Invariance of the generalized oscillator 
under linear transformation of 
the related system of orthogonal polynomials

We consider two families of polynomials $\mathbb{P} = \{ P_n(x) \}_{n=0}^{\infty}$ and $\mathbb{Q} = \{ Q_n(x) \}_{n=1}^{\infty}$ orthogonal on the real line with respect to probability measures $\mu$ and $\nu$ respectively. Let $\{ Q_n(x) \}_{n=0}^{\infty}$ and $\{ P_n(x) \}_{n=0}^{\infty}$ connected by the linear relations

$$Q_n(x) = P_n(x) + a_1 P_{n-1}(x) + \ldots + a_k P_{n-k}(x).$$

Let us denote $\mathfrak{A}_P$ and $\mathfrak{A}_Q$ generalized oscillator algebras associated with the sequences $\mathbb{P}$ and $\mathbb{Q}$. In the case $k = 2$ we describe all pairs $(\mathbb{P}, \mathbb{Q})$, for which the algebras $\mathfrak{A}_P$ and $\mathfrak{A}_Q$ are equal. In addition, we construct corresponding algebras of generalized oscillators for arbitrary $k \geq 1$.

1 Introduction

Let $\mathbb{P} = \{ P_n(x) \}_{n=0}^{\infty}$ is a family of polynomials orthogonal on the real line with respect to the probability measure $\mu$. Consider the sequence of polynomials $\mathbb{Q} = \{ Q_n(x) \}_{n=0}^{\infty}$ such that

$$Q_n(x) = P_n(x) + a_1 P_{n-1}(x) + \ldots + a_k P_{n-k}(x), \quad n > k - 1.$$  

The family of orthogonal polynomials associated with such linear relation was discussed in several works (see e.g. [1] - [5]). In particular the necessary and sufficient conditions for the orthogonality of the sequence $\{ Q_n(x) \}_{n=0}^{\infty}$ with respect to a positive Borel measure $\nu$ on the real line are given in the article [5].

It is known [6] that every sequence of polynomials $\{ P_n(x) \}_{n=0}^{\infty}$ orthogonal with respect to positive Borel measures $\mu$ on the real line generates the generalized oscillator algebra $\mathfrak{A}_P$. In this work, we investigate the question under what conditions algebras $\mathfrak{A}_P$ and $\mathfrak{A}_Q$, generated by such linearly related polynomials, coincide

$$\mathfrak{A}_P = \mathfrak{A}_Q.$$  

This problem was considered in [8] for the simplest case $k = 1$. In this paper we discuss the case $k = 2$.

Below we will need the following results ([5]-[7]). Let $u$ is a linear functional on the linear space of polynomials with real coefficients. The polynomials $\{ P_n(x) \}_{n=0}^{\infty}$ are called orthogonal with respect to $u$, if

$$\langle u, P_n P_m \rangle = 0, \forall n \neq m \quad \text{and} \quad \langle u, P_n^2 \rangle \neq 0, \forall n = 0, 1, \ldots.$$  

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3Here and below we consider only monic polynomials.
Let $H = \{u_{i+j}\}_{i,j \geq 0}$ is the Hankel matrix associated with the functional $u$, where $u_k = \langle u, x^k \rangle$, $k \geq 0$. The linear functional $u$ is called the quasi-definite (positive definite) functional, if the leading submatrices $H_n$ of the matrix $H$ are nonsingular (positive definite) for all $n \geq 0$.

Favard’s theorem gives a description of quasi-definite (positively definite) linear functional in terms of the three-term recurrence relations that satisfied by the polynomials $\{P_n(x)\}_{n=0}^\infty$:

$$x P_n(x) = P_{n+1}(x) + \beta_n P_n(x) + \gamma_n P_{n-1}(x)$$

$$P_0(x) = 1, \quad P_1(x) = x - \beta_0,$$

where $\gamma_n \neq 0$ (respectively $\gamma_n > 0$).

If $u$ is a positive definite linear functional, then there exists the positive Borel measure $\mu$ (supported by an infinite subset of $\mathbb{R}$) such that

$$\langle u, q \rangle = \int_{\mathbb{R}} q d\mu, \quad \forall q \in \mathbb{R}.$$

Bellow we will use the following theorem.

**Theorem 1.1.** Let $\{P_n(x)\}_{n=0}^\infty$ is the sequence of orthogonal polynomials with recurrence relations

$$x P_n(x) = P_{n+1}(x) + \beta_n P_n(x) + \gamma_n P_{n-1}, \quad (\gamma_n \neq 0) \quad (1)$$

$$P_0(x) = 1, \quad P_1(x) = x - \beta_0.$$ 

Let $a_1$ and $a_2$ are real numbers such that $a_2 \neq 0$, and $Q_n(x)$ are polynomials defined by the relations

$$Q_n(x) = P_n(x) + a_1 P_{n-1}(x) + a_2 P_{n-2}, \quad n \geq 3. \quad (2)$$

Then the orthogonality of the sequence $\{Q_n(x)\}_{n=0}^\infty$ depends on the choice of the coefficients $a_1$ and $a_2$. More precisely, $\{Q_n(x)\}_{n=0}^\infty$ is a family of orthogonal polynomials with recurrence relations

$$x Q_n(x) = Q_{n+1}(x) + \tilde{\beta}_n Q_n(x) + \tilde{\gamma}_n Q_{n-1}, \quad n \geq 1, \quad (\gamma_n \neq 0) \quad (3)$$

if and only if $\gamma_3 + a_1 (\beta_2 - \beta_3) \neq 0$ and

(i) if $a_1 = 0$, then for $n \geq 4$, $\beta_n = \beta_{n-2}$, $\gamma_n = \gamma_{n-2}$;

(ii) if $a_1 \neq 0$ and $a_1^2 = 4a_2$, then for $n \geq 2$

$$\beta_n = A + Bn + Cn^2, \quad \gamma_n = D + En + Fn^2,$$

with $a_1 C = 2F$, $a_1 B = 2E - 2F$, $A, B, C, D, E, F \in \mathbb{R}$;

(iii) if $a_1 \neq 0$ and $a_1^2 > 4a_2$, then for $n \geq 2$

$$\beta_n = A + B\lambda^n + C\lambda^{-n}, \quad \gamma_n = D + E\lambda^n + F\lambda^{-n}$$

with $a_1 C = (1 + \lambda)F$, $a_1 \lambda B = (1 + \lambda)E$, $A, B, C, D, E, F \in \mathbb{R}$, where $\lambda$ is the unique solution in $(-1, 1)$ of the equation

$$a_1^2 \lambda = a_2 (1 + \lambda)^2;$$
(iv) if \( a_1 \neq 0 \) and \( a_1^2 < 4a_2 \), and we let \( \lambda = e^{i\theta}, \theta \in (0, \pi) \) be the unique solution of the equation \( a_1^2 \lambda = a_2(1 + \lambda)^2 \), then for \( n \geq 2 \)

\[
\beta_n = A + Be^{i\theta} + \overline{B}e^{-i\theta}, \quad \gamma_n = D + Ee^{i\theta} + \overline{E}e^{-i\theta},
\]

with \( a_1 \lambda B = (1 + \lambda)E \quad (A, D \in \mathbb{R}, B, F \in \mathbb{C}) \).

Let us give the definition of the generalized oscillator connected with the family of orthogonal polynomials [6]. Let \( \mu \) is a probability measure on \( \mathbb{R} \) with finite moments of all orders

\[
\mu_n = \int_{-\infty}^{+\infty} x^n d\mu < \infty, \quad n \geq 0.
\]

These moments define uniquely two sequences of real numbers \( \{a_n\}_{n=0}^{\infty}, \{b_n\}_{n=0}^{\infty} \) and the system of orthogonal polynomials \( \{\\Psi_n(x)\}_{n=0}^{\infty} \) which satisfy recurrence relations

\[
x\\Psi_n(x) = b_n\\Psi_{n+1}(x) + a_n\\Psi_n(x) + b_{n-1}\\Psi_{n-1}(x), \quad (4)
\]

for \( n \geq 0 \) and also initial conditions

\[
\\Psi_0(x) = 1, \quad \Psi_1(x) = \frac{x-a_0}{b_0}.
\]

These polynomials form an orthonormal basis in the Hilbert space \( \mathcal{H} = L_2(\mathbb{R}; \mu) \).

It is necessary to distinguish two cases in the Hilbert space \( \mathcal{H} \) we define the ladder operators \( a^\pm \) and the number operator \( N \) by the formulas

\[
a^+\\Psi_n(x) = \sqrt{2}b_n\\Psi_{n+1}(x), \quad a^-\\Psi_n(x) = \sqrt{2}b_{n-1}\\Psi_{n-1}(x),
\]

\[
N\\Psi_n(x) = n\\Psi_n(x), \quad n \geq 0.
\]

Let \( B(N) \) be an operator-valued function such that

\[
B(N)\\Psi_n(x) = b_{n-1}^2\\Psi_n(x), \quad B(N + I)\\Psi_n(x) = b_n^2\\Psi_n(x), \quad n \geq 0.
\]

The next theorem is faithful.

Theorem 1.2. [6] The operators \( a^\pm, N, I \) satisfy the following relations

\[
a^-a^+ = 2B(N + I), \quad a^+a^- = 2B(N), \quad [N, a^\pm] = \pm a^\pm.
\]
Definition. The associative algebra \( \mathfrak{A}_\Psi \) generated by the operators \( a^\pm, N, I \) satisfying the relations of theorem 1.2 and by the commutators of these operators is called the generalized oscillator algebra corresponding to the orthonormal system of polynomials \( \{ \Psi_n(x) \}_{n=0}^\infty \) with recurrence relations (4).

We will give one useful consequence of the previous theorem. Let \( \mathfrak{A}_s \) is the algebra of generalized oscillator corresponding to recurrence relations (4) in the symmetric case \((a_n = 0)\) and \( \mathfrak{A}_a \) is the algebra of generalized oscillator corresponding recurrence relations (4) in an asymmetric case \((a_n \neq 0)\). Then \( \mathfrak{A}_s = \mathfrak{A}_a \).

Now we are ready to formulate the problem under consideration. We suppose that there are two families of polynomials \( P = \{ P_n(x) \}_{n=0}^\infty \) and \( Q = \{ Q_n(x) \}_{n=0}^\infty \) orthogonal with respect to probability measures \( \mu \) and \( \nu \) respectively. We suppose that these polynomials satisfy the conditions of theorem 1.1 and \( \beta_n, \tilde{\beta}_n, \gamma_n, \tilde{\gamma}_n \in \mathbb{R} \). Let us denote \( \mathfrak{A}_P \) and \( \mathfrak{A}_Q \) the corresponding algebra of generalized oscillators.

Problem. We want to describe all pairs of families of orthogonal polynomials \( (P, Q) \) for which \( \mathfrak{A}_P = \mathfrak{A}_Q \).

2 Jacobi matrices and the main result

Let \( P \) and \( Q \) are defined in Hilbert spaces \( H_\mu = L^2(\mathbb{R}; \mu) \) and \( H_\nu = L^2(\mathbb{R}; \nu) \), respectively. Let these polynomials satisfy the recurrence relations (1) and (3), respectively. In addition, we assume that these polynomials are related to each other by the relation (2). We will also suppose that \( \beta_n, \tilde{\beta}_n, \gamma_n, \tilde{\gamma}_n \in \mathbb{R} \) and the coefficients \( \beta_n, \gamma_n \) satisfies the conditions of theorem 1.1.

The Jacobi matrices \( J_P, J_Q \) corresponding to the RR (1), (3), respectively, can be written in the form

\[
J = \begin{bmatrix} A & I_1 \\ I_2 & B \end{bmatrix}, \quad I_1 = \begin{bmatrix} 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & \cdots \\ 1 & 0 & 0 & \cdots \end{bmatrix}, \quad I_2 = \begin{bmatrix} 0 & 0 & \gamma_3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ \cdots & \cdots & \cdots \end{bmatrix}
\]

where the matrix \( A \) for sequences \( P \) and \( Q \) have the following form

\[
A_P = \begin{bmatrix} \beta_0 & 1 & 0 \\ \gamma_1 & \beta_1 & 1 \\ 0 & \gamma_2 & \beta_2 \end{bmatrix}, \quad A_Q = \begin{bmatrix} \tilde{\beta}_0 & 1 & 0 \\ \tilde{\gamma}_1 & \tilde{\beta}_1 & 1 \\ 0 & \tilde{\gamma}_2 & \tilde{\beta}_2 \end{bmatrix},
\]

and the matrix \( B \) equals to

\[
B = \begin{bmatrix} \beta_3 & 1 & 0 & 0 & 0 & \cdots \\ \gamma_2 & \beta_2 & 1 & 0 & 0 & \cdots \\ 0 & \gamma_3 & \beta_3 & 1 & 0 & \cdots \\ 0 & 0 & \gamma_2 & \beta_2 & 1 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{bmatrix}
\]
Let us note that elements \((\beta_0, \beta_1, \beta_2, \beta_3)\) and \((\gamma_1, \gamma_2, \gamma_3)\) of the matrix \(J_P\) are given, while the elements \((\tilde{\beta}_0, \tilde{\beta}_1, \tilde{\beta}_2)\) as well as coefficients \(a_1, a_2\) in the equation (2) should be defined.

We will consider all 4 cases of the theorem 1.1. General relations valid for all four cases have the following form:

\[
\begin{align*}
\tilde{\gamma}_n &= \gamma_n, \ n \geq 1; \\
\beta_{2n+1} &= \beta_3, \ n \geq 1; \\
\beta_{2n} &= \beta_2, \ n \geq 1; \\
\gamma_n &\neq 0; \\
\tilde{\beta}_n &= \beta_n, \ n \geq 3; \\
\gamma_{2n} &= \gamma_2, \ n \geq 1; \\
\gamma_{2n+1} &= \gamma_3, \ n \geq 1;
\end{align*}
\]

(6)

We use the following notation

\[
\begin{align*}
s_1 &= \frac{\gamma_3 - \gamma_1 - (\beta_2 - \beta_1)(\beta_3 - \beta_1)}{\gamma_3}, \\
s_2 &= \frac{\beta_3 - \beta_1}{\beta_3 - \beta_1}, \ s_3 = \frac{\gamma_3}{\beta_3 - \beta_1}, \\
w &= \frac{a_1}{4s_3} - \frac{\gamma_2}{\gamma_3}, \ w_\lambda = \frac{\lambda}{(1 + \lambda)^2} \frac{a_1}{s_3} - \frac{\gamma_2}{\gamma_3}.
\end{align*}
\]

(7)

We now formulate our main result in terms of Jacobi matrices. Namely, we can prove that all pairs of orthogonal polynomial systems \(\{P_n(x)\}_{n=0}^\infty\) and \(\{Q_n(x)\}_{n=0}^\infty\) connected by the linear relation (2), which generate the same algebra of generalized oscillator, can be divided into following eight groups:

**The case I** \(a_1 = 0, \ \beta_1 \neq \beta_3;\)

In this case the matrix \(A_Q\) and the coefficient \(a_2\) are defined uniquely by the relations

\[
\begin{align*}
\tilde{\beta}_0 &= \beta_0 + \frac{(\beta_3 - \beta_1 - \beta_0)\gamma_1}{\gamma_2\gamma_3}a_2, \ a_2 = -s_1s_3^2, \\
\tilde{\beta}_1 &= \beta_1 + \frac{a_2}{s_3} - \tilde{\beta}_0, \ \tilde{\beta}_2 = -\frac{a_2}{s_3}, \\
\tilde{\gamma}_n &= \gamma_n, \ n \geq 1, \ \tilde{\beta}_n = \beta_n, \ n \geq 3.
\end{align*}
\]

(8)

**The case II** \(a_1 = 0, \ \beta_1 = \beta_3, \ \gamma_3 = \gamma_1, \ \beta_2 \neq \beta_0;\)

In this case the matrix \(A_Q\) and the coefficient \(a_2\) are defined uniquely by the relations

\[
\begin{align*}
\tilde{\beta}_0 &= \beta_1 - \frac{\gamma_2}{\beta_2 - \beta_0}, \\
\tilde{\beta}_1 &= \beta_0 + \frac{\gamma_2}{\beta_2 - \beta_0}, \\
\tilde{\beta}_2 &= \beta_2, \ a_2 = \gamma_2 \frac{\beta_1 - \beta_0}{\beta_2 - \beta_0} - \frac{\gamma_2^2}{(\beta_2 - \beta_0)^2}, \\
\tilde{\gamma}_n &= \gamma_n, \ n \geq 1, \ \tilde{\beta}_n = \beta_n, \ n \geq 3.
\end{align*}
\]

(9)
The case III  
\( a_1 \neq 0, \beta_1 \neq \beta_3, \ a_2 = \frac{1}{4} a_1^2. \)

We denote by \( w \) the solution of the equation

\[
16 s_3^2 w^4 + 32 s_2 s_3 w^3 + (16 s_2^2 + 4 s_3^2) w^2 +
(4 s_2 + s_1 s_3) w + s_1 s_2 + \frac{\gamma_2}{\gamma_3} (\beta_2 - \beta_1) = 0
\]

such that \( a_1 = (4 s_3 w + 4 s_2) \in \mathbb{R} \) and introduce the quantity

\[ C_{\beta, \gamma} = a_2 \left[ -\frac{\gamma_1}{\gamma_3} (\beta_2 - a_1 (w + 1)) + \beta_0 (4 w^3 + 4 w + 1) \right] - a_1 w [\beta_0 (\beta_1 + \beta_2) + \gamma_1]. \tag{10} \]

In this case, for given \( w \), the matrix \( A_Q \) and the coefficients \( a_1, a_2 \) are defined uniquely by the relations

\[
\tilde{\beta}_0 = \beta_0 - \frac{C_{\beta, \gamma}}{\gamma_2}, \quad \tilde{\beta}_1 = \beta_1 + \frac{C_{\beta, \gamma}}{\gamma_2} + a_1 w, \\
\tilde{\beta}_2 = \beta_2 - a_1 (w + 1), \\
a_1 = 4 s_3 w + 4 s_2, \quad a_2 = \frac{1}{4} a_1^2. \tag{11} \]

The case IV  
\( a_1 \neq 0, \beta_1 = \beta_3, \ a_2 = \frac{1}{4} a_1^2. \)

Let \( a_1 \) be a real solution of the equation

\[
\frac{\gamma_2^2}{\gamma_3} a_1^2 - a_1 \left( \frac{\gamma_2}{\gamma_3} + \frac{\gamma_1}{4 \gamma_3} - \frac{1}{4} \right) + \frac{\gamma_2}{\gamma_3} (\beta_2 - \beta_1) = 0.
\]

In this case, the matrix \( A_Q \) is defined uniquely by the relations

\[
\tilde{\beta}_0 = \beta_0 - \frac{D_{\beta, \gamma}}{\gamma_2}, \quad \tilde{\beta}_1 = \beta_1 + \frac{D_{\beta, \gamma}}{\gamma_2} + a_1 w, \\
\tilde{\beta}_2 = \beta_2 - a_1 (w + 1), \tag{13} \]

where \( w = -\frac{\gamma_2}{\gamma_3} \) and \( D_{\beta, \gamma} \) is defined by (10) at \( w = -\frac{\gamma_2}{\gamma_3} \).

The case V  
\( a_1 \neq 0, \ a_1^2 > 4 a_2, \ a_2 = \frac{\lambda}{(1 + \lambda)^2} a_1^2, \ \beta_1 \neq \beta_3, \ \lambda \in (-1, 1). \)

Let \( w_\lambda \) be a solution of the equation

\[
\frac{(1 + \lambda)^4}{\lambda^2} s_3^2 w_\lambda^4 + 2 \frac{(1 + \lambda)^2}{\lambda} s_2 s_3 w_\lambda^3 +
\left( \frac{(1 + \lambda)^4}{\lambda^2} s_2^2 + \frac{(1 + \lambda)^2}{\lambda} s_3 \right) w_\lambda^2 +
\left( \frac{(1 + \lambda)^2}{\lambda} s_2 + s_1 s_3 \right) w_\lambda + s_1 s_2 + \frac{\gamma_2}{\gamma_3} (\beta_2 - \beta_1) = 0, \tag{14} \]
such that
\[ a_1 = \frac{(1 + \lambda)^2}{\lambda}(s_3w + s_2) \in \mathbb{R}. \]

In this case, the matrix \( A_Q \) is defined uniquely by the relations
\[ \tilde{\beta}_0 = \beta_0 - \frac{C_\lambda}{\gamma_2}, \quad \tilde{\beta}_1 = \beta_1 + \frac{C_\lambda}{\gamma_2} + a_1w, \quad \tilde{\beta}_2 = \beta_2 - a_1(w + 1), \tag{15} \]
where
\[
C_\lambda = a_2 \left[ -\frac{\gamma_1}{\gamma_3}(\beta_2 - a_1(w + 1)) + \beta_0 \left( \frac{(1 + \lambda)^2}{\lambda}(w^2 + w) + 1 \right) \right] \\
- a_1w[\beta_0(\beta_1 + \beta_2) + \gamma_1] 
\tag{16}
\]
and \( \lambda \) is a free parameter.

**The case VI** \( a_1 \neq 0, a_1^2 > 4a_2, a_2 = \frac{\lambda}{(1 + \lambda)^2}a_1^2, \beta_1 = \beta_3, \lambda \in (-1, 1). \)

Let \( a_1 \) be a real solution of the equation
\[
\frac{\gamma_2^2}{\gamma_3}a_1^2 - a_1 \left[ \frac{\gamma_2}{\gamma_3} + \frac{\lambda}{(1 + \lambda)^2} \left( \frac{\gamma_1}{\gamma_3} - 1 \right) \right] + \frac{\gamma_2^2}{\gamma_3}(\beta_2 - \beta_1) = 0, \\
\]
\[
a_2 = \frac{\lambda}{(1 + \lambda)^2}a_1^2.
\]
In this case, the matrix \( A_Q \) is defined uniquely by the relations
\[ \tilde{\beta}_0 = \beta_0 - \frac{D_\lambda}{\gamma_2}, \quad \tilde{\beta}_1 = \beta_1 + \frac{D_\lambda}{\gamma_2} + a_1w, \quad \tilde{\beta}_2 = \beta_2 - a_1(w + 1), \tag{17} \]
where \( w = -\frac{\gamma_2}{\gamma_3} \) and \( D_\lambda \) is defined by the relation \( (16) \) for \( C_\lambda \) with \( w = -\frac{\gamma_2}{\gamma_3} \).

**The cases VII and VIII** \( a_1 \neq 0, a_1^2 < 4a_2, a_2 = \frac{\lambda}{(1 + \lambda)^2}a_1^2, \lambda = e^{i\theta}, \theta \in (0, \pi). \)

The other relations in the case VII are the same as in the case V, and in the case VIII are the same as in the case VI.

## 3 Possibility of generalization to the case of \( k > 2 \)

We give few comments about construction all possible pairs of polynomial systems connected by the general linear relation
\[
Q_n(x) = P_n(x) + a_1P_{n-1}(x) + \ldots + a_kP_{n-k}, \quad k \geq 2 \tag{18}
\]
for which \( \mathcal{A}_P = \mathcal{A}_Q. \)

From results of the paper [5] we have
and

possible if

construction of the generalized oscillator algebra

orthogonal polynomials systems connected with each other by the linear relation (18).

We discuss the generalized oscillator algebra

4 The generalized oscillator algebra corresponding to the pair (\(\mathbb{P}, \mathbb{Q}\))

We define the ladder operators

\[ a^{\pm}_{\Phi} \]

and the number operator \(N_{\Phi}\) in \(\mathcal{H}_{\mu}\) by formulas

\[ a^{+}_{\Phi}\varphi_n = \sqrt{2\gamma_{n+1}}\varphi_{n+1}, \]

\[ a^{-}_{\Phi}\varphi_n = \sqrt{2\gamma_n}\varphi_{n-1}, \]

\[ N_{\Phi}\varphi_n = n\varphi_n, \quad n \geq 0. \]
Let $B_\Phi(N\Phi)$ be an operator-valued function defined by the following equalities

$$B_\Phi(N\Phi)\varphi_n = \gamma_n \varphi_n, \quad n \geq 0.$$  

$$B_\Phi(N\Phi + I_\mu)\varphi_n = \gamma_{n+1} \varphi_n, \quad n \geq 0.$$  

Then the generalized oscillator algebra $\mathcal{A}_\Phi$ is generated by the operators $a_\Phi^\pm, N\Phi, I_\mu$ satisfying the relations

$$\begin{aligned}
    a_\Phi^- a_\Phi^+ &= 2B_\Phi(N\Phi + I_\mu), \\
    a_\Phi^+ a_\Phi^- &= 2B_\Phi(N\Phi), \\
    [N\Phi, a_\Phi^\pm] &= \pm a_\Phi^\pm 
\end{aligned}$$ (21)

and by the commutators of these operators.

In this case, the quadratic Hamiltonian

$$H_\Phi = a_\Phi^- a_\Phi^+ + a_\Phi^+ a_\Phi^-$$

is a selfadjoint operator in the Hilbert space $\mathcal{H}_\mu$.

Orthonormal polynomials $\{\varphi_n(x)\}_{n=0}^\infty$ are eigenfunctions of the operator $H_\Phi$. The corresponding eigenvalues are equal to

$$\lambda_0 = 2\gamma_1, \quad \lambda_n = 2(\gamma_n + \gamma_{n+1}), \quad n \geq 1.$$  

In conclusion, let us note that since in our case

$$b_n^2 \neq (a_0 + a_2n)(1 + n),$$

then according to the results of [9], [10]

$$\dim \mathcal{A}_\Phi = \dim \mathcal{A}_\Theta = \infty.$$  

Note also that it would be interesting to study the relation of the orthogonality measures $\mu$ and $\nu$ in the spaces $\mathcal{H}_\mu$ and $\mathcal{H}_\nu$, respectively.

Acknowledgment. EVD grateful to RFBR for financial support under the grant 15-01-03148.

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