Structural Reliability Analysis of a Beam with Different Distributed Variables Via Stochastic Reduced Basis Method

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Abstract. In this paper, the reliability of a random beam structure with different distributed variables is calculated, and the scope of application of the stochastic reduced basis methods (SRBMs) method is expanded. With the help of stochastic Krylov subspace, the undetermined coefficients of the response expression under static force can be obtained. Then using Galerkin projection technology to obtain the final expression of the random response expression. Furthermore, an example is investigated in order to demonstrate the precision of the method, the results show that this method has high calculation accuracy comparing with direct Monte Carlo simulation.

Keywords. Beam; Reliability; Stochastic Krylov subspace; Galerkin projection technology

1. Introduction

Due to the uncertainty of the material or the external load used in a beam structure, the structure response is also random, so the model using random parameters will be much more realistic. At present, as we know that a key indicator to measure the safety of a structure is the reliability of structures, in other words, the calculation of the failure probability. But the calculation will become very difficult due to the non-linearity of the function or the non-Gaussian nature of the probability distribution of random variables [1-4].

In the last decades, the stochastic reduced basis methods (SRBMs) [5-7] is relatively popular to express the unknown random field or stochastic process in the series expansion methods. In this paper, based on SRBMs method, the reliability of a random beam structure with different distributions variables is calculated, the scope of application of the SRBMs method is expanded. With the help of stochastic Krylov subspace, the undetermined coefficients of the response expression under static force can be obtained. Then using Galerkin projection technology to obtain the final expression of the random response expression. Finally, comparing this method with the direct Monte Carlo method (DMC) [8, 9] to fully illustrate the SRBMs method, DMC is currently known as a classic numerical method that can test the accuracy of other methods. The calculation example shows that this method has high calculation accuracy to solve structural reliability analysis of a beam with different distributed variables.
2. Stochastic Krylov subspace

If the output response function is sufficiently smooth in the whole space of the random sample, with the increasing of the order of expansion, and when a small deformation of a linear elastic structural system occurs under an external static load, its finite element equation can be expressed as follows

$$\mathbf{Ku} = \mathbf{f}$$

where $\mathbf{K}$ is a $N \times N$ dimensional structural stiffness matrix; $\mathbf{u}$ denotes a $N \times 1$ dimensional nodal displacement vector; and $\mathbf{f}$ is a $N \times 1$ dimensional equivalent nodal load vector.

The stiffness matrix of the linear system can be expressed as follows

$$\mathbf{K} = \mathbf{K}_0 + \sum_{i=1}^{P} \mathbf{K}_i \alpha_i$$

where $\mathbf{K}_0$ is mean parameter; $\mathbf{K}_i$ denote $N \times N$ dimensional deterministic accompanying matrices; $P$ is the order of expansion (2); $\alpha_1, \alpha_2, \ldots, \alpha_n$ are variables with different distributions.

Krylov subspace [10] defined below

$$\mathcal{R}_n = \text{span} \{ \mathbf{f}, \mathbf{Kf}, \mathbf{K}^2\mathbf{f}, \ldots, \mathbf{K}^{n-1}\mathbf{f} \}$$

The first three basis vectors spanning the preconditioned stochastic Krylov subspace can be written as

$$\mathbf{u}^{(a)}(\mathbf{a}) = \beta_0 \psi_o(\mathbf{a}) + \beta_1 \psi_1(\mathbf{a}) + \cdots + \beta_n \psi_n(\mathbf{a})$$

where $\mathbf{a} = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$; $\mathbf{\beta} = \{\beta_1, \beta_2, \beta_3, \ldots\}$ is a vector of undetermined coefficients; the basis vectors of random solution $\psi = \{\psi_0, \psi_1, \psi_2, \ldots, \psi_n\}$ are

$$\psi_0(\mathbf{a}) = \mathbf{u}_0$$

$$\psi_1(\mathbf{a}) = \sum_{i=1}^{N} \mathbf{d}_i \alpha_i$$

$$\psi_2(\mathbf{a}) = \sum_{i=1}^{N} \sum_{j=1}^{N} \mathbf{e}_{ij} \alpha_i \alpha_j$$

where $\mathbf{u}_0 = \mathbf{K}_0^{-1} \mathbf{f}$, $\mathbf{d}_i = \mathbf{K}_0^{-1} \mathbf{K}_i \mathbf{u}_0$, $\mathbf{e}_{ij} = \mathbf{K}_0^{-1} \mathbf{K}_j \mathbf{d}_i$.

3. Stochastic projection scheme

The stochastic residual error of the structure can be expressed as

$$\varepsilon = \left\{ \sum_{i=1}^{P} \mathbf{K}_i \psi(\mathbf{a}) \mathbf{\beta} \right\} \cdot \mathbf{f}$$

Then using Galerkin scheme [11-14], $\varepsilon$ is orthogonal to all the stochastic basis vectors, then it is easy to get the following random algebraic equation representing a linear system,

$$\left\{ \psi^T(\mathbf{a}) \left[ \sum_{i=1}^{P} \mathbf{K}_i \right] \psi(\mathbf{a}) \mathbf{\beta} \right\} = \left\{ \psi^T(\mathbf{a}) \mathbf{f} \right\}$$

where symbol $\langle \rangle$ indicates expectation calculation.

At this time, a vector of undetermined coefficients $\mathbf{\beta}$ can be obtained by equation (7), so the final expression of displacement with random variables is finally gotten, then it is easy to calculated reliability analysis using related definition formula.

4. Example

A one-dimensional beam that is subjected to two concentrated force which are both 200kN in the Figure 1, the beam is composed of two segments which are both 3m. It is assumed that the bending stiffnesses of two segments are random. The means of the bending stiffnesses of the left part $EI_1$ and the right part $EI_2$ are constant and equal to $3 \times 10^7 \text{kN} \cdot \text{m}^2$ and $1 \times 10^7 \text{kN} \cdot \text{m}^2$ respectively.
Assuming $EI_1$ is Beta distribution and $EI_2$ is uniform distribution, the probability density function are written as below respectively,

$$f(\alpha_1) = \frac{3}{4}(1-\alpha_1^2), \quad (-1 \leq \alpha_1 \leq 1)$$  \hspace{1cm} (8)

$$f(\alpha_2) = \frac{1}{2}$$  \hspace{1cm} (9)

where $\alpha_1$ denotes $EI_1$, $\alpha_2$ is $EI_2$.

The results of the first two moments of vertical displacement at the end of beam from different methods are shown in Table 1 to Table 4. In order to compare with the results of the numerical simulation, herein, we choose the fourth order term of SRBM, and give the results of 200,000 random values of the direct Monte-Carlo simulation (DMC simulation), here $\delta_i$ is the coefficient of variation (COV) of $EI_i$.

**Table 1.** The Mean of Vertical Displacement at The End of The Beam ($\delta_i=0.05$)

| $\delta_i$ | SRBM | DMC |
|------------|------|-----|
| 0.05       | 7.143e-04 | 7.143e-04 |
| 0.10       | 7.154e-04 | 7.154e-04 |
| 0.15       | 7.173e-04 | 7.173e-04 |
| 0.20       | 7.201e-04 | 7.201e-04 |
| 0.25       | 7.241e-04 | 7.240e-04 |
| 0.30       | 7.294e-04 | 7.294e-04 |

**Table 2.** The Mean Square Error of Vertical Displacement at The End of The Beam ($\delta_i=0.05$)

| $\delta_i$ | SRBM | DMC |
|------------|------|-----|
| 0.05       | 5.111e-07 | 5.111e-07 |
| 0.10       | 5.129e-07 | 5.129e-07 |
| 0.15       | 5.159e-07 | 5.159e-07 |
| 0.20       | 5.204e-07 | 5.204e-07 |
| 0.25       | 5.268e-07 | 5.269e-07 |
| 0.30       | 5.357e-07 | 5.357e-07 |

**Table 3.** The Skewness of Vertical Displacement at The End of The Beam ($\delta_i=0.05$)

| $\delta_i$ | SRBM | DMC |
|------------|------|-----|
| 0.05       | 3.664e-10 | 3.664e-10 |
| 0.10       | 3.684e-10 | 3.684e-10 |
| 0.15       | 3.720e-10 | 3.720e-10 |
| 0.20       | 3.774e-10 | 3.774e-10 |
| 0.25       | 3.851e-10 | 3.851e-10 |
| 0.30       | 3.963e-10 | 3.963e-10 |

**Table 4.** The Kurtosis of Vertical Displacement at The End of The Beam ($\delta_i=0.05$)
Table 1 to Table 4 show that the results of the SRBMs method and the DMC simulation with the increase of the COV $\delta_2$ of the bending stiffness $EI_2$ are increasing ($\delta_2=0.05$), and the results of the first four moments of the fourth order polynomial match well with those of DMC simulation whether $\delta_2$ is large or small.

To fully illustrate the effects of the random inputs on the statistical properties of random response, the curves of probability density function (PDF) of the response of the SRBMs method of the fourth order polynomial is shown in Figure 2 to Figure 7 with the increase of the COV of random variable $\delta_2$ ($\delta_2=0.05$).

![Figure 2. PDF of displacement at the end of beam ($\delta_2=0.05$)](image1)

![Figure 3. PDF of displacement at the end of beam ($\delta_2=0.1$)](image2)
Figure 4. PDF of displacement at the end of beam (δ2=0.15)

Figure 5. PDF of displacement at the end of beam (δ2=0.2)

Figure 6. PDF of displacement at the end of beam (δ2=0.25)
Figure 7. PDF of displacement at the end of beam ($\delta_z=0.3$)

Figure 2 to figure 7 show probability density function (PDF) of displacement at the end of beam with different variable $\delta_1$ ($\delta_1=0.05$) from 0.05 to 0.35, with the larger the coefficient of variation, the PDF curve becomes shorter and fatter.

5. Conclusions

The reliability of random beam structure with different distributed variables is calculated in this paper, the scope of application of the stochastic reduced basis methods (SRBMs) method is expanded. With the help of stochastic projection scheme and stochastic Krylov subspace, the final expression of the random response expression under static force can be obtained. Furthermore, an example is investigated in order to demonstrate the precision of the method, the results show that this method has high calculation accuracy comparing with the direct Monte Carlo simulation.

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