LIMITING BEHAVIOR OF ADDITIVE FUNCTIONALS ON THE STABLE TREE

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Abstract. We consider the normalized stable tree $T$ with index $\gamma \in (1,2]$ and study the asymptotic behavior of additive functionals of the form

$$Z_{\alpha, \beta} = \int_{T} \mu(dx) \int_{0}^{H(x)} \sigma_{r,x}^{\alpha} h_{r,x}^{\beta} dr,$$

as $\max(\alpha, \beta) \to \infty$, where $\mu$ is the mass measure on $T$, $H(x)$ is the height of $x$ and $\sigma_{r,x}$ (resp. $h_{r,x}$) is the mass (resp. height) of the subtree of $T$ above level $r$ containing $x$. Such functionals arise as scaling limits of additive functionals of the size and height on conditioned Bienaymé-Galton-Watson trees [2]. We distinguish different regimes depending on the behavior of $\beta / \alpha^{1-1/\gamma}$ and describe the limiting random variable in terms of the total height $h$ of $T$ and an independent $(1-1/\gamma)$-stable subordinator.

1. Introduction

Stable trees are special instances of Lévy trees which were introduced by Le Gall and Le Jan [20] in order to generalize Aldous’ Brownian tree [4]. More precisely, stable trees are compact weighted rooted real trees depending on a parameter $\gamma \in (1,2]$, with $\gamma = 2$ corresponding to the Brownian tree, which encode the genealogical structure of continuous-state branching processes with branching mechanism $\psi(\lambda) = \lambda^\gamma$. As such, they are the possible scaling limits of conditioned Bienaymé-Galton-Watson trees with critical offspring distribution belonging to the domain of attraction of a stable distribution with index $\gamma \in (1,2]$, see Duquesne [8] and Kortchemski [19]. They also appear as scaling limits of various models of trees and graphs, see e.g. Haas and Miermont [17], and are intimately related to fragmentation and coalescence processes, see Miermont [22, 23] and Berestycki, Berestycki and Schweinsberg [5]. Stable trees can be defined via the normalized excursion of the so-called height process which is a local time functional of a spectrally positive Lévy process. We refer to Duquesne and Le Gall [9] for a detailed account. See also Duquesne and Winkel [11], Goldschmidt and Haas [14], Marchal [21] for alternative constructions.

In the present paper, we study the asymptotic behavior as $\max(\alpha, \beta) \to \infty$ of additive functionals on the normalized stable tree $T$ (i.e. the stable tree conditioned to have total mass 1) of
the form
\[ Z_{\alpha,\beta} = \int_T Z_{\alpha,\beta}(x) \mu(dx) \quad \text{with} \quad \forall x \in T, \quad Z_{\alpha,\beta}(x) = \int_0^{H(x)} \sigma_{r,x}^\alpha h_{r,x}^\beta \, dr, \]
(1.1)
where \( \mu \) is the mass measure on \( T \) which is a uniform measure supported by the set of leaves, \( H(x) \) is the height of \( x \in T \), that is its distance to the root, and \( \sigma_{r,x} \) (resp. \( h_{r,x} \)) is the mass (resp. height) of the subtree of \( T \) above level \( r \) containing \( x \). Such functionals arise as scaling limits of additive functionals of the size and height on conditioned Bienaymé-Galton-Watson trees, see Delmas, Dhersin and Sciauveau [7] or Abraham, Delmas and Nassif [2] where it is shown that \( Z_{\alpha,\beta} < \infty \) a.s. if (and only if) \( \gamma \alpha + (\gamma - 1)(\beta + 1) > 0 \), see Corollary 6.10 therein. In the present paper, we only consider \( \alpha,\beta \geq 0 \) which guarantees the finiteness of \( Z_{\alpha,\beta} \). For example, let us mention the total path length and the Wiener index which properly scaled converge respectively to \( Z_{0,0} \) and \( Z_{1,0} \). Fill and Janson [13] considered the case \( \gamma = 2 \) and \( \beta = 0 \) (i.e. functionals of the mass on the Brownian tree) and proved that there is convergence in distribution as \( \alpha \to \infty \) of \( Z_{\alpha,0} \) properly normalized to
\[ \int_0^\infty e^{-S_t} \, dt, \]
where \((S_t, t \geq 0)\) is a 1/2-stable subordinator. Their proof relies on the connection between the normalized Brownian excursion which codes the Brownian tree and the three-dimensional Bessel bridge. Our aim is twofold: we extend their result to the non-Brownian stable case \( \gamma \in (1,2) \) while also considering polynomial functionals depending on both the mass and the height. We use a different approach relying on the Bismut decomposition of the stable tree. Even though the computations are a bit involved, the underlying idea is simply explained as follows: zooming in at the root of the normalized stable tree \( T \), one gets an infinite branch on which subtrees are grafted according to a Poisson point process much like the Kesten tree. In particular, the conditioning for the total mass of \( T \) to be equal to 1 disappears when zooming at the root.

This idea to zoom in at the root of the stable tree is closely related to the small time asymptotics present in the works of Miermont [22] and Haas [16] – of the self-similar fragmentation process \( F^-(t) = (F_1^-(t), F_2^-(t), \ldots) \) obtained from the stable tree by removing vertices located under height \( t \). In fact, it is not hard to see that, at least when \( \beta = 0 \), the additive functional we consider can be expressed in terms of \( F^- \)
\[ Z_{\alpha,0} = \sum_{i \geq 1} \int_0^\infty F_i^-(t)^{\alpha+1} \, dt. \]
Once this is established, one can argue that only the largest fragment \( F_1^- \) contributes to the limit, the others being negligible, then use [16, Corollary 17] which implies that \( 1 - F_1^- \) properly normalized converges in distribution to a \((1 - 1/\gamma)\)-stable subordinator \( S \), to get the convergence of \( Z_{\alpha,0} \) to \( \int_0^\infty e^{-S_t} \, dt \). In the present paper, we do not adopt this approach as it does not allow to consider functionals of the height (that is \( \beta \neq 0 \)).

Denote by \( \mathcal{N}^{(1)} \) the distribution of the normalized stable tree with total mass 1 and by \( h \) its height, see Section 3 for a precise definition. Our main result can be stated as follows, see Theorem 4.5.

**Theorem 1.1.** Assume that \( \alpha \to \infty \), \( \beta \geq 0 \) and \( \beta/\alpha^{1-1/\gamma} \to c \in [0, \infty) \). Let \( T \) be the normalized stable tree with branching mechanism \( \psi(\lambda) = \lambda^\gamma \) where \( \gamma \in (1,2) \). Then we have the following
convergence in distribution under $\mathbb{N}^{(1)}$

$$\alpha^{1-1/\gamma} h^{-\beta} Z_{\alpha,\beta} \xrightarrow{(d)} \int_0^\infty e^{-S_t - \alpha/h} \, dt,$$

where $(S_t, t \geq 0)$ is a stable subordinator with Laplace exponent $\varphi(\lambda) = \gamma \lambda^{1-1/\gamma}$, independent of $T$.

Let us briefly explain why we get a subordinator $S$ at the limit. It is well known that $\mu$ is supported on the set of leaves of $T$. Let $x \in T$ be a leaf and recall that $\sigma_{r,x}$ is the mass of the subtree above level $r$ containing $x$. Since the total mass of the stable tree is 1, the main contribution to $Z_{\alpha,\beta}(x)$ as $\alpha \to \infty$ comes from large subtrees $T_{r,x}$ with $r$ close to 0. The height $h_{r,x}$ of such subtrees is approximately $h - r$. On the other hand, their mass is equal to 1 minus the mass we discarded from the subtrees grafted on the branch $[\emptyset, x]$ at height less than $r$. Roughly speaking, subtrees are grafted on $[\emptyset, x]$ according to a point process which is approximately Poissonian, at least close to the root $\emptyset$. Thus the mass $\sigma_{r,x}$ is approximately $1 - S_r$.

Theorem 4.5 is slightly more general: we prove joint convergence in distribution of $\alpha^{1-1/\gamma} h^{-\beta} Z_{\alpha,\beta}$ and $\alpha^{1-1/\gamma} h^{-\beta} Z_{\alpha,\beta}(U)$, where $U \in T$ is a leaf chosen uniformly at random (i.e. according to the measure $\mu$), to the same random variable. In other words, taking the average of $Z_{\alpha,\beta}(x)$ over all leaves yields the same asymptotic behavior as taking a leaf uniformly at random. This is due to the following observations: a) a uniform leaf $U$ is not too close to the root with high probability in the sense that its most recent common ancestor with $x^*$ has height greater than $\varepsilon$, where $x^* \in T$ is the heighest leaf, b) when taking the average over all leaves, the contribution of those leaves whose most recent common ancestor with $x^*$ has height less than $\varepsilon$ is negligible, and c) for those $x \in T$ whose most recent common ancestor with $x^*$ has height greater than $\varepsilon$, the main contribution to $Z_{\alpha,\beta}(x)$ comes from large subtrees $T_{r,x}$ with $r$ close to $\varepsilon$, these subtrees are common to all such leaves as $T_{r,x} = T_{r,x^*}$. This is made rigorous in Lemma 4.4.

Let us make a connection with Theorem 1.18 of Fill and Janson [13]. Recall that the normalized Brownian tree with branching mechanism $\psi(\lambda) = \lambda^2$ is coded by $\sqrt{2} B^{\text{ex}}$ where $B^{\text{ex}}$ is the normalized Brownian excursion, see [9]. Thanks to the representation formula of [7, Lemma 8.6], we see that Fill and Janson’s $Y(\alpha) = \sqrt{2} Z_{\alpha-1,0}$. Thus, we recover their result in the Brownian case $\gamma = 2$ when $\beta = 0$ (in which case $c = 0$).

Notice that as long as the exponent $\beta$ of the height does not grow too quickly, viz. $\beta/\alpha^{1-1/\gamma} \to 0$, the additional dependence on the height makes no contribution at the limit. On the other hand, if $\beta/\alpha^{1-1/\gamma} \to \infty$, we get the convergence in probability of $Z_{\alpha,\beta}$ with a different scaling. See Theorem 5.1 for a more general statement.

**Theorem 1.2.** Assume that $\beta \to \infty$, $\alpha \geq 0$ and $\alpha^{1-1/\gamma} / \beta \to 0$. Let $T$ be the normalized stable tree with branching mechanism $\psi(\lambda) = \lambda^\gamma$ where $\gamma \in (1, 2)$. Then we have the following convergence in $\mathbb{N}^{(1)}$-probability

$$\lim_{\beta \to \infty} \beta h^{-\beta} Z_{\alpha,\beta} = h.$$  

$$\text{(1.3)}$$
**Remark 1.3.** Assume that \( \alpha, \beta \to \infty \) and \( \beta/\alpha^{1-1/\gamma} \to c \in (0, \infty) \) so that Theorem 1.1 applies. Then we have the convergence in distribution under \( \mathbb{N}^{(1)} \)

\[
\beta h^{-\beta} Z_{\alpha, \beta} = \frac{\beta}{\alpha^{1-1/\gamma}} h^{-\beta} Z_{\alpha, \beta} \xrightarrow{(d) \beta \to \infty} c \int_0^\infty e^{-S_t - ct/h} \, dt = h \int_0^\infty e^{-S_{ht/c} - t} \, dt.
\]

Now letting \( c \to \infty \), the right-hand side converges to \( h \int_0^\infty e^{-t} \, dt = h \). Thus, one may view Theorem 1.2 as a special case of Theorem 1.1 by saying that, if \( \beta \to \infty \) and \( \beta/\alpha^{1-1/\gamma} \to c \in (0, \infty) \), then we have the convergence in distribution under \( \mathbb{N}^{(1)} \)

\[
\beta h^{-\beta} Z_{\alpha, \beta} \xrightarrow{(d) \beta \to \infty} c \int_0^\infty e^{-S_t - ct/h} \, dt,
\]

(1.4)

where the measure \( ce^{-ct/h} \, dt \) should be understood as the Dirac measure \( \delta_0 \) if \( c = \infty \).

We conclude the introduction by giving a decomposition of a general (compact) Lévy tree which is of independent interest. Let \( \mathbb{T} \) be the space of weighted rooted compact real trees, see Section 2 for a precise definition. Consider a Lévy tree \( \mathcal{T} \) under its excursion measure \( \mathbb{N} \) associated with a branching mechanism \( \psi(\lambda) = a\lambda + b\lambda^2 + \int_0^\infty (e^{-\lambda r} - 1 + \lambda r) \pi(dr) \) where \( a, b \geq 0 \) and \( \pi \) is a \( \sigma \)-finite measure on \((0, \infty)\) satisfying \( \int_0^\infty (r \wedge r^2) \pi(dr) < \infty \). We further assume that \( \int_0^\infty d\lambda/\psi(\lambda) < \infty \) which is equivalent to the compactness of the Lévy tree. We refer to [9, Section 1] for a complete presentation of the subject. For every \( x \in \mathcal{T} \) and every \( 0 \leq r < r' \leq H(x) \), we let \( \mathcal{T}_{[r, r')} = (\mathcal{T}_{r, x} \setminus \mathcal{T}_{r', x}) \cup \{x_{r'}\} \) where \( x_{r'} \) is the unique ancestor of \( x \) at height \( H(x_{r'}) = r' \). The following result states that, when \( x \in \mathcal{T} \) and \( 0 = r_0 < r_1 < \ldots < r_n < r_{n+1} := H(x) \) are chosen “uniformly” at random under \( \mathbb{N} \), then the random trees \( \mathcal{T}_{[r_{i-1}, r_i), x}, 1 \leq i \leq n + 1 \) are independent and distributed as \( \mathcal{T} \) under \( \mathbb{N}[\sigma] \), see Figure 1. In particular, this generalizes [2, Lemma 6.1] which corresponds to \( n = 1 \).

**Figure 1.** The decomposition of \( \mathcal{T} \) under \( \mathbb{N} \) into \( n+1 \) subtrees along the ancestral line of a uniformly chosen leaf \( x \).
Theorem 1.4. Let $T$ be the Lévy tree with a general branching mechanism under its excursion measure $\mathbb{N}$. Then for every $n \geq 1$ and every nonnegative measurable functions $f_i$, $1 \leq i \leq n + 1$ defined on $\mathbb{T}$, we have

$$\mathbb{N} \left[ \int_T \mu(dx) \int_{0<r_1<\ldots<r_n<h(x)} \prod_{i=1}^{n+1} f_i(T_{[r_{i-1},r_i]},x) \prod_{i=1}^{n} dr_i \right] = \prod_{i=1}^{n+1} \mathbb{N}[\sigma f_i(T)].$$

The paper is organized as follows. Section 2 defines the space of real trees and the Gromov-Hausdorff-Prokhorov topology. In Section 3, we introduce the stable tree, recall some of its properties and prove Theorem 1.4 as well as some other useful results. Sections 4 and 5 treat the cases $\beta/\alpha^{1-1/\gamma} \to c \in [0, \infty)$ and $\beta/\alpha^{1-1/\gamma} \to \infty$ respectively. Finally, we gather some technical proofs in Section 6.

2. Real trees and the Gromov-Hausdorff-Prokhorov topology

2.1. Real trees. We recall the formalism of real trees, see [12]. A metric space $(T,d)$ is a real tree if the following two properties hold for every $x, y \in T$.

(i) (Unique geodesics). There exists a unique isometric map $f_{x,y} : [0,d(x,y)] \to T$ such that $f_{x,y}(0) = x$ and $f_{x,y}(d(x,y)) = y$.

(ii) (Loop-free). If $\varphi$ is a continuous injective map from $[0,1]$ into $T$ such that $\varphi(0) = x$ and $\varphi(1) = y$, then we have

$$\varphi([0,1]) = f_{x,y}([0,d(x,y)]).$$

A weighted rooted real tree $(T, \emptyset, d, \mu)$ is a real tree $(T,d)$ with a distinguished vertex $\emptyset \in T$ called the root and equipped with a nonnegative finite measure $\mu$. Let us consider a weighted rooted real tree $(T, \emptyset, d, \mu)$. The range of the mapping $f_{x,y}$ described above is denoted by $\llbracket x, y \rrbracket$ (this is the line segment between $x$ and $y$ in the tree). In particular, $\llbracket 0, x \rrbracket$ is the path going from the root to $x$ which we will interpret as the ancestral line of vertex $x$. We define a partial order on the tree by setting $x \preceq y$ ($x$ is an ancestor of $y$) if and only if $x \in \llbracket 0, y \rrbracket$. If $x, y \in T$, there is a unique $z \in T$ such that $\llbracket 0, x \rrbracket \cap \llbracket 0, y \rrbracket = \llbracket 0, z \rrbracket$. We write $z = x \wedge y$ and call it the most recent common ancestor to $x$ and $y$. For every vertex $x \in T$, we define its height by $H(x) = d(\emptyset, x)$. The height of the tree is defined by $h(T) = \sup_{x \in T} H(x)$. Note that if $(T,d)$ is compact, then $h(T) < \infty$.

Let $x \in T$ be a vertex. For every $r \in [0, H(x)]$, we denote by $x_r \in T$ the unique ancestor of $x$ at height $r$. Furthermore, we define the subtree $T_{r,x}$ of $T$ above level $r$ containing $x$ as

$$T_{r,x} = \{ y \in T : H(x \wedge y) \geq r \}.$$

Equivalently, $T_{r,x}$ is the subtree of $T$ above $x_r$. Then $T_{r,x}$ can be naturally viewed as a weighted rooted real tree, rooted at $x_r$ and endowed with the distance $d$ and the measure $\mu|_{T_{r,x}}$. Note that $T_{0,x} = T$ and $T_{H(x),x} = T_x$ is the subtree of $T$ above $x$. Denote by

$$\sigma_{r,x}(T) = \mu(T_{r,x}) \quad \text{and} \quad h_{r,x}(T) = h(T_{r,x}).$$

(2.2)
Next, for any nonnegative finite measure $m$ defined as 

$$Z_{\alpha,\beta}(x) = \int_0^{H(x)} \sigma_{r,x}(T)^{\alpha} \beta_{r,x}(T)^{\beta} \, dr, \quad \forall x \in T. \quad (2.3)$$

We shall omit the dependence on $T$ when there is no ambiguity, simply writing $\sigma_{r,x}$, $\beta_{r,x}$ and $Z_{\alpha,\beta}(x)$. For every $0 \leq r < r' \leq H(x)$, we also introduce the notation

$$T_{[r,r'),x} = (T_{r,x} \setminus T_{r',x}) \cup \{x_r\} = \{y \in T : r \leq H(x \land y) < r'\} \cup \{x_r\}, \quad (2.4)$$

which defines a weighted rooted real tree, equipped with the distance and the measure it inherits from $T$ and naturally rooted at $x_r$.

The next lemma, whose proof is elementary, relates $\beta_{r,x}(T)$, the height of the subtree of $T$ above level $r$ containing $x$, to the total height $\beta(T)$.

**Lemma 2.1.** Let $T$ be a compact real tree. For every $x \in T$ and $r \in [0, H(x)]$, we have

$$\beta(T) \geq \beta_{r,x}(T) + r. \quad (2.5)$$

Furthermore, if $x^* \in T$ is such that $H(x^*) = \beta(T)$, then for every $r \in [0, H(x^* \land x)]$, we have

$$\beta(T) = \beta_{r,x}(T) + r. \quad (2.6)$$

**2.2. The Gromov-Hausdorff-Prokhorov topology.** We denote by $T$ the set of (measure-preserving, root-preserving isometry classes of) compact real trees. We will often identify a class with an element of this class. So we shall write $(T, \emptyset, d, \mu) \in T$ for a weighted rooted compact real tree.

Let us define the GHP topology on $T$. Let $(T, \emptyset, d, \mu), (T', \emptyset', d', \mu') \in T$ be two compact real trees. Recall that a correspondence between $T$ and $T'$ is a subset $\mathcal{R} \subset T \times T'$ such that for every $x \in T$, there exists $x' \in T'$ such that $(x, x') \in \mathcal{R}$, and conversely, for every $x' \in T'$, there exists $x \in T$ such that $(x, x') \in \mathcal{R}$. In other words, if we denote by $p : T \times T' \to T$ (resp. $p' : T \times T' \to T'$) the canonical projection on $T$ (resp. on $T'$), a correspondence is a subset $\mathcal{R} \subset T \times T'$ such that $p(\mathcal{R}) = T$ and $p'(\mathcal{R}) = T'$. If $\mathcal{R}$ is a correspondence between $T$ and $T'$, its distortion is defined by

$$\text{dis}(\mathcal{R}) = \sup \{d(x, y) - d'(x', y') : (x, x'), (y, y') \in \mathcal{R} \}.$$ 

Next, for any nonnegative finite measure $m$ on $T \times T'$, we define its discrepancy with respect to $\mu$ and $\mu'$ by

$$D(m; \mu, \mu') = d_{\text{TV}}(m \circ p^{-1}, \mu) + d_{\text{TV}}(m \circ p'^{-1}, \mu'),$$

where $d_{\text{TV}}$ denotes the total variation distance. Then the GHP distance between $T$ and $T'$ is defined as

$$d_{\text{GHP}}(T, T') = \inf \left\{ \frac{1}{2} \text{dis}(\mathcal{R}) \lor D(m; \mu, \mu') \lor m(\mathcal{R}^c) \right\}, \quad (2.7)$$

where the infimum is taken over all correspondences $\mathcal{R}$ between $T$ and $T'$ such that $(\emptyset, \emptyset') \in \mathcal{R}$ and all nonnegative finite measures $m$ on $T \times T'$. It can be verified that $d_{\text{GHP}}$ is indeed a distance on $T$ and that the space $(T, d_{\text{GHP}})$ is complete and separable, see e.g. [3].
The next lemma gives an upper bound for the GHP distance between a tree \((T, \emptyset, d, \mu) \in \mathbb{T}\) and the tree \((T, \emptyset, ad, b\mu)\) obtained from \(T\) by multiplying all distances by \(a > 0\) and the measure \(\mu\) by \(b > 0\). The proof is elementary and is left to the reader.

**Lemma 2.2.** For every \(T \in \mathbb{T}\) and \(a, b > 0\), we have
\[
d_{GHP}((T, \emptyset, d, \mu), (T, \emptyset, ad, b\mu)) \leq 2|a - 1|h(T) + |b - 1|\mu(T). \tag{2.8}
\]

3. The stable tree

Here, we define the stable tree and recall some of its properties. We refer to \([10]\) for background.

We shall work with the stable tree \(\mathcal{T}\) with branching mechanism \(\psi(\lambda) = \lambda^\gamma\) where \(\gamma \in (1, 2]\) under its excursion measure \(\mathbb{N}\): more explicitly, using the coding of compact real trees by height functions, one can define a \(\sigma\)-finite measure \(\mathbb{N}\) on \(\mathbb{T}\) with the following properties.

(i) **Mass measure.** \(\mathbb{N}\)-a.e. the mass measure \(\mu\) is supported by the set of leaves \(\text{Lf} (\mathcal{T})\) and the distribution on \((0, \infty)\) of the total mass \(\sigma := \mu(\mathcal{T})\) is given by
\[
\mathbb{N}[\sigma \in da] = \frac{1}{\gamma \Gamma(1 - 1/\gamma)} \frac{da}{a^{1+1/\gamma}}.
\]

(ii) **Height.** \(\mathbb{N}\)-a.e. there exists a unique leaf \(x^* \in \mathcal{T}\) realizing the height, that is \(H(x^*) = h(\mathcal{T})\), and the distribution on \((0, \infty)\) of the height \(h := h(\mathcal{T})\) is given by
\[
\mathbb{N}[h \in da] = (\gamma - 1)^{-\gamma/(\gamma - 1)} \frac{da}{a^{\gamma/(\gamma - 1)}}.
\]

We will also deal with regular versions of the probability measures \(\mathbb{N}[\bullet | \sigma = a]\) for \(a > 0\). Using the scaling property of the stable tree, one can define a regular conditional probability measure \(\mathbb{N}^{(a)} = \mathbb{N}[\bullet | \sigma = a]\) such that \(\mathbb{N}^{(a)}\)-a.s. \(\sigma = a\) and
\[
\mathbb{N}[\bullet] = \frac{1}{\gamma \Gamma(1 - 1/\gamma)} \int_0^\infty \mathbb{N}^{(a)}[\bullet] \frac{da}{a^{1+1/\gamma}}.
\]

Informally, \(\mathbb{N}^{(a)}\) can be seen as the distribution of the stable tree \(\mathcal{T}\) with total mass \(a\).

We will make extensive use of the scaling property of the stable tree under \(\mathbb{N}\). Define the mapping \(R\): \(\mathbb{T} \times (0, \infty) \rightarrow \mathbb{T}\) by
\[
R((T, \emptyset, d, \mu), a) = (T, \emptyset, a^{1-1/\gamma}d, a\mu). \tag{3.1}
\]

In words, \(R((T, \emptyset, d, \mu), a)\) is the tree obtained from \((T, \emptyset, d, \mu)\) by multiplying all distances by \(a^{1-1/\gamma}\) and all masses by \(a\). Note that if \(T\) has total mass \(\sigma\) and height \(h\) then \(R(T, a)\) has total mass \(a\sigma\) and height \(a^{1-1/\gamma}h\). Furthermore, it is straightforward to show that
\[
\sigma_{r,x}(R(T, a)) = a\sigma_{a^{1-1/\gamma}r,x}(T),
\]
\[
h_{r,x}(R(T, a)) = a^{1-1/\gamma}h_{a^{1-1/\gamma}r,x}(T),
\]
\[
Z_{\alpha,\beta}^{R(T, a)}(x) = a^{\alpha + (\beta + 1)(1-1/\gamma)}Z_{\alpha,\beta}^T(x). \tag{3.2}
\]
Lemma 3.1. Let $\mathcal{T}$ be the stable tree with branching mechanism $\psi(\lambda) = \lambda^\gamma$ where $\gamma \in (1, 2]$.

(i) For every measurable function $F: \mathbb{T} \to [0, \infty]$, we have

$$\mathbb{N}^{(1)}[F(\mathcal{T})] = \Gamma(1-1/\gamma)\mathbb{N}[\{\sigma > 1\}F(R(\mathcal{T}, \sigma^{-1}))].$$

(ii) Under $\mathbb{N}^{(a)}$, the random tree $\mathcal{T}$ is distributed as $R(\mathcal{T}, a)$ under $\mathbb{N}^{(1)}$ for every $a > 0$.

We shall need Bismut’s decomposition of the stable tree on several occasions. This is a decomposition of the tree along the ancestral line of a uniformly chosen leaf. We refer the reader to [10, Theorem 4.5] and [1, Theorem 2.1] for more details. Although in this paper we are only interested in the stable case $\psi(\lambda) = \lambda^\gamma$, we state the next two results in the general Lévy case. Let $\mathcal{T}$ denote a Lévy tree under its excursion measure $\mathbb{N}$ associated with a branching mechanism

$$\psi(\lambda) = a\lambda + b\lambda^2 + \int_0^\infty (e^{-\lambda r} - 1 + \lambda r) \pi(dr)$$

(3.3)

where $a, b \geq 0$ and $\pi$ is a $\sigma$-finite measure on $(0, \infty)$ satisfying $\int_0^\infty (r \wedge r^2) \pi(dr) < \infty$. We further assume that $\int_0^\infty d\lambda/\psi(\lambda) < \infty$ so that the Lévy tree is compact. Notice that the Brownian case $\gamma = 2$ corresponds to $a = 0$, $b = 1$ and $\pi = 0$ while the non-Brownian stable case $\gamma \in (1, 2)$ corresponds to $a = b = 0$ and

$$\pi(dr) = \frac{\gamma(\gamma-1)}{\Gamma(2-\gamma)} \frac{dr}{r^{1+\gamma}}.$$  

(3.4)

We will also need the probability measure $\mathbb{P}_r$ on $\mathbb{T}$ which is the distribution of the Lévy tree starting from $r > 0$ individuals. More precisely, take $\sum_{i \in I} \delta_{T_i}$ a Poisson point measure on $\mathbb{T}$ with intensity $r \mathbb{N}$ and define $\mathbb{P}_r$ as the distribution of the random tree $\mathcal{T}$ obtained by gluing together the trees $T_i$ at their root. See [1, Section 2.6] for further details.

Before stating the result, we first introduce some notations. Let $(T, \emptyset, d, \mu) \in \mathbb{T}$ be a compact real tree and let $x \in T$. Denote by $(x_i, i \in I_x)$ the branching points of $T$ which lie on the branch $[\emptyset, x]$, that is those points $y \in [\emptyset, x]$ such that $T \setminus \{y\}$ has at least three connected components. For every $i \in I_x$, define the tree grafted on the branch $[\emptyset, x]$ at $x_i$ by $T_i = \{y \in T : x \wedge y = x_i\}$. We consider $T_i$ as an element of $\mathbb{T}$ in the obvious way. Let $h_i = H(x_i)$ and define a point measure on $[0, \infty) \times \mathbb{T}$ by

$$\mathcal{M}_x^T = \sum_{i \in I_x} \delta_{(h_i, T_i)}.$$

We can now state Bismut’s decomposition, see [10, Theorem 4.5] or [1, Theorem 2.1].

Theorem 3.2. Let $\mathcal{T}$ be the Lévy tree with a general branching mechanism (3.3) under its excursion measure $\mathbb{N}$. For every $\lambda \geq 0$ and every nonnegative measurable function $\Phi$ on $[0, \infty) \times \mathbb{T}$, we have

$$\mathbb{N}\left[\int_T \mu(dx)e^{-\lambda H(x) - \langle M_x^T, \Phi \rangle}\right] = \int_0^\infty dt e^{-(\lambda+a)t} \mathbb{E}\left[e^{-\sum_{0 \leq s \leq t} \Phi(s, T_s)}\right],$$

(3.5)

where $(T_s, 0 \leq s \leq t)$ is a Poisson point process with intensity $2b\mathbb{N}[dT] + \int_0^\infty r \pi(dr)\mathbb{P}_r(dT)$. 


Remark 3.3. Bismut’s decomposition states the following: let \( \mathcal{T} \) be the Lévy tree under its excursion measure \( N \) and, conditionally on \( \mathcal{T} \), let \( U \) be a leaf chosen uniformly at random, i.e. according to the distribution \( \sigma^{-1} \mu \). Then, under \( \mathbb{N}[\sigma \bullet] \), the random variable \( H(U) \) has “distribution” \( e^{-ut} \, dt \) on \( (0, \infty) \) and, conditionally on \( H(U) = t \), the point measure \( M_{t}^{\mathcal{T}} \) is distributed as \( (T_{s}, 0 \leq s \leq t) \).

One can make this claim rigorous by introducing the space of compact weighted rooted real trees with an additional marked vertex and considering the semidirect product measure \( N \times \sigma^{-1} \mu \) on it which corresponds to the distribution of the pair \((\mathcal{T}, U)\).

Let \((T_{s}, 0 \leq s \leq t)\) be a Poisson point process as in Theorem 3.2 and denote by
\[
T_{r}^{\downarrow} := [t-r,t] \otimes_{t-r \leq s \leq t} (T_{s}, s) \quad \forall 0 \leq r \leq t
\]
the random real tree obtained by grafting \( T_{s} \) on a branch \([t-r,t]\) at height \( s \) for every \( t-r \leq s \leq t \) and rooted at \( t-r \), see Figure 2. We refer the reader to [1, Section 2.4] for a precise definition of the grafting procedure. Let
\[
\tau_{r} := \mu(T_{r}^{\downarrow}) = \sum_{t-r \leq s \leq t} \mu(T_{s}) \quad \text{and} \quad \eta_{r} := h(T_{r}^{\downarrow}) = \max_{t-r \leq s \leq t} (h(T_{r}) + s - (t-r))
\]
denote its mass and height. Finally, let
\[
S_{r} := \sum_{s \leq r} \mu(T_{s}).
\]

It is shown in the proof of [7, Lemma 4.6], see Section 8.6 and more precisely (8.20) therein, that in the stable case \( \psi(\lambda) = \lambda^\gamma \), both \( \tau \) and \( S \) are subordinators defined on \([0,t]\) with Laplace exponent
\[
\varphi(\lambda) = \gamma \lambda^{1-1/\gamma}.
\]

We now give the following form of Bismut’s decomposition which we will use throughout the paper. Denote by \( D[0,\infty) \) the space of càdlàg functions on \([0,\infty)\) endowed with the Skorokhod \( J_{1} \) topology. For every measurable function \( F: [0,\infty)^{3} \times \mathbb{T} \times D[0,\infty)^{2} \to [0,\infty] \), we have
\[
\mathbb{N} \left[ \int_{\mathcal{T}} \mu(dx) F \left( H(x), \sigma, h, T, \left( \sigma_{H(x)-r} x, 0 \leq r \leq H(x) \right), \left( h_{H(x)-r} x, 0 \leq r \leq H(x) \right) \right) \right]
\]
\[ = \int_0^\infty dt \mathbb{E} \left[ F \left( t, \tau_t, \eta_t, T_t^1, (\tau_r, 0 \leq r \leq t), (\eta_r, 0 \leq r \leq t) \right) \right]. \]  

(3.10)

Notice that by definition \( \tau_t = S_t \) and \( S_{t-r} = \tau_t - \tau_{t-r} \) for every \( r \in [0, t] \). This will be used implicitly in the sequel. In particular, the following computation will be useful

\[ \int_0^\infty \mathbb{E} \left[ \frac{1}{S_t} \mathbf{1}_{\{S_t > 1\}} \right] dt = \int_0^\infty \mathbb{E} \left[ \frac{1}{\tau_t} \mathbf{1}_{\{\tau_t > 1\}} \right] dt = \mathbb{N} [\sigma > 1] = \frac{1}{\Gamma(1 - 1/\gamma)}, \]

(3.11)

where in the last equality we used Lemma 3.1-(i) with \( F \equiv 1 \).

As a first application of Bismut’s decomposition, we give a decomposition of the stable tree into \( n + 1 \) subtrees, which says that if we choose a leaf \( x \in \mathcal{T} \) and \( 0 = r_0 < r_1 < \ldots < r_n < r_{n + 1} := H(x) \) “uniformly” at random under \( \mathbb{N} \), then the random trees \( \mathcal{T}_{[r_i, r_{i+1})}, x, 1 \leq i \leq n + 1 \) are independent and distributed as \( \mathcal{T} \) under \( \mathbb{N}[\sigma \bullet] \). In particular, this generalizes [2, Lemma 6.1] which corresponds to \( n = 1 \).

**Theorem 3.4.** Let \( \mathcal{T} \) be the Lévy tree with a general branching mechanism (3.3) under its excursion measure \( \mathbb{N} \). Then for every \( n \geq 1 \) and every nonnegative measurable functions \( f_i, 1 \leq i \leq n + 1 \) defined on \( \mathbb{T} \), we have

\[ \mathbb{N} \left[ \int_\mathcal{T} \mu(dx) \int_{0 < r_1 < \ldots < r_n < H(x)} \prod_{i=1}^{n+1} f_i \left( \mathcal{T}_{[r_i, r_{i+1})}, x \right) \prod_{i=1}^{n} dr_i \right] = \prod_{i=1}^{n+1} \mathbb{N} [\sigma f_i(\mathcal{T})]. \]  

(3.12)

**Proof.** Recall from (3.6) the definition of \( T^1 \). By Theorem 3.2, we have

\[ \mathbb{N} \left[ \int_\mathcal{T} \mu(dx) \int_{0 < r_1 < \ldots < r_n < H(x)} \prod_{i=1}^{n+1} f_i \left( \mathcal{T}_{[r_i, r_{i+1})}, x \right) \prod_{i=1}^{n} dr_i \right] = \int_0^\infty dt \mathbb{E} \left[ \int_{0 < r_1 < \ldots < r_n < t} \prod_{i=1}^{n+1} f_i \left( \mathcal{T}_{[r_i, r_{i+1})} \right) \prod_{i=1}^{n} dr_i \right], \]

where we set \( \mathcal{T}_{[r, r')} = (\mathcal{T}_{t-r} \setminus \mathcal{T}_{t-r'}) \cup \{t - r'\} \) for every \( 0 < r < r' < t \). Since \( \mathcal{T}_s, 0 \leq s \leq t \) is a Poisson point process, we get that the \( \mathcal{T}_{[r_i, r_{i+1})} \) are independent and distributed as \( \mathcal{T}_{[0, r_{i+1}-r_i]} \). We deduce that

\[ \mathbb{N} \left[ \int_\mathcal{T} \mu(dx) \int_{0 < r_1 < \ldots < r_n < H(x)} \prod_{i=0}^{n} f_i \left( \mathcal{T}_{[r_i, r_{i+1})}, x \right) \prod_{i=1}^{n} dr_i \right] = \int_{(0, \infty)^{n+1}} \prod_{i=1}^{n+1} \mathbb{E} \left[ f_i \left( \mathcal{T}_{[0, s_i]} \right) \right] ds_i \]

\[ = \prod_{i=1}^{n+1} \int_0^\infty \mathbb{E} \left[ f_i \left( \mathcal{T}_{[0, t]} \right) \right] dt \]

\[ = \prod_{i=1}^{n+1} \mathbb{N} [\sigma f_i(\mathcal{T})], \]

where we made the change of variables \( (s_1, s_2, \ldots, s_{n+1}) = (r_1, r_2 - r_1, \ldots, r_{n+1} - r_n) \) for the second equality and used Bismut’s decomposition together with the fact that \( \mathcal{T}_{[0, t]} = T_t^1 \mathbb{P}-a.s. \) for the last. \[ \square \]
From now on, we restrict ourselves to the stable case $\psi(\lambda) = \lambda^\gamma$ where $\gamma \in (1, 2]$.

**Proposition 3.5.** Let $T$ be the stable tree with branching mechanism $\psi(\lambda) = \lambda^\gamma$ where $\gamma \in (1, 2]$. For every $n \geq 1$ and every nonnegative measurable functions $f_i$, $1 \leq i \leq n + 1$ defined on $\mathbb{T}$, we have

$$N^{(n)} \left[ \int_T \mu(dx) \int_{0<r_1<\ldots<r_n<\varnothing(x)} \prod_{i=1}^{n+1} f_i(T_{(r_{i-1}, r_i)}, x) \prod_{i=1}^{n} dr_i \right]$$

$$= \frac{1}{\gamma^n \Gamma(1 - 1/\gamma)^n} F_1 \ast \cdots \ast F_{n+1}(1),$$

where $F_i(a) = a^{-1/\gamma} N^{(n)} [f_i \circ R(T, a)]$, \quad (3.13)

where $R$ is defined in (3.1). In particular, for every $n \geq 1$ and every nonnegative measurable functions $g_i$, $1 \leq i \leq n+1$ defined on $[0, 1]$, we have

$$N^{(n)} \left[ \int_T \mu(dx) \int_{0<r_1<\ldots<r_n<H(x)} \prod_{i=1}^{n+1} g_i(\sigma_{(r_{i-1}, x)} - \sigma_{(r_i, x)}) \prod_{i=1}^{n} dr_i \right]$$

$$= \frac{1}{\gamma^n \Gamma(1 - 1/\gamma)^n} G_1 \ast \cdots \ast G_{n+1}(1),$$

where $G_i(a) = a^{-1/\gamma} g_i(a)$. \quad (3.14)

**Proof.** Let $f_i : \mathbb{T} \rightarrow \mathbb{R}$ be continuous and bounded for $1 \leq i \leq n+1$. By Theorem 3.4, we have for $\lambda > 0$

$$\prod_{i=1}^{n+1} N \left[ \sigma e^{-\lambda \sigma} f_i(T) \right]$$

$$= N \left[ \int_T \mu(dx) \int_{0<r_1<\ldots<r_n<H(x)} \prod_{i=1}^{n+1} \exp \left( -\lambda \mu(T_{(r_{i-1}, r_i), x}) f_i(T_{(r_{i-1}, r_i), x}) \prod_{i=1}^{n} dr_i \right) \prod_{i=1}^{n+1} f_i(T_{(r_{i-1}, r_i), x}) \prod_{i=1}^{n} dr_i \right].$$

Disintegrating with respect to $\sigma$ and using the scaling property from Lemma 3.1-(ii), we have

$$N \left[ \sigma e^{-\lambda \sigma} f_i(T) \right] = \frac{1}{\gamma \Gamma(1 - 1/\gamma)} \int_{0}^{\infty} ae^{-\lambda a} N^{(n)}(a) f_i(T) \frac{da}{a^{1+1/\gamma}}$$

$$= \frac{1}{\gamma \Gamma(1 - 1/\gamma)} \mathcal{L} F_i(\lambda),$$

where $\mathcal{L}$ denotes the Laplace transform on $[0, \infty)$ and $F_i(a) = a^{-1/\gamma} N^{(n)} [f_i \circ R(T, a)]$.

On the other hand, again disintegrating with respect to $\sigma$, we have

$$\gamma \Gamma(1 - 1/\gamma) N \left[ e^{-\lambda \sigma} \int_T \mu(dx) \int_{0<r_1<\ldots<r_n<H(x)} \prod_{i=1}^{n+1} f_i(T_{(r_{i-1}, r_i), x}) \prod_{i=1}^{n} dr_i \right]$$
3.15

Now notice that

\[
\int_0^\infty e^{-\lambda_n} \mathbb{N}^{(a)} \left[ \int_T \mu(dx) \int_{0<r_1<\ldots<r_n<H(x)} \prod_{i=1}^{n+1} f_i \left( \mathcal{T}_{[r_{i-1}, r_i], x} \right) \prod_{i=1}^{n} dr_i \right] \frac{da}{a^{1+1/\gamma}}
\]

\[
= \int_0^\infty e^{-\lambda_n} \mathbb{N}^{(1)} \left[ \int_T \mu(dx) \int_{0<r_1<\ldots<r_n<H(x)} \prod_{i=1}^{n+1} f_i \left( R(T, a)_{[r_{i-1}, r_i], x} \right) \prod_{i=1}^{n} dr_i \right] \frac{da}{a^{1+1/\gamma}} \quad (3.17)
\]

This together with (3.15), (3.16) and (3.17) gives

\[
\frac{1}{\gamma^n \Gamma(1 - 1/\gamma)^n} \mathcal{L}(F_1 * \ldots * F_{n+1})(\lambda) = \frac{1}{\gamma^n \Gamma(1 - 1/\gamma)^n} \prod_{i=1}^{n+1} \mathcal{L}F_i(\lambda)
\]

\[
= \int_0^\infty e^{-\lambda_n} a^{(n+1)(1-1/\gamma)-2} \mathbb{N}^{(1)} \left[ \int_T \mu(dx) \int_{0<r_1<\ldots<r_n<H(x)} \prod_{i=1}^{n+1} f_i \circ R \left( \mathcal{T}_{[r_{i-1}, r_i], x}, a \right) \prod_{i=1}^{n} dr_i \right] da.
\]

Since this holds for every \( \lambda > 0 \), we deduce that \( da \)-a.e. on \((0, \infty)\),

\[
\frac{1}{\gamma^n \Gamma(1 - 1/\gamma)^n} F_1 * \ldots * F_{n+1}(a)
\]

\[
= a^{(n+1)(1-1/\gamma)-2} \mathbb{N}^{(1)} \left[ \int_T \mu(dx) \int_{0<r_1<\ldots<r_n<H(x)} \prod_{i=1}^{n+1} f_i \circ R \left( \mathcal{T}_{[r_{i-1}, r_i], x}, a \right) \prod_{i=1}^{n} dr_i \right] \quad (3.18)
\]

Thanks to Lemma 2.2, the mapping \( a \mapsto R(T, a) \) is continuous on \((0, \infty)\) for every \( T \in \mathbb{T} \). We deduce from the dominated convergence theorem that the \( F_i \) are continuous on \((0, \infty)\) and thus \( F_1 * \ldots * F_{n+1} \) too. Similarly, the right-hand side of (3.18) is continuous with respect to \( a \). Therefore the equality holds for every \( a \in (0, \infty) \). In particular, taking \( a = 1 \) proves (3.13) for continuous bounded functions \( f_i: \mathbb{T} \to \mathbb{R} \). Finally, the monotone class theorem extends this to measurable functions \( f_i: \mathbb{T} \to \mathbb{R} \). \[ \square \]

In particular, the following corollary will be useful.

**Corollary 3.6.** We have

\[
\sup_{\alpha \geq 0} \alpha^{2-2/\gamma} \mathbb{N}^{(1)} \left[ \int_T \mu(dx) \left( \int_0^{H(x)} \sigma_{r,x}^n \, dr \right)^2 \right] < \infty. \quad (3.19)
\]
Proof. Applying (3.14) with \( n = 2 \), \( g_1(a) = g(1 - a) \), \( g_2(a) = 1 \) and \( g_3(a) = g(a) \) yields, for every measurable function \( g : [0, 1] \to [0, \infty] \),

\[
\mathbb{N}^{(1)} \left[ \int_{T} \mu(dx) \left( \int_{0}^{H(x)} g(\sigma_{r,x}) \, dr \right)^2 \right] = \frac{2}{\gamma^2 \Gamma(1 - 1/\gamma)^2} \int_{0}^{1} g(y)(1 - y)^{-1/\gamma} \, dy \int_{0}^{y} g(z)z^{-1/\gamma}(y - z)^{-1/\gamma} \, dz. \tag{3.20}
\]

Taking \( g(a) = a^\alpha \), we get

\[
\alpha^{2 - 2/\gamma} \mathbb{N}^{(1)} \left[ \int_{T} \mu(dx) \left( \int_{0}^{H(x)} \sigma_{r,x}^\alpha \, dr \right)^2 \right] = \frac{2\alpha^{2 - 2/\gamma}}{\gamma^2 \Gamma(1 - 1/\gamma)^2} \int_{0}^{1} y^\alpha(1 - y)^{-1/\gamma} \, dy \int_{0}^{y} z^{\alpha - 1/\gamma}(y - z)^{-1/\gamma} \, dz
\]

\[
= \frac{2\alpha^{2 - 2/\gamma}}{\gamma^2 \Gamma(1 - 1/\gamma)^2} \text{B} \left( \frac{2\alpha + 2 - 2/\gamma, 1 - 1/\gamma} \right) \text{B} \left( \alpha + 1 - 1/\gamma, 1 - 1/\gamma \right),
\]

where \( \text{B} \) is the Beta function. Using that \( \text{B}(x, 1 - 1/\gamma) \sim \Gamma(1 - 1/\gamma)x^{-1 + 1/\gamma} \) as \( x \to \infty \), (3.19) readily follows. \( \square \)

Another application of Theorem 3.2 is the following result giving the moments of the height \( H(U) \) of a uniformly distributed leaf \( U \in T \) \( \text{i.e.} \) according to \( \mu \). In particular, this allows to give a nontrivial upper bound for the size of the ball with radius \( \varepsilon > 0 \) centered around the root of the stable tree. Let us mention that the distribution of \( H(U) \) is known: in the Brownian case \( \gamma = 2 \), \( H \) is distributed as \( \sqrt{2} e \) where \( e \) is the Brownian excursion so \( \sqrt{2} H(U) \) has Rayleigh distribution; in the case \( \gamma \in (1, 2) \), \( H(U) \) is distributed as a multiple of the local time at 0 of the Bessel bridge of dimension \( 2/\gamma \), see [18, Corollary 10].

Lemma 3.7. Let \( T \) be the normalized stable tree with branching mechanism \( \psi(\lambda) = \lambda^{\gamma} \) where \( \gamma \in (1, 2) \). For every \( p < 2 \), we have

\[
\mathbb{N}^{(1)} \left[ \int_{T} H(x)^{-p} \mu(dx) \right] = \frac{(\gamma - 1)^{\gamma p - 1} \Gamma(1 - 1/\gamma) \Gamma(2 - p) \Gamma(1 - (p - 1)(1 - 1/\gamma))}{\Gamma(1 - (p - 1)(1 - 1/\gamma))} < \infty. \tag{3.21}
\]

Remark 3.8. Conditionally on \( T \), let \( U \in T \) be a uniformly distributed leaf. Then we can rewrite (3.21) as follows:

\[
\frac{1}{c_\gamma} \mathbb{N}^{(1)} \left[ \frac{1}{H(U)} \left( \gamma H(U) \right)^p \right] = \frac{\Gamma(p + 1)}{\Gamma(p(1 - 1/\gamma) + 1)}, \quad \forall p > -1, \tag{3.22}
\]

where \( c_\gamma = (\gamma - 1)\Gamma(1 - 1/\gamma) \). This implies that, under the probability measure \( c_\gamma^{-1} \mathbb{N}^{(1)}[H(U)^{-1}] \), the random variable \( \gamma H(U) \) has Mittag-Leffler distribution with index \( 1 - 1/\gamma \), see [24, Eq. (0.42)].

Proof. Using Bismut’s decomposition (3.10), we have for every \( \lambda > 0 \)

\[
\mathbb{N} \left[ se^{-\lambda \sigma} \int_{T} H(x)^{-p} \mu(dx) \right] = \int_{0}^{\infty} t^{-p} \mathbb{E} \left[ \tau_t e^{-\lambda \sigma} \right] \, dt = \varphi'(\lambda) \int_{0}^{\infty} t^{1-p} e^{-t \varphi(\lambda)} \, dt.
\]
On the other hand, disintegrating with respect to \( \sigma \) and using Lemma 3.1-(ii), we have
\[
N \left[ \sigma e^{-\lambda \sigma} \int_T H(x)^{-p} \mu(dx) \right] = \frac{1}{\gamma(1-1/\gamma)} \int_0^\infty a e^{-\lambda a} N^{(\alpha)} \left[ \int_T H(x)^{-p} \mu(dx) \right] \frac{da}{a^{1+1/\gamma}}
\]
\[
= \frac{1}{\gamma(1-1/\gamma)} \int_0^\infty e^{-\lambda a} \frac{da}{a^{(p-1)(1-1/\gamma)}} N^{(1)} \left[ \int_T H(x)^{-p} \mu(dx) \right]
\]
\[
= \frac{\Gamma(1-(p-1)(1-1/\gamma))}{\Gamma(1-1/\gamma)\lambda^{1-(p-1)(1-1/\gamma)}} N^{(1)} \left[ \int_T H(x)^{-p} \mu(dx) \right].
\]
It follows that
\[
N^{(1)} \left[ \int_T H(x)^{-p} \mu(dx) \right] = \frac{\gamma \Gamma(1-1/\gamma) \lambda^{1-(p-1)(1-1/\gamma)} \varphi'(\lambda)}{\Gamma(1-(p-1)(1-1/\gamma))} \int_0^\infty t^{1-p} e^{-t\varphi(\lambda)} dt
\]
\[
= \frac{(\gamma-1)\lambda^{1-p} \Gamma(1-1/\gamma) \Gamma(2-p)}{\Gamma(1-(p-1)(1-1/\gamma))}.
\]

\[\square\]

4. The case \( \beta/\alpha^{1-1/\gamma} \rightarrow c \in [0, \infty) \)

We start by showing that if \( U \in T \) is a leaf chosen uniformly at random, \( Z_{\alpha,\beta}(U) \) properly rescaled converges in distribution.

**Proposition 4.1.** Assume that \( \alpha \rightarrow \infty \), \( \beta \geq 0 \) and \( \beta/\alpha^{1-1/\gamma} \rightarrow c \in [0, \infty) \). Let \( T \) be the normalized stable tree with branching mechanism \( \psi(\lambda) = \lambda^\gamma \) where \( \gamma \in (1, 2] \). Conditionally on \( T \), let \( U \) be a \( T \)-valued random variable with distribution \( \mu \) under \( N^{(1)} \). Then we have the following convergence in distribution
\[
\left( T, H(U), \alpha^{1-1/\gamma} h^{-\beta} Z_{\alpha,\beta}(U) \right) \xrightarrow{(d)} \left( T, H(U), \int_0^\infty e^{-S_t - ct/h} dt \right), \tag{4.1}
\]
where \( (S_t, t \geq 0) \) is a stable subordinator with Laplace exponent \( \varphi \) given by (3.9), independent of \( (T, H(U)) \).

**Proof.** Let \( f : T \rightarrow \mathbb{R} \) and \( g, h : [0, \infty) \rightarrow \mathbb{R} \) be Lipschitz-continuous and bounded. Set \( \varepsilon = \varepsilon(\alpha) := \alpha^{(\delta-1)(1-1/\gamma)} \) with \( \delta \in (0, 1/3) \) so that \( \varepsilon \rightarrow 0 \) as \( \alpha \rightarrow \infty \). Recall from (3.1) the definition of \( R \). Applying Lemma 3.1-(i) and using (3.2), we have
\[
\frac{1}{\Gamma(1-1/\gamma)} N^{(1)} \left[ f(T) g(H(U)) h \left( \alpha^{1-1/\gamma} h^{-\beta} Z_{\alpha,\beta}(U) \right) \right]
\]
\[
= \frac{1}{\Gamma(1-1/\gamma)} N^{(1)} \left[ \int_T f(T) g(H(x)) h \left( \alpha^{1-1/\gamma} h^{-\beta} Z_{\alpha,\beta}(x) \right) \mu(dx) \right]
\]
\[
= N \left[ \frac{1}{\sigma} \int f \circ R \left( T, \sigma^{-1} \right) g \left( \sigma^{-1} H(x) \right) h \left( \frac{\alpha}{\sigma} \right)^{1-1/\gamma} \sigma^{-\alpha h^{-\beta} Z_{\alpha,\beta}(x)} \mu(dx) \right].
\]

Write
\[ h \left( \left( \frac{\alpha}{\sigma} \right)^{1-1/\gamma} \sigma^{-\alpha} b^{-\beta} Z_{\alpha,\beta}(x) \right) = I_\alpha(x) + J_\alpha(x), \]

where \( I_\alpha(x) = 1_{\{H(x) \geq \varepsilon\}} h \left( \left( \frac{\alpha}{\sigma} \right)^{1-1/\gamma} \int_0^\varepsilon \left( \frac{\sigma_{\tau x}}{\sigma} \right)^\alpha e^{-\beta r/\eta} \, dr \right) \).

**Lemma 4.2.** We have that \( \mathbb{N} \)-a.e. \( \mu(dx) \)-a.e., \( \lim_{\alpha \to \infty} J_\alpha(x) = 0 \).

The proof is postponed to Section 6.1. Now it is clear that
\[
|f \circ R \left( T, \sigma^{-1} \right) g \left( \sigma^{-1+1/\gamma} H(x) \right) J_\alpha(x)| \leq 2 \|f\|_\infty \|g\|_\infty \|h\|_\infty.
\]

Thus, by Lemma 4.2 and the dominated convergence theorem, we get that \( \mathbb{N} \)-a.e.
\[
\lim_{\alpha \to \infty} \int_T \left| f \circ R \left( T, \sigma^{-1} \right) g \left( \sigma^{-1+1/\gamma} H(x) \right) J_\alpha(x) \right| \mu(dx) = 0.
\]

Since
\[
\frac{1}{\sigma} \mathbb{1}_{\{\sigma > 1\}} \int_T \left| f \circ R \left( T, \sigma^{-1} \right) g \left( \sigma^{-1+1/\gamma} H(x) \right) J_\alpha(x) \right| \mu(dx) \leq 2 \|f\|_\infty \|g\|_\infty \|h\|_\infty \mathbb{1}_{\{\sigma > 1\}}
\]

where the right-hand side is integrable with respect to \( \mathbb{N} \), the dominated convergence theorem gives
\[
\lim_{\alpha \to \infty} \mathbb{N} \left[ \frac{1}{\sigma} \mathbb{1}_{\{\sigma > 1\}} \int_T \left| f \circ R \left( T, \sigma^{-1} \right) g \left( \sigma^{-1+1/\gamma} H(x) \right) J_\alpha(x) \right| \mu(dx) \right] = 0.
\]

It follows that
\[
\lim_{\alpha \to \infty} \mathbb{N} \left[ f(T) \mathbb{g}(H(U)) h \left( \alpha^{1-1/\gamma} b^{-\beta} Z_{\alpha,\beta}(U) \right) \right] = \Gamma(1-1/\gamma) \lim_{\alpha \to \infty} \mathbb{N} \left[ \frac{1}{\sigma} \mathbb{1}_{\{\sigma > 1\}} \int_T f \circ R \left( T, \sigma^{-1} \right) g \left( \sigma^{-1+1/\gamma} H(x) \right) I_\alpha(x) \mu(dx) \right]. \quad (4.2)
\]

For every \( t > 0 \), set
\[
X_t = f \circ R \left( T_{\tau_t}, \tau_{-1} \right) g \left( \tau_{-1+1/\gamma} t \right). \quad (4.3)
\]

Using Bismut’s decomposition (3.10), we get
\[
\mathbb{N} \left[ \frac{1}{\sigma} \mathbb{1}_{\{\sigma > 1\}} \int_T f \circ R \left( T, \sigma^{-1} \right) g \left( \sigma^{-1+1/\gamma} H(x) \right) I_\alpha(x) \mu(dx) \right] = \int_0^\infty \mathbb{E} \left[ \frac{1}{\tau_t} \mathbb{1}_{\{\tau_t > 1\}} X_t h \left( \left( \frac{\alpha}{\tau_t} \right)^{1-1/\gamma} \int_0^\varepsilon \left( 1 - \frac{S_r}{\tau_t} \right)^\alpha e^{-\beta r/\eta} \, dr \right) \right] \, dt
\]
\[
= \int_0^\infty \mathbb{E} \left[ L_\alpha(t) + M_\alpha(t) + N_\alpha(t) \right] \, dt, \quad (4.4)
\]

where
\[
L_\alpha(t) = \frac{1}{\tau_{t-\varepsilon}} \mathbb{1}_{\{\tau_{t-\varepsilon} < 1, t \geq \varepsilon\}} X_{t-\varepsilon} h \left( \left( \frac{\alpha}{\tau_{t-\varepsilon}} \right)^{1-1/\gamma} \int_0^{\varepsilon} e^{-\alpha S_r / \tau_{t-\varepsilon}} e^{-\beta r / \eta \tau_{t-\varepsilon}} \, dr \right), \quad (4.5)
\]
\[ M_\alpha(t) = \left\{ \frac{1}{\tau_t} 1_{\{\tau_t > 1, t > \varepsilon\}} - \frac{1}{\tau_{t-\varepsilon}} 1_{\{\tau_{t-\varepsilon} > 1, t > \varepsilon\}} \right\} X_{t-\varepsilon} \]

\[ \times h \left( \frac{\alpha}{\tau_{t-\varepsilon}} \right)^{1-1/\gamma} \int_0^\varepsilon e^{-\alpha S_r/\tau_{t-\varepsilon} - \beta r/\eta_{t-\varepsilon}} \, dr \right), \]

\[ N_\alpha(t) = \frac{1}{\tau_t} 1_{\{\tau_t > 1, t > \varepsilon\}} \left\{ X_t h \left( \frac{\alpha}{\tau_t} \right)^{1-1/\gamma} \int_0^\varepsilon \left( 1 - \frac{S_{r'}}{\tau_t} \right)^{\alpha-1} e^{-\beta r/\eta} \, dr \right) \]

\[ - X_{t-\varepsilon} h \left( \frac{\alpha}{\tau_{t-\varepsilon}} \right)^{1-1/\gamma} \int_0^\varepsilon e^{-\alpha S_r/\tau_{t-\varepsilon} - \beta r/\eta_{t-\varepsilon}} \, dr \right). \]

The proof of the next lemma is postponed to Section 6.2.

**Lemma 4.3.** We have

\[ \lim_{\alpha \to \infty} \int_0^\infty E[M_\alpha(t) + N_\alpha(t)] \, dt = 0. \]

We deduce from (4.2) and (4.4) that

\[ \lim_{\alpha \to \infty} \mathbb{N}^{(1)}_\alpha \left[ f(T) g(H(U)) h \left( \alpha^{1-1/\gamma} \eta^{-\beta} Z_{\alpha,\beta}(U) \right) \right] = \Gamma(1 - 1/\gamma) \lim_{\alpha \to \infty} \int_0^\infty E[L_\alpha(t)] \, dt. \]

Now take \((S'_r, r \geq 0)\) a subordinator independent of \((T_s, 0 \leq s \leq t)\) and having the same distribution as \(S\). Notice that, since \((T_s, 0 \leq s \leq t)\) is a Poisson point process, \(T_{t-\varepsilon}^\dagger\) and \((S'_r, 0 \leq r \leq \varepsilon)\) are independent. Therefore we get the identity

\[ \left( T_{t-\varepsilon}^\dagger, (S'_r, 0 \leq r \leq \varepsilon) \right) \overset{(d)}{=} \left( T_{t-\varepsilon}^\dagger, (S'_r, 0 \leq r \leq \varepsilon) \right). \]

It follows that

\[ \int_0^\infty E[L_\alpha(t)] \, dt \]

\[ = \int_\varepsilon^\infty E \left[ \frac{1}{\tau_{t-\varepsilon}} 1_{\{\tau_{t-\varepsilon} > 1\}} X_{t-\varepsilon} h \left( \frac{\alpha}{\tau_{t-\varepsilon}} \right)^{1-1/\gamma} \int_0^\varepsilon e^{-\alpha S'_r/\tau_{t-\varepsilon} - \beta r/\eta_{t-\varepsilon}} \, dr \right] \, dt \]

\[ = \int_\varepsilon^\infty E \left[ \frac{1}{\tau_{t-\varepsilon}} 1_{\{\tau_{t-\varepsilon} > 1\}} X_{t-\varepsilon} h \left( \frac{\alpha}{\tau_{t-\varepsilon}} \right)^{1-1/\gamma} \int_0^\varepsilon \exp \left( \frac{-\beta r/\eta_{t-\varepsilon}}{\alpha^{1-1/\gamma} \eta_{t-\varepsilon}} \right) \, dr \right] \, dt \]

\[ = \int_0^\infty E \left[ \frac{1}{\tau_u} 1_{\{\tau_u > 1\}} X_u h \left( \int_0^{(\alpha/\tau_u)^{1-1/\gamma}} e^{-S'_r} \exp \left( \frac{-\beta r/\eta_u}{\alpha^{1-1/\gamma} \eta_u} \right) \, dr \right) \right] \, du, \]

where for the second equality we used the fact that \(X_{t-\varepsilon}\) defined in (4.3) is a measurable function of \(T_{t-\varepsilon}^\dagger\), together with the identity

\[ \left( T_{t-\varepsilon}^\dagger, (\alpha S'_r/\tau_{t-\varepsilon}, r \geq 0) \right) \overset{(d)}{=} \left( T_{t-\varepsilon}^\dagger, (S'_r (\alpha/\tau_{t-\varepsilon})^{1-1/\gamma}, r \geq 0) \right), \]
which holds since $S'$ is stable with index $1 - 1/\gamma$ and is independent of $(T_{\lambda-\varepsilon}^1, \tau_{\lambda-\varepsilon})$. Now notice that $\mathbb{P}$-a.s. for every $u > 0$, we have

$$\lim_{\alpha \to \infty} \varepsilon \left( \frac{\alpha}{\tau_u} \right)^{1-\gamma} = \infty \quad \text{and} \quad \lim_{\alpha \to \infty} \frac{\beta \tau_u^{1-\gamma}}{\alpha^{1-\gamma} \eta_u} = \frac{c \tau_u^{1-\gamma}}{\eta_u}.$$ 

Hence the dominated convergence theorem together with (4.8) and (4.9) yields

$$\lim_{\alpha \to \infty} N^{(1)} \left[ f(T)g(H(U))h \left( \alpha^{1-\gamma} \beta Z_{\alpha,\beta}(U) \right) \right] = \Gamma(1 - 1/\gamma) \int_0^\infty \mathbb{E} \left[ \frac{1}{\tau_u} 1_{\{\tau_u > 1\}} X_u h \left( \int_0^\infty e^{-S_u^{\varepsilon} - cv \tau_u^{1-\gamma}/\eta_u} \, dv \right) \right] \, du. \quad (4.10)$$

Since $S'$ is independent of $(X_u, \tau_u, \eta_u)$, using the definition of $X_u$ for the first equality, Bismut’s decomposition (3.10) and Lemma 3.1-(i) for the second, we have

$$\int_0^\infty \mathbb{E} \left[ \frac{1}{\tau_u} 1_{\{\tau_u > 1\}} X_u h \left( \int_0^\infty e^{-S_u^{\varepsilon} - cv \tau_u^{1-\gamma}/\eta_u} \, dv \right) \right] \, du$$

$$= \int_0^\infty \mathbb{E} \left[ \frac{1}{\tau_u} 1_{\{\tau_u > 1\}} f \circ R \left( T_u^1, \tau_u^{-1} \right) g \left( \tau_u^{1 - 1/\gamma} \right) h \left( \int_0^\infty e^{-S_u^{\varepsilon} - cv \tau_u^{1-\gamma}/\eta_u} \, dv \right) \right] \, du$$

$$= \frac{1}{\Gamma(1 - 1/\gamma)} N^{(1)} \left[ \int_T f(T)g(H(x))h \left( \int_0^\infty e^{-S_u^{\varepsilon} - cv \eta_u} \, dv \right) \, \mu(dx) \right] \right|_{S'}$$

$$= \frac{1}{\Gamma(1 - 1/\gamma)} N^{(1)} \left[ f(T)g(H(U))h \left( \int_0^\infty e^{-S_u^{\varepsilon} - cv \eta_u} \, dv \right) \right] \, \mu(dx) \right|_{S'} \quad (4.11)$$

where, with a slight abuse of notation, we denote by $S'$ a subordinator with Laplace exponent given by (3.9) under $N^{(1)}$, independent of $(T, H(U))$.

Combining (4.10) and (4.11), we deduce that

$$\lim_{\alpha \to \infty} N^{(1)} \left[ f(T)g(H(U))h \left( \alpha^{1-\gamma} \beta Z_{\alpha,\beta}(U) \right) \right] = N^{(1)} \left[ f(T)g(H(U))h \left( \int_0^\infty e^{-S_u^{\varepsilon} - cv \eta_u} \, dv \right) \right].$$

This completes the proof. \hfill \Box

The next lemma, whose proof is postponed to Section 6.3, states that taking a leaf uniformly at random or taking the average over all leaves yields the same limiting behavior for $Z_{\alpha,\beta}(x)$. Recall from (1.1) the definition of $Z_{\alpha,\beta}$.

**Lemma 4.4.** Conditionally on $T$, let $U$ be a $T$-valued random variable with distribution $\mu$ under $N^{(1)}$. Then, under the assumptions of Theorem 4.1, we have the convergence in $N^{(1)}$-probability

$$\lim_{\alpha \to \infty} \alpha^{1-\gamma} \beta Z_{\alpha,\beta}(U) - Z_{\alpha,\beta} = 0. \quad (4.12)$$

Combining Proposition 4.1 and Lemma 4.4, we get the following result using Slutsky’s theorem.

**Theorem 4.5.** Assume that $\alpha \to \infty$, $\beta \geq 0$ and $\beta/\alpha^{1-\gamma} \to c \in [0, \infty)$. Let $T$ be the stable tree with branching mechanism $\psi(\lambda) = \lambda^\gamma$ where $\gamma \in (1, 2]$. Conditionally on $T$, let $U$ be a $T$-valued random variable with distribution $\mu$ under $N^{(1)}$. Then we have the following convergence in
distribution
\[
\left( \mathcal{T}, H(U), \alpha^{1-1/\gamma} h^{-\beta} Z_{\alpha,\beta}(U), \alpha^{1-1/\gamma} h^{-\beta} Z_{\alpha,\beta} \right) 
\xrightarrow{\text{(d)}} 
\left( \mathcal{T}, H(U), \int_0^\infty e^{-S_t - ct/h} dt, \int_0^\infty e^{-S_t - ct/h} dt \right), \quad (4.13)
\]
where \( S \) is a stable subordinator with Laplace exponent \( \varphi \) given by (3.9), independent of \((\mathcal{T}, H(U))\).

5. The case \( \beta/\alpha^{1-1/\gamma} \to \infty \)

We treat the case \( \beta/\alpha^{1-1/\gamma} \to \infty \). Intuitively, this assumption guarantees that \( h^\beta \) dominates \( \sigma^{\alpha,\gamma} \), thus we get a different asymptotic behavior and there is no longer a subordinator in the limit.

**Theorem 5.1.** Assume that \( \beta \to \infty \), \( \alpha \geq 0 \) and \( \alpha^{1-1/\gamma}/\beta \to 0 \). Let \( \mathcal{T} \) be the stable tree with branching mechanism \( \psi(\lambda) = \lambda^\gamma \) where \( \gamma \in (1, 2] \). Then we have the following convergence in \( \mathbb{N}^{(1)} \)-probability
\[
\lim_{\beta \to \infty} \beta h^{-\beta} Z_{\alpha,\beta} = h. \tag{5.1}
\]
Furthermore, if \( \alpha^{1-1/\gamma}/\beta^\rho \to 0 \) for some \( \rho \in (0, 1) \), then the convergence holds \( \mathbb{N}^{(1)} \)-almost surely.

**Proof.** We start by assuming that \( \alpha \to \infty \) and \( \alpha^{1-1/\gamma}/\beta \to 0 \) (the case \( \alpha \) bounded above is covered by the second part of the theorem). Setting \( \varepsilon = (\alpha^{1-1/\gamma}/\beta)^{-1/2} \), it is straightforward to check that \( \varepsilon \to 0 \), \( \beta \varepsilon \to \infty \) and \( \alpha^{1-1/\gamma} \varepsilon \to 0 \). Write
\[
\beta h^{-\beta} Z_{\alpha,\beta} = E_\beta + \sum_{i=1}^4 F^i_\beta \tag{5.2}
\]
where
\[
E_\beta = \beta \int_\mathcal{T} 1_{\{H(x) \geq 2\varepsilon\}} \mu(dx) \int_0^\varepsilon \sigma_{r,x}^\alpha e^{-r/h} dr,
\]
\[
F^1_\beta = \beta \int_\mathcal{T} 1_{\{H(x) < 2\varepsilon\}} \mu(dx) \int_0^{H(x)} \sigma_{r,x}^\alpha \left( \frac{b_{r,x}}{h} \right)^{\beta+\mu} dr,
\]
\[
F^2_\beta = \beta \int_\mathcal{T} 1_{\{H(x) \geq 2\varepsilon\}} \mu(dx) \int_\varepsilon^{H(x)} \sigma_{r,x}^\alpha \left( \frac{b_{r,x}}{h} \right)^{\beta} dr,
\]
\[
F^3_\beta = \beta \int_\mathcal{T} 1_{\{H(x) \geq 2\varepsilon\}} \mu(dx) \int_0^\varepsilon \sigma_{r,x}^\alpha \left[ \left( \frac{b_{r,x}}{h} \right)^{\beta} - \left( 1 - \frac{r}{h} \right)^{\beta} \right] dr,
\]
\[
F^4_\beta = \beta \int_\mathcal{T} 1_{\{H(x) \geq 2\varepsilon\}} \mu(dx) \int_0^\varepsilon \sigma_{r,x}^\alpha \left[ \left( 1 - \frac{r}{h} \right)^{\beta} - e^{-r/h} \right] dr.
\]
We shall prove that \( \lim_{\beta \to \infty} F^i_\beta = 0 \) in \( \mathbb{N}^{(1)} \)-probability for every \( i \in \{1, 2, 3, 4\} \).
Let \( p \in (1, 2) \). Using that \( \sigma_{r,x} \leq 1 \) and \( h_{r,x} \leq h \) and applying the Markov inequality, it is clear that
\[
F_\beta^1 \leq 2\beta \varepsilon \int_T 1_{\{H(x) < 2\varepsilon\}} \mu(dx) \leq 2^{1+p} \beta \varepsilon^{1+p} \int_T H(x)^{-p} \mu(dx).
\]
Since the last integral has a finite first moment by Lemma 3.7 and \( \beta \varepsilon^{1+p} \to 0 \), we deduce that \( N^{(1)}\text{-a.s. } \lim_{\beta \to \infty} F_\beta^1 = 0 \).

Next, using (2.5), we get
\[
F_\beta^2 = \beta \int_T 1_{\{H(x) \geq 2\varepsilon\}} \mu(dx) \int_0^{H(x)} \sigma_{r,x}^{\alpha} \left( \frac{h_{r,x}}{h} \right)^{\beta} dr
\leq \beta \left( 1 - \frac{\varepsilon}{h} \right)^{\beta} \int_T \mu(dx) \int_0^{H(x)} \sigma_{r,x}^{\alpha} dr.
\]
(5.3)

By [2, Corollary 6.6], we have
\[
N^{(1)} \left[ \int_T \mu(dx) \int_0^{H(x)} \sigma_{r,x}^{\alpha} dr \right] = \frac{1}{|\Gamma(-1/\gamma)|} B(\alpha + 1 - 1/\gamma, 1 - 1/\gamma),
\]
where \( B \) is the beta function. Using that \( B(x, 1-1/\gamma) \sim \Gamma(1-1/\gamma)x^{-1+1/\gamma} \) as \( x \to \infty \), we deduce that
\[
\sup_{\alpha > 0} N^{(1)} \left[ \alpha^{-1/\gamma} \int_T \mu(dx) \int_0^{H(x)} \sigma_{r,x}^{\alpha} dr \right] < \infty.
\]
(5.4)

On the other hand, let \( \theta > 1 \). Since the function \( x \mapsto x^{1+\theta} e^{-x} \) is bounded on \([0, \infty)\), it follows that
\[
\frac{\beta}{\alpha^{1-1/\gamma}} \left( 1 - \frac{\varepsilon}{h} \right)^{\beta} \leq \frac{\beta}{\alpha^{1-1/\gamma}} e^{-\beta \varepsilon/h} \leq C \frac{h^{1+\theta}}{\beta^{1+\theta} \varepsilon^{1+\theta} \alpha^{1-1/\gamma}}
\]
(5.5)

for some constant \( C > 0 \). Notice that \( \beta^{1+\theta} \alpha^{1-1/\gamma} \to \infty \) since \( \theta > 1 \), thus the right-hand side of (5.5) goes to 0 almost surely. Now putting together (5.3), (5.4) and (5.5), we deduce that \( \lim_{\beta \to \infty} F_\beta^2 = 0 \) in \( N^{(1)}\)-probability.

Let \( x \in T \). Recall from (2.5) and (2.6) that \( h_{r,x} \leq h - r \) for every \( r \in [0, H(x)] \) and that the equality holds for \( r \in [0, H(x \wedge x^*)] \). Therefore, we get
\[
|F_\beta^3| = \beta \int_T 1_{\{H(x) \geq 2\varepsilon, H(x \wedge x^*) < \varepsilon\}} \mu(dx) \int_0^{\varepsilon} \sigma_{r,x}^{\alpha} \left[ \left( 1 - \frac{r}{h} \right)^{\beta} - \left( \frac{h_{r,x}}{h} \right)^{\beta} \right] dr
\leq \beta \int_T 1_{\{H(x) \geq 2\varepsilon, H(x \wedge x^*) < \varepsilon\}} \mu(dx) \int_0^{\varepsilon} \left( 1 - \frac{r}{h} \right)^{\beta} dr
\leq \beta \int_T 1_{\{H(x) \geq 2\varepsilon, H(x \wedge x^*) < \varepsilon\}} \mu(dx) \int_0^{\varepsilon} e^{-\beta r/h} dr
\leq h \int_T e^{-\beta H(x \wedge x^*)/h} \mu(dx).
\]
Now a simple application of the dominated convergence theorem gives that \( N^{(1)}\text{-a.s. } \lim_{\beta \to \infty} F_\beta^3 = 0 \).
Furthermore, using the inequality $|e^b - e^a| \leq |b - a|e^b$ for $a \leq b$ together with the fact that $j: y \mapsto -(y + \log(1 - y))/y^2$ is increasing on $[0, 1]$, we get for $r \in [0, \varepsilon]$

$$|e^{-\beta r/\theta} - \left(1 - \frac{r}{\theta}\right)| \leq \beta \left(\frac{r}{\theta} + \log \left(1 - \frac{r}{\theta}\right)\right) e^{-\beta r/\theta} \leq \beta \left(\frac{r}{\theta}\right)^2 e^{-\beta r/\theta} j \left(\frac{\varepsilon}{\theta}\right).$$

Therefore, we deduce that

$$|F^4_{\beta}| \leq j \left(\frac{\varepsilon}{\theta}\right) \int_{\mathcal{T}} 1_{\{H(x) \geq 2\varepsilon\}} \mu(dx) \int_{0}^{\varepsilon} \left(\frac{\beta r}{\theta}\right)^2 e^{-\beta r/\theta} \, dr \leq C_j \left(\frac{\varepsilon}{\theta}\right) \varepsilon,$$

where we used that $y \mapsto y^2 e^{-y}$ is bounded on $[0, \infty)$ by some constant $C < \infty$ for the second inequality. Since $\lim_{y \to 0} j(y) = 1/2$, we get $\mathbb{N}^{(1)}$-a.s. $\lim_{\beta \to \infty} F^4_{\beta} = 0$.

We deduce that

$$\lim_{\beta \to \infty} \beta^{-1} \int_{\mathcal{T}} Z_{\alpha, \beta}(x) \mu(dx) = \lim_{\beta \to \infty} E_{\beta}. \quad (5.6)$$

Notice that

$$E_{\beta} \leq \beta \int_{\mathcal{T}} 1_{\{H(x) \geq 2\varepsilon\}} \mu(dx) \int_{0}^{\varepsilon} e^{-\beta r/\theta} \, dr = h(1 - e^{-\beta \varepsilon/\theta}) \int_{\mathcal{T}} 1_{\{H(x) \geq 2\varepsilon\}} \mu(dx) \leq h. \quad (5.7)$$

On the other hand, using that $\sigma_{r, x} \geq \sigma_{\varepsilon, x}$ for every $x \in \mathcal{T}$ such that $H(x) \geq 2\varepsilon$ and every $r \in [0, \varepsilon]$, we get

$$E_{\beta} \geq h \left(1 - e^{-\beta \varepsilon/\theta}\right) \int_{\mathcal{T}} 1_{\{H(x) \geq 2\varepsilon\}} \sigma_{\varepsilon, x}^\alpha \mu(dx). \quad (5.8)$$

We now shall prove the following convergence in $\mathbb{N}^{(1)}$-probability

$$\lim_{\beta \to \infty} \int_{\mathcal{T}} 1_{\{H(x) \geq 2\varepsilon\}} \sigma_{\varepsilon, x}^\alpha \mu(dx) = 1. \quad (5.9)$$

Using Lemma 3.1-(i) and Bismut’s decomposition (3.10), we have

$$\mathbb{N}^{(1)} \left[\int_{\mathcal{T}} 1_{\{H(x) \geq 2\varepsilon\}} \sigma_{\varepsilon, x}^\alpha \mu(dx)\right] = \Gamma(1 - 1/\gamma) \mathbb{N} \left[\frac{1}{\sigma} 1_{\{\sigma > 1\}} \int_{\mathcal{T}} 1_{\{\sigma^{-1+1/\gamma} H(x) \geq 2\varepsilon\}} \left(\frac{\sigma_{\sigma^{-1+1/\gamma} x, x}^\alpha}{\sigma}\right) \mu(dx)\right]$$

$$= \Gamma(1 - 1/\gamma) \int_{0}^{\infty} dt \mathbb{E} \left[\frac{1}{S_{t}} 1_{\{S_{t} > 2\varepsilon S_{t}^{1-1/\gamma}\}} \left(1 - \frac{S_{t}^{1-1/\gamma}}{S_{t}}\right)\right]. \quad (5.10)$$

Recall that $S$ is a stable subordinator with index $1 - 1/\gamma$. Thus we have the following identity in distribution

$$(S_{rr}, r \geq 0) \overset{(d)}{=} (c^\gamma(\gamma - 1)S_{r}, r \geq 0).$$

Applying this, we get that

$$\alpha S(\varepsilon S_{t}^{1-1/\gamma}) \overset{(d)}{=} S(\varepsilon S_{\alpha^{-1-1/\gamma} t}^{1-1/\gamma}). \quad (5.11)$$

Now notice that

$$\varepsilon S_{\alpha^{-1-1/\gamma} t}^{1-1/\gamma} \overset{(d)}{=} \varepsilon \alpha^{-1-1/\gamma} S_{t}^{1-1/\gamma}.$$
Since $\varepsilon \alpha^{1-1/\gamma} \to 0$, this clearly implies that $\varepsilon S_{\alpha-1/\gamma t}^{1-1/\gamma} \to 0$ in probability. As $S$ is a.s. continuous at 0, we deduce that $S \left( \varepsilon S_{\alpha-1/\gamma t}^{1-1/\gamma} \right) \to 0$ in probability. Thus, it follows from (5.11) that

$$
\alpha \log \left( 1 - \frac{S_{\varepsilon S_{t}^{1-1/\gamma}}}{S_{t}} \right) \sim -\alpha \frac{S_{\varepsilon S_{t}^{1-1/\gamma}}}{S_{t}} \to 0.
$$

In particular, this implies the following convergence in probability for every $t > 0$

$$
\frac{1}{S_{t}} \mathbf{1}_{\left\{ \varepsilon S_{t}^{1-1/\gamma} \right\}} \left( 1 - \frac{S_{\varepsilon S_{t}^{1-1/\gamma}}}{S_{t}} \right)^{\alpha} \xrightarrow{\mathbb{P}} \frac{1}{S_{t}} \mathbf{1}_{\left\{ S_{t} > 1 \right\}}.
$$

Since we have the inequality

$$
\frac{1}{S_{t}} \mathbf{1}_{\left\{ \varepsilon S_{t}^{1-1/\gamma} \right\}} \left( 1 - \frac{S_{\varepsilon S_{t}^{1-1/\gamma}}}{S_{t}} \right)^{\alpha} \leq \frac{1}{S_{t}} \mathbf{1}_{\left\{ S_{t} > 1 \right\}}
$$

where the right-hand side is integrable with respect to $\mathbf{1}_{(0, \infty)}(t) \, dt \otimes \mathbb{P}$ thanks to (3.11), the dominated convergence theorem yields

$$
\int_{0}^{\infty} dt \mathbb{E} \left[ \frac{1}{S_{t}} \mathbf{1}_{\left\{ \varepsilon S_{t}^{1-1/\gamma} \right\}} \left( 1 - \frac{S_{\varepsilon S_{t}^{1-1/\gamma}}}{S_{t}} \right)^{\alpha} \right] \to \int_{0}^{\infty} dt \mathbb{E} \left[ \frac{1}{S_{t}} \mathbf{1}_{\left\{ S_{t} > 1 \right\}} \right] = \frac{1}{\Gamma(1-1/\gamma)}.
$$

Together with (5.10) and the fact that

$$
\int_{T} \mathbf{1}_{\left( H(x) \geq 2 \varepsilon \right)} \sigma_{\varepsilon, x}^{\alpha} \mu(dx) \leq 1,
$$

this proves (5.9).

Finally, since $\beta \varepsilon \to \infty$, it is clear that $\mathfrak{h}(1 - e^{-\beta \varepsilon / \mathfrak{h}}) \to \mathfrak{h}$ almost surely. In conjunction with (5.9), this gives the following convergence in $\mathbb{N}^{(1)}$-probability

$$
\mathfrak{h} \left( 1 - e^{-\beta \varepsilon / \mathfrak{h}} \right) \int_{T} \mathbf{1}_{\left( H(x) \geq 2 \varepsilon \right)} \sigma_{\varepsilon, x}^{\alpha} \mu(dx) \to \mathfrak{h}.
$$

Thus, using this together with (5.6), (5.7) and (5.8) yields the convergence in $\mathbb{N}^{(1)}$-probability

$$
\lim_{\beta \to \infty} \beta^{-1} \mathfrak{h}^{-1} \mathfrak{Z}_{\alpha, \beta} = \mathfrak{h}.
$$

This proves the first part of the theorem.

Next, we treat the case $\alpha^{1-1/\gamma} / \beta^{\rho} \to 0$ for some $\rho \in (0, 1)$. The proof is similar and we only highlight the differences. Notice that there exists $p, q \in (0, 1)$ and $\theta \in (0, \gamma / (\gamma - 1))$ such that $(1+p)q > 1$ and $q \theta > \rho \gamma / (\gamma - 1)$. Taking $\varepsilon = \beta^{-q}$, it is straightforward to check that $\varepsilon \to 0$, $\beta \varepsilon \to \infty$, $\beta \varepsilon^{1+p} \to 0$ and $\alpha e^{\theta} \to 0$. As in the first part, we have that $\mathbb{N}^{(1)}$-a.s. $\lim_{\beta \to \infty} F_{\beta}^{3} + F_{\beta}^{3} + F_{\beta}^{3} = 0$.

Furthermore, using that $\sigma_{r, x} \leq 1$, it follows from (5.3) that

$$
F_{\beta}^{2} \leq \beta \left( 1 - \frac{\varepsilon}{\mathfrak{h}} \right)^{\beta} \mathfrak{h} \leq \beta e^{-\beta \varepsilon / \mathfrak{h}} \mathfrak{h} = \beta e^{-\beta^{-1} - \beta / \mathfrak{h}}.
$$
This proves that $\mathbb{N}^{(1)}$-a.s. $\lim_{\beta \to \infty} F_\beta^2 = 0$.

Now we shall prove that $\mathbb{N}^{(1)}$-a.s. $\mu(dx)$-a.s.

$$\lim_{\beta \to \infty} 1_{\{H(x) \geq 2\varepsilon\}} \sigma_{\varepsilon, x}^\alpha = 1. \quad (5.12)$$

Using the same computation as in (5.10), we have the following identity in distribution

$$\left( 1_{\{H(x) \geq 2\varepsilon\}} \sigma_{\varepsilon, x}^\alpha, \varepsilon > 0 \right) \text{ under } \mathbb{N}^{(1)}$$

$$= \left( 1_{\left\{ t > 2\varepsilon S_t^{1-1/\gamma} \right\}} \left( 1 - \frac{S_t^{1-1/\gamma}}{S_t} \right)^\alpha, \varepsilon > 0 \right) \text{ under } \int_0^\infty dt \mathbb{E} \left[ \frac{1}{S_t} 1_{\{S_t > 1\}} \right]. \quad (5.13)$$

Since $\theta < \gamma / (\gamma - 1)$, [6, Chapter III, Theorem 9] guarantees that $\mathbb{P}$-a.s. $\lim_{\varepsilon \to 0} \sup_r r^{-\theta} S_r = 0$. By composition, it follows that $\mathbb{P}$-a.s. for every $t > 0$, $\lim_{\varepsilon \to 0} \varepsilon^{-\theta} S_t^{1-1/\gamma} = 0$. Thus we deduce that

$$\alpha \log \left( 1 - \frac{S_t^{1-1/\gamma}}{S_t} \right) \sim -\alpha S_t^{1-1/\gamma} = -\alpha \varepsilon^\theta \varepsilon^{-\theta} S_t^{1-1/\gamma} \to 0$$

since $\alpha \varepsilon^\theta \to 0$. This proves that the process in the right-hand side of (5.13) goes to 1 as $\varepsilon \to 0$, thus (5.12) follows.

Thanks to (5.12), since $\sigma_{\varepsilon, x} \lesssim 1$, a simple application of the dominated convergence theorem gives that $\mathbb{N}^{(1)}$-a.s.

$$\lim_{\beta \to \infty} \int_T 1_{\{H(x) \geq 2\varepsilon\}} \sigma_{\varepsilon, x}^\alpha \mu(dx) = 1.$$ 

This, together with the estimates (5.7) and (5.8) yields the $\mathbb{N}^{(1)}$-a.s. convergence $\lim_{\beta \to \infty} E_\beta = h$ which concludes the proof of the second part of the theorem.

\[ \square \]

6. Technical lemmas

6.1. Proof of Lemma 4.2. Recall that

$$J_\alpha(x) = h \left( \left( \frac{\alpha}{\sigma} \right)^{1-1/\gamma} \int_0^{H(x)} \left( \frac{\sigma_{r,x}}{\sigma} \right)^{\alpha} \left( \frac{h_{r,x}}{h} \right)^{\beta} dr \right) - 1_{\{H(x) \geq \varepsilon\}} h \left( \left( \frac{\alpha}{\sigma} \right)^{1-1/\gamma} \int_0^\varepsilon \left( \frac{\sigma_{r,x}}{\sigma} \right)^{\alpha} e^{-r/h} dr \right).$$

Write $J_\alpha(x) = \sum_{i=1}^4 J^i_\alpha(x)$ where

$$J^1_\alpha(x) = 1_{\{H(x) < \varepsilon\}} h \left( \left( \frac{\alpha}{\sigma} \right)^{1-1/\gamma} \int_0^{H(x)} \left( \frac{\sigma_{r,x}}{\sigma} \right)^{\alpha} \left( \frac{h_{r,x}}{h} \right)^{\beta} dr \right),$$

$$J^2_\alpha(x) = 1_{\{H(x) \geq \varepsilon\}} \left\{ h \left( \left( \frac{\alpha}{\sigma} \right)^{1-1/\gamma} \int_0^{H(x)} \left( \frac{\sigma_{r,x}}{\sigma} \right)^{\alpha} \left( \frac{h_{r,x}}{h} \right)^{\beta} dr \right) \right\},$$

$$J^3_\alpha(x) = -h \left( \left( \frac{\alpha}{\sigma} \right)^{1-1/\gamma} \int_0^\varepsilon \left( \frac{\sigma_{r,x}}{\sigma} \right)^{\alpha} \left( \frac{h_{r,x}}{h} \right)^{\beta} dr \right),$$

$$J^4_\alpha(x) = 1_{\{H(x) \geq \varepsilon\}} h \left( \left( \frac{\alpha}{\sigma} \right)^{1-1/\gamma} \int_0^\varepsilon \left( \frac{\sigma_{r,x}}{\sigma} \right)^{\alpha} e^{-r/h} dr \right).$$
It is clear that $\mathbb{N}$-a.e. for every $x \in \mathcal{T}$, $\lim_{\alpha \to \infty} J_\alpha^1(x) = 0$.

Next, we shall prove that $\mathbb{N}$-a.e. $\mu(dx)$-a.e.

$$\lim_{\alpha \to \infty} 1_{\{H(x) \geq \epsilon\}} \alpha^{1-1/\gamma} \int_{\epsilon}^{H(x)} \left( \frac{\sigma_{r,x}}{\sigma} \right)^{\alpha} \, dr = 0. \quad (6.1)$$

Using Bismut's decomposition (3.10), we get that

$$\begin{align*}
\mathbb{N} \left[ \mu \left( x \in \mathcal{T} : \limsup_{\alpha \to \infty} 1_{\{H(x) \geq \epsilon\}} \alpha^{1-1/\gamma} \int_{\epsilon}^{H(x)} \left( \frac{\sigma_{r,x}}{\sigma} \right)^{\alpha} \, dr > 0 \right) \right] &= \int_0^\infty \mathbb{P} \left( \limsup_{\alpha \to \infty} 1_{\{t \geq \epsilon\}} \alpha^{1-1/\gamma} \int_{\epsilon}^{t} \left( 1 - \frac{S_r}{\tau_t} \right)^{\alpha} \, dr > 0 \right) \, dt \\
&\leq \int_0^\infty \mathbb{P} \left( \limsup_{\alpha \to \infty} 1_{\{t \geq \epsilon\}} \alpha^{1-1/\gamma} \int_{\epsilon}^{t} e^{-\alpha S_r/\tau_t} \, dr > 0 \right) \, dt \\
&\leq \int_0^\infty \mathbb{P} \left( \limsup_{\alpha \to \infty} t^{1-1/\gamma} e^{-\alpha S_r/\tau_t} > 0 \right) \, dt, \quad (6.2)
\end{align*}$$

where $(S_r, 0 \leq r \leq t)$ is a subordinator with Laplace exponent given by (3.9). According to [6, Chapter III, Theorem 11], we have that $\mathbb{P}$-a.s.

$$\lim_{\epsilon \to 0} \frac{S_\epsilon}{h(\epsilon)} = \gamma - 1 > 0,$$

where $h(r) = r^{\gamma/(\gamma-1)} \log(\log r)^{-1/(\gamma-1)}$. As a consequence, there exist a random variable $\rho = \rho(\omega) > 0$ and a constant $c > 0$ such that $\mathbb{P}$-a.s. $S_\epsilon \geq c h(\epsilon)$ for every $\epsilon \in (0, \rho)$. We deduce that $\mathbb{P}$-a.s.

$$\limsup_{\alpha \to \infty} t^{1-1/\gamma} e^{-\alpha S_r/\tau_t} \leq \limsup_{\alpha \to \infty} t^{1-1/\gamma} e^{-\alpha h(\epsilon)/\tau_t} = \limsup_{\alpha \to \infty} t^{1-1/\gamma} e^{-\alpha \delta \log(\log \epsilon)^{-1}/\tau_t} = 0.$$

In conjunction with (6.2), this yields (6.1). Since $h_{r,x} \leq b$, we have

$$\left| J_\alpha^2(x) \right| \leq \|h\|_{L} 1_{\{H(x) \geq \epsilon\}} \left( \frac{\alpha}{\sigma} \right)^{1-1/\gamma} \int_{\epsilon}^{H(x)} \left( \frac{\sigma_{r,x}}{\sigma} \right)^{\alpha} \left( \frac{h_{r,x}}{b} \right)^{\beta} \, dr$$
\[ \|h\|_L \mathbf{1}_{\{H(x) \geq \varepsilon\}} \left( \frac{\alpha}{\sigma} \right)^{1-1/\gamma} \int_{\varepsilon}^{H(x)} \left( \frac{\sigma r x}{\sigma} \right)^\alpha \, dr. \]

Thanks to (6.1), we get that \( N \)-a.e. \( \mu(dx) \)-a.e. \( \lim_{\alpha \to \infty} J^2_\alpha(x) = 0 \).

Under \( N \), let \( x^* \) be the unique leaf realizing the total height, that is the unique \( x \in T \) such that \( H(x) = \mathbf{h} \). Then \( N \)-a.e. for every \( x \in T \setminus \{\emptyset\} \), we have \( H(x \wedge x^*) > 0 \) and, thanks to (2.6), \( \mathbf{h}_{r,x} = \mathbf{h} - r \) for every \( r \in [0, \varepsilon] \) if \( \varepsilon > 0 \) is small enough (depending on \( x \)). In particular, this implies that \( N \)-a.e. for every \( x \in T \setminus \{\emptyset\} \),

\[ \int_0^{\varepsilon} \left( \frac{\sigma r x}{\sigma} \right)^\alpha \left( \frac{\mathbf{h}_{r,x}}{\mathbf{h}} \right)^\beta \, dr = \int_0^{\varepsilon} \left( \frac{\sigma r x}{\sigma} \right)^\alpha \left( 1 - \frac{r}{\mathbf{h}} \right)^\beta \, dr \]

for small enough \( \varepsilon > 0 \). As a consequence, we get that \( \lim_{\alpha \to \infty} J^3_\alpha(x) = 0 \) \( N \)-a.e. for every \( x \in T \setminus \{\emptyset\} \).

Finally, we have

\[ |J^4_\alpha(x)| \leq \|h\|_L \mathbf{1}_{\{H(x) \geq \varepsilon\}} \left( \frac{\alpha}{\sigma} \right)^{1-1/\gamma} \int_{\varepsilon}^{H(x)} \left( \frac{\sigma r x}{\sigma} \right)^\alpha \left[ e^{-\beta r/\mathbf{h}} - \left( 1 - \frac{r}{\mathbf{h}} \right)^\beta \right] \, dr \]

\[ \leq \|h\|_L \mathbf{1}_{\{H(x) \geq \varepsilon\}} \left( \frac{\alpha}{\sigma} \right)^{1-1/\gamma} \beta \int_{\varepsilon}^{H(x)} \left( \frac{\sigma r x}{\sigma} \right)^\alpha \left[ \beta \left( \frac{\sigma r x}{\sigma} \right)^\alpha + \log \left( 1 - \frac{r}{\mathbf{h}} \right) \right] \, e^{-\beta r/\mathbf{h}} \, dr \]

\[ \leq C \|h\|_L \frac{1}{\sigma^{1-1/\gamma} \mathbf{h}^2} J \left( \frac{\varepsilon}{\mathbf{h}} \right) \mathbf{1}_{\{H(x) \geq \varepsilon\}} \left( \frac{\mathbf{h}}{\sigma} \right)^\alpha \left( \mathbf{h} \mathbf{h} \right)^{\alpha^{1-1/\gamma}} \int_{\varepsilon}^{H(x)} \left( \frac{\sigma r x}{\sigma} \right)^\alpha \, dr \]

\[ \leq C \|h\|_L \frac{1}{\sigma^{1-1/\gamma} \mathbf{h}^2} j \left( \frac{\varepsilon}{\mathbf{h}} \right) \varepsilon^3 \alpha^{2(1-1/\gamma)}, \]

where we used that \( |e^b - e^a| \leq |b - a|e^b \) for \( a \leq b \) for the second inequality and that the function \( j : y \mapsto -(y + \log(1 - y))/y^2 \) is increasing on \([0, 1]\) together with the fact that \( \beta/\alpha^{1-1/\gamma} \) is bounded by some constant \( C > 0 \) for the last. Notice that \( \varepsilon^3 \alpha^{2(1-1/\gamma)} = \alpha^{(3\delta - 1)(1-1/\gamma)} \to 0 \) as \( \delta < 1/3 \). Since \( \lim_{\gamma \to 0} j(y) = 1/2 \), we deduce that \( N \)-a.e. \( \mu(dx) \)-a.e. \( \lim_{\alpha \to \infty} J^4_\alpha(x) = 0 \). This concludes the proof.

6.2. Proof of Lemma 4.3. Recall from (4.6) the definition of \( M_\alpha(t) \) and notice that

\[ \int_0^\infty \mathbb{E} [M_\alpha(t)] \, dt \leq \|f\|_\infty \|g\|_\infty \|h\|_\infty \int_\varepsilon^\infty \mathbb{E} \left[ \frac{1}{\tau_t} \mathbf{1}_{\{\tau_t < \tau_{t-\varepsilon}\}} + \frac{\tau_t - \tau_{t-\varepsilon}}{\tau_t \tau_{t-\varepsilon}} \mathbf{1}_{\{\tau_{t-\varepsilon} > 1\}} \right] \, dt. \quad (6.3) \]

Thanks to (3.11) and the dominated convergence theorem, it is clear that

\[ \lim_{\alpha \to \infty} \int_\varepsilon^\infty \mathbb{E} \left[ \frac{1}{\tau_t} \mathbf{1}_{\{\tau_t < \tau_{t-\varepsilon}\}} \right] \, dt = 0 \quad (6.4) \]

as the process \( \tau \) is a.s. continuous at \( t \). On the other hand, using the inequality

\[ \frac{\tau_t - \tau_{t-\varepsilon}}{\tau_t \tau_{t-\varepsilon}} \mathbf{1}_{\{\tau_{t-\varepsilon} > 1\}} \leq \left( \frac{\tau_t - \tau_{t-\varepsilon}}{\tau_t} \right)^{1-q} \left( \frac{\tau_t - \tau_{t-\varepsilon}}{\tau_t} \right)^q \left( \frac{\tau_t - \tau_{t-\varepsilon}}{\tau_t} \right)^{1-q} \mathbf{1}_{\{\tau_{t-\varepsilon} > 1\}} \leq \left( \frac{\tau_t - \tau_{t-\varepsilon}}{\tau_t} \right)^{1-q} \left( \frac{\tau_t - \tau_{t-\varepsilon}}{\tau_t} \right)^q \mathbf{1}_{\{\tau_{t-\varepsilon} > 1\}} \]

where \( q \in (0, 1 - 1/\gamma) \), we get that

\[
\int_\varepsilon^\infty \mathbb{E} \left[ \frac{\tau_t - \tau_{t-\varepsilon}}{\tau_t \tau_{t-\varepsilon}} \mathbf{1}_{\{\tau_{t-\varepsilon} > 1\}} \right] dt \leq \int_\varepsilon^\infty \mathbb{E} \left[ \frac{(\tau_t - \tau_{t-\varepsilon})^q}{\tau_{t-\varepsilon}^{1+q}} \mathbf{1}_{\{\tau_{t-\varepsilon} > 1\}} \right] dt
\]

\[
= \mathbb{E} [\tau_1^q] \int_\varepsilon^\infty \mathbb{E} \left[ \frac{1}{\tau_{t-\varepsilon}^{1+q}} \mathbf{1}_{\{\tau_{t-\varepsilon} > 1\}} \right] dt
\]

\[
= \mathbb{E} [\tau_1^q] \int_0^\infty \mathbb{E} \left[ \frac{1}{\tau_r^{1+q}} \mathbf{1}_{\{\tau_r > 1\}} \right] dr
\]

\[
= \varepsilon^{q/(\gamma - 1)} \mathbb{E} [\tau_1^q] \int_0^\infty \mathbb{E} \left[ \frac{1}{\tau_r^{1+q}} \mathbf{1}_{\{\tau_r > 1\}} \right] dr,
\]  \hspace{1cm} (6.5)

where we used that \( \tau_t - \tau_{t-\varepsilon} \) is independent of \( \tau_{t-\varepsilon} \) and is distributed as \( \tau_\varepsilon \) for the first equality and that \( \tau_\varepsilon \overset{d}{=} \varepsilon^{q/(\gamma - 1)} \tau_1 \) for the last. Notice that, thanks to (3.11), we have

\[
\int_0^\infty \mathbb{E} \left[ \frac{1}{\tau_r^{1+q}} \mathbf{1}_{\{\tau_r > 1\}} \right] dr < \infty.
\]

Moreover, by [25, Eq. (2.1.8)], we have \( \mathbb{E} [\tau_1^q] < \infty \). Thus, it follows from (6.5) that

\[
\lim_{\alpha \to \infty} \int_\varepsilon^\infty \mathbb{E} \left[ \frac{\tau_t - \tau_{t-\varepsilon}}{\tau_t \tau_{t-\varepsilon}} \mathbf{1}_{\{\tau_{t-\varepsilon} > 1\}} \right] dt = 0.
\]  \hspace{1cm} (6.6)

Combining (6.3), (6.4) and (6.6), we deduce that

\[
\lim_{\alpha \to \infty} \int_0^\infty \mathbb{E} [M_\alpha(t)] dt = 0.
\]  \hspace{1cm} (6.7)

Next, recall from (4.7) the definition of \( N_\alpha(t) \) and from (4.3) that of \( X_t \). Write \( N_\alpha(t) = \sum_{i=1}^6 N^i_\alpha(t) \), where

\[
N^1_\alpha(t) = \frac{1}{\tau_t} \mathbf{1}_{\{\tau_{t+1} > t, \tau_{t+\varepsilon} > t\}} X_t \left\{ h \left( \frac{(\alpha \tau_t)^{1-1/\gamma}}{\int_0^\varepsilon \left( 1 - \frac{S_{t+\varepsilon}}{\tau_t} \right) e^{-\beta r / \eta t} dr \right) \right. \\
- h \left( \frac{(\alpha \tau_t)^{1-1/\gamma}}{\int_0^\varepsilon e^{-\alpha S_{t+\varepsilon} / \tau_t e^{-\beta r / \eta t} dr} \right),
\]

\[
N^2_\alpha(t) = \frac{1}{\tau_t} \mathbf{1}_{\{\tau_{t+1} > t, \tau_{t+\varepsilon} > t\}} X_t \left\{ h \left( \frac{(\alpha \tau_t)^{1-1/\gamma}}{\int_0^\varepsilon e^{-\alpha S_{t+\varepsilon} / \tau_t e^{-\beta r / \eta t} dr} \right) \right. \\
- h \left( \frac{(\alpha \tau_t)^{1-1/\gamma}}{\int_0^\varepsilon e^{-\alpha S_{t+\varepsilon} / \tau_t e^{-\beta r / \eta t} dr} \right),
\]

\[
N^3_\alpha(t) = \frac{1}{\tau_t} \mathbf{1}_{\{\tau_{t+1} > t, \tau_{t+\varepsilon} > t\}} X_t \left\{ h \left( \frac{(\alpha \tau_t)^{1-1/\gamma}}{\int_0^\varepsilon e^{-\alpha S_{t+\varepsilon} / \tau_t e^{-\beta r / \eta t} dr} \right) \right. \\
- h \left( \frac{(\alpha \tau_t)^{1-1/\gamma}}{\int_0^\varepsilon e^{-\alpha S_{t+\varepsilon} / \tau_t e^{-\beta r / \eta t} dr} \right),
\]
\[ \begin{align*}
N_\alpha^4(t) &= \frac{1}{\tau_t} \mathbf{1}_{\{\tau_t > t > \varepsilon\}} X_t \left\{ h \left( \left( \frac{\alpha}{\tau_t} \right)^{1-1/\gamma} \int_0^\varepsilon e^{-aS_r/\tau_t - \alpha \tau_t - \varepsilon} e^{-\beta r/\eta} \, dr \right) 
- h \left( \left( \frac{\alpha}{\tau_t} \right)^{1-1/\gamma} \int_0^\varepsilon e^{-aS_r/\tau_t - \alpha \tau_t - \varepsilon} e^{-\beta r/\eta} \, dr \right) \right\}, \\
N_\alpha^5(t) &= \frac{1}{\tau_t} \mathbf{1}_{\{\tau_t > t > \varepsilon\}} \left\{ f \circ R \left( T_{t-\varepsilon}^4, \tau_t^{-1} \right) - f \circ R \left( T_{t-\varepsilon}^4, \tau_t^{-1} \varepsilon \right) \right\} g \left( \tau_t^{\varepsilon + 1/\gamma} t \right) 
\times h \left( \left( \frac{\alpha}{\tau_t} \right)^{1-1/\gamma} \int_0^\varepsilon e^{-aS_r/\tau_t - \alpha \tau_t - \varepsilon} e^{-\beta r/\eta} \, dr \right), \\
N_\alpha^6(t) &= \frac{1}{\tau_t} \mathbf{1}_{\{\tau_t > t > \varepsilon\}} f \circ R \left( T_{t-\varepsilon}^4, \tau_t^{-1} \varepsilon \right) \{ g \left( \tau_t^{\varepsilon + 1/\gamma} \right) - g \left( \tau_t^{\varepsilon + 1/\gamma} (t - \varepsilon) \right) \} 
\times h \left( \left( \frac{\alpha}{\tau_t} \right)^{1-1/\gamma} \int_0^\varepsilon e^{-aS_r/\tau_t - \alpha \tau_t - \varepsilon} e^{-\beta r/\eta} \, dr \right), 
\end{align*} \]

We have
\[ \begin{align*}
\left| N_\alpha^4(t) \right| &\leq \|f\|_\infty \|g\|_\infty \|h\|_L \frac{1}{\tau_t^{2-1/\gamma}} \mathbf{1}_{\{\tau_t > 1\}} \alpha^{1-1/\gamma} \int_0^\varepsilon \left( 1 - \frac{S_r}{\tau_t} \right)^\alpha e^{-\alpha S_r/\tau_t} e^{-\beta r/\eta} \, dr \\
&\leq \|f\|_\infty \|g\|_\infty \|h\|_L \alpha^{2-1/\gamma} \int_0^\varepsilon \left| \log \left( 1 - \frac{S_r}{\tau_t} \right) + \frac{S_r}{\tau_t} \right| e^{-\alpha S_r/\tau_t} e^{-\beta r/\eta} \, dr \\
&\leq \|f\|_\infty \|g\|_\infty \|h\|_L j \left( \frac{S_r}{\tau_t} \right) \alpha^{2-1/\gamma} \int_0^\varepsilon \left( \frac{S_r}{\tau_t} \right)^2 e^{-\alpha S_r/\tau_t} \, dr \\
&\leq C \|f\|_\infty \|g\|_\infty \|h\|_L j \left( \frac{S_r}{\tau_t} \right) \alpha^{-1/\gamma} \varepsilon,
\end{align*} \]

where we used that \(|e^b - e^a| \leq |b - a|e^b| for a \leq b for the second inequality, that the function \(j: x \mapsto -(x + \log(1 - x))/x^2\) is increasing on \([0, 1]\) for the third and that the function \(x \mapsto x^2e^{-x}\) is bounded on \([0, \infty)\) for the last. Since \(\lim_{x \to 0} j(x) = 1/2\) and \(\alpha^{-1/\gamma} \varepsilon \to 0\) we deduce that \(\mathbb{P}\)-a.s. for every \(t > 0\)
\[ \lim_{\alpha \to \infty} N_\alpha^4(t) = 0. \quad (6.8) \]

Next, we have
\[ \begin{align*}
\left| N_\alpha^5(t) \right| &\leq \|f\|_\infty \|g\|_\infty \|h\|_L \frac{1}{\tau_t^{2-1/\gamma}} \mathbf{1}_{\{\tau_t > 1, t > \varepsilon\}} \alpha^{1-1/\gamma} \int_0^\varepsilon \left| e^{-\alpha S_r/\tau_t} - e^{-\alpha S_r/\tau_t - \alpha \tau_t - \varepsilon} \right| e^{-\beta r/\eta} \, dr \\
&\leq \|f\|_\infty \|g\|_\infty \|h\|_L \mathbf{1}_{\{t > \varepsilon\}} \alpha^{2-1/\gamma} \int_0^\varepsilon \frac{S_r (\tau_t - \tau_t - \varepsilon)}{\tau_t \tau_t} e^{-\alpha S_r/\tau_t} \, dr \\
&\leq C \|f\|_\infty \|g\|_\infty \|h\|_L \mathbf{1}_{\{t > \varepsilon\}} \alpha^{1-1/\gamma} \varepsilon \frac{\tau_t - \tau_t - \varepsilon}{\tau_t - \varepsilon},
\end{align*} \]

where we used that \(|e^b - e^a| \leq |b - a|e^b| for a \leq b for the second inequality and that the function \(x \mapsto xe^{-x}\) is bounded on \([0, \infty)\) for the third. Now it is clear from (3.7) and (3.8) that \(\tau_t - \tau_t - \varepsilon = \)
Setting \( \theta = \delta/(1 - \delta) \), we get
\[
\alpha^{1 - 1/\gamma} \varepsilon (\tau_t - \tau_{t - \varepsilon}) = \varepsilon^{-\theta} S_{\varepsilon -} \tag{6.9}
\]
which goes to 0 as \( \varepsilon \to 0 \) by \([6, \text{Chapter III, Theorem 9}]\) since \( \theta < \gamma/(\gamma - 1) \). It follows that \( \mathbb{P}\text{-a.s.} \) for every \( t > 0 \)
\[
\lim_{\alpha \to \infty} N^3_{\alpha}(t) = 0. \tag{6.10}
\]

Similarly, we have
\[
\left| N^3_{\alpha}(t) \right| \leq \| f \|_{\infty} \| g \|_{\infty} \| h \|_L \frac{1}{\tau_t^{2 - 1/\gamma}} \alpha^{1 - 1/\gamma} \left( \int_0^\varepsilon e^{-\alpha S_{\varepsilon}/\tau_{t - \varepsilon}} e^{-\beta r/\eta_t} - e^{-\beta r/\eta_{t - \varepsilon}} \, dr \right)
\]
\[
\leq \| f \|_{\infty} \| g \|_{\infty} \| h \|_L \alpha^{1 - 1/\gamma} \left( \int_0^\varepsilon \frac{\eta_t - \eta_{t - \varepsilon}}{\eta_t \eta_{t - \varepsilon}} e^{-\beta r/\eta_t} \, dr \right)
\]
\[
\leq \| f \|_{\infty} \| g \|_{\infty} \| h \|_L \alpha^{1 - 1/\gamma} \varepsilon \frac{\eta_t - \eta_{t - \varepsilon}}{\eta_{t - \varepsilon}}.
\]

But using (2.6) and the definition of \( \eta \), we get that \( \eta_t - \eta_{t - \varepsilon} = \varepsilon \) for \( \varepsilon > 0 \) small enough. Since \( \alpha^{1 - 1/\gamma} \varepsilon^2 = \alpha^{(2\delta - 1)(1 - 1/\gamma)} \to 0 \) as \( \delta < 1/3 \), it follows that \( \mathbb{P}\text{-a.s.} \) for every \( t > 0 \)
\[
\lim_{\alpha \to \infty} N^3_{\alpha}(t) = 0. \tag{6.11}
\]

Next, we have
\[
\left| N^4_{\alpha}(t) \right| \leq \| f \|_{\infty} \| g \|_{\infty} \| h \|_L \frac{1}{\tau_t} \alpha^{1 - 1/\gamma} \left( \int_0^\varepsilon e^{-\alpha S_{\varepsilon}/\tau_{t - \varepsilon}} e^{-\beta r/\eta_t} \, dr \right)
\]
\[
\leq \| f \|_{\infty} \| g \|_{\infty} \| h \|_L \alpha^{1 - 1/\gamma} \varepsilon \frac{\eta_t - \eta_{t - \varepsilon}}{\eta_{t - \varepsilon}}.
\]

Using that \( \tau_t - \tau_{t - \varepsilon} = S_{\varepsilon -} \), we have as \( \alpha \to \infty \)
\[
\varepsilon^{-\theta} \frac{\tau_t^{1 - 1/\gamma} - \tau_{t - \varepsilon}^{1 - 1/\gamma}}{\tau_t^{1 - 1/\gamma} \tau_{t - \varepsilon}^{1 - 1/\gamma}} \sim \varepsilon^{-\theta} \frac{\tau_t^{1 - 1/\gamma} - \tau_{t - \varepsilon}^{1 - 1/\gamma}}{\tau_t^{1 - 1/\gamma} \tau_{t - \varepsilon}^{1 - 1/\gamma}}.
\]
where the last convergence is the same as for (6.9). We deduce that \( \mathbb{P}\text{-a.s.} \) for every \( t > 0 \)
\[
\lim_{\alpha \to \infty} N^4_{\alpha}(t) = 0. \tag{6.12}
\]

Furthermore, we have
\[
\left| N^5_{\alpha}(t) \right| \leq \| f \|_L \| g \|_{\infty} \| h \|_L \frac{1}{\tau_t} \left( \int_0^\varepsilon d_{GHP} \left( R \left( \mathcal{T}_t^\uparrow, \tau_t^{-1} \right), R \left( \mathcal{T}_t^{-\varepsilon}, \tau_{t - \varepsilon}^{-1} \right) \right) \right)
\]
\[
\leq \| f \|_L \| g \|_{\infty} \| h \|_L \frac{1}{\tau_t} \left[ d_{GHP} \left( R \left( \mathcal{T}_t^\uparrow, \tau_t^{-1} \right), R \left( \mathcal{T}_t^{-\varepsilon}, \tau_{t - \varepsilon}^{-1} \right) \right) \right.
\]
\[
+ d_{GHP} \left( R \left( \mathcal{T}_t^{-\varepsilon}, \tau_{t - \varepsilon}^{-1} \right), R \left( \mathcal{T}_t^\uparrow, \tau_{t - \varepsilon}^{-1} \right) \right) \right].
\]
Notice that, by construction, the tree $T^{\perp}_t$ is obtained from $T^{\perp}_{t-\varepsilon}$ by adding to the root a branch $[0, \varepsilon)$ onto which we graft $T_s$ at height $0 \leq s < \varepsilon$. It is clear that the added part has mass $\sum_{s<\varepsilon} \mu(T_s) = S_{\varepsilon-}$ and height at most $\max_{s<\varepsilon} h(T_s) + \varepsilon$. Thus, by definition (3.1) of the mapping $R$, we deduce that

$$d_{\text{GHP}} \left( R \left( T^{\perp}_{t-\varepsilon}, \tau_t^{-1} \right), R \left( T^{\perp}_t, \tau_t^{-1} \right) \right) \leq \tau_t^{-1} S_{\varepsilon-} + \tau_t^{-1+1/\gamma} \left( \max_{s<\varepsilon} h(T_s) + \varepsilon \right). \tag{6.13}$$

Moreover, using Lemma 2.2 and again the definition of $R$, we get

$$d_{\text{GHP}} \left( R \left( T^{\perp}_{t-\varepsilon}, \tau_t^{-1} \right), R \left( T^{\perp}_t, \tau_t^{-1} \right) \right) \leq 2 \left( \tau_t^{-1+1/\gamma} - \tau_t^{-1+1/\gamma} \right) h \left( T^{\perp}_{t-\varepsilon} \right) + \left( \tau_t^{-1} - \tau_t^{-1} \right) \mu \left( T^{\perp}_{t-\varepsilon} \right). \tag{6.14}$$

From (6.13) and (6.14), we deduce that

$$\left| N^5_\alpha(t) \right| \leq \|f\|_L \|g\|_\infty \|h\|_\infty \left[ S_{\varepsilon-} + \max_{s<\varepsilon} h(T_s) + \varepsilon \right] + 2 \left( \tau_t^{-1+1/\gamma} - \tau_t^{-1+1/\gamma} \right) h \left( T^{\perp}_{t-\varepsilon} \right) + \left( \tau_t^{-1} - \tau_t^{-1} \right) \mu \left( T^{\perp}_{t-\varepsilon} \right).$$

Therefore it follows that $\mathbb{P}$-a.s. for every $t > 0$

$$\lim_{\alpha \to \infty} N^5_\alpha(t) = 0. \tag{6.15}$$

Finally, it is clear that

$$\left| N^6_\alpha(t) \right| \leq \|f\|_\infty \|g\|_L \|h\|_\infty \left| \tau_t^{-1+1/\gamma} t - \tau_t^{-1+1/\gamma} (t - \varepsilon) \right|.$$  

Thus, we have $\mathbb{P}$-a.s. for every $t > 0$

$$\lim_{\alpha \to \infty} N^6_\alpha(t) = 0. \tag{6.16}$$

Combining (6.8), (6.10)–(6.12), (6.15) and (6.16), we get that $\mathbb{P}$-a.s. for every $t > 0$, $\lim_{\alpha \to \infty} N_\alpha(t) = 0$. Notice that

$$\lim_{\alpha \to \infty} \int_0^\infty \mathbb{E} \left[ N_\alpha(t) \right] dt = 0. \tag{6.17}$$

6.3. Proof of Lemma 4.4. It is enough to show that for every Lipschitz-continuous and bounded function $f : [0, \infty) \to \mathbb{R}$

$$\lim_{\alpha \to \infty} N^{(1)} \left[ \int_T \mu(dx) f \left( \alpha^{1-1/\gamma} h^{-\beta} (Z_{\alpha,\beta}(x) - Z_{\alpha,\beta}) \right) \right] = f(0).$$

Let $\varepsilon = \alpha^{(\delta-1)(1-1/\gamma)}$ with $\delta \in (0, 1/2)$. For every $x \in T$ such that $H(x) \geq \varepsilon$, set

$$Z^{\varepsilon}_{\alpha,\beta}(x) = \int_0^\varepsilon \sigma_{r,x}^\alpha h^\beta_r dr \quad \text{and} \quad Z^{\varepsilon}_{\alpha,\beta} = \int_T 1_{\{H(x) \geq \varepsilon\}} Z^{\varepsilon}_{\alpha,\beta}(x) \mu(dx).$$
Let $x^* \in \mathcal{T}$ be the unique leaf realizing the height, that is \( H(x^*) = \mathfrak{h} \). Using that \( \mathfrak{h} \geq H(x \wedge x^*) \) and that \( Z^n_{\alpha,\beta}(x) = Z^\varepsilon_{\alpha,\beta}(x^*) \) if \( \varepsilon \leq H(x \wedge x^*) \), write

\[
\int_{\mathcal{T}} \mu(dx) f \left( \alpha^{1-1/\gamma} \mathfrak{h}^{-\beta} (Z_{\alpha,\beta}(x) - Z_{\alpha,\beta}) \right) = \sum_{i=1}^{4} A^i_{\alpha} + B_{\alpha},
\]

where

\[
A^1_{\alpha} = \int_{\mathcal{T}} \mu(dx) 1_{\{H(x \wedge x^*) \leq \varepsilon\}} f \left( \alpha^{1-1/\gamma} \mathfrak{h}^{-\beta} (Z_{\alpha,\beta}(x) - Z_{\alpha,\beta}) \right),
\]

\[
A^2_{\alpha} = \int_{\mathcal{T}} \mu(dx) 1_{\{H(x \wedge x^*) \geq \varepsilon\}} \left\{ f \left( \alpha^{1-1/\gamma} \mathfrak{h}^{-\beta} (Z_{\alpha,\beta}(x) - Z_{\alpha,\beta}) \right) - f \left( \alpha^{1-1/\gamma} \mathfrak{h}^{-\beta} (Z^n_{\alpha,\beta}(x) - Z^n_{\alpha,\beta}) \right) \right\},
\]

\[
A^3_{\alpha} = \int_{\mathcal{T}} \mu(dx) 1_{\{H(x \wedge x^*) \geq \varepsilon\}} \left\{ f \left( \alpha^{1-1/\gamma} \mathfrak{h}^{-\beta} (Z^n_{\alpha,\beta}(x) - Z_{\alpha,\beta}) \right) - f \left( \alpha^{1-1/\gamma} \mathfrak{h}^{-\beta} (Z^n_{\alpha,\beta}(x) - Z^n_{\alpha,\beta}) \right) \right\},
\]

\[
A^4_{\alpha} = - \int_{\mathcal{T}} \mu(dx) 1_{\{H(x \wedge x^*) \leq \varepsilon\}} f \left( 1_{\{h \geq \varepsilon\}} \alpha^{1-1/\gamma} \mathfrak{h}^{-\beta} (Z^n_{\alpha,\beta}(x^*) - Z^n_{\alpha,\beta}) \right),
\]

\[
B_{\alpha} = f \left( 1_{\{h \geq \varepsilon\}} \alpha^{1-1/\gamma} \mathfrak{h}^{-\beta} (Z^n_{\alpha,\beta}(x^*) - Z^n_{\alpha,\beta}) \right).
\]

We have

\[
\lim_{\alpha \to \infty} N^{(1)} \left[ |A^1_{\alpha} + A^4_{\alpha}| \leq 2 \|f\|_\infty \lim_{\alpha \to \infty} N^{(1)} \left[ \int_{\mathcal{T}} \mu(dx) 1_{\{H(x \wedge x^*) \leq \varepsilon\}} \right] = 0 \right.
\]

by the dominated convergence theorem.

Next, notice that

\[
N^{(1)} \left[ |A^2_{\alpha}| \right] \leq \|f\|_L N^{(1)} \left[ \alpha^{1-1/\gamma} \mathfrak{h}^{-\beta} \int_{\mathcal{T}} \mu(dx) 1_{\{H(x \wedge x^*) \geq \varepsilon\}} (Z_{\alpha,\beta}(x) - Z^n_{\alpha,\beta}(x)) \right]
\]

\[
\leq \|f\|_L N^{(1)} \left[ \alpha^{1-1/\gamma} \int_{\mathcal{T}} \mu(dx) 1_{\{H(x) \geq \varepsilon\}} \int_{\varepsilon}^{H(x)} \sigma^\alpha_{r,x} dr \right],
\]

where we used that \( H(x \wedge x^*) \leq H(x) \) and \( \mathfrak{h}_{r,x} \leq \mathfrak{h} \) for the second inequality. Let us show that \( N^{(1)} \)-a.s. \( \mu(dx) \)-a.s.

\[
\lim_{\alpha \to \infty} \alpha^{1-1/\gamma} \int_{\varepsilon}^{H(x)} \sigma^\alpha_{r,x} dr = 0.
\]

Using Lemma 3.1-(i), we have

\[
N^{(1)} \left[ \mu \left( x \in \mathcal{T} : \lim \sup_{\alpha \to \infty} \alpha^{1-1/\gamma} 1_{\{H(x) \geq \varepsilon\}} \int_{\varepsilon}^{H(x)} \sigma^\alpha_{r,x} dr > 0 \right) \right]
\]

\[
= \Gamma(1-1/\gamma) N \left[ \frac{1}{\sigma} 1_{\sigma > 1} \mu \left( x \in \mathcal{T} : \lim \sup_{\alpha \to \infty} \left( \frac{\alpha}{\sigma} \right)^{1-1/\gamma} 1_{\{H(x) \geq \sigma^{1-1/\gamma} \varepsilon\}} \int_{\sigma^{1-1/\gamma} \varepsilon}^{H(x)} \left( \frac{\sigma_{r,x}}{\sigma} \right) \alpha dr > 0 \right) \right].
\]

But, on the event \( \{\sigma > 1\} \), the following inequality holds

\[
\left( \frac{\alpha}{\sigma} \right)^{1-1/\gamma} 1_{\{H(x) \geq \sigma^{1-1/\gamma} \varepsilon\}} \int_{\sigma^{1-1/\gamma} \varepsilon}^{H(x)} \left( \frac{\sigma_{r,x}}{\sigma} \right)^{\alpha} dr \leq \alpha^{1-1/\gamma} 1_{\{H(x) \geq \varepsilon\}} \int_{\varepsilon}^{H(x)} \left( \frac{\sigma_{r,x}}{\sigma} \right)^{\alpha} dr.
\]

We deduce that

\[
N^{(1)} \left[ \mu \left( x \in \mathcal{T} : \lim \sup_{\alpha \to \infty} \alpha^{1-1/\gamma} 1_{\{H(x) \geq \varepsilon\}} \int_{\varepsilon}^{H(x)} \sigma^\alpha_{r,x} dr > 0 \right) \right]
\]

\[
= \Gamma(1-1/\gamma) N \left[ \frac{1}{\sigma} 1_{\sigma > 1} \mu \left( x \in \mathcal{T} : \lim \sup_{\alpha \to \infty} \left( \frac{\alpha}{\sigma} \right)^{1-1/\gamma} 1_{\{H(x) \geq \sigma^{1-1/\gamma} \varepsilon\}} \int_{\sigma^{1-1/\gamma} \varepsilon}^{H(x)} \left( \frac{\sigma_{r,x}}{\sigma} \right)^{\alpha} dr > 0 \right) \right].
\]
\[
\leq \Gamma (1 - 1/\gamma) \mathbb{N} \left[ \frac{1}{\sigma} \mathbf{1}_{\{\sigma > 1\}} \mu \left( x \in \mathcal{T} : \limsup_{\alpha \to \infty} \alpha^{1-1/\gamma} \mathbf{1}_{\{H(x) \geq \varepsilon\}} \int_{\varepsilon}^{H(x)} \left( \frac{\sigma_{r,x}}{\sigma} \right)^{\alpha} \, dr > 0 \right) \right], \]

where the right-hand side vanishes by (6.1). This proves (6.20). Furthermore, applying Lemma 3.6, we have

\[
\sup_{\alpha \geq 0} \alpha^{2-2/\gamma} \mathbb{N}^{(1)} \left[ \int_{\mathcal{T}} \mu(dx) \left( \mathbf{1}_{\{H(x) > \varepsilon\}} \int_{\varepsilon}^{H(x)} \sigma_{r,x}^{\alpha} \, dr \right)^2 \right] \leq \sup_{\alpha > 0} \alpha^{2-2/\gamma} \mathbb{N}^{(1)} \left[ \int_{\mathcal{T}} \mu(dx) \left( \int_{0}^{H(x)} \sigma_{r,x}^{\alpha} \, dr \right)^2 \right] < \infty.
\]

We deduce that the family

\[
\left( \alpha^{1-1/\gamma} \mathbf{1}_{\{H(x) \geq \varepsilon\}} \int_{\varepsilon}^{H(x)} \sigma_{r,x}^{\alpha} \, dr, \alpha \geq 0 \right)
\]

is uniformly integrable under the measure \( \mathbb{N}^{(1)}[d\mathcal{T}]\mu(dx) \). In conjunction with (6.20), this gives

\[
\lim_{\alpha \to \infty} \mathbb{N}^{(1)} \left[ \alpha^{1-1/\gamma} \int_{\mathcal{T}} \mathbf{1}_{\{H(x) > \varepsilon\}} \mu(dx) \int_{\varepsilon}^{H(x)} \sigma_{r,x}^{\alpha} \, dr \right] = 0, \tag{6.21}
\]

which, thanks to (6.19), implies that

\[
\lim_{\alpha \to \infty} \mathbb{N}^{(1)}[\|A_{\alpha}^2\|] = 0. \tag{6.22}
\]

We have

\[
\mathbb{N}^{(1)}[\|A_{\alpha}^3\|] \leq \|f\|_{L} \mathbb{N}^{(1)} \left[ \alpha^{1-1/\gamma} \mathbf{1}_{\{H(x) > \varepsilon\}} \mu(dx) \int_{\mathcal{T}} \left( Z_{\alpha,\beta} - Z_{\alpha,\beta}^{\varepsilon} \right) \right] \\
\leq \|f\|_{L} \mathbb{N}^{(1)} \left[ \alpha^{1-1/\gamma} \mathbf{1}_{\{H(x) > \varepsilon\}} \left( Z_{\alpha,\beta} - Z_{\alpha,\beta}^{\varepsilon} \right) \right] \\
\leq \|f\|_{L} \mathbb{N}^{(1)} \left[ \alpha^{1-1/\gamma} \int_{\mathcal{T}} \mathbf{1}_{\{H(x) > \varepsilon\}} \mu(dx) \int_{\varepsilon}^{H(x)} \sigma_{r,x}^{\alpha} \, dr \right] \\
+ \|f\|_{L} \mathbb{N}^{(1)} \left[ \alpha^{1-1/\gamma} \int_{\mathcal{T}} \mathbf{1}_{\{H(x) < \varepsilon\}} \mu(dx) \int_{0}^{H(x)} \sigma_{r,x}^{\alpha} \, dr \right], \tag{6.23}
\]

where we used that \( h_{r,x} \leq h \) for the last inequality. Let \( p \in (1, 2) \) so that \( \varepsilon^{1+p}\alpha^{1-1/\gamma} \to 0 \). Using that \( \sigma_{r,x} \leq 1 \) together with the Markov inequality, we get

\[
\mathbb{N}^{(1)} \left[ \alpha^{1-1/\gamma} \int_{\mathcal{T}} \mathbf{1}_{\{H(x) > \varepsilon\}} \mu(dx) \int_{0}^{H(x)} \sigma_{r,x}^{\alpha} \, dr \right] \leq \mathbb{N}^{(1)} \left[ \varepsilon \alpha^{1-1/\gamma} \int_{\mathcal{T}} \mathbf{1}_{\{H(x) < \varepsilon\}} \mu(dx) \right] \\
\leq \varepsilon^{1+p}\alpha^{1-1/\gamma} \mathbb{N}^{(1)} \left[ \int_{\mathcal{T}} H(x)^{-p} \mu(dx) \right].
\]

By Lemma 3.7, the last term is finite. This, in conjunction with (6.21) and (6.23), implies that

\[
\lim_{\alpha \to \infty} \mathbb{N}^{(1)}[\|A_{\alpha}^3\|] = 0. \tag{6.24}
\]
It remains to show that \( \lim_{\alpha \to \infty} N^{(1)}[B_{\alpha}] = f(0) \), which is equivalent to the following convergence in \( N^{(1)} \)-probability

\[
\lim_{\alpha \to \infty} \mathbf{1}_{\{b \geq \varepsilon\}} \alpha^{1-\gamma} h^{-\beta} \left( Z_{\alpha,\beta}^\varepsilon(x^*) - Z_{\alpha,\beta}^\varepsilon \right) = 0. \tag{6.25}
\]

Again using that \( Z_{\alpha,\beta}(x) = Z_{\alpha,\beta}(x^*) \) if \( \varepsilon \leq H(x \wedge x^*) \), we write

\[
\mathbf{1}_{\{b \geq \varepsilon\}} \alpha^{1-\gamma} h^{-\beta} \left( Z_{\alpha,\beta}^\varepsilon(x^*) - Z_{\alpha,\beta}^\varepsilon \right) = B^1_{\alpha} + B^2_{\alpha},
\]

where

\[
B^1_{\alpha} = \alpha^{1-\gamma} h^{-\beta} \left( \mathbf{1}_{\{b \geq \varepsilon\}} Z_{\alpha,\beta}^\varepsilon(x^*) - \int_\mathcal{T} \mu(dx) \mathbf{1}_{\{H(x \wedge x^*) \geq \varepsilon\}} Z_{\alpha,\beta}^\varepsilon(x) \right),
\]

\[
B^2_{\alpha} = \alpha^{1-\gamma} h^{-\beta} \left( \int_\mathcal{T} \mu(dx) \mathbf{1}_{\{H(x \wedge x^*) \geq \varepsilon\}} Z_{\alpha,\beta}^\varepsilon(x) - \mathbf{1}_{\{b \geq \varepsilon\}} Z_{\alpha,\beta}^\varepsilon \right).
\]

Recall that \( \varepsilon = \alpha^{(d-1)(1-\gamma)} \to 0 \) as \( \alpha \to \infty \). Fix \( \eta > 0 \) and let \( \alpha_0 > 0 \) be large enough so that for every \( \alpha \geq \alpha_0 \)

\[
N^{(1)} \left[ \int_\mathcal{T} \mu(dx) \mathbf{1}_{\{H(x \wedge x^*) < \varepsilon\}} \right] \leq \eta.
\]

Then we have for every \( \alpha \geq \alpha_0 \) and \( C > 0 \)

\[
N^{(1)} \left[ \alpha^{1-\gamma} h^{-\beta} Z_{\alpha,\beta}^\varepsilon(x^*) \mathbf{1}_{\{b \geq \varepsilon\}} \right] \geq C
\]

\[
\leq N^{(1)} \left[ \int_\mathcal{T} \mu(dx) \mathbf{1}_{\{ \alpha^{1-\gamma} h^{-\beta} Z_{\alpha,\beta}^\varepsilon(x) \geq C, H(x \wedge x^*) \geq \varepsilon\}} \right] + N^{(1)} \left[ \int_\mathcal{T} \mu(dx) \mathbf{1}_{\{H(x \wedge x^*) < \varepsilon\}} \right]
\]

\[
\leq \frac{\alpha^{2-2/\gamma}}{C^2} N^{(1)} \left[ \int_\mathcal{T} \mu(dx) \mathbf{1}_{\{H(x \wedge x^*) \geq \varepsilon\}} \left( h^{-\beta} Z_{\alpha,\beta}^\varepsilon(x) \right)^2 \right] + \eta
\]

\[
\leq \frac{\alpha^{2-2/\gamma}}{C^2} N^{(1)} \left[ \int_\mathcal{T} \mu(dx) \left( \int_0^{H(x)} \sigma_{r,x}^\alpha dr \right)^2 \right] + \eta
\]

\[
\leq \frac{M}{C^2} + \eta \tag{6.26}
\]

for some constant \( M > 0 \), where we used that \( Z_{\alpha,\beta}^\varepsilon(x^*) = Z_{\alpha,\beta}^\varepsilon(x) \) for every \( x \in \mathcal{T} \) such that \( H(x \wedge x^*) \geq \varepsilon \) for the first inequality, the Markov inequality for the second and Lemma 3.6 for the last. Thus, we get that the family \( \left( \mathbf{1}_{\{b \geq \varepsilon\}} \alpha^{1-\gamma} h^{-\beta} Z_{\alpha,\beta}^\varepsilon(x^*), \alpha \geq \alpha_0, \beta \geq 0 \right) \) is tight. Since \( N^{(1)} \)-a.s.

\[
\lim_{\alpha \to \infty} \int_\mathcal{T} \mu(dx) \mathbf{1}_{\{H(x \wedge x^*) < \varepsilon\}} = 0,
\]

we deduce the following convergence in \( N^{(1)} \)-probability

\[
\lim_{\alpha \to \infty} B^1_{\alpha} = \lim_{\alpha \to \infty} \mathbf{1}_{\{b \geq \varepsilon\}} \alpha^{1-\gamma} h^{-\beta} Z_{\alpha,\beta}^\varepsilon(x^*) \int_\mathcal{T} \mu(dx) \mathbf{1}_{\{H(x \wedge x^*) < \varepsilon\}} = 0.
\]

Furthermore, we have

\[
N^{(1)}[|B^2_{\alpha}|] \leq \alpha^{1-\gamma} N^{(1)} \left[ \int_\mathcal{T} \mu(dx) \mathbf{1}_{\{H(x) \geq \varepsilon, H(x \wedge x^*) < \varepsilon\}} h^{-\beta} Z_{\alpha,\beta}^\varepsilon(x) \right]
\]
\[
\alpha^{1-1/\gamma} N^{(1)} \left[ \int_T \mu(dx) \left( \mathbf{1}_{\{H(x^{\vee}, x^{\wedge}) < \varepsilon\}} \int_0^{H(x)} \sigma^{\alpha}_{r,x} dr \right) \right] \\
\leq \alpha^{1-1/\gamma} N^{(1)} \left[ \int_T \mu(dx) \left( \int_0^{H(x)} \sigma^{\alpha}_{r,x} dr \right)^2 \right]^{1/2} N^{(1)} \left[ \int_T \mu(dx) \mathbf{1}_{\{H(x^{\vee}, x^{\wedge}) < \varepsilon\}} \right]^{1/2} \\
\leq C N^{(1)} \left[ \int_T \mu(dx) \mathbf{1}_{\{H(x^{\vee}, x^{\wedge}) < \varepsilon\}} \right]^{1/2}
\]

for some constant \(C > 0\), where we used the Cauchy–Schwarz inequality for the third inequality and (3.19) for the last. It follows from the dominated convergence theorem that \(\lim_{\alpha \to \infty} N^{(1)}[|B_\alpha^2|] = 0\). This finishes the proof of (6.25).

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