Approximations of Kolmogorov Complexity

Samuel Epstein
samepst@icloud.com
January 30, 2020

Abstract

In this paper we show that the approximating the Kolmogorov complexity of a set of numbers is equivalent to having common information with the halting sequence. The more precise the approximations are, and the greater the number of approximations, the more information is shared with the halting sequence. An encoding of the $2^N$ unique numbers and their Kolmogorov complexities contains at least $\gtrsim N$ mutual information with the halting sequence. We also provide a generalization of the “Sets have Simple Members” theorem to conditional complexity.

1 Introduction

The Kolmogorov complexity of a string $x$, $K(x)$, is the size of the smallest program that outputs $x$ with respect to a universal prefix-free program. It is a well known fact that Kolmogorov complexity $K$ is uncomputable (see [Kol65] and [Sol64]). In fact any computable function $f : \mathbb{N} \to \mathbb{N}$ that is not greater than $K$ is bounded by a constant. This is because for each $n \in \mathbb{N}$, you can find an $x_n$ such that $f(x_n) > n$. Thus $x_n$ can be identified with $f$ and $n$, so $K(x_n) < O(\log n)$. However since $f \leq K$, we have $n < K(x_n)$, thus causing a contradiction for large enough $n$.

The authors in [BFNV05], using expanding graphs, introduced an algorithm that when given a non-random string, outputs a small list of strings of the same length containing a string with higher complexity. In [Zim16], an algorithm was presented that when given a non-random string, outputs a large list of strings of the same length where 99% of the outputted strings have higher complexity. Given a universal machine $U$, a $c$-short program for $x$ is a string $p$ such that $U(p) = x$ and the length of $p$ is bounded by $c + K(x)$. The authors in [BMVZ13] showed that there exists a computable function that maps every $x$ to a list of size $|x|^2$ containing a $O(1)$-short program for $x$.

In this paper, we show that the approximate knowledge about the Kolmogorov complexity of a finite number of strings is equivalent to sharing a certain amount of information with the halting sequence. The more strings in the collection and the better their approximation to $K$, the more information this collection has with the halting sequence. The mutual information between an encoding of $2^N$ unique numbers alongside their Kolmogorov complexity and the halting sequence is at least $\gtrsim N$. Due to information non-growth laws, there is no (randomized) algorithmic means to produce information with the halting sequence.

We also provide a generalization of “Sets Have Simple Members theorem, first seen in [EL11], to conditional Kolmogorov complexity and conditional algorithmic probability. The theorem states that the minimum conditional complexity, over pairs specified by an binary relation is less than the negative log of the combined conditional algorithmic probability of all pairs in the enumeration.

2 Conventions

We use $\mathbb{N}$, $\mathbb{Q}$, $\mathbb{R}$, $\Sigma$, $\Sigma^*$, $\Sigma^\infty$ to denote natural numbers, rational numbers, real numbers, bits $\{0,1\}$, finite strings, and infinite sequences. We use $X_{>0}$ and $X_{\geq 0}$ to denote the positive and non-negative
elements of set $X$. The $i$th bit a string $x$ is $x[i]$. For string $x \in \Sigma^*$, $x0^- = x1^- = x$. The length of a string $x$ is $|x|$. The size of a set $D \subseteq \Sigma^*$ is $|D|$. For $x \in \Sigma^*$ and $y \in \Sigma^* \cup \Sigma^\infty$, we use $x \subseteq y$ iff $x = y$ or there is some string $z \in \Sigma^* \cup \Sigma^\infty$ where $xz = y$. We say $x \sqsupset y$ iff $x \neq y$ and $x \subseteq y$. The self delimiting code of a string $x \in \Sigma^*$ is $\langle x \rangle = 1|\#|0x$. The encoding of (a possibly ordered) set $\{x_1, \ldots, x_m\} \subset \Sigma^*$, is $(m\langle x_1 \rangle \ldots \langle x_m \rangle)$.

A (discrete) measure $Q$ is a function $Q : \Sigma^* \to \mathbb{R}_{\geq 0}$. Measure $Q$ is a semi-measure iff $\sum_x Q(x) \leq 1$. Measure $Q$ is a probability measure iff $\sum_x Q(x) = 1$. The support of a measure $Q$ is $\text{supp}(Q) = \{x : Q(x) > 0\}$. Probability measure $Q$ is elementary if $|\text{supp}(Q)| < \infty$ and $\text{Range}(Q) \subset \mathbb{R}_{\geq 0}$. Elementary probability measures $Q$ with $\{x_1, \ldots, x_m\} = \text{supp}(Q)$ are encoded by finite strings of the $\langle Q \rangle = \langle\{x_1, Q(x_1), \ldots, x_m, Q(w_m)\}\rangle$. For semi-measure $Q$, we say function $d : \Sigma^* \to \mathbb{R}_{\geq 0}$ is a $Q$ test, iff $\sum_x Q(x)2^{d(x)} \leq 1$.

For nonnegative real function $f$, we use $\leq^+ f$, $\geq^+ f$, $=^+ f$ to denote $< f + O(1)$, $> f - O(1)$, and $= f + O(1)$. We also use $\leq^{\text{log}} f$ and $\geq^{\text{log}} f$ to denote $< f + O(\log(f + 1))$ and $> f - O(\log(f + 1))$.

We use algorithms $T_\alpha(x)$ on input programs $x \in \Sigma^*$ and auxilliary inputs $\alpha \in \Sigma^* \cup \Sigma^\infty$. $T$ is a prefix free algorithm if for all $\alpha \in \Sigma^* \cup \Sigma^\infty$ and $x, s \in \Sigma^*$, $s \neq \emptyset$, $T_\alpha(x)$ does not halt or $T_\alpha(xs)$ does not halt. There exists a universal prefix algorithm $U$ where for all prefix algorithm $T$ there exists a $t \in \Sigma^*$, where for all $x \in \Sigma^*$ and $\alpha \in \Sigma^* \cup \Sigma^\infty$, $U_\alpha(tx) = T_\alpha(x)$. As is standard, we define Kolmogorov complexity with respect to $U$, with for $x, y \in \Sigma^*$, $K(x|y) = \min\{||p|| : U_y(p) = x\}$. The universal probability $m$ is defined as $m(x|y) = \sum\{2^{-||p||} : U_y(p) = x\}$. By the coding theorem $K(x|y) = +^+ K(x) + K(y|x, K(x))$.

Let $F : D \to \mathbb{N}$ be a function with a finite domain $D \subseteq \Sigma^*$, $|D| < \infty$. Then $\langle F \rangle$, where $D = \{x_1, \ldots, x_m\}$, is $\langle\{x_1, F(x_1), \ldots, x_m, F(x_m)\}\rangle$. $K(F) = K(\langle F \rangle)$. The complexity of general partial computable functions is defined as the length of the shortest $U$-program that computes it. The halting sequence $H \in \Sigma^\infty$ is the unique infinite sequence where $H[i] = 1$ iff $U(i)$ halts. The information that $H$ has about $x \in \Sigma^*$ is $I(x : H) = K(x) - K(x|H)$.

3 Left-Total Machines

This paper uses notions of total strings and left-total machines. A string $x$ is total if all sufficiently long extensions of $x$ will cause the universal Turing machine $U$ to halt. More formally $x$ is total if and only if there exists a finite prefix free set of strings $G \subset \Sigma^*$ such that $\sum\{2^{-||p||} : y \in G\} = 1$, and for all $y \in G$, $U(xy)$ halts. Along with totality, we introduce the notion of leftness. We say $x \in \Sigma^*$ is to the left of $y \in \Sigma^*$, $x < y$, iff there exists a string $z \in \Sigma^*$ such that $z0 \sqsubseteq x$ and $z1 \sqsubseteq y$. We say the universal Turing machine $U$ is left total if for all strings $x, y \in \Sigma^*$, with $x < y$, if $U(y)$ halts then $x$ is total. An example of the domain of a left total machine can be seen in Figure 1. This example also illustrates the reason for using “left” in the definition.

Without loss of generality, we can assume that the universal Turing machine $U$ is left-total. We refer the readers to [Eps19b] on the explicit construction of a left-total universal Turing machine.

The border sequence $B \in \Sigma^\infty$ is the unique sequence where if $x \sqsupset B$ then $x$ has both total and non-total extensions. The sequence is called “border” because if $x \sqsupset B$ then $x$ is total and if $B \sqsupset x$, then $U$ will never halt when given $x$ as the starting input.

For total string $b$, we define the following function $\text{bbtime}(b) = \{t : U(p) \text{ runs in time } t, p \sqsubseteq b \text{ or } p \sqsupseteq b\}$ as the longest running time of a program that is to the left of $b$ or extends $b$. If $b$ and $b^-$ are total, then $\text{bbtime}(b) \leq \text{bbtime}(b^-)$. For total string $b \in \Sigma^*$, and $x, y \in \Sigma^*$, let $m_b(x|y)$ be the algorithmic weight of $x$ from programs conditioned on $y$ in time $\text{bbtime}(b)$. More formally,

$$m_b(x|y) = \sum\{2^{-||p||} : U_y(p) = x \text{ in time } \text{bbtime}(b)\}.$$
Figure 1: The above diagram represents the domain of a left total machine with the 0 bits branching to the left and the 1 bits branching to the right, with \( y = 110 \). For \( i \in \{1, \ldots, 5\} \), \( x_i \triangleleft x_{i+1} \) and \( x_i \triangleleft y \). Assuming \( T(y) \) halts, each \( x_i \) is total. This also implies each \( x_i^- \) is total as well.

The term \( m_b(x|y) \) is 0 if \( b \) is not total. If \( b \) and \( b^- \) are total, then \( m(b) \leq m(b^-) \).

## 4 Stochasticity

This paper uses the notion of stochasticity, which is a part of algorithmic statistics. For a comprehensive survey of algorithmic statistics, see [VS17]. A string is stochastic if it is typical of a simple probability measure. Typicality is measured by the deficiency of randomness \( d \). The deficiency of randomness of a string \( x \in \Sigma^* \), with respect to a probability measure \( Q \), conditioned on auxiliary string \( v \in \Sigma^* \), is

\[
d(x|Q, v) = \lfloor -\log Q(x) \rfloor - K(x|v).
\]

The deficiency of randomness measures the difference between the length of the \( Q \)-Shannon-Fano code for \( x \) and the shortest description of \( x \) (given \( v \)). If \( x \) is typical, then its \( d \) measure will be small. We say a string is \((j, k)\) stochastic conditional to \( y \), for \( j, k \in \mathbb{N} \) and \( y \in \Sigma^* \), if there exists a program \( v \in \Sigma^j \) of length \( j \) where \( U_y(p) = \langle Q \rangle \), and \( Q \) is an elementary probability measure, and \( d(x|Q, \langle v, y \rangle) \leq k \). The stochasticity measure of a string \( x \in \Sigma^* \), conditional on auxiliary information \( y \in \Sigma^* \) is

\[
\Lambda(x|y) = \min\{j + 3 \log k : x \text{ is } (j, k) \text{ stochastic conditional to } y\}.
\]

The following lemma is from [EL11]. It states that strings that have high stochasticity measures are exotic, in that they have high mutual information with the halting sequence. Another version of the lemma can be found in [Eps19b].

**Lemma 1** For \( x, y \in \Sigma^* \), \( \Lambda(x|y) \leq \log I(x : H|y) \).

The following lemma is from [Eps19a]. A variant of the same idea can be found in Proposition 5 of [VS17]. It states that there is no total computable function can increase the stochasticity of a string by more than a constant factor (dependent on the complexity of the function).

**Lemma 2** Given total recursive function \( g : \Sigma^* \to \Sigma^* \), \( \Lambda(g(a)) < \Lambda(a) + K(g) + O(\log K(g)) \).
The following lemma is from [Eps19b]. If a string $b$ is total and $b^-$ is not total, then $b^- \subset B$.

This is because the border sequence $B$ is defined by the unique sequence whose prefixes have total and non-total extensions. Since $b$ is total and $b^-$ is not total, $b^-$ has total and non-total extensions.

The following lemma states that if a prefix of border is simple relative to a string $x$ and its own length, then it will be a part of the common information of $x$ and $H$.

**Lemma 3** If $b \in \Sigma^*$ is total and $b^-$ is not, and $x \in \Sigma^*$, then $K(b) + I(x; H|b) \leq \log(I(x:H) + K(b|x, |b|)).$

The following theorem is from [Eps19b]. It states that given two (not necessarily probabilistic) measures $W$ and $\eta$ with certain summation requirements, if the combined $\eta$-score of elements of set $D$ is large, then there exist an element in $D$ can be identified by low $W$-code.

**Theorem 1** Relativized to computable $W: \mathbb{N} \to \mathbb{R}_{\geq 0}$ an $\eta: \mathbb{N} \to \mathbb{R}_{\geq 0}$ with $\sum_{a \in \mathbb{N}} W(a)\eta(a) \leq 1$, if for some finite set $D \subset \mathbb{N}$, $\log \sum_{a \in D} \eta(a) \geq s \in \mathbb{N}$, then there exists $a \in D$ with $K(a) < -\log W(a) - s + \Lambda(D) + O(K(s)).$

## 5 Uncomputability of $K$

Corollary 1 shows that an encoding of any $2^n$ unique pairs $\langle b, K(b) \rangle$ has more than $\sim n$ bits of mutual information with the halting sequence $H$. So all such large sets are exotic.

**Theorem 2** For any finite set $D$ of natural numbers and $L: D \to \mathbb{N}$, where $s = [\log |D|]$, we have $s < 2 \max_{a \in D} (|L(a) - K(a)|) + I(L: H) + O(K(s) + \log I(L:H)).$

**Proof.** Let $j = \max_{a \in D} |L(a) + [\log m(a)]|$. Note that by the coding theorem $K(a) = -\log m(a).$

Let $b$ be the shortest total string with $\max_{a \in D} |L(a) + [\log m_b(a)]| \leq j + 1.$

$$K(b|\langle|b|, L\rangle) \leq^? K(j), \tag{1}$$

as there is a program that when given $\langle|b|, L\rangle$ and $j$, enumerates all total strings of length $|b|$ and returns the first $x$ where $\max_{a \in D} |L(a) + [\log m_b(a)]| \leq j + 1$, which we call satisfying property $A$. This is equal to $b$, otherwise there is a $b' \subset b$, $|b'| = |b|$ that satisfies property $A$. This implies that $b^-$ is total and satisfies property $A$. This implies a contradiction for $b$ being the smallest total string satisfying property $A$. The same arguments can be used if $b \subseteq b'$. This also implies $b^-$ is not total. A graphical depiction of this argument can be seen in Figure 2.

So for all $a \in D$, $-\log m_b(a) - K(a) \leq^? 2j$. Let $\eta(a) = 1$, and $W(a) = m_b(a)$. $K(\langle W, \eta \rangle|b) = O(1)$. Theorem 1, relativized to $b$, gives $a \in D$ where $K(a|b) < -\log m_b(a) - s + \Lambda(D|b) + O(K(s)).$

So

$$s < -\log m_b(a) - K(a|b) + \Lambda(D|b) + O(K(s))$$

$$< -\log m_b(a) - K(a) + K(b) + \Lambda(D|b, s)) + O(K(s))$$

$$< \log(m(a)/m_b(a)) + K(b) + \Lambda(D|b) + O(K(s))$$

$$< 2j + K(b) + \Lambda(D|b) + O(K(s)).$$

Let $f$ be a total computable function that when given an encoding of a function $G: R \to \mathbb{N}$ for finite $R \subset \mathbb{N}$, outputs $R$. Thus $D = f(L)$. Due to Lemma 2, conditioned on $b$, $s < 2j + K(b) + \Lambda(L|b) + O(K(s))$. Due to Lemma 1,

$$s < 2j + K(b) + I(L: H|b) + O(log I(L: H|b) + K(s)).$$
illustrates this point. Each path represents a string, with 0s branching to the left and 1s branching to the right. If another string $b'$ exists with the desired $m_y'$ property, and it is to the left of $b$, then its prefix $b'^-$ will also be total and have the desired $m_y^-$ property, causing a contradiction.

Figure 2: A graphical argument for why the total string $b$ in the proof of Theorem 2 is unique. Each path represents a string, with 0s branching to the left and 1s branching to the right.

Let $h_x = I(L : H|x)$. Due to Lemma 3 and Equation 1, $K(b) + h_b \leq \log h_\emptyset + K(b\langle L, \|b\|\rangle) \leq \log h_\emptyset + K(j)$. This implies

$$s \leq 2j + h_\emptyset + O(K(s) + K(j) + \log h_\emptyset).$$

If $2j \geq s$, then the theorem is trivially solved. So, assuming $s > 2j$, we have $K(s - 2j) < O(\log(s - 2j)) < O(\log(K(s) + K(j) + h_\emptyset))$. So $K(j) \leq s - 2j < O(K(s) + \log(K(j) + h_\emptyset))$. Therefore $K(j) < O(K(s) + \log h_\emptyset)$. So $s < 2j + h_\emptyset + O(K(s) + \log h_\emptyset)$.

**Corollary 1** Any set $X \subseteq \Sigma^*$ of $2^n$ unique pairs $\langle b, K(b) \rangle$ has $n \leq \log I(X : H)$.

### 6 Exotic Binary Relations

For $U$-programs $p \in \Sigma^*$ that enumerate a (potentially infinite) binary relation and total string $b$, we use $p[b] \subseteq N \times N$ to denote the finite binary relation enumerated by $p$ in $\text{bbtime}(b)$ steps. We use $p[\infty] \subseteq N \times N$ to denote the entire binary relation enumerated by $p$.

**Theorem 3** For $U$-program $p \in \Sigma^*$ that enumerates a binary relation, with

$$i = \max\{-\log \sum_{(x,y)\in p[\infty]} 1\} \text{ and } h = I(p : H),$$

$$\min_{(x,y)\in p[\infty]} K(x|y) < i + h + O(K(i) + \log h).$$

**Proof.** Let $b \in \Sigma^*$ be the shortest total string where $[-\log \sum_{(x,y)\in p[b]} m_b(x|y)] \leq i + 1$. We have the inequality $K(b|p, \|b\|) \leq^+ K(i)$ because there is a program that when given $\|b\|$, $p$, and $i$, can enumerate all total strings $c$ of length $\|b\|$ and all pairs $(x, y) \in p[c]$, and return the first total string $c$ where $[-\log \sum_{(x,y)\in p[c]} m_c(x|y)] \leq i + 1$, which we call satisfying property $A$. This string is unique, otherwise there exists a string $b' \neq b$, $\|b'\| = \|b\|$ which satisfies property $A$. If $b' \prec b$, then $b'^-$ is total and satisfies property $A$, contradicting the definition of $B$ being the shortest total string satisfying property $A$. Similar reasoning can be used for when $b \prec b'$. Therefore $b' = b$, and $b$ is unique. Figure 2 illustrates this point.
Let $v' \in \Sigma^*$, $Q'$ be the program and elementary probability measure that minimize the stochasticity of $p$ conditional on $\langle b, i \rangle$, $\Lambda(p|\langle b, i \rangle)$, where $U_{\langle b, i \rangle}(v') = \langle Q' \rangle$, and

$$\Vert v' \Vert + 3 \log \max \{ \mathbf{d}(p|Q', \langle v', b, i \rangle), 1 \} = \Lambda(p|\langle b, i \rangle).$$

Let $Q$ be an elementary probability measure equal to $Q'$ conditioned on the largest set of programs $q$ that enumerate binary relations where $- \log \sum_{x,y \in q[b]} m_b(x|y) \leq i + 1$, which we call satisfying property $B$. Thus $Q(q) = \{ q \in T|Q'(q)/Q(T), \text{ where } T = \{ q : q \in \text{Supp}(Q'), q \text{ satisfies property } B \}$.

Let $v \in \Sigma^*$, $U_{\langle b, i \rangle}(v) = \langle Q \rangle$, with $v = v_0v'$, where $v_0 \in \Sigma^*$ is helper code of size $O(1)$. Thus $K(v|v', b, i) = O(1)$ which implies $- K(q|v, b, i) \leq + K(q|v', b, i)$. Let $d = \max \{ \mathbf{d}(q|Q, \langle v, b, i \rangle), 1 \}$. So

$$\Vert v \Vert \leq^+ \Vert v' \Vert + 3 \log d \leq^+ \Vert v' \Vert + 3 \log d \\
=^+ \Vert v' \Vert + 3 \log (\max \{ - \log Q(q) - K(q|v, b, i), 1 \}) \\
\leq^+ \Vert v' \Vert + 3 \log (\max \{ - \log Q'(q) - K(q|v', b, i), 1 \}) \\
\leq^+ \Vert v' \Vert + 3 \log (\max \{ - \log Q'(q) - K(q|v', b, i), 1 \}) \\
\leq^+ \Lambda(q|\langle b, i \rangle). \quad (2)$$

Let $S = \bigcup \{ y : (x, y) \in q[b], q \in \text{supp}(Q) \}$, which is finite. Let $\delta_y$ be a set of random vectors, indexed by $y \in S$, each of size $(c + d)2^i + 1$. The number $c \in \mathbb{N}$ is a constant solely dependent on $U$ to be determined later. Each element of the vector $\delta_y$ is chosen with probability $m_b(\cdot|y)$, and $\emptyset$ is chosen with probability $1 - \sum_{x \in \Sigma^*} m_b(x|y)$. Let $t_{H_y} : \Sigma^* \rightarrow \mathbb{R}_{\geq 0}$, be a nonnegative function over strings, parameterized by sets of strings $H_y$, each of size $(c + d)2^i + 1$, each indexed by a string $y \in S$. For an enumerative program $q$, $t_{H_y}(q) = 0$, if there exists $(x, y) \in q[b]$ where $x \in H_y$. Otherwise $t_{H_y}(q) = e^{d+c-1}$. So, using the fact $(1 - m)e^m \leq 1$ for $m \in [0, 1],

$$E_{\delta_y}[Q(t_{\delta_y})] = \sum_{q} Q(q) \prod_{y \in S} (1 - \sum_{x : (x, y) \in q[b]} m_b(x|y))(c + d)2^i + 1 e^{c+d-1} \\
\leq \sum_{q} Q(q) \prod_{y \in S} e^{-\sum_{x : (x, y) \in q[b]} m_b(x|y)(c + d)2^i + 1} e^{c+d-1} \\
\leq \sum_{q} Q(q)e^{-(\sum_{x : (x, y) \in q[b]} m_b(x|y))(c + d)2^i + 1} e^{c+d-1} \\
\leq \sum_{q} Q(q)e^{-2^{-i}(c + d)2^i + 1} e^{c+d-1} \\
= e^{-1} < 1.$$

Thus there exists a collection of sets $G_y$, indexed by $y \in S$, where $Q(t_{G_y})) \leq 1$. This collection can be found using brute search given $v, d, c,$ and $\langle b, i \rangle$, with $K(G_y|v, d, c, b, i) = O(1)$.

There exists $y \in S$, and $(x, y) \in p[b]$ where $x \in G_y$. Otherwise $t_{G_y}(p) = e^{d+c-1}$, and for proper choice of $c$, solely dependent on $U$, we have

$$d > - \log Q(p) - K(p|v, b, i) - O(1) \\
> - \log Q(p) - (- \log t_{G_x}(p))Q(p) + K(t_{G_x}(\cdot)|Q(\cdot)|v, b, i) - O(1) \\
> - \log Q(p) - (- \log t_{G_x}(p))Q(p) + K(G_x, Q|v, b, i) - O(1) \\
> (\log e)(c + d) - K(d, c) - O(1) \\
> d,$$
causing a contradiction. We roll $c$ into the additive constants for the rest of the theorem. So $t_{G_y}(p) = 0$, and there exists an $(x,y) \in p[b]$, where $x \in G_y$. So

$$
K(x|y,b,i) \leq^+ \log |G_y| + K(G_y|v,d,b,i) + K(v,d|b,i)
$$

$$
\leq^+ i + 3 \log d + ||v||
$$

$$
\leq^+ i + \Lambda(p|i, b)
$$

$$
< i + I(p : H|i, b) + O(\log I(p : H|i, b))
$$

Equation 3 is due to the fact that $v$ is a $U$ program (conditioned on $\langle b, i \rangle$). So its conditional complexity is not more than its length. Equation 4 is due to Equation 2. Equation 5 is due to Lemma 1. Equation 6 is due to Lemma 3. Equation 7 is to the inequality $K(b|p, ||b||) \leq^+ K(i)$.

**Corollary 2** For finite binary relation $B \subset N \times N$, with $i = \max\{ \lfloor - \log \sum_{(x,y) \in B} m(x|y) \rfloor, 1 \}$, \min_{(x,y)\in B} K(x|y) < i + I(B : H) + O(K(i) + \log I(B : H)).

**Corollary 3** For partial computable function $f$ with $i = \max\{ \lfloor - \log \sum_{x \in \text{Dom}(f)} m(f(x)|x) \rfloor, 1 \}$, \min_x K(f(x)|x) < i + I(f : H) + O(K(i) + \log I(f : H)).

**References**

[BFNV05] H. Buhrman, L. Fortnow, I. Newman, and N. Vereshchagin. Increasing kolmogorov complexity. In *STACS 2005*, pages 412–421, 2005.

[BMVZ13] B. Bauwens, A. Makhlin, N. Vereshchagin, and M. Zimand. Short lists with short programs in short time. In *2013 IEEE Conference on Computational Complexity*, pages 98–108, 2013.

[EL11] Samuel Epstein and Leonid Levin. On sets of high complexity strings. *CoRR*, abs/1107.1458, 2011.

[Eps19a] S. Epstein. On the Complexity of Completing Binary Predicates. *arXiv e-prints*, page arXiv:1907.04776, 2019.

[Eps19b] Samuel Epstein. All sampling methods produce outliers. *CoRR*, abs/1304.3872, 2019.

[Kol65] A. N. Kolmogorov. Three approaches to the quantitative definition of information. *Problems in Information Transmission*, 1:1–7, 1965.

[Sol64] R. J. Solomonoff. A Formal Theory of Inductive Inference, Part 1. *Information and Control*, 7:1–22, 1964.

[VS17] Nikolay K. Vereshchagin and Alexander Shen. Algorithmic statistics: Forty years later. In *Computability and Complexity*, pages 669–737, 2017.

[Zim16] Marius Zimand. List approximation for increasing kolmogorov complexity. *CoRR*, abs/1609.05984, 2016.