A uniqueness and periodicity result for solutions of elliptic equations in unbounded domains

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Abstract

We proof a uniqueness and periodicity theorem for bounded solutions of uniformly elliptic equations in certain unbounded domains.

1. Introduction

In this note we study solutions $u \in C^2(\Omega, \mathbb{R}) \cap C^0(\overline{\Omega}, \mathbb{R})$ of the Dirichlet problem

$$a^{ij}(x)\partial_{ij} u + b^i(x)\partial_i u + c(x)u = f \quad \text{in } \Omega, \quad u = g \quad \text{on } \partial\Omega. \quad (1)$$

(using the sum convention) assuming the differential equation to be elliptic, i.e. at each point $x \in \Omega$ the matrix $a_{ij}(x)$ is symmetric and positive definite. In addition, we require the sign condition $c(x) \leq 0$.

If the domain $\Omega$ is bounded, the well known classical maximum principle (see [1, Theorem 3.3]) asserts that (1) admits at most one solution. In contrast, such a result does in general not hold for unbounded domains $\Omega$ and examples are given below. First, let us make the following assumptions on the coefficients: Let $a^{ij}, b^i, c \in C^0(\overline{\Omega}, \mathbb{R})$ and satisfy

$$||a^{ij}||_{C^0(\Omega)} + ||b^i||_{C^0(\Omega)} + ||c||_{C^0(\Omega)} \leq H \quad \text{for } i, j = 1, \ldots, n \quad \text{and} \quad c(x) \leq 0 \quad \text{in } \Omega \quad (2)$$

with some constant $H$. Additionally, we have to require a uniform ellipticity condition

$$\frac{1}{\Lambda}||\xi||^2 \leq a^{ij}(x)\xi_i\xi_j \leq \Lambda||\xi||^2 \quad \text{for all } x \in \Omega, \xi \in \mathbb{R}^n \quad (3)$$

with constant $\Lambda < \infty$. Our first result is the following

Theorem 1: Additionally to (2) and (3), assume the following

a) The unbounded domain $\Omega \subset \mathbb{R}^n$ has bounded thickness, i.e. $\sup_{x \in \overline{\Omega}} \text{dist}(x, \partial\Omega) < +\infty$.

b) Let $u \in C^2(\Omega, \mathbb{R}) \cap C^0(\overline{\Omega}, \mathbb{R})$ be a bounded solution of (1) for the right side $f \equiv 0$ and boundary values $g \equiv 0$.

c) Let $u$ satisfy the following uniform boundary condition: For any sequence $x_k \in \Omega$ with $\text{dist}(x_k, \partial\Omega) \to 0$ for $k \to \infty$ it follows that $u(x_k) \to 0$ as $k \to \infty$.

Then we must have $u \equiv 0$ in $\Omega$. 

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As the proof of Theorem 1 reveals, this result remains true for weak solutions \( u \) of regularity class \( W^{2,n}_{loc}(\Omega, \mathbb{R}) \cap C^0(\overline{\Omega}, \mathbb{R}) \).

Let us now demonstrate the necessity of the assumption a), b) and c) by considering the following examples.

**Example 1:** For some \( k \in \mathbb{N} \) take the domain \( \Omega = \{ re^{i\varphi} \in \mathbb{C} \mid 0 < \varphi < \frac{\pi}{k} \} \) and the harmonic function \( u(x, y) := \text{Re}\{(x + iy)^k\} \) with \( u = 0 \) on \( \partial \Omega \). This very simple example already shows that certain assumption on the solution \( u \) or the domain \( \Omega \) are needed for a uniqueness theorem to hold. Note that for this example all of the assumptions a), b) and c) of Theorem 1 are not satisfied.

**Example 2:** As domain we take \( \Omega = \{(x, y) \in \mathbb{R}^2 \mid 0 < y < \pi\} \) and consider the unbounded, harmonic function \( u(x, y) = e^x \sin y \) with \( u = 0 \) on \( \partial \Omega \). Here, assumption a) of Theorem 1 is satisfied while assumptions b) and c) are not.

**Example 3:** Now consider the domain \( \Omega = \{ x \in \mathbb{R}^n : |x| > 1 \} \), \( n \geq 3 \) and the bounded, harmonic function \( u(x) = 1 - |x|^{2-n} \) with \( u = 0 \) on \( \partial \Omega \). Here, assumptions b) and c) of Theorem 1 are satisfied while assumption a) is not.

Let us make a remark on assumption c): If the domain \( \Omega \) had a compact boundary \( \partial \Omega \), then assumption c) would directly follow from \( u \in C^0(\overline{\Omega}, \mathbb{R}) \) together with \( u = 0 \) on \( \partial \Omega \). However, note that an unbounded domain cannot both have a compact boundary and at the same time satisfy assumption a). Assumption c) will hold provided that the solution \( u \) is uniformly continuous in \( \Omega \).

By suitably restricting the domain \( \Omega \), we can actually show a uniform continuity of the solution.

**Theorem 2:** Additionally to (2) and (3), assume the following

a) The unbounded domain \( \Omega \subset \mathbb{R}^n \) has bounded thickness, i.e. \( \sup_{x \in \Omega} \text{dist}(x, \partial \Omega) < +\infty \).

b) Let \( u_k \in C^2(\Omega, \mathbb{R}) \cap C^0(\overline{\Omega}, \mathbb{R}) \) be two solutions of (1) for some right side \( f \) and some boundary values \( g \). Assume difference \( |u_1(x) - u_2(x)| \) is uniformly bounded in \( \Omega \).

c) The domain \( \Omega \) satisfies a uniform exterior sphere condition.

Then it follows \( u_1 \equiv u_2 \) in \( \Omega \).

By uniform exterior sphere condition we mean the following: There exists some \( r > 0 \) such that for each \( x_0 \in \partial \Omega \) there exists some \( x_1 \in \mathbb{R}^n \) such that \( B_r(x_1) \cap \overline{\Omega} = \{x_0\} \).

Finally, we want to point out that uniqueness results for partial differential equations also imply symmetry properties of the solutions. To illustrate this by an example, we have the following result.

**Corollary 1:** Assume that (4) and (5) hold. Moreover, assume that \( \Omega \) satisfies a uniform exterior sphere condition and can be decomposed into \( \Omega = \mathbb{R} \times \Omega' \) for some bounded domain \( \Omega' \subset \mathbb{R}^{n-1} \). We require the coefficients \( a^ij, b^i, c \), the right side \( f \) and the boundary values \( g \) to be periodic w.r.t. the \( x_1 \)-variable with one and the same period length \( L > 0 \).

Then any bounded solution \( u \in C^2(\Omega, \mathbb{R}) \cap C^0(\overline{\Omega}, \mathbb{R}) \) of (7) is periodic w.r.t. the \( x_1 \)-variable.

Note that in Example 2 we found a solution not being periodic. There, all of the assumptions of Corollary 1 are satisfied except for the boundedness of the solution. Hence, also for Corollary 1 it is crucial only to consider bounded solutions.
2. The proof of Theorem 1

For the proof of Theorem 1 we first need the following lemma, which may be of independent interest. It is a generalisation of the strong maximum principle.

**Lemma 1:** Let \( u_k \in C^2(\Omega, \mathbb{R}) \) be a sequence of solutions of

\[
a^{ij}_k(x) \partial_{ij} u_k + b^i_k(x) \partial_i u_k + c_k(x) u_k = 0 \quad \text{in } \Omega.
\]

Let the coefficients \( a^{ij}_k, b^i_k, c_k \) satisfy (2) and (3) with constants \( \Lambda, H \) independent of \( k \) and \( c_k(x) \leq 0 \) in \( \Omega \). Assume that \( u_k \) converge uniformly in \( \Omega \) to some \( u \in C^0(\Omega, \mathbb{R}) \). For some \( x_* \in \Omega \) and \( M \in \mathbb{R} \) let

\[
u_k(x) \leq M \quad \text{in } \Omega \quad \text{and} \quad \lim_{k \to \infty} u_k(x_*) = M.
\]

Then it follows \( u \equiv M \) in \( \Omega \).

**Proof:**

Consider the set

\[
\Theta := \{ x \in \Omega \mid u(x) = M \}
\]

which is not empty because of \( x_* \in \Theta \). Now \( \Theta \) is closed within \( \Omega \) due to the continuity of \( u \). We now show that \( \Theta \) is also open implying \( \Theta = \Omega \) and proving the lemma. For \( x_0 \in \Theta \) choose \( r > 0 \) small enough such that \( \overline{B}_r(x_0) \subset \Omega \). Now consider the function \( v_k(x) := M - u_k(x) \) for \( x \in \Omega \) with \( v_k(x) \geq 0 \) in \( \Omega \). Because of \( c_k \leq 0 \) this \( v_k \) is a solution of the differential inequality

\[
a^{ij}_k(x) \partial_{ij} v_k + b^i_k(x) \partial_i v_k \leq 0 \quad \text{in } \Omega.
\]

We now apply the Harnack type inequality \([1\text{, Theorem 9.22}]\) on the domain \( B_{2r}(x_0) \): There exist constants \( p > 0 \) and \( C < \infty \) only depending on \( r, H, \Lambda \) and \( n \) such that

\[
\left\{ \int_{B_r(x_0)} v_k(x)^p \, dx \right\}^{1/p} \leq C \inf_{B_r(x_0)} v_k(x) \leq C v_k(x_0) = C \left( M - u_k(x_0) \right).
\]

(4)

Noting that \( u_k(x_0) \to u(x_0) \) for \( k \to \infty \) and \( u(x_0) = M \) because of \( x_0 \in \Theta \), passing to the limit in (4) then yields

\[
\left\{ \int_{B_r(x_0)} (M - u(x))^p \, dx \right\}^{1/p} = 0.
\]

Together with \( u(x) \leq M \) in \( \Omega \) this implies \( u(x) = M \) in \( B_r(x_0) \) proving that \( \Theta \) is open. \( \square \)

**Remarks:**

1.) The lemma remains true for weak solutions of regularity \( W^{2,n}(\Omega, \mathbb{R}) \cap C^0(\Omega, \mathbb{R}) \).

2.) The proof of this lemma is similar to the proof of the strong maximum principle for weak solutions (see \([1\text{, Theorem 8.19}]\)). In case of \( u_k(x) = u(x) \) for all \( k \) the statement of the lemma reduces to the classical strong maximum principle.

**Proof of Theorem 1:**

Given a solution \( u \) of (1) for \( f \equiv 0 \) and \( g \equiv 0 \), we will show \( u \equiv 0 \) in \( \Omega \) as follows: Assume to the contrary that \( u(x_0) \neq 0 \) for some \( x_0 \in \Omega \), say \( u(x_0) > 0 \). Defining \( M := \sup_{\Omega} u(x) > 0 \) we have \( M < +\infty \) by the boundedness assumption b) of Theorem 1. We can now find a sequence \( x_k \in \Omega \) such that \( u(x_k) \to M \) for \( n \to \infty \). Now, for each \( k \in \mathbb{N} \) let us define

\[
r_k := \text{dist}(x_k, \partial \Omega).
\]
We claim that there exist constants \( \varepsilon > 0 \) and \( R < \infty \) such that
\[
\varepsilon < r_k < R \quad \text{for all } k \in \mathbb{N}.
\] (5)

In fact, the right inequality follows directly from assumption a) of Theorem 1 if we define \( R = \sup_{x \in \Omega} \text{dist}(x, \partial \Omega) \). The left inequality follows from assumption c) together with \( u(x_k) \to M > 0 \). On the ball \( B := \{ x \in \mathbb{R}^n : |x| < 1 \} \), let us now consider the shifted and rescaled functions
\[
v_k : \overline{B} \to \mathbb{R} \quad , \quad v_k(x) := u(x_k + r_k x).
\]

By the definiton of \( r_k \), for each \( k \in \mathbb{N} \) we can find some \( y_k \in \partial B \) with \( x_k + r_k y_k \in \partial \Omega \) implying \( v_k(y_k) = u(x_k + r_k y_k) = 0 \). Since \( u \) is solution of (1), \( v_k \) will then be solution of
\[
a^{ij}_k(x) \partial_{ij} v_k + b^i_k(x) \partial_i v_k + c_k(x) v_k = 0 \quad \text{in } B \quad \text{for } k \in \mathbb{N}
\]
with coefficients \( a^{ij}_k(x) := r_k^{-2} a^{ij}(x_k + r_k x) \) and \( b^i_k, c_k \) defined similarly. By (5) together with the assumptions on \( a^{ij}, b^i, c \) there is a uniform \( C^0 \)-bound
\[
\sup_{k \in \mathbb{N}} \left( ||a^{ij}_k||_{C^0(B)} + ||b^i_k||_{C^0(B)} + ||c_k||_{C^0(B)} \right) < +\infty \quad \text{for all } i, j = 1, \ldots, n.
\]

Using the interior Hölder estimate [1, Theorem 9.26] for weak solutions we get
\[
\sup_{k \in \mathbb{N}} ||v_k||_{C^0(B)} < +\infty \quad \text{for all } 0 < s < 1
\]
with some Hölder exponent \( \alpha = \alpha(s) \in (0, 1) \) independent of \( k \). After extracting some subsequence we obtain the uniform convergence
\[
v_k \to v \quad \text{in } C^0(B_s, \mathbb{R}) \quad \text{for } k \to \infty
\]
(6)
for each \( s < 1 \) with some limit function \( v \in C^0(B, \mathbb{R}) \) satisfying
\[
v(x) \leq M \quad \text{in } B \quad \text{and } \quad v(0) = M.
\]

By Lemma (applied to \( \Omega = B_s \)) we have \( v(x) = M \) in \( B_s \) for each \( s < 1 \) and hence \( v(x) = M \) in \( B \).

On the other hand, from \( v_k(y_k) = 0 \) together with \( v_k(0) \to M \) we conclude that, for sufficiently large \( k \), there exists some \( z_k = t_k y_k \in B \) with \( t_k \in (0, 1) \) such that \( v_k(z_k) = M/2 = u(x_k + r_k z_k) \).

We may assume that \( t_k \to t_* \in [0, 1] \) and \( z_k \to z_* \in \overline{B} \) as \( k \to \infty \). We now claim that \( t_* < 1 \). Otherwise we would have \( t_k \to 1 \) for \( k \to \infty \). However, we would then have
\[
\text{dist}(x_k + r_k z_k, \partial \Omega) \leq |x_k + r_k z_k - (x_k + r_k y_k)| = r_k |y_k|(1 - t_k) \leq R(1 - t_k) \to 0 \quad \text{for } k \to \infty
\]
contradicting assumption c) together with \( u(x_k + r_k z_k) = M/2 \), proving the claim. Using the uniform convergence (6) in the ball \( B_{t_*} \), together with \( M/2 = v_k(z_k) \) we obtain \( v(z_*) = M/2 \), contradicting \( v(x) \equiv M \) in \( B \).

\[
3. \quad \text{The proof of Theorem 2 and Corollary 1}
\]

We start with

\[
\text{Proof of Theorem 2:}
\]
Consider two bounded solutions \( u_1, u_2 \) of (1). Then the difference function \( u(x) := u_1(x) - u_2(x) \)
will be solution of (1) for the right side \( f \equiv 0 \) and boundary values \( u \equiv 0 \) on \( \partial \Omega \). By assumption b) of Theorem 1, \( u \) is bounded in \( \Omega \), hence \( |u(x)| \leq M \) for some \( M > 0 \). We want to apply Theorem 1 to \( u \), but we first have to check whether the uniform boundary condition, assumption c) of Theorem 1, is satisfied by \( u \). As described in Remark 3 of [1, Chapter 6.3] we can construct a uniform barrier at each boundary point, using the uniform exterior sphere condition.

Let \( \Theta = \{ x \in \Omega : |x - y| < R + 1 \} \). By choosing \( \sigma = \sigma(\Lambda, R, n) > 0 \) sufficiently large, we obtain

\[
a^{ij}(x)\partial_{ij}w + b^i(x)\partial_iw + c(x)w \leq 0 \quad \text{in} \quad \Theta.
\]

We now define

\[
\tau := \frac{M}{R^{-\sigma} - (R + 1)^{-\sigma}} > 0
\]

and note that \(-\tau w(x) \leq u(x) \leq \tau w(x)\) on \( \partial \Theta \). From the maximum principle we conclude that \(-\sigma w(x) \leq u(x) \leq \sigma w(x)\) in \( \Theta \). Using \( |x_0 - y| = R \) this yields

\[
|u(x)| \leq \tau|w(x)| = \tau\left(R^{-\sigma} - |x - y|^{-\sigma}\right) \\
\leq \tau\left(R^{-\sigma} - (|x - x_0| + R)^{-\sigma}\right) \quad \text{for all} \quad x \in \Omega, \quad x_0 \in \partial \Omega \quad \text{with} \quad |x - x_0| < 1.
\]

In particular, for \( |x_0 - x| = \text{dist}(x, \partial \Omega) \) we obtain

\[
|u(x)| \leq \tau\left(R^{-\sigma} - (\text{dist}(x, \partial \Omega) + R)^{-\sigma}\right) \quad \text{for all} \quad x \in \Omega \quad \text{with} \quad \text{dist}(x, \partial \Omega) < 1.
\]

As the constants \( R, \sigma \) and \( \tau \) are independent of the chosen boundary point \( x_0 \in \partial \Omega \), we see that assumption c) of Theorem 1 is satisfied by \( u \). \( \square \)

We finally give the

**Proof of Corollary 1**

Let \( \Omega = \mathbb{R} \times \Omega' \) for some bounded domain \( \Omega' \subset \mathbb{R}^{n-1} \). Note that such a domain \( \Omega \) satisfies the uniform thickness condition \( \sup_\Omega \text{dist}(x, \partial \Omega) \leq d \) with \( d := \text{diam}(\Omega') \). Let \( u \in C^2(\Omega, \mathbb{R}) \cap C^0(\overline{\Omega}, \mathbb{R}) \) be a bounded solution of (1). For some \( k \in \mathbb{Z} \) let us define a translation of \( u \) by

\[
\tilde{u}(x) \in C^2(\Omega, \mathbb{R}) \cap C^0(\overline{\Omega}, \mathbb{R}) \quad \text{and} \quad \tilde{u}(x_1, \ldots, x_n) := u(x_1 + kL, x_2, \ldots, x_n) \quad \text{for} \quad x \in \overline{\Omega}.
\]

Note that \( \tilde{u} \) is bounded just as \( u \) is. By the periodicity assumptions on the data \( a^{ij}, b^i, c, f \) and \( g \) this \( \tilde{u} \) will be solution of the same problem (1) as \( u \). By Theorem 2 we obtain \( \tilde{u}(x) = u(x) \) in \( \overline{\Omega} \) proving the periodicity of \( u \). \( \square \)
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