Enhancing complex-network synchronization

Adison E. Motter,† Changsong Zhou,‡ and Jürgen Kurths‡

†Max Planck Institute for the Physics of Complex Systems, Nöthnitzer Strasse 38, 01187 Dresden, Germany
‡Institute of Physics, University of Potsdam PF 601553, 14415 Potsdam, Germany

Heterogeneity in the degree (connectivity) distribution has been shown to suppress synchronization in networks of symmetrically coupled oscillators with uniform coupling strength (unweighted coupling). Here we uncover a condition for enhanced synchronization in directed networks with weighted coupling. We show that, in the optimum regime, synchronizability is solely determined by the average degree and does not depend on the system size and the details of the degree distribution. In scale-free networks, where the average degree may increase with heterogeneity, synchronizability is drastically enhanced and may become positively correlated with heterogeneity, while the overall cost involved in the network coupling is significantly reduced as compared to the case of unweighted coupling.

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Networks of dynamical elements serve as natural models for a variety of systems, with examples ranging from cell biology to epidemiology to the Internet [1]. Many of these complex networks display common structural features, such as the small-world [2] and scale-free properties [3]. Small-world networks (SWNs) exhibit short average distance between nodes and high clustering [2], while scale-free networks (SFNs) are characterized by an algebraic, highly heterogeneous distribution of degrees (number of links per node) [3]. The interplay between structure and dynamics has attracted a great deal of attention, especially in connection with the problem of synchronization of coupled oscillators [4, 5, 6, 7]. The ability of a network to synchronize is generally enhanced in both SWNs and random SFNs as compared to regular lattices [4, 5, 6, 7, 8]. This enhancement was previously believed to be due to the decrease of the average distance between oscillators. Recently, it was shown that random networks with strong heterogeneity in the degree distribution, such as SFNs, are much more difficult to synchronize than random homogeneous networks [7], even though the former display smaller average path length [10]. This suggests that, although structurally advantageous [11], the scale-free property may be dynamically detrimental. Here we present a solution to this problem.

A basic assumption of previous work is that the oscillators are coupled symmetrically and with the same coupling strength. Under the assumption of symmetric coupling, the maximum synchronizability is indeed achieved when the coupling strength is uniform [12]. But to get a better synchronizability the couplings are not necessarily symmetrical. Many realistic networks are actually directed [11] and weighted [13]. In particular, the communication capacity of a node is likely to saturate when the degree becomes large.

In this Letter, we study the impact that asymmetry and saturation of connection strength have on the synchronization dynamics on complex networks. As a prime example, we consider complete synchronization of linearly coupled identical oscillators, namely

\[ \dot{x}_i = f(x_i) - \alpha \sum_{j=1}^{N} G_{ij} h(x_j), \quad i = 1, \ldots, N, \]  

where \( f = f(x) \) describes the dynamics of each individual oscillator, \( h = h(x) \) is the output function, \( G = (G_{ij}) \) is the coupling matrix, and \( \alpha \) is the overall coupling strength. In the case of symmetrically coupled oscillators with uniform coupling strength, \( G \) is the usual (symmetric) Laplacian matrix \( L = (L_{ij}) \) [14]. For \( G_{ij} = L_{ij} \), heterogeneity in the degree distribution suppresses synchronization [7]. In order to enhance the synchronizability of heterogeneous networks, we propose to scale the coupling strength by the degrees of the nodes. For specificity, we consider

\[ G_{ij} = \frac{L_{ij}}{k_i^{\beta}}, \]

where \( k_i \) is the degree of node \( i \) and \( \beta \) is a tunable parameter. The underlying network associated with the Laplacian matrix \( L \) is undirected and unweighted, but with the introduction of the weights in eq. (2), the network of couplings becomes not only weighted but also directed because the resulting matrix \( G \) is in general asymmetric. We say that the network or coupling is weighted when \( \beta \neq 0 \) and unweighted when \( \beta = 0 \).

The variational equations governing the linear stability of a synchronized state \( \{x_i(t) = s(t), \forall i\} \) can be diagonalized into \( N \) blocks of the form \( \dot{\eta} = [Df(s) - \alpha Dh(s)] \eta \), where \( \alpha = \sigma \lambda_i \), and \( \lambda_i \) are the eigenvalues of the coupling matrix \( G \), ordered as \( 0 = \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_N \) (see below). The largest Lyapunov exponent \( \Lambda(\alpha) \) of this equation can be regarded as a master stability function, which determines the linear stability of the synchronized state [15]: the synchronized state is stable if \( \Lambda(\sigma \lambda_i) < 0 \) for \( i = 2, \ldots, N \). (The eigenvalue \( \lambda_1 \) corresponds to a mode parallel to the synchronization manifold.) For many widely studied oscillatory systems [4, 13], the master stability function \( \Lambda(\alpha) \) is negative in a finite interval \( (\alpha_1, \alpha_2) \). Therefore, the network is synchronizable for some \( \sigma \) when the eigenratio \( R = \lambda_N / \lambda_2 \) satisfies \( R < \alpha_2 / \alpha_1 \). The ratio \( \alpha_2 / \alpha_1 \) depends only on the dynamics \( f, h, \) and \( s \), while the eigenratio \( R \) depends only on the coupling matrix \( G \). The problem of synchronization is then reduced to the analysis of

\[ \sigma \]
eigenvalues of the coupling matrix \([\beta]\): the smaller the eigenratio \(R\) the more synchronizable the network.

Here we show that, as a function of \(\beta\), the eigenratio \(R\) has a global minimum at \(\beta = 1\). In large networks with some degree of randomness, the eigenratio at \(\beta = 1\) is primarily determined by the average degree \(k\) of the network and does not depend on the degree distribution and system size, in sharp contrast with the case of unweighted coupling (\(\beta = 0\)), where synchronization is strongly suppressed as the heterogeneity or number of oscillators is increased. Furthermore, we show that the total cost involved in the network coupling is significantly reduced for \(\beta = 1\) when compared to \(\beta = 0\). As a result, structural robustness \([\ref{footnote:11}]\) and improved synchronizability can coexist in scale-free and other heterogeneous networks.

In matrix notation, eq. (2) can be written as \(G = D^{-\beta}L\), where \(D = \text{diag}\{k_1, k_2, \ldots, k_N\}\) is the diagonal matrix of degrees. From the identity \(\det(D^{-\beta/2}L - \lambda I) = \det(D^{-\beta/2}LD^{-\beta/2} - \lambda I)\), valid for any \(\lambda\), we have that the spectrum of eigenvalues of matrix \(G\) is equal to the spectrum of a symmetric matrix defined as \(H = D^{-\beta/2}LD^{-\beta/2}\). As a result, all the eigenvalues of matrix \(G\) are real. Moreover, because \(H\) is positive semidefinite, all the eigenvalues are nonnegative and, because the rows of \(G\) have zero sum, the smallest eigenvalue \(\lambda_1\) is always zero, as assumed above. If the network is connected, \(\lambda_2 > 0\) for any finite \(\beta\). For spectral properties of unweighted networks, see refs. \([\ref{footnote:16}, \ref{footnote:17}, \ref{footnote:18}, \ref{footnote:19}]\).

We first examine the dependence on \(\beta\). Physically, we expect the synchronizability to be strongly influenced by the strength of the input coupling at each oscillator. When \(\beta < 1\), oscillators with larger degree are more strongly coupled than oscillators with smaller degree. When \(\beta > 1\), the opposite happens. Because \((\alpha_1,\alpha_2)\) is finite, for the network to synchronize, the overall coupling strength \(\sigma\) must be large enough to synchronize the least coupled oscillators and small enough to synchronize the most coupled ones (i.e., the synchronizability of these oscillators is expected to be primarily determined by the modes associated with the eigenvalues \(\lambda_2\) and \(\lambda_N\), respectively). Therefore, for both \(\beta < 1\) and \(\beta > 1\), some oscillators are more strongly coupled than others, and the ability of the network to synchronize is limited by those oscillators that are least and most strongly coupled. We then expect the network to achieve maximum synchronizability at \(\beta = 1\). In fig. \(\ref{fig:1}\), we show the numerical verification of this hypothesis on three different models of SFNs, defined as follows:

(A) Random SFNs \([\ref{footnote:20}]\) — Each node is assigned to have a number \(k_i\) of “half-links” according to the probability distribution \(P(k) \sim k^{-\gamma}\), where \(\gamma\) is a scaling exponent and \(k \geq k_{\text{min}}\). The network is generated by randomly connecting these half-links to form links, prohibiting self- and repeated links. In the limit \(\gamma = \infty\), all the nodes have the same degree \(k = k_{\text{min}}\).

(B) Networks with expected scale-free sequence \([\ref{footnote:18}]\) — The network is generated from a sequence \(k_1, k_2, \ldots, k_N\), where \(\max_i k_i^2 < \sum_i k_i\), so that links are independently assigned to each pair of nodes \((i, j)\) with probability \(p_{ij} = k_i k_j / \sum_i k_i\). When the expected degrees \(\bar{k}_i \geq k_{\text{min}}\) follow the distribution \(P(\bar{k}) \sim \bar{k}^{-\gamma}\), we have a network with expected scale-free sequence.

(C) Growing SFNs \([\ref{footnote:21}]\) — We start with a fully connected network with \(m\) nodes and at each time step a new node with \(m\) links is added to the network. Each new link is connected to a node \(i\) in the network with probability \(\Pi_i \sim (1 - p)k_i + p\), where \(0 \leq p \leq 1\) is a tunable parameter. For large degrees, the scaling exponent of the resulting network is \(\gamma = 3 + p/m(1 - p)^{-1}\). For \(p = 0\), we recover the Barabási–Albert model \([\ref{footnote:3}]\).

As shown in fig. \(\ref{fig:1}\) a pronounced minimum for the eigenratio \(R\) at \(\beta = 1\) is observed in each case. A similar minimum for \(R\) at \(\beta = 1\) is also observed in many other models of complex networks, including the Watts-Strogatz model \([\ref{footnote:22}]\) of SWNs. The only exception is the class of homogeneous networks, where all the nodes have the same degree \(k\). In this case, the weights can be factored out and \(R\) is independent of \(\beta\), as shown in fig. \(\ref{fig:1}\) a for random homogeneous networks with \(k = 10\) (solid line).

In heterogeneous networks, the synchronizability is significantly enhanced when the coupling is suitably weighted, as shown in fig. \(\ref{fig:2}\) for SFNs with \(\beta = 1\). In SFNs, the heterogeneity (variance) of the degree distribution increases as the scaling exponent \(\gamma\) is reduced. When the coupling is unweighted (\(\beta = 0\)), the eigenratio \(R\) increases with heterogeneity, but the eigenratio does not increase and may even decrease with heterogeneity when the coupling is weighted (\(\beta = 1\)), as shown in figs. \(\ref{fig:2}\) a-c. The enhancement is particularly large for small \(\gamma\), where the networks are highly heterogeneous (note the logarithmic scale in fig. \(\ref{fig:2}\)). The networks become more homogeneous as \(\gamma\) is increased. In the limit \(\gamma = \infty\), random SFNs converge to random homogeneous networks with the same degree \(k_{\text{min}}\) for all nodes [fig. \(\ref{fig:2}\) a], while networks with expected scale-free sequence converge to Erdős-Rényi random networks \([\ref{footnote:23}]\), which have links assigned with the same probability between each pair of nodes [fig. \(\ref{fig:2}\) b]. As one can see from fig. \(\ref{fig:2}\) b, the synchronizability is strongly enhanced even in the relatively homogeneous Erdős-Rényi model; such an enhancement occurs also in growing networks [fig. \(\ref{fig:2}\) c]. Surprisingly, for \(\beta = 1\), the eigenratio \(R\) turns out to be well approximated by the corresponding eigenratio of random homogeneous networks with the same average degree [figs. \(\ref{fig:2}\) a-
The eigenratio $R$ for random SFNs with the same average degree $\bar{k}$, as defined in the model (B) above, increases with increasing clustering and assortativity in correlated networks. However, a pronounced global minimum for the eigenratio is bounded by a function of the average degree, which does not depend on the system size, in agreement with the results in figs. 2(d-f). Moreover, we expect $R$ to approach the upper bound in eq. (4) because the semicircle law holds and the spectrum is symmetric around 1 for $\bar{k}_{\text{min}} \gg \sqrt{\bar{k}}$ in the thermodynamical limit $\beta \gg 1$. This prediction is confirmed numerically under much weaker conditions, as shown in figs. 2(a-c), where one can see a remarkable agreement between the approximate and exact values of $R$ for all three models of SFNs. Since the bound in eq. (4) does not depend on the degree distribution, this result also explains the agreement between the eigenratio for weighted SFNs [figs. 2(a-c), ] and the eigenratio for random homogeneous networks with the same average degree [figs. 2(a-c), ]. A similar agreement is observed in many other complex networks.

It is also important to study the influence of degree correlation and clustering on the synchronizability of the networks. Our extensive numerical computation on the models of refs. [23][24] shows that the eigenratio $R$ generally increases with increasing clustering and assortativity in correlated networks. However, a pronounced global minimum for $R$ as a function of $\beta$ is always observed at $\beta = 1$. In addition, weighted networks at $\beta = 1$ are much more insensitive to the effects of correlation than their unweighted counterparts. The same tendency is observed in the growing model with aging of ref. [25], which has nontrivial clustering and correlation. All together, these suggest that our results are quite robust and expected to hold on real-world networks as well.

Now we address the important problem of the cost involved in the connections of the network. The cost $C$ is naturally defined as the total strength of all the directed links, i.e., $C = \sigma_{\text{min}} \sum_i k_i^{1-\beta}$, where $\sigma_{\text{min}} = \alpha_1/\lambda_2$ is the minimum overall coupling strength for the network to synchronize.

**FIG. 2:** (a-c) Eigenratio $R$ as a function of the scaling exponent $\gamma$: (a) random SFNs, (b) networks with expected scale-free sequence, and (c) growing SFNs, for $\beta = 1$ (•) and $\beta = 0$ (◦). The other parameters are the same as in fig. [1] Also plotted are the bound of eq. (4) (solid line) and $R$ at $\gamma = \infty$ for $\beta = 1$ (dashed line) and $\beta = 0$ (dot-dashed line). The • symbols correspond to random homogeneous networks with the same average degree of the corresponding SFNs, as indicated in the figure. (d-f) $R$ as a function of the system size for $\gamma = 3$ and the models in (a-c), respectively. The legend is the same as in (a-c).

The system size for $\beta = 1$ in large SFNs, in contrast to the unweighted case, where $R$ increases strongly with the number of oscillators [figs. 2(d-f)]. Moreover, the eigenratio $R$ appears to be independent of the system size for $\beta = 1$ in large SFNs, in contrast to the unweighted case, where $R$ increases strongly with the number of oscillators [figs. 2(d-f)].

We now present an approximation for the eigenratio $R$ that supports and extends our numerical observations. In what follows we focus on the case $\beta = 1$. Based on results of ref. [18] for random networks with arbitrary expected degrees, which includes important SFNs, we get

$$\max \{1 - \lambda_2, \lambda_N - 1\} = [1 + o(1)] \frac{2}{\sqrt{k}},$$

(3)

where $\bar{k}$ is the average expected degree. This result is rigorous for networks with a given expected degree sequence $\bar{k}_1, \bar{k}_2, \ldots, \bar{k}_N$, as defined in the model (B) above. The assumption for this result is $\bar{k}_{\text{min}} \equiv \min \bar{k}_i$ to be large as compared to $\sqrt{k} \ln^3 N$, but our numerical simulations suggest that this assumption can be released considerably because eq. (3) is observed to hold for $\bar{k}_{\text{min}}$ as small as $2\sqrt{k}$. Having released this assumption, from eq. (3) we have the following explicit upper bound for the eigenratio in large networks:

$$R \leq \frac{1 + 2/\sqrt{k}}{1 - 2/\sqrt{k}}.$$  

(4)

Therefore, the eigenratio is bounded by a function of the average degree, which does not depend on the system size, in agreement with the results in figs. 2(d-f). Moreover, we expect $R$ to approach the upper bound in eq. (4) because the semicircle law holds and the spectrum is symmetric around 1 for $\bar{k}_{\text{min}} \gg \sqrt{\bar{k}}$ in the thermodynamical limit $\beta \gg 1$. This prediction is confirmed numerically under much weaker conditions, as shown in figs. 2(a-c), where one can see a remarkable agreement between the approximate and exact values of $R$ for all three models of SFNs. Since the bound in eq. (4) does not depend on the degree distribution, this result also explains the agreement between the eigenratio for weighted SFNs [figs. 2(a-c), •] and the eigenratio for random homogeneous networks with the same average degree [figs. 2(a-c), ◦]. A similar agreement is observed in many other complex networks.

**FIG. 3:** Normalized cost as a function of the scaling exponent $\gamma$ for random SFNs with $\beta = 1$ (•) and $\beta = 0$ (◦), and for random homogeneous networks with the same average degree (○). The solid line corresponds to $\gamma = \infty$. Inset: Ratio $r = C_0/C_1$ of the cost for $\beta = 0$ ($C_0$) and $\beta = 1$ ($C_1$) as a function of $\gamma$. The other parameters are the same as in fig. [1]

\[ R \leq \frac{1 + 2/\sqrt{k}}{1 - 2/\sqrt{k}}. \]
Strikingly, in heterogeneous networks, the cost for $\beta = 1$ is considerably smaller than the cost for $\beta = 0$ [fig. 3]. The cost for $\beta = 1$ is well approximated by the cost for random homogeneous networks with the same average degree $k$, as indicated in fig. 3. In this case, from above we have $C/(N\alpha_1) = 1/\beta_2 \approx 1/(1 - 2/\sqrt{k})$, and the cost is reduced as $k$ is increased, approaching $C/(N\alpha_1) = 1$ for large globally coupled networks. Therefore, cost reduction is another important advantage of the weighted coupling.

In summary, we have introduced a model of directed networks with weighted coupling which, we believe, can serve as a paradigm to address various issues regarding dynamics on complex networks. Within this model, we have shown that suitably weighted networks display significantly improved synchronizability and lower cost. An important implication of our findings is that weighted SFNs can exhibit enhanced complete synchronization. We expect our results to be relevant for both network design and the understanding of dynamics in natural systems, such as synchronization in neural networks and synchronization of epidemic outbreaks in networks of cities [27], where weighted couplings are expected to play an important role [28].

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