Abstract—Nowadays data compressors are applied to many problems of text analysis, but many such applications are developed outside of the framework of mathematical statistics. In this paper we overcome this obstacle and show how several methods of classical mathematical statistics can be developed based on applications of the data compressors.

keywords: data compression, hypothesis testing, homogeneity test, classification, universal code.

I. INTRODUCTION

Data compression methods (or universal codes) were discovered in the 1960’s and nowadays they are widely used to compress texts for their storage or transmission. In the last thirty years, it was recognized that data compressors can be used for many purposes which are far from file compaction. In particular, it was shown that methods of data compression can be used for prediction and hypothesis testing for time series in the framework of classical mathematical statistics, see [15] and a review there. Later, several authors applied data compressors to problems which are close, in spirit, to homogeneity testing, estimation of correlation and covariance, classifications, clustering and some others; see [3], [4], [10], [11], [18]. The main idea of their approach can be understood from the following example. Suppose that there are three sequences of letters $x_1x_2...x_n$, $y_1y_2...y_k$ and combined ones $x_1x_2...x_nz_1z_2...z_m$ and $y_1y_2...ykz_1z_2...z_m$, the difference $|\varphi(x_1x_2...x_nz_1z_2...z_m)| - |\varphi(x_1x_2...x_n)|$ will be less than $|\varphi(y_1y_2...ykz_1z_2...z_m)| - |\varphi(y_1y_2...yk)|$ (Here $|U|$ is the length $U$.) For instance, let $x_1x_2...x_n$, $z_1z_2...z_m$ be texts in English, whereas $y_1y_2...y_k$ is in German. Then the English text $z_1z_2...z_m$ will be compressed better after the text in the same language $(x_1x_2...x_n)$ than after the text in German $(y_1y_2...yk)$, i.e. the first difference will be less than the second one.

This natural approach was used for diagnostic of the authorship of literary and musical texts, for estimation of closeness of DNA sequences, construction of phylogenetic trees and many other problems ( [3], [4], [6], [10], [11], [18], [19]). Many papers (see [11] and review there) were devoted to the measurement of the interdependence between sequences (or the association, similarity, closeness, etc.), because such measures plays an important rule in clustering, classification and some other methods of text analysis. It is important to note, that their approaches are outside of the framework of mathematical statistics and, in particular, do not give a possibility to reason about consistency of estimates, tests, classifiers, clustering, etc.

It is important that the modern data compressors are based on so-called universal codes and the main properties of universal codes are valid for them (as far as asymptotic properties can be valid for a real computer program). A formal definition of universal codes is given in Appendix 1, but here we informally note that universal codes can compress sequences generated by a source with unknown statistics till its Shannon entropy, which, in turn, is a lower limit on lossless compression. Note that nowadays there are many classes of universal

Using data-compressors for statistical analysis of problems on homogeneity testing and classification

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codes which are based on different ideas and showed their practical efficiency. Out of the most popular we mention the PPM algorithm, which is used along with the
arithmetic code ([1], [14]), Lempel-Ziv codes ([20],
[21]), Borrow-Wille transformation ([2], [12]) which is used along with the “book stack” (or MTF) code ([16], [17]) and some others (see for review ([13], [15]).

In this paper we show how the idea of compression of combined texts described above can be used for solving problems of homogeneity testing, classification, and estimation of a measure of interdependence, or the association. A distinction of the suggested method from other approaches is that it belongs to the framework of mathematical statistics.

II. DEFINITIONS AND PROBLEM FORMULATIONS

First we briefly consider the main properties of so-called universal codes, paying the main attention to their meaning whereas formal definitions of codes, stationary ergodic sources and Shannon entropy are given in Appendix 1 and can be found in [5], [15].

Here we only note that we consider so-called lossless codes \( \varphi \) which encode words over alphabet \( A \) using words from the binary alphabet \( \{0, 1\} \) in such a way that any word can be decoded without mistakes, i.e. there exists a map \( \varphi^{-1} \) such that for any word \( w \) over \( A \)

\[
\varphi^{-1}(\varphi(w)) = w.
\]

Let a code \( \varphi \) be applied to encode sequences generated by a stationary ergodic source \( \mu \). The value \( R_\ell(\varphi, \mu) = \frac{1}{\ell} |\varphi(x_1...x_\ell)| - h_\infty(\mu) \) is called the redundancy, where \( h_\infty(\mu) \) is the limit entropy, see Appendix 1. (It is known that \( h_\infty(\mu) \) is an attainable lower limit for lossless codes, that is why the difference is called the redundancy.) By definition, the code \( \varphi \) is called universal if the redundancy goes to 0 as \( \ell \) grows, i.e. \( \lim_{\ell \to \infty} R_\ell(\varphi, \mu) = 0 \). (In other words, a universal code compresses sequences generated by any stationary ergodic source till the limit value.)

The main goal of the paper is to give a compression-based solution for the following problems:

i) Homogeneity test, where there are several sequences

\[
\begin{align*}
&x_1^1x_1^2...x_{n_1}^1, \ldots, x_1^kx_2^k...x_{n_k}^k, \\
& y_1^1y_1^2...y_{m_1}^1, \ldots, y_1^ny_2^ny_{m_n}^n,
\end{align*}
\]

generated either by a single source or by two different ones, and two corresponding hypotheses. We also consider the more general case where there are more than two different sets of sequences.

ii) Classification problems, where there are samples

\[
\begin{align*}
&x_1^1x_2^1...x_{n_1}^1, \ldots, x_1^kx_2^k...x_{n_k}^k, \\
& y_1^1y_1^2...y_{m_1}^1, \ldots, y_1^ny_2^ny_{m_n}^n,
\end{align*}
\]

generated two different (but unknown) sources and \( z_1...z_2 \) is generated by one of the two. The goal is to determine which of them generated \( z_1...z_2 \).

iii) Estimation of a so-called measurement of interdependence, or the association.

III. THE MAIN THEOREMS

First we present a theorem which can be considered as a theoretical basis for application of data compressors for solving the problems described above. First, for two words \( v_1...v_l \) and \( u_1...u_s \) we define the following value:

\[
|\varphi(v_1...v_l/u_1...u_s)| = |\varphi(u_1...u_sv_1)| - |\varphi(u_1...u_s)|,
\]

where \( s, l \) are integers. Informally, it is the length of the codeword for \( v_1...v_l \) if it is encoded with the word \( u_1...u_s \).

**Theorem 1.** Let \( \mu_x \) and \( \mu_y \) be stationary ergodic sources generating letters from a finite alphabet \( A \) and let their memory be upper-bounded by a certain constant \( M \). Suppose that \( ...y_{t-1}...y_{t+l}...y_{t+k}...y_{t-1}y_0 \) is generated by \( \mu_y \), whereas \( ...x_{t-1}x_{t-1}1...x_{t+1}x_0 \) and \( u_1...u_s, m \geq 1 \), are generated by \( \mu_x \) in such a way that all three sequences are independent, and let \( \varphi \) be a universal code. Define

\[
\Delta_{t,k,m} = |\varphi(w_1...w_m/y_{t-k}...y_{t-k+1}...y_0)| - |\varphi(w_1...w_m/x_{t-l}x_{t-l+1}...x_{t-1}x_0)|
\]

where \( |\varphi(v_1...v_l/u_1...u_s)| \) is defined in [7]. Then there exists such a constant \( \lambda \) and an integer \( m_0 \) that for any \( m > m_0 \) there is such an integer \( L \) that, for any \( t > m_0 + l, k > m + L \),

\[
E_{\mu_x}E_{\mu_y}(\Delta_{t,k,m}) \geq \lambda m + o(m),
\]

and, if \( \mu_x = \mu_y \) then \( \lambda = 0 \), otherwise \( \lambda > 0 \). Here \( E_\nu() \) is the expectation with respect to the measure \( \nu \).

The proof is given in Appendix 2, whereas here we give some explanations of the theorem. The theorem compares two cases: the word \( w_1...w_m \) is compressed either together with \( x_{t-l}x_{t-l+1}...x_{t-1}x_0 \) or with \( y_{t-k}...y_{t-k+1}...y_0 \). The word \( w_1...w_m \) is generated by \( \mu_x \), hence, it should be compressed better with \( x_{t-l}x_{t-l+1}...x_{t-1}x_0 \) and the value

\[
|\varphi(w_1...w_l/x_{t-l}x_{t-l+1}...x_{t-1}x_0)| - |\varphi(w_1...w_l/y_{t-k}...y_{t-k+1}...y_0)|
\]

should be less than \( |\varphi(w_1...w_l/y_{t-k}...y_{t-k+1}...y_0)| \). The theorem shows, that, indeed, the word \( w_1...w_m \) is compressed better together with \( x_{t-l}x_{t-l+1}...x_{t-1}x_0 \) if the lengths of sequences \( w_1w_2...w_m, x_{t-l}x_{t-l+1}...x_{t-1}x_0 \) and \( y_{t-k}...y_{t-k+1}...y_0 \) are sufficiently large.

Let there be two sets of sequences: \( X = \{x_1^1x_2^1...x_{n_1}^1, x_1^2x_2^2...x_{n_2}^2, \ldots, x_1^kx_2^k...x_{n_k}^k\} \) and \( Y = \{y_1^1y_2^1...y_{m_1}^1, y_1^2y_2^2...y_{m_2}^2, \ldots, y_1^ny_2^ny_{m_n}^n\} \), generated by possibly different measures \( \mu_x \) and \( \mu_y \). We consider two hypotheses

\[
H_0 = \{\mu_x = \mu_y\} \text{ and } H_1 = \{\mu_x \neq \mu_y\}
\]

and our goal is to develop a statistical test for them using the sets \( X \) and \( Y \). First we give an informal description of the suggested test, which will be based on data compression.
Combine a half of the sequences from the set \( X \) into \( X^* \)
(say, \( x_1^1, x_1^2, ... x_n^1, x_1^2, x_2^2, ... x_n^2, ... x_{n/2}^{k/2} ... x_{n(k/2)}^{k/2} \))
and half of \( Y \) (say and \( y_1^1, y_1^2, ... y_{m_1}^1, y_1^2, y_{m_2}^2, ... y_{m(s/2)}^{s/2}, ... y_{m(s/2)}^{m(s/2)} \))
into \( Y^* \). Then compress all other sequences using a universal code \( \varphi \) along with \( X^* \) and
\( Y^* \). If \( H_0 \) is true, the values \( |\varphi(x_1^1, x_1^2, ... x_n^1) / Y^*| \),
\( i = [k/2] + 1, ..., k \) and \( |\varphi(y_1^1, y_1^2, ... y_{m_1}^1) / X^*| \),
\( j = [s/2] + 1, ..., s \) should be evenly mixed. Otherwise, if \( H_1 \)
true, then, on average, the numbers from the second
set should be larger than those from the first one, be-
cause the probability distributions of the sequences from
\( y_1^1, y_1^2, ... y_{m_1}^1 \) and \( X^* \) are different, whereas the probability
distributions of the sequences \( x_1^1, x_1^2, ... x_n^1 \) and \( X^* \) are the
same (here and below \( |U| \) is the number of elements in \( U \)).

The test. A more formal description of the suggested
test is as follows: i) Denote the set \( X \setminus X^* \) by \( \tilde{X} \) and
\( Y \setminus Y^* \) by \( \tilde{Y} \) and calculate for any \( x_1^1, x_1^2, ... x_n^1 \) from \( \tilde{X} \)
the values
\[
\gamma_i = |\varphi(x_1^1, x_1^2, ... x_n^1) / Y^*| - |\varphi(x_1^1, x_1^2, ... x_n^1) / X^*|
\]
and for any \( y_1^1, y_1^2, ... y_{m_1}^1 \) from \( \tilde{Y} \)
\[
\delta_j = |\varphi(y_1^1, y_1^2, ... y_{m_1}^1) / X^*| - |\varphi(y_1^1, y_1^2, ... y_{m_1}^1) / Y^*|
\]
Define
\[
n_{1,1} = \{i : \gamma_i < 0\}, \quad n_{1,2} = \{i : \gamma_i > 0\},
\]
\[
n_{2,1} = \{i : \delta_i < 0\}, \quad n_{2,2} = \{i : \delta_i > 0\}.
\]
ii) Apply the test of the independence for the \( 2 \times 2 \)
table to
\[
N_{2,2} = \begin{vmatrix} n_{1,1} & n_{1,2} \\ n_{2,1} & n_{2,2} \end{vmatrix}
\]
A detailed analysis of this problem is carried out in [7],
part 33. In particular, there is a description of efficient
tests for homogeneity problem for the \( 2 \times 2 \) table (see part
33.22). We denote this test as \( \Psi_\alpha \), where \( \alpha \) is the level
of significance. Note, that there are some requirements
for values of \( n_{1,1}, n_{1,2}, n_{2,1}, n_{2,2} \) which should be valid
if \( \Psi_\alpha \) is used (see [7]).

**Theorem 2.** Let \( \mu_x \) and \( \mu_y \) be stationary ergodic mea-
asures whose memory is finite (but, possibly, unknown). If
the above described test is applied for testing \( H_0 \) against
\( H_1 \) along with the test \( \Psi_\alpha \) and the requirements of \( \Psi_\alpha \)
for values of \( n_{1,1}, n_{1,2}, n_{2,1}, n_{2,2} \) are valid, then for any code \( \varphi \) the Type I error is not grater than \( \alpha \).

If \( \varphi \) is a universal code, \(|X|\) and \(|Y|\) go to infinity
and the lengths of all sequences from \( X \) and \( Y \) go to
infinity in such a way that for all sequences \( x, x^* \in X \),
\( y, y^* \in Y \) the ratios \( |x| / |x^*| \) and \( |y| / |y^*| \) are upper-
bounded by a certain constant, then, with probability 1, the
Type II error goes to 0.

The proof is given in Appendix 2, but here we give
some comments. First, note that there are many tests
of homogeneity for \( 2 \times 2 \) tables and, in principle, any
of them can be used. That is why, we do not describe
the test \( \Psi_\alpha \) on sizes of \( n_{1,1}, n_{1,2}, n_{2,1}, n_{2,2} \). Second,
the described method can be easily extended from the
two-sample problem to the \( s \)-sample problem, \( s > 2 \).
Namely, let there be \( s, s > 2 \), sets \( V_1, V_2, ..., V_s \) of
sequences generated by stationary ergodic sources \( \mu_1, \mu_2 
..., \mu_s \). Let there be the hypotheses \( H_0 = \{ \mu_1 = \mu_2 
= ... = \mu_s \} \) and \( H_1 = H_0 \). In this case we carry
out calculations based on the scheme described above
in order to obtain a so-called \( s \times s \) table and then apply
a test for homogeneity, see ([7]).

**Measurement of the interdependence and association.**
If the hypothesis of homogeneity is rejected, it is
natural to measure interdependence. We suggest to
measure interdependence between two sets of sequences
\( X \) and \( Y \) (and the corresponding sources) based on
the above described \( 2 \times 2 \) tables. (In the case of the
\( s \)-sample problem, \( s > 2 \), the measures will be
based on the \( s \times s \) table.) This problem is well-
investigated in the mathematical statistics, see, for
example, ([7], part 33). That is why, we mention such
measures only briefly. For \( 2 \times 2 \) tables we mention the
coefficient of association, \( Q \), defined by the equation
\[
Q = (n_{1,1} n_{2,2} - n_{1,2} n_{2,1}) / n_{1,1} n_{2,2} + n_{1,2} n_{2,1}
\]
and the coefficient \( \psi = (n_{1,1} n_{2,2} - n_{1,2} n_{2,1}) /
\sqrt{(n_{1,1} + n_{2,2}) (n_{1,1} + n_{2,1}) (n_{1,2} + n_{2,2}) (n_{1,2} + n_{2,1})}
\), see, ([7], part 33). It is important to note that there are
well-known methods of building standard errors and
confidence interval for \( Q \) and \( \psi \) ([7], part 33).

Let there be sequences \( \tilde{w}^1 = w_1^1 w_1^2 ... w_1^{m_1}, \tilde{w}^2 = 
w_1^2 w_2^2 ... w_{m_2}^2, ..., \tilde{w}^k = w_1^k w_2^k ... w_{m_k}^k \), generated by
stationary ergodic sources \( \nu_1, ..., \nu_k \), correspondingly,
where \( k \geq 2 \). There is a new sequence \( \tilde{w} = u_1 u_2 ... u_n, n \geq 2 \), and it is known beforehand that it is generated
by one of the sources from \( \{ \nu_1, ..., \nu_k \} \). The problem
of classification is to determine which source generated
the sequence \( \tilde{u} = u_1 u_2 ... u_n \). By definition, a method of
classification is called asymptotically consistent if, with
probability 1, the method finds \( \nu \) which generated the
sequence \( u_1 u_2 ... u_n \) when \( \min(n, m_1, m_2, ..., m_k) \) goes
to infinity.

**Method of classification.** We suggest the follow-
ing method of classification: decide that the sequence
\[ u_1 u_2 \ldots u_n \text{ was generated the source } \nu_j \text{ for which} \]
\[ j = \arg \min_{i=1, \ldots, k} \| \varphi(u_1 u_2 \ldots u_n / w_i^1 w_i^2 \ldots w_i^{m_k}) \|. \]  

(7)

**Theorem 3.** Let \( \nu_1, \ldots, \nu_k \) be stationary ergodic measures whose memories are finite (but, possibly, unknown). If \( \varphi \) is a universal code and the lengths of all sequences \( \hat{u}, \hat{w}_1, \hat{w}_2, \ldots, \hat{w}_k \) go to infinity in such a way that
\[
\lim_{|u| \to \infty} |\hat{u}| / \min_j |\hat{w}_j| = 0,
\]
then the described method is asymptotically consistent.

IV. APPENDIX 1

Let \( \tau \) be a stationary ergodic source generating letters from a finite alphabet \( A \). (Definitions can be found in [5].) The \( m \)-order (conditional) Shannon entropy and the limit Shannon entropy are defined as follows:
\[
h_m(\tau) = - \sum_{v \in A^m} \tau(v) \sum_{a \in A} \tau(a|v) \log \tau(a|v),
\]
\[
h(\tau) = \lim_{m \to \infty} h_m(\tau).
\]

(9)

It is known that for any integer \( m \)
\[
h_m(\tau) \geq h_{m+1}(\tau) \geq h(\tau),
\]
see [5]. Now we define codes. Let \( A^m \) be the set of all infinite words \( x_1 x_2 \ldots \) over the alphabet \( A \). A data compression method (or code) \( \varphi \) is defined as a set of mappings \( \varphi_n \) such that \( \varphi_n : A^n \to \{0, 1\}^* \), \( n = 1, 2, \ldots \) and for each pair of different words \( x, y \in A^n \varphi_n(x) \neq \varphi_n(y) \). Informally, it means that the code \( \varphi \) can be applied for compression of each message of any length \( n \) over the alphabet \( A \) and the message can be decoded if its code is known. It is also required that each sequence \( \varphi_n(u_1) \varphi_n(u_2) \ldots \varphi_n(u_r), r \geq 1 \), of encoded words from the set \( A^n \), \( n \geq 1 \), can be uniquely decoded into \( u_1 u_2 \ldots u_r \). Such codes are called uniquely decodable. For example, let \( A = \{a, b\} \), the code \( \psi_1(a) = 0, \psi_3(b) = 00 \), obviously, is not uniquely decodable. It is well known that if a code \( \varphi \) is uniquely decodable then the lengths of the codewords satisfy the following inequality (Kraft inequality): \( \sum_{u \in A^n} 2^{-|\varphi(u)|} \leq 1 \), see, for ex., [5]. Moreover, if the sum \( \sum_{u \in A^n} 2^{-|\varphi(u)|} \) is less than 1, there exists such a code \( \varphi^* \) that i) \( \sum_{u \in A^n} 2^{-|\varphi^*(u)|} = 1 \), and \( |\varphi^*(u)| \leq |\varphi(u)| \) for any word \( u \), [5]. (Informally, it means, that \( \varphi^* \) compresses better.) So, we can consider only codes for which
\[
\sum_{u \in A^n} 2^{-|\varphi(u)|} = 1.
\]

(11)

We will use a so-called Kullback-Leibler (KL) divergence, which is defined by
\[
D(P||Q) = \sum_{b \in B} P(b) \log \frac{P(b)}{Q(b)}.
\]

(12)
log $\mu_x(w_1...w_m|x_{-t}x_{-t+1}...x_0) = (m - M)\lambda + O(1)$, (18)

where $M$ is a upper bound of memories of $\mu_x$ and $\mu_y$, and $\lambda$ is such a constant that $\lambda = 0$ if $\mu_x = \mu_y$ and $\lambda > 0$ otherwise.

Proof of the claim. The left side can be presented as follows:

$$E_{\mu_x}E_{\mu_y} \sum_{w_1 w_2 ... w_m} \mu_x(w_1 w_2 ... w_m)$$

$$\log \frac{\mu_x(w_1...w_m|x_{-t}x_{-t+1}...x_0)}{\mu_y(w_1|y_ky_{k+1}...y_0)} =$$

$$E_{\mu_x}E_{\mu_y} \left( \sum_{w_1 w_2 ... w_{m-1}} \mu_x(w_1 w_2 ... w_{m-1}) \right)$$

$$\sum_{w_m} \mu_x(w_m | w_1 w_2 ... w_{m-1})$$

$$\log \frac{\mu_x(w_m | x_{-t}x_{-t+1}...x_0)}{\mu_y(w_m | y_ky_{k+1}...y_0)}$$

$$+ \sum_{w_1 w_2 ... w_{m-2}} \mu_x(w_1 w_2 ... w_{m-2})$$

$$\sum_{w_{m-1}} \mu_x(w_{m-1} | w_1 w_2 ... w_{m-2})$$

$$\log \frac{\mu_x(w_{m-1} | x_{-t}x_{-t+1}...x_0)}{\mu_y(w_{m-1} | y_ky_{k+1}...y_0)}$$

$$+ ... + \sum_{w_1} \mu_x(w_1) \log \frac{\mu_x(w_1 | x_{-t}x_{-t+1}...x_0)}{\mu_y(w_1 | y_ky_{k+1}...y_0)}. \tag{19}$$

It is supposed that the memory of $\mu_x$ and $\mu_y$ is upper-bounded by $M$. Hence, by definition, it means that $\mu_x(v|u_1...u_j) = \mu_y(v|\bar{u}_j-M+1...\bar{u}_j-M+2...u_j)$, where $j \geq i+M-1$. From this equation and the previous one we obtain the following equation:

$$E_{\mu_x}E_{\mu_y} \sum_{w_1 w_2 ... w_m} \mu_x(w_1 w_2 ... w_m)$$

$$\log \frac{\mu_x(w_1...w_m|x_{-t}x_{-t+1}...x_0)}{\mu_y(w_1|y_ky_{k+1}...y_0)} =$$

$$\sum_{w_1 w_2 ... w_{m-1}} \mu_x(w_1 w_2 ... w_{m-1}) \sum_{w_m} \mu_x(w_m | w_1 w_2 ... w_{m-1})$$

$$\log \frac{\mu_x(w_m | w_{m-M+1}...w_{m-1})}{\mu_y(w_m | w_{m-M+1}...w_{m-1})}$$

$$+ \sum_{w_1 w_2 ... w_{m-2}} \mu_x(w_1 w_2 ... w_{m-2}) \sum_{w_{m-1}} \mu_x(w_{m-1} | w_1 w_2 ... w_{m-2})$$

$$\log \frac{\mu_x(w_{m-M+1}...w_{m-2})}{\mu_y(w_{m-M+1}...w_{m-2})} + \ldots + \sum_{w_1 w_2 ... w_{m-M+1}} \mu_x(w_1 w_2 ... w_{m-M+1}) \sum_{w_{M+1}} \mu_x(w_{M+1} | w_1 w_2 ... w_{M+1})$$

$$\log \frac{\mu_x(w_{M+1} | w_1 w_2 ... w_{M+1})}{\mu_y(w_{M+1} | w_1 w_2 ... w_{M+1})}$$

$$+ E_{\mu_x}E_{\mu_y} \left( \sum_{w_1 w_2 ... w_{M-1}} \mu_x(w_1 w_2 ... w_{M-1}) \right)$$

$$\sum_{w_{M}} \mu_x(w_{M} | x_{-t}x_{-t+1}...x_0)$$

$$\log \frac{\mu_x(w_{M} | x_{-t}x_{-t+1}...x_0)}{\mu_y(w_{M} | y_ky_{k+1}...y_0)} + \ldots + \sum_{w_1} \mu_x(w_1) \log \frac{\mu_x(w_1 | x_{-t}x_{-t+1}...x_0)}{\mu_y(w_1 | y_ky_{k+1}...y_0)}. \tag{20}$$

For any function $\psi(u_{e+1}, ..., u_{e+b})$ and any measure $\nu(u_1 u_2 ... u_{e+1} u_{e+1} ... u_{e+b})$

$$\sum_{u_1 u_2 ... u_{e+1} u_{e+1} ... u_{e+b}} \nu(u_1 u_2 ... u_{e+1} u_{e+1} ... u_{e+b})$$

$$\psi(u_{e+1}, ..., u_{e+b})$$

$$= \sum_{u_{e+1} ... u_{e+b}} \nu(u_{e+1} ... u_{e+b}) \psi(u_{e+1}, ..., u_{e+b}),$$

where $e$ and $b$ are integers. Having taken into account this equation, stationarity $\mu_x$ and $\mu_y$, equations (19), (20) and (19), we obtain

$$E_{\mu_x}E_{\mu_y} \sum_{w_1 w_2 ... w_m} \mu_x(w_1 w_2 ... w_m)$$

$$\log \frac{\mu_x(w_1...w_m|x_{-t}x_{-t+1}...x_0)}{\mu_y(w_1|y_ky_{k+1}...y_0)} =$$

$$(m - M) \sum_{w_1 w_2 ... w_{M-2}} \mu_x(w_1 w_2 ... w_{M-2})$$

$$\sum_{w_{M-1}} \mu_x(w_{M-1} | w_1 w_2 ... w_{M-2})$$

$$\log \frac{\mu_x(w_{M-1} | w_{M-2} ... w_1)}{\mu_y(w_{M-1} | w_{M-2} ... w_1)} + O(1).$$

From properties of K-L divergence (see (12)) we can see that $\lambda > 0$ if $\mu_x \neq \mu_y$ and, obviously, $\lambda = 0$, if $\mu_x = \mu_y$. The claim is proven.

Let us proceed with proof of the theorem. Having taken into account the definitions (16), (17) and (18), we obtain that

$$E_{\mu_x}E_{\mu_y}(\Delta_{k,M}) =$$

$$\sum_{u_{-t}u_{-t+1}...u_0 \in A^{t+1}} \sum_{v_{-k}v_{-k+1}...v_0 \in A^{k+1}} \mu_x(u_{-t}u_{-t+1}...u_0) \mu_x(v_{-k}v_{-k+1}...v_0) \mu_x(u_{-t}u_{-t+1}...u_0) \mu_x(v_{-k}v_{-k+1}...v_0).$$
Theorem 1. From this theorem we can see that, with probability 1, the value $u$ goes to 0. The following equation is obvious:

$$\log \frac{\pi_x(w_1 \cdots w_m | u_{-1} u_{-1} + \cdots u_0)}{\pi_y(w_1 \cdots w_m | v_{-k} v_{-k} \cdots v_0)} = (21)$$

The following equation is obvious:

$$\log \frac{\pi_{x^+}(w_1 \cdots w_m | u_{-1} u_{-1} \cdots u_0)}{\pi_{y^+}(w_1 \cdots w_m | v_{-k} v_{-k} \cdots v_0)} + \log \frac{\pi_{x^+}(w_1 \cdots w_m | u_{-1} u_{-1} \cdots u_0)}{\pi_{y^+}(w_1 \cdots w_m | v_{-k} v_{-k} \cdots v_0)} + \log \frac{\pi_{y^+}(w_1 \cdots w_m | u_{-1} u_{-1} \cdots u_0)}{\pi_{y^+}(w_1 \cdots w_m | v_{-k} v_{-k} \cdots v_0)}$$

The following equation is obvious:

$$\log \frac{\pi_x(w_1 \cdots w_m | u_{-1} u_{-1} + \cdots u_0)}{\pi_y(w_1 \cdots w_m | v_{-k} v_{-k} \cdots v_0)} = (22)$$

The first term is estimated in the claim, see (17), whereas the second and the third terms can be estimated based on (17). So, from the claim and (17), we can see that, with probability 1, $E_{x^+} E_{y^+} (\Delta t,k,m) = (m - M) \lambda + O(1)$. The theorem is proven.

Proof of the Theorem 2 First we consider the case where $H_0$ is true. It means that the sequences from $X$ and $Y$ obey the same distribution. Hence, $\gamma_i$ and $\delta_j$ have the same distribution, too, and the above mentioned test $\Psi_\alpha$ from (7), part 33, can be applied. Now we consider the case where $H_1$ is true. In this case the length of any sequence grows, so the length will be grater than $m_0$ from Theorem 1. The number of sequences grows to infinity and the total length of a half of them goes to infinity in such a way that for any integer $L$ the total length will be grater than the sum $m + L$ from Theorem 1. From this theorem we can see that $n_{1,2}$ and $n_{2,1}$ goes to 0 and, hence, the Type II error goes to 0.

Proof of the Theorem 3 Suppose that the sequence $u_1 u_2 \cdots u_n$ was generated by $\nu_j$. Then, we can see from Theorem 1 that, with probability 1, the value $|\varphi(u_1 u_2 \cdots u_n / w_1^i w_2^i \cdots w_{m_i}^i)|$ grows as $\lambda_i n + o(n)$, $\lambda_i > 0$, if $i \neq j$, (i.e. $w_1^i w_2^i \cdots w_{m_i}^i$ is generated by $\nu_i$). On the other hand, $|\varphi(u_1 u_2 \cdots u_n / w_1^j w_2^j \cdots w_{m_j}^j)| = o(n)$. Hence, $|\varphi(u_1 u_2 \cdots u_n / w_1^i w_2^j \cdots w_{m_i}^j)|$ is minimal when $i = j$ (i.o., $u_1 u_2 \cdots u_n$ is generated by $\nu_j$). The theorem is proven.

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