1 Introduction

Post correspondence problem is a basic undecidable problem [1, 2, 3]. To establish the algorithmic unsolvability of a specific problem, researchers reduce it in many cases to the Post correspondence problem.

We introduce the concept of a threefold Post correspondence system (3PCS for short) and we consider it as an instance of the threefold Post correspondence problem. With each 3PCS, we associate three Post correspondence systems, i.e., three instances of the Post correspondence problem. We conjecture that for each 3PCS, the question of the threefold Post correspondence problem or for some associated Post correspondence system the question of the Post correspondence problem is decidable.

In Sections 2 and 3 we present an intuitive and a formal description, respectively, of the concepts and the conjecture.

2 Intuitive description of concepts and conjecture

We define a domino as an ordered triple of strings written in the top, middle, and bottom third of the domino, see Figure 1. A threefold Post correspondence system

\[
\begin{pmatrix}
abba \\
- - - \\
bbb \\
- - - \\
aa
\end{pmatrix}
\]

Figure 1: A domino.
(3PCS for short) is a finite set of dominoes of this kind, see Figure 2. A list of dominoes may include any dominoes any number of times. A list of dominoes yields three words: one at the top, one at the middle, and one at the bottom, when we read across from left to right. With a 3PCS, i.e. a finite set of dominoes, we associate a quadruple of solitaire games played with the dominoes: Games Top-Middle-Bottom, Top-Middle, Top-Bottom, and Middle-Bottom.

**Game Top-Middle-Bottom:** The way to win the game is to find a list of dominoes where the same word appears on the top thirds as on the middle thirds and as on the bottom thirds of the dominoes when we read across from left to right.

**Game Top-Middle:** The way to win the game is to find a list of dominoes where the same word appears on the top thirds as on the middle thirds of the dominoes when we read across from left to right.

**Game Top-Bottom:** The way to win the game is to find a list of dominoes where the same word appears on the top thirds as on the bottom thirds of the dominoes when we read across from left to right.

**Game Middle-Bottom:** The way to win the game is to find a list of dominoes where the same word appears on the middle thirds as on the bottom thirds of the dominoes when we read across from left to right.

By definition, if we win the game Top-Middle-Bottom, then we win the games Top-Middle, Top-Bottom, and Middle-Bottom as well.

**Figure 3:** For the list 1, 2, 3 of dominoes, the word $ababbb$ appears on the top thirds and the middle thirds and the bottom thirds of the dominoes when we read across from left to right.

Consider the 3PCS shown on Figure 2. Figure 3 and Figure 4 show how to win the game Top-Middle-Bottom. Figure 5 shows how to win the game Top-Middle in another way.
Figure 4: For the list 1, 2, 3, 1, 2, 3 of dominoes, the word \( ababbbababbb \) appears on the top thirds and the middle thirds and the bottom thirds of the dominoes when we read across from left to right.

Figure 5: For the list 3 of dominoes, the word \( b \) appears on the top third and the middle third of the domino.

We conjecture that for each 3PCS, we can decide for some game in the associated quadruple of solitaire games whether we can win it.

3 Formal description of concepts and conjecture

A Post correspondence system (PCS for short) over an alphabet \( \Delta \) is a pair \( \langle w, z \rangle = \langle (w_1, \ldots, w_n), (z_1, \ldots, z_n) \rangle, n \geq 1 \), of lists of words from the alphabet \( \Delta \).

Post correspondence problem

Instance: a PCS \( \langle w, z \rangle \).

Question: Do exist \( k \geq 1 \) and \( i_1, \ldots, i_k \), where \( 1 \leq i_1, \ldots, i_k \leq n \), such that

\[
\begin{align*}
  w_{i_1} \ldots w_{i_k} &= z_{i_1} \ldots z_{i_k} ?
\end{align*}
\]

Here we call the index sequence \( i_1, \ldots, i_k \) a match of the PCS \( \langle w, z \rangle \).

Proposition 3.1 [1, 2, 3] The Post correspondence problem is unsolvable. That is, there is no algorithm which takes a PCS \( \langle w, z \rangle \) as input and determines whether or not there is a match of the PCS \( \langle w, z \rangle \).

A Threefold Post correspondence system (3PCS for short) over an alphabet \( \Delta \) is a triple \( \langle u, w, z \rangle = \langle (u_1, \ldots, u_n), (w_1, \ldots, w_n), (z_1, \ldots, z_n) \rangle, n \geq 1 \), of lists of words from the alphabet \( \Delta \).

Threefold Post correspondence problem

Instance: a 3PCS \( \langle u, w, z \rangle \).

Question: Do exist \( k \geq 1 \) and \( i_1, \ldots, i_k \), where \( 1 \leq i_1, \ldots, i_k \leq n \) such that

\[
\begin{align*}
  u_{i_1} \ldots u_{i_k} &= w_{i_1} \ldots w_{i_k} = z_{i_1} \ldots z_{i_k} ?
\end{align*}
\]

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Here we call the index sequence \( i_1, \ldots, i_k \) a match of the 3PCS \( \langle u, w, z \rangle \).

To a 3PCS \( \langle u, w, z \rangle \), we assign the three PCSs \( \langle u, w \rangle, \langle u, z \rangle, \langle w, z \rangle \).

**Conjecture 3.2** There is an algorithm which takes a 3PCS \( \langle u, w, z \rangle \) as input and decides

- the question of the Post correspondence problem for the instance \( \langle u, w \rangle \) or \( \langle u, z \rangle \) or \( \langle w, z \rangle \), or
- the question of the threefold Post correspondence problem for the instance \( \langle u, w, z \rangle \).

**References**

[1] M. D. Davis, R. Sigal, E. J. Weyuker, Computability, Complexity, and Languages, (Academic Press, New York, 1994).

[2] Post correspondence problem. G. Rozenberg, A. Salomaa (originator), Encyclopedia of Mathematics.

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[3] Post correspondence problem From Wikipedia, the free encyclopedia.

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