Skew Invariant Theory of Symplectic Groups, Pluri-Hodge Groups and 3-Manifold Invariants

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Introduction

This paper is concerned with four subjects. The second and third of these appear to be, at first sight, quite disparate. The fourth, however, shows how they are, somewhat surprisingly, related.

The first has to do with skew invariant theory. Let $V$ be a finite dimensional complex vector space with a skew symmetric non-degenerate form and let $\text{Sp}(V)$ be the corresponding symplectic group. For a complex vector space $W$ we consider the action of $\text{Sp}(V)$ on the exterior algebra $\Lambda(V \otimes W)$ (the action being trivial on $W$). In Theorem 2.3 we give generators and relations for the algebra of $\text{Sp}(V)$ invariants for this action ("Fundamental theorem for skew invariant theory"). The relations are the so-called "$P_n$" relations which appear in the study of certain 3-manifold invariants.

Next we show that there is a natural action of $\text{Sp}(g)$, where $g = \text{dim}_\mathbb{C} W$, on $\Lambda(V \otimes W)$, by considering the spin representation of the orthogonal algebra associated to $V \otimes (W \oplus W^*)$. The action of $\text{Sp}(V) \times \text{Sp}(g)$ on $\Lambda(V \otimes W)$ is multiplicity free and we determine the highest weights of the representations which occur, Theorem 2.4.

The second subject is concerned with the study of Dolbeault cohomology groups with values in specific vector bundles. Let $X$ be a compact closed complex manifold. Let $\Omega^p_X$ be the sheaf of holomorphic $p$-forms on $X$. The cohomology groups of interest are

$$H^q(X, \Omega^p_X \otimes \ldots \otimes \Omega^p_X).$$

These cohomology groups are not exotic. For example, the pluri-canonical sections, $H^0(X, K_X^\otimes m)$, are well studied. In keeping with this terminology I call these cohomology groups, pluri-Hodge groups.

Some of the motivation for studying pluri-Hodge groups, when $X$ is holomorphic symplectic, comes from physics. In section 3.1 I give an alternative motivation. Namely one can show that knowing the dimensions of these groups (actually one needs somewhat less) one knows all the Chern numbers of $X$. Theorem 3.3 gives a precise formula for the Chern numbers in terms of the dimensions of the pluri-Hodge groups. Along the way a pluri-$\chi_g$ genus is defined.
In section 4 one specializes to holomorphic symplectic manifolds. In this setting physics suggests that there is an $Sp(g)$ action on the direct sum of these pluri Hodge spaces. This is a straightforward consequence of the preparational material in section 1. The case $g = 1$ has been studied by Fujiki [5]. The multiplicities of the irreducible components of the total representations are determined in Theorem 4.4. Also in this section Theorem 4.3 shows that the graded trace of an $Sp(g)$ element is essentially (up to normalization) the pluri $\chi_y$ genus. This generalizes the situation for the usual $\chi_y$ genus.

For a hyper-Kähler manifold $X$ there are good reasons (see below) to believe that the dimensions of the pluri-Hodge groups do not depend on the intrinsic $\mathbb{P}^1$ of complex structures associated with $X$. In Theorem 5.1 the dimensions of the pluri-Hodge groups are determined in the case of K3 surfaces and this shows that indeed the dimensions do not depend on the specific K3 surface.

Obviously, one unifying theme between the two subjects is the group $Sp(g)$. Surprisingly, though, another thread through these subjects has to do with a class of 3-manifold invariants. A special case of Theorem 2.4 tells us that the space of $Sp(V)$ invariants carries an irreducible representation of $Sp(g)$ (of type $(n, \ldots, n)$ where $2n = \dim_{\mathbb{C}} V$) while the fundamental theorem of skew invariants Theorem 2.3 says that the space of invariants is determined by the $P_{n+1}$ relations. On the other hand the $P_n$ relations appear in the study of certain 3-manifold invariants.

For 3-manifolds without boundary these are the LMO [12] invariants and for 3-manifolds with boundary they are the MO [16] invariants. In these theories the $P_n$ relations have been implemented on spaces of chord diagrams $A(\Gamma_g)$. There is an equivalent formulation, for the MO invariants, employing spaces of uni-trivalent graphs $B_g$ introduced by J. Sawon [21]. His formulation is used, in Theorem 6.2 to show that the resulting quotient spaces carry representations of symplectic groups. Theorem 6.4 determines the irreducible representations that occur in the simplest case.

We come to the last topic. There is a path integral formulation of LMO type invariants due to Rozansky and Witten [20]. For closed 3-manifolds there is strong evidence that there is a weight system which when applied to the LMO invariants produces the Rozansky-Witten invariants [7]. The pluri-Hodge groups for a hyper-Kähler manifold make an appearance in the Rozansky-Witten theory. This theory assigns to a connected genus $g$ surface $\Sigma_g$ the Hilbert space of states,

$$\mathcal{H}_g(X) = \bigoplus_{q,p_1,\ldots,p_g} H^q(X, \Omega^{p_1}_X \otimes \ldots \otimes \Omega^{p_g}_X).$$

Sawon showed that there is a weight system which is a homomorphism of vector spaces $W_X : B_g \to \mathcal{H}_g(X)$ (this weight system appears also in [25] in a more limited situation). Theorem 7.2 states that $W_X$ preserves the symplectic group actions on the spaces. In this way one explains the ubiquitous appearance of the representation theory of the symplectic group. The Rozansky-Witten theory then acts as a unifying thread for the symplectic group actions in this paper.
To finish this introduction it should be pointed out how the RW theory gives rise to some expectations that are expressed in the text. Proposition 7.3 states that the homomorphism $W_X$ has as image a much smaller space than $H_g(X)$. However, the generalized $\chi_g(X)$ genera for holomorphic symplectic manifolds also arise in the RW theory for certain 3-manifolds that are mapping tori [24]. These ‘see’ a lot more of the pluri-Hodge groups than allowed by the proposition. This should help in formulating a TQFT for these theories. For closed 3-manifolds the RW theory can be expressed in terms of a given hyper-Kähler manifold $X$ without reference to a preferred complex structure in the $\mathbb{P}^1$ of complex structures. Since in the physical theory the RW invariant arises as pairings of vectors in $H_g(X)$ these pairings should not depend on the choice of complex structure implicit in $H_g(X)$. With this in mind it is natural to expect that the dimensions of the pluri-Hodge groups are deformation invariants.

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1 Special Representations of Products of Symplectic Groups

Let $V$ be a symplectic vector space of complex dimension $2n$ with symplectic form $\epsilon$. We will define in this section a natural representation of $Sp(g)$ (with $Sp(1) = SL(2, \mathbb{C})$) on the exterior algebra of $V \otimes \mathbb{C}^g$ and decompose this space under the action of $Sp(V) \times Sp(g)$. For convenience of notation let $W$ be a vector space of complex dimension $g$. We have on $W \oplus W^*$ a natural symplectic form, $\alpha$, defined by 

$$\alpha ((w_1, w^*_1), (w_2, w^*_2)) = w^*_2(w_1) - w^*_1(w_2).$$

Now $h = \epsilon \otimes \alpha$ defines a non-degenerate symmetric bi-linear form on $H = V \otimes (W \oplus W^*)$ and let $o(H, h)$ be the Lie algebra of the corresponding orthogonal group. Note that $Sp(V) \times Sp(W)$ leaves $h$ invariant. It is now not too surprising that the spin representation makes an appearance.

Observe that $V \otimes W$ is a maximal isotropic space of $H$ with respect to $h$ since $\alpha$ on $W$ vanishes. Thus we have a spin representation of $o(H, h)$ on $\Lambda(V \otimes W)$ [4]. By restriction we get a representation of the product of the Lie algebras of the symplectic groups and, as these groups are simply connected, we indeed get a representation of the product group. Furthermore, we observe for later use, that the Lie algebras of $Sp(V)$
and \( Sp(W \oplus W^*) \) form a dual reductive pair in the Lie algebra of \( O(H) \) in the sense of Howe \[9\].

### 1.1 Spin Representations

The following comes from \[4\] (see in particular Lemma 1 p. 195). Let \((H, h)\) be an even dimensional orthogonal vector space which can be decomposed as

\[ H = F \oplus F^* \]

for \( F \) a finite dimensional vector space over \( \mathbb{C} \) and \( F^* \) its dual. The symmetric bilinear form \( h \) is given by \( h(x_1 + x_1^*, y_1 + y_1^*) = (y_1, x_1^*) + (x_1, y_1^*) \) where \( x_1, y_1 \in F, x_1^*, y_1^* \in F^* \)

and \( \langle , \rangle \) denotes the pairing between \( F \) and \( F^* \). Given \( e_a \) as a basis for \( F \) and \( \bar{e}^b \) a dual basis for \( F^* \), the symmetric form is such that

\[ h(e_a, e_b) = 0, \quad h(e_a, \bar{e}^b) = \delta_a^b, \quad h(\bar{e}^a, \bar{e}^b) = 0. \]

Denote by \( C(Q) \) the Clifford algebra of \( H \) with respect to the quadratic form \( Q \) defined by \( Q(v) = h(v, v)/2 \). We have a canonical map \( f_0 : o(H, h) \rightarrow C(Q) \) defined as the composition of maps

\[ o(H, h) \rightarrow \text{End}(H) \rightarrow H \otimes H^* \rightarrow H \otimes H \rightarrow C(Q). \]

Here the last map is given by multiplication in the Clifford algebra, and the third map by using the canonical isomorphism of \( H^* \) with \( H \) given by the bilinear form \( h \). Put \( f = f_0/2 \), then \( f \) is a Lie algebra homomorphism of \( o(H, h) \) into the Lie algebra corresponding to the associative algebra \( C(Q) \). Indeed, for \( X \in o(H, h) \),

\[ f(X) = \frac{1}{2} \sum_a [(Xe_a) \otimes \bar{e}^a + (X\bar{e}^a) \otimes e_a]. \quad (1.1) \]

Let \( \Lambda F \) be the exterior algebra of \( F \) and \( \Lambda F^* \) be the exterior algebra of \( F^* \). Let \( \lambda : C(Q) \rightarrow \text{End}(\Lambda F) \) and \( \lambda^* : C(Q) \rightarrow \text{End}(\Lambda F^*) \) be the two spin representations of \( C(Q) \). Concretely, on \( \Lambda F \) (resp. \( \Lambda F^* \)) the defining property of \( \lambda \) (resp. \( \lambda^* \)) is to send \( \bar{e}^b \) (resp. \( e_b \)) to interior multiplication, \( i_{e_b} \) (resp. \( i_{\bar{e}^b} \)), while \( e_a \) (resp. \( \bar{e}^a \)) is understood to be exterior multiplication (wedging on the left), these operations then give spin representations since, on the respective spaces, one has

\[ \{i_{e_b}, e_a\} = \delta_{ab}, \quad \{i_{\bar{e}^b}, \bar{e}^a\} = \delta_{ab}. \]

### 1.2 Restriction of Spin to \( Sp(g) \)

The spin representations of \( o(H, h) \) are \( \rho := \lambda.f : o(H, h) \rightarrow \text{End}(\Lambda F) \) and \( \rho^* := \lambda^*.f : o(H, h) \rightarrow \text{End}(\Lambda F^*) \). Note that the Lie algebra of the orthogonal group is

\[ o(H, h) = S^2(F \oplus F^*) = S^2 F \oplus S^2 F^* \oplus F \otimes F^*. \]
Let $S : F \to F$ be a linear map and let $\tilde{S} : F \oplus F^* \to F \oplus F^*$ be defined by $\tilde{S} = (S, -S^T)$ then $\tilde{S} \in o(H, h)$. In particular $\tilde{S} \in F \otimes F^*$.

Furthermore, 

$$f(\tilde{S}) = \frac{1}{2} \sum_a [(Se_a) \otimes \tilde{e}^a - (S^T \tilde{e}^a) \otimes e_a].$$

We have

**Lemma 1.1** Let $S : F \to F$ be a linear map and $\tilde{S} = (S, -S^T)$. Then

1. $\rho(\tilde{S}) = (S) - \frac{1}{2} \text{Tr}(S) \mathbb{I}$
2. $\rho^*(\tilde{S}) = (-S^T) + \frac{1}{2} \text{Tr}(S) \mathbb{I}$

where $(S)$ denotes the extension to the exterior algebra $\Lambda F$ as derivation and $( -S^T)$ denotes the unique derivation of $\Lambda F^*$ which coincides with $(-S^T)$ on $F^*$.

**Proof:** We show the first statement, the proof of the second is similar. In $C(Q)$ we have $\tilde{e}^a \otimes e_a = \mathbb{I} - e_a \otimes \tilde{e}^a$, and after applying $\lambda$ we understand that $\tilde{e}^a = \iota e_a$ etc. so that,

$$\rho(\tilde{S}) = \sum_a \left[ (Se_a) \cdot \iota e_a - \frac{1}{2} \text{Tr}(S) \mathbb{I} \right].$$

Note that $Gl(W^*)$ has a natural action, $\rho_0$, on $\Lambda(V \otimes W)$ with its action on $V \otimes W$ being $\text{Id}_V \otimes (M^T)^{-1}$ with $M \in Gl(W^*)$. On the other hand $Gl(W^*)$ is a subgroup of $Sp(W \oplus W^*) \equiv Sp(g)$. However, the restriction to $Gl(W^*)$ of the action of $Sp(g)$ on $\Lambda(V \otimes W)$ is the natural action twisted by a character of $Gl(W^*)$. More precisely we have, on applying Lemma [1.1]

**Proposition 1.1** Let $\rho$ be the restriction of the spin representation to $Gl(W^*)$ and $\rho_0$ be the natural representation on $\Lambda(V \otimes W)$, $M \in Gl(W^*)$ then we have

$$\rho(M) = (\det M)^n \rho_0(M).$$

**Proof:** Apply Lemma [1.1] with $F = V \otimes W$, and $S = \text{Id}_V \otimes -S_0^T$ to obtain the formula of interest at the the Lie algebra level with $S_0 \in gl(W^*)$. Exponentiation yields the proposition.

**1.3 Explicit Formulae**

By considering $X \in S^2 F$ and $X \in S^2 F^*$ as well one can be more explicit about the representations. Any element of $sp(g)$, in the defining representation, can be written as
a block matrix
\[
\begin{pmatrix}
  a & b \\
  c & -a^T
\end{pmatrix}
\] (1.2)
of \(g \times g\) matrices, where \(a \in \text{gl}(g)\) and \(b\) and \(c\) are symmetric matrices. Let \(E_{ij}\) be the \(g \times g\) matrix whose only non-zero entry is the one at the \(i\)-th row and \(j\)-th column and that entry is 1. The identification to be made is the following
\[
h^i_j = \begin{pmatrix}
  E_{ji} & 0 \\
  0 & -E_{ij}
\end{pmatrix} - n\delta^i_j,
\]
with the shift by the identity matrix taking into account the twisting by a character, and
\[
L_{ij} = \begin{pmatrix}
  0 & \frac{1}{2}(E_{ij} + E_{ji}) \\
  0 & 0
\end{pmatrix}, \quad \Lambda_{ij} = \begin{pmatrix}
  0 & 0 \\
  \frac{1}{2}(E_{ij} + E_{ji}) & 0
\end{pmatrix}.
\]
These clearly span the matrices of the type (1.2).

Set \(V \otimes \mathbb{C}^g = V_1 \oplus \ldots \oplus V_g\) where all of the spaces \(V_i = V\). Introduce on each \(V_i\) a basis of \(\Lambda^1 V_i, v^I_i\), where \(I = 1, \ldots, 2n\) and \(i = 1, \ldots, g\). Introduce the interior multiplication, \(i^I_j\), with respect to the previous basis
\[
i^I_j v^I_i = \delta^I_i \delta^J_j
\]
and let the symplectic form, \(\epsilon\), on \(V\) be given by \(\epsilon = \frac{1}{2} \sum_{I,J} \epsilon_{IJ} v^I_i \wedge v^J_l\). Since the symplectic form is non-degenerate there exists an inverse matrix, \(\epsilon^{IJ}\), such that
\[
\sum_{J=1}^{2n} \epsilon^{IJ} \epsilon_{JK} = \delta^K_I.
\]
Define the following operations on \(\Lambda(V \otimes W)\), let \(\omega \in \Lambda(V \otimes W),\)

**Definition 1.1**
\[
L_{ij}(\omega) = \frac{1}{2} \sum_{I,J} \epsilon_{IJ} v^I_i \wedge v^J_l \wedge \omega, \quad \Lambda^{ij}(\omega) = \frac{1}{2} \sum_{I,J} \epsilon^{IJ} i^I_j i^J_l(\omega), \quad h^i_j(\omega) = \sum J v^J_j \wedge i^J_l(\omega)
\]
Note that this definition is actually an application of (1.1) on the appropriate Lie algebra elements with \(\text{sp}(g) \hookrightarrow o(H, h)\).

**Proposition 1.2** The operators, \(L_{ij}, \Lambda^{ij}\) and \(h^i_j\) satisfy the relations
\[
\begin{align*}
[\Lambda^{ij}, L_{kl}] &= \frac{n}{2} \left( \delta^i_k \delta^j_l + \delta^i_l \delta^j_k \right) - \frac{1}{4} \left( \delta^i_k h^j_l + \delta^i_l h^j_k + \delta^j_k h^i_l + \delta^j_l h^i_k \right) \\
[h^k_l, L_{ij}] &= \delta^k_i L_{lj} + \delta^k_j L_{il}, \quad [h^k_l, \Lambda^{ij}] = -\delta^k_i \Lambda^{kj} - \delta^k_j \Lambda^{ik} \\
[h^i_j, h^k_l] &= \delta^i_l h^k_j - \delta^i_j h^k_l, \quad [\Lambda^{ij}, \Lambda^{kl}] = 0, \quad [L_{ij}, L_{kl}] = 0
\end{align*}
\]
and this is a representation of the Lie algebra \(\text{sp}(g)\).
Proof: That the generators satisfy the algebra and that this is \(sp(g)\) follows from what has been said. Alternatively the algebra can be checked by a short calculation and can be put in standard form by sending \(h_{j}^{i} \rightarrow h_{j}^{i} + n\delta_{j}^{i}\).

Let \(\alpha \in \Lambda^{p_{1}}V \otimes \ldots \otimes \Lambda^{p_{s}}V\) and make the dependence on the basis of \(\Lambda^{p_{1}}V \otimes \ldots \otimes \Lambda^{p_{s}}V\) explicit as \(\alpha(v_{k}^{j})\). Also set \(|p| = \sum_{i=1}^{g} p_{i}\).

The following theorem allows us to write the \(Sp(g)\) action explicitly, and represents the exponentiation alluded to in Proposition [1.1].

**Theorem 1.3** The following vector spaces are representations of \(Sp(g)\)

\[
\mathcal{H}_{+}(V) = \bigoplus_{\sum p_{i} = 0 \mod 2} \Lambda^{p_{1}}V \otimes \ldots \otimes \Lambda^{p_{s}}V, \quad \text{and}
\]

\[
\mathcal{H}_{-}(V) = \bigoplus_{\sum p_{i} = 1 \mod 2} \Lambda^{p_{1}}V \otimes \ldots \otimes \Lambda^{p_{s}}V.
\]

In particular for \(U \in Sp(g)\) and \(\alpha \in \Lambda^{p_{1}}V \otimes \ldots \otimes \Lambda^{p_{s}}V\) the action is given by

\[
\alpha(v_{k}^{j})U = (\text{Det} A)^{n}\alpha(e^{lI.C.iJ + D.v'}). \exp (-\text{Tr}(L.(A^{-1}.B))),
\]

where

\[
U = \begin{pmatrix}
A & B \\
C & D
\end{pmatrix}.
\]

Proof: By the previous proposition we know that the space

\[
\mathcal{H}(V) = \mathcal{H}_{+}(V) + \mathcal{H}_{-}(V)
\]

furnishes a representation space for the Lie algebra \(sp(g)\). Indeed, as the generators \(L, A, h\) all change \(|p|\) by \(0 \mod 2\) we see that the individual spaces, \(\mathcal{H}_{+}(V)\) and \(\mathcal{H}_{-}(V)\), form representation spaces for the Lie algebra. Since the symplectic group is simply connected we know that one can obtain a representation from the algebra by exponentiation. However, this is not to say that the action of the group is easy to write down explicitly. That the formula advocated is a representation of \(Sp(g)\) can be shown by straightforward but rather tedious algebra which the reader is spared. However, it is possible to easily establish its validity on the subgroup of \(Sp(g)\) where the bottom left hand block \((C)\) is zero. Let \(U_{1}\) and \(U_{2}\) be elements of the subgroup, then

\[
(\alpha.U_{1}).U_{2} = (\text{Det} A_{1}.A_{2})^{n}\alpha(D_{1}.D_{2}.v') \exp (-\text{Tr}(L.[D_{2}^{T}.A_{1}^{-1}.B_{1}.D_{2} + A_{2}^{-1}.B_{2}]))
\]

The product \(U_{3} = U_{1}U_{2}\) is,

\[
U_{3} = \begin{pmatrix}
A_{3} & B_{3} \\
0 & D_{3}
\end{pmatrix} = \begin{pmatrix}
A_{1}.A_{2} & A_{1}.B_{2} + B_{1}.D_{2} \\
0 & D_{1}.D_{2}
\end{pmatrix},
\]

and we note that \(A_{3}^{-1}.B_{3} = A_{2}^{-1}.B_{2} + A_{2}^{-1}.A_{1}^{-1}.B_{1}.D_{2}\). However, for \(U_{2}\) to be an element of \(Sp(g)\), we must have that \(D_{2}^{T} = A_{2}^{-1}\) so that we have shown \((\alpha.U_{1}).U_{2} = \alpha.(U_{1}.U_{2})\) as required.
The action of the group on $\alpha$ also preserves $(-1)^{|p|}$ since all the terms appearing in the formula shift $|p|$ by multiples of 2 and so preserves the splitting of $\mathcal{H}(V)$.

Remark 1.1 The action of minus the identity, $-I$ the generator of the centre of $Sp(g)$, is as multiplication by $(-1)^{ng-|p|}$. Hence, it is $Sp(g)/\pm I$ which acts on $\mathcal{H}_{\pm}(V)$ when $ng = 0 \mod 2$ and on $\mathcal{H}_{\mp}(V)$ when $ng = 1 \mod 2$.

2 Skew Invariant Theory

Let $V$ and $W$ be finite dimensional vector spaces of dimension $m$ and $g$ over $\mathbb{C}$ respectively.

Proposition 2.1 Under the action of $GL(V) \times GL(W)$ the exterior algebra $\Lambda(V \otimes W)$ decomposes as

$$\Lambda(V \otimes W) = \bigoplus_{\lambda} \left( S_\lambda(V) \otimes S_{\bar{\lambda}}(W) \right)$$

where $S_\lambda$ (respectively $S_{\bar{\lambda}}$) is the Schur functor corresponding to the Young diagram $\lambda$ (respectively to the Schur functor of the dual partition $\bar{\lambda}$) and the partition $\lambda$ has at most $m$ columns and $\bar{\lambda}$ has at most $g$ rows.

Proof: See for example Theorem (8.4.1) in [19].

Proposition 2.2 Suppose that $V$ is of complex dimension $2n$ and comes equipped with a non-degenerate skew symmetric form. Denote by $Sp(V)$ the corresponding symplectic group, then, the space $I$ of $Sp(V)$ invariants in $\Lambda(V \otimes W)$ has the following decomposition under $GL(W)$.

$$I = \bigoplus S_\mu(W)$$

where

1. each row of $\mu$ has an even number of elements
2. the first row has $\leq 2n$ elements and
3. the number of rows is at most $g$.

Proof: For a partition $\lambda$, the space of $Sp(V)$ invariants in $S_\lambda(V)$, is denoted $S_\lambda(V)^{Sp(V)}$. We have, using proposition 2.1

$$I = \left( \bigoplus S_\lambda(V) \otimes S_{\bar{\lambda}}(W) \right)^{Sp(V)} = \bigoplus \left( S_\lambda(V)^{Sp(V)} \otimes S_{\bar{\lambda}}(W) \right).$$
On the other hand, (see the following remark)

$$\dim S_\lambda(V)^{Sp(V)} = \begin{cases} 1, & \text{if each column of } \lambda \text{ has an even number of elements} \\ 0, & \text{otherwise} \end{cases} \quad (2.1)$$

Now this proposition follows from the previous one.

$$\square$$

**Remark 2.1** While (2.1) is apparent I have not been able to find a reference for it. The result follows from the restriction rules of [11], page 443, together with the conditions on the Littlewood-Richardson coefficients given in [13] page 142.

**Theorem 2.3** (The fundamental theorem of skew invariant theory)

The (commutative) algebra $\mathcal{I}$ of $Sp(V)$ invariants in $\Lambda(V \otimes W)$ has the following presentation. Let $S(S^2W)$ denote the symmetric algebra of the second symmetric power of $W$. Consider the natural embedding, $P_{n+1}$, of $S^{2n+2}(W)$ into $S(S^2W)$ given by the inclusion $S^{2n+2}(W) \hookrightarrow S^{n+1}(S^2W)$. See for example (4.3.3) in [6]. Then $\mathcal{I}$ is isomorphic to the quotient algebra of $S(S^2W)$ by the ideal generated by the image of $S^{2n+2}(W)$ under $P_{n+1}$.

**Proof:** From Proposition 2.2 we see that $S^2W$ is contained in $\mathcal{I}$. Consequently there is an algebra homomorphism, $\phi$, from $S(S^2(W))$ to $\mathcal{I}$ which is in fact $GL(W)$ equivariant. Indeed $\phi$ is surjective which we know in any case from Theorem 2 of [9]. To proceed we use the following results,

1. $S(S^2W) = \bigoplus_{\mu} S_{\mu}W$ where $\mu$ is as in Proposition 2.2 except the second restriction is lifted, Proposition 1 in [9] page 562.

2. The ideal generated by $S^{2k}W$ is equal to $\bigoplus_{\mu} S_{\mu}(W)$ with $\mu$ as in Proposition 2.2 except that the second condition is replaced with the first row having $\geq 2k$ elements, observing that the representation $S^{2k}W$ corresponds to a Young diagram with only one row and it consists of $2k$ boxes. (A special case of Theorem 3.1. of S. Abeasis [11])

It follows that the ideal generated by $S^{2n+2}W$ is mapped to zero under $\phi$ by using Proposition 2.2 (2) and the $GL(W)$ equivariance of $\phi$. Now using statement 1) above, Proposition 2.2 again and the surjectivity of $\phi$ we see that the ideal generated by $S^{2n+2}W$ is precisely the kernel of $\phi$. This completes the proof.

$$\square$$

**Remark 2.2** Let us write explicitly the map $P_k : S^{2k}W \rightarrow S^k(S^2W)$. Fix a basis for $W$, $e_1, \ldots, e_g$, let us also denote by $L_{ij} = e_i.e_j = L_{ji}$. Let $e_{i_1} \ldots e_{i_{2k}}$ with $i_j \in \{1, \ldots, g\}$,
be a monomial in $S^{2k}W$. Then $P_k(e_{i_1}, \ldots, e_{i_{2k}})$ is the sum over all partitions in pairs $(i_1, i_2), \ldots, (i_{2k-1}, i_{2k})$ of the product $L_{i_1, i_2} \cdots L_{i_{2k-1}, i_{2k}}$.

**Definition 2.1** Given an integer $m$ and $\underline{\mu} = (\mu_1, \ldots, \mu_m)$ with $\mu_1 \geq \ldots \geq \mu_m \geq 0$ we denote by $R_{\mu}(Sp(m))$ the irreducible representation of $Sp(m)$ with highest weight $\mu = \mu_1 \lambda_1 + \ldots + \mu_m \lambda_m$, where the $\lambda_i$ are the fundamental weights. If it is clear which symplectic group one is referring to then we may write $R(\mu_1, \ldots, \mu_m)$ for the irreducible representation.

**Theorem 2.4** We have the decomposition of $\Lambda(V \otimes W)$ under $Sp(V) \times Sp(g)$ as

$$\Lambda(V \otimes W) = \bigoplus \mu R_{\mu}(Sp(V)) \otimes R_{\mu}(Sp(g))$$

where the allowed $\mu$ are $\underline{\mu} = (n - a_g, \ldots, n - a_1)$, with $n \geq a_1 \geq \ldots \geq a_g \geq 0$ and $\tilde{\mu}$ is determined in terms of $\mu$ and given by

$$\tilde{\mu} = \tilde{\omega}_{a_1} + \ldots + \tilde{\omega}_{a_g}$$

with $\tilde{\omega}_i$ is the $i$'th fundamental weight of $Sp(V)$, alternatively one has

$$\tilde{\mu} = (\sum_{i=1}^g \theta(a_i - 1), \ldots, \sum_{i=1}^g \theta(a_i - n))$$

with $\theta(x) = 0$ if $x < 0$ and $\theta(x) = 1$ if $x \geq 0$. In particular, setting $\underline{\mu} = (n, \ldots, n)$ so that $\tilde{\mu} = 0$, we have

$$\Lambda(V \otimes W)_{Sp(V)} = R_{(n, \ldots, n)}(Sp(g)),$$

that is the space of $Sp(V)$ invariants in $\Lambda(V \otimes W)$ carries the irreducible representation of $Sp(g)$ with highest weight $(n, \ldots, n)$.

**Proof:** This is implicit in the proof of Lemma 3.7 (especially pages 706-707) of [3]. However, note that the roles of the numbers ’$g$’ and ’$n$’ are interchanged. As on p. 706 we find, for $\underline{a} = (a_1, \ldots, a_g)$ with $n \geq a_g \geq \ldots \geq a_1 \geq 0$, a vector $\phi_{\underline{a}} \in \Lambda(V \otimes W)$ such that the 1-dimensional subspace spanned by $\phi_{\underline{a}}$ is invariant under $B_1 \times B_2$ where $B_1$ and $B_2$ are suitable Borel subgroups of $Sp(V)$ and $Sp(g)$ respectively. (It is proven there that this subspace is invariant under $B_1 \times B_0$ where $B_0$ is a Borel subgroup of $GL(W^*)$ (indeed the upper triangular matrices of $GL(W^*)$) and also that $\phi_{\underline{a}} \in \mathcal{H}$ in the notation of [3]. Since $\mathcal{H}$ is the space annihilated by the $L_{ij}$, defined in section 1.3, this in fact implies invariance under $B_1 \times B_2$ as $B_2$ is the subgroup generated by $B_0$ and the $L_{ij}$.)

The subrepresentation $\rho_{\underline{a}}$ spanned by the transforms of $\phi_{\underline{a}}$ by $Sp(V) \times Sp(g)$ is irreducible with highest weight $\mu = (n - a_1, \ldots, n - a_g)$ for $Sp(g)$ and $\tilde{\mu}$ for $Sp(V)$. As on page 707 of [3] we can show that the representations $\rho_{\underline{a}}$ for $\underline{a} = (a_1, \ldots, a_g)$ satisfying $n \geq a_g \geq \ldots \geq a_1 \geq 0$ exhaust all irreducible representations contained in $Sp(V) \times Sp(g)$. (Here
we apply Theorem 8 of [9] to the reductive dual pair \( \text{Sp}(V) \) and \( \text{Sp}(g) \) in the orthogonal algebra.)

\[ \square \]

**Remark 2.3** By varying \( W \) all irreducible representations of \( \text{Sp}(V) \) are realized on \( \Lambda(V \otimes W) \).

**Remark 2.4** The case \( g = 1 \) of the theorem is well known [6] (page 207 Corollary 4.5.9).

**Theorem 2.5** Let \( F \) be a representation space of \( \text{Sp}(g) \) with highest weight \((a_1, \ldots, a_g)\) such that \( n \geq a_1 \geq \ldots \geq a_g \geq 0 \) then the ideal in \( S(S^2W) \) generated by the image of \( P_{n+1} \) annihilates \( F \).

**Remark 2.5** Thus, as an ideal, the \( P_{n+1} \) relations do not pick out a single irreducible representation, but rather, all those irreducible representations that occur in \( \Lambda(V \otimes \mathbb{C}^g) \).

**Proof:** Let \( v \in F \) be a highest weight of some irreducible representation \((a_1, \ldots, a_g)\) with \( n \geq a_1 \geq \ldots \geq a_g \geq 0 \). We have at our disposal, by Theorem 2.4, all irreducible representations satisfying \( a_1 \leq n \) (and ultimately all irreducible representations by taking symplectic vector spaces \( V \) of larger and larger dimension). As in section 1.3 the \( L_{ij} \) are realized as wedging with respect to the symplectic form. In this case we may write

\[ P_{n+1}(e_{i_1}, \ldots, e_{i_{2n+2}}) = L_{n+1}(I_1, \ldots, I_{2n+2}) v_i^{I_i} \wedge \ldots \wedge v_i^{I_{2n+2}}. \]

As \( P_{n+1}(e_{i_1}, \ldots, e_{i_{2n+2}}) \) is totally symmetric in the labels \( i_1, \ldots, i_{2n+2} \), \( L_{n+1}(I_1, \ldots, I_{2n+2}) \) must be totally anti-symmetric in the labels \( I_1, \ldots, I_{2n+2} \), i.e. \( L_{n+1} \in \Lambda^{2n+2}V \) but as the dimension of \( V \) is \( 2n \) there is no such tensor.

\[ \square \]

### 3 Complex Manifolds

Throughout this section \( X \) is a compact closed complex manifold and no extra structure is assumed. The dimension of \( X \) is \( \dim_{\mathbb{C}}(X) = n \). Let \( \Omega^p_X \) denote the sheaf of holomorphic \( p \)-forms on \( X \). The objects of main interest for us are the Dolbeault cohomology groups of \( X \) with values in

\[ \Omega^p_X = \Omega_X^{p_1} \otimes \ldots \otimes \Omega_X^{p_g} \]

where \( \underline{p} \) is a \( g \)-tuple of natural numbers \( \underline{p} = (p_1, \ldots, p_g) \in \mathbb{N}^g \), where \( \mathbb{N} = \{0, 1, 2, \ldots \} \).

**Definition 3.1** The pluri-Hodge groups of \( X \) of length \( g \) are

\[ \mathcal{H}^q(X, \Omega^\underline{p}_X) \quad \xrightarrow{\text{def}} \quad \mathcal{H}^q(X, \Omega_X^{p_1} \otimes \ldots \otimes \Omega_X^{p_g}). \]
Remark 3.1 When $g - 1$ of the $p_i$ are zero ($\Omega_X^0 = \mathcal{O}_X$) the generalized Hodge groups are the Hodge cohomology groups $H^{(p,q)}(X)$.

Definition 3.2 The degree of $\alpha \in H^q(X, \Omega_X^p)$ is $q + |p|$.

Definition 3.3 The dimensions of the pluri-Hodge groups are
\[ h^{(p,q)}(X) = \dim H^q(X, \Omega_X^p). \]

3.1 Pluri $\chi_y$ Genera determine Chern Numbers

Now we motivate the introduction of pluri-Hodge groups by showing that they determine the Chern numbers of $X$. To do this one groups the dimensions $h^{(p,q)}(X)$ into a form generalizing the usual $\chi_y$ genus. Then an application of Riemann-Roch yields the result.

Definition 3.4 The pluri-$\chi_y$ genus (of length $g$) is,
\[ \chi_y(X) = \sum_{q, p} (-1)^{q + |p|} h^{(p,q)}(X) y_1^{p_1} \cdots y_g^{p_g} \]

Proposition 3.1
\[ \prod_{i=1}^n (a_i + te_i) = \sum_{p=0}^n t^p \sum_{i_1 < \ldots < i_p} e_{i_1} \ldots e_{i_p} \prod_{k \notin \{i\}} a_k. \]

where $\{i\}$ is the set $(i_1, \ldots, i_p)$.

No proof required.

Proposition 3.2 Let $x_r$ denote the Chern roots of $X$ and $c_i$ the $i$-th Chern classes of the holomorphic tangent bundle of $X$. Chern classes are given by
\[ c_i = \sum_{j_1 < \ldots < j_{n-i}} \frac{c_n}{x_{j_1} \cdots x_{j_{n-i}}} \]
\[ = \sum_{j_1 < \ldots < j_{n-i}} \prod_{k \notin \{j\}} x_k. \]

Proof: We start with the definition in terms of Chern roots,
\[ \sum_{i=0}^n c_i t^i = \prod_{i=1}^n (1 + tx_i) \]
so that
\[ \sum_{i=0}^n c_i t^{n-i} = t^n \prod_{i=1}^n (1 + t^{-1}x_i) = \prod_{i=1}^n (t + x_i) = c_n \prod_{i=1}^n (tx_i^{-1} + 1). \]
We have, by proposition 3.1
\[ \sum_{i=0}^{n} c_i t^{n-i} = c_n \sum_{p=0}^{n} t^p \sum_{i_1 < \ldots < i_p} x_{i_1}^{-1} \ldots x_{i_p}^{-1}. \]

\[ \square \]

**Theorem 3.3** The coefficient of \((1 - y_1)^{n-q_1} \ldots (1 - y_g)^{n-q_g}\) with \(\sum_{i=1}^{g} q_i = \dim_{\mathbb{C}}(X) = n\) in the \(\chi_{-y}\) genus of length \(g\) is \(c_{q_1} \ldots c_{q_g} [X]\).

**Proof:** The Riemann-Roch theorem, in the form presented in [17], tells us that
\[ \chi_{-y} = \text{Todd} \left( \prod_{i=1}^{n} (a_i + (1 - y_1)e_i) \right) \ldots \left( \prod_{i=1}^{n} (a_i + (1 - y_g)e_i) \right) [X], \]
where, Todd is the Todd class of the tangent bundle of \(X\) and
\[ e_i = e^{-x_i}, \quad a_i = 1 - e_i. \]

Now from Proposition 3.1 we have that
\[ \chi_{-y} = \text{Todd} \sum_{p} \left[ (1 - y_1)^{p_1} \sum_{i_1 < \ldots < i_{p_1}} e_{i_1} \ldots e_{i_{p_1}} \prod_{k \notin \{i\}} a_k \ldots \right] \left[ (1 - y_g)^{p_g} \sum_{j_1 < \ldots < j_{p_g}} e_{j_1} \ldots e_{j_{p_g}} \prod_{l \notin \{j\}} a_l \right] [X]. \]

Set \(p_i = n - q_i\), then the coefficient of \((1 - y_1)^{n-q_1} \ldots (1 - y_g)^{n-q_g}\) is
\[ \text{Todd} \left[ \sum_{i_1 < \ldots < i_{n-q_1}} e_{i_1} \ldots e_{i_{n-q_1}} \prod_{k \notin \{i\}} a_k \ldots \sum_{j_1 < \ldots < j_{n-q_g}} e_{j_1} \ldots e_{j_{n-q_g}} \prod_{l \notin \{j\}} a_l \right] [X]. \] (3.1)

Note also that \(a_i = x_i + \ldots\) where the ellipses represent forms of higher degree. Consequently,
\[ \prod_{k \notin \{i\}} a_k \ldots \prod_{l \notin \{j\}} a_l, \] (3.2)
has form degree greater than or equal to \((n - p_1) + \ldots + (n - p_g) = \sum_{i=1}^{g} q_i\). Fix from now on \(\sum q_i = n\). The product (3.2) is then a top degree form, and we may as well take \(a_i = x_i\), since all improvements yield an even higher degree form. Since the products of the \(a_i\) in (3.1) already constitute a form of highest possible degree we can set Todd = 1 and \(e_i = 1\). Hence (3.1) simplifies to
\[ \left[ \sum_{i_1 < \ldots < i_{n-q_1}} \prod_{k \notin \{i\}} x_k \ldots \sum_{j_1 < \ldots < j_{n-q_g}} \prod_{l \notin \{j\}} x_l \right] [X]. \]
By Proposition 3.2 this is
\[ c_{q_1} \cdots c_{q_g} [X] \]

**Corollary 3.3.1** Two complex manifolds of complex dimension \( n \) are complex cobordant iff their generalized \( \chi_{-y} \) genera of length \( n \) agree.

**Proof:** Two complex manifolds are complex cobordant if all of their Chern numbers agree. By Theorem 3.3 all Chern numbers are determined by the generalized \( \chi_{-y} \) genus of length \( n \). Conversely the Riemann-Roch theorem shows that the generalized \( \chi_{-y} \) genera of any length are known once all Chern numbers are.

**Remark 3.2** The theorem is by no means sharp. For a complex surface the \( \chi_y \) genus determines and is determined by the Chern numbers while the pluri-\( \chi_{-y} \) genus of length 2 will do for up to complex 5-folds. Also for a complex manifold with \( c_1 = 0 \), the pluri-\( \chi_y \) genus of length \( n/2 \) suffices as the longest strings of Chern numbers are either \( c_2^n [X] \) for \( n = 2m \) or \( c_2^{m-1} c_3 [X] \) for \( n = 2m + 1 \).

**Remark 3.3** While the \( \chi_{-y} \) genus tells us quite a bit about the pluri-Hodge groups it is not enough to establish that the dimensions are in fact deformation invariants or otherwise of \( X \). Clearly the combinations that appear in \( \chi_{-y} \) are invariant but we know that for non-Kähler manifolds even the usual Hodge numbers need not be invariants.

## 4 Holomorphic Symplectic Manifolds

Let \( X \) be a compact holomorphic symplectic manifold. This means that \( X \) is a compact complex manifold with a nowhere degenerate holomorphic 2-form \( \epsilon \). Note that we do not assume that \( \epsilon \) is closed. No extra structure is assumed. In local coordinates write the components of \( \epsilon \) as \( \epsilon_{IJ} \), since it is pointwise non-degenerate there exists an inverse matrix at each point of \( X \) which we denote by \( \epsilon^{IJ} \),

\[
\sum J \epsilon^{IJ} \epsilon_{JK} = \delta^I_K.
\]

### 4.1 Representations of the Symplectic Group on Sums of Pluri-Hodge Groups

Set
\[
\mathcal{H}^q(X) = \bigoplus_p H^q(X, \Omega_X^p) = \mathcal{H}^q_+ (X) \oplus \mathcal{H}^q_- (X)
\]

with
\[
\mathcal{H}^q_+ (X) = \bigoplus_{|p| = 0 \mod 2} H^q(X, \Omega_X^p), \quad \mathcal{H}^q_- (X) = \bigoplus_{|p| = 1 \mod 2} H^q(X, \Omega_X^p).
\]
We shall now show that there is a natural action of $Sp(g)$ on the holomorphic bundle $\Lambda(\Omega^1_X \otimes \mathbb{C}^g)$ and hence on $\mathcal{H}_{\pm}^q(X)$ in case $X$ has a holomorphic symplectic structure.

**Theorem 4.1** The holomorphic vector bundle $\Lambda(\Omega^1_X \otimes \mathbb{C}^g)$ and the vector spaces $\mathcal{H}_{\pm}^q(X)$ of pluri-Hodge groups of length $g$ form representations of $Sp(g)$.

**Proof:** The symplectic structure of $X$ gives a holomorphic reduction of the cotangent bundle to the symplectic group $Sp(n)$ where $\dim_{\mathbb{C}} X = 2n$. Consider the action of $Sp(n) \times Sp(g)$ on the vector space $\Lambda(\mathbb{C}^{2n} \otimes \mathbb{C}^g)$ discussed in section 1. Since the actions of $Sp(n)$ and $Sp(g)$ commute we obtain an action of $Sp(g)$, which is trivial on $X$, on the associated vector bundle with typical fibre $\Lambda(\mathbb{C}^{2n} \otimes \mathbb{C}^g)$ which is the vector bundle $\Lambda(\Omega^1_X \otimes \mathbb{C}^g)$ where $\mathbb{C}^g$ denotes the trivial vector bundle. Hence we obtain an action of $Sp(g)$ on $\mathcal{H}^q(X, \Lambda(\Omega^1_X \otimes \mathbb{C}^g)) = \mathcal{H}^q(X)$. Since this action preserves the degree we get an action on $\mathcal{H}_{\pm}^q(X)$.

\[ \square \]

**Remark 4.1** In other words: This is a direct application of the construction of section 1 to obtain an $Sp(g)$ action on $\Omega^p_X \otimes \ldots \otimes \Omega^g_X$. The only change to the notation that is required is that $v^I_i \rightarrow dz^I_i$ where $dz^I_i$ are a local coordinate basis for $\Omega^1_{I}$ with $I = 1, \ldots, n$ and $i = 1, \ldots, g$. We have therefore established that the operators, $L_{ij}$, $A^I_j$ and $h^I_j$ acting on $\Omega^p_X \otimes \ldots \otimes \Omega^g_X$ satisfy the algebra of $sp(g)$ as given in Proposition 1.2 with $\epsilon_{IJ}$ now being the components of the holomorphic symplectic 2-form. These operators commute with $\overline{\partial}$, since they are made out the holomorphic 2-form, and so the action extends to the pluri-Hodge groups.

**Remark 4.2** The $Sp(g)$ action on $\alpha \in \mathcal{H}^q(X, \Omega^p_X)$ is given by the formula in Theorem 1.3 with $v^I_i \rightarrow dz^I_i$ and generalizes the formulae with $n = 1$ for the Hodge groups ($g = 1$) given in [20].

**Remark 4.3** Notice that the proof of the Lefschetz $sp(1)$ action with respect to the Kähler form is much more involved. This requires an application of Hodge theory since the corresponding Lie algebra generators do not all commute with $\overline{\partial}$.

### 4.2 The Geometric Meaning of the Genus

In this section we assume that $X$ is compact. In [24] it was shown that the $\chi_y$ genus is essentially the same as the graded trace of an element of the $Sp(1)$ action on the Hodge groups. Here this result is strengthened and the proof is simplified. For ease of notation in the following theorem denote the condition $|p| = 0 \mod 2$ by $|p|_+$ and the condition $|p| = 1 \mod 2$ by $|p|_-$. 15
Proposition 4.2 Let $X$ be a compact holomorphic symplectic manifold of real dimension $4n$. Let $U \in \text{Sp}(g)$ and $y_1, \ldots, y_g, y_1^{-1}, \ldots, y_g^{-1}$ be eigenvalues of $U$, in the $2g$ dimensional defining representation, then

$$\text{Tr}_{H^q_\pm(X)} U = \frac{1}{y_1^n \cdots y_g^n} \sum_{|p|\pm} h^{p,q}(X) y_1^{p_1} \cdots y_g^{p_g}.$$ 

Proof: Since we have a representation the trace makes sense and will give us a sum of characters of the group element. Furthermore, since the space of diagonalizable elements in $\text{Sp}(g)$ includes an open dense set, to establish the formula we need only consider diagonalizable matrices. For diagonalizable matrices, as the characters are class functions, one need only consider their diagonalized form. Consequently, we may as well set

$$U = \text{diag}(y_1^{-1}, \ldots, y_g^{-1}, y_1, \ldots, y_g).$$

By Theorem 4.1, any form $\alpha \in H^q(X, \Omega^2_X)$ maps to

$$y_1^{-n} \cdots y_g^{-n} \cdot y_1^{p_1} \cdots y_g^{p_g} \cdot \alpha$$

so the action on the cohomology groups is just multiplicative and we get such a factor for each element in $H^q(X, \Omega^2_X)$, that is we get such a factor precisely $h^{p,q}(X)$ times.

Remark 4.4 The righthand side of the formula is invariant under sending any of the eigenvalues $y_i$ to their inverse $1/y_i$ since, by the triviality of the canonical bundle, we have $h^{p_1,\ldots,p_n,\ldots,p_g,q}(X) = h^{p_1,\ldots,2n-p_1,\ldots,p_g,q}(X)$.

Definition 4.1 Let $U \in \text{Sp}(g)$, the graded trace or super trace on $\mathcal{H}(X) = \bigoplus_q \mathcal{H}^q(X)$ is

$$\text{STr}_{\mathcal{H}(X)} U = \sum_{q=0}^{2n} (-1)^q \left( \text{Tr}_{\mathcal{H}^q_+(X)} U - \text{Tr}_{\mathcal{H}^q_-(X)} U \right).$$

Theorem 4.3 Let $X$ be an irreducible holomorphic symplectic manifold of real dimension $4n$. Let $U \in \text{Sp}(g)$ and $y_1, \ldots, y_g$ be eigenvalues of $U$, in the $2g$ dimensional defining representation, then

$$\text{STr}_{\mathcal{H}(X)} U = \frac{\chi - y}{y_1^n \cdots y_g^n}.$$ 

Proof: This is immediate given the previous proposition.

4.3 Decomposition of $Sp(g)$ Representations on Pluri-Hodge Groups

Fujiki [5] gives a holomorphic version of the Lefschetz decomposition theorem. This section is devoted to extending that result to the pluri-Hodge groups.
Definition 4.2 Denote the multiplicity of the representation \( R(a_1, \ldots, a_g) \) in \( \mathcal{H}_q^\pm(X) \) by \( m_q^\pm(a_1, \ldots, a_g) \).

Definition 4.3 The Weyl group of \( Sp(g) \) or the ‘octahedral’ group of permutations and sign changes of a \( g \)-tuple \((m_1, \ldots, m_g) \in \mathbb{Z}^g \), as Weyl calls it, is denoted by \( W_g \). For \( w \in W_g \) set \([w]\) to equal the number of sign changes plus 1 if \( w \) induces an odd permutation or simply equal to the number of sign changes if \( w \) induces an even permutation.

Theorem 4.4 Let \( \underline{n} = (n, \ldots, n) \), \( \underline{\rho} = (g, \ldots, 1) \) and \( \underline{a} = (a_1, \ldots, a_g) \). We have, with the notation of Definition 2.1 and Definition 3.3,

\[
m_q^\pm(a_1, \ldots, a_g) = \sum_{w \in W_g} (-1)^{[w]} h^{(a_1, a_2, \ldots, a_g + \rho, \rho)}(X).
\]

Proof: The proof rests on an application of the decomposition theorem of Weyl, (7.10.A) in [28] (this is also known as outer multiplicity, see Corollary 7.1.5 in [6]), combined with Proposition 4.2 above. Since the proof is the same for either \( \mathcal{H}_q^+ \) and \( \mathcal{H}_q^- \) we drop the distinction here. By Proposition 4.2 and the remark after it we can write

\[
\text{Tr}_{\mathcal{H}_q(X)} U = \sum_m h^{(m+m, q)}(X) y_1^{m_1} \ldots y_g^{m_g}.
\]

According to the theorem of Weyl, given a representation \( V \) in which the character is a polynomial of the form

\[
\chi_V(U) = \sum_{\underline{\lambda}} k_{\underline{\lambda}} y_1^{\lambda_1} \ldots y_g^{\lambda_g},
\]

the multiplicity of the representation \( R(a_1, \ldots, a_g) \) is given by

\[
m_{\underline{a}} = \sum_{w \in W_g} (-1)^{[w]} k_w(a_1, a_2, \ldots, a_g + \rho, \rho)
\]

on taking \( V = \mathcal{H}_q(X) \) and comparing the two forms for the character we have finished.

While the formula for the multiplicities is quite succinct it is perhaps worth spelling it out for certain cases. The following proposition is arrived at in the most pedestrian manner so the proof is omitted.

Proposition 4.5 The following multiplicities hold

1. For any \( n \) and \( g \) we have

\[
m^q(n, \ldots, n, n - p) = h^{(p, q)}(X) - h^{(p-2, q)}(X), \quad 0 \leq p \leq n.
\]

This includes the special case \( g = 1 \)

\[
m^q(n - p) = h^{(p, q)}(X) - h^{(p-2, q)}(X),
\]

which is one of the results of Fujiki.
2. We find for \( Sp(2) \)

\[
m^q(n-p_1, n-p_2) = h^{(p_1, p_2, q)}(X) - h^{(p_1, p_2-2, q)}(X) - h^{(p_1-4, p_2, q)}(X)
+ h^{(p_1-4, p_2-2, q)}(X) - h^{(p_2+1, p_1-1, q)}(X)
+ h^{(p_2+1, p_1-3, q)}(X) + h^{(p_2-3, p_1-1, q)}(X)
- h^{(p_2-3, p_1-3, q)}(X).
\]

\[5 \text{ K3 Surfaces}\]

5.1 The Pluri-Hodge Groups on a K3 Surface

One expects, from physics, that the dimensions of the pluri-Hodge groups for a holomorphic symplectic manifold \( X \) are deformation invariants of \( X \). For \( X = K3 \) one can show this by explicitly computing the dimensions. The following theorem is provided by M.S. Narasimhan.

**Theorem 5.1** Let \( X \) be a K3 surface then the dimensions \( h^{(p, q)}(X) \) are independent of \( X \). Set \( m \) to equal the number of ones that appear in \( p \). The dimensions are given by, when \( m \) is odd

\[
h^{(p, 2)}(X) = h^{(p, 0)}(X) = 0 \quad \text{and} \quad h^{(p, 1)}(X) = 2^{m+1}[6m - 1],
\]

while for \( m \) even \( m = 2k \),

\[
h^{(p, 2)}(X) = h^{(p, 0)}(X) = \frac{(2k)!}{k!(k+1)!} \quad \text{and} \quad h^{(p, 1)}(X) = 2 \frac{(2k)!}{k!(k+1)!} + 2^{m+1}[6m - 1].
\]

**Proof:** We will first show that the \( h^{(p, 0)}(X) \) is constant. Choose a Kähler metric on \( X \) with Ricci curvature equal to zero. Recall that the holonomy group is \( SU(2) \). By Theorem 1.34 in Kobayashi [10] every holomorphic section of \( \Omega_X^{1 \otimes m} \) is covariantly constant.

Conversely every covariantly constant section of \( (T^*X)^{\otimes m} \) defines a holomorphic section of \( (T^*X)^{\otimes m} \). In fact the connection \( D \) on \( (T^*X)^{\otimes m} \) is the “Chern connection” on the holomorphic bundle \( (T^*X)^{\otimes m} \) so that if \( D \) is the associated covariant differential we have \( D^0,1 = \overline{\partial} \). Hence if \( D\alpha = 0 \), \( \alpha \) a section of \( (T^*X)^{\otimes m} \), then \( \overline{\partial}\alpha = 0 \).

Observe that in our case the allowed values of \( p_i \) are 0, 1, and 2. However, when \( p_i = 0 \) or 2, \( \Omega_X^{p_i} \) is the structure sheaf and the theorem quoted above applies. Hence the sections of the bundle are covariantly constant. Thus \( h^{(p, 0)}(X) \) is equal to the dimension of the space of invariants of \( SU(2) \) in the \( m \)-fold tensor product of the defining representation of \( SU(2) \) where \( m \) is equal to the number of \( p_i = 1 \). This is clearly independent of \( X \). \[1\]

\[1\] More generally if \( X \) is a compact Ricci flat Kähler manifold any holomorphic section of \( \Omega_X^2 \) is covariantly constant and in principle can be calculated from the holonomy representations.
As the canonical bundle is trivial $\Omega^p_X$ is self dual and so by the duality theorem we see that $h^{(p,2)}(X) = h^{(p,0)}(X)$. Consequently $h^{(p,1)}(X)$ is also constant.

Now by calculating the dimension of the invariants and applying the Riemann-Roch theorem we can calculate $h^{(p,1)}(X)$. Riemann-Roch yields

$$\int_X \text{Todd}(X) \text{ch}(\Omega^1_X) = 2^{m+1}(1-6m),$$

with $c_2(X) = 24$.

On the other hand the space of invariants of $SU(2)$ on the $m$-fold tensor product of the defining representation can be determined by picking out the term in the character expansion of

$$(e^x + e^{-x})^m, \tag{5.1}$$

which corresponds to the trivial representation. We want the dimensions of the $SU(2)$ invariants in $W^\otimes m$ where $W$ is the defining representation. Use the following fact:

(see e.g. [23]) If $\text{ch}(\lambda, V)$ is the character of a representation of $SU(2)$ in a vector space $V$, then the dimension of the invariants in $V$, (i.e. the multiplicity of the trivial representation) is given by the co-efficient of $e^\lambda$ in $(e^{\lambda} - e^{-\lambda})\text{ch}(V)$.

We have $\text{ch}(W) = e^\lambda + e^{-\lambda}$ and $\text{ch}(W^\otimes m) = (e^\lambda + e^{-\lambda})^m$, hence

$$(e^\lambda - e^{-\lambda})\text{ch}(W^\otimes m) = (e^\lambda - e^{-\lambda})(e^\lambda + e^{-\lambda})^m = \sum_{r=0}^m \binom{m}{r} e^{\lambda(m+1-2r)} - \sum_{s=0}^m \binom{m}{s} e^{\lambda(m-1-2s)}.$$

Now we can get $e^\lambda$ either from $m = 2r$ or $m = 2s + 2$ in either case $m$ is necessarily even. Thus if $m$ is odd the trivial representation does not occur in $W^\otimes m$. On the other hand when $m = 2k$ the coefficient of $e^\lambda$ is

$$\binom{2k}{k} - \binom{2k}{k-1} = \frac{(2k)!}{k!(k+1)!}.$$

\[\square\]

6 Graph Representations of $Sp(g)$

We saw in previous sections that the $P_{n+1}$ relations determine certain representations of $Sp(g)$. It is quite remarkable that the same relations appear in the construction of the LMO invariant. They are imposed as a set of relations so that the second Kirby move be respected.

In this section a certain amount of pictorial gymnastics is unavoidable. For a more complete account of chord diagrams and the $IHX$, $AS$ (and orientation), $STU$, $P_n$, $O_n$ and branching relations $B$, I refer the reader to [12, 16].
Definition 6.1 A chain graph, \( \Gamma_g \), with \( g \) copies of \( S^1 \), is depicted in Figure 1.

A chord diagram \( D \) with support \( \Gamma_g \) is \( \Gamma_g \) together with an oriented uni-trivalent graph with the univalent vertices lying in \( \Gamma_g \). Denote by \( \mathcal{C}(\Gamma_g) \) the \( \mathbb{C} \)-linear space of chord diagrams with support \( \Gamma_g \) including trivial circles of dashed lines.

Definition 6.2 Denote the number of trivalent vertices by \( D^i \) (i-for internal) and let \( D^i_{>n} \) be the relation such that any chord diagram with internal degree more than \( n \) is zero. Let \( D_T \), the degree, be half the sum of trivalent and univalent vertices.

Definition 6.3 We set \( \mathcal{A}(\Gamma_g) = \mathcal{C}(\Gamma_g)/IHX, AS, STU, B \) and furthermore let \( \mathcal{A}^{(n)}(\Gamma_g) = \mathcal{A}(\Gamma_g)/O_n, P_{n+1}, D^i_{>2n} \).

Remark 6.1 The \( STU \) relations do not preserve the internal degree \( D^i \) even though they do preserve the total degree \( D_T \) which counts trivalent as well as univalent vertices. We will need a slightly different presentation of chord diagrams which does not make use of the \( STU \) relations. Such a space of uni-trivalent graphs was introduced by Sawon [21].

Definition 6.4 A marked uni-trivalent graph is an oriented uni-trivalent graph whose univalent vertices carry labels 1 to \( g \) and repetitions are allowed. Denote by \( \mathcal{B}_g \) the space of linear combinations of marked uni-trivalent graphs modulo \( AS \) and \( IHX \) and including trivial circles of dashed lines. We also denote the completion of \( \mathcal{B}_g \) with respect to the total degree by the same symbol.

Remark 6.2 Sawon did not include dashed circles in his definition, we will need this generalization below.

Proposition 6.1 [21] There is an isomorphism \( \tau : \mathcal{B}_g \to \mathcal{A}(\Gamma_g) \).

Remark 6.3 The isomorphism requires that \( \mathcal{B}_g \) include the ‘empty’ uni-trivalent graph, \( e_g \), which corresponds to \( \Gamma_g \in \mathcal{A}(\Gamma_g) \). Furthermore, it is not difficult to show that the isomorphism between \( \mathcal{B}_g \) and \( \mathcal{A}(\Gamma_g) \) given in [21] preserves the \( P_n \) and \( O_n \) equivalence relations.

We now see that there is a natural \( Sp(g) \) action on these spaces.
**Theorem 6.2** The space of graphs $B_g/O_n$ is a (reducible) representation space of $Sp(g)$.

**Proof:** The manner in which the Lie algebra of $Sp(g)$ is to be realised on the space of graphs $B_g/O_n$ is as follows:

$L_{ij}$ introduces a dashed line connecting two univalent vertices one with the label $i$ and the other with the label $j$. $h^i_j$ relabels an existing univalent vertex with the label $i$ with a $j$ and one repeats this summing over all univalent vertices with the label $i$. The action of $\Lambda^{ij}$ is best described as follows. Consider that $i \neq j$ first. If there are no univalent vertices marked either $i$ or $j$ the action of $\Lambda^{ij}$ is to give zero. So suppose that there are such marked univalent vertices. For fixed such marked univalent vertices $\Lambda^{ij}$ acts to eliminate both the univalent vertices and it joins the dashed lines that ended on those vertices and multiplies by $-1/4$. The action of $\Lambda^{ij}$ on the uni-trivalent graph is to sum over the operation just described on all possible pairs of univalent vertices with the markings $i$ and $j$. $\Lambda^{ii}$ is defined in the same way except that the multiplicative factor is $-1/2$.

![Figure 2: $L_{ij}$ acting on a uni-trivalent graph.](image)

One can now show graphically that these maps satisfy the algebra given in Proposition 1.2. For example in Figures 2, 3 and 4 the filled in ellipse designates the rest of a uni-trivalent graph, the labels $p$, $q$ and $r$ indicate the number of vertical dashed lines connected to univalent vertices marked with $i$, $j$ and $k$ respectively and if a dashed line is not strictly vertical it is not counted. One checks that, on these figures, $[\Lambda^{ij}, L_{jk}] = -1/4h^i_k$, $[h^i_j, h^j_k] = -h^i_k$ and so on. In this manner, running through the various possibilities, one establishes the claim that the space of uni-trivalent graphs $B_g/O_n$ is a representation space of $Sp(g)$.

![Figure 3: $\Lambda^{ik}$ acting on a uni-trivalent graph with the sum being over all pairs of univalent vertices marked $i$ and $j$.](image)

$\square$
Figure 4: The action of $h_i^j k$ on a uni-trivalent graph, the sum is over the $p$ univalent vertices marked $i$.

**Definition 6.5** Let the vector space $B_{g,m}^n$ denote the part of $B_g/P_{n+1}, O_n$ with internal degree $m$, and set $B_g^{(n)} = B_g/P_{n+1}, O_n, D_{>2n}^i$ so that $B_g^{(n)} = \sum_{m=0}^{2n} \oplus B_{g,m}^n$.

**Lemma 6.1** Let $D$ be a uni-trivalent graph in $B_g$. Then, modulo the equivalence relation $P_{n+1}$,

1. The uni-trivalent graph $D$ is equivalent to a linear sum of uni-trivalent graphs which have at most $2n$ univalent vertices labeled $i$ for all $i \in \{1, \ldots, g\}$ and

2. The uni-trivalent graph $D$ becomes equivalent to a linear sum of uni-trivalent graphs each of which has either $n$ dashed chords joining vertices with the same labels or at most $2n - 1$ univalent vertices with a given marking $i$.

**Proof:** This is a transcription to $B_g$ of Lemma 3.1 in [12]. The proof there can be adapted to the current situation as follows. All vertices with the same marking can be thought of as being connected to the component $C$ in that paper. Since ordering of univalent vertices with the same marking is immaterial in $B_g$ the univalent vertices on $C$ obey a $TU$ relation, see Figure 5, rather than the $STU$ relation. Consequently, the only difference with [12] is that now we do not generate ‘lower order terms’ and the proof goes through.

**Proposition 6.3** $B_{g,2}^1 \cong \theta \sqcup B_{g,0}^1$ where $\sqcup$ is disjoint union and $\theta$ is the graph

\[
\theta = \includegraphics[scale=0.5]{theta_graph.png}
\]
Proof: There are two trivalent vertices. If they are not connected to any univalent vertex then they form a $\theta$. Now suppose that the trivalent vertices do not form a $\theta$ diagram. There are a number of different situations to consider. By Lemma 6.1 we only need consider that if a univalent vertex marked $i$ is connected to a trivalent vertex then, it is the unique such marked univalent vertex. Consider the situation where the two trivalent vertices are joined by two dashed lines, ($g \geq 2$) as in Figure 6. An application of $P_2$ and IHX shows that such a graph is equivalent to $-1/2 \theta \sqcup$ a dashed chord joining the $i$ and $j$ univalent vertices. Two trivalent vertices can be connected by just one dashed line if $g \geq 4$, as shown in Figure 7. An application of $P_2$ and IHX turns such uni-trivalent graphs into sums of graphs of the form of $\theta$ union chord diagrams as in Figure 8. Lastly if $g \geq 6$ we can have uni-trivalent graphs where the two trivalent vertices are not connected at all by a dashed line as in Figure 9. By repeated use of $P_2$ such a uni-trivalent graph can be expressed as a sum of terms of the form shown in Figure 10. However, these graphs have already been dealt with successfully and so we are done.

Remark 6.4 From the work of J. Murakami [15] we see that $A^{(1)}(\Gamma_1)$ and $A^{(1)}(\Gamma_2)$ are two and five-dimensional spaces as $A^{(1)}(\emptyset)$-modules. We note that $\dim R(1) = 2$ and that $\dim R(1,1) = 5$ where $R$ is a representation of $Sp(g)$. We are now in a position to show that this numerical agreement is no accident.

Theorem 6.4 The space of uni-trivalent graphs $\mathcal{B}^{(1)}_g$ is a representation space for $Sp(g)$. In fact, as $Sp(g)$ representation spaces, $\mathcal{B}^{(1)}_{g,0} = R(n,\ldots,n)$ and $\mathcal{B}^{(1)}_{g,1} = R(n,\ldots,n,0,0,0)$. 

\[\begin{array}{c}
i\\\longrightarrow\\\cdot\\\longrightarrow\\j\\\end{array}\]

Figure 6: Two marked univalent vertices joined to two trivalent vertices

\[\begin{array}{c}
i\\\bullet\\j\\\bullet\\k\\\bullet\\l\\\bullet\\\end{array}\]

Figure 7: Four marked univalent vertices joined to two trivalent vertices

\[\begin{array}{c}
i\\\bullet\\j\\\bullet\\k\\\bullet\\l\\\bullet\\\end{array}\]

Figure 8: A pair of chords joining marked univalent vertices.
Proof: By the previous theorem \( \mathcal{B}_g/O_n \) is a representation space for \( Sp(g) \), we will now see that the \( P_{n+1} \) equivalence relations pick out certain irreducible representations.

For \( \mathcal{B}_{g_0}^n \) any uni-trivalent graph can be obtained from the empty graph, \( e_g \), by successive application of \( L_{ij} \) for various \( i, j \). Consequently, all the uni-trivalent graphs belong to one irreducible representation of \( Sp(g) \) since, for a semi-simple Lie algebra, a finite dimensional reducible representation decomposes into irreducible representations. The representation is finite dimensional since the \( P_{n+1} \) relations imply that no more than \( 2n \) univalent vertices have the same marking, \( D^i = 0 \) says that there are no trivalent vertices and \( O_n \) ensures that dashed circles are set to be multiplication by \( -2n \).

e_g \) is a highest weight state with weight \((n, \ldots, n)\). To see this we apply Theorem 4.7.3 in [26]. To adapt to the notation of that reference let \( Y_g = L_{gg} \), \( X_g = \Lambda^{gg} \) and \( Y_i = h_i^{g+1} \) and \( X_i = h_i^{g+1} \) for \( i = 1, \ldots, g - 1 \) and set \( H_i = n - h_i^{g+1} \). Clearly \( X_i e_g = 0 \), \( H_i e_g = n e_g \) and \( Y_i^{g+1} e_g = 0 \), proving that \( e_g \) is a highest weight vector with the advertised weight.

In the case of \( \mathcal{B}_{g_1}^n \), by Lemma 6.1, every uni-trivalent graph can be obtained as follows. Begin by connecting by dashed lines the unique trivalent vertex to three distinctly marked univalent vertices (if any two of the univalent vertices have the same marking \( AS \) assures us that this vanishes). Then add an even number of univalent vertices (of any marking) to the graph and join these in pairs by dashed lines.

Let \( D_{g_1}^1(i, j, k) \) be the uni-trivalent graph consisting of a trivalent vertex connected by dashed lines to three univalent vertices with markings \( i, j \) and \( k \). Acting with \( L_{mm} \) one generates part of the span of \( \mathcal{B}_{g_1}^n \) just described and by summing over all possible \( i, j \) and \( k \) one obtains the complete span of \( \mathcal{B}_{g_1}^n \). However, as \( D_{g_1}^1(i, j, k) = h_i^{g-2} h_j^{g-1} h_k^g D_g^1(g - 2, g - 1, g) \) one can obtain any uni-trivalent graph in \( \mathcal{B}_{g_1}^n \) starting from \( D_g^1(g - 2, g - 1, g) \). This uni-trivalent graph is a highest weight with \( \lambda(H_g) = \lambda(H_{g-1}) = \lambda(H_{g-2}) = 0 \) and \( \lambda(H_i) = n \).
7 Rozansky-Witten Topological Field Theory

I now want to make contact between the pluri-Hodge groups, the graph representations of the previous section and so of the surprising omnipresence of the skew representation theory of \( Sp(g) \). The pluri-Hodge groups and uni-trivalent graphs are brought together in the context of the theory of Rozansky and Witten [20].

**Remark 7.1** On any complex compact closed manifold, \( Y \), the Atiyah class \( \alpha \in H^1(Y, \Omega^1_Y \otimes \Omega^1_Y \otimes T_Y) \) measures the obstruction to having a global holomorphic section of the holomorphic tangent bundle \( T_Y \) of \( Y \). When \( Y = X \) is holomorphic-symplectic there is an identification \( T_X \cong \Omega^1_X \) (through the holomorphic 2-form) and we also denote by \( \alpha \) its image in \( H^1(X, T_X \otimes T_X \otimes T_X) \). Indeed \( \alpha \in H^1(X, \text{Sym}^3 T_X) \).

**Proposition 7.1** [21] Let \( X \) be a compact closed holomorphic symplectic manifold with \( \dim_{\mathbb{C}} X = 2n \). There is a homomorphism of vector spaces

\[
W_X : B_g/O_n \to \mathcal{H}_g(X).
\]

If a marked uni-trivalent graph \( D \) has \( q \) trivalent vertices and \( p_i \) univalent vertices marked by \( i \) then

\[
W_X(D) \in H^q(X, \Omega^{p_1}_X \otimes \ldots \otimes \Omega^{p_g}_X).
\]

**Definition 7.1** \( W_X \) is defined as follows: Suppose there are \( q \) trivalent vertices in the graph. To each such vertex we associate the Atiyah class \( \alpha \), and cup product is understood between the Atiyah classes. Each dashed leg of the vertex is understood to correspond to one of the fibre factors \( T_X \) in \( T_X \otimes T_X \otimes T_X \). If there is a dashed line between two trivalent vertices, then one contracts the fibre factors of the contracted legs of the two different trivalent vertices as follows \( T_X \otimes T_X \cong T_X \otimes \Omega^1_X \to O_X \). If a trivalent vertex is connected to a univalent vertex marked \( i \) then the \( T_X \) factor of the leg attached to the trivalent vertex is replaced with the \( i \)'th \( \Omega^1_X \). If two univalent vertices, marked \( i \) and \( j \) respectively, are connected by a dashed line then this is mapped to \( \epsilon_{ij} \) and the cup product is understood throughout. \( W_X(e_g) = 1 \in H^0(X, O_X) \) and the circle of a dashed line maps to \(-2n\).

**Remark 7.2** That \( W_X \) is compatible with the \( AS \) and \( IHX \) relations when there are no univalent vertices was shown in [20], but perhaps a more accessible reference is [8]. That the image of \( W_X \) is in \( \mathcal{H}_g(X) \) as well as why \( W_X \) is compatible with the \( AS \) and \( IHX \) relations when there are marked univalent vertices is explained in some detail in [21]. Essentially, in local coordinates we have

\[
W_X(D) = \sum F_{i_1 \ldots j_p} dz_1^{i_1} \wedge \ldots \wedge dz_p^{i_p} \otimes \ldots \otimes dz_1^{j_1} \wedge \ldots \wedge dz_g^{j_g}
\]

where the coefficients \( F_{i_1 \ldots j_p} \) (which are themselves \( (q, 0) \)-forms) are totally anti-symmetric in all labels not just within each marking.
Theorem 7.2 The weight system $W_X : \mathcal{B}_g/O_n \to \mathcal{H}_g(X)$ commutes with the respective $Sp(g)$ actions on $\mathcal{B}_g/O_n$ and $\mathcal{H}_g(X)$.

Proof: Let $D$ be some uni-trivalent graph, then by the definition of $W_X$ we have that $W_X(L_{ij}D) = \epsilon_{ij} \wedge W_X(D)$. However, wedging with $\epsilon_{ij}$ is the same as acting with the $Sp(g)$ generators $L_{ij}$ of section 4, see Remark 4.1. It is not difficult to see, given Remark 7.2, that the rest of the generators map correctly. Compatibility with the $O_n$ relation is a property of the holomorphic 2-form, namely that $\sum_{I,J} \epsilon^{IJ} \epsilon_{IJ} = -2n$.

Corollary 7.2.1 Let $X$ be a K3 surface, then $W_X : \mathcal{B}_{g_0}^1 \to \mathcal{H}_0^0(X)$ and $W_X : \theta \sqcup \mathcal{B}_{g_0}^1 \to \mathcal{H}_2^1(X)$ for $g \geq 3$.

Proof: This follows from the theorem and Theorem 6.4. Note that $W_X(\theta) \in \mathcal{H}^2(X, \mathcal{O}_X)$ a non-zero multiple of the anti-holomorphic 2-form.

Remark 7.3 The multiplicity of the trivial representation in $\mathcal{H}^q(K3)$ with $g = 3$ can be deduced from Theorem 4.4 to be $h^{(1,1,1,q)}(K3) - 2h^{(1,q)}(K3)$ and by Theorem 5.1 this is 232 for $q = 1$ and zero otherwise. Of this large number of possibilities it is $\mathbb{C}.\alpha$ that is the trivial $Sp(3)$ representation of the corollary and more generally $\alpha \in \mathcal{H}^1(X, \Omega^2_X)$ with $p_g = p_{g-1} = p_{g-2} = 1$ and all other $p_i = 0$ is the highest weight state of $R(1, \ldots, 1, 0, 0, 0)$ for $g \geq 3$.

Remark 7.4 Sawon [21] noted that $W_X$ is not surjective (as is evident from Corollary 7.2.1). Indeed we will now see that for any holomorphic symplectic manifold $X$ it misses at least ‘1/2’ of the pluri-Hodge groups.

Proposition 7.3 The pluri-Hodge groups $\mathcal{H}^q(X, \Omega^2_X)$ are not in the image of $W_X$ for $q + |\underline{p}| \notin 2\mathbb{N}$ or for $q = |\underline{p}| = 1$.

Proof: If there are $q$ trivalent vertices in a graph $D$ these have $3q$ legs. Each univalent vertex can be thought of as a ‘leg’ so there are $|\underline{p}|$ of these. All legs must be joined in pairs so that $3q + |\underline{p}|$ must be even to have such a graph in $\mathcal{B}_g$. When $q = 1$ the AS relation ensures that $D$ is equivalent to zero if there is only one univalent vertex.

Remark 7.5 The chord diagrams, $\mathcal{A}(\Gamma_g)$, of the previous section arise in the topological quantum field theory (TQFT) of Murakami and Ohtsuki. Sawon introduced $W_X$ as a ‘hyper-Kähler weight system’ for the MO invariants. This is the map $W_X \circ \tau^{-1} : \mathcal{A}(\Gamma_g) \to \mathcal{H}_g(X)$. The natural hope being that this would be the correct TQFT approach to the Rozansky-Witten theory [20]. One can show that this expectation is
borne out for $X$ a $K3$ surface. Let $M$ be a rational homology 3-sphere and $K$ a null-homologous knot in $M$. Denote the complement of a tubular neighbourhood of $K$ in $M$ by $M\setminus K$. Rozansky and Witten determine (using path integrals) the associated vector in $\mathcal{H}_1(K3)$; this is (5.68) in [20] and denoted by $|\text{M}\setminus K\rangle$ there. The Murakami invariant $\Lambda_1(M, K)$, is the first MO invariant in a special normalization. For $K$ with zero framing we have $W_{K3} \circ \tau^{-1}(\Lambda_1(M, K)) = |\text{M}\setminus K\rangle$ [25] (though the formulae for both $|\text{M}\setminus K\rangle$ and $\Lambda_1(M, K)$, given in [20] and [14] respectively, require some minor corrections).

Remark 7.6 Physics implies the much more interesting result that the pluri-Hodge groups provide a representation of the mapping class group $\text{MCG}_g$ of a Riemann surface $\Sigma_g$ of genus $g$. J. Murakami [15] gives representations of $\text{MCG}_g$ on $\mathcal{A}^{(1)}(\Gamma_g)$ for $g = 1$ and $g = 2$ with a non-trivial Torelli subgroup action. The map $W_{K3} \circ \tau^{-1}$ induces representations of $\text{MCG}_g$ on $\mathcal{H}^0(K3) \oplus \mathcal{H}^2_+(K3)$ (which mixes the two spaces), with a non-trivial Torelli action [22, 18].

References

[1] S. Abeasis, The $\text{GL}(V)$-Invariant Ideals in $S(S^2V)$, Rend. Mat. (6) 13, (1980) 235-262.
[2] Atiyah, Complex Analytic Connections in Fibre Bundles, Trans. Am. Math. Soc 85 (1957), 181-207.
[3] I. Biswas and M.S. Narasimhan, Hodge Classes of Moduli Spaces of Parabolic Bundles over the General Curve, J. Alg. Geom. 6 (1997), 697-715.
[4] N. Bourbaki, Groupes et Algebras de Lie, Ch. 7 et 8, VIII, Paris, Herman (1975).
[5] A. Fujiki, On the De Rham Cohomology Group of a compact Kähler Symplectic Manifold, Advanced Studies in Pure Mathematics, 10, T. Oda ed. Amsterdam: North Holland (1987).
[6] R. Goodman and N. Wallach, Representations and Invariants of the Classical Groups, Encyclopedia of mathematics and its applications, C.U.P., (1998).
[7] N. Habegger and G. Thompson, The Universal Perturbative Quantum 3-Manifold Invariant, the Rozansky-Witten Invariants and the Generalised Casson Invariant, preprint math.GT/9911049.
[8] N. Hitchin and J. Sawon, Curvature and Characteristic Numbers of Hyperk”ahler Geometry, Duke Math. Jour. 106 (2001) 599-615.
[9] R. Howe, Remarks on Classical Invariant Theory, Trans. Amer. Math. Soc. 313, 539-570 (1989).
[10] S. Kobayashi, Differential Geometry of Complex Vector Bundles, Princeton University Press (1987).
[11] K. Koike and I. Terada, Young-Diagrammatic Methods for the Representation Theory of the Groups $Sp$ and $SO$, Proc. Symp. in Pure Math. 47 (1987) 437-447.
[12] T. Le, J. Murakami and T. Ohtsuki, On a universal perturbative invariant of 3-manifolds, Topology 37 (1998) 539-574.

[13] I.G. Macdonald, Symmetric Functions and Hall Polynomials, Oxford University Press, Oxford (1995).

[14] J. Murakami, The Casson Invariant for a Knot in a 3-Manifold, Geometry and Physics, Eds: Anderson, Dupont Pedersen and Swan, LNPAM 184 (1996) 459-469.

[15] J. Murakami, Representation of Mapping Class Groups via the Universal Perturbative Invariant, Knots ’96, S. Suzuki ed., World Scientific (1997) 573-586.

[16] J. Murakami and T. Ohtsuki, Topological Quantum Field Theory for the Universal Quantum Invariant, Commun. Math. Phys. 188 (1997) 501-520.

[17] M.S. Narasimhan and S. Ramanan, Generalized Prym Varieties as Fixed Points, J. Indian Math Soc. 39 (1975) 1-19.

[18] M.S. Narasimhan and G. Thompson, unpublished.

[19] C. Procesi, Lie Groups: An Approach through Invariants and Representations, Springer Universitext, Springer USA (2007).

[20] L. Rozansky and E. Witten Hyper-Kähler Geometry and Invariants of Three Manifolds, Selecta Math. 3 (1997) 401-458, hep-th9612216.

[21] J. Sawon, Topological Quantum Field Theory and Hyper-Kähler Geometry, Turkish J. Math. 25 (2001) 169-194.

[22] J. Sawon, Actions of the Mapping Class Groups on Hyperkähler Cohomology, unpublished (2000), available at http://www.math.colostate.edu/~sawon/mapping.shtml

[23] J.P. Serre, Algebras des Lie Semi-Simple Complexes, Cor 2 in VII -18, New York, Benjamin (1966).

[24] G. Thompson, A Geometric Interpretation of the $\chi_y$ Genus on Hyper-Kähler Manifolds, Commun. Math. Phys. 212 (2000) 649-652.

[25] G. Thompson, Murakami’s Knot Invariant, the Murakami-Ohtsuki TQFT and the Rozansky-Witten Invariant of a Knot, (2001) unpublished.

[26] V. S. Varadarajan, Lie Groups, Lie Algebras and their Representations, Prentice-Hall, New Jersey (1974).

[27] R.O. Wells, Differential Analysis on Complex Manifolds, Graduate texts in Mathematics 65, Springer-Verlag (1979).

[28] H. Weyl, The Classical Groups, Princeton University Press (1946).