James-Stein Type Estimators Shrinking towards Projection Vector When the Norm is Restricted to an Interval

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Abstract

Consider the problem of estimating a $p \times 1$ mean vector $\mathbf{\theta} (p - q \geq 3)$, with a projection matrix $P$, under the quadratic loss, based on a sample $X_1, X_2, \cdots, X_n$. We find a James-Stein type decision rule which shrinks towards projection vector when the underlying distribution is of a variance mixture of normals and when the norm $\| \mathbf{\theta} - P_{\mathbf{\ell}} \mathbf{\theta} \|$ is restricted to a known interval, where $P_{\mathbf{\ell}}$ is an idempotent and projection matrix and $\text{rank}(P_{\mathbf{\ell}}) = q$. In this case, we characterize a minimal complete class within the class of James-Stein type decision rules. We also characterize the subclass of James-Stein type decision rules that dominate the sample mean.

Keywords: James-Stein Type Decision Rule, Mean Vector, Quadratic Loss, Underlying Distribution

1. Introduction

The problem considered is that of estimating with quadratic loss function the mean vector of a compound multinormal distribution when the norm $\| \mathbf{\theta} - P_{\mathbf{\ell}} \mathbf{\theta} \|$ is restricted to a known interval. The class of estimation rules considered will consist of Lindley type estimators only. Such a class was introduced by James-Stein[1] and Lindley[2] in order to prove that some of its members dominate the sample mean in the multinormal case. Strawderman[3] also derived a similar result for the more general case considered in this paper of a compound multinormal distribution. The problem of estimation of a mean under constraint has an old origin and recently focussed again in the context of curved model in the works of Amari[4], Kariya[5], Perron and Giri[6], Merchand and Giri[7], and Baek[8] among others. A study of compound multinormal distributions and the estimation of their location vectors was carried out by Berger[9].

In section 2, we present the general setting of our problem and develop necessary notations. In section 3, we examine the estimation problem based on a Lindley type decision rule when the norm $\| \mathbf{\theta} - P_{\mathbf{\ell}} \mathbf{\theta} \|$ is restricted to a known interval. In this case, we give to the subclass of Lindley type estimators which dominate the sample mean when the norm is restricted to a known interval.

2. Notation and Preliminaries

Let $\mathbf{x} = (x_1, \cdots, x_p)'$, $p - q \geq 3$, be an observation from a compound multinormal distribution with unknown location parameter $\mathbf{\theta} (p \times 1)$ and mixture parameter $H(\cdot)$, where $H(\cdot)$ represents a known c.d.f defined on the interval $(0, \infty)$. In other words, we assume that the random variable $X$ generating our observation $\mathbf{x}$ admits the representation,

$$L(\mathbf{x} | \mathbf{z} = z) = N_{p'}(\mathbf{\theta}, z\mathbf{L}) \forall z > 0$$  \hspace{1cm} (2.1)

where $\mathbf{z}$ being the positive random variable with c.d.f. $H(\cdot)$. Our problem concerns the estimation of the location parameter $\mathbf{\theta}$ with loss function.

$$L(\mathbf{\theta}, \delta(\mathbf{x})) = (\mathbf{\delta}(\mathbf{x}) - \mathbf{\theta})'(\mathbf{\delta}(\mathbf{x}) - \mathbf{\theta})$$, \hspace{1cm} with \hspace{1cm} $\mathbf{\theta} \in \Theta_{\lambda_{\mathbf{\theta}}} = \{ \theta \in \mathbb{R}^p | \theta - P_{\mathbf{\ell}} \theta \in [\lambda_1, \lambda_2], 0 \leq \lambda_1 \leq \lambda_2 \leq \infty \}$ where $P_{\mathbf{\ell}}$ is an idempotent and projection matrix with $\text{rank}(P_{\mathbf{\ell}}) = q$ and the decision rule $\delta, \delta(\cdot) : \mathbb{R}^p \rightarrow \mathbb{R}^p$, is of the form
\[ \delta(x) = P_{\theta}x + \left(1 - \frac{c}{(x - P_{\theta}x) (x - P_{\theta}x)}\right)(x - P_{\theta}x), \]

where the parameter space is of the form \( \Theta = \{\theta \mid \|\theta - P_{\theta}\| = \lambda, \lambda > 0\} \). Then under the assumptions \( \theta \in \Theta_{\lambda} \), \( \lambda > 0 \), we can show that

\[ R(\theta, \delta) = E_{\theta}[[\delta(x) - \delta(x')][\delta(x) - \delta(x')]] \]

where \( \theta \in \Theta_{\lambda} \) and, using the method by Baek[8],

\[ \text{The natural estimator } \delta^*(\lambda) = \lambda \text{ is a member of the Lindley class and has a constant risk function equal to } \lambda. \]

using the expression (2.5), we can verify that the Lindley type estimator \( \delta^* \) dominates the natural estimator \( \delta^* \) if and only if \( 0 < c < 2 < c^*(\lambda) \) for the Lindley type estimator \( \delta^* \) if and only if \( 0 < c < 2 < c^*(\lambda) \) for \( \theta \in \Theta_{\lambda} \).

3. Estimation when the Norm is Restricted to an Interval

In this section, we study the case where the mean \( \theta \) is restricted to a known interval \( [\lambda_1, \lambda_2] \), no optimal Lindley type decision rule will exist whenever \( \lambda_1 \leq \lambda_2 \) (but see the discussion following Corollary 3.7 for asymptotic considerations). We can also characterize the subclass of Lindley type decision rules that dominate the natural estimator \( \delta^* \) when \( \theta \in \Theta_{\lambda_1} \). In the following, we will denote \( c^* \) by an interval \( [\lambda_1, \lambda_2] \) of \( \lambda \) and \( c^* \) as an interval \( [\lambda_1, \lambda_2] \) of \( \lambda \)

**Theorem 3.1** Let \( x \) be a single observation from a p-dimensional location parameter with p.d.f. of the form given by expression (2.1). Under the assumptions

\[ \theta \in \Theta_{\lambda}, \quad 0 \leq \lambda_1 \leq \lambda_2 < \infty; \quad p - q \geq 3 \]

and

\[ E[Z] < \infty, \]

then

\[ R(\theta, \delta) = E_{\theta}[[\delta(x) - \delta(x')][\delta(x) - \delta(x')]] \]

and its risk is

\[ \lambda^2 \int_{[\lambda_1, \lambda_2]} \left(1 - \frac{c}{(x - P_{\theta}x) (x - P_{\theta}x)}\right)(x - P_{\theta}x), \]
$E(Z) < \infty$, (a) the subclass
\[
\{ \delta \in \mathcal{D}_{\text{ind}} | c \leq c' \leq \bar{c}[\lambda_1, \lambda_2] \}
\]
is a minimal complete class within the class $\mathcal{D}_{\text{ind}}$ and (b) the decision rule $\delta'$ will be the natural estimator $\delta'$ if
\[
0 < c < c'[\lambda_1, \lambda_2].
\]

**Proof.** (a) Let $c_0$ be a real number such that
\[
c_0 \leq \frac{c}{c}[\lambda_1, \lambda_2].
\]
Then, using expression (2.6), if $c_0 \leq c'[\lambda_1, \lambda_2]$, we may write the difference in risks
\[
R(\theta, \delta') - R(\theta, \delta'_{c_0}[\lambda_1, \lambda_2])
\]
\[
= \left[ R(\theta, \delta') - R(\theta, \delta'_{c_0}[\lambda_1, \lambda_2]) \right] - \left[ R(\theta, \delta'_{c_0}[\lambda_1, \lambda_2]) - R(\theta, \delta'_{c_0}([\theta - P_1 \theta])) \right]
\]
\[
= \left\{ c_0 - c' \| \theta - P_1 \theta \| \right\}^2 - c'[\lambda_1, \lambda_2] - c' \| \theta - P_1 \theta \| \right\}^2
\]
this last expression being positive for all $\theta \in \Theta_{\lambda_1}$ given that $c_0 \leq c'[\lambda_1, \lambda_2]$. In the same manner, the decision rule $\delta$ with $c = c'[\lambda_1, \lambda_2]$ will dominate the decision rule $\delta'$ if $c_0 > c'[\lambda_1, \lambda_2]$, the intermediate value theorem ($c'(\lambda)$ is easily shown to be continuous) assures us that $R(\theta, \delta') - R(\theta, \delta') \geq 0$, $\forall c \neq c_0$, when $c'(\| \theta - P_1 \theta \| ) = c_0$. These last results guarantee that all the rules $\delta$ with $c \leq c'[\lambda_1, \lambda_2]$, $c^* \leq c'[\lambda_1, \lambda_2]$ are inadmissible within the class $\mathcal{D}_{\text{ind}}$ and the rules $\delta$ with $c$ belonging to the interval $\left[ c'[\lambda_1, \lambda_2], c^* \right]$ cannot be improved upon by another rule of the class $\mathcal{D}_{\text{ind}}$. Thus, the result of part (a) follows.

(b) Similar to last part in Section 2, the decision rule $\delta$ will dominate the decision rule $\delta'$ if
\[
R(\theta, \delta) < R(\theta, \delta'), \quad \forall \theta \in \Theta_{\lambda_1}
\]
\[
\Leftrightarrow 0 < c < 2c'[\| \theta - P_1 \theta \| ],
\]
\[
\Leftrightarrow 0 < c < 2c'[\lambda_1, \lambda_2]
\]

It may also be remarked that the rules $\delta$ with $c = 2c'[\lambda_1, \lambda_2]$ also dominate $\delta'$ under the conditions of the theorem when $\lambda_1 < \lambda_2$ and that all the decisions rules $\delta$ with $c > 2c'[\lambda_1, \lambda_2]$ do not dominate $\delta'$ under the conditions of the theorem. The results above would be more explicit if the function $c'[\lambda_1, \lambda_2] = c'(\lambda_1)$ and $c^*[\lambda_1, \lambda_2] = c'(\lambda_2)$.

The case with no restrictions on the norm $\| \theta - P_1 \theta \|$ (i.e., $\lambda_1 = 0$ and $\lambda_2 = \infty$) can be expanded using by Strawderman’s result[3] and it can be shown that the decision rules $\delta$ with $0 \leq c \leq 2(p - q - 2)(Z^{-1})$ are minimax rules by showing that their risk functions are uniformly less than or equal to the risk function $(= pE(Z))$ of the minimax decision rule $\delta$. This result is derived below as a particular case of Theorem 3.1. To do so, we need to determine the quantity $c^*[0, \infty]$. The following three Lemmas will prove useful in determining $c^*[0, \infty]$ and, also, $c'[\lambda_1, \lambda_2]$.

**Lemma 3.2.** Let $X$ be an arbitrary random variable and let $f$ and $g$ be two real nondecreasing functions on the support of $X$. Then, if the quantities $E[f(x)]$ and $E[g(x)]$ exist, $Cov(f(x), g(x)) \geq 0$ with the inequality being strict if $f$ and $g$ are strictly increasing and $X$ is nondegenerate.

**Proof.** A neat proof of Lemma 3.2. is given by Chow and Wang[10].

**Lemma 3.3.** Let $L$ be a Poisson random variable with mean $\gamma(>0)$ and $f_p'(\gamma) = E'[(p - q + 2L - 2)^{-1}]$, $p \geq 4$ then
\[
(i) \quad f_{p - q}(\gamma) = e^{-\gamma} \int_{[0,\gamma]} \gamma^{p - q - 3} e^\gamma'$ \quad \text{and}\quad (ii) \quad f_{p - q + 2}(\gamma) = (2\gamma)^{-1} [1 - (p - q - 2)f_{p - q}'(\gamma)]
\]

**Proof.** We can prove this lemma using the method by Egerton and Laycock[11].

**Lemma 3.4.** Let $f_p'(\gamma)$, $p \geq 4$ be a function defined on $[0, \infty]$ and equal to $f_p'(\gamma) = E'[(p - q + 2L - 2)^{-1}]$, $\gamma \geq 0$, where $L$ is a Poisson random variable with mean $\gamma$. Then,
\[
(i) \quad f_{p - q}'(\gamma) \quad \text{is a strictly decreasing function},
\]

J. Chosun Natural Sci., Vol. 10, No. 1, 2017
(ii) \( \lim_{\gamma \to -0} f_{p-q}^\gamma (\gamma) = (p-q-2)^{-1} \), \( \lim_{\gamma \to -0} f_{p-q}^{-\gamma} (\gamma) = 0 \)

(iii) if \( p \geq 5 \), \( \gamma f_{p-q}^\gamma (\gamma) \) is strictly increasing function for \( \gamma \geq 0 \).

Proof. (i) Using part (i) of Lemma 3.3, we have for \( \gamma_2 > \gamma_1 > 0 \),
\[
\int_{[\gamma_1, \gamma_2]} e^{-(p-q-3)\gamma} d\gamma = \int_{[\gamma_1, \gamma_2]} e^{-(p-q-2)\gamma} d\gamma < 0.
\]

(ii) By the dominated convergence theorem,
\[
\lim_{\gamma \to -0} f_{p-q}^\gamma (\gamma) = \lim_{\gamma \to -0} \int_{[0, 1]} e^{-(p-q-3)(\gamma)} d\gamma = \int_{[0, 1]} e^{-(p-q-3)(\gamma)} d\gamma = (p-q-2)^{-1}
\]

and
\[
\lim_{\gamma \to -0} f_{p-q}^{-\gamma} (\gamma) = \lim_{\gamma \to -0} \int_{[0, 1]} e^{-(p-q-3)(\gamma)} d\gamma = \int_{[0, 1]} e^{-(p-q-3)(\gamma)} d\gamma = 0.
\]

(iii) Using Lemma 3.3, we have \( \gamma f_{p-q}^\gamma (\gamma) = \frac{1}{2} (1-e^{-\gamma}) \), which is easily seen to be strictly increasing. For \( p \geq 6 \) we obtain by the recurrence formula given by expression (3.1),
\[
\gamma f_{p-q}^\gamma (\gamma) = \frac{1}{2} (1-(p-q-4)f_{p-q-2}^{-\gamma} (\gamma)), \gamma > 0.
\]

which must be strictly increasing given that function \( f_{p-q-2}^{-\gamma} (\cdot) \) is strictly decreasing by part (i).

In the following, we will set \( E^{-1}[Z^{-1}] \) equal to zero if the expectation \( E[Z^{-1}] = \infty \).

**Theorem 3.5.** The function \( \hat{c} (\cdot) \) defined by expression (2.4) satisfies the following properties:

(a) \( \infty \hat{c} (\lambda) = (p-q-2) E[Z^{-1}] \lambda \geq 0 \)

(b) \( \hat{c} (\gamma) = k \Rightarrow Z \) is constant with probability one and,

(c) for \( p \geq 5 \),

Proof. (a) Expression (2.4) can be rewritten as
\[
\hat{c} (\lambda) = (p-q-2) E[Z^{-1}] f_{p-q}^{-\gamma} (\lambda, Z), \lambda \geq 0.
\]

By applying Lemma 3.2 to the functions \( f_{p-q} (\lambda, Z) \) and \( Z^{-1} \), the function \( f_{p-q}^{-\gamma} (\lambda, Z) \) being an increasing function by part (i) of Lemma 3.4, we have for \( \lambda \geq 0 \),
\[
\text{Cov} (f_{p-q}^{-\gamma} (\lambda, Z), -Z^{-1}) \geq 0
\]

\[
\Rightarrow E[Z^{-1} f_{p-q}^{-\gamma} (\lambda, Z)] \geq E[Z^{-1}] E[f_{p-q}^{-\gamma} (\lambda, Z)]
\]

\[
\Rightarrow \hat{c} (\lambda) \geq (p-q-2) E^{-1}[Z^{-1}]
\]

\[
\Rightarrow \lambda \geq 0 \hat{c} (\lambda) \geq (p-q-2) E^{-1}[Z^{-1}]
\]

The reverse inequality is obtained by observing that \( \hat{c} (0) = (p-q-2) E^{-1}[Z^{-1}] \).

(b) The constancy of \( \hat{c} (\lambda) \) implies
\[
\hat{c} (\lambda) = k = \hat{c} (0) = (p-q-2) E^{-1}[Z^{-1}]
\]

\( \forall \lambda > 0 \),

and \( \int_{(0, \infty)} \left[ p-q-1 - \frac{k}{z} \right] f_{p-q}^{-\gamma} (\lambda, z) dH(z) = 0 \)

Since both \( f_{p-q} (\lambda, Z) \) and \( -kz^{-1} \) are strictly increasing function of \( z \), we have by Lemma 3.2, for nondegenerate \( Z \),
\[
\text{Cov} (f_{p-q} (\lambda, Z), p-q-2-kz^{-1}) \geq 0
\]

\( \Rightarrow E[p-q-2-kz^{-1}] f_{p-q} (\lambda, Z) > E[p-q-2-kz^{-1}] E[f_{p-q} (\lambda, Z)] = 0 \)

which results in a contradiction implying \( Z \) is constant with probability one.

(c) By applying Lemma 3.2 to the functions \( -z^{-1} f_{p-q} (\lambda, Z) \) and \( z \), the function \( -z^{-1} f_{p-q} (\lambda, Z) \) being an increasing function by virtue of part (iii) of Lemma 3.4, we have for \( p \geq 5 \) and \( \lambda \geq 0 \),
\[
\text{Cov} (-Z^{-1} f_{p-q} (\lambda, Z), Z) \geq 0
\]

\( \Rightarrow E[Z^{-1} f_{p-q} (\lambda, Z)] \leq E[Z^{-1}] E[f_{p-q} (\lambda, Z)] E[Z]
\]

\( \Rightarrow \hat{c} (\lambda) \leq (p-q-2) E[Z]
\]

\( \Rightarrow \sup_{\lambda \geq 0} \hat{c} (\lambda) \leq (p-q-2) E[Z] \)
The reverse inequality is obtained by verifying that
\[
\lim_{\lambda \to 0} c^*(\lambda) = (p-q-2)E[Z] \quad \text{whenever } p \geq 5. \quad \text{To do so, it will be useful to express the function } c^*(\cdot) \quad \text{in the following way,}
\]
\[
c^*(\lambda) = (p-q-2) \int_{(0,\infty)^{p-q}} \frac{e^{\frac{\lambda}{2} \sum z^2}}{p-q+2\lambda} z \, dH(z)
\]
\[
\int_{(0,\infty)^{p-q}} \frac{e^{\frac{\lambda}{2} \sum z^2}}{p-q+2\lambda} \, dH(z)
\]
\[
\lambda > 0.
\]
Moreover, we can write
\[
\lim_{\lambda \to 0} c^*(\lambda) = (p-q-2) \int_{(0,\infty)^{p-q}} \frac{e^{\frac{\lambda}{2} \sum z^2}}{p-q+2\lambda} \, dH(z)
\]
\[
\lim_{\lambda \to 0} \int_{(0,\infty)^{p-q}} \frac{e^{\frac{\lambda}{2} \sum z^2}}{p-q+2\lambda} \, dH(z)
\]
if both limits exist and the denominator is not equal to zero. By the dominated convergence theorem, we can then write \( \lim_{\lambda \to 0} c^*(\lambda) \) as
\[
(p-q-2) \int_{(0,\infty)^{p-q}} \frac{2L}{p-q+2L-4} 1_{\{1,2,\ldots,L\}}(L) \, dH(z)
\]
\[
\lim_{\lambda \to 0} \int_{(0,\infty)^{p-q}} \frac{2L}{p-q+2L-4} 1_{\{1,2,\ldots,L\}}(L) \, dH(z)
\]
where, for \( z > 0, L \) is a Poisson random variable with mean \( \lambda^2/2z \). Finally by noting that,
\[
\forall z > 0, \lim_{\lambda \to 0} \int_{(0,\infty)^{p-q}} \frac{2L}{p-q+2L-4} 1_{\{1,2,\ldots,L\}}(L) = 1
\]

because the integrand tends to one when \( L_1 \to \infty \) we obtain
\[
\lim_{\lambda \to 0} c^*(\lambda) = (p-q-2) \int_{(0,\infty)^{p-q}} \, dH(z) = (p-q-2)E[Z]
\]

Having evaluated the quantities \( e^* [0,\infty] \) and \( e^* [0,\infty] \), and Theorem 3.1 yields the following result.

Corollary 3.6. Let \( x \) be a single observation from a \( p \)-dimensional location parameter family with p.d.f. of the form given by expression (2.1), with \( p-q \geq 3 \), and under the assumption \( \theta \equiv R\theta \) and \( E[Z] < \infty \),

(a) the subclass
\[
\{\theta \in D_{\text{lin}} \mid (p-q-2)E[Z] \leq c \leq (p-q-2)E[Z] \}
\]
for \( p-q \geq 4 \),

(b) the decision rule \( \hat{\delta} \) will dominate the decision rule \( \delta^* \) if \( 0 < c < 2(p-q-2)E[Z^2] \).

Proof. These results above are a direct application of Theorem 3.1 and 3.5. We pursue with some remarks.

Remark 3.1. Under the conditions of Corollary 3.6, the decision rule \( \hat{\delta} \) is a minimax rule if and only if \( 0 \leq c \leq 2(p-q-2)E[Z^2] \). This condition can also be obtained using part (a) of Theorem 3.5 and similar to last part in Section 2 which, under the same conditions, would specify that
\[
R(\theta ; \hat{\delta}) \leq p \quad \text{if } p \quad \text{if } 0 \leq c \leq 2c^* \left( \| \theta - P_v \theta \| \right) .
\]

It is interesting to note that the natural estimator \( \hat{\delta} \) represents the only minimax rule within the class \( D_{\text{lin}} \) when the quantity \( E[Z^2] \) does not exist.

Remark 3.2. The results above of Theorem 3.1 and Corollary 3.6 can be extended to the case where the experimental information consist of a sample \( X_1, \ldots, X_n \), with p.d.f. of the form in (2.1) and the class of decision rules considered consists of the decision rules of the form
\[
\delta(\lambda_1, \ldots, \lambda_n), \, \lambda_1, \ldots, \lambda_n \in R
\]
\[
= P_{\lambda} \bar{X} + \left[ 1 - \frac{c}{\left( \bar{X} - P_{\lambda} \bar{X} \right)^2} \right] (\bar{X} - P_{\lambda} \bar{X})
\]
where \( \bar{X} \) is the sample mean and \( P_{\lambda} \) is an idempotent and projection matrix. This can be seen by nothing that the probability law of sample mean \( \bar{X} = n^{-1} \sum_i X_i \); \( X_1, \ldots, X_n \) being \( n \) independently and identically distributed random vectors admitting the representations.

\[
L(\theta \mid z_i) = N_p(\theta, z_i) \quad j = 1, \ldots, n
\]

for all values \( z_1, \ldots, z_n \) of \( n \) independent copies \( Z_1, \ldots, Z_n \) of a positive random variable \( Z \); admits the representation.
for $p - q \geq 4$, the subclass

$$\mathcal{S} \subseteq D_{\mathcal{N}} \mid n^{-1}(p - q - 2) E^{-1}
\left(\left(\sum_{i=1}^{n} Z_i\right)^{-1}\right) \leq c \leq n^{-1}(p - q - 2) E[Z]$$

is a minimal complete class with the class $D_{\mathcal{N}}$ and
(b) the decision rule $\mathcal{S}$ will dominate the sample mean

\[ 0 < c < 2n^{-2}(p - q - 2) E^{-1}\left(\left(\sum_{i=1}^{n} Z_i\right)^{-1}\right) \]

**Proof.** These results are a direct application of Corollary 3.6 and the discussion above expression (3.2).

However, the results concerning the minimax criteria given by Strawerman cannot be applied to the decision rules $\mathcal{S}(z)$ since the statistic $\bar{X}$ does not represent in general a sufficient statistic (the multinormal case being a well known exception). Finally it is interesting to note that,

$$E^{-1}\left(\left(\sum_{i=1}^{n} Z_i\right)^{-1}\right) = E\left(\sum_{i=1}^{n} Z_i\right) = nE[Z],$$

(the above inequality can be seen as a consequence of Lemma 3.2), implying that the interval

$$\left(0, 2n^{-2}(p - q - 2) E^{-1}\left(\left(\sum_{i=1}^{n} Z_i\right)^{-1}\right) \right) \rightarrow \varnothing \text{ as } n \rightarrow \infty$$

which, by expression (3.3), indicates that the subclass of Lindley type decision rules dominating the sample mean can be made arbitrarily small by increasing the sample size $n$.

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