PRESCRIBING THE BEHAVIOR OF WEIL-PETERSSON GEODESICS IN THE MODULI SPACE OF RIEMANN SURFACES

BABAK MODAMI

Abstract. We study the Weil-Petersson (WP) geodesics with narrow end invariant and develop techniques to control length-functions and twist parameters along them and prescribe their itinerary in the moduli space of Riemann surfaces. This class of geodesics is rich enough to provide for examples of closed WP geodesics in the thin part of moduli space, as well as divergent WP geodesic rays with minimal filling ending lamination.

As an intermediate step we prove that hierarchy resolution paths between narrow pairs of partial markings or laminations are stable in the pants graph of the surface.

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1. INTRODUCTION

The Weil-Petersson (WP) metric on the moduli space of Riemann surfaces is an incomplete Riemannian metric with negative sectional curvatures asymptotic to both 0 and $-\infty$ in the completion. The WP metric is a metric of higher rank in the sense of Gromov (see [BP06]). The WP geodesic flow is not uniformly hyperbolic. These features prevent applying most of the standard techniques available in the study of the global geometry and dynamics of complete metrics with negative curvatures bounded away from zero and uniformly hyperbolic dynamical systems to the WP metric and its geodesic flow. For some of these techniques see [PP10], [Ebe72], [KH95]. The main theme of this paper and the pioneer work of Brock, Masur and Minsky in [BM08], [BMM10], [BMM11] is to apply combinatorial techniques from surface theory to study the global behavior of WP geodesics.

In [BMM10] the authors introduce a notion of ending lamination for WP geodesic rays. They show that the ending lamination determines the strong asymptotic class of a WP geodesic ray recurrent to a compact subset of moduli space. In [BMM11] a more explicit connection between the combinatorics of the ending laminations of a WP geodesic and its behavior was established. They provide a necessary and sufficient combinatorial condition for a WP geodesic to stay in the compact part of moduli space. In this paper we prove the following two results about the behavior of WP geodesics in the moduli space of Riemann surfaces:

**Theorem 1.1.** (Closed geodesic in the thin part) Given any compact subset of moduli space $K$, there are infinitely many closed Weil-Petersson geodesics not intersecting $K$.

**Theorem 1.2.** (Divergent geodesic) Starting from any point in the moduli space there are uncountably many divergent WP geodesic rays with minimal filling ending lamination.

The WP volume of moduli space is finite, so by the Poincaré recurrence theorem almost every WP geodesic ray is recurrent to a compact subset of
moduli space. However the second theorem above exhibits the abundance of WP geodesic rays divergent in the moduli space. These geodesics diverge in the moduli space by getting closer and closer to a chain of completion strata.

Given a WP geodesic \( g : (a, b) \to \text{Teich}(S) \) denote the end invariant associated to its forward trajectory by \( \nu^+ = \nu^+(g) \) and the one associated to its backward trajectory by \( \nu^- = \nu^-(g) \). Here \( \nu^- \) and \( \nu^+ \) are either lamination or (partial) marking.

To each subsurface \( Z \subseteq S \) that is not a three holed sphere, there is an associated \textit{subsurface coefficient} denoted by

\[
d_Z(\nu^-, \nu^+)
\]

which is the distance in the curve complex of \( Y \) between the projections of \( \nu^- \) and \( \nu^+ \), for more details see [2].

Subsurface coefficients are an analogue of continued fraction expansions which provide for a coding of geodesics on the modular surface which is the moduli space of one hold tori \( \mathcal{M}(S_{1,1}) \), see for example [Ser85]. Conjecturally these coefficients provide for extensive information about the behavior of Weil-Petersson geodesics in the moduli space. The following conjecture was proposed in \[BMM10\]:

**Conjecture 1.3.** (Short Curve) Let \( g \) be a Weil-Petersson geodesic with end invariant \((\nu^-, \nu^+)\). Then

1. For every \( \epsilon > 0 \) there is an \( A > 0 \), such that if \( d_Z(\nu^-, \nu^+) > A \) then
   \[
   \inf_t \ell_\alpha(g(t)) \leq \epsilon \text{ for every } \alpha \in \partial Z.
   \]
2. For every \( A > 0 \) there is an \( \epsilon > 0 \), such that if \( \inf_t \ell_\alpha(g(t)) \leq \epsilon \) then
   there is a subsurface \( Z \subset S \) such that \( \alpha \in \partial Z \) and \( d_Z(\nu^-, \nu^+) > A \).

Furthermore, it would be very useful to have estimates on the length of the time interval that a curve is short (has length less than a given \( \epsilon \)) along a WP geodesic using subsurface coefficients of the end invariants.

The WP metric exhibits different features in the thick and thin parts of Teichmüller space. For instance in the thick part the sectional curvatures are all bounded away from both 0 and \(-\infty\), while in the thin part the WP metric is almost a product metric. Therefore, to answer questions about the global geometry and dynamics of the WP metric often one needs to determine the \textit{itinerary of geodesics}. By this we mean the thin regions of the Teichmüller space that a WP geodesic \( g \) visits, the order \( g \) visits the regions and the time \( g \) spends in each one of these regions.

After the work of Masur-Minsky [MM99], [MM10] and Rafi [Rafi05], [Rafi], the above conjecture holds for Teichmüller geodesics and provides for a complete picture of the itinerary of Teichmüller geodesics in the moduli space. An important ingredient which is missing here is an explicit description of the Riemann surface along WP geodesics.

The underlying machinery to obtain control on the behavior of WP geodesics is the Masur-Minsky machinery of hierarchies and their resolutions in pants
and marking graphs introduced in [MM00]. Given a pair of partial markings or laminations a hierarchy (resolution) paths $\rho : [m, n] \to P(S), [m, n] \subseteq Z \cup \{\pm \infty\}$, is a quasi-geodesic between them in the pants graph with certain properties encoded in the pair and their subsurface coefficients. For example corresponding to any subsurface $Z$ with big enough subsurface coefficient, there is a subinterval of $[m, n]$ denoted by $J_Z$ such that $\partial Z \subseteq \rho(i)$, for every $i \in J_Z$. In Theorem 2.13 the list of properties is listed.

By a result of Brock (Theorem 3.3) the Bers pants decompositions along a WP geodesic trace a quasi-geodesic in the pants graph of the surface. When such a quasi-geodesic and a hierarchy path $\rho$ fellow travel each other (see Definition 5.23) there is a correspondence between the parameters of the hierarchy path and the parameters of the WP geodesic, which is roughly the nearest point correspondence of fellow traveling paths. In this situation we use the hierarchy machinery to determine the itinerary of WP geodesics. An example is visiting the region where the curves in $\partial Z$ are short over the time interval corresponding to $J_Z$.

An $A$–narrow condition on the end invariant $(\nu^-, \nu^+)$ is a constraint on the set of subsurfaces with big subsurface coefficient. More precisely, the pair is $A$–narrow if every non-annular subsurface $Z \subseteq S$ with

$$d_Z(\nu^-, \nu^+) > A$$

is a large subsurface i.e. is a connected subsurface with complement consisting of only annuli and three holed spheres.

The $A$–narrow condition on end invariant implies uniform fellow traveling, depending only on $A$, of a WP geodesic segment and a hierarchy path with end points equal to the pair. Heuristically hierarchy paths with narrow end invariant avoid quasi-flats in the pants graph corresponding to separating multi-curves on a surface, and WP geodesics with narrow end invariant avoid asymptotic quasi-flats in the WP metric corresponding to separating multi-curves on a surface. In this paper we develop a control for length-functions and twist parameters along geodesics with narrow end invariant and show that their itinerary mimic combinatorial properties of hierarchy paths. In order to prove our main technical results we introduce the following notions:

Let $Z \subseteq S$ be a subsurface with $d_Z(\nu^-, \nu^+)$ sufficiently big. We say that $Z$ has $(R, R')$–bounded combinatorics over a subinterval $[i_1, i_2] \subset J_Z \subset [m, n]$ if for every non-annular subsurface $Y \subset Z$ the subsurface coefficient be bounded as

$$d_Y(\rho(i_1), \rho(i_2)) \leq R$$

and for every annular subsurface $A(\gamma)$ with core curve $\gamma$ inside $Z$ the subsurface coefficient be bounded as

$$d_\gamma(\rho(i_1), \rho(i_2)) \leq R'$$
This condition is a local version of the bounded combinatorics of the end invariant in [BMM11]. There is shown that a geodesic with bounded combinatorics end invariant stays in the thick part of moduli space. In the direction of Conjecture 1.3 we prove:

**Theorem 1.5.** (Short Curve)

Given $A, R, R' > 0$ and a sufficiently small $\epsilon > 0$, there is $\bar{w} = \bar{w}(A, R, R', \epsilon)$ with the following property. Let $g : [a, b] \to \mathrm{Teich}(S)$ be a WP geodesic segment with $A$–narrow end invariant $(\nu^-, \nu^+)$. Let $\rho : [m, n] \to P(S)$ be a hierarchy between $\nu^-$ and $\nu^+$. Assume that a large component domain of $\rho$, $Z$ has $(R, R')$–bounded combinatorics over an interval $[m', n'] \subset J_Z$.

If $n' - m' \geq 2 \bar{w}$, then for every $\alpha \in \partial Z$ we have

$$\ell_\alpha(g(t)) \leq \epsilon$$

for every $t \in [a', b']$, where $a'$ and $b'$ are the corresponding times to $m' + \bar{w}$ and $n' - \bar{w}$, respectively.

Here $N$ is the parameter map from $[m, n]$ to $[a, b]$ which comes from the fellow traveling of a the Bers curves along a WP geodesic with narrow end invariant $(\nu^-, \nu^+)$ and a hierarchy path between $(\nu^-, \nu^+)$, see §5.3.

In §6 we use bounded combinatorics intervals to isolate annular subsurface along a hierarchy path. The twist parameter developed about an isolated annular subsurface along a WP geodesic is comparable to the one along a fellow traveling hierarchy path. This together with the length-function versus twist parameter controls over uniformly bounded length WP geodesic segments we develop in §4 are the main technical tools in §6 where we prove the above theorem.

Using the control we develop on length-functions along WP geodesic segments and by extracting limits of WP geodesic segments with narrow end invariant we provide WP geodesics with prescribed itinerary in the moduli space (Theorem 8.5). Itineraries of these rays mimic the combinatorial properties of hierarchy paths encoded in the end points and the associated subsurface coefficients. Using these rays and our constructions of pairs of laminations/markings with prescribed list of subsurface coefficients in §7 we prove theorems 1.1 and 1.2 in §8. Jeffery Brock has communicated us that he also constructs divergent WP geodesics, [Bro]. Our constructions of pairs of laminations/markings with prescribed list of subsurface coefficients in §7 is a kind of symbolic coding for laminations using subsurface coefficients analogue of continued fraction expansions.

Fellow traveling property of hierarchy paths is a crucial part of the combinatorial framework to control length-functions along WP geodesics. In §5 we prove the following stability result for hierarchy paths in the pants graph of a surface

**Theorem 5.11.** (Stable hierarchy path) Given $A > 0$ there is a function $d_A$ such that any hierarchy path with $A$–narrow end points is $d_A$–stable in the pants graph.
Here $d_A : \mathbb{R}_{\geq 1} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is the quantifier of the stability, see Definition 5.1.

This theorem implies that the Bers pants along a WP geodesic and a hierarchy path connecting the end invariants of the geodesic uniformly fellow travel each other in the pants graph.

1.1. Outline of the paper. Section 2 is devoted to the background about curve complexes and some important notions and results in the setting of pants and marking graphs. Here we recall hierarchical structures on pants and marking graphs and their resolutions introduced by Masur and Minsky in [MM00]. In Theorem 2.13 we list the properties of hierarchy resolution paths. Moreover, we recall the $\Sigma-$hulls and their projections from [BKMM12]. In Section 3 we give some background about the WP metric and synthetic properties of WP geodesics.

In Section 4 we provide the proofs of refined version of some of the key results in [Wol03] and [BMM11] which we need in this work. These are mainly based on compactness arguments in the WP completion of Teichmüller space and are consequences of Wolpert’s geodesic limit theorem. These results give us a kind of control on development of Dehn twists versus change off length-functions along uniformly bounded length WP geodesic segments.

In Section 5 we prove that hierarchy paths between narrow pairs are stable. The proof will be through $\Sigma-$hulls and their stability properties.

In Section 6 we develop some new techniques to control length-functions and twist parameters along WP geodesics fellow traveling hierarchy paths. In Lemma 6.6 we develop some new techniques to control length-functions and twist parameters along WP geodesics fellow traveling hierarchy paths. In Theorem 6.1 using convexity of length-functions and an inductive argument on the complexity of subsurfaces we sharpen these bounds.

In Section 7 we provide pairs of marking or laminations with a prescribed list of subsurface coefficients. This is a kind of symbolic coding for laminations in terms of subsurface coefficients similar to continued fraction expansions.

In Section 8 we provide examples of WP geodesics with prescribed behaviors. These results could be considered as a kind of symbolic coding for WP geodesics. Here we use the control on length-functions from [6] and infinite stable hierarchy paths with a prescribed list of subsurface coefficients from [7].

Notation 1.4. Given $K \geq 1$ and $C \geq 0$. Let $f, g : X \to \mathbb{R}$ be two functions on a set $X$. Then $f \asymp_{K,C} g$ means that for every $x \in X$ we have

$$\frac{1}{K} g(x) - C \leq f(x) \leq K g(x) + C$$

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2. Curve complexes and hierarchical structures

Curve complex of a surface: Let $S = S_{g,b}$ be a connected, compact, oriented surface with genus $g$ and $b$ boundary components. We define $\xi(S) = 3g - 3 + b$ as the complexity of the surface.

The curve complex of $S$, denoted by $C(S)$, serves to organize the isotopy classes of essential, simple closed curves on $S$. Let $S$ be a surface with $\xi(S) \geq 1$. To each isotopy class of essential simple closed curves on $S$ (neither isotope to boundary nor a point) is associated a vertex (0-simplex) in $C(S)$. When $\xi(S) > 1$, an edge is associated to disjoint pair of isotopy classes of curves. Similarly, a $k$-simplex is associated to any $k+1$ pairwise disjoint isotopy classes of simple closed curves. Here two isotopy classes are disjoint if there are curves in each of them which are disjoint on $S$. We denote the $k$-skeleton of this complex by $C_k(S)$. When $\xi(S) = 1$, $S$ is either a one holed torus or a four holed sphere. Then 1-simplices (edges) correspond to curves intersecting respectively once and twice, which are the minimum possible intersection number of curves on $S$, respectively.

We equip the curve complex with a distance by making each simplex Euclidean with side lengths 1, and denote the distance by $d_S = d_{C(S)}$. By the main result of Masur-Minsky in [MM99] $C(S)$ equipped with this distance is a $\delta$-hyperbolic space in the sense of Gromov, where $\delta$ only depends on topological type of $S$.

An annular subsurface is an annulus $Y$ with essential core curve in $S$. The purpose of defining complexes for annuli is to keep track of Dehn twisting about their core curves. These complexes are quasi-isometric to $\mathbb{Z}$. Let $Y$ be an annulus with core curve. Let $Y = S/\langle \alpha \rangle$ be the annular cover of $S$ to which $Y$ lifts homeomorphically. There is a natural compactification of $\tilde{Y}$ to a closed annulus $\hat{Y}$ which is the quotient by $\langle \alpha \rangle$ of the compactification of the universal cover of $S$, $\tilde{S} = \mathbb{D}^2$ (the Poincaré disk) by the closed disk. A vertex of $C(Y)$ is associated to a path connecting the two boundary components of $\tilde{Y}$ modulo isotopies that fix the endpoints (isotopy classes of arcs relative to the boundary). An edge is associated to two vertices which have representatives with disjoint interiors. Curve complexes can be made metr with a metric by assigning length 1 to each edge. Let $\alpha \in C_0(S)$ be the core curve of $Y$. We write $Y = A(\alpha)$ and $C(\alpha) = C(Y)$.

Notation 2.1. A curve on $S$ refers to the isotopy class of an essential simple closed curve on $S$, that is a vertex of $C_0(S)$. If there are representatives of the isotopy classes of curves $\alpha$ and $\beta$ which are disjoint, then $\alpha$ and $\beta$ are disjoint. Otherwise, $\alpha$ and $\beta$ overlap each other which is denoted by $\alpha \pitchfork \beta$. A multi-curve consists of the vertices of a simplex in $C(S)$, which is a collection of pair-wise disjoint curves. A curve system is a finite diameter subset of $C_0(S)$. Two multi-curves $\sigma$ and $\sigma'$ are disjoint if any pair of curves $\alpha \in \sigma$ and $\alpha' \in \sigma'$ are disjoint, otherwise $\sigma$ and $\sigma'$ overlap.
Filling curve systems: We say that a curve system $\mu$ on $S$ (a finite diameter subset of $C_0(S)$) fills $S$ if $S \setminus \mu$ consists of only topological disk and annuli. Given a filling system $\mu$ any $\gamma \in C_0(S)$ overlaps $\mu$.

Let $\alpha, \beta \in C_0(S)$. If $d_S(\alpha, \beta) \geq 3$ then $\alpha$ and $\beta$ fill $S$. To see this suppose that $\alpha$ and $\beta$ do not fill $S$. Then there is a curve $\gamma$ disjoint from both $\alpha$ and $\beta$ which implies that $d_S(\alpha, \beta) \leq 2$.

Laminations: Let $S$ be a surface equipped with a complete hyperbolic metric. A geodesic lamination on $S$ is a closed subset $\lambda$ of $S$ consisting of disjoint, complete, simple geodesics. We denote the space of geodesic laminations of a surface $S$ by $GL(S)$. A geodesic lamination $\lambda$ can be equipped with transversal measures: a transversal measure is a measure on arcs transversal to $\lambda$ which is invariant under isotopies of the surface $S$ relative to $\lambda$. A measure (geodesic) lamination $L = (\lambda, m)$ is the pair of a geodesic lamination $\lambda$, called the support of $L$ and $m$ a transverse measure of $\lambda$. The space of measured geodesic laminations of $S$ equipped with the weak$^*$ topology is denoted by $ML(S)$. For more detail see \[CEGS87\] and \[Bon01\].

$\mathbb{R}^+$ acts on $ML(S)$ by rescaling the measure, that is $s(\lambda, m) = (\lambda, sm)$.

Then the quotient, $PML(S) := ML(S)/\mathbb{R}$ is the space of projective of a measured laminations, the projective class of a measured lamination $L$ is denoted by $[L]$. We equip $PML(S)$ with the quotient of the weak$^*$ topology on $ML(S)$.

A geodesic lamination $\lambda$ is filling $S$ if the connected components of $S \setminus \lambda$ are only topological disks or annuli. $\lambda$ is minimal if any geodesic ray in $\lambda$ is dense in $\lambda$. Denote by $E(L)$ the set of projective classes of measured laminations supported on minimal filling laminations. Equip $E(L)$ with the topology induced from the topology of $PML(S)$.

The Gromov boundary of a $\delta$–hyperbolic space has a standard topology, see for example §III.H.3 of \[BH99\]. Klarriech in \[Kla\] proves that

**Theorem 2.2.** There is a homeomorphism $\Phi$ from the Gromov boundary of $C(S)$ to $E(L(S))$.

Furthermore Klarriech describes the relation between a sequence of curves going off to the Gromov boundary of $C(S)$ and the accumulation points of the sequence in $PML(S)$.

**Theorem 2.3.** (Theorem 1.4 of \[Kla\]) If a sequence of vertices in $C_0(S)$ converges to a point $\xi$ in the Gromov boundary of $C(S)$ then regarding each element of the sequence as a projective measured lamination every accumulation point of the sequence in $PML(S)$ is topologically equivalent to $\Phi(\xi)$. This means that their supporting geodesic laminations are topologically equivalent.

The following notions of subsurface projection and subsurface coefficient are basic in the Masur-Minsky machinery of curve complexes and hierarchical structures on the pants and marking graphs.
Subsurface projection: Given a non-annular subsurface $Y \subseteq S$ define the subsurface projection
\[
\pi_Y : \mathcal{GL}(S) \rightarrow \mathcal{P}(C_0(Y))
\]
where $\mathcal{P}(C_0(Y))$ is the power set of $C_0(S)$ as follows: Given $\lambda \in \mathcal{GL}(S)$, let $\lambda \cap Y$ be the collection of all essential curves and arcs of the intersection of $\lambda$ and $Y$ after identifying all the arcs and curves which are isotopic to each other. Here the end points of arcs are allowed to move within the boundary of $Y$. $\pi_Y(\lambda)$ consists of the boundary of a regular neighborhood of any arc $a$ in $\lambda \cap Y$ and $\partial Y$ and all of the closed curves in $\lambda \cap Y$. If $\lambda$ does not intersect $Y$ essentially then $\pi_Y(\lambda) = \emptyset$.

Note that since $C_0(S) \subset \mathcal{GL}(S)$, the above projection in particular gives a projection
\[
\pi_Y : C_0(S) \rightarrow \mathcal{P}(C_0(Y)).
\]

The projection for an annular subsurface $Y$ is defined as follows: If $\lambda \in \mathcal{GL}(S)$ crosses the core of $Y$ transversely, then the lift of $\lambda$ to $\hat{Y}$ (the compactification of $\tilde{Y}$ the annular cover of $Y$) has at least one component that connects the two boundaries of $\hat{Y}$. Then together these components make up a set of diameter 1 in $\mathcal{C}(Y)$. Let $\pi_Y(\lambda)$ be this set. If $\lambda$ does not intersect $Y$ essentially (including the case that $\lambda$ is the core of $Y$) then $\pi_Y(\gamma) = \emptyset$.

The projection of a curve system $\mu$ to a subsurface $Y$ is the union of $\pi_Y(\alpha)$ for all $\alpha \in \mu$.

Subsurface coefficient: Let $\mu$ and $\mu'$ be two laminations or curves systems. Let $Y \subseteq S$ be a subsurface. We consider the following notion of distance
\[
(2.1) \quad d_Y(\mu, \mu') := \min \{d_Y(\gamma, \gamma') : \gamma \in \pi_Y(\mu), \gamma' \in \pi_Y(\mu')\}
\]
which provides for a useful notion of complexity of $\mu$ and $\mu'$ from the point of view of the subsurface $Y$. We call it the $Y$ subsurface coefficient of $\mu$ and $\mu'$.

For an annular subsurface $Y$ with core curve $\alpha$ we denote $\pi_\alpha := \pi_Y$ and define the annular coefficients of geodesic laminations/curve systems $\mu$ and $\mu'$ by (2.1) and denote it by $d_\alpha$.

Subsurface coefficients play the role of continued fraction expansions for coding of laminations (see 7).

The triangle inequality: Given a curve system $\mu \subset \mathcal{C}(Y)$ (a finite diameter subsets of $C_0(Y)$) denote by $\text{diam}_Y(\mu)$ the diameter of $\pi_Y(\mu)$ as a subset of $\mathcal{C}(Y)$.

Let $\mu_1, \mu_2, \mu_3 \subset \mathcal{C}(Y)$ be curve systems. Suppose that $d_Y(\mu_1, \mu_2) = d_Y(\alpha, \beta)$ and $d_Y(\mu_2, \mu_3) = d_Y(\beta', \gamma)$, where $\alpha \in \mu_1$, $\beta, \beta' \in \mu_2$ and $\gamma \in \mu_3$. Then by the triangle inequality we have that $d_Y(\alpha, \beta) + d_Y(\beta', \gamma) \geq d_Y(\alpha, \gamma)$. Now since $d_Y(\beta, \beta') \leq \text{diam}_Y(\mu_2)$ and $d_Y(\alpha, \gamma) \geq$
\[ d_Y(\mu_1, \mu_2) \] we conclude that
\[ d_Y(\mu_1, \mu_2) + d_Y(\mu_2, \mu_3) + \text{diam}_Y(\mu_2) \geq d_Y(\mu_1, \mu_3). \]
We refer to this inequality as the triangle inequality.

**Hausdorff convergence**: Let \( \lambda_k \) be a sequence of geodesic laminations and \( \xi \) be any accumulation point of \( \lambda_k \)'s in the Hausdorff topology of closed subsets of \( S \). The Hausdorff convergence and the definition of subsurface coefficient imply that given subsurface \( Z \subseteq S \) and \( \lambda' \in GL(S) \)
\[ d_Z(\lambda', \xi) \approx 1, \]
for all \( k \) sufficiently large.

**Proposition 2.4.** Let \( L_i = (\lambda_i, m_i) \) be a sequence of measured laminations. Suppose that \( L_i \) converge to \( L = (\lambda, m) \) in the weak* topology. Let \( \xi \) be the limit of a subsequence \( \lambda_i \)'s in the Hausdorff topology of closed subset of the surface. Then \( \lambda \subseteq \xi \).

**Proof.** Let \( x \in \lambda \). Let \( U \) be an open neighborhood of \( x \). Let \( a \subseteq U \) be an arc transversal to \( \lambda \). By weak* convergence \( \int_a L_i \to \int_a L \) as \( i \to \infty \). Further \( \int_a L > 0 \), which implies that for all \( i \) sufficiently large \( \int_a L_i > 0 \). Then \( \lambda_i \cap a \), for otherwise \( \int_a L_i = 0 \). So there is \( x_i \in \lambda_i \) so that \( x_i \in U \). This implies that \( x \) is in the Hausdorff limit of any convergent subsequence of \( \lambda_i \)'s. \( \square \)

**Notation 2.5.** A subsurface refers to the isotopy class of a subsurface. If a subsurface \( Y \) has a representative which is a subset of a representative of a surface \( Z \), then \( Y \) is a subsurface of \( Z \), we denote it by \( Y \subseteq Z \). Given a multi-curve \( \sigma \), \( \sigma \nparallel Y \) means that at least one of the curves in \( \sigma \) is not isotopic to curve in the complement of \( Y \). If \( \partial Y \nparallel Z \) and \( \partial Z \nparallel Y \), we say \( Y \) and \( Z \) overlap, and denote it by \( Y \nparallel Z \).

2.1. **The pants and marking graph.** A pants decomposition \( P \) on a surface \( S \) is a maximal collection of pairwise disjoint simple closed curves. A (partial) marking \( \mu \) consists of a pants decomposition which is the base of the marking and is denoted by \( \text{base}(\mu) \) together with an element of \( C(\alpha) \) for (some) every \( \alpha \in \text{base}(\mu) \). The element in \( C(\alpha) \) can be represented by a transversal curve \( \beta(\alpha) \) to \( \alpha \) on \( S \). A clean marking is a marking such that each transversal curve \( \beta(\alpha) \) (\( \alpha \) is in the base) does not intersect any curve in \( P - \alpha \) and has minimal intersection number 1 or 2 with \( \alpha \) depending on that the subsurface \( S \backslash \{P - \alpha\} \) is a once punctured torus or a four holed sphere.

**Pants and marking graph:** Each vertex of the pants graph is a pants decomposition, each edge corresponds to two pants decompositions which differ by an elementary move. Pants decomposition \( P \) and \( P' \) differ by an elementary move if \( P' \) is obtained from \( P \) by replacing one curve \( \alpha \in P \) with a curve \( \alpha' \in C_0(S \backslash P - \alpha) \) with minimal intersection number (1 or 2) with \( \alpha \) and fixing the rest of curves in \( P \). Assigning length 1 to each edge defines the distance \( d \) on \( P(S) \) and makes it a metric graph.
Each vertex of the marking graph is a marking, each edge corresponds to a pair of markings which differ by an elementary move. An elementary move on a marking roughly speaking is either an elementary move on the base of the marking or is an interchange of a curve in the base and its transversal curve. For more detail see [MM00]. Assigning length one to each edge defines the distance \( \hat{d} \) and makes the marking graph a metric graph.

The following theorem plays an important role in organizing the so called tight geodesics in curve complexes of a surface and its subsurfaces and inductive construction of hierarchies in the pants and marking graphs of a surface.

**Theorem 2.6. (Bounded Geodesic Image)** [MM00] There exists a constant \( G > 0 \) depending only on the topological type of \( S \) with the following property. Let \( Y \subseteq S \) be an essential, connected subsurface and let \( g : I \to \mathcal{C}(S) \) be a geodesic such that for every \( i \in I \), \( \pi_Y g(i) \neq \emptyset \) i.e. \( g(i) \pitchfork Y \). Then we have

\[
\text{diam}_Y(\{g(i)\}_{i \in I}) \leq G
\]

In the rest of this subsection we recall some results in the context of pants and marking graphs which we use often in this paper.

**Distance formula:** Let \( A \geq 0 \), define the cut-off function \( \{\} : \mathbb{R} \to \mathbb{R} \) by

\[
\{a\} = \begin{cases} a & \text{if } a \geq A \\ 0 & \text{if } a < A \end{cases}
\]

Masur-Minsky in [MM00] proved the following quasi distance formula for the pants and marking graph distance.

There exists a constant \( M_1 > 0 \) such that for any \( A \geq M_1 \), there are \( K \geq 1 \) and \( C \geq 0 \) such that the distance between any two pants decompositions \( P \) and \( Q \) is given by

\[
d(P, Q) \asymp_{K,C} \sum_{\text{non-annular } Y \subseteq S} \{d_Y(P, Q)\}_A.
\]

Note that the sum is only over non-annular subsurfaces. We call \( A \) the threshold constant and say that \( K, C \) are the constants corresponding to the threshold constant \( A \).

**Remark 2.7.** Given partial markings \( \mu \) and \( \mu' \) if in (2.2) we sum over all subsurfaces coefficients, including the annular subsurface coefficients, we get the marking distance of \( \mu \) and \( \mu' \).

**Theorem 2.8. (Behrstock Inequality)** [Beh06] There exists a positive constant \( B_0 \) with the property that for subsurfaces \( Y, Z \subseteq S \) such that \( Y \pitchfork Z \) and a partial marking \( \mu \) such that \( \mu \pitchfork Y \) and \( \mu \pitchfork Z \) we have

\[
\min\{d_Y(\partial Z, \mu), d_Z(\partial Y, \mu)\} \leq B_0.
\]
We recall the Consistency Theorem of Behrstock and Minsky from [BKMM12]. Note that there this theorem is stated and proved for markings. It is straightforward to verify that all of their arguments go through for pants decompositions, considering only non-annular subsurfaces (excluding all annular subsurfaces).

**Theorem 2.9.** (Consistency) ([BKMM12] Theorem 4.3) Given $F_1, F_2 \geq 1$, there is a constant $F > 0$ with the following property. Let $(x_Y)_{Y \subseteq S}$ be a tuple where $x_Y \in \mathcal{C}(Y)$ for each non-annular subsurface $Y \subseteq S$. Suppose that $(x_Y)_{Y \subseteq S}$ satisfies the following two conditions:

1) If $Y \cap Z$ then $\min\{d_Y(x_Y, \partial Z), d_Z(x_Z, \partial Y)\} \leq F_1$, and
2) If $Y \subseteq Z$ then $d_Y(x_Y, \pi_Y(x_Z)) \leq F_2$.

Then there is a $P \in P(S)$ such that for every subsurface $Y \subseteq S$ we have:

$$d_Y(P, x_Y) \leq F.$$

**Remark 2.10.** Note that given a partial marking $\mu$, the tuple of $x_Y = \pi_Y(\mu)$ satisfies the above two conditions with $F_1 = B_0$ ($B_0$ is the constant from Theorem 2.8) and $F_2 = 1$.

We recall the following relation on subsurfaces of $S$ from [BKMM12].

**Definition 2.11.** (Partial order in pants and marking graphs) Fix a tuple $(x_Y)_{Y \subseteq S}$ where $x_Y \in \mathcal{C}(S)$ satisfying the consistency conditions in Theorem 2.9. Assume that $F_1 > \min\{F_2, B_0, G\}$, where $B_0$ is the constant from Theorem 2.8. We define the following two relations on proper, connected, essential subsurfaces of $S$:

1. Given a positive integer $k$, $Y \prec_k Z$, if $Y \cap Z$ and $d_Y(x_Y, \partial Z) \geq k(F_1 + 4)$.
2. Given a positive integer $k$, $Y \ll_k Z$, if $Y \cap \partial Z$ and $d_Y(x_Y, \partial Z) \geq k(F_1 + 4)$

$Y \cap Z$ if $\partial Y \cap Z$ and $\partial Z \cap Y$, so it is immediate from the definition that $Y \prec_k Z \implies Y \ll_k Z$, but not the other way round. Moreover, if $k > l$ then $Y \ll_k Z \implies Y \ll_l Z$, and similarly $Y \ll_k Z \implies Y \ll_k Z$. These notions of partial order are closely related to the partial order of subsurface in Proposition 2.15. The following theorem provides for some useful transitivity properties of the partial order defined above.

**Theorem 2.12.** ([BKMM12] Lemma 4.1)

Given an integer $k > 1$, let $(x_Y)_{Y \subseteq S}$ be a tuple satisfying the consistency conditions and consider the partial order defined by it.

Let $U, V$ and $W$ be subsurfaces such that $x_U, x_V, x_W \neq \emptyset$ then we have:

1. If $U \prec_k V$ and $V \ll_2 W$ then $U \prec_{k-1} W$, also if $\mu$ is a partial marking, and if $U \prec_k V$ and $V \ll_2 \mu$, then $U \prec_{k-1} \mu$.
2. If $U \ll_k V$ and $V \ll_2 W$ then $U \ll_{k-1} W$, also if $\mu$ is a partial marking, and if $U \ll_k V$ and $V \ll_2 \mu$ then $U \ll_{k-1} \mu$.
3. If $U \cap V$ and both $U \ll_k \mu$ and $V \ll_k \mu$ for a partial marking $\mu$, then $U$ and $V$ are $\ll_k \mu$-ordered, that is, either $U \ll_{k-1} \mu$ or $V \ll_{k-1} \mu$. 

2.2. Hierarchies and their resolutions in the pants and marking graph. The hierarchies of *tight geodesics* of curve complexes of subsurfaces of a surface were introduced by Masur and Minsky in [MM00]. See also [Min10], [BKMM12]. Hierarchy paths are quasi-geodesics in the pants and marking graph of a surface, with quantifiers depending only on the topological type of $S$, obtained by resolving hierarchies. Although in most of this paper we use only hierarchy resolution paths and their properties given in Theorem 2.13, a good understanding of the structure of hierarchies themselves will be extremely useful for the reader to follow our arguments. Given a geodesic $h$ in the curve complex of a subsurface $Y$, we refer to $Y$ as the support of $h$ and denote it by $D(h)$. Let $(\mu^-, \mu^+)$ be a pair of partial makings or laminations in $S$, a complete Hierarchy of tight geodesics $H$ inductively associates to the pair $(\mu^-, \mu^+)$ a collections of tight geodesics in the curve complex of $S$ and the curve complexes of its subsurfaces. A tight geodesic in $C(Y)$, is a geodesic with the property that any three of its consecutive vertices fill the surface $Y$. Here we list the properties which describe the inductive and layered structure of the hierarchy $H$, for more detail see the above references.

(1) There is a unique main geodesic $g_S$ with $D(g_S) = S$, whose endpoints lie on base($\mu^-$) and base($\mu^+$).

(2) For any geodesic $h \in H$ other than $g_S$, there exists another geodesic $k \in H$ such that, for some simplex $v \in k$, $D(h)$ is either a component of $D(k) \setminus v$, or an annulus whose core is a component of $v$. We say that $D(h)$ is a component domain of $k$.

(3) A subsurface in $S$ can occur as the domain of at most one geodesic in $H$.

**Infinite hierarchies:** An infinite hierarchy is a hierarchy $H$ with associated infinite tight geodesic rays or lines in the curve complexes of subsurfaces. Here each of the end points $\mu^\pm$ is a union of minimal, filling laminations supported on disjoint subsurfaces. Each lamination is the point at infinity of an infinite geodesic of the hierarchy in the curve complex of the subsurface supporting the geodesic (see Theorem 2.2). The existence of infinite hierarchies is proved in [Min10].

**Hierarchy resolution paths:** These are path which comprise for a set of transitive quasi-geodesics in the pants (marking) graphs of the surfaces $S$ with constants depending only on the topological type of the surface. In §5 we prove a condition for stability of hierarchy resolution paths in the pants graph.

A pants graph resolution of a hierarchy $H$ is a path in the pants graph denoted by $\rho : [n, m] \rightarrow P(S)$, where $[m, n] \subseteq \mathbb{Z} \cup \{\pm \infty\}$. For any $i \in [m, n]$, $\rho(i + 1)$ is obtained from $\rho(i)$ by an elementary move ($\rho(i + 1)$ and $\rho(i)$ have pants distance 1). Similarly a marking graph resolution of $H$ is a path of clean markings such that any two consecutive markings differ by an
elementary move. Given a hierarchy path $\rho$ we denote

$$|\rho| = \{\rho(i) : i \in [n, m]\}$$

which is a subset of the pants graph. A resolution path of a hierarchy consists of slices of the hierarchy. Each slice is the union of vertices of geodesics in curve complexes of subsurfaces of $S$. The supporting domains of these geodesics consist of a tower of nested domains of $H$. In the following theorem we state some of the list properties of these quasi-geodesics (see also [BMM11], [BM08]) which we will use frequently in this paper. The main feature of these properties is that they are encoded in the subsurface coefficients of the pair.

**Theorem 2.13. (Properties of pants hierarchy resolution paths)**

Given partial markings or laminations $(\mu^-, \mu^+)$, there are hierarchy (resolution) paths $\rho : [m, n] \to P(S) \subseteq \mathbb{Z} \cup \{\pm \infty\}$ with $\rho(m) = \mu^-$ and $\rho(n) = \mu^+$ satisfying the following list of properties. In what follows $M_1, M_2 > 0$ ($M_1 \geq 2M_2$) are constants depending only on the topological type of $S$.

1. Given a component domain $Y$, there is a connected interval $J_Y \subseteq [m, n]$ and a geodesic $g_Y \subset C(Y)$ such that for each $j \in J_Y$, $\partial Y \subset \rho(i)$ and there is a simplex $v \in g_Y$ such that $v \in \rho(j)$ ($v \subset g_Y \cap \rho(j)$).
2. There is a constant $M_1 > 0$, depending only on the topological type of $S$, such that an essential, non-annular subsurface $Y \subseteq S$ with $d_Y(\mu^-, \mu^+) > M_1$ is a component domain of $\rho$.
3. (Monotonicity) Let $i, j \in J_Y$, $v = \rho(i) \cap g_Y$ and $w = \rho(j) \cap g_Y$. Then $i \leq j$ if and only if $v \leq w$ as vertices along $g_Y$.
4. (Bounded projection) Let $Y$ be a component domain of $\rho$ and $J_Y = [j^-, j^+]$. If $i > j^+$ then $d_Y(\rho(i), \rho(j^+)) \leq M_2$, and if $i < j^-$ then $d_Y(\rho(i), \rho(j^-)) \leq M_2$.
5. (Hausdorff distance bound) Let $W \subseteq S$ be a subsurface. For any $i \in [m, n]$ there is an $x \in \text{hull}_W(\mu^-, \mu^+)$ such that

$$d_W(\rho(i), x) \leq M_2$$

also for any $x \in \text{hull}_W(\mu^-, \mu^+)$ there is an $i \in [m, n]$ such that the above bound holds. Here $\text{hull}_W(\mu^-, \mu^+)$ is the convex hull of $\pi_W(\mu^-)$ and $\pi_W(\mu^+)$ in $C(W)$. In other words, the Hausdorff distance of $\text{hull}_W(\mu^-, \mu^+)$ and $\pi_W(\rho)$ in $C(W)$ is less than $M_2$. Note that here we do not necessarily assume that $W$ is component domain of $\rho$.

When $W$ is a component domain the statement follows from (7).

6. (No backtracking) Let $i, j, k \in [m, n]$ with $i \leq j \leq k$. Then for any subsurface $Y \subseteq S$, $d_Y(\rho(i), \rho(k)) + M_2 \geq d_Y(\rho(i), \rho(j)) + d_Y(\rho(j), \rho(k))$.

In this theorem $J_Y \subset [m, n]$ consists of all $j \in [m, n]$ such that $\rho(j)$ is a slice of $H(\mu^-, \mu^+)$ containing $(v, g_Y)$, where $g_Y \subset H$ is the tight geodesic supported on $Y$ and $v \in g_Y$. The fact that $J_Y$ is an interval was proved in Lemma 4.9 of [BCM12]. This explains property (1). Property (2) is
Lemma 6.1 (Large Link Lemma) in [MM00]. Property (4) is a consequence of the Bounded Geodesic Image Theorem and is established in the proof of Large Link Lemma. Property (5) is established in the proof of Lemma 5.14 in [Min10]. Property (6) is a straightforward consequence of properties (4) and (5). Property (3) is a consequence of the definition of slices of hierarchy and the partial order of them defined using the partial order of point geodesics of $H$ in §5 of [MM00].

Remark 2.14. Given $(\mu^-, \mu^+)$, a marking hierarchy resolution path $\tilde{\rho} : [m, n] \to M(S) ([m, n] \subseteq Z \cup \{\pm \infty\})$ between $\mu^-$ and $\mu^+$ satisfies the same list of properties as pants hierarchy resolution paths, besides in properties (1)-(6) subsurfaces $Y$ and $W$ can be annular subsurfaces as well.

Partial order on subsurfaces along hierarchies is introduced in [MM00]. It roughly gives the order in which the intervals $J_Y$ appear along the component domains of a resolution of the hierarchy. In this paper we mainly need the following weaker version of it.

Proposition 2.15. (Order of subsurfaces) Fix the constant $M = M_1 + B_0 + 4$. Let $Y, W \subseteq S$ be subsurfaces. Suppose that $Y \pitchfork W$, $d_Y(\mu, \mu') > 4M$ and $d_W(\mu, \mu') > 4M$. Then one and only one of the following inequalities holds.

(a) $d_Y(\mu, \partial W) \geq 2M$ and $d_W(\mu', \partial Y) > 2M$.
(b) $d_W(\mu, \partial Y) \geq 2M$ and $d_Y(\mu', \partial W) > 2M$.

If (a) holds we denote $Y < W$. If (b) holds we denote $W < Y$.

Furthermore, the relation $<$ has the following properties.

- It is transitive.
- Suppose that $W < Y$, $j \in J_Y - J_W$ and $i \in J_W - J_Y$, then $i \leq j$.

Proof. Since $d_Y(\mu, \mu') > 4M$, by the triangle inequality either $d_Y(\mu, \partial W) \geq 2M$ or $d_Y(\mu', \partial W) \geq 2M$. First assume that $d_Y(\mu, \partial W) \geq 2M$ i.e. (a) holds. Then by Theorem 2.8 (Behrstock Inequality) we get $d_W(\mu, \partial Y) \leq B_0 < M$. So (b) cannot hold. Furthermore, by the assumption of the proposition, $d_W(\mu, \mu') > 4M$. The last two inequalities combined by the triangle inequality imply that $d_W(\mu', \partial Y) > 3M - \text{diam}_W(\mu) \geq 3M - 2 > 2M$. When (b) holds a similar argument shows that $d_Y(\mu, \partial W) \leq B_0$, thus (a) does not hold and using the assumption $d_Y(\mu, \mu') > 4M$ we get $d_Y(\mu', \partial W) > 2M$.

To prove the transitivity of $<$, let $W < Y$ and $Y < Z$. Then as we saw above $d_Y(\mu', \partial W) > 2M$ and $d_Z(\mu', \partial Y) > 2M$. The second inequality and the Behrstock inequality imply that $d_Y(\mu, \partial Z) \leq M$. This and first inequality combined by the triangle inequality imply that

$$d_Y(\partial W, \partial Z) > M - 2.$$ 

But $M - 2 \geq 2$, thus $d_Y(\partial W, \partial Z) \geq 3$. So $\partial Y$ and $\partial Z$ fill $W$. Thus $\partial Y \cap \partial Z$ and $Y \cap Z$. $M - 2 > B_0$ so by the above inequality we also have that $d_Y(\partial W, \partial Z) > B_0$. Then by the Behrstock inequality $d_W(\partial Y, \partial Z) \leq B_0$. 


Now we have
\[ d_W(\mu, \partial Z) \geq d_W(\mu, \mu') - d_W(\partial Y, \mu') - d_W(\partial Y, \partial Z) - \text{diam}_W(\partial Y) - \text{diam}_W(\mu') \geq 4M - 2B_0 - 3 > 2M. \]
This finishes proving that \( W < Z \).

We proceed to prove the last assertion of the proposition. Since \( W < Y \),
\[ d_W(\partial Y, \mu') \geq 2M \] so by Theorem 2.8, \( d_Y(\partial W, \mu) \leq M \). Moreover, \( i \in J_W - J_Y \) so \( \rho(i) \supset \partial W \). Thus
\[ d_Y(\rho(i), \mu) \leq M \quad (\ast) . \]

Now suppose that \( i > j \), then since \( j \in J_Y - J_W \), \( j \) is greater than the right end point of the interval \( J_Y \). So Theorem 2.13 implies that
\[ d_Y(\rho(i), \mu') \leq M. \] This inequality combined by \( d_Y(\mu, \mu') > 4M \) with the triangle inequality implies that
\[ d_Y(\rho(i), \mu) > 3M - 2 > M \]
which contradicts the upper bound \((\ast)\). Thus \( i \leq j \) as was desired. \( \Box \)

In this paper we deduce almost all of the properties of hierarchy paths we need from the Theorem 2.13 and Proposition 2.15. In a couple of occasions we need some finer properties of hierarchies and their resolutions where we provide a reference.

2.3. \( \Sigma \)-hulls and their projections. In this section we recall the subsets of pants graph called \( \Sigma \)-hull and their projection introduced in \cite{BKMM12}. The projection is coarsely the closest point projection on the \( \Sigma \)-hull as a subset of the pants graph of the surface. Note that in \cite{BKMM12} these notions are introduced in the context of marking graphs. Other places were variations of this projection is used are \cite{Beh06, BM08, BMM11}. Given a pair of partial markings or laminations \((\mu^-, \mu^+)\) and \( \epsilon > 0 \) define
\[ \Sigma_\epsilon(\mu^-, \mu^+) := \{ P \in P(S) : d_Y(P, \text{hull}_Y(\mu^-, \mu^+)) \leq \epsilon \} \]
for every non-annular subsurface \( W \subseteq S \).

Here \( \text{hull}_Y(\mu^-, \mu^+) \) is the set of all geodesics in \( C(Y) \) connecting \( \pi_Y(\mu^-) \) to \( \pi_Y(\mu^+) \). Note that since \( C(Y) \) is \( \delta_Y \)-hyperbolic all of the geodesics connecting \( \pi_Y(\mu^-) \) and \( \pi_Y(\mu^+) \) uniformly fellow travel each other.

**Theorem 2.16.** (\cite{BKMM12} Proposition 5.2) There is an \( F > 0 \) depending only on the topological type of the surface such that for every \( \epsilon > F \) there is a coarse map (projection)
\[ \Pi : P(S) \to \Sigma_\epsilon(\mu^-, \mu^+) \]
with the following properties:

1. For every non-annular subsurface \( Y \subseteq S \) we have
\[ d_Y(\Pi P, \text{hull}_Y(\mu^-, \mu^+)) \leq F \]
2. \( \Pi|_{\Sigma_\epsilon(\mu^-, \mu^+)} \) is uniformly close to the identity.
3. \( \Pi \) is coarse-Lipschitz.
Remark 2.17. This theorem in [BKMMM12] is stated and proved in the context of marking graph. But it is straightforward to verify that all of their arguments go through in the context of pants graph excluding all annular subsurfaces.

The coarse Lipschitz refers to the fact that the projection is defined on the vertices of the pants graph and does not say anything when two points are within distance 1.

Using the distance formula (2.2) II P coarsely minimizes the distance between P and the $\Sigma_{\epsilon}(\mu^-, \mu^+)$ in the pants graph.

The main ingredient of the proof of Theorem 2.16 is to prove that there are positive constants $F_1$ and $F_2$, depending only on the topological type of the surface, such that the tuple $(x_Y)_{Y \subseteq S}$, where each $x_Y \in C(Y)$ is a nearest point to $\pi_Y(P)$ on $\text{hull}_Y(\mu^-, \mu^+)$, satisfies the consistency conditions of Theorem 2.9 (Consistency Theorem). Then the consistency theorem implies that there is a constant $F > 0$ and a pants decomposition, denoted by II P, such that $d_Y(\text{II P}, \text{hull}_Y(\mu^-, \mu^+)) \leq F$ for every $Y \subseteq S$.

3. The Weil-Petersson metric and its synthetic properties

We start with some basic facts about Teichmüller theory and the Weil-Petersson (WP) metric, and through out will set up our notation. Let $S$ be a surface with genus $g$ and $b$ boundary components. A point in the Teichmüller space of $S$, denoted by Teich$(S)$, is a complete, finite area hyperbolic surface $x$ equipped with a diffeomorphism $h : S \to x$. The map $h$ is a marking for $x$. Two marked surfaces $h_1 : S \to x_1$ and $h_2 : S \to x_2$ define the same point in Teich$(S)$ if and only if $h_2 \circ h_1^{-1} : x_1 \to x_2$ is isotopic to an isometry. The mapping class group of the surface $S$, denoted by Mod$(S)$, is the group of isotopy classes of orientation preserving self diffeomorphisms of $S$. Mod$(S)$ acts on Teich$(S)$ by remarking as follows: an element $f \in \text{Mod}(S)$ maps a marked surface $h : S \to x$ to the marked surface $h \circ f : S \to x$. The quotient $\text{Teich}(S)/\text{Mod}(S)$ is the moduli space of $S$ denoted by $\mathcal{M}(S)$. Given a point $f : S \to x$ in the Teichmüller space we usually drop the marking map and denote it by $x$. We denote the point in moduli space corresponding to the Mod$(S)$ orbit of $x$ by $\hat{x}$.

Given $\epsilon > 0$, the $\epsilon$–thick part of Teichmüller space is $\{x \in \text{Teich}(S) : \text{inj}(x) \geq \epsilon\}$. Here inj$(x)$ is the injectivity radius of the hyperbolic surface $x$. The $\epsilon$–thin part is $\{x \in \text{Teich}(S) : \text{inj}(x) \leq \epsilon\}$. Let $\epsilon > 0$ be small enough such that by the Collar lemma ($\S$4.1 of [Bus10]) on any complete hyperbolic surface there is not any pair of intersecting closed geodesic with length less than or equal of $\epsilon$. Let $\epsilon' > 0$. Given a multi-curve $\sigma$ (a simplex in $C(S)$) we define the following regions in the Teichmüller space:

- $U_\epsilon(\sigma) := \{x \in \text{Teich}(S) : \ell_\alpha(x) \leq \epsilon \text{ for any } \alpha \in \sigma\}$
- $U_{\epsilon, \epsilon'}(\sigma) := \{x \in \text{Teich}(S) : \ell_\alpha(x) \leq \epsilon \text{ for any } \alpha \in \sigma \text{ and } \ell_{\alpha'}(x) > \epsilon' \text{ for every } \alpha' \notin \sigma\}$
Here we recall some properties of the Weil-Petersson metric and its geodesics which will be used in this paper. References for these material are [Wol03], [Wol08], [Wol10], see them also for further references.

Given holomorphic quadratic differentials $\varphi, \psi \in T^*_x \text{Teich}(S)$ the Weil-Petersson $L^2$ co-product is defined by

$$\mathcal{R}e(\int_x \varphi \overline{\psi} \rho^{-2})$$

where $\rho(z)^2|dz|^2$ is the hyperbolic metric of the marked hyperbolic surface $x$. This co-product induces a norm on Teichm"uller space via the standard pairing of quadratic differentials and measurable Beltrami differentials on $x$ which is defined by $\int_x \varphi \mu$. Any measurable Beltrami differential presents a vector in $T_x \text{Teich}(S)$ and the Weil-Petersson metric on Teichm"uller space is defined by the polarization of this norm. In this paper we study the global behavior of geodesics of this metric.

The Weil-Petersson metric is a Riemannian metric with negative sectional curvatures which is invariant under the action of the mapping class group of the surface. It is an incomplete metric, however it is geodesically convex. The negative curvature and convexity imply that the completion of Teichm"uller space with respect to the WP metric $\text{Teich}(S)$ is a CAT(0) space. For background about CAT(0) space see for example [BH99].

**Length-functions:** Given any $\alpha \in \mathcal{C}_0(S)$ the $\alpha-$length-function

$$\ell_\alpha : \text{Teich}(S) \to \mathbb{R}^+$$

assigns to $x \in \text{Teich}(S)$ the length of the geodesic representative of $\alpha$ on the marked hyperbolic surface $x$.

The notion of length-function has a natural extension to the space of measured geodesic laminations (see [Bon01]). Given a measured geodesic lamination $L$ we denote the $L-$length-function by $\ell_L(x)$.

**Fenchel-Nielsen coordinates and twist parameters:** Given a pants decomposition $P$, a Fenchel-Nielsen (FN) coordinate system $(\ell_\gamma, \theta_\gamma)_{\gamma \in P}$ maps $\text{Teich}(S)$ to $\prod_{\gamma \in P} \mathbb{R}^+ \times \mathbb{R}$. The first coordinate of the pair $\mathbb{R}^+ \times \mathbb{R}$ is the $\gamma-$length-function and the second coordinate is a twist parameter about $\gamma$. For more detail about Fenchel-Nielsen coordinates and twist parameters see §3 of [Bus10]. We denote the Dehn twist about a curve $\gamma$ by $\mathcal{D}_\gamma$, it is defined as follows: Let $x \in \text{Teich}(S)$. Given a pants decomposition $P$ with $\gamma \in P$ fix a FN coordinate system, $\mathcal{D}_\gamma(x)$ is the point with all coordinates equal to that of $x$ except $\theta_\gamma(\mathcal{D}_\gamma(x)) = \theta_\gamma(x) + 2\pi$.

The Weil-Petersson completion of Teichmüller space and the completion strata: The incompleteness of the Weil-Petersson metric is due to existence of finite length paths in Teichmüller space along which length of a curve converges to zero, [Wol10]. In [Mas76], Masur gives a concrete description of the completion as the augmented Teichmüller space. The augmented Teichmüller space consists of strata: Let $\sigma$ be a simplex in
the augmented curve complex $\hat{C}(S) = C(S) \cup \emptyset$, a point in the $\sigma$–stratum is a collection of marked hyperbolic metrics of connected components of $S \setminus \sigma$, where for each curve in $\sigma$ a pair of cusps is introduced. The topology is described via extended Fenchel-Nielsen coordinate systems as follows: Given a pants decomposition $P$, the FN coordinate system maps $Teich(S)$ to $\prod_{\gamma \in P} \mathbb{R} \times \mathbb{R}_+$. We extend the FN coordinate system $(\ell_\gamma, \theta_\gamma)_{\gamma \in P}$ to allow length-functions take value 0 as well. Now take the quotient of $\prod_{\gamma \in P} \mathbb{R} \times \mathbb{R}_+$ by identifying $(0, \theta) \sim (0, \theta')$ in each $\mathbb{R}_+ \times \mathbb{R}_+$ factor. Let $\sigma \subset P$ then the topology near any point of the $\sigma$–stratum is such that the map defined by the FN coordinate system is a homeomorphism near that point.

In this topology each stratum $S(\sigma)$ is the product of the lower dimensional Teichmüller spaces of the connected components of $S \setminus \sigma$.

**Continuity of length-functions:** In this paper we refer to the following theorem as the continuity of length-functions. It is consequence of the fact that the topology induced by the Weil-Petersson metric and the Chaubaty topology of the Teichmüller space are the same. The Chaubaty topology is defined using the fact that each point in Teichmüller space is the conjugacy class of a representation of $\pi_1(S)$ into $PSL_2(\mathbb{R})$. For more detail see the beginning of §4 of [Wol08].

**Theorem 3.1.** *(Continuity of length-functions)* Suppose that a sequence of points $x_n \to x$ as $n \to \infty$ in the completion of Teichmüller space with respect to the WP metric. Then for every $\alpha \in C_0(S)$, $\ell_\alpha(x_n) \to \ell_\alpha(x)$ as $n \to \infty$.

**Non-refraction of strata:**

**Theorem 3.2.** *(Non-refraction)* [DW03], [Wol03], [Yam04]

Let $\zeta : [0, T] \to \overline{Teich(S)}$ be a WP geodesic segment, and let $\sigma^-$ and $\sigma^+$ be the maximal simplicies in $C(S)$ such that $\zeta(0) \in S(\sigma^-)$ and $\zeta(T) \in S(\sigma^+)$ then

$$int(\zeta) \subset S(\sigma^- \cap \sigma^+)$$

As a consequence of the Non-refraction Theorem Daskalopoulos and Wentworth in [DW03] and Wolpert in [Wol03] show that any pseudo-Anosov element of the mapping class group $f$ has an axis in the Teichmüller space equipped with the WP metric. The axis is a bi-infinite WP geodesic $Ax_f \subset \overline{Teich(S)}$ such that

$$d_{WP}(x, fx) = \inf_{y \in \overline{Teich(S)}} d_{WP}(y, fy)$$

for every $x \in Ax_f$. The axis projects to a closed geodesic in the moduli space.

**Bers pants decomposition and Bers marking:** By a result of Bers (see §3 of [Bus10]) given a surface $S$ with $\chi(S) < 0$, there is a constant $L_S > 0$ *(Bers constant)* depending only on the topological type of $S$ such
that any complete finite area hyperbolic metric on \( S \) possesses a pants decomposition (Bers pants decomposition) with the property that the geodesic representative of any curve in the pants decomposition has length at most \( L_S \). We call any curve in a Bers pants decomposition a Bers curve. By the Collar Lemma there are only finitely many Bers curves and consequently Bers pants decompositions on a complete hyperbolic surface. Given \( x \in \overline{\Teich(S)} \), suppose that \( x \in S(\sigma) \) (\( \sigma \) is a simplex in \( C(S) \)). Then a Bers pants decomposition of \( x \) is the union of Bers pants decompositions of each of the connect components of \( S\setminus \sigma \) and \( \sigma \). A Bers marking is a (partial) marking obtained from a Bers pants decomposition by adding transversal curves with representatives at \( x \) of minimal length. We denote a Bers marking of \( x \in \overline{\Teich(S)} \) by \( \mu(x) \). Given \( x \in S(\sigma) \) the partial marking does not have any transversal to the curves in \( \sigma \).

By the following theorem of Brock the hierarchies of curves complexes and their resolution in pants and marking graphs would play an essential role in our study of the global behavior of WP geodesics:

**Theorem 3.3.** (Quasi-isometric model)\(^{[Bro03]}\) There are constants \( K_{WP} \geq 1 \) and \( C_{WP} \geq 0 \) depending only on the topological type of \( S \), such that the coarsely defined map

\[
Q : \overline{\Teich(S)} \to P(S)
\]

which assigns to \( x \) a Bers pants decomposition \( Q(x) \) is a \( (K_{WP}, C_{WP}) \)-quasi-isometry.

**Gradient of length-functions:** Wolpert gives the following estimate for the pairing of the gradients of length-functions:

**Lemma 3.4.**\(^{[Wol08]}\) The WP pairing of length-function gradients of curves \( \alpha, \beta \) with disjoint geodesic representatives satisfies

\[
0 < \langle \text{grad} \ \ell_\alpha, \text{grad} \ \ell_\beta \rangle - \frac{2}{\pi} \ell_\alpha \delta_{\alpha \beta} = O(\ell_\alpha^2 \ell_\beta^2)
\]

where the constant of the \( O \) notation depends only on \( c_0 > 0 \) with \( \ell_\alpha, \ell_\beta \leq c_0 \).

**Corollary 3.5.** Given \( c_0 > 0 \), there is a two variable function \( d \) with the following property. Given \( l, a \in [0, c_0] \) such that \( l > a \geq 0 \). Let \( x, x' \in \Teich(S) \) be such that \( \ell_\alpha(x) \leq l - a \) and \( \ell_\alpha(x') \geq l \). Then \( d_{WP}(x', x) \geq d(l, a) \).

**Proof.** By Lemma 3.4 at \( y \in \Teich(S) \) with \( \ell_\alpha(y) \leq c_0 \), \( ||\text{grad} \ \ell_\alpha(y)|| \leq \left( \frac{2}{\pi} \ell_\alpha(y) + O(\ell_\alpha(y)^4)^{1/2} \right) \) \( (*) \) where the \( O \) notation constant depends only on \( c_0 \). Let \( u \) be the WP geodesic segment from \( x \) to \( x' \), parametrized by arc-length. Let \( t^* \) be the first time that \( \ell_\alpha(u(t)) = l \). Then \( \ell_\alpha(u(t)) \leq l \) for every \( t \in [0, t^*] \). Using this bound and integrating \( (*) \) we get

\[
a \leq |\ell_\alpha(u(t^*)) - \ell_\alpha(u(0))| \leq \int_0^{t^*} ||\text{grad} \ \ell_\alpha(u(t))||dt \leq \left( \frac{2}{\pi} l + O(l^4)^{1/2} \right) t^*.
\]

Hence \( t^* \geq d(l, a) := \frac{a}{\left( \frac{2}{\pi} l + O(l^4)^{1/2} \right)^{1/2}} \). \( \square \)
Using the estimates on pairings of the gradient of length-functions Wolpert gives an asymptotic expansion for the WP metric near completion strata. These expansion shows that there are asymptotic quasi-flats transverse to strata corresponding to any pair of pinching curves. He also gives the following estimate for the distance of a point in Teichmüller space and stratum.

Proposition 3.6. (Corollary 4.10 of [Wol08]) Let \( x \in \text{Teich}(S) \) and \( \sigma \) be a multi-curve, then 
\[
d_{\text{WP}}(x, S(\sigma)) \leq (2\pi \sum_{\alpha \in \sigma} \ell_\alpha(x))^{1/2}.
\]

Tangent cones of the Weil-Petersson completion of Teichmüller space: The completion of Teichmüller space with the Weil-Petersson metric is a CAT(0) space. Assigned to any point \( p \) of a CAT(0) space there is \( AC_p \), the Alexandrov tangent cone consisting of equivalence classes of geodesics \( \zeta \) starting at the point \( p \). Two geodesics \( \zeta \) and \( \zeta' \) starting at \( p \) are equivalent if their angle at \( p \) in the sense of Alexandrov is equal to 0. For more detail about tangent cones see [BH99].

Given a multi-curve \( \sigma \) on \( S \) let \( \chi \) be a full marking on \( S \setminus \sigma \) and consider the map 
\[
L(\zeta(t)) = (\ell_\alpha^{1/2}(\zeta(t)), \ell_\beta^{1/2}(\zeta(t)))_{\alpha \in \sigma, \beta \in \chi}
\]
where \( \zeta : [0, T] \to \overline{\text{Teich}(S)} \) is a geodesic segment with \( \zeta(0) = p \in S(\sigma) \). Then define \( \Lambda : AC_p \to \mathbb{R}^+ \) by
\[
\Lambda(\zeta) = (2\pi)^{1/2} \frac{d}{dt} \bigg|_{t=0} L(\zeta(t)).
\]
Wolpert gives the following description of the WP Alexandrov tangent cone of Teichmüller space at given point \( p \), Theorem 4.18 of [Wol08].

Proposition 3.7. (WP tangent cone) The map \( \Lambda \) from the tangent cone of the WP metric at \( p \) to \( \mathbb{R}^+_{\geq 0} \times T_p \text{Teich}(S \setminus \sigma) \) is an isometry of tangent cones with restriction of inner products. A WP geodesic \( \zeta \) with \( \zeta(0) = p \) and root length-function initial derivative \( \frac{d}{dt}\big|_{t=0} \ell_\alpha^{1/2}(\zeta(t)) \) vanishing is contained in the stratum \( \{ \ell_\alpha = 0 \} \), \( S(\sigma) \subset \{ \ell_\alpha = 0 \} \).

3.1. End invariant. In this subsection we recall the notion of end invariant for WP geodesics introduced by Brock, Masur and Minsky in [BMM11].

Theorem 3.8. (Convexity of length-functions) [Wol08] Given \( \epsilon > 0 \) there is \( c = c(\epsilon) \) with the following property. Let \( g : (a,b) \to \text{Teich}(S) \) be a WP geodesic parametrized by arc-length and \( \alpha \in C_0(S) \). If for some \( t \in (a,b) \) \( \text{inj}(g) \geq \epsilon \) (\( g(t) \) is a point in the \( \epsilon \)-thick part of Teichmüller space), then we have
\[
(3.1) \quad \ell_\alpha(g(t)) \geq c \ell_\alpha(g(t)).
\]
Similar inequality holds for the length of any measured lamination \( \mathcal{L} \), which is
\[
\ell_{\mathcal{L}}(g(t)) \geq c \ell_{\mathcal{L}}(g(t)).
\]
Remark 3.9. The above estimates are local and only depend on the injectivity radius of the surface $g(t)$.

Definition 3.10. (Ending measured lamination) The weak* limit in $\mathcal{ML}(S)$ of any weighted sequence of distinct Bers curves along a WP geodesic ray $r$ is an ending measured lamination of $r$.

In [BMM11] the following notion of ending lamination for WP geodesic rays is introduced, its existence relies on the convexity of length-functions along WP geodesics and properties of CAT(0) spaces.

Definition 3.11. (Ending Lamination) The union of pinching curves along a WP geodesic ray and the geodesic laminations arising as supports of all ending measured laminations of $r$ is the ending lamination of $r$.

Definition 3.12. (End invariant of Weil-Petersson geodesics) To each open end of a geodesic $g : (a, b) \to \text{Teich}(S)$ we associate an end invariant which is a partial marking or a lamination. If the forward trajectory $g|_{[0,b)}$ can be extended to $b$ such that $g(b) \in \overline{\text{Teich}(S)}$ then the forward end invariant $\nu^+(g)$ is any Bers marking $\nu(g(b))$ (there are finitely many of them). Otherwise, $\nu^+(g)$ is the ending lamination of the forward trajectory ray $g|_{[0,b)}$ which was defined above. We define the backward end invariant $\nu^-(g)$ similarly by considering the backward trajectory $g|_{(a,0]}$. We call the pair $(\nu^-, \nu^+)$, the end invariant of $g$.

Here we recall two properties of the ending measured laminations proved in [BMM10]:

Lemma 3.13. (Decreasing of length-functions along WP geodesic rays) Let $\mathcal{L}$ be any ending measured lamination of a WP geodesic ray $r$, then $\ell_{\mathcal{L}}(r(t))$ is a decreasing function.

Lemma 3.14. Let $r_n \to r$ be a convergent sequence of rays in the WP visual sphere at $x$. Then if $\mathcal{L}_n$ is any sequence of ending measured laminations or weighted pinching curves for $r_n$, any representative $\mathcal{L} \in \mathcal{ML}(S)$ of the limit of the projective classes $[\mathcal{L}_n]$ in $\mathcal{PML}(S)$ has bounded length along the ray $r$.

4. LENGTH-FUNCTION CONTROL ALONG UNIFORMLY BOUNDED LENGTH WEIL-PETERSSON GEODESICS

In this section we study length-functions and twist parameters along sequences of bounded length WP geodesic segments in the WP completion of Teichmüller space.

In §4.2 we will prove a modified version of Lemma 4.5 in [BMM11] about the development of Dehn twists along sequences of uniformly bounded length WP geodesic segments (Theorem 4.6). Corollaries 4.13 and 4.12 are somewhat quantified versions of this theorem which provide us with a kind of twist parameter versus length-function control along WP geodesic segments. This
control plays an important role in where we study the itinerary of WP geodesics fellow traveling hierarchy paths.

The proof of Theorem 4.6 uses Wolpert’s characterization of limits of sequences of uniformly bounded length WP geodesic segments in the Weil-Petersson completion of Teichmüller space. In §4.1 we state Wolpert’s geometric limit theorem and using suggestions of Jeff Brock will give an improved version of it (Theorem 4.5). This improved version is crucial to prove our results in §4.2.

4.1. Limits of sequences of uniformly bounded length WP geodesic segments. In this subsection we provide a modified version of Wolpert’s geodesic limit theorem. Given a multi-curve σ, denote by \( \text{tw}(\sigma) \) the subgroup of \( \text{Mod}(S) \) generated by the Dehn twists about the curves in \( \sigma \). Using the non-refraction property of the Weil-Petersson completion strata (Theorem 3.2) and the fact that the quotient of any \( U_i(\sigma) \) by the action of \( \text{tw}(\sigma) \) is compact, Wolpert provides the following characterization of the limits of uniformly bounded length WP geodesic segments in the Teichmüller space. See also [BMM11].

**Theorem 4.1.** [Wol03] Let \( \zeta_n : [0, T] \to \overline{\text{Teich}(S)} \) be a sequence of WP geodesic segments parametrized by arc-length of length \( T \) in the WP completion of the Teichmüller space. Then after possibly passing to a subsequence there exist a partition of the interval \( [0, T] \) by \( 0 = t_0 < t_1 < t_2 < \ldots < t_k < t_{k+1} = T \), simplices \( \sigma_0, ..., \sigma_{k+1} \), and simplices \( \tau_i = \sigma_{i-1} \cap \sigma_i \) (\( i = 1, ..., k+1 \)) in \( \hat{C}(S) \) where \( \sigma_i \subset \tau_i \) for each \( 1 \leq i \leq k \), and a piecewise geodesic \( \hat{\zeta} : [0, T] \to \overline{\text{Teich}(S)} \)

with the following properties

1. \( \hat{\zeta}([t_{i-1}, t_i)) \subset S(\tau_i) \), for \( i = 1, ..., k + 1 \),
2. \( \hat{\zeta}(t_i) \in S(\sigma_i) \), for \( i = 0, ..., k + 1 \),
3. There are elements \( \psi_n \in \text{Mod}(S) \) and \( \mathcal{T}_{i,n} \in \text{tw}(\sigma_i - \tau_i \cup \tau_{i+1}) \), for \( i = 1, ..., k \), such that after possibly passing to a subsequence \( \psi_n \circ \zeta_n(t) \) converges to \( \hat{\zeta}(t) \) for every \( t \in [0, t_1] \) in \( \overline{\text{Teich}(S)} \) and for each \( i = 1, ..., k \), and \( t \in [t_i, t_{i+1}] \),

\[
\mathcal{T}_{i,n} \circ ... \circ \mathcal{T}_{1,n} \circ \psi_n \circ \zeta_n(t) \to \hat{\zeta}(t)
\]

as \( n \to \infty \). For convenience for each \( i = 0, 1, ..., k+1 \) we define

\[
\varphi_{i,n} = \mathcal{T}_{i,n} \circ ... \circ \mathcal{T}_{1,n} \circ \psi_n
\]

4. The elements \( \psi_n \) are either trivial or unbounded and the elements \( \mathcal{T}_{i,n} \) are unbounded.
5. The piecewise geodesic \( \hat{\zeta} \) is the minimal length path in \( \overline{\text{Teich}(S)} \) joining \( \zeta(0) \) to \( \zeta(T) \) and intersecting the strata \( S(\sigma_1), S(\sigma_2), ..., S(\sigma_k) \) in order.
The following two lemmas which were suggested to us by Jeff Brock help us to considerably improve the above picture of limits of uniformly bounded length WP geodesic segments (see Theorem [4.5]).

Lemma 4.2. Given a sequence of WP geodesic segments $\zeta_n : [0, T] \rightarrow \text{Teich}(S)$, let the simplices $\tau_i$, $i = 1, ..., k + 1$, be as in Theorem 4.1. Then $\tau_1 = ... = \tau_{k+1}$. We denote

\[ \hat{\tau} = \tau_i, \quad i = 1, ..., k \]

Proof. Let the piecewise geodesic path $\hat{\zeta} : [0, T] \rightarrow \text{Teich}(S)$, the partition $0 = t_0 < t_1 < ... < t_{k+1} = T$ and simplices $\sigma_i$, $i = 0, ..., k + 1$ be as in Theorem 4.1. Let $\delta < \min_{i=1}^{k+1} \frac{t_{i+1} - t_i}{2}$. Let $0 \leq i \leq k$. By Theorem 4.1 (1), $\hat{\zeta}|_{[t_i - \delta, t_i)} \subset S(\tau_i)$ and $\hat{\zeta}|_{[t_i, t_i + \delta]} \subset S(\tau_{i+1})$. Moreover by Theorem 4.1 (5) the concatenation of $\hat{\zeta}|_{[t_i - \delta, t_{i+1}]}$ and $\hat{\zeta}|_{[t_i, t_i + \delta]}$ is the distance minimizing path in $\text{Teich}(S)$ joining $\hat{\zeta}(t_i - \delta)$ to $\hat{\zeta}(t_i + \delta)$ and intersecting $S(\sigma_i)$. Recall that $\tau_i \subseteq \sigma_i$, so $\alpha \in \sigma_i$, then as Wolpert shows on page 328 of [Wol08] the following equality of the one-sided derivatives of the square root of the $\alpha$-length-function holds at $t = t_i$,

\[ \frac{d}{dt} \bigg|_{t=t_i} \ell_\alpha^{1/2}(\hat{\zeta}|_{[t_i, t_i + \delta]}) = -\frac{d}{dt} \bigg|_{t=t_i} \ell_\alpha^{1/2}(\hat{\zeta}|_{[t_i - \delta, t_i]}). \]

Then by (4.3) $\hat{\zeta}|_{[t_i, t_i + \delta]} \subset S(\tau_{i+1})$, so $\ell_\alpha^{1/2}(\hat{\zeta}(t)) = 0$ for all $t \in [t_i, t_i + \delta]$ and thus $\frac{d}{dt} \bigg|_{t=t_i} \ell_\alpha^{1/2}(\hat{\zeta}|_{[t_i, t_i + \delta]}) = 0$. Then by (4.3) $\frac{d}{dt} \bigg|_{t=t_i} \ell_\alpha^{1/2}(\hat{\zeta}|_{[t_i - \delta, t_i]}) = 0$. So by Proposition 3.7, $\hat{\zeta}|_{[t_i - \delta, t_i]} \subset S(\alpha)$. Moreover, again by Theorem 4.1 (1) $\hat{\zeta}|_{[t_i - \delta, t_i]} \subset S(\tau_i)$. The last two inclusions imply that $\alpha \in \tau_i$. This holds for every $\alpha \in \tau_i$, so we conclude that $\tau_i \subseteq \tau_{i+1}$. Exchanging the role of $\tau_i$ and $\tau_{i+1}$ a similar argument implies that $\tau_i \subseteq \tau_{i+1}$. Thus $\tau_i = \tau_{i+1}$ for $i = 1, ..., k$ and we get $\tau_1 = ... = \tau_{k+1}$ as desired. \qed

Lemma 4.3. Let $\zeta_n : [0, T] \rightarrow \text{Teich}(S)$ be a sequence of WP geodesic segments parametrized by arc-length. Let $\mathcal{T}_{i,n}$, $i = 1, ..., k$, and the simplices $\sigma_i$, $i = 0, ..., k + 1$, be as in Theorem 4.1 and the simplex $\tilde{\tau}$ be as in (4.2). Let $1 \leq i \leq k$. Suppose that $\sigma_i - \tilde{\tau} \neq \emptyset$ and $\gamma \in \sigma_i - \tilde{\tau}$, then the power of $D_\gamma$ in $\mathcal{T}_{i,n}$ goes to $\infty$ as $n \rightarrow \infty$.

Proof. Let the elements of mapping class group $\varphi_{i,n}$, $i = 1, ..., k + 1$, be as in (4.1). For $i = 1, ..., k$ define

$\sigma_{i,n} = \varphi_{i-1,n}^{-1}(\sigma_i)$

also define the geodesic segments

$\zeta_{i,n}(t) = \varphi_{i-1,n} \circ \zeta_n(t)$ for $t \in [t_{i-1}, t_{i+1}]$

where $0 = t_0 < t_1 < ... < t_k < t_{k+1} = T$ is the partition from Theorem 4.1. We claim that
from (ii') we obtain that the injectivity radius of the point $T_i(T_i - \delta)$ is bounded below by $\epsilon_1$ away from the collars of the curves in $\hat{\tau}$.

Moreover, by Theorem 4.1 (3) for every $t \in [t_{i-1}, t_i]$ the bounds (i) and (ii) at $\zeta_{i,n}(t_i - \delta)$ follow from (i') and (ii'), respectively.

The claim follows from the limit picture of geodesics $\zeta_n$ and continuity of length-functions. To see this, let $\delta < \min_{i=1,\ldots,k+1} \frac{t_{i-1} - t_i}{2}$. Fix $1 \leq i \leq k$. By Lemma 4.2 the two points $\hat{\zeta}(t_i \pm \delta)$ are in the stratum $S(\hat{\tau})$ and by Theorem 4.1 (2), $\zeta(t_i) \in S(\sigma_i)$. Thus there are $\epsilon'_1, \epsilon'_2 > 0$ such that:

(i') $\ell_{\gamma}(\hat{\zeta}(t_i \pm \delta)) \leq \epsilon'_2$ for every $\gamma \in \sigma_i$, and

(ii') The injectivity radius of the points $\zeta_{i,n}(t_i \pm \delta)$ is bounded below by $\epsilon'_1$ away from the collars of the curves in $\hat{\tau}$.

Moreover, by Theorem 4.1 (3) for every $t \in [t_{i-1}, t_i]$, $\zeta_{i,n}(t_i) \to \hat{\zeta}(t)$ as $n \to \infty$. Thus by the continuity of length-functions for $n$ sufficiently large the bounds (i) and (ii) at $\zeta_{i,n}(t_i - \delta)$ for $\epsilon_1 = \epsilon'_1$ and $\epsilon_2 = 2\epsilon'_2$ follow from the bounds in (i') and (ii'), respectively.

Since $\varphi_{i,n} = T_{i,n} \circ \varphi_{i-1,n}$, by Theorem 4.1 (3) for every $t \in [t_i, t_{i+1}]$, $T_{i,n} \circ \zeta_{i,n}(t_i) \to \hat{\zeta}(t)$ as $n \to \infty$. Then by the continuity of length-functions from (i') we obtain that $\ell_{\gamma}(T_{i,n} \circ \zeta_{i,n}(t_i + \delta)) \leq \epsilon'_2$ for every $\gamma \in \sigma_i$, and from (ii') we obtain that the injectivity radius of the point $\zeta_{i,n}(t_i - \delta)$ is bounded below by $\epsilon'_1$ away from the collars of the curves in $\hat{\tau}$. It follows form Theorem 4.1 (3) and Lemma 4.2 that for each $n$, $T_{i,n} \in tw(\sigma_i - \hat{\tau})$. So applying $T_{i,n}$ does not change the length of every curve in $\sigma_i$ and injectivity radius. Thus the bounds (i) and (ii) at $\zeta_{i,n}(t_i + \delta)$ follow from the ones we just established. This finishes the proof of the claim.

We proceed to prove the lemma. Fix $\gamma \in \sigma_i - \hat{\tau}$. Let $h_1 : S \to \zeta_{i,n}(t_i - \delta)$ be the marking of the surface $\zeta_{i,n}(t_i - \delta)$. Let $\beta \in C_0(S' - \hat{\tau})$ be a curve which has minimal intersection number (1 or 2) with $\gamma$ and does not intersect any curve in $\sigma_i - \gamma$ so that $h_1(\beta)$ has minimal length at $\zeta_{i,n}(t_i - \delta)$, see Figure 1.

Realize the curves $h_1(\beta)$ and $h_1(\gamma)$ as geodesics on $\zeta_{i,n}(t_i - \delta)$. Denote the collar of $h_1(\gamma)$ by $C(h_1(\gamma))$. Denote the length of each component of

**Figure 1.** $\beta \in C_0(S' - \hat{\tau})$ is a curve which has minimal intersection number one or two with $\gamma$ and does not intersect any curve in $\sigma_i - \gamma$ so that has minimal length at $\zeta_{i,n}(t_i - \delta)$.
the boundary of the collar by \( C \). Further let \( w \) be the width of the collar. Lifting the picture to the universal cover the length of \( h_1(\beta) \cap C(h_1(\gamma)) \) is bounded above by \( w + 2C \), see Figure 2. Further the length of \( h_1(\beta) \) outside \( C(h_1(\gamma)) \) is bounded above by the diameter of \( \zeta_{i,n}(t_i - \delta) \) out side the collars of curves in \( \hat{\tau} \) plus 2\( C \). By a compactness argument the diameter is bounded above by a constant depending only on the injectivity radius of the surface out side the collars of the curves in \( \hat{\tau} \) and \( C \). Then the length of \( h_1(\gamma) \) is bounded above by \( w + 2C + 2C \) plus the diameter. Claim 4.4(ii) provides the lower bound \( \epsilon_1 \) for the length of \( \gamma \) and consequently an upper bound for \( C \) and \( w \). Part (i) of the claim provides the upper bound \( \epsilon_2 \) for the injectivity radius and consequently an upper bound for the diameter. Thus there is \( L \) depending only on \( \epsilon_1, \epsilon_2 \) so that

\[
(4.4) \quad \ell_\beta(\zeta_{i,n}(t_i - \delta)) \leq L.
\]

Let \( h_2 : S \to \zeta_{i,n}(t_i + \delta) \) be the marking of \( \zeta_{i,n}(t_i + \delta) \). \( h_2(\beta) = h_1 \circ \tau_{i,n}(\beta) \) and \( h_2(\gamma) = h_1(\gamma) \). Denote by \( D_\gamma \) the Dehn twist about \( \gamma \). Let \( m_i \) be the power of \( D_\gamma \) in \( \tau_{i,n} \). Realize \( h_2(\gamma) \) and \( h_2(D_\gamma^{m_i}(\beta)) \) as geodesics. Lifting the picture to the universal cover the length of \( D_\gamma^{m_i}(\beta) \cap C(h_2(\gamma)) \) is bounded above by \( |m| \ell_\gamma(\zeta_{i,n}(t_i + \delta)) + w + 2C \), see Figure 2. Further its length outside the collar is bounded by the diameter of \( \zeta_{i,n}(t_i + \delta) \) out side the collars of the curves in \( \hat{\tau} \) plus 2\( C \). Then the length of \( D_\gamma^{m_i}(\beta) \) is bounded above by \( w + |m_i| \ell_\gamma(\zeta_{i,n}(t_i + \delta)) + 2C + 2C \) plus the diameter. Suppose that \( |m_i| \) is bounded by some \( N > 0 \). Then as we saw above Claim 4.4 gives us \( L' \) depending only on \( \epsilon_1, \epsilon_2 \) and \( N \) so that

\[
(4.5) \quad \ell_\beta(\zeta_{i,n}(t_i + \delta)) \leq L'.
\]

On the other hand, since \( \ell_\gamma(\zeta_{i,n}(t_i)) \to 0 \) as \( n \to \infty \) and \( \beta \cap \gamma \), by the Collar lemma (§4.1 of [Bus10]) we have that

\[
(4.6) \quad \ell_\beta(\zeta_{i,n}(t_i)) \to \infty
\]

as \( n \to \infty \).

For \( n \) sufficiently large, by (4.6), \( \ell_\beta(\zeta_{i,n}(t_i)) > \max\{L, L'\} \). But \( t_i - \delta < t_i < t_i + \delta \), then this bound, (4.4) and (4.5) violate the convexity of the \( \beta \)-length-function along the WP geodesic segment \( \zeta_{i,n} \). This contradiction shows that the power of \( D_\gamma \) in \( \tau_{i,n} \) is unbounded. \( \square \)

Here for the purpose of reference in this paper we state the following strength version of the geodesic limit theorem which essentially contains the properties listed in Theorem 4.1 modified to incorporate Lemmas 4.2 and 4.3.

**Theorem 4.5.** (Geodesic Limit) Let \( \zeta_n : [0, T] \to \overline{\text{Teich}(S)} \) be a sequence of WP geodesic segments of length \( T \). Then after possibly passing to a subsequence there exists a partition of the interval \([0, T]\) by \( 0 = t_0 < t_1 < \ldots < t_k < t_{k+1} = T \), and simplices \( \sigma_0, \ldots, \sigma_{k+1} \) in \( \overline{C}(S) \) such that \( \sigma_i \cap \sigma_{i+1} = \hat{\tau} \),
for $i = 0, \ldots, k$, and a piecewise geodesic

$$\hat{\zeta} : [0, T] \to \Teich(S)$$

with the following properties

1. $\hat{\zeta}((t_{i-1}, t_i)) \subset S(\hat{\tau})$, for $i = 1, \ldots, k + 1$,
2. $\hat{\zeta}(t_i) \in S(\sigma_i)$, for $i = 0, \ldots, k + 1$,
3. There are elements $\psi_n \in \Mod(S)$ and $T_{i,n} \in \text{tw}(\sigma_i - \hat{\tau})$, for $i = 1, \ldots, k$, such that for every $t \in [0, t_1]$, $\psi_n \circ \zeta_n(t)$ converges to $\hat{\zeta}(t)$, and for each $i = 1, \ldots, k$, and every $t \in [t_i, t_{i+1}]$, $T_{i,n} \circ \cdots \circ T_{1,n} \circ \psi_n \circ \zeta_n(t) \to \hat{\zeta}(t)$
4. The elements $\psi_n$ are either trivial or unbounded. Moreover, for any $1 \leq i \leq k$ and $\gamma \in \sigma_i$ the power of $D_\gamma$ in the element $T_{i,n}$ goes to $\infty$ as $n \to \infty$.
5. The piecewise geodesic $\hat{\zeta}$ is the minimal length path in $\Teich(S)$ joining $\hat{\zeta}(0)$ to $\hat{\zeta}(T)$ and intersecting the strata $S(\sigma_1), S(\sigma_2), \ldots, S(\sigma_k)$ in order.

By Lemma 4.2 $\hat{\tau} \equiv \sigma_i \cap \sigma_{i+1}$ for $i = 0, \ldots, k$. Then part (1) follows from this and part (1) of Theorem 4.1. Part (4) follows from part (4) in Theorem 4.1 and Lemma 4.3.
4.2. Length-function versus twist parameter control. In this subsection we show that, roughly speaking, provided a lower bound for the length of a curve $\gamma$ at the end points of a uniformly bounded length WP geodesic segment $\zeta$, the higher Dehn twist about $\gamma$ forces $\gamma$ to get shorter along $\zeta$ (Corollary 4.12). Moreover, in Corollary 4.13 we show that the shorter $\gamma$ gets along $\zeta$ the higher Dehn twist develops about $\gamma$.

The main technical part of this subsection is the following modification of Lemma 4.5 in [BMM11].

**Theorem 4.6.** Given $\epsilon_0, T$ and $s$ positive, let $\zeta_n : [0, T_n] \to \text{Teich}(S)$ be a sequence of Weil-Petersson geodesic segments parametrized by arc-length of length $2s \leq T_n \leq T$. Then given a sequence of curves $\alpha_n$ we have the following:

1. If there are subintervals $J_n \subseteq [s, T_n - s]$ such that
   - (a) $\sup_{t \in J_n} \ell_{\alpha_n}(\zeta_n(t)) \geq \epsilon_0$, and
   - (b) $\inf_{t \in J_n} \ell_{\alpha_n}(\zeta_n(t)) \to 0$ as $n \to \infty$
   then
   $$d_{\alpha_n}(\mu(\zeta_n(0)), \mu(\zeta_n(T_n))) \to \infty$$
   as $n \to \infty$.

2. If
   - (a) $\sup_{t \in [0, T_n]} \ell_{\alpha_n}(\zeta_n(t)) \geq \epsilon_0$, and
   - (b) $d_{\alpha_n}(\mu(\zeta_n(0)), \mu(\zeta_n(T_n))) \to \infty$ as $n \to \infty$
   then
   $$\inf_{t \in [0, T_n]} \ell_{\alpha_n}(\zeta_n(t)) \to 0$$
   as $n \to \infty$.

**Remark 4.7.** Here we prove that the conditions in parts (1) and (2) hold for the sequence of curves $\alpha_n$, then the conclusions hold for the sequence $\alpha_n$ itself. In Lemma 4.5 of [BMM11] the conclusions are proved for some sequence of curves $\beta_n$ such that each $\beta_n$ does not overlap $\alpha_n$.

**Proof.** Trimming the intervals slightly and changing the parameters $s$ and $T$ we may assume that $T_n \equiv T$ for some $T \geq 2s$. After possibly passing to a subsequence by Theorem 4.5 there exist a partition of $[0, T]$ with $0 = t_0 < t_1 < \ldots < t_{k+1} = T$, simplices $\sigma_i$ for $i = 0, \ldots, k + 1$ in $\mathcal{C}(S)$ and a simplex $\hat{\tau}$, and a piecewise geodesic path

$$\hat{\zeta} : [0, T] \to \text{Teich}(S)$$

so that $\hat{\zeta}([t_i, t_{i+1}])$ is a geodesic segment in $\mathcal{S}(\hat{\tau})$ joining the stratum $\mathcal{S}(\sigma_i)$ to $\mathcal{S}(\sigma_{i+1})$. Let $\varphi_{i,n} = \mathcal{T}_{i,n} \circ \ldots \circ \mathcal{T}_{i,1} \circ \psi_n$ be as in (4.7), where $\mathcal{T}_{i,n} \in \text{tw}(\sigma_i - \hat{\tau})$ and $\psi_n \in \text{Mod}(S)$ is either trivial or unbounded.

We start by setting up some notation. For each $i = 0, \ldots, k + 1$ and $n \geq 1$ let

$$\sigma_{i,n} = \varphi_{i,n}^{-1}(\sigma_i) = \varphi_{i-1,n}^{-1}(\sigma_i)$$
be the pull backs of $\sigma_i$ to the $\zeta_n$ picture. For each $i = 1, \ldots, k$ and $n \geq 1$ let

$$\tau_{i,n} = \varphi_{i-1,n}(\hat{\tau})$$

For each $i = 0, \ldots, k + 1$, choose a partial marking $\mu_i$ such that

1. $\sigma_i \subset \text{base}(\mu_i)$, and
2. $\mu_i$ restricts to a full marking of each connected component $Y \subseteq S \setminus \hat{\tau}$ with complexity at least one.

For each $i = 1, \ldots, k + 1$ and $n \geq 1$ define the pullback marking

$$\mu_{i,n} = \varphi_{i-1,n}(\mu_i)$$

and for each $i = 0, \ldots, k$ and $n \geq 1$ the pullback marking

$$\mu_{i,n}^+ = \varphi_{i,n}(\mu_i)$$

Let $1 \leq i \leq k$ and let $\gamma \in \sigma_i - \hat{\tau}$. In the following three claims we will measure the twisting of these markings relative to $\gamma_n = \varphi_{i,n}^{-1}(\gamma)$ and prove that

$$d_{\gamma_n}(\mu(\zeta(0)), \mu(\zeta_n(T))) \rightarrow \infty$$

as $n \rightarrow \infty$.

Claim 4.8. $d_{\gamma_n}(\mu_{j,n}^-, \mu_{j,n}^+)$ is bounded for any $j = 1, \ldots, k$ with $j \neq i$.

First note that $\text{base}(\mu_{j,n}^+)$ and $\text{base}(\mu_{j,n}^-)$ both contain $\sigma_{j,n} = \varphi_{j,n}^{-1}(\sigma_j) = \varphi_{j-1,n}^{-1}(\sigma_j)$

Now we verify that

$$\gamma_n \notin \sigma_{j,n}, \text{ for any } j \neq i$$

Otherwise, the length of $\gamma_n$ would converge to 0 at $\zeta_n(t_i)$ and $\zeta_n(t_j)$, and hence by the convexity of the length-functions on all of $[t_{i-1}, t_i]$ or $[t_i, t_{i+1}]$ (the first if $j < i$ and the second if $j > i$). So by Theorem 4.5 (1), $\gamma \in \hat{\tau}$. But this contradicts the choice of $\gamma \in \sigma_i - \hat{\tau}$. So (4.9) holds.

$\mu_{j,n}^-$ and $\mu_{j,n}^+$ restrict to a full marking on $S \setminus \sigma_{j,n}$. So by (4.9) $\gamma_n$ intersects $\mu_{j,n}^-$ and $\mu_{j,n}^+$ nontrivially, and consequently $\pi_{\gamma_n}(\mu_{j,n}^\pm)$ is nonempty.

Let $\hat{T}_{j,n} = \varphi_{j,n}^{-1} \circ T_{j,n} \circ \varphi_{j,n}$, then $\mu_{j,n}^- = \hat{T}_{j,n}(\mu_{j,n}^-)$. Furthermore, since $T_{j,n} \in \text{tw}(\sigma_j)$, $\hat{T}_{j,n}$ is an element of $\text{tw}(\sigma_{j,n})$. So $\mu_{j,n}^+$ differs from $\mu_{j,n}^-$ by composition of Dehn twists about the curves in $\sigma_{j,n}$. By the definition of annular subsurface coefficients the annular subsurface coefficient of a sequence of curves intersecting both $\mu_{j,n}^-$ and $\mu_{j,n}^+$ could go to $\infty$, only if they are in $\sigma_{j,n}$. But by (4.9) it is not the case. The claimed bound follows from this contradiction.

Claim 4.9. $d_{\gamma_n}(\mu_{j,n}^+, \mu_{j+1,n}^-)$ is bounded for $j = 1, \ldots, k$. 

\[ \mu_j \text{ and } \mu_{j+1} \text{ are full markings of } S \setminus \hat{\gamma}, \text{ where their marking distance is some finite number. Hence we may connect them with a finite sequence of full markings of } S \setminus \hat{\gamma}. \] Applying \( \varphi_{j,n}^{-1} \) to this sequence we obtain a sequence of the same length connecting \( \mu_{j,n} \) to \( \mu_{j+1,n} \) through full markings of \( S \setminus \tau_{j+1,n} \). Moreover, \( \gamma_n \notin \tau_{j+1,n} \) (because \( \gamma \notin \hat{\tau} \)), so all of the markings in the connecting sequence intersect \( \gamma_n \) nontrivially. Any two consecutive markings in the sequence differ by an elementary move and each elementary move increases the \( A(\gamma_n) \) subsurface coefficient by at most one, then the claimed bound follows.

**Claim 4.10.**

\[ d_{\gamma_n}(\mu_{i,n}^{-}, \mu_{i,n}^{+}) \to \infty \text{ as } n \to \infty \]

\[ \varphi_{i,n}(\mu_{i,n}^{-}) = \mathcal{T}_{i,n}(\mu_i), \text{ so after applying } \varphi_{i,n} \text{ to all of the curves in the subsurface coefficient in } (4.10) \text{ we get} \]

\[ d_{\gamma}(\mathcal{T}_{i,n}(\mu_i), \mu_i) \]

Now \( \mu_i \) is a fixed marking which contains \( \gamma \) as well as a transversal curve for \( \gamma \). By Theorem 4.5 (4) \( \mathcal{T}_{i,n} \) contains an arbitrarily large power of \( D_{\gamma} \). So we obtain the claimed bound.

Combining the bounds established in claims 4.8, 4.9 and 4.10 with the triangle inequality the bound (4.8) follows. Having this bound we proceed by proving our theorem.

**Proof of part (1):** We show that after possibly passing to a subsequence there is an \( 1 \leq i \leq k \) and a curve \( \gamma \in \sigma_i - \hat{\tau} \), such that \( \alpha_n = \varphi_{i,n}^{-1}(\gamma) \). Part (1) then follows from (4.8).

Let \( J_n \subset [s, T - s] \) be the subintervals in the statement of part (1). Passing to a subsequence we may assume that \( J_n \)'s converge to a subinterval \( J \).

Since each \( J_n \subset [s, T - s] \), then \( J \subset [s, T - s] \).

For each \( i = 0, \ldots, k + 1, \varphi_{i,n} \circ \zeta_n|_{[t_i, t_{i+1}]} \to \hat{\zeta}|_{[t_i, t_{i+1}]} \) and \( \varphi_{i,n} \) is isometry of the WP metric. So the length of \( \zeta_n(J_n) \)'s converge to the length of \( \hat{\zeta}(J) \). Now since \( \zeta_n \)'s and \( \hat{\zeta} \) are (piece-wise) geodesics parametrized by arc-length it follows that the length of the intervals \( J_n \)'s converge to the length of the interval \( J \). Now we show that the length of all \( J_n \)'s are uniformly bounded below. For each \( n \geq 1 \), by (1(a)), \( \ell_{\alpha_n}(\zeta_n(t)) \) achieves the value \( e_0 \) in \( J_n \) and by (1(b), for \( n \) sufficiently large large \( \inf_{t \in J_n} \ell_{\alpha_n}(\zeta_n(t)) \leq \frac{e_0}{2} \). Thus by Corollary 3.5 the length of \( J_n \) is at least \( d(\epsilon_0, \frac{e_0}{2}) > 0 \). This uniform lower bound for the length of \( J_n \) for all \( n \) sufficiently large and the convergence of the length of \( J_n \)'s to that of \( J \) implies that \( J \) has length at least \( d(\epsilon_0, \frac{e_0}{2}) \).

For each \( n \geq 1 \), let \( t_n^{*} \in J_n \) be the time where \( \inf_{t \in J_n} \ell_{\alpha_n}(\zeta_n(t)) \) is realized. There is an \( 0 \leq i \leq k + 1 \) such that for \( n \) sufficiently large \( t_n^{*} \in [t_i, t_{i+1}] \) and \( t_n^{*} \) converge to some \( t^{*} \in J \cap [t_i, t_{i+1}] \).
First suppose that \( t^* \neq t_i, t_{i+1} \). By 1(b) \( \ell_{\alpha_n}(\zeta_n(t^*_n)) \to 0 \) as \( n \to \infty \), so applying \( \varphi_{i,n} \) to \( \ell_{\alpha_n}(\zeta_n(t^*_n)) \) we have

\[
(4.11) \quad \ell_{\varphi_{i,n}(\alpha_n)}(\varphi_{i,n} \circ \zeta_n(t^*_n)) \to 0 \text{ as } n \to \infty.
\]

By Theorem 4.5 (3), \( \varphi_{i,n} \circ \zeta_n(t^*_n) \to \zeta(t^*) \) as \( n \to \infty \). Moreover, the only curves with length 0 at \( \zeta(t^*) \) are the ones in \( \hat{\tau} \). Thus Theorem 3.1 (Continuity of length-functions) implies that the curves in \( \hat{\tau} \) are the only ones whose length at \( \varphi_{i,n} \circ \zeta_n(t^*_n) \) converge to 0. Then by (4.11) after possibly passing to a subsequence \( \varphi_{i,n}(\alpha_n) = \beta \) for some \( \beta \in \tau \). For each \( l = 0, \ldots, k + 1, \hat{\tau} \subseteq \sigma_l \), so \( \beta \in \sigma_n \). Now given \( j = 0, \ldots, k + 1, \varphi_{j,n} \circ \varphi_{j-1}^{-1} \) is a composition of Dehn twists about curves in \( \sigma_l, l = j + 1, \ldots, i \) (see 4.7), so preserves \( \beta \). Thus \( \varphi_{j,n}(\alpha_n) = \beta \).

Given \( t \in [0, T] \), \( t \in [t_j, t_{j+1}] \) for some \( 0 \leq j \leq k \). By Theorem 4.5 (3) \( \varphi_{j,n}(\zeta_n(t)) \to \zeta(t) \) as \( n \to \infty \). Moreover, \( \ell_\beta(\zeta(t)) \equiv 0 \) for all \( t \in [0, T] \). Thus the continuity of length-functions implies that \( \ell_\beta(\varphi_{j,n}(\zeta_n(t))) \to 0 \) as \( n \to \infty \). As we saw above \( \varphi_{j,n}(\alpha_n) = \beta \), so \( \ell_{\varphi_{j,n}(\alpha_n)}(\varphi_{j,n}(\zeta_n(t))) \to 0 \) as \( n \to \infty \), then applying \( \varphi_{j-1}^{-1} \) we have that \( \ell_{\alpha_n}(\zeta_n(t)) \to 0 \) as \( n \to \infty \). But this contradicts 1(a).

So \( t^*_n \) converges to either \( t_i \) or \( t_{i+1} \). Let \( t^*_n \to t_i(t_{i+1}) \) as \( n \to \infty \). Then \( \varphi_{i,n}(\zeta_n(t)) \to \zeta(t_i) \) \( (\varphi_{i,n}(\zeta_n(t))) \to \zeta(t_{i+1}) \) as \( n \to \infty \). Note that the only curves with length 0 at \( \zeta(t_i) \) are the ones in \( \sigma_i(\hat{\tau}_{i+1}) \). Thus (4.11) and the convergence of length-functions imply that \( \alpha_n = \varphi_{i,n}(\gamma)(\varphi_{i+1,n}^{-1}(\gamma)) \) for some \( \gamma \in \sigma_i - \hat{\tau}(\sigma_{i+1} - \hat{\tau}) \), as was desired.

**Proof of part (2):** Suppose that for an \( i \) with \( \sigma_i \neq \emptyset \), \( \alpha_n \in \sigma_i \) for all \( n \geq 1 \). Applying \( \varphi_{i,n} \) to \( \ell_{\alpha_n}(\zeta_n(t_i)) \) we get \( \ell_{\varphi_{i,n}(\alpha_n)}(\varphi_{i,n}(\zeta_n(t_i))) \). By Theorem 4.5 (3) we have that \( \varphi_{i,n}(\zeta_n(t_i)) \to \zeta(t_i) \) as \( n \to \infty \). Since \( \alpha_n \in \sigma_i \), \( \varphi_{i,n}(\alpha_n) \in \sigma_i \). Further the length of every curve in \( \sigma_i \) is 0 at \( \zeta(t_i) \). So the continuity of length-functions implies that \( \ell_{\varphi_{i,n}(\alpha_n)}(\varphi_{i,n}(\zeta_n(t_i))) \to 0 \) as \( n \to \infty \). Consequently, \( \ell_{\alpha_n}(\zeta_n(t_i)) \to 0 \) as \( n \to \infty \). So the proof of part (2) would be complete if we show that for some \( i \) with \( \sigma_i \neq \emptyset \), \( \alpha_n \in \sigma_i \) for all \( n \geq 1 \).

In contrary suppose that after possibly passing to a subsequence \( \alpha_n \notin \sigma_i \) for all \( i = 0, \ldots, k + 1 \) and \( n \geq 1 \). Then \( \alpha_n \) intersects \( \mu_{i,n}^- \) for \( i = 1, \ldots, k + 1 \), and \( \mu_{i,n}^+ \) for \( i = 0, \ldots, k \). For each \( i \), let \( \tilde{T}_{i,n} = \varphi_{i,n}^{-1} \circ T_{i,n} \circ \varphi_{i,n} \), as before, as we saw earlier \( \tilde{T}_{i,n} \in \text{tw}(\sigma_{i,n}) \) and \( \mu_i^- = \tilde{T}_{i,n}(\mu_{i,n}^-) \). So the only annular subsurfaces of \( \mu_i^- \) and \( \mu_i^+ \) which grow as \( n \to \infty \) are the ones with core curve in \( \sigma_{i,n} \). Thus \( d_{\alpha_n}(\mu_{i,n}^-, \mu_{i,n}^+) \) is uniformly bounded for \( i = 1, \ldots, k \) and all \( n \geq 1 \).

Moreover, as we saw in the proof of Claim 4.9 the fact that for each \( i = 0, \ldots, k + 1 \), \( \alpha_n \) intersects \( \mu_{i,n}^+ \) and \( \mu_{i+1,n}^- \) implies that \( d_{\alpha_n}(\mu_{i,n}^+, \mu_{i+1,n}^-) \) is uniformly bounded for \( i = 1, \ldots, k \) and all \( n \geq 1 \).
Combining the bounds form the above two paragraphs by the triangle inequality we conclude that $d_{\alpha_n}(\mu_{0,n}, \mu_{k,n}^+)$ is uniformly bounded above for all $n$. But $\mu(\zeta_n(0)) = \mu_{0,n}^-$ and $\mu(\zeta_n(T)) = \mu_{k,n}^+$, so this bound contradicts assumption 2(b).

\[\Box\]

**Remark 4.11.** Fix a pants decomposition $P$ and let $(\ell_\gamma, \theta_\gamma)_{\gamma \in P}$ be a corresponding Fenchel-Nielsen coordinates. Let $x \in \text{Teich}(S)$ be a point with $\theta_\alpha = \theta_0$ for some $\alpha \in P$. Let $\epsilon_n \to 0$. Consider the following two sequences of WP geodesic segments:

- Let $x_n (n \geq 1)$ be a point with all coordinates equal to that of $x$, except for $\theta_\alpha(x_n) = \theta_0 + 2\pi n$ and $\ell_\alpha(x_n) = \epsilon_n$. Then the length of the WP geodesic segments $[x, x_n]$ is uniformly bounded and $d_{\alpha}(\mu(x), \mu(x_n)) \to \infty$ as $n \to \infty$.

- Let $x_n (n \geq 1)$ be a point with all coordinates equal to that of $x$, except for $\theta_\alpha = \theta_0$ and $\ell_\alpha(x_n) = \epsilon_n$. Then the length of WP geodesic segment $[x, x_n]$ is uniformly bounded and $d_{\alpha}(\mu(x), \mu(x_n)) = 0$.

These two examples show that having a sequence of curves $\alpha_n$ whose lengths converge to zero at the end points of a sequence of WP geodesic segments, a-priori does not give any information about the growth of the $A(\alpha_n)$ subsurface coefficients of the Bers markings at the end points of the geodesic segments. Thus the above theorem is the sharpest control of length versus annular coefficient one could expect.

We are ready to prove the following two corollaries which are somewhat quantified versions of Theorem 4.6. These corollaries provide us with a kind of length-function versus twist parameter bounds over uniformly bounded length WP geodesic segments which often will be used in §6.

**Corollary 4.12.** (large twist $\implies$ short curve) Given $T, \epsilon_0$ and $N$ positive there is $\epsilon < \epsilon_0$ with the following property. Let $\zeta : [0, T'] \to \text{Teich}(S)$ be a WP geodesic segment of length $T' \leq T$ such that

$$\sup_{t \in [0, T']} \ell_\gamma(\zeta(t)) \geq \epsilon_0$$

If $d_{\gamma}(\mu(\zeta(0)), \mu(\zeta(T'))) > N$ then we have

$$\inf_{t \in [0, T']} \ell_\gamma(\zeta(t)) \leq \epsilon$$

**Proof.** The proof is by contradiction. Assume that the corollary does not hold. Then there is a sequence of WP geodesic segments $\zeta_n : [0, T_n] \to \text{Teich}(S)$ parametrized by arc-length with lengths $T_n \leq T$ and curves $\gamma_n \in \zeta_0(S)$, such that

(a) $\sup_{t \in [0, T_n]} \ell_{\gamma_n}(\zeta_n(t)) \geq \epsilon_0$ for every $n$,

(b) $d_{\gamma_n}(\mu(\zeta_n(0)), \mu(\zeta_n(T_n))) \to \infty$ as $n \to \infty$, ...
and \( \inf_{t \in [0, T_n]} \ell_{\gamma_n}(\zeta_n(t)) > \epsilon \) for every \( n \). But this contradicts Theorem 4.6 (2).

\[ \square \]

Corollary 4.13. (short curve \( \implies \) large twist)

Given \( \epsilon, T, s \) and \( \epsilon < \epsilon_0 \) positive with \( T > 2s \), there is an integer \( N > 0 \) with the following property. Let \( \zeta : [0, T'] \to \text{Teich}(S) \) be a WP geodesic segment parametrized by arc-length of length \( T' \in [2s, T] \), \( \gamma \in \mathcal{C}_0(S) \) and \( J \subseteq [s, T' - s] \) be a subinterval such that:

\[ \sup_{t \in [0, T']} \ell_{\gamma}(\zeta(t)) \geq \epsilon_0 \]

If \( \inf_{t \in J} \ell_{\gamma}(\zeta(t)) \leq \epsilon \), then

\[ d_{\gamma}(\mu(\zeta(0)), \mu(\zeta(T'))) > N \]

Proof. The proof is again by contradiction. Assume that the corollary does not hold. Then there is a sequence of WP geodesic segments \( \zeta_n : [0, T_n] \to \text{Teich}(S) \) parametrized by arc-length of length \( 2s \leq T_n \leq T \), \( \gamma_n \in \mathcal{C}(S) \), subintervals \( J_n \subset [s, T_n - s] \) such that

(a) \( \sup_{t \in J_n} \ell_{\gamma_n}(\zeta_n(t)) \geq \epsilon_0 \) for every \( n \),

(b) \( \inf_{t \in J_n} \ell_{\gamma_n}(\zeta_n(t)) \to 0 \) as \( n \to \infty \),

and \( d_{\gamma_n}(\mu(\zeta_n(0)), \mu(\zeta_n(T_n))) \leq N \) for every \( n \). But this contradicts Theorem 4.6 (1). \( \square \)

Remark 4.14. Note that these results have been proved using only compactness arguments in the WP completion of Teichmüller space.

5. Stable hierarchy paths

In this section we show that a certain class of hierarchy paths are stable in the pants graph of the surface.

Definition 5.1. (\( d \)--stable subset) Given a function \( d : \mathbb{R}^{\geq 1} \times \mathbb{R}^{>0} \to \mathbb{R}^{\geq 0} \) a subset \( \mathcal{Y} \) of a metric space \( \mathcal{X} \) is \( d \)--stable if for any \( K \geq 1 \) and \( C \geq 0 \) every \( (K, C) \)--quasi-geodesic \( h \) with end points in \( \mathcal{Y} \) is contained in the \( d(K, C) \) neighborhood of \( \mathcal{Y} \). We call the function \( d \) the quantifier of the stability.

Here we summarize some of the results about stability of subsets of pants graph of surfaces: Brock and Masur in [BM08] prove that when \( \xi(S) = 3 \) the pants graph of \( S \) is strongly relatively hyperbolic with respect to the quasi-flats corresponding to separating curves. The main ingredient of their proof is that given a hierarchy path \( \rho : [m, n] \to P(S) \) the subset of the pants graph \( X(\rho) = |\rho| \cup W \{P(W) \times P(W^c)\} \), where \( W \) or \( W^c = S \setminus W \) is a component domain of \( \rho \), is a stable subset of the pants graph. Behrstock, Drutu and Mosher in [BDM09] study thick metric spaces. These are metric spaces with rank at least 2 where any two quasi-flats are connected through a chain of quasi-flats with the property that any two consecutive quasi-flats in the chain has coarse intersection of infinite diameter. They show that thick
metric spaces fail to be relatively hyperbolic with respect to any collection of quasi-flats. Moreover, they observe that \( P(S) \) for \( \xi(S) > 3 \) is a thick metric space and consequently is not relatively hyperbolic with respect to any collection of quasi-flats. In [BMM11] it is proved that hierarchy paths with bounded combinatorics end points are stable.

Here we show that restricting the subsurfaces of a pair of partial markings or laminations with subsurface coefficient bigger than a given \( A > 0 \) to large subsurfaces, implies stability of any hierarchy path \( \rho \) between the pair in the pants graph. We call such a pair \( A \)-narrow. Heuristically, these hierarchy paths avoid quasi-flats in the pants graph corresponding to separating multicurves on the surface.

To be able to save considerable amount of work using results in the context of \( \Sigma \)-hulls (see \S 2.3) and present our results in a more general setting we prove that \( \Sigma \)-hulls with the mentioned constraint on the subsurface coefficients of their end points are stable. In \S 5.2 we prove that for any \( \epsilon > F \) the \( \Sigma_\epsilon \)-hull of an \( A \)-narrow pair is \( d_A \)-stable. Then the stability of hierarchy paths between the \( A \)-narrow pair follows from the fact that the Hausdorff distance of a hierarchy path between an \( A \)-narrow pair and the \( \Sigma_\epsilon \)-hull (\( \epsilon \) sufficiently large) of the pair is bounded depending only on \( A \) and \( \epsilon \). This is proved in Theorem 5.5.

5.1. Narrow pairs. In this subsection first we introduce the notion of an \( A \)-narrow pair of makings or laminations. Then we will show that any hierarchy path between a narrow pair and the \( \Sigma_\epsilon \)-hull (\( \epsilon > 0 \) is sufficiently large) of the pair have finite Hausdorff distance depending only on \( A \) and \( \epsilon \).

**Definition 5.2.** (Large subsurface) A connected essential subsurface \( Z \subseteq S \) is called large if any connected component of \( S \setminus Z \) is either an annulus or a three holed sphere.

**Remark 5.3.** If \( Z \) is not connected or \( S \setminus Z \) has connected components other than annuli and three holed spheres then \( Z \) is not a large subsurface.

**Definition 5.4.** (\( A \)-narrow) A pair of partial markings or laminations \((\mu^-, \mu^+)\) is called \( A \)-narrow if every non-annular subsurface \( Z \subseteq S \) with the property that

\[
d_Z(\mu^-, \mu^+) > A
\]

is a large subsurface of \( S \).

Recall the constants \( M_1, M_2 \) from Theorem 2.13 and \( B_0 \) from Theorem 2.8 We fix the constant \( M = M_1 + B_0 + 4 \) in this subsection. Note that since \( M_1 \geq 2M_2 \), \( M \geq M_2 \).

**Theorem 5.5.** (\( \Sigma \)-hull of narrow pair)

Given \( \epsilon > M \) and \( A > 4M + 2\epsilon + 12 \), there is a constant \( \Delta = \Delta(A, \epsilon) \) with the following property. Given an \( A \)-narrow pair \((\mu^-, \mu^+)\), the Hasudorff distance of the \( \Sigma_\epsilon(\mu^-, \mu^+) \) and any hierarchy path between \((\mu^-, \mu^+)\) is less than \( \Delta \).
Lemma 5.6. Given \( \epsilon > M \) and \( A > 4M + 2\epsilon + 12 \), there is \( d = d(A, \epsilon) \) with the property that given \( P \in \Sigma_e(\mu^-, \mu^+) \) there is \( j \in [m, n] \) such that for every non-annular subsurface \( X \subseteq S \) we have

\[
d_X(p(j), P) \leq d
\]

Proof. Let \( X \subseteq S \) be a non-annular subsurface. \( P \in \Sigma_e(\mu^-, \mu^+) \), so there is \( x_X \in \text{hull}_X(\mu^-, \mu^+) \) such that \( d_X(P, x_X) \leq \epsilon \). By Theorem 2.13 (5), there is \( i \in [m, n] \) such that \( \rho(i) \in \Sigma_e(\mu^-, \mu^+) \). Consequently

\[
|\rho| \subseteq \Sigma_e(\mu^-, \mu^+)
\]

We proceed to prove that \( \Sigma_e(\mu^-, \mu^+) \) is contained in a neighborhood of \( |\rho| \).

Claim 5.7. Let \( X \) be a component domain of \( \rho \). Then

\[
E_X \cap J_X \neq \emptyset
\]

\( P \in \Sigma_e(\mu^-, \mu^+) \), so there is \( x_X \in \text{hull}_X(\mu^-, \mu^+) \) such that \( d_X(P, x_X) \leq \epsilon \). \( X \) is a component domain of \( \rho \), so by Theorem 2.13 (4), there is \( x'_X \in g_X \) such that \( d_X(x_X, x'_X) \leq M_2 \). Then by the triangle inequality we have

\[
d_X(x'_X, P) \leq M + \epsilon.
\]

Let \( e = 2A \). Then the subset of parameters \( E_X(P) = \{ i \in [m, n] : d_X(P, \rho(i)) \leq \epsilon \} \) is non-empty. Because any \( j \) as above is in \( E_X \). Denote the minimum and maximum of the set \( E_X(P) \) by \( e^X_{\downarrow} \) and \( e^X_{\uparrow} \) respectively. Let \( E_X := E_X(P) = [e^X_{\downarrow}, e^X_{\uparrow}] \).

Recall the interval \( J_X \) from Theorem 2.13 (4).

Claim 5.7. Let \( X \) be a component domain of \( \rho \). Then

\[
E_X \cap J_X \neq \emptyset
\]

\( P \in \Sigma_e(\mu^-, \mu^+) \), so there is \( x_X \in \text{hull}_X(\mu^-, \mu^+) \) such that \( d_X(P, x_X) \leq \epsilon \). \( X \) is a component domain of \( \rho \), so by Theorem 2.13 (4) there is \( x'_X \in g_X \) such that \( d_X(x_X, x'_X) \leq M_2 \). Then by the triangle inequality we have

\[
d_X(x'_X, P) \leq M + \epsilon.
\]

Let \( j \in J_X \) be such that \( \rho(j) \supset x'_X \). Then

\[
d_X(p(j), P) \leq M + \epsilon.
\]

Now since \( M + \epsilon > e \) we have that \( j \in E_X \). The claim is proved.
Similarly, we get the above bound assuming that $d_W(v, w) \leq e + 2$.

Now suppose that $X$ is not a component domain of $\rho$. Then $d_X(\mu^-, \mu^+) \leq M$. Furthermore, by Theorem 2.13, the Hausdorff distance of $\pi_X(|\rho|)$ and $\text{hull}_X(\mu^-, \mu^+)$ is bounded above by $M$. Therefore, $\text{diam}_X(|\rho|) \leq 3M + 2$. Thus

$$d_X(\rho(j), P) \leq e + 3M + 2$$

(5.6)

To prove the lemma it suffices to show that

$$\bigcap_{X \subseteq S \text{ non-annular}} E_X \neq \emptyset.$$  (5.7)

To see this let $\hat{j} \in \bigcap_{X \subseteq S \text{ non-annular}} E_X$. Then by the bounds (5.5) and (5.6) for every non-annular subsurface $X \subseteq S$,

$$d_X(\rho(\hat{j}), P) \leq \max\{2e + 2, e + 3M + 2\},$$

which is the desired bound in (5.1).

Our strategy to prove that (5.7) holds is to verify that for any two non-annular subsurfaces $Y$ and $W$ we have that

$$E_Y \cap E_W \neq \emptyset.$$  (5.8)

Then Helly’s Theorem in one dimension (see [Eck93]) implies that the intersection of all of the intervals $\{E_X\}_{X \subseteq S \text{ non-annular}}$ is nonempty.

If $d_Y(\mu^-, \mu^+) \leq A$, then $\text{diam}_Y(\text{hull}_Y(\mu^-, \mu^+)) \leq A + 2$. $P \in \Sigma_e(\mu^-, \mu^+)$ so there is $x_Y \in \text{hull}_Y(\mu^-, \mu^+)$ with $d_Y(P, x_Y) \leq e$. By Theorem 2.13, for every $i \in [m, n]$ there is $y \in \text{hull}_Y(\mu^-, \mu^+)$ such that $d_Y(\rho(i), y) \leq M$. By the bound on the diameter of the hull $d_Y(x_Y, y) \leq A + 2$. The last three bounds combined by the triangle inequality give us

$$d_Y(\rho(i), P) \leq e + M + A + 2 \leq e.$$  

Thus $E_Y = [m, n]$, which obviously intersects $E_W$. If $d_W(\mu^-, \mu^+) \leq A$, similarly we can conclude that $E_W = [m, n]$, which implies that $E_W \cap E_Y \neq \emptyset$. Therefore, in the rest of the proof we may assume that

- $d_Y(\mu^-, \mu^+) > A$, and
- $d_W(\mu^-, \mu^+) > A$.

In particular, since $A > M$ both $Y$ and $W$ are component domains of $\rho$.

Now consider the following collection of subsurfaces:

- $\mathcal{L} := \mathcal{L}_A(\mu^-, \mu^+) = \{X \subseteq S \text{ non-annular subsurface} : d_X(\mu^-, \mu^+) > A\}$

For each $X \in \mathcal{L}$ define

- $i_X^- = \max\{i \in [m, n] : d_X(\rho(i), \mu^-) \leq M\}$ and
- $i_X^+ = \min\{i \in [m, n] : d_X(\rho(i), \mu^+) \leq M\}$.
Claim 5.8. Let \( J_X = [j^-_X, j^+_X] \). Then \( j^-_X \leq i^-_X \leq i^+_X \leq j^+_X \) and we may write (5.9) 
\[ [i^-_X, i^+_X] \subseteq J_X \]

By Theorem 2.13(4), \( d_X(\mu^-, \rho(j^-_X)) \leq M \) and \( d_X(\mu^+, \rho(j^+_X)) \leq M \). So we have that \( i^-_X \geq j^-_X \) and \( i^+_X \leq j^+_X \). Then since \( J_X \) is an interval, \( i^-_X, i^+_X \in J_X \). Now we show that \( i^-_X \leq i^+_X \). As we just said, \( d_X(\rho(j^-_X), \mu^-) \leq M \). By the definition of \( i^-_X \), \( d_X(\rho(i^-_X), \mu^-) \leq M \). The last two inequalities combined by the triangle inequality imply that 
\[(5.10) \quad d_X(\rho(i^-_X), \rho(j^-_X)) \leq 2M + 2.\]

Similarly we have that 
\[(5.11) \quad d_X(\rho(i^+_X), \rho(j^+_X)) \leq 2M + 2.\]

Furthermore, \( d_X(\mu^-, \mu^+) > A > 6M + 12 \), so by the bounds \( d_X(\mu^-, \rho(j^-)) \leq M \) and \( d_X(\mu^+, \rho(j^+)) \leq M \) we have that 
\[(5.12) \quad d_X(\rho(j^-_X), \rho(j^+_X)) > 4M + 12.\]

Let \( u = \rho(j^-_X) \cap g_X, w = \rho(j^+_X) \cap g_X, v = \rho(i^-_X) \cap g_X \) and \( v' = \rho(i^+_X) \cap g_X \). By (5.10), 
\[ d_X(u, v) \leq d_X(\rho(i^-_X), \rho(j^-_X)) + \text{diam}_X(\rho(i^-_X)) = 2M + 10. \]

By (5.11) \( d_X(w, v') \leq 2M + 4 \) and by (5.12) \( d_X(u, w) \geq 4M + 10 \). Then since \( u < w \) as vertices along \( g_X \) the last there inequalities imply that \( v < v' \). Then Theorem 2.13(3) (Monotonicity) implies that \( i^-_X \leq i^+_X \).

Claim 5.9. Let \( X \in \mathcal{L} \). If \( i \in [m, i^-_X] \) then \( d_X(\rho(i), \mu^-) \leq 2M + 3 \). If \( i \in [i^+_X, n] \) then \( d_X(\rho(i), \nu^+) \leq 2M + 3 \).

We prove the first part of the claim. The proof of the second part is similar. Let \( J_X = [j^-_X, j^+_X] \). By Claim 5.8 \( j^-_X \leq i^-_X \). If \( i \leq j^-_X \) then by Theorem 2.13(4) we have \( d_X(\mu^-, \rho(i)) \leq M < 2M + 3 \), which is the desired bound. Otherwise, \( j^-_X \leq i \leq i^-_X \). Note that \( j^-_X, i, i^-_X \in J_X \). Then let \( u = \rho(j^-_X) \cap g_X, v = \rho(i) \cap g_X \) and \( w = \rho(i^-_X) \cap g_X \). Then by (5.10) we have 
\[ d_X(u, w) \leq 2M + 2 + \text{diam}_X(\rho(j^-_X)) + \text{diam}_X(\rho(i^-_X)) = 2M + 4. \]

Moreover, since \( j^-_X \leq i \leq i^-_X \), by Theorem 2.13(3) (Monotonicity) \( u \leq v \leq w \) as vertices along the geodesic \( g_X \subseteq C(X) \). So by the above inequality either \( d_X(u, v) \leq M + 2 \) or \( d_X(v, w) \leq M + 2 \). The former inequality implies that \( d_X(\rho(j^-_X), \rho(i)) \leq M + 2 \). This inequality and \( d_X(\rho(j^-_X), \mu^-) \leq M \) (Theorem 2.13(4)) combined by the triangle inequality imply that \( d_X(\rho(i), \mu^-) \leq 2M + 2 + \text{diam}_X(\rho(j^-_X)) \leq 2M + 3 \). The later inequality implies that \( d_X(\rho(i), \rho(j^-_X)) \leq M + 2 \). This inequality and \( d_X(\mu^-, \rho(j^-_X)) \leq M \) (Theorem 2.13(4)) combined by the triangle inequality imply that \( d_X(\rho(i), \mu^-) \leq 2M + 3 \). The first part of the claim is proved.

Given \( Y, W \in \mathcal{L} \) the \( A \)-narrow condition implies that either

1. \( Y \cap W \),
(2) \( Y \supseteq W \), or
(3) \( Y \subseteq W \).

In what follows we discuss these three cases and in each case verify that (5.8) holds.

**Case 1:** \( Y \nsubseteq W \).

\( Y, W \in \mathcal{L} \) and \( A > 4M \). So \( d_Y(\mu^-, \mu^+) > 4M \) and \( d_W(\mu^-, \mu^+) > 4M \). Then by Proposition 2.15 either \( Y < W \) or \( W < Y \) (not both). Assume that \( Y < W \). \( W < Y \) can be treated similarly. Then by the proposition we have the following two inequalities

\[
\tag{5.13} d_Y(\mu^-, \partial W) > M, \quad \text{and} \quad \tag{5.14} d_W(\mu^+, \partial Y) > M.
\]

We proceed to discuss the following three subcases depending on the values of \( d_Y(P, \mu^-) \) and \( d_Y(P, \mu^+) \). In each case we verify that (5.8) holds.

**Case 1.1:**

\[
\tag{5.15} d_Y(P, \mu^-) \leq 3M + \epsilon + 4.
\]

We show that the following inclusion of intervals holds

\[
\tag{5.16} E_Y \supseteq [m, i_Y^-]
\]

Let \( i \in [m, i_Y^-] \), then by the first part of Claim 5.9 we have \( d_Y(\rho(i), \mu^-) \leq 2M + 3 \). This inequality and (5.15) combined by the triangle inequality give us

\[
\tag{5.17} d_Y(P, \rho(i)) \leq 5M + \epsilon + 7 + \text{diam}(\mu^-) \leq 5M + \epsilon + 9
\]

Furthermore, \( e = 2A > 5M + \epsilon + 9 \), so by the definition of \( E_Y \) we have \( i \in E_Y \).

Now we show that the inclusion of intervals

\[
\tag{5.18} E_W \supseteq [m, i_W^-]
\]

holds. To see this, note that since \( Y \in \mathcal{L} \) we have \( d_Y(\mu^-, \mu^+) > A > 4M + 2\epsilon + 12 \). By (5.14) and Theorem 2.8 (Behrstock Inequality) we have, \( d_Y(\partial W, \mu^+) \leq M \). The last two bounds and (5.15) combined by the triangle inequality imply that \( d_Y(P, \partial W) > \epsilon > M \). Thus by the Behrstock inequality we have

\[
\tag{5.19} d_W(\partial Y, P) \leq M.
\]

By (5.13) and the Behrstock inequality we have that \( d_W(\partial Y, \mu^-) \leq M \). This inequality and (5.19) combined by the triangle inequality give us

\[
\tag{5.20} d_W(P, \mu^-) \leq 2M + 1
\]

Let \( i \in [m, i_W^-] \), by the first part of Claim 5.9 \( d_W(\mu^-, \rho(i)) \leq 2M + 3 \). Combining this inequality and (5.19) by the triangle inequality we get

\[
\tag{5.21} d_W(P, \rho(i)) \leq 4M + 4.
\]

Now since \( e > 4M + 4 \), by the definition of \( E_W \), \( i \in E_W \). So we conclude that \( E_W \supseteq [m, i_W^-] \).
The inclusion of intervals (5.16) and (5.17) together imply that $E_W \cap E_Y \neq \emptyset$.

Case 1.2:

(5.20) \quad \min\{d_Y(P, \mu^-), d_Y(P, \mu^+)\} > 3M + \epsilon + 4

We show that

(5.21) \quad E_Y \cap [i^-_Y, i^+_Y] \neq \emptyset

To see this, note that $P \in \Sigma(\mu^-, \mu^+)$, then by (5.4), there is $i \in [m, n]$ such that

(5.22) \quad d_Y(P, \rho(i)) \leq M + \epsilon

Now since $e > M + \epsilon$, by the definition of $E_Y$, $i \in E_Y$.

By (5.20), $d_Y(P, \mu^-) > 3M + \epsilon + 4$. Combining this inequality and (5.22) by the triangle inequality we have that $d_Y(\rho(i), \mu^-) > 2M + 4 - \text{diam}_Y(P) = 2M + 3$. Then by the contrapositive of the first part of Claim 5.9 we conclude that $i > i^-_Y$. By (5.20), $d_Y(P, \mu^+) > 3M + \epsilon + 4$. Combining this inequality and (5.22) by the triangle inequality we have that $d_Y(\rho(i), \mu^+) > 2M + 3$. Then by the contrapositive of the second part of Claim 5.9 we conclude that $i < i^+_Y$. Therefore $i \in [i^-_Y, i^+_Y]$.

Now we show that the inclusion of intervals

(5.23) \quad E_W \supseteq [m, i^-_W]

holds. By (5.14) and Theorem 2.8 (Behrstock Inequality), $d_Y(\partial W, \mu^+) \leq M$. Moreover, by (5.20), $d_Y(P, \mu^+) > 3M + \epsilon + 4$. These two bounds combined by the triangle inequality give us $d_Y(P, \partial W) > 2M + \epsilon + 4 - \text{diam}_Y(\mu^+) > M$. Therefore, by the Behrstock inequality

(5.24) \quad d_Y(P, \partial Y) \leq M

Having (5.24), the rest of the proof of the inclusion of intervals (5.23) follows from exact the same lines given after (5.18) to prove the inclusion of intervals (5.17).

Claim 5.10. $i^+_Y \leq i^-_W$.

By (5.14) and the Behrstock inequality, $d_Y(\partial W, \mu^+) \leq M$. By (5.9) $i^-_W \in J_W$, so $\rho(i^-_W) \supset \partial W$. Thus $d_Y(\rho(i^-_W), \mu^+) \leq M$. Then since $i^+_Y$ is the maximal time such that $d_Y(\rho(i), \mu^+) \leq M$ we have $i^+_Y \leq i^-_W$.

By Claim 5.10 $[i^-_Y, i^+_Y] \subseteq [m, i^-_W]$. Then by (5.23), $[i^-_Y, i^+_Y] \subseteq E_W$. So (5.21) implies that $E_Y \cap E_W \neq \emptyset$.

Case 1.3:

(5.25) \quad d_Y(P, \mu^+) \leq 3M + \epsilon + 4

(5.25) and the second part of Claim 5.9 using an argument similar to the one for the proof of (5.16) in Case 1.1 imply that

(5.26) \quad E_Y \supseteq [i^+_Y, n]
For every \( j \in J_W \), \( \rho(j) \supset \partial W \). By (5.14) and the Behrstock inequality, \( d_Y(\partial W, \mu^+) \leq M \). So \( d_Y(\rho(j), \mu^+) \leq M \). Thus by the definition of \( i^+_W \), \( j \geq i^+_W \). Therefore \( J_W \subseteq [i^+_W, \infty) \). Then by (5.26), \( J_W \subseteq E_Y \). Moreover, by (5.3) \( J_W \cap E_W \neq \emptyset \). So we conclude that \( E_W \cap E_Y \neq \emptyset \).

**Case 2:** \( W \subseteq Y \).

Recall the constant \( e = 2A \). We consider the following two subcases depending on the value of \( d_Y(\partial W, P) \).

**Case 2.1:** \( d_Y(\partial W, P) \leq e \).

Let \( i \in J_W \), by Theorem 2.13 [1] \( \partial W \subseteq \rho(i) \), so \( d_Y(\rho(i), P) \leq d_Y(\partial W, P) \leq e \). Recall that \( E_Y = \{ i : d_Y(\rho(i), P) \leq e \} \), thus \( J_W \subseteq E_Y \). Moreover, by (5.3) \( J_W \cap E_W \neq \emptyset \). Thus \( E_W \cap E_Y \neq \emptyset \).

**Case 2.2:**

(5.27) \[ d_Y(\partial W, P) > e. \]

\( P \in \Sigma_e(\mu^-, \mu^+) \), then as we saw in the paragraph before (5.4), there is \( x_Y \in \text{hull}(\mu^-, \mu^+) \) such that \( d_Y(x_Y, P) \leq e \) and by Theorem 2.13 [5], there is \( x'_Y \) on \( g_Y \) such that \( d_Y(x'_Y, x_Y) \leq M \). Then by the triangle inequality,

(5.28) \[ d_Y(x'_Y, P) \leq M + e. \]

Let \( h \) be a geodesic in \( C(Y) \) connecting \( \pi_Y(P) \) to \( x'_Y \). We claim that \( h \) does not intersect the \( 1 \)-neighborhood of \( \partial W \). Otherwise, there is a vertex \( z \in h \) with \( d_Y(z, \partial W) \leq 1 \), see Figure 3. Then we have

\[ d_Y(P, \partial W) \leq d_Y(P, z) + d_Y(z, \partial W) \leq d_Y(P, x'_Y) + 1 \leq (M + e) + 1 \]

The first inequality is the triangle inequality and the second inequality follows since \( d_Y(P, z) \leq d_Y(P, x'_Y) \) (see Figure 3) and \( d_Y(z, \partial W) \leq 1 \). The third inequality holds by (5.28). But this upper bound contradicts the lower bound (5.27) given as the assumption of Case 2.2.

By the above claim \( \partial W \) intersects every vertex of \( h \), so Theorem 2.6 (Bounded Geodesic Image Theorem) implies that

(5.29) \[ d_W(P, x'_Y) \leq G. \]

\( W \subseteq Y \) and both \( W \) and \( Y \) are component domains of \( \rho \). Let \( \phi_{g_Y}(W) \) be the footprint of \( W \) on \( g_Y \) consisting of the vertices of \( g_Y \) which do not overlap \( W \), see §4 of [MM00] and Definition 4.9 there. Then by Lemma 4.10 of [MM00], \( \phi_{g_Y}(W) \) is a sequence of 1, 2 or 3 consecutive vertices of \( g_Y \).

If \( x'_Y \in \phi_{g_Y}(W) \) then by the definition of foot print it does not intersect \( \partial W \), so \( d_Y(\partial W, x'_Y) \leq 1 \). Furthermore, by (5.28), \( d_Y(x'_Y, P) \leq M + e \). These two bounds combined by the triangle inequality imply that \( d_Y(\partial W, P) \leq e + M + 1 \). But this contradicts (5.27) because \( M + e + 1 < e \). So we conclude that \( x'_Y \notin \phi_{g_Y}(W) \).
Figure 3. **Case 2.2:** If $h$ intersects the 1–neighborhood of $\partial W$ then the distance between $\pi_Y(\partial W)$ and $\pi_Y(P)$ would be less than the lower bound in the assumption of Case 2.2.

Therefore, either $x'_Y > \max \phi_{g_Y}(W)$ or $x'_Y < \min \phi_{g_Y}(W)$ as vertices on the geodesic $g_Y \subset \mathcal{C}(S)$. We proceed to discuss these two cases. Let $u$ be the initial vertex of the geodesic $g_Y$ and $v$ be its final vertex.

**Case 2.2.1:** $x'_Y < \min \phi_{g_Y}(W)$.

By the definition of foot print $\partial W$ intersects every vertex of $g_Y$ between $u$ and $x'_Y$. Hence by Theorem 2.6

$$d_W(x'_Y, u) \leq G.$$ 

Let $j^*_W$ be the initial parameter of $J_W$. By condition S3 of slices at the beginning of §5 of [MM00] there is $w \in \phi_{g_Y}(W)$ such that $w \subset \rho(j^*_W)$. Let $i \in J_Y$ be such that $\rho(i) \supset u$. Then since $u \leq w$ on $g_Y$, Theorem 2.13 (3) (Monotonicity) implies that $i \leq j^*_W$. Then Theorem 2.13 (4) implies that

$$d_W(u, \mu^-) \leq M_2.$$ 

Combining the above two subsurface coefficient bounds and (5.29) by the triangle inequality we get

$$d_W(P, \mu^-) \leq M_2 + 2G \leq 2M$$ (5.30)

The second inequality above follows from the fact that $M \geq 2G$. Because $M_2 \geq G$ (see Lemma 6.1 (Sigma projection) in [MM00]), $M_1 \geq 2M_2$ and $M \geq M_1$.

Let $i \in [m, i^*_W]$. Then by the first part of Claim 5.9, $d_W(\rho(i), \mu^-) \leq 2M + 3$. This bound and the bound (5.30) combined by the triangle inequality
imply that \( d_W(\rho(i), P) \leq 4M + 3 \). Now since \( e > 4M + 3 \), by the definition of \( E_W \) we have \( i \in E_W \) so
\[
(5.31) \quad [m, i_W] \subseteq E_W.
\]
By (5.9) \( i_W \in J_W \), so by condition S3 of slices in §5 of [MM00], as before, there is \( w \in \pi_{g_Y}(W) \) such that \( w \subset \rho(i_W) \). Let \( j \in J_Y \) with \( \rho(j) \supseteq x'_Y \). \( x'_Y \leq w \) as vertices along \( g_Y \). Hence by Theorem 2.13 (3) (Monotonicity) we have that \( j \leq i_W \). So by (5.31), \( j \in E_W \).

Furthermore, \( d_Y(\rho(j), P) \leq d_Y(x'_Y, P) \leq \epsilon + M \) and \( e > \epsilon + M \) so by the definition of \( E_Y \) any \( j \) as above is in the interval \( E_Y \). Therefore \( E_W \cap E_Y \neq \emptyset \).

**Case 2.2.2:** \( x'_Y > \max \phi_{g_Y}(W) \).

In this case similar to Case 2.2.1 we can first show that \( d_W(P, \mu^+) \leq 2M \), which again using a similar argument implies that \( E_W \supseteq [i_W, n] \). Then following the exact same lines we can conclude that \( E_W \cap E_Y \neq \emptyset \).

In summary in all of the above cases we verified that (5.8) holds. Thus as we explained earlier the lemma follows from Helly’s Theorem in dimension one.

We can now complete the proof of Theorem 5.5. For any \( P \in \Sigma_c(\mu^-, \mu^+) \) Lemma 5.6 provides \( d > 0 \) and \( \hat{j} \in [m, n] \) such that the inequality (5.1)
\[
d_X(\rho(\hat{j}), P) \leq d
\]
holds for every non-annular subsurface \( X \subseteq S \). Let the threshold constant in the distance formula (2.2) be \( \max \{M_1, d\} \). Let \( \Delta \) be the additive constant in the distance formula corresponding to this threshold constant. Then
\[
d(P, \rho(\hat{j})) \leq \Delta.
\]
The above bound shows that \( \Sigma_c(\mu^-, \mu^+) \) is contained in the \( \Delta \) neighborhood of \( |\rho| \). We earlier proved that \( |\rho| \subset \Sigma_c(\mu^-, \mu^+) \). These facts together imply that the Hausdorff distance of \( |\rho| \) and \( \Sigma_c(\mu^-, \mu^+) \) is bounded by \( \Delta \). Note that \( d \) depends only on \( A \) and \( \epsilon \), so \( \Delta \) depends only on \( A \) and \( \epsilon \).

**5.2. Stability.** The following theorem is the main result of this subsection

**Theorem 5.11.** (Stable hierarchy resolution path) Given \( A > 0 \) there is a quantifier function \( d_A : \mathbb{R}^{\geq 0} \times \mathbb{R}^{\geq 1} \rightarrow \mathbb{R}^{\geq 0} \) such that any hierarchy path with \( A \)-narrow end points \( \mu^- \) and \( \mu^+ \) is \( d_A \)-stable in the pants graph.

First note that by Theorem 5.5 given an \( A \)-narrow pair \( (\mu^-, \mu^+) \) the Hausdorff distance of \( \Sigma_c(\mu^-, \mu^+) \) \( (\epsilon > M) \) and a hierarchy path \( \rho \) between \( \mu^- \) and \( \mu^+ \) is bounded by the constant \( \Delta \) depending only on \( A \) and \( \epsilon \). Then if \( \Sigma_c(\mu^-, \mu^+) \) is \( d \)-stable the hierarchy path \( \rho \) is stable with quantifier function \( d + \Delta \). So it suffices to prove that \( \Sigma_c(\mu^-, \mu^+) \) is stable. Our strategy to prove the stability of the \( \Sigma_c \)-hull is to show that the projection map \( \Pi \) onto the \( \Sigma_c \)-hull \( (\epsilon > F) \) defined in Theorem 2.16 has the following contraction property:
Definition 5.12. (Contraction property) Given \( R, B \geq 0 \) and \( 0 < \eta \leq 1 \) a subset \( Y \) of a metric space \( X \) is \((R, B, \eta)\)-contracting if there is a map \( \Pi : X \to Y \) with the following property. For every \( x, y \in X \) if \( d(x, \Pi x) > R \) then

\[
d(x, y) \leq \eta d(x, \Pi y) \implies d(\Pi x, \Pi y) \leq B
\]

If a map \( \Pi : X \to Y \) satisfies the contraction property, the coarse Lipschitz property, and coarsely preserves \( Y \), then for any \( K \geq 1 \) and \( C \geq 0 \) the standard Morse lemma argument as is in the proof of Lemma 7.1 of [MM99], gives a \( d > 0 \), such that a \((K, C)\)-quasi-geodesic with end points in \( Y \) stays in the \( d\)–neighborhood of \( Y \). In this way we get a quantifier function \( d : \mathbb{R}^{\geq 1} \times \mathbb{R}^{\geq 0} \to \mathbb{R}^{\geq 0} \), depending only on \( R, B \) and \( \eta \) such that \( Y \) is \( d\)–stable in \( X \).

We take from [Mah10] the following properties of \( \delta\)–hyperbolic spaces which are not necessarily locally compact (for example the curve complex). These properties will be used in the proof of Lemma 5.18. Let \( X \) be a \( \delta\)–hyperbolic space which is not necessarily locally compact.

Proposition 5.13. Let \( \zeta \) be a geodesic in \( X \). Given points \( x, y \) in \( X \) or its Gromov boundary. Let \( x', y' \) be closet points to \( x \) and \( y \) on \( \zeta \), respectively. When \( x \) is at the boundary, let \( \xi_i \in X \) be a sequence of points with \( \xi_i \to x \) as \( i \to \infty \) and let \( x' \) be the limit of \( \xi_i \) 's. Similarly for \( y \) and \( y' \). Then

(i) A geodesic \([x, w]\) connecting \( x \) to \( w \) any point on \( \zeta \) intersects the \( 3\delta \) neighborhood of \( x' \).
(ii) \( d(x', y') \leq d(x, y) + 12\delta \).
(iii) (Tree like) Let \( K = 14\delta \) and \( \delta' = 24\delta \), suppose that \( d(x', y') > K \) then we have

\[
d(x, x') + d(x', y') + d(y', y) \leq d(x, y) + \delta'
\]

There is a function \( b(a, \delta) \) increasing in both \( a \) and \( \delta \) with the following property. Let \([x, x']\) and \([y, y']\) be geodesics connecting \( x \) to \( x' \) and \( y \) to \( y' \), respectively. Let \( \mathcal{N}_a \) denote the \( a\)–neighborhood of subset of \( X \).

(iv) If \( \mathcal{N}_a([x, x']) \cap \mathcal{N}_a([y, y']) \neq \emptyset \), then \( d(x, x') \leq b \).

(i) and (iii) are respectively propositions 3.2 and 3.4 of [Mah10]. (iv) can be proved by a slight modification of the proof given for Proposition 3.4 in [Mah10].

Proof of part (ii). By part (i) there is a point \( z \) on \([x, y']\) with \( d(z, x') \leq 3\delta \) (1) and there is a point \( z' \) with \( d(z', y') \leq 3\delta \) (2). Without loss of generality suppose that \( d(x, z) \geq d(y, z') \) (3). We claim that

\[
d(z, z') \leq d(x, y) + 6\delta.
\]

Otherwise, \( d(z, z') > d(x, y) + 6\delta \) (4). Now we have

\[
\begin{align*}
d(x, y') & \leq d(x, y) + d(y, z') + d(z', y') < d(z, z') - 6\delta + d(x, z) + 3\delta \\
& \leq d(z, z') + d(x, z) - d(z', y') \leq d(y', z) + d(x, z) = d(x, y')
\end{align*}
\]
The second inequality follows from (2), (3) and (4). The third inequality follows from (2). The fourth inequality is the triangle inequality.

But then we have \( d(x, y') < d(x, y) \), which is contradictory and we obtain the claimed bound.

Finally, we have that

\[
d(x', y') \leq d(x', z) + d(z, z') + d(z', y') \leq d(x, y) + 12\delta,
\]

where the first inequality is the triangle inequality and the second one follows from (1), (5.33) and (2). □

The constants \( K \) and \( \delta' \) depend only on \( \delta \) the hyperbolicity constant of the metric space \( X \). Given \( Y \subseteq S \), denote by \( K_Y, \delta_Y \) the corresponding constants of the curve complex of \( Y \) which depend only on \( \delta_Y \) and consequently the topological type of \( Y \).

We will also need the following elementary lemmas.

**Lemma 5.14.** Given a point \( z \) and a geodesic \( \zeta \) in \( X \). Let \( z' \) be a nearest point to \( z \) on \( \zeta \). Let \( x \) be a point on \( \zeta \) and \( x' \) be a nearest point to \( x \) on \([z, z']\) the geodesic connecting \( z \) to \( z' \). Then \( d(z', x') \leq 6\delta \).

**Proof.** By Proposition 5.13 (i) the geodesic segment \([x, z']\) intersects the \( 3\delta \) neighborhood of \( x' \) at a point \( f \). We claim that \( d_Z(f, x') \leq 3\delta \). For otherwise the path \([x, f] \cup [f, z']\) would have length less than the length of \([x, x']\) which contradicts the fact that \([x, x']\) minimizes the distance between \( x \) and \( x' \). Then by the triangle inequality \( d_Z(x', z') \leq d(x', f) + d(f, z') \leq 3\delta + 3\delta = 6\delta \). □

**Lemma 5.15.** Given a point \( z \) and a geodesic \( \zeta \) in \( X \). Let \( z' \) be a closest point to \( z \) on \( \zeta \), then for any \( p \) on \( \zeta \), \( d(p, z) + 6\delta \geq d(p, z') \).

**Proof.** Extend \( \zeta \) to an infinite geodesic \( \hat{\zeta} \). By Proposition 5.13 (ii) the diameter of the set of nearest points to \( z \) on \( \hat{\zeta} \) is at most \( 12\delta \). Let \( p' \) be the point on \( \hat{\zeta} \) so that the distance of \( p' \) and the set is the same as the distance of \( p \) and the set. By symmetry the \( d(p, z) = d(p', z) \) and by the triangle inequality \( d(p, z) + d(z, p') \geq d(p, p') \). Thus \( 2d(p, z) \geq d(p, p') \). But \( d(p, p') + 12\delta \geq 2d(p, z') \). So we get \( 2d(p, z) + 12\delta \geq 2d(p, z) \), dividing both sides of this inequality by 2 we get the desired inequality. □

**Lemma 5.16.** Let geodesics \( f \) and \( f' \) \( D \) fellow travel each other in \( X \). Let \( u \) and \( \hat{u} \) be nearest points to a point \( p \) on \( f \) and \( f' \) respectively. Then \( d(\hat{u}, u) \leq 3D + 6\delta \).

**Proof.** Since \( f \) and \( f' \) \( D \) fellow travel each other, the nearest point to \( u \) on \( f' \) has distance at most \( d(u, p) + D \) to \( p \). Moreover, \( \hat{u} \) is a nearest point to \( p \) on \( f' \) so \( d(p, \hat{u}) \leq d(u, p) + D \). Similarly \( d(p, u) \leq d(\hat{u}, p) + D \).

Let \( u' \) be a nearest point to \( u \) on \( f' \), then \( d(\hat{u}', u) \leq d \). Now

\[
d(p, u') \leq d(p, u) + d(u, u') \leq d(p, u) + 2D.
\]
Now we prove that \( d(u, u') \leq 3\delta + 2D \). By Proposition 5.13 (i), there is a point \( z \) on \([p, u']\) such that \( d(z, \hat{u}) \leq 3\delta \). Then by the triangle inequality \( d(z, p) \geq d(p, \hat{u}) - d(\hat{u}, z) \geq d(p, \hat{u}) - 3\delta \). Thus
\[
\begin{align*}
    d(z, u') &= d(p, u') - d(p, z) \leq d(p, u') - (d(p, \hat{u}) - 3\delta) \\
    &= d(p, u') - d(p, \hat{u}) + 3\delta \leq D + 3\delta.
\end{align*}
\]

Then by the triangle inequality \( d(u', u) \leq d(u', z) + d(z, u) \leq 3\delta + 2D \). Finally \( d(u, \hat{u}) \leq d(\hat{u}, u') + d(u', u) \leq D + 3\delta + 3D = 6\delta + 3D \). \( \square \)

Let \( F \) be the constant from Theorem 2.16.

**Theorem 5.17.** (Narrow hulls are contracting) Given \( A > 0 \) and \( \epsilon > F \), there are \( R, B > 0 \) and \( 0 < \eta \leq 1 \) with the following property. Let \((\mu^-, \mu^+)\) be an \( A \)-narrow pair, then \( \Sigma_\epsilon(\mu^-, \mu^+) \) has the contraction property with constants \( R, B \) and \( \eta \).

**Proof.** We prove that the projection map onto \( \Sigma_\epsilon(\mu^-, \mu^+) \) defined in Theorem 2.16 has the contraction property. Note that to have a projection onto the hull the theorem requires that \( \epsilon > F \). Indeed we prove the contrapositive of the contraction property which says that for \( B, R > 0 \), if \( d(P, \Pi P) > R \) then \( d(\Pi P, \Pi Q) > B \implies d(P, \Pi P) > \eta d(P, Q) \).

**Lemma 5.18.** Given \( A > 0 \) and \( \epsilon > F \), there are \( B > 0 \) and \( q = q(A) > 0 \) with the following properties. Let \((\mu^-, \mu^+)\) be an \( A \)-narrow pair and \( \Pi : P(S) \to \Sigma_\epsilon(\mu^-, \mu^+) \) be the projection onto the \( \Sigma \)-hull. Let \( P, Q \in P(S) \) be such that \( d(\Pi P, \Pi Q) > B \), then
\[
(5.34) \quad d_Z(P, Q) \geq d_Z(P, \Pi P) - q
\]
for every subsurface \( Z \subseteq S \).

Let us first see how this lemma implies the contraction property. Assume that (5.34) holds for every subsurface \( Z \subseteq S \) then by the distance formula (2.2) we have
\[
\begin{align*}
    d(P, Q) &\asymp_{K,C} \sum_{Z \subseteq S \non-\text{annular}} \{d_Z(P, Q)\}_A \\
    &\geq \sum_{Z \subseteq S \non-\text{annular}} \{d_Z(P, \Pi P) - q\}_A
\end{align*}
\]
Recalling the definition of \( \{\cdot\}_A \), for any term in the last sum above we have
\[
\{d_Z(P, \Pi P) - q\}_A \geq \frac{A}{A+q} \{d_Z(P, \Pi P)\}_A + q
\]
Moreover, for the threshold constant \( A_1 = A + q \) there are constants \( K_1, C_1 \) such that the distance formula (2.2) is written as
\[
\sum_{Z \subseteq S \non-\text{annular}} \{d_Z(P, \Pi P)\}_A + q \asymp_{K_1,C_1} d(P, \Pi P)
\]
Therefore, we obtain
\[ d(P, Q) > \eta' d(P, P) - c \]
where \( \eta' = \frac{A}{K(A+q)K_1} \) and \( c = \frac{AC}{K_1(A+q)K} + C_1. \)

Now let \( R \) be large enough such that \( \eta = \eta' - \frac{c}{R} > 0. \) Then for any \( P \in P(S) \) such that \( d(P, \Pi P) > R, \) and any \( Q \in P(S), \) we have
\[ d(P, Q) > \eta' d(P, P) - c = \left[ (\eta' - \frac{c}{R}) d(P, P) \right] - c + \left( \frac{c}{R} d(P, P) \right) > \eta d(P, P) \]

This shows that for \( R \) and \( \eta \) as above the projection map \( \Pi : P(S) \to \Sigma_\epsilon(\mu^-, \mu^+) \) provided that \( \epsilon > F \) and \( (\mu^-, \mu^+) \) is \( A \)-narrow satisfies the contrapositive of the contraction property. So it has the contraction property. \( \square \)

**Proof of Lemma 5.18.** We set
\[ A' := A + 12\delta + \delta' + K + 2F + 2M + 2(F_1 + 4) + G + 2(3D + 6\delta) + 2(D_1 + 1), \]
as the threshold constant of the distance formula (2.2) in this proof.

\[ \delta = \max_{Y \subset S} \delta_Y, \quad K = \max_{Y \subset S} K_Y \quad \text{and} \quad \delta' = \max_{Y \subset S} \delta'_Y, \]
where \( \delta_Y \) is the hyperbolicity constant of the curve complex of a subsurface \( Y \subset C(Y). \) \( K_Y \) and \( \delta'_Y \) are the constants from Proposition 5.13 (iv) for \( C(Y). \) Note that \( \delta_Y, K_Y \) and \( \delta'_Y \) depend only on the topological type of \( Y, \) so the above maxima exist. \( G \) is the bound from Theorem 2.6 (Bounded Geodesic Image Theorem). \( M = M_1 + B_0 + 4, \) where the constants \( M_1 \) is from Theorem 2.13 (2) and \( B_0 \) is from Theorem 2.8 (Behrstock Inequality). \( F \) and \( F_1 \) are the constants in Theorem 2.16 (Consistency Theorem).

Further, \( D = \max_{Y \subset S} D_Y, \) where \( D_Y \) is the fellow traveling distance of two geodesics in \( C(Y) \) with end points are within distance 2 of each other. In particular, since the diameter of the projection of a marking or a lamination to a subsurface is at most 2, \( D \) is the fellow traveling distance of two geodesics in the convex hull of a pair of markings/laminations in any subsurface. Let \( D_1 \) be the maxima over all subsurfaces \( Y \) of the fellow traveling distance of two geodesics with end points within distance \( F + (3D + 6\delta) \) in \( C(Y). \) For the fellow traveling property of geodesics in \( \delta \)-hyperbolic spaces see part III.H of [BH99].

First note that if
\[ d_Z(P, P) \leq A' + 5(F_1 + 4) \]
then for \( q = A' + 5(F_1 + 4), \) (5.34) holds. Thus in the rest of the proof we will assume that \( Z \) is a subsurface with
\[ d_Z(P, P) > A' + 5(F_1 + 4) \]
(5.36)

We proceed to discuss the following two cases depending on the value of \( d_Z(P, P). \)
Caes 1:

\[ d_Z(\Pi P, \Pi Q) > A'. \]

Let \( \hat{u} \) be a nearest point to \( \pi_Z(P) \) on \( \text{hull}_Z(\mu^-, \mu^+) \) and \( \hat{w} \) be a nearest point to \( \pi_Z(Q) \) on the hull. Then by Theorem 2.16 \( d_Z(\hat{u}, \Pi P) \leq F \) and \( d_Z(\hat{w}, \Pi Q) \leq F \). Let \( f \) be a geodesic in \( \text{hull}_Z(\mu^-, \mu^+) \). Let \( u \) and \( w \) be nearest points to \( \pi_Z(P) \) and \( \pi_Z(Q) \) on \( f \), respectively. \( \hat{u} \) lies on a geodesic \( \hat{f} \) and \( \hat{f} \) and \( f \) \( \mathcal{D} \) fellow travel each other. Then Lemma 5.16 applied to geodesics \( f, \hat{f} \) and the point \( \pi_Z(P) \) implies that \( d_Z(\hat{u}, u) \leq 3D + 6\delta \). Similarly we can show that \( d_Z(\hat{w}, w) \leq 3D + 6\delta \).

The four bounds in the previous paragraph combined by the triangle inequality imply that

\[
d_Z(u, w) \geq d_Z(\Pi P, \Pi Q) - 2F - 2(3D + 6\delta) - \text{diam}_W(\Pi P) - \text{diam}_W(\Pi Q) \geq A' - 2F - 2(3D + 6\delta) - 2
\]

where the second inequality follows from (5.37). Then by the choice of \( A' \) (5.35), \( A' - 2F - 2(3D + 6\delta) - 2 \geq K_Z + 1 \). Thus by the tree like property (5.32) we have that

\[
d_Z(P, Q) \geq d_Z(P, u) - \delta_Z - \text{diam}_Z(P) \geq d_Z(P, u) - \delta' - 1.
\]

Furthermore, by the triangle inequality and the bounds \( d_Z(\Pi P, \hat{u}) \leq F \) and \( d_Z(\hat{u}, u) \) \( 3D + 6\delta \) above we have

\[
d_Z(P, u) \geq d_Z(P, \Pi P) - d_Z(\Pi P, \hat{u}) - d_Z(\hat{u}, u) - \text{diam}(\Pi P) \geq d_Z(P, \Pi P) - F - (3D + 6\delta) - 1.
\]

Plugging the above inequality into the one before it we get

\[
d_Z(P, Q) \geq d_Z(P, \Pi P) - \delta' - F - (3D + 6\delta) - 1.
\]

Thus (5.34) holds for \( q = \delta' + F + (3D + 6\delta) + 1 \).

Case 2:

\[ d_Z(\Pi P, \Pi Q) \leq A'. \]

Since the threshold constant of the distance formula is \( A' \), \( d_Z(\Pi P, \Pi Q) \) has no contribution to \( d(\Pi P, \Pi Q) \). Let \( K' \) and \( C' \) be the constants in the distance formula corresponding to the threshold constant \( A' \). Set \( B = K' + C' \) as the projection bound in the statement of Theorem 5.17. Then by the assumption of the lemma we have that

\[
d(\Pi P, \Pi Q) > K' + C'.
\]

Then by the distance formula (2.2) there must be a subsurface \( W \subseteq S \) with

\[ d_W(\Pi P, \Pi Q) > A'. \]

Let \( \hat{v} \) be a nearest point to \( \pi_W(P) \) on \( \text{hull}_W(\mu^-, \mu^+) \) and \( \hat{z} \) be a nearest point to \( \pi_W(Q) \) on the hull. Then by Theorem 2.16 \( d_W(\Pi P, \hat{v}) \leq F \) and \( d_W(\Pi Q, \hat{z}) \leq F \). Let \( f \) be a geodesic in \( \text{hull}_W(\mu^-, \mu^+) \). Let \( v \) and \( z \) be nearest points to \( \pi_W(P) \) and \( \pi_W(Q) \) on \( f \), respectively. \( \hat{v} \) lies on a geodesic \( \hat{f} \) in the hull. Then applying Lemma 5.16 to the geodesics \( f \) and \( \hat{f} \) and
the point \( \pi_W(P) \) implies \( d_W(v, \hat{v}) \leq 3D + 6\delta \). Similarly we can show that \( d_W(z, \hat{z}) \leq 3D + 6\delta \).

The four bounds in the above paragraph and (5.39) combined by the triangle inequality imply that

\[
d_W(v, z) > A' - 2F - 2(3D + 6\delta) - \text{diam}_W(\Pi P) - \text{diam}_W(\Pi Q) \geq A' - 2F - 2(3D + 6\delta) - 2.
\]

Now let \( f \) be the geodesic in the hull realizing the distance between \( \pi_W(\mu^-) \) and \( \pi_W(\mu^+) \). Let \( v \) and \( z \) be nearest points to \( \pi_W(P) \) and \( \pi_W(Q) \) on \( f \), respectively. Then by (5.40) and since \( d_W(\mu^-, \mu^+) \geq d_W(v, z) \), we get

\[
d_W(\mu^-, \mu^+) \geq A' - 2F - 2 - 2(3\delta + 6\delta).
\]

By the choice of \( A' \) in (5.35), \( A' - 2F - 2 - 2(3\delta + 6\delta) > A \), so we get

\[
d_W(\mu^-, \mu^+) > A
\]

By the assumption the lemma of the pair \((\mu^-, \mu^+)\) is \( A \)-narrow, so the above inequality implies that the subsurface \( W \) is a large subsurface. This excludes the possibility that \( W \) and \( Z \) are disjoint subsurfaces. Thus we need to discuss the following three subcases:

(2.1) \( W \cap Z \),
(2.2) \( W \subsetneq Z \) and
(2.3) \( Z \subsetneq W \).

Case 2.1: \( W \cap Z \).

Let

- \( k := k(P) = \left\lfloor \frac{d_Z(\Pi P, \Pi Q)}{F_1 + 4} - 2 \right\rfloor \) and
- \( k' = \left\lfloor \frac{A'}{F_1 + 4} \right\rfloor \).

Here \( \left\lfloor x \right\rfloor \) is the floor function, which assigns to \( x \in \mathbb{R} \) the largest integer less than or equal to \( x \).

Dividing both sides of the inequality (5.36) by \( F_1 + 4 \) and subtracting \( 2 \) from both sides we get

\[
\frac{d_Z(P, \Pi P) - 2}{F_1 + 4} \geq \frac{A'}{F_1 + 4} + 5 - \frac{2}{F_1 + 4}.
\]

Now since \( 0 < \frac{2}{F_1 + 4} \leq 1 \), taking the floor of both sides of the above inequality we have that

\[
k \geq k' + 4
\]

We claim that

\[
d_Z(\Pi P, \partial W) \leq (k' + 2)(F_1 + 4).
\]

Otherwise,

\[
d_Z(\Pi P, \partial W) > (k' + 2)(F_1 + 4),
\]

then using \( \Pi P \) to define a partial order, which means that we let \( x_Y = \pi_Y(\Pi P) \) in Definition 2.11 the last inequality can be written as

\[
Z \preccurlyeq_{k' + 2} W.
\]
Moreover, by (5.39) and since \( A' > 2(F_1 + 4) \) we have
\[
W \ll_2 \Pi Q.
\]
Having (5.44) and (5.45), by the transitivity property of \( \ll \) (Theorem 2.12 (2)) we deduce that
\[
Z \ll_{k' + 1} \Pi Q,
\]
which means that
\[
d_Z(\Pi Q, \Pi P) \geq (k' + 1)(F_1 + 4) > A'.
\]
But this lower bound contradicts (5.38) and our claim follows. Therefore, in the rest of Case 2.1 we may assume that (5.43) holds.

By the choice of \( k \) we have
\[
d_Z(\Pi P, P) - 2 \geq k(F_1 + 4).
\]
This inequality and (5.43) combined by the triangle inequality imply that
\[
d_Z(P, \partial W) \geq (k' - k' - 2)(F_1 + 4) + 2 - \text{diam}_Z(\Pi P) \geq (k' - k' - 2)(F_1 + 4)
\]
Now using \( P \) to define a partial order, which means that we let \( x = \pi_Y P \) in Definition 2.11 (5.46) can be written as
\[
Z \ll_{k' - k' - 2} W
\]
Note that by (5.42) \( k' - k' - 2 \geq 1 \).

Let \( f \) be a geodesic in the hull. Let \( v \) and \( z \), as before, be nearest points to \( \pi_W(P) \) and \( \pi_W(Q) \) on \( f \), respectively. Then by (5.40) and the choice of \( A' \), \( d_W(v, z) \geq A' - 2F - 2 - 2(3D + 6\delta) > K > K_W \). Therefore, the tree like property (5.32) implies that
\[
d_W(P, Q) > d_W(v, z) - \delta' - \text{diam}_W(P) - \text{diam}_W(Q) > A' - 2F - \delta' - 2(3D + 6\delta) - 4.
\]
Thus we have
\[
d_Z(P, Q) \geq (\lfloor \frac{A' - 2F - \delta' - 2(3D + 6\delta) - 4}{F_1 + 4} \rfloor - k' - 3)(F_1 + 4)
\]
for \( m = \lfloor \frac{A' - 2F - \delta' - 2(3D + 6\delta) - 4}{F_1 + 4} \rfloor \). Note that by the choice of \( A' \) (5.35), \( m \geq 2 \).

Having (5.47) and (5.48), by the transitivity property of \( \ll \) (Theorem 2.12 (2)) we deduce that
\[
W \ll_m Q
\]
where by (5.42), \( k - k' - 3 \geq 1 \). Therefore,
\[
d_Z(P, Q) \geq (\lfloor \frac{d_Z(P, \Pi P) - 2}{F_1 + 4} \rfloor - k' - 3)(F_1 + 4) \geq d_Z(P, \Pi P) - (k' + 4)(F_1 + 4) - 2.
\]

So the inequality (5.34) holds for \( q = (k' + 4)(F_1 + 4) + 2 \).

**Case 2.2:** \( W \subset Z \).

Let \( \rho' \) be a hierarchy path between \( P \) and \( \Pi P \). By (5.36) we have that \( d_Z(P, \Pi P) > A' > M_1 \), so by Theorem 2.13 (2) \( Z \) is a component domain of
Figure 4. Case 2.2: Left diagram: $x$ is a nearest point to $\pi_Z(Q)$ on $\text{hull}_Z(P, \Pi P)$ and $T$ is a slice of $\rho'$ (a hierarchy path between $P$ and $\Pi P$) with $d_Z(T, x) \leq M_2$. $\pi_Z(\partial W)$ is in the 1-neighborhood of the geodesic $h$ connecting $\pi_Z(Q)$ to $\pi_Z(T)$ and any geodesic $k$ in $\text{hull}_Z(\mu^-, \mu^+)$. Right diagram: $y$ on $\text{hull}_W(P, \Pi P)$ is such that $d_W(T, y) \leq M_2$ and $y'$ is a nearest point to $y$ on the geodesic $l$ connecting $\pi_W(P)$ to $v$ a nearest point to $\pi_W(P)$ on the $f$.

Let $x$ be a nearest point to $\pi_Z(Q)$ on $\text{hull}_Z(P, \Pi P)$. By Theorem 2.13 (5) there is a pants decomposition $T$ of $\rho'$ such that $d_Z(T, x) \leq M_2$. Let $h$ be a geodesic in $C(Z)$ connecting $\pi_Z(Q)$ to $\pi_Z(T)$, see the left diagram of Figure 4.

Since $T \in [\rho']$, by Theorem 2.13 (5) there is $y \in \text{hull}_W(P, \Pi P)$ such that

\[(5.49) \quad d_W(T, y) \leq M_2 \leq M,\]

see the right diagram of Figure 4.

Let $\hat{v}$ and $\hat{z}$, as before, be nearest points to $\pi_W(P)$ and $\pi_W(Q)$ on $\text{hull}_W(\mu^-, \mu^+)$ respectively. Let $f$ be a geodesic in $\text{hull}_W(\mu^-, \mu^+)$ and $v$ and $z$, as before, be nearest points to $\pi_W(P)$ and $\pi_W(Q)$ on $f$, respectively.

$y$ is a geodesic connecting $\pi_W(P)$ to $\pi_W(\Pi P)$. By the triangle inequality $d_W(v, \Pi P) \leq d_W(v, \hat{v}) + d_W(\hat{v}, \Pi P) \leq (3D + 6\delta) + F$, so this geodesic and any geodesic $l$ connecting $\pi_Y(P)$ to $v$, $D_1$ fellow travel each other ($D_1$ is the fellow traveling distance of two geodesics with end pints within distance $(3D + 6\delta) + F$ of each other). Let $y'$ be a nearest point to $y$ on $l$, then by the fellow traveling we have that

\[d_W(y, y') \leq D_1.\]

Furthermore, by Proposition 5.13 (ii),

\[d_W(y', Q) \geq d_W(v, z) - 12\delta_W - \text{diam}_W(Q) \geq d_W(v, z) - 12\delta - 1.\]
Combing the above two inequalities with the triangle inequality we get
\[ d_W(Q, y) \geq d_W(v, z) - 12\delta - D_1 - 1, \]
see the right diagram of Figure 4. By (5.40) \[ d_W(v, z) \geq A' - 2F - 2(3D + 6\delta) - 2, \] so we get
\[ (5.50) \quad d_W(Q, y) \geq A' - 2F - 2(3D + 6\delta) - 12\delta - D_1 - 3. \]

Now combining (5.49) and (5.50) by the triangle inequality we get
\[ d_W(Q, T) \geq A' - 2F - 2(3D + 6\delta) - D_1 - 12\delta - 3 - M. \]
Then by the choice of \( A' \) (5.35) we have that
\[ (5.51) \quad d_W(Q, T) > G. \]

Note that since \( W \subset Z \) we have \( \partial W \cap Z \). We claim that

**Claim 5.19.** \( \pi_Z(\partial W) \) is in the 1–neighborhood of \( h \) in \( C(Z) \).

Otherwise, \( \partial W \) would intersect every vertex of the geodesic \( h \), which connects \( \pi_Z(T) \) to \( \pi_Z(Q) \). Then by Theorem 2.6 (Bounded Geodesic Image) \( \text{diam}_W(h) \leq G \). This contradicts the lower bound (5.51) and the claim follows.

Let \( k \subset \text{hull}_Z(\mu^-, \mu^+) \) be any geodesic connecting \( \pi_Z(\mu^-) \) to \( \pi_Z(\mu^+) \). We claim that

**Claim 5.20.** \( \pi_Z(\partial W) \) is in the 1–neighborhood of \( k \).

Otherwise, \( \partial W \) would intersect every vertex of \( k \), so Theorem 2.6 implies that
\[ (5.52) \quad d_W(\mu^-, \mu^+) \leq G. \]
On the other hand, by (5.41) and the choice of \( A' \) in (5.35) we have \( d_W(\mu^-, \mu^+) \geq A' - 2F - 2(3D + 6\delta) - 2 > G \). But this contradicts the upper bound (5.52) and the claim follows.

Let \( \hat{u} \) be a closest point to \( \pi_Z(P) \) on \( \text{hull}_Z(\mu^-, \mu^+) \). Suppose that \( \hat{u} \) is on a geodesic \( k \) in the hull. Let \( n \) be a geodesic connecting \( \pi_Z(P) \) to \( \hat{u} \). Let \( m \) be a nearest point to \( \pi_Z(\mu^+) \) on the geodesic \( n \). When \( \pi_Z(\mu^+) \) is at infinity of \( C(Z) \), let \( \xi_i \in C(Z) \) be a sequence of points with \( \xi_i \to \pi_Z(\mu^+) \) as \( i \to \infty \) and let \( m \) be the limit of \( \xi_i \)'s. See the left diagram of Figure 4. Then by Lemma 5.14 \( d_Z(\hat{u}, m) \leq 6\delta_Z \leq 6\delta \). This implies that \( k \) and \( t \) a geodesic between \( m \) and \( \pi_Z(\mu^+) \), \( D_1 \) fellow travel each other. By Claim 5.20 there is a point on \( k \) within distance 1 of \( \pi_Z(\partial W) \). Then the fellow traveling implies that \( \pi_Z(\partial W) \) is within distance \( D_1 + 1 \) of \( t \).

Let \( x' \) be a nearest point to \( \pi_Z(Q) \) on \( n \) and let \( h' \) be a geodesic connecting \( \pi_Z(Q) \) to \( x' \), see the left diagram of Figure 4. By the choice of \( T \), \( d_Z(x, T) \leq M \). The fact that the end points of \( n \) and any geodesic between \( \pi_Z(P) \) and \( \pi_Z(\text{IP}) \) are within distance \( F \) of each other implies that they \( D_1 \) fellow travel each other. Then \( r \) the geodesic in \( \text{hull}_Z(P, \text{IP}) \) on which \( x \) lies and the geodesic \( n \) on which \( x' \) lies, \( D_1 \) fellow travel each other. Moreover \( x \)
is a nearest point to \( \pi_Z(Q) \) on \( r \) and \( x' \) a nearest point to \( \pi_Z(Q) \) on the geodesic \( n \). Then Lemma \[5.16\] implies that \( d_Z(x, x') \leq 3D_1 + 6\delta \). Then by the triangle inequality we get

\[
d_Z(T, x') \leq d_Z(T, x) + d_Z(x, x') \leq M + 3D_1 + 6\delta.
\]

This bound on the distance of end points of \( h \) and \( h' \) implies that they \( D_2 \) fellow travel each other. Here \( D_2 \) is the maximum over all subsurfaces of the fellow traveling distance of two geodesics with end point within distance \( M + 3D_1 + 6\delta \) of each other. By Claim \[5.19\] there is a point on \( h \) within distance 1 of \( \pi_Z(\partial W) \). Then the fellow traveling implies that \( \pi_Z(\partial W) \) is within distance \( D_2 + 1 \) of \( h' \).

Let \( a := \max\{D_1 + 1, D_2 + 1\} \). Then by the conclusions of the above two paragraphs \( \pi_Z(\partial W) \) is in the \( a \) neighborhood of \( h' \) and \( t \), see the left diagram of Figure \[4\]. So Proposition \[5.13\] (iv) implies that \( d_Z(m, x') \leq b(\delta_Z, a) \leq b(\delta, a) \). Further recall the bounds \( d_Z(\hat{u}, \Pi P) \leq F \), \( d_Z(x', x) \leq 3D_1 + 6\delta \) and \( d_Z(\hat{u}, m) \leq 6\delta \) we established earlier. These bounds combined by the triangle inequality imply that

\[
(5.53) \quad d_Z(x, \Pi P) \leq 2 + (3D_1 + 6\delta) + F + b + 6\delta
\]

\( x \) is a nearest point to \( \pi_Z(Q) \) on the geodesic \( r \subset \text{hull}_Z(P, \Pi P) \), then Lemma \[5.15\] applied to the geodesic \( r \) and the points \( \pi_Z(Q) \) and \( \pi_Z(P) \) implies that

\[
(5.54) \quad d_Z(P, Q) + 6\delta \geq d_Z(P, x) - \text{diam}_Z(P) \geq d_Z(P, x) - 1
\]

Now we have

\[
d_Z(P, Q) \geq d_Z(P, x) - 6\delta - 1 \geq d_Z(P, \Pi P) - d_Z(\Pi P, x) - \text{diam}_Z(\Pi P) - 6\delta - 2 \geq d_Z(P, \Pi P) - F - (3D_1 + 6\delta - b - 6\delta - 6\delta - 3)
\]

The first inequality is \[5.54\]. The second one is the triangle inequality. The third one follows from the bound \[5.53\]. Consequently, \[5.34\] holds for \( q = F + (3D_1 + 6\delta) + 12\delta + b + 3 \).

Case 2.3: \( Z \subset W \).

Note that \( \partial Z \cap W \). We claim that

Claim 5.21. \( \pi_W(\partial Z) \) is in the 1–neighborhood of any geodesic \( l \subset \text{hull}_W(P, \Pi P) \) connecting \( \pi_W(P) \) to \( \pi_W(\Pi P) \).

Otherwise, \( \partial Z \) would intersect every vertex of \( l \). Then Theorem \[2.6\] implies that \( d_Z(P, \Pi P) \leq G < A' \), but this contradicts \[5.36\].

Let \( f \) be a geodesic in \( \text{hull}_W(\mu^-, \mu^+) \). Let \( v \) and \( z \) be nearest points to \( \pi_W(P) \) and \( \pi_W(Q) \) on \( f \), respectively. Let \( k \) be a geodesic connecting \( \pi_W(Q) \) to \( \pi_W(\Pi Q) \), and \( k' \) be a geodesic connecting \( \pi_W(Q) \) to \( z \). Let \( l \) be a geodesics connecting \( \pi_W(P) \) to \( \pi_W(\Pi P) \), and \( l' \) be a geodesics connecting \( \pi_W(P) \) to \( v \), see the left diagram of Figure \[5\]. The end points of \( k \) and \( k' \) are within distance \( F \) of each other, so \( D_1 \) fellow travel each other. Similarly \( l \)
and $l', D_1$ fellow travel each other. By Claim 5.21 and the fellow traveling
of $l$ and $l'$ there is a point $\hat{p}$ on $l'$ with

\begin{equation}
(5.55) \quad d_W(\partial Z, \hat{p}) \leq 1 + D_1.
\end{equation}

By (5.40) and the choice of $A'$ in (5.35), $d_W(v, z) \geq A' - 2F - 2(3D + 6\delta) > K_W$. Then the tree like property (5.32) implies that for every $q$ on $k'$ we have that

\begin{equation}
(5.55) \quad d_W(\hat{p}, q) \geq d_W(v, z) - \delta_w.
\end{equation}

Since $q$ on $k'$ was arbitrary, $d_W(\hat{p}, k') > 2D_1 + 2$. This and (5.55) imply that $d_W(\partial Z, k') > D_1 + 1$. Finally, the $D_1$ fellow traveling of $k$ and $k'$ implies that $d_W(\partial Z, k) > 1$. Therefore, $\partial Z$ intersects every vertex of $k$. So Theorem 2.6 implies that

\begin{equation}
(5.56) \quad d_Z(Q, \Pi Q) \leq G.
\end{equation}

By the triangle inequality $d_Z(P, Q) \geq d_Z(P, \Pi Q) - d_Z(\Pi Q, Q) - 1$. Replacing (5.56) in this inequality we get

\begin{equation}
(5.57) \quad d_Z(P, Q) \geq d_Z(P, \Pi Q) - G - 1.
\end{equation}

Let $f$ be a geodesics in hull$_Z(\mu^-, \mu^+)$. Let $u$ and $w$ be nearest points to respectively $\pi_Z(P)$ and $\pi_Z(Q)$ on $f$. Since $u$ is a nearest point to $\pi_Z(P)$ on $f$ and $w$ is on $f$ we have $d_Z(P, w) \geq d_Z(P, u)$. Furthermore, $d_Z(\Pi P, u) \leq d_Z(\Pi P, \hat{u}) - d_Z(\hat{u}, u) \leq F + (3D + 6\delta)$ and $d_Z(\Pi Q, w) \leq F + (3D + 6\delta)$. So

\begin{equation}
(5.57) \quad d_Z(P, \Pi Q) \geq d_Z(P, \Pi P) - 2F - 2(3D + 6\delta).
\end{equation}

Plugging the last inequality into (5.57) we get

\begin{equation}
(5.58) \quad d_Z(P, Q) \geq d_Z(P, \Pi P) - G - 2F - 2(3D + 6\delta) - 1
\end{equation}

(see the right diagram of Figure 5). Thus (5.34) holds for $q = G + 2F + 2(3D + 6\delta) + 1$.

Establishing (5.34) in cases (1) and (2), we may conclude that it holds for $q$ the maximum of the $q$'s we obtained in these two cases. This finishes the proof of the lemma. \hfill \square

Remark 5.22. $R \to \infty$, $\eta \to 0$ and $B \to \infty$, as $A \to \infty$, so applying the Morse lemma argument we get $\Sigma$–hulls with worse and worse stability property. More precisely, there are $K \geq 1$ and $C \geq 0$ such that $d_A(K, C) \to \infty$ as $A \to \infty$. 
Figure 5. **Case 2.3:** Left diagram: $\partial Z$ is in the 1–neighborhood of $l$. The geodesics $l$ and $l'$, and $k$ and $k'$, respectively, $D_1$–fellow travel each other. $d_{W}(v, z) > 2D_1 + 2 + G$. Thus $\partial Z$ intersects every vertex of the geodesic $k$ connecting $\pi_{W}(Q)$ to $\pi_{W}(\Pi Q)$. Then the Bounded Geodesic Image Theorem implies that $d_{Z}(Q, \Pi Q) \leq G$.

5.3. **Fellow traveling.** We start by the definition of fellow traveling of parametrized quasi-geodesics in a metric space.

**Definition 5.23.** (Fellow traveling) Given $D \geq 0$. Let $h_1 : I_1 \rightarrow X$ and $h_2 : I_2 \rightarrow X$ be two parametrized quasi-geodesics. We say that $h_1$ and $h_2$, $D$–fellow travel if

- For every $i \in I_1$ there is an $i' \in I_2$ such that $d(h_1(i), h_2(i')) \leq D$ and vice versa. In other words, the Hausdorff distance of $h_1(I_1)$ and $h_2(I_2)$ is bounded by $D$.

Given a subinterval $I'_1 \subset I_1$, we say that $h_1$, $D$–fellow travels $h_2$ over $I'_1$ if there is a subinterval $I'_2 \subset I_2$ such that $h_1|_{I'_1}$ and $h_2|_{I'_2}$, $D$–fellow travel as above.

Let $h_1$ be a $(K_1, C_1)$–quasi-geodesic and $h_2$ be a $(K_2, C_2)$. Let $N_{h_1, h_2} : I_1 \rightarrow I_2$, be the map which assigns to each $i \in I_1$ any $h_2(i')$ where $d(h_1(i), h_2(i')) \leq D$. Then $N_{h_1, h_2}$ is a reparametrization of $h_2$ and for any $i, j \in I_1$,

$$d_{Haus}(N_{h_1, h_2}(i), N_{h_1, h_2}(j)) \asymp_{K, C} |i - j|$$

Here $d_{Haus}$ denotes the Hausdorff distance of subintervals of the real line, $\mathbb{R}$, $K = K_1 K_2$ and $C = \max\{K_2 C_1 + C_2 + 2K_2 D, K_1 C_2 + C_1 + 2K_1 D\}$. In particular, the diameter of $N_{h_1, h_2}(i)$ is bounded above by $C$. Also the same holds exchanging $h_1$ and $h_2$.

**Theorem 5.24.** Given $A > 0$ there is a constant $D = D(A)$ with the following property. Let $g : [a, b] \rightarrow \text{Teich}(S)$ be a WP geodesic segment with
A–narrow end invariant \((\nu^-, \nu^+)\), and let \(\rho\) be a hierarchy path between \((\nu^-, \nu^+)\). Then \(\rho\) and \(Q(g)\), \(D\)–fellow travel.

**Proof.** \(Q(g)\) is a \((K_{WP}, C_{WP})\)–quasi geodesic in \(P(S)\) where \(K_{WP}\) and \(C_{WP}\) depend only on the topological type of \(S\) (Theorem 3.3). The hierarchy path \(\rho\) is a \((k, c)\)–quasi-geodesic in \(P(S)\) where \(k\) and \(c\) depend only on the topological type of \(S\) (see §2). By Theorem 5.11 \(\rho\) is \(d_A\)–stable. So the Hausdorff distance of \(Q(g)\) and \(|\rho|\) is bounded by \(D = d_A(K_{WP}, C_{WP})\) in \(P(S)\). These are the conditions required by Definition 5.23 so that \(\rho\) and \(Q(g)\), \(D\) fellow travel each other. \(\square\)

Let \(g : [a, b] \to \text{Teich}(S)\) and \(\rho : [m, n] \to P(S)\). Suppose that \(Q(g)\) and \(\rho\), \(D\)–fellow travel. Given \(i \in [m, n]\) take the smallest interval \(I_i\) containing every \(t \in [a, b]\) such that \(d(Q(g(t)), \rho(i)) \leq D\). Then define the coarse map \(N_{\rho, g} : [m, n] \to [a, b]\) so that \(N_{\rho, g}(i)\) is any \(t \in [a, b]\) in the \(K_{WP}(2D+1)+C_{WP}\) neighborhood \(I_i\). The following proposition is a straightforward consequence of the definition of \(N_{\rho, g}\).

**Proposition 5.25.** The coarse map \(N_{\rho, g} : [m, n] \to [a, b]\) has the following properties:

- \(|N_{\rho, g}(i)|\) is bounded by a constant depending only on the fellow traveling distance \(D\) and topological type of the surface.
- \(\bigcup_{i \in [m, n]} N_{\rho, g}(i)\) covers \([a, b]\),
- There are \(K \geq 1\) and \(C \geq 0\) depending only on \(D\) and the topological type of \(S\), such that for any \(i, j \in [m, n]\) we have

\[
d_{Haus}(N_{\rho, g}(i), N_{\rho, g}(j)) \lesssim_{K, C} |i - j|.
\]

By Theorem 5.24 this proposition in particular applies to a WP geodesic segment with narrow end invariant and a hierarchy path with the same end points.

6. **Itinerary of a Weil-Petersson geodesic segment**

Itinerary of a WP geodesic \(g\) in Teichmüller space refers to the list of short curves (curves with length less than a sufficiently small \(\epsilon > 0\)), the time intervals along \(g\) over which each curve is short and the order in which these intervals appear along \(g\).

In this section we present our results on the control of length-functions and twist parameters along WP geodesics with narrow end invariant.

Our main result asserts the following: Suppose that over an interval all of the subsurface coefficients are bounded, except possibly those of some annular subsurfaces whose core curves consist the boundary of a large subsurface \(Z\). Then the length of these curves are arbitrary short over a suitably shrunk subinterval of the interval.

**Theorem 6.1.** (Short Curve)

Given \(A, R, R' > 0\) and a sufficiently small \(\epsilon > 0\), there is a constant \(\bar{w} = \bar{w}(A, R, R', \epsilon)\) with the following property. Let \(g : [a, b] \to \text{Teich}(S)\)
be a WP geodesic segment with $A$--narrow end invariant $(\nu^-, \nu^+)$. Let $\rho : [m, n] \to P(S)$ be a hierarchy path between $\nu^-$ and $\nu^+$. Assume that a large domain $Z$ has $(R, R')$--bounded combinatorics over $[m', n'] \subset J_Z$.

If $m' - n' \geq 2\bar{w}$, then for every $\alpha \in \partial Z$ we have
\[
\ell_\alpha(g(t)) \leq \epsilon
\]
for every $t \in [a', b']$, where $a' \in N(m' + \bar{w})$ and $b' \in N(n' - \bar{w})$. Here $N := N_{\rho, g}$ be the parameter map from Proposition 5.25.

We prove this theorem at the end of §6.2. This theorem is a partial itinerary for WP geodesic segments with narrow end invariant.

6.1. Isolated annular subsurfaces. In this subsection we prove two combinatorial lemmas which together with the fellow traveling property of hierarchy paths between narrow pairs (Theorem 5.24) provide us with a combinatorial frame work in which we will be able to control length-functions along WP geodesics.

**Bounded combinatorics:** Given $R, R' > 0$ and a subsurface $Z$, we say that $Z$ has $(R, R')$--bounded combinatorics between a pair of partial markings/laminations $\mu_1$ and $\mu_2$ if for any essential, proper, non-annular subsurface $Y \subseteq Z$ the subsurface projection be bounded as
\[
d_Y(\mu_1, \mu_2) \leq R,
\]
and for any annular subsurface with core curve $\gamma \in C_0(Z)$ which intersects both $\mu_1$ and $\mu_2$ the subsurface coefficient be bounded as
\[
d_\gamma(\mu_1, \mu_2) \leq R'.
\]
If only the first bound holds we say that the $Z$ has non-annular $R$--bounded combinatorics and if only the second bound holds we say that $Z$ has annular $R'$--bounded combinatorics.

**Lemma 6.2.** (No backtracking) Let $\rho : [m, n] \to P(S)$ be a hierarchy path. Let $[i_1, i_2] \subseteq [m, n]$ and $i, j \in [i_1, i_2]$ with $i < j$. Then for every subsurface $Y \subseteq S$ we have that
\[
d_Y(\rho(i_1), \rho(i_2)) \geq d_Y(\rho(i), \rho(j)) - 2M_2.
\]

**Proof.** We have that
\[
d_Y(\rho(i_1), \rho(i_2)) \geq d_Y(\rho(i), \rho(j)) + d_Y(\rho(j), \rho(i_2))
\]
\[
\leq d_Y(\rho(i_1), \rho(j)) + d_Y(\rho(j), \rho(i_2)) + M_2
\]
\[
\leq d_Y(\rho(i_1), \rho(i_2)) + 2M_2
\]
The first inequality follows from Theorem 2.13 (6) (no backtracking) for $i_1 < i < j$. The second inequality follows from no backtracking by considering $i < j < i_2$. All of the terms in (6.2) are non negative, thus $d_Y(\rho(i_1), \rho(i_2)) \geq d_Y(\rho(i), \rho(j)) - 2M_2$. \hfill $\Box$
Now let $Z$ be a component domain of $\rho$ and $[i_1, i_2] \subset J_Z$. If $Z$ has $(R, R')$–bounded combinatorics between $\rho(i_1)$ and $\rho(i_2)$ then by (6.1), $d_Y(\rho(i), \rho(j)) \leq R + 2M_2$ for any $Y \subseteq S$ and $d_Y(\rho(i), \rho(j)) \leq R' + 2M_2$ for any $\gamma \in C_0(Z)$. In this situation we say that $Z$ has $(R, R')$–bounded combinatorics over the interval $[i_1, i_2]$.

In the following lemma we show that over a subinterval of a hierarchy path where a large subsurface $Z$ has non-annular bounded combinatorics $\pi_Z \circ \rho$ ($\pi_Z$ is the $Z$ subsurface projection) is a parametrization of the geodesic $g_Z \subset C(Z)$ (see Theorem 2.13) as a quasi-geodesic with constants depending only on $R$.

**Lemma 6.3.** Given $R > 0$, there are $K_R \geq 1$ and $C_R \geq 0$ with the following properties. Let $\rho : [m, n] \to P(S)$ be a hierarchy path and let $Z$ be a large, non-annular domain with non-annular $R$–bounded combinatorics over $[i_1, i_2] \subset J_Z \subset [m, n]$. Then for any $i, j \in [i_1, i_2]$ we have

$$d(\rho(i), \rho(j)) \asymp_{K_R, C_R} d_Z(\rho(i), \rho(j))$$

**Proof.** Given a threshold constant $A \geq M_1$ for distance formula (2.2), then we have

$$d(\rho(i), \rho(j)) \asymp_{K_R, C_R} \sum_{\substack{Y \subseteq S \text{ non-annular}}} \{d_Y(\rho(i), \rho(j))\} A$$

Note that since $Z$ has complement consisting of only annuli and three holed spheres, every subsurface $Y$ contributing to the above sum either is a subsurface of $Z$ or overlaps $Z$. Suppose that $Y \cap Z$. Since $i, j \in J_Z$, by Theorem 2.13, $\rho(i) \supset \partial Z$ and $\rho(j) \supset \partial Z$. Then $\rho(i) \cap Y$ and $\rho(i) \cap Y$, so by the triangle inequality

$$d_Y(\rho(i), \rho(j)) \leq d_Y(\rho(i), \partial Y) + d_Y(\partial Y, \rho(j)) + 2 \text{diam}_Y(\partial Z) \leq 4.$$

Now suppose that $Y \cap Z$. With out loss of generality we may assume that $i_1 < i < j < i_2$. Then the non-annular $R$–bounded combinatorics of $Z$ over $[i_1, i_2]$ and no backtracking (6.1) imply that

$$d_Y(\rho(i), \rho(j)) \leq d_Y(\rho(i_1), \rho(i_2)) + 2M_2 \leq R + 2M_2.$$

Having the bounds (6.4) and (6.5) on the subsurface coefficients contributing to the sum (6.3) if we let the threshold constant be $A_R = \max\{R + 2M_2, 4\}$ (note that it is larger than $M_1$) we get

$$d(\rho(i), \rho(j)) \asymp_{K_R, C_R} d_Z(\rho(i), \rho(j))$$

where $K_R$ and $C_R$ are the constants corresponding to the threshold constant $A_R$ in the distance formula. \qed

Let $\rho$ and $Q(g)$, $D$–fellow travel each other and $N = N_{g, \rho}$ be the parameter map from Proposition 5.25. Given $i, j \in [m, n]$ let $r \in N(i)$ and $s \in N(j)$. Then the $D$–fellow traveling implies that $d(\rho(i), Q(g(r))) \leq D$ and $d(\rho(j), Q(g(s))) \leq D$. An elementary move on a pants decomposition $P$,
replaces a curve $\alpha \in P$ with a curve $\alpha'$ with distance 1 in the complexity 1 subsurface which $\alpha$ and $\alpha'$ fill and does not change curves in $P - \alpha$. Thus an elementary move changes the value of a non-annular subsurface coefficient by at most 1. Thus for any $P, Q \in P(S)$ and any non-annular subsurface $Y \subseteq S$ we have that

\begin{equation}
(6.6)
|d_Y(P, Q)| \leq d(P, Q)
\end{equation}

Therefore, $d_Y(\rho(i), Q(g(r))) \leq D$ and $d_Y(\rho(j), Q(g(s))) \leq D$. So by the triangle inequality we obtain the comparison of subsurface coefficients

$$d_Y(\rho(i), \rho(j)) \asymp_{1,2D} d_Y(Q(g(r)), Q(g(s))).$$

But such a comparison a-priori does not hold for annular subsurfaces, so we consider isolated annular subsurfaces along hierarchy paths. In Lemma 6.5 we prove a comparison for the subsurface coefficient of an isolated annular subsurface along $\rho$ and a $D-$fellow traveling WP geodesic depending only on $D$.

**Definition 6.4.** (Isolated annular subsurface) Given $w, r > 0$. Let $\rho : [m, n] \to P(S)$ be a hierarchy path. We say that an annular subsurface $A(\gamma)$ with core curve $\gamma$ is $(w, r)-$isolated at $i \in [m, n]$, if there is a pants decomposition $\hat{Q}$ such that $\gamma \in \hat{Q}$ and $d(\hat{Q}, \rho(i)) \leq r$. Moreover, there are large, non-annular component domains of $\rho$, $Z_1$ and $Z_2$ with $\gamma \notin \partial Z_1$ and $\gamma \notin \partial Z_2$ and intervals $I_1 \subseteq J_{Z_1}$ and $I_2 \subseteq J_{Z_2}$ with $|I_1|, |I_2| \geq w$ and \max $I_1 < i < \min I_2$ such that $Z_1$ and $Z_2$ have non-annular $R-$bounded combinatorics over $I_1$ and $I_2$, respectively. For the illustration of isolation see Figure 6.

**Lemma 6.5.** (Annular coefficient comparison) Given $D, r$ and $R$ positive, there are constants $w = w(D, r, R)$ and $B = B(D)$ with the following properties. Let $\rho : [m, n] \to P(S)$ be a hierarchy path. Assume that $A(\gamma)$ is $(w, r)-$isolated at $i$. Let $I_1, I_2$ be as in the definition of isolated annular subsurface and $i_1 < \min I_1$ and $i_2 > \max I_2$. Let $g : [a, b] \to \text{Teich}(S)$ be a WP geodesic parametrized by arc-length such that $Q(g) D-$fellow travels $\rho$. Let $t_1 \in N(i_1)$ and $t_2 \in N(i_2)$, where $N := N_{\rho g}$ be the parameter map from Proposition 5.25. Then we have the annular coefficient comparison

\begin{equation}
(6.7)
d_\gamma(Q(g(t_1)), Q(g(t_2))) \asymp_{1,B} d_\gamma(\rho(i_1), \rho(i_2)).
\end{equation}

Furthermore, we have the following lower bound on the length of $\gamma$

\begin{equation}
(6.8)
\min \{\ell_\gamma(g(t_1)), \ell_\gamma(g(t_2))\} \geq \omega(L_S).
\end{equation}

Here $\omega(l)$ denotes the width of the collar of a simple closed geodesic with length $l$ on a hyperbolic surface.

**Proof.** Since $\gamma$ is isolated at $i$, there is a pants decomposition $\hat{Q}$ with $\gamma \in \hat{Q}$ and $d(\hat{Q}, \rho(i)) \leq r$. By the $D-$fellow traveling $d(\rho(i_1), P) \leq D$ for every $P$ on the geodesic connecting $\rho(i_1)$ to $Q(g(t_1))$. Thus by (6.6) these two bounds
imply that $d_Y(\hat{Q}, \rho(i)) \leq r$ and $d_Y(Q(g(t_1)), \rho(i_1)) \leq D$, respectively, for every non-annular subsurface $Y \subseteq S$ (*).

Let $w(D, r, R) = kK_R(D + r + C_R + 3 + 2M_2) + kc$, where $K_R$ and $C_R$ are the constants form Lemma 6.3 and $k, c$ are the quasi-geodesic constants of $\rho$ depending only on the topological type of $S$. Let $P$ be a pants decomposition on a geodesic in $P(S)$ connecting $\rho(i_1)$ to $Q(g(t_1))$, see Figure 6. Let $I_1 = [j, j']$. $\rho$ is a $(k, c)$-quasi-geodesic so $d(\rho(j), \rho(j')) \geq \frac{1}{k}|I_1| - c$.

Then by the assumption of the lemma about $I_1$, Lemma 6.3 implies that $d_{Z_1}(\rho(j), \rho(j')) \geq \frac{1}{K_R}(\frac{1}{k}|I_1| - c) - C_R$. Now no backtracking (6.1) applied to $i_1, j, j'$ and $i$ implies that $d_{Z_1}(\rho(i), \rho(i_1)) \geq \frac{1}{K_R}(\frac{1}{k}|I_1| - c) - C_R - 2M_2$.

Then we have

$$d_{Z_1}(P, \hat{Q}) \geq d_{Z_1}(\rho(i_1), \rho(i)) - d_{Z_1}(\hat{Q}, \rho(i)) - d_{Z_1}(\rho(i_1), P)
\geq \frac{1}{K_R}(\frac{1}{k}|I_1| - c) - 2M_2 - C_R - r - D \geq 3.$$

The first inequality is the triangle inequality. The second inequality follows from the inequality we established above and the bounds (*). The last inequality follows from the choice of $w$.

Now since $\gamma \in \hat{Q}$ and $\gamma \not\in \partial Z_1$, the above inequality implies that any curve in $P$ and $\gamma$ fill the subsurface $Z_1$ and intersect each other. Hence the projection to the annular subsurface $A(\gamma)$ of each two consecutive pants decompositions $P, P'$ on the geodesic connecting $\rho(i_1)$ to $Q(g(t_1))$ is non-empty. Then by [6.6] $d(P, P') \leq 1$. Thus the projection of the geodesic to $A(\gamma)$ has diameter bounded above by $D$, and consequently $d_\gamma(Q(g(t_1)), \rho(i_1)) \leq D$. Similarly, replacing $i_1$ by $i_2$, $t_1$ by $t_2$ and $Z_1$ by $Z_2$ we get $d_\gamma(Q(g(t_2)), \rho(i_2)) \leq D$.

These two bounds and the triangle inequality imply that the annular coefficient comparison [6.7] holds for $B = 2D$.

As we saw above $\gamma \cap Q(g(t_1))$ and $\gamma \cap Q(g(t_2))$. Now since every curve in the pants decompositions $Q(g(t_1))$ and $Q(g(t_2))$ has length at most the Bers constant $L_S$, the lower bound for the length of $\gamma$ (6.8) follows from the Collar lemma.
6.2. **Length-function control.** Corollaries \ref{cor:local-control} and \ref{cor:global-control} in \cite{Babak} provide a local control on development of Dehn twists about a curve versus the change of its length along WP geodesic segments. The control is uniform in the length of the geodesic segment and the supremum of the length-function along the geodesic segment. Lemma \ref{lem:comparison-local} provides a comparison for the sub-surface coefficient of an isolated annular subsurface between two point of a hierarchy path and the corresponding points of a fellow traveling WP geodesic. The comparison is uniform in the fellow traveling distance. The comparison lets us to pull back the annular coefficient from the geodesic to the hierarchy path and use the combinatorial properties of the hierarchy to control the length-functions along the geodesic using its end invariant. As we will see in \[8\] the control of length-functions would lead to the control of the global behavior of WP geodesics.

**Lemma 6.6. (Rough bounds)**

Given \( D, R, R' > 0 \), there are constants \( w = w(D, R), \bar{\epsilon} = \bar{\epsilon}(D, R') \) and \( l = l(D, R) \) with the following properties. Let \( \rho : [m, n] \rightarrow P(S) \) be a hierarchy path. Assume that a large domain \( Z \) has \((R,R')\)-bounded combinatorics over \([m',n']\subset J_Z\). Let \( g : [a, b] \rightarrow \text{Teich}(S) \) be a WP geodesic parametrized by arc-length such that \( \rho \) and \( Q(g) \), \( D \)-fellow travel. If \( m' - n' > 2w \) then

(1) \( \ell_\alpha(g(t)) \leq l \) for every \( \alpha \in \partial Z \) and

(2) \( \ell_\gamma(g(t)) > \bar{\epsilon} \) for every \( \gamma \notin \partial Z \)

for every \( t \in [a', b'] \), where \( a' \in N(m' + w) \) and \( b' \in N(n' - w) \). Here \( N := N_{\rho, g} \) is the parameter map from Proposition \ref{prop:hierarchy}

**Proof.** We start by establishing the lower bound (2). Let \( r = K_{WP} D + C_{WP} \). Let \( w = w(D, r, R) \), where \( w \) is the constant form Lemma \ref{lem:comparison-local}. Let \( s = K w + C \), where \( K \) and \( C \) are the constants for the parameter map \( N \) in Proposition \ref{prop:hierarchy}. Assume that \( n' - m' > 2w \). Let \( i_0 \in [m' + w, n' - w] \) and \( t_0 \in N(i_0) \). Pick \( t_0^+ \in N(i_0 + w) \) such that \( t_0^+ - t_0 = s \) and \( t_0^- \in N(i_0 - w) \) such that \( t_0 - t_0^- = s \). See Figure \ref{fig:WPgeodesic}. We denote by \( \mu(g(t_0)) \) a (partial) Bers marking of the surface \( g(t_0) \). Similarly, we denote (partial) Bers markings of \( g(t_0^-) \) and \( g(t_0^+) \) by \( \mu(g(t_0^-)) \) and \( \mu(g(t_0^+)) \), respectively.

Let \( \gamma \notin \partial Z \). If \( \ell_\gamma(g(t_0)) \geq L_S \) then we already have the lower bound at \( g(t_0) \). Otherwise, there is \( Q_0 \) a Bers pants decomposition of \( g(t_0) \) such that \( \gamma \in Q_0 \). Then according to Definition \ref{def:Bers-marking}, by the assumption of the lemma, \( A(\gamma) \) is \((w, r)\)-isolated at \( i_0 \) where the non-annular bounded combinatorics domain on both sides is \( Z \). So by Lemma \ref{lem:comparison-local} we have

\begin{equation}
\min\{\ell_\gamma(g(t_0^-)), \ell_\gamma(g(t_0^+))\} \geq \omega(L_S).
\end{equation}

Let \( B = B(D) \) be the constant from Lemma \ref{lem:comparison-local}. By the choice of \( t_0^- \) and \( t_0^+ \), \( |t_0^+ - t_0^-| = 2s \). Then by Corollary \ref{cor:global-control} there is \( \bar{\epsilon} \leq L_S \) depending
only on $s$ and $\omega(L_S)$ such that if
\[
\inf_{t \in [t_0^-, t_0^+]} \ell_\gamma(g(t)) \leq \bar{\epsilon}
\]
then
\[(6.10) \quad d_\gamma(\mu(g(t_0^-)), \mu(g(t_0^+))) > R' + 2M_2 + B + 8.\]

Let $\mu_1$ and $\mu_2$ be marking slices of $H(\mu^-, \mu^+)$ such that $\text{base}(\mu_1) = \rho(i_0 - w)$ and $\text{base}(\mu_2) = \rho(i_0 + w)$. Then $d_\gamma(\mu_1, \mu_2) + \text{diam}_\gamma(\mu_1) + \text{diam}_\gamma(\mu_2) \geq d_\gamma(\rho(i_0 - w), \rho(i_0 + w))$. Further $A(\gamma)$ is $(w, r)$-isolated at $i_0$ so again by Lemma 6.5\[6.7],
\[
d_\gamma(\mu_1, \mu_2) \geq d_\gamma(\mu(g(t_0^-)), \mu(g(t_0^+))) - B - 4.
\]
Then by (6.10) we have
\[(6.11) \quad d_\gamma(\mu_1, \mu_2) > R' + 2M_2 + 4.\]

Since $m' \leq i_0 - w \leq i_0 + w \leq n'$ by no backtracking (6.1) we have that $d_\gamma(\rho(m'), \rho(n')) \geq d_\gamma(\rho(i_0 - w), \rho(i_0 + w)) - 2M$. Further $\rho(i_0 - w) \subset \mu_1, \rho(i_0 + w) \subset \mu_2$, so
\[
d_\gamma(\rho(m'), \rho(n')) \geq d_\gamma(\mu_1, \mu_2) - 2M - \text{diam}_\gamma(\mu_1) - \text{diam}_\gamma(\mu_2).
\]
Thus (6.11) implies that $d_\gamma(\rho(m'), \rho(n')) > R'$. But this contradicts the assumption of the lemma that the $\gamma$ annular coefficient is bounded above by $R'$. The lower bound $\bar{\epsilon}$ for the length of $\gamma$ at $g(t_0)$ follows form this contradiction.

Moreover by Proposition 5.25 \[5.25], $\bigcup_{i \in [m' + w, n' - w]} N(i)$ covers $[a', b']$. Thus $\ell_\gamma(g(t)) > \bar{\epsilon}$ for any $t \in [a', b']$. This is the desired lower bound (2). Note that $\bar{\epsilon}$ depends only on $s$ and is uniform along $[a', b']$ so $\bar{\epsilon}$ is uniform over $[a', b']$ and does not depend on the value of the parameter $t_0$.

We proceed to establish the rough upper bound (1). We choose a hyperbolic surface $x \in V_{L_S}(\rho(i_0))$ as follows: Pick $\alpha \in \rho(i_0) - \partial Z$ and let $\ell_\alpha(x) = L_S$ for every $\alpha' \in \rho(i_0) - \alpha$. If $\ell_\alpha(g(t_0)) \geq \frac{3L_S}{4}$, let $\ell_\alpha(x) = \frac{L_S}{2}$. Otherwise, $\ell_\alpha(g(t_0)) < \frac{3L_S}{4}$, let $\ell_\alpha(x) = L_S$. Observe that
(a) $\rho(i_0)$ is a Bers marking of $x$.
(b) $d_{\text{WP}}(g(t_0), x) \geq d_0$.
(c) Let $d$ be the function in Corollary 3.5, let $d_0 = \min\{d(L_S, \frac{L_S}{4}), d(\frac{3L_S}{4}, \frac{L_S}{4})\}$. Then we have
\[d_{\text{WP}}(g(t_0), x) \geq d_0.
\]
(d) Finally, observe that by changing the twist parameters about the curves in $\rho(i_0)$ we can obtain $x$ such that $\mu_x$ is equal to a marking slice of any hierarchy $H(\mu^-, \mu^+)$ and $\text{base}(\mu_x) = \rho(i_0)$. Here $\mu^-$ and $\mu^+$ are the end points of $\rho$. There may be several such surfaces and we choose one.
(e) It follows from (a) that $x$ has a definite positive injectivity radius $\text{inj}(x)$ only depending on $L_S$.\]
Let \([x, g(t_0)]\) be the WP geodesic connecting \(x\) to \(g(t_0)\). Let \(u : [0, d] \rightarrow \text{Teich}(S)\) be the parametrization of \([x, g(t_0)]\) by arc-length such that \(u(0) = x\) and \(u(d) = g(t_0)\). Then by the \(D\)–fellow traveling of \(\rho\) and \(Q(g)\) the length of \(u\) is bounded by \(d_1 = K_{WP}D + C_{WP}\). Thus \(d \leq d_1\). On the other hand, by (b) \(d \geq d_0\).

To get a rough upper bound for lengths of boundary components of \(Z\) at \(g(t_0)\) first we establish a lower bound for the length of every \(\gamma \notin \partial Z\) along \(u\). Then the rough upper bound will follow from a compactness argument.

Let \(D' = K_{WP}d_1 + C_{WP}\), then \(D'\) bounds form above the distance between \(Q(x) = Q(u(0))\) and \(Q(u(t))\) for any \(t \in [0, d]\).

In the rest of the proof let \(w = w(D, D', R)\). Let \(\bar{e} = \bar{e}(D, R)\) be the lower bound we established earlier for \(\ell_{\gamma}(g(t))\) for each \(\gamma \notin \partial Z\) and every \(t \in [a' + s, b' - s]\). Here we make the following two choices:

(i) Let \(I^- = [t_0^-, t_0]\) and \(I^+ = [t_0, t_0^+]\). By (6.9), \(\ell_{\gamma}(g(t_0^-)) \geq \omega(L_S)\). Thus \(\sup_{t \in I^-} \ell_{\gamma}(g(t)) \geq \omega(L_S)\). Moreover, \(|I^-| = s\). Then Corollary 4.12 applied to \(g|_{I^-}\) implies that there is \(N_0\) depending only on \(s\) and \(\omega(L_S)\) such that

\[
d_\gamma(\mu(g(t_0^-)), \mu(g(t_0))) > N_0
\]

then

\[
\inf_{t \in I^-} \ell_{\gamma}(g(t)) \leq \bar{e}.
\]

Similarly, \(\sup_{t \in I^+} \ell_{\gamma}(g(t)) \geq \omega(L_S)\) and \(|I^+| = s\). So the corollary applied to \(g|_{I^+}\) implies that if \(d_\gamma(\mu(g(t_0^+)), \mu(g(t_0))) > N_0\) then \(\inf_{t \in I^+} \ell_{\gamma}(g(t)) \leq \bar{e}\).

(ii) Fix \(\gamma \notin \partial Z\). By the lower bound (2), \(\ell_{\gamma}(g(t_0)) \geq \bar{e}\) and by (d) \(\ell_{\gamma}(x) \geq 2\inf(x)\). The length of \(u\) is at least \(d_0\). Let \(e = \min\{\inf(x), \frac{\bar{e}}{2}\}\). Then Corollary 3.5 implies that there is \(s_0\) with \(2s_0 < d_0 < \bar{e}\) with the property that if \(r \in [d - s_0, d]\) then \(\ell_{\gamma}(u(r)) \geq e\), and if \(r \in [0, s_0]\) then \(\ell_{\gamma}(u(r)) \geq e\).
By what we just said $\sup_{r \in [0,d]} \ell_{\gamma}(u(r)) \geq e$. Then Corollary 4.13 applied to $u$ implies that there is $\epsilon_2 < e$ such that if
\begin{equation}
\inf_{r \in [s_0, d - s_0]} \ell_{\gamma}(u(r)) \leq \epsilon_2
\end{equation}
then
\begin{equation}
d_{\gamma}(\mu_x, \mu(g(t_0))) > N_0 + R' + 2M_2 + B
\end{equation}
In what follows we prove the lower bound
\[ \inf_{t \in [0,d]} \ell_{\gamma}(u(t)) > \epsilon_2 \]
for every $\gamma \notin \partial Z$, where $\epsilon_2$ is the constant we chose in (ii). Indeed, we rule out the possibility that $\inf_{r \in [0,d]} \ell_{\gamma}(u(t)) \leq \epsilon_2$ for any $\gamma \notin \partial Z$.

The proof is by contradiction. First note that $\mu_x$ is a Bers marking of $u(0) = x$ and $\mu(g(t_0))$ is a Bers marking of $u(d) = g(t_0)$. Moreover, by (6.12) there is a point $y \in u([s_0, d - s_0])$ such that $\gamma \in Q(y)$.

Since the length of $u$ is less than $d_1$ we have $d(Q(y), \rho(i_0)) \leq D'$. Recall that $D' = K_{WP}d_1 + C_{WP}$.

Let $\mu_1$ and $\mu_2$, as before, be marking slices of $H(\mu^-, \mu^+)$ with base($\mu_1$) = $\rho(i_0 - w)$ and base($\mu_2$) = $\rho(i_0 + w)$. Then as we saw in the proof of Lemma 6.5 there is $B := B(D)$ such that
\begin{equation}
d_{\gamma}(\mu_1, \mu(g(t^-_0))) \leq B.
\end{equation}
By (c) base($\mu_x$) = $\rho(i_0)$. Thus (6.1) and the assumption of the lemma that the annular coefficients are bonded over $[m', n']$ imply that
\begin{equation}
d_{\gamma}(\mu_x, \mu_1) \leq R' + 2M_2
\end{equation}
(6.13), (6.14) and (6.15) combined by the triangle inequality give us
\[ d_{\gamma}(\mu(g(t_0)), \mu(g(t^-_0))) > N_0 \]
So the choice of $N_0$ in (i) implies that $\inf_{t \in [t^-, t_0]} \ell_{\gamma}(g(t)) \leq \bar{c}$. Moreover, $A(\gamma)$ is $(w, D')$–isolated at $i_0$ where the isolating domain on both sides is $Z$. Then by Lemma 6.5 (6.7), $d_{\gamma}(\mu_2, \mu_1) \geq d_{\gamma}(\mu(g(t^-_0)), \mu(g(t^+_0))) - B - 4$. So by (6.10) we have
\[ d_{\gamma}(\mu_2, \mu_1) > R' + 2M_2 + 4 \]
Recall that $\rho(i_0 - w) \subset \mu_1$ and $\rho(i_0 + w) \subset \mu_2$. Then since $m' < i_0 - w < i_0 + w < n'$ no backtracking (6.1) and the above bound imply that $d_{\gamma}(\rho(m'), \rho(n')) > R'$. This contradicts the assumption of the lemma that the $\gamma$ annular coefficient over $[m', n']$ is bounded above by $R'$.

Note that $\epsilon_2$ does not depend on the value of the parameters $i_0 \in [m' + w, n' - w]$ and $t_0 \in N(i_0)$, therefore it is uniform along $[m' + w, n' - w]$. We will finish establishing of the upper bound (2) by a compactness argument. Let $u_n : [0, d_n] \rightarrow \text{Teich}(S)$ be a sequence of WP geodesic segments parametrized by arc-length with $d_0 \leq d \leq d_1$. Let subsurfaces $Z_n$ be such that $\inf_{t \in [0,d_n]} \ell_{\gamma}(u_n(t)) \geq \epsilon_2$ for every $\gamma \notin \partial Z_n$. Moreover, let $t_n^* \in [0,d_n]$
and $\hat{\alpha}_n \in \partial Z_n$ be such that $\ell_{\hat{\alpha}_n}(u_n(t^n_n)) \to \infty$ as $n \to \infty$. After possibly passing to a subsequence and remarking (applying elements of the mapping class group) we may assume that there is a subsurface $Z$ such that $\inf_{t \in [0,d_n]} \ell_\gamma(u_n(t)) \geq \epsilon_2$ for every $\gamma \notin \partial Z$, $\hat{\alpha} \in \partial Z$ and a sequence $t^n_n$ such that $\ell_\alpha(u_n(t^n_n)) \to \infty$ as $n \to \infty$. We proceed to get a contradiction. Let $\hat{u} : [0,d] \to \text{Teich}(S)$ be the geodesic limit of $u_n$’s as in Theorem 4.5 (Geodesic Limit Theorem). Let the partition $0 = t_0 < t_1 < \ldots < t_{k+1} = T$, simplices $\sigma_0, \ldots, \sigma_{k+1}$, simplex $\hat{\tau}$ and the elements of mapping class group $\varphi_{i,n} = T_{i,n} \circ \ldots \circ T_{i,n} \circ \psi_n$ be as in the theorem. We claim that

- $\ell_\alpha(\psi_n(x)) = \ell_\alpha(x)$ for every $\alpha \in \partial Z$,
- $\sigma_i \subseteq \partial Z$ for $i = 1, \ldots, k + 1$.

In the proof of Geodesic Limit Theorem $\psi_n$ is applied to keep $u_n(0)$ in a compact subset of Teichmüller space. Now since $\ell_\alpha(u_n(0)) = L_S$ for every $\alpha \in \partial Z$, we may choose each $\psi_n$ so that is supported on $S \setminus \partial Z$ i.e. is identity in a regular neighborhood of $\partial Z$ and $\psi_n \circ u_n(0)$ be in a compact subset of Teichmüller space where injectivity radius of surfaces is at least $L_S$. We have the first bullet. We proceed to prove the second bullet inductively. In the Geodesic Limit Theorem for each $i = 1, \ldots, k + 1$, $\sigma_i$ is the multi-curve determining the stratum that the limit of $\varphi_{i-1,n} \circ u_n|[t_{i-1},d]$ after possibly passing to a subsequence intersects. By the lower bound on the length of each $\gamma \notin \partial Z$ along $u_n([0,d])$ and since $\psi_n$ is supported on $S \setminus \partial Z$, the limit of $\psi_n \circ u_n|[0,d]$ intersects the stratum of a multi-curve $\sigma_1 \subseteq \partial Z$. Now let $i > 1$. Suppose that $\sigma_j \subseteq \partial Z$ for every $0 \leq j \leq i$. Then $T_{j,n} \in \text{tw}(\sigma_j - \hat{\tau})$ for each $j = 1, \ldots, i$, their composition does not change the isotopy class of any curve in $\partial Z$, so $\varphi_{i,n}$ does not change the isotopy class of every curve in $\partial Z$. Furthermore, the length of each $\gamma \notin \partial Z$ along $u_n([t_i,d])$ is bounded below. So we may conclude that the limit of $\varphi_{i,n} \circ u_n|[t_i,d]$ as $n \to \infty$ intersects the stratum of a multi-curve $\sigma_{i+1} \subseteq \partial Z$. 

**Figure 8.** $U_{L_S}(\rho(i_0))$ is the Bers region of $\rho(i_0)$ in $\text{Teich}(S)$. $g$ intersects this region. By (b) there is $x$ in $U_{L_S}(\rho(i_0))$ so that $d_{\text{WP}}(x,g(t_0)) > d_0$. Thus the length of the WP geodesic segment connecting $x$ to $g(t_0)$ is at least $d_0$. So we may choose $s_0$ with $2s_0 < d_0$ in (ii) independent of $t_0$ (uniformly along $g$). Then by the lower bounds for the injectivity radius of $g(t_0)$ (part (2) of the theorem) and $x$ (d) Corollary 4.13 applies to the geodesic segment $u$ and gives a uniform upper bound for the length of the curves in $\partial Z$ at $g(t_0)$.
The boundary curves of $Z$ are the only curves with possibly length 0 along $\hat{u}$ and $\hat{a}$ does not intersect any of them. So the length of $\hat{a}$ is bounded along $\hat{u}$ by some $l_0 > 0$. By the second bullet above, each $\varphi_{i,n}$ is the composition of $\psi_n$ and powers of Dehn twists about curves in $\partial Z$. So the isotopy class of each curve in $\partial Z$ is preserved by $\varphi_{i,n}$. After possibly passing to a subsequence we may assume that $t^*_n \in [t_i, t_{i+1}]$ for all $n$. Then by the third part of the Geodesic Limit Theorem, $\varphi_{i,n} \circ u_n(t^*_n) \to \hat{u}(t^*)$ as $n \to \infty$, so $\ell_{\hat{a}}(u_n(t^*_n)) = \ell_{\hat{a}}(\varphi_{i,n} \circ u_n(t^*_n))$. So for $n$ sufficiently large $\ell_{\hat{a}}(u_n(t^*_n)) \leq 2l_0$. This contradicts the assumption that $\ell_{\hat{a}}(u_n(t^*_n)) \to \infty$ as $n \to \infty$.

\[ \square \]

**Proposition 6.7.** Given $A, R, R' > 0$, there are constants $w = w(A, R)$ and $\bar{\epsilon} = \bar{\epsilon}(A, R, R')$ with the following properties. Let $g : [a, b] \to \overline{\text{Teich}(S)}$ be a WP geodesic segment parametrized by arc-length with $A$–narrow end invariant $(\nu, \nu^+)$ and $\rho : [m, n] \to P(S)$ be a hierarchy path between $\nu^-$ and $\nu^+$. Suppose that $S$ has $(R, R')$–bounded combinatorics over $[m', n'] \subseteq [m, n]$ and $m' - n' > 2w$, then

$$\text{inj}(g(t)) \geq \frac{\bar{\epsilon}}{2}$$

for every $t \in [a', b']$, where $a' \in N(m' + w)$ and $b' \in N(n' - w)$.

**Proof.** Let $D = D(A)$ be the fellow traveling distance of $g$ and the hierarchy path between $(\nu^-, \nu^+)$. By the assumption that $S$ is the subsurface with $(R, R')$–bounded combinatorics Lemma 6.6 implies that for every $\gamma \in C(S)$ we have $\ell_{\gamma}(g(t)) \geq \bar{\epsilon}$ for every $t \in [a', b']$. So $\text{inj}(g(t)) \geq \frac{\bar{\epsilon}}{2}$ on this interval. \[ \square \]

**Remark 6.8.** Compare the above corollary with the main result of [BMM11] which asserts that given $R > 0$ there is an $\epsilon > 0$ such that if the end invariant of a WP geodesic $g$ is $(R, R')$–bounded combinatorics then $g$ stays in the $\epsilon$–think part of Teichm"uller space.

**Proof of Theorem 6.1.** To get arbitrary short boundary curves we sharpen the rough upper bound obtained in Lemma 6.6. This would be done in the following lemma. Here we use convexity of length-functions along WP geodesics to get an arbitrary short curve in the boundary of $Z$ over an arbitrary long interval. Then using the fact that over this interval the geodesic is close to a stratum an inductive argument gives us the upper bound for the length of all of the curves in the boundary of $Z$.

**Lemma 6.9.** Given $\ell, \bar{\epsilon}$ and $\epsilon \leq \bar{\epsilon}$, there is $\bar{s} > 0$ with the following property. Let $g : [a', b'] \to \overline{\text{Teich}(S)}$ be a WP geodesic such that the length-function bounds

1. $\ell_\alpha(g(t)) \leq \ell$ for every $\alpha \in \partial Z$, and
2. $\ell_{\gamma}(g(t)) \geq \bar{\epsilon}$ for every $\gamma \notin \partial Z$
hold for every $t \in [a', b']$. Furthermore, assume that $b' - a' > 2s$. Then for every $\alpha \in \partial Z$ we have
\begin{equation}
(6.16) \quad \ell_\alpha(g(t)) \leq \epsilon
\end{equation}
for every $t \in [a' + \bar{s}, b' - \bar{s}]$.

The theorem follows from this lemma. Let $D = D(A)$ be the fellow traveling distance from Theorem 5.27. Then $Q(g|[a', b'])$ and $\rho|[m', n']$, $D$–fellow travel. Furthermore, $Z$ has $(R, R')$–bounded combinatorics over $[m', n']$. Then by Lemma 6.6 there are $l = l(D, R)$, $\bar{\epsilon} = \bar{\epsilon}(D, R')$ and $w = w(D, R)$ with the property that if $m' - n' \geq 2w$ we have

(1) \quad \ell_\alpha(g(t)) \leq l \text{ for every } \alpha \in \partial Z,
(2) \quad \ell_\gamma(g(t)) \geq \bar{\epsilon} \text{ for every } \gamma \notin \partial Z.

for every $t \in [a', b']$, where $a' \in N(m' + w) \text{ and } b' \in N(n' - w)$. Now let $\bar{w} = K\bar{s} + C$. Then $m' - n' > 2\bar{w}$ guarantees that $b' - a' > 2\bar{s}$. Then by the above lemma if $b' - a' > 2\bar{s}$ we have the asserted length-function bound of the theorem.

Proof of Lemma 6.9. The proof is by induction on $|\partial Z|$ the number of boundary components of $Z$. When $\partial Z = \emptyset$, the lemma holds vacuously and provides us with the base of induction.

Claim 6.10. Given $\epsilon < \bar{\epsilon}$, there is $T > 0$ such that if $[c, d] \subseteq [a', b']$ is a subinterval with $d - c > 2T$ then there is a curve $\alpha \in \partial Z$ such that
\begin{equation}
(6.17) \quad \ell_\alpha(g(t)) \leq \epsilon
\end{equation}
for some $t \in [c + T, d - T]$.

We show that if $g|[c, d]$ stays in the $\epsilon$–thick part of Teichmüller space then there is an upper bound $T = T(l, \epsilon)$ for the length of the interval $[c, d]$. Fix $\hat{\alpha} \in \partial Z$. $g|[c, d]$ is in the $\epsilon$–thick part, thus by Theorem 3.8 (3.1) there is $c(\epsilon) > 0$ for which the differential inequality
\[ \hat{\ell}_\hat{\alpha}(g(t)) \geq \epsilon c(\epsilon) \]
holds on the interval $[c, d]$. By (1) above $\ell_\hat{\alpha}(g(t)) \leq l$ for every $t \in [c, d]$. Thus by the Mean-value Theorem if $d - c > \frac{2T}{\epsilon\sqrt{2c(\epsilon)}}$ then there is $t^* \in [c, d]$ such that $|\hat{\ell}_\hat{\alpha}(g(t^*))| < \epsilon\sqrt{2c(\epsilon)}$. Integrating the above differential inequality we get
\begin{equation}
(6.17) \quad \ell_\hat{\alpha}(g(t)) \geq \ell_\hat{\alpha}(g(t^*)) + \hat{\ell}_\hat{\alpha}(g(t^*))(t - t^*) + \frac{1}{2} \epsilon c(\epsilon)(t - t^*)^2
\end{equation}
and $|\hat{\ell}_\hat{\alpha}(g(t^*))| > \epsilon$ and $|\hat{\ell}_\hat{\alpha}(g(t^*))| < \epsilon\sqrt{2c(\epsilon)}$. So for $\Delta$ the discriminant of the quadratic function on the right hand side of (6.17) we have
\[ \Delta = (\hat{\ell}_\hat{\alpha}(g(t^*)) - 4\frac{\epsilon c(\epsilon)}{2}\hat{\ell}_\hat{\alpha}(g(t^*)) \leq (\epsilon\sqrt{2c(\epsilon)}/2)^2 - 4\frac{1}{2} \epsilon^2 c(\epsilon) = 0. \]
This guarantee that the quadratic function is positive on $R$. 

Claim 6.12. The pigeon-hole principle implies that there is a curve \( \alpha \) such that \( \alpha \subset \partial Z \) and indices \( i_1, i_2, i_3 \) with \( i_1 < i_2 < i_3 \) such that \( \ell_\alpha(g(t_{i_1})), \ell_\alpha(g(t_{i_2})) \) and \( \ell_\alpha(g(t_{i_3})) \) are less than \( \epsilon \). Then by the convexity of the \( \alpha \)-length-function along \( g \), \( \ell_\alpha(g(t)) \leq \epsilon \) on \( [t_{i_1}, t_{i_3}] \). Moreover \( t_{i_3} - t_{i_1} \geq |I_{i_2}| \geq L \). So \( [t_{i_1}, t_{i_3}] \) is the claimed subinterval.

As before \( \ell_\alpha(g(t)) < l \) for every \( t \in [c, d] \). Then by the completing square we get \( l \geq (\sqrt{c(e)}(t - t^*) - \frac{\ell_\alpha(g(t^*))}{\sqrt{2c(e)}})^2 - \frac{\Delta}{2c(e)} \). So using \( |\ell_\alpha(g(t^*))| < \epsilon \sqrt{2c(e)} \) and \( \Delta \leq 0 \) we get \( |t - t^*| \leq \frac{\sqrt{\Delta}}{\sqrt{c(e)}} + \frac{\sqrt{2\ell_\alpha(g(t^*))}}{\sqrt{c(e)}} \). Consequently \( b' - a' \leq \max\left\{ \frac{2l}{\epsilon \sqrt{2c(e)}}, \frac{\sqrt{\Delta}}{\sqrt{c(e)}} + \frac{\sqrt{2\ell_\alpha(g(t^*))}}{\sqrt{c(e)}} \right\} \). So the claim holds for \( T := T(l, \epsilon) = \max\left\{ \frac{2l}{\epsilon \sqrt{2c(e)}}, \frac{\sqrt{\Delta}}{\sqrt{c(e)}} + \frac{\sqrt{2\ell_\alpha(g(t^*))}}{\sqrt{c(e)}} \right\} \).

Remark 6.11. For \( \epsilon \) sufficiently small, \( T(l, \epsilon) = \frac{2l}{\epsilon \sqrt{2c(e)}} \).

The contrapositive of what we just proved is that if \( d - c > T \), then there are curves which get shorter than \( \epsilon \) at some time along \( g|_{[c, d]} \). Moreover, \( [c, d] \subseteq [a, b] \) so by the bound (2) component curves of \( \partial Z \) are the only curves which can get shorter than \( \epsilon < \epsilon \) along \( g|_{[c, d]} \). Thus we conclude that if \( d - c > T \), then there is a time \( t \in [c, d] \) and a curve \( \alpha \in \partial Z \) such that \( \ell_\alpha(g(t)) \leq \epsilon \) as was desired.

Claim 6.12. Given \( L > 0 \), if \( b' - a' > (2|\partial Z| + 1)L + 2T \) then there is a curve \( \alpha \in \partial Z \) such that \( \ell_\alpha(g(t)) \leq \epsilon \) on a subinterval of \( [a', b'] \) of length at least \( L \).

Let the intervals \( I_1, \ldots, I_{2|\partial Z|+1} \) with \( |I_i| = 1 \) for \( i = 1, \ldots, 2|\partial Z| + 1 \) consist of a partition of the interval \( [a' + T, b' + T] \) into \( 2|\partial Z| + 1 \) subintervals. Claim 6.10 applied to each interval \( [\min I_i - T, \max I_i + T] \) implies that there is a time \( t_i \in I_i \) at which a component curve of \( \partial Z \) is shorter than \( \epsilon \). Now the pigeon-hole principle implies that there is a curve \( \alpha \in \partial Z \) and indices \( i_1, i_2, i_3 \) with \( i_1 < i_2 < i_3 \) such that \( \ell_\alpha(g(t_{i_1})), \ell_\alpha(g(t_{i_2})) \) and \( \ell_\alpha(g(t_{i_3})) \) are less than \( \epsilon \). Then by the convexity of the \( \alpha \)-length-function along \( g \), \( \ell_\alpha(g(t)) \leq \epsilon \) on \( [t_{i_1}, t_{i_3}] \). Moreover \( t_{i_3} - t_{i_1} \geq |I_{i_2}| \geq L \). So \( [t_{i_1}, t_{i_3}] \) is the claimed subinterval.
Let \(\epsilon' < \min\{\epsilon, \epsilon\}\), which will be determined. Let \(L = \bar{s}' + 2\sqrt{2\pi\epsilon'}\), where \(\bar{s}'\) will be determined too.

Now by Claim 6.12 if \(b' - a' > (2|\partial Z| + 1)L + 2T\), then there is a curve \(\alpha \in \partial Z\) such that

\[
\ell_\alpha(g(t)) \leq \epsilon'
\]

on an interval of length at least \(L\). Denote this interval by \([t^-_\alpha, t^+_\alpha]\). Let \(x, y\) be the nearest points to \(g(t^-_\alpha)\) and \(g(t^+_\alpha)\) on the \(\alpha\)-stratum, respectively. Since the \(\alpha\)-stratum is convex, there is a WP geodesic segment \(g' : [a', b'] \to S(\alpha)\) parametrized by arc-length connecting \(x\) to \(y\) (see Figure 9).

Since \(\ell_\alpha(g(t^-_\alpha)) \leq \epsilon'\) and \(\ell_\alpha(g(t^+_\alpha)) \leq \epsilon'\), Proposition 3.6 gives us the upper bounds \(d_{WP}(g(t^-_\alpha), x) \leq \sqrt{2\pi\epsilon}\) and \(d_{WP}(g(t^+_\alpha), y) \leq \sqrt{2\pi\epsilon}\). Moreover, \(\text{Teich}(\mathcal{S})\) equipped with the WP metric is a CAT(0) space. Therefore by the CAT(0) comparison the distance between any point on \(g([t^-_\alpha, t^+_\alpha])\) and its nearest point on \(g'\) is less than \(\sqrt{2\pi\epsilon}\). Here we choose \(\epsilon'\) such that:

- \(\sqrt{2\pi\epsilon} \leq \min\{d(\epsilon, \frac{\epsilon}{2}), d(2l, l), d(\epsilon, \frac{\epsilon}{2})\}\).

Here \(d\) is the function from Corollary 3.5. By the choice of \(\epsilon'\) we have the following length-function bounds

\[
\begin{align*}
(1') & \quad \ell_\alpha(g'(t)) \leq 2l \text{ for every } \alpha' \in \partial Z' \\
(2') & \quad \ell_{\gamma'}(g'(t')) \geq \frac{\epsilon}{2} \text{ for every } \gamma' \notin \partial Z'
\end{align*}
\]

for every \(t \in [a'', b'']\). Here \(Z' = Z \cup A(\alpha)\).

Now since \(Z'\) is a large subsurface, by the above two length-function bounds the assumption of the induction for the geodesic \(g' : [a'', b''] \to \text{Teich}(\mathcal{S})\) implies that there is \(s'\) such that if \(b'' - a'' > 2s'\) then for every \(\alpha' \in \partial Z' = \partial Z - \{\alpha\}\)

\[
\ell_{\alpha'}(g'(t)) \leq \frac{\epsilon}{2}
\]

for every \(t \in [a'' + s', b'' - s']\).

Let \(\bar{s}' = s' + 2\sqrt{2\pi\epsilon}\). Given \(t \in [t^-_\alpha + \bar{s}, t^+_\alpha - \bar{s}]\), let \(t' \in [a'', b'']\) be the nearest point to \(g(t)\) on \(g'|_{[a'', b']}\). Since \(d_{WP}(g(t^-_\alpha), g'(a'')) \leq \sqrt{2\pi\epsilon}\) and \(d_{WP}(g(t^+_\alpha), g'(b'')) \leq \sqrt{2\pi\epsilon}\), the CAT(0) comparison implies that \(d_{WP}(g(t), g'(t')) \leq \sqrt{2\pi\epsilon}\). Furthermore by the triangle inequality \(d_{WP}(g'(t'), g'(a'')) \geq d_{WP}(g(t), g(t^-_\alpha)) - d_{WP}(g(t), g'(t')) - d_{WP}(g(t^-_\alpha), g'(a''))\), so \(t'' - a'' \geq (t - t^-_\alpha) - 2\sqrt{2\pi\epsilon} \geq s'\) and similarly \(b'' - t' \geq (t^+_\alpha - t) - 2\sqrt{2\pi\epsilon} \geq s'\). Thus \(t' \in [a'' + s', b'' - s']\). Then by the bound (6.19) and the choice of \(\epsilon'\), Corollary 3.5 implies that for every curve \(\alpha' \in \partial Z'\), \(\ell_{\alpha'}(g(t)) \leq \epsilon\) for every \(t \in [a + \bar{s}, b' - \bar{s}]\). Moreover \([a' + \bar{s}, b' - \bar{s}] \subset [t^-_\alpha, t^+_\alpha]\), so by (6.18), \(\ell_\alpha(g(t)) \leq \epsilon\) on this interval. We established the bound for the length of all of the component curves of \(\partial Z\) on the interval \([a' + \bar{s}, b' - \bar{s}]\). This finishes the step of induction.

\[\square\]
7. Laminations with prescribed subsurface coefficients

Our purpose in this section is to construct pair of partial markings or laminations with a given list of subsurface coefficients. More precisely, given a sequence of integers \( \{e_i\} \), we will construct a pair of laminations/markings \((\mu_I, \mu_T)\) such that there is a list of large subsurfaces \( \{Z_i\} \) with \( d_{Z_i}(\mu_I, \mu_T) \sim K_1, C_1 \) \( |e_i| \) where \( K_1, C_1 \) depend on certain initial choices. Furthermore, there are constants \( m \) and \( m' \) depending on the initial choices such the subsurface coefficient of any subsurface which is not in the list of \( Z_i \)'s is bounded above by \( m \), and all annular subsurface coefficients are bounded above by \( m' \). This is a kind of symbolic coding for laminations using subsurface coefficients. Here we restrict the set of subsurface with a big subsurface coefficient. This can be thought of as continued fraction expansions with a specific pattern of decimal numbers. We will use these constructions in §8 to provide examples of WP geodesics with certain behavior in the moduli space.

The construction uses compositions of powers of (partial) pseudo-Anosov maps. A partial pseudo-Anosov map \( f \) is a reducible element of mapping class group which preserves the isotopy class of a collection of curves \( \{\delta_j\} \) on the surface \( S \) and does not rearrange connected components of \( S \setminus \{\delta_j\} \). Moreover, the restriction of \( f \) to each of the connected components of \( S \setminus \{\delta_j\} \) is a pseudo-Anosov maps. We say that this partial pseudo-Anosov map is supported on \( S \setminus \sigma \).

We start by some background about the action of (partial) pseudo-Anosov maps on the curve complex of a surface and its subsurfaces also the space of projective measured laminations.

The following proposition is a straightforward consequence of Proposition 4.6 in [MM99].

**Proposition 7.1.** Let \( f \) be a (partial) pseudo-Anosov map supported on a subsurface \( Y \subseteq S \). There is \( \tau_f > 0 \) such that for every \( \alpha \in C(Y) \) and every integer \( e \) we have

\[
d_Y(\alpha, f^e \alpha) \geq \tau_f |e|.
\]

**Lemma 7.2.** Let \( f \) be a (partial) pseudo-Anosov map supported on a subsurface \( X \). There is a constant \( \bar{\tau}_f > 0 \) such that for every \( \alpha \in C(X) \),

\[
\limsup_{n \to \infty} \frac{d_X(\alpha, f^n \alpha)}{n} = \bar{\tau}_f.
\]

**Proof.** Using the fact that \( f \) is an isometry of \( C(X) \) and the triangle inequality we have that for any positive integer \( n \), \( d_X(\alpha, f^n \alpha) \leq \sum_{i=0}^{n-1} d_X(f^i \alpha, f^{i+1} \alpha) \leq nd_X(\alpha, f \alpha) \). So \( \frac{d_X(\alpha, f^n \alpha)}{n} \leq d_X(\alpha, f \alpha) \). Thus \( \limsup_{n \to \infty} \frac{d_X(\alpha, f^n \alpha)}{n} \) is a finite number. Furthermore, for any \( \beta \in C(X) \) with \( \alpha \neq \beta \), by the triangle inequality we have \( d_X(f^n \beta, \alpha) \leq d_X(\beta, f^n \beta) \leq d_X(f^n \beta, \alpha) + d_X(\alpha, f^n \beta) \). Now diving both sides of each of the two inequalities by \( n \) and taking \( \limsup \) we see that \( \bar{\tau}_f \) is the same for \( \alpha \) and \( \beta \) and consequently independent of the choice of \( \alpha \). Finally, by Proposition 7.1, \( \bar{\tau}_f \geq \tau_f > 0 \). \( \square \)
We continue by reviewing some facts about the action of (partial) pseudo-Anosov maps on $\mathcal{PML}(S)$. We essentially follow exposé 11 of [PLP79] and §3 and appendix A of [Iva92]. Here we replace measured geodesic laminations with measured geodesic foliations used in these two references. The correspondence of measured foliations and measured geodesic lamination is explained in [CBS88]. Given a reducible element of the mapping class group by Theorem 11.7 of [PLP79] there is a finite collection of simple closed curves $\{\delta_j\}_{j=1}^m$ and subsurfaces $\{X_i\}_{i=1}^n$ such that $S\setminus\{\delta_j\}_j = \sqcup_i X_i$ and the restriction of $f$ to each $X_i$ is either pseudo-Anosov or periodic. For a partial pseudo-Anosov map we further suppose that the map does not rearrange connected components of $S\setminus\{\delta_j\}_j$ and its restriction to each connected component is a pseudo-Anosov map. Then as in exposé 11 of [PLP79] for each $i$, there are measured laminations $L_i^\pm = (\lambda_i^\pm, m_i^\pm)$ the attracting and repelling measured laminations of $f|X_i$ and real numbers $s_i > 1$ such that for each $i$

- $f(\lambda_i^+) = \lambda_i^+$ and $f m_i^+ \geq s_i m_i^+$,
- $f(\lambda_i^-) = \lambda_i^-$ and $f m_i^- \leq s_i^{-1} m_i^-$. 

Moreover, both $\lambda_i^\pm$ are uniquely ergodic laminations on $X_i$. In particular the support of $\lambda_i^\pm$ is minimal filling on $X_i$.

Let $i : \mathcal{ML}(S) \times \mathcal{ML}(S) \to \mathbb{R}$ be the intersection number defined for any pair of measured laminations (see §2.7 of [Iva92]). Given a complete hyperbolic metric on $S$, let $\ell : \mathcal{ML}(S) \to \mathbb{R}_{\geq 0}$ be the length-function (for the definition see [Bon01]). Note that both $i$ and $\ell$ are homogeneous of degree one in each of their variables. For example, $i(sL, L') = si(L, L')$.

As in Appendix A of [Iva92] (see also §3 of the book) let $\Delta_f^+$ be the set of projective classes of measured geodesic laminations $\{\sum_{i=1}^n t_i L_i^+ : t_i \geq 0, \sum_i t_i > 0\}$. Also let $\Psi_f^+$ be the set of projective classes of measured geodesic laminations $\{L \neq 0 : i(L, L_i^+) = 0 \text{ for all } i\}$. Note that $\Delta_f^+ \subseteq \Psi_f^+$.

Similarly, define the sets $\Delta_f^-$ and $\Psi_f^-$ and note that $\Delta_f^- \subseteq \Psi_f^-$. 

Define the functions $L^\pm : \mathcal{PML}(S) \to \mathbb{R}_{\geq 0}$ by $L^\pm([L]) = \frac{\ell(L)}{i(L, L_i^+)} + \sum_{j=1}^m i(L, \delta_j)$. Note that $(L^+)^{-1}(0) = \Delta^-$ and $(L^-)^{-1}(0) = \Delta^+$. 

In §3 of [Iva92] is shown that given a compact subset $K \subset \mathcal{PML}(S) \setminus \Psi^+$, there are constants $c_1, d_1, d_1', c_2, d_2, d_2'$, depending only on $f$ and $K$ such that any integer $n \geq 1$:

\[ \frac{i(f^n(L), \delta_i)}{\ell(f^n(L))} \leq \frac{d_1'}{c_1 n - d_1} \text{ and } \frac{i(f^n(L), L_i^+)}{\ell(f^n(L))} \leq \frac{s_i d_2'}{c_2 n - d_2}. \]

Similarly, given a compact subset $K \subset \mathcal{PML}(S) \setminus \Psi^-$ there are constants $c_1, d_1, d_1'$ and $c_2, d_2, d_2'$, depending only on $f$ and $K$ such that for any $n \geq 1$,

\[ \frac{i(f^n(L), \delta_i)}{\ell(f^n(L))} \leq \frac{d_1'}{c_1 n - d_1} \text{ and } \frac{i(f^n(L), L_i^-)}{\ell(f^n(L))} \leq \frac{s_i d_2'}{c_2 n - d_2}. \]

Consequently, for any integer $n \geq 1$, $L^+(f^n([L])) \leq \frac{s_i n d_2'}{c_2 n - d_2} + \frac{d_1'}{c_2 n - d_2}$. Similarly, for any integer $n \geq 1$, $L^-(f^{-n}([L])) \leq \frac{s_i n d_2'}{c_2 n - d_2} + \frac{d_1'}{c_2 n - d_2}$.

Using the above bounds for the functions $L^\pm$ and the fact that they are continuous one may easily verify that the action of $f$ on $\mathcal{PML}(S) \setminus \Psi_f^+ \cup \Psi_f^-$ has a compact fundamental domain, denoted by $K_f$. Furthermore, Ivanov
proves in Theorem A.2 of Appendix A of [Iva92] (see also Theorem 3.5 in §3 of the book) that

**Theorem 7.3.** Let $U$ be an open subset and $K$ be a compact subset of $\mathcal{PLM}(S)$. If $\Delta_f^+ \subset U$ and $K \subset \mathcal{PML}(S) \setminus \Psi_f^+$, then there is $N > 0$ such that $f^n(K) \subset U$ for any $n \geq N$. If $\Delta_f^− \subset U$ and $K \subset \mathcal{PML}(S) \setminus \Psi_f^-$, then there is $N > 0$ such that $f^{-n}(K) \subset U$ for any $n \geq N$.

We proceed to prove a lemma which will give us certain upper bounds for subsurface coefficients in §7.1 and 7.2 where we construct laminations with prescribed subsurface coefficients.

**Lemma 7.4.** Let $f$ be a partial pseudo-Anosov map supported on a large subsurface $X \subseteq S$. Given a compact subset $K \subset \mathcal{PML}(S) \setminus \Psi_f^- \cup \Psi_f^+$ there is a constant $m = m(f,K)$, depending only on $f$ and $K$, such that for every $\gamma \in K$, every subsurface $W \pitchfork \gamma$ which is neither $X$ nor an annulus with core curve a component of $\partial X$ and every integer $e$ we have

\[(7.1)\quad d_W(\gamma, f^e(\gamma)) \leq m\]

*Proof.* First suppose that $W$ is a non-annular subsurface. Since $X$ is a large subsurface either $W \pitchfork X$, $W \subseteq X$ or $W \supseteq X$.

We proceed to establish the bound (7.1) when $W \pitchfork X$ or $W \subseteq X$. In this situation there are component curves of $\partial W$ which overlap $X$.

In this proof let $K := K_f$ be the fundamental domain for the action of $f$ on $\mathcal{PML}(S) \setminus \Psi_f^- \cup \Psi_f^+$. Then applying an appropriate power of $f$ to $W$ we may assume that the projective class of all of the component curves of $\partial W$ which overlap $X$ are in $K$. Applying this power of $f$ to the subsurface coefficient in (7.1) we get the subsurface coefficient

\[(7.2)\quad d_W(f^{e_1}(\gamma), f^{e_2}(\gamma))\]

for some $e_1$ and $e_2$. So we need to bound (7.2) for any pair of integers $e_1$ and $e_2$.

Fix a complete hyperbolic metric on $S$. Realize all curves and laminations as geodesics in this metric. We claim that there is a constant $l_1 > 0$ and a positive integer $N_1$ depending only on $K$ and $f$ such that for any $\gamma \in K$ with $\gamma \pitchfork W$ and any $n \geq N_1$, $f^n(\gamma) \cap W$ has length less than $l_1$. We proceed to get the bound by essentially following the compactness argument given by Minsky in [Min00] (see also Theorem 3.9 of [KL08]). Suppose that the claim does not hold. Then after possibly passing to a subsequence we may assume that for each $n$ there is a subsurface $W_n$, a curve $\gamma_n \in K$ with $\gamma_n \pitchfork W_n$ such that the length of the collection of arcs $\alpha_n := f^n(\gamma_n) \cap W_n$ goes to $\infty$ as $n \to \infty$. Theorem 7.3 applied to the fundamental domain of the action of $f$, $K$ and arbitrary open subsets $U \supset \Delta_f^+$ implies that $[\alpha_n]$ (as $\alpha_n$ is equipped with the measure $i(\alpha_n,.)$) converges into the subset $\Delta_f^+$. Since $X$ is a large subsurface, $\Psi_f^+ = [\mathcal{L}^+]$, where $\mathcal{L}^+$ is the attracting lamination of $f|_X$. Note that $\lambda^+$ the support of $\mathcal{L}^+$ is minimal and fills $X$. 


Let $β_n$ be a component curve of $∂W_n$ which overlaps $X$ and consider the projective measured laminations $[β_n] ∈ K$ ($β_n$ is equipped with the measure $i(β_n, )$). $K$ is a compact subset of $PML(S) \setminus Ψ_f^− \cup Ψ_f^+$, so after possibly passing to a subsequence $[β_n]$ converge to a projective lamination $[L']$. Denote the support of $L'$ by $ξ$. For each $n$, $α_n$ is disjoint from $β_n$, so $ξ$, the support of $L'$, is disjoint from $λ^+$, the support of $L$'. Furthermore, $λ^+ \cup ∂X$ fills $S$, so $ξ$ is a sublamination of $λ^+ \cup ∂X$. But then $[L']$ is in $Ψ_f^+$, which contradicts the fact that $K$ misses $Ψ^+$. So we conclude that the integer $N_1$ and the bound $l_1$ exist.

By a similar argument there exist a constant $l_2 > 0$ and a positive integer $N_2$, depending only on $K$ and $f$, such that for any $γ ∈ K$ with $γ \cap W$ and any $n ≥ N_2$, $f^{−n}(γ) \cap W$ has length bounded above by $l_2$. Here after possibly passing to a subsequence $f^{−n}(γ)$ converges into $Ψ_f^−$ and the rest of argument would be similar to the above situation.

Furthermore, the set $∪_{e=−N_2}^{N_1} f^e(K)$ is a compact subset of $PML(S)$ and the length function is continuous on $PML(S)$. So there is an upper bound $l_3$ for the length of laminations in this set.

By the bounds we established above we conclude that for any $γ ∈ K$ the length of $f^{n}(γ) \cap W$ is bounded above by $l = \max\{l_1, l_2, l_3\}$. Here $l$ only depends on $f$ and $K$.

Since $X$ is a large subsurface each component of $∂W$ is either in $K$ (a compact subset of $PML(S)$) or $∂X$. Then since the length-functions are continuous on $PML(S)$ we have that the length of $∂W$ in the hyperbolic metric we fixed on $S$ is bounded above by an $L$ depending only on $K$ and the subsurface $X$. Furthermore, for any integer $e$ as we saw above the length of $f^e(γ)$ is bounded above by $l$. By the Collar lemma (§4.1 [Bus10]) upper bound on the length of two curves gives an upper bound on their intersection number. Then $i(f^e(γ), ∂W) ≤ i_0$ for some $i_0$. This implies that the number of arcs in $f^e(γ) \cap W$ is bounded above by $i_0$ as well.

By the definition of subsurface projection from [2] the length of the curve in $π_W(f^e(γ)) ⊂ C_0(W)$ corresponding to an arc of $f^e(γ) \cap W$ with length at most $l$ is bounded above by $2l$ plus the length of $∂W$. As we saw above the number of arcs in $f^e(γ) \cap W$ is bounded by $i_0$ and the length of $∂W$ is bounded above by $L$. Thus the length of $π_W(f^e(γ))$ is bounded by $i_0(2l+L)$.

Given $e_1, e_2$ let $γ_1 = f^{e_1}(γ)$ and $γ_2 = f^{e_2}(γ)$. As we saw above the length of $π_W(γ_1)$ and $π_W(γ_2)$ are bounded above by $i_0(2l+L)$. Then by the Collar lemma $i(γ_1, γ_2) ≤ i_1$ for some $i_1$. Further by Lemma 2.1 of [MM99] we have that $d_W(γ_1, γ_2) ≤ 2i(γ_1, γ_2) + 1$. So (7.2) is bounded above by $2i_1 + 1$.

Now we proceed to establish the bound (7.1) for non-annular subsurface $W ⊃ X$. Here $∂X$ has at least one component curve $β ∈ C(W)$, $γ \cap W$, so for any integer $e$, $f^e(γ) \cap W$. Then by the triangle inequality we have

$$d_W(f^e(γ), γ) ≤ d_W(f^e(γ), β) + d_W(β, γ).$$
Applying $f^{-e}$, $d_W(f^e\gamma,\gamma) = d_{f^{-e}W}(\gamma,\beta)$. Here we use the fact that since $\beta$ is a component curve of $\partial X$ and $f$ is supported on $X$, $f^{-e}(\beta) = \beta$. Lemma 2.1 of [MM99] we have $d_{f^{-e}W}(\gamma,\beta) \leq 2i(\gamma,\beta) + 1$. So the first subsurface coefficient above is bounded by $2i(\gamma,\beta) + 1$. Further by Lemma 2.1 of [MM99] the second subsurface coefficient is bounded above by $2i(\gamma,\beta) + 1$. Now $\beta,\gamma \in K$ and $K$ is a compact subset of $\mathcal{PML}(S)$. Then since the intersection number $i$ is a continuous function, $i(\gamma,\beta) \leq i_2$ for some $i_2$ depending only on $K$. Then (7.2) is bounded above by $2(2i_2 + 1)$.

Finally suppose that $W$ is an annular subsurface whose core curve is not a boundary curve of $X$. Applying an appropriate power of $f$ to $W$ we may assume that the projective class of the core curve of $W$ overlaps $X$ is in $K$ the fundamental domain of the action of $f$. Applying this power of $f$ to the subsurface coefficient in (7.1) we get the subsurface coefficient (7.2) for some $e_1$ and $e_2$. So we need to bound (7.2) for any pair of integers $e_1$ and $e_2$.

Denote the core curve of $W$ by $\beta$. Define the angle between two curves or laminations realized as geodesics in the metric we fixed on $S$ to be the minimum of the smaller angle between them at their intersection points. We proceed by a compactness argument similar to the one we gave to bound the length of $f^e(\gamma) \cap W$ to show that there exists a constant $\theta_1 > 0$ which bounds from below the angle between the curve $f^n(\gamma)$ and $\beta$ for any $n \geq 0$. If there is not such a $\theta_1$, there is a sequence of annular subsurfaces $W_n$ with core curve $\beta_n$ and a sequence of curves $\gamma_n \in K$ such that the angle between $\beta_n$ and $f^n(\gamma_n)$ goes to 0 as $n \to \infty$. Theorem 7.3 implies that $[f^n(\gamma_n)] \to [\mathcal{L}^+]$ as $n \to \infty$. Further since $K$ is compact after possibly passing to a subsequence, $[\beta_n] \to [\mathcal{L}']$ in $K$. Denote the support of $\mathcal{L}'$ by $\xi$. But then since the angle between $\beta_n$ and $f^n(\gamma_n)$ goes to 0, $\xi$ would be a sub-lamination of $\lambda^+ \cup \partial X$. Recall that $\mathcal{L}^+$ is the attracting lamination of $f|_X$ and its support $\lambda^+$ fills $X$. Further $X$ is a large subsurface, so $\lambda^+ \cup \partial X$ fills $S$. Consequently $[\mathcal{L}']$ is in $\Psi_f^\perp$. But this contradicts the fact that $K$ misses $\Psi_f^-$. Similarly we can show that the angle between $\beta$ and $f^{-n}(\gamma)$, $n \geq 0$ is bounded below by some $\theta_2 > 0$. Thus the angle between $f^e(\gamma)$ and $\beta$ is bounded below by some $\theta_0 := \min\{\theta_1, \theta_2\}$ for any integer $e$.

Having the lower bound $\theta_0$ for the angle, Lemma 2.6 of [KL08] applied to the curves $f^{e_1}(\gamma)$ and $f^{e_2}(\gamma)$ and the annular subsurface $W$ with core curve $\beta$ gives us an upper bound for $d_\beta(f^{e_1}(\gamma), f^{e_2}(\gamma))$, depending only on $\theta_0$ and the lower bound for the length of $\beta$. The length of $\beta$ is bounded below by twice of the injectivity radius of the hyperbolic metric which was fixed on the surface. So we get the desired bound on (7.2) for annular subsurfaces as well. 

\[\square\]

7.1. **Scheme I.** The construction of this subsection will be used in §8.2 to provide examples of divergent WP geodesic rays and in §8.3 to provide examples of closed WP geodesics in the thin part of moduli space.
Let \( \alpha \) and \( \beta \) be two disjoint curves on \( S \) such that \( S \setminus \alpha \), \( S \setminus \beta \) and \( S \setminus \{\alpha, \beta\} \) are large subsurfaces. Consider indexed large subsurfaces \( X_0 = S \setminus \{\alpha, \beta\} \), \( X_1 = S \setminus \alpha \), \( X_2 = S \setminus \{\alpha, \beta\} \) and \( X_3 = S \setminus \beta \). Note that \( X_0 \) and \( X_2 \) are both the same subsurface \( S \setminus \{\alpha, \beta\} \) with different indices. Let \( f_0, f_1, f_2 \) and \( f_3 \) be partial pseudo-Anosov maps supported on \( X_0, X_1, X_2 \) and \( X_3 \), respectively, where \( f_0 = f_2 \) (the same partial pseudo-Anosov maps with different indices). Then in particular, \( f_a, a = 0, 1, 2, 3 \), preserves each component of \( \partial X_a \). Furthermore, suppose that for \( a = 0, 1, 2, 3 \), if \( \delta \in \partial X_a \) then we have

\[
(7.3) \quad d_\delta(\gamma, f_a^\delta \gamma) \leq 2
\]

for every \( \gamma \) in \( \partial X_a \) and any integer \( e \).

To see that partial pseudo-Anosov maps as above exist we have that: Given a partial pseudo-Anosov map \( g \) supported on \( X \) by Theorem 11.7 in exposé 11 of [FLP79], \( g \) is isotopic to some \( f \) such that the restriction of \( f \) to all of the component curves of \( \partial X \) is identity. This \( f \) satisfies the above annular coefficient bound.

Let \( q_0 : \mathbb{N} \to \{0, 1, 2, 3\} \) be the function \( q_0(i) \equiv i \) (mod 4). Let \( q_1(i) = q_0(i + 1) \), \( q_2(i) = q_0(i + 2) \) and \( q_3(i) = q_0(i + 3) \). Let \( q \) denote any of the functions \( q_0, q_1, q_2 \) and \( q_3 \) or the restriction of any of them to the set \( \{1, \ldots, k\} \), where \( k \) is a positive integer.

Let \( q \) be as above. When the domain of \( q \) is \( \mathbb{N} \) let \( \{\epsilon_i\}_i \) be an infinite sequence of integers and when the domain of \( q \) is \( \{1, \ldots, k\} \) let \( \{\epsilon_i\}_i \) be a sequence of integers with \( k \) elements. For simplicity of notation we some times denote the sequence \( \{\epsilon_i\}_i \) by \( e \).

For any \( i \) in the domain of \( q \) set the subsurface

\[
Z_i(q, e) = f_{q(1)}^{\epsilon_1} \cdots f_{q(i-1)}^{\epsilon_{i-1}} X_{q(i)}
\]

Let \( \mu_T(q, e) \) be a marking whose base contains \( \partial X_a \) for \( a = 0, 1, 2, 3 \) = \{\alpha, \beta\}. Throughout the following lemmas and propositions we assume that the domain of \( q \) is \( \{1, \ldots, k\} \) for some \( k \geq 1 \). We let \( \mu_T(q, e) = f_{q(1)}^{\epsilon_1} \cdots f_{q(k)}^{\epsilon_k} \mu_T(q, e) \) and establish several bounds on the subsurface coefficients of \( \mu_T(q, e) \) and \( \mu_I(q, e) \).

When there is no confusion we drop the reference to \( (q, e) \). For example we denote \( Z_i(q, e) \) by \( Z_i \).

**Remark 7.5.** The construction of this subsection and the estimates on subsurface coefficients can be carried out in a more general setting. Here we restrict ourself to be able to provide detailed step by step estimates and complete arguments.

**Lemma 7.6.** There are constants \( K'_1 > 0, C'_1 \geq 0 \) and \( E_1 > 0 \), depending only on the partial pseudo-Anosov maps \( f_0, f_1, f_2 \) and \( f_3 \), and \( \mu_I \) with the following properties. Given \( q \) and \( \{\epsilon_i\}_i \) such that \( |\epsilon_i| > E_1 \) for any \( i \in \{1, \ldots, k\} \), we have

(i) For any \( i \in \{1, \ldots, k\} \),

\[
(7.4) \quad d_{Z_i(q, e)}(\mu_I(q, e), \mu_T(q, e)) \geq K'_1|\epsilon_i| - C'_1
\]
(ii) Let \( k \geq 3 \). Let \( i, j \in \{1, \ldots, k\} \) and \( j \geq i + 2 \). Then \( Z_i(q, e) < Z_j(q, e) \) between \( \mu_I(q, e) \) and \( \mu_T(q, e) \).

**Proof.** Our proof modifies the proof of Theorem 5.2 of [CLM12]. There the authors assume that any two of the subsurfaces which support partial pseudo-Anosov maps either overlap or are disjoint. But here \( X_0 = X_2 \) is a subsurface of \( X_1 \) and \( X_3 \). As a result their argument does not go through completely to prove the lemma and needs some modification. Furthermore, our set up is different.

**Proof of part part (ii).** The proof is by induction on \( k \). Denote \( \mu_I \) by \( \mu \). Set the constant

\[
K'_1 = \min\{\tau_a : a = 0, 1, 2, 3\}.
\]

Here \( \tau_a = \tau_{f_a}, a = 0, 1, 2, 3, \) is the constant from Proposition 7.1 for the partial pseudo-Anosov map \( f_a \) supported on \( X_a \).

Let \( \eta = \max\{d_{X_a}(\mu, f_b^a \mu) : a, b \in \{0, 1, 2, 3\}, X_a \neq X_b \) and \( e \in \mathbb{Z}\} \). Note that since \( \mu \) is fixed and \( X_a \neq X_b \) by Lemma 7.4 the maxima exists. Set the constant

\[
C'_1 = 2(B_0 + \eta + 1)
\]

Here \( B_0 \) is the constant in Theorem 2.8 (Behrstock Inequality).

Let \( \omega = \max\{d_W(\mu, \partial X_a) : W \subseteq \mathcal{S} \) and \( a = 0, 1, 2, 3\} \). Note that the marking \( \mu \) and subsurfaces \( \{X_a\}_{a=0,1,2,3} \) are fixed so the maxima exists. Set the constant

\[
E_1 = B_0 + \omega + 4M + 4 + C'_1
\]

By Lemma 7.4 \( d_{X_a}(f^e \mu, \mu) \geq \tau_a |e| \geq K'_1 |e|, a = 0, 1, 2, 3, \) so we have the base of induction for \( k = 1 \).

Suppose that for any function \( q : \{1, \ldots, k'\} \rightarrow \{0, 1, 2, 3\} \) with \( k' < k \) as the beginning of this section the proposition holds. Fix \( i \in \{1, \ldots, k\} \) and let \( g = f_{q(1)}^{e_1} \cdots f_{q(i-1)}^{e_{i-1}} \). Applying \( g^{-1} \) to \( d_{Z_i}(\mu_I, \mu_T) \) we get

\[
d_{Z_i}(\mu_I, \mu_T) = d_{X_{q(i)}(g^{-1} \mu, f_{q(i)}^{e_{i}} h \mu)}
\]

where \( h = f_{q(i+1)}^{e_{i+1}} \cdots f_{q(k)}^{e_{k}} \). By the triangle inequality the left hand side is bounded below by

\[
d_{X_{q(i)}(h \mu, f_{q(i)}^{e_{i}} h \mu)} - d_{X_{q(i)}(g^{-1} \mu, h \mu)} - 2
\]

By Proposition 7.1 the first term of (7.5) is bounded below by \( K'_1 |e_i| \). This gives us the multiplicative constant in (7.4). To get the additive constant in (7.4) we proceed to bound the second term of (7.5). By the triangle inequality it is bounded above by

\[
d_{X_{q(i)}(g^{-1} \mu, \mu)} + d_{X_{q(i)}(\mu, h \mu)} + 2
\]
First we show that 
\[ d_{X_{q(i)}}(\mu, h\mu) \leq \frac{C_1}{4} + 1. \]
Let \( q'(j) = q(j + i) \) for \( j = 1, \ldots, k - i \) and \( e'(j) = e(j + i) \) for \( j = 1, \ldots, k - i \). Then by the definition 
\[ Z_2(q', e') = f_{q(i+1)}^{\varepsilon_{i+1}} X_{q(i+2)} \]
so 
\[ d_{f_{q(i+1)}^{\varepsilon_{i+1}} X_{q(i+2)}}(\mu, h\mu) = d_{Z_2(q', e')}(\mu, h\mu) \]

The assumption of induction applied to \( q' \) implies that the right hand side subsurface coefficient is greater than or equal to \( K'_1|e_{i+1}| - C'_1 \). So we have

\[ (7.7) \quad d_{f_{q(i+1)}^{\varepsilon_{i+1}} X_{q(i+2)}}(\mu, h\mu) \geq K'_1|e_{i+1}| - C'_1. \]

We claim that

\[ \text{Claim 7.7.} \quad \partial X_{q(i)}(\mu) \cap f_{q(i+1)}^{\varepsilon_{i+1}} X_{q(i+2)}. \]

To see this, first suppose that \( q(i) = 1 \) then \( X_{q(i)} = S\backslash\alpha \). Further by the definition of \( q, X_{q(i)} = S\backslash\beta \) and \( X_{q(i+1)} = S\backslash\{\alpha, \beta\} \). \( f_{q(i+1)} \) preserves each component of \( \partial X_{q(i+1)} = \{\alpha, \beta\} \), so \( f_{q(i+1)}^{\varepsilon_{i+1}} X_{q(i+2)} = S\backslash\beta \). Further \( \partial X_{q(i)} = \alpha \). Then since \( \alpha \neq \beta \), \( \partial X_{q(i)} \cap X_{q(i+1)} \).

If \( q(i) = 3 \) then \( X_{q(i)} = S\\backslash\beta \) and the claim follows from a similar argument replacing \( \alpha \) by \( \beta \).

Now suppose that \( q(i) = 0(2) \), then \( X_{q(i)} = S\\{\alpha, \beta\} \). Further by the definition of \( q, X_{q(i+2)} = S\\{\alpha, \beta\} \). Moreover, \( f_{q(i+1)} \) is supported on \( X_{q(i+1)} \), which is \( S\\backslash\alpha(S\\backslash\beta) \). Note that \( \partial X_{q(i)} \cap X_{q(i+1)} \), because \( \{\alpha, \beta\} \cap S\\backslash\alpha(S\\backslash\beta) \).

- If \( f_{q(i+1)} \) is supported on \( S\\backslash\beta \), then by Proposition 7.1

\[ d_{X_{q(i+1)}}(f_{q(i+1)}^{\varepsilon_{i+1}} \partial X_{q(i)}), \partial X_{q(i)}) > K'_1|e_{i+1}| > K'_1E_1 > 4. \]

This implies that \( f_{q(i+1)}^{\varepsilon_{i+1}} \alpha \) and \( \alpha \) overlap. Now \( f_{q(i+1)}^{\varepsilon_{i+1}} \alpha \in f_{q(i+1)}^{\varepsilon_{i+1}} \partial X_{q(i)} \) so \( \partial X_{q(i)} \) overlaps \( f_{q(i+1)}^{\varepsilon_{i+1}} X_{q(i)} \).

- If \( f_{q(i+1)} \) is supported on \( S\\alpha \), then by a similar argument \( f_{q(i+1)}^{\varepsilon_{i+1}} \beta \) and \( \beta \) overlap. Now \( f_{q(i+1)}^{\varepsilon_{i+1}} \beta \in f_{q(i+1)} \partial X_{q(i)} \), so \( \partial X_{q(i)} \cap f_{q(i+1)}^{\varepsilon_{i+1}} X_{q(i+2)} \).

The proof of the claim is complete.

By the above claim in hand we may write

\[ (7.8) \quad d_{f_{q(i+1)}^{\varepsilon_{i+1}} X_{q(i+2)}}(\partial X_{q(i)}, h\mu) \]

This subsurface coefficient by the triangle inequality is bounded below by

\[ (7.9) \quad d_{f_{q(i+1)}^{\varepsilon_{i+1}} X_{q(i+2)}}(h\mu, \mu) - d_{f_{q(i+1)}^{\varepsilon_{i+1}} X_{q(i+2)}}(\mu, \partial X_{q(i)}) - 2 \]

The second term of (7.9) is bounded above by \( \omega \) and by (7.7) the first term of (7.9) is greater than or equal to \( K'_1|e_{i+1}| \). So if \( |e_{i+1}| > E_1 \) then (7.9) is at least \( B_0 \) and consequently (7.8) is at least \( B_0 \). So by Theorem 2.8 (Behrstock Inequality) we get

\[ (7.10) \quad d_{X_{q(i)}}(f_{q(i+1)}^{\varepsilon_{i+1}} \partial X_{q(i+2)}, h\mu) \leq B_0. \]
Now by the triangle inequality
\[ d_{X_q(i)}(\mu, \mu) \leq d_{X_q(i)}(\mu, f_{q(i+1)}^e \partial X_{q(i+2)}) + d_{X_q(i)}(f_{q(i+1)}^{e+1} \partial X_{q(i+2)}, \mu) + 1 \]
(7.11) \[ \leq \eta + B_0 \leq C_1'. \]

The second inequality follows from the choice of \( \eta \) and the bound (7.10).
The third one follows from the choice of \( C_1' \).

Further, let \( q'(j) = q(i+1-j) \) for \( j = 1, \ldots, i \) and \( e'(j) = -e(i+1-j) \)
for \( j = 1, \ldots, i \). Then the exact same lines we gave above using this \( q' \) and \( e' \)
implies that \( d_{X_q(i)}(g^{-1}\mu, \mu) \leq C_1' \frac{1}{2} \). This bound and (7.11) give us the bound \( C_1' + 2 \) for (7.6).
Plugging this into (7.5) we get the bound \( C_1' \) as the additive constant for (7.4).

**Proof of part (ii).** The proof is by induction on \( k \). The base of induction for \( k = 3 \) is obtained as follows.

We would like to show that \( Z_1 < Z_3 \). We need to verify that the conditions of Proposition 2.15 hold for \( \mu_I, \mu_T \) and subsurfaces \( Z_1 \) and \( Z_3 \). First note that by part (i),
\[ d_{Z_1}(\mu_I, \mu_T) > K_1' |e_1| - C_1' > K_1' E_1 - C_1' > 4M \]
and similarly \( d_{Z_3}(\mu_I, \mu_T) > 4M \).

We proceed to show that \( Z_1 \pitchfork Z_3 \). If \( q(1) = 1 \), then by the definition of \( q \), \( Z_1(q, e) = S(\alpha, Z_2(q, e) = f_1^e S(\alpha, \beta), Z_3(q, e) = f_2^e S(\alpha, \beta) \). Furthermore, \( f_1 \) preserves both \( \alpha \) and \( \beta \). So \( Z_2(q, e) = S(\alpha, f_1^e S(\alpha, \beta) \). Then applying \( f_1^{-e} \) to \( \partial Z_1 = \alpha \) and \( \partial Z_3 = f_1^e \beta \), we get \( \alpha \) and \( \beta \). Moreover, \( S(\alpha, \beta) \), so \( Z_1 \pitchfork Z_3 \).

If \( q(1) = 3 \) a similar argument implies that \( Z_1 < Z_3 \). If \( q(1) = 0(2) \), then \( X_q(1) = S(\alpha, \beta) \), \( X_{q(2)} = S(\alpha, \beta) \) and \( X_{q(3)} = S(\alpha, \beta) \). So \( Z_1(q, e) = S(\alpha, \beta), Z_2(q, e) = f_0^e S(\alpha, f_2^e S(\alpha, \beta) \) and \( Z_3(q, e) = f_0^e f_2^e S(\alpha, \beta) \). Furthermore, \( f_{q(1)} = f_0(f_2) \) preserves both \( \alpha \) and \( \beta \), and \( f_{q(2)} = f_1(f_3) \) preserves \( \alpha \). So \( Z_1(q, e) = S(\alpha, \beta), Z_2(q, e) = S(\alpha, \beta) \) and \( Z_3(q, e) = S(\alpha, f_2^e S(\alpha, \beta) \). Apply \( f_1^{-e} f_0^{-e} f_2^{-e} \) to \( \partial Z_1 \) and \( \partial Z_3 \). The resulting multi curves contain the curves \( \beta \) and \( f_1^{-e}\beta \alpha \) and \( f_3^{-e}\alpha \), respectively. Moreover, by Proposition 7.4, \( d_{S(\alpha, \beta)}(\beta, f_1^e \beta) > \tau_1 |e_1| > \tau_1 E_1 > 4 \) \( (d_{S(\alpha, \beta)}(\alpha, f_3^{-e}\alpha) > \tau_3 E_1 > 4) \), so \( \beta \) and \( f_1^{-e} \beta \alpha \) overlap \( \alpha \) and \( f_3^{-e}\alpha \) overlap. This implies that \( Z_1 \pitchfork Z_3 \).

Now we may write \( d_{Z_1}(\mu_I, \partial Z_3) \). Then since \( \mu_I \pitchfork \partial Z_3 \) by (7.15) we get
\[ d_{Z_1}(\mu_I, \partial Z_3) \geq d_{Z_1}(\mu_I, \mu_T) - \text{diam}_{Z_1}(\mu_I) \geq K_1' |e_1| - C_1' - 2 > K_1' E_1 - C_1' - 2 > 2M. \]
Consequently by Proposition 2.15, \( Z_1 < Z_3 \). This finishes establishing of the base of induction.

Suppose that (ii) holds for every \( q : \{1, \ldots, k'\} \rightarrow \{0, 1, 2, 3\} \), with \( k' < k \).

Let \( q_{\text{init}}(l) = q(l) \) for \( l = 1, \ldots, k-1 \) and \( e_{\text{init}, l} = e_l \) for \( l = 1, \ldots, k-1 \). Then \( \mu_I(q_{\text{init}}, e_{\text{init}}) = \mu_I(q, e) \) and \( Z_j(q_{\text{init}}, e_{\text{init}}) = Z_j(q, e) \) for \( j = 1, \ldots, k-1 \).
Let \( q_{\text{term}}(l) = q(l + 1) \) for \( l = 1, \ldots, k - 1 \) and \( e_{\text{term}, l} = e_{l+1} \) for \( l = 1, \ldots, k - 1 \).

First suppose that \( i, j \in \{1, \ldots, k\} \) and \( j \geq i + 2 \). If \( j < k \), then the assumption of induction applied to \( q_{\text{init}} \) and \( e_{\text{init}} \) implies that \( Z_i(q_{\text{init}}, e_{\text{init}}) < Z_j(q_{\text{init}}, e_{\text{init}}) \), which by Proposition 2.15 means that \( Z_i(q_{\text{init}}, e_{\text{init}}) \cap Z_j(q_{\text{init}}, e_{\text{init}}) \) and \( d_{Z_i(q_{\text{init}}, e_{\text{init}})}(\mu_1(q_{\text{init}}, e_{\text{init}}), \partial Z_i(q_{\text{init}}, e_{\text{init}})) > 2M \). But this implies that \( Z_i(q, e) < Z_j(q, e) \).

If \( i > 1 \), then by the assumption of the induction \( Z_{i-1}(q_{\text{term}}, e_{\text{term}}) < Z_{j-1}(q_{\text{term}}, e_{\text{term}}) \). This means that \( Z_{i-1}(q_{\text{term}}, e_{\text{term}}) \cap Z_j(q, e) \) and

\[
d_{Z_{j-1}(q_{\text{term}}, e_{\text{term}})}(\mu_T(q_{\text{term}}, e_{\text{term}}), \partial Z_{i-1}(q_{\text{term}}, e_{\text{term}})) > 2M
\]

Now applying \( f^e_{q(1)} \) to the subsurfaces \( Z_{i-1}(q_{\text{term}}, e_{\text{term}}), Z_{j-1}(q_{\text{term}}, e_{\text{term}}) \) and the marking \( \mu_T(q_{\text{term}}, e_{\text{term}}) \) we get the subsurfaces \( Z_i(q, e), Z_j(q, e) \) and the marking \( \mu_T(q, e) \). Thus \( Z_i(q, e) \cap Z_j(q, e) \) and

\[
d_{Z_j(q, e)}(\mu_T(q, e), \partial Z_i(q, e)) > 2M
\]

So by Proposition 2.15 \( Z_i(q, e) < Z_j(q, e) \).

Now suppose that \( i = 1 \) and \( j = k \). When \( k = 4 \), we may proceed as in the base of induction and prove directly that \( Z_1 \cap Z_4 \) and \( d_{Z_i}(\mu_1(q, e), \partial Z_4) > 2M \), which implies that \( Z_1 < Z_4 \). The details are similar so we skip them. When \( k \geq 5 \), let \( l \in \{3, \ldots, k - 2\} \). Then as we saw above \( Z_1 < Z_l \) and \( Z_l < Z_k \). So by the transitivity of the relation < on subsurfaces (Proposition 2.15) we conclude that \( Z_1 < Z_k \).

\[\square\]

The second part of the above lemma does not say anything about the order of two consecutive subsurfaces \( Z_i \) and \( Z_{i+1} \). The following lemma gives an order for times in the two consecutive intervals \( J_{Z_i} \) and \( J_{Z_{i+1}} \) (see Theorem 2.13).

**Lemma 7.8.** Given \( q \) and \( e \). Let \( i \) be such that \( q(i) = 1 \) or \( 3 \) (\( Z_i \) has one boundary curve). If \( i > 1 \) and \( j \in J_{Z_i} \), then \( j \geq \min J_{Z_{i-1}} \). If \( j \in J_{Z_{i-1}} \) then \( j \leq \max J_{Z_{i+1}} \).

**Proof.** By the definition of \( q \) and the subsurfaces \( Z_i(q, e) \) we have \( \partial Z_i = \partial Z_{i-1} \cap \partial Z_{i+1} \).

By Lemma 7.6, \( Z_{i-1} < Z_{i+1} \), so by Proposition 2.15 \( d_{Z_{i+1}}(\mu_T, \partial Z_{i-1}) > 2M \). Then Theorem 2.8 implies that \( d_{Z_{i-1}}(\mu_T, \partial Z_{i+1}) \leq M \). Then since \( \partial Z_i \subset \partial Z_{i+1} \) we obtain \( d_{Z_{i-1}}(\mu_T, \partial Z_i) = M + \text{diam}_{Z_{i-1}}(\partial Z_{i+1}) \leq M + 1 \). Since \( j \in J_{Z_i} \), \( \rho(j) \supset \partial Z_i \), so we get that \( d_{Z_{i-1}}(\mu_T, \rho(j)) \leq M + 1 \). This inequality and \( d_{Z_{i-1}}(\mu_1, \mu_T) > 4M \), combined by the triangle inequality imply that \( d_{Z_{i-1}}(\mu_T, \rho(j)) > 3M - 1 \).

Now assume that \( j \leq \min J_{Z_{i-1}} \), then by Theorem 2.13 \( d_{Z_{i-1}}(\mu_T, \rho(j)) \leq M \), which contradicts the lower bound we just proved. Thus \( j \geq \min J_{Z_{i-1}} \).

The proof of \( j \leq \max J_{Z_{i+1}} \) is similar. \[\square\]
Proposition 7.9. There are constants $m, m' > 4M$ and $E_2 > E_1$, only depending on $f_0, f_1, f_2$ and $f_3$ and $\mu_I$ with the following properties. Given $q$ and $\{e_i\}_i$ such that $|e_i| > E_2$ for any $i \in \{1, \ldots, k\}$, we have

(i) For any non-meridional subsurface $W$ which is neither $Z_i(q,e)$ for some $i$ nor $S$ we have $d_W(\mu_I(q,e), \mu_T(q,e)) \leq m$.
(ii) Given $\gamma \in C_0(S)$ we have $d_\gamma(\mu_I(q,e), \mu_T(q,e)) \leq m'$.

Proof. Proof of part (ii). If $d_W(\mu_I, \mu_T) \leq 4M$, then we already have the upper bound. If not, $d_W(\mu_I, \mu_T) > 4M$.

Let $l \in \{1, \ldots, k\}$ with $q(l) = 1$ or $3$. Then $Z_l$ is a subsurface with one boundary curve. We claim that $\partial Z_{l-1}$ and $\partial Z_{l+1}$ fill $Z_l$. To see this, suppose that $X_{q(l)} = S \setminus \alpha$. When $X_{q(l)} = S \setminus \beta$ the proof is similar. Let $g = f_{q(l)}^e \cdots f_{q(t-1)}^e$. We have $g^{-1}Z_l = S \setminus \alpha$. By the definition of $q$, $X_{q(l-1)} = S \setminus \{\alpha, \beta\}$. Then $g^{-1}\partial Z_{l-1} = f_{q(l-1)}^{-e} \partial X_{q(l-1)} = \{f_{q(l-1)}^{-e_1}, f_{q(l-1)}^{-e_2}\}$. Furthermore, $f_{q(l-1)}$ is supported on $X_{q(l-1)}$, so preserves each component of $\partial X_{q(l-1)} = \{\alpha, \beta\}$ i.e. $f_{q(l-1)}^e \alpha = \alpha$ and $f_{q(l-1)}^e \beta = \beta$. Thus $g^{-1}\partial Z_{l-1} = \{\alpha, \beta\}$. Again by the definition of $q$, $X_{q(l+1)} = S \setminus \{\alpha, \beta\}$, so $g^{-1}\partial Z_{l+1} = \{\alpha, f_{q(l+1)}^e \beta\}$. Now $\beta \cap S \setminus \alpha$, so by Proposition 7.1 for $|e_l| > E_1$, $d_{S \setminus \alpha}(\beta, f_{q(l+1)}^e \beta) \geq \tau_1 |e_l| > 3$. So $\beta$ and $f_{q(l+1)}^e \beta$ fill $S \setminus \alpha$. Recall that if two curves have distance at least $3$ in the curve complex of a surface then their union fills the subsurface (see §2). Thus $g^{-1}\partial Z_{l-1}$ and $g^{-1}\partial Z_{l+1}$ fill $g^{-1}Z_l$, and consequently $\partial Z_{l-1}$ and $\partial Z_{l+1}$ fill $Z_l$.

By the definition of $q$ and the subsurfaces $Z_l$ we have $\partial Z_l = \partial Z_{l-1} \cap \partial Z_{l+1}$. If $\partial W \subseteq \partial Z_l$ then since $Z_l$ has one boundary curve, $W = Z_l$. This is excluded by the assumption that $W$ is not a subsurface in the list of $Z_i$'s. As we saw in the previous paragraph $\partial Z_{l-1}$ and $\partial Z_{l+1}$ fill $Z_l$, so either $W \cap Z_{l-1}$ or $W \cap Z_{l+1}$. Since $|e_l| > E_1$ ($E_1$ is the constant from Proposition 7.11) by Lemma 7.6 for each $l$, $d_{Z_l}(\mu_I, \mu_T) > 4M$. Also $d_W(\mu_I, \mu_T) > 4M$. So by Proposition 2.15 $W$ is ordered with respect to either $Z_{l-1}$ or $Z_{l+1}$. Similarly since $\partial Z_{l+1}$ and $\partial Z_{l+3}$ fill $Z_{l+2}$, $W$ is ordered with respect either $Z_{l+1}$ or $Z_{l+3}$. If the subsurfaces $W$ is not ordered with respect to two subsurfaces with different indices among $l - 1, l + 1$ and $l + 3$, then $W$ is ordered only with respect to $Z_{l+1}$. Now since $\partial Z_{l+3}$ and $\partial Z_{l+5}$ fill the subsurface $Z_{l+4}$ a similar argument shows that $W$ is ordered with respect to either $Z_{l+3}$ or $Z_{l+5}$. If the subsurface is not ordered with respect to two subsurfaces with different indices among $l + 1, l + 3$ and $l + 5$ then it is ordered only with respect to $Z_{l+3}$. But then it would be ordered with respect to $Z_{l+1}$ and $Z_{l+3}$. Thus we conclude that $W$ is ordered with respect to either both $Z_{l-1}$ and $Z_{l+1}$, $Z_{l+1}$ and $Z_{l+3}$, $Z_{l-1}$ and $Z_{l+3}$, $Z_{l+1}$ and $Z_{l+3}$, $Z_{l+1}$ and $Z_{l+5}$ or $Z_{l+3}$ and $Z_{l+5}$.

Repeating this argument for every $l$ with $q(l) = 1$ or $3$ we conclude that $W$ is ordered in the list of $Z_i$'s as one of the following cases.

1. There is an index $i$ such that either $Z_{i-1} < W < Z_{i+1}$ or $Z_{i-2} < W < Z_{i+2}$,
(2) $W < Z_i$, where $i = 1$ or 2,
(3) $Z_i < W$, where $i = k - 1$ or $k - 2$.

We proceed to establish the upper bound on $d_W(\mu_I, \mu_T)$ in Case (1). First suppose that $Z_{i-2} < W < Z_{i+2}$.

Since $Z_{i-2} < W$, by Proposition 2.15, $d_W(\partial Z_{i-2}, \mu_I) \leq M$. Similarly, since $W < Z_{i+2}$, $d_W(\partial Z_{i+2}, \mu_T) \leq M$. Then by the triangle inequality and these two bounds we have

$$d_W(\mu_I, \mu_T) \leq d_W(\mu_I, \partial Z_{i-2}) + d_W(\partial Z_{i-2}, \partial Z_{i+2}) + d_W(\partial Z_{i+2}, \mu_T) + 2$$

$$\leq d_W(\partial Z_{i-2}, \partial Z_{i+2}) + 2M + 2.$$

Let $g = f_{q(i-1)}^{\epsilon_{i-1}} \ldots f_{q(i-2)}^{\epsilon_{i-1}}$. Applying $g^{-1}$ to the last subsurface coefficient above we get

$$d_{g^{-1}W}(f_{q(i-1)}^{\epsilon_{i-1}} f_{q(i-2)}^{\epsilon_{i-2}} \partial X_{q(i-2)}, f_{q(i)}^{\epsilon_i} f_{q(i+1)}^{\epsilon_{i+1}} \partial X_{q(i+2)})$$

Recall that $\mu_I \supset \{\partial X_a\}_{a=0,1,2,3}$. Denote $\mu_I$ by $\mu$. Since diam$_{g^{-1}W}(\mu) = 2$ the above subsurface coefficient is bounded above by

$$d_{g^{-1}W}(f_{q(i-1)}^{\epsilon_{i-1}} f_{q(i-2)}^{\epsilon_{i-2}} \mu, f_{q(i)}^{\epsilon_i} f_{q(i+1)}^{\epsilon_{i+1}} \mu) + 4.$$

By the triangle inequality this subsurface coefficient is bounded above by

$$d_{g^{-1}W}(f_{q(i-1)}^{\epsilon_{i-1}} f_{q(i-2)}^{\epsilon_{i-2}} \mu, f_{q(i)}^{\epsilon_i} f_{q(i+1)}^{\epsilon_{i+1}} \mu) + d_{g^{-1}W}(f_{q(i-2)}^{\epsilon_{i-2}} \mu, f_{q(i)}^{\epsilon_i} f_{q(i+1)}^{\epsilon_{i+1}} \mu) + 6.$$  

(7.12)

Note that we replaced $\partial X_a$ by the marking $\mu$ because then the markings in the above sum overlap $g^{-1}W$ and each subsurface coefficient makes sense.

**Claim 7.10.** For any $a \in \{0, 1, 2, 3\}$, $g^{-1} W \neq X_a$ where $g = f_{q(1)}^{\epsilon_1} \ldots f_{q(i-1)}^{\epsilon_{i-1}}$.

First assume that $X_{q(i)} = S \setminus \alpha$. If $g^{-1} W = X_a$ for some $a \in \{0, 1, 2, 3\}$ we get contradiction as follows:

- $g^{-1} W = S \setminus \alpha$. Since $X_{q(i)} = S \setminus \alpha$, $W = gX_{q(i)} = Z_i$. This contradicts the fact that $W$ is not in the list of $Z_i$’s.
- $g^{-1} W = S \setminus \beta$. Since $X_{q(i)} = S \setminus \alpha$ by the definition of $q$, $X_{q(i-1)} = S \setminus \{\alpha, \beta\}$ and $X_{q(i-2)} = S \setminus \beta$. We have $\partial Z_{i-2} = g f_{q(i-1)}^{\epsilon_{i-1}} f_{q(i-2)}^{\epsilon_{i-2}} \beta$. Now $f_{q(i-1)}$ preserves each component of $\partial X_{q(i-1)} = \{\alpha, \beta\}$ (because is supported on $X_{q(i-1)}$) and $f_{q(i-2)}$ preserves $\partial X_{q(i-2)} = \beta$. So $\partial Z_{i-2} = g \beta = \partial W$. Thus $W = Z_{i-2}$. This contradicts the fact that $W$ is not in the list of $Z_i$’s.
- $g^{-1} W = S \setminus \{\alpha, \beta\}$. By the definition of $q$, $X_{q(i+1)} = S \setminus \{\alpha, \beta\}$, so $\partial Z_{i+1} = \{g f_{q(i+1)}^{\epsilon_{i+1}} \alpha, g f_{q(i+1)}^{\epsilon_{i+1}} \beta\}$. Moreover, $f_{q(i+1)}$ preserves each component of $\partial X_{q(i+1)} = \{\alpha, \beta\}$, so $\partial Z_{i+1} = \{g \alpha, g \beta\} = \partial W$. Thus $W = Z_{i+1}$, which contradicts the fact that $W$ is not in the list of $Z_i$’s.

If we assume that $X_{q(i)} = S \setminus \beta$, then $g^{-1} W = X_a$ for any $a \in \{0, 1, 2, 3\}$ gives a contradiction as above. Finally we assume that $X_{q(i)} = S \setminus \{\alpha, \beta\}$. If $g^{-1} W = X_a$ for some $a \in \{0, 1, 2, 3\}$ we get contradiction as follows:
• $g^{-1}W = S\{\alpha, \beta\}$. Since $X_{q(i)} = S\{\alpha, \beta\}$, $W = Z_i$. This contradicts the fact that $W$ is not in the list of $Z_i$'s.

• $g^{-1}W = S\alpha$. Since $X_{q(i)} = S\{\alpha, \beta\}$ by the definition of $q$, either $X_{q(i+1)} = S\alpha$ or $X_{q(i-1)} = S\alpha$.

First assume that $X_{q(i+1)} = S\alpha$. By the definition of $q$, $\partial X_{q(i+2)} = \{\alpha, \beta\}$. We have $\partial Z_{i+2} = g f_{q(i)}^e f_{q(i)+1}^e X_{q(i+2)}$. $f_{q(i)}$ preserves each component of $\partial X_{q(i)} = \{\alpha, \beta\}$ and $f_{q(i)+1}$ preserves $\partial X_{q(i+1)} = \alpha$, so we get $\partial Z_{i+2} = \{g\alpha, g f_{q(i+1)}^e \beta\}$. Now $\partial W = g\alpha$, therefore $Z_{i+2} \subsetneq W$. But this contradicts the fact that $W \cap Z_{i+2}$.

Now assume that $X_{q(i-1)} = S\alpha$. $\partial Z_{i-2} = g f_{q(i-1)} f_{q(i)-2} X_{q(i-2)}$. $f_{q(i-1)}$ preserves $\partial X_{q(i-1)} = \alpha$ and $f_{q(i-2)}$ preserves $\partial X_{q(i-2)} = \{\alpha, \beta\}$, so we get $\partial Z_{i-2} = \{g\alpha, g f_{q(i-1)}^e \beta\}$. Now $\partial W = g\alpha$, so $Z_{i-2} \subsetneq W$. But this contradicts the fact that $W \cap Z_{i-2}$.

• $g^{-1}W = S\beta$. We may get a contradiction similar to the previous bullet.

We proceed to bound the terms of (7.12). For this purpose for partial pseudo-Anosov maps $f_a, a = 0, 1, 2, 3$, let $\Psi^+_a = \Psi^+_f a$ and $\Delta^+_a = \Delta^+_f a$ be the subsets of $PML(S)$ defined at the beginning of [7]. We fix $U^+_a$ neighborhoods of $\Delta^+_a$ in $PML(S)$, $a = 0, 1, 2, 3$, such that $\overline{U_a^-}$ the closure of $U_a^-$ and $\overline{U_a^+}$ the closure of $U_a^+$ are disjoint from $\Psi^-_0 \cup \Psi^+_0$ for any $a, b \in \{0, 1, 2, 3\}$ with $X_a \neq X_b$. Since $PML(S)$ is compact (see [CBSS]) each set $\overline{U^+_a}, a = 0, 1, 2, 3$, is compact. Applying Theorem 7.3 to the pseudo-Anosov map $f_a$ and compact sets $\pi X_a (\mu), U^+_a$ and $U^-_a$, for any $a, b \in \{0, 1, 2, 3\}$ with $X_a \neq X_b$, there exists $E_2 > E_1 (E_1)$ is the constant from Lemma 7.6 such that

• $f^a_n (\pi X_a (\mu)) \subset U^+_a$ for all $n \geq E_2$ and $f^a_{-n} (\pi X_a (\mu)) \subset U^-_a$ for all $n \geq E_2$ and
• $f^a_n U^+_a \subset U^+_a$ for all $n \geq E_2$ and $f^a_{-n} U^-_a \subset U^-_a$ for all $n \leq E_2$.

Bounding the second term of (7.12). Because $q(i - 2) \in \{0, 1, 2, 3\}$ by Claim 7.10 $f_{q(i-2)}$ is not supported on $g^{-1}W$. Lemma 7.4 applied to the curves in $\mu$ which overlap $W$ implies that the term is bounded above by $m_1 = \max_{a=0, 1, 2, 3} m(f_a, \mu)$. Note that $m_1$ only depends on the pseudo-Anosov maps $f_0, f_1, f_2$ and $f_3$ and the marking $\mu$. Similarly, we can show that the third term of (7.12) is bounded by $m_1$. The first term of (7.12):

By the choice of $E_2$ for $|e_{-2}| > E_2$, $f_{q(i-2)}^e X_{q(i-2)} \subset U^+_q X_{q(i-2)}$. But $\overline{U^-_{q(i-2)}}$ and $U^+_q X_{q(i-2)}$ are disjoint from $\Psi^-_{q(i-1)} \cup \Psi^+_{q(i-1)}$. Further by Claim 7.10 $f_{q(i-1)}$ is not supported on $g^{-1}W$. Then Lemma 7.4 applied to the compact subsets $\overline{U^-_{q(i-2)}} \subset \overline{PML(S)} \setminus \Psi^-_{q(i-1)} \cup \Psi^+_{q(i-1)}$ implies that the term is bounded above by $m_2 = \max_{a=0, 1, 2, 3} m(f_a, \overline{U^+_a})$. Similarly, we may get the bound $m_2$ on the fourth term of (7.12).

Then the sum in (7.12) is bounded above by $4 \max\{m_1, m_2\} + 6$. Therefore, $d_W(\mu, \mu_T) \leq 4 \max\{m_1, m_2\} + 2M + 12$. 

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We proceed to establish the bound for $d_W(\mu_1, \mu_T)$ in Case (1) when $Z_{i-1} < W < Z_{i+1}$. By Proposition 2.15 since $Z_{i-1} < W$, $d_W(\mu_1, \partial Z_{i-1}) \leq M$ and since $W < Z_{i+1}$, $d_W(\partial Z_{i+1}, \mu_T) \leq M$. Then by the triangle inequality we obtain

$$d_W(\mu_1, \mu_T) \leq d_W(\partial Z_{i-1}, \partial Z_{i+1}) + 2M + 2.$$ 

Let $i \in \{1, \ldots, k\}$ and $g = f_{q(1)}^{e_1} \cdots f_{q(i-1)}^{e_{i-1}}$. Applying $g^{-1}$ to the subsurface coefficient on the right hand side we get

$$d_{g^{-1}}(f_{q(i-1)}^{e_{i-1}}, f_{q(i)}^{e_i}) \leq d_{g^{-1}}(\partial X_{q(i-1)}, f_{q(i)}^{e_i}) \partial X_{q(i+1)}(\partial X_{q(i-1)}) + 2M + 2.$$ 

Note that $\mu \supset \{\partial X_a\}_{a=0,1,2,3}$ so the above subsurface coefficient is bounded above by $d_{g^{-1}}(f_{q(i-1)}^{e_{i-1}}) + 4$ by the triangle inequality this subsurface coefficient is bounded above by

$$d_{g^{-1}}(f_{q(i-1)}^{e_{i-1}}) + d_{g^{-1}}(\mu, f_{q(i)}^{e_i}) + 4.$$ 

By Claim 7.10, $f_{q(i-1)}$ and $f_{q(i)}$ are not supported on $g^{-1}W$. Then we may bound the two terms above as the second term of (7.12) by $m_1$. Then we get the bound $2m_1 + 4$ for the sum. Therefore, $d_W(\mu_1, \mu_T) \leq 2m_1 + 2M + 6$.

We are left with cases (2) and (3).

Case (2): First assume that $W < Z_2$, $W < Z_1$ would be treated similarly. By Proposition 2.15, $d_W(\partial Z_2, \mu_T) \leq M$. Then by the triangle inequality and this bound we get

$$d_W(\mu_1, \mu_T) \leq d_W(\mu_1, \partial Z_2) + M + 1 = d_W(\mu_1, f_{1}^{e_1}) \partial X_2) + M + 1.$$ 

Since $\mu := \mu \supset \{\partial X_a\}_{a=0,1,2,3}$ and $\text{diam}_W(\mu) \leq 2$ the last subsurface coefficient is bounded by $d_W(\mu, f_{1}^{e_1}) + 2$. Then since $f_1$ is not supported on $W$, Lemma 7.4 gives us

$$d_W(\mu, f_{1}^{e_1}) \leq m_1 + M + 1,$$

where $m_1 = \max_{a=0,1,2,3} m(f_a, \mu)$.

Therefore we obtain the bound $d_W(\mu_1, \mu_T) \leq m_1 + 2M + 5$.

Case (3) can be treated exactly similar to Case (2) by considering $q'$ defined by $q'(i) = q(k - i + 1)$ for $i = 1, \ldots, k$ and the sequence $e'_i = e_{k-i+1}$ for $i = 1, \ldots, k$. Then we obtain the upper bound $d_W(\mu_1, \mu_T) \leq m_1 + M + 5$.

Establishing the bounds in cases (1), (2) and (3), we may conclude that $m = 4 \max\{m_1, m_2\} + 4M + 12$ is the desired bound in (i) for the subsurface coefficients of non-annular subsurfaces which are not in the list of $Z_i$’s.

We proceed to obtain the bound on annular subsurface coefficients.

Proof of part (ii). If $d_{\gamma}(\mu_1, \mu_T) \leq 4M$ we already have the bound. If not, then $d_{\gamma}(\mu_1, \mu_T) > 4M$. 

We show that if $q(l) = 1$ or 3 (when $\partial Z_l$ consists of one curve), then $\partial Z_{l-3}$ and $\partial Z_{l-5}$ fill $Z_l$.

$X_{q(l)}$ is either $S \backslash \alpha$ or $S \backslash \beta$. First suppose that $X_{q(l)} = S \backslash \alpha$. By the definition of $q$, $X_{q(l-3)} = S \{ \alpha, \beta \}$. Let $g = f_{q(l-3)}^{e_1} \ldots f_{q(l-4)}^{e_1}$, then $g^{-1} \partial Z_{l-3} = \{ \alpha, \beta \}$. Furthermore, by the definition of $q$, $X_{q(l-5)} = S \{ \alpha, \beta \}$, then

$$g^{-1} \partial Z_{l-5} = \{ f_{q(l-4)}^{-e_1} f_{q(l-5)}^{-e_1} \alpha, f_{q(l-4)}^{-e_1} f_{q(l-5)}^{-e_1} \beta \}.$$ 

$f_{q(l-5)}$ preserves $\partial X_{q(l-5)} = \{ \alpha, \beta \}$. So $g^{-1} \partial Z_{l-5}$ contains the curve $f_{q(l-4)}^{-e_1} \beta$.

We have $\beta \not\subset S \backslash \alpha$. Furthermore by the definition of $q$, $f_{q(l-4)}$ is supported on $S \backslash \alpha$. So by Lemma 7.6 and the choice of $E_2 > E_1$, if $|e_l| > E_2$, then $d_{S \backslash \alpha} (\beta, f_{q(l-4)}^{-e_1} \beta) > 4$. This implies that $\beta$ and $f_{q(l-4)}^{-e_1} \beta$ fill $S \backslash \alpha$. So $g^{-1} \partial Z_{l-3}$ and $g^{-1} \partial Z_{l-5}$ fill $g^{-1} Z_l$. This implies that $\partial Z_{l-3}$ and $\partial Z_{l-5}$ fill $Z_l$. If $g^{-1} Z_l = S \backslash \beta$, a similar argument implies that $\partial Z_{l-3}$ and $\partial Z_{l-5}$ fill $Z_l$. Thus given $\gamma$ we have

- $\gamma \not\subset \partial Z_{l-3}$ or $\gamma \not\subset \partial Z_{l-5}$

Similarly we can prove that $\partial Z_{l+3}$ and $\partial Z_{l+5}$ fill $Z_l$. Thus given $\gamma$ we have that

- $\gamma \not\subset \partial Z_{l+3}$ or $\gamma \not\subset \partial Z_{l+5}$.

Given $l$ with $q(l) = 1$ or 3 since $|e_l| > E$ by Lemma 7.6, $d_{Z_l} (\mu_l, \mu_T) > 4M$. Also $d_{S \backslash \alpha} (\mu_l, \mu_T) > 4M$. Then by the above two bullets Proposition 2.15 implies that $\gamma$ is ordered with respect to $Z_j$ and $Z_{j'}$, where $j = l - 3$ or $l - 5$ and $j' = l + 3$ or $l + 5$. If $Z_j < \gamma < Z_{j'}$ we are in Case (1) below. If $Z_{j'} < \gamma$ by transitivity of $\lessgtr$, $Z_j < \gamma$, then if $Z_{j'} < \gamma < Z_{j''}$, where $j'' = j' + 3$ or $j' + 5$, we are again in Case (1). Otherwise we repeat the comparison until we end up in either Case (1) for some $j$ or Case (3) below. If $\gamma < Z_l$, similarly by repeating the comparison we will end up either in Case (1) for some $j$ or Case (2).

1. $Z_j < \gamma < Z_{j'}$, where $j = i - 3$ or $i - 5$ and $j' = i + 3$ or $i + 5$,
2. $\mu_l < \gamma < Z_i$, where $i = 3$ or $5$,
3. $Z_i < \gamma < \mu_T$, where $i = k - 3$ or $k - 5$.

We proceed by establishing the bound in each of these cases.

Case (1). By Proposition 2.15 $d_{S \backslash \alpha} (\mu_l, \partial Z_{i-3}) \leq M$ and $d_{S \backslash \alpha} (\mu_T, \partial Z_{i+3}) \leq M$. Having these bounds, by the triangle inequality we get

$$d_{S \backslash \alpha} (\mu_l, \mu_T) \leq d_{S \backslash \alpha} (\partial Z_{i-3}, \partial Z_{i+3}) + 2M + 2$$

So we only need to bound $d_{S \backslash \alpha} (\partial Z_{i-3}, \partial Z_{i+3})$. Let $g = f_{q(l-3)}^{e_1} \ldots f_{q(l-1)}^{e_1}$. Applying $g^{-1}$ to the subsurface coefficient on the right hand side, we get

$$d_{S \backslash \alpha} (f_{q(l-3)}^{-e_1} f_{q(l-2)}^{-e_1} f_{q(l-1)}^{-e_1} \partial X_{q(l-3)}, f_{q(l)}^{e_1} f_{q(l+1)}^{e_1} f_{q(l+2)}^{e_1} \partial X_{q(l+3)})$$

Recall that $\mu = \mu_l \cup \{ \partial X_a \}_{a=0,1,2,3}$. Then since $diam_{g^{-1} \gamma} (\mu) \leq 2$ the above subsurface coefficient is bounded above by $d_{g^{-1} \gamma} (f_{q(l-3)}^{-e_1} f_{q(l-2)}^{-e_1} f_{q(l-1)}^{-e_1} \mu, f_{q(l)}^{e_1} f_{q(l+1)}^{e_1} f_{q(l+2)}^{e_1} \mu)$.
4. By the triangle inequality this subsurface coefficient is bounded above by

\[ d_{g^{-1}}(f_{q(i-1)}^{-\varepsilon_1} f_{q(i-2)}^{-\varepsilon_2} f_{q(i-3)}^{-\varepsilon_3} \mu, f_{q(i-2)}^{-\varepsilon_2} f_{q(i-1)}^{-\varepsilon_1} \mu) + d_{g^{-1}}(f_{q(i-1)}^{-\varepsilon_1} f_{q(i-2)}^{-\varepsilon_2} f_{q(i-3)}^{-\varepsilon_3} \mu, f_{q(i-2)}^{-\varepsilon_2} f_{q(i-1)}^{-\varepsilon_1} \mu) + d_{g^{-1}}(f_{q(i-1)}^{-\varepsilon_1} f_{q(i-2)}^{-\varepsilon_2} f_{q(i-3)}^{-\varepsilon_3} \mu, \mu) \]

\[ (7.14) d_{g^{-1}}(f_{q(i)}^{\varepsilon_i} \mu, f_{q(i)}^{\varepsilon_i} f_{q(i+1)}^{\varepsilon_{i+1}} \mu) + d_{g^{-1}}(f_{q(i)}^{\varepsilon_i} f_{q(i+1)}^{\varepsilon_{i+1}} \mu, \mu) + 10 \]

We proceed to bound the terms of (7.14). Let the subsets \( \Psi^\pm_a, \Delta^\pm_a \) and open subsets of \( \mathcal{PML}(S) U^\pm_a \supset \Delta^\pm_a, a = 0, 1, 2, 3 \), be as in the proof of part [i]. The third term: \( f_{q(i)} \) is supported on \( X_{q(i)} \). If \( g^{-1} \gamma \) is a boundary curve of \( X_{q(i)} \) then by (7.3) this term is bounded above by 2. Otherwise, \( A(g^{-1} \gamma) \) is not equal to \( X_{q(i)} \), because \( X_a, a = 0, 1, 2, 3 \), is a non-annular subsurface. Then applying Lemma 7.4 to the curves in \( \mu \) which overlap \( A(g^{-1} \gamma) \), we get the upper bound \( m_2 \) as was explained in the proof of part [i] where the subsurface was non-annular. We may bound the second term by \( m_1 \) similarly. The term fourth: If \( g^{-1} \gamma \) is a boundary curve of \( X_{q(i+1)} \) then by (7.3) this term is bounded by 2. Otherwise, \( A(g^{-1} \gamma) \) is not equal to \( X_{q(i+1)} \). Then for \( |e_i| > E_2, f_{q(i)}^{\varepsilon_i} \mu \in U^\pm \) we may get the bound \( m_2 \) as in part [i].

Thus we obtain the bound \( 6 \max \{2, m_1, m_2\} + 2M + 6 \) for the sum (7.13).

In all of the other cases \( d_{\gamma}(\mu_I, \mu_T) \) is bounded above by a sum like (7.14) with at most 10 terms. Then we may bound each term of the sum as the above paragraph. Since the number of terms of the sum in each case is at most 10 (7.13) is bounded above by \( 10 \max \{2, m_1, m_2\} + 2M + 24 \).

Case (2). Suppose that \( i = 3 \). By Proposition 2.15 \( d_{\gamma}(\mu_T, \partial Z_3) \leq M \). Then by the triangle inequality

\[ d_{\gamma}(\mu_I, \mu_T) \leq d_{\gamma} (\partial Z_3, \mu_T) + d_{\gamma}(\mu_I, \partial Z_3) \leq d_{\gamma}(\mu_I, \partial Z_3) + M + 1. \]

Now \( \mu \supset \{ \partial X_a \}_{a=0,1,2,3} \) and \( \text{diam}_{\gamma}(\mu) \leq 2 \) so \( d_{\gamma}(\mu_I, \partial Z_3) \leq d_{\gamma}(\mu, f_{q(1)}^{\varepsilon_1} f_{q(2)}^{\varepsilon_2}) + 2 \). Then by the triangle inequality we have

\[ d_{\gamma}(\mu_I, \partial Z_3) \leq d_{\gamma}(\mu, f_{q(2)}^{\varepsilon_2} \mu) + d_{\gamma}(f_{q(2)}^{\varepsilon_2} \mu, f_{q(1)}^{\varepsilon_1} f_{q(2)}^{\varepsilon_2} \mu) + 2 + 2 \]

The first term above can be bounded as the third term of (7.13) by 2 or \( m_1 \). The second term can be bounded as the fourth term of (7.13) by 2 or \( m_2 \). So \( d_{\gamma}(\mu_I, \mu_T) \leq 2 \max \{2, m_1, m_2\} + M + 5 \). When \( i = 5 \), with a similar argument we can obtain the bound \( d_{\gamma}(\mu_I, \mu_T) \leq 4 \max \{2, m_1, m_2\} + M + 9 \).

Case (3). Considering \( q'(i) = q(k - i + 1) \) for \( i = 1, \ldots, k \) and \( e'_i = e_{k-i+1} \) for \( i = 1, \ldots, k \) the argument of Case (2) for \( q' \) and \( e'_i \) gives us the bound \( d_{\gamma}(\mu_I, \mu_T) \leq 5 \max \{2, m_1, m_2\} + M + 9 \).

Establishing the bounds in cases (1), (2) and (3) we conclude that \( m' = 10 \max \{2, m_1, m_2\} + 4M + 24 \) is the desired bound in [ii].
Proposition 7.11. There are $K_1 \geq 1$ and $C_1 \geq 0$ and $E > E_2$, depending only on the partial pseudo-Anosov maps $f_0, f_1, f_2$ and $f_3$ and $\mu_I$ with the following properties. Given $q$ and $\{e_i\}_i$ such that $|e_i| > E$ for any $i \in \{1, ..., k\}$, we have

\begin{equation}
    d_{Z_i(q,e)}(\mu_I(q,e), \mu_T(q,e)) \asymp_{K_1,C_1} |e_i|
\end{equation}

Proof. By Lemma 7.6 there are $K'_1, C'_1$ and $E_1$, such that when $|e_i| > E_1$ for all $i \in \{1, ..., k\}$, we have $d_{Z_i(\mu_I, \mu_T)} \geq K'_1 |e_i| - C'_1$.

Let $\mu = \mu_I(q,e)$. Let $i \in \{1, ..., k\}$ and $g = f^{e_1(q)}_{q(1)} ... f^{e_{i-1}(q)}_{q(i-1)}$. Applying $g^{-1}$, $d_{Z_i(\mu, \mu_T)} = d_{X_i(i)}(g^{-1}\mu, f^{e_i}_{q(i)}h\mu)$, where $h = f^{e_{i+1}}_{q(i+1)} ... f^{e_k}_{q(k)}$. Now by the triangle inequality

\begin{equation}
    d_{X_i(i)}(g^{-1}\mu, f^{e_i}_{q(i)}h\mu) \leq d_{X_i(i)}(f^{e_i}_{q(i)}h\mu, h\mu) + d_{X_i(i)}(h\mu, g^{-1}\mu).
\end{equation}

In Lemma 7.6 we proved that $d_{X_i(i)}(h\mu, g^{-1}\mu) \leq C'_1$. So here we only need to show that for some $K_1 \geq 1$, $d_{X_i(i)}(f^{e_i}_{q(i)}h\mu, h\mu) \geq K_1 |e_i|$.

Define $q'(j) = q(j + i - 1)$ for $j = 1, ..., k - i + 1$ and $e'_j = e_{j+i-1}$ for $j = 1, ..., k - i + 1$. Then the subsurface $X_i(i)$ is not in the list of subsurfaces $Z_j(q', e')$. So by Proposition 7.9 there are $E_2$ and $m$ such that: if $|e_i| > E_2$ then $d_{X_i}(\mu, h\mu) \leq m$. Note that this bound does not depend on $e_i$.

Let $\tau_a := \tau_{a,n} = \limsup_{n \to \infty} d_{X_a}(\delta, f^n\delta)$ for every $\delta \in C(X_a)$, $a = 0, 1, 2, 3$, as in Lemma 7.2. Let $a \in \{0, 1, 2, 3\}$. Given $\delta \in C(X_a)$, there is $N = N(\delta)$ such that if $n \geq N$, then $d_{X_a}(\delta, f^n\delta) \leq (\tau_a + 1)n$. Moreover, note that for $\delta' \neq \delta$, $|N(\delta) - N(\delta')|$ is bounded by a constant depending only on the distance of $\delta$ and $\delta'$ in $C(X_a)$ (see the proof of Lemma 7.2). Thus $N$ is the same for all $\delta$ in the $\mathbf{m}$-neighborhood of $\pi_{X_a}(\mu) \in C(X_a)$. Then there exists $E > E_2$ (recall that $E_2 > E_1$) such that for any $n \geq E$, $d_{X_a}(\delta, f^n\delta) \leq (\tau_a + 1)n$ for every curve $\delta$ in the $\mathbf{m}$-neighborhood of $\pi_{X_a}(\mu)$ in $C(X_a)$ and $a = 0, 1, 2, 3$. Now since $\pi_{X_a(i)}(h\mu)$ is in the $\mathbf{m}$-neighborhood of $\pi_{X_a(i)}(\mu)$, if $|e_i| > E$, we have $d_{X_a(i)}(h\mu, f^{e_i}_{q(i)}h\mu) \leq (\tau_a + 1)|e_i|$.

Let $K_1 = \max \{\tau_a + 1,\frac{1}{\tau_a} : a = 0, 1, 2, 3\}$ then as we saw above $d_{X_i(i)}(\mu, h\mu) \leq K_1|e_i|$. Further, by Lemma 7.6, $d_{X_i(i)}(\mu, h\mu) \geq K'_1|e_i| - C_1 \geq \frac{1}{K_1}|e_i| - C'_1$. Thus for constants $K_1, C_1 = C_1$ and $E$ the proposition holds.

Lemma 7.6 and propositions 7.9 and 7.11 together prescribe the list of all subsurface coefficients of a hierarchy path between $\mu_I(q,e)$ and $\mu_T(q,e)$ (corresponding to $q$ and $\{e_i\}_i$). To generalize this construction to infinite hierarchy paths, let $q : N \to \{0, 1, 2, 3\}$ be as the beginning of this subsection and let $e_i$ be a sequence of integers. For each $i \geq 1$ set the subsurface $Z_i(q,e) = f^{e_1}_{q(1)} ... f^{e_{i-1}_{q(i-1)}}_{q(i-1)}\partial X_{q(i)}$. Define $q^k : \{1, ..., k\} \to \{0, 1, 2, 3\}$ by $q^k(i) = q(i)$ for $i = 1, ..., k$, and define the sequence $e^k_i = e_i$ for $i = 1, ..., k$. Let $\mu \equiv \mu_I(q^k, e^k)$ be a marking containing $\{\partial X_a\}_{a=0,1,2,3}$ and let $\mu_k = \mu_T(q^k, e^k)$ be defined as before. For each $k \geq 1$, let $\delta_k$ be a curve in the base of the marking $\mu_k$, after possibly passing to a subsequence $\delta_k$'s converge to a
lamination \( \lambda \) in the Hausdorff topology of closed subsets of the surface \( S \).

Note that by Proposition 2.4 \( \lambda \) contains the support of any accumulation point of the projective classes \( [\delta_k] \) in the \( \mathcal{PLM}(S) \) topology. Here each \( \delta_k \) is equipped with the transversal measure \( i(\delta_k,.) \).

**Proposition 7.12.** There are constants \( m, m' > 4M, C_1 \geq 0, K_1 \geq 1 \), depending only on \( f_0, f_1, f_2 \) and \( f_3 \), and \( \mu_T \) with the following properties. Given \( q \) and \( \{ e_i \}_i \) such that \( |e_i| \geq E \) for any \( i \) in the domain of \( q \), we have

(i) For any integer \( i \) in the domain of \( q \), \( d_{Z_i(q,e)}(\mu_T(q,e)) \simeq K_1 C_1 |e_i| \).

(ii) For any non-annular subsurface \( W \) which is neither \( Z_i \) for some \( i \) nor \( S \) we have \( d_W(\mu_T, \mu_T) \leq m \).

(iii) For any \( \gamma \in C_0(S) \) we have \( d_\gamma(\mu_T, \mu_T) \leq m' \).

(iv) Given \( i, j \geq 1 \), if \( j \geq i+2 \), then \( Z_i < Z_j \).

(v) Let \( i \) be such that \( q(i) = 1 \) or \( 3 \). If \( j \in J_{Z_i} \) then \( j \geq \min J_{Z_i} \) and \( j \leq \max J_{Z_i} \).

When the domain of \( q \) is \( \mathbb{N} \), \( \mu_T(q,e) \) is a minimal filling lamination.

**Proof.** When the domain of \( q \) is \( \{1,...,k\} \) for some \( k \geq 1 \) the subsurface coefficient bounds (ii) and (iii) are already established Proposition 7.9 and (iv) in Propositions 7.11. Further, the order of domains (iv) is established in Lemma 7.6 (ii) and (v) is proved in Lemma 7.8.

We proceed to establish the bounds when the domain of \( q \) is \( \mathbb{N} \). Given \( i \geq 1 \), let \( Z_i = f_1^{q(1)}...f_i^{q(i-1)}X_{q(i)} \). By Proposition 7.11 we have that for every \( k \geq i \), \( d_{Z_i}(\mu, \mu_k) \asymp c_1 \). Then the Hausdorff convergence of \( \delta_k \)'s to \( \mu_T(q,e) \) (\( \delta_k \) is a curve in the base of \( \mu_k \)) implies that for all \( k \) sufficiently large \( d_{Z_i}(\mu, \mu_k) \asymp c_{1,1} \) by Proposition 7.11. (ii) is established.

Let \( W \) be a subsurface which is neither \( Z_i \) for some \( i \geq 1 \) nor \( S \). By Proposition 7.9 (ii) for any \( k \geq 1 \) we have that \( d_W(\mu, \mu_k) \leq m \). Then the Hausdorff convergence of \( \delta_k \)'s to \( \mu_T(q,e) \) implies that for all \( k \) sufficiently large \( d_W(\mu, \mu_k) \asymp c_{1,1} \) by Proposition 7.11. (iii) is established. Similarly Proposition 7.9 (ii) and the Hausdorff convergence imply that \( d_{\gamma}(\mu, \mu_k) \leq m' \), (iii) is established.

Given \( k \geq 1 \) and \( i, j \leq k \) by Lemma 7.6 (ii), the order of the subsurfaces \( Z_i \) and \( Z_j \) with respect to \( \mu \) and \( \mu_k \) is: \( Z_i < Z_j \) if \( j \geq i+2 \). By Proposition 2.15 this means that \( Z_i \nsubseteq Z_j \) and the subsurface coefficient bounds \( d_{Z_i}(\mu, \mu_k) > 4M \), \( d_{Z_j}(\mu, \mu_k) > 4M \) and \( d_{Z_j}(\mu, \partial Z_j) > 2M \) hold. Then the Hausdorff convergence of \( \mu_T \) to \( \mu_k \) implies that the first two subsurface coefficient lower bounds hold for \( \mu \) and \( \mu_T(q,e) \) and (iv) follows for \( \mu \) and \( \mu_T(q,e) \).

The proof of (v) in Lemma 7.8 uses the fact that the \( Z_i \) subsurface coefficients are greater than or equal to \( 2M \), and the way that the subsurfaces \( Z_{i-1} \) and \( Z_i \) intersect. By part (ii) the \( Z_i \) subsurface coefficients of \( \mu \) and \( \mu_T \) are greater than \( 2M \). Moreover, their pattern of intersection is the same as the lemma. Then the proof of Lemma 7.8 goes through and gives us (v) when the domain of \( q \) is \( \mathbb{N} \) holds as well.
For each \( k \geq 1 \) let \( \rho_k \) be a hierarchy path between \( \mu \) and \( \mu_k \). Let \( \rho \) be a hierarchy path between \( \mu \) and \( \mu_T(q,e) \). For any \( i \) with \( q(i) = 1 \) or \( 3 \), by part (ii), \( d_{Z_i}(\mu, \mu_T(q,e)) > M \), so \( Z_i \) is a component domain of \( \rho \) with one boundary curve. Thus \( \partial Z_i \) is a curve on the main geodesic of \( \rho \). Furthermore, \( Z_i < Z_j \) implies that \( \partial Z_i \) is before \( \partial Z_j \) along the main geodesic of \( \rho \). To see this, note that since \( \partial Z_i \) and \( \partial Z_j \) are on the main geodesic and each consist of one curve, the tight geodesics \( g_{Z_i} \) and \( g_{Z_j} \) are time ordered as is defined in §4 of [MM00]. Then if \( \partial Z_i \) is after \( \partial Z_j \) there would be an \( m \in J_{Z_i} \) so that \( m > \max J_{Z_j} \) (see §5 of [MM00]), but this contradicts the second bullet of Proposition 2.15. Therefore, the curves \( \partial Z_i \) converge to a point \( \xi \) in the Gromov boundary of \( \mathcal{C}(S) \) as \( i \to \infty \). This point by Theorem 2.2 determines a projective measured lamination \([\mathcal{E}]\) with minimal filling support. Furthermore, given \( k \geq 1 \), for each \( 1 \leq i \leq k \), \( d_{Z_i}(\mu, \mu_k) > M \), so \( \partial Z_i \) is on the main geodesic of \( \rho_k \) as well. This implies that as \( k \to \infty \), \( \delta_k \) is \( \text{base}(\mu_k) \) converge to \( \xi \) in the Gromov boundary of \( \mathcal{C}(S) \). By Theorem 2.3 the projective classes \([\delta_k]\) after possibly passing to a subsequence converge to a lamination with support equivalent to the support of \( \mathcal{E} \). Then by Proposition 2.4, is contained in \( \mu_T(q,e) \) which implies that minimal filling.

\[ \Box \]

7.2. Scheme II. The construction of this subsection will be used in §8.4 to provide examples of recurrent WP geodesic rays.

Let \( \alpha \) be a curve such that \( S \setminus \alpha \) is a large subsurface. Consider the indexed subsurfaces \( X_0 = S \) and \( X_1 = S \setminus \alpha \). Let \( f_0, f_1 \) be partial pseudo-Anosov maps supported on \( X_0 \) and \( X_1 \), respectively. Define functions \( q_0(i) \equiv i \) (mod 2) and \( q_1(i) = q_0(i + 1) \). Let \( q \) be any of \( q_0 \) and \( q_1 \) or their restriction to \( \{1, \ldots, k\} \), where \( k \) is a positive integer.

Let \( q \) be as above. When the domain of \( q \) is \( \mathbb{N} \) let \( \{e_i\}_i \) be an infinite sequence of integers and when the domain of \( q \) is \( \{1, \ldots, k\} \) let \( \{e_i\}_i \) be a sequence of integers with \( k \) elements.

For any \( i \) in the domain of \( q \) set the subsurface

\[ Z_i(q, e) = f_{q(1)}^{e_1} \cdots f_{q(i-1)}^{e_{i-1}} X_q(i) \]

Let \( \mu_I(q, e) \) be a marking whose base contains \( \partial X_0 \) and \( \{1, \ldots, k\} \). When the domain of \( q \) is \( \{1, \ldots, k\} \) for some \( k \geq 1 \) let \( \mu_T(q, e) = f_{q(1)}^{e_1} \cdots f_{q(k)}^{e_k} \mu_I(q, e) \).

When the domain of \( q \) is \( \mathbb{N} \) define \( q^k(i) = q(i) \) for \( i = 1, \ldots, k \) and the sequence \( e_i^k = e_i \) for \( i = 1, \ldots, k \). Let \( \delta_k \) be a curve in the base of \( \mu_T(q^k, e^k) \), after possibly passing to a subsequence \( \delta_k \) converge to a lamination \( \mu_T(q, e) \) in the a Hausdorff topology of closed subsets of the surface.

In the following proposition we establish several bounds on the subsurface coefficients of \( \mu_I(q, e) \) and \( \mu_T(q, e) \).

Remark 7.13. The construction of this subsection and the estimates on subsurface coefficients can be carried out in a more general setting. Here
we restrict ourself to be able to provide detailed step by step estimates and complete arguments.

**Proposition 7.14.** There are constants $m, m' > 4M, C_1 \geq 0$ and $K_1 \geq 1$ depending only on $f_0, f_1$ and $\mu_I$ with the following properties. Given $q$ and $\{e_i\}_i$ such that $|e_i| > E$ for any $i$ in the domain of $q$, we have

(i) For any integer $i$ in the domain of $q$ with $q(i) = 1$,\n\[ d_{Z,(q,e)}(\mu_I(q,e), \mu_T(q,e)) \geq K_1, C_1 |e_i| \].

(ii) Given $i, j \geq 1$, if $i < j$ and $q(i) = q(j) = 0$ then $Z_i < Z_j$. If $j < i$ and $q(i) = q(j) = 0$ then $Z_j < Z_i$.

(iii) For any non-annular subsurface $W$ which is neither $Z_i$ for some $i$ nor $S$ we have $d_W(\mu_I, \mu_T) \leq m$.

(iv) For any $\gamma \in \mathcal{C}_0(S)$ we have $d_2(\mu_I, \mu_T) \leq m'$.

When the domain of $q$ is $\mathbb{N}$ we have

(v) $\mu_T(q,e)$ is a minimal filling lamination.

**Proof.** First we prove the statements (i) to (iv) when for some $k \geq 1$, $q : \{1, \ldots, k\} \to \{0,1\}$ and $\{e_i\}_{i=1}^k$ is a sequence of integers with $k$ elements. Most of the details are similar to the ones given in §7.1 (Scheme I) so here we mainly sketch them and explain the necessary modifications.

We set the constants: $K'_1 = \min_{a=0,1} \{\tau_a\}$ where the constant $\tau_a := \tau_{f_a}$ is from the Proposition 7.1, $C'_1 = 2(B_0 + \eta)$, where $\eta = \max\{d_{X_i}(f^e_{\alpha}, \mu, \mu) : a, b \in \{0,1\}$ and $a \neq b\}$, and $E_1 = \frac{B_0 + 4 + 4\omega + C'_1}{K'_1}$, where $\omega = \max\{d_W(\mu, \partial X) : W \subseteq S$ and $a = 0,1\}$.

Let $l \in \{1, \ldots, k\}$ be such that $q(l) = 1$. Let $g = f^{e_{l-1}}_{q(1)} \cdots f^{e_{l-1}}_{q(l-1)} f^{e_l}_{q(l)}$ be supported on $S \setminus \alpha$ and preserves $\alpha$ and by the definition of $q$, $X_{q(l+2)} = S \setminus \alpha$. Then applying $g^{-1}$ to $d_S(\partial Z_l, \partial Z_{l+2})$ we get $d_S(\alpha, f^{e_{l+1}}_{q(l+1)} \alpha) = f^{e_{l+1}}_{q(l+1)}$ is supported on $S$, so by Proposition 7.1, $d_S(\alpha, f^{e_{l+1}}_{q(l+1)} \alpha) \geq \tau_0 |e_{l+1}| > \tau_0 E_1 \geq 3$.

Therefore,
\[ d_S(\partial Z_l, \partial Z_{l+2}) \geq \tau_0 |e_{l+1}|. \]

Let $i \in \{1, \ldots, k\}$ be such that $q(i) = 1$. Then the proof of Lemma 7.6 (ii) goes through line by line and gives us
\[ d_{Z,(q,e)}(\mu_I(q,e), \mu_T(q,e)) \geq K'_1 |e_i| - C'_1. \]

The only difference is that Claim 7.7 which asserts that $\partial X_{q(i)} \cap f_{q(i+1)}^{e_{i+1}} X_{q(i+2)}$ here is proved as follows: $\partial X_{q(i)} = \alpha$ and by the definition of $q$, $\partial X_{q(i+2)} = \alpha$. Then as we saw above $d_S(\alpha, f^{e_{i+1}}_{q(i+1)}(\alpha)) \geq 3$ which implies that $\alpha \cap f^{e_{i+1}}_{q(i+1)}(\alpha)$.

Then as part (ii) of Lemma 7.6 we may show that the subsurfaces with $q(i) = 1$ are ordered. Here we establish the base of induction as follows: By (7.16) since $|e_3| > E_1$ we have that $d_S(\partial Z_1, \partial Z_3) \geq 3$ and thus $\partial Z_1 \cap \partial Z_3$. Further by (7.17) $d_{Z_1}(\mu_I, \partial Z_3) > 2M$. Then by Proposition 2.15 $Z_1 < Z_3$.

**Proof of part (iii).** Suppose that $d_W(\mu_I, \mu_T) > 4M$ (otherwise we already have the bound). Let $l$ be such that $q(l) = 1$. Since $|e_{l+1}| > E_1$, it follows
from (7.16) that $\partial Z_l$ and $\partial Z_{l+2}$ fill $S$. Thus $W$ is ordered with respect to either $Z_l$ or $Z_{l+2}$. If $Z_l < W < Z_{l+2}$ then we are in case (1) below. Otherwise, either $Z_l < Z_{l+2} < W$ or $W < Z_l < Z_{l+2}$. In the former situation compare $W$ with $Z_{l+2}$ and $Z_{l+4}$ and in the later situation with $Z_l$ and $Z_{l-2}$. Repeating this comparison we may conclude that $W$ is ordered in the list of subsurfaces $Z_i$ as one of the following cases.

1. There is an index $i$ such that either $Z_{i-1} < W < Z_{i+1}$ or $Z_{i-2} < W < Z_{i+2}$.
2. $W < Z_i$, where $i = 1$ or 2.
3. $Z_i < W$, where $i = k-1$ or $k-2$.

First suppose that $Z_{i-2} < W < Z_{i+2}$. Then

$$d_W(\mu, \mu_T) \leq d_W(\partial Z_{i-2}, \partial Z_{i+2}) + 2M.$$ 

Let $g = f_{q(1)}^{e_1} \cdots f_{q(i-1)}^{e_{i-1}}$. Applying $g^{-1}$ to the subsurface on the left hand side we obtain

$$d_{g^{-1}W}(f_{q(i-1)}^{-e_{i-1}} f_{q(i-2)}^{e_{i-2}} \partial X_{q(i-2)}, f_{q(i-1)}^{e_{i-1}} f_{q(i-2)}^{e_{i-2}} \partial X_{q(i-2)}).$$

Recall that $\mu \supset \{\partial X_a\}_{a=0,1,2,3}$. Denote $\mu$ by $\mu$. Since $\text{diam}_{g^{-1}W}(\mu) \leq 2$,

$$d_{g^{-1}W}(f_{q(i-1)}^{-e_{i-1}} f_{q(i-2)}^{e_{i-2}} \mu, f_{q(i-1)}^{e_{i-1}} f_{q(i-2)}^{e_{i-2}} \mu) + 4$$

bounds the last subsurface coefficient. By the triangle inequality this is bounded above by

$$d_{g^{-1}W}(f_{q(i-1)}^{-e_{i-1}} f_{q(i-2)}^{e_{i-2}} \mu, f_{q(i-2)}^{e_{i-2}} \mu) + d_{g^{-1}W}(f_{q(i-1)}^{e_{i-1}} \mu, \mu) + d_{g^{-1}W}(\mu, f_{q(i-1)}^{e_{i-1}} \mu)$$

(7.18)
• $f_a^n(\pi X_i(\mu)) \subset U_b^+$ for all $n \geq E_2$ and $f_a^{-n}(\pi X_i(\mu)) \subset U_b^-$ for all $n \geq E_2$, and
• $f_a^n U_b^+, \delta \subset U_b^+$ for all $n \geq E_2$ and $f_a^{-n} U_b^+, \delta \subset U_b^-$ for all $n \geq E_2$.

Now using Claim \ref{claim-7.15} we may proceed as in the proof of part (i) of Proposition \ref{prop-7.9} and bound all of the terms in \ref{eq-7.18} by either $m_1 = \max_{a=0,1} m(f_a, \mu)$ or $m_2 = \max_{a=0,1} m(f_a, U_b^+, \delta)$.

Proof of part (iv). By \ref{eq-7.16} $d_S(\partial Z_{i-2}, \partial Z_i) \geq 3$ so $\partial Z_i$ and $\partial Z_{i-2}$ fill $S$. Similarly $d_S(\partial Z_{i+2}, \partial Z_i) \geq 3$ and $\partial Z_i$ and $\partial Z_{i+2}$ fill $S$. Thus
• $\gamma \cap \partial Z_i$ or $\gamma \cap \partial Z_{i-2}$, and
• $\gamma \cap \partial Z_i$ or $\gamma \cap \partial Z_{i+2}$

Then similar to the proof of part (ii) of Proposition \ref{prop-7.9} we can show that $\gamma$ is ordered in the list of $Z_i$’s as one of the following cases:

(1) There is an index $i$ such that either $Z_{i-1} < W < Z_{i+1}$ or $Z_{i-2} < W < Z_{i+2}$,
(2) $\mu_T < \gamma < Z_i$, where $i = 1$ or 2,
(3) $Z_i < \gamma < \mu_T$, where $i = k - 1$ or $k - 2$.

Then similar to the proof of part (iii) of Proposition \ref{prop-7.9} we may establish the upper bound $m' = 4 \max\{m_1, m_2\} + 4M + 16$ for $d_S(\mu_T, \mu_T)$.

The lower bound \ref{eq-7.17} and part (iii) together as in the proof of Proposition \ref{prop-7.11} give us (i). Here we set $E > E_2$ such that for any integer $e$ with $|e| > E$, $d_{X_n}(\delta, f_a^n(\delta)) \leq (\tau_a + 1)|e|$ for every $\delta$ in the $m$–neighborhood of $\pi X_n(\mu)$ in $C(X_a)$ and $a = 0, 1$. Then for $K_1 = \max\{\frac{1}{\tau_a}, \tau_a + 1\}$, $C_1 = C'_1$ and $E$ \ref{eq-7.11} holds.

Establishing all of the bounds when the domain of $q$ is $\{1, \ldots, k\}$ for some $k \geq 1$, the bounds when the domain of $q$ is $\mathbb{N}$ and the fact that $\mu_T$ is minimal filling, part (i), follow from the limiting argument we gave in the proof of Proposition \ref{prop-7.12}. The order of subsurface, part (ii), also follows from the one when the domain of $q$ is finite we established above and the limiting argument we gave in the proof of Proposition \ref{prop-7.12}.

\qed

8. Weil-Petersson Geodesics

In this section we use the control on length-functions along WP geodesic segments from \ref{prop-8.6} and the pair of laminations/markings with prescribed list of subsurface coefficients from \ref{prop-8.7} to provide examples of Weil-Petersson geodesics with certain behavior in the moduli space. For example divergent rays in the moduli space and closed geodesics in the thin part of moduli space. We also provide a recurrence condition for WP geodesics in terms of
ending laminations. Our results in this section can be considered as a kind of symbolic coding of WP geodesics.

In [PWW10], the authors construct WP geodesics which are dense in the moduli space when $\xi(S) = 1$. Jeff Brock produces examples of divergent WP geodesic rays with minimal filling ending lamination in any complexity. Both constructions start with a piecewise geodesic in the Weil-Petersson completion of Teichmüller space. Then applying high Dehn twists about curves in the multi-curves which determine the strata that the piecewise geodesic intersects manage to replace the piecewise geodesic with a piecewise geodesic in the completion of Teichmüller space with arbitrary small exterior angles. Then using a kind of shadowing lemma perturb it to a single WP geodesic in the Teichmüller space. In these examples the relation between the itinerary of the ray the end invariants and their associated subsurface coefficients is not explicit.

These constructions are analogue of the ones for Teichmüller geodesics. Cheung and Masur in [CM06] give examples of divergent Teichmüller geodesic rays with uniquely ergodic vertical lamination. Rafi using the control on length functions in terms of subsurface coefficients he developed along Teichmüller geodesic constructs closed Teichmüller geodesics staying in the thin part and divergent geodesic rays.

8.1. Weil-Petersson geodesic rays with prescribed itinerary. In this subsection by extracting limits of WP geodesic segments with end invariants on a single infinite hierarchy path with narrow end points we construct WP geodesics whose behavior mimic the combinatorial properties of hierarchy paths.

Let $\nu \in \mathcal{GL}(S)$. Let $Z_a$, $a = 1, \ldots, m$, be the connected components of $S \setminus \{\text{closed leaves of } \nu\}$ with $\xi(Z_a) \geq 1$ such that $\nu_a$ the restriction of $\nu \setminus \{\text{closed leaves of } \nu\}$ to $Z_a$ is minimal filling on $Z_a$. For $a = 1, \ldots, m$ fix measures $\mathcal{L}_a$ supported on $\nu_a$. By Theorem 2.2 the projective class of the measure $\mathcal{L}_a$ supported on $\nu_a$ is a point in the Gromov boundary of $\mathcal{C}(Z_a)$. Let $\gamma_{n,a} \in \mathcal{C}(Z_a)$ be a sequence of curves which converges to $\mathcal{L}_a$ as $n \to \infty$ in the weak$^*$ topology of $\mathcal{ML}(Z_a)$. For $n \geq 1$ let $Q_n$ be a pants decomposition containing $\bigcup_{a=1}^m \partial Z_a \cup \gamma_{n,a}$. Let $[x, c_n]$ be the WP geodesic segment connecting a base point $x$ in the interior of Teichmüller space to $c_n$, a maximally nodal hyperbolic surface at $Q_n$. Denote the parametrization of $[x, c_n]$ by arc-length by $r_n$. The proof of the following lemma essentially follows the proof of surjectivity of weighted ending laminations of WP geodesic rays in the Teichmüller space of surface with complexity 5 given in §4 of [BM08].

Lemma 8.1. (Infinite ray) After possibly passing to a subsequence the geodesic segments $r_n$ converge to an infinite ray $r$ in the Weil-Petersson visual sphere at $x$. Furthermore the forward ending lamination of $r$ contains $\nu_a$, for all $a = 1, \ldots, m$, and the length of every curve $\alpha \in \partial Z_a$, $a = 1, \ldots, m$, is decreasing along $r$. 
Proof. By Theorem 3.2 (Non-refraction Theorem) the interior of each one of the WP geodesic segments \( r_n \) is inside the Teichmüller space. Moreover, by the local compactness of the WP metric at \( x \) after possibly passing to a subsequence the initial parts of \( r_n \)’s converge to a geodesic segment \( r_\infty \) starting at \( x \). Let \( r \) be the maximal geodesic ray in Teichmüller space with initial part \( r_\infty \). We prove that \( r \) is an infinite ray.

Let \( s_n = \frac{1}{\ell_{Q_n}(x)} \), where \( \ell_{Q_n}(x) = \max_{t \in Q_n} \ell_t(x) \). Then for each integer \( n \), and \( t \) in the domain of \( r_n \) we have that \( \ell_{s_n Q_n}(t(t)) \leq 1 \). This follows from the convexity of length-functions along WP geodesics and the observation that \( \ell_{s_n Q_n}(x) = 1 \) and \( \ell_{s_n Q_n}(c_n) = 0 \). Now assuming that \( r(t) \) has finite length \( T \) we get a contradiction.

After possibly passing to a subsequence \( s_n Q_n \) converge to some \( \mathcal{L} \in \mathcal{ML}(S) \) in the weak* topology of \( \mathcal{ML}(S) \). Let \( \mathcal{L}_a \) be the restriction of \( \mathcal{L} \) to \( Z_a \), \( a = 1, \ldots, m \). By the way we chose \( Q_n \)’s, the projective class of \( \pi_{Z_a}(Q_n) \) converge to the projective class of a measure supported on \( \nu_a \). Then by Theorem 2.3 the support of \( \mathcal{L}_a \) is a measured lamination equivalent to \( \nu_a \) and in particular fills \( Z_a \). Now let \( \gamma \) be a curve with \( i(\gamma, \mathcal{L}) = 0 \). Then it is disjoint from \( \mathcal{L}_a \)’s and since for each \( a \), \( \mathcal{L}_a \) fills \( Z_a \), \( \gamma \) is disjoint from all \( Z_a \)’s.

The length-function \( \ell(\mathcal{L}) : \text{Teich}(S) \times \mathcal{ML}(S) \to \mathbb{R}^{\geq 0} \) is continuous in both \( x \) and \( \mathcal{L} \) variables, which implies that \( \ell(\mathcal{L})(r(t)) \leq 1 \) for any \( t < T \). Thus

\[
\lim_{t \to T} \ell(\mathcal{L})(r(t)) \leq 1.
\]

Let \( \sigma \) be the maximal multi-curve such that \( r(T) \in S(\sigma) \). Then for each simple closed curve \( \gamma \in \sigma \) we have that \( i(\gamma, \mathcal{L}) = 0 \). Otherwise, since \( \ell(\gamma(r(t))) \to 0 \) as \( t \to T \), we would have that \( \ell(\gamma)(r(t)) \to \infty \) as \( t \to T \), which contradicts the bound \( \underline{(8.1)} \). This as we saw in the previous paragraph implies that \( \sigma \) is disjoint from all \( Z_a \)’s, and thus \( \sigma \subseteq \bigcup_{a=1}^{m} \partial Z_a \).

Claim 8.2. \( d_{\text{WP}}(r(T), c_n) \to \infty \) as \( n \to \infty \).

Let \( 1 \leq a \leq m \). By the choice of \( Q_n \)’s \( [Q_n] \to [\mathcal{L}_a] \) as \( n \to \infty \) and \( [\mathcal{L}_a] \) is in the Gromov boundary of \( C(Z_a) \) so \( d_{Z_a}(Q_n, Q(r(T))) \to \infty \) as \( n \to \infty \). So by the distance formula \( \underline{(2.2)} \) \( d(Q(c_n), Q(r(T))) \to \infty \) as \( n \to \infty \). Then by Theorem 3.3 (Quasi-Isometric Model) \( d_{\text{WP}}(r(T), c_n) \to \infty \) as \( n \to \infty \).

Claim 8.3. There is \( d > 0 \) such that \( d_{\text{WP}}(c_n, S(\sigma)) \) for all \( n \) sufficiently large.

Let \( P_n \) be the pants decomposition consisting of the curves in \( \pi_{S \setminus \sigma}(Q_n) \) and \( \sigma \). First we show that \( d(Q_n, P_n) \) is uniformly bounded. Let \( W \subseteq S \) be a non-annular subsurface. Either \( W \cap S \setminus \sigma \) or \( W \cap \sigma \). Suppose that \( W \cap S \setminus \sigma \), then let \( \alpha \in \pi_{S \setminus \sigma}(Q_n) \) be such that \( \alpha \cap W \). Then since \( \alpha \) is in both \( P_n \) and \( Q_n \) by the triangle inequality we have \( d_W(P_n, Q_n) \leq d_W(P_n, \alpha) + d_W(\alpha, Q_n) \leq 2 \). If not, then \( W \cap \sigma \). As we saw above \( \sigma \subseteq \bigcup_{a=1}^{m} \partial Z_a \) and by the choice of \( Q_n \)’s for \( n \) sufficiently large \( \bigcup_{a=1}^{m} \partial Z_a \subset Q_n \), so \( \sigma \subseteq Q_n \). Thus again by the triangle inequality \( d_W(P_n, Q_n) \leq d_W(P_n, \sigma) \).
$d_W(\sigma, Q_n) + \text{diam}_W(\sigma) \le 3$. Thus $d_W(P_n, Q_n) \le 3$ for every $W \subseteq S$. Let $C$ be the additive constant corresponding to the threshold constant 3 in the distance formula (2.2). Then $d(P_n, Q_n) \le C$ for all $n$ sufficiently large.

Let $x_{P_n} \in \text{Teich}(S)$ be a point with a Bers pants decomposition $P_n$. By the bound from the previous paragraph Theorem 3.3 implies that $d_{WP}(c_n, x_{P_n}) \le K_{WP}C + C_{WP}$. Furthermore, $\sigma \in P_n$ so by Proposition 3.6 $d_{WP}(x_{P_n}, \mathcal{S}(\sigma)) \le \sqrt{2\pi \sum_{\gamma \in \sigma} \ell_{\gamma}} \le \sqrt{2\pi \xi(S)L_S}$. Then by the triangle inequality $d_{WP}(c_n, \mathcal{S}(\sigma)) \le d$, where $d$ is the sum of the two upper bounds.

**Claim 8.4.** For any $t' > T$ the points $r_n(t')$ converge to $\overline{\mathcal{S}(\sigma)}$.

Let $c_n'$ be the nearest point to $c_n$ on the $\sigma$-stratum. Let $\eta_n$ be the parametrization of $[r(T), c_n]$ by arc-length with $\eta_n(0) = r(T)$ and $\eta_n'$ be the parametrization of $[r(T), c_n']$ by arc-length with $\eta_n'(0) = r(T)$. Since $\mathcal{S}(\sigma)$ is geodesically convex $\eta_n' \subset \mathcal{S}(\sigma)$. By Claim 8.2 $d_{WP}(r(T), c_n) \to \infty$ as $n \to \infty$. By Claim 8.3 $d_{WP}(c_n, \mathcal{S}(\sigma)) \le d$. Then the CAT(0) comparison for the triangles with vertices $r(T), c_n$ and $c_n'$ implies that given any $s$, $\eta_n(s) \to \eta_n'(s)$ as $n \to \infty$.

Further $\eta_n$ and $[r_n(T), c_n]$ have the same end point $c_n$, and $r_n(T)$ converges to $r(T)$ as $n \to \infty$. Then the CAT(0) comparison for the triangle with vertices $r(T), r_n(T)$ and $c_n$ implies that the Hausdorff distance between $\eta_n$ and $[r_n(T), c_n]$ tends zero as $n \to \infty$. This together with the previous paragraph imply that $r_n(t')$ converges into the $\sigma$-stratum.

By Claim 8.4 the points $r_n(t')$ converge to a point $y \in \overline{\mathcal{S}(\sigma)}$. By the Non-refraction Theorem the interior of $[x, y]$ is inside the maximal stratum containing its endpoints which is the Teichmüller space. By the CAT(0) comparison the Hausdorff distance of $r|_{[0, t']}$ and $[x, y]$ goes to 0 as $n \to \infty$. So $r_n(T)$ converges to a point inside the Teichmüller space. But this contradicts the assumption that $r_n(T)$ converge to a point in the $\sigma$-stratum.

This finishes the proof of that the geodesic $r$ is an infinite geodesic ray.

We proceed to show that the forward ending lamination of $r$ contains $\nu_a$ for $a = 1, \ldots, m$. Fix an $a$. Let $t_n \in [0, \infty)$ be a sequence of times such that $d_{WP}(r(t_n), c_n) \le d$. Let $D = K_{WP}d + C_{WP}$, then $d(Q(r(t_n)), Q(c_n)) \le D$. Let $\alpha_n \in Q(r(t_n))$ be such that $\alpha_n \pitchfork Z_a$, then by (6.6) $d_{Z_a}(\alpha_n, \gamma_n) \le D$. Furthermore, $[\gamma_n]$ converge to the projective class of measured lamination $\mathcal{E}$ with support $\nu_a$ in the Gromov boundary of $\mathcal{C}(Z_a)$. Then the bound (*) implies that after possibly passing to a subsequence $[\alpha_n]$ converges to $[\mathcal{E}]$. This follows for example from the definition of convergence of sequences in the union of a $\delta$–hyperbolic space and its boundary using the Gromov inner product, see §III.H.3 of [BH99]. Now $\alpha_n$ is a sequence of distinct Bers curves along $r$ so by the definition of forward ending lamination, $\nu_a$ is contained in the forward ending lamination of $r$.

To prove the last statement of the lemma fix $1 \le a \le m$. Let $\alpha \in \partial Z_a$. For $n$ sufficiently large $\partial Z_a \subseteq Q_n$, so $\alpha$ is a pinching curve of $r_n$. 


Furthermore, \( r_\nu \)'s converge to \( r \). Then Lemma 3.14 implies that the length of \( \alpha \) is decreasing along \( r \).

Let \((\nu^-, \nu^+)\) be a narrow pair, where \( Z = S\setminus \{\text{closed leaves of } \nu^+\} \) is a large subsurface and \( \nu' = \nu^+\setminus \{\text{closed leaves of } \nu^+\} \) is a minimal filling lamination on \( Z \). Let \( \rho \) be any infinite hierarchy path between \( \nu^- \) and \( \nu^+ \) as above. Recall that by Theorem 5.5 any two such resolution paths fellow travel each other in the pants graph. The above lemma provides us with an infinite ray, denoted the ray by \( r_{\nu^\pm} \), with forward ending lamination containing \( \nu' \). The following theorem shows that the behavior of \( r_{\nu^\pm} \) mimics the combinatorial properties of hierarchy paths encoded in the end invariant listed in Theorem 2.13.

**Theorem 8.5.** (Infinite ray with prescribed itinerary) Given \( A, R, R' > 0 \).
Let \((\nu^-, \nu^+)\) be an \( A \)-narrow pair. Let \( \rho \) be a hierarchy path between \( \nu^- \) and \( \nu^+ \). Let \( r_{\nu^\pm} : [0, \infty) \to \text{Teich}(S) \) be the corresponding infinite WP geodesic ray.

Let \( \bar{\epsilon} = \bar{\epsilon}(A, R) \) be the constant from Lemma 6.6 and for an \( \epsilon \leq \bar{\epsilon} \) let \( \bar{w} = \bar{w}(A, R, R', \epsilon) \) be the constant form Theorem 6.7. Assume that \( Z \) a large component domain of \( \rho \) has \((R, R')\)-bounded combinatorics over an interval \([m', n'] \subset J_Z \) with \( n' - m' > 2\bar{w} \). Then

1. \( \ell_\gamma(r_{\nu^\pm}(t)) > \epsilon \) for every \( \gamma \notin \partial Z \), and
2. \( \ell_\alpha(r_{\nu^\pm}(t)) \leq \epsilon \) for every \( \alpha \in \partial Z \)

for every \( t \in [a', b'] \), where \( a' \in N(m' + \bar{w}) \) and \( b' \in N(n' - \bar{w}) \). Here \( N := N_{\rho, \gamma} \) is the parameter map from Proposition 5.27.

Moreover, if \( Z_1 \) and \( Z_2 \) are subsurfaces as above, \( n_1' < m_2' \) implies that \( b_1' < a_2' \).

**Proof.** By the narrow assumption given any subsurface \( Y \subseteq S \) which is not large, \( d_Y(\nu^-, \nu^+) \leq A \). So for any \( n \) Theorem 2.13[0] (no backtracking) implies that \( d_Y(\rho(0), \rho(n)) \leq A + 2M_2 \). Thus the end invariant of the geodesic segment \([x, c_n]\) is \( A + 2M_2 \) narrow. Since \( Z \) has \((R, R')\)-bounded combinatorics over \([m, n]\), by Lemma 6.6, there are \( \bar{\epsilon} \) and \( w \) such that for any \( \gamma \notin \partial Z \), \( \ell_\gamma(r_n(t)) \) for every \( t \in N(j) \) where \( j \in [m' + w, n' - w] \). Moreover, since \( w \leq \bar{w} \) the bounds hold on the interval \([a', b']\). Furthermore, by Theorem 6.1 for every \( \alpha \in \partial Z \), \( \ell_\alpha(r_n(t)) \leq \epsilon \) for every \( t \in [a', b'] \). Since the geodesic rays \( r_n \) converge to \( r_{\nu^\pm} \) point-wise, by Theorem 3.1 (Continuity of length-functions) on Teichm"uller space, the same bounds on length-functions hold along \( r_{\nu^\pm} \). Finally, the statement about order of intervals follows from fellow traveling.

We will refer to this ray as a ray with prescribed itinerary.

### 8.2. Divergent geodesic rays.

In this subsection we construct divergent Weil-Petersson geodesics in the moduli space. A ray is divergent if eventually leaves every compact subset of moduli space. The existence of uncountably
many divergent rays starting at a given point with minimal filling ending lamination is a consequence of our construction in this subsection.

**Definition 8.6.** A geodesic ray \( r : [0, \infty) \to \mathcal{M}(S) \) is recurrent to a compact subset \( K \subset \mathcal{M}(S) \), if there is a sequence of times \( t_i \to \infty \) as \( i \to \infty \) such that \( r(t_i) \in K \). A ray \( r : [0, \infty) \to \mathcal{M}(S) \) is divergent if it is not recurrent to any compact subset of moduli space. In other words, \( r \) is divergent if for every compact set \( K \subset \mathcal{M}(S) \), there is a \( T \geq 0 \) such that \( r([T, \infty)) \) does not intersect \( K \).

**Proof of Theorem 1.2.** Let the indexed subsurfaces \( X_0, X_1, X_2 \) and \( X_3 \), and partial pseudo-Anosov maps \( f_0, f_1, f_2 \) and \( f_3 \) supported on them respectively be as in §7.1. Recall that \( X_0 \) and \( X_2 \) are the same subsurface with different indices. Let the function \( q = q_0 \) where \( q_0(i) \equiv i \) (mod 4) and the sequence of integers \( e_i > E \) be as in Proposition 7.12. For each \( i \geq 1 \) let

\[ Z_i = f_{q(i)}^{i-1} \cdots f_{q(1)}^{i-1} X_{q(i)}. \]

Fix a marking \( \mu_\ell \) containing \( \{\partial X\}_{n=0,1,2,3} \) and let \( \mu_T \) be as in Proposition 7.12. Then by part (i) of the proposition there are constants \( K_1, C_1 \) and \( E > 0 \) such that if \( |e_i| > E \) then

\[ d_{Z_i}(\mu_\ell, \mu_T) \asymp_{K_1, C_1} |e_i|. \]

In particular, by the choice of \( E \),

\[ d_{Z_i}(\mu_\ell, \mu_T) > 4M. \]

By Proposition 7.12 (ii) every subsurface \( Z \) with \( d_{Z}(\mu_\ell(q,e), \mu_T(q,e)) > m \) is in the list of the subsurfaces \( Z_i \) and consequently is a large subsurface. So the pair \( (\mu_\ell, \mu_T) \) is \( m \)-narrow. Let \( \rho : [0, \infty) \to P(S) \) be a hierarchy path between \( \mu_\ell \) and \( \mu_T \). Let \( r : [0, \infty) \to \text{Teich}(S) \) be the WP geodesic ray with end invariant \( (\mu_\ell, \mu_T) \) and prescribed itinerary as in Theorem 8.5. By Proposition 5.25 we have the parameter map \( N \) from the parameters of \( \rho \) to the parameters of \( r \).

For each \( i \) odd, let \( k_i = \max J_{Z_i} \cap J_{Z_{i-1}} \) and \( l_i = \min J_{Z_i} \cap J_{Z_{i+1}} \) and suppose that \( l_i > k_i \). For each \( i \) even, let \( k_i = \min J_{Z_i} \) and \( l_i = \max J_{Z_i} \). Note that when \( i \) is even \( J_{Z_i} = [k_i, l_i] \).

First we collect some subsurface coefficient bounds.

**i is even.** Let \( J_{Z_i} = [j^-, j^+] \) (when \( i \) is even \( k_i = j^- \) and \( l_i = j^+ \)). By Proposition 7.12 (ii) and no backtracking (6.1) for every subsurface which is not in the list of \( Z_i \)'s we have that

\[ d_{W}(\rho(j^-), \rho(j^+)) \leq m + 2M. \]

By Proposition 7.12 (iv), for all \( j \geq 1 \) with \( j - i \geq 2 \), \( Z_i < Z_j \) and for all \( j \) with \( i - j \geq 2 \), \( Z_j < Z_i \). Suppose that \( Z_i < Z_j \). Then by Proposition 2.15 \( d_{Z_i}(\mu_\ell, \partial Z_i) \leq M \). Furthermore \( \partial Z_i \subset \rho(j^-) \) and \( \partial Z_i \subset \rho(j^+) \). So \( d_{Z_i}(\mu_\ell, \rho(j^-)) \leq M + \text{diam}_{Z_i}(\rho(j^-)) = M + 1 \) and \( d_{Z_i}(\mu_\ell, \rho(j^+)) \leq M + 1 \). Then by the triangle inequality

\[ d_{Z_i}(\rho(j^-), \rho(j^+)) \leq d_{Z_i}(\mu_\ell, \rho(j^+)) + d_{Z_i}(\mu_\ell, \rho(j^-)) + 4 \leq 2M + 4. \]
If $Z_j < Z_i$, similarly we can get the same bound. We recored these bounds

(8.4)  $d_{Z_j}(\rho(j^-), \rho(j^+)) \leq 2M + 4$.

$i$ is odd. Let $J_{Z_i} = [j^-, j^+]$. By Proposition 7.12 (iii) and no backtracking [6.1] for every subsurface $W$ which is not in the list of $Z_i$’s,

(8.5)  $d_{W}(\rho(j^-), \rho(j^+)) \leq m + 2M$.

and

(8.6)  $d_{W}(\rho(k_i), \rho(l_i)) \leq m + 2M$.

By Proposition 7.12 (iv), for all $j \geq 1$ with $j - i \geq 2$, $Z_i < Z_j$ and for all $j$ with $i - j \geq 2$, $Z_j < Z_i$. Then similar to the proof of (8.4) we can get

(8.7)  $d_{Z_j}(\rho(j^-), \rho(j^+)) \leq 2M + 4$,

and

(8.8)  $d_{Z_j}(\rho(k_i), \rho(l_i)) \leq 2M + 4$.

By Theorem 2.13 (6)

$$d_{Z_{i-1}}(\rho(k_i), \rho(l_i)) + d_{Z_{i-1}}(\rho(l_i), \mu T) \leq d_{Z_{i-1}}(\rho(k_i), \mu T) + M.$$  

Now by Theorem 2.13 (4), $d_{Z_{i-1}}(\mu T, \rho(k_i)) \leq M$. Then by the above inequality

(8.9)  $d_{Z_{i-1}}(\rho(k_i), \rho(l_i)) \leq 2M$.

Similarly,

$$d_{Z_{i+1}}(\rho(k_i), \mu T) + d_{Z_{i+1}}(\rho(k_i), \rho(l_i)) \leq d_{Z_{i+1}}(\rho(l_i), \mu T) + M,$$

then since $d_{Z_{i+1}}(\mu T, \rho(l_i)) \leq M$ we get

(8.10)  $d_{Z_{i+1}}(\rho(k_i), \rho(l_i)) \leq 2M$.

Finally, for any $i$ by Proposition 7.12 (iii) and no backtracking for any $\gamma \in C_0(S)$,

(8.11)  $d_{\gamma}(\rho(k_i), \rho(l_i)) \leq m' + 2M$.

Now we proceed to estimate the length of $J$ intervals (see Theorem 2.13 (1)) using the above subsurface coefficient bounds.

For each $i$ even by the bounds (8.3) and (8.4) all of the subsurface coefficients of $\rho(j^-)$ and $\rho(j^+)$ except that of $Z_i$ are bounded above by $m + 2M$. Let the threshold constant in the distance formula (2.2) be $m + 2M$ (note that it is larger than $M_1$) and let $K_2, C_2$ be the constants corresponding to this threshold constant. Then we have that

$$d(\rho(j^-), \rho(j^+)) \approx_{K_2, C_2} d_{Z_i}(\rho(j^-), \rho(j^+)).$$

Similarly for each $i$ odd by the bounds (8.5) and (8.7) and the distance formula we have that

$$d(\rho(j^-), \rho(j^+)) \approx_{K_2, C_2} d_{Z_{i-1}}(\rho(j^-), \rho(j^+)) + d_{Z_i}(\rho(j^-), \rho(j^+)) + d_{Z_{i+1}}(\rho(j^-), \rho(j^+)).$$
By the no backtracking \textcircled{(6.1)} and Theorem \textcircled{2.13} \textcircled{4} each subsurface coefficient on the right hand side is \((1,2M)\) comparable with the corresponding subsurface coefficient of \(\mu_T\) and \(\mu_T\). For example \(\tilde{d}_{Z_{i-1}}(\rho(j^-), \rho(j^+)) \asymp_{1,2M} \tilde{d}_{Z_{i-1}}(\mu_T, \mu_T)\). Moreover \(\rho\) is a \((k,c)\)-quasi-geodesic where \(k,c\) depend only on the topological type of \(S\). Let \(K_3 = kK_2\) and \(C_3 = k(C_2 + 2M) + c\). Then we have:

For each \(i\) even
\begin{equation}
|J_{Z_i}| \asymp_{K_3,C_3} d_{Z_i} \left( \mu_T, \mu_T \right)
\end{equation}

and for each \(i\) odd,
\begin{equation}
|J_{Z_i}| \asymp_{K_3,C_3} d_{Z_{i-1}}(\mu_T, \mu_T) + d_{Z_i}(\mu_T, \mu_T) + d_{Z_{i+1}}(\mu_T, \mu_T)
\end{equation}

We proceed to set the sequence \(\{e_i\}_i\) and using the above estimates finish the construction of divergent WP geodesic rays.

Let \(\epsilon_j \to 0\) be a decreasing sequence. Let \(R = m + 2M\) and \(R' = m' + 2M\). For each \(j \geq 1\) let
\begin{equation}
y_j \geq 2\bar{w}(m, R, R', \epsilon_j)
\end{equation}
where \(\bar{w}\) is the constant from Theorem \textcircled{8.5}. Let \(K = K_3K_1\) and \(C = K_1C_1 + K_1K_2C_3\). Define the sequence \(\{e_i\}_i\) as follows: for \(i\) even let \(e_i = Ky_i + KC\), and for \(i\) odd let \(e_i = (K^3y_{i-1} + K^2C) + (K^3y_{i+1} + K^3C)\).

We investigate the pattern that the \(J\) intervals overlap and estimate the length of each interval \([k_i, l_i]\).

\(i\) is even. Recall that when \(i\) is an even integer \([k_i, l_i] = J_{Z_i}\). Then \(\textcircled{(8.12)}\) implies that \(l_i - k_i \geq K_3d_{Z_i}(\mu_T, \mu_T) - C_3\). Then by \(\textcircled{(8.2)}\), \(l_i - k_i \geq K_3E_i - C_3\). Therefore, \(l_i - k_i \geq y_{i-1}^+\).

\(i\) is odd. By \(\textcircled{(8.12)}\) and \(\textcircled{(8.2)}\), \(|J_{Z_{i-1}}| \leq K|e_{i-1}| + C\), so \(|J_{Z_i} \cap J_{Z_{i-1}}| \leq K|e_{i-1}| + C\). Similarly, \(|J_{Z_{i+1}}| \leq K|e_{i+1}| + C\) and \(|J_{Z_i} \cap J_{Z_{i+1}}| \leq K|e_{i+1}| + C\). Furthermore, by \(\textcircled{(8.13)}\) and \(\textcircled{(8.2)}\), \(|J_{Z_i}| \geq \frac{1}{K}(|e_{i-1}| + |e_i| + |e_{i+1}|) - C\).

By Proposition \textcircled{7.12}(v) for every \(j \in J_{Z_i}, j \geq \min J_{Z_{i-1}}, \) so \(J_{Z_{i-1}}\) and \(J_{Z_i}\) intersect as in Figure \textcircled{10}. Furthermore, by the choice of \(e_i\)'s and the bounds above \(|J_{Z_i}| > |J_{Z_i} \cap J_{Z_{i-1}}|\), thus \(l_i - 1 < k_i\). Similarly by Proposition \textcircled{7.12}(v) for every \(j \in J_{Z_i}, j \geq \max J_{Z_{i+1}}\) so \(J_{Z_{i+1}}\) and \(J_{Z_i}\) intersect as in Figure \textcircled{10}. Furthermore, \(|J_{Z_i}| > |J_{Z_i} \cap J_{Z_{i-1}}|\) and \(l_i < k_i+1\).

Finally putting together the bounds on the length of \(J_{Z_i}\) and its intersection with the intervals \(J_{Z_{i-1}}\) and \(J_{Z_{i+1}}\) above, we obtain \(|J_{Z_i} - (J_{Z_{i-1}} \cup J_{Z_{i+1}})| = |J_{Z_i} - |J_{Z_i} \cap J_{Z_{i-1}}| - |J_{Z_i} \cap J_{Z_{i+1}}| \geq |y_{i-1}^+|\). Therefore, \(l_i - k_i \geq y_{i-1}^+\) and \(l_i > k_i\). See Figure \textcircled{10}.

Let \(R = m + 2M\) and \(R' = m' + 2M\). When \(i\) is even by the subsurface coefficient bounds \(\textcircled{(8.3)}, \textcircled{(8.4)}\) and \(\textcircled{(8.11)}\) \(Z_i\) has \((R, R')\)-bounded combinatorics over the interval \([k_i, l_i]\). When \(i\) is odd by the bounds \(\textcircled{(8.6)}, \textcircled{(8.8)}, \textcircled{(8.9)}, \textcircled{(8.10)}\) and \(\textcircled{(8.11)}\) the subsurface \(Z_i\) has \((R, R')\)-bounded combinatorics over the interval \(J_{Z_i} = [k_i, l_i]\).
Figure 10. The intersection pattern of the intervals $J_{Z_i}$, $J_{Z_{i-1}}$, and $J_{Z_{i+1}}$ when $i$ is even. Over the red subintervals of $J$ intervals the corresponding domain has bounded combinatorics. $k_i = \max J_{Z_{i-1}} \cap J_{Z_i}$ and $l_i = \min J_{Z_{i+1}} \cap J_{Z_i}$.

For each $i \geq 1$ let $t_i \in N(j)$ for some $j \in [k_i + \bar{w}, l_i - \bar{w}]$. For any $i \geq 1$, as we saw above, $l_i < k_{i+1}$ so we have that $t_i < t_{i+1}$. Theorem 8.5 applied to the interval $[k_i, l_i]$ implies that

$$\ell_{\partial Z_i}(r(t_i)) \leq \epsilon_{i/2},$$

$$\ell_{\partial Z}(x) = \max \{\ell_\alpha(x) : \alpha \in \partial Z\}.$$ 

By the definition of $q$ any two consecutive domains $Z_i$ and $Z_{i+1}$ have a boundary curve in common $\delta_i := \partial Z_i \cap \partial Z_{i+1}$. Then by the convexity of length-functions we have

$$\ell_{\delta_i}(r(t)) \leq \epsilon_{i/2}$$

for every $t \in [t_i, t_{i+1}]$.

Now since the intervals $[t_i, t_{i+1}]$, $i \geq 1$, cover $[0, \infty)$, the domain of $r$, and $\epsilon_i \to 0$, the systole of the surfaces along the WP geodesic ray $r$ decreases and goes to 0.

By the Mumford’s compactness criterion every compact subset of moduli space is contained in some $\epsilon$–thick part. So $\hat{r}$ the projection of $r$ to the moduli space is a divergent geodesic ray. Finally, by Proposition 7.12 (vi), $\mu_T(q, \epsilon)$ is a minimal filling lamination, so the forward ending lamination of $\hat{r}$ is minimal filling.

Note that there are uncountably sequences $\{y_j\}_{j \geq 1}$ which satisfy inequality (8.14). Consequently, we have uncountably many divergent rays starting from a given point in the moduli space. These geodesics are distinct because have different forward ending lamination. □

8.3. Closed geodesics in the thin part. In this subsection we provide examples of closed Weil-Petersson geodesics which stay in the thin part of moduli space.

Proof of Theorem 1.1. Let the indexed subsurfaces $X_0, X_1, X_2$ and $X_3$, and partial pseudo-Anosov maps $f_0, f_1, f_2$ and $f_3$ supported on them respectively be as in §7.1. Recall that $X_0$ and $X_2$ are the same subsurfaces with different indices. Let the function $q = q_0$ and the sequence of integers $e_i > E$ ($i \geq 1$) be as in Proposition 7.12. For each $i \geq 1$ let

$$Z_i = \frac{f_{q(1)}^{e_1} \cdots f_{q(i-1)}^{e_{i-1}}}{X_q(i)}.$$
Fix a marking \( \mu_I \) containing \( \{ \partial X \}_{\alpha=0,1,2,3} \) and let \( \mu_T \) be as in Proposition 7.12.

Then we are in the set up of the proof of Theorem 1.2 in [8.2] as we saw there the pair \((\mu_I, \mu_T)\) is \( m \)-narrow where \( m \) is the constant form Proposition 7.12. Let \( \rho : [0, \infty] \rightarrow P(S) \) be a hierarchy path between \( \mu_I \) and \( \mu_T \). Let \( r : [0, \infty) \rightarrow \text{Teich}(S) \) be the WP geodesic ray with end invariant \((\mu_I, \mu_T)\) and prescribed itinerary as in Theorem 8.5. Let \( N \) be the parameter map from Proposition 5.25.

As in the proof of Theorem 1.2 for each \( i \) odd let \( k_i = \max J_{Z_{i-1}} \cap J_{Z_i} \) and \( l_i = \min J_{Z_{i+1}} \cap J_{Z_i} \). For each \( i \) even, let \( k_i = \min J_{Z_i} \) and \( l_i = \max J_{Z_i} \).

Then all of the subsurface coefficient bounds (8.3)-(8.11) in the proof of Theorem 1.2 hold.

Let \( R = m + 2M \) and \( R' = m' + 2M \). Given \( \epsilon > 0 \) let
\[
y \geq 2\bar{w}(m, R, R', \epsilon).
\]
where \( \bar{w} \) is the constant from Theorem 8.5.

Let the constants \( K \geq 1 \) and \( C \geq 0 \) be as in the proof of Theorem 1.2. Define the periodic sequence \( e_i \) with \( e_i = e_{j} \) whenever \( i \equiv j \pmod{4} \) and the first four terms \( e_1 = 2(K^3y + K^3C), e_2 = KyK^C, e_3 = 2(K^3y + K^3C), e_4 = Ky + K^C \).

The estimates (8.12) and (8.13) on the length of \( J_{Z_i} \) intervals hold. Then similar to the proof of Theorem 1.2 we can show that the \( J \) intervals intersect as in Figure 10. \( l_i < k_{i+1} \) and \( l_i - k_i \geq y \) for all \( i \geq 1 \).

When \( i \) is even by the subsurface coefficient bounds (8.3), (8.4) and (8.11) \( Z_i \) has \((R, R')\)-bounded combinatorics over the interval \([k_i, l_i]\). When \( i \) is odd by the bounds (8.6), (8.8), (8.9), (8.10) and (8.11) the subsurface \( Z_i \) has \((R, R')\)-bounded combinatorics over the interval \([k_i, l_i]\). Let \( t_i \in \mathcal{N}(j) \) where \( j \in [k_i + \bar{w}, l_i - \bar{w}] \). Theorem 8.5 applied to the interval \([k_i, l_i]\) implies that
\[
\ell_{\partial Z_i}(r(t_i)) \leq \epsilon
\]
Furthermore since \( l_i < k_{i+1}, t_{i+1} > t_i \).

By the definition of \( q \) any two consecutive domains \( Z_i \) and \( Z_{i+1} \) have a boundary curve in common \( \delta_i = \partial Z_i \cap \partial Z_{i+1} \). Then by the convexity of length-functions we have that
\[
\ell_{\delta_i}(r(t)) \leq \epsilon \text{ for every } t \in [t_i, t_{i+1}]
\]
Now since the intervals \([t_i, t_{i+1}], i \geq 1\), cover \([0, \infty)\), at any time the systole of the surface along \( r \) is less than \( \epsilon \). Therefore, \( r \) stays in the \( \epsilon \)-thin part of Teichmüller space. Consequently, \( \hat{r} \) the projection of \( r \) to the moduli space stays in the \( \epsilon \)-thin part of moduli space. Furthermore, since \( q \) and \( \{e_i\}_i \) are periodic \( \hat{r} \) is a closed geodesic.

By Mumford’s compactness criterion there is \( \epsilon_0 > 0 \) such that \( \mathcal{K} \), the compact subset of moduli space, is contained in the \( \epsilon_0 \)-thick part of the moduli space and consequently is disjoint from the \( \epsilon_0 \)-thin part of the moduli space. Let \( y \) be such that (8.15) holds for \( \epsilon_0 \). Then our construction produces
closed WP geodesics not intersecting $\mathcal{K}$. There are infinitely many integers $y$ which satisfy this condition and consequently there are infinitely many closed geodesics not intersecting $\mathcal{K}$. These geodesics are distinct because they have different forward ending lamination. 

8.4. A recurrence condition. Given $A, R$ and $R'$ positive. Let the constants $w = w(A, R)$ and $\bar{\varepsilon} = \bar{\varepsilon}(A, R, R')$ be form Proposition 6.7. The following theorem is a straightforward consequence of the proposition.

Theorem 8.7. (Recurrence condition) Let $(\mu^-, \mu^+)$ be an $A$–narrow pair. Let $\rho$ be a hierarchy path between $\mu^-$ and $\mu^+$. Let $[k_i, l_i], i \geq 1$, be a sequence of intervals with $l_i - k_i \geq 2w$ and $l_i < k_{i+1}$. Furthermore suppose that over each interval $[k_i, l_i], S$ has $(R, R')$–bounded combinatorics. Let $r$ be the ray with prescribed itinerary with end invariant $(\mu^-, \mu^+)$. Then $\text{inj}(t_i) \geq \frac{\varepsilon}{2}$, where $t_i \in N(j)$ for some $j \in [k_i + w, l_i - w]$.

The WP volume of the moduli space is finite. It follows for example from the fact that given $\bar{\varepsilon} > 0$ the WP metric extends to the Delinge-Mumford compactification of the moduli space, [Mas76]. So the Poincaré Recurrence Theorem implies that almost every WP geodesic is recurrent to the $\bar{\varepsilon}$–thick part of moduli space. But here our construction of recurrent WP geodesics to the $\bar{\varepsilon}$–thick part of moduli space only uses the combinatorial control we developed in this paper.

Theorem 8.8. There are WP geodesic rays recurrent to the $\bar{\varepsilon}$–thick part of moduli space.

Proof. Let the subsurfaces $X_0 = S$ and $X_1$ and the pseudo-Anosov maps $f_0$ and $f_1$ supported on $X_0$ and $X_1$, respectively be as in §7.2. Let the function $q = q_0$ and the sequence of $e_i > E$ be as in Proposition 7.14. For each $i \geq 1$ let $Z_i = f_{q(1)}^e \ldots f_{q(i-1)}^{e_{i-1}} X_{q(i)}$. Let $\mu_I$ be a marking whose base contains $\{\partial X_0\}_{a=0,1}$ and $\mu_T$ be as in Proposition 7.14.

By Proposition 7.14(iii) we have that

\begin{equation}
(8.16) \quad d_{Z_i}(\mu_I, \mu_T) \asymp_{K_1, C_1} |e_i|.
\end{equation}

By Propositions 7.14(iii) every subsurface $Z$ with $d_{Z}(\mu_I(q,e), \mu_T(q,e)) > m$ is a large subsurface. So the pair $(\mu_I, \mu_T)$ is $m$–narrow.

Let $r$ be the ray with the end invariant $\mu_I$ and $\mu_T$. Let $N$ be the parameter map form Proposition 5.25.

For $i$ even let $k_i = \max J_{Z_{i-1}}$ and $l_i = \min J_{Z_{i+1}}$. For $i$ odd, let $k_i = \min J_{Z_{i}}$ and $l_i = \max J_{Z_{i}}$. Note that when $i$ is odd $J_{Z_{i}} = [k_i, l_i]$.

We have the following subsurface coefficient bounds:

- **$i$ is odd**: Proposition 7.14(iii) and (6.1) (no backtracking) imply that for every non-annular subsurface $W$ which is not in the list of $Z_i$’s,

\begin{equation}
(8.17) \quad d_{W}(\rho(k_i), \rho(l_i)) \leq m + 2M.
\end{equation}
By Proposition 7.14 (iii) for every odd integer \( j > i \), \( Z_i < Z_j \), also for every odd integer \( j < i \), \( Z_j < Z_i \). Then an argument similar to the one we gave to prove (8.4) in the proof of Proposition 7.14 we obtain the bound

(8.18) \[ d_{Z_j}(\rho(k_i), \rho(l_i)) \leq 2M + 4 \]

\( i \) is even: Proposition 7.14 (iii) and (6.1) (no backtracking) imply that for every non-annular subsurface \( W \) which is not in the list of \( Z_i \)'s,

(8.19) \[ d_W(\rho(k_i), \rho(l_i)) \leq m + 2M. \]

\( k_i = \max J_{Z_{i-1}} \) and \( l_i = \min J_{Z_{i+1}} \), so \( J_{Z_{i-1}} \cap [k_i, l_i] = \emptyset \) and \( J_{Z_{i+1}} \cap [k_i, l_i] = \emptyset \). Then by Theorem 2.13 (4),

(8.20) \[ d_{Z_{i-1}}(\rho(k_i), \rho(l_i)) \leq 2M, \text{ and } d_{Z_{i+1}}(\rho(k_i), \rho(l_i)) \leq 2M. \]

Further, for any odd \( j < i - 1 \), \( Z_j < Z_i \) and for any odd \( j > i + 1 \), \( Z_i < Z_j \). Then an argument similar to one for the proof of (8.4) gives us

(8.21) \[ d_{Z_j}(\rho(k_i), \rho(l_i)) \leq 2M + 4 \]

By Propositions 7.14 (iv) and the no backtracking for every annular subsurface \( A(\gamma) \),

(8.22) \[ d_{\gamma}(\rho(k_i), \rho(l_i)) \leq m' + 2M. \]

We proceed to estimate the length of the intervals \([k_i, l_i]\).

\( i \) is odd: By the bounds (8.17) and (8.18) all of the subsurface coefficients of \( \rho(k_i) \) and \( \rho(l_i) \) except that of \( Z_i \) are bounded above by \( m + 2M \). Let the threshold constant in the distance formula (2.2) be \( m + 2M \) and let \( K_2, C_2 \) be the constants corresponding to this threshold constant. Then we get

\[ d(\rho(k_i), \rho(l_i)) \asymp_{K_2, C_2} d_{Z_i}(\rho(k_i), \rho(l_i)) \]

By the no backtracking \( d_{Z_i}(\rho(k_i), \rho(l_i)) \asymp_{1,2M} d_{Z_i}(\mu_I, \mu_T) \). Furthermore, \( \rho \) is a \((k, c)\)-quasi-geodesic where \( k \) and \( c \) only depend on the topological type of the surface. Let \( K_3 = kK_2, C_3 = k(C_2 + 2M) + c \). Then

(8.23) \[ l_i - k_i \asymp_{K_3, C_3} d_{Z_i}(\mu_I, \mu_T). \]

\( i \) is even: By the bound (7.16) in the proof of Proposition 7.14 we have that

(8.24) \[ d_S(\rho(k_i), \rho(l_i)) \geq \frac{1}{K_1'} |e_i|. \]

where as in the proposition \( K_1' = \min\{\tau_a : a = 0, 1\} \).

By the bounds (8.19), (8.20) and (8.21) all of the subsurface coefficients of the pair \( \rho(k_i) \) and \( \rho(l_i) \) are bounded above by \( m + 2M \). Let \( K_2, C_2 \) be the constants corresponding to the threshold constant \( m + 2M \) in the distance formula (2.2). Then we have

\[ d(\rho(k_i), \rho(l_i)) \asymp_{K_2, C_2} d_S(\rho(k_i), \rho(l_i)). \]
Note that $\rho$ is a $(k, c)$–quasi-geodesic. Let $K'_i = kK_i$ and $C'_i = kC_i + c$. Then we have

\begin{equation}
(8.25) \quad l_i - k_i \approx_{K'_i, C'_i} d_S(\rho(k_i), \rho(l_i)).
\end{equation}

We proceed to set the sequence $\{e_i\}_i$ and using the above estimates finish the construction of recurrent WP geodesic rays.

Let $R = m + 2M$ and $R' = m' + 2M$. For any $i$ even let

$$y_i \geq 2w(m, R)$$

Let $K' = K'_1K'_3$ and $C' = K'_1K'_3C'_1$. Define the sequence $\{e_i\}_i$ such that for any $i$ even $e_i = K'y_i + C'$. For the next paragraph $e_i$ could be any integer when $i$ is odd.

Then the bounds (8.24) and (8.25) imply that $l_i - k_i \geq 2w$.

Furthermore, by the bounds (8.19), (8.18) and (8.22) $S$ has $(R, R')$–bounded combinatorics over the interval $[k_i, l_i]$. Let $t_i \in N(j)$ where $j \in [k_i, l_i]$. Applying Theorem 8.7 (Recurrence condition) the geodesic ray $r$ with end invariant $\mu_T$ and $\mu_T$ has injectivity radius at least $\frac{1}{2} \epsilon = \frac{1}{2} \epsilon(m, R, R')$ at each $t_i$. Consequently, $\hat{r}$ the projection of $r$ to the moduli space is recurrent the $\frac{\epsilon}{2}$–thick part of the moduli space.

Let $R = m + 2M$ and $R' = m' + 2M$, as before. Let $\epsilon_j \to 0$ as $j \to \infty$. For each $i$ odd let

$$y_i \geq 2\bar{w}(m, R, R', \epsilon_{\frac{1}{2}j})$$

Let $K = K_1K_3$, $C = K_1C_1 + K_1K_3C_3$. Now in the sequence $\{e_i\}_i$ we set above let for each $i$ odd, $e_i \geq K y_i + C$.

Then the bounds (8.23) and (8.16) imply that $l_i - k_i > 2\bar{w}$.

Furthermore, by the bounds (8.17), (8.20), (8.21) and (8.22), $Z_i$ has $(R, R')$–bounded combinatorics over the interval $[k_i, l_i]$. Let $t_i \in N(j)$ where $j \in [k_i + \bar{w}, l_i - \bar{w}]$. Then Theorem 8.5 applied to the interval $[k_i, l_i]$ implies that $\ell_0Z_i(r(t_i)) \leq \epsilon_{\frac{1}{4}j}$. Thus since $\epsilon_j \to 0$ as $j \to \infty$ the ray $\hat{r}$ is not contained in any thick part of the moduli space. As a result $\hat{r}$ is recurrent to the $\frac{\epsilon}{2}$–thick part of the moduli space and is not contained in any compact subset of the moduli space.

\[\Box\]

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