Tridiagonal matrices with nonnegative entries

Kazumasa Nomura and Paul Terwilliger

Abstract

In this paper we characterize the nonnegative irreducible tridiagonal matrices and their permutations, using certain entries in their primitive idempotents. Our main result is summarized as follows. Let \( d \) denote a nonnegative integer. Let \( A \) denote a matrix in \( \text{Mat}_{d+1}(R) \) and let \( \{\theta_i\}_{i=0}^d \) denote the roots of the characteristic polynomial of \( A \). We say \( A \) is multiplicity-free whenever these roots are mutually distinct and contained in \( R \). In this case \( E_i \) will denote the primitive idempotent of \( A \) associated with \( \theta_i \) (0 \( \leq \) \( i \) \( \leq \) \( d \)). We say \( A \) is symmetrizable whenever there exists an invertible diagonal matrix \( \Delta \in \text{Mat}_{d+1}(R) \) such that \( \Delta A \Delta^{-1} \) is symmetric. Let \( \Gamma(A) \) denote the directed graph with vertex set \( \{0, 1, \ldots , d\} \), where \( i \leftrightarrow j \) whenever \( i \neq j \) and \( A_{ij} \neq 0 \).

**Theorem.** Assume that each entry of \( A \) is nonnegative. Then the following are equivalent for \( 0 \leq s, t \leq d \).

(i) The graph \( \Gamma(A) \) is a bidirected path with endpoints \( s, t \):

\[ s \leftrightarrow * \leftrightarrow * \leftrightarrow \cdots \leftrightarrow * \leftrightarrow t. \]

(ii) The matrix \( A \) is symmetrizable and multiplicity-free. Moreover the \( (s, t) \)-entry of \( E_i \) times

\[ (\theta_i - \theta_0) \cdots (\theta_i - \theta_{i-1})(\theta_i - \theta_{i+1}) \cdots (\theta_i - \theta_d) \]

is independent of \( i \) for \( 0 \leq i \leq d \), and this common value is nonzero.

Recently Kurihara and Nozaki obtained a theorem that characterizes the \( Q \)-polynomial property for symmetric association schemes. We view the above result as a linear algebraic generalization of their theorem.

1 Introduction

Recently Kurihara and Nozaki gave the following characterization of the \( Q \)-polynomial property for symmetric association schemes (see Section 4 for definitions).

**Theorem 1.1** [3, Theorem 1.1] Let \( X \) denote a \( d \)-class symmetric association scheme with adjacency matrices \( \{A_i\}_{i=0}^d \). Let \( E \) and \( F \) denote primitive idempotents of \( X \) with \( E \) nontrivial. For \( 0 \leq i \leq d \) let \( \theta_i^* \) denote the dual eigenvalue of \( E \) for \( A_i \). Then the following are equivalent.

(i) \( X \) is \( Q \)-polynomial relative to \( E \), and \( F \) is the last primitive idempotent in this \( Q \)-polynomial structure.

(ii) \( \{\theta_i^*\}_{i=0}^d \) are mutually distinct, and for \( 0 \leq i \leq d \) the eigenvalue of \( A_i \) for \( F \) is

\[ \frac{(\theta_0^* - \theta_1^*)(\theta_0^* - \theta_2^*) \cdots (\theta_0^* - \theta_d^*)}{(\theta_i^* - \theta_0^*) \cdots (\theta_i^* - \theta_{i-1}^*)(\theta_i^* - \theta_{i+1}^*) \cdots (\theta_i^* - \theta_d^*)}. \]

As suggested by [3], there is a “dual” version of Theorem 1.1 in which the \( Q \)-polynomial structure is replaced by a \( P \)-polynomial structure. We now state this dual version.
Theorem 1.2 Let $\mathcal{X}$ denote a $d$-class symmetric association scheme with primitive idempotents $\{E_i\}_{i=0}^d$. Let $B$ and $C$ denote adjacency matrices of $\mathcal{X}$ with $B$ nontrivial. For $0 \leq i \leq d$ let $\theta_i$ denote the eigenvalue of $B$ for $E_i$. Then the following are equivalent.

(i) $\mathcal{X}$ is $P$-polynomial relative to $B$, and $C$ is the last adjacency matrix in this $P$-polynomial structure.

(ii) $\{\theta_i\}_{i=0}^d$ are mutually distinct, and for $0 \leq i \leq d$ the dual eigenvalue of $E_i$ for $C$ is

$$\frac{(\theta_0 - \theta_1)(\theta_0 - \theta_2)\cdots(\theta_0 - \theta_d)}{(\theta_i - \theta_0)(\theta_i - \theta_{i+1})\cdots(\theta_i - \theta_d)}.$$  \hfill (2)

In this paper we show that Theorems [1.1] and [1.2] follow from a linear algebraic result concerning matrices with nonnegative entries. We give two versions of the result, which are Theorems [1.3] and [1.4] below. Theorem [1.4] is the general version, and Theorem [1.3] is about an attractive special case. Before presenting these theorems we recall some concepts from linear algebra.

Throughout the paper $\mathbb{R}$ denotes the field of real numbers, $d$ denotes a nonnegative integer, and $\text{Mat}_{d+1}(\mathbb{R})$ denotes the $\mathbb{R}$-algebra consisting of the $(d+1) \times (d+1)$ matrices that have all entries in $\mathbb{R}$. We index the rows and columns by $0, 1, \ldots, d$. Let $V = \mathbb{R}^{d+1}$ denote the vector space over $\mathbb{R}$ consisting of the $(d+1) \times 1$ matrices that have all entries in $\mathbb{R}$. We index the rows by $0, 1, \ldots, d$. Observe that $\text{Mat}_{d+1}(\mathbb{R})$ acts on $V$ by left multiplication.

Let $A$ denote a matrix in $\text{Mat}_{d+1}(\mathbb{R})$. We say $A$ is nonnegative whenever each entry of $A$ is nonnegative. We say $A$ is symmetric whenever $A^t = A$, where $t$ denotes transpose. We say $A$ is symmetrizable whenever there exists an invertible diagonal matrix $\Delta \in \text{Mat}_{d+1}(\mathbb{R})$ such that $\Delta A \Delta^{-1}$ is symmetric. A subspace $W \subseteq V$ is called an eigenspace of $A$ whenever $W \neq 0$ and there exists $\theta \in \mathbb{R}$ such that $W = \{v \in V \mid Av = \theta v\}$; in this case $\theta$ is the eigenvalue of $A$ associated with $W$. We say $A$ is diagonalizable whenever its eigenspaces span $V$. We say that $A$ is multiplicity-free whenever $A$ is diagonalizable and its eigenspaces all have dimension $1$. Assume $A$ is multiplicity-free and let $\{\theta_i\}_{i=0}^d$ denote an ordering of the eigenvalues of $A$. For $0 \leq i \leq d$ let $V_i$ denote the eigenspace of $A$ associated with $\theta_i$. For $0 \leq i \leq d$ define $E_i \in \text{Mat}_{d+1}(\mathbb{R})$ such that $(E_i - I)V_i = 0$ and $E_iV_j = 0$ for $j \neq i$ ($0 \leq j \leq d$). Here $I$ denotes the identity matrix in $\text{Mat}_{d+1}(\mathbb{R})$. We call $E_i$ the primitive idempotent of $A$ associated with $V_i$ (or $\theta_i$). Observe that (i) $I = \sum_{i=0}^d E_i$; (ii) $E_iE_j = \delta_{i,j}E_i$ ($0 \leq i, j \leq d$); (iii) $V_i = E_iV$ ($0 \leq i \leq d$); (iv) $A = \sum_{i=0}^d \theta_i E_i$. Using these facts we find

$$E_i = \prod_{0 \leq j \leq d \atop j \neq i} \frac{A - \theta_j I}{\theta_i - \theta_j} \quad (0 \leq i \leq d).$$  \hfill (3)

Again let $A$ denote a matrix in $\text{Mat}_{d+1}(\mathbb{R})$. We say that $A$ is tridiagonal whenever each nonzero entry lies on either the diagonal, the subdiagonal, or the superdiagonal. Assume for the moment that $A$ is tridiagonal. Then $A$ is said to be irreducible whenever each entry on the subdiagonal is nonzero and each entry on the superdiagonal is nonzero. Let $\Gamma(A)$ denote the directed graph with vertex set $\{0, 1, \ldots, d\}$, where $i \rightarrow j$ whenever $i \neq j$ and $A_{ij} \neq 0$. Observe that the following are equivalent: (i) $A$ is irreducible tridiagonal; (ii) $\Gamma(A)$ is the bidirected path $0 \leftrightarrow 1 \leftrightarrow 2 \leftrightarrow \cdots \leftrightarrow d$. More generally the following
are equivalent: (i) there exists a permutation matrix $\Lambda \in \text{Mat}_{d+1}(\mathbb{R})$ such that $\Lambda A \Lambda^{-1}$ is irreducible tridiagonal; (ii) $\Gamma(A)$ is a bidirected path. We now state our first main result.

**Theorem 1.3** Let $A$ denote a nonnegative matrix in $\text{Mat}_{d+1}(\mathbb{R})$. Then the following are equivalent for $0 \leq s, t \leq d$.

(i) The graph $\Gamma(A)$ is a bidirected path with endpoints $s, t$:

$$s \leftrightarrow * \leftrightarrow * \leftrightarrow \cdots \leftrightarrow * \leftrightarrow t.$$  

(ii) The matrix $A$ is symmetrizable and multiplicity-free. Moreover the $(s, t)$-entry of $E_i$ times

$$(\theta_i - \theta_0) \cdots (\theta_i - \theta_{i-1})(\theta_i - \theta_{i+1}) \cdots (\theta_i - \theta_d)$$

is independent of $i$ for $0 \leq i \leq d$, and this common value is nonzero.

The proof of Theorem 1.3 is given in Section 3. Before stating our second main result, we make a few comments. For $A \in \text{Mat}_{d+1}(\mathbb{R})$ we say that $A$ is (upper) Hessenberg whenever each entry below the subdiagonal is zero and each entry on the subdiagonal is nonzero. Observe that the following are equivalent: (i) $A$ is Hessenberg; (ii) in the graph $\Gamma(A)$, for all vertices $i, j$ we have $i \to j$ if $i - j = 1$ and $i \not\to j$ if $i - j > 1$. An ordering $\{x_i\}_{i=0}^d$ of the vertices of $\Gamma(A)$ is called Hessenberg whenever for $0 \leq i, j \leq d$, $x_i \to x_j$ if $i - j = 1$ and $x_i \not\to x_j$ if $i - j > 1$. We recall the directed distance function $\partial$ for $\Gamma(A)$. Given vertices $s, t$ of $\Gamma(A)$ and an integer $i$ $(0 \leq i \leq d)$, we have $\partial(s, t) = i$ whenever there exists a directed path in $\Gamma(A)$ from $s$ to $t$ that has length $i$, and there does not exist a directed path in $\Gamma(A)$ from $s$ to $t$ that has length less than $i$. For all vertices $s, t$ in $\Gamma(A)$ the following are equivalent: (i) there exists a Hessenberg ordering $\{x_i\}_{i=0}^d$ of the vertices of $\Gamma(A)$ such that $x_0 = t$ and $x_d = s$; (ii) $\partial(s, t) = d$. We now state our second main result.

**Theorem 1.4** Let $A$ denote a nonnegative matrix in $\text{Mat}_{d+1}(\mathbb{R})$. Then the following are equivalent for $0 \leq s, t \leq d$.

(i) The matrix $A$ is diagonalizable, and $\partial(s, t) = d$ in $\Gamma(A)$.

(ii) The matrix $A$ is multiplicity-free. Moreover the $(s, t)$-entry of $E_i$ times

$$(\theta_i - \theta_0) \cdots (\theta_i - \theta_{i-1})(\theta_i - \theta_{i+1}) \cdots (\theta_i - \theta_d)$$

is independent of $i$ for $0 \leq i \leq d$, and this common value is nonzero.

The proof of Theorem 1.4 is given in Section 2. In Sections 4–6 we apply Theorem 1.3 to symmetric association schemes. In Sections 4 and 5 we give some basic facts about these objects. In Section 6 we use these facts and Theorem 1.3 to prove Theorems 1.1 and 1.2.
2 Hessenberg matrices

In this section we prove Theorem 1.4.

**Lemma 2.1** Let \( A \) denote a Hessenberg matrix in \( \text{Mat}_{d+1}(\mathbb{R}) \). Then for \( 0 \leq r \leq d \) the entries of \( A^r \) are described as follows. For \( 0 \leq i, j \leq d \) the \((i, j)\)-entry is nonzero if \( i - j = r \) and zero if \( i - j > r \).

**Proof.** Use matrix multiplication and the definition of Hessenberg. \( \square \)

**Corollary 2.2** Let \( A \) denote a Hessenberg matrix in \( \text{Mat}_{d+1}(\mathbb{R}) \). Then the matrices \( \{A^r\}_{r=0}^{d} \) are linearly independent.

**Proof.** For \( 0 \leq r \leq d \) let \( u_r \in \mathbb{R}^{d+1} \) denote the 0th column of \( A^r \). By Lemma 2.1 the \( i \)-th entry of \( u_r \) is nonzero for \( i = r \) and zero for \( r + 1 \leq i \leq d \). Therefore \( \{u_r\}_{r=0}^{d} \) are linearly independent. The result follows. \( \square \)

**Lemma 2.3** Let \( A \) denote a Hessenberg matrix in \( \text{Mat}_{d+1}(\mathbb{R}) \). Then the minimal polynomial of \( A \) equals the characteristic polynomial of \( A \).

**Proof.** By construction the characteristic polynomial of \( A \) is monic with degree \( d + 1 \). By elementary linear algebra the minimal polynomial of \( A \) is monic and divides the characteristic polynomial of \( A \). By Corollary 2.2 the minimal polynomial of \( A \) has degree \( d + 1 \). The result follows. \( \square \)

**Lemma 2.4** Let \( A \) denote a diagonalizable Hessenberg matrix in \( \text{Mat}_{d+1}(\mathbb{R}) \). Then \( A \) is multiplicity-free.

**Proof.** Let \( \{\theta_i\}_{i=0}^{d} \) denote the roots of the characteristic polynomial of \( A \). We have \( \theta_i \in \mathbb{R} \) \((0 \leq i \leq d)\) since \( A \) is diagonalizable. Moreover \( \{\theta_i\}_{i=0}^{d} \) are mutually distinct since the minimal polynomial of \( A \) equals the characteristic polynomial of \( A \) by Lemma 2.3 and since the roots of the minimal polynomial are mutually distinct. Thus \( A \) is multiplicity-free. \( \square \)

For \( A \in \text{Mat}_{d+1}(\mathbb{R}) \) let \( \Gamma(A) \) denote the directed graph with vertex set \( \{0, 1, \ldots, d\} \), where \( i \rightarrow j \) whenever \( A_{ij} \neq 0 \).

**Lemma 2.5** Let \( A \) denote a nonnegative matrix in \( \text{Mat}_{d+1}(\mathbb{R}) \). Then the following are equivalent for \( 0 \leq r, s, t \leq d \).

(i) The \((s, t)\)-entry of \( A^r \) is nonzero.

(ii) In the graph \( \Gamma(A) \) there exists a directed path of length \( r \) from \( s \) to \( t \).

**Proof.** Consider the \((s, t)\)-entry of \( A^r \) using matrix multiplication. \( \square \)
Lemma 2.6 The following are equivalent for all \( A \in \text{Mat}_{d+1}(\mathbb{R}) \) and \( 0 \leq r, s, t \leq d \).

(i) \( \partial(s, t) = r \) in \( \Gamma(A) \).

(ii) \( \partial(s, t) = r \) in \( \Gamma_t(A) \).

Proof. Routine verification. \( \square \)

Lemma 2.7 Let \( A \) denote a nonnegative matrix in \( \text{Mat}_{d+1}(\mathbb{R}) \). Then the following are equivalent for \( 0 \leq s, t \leq d \).

(i) The \((s, t)\)-entry of \( A^r \) is nonzero if \( r = d \) and zero if \( r < d \) \( (0 \leq r \leq d) \).

(ii) \( \partial(s, t) = d \) in \( \Gamma(A) \).

Proof. Follows from Lemmas 2.5 and 2.6 \( \square \)

Let \( \lambda \) denote an indeterminate and let \( \mathbb{R}[\lambda] \) denote the \( \mathbb{R} \)-algebra consisting of the polynomials in \( \lambda \) that have all coefficients in \( \mathbb{R} \).

Lemma 2.8 For \( 0 \leq i \leq d \) let \( f_i \in \mathbb{R}[\lambda] \) be monic with degree \( d \), and assume \( \{f_i\}_{i=0}^d \) are linearly independent. Then the following are equivalent for all \( A \in \text{Mat}_{d+1}(\mathbb{R}) \) and \( 0 \leq s, t \leq d \).

(i) The \((s, t)\)-entry of \( A^r \) is zero for \( 0 \leq r \leq d - 1 \).

(ii) The \((s, t)\)-entry of \( f_i(A) \) is equal to the \((s, t)\)-entry of \( A^d \) for \( 0 \leq i \leq d \).

(iii) The \((s, t)\)-entry of \( f_i(A) \) is independent of \( i \) for \( 0 \leq i \leq d \).

Proof. (i)\(\Rightarrow\) (ii): Since \( f_i \) is monic with degree \( d \).

(ii)\(\Rightarrow\) (iii): Clear.

(iii)\(\Rightarrow\) (i): For \( 1 \leq i \leq d \) define \( g_i = f_i - f_0 \) and observe that \( g_i \) has degree at most \( d - 1 \). Note that \( \{g_i\}_{i=1}^d \) are linearly independent. So \( \{g_i\}_{i=1}^d \) form a basis for the subspace of \( \mathbb{R}[\lambda] \) consisting of the polynomials with degree at most \( d - 1 \). So for \( 0 \leq r \leq d - 1 \), \( \lambda^r \) is a linear combination of \( \{g_i\}_{i=1}^d \). By construction the \((s, t)\)-entry of \( g_i(A) \) is zero for \( 1 \leq i \leq d \). By these comments the \((s, t)\)-entry of \( A^r \) is zero for \( 0 \leq r \leq d - 1 \). \( \square \)

Referring to Lemma 2.8 we now make a specific choice for the polynomials \( \{f_i\}_{i=0}^d \).

Lemma 2.9 Assume \( A \in \text{Mat}_{d+1}(\mathbb{R}) \) is multiplicity-free with eigenvalues \( \{\theta_i\}_{i=0}^d \). For \( 0 \leq i \leq d \) define a polynomial \( f_i \in \mathbb{R}[\lambda] \) by

\[
    f_i = (\lambda - \theta_0) \cdots (\lambda - \theta_{i-1}) (\lambda - \theta_{i+1}) \cdots (\lambda - \theta_d).
\]

Then

(i) \( f_i(A) = f_i(\theta_i)E_i \) for \( 0 \leq i \leq d \).

(ii) \( f_i \) is monic with degree \( d \) for \( 0 \leq i \leq d \).

(iii) \( \{f_i\}_{i=0}^d \) are linearly independent.

Proof. (i): Compare (3) and (4).
(ii): Clear.

(iii): For $0 \leq i, j \leq d$ the scalar $f_i(\theta_j)$ is zero if $i \neq j$ and nonzero if $i = j$. □

Proof of Theorem 1.4.

(i)⇒(ii): By the comments above Theorem 1.4 there exists a Hessenberg ordering \{x_i\}_{i=0}^d of the vertices of $\Gamma(A)$ such that $x_0 = t$ and $x_d = s$. Let $\Lambda \in \text{Mat}_{d+1}(\mathbb{R})$ denote the permutation matrix that corresponds to the permutation $i \mapsto x_i$ ($0 \leq i \leq d$). Then $\Lambda \Lambda A \Lambda^{-1}$ is Hessenberg. We assume $A$ is diagonalizable so $\Lambda \Lambda A \Lambda^{-1}$ is diagonalizable. Now $\Lambda \Lambda A \Lambda^{-1}$ is multiplicity-free by Lemma 2.4 so $A$ is multiplicity-free. Define the polynomials \{f_i\}_{i=0}^d as in Lemma 2.9. By Lemma 2.7 for $0 \leq r \leq d$ the $(s, t)$-entry of $A^r$ is nonzero if $r = d$ and zero if $r < d$. By this and Lemma 2.8 the $(s, t)$-entry of $f_i(A)$ is independent of $i$ for $0 \leq i \leq d$, and this common value is nonzero. By this and Lemma 2.9(i), the $(s, t)$-entry of $E_i$ times $f_i(\theta_i)$ is independent of $i$ for $0 \leq i \leq d$, and this common value is nonzero.

(ii)⇒(i): The matrix $A$ is diagonalizable since it is multiplicity-free. Define \{f_i\}_{i=0}^d as in Lemma 2.9. By assumption, the $(s, t)$-entry of $E_i$ times $f_i(\theta_i)$ is independent of $i$ for $0 \leq i \leq d$, and this common value is nonzero. By this and Lemma 2.9(i), the $(s, t)$-entry of $f_i(A)$ is independent of $i$ for $0 \leq i \leq d$, and this common value is nonzero. By this and Lemma 2.8 for $0 \leq r \leq d$ the $(s, t)$-entry of $A^r$ is nonzero if $r = d$ and zero if $r < d$. By this and Lemma 2.7 we find $\partial(s, t) = d$. □

3 Tridiagonal matrices

In this section we prove Theorem 1.3.

Lemma 3.1 Let $A$ denote a symmetrizable matrix in $\text{Mat}_{d+1}(\mathbb{R})$. Then $A_{ij} = 0$ if and only if $A_{ji} = 0$ ($0 \leq i, j \leq d$).

Proof. Since $A$ is symmetrizable, there exists an invertible diagonal matrix $\Delta \in \text{Mat}_{d+1}(\mathbb{R})$ such that $\Delta A \Delta^{-1}$ is symmetric. Comparing the $(i, j)$-entry and the $(j, i)$-entry of $\Delta A \Delta^{-1}$ we find $\Delta_{ii}A_{ij}\Delta^{-1}_{jj} = \Delta_{jj}A_{ji}\Delta^{-1}_{ii}$. The result follows. □

Lemma 3.2 Let $A$ denote a symmetrizable matrix in $\text{Mat}_{d+1}(\mathbb{R})$. Then $A$ is diagonalizable.

Proof. Since $A$ is symmetrizable, there exists an invertible diagonal matrix $\Delta \in \text{Mat}_{d+1}(\mathbb{R})$ such that $\Delta A \Delta^{-1}$ symmetric. By [3] Corollary 3.3.1 every symmetric matrix in $\text{Mat}_{d+1}(\mathbb{R})$ is diagonalizable. Therefore $\Delta A \Delta^{-1}$ is diagonalizable, so $A$ is diagonalizable. □

Lemma 3.3 Let $A \in \text{Mat}_{d+1}(\mathbb{R})$ denote a symmetrizable matrix. Then $\Lambda A \Lambda^{-1}$ is symmetrizable for every permutation matrix $\Lambda \in \text{Mat}_{d+1}(\mathbb{R})$.

Proof. Since $A$ is symmetrizable, there exists an invertible diagonal matrix $\Delta \in \text{Mat}_{d+1}(\mathbb{R})$ such that $\Delta A \Delta^{-1}$ is symmetric. Set $\Delta' = \Lambda \Lambda \Lambda^{-1}$, and observe that $\Delta'$ is invertible diagonal. Using $\Lambda^{-1} = \Lambda'$ we find $\Delta' \Lambda A \Lambda^{-1} (\Delta')^{-1}$ is symmetric. Now $\Lambda A \Lambda^{-1}$ is symmetrizable. □
Lemma 3.4 Let $A$ denote a nonnegative irreducible tridiagonal matrix in $\text{Mat}_{d+1}(\mathbb{R})$. Then $A$ is symmetrizable and multiplicity-free.

**Proof.** We first show that $A$ is symmetrizable. Since $A$ is irreducible and nonnegative we have $A_{i,i-1} > 0$ and $A_{i-1,i} > 0$ for $1 \leq i \leq d$. For $0 \leq i \leq d$ define

$$\kappa_i = \frac{A_{01}A_{12} \cdots A_{i-1,i}}{A_{10}A_{21} \cdots A_{i,i-1}}$$

and note that $\kappa_i > 0$. Define a diagonal matrix $K \in \text{Mat}_{d+1}(\mathbb{R})$ with $(i,i)$-entry $\kappa_i$ for $0 \leq i \leq d$. Define a diagonal matrix $\Delta \in \text{Mat}_{d+1}(\mathbb{R})$ with $(i,i)$-entry $\sqrt{\kappa_i}$ for $0 \leq i \leq d$, so that $\Delta^2 = K$. By this and $KA = A^tK$ one finds that $\Delta A \Delta^{-1}$ is symmetric. Therefore $A$ is symmetrizable. Now $A$ is diagonalizable by Lemma 3.2 and multiplicity-free by Lemma 2.4. □

Lemma 3.5 Let $A$ denote a nonnegative matrix in $\text{Mat}_{d+1}(\mathbb{R})$. Then the following are equivalent for $0 \leq s, t \leq d$.

(i) The graph $\Gamma(A)$ is a bidirected path with endpoints $s, t$.

(ii) The matrix $A$ is symmetrizable, and $\partial(s,t) = d$ in $\Gamma(A)$.

**Proof.** (i)⇒(ii): We first show that $A$ is symmetrizable. By the observation above Theorem 1.3 there exists a permutation matrix $\Lambda \in \text{Mat}_{d+1}(\mathbb{R})$ such that $\Lambda A \Lambda^{-1}$ is irreducible tridiagonal. We assume $A$ is nonnegative so $\Lambda A \Lambda^{-1}$ is nonnegative. So $\Lambda A \Lambda^{-1}$ is symmetrizable in view of Lemma 3.4. Now $A$ is symmetrizable by Lemma 3.3. By construction $\partial(s,t) = d$ in $\Gamma(A)$.

(ii)⇒(i): Routine using Lemma 3.1. □

Proof of Theorem 1.3. (i)⇒(ii): $A$ is symmetrizable by Lemma 3.5 and $A$ is diagonalizable by Lemma 3.2. By Lemma 3.5 $\partial(s,t) = d$ in $\Gamma(A)$. The result follows in view of Theorem 1.4.

(ii)⇒(i): By Theorem 1.4 $\partial(s,t) = d$ in $\Gamma(A)$. By this and Lemma 3.5 the graph $\Gamma(A)$ is a bidirected path with endpoints $s, t$. □

4 Symmetric association schemes

In this section we review some definitions and basic concepts concerning symmetric association schemes. For more information we refer the reader to [1, 2, 5].

A $d$-class symmetric association scheme is a pair $\mathcal{X} = (X, \{R_i\}_{i=0}^d)$, where $X$ is a finite nonempty set and $\{R_i\}_{i=0}^d$ are nonempty subsets of $X \times X$ that satisfy

(i) $R_0 = \{(x,x) \mid x \in X\}$;

(ii) $X \times X = R_0 \cup R_1 \cup \cdots \cup R_d$ (disjoint union);

(iii) $R_i^t = R_i$ for $0 \leq i \leq d$, where $R_i^t = \{(x,y) \mid (x,y) \in R_i\}$;
(iv) there exist integers $p_{ij}^h$ ($0 \leq h, i, j \leq d$) such that, for every $(x, y) \in R_h$,$$
p_{ij}^h = |\{z \in X \mid (x, z) \in R_i, (z, y) \in R_j\}|.$$
The parameters $p_{ij}^h$ are called the intersection numbers of $\mathcal{X}$.

From now on let $\mathcal{X} = (X, \{R_i\}_{i=0}^d)$ denote a $d$-class symmetric association scheme. Observe by (iii) that $p_{ij}^h = p_{ji}^h$ for $0 \leq h, i, j \leq d$. For $0 \leq i \leq d$ define $k_i = p_{ii}^0$, and observe
$$k_i = |\{y \in X \mid (x, y) \in R_i\}| \quad (x \in X).$$

Note that $k_i > 0$. By [1, Proposition II.2.2],
$$k_i p_{ij}^h = k_j p_{ih}^j \quad (0 \leq h, i, j \leq d).$$

We recall the Bose-Mesner algebra of $\mathcal{X}$. Let $\text{Mat}_X(\mathbb{R})$ denote the $\mathbb{R}$-algebra consisting of the matrices whose rows and columns are indexed by $X$ and whose entries are in $\mathbb{R}$. For $0 \leq i \leq d$ let $A_i$ denote the matrix in $\text{Mat}_X(\mathbb{R})$ with $(x, y)$-entry
$$(A_i)_{x,y} = \begin{cases} 1 & \text{if } (x, y) \in R_i \\
0 & \text{if } (x, y) \notin R_i \end{cases} \quad (x, y \in X).$$

We call $\{A_i\}_{i=0}^d$ the adjacency matrices of $\mathcal{X}$. Note that $A_0 = I$, where $I$ denotes the identity matrix in $\text{Mat}_X(\mathbb{R})$. We call $A_0$ the trivial adjacency matrix. Observe $A_i^d = A_i$ for $0 \leq i \leq d$. The matrices $\{A_i\}_{i=0}^d$ are linearly independent since they have nonzero entries which are in disjoint positions. Observe
$$A_i A_j = \sum_{h=0}^d p_{ij}^h A_h \quad (0 \leq i, j \leq d).$$

By $p_{ij}^h = p_{ji}^h$, we find $A_i A_j = A_j A_i$ for $0 \leq i, j \leq d$. Using these facts we find $\{A_i\}_{i=0}^d$ is a basis for a commutative subalgebra $M$ of $\text{Mat}_X(\mathbb{R})$. We call $M$ the Bose-Mesner algebra of $\mathcal{X}$.

By [1, Section II.2.3] $M$ has a second basis $\{E_i\}_{i=0}^d$ such that (i) $E_0 = |X|^{-1} J$; (ii) $I = \sum_{i=0}^d E_i$; (iii) $E_i^d = E_i$ ($0 \leq i \leq d$); (iv) $E_i E_j = \delta_{i,j} E_i$ ($0 \leq i, j \leq d$). We call $\{E_i\}_{i=0}^d$ the primitive idempotents of $\mathcal{X}$. We call $E_0$ the trivial primitive idempotent. For $0 \leq i \leq d$ let $m_i$ denote the rank of $E_i$. Note that $m_i > 0$.

We recall the matrices $P$ and $Q$. We mentioned above that $\{A_i\}_{i=0}^d$ and $\{E_i\}_{i=0}^d$ are bases for $M$. Define $P \in \text{Mat}_{d+1}(\mathbb{R})$ such that
$$A_j = \sum_{i=0}^d P_{ij} E_i \quad (0 \leq j \leq d).$$

Define $Q \in \text{Mat}_{d+1}(\mathbb{R})$ such that
$$E_j = |X|^{-1} \sum_{i=0}^d Q_{ij} A_i \quad (0 \leq j \leq d).$$
Observe that $PQ = QP = |X|I$. Setting $j = 0$ and $A_0 = I$ in (7) we find $P_{i0} = 1$ for $0 \leq i \leq d$. Setting $j = 0$ and $E_0 = |X|^{-1}J$ in (8) we find $Q_{i0} = 1$ for $0 \leq i \leq d$.

We recall the $P$-polynomial property. Let \( \{A_i\}_{i=1}^d \) denote an ordering of the nontrivial adjacency matrices of $X$. This ordering is said to be $P$-polynomial whenever for $0 \leq i, j \leq d$ the intersection number $p_{ij}^1$ is zero if $|i - j| > 1$ and nonzero if $|i - j| = 1$. Let $A$ denote a nontrivial adjacency matrix of $X$. We say $X$ is $P$-polynomial relative to $A$ whenever there exists a $P$-polynomial ordering $\{A_i\}_{i=1}^d$ of the nontrivial adjacency matrices such that $A_1 = A$. In this case we call $A_d$ the last adjacency matrix in this $P$-polynomial structure.

We recall the Krein parameters. Let $\circ$ denote the entrywise product in Mat\(_X(\mathbb{R})\). Observe $A_i \circ A_j = \delta_{i,j}A_i$ for $0 \leq i, j \leq d$, so $M$ is closed under $\circ$. Thus there exist $q_{ij}^h \in \mathbb{R}$ ($0 \leq h, i, j \leq d$) such that

\[
E_i \circ E_j = |X|^{-1} \sum_{h=0}^d q_{ij}^h E_h \quad (0 \leq i, j \leq d).
\] (9)

The parameters $q_{ij}^h$ are called the Krein parameters of $X$. By [1, Theorem II.3.8] the Krein parameters are nonnegative. By (9) we have $q_{ij}^h = q_{ji}^h$ for $0 \leq h, i, j \leq d$. Setting $j = 0$ and $E_0 = |X|^{-1}J$ in (9) we find $q_{i0}^h = \delta_{h,i}$ for $0 \leq h, i \leq d$. By [1] Proposition II.3.7,

\[
m_h q_{ij}^h = m_j q_{ih}^j \quad (0 \leq h, i, j \leq d).
\] (10)

We recall the $Q$-polynomial property. Let $\{E_i\}_{i=1}^d$ denote an ordering of the nontrivial primitive idempotents of $X$. This ordering is said to be $Q$-polynomial whenever for $0 \leq i, j \leq d$ the Krein parameter $q_{ij}^1$ is zero if $|i - j| > 1$ and nonzero if $|i - j| = 1$. Let $E$ denote a nontrivial primitive idempotent of $X$. We say $X$ is $Q$-polynomial relative to $E$ whenever there exists a $Q$-polynomial ordering $\{E_i\}_{i=1}^d$ of the nontrivial primitive idempotents such that $E_1 = E$. In this case we call $E_d$ the last primitive idempotent in this $Q$-polynomial structure.

We recall the dual Bose-Mesner algebra. For the rest of the paper fix $x \in X$. For $0 \leq i \leq d$ let $E_i^*$ denote the diagonal matrix in Mat\(_X(\mathbb{R})\) with $(y, y)$-entry

\[
(E_i^*)_{yy} = \begin{cases} 1 & \text{if } (x, y) \in R_i \\ 0 & \text{if } (x, y) \notin R_i \end{cases} \quad (y \in X).
\]

For $y \in X$ the $(y, y)$-entry of $E_i^*$ coincides with the $(x, y)$-entry of $A_i$. Observe $E_i^*E_j^* = \delta_{i,j}E_i^*$ ($0 \leq i, j \leq d$) and $I = \sum_{i=0}^d E_i^*$. We call $\{E_i^*\}_{i=0}^d$ the dual primitive idempotents of $X$. By the above comments $\{E_i^*\}_{i=0}^d$ is a basis for a commutative subalgebra $M^*$ of Mat\(_X(\mathbb{R})\). We call $M^*$ the dual Bose-Mesner algebra of $X$. For $0 \leq i \leq d$ let $A_i^*$ denote the diagonal matrix in Mat\(_X(\mathbb{R})\) with $(y, y)$-entry $|X|(E_i)_{x,y}$ for $y \in X$. We call $\{A_i^*\}_{i=0}^d$ the dual adjacency matrices of $X$. Using (8),

\[
A_j^* = \sum_{i=0}^d Q_{ij}E_i^* \quad (0 \leq j \leq d).
\] (11)
Using (7),
\[ E_j^* = |X|^{-1} \sum_{i=0}^{d} P_{ij} A_i^* \quad (0 \leq j \leq d). \] (12)

Using (9),
\[ A_i^* A_j^* = \sum_{h=0}^{d} q_{ij}^h A_h^* \quad (0 \leq i, j \leq d). \] (13)

For \( 0 \leq i, j \leq d \) the scalar \( P_{ij} \) (resp. \( Q_{ij} \)) is known as the eigenvalue of \( A_j \) for \( E_i \) (resp. dual eigenvalue of \( E_j \) for \( A_i \)).

5 The subconstituent algebra and its primary module

We continue to discuss the symmetric association scheme \( \mathcal{X} = (X, \{ R_i \}_{i=0}^{d}) \) from Section 4. Let \( \mathbb{R}^X \) denote the vector space over \( \mathbb{R} \) consisting of column vectors with entries in \( \mathbb{R} \) and coordinates indexed by \( X \). Observe that Mat\(_X\)(\( \mathbb{R} \)) acts on \( \mathbb{R}^X \) by left multiplication. For all \( y \in X \) let \( \hat{y} \) denote the vector in \( \mathbb{R}^X \) that has \( y \)-coordinate 1 and all other coordinates 0. Note that \( \{ \hat{y} | y \in X \} \) is a basis for \( \mathbb{R}^X \). Let \( T = T(x) \) denote the subalgebra of Mat\(_X\)(\( \mathbb{R} \)) generated by \( M \) and \( M^* \). We call \( T \) the subconstituent algebra of \( \mathcal{X} \) with respect to \( x \) [5, Definition 3.3]. We now describe a certain irreducible \( T \)-module known as the primary module. Let \( 1 = \sum_{y \in X} \hat{y} \) denote the “all 1’s” vector in \( \mathbb{R}^X \). By construction, for \( 0 \leq i \leq d \) we have \( E_i^* 1 = A_i \hat{x} \) and \( A_i^* 1 = |X| E_i \hat{x} \). Therefore \( M \hat{x} = M^* 1 \). Denote this common space by \( W \) and observe that \( W \) is a \( T \)-module. This \( T \)-module is said to be primary. The \( T \)-module \( W \) is irreducible by [5, Lemma 3.6]. We now describe two bases for \( W \). For \( 0 \leq i \leq d \) define
\[ 1_i = E_i^* 1 = A_i \hat{x}, \quad 1_i^* = A_i^* 1 = |X| E_i \hat{x}. \] (14)

Then each of \( \{ 1_i \}_{i=0}^{d} \) and \( \{ 1_i^* \}_{i=0}^{d} \) is a basis for \( W \). We now describe the transition matrices between these bases. Using (7) and (14) we find
\[ 1_j = |X|^{-1} \sum_{i=0}^{d} P_{ij} 1_i^* \quad (0 \leq j \leq d). \] (15)

By (15) and \( PQ = |X| I \),
\[ 1_j^* = \sum_{i=0}^{d} Q_{ij} 1_i \quad (0 \leq j \leq d). \] (16)

**Definition 5.1** For all \( B \in T \) let \( \rho(B) \in \text{Mat}_{d+1}(\mathbb{R}) \) denote the matrix that represents \( B \) with respect to the basis \( \{ 1_i \}_{i=0}^{d} \). Thus
\[ B 1_j = \sum_{i=0}^{d} \rho(B)_{ij} 1_i \quad (0 \leq j \leq d). \] (17)

This defines an \( \mathbb{R} \)-algebra homomorphism \( \rho : T \to \text{Mat}_{d+1}(\mathbb{R}) \).
Definition 5.2  For all $B \in T$ let $\rho^*(B) \in \text{Mat}_{d+1}(\mathbb{R})$ denote the matrix that represents $B$ with respect to the basis $\{1_i^*\}_{i=0}^d$. Thus

$$B1_j^* = \sum_{i=0}^d \rho^*(B)_{ij}1_i^*$$

$(0 \leq j \leq d)$.  

This defines an $\mathbb{R}$-algebra homomorphism $\rho^*: T \to \text{Mat}_{d+1}(\mathbb{R})$.

Lemma 5.3  The following hold for all $B \in T$.

(i) $P \rho(B)P^{-1} = \rho^*(B)$.

(ii) $Q \rho^*(B)Q^{-1} = \rho(B)$.

Proof.  (i): By (15) and elementary linear algebra.

(ii): Follows from (i) and $PQ = |X|I$. □

Lemma 5.4  The following hold for $0 \leq h, i, j \leq d$.

(i) $\rho(A_i)$ has $(h, j)$-entry $p^h_{ij}$.

(ii) $\rho^*(A_i^*)$ has $(h, j)$-entry $q^h_{ij}$.

Proof.  (i): Using (6) and (14) we argue $A_i1_j = A_iA_j\hat{x} = \sum_{h=0}^d p^h_{ij}A_h\hat{x} = \sum_{h=0}^d p^h_{ij}1_h$.

(ii): Similar to the proof of (i). □

Lemma 5.5  The following hold for $0 \leq i \leq d$.

(i) $\rho(A_i^*)$ is diagonal with $(j, j)$-entry $Q_{ji}$ for $0 \leq j \leq d$.

(ii) $\rho^*(A_i)$ is diagonal with $(j, j)$-entry $P_{ji}$ for $0 \leq j \leq d$.

Proof.  (i): Using (14) we argue $A_i^*1_j = A_i^*E_j^*1 = Q_{ji}E_j^*1 = Q_{ji}1_j$.

(ii): Similar to the proof of (i). □

Lemma 5.6  The following hold for $0 \leq i \leq d$.

(i) $\rho(E_i^*)$ has $(i, i)$-entry 1 and all other entries 0.

(ii) $\rho^*(E_i)$ has $(i, i)$-entry 1 and all other entries 0.

Proof.  (i): Using (14) we argue $E_i^*1_j = E_i^*E_j^*1 = \delta_{i,j}E_i^*1 = \delta_{i,j}1_i$.

(ii): Similar to the proof of (i). □
Lemma 5.7 The following hold for $0 \leq h, i, j \leq d$.

(i) $\rho(E_i)$ has $(h, j)$-entry $|X|^{-1}Q_{hi}P_{ij}$.

(ii) $\rho^*(E_i^*)$ has $(h, j)$-entry $|X|^{-1}P_{hi}Q_{ij}$.

Proof. (i): By Lemma 5.3(ii) $\rho(E_i) = Q\rho^*(E_i)Q^{-1}$. By $PQ = |X|I$ we have $Q^{-1} = |X|^{-1}P$. By Lemma 5.6(ii) $\rho^*(E_i)$ has $(i, i)$-entry 1 and all other entries 0. The result follows from these comments.

(ii): Similar to the proof of (i). □

Lemma 5.8 For $0 \leq i \leq d$ each of $\rho(A_i)$ and $\rho^*(A_i^*)$ is nonnegative and symmetrizable.

Proof. Concerning $\rho(A_i)$, observe it is nonnegative by Lemma 5.4(i). Define a diagonal matrix $\Delta \in \text{Mat}_{d+1}(\mathbb{R})$ with $(i, i)$-entry $\sqrt{k_i}$ for $0 \leq i \leq d$. Using (5) we routinely find that $\Delta \rho(A_i)\Delta^{-1}$ is symmetric. Therefore $\rho(A_i)$ is symmetrizable. The proof for $\rho^*(A_i^*)$ is similar using (10) and Lemma 5.4(ii). □

We will need the following well-known facts.

Lemma 5.9 [1, Proposition III.1.1] The following are equivalent.

(i) The ordering $\{A_i\}_{i=1}^d$ is $P$-polynomial.

(ii) The matrix $\rho(A_1)$ is irreducible tridiagonal.

(iii) The graph $\Gamma(\rho(A_1))$ is the bidirected path $0 \leftrightarrow 1 \leftrightarrow 2 \leftrightarrow \cdots \leftrightarrow d$.

Proof. In the definition of $P$-polynomial, $p^1_{ij}$ is nonzero if and only if $p^1_{ji}$ is nonzero by (5) and since $k_h \neq 0$. Now we obtain the result using Lemma 5.4(ii). □

Lemma 5.10 [1, Section III.1] The following are equivalent.

(i) The ordering $\{E_i\}_{i=1}^d$ is $Q$-polynomial.

(ii) The matrix $\rho^*(A_1^*)$ is irreducible tridiagonal.

(iii) The graph $\Gamma(\rho^*(A_1^*))$ is the bidirected path $0 \leftrightarrow 1 \leftrightarrow 2 \leftrightarrow \cdots \leftrightarrow d$.

Proof. In the definition of $Q$-polynomial, $q^1_{ij}$ is nonzero if and only if $q^1_{ji}$ is nonzero by (10) and since $m_h \neq 0$. Now we obtain the result using Lemma 5.4(ii). □

Lemma 5.11 For $1 \leq i \leq d$ consider the graph $\Gamma(\rho(A_i))$ and $\Gamma(\rho^*(A_i^*))$. In either case,

(i) $h \rightarrow j$ if and only if $j \rightarrow h$ $(0 \leq h, j \leq d)$.

(ii) $0 \rightarrow i$.

(iii) $0 \not\rightarrow h$ if $h \neq i$ $(0 \leq h \leq d)$.

Proof. (i): By Lemmas 5.1 and 5.8

(ii), (iii): By Lemma 5.4 and since $p^0_{ih} = \delta_{h,i}$, $q^0_{ih} = \delta_{h,i}$. □
Lemma 5.12  For $1 \leq i \leq d$ the following are equivalent.

(i) $X$ is $P$-polynomial relative to $A_i$.

(ii) The graph $\Gamma(\rho(A_i))$ is a bidirected path.

Suppose (i) and (ii) hold. Then the above graph is $0 \leftrightarrow i \leftrightarrow * \leftrightarrow \cdots * \leftrightarrow s$, where $A_s$ is the last adjacency matrix in the $P$-polynomial structure.

Proof. Use Lemmas 5.9 and 5.11.

Lemma 5.13  For $1 \leq i \leq d$ the following are equivalent.

(i) $X$ is $Q$-polynomial relative to $E_i$.

(ii) The graph $\Gamma(\rho^*(A^*_i))$ is a bidirected path.

Suppose (i) and (ii) hold. Then the above graph is $0 \leftrightarrow i \leftrightarrow * \leftrightarrow \cdots * \leftrightarrow s$, where $E_s$ is the last primitive idempotent in the $Q$-polynomial structure.

Proof. Use Lemmas 5.10 and 5.11.

6 Proof of Theorems 1.1 and 1.2

For convenience we first prove Theorem 1.2.

Proof of Theorem 1.2. Fix an ordering $\{A_i\}_{i=1}^d$ of the nontrivial adjacency matrices such that $A_1 = B$, and let $C = A_s$. For $0 \leq i \leq d$ the scalar $\theta_i = P_{i1}$ is the eigenvalue of $B$ for $E_i$, and $Q_{si}$ is the dual eigenvalue of $E_i$ for $C$. For $0 \leq i \leq d$ define a polynomial $f_i \in \mathbb{R}[\lambda]$ by (4). Let the map $\rho$ be as in Definition 5.1. By Lemma 5.8 the matrix $\rho(B)$ is nonnegative, so we can apply Theorem 1.3 with $A = \rho(B)$. We will do this after a few comments. Combining Lemma 5.3(i) and Lemma 5.5(ii) we find $P\rho(B)P^{-1} = \text{diag}(\theta_0, \theta_1, \ldots, \theta_d)$. Therefore $\rho(B)$ is multiplicity-free if only if $\{\theta_i\}_{i=0}^d$ are mutually distinct, and in this case $\rho(E_i)$ is the primitive idempotent of $\rho(B)$ for $\theta_i$ ($0 \leq i \leq d$). For $0 \leq i \leq d$ the $(s, 0)$-entry of $\rho(E_i)$ is given in Lemma 5.7(i). This entry is $|X|^{-1}Q_{si}$ since $P_{i0} = 1$.

(i)$\Rightarrow$(ii): By Lemma 5.12 the graph $\Gamma(\rho(B))$ is a bidirected path with endpoints $s, 0$. Therefore $A = \rho(B)$ satisfies Theorem 1.3(i) with $t = 0$. Applying Theorem 1.3 we draw two conclusions. First, $\rho(B)$ is multiplicity-free, so $\{\theta_i\}_{i=0}^d$ are mutually distinct. Second, the $(s, 0)$-entry of $\rho(E_i)$ times $f_i(\theta_i)$ is independent of $i$ for $0 \leq i \leq d$. By this and our above comments, $f_i(\theta_i)Q_{si}$ is independent of $i$ for $0 \leq i \leq d$. Therefore $f_i(\theta_i)Q_{si} = f_0(\theta_0)Q_{s0}$ for $0 \leq i \leq d$. By this and since $Q_{s0} = 1$, we find $Q_{si} = f_0(\theta_0)f_i(\theta_i)^{-1}$ for $0 \leq i \leq d$. In other words, for $0 \leq i \leq d$ the dual eigenvalue of $E_i$ for $C$ is equal to $f_i(\theta_i)$. (ii)$\Rightarrow$(i): Observe that $\rho(B)$ is symmetrizable by Lemma 5.8 and multiplicity-free since $\{\theta_i\}_{i=0}^d$ are mutually distinct. For $0 \leq i \leq d$ the dual eigenvalue of $E_i$ for $C$ is $Q_{si}$, and this is equal to $f_0(\theta_0)f_i(\theta_i)^{-1}$ by (2). By this and our above comments, for $0 \leq i \leq d$ the $(s, 0)$-entry of $\rho(E_i)$ is equal to $|X|^{-1}f_0(\theta_0)f_i(\theta_i)^{-1}$. So the $(s, 0)$-entry of $\rho(E_i)$ times $f_i(\theta_i)$ is independent of $i$ for $0 \leq i \leq d$, and this common value is nonzero. Therefore
\( \rho(B) \) satisfies Theorem 1.1(ii) with \( t = 0 \). Now by Theorem 1.3 the graph \( \Gamma(\rho(B)) \) is a bidirected path with endpoints \( s, 0 \). Now by Lemma 5.12 \( \mathcal{X} \) is \( P \)-polynomial relative to \( B \), and \( C = A_s \) is the last adjacency matrix in this \( P \)-polynomial structure. \( \square \)

The proof of Theorem 1.1 is similar to the proof of Theorem 1.2. We give a precise proof for completeness.

**Proof of Theorem 1.1.** Fix an ordering \( \{E_i\}_{i=1}^d \) of the nontrivial primitive idempotents such that \( E_1 = E \), and let \( F = E_s \). For \( 0 \leq i \leq d \) the scalar \( \theta^*_i = Q_{ii} \) is the dual eigenvalue of \( E \) for \( A_i \), and \( P_{si} \) is the eigenvalue of \( A_i \) for \( F \). For \( 0 \leq i \leq d \) define a polynomial \( f^*_i \in \mathbb{R}[\lambda] \) by

\[
f^*_i = (\lambda - \theta^*_0) \cdots (\lambda - \theta^*_{i-1})(\lambda - \theta^*_{i+1}) \cdots (\lambda - \theta^*_d).
\]

For \( 0 \leq i \leq d \) let \( E^*_i \) (resp. \( A^*_i \)) denote the dual primitive idempotent (resp. dual adjacency matrix) corresponding to \( A_i \) (resp. \( E_i \)). Let the map \( \rho^* \) be as in Definition 5.2. By Lemma 5.13 the matrix \( \rho^*(A^*_i) \) is nonnegative, so we can apply Theorem 1.3 with \( A = \rho^*(A^*_1) \). We will do this after a few comments. Combining Lemma 5.3(ii) and Lemma 5.5(i) we find \( Q \rho^*(A^*_1) Q^{-1} = \text{diag}(\theta^*_0, \theta^*_1, \ldots, \theta^*_d) \). Therefore \( \rho^*(A^*_i) \) is multiplicity-free if only if \( \{\theta^*_i\}_{i=0}^d \) are mutually distinct, and in this case \( \rho^*(E^*_i) \) is the primitive idempotent of \( \rho^*(A^*_1) \) for \( \theta^*_i \) \( (0 \leq i \leq d) \). For \( 0 \leq i \leq d \) the \( (s,0) \)-entry of \( \rho^*(E^*_i) \) is given in Lemma 5.7(ii). This entry is \( |X|^{-1} P_{si} \) since \( Q_{i0} = 1 \).

(i) \( \Rightarrow \) (ii): By Lemma 5.13 the graph \( \Gamma(\rho^*(A^*_i)) \) is a bidirected path with endpoints \( s, 0 \). Therefore \( A = \rho^*(A^*_1) \) satisfies Theorem 1.3(i) with \( t = 0 \). Applying Theorem 1.3 we draw two conclusions. First, \( \rho^*(A^*_i) \) is multiplicity-free, so \( \{\theta^*_i\}_{i=0}^d \) are mutually distinct. Second, the \( (s,0) \)-entry of \( \rho^*(E^*_i) \) times \( f^*_i(\theta^*_i) \) is independent of \( i \) for \( 0 \leq i \leq d \). By this and our above comments, \( f^*_i(\theta^*_i) P_{si} \) is independent of \( i \) for \( 0 \leq i \leq d \). Therefore \( f^*_i(\theta^*_i) P_{si} = f^*_0(\theta^*_0) P_{s0} \) for \( 0 \leq i \leq d \). By this and since \( P_{s0} = 1 \), we find \( P_{si} = f^*_0(\theta^*_0) f^*_i(\theta^*_i)^{-1} \) for \( 0 \leq i \leq d \). In other words, for \( 0 \leq i \leq d \) the eigenvalue of \( A_i \) for \( F \) is equal to (1).

(ii) \( \Rightarrow \) (i): Observe that \( \rho^*(A^*_i) \) is symmetrizable by Lemma 5.8 and multiplicity-free since \( \{\theta^*_i\}_{i=0}^d \) are mutually distinct. For \( 0 \leq i \leq d \) the eigenvalue of \( A_i \) for \( F \) is \( P_{si} \), and this is equal to \( f^*_0(\theta^*_0) f^*_i(\theta^*_i)^{-1} \) by (1). By this and our above comments, for \( 0 \leq i \leq d \) the \( (s,0) \)-entry of \( \rho^*(E^*_i) \) is equal to \( |X|^{-1} f^*_0(\theta^*_0) f^*_i(\theta^*_i)^{-1} \). So the \( (s,0) \)-entry of \( \rho^*(E^*_i) \) times \( f^*_i(\theta^*_i) \) is independent of \( i \) for \( 0 \leq i \leq d \), and this common value is nonzero. Therefore \( \rho^*(A^*_i) \) satisfies Theorem 1.3(ii) with \( t = 0 \). Now by Theorem 1.3 the graph \( \Gamma(\rho^*(A^*_i)) \) is a bidirected path with endpoints \( s, 0 \). Now by Lemma 5.13 \( \mathcal{X} \) is \( Q \)-polynomial relative to \( E \), and \( F = E_s \) is the last primitive idempotent in this \( Q \)-polynomial structure. \( \square \)

**References**

[1] E. Bannai, T. Ito, Algebraic Combinatorics I: Association Schemes, Benjamin/Cummings, London, 1984.

[2] A. E. Brouwer, A. M. Cohen, A. Neumaier, Distance-Regular Graphs, Springer-Verlag, Berlin, 1989.

[3] H. Kurihara, H. Nozaki, An equivalent condition of the \( Q \)-polynomial property on the spherical embedding of symmetric association schemes, preprint, arXiv:1007.0473.
[4] D. Serre, Matrices: Theory and Applications, Graduate Texts in Mathematics 216, Springer-Verlag, New York, 2002.

[5] P. Terwilliger, The subconstituent algebra of an association scheme I, J. Algebraic Combin. 1 (1992) 363–388.

Kazumasa Nomura
Professor Emeritus
Tokyo Medical and Dental University
Kohnodai, Ichikawa, 272-0827 Japan
e-mail: knomura@pop11.odn.ne.jp

Paul Terwilliger
Department of Mathematics
University of Wisconsin
480 Lincoln Drive
Madison, Wisconsin, 53706 USA
e-mail: terwilli@math.wisc.edu

Keywords. Hessenberg matrix, tridiagonal matrix, association scheme.

2010 Mathematics Subject Classification. 05E30, 15A30, 16S50.