{3} \omega_0^3 + \phi_1 (dB_1 + [\omega_0, B_1]) - \Omega_0 \phi_1^2 \right], \]
\[ S_4^g(\psi = 0) = \int \text{Tr} \left[ - B_1 (d\omega_0 + \omega_0^2) - \phi_1 (d\Omega_0 + \{\omega_0, \Omega_0\} + B_1^2) - H_1 \phi_1^2 \right], \]

which are invariant under the following generalized gauge transformations:

\[ \delta \phi_1 = [\phi_1, v_0], \]
\[ \delta \omega_0 = dv_0 + [\omega_0, v_0] + \{\phi_1, u_1\}, \]
\[ \delta B_1 = -du_1 - \{\omega_0, u_1\} + [\phi_1, b_0] + [B_1, v_0], \]
\[ \delta \Omega_0 = db_0 + [\omega_0, b_0] + \{\phi_1, U_1\} + \{B_1, u_1\} + [\Omega_0, v_0], \]
\[ \delta H_1 = -dU_1 - \{\omega_0, U_1\} + [\phi_1, h_0] + [B_1, b_0] - \{\Omega_0, u_1\} + [H_1, v_0]. \]

One can easily see that these generalized actions are natural generalizations of the standard Chern-Simons action in 3 and 4 dimensions that include all the degrees of differential form. The 4-dimensional generalized Chern-Simons action includes the $BF$ action as a leading term. We can interpret this formulation as a generalization of the Chern-Simons action to arbitrary dimensions and thus we call these actions generalized Chern-Simons actions. It should be noted here that in the generalized gauge transformations (34) commutators and anti-commutators are mixed. However after taking into account the odd-form nature and the odd-grading nature of the suffix “1”, all the anti-commutators turn into commutators so that algebra is closed within the Lie algebra as it was already noted after Eqs. (21).

It is important to realize that all the even-form generalized gauge fields in (33) and all the odd-form generalized gauge parameters in (34) carry the suffix 1. The even-form gauge fields and the odd-form gauge parameters have a hidden grading structure, whose nature is not completely determined and could also be interpreted as a grading of fermionic nature.

### 8 Equations of motions for generalized Chern-Simons actions

The equations of motion for the generalized Chern-Simons actions can be derived and are given by the vanishing condition for the generalized curvature:

\[ \mathcal{F} = QA + A^2 = 1 \mathcal{F}^1 + i \mathcal{F}^i + j \mathcal{F}^j + k \mathcal{F}^k = 0, \]
with

\[ \mathcal{F}^1 = -dA_0 - A_0^2 - \dot{A}_1^2 + \psi_1^2 - \dot{\psi}_0^2 = 0, \]
\[ \mathcal{F}^i = d\dot{A}_1 + [A_0, \dot{A}_1] + \{\psi_1, \dot{\psi}_0\} = 0, \]
\[ \mathcal{F}^j = dv_1 + \{A_0, v_1\} + [\dot{A}_1, \dot{\psi}_0] = 0, \]
\[ \mathcal{F}^k = -dv_0 - \{A_0, \dot{v}_0\} + [\dot{A}_1, v_1] = 0. \]

Notice that the first line of (35) is a generalization of the Maurer-Cartan equation and it admits the solution

\[ A = G^{-1} QG, \]
with
\[ G = e^{i\mathcal{V}}, \] (37)
where \( \mathcal{V} \) is a generalized parameter as in Eq. (17).

Component expressions for the equations of motion can be derived by equating to zero separately the sectors of the generalized curvature with different form degree. Bosonic equations of motion for 0-, 1-, 2-, 3-, 4-form sectors are given by:

\[-\phi^2 - (\hat{\psi}^{(0)})^2 = 0,\]
\[d\phi_1 + [\omega_0, \phi_1] + \{\psi_1^{(1)}, \hat{\psi}^{(0)}\} = 0,\]
\[d\omega_0 - \omega_0^2 - \{\phi_1, B_1\} + (\hat{\psi}_1^{(1)})^2 - \{\hat{\psi}_0^{(0)}, \hat{\psi}_0^{(2)}\} = 0,\] (38)
\[dB_1 + [\omega_0, B_1] + [\Omega_0, \phi_1] + \{\psi_1^{(1)}, \hat{\psi}_0^{(2)}\} + \{\psi_1^{(3)}, \hat{\psi}_0^{(0)}\} = 0,\]
\[d\Omega_0 - \{\omega_0, \Omega_0\} - \{\phi_1, H_1\} - B_1^2 + \{\psi_1^{(1)}, \hat{\psi}_1^{(3)}\} - \{\hat{\psi}_0^{(0)}, \hat{\psi}_0^{(4)}\} - (\hat{\psi}_0^{(2)})^2 = 0.\]

And corresponding fermionic equations of motion are

\[\{\phi_1, \hat{\psi}^{(0)}_0\} = 0,\]
\[-d\hat{\psi}^{(0)}_0 - [\omega_0, \hat{\psi}^{(0)}_0] + \{\phi_1, \hat{\psi}^{(1)}_0\} = 0,\]
\[d\hat{\psi}^{(1)}_0 + \{\omega_0, \hat{\psi}^{(1)}_0\} + [\phi_1, \hat{\psi}^{(2)}_0] + [B_1, \hat{\psi}^{(0)}_0] = 0,\] (39)
\[-d\hat{\psi}^{(2)}_0 - [\omega_0, \hat{\psi}^{(2)}_0] + \{\phi_1, \hat{\psi}^{(3)}_1\} + [B_1, \hat{\psi}^{(1)}_1] - [\Omega_0, \hat{\psi}^{(0)}_0] = 0,\]
\[d\hat{\psi}^{(3)}_1 + \{\omega_0, \hat{\psi}^{(3)}_1\} + [\phi_1, \hat{\psi}^{(4)}_0] + [B_1, \hat{\psi}^{(2)}_0] + [\Omega_0, \hat{\psi}^{(1)}_1] + [H_1, \hat{\psi}^{(0)}_0] = 0.\]

The equations of motion of 3- and 4-dimensional generalized Chern-Simons actions in (33) are simply given by setting \( \psi = 0 \) in (38). It is interesting to realize that these generalized equations of motion have the generalized gauge invariance.

9 Generalized topological relations

A generalized gauge theory version of the second Chern character can be defined and it is related to the generalized Chern-Simons action in analogy with the standard gauge theory:

\[
\int \text{Tr}(F^2) = \int \text{Tr} Q \left( A Q A + \frac{2}{3} A^3 \right) = 1G^1 + iG^i + jG^j + kG^k, \] (40)

where \( F, A \) and \( Q \) are defined in (35), (28) and (16), respectively.

From the sector-wise equivalence for the quaternion in (40) we can obtain the following new topological relations:

\[
G^1 \equiv \int \text{Tr} \left( (F^1)^2 - (F^i)^2 - (F^j)^2 - (F^k)^2 \right) = -2 \int dL_{GCS}^1,
\]
\[
G^i \equiv \int \text{Tr} \left( \{F^1, F^i\} + [F^j, F^k] \right) = 2 \int dL_{GCS}^k,
\]
\[
G^j \equiv \int \text{Tr} \left( \{F^1, F^j\} + [F^k, F^i] \right) = 2 \int dL_{GCS}^1,
\]
\[
G^k \equiv \int \text{Tr} \left( \{F^1, F^k\} + [F^i, F^j] \right) = -2 \int dL_{GCS}^i, \] (41)

10
where $F^A (A = 1, i, j, k)$ are given in (35) and $L^A_{GCS} (A = 1, i, j, k)$ are given by the following relations:

$$S^A = \int \text{Tr} L^A_{GCS}, \quad (A = 1, i, j, k),$$

with $S^A$ given in (20).

It should be recognized here that $\text{Tr}(F^2)$ and correspondingly the right hand side in (40) have the $V$-type quaternion structure. Therefore $G_1$ and $G_i$ are bosonic even- and odd-form sectors while $G^j$ and $G^k$ are fermionic even- and odd-form sectors respectively.

Reduction to 2-grading formulation which includes all degrees of differential forms for bosonic gauge fields with hidden grading for even-forms, can be easily obtained by simply setting $\psi = 0$ in $G_1$ and $G_i$ while $G^j$ and $G^k$ disappear. For a given space-time dimension the expressions of $G^A$ can be derived by extracting in (41) the sector of the corresponding from degree. For example 4-dimensional counterpart of the generalized topological relation without fermions is given by

$$G^1_4 = \int \text{Tr} \left[ (d\omega_0 + \omega_0^2 + \{\phi_1, B_1\})^2 + \{\phi_1^2, d\Omega_0 + \{\omega_0, \Omega_0\} + B_1^2 + \{\phi_1, H_1\} \right]$$

$$= \int \text{Tr} d \left[ \omega_0 d\omega + \frac{2}{3} \omega_0^3 - 2\phi_1 (dB_1 + [\omega_0, B_1]) + 2\Omega_0 \phi_1^2 \right],$$

where the last term is the exterior derivative times the 3-dimensional generalized Chern-Simons action in (33). It should be noted that 0-, 2-, 4-form gauge fields $\phi_1$, $B_1$ and $H_1$ carry a hidden $Z_2$-grading. Notice also that some nontrivial cancellations occur in going from the second to the third term due to the hidden grading. It is interesting to realize that the topological Yang-Mills action is included as the first term in the action in (43).

### 10 Generalized differential operators

In the component expressions for the actions given in (29) we have specifically assigned the first and the second grading as (fermion, boson) and (even form, odd form) so that the second grading is related to the space-time structure of the generalized fields and parameters. There is some freedom in the interpretation of gradings. A physical meaning of the third grading is unspecified in this paper. This assignment of gradings automatically generates an asymmetric nature of exterior derivative operator with respect to the quaternion structure as in (36). This special assignment is the origin of the difference between the action $S^j$ and the rest of the actions and the similarity of $S^1$, $S^i$, and $S^k$, as is already pointed out.

We have introduced fermionic fields which have an anti-symmetric space-time structure. There is no spinor introduced. One may wonder what kind of fermions they are. It was shown that these fermions can be identified as ghost fields of quantized generalized Chern-Simons, which turned out to be an infinitely reducible gauge system [6]. It was noticed that the generalized differential operator can be extended to:

$$Q = -is + jd,$$

where $s$ is a fermionic zero form and can be interpreted as a BRST operator.

The generalized Chern-Simons action (18) has the gauge invariance with the generalized gauge parameters which turn into ghosts after quantization. It is surprising to find that $V$-type gauge parameters turn into ghosts as $A$-type gauge fields due to the change into alternative
grading for fermion and boson. This structure is realized as a BRST transformation by the addition of $-i\sigma$ term for the $Q$-operator in (44). It is fundamentally reflected from the fact that the generalized gauge transformation is infinitely reducible [6].

The quantization of the generalized Chern-Simons action [6] was completed by BV and BFV formalism [7, 8]. BV formalism is the special realization of Q-P manifold, à la AKSZ formalism [9]. The connection of the current formulation of the generalized gauge theory and the AKSZ formalism is of a great interest. It is expected that the topological particle field theory formulation which was discovered to be equivalent to the quaternion formulation of the generalized gauge theory plays an important role [10].

As a generalization of the $Q$ operator (44) we propose the most symmetric differential operator:

$$Q = -1s d\sigma - is + jd + k\sigma,$$

where $\sigma$ is an exterior derivative operator corresponding to the 3rd grading. This $Q$ operator satisfies the item 1. and 2. in section 4 and thus define a more general and symmetric generalized gauge theory.

11 Conclusions and discussions

The generalized Chern-Simons gauge theory to accommodate all the degrees of differential forms as Lie algebra valued gauge fields and parameters is explicitly formulated for arbitrary even and odd dimensions. The role of quaternion in the formulation is clarified as an alternative presentation for three grading structure of the generalized gauge fields and parameters. The connection of the Lie algebra formulation with the original graded Lie algebra formulation is also clarified.

In considering a gauge system of three physically independent gradings, it is natural to assume that these gradings commute each other. For example the differential form grading of space-time and the quantized ghost grading have no physical connection in general. Thus we assume that these gradings are commutative as in (2). In this physical system the quaternion accompanied formulation of the generalized gauge theory is necessary. On the other hand this formulation can be equivalently described by the formulation of the total three gradings of (3).

There has been recently a growing interest in higher form gauge theory as a way of finding a unified formulation of extended objects such as D-branes [11, 12, 13] and of getting a deeper understanding of duality related prescriptions for gauge theories [14]. We consider that the present formulation as well as its original version [2] have possible applications in this new context and also in the former formulations of topological field theories [15]. Chern-Simons related gravities [11, 15, 16], and higher spin formulations [17]. It is also known that the standard Chern-Simons and BF actions lead simplicial gravities of Ponzano-Regge type in 3 dimensions and 15-$j$ topological gravity type in 4 dimensions [18, 19]. One may wonder what is the role of the extra differential forms other than the ones constructing gravity background. The generalized gauge theory formulation suggests a hope that we may be able to formulate a simplicial gravity theory with matter introduced naturally on the simplicial lattice by simplex-form correspondence [20].

Finally we would like to point out that physical identification of the unknown third grading would be very important for finding the generalized gauge symmetry.

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