Entanglement production with multimode Bose-Einstein condensates in optical lattices

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Abstract

Deep optical lattices are considered, in each site of which there are many Bose-condensed atoms. By the resonant modulation of trapping potentials it is possible to transfer a macroscopic portion of atoms to the collective nonlinear states corresponding to topological coherent modes. Entanglement can be generated between these modes. By varying the resonant modulating field it is possible to effectively regulate entanglement production in this multimode multitrap system of Bose condensates.
Entanglement is assumed to play an important role in quantum computation and quantum information processing [1–5]. As possible candidates for engineering entanglement, one considers spin assemblies, trapped ions, or Bose-condensed trapped atoms. Here we shall concentrate our attention on atomic Bose-Einstein condensates [6–9]. We shall study entanglement production that can be realized between different topological coherent modes of nonequilibrium Bose condensates. A general theory of the resonant generation of these coherent modes was advanced in Ref. [10]. The properties of these modes have been intensively investigated [10–30]. A brief survey is given in a recent publication [31].

Let us, first, recall the meaning of the topological coherent modes [10]. We consider a dilute Bose gas, with the effective local interaction

$$\Phi(r) = \Phi_0 \delta(r), \quad \Phi_0 \equiv 4\pi a_s/m,$$

where $a_s$ is a scattering length, $m$ is atomic mass. When an atomic system is confined in a trap and cooled down to very low temperatures, then almost all atoms can be piled down to a single quantum state, thus creating Bose-Einstein condensate. The latter forms a coherent system, which is described by the Gross-Pitaevskii equation

$$\hat{H}[\eta_k] \eta_k(r) = E_k \eta_k(r),$$

with the nonlinear Schrödinger Hamiltonian

$$\hat{H}[\eta] \equiv -\frac{\nabla^2}{2m} + U(r) + \Phi_0|\eta|^2,$$

in which $U(r)$ is a trapping potential. The solutions to the eigenproblem (2), labelled by a set of quantum numbers $k$, are the topological coherent modes $\eta_k(r)$. These are not compulsorily orthogonal, but can always be normalized to the total number of atoms in the trap,

$$||\eta_k||^2 = (\eta_k, \eta_k) = N.$$

The lowest-energy mode $\eta_0(r)$ represents the usual Bose-Einstein condensate, while the higher-energy modes describe nonground-state condensates.

For each coherent mode $\eta_k(r)$, one can construct the coherent state

$$|\eta_k> = \left[ \frac{e^{-N/2}}{\sqrt{n!}} \prod_{i=1}^{n} \eta_k(r_i) \right],$$

which is a column with elements labelled by the index $n$. The coherent states are normalized to unity, $< \eta_k | \eta_k > = 1$. Though, in general, coherent states are not orthogonal, it is possible to show [32] that the states (4) are asymptotically orthogonal, in the sense that

$$\lim_{N \to \infty} < \eta_k | \eta_p > = \delta_{kp}.$$

The totality of all $|\eta_k>$, that is, the closed linear envelope $\mathcal{L}\{|\eta_k>\}$, forms the space of states for the trapped atomic system.

Now, let us consider a set of traps, in each of which there are many Bose-condensed atoms. Such a setup can be realized by creating a deep optical lattice. For example, cold
rubidium $^{87}\text{Rb}$ atoms have been loaded [33] into an optical lattice, with adjacent sites spaced by $a = 5.3 \times 10^{-4} \text{ cm}$. Lattice sites were practically independent, with the tunneling time between sites above $10^{18} \text{ s}$. The total number of Bose-condensed atoms in the optical lattice was about 7000. The number of lattice sites was typically between 5 to 35. So that the number of condensed atoms in each site could be varied between about 200 to 2000.

Nowadays there exists a variety of optical lattices, one-dimensional, two-dimensional, or three-dimensional [34,35]. Multiphoton processes [36] can be employed for creating asymmetric lattice potentials [37]. Optical lattice potentials can be made spin-dependent [38]. Being separated in different lattice sites, atoms are practically independent. But if there occurs small, though finite, tunneling, phase coherence may persist on short length scales even deep in the insulating state [39].

Assume that in each lattice site of an optical lattice there are many Bose-condensed atoms, so that each site plays the role of a trap. The space of states for a $j$-trap is

$$\mathcal{H}_j = \mathcal{L}\{|\eta_{jk}\rangle\}.$$  \hspace{1cm} (5)

The total space of states for the whole lattice is

$$\mathcal{H} = \otimes_j \mathcal{H}_j.$$  \hspace{1cm} (6)

The set of disentangled states,

$$\mathcal{D} \equiv \{\otimes_j |\varphi_j\rangle: |\varphi_j\rangle \in \mathcal{H}_j\},$$  \hspace{1cm} (7)

consists of the states having the form of the tensor products. An arbitrary state of the considered system of traps can be represented as

$$|\eta(t)\rangle = \sum_k c_k(t) \otimes_j |\eta_{jk}\rangle,$$  \hspace{1cm} (8)

whose coefficients define the mode probabilities, or the fractional mode populations

$$n_k(t) \equiv |c_k(t)|^2.$$  \hspace{1cm} (9)

Let the number of trapping sites be $L$, with $M$ coherent modes each. If the coefficients $c_k(t)$ in state (8) can be varied, then different entangled states can be created, such as the Bell states

$$|B\rangle = \frac{1}{\sqrt{2}}(|00\rangle \pm |11\rangle)$$

for $L = 2$ and $M = 2$, the multicat states

$$|MC\rangle = \frac{1}{\sqrt{2}}(|00\ldots0\rangle \pm |11\ldots1\rangle)$$

for $L > 2$ and $M = 2$, or the multicat multimode states for $L > 2$ and $M > 2$.

When the system is in a pure statistical state, then its statistical operator is

$$\hat{\rho}(t) = |\eta(t)\rangle \langle \eta(t)|.$$  \hspace{1cm} (10)

In the case of a mixed statistical state, the statistical operator becomes
\[ \hat{\rho}(t) = \sum_k |c_k(t)|^2 \otimes_j |\eta_{jk} > < \eta_{jk}|. \]  \hfill (11)

Entanglement, generated by a statistical operator \( \hat{\rho}(t) \), is quantified \([40,41]\) by the measure of entanglement production

\[ \varepsilon(\hat{\rho}(t)) \equiv \log \frac{||\hat{\rho}(t)||_D}{||\hat{\rho} \otimes(t)||_D}, \] \hfill (12)

in which the logarithm is to the base 2,

\[ ||\hat{\rho}(t)||_D \equiv \sup_{f \in D} ||\hat{\rho}(t)f|| \quad (||f|| = 1), \]

\[ \hat{\rho} \otimes(t) \equiv \otimes_j \hat{\rho}_j(t), \quad \hat{\rho}_j(t) \equiv \text{Tr}_{H_{i \neq j}} \hat{\rho}(t). \]

For \( L \) trapping sites of a lattice, we find

\[ \varepsilon(\hat{\rho}(t)) = (L - 1)\varepsilon_2(t), \] \hfill (13)

where

\[ \varepsilon_2(t) = -\log \sup_k n_k(t). \] \hfill (14)

In order to generate higher coherent modes, it is necessary to apply an external resonant field \([10,31] \). We assume that the same modulating field acts on all lattice sites. The field has the form

\[ V(r, t) = \zeta(t) [V_1(r) \cos(\omega t) + V_2(r) \sin(\omega t)]. \] \hfill (15)

Here \( \zeta(t) = 0, 1 \) is a switching function that allows one to switch on and off the resonant field (15). The frequency \( \omega \) is tuned close to the transition frequency

\[ \omega_0 \equiv E_1 - E_0, \] \hfill (16)

in which \( E_0 \) is the energy of the ground-state mode and \( E_1 \) is the energy of the desired mode to be generated. The resonance condition is implied, such that

\[ \frac{\Delta \omega}{\omega} \ll 1 \quad (\Delta \equiv \omega - \omega_0). \] \hfill (17)

With the resonant modulating field (15), the time-dependent Gross-Pitaevskii equation becomes

\[ i \frac{\partial}{\partial t} \varphi(r, t) = (\hat{H}[\varphi] + \hat{V}) \varphi(r, t), \] \hfill (18)

where the nonlinear Schrödinger Hamiltonian is

\[ \hat{H}[\varphi] = -\frac{\nabla^2}{2m} + U(r) + \Phi_0 N|\varphi|^2. \]
and the function \( \varphi(r, t) \) is normalized to one, \( \| \varphi \|^2 = 1 \).

The solution to Eq. (18) can be represented as the mode expansion

\[
\varphi(r, t) = \sum_k c_k(t) \varphi_k(r)e^{-iE_k t},
\]

in which the coefficient functions \( c_k(t) \) define the fractional mode populations (9). These functions can be found as the projections

\[
c_k(t) = \frac{1}{T} \int_t^{t+T} \varphi^*_k(r)e^{iE_k t}\varphi(r, t) \, dt \quad \left( T = \frac{2\pi}{\omega} \right).
\]

The dynamics of the mode populations essentially depends on the strength of atomic interactions and on the amplitude of the resonant field \([10,31]\). We shall denote by \( b \) the dimensionless amplitude of the modulating resonant field, reduced to the strength of atomic interactions. Respectively, the temporal behaviour of the entanglement production measure (14), which for the two-mode case takes the form

\[
\varepsilon_2(t) = -\log_2 \sup\{n_0(t), n_1(t)\},
\]

strongly depends on the properties of the resonant field.

We assume that at the initial time \( t = 0 \) solely the ground-state is available, so that \( n_0(0) = 1 \) and \( n_1(0) = 0 \). Then the resonant field (15) is switched on, generating an excited coherent mode and changing the entanglement production measure (20). There are two ways of regulating the amount of entanglement production.

First, one can vary the amplitude and frequency of the resonant pumping field, choosing by this the required parameters, whose values define two main regimes of an oscillatory behaviour of \( \varepsilon_2(t) \). These are the mode-locked and mode-unlocked regimes \([31]\). We solve numerically the evolution equations and calculate the measure of entanglement production (20). Keeping in mind that the detuning can always be made small, we set it to zero. Then the value \( b_c = 0.497764 \) is the critical point for the change of the dynamical regimes. Below \( b_c \), the measure (20) oscillates with time, never reaching one, as is shown in Fig. 1. When the parameter \( b \) reaches the critical point \( b = b_c \), then the oscillating \( \varepsilon_2(t) \) reaches one, as is demonstrated in Fig. 2. For the dimensionless amplitude of the pumping field \( b > b_c \), the oscillations of \( \varepsilon_2(t) \) are always in the interval between 0 and 1. But the oscillation period sensitively depends on the value of \( b \). Thus, for \( b = 0.5 \), just a little above \( b_c \), the period of oscillations, shown in Fig. 3, is more than twice shorter than that in Fig. 2 for \( b = b_c \). The period for \( b = 0.7 \), as is demonstrated in Fig. 4, is about eight times shorter than in Fig. 2. Thus, by varying the amplitude of the pumping resonant field, we can strongly influence the evolution of \( \varepsilon_2(t) \) both in its amplitude and period of oscillations.

There is one more very interesting way of regulating entanglement generation, which can be done by switching on and off the applied resonant field. Recall that this alternating field can be easily produced by modulating the magnetic field forming the trapping potential in magnetic traps or by varying the laser intensity in optical traps. Then, it is possible to create various sequences of pulses for \( \varepsilon_2(t) \). For example, by switching on and off the resonant field in a periodic manner, we may form equidistant pulses of \( \varepsilon_2(t) \), with all pulses having the same shape, as is demonstrated in Fig. 5. But we can also switch on and off the pumping field at different time intervals, thus, forming nonequidistant pulses, as is shown in
Fig. 6. The possibility of creating very different pulses is illustrated in Fig. 7. Regulating entanglement production by means of a manipulation with the resonant pumping field, it is feasible to organize a kind of the Morse alphabet.

Here we have considered entanglement production in a multimode Bose-Einstein condensate loaded in a deep optical lattice. The arising entanglement production occurs for different coherent modes generated by a resonant external field. Recently ultra-cold fermions in optical lattices have attracted great attention (see survey [42]). Entanglement production for fermions in optical lattices can also be considered, though requiring different techniques. Varying the interaction between fermions by means of the Feshbach resonance methods, bound fermionic states can be achieved, forming bosonic molecules. The latter can be Bose-condensed (see [43,44] and references therein). Therefore the coherent modes of molecules can be created, similarly to those of atoms. Then the entanglement production for molecular coherent modes can be studied in the way analogous to that for atomic coherent modes.

The measure of entanglement production (14) or (20) is directly connected with the fractional mode populations. The latter define the spatial features of atomic clouds inside each lattice site. Thus, analyzing the spatial distribution of atoms, one can make conclusions on the mode entanglement production. The spatial distribution can be studied by means of scattering experiments. Wave scattering on periodic structures is known to possess a number of interesting properties [45]. Another possibility of studying the spatial characteristics of atomic clouds is through the time-of-flight experiments, by releasing atoms from the trapping potentials and observing the atom expansion and interference.

In conclusion, a multitrap ensemble of multimode Bose-Einstein condensates, subject to the action of a common resonant field, is analogous to a system of finite-level atoms in a common resonant electromagnetic field. A multitrap system can be formed, e.g., as an optical lattice with deep potential wells, incorporating many atoms around each lattice site. In the multitrap multimode condensate a high level of entanglement can be achieved. By varying the amplitude and frequency of the pumping resonant field, different regimes of evolutional entanglement can be realized. Moreover, by switching on and off the pumping field in various ways, it is feasible to create entanglement pulses of arbitrary length and composing arbitrary sequences of punctuated entanglement generation. Such a high level of admissible manipulation with and regulating of entanglement can, probably, be useful for information processing and quantum computing.

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Figure Captions

**Fig. 1.** Evolutional entanglement production, quantified by the measure $\varepsilon_2(t)$, in the mode-locked regime, with $b = 0.3$. In this and in all following figures, time is measured in units of $\alpha^{-1}$.

**Fig. 2.** The measure $\varepsilon_2(t)$ for the boundary between the mode-locked and mode-unlocked regimes, when $b = b_c = 0.497764$.

**Fig. 3.** Entanglement production in the mod-unlocked regime, with $b = 0.5$.

**Fig. 4.** Drastic shortenning of the period of $\varepsilon_2(t)$ for $b = 0.7$.

**Fig. 5.** Regulated equidistant pulses of $\varepsilon_2(t)$, formed by switching on and off the resonant field, with $b = 0.7$, so that $\varepsilon_2(t)$ equals one during the time intervals $\Delta t = 7.35$ (in units of $\alpha^{-1}$), and it equals zero during the same intervals $\Delta t = 7.35$.

**Fig. 6.** Nonequidistant pulses of $\varepsilon_2(t)$, created by switching on and off the pumping field, with $b = 0.7$, at nonequal time intervals.

**Fig. 7.** Regulated pulses of $\varepsilon_2(t)$, for the same $b = 0.7$, as in Fig. 6, but for essentially different moments of switching on and off the pumping field.
FIG. 1. Evolutional entanglement production, quantified by the measure $\varepsilon_2(t)$, in the mode-locked regime, with $b = 0.3$. In this and in all following figures, time is measured in units of $\alpha^{-1}$. 
FIG. 2. The measure $\varepsilon_2(t)$ for the boundary between the mode-locked and mode-unlocked regimes, when $b = b_c = 0.497764$. 
FIG. 3. Entanglement production in the mod-unlocked regime, with $b = 0.5$. 
FIG. 4. Drastic shortening of the period of $\varepsilon_2(t)$ for $b = 0.7$. 

$\varepsilon_2(t)$
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