ABSTRACT. In this article we dwell into the class of so called ill-posed Linear Inverse Problems (LIP) which simply refers to the task of recovering the entire signal from its relatively few random linear measurements. Such problems arise in variety of settings with applications ranging from medical image processing, recommender systems, etc. We propose a slightly generalised version of the error constrained linear inverse problem and obtain a novel and equivalent convex-concave min-max reformulation by providing an exposition to its convex geometry. Saddle points of the min-max problem are completely characterised in terms of a solution to the LIP, and vice versa. Applying simple saddle point seeking ascend-descent type algorithms to solve the min-max problems provides novel and simple algorithms to find a solution to the LIP. Moreover, reformulation of an LIP as the min-max problem provided in this article is crucial in developing methods to solve the dictionary learning problem with almost sure recovery constraints.

Keywords: Linear Inverse Problems, Fenchel Duality, min-max problems, Dictionary Learning.

1. Introduction

A Linear Inverse Problem (LIP) is, simply stated, the recovery of a signal from its linear measurements. Signals encountered in practise tend to be very high dimensional; for example, audio signals and images typically have ambient dimension ranging from a few thousands to millions. However, the number of linear measurements that are typically available to recover the entire signal from, are relatively few compared to their ambient dimension. This makes such an LIP ill-posed. Fortunately, high dimensional data of the present day and age often contain underlying low dimensional characteristics, which if taken into consideration, often suffice to overcome the ill-posedness of the problem.
One of the early instances that gave recognition to linear inverse problems is compressed sensing \([7, 3, 2, 5]\), where a given signal \(f'\) is assumed to be sparse in some known basis. So, given the partial information of the signal in the form of a collection of linear measurements \(x = \phi(f')\), the objective is to recover the entire signal almost accurately. Since the given signal is known to be sparse, one would expect that the true signal can be recovered accurately by finding a sparsest solution to the under determined system of linear equations \(x = \phi(f)\) given by the linear measurements. However, finding sparsest solutions (i.e., having minimum \(\ell_0\) “norm”) to linear equations is NP hard and therefore, not practical in the intended applications due to the size of the data typically encountered there. Fortunately, it is now well established that under mild conditions, the simple convex heuristic of minimizing the \(\ell_1\)-norm

\[
\begin{aligned}
\text{minimize} & \quad \|f\|_1 \\
\text{subject to} & \quad x = \phi(f),
\end{aligned}
\]

instead of the \(\ell_0\)-penalty finds the sparsest solution almost always. Thus, the true signal can be recovered exactly by simply solving a convex optimization problem. Moreover, even if the linear measurements are noisy, recovery done via minimizing the \(\ell_1\)-penalty is reasonably accurate.

Similar to compressed sensing is the problem of low rank matrix recovery or completion \([4, 14, 6]\), where the objective is to reconstruct an entire matrix \(M'\) from only a few of its entries \([M']_{ij}\) for \((i, j) \in I\), where the cardinality of \(I\) is “small” compared to the size of the matrix \(M'\). Matrix recovery or completion problems arise regularly in recommender systems, and the Netflix challenge case in point. Since the unknown matrix is known to be of low rank, one expects that the true matrix is the solution to the rank minimization problem:

\[
\begin{aligned}
\text{minimize} & \quad \text{rank}(M) \\
\text{subject to} & \quad [M']_{ij} = [M]_{ij} \text{ for } (i, j) \in I.
\end{aligned}
\]

However, minimizing the rank exactly, is extremely hard and impractical for most applications. Analogous to the \(\ell_1\)-minimization, it is now well established \([14]\) that under mild conditions, the simple convex heuristic of minimizing the matrix nuclear norm \(\|\cdot\|_*\) recovers the true low rank matrix.

\[
\begin{aligned}
\text{minimize} & \quad \|M'\|_* \\
\text{subject to} & \quad [M']_{ij} = [M]_{ij} \text{ for } (i, j) \in I.
\end{aligned}
\]

Often, signals that are encountered in practice can be written as a linear combination of relatively few elements from some atomic set \(\mathcal{A}\) which depends on the low dimensional characteristics present in the signal. For instance, in compressed sensing, since the signal is assumed to be sparse, the atomic set \(\mathcal{A}\) is considered to be the standard Euclidean basis of appropriate dimension. In the matrix recovery problem, since the unknown matrix is assumed to be of low rank, it can be written as the sum of a few rank-1 matrices, and thus the atomic set \(\mathcal{A}\) is the set of all rank-1 matrices of appropriate dimensions. So, given such a signal with the corresponding atomic set \(\mathcal{A}\), an LIP attempts to find a linear combination of few elements from the atomic set \(\mathcal{A}\) that agree with the given linear measurements of the signal. However, as evident in the compressed sensing and matrix recovery
problems, finding such a linear combination by simply searching the atomic set is impractical.

It is to be observed that the $\ell_1$ and the nuclear norms are the guage functions corresponding to the convex hull of the standard Euclidean basis (atomic set in compressed sensing) and the set of rank-1 matrices (atomic set in matrix recovery problem) respectively. By minimizing such convex functions subject to the linear measurement constraints, guarantees have been obtained for fruitful signal recovery in compressed sensing and matrix recovery problems. Motivated by this observation, it was established in [6] that for a generic LIP with a generic atomic set $\mathcal{A}$, the analogous convex heuristic of minimizing the guage function corresponding to the set $\text{conv}(\mathcal{A})$ provides exact recovery under mild conditions. Thus, given linear measurements $x = \phi(f)$ of a signal $f$, an LIP seeks to solve

$$
\begin{align*}
\begin{cases}
\text{minimize} & c(f') \\
\text{subject to} & x = \phi(f'),
\end{cases}
\end{align*}
$$

where $c$ is a positively homogenous convex cost function such that $\text{conv}(\mathcal{A}) = \{f : c(f) \leq 1\}$. If the observed linear measurements are noisy, i.e., $x = \phi(f') + \xi$, for some measurement noise $\xi$. We solve

$$
\begin{align*}
\begin{cases}
\text{minimize} & c(f') \\
\text{subject to} & \|x - \phi(f')\| \leq \epsilon,
\end{cases}
\end{align*}
$$

where $\epsilon \geq 0$ is chosen based on the statistical properties of the measurement noise $\xi$. If $f$ is a linear combination of only a “few” elements of $\mathcal{A}$, the true signal can be recovered from only the linear measurements by solving the LIP (4). A great body of literature [7, 3, 2, 5, 4, 14, 6] exists on linear inverse problems focusing primarily towards providing quantitative analysis of the number of measurements required, and the constraints on the type of measurements suited for a given atomic set in order to guarantee fruitful recovery. However, our objective in studying the linear inverse problems is not directed towards this cause.

The main motivation for our work in this article comes from the related problem of Dictionary Learning, which is another well known machine learning problem. The objective in dictionary learning is to find a standard database of vectors called the dictionary such that samples of the given data $(x_t)_t$ can be expressed as linear combinations of the dictionary vectors with desirable features, an important one being sparsity. Due to the many benefits of sparse representation in applications such as compression, robustness, clustering etc., there is an ever increasing demand to learn good dictionaries that offer maximally sparse but also reasonably accurate representation of the data. A brief overview on the relevance of the dictionary learning problem and methods used to learn a ‘good’ dictionary are given in [17].

To this end, let $D = (d_1 \ d_2 \ \cdots \ d_K)$ denote a dictionary of $K$ vectors, where $K$ is some positive integer. Let $f_t$ denote the representation of sample $x_t$ for every
\( t \in 1, 2, \ldots, T \). Then the dictionary learning problem that we aim to solve is

\[
(5) \quad \begin{cases} 
\text{minimize} & \frac{1}{T} \sum_{t=1}^{T} \| f_t \|_1 \\
\text{subject to} & D \in \mathcal{D}, \\
& \| x_t - D f_t \|_2 \leq \epsilon_t \quad \text{for every } t = 1, 2, \ldots, T,
\end{cases}
\]

where \( \mathcal{D} \) is some convex subset of \( \mathbb{R}^{n \times K} \) and \( (\epsilon_t)_t \) is a given sequence of non-negative real numbers. In image processing applications like denoising etc., \( \epsilon_t \) corresponds to the bounds on the additive noise in the noisy data. It is to be noted that for a fixed dictionary \( D \), the optimization over \( (f_t)_t \) simply consists of solving the LIP (4) for each \( t \). Minimization of the \( \ell_1 \)-penalty is to enforce sparsity in the representation vectors \( (f_t)_t \). For different applications, the dictionary can be learned to optimize a task-specific but otherwise generic cost function \( c(\cdot, \cdot) \) instead of the \( \ell_1 \)-norm.

Conventionally, the dictionary learning problem is addressed by solving the optimization problem

\[
(6) \quad \text{minimize}_{(f_t)_t, D \in \mathcal{D}} \quad \frac{1}{T} \sum_{t=1}^{T} \left( \| f_t \|_1 + \gamma \| x_t - D f_t \|_2^2 \right)
\]

where \( \gamma > 0 \) is the regularization parameter. It should be noted that the cost function in (6) is a weighted sum of the sparsity inducing \( \ell_1 \)-penalty \( \| f_t \|_1 \) and the error term \( \| x_t - D f_t \|_2^2 \). The regularization parameter \( \gamma \) influences the tradeoff between the level of sparsity and the error, and for a given value of \( \gamma \), this tradeoff is specific to a given distribution or data set. However, the precise relation between the value of regularization parameter \( \gamma \) and the tradeoff is not straightforward. Thus, a priori one does not know which value of the regularization parameter \( \lambda \) to pick for a given distribution or data set; it is a tuning parameter that needs to be learned from the data.

The dictionary learning problem (5) differs from the mainstream one (6) in that it imposes constraint on every sample to be reconstruction within limits. Such a formulation arises naturally in many image processing applications like compressed sensing [7, 3], inpainting, denoising problems in image processing [8] etc., where, good estimates of \( (\epsilon_t)_t \) to be used in (5) are known a priori. In contrast, if we were to learn the dictionary for the same applications but by solving (6) instead, the appropriate value of the regularization parameter to be used is not known and needs to be learnt from other techniques like cross validation, which poses additional computational challenges. Furthermore, with a single parameter the problem formulation (6) does not provide the level of customisability that is available in (5). Therefore, learning dictionaries by solving (5) is more appealing in situations where good estimates of \( (\epsilon_t)_t \) to be used are known beforehand or in situations where the user has the liberty of specifying it.

Most of the existing techniques [12, 1] that learn a dictionary by solving (6), do so by alternating the minimization over the variables \( (f_t)_t \) and \( D \) keeping the other
We note that, individually both the problems in (7) are convex, in particular, the optimization over the dictionaries is a QP.

In contrast to (6), an alternating minimization technique like (7) is completely ineffective in order to solve (5). Indeed, once the variables \(f_t\) and \(x_t\) are fixed, there is no evident way to update the dictionary variable such that the resulting dictionary minimizes the cost. This makes the dictionary learning problem (5) ill-posed and more challenging than the conventional regularized formulation. We observe that the objective function in (6) depends directly on the dictionary variable \(D\), whereas, it affects the objective function of (5) indirectly by only changing the set of feasible representations. This is the key reason which makes updating the dictionary in (5) such a difficult task.

In this article, we propose a slight modification to the original linear inverse problem (3), involving an additional parameter, which adds regularity to the non-regularised problem (3) when positive. Convex duality of the proposed problem is studied by exposing the underlying geometry. We propose convex-concave min-max problems and establish their equivalence to the modified LIP. Mathematical guarantees based on Fenchel duality relating the optimal solution of the LIP with the saddle points of the min-max problems are provided. This equivalent reformulation of the LIP as a min-max problem precisely addresses the issue of ill-posedness in the dictionary learning problem (5). In the dictionary learning problem (5), replacing the optimization over variables \((f_t)_t\), which is a collection of linear inverse problems, with their respective min-max problems, gives us another min-max problem equivalent to (5), but well-posed. Making use of this reformulation, we have provided novel dictionary learning algorithm in [16] to solve (5). To the best of our knowledge, these are the first set of results that effectively solve the dictionary learning problem for situations where solving the formulation (5) is natural.

In addition to the importance of the reformulations of the LIP in dictionary learning, the min-max forms also provide a new approach to solve an LIP, which is of independent interest. Gradient based algorithms to compute saddle points of the min-max problem give rise to simple and easy to implement algorithms to solve an LIP. Due to the relevance of large dimensional linear inverse problems, simple to implement yet reasonably fast and efficient algorithms to solve them are always desirable. One of the objectives of this article is to take a step towards this direction. Furthermore, the theory and mathematical guarantees provided in this article are fairly generic and easily carry forward to the extended class of optimization problems which optimize a convex gauge function over convex sets, this includes many relevant problems like projections onto convex sets, LASSO etc., thereby yielding new algorithms for all such problems at once.
The article unfolds as follows: In Section 2 we formally introduce the LIP in a more generalised form and provide the main results including the equivalent convex-concave min-max reformulations, algorithms to solve them and applications to some specific problems of interest. In Section 3, we expose the duality by studying the underlying convex geometry of the LIP and provide proofs for all the results. We employ standard notations, and specific ones are explained as they appear.

2. Formal problem statement and main results

Let $n$ be a positive integer, $\mathbb{H}_n$ be an $n$-dimensional Hilbert space equipped with an inner product $\langle \cdot , \cdot \rangle$ and its associated norm $||\cdot||$. For every $z \in \mathbb{H}_n$ and $r > 0$, let $B(z, r) := \{ y \in \mathbb{H}_n : ||x - y|| < \epsilon \}$ and let $B[z, r] := \{ y \in \mathbb{H}_n : ||x - y|| \leq \epsilon \}$. Let $c : \mathbb{R}^K \rightarrow [0, + \infty]$ be a cost function such that it satisfies the following assumption.

**Assumption 2.1.** The cost function $c : \mathbb{R}^K \rightarrow [0, + \infty]$ has the following properties

- **Positive Homogeneity**: There exists a positive real number $p$ such that for every $\alpha \geq 0$ and $f \in \mathbb{R}^K$, we have $c(\alpha f) = \alpha^p c(f)$.
- **Pseudo-Convexity**: The unit sublevel set $V_c := \{ f \in \mathbb{R}^K : c(f) \leq 1 \}$ is convex.
- **Inf-Compactness**: The unit sublevel set $V_c$ is compact.

Let $x \in \mathbb{H}_n$, non-negative real numbers $\epsilon$ and $\delta$, and the linear map $\phi : \mathbb{H}_n \rightarrow \mathbb{R}^K$ be given. We consider the following general formulation of the linear inverse problem

$$
\begin{align*}
\text{minimize} \quad & \quad c^p \\
\text{subject to} \quad & \quad \left( c(f) \right)^{1/p} \leq c \\
& \quad ||x - \phi(f)|| \leq \epsilon + \delta c.
\end{align*}
$$

When $\delta = 0$, we see that the feasible collection of $f$ is independent from the variable $c$. As a consequence we see that for every feasible $f \in \mathbb{R}^K$, the minimization over the variable $c$ is achieved for $c = c(f)$. Thus the linear inverse problem (8) reduces to the following more familiar formulation.

$$
\begin{align*}
\text{minimize} \quad & \quad c(f) \\
\text{subject to} \quad & \quad ||x - \phi(f)|| \leq \epsilon.
\end{align*}
$$

The non-negative real number $\delta$ acts as a regularization parameter. If $\delta > 0$, by considering $c > \left( ||x||/\delta \right)^p$ and $f = 0$, we see that the linear inverse problem (8) is always feasible. On the contrary, if $\delta = 0$, it is immediately apparent that (8) is feasible if and only if $B[x, \epsilon] \cap \text{image}(\phi) \neq \emptyset$.

It might be surprising at first to see the rather unusual formulation (8) of the linear inverse problem. Our formulation makes way for the possibility of $\delta$ to take positive values also. BY considering a positive value for the regularization parameter $\delta$, we obtain several advantages:

- A positive value of $\delta$ amounts to having the effect of regularization in the problem. Thus, one can harvest all the advantages that come from regularization like numerical stability in algorithms, well conditioning etc.
Whenever \( \delta > 0 \), the LIP (8) is always strictly feasible. This is a crucial feature in the initial stages of dictionary learning, in particular, when the data lies in a subspace of lower dimension \( m \), such that \( m, K \ll n \), where \( K \) is the number of dictionary vectors. Having a positive value of \( \delta \) eliminates the pathological cases that arise in dictionary learning and provides guarantees for convergence of dictionary learning algorithms. Moreover, it leads to useful fixed point characterization of the optimal dictionary, which in turn lead to simple online dictionary update algorithms.

We observe that the mapping \( f \mapsto (c(f))^{1/p} \) is an inf-compact, convex and positively homogeneous of order 1. Therefore, it is immediate that the constraints of the LIP (8) are convex. Furthermore, the objective function is also convex whenever \( p \geq 1 \). Thus, it is apparent that the LIP (8) is a convex problem when \( p \geq 1 \). When \( p < 1 \), we highlight that \( [0, +\infty] \ni (\cdot) \mapsto (\cdot)^p \) is an increasing function, and therefore, minimizing \( c^p \) is equivalent to minimizing \( c \). Thus, the LIP (8) has an underlying convex problem (except that the objective function is a non-convex power).

We emphasise that whenever the optimization problem (8) is feasible, the feasible set is closed and the cost function is continuous and coercive.\(^1\) Therefore, from the Weierstrass theorem [15, Theorem 4.16] we conclude that whenever (8) is feasible, it admits an optimal solution. To this end, let

\[
(C_\delta(\phi, x, \epsilon))^{1/p} := \arg\min_{(c, f) \in \mathbb{R} \times \mathbb{R}^K} \left\{ \begin{array}{ll}
\text{subject to} & (c(f))^{1/p} \leq c \\
\|x - \phi(f)\| \leq \epsilon + \delta c.
\end{array} \right.
\]

Note that \( C_\delta(\phi, x, \epsilon) \) is also the optimal value achieved in (8). In view of this, we slightly abuse the definition (10), and follow the convention that if (8) is infeasible, then \( C_\delta(\phi, x, \epsilon) := +\infty \) and \( F_\delta(\phi, x, \epsilon) := \emptyset \).

The optimization problem (8) depending on the parameters \( x, \phi, \epsilon, \delta \) could potentially have multiple solutions. However, in signal recovery from ill-posed linear inverse problems, if there are sufficiently many linear measurements, and of correct type, the LIP (8) admits a unique solution and the set \( F_\delta(\phi, x, \epsilon) \) is then a singleton containing the true signal to be recovered. In other situations like sparse encoding, the LIP (8) could have multiple solutions, and if it does, we see that \( F_\delta(\phi, x, \epsilon) \) is a convex set.

**Definition 2.2.** Let \( \phi : \mathbb{R}^K \rightarrow \mathbb{H}_n \) be a linear map, and let \( \epsilon, \delta \geq 0 \). A vector \( x \in \mathbb{H}_n \) is said to be \( (\phi, \epsilon, \delta) \)-feasible if \( C_\delta(\phi, x, \epsilon) < +\infty \).

**Remark 2.3.** We see that \( x \in \mathbb{H}_n \) is \( (\phi, \epsilon, \delta) \)-feasible if and only if at least one of the following holds:

- \( \delta > 0 \),
- \( B[x, \epsilon] \cap \text{image}(\phi) \neq \emptyset \).

\(^1\)Recall that a continuous function \( c \) defined over an unbounded set \( U \) is said to be coercive in the context of an optimization problem, if \( \lim_{\|u\| \to \infty} c(u) = +\infty \) (\( -\infty \)), in the context of minimization (maximization) of \( c \) and the limit is considered from within the set \( U \).
**Definition 2.4.** For the linear map \( \phi : \mathbb{H}_n \to \mathbb{R}^K \), non-negative real number \( \delta \) and the cost function \( c : \mathbb{H}_n \to [0, +\infty[ \) satisfying Assumption 2.1, let us define

\[
S_\delta(\phi, 1) := \{ z \in \mathbb{H}_n : \text{there exists } f \in V_c \text{ satisfying } \| z - \phi(f) \| \leq \delta \},
\]

where \( V_c \) is the unit sub level set of the cost function \( c \).

By denoting \( S' := \{ \phi(f) : f \in V_c \} \), it is clear that \( S' \) is the image of the compact and convex set \( V_c \) under the linear map \( \phi \), and is therefore, compact and convex. Moreover, since the set \( S_\delta(\phi, 1) \) is the image of the linear map \( S' \times B[0, \delta] \ni (z', y) \mapsto z' + y \), \( S_\delta(\phi, 1) \) is also compact and convex. Furthermore, for every \( r \geq 0 \) the set \( S_\delta(\phi, r) := r \cdot S_\delta(\phi, 1) \), obtained by linearly scaling \( S_\delta(\phi, 1) \) by an amount of \( r \), is also compact and convex.

The **guage function** \( \| \cdot \|_\phi : \mathbb{H}_n \to [0, +\infty[ \) corresponding to the set \( S_\delta(\phi, 1) \) is given by

\[
\| z \|_\phi := \min \{ r \geq 0 : z \in S_\delta(\phi, r) \}.
\]

When \( \delta > 0 \), we know that \( S_\delta(\phi, 1) \) has non-empty interior, therefore, \( \| z \|_\phi < +\infty \) for every \( z \in \mathbb{H}_n \). Similarly when \( \delta = 0 \), \( \| z \|_\phi < +\infty \) if and only if \( z \in \text{image}(\phi) \). Moreover, due to the set \( S_\delta(\phi, r) \) being compact for every \( r \geq 0 \), the minimization in the definition of the guage function is always achieved. In other words, for every \( z \in \mathbb{H}_n \) such that \( \| z \|_\phi < +\infty \), we have \( z \in S_\delta(\phi, \| z \|_\phi) \).

The underlying convexity of the linear inverse problem (8) gives rise to an interplay of the convex bodies \( B[x, \epsilon] \) and \( S_\delta(\phi, 1) \). As a result, we obtain the relation between the optimal cost \( C_\delta(\phi, x, \epsilon) \), the guage function \( \| \cdot \|_\phi \) and the set \( B[x, \epsilon] \).

**Lemma 2.5.** Consider the LIP (8) for the linear map \( \phi \), cost function \( c \), non-negative real numbers \( \epsilon, \delta \) and \( x \in \mathbb{H}_n \). The optimal cost \( C_\delta(\phi, x, \epsilon) \) of the LIP (8) and the guage function \( \| \cdot \|_\phi \) satisfy

\[
(C_\delta(\phi, x, \epsilon))^{1/p} = \min_{y \in B[x, \epsilon]} \| y \|_\phi.
\]

### 2.1. Duality

The guage function \( \| \cdot \|_\phi \) gives rise to its corresponding dual function \( \| \cdot \|'_\phi : \mathbb{H}_n \to [0, +\infty[ \) defined by:

\[
\| \lambda \|'_\phi := \sup_{\| z \|_\phi \leq 1} \langle \lambda, z \rangle = \sup_{z \in S_\delta(\phi, 1)} \langle \lambda, z \rangle.
\]

Let \( \lambda, y \in \mathbb{H}_n \), we recall that \( y \in S_\delta(\phi, \| y \|_\phi) \), and consequently, we have the following Holder like inequality

\[
\langle \lambda, y \rangle \leq \sup_{z \in S_\delta(\phi, \| y \|_\phi)} \langle \lambda, z \rangle = \| y \|_\phi \sup_{z \in S_\delta(\phi, 1)} \langle \lambda, z \rangle = \| y \|_\phi \| \lambda \|'_\phi.
\]

This gives rise to strong duality between the guage function \( \| \cdot \|_\phi \) and its associated dual function \( \| \cdot \|'_\phi \) in the following way

\[
\| y \|_\phi = \left\{ \begin{array}{ll}
\sup_{\lambda} \langle \lambda, y \rangle & \quad \text{subject to } \| \lambda \|'_\phi \leq 1.
\end{array} \right.
\]

\(^2\text{Considering, for instance, } c(\cdot) = \| \cdot \|_1 \text{ and the linear map } \phi \text{ given by the matrix } D = \begin{pmatrix} d_1 & d_2 & \cdots & d_K \end{pmatrix} \in \mathbb{R}^{n \times K}, \text{ we see that } V_c \text{ is the } \ell_1\text{-closed ball in } \mathbb{R}^K \text{ and } S' = \text{conv}(\pm d_i)_{i=1}^K.\)
By replacing the gauging function \( \| \cdot \|_\phi \) in (13) with its equivalent sup formulation provided in (15), we obtain the Convex Dual of the LIP (8). First, we will define the set \( \Lambda_\delta(\phi, x, \epsilon) \) which is the collection of optimal dual variables.

**Definition 2.6.** Let the linear map \( \phi \) and \( \epsilon, \delta \geq 0 \) and a cost function \( c \) satisfying Assumption 2.1 be given. Then for every \( x \in \mathbb{H}_n \) that is \( (\phi, \epsilon, \delta) \)-feasible, let \( \Lambda_\delta(\phi, x, \epsilon) \subset \mathbb{H}_n \) denote the collection of points \( \lambda \in \mathbb{H}_n \setminus B[0, \epsilon] \) that satisfy the following two conditions simultaneously:

- \( \| \lambda \|_\phi' = 1 \), and
- \( \langle \lambda, x \rangle - \epsilon \| \lambda \| = (C_\delta(\phi, x, \epsilon))^{1/p} \).

**Theorem 2.7.** Let the linear map \( \phi: \mathbb{R}^K \rightarrow \mathbb{H}_n \), real numbers \( \epsilon, \delta \geq 0 \) and \( x \in \mathbb{H}_n \) be given. Consider the linear inverse problem (8) and its convex dual problem:

\[
\begin{align*}
\sup_{\lambda} & \quad \langle \lambda, x \rangle - \epsilon \| \lambda \| \\
\text{subject to} & \quad \| \lambda \|_\phi' \leq 1.
\end{align*}
\]

(i) **Strong Duality:** The supremum value in (16) is finite if and only if \( x \) is \( (\phi, \epsilon, \delta) \)-feasible, and is equal to the optimal cost \((C_\delta(\phi, x, \epsilon))^{1/p}\).

(ii) **Existence and description of an optimal solution to (16).**

(a) Irrespective of the value of \( \delta \), for any \( \epsilon \geq 0 \) if \( \| x \| \leq \epsilon \), then \( \lambda^* = 0 \) is an optimal solution.

(b) Whenever \( \| x \| > \epsilon \), the optimization problem (16) admits an optimal solution if and only if the set \( \Lambda_\delta(\phi, x, \epsilon) \) defined in 2.6 is non-empty and \( \lambda^* \) is a solution if and only if \( \lambda^* \in \Lambda_\delta(\phi, x, \epsilon) \). As a result, the supremum is indeed a maximum and it is achieved at \( \lambda^* \).

(c) Whenever \( \| x \| > \epsilon \) and the set \( \Lambda_\delta(\phi, x, \epsilon) \) is empty, the optimization problem (16) does not admit an optimal solution even though the value of the supremum is finite.

**Remark 2.8.** We provide a complete description of the set \( \Lambda_\delta(\phi, x, \epsilon) \) in Proposition 3.10. It turns out that the optimal solution to the dual problem (16) can be entirely characterization in terms of the optimal solution \((C_\delta(\phi, x, \epsilon), F_\delta(\phi, x, \epsilon))\) to the LIP (8) itself. Therefore, if we only have access to a black box that produces an optimal solution to the LIP (8), a corresponding dual optimal solution can be easily computed from the solutions to the LIP itself, and we need not solve the dual problem again separately. This is very advantageous in dictionary learning, where the optimal value of dual variable is required to compute a better dictionary.

**Remark 2.9.** We look ahead at Proposition 3.10 and see that the dual problem does not admit any optimal solution only when \( \delta = 0 \) and \( B(x, \epsilon) \cap \text{image}(\phi) = \emptyset \). Interestingly, in that case, we also observe that the corresponding primal problem is not strictly feasible.

### 2.2. Equivalent min-max problems.

Whenever \( \| x \| \leq \epsilon \), we immediately see that the pair \( \mathbb{R}_+ \times \mathbb{R}^K \ni (c_x, f_x) := (0, 0) \) is feasible for (8). Moreover, since \( c(f) > 0 \) for every \( f \neq 0 \) due to \( \inf \)-compactness and positive homogeneity, we conclude that:

\[
C_\delta(\phi, x, \epsilon) = 0 \quad \text{if and only if} \quad \| x \| \leq \epsilon.
\]
Therefore, the case: \(\|x\| \leq \epsilon\) is uninteresting and inconsequential. In fact, in dictionary learning, since every such sample can be effectively represented by the zero vector, irrespective of the dictionary. The average cost of representation then depends only on the samples that satisfy \(\|x\| > \epsilon\). Therefore, for the convex-concave min-max formulation of the LIP (8), we consider only the case when \(\|x\| > \epsilon\).

**Constrained formulation.**

**Theorem 2.10.** Let the linear map \(\phi : \mathbb{R}^K \rightarrow \mathbb{H}_n\), real numbers \(\epsilon, \delta \geq 0\), \(q \in ]0, 1[\), \(r > 0\) and \(x \in \mathbb{H}_n \setminus B[x, \epsilon]\) be given. Consider the linear inverse problem (8) and the following inf-sup problem:

\[
\inf_{h} \sup_{\lambda} \left\{ \begin{array}{ll}
\min & r \left( \langle \lambda, x \rangle - \epsilon \|\lambda\| \right) \frac{q}{q} - \left( \delta \|\lambda\| + \langle \lambda, \phi(h) \rangle \right) \\
\text{subject to} & \langle \lambda, x \rangle - \epsilon \|\lambda\| > 0
\end{array} \right. \]

The following assertions hold with regards to the coding problem (8) and its equivalent (18).

(i) The optimal value of (18) is equal to: \(s(r, q) (C_\delta(\phi, x, \epsilon))^{\frac{q}{q-p}}\), and therefore finite if and only if \(x\) is \((\phi, \epsilon, \delta)\)-feasible.\(^3\)

(ii) Existence and description of an optimal solution to (18).

(a) The minimization over variables \(h\) in (18) is achieved, and

\[h^* \in \arg\min_{h \in V_c} \left\{ \sup_{\lambda} \left( \begin{array}{ll}
\min & r \left( \langle \lambda, x \rangle - \epsilon \|\lambda\| \right) \frac{q}{q} - \left( \delta \|\lambda\| + \langle \lambda, \phi(h) \rangle \right) \\
\text{subject to} & \langle \lambda, x \rangle - \epsilon \|\lambda\| > 0
\end{array} \right. \right. \]

if and only if \(h^* \in \frac{1}{(C_\delta(\phi, x, \epsilon))^{1/p}} \cdot F_\delta(\phi, x, \epsilon)\).

(b) In addition, the inf-sup problem (18) admits a saddle point solution if and only if the set \(L_\delta(\phi, x, \epsilon)\) is non-empty, then a pair \((h^*, \lambda^*)\) \(\in V_c \times \mathbb{H}_n\) is a saddle point solution to (18) if and only if

\[h^* \in \frac{1}{(C_\delta(\phi, x, \epsilon))^{1/p}} \cdot F_\delta(\phi, x, \epsilon) \text{ and } \lambda^* \in (rq)^{\frac{1}{1-q}} (C_\delta(\phi, x, \epsilon))^{\frac{q}{q-p}} \cdot L_\delta(\phi, x, \epsilon)\).

**Corollary 2.11.** By considering \(r = r(p) := (1 + p)p^{\frac{1}{Q(p)}}\) and \(q = q(p) := \frac{r}{1+r}\), we get

\[C_\delta(\phi, x, \epsilon) = \left\{ \begin{array}{ll}
\min_{h \in V_c} \sup_{\lambda} \left( \begin{array}{ll}
\min & r(p) \left( \langle \lambda, x \rangle - \epsilon \|\lambda\| \right) \frac{q}{q} - \left( \delta \|\lambda\| + \langle \lambda, \phi(h) \rangle \right) \\
\text{subject to} & \langle \lambda, x \rangle - \epsilon \|\lambda\| > 0
\end{array} \right. \right. \]

In particular, if the cost function \(c(\cdot)\) is positively homogeneous of order 1 (like any norm), we have

\[C_\delta(\phi, x, \epsilon) = \left\{ \begin{array}{ll}
\min_{h \in V_c} \sup_{\lambda} \left( \begin{array}{ll}
\min & 2 \sqrt{\langle \lambda, x \rangle - \epsilon \|\lambda\| - \left( \delta \|\lambda\| + \langle \lambda, \phi(h) \rangle \right)} \\
\text{subject to} & \langle \lambda, x \rangle - \epsilon \|\lambda\| > 0
\end{array} \right. \right. \]

\(^3\) \(s(r, q) := \left( (1 - q)(q/r) \right)^{\frac{1}{q-p}}\)
Remark 2.12. In the case of signal recovery from its linear measurements, it is assumed that the true signal is a linear combination of only a few elements of some atomic set $A$. The signal is recovered by solving the linear inverse problem (8), by considering the cost function $c$ such that $V_c = \text{conv}(A)$ and $p = 1$. If the conditions on minimum number and type of measurements for fruitful recovery are satisfied, then the set $F_{\delta}(\phi, x, \epsilon)$ is a singleton and contains the signal to be recovered.

Remark 2.13. In several scenarios like the problem of non-negative matrix factorization [11, 10, 9], one has to solve (8) with the additional constraint that $f \in Q$, where $Q \subset \mathbb{R}^K$ is a convex cone. From similar analysis provided in this chapter, it can be easily verified that the resulting equivalent min-max formulation analogous to (18) is

$$
\begin{align*}
\left\{ \min_{h \in V_c \cap Q} & \sup_{\lambda} \lambda \left( \langle \lambda, x \rangle - \epsilon \| \lambda \| \right)^q - \left( \delta \| \lambda \| + \langle \lambda, \phi(h) \rangle \right) \right. \\
\text{subject to} & \langle \lambda, x \rangle - \epsilon \| \lambda \| > 0.
\end{align*}
$$

It should be noted that the definition of the set $S_{\delta}(\phi, 1)$ then changes to $S_{\delta}(\phi, 1) = \{ z \in \mathbb{H}_n : \text{there exists } f \in V_c \cap Q \text{ satisfying } \| x - \phi(f) \| \leq \delta \}$, and the quantities $\| \cdot \|_\phi$, $\| \cdot \|'_\phi$ and $\Lambda_{\delta}(\phi, x, \epsilon)$ are accordingly defined w.r.t. the appropriate definition of the set $S_{\delta}(\phi, 1)$.

Remark 2.14. The constraint $\langle \lambda, x \rangle - \epsilon \| \lambda \| > 0$ is an inactive constraint, and is insignificant for theoretical purpose. However, in practice, one must ensure that the inequality is satisfied so that the quantity $\left( \langle \lambda, x \rangle - \epsilon \| \lambda \| \right)^q$ is well defined for $q \in [0, 1]$. This is easily ensured by initialising the variable $\lambda$ so that it satisfies the inequality (for e.g., $\lambda_0 = x$), and then properly selecting the step sizes in subsequent iterations. Suppose if the variable $\lambda$ is updated as $\lambda \leftarrow \lambda + \alpha \lambda'$, then we must ensure that the inequality $0 < \langle \lambda + \alpha \lambda', x \rangle - \epsilon \| \lambda + \alpha \lambda' \|$ is satisfied. We see that

$$
\begin{align*}
\langle \lambda + \alpha \lambda', x \rangle - \epsilon \| \lambda + \alpha \lambda' \| & \geq \langle \lambda + \alpha \lambda', x \rangle - \epsilon \left( \| \lambda \| + \alpha \| \lambda' \| \right) \\
& \geq \left( \langle \lambda, x \rangle - \epsilon \| \lambda \| \right) + \alpha \left( \langle \lambda', x \rangle - \epsilon \| \lambda' \| \right).
\end{align*}
$$

If $0 \leq \langle \lambda', x \rangle - \epsilon \| \lambda' \|$, then the required inequality trivially holds, whereas if $0 > \langle \lambda', x \rangle - \epsilon \| \lambda' \|$, then by selecting $\alpha < \frac{\langle \lambda, x \rangle - \epsilon \| \lambda \|}{\| \langle \lambda', x \rangle - \epsilon \| \lambda' \| \|}$, it is easily verified that $0 < \langle \lambda + \alpha \lambda', x \rangle - \epsilon \| \lambda + \alpha \lambda' \|$. 

Remark 2.15. Minimizing over $h$ in the min-max problem of Corollary 2.11, we get

$$
C_{\delta}(\phi, x, \epsilon) = \sup_{\lambda} \left\{ r(p) \left( \langle \lambda, x \rangle - \epsilon \| \lambda \| \right)^{\frac{2}{p+1}} - \| \lambda \|'_x \right. \\
\left. \right\}.
$$

A crucial observation to be made here is that since $\frac{p}{p+1} \in [0, 1]$, the sublinear component $\left( \langle \lambda, x \rangle - \epsilon \| \lambda \| \right)^{\frac{2}{p+1}}$ initially grows faster than the linear component $\| \lambda \|'_x$ but is eventually overpowered. Alternatively, when $p > 1$, analysis similar to the proof of Lemma 3.14 shows that

$$
C_{\delta}(\phi, x, \epsilon) = \sup_{\lambda} \left\{ p(p-1)^{p-1} \left( \langle \lambda, x \rangle - \epsilon \| \lambda \| \right) - \| \lambda \|'_x^{\frac{p}{p+1}} \right. \\
\left. \right\}.
$$

If $\delta = 0$ and we were to replace the error constraint $\phi(f) \in B[x, \epsilon]$ with $\phi(f) \in B_x$, where $B_x \subset \mathbb{H}_n$ is some compact convex subset. By replacing $\langle \lambda, x \rangle - \epsilon \| \lambda \|$ with
the quantity \( \min_{y \in B_x} \langle \lambda, y \rangle \) in (19), we arrive at the corresponding min-max problem

\[
\min_{y \in B_x} \sup_{\lambda \in \mathbb{R}_+} p(p-1)^{p-1} \langle \lambda, y \rangle - \| \lambda \|_{\phi}^{\frac{p}{p-1}}.
\]

This allows us to write equivalent min-max formulations for problems like LASSO as seen in Remark 2.16.

**Remark 2.16.** Another optimization problem of interest and related to the linear inverse problem (8) is:

\[
\min_{f \in S} c(x - \psi(f)),
\]

where \( c(\cdot) \) is a cost function satisfying Assumption 2.1 with order of homogeneity \( p > 1 \), \( \psi : \mathbb{R}^K \to \mathbb{H}_n \) is some linear map and \( S \) is some compact convex subset of \( \mathbb{R}^K \). A frequent example of such a problem is LASSO, where \( c(\cdot) = \| \cdot \|^2 \) and \( S = \{ z : \| z \|_1 \leq \tau \} \).

Redefining \( y := \psi(f) - x \), the problem (21) simply becomes \( \min_{y \in \psi(S) - \{ x \}} c(y) \), whose equivalent min-max form using (20) is

\[
\min_{y \in \psi(S) - \{ x \}} \max_{\lambda \in \mathbb{H}_n} p(p-1)^{p-1} \langle \lambda, y \rangle - \| \lambda \|_{\phi}^{\frac{p}{p-1}},
\]

where \( \| \lambda \|_c := \max_{h \in V_c} \langle \lambda, h \rangle \). Replacing the minimization over variable \( y \) with \( f \), we get the final equivalent min-max form to (21).

\[
\min_{f \in S} \max_{\lambda \in \mathbb{H}_n} p(p-1)^{p-1} \left( \langle \lambda, x - \psi(f) \rangle \right) - \| \lambda \|_{\phi}^{\frac{p}{p-1}}.
\]

**Remark 2.17.** When the error constraint \( \| x - \phi(f) \| \leq \epsilon + \delta c \) is measured using a generic norm \( \| \cdot \| \), the min-max problem is written using the corresponding dual norm \( \| \cdot \|' \) as

\[
\min_{f \in S} \max_{\lambda \in \mathbb{H}_n} r(p) \left( \langle \lambda, x - \epsilon \| \lambda \|' \rangle \right)^{g(p)} - \left( \delta \| \lambda \|' + \langle \lambda, \phi(h) \rangle \right).
\]

**Unconstrained formulation.** The reformulation of the LIP (8) as the convex-concave min-max problem (18) provides a way to obtain a solution to the LIP (8) by solving the min-max problem instead. Even though (18) is a convex problem, it is to be observed that the minimization variable \( h \) is constrained. Therefore, updating the iterates of the minimization variable involves computing projections at each iterations, which could be an expensive task. So, in cases, where these projections are expensive, we seek to obtain a similar min-max reformulation which bypasses this computational bottleneck due to projections.

**Proposition 2.18.** Let \( \epsilon, \delta \geq 0 \) and \( x \in \mathbb{H}_n \backslash B[0, \epsilon] \), and a positively homogeneous cost function \( c(\cdot) \) of order 1 be given. Then the following assertions hold in view of the linear inverse problem (8) and the min-max problem:

\[
\min_{f \in \mathbb{R}^K} \max_{\lambda \in \mathbb{H}_n} c(f) \left( 1 - \delta \| \lambda \| \right) + \left( \langle \lambda, x \rangle - \epsilon \| \lambda \| \right) - \langle \lambda, \phi(f) \rangle
\]

subject to

\[
\| \lambda \| \leq \frac{1}{2}, \quad \langle \lambda, x \rangle - \epsilon \| \lambda \| > 0.
\]
(i) the optimal value of the min-max problem is equal to $C_δ(φ, x, ε)$
(ii) the minimization over variable $f$ is achieved and $F_δ(φ, x, ε)$ is the set of minimizers
(iii) the min-max problem (23) admits a saddle point solution $(f^*, λ^*)$ if and only if the set $Λ_δ(φ, x, ε)$ is non-empty, then a pair $(f^*, λ^*) ∈ \mathbb{R}_K × E_n$ is saddle point solution if and only if $f^* ∈ F_δ(φ, x, ε)$ and $λ^* ∈ Λ_δ(φ, x, ε)$.

Even though the maximization variable is constrained, the reference to the min-max problem (23) as “unconstrained” is due to the fact that projecting onto the feasible set $\{λ : \|λ\| ≤ 1/δ\}$ only requires normalizing the iterates $λ$ by the factor $\frac{1}{δ\|λ\|}$, which is easy. Moreover, if $δ = 0$, the min-max is truly unconstrained, justifying its name.

On the one hand, solving (18) involves projecting the iterates $h$ onto the level set $V_c$ at each iteration, which is generally a demanding task. Whereas, solving (23) requires the relatively easier task of computing gradient of the cost function $c(\cdot)$. However, it is observed that it takes fewer iterations to compute a saddle point solution to the constrained formulation (18) compared to that of the unconstrained formulation (23). Therefore, if we were to find a solution to the LIP (8) by solving the min-max problems (18) or (23), the user has to decide between solving (18) with fewer but expensive iterations or solving (23) with relatively easier but more iterations. This tradeoff depends on the given cost function and could be equally expensive like in the case of minimizing Nuclear norm.

Remark 2.19. Observe that the order of homogeneity $p$ in the LIP (8) is assumed to be 1 in Proposition 2.18. If $p > 1$ and when $δ = 0$, the equivalent unconstrained min-max reformulation similar to (23) is

$$
\left\{ \begin{array}{l}
\min_{f ∈ \mathbb{R}_K} \sup_{λ ∈ H_n} \ c(f) + \left( p^{\frac{1+2p}{p}} \right) \left( \langle λ , x \rangle - ε\|λ\| \right) - \langle λ , φ(f) \rangle .
\end{array} \right.
$$

Remark 2.20. If $δ = 0$ and we were to replace the error constraint $φ(f) ∈ B[x, ε]$ with $φ(f) ∈ B_x$, where $B_x ⊂ H_n$ is some compact convex subset. The min-max problem written using (23) is

$$
\left\{ \begin{array}{l}
\min_{f ∈ \mathbb{R}_K} \sup_{y ∈ B_x} \ c(f) - \langle λ , φ(f) \rangle + \left( p^{\frac{1+2p}{p}} \right) \langle λ , y \rangle .
\end{array} \right.
$$

2.3. Algorithms. We propose to solve the LIP (8) by computing saddle point solutions using existing algorithms to solve the min-max problems (18) or (23). There are many techniques available for solving such min-max problems, notably among them are the vanilla Gradient Descent Ascent (GDA), Optimistic Gradient Descent Ascent (OGDA), Proximal Point (PP) and Extra Gradient (EG) algorithms. A quick review of these algorithms can be found in [13]. The performance of the algorithm to solve LIP depends on the convergence attributes of the algorithm chosen to solve the min-max problems.

For the constrained formulation (18), Algorithm 1 performs projected gradient descent on the minimization variable $h$ for each iteration, and it computes gradients by solving the maximization over $λ$ by keeping the variable $h$ fixed. This can also be done alternatively by performing gradient ascent on the maximization
variable over a faster time scale and performing projected gradient descent on the minimization variable over a slower time scale. The unconstrained min-max problem is solved in Algorithm 2 by performing momentum based Optimistic Gradient Descent Ascent (OGDA) update.\footnote{In the algorithms, $\phi^\dagger$ indicates the conjugate of the linear map $\phi$, $\pi_c(\cdot)$ is the projection operator onto the level set $V_c$, and $\partial c(f)$ is the subdifferential of the cost function $c$ evaluated at $f$ (if the function $c$ is differentiable at $f$, we slightly abuse the notation and use the same notation $\partial c(f)$ to refer to the gradient).} A comparison between Algorithms 1 and 2 for $\ell_1$-minimization problems may be found in Section 2.4. We would like to emphasise that the particular algorithms 1 and 2 to solve LIP provided in this chapter are for representation purpose only. In practice, depending on specifics of the given LIP the user has to select the appropriate saddle point seeking algorithm to solve the equivalent min-max problems.

**Algorithm 1:** Projected gradient descent algorithm for constrained min-max problem (18).

**Input:** Problem data: $x$, $\phi$, $\epsilon$, $\delta$, $c$.

**Output:** An optimal solution $f \in F_\delta(\phi, x, \epsilon)$ and the optimal value $C_\delta(\phi, x, \epsilon)$.

1. Proceed only if $\|x\| > \epsilon$, else output 0.
2. Initialize $t = 0$, $h_0$ and $\lambda_0$.
3. Iterate till convergence
   - Initialise $\lambda = \lambda_t$ and iterate $M$ times
     $$\lambda \leftarrow \lambda + \alpha \left( \frac{\partial^+ p(x - \frac{\epsilon}{\|\lambda\|}\lambda)}{(\langle \lambda, x \rangle - \epsilon \|\lambda\|)} \right) - \frac{\delta}{\|\lambda\|} \lambda - \phi(h_t)$$
   - Update $: h_{t+1} = \pi_c(h_t + \beta_t(\phi^\dagger(\lambda_t)))$ and $\lambda_{t+1} = \lambda$
   - $t \leftarrow t + 1$
4. Repeat
5. **Output:** $C_\delta(\phi, x, \epsilon) = \langle \lambda_t, \phi(h_t) \rangle$ and $f = C_\delta(\phi, x, \epsilon) \cdot h_t$.

**Algorithm 2:** OGDA algorithm for unconstrained min-max problem (23) when $\delta = 0$.

**Input:** Problem data: $x$, $\phi$, $\epsilon$, $c$.

**Output:** An optimal solution $f \in F_\delta(\phi, x, \epsilon)$.

1. Proceed only if $\|x\| > \epsilon$, else output 0.
2. Initialize $t = 0$, $(f_0, \lambda_0)$ and $(\Gamma_0, \gamma_0)$.
3. Iterate till convergence
   - Compute $\gamma_t \in \left(\partial c(f_t) - \phi^\dagger(\lambda_t)\right)$ and $\Gamma_t = \left(x - \phi(f_t) - \frac{\epsilon}{\|\lambda_t\|}\lambda_t\right)$.
   - Update $: f_{t+1} = f_t - \alpha_t \left(2\gamma_t - \gamma_{t-1}\right)$ and $\lambda_{t+1} = \lambda_t + \alpha_t \left(2\Gamma_t - \Gamma_{t-1}\right)$
   - $t \leftarrow t + 1$
4. Repeat
5. **Output:** $f_t$. 

4
2.4. **Applications to standard problems.** In this section, we will discuss two linear inverse type problems namely *Basis Pursuit Denoising* (BPDN) and the *Quadratic Program* (QP). We will discuss the corresponding min-max forms for these problems, and implement the corresponding algorithms on image denoising problems. Finally, we will also discuss how the min-max forms help us in overcoming the ill-posedness of the dictionary learning problem (5).

*Basis Pursuit Denoising (BPDN).* One of the most practical example of an LIP is the classical *Basis Pursuit Denoising* problem [8, 3], which arises in various scenarios of compressed sensing and image processing like denoising, deblurring etc.

\[
\begin{aligned}
\min_{f \in \mathbb{R}^K} & \quad \|f\|_1 \\
\text{subject to} & \quad \|x - \phi(f)\|_2 \leq \epsilon.
\end{aligned}
\]

In such image processing problems, instead of solving the problem directly on the entire image, it is often done on a collection of smaller patches (typically of size $8 \times 8$) that cover the entire image. Since natural images are reasonably sparse in 2d-DCT basis, a common choice for the linear map $\phi$ is the inverse 2d-DCT operator for $8 \times 8$ images.

We implement BPDN based image denoising on two images by solving (25) for all non-overlapping patches of size $8 \times 8$. Figure 1 shows the denoising results for the standard cameraman image, which is of size $256 \times 256$. From left to right, we have the original image, noisy image and the recovered image in order. The noisy image is obtained by adding a mean zero Gaussian noise of standard deviation 0.0065 using `imnoise` function in MATLAB resulting in a PSNR of 22.0741dB. To recover the image, BPDN was solved with $\epsilon = 0.3$ for every non-overlapping $8 \times 8$ patch using Algorithm 1 to compute the saddle point of the equivalent constrained min-max problem

\[
\begin{aligned}
\min_{\|h\|_1 \leq 1} & \quad 2\sqrt{\lambda^\top x - \epsilon \|\lambda\|_2} - \lambda^\top \phi(h) \\
\text{subject to} & \quad \lambda^\top x - \epsilon \|\lambda\|_2 > 0.
\end{aligned}
\]

The recovered image has a PSNR value of 26.8119dB.

Similarly, in Figure 2 the results for denoising the flower image are shown, where the noisy image is obtained by adding a mean zero Gaussian noise of standard deviation 0.005, resulting in a PSNR of 23.0954dB. To denoise the image, BPDN was solved with $\epsilon = 0.385$ for every non-overlapping $8 \times 8$ patch using Algorithm 2 to compute saddle points of the equivalent unconstrained min-max problem

\[
\begin{aligned}
\min_{f} & \quad \|f\|_1 + \lambda^\top (x - \phi(f)) - \epsilon \|\lambda\|_2.
\end{aligned}
\]

The recovered image has a PSNR value of 28.5362dB.

For comparison purpose, we solve a BPDN problem for a randomly selected $8 \times 8$ patch from the noisy flower image in Figure 2 via both Algorithm 1 and 2. In Algorithm 1, we choose the step size sequence $\alpha_t = 5/(20 + t)$, $\epsilon = 0.385$ and for initialisation, $h_0 = \frac{\phi^+(x)}{\|\phi^+(x)\|_1}$ and $f_0 = x$. To show convergence to a saddle point in
Figure 1. In order from left to right, we have the original image, noisy image, and the recovered image. The image recovery is done by employing Algorithm 1 to solve the BPDN problem (25) on each $8 \times 8$ patch. The PSNR value of the noisy image is 22.0741dB, and that of the recovered image is 26.8119dB, both w.r.t. the original image.

Figure 2. In order from left to right, we have the original image, noisy image, and the recovered image. The image recovery is done by employing Algorithm 2 to solve the BPDN problem (25) on each $8 \times 8$ patch. The PSNR value of the noisy image is 23.0954dB, and that of the recovered image is 28.5362dB, both w.r.t. the original image.

Figure 3, we plot the quantities

\begin{align*}
G_t &= \left\| \frac{(x - \frac{\epsilon}{\|\lambda_t\|} \lambda_t)}{\sqrt{\langle \lambda_t, x \rangle - \epsilon \|\lambda_t\|}} - \phi(h_{t-1}) \right\| \\
g_t &= \|\phi^\perp(\lambda_t)\|_\infty - \langle \lambda_t, \phi(h_t) \rangle.
\end{align*}
Note that $G_t = 0$ guarantees the maximization condition, and $g_t = 0$ guarantees the minimization condition in the constrained min-max problem (26). Therefore, if $g_t = G_t = 0$ for some $t$, it implies that $(\lambda_t, h_t)$ is a saddle point to (26). Therefore, the sequences $(g_t)_t$ and $(G_t)_t$ act as certificates for the saddle point condition for the min-max problem (26).

Similarly, to compute a saddle point of the unconstrained min-max problem (27) via Algorithm 2, we use the step size sequence $\alpha_t = 2.5/(10 + t)$, $\epsilon = 0.385$ and for initialisation, $f_0 = \phi^+(x)$ and $\lambda_0 = x$. To show convergence to a saddle point in Figure 4, we have plot the certificates

$$G_t' = \alpha_t \|2\Gamma_t - \Gamma_{t-1}\|$$
$$g_t' = \alpha_t \|2\gamma_t - \gamma_{t-1}\|.$$

Evidently, the constrained formulation requires fewer iterations to converge to a reasonable solution, in comparison to the number of iterations required to solve (27). However, each iteration to solve (26) involves computing projections onto $\ell_1$-ball, which makes each iteration expensive.

**Figure 3.** Saddle point certificates to solve unconstrained min-max problem (26) via Algorithm 1.

**Projection onto Convex sets and Quadratic Programs.** Another optimization problem that arises regularly is the Quadratic Program (QP). Particularly, the problem of projecting a given point $x$ onto some given compact-convex set $S$. Projection of points onto convex sets arises in almost every practical optimization problem where the iterates have to be projected onto the respective feasible sets after they are updated using a gradient descent like technique to minimize the cost. We consider
Figure 4. Saddle point certificates to solve unconstrained min-max problem (27) via Algorithm 2.

the following QP
\[(30) \quad \min_{y \in S} \langle y - x, Q(y - x) \rangle,
\]
where \(Q\) is some given positive definite matrix. In terms of new variable \(f := y - x\), the QP (30) becomes
\[(31) \quad \min_{f \in S - \{x\}} \langle f, Qf \rangle,
\]
which is in the form of the LIP (8) with \(\delta = 0\), \(c(f) = \langle f, Qf \rangle\), \(\phi\) being the identity map and the constraint \(f \in S - \{x\}\) instead of \(f \in B[x, \epsilon]\). Using the min-max reformulation (18) of the LIP, and replacing the quantity \(\langle \lambda, x - \epsilon \| \lambda \|\) with \(\min_{y' \in S - \{x\}} \langle \lambda, y' \rangle\) we obtain the following equivalent min-max formulation for the QP (30).

\[(32) \quad \begin{cases} 
\min_{\langle h, Qh \rangle \leq 1} \max_{\lambda} & 1.889 \left( \min_{y \in S} \langle \lambda, y - x \rangle \right)^{2/3} - \langle \lambda, h \rangle \\
\text{subject to} & 0 < \min_{y \in S} \langle \lambda, y - x \rangle.
\end{cases}
\]

Since the set \(S\) is compact, the QP (30) always admits a solution and the corresponding min-max problem (32) admits a saddle point solution \((h^*, \lambda^*)\). From first order conditions, we know that the optimal value of the min-max problem is \(0.5 \langle \lambda^*, h^* \rangle\). Then, simple algebra shows that the optimal solution \(f^*\) to (31) is given by \(0.5 \langle \lambda^*, h^* \rangle h^*\), and the optimal solution \(y^*\) to the QP (30) is \(y^* = x + f^*\).

The advantage lies in the fact that the projection map \(\pi_Q : \mathbb{H}_n \longrightarrow \mathbb{H}_n\) onto the level set \(\{h : \langle h, Qh \rangle \leq 1\}\) is a lot simpler than projecting onto the given
Algorithm 3: Projected gradient descent algorithm for constrained min-max problem (18).

**Input:** Problem data: $x$, $Q$ and the set $S$.

**Output:** The solution $y^*$ to QP (30).

1. Initialize $t = 0$ and $\lambda_0$.

2. **Iterate till convergence**

   Compute $y_t \in \underset{y \in S}{\text{argmin}} \langle \lambda_t \; , \; y \rangle$, and
   
   \[
   \begin{align*}
   \lambda_{t+1} &= \lambda_t + \alpha_t \left( \frac{1.26(y_t - x)}{\langle \lambda_t\; , \; y_t - x \rangle^{1/3}} - h_t \right) \\
   h_{t+1} &= \pi_Q(h_t + \alpha_t \lambda_t)
   \end{align*}
   \]

   $t \leftarrow t + 1$

3. **Repeat**

4. **Output:** $y^* = x + \frac{1}{2}Q^{-1}\lambda_t$.

set $S$ itself. For example, if $Q$ is identity matrix, then $\pi_Q(h)$ simply normalizes $h$ to have unit norm. Since in most relevant cases, the function $\lambda \mapsto \min_{y \in S} \langle \lambda \; , \; y \rangle$ is not differentiable, the step sizes $\alpha$ have to be diminishing for the subgradient descent-ascent to converge.

Alternatively, since the minimization $\min_{\langle h \; , \; Qh \rangle \leq 1} \langle \lambda \; , \; h \rangle$ admits a unique solution for every $\lambda$, the saddle point solution to the min-max problem can be computed by solving the following maximization problem alone.

\[
(33) \quad \sup_{\lambda} \; 1.889 \left( \min_{y \in S} \langle \lambda \; , \; y - x \rangle \right)^{2/3} - \|Q^{-\frac{1}{2}}\lambda\|,
\]

using subgradient ascent type algorithms. If $(h^*, \lambda^*)$ is a saddle point solution to the min-max problem (32), then

\[
\begin{align*}
\lambda^* &= \frac{Q^{-1}\lambda^*}{\sqrt{\langle \lambda^* \; , \; Q^{-1}\lambda^* \rangle}} \\
\lambda^* &= \max_{\langle h \; , \; Qh \rangle \leq 1} \langle \lambda^* \; , \; h \rangle.
\end{align*}
\]

Finally, the optimal solution $y^*$ to the QP (30) is $y^* = x + \frac{1}{2}Q^{-1}\lambda^*$.

As an exercise, we solve the orthogonal projection problem of projecting a point $x$ onto the $\ell_1$-ball, which if we recall, arises often while using Algorithm 1 to solve the LIP when $c(\cdot) = \|\cdot\|_1$. The point $x$ is drawn uniformly randomly from the $\ell_\infty$-ball of $\mathbb{R}^{1000}$, and we solve (30) with $Q$ being the identity operator using Algorithm 3. Figure 5 shows the progressive distance between the original point $x$ and the computed projections $x + 0.5\lambda_t$ at each iteration of Algorithm 3.

The dictionary learning problem. The setup is that every vector $x \in \mathbb{H}_n$ is encoded as a vector $f(x)$ in $\mathbb{R}^K$ via the encoder map $f : \mathbb{H}_n \to \mathbb{R}^K$. We shall refer to $f(x)$ as the representation of $x$ under the encoder $f$. The reconstruction of the encoded samples from the representation $f(x)$ is done by taking the linear combination
For projecting a point $x$ onto the $\ell_1$-ball, the plot shows the distance between the iterates $(x + 0.5\lambda_t)$ in Algorithm 3 to the original point $x$.

$$\sum_{i=1}^{K} f_i(x) d_i$$ with some standard collection of vectors $D := (d_1 \ d_2 \ \ldots \ d_K)$ referred to as the dictionary. Since the reconstruction has to be a good representative of the true vector $x$, we constrain the error $\|x - Df(x)\|$ to be small.

Given a dictionary $D$, we encode every vector $x$ by solving the LIP (8) with an appropriate cost function $c$. This cost function determines the desirable characteristics in the representation. In other words, the optimal encoder map $f_D : \mathbb{H}_n \rightarrow \mathbb{R}^K$ corresponding to the dictionary $D$ is such that $f_D(x) \in F_\delta(D, x, \epsilon)$ for every $x$. Our objective is to find dictionaries such that the corresponding encoder map $f_D$ has desirable features like sparsity, robustness with respect to loss of coefficients etc., in the representation. We refer to the task of finding such a dictionary as the dictionary learning problem.

Formally, let $P$ be a distribution on $\mathbb{H}_n$ and $X$ be a $P$ distributed random variable. Let $c : \mathbb{R}^K \rightarrow \mathbb{R}_+$ be a given cost function that satisfies Assumption 2.1, $\epsilon : \mathbb{H}_n \rightarrow [0, +\infty]$ be a given error threshold function and $\delta$ be a non-negative real number. Given a dictionary $D$, since the random variable $X$ is encoded as $f_D(X) \in F_\delta(D, X, \epsilon(X))$, we consider the cost incurred to encode to be $C_\delta(D, X, \epsilon(X))$. Our objective is to find a dictionary that facilitates optimal encoding of the data, which are the samples drawn from $P$. Therefore, we consider the following dictionary learning problem:

$$\minimize_{D \in \mathcal{D}} \mathbb{E}_P [C_\delta(D, X, \epsilon(X))],$$

where $\mathcal{D} \subset \mathbb{R}^{n \times K}$ is some known compact convex subset.
For a large integer $T$, let $[X : T] := (x_t)_{t=1}^T$ be a collection of samples drawn from the distribution $P$. Let us consider the dictionary learning problem for the sampled data, given by:

$$
(35) \quad \min_{D \in D} \frac{1}{T} \sum_{t=1}^T C_\delta(D, x_t, \epsilon(x_t)).
$$

For the special case of $\delta = 0$, the dictionary learning problem (35) can be restated using the definition of the encoding cost $C_\delta(D, x_t, \epsilon(x_t))$ in the more conventional form as:

$$
(36) \quad \begin{aligned}
&\min_{D, (f_t)} \frac{1}{T} \sum_{t=1}^T c(f_t) \\
&\text{subject to}\quad D \in D, \\
&\quad f_t \in \mathbb{R}^K, \\
&\quad \|x_t - D f_t\| \leq \epsilon(x_t) \quad \text{for all } t = 1, 2, \ldots, T.
\end{aligned}
$$

The dependence of the encoding cost $C_\delta(D, x_t, \epsilon)$ on the dictionary variable $D$ is not immediately evident. Therefore, it is replaced in the dictionary learning problem (35) with the min-max problem provided in Corollary 2.11 to obtain

$$
\min_{D \in D} \min_{(h_t) \in V_c} \sup_{\lambda_t \in V^c} \left( \frac{1}{T} \sum_{t=1}^T \left( r(p) \left( \langle \lambda_t, x_t \rangle - \epsilon \|\lambda_t\| \right)^{\frac{p}{p+1}} - \left(\delta \|\lambda_t\| + \langle \lambda_t, Dh_t \rangle \right) \right) \\
\text{s.t.} \quad \langle \lambda_t, x_t \rangle - \epsilon \|\lambda_t\| > 0,
$$

where $r(p) = (1 + p)^{\frac{1}{p+1}}$. The dictionary learning problem (35) is then solved by alternating the optimization over $D$ and $(h_t)$, keeping the other one fixed. It is to be noted that each of these optimization problem is a min-max problem in variables $(D, (\lambda_t))$ and $(h_t)$ respectively. In particular, for a given sequence $(h_t) \in V_c$, the dictionary is updated by solving the following min-max problem

$$
\begin{aligned}
&\min_{D \in D} \sup_{\lambda_t \in V^c} \left( \frac{1}{T} \sum_{t=1}^T \left( r(p) \left( \langle \lambda_t, x_t \rangle - \epsilon \|\lambda_t\| \right)^{\frac{p}{p+1}} - \left(\delta \|\lambda_t\| + \langle \lambda_t, Dh_t \rangle \right) \right) \\
&\quad \text{s.t.} \quad \langle \lambda_t, x_t \rangle - \epsilon \|\lambda_t\| > 0.
\end{aligned}
$$

It is shown in [16] that if $\delta > 0$ the above min-max problem always admits a saddle point solution. Moreover, we observe the objective function of the min-max problem is linear w.r.t. the dictionary variable $D$ and concave w.r.t. $\lambda$. Thus, the saddle point solution can be computed efficiently by simple ascent-descent type iterations. The novelty and the convergence attributes of learning a dictionary to solve (35) can be attributed to the reformulations (18), (23) of the LIP provided in this article.

### 3. Theory, discussion, and proofs.

In this section we investigate the linear inverse problem (8) in detail and its dual with special emphasis on the underlying convex geometry. Based on the principle of separation of convex bodies by linear functionals, we obtain the dual problem (16) of the linear inverse problem which then leads to the convex-concave min-max problem (16). We later provide the proof of Theorem 2.10 establishing that...
the optimal value of this min-max problem is proportional to the optimal cost $C_\delta(\phi, x, \epsilon)$ of the LIP (8).

**Lemma 3.1.** For a linear map $\phi : \mathbb{R}^K \rightarrow \mathbb{H}_n$ and $\delta, r \geq 0$, let $S_\delta(\phi, r) := r \cdot S_\delta(\phi, 1)$, we have

$$S_\delta(\phi, r) = \{ z \in \mathbb{H}_n : C_\delta(\phi, z, 0) \leq r^p \}. \quad (37)$$

**Proof.** On the one hand, it follows from the definition (11) of $S_\delta(\phi, 1)$ that for every $z \in S_\delta(\phi, r)$, there exists $f_z \in \mathbb{R}^K$ such that $c(f_z) \leq r^p$ and $\| z - \phi(f_z) \| \leq \delta r$. Thus, considering $\epsilon = 0$ in (8), we see that the pair $(r, f_z)$ is a feasible point and hence we have $C_\delta(\phi, z, 0) \leq r^p$.

On the other hand, if $z \in \mathbb{H}_n$ is such that $C_\delta(\phi, z, 0) \leq r^p$, we know that there exists a pair $(\epsilon, \delta)$ such that it intersects with $S_\delta(x, \epsilon)$ and is proportional to the optimal cost $C_\delta(\phi, x, \epsilon)$ of the membership $z \in S_\delta(\phi, r)$. Collecting the two assertions we arrive at (37). $\square$

It is easy to see that $S_\delta(\phi, 1) = \bigcup_{h \in V_n} B[\phi(h), \delta]$. Thus, if $\delta > 0$, the set $S_\delta(\phi, 1)$ has non-empty interior, and is therefore an absorbing set of $\mathbb{H}_n$.\footnote{A set $S$ is an absorbing set of a vector space $H$ if for every $z \in H$ there exists $r_z \geq 0$ such that $z \in r_z \cdot S$.} When $\delta = 0$, we immediately see that $S_\delta(\phi, r)$ has a non-empty interior for every $r \geq 0$. Furthermore, for every $z \in \text{image}(\phi)$ we know that there exists $f \in \mathbb{R}^K$ such that $z = \phi(f)$, and therefore $z \in S_0(\phi, (c(f))^{1/p})$. Consequently, we obtain:

$$\lim_{r \rightarrow +\infty} S_\delta(\phi, r) = \begin{cases} \mathbb{H}_n & \delta > 0, \\ \text{image}(\phi) & \delta = 0. \end{cases}$$

As the scaling factor $r$ increases, the set $S_\delta(\phi, r)$ scales linearly by absorbing every $(\phi, \epsilon, \delta)$-feasible point in the set $\mathbb{H}_n$. In particular, the set $B[x, \epsilon]$ eventually intersects with $S_\delta(\phi, r)$ for some $r \geq 0$. We shall see that the optimal cost $C_\delta(\phi, x, \epsilon)$ is proportional to the minimum amount by which the set $S_\delta(\phi, 1)$ needs to be scaled so that it intersects with $B[x, \epsilon]$.

**Lemma 3.2.** For a given linear map $\phi : \mathbb{R}^K \rightarrow \mathbb{H}_n$ and non-negative real numbers $\epsilon, \delta$, let $x \in \mathbb{H}_n$ be $(\phi, \epsilon, \delta)$-feasible in the sense of Definition 2.2, then we have

$$C_\delta(\phi, x, \epsilon) = \begin{cases} \min_{r \geq 0} & r^p \\ \text{subject to} & S_\delta(\phi, r) \cap B[x, \epsilon] \neq \emptyset. \end{cases} \quad (38)$$

**Proof.** Let $r \geq 0$ be such that $S_\delta(\phi, r) \cap B[x, \epsilon] \neq \emptyset$. Then on the one hand, there exists $y_r \in B[x, \epsilon]$ and $f_r \in \mathbb{R}^K$ such that $\| y_r - \phi(f_r) \| \leq \delta r$ and $c(f_r) \leq r^p$. From this we get

$$\| x - \phi(f_r) \| \leq \| x - y_r \| + \| y_r - \phi(f_r) \| \leq \epsilon + \delta r,$$

where

\[ \epsilon = 0 \quad \text{and} \quad \delta = \sigma(x). \]

Thus, $\| x - \phi(f_r) \| \leq \epsilon + \delta r$.

Hence, we have $C_\delta(\phi, x, \epsilon) \leq r_0^p = r$. In particular, $\epsilon$ is such that $C_\delta(\phi, x, \epsilon) = r^p$. Therefore, $S_\delta(\phi, r) \cap B[x, \epsilon] \neq \emptyset$.
which implies that the pair \((r, f, x)\) is feasible for (8), and as a result we get \(C_\delta(\phi, x, \epsilon) \leq r^p\). By minimizing over \(r \geq 0\) such that \(S_\delta(\phi, r) \cap B[x, \epsilon] \neq \emptyset\) we get our first inequality:

\[
C_\delta(\phi, x, \epsilon) \leq \min_{r \geq 0} \begin{cases} r^p \\
\text{subject to } S_\delta(\phi, r) \cap B[x, \epsilon] \neq \emptyset.
\end{cases}
\]

On the other hand, for every pair \((r, f)\) that is feasible for (8), by defining

\[y := x \mathbf{1}_{[0]}(\epsilon) + \frac{\epsilon \phi(0) + \delta x}{\epsilon + \delta r} \mathbf{1}_{[0, \infty)}(\epsilon),\]

we shall establish that \(y \in B[x, \epsilon] \cap S_\delta(\phi, r)\).

Whenever \(\epsilon = 0\) we have \(y = x\), and from the feasibility of the pair \((r, f)\) it easily follows that \(\|y - \phi(f)\| \leq \|x - \phi(f)\| \leq \epsilon + \delta r = \delta r\) and \(c(f) \leq r^p\). Thus, \(C_\delta(y, \phi, 0) \leq r^p\) and from (37) the membership \(y \in S_\delta(\phi, r)\) holds. Similarly, if \(\epsilon = 0\), we see that \(y = x = B[x, 0]\). Therefore, \(y \in B[x, \epsilon] \cap S_\delta(\phi, r)\), and the intersection is non-empty.

When \(\epsilon > 0\), we see that

\[
\|x - y\| = \left\| x - \frac{\epsilon \phi(0) + \delta x}{\epsilon + \delta r} \right\| = \frac{\epsilon}{\epsilon + \delta r} \|x - \phi(f)\| \leq \epsilon,
\]

and

\[
\|y - \phi(f)\| = \left\| \frac{\epsilon \phi(0) + \delta x}{\epsilon + \delta r} - \phi(f) \right\| = \frac{\delta r}{\epsilon + \delta r} \|x - \phi(f)\| \leq \delta r.
\]

These inequalities, along with the fact that \(c(f) \leq r^p\) imply that \(y \in B[x, \epsilon] \cap S_\delta(\phi, r)\) and in particular that \(B[x, \epsilon] \cap S_\delta(\phi, r) \neq \emptyset\). As a consequence, the inequality:

\[
r \geq \min_{r \geq 0} \begin{cases} r^p \\
\text{subject to } S_\delta(\phi, r) \cap B[x, \epsilon] \neq \emptyset.
\end{cases}
\]

holds for every pair \((r, f)\) that is feasible for (8). By minimizing over all the pairs \((r, f)\) that are feasible for (8), we obtain the converse inequality

\[
C_\delta(\phi, x, \epsilon) \geq \min_{r \geq 0} \begin{cases} r^p \\
\text{subject to } S_\delta(\phi, r) \cap B[x, \epsilon] \neq \emptyset.
\end{cases}
\]

This completes the proof. □

**Remark 3.3.** An interesting viewpoint to take from this in dictionary learning problem is that every dictionary \(D\) gives rise to an atomic set \(S_\delta(D, 1)\), and the encoding cost \(C_\delta(D, x, \epsilon)\) of a vector \(x\) is proportional to the approximate Minkowski gauge function with respect to this set.\(^6\) The corresponding dictionary learning problem can be viewed as the task of finding a ‘good’ atomic set arising from a dictionary.

**Intersection of the convex bodies.** Lemma 3.2 gives us the first required connection between the LIP (8) and the underlying convex geometry. It asserts that the value \((C_\delta(\phi, x, \epsilon))^{1/p}\) is the minimum amount by which the set \(S_\delta(\phi, 1)\) has to be scaled linearly so that it intersects with \(B[x, \epsilon]\). To this end, let us define:

\[
S_\delta(\phi, x, \epsilon) := (C_\delta(\phi, x, \epsilon))^{1/p} \cdot S_\delta(\phi, 1).
\]

\(^6\) We say “approximate” in the sense that we do not scale the atomic set \(S_\delta(D, 1)\) so as to absorb \(x\). Instead, we scale it only until it intersects with a given neighborhood of \(x\).
Lemma 3.4. Let \( x \in \mathbb{H}_n \) be \((\phi, \epsilon, \delta)\)-feasible in the sense of Definition 2.2. Let \((c_x, f_x) \in [0, +\infty[^n \times \mathbb{R}^K\) be an optimal solution to the coding problem (8), i.e., \( c_x = (C_\delta(\phi, x, \epsilon))^1/p \) and \( f_x \in F_\delta(\phi, x, \epsilon) \). Then the sets \( B[x, \epsilon] \) and \( S_\delta(\phi, x, \epsilon) \) intersect at a unique point \( y^* \) given by:
\[
B[x, \epsilon] \cap S_\delta(\phi, x, \epsilon) =: y^* = x \mathbb{1}_{[0, \epsilon]}(\epsilon) + \frac{\epsilon \phi(f_x) + \delta c_x x}{\epsilon + \delta c_x} \mathbb{1}_{[\epsilon, +\infty[}(\epsilon).
\]
As a consequence, we assert that
- whenever \( \|x\| > \epsilon \), every \( f_x \in F_\delta(\phi, x, \epsilon) \) satisfies \( \|x - \phi(f_x)\| = \epsilon + \delta c_x \); 
- for every \( f_x, g_x \in F_\delta(\phi, x, \epsilon) \), we have \( \phi(f_x) = \phi(g_x) \).

Remark 3.5. In dictionary learning, for a given dictionary \( D \), since every sample vector \( x \in \mathbb{H}_n \) is represented by some vector \( f_D(x) \in F_\delta(\phi, D, x, \epsilon) \), the representation is not unique whenever the set \( F_\delta(D, x, \epsilon) \) is not a singleton. In such situations, even though the representation need not be unique, we emphasize that the reconstruction \( x_{\text{rec}} := Df_D(x) \) of the vector \( x \) obtained from its representation \( f_D(x) \) is unique.

Remark 3.6. When \( x \notin B[0, \epsilon] \) since \( c_x > 0 \), it is easily verified that the unique point of intersection \( y \) in Lemma 3.4 can also be written as:
\[
y^* = \phi(f_x) + \frac{\delta c_x}{\epsilon + \delta c_x}(x - \phi(f_x)) \mathbb{1}_{[\epsilon, +\infty[}(\epsilon).
\]

Lemma 3.4. We note that if \( \epsilon = 0 \), \( B[x, \epsilon] = x \), and since by definition, the set \( S_\delta(\phi, x, \epsilon) \) intersects with \( B[x, \epsilon] \). The intersection happens at the point \( x \) which is unique. We shall establish (39) by considering the remaining cases.

- \( 0 < \|x\| \leq \epsilon \): From (17), we know that \( C_\delta(\phi, x, \epsilon) = 0 \) and \( F_\delta(\phi, x, \epsilon) = \{0\} \). This implies that \( S_\delta(\phi, x, \epsilon) = \{0\} \). In addition, we see that \( 0 \in B[x, \epsilon] \) whenever \( \|x\| \leq \epsilon \). As a result, we obtain that \( B[x, \epsilon] \cap S_\delta(\phi, x, \epsilon) = \{0\} \). Now, by using the fact that \((c_x, f_x)\) is an optimal solution to the coding problem (8) if and only if \((c_x, f_x) = (0, 0)\), we see that \( y^* \) in (39) evaluates to 0 confirming (39).

- \( 0 < \epsilon < \|x\| \): We shall prove by contradiction that the sets \( B[x, \epsilon] \) and \( S_\delta(\phi, x, \epsilon) \) intersect at a unique point. Let \( y_1 \neq y_2 \) be such that \( y_1, y_2 \in B[x, \epsilon] \cap S_\delta(\phi, x, \epsilon) \). Since \( B[x, \epsilon] \) is a strictly convex set, \( \frac{1}{2}(y_1 + y_2) \in B(x, \epsilon) \). However, since \( B(x, \epsilon) \) is an open set, one can find \( \rho > 0 \) such that \( B[\frac{1}{2}(y_1 + y_2), \rho] \subset B(x, \epsilon) \). Since \( 0 \notin B(x, \epsilon) \), we conclude that \( 2\rho < ||y_1 + y_2|| \); Defining \( \theta := \left(1 - \frac{2\rho}{\|y_1 + y_2\|} \right) \), we see that \( \theta \in ]0, 1[ \). It is easily verified that \( \frac{1}{2}(y_1 + y_2) - \frac{\theta}{2}(y_1 + y_2) \in B[\frac{1}{2}(y_1 + y_2), \rho] \subset B(x, \epsilon) \). In addition, we note that the set \( S_\delta(\phi, x, \epsilon) \) is also convex, which means that \( \frac{1}{2}(y_1 + y_2) \in S_\delta(\phi, x, \epsilon) \). Since \( S_\delta(\phi, x, \epsilon) \) scales linearly, we conclude that \( \frac{1}{2}(y_1 + y_2) \in \theta \cdot S_\delta(\phi, x, \epsilon) \). From these two inclusions, it is clear that
\[
\frac{\theta}{2}(y_1 + y_2) \in B[x, \epsilon] \cap \theta \cdot S_\delta(\phi, x, \epsilon) = B[x, \epsilon] \cap S_\delta(\phi, \theta c_x),
\]
and equivalently, \( B[x, \epsilon] \cap S_\delta(\phi, \theta c_x) \neq \emptyset \). This, however, contradicts the assertion of Lemma 3.2 since \( \theta < 1 \).
To summarize, we have established that if the intersection of the sets \( B[x, \epsilon] \) and \( S_\delta(\phi, x, \epsilon) \) is not a singleton, we can slightly shrink the set \( S_\delta(\phi, x, \epsilon) \) such that it still intersects \( B[x, \epsilon] \) nontrivially. This is a contradiction in view of Lemma 3.2.

To prove that \( y^* \) defined in (39) is indeed the unique point of intersection, it suffices to show that \( y^* \in B[x, \epsilon] \cap S_\delta(\phi, x, \epsilon) \). We observe that:

\[
\|x - y^*\| = \left\| x - \frac{\epsilon \phi(f_x) + \delta c_x x}{\epsilon + \delta c_x} \right\| = \frac{\epsilon}{\epsilon + \delta c_x} \|x - \phi(f_x)\| \leq \epsilon, \quad \text{and} \\
\|y^* - \phi(f_x)\| = \left\| \frac{\epsilon \phi(f_x) + \delta c_x x}{\epsilon + \delta c_x} - \phi(f_x) \right\| = \frac{\delta c_x}{\epsilon + \delta c_x} \|x - \phi(f_x)\| \leq \delta c_x.
\]

These inequalities, along with the fact that \( c(f_x) \leq c^p \), imply that \( y^* \in B[x, \epsilon] \cap S_\delta(\phi, x, \epsilon) \). This establishes (39).

We proceed to establish the two consequences. To see the first, let us prove that the error constraint is active at the optimal solution \((c_x, f_x)\) whenever \( \|x\| > \epsilon \geq 0 \). If \( \epsilon = \delta = 0 \), then the error constraint is trivially active since \( \|x - \phi(f_x)\| \leq 0 \) implies that \( \|x - \phi(f_x)\| = 0 \). If at least one of the parameters \( \epsilon \) and \( \delta \) is positive, we know that \( c_x > 0 \) for every \( \|x\| > \epsilon \). Therefore, we have \( \epsilon + \delta c_x > 0 \), and the quantity \( y^* := \frac{\epsilon \phi(f_x) + \delta c_x x}{\epsilon + \delta c_x} \) is well defined for every \( f_x \in F_\delta(\phi, x, \epsilon) \). We know from the previous assertion of the lemma that \( y^* \in S_\delta(\phi, x, \epsilon) \). However, since \( S_\delta(\phi, x, \epsilon) \) is a convex set that contains 0, we conclude that

\[
\theta y^* \in S_\delta(\phi, x, \epsilon) \text{ for every } \theta \in [0, 1].
\]

If we suppose that \( \|x - \phi(f_x)\| < \epsilon + \delta c_x \), it is easily verified that \( \|x - y^*\| < \epsilon \), and thus \( y^* \in B[x, \epsilon] \). As a result, one can find \( \rho > 0 \) such that \( B[y^*, \rho] \subset B[x, \epsilon] \).

Since \( 0 \not\in B[x, \epsilon] \), we see at once that \( \rho < \|y^*\| \), and conclude that

\[
\alpha y^* \in B[y^*, \rho] \subset B[x, \epsilon] \text{ for every } \alpha \text{ such that } \left( 1 - \frac{\rho}{\|y^*\|} \right) \leq \alpha \leq 1.
\]

These two inclusions together contradict that the sets \( B[x, \epsilon] \) and \( S_\delta(\phi, x, \epsilon) \) intersect at a unique point.

It remains to prove the final assertion that for every \( f_x, g_x \in F_\delta(\phi, x, \epsilon) \), the equality \( \phi(f_x) = \phi(g_x) \) holds. Indeed, whenever \( \|x\| \leq \epsilon \), we have \( C_\delta(\phi, x, \epsilon) = 0 \) and \( F_\delta(\phi, x, \epsilon) = \{0\} \). This implies that \( f_x = g_x = 0 \), and thus \( \phi(f_x) = \phi(g_x) \).

Let us consider the case when \( \|x\| > \epsilon \), and suppose that \( \phi(f_x) \neq \phi(g_x) \) for some \( f_x, g_x \in F_\delta(\phi, x, \epsilon) \). Then it follows that \( \frac{1}{2}(f_x + g_x) \) satisfies the error constraint

\[
\left\| x - \frac{1}{2}(f_x + g_x) \right\| = \left\| \frac{1}{2}(x - \phi(f_x)) + \frac{1}{2}(x - \phi(g_x)) \right\| \leq \epsilon + \delta c_x.
\]

Moreover, we know that the level sets of \( c \) are convex and since \( f_x, g_x \in c_x \cdot V_c \), we have \( \frac{1}{2}(f_x + g_x) \in c_x \). Therefore, \( c(\frac{1}{2}(f_x + g_x)) \leq c^p = C_\delta(\phi, x, \epsilon) \), we conclude that \( \frac{1}{2}(f_x + g_x) \in F_\delta(\phi, x, \epsilon) \). However, since \( \phi(f_x) \neq \phi(g_x) \), the triangle inequality implies that the above error constraint is satisfied strictly. This contradicts our earlier assertion that the error constraint is active for every \( f_x \in F_\delta(\phi, x, \epsilon) \). The proof is complete.
Lemma 3.7. Let the linear map $\phi : \mathbb{R}^K \rightarrow \mathbb{H}_n$ and non-negative real numbers $\epsilon, \delta$ be given, then for every $\lambda \in \mathbb{H}_n$, we have

$$
\|\lambda\|_{\phi}^* := \max_{z \in S_\delta(\phi, 1)} \langle \lambda, z \rangle = \delta \|\lambda\| + \max_{h \in V_c} \langle \lambda, \phi(h) \rangle.
$$

Furthermore,

(i) If $\delta > 0$, then $\|\lambda\|_{\phi}^* > 0$ for every $\lambda \in \mathbb{H}_n \setminus \{0\}$.

(ii) If $\delta = 0$, and $\lambda \in \mathbb{H}_n \setminus \{0\}$ satisfies $\|\lambda\|_{\phi}^* = 0$, then $\langle \lambda, \epsilon \rangle - \epsilon \|\lambda\| \leq 0$ for every $(\phi, \epsilon, 0)$-feasible vector $x \in \mathbb{H}_n$.

Proof. We recall from the definition (11) that the set $S_\delta(\phi, 1)$ is the image of the linear map: $B[0, \delta] \times V_c \ni (z', h) \rightarrow z' + \phi(h)$. This allows us to write the optimization problem $\max_{z \in S_\delta(\phi, 1)} \langle \lambda, z \rangle$ equivalently as:

$$
\max_{h, z'} \langle \lambda, z' + \phi(h) \rangle \quad \text{subject to} \quad h \in V_c, \ z' \in B[0, \delta].
$$

It is easily seen that the above optimization problem is separable into maximization over individual variables, and using the fact that $\max_{z' \in B[0, \delta]} \langle \lambda, z' \rangle = \delta \|\lambda\|$ for every $\lambda \in \mathbb{H}_n$ (40) follows at once. Moreover, since $0 \in V_c$, we have $0 \leq \max_{h \in V_c} \langle \lambda, \phi(h) \rangle$ for every $\lambda \in \mathbb{H}_n$. Applying this inequality in (40), assertion (i) of the lemma follows immediately.

Finally, let $\delta = 0$ and $\lambda \in \mathbb{H}_n \setminus \{0\}$ satisfy $\|\lambda\|_{\phi}^* = 0$. Since $S_0(\phi, 1)$ is an absorbing set to image($\phi$), we conclude from the definition (14) of the dual function that $\langle \lambda, y \rangle \leq 0$ for every $y \in \text{image}(\phi)$. If $x \in \mathbb{H}_n$ is $(\phi, \epsilon, 0)$-feasible, we know that $B[x, \epsilon] \cap \text{image}(\phi) \neq \emptyset$. Let $y' \in B[x, \epsilon] \cap \text{image}(\phi)$, then

$$
\langle \lambda, x \rangle - \epsilon \|\lambda\| = \min_{y \in B[x, \epsilon]} \langle \lambda, y \rangle \leq \langle \lambda, y' \rangle \leq 0.
$$

This completes the proof. \qed

Separation of sets $B[x, \epsilon]$ and $S_\delta(\phi, x, \epsilon)$. We recall that both the sets $B[x, \epsilon]$ and $S_\delta(\phi, x, \epsilon)$ are compact convex subsets that intersect at the unique point $y^*$. As a result, we know from the Hahn-Banach separation principle that there exists a $\lambda^* \in \mathbb{H}_n$ such that the linear functional $\langle \lambda^*, \cdot \rangle$ satisfies

$$
\max_{z \in S_\delta(\phi, x, \epsilon)} \langle \lambda^*, z \rangle = \langle \lambda^*, y^* \rangle = \min_{y \in B[x, \epsilon]} \langle \lambda^*, y \rangle.
$$

In other words, the linear functional $\langle \lambda^*, \cdot \rangle$ separates the convex sets $B[x, \epsilon]$ and $S_\delta(\phi, x, \epsilon)$, and supports them at their unique point of intersection $y^*$. This fact, is central in establishing strong duality and explicitly characterizing the optimal dual variables.

Lemma 3.8. Consider (41) where at least one of $\epsilon, \delta$ is positive. If $0 \neq \lambda' \in \mathbb{H}_n$ satisfies (41), then $\lambda' = \alpha(x - \phi(f_x))$ for some $\alpha > 0$ and $f_x \in F_\delta(\phi, x, \epsilon)$. Consequently, (41) is satisfied by $\alpha(x - \phi(f_x))$ for every $\alpha > 0$.

Proof. We recall from the Remark 3.6 that the sets $B[x, \epsilon]$, $S_\delta(\phi, x, \epsilon)$ intersect at the unique point $y^*$, given by

$$
y^* = \phi(f_x) + \frac{\delta c_x}{\epsilon + \delta c_x} (x - \phi(f_x)) 1_{[0, +\infty)}(\delta),
$$
where \( c_\varepsilon := (C_\delta(\phi, x, \varepsilon))^{1/p} \) and \( f_x \in F_\delta(\phi, x, \varepsilon) \).

- On the one hand, if \( \varepsilon > 0 \) and \( \lambda' \neq 0 \) satisfies: 
  \[
  \langle \lambda', y^* \rangle = \min_{y \in B[x, \varepsilon]} \langle \lambda', y \rangle,
  \]
  then necessarily \( \lambda' = \alpha'(x - y^*) \) for some \( \alpha' > 0 \).

- On the other hand, if \( \delta > 0 \), and \( \lambda' \neq 0 \) satisfies: 
  \[
  \langle \lambda', y^* \rangle = \max_{z \in S_\delta(\phi, x, \varepsilon)} \langle \lambda', z \rangle,
  \]
  then due to the fact that \( y^* \in B[\phi(f_x), \delta c_\varepsilon] \subset S_\delta(\phi, x, \varepsilon) \) \( \lambda' \) also satisfies: 
  \[
  \langle \lambda', y^* \rangle = \max_{z \in B[\phi(f_x), \delta c_\varepsilon]} \langle \lambda', z \rangle.
  \]
  It follows that: \( \lambda' = \alpha''(y^* - \phi(f_x)) \) for some \( \alpha'' > 0 \).

By substituting for \( y^* \) and simplifying, we easily deduce that in both the cases \( \lambda' = \alpha(x - \phi(f_x)) \) for some \( \alpha > 0 \).

Suppose that (41) is true for some \( \alpha' > 0 \), then for any \( \alpha > 0 \), the inequalities in (41) are preserved by multiplying throughout by the positive quantity \( \frac{\alpha}{\alpha'} \). Thus, (41) is satisfied by \( \alpha(x - \phi(f_x)) \) for every \( \alpha > 0 \).

**Lemma 3.9.** Let the linear map \( \phi : \mathbb{R}^K \rightarrow \mathbb{H}_n \) and non-negative real numbers \( \varepsilon, \delta \) be given, and \( x \in \mathbb{H}_n \setminus B[0, \varepsilon] \) be any \((\phi, \varepsilon, \delta)\)-feasible vector such that \( \Lambda_\delta(\phi, x, \varepsilon) \neq \emptyset \). Then every \( \lambda^* \in \Lambda_\delta(\phi, x, \varepsilon) \) satisfies (41).

**Proof.** We first recall that \( S_\delta(\phi, x, \varepsilon) = (C_\delta(\phi, x, \varepsilon))^{1/p} \cdot S_\delta(\phi, 1) \). Thus, for every \( \lambda^* \in \Lambda_\delta(\phi, x, \varepsilon) \), the following relations hold:

\[
\begin{align*}
\max_{z \in S_\delta(\phi, x, \varepsilon)} \langle \lambda^*, z \rangle &= (C_\delta(\phi, x, \varepsilon))^{1/p} \max_{z \in S_\delta(\phi, 1)} \langle \lambda^*, z \rangle = (C_\delta(\phi, x, \varepsilon))^{1/p}, \quad \text{and} \\
\min_{y \in B[x, \varepsilon]} \langle \lambda^*, y \rangle &= \langle \lambda^*, x \rangle - \varepsilon \|\lambda^*\| = (C_\delta(\phi, x, \varepsilon))^{1/p}.
\end{align*}
\]

In other words, the linear functional \( \langle \lambda^*, \cdot \rangle \) separates the sets \( B[x, \varepsilon] \) and \( S_\delta(\phi, x, \varepsilon) \). Moreover, both these sets are compact and convex, and we know from Lemma 3.4 that they intersect at a unique point \( y^* \). Therefore, the linear functional \( \langle \lambda^*, \cdot \rangle \) must support both these sets at their intersection point \( y^* \), and (41) follows at once.

**Proposition 3.10.** Let the linear map \( \phi : \mathbb{R}^K \rightarrow \mathbb{H}_n \) and non-negative real numbers \( \varepsilon, \delta \) be given, and \( x \in \mathbb{H}_n \setminus B[0, \varepsilon] \) be a \((\phi, \varepsilon, \delta)\)-feasible in the sense of Def. 2.2. The set \( \Lambda_\delta(\phi, x, \varepsilon) \) is completely described in the following.

(i) If \( \delta = 0, \varepsilon = 0 \), then the set \( \Lambda_0(\phi, x, 0) \neq \emptyset \), and in particular, \( \Lambda_0(\phi, x, 0) \cap \text{image}(\phi) \neq \emptyset \). A vector \( \lambda^* \in \Lambda_0(\phi, x, 0) \) if and only if the linear functional \( \langle \lambda^*, \cdot \rangle \) supports the set \( S_0(\phi, x, 0) \) at \( x \), and satisfies \( \|\lambda^*\| = 1 \).

(ii) If at least one of the following is true

- \( \delta > 0 \)
- \( \delta = 0 \) and \( \varepsilon > 0 \) with \( B(\varepsilon, x) \cap \text{image}(\phi) \neq \emptyset \)

\[\text{footnote {If image}(\phi) \text{ is a proper subspace of } \mathbb{H}_n, \text{ then every } \lambda \text{ in the orthogonal complement of image}(\phi) \text{ supports the set } S_0(\phi, 1) \text{ at every point, and in particular at } x. \text{ However, such a } \lambda \text{ doesn't satisfy the condition } \langle \lambda, x \rangle - \varepsilon \|\lambda\| = (C_\delta(\phi, x, \varepsilon))^{1/p}.} \]
then the set \( \Lambda_{\delta}(\phi, x, \epsilon) \) consists of a unique element \( \lambda^* \) given by

\[
\lambda^* = \frac{x - \phi(f_x)}{\|x - \phi(f_x)\|_\phi} \quad \text{for any } f_x \in F_\delta(\phi, x, \epsilon). \tag{42}
\]

(iii) If \( \delta = 0 \) and \( \epsilon > 0 \) such that \( B(x, \epsilon) \cap \text{image}(\phi) = \emptyset \), then \( \Lambda_{\delta}(\phi, x, \epsilon) = \emptyset \).

Proof. If \( \epsilon = \delta = 0 \), then \( B[x, \epsilon] = \{ x \} \). In view of Lemma 3.9 we know that \( \lambda^* \in \Lambda_0(\phi, x, 0) \) if and only if the linear functional \( \langle \lambda^*, \cdot \rangle \) supports the set \( S_0(\phi, x, 0) \) at \( x \), and satisfies \( \|\lambda^*\|_\phi' = 1 \). It remains to be shown that the set \( \Lambda_0(\phi, x, 0) \) is non-empty, and we do so by showing that there exists \( \lambda_0 \in \Lambda_0(\phi, x, 0) \cap \text{image}(\phi) \). Since \( x \) is \( (\phi, 0, 0) \)-feasible, we have \( x \in \text{image}(\phi) \). We note from Lemma 3.2 that \( C_0(\phi, x, 0) \) is the least amount by which the set \( S_0(\phi, 0) \) has to be linearly scaled so that it contains \( x \). This implies that \( x \) lies on the boundary of the set \( S_0(\phi, x, 0) \), i.e., \( x \notin \text{relint}(S_0(\phi, x, 0)) \). In addition, since \( S_0(\phi, x, 0) \) is a convex subset of \( \text{image}(\phi) \), we know that there exists \( 0 \neq \lambda_0 \in \text{image}(\phi) \) such that the linear functional \( \langle \lambda_0, \cdot \rangle \) supports the set \( S_0(\phi, x, 0) \) at the boundary point \( x \). As result, we obtain:

\[
\langle \lambda_0, x \rangle = \max_{z \in S_0(\phi, x, 0)} \langle \lambda_0, z \rangle = (C_0(\phi, x, \epsilon))^{1/p} \max_{z \in S_0(\phi, 1)} \langle \lambda_0, z \rangle = (C_0(\phi, x, \epsilon))^{1/p} \|\lambda_0\|_\phi'.
\]

Since \( S_0(\phi, 1) \) is an absorbing set to \( \text{image}(\phi) \) we have \( 0 \in \text{relint}(S_0(\phi, 1)) \) and therefore \( 0 < \|\lambda_0\|_\phi' \). Thus, defining \( \lambda^* := (1/\|\lambda_0\|_\phi')\lambda_0 \) it readily follows that \( \lambda^* \in \Lambda_0(\phi, x, 0) \). This establishes the assertion (i) of the proposition.

If either \( \epsilon > 0 \) or \( \delta > 0 \), on the one hand we know from Lemma 3.8 that \( (x - \phi(f_x)) \) satisfies

\[
\langle x - \phi(f_x), x \rangle - \epsilon \|x - \phi(f_x)\| = \max_{z \in S_0(\phi, x, \epsilon)} \langle x - \phi(f_x), z \rangle = (C_0(\phi, x, \epsilon))^{1/p} \|x - \phi(f_x)\|_\phi'.
\]

We immediately see that if \( \|x - \phi(f_x)\|_\phi' > 0 \), then \( \frac{x - \phi(f_x)}{\|x - \phi(f_x)\|_\phi'} \in \Lambda_\delta(\phi, x, \epsilon) \). On the other hand, if \( \lambda^* \in \Lambda_\delta(\phi, x, \epsilon) \) then Lemma 3.9 implies that \( \lambda^* \) must satisfy (41), and from Lemma 3.8 we infer that \( \lambda^* = \alpha(x - \phi(f_x)) \) for some \( \alpha > 0 \). From Definition 2.6 it immediately implies that if \( \alpha(x - \phi(f_x)) \in \Lambda_\delta(\phi, x, \epsilon) \), then \( \|x - \phi(f_x)\|_\phi' > 0 \) and \( \alpha = \frac{1}{\|x - \phi(f_x)\|_\phi'}. \) Thus the set \( \Lambda_\delta(\phi, x, \epsilon) \) is non-empty, and is the singleton

\[
\left\{ \frac{x - \phi(f_x)}{\|x - \phi(f_x)\|_\phi'} \right\}
\]

if and only if \( \|x - \phi(f_x)\|_\phi' > 0 \).

We complete the proof by showing that \( \|x - \phi(f_x)\|_\phi' = 0 \) if and only if \( \delta = 0 \) and \( B(x, \epsilon) \cap \text{image}(\phi) = \emptyset \). On the one hand, if \( \delta = 0 \) and \( B(x, \epsilon) \cap \text{image}(\phi) = \emptyset \), then we have

\[
\|x - \pi_\phi(x)\| = \min_{z \in \text{image}(\phi)} \|x - z\| \geq \epsilon.
\]

However, from Lemma 3.4 we know that \( \|x - \phi(f_x)\| = \epsilon \), and since \( \phi(f_x) \in \text{image}(\phi) \), we deduce that \( \pi_\phi(x) = \phi(f_x) \). Due to orthogonality of projection,

\footnote{Even though the set \( F_\delta(\phi, x, \epsilon) \) may contain multiple elements, \( \lambda^* \) is unique due to the fact that \( \phi(f_x) \) is unique.}

\footnote{\( \pi_\phi : \mathbb{H}_n \rightarrow \text{image}(\phi) \) is the orthogonal projection operator onto \( \text{image}(\phi) \).}
\[ \langle x - \phi(f_x), z \rangle = 0 \text{ for all } z \in \text{image}(\phi). \] Since \( S_0(\phi, 1) \subset \text{image}(\phi) \), we obtain
\[ \| x - \phi(f_x) \|' = \max_{z \in S_0(\phi, 1)} \langle x - \phi(f_x), z \rangle = 0 \]

One the other hand, if \( \| x - \phi(f_x) \|' = 0 \), Lemma 3.7(i) implies that \( \delta = 0 \). Moreover, since \( S_0(\phi, 1) \) is an absorbing set to image(\( \phi \)), we conclude from the definition (14) of the dual function that \( \langle x - \phi(f_x), z \rangle = 0 \) for all \( z \in \text{image}(\phi) \). Furthermore, since \( \phi(f_x) \in \text{image}(\phi) \) it implies from the orthogonality principle that \( \pi_\phi(x) = \phi(f_x) \). Consequently,
\[ \min_{z \in \text{image}(\phi)} \| x - z \| = \| x - \pi_\phi(x) \| = \| x - \phi(f_x) \| = \epsilon. \]

In other words, we have \( \mathcal{B}(x, \epsilon) \cap \text{image}(\phi) = \emptyset \). The proof is now complete. \( \square \)

**Lemma 3.11.** Let the linear map \( \phi: \mathbb{R}^K \to \mathbb{H}_n \) and non-negative real numbers \( \epsilon, \delta \) be given, and \( x \in \mathbb{H}_n \setminus B[0, \epsilon] \) be any \((\phi, \epsilon, \delta)\)-feasible vector such that \( \Lambda_\delta(\phi, x, \epsilon) \neq \emptyset \). Then for every \( \lambda^* \in \Lambda_\delta(\phi, x, \epsilon) \) and \( h^* \in \left( C_\epsilon(\phi, x, \epsilon) \right)^n \cdot F_\delta(\phi, x, \epsilon) \), we have
\[ \langle \lambda^*, \phi(h^*) \rangle = \max_{h \in V} \langle \lambda^*, \phi(h) \rangle = 1 - \delta \| \lambda^* \|. \]

**Proof.** Applying (40) directly to \( \lambda^* \in \Lambda_\delta(\phi, x, \epsilon) \) gives us
\[ \max_{h \in V} \langle \lambda^*, \phi(h) \rangle = -\delta \| \lambda^* \| + \| \lambda^* \|' = 1 - \delta \| \lambda^* \|. \]

By denoting \( c_x = (C_\delta(\phi, x, \epsilon))^{1/p} \), we know from (41) that
\[ \langle \lambda^*, y^* \rangle = \max_{z \in S_0(\phi, x, \epsilon)} \langle \lambda^*, z \rangle = c_x \max_{z \in S_0(\phi, 1)} \langle \lambda^*, z \rangle = c_x \| \lambda^* \|' = c_x. \]

On substituting for \( y^* \) by considering \( f_x = c_x h^* \) in Remark 3.6, we get
\[ c_x = \langle \lambda^*, y^* \rangle = c_x \langle \lambda^*, \phi(h^*) \rangle + \frac{\delta c_x}{\epsilon + \delta c_x} \langle \lambda^*, \phi(h^*) \rangle \mathbb{1}_{[0, +\infty]}(\delta). \]

Whenever \( \delta > 0 \) we know from Proposition 3.10 that \( \lambda^* \) and \( (x - c_x \phi(h^*)) \) are co-linear. Thus, we obtain that:
\[ \langle \lambda^*, (x - c_x \phi(h^*)) \rangle = \| \lambda^* \| \| x - c_x \phi(h^*) \| = (\epsilon + \delta c_x) \| \lambda^* \|, \]

where the last equality follows from Lemma 3.4. Note that \( c_x > 0 \) since \( \| x \| > \epsilon \). Therefore, cancelling \( c_x \) throughout in (45) and simplifying for \( \langle \lambda^*, \phi(h^*) \rangle \) yields
\[ \langle \lambda^*, \phi(h^*) \rangle = 1 - \left( \delta \| \lambda^* \| \mathbb{1}_{[0, +\infty]}(\delta) \right) = 1 - \delta \| \lambda^* \|, \]

(43) follows at once from (44) and (46). \( \square \)

**Lemma 2.5.** If \( x \) is not \((\phi, \epsilon, \delta)\)-feasible, then we know that \( \delta = 0 \) and \( B[x, \epsilon] \cap \text{image}(\phi) = \emptyset \). Consequently, \( \| y \|_\phi = +\infty \) for all \( y \in B[x, \epsilon] \). Therefore, the assertion holds since \( C_\delta(\phi, x, \epsilon) = +\infty \).

If \( x \) is \((\phi, \epsilon, \delta)\)-feasible, then from Lemma 3.4, we know that the sets \( B[x, \epsilon] \) and \( S_\delta(\phi, x, \epsilon) \) intersect at a unique point \( y^* \). Thus we have
\[ \min_{y \in B[x, \epsilon]} \| y \|_\phi \leq \| y^* \|_\phi \leq (C_\delta(\phi, x, \epsilon))^{1/p}, \]
where the first inequality follows from the fact that $y^* \in B[x, \epsilon]$ and the second one follows from $y^* \in (C_{\delta}(\phi, x, \epsilon))^{1/p} \cdot S_\delta(\phi, 1)$ and the definition (12) of the gauge function $\|\cdot\|_\phi$.

On the one hand, for $y \in B[x, \epsilon]$ such that $\|y\|_\phi = +\infty$, the inequality $(C_{\delta}(\phi, x, \epsilon))^{1/p} \leq \|y\|_\phi$ holds trivially. On the other hand, for $y \in B[x, \epsilon]$ such that $\|y\|_\phi < +\infty$, we know from the definition (12) that $y \in S_\delta(\phi, \|y\|_\phi)$. Thus, $B[x, \epsilon] \cap S_\delta(\phi, \|y\|_\phi) = \emptyset$, and in view of Lemma 3.2, we get $(C_{\delta}(\phi, x, \epsilon))^{1/p} \leq \|y\|_\phi$. Combining the two facts, we conclude

$$(C_{\delta}(\phi, x, \epsilon))^{1/p} \leq \min_{y \in B[x, \epsilon]} \|y\|_\phi.$$ 

Collecting the two inequalities, (13) follows at once. □

**Remark 3.12.** The proof of the lemma also implies that $\|y^*\|_\phi = (C_{\delta}(\phi, x, \epsilon))^{1/p}$, and therefore, $y^*$ is a minimizer in the problem (13). Furthermore, if $y' \neq y^*$ is also a minimizer, then we have $\|y'\|_\phi = (C_{\delta}(\phi, x, \epsilon))^{1/p}$ and $y' \in B[x, \epsilon]$. Then it follows that $y' \in S_\delta(\phi, \|y'\|_\phi) = S_\delta(\phi, x, \epsilon)$, and thus $y' \in B[x, \epsilon] \cap S_\delta(\phi, x, \epsilon)$. From Lemma 3.4, we then have $y' = y^*$. Which is a contradiction. Thus, $y^* = \arg\min_{y \in B[x, \epsilon]} \|y\|_\phi$.

**Theorem 2.7.** Combining (13) and (15), we obtain

$$\begin{align*}
(C_{\delta}(\phi, x, \epsilon))^{1/p} & \leq \min_{y \in B[x, \epsilon]} \sup_{\|\lambda\|_\phi \leq 1} \langle \lambda, y \rangle \\
& \geq \sup_{\|\lambda\|_\phi \leq 1} \min_{y \in B[x, \epsilon]} \langle \lambda, y \rangle \\
& \geq \left\{ \begin{array}{ll}
\langle \lambda, x \rangle - \epsilon \|\lambda\| & \text{subject to } \|\lambda\|_\phi \leq 1.
\end{array} \right.
\end{align*}$$

(47)

Therefore, $(C_{\delta}(\phi, x, \epsilon))^{1/p}$ is an upper bound to the optimal value of (16). We shall establish the proposition by considering all the possible cases and showing that the upper bound is indeed the supremum.

**Case 1:** When $x$ is not $(\phi, \epsilon, \delta)$-feasible. We know that this happens only if $\delta = 0$ and $B[x, \epsilon] \cap \text{image}(\phi) = \emptyset$. Denoting $\pi_\phi(x)$ to be the orthogonal projection of $x$ onto $\text{image}(\phi)$, we have $\langle x - \pi_\phi(x) , z \rangle = 0$ for every $z \in \text{image}(\phi)$.

Since $\delta = 0$ we have $S_0(\phi, 1) \subset \text{image}(\phi)$. Thus, for every $\alpha \geq 0$, letting $\lambda'_\alpha := \alpha(x - \pi_\phi(x))$ we see that $\langle \lambda'_\alpha , z \rangle = 0$ for every $z \in S_0(\phi, 1)$. In other words, we have $\|\lambda_0\|_\phi' = 0$, and therefore, $\lambda'_\alpha$ is a feasible point in (16) for every $\alpha \geq 0$. Moreover, since $B[x, \epsilon] \cap \text{image}(\phi) = \emptyset$ we see that $\|x - \pi_\phi(x)\| \geq \epsilon + \rho$ for some $\rho > 0$. Therefore, the objective function of (16) evaluated at $\lambda_\alpha$ satisfies

$$\begin{align*}
\langle \lambda'_\alpha , x \rangle - \epsilon \|\lambda'_\alpha\| & = \alpha \left( \langle x - \pi_\phi(x) , x \rangle - \epsilon \|x - \pi_\phi(x)\| \right) \\
& = \alpha \left( \|x - \pi_\phi(x)\|^2 + \langle x - \pi_\phi(x) , \pi_\phi(x) \rangle - \epsilon \|x - \pi_\phi(x)\| \right) \\
& = \alpha \|x - \pi_\phi(x)\| \left( \|x - \pi_\phi(x)\| - \epsilon \right) \\
& \geq \alpha(\epsilon + \rho)\rho.
\end{align*}$$
By considering arbitrarily large value of α, we observe that the cost function in (16) attains arbitrarily large values for λ∗, i.e., the supremum is +∞.

**Case 2:** When 0 ≤ ||x|| < ε. We know that the optimal cost Cδ(φ, x, ε) is identically equal to zero, and we shall conclude that so is the value of the supremum in (16). Indeed, since 0 ∈ B[x, ε], for every λ ∈ ℜn we have

\[ \langle \lambda , x \rangle - \epsilon \| \lambda \| = \min_{y \in B[x, \epsilon]} \langle \lambda , y \rangle \leq \langle \lambda , 0 \rangle = 0. \]

Thus, zero is an upper bound for the supremum in (16). Moreover, for λ∗ = 0, we have \|λ∗\|₀ = 0 and \langle λ∗ , x \rangle - ε\|λ∗\| = 0. Thus, the value of the supremum is achieved, and λ∗ = 0 is an optimal solution.  

**Case 3:** When x is a (φ, ε, δ)-feasible, and ||x|| > ε with Aδ(φ, x, ε) = ∅. We know that there exists a λ∗ ∈ Aδ(φ, x, ε) and the following two conditions hold simultaneously:

\[ \|\lambda^*\|_φ' = 1, \text{ and} \]
\[ \langle \lambda^* , x \rangle - \epsilon \| \lambda^* \| = (C_δ(\phi, x, \epsilon))^{1/p}. \]

The first equality implies that λ∗ is a feasible point to (16), and the latter, in conjunction with (47) implies that the upper bound of (C₀(φ, x, ε))₁/ᵖ is achieved at λ∗. Thus, (C₀(φ, x, ε))₁/ᵖ is the optimum value of (16), and that every λ∗ ∈ A₀(φ, x, ε) is an optimal solution to (16).

Conversely, if λ∗ is an optimal solution to (16), then readily we get \langle λ∗ , x \rangle - ε\|λ∗\| > 0. Since (C₀(φ, x, ε))₁/ᵖ > 0, we have \langle λ∗ , x \rangle - ε\|λ∗\| > 0. Therefore, from the assertions (i) and (ii) of Lemma 3.7, we conclude that \|λ∗\|_φ > 0. Moreover, if \|λ∗\|_φ < 1, then λ′ := \frac{1}{\|λ∗\|_φ}λ∗ is also a feasible point to (16). However, the cost function evaluated at λ′ satisfies

\[ \langle \lambda' , x \rangle - \epsilon \| \lambda' \| = \frac{1}{\|\lambda^*\|_φ} \left( \langle \lambda^* , x \rangle - \epsilon \| \lambda^* \| \right) > (C_δ(\phi, x, \epsilon))^{1/p}, \]

which is a contradiction. Therefore, it follows at once that λ∗ ∈ A₀(φ, x, ε).

**Case 4:** When x is a (φ, ε, δ)-feasible vector and ||x|| > ε with Aδ(φ, x, ε) = ∅. We know from Proposition 3.10 that this happens only if δ = 0 and B(x, ε) ∩ image(φ) = ∅. Since x is (φ, ε, δ)-feasible, the set B[x, ε] intersects image(φ) only at the point πφ(x) - the orthogonal projection of x onto image(φ). Since no point other than πφ(x) in B[x, ε] intersects with image(φ), the LIP (8) reduces to the following:

\[
\begin{align*}
\text{minimize} & \quad c(f) \\
\text{subject to} & \quad \phi(f) = \pi_φ(x),
\end{align*}
\]

which simply is another LIP with parameters πφ(x), φ and ε = δ = 0. Since, πφ(x) ∈ image(φ), πφ(x) is (0, 0, 0)-feasible. Therefore, C₀(φ, x, ε) = C₀(φ, πφ(x), 0) and F₀(φ, x, ε) = F₀(φ, πφ(x), 0). In addition, from the Proposition 3.10 it follows

---

10It is to be noted that whenever ||x|| = ε, there could be non-zero optimal solutions, for e.g., \( \lambda^* = \alpha x \) for every \( \alpha \geq 0 \).
that the set $\Lambda_0(\phi, \pi_{\phi}(x), 0)$ is non-empty, and there exists $\lambda' \in \text{image}(\phi)$ such that the following two conditions hold simultaneously.

$$\langle \lambda', \pi_{\phi}(x) \rangle = (C_0(\phi, \pi_{\phi}(x), 0))^{1/p} = (C_0(\phi, x, \epsilon))^{1/p}$$

and

$$\lambda' = 1$$

Using the above facts, we shall first establish that the value $(C_0(\phi, x, \epsilon))^{1/p}$ is not just an upper bound but is indeed the supremum in (16).

For every $\alpha \geq 0$ let $\lambda(\alpha) := \lambda + \alpha(x - \pi_{\phi}(x))$. Since the linear functional $\langle x - \pi_{\phi}(x), \cdot \rangle$ vanishes on $\text{image}(\phi)$, for every $z \in \text{image}(\phi)$ we have

$$\langle \lambda(\alpha), z \rangle = \langle \lambda', z \rangle + \alpha \langle x - \pi_{\phi}(x), z \rangle = \langle \lambda', z \rangle,$$

and therefore,

$$\|\lambda(\alpha)\|_{\phi}^\prime = \max_{z \in S_{0}(\phi,1)} \langle \lambda(\alpha), z \rangle = \max_{z \in S_{0}(\phi,1)} \langle \lambda', z \rangle = \|\lambda'\|_{\phi} = 1.$$

Thus, $\lambda(\alpha)$ is a feasible point to (16), and the cost function evaluated at $\lambda(\alpha)$ satisfies:

$$\langle \lambda(\alpha), x \rangle - \epsilon \|\lambda(\alpha)\| = \langle \lambda(\alpha), \pi_{\phi}(x) \rangle + \langle \lambda(\alpha), x - \pi_{\phi}(x) \rangle - \epsilon \|\lambda(\alpha)\|$$

$$= \langle \lambda', \pi_{\phi}(x) \rangle + \alpha \|x - \pi_{\phi}(x)\|^2 - \epsilon \sqrt{\|\lambda'\|^2 + \alpha^2 \|x - \pi_{\phi}(x)\|^2}$$

$$= (C_0(\phi, x, \epsilon))^{1/p} + \epsilon \left(\alpha - \sqrt{\|\lambda'\|^2 + \alpha^2}\right).$$

Since $\lambda(\alpha)$ is feasible in (16) for every $\alpha \geq 0$, the supremum in (16) is sandwiched between $\sup_{\alpha \geq 0} \langle \lambda(\alpha), x \rangle - \epsilon \|\lambda(\alpha)\|$ and the optimal cost $(C_0(\phi, x, \epsilon))^{1/p}$. However, we see that:

$$\sup_{\alpha \geq 0} \langle \lambda(\alpha), x \rangle - \epsilon \|\lambda(\alpha)\| \geq \lim_{\alpha \to +\infty} \langle \lambda(\alpha), x \rangle - \epsilon \|\lambda(\alpha)\| = (C_0(\phi, x, \epsilon))^{1/p}.$$\footnote{For any $b > 0$, we have that $\lim_{\alpha \to +\infty} (a - \sqrt{b + \alpha^2}) = \lim_{\alpha \to +\infty} \frac{(a - \sqrt{b + \alpha^2}) (a + \sqrt{b + \alpha^2})}{(a + \sqrt{b + \alpha^2})} = \lim_{\alpha \to +\infty} \frac{-b}{a + \sqrt{\alpha^2}} = 0.$}

This implies that the supremum in (16) is indeed equal to $(C_0(\phi, x, \epsilon))^{1/p}$.

Now that we know the value of the supremum, it suffices to establish that (16) does not admit an optimal solution in this case. If there were any $\lambda'$ that is an optimal solution to (16), then from the arguments provided in the proof of necessary implication for case 3, it follows that $\lambda' \in \Lambda_0(\phi, x, \epsilon)$. This contradicts the premise $\Lambda_0(\phi, x, \epsilon) = \emptyset$. Therefore, (16) admits no solution whenever $\Lambda_0(\phi, x, \epsilon) = \emptyset$. \hfill \Box

**Lemma 3.13.** Let the linear map $\phi : \mathbb{R}^K \longrightarrow \mathbb{H}_n$, non-negative real numbers $\epsilon, \delta$ and $x \in \mathbb{H}_n \setminus B[0, \epsilon]$ be given. For every $h \in V_\epsilon$, consider the optimization problem

$$\begin{align*}
\sup_{\lambda} & \langle \lambda, x \rangle - \epsilon \|\lambda\| \\
\text{subject to} & \begin{cases} 
\langle \lambda, x \rangle - \epsilon \|\lambda\| > 0, \\
\langle \lambda, \phi(h) \rangle + \delta \|\lambda\| \leq 1
\end{cases}
\end{align*}$$

(i) The optimal value of (48) is equal to

$$\eta_h := \min \{\theta \geq 0 : B[x, \epsilon] \cap B[\phi(\theta h), \theta \delta] \neq \emptyset\}.$$
(ii) \( \eta_h \geq (C_h(\phi, x, \epsilon))^{1/p} \) and equality holds if and only if \( h \in \frac{1}{(C_h(\phi, x, \epsilon))^{1/p}} F_h(\phi, x, \epsilon) \).

(iii) \( \eta_h = +\infty \) if and only if there exists a \( \lambda' \in \mathbb{H}_n \) that simultaneously satisfies the conditions
- \( \langle \lambda', x \rangle - \epsilon \| x \| > 0 \)
- \( \langle \lambda', \phi(h) \rangle + \delta \| \lambda \| \leq 0 \).

Proof. Let the map \( L(\eta, h) : [0, +\infty] \times V_c \longrightarrow [0, +\infty] \) be defined by

\[
L(\eta, h) := \left\{ \sup_{\lambda} \left( \langle \lambda, x \rangle - \epsilon \| \lambda \| \right) - \eta \left( \langle \lambda, \phi(h) \rangle + \delta \| \lambda \| \right) \right. \\
\left. \text{subject to } \langle \lambda, x \rangle - \epsilon \| \lambda \| > 0. \right\}
\]

(50)

For every \( \eta \geq 0 \), let us define the set \( S'(\eta) := \bigcup_{\theta \in [0, \eta]} B[\phi(\theta h), \delta \theta] \). Clearly \( S'(\eta) \) is a convex-compact subset of \( \mathbb{H}_n \) and monotonic, i.e., \( S'(\eta) \subset S'(\eta') \) for every \( \eta \leq \eta' \).

For every \( h \in V_c \) and \( \theta \geq 0 \), we observe that \( B[\phi(\theta h), (\delta \theta)] = \theta \cdot B[\phi(h), \delta] \).

Since the sets \( B[x, \epsilon] \) and \( B[\phi(h), \delta] \) are compact, the minimization over \( \theta \geq 0 \) in (49) is achieved. Therefore, we have \( B[x, \epsilon] \cap B[\phi(\eta h), (\eta \delta)] \neq \emptyset \).

On the one hand, for \( 0 \leq \eta < \eta_h \leq +\infty \), we know that the convex sets \( B[x, \epsilon] \) and \( S'(\eta) \) do not intersect. Therefore, there exists a non-zero \( \lambda' \in \mathbb{H}_n \) such that the linear functional \( \langle \lambda', \cdot \rangle \) separates them. In other words, we have

\[
\min_{y \in B[x, \epsilon]} \langle \lambda', y \rangle \geq \max_{z \in S'(\eta)} \langle \lambda', z \rangle.
\]

Observing the following equalities

\[
\min_{y \in B[x, \epsilon]} \langle \lambda', y \rangle = \langle \lambda', x \rangle - \epsilon \| \lambda' \|,
\]

and

\[
\max_{z \in S'(\eta)} \langle \lambda', z \rangle = \max\left\{ 0, \max_{z \in B[\phi(\eta h), \delta \eta]} \langle \lambda', z \rangle \right\} = \max\left\{ 0, \eta \left( \langle \lambda', \phi(h) \rangle + \delta \| \lambda \| \right) \right\},
\]

we get

\[
\langle \lambda', x \rangle - \epsilon \| \lambda' \| \geq \max\left\{ 0, \eta \left( \langle \lambda', \phi(h) \rangle + \delta \| \lambda \| \right) \right\}.
\]

(51)

It follows at once that for every \( \alpha \geq 0 \), \( \lambda'_\alpha := \alpha \lambda' \) is a feasible point in (50), and thus, we have

\[
L(\eta, h) \geq \sup_{\alpha \geq 0} \left( \langle \lambda'_\alpha, x \rangle - \epsilon \| \lambda'_\alpha \| \right) - \eta \left( \langle \lambda'_\alpha, \phi(h) \rangle + \delta \| \lambda'_\alpha \| \right)
\]

\[
= \left( \langle \lambda', x \rangle - \epsilon \| \lambda' \| \right) - \eta \left( \langle \lambda', \phi(h) \rangle + \delta \| \lambda \| \right) \left( \sup_{\alpha \geq 0} \alpha \right)
\]

\[
= +\infty.
\]

On the other hand, for \( \eta_h \leq \eta < +\infty \), we know that \( B[x, \epsilon] \cap S'(\eta) \neq \emptyset \). Due to convexity, we know that for every \( \lambda \in \mathbb{H}_n \), we have

\[
\langle \lambda, x \rangle - \epsilon \| \lambda \| = \min_{y \in B[x, \epsilon]} \langle \lambda', y \rangle \leq \max_{z \in S'(\eta)} \langle \lambda', z \rangle = \max\left\{ 0, \eta \left( \langle \lambda', \phi(h) \rangle + \delta \| \lambda \| \right) \right\}.
\]

Therefore, for every \( \lambda \) such that \( \langle \lambda, x \rangle - \epsilon \| \lambda \| > 0 \), we obtain that \( \langle \lambda, \phi(h) \rangle + \delta \| \lambda \| > 0 \) and

\[
\langle \lambda, x \rangle - \epsilon \| \lambda \| \leq \eta \left( \langle \lambda, \phi(h) \rangle + \delta \| \lambda \| \right).
\]
By taking the supremum over all $\lambda$, we obtain $L(\eta, h) \leq 0$. However, by picking any $\lambda$ such that $\langle \lambda , x \rangle - \delta \| \lambda \| > 0$, and defining $\lambda_{\alpha} := \alpha \lambda$ for every $\alpha > 0$, we immediately see that $\langle \lambda_{\alpha} , x \rangle - \delta \| \lambda_{\alpha} \| > 0$ and

$$0 = \lim_{\alpha \to 0} \left( \langle \lambda_{\alpha} , x \rangle - \delta \| \lambda_{\alpha} \| \right) - \eta \left( \langle \lambda_{\alpha} , \phi(h) \rangle + \delta \| \lambda_{\alpha} \| \right).$$

Therefore, $L(\eta, h) = 0$. Summarizing, we have:

$$L(\eta, h) = \begin{cases} +\infty & \text{if } 0 \leq \eta < \eta_h \\ 0 & \text{if } \eta_h \leq \eta < \infty. \end{cases}$$

Let us consider the Lagrange dual of (48), which is written in the following inf-sup formulation.

$$\inf_{\eta \geq 0} \sup_{\lambda} \left\{ \langle \lambda , x \rangle - \delta \| \lambda \| - \eta \left( \delta \| \lambda \| + \langle \lambda , \phi(h) \rangle - 1 \right) \right\} \text{ subject to } \langle \lambda , x \rangle - \delta \| \lambda \| > 0. \tag{52}$$

Solving for the supremum over $\lambda$, the inf-sup problem (52) reduces to $\inf_{\eta \geq 0} \eta + L(\eta, h)$. It is immediate that the optimal value of the inf-sup problem (52) is equal to $\eta_h$.

We observe that the optimization problem (48), is a convex program. Moreover, since $\| x \| > \epsilon$, we see that $\lambda' := \alpha x$ is a strictly feasible point in (48) for every $0 < \alpha < \frac{1}{\| x \| \cdot \phi(x) + \delta \| x \|}$. Therefore, strong duality holds for the convex problem (48), and the optimal value of (48) is indeed equal to $\eta_h$. This establishes the assertion (i) of the lemma.

Since $B[\phi(h), \delta] \subset S_\delta(\phi, 1)$ we see that

$$B[\phi(\eta_h h), \delta \eta_h] = \eta_h \cdot B[\phi(h), \delta] \subset \eta_h \cdot S_\delta(\phi, 1) \subset S_\delta(\phi, \eta_h).$$

Combining this with the fact that $B[x, \epsilon] \cap B[\phi(\eta_h h), (\eta_h \delta)] \neq \emptyset$, we immediately infer that $B[x, \epsilon] \cap S_\delta(\phi, \eta_h) \neq \emptyset$. In view of Lemma 3.2, we have $\eta_h \geq (C_\delta(\phi, x, \epsilon))^{1/p}$. It is a straightforward exercise to verify that $\eta_{h'} = (C_\delta(\phi, x, \epsilon))^{1/p}$ for some $h' \in V_c$, if and only if $(C_\delta(\phi, x, \epsilon))^{1/p} h' \in F_\delta(\phi, x, \epsilon)$. This establishes assertion (ii) of the lemma.

If there exists a $\lambda' \in \mathbb{H}_n$ such that the conditions $\langle \lambda' , x \rangle - \delta \| \lambda' \| > 0$ and $\langle \lambda' , \phi(h) \rangle + \delta \| \lambda' \| \leq 0$ hold simultaneously. Then for every $\alpha > 0$, $\lambda_{\alpha} := \alpha \lambda'$ is a feasible point in (48). Therefore, we have

$$\eta_h \geq \sup_{\alpha > 0} \langle \lambda_{\alpha} , x \rangle - \delta \| \lambda_{\alpha} \| = +\infty.$$

Conversely let $\eta_h = +\infty$, then we know that the compact-convex set $B[x, \epsilon]$ does not intersect with the closed convex-cone $S' := \bigcup_{\delta \in [0, +\infty]} B[\phi(\theta h), \delta \theta]$. Since one of the sets involved is compact, there exists a $\lambda' \in \mathbb{H}_n$ such that the linear functional $\langle \lambda' , \cdot \rangle$ separates these sets strictly. Thus, we have

$$\max_{z \in S'} \langle \lambda' , z \rangle < \min_{y \in B[x, \epsilon]} \langle \lambda' , y \rangle = \langle \lambda' , x \rangle - \epsilon \| \lambda' \| .$$

We note that the quantity $\langle \lambda' , x \rangle - \epsilon \| \lambda' \|$ is a minimum of a linear functional over a compact set, and thus finite. On the contrary, $\max_{z \in S'} \langle \lambda' , z \rangle$ is a maximum.

---

$^{12}$It is to be noted that if $\eta_h = +\infty$, then $L(\eta, h) = +\infty$ for every $\eta \in [0, +\infty[$.
of the linear functional $\lambda'$ over the cone $S'$. Therefore, it can be either 0 or $+\infty$.

However, since $\langle \lambda', x \rangle - \epsilon \| \lambda' \|$ is an upper bound to this maximum, we have $0 = \max_{z \in S'} \langle \lambda', z \rangle$. Therefore, we get $\langle \lambda', x \rangle - \epsilon \| \lambda' \| > 0$, and since $B[\phi(h), \delta] \subset S'$ we also have $\langle \lambda', \phi(h) \rangle + \delta \| \lambda' \| \leq 0$. This completes the proof. □

Lemma 3.14. Let the linear map $\phi$, real numbers $\epsilon, \delta \geq 0$, $q \in [0, 1]$, $r > 0$ and $x \in \mathbb{R}_n \setminus B[x, \epsilon]$ be given. For every $h \in V$, let us consider the following optimization problem:

$$
(53) \quad \begin{cases}
\sup_{\lambda, \alpha > 0} & r \left( \langle \lambda, x \rangle - \epsilon \| \lambda \| \right)^q - \left( \delta \| \lambda \| + \langle \lambda, \phi(h) \rangle \right) \\
\text{subject to} & \langle \lambda, x \rangle - \epsilon \| \lambda \| > 0.
\end{cases}
$$

(i) If $q \in [0, 1]$, the optimal value of $(53)$ is $s(r, q) \eta_h^{-\frac{1}{q}}$, where $\eta_h$ is as defined in $(49)$ and $s(r, q)$ is some constant.\(^{13}\)

(ii) If $q = 1$, the optimal value of $(53)$ is finite and equal to 0 if and only if $\eta_h \leq \frac{1}{r}$.

Proof. We begin by considering the case when $\eta_h < +\infty$. From the assertion (iii) of Lemma 3.13, it follows that $\langle \lambda, \phi(h) \rangle + \delta \| \lambda \| > 0$ for every $\lambda \in \mathbb{R}_n$ satisfying $\langle \lambda, x \rangle - \epsilon \| \lambda \| > 0$. Then, the optimization problem can be equivalently written as

$$
\begin{cases}
\sup_{\lambda, \alpha > 0} & r \left( \langle \lambda, x \rangle - \epsilon \| \lambda \| \right)^q - \alpha \\
\text{subject to} & \langle \lambda, \phi(h) \rangle + \delta \| \lambda \| = \alpha \\
& \langle \lambda, x \rangle - \epsilon \| \lambda \| > 0.
\end{cases}
$$

Redefining new variables $\lambda' := \frac{1}{\alpha} \lambda$, the above optimization problem is written as

$$
\begin{cases}
\sup_{\lambda', \alpha > 0} & \alpha^q r \left( \langle \lambda', x \rangle - \epsilon \| \lambda' \| \right)^q - \alpha \\
\text{subject to} & \langle \lambda', \phi(h) \rangle + \delta \| \lambda' \| = 1 \\
& \langle \lambda', x \rangle - \epsilon \| \lambda' \| > 0.
\end{cases}
$$

By keeping a feasible $\lambda'$ fixed, one can explicitly optimize over $\alpha > 0$. In fact, for any $r' > 0$ we know that

$$
\sup_{\alpha > 0} \left( r' \alpha^q - \alpha \right) = \left( r' \right)^{\frac{1}{1-q}} \frac{1}{q} \left( 1 - q \right).
$$

Substituting $r' = r \left( \langle \lambda', x \rangle - \epsilon \| \lambda' \| \right)^q$, we see that $(53)$ simplifies to

$$
\begin{cases}
\sup_{\lambda'} & s(r, q) \left( \langle \lambda', x \rangle - \epsilon \| \lambda' \| \right)^{\frac{q}{1-q}} \\
\text{subject to} & \langle \lambda', \phi(h) \rangle + \delta \| \lambda' \| = 1 \\
& \langle \lambda', x \rangle - \epsilon \| \lambda' \| > 0.
\end{cases}
$$

Since, $\eta_h < +\infty$, we know that $\langle \lambda', \phi(h) \rangle + \delta \| \lambda' \| > 0$ for every $\lambda'$ satisfying $\langle \lambda', x \rangle - \epsilon \| \lambda' \| > 0$. Moreover, if $\langle \lambda', \phi(h) \rangle + \delta \| \lambda' \| < 1$ also holds for $\lambda'$, we see

\(^{13}\) $s(r, q) := \left( (1-q)(q^q r)^{\frac{1}{1-q}} \right)$
that its scaled version \( \lambda' := \frac{1}{(\lambda, \phi(h)) + \delta \|\lambda'\|} \lambda' \), satisfies
\[
\langle \lambda'', x \rangle - \epsilon \|\lambda''\| > \langle \lambda', x \rangle - \epsilon \|\lambda'\| \quad \text{and} \quad \langle \lambda'', \phi(h) \rangle + \delta \|\lambda''\| = 1.
\]

Therefore, the equality constraint \( \langle \lambda', \phi(h) \rangle + \delta \|\lambda'\| = 1 \) can be relaxed to an inequality without changing the value of the supremum. Thus, we obtain the following problem equivalent to (53).
\[
\begin{align*}
\sup_{\lambda'} & \quad s(r, q) \left( \langle \lambda', x \rangle - \epsilon \|\lambda'\| - \frac{1}{r} \right) \\
\text{subject to} & \quad \langle \lambda', \phi(h) \rangle + \delta \|\lambda'\| \leq 1 \quad \langle \lambda', x \rangle - \epsilon \|\lambda'\| > 0.
\end{align*}
\]

Finally, we observe that \([0, \infty] \ni \eta \longrightarrow \eta^\frac{q}{q-1} \in [0, +\infty]\) is an increasing function for every \( q \in [0, 1] \). Then it follows at once from Lemma 3.13 that the optimal value of (53) is equal to \( s(r, q) \eta_h^\frac{q}{q-1} \).

Finally if \( q = 1 \), employing similar technique as before, reduces the problem (53) to
\[
\begin{align*}
\sup_{\lambda', \alpha > 0} & \quad \alpha r \left( \langle \lambda', x \rangle - \epsilon \|\lambda'\| - \frac{1}{r} \right) \\
\text{subject to} & \quad \langle \lambda', \phi(h) \rangle + \delta \|\lambda'\| = 1 \quad \langle \lambda', x \rangle - \epsilon \|\lambda'\| > 0,
\end{align*}
\]
which simplifies to: \( \sup_{\alpha > 0} \alpha r (\eta_h - \frac{1}{r}) \). It then follows at once that the optimal value of the sup problem (53) is finite and equal to 0 if and only if \( \eta_h \leq \frac{1}{r} \). The proof if now complete.

**Theorem 2.10.** Solving for the supremum over \( \lambda \) for every \( h \in V_c \) in the min-sup problem (18), we deduce from Lemma 3.14 that (18) reduces to
\[
\min_{h \in V_c} s(r, q) \eta_h^{\frac{q}{q-1}}.
\]

Since \([0, +\infty] \ni \eta \longrightarrow \eta^\frac{q}{q-1} \in [0, 1] \), in view of the assertion (ii) of Lemma 3.13, we conclude that the minimization over the variable \( h \) is achieved at \( h^* \) such that \( \eta_{h^*} = (C_\delta(\phi, x, \epsilon))^{1/p} \). Therefore, the optimal value of the min-sup problem (18) is equal to \( s(r, q) (C_\delta(\phi, x, \epsilon))^{\frac{q}{q-1}} \) and the set of minimizers is \( \frac{1}{(C_\delta(\phi, x, \epsilon))^{1/p}} F_\delta(\phi, x, \epsilon) \). This establishes the assertions (i) and (ii)-(a) of the theorem.

**Necessary condition for \((h^*, \lambda^*)\) to be a saddle point solution.**

Suppose that \((h^*, \lambda^*) \in V_c \times \mathbb{H}_n\) is a saddle point solution to the min-sup problem (18). Then necessarily, we have
\[
h^* \in \arg\min_{h \in V_c} \left\{ \sup_{\lambda} r \left( \langle \lambda , x \rangle - \epsilon \|\lambda\| \right)^q - \left( \delta \|\lambda\| + \langle \lambda , \phi(h) \rangle \right) \right. \\
\left. \text{subject to} \quad \langle \lambda , x \rangle - \epsilon \|\lambda\| > 0, \right\}
\]
which implies that \( h^* \in \frac{1}{(C_{\phi}(\phi, x, \epsilon))^{1/q}} \cdot F_\delta(\phi, x, \epsilon) \). Moreover, we also have

\[
\lambda^* \in \left\{ \begin{array}{l}
\arg\max_{\lambda} \quad \min_{h \in V_c} \left\{ \langle \lambda, x \rangle - \epsilon \|\lambda\| \right\}^q - \left( \delta \|\lambda\| + \langle \lambda, \phi(h) \rangle \right) \\
\text{subject to} \quad \langle \lambda, x \rangle - \epsilon \|\lambda\| > 0.
\end{array} \right.
\]

The minimization over \( h \) can be solved explicitly, and simplifying using (40), we have

\[
\lambda^* \in \left\{ \begin{array}{l}
\arg\max_{\lambda} \quad r \left( \langle \lambda, x \rangle - \epsilon \|\lambda\| \right)^q - \|\lambda\|^n
\text{subject to} \quad \langle \lambda, x \rangle - \epsilon \|\lambda\| \geq 0.
\end{array} \right.
\]

By defining the new variables \( \alpha := \|\lambda\|_\phi \), and \( \lambda' := \frac{1}{\|\lambda\|_\phi} \lambda \), and writing the above optimization problem in terms of the variables \( (\lambda', \alpha) \), we obtain

\[
\left( \frac{\lambda^*}{\|\lambda^*\|_\phi}, \|\lambda^*\|_\phi' \right) \in \left\{ \begin{array}{l}
\arg\max_{(\lambda', \alpha)} \quad r \alpha^q \left( \langle \lambda', x \rangle - \epsilon \|\lambda'\| \right)^q - \alpha
\text{subject to} \quad \{ \begin{array}{l}
\alpha > 0, \\
\|\lambda'\|_\phi' = 1, \\
\langle \lambda', x \rangle - \epsilon \|\lambda'\| > 0.
\end{array} \right.
\end{array} \right.
\]

Observe that for every feasible \( \lambda' \), the optimization over the variable \( \alpha \) can be solved explicitly. Then from arguments similar to the ones provided in the proof of Lemma 3.14, we conclude that

\[
\frac{1}{\|\lambda^*\|_\phi'} \lambda^* \in \left\{ \begin{array}{l}
\arg\max_{\lambda'} \quad \max_{\alpha > 0} \left\{ r \alpha^q \left( \langle \lambda', x \rangle - \epsilon \|\lambda'\| \right)^q - \alpha \right\}
\text{subject to} \quad \{ \begin{array}{l}
\|\lambda'\|_\phi' = 1, \\
\langle \lambda', x \rangle - \epsilon \|\lambda'\| > 0, \\
\|\lambda'\|_\phi' = 1, \\
\langle \lambda', x \rangle - \epsilon \|\lambda'\| > 0.
\end{array} \right.
\end{array} \right.
\]

\[
= \Lambda_\delta(\phi, x, \epsilon). \quad \text{(55)}
\]

Since every \( \lambda \in \Lambda_\delta(\phi, x, \epsilon) \) satisfies \( \langle \lambda, x \rangle - \epsilon \|\lambda\| = C_\delta(\phi, x, \epsilon) \frac{1}{r} \), we conclude from (55) that the following also holds.

\[
\|\lambda^*\|_\phi' \in \arg\max_{\alpha > 0} \left\{ r \alpha^q C_\delta(\phi, x, \epsilon) \frac{1}{r} - \alpha \right\}
= (rq)^{\frac{1}{1-q}} (C_\delta(\phi, x, \epsilon))^{\frac{q}{1-q}}.
\]

\[
\text{Since } \|x\| > \epsilon, \text{ we know that the optimal value achieved in (16) is positive. Therefore, adding the additional constraint } \langle \lambda', x \rangle - \epsilon \|\lambda'\| > 0 \text{ neither changes the optimizer nor the optimal value. Then the last last equality follows from Theorem 2.7.}
\]
Therefore, \( \lambda^* \in (rq)^{\frac{1}{2-p}} \left( C_\delta(\phi, x, \epsilon) \right)^{\frac{q}{1-q-p}} \cdot \Lambda_\delta(\phi, x, \epsilon) \), and the necessary conditions hold.

Sufficient condition for \((h^*, \lambda^*)\) to be a saddle point solution.

Since \( 0 < (rq)^{\frac{1}{2-p}} \left( C_\delta(\phi, x, \epsilon) \right)^{\frac{q}{1-q-p}} \), we conclude from (43), that \( \langle \lambda^*, \phi(h^*) \rangle = \max_{h \in V_c} \langle \lambda^*, \phi(h) \rangle \). Then it immediately follows that

\[
\text{argmin}_{h \in V_c} r \left( \langle \lambda^*, x \rangle - \epsilon \|\lambda^*\| \right)^q - \left( \delta \|\lambda^*\| + \langle \lambda^*, \phi(h) \rangle \right).
\]

From Lemma 3.13(ii), we note that \( \eta_{h^*} = C_\delta(\phi, x, \epsilon)^{1/p} \). Therefore, from Lemma 3.14 we have

\[
s(r, q)C_\delta(\phi, x, \epsilon)^{\frac{q}{1-q-p}} = \left\{ \sup_{\lambda} \frac{r \left( \langle \lambda, x \rangle - \epsilon \|\lambda\| \right)^q - \left( \delta \|\lambda\| + \langle \lambda, \phi(h^*) \rangle \right)}{\|\lambda\| > 0, \langle \lambda, \phi(h^*) \rangle} \right. \}
\]

Moreover, from (43) and the Definition 2.6 of the set \( \Lambda_\delta(\phi, x, \epsilon) \), it is a straightforward exercise to verify that

\[
s(r, q)C_\delta(\phi, x, \epsilon)^{\frac{q}{1-q-p}} = r \left( \langle \lambda^*, x \rangle - \epsilon \|\lambda^*\| \right)^q - \left( \delta \|\lambda^*\| + \langle \lambda^*, \phi(h^*) \rangle \right).
\]

Since it is obvious from the Definition 2.6 that \( \langle \lambda^*, x \rangle - \epsilon \|\lambda^*\| > 0 \), we get at once that

\[
\lambda^* \in \left\{ \argmax_{\lambda} \frac{r \left( \langle \lambda, x \rangle - \epsilon \|\lambda\| \right)^q - \left( \delta \|\lambda\| + \langle \lambda, \phi(h^*) \rangle \right)}{\|\lambda\| > 0, \langle \lambda, \phi(h^*) \rangle} \right. \}
\]

Collecting (56) and (57), we conclude that \((h^*, \lambda^*)\) is indeed a saddle point solution to (18). The proof is now complete.

**Proposition 2.18.** If \( x \) is not \((\phi, \epsilon, \delta)\)-feasible, then we know that \( \delta = 0 \) and \( B[x, \epsilon] \cap \text{image}(\phi) = \emptyset \). Consequently, there exists \( H_n \ni \lambda' \perp \text{image}(\phi) \) such that \( \langle \lambda', x \rangle - \epsilon \|\lambda'\| > 0 \). Therefore, for every \( \alpha > 0 \), \( \lambda_\alpha := \alpha \lambda' \) is a feasible point, and by considering arbitrarily large values of \( \alpha \) we see that

\[
+\infty = \sup_{\lambda \in H_n} \left\{ c(f) + \left( \langle \lambda, x \rangle - \epsilon \|\lambda\| \right) - \langle \lambda, \phi(f) \rangle \right. \}
\]

for every \( f \in \mathbb{R}^K \). Observe that since \( \delta = 0 \), the constraint \( \|\lambda\| \leq \frac{1}{\delta} \) in (23) can be omitted. Thus, if \( x \) is not \((\phi, \epsilon, \delta)\)-feasible, the optimal value of the min-max problem (23) is \( +\infty \).

For every \( f \in \mathbb{R}^K \) define

\[
\eta_f := \sup_{\|\lambda\| < \frac{1}{\delta}} c(f) \left( 1 - \delta \|\lambda\| \right) + \left( \langle \lambda, x \rangle - \epsilon \|\lambda\| \right) - \langle \lambda, \phi(f) \rangle.
\]

To complete the proof of assertions (1) and (ii) of the proposition, we shall consider \( x \) to be \((\phi, \epsilon, \delta)\)-feasible and establish that \( C_\delta(\phi, x, \epsilon) = \min_{f \in \mathbb{R}^K} \eta_f \), where the set of minimizers is \( F_\delta(\phi, x, \epsilon) \). We begin by first showing that the inequality \( C_\delta(\phi, x, \epsilon) \leq \frac{1}{\delta} \|\lambda\| \) holds.
It is easily verified that Case 3 immediately from the feasibility of \( f \).

\[
\eta_f = c(f) + \sup_{\alpha \in [0, \frac{1}{\delta}]} \alpha \left( \|x - \phi(f)\| - (\epsilon + \delta c(f)) \right).
\]

Case 1: If \( \delta = 0 \) and \( \|x - \phi(f)\| > \epsilon \). Since \( \delta = 0 \), \( \alpha \) is unconstrained in the maximization problem of (58), and therefore, \( \eta_f = +\infty \). Since \( x \) is \( (\phi, \epsilon, \delta) \)-feasible we have \( C_\delta(\phi, x, \epsilon) < +\infty \), and the inequality \( C_\delta(\phi, x, \epsilon) \leq \eta_f \) follows.

Case 2: If \( \delta = 0 \) and \( \|x - \phi(f)\| \leq \epsilon \). It is immediate that
\[
\eta_f = c(f) + \frac{1}{\delta} \max \left\{ 0, \|x - \phi(f)\| - (\epsilon + \delta c(f)) \right\}.
\]
Observe that \( c(f) \leq \eta_f \) follows trivially, and moreover,
\[
\|x - \phi(f)\| = (\epsilon + \delta c(f)) + \left( \|x - \phi(f)\| - (\epsilon + \delta c(f)) \right)
\leq (\epsilon + \delta c(f)) + \max \left\{ 0, \|x - \phi(f)\| - (\epsilon + \delta c(f)) \right\}
= (\epsilon + \delta c(f)) + \delta (\eta_f - c(f))
= \epsilon + \delta \eta_f.
\]
Therefore, the pair \((\eta_f, f)\) is a feasible point in the LIP (8), and consequently, the inequality \( C_\delta(\phi, x, \epsilon) \leq \eta_f \) follows.

Let us consider \( f^* \in F_\delta(\phi, x, \epsilon) \), to establish that the inequality \( C_\delta(\phi, x, \epsilon) \leq \min_{f \in \mathbb{R}^K} \eta_f \) is indeed satisfied with the equality, it suffices to show that \( \eta_{f^*} = C_\delta(\phi, x, \epsilon) \). If \( \delta = 0 \), then indeed \( \eta_{f^*} = c(f^*) = C_\delta(\phi, x, \epsilon) \). If \( \delta > 0 \), we know that \( \eta_{f^*} \geq c(f^*) \), whereby we have
\[
\|x - \phi(f^*)\| - (\epsilon + \delta c(f^*)) \geq \|x - \phi(f^*)\| - (\epsilon + \delta \eta_{f^*}) \geq 0.
\]
Using \( \eta_{f^*} = c(f^*) + \frac{1}{\delta} \max \left\{ 0, \|x - \phi(f^*)\| - (\epsilon + \delta c(f^*)) \right\} \) and simplifying, we get
\[
\eta_{f^*} = \frac{1}{\delta} \left( \|x - \phi(f^*)\| - \epsilon \right) = C_\delta(\phi, x, \epsilon),
\]
where the last equality follows from the assertion (i) of Lemma 3.4. Furthermore, if there exists \( f' \in \mathbb{R}^K \) such that \( \eta_{f'} = C_\delta(\phi, x, \epsilon) \), we know that the pair \((\eta_{f'}, f')\) is a feasible point in the LIP (8) it readily follows that \( f' \in F_\delta(\phi, x, \epsilon) \). This completes the proof of assertions (i) and (ii) of the proposition.

**Necessary condition for the pair \((f^*, \lambda^*)\) to be a saddle point in (23).**
The fact that \( f^* \in F_\delta(\phi, x, \epsilon) \) follows at once from assertion (ii) of the proposition.
To prove that \( \lambda^* \in \Lambda_{\delta}(\phi, x, \epsilon) \), first we observe that

\[
\min_{f \in \mathbb{R}^N} c(f)(1 - \delta \|\lambda\|) - \langle \lambda, \phi(f) \rangle = \min_{\eta \geq 0, \lambda \in V_{\epsilon}} \eta(1 - \delta \|\lambda\|) - \langle \lambda, \phi(\eta) \rangle
\]

\[
= \min_{\eta \geq 0} \eta(1 - \|\lambda\|\phi)
\]

(60)

\[
\begin{cases} 
-\infty & \text{if } \|\lambda\|_\phi > 1, \\
0 & \text{if } \|\lambda\|_\phi \leq 1.
\end{cases}
\]

Therefore, if \((f^*, \lambda^*)\) is a saddle point in (23), we have

\[
\lambda^* \in \text{argmax} \left\{ \min_{\|\lambda\| \leq \frac{1}{\delta}} c(f)(1 - \delta \|\lambda\|) + (\langle \lambda, x \rangle - \epsilon \|\lambda\|) - \langle \lambda, \phi(f) \rangle \right\}
\]

\[
\in \text{argmax} \left\{ \langle \lambda, x \rangle - \epsilon \|\lambda\| \right\} \quad \text{because, } \delta \|\lambda\| \leq \|\lambda\|_\phi
\]

\[
\in \Lambda_{\delta}(\phi, x, \epsilon).
\]

**Sufficient condition for the pair \((f^*, \lambda^*)\) to be a saddle point in (23).**

Let \(\lambda^* \in \Lambda_{\delta}(\phi, x, \epsilon)\) and \(f^* \in F_{\delta}(\phi, x, \epsilon)\). From Definition 2.6, we see that \(\|\lambda^*\|_\phi = 1\) and \((1 - \delta \|\lambda^*\|) \geq 0\). Therefore, from (60)

\[
0 = \min_{f \in \mathbb{R}^N} c(f)(1 - \delta \|\lambda^*\|) - \langle \lambda^*, \phi(f) \rangle
\]

\[
\leq c(f^*)(1 - \delta \|\lambda^*\|) - \langle \lambda^*, \phi(f^*) \rangle
\]

\[
\leq C_{\delta}(\phi, x, \epsilon)(1 - \delta \|\lambda^*\|) - \langle \lambda^*, \phi(h^*) \rangle
\]

\[
= 0,
\]

where \(h^* := \frac{1}{C_{\delta}(\phi, x, \epsilon)}f^*, \) and the last equality follows from (43). Therefore, all the inequalities are satisfied with equality, and we have \(0 = c(f^*)(1 - \delta \|\lambda^*\|) - \langle \lambda^*, \phi(f^*) \rangle\). Moreover, it also immediately implies that

\[
(61) \quad f^* \in \text{argmin}_{f \in \mathbb{R}^N} c(f)(1 - \delta \|\lambda^*\|) + (\langle \lambda^*, x \rangle - \epsilon \|\lambda^*\|) - \langle \lambda^*, \phi(f) \rangle.
\]

From Definition 2.6 we know that \(\langle \lambda^*, x \rangle - \epsilon \|\lambda^*\| = C_{\delta}(\phi, x, \epsilon)\) and \(\delta \|\lambda^*\| \leq 1 - \max_{h \in V_{\epsilon}} \langle \lambda^*, x \rangle \leq 1\). Moreover, from the fact that \(0 = c(f)(1 - \delta \|\lambda^*\|) - \langle \lambda^*, \phi(f) \rangle\) we have

\[
\left( c(f)(1 - \delta \|\lambda^*\|) - \langle \lambda^*, \phi(f) \rangle \right) + \left( \langle \lambda^*, x \rangle - \epsilon \|\lambda^*\| \right) = 0 + C_{\delta}(\phi, x, \epsilon).
\]

Recalling from (59) that \(\eta_{f^*} = C_{\delta}(\phi, x, \epsilon),\) we immediately get

\[
(62) \quad \lambda^* \in \text{argmax}_{\|\lambda\| \leq \frac{1}{\delta}} c(f^*)(1 - \delta \|\lambda\|) + (\langle \lambda, x \rangle - \epsilon \|\lambda\|) - \langle \lambda, \phi(f^*) \rangle.
\]

Collecting (61) and (62), we conclude that \((f^*, \lambda^*) \in F_{\delta}(\phi, x, \epsilon) \times \Lambda_{\delta}(\phi, x, \epsilon)\) is indeed a saddle point solution to the min-max problem (23), and the proof is now complete. \(\square\)
4. Conclusion

In this article, we have proposed a slightly generalised formulation of the error constrained linear inverse problem and provide an exposition to its underlying convex geometry. Novel convex-concave min-max problems have been proposed and their equivalence to the LIP is mathematically established. These equivalent reformulations are crucial in overcoming the ill-posedness of the error constrained dictionary learning problem. Furthermore, complete characterization of the saddle points of the min-max problems is also provided in terms of a solution to the LIP, and vice versa. Consequently, a solution to the LIP can be computed by applying saddle point seeking methods to its equivalent min-max problems, which gives rise to simple algorithms to solve linear inverse problems and problems alike. Of course, the intent of this article is to only show that the min-max forms can also be used to obtain algorithms for an LIP, comparison of the resulting algorithms with the existing methods needs a separate and thorough investigation of its own, and will be reported in subsequent articles.

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