RATIONAL WEAK MIXING IN INFINITE MEASURE SPACES

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Abstract. Rational weak mixing is a measure theoretic version of Krickeberg’s strong ratio mixing property for infinite measure preserving transformations. It requires “density” ratio convergence for every pair of measurable sets in a dense hereditary ring. Rational weak mixing implies weak rational ergodicity and (spectral) weak mixing. It is enjoyed for example by Markov shifts with Orey’s strong ratio limit property. The power, subsequence version of the property is generic.

§0 Introduction: Hopf’s example

E. Hopf gave an example in [H] of a transformation of the infinite strip $\mathbb{R}_+ \times [0,1]$, preserving Lebesgue measure $m$ which satisfies the ratio mixing property:

$$\left( \star \right) \quad \lim_{n \to \infty} \frac{m(A \cap T^{-n}B)}{u_n} = m(A)m(B)$$

\forall A, B bounded with $m(\partial A) = m(\partial B) = 0$

where $u_n = \sqrt{\frac{2}{\pi n}}$.

Hopf mentioned that if (\star) could be established for every bounded measurable set, this would imply ergodicity of $T$. This latter property is also invariant under isomorphism.

The theory of weakly wandering sets as in [HK] shows that (\star) cannot hold for every pair of sets in any dense, hereditary collection (see below) in the absence of absolutely continuous, invariant probabilities (and this cannot be used to establish ergodicity of Hopf’s example).

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We show here that Hopf’s example is rationally weakly mixing in a sense which implies that for every pair of bounded measurable sets, (✠) (as on page 1) is satisfied on a subsequence of full density.

**Organization of the paper.**

In §1, we give definitions and preliminary discussions. The main results are stated in §2. In §3, we study the modes of convergence involved in the rational weak mixing properties. §4 contains the proof of the “basic” proposition 0 and the “density convergence” theorem A. In §5, we establish sufficient conditions for rational weak mixing (Lemmas B and C). We collect in §6 some facts on the mean ergodic theorem with weighted averages for use in §7 to prove theorem D which connects subsequence rational weak mixing with other mixing properties. Markov shifts are treated in §8 (where there is some discussion of smoothness of renewal sequences) and Gibbs-Markov towers are studied in §9 via their local limit properties. We “make categorical statements” (theorem F) in §10 and some closing remarks in §11.

§1 Definitions and preliminaries

**Notation and basics.**

In this paper $(X, \mathcal{B}, m, T)$ denotes a measure preserving transformation $T$ of a non-atomic, $\sigma$-finite, standard measure space $(X, \mathcal{B}, m)$.

Unless otherwise stated, the measure will be infinite ($m(X) = \infty$). Measure preserving transformations of finite measure spaces are referred to as probability preserving transformations.

A standing assumption on $(X, \mathcal{B}, m, T)$ is conservativity:

$$m(A \setminus \bigcup_{n=1}^{\infty} T^{-n} A) = 0 \ \forall \ A \in \mathcal{B}.$$ 

The collection of measurable subsets of $X$ with finite measure is denoted $\mathcal{F} := \{ A \in \mathcal{B} : m(A) < \infty \}$ and for any $\mathcal{C} \subset \mathcal{B}$ the “positive elements” of $\mathcal{C}$ are denoted

$$\mathcal{C}_+ := \{ A \in \mathcal{C} : m(A) > 0 \}.$$ 

**Krickeberg mixing.**

Krickeberg ([Kri1]) noted that Hopf’s example is isomorphic to the Markov shift of the simple symmetric random walk on $\mathbb{N}$ with reflecting barrier at 1 which has irreducible, recurrent transition matrix ([KM]) and is therefore conservative, ergodic ([HR]). He also formulated a concept of topological ratio mixing for transformations preserving infinite measures:
Let \((X, \mathcal{B}, m, T)\) be a measure preserving transformation and let \(\alpha \subset \mathcal{B}\) be a countable partition, generating \(\mathcal{B}\) under \(T\) in the sense that \(\sigma(\bigcup_{n \in \mathbb{Z}} T^n \alpha) = \mathcal{B}\). The measure preserving transformation \((X, \mathcal{B}, m, T)\) is called Krickeberg \(\alpha\)-mixing if

\[\exists u_n > 0 \ (n \geq 1) \text{ such that } (\star) \text{ (as on page 1)} \text{ is satisfied } \forall A, B \in \mathcal{C}_\alpha,\]

the collection of \((\alpha, T)\)-cylinder sets defined by

\[\mathcal{C}_\alpha = \mathcal{C}_\alpha(T) := \{[a_1, \ldots, a_N]_k : N \in \mathbb{N}, k \in \mathbb{Z}, a_1, \ldots, a_N \in \alpha\};\]

and hence ([Kri1]) \(\forall A, B \text{ with } m(\partial A) = m(\partial B) = 0 \text{ when } X \text{ is considered equipped with the product topology from } \alpha^\mathbb{Z}.

Markov shifts with the strong ratio limit property (SRLP) as in [O] (e.g. Hopf’s example) are Krickeberg \(\alpha\)-mixing with \(\alpha\) the natural partition according to the state occupied at time 0 ([Kri1]). Examples of Krickeberg \(\alpha\)-mixing measure preserving transformations are also given in [Fri], [Pap], [T] and [MT]. Other definitions of mixing are discussed in [L].

It follows from theorem 8.1 that Markov shifts whose associated renewal sequences have the strong ratio limit property are \textit{rationally weakly mixing}.

\textbf{Hereditary rings.}

Let \((X, \mathcal{B}, m)\) be a \(\sigma\)-finite measure space. A collection \(\mathcal{C} \subset \mathcal{B}\) is called \textit{hereditary} if

\[\mathcal{H}(\mathcal{C}) := \{A \in \mathcal{B}, A \subseteq B \in \mathcal{C}\} = \mathcal{C}\]

A \textit{hereditary ring} \(\mathcal{H} \subset \mathcal{B}\) is a hereditary collection which is closed under finite union. It is \textit{dense} if

\[\forall A \in \mathcal{F} := \{F \in \mathcal{B}: m(F) < \infty\}, \epsilon > 0 \exists H \in \mathcal{H}, m(A \setminus H) < \epsilon.\]

For example, both \(\mathcal{F}\) and the collection \(\mathcal{R}_b\) of bounded measurable subsets of the infinite strip \(\mathbb{R}_+ \times [0,1]\) are dense hereditary rings. The collection of null sets is a hereditary ring which is not dense. We’ll denote the minimal hereditary ring containing the collection \(\mathcal{C} \subset \mathcal{B}\) by \(\mathcal{H}\mathcal{R}(\mathcal{C})\).

Any two dense, hereditary rings in the same measure space intersect and thus many ergodic properties involving such are isomorphism invariant (e.g. rational weak mixing).
Standard measure spaces.

We assume that all measure preserving transformations are defined on standard measure spaces. The $\sigma$-finite measure space $(X, B, m)$ is **standard** if $X$ is a Polish space, $B$ is the collection of Borel sets and $m$ is non-atomic. The standardness assumption is used as follows:

- If $(X, B, m, T)$ is a conservative, ergodic, measure preserving transformations of a standard measure space then $\exists$ a countable partition $\alpha \subset F = \{A \in B : m(A) < \infty\}$ which generates $B$ under $T$ and, up to isomorphism, $T$ is the shift on $X = \alpha\mathbb{Z}$ equipped with the product topology (a homeomorphism).

Weights.

Our results involve averaging techniques using certain non-negative, bounded weight sequences. We call a bounded sequence $u = (u_0, u_1, \ldots)$ an **admissible weight sequence** (abbr. to weight) if

$$u_n \geq 0 \ \forall \ n \geq 1 \ \& \ a_u(n) := \sum_{k=1}^{n} u_k \rightarrow \infty$$

and denote the collection of weights by $\mathcal{W}$.

We’ll denote, for (eventually) positive sequences $u = (u_0, u_1, \ldots)$ & $w = (w_0, w_1, \ldots)$:

- $u_n \sim w_n$ if $\lim_{n \to \infty} \frac{u_n}{w_n} = 1$;

  and for non-negative sequences $u = (u_0, u_1, \ldots)$ & $w = (w_0, w_1, \ldots)$:

- $u_n \ll w_n$ if $\exists M > 0$ such that $u_n \leq Mw_n \ \forall \ n \geq 0$;
- $u_n \gg w_n$ if $u_n \ll w_n$ and $u_n \gg w_n$.

Given a subsequence $K \subset \mathbb{N}$, we call weights $u, w \in \mathcal{W}$

- $K$-asymptotic ($u \approx w$) if $\lim_{n \to \infty, n \in K} \frac{1}{a_u(n)} \sum_{k=1}^{n} |u_k - w_k| = 0$.

  Evidently,

$$u \approx w \implies \frac{a_u(n)}{a_w(n)} = 1 \quad \text{and} \quad \lim_{n \to \infty, n \in K} \frac{a_u(n)}{a_w(n)} = 1.$$

The converse implication sometimes holds and will be discussed in the sequel.

We call $u \in \mathcal{W}$

- $K$-smooth if $\lim_{n \to \infty, n \in K} \frac{1}{a_u(n)} \sum_{k=1}^{n} |u_k - u_{k+1}| = 0$; equivalently $(u_0, u_1, \ldots) \approx (u_1, u_2, \ldots)$.

We’ll say that weights $u, w \in \mathcal{W}$ are asymptotic ($u \approx w$) if $u \approx w$, that $u \in \mathcal{W}$ is smooth if it is $\mathbb{N}$-smooth and subsequence smooth if it is $K$-smooth for some subsequence $K \subset \mathbb{N}$. 
Intrinsic weights.

For \((X, B, m, T)\) a conservative, ergodic, measure preserving transformation and \(E, F \in \mathcal{F}_+\) the intrinsic weight \(u(E, F) \in \mathfrak{W}\) is defined by
\[
u_n(E, F) := \frac{m(F \cap T^{-n}F)}{m(E)m(F)}.
\]
We denote \(a_n(E, F) = a_u(E, F)(n) := \sum_{k=0}^{n-1} u_k(E, F)\) and write \(u(F) := u(F, F)\).

Rational weak mixing.

Let \(\mathfrak{R} \subset \mathbb{N}\) be a subsequence. We'll call the conservative, ergodic, measure preserving transformation \((X, B, m, T)\) rationally weakly mixing along \(\mathfrak{R}\) if \(\exists F \in \mathcal{F}_+\) so that
\[
m(A \cap T^{-n}B) \equiv m(A)m(B)u_n(F) \quad \forall A, B \in B \cap F.
\]
We call the measure preserving transformation \((X, B, m, T)\)
• rationally weakly mixing if it is rationally weakly mixing along \(\mathbb{N}\);
and
• subsequence rationally weakly mixing if it is rationally weakly mixing along some \(\mathfrak{R} \subset \mathbb{N}\).

Weak rational ergodicity.

Again for \(\mathfrak{R} \subset \mathbb{N}\) a subsequence, the conservative, ergodic, measure preserving transformation \((X, B, m, T)\) is called weakly rationally ergodic along \(\mathfrak{R}\) if \(\exists F \in \mathcal{F}_+\) so that
\[
\left(\bigstar_{\mathfrak{R}}\right) \frac{1}{a_n(F)} \sum_{k=0}^{n-1} m(B \cap T^{-k}C) \rightarrow m(B)m(C) \quad \forall B, C \in B \cap F
\]
where \(a_n(F) := \frac{1}{m(F)^2} \sum_{k=0}^{n-1} m(F \cap T^{-k}F)\). Weak rational ergodicity entails conservativity and ergodicity.

The proof of theorem 3.3 in [FL] easily adapts to show that \(F \in \mathcal{F}_+\) satisfies \(\left(\bigstar_{\mathfrak{R}}\right)\) if and only if
\[
\left\{ \frac{S_n^{(T)}(1_F)}{a_n(F)} : n \in \mathfrak{R} \right\} \quad \text{is uniformly integrable on } F.
\]
A useful sufficient condition for this is
\[
\sup_{n \in \mathfrak{R}} \frac{1}{a_n(F)^2} \int_F S_n(1_F)^2dm < \infty
\]
and \((X, B, m, T)\) is called rationally ergodic along \(\mathfrak{R}\) if \(\exists F \in \mathcal{F}_+\) with this property. See [A], [A1] (for the special case \(\mathfrak{R} = \mathbb{N}\)).

In case \(T\) is weakly rationally ergodic along \(\mathfrak{R}\):
• the collection of sets \(R_\mathfrak{R}(T)\) satisfying \(\left(\bigstar_{\mathfrak{R}}\right)\) is a hereditary ring;
\[ \exists a_n(T) \text{ (the return sequence along } \mathfrak{R}) \text{ such that} \]
\[ \lim_{n \to \infty} \frac{a_n(A)}{a_n(T)} = 1 \quad \forall A \in R_{\mathfrak{R}}(T); \]

- for conservative, ergodic \( T \), \( R_{\mathfrak{R}}(T) = \mathcal{F} \) only when \( m(X) < \infty \). The proofs of these statements are analogous to those in [A], [A1] (for the special case \( \mathfrak{R} = \mathbb{N} \)).

We’ll call a measure preserving transformation \((X, \mathcal{B}, m, T)\) :

- \([\text{weakly}] \text{ rationally ergodic} \) if it is \([\text{weakly}] \) rationally ergodic along \( \mathcal{N} \) and set \( R(T) := R_{\mathcal{N}}(T) \) (as in [A], [A1]); and
- \( \text{subsequence [weakly] rationally ergodic} \) if it is \([\text{weakly}] \) rationally ergodic along some \( \mathfrak{R} \subset \mathbb{N} \).

For example, conservative, ergodic Markov shifts are rationally ergodic. For further examples, see [A].

We’ll see that rational weak mixing along \( \mathfrak{R} \) implies weak rational ergodicity along \( \mathfrak{R} \) and that for \( T \) rationally weakly mixing along \( \mathfrak{R} \),

\[ \{ F \in \mathcal{F}_+ : (\star_{\mathfrak{R}}) \text{ holds} \} = R_{\mathfrak{R}}(T) \]

where \((\star_{\mathfrak{R}})\) is as on page 5.

**Weak mixing.**

For a measure preserving transformation \((X, \mathcal{B}, m, T)\) of a \( \sigma \)-finite measure space (as shown in [ALW]) the following conditions are equivalent:

(i) \( f \in L^\infty, \lambda \in \mathbb{S}^1, f \circ T = \lambda f \text{ a.e. } \Rightarrow f \text{ is constant a.e.} \)

(ii) \( T \times S \text{ is ergodic } \forall \text{ ergodic, probability preserving } S; \)

(iii) \( \frac{1}{n} \sum_{k=0}^{n-1} \int_X u f \circ T^k dm \to 0 \quad \forall \ u \in L^1_0, \ f \in L^\infty. \)

We’ll call a measure preserving transformation satisfying (any one of) them spectrally weakly mixing. This in the interest of disambiguation. In [ALW] and elsewhere “spectral weak mixing” is called ”weak mixing”.

We’ll see that subsequence rational weak mixing \( \implies \) spectral weak mixing.
Categorical statements.
Let \((X, \mathcal{B}, m)\) be a standard \(\sigma\)-finite, non-atomic, infinite measure space and consider \(\text{MPT}(X, \mathcal{B}, m)\), the collection of invertible measure preserving transformations of \((X, \mathcal{B}, m)\) equipped with the \textit{weak operator topology} defined by \(T_n \to T\) if
\[
m(TA \Delta T_n A) + m(T^{-1}A \Delta T_n^{-1}A) \xrightarrow{n \to \infty} 0 \quad \forall \ A \in \mathcal{F}.
\]
It follows that \(\text{MPT}(X, \mathcal{B}, m)\) is a Polish group. A \textit{categorical statement} is a statement concerning the Baire category of a subset of \(\text{MPT}\). For a review of this subject, see [CP].

We’ll see that (the \textit{power} version of) the subsequence rational weak mixing elements of \(\text{MPT}\) form a residual set in \(\text{MPT}\).

§2 Results

Proposition 0 (basics)
Let \(\mathcal{R} \subset \mathbb{N}\) be a subsequence and suppose that \((X, \mathcal{B}, m, T)\) is rationally weakly mixing along \(\mathcal{R}\); then

(i) \(T\) is weakly rationally ergodic along \(\mathcal{R}\);

(ii) \(\{ F \in \mathcal{F}_+ : (\star_{\mathcal{R}}) \text{ holds} \} = R_{\mathcal{R}}(T)\)

where (\(\star_{\mathcal{R}}\)) is as on page 22.

(iii) \(u(F, G)\) is \(\mathcal{R}\)-smooth \(\forall \ F, G \in R_{\mathcal{R}}(T), \ m(F), \ m(G) > 0\).

(iv) for each \(p \in \mathbb{Z}, \ p \neq 0\), \(T^p\) is rationally weakly mixing along \(\frac{k}{|p|} \mathcal{R} := \{ \lfloor \frac{k}{|p|} \rfloor : k \in \mathbb{R} \}\) and \(R_{\frac{1}{|p|} \mathcal{R}}(T^p) = R_{\mathcal{R}}(T)\).

Corollary: Isomorphism invariance.
It follows from Proposition 0 that if \((X, \mathcal{B}, m, T)\) is rationally weakly mixing along \(\mathcal{R}\), and is isomorphic by measure preserving transformation to \((X', \mathcal{B}', m', T')\), then \((X', \mathcal{B}', m', T')\) is also rationally weakly mixing along \(\mathcal{R}\), and the intrinsic weights of \(T'\) are \(\mathcal{R}\)-asymptotic to those of \(T\).

Theorem A (density convergence)
Suppose that \((X, \mathcal{B}, m, T)\) is rationally weakly mixing and that

\(\exists \ E \in R_{\mathcal{R}}(T) \ s.t. u(E) \approx v\) where \(v \in \mathbb{W}\) is regularly varying with index \(s \in (-1, 0)\) \(\frac{d}{n} \to \lambda^s \ \forall \ \lambda > 0\), then for \(F \in R_{\mathcal{R}}(T)_+\)
\[
\frac{m(A \cap T^{-n}B)}{m_n(F)} \xrightarrow{n \to \infty} m(A)m(B) \quad \forall \ A, B \in R_{\mathcal{R}}(T).
\]
Here $s_n \xrightarrow{\text{density}} L$ means $s_n \xrightarrow{n \to \infty, n \in K} L$ where $K \subset \mathbb{N}$ has zero density in the sense that $\#(K \cap [1, n]) = o(n)$ as $n \to \infty$.

**Lemma B (sufficient conditions)** Let $\mathcal{K} \subset \mathbb{N}$ be a subsequence. Suppose that $X$ is a Polish space and

(i) $(X, \mathcal{B}, m, T)$ is an invertible, measure preserving transformation, weakly rationally ergodic along $\mathcal{K}$;

(ii) $\exists \Omega \in R_{\mathcal{K}}(T)$ open in $X$ and a countable base $\mathcal{C}$ for the topology of $\Omega$ such that

(a) $\forall \{C_i\}_{i=1}^k \subset \mathcal{C} \exists \{D_j\}_{j=1}^\ell \subset \mathcal{C}$, $m(D_i \cap D_j) = 0 \forall i \neq j$; $\bigcup_{i=1}^k C_i = \bigcup_{j=1}^\ell D_j$;

(b) $m(A \cap T^{-n}B) \underset{\mathcal{K}}{\approx} m(A)m(B)u_n(\Omega) \forall A, B \in \mathcal{C}$.

then $(X, \mathcal{B}, m, T)$ is rationally weakly mixing along $\mathcal{K}$.

**Lemma C (sufficient conditions)** Let $\mathcal{K} \subset \mathbb{N}$ be a subsequence. Suppose that

(i) $(X, \mathcal{B}, m, T)$ is an invertible, measure preserving transformation, weakly rationally ergodic along $\mathcal{K}$;

(ii) $\exists$ a countable generating partition $\alpha \subset R_{\mathcal{K}}(T)$ and $\Omega \in C_\alpha$ such that

$m(A \cap T^{-n}B) \underset{\mathcal{K}}{\approx} m(A)m(B)u_n(\Omega) \forall A, B \in C_\alpha$

where $u = u(\Omega)$, then $(X, \mathcal{B}, m, T)$ is rationally weakly mixing along $\mathcal{K}$.

**Theorem D (mixing properties)** Let $\mathcal{K} \subset \mathbb{N}$ be a subsequence and suppose that $(X, \mathcal{B}, m, T)$ is rationally weakly mixing along $\mathcal{K}$, then

(i) $(X, \mathcal{B}, m, T)$ is spectrally weakly mixing;

(ii) $T \times S$ is rationally weakly mixing along $\mathcal{K} \forall$ weakly mixing, probability preserving transformation $(\Omega, \mathcal{F}, P, S)$.

Invertible rationally weakly mixing measure preserving transformations of infinite measure spaces are obtained via

**Corollary E**
The natural extension of a measure preserving transformation, rationally weakly mixing along $\mathcal{R}$, is also rationally weakly mixing along $\mathcal{R}$ with $\mathcal{R}$-asymptotic intrinsic weights.

Let
\[
\text{RWM}(X) := \{ T \in \text{MPT}(X) : T \text{ is rationally weakly mixing} \}
\]
\[
\text{SRWM}(X) := \{ T \in \text{MPT}(X) : T \text{ is subsequence rationally weakly mixing} \}.
\]

For $T \in \text{MPT}(X)$, $\Delta \geq 1$ and $(\kappa_1, \ldots, \kappa_\Delta) \in \mathbb{Z}^\Delta$, let
\[
T^{(\kappa_1, \ldots, \kappa_\Delta)} := T^{\kappa_1} \times T^{\kappa_2} \times \ldots \times T^{\kappa_\Delta} \in \text{MPT}(X^\Delta).
\]
Call $T \in \text{MPT}$ power, subsequence rationally weakly mixing if $T^{(\kappa_1, \ldots, \kappa_\Delta)}$ is subsequence rationally weakly mixing $\forall \Delta \geq 1$ & $(\kappa_1, \ldots, \kappa_\Delta) \in (\mathbb{Z} \setminus \{0\})^\Delta$. Let
\[
\text{PSRWM}(X) := \{ T \in \text{MPT}(X) : T^{(\kappa_1, \ldots, \kappa_\Delta)} \in \text{SRWM}(X^\Delta) \ \forall \Delta \geq 1, (\kappa_1, \ldots, \kappa_\Delta) \in (\mathbb{Z} \setminus \{0\})^\Delta \}.
\]

**Theorem F** (Baire category)
(i) The collection RWM is meagre in MPT.
(ii) The collection PSRWM(X) is residual in MPT(X).

§3 Convergence

In this section we study the modes of convergence involved in the rational weak mixing properties. Let $\mathcal{R} \subset \mathbb{N}$ be a subsequence.

$(u, \mathcal{R})$-small sets.

Let $u \in \mathcal{W}$. We’ll say that the set $K \subset \mathbb{N}$

- is $(u, \mathcal{R})$-small if $\frac{a_u(K,n)}{a_u(n)} \to 0$ where $a_u(K,n) := \sum_{k \in K \cap [1,n]} u_k$; and $(u, \mathcal{R})$-large if $K^c$ is $(u, \mathcal{R})$-small.

We’ll call a set $u$-small if it is $(u, \mathbb{N})$-small.

Recall that the set $K \subset \mathbb{N}$

- has density $d(K)$ if $\frac{1}{n} \#(K \cap [1,n]) \to d(K)$ as $n \to \infty$;
- and has zero density if $d(K) = 0$ (equivalently: $K$ is $1$-small where $1 \in \mathcal{W}$, $\frac{1}{n} = 1 \ \forall \ n$).
The following remark collects some elementary facts about \((u, \mathcal{R})\)-smallness:

**Remark 3.1.**

Suppose that \(u \in \mathcal{W}\), then

(i) a finite union of \((u, \mathcal{R})\)-small sets is itself \((u, \mathcal{R})\)-small;
(ii) if \(v \in \mathcal{W}\) satisfies \(u_n \asymp v_n\) then \(K \subset \mathbb{N}\) is \((u, \mathcal{R})\)-small iff it is \((v, \mathcal{R})\)-small;
(iii) if \(K_1 \subset K_2 \subset \ldots\) is an increasing sequence of \((u, \mathcal{R})\)-small sets, then \(\exists N_1 < N_2 < \ldots\) so that \(K_\infty := \bigcup_{j=1}^{\infty} K_j \cap [N_j + 1, N_{j+1}]\) is a \((u, \mathcal{R})\)-small set.

**Proof** A suitable sequence is obtained by choosing \(N_j \uparrow\) such that 

\[
\frac{a_u(K_j, n)}{a_u(n)} < \frac{1}{j} \quad \forall\ n \geq N_j,\ n \in \mathcal{R}.
\]

(iv) if \(u, v \in \mathcal{W}\) and \(u \asymp v\), then \(K \subset \mathbb{N}\) is \((u, \mathcal{R})\)-small iff it is \((v, \mathcal{R})\)-small.

**Proof** \(|a_u(K, n) - a_v(K, n)| \leq \sum_{k=0}^{N} |u_k - v_k| = o(a_u(n))\) along \(\mathcal{R}\).

(v) if \(u \in \mathcal{W}\) is \(\mathcal{R}\)-smooth, then \(K \subset \mathbb{N}\) is \((u, \mathcal{R})\)-small iff \(K + 1\) is \((u, \mathcal{R})\)-small.

(vi) if \(u \in \mathcal{W}\) is \(\mathcal{R}\)-smooth, \(p \geq 1\), and \(u^{(p)}_n := u_{pn}\), then \(a_u^{(p)}(n) - \frac{1}{p} a_u(pn)\) along \(\frac{1}{p} \mathcal{R}\) and \(u^{(p)}\) is \(\frac{1}{p} \mathcal{R}\)-smooth where \(\frac{1}{p} \mathcal{R} := \{\lfloor \frac{j}{p}\rfloor : j \in \mathcal{R}\}\).

**Proof** This follows from

\[
\sum_{k=1}^{n} |u^{(p)}_k - u^{(p)}_{k+1}| \leq p \sum_{k=1}^{pn} |u_k - u_{k+1}| = o(a_u(pn)) \quad \&
\]

\[
|pa_u^{(p)}(n) - a_u(pn)| \leq \sum_{r=1}^{p-1} \sum_{k=1}^{n} |u_{pk} - u_{pk+r}| \leq p^2 \sum_{k=1}^{pn} |u_k - u_{k+1}| = o(a_u((p+1)n)).
\]

**Proposition 3.1**

Suppose that \(u \in \mathcal{W}\) and \(u_n \asymp \frac{a_u(n)}{n}\) and \(u_n \asymp v_n\) where \(v \in \mathcal{W}\) and \(v_n \downarrow\), then a set is \(u\)-small iff it has zero density.

**Proof** By remark 3.1(ii), there is no loss of generality in assuming \(u_n \downarrow\).

**Proof** of \(\Rightarrow\):
Suppose that $u_n \geq \frac{\eta a_u(n)}{n}$. Since $u_n \downarrow$,

$$a_u(K, n) = \sum_{k \in [1, n] \cap K} u_k \geq u_n |K \cap [1, n]| \geq \frac{\eta a_u(n)}{n} |K \cap [1, n]|$$

whence

$$\frac{|K \cap [1, n]|}{n} \leq \frac{a_u(K, n)}{\eta a_u(n)}. \quad \Box$$

**Proof** of $\Leftarrow$: We show first that

$$\lim_{n \to \infty, \epsilon \to 0} \frac{a_u(\epsilon n)}{a_u(n)} = 0.$$

To do this, it suffices to show that

$$a_u(2n) \geq (1 + \eta \log 2) a_u(n)$$

where $u_n \geq \frac{\eta a_u(n)}{n}$.

Indeed,

$$a_u(2n) - a_u(n) = \sum_{k=n+1}^{2n} u_k \geq \eta \sum_{k=n+1}^{2n} \frac{a_u(k)}{k} \geq \eta a_u(n) \sum_{k=n+1}^{2n} \frac{1}{k} \geq \eta \log 2 \cdot a_u(n). \quad \Box$$

Next, since $u_n \downarrow$, $u_n \leq M \frac{a_u(n)}{n}$, and

$$a_u(K, n) = \sum_{k \in [1, n] \cap K} u_k \leq a_u(\epsilon n) + \sum_{k \in [\epsilon n, n] \cap K} u_k$$

$$\leq a_u(\epsilon n) + u_n |[1, n] \cap K|$$

$$\leq a_u(\epsilon n) + \frac{a_u(\epsilon n)}{\epsilon n} |[1, n] \cap K|$$

whence

$$\frac{a_u(K, n)}{a_u(n)} \leq \frac{a_u(\epsilon n)}{a_u(n)} + \frac{1}{\epsilon} \cdot \frac{|K \cap [1, n]|}{n}. \quad \Box$$

**Remark 3.2.**

(i) In case $u \in \mathcal{W}$ and $u_n$ is *regularly varying with index* $s \in (-1, 0)$ (i.e. $u_n^{1+s} \to \lambda^s$ as $n \to \infty$) then (see e.g. [BGT]) $u_n \sim (1+s) a_u(n)$ and $\exists v \in \mathcal{W}$ such that $v_n \downarrow$, $v_n \sim u_n$. Thus, proposition 3.1 applies.

(ii) The conclusion of proposition 3.1 fails for $u_n = \frac{1}{n+1}$. The set $K := \bigcup_{k=1}^{\infty} [2k^2, k2^2] \cap \mathbb{N}$ is $u$-small, but $\lim_{n \to \infty} \frac{|K \cap [1, n]|}{n} = 1$. 
(\(u, \mathfrak{R}\))-density and \((u, \mathfrak{R})\)-strong Cesaro convergence.

Let \(\mathfrak{R} \subseteq \mathbb{N}\) be a subsequence and let \(u \in \mathfrak{W}\).

We’ll say that a sequence \(s_n\):

- **converges in \((u, \mathfrak{R})\)-density** to \(L \in \mathbb{R}\) \(s_n \xrightarrow{\text{\(n \to \infty\)}}^\text{\((u, \mathfrak{R})\)-d.} L\) if \(\exists K \subseteq \mathbb{N}\) \((u, \mathfrak{R})\)-small such that \(s_n \xrightarrow{\text{\(n \to \infty\)}}^\text{\(n \notin K\)}} L\);

and that

- \(s_n\) **converges \((u, \mathfrak{R})\)-strongly Cesaro** to \(L \in \mathbb{R}\) \(s_n \xrightarrow{\text{\(n \to \infty\)}}^\text{\((u, \mathfrak{R})\)-s.C.} L\) if

\[
\frac{1}{a_u(n)} \sum_{k=0}^{n} u_k \left| s_k - L \right| \xrightarrow{\text{\(n \to \infty\)}} 0.
\]

**Remark 3.3.**

Let \(\mathfrak{R} \subseteq \mathbb{N}\) be a subsequence and let \(u \in \mathfrak{W}\), \(s = (s_1, s_2, \ldots) \in \mathbb{R}^\mathbb{N}\) and \(L \in \mathbb{R}\).

(i) \(s_n \xrightarrow{\text{\(n \to \infty\)}}^\text{\((u, \mathfrak{R})\)-d.} L\) if and only if

\[K_\epsilon := \{n \in \mathbb{N} : |s_n - L| > \epsilon\} \text{ is } (u, \mathfrak{R})\text{-small } \forall \epsilon > 0.\]

**Proof:** Evidently \(s_n \xrightarrow{\text{\(n \to \infty\)}}^\text{\((u, \mathfrak{R})\)-d.} L \Rightarrow K_\epsilon \text{ is } (u, \mathfrak{R})\text{-small } \forall \epsilon > 0.\) To see the reverse implication, if \(K_\epsilon \text{ is } (u, \mathfrak{R})\text{-small } \forall \epsilon > 0,\) then by remark 3.1(iii) (on p. 10), \(\exists N_1 < N_2 < \ldots \) so that \(K := \bigcup_{j=1}^{\infty} K_{1/j} \cap [N_j + 1, N_{j+1}]\) is a \((u, \mathfrak{R})\)-small set and and \(s_n \xrightarrow{\text{\(n \to \infty\)}}^\text{\(n \notin K\)}} 0.\)

(ii) if \(v \in \mathfrak{W}, u \equiv v,\) then

\(s_n \xrightarrow{\text{\(n \to \infty\)}}^\text{\((u, \mathfrak{R})\)-d.} L\) if and only if \(s_n \xrightarrow{\text{\(n \to \infty\)}}^\text{\((v, \mathfrak{R})\)-d.} L.\)

(iii) \(s_n \xrightarrow{\text{\(n \to \infty\)}}^\text{\((u, \mathfrak{R})\)-s.C.} L \Rightarrow s_n \xrightarrow{\text{\(n \to \infty\)}}^\text{\((u, \mathfrak{R})\)-d.} L.\)

**Proof**

Suppose that \(s_n \geq 0, L = 0\) and that \(\frac{1}{a_u(n)} \sum_{k=1}^{n} u_k s_k \xrightarrow{\text{\(n \to \infty\)}} 0.\) By the Chebyshev-Markov inequality,

\[
\frac{a_u(K_\epsilon, n)}{a_u(n)} = \frac{1}{a_u(n)} \sum_{k=1}^{n} u_k \leq \frac{1}{\epsilon a_u(n)} \sum_{k=1}^{n} u_k s_k \xrightarrow{\text{\(n \to \infty\)}} 0. \quad \square
\]

We call a sequence \(x = (x_1, x_2, \ldots) \in \mathbb{R}^\mathbb{N}\) one-sidedly bounded if it is either bounded above, or below (or both).
Proposition 3.2 Let $\mathcal{R} \subset \mathbb{N}$ be a subsequence and let $u \in \mathcal{M}$. Suppose that $x = (x_1, x_2, \ldots) \in \mathbb{R}^\mathbb{N}$ is one-sidedly bounded, and $L \in \mathbb{R}$, then

\[
\lim_{n \to \infty} x_n^{(u,\mathcal{R})-s.c.} = L
\]

if and only if

\[
\lim_{n \to \infty} \frac{1}{a_u(n)} \sum_{k=0}^{n} u_k x_k = L \quad \text{and} \quad \lim_{n \to \infty} x_n^{(u,\mathcal{R})-d.} = L.
\]

Proof of (†) ⇒ (‡):

We assume (without loss in generality) that $x_n \geq 0$ $\forall$ $n \geq 1$ and $L \geq 0$. Fix $\epsilon > 0$ and set

\[
K_{+,\epsilon} := \{ n \geq 1, \ x_n > L + \epsilon \}, \ K_{-,\epsilon} := \{ n \geq 1, \ x_n < L - \epsilon \}, \ K_\epsilon := K_{+,\epsilon} \cup K_{-,\epsilon}.
\]

By our assumptions $\exists \ N_\epsilon > 1$ such that

\[
a_u(K_\epsilon, [1, n]) < \epsilon a_u(n) \quad \text{and} \quad L - \epsilon < \frac{1}{a_u(n)} \sum_{k=0}^{n} u_k x_k < L + \epsilon \quad \forall \ n > N_\epsilon, \ n \in \mathcal{R}.
\]

For large enough $n > N_\epsilon, \ n \in \mathcal{R}$,

\[
\sum_{k=1}^{n} u_k |x_k - L| \leq \epsilon \sum_{k \in K_\epsilon^{c} \cap [1, n]} u_k + \sum_{k \in K_{+,\epsilon} \cap [1, n]} u_k (x_k - L) + \sum_{k \in K_{-,\epsilon} \cap [1, n]} u_k (L - x_k)
\]

\[
\leq \epsilon a_u(n) + L a_u(K_\epsilon, n) + \sum_{k \in K_\epsilon^{c} \cap [1, n]} u_k x_k
\]

\[
< (1 + L) \epsilon a_u(n) + \sum_{k \in K_\epsilon^{c} \cap [1, n]} u_k x_k
\]

Now (for large enough $n > N_\epsilon, \ n \in \mathcal{R}$),

\[
\sum_{k \in K_\epsilon^{c} \cap [1, n]} u_k x_k = \sum_{k \in [1, n]} u_k x_k - \sum_{k \in K_\epsilon^{c} \cap [1, n]} u_k x_k
\]

\[
< (L + \epsilon) a_u(n) - (L - \epsilon) a_u(K_\epsilon^{c}, n)
\]

\[
= (L + \epsilon) a_u(K_\epsilon, n) + 2 \epsilon a_u(K_\epsilon^{c}, n)
\]

\[
< \epsilon (L + \epsilon) a_u(n) + 2 \epsilon a_u(n)
\]

\[
= (L + 2 + \epsilon) \epsilon a_u(n).
\]

Reassembling,

\[
\sum_{k=1}^{n} u_k |x_k - L| < (2L + 3 + \epsilon) \epsilon a_u(n). \qed
\]

A version of the following proposition is implicit in [GL]:
Proposition 3.3
Let \( \mathcal{A} \subset \mathbb{N} \) be a subsequence and let \( u \in \mathcal{W} \). Suppose that \( x = (x_1, x_2, \ldots) \in \mathbb{R}^\mathbb{N} \) and that \( L \in \mathbb{R} \).

If \( \frac{1}{a_u(n)} \sum_{k=0}^{n} u_k x_k \underset{n \to \infty}{\longrightarrow} L \) and
either (i) \( x = (x_1, x_2, \ldots) \) is bounded below and \( \exists \) a \((u, \mathcal{A})\)-small set
\( K_0 \subset \mathbb{N} \) such that \( \lim_{n \to \infty, n \notin K_0} x_n \geq L \);
or (ii) \( x = (x_1, x_2, \ldots) \) is bounded above and \( \exists \) a \((u, \mathcal{A})\)-small set
\( K_0 \subset \mathbb{N} \) such that \( \lim_{n \to \infty, n \notin K_0} x_n \leq L \); then \( x_n \underset{n \to \infty}{\longrightarrow} L \).

Proof
By proposition 3.2, it suffices to prove that \( x_n \underset{n \to \infty}{\longrightarrow} L \). By symmetry it suffices to prove the proposition under assumption (i). By possibly translating \( x \) with a constant sequence, we reduce to the case \( x_n \geq 0 \ \forall \ n \geq 1 \ & L \geq 0 \).

For \( \epsilon > 0 \), set \( K_\epsilon := \{ n \notin K_0 : x_n > L + \epsilon \} \). It suffices to prove that \( K_\epsilon \)
is \((u, \mathcal{A})\)-small \( \forall \ 0 < \epsilon < \frac{1}{2} \).

To see this, fix \( 0 < \epsilon < \frac{1}{2} \) let \( N_\epsilon \) be so that
\[ x_n > L - \epsilon^2 \ \forall \ n \geq N_\epsilon, \ n \notin K_0, \]
then for large \( n \gg N_\epsilon, \ n \in \mathcal{A}, \)
\[ (L + \epsilon^2)a_u(n) > \sum_{k=N_\epsilon}^{n} x_k u_k \]
\[ \geq \sum_{k \notin K_\epsilon \cap [N_\epsilon, n]} x_k u_k + \sum_{k \in K_\epsilon \cap [N_\epsilon, n]} x_k u_k \]
\[ > (L + \epsilon)a_u(K_\epsilon, n) + (L - \epsilon^2)a_u(K_\epsilon^c \cap K_0^c, n) - (2L + \epsilon^2)a_u(N_\epsilon) \]
\[ > (L + \epsilon)a_u(K_\epsilon, n) + (L - \epsilon^2)a_u(K_\epsilon^c, n) - a_u(K_0, n) - (2L + 1)a_u(N_\epsilon) \]
\[ = (L + \epsilon)a_u(K_\epsilon, n) + (L - \epsilon^2)a_u(K_\epsilon^c, n) - \mathcal{E}_n \]
where \( \mathcal{E}_n := a_u(K_0, n) + (2L + 1)a_u(N_\epsilon) \).

Writing
\[ (L + \epsilon^2)a_u(n) = (L + \epsilon^2)a_u(K_\epsilon, n) + (L + \epsilon^2)a_u(K_\epsilon^c, n) \]
we see that
\[ (\epsilon - \epsilon^2)a_u(K_\epsilon, n) \leq 2\epsilon^2a_u(K_\epsilon^c, n) + \mathcal{E}_n. \]
For large \( n \in \mathcal{A}, \mathcal{E}_n < \epsilon^2a_u(n) \) whence
\[ a_u(K_\epsilon, n) \leq \frac{3\epsilon}{1 - \epsilon} \cdot a_u(n) < 6\epsilon a_u(n). \]
Corollary 3.4

Let \( K \subset \mathbb{N} \) be a subsequence and let \( u \in \mathcal{W} \). If \( \frac{u_{n+1}}{u_n} \stackrel{(u,\mathcal{R})-d.}{\rightarrow} 1 \), then \( u \) is \( K \)-smooth.

Proof. If \( u_n > 0 \ \forall \ n \geq 0 \), this follows from proposition 3.2 since
\[
\sum_{k=0}^{n} u_k \frac{u_{k+1}}{u_k} - 1 = \sum_{k=0}^{n} |u_{k+1} - u_k|.
\]
If this is not the case, define \( v \in \mathcal{W} \) by
\[
v_n \begin{cases} 
    u_n & u_n > 0, \\
    \frac{1}{2n} & u_n = 0.
\end{cases}
\]
Evidently \( a_v(n) \leq a_u(n) \leq a_v(n) + 2 \) so \( a(n) := a_u(n) - a_v(n) \).
Moreover, if \( K \subset \mathbb{N} \), then \( a_u(K,n) \leq a_v(K,n) \leq a_u(K,n) + 2 \) and so
\( K \) is \( (u,\mathcal{R}) \)-small iff it is \( (v,\mathcal{R}) \)-small.

Next, if \( \frac{1}{2} < \frac{u_{n+1}}{u_n} < 2 \), then \( v_n = u_n \) and \( v_{n+1} = u_{n+1} \), \( \frac{v_{n+1}}{v_n} = \frac{u_{n+1}}{u_n} \). Thus
\[
\frac{v_{n+1}}{v_n} \stackrel{\text{d.}}{\rightarrow} 1 \quad \text{and by proposition 3.2 (as above),}
\]

\[
\frac{1}{a(n)} \sum_{k=0}^{n} |v_{k+1} - v_k| \rightarrow 0, \quad n \in \mathbb{R}.
\]
Finally,
\[
\frac{1}{a(n)} \sum_{k=0}^{n} |u_{k+1} - u_k| \leq \frac{1}{a(n)} \sum_{k=0}^{n} |v_{k+1} - v_k| + \frac{2}{a(n)} \rightarrow 0. \quad \Box
\]

§4 Proofs of Proposition 0 and Theorem A

Proof of Proposition 0
Fix \( F \in \mathcal{F} \) satisfying \( (\bigstar_{\mathcal{R}}) \). Evidently, for \( A, B \in \mathcal{B} \cap F \),

\[
\frac{1}{a_n(F)} \sum_{k=0}^{n-1} m(A \cap T^{-k}B) \rightarrow m(A)m(B) \quad \forall \ A, B \in \mathcal{B} \cap F.
\]
This shows that \( F \in R_{\mathcal{R}}(T) \) and that \( T \) is weakly rationally ergodic along \( \mathcal{R} \); proving (i).

To prove (ii), let \( F \in \mathcal{F} \) satisfy \( (\bigstar_{\mathcal{R}}) \). It suffices to show that
\[
(\bigstar) \quad m(B \cap T^{-n}C) \stackrel{\mathcal{R}}{\approx} m(B)m(C)u_n(F) \quad \forall \ B, C \in R_{\mathcal{R}}(T).
\]
Proof of (\( \bigstar \)):
Fix \( B, C \in R_{\mathcal{R}}(T) \), then \( G := B \cup C \in R_{\mathcal{R}}(T) \) and we claim:
\[ \exists K \subset \mathbb{N} \ (u(F), \mathcal{R}) \text{-small such that} \]
\[ \lim_{n \to \infty, \ n \notin K} \frac{m(B \cap T^{-n}C)}{u_n(F)} \geq m(B)m(C). \]

**Proof of (1):**
Let \( \epsilon > 0 \), then \( \exists B_0, \ldots, B_N, C_0, \ldots, C_N \in \mathcal{B} \cap \mathcal{F} \) so that
\[ B' := \bigcup_{k=0}^{N} T^{-k}B_k \subset B, \ C' := \bigcup_{k=0}^{N} T^{-k}C_k \subset C, \ m(B \setminus B') < \epsilon, \ m(C \setminus C') < \epsilon. \]
Using (\Star\mathcal{R}) (as on page 10)
\[ \frac{m(B \cap T^{-n}C)}{u_n(F)} \geq \frac{m(B' \cap T^{-n}C')}{u_n(F)} = \sum_{k, \ell=0}^{N} \frac{m(T^{-k}B_k \cap T^{-n-\ell}C_\ell)}{u_n(F)} \]
\[ (u(F), \mathcal{R}) \rightarrow \mathcal{F}, \]
\[ \sum_{k, \ell=0}^{N} m(T^{-k}B_k)m(T^{-\ell}C_\ell) = m(B')(C') > (m(B) - \epsilon)(m(C) - \epsilon). \]

Choose \( \epsilon_n \downarrow 0 \). By the above \( \exists K_1 \subset K_2 \subset \cdots \subset \mathbb{N} \), each \( K_\nu \) being (\( u(F), \mathcal{R} \))-small, such that
\[ \lim_{n \to \infty, \ n \notin K_\nu} \frac{m(B \cap T^{-n}C)}{u_n(F)} \geq m(B)m(C) - \epsilon_\nu \ \forall \ \nu \geq 1. \]

By remark 3.1(iii) (on p. 10) \( \exists K \subset \mathbb{N} \) realizing (1).

By weak rational ergodicity along \( \mathcal{R} \),
\[ \frac{1}{a_n(F)} \sum_{k=0}^{n} m(B \cap T^{-k}C) \rightarrow_{n \to \infty, \ n \notin \mathcal{R}} m(B)m(C) \]
so by proposition 3.3(i),
\[ \frac{m(B \cap T^{-n}C)}{u_n(F)} \rightarrow_{n \to \infty, \ n \notin \mathcal{R}} m(B)m(C) \ \forall \ B, \ C \in \mathcal{B} \cap A; \]
equivalently \( m(B \cap T^{-n}C) \overset{\mathcal{R}}{\sim} m(B)m(C)u_n(F). \ \Box(\mathcal{R}) \)

This proves (ii).
To see (iii) (\( \mathcal{R} \)-smoothness of \( u(F) \) for \( F \in R_\mathcal{R}(T) \)), take \( B = F, \ C = T^{-1}F \) in (\( \mathcal{R} \)).

To prove (iv), fix \( p \in \mathbb{N} \). To see rational weak mixing of \( T^p \) along \( \frac{1}{p} \mathcal{R} \), let \( A \in R_\mathcal{R}(T) \). By remark 3.1(vi), \( a_n^{(T^p)}(A) \sim \frac{1}{p}a_{np}^{(T)}(A) \) along \( \frac{1}{p} \mathcal{R} \). It
also follows that for \( B, C \in \mathcal{B} \cap A, \)
\[
\sum_{k=0}^{n} |m(B \cap T^{-kp}C) - m(B)m(C)u_k^{(p)}| = \sum_{k=0}^{n} |m(B \cap T^{-kp}C) - m(B)m(C)u_k| \\
\leq \sum_{k=0}^{pm} |m(B \cap T^{-k}C) - m(B)m(C)u_k| \\
= o(a_{pm}(A)) \text{ along } \frac{1}{p} \mathbb{R}.
\]
This shows that \( R_{\mathbb{R}}(T) \subset R_{\frac{1}{p} \mathbb{R}}(T^p). \)

The other inclusion follows from results in [FL]. The proof of theorem 3.3 there shows that
\( A \in R_{\mathbb{R}}(T) \iff \{ S_n^{(T)}(1_A) : n \in \mathbb{R} \} \text{ is uniformly integrable on } A. \)

Now \[
S_{pn}^{(T)}(1_A) = \sum_{k=0}^{pn-1} 1_A \circ T^k = \sum_{\nu=0}^{p-1} \sum_{k=0}^{n-1} 1_A \circ T^{kp+\nu} = \sum_{\nu=0}^{p-1} S_n^{(T^\nu)}(1_A) \circ T^\nu
\]
whence
\[
A \in R_{\frac{1}{p} \mathbb{R}}(T^p) \iff \{ S_n^{(T^\nu)}(1_A) : n \in \frac{1}{p} \mathbb{R} \} \text{ is uniformly integrable on } A \\
\implies \{ S_{pn}^{(T^\nu)}(1_A) : n \in \frac{1}{p} \mathbb{R} \} \text{ is uniformly integrable on } A \\
\implies \{ S_n^{(T)}(1_A) : n \in \mathbb{R} \} \text{ is uniformly integrable on } A \\
\implies A \in R_{\mathbb{R}}(T). \quad \Box
\]

\[\Box\]

**Proof of theorem A** This follows from proposition 0(ii) via proposition 3.1. \[\Box\]

§5 Proof of Lemmas B and C, and corollary E

**Proof of Lemma B**
Let \( \mathcal{U} \) be the collection of finite unions sets in \( \mathcal{C}. \) It follows from assumptions (ii) (a) and (b) that
\[
(\bigotimes) \quad \frac{m(A \cap T^{-k} B)}{u_k} \xrightarrow{k \to \infty} m(A)m(B) \quad \forall \ A, B \in \mathcal{U}.
\]
Let \( \mathcal{K} := \{ K \subset \Omega : K \text{ compact} \}. \)
We claim first that  
\[ m(A \cap T^{-k}B) \underset{u_k}{\sim} m(A)m(B) \quad \forall \ A, \ B \in \mathcal{K}, \ A, \ B \in \Omega. \]

**Proof** We show first that  
\[ \lim_{k \to \infty, \ k \in K_1} m(A \cap T^{-k}B) \leq m(A)m(B). \]

To see this, we show first that  
\[ \forall \ \epsilon > 0, \ \exists \ U, V \in \mathcal{U} \text{ so that } A \subset U, \ B \subset V, \ m(U \setminus A), \ m(V \setminus B) < \epsilon. \]

Given \( \epsilon > 0 \), the Borel property of the measure \( m \) ensures open sets \( U, V \) satisfying (2). Each of these is a countable union of members of \( \mathcal{C} \). By compactness of \( A, B \) we can reduce to finite unions \( U, V \in \mathcal{U} \).

By (2) \( \exists K_1 \subset \mathbb{N} \) -small, so that  
\[ \lim_{k \to \infty, \ k \in K_1} m(A \cap T^{-k}B) \leq m(A)m(B). \]

Fix \( N_{\nu} \ (\nu \geq 1) \) so that  
\[ a_u(K_{\frac{1}{\nu}}, n) < \frac{1}{\nu} a_u(\Omega) \quad \forall \ n \in \mathfrak{K}, \ n \geq \nu \]
and  
\[ \frac{m(A \cap T^{-k}B)}{u_k} < (m(A) + \frac{1}{\nu})(m(B) + \frac{1}{\nu}) \quad \forall \ k \in K_{\frac{1}{\nu}} \cap [N_{\nu}, \infty). \]

The set  
\[ K_1 := \bigcup_{\nu \geq 1} K_{\frac{1}{\nu}} \cap [N_{\nu}, N_{\nu+1}) \]
is as required for (1).

Now fix \( \forall \ A, B \in \mathcal{K}, \ A, B \in \Omega. \) Since \( \Omega \in R_{\mathfrak{K}}(T), \)
\[ \sum_{k=0}^{n-1} m(A \cap T^{-k}B) \sim m(A)m(B)a_u(\Omega) \quad \text{as } n \to \infty, \ n \in \mathfrak{K}, \]
and the claim follows from (1) and proposition 3.3(ii). \( \Box(\mathfrak{K}) \)

To complete the proof of that \( \Omega \) satisfies \( \bigstar_{\mathfrak{K}} \), let \( A, B \in \mathcal{B} \cap \Omega, \) then \( \exists E_N, F_N \in \mathcal{K} \) such that \( \text{mod } m: \)
\[ E_N \uparrow A \quad \& \quad F_N \uparrow B \]
whence by (2), \( \exists K_N \ (u, \mathfrak{K}) \)-small so that \( \forall \ N \geq 1, \)
\[ \lim_{n \to \infty, \ n \notin K_N} \frac{m(A \cap T^{-n}B)}{u_n} \geq m(E_N)m(F_N). \]
As above, \( \exists K \subset \mathbb{N} \) \((u, R)\)-small such that
\[
\lim_{n \to \infty, \ n \notin K} \frac{m(A \cap T^{-n}B)}{u_n} \geq m(A)m(B)
\]
and \((\star_R)\) follows from proposition 3.3(i). \( \Box \)

**Proof of Lemma C**

By standardness, up to isomorphism, \( X = \alpha Z \), \( T : X \to X \) is the shift and the collection \( C_\alpha \) of \((\alpha, T)\)-cylinder sets forms a base of clopen sets for the Polish topology on \( X \). Thus \( C := C_\alpha \cap \Omega \) satisfies assumptions (ii) of lemma B and lemma C follows. \( \Box \)

**Proof of Corollary E**

Let \((X, B, m, T)\) be rationally weakly mixing along \( R \), and let \( \pi : (X', B', m', T') \to (X, B, m, T) \) be its natural extension, that is:

- \( T' \) invertible, \( \pi \circ T' = T \circ \pi \), \( \pi^{-1}B \subset B' \), \( m' \circ \pi^{-1} = m \) \& \( \bigvee_{n \geq 0} T^m \pi^{-1}B = B' \).

It follows from uniform integrability considerations (as in theorem 3.3 of [FL]) that \( T' \) is weakly rationally ergodic along \( R \) with \( R_R(T') \supseteq \mathcal{HR}(\pi^{-1}R_R(T)) \).

To see that \( T' \) is rationally weakly mixing along \( R \), fix a countable, one-sided \( T \)-generator \( \alpha \subset R_R(T) \), then \( \alpha' := \pi^{-1}\alpha \subset R_R(T') \) is a countable, two-sided \( T' \)-generator.

Fix \( \Omega \in \alpha' \) and let \( u := u(\Omega) \). By rational weak mixing of \( T \) along \( R \),
\[
\frac{m'(A \cap T'^{-n}B)}{u_n} \xrightarrow{\substack{\text{\((u,R)\)-d.} \\ \text{\(n \to \infty\)}}} m'(A)m'(B) \quad \forall \ A, B \in C_{\alpha'} = \pi^{-1}C_\alpha
\]
whence by lemma C, \((X', B', m', T')\) is rationally weakly mixing along \( R \). \( \Box \)

**§6 Mean ergodic theorem for weighted averages**

Let \( R \subset \mathbb{N} \) be a subsequence. Call a weight \( u \in \mathfrak{W} \) (good for the) mean ergodic theorem along \( R \) (abbr. \( \text{MET}_R \)) if for any ergodic, probability preserving transformation \((\Omega, \mathcal{A}, P, S)\), we have that
\[
(\text{MET}_R) \quad \frac{1}{a_u(n)} \sum_{k=0}^{n-1} u_k f \circ S^k \xrightarrow{\substack{\text{\(L^2(P)\)} \\ \text{\(n \to \infty, \ n \in R\)}}} E(f) \quad \forall \ f \in L^2(P).
\]

We let \( \text{MET} := \text{MET}_\mathbb{N} \).

Let \( R \subset \mathbb{N} \) be a subsequence.

Using the spectral theorem for unitary operators, it can be shown (see [Kre]) that the following conditions are equivalent for \( u \in \mathfrak{W} \):
• $u$ is MET$_R$;

• \[ \frac{1}{a_u(n)} \sum_{k=0}^{n-1} u_k z^k \rightarrow 0 \text{ for } z \in \mathbb{C}, \ |z| = 1, \ z \neq 1; \]

• $u$ is (good for the) weak ergodic theorem along $\mathcal{R}$ (abbr. WET$_R$) in the sense that for any ergodic, probability preserving transformation $(\Omega, \mathcal{A}, P, S)$,

\[
(WET_R) \quad \frac{1}{a_u(n)} \sum_{k=0}^{n-1} u_k f \circ S^k \rightarrow E(f) \text{ weakly in } L^2(P) \ \forall \ f \in L^2(P).
\]

The recurrent, renewal sequences form an important subclass of weights. A weight $u \in \mathcal{W}$ is a recurrent renewal sequence if $u_0 = 1$ and $\exists \ f \in \mathcal{P}(\mathbb{N})$, called the associated lifetime distribution satisfying the renewal equation

\[ u_n = \sum_{k=1}^{n} f_k u_{n-k} \ (n \geq 1). \]

The renewal sequence is $u$ is called aperiodic if $\{n \in \mathbb{N} : u_n > 0\} = \mathbb{Z}$.

It follows from the renewal equation that any aperiodic, recurrent renewal sequence satisfies $|\sum_{k=0}^{n} u_k z^k| < \infty$ for $z \in \mathbb{C}, \ |z| = 1, \ z \neq 1$ and hence is MET. Proposition 6.2 (below) generalizes this.

Let $\mathcal{R} \subset \mathbb{N}$ be a subsequence. Any $\mathcal{R}$-smooth weight $u \in \mathcal{W}$ is MET$_R$ (see [HP], [Kre] and references therein). A weight $u \in \mathcal{W}$ which is MET and not $\mathcal{R}$-smooth for any subsequence $\mathcal{R} \subset \mathbb{N}$ is exhibited in [HP].

We'll need

**Lemma 6.1**

Let $\mathcal{R} \subset \mathbb{N}$ be a subsequence and suppose that $u \in \mathcal{W}$ is MET$_R$, and that $(\Omega, \mathcal{A}, P, S)$ is a weakly mixing probability preserving transformation, then

\[ P(A \cap S^{-n}B)^{(u,R)-d.} \rightarrow_{n \rightarrow \infty} P(A)P(B) \ \forall \ A, B \in \mathcal{A}. \]

**Proof**

It follows from $(WET_R)$ for $S$ and $A, B \in \mathcal{A}$, that

\[ \frac{1}{a_u(n)} \sum_{k=0}^{n-1} u_k P(A \cap S^{-k}B) \rightarrow_{n \rightarrow \infty, n \in \mathcal{R}} P(A)P(B), \]
and it follows from (X) for $S \times S$ (which is ergodic) and $A \times A, B \times B \in \mathcal{A} \otimes \mathcal{A}$ that

\[(\mathcal{Z}) \quad \frac{1}{a_u(n)} \sum_{k=0}^{n-1} u_k P(A \cap S^{-k} B)^2 = \frac{1}{a_u(n)} \sum_{k=0}^{n-1} u_k P \times P(A \times A \cap (S \times S)^{-k} B \times B) \rightarrow_{n \to \infty, n \in \mathbb{N}} P \times P(A \times A) \cdot P \times P(B \times B) = P(A)^2 P(B)^2.
\]

Using (X) and (Z)

\[\frac{1}{a_u(n)} \sum_{k=0}^{n-1} u_k (P(A \cap S^{-k} B) - P(A) P(B))^2 = \]

\[\frac{1}{a_u(n)} \sum_{k=0}^{n-1} u_k \left(P(A \cap S^{-k} B)^2 - 2P(A)P(B)P(A \cap S^{-k} B) + P(A)^2 P(B)^2\right) \rightarrow_{n \to \infty, n \in \mathbb{N}} 0\]

whence $P(A \cap S^{-n} B) \rightarrow_{n \to \infty, n \in \mathbb{N}}^d P(A) P(B)$. \(\Box\)

**Proposition 6.2**

Let $\mathfrak{R} \subset \mathbb{N}$ be a subsequence. Suppose that $(X, \mathcal{B}, m, T)$ is weakly rationally ergodic along $\mathfrak{R}$ and spectrally weakly mixing, then $u(E, F)$ is $\text{MET}_\mathfrak{R} \forall E, F \in R_\mathfrak{R}(T)_+$. 

**Proof**

Let $(\Omega, \mathcal{A}, P, S)$ be an ergodic, probability preserving transformation.

It follows from the assumptions that $T \times S$ is weakly rationally ergodic along $\mathfrak{R}$ and $R_\mathfrak{R}(T \times S) \ni R_\mathfrak{R}(T) \times \Omega$.

It suffices to show that for $E, F \in R_\mathfrak{R}(T)_+$,

$(\text{WET}_\mathfrak{R}) \quad A_n f \rightarrow_{n \to \infty, n \in \mathbb{N}} E(f) \text{ weakly in } L^2(P) \forall f \in L^2(P)$

where $A_n f := \frac{1}{a_u(E, F)} \sum_{k=0}^{n-1} u_k (E, F) f \circ S^k$.

Since $E \times \Omega, F \times \Omega \in R_\mathfrak{R}(T \times S)$,

\[\frac{1}{a_u} \sum_{k=0}^{n-1} u_k E, F) P(C \cap S^{-k} D) \rightarrow_{n \to \infty} P(C) P(D) \forall C, D \in \mathcal{B}(\Omega).
\]

This shows $(\text{WET}_\mathfrak{R})$ for indicators, whence for simple functions $f$. By the triangle inequality $\|A_n f\|_2 \leq \|f\|_2 \forall f \in L^2(P)$ and $(\text{WET}_\mathfrak{R})$ follows by approximation. \(\Box\)
§7 Proof of theorem D

We assume that $T$ is invertible. By Corollary E, this involves no loss in generality.

**Proof of theorem D(i):**
We’ll prove spectral weak mixing of $T$ by showing that $T \times S$ is weakly rationally ergodic along $K$ for any ergodic, probability preserving transformation $(\Omega, \mathcal{A}, P, S)$. To this end, let $(\Omega, \mathcal{A}, P, S)$ be an ergodic, probability preserving transformation. We claim first that

$$
\frac{1}{a_n(T)} \sum_{k=0}^{n-1} m(A \cap T^{-k} B) P(C \cap S^{-k} D)
\xrightarrow{n \in \mathbb{R}, n \to \infty} m(A) m(B) P(C) P(D) \quad \forall \ A, B \in R_\mathbb{R}(T), \ C, D \in \mathcal{A}.
$$

**Proof of (Ⅲ):**
Fix $A, B \in R_\mathbb{R}(T)$, and set $v = u(A, B)$. By proposition 0(iii), $v$ is smooth whence $\text{MET}_\mathbb{R}$; and (Ⅲ) follows from (WET$^\mathbb{R}$) for $C, D \in \mathcal{A}$. $\Box$

Let $\mu := m \times P$, $\mathcal{C} := \mathcal{B} \otimes \mathcal{A}$ and $\tau := T \times S$.

We claim next that for $F \in R_\mathbb{R}(T)$, $A \in \mathcal{B} \cap F$, $B \in \mathcal{A}$, $C \in \mathcal{C} \cap (F \times \Omega)$,

$$
\frac{1}{a_n} \sum_{k=0}^{n-1} \mu(C \cap \tau^{-k}(A \times B)) \xrightarrow{n \to \infty, n \in \mathbb{R}} \mu(C) \mu(A \times B).
$$

**Proof of (Ⅳ):**
By weak rational ergodicity along $\mathbb{R}$ the collection \{\frac{1}{a_n} \sum_{k=0}^{n-1} 1_{A \times B} \circ \tau^k : n \in \mathbb{R}\} is uniformly integrable on $F$. It follows that the collection \{\frac{1}{a_n} \sum_{k=0}^{n-1} 1_{A \times B} \circ \tau^k : n \in \mathbb{R}\} is uniformly integrable on $F \times \Omega$.

Let $\Phi \in \{\frac{1}{a_n} \sum_{k=0}^{n-1} 1_{A \times B} \circ \tau^k : n \in \mathbb{R}\}'$ be a weak limit, then by (Ⅲ),

$$
\int_{C \times D} \Phi d\mu = \mu(A \times B) \mu(C \times D) \quad \forall \ C \in \mathcal{B} \cap F, \ D \in \mathcal{A}.
$$

It follows that $\Phi \equiv \mu(A \times B)$ whence $\frac{1}{a_n} \sum_{k=0}^{n-1} 1_{A \times B} \circ \tau^k \xrightarrow{n \to \infty, n \in \mathbb{R}} \mu(A \times B)$ weakly in $L^1(F \times \Omega)$ and (Ⅳ) follows. $\Box$

Finally we complete the proof of theorem D(i) by showing that

$$
F \times \Omega \in R_\mathbb{R}(\tau);
$$
namely, for $F \in R_{R}(T), \ C, D \in C \cap (F \times \Omega)$,

$$\left(\mathcal{A}\right) \quad \frac{1}{a_{n}(F)} \sum_{k=0}^{n-1} \mu(C \cap \tau^{-k} D) \xrightarrow[n \to \infty, n \in \mathbb{N}]{} \mu(C) \mu(D).$$

**Proof of (\mathcal{A}):**

Since $D \subset F \times \Omega$, the collection $\{\frac{1}{a_{n}} \sum_{k=0}^{n-1} 1_{D} \circ \tau^{k} : n \in \mathbb{N}\}$ is uniformly integrable on $F \times \Omega$. Let $\Psi \in \{\frac{1}{a_{n}} \sum_{k=0}^{n-1} 1_{D} \circ \tau^{k} : n \in \mathbb{N}\}'$ be a weak limit, then by (\mathcal{B}) for $\tau^{-1}$,

$$\int_{A \times B} \Phi d\mu = \mu(A \times B) \mu(D) \quad \forall \ A \in B \cap F, \ B \in A$$

whence $\frac{1}{a_{n}} \sum_{k=0}^{n-1} 1_{D} \circ \tau^{k} \xrightarrow[n \to \infty, n \in \mathbb{N}]{} \mu(D)$ weakly in $L^{1}(F \times \Omega)$ and (\mathcal{A}) follows. \(\Box\)

**Remark.** Spectral weak mixing alone does not imply subsequence rational weak mixing. See [ALV] for *squashable*, spectrally weakly mixing, transformations. These are not even subsequence weakly rationally ergodic. We do not know whether weak rational ergodicity and spectral weak mixing together imply subsequence rational weak mixing.

**Proof of theorem D(ii):**

Fix a countable, $\mathcal{B}$-generating partition $\alpha \subset R_{R}(T)$. By standardness, up to isomorphism, $X = \alpha^{\mathbb{Z}}$ and $T : X \to X$ is the shift. The collection $C_{\alpha}$ of $(\alpha, T)$-cylinder sets forms a base of clopen sets for the $T$-invariant, measurable, Polish topology on $X$.

Let $(\Omega, \mathcal{A}, P, S)$ be a weakly mixing, probability preserving transformation. Fix a compact $S$-invariant, completely disconnected, measurable topology on $\Omega$ generating $\mathcal{A}$.

We must show that the measure preserving transformation

$$(Z, \mathcal{C}, \mu, \tau) := (X \times \Omega, \mathcal{B} \otimes \mathcal{A}, m \times P, T \times S)$$

is rationally weakly mixing along $R_{R}$.

For this, it suffices to show that for $F \in \alpha$, $m(F) > 0$, $F \times \Omega$ satisfies (\mathcal{E}) with respect to $\tau$.

By Lemma B, it suffices to establish

$$\left(\mathcal{E}\right) \quad \frac{m(A \cap T^{-n} B) P(C \cap S^{-n} D)}{u_{n}(F)} \xrightarrow[n \to \infty]{} m(A)m(B)P(C)P(D) \quad \forall \ A, B \in \mathcal{B} \cap F, \ C, D \in \mathcal{A}.$$

**Proof of (\mathcal{E}):**
By proposition 0(iii), $u(F)$ is $\mathcal{R}$-smooth, whence $\text{MET}_\mathcal{R}$ and by lemma 6.1
\[ P(C \cap S^{-n}D) \xrightarrow{n \to \infty} P(C)P(D) \quad \forall \ C, D \in \mathcal{A}. \]
Thus, since $F$ satisfies ($\star_\mathcal{R}$), for $A, B \in \mathcal{B} \cap F$
\[ \frac{m(A \cap T^{-n}B)}{u_n(F)} \xrightarrow{n \to \infty} m(A)m(B). \]
These two $(u(F), \mathcal{R})$-density convergences imply ($\mathcal{G}$), and (via lemma B) theorem D(ii).

\[ \Box \]

§8 Markov shift examples

Let $S$ be a countable set (the state space) and let $P : S \times S \to [0, 1]$ be a stochastic matrix (the transition matrix) on $S$ ($\sum_{t \in S} p_{s,t} = 1 \ \forall s \in S$) with an invariant distribution $\pi : S \to \mathbb{R}_+$ ($\sum_{s \in S} \pi_s p_{s,t} = \pi_t$).

The stationary, two-sided Markov shift of $(P, \pi)$ is the quadruple $(S^\mathbb{Z}, \mathcal{B}, m, T)$, where
\[ T : S^\mathbb{Z} \to S^\mathbb{Z} \]
the measure preserving transformation.

As shown in [HR], $T$ is
\begin{itemize}
  \item conservative iff $P$ is recurrent ($\sum_{n=0}^{\infty} p_{s,s}^{(n)} = \infty \ \forall \ s \in S$) and in this case, $T$ is
  \item ergodic iff $P$ is irreducible ($\forall s, t \in S, \ \exists n \in \mathbb{N} \ \exists p_{s,t}^{(n)} > 0$).
\end{itemize}

The (stationary) one-sided, Markov shift is $(S^n, \mathcal{B}_+, m_+, \tau)$, where $\tau : S^n \to S^n$ is the shift,
\[ \mathcal{B}_+ := \sigma(\{\text{one-sided cylinders}\}), \]
a one-sided cylinder being a set of form
\[ [s_1, \ldots, s_n] := \{x = (x_1, x_2, \ldots) \in S^n : x_j = s_j \ \forall \ 1 \leq j \leq n\} \]
\((s_1, \ldots, s_n \in S^n, k \in \mathbb{Z}, n \in \mathbb{N})\); and the measure \(m_+\) is defined by

\[ m_+([s_1, \ldots, s_n]) = \pi_{s_1}p_{s_1,s_2} \cdots p_{s_{n-1},s_n} \forall s_1, \ldots, s_n \in S^n, n \in \mathbb{N}. \]

As shown in [BF], if the stochastic matrix \(P\) is irreducible, recurrent and \textit{aperiodic} \((\gcd \{ n \geq 1 : p^{(n)}_{s,s} > 0 \} = 1 \text{ for some and hence all } s \in S)\), then \(T\) is a conservative K-automorphism (natural extension of an exact endomorphism), whence (see [ALW]) spectrally weakly mixing.

As shown in [A1], a conservative, ergodic Markov shift \((S^\mathbb{N}, \mathcal{B}, m, T)\) is rationally ergodic with \(R_K(T) \supset H\mathcal{R}(C, \alpha)\) where \(\alpha : \{ [s]_0 : s \in S \} \) with \(a_n(T) = a_n(P) \sim \frac{1}{\pi_s} \sum_{k=0}^{n-1} p_s^{(k)} (\forall s \in S)\).

\textbf{Theorem 8.1}

Let \(K \subset \mathbb{N}\) be a subsequence. The Markov shift \((S^\mathbb{Z}, \mathcal{B}, m, T)\) of the irreducible, recurrent, aperiodic transition matrix \(P\) on state space \(S\) is rationally weakly mixing along \(K\) iff \(\exists s \in S\) with \(u([s]_0)\) is \(K\)-smooth.

\textbf{Proof} By proposition 0(iii), if \(T\) is rationally weakly mixing along \(K\), then \(u([s]_0)\) is \(K\)-smooth \(\forall s \in S\).

To prove the other implication, we'll need the following lemma:

\textbf{Lemma 8.2}

Let \(S\) be a countable set and let \(P : S \times S \to [0, 1]\) be an irreducible, recurrent, aperiodic stochastic matrix with the property that for some \(s \in S, u = u([s]_0)\) is \(K\)-smooth, then

\[
\left( K \right) \quad \frac{P^{(n+\ell)}_{r,t}}{u_n} \xrightarrow{(u, K) - \text{a.c.}} \frac{1}{\pi_t} \quad \forall r, t \in S, \ell \in \mathbb{Z}.
\]

Lemma 8.2 is a \((u, K)\)-density version of lemma 1 in [O].

\textbf{Proof of lemma 8.2:}

Recall from [Ch] that the \(P\)-stationary distribution \(\pi : S \to \mathbb{R}_+\) with \(\pi_s = 1\) is given by

\[
\pi_t = \sum_{n=1}^{\infty} s^{(n)}_{P, s,t}
\]

where

\[
s^{(1)}_{P, s,t} := p_{s,t}, \quad s^{(n+1)}_{P, s,t} := \sum_{r \in S \setminus \{ s \}} s^{(n)}_{P, s,t} p_{r,t}.
\]
As shown in [Ch], \( \forall \ r, \ t \in S \),
\[
\frac{1}{a_u(n)} \sum_{k=0}^{n-1} p_r^{(k)} (n) \to_{n \to \infty} \pi_t.
\]
In view of this, to show (\( \mathfrak{R} \)), it suffices by proposition 3.3(i) to show that \( \forall \ r, \ t \in S, \ \ell \in \mathbb{Z} \exists K_{r,t,\ell} \subset \mathbb{N}, \ (u, \mathfrak{R}) \)-small such that
\[
(\mathfrak{R}^{'}) \quad \lim_{n \to \infty, \ n \in K_{r,t,\ell}} \frac{p_r^{(n+\ell)} (n)}{u_n} \geq \pi_t.
\]
Let \( K_0 \subset \mathbb{N} \) be \((u, \mathfrak{R})\)-small such that \( \frac{a_u(n)}{u_n} \to_{n \to \infty, \ n \notin K_0} 1 \ \forall \ k \in \mathbb{Z} \).

To see (\( \mathfrak{R}^{'}) \) for \( r = s \& t \in S, \ \ell \in \mathbb{Z} \),
\[
p_{s,t}^{(n+\ell)} = \sum_{k=0}^{n+\ell-1} u_{n+\ell-k} s p_{s,t}^{(k)} \geq \sum_{k=0}^{N-1} u_{n+\ell-k} s p_{s,t}^{(k)} \ \forall \ n + \ell > N \geq 1.
\]
Thus
\[
\frac{p_{s,t}^{(n+\ell)}}{u_n} \geq \sum_{k=0}^{N-1} s p_{s,t}^{(k)} \frac{u_{n+\ell-k}}{u_n} \to_{n \to \infty, \ n \notin K_0} \sum_{k=0}^{N-1} s p_{s,t}^{(k)}
\]
and (\( \mathfrak{R}^{'}) \) holds with \( K_{s,t} = K_0 \). As mentioned above we now have (\( \mathfrak{R} \)) with \( r = s \).

To see (\( \mathfrak{R}^{'}) \) for general \( r, \ t \in S \), fix first \( K_t \ (u, \mathfrak{R})\)-small such that
\[
p_{s,t}^{(n+k)} \to_{n \to \infty, \ n \notin K_t} \pi_t \ \forall \ k \in \mathbb{Z}.
\]
Next,
\[
p_{r,t}^{(n+\ell)} = \sum_{k=0}^{n-1} s p_{r,s}^{(k)} p_{s,t}^{(n+\ell-k)} \geq \sum_{k=0}^{N-1} s p_{r,s}^{(k)} p_{s,t}^{(n+\ell-k)} \ \forall \ n + \ell > N \geq 1,
\]
and
\[
\frac{p_{r,t}^{(n+\ell)}}{u_n} \geq \sum_{k=0}^{N-1} s p_{r,s}^{(k)} \frac{p_{s,t}^{(n+\ell-k)}}{u_n} \to_{n \to \infty, \ n \notin K_t} \sum_{k=0}^{N-1} s p_{r,s}^{(k)} \cdot \pi_t \to_{N \to \infty} \pi_t
\]
\((\because \ \sum_{k=0}^{\infty} s p_{r,s}^{(k)} = 1) \) and (\( \mathfrak{R}^{'}) \) holds with \( K_{r,t} = K_t \). \( \Box \) (\( \mathfrak{R} \)).

Proof of theorem 8.1:
Suppose that \( A = [a_1, \ldots, a_I]_k \& B = [b_1, \ldots, b_J]_\ell \in C_\alpha \), then for \( n \in \mathbb{Z} \),
\[A \cap T^{-n} B = \{ x \in S^Z : x_{k+i} = a_i \ \forall \ 1 \leq i \leq I \& x_{n+\ell+j} = b_j \ \forall \ 1 \leq j \leq J \}\]
and for \( n > k + I - \ell \),
\[ m(A \cap T^{-n} B) = \pi_{a_1} p_{a_1, a_2} \cdots p_{a_{i-1}, a_i} p_{a_i, b_1}^{(n+\ell-I)} p_{b_1, b_2} \cdots p_{b_j-1, b_j} \]
\[ = m(A) m(B) \cdot \frac{p_{a_i, b_1}^{(n+\ell-I)}}{\pi_{b_1}} \]
whence
\[ m(A \cap T^{-n} B) = \frac{m(A)m(B)}{\pi_{b_1}} \cdot \frac{p_{a_i, b_1}^{(n+\ell-I)}}{u_n} \xrightarrow{n \to \infty} m(A)m(B). \]
Rational weak mixing follows from lemma C. \(\Box\)

**Smoothness of renewal sequences.**

**Remark 8.1.**

If \(u\) is a recurrent, aperiodic renewal sequence, whose associated lifetime distribution \(f \in \mathcal{P}(\mathbb{N})\) has tails \(f([n, \infty))\) which are \((-\gamma)\)-regularly varying with \(\gamma \in (0, 1]\), then

\[ \frac{nu_n}{a_n} \begin{cases} \to 1 & \gamma = 1 \text{ by [E]}; \\ \to \gamma & \frac{1}{2} < \gamma < 1 \text{ by [GL]}; \\ \text{density} & \gamma \leq \frac{1}{2} \text{ by [GL]}. \end{cases} \]

By proposition 3.1, the convergence in the third case (which follows from Lemma 9.2 below) is also in \(u\)-density. In all cases, \(u\) is smooth and any corresponding Markov chain is rationally weakly mixing by theorem 8.1.

**Proposition 8.3**

Suppose that \(u = (u_0, u_1, \ldots)\) is an aperiodic, recurrent, renewal sequence with lifetime distribution \(f \in \mathcal{P}(\mathbb{N})\). Let \(L(n) := \sum_{k=1}^{n} f([k, \infty))\) and \(V(t) := \sum_{1 \leq n \leq t} n^2 f_n\).

(i) If for some \(N \geq 1\), \(\sum_{n=N}^{\infty} \frac{1}{V(n)^2} < \infty\), then \(\sum_{n=1}^{\infty} (u_n - u_{n+1})^2 < \infty\).

(ii) If, in addition, \(\frac{L(n)}{\sqrt{n}} \xrightarrow{n \to \infty} 0\), then \(u\) is smooth.

**Proof** of (i): By Parseval’s formula, and the renewal equation,

\[ \int_{-\pi}^{\pi} \frac{|\theta|^2 d\theta}{|1 - f(\theta)|^2} < \infty \iff \sum_{n=1}^{\infty} (u_n - u_{n+1})^2 < \infty \]
where \( f(\theta) := \sum_{n=1}^{\infty} f_n e^{in\theta} \). By aperiodicity, \( \sup_{\epsilon \leq |\theta| \leq \pi} |f(\theta)| < 1 \) \( \forall \epsilon > 0 \) whence (using symmetry)

\[
\int_{-\pi}^{\pi} \frac{|\theta|^2 d\theta}{|1 - f(\theta)|^2} < \infty \iff \int_{0}^{\epsilon} \frac{\theta^2 d\theta}{|1 - f(\theta)|^2} < \infty \text{ for some } \epsilon > 0.
\]

Next,

\[
|1 - f(\theta)| \geq \Re(1 - f(\theta)) = 2 \sum_{n=1}^{\infty} f_n \sin^2 \left(\frac{n\theta}{2}\right)
\]

\[
\geq 2 \sum_{1 \leq n \leq \frac{\pi}{\theta}} f_n \sin^2 \left(\frac{n\theta}{2}\right) \geq \frac{2\theta^2}{\pi^2} \sum_{1 \leq n \leq \frac{\pi}{\theta}} n^2 f_n =: C\theta^2 V\left(\frac{\pi}{\theta}\right).
\]

For large \( N \), \( V(N) > 0 \) and

\[
\int_{0}^{\frac{\pi}{\theta}} \frac{\theta^2 d\theta}{|1 - f(\theta)|^2} \leq \sum_{n=N}^{\infty} \int_{\frac{\pi}{n+1}}^{\frac{\pi}{n}} \frac{d\theta}{(C\theta^2 V(\frac{\pi}{\theta}))^2} \leq C' \sum_{n=N}^{\infty} \frac{1}{V(n)^2}.
\]

\( \Box (i) \)

**Proof of (ii):**

It follows from the renewal equation (see lemma 3.8.5 of [A]) that \( a_{u}(n) \asymp \frac{1}{L(n)} \) whence

\[
\frac{\sqrt{n}}{a_{u}(n)} \asymp \sqrt{n} \cdot \frac{L(n)}{\sqrt{n}} = \frac{L(n)}{\sqrt{n}} \xrightarrow{n \to \infty} 0
\]

whence, by (i)

\[
\frac{1}{a_{u}(n)} \sum_{k=1}^{n} |u_k - u_{k+1}| \leq \frac{\sqrt{n}}{a_{u}(n)} \sqrt{\sum_{n \geq 1} |u_n - u_{n+1}|^2} \xrightarrow{n \to \infty} 0. \quad \Box (ii)
\]

For example, let \( f \in \mathcal{P}(\mathbb{N}) \) be the winnings distribution in the St Petersburg game:

\[
f_k = \begin{cases} 
\frac{1}{2^k}, & k = 2^n \ (n \geq 0), \\
0, & \text{else}.
\end{cases}
\]

The associated aperiodic, recurrent renewal sequence is smooth by proposition 8.3 (remark 8.1 above does not apply).

The following is “extends” Dyson’s example (on p. 55 of [Ch]) of an aperiodic renewal sequence without the strong ratio limit property:

**Proposition 8.4**

There is a subsequence smooth, recurrent, renewal sequence which does not have the strong ratio limit property.
Proof. We consider $P := \{ f \in P(\mathbb{N}) : f_1 > 0 \}$ metrized by
$$d(f,g) := \frac{1}{f_1} + \frac{1}{g_1} + \sum_{n \geq 1} |f_n - g_n|.$$ This space is Polish (complete and separable).

For $f \in P$, let $u^{(f)}$ be the associated (aperiodic, recurrent) renewal sequence. Let
$$P_{SRLP} := \{ f \in P : u^{(f)} \text{ has the strong ratio limit property} \}$$
and
$$P_{ss} := \{ f \in P : \exists K \subset \mathbb{N}, u^{(f)} \text{ is } K\text{-smooth} \}.$$ We show that
$$P_{ss} \setminus P_{SRLP}$$
is residual in $P$ (and therefore not empty).

By Baire’s theorem, it suffices to show residuality of $P_{ss}$ and $P \setminus P_{SRLP}$.

Proof that $P_{ss}$ is residual. For each $n \geq 1$, the function $f \mapsto u_n^{(f)}$ is continuous ($P \to \mathbb{R}$), being a polynomial function of $(f_1, f_2, \ldots, f_n)$. Thus
$$P_{ss} = \bigcap_{k=1}^{\infty} \bigcup_{N=k}^{\infty} \{ f \in P : \sum_{j=1}^{N} |u_j^{(f)} - u_j^{(f)}| < \frac{1}{k} \sum_{j=1}^{N} u_j^{(f)} \}$$
is a $G_\delta$ set. By the renewal theorem $P_{ss} \supset P_+ := \{ f \in P : \sum_{n \geq 1} n f_n < \infty \}$ which is dense in $P$. \qed

Proof that $P \setminus P_{SRLP}$ is residual. For each $k \in \mathbb{N}$,
$$\Pi_k := \{ f \in P : \exists N > k \text{ such that } u_N^{(f)} < \frac{u_N^{(f)}}{k} \}$$
is open.

Since $P \setminus P_{SRLP} \supseteq \bigcap_{k \geq 1} \Pi_k$, it suffices to prove that each $\Pi_k$ is dense. To this end, fix $k \geq 1$, $f \in P \& \epsilon > 0$. We’ll show that $\exists g \in \Pi_k, \; d(f,g) < 2\epsilon$. To this end note first that $\exists h \in P$ so that $d(f,h) < \epsilon$ and so that the set $\{ n \in \mathbb{N} : h_n > 0 \}$ is infinite. Using this, find $\ell > k$ so that
$$0 < 1 - H := \sum_{j=\ell+1}^{\infty} h_j < \epsilon.$$ For $L > \ell$ define $g^{(L)} \in P$ by
$$g_n^{(L)} := \begin{cases} h_n & n \leq \ell; \\ \sum_{j=\ell+1}^{\infty} h_j & n = L; \\ 0 & \text{else.} \end{cases}$$ We claim that $\forall L$ large, $u_L^{(g^{(L)})} < \frac{1 - H}{k} < \frac{u_L^{(g^{(L)})}}{k},$ whence $g^{(L)} \in \Pi_k.$
To see this define the defective renewal sequence \( v \) by

\[
v_0 = 1, \quad v_n := \sum_{k=1}^{n \wedge \ell} h_k v_{n-k},
\]

then \( u^{(g(L))} = v_j \quad \forall \ 1 \leq j < L. \)

Let \( V_r := \max_{\nu \geq r \ell+1} v_{\nu}. \) For \( j \geq 1, \)

\[
v_{r \ell+j} = \sum_{i=1}^{\ell} h_i v_{r \ell+j-i} = \sum_{i=1}^{\ell} h_i v_{(r-1)\ell+j-i} \leq HV_{r-1}
\]

whence \( V_r \leq HV_{r-1} \) and \( v_{r \ell} \leq H^{r-1} v_{\ell}. \) Now fix \( L > \ell \) so that \( v_{L-1} < \frac{1-H}{k}. \)

It follows as above that \( g(L) \in \Pi_k \ & d(f,g(L)) < 2\epsilon. \) \( \Box \)

§9 Examples with Local limit sets

In this section, we prove a generalization of part of theorem 1.1 in [GL] thereby establishing sufficient conditions for rational weak mixing. It is necessary to deal with essentially non-invertible transformations. By corollary E, rational weak mixing passes to the natural extensions of these non-invertible transformations.

Suppose that \((X, \mathcal{B}, m, T)\) is a pointwise dual ergodic, measure preserving transformation (as in [A]) with \(\gamma\)-regularly varying return sequence \(a(n) = a_n(T) \ (0 < \gamma < 1)\). As shown in [A] (chapter 3), \(T\) is rationally ergodic, and \(T\) is not invertible.

By the Darling Kac theorem ([DK], see also chapter 3 in [A])

\[
\frac{1}{a(n)}S_n(f) \xrightarrow{\mathcal{D}} X_{\gamma} m(f) \quad \forall \ f \in L^1_+
\]

on \((X, \mathcal{B}, m)\) where \(X_{\gamma}\) is the Mittag-Leffler distribution of order \(\gamma\) normalized so that \(E(X_{\gamma}) = 1, \ m(f) := \int_X f \ dm\) and \(\xrightarrow{\mathcal{D}}\) on \((X, \mathcal{B}, m)\) denotes convergence in distribution with respect to all \(m\)-absolutely continuous probabilities.

Let \(\Omega \in \mathcal{B}, \ m(\Omega) = 1\) (the normalization \(m(\Omega) = 1\) is not necessary, but convenient).

The return time function to \(\Omega\) is \(\varphi = \varphi_{\Omega} : \Omega \to \mathbb{N}\) defined by \(\varphi(\omega) := \min\{n \geq 1 : T^n \omega \in \Omega\} < \infty\ \text{a.s. by conservativity. The induced transformation on} \ \Omega\ \text{is} T_{\Omega} : \Omega \to \Omega \ \text{defined by} T_{\Omega}(\omega) := T^{\varphi(\omega)}(\omega). \) As is well known, \(T_{\Omega}\) is an ergodic, probability preserving transformation of \((\Omega, \mathcal{B}(\Omega), m_{\Omega})\).

The return time process on \(\Omega\) satisfies the stable limit theorem. Indeed, by proposition 1 in [A2],

\[
\frac{1}{B(n)} \varphi_n \xrightarrow{\mathcal{D}} Z_{\gamma}
\]
on \((\Omega, B(\Omega), m_\Omega)\) where \(B(n) := a^{-1}(n), \ Z_\gamma = Y_\gamma^{-1/\gamma}\) is the stable random variable of order \(\gamma\) and \(\varphi_n := \sum_{k=0}^{n-1} \varphi \circ T_{\Omega}^k\).

The above is true for any \(\Omega \in B, \ m(\Omega) = 1\). By “choosing” \(\Omega\) carefully, it may be possible to obtain stronger properties.

Accordingly, in the above situation, we call \(\Omega \in R(\gamma)\) a local limit set \((\text{LLT})\) if \(\exists\) a countable, partition \(\beta \subset B(\Omega)\) generating \(B(\Omega)\) under \(T_\Omega\) such that \(\varphi^{-1}_\Omega\{n\} \in \sigma(\beta) \ \forall \ n \geq 1\) and such that \(\forall \ A, B \in C(\beta(T_\Omega))\),

\[
B(n) m(A \cap T_\Omega^{-n} B \cap [\varphi_n = k_n]) \xrightarrow{n \to \infty} f_{Z_\gamma}(x) m(A) m(B).
\]

uniformly in \(x \in [c, d]\) whenever \(0 < c < d < \infty\), where \(f = f_{Z_\gamma}\) is the probability density function of \(Z_\gamma\).

To be a LLT set, essentially, the return time stochastic process to \(\Omega\) needs to satisfy the conditional, \(\gamma\)-stable, local limit theorem.

**Examples 9.1.**

If \((X, B, m, T)\) is the tower over the a Gibbs Markov fibred system (as in [AD]), or an AFU fibred system (as in [ADSZ]) \((\Omega, A, P, S, \alpha)\) with \(\alpha\)-measurable height function \(\varphi\) satisfying \(E(\varphi \wedge t)\) regularly varying at infinity with index in \((0, 1)\), then \((X, B, m, T)\) is pointwise dual ergodic, \(\Omega \in R(T)\) with \(a_n(T) = a_n(\Omega) \propto \frac{n}{E(\varphi \wedge n)}\) and the return time stochastic process to \(\Omega\) satisfies the conditional, \(\gamma\)-stable, local limit theorem. See [AD] and [ADSZ] respectively. Thus, \(\Omega\) is a LLT set.

**Theorem 9.1**

*Suppose that \((X, B, m, T)\) is pointwise dual ergodic with \(a(n) = a_n(T)\) \(\gamma\)-regularly varying \((\gamma \in (0, 1))\) and which has a LLT set, then \((X, B, m, T)\) is rationally weakly mixing.*

*Proof*

Let \(\Omega \in R(T)\) be a LLT set with accompanying \(T_\Omega\)-generating partition \(\beta\). By standardness, up to isomorphism, \(\Omega = \alpha^N, \ T_\Omega : \Omega \to \Omega\) is the shift and the collection \(C_\beta(T_\Omega)\) of \((\beta, T_\Omega)\)-cylinder sets forms a base of clopen sets for the Polish topology on \(\Omega\). The proof is via lemma C, whose use is enabled by the following lemma 9.2, which is a version of the “local limit” proof of theorem 1.1 of [GL]. Analogous results are established in [MT].

**Lemma 9.2**
Suppose that \((X,B,m,T)\) is pointwise dual ergodic with return sequence \(a(n) = a_n(T)\) which is \(\gamma\)-regularly varying (\(\gamma \in (0,1)\)) and which has a LLT set \(\Omega \in R(T)\), \(m(\Omega) = 1\), then

\[
(\text{GL}) \quad \lim_{n \to \infty} \frac{m(A \cap T^{-n}B)}{u_n} \geq m(A)m(B) \quad \forall \ A, \ B \in C_\beta(T_\Omega)
\]

where \(u_n := \frac{\gamma a(n)}{n}\) and \(\beta\) is the accompanying \(T_\Omega\)-generating partition.

Proof (as in [GL]):

Fix \(A, B \in C_\beta(T_\Omega)\) and \(0 < c < d < \infty\). Writing \(x_{k,n} := \frac{n}{B(k)}\) for \(1 \leq k \leq n\) and using the LLT property of \(\Omega\), we have,

\[
m(A \cap T^{-n}B) = \sum_{k=1}^{n} m(A \cap T^{-k}_\Omega B \cap [\varphi_k = n]) \\
\geq \sum_{1 \leq k \leq n, x_{k,n} \in [c,d]} m(A \cap T^{-k}_\Omega B \cap [\varphi_k = x_{k,n} B(k)]) \\
\sim \sum_{1 \leq k \leq n, x_{k,n} \in [c,d]} \frac{f(x_{k,n})}{B(k)} m(A)m(B)
\]
as \(n \to \infty\) since \(\Omega\) is a LLT set. We are going to show that the last sum is in fact a Riemann sum.

Now,

\[
x_{k,n} - x_{k+1,n} = \frac{n}{B(k)} - \frac{n}{B(k+1)} \sim \frac{n}{B(k)} k
\]
as \(k, n \to \infty\), \(x_{k,n} \in [c,d]\) since \(B = a^{-1}\) is \(\frac{1}{\gamma}\)-regularly varying.

Also

\[
a(n) = a(x_{k,n} B(k)) \sim x_{k,n,}^\gamma a(B(k)) \sim x_{k,n}^\gamma k
\]
as \(k, n \to \infty\), \(x_{k,n} \in [c,d]\) by the uniform convergence theorem for regularly varying functions. so

\[
\frac{1}{B(k)} \sim \frac{\gamma k}{n}, (x_{k,n} - x_{k+1,n}) \sim \frac{\gamma a(n)}{n}, \frac{x_{k,n} - x_{k+1,n}}{x_{k,n}^\gamma}
\]
whence, as \(n \to \infty\),

\[
\sum_{1 \leq k \leq n, x_{k,n} \in [c,d]} \frac{f(x_{k,n})}{B(k)} \sim \frac{\gamma a(n)}{n} \sum_{1 \leq k \leq n, x_{k,n} \in [c,d]} \frac{(x_{k,n} - x_{k+1,n})}{x_{k,n}^\gamma} f(x_{k,n}) \\
\sim \frac{\gamma a(n)}{n} \int_{[c,d]} \frac{f(x)}{x^\gamma} dx = \frac{\gamma a(n)}{n} \mathbb{E}(1_{[c,d]}(Z_\gamma) Z_\gamma^\gamma).
\]

Now

\[
\mathbb{E}(1_{[c,d]}(Z_\gamma) Z_\gamma^\gamma) = \mathbb{E}(1_{[c,d]}(X_\gamma^{-1/\gamma}) X_\gamma) \xrightarrow{c \to 0, d \to \infty} \mathbb{E}(X_\gamma) = 1,
\]

\[
\therefore m(A \cap T^{-n}B) \geq \frac{\gamma a(n)}{n} m(A)m(B). \quad \square
\]
Theorem 9.1 now follows from lemma C.

Remark 9.1.
In some cases, \(\lim\) in lemma 9.2 is actually \(\lim\) and the transformation has Krickeberg’s mixing property. This occurs in:

(i) the Markov case when \(\gamma \in \left(\frac{1}{2}, 1\right]\) (in remark 8.1), see [GL] for \(\gamma \in \left(\frac{1}{2}, 1\right)\) and [E] for \(\gamma = 1\) (see also [Fre]);
(ii) examples 9.1 when \(\gamma \in \left(\frac{1}{2}, 1\right)\) and sometimes when \(\gamma = 1\) (in theorem 9.1), see [MT].

§10 Proof of theorem F

Proof of (i) Recall from [A2] that for \(T \in \text{MPT}\), \(d_k > 0\) & \(Y\) a random variable on \([0, \infty]\), \(\frac{S_{nk}(T)}{d_k}\) \(\Rightarrow\) \(Y\) if

\[
g\left(\frac{S_{nk}(f)}{d_k}\right) \xrightarrow[k \to \infty]{} E(g(Y \int_X f dm))\text{ weak-\* in } L^\infty \quad \forall \ g \in C([0, \infty]).
\]

The sequence \(\{m_j\}\) is called a loose sequence for \(T\) if

\[n_k = m_{j_k} \to \infty, \ d_k > 0, \ \frac{S_{nk}(T)}{d_k} \xrightarrow{\triangleright} Y \implies \text{Prob}(Y \in (0, \infty)) = 0.
\]

As shown in the proof of theorem 2 in [A2], the collection

\(\{T \in \text{MPT} : T \text{ has a loose sequence}\}\)

is residual in \(\text{MPT}\). No weakly rationally ergodic transformation has a loose sequence and so the collection of these is meagre in \(\text{MPT}\). Thus \(\text{RWM}\) is contained in a meagre collection.

We commence the proof of (ii) by showing:

Subsequence, rational, weak mixing is residual

We’ll use the

Conjugacy Lemma (see e.g. [A], [Kri2], [S])

For aperiodic \(T \in \text{MPT}\),

\(\{\psi^{-1} \circ T \circ \psi : \psi \in \text{MPT}\}\)

is dense in \(\text{MPT}\).

By the isomorphism theorem, we may assume WLOG that \((X, B, m)\)
is as in Hopf’s example:

\(X = \mathbb{R}_+ \times [0, 1], \ B = \mathcal{B}(\mathbb{R}_+ \times [0, 1]) \& m = \text{Leb}\)
A dyadic square in $X$ is a square $S = I \times J$ with $I, J$ dyadic intervals in $\mathbb{R}$ (i.e. $\partial I, \partial J \in \mathbb{Q}_2$) of the same length. A dyadic set in $X$ is a finite union of dyadic squares. Let $\mathcal{D} := \{\text{dyadic sets in } X\}$.

We'll need the (standard) result that for $N \geq 2$ there is a measure space isomorphism $\Phi_N : X^N \to X$ so that

$$\Phi_{N}^{-1}(D) = \{\text{finite unions of sets in } \mathcal{D} \otimes \cdots \otimes \mathcal{D}\}.$$ 

Permutations.

An automorphism $T \in \text{MPT}(X, \mathcal{B}, m)$ is called a permutation if there exist finitely many disjoint dyadic squares $E_1, \ldots, E_N$ and a permutation $\sigma : \{1, \ldots, N\} \to \{1, \ldots, N\}$ so that

- $T$ maps each $E_i$ onto $E_{\sigma(i)}$;
- $T(x) = x \forall x \notin \bigcup_{i=1}^N E_i$.

The proof of Satz 2 in [Kri2] applies to show that the collection $\Pi := \{\text{permutations in MPT}\}$ is dense in MPT. This immediately implies the

Permutation Conjugacy Lemma

For aperiodic $T \in \text{MPT}$,

$$\{\psi^{-1} \circ T \circ \psi : \psi \in \Pi\}$$

is dense in MPT.

Note that $\psi \mathcal{R}_b = \mathcal{R}_b$ (the collection of bounded measurable sets) for $\psi \in \Pi$, but not for arbitrary $\psi \in \text{MPT}$.

Markov shifts in MPT($X$).

We show that any conservative, ergodic, stationary Markov shift with infinite stationary distribution is isomorphic to a piecewise affine transformation $T \in \text{MPT}(X)$ with a Markov partition whose cylinder sets are bounded rectangles in $X$.

We consider (WLOG) only Markov chains with state space $\mathbb{N}$.

Let $P : \mathbb{N} \times \mathbb{N} \to [0, 1]$ be a stochastic matrix with infinite stationary distribution $\pi : \mathbb{N} \to \mathbb{R}_+$.

We show first that the one-sided shift of $(P, \pi)$ is isomorphic to a measure preserving, piecewise affine map $\tau_{(P, \pi)} : \mathbb{R}_+ \to \mathbb{R}_+$. To this end, let

- $\alpha = \{a_k : k \in \mathbb{N}\}$ be a partition mod 0 of $\mathbb{R}_+$ into open intervals so that $\lambda(a_s) = \pi_s \forall s \in \mathbb{N}$ where $\lambda$ denotes Lebesgue measure on $\mathbb{R}_+$; and

- for each $s \in \mathbb{N}$ let $\{a_{s,t} : t \in \mathbb{N}, p_{s,t} > 0\}$ be a partition mod 0 of $a_s$ into open intervals so that $\lambda(a_{s,t}) = \pi_{s} p_{s,t} \forall t \in \mathbb{N}$.
Now define $\tau : \mathbb{R}_+ \to \mathbb{R}_+$ by

$$\tau(x) := \frac{\pi_t}{\pi_s p_{s,t}} \cdot x + \gamma_{s,t} \quad x \in \{s, t \in \mathbb{N}, p_{s,t} > 0\}$$

where $\gamma_{s,t}$ is chosen so that $\tau a_{s,t} = a_t$.

It is standard to show that $\tau_{(P, \pi)}$ preserves $\lambda$ and is isomorphic to the one-sided shift of $(P, \pi)$.

To obtain the two-sided shift of $(P, \pi)$, we represent the natural extension of $\tau$ on $\mathbb{R}_+ \times [0, 1]$. For $s \in \mathbb{N}$, define $v_{a_s} : \tau a_s \to a_s$ by

$$v_{a_s}(y) := \frac{\pi_s p_{s,t}}{\pi_t} (y - \gamma_{s,t}) \quad y \in a_t \subseteq \tau(a_s).$$

Note that $v'_{a_s} = \sum_{t \in \mathbb{N}, p_{s,t} > 0} \frac{\pi_s p_{s,t}}{\pi_t} 1_{a_t}$.

Define for $x \in \mathbb{R}_+$

$$q_0(x) := 0, \quad q_k(x) := \sum_{1 \leq j \leq k} 1_{\tau a_j}(x) v'_{a_j}(x) \quad (k \geq 1)$$

and let $F_{x,a_k} : [0, 1] \to [q_k(x), q_{k+1}(x)]$ be the increasing affine map

$$F_{x,a_k}(y) := 1_{\tau a_k}(x) v'_{a_k}(x)y + q_{k+1}(x).$$

Now define $T = T_{(P, \pi)} : \mathbb{R}_+ \times [0, 1] \to \mathbb{R}_+ \times [0, 1]$ by

$$T(x, y) := (\tau(x), F_{x,\alpha(x)}(y)) \text{ where } x \in \alpha(x) \in \alpha.$$

It is standard to show that $T_{(P, \pi)} \in \text{MPT}(X)$ is a natural extension of $\tau$, whence isomorphic to the two-sided shift of $(P, \pi)$. The partition $\beta := \alpha \times [0, 1]$ is a Markov partition whose cylinder sets are finite unions of bounded rectangles wherein $\mathcal{HR}(\mathcal{C}_\beta) = \mathcal{R}_\beta$.

Let $P : S \times S \to [0, 1]$ be a stochastic matrix on the state space $S$ with invariant distribution $\pi : S \to \mathbb{R}_+$ and let $T_{(P, \pi)}$ be a Markov shift in $\text{MPT}$ isomorphic to the stationary Markov shift of $(P, \pi)$. Fix $s \in S$ and let $u = u([s]_0), \quad a(n) := \sum_{k=0}^{n-1} u_k$.

Assume that $u$ is smooth, then $T = T_{(P, \pi)}$ is rationally weakly mixing with $R(T) \supset \mathcal{HR}(\mathcal{C}_\beta) = \mathcal{R}_\beta$, whence

$$(\star) \quad \frac{1}{a(n)} \sum_{k=0}^{n-1} \left| m(D \cap T^{-n} D') - u_n m(D)m(D') \right| \longrightarrow 0 \forall \ D, D' \in \mathcal{D};$$

which implies

$$\frac{1}{a(n)} \int_D S_n^{(T)}(1_D) dm \longrightarrow m(D)^2 \quad \forall \ D \in \mathcal{D}.$$
We claim that also
\[
\lim_{n \to \infty} \frac{1}{a(n)^2} \int_D S_n^{(T)}(1_D)^2 \, dm \leq 2m(D)^3 \quad \forall \; D \in \mathcal{D}.
\]

**Proof of (♠) ** Let $\beta = \alpha \times [0,1]$, the Markov partition of $T$. We first show (♠) for $A \in \mathcal{U}_\beta$. Let $\tau = \tau_{(P,\pi)} : \mathbb{R}_+ \to \mathbb{R}_+$ be as above (isomorphic to the one-sided Markov shift of $(P,\pi)$). It is pointwise dual ergodic in the sense that
\[
\frac{1}{a(n)} \sum_{k=0}^{n-1} \tilde{\tau}^k 1_A \xrightarrow{n \to \infty} \lambda(A) \quad \forall \; A \in \mathcal{F}
\]
where $\tilde{\tau} : L^1(\lambda) \to L^1(\lambda)$ is the transfer operator defined by
\[
\int_{\mathbb{R}_+} \tilde{\tau} f \cdot g \, d\lambda = \int_{\mathbb{R}_+} f \cdot g \circ \tau \, d\lambda
\]
(see §3.7 in [A]).

Now $\sup_{\mathbb{R}_+} \sum_{k=0}^{n-1} \tilde{\tau}^k 1_A = \sup_A \sum_{k=0}^{n-1} \tilde{\tau}^k 1_A$. For $A \in \mathcal{C}_\alpha$, the convergence (a) is uniform on $A$, whence
\[
\frac{1}{a(n)} \sup_{\mathbb{R}_+} \sum_{k=0}^{n-1} \tilde{\tau}^k 1_A = \frac{1}{a(n)} \sum_{k=0}^{n-1} \tilde{\tau}^k 1_A \xrightarrow{n \to \infty} m(A) \quad \forall \; A \in \mathcal{C}_\alpha.
\]

From (b) we see that
\[
\lim_{n \to \infty} \frac{1}{a(n)} \sup_{\mathbb{R}_+} \sum_{k=0}^{n-1} \tilde{\tau}^k 1_A \leq m(A) \quad \forall \; A \in \mathcal{C}_\alpha.
\]

The statement (b') holds
\[
\forall \; A \in \mathcal{U}_\alpha := \{ \bigcup_{k=1}^N C_k : N \geq 1, \; C_1, \ldots, C_N \in \mathcal{C}_\alpha \}
\]
and it follows that for $A \in \mathcal{U}_\alpha$, and $n$ large so that $\sum_{k=0}^{n-1} \tilde{\tau}^k 1_A \leq 2m(A)a(n)$,
\[
\int_A S_n(1_A)^2 d\lambda \leq 2 \sum_{0 \leq i \leq j \leq n-1} \lambda(A \cap \tau^{-i} A \cap \tau^{-j} A)
\]
\[
= 2 \int_A \sum_{0 \leq i \leq n-1} \tilde{\tau}^i 1_A S_{n-i}(1_A) \circ \tau^i d\lambda
\]
\[
\leq \int_A S_n(1_A) \sum_{k=0}^{n-1} \tilde{\tau}^k 1_A d\lambda
\]
\[
\leq 4m(A)a(n) \int_A S_n(1_A) d\lambda
\]
\[
\leq 8m(A)^3 a(n)^2.
\]
If \( \psi : X = \mathbb{R}_+ \times [0, 1] \to \mathbb{R}_+ \) is the projection \( \psi(x, y) = x \), then
\[
U_\beta = \bigcup_{n \geq 1} T^n \psi^{-1} U_\alpha
\]
and \( (\star) \) follows for \( A \in U_\beta \). Dyadic sets can be monotonically approximated by sets in \( U_\beta \) and \( (\star) \) follows.

Now enumerate \( D := \{D_i : i \in \mathbb{N}\} \) and define
\[
G := \bigcap \bigcup \bigcap_{k \geq 1} N \geq k \{U(N, k, i, j) : 1 \leq i, j \leq k \}
\]
where
\[
U(N, k, i, j) := \{T \in \text{MPT} : \sum_{\nu=0}^{N-1} |m(D_i \cap T^{-\nu} D_j) - m(D_i)m(D_j)u_\nu| < \frac{a(N)}{k} ; \}
& \int_{D_i} S_N(1_{D_i})^2 dm < 8m(D_i)^3 a(N)^2 \}.
\]
Evidently each \( U(N, k, i, j) \) is open in \( \text{MPT} \), whence the set \( G \) is a \( G_\delta \) set. We’ll complete the proof of residuality of \( \text{SRWM} \) by showing that
- \( G \) is dense in \( \text{MPT} \) and
- each \( T \in G \) is subsequence rationally weak mixing.

**Proof of density of \( G \)**

By \( (\star) \) and \( (\star) \) (as on page 35), \( T(P, \pi) \in G \). Since \( \psi \mathcal{R}_b = \mathcal{R}_b \forall \psi \in \Pi \),
\[
\{\psi^{-1} \circ T(P, \pi) \circ \psi : \psi \in \Pi\} \subset G.
\]
Since \( T(P, \pi) \) is ergodic, by the permutation conjugacy lemma,
\[
G \supset \{\psi^{-1} \circ T(P, \pi) \circ \psi : \psi \in \Pi\} = \text{MPT}. \]

**Proof of subsequence rational weak mixing of elements of \( G \)**

Let \( T \in G \), then \( \exists \) a subsequence \( \mathfrak{R} \subset \mathbb{N} \) such that
(a)
\[
\frac{1}{a(N)} \sum_{\nu=0}^{N-1} |m(D \cap T^{-\nu} D') - m(D)m(D')u_\nu| \to 0 \forall D, D' \in \mathcal{D};
\]
(b)
\[
\int_D S_N(1_D)^2 dm < 8m(D)^3 a(N)^2 \forall D \in \mathcal{D}, N \in \mathfrak{R}.
\]
It follows from (a) that
\[
\frac{a_N(T)(D)}{a(N)} \to m(D)^2 \forall D \in \mathcal{D}
\]
whence by (b), $T$ is weakly rationally ergodic along $R$ with return sequence $a(n)$ along $R$ and $D \subset R(T)$. This enables use of (a) and lemma C to show that $T$ is rationally weakly mixing along $R$. $\Box$

**Proof of (ii)**

For $\kappa = (\kappa_1, \ldots, \kappa_\Delta) \in (\mathbb{Z} \setminus \{0\})^\Delta$, define $\psi_{\kappa} : \text{MPT}(X) \to \text{MPT}(X)$ by

$$\psi_{\kappa}(T) := \phi_\Delta \circ T(\kappa_1, \ldots, \kappa_\Delta) \circ \phi^{-1}_\Delta \in \text{MPT}(X)$$

where as above, $\Phi_\Delta : X^\Delta \to X$ so that

$$\Phi_\Delta^{-1}(D) = \{\text{finite unions of sets in } D \otimes \cdots \otimes D\}.$$  

If $\psi_{\kappa}(T) \in G$, then $T(\kappa_1, \ldots, \kappa_\Delta) \in \text{SRWM}$. Thus it suffices to show that

$$G_{\text{power}} := \bigcap_{\Delta=1}^\infty \bigcap_{(\kappa_1, \ldots, \kappa_\Delta) \in (\mathbb{Z} \setminus \{0\})^\Delta} \psi_{\kappa}^{-1}G$$

is residual.

It is not hard to see that:

- each $\psi_{\kappa} : \text{MPT}(X) \to \text{MPT}(X)$ is a continuous homomorphism, whence $G_{\text{power}}$ is a $G_\delta$ set in $\text{MPT}(X)$; and that
- $\psi_{\kappa}(\Pi) = \Pi$, whence $\psi^{-1} \circ T \circ \psi \in G_{\text{power}} \forall T \in G_{\text{power}}, \psi \in \Pi$, because for $T \in G_{\text{power}} \& \pi \in \Pi$, $\psi_{\kappa}(\pi)D = D$ and

$$\psi_{\kappa}(\pi^{-1} \circ T \circ \pi) = \psi_{\kappa}(\pi)^{-1} \circ \psi_{\kappa}(T) \circ \psi_{\kappa}(\pi) \in G.$$

To prove density of $G_{\text{power}}$ (and thus complete the proof of (ii)) it suffices to exhibit $T \in G_{\text{power}}$ for then $T$ is ergodic and

$$\overline{G_{\text{power}}} = \overline{\{\pi^{-1} \circ T \circ \pi : \pi \in \Pi\}} = \text{MPT}$$

by the permutation conjugacy lemma.

**Renewal shifts.** Let $u$ be a recurrent, renewal sequence with lifetime distribution $f \in \mathcal{P}(\mathbb{N})$. Define (as in [Ch]) a stochastic matrix $P = P_u$ on $\mathbb{N}$ by

$$P_{1,n} := f_n \quad \& \quad P_{n+1,n} = 1 \quad \forall \ n \in \mathbb{N}.$$ 

This has stationary distribution $\pi = \pi_u$ defined by $\pi_n := \sum_{k=n}^\infty f_k$ and $P_{1,1} = u_n$. The Markov shift of $(P,\pi)$ is called the renewal shift of $u$. Let $T_u := T_{(P,\pi)} \in \text{MPT}$.

If $u$ is smooth, then $T_u \in \text{RWM}$.

Now suppose that $\Delta \geq 1$, $\kappa = (\kappa_1, \ldots, \kappa_\Delta) \in (\mathbb{Z} \setminus \{0\})^\Delta$, then as evidently $T_u^{-1} \simeq T_u$,

$$T_u^{\kappa_1} \times \cdots \times T_u^{\kappa_\Delta} \simeq T_u^{[\kappa_1]} \times \cdots \times T_u^{[\kappa_\Delta]}$$
and we may assume WLOG that $\kappa \in \mathbb{N}^\Delta$. Now $T_u^{\kappa_1} \times \ldots \times T_u^{\kappa_\Delta}$ is also the Markov shift of an irreducible, aperiodic, stochastic matrix with renewal sequence $u(\omega)$ defined by

$$u_n^{(\kappa)} := \prod_{j=1}^{\Delta} u_{\kappa_j,n}.$$ 

If $u$ is smooth and $u(\omega)$ is recurrent, then $u(\omega)$ is also smooth, $T_u^{\kappa_1} \times \ldots \times T_u^{\kappa_\Delta}$ is rationally weakly mixing and $\psi_\omega(T_u) \in \text{RWM}$. 

Now let $u$ be the sequence defined by

$$u_n := \frac{1}{\log(n+e)} \quad (n \geq 0),$$

then $u$ is a Kaluza sequence in the sense that $u_0 = 1$ & $\frac{u_{n+1}}{u_n} \uparrow 1$ and hence a smooth, recurrent renewal sequence. 

As can be easily checked, so is $u(\omega) \forall \Delta \geq 1$, $\kappa = (\kappa_1, \ldots, \kappa_\Delta) \in \mathbb{N}^\Delta$. 
It follows that $T_u \in \mathcal{G}_{\text{power}}$. ☐

§11 Closing Remarks

All infinite, rationally weak mixing examples in this paper are of form $T \times S$ where $T$ is an infinite K-automorphism and $S$ is a weakly mixing probability preserving transformation. 
Their Koopman operators all have countable Lebesgue spectrum. 
This is shown in [Par] for K-automorphisms and a simple argument shows that multiplying by a weakly mixing probability preserving transformation does not change this. 

The transformation $T \in \text{MPT}$ is called rigid if $\exists \mathcal{L} \subset \mathbb{N}$ so that

$$m(A\Delta T^{-n}A) \underset{n \to \infty, n \in \mathcal{L}}{\to} 0 \quad \forall \ A \in \mathcal{F}.$$ 

The spectrum of a rigid transformation is Dirichlet, and hence singular. 

As shown in [AS], the collection $\text{RIGID}$ of rigid transformations in $\text{MPT}$ is residual. 

By Theorem F, so is $\text{PSRWM} \circ \text{RIGID}$ and so there is a rigid, power, subsequence, rationally weak mixing, measure preserving transformation with singular spectrum.

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