ON LOCAL BEHAVIOR OF ANALYTIC FUNCTIONS

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Abstract

We prove local inequalities for analytic functions defined on a convex body in \( \mathbb{R}^n \) which generalize well-known classical inequalities for polynomials.

1. Introduction.

1.1. The classical Chebyshev inequality estimates the supremum norm of a univariate real polynomial \( p \) on an interval \( I \) by its norm on a subinterval \( I_1 \) up to a multiplicative constant depending on the degree of \( p \) and the ratio of lengths \( |I|/|I_1| \) only. In the 1930’s Remez [R] proved a generalization of the Chebyshev inequality replacing \( I_1 \) by any measurable subset. A multivariate inequality of such a kind (which coincides with the Remez inequality in the one-dimensional case) was proved by Yu.Brudnyi and Ganzburg [BG] in the 1970’s. To formulate the result let \( P_{k,n}(\mathbb{R}) \subset \mathbb{R}[x_1,\ldots,x_n] \) denote the space of real polynomials of degree at most \( k \) and \( |U| \) denote the Lebesgue measure of \( U \subset \mathbb{R}^n \).

Brudnyi-Ganzburg inequality. Let \( V \subset \mathbb{R}^n \) be a bounded convex body and \( \omega \subset V \) be a measurable subset. For every \( p \in P_{k,n} \) the inequality

\[
\sup_V |p| \leq T_k \left( \frac{1 + \beta_n(\lambda)}{1 - \beta_n(\lambda)} \right) \sup_\omega |p| \tag{1.1}
\]

holds. Here \( \lambda := |\omega|/|V| \) and \( \beta_n = (1 - \lambda)^{1/n} \) and \( T_k(x) = \frac{(x+\sqrt{x^2-1})^k+(x-\sqrt{x^2-1})^k}{2} \) is the Chebyshev polynomial of degree \( k \).

From the above inequality one obtains

\[
\sup_V |p| \leq \left( \frac{4n|V|}{|\omega|} \right)^k \sup_\omega |p| . \tag{1.2}
\]

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Inequalities of this kind are usually referred to as Bernstein-type inequalities. They play an important role in the area of Approximation Theory which investigates interrelation between analytic, approximative and metric properties of functions. The purpose of this paper is to prove inequalities similar to (1.2) for analytic functions.

We define the local degree of an analytic function which expresses its geometric properties and generalizes the degree of a polynomial. This notion is central for our consideration. It allows us to obtain better constants in Bernstein-type inequalities even in the standard (polynomial) case. It is worth pointing out that in recent years an essential progress was done in studying Bernstein- and Markov-type inequalities for algebraic and analytic functions. Such inequalities proved to be important in different areas of modern analysis, see, e.g., [B], [BG], [BMLT], [Br1], [Br2], [FN1], [FN2], [FN3], [G], [KY], [RY], [S]. We hope that the inequalities established in this paper would also have various applications in the fields that make use of the classical polynomial inequalities (Approximation Theory, trace and embedding theorems, signal processing, PDE etc). We proceed to formulation of the main result of the paper.

1.2. A generalized Chebyshev inequality. Let $B_c(0,1) \subset B_c(0,r) \subset \mathbb{C}^n$ be the pair of open complex Euclidean balls of radii 1 and $r$ centered at 0. Denote by $O_r$ the set of holomorphic functions defined on $B_c(0,r)$. Let $l_x \subset \mathbb{R}^{2n}$ be a real straight line passing through $x \in B_c(0,1)$. Further, let $I \subset l_x \cap B_c(0,1)$ be an interval and $\omega \subset I$ be a measurable subset.

**Theorem 1.1** For any $f \in O_r$ there is a constant $d = d(f,r) > 0$ such that for any $\omega \subset I \subset l_x \cap B_c(0,1)$

$$
\sup_I |f| \leq \left( \frac{4|I|}{|\omega|} \right)^d \sup_\omega |f| .
$$

(1.3)

**Example 1.2** As an application of the above theorem we obtain local inequalities for quasipolynomials.

**Definition 1.3** Let $f_1, \ldots, f_k \in (\mathbb{C}^n)^*$ be complex linear functionals. A quasipolynomial with the spectrum $f_1, \ldots, f_k$ is a finite sum

$$f(z) = \sum_i p_i(z)e^{f_i(z)},$$

where $p_i \in \mathbb{C}[z_1, \ldots, z_n]$. Expression $\sum_i (1 + \deg p_i)$ is said to be the degree of $f$.

**Proposition 1.4** Let $f$ be a quasipolynomial of degree $m$ and $l_x$ be a real straight line passing through $x \in B_c(0,1)$. Then there is an absolute constant $c > 0$ such that the inequality

$$
\sup_I |f| \leq \left( \frac{4|I|}{|\omega|} \right)^c \sqrt[k]{M^{m+1}}
$$

holds for any interval $I \subset l_x \cap B_c(0,1)$ and any measurable subset $\omega \subset I$. Here $M := \max_i \{||f_i||_{L^2(\mathbb{C}^n)}\}$. 

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Definition 1.5 The best constant \( d \) in inequality (1.3) will be called the Chebyshev degree of the function \( f \in \mathcal{O}_r \) in \( B_\mathbb{C}(0,1) \) and will be denoted by \( d_f(r) \).

All constants in inequalities formulated below depend upon the possibility to obtain an effective bound of Chebyshev degree in (1.3). The following result gives such a bound in terms of the local geometry of \( f \).

We say that a univariate holomorphic function \( f \) defined in a disk is \( p \) -valent if it assumes no value more than \( p \)-times there. We also say that \( f \) is 0-valent if it is a constant. For any \( t \in [1, r) \) let \( L_t \) denote the set of one-dimensional complex affine spaces \( l \subset \mathbb{C}^n \) such that \( l \cap B_\mathbb{C}(0,t) \neq \emptyset \).

Definition 1.6 Let \( f \in \mathcal{O}_r \). The number

\[
v_f(t) := \sup_{l \in L_t} \{ \text{valency of } f|_{l \cap B_\mathbb{C}(0,t)} \}
\]

is said to be the valency of \( f \) in \( B_\mathbb{C}(0,t) \).

Proposition 1.7 For any \( f \in \mathcal{O}_r \) and any \( t, 1 \leq t < r \), the valency \( v_f(t) \) is finite. There is a constant \( c = c(r) > 0 \) such that \( d_f(r) \leq cv_f(1+\frac{r^2}{2}) \).

Remark 1.8 For any holomorphic polynomial \( p \in \mathbb{C}[z_1, \ldots, z_n] \) of degree at most \( k \) the classical Remez inequality implies \( d_p(r) \leq k \) while in many cases Proposition 1.7 yields a sharper estimate.

1.3. In this section we formulate a generalization of inequality (1.2). Let \( B(0,1) \subset B_\mathbb{C}(0,1) \) be the real Euclidean unit ball.

Theorem 1.9 For any convex body \( V \subset B(0,1) \), any measurable subset \( \omega \subset V \) and any \( f \in \mathcal{O}_r \), the inequality

\[
\sup_V |f| \leq \left( \frac{4n|V|}{|\omega|} \right)^{d_f(r)} \sup_\omega |f|
\]

holds.

The following corollary is a version of the log-BMO-property for analytic functions (cf. [St] and [Br2]).

Corollary 1.10 Under the hypothesis of Theorem 1.9 the inequality

\[
\frac{1}{|V|} \int_V \left| \log \frac{|f|}{||f||_V} \right| \, dx \leq Cd_f(r) \log n
\]

holds with an absolute constant \( C \), where \( ||f||_V := \sup_V |f| \).

Our next application of inequality (1.3) is a generalization of Bourgain’s polynomial inequality [B].
Theorem 1.11 Let \( V \subset B(0,1) \) be a convex body and \( \tilde{d}_f(r) \) be the smallest integer \( \geq d_f(r) \). There are positive absolute constants \( c_1, c_2 \) such that the following inequality

\[
\left| \{ x \in V : |f(x)| > \lambda \right| \leq c_1 \exp(-\lambda^{c_2/\tilde{d}_f(r)}) |V|
\]  

holds for any \( f \in \mathcal{O}_r \). In particular,

\[
\|f\|_{L^s(V, dx)} \leq (c_1 + 1) \|f\|_{L^1(V, dx)},
\]

where \( L^\Phi \) refers to the Orlicz space with the Orlicz function \( \Phi(t) = \exp(t^{c_2/\tilde{d}_f(r)}) - 1 \).

Remark 1.12 The original Bourgain’s inequality for polynomials contains the degree of the polynomial instead of \( \tilde{d}_f(r) \).

As a corollary we also obtain the reverse Hölder inequality with the constant which does not depend on the dimension (this result does not follow from Theorem 1.9).

Corollary 1.13

\[
\left( \frac{1}{|V|} \int_V |f(x)|^s dx \right)^{1/s} \leq c(\tilde{d}_f(r), s) \frac{1}{|V|} \int_V |f(x)| dx \quad (f \in \mathcal{O}_r, s \in \mathbb{Z}_+) \, .
\]

The following example shows that in the polynomial case our inequalities might be sharper than those of [BG] and [B].

Example 1.14 Let \( f \in \mathcal{O}_r \) be such that \( \sup_{B_c(0,s)} |f| < 1 \). Let \( \phi \) be a holomorphic non-polynomial function univalent in an open neighbourhood \( U \) of \( \mathbb{D} = \{ z \in \mathbb{C} : |z| \leq 1 \} \). Then using Proposition [1.7] and Proposition [3.1] below yields \( d_{\phi \circ f}(r) \leq c(r) v_f \left( \frac{1}{d_{\phi \circ f}(r)} \right) \). Consider a polynomial approximation \( h_k \) of \( \phi \) such that \( \deg h_k = k \) and \( h_k \) is also univalent on \( \mathbb{D} \). Assume now that \( f \in \mathcal{O}_r \) is a polynomial. Then \( \deg(h_k \circ f) = k \cdot \deg f \). Further, apply Brudnyi-Ganzburg and Bourgain’s polynomial inequalities to the polynomial \( h_k \circ f \). Then the exponents in these inequalities will be equivalent to \( k \cdot \deg f \) and \( 1/(k \cdot \deg f) \), respectively. However, in our generalizations of the above inequalities these exponents contain numbers \( d_{h_k \circ f}(r) \) and \( 1/\tilde{d}_{h_k \circ f}(r) \) with \( d_{h_k \circ f}(r) \leq c(r) \deg f \) and this is essentially better for all sufficiently large \( k \).

2. Proofs of Theorem [1.1] and Proposition [1.7].

2.1. We begin with auxiliary results used in the proof.

Parametrization of straight lines in the ball. Let \( B_c(0,s) \), \( 1 < s < r \), be an open complex Euclidean ball. For any \( x \in B_c(0,s) \) consider the complex straight line \( l_{x,v} = \{ x + vz \sqrt{s^2 - |x|^2} : \langle x, v \rangle = 0, |v| = 1, z \in \mathbb{C} \} \) passing through \( x \). Here \( | \cdot | \) denotes the Euclidean norm and \( \langle \cdot, \cdot \rangle \) the inner product on \( \mathbb{C}^n \). In this way
we parametrize the set $L_s$ of all complex straight lines passing through points of $B_c(0, s)$. Let $f$ be a holomorphic function from $O_r$. Consider the function

$$F(z, x, v, s) = f(x + vz\sqrt{s^2 - |x|^2}) \quad (z \in \mathbb{D}) \tag{2.1}$$

Then $F(\cdot, x, v, s)$ is the restriction of $f$ to $l_{x,v} \cap B_c(0, s)$. Note also that for any $t < s$ the inequality

$$\frac{s^2 - |x|^2}{t^2 - |x|^2} \geq (s/t)^2 \tag{2.2}$$

holds. This implies that the set $\{x + vz\sqrt{s^2 - |x|^2} : \langle x, v \rangle = 0, |v| = 1, z \in \mathbb{D}\}$ contains disk $l_{x,v} \cap B_c(0, t)$. Set

$$M(x, v, s, t) = \sup_{\mathbb{D}} \log |F(\cdot, x, v, s)|.$$ 

**Definition 2.1** The number

$$b_f(s, t, r) := \sup_{x,v} \{M(x, v, s, t) - M(x, v, s, 1)\}$$

is said to be the Bernstein index of $f \in O_r$.

**Bernstein index and Remiz inequality.** Assume that $F(\cdot, x, s) (= f|_{l_{x,v} \cap B_c(0, s)})$ has valency $m$ on $\frac{1}{s}\mathbb{D}$. Assume also that $1 < t < s$. By Theorem 2.1.3 and Corollary 2.3.1 of [RY] (see also [Br2, Lemma 3.1]), there is a constant $A = A(t) > 0$ such that

$$M(x, v, s, (1 + t)/2) - M(x, v, s, 1) \leq Am. \tag{2.3}$$

Then we apply the main inequality of Theorem 1.1 of [Br2] to the function $|F|$ obtaining that there is a constant $c = c(t, A) > 0$ such that the inequality

$$\sup_{I'} |F| \leq \left(\frac{4|I'|}{|\omega'|}\right)^{cm} \sup_{\omega'} |F| \tag{2.4}$$

is valid for any interval $I' \subset [-1/s, 1/s]$ and any measurable set $\omega' \subset I'$. Since $l_{x,v} \cap B_c(0, 1) \subset \{x + vz\sqrt{s^2 - |x|^2} : \langle x, v \rangle = 0, |v| = 1, z \in \frac{1}{s}\mathbb{D}\}$, (2.4) implies inequality \([1,3]\) with exponent $cm$ for $f$ restricted to the real straight line $l_x \subset l_{x,v}$.

**2.2. Proofs of Proposition 1.7 and Theorem 1.1.** Let $1 < t < r$ and $f \in O_r$. First we prove inequality $v_f(t) < \infty$.

Fix a number $s$ satisfying $t < s < r$. For any $x \in B_c(0, s)$ consider complex straight line $l_{x,v} = \{x + vz\sqrt{s^2 - |x|^2} : \langle x, v \rangle = 0, |v| = 1, z \in \mathbb{C}\}$ passing through $x$. Let $K := \{(x, v) \in B_c(0, s) \times \mathbb{S}^{2n-1}, \langle x, v \rangle = 0\}$. Further, for $f \in O_r$ consider the function $F$ defined by (2.1). Then $F$ is analytic on $\mathbb{D} \times K$ and $F(\cdot, x, v, s)$ is holomorphic on $\mathbb{D}$ for any $(x, v) \in K$. Let $K_1 \subset K$ be a compact subset that consists of points with the first coordinate from $B_c(0, t)$. In particular, the set of lines $l_{x,v}$ with $x \in B_c(0, t)$ coincides with $L_t$ (defined just before Definition \([1,0]\) of $\mathbb{D}$. Assume
without loss of generality that \( \sup_{B_r(0, s)} |f| = 1 \) and consider the analytic function 
\( F(z, x, v, s, w) = F(z, x, v, s) - w \) defined on \( \mathbb{D} \times K \times 2\mathbb{D} \). Set 
\[
 f_1(x, v, r, w) = \sup_{z \in \frac{1}{x+r} \mathbb{D}} \log |F(z, x, v, s, w)|, \quad f_2(x, v, r, w) = \sup_{z \in \frac{1}{x+r} \mathbb{D}} \log |F(z, x, v, s, w)| .
\]

Fix \((x, v, w) \in K_1 \times \mathbb{D}\). If \( F(\cdot, x, v, s, w) \) is not a constant then the number of its zeros in \( \frac{1}{x} \mathbb{D} \) is estimated by the Jensen inequality
\[
\# \{ z \in \frac{1}{x} \mathbb{D} : F(z, x, v, s, w) = 0 \} \leq c' (f_1(x, v, r, w) - f_2(x, v, r, w))
\]
with \( c' = c'(s, t) > 0 \). Note also that by (2.2), the above number of zeros gives an upper bound for the number of points \( y \in I_{x,v} \cap B_c(0, t) \) such that \( f(y) = w \). Since \( K_1 \times \mathbb{D} \) is a compact, the Bernstein theorem of [FN3] and the Hadamard three circle theorem imply that there is a constant \( C = C(F, K_1 \times \mathbb{D}) > 0 \) such that
\[
 f_1(x, v, r, w) - f_2(x, v, r, w) \leq C
\]
for any \((x, v, w) \in K_1 \times \mathbb{D}\). This inequality yields \( v_f(t) \leq c' C \) (see Definition 1.4).

It remains to prove inequality \( d_f(r) \leq c(r) v_f((1+r)/2) \). We will do it in a parallel way with the proof of Theorem 1.1.

Let \( x \in B_c(0, 1) \) and \( l_x \subset \mathbb{C}^v \) be a real straight line passing through \( x \). Let \( I \subset l_x \cap B_c(0, 1) \) be an interval and \( \omega \subset I \) be a measurable subset. Set \( s = \frac{1+x}{2}, t = \frac{1+s}{2} \) and denote by \( l_x = \{ y + vz \sqrt{s^2 - |y|^2} : \langle y, v \rangle = 0, \ |v| = 1, \ z \in \mathbb{C} \} \) the complex straight line containing \( l_x \), where \( y \in l_x \) is such that \( \text{dist}(0, l_x) = |y| \). By definition function \( F(\cdot, y, v, s) = f|_{l_x \cap B_c(0, s)} \) determined by (2.1) has valency \( \leq v_f(s) \) on \( \frac{1}{x} \mathbb{D} \). Therefore Bernstein index \( b_f(s, \frac{14}{r}, r) \leq A v_f(s) \) for \( A = A(r) > 0 \) (see section 2.1). Finally, inequality (2.4) and arguments of section 2.1 show that the inequality of Theorem 1.1 is valid with \( d \leq c v_f(s), \ c = c(r) > 0 \). This implies that
\[
 d_f(r) \leq c v_f((1+r)/2) \quad \square
\]

Remark 2.2 In order to estimate Chebyshev degree we can also use instead of \( v_f((1+r)/2) \) an appropriate Bernstein index \( b_f(r) = b_f(s(r), t(r), r) \). Then \( d_f(r) \leq c_b f(r) \leq c v_f((1+r)/2) \) with some \( c = c(r) > 0 \).

3. Properties of Chebyshev Degree.

We formulate further inequalities between Chebyshev degree and valency. In the following proposition the constant \( c = c(r) \) is the same as in Proposition 1.7.

Proposition 3.1 (a) Let \( f \in \mathcal{O}_r \) and \( f(B_c(0, r)) \subset \mathbb{D} \subset \mathbb{C} \). Let \( \phi \) be a holomorphic function defined in an open neighbourhood \( U \supset \mathbb{D} \). Assume the \( \phi \) has valency \( k \) in \( U \). Then
\[
 d_{\phi \circ f}(r) \leq c k v_f((1+r)/2) .
\]
(b) Let \( h := e^g \in \mathcal{O}_r \). Then
\[
d_{1/h}(r) \leq cv_h((1 + r)/2)
\]

(c) There is a constant \( c_1 = c_1(r) > 0 \) such that
\[
d_{fg}(r) \leq c_1(v_f((1 + r)/2) + v_g((1 + r)/2))
\]
for any \( f, g \in \mathcal{O}_r \).

Consider differential operator \((a, D) = \sum_{i=1}^n a_i D_i\), where \( a = (a_1, \ldots, a_n) \in \mathbb{C}^n\), \( D_i := \frac{d}{dz_i}, i = 1, \ldots, n \) and \( z_1, \ldots, z_n \) are coordinates on \( \mathbb{C}^n \). Set \( f_{m,a} := (a, D)^m(f) \).

**Proposition 3.2** (The Rolle Theorem). Let \( f \in \mathcal{O}_r \). Assume that for any \( a \in \mathbb{C}^n \) the valency of \( f_{m,a} \) satisfies \( v_{f_{m,a}}(\frac{1+r}{2}) \leq M \). Then there is a constant \( c_2 = c_2(r) > 0 \) such that
\[
d_f(r) \leq c_2(m + M) \]

**Proof of Proposition 3.1.**

(a) According to the definition of the valency we have \( v_{\phi \circ f}(\frac{1+r}{2}) \leq kv_f(\frac{1+r}{2}) \), where \( k \) is valency of \( \phi \). Then \( d_{\phi \circ f}(r) \leq ckv_f(\frac{1+r}{2}) \) by Proposition 1.7.

(b) The statement follows from Proposition 1.7 and the identity \( v_{1/h}(\frac{1+r}{2}) = v_h(\frac{1+r}{2}) \) for \( h = e^g \).

(c) According to results of section 2.1 it suffices to prove the statement for univariate holomorphic functions \( F(., x, v, s) = f|_{x,v} \) and \( G(., x, v, s) = g|_{x,v} \). We consider more general situation.

Assume that \( D_{r_1} \subset D_{r_2} \subset \mathbb{C} \), \( r_1 < r_2 \), are disks centered at 0 of radii \( r_1, r_2 \), respectively. Further, assume that \( f, g \) are holomorphic in \( D_{r_2} \) of valency \( a \) and \( b \), respectively. We prove that there is a constant \( c = c(r_1, r_2) > 0 \) such that Chebyshev degree \( d_{fg}(r_1) \) of \( fg \) in \( D_{r_1} \leq c(a + b) \). Let \( K = \{ z \in \mathbb{C} : \frac{r_1 + r_2}{2} \leq |z| \leq \frac{r_1 + 3r_2}{4} \} \) be an annulus in \( D_{r_2} \) and

\[
g' = \frac{\log |g| - \sup_{D_{r_2}} \log |g|}{\sup_{D_{r_2}} \log |g| - \sup_{D_{r_1}} \log |g|}.
\]

Repeating word-for-word the arguments of Lemma 2.3 of [Br2] we can find a number \( C = C(r_1, r_2) > 0 \) and a circle \( S \subset K \) centered at 0 such that
\[
\inf_S g' \geq -C
\]

(another relatively simple proof of this result can be done by Cartan’s estimates for holomorphic functions, see, e.g. [L, p. 21]). Going back to \( |g| \) we obtain
\[
\inf_S |g| \geq \sup_{D_{r_2}} |g| \left( \frac{\sup_{D_{r_1}} |g|}{\sup_{D_{r_2}} |g|} \right)^C.
\]
This implies
\[
\frac{\sup_{D_{r_2}} |fg|}{\sup_{S} |fg|} \leq \frac{\sup_{D_{r_2}} |f| \sup_{D_{r_2}} |g|}{\sup_{S} |f| \inf_{S} |g|} \leq \frac{\sup_{D_{r_2}} |f|}{\sup_{S} |f|} \cdot \left( \frac{\sup_{D_{r_1}} |g|}{\sup_{D_{r_1}} |g|} \right)^C.
\]

Finally, according to Lemma 3.1 of [Br2] (see also section 2.1 above), there is a constant \( B = B(r_1, r_2) > 0 \) such that
\[
\frac{\sup_{D_{r_2}} |f|}{\sup_{D_{r_1 + r_2}} |f|} \leq B^a \quad \text{and} \quad \frac{\sup_{D_{r_2}} |g|}{\sup_{D_{r_1}} |g|} \leq B^b.
\]

Thus we get
\[
\frac{\sup_{D_{r_2}} |fg|}{\sup_{D_{r_1 + r_2}} |fg|} \leq \frac{\sup_{D_{r_2}} |fg|}{\sup_{S} |fg|} \leq \tilde{B}^{a+b},
\]
with \( \tilde{B} = \tilde{B}(r_1, r_2, B) > 0 \). Then inequality (2.4) applied to \(|fg|\) implies the inequality of Theorem 1.1 with exponent \( c(a + b) \), \( c = c(r_1, r_2, \tilde{B}) > 0 \). Therefore \( df_g(r_1) \leq c(a + b) \).

In the multivariate case the above arguments estimate an appropriate Bernstein index of \( fg \) by sum of Bernstein indices of \( f \) and \( g \). These indices can be estimated by \( c_1 v_f(\frac{1+r}{2}) \) and \( c_1 v_g(\frac{1+r}{2}) \) with some \( c_1 = c_1(r) > 0 \). Thus according to Remark 2.2, \( df_g(r) \leq c(r)(v_f(\frac{1+r}{2}) + v_g(\frac{1+r}{2})) \). This completes the proof of (c).

Proposition 3.1 is proved. \( \square \)

**Proof of Proposition 3.2.** We, first, recall the relation between Bernstein index and Bernstein classes (see [RY]).

**Definition 3.3** Let \( f(z) = \sum_{i=0}^{\infty} a_i z^i \) be holomorphic in the disk \( D_R, R > 1 \). We say that \( f \) belongs to the Bernstein class \( B_{N,R,c}^2 \), if for any \( j > N \),
\[
|a_j|R^j \leq c \max_{0 \leq i \leq N} |a_i|R^i.
\]

According to Corollary 2.3.1 of [RY], if the \( m \)th derivative \( f^{(m)} \) of \( f \) is \( M \)-valent then \( f^{(m+1)} \in B_{M-1,\frac{1+3r}{4},c}^2 \) with \( c := c(R) > 0 \). Moreover, from Definition 3.3 it follows that \( f \in B_{m+M,\frac{1+3r}{4},c}^2 \). Then Theorem 2.1.3 of [RY] based on the last implication yields
\[
\sup_{D_{\frac{1+3r}{4}}} |f| \leq a^{m+M} \sup_{D_{\frac{1+3r}{4}}} |f| \quad \text{(3.1)}
\]
for some constant \( a = a(R) > 1 \).

We proceed with the proof of the proposition. As in the proof of Proposition 3.1 it suffices to prove the result for restriction \( F_l \) of \( f \) to a complex line \( l \) passing through a point of \( B_c(0, 1) \). Then the condition of the proposition implies that \( m \)th derivative \( F_l^{(m)} \) of \( F_l \) has valency at most \( M \) in the larger disk \( l \cap B_c(0, \frac{1+3r}{4}) \). Therefore the required result follows immediately from inequality (3.1) (an estimate for Bernstein index) and arguments of section 2.1.

The proof of proposition is complete. \( \square \)
4. Proofs.

Proof of Proposition 1.4. Let \( l_y^c = \{ y + vz \sqrt{4 - |y|^2} : \langle y, v \rangle = 0, |v| = 1, z \in \mathbb{C} \} \) be a complex straight line passing through a point \( y \in B_c(0, 1) \). Consider restriction \( F \) of the quasipolynomial \( f(z) = \sum_{i=1}^{k} p_i(z) e^{I_i(z)} \) to \( l_y^c \). Then \( F \) is a univariate quasipolynomial of the form
\[
F(z) = q_i(z)e^{I_i(y)}e^{z\sqrt{4 - |y|^2}f_i(v)} \quad (q_i \in \mathbb{C}[z])
\]
of degree \( \leq m \). We estimate valency of \( F \) in disk \( \mathbb{D}_2 := 2\mathbb{D} \) (i.e. we estimate the number of zeros of \( F + c \) for any \( c \in \mathbb{C} \)). Note that \( F + c \) is also a quasipolynomial of degree \( \leq m + 1 \). Further, by definition \( \max_i \{|f_i(v)|\} \leq M \) implying \( \sqrt{4 - |y|^2}f_i(v) \in \mathbb{D}_{2M} \) for any \( i \). Then by Theorem 2 in [KY] the number of zeros of \( F + c \) in \( \mathbb{D}_2 \) less than or equal to \( m + \frac{2}{\pi}(\sqrt{k + 1} + 1).16M < 32(\sqrt{k + 1}M + m) \). This and Proposition 1.7 yield
\[
d_f(2) \leq c v_f(3/2) \leq c' (\sqrt{k + 1}M + m)
\]
with an absolute constant \( c' > 0 \). The required inequality follows from the definition of Chebyshev degree. □

Proof of Theorem 1.9. Let \( V \subset B(0, 1) \) be a convex body, \( \lambda \subset V \) be a measurable subset and \( f \in \mathcal{O}_r \). Take a point \( x \in V \) such that
\[
|f(x)| = \sup_{V}|f|.
\]
(Without loss of generality we may assume that \( x \) is an interior point of \( V \); for otherwise, apply the arguments below to an interior point \( x_{\epsilon} \in V, \epsilon > 0 \), such that \( |f(x_{\epsilon})| > \sup_{V}|f| - \epsilon \) and then take the limit when \( \epsilon \to 0 \).) According to Lemma 3 of [BG] there is a ray \( l \) with origin at \( x \) such that
\[
\frac{mes_1(l \cap V)}{mes_1(l \cap \lambda)} \leq \frac{n|V|}{|\lambda|}. \quad (4.1)
\]
Let \( l' \) be the real straight line containing \( l \). Applying inequality (4.1) to \( f|_\nu \) with \( I := l \cap V \) and \( \omega := l \cap \lambda \) and then inequality (4.1) lead to the required result. □

Remark 4.1 Assume that \( \omega \subset V \) is a pair of Euclidean balls of radii \( R_1 \) and \( R_2 \), respectively. Then the ray \( l \) in (4.1) can be chosen such that the constant in the inequality of Theorem 1.9 will be \( \left( \frac{4R_1}{R_2} \right)^{d_f} \).

Proof of Corollary 1.10. Let \( V \subset B(0, 1) \) be a convex body and \( f \in \mathcal{O}_r \). For the distribution function \( D_f(t) := \text{mes}\{ x \in V : |f(x)| \leq t \} \) the inequality of Theorem 1.9 acquires the form
\[
D_f(t) \leq 4n|V| \left( \frac{t}{||f||_V} \right)^{1/d_f(r)}.
\]
The required result follows from the above inequality and the identity
\[
\int_V \log \frac{|f|}{||f||_V} \ dx = \int_0^{||f||_V} \log \frac{f_s}{||f||_V} \ ds.
\]
where \( f_* = \inf \{ s : D_f(s) \geq t \} \) (cf. [Br1, Th. 5.1]). □

**Proof of Theorem 1.11.** Let \( V \subset B(0, 1) \) be a convex body. For a real straight line \( l, l \cap V \neq \emptyset \), and an interval \( I \subset l \cap V \) inequality (L3) implies

\[
\text{mes} \{ t \in I : |f(t)| \geq 10^{-d_f(r)} ||f||_I \} > |I|/2
\]

holds for any \( f \in \mathcal{O}_r \) with \( ||f||_I = \sup_I |f| \). Applying the same arguments as in the original proof of Bourgain’s inequality for polynomials [B] but based on the above inequality instead of that of Lemma 3.1 of [B] one obtains the required result.

The second part of Theorem 1.11 follows from the distributional inequality of the theorem and the definition

\[
||f||_{L^\Phi(V, dx)} := \inf \{ A \geq 0 : \int_V \Phi(|f|/A) dx \leq 1 \}
\]

□

**Proof of Corollary 1.13.** The reverse Hölder inequality (1.5) follows straightforwardly from the distributional inequality of Theorem 1.11. □

5. Concluding Remarks.

5.1. Consider a uniformly bounded sequence of functions \( \{f_i\}_{i \in I} \subset \mathcal{O}_r \) and define

\[
h_i = (||f_i||)^{1/d_{f_i}(r)}.
\]

Let

\[
h = (\lim_{i \to \infty} h_i)^*,
\]

where \( g^* \) denotes upper semicontinuous regularization of \( g \). Clearly \( h \) is logarithmically plurisubharmonic. Then one can show that inequalities of Theorems 1.10 and 1.11 hold for \( h \) with exponents 1 and \( c_2 \) instead of \( d_f(r) \) and \( c_2/d_f(r) \), respectively.

Assume that a plurisubharmonic function \( u \) is taken from the class \( L \), i.e. satisfies

\[
u(z) \leq \alpha + \log(1 + |z|) \quad (z \in \mathbb{C}^n)
\]

for some \( \alpha \in \mathbb{R} \). Then inequalities of Theorems 1.9 and 1.11 are valid for \( e^u \) restricted to a convex body \( V \subset \mathbb{R}^n \) with the constants which contain exponents 1 and \( c \) (absolute constant), respectively. It follows from the fact \( u = (\lim_{i \to \infty} (\log |p_i|)/deg p_i)^* \), where \( \{p_i\} \) is a sequence of holomorphic polynomials on \( \mathbb{C}^n \) (for the proof see, e.g. [K]).

5.2. Inequalities of Theorems 1.9 and 1.11 can also be written in the same form for convex bodies in \( B_e(0, 1) \), where one replaces coefficient \( 4n \) by \( 8n \) in the first inequality.

5.3. If \( f_1, ..., f_k \) are functions from \( \mathcal{O}_r \) and \( p \) is a holomorphic polynomial of degree \( d \) then for \( h = p(f_1, ..., f_k) \) its degree \( d_h(r) \) is bounded by a constant depending on \( d, r \) and \( f_1, ..., f_k \). It follows, e.g., from results of [FN3] and arguments used in the proof of Proposition 1.7. However, it is difficult to obtain an explicit estimate for \( d_h(r) \) even in the case of naturally defined functions \( f_i \) (e.g., taken as solutions of some systems of ODEs). Assume, e.g., that \( f_1 = z_1, ..., f_n = z_n \) are coordinate functions on \( \mathbb{C}^n \) and \( k \geq n \). Then inequality \( d_h(r) \leq cd \) holds for any polynomial \( p \) of degree \( d \) with \( c \) which does not depend on \( d \) if and only if \( f_{n+1}, ..., f_k \) are algebraic functions, see [S] and [Br2, Th.1.3].
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