Correlators of Polynomial Processes

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Abstract

A process is polynomial if its extended generator maps any polynomial to a polynomial of equal or lower degree. Then its conditional moments can be calculated in closed form, up to the computation of the exponential of the so-called generator matrix. In this article, we provide an explicit formula to the problem of computing correlators, that is, computing the expected value of moments of the process at different time points along its path. The strength of our formula is that it only involves linear combinations of the exponential of the generator matrix, as in the one-dimensional case. The framework developed allows then for easy-to-implement solutions when it comes to financial pricing, such as for path-dependent options or in a stochastic volatility models context.

Keywords Stochastic process; Polynomial process; Generator matrix; Correlators; Eliminating and Duplicating matrix.

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1 Introduction

A jump-diffusion process is called polynomial if its extended generator maps any polynomial function to a polynomial function of equal or lower degree. As a consequence, the expectation of any polynomial of the future state of the process, conditioned on the information up to the current state, is given by a polynomial of the current state. This means that the conditional moments can be calculated in closed form, up to the computation of the exponential of the so-called generator matrix, which is nothing but the linear representation of the action of the extended generator on the basis vector of monomials. No knowledge of the probability distribution nor of the characteristic function of the process is required in order to calculate the moments. The class of polynomial processes includes exponential Lévy processes, affine processes and processes of Pearson diffusion type, with the Ornstein-Uhlenbeck processes as a canonical example. Some of the first time-homogeneous Markov jump-diffusion treatments in a polynomial context can be found in [7] and [9], while in [11] a mathematical analysis for polynomial diffusions is carried out. More recently, in [12], the authors have studied polynomial jump-diffusions in a semi-martingale context, inspired by the existence of non-Markovian polynomial jump-diffusions, as well as by the tractability of semi-martingale processes in practical applications.

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Because of their closed moments formula, polynomial processes find large application in finance and pricing. In particular, one of the first applications of polynomial diffusions in financial modelling is addressed to [13]. In the literature, we find examples on interest rates in [13] and in [10], examples on stochastic volatility models in [1], [3] and in [11]. In [14], for example, the authors derive one- and two-factor models for the spot price of electricity, showing with numerical experiments the power of this class of processes also in this applied field. We can also find examples on option pricing, such as in [12] and [2]. In [9] the properties of jump-diffusion processes are exploited to improve the performance of computational and statistical methods, such as the generalized method of moments, and for variance reduction techniques in Monte Carlo methods. Further examples cover the stochastic portfolio theory, such as in [8].

Given a stochastic basis \((\Omega, \mathcal{F}, \mathbb{P})\) with a filtration \(\{\mathcal{F}_t\}_{t \geq 0}\), for a jump-diffusion process \(Y\) we study conditional expectations of the following form:

\[
\mathbb{E}[p_m (Y(s_0)) p_{m-1} (Y(s_1)) \cdots p_0 (Y(s_m)) \mid \mathcal{F}_t].
\]  

(1.1)

Here, \(p_k\) is a polynomial function of degree \(n_k \in \mathbb{N}\), \(k = 0, \ldots, m\), and \(t < s_0 < s_1 < \cdots < s_m < T < \infty\). We refer to expectations such as the one in equation (1.1) as \((m + 1)\)-points correlators. We denote by \(n := \max \{n_0, \ldots, n_m\}\) the maximal degree for the \(m+1\) polynomials. Considering only one polynomial function, i.e. \(m = 0\), equation (1.1) corresponds to what was discussed above, thus the formula for this particular case is already provided and it is what makes polynomial processes applicable. We recall this result in Theorem 2.3.

However, when considering \(m > 0\), calculations become much more challenging. In particular, the key in order to get the one-polynomial \((m = 0)\) moments formula lies in the existence of the so-called generator matrix, that we state in Theorem 2.2. Such a matrix can be seen as the linear representation of the action of the extended generator on the basis vector of monomials defined by \(H_n(x) := (1, x, x^2, \ldots, x^n)^\top\). More precisely, if \(\mathcal{G}\) denotes the extended generator and \(G_n\) is the generator matrix for a fixed \(n\), then \(G_n\) is the matrix satisfying the equality \(\mathcal{G}H_n(x) = G_nH_n(x)\). However, already with two polynomials, i.e. \(m = 1\), we encounter in equation (1.1) a first problem. Each polynomial can indeed be represented in terms of the basis vector \(H_n(x)\). Then, taking the product of the two polynomials we get an object of the form \(H_n(x)H_n(x)^\top\), which can also be expressed using the Kronecker product (see Definition 3.3) as \(H_n(x)^\top \otimes H_n(x)\), where this second formulation is crucial when it comes to generalizing our results to \(m \geq 1\). The difference between the case \(m = 0\) and the \(m = 1\) case, is that now we must deal with a matrix, namely \(H_n(x)H_n(x)^\top\), instead of the vector \(H_n(x)\). It can be proved by a simple counter-example (see Example 3.1) that we cannot construct a generator matrix for \(H_n(x)H_n(x)^\top\). The solution that we propose is to consider the vectorization of \(H_n(x)H_n(x)^\top\) instead, namely the column vector obtained by stacking the columns of the matrix \(H_n(x)H_n(x)^\top\) into a vector (see Definition 3.1), leading to a framework similar to what encountered for the one-polynomial case.

In the last sentence, we intentionally used the word similar. Indeed, once considered the vectorization of \(H_n(x)H_n(x)^\top\), the new generator matrix can in principle be constructed. However, the matrix \(H_n(x)H_n(x)^\top\), and hence its vectorization, contains redundant terms, namely repeating powers of \(x\), which implies that the new generator matrix contains equal rows and/or zero columns (see Example 3.2). We resolve this issue by introducing two linear operators, the first of which we call the L-eliminating matrix. This matrix eliminates from the vectorization of \(H_n(x)H_n(x)^\top\) all the redundant powers of \(x\). It is not difficult to notice that this operator leaves us with a vector that is nothing but \(H_{2n}(x)\), for which there exists the generator matrix \(G_{2n}\). Using the inverse operator, called the L-duplicating matrix, we then recover the full-
dimensional vector, and, finally, via inverse-vectorization we get the linear operator required, which allows to find the moments formula for \( m = 1 \) (stated in Theorem 3.14). In particular, we summarize the steps in the following graph:

These steps work also when increasing further the number of polynomials in equation (1.1). For \( m + 1 > 1 \) polynomial functions, we must deal with \( m + 1 > 1 \) basis vectors \( H_n(x) \). Thanks to some manipulations, we are led to study an object of the form \((H_n(x)^{\otimes m})^\top \otimes H_n(x)\), which is still a matrix (here \( H_n(x)^{\otimes m} \) denotes the \( m \)-th power of \( H_n(x) \) in the Kronecker sense). However, its structure is more complex and it is analysed in details in Section 4. In particular, it requires the appropriate eliminating and duplicating matrices, for which we prove a recursion formula in the number of polynomials \( m \geq 1 \). Then, the solution to the general case can be stated in a similar way to what was done for the two-polynomials case \( (m = 1) \).

The strength of our developed framework lies in two main facts. First of all, the L-eliminating and L-duplicating matrices are highly sparse. This means that, even if the dimensions of the problem get large, the computational cost remains low. Secondly, we are now able to provide an explicit formula to equation (1.1) which involves only linear combinations of the matrix exponential of the original generator matrix \( G_n \). More precisely, for \( m + 1 \) polynomials, we need \( m + 1 \) matrix exponentials of the generator matrices \( G_{n(r+1)} \), for \( r = 0, \ldots, m \). Up to our knowledge, no rigorous study has been developed yet concerning the generator matrix and its exponential. In order to facilitate numerical applications, we then provide a first recursion formula which allows to compute the generator matrix of order \( n + 1 \), given the generator matrix of order \( n \), and a second recursion formula, consequence of the first one, for the matrix exponential, which holds under some conditions on the eigenvalues of the generator matrix. Thanks to these two recursion formulas and to the high sparsity of the eliminating and duplicating matrices, the framework developed allows for easy-to-implement solutions when it comes to numerical applications.

The rest of the paper is organized as follows: in Section 1.1 we give some financial motivations for studying expectations such as the one in equation (1.1) and we clarify the name correlators. In Section 2 we give the definition of polynomial processes and introduce the concept of generator matrix. In Section 3 we solve the problem for the two-points correlators, presenting the main tools and framework which will allow to solve the more general problem for the \((m + 1)\)-points correlators, in Section 4. Finally, in Section 5 we provide the above mentioned recursion for the generator matrix itself and the one for its matrix exponential.

### 1.1 Motivation and background

In this section we introduce the concept of correlators, as well as two possible applications of our theory, the pricing of Asian options, and the pricing under stochastic volatility models. The intent is to motivate our analysis, leaving details aside for future work.

In [5, Section 9.3] the authors define the concept of correlators, a standard tool in turbulence
theory. For \( t < s_0 < s_1 < T < \infty \) and \( k_0, k_1 \in \mathbb{N} \), the correlator of order \((k_0, k_1)\) between \( Y(s_0) \) and \( Y(s_1) \) is defined by

\[
\text{Corr}_{k_0, k_1}(s_0, s_1; t) = \frac{\mathbb{E} \left[ Y(s_0)^{k_0} Y(s_1)^{k_1} \mid \mathcal{F}_t \right]}{\mathbb{E} \left[ Y(s_0)^{k_0} \mid \mathcal{F}_t \right] \mathbb{E} \left[ Y(s_1)^{k_1} \mid \mathcal{F}_t \right]}.
\]

(1.2)

Equation (1.2) represents a generalization of the autocorrelation, which can be recovered for \((k_0, k_1) = (1, 1)\). In particular, in order to compute the numerator in equation (1.2), we need the conditional expectation of the product between two powers of the process \( Y \) evaluated in two different time points. Exploiting the properties of polynomial processes, we will construct the framework and the tools in order to provide a closed formula to such a problem, which can be also extended looking, for \( m \geq 0 \), at expectations of the form

\[
\mathbb{E} \left[ Y(s_0)^{k_0} Y(s_1)^{k_1} \cdots Y(s_m)^{k_m} \mid \mathcal{F}_t \right],
\]

(1.3)

\( Y \) being a polynomial process, \((k_0, k_1, \ldots, k_m) \top \in \mathbb{N}^{m+1} \) and \( t < s_0 < s_1 < \cdots < s_m < T < \infty \). It can be noticed that equation (1.3) is a particular instance of equation (1.1), obtained considering the polynomial functions \( p_j \) to be monomials, namely \( p_j(x) = x^{k_j} \), for \( j = 0, \ldots, m \).

In this article we extend the definition of correlators introduced in [5, Section 9.3], using it to indicate any expectation of the form as in equation (1.1).

Path-dependent options

Let us consider a path-dependent option, such as an Asian option. Path-dependent means that the entire path of the price process within the settlement period for the option, say \([t, T]\), is taken into account by the pay-off function, and hence it must be considered for the pricing.

We refer to [16] for more details. In particular, if \( Y \) is the risk-neutral price dynamics of the underlying asset and \( \varphi \) is the pay-off function, we can define the price at time \( t \) for an Asian option as given by the following conditional expectation:

\[
\Pi(t) = \mathbb{E} \left[ \varphi \left( \int_t^T Y(s) \, ds \right) \mid \mathcal{F}_t \right].
\]

(1.4)

For \( \varphi \) a real-valued continuous function on a bounded interval, we can consider \( \hat{\varphi} \) as the polynomial approximation of the pay-off function \( \varphi \). Different choices can be considered, such as the Hermite polynomials, the Bernstein polynomials or a Taylor expansion among others, the choice depending on the nature of the function \( \varphi \).

If one then considers a discrete sampling for the time interval \([t, T]\) of the form \( t = s_0 < s_1 < \cdots < s_m = T \), the integral \( \int_t^T Y(s) \, ds \) can be reasonably well approximated with the sum \( \sum_{s=s_0}^{s_m} Y(s) \), when the time increments are small. In such a framework, the price for the Asian option in equation (1.4) can be found by

\[
\hat{\Pi}(t) \approx \mathbb{E} \left[ \hat{\varphi} \left( \sum_{s=s_0}^{s_m} Y(s) \right) \mid \mathcal{F}_t \right].
\]

(1.5)

It is easy to notice that equation (1.5) leads to linear combinations of expectations of the form such as the one in equation (1.3). We also remark that the discretized Asian option pay-off constitutes a Bermudan option.
Stochastic volatility models

Let us consider now for $0 \leq t \leq T$ the process $X$ defined by $X(T) = \int_t^T \sigma(s) dB(s)$, with $B$ being a standard Brownian motion and $\sigma$ a volatility process, which we assume to be independent from $B$. If $\varphi$ is the pay-off function, we want to price a financial derivative like follows:

$$\Pi(t) = \mathbb{E} [\varphi(X(T)) | \mathcal{F}_t].$$ (1.6)

One possible approach, as suggested in [3], is to consider the Fourier transform $\hat{\varphi}$ of $\varphi$. Then, under appropriate integrability conditions on the pay-off function $\varphi$, we can write that $\varphi(x) = \int_{-\infty}^{\infty} \hat{\varphi}(z) e^{2\pi i x z} \, dz$, and equation (1.10) becomes

$$\Pi(t) = \mathbb{E} \left[ \int_{-\infty}^{\infty} \hat{\varphi}(z) e^{2\pi i X(T) z} \, dz \bigg| \mathcal{F}_t \right].$$ (1.7)

For $\sigma$ and $B$ independent, by means of the tower rule, we can condition in equation (1.7) with respect to the filtration $\mathcal{F}_t$ generated by the volatility $\sigma$ up to time $T$, namely

$$\Pi(t) = \mathbb{E} \left[ \mathbb{E} \left[ \int_{-\infty}^{\infty} \hat{\varphi}(z) e^{2\pi i X(T) z} \, dz \bigg| \mathcal{F}_T \right] \bigg| \mathcal{F}_t \right] = \mathbb{E} \left[ \int_{-\infty}^{\infty} \hat{\varphi}(z) \mathbb{E} \left[ e^{2\pi i X(T) z} \bigg| \mathcal{F}_T \right] \, dz \bigg| \mathcal{F}_t \right],$$

where the last equality is due to the fact that, knowing the entire path of the volatility given by $\mathcal{F}_T$, the process $X(T)$ has a Gaussian distribution with mean 0 and variance $\int_t^T \sigma^2(s) \, ds$. Thus, in order to price $\Pi(t)$, we must find an expectation of the form

$$\mathbb{E} \left[ e^{\lambda \int_t^T \sigma^2(s) \, ds} \bigg| \mathcal{F}_t \right],$$ (1.8)

for $\lambda \leq 0$. If we now consider the Taylor expansion for the exponential function, equation (1.8) becomes

$$\mathbb{E} \left[ e^{\lambda \int_t^T \sigma^2(s) \, ds} \bigg| \mathcal{F}_t \right] = \mathbb{E} \left[ \sum_{k=0}^{\infty} \frac{1}{k!} \left( \int_t^T \sigma^2(s) \, ds \right)^k \bigg| \mathcal{F}_t \right] = \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} \mathbb{E} \left[ \left( \int_t^T \sigma^2(s) \, ds \right)^k \bigg| \mathcal{F}_t \right],$$ (1.9)

that is, we need to find the moments of the integrated volatility, $\int_t^T \sigma^2(s) \, ds$. For $Y(s) := \sigma^2(s)$, it can be proved, using iteratively the fundamental theorem of calculus, that, for every $k \geq 1$, the $k$-th power of $\int_t^T \sigma^2(s) \, ds$ can be rewritten in terms of a $k$-th order integral, namely

$$\left( \int_t^T \sigma^2(s) \, ds \right)^k = \left( \int_t^T Y(s) \, ds \right)^k = k! \int_t^T \int_t^{s_1} \cdots \int_t^{s_{k-1}} Y(s_1) Y(s_2) \cdots Y(s_k) \, ds_1 \cdots ds_k,$$ (1.10)

where $t \leq s_1 \leq s_2 \leq \cdots \leq s_k \leq T$ is a partition of the interval $[t, T]$. Combining equations (1.9) and (1.10), we get that

$$\mathbb{E} \left[ e^{\lambda \int_t^T Y(s) \, ds} \bigg| \mathcal{F}_t \right] = \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} \int_t^T \int_t^{s_1} \cdots \int_t^{s_{k-1}} \mathbb{E} \left[ Y(s_1) Y(s_2) \cdots Y(s_k) \bigg| \mathcal{F}_t \right] \, ds_1 \cdots ds_k,$$

so that for every $k \geq 1$ we need to study expectations of the form $\mathbb{E} \left[ Y(s_1) Y(s_2) \cdots Y(s_k) \bigg| \mathcal{F}_t \right]$. Considering $Y$ a polynomial jump-diffusion process, we will provide an explicit formula for
these expectations, in terms of the the generator matrix associated to the polynomial process.

2 Polynomial processes

A polynomial on \( \mathbb{R} \) is a function \( p : \mathbb{R} \to \mathbb{R} \) of the form \( p(x) = \sum_{k=0}^{n} p_k x^k \) for some real coefficients \( p_k \in \mathbb{R} \), \( k = 0, \ldots, n \). For \( p_n \neq 0 \), we say that \( n \) is the degree of the polynomial \( p \). We denote with \( \text{Pol}_n(\mathbb{R}) \) the space of all polynomials of degree less or equal than \( n \) on \( \mathbb{R} \). Then \( \{1, x, x^2, \ldots, x^n\} \) forms a basis for \( \text{Pol}_n(\mathbb{R}) \), and we can introduce the vector valued function

\[
H_n : \mathbb{R} \to \mathbb{R}^{n+1}, \quad H_n(x) = (1, x, x^2, \ldots, x^n)^	op,
\]

with \( \top \) the transpose operator, so that every polynomial function \( p \in \text{Pol}_n(\mathbb{R}) \) with vector of coordinates \( \vec{p}_n = (p_0, p_1, \ldots, p_n)^	op \in \mathbb{R}^{n+1} \) can be represented by

\[
p(x) = \vec{p}_n H_n(x) = H_n(x)^	op \vec{p}_n. \tag{2.2}
\]

We consider now a standard one-dimensional Brownian motion \( B \) and a compensated Poisson random measure \( \tilde{N}(dt, dz) \) with compensator \( \ell(dz)dt \), and we introduce the jump-diffusion stochastic differential equation (SDE) of the form

\[
dY(t) = b(Y(t))dt + \sigma(Y(t))dB(t) + \int_{\mathbb{R}} \delta(Y(t^-), z)\tilde{N}(dt, dz), \tag{2.3}
\]

for some measurable maps \( b : \mathbb{R} \to \mathbb{R} \), \( \sigma : \mathbb{R} \to \mathbb{R} \) and \( \delta : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \), such that equation (2.3) has a unique strong solution which we assume to be càdlàg, see [4] for more details. Here \( Y(t^-) \) denotes the left-limit defined by \( Y(t^-) := \lim_{s \uparrow t^-} Y(s) \).

Starting from a jump-diffusion process \( Y \) such as the one in equation (2.3), in accordance with [1], we introduce the following definition for the extended generator associated to \( Y \), that will be used to characterize polynomial processes:

**Definition 2.1** (Extended generator). An operator \( \mathcal{G} \) with domain \( D_\mathcal{G} \) is called the *extended generator* for a process \( Y \), if \( D_\mathcal{G} \) consists of those Borel measurable functions \( f : \mathbb{R} \to \mathbb{R} \) for which there exists a function \( \mathcal{G}f \) such that the process

\[
f(Y(t)) - f(y) - \int_0^t \mathcal{G}f(Y(s))ds
\]

is a \((\mathcal{F}_t, \mathbb{P}_y)\)-local martingale for every \( y \in \mathbb{R} \).

In particular, for every bounded function \( f \in C^2(\mathbb{R}) \), namely, with continuous second derivative, we can associate to the jump-diffusion process \( Y \) defined by equation (2.3) its extended generator, that is given by

\[
\mathcal{G}f(x) = b(x)f'(x) + \frac{1}{2} \sigma^2(x)f''(x) + \int_{\mathbb{R}} \left( f(x + \delta(x, z)) - f(x) - f'(x)\delta(x, z) \right) \ell(dz), \tag{2.4}
\]

and we are then able to give the following definition of a polynomial process:

**Definition 2.2** (Polynomial process). We call the stochastic process \( Y \) an *m-polynomial process* if for every \( k \in \{0, \ldots, m\} \), the generator \( \mathcal{G} \) introduced in equation (2.4) maps \( \text{Pol}_k(\mathbb{R}) \) into itself. If \( Y \) is \( m \)-polynomial for every \( m \geq 0 \), then it is called a *polynomial process*.

From [12], we can also give a characterization of the dynamics of the process \( Y \) in terms of its extended generator. Indeed, in order \( \mathcal{G} \) to be polynomial, the following conditions must
The result stated in Theorem 2.1 is one of the reasons why polynomial processes find large application in finance and option pricing. Indeed, it means that all increments of $f(Y(t)) - f(y) - \int_y^t G f(Y(s)) \, ds$ have vanishing expectation, i.e. for every $u < t$

$$\mathbb{E}_y \left[ f(Y(t)) - f(Y(u)) - \int_u^t G f(Y(s)) \, ds \right] = 0,$$

and this is, in particular, the main ingredient to prove the moments formula.

### 2.1 The generator matrix

We state now the following results from [12] Theorem 2.5 for a polynomial process $Y$. Even if the results are stated and proved in [12], we decided to include the proofs, as they will be useful in our analysis in Section 3 when increasing the order $m$, namely, the number of polynomial functions considered.

**Theorem 2.2.** For every $n \geq 1$, given $Y$ polynomial process with extended generator $G$ as defined in equation (2.4), there exists a matrix $G_n \in \mathbb{R}^{(n+1) \times (n+1)}$ such that

$$G H_n(x) = G_n H_n(x).$$

We call the matrix $G_n$ the generator matrix associated to the process $Y$.

**Proof.** Let us consider $f(x) = x^k$ for $0 \leq k \leq n$ in equation (2.4). Since $Y$ is a polynomial process, there exists a vector $\vec{q}_k \in \mathbb{R}^{k+1}$ such that $G x^k = \vec{q}_k^T H_n(x)$, or, more generally, a vector $\vec{q}_k \in \mathbb{R}^{n+1}$ such that $G x^k = \vec{q}_k^T H_n(x)$. As this holds for every $0 \leq k \leq n$, we can then construct a matrix $G_n \in \mathbb{R}^{(n+1) \times (n+1)}$ such that $G H_n(x) = G_n H_n(x)$. \qed

**Theorem 2.3.** Given $Y$ polynomial process and $p \in \text{Pol}_n(\mathbb{R})$, for $0 \leq t \leq T$, the moment formula holds

$$\mathbb{E} \left[ p(Y(T)) \mid \mathcal{F}_t \right] = \bar{p}_n^T e^{G_n(T-t)} H_n(Y(t)),$$

where $\bar{p}_n \in \mathbb{R}^{n+1}$ being a vector of coefficients as in equation (2.2), while the matrix $G_n \in \mathbb{R}^{(n+1) \times (n+1)}$ is implicitly defined by equation (2.6).

**Proof.** By Theorem 2.1 for $p \in \text{Pol}_n(\mathbb{R})$, we can write that

$$\mathbb{E}[p(Y(T)) \mid \mathcal{F}_t] = p(Y(t)) + \int_t^T \mathbb{E}[G p(Y(s)) \mid \mathcal{F}_t] \, ds.$$

(2.7)
Moreover, by means of Theorem 2.2 for $p(Y(T)) = \overline{\mathcal{G}}_n H_n(Y(T))$, we can focus on the vector function $H_n(Y(T))$ and equation (2.7) becomes

$$
\mathbb{E}[H_n(Y(T)) \mid \mathcal{F}_t] = H_n(Y(t)) + G_n \int_t^T \mathbb{E}[H_n(Y(s)) \mid \mathcal{F}_t] ds.
$$

Let us now introduce $Z(s) := \mathbb{E}[H_n(Y(s)) \mid \mathcal{F}_1]$; then equation (2.8) can be written in differential form as $dZ(s) = G_n Z(s) ds$, whose solution, by separation of variables, takes the form $Z(T) = e^{G_n(T-t)} Z(t)$. Going back to the definition of $Z$ and combining the result with equation (2.2), we get the statement of the theorem.

Theorem 2.3 tells us that $\mathbb{E}[p(Y(T)) \mid \mathcal{F}_1]$ is a polynomial function in $Y(t)$ for every $p \in \text{Pol}_n(\mathbb{R})$. In particular, it means that the conditional moments of a polynomial process can be found in closed form up to the exponential of the generator matrix $G_n$. We conclude this section with some examples for the generator matrix as introduced in Theorem 2.2.

**Example 2.1.** Let $n = 1$ in equation (2.6). Then $H_1(x) = (1, x)^\top$, and $\mathcal{G} H_1(x) = (\mathcal{G} 1, \mathcal{G} x)^\top$. In particular, from the definition of the extended generator $\mathcal{G}$ in equation (2.4), together with the conditions listed in equation (2.5), we get that $\mathcal{G} 1 = 0$ and $\mathcal{G} x = b_0 + b_1 x$. We can then easily see that the generator matrix $G_1 \in \mathbb{R}^{2 \times 2}$ satisfying (2.6) is of the following form:

$$
G_1 = \begin{pmatrix}
0 & 0 \\
0 & 0
\end{pmatrix}.
$$

**Example 2.2.** Let now $n = 2$. Then the basis vector to be considered is $H_2(x) = (1, x, x^2)^\top$, and when applying the extended generator to it, we get $\mathcal{G} H_2(x) = (\mathcal{G} 1, \mathcal{G} x, \mathcal{G} x^2)^\top = (\mathcal{G} 1, \mathcal{G} x, \mathcal{G} x^2)^\top$. In particular, we need to find $\mathcal{G} x^2$, that, by means of equations (2.4) and (2.5), is given by

$$
\mathcal{G} x^2 = \left( \sigma_0 + \int_\mathbb{R} \delta^2_0(z) \ell(dz) \right) + \left( \sigma_1 + 2 b_0 + 2 \int_\mathbb{R} \delta_0(z) \delta_1(z) \ell(dz) \right) x + \left( \sigma_2 + 2 b_1 + \int_\mathbb{R} \delta^2_1(z) \ell(dz) \right) x^2.
$$

Then, one easily finds that the generator matrix $G_2 \in \mathbb{R}^{3 \times 3}$ satisfying (2.3) is of the following form:

$$
G_2 = \begin{pmatrix}
0 & 0 & 0 \\
0 & b_0 & b_1 \\
\sigma_0 + \int_\mathbb{R} \delta^2_0(z) \ell(dz) & \sigma_1 + 2 b_0 + 2 \int_\mathbb{R} \delta_0(z) \delta_1(z) \ell(dz) & \sigma_2 + 2 b_1 + \int_\mathbb{R} \delta^2_1(z) \ell(dz)
\end{pmatrix}.
$$

In particular, we notice that the matrix $G_2$ contains $G_1$ in the left-upper corner, and we can prove that this gives us a recursion in order to construct the matrix $G_n$ given $G_{n-1}$, for $n \geq 2$. This will be studied in details in Section 3.

### 3 Two-points correlators

Aiming at solving the $(m+1)$-points correlators problem stated in equation (1.11), we start the analysis for $m = 1$, that is the case with two polynomials, $p_0 \in \text{Pol}_{n_0}(\mathbb{R})$ and $p_1 \in \text{Pol}_{n_1}(\mathbb{R})$, because the steps and ideas developed to solve the $m = 1$ case are crucial in order to understand the framework that will be introduced for a general number of polynomials, $m + 1$.

In particular, in this case, equation (1.11) reads like

$$
\mathbb{E}[p_1(Y(s_0)) p_0(Y(s_1)) \mid \mathcal{F}_1],
$$

8
and, for \( n := \max(n_0, n_1) \), we then start with the following result:

**Proposition 3.1.** For \( Y \) polynomial process, the conditional expectation of the product of two polynomial functions, \( p_0 \) and \( p_1 \), in, respectively, \( Y(s_0) \) and \( Y(s_1) \), for \( t < s_0 < s_1 \), is

\[
\mathbb{E}[p_1(Y(s_0))p_0(Y(s_1)) \mid \mathcal{F}_t] = \tilde{p}_{1,n}^T\mathbb{E}
\begin{bmatrix}
H_n(Y(s_0))H_n(Y(s_1))
\end{bmatrix}^T \mid \mathcal{F}_t \ e^{G_n^+(s_1-s_0)}\tilde{p}_{0,n}, \tag{3.1}
\]

where the last equality is obtained by simple transposition. Plugging this result into equation (3.2), gives us

\[
\tilde{p}_{0,n}, \tilde{p}_{1,n} \in \mathbb{R}^{n+1} \text{ being the vectors of coefficients referred to the polynomial function } p_0, \text{ respectively}, p_1, \text{ and } H_n \text{ the basis vector defined in equation (2.1)}.
\]

**Proof.** From equation (2.2), we can represent the two polynomial functions \( p_0 \) and \( p_1 \) by means of the basis function, namely \( p_0(x) = \tilde{p}_{0,n}^T H_n(x) \) and \( p_1(x) = \tilde{p}_{1,n}^T H_n(x) \). Then the left hand side of equation (3.1) becomes:

\[
\mathbb{E}[p_1(Y(s_0))p_0(Y(s_1)) | \mathcal{F}_t] = \mathbb{E}
\begin{bmatrix}
\tilde{p}_{1,n}^T H_n(Y(s_0))\tilde{p}_{0,n}^T H_n(Y(s_1))
\end{bmatrix} \mid \mathcal{F}_t.
\]

Since \( \mathcal{F}_t \subseteq \mathcal{F}_{s_0} \), for \( t < s_0 \), we can apply the tower rule to obtain

\[
\mathbb{E}[\tilde{p}_{1,n}^T H_n(Y(s_0))\tilde{p}_{0,n}^T H_n(Y(s_1)) | \mathcal{F}_t] = \mathbb{E}[\tilde{p}_{1,n}^T H_n(Y(s_0)) | \mathcal{F}_s_0] \ E[\tilde{p}_{0,n}^T H_n(Y(s_1)) | \mathcal{F}_s_0] | \mathcal{F}_t. \tag{3.2}
\]

Moreover, from Theorem 2.3, we know that

\[
\mathbb{E}[\tilde{p}_{0,n}^T H_n(Y(s_1)) | \mathcal{F}_{s_0}] = \tilde{p}_{0,n}^T e^{G_n^+(s_1-s_0)} H_n(Y(s_0)) = H_n(Y(s_0))^T e^{G_n^+(s_1-s_0)} \tilde{p}_{0,n},
\]

where the last equality is obtained by simple transposition. Plugging this result into equation (3.2), gives us

\[
\mathbb{E}[\tilde{p}_{1,n}^T H_n(Y(s_0))\tilde{p}_{0,n}^T H_n(Y(s_1)) | \mathcal{F}_t] = \tilde{p}_{1,n}^T \mathbb{E}
\begin{bmatrix}
H_n(Y(s_0))H_n(Y(s_0))^T
\end{bmatrix} \mid \mathcal{F}_t \ e^{G_n^+(s_1-s_0)}\tilde{p}_{0,n},
\]

which concludes the proof. \( \square \)

Proposition 3.1 tells us that the conditional expectation of the product of two polynomial functions reduces to the conditional expectation of the outer product of the basis function \( H_n \) with itself, namely \( \mathbb{E}[H_n(Y(s_0))H_n(Y(s_0))^T | \mathcal{F}_t] \). In particular, the product \( H_n(x)H_n(x)^T \in \mathbb{R}^{(n+1) \times (n+1)} \) is a matrix of the form

\[
H_n(x)H_n(x)^T =
\begin{pmatrix}
1 & x & x^2 & \ldots & x^n \\
0 & x & x^2 & \ldots & x^{n+1} \\
0 & 0 & x^2 & \ldots & x^{n+2} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \ldots & x^{2n}
\end{pmatrix}. \tag{3.3}
\]

In order to calculate the expectation \( \mathbb{E}[H_n(Y(s_0))H_n(Y(s_0))^T | \mathcal{F}_t] \), we can notice that, by means of Theorem 2.4, it satisfies the following identity:

\[
\mathbb{E}
\begin{bmatrix}
H_n(Y(s_0))H_n(Y(s_0))^T
\end{bmatrix} | \mathcal{F}_t
\]

\[
= H_n(Y(t))H_n(Y(t))^T + \int_t^{s_0} \mathbb{E}
\begin{bmatrix}
G(H_n(Y(s))H_n(Y(s))^T)
\end{bmatrix} | \mathcal{F}_s \ ds. \tag{3.4}
\]
Then, in the spirit of the proof of Theorem 2.3 we seek a linear operator $G^{(1)}_n : \mathbb{R}^{(n+1) \times (n+1)} \rightarrow \mathbb{R}^{(n+1) \times (n+1)}$ such that

$$G \left( H_n(x)H_n(x)^\top \right) = G^{(1)}_n \left( H_n(x)H_n(x)^\top \right),$$

that should be the equivalent linear operator in the two-dimensional setting ($m = 1$) to the generator matrix $G_n$ defined in Section 2.1. However, the following example shows us that $G_n^{(1)}$ can not be represented by a matrix:

**Example 3.1.** The linear operator $G_n^{(1)}$ satisfying equation (3.5), does not belong to the space of matrices in $\mathbb{R}^{(n+1) \times (n+1)}$. In order to see that, we can, as example, consider $n = 1$. Then we are looking for $G_1^{(1)} \in \mathbb{R}^{2 \times 2}$ such that $G \left( H_1(x)H_1(x)^\top \right) = G_1^{(1)} \left( H_1(x)H_1(x)^\top \right)$, that is $G_1^{(1)} = \begin{pmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{pmatrix}$, for some $h_{11}, h_{12}, h_{21}, h_{22} \in \mathbb{R}$, such that

$$\begin{pmatrix} G_1 & Gx \\ Gx & Gx^2 \end{pmatrix} = \begin{pmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{pmatrix} \begin{pmatrix} 1 & x \\ x & x^2 \end{pmatrix} = \begin{pmatrix} h_{11} + h_{12}x & h_{11}x + h_{12}x^2 \\ h_{21} + h_{22}x & h_{21}x + h_{22}x^2 \end{pmatrix}.$$

For $G_1, Gx, Gx^2$ found in Example 2.1 and Example 2.2 we see that there is no solution to such a problem. Indeed, for example, $Gx^2$ is of the form $Gx^2 = a_2^2 + a_2^2x + c_2x^2$ by Theorem 5.1 while the element in position $(2, 2)$ of the right matrix above is of the form $h_{21}x + h_{22}x^2$, which obviously cannot match, unless $a_2^2 = 0$.

This means that we can not represent $G_n^{(1)}$ with a matrix. However, what we can notice is that the generator matrix $G_n$ introduced in Section 2.1 is mapping a vector into a vector, while we are asking $G_n^{(1)}$ to map a matrix into a matrix. As the first case has a solution, the idea is to transform our matrix-matrix problem into a vector-vector problem and try to exploit the results obtained in Section 2.1 for $G_n$, to construct the linear operator $G_n^{(1)}$. In order to do this, we need first to introduce the following operator for a general matrix $A \in \mathbb{R}^{n \times m}$:

**Definition 3.1** (Vectorization). Given a matrix $A \in \mathbb{R}^{n \times m}$, denoting by $A_j$ the $j$-th column of $A$, we define $vec : \mathbb{R}^{n \times m} \rightarrow \mathbb{R}^{nm}$ as the operator that associates to the matrix $A$ the $nm$-column vector

$$vec(A) = (A_1, A_2, \ldots, A_n)^\top,$$

which is called the vectorization of $A$.

The $vec$ operator transforms a matrix in $\mathbb{R}^{n \times m}$ into a vector of length $nm$. For a vector of given length $l \in \mathbb{N}$, the inverse operation is obviously not unique, since we can construct as many matrices of different size as the number of factorizations of the integer $l$ into two non trivial, namely different from 1, integers. With the following definition, we mean to introduce the inverse-vectorization as associated to a specific vectorization call, that is the operation that starting from $vec(A)$, for $A \in \mathbb{R}^{n \times m}$, returns the matrix $A$ itself:

**Definition 3.2** (Inverse-vectorization). Given a vector $v \in \mathbb{R}^{nm}$, we define $vec^{-1} : \mathbb{R}^{nm} \rightarrow \mathbb{R}^{n \times m}$ as the operator that associates to the vector $v$ the $n \times m$ matrix $A = vec^{-1}(v)$, such that $[A]_{i,j} = v_{n(j-1)+i}$, for $i = 1, \ldots, n$ and $j = 1, \ldots, m$. In this case, we say that $A$ is the inverse-vectorization of the vector $v$.

Introducing

$$X_n(x) := H_n(x)H_n(x)^\top$$

(3.6)
Proof. The equation (3.7) for the matrix $\tilde{G}$ as illustrated in the following theorem:

$$G \text{vec}(X_n(x)) = \tilde{G}^{(1)} \text{vec}(X_n(x)). \tag{3.7}$$

Once found $\tilde{G}^{(1)}$, then $G^{(1)}$ can be constructed by composition with the vec and vec$^{-1}$ operators, as illustrated in the following theorem:

**Theorem 3.2.** The linear operator $G^{(1)}_n$ satisfying equation (3.5) is of the form

$$G^{(1)}_n = \text{vec}^{-1} \circ \tilde{G}^{(1)} \circ \text{vec},$$

$\tilde{G}^{(1)}_n \in \mathbb{R}^{(n+1)^2 \times (n+1)^2}$ being the matrix that transforms vec$(X_n(x))$ into Gvec$(X_n(x))$, as in equation (3.7).

**Proof.** The proof is a straightforward consequence of the discussion above.

Let us consider an example to clarify the situation:

**Example 3.2.** Let $n = 1$. Then we seek a matrix $G^{(1)}_1 \in \mathbb{R}^{4 \times 4}$ such that $G \text{vec}(X_1(x)) = \tilde{G}^{(1)}_1 \text{vec}(X_1(x))$, that is

$$\begin{pmatrix} G1 \\ Gx \\ Gx \times X^{-1} \\ Gx \times x \end{pmatrix} = \tilde{G}^{(1)}_1 \begin{pmatrix} 1 \\ x \\ x \times x \end{pmatrix}.$$

In Example 2.1 and Example 2.2 we calculated $G1, Gx, Gx \times x$. Then a possible $\tilde{G}^{(1)}_1$ is given by

$$\tilde{G}^{(1)}_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ b_0 & b_1 & 0 & 0 \\ b_0 & b_1 & 0 & 0 \\ \sigma_0 + \int_{\mathbb{R}} \delta_0(z) \ell(dz) & \gamma & 0 & \sigma_2 + 2b_1 + \int_{\mathbb{R}} \delta_1(z) \ell(dz) \end{pmatrix},$$

but also by

$$\tilde{G}^{(1)}_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ b_0 & b_1/2 & b_1/2 & 0 \\ b_0 & b_1/2 & b_1/2 & 0 \\ \sigma_0 + \int_{\mathbb{R}} \delta_0(z) \ell(dz) & \gamma/2 & \gamma & \sigma_2 + 2b_1 + \int_{\mathbb{R}} \delta_1(z) \ell(dz) \end{pmatrix},$$

with $\gamma := \sigma_1 + 2b_0 + 2 \int_{\mathbb{R}} \delta_0(z) \delta_1(z) \ell(dz)$.

What we notice from Example 3.2 is that the first $\tilde{G}^{(1)}_1$ has two rows that coincide and a null column, while the second $\tilde{G}^{(1)}_1$ provided has both two rows and two columns that coincide. The problem is in the double presence of the term $Gx$ in vec$(GX_1(x))$, or, analogously, the double presence of the term $x$ in vec$(X_1(x))$. Increasing the value of $n$, the number of redundant terms in vec$(GX_n(x))$ and vec$(X_n(x))$ increases as well. For this reason, to find a recursion for the matrix $G^{(1)}_n$ as done for the generator matrix $G_n$ in the previous section, seems not an easy task. Moreover, we would like to write the matrix $G^{(1)}_n$ in terms of $G_n$. We need then to introduce a new operator in order to remove the redundant terms in the vector vec$(X_n(x))$. 

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3.1 The L-eliminating and L-duplicating matrices

For this section, we place ourselves in a general setting, that is in the space of matrices in \( \mathbb{R}^{n \times m} \). Let us start by recalling the Kronecker product:

**Definition 3.3** (Kronecker product). The **Kronecker product** of a matrix \( A \in \mathbb{R}^{n \times m} \) with elements \( [A]_{i,j} = a_{i,j} \), for \( 1 \leq i \leq n \) and \( 1 \leq j \leq m \), and a matrix \( B \in \mathbb{R}^{r \times s} \), is the matrix \( A \otimes B \in \mathbb{R}^{nr \times ms} \) given by

\[
A \otimes B = \begin{pmatrix}
a_{1,1}B & \cdots & a_{1,m}B \\
a_{n,1}B & \cdots & a_{n,m}B
\end{pmatrix}.
\]

Several authors have studied the Kronecker product and its properties, \([15]\) and \([17]\) among others. In the following proposition we report some of the properties, which will be useful for our analysis. We denote with \( \vec{e}_{k,j} \) the \( j \)-th canonical basis vector in \( \mathbb{R}^k \).

**Proposition 3.3.** Given the matrices \( A, B \in \mathbb{R}^{n \times m} \) with elements, respectively, \( [A]_{i,j} = a_{i,j} \) and \( [B]_{i,j} = b_{i,j} \) for \( 1 \leq i \leq n \) and \( 1 \leq j \leq m \), and \( \vec{x}, \vec{y} \) vectors of any order, we can state the following properties:

\[
A = \sum_{i=1}^{n} \sum_{j=1}^{m} a_{i,j} \vec{e}_{n,i} \vec{e}_{m,j}^\top \tag{3.8a}
\]

\[
(vec(A))^\top vec(B) = tr(A^\top B) \tag{3.8b}
\]

\[
\vec{x} \otimes \vec{y} = vec(\vec{y}^\top \vec{x}^\top) \tag{3.8c}
\]

\[
\vec{x} \otimes \vec{y}^\top = \vec{x} \vec{y}^\top \otimes \vec{x} \tag{3.8d}
\]

Here \( tr \) denotes the trace operator. Moreover, for every \( A \in \mathbb{R}^{p \times q} \), \( B \in \mathbb{R}^{r \times s} \), \( C \in \mathbb{R}^{q \times k} \) and \( D \in \mathbb{R}^{s \times l} \) the mixed-product property holds:

\[
(A \otimes B) (C \otimes D) = (AC) \otimes (BD). \tag{3.9}
\]

**Proof.** We refer to \([17]\) Section 2 and \([15]\) Lemma 4.2.10].

In the previous section, we argued for removing the redundant terms occurring in \( vec(X_n(x)) \), or rather in \( X_n(x) \) itself. Looking at equation (3.3), we can notice that a possible way, among others, to select from the matrix \( X_n(x) \) all the elements without repetition (that is equivalent to select all the powers of \( x \) from 0 to \( 2n \), without repetition) is to select the first column of \( X_n(x) \) together with the last row. We introduce the following operator:

**Definition 3.4** (L-vectorization). Given a matrix \( A \in \mathbb{R}^{n \times m} \) with elements \( [A]_{i,j} = a_{i,j} \), for \( 1 \leq i \leq n \) and \( 1 \leq j \leq m \), we define the **L-vectorization** of \( A \) as the operator \( vecL : \mathbb{R}^{n \times m} \rightarrow \mathbb{R}^{n+m-1} \) that associates to the matrix \( A \) the \( (n + m - 1) \)-column vector obtained by selecting only the first column and the last row of \( A \), namely

\[
vecL(A) = (a_{1,1} \ a_{2,1} \ \cdots \ a_{n,1} \ a_{n,2} \ \cdots \ a_{n,m})^\top.
\]

Intuitively, the \( vecL \) operator selects the elements from the matrix \( A \) which are in those positions that all together take the biggest "L" shape inscribed in the matrix \( A \) itself. In \([17]\), the authors introduce the half-vectorization operator, which, starting from a matrix \( A \), returns the column vector obtained by stacking together the columns of the lower-triangular matrix.
We will prove the existence of the matrix defined in equation (3.11), finalizing the proof. Finally, we can rewrite (3.13) by means of equations (3.8c) and (3.8d): eliminating matrix \(E\) of all, we need to give a characterization of results and plugging them into equation (3.12), we get Theorem 3.4. For every \(n,m \geq 1\) and for every matrix \(A \in \mathbb{R}^{n \times m}\), there exists an L-eliminating matrix \(E_{n,m} \in \mathbb{R}^{(n+m-1) \times nm}\) such that

\[E_{n,m}vec(A) = vec(L(A)).\] (3.10)

In particular, \(E_{n,m}\) can be represented as

\[E_{n,m} = \sum_{i=1}^{n} \tilde{e}_{n+m-1,i} \otimes \tilde{e}_{m,1}^\top \otimes \tilde{e}_{n,1}^\top + \sum_{i=2}^{m} \tilde{e}_{n+m-1,n+i-1} \otimes \tilde{e}_{m,i}^\top \otimes \tilde{e}_{n,n}^\top.\] (3.11)

Proof. We will prove the existence of the matrix \(E_{n,m}\) by proving its explicit definition. First of all, we need to give a characterization of \(vec(L(A))\) in terms of the unitary vectors. By means of equation (3.8a), one can easily see that

\[vec(L(A)) = \sum_{i=1}^{n} \tilde{a}_{i,1} \tilde{e}_{n+m-1,i} + \sum_{i=2}^{m} \tilde{a}_{n,i} \tilde{e}_{n+m-1,n+i-1}.\] (3.12)

In particular, \(a_{i,1} = e_{n,i}^\top A e_{m,1} = tr(e_{n,i} e_{m,1}^\top A)\), and, similarly, \(a_{n,i} = e_{n,n}^\top A e_{m,i} = tr(e_{m,i} e_{n,n}^\top A)\). Moreover, by property (3.8b), we can write that

\[tr(e_{m,i} e_{n,n}^\top A) = tr((e_{n,n}^\top e_{m,i})^\top A) = vec(e_{n,n}^\top e_{m,i})^\top vec(A),\]

and, similarly, \(tr(e_{m,i} e_{n,n}^\top A) = tr((e_{n,n}^\top e_{m,i})^\top A) = vec(e_{n,n}^\top e_{m,i})^\top vec(A)\). Combining these results and plugging them into equation (3.12), we get

\[vec(L(A)) = \left(\sum_{i=1}^{n} \tilde{e}_{n+m-1,i} vec(e_{n,i} e_{m,1}^\top) + \sum_{i=2}^{m} \tilde{e}_{n+m-1,n+i-1} vec(e_{n,n} e_{m,i}^\top)\right) vec(A).\] (3.13)

Finally, we can rewrite (3.13) by means of equations (3.8c) and (3.8d):

\[vec(L(A)) = \left(\sum_{i=1}^{n} \tilde{e}_{n+m-1,i} \otimes \tilde{e}_{m,1}^\top \otimes \tilde{e}_{n,1}^\top + \sum_{i=2}^{m} \tilde{e}_{n+m-1,n+i-1} \otimes \tilde{e}_{m,i}^\top \otimes \tilde{e}_{n,n}^\top\right) vec(A).\]

Then the L-eliminating matrix \(E_{n,m}\) satisfying the implicit definition in equation (3.10), is exactly the one defined in equation (3.11), finalizing the proof.

Let us notice that in Theorem 3.3 we have defined the L-eliminating matrix to transform the vectorization of a generic matrix \(A \in \mathbb{R}^{n \times m}\) into the L-vectorization of the matrix \(A\) itself. We are not claiming uniqueness of \(E_{n,m}\), even if there is only one way to select from a matrix \(A\) the elements positioned in the L-shape above defined, and to construct a vector of this. What might not be unique is the way of representing such eliminating matrix, namely equation (3.11). The reason for choosing such a representation is that it involves only basis vectors of the three different spaces \(\mathbb{R}^n\), \(\mathbb{R}^m\) and \(\mathbb{R}^{n+m-1}\), but other alternatives may be possible. The matrix \(E_{n,m}\)
is moreover a sparse matrix, meaning that, even with \( n \) and \( m \) large, the computational cost for numerical calculations remains low. The following result is indeed straightforward:

**Lemma 3.5.** For \( E_{n,m} \in \mathbb{R}^{(n+m-1)\times nm} \), the number of non-zero elements is exactly \( n + m - 1 \).

**Proof.** The total elements of \( E_{n,m} \) is equal to \( nm(n + m - 1) \). When multiplying \( E_{n,m} \) with a matrix \( A \in \mathbb{R}^{n\times m} \), we select the elements of \( A \) in the first column and last row, that account for exactly \( n + m - 1 \) terms. That means that \( E_{n,m} \) must have exactly \( n + m - 1 \) elements equal to 1 and the rest must be zeros. \( \square \)

We now show how the matrix \( E_{n,m} \) looks like in a couple of examples.

**Example 3.3.** Let \( n = m = 2 \). Then equation (3.11) becomes

\[
E_{2,2} = \sum_{i=1}^{2} e_{3,i}^{2} \otimes e_{2,1}^{\top} \otimes e_{2,i}^{\top} + \sum_{i=2}^{2} e_{3,2+i}^{2} \otimes e_{2,i}^{\top} \otimes e_{2,2}^{\top},
\]

which, by calculations, is

\[
E_{2,2} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \otimes (1 \ 0) \otimes (1 \ 0) + \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \otimes (1 \ 0) \otimes (0 \ 1) + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \otimes (0 \ 1) \otimes (0 \ 1) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.
\]

We notice that the number of non-zero elements is exactly \( n + m - 1 = 3 \), as stated in Lemma 3.5. Let us now consider \( A \in \mathbb{R}^{2\times 2} \) of the form \( A = \begin{pmatrix} a_1 & a_3 \\ a_2 & a_4 \end{pmatrix} \), whose vectorization is \( vec(A) = (a_1, a_2, a_3, a_4)^{\top} \), while the L-vectorization is \( vecL(A) = (a_1, a_2, a_4) \). We can easily verify that \( E_{2,2} \) satisfies the definition of an L-eliminating matrix in equation (3.10):

\[
E_{2,2}vec(A) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{pmatrix} = vecL(A).
\]

Moreover, when applied to \( X_1(x) = \begin{pmatrix} 1 \\ x \\ x^2 \end{pmatrix} \), it exactly eliminates the duplicated value, \( x \).

**Example 3.4.** Let \( n = m = 3 \). Then, after some technical calculations, we get

\[
E_{3,3} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}.
\]

The number of non-zero elements is \( n + m - 1 = 5 \). Moreover, \( E_{3,3} \) satisfies (3.11) when applied to \( A \in \mathbb{R}^{3\times 3} \), as well as to \( X_2(x) \in \mathbb{R}^{3\times 3} \).

Next, we want to define an inverse operator to \( E_{n,m} \), that is, a linear mapping transforming the L-vectorization of a matrix \( A \) into the vectorization of \( A \) itself. However, this inverse operation is not well defined in the space of matrices in \( \mathbb{R}^{n\times m} \), and the reason is the following. When we apply \( E_{n,m} \) to \( vec(A) \), we go from a space of dimension \( nm \) to a space of lower dimension, \( n + m - 1 \), loosing in such a way information. Then the inverse transformation in general does not exist, as there is no way to recover such lost information. In [17], the
Following two forms: two cases corresponding to whether those matrices whose elements on the same anti-diagonal coincide. In particular, we distinguish Theorem 3.6. For every \( a \) for some coefficients \( a \) for some coefficients \( a \), \( a \)

\[
\begin{pmatrix}
(a_1 & a_2 & \cdots & a_m \\
 a_2 & & & \ddots \\
 & a_m & & \\
 & & \ddots & \\
 & & & a_n \\
\end{pmatrix}
\]

for \( a_1, \ldots, a_{n+m-1} \in \mathbb{R} \). We call \( A \in \mathcal{A}_{n,m} \) an anti-diagonal matrix.

In order to clarify Definition 3.5 we consider the following example:

**Example 3.5.** Let \( A_1 \in \mathcal{A}_{3,3} \) and \( A_2 \in \mathcal{A}_{3,5} \). Then \( A_1 \), respectively, \( A_2 \) are of the form

\[
A_1 = \begin{pmatrix}
(a_1 & a_2 & a_3) \\
 a_2 & a_3 & a_4 \\
 a_3 & a_4 & a_5 \\
 a_4 & a_5 & a_6 \\
 a_5 & a_6 & a_7
\end{pmatrix}
\quad \text{and} \quad
A_2 = \begin{pmatrix}
(a_1 & a_2 & a_3 & a_4 & a_5) \\
 a_2 & a_3 & a_4 & a_5 & a_6 \\
 a_3 & a_4 & a_5 & a_6 & a_7 \\
\end{pmatrix}
\]

for some coefficients \( a_1, \ldots, a_7 \in \mathbb{R} \).

From Example 3.5, we notice that for \( A \in \mathcal{A}_{n,m} \) the maximum number of different values in \( A \) equals the number of anti-diagonals of \( A \), that is \( n + m - 1 \). In particular, the L-vectorization of such matrix gives exactly all the different elements of \( A \) without repetition, as it can be noticed in Example 3.5. We can also see that \( X_n(x) \) defined in equation (3.6) belongs to the space \( \mathcal{A}_{n+1,n+1} \).

We can now define the inverse operator of \( E_{n,m} \) on the space \( \mathcal{A}_{n,m} \):

**Theorem 3.6.** For every \( n, m \geq 1 \), and for every matrix \( A \in \mathcal{A}_{n,m} \), there exists an L-duplicating matrix \( D_{n,m} \in \mathbb{R}^{nm \times (n+m-1)} \) such that:

\[
D_{n,m} \text{vec} L(A) = \text{vec}(A).
\]

In particular, \( D_{n,m} \) can be represented as

\[
D_{n,m} = \sum_{i=1}^{n} \sum_{j=1}^{m} \tilde{e}_{n+m-1,i+j-1} \otimes \tilde{e}_{m,j} \otimes \tilde{e}_{n,i}.
\]
Proof. As in Theorem 3.4 for the L-eliminating matrix, we construct the matrix $D_{n,m}$ explicitly. Since $A \in \mathbb{A}_{n,m}$, then the elements of $A$ along the anti-diagonals coincide. Moreover, as pointed out previously, in the matrix $A$ there are $n+m-1$ anti-diagonals, leading to at most $n+m-1$ different values. In the notation of Definition 3.3 let $a_k$, $k = 1, \ldots, (n + m - 1)$, such that $vecL(A) = (a_1, a_2, \ldots, a_{n+m-1})^\top \in \mathbb{R}^{n+m-1}$. For $1 \leq i \leq n$ and $1 \leq j \leq m$, one can easily check that

$$[A]_{i,j} = a_{i+j-1} = [vecL(A)]_{i+j-1} = vecL(A)^\top \vec{e}_{n+m-1,i+j-1} = e_{n+m-1,i+j-1}^\top vecL(A). \quad (3.16)$$

Moreover, let us notice that $\vec{e}_{m,j} \otimes \vec{e}_{n,i}$ is the unitary vector in $\mathbb{R}^{nm}$ with 1 in position $n(j-1)+i$ and 0 elsewhere. We can use this fact together with equation (3.16) in order to express the vectorization of $A$ as follows:

$$vec(A) = \sum_{i=1}^{n} \sum_{j=1}^{m} a_{i+j-1} \vec{e}_{m,j} \otimes \vec{e}_{n,i} = \left( \sum_{i=1}^{n} \sum_{j=1}^{m} (\vec{e}_{m,j} \otimes \vec{e}_{n,i}) e_{n+m-1,i+j-1}^\top \right) vecL(A).$$

By means of equation (3.8d), we define the matrix $D_{n,m}$ as in equation (3.11), which proves the theorem.

For $A \in \mathbb{A}_{n,m}$, the matrix $D_{n,m}$ defined in Theorem 3.3 is basically duplicating each element $a_k$ in $vecL(A)$ as many times as the number of elements in the $k$-th anti-diagonal of $A$, for $k = 1, \ldots, n + m - 1$. This means that, in the $k$-th column of $D_{n,m}$ there are as many 1’s as the number of elements in the $k$-th anti-diagonal of $A$, while the remaining elements are all 0’s. We make this precise with the following result:

**Lemma 3.7.** For every $n, m \geq 1$, $D_{n,m}$ is a sparse matrix. Moreover, if $n \neq m$, then the number of 1’s in the $k$-th column, for $k = 1, \ldots, n + m - 1$, is given by

$$\begin{cases} k & \text{for } 1 \leq k \leq \min(n, m) - 1 \\ \min(n, m) & \text{for } \min(n, m) \leq k \leq \max(n, m) - 1 \\ n + m - k & \text{for } \max(n, m) \leq k \leq n + m - 1 \end{cases}. \quad (3.17)$$

If $n = m$, then the following formula holds instead:

$$\begin{cases} k & \text{for } 1 \leq k \leq n - 1 \\ 2n - k & \text{for } n \leq k \leq 2n - 1 \end{cases}. \quad (3.18)$$

In particular, if $n = m$, then the number of 1’s in the $k$-th column corresponds to the coefficient of the $(k-1)$-th power of $x$ in the power expansion $\left( \sum_{\alpha=0}^{n-1} x^\alpha \right)^2$.

**Proof.** Let $n \neq m$. As pointed out previously, in the $k$-th column of $D_{n,m}$ there are as many 1’s as the number of elements in the $k$-th anti-diagonal of $A$, for $A \in \mathbb{A}_{n,m}$. Indicating with $a_k$ the value of the elements on the $k$-th anti-diagonal of $A$ as in Definition 3.3 for $k = 1, \ldots, n + m - 1$, equation (3.17) gives then the cardinality of each $a_k$, and the sum of such numbers should give the total number of elements in $A$, that is $nm$. Summing up the elements in equation (3.17)
we get:

\[
\begin{align*}
&\min(n, m) - 1 + \sum_{k=1}^{\min(n, m) - 1} k + \sum_{k=\min(n, m)}^{\min(n, m) - 1} \min(n, m) + \sum_{k=\max(n, m)}^{n+m-1} (m + n - k) \\
&= \frac{\min(n, m) (\min(n, m) - 1)}{2} + \min(n, m) (\max(n, m) - \min(n, m)) + \\
&\quad + \frac{\min(n, m) (\min(n, m) + 1)}{2}
\end{align*}
\]

\[
= \max(n, m) \min(n, m) = nm,
\]

where for the third sum we used the change of variables \( k' := m + n - k \) so that \( \sum_{k=\max(n, m)}^{n+m-1} (m + n - k) = \sum_{k'=1}^{n+m-\max(m, n)} k' \), and the fact that \( m + n - \max(n, m) = \min(n, m) \). The case \( n = m \) is similar, so that the first part of the lemma is proved.

We need now to prove that the numbers in equation (3.18) correspond also to the coefficients in the power expansion \((\sum_{\alpha=0}^{n-1} x^\alpha)^2\). In order to do that, we proceed by induction on the matrix dimension \( n \geq 2 \) (the case \( n = 1 \) is trivial).

- \( n = 2 \): a matrix \( A \in \mathcal{A}_{2,2} \) is of the form \( A = \begin{pmatrix} a_1 & a_2 \\ a_2 & a_3 \end{pmatrix} \) and the cardinality of the entries \( a_k, k = 1, 2, 3 \), is \( 1 - 2 - 1 \), which correspond to the coefficients of the polynomial \((\sum_{\alpha=0}^{n-1} x^\alpha)^2 = (1 + x)^2 = 1 + 2x + x^2\).

- \( n \rightarrow n + 1 \): let us indicate with \( A_n \) a general matrix in \( \mathcal{A}_{n,n} \), and with \( A_{n+1} \) a general matrix in \( \mathcal{A}_{n+1,n+1} \). Then \( A_n \) and \( A_{n+1} \) can be represented in the following way:

\[
A_n = \begin{pmatrix}
a_1 & a_2 & \ldots & a_n \\
a_2 & a_3 & \ldots & a_{n+1} \\
a_3 & \ldots & \ldots & a_{n+1} \\
a_n & a_{n+1} & \ldots & a_{2n-1}
\end{pmatrix}, \quad A_{n+1} = \begin{pmatrix}
a_1 & a_2 & \ldots & a_n & a_{n+1} \\
a_2 & a_3 & \ldots & a_{n+1} & \ldots \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
a_n & a_{n+1} & \ldots & a_{2n-1} & a_{2n+1} \\
a_{n+1} & a_{n+1} & \ldots & a_{2n-1} & a_{2n+1}
\end{pmatrix}.
\]

In particular, \( A_n \) contains the entries \( a_k \) from \( k = 1 \) to \( k = 2n - 1 \), whose cardinality, by induction hypothesis, corresponds to the coefficients of the polynomial \((\sum_{\alpha=0}^{n-1} x^\alpha)^2\). Moreover, the entries \( a_k \) from \( k = n + 1 \) to \( k = 2n - 1 \) appears two extra times in \( A_{n+1} \), once in the last row and once in the last column. Finally, in \( A_{n+1} \) we have two additional entries, \( a_{2n} \) and \( a_{2n+1} \), that are not in \( A_n \), and whose cardinality is, respectively, 2 and 1. To summarize, we can say that the cardinality of the entry \( a_k \) in \( A_{n+1} \), for
\( k = 1, \ldots, 2n + 1 \), corresponds to the \((k - 1)\)-th power of \(x\) in the following polynomial:

\[
\left( \sum_{\alpha=0}^{n-1} x^\alpha \right)^2 + 2 \left( x^n + \cdots + x^{2n-2} \right) + 2x^{2n-1} + x^{2n}
\]

\[
= \left( \sum_{\alpha=0}^{n-1} x^\alpha \right)^2 + 2x^n \left( 1 + \cdots + x^{n-1} \right) + x^{2n}
\]

\[
= \left( \sum_{\alpha=0}^{n-1} x^\alpha \right)^2 + 2x^n \left( \sum_{\alpha=0}^{n-1} x^\alpha \right) + x^{2n} = \left( \sum_{\alpha=0}^{n} x^\alpha \right)^2,
\]

which concludes the proof.

\[\Box\]

We now show how the matrix \(D_{n,m}\) looks like with a couple of examples.

**Example 3.6.** Let \(n = m = 2\). Then equation (3.15) becomes

\[
D_{2,2} = \sum_{i=1}^{2} \sum_{j=1}^{2} \vec{e}_{3,i+j-1}^\top \otimes \vec{e}_{2,j} \otimes \vec{e}_{2,i},
\]

which, by calculations, equals

\[
D_{2,2} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix}
\]

We can notice that the number of 1’s in each column is respectively \(1 - 2 - 1\), in accordance with Lemma 3.7. For \(A \in A_{2,2}\) of the form \(A = \begin{pmatrix} a_1 & a_2 \\ a_2 & a_3 \end{pmatrix}\), whose vectorization is \(\text{vec}(A) = (a_1, a_2, a_2, a_3)^\top\), while the L-vectorization is \(\text{vecL}(A) = (a_1, a_2, a_3)^\top\), we can verify that \(D_{2,2}\) satisfies the definition of an L-duplicating matrix, equation (3.14)

\[
D_{2,2}\text{vecL}(A) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \text{vec}(A).
\]

Moreover, when applied to \(X_1(x) \in A_{2,2}\) of the form \(X_1(x) = \begin{pmatrix} 1 & x \\ x & x^2 \end{pmatrix}\), it duplicates the value \(x\) omitted in the L-vectorization.
Proof. By equations (3.11) and (3.15), we can write that

\[ D_{3,3} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}. \]

For every \( \mathbf{X} \) according to Lemma 3.7. It is easy to check that \( D_{3,3} \) satisfies (3.14) when applied to \( A \in \mathcal{A}_{3,3} \), as well as to \( X_2(x) \in \mathcal{A}_{3,3} \).

The next result tells us that the matrix \( D_{n,m} \) found in Theorem 3.6 is the right-inverse of the matrix \( E_{n,m} \):

**Proposition 3.8.** For every \( n, m \geq 1 \) the product \( E_{n,m}D_{n,m} \) gives the identity matrix in the space \( \mathbb{R}^{(n+m-1)\times(n+m-1)} \).

**Proof.** By equations (3.11) and (3.15), we can write that

\[
E_{n,m}D_{n,m} = \sum_{i=1}^{n} \sum_{j=1}^{m} \sum_{k=1}^{n} (\bar{c}_{n+m-1,k} \otimes \bar{c}_{m,1}^T \otimes \bar{c}_{n,k}^T) \left( \bar{c}_{n+m-1,i+j-1}^T \otimes \bar{c}_{m,j} \otimes \bar{c}_{n,i} \right) + \\
+ \sum_{i=1}^{n} \sum_{j=1}^{m} \sum_{k=2}^{m} (\bar{c}_{n+m-1,n+k-1} \otimes \bar{c}_{m,k}^T \otimes \bar{c}_{n,n}^T) \left( \bar{c}_{n+m-1,i+j-1}^T \otimes \bar{c}_{m,j} \otimes \bar{c}_{n,i} \right). \tag{3.19}
\]

Let us focus on the first sum. We notice that:

- \( \bar{c}_{m,1}^T \otimes \bar{c}_{n,k}^T \in \mathbb{R}^{nm} \) is the row unitary vector with 1 in position \( k \) and 0 elsewhere, namely \( \bar{c}_{nm,k}^T \);
- \( A_k := \bar{c}_{n+m-1,k} \otimes \bar{c}_{m,1}^T \otimes \bar{c}_{n,k}^T \in \mathbb{R}^{(n+m-1)\times nm} \) is the matrix with 1 in position \( (k, k) \) and 0 elsewhere;
- \( \bar{c}_{m,j} \otimes \bar{c}_{n,i} \in \mathbb{R}^{nm} \) is the column unitary vector with 1 in position \( n(j - 1) + i \) and 0 elsewhere, namely \( \bar{c}_{nm,n(j-1)+i} \);
- \( B_{k,j} := \bar{c}_{n+m-1,i+j-1}^T \otimes \bar{c}_{m,j} \otimes \bar{c}_{n,i} \in \mathbb{R}^{nm \times (n+m-1)} \) is the matrix with 1 in position \( (n(j - 1) + i, i + j - 1) \) and 0 elsewhere.

Then we can easily see that the product \( A_kB_{i,j} \in \mathbb{R}^{(n+m-1)\times(n+m-1)} \) gives a matrix with 1 in position \( (k, k) \) and 0 elsewhere. Similarly, looking at the second sum in equation (3.19), we notice that

- \( \bar{c}_{m,k}^T \otimes \bar{c}_{n,n}^T \in \mathbb{R}^{nm} \) is the row unitary vector with 1 in position \( kn \) and 0 elsewhere, namely \( \bar{c}_{nm,kn}^T \).
• \( \hat{A}_k := \tilde{e}_{n+m-1,n+k-1} \otimes \tilde{e}_{m,k}^T \otimes \tilde{e}_{n,n}^T \in \mathbb{R}^{(n+m-1) \times nm} \) is the matrix with 1 in position 
\((n + k - 1, kn)\) and 0 elsewhere. Then we can easily see that the product \( \hat{A}_k B_{i,j} \in \mathbb{R}^{(n+m-1) \times (n+m-1)} \) gives us a matrix with 1 in position \((n + k - 1, n + k - 1)\) and 0 elsewhere. Indicating with \( I_i \in \mathbb{R}^{(n+m-1) \times (n+m-1)} \) the matrix with 1 in position \((i, i)\) and 0 elsewhere, and with \( I_{i,j} \in \mathbb{R}^{(n+m-1) \times (n+m-1)} \) the matrix with 1 in position \((k, k)\) for \( i \leq k \leq j \), combining these results into equation (3.19) we get

\[
E_{n,m}D_{n,m} = \sum_{i=1}^{n} \sum_{j=1}^{m} \sum_{k=1}^{n} A_k B_{i,j} + \sum_{i=1}^{n} \sum_{j=1}^{m} \sum_{k=2}^{m} \hat{A}_k B_{i,j}
\]

\[
= \sum_{k=1}^{n} I_k + \sum_{k=2}^{m} I_{k+1} = I_{1,n} + I_{n+1,n+m-1} = I_{n+m-1},
\]

that concludes the proof.

Proposition 3.8 shows that \( E_{n,m}D_{n,m} = I_{n+m-1} \), which means \( D_{n,m} \) is the right-inverse of \( E_{n,m} \). However, the opposite is not true, namely \( D_{n,m}E_{n,m} \neq I_{nm} \). Consider the following counterexample:

Example 3.8. Let \( n = m = 2 \). From Example 3.3 and Example 3.6 we get

\[
D_{2,2}E_{2,2} = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix} ^T \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix} = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix},
\]

which is clearly not the identity matrix in \( \mathbb{R}^{4 \times 4} \). However, for \( A \in A_{2,2} \), in the notation of Definition 3.3 the vector \( \text{vec}(A) \) is of the shape \( \text{vec}(A) = (a_1, a_2, a_3)^T \). We can then notice that \( D_{2,2}E_{2,2} \) applied to \( \text{vec}(A) \) gives back the vector \( \text{vec}(A) \) itself. Hence, even if \( D_{2,2}E_{2,2} \) is not the identity matrix, when applied on the elements in \( A_{2,2} \), it behaves like an identity operator.

We can generalize what we observed in Example 3.8 for every \( n, m \geq 1 \).

**Proposition 3.9.** For every \( n, m \geq 1 \) and for every \( A \in A_{n,m} \), the product \( D_{n,m}E_{n,m} \in \mathbb{R}^{nm \times nm} \) acts on \( \text{vec}(A) \) like an identity operator, namely

\[
D_{n,m}E_{n,m} \text{vec}(A) = \text{vec}(A).
\]

**Proof.** The proof is straightforward by definition of \( E_{n,m} \) and \( D_{n,m} \), namely combining equation (3.10) and (3.11). Indeed, we can resume the situation as follows:

\[
\text{vec}(A) \xrightarrow{E_{n,m}} \text{vec}L(A) \xrightarrow{D_{n,m}} \text{vec}(A),
\]

so that the product \( D_{n,m}E_{n,m} \in \mathbb{R}^{nm \times nm} \) acts like an identity operator on \( \text{vec}(A) \).

**3.1.1 The squared case**

Let us remember that we started this discussion with the aim of removing the redundant terms in the vectorization of the matrix \( X_n(x) \in \mathbb{R}^{(n+1) \times (n+1)} \) defined in equation (3.6). From the previous section, we get the L-eliminating matrix and the L-duplicating matrix for this particular case:
Corollary 3.10. For every \( n \geq 1 \), the L-eliminating matrix \( E_{n+1} \in \mathbb{R}^{(2n+1) \times (n+1)^2} \) transforming \( \text{vec}(X_n(x)) \) into \( \text{vecL}(X_n(x)) \) is given by

\[
E_{n+1} := E_{n+1,n+1} = \sum_{i=1}^{n+1} e_{2n+1,i}^T \otimes e_{n+1,i}^T + \sum_{i=2}^{n+1} e_{2n+1,n+i} \otimes e_{n+1,i}^T \otimes e_{n+1,n+1}. \tag{3.20}
\]

Proof. This is a particular case of Theorem 3.4. \( \square \)

Corollary 3.11. For every \( n \geq 1 \), the L-eliminating matrix \( D_{n+1} \in \mathbb{R}^{(n+1)^2 \times (2n+1)} \) transforming \( \text{vecL}(X_n(x)) \) into \( \text{vec}(X_n(x)) \) is given by

\[
D_{n+1} := D_{n+1,n+1} = \sum_{i=1}^{n+1} \sum_{j=1}^{n+1} e_{2n+1,i}^T \otimes e_{n,i} \otimes e_{n,j}. \tag{3.21}
\]

Proof. This is a particular case of Theorem 3.0. \( \square \)

3.2 The generator for correlators

Let us now go back to our original problem: by means of Theorem 3.2 we seek a linear operator \( \tilde{G}_n^{(1)} \) transforming \( \text{vec}(X_n(x)) \) into \( \mathcal{G}\text{vec}(X_n(x)) \). As pointed out before, focusing on \( X_n(x) = H_n(x)H_n(x)^\top \) in equation (3.3), we can see that the elements lying on the left-bottom L-shape are nothing but all the powers of \( x \) from 0 to 2\( n \). The following result is then straightforward:

Lemma 3.12. For every \( n \geq 1 \), the following identity is satisfied:

\[
\text{vecL}(H_n(x)H_n(x)^\top) = H_{2n}(x).
\]

Proof. The proof is trivial. \( \square \)

By means of Lemma 3.12 by transforming the vectorization of \( H_n(x)H_n(x)^\top \) into its L-vectorization, we are addressing the problem of finding the generator matrix for \( H_n(x)H_n(x)^\top \), to the problem of finding the generator matrix for \( H_{2n}(x) \). However, this latter problem has been already solved in Section 2.1. We can then prove the following result:

Theorem 3.13. For every \( n \geq 1 \), the matrix \( \tilde{G}_n^{(1)} \) satisfying property (3.7) can be decomposed into the following matrix representation:

\[
\tilde{G}_n^{(1)} = D_{n+1}G_{2n}E_{n+1},
\]

\( E_{n+1} \) and \( D_{n+1} \) being the L-eliminating matrix, respectively, the L-duplicating matrix, given in equation (3.20), respectively, in equation (3.21), while \( G_{2n} \) is the generator matrix defined in equation (2.6).

Proof. The proof is straightforward from the definition of \( E_{n+1} \), \( D_{n+1} \) and \( G_{2n} \) as mentioned in the statement, and from Lemma 3.12. We know indeed that for every \( r \geq 1 \) there exists a matrix \( G_r \) such that \( \mathcal{G}H_r(x) = G_rH_r(x) \). If then we take \( r = 2n \), we get \( G_{2n} \) such that \( \mathcal{G}H_{2n}(x) = G_{2n}H_{2n}(x) \). By Lemma 3.12 we also know that \( H_{2n}(x) = \text{vecL}(X_n(x)) \), for \( X_n(x) = H_n(x)H_n(x)^\top \). Hence \( G_{2n} \) satisfies

\[
\text{vecL}(\mathcal{G}X_n(x)) = G_{2n}\text{vecL}(X_n(x)).
\]
By definition of $E_{n+1}$, we can rewrite the last equation as

$$E_{n+1}\text{vec}(GX_n(x)) = G_{2n}E_{n+1}\text{vec}(X_n(x)),$$

and multiplying both sides by $D_{n+1}$ from the left, we get

$$D_{n+1}E_{n+1}\text{vec}(GX_n(x)) = D_{n+1}G_{2n}E_{n+1}\text{vec}(X_n(x)), \quad (3.22)$$

where, in particular, $D_{n+1}E_{n+1}\text{vec}(GX_n(x)) = \text{vec}(GX_n(x))$ because of Proposition 3.9. Then we see that the matrix $\tilde{G}_n^{(1)}$ satisfying equation (3.7) is given by the product of the three matrices on the right hand side of equation (3.22).

Let us remind that the reason why we seek $\tilde{G}_n^{(1)}$ is to provide an explicit expression to the conditional expectation of the product of two polynomial functions of a polynomial process evaluated at two different time points. Due to Proposition 3.1, we need to compute the conditional expectation $\mathbb{E}\left[H_n(Y(s_0))H_n(Y(s_0))^\top \mid \mathcal{F}_t\right]$, which is also a solution of the ordinary differential equation (3.4). With the tools constructed, we are now able to provide a solution to that problem.

**Theorem 3.14.** The solution of equation (3.4) is given by

$$\mathbb{E}\left[H_n(Y(s_0))H_n(Y(s_0))^\top \mid \mathcal{F}_t\right] = \text{vec}^{-1} \circ D_{n+1}e^{G_{2n}(s_0-t)}E_{n+1} \circ \text{vec}\left(H_n(Y(t))H_n(Y(t))^\top\right).$$

**Proof.** Starting from equation (3.4), and applying the operator vec on both sides, we get

$$\mathbb{E}\left[\text{vec}\left(H_n(Y(s_0))H_n(Y(s_0))^\top\right) \mid \mathcal{F}_t\right] = \text{vec}\left(H_n(Y(t))H_n(Y(t))^\top\right) + \int_t^{s_0} \mathbb{E}\left[\text{vec}\left(H_n(Y(s))H_n(Y(s))^\top\right) \mid \mathcal{F}_t\right] ds. \quad (3.23)$$

By equation (3.7) and Theorem 3.13, equation (3.23) becomes

$$\mathbb{E}\left[\text{vec}\left(H_n(Y(s_0))H_n(Y(s_0))^\top\right) \mid \mathcal{F}_t\right] = \text{vec}\left(H_n(Y(t))H_n(Y(t))^\top\right) + D_{n+1}G_{2n}E_{n+1} \int_t^{s_0} \mathbb{E}\left[\text{vec}\left(H_n(Y(s))H_n(Y(s))^\top\right) \mid \mathcal{F}_t\right] ds. \quad (3.24)$$

In the spirit of the proof of Theorem 2.3, we introduce $Z(s) := \mathbb{E}[\text{vec}\left(H_n(Y(s))H_n(Y(s))^\top\right) \mid \mathcal{F}_t]$, so that equation (3.24) can be written in differential form as

$$dZ(s) = D_{n+1}G_{2n}E_{n+1}Z(s)ds,$$

and the solution, by separation of variables, is given by

$$Z(s_0) = e^{D_{n+1}G_{2n}E_{n+1}(s_0-t)} Z(t) = D_{n+1}e^{G_{2n}(s_0-t)}E_{n+1}Z(t),$$

where for the last equality we refer to Lemma 3.16 below. Finally, going back to the definition of $Z(s)$, and applying the $\text{vec}^{-1}$ operator, we get the statement of the theorem.

In the proof of Theorem 3.14, we used the equality $e^{D_{n+1}G_{2n}E_{n+1}t} = D_{n+1}e^{G_{2n}t}E_{n+1}$, for $t \geq 0$. This can be proved in two steps, given in the two following lemmas:
Lemma 3.15. For every $k \geq 1$, the following power formula holds:

$$(D_{n+1}G_{2n}E_{n+1})^k = D_{n+1}(G_{2n})^k E_{n+1}.$$ 

Proof. We can proceed by induction on the exponent $k \geq 1$.

- $k = 1$: trivial.
- $k - 1 \to k$: assuming the statement holds for $k - 1$, we get

$$
(D_{n+1}G_{2n}E_{n+1})^k = (D_{n+1}G_{2n}E_{n+1})^{k-1}(D_{n+1}G_{2n}E_{n+1}) = D_{n+1}(G_{2n})^{k-1}E_{n+1}D_{n+1}G_{2n}E_{n+1} = D_{n+1}(G_{2n})^k E_{n+1},
$$

where we used the fact that $D_{n+1}$ is the right inverse of $E_{n+1}$, as stated in Proposition 3.8. This concludes the proof.

With the power formula in Lemma 3.15 we can then define the matrix exponential of the matrices product $D_{n+1}G_{2n}E_{n+1}$:

Lemma 3.16. For every $t \geq 0$, the matrix exponential of the product $D_{n+1}G_{2n}E_{n+1}$ is given by the matrix exponential of $G_{2n}$ multiplied on the left by $D_{n+1}$, and on the right by $E_{n+1}$, namely

$$e^{D_{n+1}G_{2n}E_{n+1}t} = D_{n+1}e^{G_{2n}t}E_{n+1}.$$ 

Proof. As done previously, we can consider the definition of exponential function as the infinite sum of powers, namely

$$e^{D_{n+1}G_{2n}E_{n+1}t} = \sum_{k=0}^{\infty} \frac{t^k}{k!} (D_{n+1}G_{2n}E_{n+1})^k.$$ 

Then, by Lemma 3.15 we get that

$$
\sum_{k=0}^{\infty} \frac{t^k}{k!} (D_{n+1}G_{2n}E_{n+1})^k = \sum_{k=0}^{\infty} \frac{t^k}{k!} (D_{n+1}(G_{2n})^k E_{n+1})
$$

$$
= D_{n+1} \left( \sum_{k=0}^{\infty} \frac{(G_{2n}t)^k}{k!} \right) E_{n+1} = D_{n+1} \left( e^{G_{2n}t} \right) E_{n+1},
$$

which concludes the proof.

With Lemma 3.16 we have provided all the tools for the solution of Theorem 3.14, that is for the conditional expectation (1.1) in the case of two polynomials ($m = 1$). We want now to extend this result to every $m \geq 1$.

4 Higher order correlators

In Section 1 we introduced the $(m+1)$-points correlator as the expectation for the product of $m+1$ polynomial functions

$$
\mathbb{E} \left[ p_{m}(Y(s_0)) p_{m-1}(Y(s_1)) \cdots p_{0}(Y(s_m)) \mid \mathcal{F}_t \right],
$$

(4.1)
In particular, following matrix function

\[
\begin{align*}
    &d-Kronecker product \quad A \in \mathbb{R}^{n \times m} \quad B \in \mathbb{R}^{r \times s}, \quad \text{as the } d\text{-th Kronecker power of } A \text{ multiplied in the Kronecker sense with } B, \quad d \geq 1, \\
    &\quad \text{and a matrix } R, \quad \text{or equal to } B \text{ for } d = 0, \text{ namely} \\
    &\quad \left\{ \begin{aligned}
        A \otimes^d B &= A^{\otimes^d} \otimes B & d \geq 1 \\
        A \otimes^0 B &= B & d = 0
      \end{aligned} \right.
\end{align*}
\]

In accordance with Definition 4.1, for every \( n \geq 1 \) and for every \( r \geq 0 \), we introduce the following matrix function

\[
X_n^{(r)}(x) := H_n(x)^\top \otimes^r H_n(x). \tag{4.2}
\]

In particular,

\begin{itemize}
  \item for \( r = 0 \): we get \( X_n^{(0)}(x) = H_n(x) \);
  \item for \( r = 1 \): by property (3.8c), we get
    \[
    X_n^{(1)}(x) = H_n(x)^\top \otimes^1 H_n(x) = H_n(x) H_n(x)^\top = X_n(x), \tag{4.3}
    \]
    for \( X_n(x) \) introduced in equation (3.6), that is a squared matrix of dimension \((n + 1) \times (n + 1)\) which belongs to \( \mathcal{A}_{n+1,n+1} \), as noticed previously;
  \item for \( r = 2 \): by the associativity property of the Kronecker product,
    \[
    X_n^{(2)}(x) = H_n(x)^\top \otimes^2 H_n(x) = H_n(x)^\top \otimes \left( H_n(x)^\top \otimes^1 H_n(x) \right) = H_n(x)^\top \otimes X_n^{(1)}(x)
    \]
    is the following block matrix
    \[
    X_n^{(2)}(x) = \begin{pmatrix}
        X_n^{(1)}(x)^\top & x X_n^{(1)}(x)^\top & x^2 X_n^{(1)}(x)^\top & \cdots & x^n X_n^{(1)}(x)^\top
    \end{pmatrix}
    \]
    composed by \( n + 1 \) blocks, where the \( k\)-th block is a squared matrix of dimension \((n + 1) \times (n + 1)\) of the form
    \[
    B_{n,2}^{(k)} = x^{k-1} X_n^{(1)}(x),
    \]
    for \( k = 1, \ldots, (n + 1) \), and, in particular, it is easy to see that such blocks are all different from each others and belong to \( \mathcal{A}_{n+1,n+1} \);
  \item for \( r = 3 \): as before, we can write that
    \[
    X_n^{(3)}(x) = (H_n(x)^\top)^{\otimes^2} \otimes X_n^{(1)}(x). \tag{4.4}
    \]
    In particular, \( (H_n(x)^\top)^{\otimes^2} \) is a block row vector of length \((n + 1)^2\) of the form
    \[
    \begin{pmatrix}
        1 & x & x^2 & \cdots & x^n \\
        x & x^2 & x^3 & \cdots & x^{n+1} \\
        \vdots & \vdots & \vdots & \ddots & \vdots \\
        x^n & x^{n+1} & x^{n+2} & \cdots & x^{2n}
    \end{pmatrix},
    \]
    so that \( X_n^{(3)}(x) \) is a block matrix composed by \((n + 1)^2\) blocks. It is easy to see that for each block \( B_{n,3}^{(k)} \), \( k = 1, \ldots, (n + 1)^2 \), there exists an index \( j_k \in \{0, \ldots, 2n\} \) such that
    \[
    B_{n,3}^{(k)} = x^{j_k} X_n^{(1)}(x).
    \]
    Moreover, \( B_{n,3}^{(k)} \in \mathcal{A}_{n+1,n+1} \). The difference from the previous case is that, now, some of the blocks are repeated, as a consequence of the fact that in the
By definition of d-Kronecker product, since \( x \in R^k \) particular, for each \( R \) block matrix in Proposition 4.1.

Generalizing, we can state the following result:

**Proposition 4.1.** For every \( n, r \geq 1 \), \( X_n^{(r)}(x) \) introduced in equation (4.2) is a rectangular block matrix in \( \mathbb{R}^{(n+1) \times (n+1)^r} \), which is composed by \((n+1)^{r-1}\) blocks, \( B_n^{(r)}(x) \in A_{n+1,n+1} \). In particular, for each \( k = 1, \ldots, (n+1)^{r-1} \) there exists an index \( j_k \in \{0, \ldots, (r-1)n\} \), such that

\[
B_n^{(r)}(x) = x^{j_k} X_n^{(1)}(x).
\]

Each block of the form \( x^j X_n^{(1)}(x) \), for \( j = 0, \ldots, (r-1)n \), is repeated with cardinality \( \beta_n^{(j)} := \#\{k : j_k = j\} \), that is equal to the coefficient of the \( j \)-th power of \( x \) in the polynomial expansion \((\sum_{\alpha=0}^{n} x^\alpha)^{r-1}\).

**Proof.** By definition of d-Kronecker product, since \( H_n(x) \in \mathbb{R}^{n+1} \), one can verify that \( X_n^{(r)}(x) \in \mathbb{R}^{(n+1) \times (n+1)^r} \). We then proceed by induction on \( r \geq 1 \).

- \( r = 1 \): we get that \( X_n^{(1)}(x) = H_n(x)^\top \otimes 1 H_n(x) = x^0 X_n^{(1)}(x) \in A_{n+1,n+1} \) is composed by only one block, and, in particular, the polynomial \((\sum_{\alpha=0}^{n} x^\alpha)^0 = 1\) has the only coefficient \( 1 \), that is the cardinality of the unique block.

- \( r \to r+1 \): assuming the statement holds for \( r \), since the Kronecker product is associative, we can write that

\[
X_n^{(r+1)}(x) = H_n(x)^\top \otimes^{r+1} H_n(x) = H_n(x)^\top \otimes \left( H_n(x)^\top \otimes^r H_n(x) \right) = H_n(x)^\top \otimes X_n^{(r)}(x).
\]

We then need to multiply the row vector \( H_n(x)^\top = (1, x, \ldots, x^n) \) in the Kronecker sense with the matrix \( X_n^{(r)}(x) = H_n(x)^\top \otimes^r H_n(x) \), which we know satisfies the statement of the proposition. That means that each of the \((n+1)^{r-1}\) blocks \( B_n^{(r)}(x) = x^{j_k} X_n^{(1)}(x) \), for \( j_k \in \{0, \ldots, (r-1)n\} \), has to be multiplied for each of the elements of the vector \( H_n(x) \), namely for each power \( x^\alpha, \alpha = 0, \ldots, n \). We can then say that for each block \( B_n^{(r)}(x) \) there exists an index \( \gamma_k \in \{0, \ldots, rn\} \) such that \( B_n^{(r)}(x) = x^{\gamma_k} X_n^{(1)}(x) \), and we have \((n+1)^r\) of such blocks, which belong to the space \( A_{n+1,n+1} \).

We need next to prove that for each \( j = 0, \ldots, rn \), the cardinality of the block of the form \( x^j X_n^{(1)}(x) \) corresponds to the coefficient of the \( j \)-th power of \( x \) in the polynomial expansion \((\sum_{\alpha=0}^{n} x^\alpha)^r\). From equation (4.4), we can represent \( X_n^{(r+1)}(x) \) as follows:

\[
X_n^{(r+1)}(x) = \left( X_n^{(r)}(x) \right) \left( x^1 X_n^{(r)}(x) \right) \left( x^2 X_n^{(r)}(x) \right) \cdots \left( x^n X_n^{(r)}(x) \right).
\]

In particular, we know by the induction hypothesis that each block of \( X_n^{(r)}(x) \) of the form \( x^j X_n^{(1)}(x) \), for \( j = 0, \ldots, (r-1)n \), has cardinality according to the \( j \)-th coefficient of the polynomial expansion \((\sum_{\alpha=0}^{n} x^\alpha)^{r-1}\). It is clear that such cardinality shifts to the upper
power when we multiply $X_n^{(r)}(x)$ by $x$, and it shifts by two positions when we multiply $X_n^{(r)}(x)$ by $x^2$, and so on (for example, if the block $x^j X_n^{(1)}(x)$ has cardinality $\beta_{n,j}$ in $X_n^{(r)}(x)$, then $\beta_{n,j}$ will be the cardinality of $x^{j+1} X_n^{(1)}(x)$ in $x X_n^{(r)}(x)$, of $x^{j+2} X_n^{(1)}(x)$ in $x^2 X_n^{(r)}(x)$, and so on). To summarize the situation, we can say that in $X_n^{(r+1)}(x)$ the cardinality for the block of the form $x^j X_n^{(1)}(x)$ corresponds to the coefficient of the $j$-th power in the following polynomial:

$$(\sum_{\alpha=0}^{n} x^\alpha)^{r-1} + x (\sum_{\alpha=0}^{n} x^\alpha)^{r-1} + \cdots + x^n (\sum_{\alpha=0}^{n} x^\alpha)^{r-1} = (\sum_{\alpha=0}^{n} x^\alpha)^r,$$

which concludes the proof.

In Proposition 4.1 we proved, for every $n \geq 1$ and $r \geq 1$, the existence of an index $j_k \in \{0, \ldots, (r-1)n\}$ such that the $k$-th block of $X_n^{(r)}(x)$ can be expressed as $B_n^{(k)}(x) = x^{j_k} X_n^{(1)}(x)$. In the next lemma we give an explicit formula for the index $j_k$ as a function of the degree $n$ and the number of polynomial functions $r$. We denote by mod and $\%$ the operators which, respectively, return the remainder and the quotient of the division between two natural numbers, namely for $a, b, c, d \in \mathbb{N}$, $c = a \mod b$ and $d = a\%b$ means that $a = bd + c$.

**Lemma 4.2.** For every $n, r \geq 1$, the matrix function $X_n^{(r)}(x)$ is composed by $(n+1)^{r-1}$ blocks of the form $B_n^{(k)}(x) = x^{\gamma_{n,r}(k)} X_n^{(1)}(x)$, for $\gamma_{n,r}(k) \in \{0, \ldots, (r-1)n\}$ given by the following formula:

$$\gamma_{n,r}(k) = \sum_{j=0}^{r-1} (k - 1) \mod (n + 1)^{r-j} \% (n + 1)^{r-1-j}, \quad (4.5)$$

for $k = 1, \ldots, (n + 1)^{r-1}$.

**Proof.** By means of Proposition 4.1 we only need to prove the power formula (4.5). In particular, since the Kronecker product is associative, we can rewrite $X_n^{(r)}(x)$ by $X_n^{(r)}(x) = (H_n(x)^\top)^{\otimes (r-1)} \otimes X_n^{(1)}(x)$, where $(H_n(x)^\top)^{\otimes (r-1)}$ is a row vector in $\mathbb{R}^{(n+1)^{r-1}}$, whose elements are the monomials $x^{\gamma_{n,r}(k)}$ whose exponents we want to study, $k = 1, \ldots, (n + 1)^{r-1}$. In particular, we focus on the vector $(H_n(x)^\top)^{\otimes r}$ for simplicity, meaning that we want to prove that the $k$-th element of $(H_n(x)^\top)^{\otimes r}$ is a monomial with exponent given by

$$P_{n,r}^{(k)} = \sum_{j=0}^{r} (k - 1) \mod (n + 1)^{r-j} \% (n + 1)^{r-j}, \quad (4.6)$$

holding for every $k = 1, \ldots, (n + 1)^r$. The result will then follow noticing that $\gamma_{n,r}(k) = P_{n,r-1}^{(k)}$. We proceed by induction on $r \geq 1$.

- $r = 1$: for the vector $(H_n(x)^\top)^{\otimes 1} = H_n(x)^\top = (1, x, \ldots, x^n)^\top$ we can easily notice that the exponent of the $k$-th term equals $k - 1$, for $k = 1, \ldots, (n + 1)$. Let us now look at
equation (4.6):

\[ P_{n,1}^{(k)} = \sum_{j=0}^{1} \{(k - 1) \mod (n + 1)^{2-j}\} \% (n + 1)^{1-j} \]

\[ = \{(k - 1) \mod (n + 1)^{2}\} \% (n + 1) + \{(k - 1) \mod (n + 1)\} \% (n + 1)^0. \quad (4.7) \]

Since \(0 \leq k - 1 \leq n\), the first term in equation (4.7) returns 0, while the second term returns exactly \(k - 1\).

- \(r \rightarrow r + 1\): Let us now assume formula (4.6) holds for \(r\). By the associativity property, 
\[(H_n(x)\top)^{(r+1)} = H_n(x)\top \otimes (H_n(x)\top)^{\otimes r},\]
so that equation (4.6) becomes
\[(H_n(x)\top)^{(r+1)} = \left( (H_n(x)\top)^{\otimes r}, x (H_n(x)\top)^{\otimes r}, x^2 (H_n(x)\top)^{\otimes r}, \ldots, x^n (H_n(x)\top)^{\otimes r} \right), \]
and for each element in \((H_n(x)\top)^{(r+1)}\), the exponent is given by a component corresponding to \(P_{n,r}^{(k)}\) plus an integer \(\alpha \in \{0, \ldots, n\}\). However, we can notice that 
\((H_n(x)\top)^{\otimes r}\) has index \(k = 1, \ldots, (n + 1)^r\), while 
\((H_n(x)\top)^{(r+1)}\) has index \(\hat{k} = 1, \ldots, (n + 1)^{r+1}\). Then in formula (4.8) we must substitute \((k - 1) = (\hat{k} - 1) \mod (n + 1)^r\). Moreover, 
\(\alpha = (\hat{k} - 1) \% (n + 1)^r\). Putting all these considerations together, we can write that (we omit the \(\hat{\cdot}\) on the index \(k\):

\[ P_{n,r+1}^{(k)} = (k - 1) \% (n + 1)^r + \sum_{j=0}^{r} \{(k - 1) \mod (n + 1)^r \mod (n + 1)^{r-j+1}\} \% (n + 1)^{r-j}. \]

In particular, it is easy to see that

\[(k - 1) \mod (n + 1)^r \mod (n + 1)^{r-j+1} = \begin{cases} (k - 1) \mod (n + 1)^{r-j+1} & j \geq 1 \\ (k - 1) \mod (n + 1)^r & j = 0 \end{cases}, \]

so that equation (4.8) becomes

\[ P_{n,r+1}^{(k)} = (k - 1) \% (n + 1)^r + (k - 1) \mod (n + 1)^r \% (n + 1)^r + \]
\[ + \sum_{j=1}^{r} \{(k - 1) \mod (n + 1)^{r-j+1}\} \% (n + 1)^{r-j}, \quad (4.9) \]

and, since \(1 \leq k \leq (n + 1)^{r+1}\), we can also notice that

\[(k - 1) \% (n + 1)^r = (k - 1) \mod (n + 1)^{r+1} \% (n + 1)^r, \]
\[(k - 1) \mod (n + 1)^r \% (n + 1)^{r+1} = (k - 1) \mod (n + 1)^{r+2} \% (n + 1)^{r+1}, \]
so that equation (4.9) can be rewritten as

\[ P_{n,r+1}^{(k)} = \sum_{j=0}^{r+1} \left\{ (k - 1) \mod (n + 1)^{r+j+1} \right\} \% (n + 1)^{r-j}, \]

which concludes the proof.

A straightforward consequence of Proposition 4.1 is that the matrix \( X_n^{(r)}(x) \) contains all the powers of \( x \) from 0 to \((r + 1)n\). This means that, being able to select such powers from \( X_n^{(r)}(x) \) by removing all the duplicates, we would be left with the vector \( H_{(r+1)n}(x) \). From Proposition 4.1 we also get that \( X_n^{(r)}(x) \) is a block matrix and each of the blocks belongs to \( \mathcal{A}_{n+1,n+1} \). However, the matrix \( X_n^{(r)}(x) \) itself does not belong to \( \mathcal{A}_{n+1,(n+1)^r} \) as it can be easily seen in the following example.

**Example 4.1.** Let \( n = 2 \) and \( r = 2 \). Then we get the following block matrix

\[
X_2^{(2)}(x) = \begin{pmatrix}
1 & x & x^2 & x & x^2 & x^3 & x^2 & x^3 & x^4 \\
1 & x^2 & x^3 & x^2 & x^3 & x^4 & x^2 & x^3 & x^4 \\
1 & x^3 & x^4 & x^3 & x^4 & x^5 & x^3 & x^4 & x^5 \\
1 & x^4 & x^5 & x^4 & x^5 & x^6 & x^4 & x^5 & x^6
\end{pmatrix},
\]

whose blocks belong to \( \mathcal{A}_{3,3} \), but the whole matrix does not belong to \( \mathcal{A}_{3,9} \).

In particular, since \( X_n^{(r)}(x) \notin \mathcal{A}_{n+1,(n+1)^r} \), we can not use the L-eliminating matrix defined in Section 3 in order to remove all the duplicates from \( X_n^{(r)}(x) \). We need instead to create a new operator, and the same holds for the L-duplicating matrix. Before doing that, let us state the following two lemmas:

**Lemma 4.3.** It holds that

\[ \text{vec}(X_n^{(m)}(x)) = H_n(x)^{\otimes m+1}. \]

Moreover, after removing all the duplicates from \( \text{vec}(X_n^{(m)}(x)) \), we are left with \( H_{n(m+1)}(x) \).

**Proof.** The result follows from a straightforward verification.

**Lemma 4.4.** There exist an L-eliminating matrix \( E_{nm+1,n+1} \) and an L-duplicating matrix \( D_{nm+1,n+1} \) such that

\[ E_{nm+1,n+1} (H_n(x) \otimes H_{nm}(x)) = H_{n(m+1)}(x), \]  \hspace{1cm} (4.10a)

\[ D_{nm+1,n+1} H_{n(m+1)}(x) = H_n(x) \otimes H_{nm}(x). \]  \hspace{1cm} (4.10b)

**Proof.** From a straightforward verification, it can be seen that the vectorization of the product \( H_n(x)^\top \otimes H_{nm}(x) \) equals \( H_n(x) \otimes H_{nm}(x) \), while its L-vectorization equals \( H_{n(m+1)}(x) \), namely

\[ \text{vec}(H_n(x)^\top \otimes H_{nm}(x)) = H_n(x) \otimes H_{nm}(x), \]

\[ \text{vec}L(H_n(x)^\top \otimes H_{nm}(x)) = H_{n(m+1)}(x). \]  \hspace{1cm} (4.11)

Then, from Theorem 3.3, we know there exists an L-eliminating matrix \( E_{nm+1,n+1} \) transforming the vectorization of \( H_n(x)^\top \otimes H_{nm}(x) \) into its L-vectorization. By means of equation (4.11),
this is equivalent to saying that $E_{n+1,n+1}$ maps $H_n(x) \otimes H_{nm}(x)$ into $H_{n(m+1)}(x)$, which is what is claimed in equation (4.10a). Similarly, by means of Theorem 3.6, there exists an L-duplicating matrix $D_{nm+1,n+1}$ satisfying equation (4.10b).

We have now the tools to construct the $m$-th L-eliminating matrix and the $m$-th L-duplicating matrix.

**Proposition 4.5.** For every $n, m \geq 1$, there exists an $m$-th L-eliminating matrix $E^{(m)}_{n+1} \in \mathbb{R}^{(n(m+1)+1) \times (n+1)^{m+1}}$ removing from $\text{vec}(X^{(m)}_n(x))$ all the duplicates, namely returning $H_{n(m+1)}(x)$. In particular, $E^{(m)}_{n+1}$ is defined by the following recursion formula:

$$
\begin{cases}
E^{(1)}_{n+1} = E_{n+1} \\
E^{(m)}_{n+1} = E_{nm+1,n+1} \left( I_{n+1} \otimes E^{(m-1)}_{n+1} \right) & m \geq 2
\end{cases}
$$

(4.12)

**Proof.** We proceed by induction on $m \geq 1$.

- $m = 1$: see Corollary 3.10 and equation (4.3).
- $m - 1 \rightarrow m$: let us now assume the statement holds for $m - 1$, which means that there exists a matrix $E^{(m-1)}_{n+1}$ that applied to $\text{vec}(X^{(m-1)}_n(x))$ removes all the duplicates. By means of Lemma 4.3, this can be translated in

$$
E^{(m-1)}_{n+1} H_n(x)^\otimes m = H_{nm}(x).
$$

We now multiply both sides in the Kronecker sense by $H_n(x)$, and successively apply on the left the matrix $E_{nm+1,n+1}$, obtaining that

$$
E_{nm+1,n+1} \left( H_n(x) \otimes \left( E^{(m-1)}_{n+1} H_n(x)^\otimes m \right) \right) = E_{nm+1,n+1} \left( H_n(x) \otimes H_{nm}(x) \right).
$$

(4.13)

Considering the identity $H_n(x) = I_{n+1} H_n(x)$, and applying (3.9) for the mixed-product property of the Kronecker product on the left hand side of equation (4.13), and equation (4.10a) on the right hand side, we get

$$
E_{nm+1,n+1} \left( \left( I_{n+1} \otimes E^{(m-1)}_{n+1} \right) (H_n(x) \otimes H_n(x)^\otimes m) \right) = H_{n(m+1)}(x).
$$

Since $H_n(x) \otimes H_n(x)^\otimes m = H_n(x)^\otimes (m+1) = \text{vec}(X^{(m)}_n(x))$ by Lemma 4.3 the matrix $E^{(m)}_{n+1} = E_{nm+1,n+1} \left( I_{n+1} \otimes E^{(m-1)}_{n+1} \right)$ is exactly the one removing all the duplicates from $\text{vec}(X^{(m)}_n(x))$.

The corresponding duplicating matrix takes the following form:

**Proposition 4.6.** For every $n, m \geq 1$, there exists an $m$-th L-duplicating matrix $D^{(m)}_{n+1} \in \mathbb{R}^{(n+1)^{m+1} \times (n(m+1)+1)}$ transforming $H_{n(m+1)}(x)$ into $\text{vec}(X^{(m)}_n(x))$. In particular, $D^{(m)}_{n+1}$ is given by the following recursion formula:

$$
\begin{cases}
D^{(1)}_{n+1} = D_{n+1} \\
D^{(m)}_{n+1} = \left( I_{n+1} \otimes D^{(m-1)}_{n+1} \right) D_{nm+1,n+1} & m \geq 2
\end{cases}
$$

(4.14)
Proof. We proceed by induction on $m \geq 1$.

- $m = 1$: see Corollary 3.11 and equation (4.3).
- $m - 1 \to m$: let us now assume the statement holds for $m - 1$. Then, starting from equation (4.10b) and multiplying both sides with $I_{n+1} \otimes D^{(m-1)}_{n+1}$ we get

$$
\left( I_{n+1} \otimes D^{(m-1)}_{n+1} \right) (D_{nm+1,n+1} H_{n,m+1}(x)) = \left( I_{n+1} \otimes D^{(m-1)}_{n+1} \right) (H_n(x) \otimes H_{nm}(x)).
$$

(4.15)

By means of (3.9) for the mixed-product property of the Kronecker product, the right hand side of equation (4.15) becomes

$$
\left( I_{n+1} \otimes D^{(m-1)}_{n+1} \right) (H_n(x) \otimes H_{nm}(x)) = (I_{n+1} H_n(x)) \otimes \left( D^{(m-1)}_{n+1} H_{nm}(x) \right).
$$

Moreover, by the induction hypothesis, $D^{(m-1)}_{n+1}$ is the matrix transforming $H_{nm}(x)$ into $vec(X_n^{(m-1)}(x))$, that is $D^{(m-1)}_{n+1} H_{nm}(x) = vec(X_n^{(m-1)}(x))$, and, by means of Lemma 4.3 we also have $vec(X_n^{(m-1)}(x)) = H_n(x) \otimes m$. Then equation (4.15) becomes

$$
\left( I_{n+1} \otimes D^{(m-1)}_{n+1} \right) D_{nm+1,n+1} H_{n,m+1}(x) = H_n(x) \otimes H_n(x) \otimes m = H_n(x) \otimes (m+1),
$$

and the matrix $D^{(m)}_{n+1} = \left( I_{n+1} \otimes D^{(m-1)}_{n+1} \right) D_{nm+1,n+1}$ is exactly the one required.

\[\square\]

We want to clarify the shape of $E(m)_{n+1}$ and $D(m)_{n+1}$ with the following example.

Example 4.2. Let $n = 2$ and $m = 2$. Then by Proposition 4.5 $E^{(1)}_3 = E_3 = E_{3,3}$ as found in Example 3.3 while $E^{(2)}_3$ is of the form

$$
E^{(2)}_3 = E_{5,3} \left( I_3 \otimes E^{(1)}_3 \right) = E_{5,3} \left( \begin{array}{ccc}
E^{(1)}_3 & 0 & 0 \\
0 & E^{(1)}_3 & 0 \\
0 & 0 & E^{(1)}_3
\end{array} \right),
$$

$E_{5,3}$ being the L-eliminating matrix acting on matrices in $\mathbb{R}^{3 \times 3}$ according to the definition in equation (3.11). To understand the situation, we can look at the vectorization of $X^{(2)}_2(x)$ in Example 4.1 as follows:

$$
vec(X^{(2)}_2(x)) = vec \left( \begin{array}{ccc}
1 & x & x^2 \\
0 & x^2 & x^3 \\
0 & 0 & x^3
\end{array} \right) vec \left( \begin{array}{ccc}
1 & x & x^2 \\
0 & x^2 & x^3 \\
0 & 0 & x^3
\end{array} \right) vec \left( \begin{array}{ccc}
1 & x & x^2 \\
0 & x^2 & x^3 \\
0 & 0 & x^3
\end{array} \right),
$$

so that, by applying $I_3 \otimes E^{(1)}_3$, we are selecting from each of the three blocks of $X^{(2)}_2(x)$ their L-vectorization (remembering that the L-eliminating matrix acts on the vectorization of a matrix and returns the L-vectorization of the matrix itself), elements which we have marked with a
circle. Namely, we are left with
\[
\left( I_3 \otimes E_3^{(1)} \right) \text{vec}(X_2^{(2)}(x)) = \text{vec} \begin{pmatrix}
1 & x & x^2 \\
\oplus & x^2 & x^3 \\
\oplus & x^3 & x^4 \\
\oplus & x^4 & x^5 \\
\end{pmatrix}.
\tag{4.16}
\]

We can easily notice that the matrix in equation (4.16) lies in \( \mathbb{R}^{5 \times 3} \) and that the elements we need are exactly on the biggest "L" in the left-bottom corner, namely the ones marked with a circle. Then, applying \( E_{5,3} \) gives us \( H_6(x) = H_{2(2+1)}(x) \). Moreover, the matrix on the right hand side of equation (4.16) belongs to \( \mathcal{A}_{5,3} \). Then the corresponding L-duplicating matrix is given by Proposition 4.6 and it looks like the following:

\[
D_3^{(2)} = \left( I_3 \otimes D_3^{(1)} \right) D_{5,3} = \begin{pmatrix}
D_3^{(1)} & 0 & 0 \\
0 & D_3^{(1)} & 0 \\
0 & 0 & D_3^{(1)}
\end{pmatrix} D_{5,3},
\]

\( D_{5,3} \) being the L-eliminating matrix acting on matrices in \( \mathbb{R}^{5 \times 3} \), according to the definition in equation (3.15), while \( D_3^{(1)} = D_3 = D_{5,3} \) as found in Example 3.7. In particular, \( D_{5,3} \) acting on \( H_6(x) \) returns exactly the matrix on the right hand side of equation (4.16), as this latter one belongs to \( \mathcal{A}_{5,3} \), while \( D_3^{(1)} \), acting singularly on each column because of the multiplication with \( I_3 \), namely \( I_3 \otimes D_3^{(1)} \), returns \( X_2^{(2)}(x) \), showing that \( D_3^{(2)} \) is the inverse operator of \( E_3^{(2)} \).

As noticed in Example 4.2 similarly to what proved in Proposition 3.8 and Proposition 3.9 we can prove that the \( m \)-th L-duplicating matrix is the right-inverse of the corresponding L-eliminating matrix and that the composition \( D_{n+1}^{(m)} E_{n+1}^{(m)} \) acts like an identity operator:

**Proposition 4.7.** For every \( n, m \geq 1 \), the matrix \( D_{n+1}^{(m)} \), defined by the recursion (4.11), is the right-inverse of \( E_{n+1}^{(m)} \), defined by (4.12), and the product \( D_{n+1}^{(m)} E_{n+1}^{(m)} \) acts on \( \text{vec}(X_n^{(m)}(x)) \) like an identity operator.

**Proof.** The results follow as a direct consequence of Proposition 3.8 and Proposition 3.9 together with the recursive definitions of, respectively, \( E_{n+1}^{(m)} \) and \( D_{n+1}^{(m)} \).

We have now all the tools to prove the following important result:

**Theorem 4.8.** If \( t \leq s \), then for every \( n \geq 1 \) and \( r \geq 0 \), there exists a matrix \( \tilde{G}_n^{(r)} \in \mathbb{R}^{(n+1)^{r+1} \times (n+1)^{r+1}} \) such that the following expectation formula holds:

\[
\mathbb{E} \left[ H_n(Y(s))^\top \otimes^r H_n(Y(s)) \mid F_t \right] = \text{vec}^{-1} \circ e^{G_n^{(r)}(s-t)} \circ \text{vec} \left( H_n(Y(t))^\top \otimes^r H_n(Y(t)) \right).
\]

In particular, \( \tilde{G}_n^{(r)} \) is given by \( \tilde{G}_n^{(r)} = D_{n+1}^{(r)} \bigcirc G_{n(r+1)}^{(r)} E_{n+1}^{(r)} \), where \( G_{n(r+1)}^{(r)} \) is the generator matrix introduced in Theorem 2.2 and its matrix exponential is given by \( e^{\tilde{G}_n^{(r)}} = D_{n+1}^{(m)} e^{G_{n(r+1)}^{(r)}} E_{n+1}^{(m)} \).

**Proof.** The idea for the proof is similar to that of Theorem 3.12. In particular, by means of Theorem 2.1 remembering that \( X_n^{(r)}(x) = H_n(x)^\top \otimes^r H_n(x) \) by Definition 4.2 we can write that

\[
\mathbb{E} \left[ X_n^{(r)}(Y(s)) \mid F_t \right] = X_n^{(r)}(Y(t)) + \int_t^s \mathbb{E} \left[ G \left( X_n^{(r)}(Y(u)) \right) \mid F_u \right] du,
\]

\[31\)
and applying the \textit{vec} operator on both sides we get
\[
\mathbb{E} \left[ \text{vec} \left( X_n^{(r)}(Y(s)) \right) \mid F_t \right] = \text{vec} \left( X_n^{(r)}(Y(t)) \right) + \int_t^s \mathbb{E} \left[ \text{vec} \left( X_n^{(r)}(Y(u)) \right) \mid F_t \right] du. \quad (4.17)
\]

By Proposition \ref{prop:eliminating_matrix}, we know that there exists an \( r \)-th L-eliminating matrix \( E_{n+1}^{(r)} \) in order to remove all the duplicates from \( \text{vec} \left( X_n^{(r)}(x) \right) \), namely such that
\[
E_{n+1}^{(r)} \text{vec} \left( X_n^{(r)}(x) \right) = H_{n(r+1)}(x). \quad (4.18)
\]

Moreover, from Theorem \ref{thm:generator_matrix}, we know that there exists a generator matrix \( G_{n(r+1)} \) such that
\[
\mathcal{G} H_{n(r+1)}(x) = G_{n(r+1)} H_{n(r+1)}(x), \quad (4.19)
\]
and, finally, by Proposition \ref{prop:duplicating_matrix} we know there exists an \( r \)-th L-duplicating matrix \( D_{n+1}^{(r)} \) such that
\[
D_{n+1}^{(r)} H_{n(r+1)}(x) = \text{vec} \left( X_n^{(r)}(x) \right). \quad (4.20)
\]

Combining equations (4.18), (4.19), (4.20) with equation (4.17) we get
\[
\mathbb{E} \left[ \text{vec} \left( X_n^{(r)}(Y(s)) \right) \mid F_t \right] = \text{vec} \left( X_n^{(r)}(Y(t)) \right) + D_{n+1}^{(r)} G_{n(r+1)} E_{n+1}^{(r)} \int_t^s \mathbb{E} \left[ \text{vec} \left( X_n^{(r)}(Y(u)) \right) \mid F_t \right] du,
\]
and, proceeding the proof as in Theorem \ref{thm:counting_formula}, we get the statement of the theorem, where, in particular, the matrix exponential of \( \tilde{G}_n^{(r)} \) is a consequence of Proposition \ref{prop:matrix_exponential} using the same argument as Lemma \ref{lem:matrix_exponential}.

We have seen that for every \( r \geq 1 \), we can find a generator matrix for \( X_n^{(r)}(x) \) in terms of the generator matrix \( G_n \) introduced for \( H_n(x) \) in Theorem \ref{thm:generator_matrix}. In what follows, we refer to the matrix treated in Section \ref{sec:polynomial_functions} with \( G_n = G_n^{(0)} = \tilde{G}_n^{(0)} \), and we will prove a general formula for the correlators as in equation (4.1). Let us remind that, for \( m \geq 1 \), we consider \( m+1 \) polynomial functions \( p_k \in \text{Pol}_{n,k}(\mathbb{R}) \), \( k = 0, \ldots, m \), in the polynomial process \( Y \), evaluated at different time points, \( t < s_0 < s_1 < \cdots < s_m \). In particular, to each polynomial function \( p_k \), we can associate a vector of coefficients, \( \vec{p}_{k,n} \in \mathbb{R}^{n+1} \), with \( n = \max\{n_0, \ldots, n_m\} \). Then the following theorem solves the general correlator problem.

\textbf{Theorem 4.9.} \textit{For every} \( m \geq 1 \), \textit{there exist} \( m+1 \) \textit{matrices} \( \tilde{G}_n^{(r)} \in \mathbb{R}^{(n+1)^{r+1} \times (n+1)^{r+1}} \) \textit{defined in Theorem} \ref{thm:matrix_expansion} \textit{with} \( n := \max\{n_0, \ldots, n_m\} \), \textit{for} \( r = 0, \ldots, m \), \textit{such that the following expectation formula holds:}
\[
\mathbb{E} \left[ p_m \left( Y(s_0) \right) p_{m-1} \left( Y(s_1) \right) \cdots p_0 \left( Y(s_m) \right) \mid F_t \right] = \vec{p}_{m,n}^{\top} \circ e^{\tilde{G}_n^{(m)}(s_0-t)} \circ \text{vec} \left( H_n(Y(t))^\top \otimes^m H_n(Y(t)) \right) \cdot \prod_{k=1}^m e^{\tilde{G}_n^{(m-k)}(s_k-s_{k-1})} \left\{ I_{n+1} \otimes^{m-k} \vec{p}_{m-k,n} \right\}. \quad (4.21)
\]

\textit{Proof.} We proceed by induction on the number of polynomial functions \( m \geq 1 \).
• $m = 1$: we combine Proposition 3.11 with Theorem 3.13 and Theorem 3.12 to get

$$\mathbb{E}[p_1(Y(s_0))p_0(Y(s_1)) \mid \mathcal{F}_t] = \overleftrightarrow{p}_{1,n} \left\{ vec^{-1} \circ e^{\tilde{G}_n^{(2)}}(s_{0-t}) \circ vec \left( H_n(Y(t))H_n(Y(t))^\top \right) \right\} e^{G_n^{(0)}(s_1-s_0)} \overleftrightarrow{p}_{0,n}.$$ 

The claim follows by $G_n = \tilde{G}_n^{(0)}$, and by property (3.8d), saying that $H_n(Y(t))H_n(Y(t))^\top = H_n(Y(t))^\top \otimes H_n(Y(t))$.

• $m \to m + 1$: let us suppose formula (4.21) holds for $m$ and let us consider $m+1$ polynomial functions. By applying the tower rule, we can write that

$$\mathbb{E}[p_{m+1}(Y(s_0))p_m(Y(s_1))p_{m-1}(Y(s_2)) \cdots p_0(Y(s_{m+1})) \mid \mathcal{F}_t] = \mathbb{E}[p_{m+1}(Y(s_0)) \mathbb{E}[p_m(Y(s_1))p_{m-1}(Y(s_2)) \cdots p_0(Y(s_{m+1})) \mid \mathcal{F}_{s_0}] \mid \mathcal{F}_t].$$

In particular, for the internal expectation, we can use the induction hypothesis which holds for $m$ polynomial functions, just shifting the index of the time points, so that

$$\mathbb{E}[p_{m+1}(Y(s_0))p_m(Y(s_1))p_{m-1}(Y(s_2)) \cdots p_0(Y(s_{m+1})) \mid \mathcal{F}_t] = \mathbb{E}\left[ \overleftrightarrow{p}_{m+1,n} H_n(Y(s_0)) \overleftrightarrow{p}_{m,n} \left\{ vec^{-1} \circ e^{\tilde{G}_n^{(m)}}(s_{1-s_0}) \circ vec \left( H_n(Y(s_0))^\top \otimes^m H_n(Y(s_0)) \right) \right\} \right].$$

Moreover, by Proposition 4.10 to be shown below, we also have the following crucial equality:

$$H_n(Y(s_0)) \overleftrightarrow{p}_{m,n} \left\{ vec^{-1} \circ e^{\tilde{G}_n^{(m)}}(s_{1-s_0}) \circ vec \left( H_n(Y(s_0))^\top \otimes^m H_n(Y(s_0)) \right) \right\} = \left\{ H_n(Y(s_0))^\top \otimes^m H_n(Y(s_0)) \right\} e^{G_n^{(m+1)}}(s_{0-s_0}) \{ I_{n+1} \otimes^m \overleftrightarrow{p}_{m,n} \}.$$ 

Combining the previous results and Theorem 4.8, we can write that

$$\mathbb{E}[p_{m+1}(Y(s_0))p_m(Y(s_1))p_{m-1}(Y(s_2)) \cdots p_0(Y(s_{m+1})) \mid \mathcal{F}_t] = \mathbb{E}\left[ \overleftrightarrow{p}_{m+1,n} \left\{ H_n(Y(s_0))^\top \otimes^m H_n(Y(s_0)) \right\} e^{G_n^{(m)}}(s_{1-s_0}) \{ I_{n+1} \otimes^m \overleftrightarrow{p}_{m,n} \} \right].$$

$$= \mathbb{E}\left[ \overleftrightarrow{p}_{m+1,n} \left\{ vec^{-1} \circ e^{\tilde{G}_n^{(m+1)}}(s_{0-t}) \circ vec \left( H_n(Y(t))^\top \otimes^m H_n(Y(t)) \right) \right\} \right].$$

and rearranging the index in the product on the right hand side of the last equation, we get the formula holding for $m + 1$ polynomial functions, and hence the proof.

☐
For the proof of the last theorem, we have used a result that we are going to state and prove in the following proposition.

**Proposition 4.10.** For every $n,m \geq 1$, the following identity holds:

$$H_n(x)\vec{v}_n^\top \left\{ \text{vec}^{-1} \circ M_n^{(m-1)} \circ \text{vec} \left( H_n(x)^\top \otimes^{m-1} H_n(x) \right) \right\} = \left\{ H_n(x)^\top \otimes^m H_n(x) \right\} M_n^{(m-1)^\top} \left\{ I_{n+1} \otimes^{m-1} \vec{v}_n \right\}, \quad (4.22)$$

$\vec{v}_n$ being a vector in $\mathbb{R}^{n+1}$ and $M_n^{(m-1)}$ a square matrix in $\mathbb{R}^{(n+1)^m \times (n+1)^m}$.

**Proof.** We proceed by induction on the order $m \geq 1$.

- $m = 1$: starting from the left hand side of the identity (4.22), for $m = 1$ we get

  $$H_n(x)\vec{v}_n^\top \left\{ \text{vec}^{-1} \circ M_n^{(0)} \circ \text{vec} \left( H_n(x)^\top \otimes^0 H_n(x) \right) \right\} = H_n(x)\vec{v}_n^\top \left\{ \text{vec}^{-1} \circ M_n^{(0)} \circ \text{vec}(H_n(x)) \right\} = H_n(x)\vec{v}_n^\top M_n^{(0)} H_n(x).$$

  Remember indeed that the $\text{vec}^{-1}$ operator transforms a vector into an object with the same dimension as the argument of the operator $\text{vec}$ previously applied. But in this case the argument of $\text{vec}$ is a vector already, hence both $\text{vec}$ and $\text{vec}^{-1}$ coincide in practise with the identity operator. Moreover, $\vec{v}_n^\top M_n^{(0)} H_n(x) \in \mathbb{R}$, and it equals its transpose. We can then perform the following trivial steps:

  $$H_n(x)\vec{v}_n^\top \left\{ \text{vec}^{-1} \circ M_n^{(0)} \circ \text{vec} \left( H_n(x)^\top \otimes^0 H_n(x) \right) \right\} = H_n(x)H_n(x)^\top M_n^{(0)^\top} \vec{v}_n = \left\{ H_n(x)^\top \otimes^1 H_n(x) \right\} M_n^{(0)^\top} \left\{ I_{n+1} \otimes^0 \vec{v}_n \right\},$$

  which proves the base case.

- $m \rightarrow m + 1$: let us assume now the identity (4.22) holds for $m$, and we consider

  $$H_n(x)\vec{v}_n^\top \left\{ \text{vec}^{-1} \circ M_n^{(m)} \circ \text{vec} \left( H_n(x)^\top \otimes^m H_n(x) \right) \right\} \cdot$$

  In particular, $M_n^{(m)} \in \mathbb{R}^{(n+1)^{m+1} \times (n+1)^{m+1}}$ can be seen as made up of $(n+1)^2$ matrices of the form $M_{i,j}^{(m-1)} \in \mathbb{R}^{(n+1)^m \times (n+1)^m}$, for $1 \leq i,j \leq n+1$, so that $M_n^{(m)}$ looks like

  $$M_n^{(m)} = \begin{pmatrix} M_{1,1}^{(m-1)} & \cdots & M_{1,n+1}^{(m-1)} \\ \vdots & \ddots & \vdots \\ M_{n+1,1}^{(m-1)} & \cdots & M_{n+1,n+1}^{(m-1)} \end{pmatrix}.$$

  The idea is then to break up the matrix $M_n^{(m)}$ into these sub-matrices, for which we know the statement holds, by the induction hypothesis. In what follows, step by step, starting from $H_n(x)^\top \otimes^m H_n(x)$ we will apply in the following order: first the $\text{vec}$ operator, then the matrix $M_n^{(m-1)}$, next the $\text{vec}^{-1}$ operator, and finally the matrix $H_n(x)\vec{v}_n^\top$. At this point we will be able to apply the induction hypothesis, and show the result required.
By Lemma 4.3 and the associativity property of the Kronecker product, we can write that

\[
vec\left( H_n(x)^\top \otimes^m H_n(x) \right) = H_n(x)^\otimes^{(m+1)} = H_n(x) \otimes H_n(x)^\otimes^m = \begin{pmatrix} H_n(x)^\otimes^m \\ xH_n(x)^\otimes^m \\ \vdots \\ x^nH_n(x)^\otimes^m \end{pmatrix},
\]

where each \( x^kH_n(x)^\otimes^m \in \mathbb{R}^{(n+1)^m}, \ k = 0, \ldots, n, \) thus \( M_n^{(m)} vec \left( H_n(x)^\top \otimes^m H_n(x) \right) \) reads like

\[
\begin{pmatrix} M_{1,1}^{(m-1)} H_n(x)^\otimes^m + M_{1,2}^{(m-1)} xH_n(x)^\otimes^m + \cdots + M_{1,n+1}^{(m-1)} x^nH_n(x)^\otimes^m \\ \vdots \\ M_{n+1,1}^{(m-1)} H_n(x)^\otimes^m + M_{n+1,2}^{(m-1)} xH_n(x)^\otimes^m + \cdots + M_{n+1,n+1}^{(m-1)} x^nH_n(x)^\otimes^m \end{pmatrix}.
\]

Applying the \( vec^{-1} \) operator to the last matrix obtained, by linearity we get:

\[
\begin{pmatrix} vec^{-1} \left( M_{1,1}^{(m-1)} H_n(x)^\otimes^m \right) + \cdots + x^nvec^{-1} \left( M_{1,n+1}^{(m-1)} H_n(x)^\otimes^m \right), \\ \vdots \\ vec^{-1} \left( M_{n+1,1}^{(m-1)} H_n(x)^\otimes^m \right) + \cdots + x^nvec^{-1} \left( M_{n+1,n+1}^{(m-1)} H_n(x)^\otimes^m \right) \end{pmatrix},
\]

where each element \( vec^{-1} \left( M_{i,j}^{(m-1)} H_n(x)^\otimes^m \right) \in \mathbb{R}^{(n+1)\times(n+1)^{m-1}}, \ 1 \leq i, j \leq n + 1. \)

Finally, multiplying the above equation by \( H_n(x)\vec{v}_n^\top \), we obtain that:

\[
H_n(x)\vec{v}_n^\top \left\{ vec^{-1} \circ M_n^{(m)} \circ vec \left( H_n(x)^\top \otimes^m H_n(x) \right) \right\} = \left( H_n(x)\vec{v}_n^\top vec^{-1} \left( M_{1,1}^{(m-1)} H_n(x)^\otimes^m \right) + \cdots + x^nH_n(x)\vec{v}_n^\top vec^{-1} \left( M_{1,n+1}^{(m-1)} H_n(x)^\otimes^m \right), \\ \vdots \\ H_n(x)\vec{v}_n^\top vec^{-1} \left( M_{n+1,1}^{(m-1)} H_n(x)^\otimes^m \right) + \cdots + x^nH_n(x)\vec{v}_n^\top vec^{-1} \left( M_{n+1,n+1}^{(m-1)} H_n(x)^\otimes^m \right) \right) \cdot \]

Since \( H_n(x)^\otimes^m = vec \left( H_n^\top (x)^\otimes^{m-1} H_n(x) \right), \) we can apply the induction hypothesis to each term, namely for \( 1 \leq i, j \leq n + 1 \) it holds:

\[
H_n(x)\vec{v}_n^\top vec^{-1} \left( M_{i,j}^{(m-1)} H_n(x)^\otimes^m \right) = \left\{ H_n(x)^\top \otimes^m H_n(x) \right\} M_{i,j}^{(m-1)} \left\{ I_{n+1} \otimes^{m-1} \vec{v}_n \right\},
\]
so that we get

\[
H_n(x) \vec{v}_n^\top \left\{ \text{vec}^{-1} \circ M_n^{(m)} \circ \text{vec} \left( H_n(x)^\top \otimes^m H_n(x) \right) \right\} \\
= \left( \{ H_n(x)^\top \otimes^m H_n(x) \} \right) M_{1,1}^{(m-1)\top} \left\{ I_{n+1} \otimes^{m-1} \vec{v}_n \right\} + \cdots \\
+ x^n \left\{ H_n(x)^\top \otimes^m H_n(x) \right\} M_{1,n+1}^{(m-1)\top} \left\{ I_{n+1} \otimes^{m-1} \vec{v}_n \right\},
\]

\[
\cdots, \\
\{ H_n(x)^\top \otimes^m H_n(x) \} M_{n+1,1}^{(m-1)\top} \left\{ I_{n+1} \otimes^{m-1} \vec{v}_n \right\} + \cdots \\
+ x^n \left\{ H_n(x)^\top \otimes^m H_n(x) \right\} M_{n+1,n+1}^{(m-1)\top} \left\{ I_{n+1} \otimes^{m-1} \vec{v}_n \right\},
\]

which can also be seen as the following matrix product:

\[
\left( \{ H_n(x)^\top \otimes^m H_n(x) \} \right) \cdots x^n \left\{ H_n(x)^\top \otimes^m H_n(x) \right\} \\
= \left( \{ H_n(x)^\top \otimes^m H_n(x) \} \right) \\
= H_n(x)^\top \otimes \left\{ H_n(x)^\top \otimes^m H_n(x) \right\} = H_n(x)^\top \otimes^{m+1} H_n(x),
\]

so that we can conclude with the expression

\[
\begin{pmatrix}
M_{1,1}^{(m-1)\top} \left\{ I_{n+1} \otimes^{m-1} \vec{v}_n \right\} & \cdots & M_{n+1,1}^{(m-1)\top} \left\{ I_{n+1} \otimes^{m-1} \vec{v}_n \right\} \\
M_{1,n+1}^{(m-1)\top} \left\{ I_{n+1} \otimes^{m-1} \vec{v}_n \right\} & \cdots & M_{n+1,n+1}^{(m-1)\top} \left\{ I_{n+1} \otimes^{m-1} \vec{v}_n \right\}
\end{pmatrix}
\]

\[
= \begin{pmatrix}
M_{1,1}^{(m-1)\top} & \cdots & M_{n+1,1}^{(m-1)\top} \\
M_{1,n+1}^{(m-1)\top} & \cdots & M_{n+1,n+1}^{(m-1)\top}
\end{pmatrix}
\begin{pmatrix}
\{ I_{n+1} \otimes^{m-1} \vec{v}_n \} & \cdots & 0 \\
0 & \cdots & \{ I_{n+1} \otimes^{m-1} \vec{v}_n \}
\end{pmatrix},
\]

where the first matrix on the right hand side is nothing but \( M_n^{(m)\top} \), while the second is exactly \( I_{n+1} \otimes \{ I_{n+1} \otimes^{m-1} \vec{v}_n \} = I_{n+1} \otimes^m \vec{v}_n \). This means we have proved

\[
H_n(x) \vec{v}_n^\top \left\{ \text{vec}^{-1} \circ M_n^{(m)} \circ \text{vec} \left( H_n(x)^\top \otimes^m H_n(x) \right) \right\} \\
= \left( \{ H_n(x)^\top \otimes^{m+1} H_n(x) \} \right) M_n^{(m)\top} \left\{ I_{n+1} \otimes^m \vec{v}_n \right\},
\]

and therefore reached the claim.
5 The recursions

In this section, we focus on the generator matrix $G_n$ defined in Theorem 2.2. In particular, we provide a recursion formula that allows the construction of $G_{n+1}$ given $G_n$, but also a second recursion for the matrix exponential $e^{G_n}$ in numerical and computational purposes: with such recursions, one can easily speed up the code when it comes to applying polynomial processes and their moments formula, as stated in Theorem 2.3. Inspired by Example 2.1 and Example 2.2, we start with the following result:

**Theorem 5.1.** For every $n \geq 2$, the generator matrix $G_n \in \mathbb{R}^{(n+1) \times (n+1)}$ satisfying (2.6) is given by the following recursion:

$$G_n = \left( \frac{G_{n-1}}{\bar{a}_n} \right) \begin{pmatrix} \bar{o}_n \\ c_n \end{pmatrix}, \quad G_1 = \begin{pmatrix} 0 & 0 \\ b_0 & b_1 \end{pmatrix}.$$  

Here $\bar{o}_n$ is a $n$-dimensional vector of 0’s, $\bar{a}_n = (a_n^0, a_n^{n-1}, \ldots, a_n^0) \in \mathbb{R}^n$ with entries defined by

$$a_n^i = \sum_{k=2}^{n} \binom{n}{k} \binom{k}{1} \int_{\mathbb{R}} \delta_0(z)\delta_1(z)^{k-1} \ell(dz) + nb_0 + \frac{1}{2} n(n-1)\sigma_1,$$
$$a_n^2 = \sum_{k=2}^{n} \binom{n}{k} \binom{k}{2} \int_{\mathbb{R}} \delta_0(z)^2\delta_1(z)^{k-2} \ell(dz) + \frac{1}{2} n(n-1)\sigma_0, \quad \text{(5.1)}$$
$$a_n^i = \sum_{k=i}^{n} \binom{n}{k} \binom{k}{i} \int_{\mathbb{R}} \delta_0(z)^{i}\delta_1(z)^{k-i} \ell(dz), \quad \text{for } i = 3, \ldots, n,$$

and $c_n \in \mathbb{R}$ is given by

$$c_n = nb_1 + \frac{1}{2} n(n-1)\sigma_2 + \sum_{k=2}^{n} \binom{n}{k} \binom{k}{0} \int_{\mathbb{R}} \delta_1(z)^k \ell(dz). \quad \text{(5.2)}$$

**Proof.** We can proceed by induction on the dimension $n \geq 2$.

- $n = 2$: see Example 2.2

- $n - 1 \rightarrow n$: let us now assume the recursion formula holds for $n - 1$. We then need to find $Gx^n$. By means of equations (2.4) and (2.5), we can write that:

$$Gx^n = n(b_0 + b_1)x^{n-1} + \frac{1}{2} n(n-1)(\sigma_0 + \sigma_1 x + \sigma_2 x^2)x^{n-2} +$$
$$\int_{\mathbb{R}} ((x + \delta_0(z) + \delta_1(z)x)^n - x^n - nx^{n-1} (\delta_0(z) + \delta_1(z)x)) \ell(dz).$$

In particular, by binomial expansion

$$\int_{\mathbb{R}} ((x + \delta_0(z) + \delta_1(z)x)^n - x^n - nx^{n-1} (\delta_0(z) + \delta_1(z)x)) \ell(dz)$$

$$= \sum_{k=2}^{n} \binom{n}{k} x^{n-k} \int_{\mathbb{R}} (\delta_0(z) + \delta_1(z)x)^k \ell(dz) = \sum_{k=2}^{n} \sum_{i=0}^{k} \binom{n}{k} \binom{k}{i} x^{n-i} \int_{\mathbb{R}} \delta_0(z)^i \delta_1(z)^{k-i} \ell(dz)$$

$$= \sum_{i=0}^{n} \sum_{k=\max(2,i)}^{n} \binom{n}{k} \binom{k}{i} \int_{\mathbb{R}} \delta_0(z)^i \delta_1(z)^{k-i} \ell(dz) x^{n-i}, \quad 37$$
so that

\[ G x^n = \left( \frac{1}{2} n (n-1) \sigma_2 \right) x^n + \left( \frac{1}{2} n (n-1) \sigma_1 - n (n-1) \sigma_0 \right) x^{n-2} + \sum_{i=0}^{n} \left[ \sum_{k=\max(2,i)}^{n} \left( \begin{array}{c} n \\ k \\ i \end{array} \right) \int_{\mathbb{R}} \delta_0(z)^i \delta_1(z)^{k-i} \ell(dz) \right] x^{n-i}, \]

that must be rearranged in order to collect the coefficients of \( x^k \), for \( k = 0, \ldots, n \), to be inserted in the last row of \( G_n \). One can see that these terms lead to \( (a_n^0, a_n^{-1}, \ldots, a_n^1, c_n)^\top \) as above defined. This concludes the proof.

\[ \square \]

**Remark 5.1.** From Theorem 5.1 it is easy to notice that for every \( n \geq 1 \) the generator matrix \( G_n \) is lower triangular. Moreover, for \( n \geq 2 \) the main diagonal of \( G_n \) is of the form

\[ \text{diag} (G_n) = (0, b_1, c_2, c_3, \ldots, c_n)^\top, \quad (5.3) \]

so that, in particular, the matrix \( G_n \) is not invertible.

**Remark 5.2.** We denote by \( d \in \mathbb{Z} \), an integer number, the order of a diagonal in \( G_n \), which means that \( d = 0 \) corresponds to the main diagonal, \( d = -1 \) corresponds to the diagonal right below the main one, and so on. In particular, since \( G_n \) is a lower triangular matrix, we take into account only the diagonals corresponding to a negative value of \( d \). Then, from equations (5.1) and (5.2), we can make the following considerations:

- the elements on the main diagonal (i.e. \( d = 0 \)) are in the span of

\[ \left\{ b_1, \sigma_2, \int_{\mathbb{R}} \delta_1(z)^2 \ell(dz), \ldots, \int_{\mathbb{R}} \delta_1(z)^n \ell(dz) \right\}; \]

- the elements on the diagonal \( d = -1 \) are in the span of

\[ \left\{ b_0, \sigma_1, \int_{\mathbb{R}} \delta_0(z) \delta_1(z) \ell(dz), \ldots, \int_{\mathbb{R}} \delta_0(z) \delta_1(z)^{n-1} \ell(dz) \right\}; \]

- the elements on the diagonal \( d = -2 \) are in the span of

\[ \left\{ \sigma_0, \int_{\mathbb{R}} \delta_0(z)^2 \ell(dz), \int_{\mathbb{R}} \delta_0(z)^2 \delta_1(z) \ell(dz), \ldots, \int_{\mathbb{R}} \delta_0(z)^2 \delta_1(z)^{n-2} \ell(dz) \right\}; \]

- the elements on the diagonal \( d = -3 \) are in the span of

\[ \left\{ \int_{\mathbb{R}} \delta_0(z)^3 \ell(dz), \int_{\mathbb{R}} \delta_0(z)^3 \delta_1(z) \ell(dz), \ldots, \int_{\mathbb{R}} \delta_0(z)^3 \delta_1(z)^{n-3} \ell(dz) \right\}; \]

- and so on.

From Remark 5.1 and Remark 5.2 we get the following result:

**Lemma 5.2.** For every \( n \geq 1 \), if \( \delta_0(z) \equiv 0 \) on \( \mathbb{R} \), then \( G_n \) is a (lower) tri-diagonal matrix.

**Proof.** The proof is a straightforward consequence from the considerations in Remark 5.1 and Remark 5.2. \( \square \)
5.1 Matrix exponential

In view of the result stated in Theorem 2.3, we need to define the exponential of the generator matrix $G_n$, namely $e^{G_n t}$, for $t \geq 0$. We want to provide a recursion formula for such matrix exponential. From now on, we denote by $I_n$ the identity matrix in the space $\mathbb{R}^{n \times n}$. Let us focus first on $n = 1$. Then we can prove:

**Lemma 5.3.** If $b_1 \neq 0$, then for every $k \geq 1$, $G_1^k = \begin{pmatrix} 0 & 0 \\ b_0 b_1^{k-1} & b_1^k \end{pmatrix}$. If $b_1 = 0$, then for every $k \geq 2$, $G_1^k = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, that is, $G_1$ is nilpotent with index 2.

**Proof.** Let $b_1 \neq 0$. We can proceed by induction on the exponent $k \geq 1$.

- **$k = 1$:** trivial.
- **$k - 1 \rightarrow k$:** let us now assume the statement holds for $k - 1$. Then we can write that:

  $$G_1^k = G_1^{k-1} G_1 = \begin{pmatrix} 0 & 0 \\ b_0 b_1^{k-2} & b_1^{k-1} \end{pmatrix} \begin{pmatrix} 0 & 0 \\ b_0 & b_1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ b_0 b_1^{k-1} & b_1^k \end{pmatrix}.$$

  which is exactly what stated in the lemma for $b_1 \neq 0$.

The case $b_1 = 0$ can be easily verified by matrix multiplication, so that we conclude the proof.

Considering the definition of the exponential function as the infinite sum of powers, namely $e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$, we can then, by means of Lemma 5.3, find the explicit expression for the exponential of $G_1$:

**Lemma 5.4.** If $b_1 \neq 0$, then $e^{G_1 t} = \begin{pmatrix} b_0 \frac{1}{b_1} (e^{b_1 t} - 1) & 0 \\ 0 & e^{b_1 t} \end{pmatrix}$. If $b_1 = 0$, then $e^{G_1 t} = \begin{pmatrix} 1 & 0 \\ b_0 t & 1 \end{pmatrix}$.

**Proof.** We can consider the definition of the matrix exponential as infinite sum of powers. Then by means of Lemma 5.3 and remembering that $G_1^0 = I_2$, we get:

$$e^{G_1 t} = \sum_{k=0}^{\infty} \frac{(G_1 t)^k}{k!} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \sum_{k=1}^{\infty} \frac{t^k}{k!} \begin{pmatrix} 0 & 0 \\ b_0 b_1^{k-1} & b_1^k \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ b_0 \sum_{k=1}^{\infty} \frac{b_0^t b_1^{k-1}}{k!} + \sum_{k=1}^{\infty} \frac{b_0^t b_1^{k-1}}{k!} \end{pmatrix}.$$

In particular,

$$b_0 \sum_{k=1}^{\infty} \frac{t^k b_1^{k-1}}{k!} = \frac{b_0}{b_1} \sum_{k=1}^{\infty} \frac{(b_1 t)^k}{k!} = \frac{b_0}{b_1} \left( \sum_{k=1}^{\infty} \frac{(b_1 t)^k}{k!} - 1 \right) = \frac{b_0}{b_1} \left( e^{b_1 t} - 1 \right),$$

and, similarly, $1 + \sum_{k=1}^{\infty} \frac{t^k b_1^k}{k!} = \sum_{k=0}^{\infty} \frac{(b_1 t)^k}{k!} = e^{b_1 t}$, which gives the statement of the lemma when $b_1 \neq 0$.

Let now $b_1 = 0$. Then, by Lemma 5.3

$$e^{G_1 t} = \sum_{k=0}^{\infty} \frac{t^k G_1^k}{k!} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + t \begin{pmatrix} 0 & 0 \\ b_0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ b_0 t & 1 \end{pmatrix},$$

which concludes the proof.
Lemma [5.4] gives us the formula for the exponential of $G_1$. We want now to provide a result for the matrix exponential $e^{S_{x,t}}$, which holds for every $n \geq 2$. Let us first start with the following result:

**Lemma 5.5.** For a fixed $n \geq 2$, the matrices of the family $\{ \Lambda_r := c_r I_r - G_{r-1} \}_{r=2}^n$, where $c_r$ is defined in equation (5.2), are invertible if the following conditions are satisfied:

\[
\begin{cases}
  c_j \neq 0 & \forall \ 2 \leq j \leq n \\
  c_j \neq c_i & \forall \ 1 \leq i < j \leq n
\end{cases}
\]  

(5.4)

**Proof.** We are asking the matrix $\Lambda_r$ to be invertible for every $2 \leq r \leq n$. Then, for a fixed $r$, the determinant of $\Lambda_r$ must be different from zero. In particular, since $G_{r-1}$ is a (lower) triangular matrix for every $2 \leq r \leq n$, then $\Lambda_r$ is a (lower) triangular matrix as well, and its determinant is given by the product of the elements on the main diagonal. By means of Theorem [5.1] and equation (5.3) we then get:

\[
\det (c_r I_r - G_{r-1}) = c_r \prod_{j=1}^{r-1} (c_r - c_j) \neq 0,
\]  

(5.5)

where we have denoted by $c_1$ the element $c_1 = b_1$. In particular, condition (5.4) is equivalent to ask that $c_r \neq 0$ and $c_r \neq c_j$ for every $1 \leq j \leq r - 1$. One can easily see that, asking these conditions to hold for every $2 \leq r \leq n$, means to ask that all the coefficients $\{ c_j \}_{j=2}^n$ are not null, and that $\{ c_j \}_{j=1}^n$ are all different among each others.

A particular case of Lemma [5.5] occurs when $\delta_1(z) \equiv 0$ on $\mathbb{R}$:

**Corollary 5.6.** If $\delta_1(z) \equiv 0$ on $\mathbb{R}$, then condition (5.4) is equivalent to

\[
b_1 \neq -\frac{k}{2} \sigma_2, \quad \forall \ 1 \leq k \leq 2(n-1).
\]

**Proof.** Let us consider the definition of the coefficients $c_j$ in equation (5.2): if $\delta_1(z) \equiv 0$, then for every $j = 2, \ldots, n$, the coefficient $c_j$ takes the form

\[
c_j = jb_1 + \frac{1}{2} j(j-1) \sigma_2,
\]

so that the first condition in (5.4), namely $c_j \neq 0 \ \forall \ 2 \leq j \leq n$, equals $b_1 \neq -\frac{j(j-1)}{2} \sigma_2$, equivalent also to

\[
b_1 \neq -\frac{k}{2} \sigma_2, \quad \forall \ 1 \leq k \leq n-1.
\]  

(5.6)

Let us look now at the second condition in (5.4): we require $c_j \neq c_i \ \forall \ 1 \leq i < j \leq n$, that is

\[
jb_1 + \frac{1}{2} j(j-1) \sigma_2 \neq ib_1 + \frac{1}{2} i(i-1) \sigma_2, \quad \forall \ 1 \leq j < i \leq n,
\]

which, after some simplifications, can be rewritten as $b_1 \neq -\frac{j(j-1)}{2} \sigma_2, \ \forall \ 1 \leq j < i \leq n$. In particular, since $3 \leq j + i \leq 2n - 1$, this is also equivalent to

\[
b_1 \neq -\frac{k}{2} \sigma_2, \quad \forall \ 2 \leq k \leq 2(n-1).
\]

Adding this to the condition previously found in equation (5.6), we conclude the proof. \qed
Remark 5.3. By means of Corollary 5.6 if $\delta_1(z) \equiv 0$ on $\mathbb{R}$, then the coefficients $b_1$ and $\sigma_2$ cannot be simultaneously equal to 0.

Under the conditions stated in Lemma 5.5, we can find the explicit formula for the matrix exponential of $G_n$, starting from its powers formula in the following lemma:

**Lemma 5.7.** Under condition 5.4, for every $n \geq 2$ and every $k \geq 1$

\[
G_n^k = \left( \frac{G_{n-1}^k}{\bar{a}_n^\top \Lambda_n^{-1}} \left( \epsilon_n^k I_n - G_{n-1}^k \right) \bar{e}_n^k \right),
\]

for $\Lambda_n$ as introduced in Lemma 5.5.

**Proof.** Let us first notice that, under condition 5.4, the inverse matrix $\Lambda_r^{-1}$ is well defined for every $2 \leq r \leq n$ by Lemma 5.5. Then, we can proceed by induction on the exponent $k \geq 1$.

- $k = 1$: trivial, by definition of $G_n$ and $\Lambda_n$.
- $k - 1 \rightarrow k$: let us now assume the formula holds for $k - 1$. We can then write the following:

\[
G_n^k = G_n^{k-1} G_n = \left( \frac{G_{n-1}^{k-1}}{\bar{a}_n^\top \Lambda_n^{-1}} \left( \epsilon_n^{k-1} I_n - G_{n-1}^{k-1} \right) \bar{e}_n^{k-1} \right) \left( \frac{G_{n-1}^{k-1}}{\bar{a}_n^\top \Lambda_n^{-1}} \left( \epsilon_n^{k-1} I_n - G_{n-1}^{k-1} \right) \bar{e}_n^{k-1} \right),
\]

and using the identity $\Lambda_n \epsilon_n^{k-1} = (\epsilon_n^k I_n - \epsilon_n^{k-1} G_{n-1})$, we conclude the proof.

Remark 5.4. Let us notice that in Lemma 5.7, by means of condition 5.4, we are asking $\Lambda_r$ to be invertible for all $2 \leq r \leq n$ and not only for $r = n$. The reason for that stays in the recursion formula we have constructed for $G_n$, and consequently for $G_n^k$. Indeed, looking at equation 5.7, we see that $G_n^k$ involves $\Lambda_r^{-1}$, but it also involves $G_{n-1}^k$, that means it involves $\Lambda_{n-1}^{-1}$, and so on. Thus the matrices $\Lambda_r^{-1}$ are all involved in $G_n^k$ for every $2 \leq r \leq n$.

**Proposition 5.8.** Under condition 5.4, the following recursion formula for the exponential of the generator matrix holds:

\[
e^{G_n t} = \left( \frac{e^{G_{n-1} t}}{\bar{a}_n^\top \Lambda_n^{-1}} \left( e^{G_{n-1} t} I_n - e^{G_{n-1} t} \right) \bar{e}_n^t \right).
\]

**Proof.** We can start from the definition of exponential function as infinite sum of powers. Then by Lemma 5.7 and remembering that $G_n^0 = I_{n+1}$, we get:

\[
e^{G_n t} = \sum_{k=0}^{\infty} \frac{(G_n t)^k}{k!} = I_{n+1} + \sum_{k=1}^{\infty} \frac{t^k}{k!} \left( \bar{a}_n^\top \Lambda_n^{-1} \left( \epsilon_n^k I_n - G_{n-1}^k \right) \bar{e}_n^k \right)
\]

\[
= \left( \frac{I_n + \sum_{k=1}^{\infty} \frac{(G_{n-1} t)^k}{k!}}{\bar{a}_n^\top \Lambda_n^{-1} \sum_{k=1}^{\infty} \frac{(G_{n-1} t)^k}{k!}} \left( \epsilon_n^k I_n - G_{n-1}^k \right) \bar{e}_n^k \right)
\]

\[
= \left( \frac{\epsilon_n^t}{\bar{a}_n^\top \Lambda_n^{-1} \left( I_n \sum_{k=1}^{\infty} \frac{(G_{n-1} t)^k}{k!} - \sum_{k=1}^{\infty} \frac{(G_{n-1} t)^k}{k!} \bar{e}_n^t \right) \bar{e}_n^t \right).
\]

In particular, using the fact that $\sum_{k=1}^{\infty} \frac{(G_{n-1} t)^k}{k!} = e^{G_{n-1} t} - 1$ and $\sum_{k=1}^{\infty} \frac{(G_{n-1} t)^k}{k!} = e^{G_{n-1} t} - I_n$, we conclude the proof.

\[\square\]
6 Conclusions

We found an explicit formula for the correlators of processes of polynomial type. This formula can be used for option pricing, such as for path-dependent options or in a stochastic volatility model context. The strength of our formula is that it consists of linear combinations of exponentials of the generator matrix associated to the polynomial process, in a similar way to the well-known moments formula for polynomial processes. To make this work, the introduction of two new linear operators, called, respectively, the L-eliminating matrix and the L-duplicating matrix, was necessary. However, due to the high sparsity properties of these two operators, and due to our recursions for the generator matrix and its exponential, the framework developed can be numerically implemented allowing for fast algorithms even when the dimensions of the problem increase. We aim at studying this in more details in a future work, building on the analysis in this article, and focusing on the performances of our formulas when applied to option pricing, in terms of both accuracy and complexity.
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