A hyperbolic slicing condition adapted to Killing fields and densitized lapses

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We study the properties of a modified version of the Bona-Masso family of hyperbolic slicing conditions. This modified slicing condition has two very important features: In the first place, it guarantees that if a spacetime is static or stationary, and one starts the evolution in a coordinate system in which the metric coefficients are already time independent, then they will remain time independent during the subsequent evolution, i.e. the lapse will not evolve and will therefore not drive the time lines away from the Killing direction. Second, the modified condition is naturally adapted to the use of a densitized lapse as a fundamental variable, which in turn makes it a good candidate for a dynamic slicing condition that can be used in conjunction with some recently proposed hyperbolic reformulations of the Einstein evolution equations.

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I. INTRODUCTION

Specifying a good foliation of spacetime is of fundamental importance when studying the dynamical evolution of systems with strong gravitational fields. In the 3+1 decomposition of General Relativity, a foliation of spacetime into spacelike hypersurfaces is described in terms of the lapse function $\alpha$ that measures the interval of proper time between neighboring hypersurfaces along their normal direction. The choice of a particular foliation represents the freedom one has to specify the time coordinate and is therefore arbitrary. In practice, however, one can not choose a foliation ahead of time since one is trying to solve for the geometry of the spacetime itself, so one must choose instead a “slicing condition”, that is, some geometric or algebraic condition that allows one to calculate the lapse function dynamically during the evolution. Slicing conditions come in many different forms, and they are usually chosen by balancing their ease of implementation with the need to obtain a well-behaved coordinate system, as well as the well-posedness of the resulting system of evolution equations.

In a recent paper [1], one of us studied a particular slicing condition that has the property of being obtained through a hyperbolic evolution equation for the lapse, and in many cases allows one to construct a foliation that avoids different types of pathological behaviors. This slicing condition, known as the Bona-Masso (BM) slicing condition [2], has been used successfully for many numerical simulations of strong gravitating systems such as black holes (see for example [3, 4, 5]). Other forms of hyperbolic slicing conditions, which generalize in one way or another the original proposal of Bona and Masso, have been suggested in the last few years, examples of which can be found in [5, 6], and most recently in [7].

In this paper we argue that the original form of the BM condition (as well as most of its generalization) has two important drawbacks: First, it is not well adapted to the presence of Killing fields in the sense that if one uses it to evolve a static spacetime, it can easily drive the time lines away from the Killing direction. Second, the standard BM condition is not well adapted to the case when one wants to use a densitized lapse as a fundamental variable. Such a densitized lapse has been recently advocated in the context of hyperbolic reformulations of the Einstein equations (see for example [8, 9]), and is therefore an important issue to consider. Here we will study a modified version of the BM slicing condition that addresses both these issues at the same time. This modified BM condition has already been used in the literature, but as far as we are aware its properties have never been studied in any detail. On this manuscript we will limit ourselves to studying the properties of this slicing condition independently of the Einstein equations, and leave the study of how this condition couples to the Einstein evolution equations for a separate work [10].

This paper is organized as follows. In Sec. II we make a brief introduction to the BM family of slicing conditions. Section III motivates the introduction of a modified BM condition from the point of view of compatibility with static and stationary solutions. We then study the relation of the modified condition both with densitized lapses and with the divergence of the coordinate time lines, and introduce a coordinate independent way of writing the condition. In Sec. IV we analyze the hyperbolicity of the modified BM condition. We start by discussing the way in which the shift vector is chosen, and later analyze the hyperbolicity of the coupled slicing-shift evolution system. Finally, Sections V and VI study under which circumstances the modified BM condition avoids focusing singularities and gauge shocks. We conclude in Sec. VII.
II. THE BONA-MASSO FAMILY OF SLICING CONDITIONS

The BM family of slicing conditions [2] has been discussed extensively in the literature (for a detailed discussion see [11] and references therein). Here we will limit ourselves to making a very brief description of its main properties. The BM slicing condition is obtained by asking for the lapse function to satisfy the following evolution equation

$$\frac{d}{dt} \alpha \equiv (\partial_t - \mathcal{L}_\beta) \alpha = -\alpha^2 f(\alpha) K,$$  \hspace{1cm} (2.1)

with $\mathcal{L}_\beta$ the Lie derivative with respect to the shift vector $\beta^i$, $K$ the trace of the extrinsic curvature and $f(\alpha)$ a positive but otherwise arbitrary function of $\alpha$. The condition above is a generalization of slicing conditions that have been used in evolution codes based on the Arnowitt-Deser-Misner (ADM) formulation [12, 13] since the early 90’s [14, 15], and was originally proposed in the context of the Bona-Masso hyperbolic re-formulation of the Einstein equations [2, 16, 17, 18, 19]. It is, however, very general and can be used with any form of the 3+1 evolution equations. Since we will later introduce a modified version of the BM slicing condition, we will refer from now on to condition (2.1) as the “standard BM condition”.

A very important property of the standard BM condition is the fact that the shift terms included through the Lie derivative in equation (2.1) are such that one is guaranteed to obtain precisely the same spacetime foliation regardless of the value of the shift vector. At first sight, this would seem to be a natural requirement for any slicing condition, but as we will see below this might not be the most important property a slicing condition must have.

The BM slicing condition can be shown to lead to a generalized wave equation for the lapse

$$\frac{d^2}{dt^2} \alpha - \alpha^2 f D^2 \alpha = -\alpha^2 f \left[K_{ij}K^{ij} - (2f + \alpha f') K^2\right].$$  \hspace{1cm} (2.2)

From this equation one can easily see that the wave speed along a specific direction $x^i$ given by

$$v_x = \alpha \sqrt{f \gamma^{ii}},$$  \hspace{1cm} (2.3)

where $\gamma_{ij}$ is the spatial metric. The above expression explains the need for $f(\alpha)$ to be positive: If it weren’t, the wave speed would not be real and the equation would be elliptic instead of hyperbolic. Notice also that the gauge speed (2.3) can be smaller or larger that the speed of light depending on the value of $f$. Contrary to what one might expect, having a gauge speed that is larger than the speed of light does not lead to any causality violations, as the superluminal speed is only related with the propagation of the coordinate system. In fact, empirically the most successful slicing conditions for the simulation of black hole spacetimes have been precisely those that allow superluminal gauge speeds (an example of this is the 1+log slicing condition mentioned below).

Reference [1] shows that, for some specific choices of the function $f(\alpha)$, the standard BM slicing condition can avoid both focusing singularities [19] and gauge shocks [20]. We will expand on these points in the following sections.

A very important particular case of the BM condition corresponds to the choice $f(\alpha) = 1$, which leads to the so-called “harmonic slicing”, for which the time coordinate obeys the simple wave equation:

$$\Box t = g^\mu\nu \Gamma^0_{\mu\nu} = 0,$$  \hspace{1cm} (2.4)

with $g_{\mu\nu}$ the spacetime metric. That is, $f = 1$ corresponds to the case when the time coordinate is a harmonic function. One can easily show that for harmonic slicing one also has the following relation

$$\alpha = h(x^i) \gamma^{i/2}_n,$$  \hspace{1cm} (2.5)

with $h(x^i)$ an arbitrary time independent function and $\gamma^{i/2}_n$ the volume element associated with observers moving normal to the hypersurfaces (which will differ from the coordinate volume element in the case of a non-vanishing shift vector).

Another particular case worth mentioning corresponds to “1+log” family, for which $f(\alpha) = N/\alpha$. In this case we find the following relation between the lapse function and the normal volume elements

$$\alpha = h(x^i) + \ln \left(\gamma^{N/2}_n\right),$$  \hspace{1cm} (2.6)

which explains the name 1+log. The case $N = 2$ has been found empirically to be particularly well behaved [3, 4, 5], something which Ref. [1] attributes (a posteriori) to the fact that this is the only member of the 1+log family that avoids gauge shocks even approximately. Notice that for $N = 2$ the gauge speed in asymptotically flat regions where $\alpha \simeq 1$ becomes $\sqrt{2} > 1$. In regions inside a black hole, where the lapse typically collapses to zero, the gauge speed can become extremely large.

Having briefly discussed the main properties of the standard BM slicing condition, we will now turn to our proposal for a modified version of this condition, motivated by the analysis of stationary spacetimes.

III. A MODIFIED BONA-MASSO SLICING CONDITION

Let us consider for a moment a static or stationary spacetime, and let us assume that we have chosen a coordinate system in which the metric coefficients are already time independent. Further, we will assume that our coordinate system is such that the shift vector does not vanish. One would think at first sight that having
a non-vanishing shift implies that the spacetime can not be truly static but is at most only stationary. This is not entirely correct, as one can write a static spacetime in a coordinate system with non-vanishing shift and where the metric coefficients are nevertheless still time independent. An example of this is the Schwarzschild metric where the metric coefficients are nevertheless still time independent. An example of this is the Schwarzschild metric written in Kerr-Schild or Painlevé-Gullstrand coordinates.

Assuming we have such a situation, the ADM evolution equations imply in particular that

\[ \partial_t \gamma_{ij} = -2\alpha K_{ij} + L_{\beta} \gamma_{ij} \]

where \( K_{ij} \) is the spacetime written in Kerr-Schild coordinates. The trace of the extrinsic curvature must therefore be given by

\[ K = \frac{D_i \beta^i}{\alpha}. \]  

(3.3)

Let us now see if the standard Bona-Masso slicing condition is compatible with the time-independent character of our spacetime in the sense of predicting a time-independent lapse as well. Substituting our expression for \( K \) into equation (2.1) we find

\[ \partial_t \alpha = \beta^i \partial_i \alpha - 2\alpha^2 f(\alpha) K \]

(3.4) 

\[ = \beta^i \partial_i \alpha - 2\alpha f(\alpha) D_i \beta^i. \] 

(3.5)

It is clear from looking at this expression that for arbitrary \( f(\alpha) \), \( \partial_t \alpha \) will generally not vanish. That is, the lapse will evolve away from its preferred value and as a consequence the spatial metric will not remain time independent. This indicates that the standard BM slicing condition is not well adapted to the evolution of stationary spacetimes.

A modified version of the BM slicing condition that is well adapted to such time independent spacetimes can nevertheless be easily obtained by just asking for

\[ \partial_t \alpha = \frac{\alpha f(\alpha)}{\gamma^{1/2}} \partial_t \gamma^{1/2}, \] 

(3.6)

which automatically guarantees that the lapse will not evolve if the spatial metric is time independent. One can integrate the last equation trivially to find

\[ \gamma^{1/2} = F(x^i) \exp \left\{ \int \frac{d\alpha}{\alpha f(\alpha)} \right\}, \] 

(3.7)

with \( F(x^i) \) a time independent function. This tells us that there is a very general functional relationship between \( \alpha \) and \( \gamma^{1/2} \). For \( f(\alpha) = \text{constant} \) this relationship is a power law, but in other cases it is more general (for example, the well known “1+log” slicing is usually obtained from \( f = 2/\alpha \), which gives us an exponential relation between \( \alpha \) and \( \gamma^{1/2} \). Functional relationships between \( \alpha \) and \( \gamma^{1/2} \) have been considered before in the context of finding hyperbolic reformulations of the 3+1 evolution equations. For example, in reference [21], Frittelli and Reula propose a general power law relation, which just mentioned is a particular form of (3.7) with \( f(\alpha) \) constant. More recently, Sarbach and Tiglio [6] have considered a completely general functional relationship with the sole restriction that \( d\alpha/d\gamma > 0 \), which clearly includes (3.7) (asking for \( f(\alpha) > 0 \) guarantees that \( d\alpha/d\gamma > 0 \)). What makes (3.7) more interesting than a very general relationship is the fact that one can learn a lot about the properties of the slicing by studying the effect of different forms of \( f(\alpha) \).

Substituting now (3.2) into (3.6) we find

\[ \partial_t \alpha = -\alpha f(\alpha) \left( \frac{\alpha K - D_i \beta^i}{\alpha} \right). \] 

(3.8)

This is the modified version of the BM slicing condition we wish to study here (compare with (2.1)). Several comments are in order here. First, we should mention that the modified BM condition defined above has been used before in the literature in Refs. [22, 23]. Both these references study the numerical evolution of a Schwarzschild spacetime written in Kerr-Schild coordinates, and attempt to maintain the static (exact) solution stable during the numerical simulation. Because of this they need to use a slicing condition that maintains the lapse equal to its initial value in the continuum limit, but allows some dynamics to respond to numerical truncation errors at finite resolutions. The modified BM condition is used in those references precisely for the reason outlined above, but no suggestion is made to use it in the more general case or to study its properties.

Second, as condition (3.8) does not include the Lie derivative terms of the lapse with respect to the shift vector, it will not give us the same foliation of spacetime for a different choice of the shift vector, i.e. the foliation of spacetime one obtains will depend on the choice of shift. We believe this is not a serious drawback since in a particular situation one would presumably want to choose a slicing condition and a shift vector that are closely inter-related.

Also, we believe that using a slicing condition that is compatible with a static solution is a necessary requirement if one looks for symmetry seeking coordinates of the type discussed by Gundlach and Garfinkle [24] and by Brady et al. [25], that will be able to find the Killing fields that static (or stationary) spacetimes have, or the approximate Killing fields that many interesting astrophysical systems will have at late times. Of course, having a slicing condition that is compatible with staticity is not enough, one also needs to have a shift condition that has the same property. Otherwise, the shift evolution will also drive us away from the frame in which the staticity is apparent. In this paper we will not deal with the issue of the shift choice, but will consider it in a future work.
does not guarantee that one can find the Killing direction if one starts in the wrong coordinate system. All that it guarantees is that if we do find this Killing direction we won’t be driven away from it again.

A. Densitized lapse

Since the early 90’s, many re-formulations of the 3+1 evolution equations of general relativity have been proposed [2, 8, 9, 17, 18, 19, 21, 26, 27, 28, 29, 30, 31, 32, 33]. The main purpose of most such re-formulations has been to recast the Einstein evolution equations as a strongly or symmetric hyperbolic system motivated by the fact that for such systems one can prove that the evolution equations are well posed, and with the hope that such well posedness will bring with it improvements in numerical simulations, both in terms of stability and in terms of allowing one to impose boundary conditions in a consistent and well behaved way.

While it is not the purpose of this paper to study the different hyperbolic formulations of the Einstein evolution equations, there is one related point that is crucial for the discussion of slicing conditions. It was realized early on that there were two main problems when trying to go from the standard ADM evolution equations to a strongly or symmetric hyperbolic formulation. One problem was associated with the existence of constraint violating solutions that spoil hyperbolicity and required for its solution the addition of multiples of the constraint equations, in a number of different ways, to the evolution equations (the constraint violating modes are not eliminated, just transformed in a way that allows a hyperbolic system to be constructed).

The second problem, more directly related to our discussion, was the observation that it is not possible to construct a strongly hyperbolic formulation of the 3+1 evolution equations if the lapse function is considered to be an a priori known function of space and time. Two different routes have been followed to solve this second problem. The first route, taken in the Bona-Masso hyperbolic re-formulation [2, 17, 18, 19], was to propose an evolution equation for the lapse (equation (2.1) of the previous section), and then construct a strongly hyperbolic system of equations where the lapse was considered just another dynamical variable. This same route has been followed very recently by Lindblom and Scheel [7], where generalizations of standard BM slicing condition and the “Γ-driver” shift condition [5] have been used to construct a symmetric hyperbolic system that includes lapse and shift as dynamical variables. The second route has been to take not the lapse, but rather the densitized lapse \( q := \alpha \gamma^{-1/2} \), to be a prescribed function of space and time (see [8, 9] and references therein). Both these routes have been successful in constructing strongly hyperbolic re-formulations of the Einstein evolution equations. The different approaches are related, but are not equivalent, as can be seen easily if we consider for a moment the BM slicing condition in the case \( f = 1 \) (the harmonic slicing case). As we have seen, in that case the lapse takes the form

\[
\alpha = h(x^i) \gamma_n^{1/2},
\]

with \( \gamma_n^{1/2} \) the volume elements associated with the normal observers, and \( h(x^i) \) an arbitrary function of space. We can in fact turn the function \( h(x^i) \) into an arbitrary function of both space and time if we add a source term to the standard BM condition in the following way

\[
(\partial_t - \mathcal{L}_{\beta}) \alpha = -\alpha^2 f(\alpha) K + H(x,t),
\]

with \( H(x,t) \) arbitrary. On the other hand, the condition for the densitized lapse to be a known function of spacetime takes the form

\[
\alpha = q(x,t) \gamma^{1/2},
\]

where \( \gamma^{1/2} \) are the coordinate volume elements. In the case of a vanishing shift vector, normal and coordinate volume elements coincide, and the BM condition can be seen as a generalization of the prescribed densitized lapse condition. But for non-zero shift vector this is no longer the case.

One can in fact rewrite the standard BM condition in terms of the densitized lapse \( q \) in the following way

\[
(\partial_t - \mathcal{L}_{\beta}) q = -q^2 \gamma^{1/2} (f - 1) K,
\]

which shows that even for \( f = 1 \), the densitized lapse \( q \) will evolve dynamically driven by the Lie derivative term.

The crucial observation here is that the modified BM condition (3.8), when written in terms of the densitized lapse, takes the form

\[
\partial_t q = -q (f - 1) \left[ q \gamma^{1/2} K - D_i \beta^i \right],
\]

It is now clear that by taking \( f = 1 \) this equation reduces to the case of a static densitized lapse (the case of a prescribed densitized lapse that is not time independent can be easily obtained by adding a source term to the above equation). The modified BM slicing condition would therefore seem to be a natural generalization of the prescribed densitized lapse, and should be well adapted to hyperbolic re-formulations of the Einstein equations that use of a densitized lapse [10].

B. Divergence of the time lines and Killing fields

From the ADM equations one can easily show that the divergence of the time lines is given by the following expression

\[
\nabla_\mu t^\mu = \frac{1}{\alpha} \left[ \partial_\mu \alpha - \alpha (\alpha K - D_i \beta^i) \right],
\]
where \( t^\mu \) is the vector tangent to the 3+1 time lines, which is defined in terms of the normal and shift vectors as

\[
t^\mu = \alpha n^\mu + \beta^\mu .
\]  

(3.15)

Equation (3.14) implies that the modified BM slicing condition can also be written as

\[
\partial_\mu \alpha = \alpha \left( \frac{f}{f + 1} \right) \nabla_\mu t^\mu .
\]  

(3.16)

The last equation shows that the evolution of the lapse is directly related to the divergence of the time lines. We then see that just as the original BM slicing condition was such that the lapse reacted to the divergence of the normal observers, the modified condition ensures that the lapse reacts to the divergence of the coordinate time lines.

Let us assume for the moment that our spacetime has a future-pointing Killing field \( v^\mu \). In that case we will have

\[
\nabla_\mu v_\nu + \nabla_\nu v_\mu = 0 \implies \nabla_\mu v^\mu = 0 .
\]  

(3.17)

If we now assume that our time lines are oriented along the Killing direction, then equation (3.16) automatically implies that the lapse function will be time independent. This, of course, we already knew since it was our initial motivation for modifying the BM condition.

C. Generalized wave equation for the time function

In reference [1] it was shown that the standard BM slicing condition can be written as a generalized wave equation for a “time function” \( \phi \) in the following way

\[
(g^{\mu\nu} - a n^\mu n^\nu) \nabla_\mu \nabla_\nu \phi = 0 ,
\]  

(3.18)

with \( n^\mu \) the unit normal vector to the spatial hypersurfaces and \( a := 1/f(\alpha) - 1 \). The different members of the spacetime foliation can then be obtained as the level sets of the time function \( \phi \).

One can also construct such a foliation equation for the modified BM condition (3.8). The corresponding equation for the time function \( \phi \) turns out to be

\[
(g^{\mu\nu} - \frac{a}{\alpha} t^\mu n^\nu) \nabla_\mu \nabla_\nu \phi = \nabla_\mu \left( \frac{\beta^\mu}{\alpha^2} \right) .
\]  

(3.19)

By writing equation (3.19) in the standard 3+1 coordinate system adapted to the foliation, it is not difficult to show that it is in fact equivalent to the slicing condition (3.8). On the other hand, notice that when written in a different coordinate system, the vector \( t^\mu \) does not have to be aligned with the time lines any longer. Also, the shift vector \( \beta^\mu \) will have a non-zero time component. However, we will still have

\[
\beta^\mu n_\mu = 0 .
\]  

(3.20)

In the case of vanishing shift, equation (3.19) reduces to equation (3.18), which is just another way of saying that in that case the original and modified BM conditions coincide. For non-vanishing shift, however, both equations differ. The foliation equation for the original BM condition makes no reference to the lapse, which implies that the foliation is independent of our choice of shift (as already mentioned above). The modified foliation equation, however, clearly depends on the shift choice, so the foliation of spacetime one obtains will depend on the shift as well.

The foliation equation (3.19) is very useful when trying to understand the properties of the slicing condition in a covariant way that is independent of the Einstein field equations.

IV. HYPERBOLICITY

The concept of hyperbolicity is of fundamental importance in the study of the evolution equations associated with a Cauchy problem. Some measure of hyperbolicity, even in a weak sense, implies that the system of equations is causal, i.e. that the solution at a given point in spacetime depends only on data in a region of compact support to the past of that point (the characteristic cone). Stronger versions of hyperbolicity can also be used to prove rigorously that the system of equations is well-posed, that is, that its solutions exist (at least locally), are unique, and are stable in the sense that small changes in the initial data will correspond to small changes in the solution. Hyperbolicity also allows one to construct well-posed initial-boundary problems, which implies that one should be able to obtain well-behaved boundary conditions for numerical simulations with artificial boundaries.

Because of this, showing that a given system of evolution equations is hyperbolic has become an important test for new formulations of the 3+1 equations. In our case, since we are studying a slicing condition that is obtained through an evolution equation for the lapse, the question of the hyperbolicity of the gauge condition becomes important. Since we want to look at this issue in a way that is independent of the Einstein equations, we will consider from now on a given background spacetime (which may or may not obey Einstein’s equations), and study our slicing condition on this fixed background.

A. Prescribed shift vector

Since the foliation equation (3.19) involves the shift vector \( \beta^\mu \), in order to analyze its hyperbolicity we must say something about how the shift vector evolves in time. The simplest approach is to assume that we have a prescribed, i.e. non-dynamical, shift vector. However, this immediately leads us into the question: What does it mean to have a prescribed shift vector in a covariant sense?
Clearly, we can not just assume that the shift vector is an a priori known vector-field in spacetime. This is because the shift vector must always be parallel to the spatial hypersurfaces making up our foliation, and those hypersurfaces are precisely what we are trying to solve for. This means that inevitably, as we solve for the hypersurfaces, the shift vector must evolve dynamically to guarantee that it remains parallel to them. At most, we can ask for the magnitude and direction of the shift vector within a given hypersurface to be prescribed functions of space and time.

We will then propose the most general evolution equation for the shift that is compatible with the fact that the shift lives on the spatial hypersurfaces. For this we proceed as follows: On a given hypersurface we can choose a basis of spatial vectors $e^i$, and express the shift vector in terms of such a basis:

$$\beta^\mu = e^i \beta^i \ .$$

(4.1)

where the $\beta^i$ are the components of the shift vector in the basis under consideration. We will now identify this basis with the standard 3+1 spatial coordinate basis. This means that the basis $e^\mu_i$, together with the time vector $t^\mu$, form a coordinate basis for the spacetime at the point under study (they form the standard 3+1 coordinate basis), which implies the following commutation relation

$$t^\nu \partial_\nu e^\mu_i = e^\nu_i \partial_\nu t^\mu \ .$$

(4.2)

Using this relation, we can obtain the following very general evolution equation for the shift vector:

$$t^\nu \partial_\nu \beta^\mu = \beta^\nu \partial_\nu t^\mu + s^\mu \ ,$$

(4.3)

where $s^\mu = e^i \mu t^\nu \partial_\nu \beta^i$. Notice that the last equation is fully covariant even if it is written in terms of partial derivatives, as the Christoffel symbols cancel out. If we now assume that the components $b^i$ of the shift in the 3+1 spatial coordinate basis are prescribed functions of spacetime, then we can consider the $s^\mu$ as source terms in the hyperbolicity analysis.

We will use equation (4.3) above as our general evolution equation for the shift. Notice that in the particular case when we restrict ourselves to the 3+1 coordinate system, for which $t^\mu = (1, 0, 0, 0)$, this equation simply reduces to

$$\partial_t \beta^i = \partial_t b^i \ ,$$

(4.4)

which is of course to be expected.

**B. Hyperbolicity of the foliation equation**

We are interested in studying the hyperbolicity of the modified BM slicing condition in a way that is independent of the Einstein equation. In order to do this we assume we have a fixed background spacetime with metric $g_{\mu \nu}$, and study the foliation equation (3.19), which we repeat here for clarity:

$$\left(\frac{g^{\mu \nu} - \frac{a}{\alpha} t^\mu n^\nu}{\alpha} \right) \nabla_\mu \nabla_\nu \phi = \nabla_\mu \left(\frac{\beta^\mu}{\alpha^2} \right) \ .$$

(4.5)

Here $\phi$ is a scalar function whose level sets identify the elements of the foliation, and $a := 1/f(\alpha) - 1$. In terms of $\phi$, the lapse function and the unit normal vector can be expressed as

$$\alpha = (\nabla_\mu \phi \nabla^\mu \phi)^{-1/2} \ ,$$

(4.6)

$$n^\mu = -\alpha \nabla_\mu \phi \ .$$

(4.7)

We will now concentrate on a point on a given slice, and construct locally flat coordinates in its neighborhood. We will further define the following first order quantities

$$\partial_t \phi \equiv \Pi \ ,$$

(4.8)

$$\partial_i \phi \equiv \Psi_i \ .$$

(4.9)

The lapse and unit normal vector then become

$$\alpha = (\Pi^2 - \Psi^2)^{-1/2} \ ,$$

(4.10)

$$n^\mu = \alpha (\Pi, -\Psi^i) \ ,$$

(4.11)

with $\Psi^2 = \sum \Psi^2$. Notice that, as we are in locally flat coordinates, there is no difference between lower and upper spatial indexes, so we will be using them indiscriminately.

On the other hand, since the shift vector is parallel to the hypersurface we must have

$$\beta^\mu n_\mu = 0 \ ,$$

(4.12)

which allows us to express the $\beta^0$ component as

$$\beta^0 = -\frac{\beta^i \Psi_i}{\Pi} \ .$$

(4.13)

With the definitions above we can now rewrite the foliation equation (4.5) as:

$$- (P \alpha^2 - ((1 + a)\Pi^2 + \Psi^2) Q) \partial_t \Pi + \frac{\Psi_i}{\alpha^2 \Pi} \partial_i \beta^i + B^m \partial_m \Pi + C^{mj} \partial_m \Psi_j - \frac{1}{\alpha^2} \partial_m \beta^m = 0 \ ,$$

(4.14)

where dot stands for the partial time derivative, and where we have defined

$$P := (1 + a) \Pi^2 - \Psi^2 \ ,$$

(4.15)

$$Q := \frac{\beta^i \Psi_i}{\Pi^2} \ ,$$

(4.16)

$$B^m := - ((1 + a)\Pi^2 + \Psi^2) \frac{\beta^m}{\Pi} + (2a \alpha^2 - (2 + a) Q) \Pi \Psi^m \ ,$$

(4.17)

$$C^{mj} := \delta^{mj} - \Psi^m (a \alpha^2 \Psi^j - (2 + a) \beta^j) \ .$$

(4.18)
Notice that to arrive at Eq. (4.14) above we have used the fact that
\[ \partial_t \Psi_i = \partial_t \Pi, \quad (4.19) \]

We can also rewrite the evolution equation for the shift introduced in the last section, Eq. (4.3), in our locally flat coordinates to find
\[ 2\alpha^2 \Psi_i \beta^m \Psi \partial_t \Pi + \Pi \partial_t \beta^i - N^{im} \partial_m \Pi \]
\[ + L^{ijm} \partial_m \Psi_j - \Psi^m \partial_m \beta^i - \frac{s^i}{\alpha^2} = 0, \quad (4.20) \]
where now
\[ N^{im} := \Pi Q \delta^{im} + 2\alpha^2 \Psi_i (Q \Psi^m + \beta^m), \quad (4.21) \]
\[ L^{ijm} := (\delta^{ij} + 2\alpha^2 \Psi_i \Psi^j) \beta^m. \quad (4.22) \]

Equations (4.14) and (4.20) can now be used to find the following evolution equations for \( \Pi \), and the spatial part of the shift vector \( \beta^i \):
\[ \partial_t \Pi = \frac{1}{T_1} \left( T_2^{ijm} \partial_m \Pi \right. \]
\[ \left. + T_3^{ijm} \partial_m \Psi_j + T_4^{ijm} \partial_m \beta^j + F^0 \right), \quad (4.23) \]
\[ \partial_t \beta^i = D_1^{ijm} \partial_m \Pi + D_2^{ijm} \partial_m \Psi_j \]
\[ + D_3^{ijm} \partial_m \beta^j + F^i, \quad (4.24) \]
where
\[ T_1 = P(Q - \alpha^2), \quad (4.25) \]
\[ T_2^0 = P \frac{\beta^m - \Pi \Psi^m}{2\alpha^2 - \frac{P Q}{\Pi^2}}, \quad (4.26) \]
\[ T_3^0 = -\delta^{im} + a \Psi_i \left( \alpha^2 \Psi^m - \beta^m \right) - \frac{\beta^i \Psi^m}{\alpha^2 \Pi}, \quad (4.27) \]
\[ T_4^{ijm} = \frac{1}{\alpha^2} \left( \delta^{ijm} - \frac{\Psi^m}{\Pi^2} \right), \quad (4.28) \]
\[ F^0 = \Psi_i \frac{s^i}{\alpha^2 \Pi}, \quad (4.29) \]
\[ D_1^{ijm} = Q \left[ \delta^{im} + 2\alpha^2 \Psi_i \left( \Psi^m + \frac{\beta^m}{Q} - \frac{T_4^{jm}}{T_1} \right) \right], \quad (4.30) \]
\[ D_2^{ijm} = -\delta^{ij} \beta^m - \frac{2\alpha^2 \Psi^i}{Q} \left( \Psi_i \beta^m + \frac{Q \Pi^2 T_4^{jm}}{T_1} \right), \quad (4.31) \]
\[ D_3^{ijm} = \frac{1}{\Pi} \left( \delta^{ij} \Psi^m - 2\alpha^2 \Pi^2 Q \Psi_i T_4^{jm} \frac{1}{T_1} \right), \quad (4.32) \]
\[ F^i = \frac{1}{\Pi} \frac{s^i}{\alpha^2} - 2\alpha^2 \Pi^2 Q \Psi_i \frac{F^0}{T_1}. \quad (4.33) \]

Equations (4.23) and (4.24), together with Eq. (4.19), form our closed set of evolution equations. Notice that these equations are only valid if \( \alpha^2 \neq Q \), the case \( \alpha^2 = Q \) being degenerate. The reason for this can be traced back to the fact that if \( \alpha^2 = Q \), then \( \beta^i \Psi_i = (\alpha \Pi)^2 \), which in implies that \( t^0 = \alpha n^0 + \beta^0 = \alpha^2 \Pi - \beta^i \Psi_i / \Pi = 0 \). This means that the vector \( t^u \) has no time component and equation (4.3) stops being an evolution equation for the shift. That this is a purely coordinate problem can be seen from the fact that we can always boost our locally flat coordinates in such a way that \( \Psi_i \) becomes zero and the problem disappears.

In order to determine whether or not the system of evolution equations is hyperbolic, we first write it in matrix notation. We start by defining the vector \( u \) in the following way
\[ u = (\Pi, \Psi_x, \Psi_y, \Psi_z, \beta_x, \beta_y, \beta_z). \quad (4.34) \]
Thus the system of equations can be expressed as
\[ \partial_t u = M^x \partial_x u + M^y \partial_y u + M^z \partial_z u + s, \quad (4.35) \]
where the Jacobian matrices \( M^i \) and the source vector \( s \) depend on the \( u \)'s but not their derivatives.

The matrix \( M^x \) has the particular form
\[ M^x = \begin{pmatrix} T_2^x/T_1 & T_3^x/T_1 & T_4^x/T_1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ D_1^{ix} & D_2^{ix} & D_3^{ix} \end{pmatrix}, \quad (4.36) \]
with \( i, j = x, y, z \). The matrices \( M^y \) and \( M^z \) have similar structures.

Having written our system of equations in the form (4.35), we can now proceed to study its hyperbolicity properties. In order to do this one should first construct the “principal symbol” \( M^j n_i \), where \( n_i \) is an arbitrary unit vector. The system is then said to be strongly hyperbolic if the eigenvalues of the principal symbol are real and there is a complete set of eigenvectors for all \( n_i \); furthermore, the system is said to be symmetric hyperbolic if the principal symbol can be symmetrized in a way that is independent of the \( n_i \) [34]. If, on the other hand, the eigenvalues are real, but there is no complete set of eigenvectors, the system is only weakly hyperbolic. Here we will use a shortcut useful when all directions have equivalent structures: We will find the eigenvalues of one of the Jacobian matrices \( M^i \) which by inspection can later be generalized to any arbitrary direction. We will concentrate on the matrix \( M^x \), as results for the other two matrices can be obtained afterward in a straightforward way.

The eigenvalues of \( M^x \) can be found to be
\[ \lambda_{\pm} = \frac{\Pi}{P} (a \Psi_x \pm R), \quad (4.37) \]
\[ \lambda^{(3)} = \frac{\beta_x}{\Pi (Q - \alpha^2)}, \quad (4.38) \]
\[ \lambda^{(4)} = \lambda^{(5)} = \frac{\Psi_x}{\Pi}, \quad (4.39) \]
\[ \lambda^{(6)} = \lambda^{(7)} = 0, \quad (4.40) \]
with \( R = \sqrt{P - a \Psi_x^2 / \alpha \Pi} \).

As expected, two of the eigenvalues are zero due to the fact that two rows of the matrix are zero. Of the remaining eigenvalues, \( \lambda^3, \lambda^4 \) and \( \lambda^5 \) are clearly real, but \( \lambda^\pm \) involve square roots through \( R \) so they require a more careful analysis. However, by inspecting the expressions for \( \lambda^\pm \) it is easy to see that, though written in a different way, they are in fact identical to the eigenvalues found in Ref. [1] for the case of the standard BM slicing condition. Since in that reference it was shown that those eigenvalues are always complex for \( f(\alpha) < 0 \), always real for \( 0 < f(\alpha) \leq 1 \) and can always be made real for \( f(\alpha) > 1 \) by an adequate orientation of the coordinate system, we conclude that our system of equations of equations is hyperbolic, at least weakly, as long as \( f > 0 \).

The fact that two of the eigenvalues turn out to be identical to those found in the case of the standard BM slicing condition is surprising. One could have expected that, since the modified BM condition depends on the choice of shift, the shift vector should have affected these eigenvalues. The fact that it leaves these eigenvalues unchanged can be more easily understood in the 3+1 coordinate frame. We carry out this analysis in the Appendix.

Having found that our equations are at least weakly hyperbolic, we now want to show that they are in fact also strongly hyperbolic. For this we must see if we have a complete set of eigenvectors. The eigenvectors associated with the matrix \( \mathbf{M}^\perp \) are:

\[
\mathbf{e}_\pm = [\lambda_\pm, 1, 0, 0, (\Psi_x B_\pm \pm R) A_\pm, \\
\Psi_y B_\pm A_\pm, \Psi_z B_\pm A_\pm], \quad (4.41)
\]

\[
\mathbf{e}^{(3)} = \left[ \frac{1}{\alpha^2} \left( \Psi_x - \frac{\beta^x}{\alpha^2} \right), \prod \left( 1 - \frac{Q}{\alpha^2} \right), 0, 0, \\
\Pi \left( 1 - \frac{Q}{\alpha^2} \right) + \Psi_x W, \Psi_y W, \Psi_z W \right], \quad (4.42)
\]

\[
\mathbf{e}^{(4)} = [0, 0, 0, 0, \frac{\Psi_x \Psi_y}{\Pi^2 - \Psi_x^2}, 1, 0], \quad (4.43)
\]

\[
\mathbf{e}^{(5)} = [0, 0, 0, 0, \frac{\Psi_x \Psi_z}{\Pi^2 - \Psi_x^2}, 0, 1], \quad (4.44)
\]

\[
\mathbf{e}^{(6)} = [0, Z^1, 1, 0, Z^2, Z^3, Z^4], \quad (4.45)
\]

\[
\mathbf{e}^{(7)} = [0, Z^5, 0, 1, Z^6, Z^7, Z^8], \quad (4.46)
\]

where

\[
B_\pm = 2 \left( \frac{\Psi_x}{\Pi} \lambda_\pm - 1 \right), \quad (4.47)
\]

\[
A_\pm = \frac{\alpha^2 \Pi^2}{(\Pi^2 - \Psi_x^2)(P - a\Psi_x^2)} \left\{ (\alpha^2 - a\Psi_x^2)Q\Psi_x \\
- (P - 2a\Psi_x^2)\beta^x + (\beta^y \Psi_y + \beta^z \Psi_z) \frac{P \lambda_\pm}{\Pi} \right\}, \quad (4.48)
\]

\[
W = 2\Pi (\beta^x - Q\Psi_x). \quad (4.49)
\]

The functions \( Z^i \) are somewhat long expressions whose explicit form is not needed in what follows, so we will not write them here.

It is well known that those eigenvectors corresponding to different eigenvalues are linearly independent. The eigenvectors associated with the degenerate eigenvalues \( \lambda^4, \lambda^5 \) and \( \lambda^6, \lambda^7 \), can be seen to be independent from one another by inspection. Therefore, the eigenvectors given above form a complete set. This, together with the fact that the eigenvalues are real for \( f > 0 \) allows us to conclude that the system of evolution equations (4.19,4.23,4.24) is strongly hyperbolic if \( f > 0 \).

V. SINGULARITY AVOIDANCE

In reference [1] it was shown that the original BM slicing condition can avoid so-called “focusing singularities” [19] depending on the form that the function \( f(\alpha) \) takes in the limit of small \( \alpha \). In particular, it was shown that if \( f(\alpha) \) behaves as \( f = A\alpha^m \) for small \( \alpha \) and the normal volume elements vanish in terms of proper time \( \tau \) as \( \gamma_n^{1/2} \sim (\tau_\alpha - \tau)^m \), one can have three different types of behavior:

1. For \( n < 0 \) the lapse vanishes before the normal spatial volume elements do, which corresponds to strong singularity avoidance

2. For \( n = 0 \) and \( mA \geq 1 \) the lapse vanishes with the normal spatial volume elements and the singularity is reached after an infinite coordinate time, corresponding to marginal singularity avoidance

3. For both \( n > 0 \), and \( n = 0 \) with \( mA < 1 \), the lapse vanishes with the normal volume elements but the singularity is still reached in a finite coordinate time, so there is no singularity avoidance.

The results summarized above, however, depended crucially on the fact that the original BM slicing condition relates the evolution of the lapse to the evolution of the normal volume elements. The modified version of the BM condition, on the other hand, relates the evolution of the lapse to the evolution of the coordinate volume elements. We must therefore see how this affects the conclusions about singularity avoidance. The first thing to notice is that all the analysis of reference [1] will follow exactly in the same way for the modified BM condition if we replace normal volume elements with coordinate volume elements. This means that the modified BM condition will avoid singularities where the coordinate volume elements vanish (i.e. coordinate focusing singularities) under the same conditions as before.

There is one very important difference between “normal” focusing singularities and “coordinate” focusing singularities. When the normal volume elements vanish, the normal direction to the hypersurfaces becomes ill-defined and the hypersurfaces become non-smooth. When the coordinate volume elements vanish, on the other hand, it is only the time lines that cross. This means that one could in principle develop a coordinate focusing singularity on a perfectly smooth hypersurface, or worse still, one could develop a normal focusing singularity for which
the time lines do not cross and the coordinate volume elements remain non-zero. The second case would be extremely problematic as our hypersurfaces would become non-smooth but the lapse would not collapse in response to this.

To see under what conditions we can have one type of focusing singularity and not the other we must look at the evolution equations for the normal and coordinate volume elements:

\[
\begin{align*}
\partial_t \ln \gamma_{nn}^{1/2} &= -\alpha K, \\
\partial_t \ln \gamma_{\beta\beta}^{1/2} &= -(\alpha K - D_i \beta^i) .
\end{align*}
\]  

From the first of these equations it is clear that for a normal focusing singularity to develop ($\gamma_{nn}^{1/2} \to 0$) we must have $K \to \infty$. From the second equation we then see that the only way in which we can have a normal focusing singularity develop while at the same time keeping a non-zero coordinate volume element is for $D_i \beta^i$ to diverge with $K$ while keeping their difference finite.

We then conclude that if the divergence of the shift remains finite, both types of focusing singularities will happen at the same time. This means that if the shift vector remains regular, the modified BM slicing condition will avoid singularities exactly in the same way in which the original condition did.

VI. GAUGE SHOCKS

In Section IV we have shown that the system of equations (4.35) for the variables (4.34) is strongly hyperbolic. We can then define a complete set of “eigenfields” $\omega_i$ in the following way:

\[ u = R \omega \implies \omega = R^{-1} u \ , \]  

where $R$ is the matrix of column eigenvectors $e_i$.

We say that the eigenfield $\omega_i$ is “linearly degenerate” if its corresponding eigenvalue $\lambda_i$ is independent of the eigenfield, that is

\[ \frac{\partial \lambda_i}{\partial \omega_i} = \sum_{j=1}^{N_u} \frac{\partial \lambda_i}{\partial u_j} \frac{\partial u_j}{\partial \omega_i} = \nabla_u \lambda_i \cdot e_i = 0 \ . \]  

Linear degeneracy guarantees that the corresponding eigenfield will not develop shocks.

Using the eigenvalues and eigenvectors found in Sec. IV, the conditions for linear degeneracy become

\[ C_5 := e_5^{(5)} \frac{\partial \lambda_5^{(5)}}{\partial \beta^z} + \frac{\partial \lambda_5^{(5)}}{\partial \beta^z} = 0 \ , \]  

\[ C_6 := C_7 = 0 \ . \]  

A straightforward calculation shows that equations (6.4)-(6.6) are satisfied identically. On the other hand, equation (6.3) is precisely the same equation found for the standard BM slicing condition in reference [5], where it was shown that it leads to the following condition on the function $f(\alpha)$

\[ 1 - f - \frac{\alpha f'}{2} = 0 \ . \]  

This means that the modified BM slicing condition is linearly degenerate under the same circumstances as the standard BM condition, and therefore it will avoid gauge shocks exactly in the same cases.

VII. CONCLUSION

We have studied a modified version of the BM slicing condition that has to important features: 1) it guarantees that if a spacetime is static or stationary, and one starts the evolution in a coordinate system in which the metric coefficients are already time independent, then they will remain time independent during the subsequent evolution, and 2) the modified condition is naturally adapted to the use of a densitized lapse as a fundamental variable.

By analyzing this modified BM condition written in covariant form on an arbitrary background spacetime, we have also shown that it is strongly hyperbolic for $f(\alpha) > 0$, just as the original BM condition was. Moreover, we have found that the characteristic speeds of the original BM condition are not modified. Finally, we have shown that as long as the shift vector remains regular, the modified BM condition avoids both focusing singularities and gauge shocks under the same conditions as the original BM condition.

Because of these results we believe that the modified BM condition might be just as useful as the original BM condition for evolving strongly gravitating systems, while at the same time having the extra benefits described above. We plan to carry out numerical experiments to test this, but since the modified BM condition leads to a different slicing of spacetime for different choices of shift, such experiments will require first that one studies different shift conditions. We are currently working on this issue, and will report our findings in a future work.

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APPENDIX

While studying the hyperbolicity of the modified BM slicing condition in Section IV we found, surprisingly, that two of the eigenvalues associated with this modified condition are identical with the eigenvalues associated with the standard BM condition found in Ref. [1]. The analysis of Section IV was done in a covariant way, which required the introduction of an evolution equation for the shift that stated the fact that the shift has to be parallel to the hypersurfaces. The introduction of the shift equation made the analysis considerably more complicated and makes it difficult to see why the eigenvalues should remain equal. Here we will do a simple analysis in the 3+1 coordinate frame for the particular case of one spatial dimension to try to understand why the characteristic speeds are not modified.

Let us then consider first the standard BM slicing condition in one spatial dimension. The evolution equation for the lapse has the form

$$\partial_t \alpha = \beta \partial_x \alpha - \alpha^2 f K.$$  (A.1)

If we now introduce the first order quantity $A := \partial_x \alpha$, we can rewrite this equation as the system

$$\partial_t \alpha = \beta A - \alpha^2 f K,$$  (A.2)
$$\partial_t A = \partial_x (\beta A - \alpha^2 f K),$$  (A.3)

with $K$ the trace of the extrinsic curvature. On the other hand, from the ADM equations we find the following evolution equation for $K$

$$\partial_t K \simeq \partial_x (\beta K - A/\gamma),$$  (A.4)

where the symbol $\simeq$ denotes equal up to principal part, and where $\gamma$ is the one-dimensional coordinate volume element. Notice that we are considering the shift $\beta$ to be a prescribed function of space and time.

The Jacobian matrix associated with the evolution equations for $(A,K)$ is then

$$M = \begin{pmatrix} -\beta & \alpha^2 f \\ 1/\gamma & -\beta \end{pmatrix}. $$  (A.5)

Notice that the overall sign chosen here is the one obtained by moving the spatial derivatives to the right hand side of the equations, as this is the sign we need if we want to associate the eigenvalues with characteristic speeds. The associated eigenvalues are

$$\lambda_{\pm} = -\beta \pm \alpha (f/\gamma)^{1/2},$$  (A.6)

with corresponding eigenvectors

$$v_{\pm} = \left(\pm \alpha (f/\gamma)^{1/2}, 1\right).$$  (A.7)

We then see that evolution equations for the pair $(A,K)$ form a strongly hyperbolic system as long as $f > 0$.

Let us now consider the modified BM slicing condition. The evolution equation for the lapse now becomes

$$\partial_t \alpha = -\alpha f \left(\alpha K - \partial_x \beta - (\beta/2\gamma) \partial_x \gamma \right).$$  (A.8)

Notice how this now includes a spatial derivative of the coordinate volume element, which can not be considered as a source since by construction the time derivative of the lapse is proportional to the time derivative of the volume element. If we now define $G := \partial_x \gamma$, we can rewrite the evolution equations for $\alpha$ and $\gamma$ as the system

$$\partial_t \alpha = -\alpha^2 f K + \alpha f \beta/2\gamma \left(G + \alpha f \partial_x \beta \right),$$  (A.9)
$$\partial_t \gamma = -2\alpha \gamma K + \beta G + 2\gamma \partial_x \beta,$$  (A.10)
$$\partial_t A \simeq \partial_x \left(-\alpha^2 f K + \alpha f \beta/2\gamma \right) \left(G \right),$$  (A.11)
$$\partial_t G \simeq \partial_x (-2\alpha \gamma K + \beta G).$$  (A.12)

We now see that the Jacobian matrix associated with the system $(A,G,K)$ takes the form

$$M = \begin{pmatrix} 0 & -\alpha f \beta/2\gamma & \alpha^2 f \\ 0 & -\beta & 2\alpha \gamma \\ 1/\gamma & 0 & -\beta \end{pmatrix}. $$  (A.13)

The eigenvalues of this matrix are easily found to be

$$\lambda_0 = 0,$$  (A.14)
$$\lambda_{\pm} = -\beta \pm \alpha (f/\gamma)^{1/2},$$  (A.15)

with associated eigenvectors

$$v_0 = \left(\beta^2 \gamma, 2\alpha \gamma, \beta \right),$$  (A.16)
$$v_{\pm} = \left(\pm \alpha (f/\gamma)^{1/2}, \pm 2 \gamma^{3/2}/f^{1/2}, 1 \right).$$  (A.17)

We can again see that two of the eigenvalues corresponding to the modified BM slicing condition coincide with the eigenvalues of the standard BM condition. However, the reason for this is now much easier to see. Notice that the Jacobian matrix (A.13) has two rows that are multiples of each other (row two can be obtained from row one by multiplying with $2\gamma/\alpha f$). This means that we can define a new variable $\Sigma := G - (2\gamma/\alpha f) A$ that will evolve only through lower order terms. We can now make a change of variables from $(A,G,K)$ to $(A,\Sigma, K)$, by replacing $G$ with $\Sigma + (2\gamma/\alpha f) A$. Since $\Sigma$ evolves only through lower order terms, its derivatives can be treated as source terms in the hyperbolicity analysis. It is then easy to see that the Jacobian matrix for the reduced system $(A,K)$ is identical to the Jacobian matrix associated with the standard BM slicing condition given in Eq. (A.5).

The important observation is that even though the modified BM slicing condition replaces the Lie derivative of $\alpha$ with respect to the shift with the divergence of the shift, since this divergence includes the Lie derivative of $\gamma$, and the time derivative of $\alpha$ and $\gamma$ are multiples of each other, we recover in the end precisely the same characteristic speeds $-\beta \pm \alpha (f/\gamma)^{1/2}$. 
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