Foamy structure of spacetime

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We examine spectrum of the physical volume operator within the non-standard loop quantum cosmology. The spectrum is discrete with equally distant levels defining a quantum of the volume. The discreteness may imply a foamy structure of spacetime at semi-classical level which may be detected in astro-cosmo observations.

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1. Introduction. Various forms of discreteness of spacetime underly many approaches in fundamental physics. Just to name a few: noncommutative geometry \cite{1}, causal sets approach \cite{2}, gravitational Wilson loops \cite{3}, Regge calculus \cite{4}, path integral over geometries \cite{5}, spin foam model \cite{6}, and categories \cite{7}. The discreteness may translate at the semi-classical level into a foamy structure of space. Such expected property of spacetime creates large activity in observational astrophysics and cosmology (see, e.g. Lorentz and CPT violation \cite{8}, dispersion of cosmic photons \cite{9}, electrons \cite{9} and neutrinos \cite{11}, birefringence effects \cite{12}).

The goal of our paper is presentation of the physics of geometry at short distances. We study the spectrum of the volume operator. We find that the spectrum is bounded from below and discrete. The minimum distance between the levels of the spectrum defines a quantum of the volume. We suggest that there may exist elementary quanta of energy connected with the multiplicity of the quantum of the volume, i.e. with a foamy like structure of space. They may have a form of elementary objects like photon, electron, proton, dark matter particle, etc.

Our results suggest that the foamy structure of space is likely to be a real property of the Universe so its identification via astro-cosmo observations has sound motivation and is important for the fundamental physics.

Our results are obtained within the non-standard loop quantum cosmology (LQC) developed recently \cite{13, 14, 15, 16, 17}. In this method one first solves classical constraints to identify the physical phase space and finds an algebra of observables, then one imposes quantum rules. Standard LQC \cite{20, 21} means first quantization of the kinematics, then imposition of constraints in the form of operators acting on the kinematical Hilbert space. Both methods should ‘commute’, i.e. give the same results. In the case of quantization of the Maxwell electrodynamics such treatment of constraints leads to equivalent results \cite{22}. Thus, another aim of our paper is of methodological nature: testing the equivalence of both methods in the case of gravitational interaction.

This Letter is meant to address a wide physical community. It popularizes and interprets the results of a ‘technical’ paper \cite{17} directed to experts in quantum cosmology, and submitted for publication elsewhere.

A direct way of testing the singularity aspects of a given cosmological model is by an examination of the energy density of matter as a function of time \cite{14}. However, the geometry of space, as a function of time, is sensitive to these aspects too. We carry out the corresponding discussion in \cite{17}.

2. Modified Hamiltonian. In what follows, for simplicity of exposition, we restrict ourselves to the quantum flat Friedmann-Robertson-Walker (FRW) model with massless scalar field.
The classical dynamics may be defined by the FRW Hamiltonian
\[ H = N\left(-\frac{3}{8\pi G\gamma^2} \beta^2 v + \frac{p_\beta^2}{2v}\right) \approx 0, \] (1)
which is known to be a dynamical constraint \[14\]: \(N\) denotes the lapse function and \(\gamma\) is the so-called Barbaro-Immirzi parameter; \((\beta, v, \phi, p_\beta)\) are the kinematical phase space variables; \(v^{1/3} \sim a\), where \(a\) is the scale factor; \(\beta \sim \dot{a}/a\) so it corresponds to the Hubble parameter; \(p_\beta\) is the momentum of the massless scalar field \(\phi\).

The Hamiltonian modified by the so-called holonomy functions specific to LQC, corresponding to \[1\], is found to be \[15\]
\[ H^{(\lambda)} = N\left(-\frac{3}{8\pi G\gamma^2} \sin^2(\lambda\beta)\beta^2 - \frac{p_\beta^2}{2v}\right) \approx 0. \] (2)
The parameter \(\lambda\) is a free parameter of the non-standard LQC method parameterizing holonomies. It is clear that in the limit \(\lambda \rightarrow 0\) the Hamiltonian \[2\] turns into \[1\]. In what follows we consider \[2\] in the gauge
\[ N^{-1} := \frac{3}{8\pi G\gamma^2 v} \left(\kappa\gamma|p_\beta| + v \frac{|\sin(\lambda\beta)|}{\lambda}\right), \] (3)
where \(\kappa^2 \equiv 4\pi G/3\). Consequently, \[2\] leads to the dynamical constraint
\[ H^{(\lambda)} := \kappa\gamma|p_\beta| - v \frac{|\sin(\lambda\beta)|}{\lambda} \approx 0. \] (4)
The FRW constraint in the gauge corresponding to \[3\], reads
\[ H := \kappa\gamma|p_\beta| - v |\beta| \approx 0. \] (5)
Since \(\sin(\cdot)\) is bounded from above, there exists \(\epsilon \in \mathbb{R}\), due to \[1\], such that \(v > \epsilon > 0\). Thus, there exists \(\epsilon \in \mathbb{R}\) such that the scale factor \(a > \epsilon > 0\). As \(\sin(\cdot)\) is a periodic function, the variable \(\beta\) which occurs in \[4\] is bounded. Thus, the Hubble parameter is bounded, which means that there is no Big-Bang. The variables \(\beta\) and \(v\) which satisfy \[5\] do not have such properties so there is Big-Bang. Thus, the modification of the classical Hamiltonian turns Big-Bang into Big-Bounce. In what follows we show that quantization of the bouncing dynamics inevitably leads to discrete spectrum of the volume operator.

It turns out that the physical phase space may be parameterized by the elementary observables \[15\],
\[ O_1 := p_\phi, \quad O_2 := \dot{\phi} - \frac{\text{sgn}(p_\phi)}{3\kappa} \text{ar\,th}(\cos(\lambda\beta)), \] (6)
i.e. functions having vanishing Poisson bracket with \[1\] on the constraint surface \(H^{(\lambda)} \approx 0\).

The classical dynamics has been solved analytically \[15\] and the explicit form of the solution for the variable \(v\), which is of interest in the present paper, is given by
\[ v(\phi) = \kappa\gamma\lambda |O_1| \cosh 3\kappa(\phi - O_2). \] (7)
The variable \(\phi\) changes monotonically with an evolution so it has been chosen to be an evolution parameter of the system \[15\].

3. Volume Operator. The variable \(v\) has the interpretation of a volume of some piece of space \[19\]. To define quantum operator corresponding to \(v\), we use the classical observables \[6\]
\[ v = |w|, \quad w := \kappa\gamma\lambda O_1 \cosh 3\kappa(\phi - O_2). \] (8)
Thus, quantization of \(v\) reduces to the quantization problem of \(w\). Quantization of the latter may be done in a standard way as follows \[17\]
\[ \hat{w} f(x) := \kappa\gamma\lambda \frac{1}{2} (\hat{O}_1 \cosh 3\kappa(\phi - \hat{O}_2) + \cosh 3\kappa(\phi - \hat{O}_2) \hat{O}_1) f(x), \] (9)
where \(f \in L^2(\mathbb{R})\), and where \(\phi\) is a scalar field used both at classical and quantum levels as an evolution parameter \[14\].

For the elementary observables \(O_1\) and \(O_2\) we use the Schrödinger representation
\[ O_1 \rightarrow \hat{O}_1 f(x) := -i\hbar \partial_x f(x), \]
\[ O_2 \rightarrow \hat{O}_2 f(x) := \hat{x} f(x) := x f(x). \] (10)
In the representation \([10]\) an explicit form of the operator \(\hat{w}\) is
\[
\hat{w} f(x) = i \frac{\kappa \gamma \lambda h}{2} \left(2 \cosh 3\kappa (\phi - x) \frac{d}{dx} - 3\kappa \sinh 3\kappa (\phi - x)\right) f(x). \tag{11}
\]

4. Eigenvalue problem. It turns out that the solution to the eigenvalue problem
\[
\hat{w} f_a(x) = a f_a(x), \quad a \in \mathbb{R}, \tag{12}
\]
reads \([17]\)
\[
f_a(x) := \sqrt{\frac{2}{\pi}} \exp \left(i \frac{2a}{3\kappa \gamma \lambda h} \arctan e^{3\kappa (\phi - x)}\right) \cosh \frac{1}{3} 3\kappa (\phi - x).
\]

The condition \(\langle f_b|f_a \rangle = 0\) leads to
\[
a - b = 6\kappa^2 \gamma \lambda h m = 8\pi G \gamma \lambda h m, \tag{14}
\]
where \(m \in \mathbb{Z}\). Thus, the set
\[
\mathcal{F}_b := \{ f_a \mid a = b + 8\pi G \gamma \lambda h m \}, \tag{15}
\]
where \(b \in \mathbb{R}\), is orthonormal. Each subspace \(\mathcal{F}_b \subset L^2(\mathbb{R})\) spans a pre-Hilbert space. The completion of each span \(\mathcal{F}_b\) in the norm of \(L^2(\mathbb{R})\) defines an infinite dimensional separable Hilbert space \(\mathcal{H}_b\). Since
\[
\langle f_b|\hat{w} f_a \rangle - \langle \hat{w} f_b|f_a \rangle = (a - b) \langle f_b|f_a \rangle, \tag{16}
\]
the operator \(\hat{w}\) is symmetric on \(\mathcal{F}_b\) for any \(b \in \mathbb{R}\). In fact, it is a self-adjoint operator on the span of \(\mathcal{F}_b\) (see, \([17]\) for a proof).

5. Spectrum. Due to the relation \([8]\) and the spectral theorem on self-adjoint operators \([26, 27]\), we may carry out quantization of the volume function on each \(\mathcal{F}_b\) as follows
\[
v = |w| \quad \longrightarrow \quad \hat{v} f_a := |a| f_a. \tag{17}
\]

It results from \([14]\) that for \(b = 0\) and \(m = 0\) the minimum eigenvalue of \(\hat{v}\) equals zero. It is a special case that corresponds to the classical situation when \(v = 0\), which due to \([4]\) means that \(p_\phi = 0\) so there is no classical dynamics (for more details see \([15]\)). Thus, we have a direct correspondence between classical and quantum levels corresponding to this very special state. It is clear that all other states describe bouncing dynamics.

6. Free parameter. There exists a fundamental problem underlying LQC (see, \([13\] and references therein), which is the unknown numerical value of the parameter \(\lambda\) \([28]\).

Determination of \(\lambda\) by standard LQC means \([18, 19]\): (a) considering eigenvalue problem for the area operator, \(\hat{A}_r = [\hat{p}]\), in kinematical phase space of standard LQC: \(\hat{A}_r |\mu\rangle = \frac{4\pi \gamma l_p^2}{3} |\mu\rangle |\mu\rangle =: ar(|\mu\rangle |\mu\rangle\) so \(ar(|\mu\rangle |\mu\rangle\) is continuous since \(\mu \in \mathbb{R}\); (b) making reference to discrete eigenvalues, \(\{0, \square, \ldots\}\), of kinematical \(\hat{A}_r\) of LQG, where \(\square := 2\sqrt{3} \pi \gamma l_p^2\); and (c) assuming that \(ar(\lambda) \equiv \square\), which leads to \(\lambda = 3\sqrt{3}/2\).

One postulates in standard LQC that a surface cannot be squeezed to the zero value due to the existence in the Universe of the quantum of area.

Physical justification for the assumption on the existence of quantum of area, offered by standard LQC, seems to be doubtful because: (d) \(\hat{A}_r\) has been examined in kinematical Hilbert space of LQG, i.e. spectrum of \(\hat{A}_r\) ignores the algebra of constraints of LQG so it has poor physical meaning; (e) discrete spectrum of LQG was used to replace continuous spectrum of standard LQC, which is the spectral discretization by hand; and (f) standard LQC is not a cosmological sector of LQG, but a quantization method inspired by LQG \([20]\).

This is why we propose to treat \(\lambda\) as a free parameter yet to be determined.

7. Conclusions. As the Universe expands a discrete spectrum of the volume operator favors a foamy structure which turns into a
continuous spacetime with time. The classical FRW model is commonly used in observational cosmology because it fits quite well the data. Thus, the detection of any cosmic events favoring the foamy spacetime would give support to the quantum FRW model.

Our non-standard LQC method gives results concerning geometrical properties of space on the physical phase space so they may be verified by the data of observational cosmology.

There exist results concerning the spectrum of the volume operator obtained within LQG (see, e.g. [30, 31]), but cannot be compared easily with our results due to the lack of a direct correspondence between LQG and LQC methods (see Sec 6f).

Both standard and non-standard LQC methods offer the resolution of the initial Big-Bang singularity in the sense that the singularity is replaced by the regular Big-Bounce (BB) transition. However, the energy scale specific to BB (the scale of unification of gravity with quantum physics) has not been determined satisfactory yet [14]. The problem reduces to the problem of determination of the minimum length [13]. Can it be solved by making use of the cosmic data? There exists speculation that the foamy structure of spacetime may lead to the dependence of the velocity of a photon on its energy. Such dependence is weak, but may sum up to give a measurable effect in the case of photons travelling over cosmological distances across the Universe [35]. Presently, available data suggest that such dispersion effects do not occur up to the energy scale $5 \times 10^{17}$ GeV [36] so this type of effects may be present, but at higher energies.

The quantum of volume may be used as a measure of a size, $\lambda_f$, of a spacetime foam. One may speculate that $\lambda_f := \Delta^{1/3} = \left(8\pi G \gamma \hbar \lambda \right)^{1/3}$. Thus, an astro-cosmo data that determine a size of spacetime ‘granularity’ $\lambda_f$ may fix the minimum length parameter $\lambda$ of LQC. That would enable making an estimate of the critical matter density $\rho_{\text{max}} = 1/2 (\kappa \gamma \lambda)^2$ corresponding to the BB [17].

The granularity of volume should lead to the granularity of energy of physical fields. We suggest, making use of the de Broglie relation, that a specific particle representing a quantum of energy may have a momentum $p_i$ corresponding to its wavelength $\lambda_i$ such that $p_i \lambda_i = \hbar$. The detection of an ultrahigh energy particle with specific $p_i$ may be used to determine $\lambda_i$, and consequently set the upper limit for the fundamental length $\lambda_f$. The set of parameters $\lambda_i$ (for a set of particles) may be treated further as multiplicities of $\lambda_f$ in which case the greatest common divisor of all $\lambda_i$ would set the lowest upper limit for $\lambda_f$.

The standard and non-standard LQC methods give comparable results as they predict the appearance of the Big-Bounce transition parameterized by a free parameter to be determined [14]. Both methods seem to commute so there exists an analogy to the case of quantum electrodynamics [22]. However, our method is fully controlled analytically as it does not require any numerical work. It may be also linked with the loop quantum gravity (LQG) by finding relation with the reduced phase space quantization [37].

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