GENERALIZED WEIGHTED COMPOSITION OPERATORS ON BERGMAN SPACES INDUCED BY DOUBLING WEIGHTS

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ABSTRACT. Bounded and compact generalized weighted composition operators acting from the weighted Bergman space \( A^p_\omega \), where \( 0 < p < \infty \) and \( \omega \) belongs to the class \( D \) of radial weights satisfying a two-sided doubling condition, to a Lebesgue space \( L^q_\nu \) are characterized. On the way to the proofs a new embedding theorem on weighted Bergman spaces \( A^p_\omega \) is established. This last-mentioned result generalizes the well-known characterization of the boundedness of the differentiation operator \( D^\alpha (f) = f^{(\alpha)} \) from the classical weighted Bergman space \( A^p_\omega \) to the Lebesgue space \( L^q_\nu \), induced by a positive Borel measure \( \mu \), to the setting of doubling weights.

1. INTRODUCTION AND MAIN RESULTS

Let \( \mathcal{H}(\mathbb{D}) \) denote the space of analytic functions in the unit disc \( \mathbb{D} = \{ z \in \mathbb{C} : |z| < 1 \} \). An integrable function \( \omega : \mathbb{D} \to [0, \infty) \) is a weight. It is radial if \( \omega(z) = \omega(|z|) \) for all \( z \in \mathbb{D} \). For \( 0 < p < \infty \) and a weight \( \omega \), the weighted Bergman space \( A^p_\omega \) consists of \( f \in \mathcal{H}(\mathbb{D}) \) such that

\[
\| f \|_{A^p_\omega} = \int_{\mathbb{D}} |f(z)|^p \omega(z) \, dA(z) < \infty,
\]

where \( dA(z) = \frac{dx \, dy}{\pi} \) is the normalized Lebesgue area measure on \( \mathbb{D} \). The corresponding Lebesgue space is denoted by \( L^p_\nu \), and thus \( A^p_\omega = L^p_\nu \cap \mathcal{H}(\mathbb{D}) \). As usual, \( A^p_\omega \) stands for the classical weighted Bergman space induced by the standard radial weight \( \omega(z) = (1 - |z|^2)^\alpha \), where \(-1 < \alpha < \infty\). For \( 0 < p \leq \infty \), the Hardy space \( H^p \) consists of functions \( f \in \mathcal{H}(\mathbb{D}) \) such that

\[
\| f \|_{H^p} = \sup_{0 < r < 1} M_p(r, f) < \infty,
\]

where

\[
M_p(r, f) = \left( \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p \, d\theta \right)^{\frac{1}{p}}, \quad 0 < p < \infty,
\]

are the \( L^p \)-means and \( M_\infty(r, f) = \max_{|z|=r} |f(z)| \) is the maximum modulus function.

For a radial weight \( \omega \), write \( \widehat{\omega}(z) = \int_{|z|=r} \omega(s) \, ds \) for all \( z \in \mathbb{D} \). A weight \( \omega \) belongs to the class \( \widehat{D} \) if there exists a constant \( C = C(\omega) \geq 1 \) such that \( \widehat{\omega}(r) \leq C \widehat{\omega}\left(\frac{1+r}{1-r}\right) \) for all \( 0 \leq r < 1 \). Moreover, if there exist \( K = K(\omega) > 1 \) and \( C = C(\omega) > 1 \) such that \( \widehat{\omega}(r) \geq C \widehat{\omega}\left(1 - \frac{1-r^2}{1+r^2}\right) \) for all \( 0 \leq r < 1 \), then we write \( \omega \in \mathcal{D} \). Weights \( \omega \) belonging to \( \mathcal{D} = \widehat{D} \cap \mathcal{D} \) are called doubling. If there exist \( C = C(\omega) > 1 \) and \( K = K(\omega) > 1 \) such that \( \omega_x \geq C \omega_{Kx} \) for all \( x \geq 1 \), we write \( \omega \in \mathcal{M} \).

Let \( \varphi \) be an analytic self-map of \( \mathbb{D} \). The function \( \varphi \) induces a composition operator \( C^{\varphi}_f \) on \( \mathcal{H}(\mathbb{D}) \) defined by \( C^{\varphi}_f = f \circ \varphi \). The weighted composition operator induced by a self-map \( \varphi \) and \( u \in \mathcal{H}(\mathbb{D}) \) is the operator \( uC^{\varphi}_f \) that sends \( f \) to the analytic function \( u \cdot f \circ \varphi \). These operators have been extensively studied in a variety of function spaces, see for example [1, 2, 3, 7, 18, 19, 22].

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The differentiation operator $D$ on $\mathcal{H}(\mathbb{D})$ is defined by $Df = f'$. Further, for $n \in \mathbb{N}$, we define $D^n f = f^{(n)}$. By following [20], the generalized weighted composition operator $D^n_{\varphi,u}$ is defined by

$$D^n_{\varphi,u}f = u \cdot f^{(n)} \circ \varphi,$$

where $\varphi$ is an analytic self-map of $\mathbb{D}$, $u \in \mathcal{H}(\mathbb{D})$ and $n \in \mathbb{N}$. Obviously, if $n = 0$ and $u \equiv 1$, then this operator reduces to the composition operator $C_\varphi$, and if $n = 0$, then we get the weighted composition operator $uC_\varphi$. If $n = 1$ and $u(z) = \varphi'(z)$, then $D^n_{\varphi,u} = DC_\varphi$ which was studied in [5] [7]. When $n = 1$ and $u \equiv 1$, then $D^n_{\varphi,u} = C_\varphi D$ which was studied in [9].

In [21], the author characterized the boundedness and compactness of the operators $D^n_{\varphi,u}$ between different standard weighted Bergman spaces. In this paper, we characterize bounded and compact operators $D^n_{\varphi,u}$ acting from the weighted Bergman space $A^p_\omega$ with $\omega \in \mathcal{D}$ to $L^p_\nu$ induced by any $0 < q < \infty$ and a positive Borel measure $\nu$.

To state main results, some more notation is needed. Let $\mu$ be a finite positive Borel measure on $\mathbb{D}$ and $h$ a measurable function on $\mathbb{D}$. For an analytic self-map $\varphi$ of $\mathbb{D}$, the weighted pushforward measure related to $h$ is defined by

$$\varphi_*(h, \mu)(M) = \int_{\varphi^{-1}(M)} h \, d\mu$$

for each measurable set $M \subset \mathbb{D}$. If $\mu$ is the Lebesgue measure, we omit the measure in the notation and write $\varphi_*(h)(M)$ for the left hand side of (1.2). Here and from now on $S(z) = \{\zeta \in \mathbb{D} : |\zeta| < |z|, \arg \zeta - \arg z < (1 - |z|)/2\}$ is the Carleson square induced by the point $z \in \mathbb{D} \setminus \{0\}$, $S(0) = \mathbb{D}$ and $\omega(E) = \int_E \omega \, dA$ for each measurable set $E \subset \mathbb{D}$. Further, $\Gamma(z) = \left\{ \zeta \in \mathbb{D} : |\zeta - \arg \zeta| < \frac{1}{2} \left(1 - \frac{1}{|z|}\right) \right\}$, $z = re^{i\theta} \in \mathbb{D} \setminus \{0\}$ is a non-tangential approach region with vertex at $z \in \mathbb{D} \setminus \{0\}$, and $T(z) = \{\zeta \in \mathbb{D} : \zeta \in \Gamma(z)\}$ is the tent induced by $z \in \mathbb{D} \setminus \{0\}$. Observe that we have $\omega(T(z)) \asymp \omega(S(z))$ for all $z \in \mathbb{D} \setminus \{0\}$ if $\omega \in \mathcal{D}$. The statement of our first result involves pseudohyperbolic discs. The pseudohyperbolic distance between two points $a$ and $b$ in $\mathbb{D}$ is $\rho(a,b) = |(a-b)/(1-\overline{a}b)|$. For $a \in \mathbb{D}$ and $0 < r < 1$, the pseudohyperbolic disc of center $a$ and of radius $r$ is $\Delta(a,r) = \{z \in \mathbb{D} : \rho(a,z) < r\}$. It is well known that $\Delta(a,r)$ is an Euclidean disk centered at $(1-r^2)a/(1-r^2|a|^2)$ and of radius $(1-|a|^2)r/(1-r^2|a|^2)$. Finally, denote $\tilde{\omega}(z) = \omega(z)/(1-|z|)$ for all $z \in \mathbb{D}$. By [13] Proposition 5] we know that

$$\|f\|_{A^p_\omega} \asymp \|f\|_{A^p_\nu}, \quad f \in \mathcal{H}(\mathbb{D}),$$

provided $\omega \in \mathcal{D}$.

Our first result reads as follows:

**Theorem 1.** Let $0 < q < p < \infty$, $\omega \in \mathcal{D}$, $n \in \mathbb{N} \cup \{0\}$, and let $\nu$ be a positive Borel measure on $\mathbb{D}$. Let $\varphi$ be an analytic self-map of $\mathbb{D}$ and $u \in L^q_\nu$. Then the following statements are equivalent:

(i) $D^n_{\varphi,u} : A^p_\omega \to L^q_\nu$ is bounded;

(ii) $D^n_{\varphi,u} : A^p_\omega \to L^q_\nu$ is compact;

(iii) $\frac{\varphi_*([u]_{L^q_\nu}(\Delta(z,r)))}{\tilde{\omega}(S(z))/(1-|z|)^{nq}}$ belongs to $L^p_{\omega,\alpha}$ for some (equivalently for all) $r \in (0,1)$.

We will need one specific tool for the proof of Theorem 1. To put it in a bigger whole, couple of words on continuous embeddings $A^p_\omega \subset L^q_\nu$ are in order. For a positive Borel measure $\mu$ on $\mathbb{D}$ and $0 < \alpha < \infty$, define the weighted maximal function

$$M_{\omega,\alpha}(\mu)(z) = \sup_{z \in S(a)} \frac{\mu(S(a))}{(\omega(S(a)))^{\alpha}}, \quad z \in \mathbb{D},$$

and write $M_{\omega}(\mu) = M_{\omega,1}(\mu)$ for short. If the identity operator $I : A^p_\omega \to L^q_\nu$ is bounded, then $\mu$ is called a $q$-Carlson measure for $A^p_\omega$. A complete characterization of such measures in the case $\omega \in \mathcal{D}$ can be found in [13], see also [11] [14]. In particular, it is known that if $q \geq p$ and
ω ∈ Ď, then μ is a q-Carleson measure for \( A^p_q \) if and only if \( M_{ω,q/p}(μ) ∈ L^∞ \), and if \( p > q \), then μ is a q-Carleson measure for \( A^p_q \) if and only if \( M_{ω}(μ) ∈ L^{∞}_{ω} \). The standard Littlewood-Paley formula implies that the bounded differentiation operators \( D^nf = f^{(n)} \) from \( A^p_q \) to \( L^q_ρ \) can be characterized once the q-Carleson measures for \( A^p_q \) are characterized. A characterization of such operators on the weighted Bergman spaces \( A^p_q \) with ω ∈ Ď can be found in [13]. In this paper, we are interested in the case of \( ω ∈ D \) and this is what we need for the proof of Theorem [1]. The following theorem is our second main result and generalizes [10, Theorem 2.2] and [9, Theorem 1] to doubling weights.

**Theorem 2.** Let \( 0 < p,q < ∞ \), \( ω ∈ D \) and \( n ∈ N \cup \{0\} \), and let μ be a positive Borel measure on \( D \).

(a) If \( 0 < p ≤ q < ∞ \), then the following statements hold:
   (i) \( D^n : A^p_q → L^q_μ \) is bounded if and only if
   \[
   \sup_{z ∈ D} \frac{μ(Δ(z,r))}{ω(S(z))^{1−r}|1−|z|^{1−q}} < ∞. \tag{1.4}
   \]
   (ii) \( D^n : A^p_q → L^q_μ \) is compact if and only if
   \[
   \lim_{|z| → 1−} \frac{μ(Δ(z,r))}{ω(S(z))^{1−r}|1−|z|^{1−q}} = 0. \tag{1.5}
   \]

(b) If \( 0 < q < p < ∞ \), then the following conditions are equivalent:
   (i) \( D^n : A^p_q → L^q_μ \) is bounded;
   (ii) \( D^n : A^p_q → L^q_μ \) is compact;
   (iii) the function
   \[
   z ↦ \frac{μ(Δ(z,r))}{ω(S(z))^{1−r}|1−|z|^{1−q}}
   \]
   belongs to \( L^{∞}_{ω} \) for some (equivalently for all) \( r ∈ (0,1) \).

When \( n = 0 \), Parts (a) and (b) of Theorem [2] convert to [6, Theorem 2] and [8, Theorem 2], respectively. If \( n ∈ N \), the method of proof of [8, Theorem 2] gives (a) immediately. Therefore our contribution consists of proving (b) for \( n ∈ N \) by establishing the implications (ii)⇒(i)⇒(iii)⇒(ii).

The next main result of this work concerns the boundedness and compactness of \( D^n_{ϕ,u} \) when \( p ≤ q \) and \( ω ∈ D \).

**Theorem 3.** Let \( 0 < p ≤ q < ∞ \) and \( ω, ν ∈ D \). Let \( ϕ \) be an analytic self-map of \( D \), \( u ∈ A^p_q \), and \( n ∈ N \cup \{0\} \). Then there exists \( γ = γ(p,ω) > 0 \) such that the following statements hold:

(i) \( D^n_{ϕ,u} : A^p_q → A^p_u \) is bounded if and only if
   \[
   \sup_{a ∈ D} \int_Ď |u(z)|^q \frac{(1−|a|)^{γq}}{|1−ϕ(z)|^{(γ+n)q}} \frac{ν(z)}{ω(S(a))^{1−r}} dA(z) < ∞; \tag{1.6}
   \]

(ii) \( D^n_{ϕ,u} : A^p_q → A^p_u \) is compact if and only if
   \[
   \lim_{|z| → 1} \int_Ď |u(z)|^q \frac{(1−|a|)^{γq}}{|1−ϕ(z)|^{(γ+n)q}} \frac{ν(z)}{ω(S(a))^{1−r}} dA(z) = 0. \tag{1.7}
   \]

If \( ω(z) = (1−|z|)^{α} \) and \( ν(z) = (1−|z|)^{β} \) with \(-1 < α, β < ∞\), then \( ω(z) ≍ (1−|z|)^{α+1} \) and \( ν(z) ≍ (1−|z|)^{β+1} \) for all \( z ∈ D \). The proof of Theorem [8] shows that the only requirement for \( γ = γ(ω,p) > 0 \) appearing in the statement is that
\[
\int_Ď \frac{ω(z)}{|1−αz|^p} dA(z) ≤ C \frac{ω(a)}{(1−|a|)^{p−1}}, \quad a ∈ D,
\]
for some constant $C = C(\omega, p, \gamma) > 0$. If $\omega(z) = (1 - |z|)^{\alpha}$, any $\gamma > \frac{\alpha + 2}{p}$ is acceptable, and the choice $2 \frac{\alpha + 2}{p}$ converts Parts (i) and (ii) of Theorem 3 to Theorems 2.6 and 2.7 in \[21\] respectively, as a simple computation shows. Therefore Theorem 4 indeed generalizes \[21\], Theorems 2.9 and 2.10 for weights in $D$.

Our last main result concerns the case in which the target space is $H^\alpha$.\[3\]

**Theorem 4.** Let $\omega \in D$, $0 < p < \infty$ and $n \in \mathbb{N} \cup \{0\}$. Let $\varphi$ be an analytic self-map of $D$ and $u \in H^\infty(D)$. Then $D_{\varphi,u}^n : A^p_\omega \to H^\infty$ is bounded if and only if

$$\sup_{z \in \mathbb{D}} \frac{|u(z)|}{\varphi(S(\varphi(z)))^{\frac{1}{p}}(1 - |\varphi(z)|)^n} < \infty. \quad (1.8)$$

Moreover, if $D_{\varphi,u}^n : A^p_\omega \to H^\infty$ is bounded, then

$$\|D_{\varphi,u}^n\| \asymp \sup_{z \in \mathbb{D}} \frac{|u(z)|}{\varphi(S(\varphi(z)))^{\frac{1}{p}}(1 - |\varphi(z)|)^n}. \quad (1.9)$$

Furthermore, with the same assumptions the following are also equivalent:

(i) $D_{\varphi,u}^n : A^p_\omega \to H^\infty$ is compact;

(ii) $\sup_{z \in \mathbb{D}} |\varphi(z)| < 1$ or

$$\lim_{|\varphi(z)| \to 1} \frac{|u(z)|}{\varphi(S(\varphi(z)))^{\frac{1}{p}}(1 - |\varphi(z)|)^n} = 0. \quad (1.10)$$

If $\omega(z) = (1 - |z|)^{\alpha}$ for some $-1 < \alpha < \infty$, then $\omega(S(\zeta)) \asymp \omega(\zeta)(1 - |\zeta|)^{2} = (1 - |\zeta|)^{2 + \alpha}$ for all $\zeta \in \mathbb{D}$. Therefore our results generalize \[21\], Theorems 2.9 and 2.10 for weights in $D$.

The rest of the paper contains the proofs of the results stated above. In the next section we go through auxiliary results. The proof of Theorem 2 is given in Section 3. In Sections 4 and 5 we characterize the boundedness and compactness of the operator $D_{\varphi,u}^n$. Finally, we show how to get the main results in Section 6.

Before going further couple of words about the notation used. The letter $C = C(\cdot)$ will denote an absolute constant whose value depends on the parameters indicated in the parenthesis, and may change from one occurrence to another. We will use the notation $a \lesssim b$ if there exists a constant $C = C(\cdot) > 0$ such that $a \leq Cb$, and $a \gtrsim b$ is understood in an analogous manner. In particular, if $a \lesssim b$ and $a \gtrsim b$, then we write $a \asymp b$ and say that $a$ and $b$ are comparable.

2. PRELIMINARIES

It is known that if $\omega \in D$, then there exist constants $0 < \alpha = \alpha(\omega) \leq \beta(\omega) < \infty$ and $C = C(\omega) \geq 1$ such that

$$\frac{1 - r}{C} \left( \frac{1 - r}{1 - t} \right)^{\alpha} \leq \frac{\omega(r)}{\omega(t)} \leq C \left( \frac{1 - r}{1 - t} \right)^{\beta}, \quad 0 \leq r < t < 1. \quad (2.1)$$

In fact, this pair of inequalities characterizes the class $D$ because the right hand inequality is satisfied if and only if $\omega \in D$ by \[14\], Lemma 2.1 while the left hand inequality describes the class $D$ in an analogous way, see \[12\] (2.27)]. The chain of inequalities \[2.1\] will be frequently used in the sequel.

Another result needed is a lemma that allows us to estimate $|f^{(n)}(z)|$ sufficiently accurately when $f \in A^p_\omega$ with $\omega \in D$.

**Lemma 5.** Let $\omega \in D$, $0 < p < \infty$ and $n \in \mathbb{N} \cup \{0\}$. If $f \in A^p_\omega$, then we have

$$|f^{(n)}(z)| \leq \frac{\|f\|_{A^p_\omega}}{(\omega(S(z)))^{\frac{1}{p}}(1 - |z|)^n}, \quad z \in \mathbb{D}. \quad (2.2)$$
Proof. By (1.3), $1 - |\xi| \asymp 1 - |z|$ for all $\xi \in \Delta(z, r)$ and $\varpi(z)(1 - |z|) \asymp \omega(S(z))$ for all $z \in \mathbb{D}$. Therefore

$$|f^{(n)}(z)|^p \leq \frac{C}{(1 - |z|)^{2np}} \int_{\Delta(z, r)} |f^{(n)}(\xi)|^p dA(\xi)$$

$$\leq \frac{C}{(1 - |z|)^{2np}} \int_{\Delta(z, r)} |f^{(n)}(\xi)|^p (1 - |\xi|)^{np} dA(\xi)$$

$$\asymp \frac{1}{\varpi(z)(1 - |z|)^{np}} \int_{\Delta(z, r)} |f^{(n)}(\xi)|^p (1 - |\xi|)^{np} \varpi(\xi) dA(\xi)$$

and hence (3.1) holds.

The lemma is proved. \hfill \Box

The next lemma follows by standard arguments, see, for example, [1 Proposition 3.11]. We omit the details of the proof.

**Lemma 6.** Let $\varphi$ be an analytic self-map of $\mathbb{D}$, $u \in \mathcal{H}(\mathbb{D})$ and $n \in \mathbb{N} \cup \{0\}$. Let $0 < p, q < \infty$, and $\omega, \nu$ in $\mathcal{D}$. Then $D^p_{\varphi, u} : A^q_{\omega} \to A^p_{\nu}$ is compact if and only if $D^p_{\varphi, u} : A^q_{\omega} \to A^p_{\nu}$ is bounded and for any bounded sequence $(f_k)_{k \in \mathbb{N}}$ in $A^q_{\omega}$, which converges to zero uniformly on compact subsets of $\mathbb{D}$, we have $\|D^p_{\varphi, u} f_k\|_{A^p_{\nu}} \to 0$ as $k \to \infty$.

### 3. Embedding Theorems

In this section we establish embeddings theorem of $A^p_{\omega}$ into $L^q_{\mu}$ with $0 < p, q < \infty$ and $\omega \in \mathcal{D}$. We begin with the case $0 < p \leq q < \infty$.

**Proposition 7.** Let $0 < p \leq q < \infty$, $\omega \in \mathcal{D}$ and $n \in \mathbb{N} \cup \{0\}$, and let $\mu$ be a positive Borel measure on $\mathbb{D}$. Then there exists $r = r(\omega) \in (0, 1)$ such that the following statements hold:

(i) $D^p : A^q_{\omega} \to L^q_{\mu}$ is bounded if and only if

$$\sup_{z \in \mathbb{D}} \frac{\mu(\Delta(z, r))}{\omega(S(z))^{\frac{1}{p}} (1 - |z|)^{nq}} < \infty. \tag{3.1}$$

(ii) $D^p : A^q_{\omega} \to L^q_{\mu}$ is compact if and only if

$$\lim_{|z| \to 1^{-}} \frac{\mu(\Delta(z, r))}{\omega(S(z))^{\frac{1}{p}} (1 - |z|)^{nq}} = 0. \tag{3.2}$$

**Proof.** When $n = 0$, the statement is proved in [4 Theorem 2]. Hence we may assume $n \in \mathbb{N}$. The necessity of the condition in (i) can be proved easily. For $a \in \mathbb{D}$, define the function

$$f_a(z) = \left(\frac{1 - |a|}{1 - \bar{a}z}\right)^{\gamma} \omega(S(a))^{\frac{1}{p}} \varpi(z), \quad z \in \mathbb{D}, \tag{3.3}$$

induced by $\omega$ and $0 < \gamma, p < \infty$. Then [12 Lemma 2.1] implies that for all $\gamma = \gamma(\omega, p) > 0$ sufficiently large we have $\|f_a\|_{A^q_{\omega}} \asymp 1$ for all $a \in \mathbb{D}$. By using [13 Lemma 8] it is easy to see that (3.1) holds.

To prove the sufficiency, we assume that (3.1) holds. Recall the known estimate

$$|f^{(n)}(z)|^p \lesssim \frac{1}{(1 - |z|)^{2np}} \int_{\Delta(z, r)} |f(\xi)|^p dA(\xi), \quad z \in \mathbb{D}, \tag{3.4}$$

...
see, for example, [10], Lemma 2.1 for details. This estimate, Minkowski’s inequality in continuous form (Fubini’s theorem in the case \( q = p \)) and [5,11] give

\[
\| f^{(n)} \|_{L^q_{\mu}}^q \lesssim \left( \int_{\mathbb{D}} |f(\zeta)|^p \frac{\omega(S(\zeta))}{(1 - |\zeta|^2)^2} dA(\zeta) \right)^{\frac{q}{p}}, \quad f \in \mathcal{H}(\mathbb{D}),
\]

see [6, Theorem 2 (i)] for details. Since \( \omega \in \mathbb{D} \) by the hypothesis, we may apply the right hand inequality in \([2,1]\) to deduce

\[
\omega(S(\zeta)) \lesssim \tilde{\omega}(\zeta)(1 - |\zeta|), \quad \zeta \in \mathbb{D}.
\]

It follows that \( \| f^{(n)} \|_{L^q_{\mu}} \lesssim \| f \|_{L^p_{\mu}} \), and hence \( \| f^{(n)} \|_{L^q_{\mu}} \lesssim \| f \|_{L^p_{\mu}} \) for all \( f \in \mathcal{H}(\mathbb{D}) \) by \([13]\). Thus \( D^n \) is bounded, and (i) is proved.

The statement concerning the compactness can be proved by following the proof of [6, Theorem 2 (ii)] line by line, and therefore we omit the details. \( \square \)

The next proposition deals with the case \( q < p \).

**Proposition 8.** Let \( 0 < q < p < \infty \), \( \omega \in \mathbb{D} \) and \( n \in \mathbb{N} \cup \{0\} \) and let \( \mu \) be a positive Borel measure on \( \mathbb{D} \). Then the following conditions are equivalent:

(i) \( D^n : A^p_\omega \to L^q_{\mu} \) is bounded;

(ii) \( D^n : A^p_\omega \to L^q_{\mu} \) is compact;

(iii) \( \frac{\mu(\Delta(z,r))}{\omega(S(z))(1 - |z|^2)^{n\lambda}} \) belongs to \( L^{\frac{q}{p-r}}_{\tilde{\omega}} \) for some (equivalently for all) \( r \in (0,1) \).

**Proof.** If \( n = 0 \), the statement reduces to [8, Theorem 2], and hence we may assume that \( n \in \mathbb{N} \). The implication (ii) \( \Rightarrow \) (i) is obvious. To prove that (i) implies (iii), assume that \( D^n : A^p_\omega \to L^q_{\mu} \) is bounded. Let \( \{z_k\} \) be a \( r \)-lattice such that \( z_k \neq 0 \) for all \( k \). By [10, Theorem 1] there exist constants \( \lambda = \lambda(p,\omega) > 1 \) and \( C = C(p,\omega) > 0 \) such that the function

\[
F(z) = \sum_k b_k \left( \frac{1 - |z_k|}{1 - |z_k|^2} \right)^{\lambda} \frac{1}{\omega(T(z_k))^{\frac{1}{\nu}}}, \quad z \in \mathbb{D},
\]

belongs to \( A^p_\omega \) and satisfies \( \| F \|_{A^p_\omega} \leq C \| b \|_{\ell^p} \) for all \( b = \{b_k\} \in \ell^p \). Since \( D^n : A^p_\omega \to L^q_{\mu} \) is bounded by the hypothesis, we deduce

\[
\| b \|_{\ell^p}^q \gtrsim \| F \|_{A^p_\omega}^q \gtrsim \int_{\mathbb{D}} \left| \sum_k b_k \left( \frac{1 - |z_k|}{1 - |z_k|^2} \right)^{\lambda} \frac{1}{\omega(T(z_k))^{\frac{1}{\nu}}} \right|^q d\mu(z), \quad b \in \ell^p.
\]

One may now complete this part of the proof by using Khinchine’s inequality and properties of an \( r \)-lattice. Namely, the argument used in [8, Proposition 7] now shows that \( \frac{\mu(\Delta(z,r))}{\omega(S(z))(1 - |z|^2)^{n\lambda}} \) belongs to \( L^{\frac{q}{p-r}}_{\tilde{\omega}} \) for some (equivalently for all) \( r \in (0,1) \).

It remains to show that (iii) implies (ii). It suffices to show that for any bounded sequence \( \{f_k\} \in A^p_\omega \) which tends to zero uniformly on compact subsets of \( \mathbb{D} \) as \( k \to \infty \), we have \( \| D^n f_k \|_{L^q_{\mu}} \to 0 \) as \( k \to \infty \). For simplicity, assume that \( \| f_k \|_{A^p_\omega} \leq 1 \) for all \( n \). By [10, Lemma 2.1], we have

\[
|f^{(n)}(z)|^q \leq \frac{C}{\omega(S(z))(1 - |z|^2)^{nq}} \int_{\Delta(z,r)} |f(\xi)|^q \tilde{\omega}(\xi) dA(\xi).
\]

By Fubini’s theorem, \([3,6]\) and \([2,1]\), we deduce

\[
\| D^n f_k \|_{L^q_{\mu}}^q \lesssim \int_{\mathbb{D}} |f_k(\xi)|^q \frac{\nu(\Delta(\xi,r))}{\omega(S(\xi))(1 - |\xi|^2)^{nq}} \tilde{\omega}(\xi) dA(\xi).
\]

Let \( \varepsilon > 0 \). The hypothesis implies that there exists an \( r \in (0,1) \) such that

\[
\int_{\mathbb{D} \setminus D(0,\varepsilon)} \left( \frac{\nu(\Delta(\xi,r))}{\omega(S(\xi))(1 - |\xi|^2)^{nq}} \right)^\frac{p}{p-q} \tilde{\omega}(\xi) dA(\xi) \leq \varepsilon^{\frac{p-q}{q}}.
\]
Proof. First assume that $\omega (z) < \epsilon$ for all $z \in \overline{D}(0, r)$. The Hölder inequality and (1.3) yield

\[
\| D^n f_k \|^q_{L^q} \lesssim \left( \frac{1}{|D(D(0,r))|} + \frac{1}{|D(D(0,r))|} \right) |f_k(x)|^q \frac{\nu(\Delta(r, r))}{\omega(S(\xi))(1 - |\xi|)^{\nu}} \omega(\xi) dA(\xi)
\]

\[
\lesssim \sup_{\xi \in \Omega} \frac{\nu(\Delta(r, r))}{\omega(S(\xi))(1 - |\xi|)^{\nu}} \int_{D(D(0,r))} |f_k(x)|^q \omega(\xi) dA(\xi)
\]

\[
+ \left( \int_{\Omega} |f_k(x)|^q \omega(\xi) dA(\xi) \right)^{\frac{1}{q}} \left( \int_{D(D(0,r))} \left( \frac{\nu(\Delta(r, r))}{\omega(S(\xi))(1 - |\xi|)^{\nu}} \omega(\xi) dA(\xi) \right)^{\frac{1}{1 - q}} \right)^{1 - \frac{q}{p}}
\]

\[
\lesssim (1 + \|f_k\|^q_{A^q_{\omega}}) \epsilon \lesssim \epsilon,
\]

and thus $D^n : A^p_{\omega} \rightarrow A^q_{\omega}$ is compact.

\[
\square
\]

4. BOUNDEDNESS

In this section, we characterize the boundedness of the generalized weighted composition operator $D^n_{\varphi, u}$ on weighted Bergman spaces.

Proposition 9. Let $\varphi$ be an analytic self-map of $\mathbb{D}$, $u \in A^p_{\omega}$, and $n \in \mathbb{N} \cup \{0\}$. Assume $0 < p < q < \infty$ and $\omega, \nu$ in $\mathbb{D}$. Then there exists $\gamma = \gamma(p, \omega) > 0$ such that $D^n_{\varphi, u} : A^p_{\omega} \rightarrow A^q_{\omega}$ is bounded if and only if

\[
\sup_{a \in \mathbb{D}} \int_{\mathbb{D}} |u(z)|^q \frac{(1 - |a|)^{\gamma q}}{1 - \overline{a} \varphi(z)} \frac{\nu(z)}{\omega(S(a))^\frac{q}{p}} dA(z) < \infty. \tag{4.1}
\]

Proof. First assume that $D^n_{\varphi, u} : A^p_{\omega} \rightarrow A^q_{\omega}$ is bounded. Then there exists a constant $C = C(\gamma, p, n)$ such that

\[
\| D^n_{\varphi, u} f_a \|^q_{A^q_{\omega}} = (C|a|^n)^q \int_{\mathbb{D}} |u(z)|^q \frac{(1 - |a|)^{\gamma q}}{1 - \overline{a} \varphi(z)} \frac{\nu(z)}{\omega(S(a))^\frac{q}{p}} dA(z)
\]

\[
\leq C \|f_a\|^q_{A^q_{\omega}} < \infty,
\]

where $f_a$ defined as in (3.3). It follows that (4.1) is satisfied.

Conversely, we assume that (4.1) holds. For each $a \in \overline{\mathbb{D}}$ and $r > 0$, the estimate $|1 - \overline{a} \varphi(z)| \asymp 1 - |a|$ for $z \in \varphi^{-1}(\Delta(a, r))$ yields

\[
\int_{\mathbb{D}} |u(z)|^q \frac{(1 - |a|)^{\gamma q}}{1 - \overline{a} \varphi(z)} \frac{\nu(z)}{\omega(S(a))^\frac{q}{p}} dA(z)
\]

\[
\lesssim \int_{\varphi^{-1}(\Delta(a, r))} |u(z)|^q \frac{(1 - |a|)^{\gamma q}}{1 - \overline{a} \varphi(z)} \frac{\nu(z)}{\omega(S(a))^\frac{q}{p}} dA(z) \geq \frac{\varphi_\ast(|u|^q \nu)(\Delta(a, r))}{\omega(S(a))^\frac{q}{p} (1 - |a|)^{\gamma q}}.
\]

and hence the function

\[
a \mapsto \frac{\varphi_\ast(|u|^q \nu)(\Delta(a, r))}{\omega(S(a))^\frac{q}{p} (1 - |a|)^{\gamma q}}
\]

belongs to $L^\infty$ for any fixed $r \in (0, 1)$. By Proposition 4 we have $\| f^{(n)} \|^q_{L^q_{\varphi_\ast(|u|^q \nu)}} \lesssim \| f \|^q_{A^q_{\omega}}$ for all $f \in A^p_{\omega}$. By the measure theoretic change of variable [1] Section 39) it follows that

\[
\| D^n_{\varphi, u} f \|^q_{L^q_{\omega}} \geq \int_{\mathbb{D}} |f^{(n)}(\varphi(z))|^q |u(z)|^q \nu(z) dA(z)
\]

\[
= \int_{\mathbb{D}} |f^{(n)}(z)|^q d\varphi_\ast(|u|^q \nu)(z)
\]

\[
= \| f^{(n)} \|^q_{L^q_{\varphi_\ast(|u|^q \nu)}} \lesssim \| f \|^q_{A^q_{\omega}}.
\]

Thus $D^n_{\varphi, u} : A^p_{\omega} \rightarrow A^q_{\omega}$ is bounded.
Next, we consider the case $0 < p < \infty$ and $q = \infty$. Then a necessary condition for the boundness of the operator $D^n_{\varphi,u} : A^p_\omega \to H^\infty$ is $u \in H^\infty$.

**Proposition 10.** Let $\omega \in D$, $0 < p < \infty$ and $n \in \mathbb{N} \cup \{0\}$. Let $\varphi$ be an analytic self-map of $D$ and $u \in H^\infty$. Then $D^n_{\varphi,u} : A^p_\omega \to H^\infty$ is bounded if and only if

$$\sup_{z \in \Delta} |u(z)| \omega(S(\varphi(z)))^{\frac{1}{p}} (1 + |\varphi(z)|)^n < \infty. \tag{4.2}$$

Furthermore, if $D^n_{\varphi,u} : A^p_\omega \to H^\infty$ is bounded, then

$$\|D^n_{\varphi,u}\| \leq \sup_{z \in \Delta} \frac{|u(z)|}{\omega(S(\varphi(z)))^{\frac{1}{p}} (1 + |\varphi(z)|)^n} \tag{4.3}$$

**Proof.** First, assume that $D^n_{\varphi,u} : A^p_\omega \to H^\infty$ is bounded. Consider the function $f_\xi$ with $\gamma = \varphi(\xi)$ defined in (3.3). Then

$$\|D^n_{\varphi,u} f_\xi\|_{A^\infty} \leq \|D^n_{\varphi,u}\| \|f_\xi\|_{A^p_\omega} \leq C \|D^n_{\varphi,u}\|,$$

that is, for any $z \in \Delta$, we have

$$|u(z)| |f_\xi^{(n)}(\varphi(z))| \leq C \|D^n_{\varphi,u}\|.$$  

In particular, letting $z = \xi$, we have

$$C(\gamma, p, n) \frac{|u(\xi)||\varphi(\xi)|^n}{\omega(S(\varphi(\xi)))^{\frac{1}{p}} (1 + |\varphi(\xi)|)^n} \leq \|D^n_{\varphi,u}\|. \tag{4.4}$$

Therefore

$$\sup_{\xi \in \Delta} \frac{|u(\xi)|}{\omega(S(\varphi(\xi)))^{\frac{1}{p}} (1 + |\varphi(\xi)|)^n} < \infty.$$  

Conversely, assume that (4.3) holds. Then for $f \in A^p_\omega$, by Lemma 5, we have

$$|(D^n_{\varphi,u} f)(z)| = |u(z)||f^{(n)}(\varphi(z))| \leq C \frac{|u(z)||f||_{A^p_\omega}}{(\omega(S(\varphi(z))))^{\frac{1}{p}} (1 + |\varphi(z)|)^n}, \quad z \in \Delta. \tag{4.5}$$

Thus $D^n_{\varphi,u} : A^p_\omega \to H^\infty$ is bounded.

By combining (4.4) and (4.5) we obtain (4.3). This completes the proof. \qed

5. COMPACTNESS

In this section we will characterize the compactness of the generalized weighted composition operator $D^n_{\varphi,u}$ on weighted Bergman spaces.

**Proposition 11.** Let $\varphi$ be an analytic self-map of $D$, $u \in A^p_{\omega}$, and $n \in \mathbb{N} \cup \{0\}$. Assume $0 < p \leq q < \infty$ and $\omega, \nu \in D$. Then there exists $\gamma = \gamma(p, \omega) > 0$ such that $D^n_{\varphi,u} : A^p_\omega \to A^p_\omega$ is compact if and only if

$$\lim_{|a| \to 1} \int_{\Delta} |u(z)|^q \frac{(1 - |a|)\gamma^q}{|1 - \overline{a}\varphi(z)|^{\gamma + n} \omega(S(a))^{\frac{1}{p}}} \mu(z) dA(z) = 0. \tag{5.1}$$

**Proof.** First, we assume that $D^n_{\varphi,u}$ is compact. Consider the function $f_a$ defined in (3.3). It is obvious that $f_a$ tends to zero uniformly on compact subsets of $D$ as $|a| \to 1^-$. Then Lemma 2.1 implies that we have $\|f_a\|_{A^p_\omega} \to 1$ for all $a \in D$ if $\gamma = \gamma(\omega, p) > 0$ sufficient large. By Lemma 0 we have $\|D^n_{\varphi,u} f_a\|_{A^p_\omega} \to 0$ as $|a| \to 1^-$. Hence
\[
\lim_{|a| \to 1^-} (C(\lambda, p, n)|a|^n)^q \int_\mathbb{D} |u(z)|^q \frac{(1 - |a|)^{\gamma q}}{|1 - \bar{\varphi}(z)|^{(\gamma + n)q} \omega(S(a))^q} dA(z)
\]
\[
= \lim_{|a| \to 1^-} \|D_{\varphi,u}^n f_a\|_{A_p^\varphi}^q = 0,
\]
and (5.1) follows.

Conversely, assume that (5.1) holds. By Proposition 12, it suffices to show that
\[
\lim_{|a| \to 1^-} \frac{\varphi_*\left(|u|^q\nu\right)(\Delta(a, r))}{\omega(S(a))^{\frac{q}{p}} (1 - |a|)^{\gamma q}} = 0.
\]
By the hypothesis and the estimate \(|1 - \varphi(z)| \cong 1 - |a|\) for \(z \in \varphi^{-1}\Delta(a, r)\) we have
\[
0 = \lim_{|a| \to 1^-} \int_\mathbb{D} |u(z)|^q \frac{(1 - |a|)^{\gamma q}}{|1 - \bar{\varphi}(z)|^{(\gamma + n)q} \omega(S(a))^q} dA(z)
\]
\[
\geq \lim_{|a| \to 1^-} \int_{\varphi^{-1}\Delta(a, r)} |u(z)|^q \frac{(1 - |a|)^{\gamma q}}{|1 - \bar{\varphi}(z)|^{(\gamma + n)q} \omega(S(a))^q} dA(z)
\]
\[
\geq \lim_{|a| \to 1^-} \frac{\varphi_*\left(|u|^q\nu\right)(\Delta(a, r))}{\omega(S(a))^{\frac{q}{p}} (1 - |a|)^{\gamma q}}.
\]
Thus \(D_{\varphi,u}^n\) is compact. The proof is complete. \(\square\)

Now we consider the compactness of the operator \(D_{\varphi,u}^n\) acting from \(A_p^\varphi\) to \(H^\infty(\mathbb{D})\).

**Proposition 12.** Let \(\omega \in \mathcal{D}, \ 0 < p < \infty\) and \(n \in \mathbb{N} \cup \{0\}\). Let \(\varphi\) be an analytic self-map of \(\mathbb{D}\) and \(u \in H^\infty(\mathbb{D})\). Then the following statements are equivalent:

(i) \(D_{\varphi,u}^n : A_p^\varphi \rightarrow H^\infty\) is compact;

(ii) \(\sup_{z \in \mathbb{D}} |\varphi(z)| < 1\)

or

\[
\lim_{|\varphi(z)| \to 1} \frac{|u(z)|}{\omega(S(\varphi(z)))^{\frac{q}{p}} (1 - |\varphi(z)|)^n} = 0. \tag{5.3}
\]

**Proof.** First, we show that (i) implies (ii). Assume that \(\sup_{\eta \in \mathbb{D}} |\varphi(\eta)| = 1\) and
\[
\lim_{|\varphi(\eta)| \to 1} \frac{|u(\eta)|}{\omega(S(\varphi(\eta)))^{\frac{q}{p}} (1 - |\varphi(\eta)|)^n} \neq 0.
\]
Then there exist \(\varepsilon > 0\) and a sequence \(\{\eta_k\}\) in \(\mathbb{D}\) such that \(\lim_{k \to \infty} |\varphi(\eta_k)| = 1\) and
\[
\frac{|u(\eta_k)|}{\omega(S(\varphi(\eta_k)))^{\frac{q}{p}} (1 - |\varphi(\eta_k)|)^n} > \varepsilon
\]
for all \(k \in \mathbb{N}\). Consider the function \(f_{\eta_k} \in A_p^\varphi\) with \(a = \varphi(\eta_k)\) defined in (5.5). Then \(f_{\eta_k} \in A_p^\varphi\), \(\|f_{\eta_k}\|_{A_p^\varphi} \leq C\) and \(f_{\eta_k} \to 0\) uniformly on compact subsets of \(\mathbb{D}\) as \(|\varphi(\eta_k)| \to 1^+\). Since \(D_{\varphi,u}^n\) is compact, we have \(\|D_{\varphi,u}^n f_{\eta_k}\|_\infty \to 0\) as \(|\varphi(\eta_k)| \to 1^-\) by Lemma 6. On the other hand
\[
\|D_{\varphi,u}^n f_{\eta_k}\|_\infty = \sup_{z \in \mathbb{D}} |u(z)| |f_{\eta_k}^{(n)}(\varphi(z))| \geq \frac{|u(\eta_k)|}{\omega(S(\varphi(\eta_k)))^{\frac{q}{p}} (1 - |\varphi(\eta_k)|)^n} > \varepsilon. \tag{5.4}
\]
This is a contradiction.

Next, we show that (ii) implies (i). It suffices to show that for any norm bounded sequence \(\{f_j\}\) in \(A_p^\varphi\) which tends to zero uniformly on compact subsets of \(\mathbb{D}\) as \(j \to \infty\), we have \(\|D_{\varphi,u}^n f_j\|_\infty \to 0\) as \(j \to \infty\). First, we assume that \(\sup_{z \in \mathbb{D}} |\varphi(z)| < 1\) and (5.3) holds. (5.4) and \(u \in H^\infty(\mathbb{D})\) imply
\[
\sup_{z \in \mathbb{D}} \frac{|u(z)|}{\omega(S(\varphi(z)))^{\frac{q}{p}} (1 - |\varphi(z)|)^n} < \infty. \tag{5.5}
\]
By Proposition [10] it follows that $D^n_{\phi,u} : A^\infty_\nu \rightarrow \mathcal{H}^\infty$ is bounded. Therefore, for any $\varepsilon > 0$, there exists $r \in (0, 1)$ such that

$$\frac{|u(z)|}{\omega(S(\varphi(z)))^{\frac{1}{p}}(1 - |\varphi(z)|)^n} < \varepsilon,$$

(5.6)

provided $r < |\varphi(z)| < 1$. For $z \in \mathbb{D}$ with $r < |\varphi(z)| < 1$, we therefore have

$$|u(z)||f_j^{(n)}(\varphi(z))| \leq \frac{|u(z)||f_j|_{A^\infty_\nu}}{(\omega(S(\varphi(z)))^{\frac{1}{p}}(1 - |\varphi(z)|)^n)} \leq C \varepsilon. \quad (5.7)$$

By Cauchy’s estimate, we see that $f_j^{(n)} \to 0$ uniformly on compact subsets of $\mathbb{D}$ as $j \to \infty$. Hence there exists a $j_0 = j_0(\varepsilon) \in \mathbb{N}$ such that $|f_j^{(n)}(\varphi(z))| < \varepsilon$ for all $j \geq j_0$ and $|\varphi(z)| \leq r$. Therefore, for all $j \geq j_0$, we have

$$\lim_{j \to \infty} ||D^n_{\phi,u} f_j||_{\infty} = \lim_{j \to \infty} \sup_{z \in \mathbb{D}} |u(z)| |f_j^{(n)}(\varphi(z))| = 0.$$

This finishes the proof.

6. PROOFS OF MAIN RESULTS

With the propositions proved in the previous two sections we can now easily obtain the main results stated in the introduction.

Proof of Theorem 1. By the measure theoretic change of variable [4, Section 39], it follows that $||D^n_{\phi,u} f||_{L^q_\nu} = ||f^{(n)}||_{L^q_{\omega(S(\varphi(z)))^{\frac{1}{p}}(1 - |\varphi(z)|)^n}}$ for each $f \in A^\infty_\nu$. Therefore the theorem is an immediate consequence of Proposition 8.

Proof of Theorem 2. This result follows by Propositions 7 and 8. Namely, these propositions characterize the $q$-Carleson measures for $A^\infty_\nu$ in the cases $0 < p \leq q < \infty$ and $0 < q < p < \infty$, respectively.

Proof of Theorems 3 and 4. Theorems 3 and 4 are immediate consequences of Propositions 9 and 11 and Propositions 10 and 12, respectively.

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