Numerical Solutions of the
Einstein-Yang-Mills System with
Cosmological Constant

Pál Géza Molnár

Institute for Theoretical Physics, University of Zürich,
Winterthurerstrasse 190, 8057 Zürich, Switzerland

Abstract: Numerical evidence for a cosmological version of the Bartnik-McKinnon family of particle-like solutions of the Einstein-Yang-Mills system is presented. Our solutions are also static, but space has the topology of a three-sphere. By adjusting the cosmological constant we found numerically a spherically symmetric solution which can be regarded as an excitation of the unique SO(4)-invariant solution. We expect that for each node number there exists such a solution without a cosmological horizon.

Diploma thesis
written under the supervision of
Prof. Dr. N. Straumann
Winter 1994/95

1Electronic address: molnar@hirschen.physik.unizh.ch
Az apámnak
Contents

1 Introduction 1

2 The EYM\(_\Lambda\) differential equations 3
   2.1 Derivation of the differential equation system 3
   2.2 An analytical solution to the system 7
   2.3 Interpretation of the solution 8
   2.4 The EYM\(_\Lambda\) equations for \( L, R, w \) 9
   2.5 Expansions 12
   2.6 Properties of the differential equation system 14

3 Numerics 22

4 Results 24

5 Time dependent cosmological solutions to the EYM system 28
1 Introduction

In recent years, relativists have been researching on what happens when gravity is coupled to nonlinear field theories, as for example the Yang-Mills (YM) fields. In the beginning, there was doubt about whether there were any solutions, since neither the vacuum-Einstein equations \[1\], nor the pure YM equations have nontrivial, static, globally regular solutions \[3, 4\].

Thus, it was even more surprising, as in 1988 Bartnik and McKinnon (BK) discovered soliton solutions to the SU(2) Einstein-Yang-Mills (EYM) system numerically \[4\]. These solutions provided for the first time solid evidence that the EYM system has particle-like solutions. Later, other authors \[5-7\] found the colored black hole solutions to the same model, which are a counterexample to the no-hair conjecture; i.e., the well-known uniqueness theorem for stationary black holes of the Einstein-Maxwell system has no natural generalisation on the non-Abelian case. Soon, the existence of both types of solution was established rigorously \[8, 10, 11\], and their instability was proven \[12, 13, 14, 15\]. Because of this instability, it was expected that it would also apply to arbitrary gauge groups. Proof was first successfully provided for the EYM solitons \[16\] and shortly afterwards for black holes as well \[17\].

There were also attempts to find corresponding solutions for other related field theories. The Einstein-Skyrme system for example has black hole solutions with hair that are at least linearly stable \[18, 19, 20, 21\]. Several authors looked at other models, notably the SU(2) Einstein-Yang-Mills-Higgs (EYMH) theory with a Higgs triplet \[22, 23, 24\], as well as the EYM-dilaton theory \[25\]. Recently, the instability of the gravitating, regular sphaleron solutions to the SU(2) EYMH system with a SU(2) Higgs doublet, that were known numerically for some time \[27\], could be shown \[26\].

In November, 1993, Ding and Hosoya (DH) presented a spherically symmetric, analytic solution to the SU(2) EYM system with a positive cosmological constant (EYM\(_\Lambda\)) \[28\]. Although, it turned out that the DH solution is not really new, but gauge equivalent \[29\] to the cosmological solution of \[30\], it triggered the work presented in this paper. It is natural to expect that the cosmological solution given in \[30\] is just the ground state of a discrete family of static spherically symmetric solutions living on a topological 3-sphere. For reasons to be explained later, one will, however, have to adjust the cosmological constant for each member of the family. In my paper I will
present the first excited solution which I found numerically. As in the BK case, further solutions were expected. In contrast to the analytic solution of Hosotani or Ding and Hosoya, $g_{00}$ is no longer constant in my solution. Furthermore, I will prove some general properties of the regular solutions, e.g. the interesting fact that solutions on $S^3$ can only exist for a non-vanishing cosmological constant.

The paper will be organized as follows: In chapter 2, I will first show a derivation of the SU(2) EYM$_\Lambda$ system and the simplest way for obtaining the DH solution. This will be followed by the above-mentioned general properties of regular solutions. In chapter 3, I will tell how I found the solutions numerically, while the conclusions and the discussion can be found in chapter 4. In conclusion, I will present a cosmological embedding of the EYM$_\Lambda$ system which remarkably simplifies the consideration of stability of the DH solution. The idea can be attributed to N. Straumann.
2 The EYM$_\Lambda$ differential equations

2.1 Derivation of the differential equation system

We write the metric for a static, spherically symmetric spacetime in the usual way:

\[ g = -e^{2a}dt^2 + e^{2b}dr^2 + r^2(d\vartheta^2 + \sin^2\vartheta \, d\varphi^2) . \] (1)

\(a\) and \(b\) are functions depending exclusively upon \(r\).

We are only interested in YM fields that are purely magnetic. According to Ref. [17], we can determine the SU(2) gauge potential \(A\) as

\[ A = (-\bar{w} \tau_1 + w \tau_2) \, d\vartheta + (\tau_3 \cot \vartheta - w \tau_1 - \bar{w} \tau_2) \, \sin \vartheta \, d\varphi . \] (2)

\(w, \bar{w}\) are dependent upon \(r\), and \(\tau_1, \tau_2, \tau_3\) are the spheric SU(2) generators with \(\vec{\tau} = \vec{\sigma}/2i\) (\(\vec{\sigma}\) being Pauli matrices). With a gauge transformation we can furthermore make \(\bar{w}\) disappear, with \(A\) finally assuming the following form:

\[ A = w \tau_2 \, d\vartheta + (\tau_3 \cos \vartheta - w \tau_1) \, \sin \vartheta \, d\varphi . \] (3)

This means we are looking for \(a = a(r), b = b(r)\) and \(w = w(r)\).

With (3), we can calculate the components of the field strength tensor \(F = dA + A \land A\)

\[ F_{12} = w' \tau_2 e^{-b} \frac{1}{r}, \quad F_{13} = -w' \tau_1 e^{-b} \frac{1}{r}, \quad F_{23} = (w^2 - 1) \tau_3 \frac{1}{r^2} . \] (4)

We made use of the often employed orthonormal basis of 1-forms

\[ \theta^0 = e^a dt, \quad \theta^1 = e^b dr, \quad \theta^2 = r \, d\vartheta, \quad \theta^3 = r \sin \vartheta \, d\varphi , \] (5)

and the notation \(\tau \equiv \partial_r\). The definition of the YM-Lagrangian leads us to

\[ L_{\text{YM}} = -\frac{1}{16\pi} \, tr(F_{\mu\nu} F^{\mu\nu}) , \]

\[ L_{\text{YM}} = \frac{1}{8\pi} \, w^2 e^{-2b} \frac{1}{r^2} + \frac{1}{16\pi} \, \frac{(w^2 - 1)^2}{r^4} . \] (6)

In the General Theory of Relativity (GR) we obtain the Einstein field equations by varying the action

\[ S = \int_D \left[ -\frac{1}{16\pi G} R - \frac{1}{16\pi} \, tr(F_{\mu\nu} F^{\mu\nu}) \right] \sqrt{-g} \, dx^4 \] (7)
with respect to the metric:

\[ G^{\mu\nu} = 8\pi G T^{\mu\nu} \quad (8) \]

\[ T^{\mu\nu} = -\frac{1}{4\pi} \left[ F_{\alpha\mu} F^{\alpha}_{\nu} - \frac{1}{2} g_{\mu\nu} (F|F) \right]. \quad (9) \]

The components of the energy-momentum tensor become

\[ T_{00} = \frac{1}{8\pi} w'^2 e^{-2b} r^2 + \frac{1}{16\pi} \frac{(w^2-1)^2}{r^4}, \]
\[ T_{11} = \frac{1}{8\pi} w'^2 e^{-2b} r^2 - \frac{1}{16\pi} \frac{(w^2-1)^2}{r^4}, \]
\[ T_{22} = T_{33} = \frac{1}{16\pi} \frac{(w^2-1)^2}{r^4}. \quad (10) \]

and the \( G_{\mu\nu} \) relative to the basis (3) are

\[ G^0_0 = -\frac{1}{r^2} + e^{-2b} \left( \frac{1}{r^2} - \frac{2b'}{r} \right), \]
\[ G^1_1 = -\frac{1}{r^2} + e^{-2b} \left( \frac{1}{r^2} + \frac{2b'}{r} \right), \]
\[ G^2_2 = G^3_3 = e^{-2b} \left( a'^2 - a'b' + a'' + \frac{a'-b'}{r} \right). \quad (11) \]

All other \( G_{\mu\nu} \) equal zero.

The spherically symmetric Einstein field equations with \( \Lambda \)-term are

\[ G_{\mu\nu} = 8\pi G T_{\mu\nu} - g_{\mu\nu} \Lambda. \quad (12) \]

We insert (10) and (11) into (12):

\[ -\frac{1}{r^2} + e^{-2b} \left( \frac{1}{r^2} - \frac{2b'}{r} \right) = -G w'^2 e^{-2b} \frac{r^2}{r^2} - G \frac{(w^2-1)^2}{2} \frac{1}{r^4} - \Lambda, \quad (13) \]
\[ -\frac{1}{r^2} + e^{-2b} \left( \frac{1}{r^2} + \frac{2b'}{r} \right) = G w'^2 e^{-2b} \frac{r^2}{r^2} - G \frac{(w^2-1)^2}{2} \frac{1}{r^4} - \Lambda. \quad (14) \]

For the YM equations, we can operate in a similar way as in obtaining in the GR the field equations from a variational principle. Here, the action is

\[ S_{YM} = \int L_{YM} \eta. \quad (15) \]

If we now vary with respect to the gauge potential \( A \), we obtain

\[ D \ast F = 0. \quad (16) \]
In addition, we can also prove the identity
\[ DF = D(dA + A \wedge A) = 0 . \] (17)

The best way to do this, is to use \( D = d + [A, .] \), as well as \([B, C] = B \wedge C - (-1)^m C \wedge B\) for \( B \in \Lambda^p(M), C \in \Lambda^q(M)\). The equations (16) and (17) are the known YM equations.

But now, back to our problem. We write the equation (16) for our \( A \) and our basis \( \{\theta^\mu\} \) in full:
\[ e^{-2b} \frac{w'' + w'(a' - b')}{r} - \frac{(w^2 - 1)w}{r^3} = 0 . \] (18)

(18) is the equation of motion of \( w \). The equations (13), (14) and (18) constitute a complete set of equations for the static spherically symmetric \( \text{SU}(2)\) EYM system. Now, we have three differential equations for \( a(r) \), \( b(r) \) and \( w(r) \). We write them down for \( m(r) \) and \( \delta(r) \) instead of for \( a(r) \) and \( b(r) \), with
\[ \delta(r) := -a(r) - b(r) , \]
\[ e^{-2b(r)} := 1 - \frac{2m(r)}{r} , \]
\[ N(r) := e^{-2b(r)} = 1 - \frac{2m(r)}{r} , \]
\[ S(r) := e^{-\delta(r)} . \]

From (13)–(14) and \( \delta' = -(a' + b') \), we obtain
\[ \delta' = -G \frac{w'^2}{r} . \] (23)

Then we differentiate (20) with respect to \( r \) and insert the result into (13) (with regard to (21)):
\[ m' = \frac{G}{2} w'^2 N(r) + \frac{G}{4} \frac{(w^2 - 1)^2}{r^2} + \frac{\Lambda}{r} r^2 . \] (24)

We set \( V := \frac{(w^2 - 1)^2}{2r^2} \), note that \( N' = \frac{2m}{r^2} - \frac{2m'}{r} \), multiply both sides by 2 and obtain
\[ 2m' = G \left( V + Nw'^2 \right) + \Lambda r^2 . \]
With \( 2m' = \frac{2m}{r} - rN' \) and \( N - 1 = -\frac{2m}{r} \) the whole turns into
\[
-r N' = G (V + N w'^2) + (N - 1) + \Lambda r^2 .
\] (25)

We write (18) in a slightly different way
\[
e^{-2b} w'' + e^{-2b} w' (a' - b') - \frac{w^2 - 1}{r^2} w = 0 . \] (26)

Then, we consider the expression \( (Ne^{-\delta} w')' \). Writing it out and multiplying by \( e^\delta \) gives
\[
\frac{1}{e^{-\delta}} (Ne^{-\delta} w')' = e^{-2b} w' (a' - b') + e^{-2b} w'' . \] (27)

Comparison of (27) with (26) gives us:
\[
(Ne^{-\delta} w')' = \frac{w^2 - 1}{r^2} w e^{-\delta} . \] (28)

The definition of \( V \) leads to:
\[
\frac{1}{2} \frac{\partial V}{\partial w} = \frac{w(w^2 - 1)}{r^2} , \quad \frac{\partial V}{\partial w} \equiv V_w .
\]

Thus, (28) becomes:
\[
(Ne^{-\delta} w')' = \frac{1}{2} V_w e^{-\delta} . \] (29)

We summarize the results:
\[
-r N' = G (V + N w'^2) + (N - 1) + \Lambda r^2 \] (30)
\[
-r \delta' = G w'^2 \] (31)
\[
(Ne^{-\delta} w')' = \frac{1}{2} V_w e^{-\delta} , \] (32)

with
\[
V = \frac{(1 - w^2)^2}{2r^2} , \quad \frac{1}{2} V_w = \frac{w(w^2 - 1)}{r^2} .
\]

(30) - (32) constitute for the functions \( N(r) \), \( \delta(r) \), \( w(r) \) our complete system of differential equations of the SU(2) EYM_\Lambda system.
2.2 An analytical solution to the system

In 1993, S. Ding and A. Hosoya presented an analytical solution to the system (30) - (32) [28]. I will give within the framework of our formulas another derivation.

We make the ansatz \( S(r) \equiv N(r)^{\alpha} \), \( \alpha = -\frac{1}{2} \). This, along with (22), gives us

\[
\delta' = \frac{1}{2} \frac{N'}{N}.
\] (33)

Our system (30) - (32) thus becomes

\[
-r N' = N - 1 + G(V + N w'^2) + \Lambda r^2, \quad \text{(34)}
\]

\[
-r N' = 2 G N w'^2, \quad \text{(35)}
\]

\[
\left(\sqrt{N} w'\right)' = \frac{1}{2\sqrt{N}} V_{,w}. \quad \text{(36)}
\]

From (34) - (35), we obtain

\[
GNw'^2 = N - 1 + GV + \Lambda r^2
\]

or

\[
G\left(\sqrt{N} w'\right)^2 = N - 1 + GV + \Lambda r^2. \quad \text{(37)}
\]

We calculate the derivative of (37) with respect to \( r \):

\[
2 G \left(\sqrt{N} w'\right)\left(\sqrt{N} w'\right)' = \left(N - 1 + GV + \Lambda r^2\right)'. \quad \text{(38)}
\]

On the other hand, (36) leads to

\[
2 G \left(\sqrt{N} w'\right)\left(\sqrt{N} w'\right)' = GV_{,w} w'. \quad \text{(39)}
\]

We compare (38) with (39), bearing in mind that \( V' = V_{,w} w' + V_{,r} = V_{,w} w' - \frac{2}{r} V \) Thus, we obtain

\[
N' = 2 G \frac{V}{r} - 2 \Lambda r. \quad \text{(40)}
\]

In (40), we eliminate \( N' \) with (35)

\[
Nw'^2 + V = \frac{\Lambda}{G} r^2. \quad \text{(41)}
\]
We insert this into (34) and obtain
\[ N' = \frac{-N}{r} + \left( \frac{1}{r} - 2\Lambda r \right). \] (42)

We clearly see that (42) is linear.

The corresponding solution is:
\[ N = \frac{c}{r} + \left( 1 - \frac{2}{3}\Lambda r^2 \right). \]

\( c \) is an integration constant. With the condition \( N(0) < \infty \), \( c \) equals zero.

Thus:
\[ N = 1 - \frac{2}{3}\Lambda r^2. \] (43)

We differentiate (43): \( N' = -\frac{4}{3}\Lambda r \), insert this into (40) and solve for \( w \):
\[ w^2 = 1 - \sqrt{2\Lambda/3G} r^2. \] (44)

If we insert (43) and (44) into (35), we see that
\[ 1 = \frac{N}{w^2}. \] (45)

According to (45), we equate (43) with (44) and obtain a quadratic equation in \( \Lambda \):
\[ \Lambda \left( \frac{2G}{3} \Lambda - 1 \right) = 0. \]

The sensible solution is \( \Lambda = \frac{3}{2G} \) which results finally in the DH solution
\[ w^2 = N = 1 - \frac{r^2}{G}. \] (46)

2.3 Interpretation of the solution

Our ansatz has been: \( S = N^{-1/2} \), which means that \( e^{2\alpha} = 1 \). With this and with (46), the metric (1) becomes
\[ g = -dt^2 + \frac{1}{1 - \frac{r^2}{G}} dr^2 + r^2 d\Omega^2. \] (47)
We want to replace $r$ by

$$r = \frac{u}{1 + \frac{u^2}{4G}}$$

and thus re-write our metric:

$$g = -dt^2 + \frac{1}{(1 + \frac{u^2}{4G})^2} \, du^2 + \frac{u^2}{(1 + \frac{u^2}{4G})^2} \, d\Omega^2. \quad (48)$$

Thus, our solution (46) describes an Einstein universe [31].

Finally, we observe

$$d\varrho = \frac{dr}{\sqrt{1 - \frac{r^2}{G}}} . \quad (49)$$

This can be solved:

$$r = \sqrt{G} \sin \left( \frac{\varrho}{\sqrt{G}} + c \right) .$$

c is an integration constant. If we require that $r(\varrho = 0) = 0$, $c$ becomes zero. Thus:

$$r = \sqrt{G} \sin \frac{\varrho}{\sqrt{G}} . \quad (50)$$

Now, the metric (47) can be written as follows:

$$g = -dt^2 + d\varrho^2 + G \sin^2 \left( \frac{\varrho}{\sqrt{G}} \right) \, d\Omega^2 . \quad (51)$$

This means that the spacetime manifold is $\mathbb{R} \times S^3$, $\varrho$ being an angular coordinate of $S^3$.

### 2.4 The EYM\(_{\Lambda}\) equations for \(L, R, w\)

We set up the EYM\(_{\Lambda}\) equations again, this time for the functions $L$, $R$, $w$ of the coordinate $\varrho$. Our metric now reads like this:

$$g = -L^2 dt^2 + d\varrho^2 + R^2(\varrho) \, d\Omega^2 , \quad (52)$$

$$R \equiv r , \quad L = \sqrt{\Lambda} N .$$
We use our old EYM$_{\Lambda}$ equations (30) - (32) and the relation
\[ \frac{dr}{d\tilde{q}} \equiv \sqrt{N} = \dot{\tilde{R}}. \] (53)

We use the notation:
\[ ' \equiv \frac{d}{dr}, \quad \cdot \equiv \frac{d}{d\tilde{q}}. \]

First, we need
\[ N' = \left( \dot{\tilde{R}}^2 \right)' = \left( \dot{\tilde{R}}^2 \right) \cdot \frac{d\tilde{q}}{dr} = 2\dot{\tilde{R}}, \] (54)
\[ w' = \dot{\tilde{w}} \cdot \frac{d\tilde{q}}{dr} = \frac{\ddot{\tilde{w}}}{\tilde{R}}. \] (55)

We insert that into (30)
\[ \ddot{\tilde{R}} = \frac{-1}{2\tilde{R}} \left[ G (V + \tilde{w}^2) + (\dot{\tilde{R}}^2 - 1) + \Lambda\tilde{R}^2 \right]. \] (56)

Next, we need
\[ S = \frac{L}{\tilde{R}} = e^{-\delta}. \] (57)

Its derivative results in
\[ -\delta' = \frac{\dot{L}}{LR} - \frac{\ddot{R}}{\tilde{R}^2}. \] (58)

(58) and (53) turn (31) into
\[ R \frac{\dot{L}}{LR} - R \frac{\ddot{R}}{\tilde{R}^2} = G \frac{\ddot{w}^2}{\tilde{R}^2} \]

and solved for $\dot{\tilde{L}}$
\[ \dot{\tilde{L}} = \frac{L\ddot{R}}{\tilde{R}} + G \frac{\ddot{w}^2 L}{RR}. \] (59)

With the help of (53), (53) and (57), (32) becomes
\[ \tilde{L} \ddot{\tilde{w}} + L \ddot{\tilde{w}} = \frac{1}{2} V_{\tilde{w}} L. \] (60)

By inserting (59) into (60), we can eliminate $L$:
\[ \ddot{\tilde{w}} = \frac{1}{2} V_{\tilde{w}} - \frac{\ddot{\tilde{w}}}{\tilde{R}} \left( G \frac{\ddot{w}^2}{\tilde{R}} + \ddot{R} \right). \] (61)
And with (56), $\dot{R}$ also disappears:

$$\ddot{w} = \frac{1}{2} V_{,w} + \frac{\dot{w}}{2 R R} \left[ G (V - \dot{w}^2) + (\dot{R}^2 - 1) + \Lambda R^2 \right].$$  \hfill (62)

Summarizing:

$$\dot{R} = -\frac{1}{2 R} \left[ G (V + \dot{w}^2) + (\dot{R}^2 - 1) + \Lambda R^2 \right]$$  \hfill (63)

$$\ddot{w} = \frac{1}{2} V_{,w} + \frac{\dot{w}}{2 R R} \left[ G (V - \dot{w}^2) + (\dot{R}^2 - 1) + \Lambda R^2 \right]$$  \hfill (64)

$$\dot{L} = \frac{L \dot{R}}{R} + G \frac{w^2 L}{R R}.$$  \hfill (65)

We know from our special solution (54) that $R(\rho)$ becomes maximal at the equator, i.e.:

$$\dot{R}(\rho) \bigg|_{\rho = \sqrt{G \frac{\pi}{2}}} = 0.$$

To regularize our equations (63) - (65), we introduce a new function that upon considering the equations is perfectly obvious:

$$\gamma(\rho) := \frac{1}{R R} \left[ G (V - \dot{w}^2) + (\dot{R}^2 - 1) + \Lambda R^2 \right].$$  \hfill (66)

Thus, (63), (64) and (65) become:

$$\ddot{R} = -\frac{1}{2} \dot{R} \gamma - G \frac{\dot{w}^2}{R},$$

$$\ddot{w} = \frac{1}{2} V_{,w} + \frac{1}{2} \dot{w} \gamma,$$

$$\dot{L} = -\frac{1}{2} L \gamma.$$

We also need $\dot{\gamma}$. After a longer, but not at all difficult calculation, we find that

$$\dot{\gamma} = \frac{1}{2} \gamma^2 - \frac{2}{R^2} \left[ 2 G V + \dot{R}^2 - 1 \right].$$

Our final system now looks like this:

$$\ddot{w} = \frac{1}{2} V_{,w} + \frac{1}{2} \dot{w} \gamma.$$  \hfill (67)
\[ \ddot{R} = -\frac{1}{2} \dot{R} \gamma - G \frac{\dot{w}^2}{R}, \]  
(68) 
\[ \dot{\gamma} = \frac{1}{2} \gamma^2 - \frac{2}{R^2} \left[ 2G V + \dot{R}^2 - 1 \right], \]  
(69) 
\[ \dot{L} = -\frac{1}{2} L \gamma, \]  
(70)

with

\[ \gamma = \frac{1}{RR} \left[ G (V - \dot{w}^2) + (\dot{R}^2 - 1) + \Lambda R^2 \right], \]  
(71) 
\[ V = \frac{(w^2 - 1)^2}{2R^2}, \]  
(72) 
\[ \frac{1}{2} V_w = \frac{w(w^2 - 1)}{R^2}. \]  
(73)

From now on, we will work only with this system.

In conclusion, I will show how the solution of Ding & Hosoya \cite{28} looks in these coordinates. We can find them quickly with the help of (50) and (46):

\[ R = \sqrt{G} \sin \left( \frac{\varrho}{\sqrt{G}} \right), \]  
(74) 
\[ w = \cos \left( \frac{\varrho}{\sqrt{G}} \right), \]  
(75) 
\[ L = \dot{R} S = \sqrt{\frac{N}{N}} = 1, \]  
(76) 
\[ \gamma = 0, \]  
(77)

\[ \varrho \in \left[ 0, \sqrt{G} \pi \right]. \]

### 2.5 Expansions

We already stated that in the case \( \varrho = 0 \) we require that \( R = 0 \). Furthermore, in \( \gamma \) (71) we encounter the term \( \Lambda R^2 \). As the spacetime manifold has the topology of \( S^3 \), \( R \) equals zero again at a certain \( \varrho \). We will call this \( \varrho_0 \). Thus, we have

\[ R(0) = 0, \quad R(\varrho_0) = 0. \]
For the solution (74) - (77), \( \varrho_0 = \sqrt{G\pi} \). Unfortunately, in our system (67) - (73) \( R \) also appears in the denominator; therefore, we have to expand our functions \( R, w, \gamma, L \) at \( \varrho = 0 \) and \( \varrho = \varrho_0 \).

Let us first take a look at (21). As we want regular solutions, \( m(r) \to 0 \) must hold for \( r \to 0 \), and thus \( N(r) \to 1 \). According to (53), that means that
\[
\dot{R}^2 \to 1 \quad (\varrho \to 0 \text{ or } \varrho \to \varrho_0)
\]

If \( R \to 0 \), then \( w \to \pm 1 \), since \( V \) otherwise becomes singular, which is obvious from (72). Equation (68) gives us
\[
R \ddot{R} = -\frac{1}{2} R \dot{R} \dot{\gamma} - G \dot{w}^2.
\]

We see immediately that \( \dot{w} \to 0 \), if \( \varrho \to 0 \) or \( \varrho \to \varrho_0 \). This corresponds to the observation made by BK [4], which says that the solutions become asymptotically to \( w = \pm 1 \), since \( \dot{w} \) must be zero there.

With enough patience in differentiation and applying the rule of Bernoulli-de l’Hôpital several times, the equations (57) - (73) provide us the important conditions:
\[
R^{(3)} \to -\frac{G\dot{w}^2}{2R} - \frac{\Lambda}{3R} \quad \varrho \to 0, \varrho \to \varrho_0.
\]

\( R^{(3)} \) and \( \dot{w} \) can be any values. We set \( R^{(3)} := \mp 2c, c > 0 \) and \( \dot{w} := -2b, b > 0 \). We summarize our results again:
\[
\begin{array}{c|c}
R & \to 0 \\
\dot{R} & \to 1 \\
\ddot{R} & \to 0 \\
R^{(3)} & \to -2c \\
w & \to 1 \\
\dot{w} & \to 0 \\
\ddot{w} & \to -2b \\
\varrho & \to 0 \\
\end{array}
\quad
\begin{array}{c|c}
R & \to 0 \\
\dot{R} & \to -1 \\
\ddot{R} & \to 0 \\
R^{(3)} & \to +2c \\
w & \to \pm 1 \\
\dot{w} & \to 0 \\
\ddot{w} & \to -2b \\
\varrho & \to \varrho_0 \\
\end{array}
\]

These conditions lead to:
\[
c = Gb^2 + \frac{\Lambda}{6}, \quad (78)
\]

13
\[ R = \varrho - \frac{c}{3} \varrho^3, \quad (79) \]
\[ w = 1 - b \varrho^2, \quad (80) \]
\[ R = (\varrho_0 - \varrho) - \frac{c}{3} (\varrho_0 - \varrho)^3, \quad (81) \]
\[ w = \mp 1 \pm b (\varrho_0 - \varrho)^2; \quad (82) \]

± in the formula (82) depends upon whether \( w \) has an even or an odd number of zeros. (78) - (82) lead to the expansion of \( \gamma \):

\[ \gamma = \left( -4Gb^2 + \frac{2}{3} \Lambda \right) \varrho, \quad \varrho \to 0 \quad (83) \]
\[ \gamma = \left( 4Gb^2 - \frac{2}{3} \Lambda \right) (\varrho - \varrho_0), \quad \varrho \to \varrho_0. \quad (84) \]

Now we can insert (83) and (84) into (70), in order to obtain \( L \):

\[ L = 1 + \left( Gb^2 - \frac{\Lambda}{6} \right) \varrho^2, \quad \varrho \to 0 \quad (85) \]
\[ L = 1 + \left( Gb^2 - \frac{\Lambda}{6} \right) (\varrho_0 - \varrho)^2, \quad \varrho \to \varrho_0. \quad (86) \]

As an example, we can expand (74) and (75) and compare them with (78) - (80). Thus, we obtain \( b = 0.25 \) and \( \Lambda = 0.75 \), if we take \( G = 2 \). When solving numerically, we need those values.

2.6 Properties of the differential equation system

We have seen in Sec. 2.5. that the function \( w \) starts at 1 and ends at \( \pm 1 \). We will now show that \( w \) cannot become higher than \( +1 \) or lower than \( -1 \) within the interval \( [0, \varrho_0] \), as it would otherwise never return and diverge.

Furthermore, we will see that any solution with \( \Lambda \equiv 0 \) is trivial, i.e. \( R \equiv 0, \ w \equiv \pm 1 \). Then I will provide a proof that \( R \) can have only maxima and no minima within the interval \( [0, \varrho_0] \). In conclusion, we will recognize that at the points where \( R \) has a saddle point\footnote{point where the first and second derivative vanishes}, \( w \) will have an extremum. All these facts combined allow for a qualitative idea of the solutions.
**Proposition 1** For any solution \( w \), the following applies:

\[
|w(\varrho)| \leq 1, \quad \forall \varrho \in ]0, \varrho_0[.
\]

**Proof:** Assume the contrary, i.e., that \( w \) is larger than +1 for certain \( \varrho \in ]0, \varrho_0[ \). In order that the behavior at \( \varrho = 0 \) and \( \varrho = \varrho_0 \) holds, \( w \) must have a maximum in that range, where it is larger than +1. That means that there is a \( \tilde{\varrho} \in ]0, \varrho_0[ \) where the following applies:

\[
w(\tilde{\varrho}) > 1, \quad \dot{w}(\tilde{\varrho}) = 0, \quad \ddot{w}(\tilde{\varrho}) \leq 0.
\]

Now we use equation (67):

\[
\ddot{w} = \frac{w(w^2 - 1)}{R^2} + \frac{1}{2} \dot{w} \gamma.
\]

\( R \) does not vanish at \( \tilde{\varrho} \) and thus, we have

\[
\ddot{w}(\tilde{\varrho}) = \frac{w(w^2 - 1)}{R^2} \bigg|_{\tilde{\varrho}},
\]

implying

\[
\ddot{w}(\tilde{\varrho}) > 0.
\]

This is a contradiction to the assumption. We can proof the case \(-1\) analogically.

\[q.e.d.\]

In order to prove that nontrivial solutions must have \( \Lambda \neq 0 \), we need several tools.

**Proposition 2** Every nontrivial solution has at least one zero of \( w \). Thus, there has to be a \( \varrho^* \in ]0, \varrho_0[ \) with \( w(\varrho^*) = 0 \).

**Proof:** We make again a counterassumption and say we have a \( w \) for which the following holds:

\[
w(\varrho^*) \neq 0, \quad \forall \varrho^* \in ]0, \varrho_0[.
\]
Without loss of generality we can chose \( w(0) = +1 \). Then \( w \) must have a minimum in order to fulfill \( w(\varrho_0) = +1 \). It cannot be equal to \(-1\), because in that case \( w \) would have a zero. Therefore, there is a \( \hat{\varrho} \in ]0, \varrho_0[ \) for which
\[
\begin{align*}
w(\hat{\varrho}) &> 0, & w(\hat{\varrho}) &< 1 \quad \text{(follows from Prop. [1])} \\
\dot{w}(\hat{\varrho}) &= 0, & \ddot{w}(\hat{\varrho}) &> 0.
\end{align*}
\]

Taking a look back at equation (67),
\[
\ddot{w}(\hat{\varrho}) = \frac{w(\hat{\varrho}) [w^2(\hat{\varrho}) - 1]}{R^2(\hat{\varrho})},
\]
we see that \( \ddot{w}(\hat{\varrho}) < 0 \).
This is a contradiction to the assumption. We can analogously treat the case \( w(0) = -1 \).

\[q.e.d.\]

For the ground state with \( w(0) = +1, w(\varrho_0) = -1 \), there is exactly one zero.

**Proposition 3** If there is a \( \tilde{\varrho} \in ]0, \varrho_0[ \) with \( \gamma(\tilde{\varrho}) = 0 \), then \( \dot{\gamma}(\tilde{\varrho}) < 0 \), if we presuppose \( \Lambda = 0 \).

**Proof:** We chose a \( \tilde{\varrho} \) with \( \gamma(\tilde{\varrho}) = 0 \). Equation (71) gives us
\[
GV(\tilde{\varrho}) - G \dot{w}^2(\tilde{\varrho}) + \dot{R}^2(\tilde{\varrho}) - 1 = 0.
\]
We solve this for \( GV \) and insert the result into (69). Thus, we obtain
\[
\dot{\gamma}(\tilde{\varrho}) = -\frac{2G}{R^2(\tilde{\varrho})} \left[ 2G \dot{w}^2(\tilde{\varrho}) - \dot{R}^2(\tilde{\varrho}) + 1 \right].
\]
Since \( \dot{w}^2 \) and \( R^2 \) are always larger than zero and \( 0 \leq \dot{R}^2 \leq 1 \), \( \dot{\gamma}(\tilde{\varrho}) < 0 \).

\[q.e.d.\]

**Proposition 4** For every solution with \( w(0) = +1 \) and \( w(\varrho_0) = -1 \) and \( \Lambda = 0 \) \( \gamma \) must vanish identically.
Proof: The equations (83) and (84) and the condition \( \Lambda = 0 \) lead us to
\[
\dot{\gamma}(q = 0) < 0 \quad \dot{\gamma}(q = q_0) < 0.
\]
However, since \( \gamma(0) = \gamma(q_0) = 0 \) still applies, \( \gamma \) must cross the abscissa at least once; e.g. at \( \tilde{q} \), where \( \dot{\gamma}(\tilde{q}) > 0 \). This, however, is a contradiction to Proposition 3, which says that \( \dot{\gamma}(\tilde{q}) < 0 \). Therefore, only \( \gamma \equiv 0 \) can apply.

\[q.e.d.\]

**Lemma 1** Every solution with \( \Lambda = 0 \) is trivial, i.e. \( R \equiv 0, w \equiv \pm 1 \).

Proof: We know from Proposition 4 that
\[
\gamma \equiv 0 \quad \forall q \in [0, q_0].
\]
Together with (69) and (71), this leads to
\[
2G V + \dot{R}^2 - 1 = 0, \\
G V - G \dot{w}^2 + \dot{R}^2 - 1 = 0.
\]
We subtract the two equations from each other and obtain:
\[
\dot{w}^2 = -V.
\]
Since \( V \geq 0 \), it must be:
\[
\dot{w} = 0, \quad \forall q \in [0, q_0].
\]
Therefore, \( w \) must be proportional to a constant. In order to keep \( V \) regular for \( R(0) = 0 \), only \( w \equiv \pm 1 \) can apply.
With (68), we can see that \( \ddot{R} \equiv 0 \). Therefore, \( R \) should equal zero or be proportional to \( q \). However, if it were proportional to \( q \), it could never equal zero at \( q_0 \). Therefore: \( R \equiv 0 \).

\[q.e.d.\]

Of course, Lemma 1 only applies if we assume that \( R \) equals zero again for a finite value of \( q_0 \). But we had already assumed this in Sec. 2.5.

Now we will take a look at the properties of \( R \).
**Proposition 5** The maximum of $R$ is positive, the minimum negative.

*Proof:*

**Maximum:** $\exists \varrho^* \in ]0, \varrho_0[,$ so that

$$\dot{R}(\varrho^*) = 0, \quad \ddot{R}(\varrho^*) < 0.$$  

We insert into equation (68):

$$\ddot{R}(\varrho^*) = -G \frac{\dot{w}^2(\varrho^*)}{R(\varrho^*)}.$$  

$\dot{w}^2$ is always positive; therefore, $R(\varrho^*) > 0.$

**Minimum:** $\exists \varrho^* \in ]0, \varrho_0[,$ so that

$$\dot{R}(\varrho^*) = 0, \quad \ddot{R}(\varrho^*) > 0.$$  

With the same argument as above, we show that: $R(\varrho^*) < 0.$

$q.e.d.$

Minima only exist when the radius $R$ at these points is negative. In order to fulfill the boundary conditions $\dot{R}(0) = +1, \ddot{R}(\varrho_0) = -1,$ $R$ must in this case have two maxima or two zeros. And for every additional minimum there must be an additional maximum. In other words:

**Number of maxima** $=$ **Number of minima** $+$ 1.

If we exclude negative values for all regular solutions to $R,$ we obtain the information that all nontrivial, regular solutions to $R$ have just one maximum.

We also know something about the saddle points.

**Proposition 6** If $R$ has saddle points, then $w$ has an extremum at this same point.

*Proof:*

If $R$ has a saddle point or a flat point, the following applies:

$$\exists \varrho \in ]0, \varrho_0[, \quad \text{so that } \dot{R}(\varrho) = 0, \ddot{R}(\varrho) = 0.$$  

We insert into (68) and obtain

$$\ddot{R}(\varrho) = -\frac{1}{2} \dot{R}(\varrho) \gamma(\varrho) - G \frac{\dot{w}^2(\varrho)}{R(\varrho)} = -G \frac{\dot{w}^2(\varrho)}{R(\varrho)} = 0.$$  

Since $R(\varrho) \neq 0,$ $\dot{w}(\varrho)$ must be equal to zero.
To provide an overall view, we summarize our results:

- All regular solutions of $w$ lie between $+1$ and $-1$.
- Nontrivial, regular solutions exist only for $\Lambda \neq 0$.
- If we exclude negative values of $R$, it has only one maximum.
- $R$ can have saddle points; at these points, $w$ has an extremum.

All these results are of particular interest when we know whether the solutions have any symmetry. For example, a reflection at the equator would be nice. We will now consider the reflection at an axis and the point reflection ($G = 1$). $\varrho$ then runs from $0$ to $\varrho_0$. Because of the symmetry the interval is divided into two new ones. We don’t want to assess where this is, but chose arbitrarily:

\[
\varrho \in [0, \varrho_0] =: I \\
\varrho' := \frac{\varrho}{\varepsilon} , \quad \varrho'' := \varrho_0 - \frac{\varepsilon - 1}{\varepsilon} \varrho , \quad \varepsilon \in R_+ \setminus \{0\} .
\]  
(87)

This leads us to

\[
\varrho' \in \left[0, \frac{\varrho_0}{\varepsilon}\right] =: I_1 , \quad \varrho'' \in \left[\varrho_0, \frac{\varrho_0}{\varepsilon}\right] =: I_2 \\
I_1 \cup I_2 = I .
\]

Our symmetries shall be:

\[
\begin{align*}
1) & \quad \begin{array}{c}
R(\varrho') = R(\varrho'') \\
w(\varrho') = w(\varrho'')
\end{array} \quad \text{(reflection)} \\
2) & \quad \begin{array}{c}
R(\varrho') = R(\varrho'') \quad \text{(reflection)} \\
w(\varrho') = -w(\varrho'') \quad \text{(point reflection)}
\end{array}
\end{align*}
\]  
(88)

First, we chose $\varepsilon = 2$. Thus, symmetry 2) describes the properties of the solution of Ding & Hosoya [28], equations (74), (75).
From (87) and (89) we have
\[ R(\dot{q}') = R(\dot{q}'') \] (90)
\[ \dot{R}(\dot{q}') = -\dot{R}(\dot{q}'') \] (91)
\[ \ddot{R}(\dot{q}') = \ddot{R}(\dot{q}'') \] (92)
\[ w(\dot{q}') = -w(\dot{q}'') \] (93)
\[ \dot{w}(\dot{q}') = \dot{w}(\dot{q}'') \] (94)
\[ \ddot{w}(\dot{q}') = -\ddot{w}(\dot{q}'') \] (95)
\[ V(\dot{q}') = V(\dot{q}'') \] (96)
\[ \frac{1}{2} V_{,w}(\dot{q}') = -\frac{1}{2} V_{,w}(\dot{q}'') \] (97)
\[ \gamma(\dot{q}') = -\gamma(\dot{q}'') \] (98)
\[ \dot{\gamma}(\dot{q}') = \dot{\gamma}(\dot{q}'') \] (99)

On the other hand, the conditions (90), (91), (93), (94), (96), (97), (98), upon insertion into (67) - (69), must lead to the equations (92), (95), (99). We will execute:
\[ \ddot{w}(\dot{q}') = \frac{1}{2} V_{,w}(\dot{q}') + \frac{1}{2} \dot{w}(\dot{q}') \gamma(\dot{q}') = \]
\[ -\frac{1}{2} V_{,w}(\dot{q}'') - \frac{1}{2} \dot{w}(\dot{q}'') \gamma(\dot{q}'') = -\ddot{w}(\dot{q}'') \]
\[ \ddot{\dot{R}}(\dot{q}') = -\frac{1}{2} \ddot{\dot{R}}(\dot{q}') \gamma(\dot{q}') - G \frac{\ddot{w}^2(\dot{q}')}{\dot{R}(\dot{q}')} = \]
\[ -\frac{1}{2} \ddot{\dot{R}}(\dot{q}'') \gamma(\dot{q}'') - G \frac{\ddot{w}^2(\dot{q}'')}{\dot{R}(\dot{q}'')} = \ddot{\dot{R}}(\dot{q}'') \]
\[ \dot{\gamma}(\dot{q}') = \frac{1}{2} \gamma^2(\dot{q}') - \frac{2}{R^2(\dot{q}')} \left[ 2GV(\dot{q}') + \dot{R}^2(\dot{q}') - 1 \right] = \]
\[ \frac{1}{2} \gamma^2(\dot{q}'') - \frac{2}{R^2(\dot{q}'')} \left[ 2GV(\dot{q}'') + \dot{R}^2(\dot{q}'') - 1 \right] = \dot{\gamma}(\dot{q}''). \]

Symmetry 1) is also easily explained. It corresponds to the solution I have found numerically, as we will see in chapter 4.

Now, we take an arbitrary \( \varepsilon \). As before, we obtain:
\[ R(\dot{q}') = R(\dot{q}'') \] (100)
\begin{align*}
\dot{R}(\vartheta') &= - (\varepsilon - 1) \dot{R}(\vartheta'') \quad (101) \\
\dot{R}(\vartheta') &= (\varepsilon - 1)^2 \dot{R}(\vartheta'') \quad (102) \\
w(\vartheta') &= - w(\vartheta'') \quad (103) \\
w(\vartheta') &= (\varepsilon - 1) \dot{w}(\vartheta'') \quad (104) \\
\ddot{w}(\vartheta') &= - (\varepsilon - 1)^2 \ddot{w}(\vartheta'') \quad (105) \\
V(\vartheta') &= V(\vartheta'') \quad (106) \\
\frac{1}{2} V_{,w}(\vartheta') &= - \frac{1}{2} V_{,w}(\vartheta'') \quad (107) \\
\gamma(\vartheta') &= \frac{1}{(\varepsilon - 1) R(\vartheta''\vartheta'')} \left[ 2G \left( V(\vartheta'') - (\varepsilon - 1)^2 \ddot{w}(\vartheta'') \right) + \\
&\quad + \left( (\varepsilon - 1)^2 \dddot{R}(\vartheta'') - 1 \right) + \Lambda R^2(\vartheta'') \right]. \quad (108)
\end{align*}

If we insert (104), (107), (108) into (67), we see that an identity with (103) can be achieved only if \(\varepsilon = 2\) applies. We don’t need to calculate any further. It won’t help if (68) and (69) apply for an arbitrary \(\varepsilon\), since (67) doesn’t. And we have already observed the case \(\varepsilon = 2\).

**Summary:**

\[
\begin{align*}
R(\frac{x}{2}) &= R(\vartheta_o - \frac{x}{2}) & R(\frac{x}{2}) &= R(\vartheta_o - \frac{x}{2}) \\
w(\frac{x}{2}) &= - w(\vartheta_o - \frac{x}{2}) & w(\frac{x}{2}) &= w(\vartheta_o - \frac{x}{2})
\end{align*}
\forall \vartheta \in [0, \vartheta_o].
\]

Together with Prop. 3 and Prop. 4, we are now able to sketch all possible solutions.
3 Numerics

As we tried to solve the system (67) - (70) by means of a standard routine for systems of ordinary differential equations, we concluded that most routines were unfit to do so. Only two of them were satisfactory: DO2BAF from the NAG library and a routine taken from Ref. [33].

I tried to solve the equations (67) - (73) with a standard shooting procedure for solving two-point boundary value problems [34], using both [33] and DO2BAF. In the program taken from [34], I just replaced the routine for solving differential equations, leaving the rest of the program unchanged. The idea of the method mentioned in [34] is to find such initial values (79) - (82) that the solutions meet at any point between the two boundary values. Unfortunately, it turned out that the procedure was only satisfactory enough when the initial values (79) - (82) were close enough to the solution. At a distance of $10^{-2}$ from the actual values, the computer was no longer able to find the solution. I tried this with the already known solution (74), (75).

So I returned to the standard program. I constructed the initial values with the expansions (79) - (86), at $\varrho = 10^{-5}$. By varying the parameters $b$ and $\Lambda$, I shot for global solutions. The tolerance was of $10^{-8}$. The program was supposed to run until $R = 0$ or $|w| > 1$. Now, I proceeded as follows: I first shot from the North Pole (i.e. at $\varrho = 0$) and directed $b$ and $\Lambda$ until the solution became regular. The initial values came from the conditions (74), (80), (83), (85). The program told me now at which $\varrho$ the South Pole was, i.e. how big $\varrho_0$ was. Then I reversed the routine and shot from the South Pole. The necessary initial values now came from (81), (82), (84) and (86). $b$ and $\Lambda$ were virtually identical. It is not until we obtain the same solution by shooting from both sides that we can be sure to have found a real solution. At this point I also employed the routine taken from [34], as the values were close enough to the actual solution. For this program, it was necessary to introduce a new independent variable $t$, so that $\varrho$ became a function. I defined:

$$\varrho(t) := \varrho t, \quad t \in [0, 1],$$

$$\varrho(0) = 0,$$

$$\varrho(1) = \varrho_0.$$

Thus, now there are six parameters; namely $b$, $\Lambda$, $\varrho$ both for the North and
the South Pole. Now I approximated very accurately the parameters to the actual values.
4 Results

As we have seen, the parameters of the solution of Ding & Hosoya [28] are

\begin{align*}
b &= 0.25 \\
\Lambda &= 0.75 \\
\varrho_0 &= \sqrt{2} \pi ,
\end{align*}

for \( G = 2 \). The functions \( R, w, \gamma, L \) are shown in Fig. [1].

If we assume \( \Lambda = 0.75 \) and increase \( b, w \) does not reach the value \(-1\); it rather moves towards zero again. An extremum is formed. If we decrease \( b, w \) becomes smaller than \(-1\) and disappears, according to Prop. [1] to infinity.

If we assume a \( \Lambda \) other than \( 0.75 \) and try to arrange \( b \) so as to allow a solution with a zero in \( w \), we can observe the following two cases:

\( \Lambda > 0.75 \): In the best-case scenario, we can arrange \( b \) such as to make \( R \) and \( w \) look rather good. On the South Pole, however, there is a peak that grows with an increasing \( \Lambda \). In Fig. [2], you can recognize the peak. In the graph, \( \dot{R} \) has been drawn against \( R \).

\( \Lambda < 0.75 \): In this case, there is a indentation at the South Pole. Again, I have plotted \( \dot{R} \) against \( R \) (see Fig. [3]).

In both cases the following applies: the farther away \( \Lambda \) is from \( 0.75 \) the farther the abscissa of the point with the ordinate \(-1\) moves off from zero. Thus, I assert that for a zero of \( w \), there is only one \( \Lambda \). In other words: for one zero, there is only one regular solution. Now the question arises whether there are also regular solutions with more than one zero in \( w \), and whether there is also just one regular solution for any number of zeros. From the analogy with the asymptotically flat case [1], we can expect that. And in fact, the same behavior was shown for two zeros, as described above. I found a solution with the parameters

\begin{align*}
b &= 0.42976 \\
\Lambda &= 0.364 \\
\varrho_0 &= 8.64 .
\end{align*}

The solution is shown in Fig. [4]. We see that the symmetry 1) is fulfilled. There seem to be no regular solutions with two zeros for other values of \( \Lambda \).
Figure 1: The DH solution
So far, I could not find any solutions with more than 2 zeros, either. We presume that either the numerics fail or a function in the system becomes singular. As an example, we observe that the functions

\[
\begin{align*}
  w &\equiv 1, \\
  R &= \sqrt{G} \sin \left( \frac{\theta}{\sqrt{G}} \right), \\
  \gamma &= \frac{2}{\sqrt{G}} \tan \left( \frac{\theta}{\sqrt{G}} \right), \\
  L &= \cos \left( \frac{\theta}{\sqrt{G}} \right)
\end{align*}
\]

solve our system (67) - (70). \( \gamma \) becomes, however, singular at the equator. Thus, in our coordinates we don’t find the solution on the computer, although it solves our system. That could mean that we cannot find all solutions in these coordinates. We must try to introduce more appropriate coordinates.
Figure 4: The solution with $b = 0.42976$, $\Lambda = 0.364$, $\varrho = 8.64$
5 Time dependent cosmological solutions to the EYM system

Our metric shall be
\[ g = a^2(t) \left[ -dt^2 + h \right] , \]  
(109)
with \( h \) being the standard metric on \( S^3 \). The radius will be taken equal to 1, which is always possible. We choose the gauge potential
\[ A = f \theta , \]  
(110)
where \( \theta \) is the Maurer-Cartan form on \( S^3 \), understood as SU(2). \( f \) shall depend exclusively upon time: \( f = f(t) \).

We set up the coupled field equations:
For the YM equations we can use the metric \( \hat{g} = (-dt^2 + h) \), since they are conformally invariant. It is helpful to express the metric \( h \) in terms of \( \theta \). Let
\[ \theta = \sum_{i=1}^{3} \theta^i \tau_i \quad (\vec{\tau} = \vec{\sigma}/i) \]  
(111)
\[ \theta^0 = dt ; \]  
(112)
then \( h = \sum (\theta^i)^2, \quad \hat{g} = \eta_{\mu\nu} \theta^\mu \otimes \theta^\nu . \)  
(113)

With the help of the Maurer-Cartan equations
\[ d\theta + \theta \wedge \theta = d\theta + \frac{1}{2}[\theta, \theta] = 0 \]  
(114)
we find the field strength tensor
\[ F = \dot{f} \theta^0 \wedge \theta + (f^2 - f) \frac{1}{2}[\theta, \theta] \]  
(115)
or
\[ F = \dot{f} \theta^0 \wedge \theta^1 \tau_1 + \dot{f} \theta^0 \wedge \theta^2 \tau_2 + \dot{f} \theta^0 \wedge \theta^3 \tau_3 + 2(f^2 - f) \left[ \theta^2 \wedge \theta^3 \tau_1 + \theta^3 \wedge \theta^1 \tau_2 + \theta^1 \wedge \theta^2 \tau_3 \right] . \]  
(116)
The upper line of (116) contains the electrical components, and the lower one the magnetic components. Now, (116) gives us very quickly
\[ (F|F) = \left[ -\dot{f}^2 + 4 \left( f^2 - f \right)^2 \right] \cdot 3 \]  
(117)
Here we have used the following normalization of the scalar product for the Lie algebra

$$\langle X, Y \rangle = -\frac{1}{2} tr(X \cdot Y) \quad \text{(i.e. } \langle \tau_i, \tau_j \rangle = \delta_{ij} \).$$

Thus, the YM equations are reduced to a 1-dimensional “mechanical problem” with the Lagrange function

$$L_{YM} = \dot{f}^2 - 4 \left( f^2 - f \right)^2 \equiv T - V .$$

The corresponding energy is $T + V = E$:

$$\dot{f}^2 + 4 \left( f^2 - f \right)^2 = E .$$

In Fig. 5, we plot the potential $V(f)$ and also the phase portrait (Fig. 6). There are two areas of stability at $f = 0$ and $f = 1$, while exactly in the middle, at $f = 1/2$, we have an unstable point, which turn out to be gauge equivalent to the DH solution as already mentioned in the introduction. Now, we explain this in short. We define the following map

$$g : S^3 \rightarrow SU(2) , \quad x \mapsto x^4 \cdot 1 + i \bar{x} \bar{\sigma} ,$$

$$x \in \mathbb{R}^4 , \quad |x| = 1 ,$$

29
and the one-form
\[ \theta = g^{-1}dg. \]

We choose the adequate coordinates
\[ r = |x|, \quad x^4 = \sqrt{1 - r^2}. \]

Then the following applies
\[ \theta = (x^4 \cdot 1 - i \vec{\sigma} \vec{x}) \left( dx^4 + i \vec{\sigma} d\vec{x} \right). \]

After a short calculation we obtain
\[ \frac{1}{i} \theta = \vec{\sigma} \left\{ \sqrt{1 - r^2} \, d\vec{x} + \vec{x} \, d\vec{x} \sqrt{1 - r^2} + \vec{x} \wedge d\vec{x} \right\} \]
and finally \((\vec{\tau} = \vec{\sigma}/i)\)
\[
-\theta = \frac{1}{\sqrt{1 - r^2}} \tau_r \, dr + r \sqrt{1 - r^2} \left( \tau_\theta \, d\theta + \tau_\varphi \sin \theta \, d\varphi \right)
- r^2 \left( \tau_\varphi \, d\theta - \tau_\varphi \sin \theta \, d\varphi \right).
\]

If we define \( r = \sin \chi \), it follows that
\[
-\theta = \tau_r \, d\chi + \sin \chi \cos \chi \left( \tau_\theta \, d\theta + \tau_\varphi \sin \theta \, d\varphi \right)
- \sin^2 \chi \left( \tau_\varphi \, d\theta - \tau_\theta \sin \theta \, d\varphi \right).
\]

The BK ansatz is
\[ \tilde{A} = \frac{w - 1}{2} \left[ \tau_\varphi \, d\theta - \tau_\theta \sin \theta \, d\varphi \right]. \]

We transform this with \( G = \cos \tilde{\chi} + \tau_r \sin \tilde{\chi} \) and obtain \((\lambda \equiv 2\tilde{\chi})\):
\[
G^{-1}AG + G^{-1}dG = \frac{1}{2} \tau_r \, d\lambda + \frac{w \cos \lambda - 1}{2} \left( \tau_\varphi \, d\theta - \tau_\theta \sin \theta \, d\varphi \right)
+ \frac{w}{2} \sin \lambda \left( \tau_\varphi \, d\theta + \tau_\varphi \sin \theta \, d\varphi \right).
\]

This is equal to \( \frac{1}{2} \theta \) for \( w = \cos \lambda \) and \( \chi = \lambda \). Thus, we find indeed
\[ \tilde{A} = \frac{1}{2} \theta. \]
We set up the Einstein equations with \( \Lambda \). Since \( tr(T) = 0 \), we consider first the trace of the Einstein equations: \( R = 4\Lambda \). Since
\[
R(g) = \frac{6}{a^2} \left[ \frac{\ddot{a}}{a} + 1 \right],
\]
we obtain
\[
\ddot{a} + a = \frac{2\Lambda}{3} a^3. \quad (120)
\]
With respect to the orthonormalized tetrad \( \{a\theta\} \) of \( g \), \( G_{00} = \frac{3}{a^2} \left[ \left( \frac{\dot{a}}{a} \right)^2 + 1 \right] \).
The energy density is given by (use (116)):
\[
8\pi T_{00} = \left[ \frac{\dot{f}^2}{a^4} \cdot 3 + \frac{4 (f^2 - f)^2}{a^4} \cdot 3 \right] = \frac{3}{a^4} (T + V). \quad (121)
\]
Thus, the Friedman equation with the \( \Lambda \)-term is:
\[
\ddot{a}^2 + a^2 = T + V + \frac{\Lambda}{3} a^4. \quad (122)
\]
For every \( T + V = E \), this is again a “mechanical problem” for \( a(t) \):
\[
\ddot{a}^2 + U(a) = E, \quad U(a) = a^2 - \frac{\Lambda}{3} a^4. \quad (123)
\]
Let us first consider the case \( \Lambda = 0 \). Then
\[
\ddot{a} + a = 0, \quad \dot{a}^2 + a^2 = E,
\]
hence
\[
a(t) = a_0 \sin t. \quad (124)
\]
When we take \( f \) as being static, the derivative of (119) will lead us to:
\[
f(f - 1)(2f - 1) = 0.
\]
We obtain three solutions:
\[
f_1 = 0, \quad f_2 = 1, \quad f_3 = \frac{1}{2}. \quad (125)
\]
For $f_1$, $A$ equals zero, and $f_2$ corresponds to a pure gauge $A = \theta$. For the interesting case $f = 1/2$, $E$ equals $1/4$. If we shift to the physical time $\tilde{t}$, \((d\tilde{t} = a \, dt)\), this leads to
\[
a(\tilde{t}) = \tilde{t}^{1/2} \left(2a_0 - \tilde{t}\right)^{1/2}. \tag{126}
\]

There is a static solution for $\Lambda \neq 0$, namely $f = 1/2$, $a = (2 \Lambda / 3)^{-1/2}$ (this follows from \((120)\)). In Fig. 7, we see the shape of $U(a)$. The $a$ above is exactly the critical point of $U(a)$. The DH solution corresponds to the local maximum of the potential $V(f)$ and is therefore unstable.

A stability analysis of the new solution, as well as other material will be published elsewhere [35].

**Acknowledgments**

During the work for my diploma thesis, some doubts arose about whether positive results would come out. In order to complete my work, I was given a further assignment. I thank Prof. Dr. N. Straumann for this support. Discussions with Othmar Brodbeck, Mikhail Volkov and George Lavrelashvili also proved to be very helpful. I am also very obliged to Markus Heusler. In the beginning, he introduced me into the matter of subject with great patience and he has helped me every time I had specific questions. Finally,
my thanks go to Ivan Colaci for his translation of my German draft into English.

References

[1] A. Lichnerowicz, in *Théories relativistes de la gravitation et de l’électromagnetisme* (Masson, Paris, 1955)

[2] S. Deser, *Phys. Lett.* B 64, 463 (1976)

[3] S. Coleman, in *New Phenomenon in Subnuclear Physics*, ed. A. Zichichi (Plenum, New York, 1975)

[4] R. Bartnik and J. McKinnon, *Phys. Rev. Lett.* 61, 141 (1988)

[5] M. S. Volkov and D. V. Galt’sov, *Prs’ma Zh. Eksp. Teor. Fiz.* 50, 312 (1989); *Sov. J. Nucl. Phys.* 51, 747 (1990)

[6] P. Bizon, *Phys. Rev. Lett.* 64, 2844 (1990)

[7] H. P. Künzle and A. K. Masoud-ul-Alan, *J. Math. Phys.* 31, 928 (1990)

[8] J. A. Smoller, A. G. Wassermann, S. T. Yau and J. B. McLeod, *Commun. Math. Phys.* 143, 115 (1992)

[9] J. A. Smoller and A. G. Wassermann, *Commun. Math. Phys.* 151, 303 (1993)

[10] J. A. Smoller, A. G. Wassermann and S. T. Yau, *Commun. Math. Phys.* 154, 377 (1993)

[11] P. Breitenlohner, P. Forgács and D. Maison, *Commun. Math. Phys.* 163, 141 (1994)

[12] N. Straumann and Z.-H. Zhou, *Phys. Lett.* B 237, 353 (1990)

[13] N. Straumann and Z.-H. Zhou, *Phys. Lett.* B 243, 33 (1990)

[14] Z.-H. Zhou and N. Straumann, *Nucl. Phys.* B 360, 180 (1991)

[15] Z.-H. Zhou, *Helv. Phys. Acta* 65, 767 (1992)
[16] O. Brodbeck and N. Straumann, *Phys. Lett.* **B 324**, 309 (1994)

[17] O. Brodbeck and N. Straumann, *Instability Proof for Einstein-Yang-Mills Solitons and Black Holes with arbitrary Gauge Groups*, Zürich University Preprint No. ZU-TH 38/1994, gr-qc/9411058

[18] S. Droz, M. Heusler and N. Straumann, *Phys. Lett.* **B 268**, 371 (1991)

[19] M. Heusler, S. Droz and N. Straumann, *Phys. Lett.* **B 271**, 61 (1991)

[20] M. Heusler, S. Droz and N. Straumann, *Phys. Lett.* **B 285**, 21 (1992)

[21] M. Heusler, N. Straumann and Z.-H. Zhou, *Helv. Phys. Acta* **66**, 614 (1993)

[22] K.-Y. Lee, V. P. Nair and E. Weinberg, *Phys. Rev. Lett.* **68**, 1100 (1992); *Phys. Rev.* **D 45**, 2751 (1992); M. E. Ortiz, *ibid.* **45**, R2586 (1992)

[23] P. Breitenlohner, P. Forgács and D. Maison, *Nucl. Phys.* **383**, 357 (1992)

[24] P. C. Aichelburg and P. Bizon, *Phys. Rev.* **D 48**, 607 (1993)

[25] E. E. Donets and D. V. Gal’tsov, *Phys. Lett.* **B 302**, 411 (1993); **312**, 391 (1993); G. Lavrelashvili and D. Maison, *Nucl. Phys.* **B 410**, 407 (1993); C. M. O’Neill, Institution Report No. CLNS-93/1246 and hep-th/9311022, 1993 (unpublished); P. Bizon, *Act. Phys. Pol.* **B 24**, 1209 (1993)

[26] P. Boschung, O. Brodbeck, F. Moser, N. Straumann and M. S. Volkov, *Phys. Rev.* **D 50**, 3842 (1994)

[27] B. R. Greene, S. D. Mathur and C. M. O’Neill, *Phys. Rev.* **D 47**, 2242 (1993)

[28] S. Ding and A. Hosoya, TIT/HEP-242/COSMO-39, 1993

[29] private communication by N. Straumann

[30] Y. Hosotani, *Phys. Lett.* **B 147**, 44 (1984)

[31] N. Straumann, *Allgemeine Relativitätstheorie und relativistische Astrophysik*, Springer-Verlag (1988), Vol. 150, p. 441
[32] H. Stöcker, *Taschenbuch mathematischer Formeln und moderner Verfahren*, Verlag Harri Deutsch (1993), p. 113

[33] L. F. Shampine, M. K. Gordon, *Computer Solutions of Ordinary Differential Equations: The Initial Value Problem*, W. H. Freeman and company, San Francisco, 1975

[34] W. H. Press et al., *Numerical Recipes*, Cambridge University Press, New York, 1992

[35] M. Heusler, G. Lavrelashvili, P. Molnár and N. Straumann, in preparation