On the kernel of the reciprocity map of simple normal crossing varieties over finite fields

Rin Sugiyama*

Graduate School of Mathematics, Nagoya University,
Furo-cho, Chikusa-ku, Nagoya 464-8602, Japan

Abstract

In this paper, we study the kernel of the reciprocity map of certain simple normal crossing varieties over a finite field and give an example of a simple normal crossing surface whose reciprocity map is not injective for any finite scalar extension.

0 Introduction

The reciprocity map of the unramified class field theory for a proper variety $X$ over a finite field $k$ is a homomorphism of the following form:

$$\rho_X : CH_0(X) \longrightarrow \pi_1^{ab}(X),$$

which sends the class of a closed point $x$ to the Frobenius substitution at $x$. Here $CH_0(X)$ is the Chow group of 0-cycles on $X$ modulo rational equivalence, and $\pi_1^{ab}(X)$ is the abelian étale fundamental group of $X$. If $X$ is normal, $\rho_X$ has dense image [11]. If $X$ is smooth, $\rho_X$ is injective [10]. We also know that there is a projective normal surface $X$ for which $\rho_X$ is not injective [12], and that there is a simple normal crossing surface $X$ over $k$ for which $\rho_X/n$ is not injective but $\rho_{X\otimes E}/n$ is injective for any sufficiently large finite extension $E/k$ and some $n > 1$ [15]. Here a normal crossing variety $X$ over $k$ is a separated scheme of finite type over $k$ which is everywhere étale locally isomorphic to

$$\text{Spec} \left( k[T_0, \ldots, T_d]/(T_0 T_1 \cdots T_r) \right) \quad (0 \leq r \leq d = \dim X).$$

*rin-sugiyama@math.nagoya-u.ac.jp
A normal crossing variety $X$ over $k$ is called simple if any irreducible component of $X$ is smooth over $k$. For any simple normal crossing variety $X$, we have an exact sequence (cf. Section 1.3)

$$H_2(\Gamma_X, \mathbb{Z}/n) \xrightarrow{\epsilon_{X,n}} CH_0(X)/n \xrightarrow{\rho_{X,n}} \pi_1^{ab}(X)/n,$$

where $\Gamma_X$ is the dual graph of $X$ which is a finite simplicial complex (cf. Section 1.1). Hence, by studying on the map $\epsilon_{X,n}$, we get an information about the injectivity of $\rho_{X,n}$. However $\epsilon_{X,n}$ is abstract and difficult to compute directly. In this paper, for a certain simple normal crossing variety $X$ over $k$, we investigate $\epsilon_{X,n}$ and the kernel of the reciprocity map $\rho_{X}$ by using the étale homology theory and the cohomological Hasse principle. We also construct a simple normal crossing surface $Y$ for which the map $\rho_{Y} \otimes F/n : CH_0(Y \otimes_k F)/n \rightarrow \pi_1^{ab}(Y \otimes_k F)/n$ is not injective for any finite extension $F/k$ and some $n > 1$.

Let $Y_0$ a projective smooth and geometrically irreducible variety over $k$ and $D$ be a simple normal crossing divisor on $Y_0$. We put $O := (0 : 1), \infty := (1 : 0) \in \mathbb{P}^1_k$. We then consider the following simple normal crossing variety:

$$Y := (Y_0 \times_k O) \cup (Y_0 \times_k \infty) \cup (D \times_k \mathbb{P}^1) \subset Y_0 \times_k \mathbb{P}^1.$$

The following map plays an important role on the kernel of the reciprocity map $\rho_{Y}$ of $Y$:

$$\delta_Y : H_1(\Gamma_D, \mathbb{Z}) \longrightarrow CH_0(Y).$$

We will construct the map $\delta_Y$ in Section 2.1 and prove the image coincides with the kernel of the reciprocity map $\rho_{Y}$. We consider the norm map $\sigma : H_1(\Gamma_D, \mathbb{Z}) \rightarrow H_1(\Gamma_D, \mathbb{Z})$, and put $G(Y)$ the image of the composite map $\delta_Y \circ \sigma$. The group $G(Y)$ relates to the following group and map (cf. Section 2.2)

$$\Theta_{\ell} := \text{Coker} \left( \bigoplus_j \pi_1^{ab}(\overline{D}_j)^{pro-\ell} \longrightarrow \pi_1^{ab}(\overline{Y_0})^{pro-\ell} \right),$$

$$\alpha^{(\ell)} : H_1(\Gamma_{\overline{D}}, \mathbb{Z}_{\ell}) \longrightarrow \Theta_{\ell}.$$

Here $\overline{Y_0} := Y_0 \otimes_k k^{sep}$ and $\overline{D}_j$ denotes irreducible component of $\overline{D} := D \otimes_k k^{sep}$, and $\pi_1^{ab}(\overline{-})^{pro-\ell}$ denotes the maximal $\ell$-quotient of $\pi_1^{ab}(\overline{-})$.

The main result of this paper is the following theorem:
Theorem 0.1 (Theorem 2.9). Let $\ell$ be an arbitrary prime number.

1. The $\ell$-primary part $G(Y)\{\ell\}$ of $G(Y)$ is a subquotient of $(\Theta_{\ell})_{\text{tors}}$.

2. Assume that
   
   (i) each connected component of $Y^{(2)}$ has a $k$-rational point,
   (ii) $G_k$ acts on $(\Theta_{\ell})_{\text{tors}}$ trivially.

Then $G(Y)\{\ell\}$ is isomorphic to the image of the map $\alpha^{(\ell)}$.

In their paper [12], Matsumi, Sato and Asakura proved a similar assertion of Theorem 0.1 for projective normal surfaces over finite fields. Theorem 0.1 shows an analogy of their result for certain simple normal crossing varieties over finite fields.

The remarkable point in Theorem 0.1 is the map $\alpha^{(\ell)}$ does not vary for finite scalar extensions and the group $\text{Im}(\alpha^{(\ell)})$ relates with $G(Y)$. Therefore we will get a information about the potential injectivity of the reciprocity map $\rho_Y$ by studying on $G(Y)$ and using the map $\alpha^{(\ell)}$. If the assumptions of Theorem 0.1(2) is satisfied and $\text{Im}(\alpha^{(\ell)})$ is not trivial, then for any finite extension $F/k$ the map $\rho_Y \otimes F$ is not injective, i.e. potentially not injective. We will give such a surface in Section 3.1.

This paper is organized as follows: In Section 1, we prepare some lemmas and theorems to prove the main result, and recall a cohomological Hasse principle and étale homology theory. In Section 2, we construct the map $\delta_Y$ and prove Theorem 0.1. In the last of this paper, we give a simple normal crossing surface over a finite field whose reciprocity map $\rho_Y$ is potentially not injective.

0.1 notation

(0.1) For an abelian group $A$ and a positive integer $n$, $A/n$ denotes the cokernel of the map $A \times^n A$. $A_{\text{tors}}$ denotes the torsion subgroup of $A$. $A^{\oplus n}$ denotes the direct sum of $n$ copies of $A$.

(0.2) For a field $k$, $k^\times$ denotes the multiplicative group, $k^{\text{sep}}$ denotes a fixed separable closure, $G_k$ denotes the absolute Galois group $\text{Gal}(k^{\text{sep}}/k)$, $G_k^{ab}$ denotes the maximal abelian quotient group of $G_k$. For a connected scheme $X$, $\pi_1^{ab}(X)$ denotes the abelian étale fundamental group. Further, for a non-connected scheme $V$, $\pi_1^{ab}(V)$ denotes $\bigoplus_i \pi_1^{ab}(V_i)$ where $V_i$ are connected components of $V$. For $k$-scheme $X$, $\pi_1^{\text{geo}}(X)$ denotes $\text{Ker}(\pi_1^{ab}(X) \rightarrow G_k^{ab})$.

(0.3) Let $k$ be a field and $X$ be a $k$-scheme. For a field extension $F/k$, $X \otimes_k F$ denotes $X \times_{\text{Spec}(k)} \text{Spec}(F)$. Especially, for a fixed separable closure $k^{\text{sep}}/k$, $\overline{X}$ denotes $X \times_{\text{Spec}(k)} \text{Spec}(k^{\text{sep}})$. 

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For a scheme $X$ and an integer $q \geq 0$, $X^q$ denotes the set of points on $X$ of codimension $q$. If $X$ is of finite type over a field, $X_q$ denotes the set of points on $X$ which $\dim(\{x\}) = q$. Put $d := \dim X$, $X_q = X^{d-q}$. For a point $x \in X$, $\kappa(x)$ denotes the residue field. For an integral scheme $X$, $k(X)$ denotes the function field. For a scheme $X$ of finite type over a field $k$ and of pure dimension $d$, we define the following group:

$$CH_0(X) := \text{Coker}(\partial_1 : \bigoplus_{x \in X^{d-1}} \kappa(x)^\times \longrightarrow \bigoplus_{x \in X^d} \mathbb{Z})$$

where $\partial_1$ is defined by the discrete valuation.

If $X$ is proper over $k$, there is the degree map $CH_0(X) \xrightarrow{\text{deg}} \mathbb{Z}$, $A_0(X)$ denotes its kernel.

(0.5) $H^r(\_, \_-)$ denotes an étale cohomology group. Especially, $H^r(F, \_-)$ denotes $H^r(\text{Spec}(F), \_-)$ for a field $F$.

Let $X$ be a scheme. For an integer $n > 1$ invertible on $X$, we denote $\mu_n$ the étale sheaf of $n$-th roots of unity. For an integer $i \geq 0$, we denote $\mathbb{Z}/n(i)$ the étale sheaf $\mu_n^\otimes i$.

Let $X$ be a smooth variety over a perfect field of positive characteristic $p$. For a positive integer $n = mp^r \ ((m, p) = 1)$ and a positive integer $i$, we put $\mathbb{Z}/n(i) := \mathbb{Z}/m(i) \bigoplus W_r\Omega^i_{\log \mathcal{X}, [\mathcal{X}], [-i]}$, where $W_r\Omega^i_{\log \mathcal{X}, [\mathcal{X}], [-i]}$ denotes the logarithmic part of the de Rham–Witt complex $W_r\Omega^i_{\log \mathcal{X}, [\mathcal{X}], [-i]}$ on $\mathcal{X}_\text{ét}$ (cf. [6]).

(0.6) We recall here Bloch–Kato conjecture.

**Conjecture 0.2** (Bloch–Kato [2]). Let $k$ be a field and $i$ be non-negative integer. Then for a positive integer $n$ prime to the characteristic $\text{ch}(k)$ of $k$, the following Galois symbol map is bijective

$$h_{k,n}^i : K_i(k)/n \longrightarrow H^i(k, \mathbb{Z}/n(i)).$$

For this conjecture, the following results is known:

**Theorem 0.3.** Conjecture 0.2 is true for the following cases:
(i) $i \leq 2$, and $n$ is arbitrary positive integer prime to $\text{ch}(k)$.
(ii) $n = 2$, and $i$ is arbitrary non-negative integer.

The case $i = 1$ follows from the Kummer theory, and the case $i = 0$ is clear. The case $i = 2$ is proved by Merkur’ev–Suslin [13]. The case (ii) is due to Voevodsky [17].
1 Preliminary

In this section, we prepare some lemmas and theorems for the proof of the main result. We also recall the cohomological Hasse principle and étale homology theory.

Through this paper, $k$ is a finite field and $n$ is a natural number.

1.1 Simple normal crossing varieties over finite fields

We here prove some lemmas about simple normal crossing varieties over finite fields. In case of curves, similar lemmas are proved in [12]. We extend these lemmas to higher dimensional cases. First we define a simple normal crossing variety over a field.

**Definition 1.1.** Let $X$ be a equidimensional scheme of finite type over a field $k$. Then we call $X$ a normal crossing variety over $k$, if $X$ is separated over $k$ and everywhere étale locally isomorphic to

$$\text{Spec}(k[T_0, \cdots , T_d]/(T_0 T_1 \cdots T_r)) \quad (0 \leq r \leq d = \dim X).$$

A normal crossing variety $X$ is called simple if any irreducible component of $X$ is smooth over $k$. For a simple normal crossing variety $X$, we use the following notation: Let $\{X_i\}_{i \in I}$ be the set of irreducible components of $X$. For a positive integer $r$, we define

$$X^{(r)} := \coprod_{\{i_1, i_2, \ldots , i_r\} \subset I} X_{i_1} \times_X X_{i_2} \times \cdots \times X_{i_r}.$$

Now we define a simplicial complex of which homology groups are very important tool in the unramified class field theory for simple normal crossing varieties over finite fields.

**Definition 1.2.** Let $X$ be a $d$-dimensional simple normal crossing variety over $k$. Then we define a simplicial complex $\Gamma_X$ called the dual graph of $X$ as follows:

Let $\{X_i\}_{i \in I}$ be the set of irreducible components of $X$. Fix an ordering on $I$. The set of $r$-simplexes $\mathcal{S}_r$ of $\Gamma_X$ is the set of irreducible components of $X^{(r)}$. We determine the orientation on $r$-simplexes inductively on $r$ by the fixed ordering on $I$ (cf. [12, §3]).

Let $F/k$ be an algebraic extension. We put $Y := X \otimes_k F$. Let $\{Y_j\}_{j \in J}$ be the set of irreducible components of $Y$. Then we define a semi-order on $J$ as follows: $j_1, j_2 \in J$,

$$j_1 < j_2 \iff \phi(j_1) < \phi(j_2),$$
where $\phi : J \rightarrow I$ is the map which sends $j$ to $\phi(j)$ when $Y_j$ lies above $X_{\phi(j)}$. By using this order on $J$, we define the homomorphism of the complexes

$$\sigma_{F/k} : \Gamma_Y \rightarrow \Gamma_X.$$ 

Then the homomorphism $H_a(\Gamma_Y, \mathbb{Z}) \rightarrow H_a(\Gamma_X, \mathbb{Z})$ induced by $\sigma_{F/k}$ is called norm map.

In the rest of this subsection, $X$ denotes a simple normal crossing variety over a finite field $k$ which is proper over $k$.

**Lemma 1.3.** (1) The degree ‘$0$-part’ $A_0(X)$ of $CH_0(X)$ is finite.
(2) Assume that each connected component of $X(2)$ has a $k$-rational point. Then the canonical map $\iota : \bigoplus_{i \in I} A_0(X_i) \rightarrow A_0(X)$ is surjective.

**Proof.** We consider the following commutative diagram with exact rows:

$$
\begin{array}{cccccc}
CH_0(X^{(2)}) & \overset{deg_X^{(2)/k}}{\longrightarrow} & \bigoplus K_0(k) & \longrightarrow & \bigoplus \mathbb{Z}/m_i & \rightarrow 0 \\
0 & \longrightarrow & \bigoplus A_0(X_i) & \longrightarrow & \bigoplus CH_0(X_i) & \overset{deg_X^{(1)/k}}{\longrightarrow} \bigoplus K_0(k) & \longrightarrow & \bigoplus \mathbb{Z}/m_i & \rightarrow 0 \\
0 & \longrightarrow & A_0(X) & \longrightarrow & CH_0(X) & \longrightarrow & K_0(k). \\
\end{array}
$$

(1.1)

Here $m_i = [\Gamma(X^{(2)}_i, \mathcal{O}_{X^{(2)}_i}) : k]$ for connected components $X^{(2)}_i$ of $X(2)$ and $m_i = [\Gamma(X_i, \mathcal{O}_{X_i}) : k]$.

(1) Since $X_i$ is smooth for any $i$, $A_0(X_i)$ is finite by a theorem of Kato-Saito [10].

On the other hand, by the diagram we have a surjective map from the kernel of $\nu$ to the cokernel of $\iota$. Hence we see that the cokernel of the map $\iota$ is finite by and that $A_0(X)$ is finite.

(2) Under the assumption, the map $deg_X^{(2)/k}$ is surjective. The assertion follows from the diagram (1.1). \[\square\]

**Lemma 1.4.** Let $\{W_s\}$ be the set of connected components of $\overline{X}$, and $\{V_j\}$ be the set of irreducible components of $\overline{X}$. Then there is an exact sequence of finite left $G_k$-modules:

$$
\bigoplus \pi_1^{ab}(V_j)/n \longrightarrow \bigoplus \pi_1^{ab}(W_s)/n \longrightarrow H_1(\Gamma_{\overline{X}}, \mathbb{Z}/n) \longrightarrow 0.
$$

(1.2)
Further the following diagram commutes:

\[
\begin{array}{ccc}
\bigoplus \pi_{1}^{ab}(W)/n & \longrightarrow & \pi_{1}^{ab}(X)/n \\
\downarrow^{(1)} & & \downarrow^{(2)} \\
H_{1}(\Gamma, \mathbb{Z}/n) & \longrightarrow & H_{1}(\Gamma, \mathbb{Z}/n).
\end{array}
\] (1.3)

**Proof.** Since we have the canonical isomorphism

\[
\bigoplus \pi_{1}^{ab}(W)/n \simeq \text{Hom} \left( H^{1}(\overline{X}, \mathbb{Z}/n), \mathbb{Q}/\mathbb{Z} \right),
\]

it is sufficient to prove the corresponding statements on étale cohomology groups and ‘cohomology’ of dual graphs. The finiteness of groups in (1.2) follows from the finiteness of étale cohomology groups [1, XVI, 5.2] and the definition of dual graph.

We consider the following exact sequence of étale sheaves on $\overline{X}_{\text{et}}$:

\[
0 \longrightarrow \mathbb{Z}/n_{\overline{X}} \longrightarrow \bigoplus \mathbb{Z}/n_{V_{j}} \longrightarrow \cdots \longrightarrow \bigoplus \mathbb{Z}/n_{\overline{X}^{(d+1)}} \longrightarrow 0.
\]

Here $\overline{X}_{i}^{(d+1)}$ denotes irreducible components of $\overline{X}^{(d+1)}$, and we have omitted the indication of direct image functors of sheaves. From this exact sequence, we obtain a spectral sequence

\[
E_{1}^{p,q} = H^{q}(\overline{X}^{(p)}, \mathbb{Z}/n) \Rightarrow H^{p+q}(\overline{X}, \mathbb{Z}/n).
\]

By computing $E_{2}$-terms, we have an exact sequence

\[
0 \longrightarrow H^{1}(\Gamma, \mathbb{Z}/n) \longrightarrow H^{1}(\overline{X}, \mathbb{Z}/n) \longrightarrow \bigoplus_{j} H^{1}(V_{j}, \mathbb{Z}/n).
\]

The Pontryagin dual of this sequence provides us with the map (1) and proves the exactness of (1.2). The commutativity of (1.3) follows from the following commutative diagram of étale sheaves on $X_{\text{et}}$

\[
\begin{array}{ccc}
0 & \longrightarrow & \mathbb{Z}/n_{X} \\
\downarrow & & \downarrow \\
0 & \longrightarrow & \bigoplus \mathbb{Z}/n_{V_{j}}.
\end{array}
\]

and the fact that the map (2) comes from the upper row.

The following lemma follows from the Hochshild-Serre spectral sequence associated with $\overline{X} \rightarrow X$ (cf. [14]).
Lemma 1.5. The canonical map \( \bigoplus \pi^1_{\text{ab}}(W_s) \longrightarrow \pi^1_{\text{ab}}(X) \) induces an isomorphism

\[
\bigoplus \pi^1_{\text{ab}}(W_s)_{G_k} \cong \pi^\text{geo}_1(X).
\]

We put \( H_1(\Gamma_X, \hat{\mathbb{Z}})_X := \lim_{\leftarrow n} H_1(\Gamma_X, \mathbb{Z}/n) \). We write \( H_1(\Gamma_X, \hat{\mathbb{Z}})_X \) for the image of the norm map \( H_1(\Gamma_X, \hat{\mathbb{Z}}) \rightarrow H_1(\Gamma_X, \hat{\mathbb{Z}})_X \). The following proposition is the ‘geometric’ part of the unramified class field theory for a simple normal crossing variety.

Proposition 1.6. There is an exact sequence

\[
A_0(X) \xrightarrow{\rho^\text{geo}_X} \pi^\text{geo}_1(X) \rightarrow H_1(\Gamma_X, \hat{\mathbb{Z}})_X \rightarrow 0.
\]

Further the following diagram commutes:

\[
\begin{array}{ccc}
\pi^\text{geo}_1(X) & \longrightarrow & \pi^1_{\text{ab}}(X) \\
\downarrow^{(\ast 3)} & & \downarrow^{(\ast 2)} \\
H_1(\Gamma_X, \hat{\mathbb{Z}})_X & \longrightarrow & H_1(\Gamma_X, \hat{\mathbb{Z}}).
\end{array}
\]

Proof. We write \( CH_0(X)^\wedge \) for \( \lim_{\leftarrow n} CH_0(X)/n, K_0(k)^\wedge \) for \( \lim_{\leftarrow n} K_0(k)/n \). We consider the following commutative diagram with exact rows:

\[
\begin{array}{cccccccccc}
0 & \longrightarrow & A_0(X) & \longrightarrow & CH_0(X)^\wedge & \longrightarrow & K_0(k)^\wedge & \longrightarrow & 0 \\
& & & \downarrow{\rho^\text{geo}_X} & & & \downarrow{\rho^\wedge_X} & & \\
0 & \longrightarrow & \pi^\text{geo}_1(X) & \longrightarrow & \pi^1_{\text{ab}}(X) & \longrightarrow & G_k & \longrightarrow & 0 \\
& & & & & \downarrow^{(\ast 2)} & & & \\
& & & \downarrow{\rho^\wedge_X} & & & & & \\
& & & H_1(\Gamma_X, \hat{\mathbb{Z}})_X & & & & & \\
\end{array}
\]

where the map \( \rho^\wedge_X \) is induced by the reciprocity map \( \rho_X \) of the unramified class field theory for \( X \). Here we used the finiteness of \( A_0(X) \) (cf. Lemma 1.3), and the fact that the pro-finite completion of an exact sequence of finitely generated abelian groups is exact. Since the map \( \rho^\wedge_k \) is injective (in fact bijective), the cokernel of the map \( \rho^\text{geo}_X \) is isomorphic to the image of \( \pi^\text{geo}_1(X) \) in \( H_1(\Gamma_X, \hat{\mathbb{Z}})_X \). Therefore Coker(\( \rho^\text{geo}_X \)) coincides with \( H_1(\Gamma_X, \hat{\mathbb{Z}})_X \) from Lemma 1.3 \( \square \)

Remark 1.7. If \( X \) is a curve, the map \( \rho^\text{geo}_X \) in the above proposition is injective by Kato–Saito [10].
1.2 Bloch–Ogus–Kato complex

In this subsection, we recall the theorem called a cohomological Hasse principle. First we recall Bloch–Ogus–Kato complex.

For $X$ be an excellent scheme and integers $r, s, n > 0$, Kato define a homological complex $C^{r,s}(X,n)$ (cf. [9, §1]):

$$
\cdots \to \bigoplus_{x \in X_i} H^{r+i}(\kappa(x), \mathbb{Z}/n(s+i)) \to \bigoplus_{x \in X_{i-1}} H^{r+i-1}(\kappa(x), \mathbb{Z}/n(s+i-1)) \to \cdots \to \bigoplus_{x \in X_1} H^{r+1}(\kappa(x), \mathbb{Z}/n(s+1)) \to \bigoplus_{x \in X_0} H^r(\kappa(x), \mathbb{Z}/n(s)) \to \cdots.
$$

The degree of the term $\bigoplus_{x \in X_i}$ is $i$. The following canonical map for a fraction field $K$ of a discrete valuation ring and its residue field $F$ plays an important role in the definition of the boundary map for the above complex

$$
H^i(K, \mathbb{Z}/n(j)) \longrightarrow H^{i-1}(F, \mathbb{Z}/n(j-1)).
$$

Remark 1.8. In the definition of $C^{r,s}(X,n)$, we assume that if $r = s + 1$, for any prime divisor $p$ of $n$ and any $x \in X_0$ such that $\text{ch}(\kappa(x)) = p$, we have $[\kappa(x) : \kappa(x)^p] \leq p^s$. The assumption is satisfied in the case we consider in this paper.

Definition 1.9. Kato homology of $X$ with coefficient $\mathbb{Z}/n$ defined by

$$
H^{r,s}_i(X,n) := H_i(C^{r,s}(X,n)).
$$

Kato conjectured the following for varieties over $k$:

Conjecture 1.10 (Kato [9]). Let $X$ be a connected projective smooth variety over $k$. Put $H^K_i(X,n) := H^{1,0}_i(X,n)$. Then we have

$$
H^K_i(X,n) \simeq \begin{cases} 
0 & \text{if } i \neq 0, \\
\mathbb{Z}/n & \text{if } i = 0.
\end{cases}
$$

If $\dim X = 1$, this is classical. This is an analogy of the following exact sequence for Brauer group of a number field $K$:

$$
0 \longrightarrow \text{Br}(K) \longrightarrow \bigoplus_v \text{Br}(K_v) \longrightarrow \mathbb{Q}/\mathbb{Z} \longrightarrow 0.
$$

The following theorem is called the cohomological Hasse principle and shows Conjecture 1.10 is true for $\dim X = 2$. Colliot-Thélène–Sansuc–Soulé [5] (in the prime to $\text{ch}(k)$ case) and Kato [9] (in the $\text{ch}(k)$-primary case) prove the theorem.
Theorem 1.11 (Colliot-Thélène–Sansuc–Soulé, Kato). Let $X$ be a proper smooth irreducible surface over $k$ and $n$ be a natural number. Let $\eta$ be the generic point of $X$. Then Bloch–Ogus–Kato complex

$$0 \to H^3(k(X), \mathbb{Z}/n(2)) \to \bigoplus_{x \in X_1} H^2(\kappa(x), \mathbb{Z}/n(1)) \to \bigoplus_{x \in X_0} H^1(\kappa(x), \mathbb{Z}/n)$$

is exact and the cokernel of the last map isomorphic to $\mathbb{Z}/n$.

We know the following theorem on the Kato Conjecture replaced coefficient $\mathbb{Z}/n$ by $\mathbb{Q}/\ell /\mathbb{Z}/\ell$ for a prime number $\ell$, which is proved by Colliot-Thélène [4] in the case $\ell$ prime to $ch(k)$, and by Suwa [16] in the case $\ell$ is a power of $ch(k)$.

Theorem 1.12 (Colliot-Thélène, Suwa). For any prime number $\ell$, $\mathbb{Q}/\ell /\mathbb{Z}/\ell$-coefficient Kato conjecture holds true for degree $i \leq 3$.

Bloch–Kato conjecture (Conjecture 0.2) relates to $\mathbb{Z}/\ell^n$-coefficient Kato conjecture and $\mathbb{Q}/\ell /\mathbb{Z}/\ell$-coefficient Kato conjecture as follows: If $\mathbb{Q}/\ell /\mathbb{Z}/\ell$-coefficient Kato conjecture holds true for degree $i \leq m$ and Bloch–Kato conjecture holds true for degree $i \leq m$, then $\mathbb{Z}/\ell^n$-coefficient Kato conjecture holds true for degree $i \leq m$.

Jannsen and Saito [7] proved the following theorem which shows a relation between homology groups of a dual graph and Kato homology groups.

Theorem 1.13 (Jannsen–Saito). Let $X$ be a $d$ dimensional simple normal crossing variety over $k$. Let $\ell$ be a prime number and $\nu$ be a natural number. Then there is a canonical homomorphism

$$\gamma_a : H^K_a(X, \ell^\nu) \to H_a(\Gamma_X, \mathbb{Z}/\ell^\nu).$$

Further, if Conjecture 1.10 holds true for degree $\leq m$, then $\gamma_a$ is isomorphism for any $a \leq m$.

1.3 Étale homology theory

We here define étale homology and compute a spectral sequence associated with that homology.

For a separated scheme $X$ of finite type over $k$ and a natural number $n$, we define the étale homology with coefficient $\mathbb{Z}/n$ to

$$H_i(X, \mathbb{Z}/n) := \text{Hom}(H^i_c(X, \mathbb{Z}/n), \mathbb{Q}/\mathbb{Z}).$$
Here $H^i_c(-,-)$ denotes an étale cohomology group with compact support. This functor $H^*_s(-,\mathbb{Z}/n)$ forms a homology theory on the category of separated schemes of finite type over $k$ and proper $k$-morphisms. By Bloch–Ogus [3], we have a niveau spectral sequence

$$E^1_{p,q} = \bigoplus_{x \in X_p} H_{p+q}(x,\mathbb{Z}/n) \implies H_{p+q}(X,\mathbb{Z}/n).$$

(1.4)

Here for $x \in X$, we define

$$H_i(x,\mathbb{Z}/n) := \lim_{\to} H_i(U,\mathbb{Z}/n),$$

and the limit is taken over all nonempty $U$ which is open in the closure $\overline{\{x\}}$ of $x$ in $X$.

This homology theory satisfies the Poincaré duality:

**Lemma 1.14.** Let $V$ be an connected smooth variety over $k$ with dim $= d$. Let $n$ be a natural number. Then we have the following isomorphism:

$$H_m(V,\mathbb{Z}/n) \simeq H^{2d+1-m}(V,\mathbb{Z}/n(d)).$$

**Proof.** If $n$ prime to $ch(k)$, this isomorphism follows from the Poincaré duality [14] for $V$ and the duality for Galois cohomology of $k$.

If $n$ is a power of $ch(k)$, this is proved in [8].

The following proposition follows from Lemma 1.14 and the above spectral sequence (1.4) (cf. [12, Prop. 2.4]).

**Proposition 1.15.** For a variety $X$ over $k$ and any $n \in \mathbb{N}$, we have a spectral sequence

$$E^1_{p,q}(X,n) := \bigoplus_{x \in X_p} H^{p+q+1}(\kappa(x),\mathbb{Z}/n(p)) \implies H_{p+q}(X,\mathbb{Z}/n).$$

(1.5)

If $p < 0$ or $q < 0$, then $E^1_{p,q} = 0$. Further, $E^1$-terms are Bloch–Ogus–Kato complexes.

$E^1_{p,q}(X)$ denotes $E^1_{p,q}(X,n)$ below.

**Proposition 1.16.** Let $X$ be a simple normal crossing variety over $k$ which is proper over $k$. For any integer $n > 1$, there is an exact sequence

$$H_2(\Gamma_X,\mathbb{Z}/n) \xrightarrow{\epsilon_{X,n}} CH_0(X)/n \xrightarrow{\rho_{X}/n} \pi^{ab}_1(X)/n \rightarrow H_1(\Gamma_X,\mathbb{Z}/n) \rightarrow 0.$$
Proof. From the spectral sequence (1.5) for $X$, we obtain the following exact sequence

$$E^2_{2,0}(X) \longrightarrow E^2_{0,1}(X) \xrightarrow{d_{0,1}} H_1(X, \mathbb{Z}/n) \longrightarrow E^2_{1,0}(X) \longrightarrow 0.$$ 

By Theorem 0.3, 1.11 and 1.12, Conjecture 1.10 holds true for degree $i \leq 2$. Therefore, from Theorem 1.13, we obtain isomorphisms

$$E^2_{2,0}(X) = H^K_2(X,n) \simeq H_2(\Gamma_X, \mathbb{Z}/n),$$

$$E^2_{1,0}(X) = H^K_1(X,n) \simeq H_1(\Gamma_X, \mathbb{Z}/n).$$

On the other hand, there is a commutative diagram

$$
\begin{array}{ccc}
E^2_{0,1}(X) & \xrightarrow{d_{0,1}} & H_1(X, \mathbb{Z}/n) \\
\downarrow \simeq & & \downarrow \simeq \\
CH_0(X)/n & \xrightarrow{\rho_X/n} & \pi^{ab}_1(X)/n.
\end{array}
$$

Here the vertical isomorphisms follow from the definition of $CH_0(X)$ and $H^1(X, \mathbb{Z}/n) = \text{Hom}(\pi^{ab}_1(X), \mathbb{Z}/n)$. Hence the proposition follows.

\[ \square \]

2 Main result

In this section, we construct the map $\delta_Y$ of introduction and prove Theorem 0.1. Let $Y_0$ be a projective smooth and geometrically irreducible variety over $k$ and let $D$ be a simple normal crossing divisor on $Y_0$. We then consider the following simple normal crossing variety

$$Y := (Y_0 \times_k O) \cup (Y_0 \times_k \infty) \cup (D \times_k \mathbb{P}^1_k) \subset Y_0 \times_k \mathbb{P}^1_k.$$ 

Here $O := (0 : 1)$, $\infty := (1 : 0) \in \mathbb{P}^1_k$.

2.1 Construction of $\delta_Y$

First, we prove the following proposition on an important homomorphism in study on $\text{Ker}(\rho_Y)$.

**Proposition 2.1.** There exists a homomorphism

$$\delta_Y : H_1(\Gamma_D, \mathbb{Z}) \longrightarrow CH_0(Y)$$

whose image coincides with $\text{Ker}(\rho_Y)$.
We prove this proposition admitting the following lemma.

**Lemma 2.2.** There exists an exact sequence:

\[ H_1(\Gamma_D, \mathbb{Z}/n) \xrightarrow{\delta} CH_0(Y)/n \xrightarrow{\rho_Y/n} \pi_1^{ab}(Y)/n. \]

**Proof of Proposition 2.1.** We consider the following diagram:

\[
\begin{array}{ccc}
H_1(\Gamma_D, \mathbb{Z}) & \xrightarrow{\rho_Y} & \pi_1^{ab}(Y) \\
\downarrow & & \downarrow \\
H_1(\Gamma_D, \hat{\mathbb{Z}}) & \xrightarrow{\delta} & \pi_1^{ab}(Y)
\end{array}
\]

where the bottom sequence is obtain by the projective limit of Lemma 2.2 and exact. Since \( A_0(Y) \) is finite (cf. Lemma 1.3(1)) and \( \text{Ker}(\rho_Y) \subset A_0(Y) \), we have \( \text{Ker}(\rho_Y) \simeq \text{Ker}(\rho_Y^\wedge) \). We define \( \delta_Y \) by the composite

\[
H_1(\Gamma_D, \mathbb{Z}) \xrightarrow{\delta} \text{Ker}(\rho_Y^\wedge) \xrightarrow{\sim} \text{Ker}(\rho_Y) \xrightarrow{\rho_Y} CH_0(Y).
\]

Then we have \( \text{Im}(\delta_Y) = \text{Ker}(\rho_Y) \), since the group \( \text{Im}(\delta) = \text{Ker}(\rho_Y^\wedge) \) is finite and the map \( H_1(\Gamma_D, \mathbb{Z}) \to H_1(\Gamma_D, \hat{\mathbb{Z}}) \) has dense image with respect to the pro-finite topology.

We consider the norm map \( \sigma : H_1(\Gamma_D, \mathbb{Z}) \to H_1(\Gamma_D, \mathbb{Z}) \) and put

\[ G(Y) := \text{Im}(\delta_Y \circ \sigma : H_1(\Gamma_D, \mathbb{Z}) \to CH_0(Y)). \]

The group \( G(Y) \) is a subgroup of \( \text{Ker}(\rho_Y) \) from Proposition 2.1. In the next subsection, we describe \( G(Y) \) with the map \( \alpha \) defined below.

We prove Lemma 2.2 admitting the following sublemma.

**Sublemma 2.3.** There are two exact sequences:

1. \( CH_0(D)/n \to CH_0(Y_0)/n^{e_2} \to CH_0(Y)/n \to 0 \),
2. \( \pi_1^{ab}(D)/n \to \pi_1^{ab}(Y_0)/n^{e_2} \to \pi_1^{ab}(Y)/n \).

**Proof of Lemma 2.2.** From Proposition 1.16 and Sublemma 2.3, we obtain the following commutative diagram with exact rows:

\[
\begin{array}{ccc}
CH_0(D)/n & \xrightarrow{\sim} & CH_0(Y_0)/n^{e_2} \\
\downarrow & & \downarrow \rho_Y/n \\
\pi_1^{ab}(D)/n & \to & \pi_1^{ab}(Y_0)/n^{e_2} \\
\downarrow & & \downarrow \rho_Y/n \\
H_1(\Gamma_D, \mathbb{Z}/n)
\end{array}
\]

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By this diagram, we have a homomorphism
\[ \delta_n : H_1(\Gamma_D, \mathbb{Z}/n) \longrightarrow CH_0(Y)/n \]
whose image coincides with Ker(\(\rho_Y/n\)). \qed

**Remark 2.4.** By Proposition 1.16, we have an exact sequence
\[ H_2(\Gamma_Y, \mathbb{Z}/n) \xrightarrow{\epsilon_{Y,n}} CH_0(Y)/n \xrightarrow{\rho_Y/n} \pi_1^{ab}(Y)/n. \]
On the other hand, from the structure of \(Y\), we obtain a suspension isomorphism
\[ H_2(\Gamma_Y, \mathbb{Z}/n) \simeq H_1(\Gamma_D, \mathbb{Z}/n). \]
Therefore we have the map
\[ H_1(\Gamma_D, \mathbb{Z}/n) \simeq H_2(\Gamma_Y, \mathbb{Z}/n) \longrightarrow CH_0(Y)/n \]
whose image coincides with Ker(\(\rho_Y/n\)). This map coincides with the map \(\delta_n\) of Lemma 2.2.

To prove Sublemma 2.3, we consider the following variety and two closed subschemes:
\[
S := (Y_0 \times_k O) \cup (Y_0 \times_k \infty) \cup (D \times_k \mathbb{P}^1),
\]
\[
Z := (Y_0 \times_k O) \cup (Y_0 \times_k \infty) \subset Y,
\]
\[
Z' := (Y_0 \times_k O) \cup (Y_0 \times_k \infty) \cup (D \times_k \{O, \infty\}) \subset S.
\]
Then we have
\[ Y \setminus Z \cong S \setminus Z' \cong D \times \mathbb{G}_m. \quad (2.2) \]

**Proof of Sublemma 2.3.** From Proposition 1.15 and (2.2), we obtain two exact sequences of complexes for fixed \(q\):
\[
0 \longrightarrow E_{*,q}^1(Z) \longrightarrow E_{*,q}^1(Y) \longrightarrow E_{*,q}^1(U) \longrightarrow 0, \quad (2.3)
\]
\[
0 \longrightarrow E_{*,q}^1(Z') \longrightarrow E_{*,q}^1(S) \longrightarrow E_{*,q}^1(U) \longrightarrow 0. \quad (2.4)
\]
From the above exact sequences \(2.3\) and \(2.4\) for \(q = 1\), we obtain the following commutative diagram with exact rows:

\[
\begin{array}{ccc}
E_1^{21}(D \times \mathbb{G}_m) & \xrightarrow{\beta} & CH_0(Z')/n \\
\downarrow & & \downarrow \\
E_1^{21}(D \times \mathbb{G}_m) & \xrightarrow{\beta} & CH_0(Z)/n \\
\end{array}
\]

(2.5)

Now we have \(CH_0(D \times \mathbb{G}_m) = 0\). We compute the kernel of the map \(\beta\). Since there are the following isomorphisms

\[
\begin{align*}
CH_0(Z') & \simeq CH_0(Y_0)^{\oplus 2} \oplus CH_0(D)^{\oplus 2}, \\
CH_0(S) & \simeq CH_0(Y_0)^{\oplus 2} \oplus CH_0(D \times \mathbb{P}^1), \\
CH_0(D \times \mathbb{P}^1) & \simeq CH_0(D),
\end{align*}
\]

we have

\[
\ker(\beta) = \{(0, 0, c, -c) | c \in CH_0(D)/n\}
\]

\(\simeq CH_0(D)/n\).

Hence, by the diagram \(2.5\) and \(CH_0(Z)/n \simeq CH_0(Y_0)/n^{\oplus 2}\), we have an exact sequence

\[
CH_0(D)/n \longrightarrow CH_0(Y_0)/n^{\oplus 2} \longrightarrow CH_0(Y)/n \longrightarrow 0.
\]

(2) Considering the localization sequence of étale homology groups, we obtain the following commutative diagram

\[
\begin{array}{ccc}
H_2(D \times \mathbb{G}_m, \mathbb{Z}/n) & \xrightarrow{\beta'} & \pi_1^{ab}(Z')/n \\
\downarrow & & \downarrow \\
H_2(D \times \mathbb{G}_m, \mathbb{Z}/n) & \xrightarrow{\beta'} & \pi_1^{ab}(Z)/n \\
\end{array}
\]

(2.6)

Similarly to (1), we have \(\ker(\beta') \simeq \pi_1^{ab}(D)/n\). Hence, by the diagram \(2.6\) and \(\pi_1^{ab}(Z)/n \simeq \pi_1^{ab}(Y_0)/n^{\oplus 2}\), we have an exact sequence

\[
\pi_1^{ab}(D)/n \longrightarrow \pi_1^{ab}(Y_0)/n^{\oplus 2} \longrightarrow \pi_1^{ab}(Y)/n.
\]

We use the following lemma to define the map \(\delta_Y^{geo}\) below.
Lemma 2.5. There are two exact sequences:

1. \( A_0(D) \to A_0(Y_0)^{\oplus 2} \to A_0(Y) \),
2. \( \pi_1^{geo}(D) \to \pi_1^{geo}(Y_0)^{\oplus 2} \to \pi_1^{geo}(Y) \).

Proof. From the projective limit of Sublemma 2.3, we have the following commutative diagram with exact rows:

\[
\begin{array}{ccccccc}
CH_0(D)^\wedge & & (CH_0(Y_0)^\wedge)^{\oplus 2} & & CH_0(Y)^\wedge & & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
0 & \to & K_0(k)^\wedge & \to & (K_0(k)^\wedge)^{\oplus 2} & \to & K_0(k)^\wedge & \to 0.
\end{array}
\]

Applying the snake lemma to this diagram, we obtain an exact sequence

\[ A_0(D) \to A_0(Y_0)^{\oplus 2} \to A_0(Y). \]

Similarly to (1), the sequence (2) is obtained by applying the snake lemma to the following commutative diagram with exact rows (cf. Sublemma 2.3):

\[
\begin{array}{ccccccc}
\pi_1^{ab}(D) & & \pi_1^{ab}(Y_0)^{\oplus 2} & & \pi_1^{ab}(Y) \\
\downarrow & & \downarrow & & \downarrow & & \\
0 & \to & G_k^{ab} & \to & G_k^{ab,\oplus 2} & \to & G_k^{ab} & \to 0.
\end{array}
\]

Now we define the map

\[ \delta_Y^{geo} : H_1(\Gamma_D, \hat{\mathbb{Z}})_{tr} \to CH_0(Y) \]

to be that induced by the following commutative diagram with exact rows (cf. Lemma 2.5, Proposition 1.6):

\[
\begin{array}{ccccccc}
A_0(D) & & A_0(Y_0)^{\oplus 2} & & A_0(Y) \\
\downarrow & & \downarrow & & \downarrow & & \\
\pi_1^{geo}(D) & & \pi_1^{geo}(Y_0)^{\oplus 2} & & \pi_1^{geo}(Y) \\
\downarrow & & \downarrow \cong & & \downarrow & & \\
H_1(\Gamma_D, \hat{\mathbb{Z}})_{tr}.
\end{array}
\]

Here the bijectivity of the middle vertical map is due to Kato–Saito [10]. The following proposition plays a key role in the proof of the main result.
Proposition 2.6. The following diagram commutes:

$$
\begin{array}{ccc}
H_1(\Gamma_D, \mathbb{Z}) & \rightarrow & H_1(\Gamma_D, \hat{\mathbb{Z}}) \\
\downarrow \sigma & & \downarrow \delta_{\gamma}^\text{geo} \\
H_1(\Gamma_D, \mathbb{Z}) & \rightarrow & \delta_Y CH_0(Y).
\end{array}
$$

(2.7)

Proof. The commutativity of this diagram follows from the constructions of $\delta_Y, \delta_{\gamma}^\text{geo}$ and the commutativity of the following diagrams (cf. Lemma 1.4, Proposition 1.6)

$$
\begin{array}{ccc}
\pi_{1}^\text{geo}(D) & \rightarrow & \pi_{1}^\text{ab}(D) \\
\downarrow & & \downarrow \\
H_1(\Gamma_D, \hat{\mathbb{Z}}) & \rightarrow & H_1(\Gamma_D, \hat{\mathbb{Z}}), \\
\pi_{1}^\text{geo}(Y) & \rightarrow & \pi_{1}^\text{ab}(Y), \\
\pi_{1}^\text{ab}(D) & \rightarrow & \pi_{1}^\text{ab}(D) \\
\downarrow & & \downarrow \\
H_1(\Gamma_D, \hat{\mathbb{Z}}) & \rightarrow & H_1(\Gamma_D, \hat{\mathbb{Z}}).
\end{array}
$$

We use the following map in the proof of the main result.

Lemma 2.7. There exists an injective homomorphism

$$
\psi : \text{Coker} \left( A_0(D) \rightarrow A_0(Y_0) \right) \rightarrow A_0(Y).
$$

Proof. We consider the following commutative diagram (cf. Lemma 2.5)

$$
\begin{array}{ccc}
A_0(D) & \rightarrow & A_0(Y_0) \\
\downarrow & & \downarrow \xi \\
A_0(D) & \rightarrow & A_0(Y_0)^{\oplus 2} \rightarrow A_0(Y),
\end{array}
$$

where $\xi$ maps an element $a$ of $A_0(Y_0)$ to an element $(a, -a)$ of $A_0(Y_0)^{\oplus 2}$. From this diagram, we obtain an injective homomorphism

$$
\psi : \text{Coker} \left( A_0(D) \rightarrow A_0(Y_0) \right) \rightarrow A_0(Y).
$$

\qed
2.2 Proof of Theorem 0.1

Let $\ell$ be a prime number. We write $\Theta_{\ell}$ for the $G_k$-module

$$\text{Coker}(\pi_{ab}^\ell(D^{(1)})_{\text{pro-}\ell} \longrightarrow \pi_{ab}^\ell(Y_0)_{\text{pro-}\ell}).$$

We consider the following $G_k$-equivariant homomorphism

$$\alpha^{(\ell)} : H_1(\Gamma_D, \mathbb{Z}_{\ell}) \longrightarrow \Theta_{\ell}$$

induced by the following commutative diagram with exact rows

$$\begin{array}{cccccc}
\pi_{ab}^\ell(D^{(1)})_{\text{pro-}\ell} & \longrightarrow & \pi_{ab}^\ell(D)_{\text{pro-}\ell} & \longrightarrow & H_1(\Gamma_D, \mathbb{Z}_{\ell}) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \pi_{ab}^\ell(Y_0)_{\text{pro-}\ell} & \longrightarrow & \pi_{ab}^\ell(Y_0)_{\text{pro-}\ell} & \longrightarrow & 0.
\end{array}$$

By the weight argument, Matsumi, Sato and Asakura [12, Thm. 3.3] proved the following:

**Lemma 2.8 (Matsumi–Sato–Asakura).** Let $\ell$ be an arbitrary prime number.

1. The image of $\alpha^{(\ell)}$ is contained in $(\Theta_{\ell})_{\text{tors}}$.

2. Assume that $G_k$ acts on $(\Theta_{\ell})_{\text{tors}}$ trivially. Then the composite of canonical maps

$$((\Theta_{\ell})_{\text{tors}})_{G_k} \longrightarrow (\Theta_{\ell})_{G_k}$$

is injective.

**Theorem 2.9 (Theorem 0.1).** Let $\ell$ be an arbitrary prime number.

1. The $\ell$-primary part $G(Y)\{\ell\}$ of $G(Y)$ is a subquotient of $(\Theta_{\ell})_{\text{tors}}$.

2. Assume that
   
   (i) each connected component of $Y^{(2)}$ has $k$-rational point,
   
   (ii) $G_k$ acts on $(\Theta_{\ell})_{\text{tors}}$ trivially.

   Then $G(Y)\{\ell\}$ is isomorphic to the image of the map $\alpha^{(\ell)}$.

**Proof.** (1) For a finite abelian group $M$, we write $M^{(\ell)}$ for the maximal $\ell$-quotient. Since $A_0(Y)$ is finite (cf. Lemma 1.3(1)), the $\ell$-primary part $G(Y)\{\ell\}$ is identified with $G(Y)^{(\ell)}$, and hence identified with the image of the composite map

$$\left(\delta_Y \circ \sigma\right)^{(\ell)} : H_1(\Gamma_D, \mathbb{Z}) \longrightarrow A_0(Y) \longrightarrow A_0(Y)^{(\ell)}.$$
From the commutativity of the diagram (2.7) in Proposition 2.6 and the constructions of $\delta_Y^{geo}$ and $\alpha^{(\ell)}$, the map $(\delta_Y \circ \sigma)^{(\ell)}$ is decomposed as follows:

$$H_1(\Gamma_{D}, \mathbb{Z}) \xrightarrow{\alpha^{(\ell)}} \Theta_{\ell} \xrightarrow{(\Theta_{\ell})_{G_k}} \eta^{(\ell)} \xrightarrow{\psi^{(\ell)}} A_0(Y)^{(\ell)},$$

where $\eta^{(\ell)}$ denotes the following composite map:

$$(\Theta_{\ell})_{G_k} \simeq \text{Coker}(\pi_1^{geo}(D^{(1)})^{pro-\ell} \to \pi_1^{geo}(Y_0)^{pro-\ell})$$

$\simeq \text{Coker}(A_0(D^{(1)})^{(\ell)} \to A_0(Y_0)^{(\ell)})$

$\to \text{Coker}(A_0(D)^{(\ell)} \to A_0(Y_0)^{(\ell)}) \xrightarrow{\psi^{(\ell)}} A_0(Y)^{(\ell)}.$

From Lemma 2.8(1), the image of $\alpha^{(\ell)}$ is contained in $(\Theta_{\ell})_{tors}$. Thus, $G(Y)^{\{\ell\}}$ is a subquotient of $(\Theta_{\ell})_{tors}$.

(2) It suffices to show that the composite of canonical maps

$$\text{Im}(\alpha^{(\ell)}) \xrightarrow{f_1} ((\Theta_{\ell})_{tors})_{G_k} \xrightarrow{f_2} (\Theta_{\ell})_{G_k} \xrightarrow{\eta^{(\ell)}} A_0(Y)^{(\ell)}$$

is injective under the assumptions. Here the first map is injective by (1). From Lemma 2.8(2), the composite map $f_2 \circ f_1$ is injective. Under the assumption (i), $\eta^{(\ell)}$ coincides with the map $\psi^{(\ell)}$ from Lemma 1.3(2), and hence is injective from Lemma 2.7.

3. Examples

We here give two examples of simple normal crossing surfaces over $k$. One is a surface $Y_1$ for which $H_2(\Gamma_{Y_1 \otimes F}, \mathbb{Z})$ is not trivial but $\rho_{Y_1 \otimes F}$ is injective for any finite extension $F/k$. Another is a surface $Y_2$ for which $\rho_{Y_2 \otimes F}$ is not injective for any finite extension $F/k$. We also see that the map $\rho_{Y_2 \otimes F}/n$ is not injective for some positive integer $n$. On the other hand, there is a simple normal crossing surface $Y_3$ for which $\rho_{Y_3}$ is not injective but $\rho_{Y_3 \otimes E}$ is injective for any sufficiently large finite extension $E/k$ (cf. [15]).

**Example 3.1.** Let $Y_0 := \mathbb{P}^1 \times_k \mathbb{P}^1$ and $D$ be the simple normal crossing divisor on $Y_0$ defining the following polynomial:

$$D : (x^2 - y^2)(z^2 - w^2) = 0 \subset Y_0.$$ 

Then, for any finite extension $F/k$, we have

$$H_1(\Gamma_{D \otimes F}, \mathbb{Z}) \simeq \mathbb{Z}.$$
On the other hand, since \( \pi_{1}^{ab}(Y_{0}) = 0 \), we have \( \Theta = 0 \). Hence, from Theorem 2.9 for the following simple normal crossing surface \( Y_{1} := (Y_{0} \times_{k} O) \cup (Y_{0} \times_{k} \infty) \cup (D \times_{k} \mathbb{P}^{1}) \), the map \( \rho_{Y_{1}} \) is injective for any finite scalar extension.

**Example 3.2** (cf. [12]). Let \( n > 1 \) be a natural number and \( (n, 6 \cdot ch(k)) = 1 \). We assume \( k \) is a finite field containing a primitive \( n \)-th root of unity \( \zeta \). We consider a Fermat surface

\[
V : T_{0}^{n} + T_{1}^{n} + T_{2}^{n} + T_{3}^{n} = 0 \subset \mathbb{P}^{3}_{k},
\]

and a free action on \( V \)

\[
\tau : (T_{0} : T_{1} : T_{2} : T_{3}) \mapsto (T_{0} : \zeta T_{1} : \zeta^{2} T_{2} : \zeta^{3} T_{3}).
\]

Then we have a projective smooth surface \( Y_{0} := V/\langle \tau \rangle \).

Now we consider \( 2n \) lines on \( V : j = 1, \ldots, n-1 \)

\[
L_{1} : T_{0} + T_{1} = T_{2} + T_{3} = 0,
L_{2} : T_{0} + T_{1} = T_{2} + \zeta T_{3} = 0,
L_{1}^{\tau_{j}} : T_{0} + \zeta^{j} T_{1} = T_{2} + \zeta^{j} T_{3} = 0,
L_{2}^{\tau_{j}} : T_{0} + \zeta^{j} T_{1} = T_{2} + \zeta^{j+1} T_{3} = 0.
\]

Then the following divisor \( L \) on \( V \) is a connected simple normal crossing divisor:

\[
L = L_{1} \cup L_{2} \cup L_{1}^{\tau} \cup \cdots \cup L_{1}^{\tau_{n-1}} \cup L_{2}^{\tau_{n-1}}.
\]

This divisor \( L \) is stable under the action of \( \langle \tau \rangle \).

Let \( \varphi : V \to Y_{0} \) and \( C_{i} = \varphi_{*}(L_{i}) \) (\( i = 1, 2 \)). Since \( C_{i} \) is isomorphic to \( L_{i} \), \( C_{i} \) is a nonsingular rational curve on \( Y_{0} \) and \( D = C_{1} \cup C_{2} \) is a simple normal crossing divisor on \( Y_{0} \). Moreover every singular points of \( D \) are \( k \)-rational.

Since \( V \) is a hypersurface in \( \mathbb{P}^{3}_{k_{sep}} \), \( \pi_{1}^{ab}(V) = 0 \) (cf. [15] Lemma 3.5). Hence we have

\[
\pi_{1}^{ab}(Y_{0}) \simeq \tau \simeq \mathbb{Z}/n.
\]

Since \( C_{i} \) is rational curves, \( \pi_{1}^{ab}(C_{i}) = 0 \). Therefore, \( G_{k} \) acts on \( \Theta_{\text{tors}} \) trivially.

On the other hand, because \( \varphi \) induces the completely splitting covering \( L \to D \),

\[
\alpha : H_{1}(\Gamma, \mathbb{Z}) \to \pi_{1}^{ab}(Y_{0})
\]

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is surjective.

We then put
\[ Y_2 := (Y_0 \times_k O) \cup (Y_0 \times_k \infty) \cup (D \times_k \mathbb{P}^1). \]

From Theorem 2.9, \( \text{Ker}(\rho_{Y_2}) \cong \mathbb{Z}/n \). Thus \( \rho_{Y_2} \) is not injective. Moreover, we have \( \text{Ker}(\rho_{Y_2 \otimes F}) \cong \mathbb{Z}/n \) for any finite extension \( F/k \), therefore the map \( \rho_{Y_2 \otimes F} \) is not injective. We also know that the map \( \rho_{Y_2 \otimes F}/n \) is not injective.

**Remark 3.3.** We can construct an example of a higher dimensional variety for which the reciprocity map is not injective for any finite scalar extension as follows; Let \( X \) be a projective smooth and geometrically irreducible variety over \( k \). We consider the fiber product \( Y_2 \times_k X \), where \( Y_2 \) is the surface of Example 3.2. Then \( \rho_{Y_2 \times_k X} \) is not injective for any finite scalar extension. This follows from the following commutative diagram:

\[
\begin{array}{ccc}
H_2(\Gamma_{Y_2 \times X}, \mathbb{Z}) & \xrightarrow{\delta_{Y_2 \times X}} & CH_0(Y_2 \times X) \\
\downarrow & & \downarrow \\
H_2(\Gamma_{Y_2}, \mathbb{Z}) & \xrightarrow{\delta_{Y_2}} & CH_0(Y_2) \\
\end{array}
\]

\[ \xrightarrow{\rho_{Y_2}} \pi_{ab}(Y_2) \]

\[ \xrightarrow{\rho_{Y_2 \otimes F}} \pi_{ab}(Y_2 \otimes F) \]

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