Numerical solution of stochastic integral equations using CAS wavelets

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Abstract
In this paper, the CAS wavelets stochastic operational matrix method is developed for the numerical solution of stochastic integral equations. Properties of CAS wavelets and its function approximation are discussed. Firstly, the CAS wavelets stochastic operational matrix of integration is generated. This stochastic operational matrix is employed for solving stochastic integral equations. Next, this technique converts the stochastic integral equation into system of algebraic equations and then solving these equations we obtain the CAS wavelet coefficients. The accuracy of the proposed method is justified through the Illustrative example and the obtained solutions are compared with those of exact solutions. Error analysis is presented to show the efficiency of the proposed method.

Keywords
Stochastic integral equations, CAS wavelets, Brownian motion.

AMS Subject Classification
60H05, 60H35, 65T60.

1. Introduction
Wavelet is newly emerging area in the field of mathematics. Wavelets are extensively used for signal processing in communications and physics, and it is one of the best mathematical tools [9]. For describing knowledge models the stochastic integral equations are important tools in applied mathematics. Since in many cases, the exact solution of these equations does not exist, the numerical approximation of these equations become necessary. There are various methods for approximating these equations and different basis functions [1, 3–6, 8, 10] are used. Here we consider the Cosine and Sine wavelets (CAS) [2] to approximate the linear stochastic integral equations. In this paper, we consider the following stochastic Volterra integral equation,

\[ y(x) = f(x) + \int_{0}^{x} k_1(x,t)y(t)dB(t), \quad t \in [0,T] \]  (1.1)

where, \( B(x) \) is a Brownian motion process, \( y(x), f(x) \) and \( k_1(x,t) \), for \( s,t \in [0,T] \), are the stochastic processes and \( y(x) \) is unknown. \( \int_{0}^{x} k_1(x,t)y(t)dB(t) \) is the Ito integral [7]. These stochastic integral equations are solved using CAS wavelets stochastic operational matrix of integration. This paper is organized as follows. In section 2, some preliminaries are given. Section 3, gives the properties of CAS wavelets and CAS wavelets stochastic operational matrix of integration. Section 4, gives the method of solution. In section 5, some examples are presented to show the efficiency of the presented method. Finally, in section 6, conclusion is drawn.

2. Preliminaries

Brownian motion

Definition 2.1. A stochastic process \( \{ W(x) : 0 \leq x \leq \infty \} \), is called a standard Brownian motion [7] if
- $W(0) = 0$

- The stochastic process $\{W(x) : 0 \leq x \leq \infty\}$ has continuous sample paths.

- The process has independent, stationary increments.

**Definition 2.2.** An $n$-dimensional process, $W(x) = (W(x)^1,...,W(x)^n)$, is a standard $n$-dimensional Brownian motion if each $W(x)^i$ is a standard Brownian motion and the $W(x)^i$’s are independent of each other.

### 3. CAS wavelets

CAS wavelets $\psi_{n,m}(x) = \psi(k,n,m,x)$ have four arguments: $n = 0,1,...,2^k - 1$, $k$ is assumed to be any positive integer. They are defined on the interval $[0,1]$ as follows:

$$\psi_{n,m}(x) = \begin{cases} 2^k \cos(2m\pi x) + \sin(2m\pi x) & \text{if } \frac{n-1}{2^k} \leq x \leq \frac{n}{2^k} \\ 0, \quad \text{Otherwise} \end{cases} \tag{3.1}$$

with

$CAS_m(x) = \cos(2m\pi x) + \sin(2m\pi x)$

where $m = -M, -(M - 1),...0,...,M-1$. For instance, for $k = 1$ and $M = 1$, we get

$$\begin{align*}
\psi_{0,-1} &= \sqrt{2}(\cos(4\pi x) - \sin(4\pi x)) \\
\psi_{0,0} &= \sqrt{2} \\
\psi_{0,1} &= \sqrt{2}(\cos(4\pi x) + \sin(4\pi x)) \\
\psi_{1,-1} &= \sqrt{2}(\cos(4\pi x) - \sin(4\pi x)) \\
\psi_{1,0} &= \sqrt{2} \\
\psi_{1,1} &= \sqrt{2}(\cos(4\pi x) + \sin(4\pi x))
\end{align*} \tag{3.2}$$

**Function approximation**

In terms of the CAS wavelets $f(x) \in L^2[0,1]$ can be written as

$$f(x) = \sum_{n=1}^{\infty} \sum_{m=-M}^{M} c_{nm} \psi_{n,m}(x) \tag{3.2}$$

Truncating the above infinite series, we get

$$f(x) \simeq \sum_{n=1}^{2^k-1} \sum_{m=-M}^{M} c_{nm} \psi_{n,m}(x) = C^T \psi(x) = f_{\hat{m}}(x) \tag{3.3}$$

where, $C$ and $\psi(x)$ are $\hat{m} \times 1$ ($\hat{m} = 2^k (2M + 1)$) matrices. For instance, for $k = 1$ and $M = 1$, we get

$$\begin{align*}
\psi(x) &= \begin{bmatrix} \psi_{0,-1}(x) \\
\psi_{0,0}(x) \\
\psi_{0,1}(x) \\
\psi_{1,-1}(x) \\
\psi_{1,0}(x) \\
\psi_{1,1}(x)
\end{bmatrix} \\
&= \begin{bmatrix} -0.5176 & -1.4142 & 1.4142 & 0 & 0 & 0 \\
1.4142 & 1.4142 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -0.5176 & -1.4142 & 1.4142 \\
0 & 0 & 0 & 1.4142 & 1.4142 & 1.4142 \\
0 & 0 & 0 & 0 & 1.4142 & 1.4142 \\
0 & 0 & 0 & 0 & 0 & 0.5176
\end{bmatrix}
\end{align*} \tag{3.4}$$

**Stochastic operational matrix of integration of CAS wavelets (SOMCASW)**

Now we derive a new SOMCASW as follows:

$$\int_0^x \psi(t) dW(t) = P_s \psi(x) \tag{3.5}$$

where $P_s$ is a $\hat{m} \times \hat{m}$ matrix and is called the stochastic operational matrix of integration of CAS wavelets. In particular, for $M = 1$ and $k = 1$, we have

$$\int_0^x \psi_{0,-1}(t) dW(t) = \begin{cases} \sqrt{2} \cos(4\pi x) + \sin(4\pi x) & 0 \leq x \leq \frac{1}{2} \\
\sqrt{2} W(\frac{1}{2}) & \frac{1}{2} \leq x \leq 1
\end{cases} \tag{3.6}$$

$$\int_0^x \psi_{0,0}(t) dW(t) = \begin{cases} \sqrt{2} W(x) & 0 \leq x \leq \frac{1}{2} \\
\sqrt{2} W(\frac{1}{2}) & \frac{1}{2} \leq x \leq 1
\end{cases} \tag{3.7}$$

$$\int_0^x \psi_{0,1}(t) dW(t) = \begin{cases} \sqrt{2} \cos(4\pi x) + \sin(4\pi x) & 0 \leq x \leq \frac{1}{2} \\
\sqrt{2} \cos(4\pi x) - \sin(4\pi x) & \frac{1}{2} \leq x \leq 1
\end{cases} \tag{3.8}$$

$$\int_0^x \psi_{1,-1}(t) dW(t) = \begin{cases} \sqrt{2} \cos(4\pi x) - \sin(4\pi x) & 0 \leq x \leq \frac{1}{2} \\
\sqrt{2} W(x) - W(\frac{1}{2}) & \frac{1}{2} \leq x \leq 1
\end{cases} \tag{3.9}$$

$$\int_0^x \psi_{1,0}(t) dW(t) = \begin{cases} \sqrt{2} \cos(4\pi x) + \sin(4\pi x) & 0 \leq x \leq \frac{1}{2} \\
\sqrt{2} W(x) + W(\frac{1}{2}) & \frac{1}{2} \leq x \leq 1
\end{cases} \tag{3.10}$$
Using equations (3.5) to (3.10), we get
\[
\psi(x) = \int_0^x \psi(t) dW(t)
\]
\[
= \begin{bmatrix}
\int_0^x \psi_0(-1)(t) dW(t) \\
\int_0^x \psi_0(0)(t) dW(t) \\
\int_0^x \psi_1(-1)(t) dW(t) \\
\int_0^x \psi_1(0)(t) dW(t) \\
\int_0^x \psi_1(1)(t) dW(t)
\end{bmatrix}
\]
The stochastic operational matrix of integration of CAS wavelets are derived here in particular for \(k = 1\) and \(M = 1\) and can extended for different values of \(k\) and \(M\).

**Remark 3.1.** If \(F\) is a \(\hat{m}\)-vector, then
\[
\psi(x) \psi^T(x) F = F \psi(x)
\]
where, \(\psi(x)\) is the CAS wavelet coefficient matrix and \(F\) is an \(\hat{m} \times \hat{m}\) matrix given by
\[
\hat{F} = \psi(x) \hat{F} \psi^{-1}(x)
\]
where \(\hat{F} = diag(\psi^{-1}(x) F)\). Also, for a \(\hat{m} \times \hat{m}\) matrix \(X\), we have
\[
\psi^T(x) X \psi(x) = \hat{X}^T \psi(x)
\]
where, \(\hat{X}^T = V \psi^{-1}(x)\) and \(V = diag(\psi^T(x) X \psi(x))\) is a \(\hat{m}\)-vector.

### 4. Method of solution

Consider the stochastic Volterra integral equation,
\[
y(x) = f(x) + \int_0^x k_1(x,t) y(t) dB(t), \quad t \in [0,T]
\]
where \(f(x) \in L^2[0,1]\), \(k_1(x,t) \in L^2[0,1] \times [0,1]\) for \(x, t \in [0,T]\), are the stochastic processes defined on the same probability space \((\Omega, F, P)\) and \(y(x)\) is unknown. Also \(B(t)\) is a Brownian motion process and \(\int_0^x k_1(x,t) y(t) dB(t),\) is the Ito integral. Approximating \(f(x), y(x)\) and \(k_1(x,t),\) with respect to CAS wavelets as follows:
\[
y(x) \simeq C^T \psi(x) = C \psi^T(x)
\]
where \(C\) is the unknown vector to be determined.
\[
f(x) \simeq F^T \psi(x) = F \psi^T(x)
\]
\[
k_1(x,t) \simeq \psi^T(x) K_1 \psi(t) = \psi^T(t) K_1^T \psi(x)
\]
where \(C\) and \(F\) are CAS wavelet coefficient vectors and \(K_1\) is the CAS wavelet matrix. Substituting 4.2, 4.3 and 4.4 in 4.1, we get
\[
C^T \psi(x) = F^T \psi(x) + \psi^T(x) K_1^T \left( \int_0^x \psi(t) \psi^T(t) C dB(t) \right)
\]
Using the remark 3.1, we get
\[
C^T \psi(x) = F^T \psi(x) + \psi^T(x) K_1^T \left( \int_0^x \hat{C} \psi(t) dB(t) \right)
\]
where \(\hat{C}\) is a \(\hat{m} \times \hat{m}\) matrix. Using the SOMCASW, we get,
\[
C^T \psi(x) = F^T \psi(x) + \psi^T(x) K_1^T \hat{C} P \psi(x)
\]
Let \(X_1 = K_1^T \hat{C} P\). Again using the remark 3.1, we get,
\[
C^T \psi(x) - X_1 = F^T
\]
where \(X_1\) is a \(\hat{m} \times \hat{m}\) matrix and are linear functions of \(C\) and this equation is applicable for all \(t \in [0,1]\), hence
\[
C^T = X_1^T
\]
Solving this linear system of equations, we get the unknown vector \(C\). Substituting this unknown vector in equation 4.2, we get the solution the stochastic volterra integral equation given in equation 4.1.

### 5. Numerical Results

**Example 5.1.** Let us consider the stochastic volterra integral equation,
\[
y(x) = f(x) + \int_0^x k_1(x,t) y(t) dB(t)
\]
where
\[
f(x) = 1 \text{ and } k_1(x,t) = \sin(t)
\]
Exact solution is
\[
y(x) = \exp \left[ -\frac{1}{4} \left( t - \cos(x) \sin(x) \right) + \int_0^x \sin(t) dB(t) \right]
\]
**Method of implementation**
\[
f(x) = 1 \quad k_1(x,t) = \sin(t)
\]
Approximating equations (5.2) and (5.3) using CAS wavelets, we obtain the vector \(F\) and the matrix \(K_1\). Substituting the obtained vector \(F\), matrix \(K_1\) is the approximated unknown solution \(C\) in (5.1) and by the use of SOMCASW, we obtain the unknown vector \(C\) as,
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Table 1. Absolute errors of the 5.1 for $k = 1$ and $M = 1$.

| $t$  | $k = 1$ and $M = 1$ |
|------|----------------------|
| 0.125 | 2.5625e-03          |
| 0.25  | 3.7100e-03          |
| 0.375 | 2.4950e-03          |
| 0.5   | 9.2550e-03          |
| 0.625 | 2.3513e-02          |
| 0.75  | 4.4790e-02          |
| 0.875 | 7.5232e-02          |

$$C = \left[ -0.0018351 \ 0.7022 \ 0.0018808 \\
-0.0015249 \ 0.69172 \ 0.0016683 \right]$$

Substituting this into $y(x) \simeq C^T \psi(x) = C \psi^T(x)$, we obtain the CAS wavelets solution as

$$y = \left[ \begin{array}{c} 0.99764 \\ 0.993 \\ 0.98854 \\ 0.98225 \\ 0.97804 \\ 0.97443 \end{array} \right]$$

Absolute errors of example of 5.1 are shown in Table 1 for $k = 1$ and $M = 1$

6. Conclusion

The CAS wavelets stochastic operational method is developed for the numerical solution of stochastic integral equations. The present method reduces an integral equation into a set of algebraic equations. For instance in the illustrative example, our results are in good accuracy with exact ones. Error analysis justifies the efficiency, validity and applicability of the present technique.

Acknowledgment

1 We thank University Grants Commission (UGC), New Delhi, for supporting this work partially through UGC-SAP DRS-III for 2016-2021: F.510/3/DRS-III/2016 (SAP-I).

2 Also, we thank Karnataka University Dharwad (KUD) for supporting this work under University Research Studentship (URS) 2016-2019: K. U. 40 (SC/ST) URS/2018-19/32/3/841 Dated: 07/07/2018.

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