A note on the topology of irreducible $\text{SO}(3)$-manifolds

Panagiotis Konstantis

Abstract

A connected, oriented 5-manifold $M$ is an irreducible $\text{SO}(3)$-manifold if there exists a rank 3 vector bundle $\eta$ over $M$ such that the tangent bundle is isomorphic to the bundle of symmetric trace-free endomorphisms of $\eta$. In this article we give necessary and sufficient topological conditions for the existence of irreducible $\text{SO}(3)$-manifolds. There we have to distinguish if the manifold is spin or not. At the end we provide some new examples of irreducible $\text{SO}(3)$-manifolds.

1 Introduction

Throughout this article let $M$ be always a smooth, connected and oriented manifold. In this paper our aim is to determine topological conditions for certain rank 3 subbundles of the tangent bundle of a 5-manifolds. A related situation (cf. Remark 1.7) was studied by E. Thomas in [Tho67a]. There he investigated the conditions to the existence of two-fields on manifolds $M$ of dimension $4k + 1$ ($k > 0$). A two-field is a pair of tangent vector fields which are linearly independent in every point of the manifold. Hence every two-field determines a trivial rank 2 subbundle of the tangent bundle of $M$. He proved the following

Theorem 1.1 ([Tho67a] Corollary 1.2). Let $M$ be a closed, connected, spin manifold of dimension $4k + 1$, $k > 0$. Then $M$ admits a two–field if and only if $w_{4k}(M) = 0$ and $\hat{\chi}(M) = 0$, where $w_{4k}(M)$ is the $4k$-th Stiefel-Whitney class of $M$ and $\hat{\chi}(M)$ is the semi-characteristic of $M$, which is defined as

$$\hat{\chi}(M) = \sum_{i=0}^{2k} \dim_{\mathbb{Z}_2} H_i(M; \mathbb{Z}_2) \mod 2.$$  

If $M$ is a spin manifold like in Theorem 1.1 with $\dim M = 5$, we can apply Wu’s formula to see that $w_4(M) = w_2(M) - w_2(M) = 0$, hence we obtain

Corollary 1.2 (see also [Tho68]). Suppose $M$ is a closed, connected spin 5-manifold. Then $M$ admits a two–field if and only if $\hat{\chi}(M) = 0$.

Later, M. Atiyah proved a more general theorem using K-theory, where $M$ has not to be spin.

Theorem 1.3 ([Ati70] Theorem 5.1). Let $M$ be a connected, closed, oriented manifold of dimension $4k + 1$, $k > 0$. Then $M$ admits a two–field if and only if $k(M) = 0$, where

$$k(M) = \sum_{i=0}^{2k} \dim_{\mathbb{R}} H^{2i}(X; \mathbb{R}) \mod 2.$$
is the Kervaire semi-characteristic of $M$.

In this article we would like to study a similar situation. A manifold $M$ of dimension 5 is called an irreducible $\text{SO}(3)$–manifold if there is a rank 3 vector bundle $\eta$ over $M$ such that the tangent bundle of $M$ is isomorphic to the bundle of symmetric trace–free endomorphisms of $\eta$ (see Remark 1.7 for an explanation why such manifolds are called irreducible). We will prove two main theorems about the topology of irreducible $\text{SO}(3)$–manifolds, which may be summarized as follows:

**Theorem 1.4.** Let $M$ be a closed, oriented and connected manifold of dimension five.

(a) Suppose $w_2(M) = 0$ (i.e. if $M$ is spin). Then $M$ is an irreducible $\text{SO}(3)$–manifold if and only if

(i) $w_4(M) = 0$ and the first Pontryagin class $p_1 \in H^4(M; \mathbb{Z})$ is divisible by five,

(ii) $\hat{\chi}(M) = 0$.

(b) Suppose $w_2(M) \neq 0$ and $H^4(M; \mathbb{Z})$ contains no element of order 4. Then $M$ is an irreducible $\text{SO}(3)$–manifold if and only if $w_4(M) = 0$ and $p_1(M)$ is divisible by five.

For a simply connected, closed 5–manifold we have $H^4(M; \mathbb{Z}) = H^4(M; \mathbb{Z}_2) = 0$ by Poincaré duality and $\hat{\chi}(M) = 1 + \dim \mathbb{Z}_2 H_2(M; \mathbb{Z}_2)$ mod 2. Hence we obtain

**Corollary 1.5.** Let $M$ be a simply connected, closed 5–manifold.

(a) Let $w_2(M) = 0$. Then $M$ is an irreducible $\text{SO}(3)$–manifold if and only if $\dim \mathbb{Z}_2 H_2(M; \mathbb{Z}_2)$ is odd.

(b) Let $w_2(M) \neq 0$. Then $M$ is an irreducible $\text{SO}(3)$–manifold.

Furthermore from Corollary 1.2 we obtain that Theorem 1.4 (a) is equivalent to

**Corollary 1.6.** Let $M$ be a closed spin 5–manifold. Then $M$ is an irreducible $\text{SO}(3)$–manifold if and only if $M$ admits a standard $\text{SO}(3)$–structure and $p_1(M)$ is divisible by five.

**Remark 1.7.** We would like to give an alternative definition of irreducible $\text{SO}(3)$–manifolds, which will be needed for proving Theorem 1.4 (a). Let $G$ be a Lie group and $\rho : G \to \text{SO}(n)$ an embedding of $G$ as a Lie subgroup of $\text{SO}(n)$. Let $\xi$ be an oriented vector bundle over a CW–complex $X$. A $G$–structure on $\xi$ is a reduction of the $\text{SO}(n)$–principal bundle of $\xi$ to a $G$–principal bundle over $X$ (where $n$ is the rank of $\xi$). If we regard $\xi$ as a map from $X$ to the classifying space $B\text{SO}(n)$, this definition is equivalent to the existence of a lift $\hat{\xi}$ for $\xi$ such that the following diagram commutes up to homotopy

$$
\begin{array}{ccc}
X & \xrightarrow{\xi} & B\text{SO}(n) \\
\downarrow & \downarrow B\rho \\
BG & \xrightarrow{\hat{\xi}} & \text{SO}(n),
\end{array}
$$

where $B\rho$ is the map induced by $\rho$ on the classifying spaces. The existence of such a lift can be decided with obstruction theory, where sometimes the obstructions can be expressed in terms of characteristic classes of $X$. Let $\tau_M$ denote the tangent bundle of $M$. We say that $M$ admits a $G$–structure if $\tau_M$ admits a $G$–structure.
The existence of a two-field is equivalent to the existence of a $\text{SO}(n-2)$–structure on $M$ with respect to the standard embedding of $\text{SO}(n-2)$ into $\text{SO}(n)$ ($n = \dim M$). We call such a structure a standard $\text{SO}(n-2)$–structure on $M$.

For $n = 5$ there is another structure of $\text{SO}(3)$ into $\text{SO}(5)$ besides the standard one (see [ABBF11]): Identify $\mathbb{R}^5$ as a vector space with the space $\text{Sym}^0(\mathbb{R}^3)$ of symmetric trace free endomorphisms of $\mathbb{R}^3$. Then for $h \in \text{SO}(3)$ and $X \in \text{Sym}^0(\mathbb{R}^3)$ we set $\rho(h)X := hXh^{-1}$. This defines the irreducible embedding

$$\rho: \text{SO}(3) \to \text{End}(\mathbb{R}^5), \quad h \mapsto \rho(h).$$

It is now easy to see that $\rho(h)$ preserves the standard metric on $\mathbb{R}^5$, which makes $\rho$ a map into $\text{SO}(5)$. It is clear that, as representations, the standard embedding and the irreducible embedding of $\text{SO}(3)$ into $\text{SO}(5)$ are not equivalent.

An oriented 5-dimensional vector bundle over a CW-complex admits an irreducible $\text{SO}(3)$–structure if its structure group can be reduced from $\text{SO}(5)$ to $\text{SO}(3)$, where $\text{SO}(3)$ is irreducible embedded into $\text{SO}(5)$. From Theorem 1.4 it follows that the existence of an irreducible $\text{SO}(3)$–manifold $M$ is equivalent to the existence of an irreducible $\text{SO}(3)$–structure of the tangent bundle of $M$.

**Historical remark.** First steps were made in [Bob06] and later in [ABBF11]. The author of [Bob06, Theorem 1.4] claimed that an irreducible $\text{SO}(3)$–structure exists if and only if the manifold admits a standard $\text{SO}(3)$–structure and the first Pontryagin class is divisible by 5. However in [ABBF11, Example 3.1] it was shown that the symmetric space $\text{SU}(3)/\text{SO}(3)$ does not have a standard $\text{SO}(3)$–structure, but it admits an irreducible $\text{SO}(3)$–structure. Nevertheless in [BF06] M. Bobienski reports that the proof of Theorem 1.4 in [Bob06] should work if one assumes the manifold is spin. Indeed we will prove in section 3 that this is true, but we will use a different approach as in [Bob06] (Bobienski tries to compare the Moore-Postnikov towers of the irreducible and the standard representation of $\text{SO}(3)$, where in this article we apply the methods of E. Thomas directly to this special case). In [ABBF11, Theorem 3.2] the authors prove some necessary conditions for the existence of an irreducible $\text{SO}(3)$–structure on a 5-manifold, using a special characterization of the tangent bundle. We will generalize this in Proposition 2.7 for oriented 5-dimensional vector bundles over an arbitrary CW-complex.

In section 2 we will make preparations for the proof of Theorem 1.4(a). In particular we will determine properties of the obstructions for lifting a map $\xi: Y \to B\text{SO}(5)$ to a map $Y \to B\text{SO}(3)$ through the fibration $Bp: B\text{SO}(3) \to B\text{SO}(5)$, where $Y$ is a CW-complex of dimension 5 (cf. Proposition 2.3). Section 3 contains the proof of Theorem 1.4(a), where we assume that $Y$ is a spin 5–manifold $M$ and $\xi$ the classifying map for the tangent bundle of $M$. In both section we will use the theory of E. Thomas, see [Tho66], [Tho67b] and [Tho67a]. The second part of Theorem 1.4 will be proved in section 4, which is a quite short proof and depends heavily on the work of Čadek and Vanžura, cf. [ČV93]. In that article the authors classify 3- and 5-dimensional vector bundles over certain 5-complexes. Note that we rely on the condition on $H^4(M; \mathbb{Z})$, since the classification of vector bundles is given by characteristic classes (see e.g. [Tho68]), where this condition is necessary (see also Remark 4.5). Finally, note that only Proposition 2.7 from section 2 is used for the proof of Theorem 1.4(b) in section 4.

The contents of the last section are examples of irreducible $\text{SO}(3)$–manifolds, where we will exploit Theorem 1.4. In particular: the symmetric space $\hat{Y} := \text{SU}(3)/\text{SO}(3)$ does not admit a standard $\text{SO}(3)$–structure (since $k(\hat{Y}) = 1$, see Theorem 1.3), but an irreducible one with $\hat{\xi}(\hat{Y}) = 0$. 

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So it was conjectured in [ABBFT11], that $\hat{\chi}(M)$ could be an obstruction to the existence of an irreducible $\text{SO}(3)$–structure. However we will prove

**Proposition 1.8.** There is a compact and connected 5–manifold $M$ such that the tangent bundle admits an irreducible $\text{SO}(3)$–structure with $\hat{\chi}(M) = 1$.

The manifold mentioned in the proposition above is constructed as the total space of a circle bundle over a simply connected, compact 4–manifold. We will also show

**Proposition 1.9.** For $i = 1, \ldots, 2k + 1$ ($k = 0, 1, 2, \ldots$) let $M_i$ be an irreducible $\text{SO}(3)$–manifold which is spin. Then

$$\#_{i=1}^{2k+1} M_i$$

is a irreducible $\text{SO}(3)$–manifold.

With proposition we will be able to consider new examples of irreducible $\text{SO}(3)$-manifolds.

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2 Preliminaries for the Proof of Theorem 1.4 (a)

We will make use of the definitions of Remark 1.7. Let $\rho: \text{SO}(3) \to \text{SO}(5)$ be the 5-dimensional irreducible representation of $\text{SO}(3)$ and $B\rho: B_5 \to B_3$ the induced map on classifying spaces where $B_k := B\text{SO}(k)$. Let $M$ be an oriented, compact 5-manifold with $\omega_2(M) = 0$ and $\tau_M: M \to B_5$ the classifying map for the tangent bundle of $M$. To prove Theorem 1.4 (a) we will compute the obstructions to find a map $M \to B_3$ such that the diagram

$$X = \text{SO}(5)/\rho(\text{SO}(3))$$

$$\downarrow$$

$$B_3$$

$$\downarrow B\rho$$

$$M \xrightarrow{\tau_M} B_5$$

commutes up to homotopy. We may assume that $B\rho$ is a fibration and let $X$ denote its homotopy fibre. Clearly the homotopy type of $X$ is given by the 7–dimensional Berger space $\text{SO}(5)/\rho(\text{SO}(3))$.

**Remark 2.1.** The space $X$ is 2-connected with $\pi_3(X) = \mathbb{Z}_{10}$ and $\pi_4(X) = \mathbb{Z}_2$. For $\pi_3(X)$ one considers the long exact homotopy sequence for $\text{SO}(3) \to \text{SO}(5) \to X$ and the fact the $\rho$ induces an isomorphism on the first homotopy groups of $\text{SO}(3)$ and $\text{SO}(5)$. By the Hurewicz theorem and the fact that $H_3(X) = \mathbb{Z}_{10}$ one obtains $\pi_3(X)$. The group $\pi_4(X)$ can be computed using Lemma 3.2 in [ABBFT11] and the long exact homotopy sequence.
In the following we will need the Serre exact sequence.

Lemma 2.2 ([? Proposition 3.2.1]). Let $E$ and $B$ be topological spaces and $p \colon E \to B$ a fibration. Let $F$ be its homotopy fibre and $i \colon F \to E$ the inclusion of $F$ as the fibre $\pi^{-1}(b)$ for $b \in B$. Suppose furthermore that $B$ is $(m-1)$-connected $(m \geq 2)$ and $F$ is $(n-1)$-connected $(n \geq 1)$. For any abelian group $G$ and $p = m + n + 1$ we have a long exact sequence

$$H^i(E;G) \xrightarrow{i^*} H^i(F;G) \xrightarrow{\partial} H^{i+1}(B;G) \xrightarrow{\delta} \cdots \xrightarrow{\delta} H^n(E;G) \xrightarrow{i^*} H^n(F;G).$$

The map $\tau \colon H^k(F;G) \to H^{k+1}(B;G)$ is called the transgression map of the fibration $F \to E \to B$.

For the rest of the section let $Y$ be a 5–dimensional CW–complex and $\xi$ an oriented rank 5 vector bundle over $Y$. We will consider $\xi$ as a map $\xi \colon Y \to B_5$.

The next proposition is central to compute the obstructions for lifting of $\xi$ to $B_3$ (for the fibration $B_\rho \colon B_3 \to B_5$).

Proposition 2.3. There is a topological space $E$ and a fibration $p \colon E \to B_5$ with the following properties

(a) There is a fibration $q \colon B_3 \to E$ which is a lift of $B_\rho$.

(b) There is a class $k_1 \in \ker(B_\rho)^* \subseteq H^4(B_5;\mathbb{Z}_{10})$ such that $\xi$ lifts to $E$ if and only if $\xi^*(k_1) = 0$.

(c) Let $\hat{\xi} \colon Y \to E$ be a lift of $\xi \colon Y \to B_5$. There is a class $k_2 \in H^5(E;\mathbb{Z}_2)$ which lies in the image of the transgression map of the fibration $q$ such that $\xi$ has a lift to $Y \to B_3$ if and only if $\hat{\xi}^*(k_2) = 0$.

Proof. From Remark 2.1 we have that $X$ is 2–connected with $\pi_3(X) \cong H_3(X) \cong \mathbb{Z}_{10}$. Hence by the Universal Coefficient Theorem $H^3(X;\pi_3(X)) \cong \text{Hom}(\pi_3(X),\pi_3(X))$. Let $\gamma_1 \in H^3(X;\pi_3(X))$ be the element which corresponds to the identity in $\text{Hom}(\pi_3(X),\pi_3(X))$.

Using the Serre exact sequence for the fibration $X \to B_3 \to B_5$ we define $k_1 := -\tau(\gamma_1) \in H^4(B_5;\pi_3(X)) = H^4(B_5;\mathbb{Z}_{10})$ where $\tau$ is the transgression map of the fibration $X \to B_3 \to B_5$. Consider $k_1$ as a map $B_5 \to K(\mathbb{Z}_{10}, 4)$ and define $E$ as the pullback of the pathspace fibration $\Omega K(\mathbb{Z}_{10}, 4) \to \mathcal{P} \to K(\mathbb{Z}_{10}, 4)$ by $k_1$. Hence one obtains a fibration $p \colon E \to B_5$ with homotopy fibre $\Omega K(\mathbb{Z}_{10}, 4)$. From [Tho66, p.3] we have that $\xi$ lifts to a map $Y \to E$ if and only if $\xi^*(k_1) = 0$. This proves part (a) and (b).

Suppose $\hat{\xi} \colon Y \to E$ is a lift of $\xi \colon Y \to B_5$ through $p \colon E \to B_5$. Note that $B_\rho^*(k_1) = 0$ since $k_1$ lies in the image of $\tau$. This means that $B_\rho \circ k_1 : B_3 \to K(\mathbb{Z}_{10}, 4)$ is homotopic to a constant map, hence there is a lift $q : B_3 \to E$ of $B_\rho$ through $p$ (cf. [Tho66, p.3]). We replace $q$ by a map which is a fibration and homotopic to $q$. Denote this fibration again by $q$. Let $F$ be its homotopy fibre. From [Tho67b, p.189] we know that $F$ is 3–connected with $\pi_4(F) \cong \pi_4(X) \cong \mathbb{Z}_2$. Let $\gamma_2 \in H^4(F;\pi_4(F)) = \text{Hom}(\pi_4(F),\pi_4(F))$ be the identity element and set $k_2 := -\tau(\gamma_2)$, where now $\tau$ is the transgression map in the Serre exact sequence for the fibration $F \to B_3 \to E$. From [Tho67b, p. 190] and since $Y$ is a complex of dimension 5, the map $\hat{\xi} : Y \to E$ has a lift to a map $Y \to B_3$ if and only if $\hat{\xi}^*(k_2) = 0$.

We fix the notation for $k_1, k_2$ and $p : E \to B$ of Proposition 2.3 for the rest of the article.

Note that if $\xi$ has a lift to a map $Y \to E$, then this lift has not to be unique. Therefore we make the

Definition 2.4. Suppose that $\xi^*(k_1) = 0$. Then, using Proposition 2.3(b), we can define a subset of $H^5(Y;\mathbb{Z}_2)$ by

$$k_2(\xi) := \bigcup_{\eta} \eta^*(k_2),$$

where the union is taken over all maps $\eta : Y \to E$ which are lifts of $\xi : Y \to B_5$ through $p : E \to B_5$.  

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The next corollary follows from Proposition 2.3(c)

Corollary 2.5. Suppose that \( \xi^*(k_1) = 0 \). Then \( \xi \) lifts to a map \( Y \to B_3 \) through \( Bp \) if and only if
\[
0 \in k_2(\xi) \subseteq H^5(Y; Z_2).
\]

In this section we will compute the cohomology class \( k_1 \) and the group \( H^5(E; Z_2) \) and derive some useful properties for the set \( k_2(\xi) \). In section 3 we will compute explicitly \( k_2(\tau_M) \) for the tangent bundle \( \tau_M \) of an oriented 5-manifold with \( w_2(M) = 0 \) where \( \tau_M^*(k_1) = 0 \).

We saw in Proposition 2.3 that \( k_3 \) lies in the kernel of \( Bp^*: H^4(B_5; Z_{10}) \to H^4(B_3; Z_{10}) \). Hence, as a first step to determine \( k_1 \) we need to compute this kernel.

Remark 2.6. The group \( Z_{10} \) is isomorphic to \( Z_5 \oplus Z_2 \). Hence for any topological space \( Y \) the groups \( H^k(Y; Z_{10}) \) are canonically isomorphic to \( H^k(Y; Z_5) \oplus H^k(Y; Z_2) \).

(a) It is known that \( H^*(B_5; Z) = Z[p_1, p_2] \oplus 2\text{-tortion} \) where \( p_1 \) and \( p_2 \) are the first and second Pontryagin classes of the universal bundle over \( B_5 \) respectively. Moreover the 2-torsion is of order 2. The long exact sequence associated to
\[
0 \to Z \xrightarrow{5} Z \to Z_5 \to 0
\]
yields the short exact sequence
\[
0 \to 5 \cdot H^4(B_5; Z) \to H^4(B_5; Z) \to H^4(B_5; Z_5) \to 0
\]
since the 5-torsion subgroup of \( H^5(B_5; Z) \) is zero. Furthermore \( 5 \cdot H^4(B_5; Z) = 5Z[p_1] \oplus 2\text{-tortion} \), hence
\[
H^4(B_5; Z_5) = Z_5[\rho_5(p_1)].
\]
where \( \rho_5(p_1) \) is the mod 5 reduction of \( p_1 \). In the same way we see \( H^5(B_5; Z_5) = Z_5[\rho_5(p_2)] \) and it follows that
\[
H^*(B_5; Z_5) = Z_5[\rho_5(p_1), \rho_5(p_2)].
\]
Same holds for \( B_3 \), i.e.
\[
H^*(B_3; Z_5) = Z_5[\rho_5(p_1)].
\]

(b) Let \( \rho^T \) be \( \rho \) restricted to the maximal Torus of \( SO(3) \), then this map is given by \( \rho^T: S^1 \to S^1 \times S^1 \), \( z \mapsto (z, z^2) \) (cf. [ABBF11] p. 69]. Hence \( B\rho^*(p_1) = 10p_1 \), \( B\rho^*(p_2) = 9p_1^2 \) thus \( B\rho^*(\rho_5(p_1)) = 0 \) and \( B\rho^*(\rho_5(p_2)) = 4\rho_5(p_1)^2 \).

(c) Finally we would like to determine the kernel of \( B\rho^* \) with coefficients in \( Z_2 \). Using [MT91] Theorem 5.9] and the explicit map \( \rho: SO(3) \to SO(5) \) given in [ABBF11] p.68] we compute
\[
B\rho^*(w_2) = w_2, \quad B\rho^*(w_3) = w_3, \quad B\rho^*(w_4) = 0, \quad B\rho^*(w_5) = 0.
\]

Remark 2.6 yields necessary conditions for the existence of an irreducible \( SO(3) \)-reduction for an arbitrary oriented vector bundles of rank 5 over a CW-complex. The authors in [ABBF11] Theorem 3.2] proved an analogous result for the tangent bundle of a 5-manifold.

Proposition 2.7. If \( \xi \) admits an irreducible \( SO(3) \)-reduction then

(a) \( p_1(\xi) \) has to be divisible by 5,
Remark 2.9. The element $k_1 = \langle -\rho_5(p_1), w_4 \rangle$ in $H^4(B_5; \mathbb{Z}_{10}) = H^4(B_5; \mathbb{Z}_5) \oplus H^4(B_5; \mathbb{Z}_2)$ is given by

$$k_1 = \langle -\rho_5(p_1), w_4 \rangle.$$  

\begin{proof}
We proceed with the computation of $k_1$.

**Proposition 2.8.** The element $k_1 \in H^4(B_5; \mathbb{Z}_{10}) = H^4(B_5; \mathbb{Z}_5) \oplus H^4(B_5; \mathbb{Z}_2)$ is given by

$$k_1 = (-\rho_5(p_1), w_4).$$

\begin{proof}
We assume coefficients in $\mathbb{Z}_{10} = \mathbb{Z}_5 \oplus \mathbb{Z}_2$. The Serre exact sequence for the fibration $X \to B_5 \to B_3$ yields

$$0 \to H^3(X) \to H^4(B_3) \to \tau^* H^4(B_3),$$

with $\tau$ the transgression map. From Remark 2.6, the kernel $\ker B\rho^*$ is given by $\mathbb{Z}_5[\rho_5(p_1)] \oplus \mathbb{Z}_2[w_4]$ as a subgroup of $H^3(B_5; \mathbb{Z}_{10})$. By the exact sequence above, the group $H^3(X)$ is mapped injectively into $\ker B\rho^* \subset H^3(B_5)$. Hence $(\gamma_1) = (\rho_5(p_1), w_4)$ where $\gamma_1 \in H^3(Y)$ represents the identity in $\text{Hom}(\pi_3(X), \pi_3(X)) \cong H^3(X)$.

\end{proof}

Remark 2.9.

(a) There is no cohomology class in $H^5(B_5; \mathbb{Z}_2)$ to which we could connect $k_2$ via $p_1$. Since $p_1$ is the principal fibration to $k_1 = \langle \rho_5(p_1), w_4 \rangle \in H^4(B_5; \mathbb{Z}_{10})$ we obtain that $p_1^*(w_4) = 0$ with $\mathbb{Z}_2$ coefficients. Hence $p_1^*(w_5) = 0$ by the Wu formula $\text{Sq}^4(w_4) = w_5$. Moreover $k_2$ can not lie in the image of $p^*: H^5(B_5; \mathbb{Z}_2) \to H^5(E; \mathbb{Z}_2)$ because in that case we would have $p_1^*(w_2w_3) = k_2$, but since $q_1(k_2) = 0 \neq B\rho^*(w_2w_3)$ this would be a contradiction.

(b) From (a) we have $\ker B\rho^* = \ker p^*$ in dimension 5 and since $B\rho^*: H^*(B_5; \mathbb{Z}_2) \to H^*(B_3; \mathbb{Z}_2)$ is surjective we obtain from [Tho66] the short exact sequence

$$0 \to H^5(E; \mathbb{Z}_2) \to H^5(KZ_{10}; 3 \times B_3) \to H^6(B_3).$$

**Proposition 2.10.** The set $k_2(\xi) \subset H^3(Y; \mathbb{Z}_2)$ is a coset of the subgroup

$$H^3(Y; \mathbb{Z}_2) \to w_2(\xi) + \text{Sq}^2 H^3(Y; \mathbb{Z}_2) \subset H^3(Y; \mathbb{Z}_2).$$

\begin{proof}
We set $\iota := \text{pr}_2^* \iota_3 \in H^3(KZ_{10}, 3, Z_2)$. Using property 5 of [Tho66, p. 16] we have the exact sequence (coefficients in $\mathbb{Z}_2$ are to be understood)

$$0 \to H^5(E) \to H^5(KZ_{10}; 3 \times B_3) \to H^6(B_3).$$

where $\nu$, $\tau_1$ are defined on p.18, p.16 of [Tho66] respectively. By the K"unneth theorem we have

$$H^5(KZ_{10}; 3 \times B_3) \cong (\mathbb{Z}_2)^3$$

generated by the elements $\iota \otimes w_2$, $\text{Sq}^2 \iota \otimes 1$, $1 \otimes w_2 \otimes w_3$. By property 2 and 3 on pp. 14 in [Tho66] we obtain that $\ker \tau_1$ is generated by $\text{Sq}^2 \iota \otimes 1 + \iota \otimes w_2$. Hence $k_2$ is the generator of $H^5(E)$ and $\nu^*(k_2) = \text{Sq}^2 \iota \otimes 1 + \iota \otimes w_2$. Let $m: KZ_{10}, 3 \times E \to E$ denote the action of $KZ_{10}, 3$ on $E$, then

$$m^*(k_2) = \text{Sq}^2 \iota \otimes 1 + \iota \otimes p^*(w_2) + 1 \otimes k_2.$$
Using Lemma 3(c) in \cite{Tho66} and the above relation we finally obtain

\[ k_2(\xi) = Sq^2 H^3(Y; \mathbb{Z}_2) + H^3(Y; \mathbb{Z}_2) \cup w_2(\xi) \subset H^5(Y; \mathbb{Z}_2). \]

\[ \blacksquare \]

**Remark 2.11.** If \( M \) is a 5-manifold with \( w_2(M) \) then the map \( Sq^2: H^3(M) \to H^5(M) \) is cupping with \( w_2 \). Hence by Proposition 2.10 the set \( k_2(\tau_M) \) consists of one element. Moreover by \( \xi \) has an irreducible \( SO(3) \)-structure if and only if this element \( k_2(\tau_M) \) is zero, cf. Corollary 2.5.

For the computation of \( k_2(\xi) \) in section 3 we have to prove the following proposition

**Proposition 2.12.** Let \( p: E \to B_5 \) and \( \tilde{p}: \tilde{E} \to B_5 \) be the principal fibrations of \( B_5 \) with respect to the elements \((w_4, p_5(p_1)) \in H^4(B_5; \mathbb{Z}_{10}) \) and \( w_4 \in H^4(B_5; \mathbb{Z}_2) \). Then \( H^5(\tilde{E}; \mathbb{Z}_2) \) is generated by a single element and there is a map \( f: E \to \tilde{E} \) which is an isomorphism on cohomology in dimension 5.

**Proof.** As in Proposition 2.10 we can prove that \( H^5(\tilde{E}; \mathbb{Z}_2) \) is generated by a single element \( k \) such that \( (k) = Sq^2 \tau_3 \otimes 1 + \tau_3 \otimes w_2 \in H^5(K(\mathbb{Z}_2, 3) \times B_3; \mathbb{Z}_2) \). Let \( pr\mathbf{2}: K(\mathbb{Z}_{10}, 3) \to K(\mathbb{Z}_2, 3) \) be the projection. This map induces a map \( f: E \to \tilde{E} \) such that \( \tilde{p} \circ f = p \). Since \( Bp^*(w_4, p_5(p_1)) = 0 \) as well as \( Bp^*(w_4) = 0 \) the map \( Bp \) lifts to \( E \) and \( \tilde{E} \). We denote those lifts by \( q \) and \( \tilde{q} \) respectively. Then \( fq \) is a lift of \( Bp \) to \( \tilde{E} \) as well. Hence there is a \( u \in H^3(B_3; \mathbb{Z}_2) \) such that \( fq = u \cdot \tilde{q} \) (see \cite{Tho66} p. 8, Lemma 3 (c)]. We obtain the commutative diagram

\[ \begin{array}{ccc}
K(\mathbb{Z}_{10}, 3) \times B_3 & \xrightarrow{mo(1 \times q)} & E \\
pr_2 \times 1 \downarrow & & \downarrow f \\
K(\mathbb{Z}_2, 3) \times B_3 & \xrightarrow{mo(1 \times u \cdot \tilde{q})} & \tilde{E}.
\end{array} \]

Since the horizontal arrows are injective (cf. \cite{Tho66} p. 16]) on cohomology level in dimension 5 we obtain that \( f^*(k) = k_2 \).

\[ \blacksquare \]

### 3 Proof of Theorem 1.4 (a)

In this section we would like to compute the cohomology class in \( k_2(\xi) \) for \( \xi \) a 5-dimensional vector bundle of rank 5 over a 5-manifold \( M \) with \( w_2(M) = 0 \) (note that by Corollary 2.5 \( k_2(\xi) \) is a set and by Remark 2.11 it contains only one element). Therefore we will connect this class to a secondary cohomology operation \( \Omega \) associated to a certain Adem relation. More precisely, we will apply the secondary cohomology operation on the Thom class of the Thom space induced by the tangent bundle of \( M \). Finally we will compute \( \Omega \) in two different ways (Corollary 3.4 and Corollary 3.7) which in turn will determine \( k_2(\xi) \). If \( k_2(\xi) \) is computed, then by Remark 2.11 (a) \( M \) admits an irreducible \( SO(3) \)-structure if and only if \( k_2(\tau_M) = 0 \). This will prove Theorem 1.4 (a). We begin with the definition of a secondary cohomology operation.

Recall that we have the Adem relation

\[ Sq^2 Sq^4 + Sq^5 Sq^1 = Sq^6. \]

Thus on integral classes of dimension \( \leq 5 \) we obtain the relation

\[ Sq^2 Sq^4 = 0. \]
Let $\Omega$ be a secondary cohomology operation associated with the above relation. Now, if $u \in H^j(Y;\mathbb{Z})$ with $j \leq 5$ then $\Omega$ is defined on $u$ provided $\text{Sq}^4 u = 0$ and the image of $\Omega$ is a coset in $H^{5+j}(Y;\mathbb{Z})$ by the subgroup

$$\text{Sq}^2 H^{5+j-2}(Y;\mathbb{Z}).$$

In our case $Y$ will be the Thom space of the tangent bundle of a 5-manifold and $u$ will be the Thom class of this bundle.

We proceed with the computation of the secondary cohomology operation applied to the Thom class of the tangent bundle of $M$. Therefore let $\gamma_5$ denote the universal vector bundle of $B_5$ and $T$ its Thom space and $U$ the Thom class of $\gamma_5$. If $g: Y \to B$ is a map, then we denote by $T_Y$ and $U_Y$ the Thom space and class of $g^*\gamma_5$. Now set $w := w_4 \in B_5$ and $\alpha = \text{Sq}^2$. Then $w$ is realizable in the sense of (6.1) in [Tho67a, p. 102] and $(w, \alpha) = (w_4, \text{Sq}^2)$ is admissible. Let $\tilde{E}$ denote the principal fibration induced by $w = w_4$. Hence we obtain the diagram (notation is taken from Proposition 2.12)

$$
\begin{array}{ccc}
K(\mathbb{Z}_2, 3) & \xrightarrow{i} & \tilde{E} \\
\downarrow & & \downarrow \rho \\
B_3 & \xrightarrow{\tilde{q}} & B_5
\end{array}
$$

Since the pair $(w, \alpha) = (w_4, \text{Sq}^2)$ is admissible we obtain

**Proposition 3.1** (Theorem 6.4 in [Tho67a]). There is an $m \in H^5(B_5)$ such that

$$U_{\tilde{E}} \sim (k + \tilde{p}^* m) \in \Omega(U_{\tilde{E}})$$

where $k$ is the generator of $H^5(\tilde{E};\mathbb{Z}_2)$.

With Proposition 2.12 and the naturality of secondary cohomology operations we conclude

**Corollary 3.2.** There is an $m \in H^5(B_5)$ such that

$$U_{\tilde{E}} \sim (k_2 + p^* m) \in \Omega(U_{\tilde{E}}).$$

**Corollary 3.3.** Suppose $\xi$ is an oriented vector bundle of rank 5 over a 5-manifold $M$ such that $w_2(\xi) = w_4(\xi) = 0$ and $p_1(\xi)$ is divisible by 5. Then there is a map $\eta: M \to E$ such that $p \circ \eta = \xi$ (where $p: E \to B_5$ is the fibration of Proposition 2.3) and

$$U_M \sim \eta^* k_2 \in \Omega(U_M).$$

**Proof.** By Proposition 2.3 and 2.8 $\xi: M \to B_5$ lifts to a map $\eta: M \to E$ with $p \circ \eta = \xi$ since $w_4(\xi) = 0$ and $p_1(\xi)$ is divisible by 5. Furthermore $H^5(B_5;\mathbb{Z}_2)$ is generated by $w_2 \sim w_3$ and $w_5$ and since $w_2(\xi) = 0$ and $w_5(\xi) = \text{Sq}^1 w_4(\xi)$ it follows that $\eta^* p_1^* m = 0$ for all $m \in H^5(B_5;\mathbb{Z}_2)$. The claim follows by naturality of $\Omega$. ■
Corollary 3.4. Let $\xi$ and $M$ be like in Corollary 3.3. Then $k_2(\xi)$ contains only one element, which we denote by the same symbol. Furthermore we obtain

$$U_M \sim k_2(\xi) = \Omega(U_M).$$

Moreover $\xi$ possess an irreducible $\text{SO}(3)$-structure if and only if $k_2(\xi) = 0$.

Proof. The indeterminacy subgroup of $\Omega(U_M)$ is given by $\text{Sq}^2 H^3(T_M)$ and that of $k_2(\xi)$ is $\text{Sq}^2 H^3(M) + w_2(M) \sim H^3(M)$ (coefficients in $\mathbb{Z}_2$ are to be understood). By Poincare duality, the Wu class of $\text{Sq}^2$ is $w_2(\xi)$, hence the indeterminacy is zero in both cases. Therefore $k_2(\xi)$ contains a single element $\kappa \in k_2(\xi)$ as well as $\Omega(U_M)$. The existence of an irreducible $\text{SO}(3)$-structure if and only if $\kappa = 0$ follows from Remark 2.11. ■

Now, because of Corollary 3.4 we can determine $k_2(\xi)$ if we compute $\Omega(U_M)$, when $M$ is a spin 5-manifold and $\xi$ the tangent bundle of $M$.

Theorem 3.5 ([Tho67a, Theorem 2.2]). Let $M$ be a closed spin manifold of dimension $m$, where $m \equiv 0$ or $1 \mod 4$ and $m > 4$. If $m$ is odd assume that $w_{m-1}(M) = 0$, while if $m$ even assume that $w_m(M) = 0$. Then the operation $\Omega$ is defined on $U_M$ and $\Omega(U_M) = U \sim (I_2 M \mu)$ with zero indeterminacy. The number $I_2 M$ is the mod 2 index of a 2-field with finite singularities on $M$ (see p. 89 [Tho67a]) and $\mu$ is the generator of $H^m(M; \mathbb{Z}_2)$.

In the next theorem Thomas found a cohomological interpretation of the number $I_2 M$

Theorem 3.6 ([Tho67a Theorem 1.1]). Let $M$ be a closed spin manifold of dimension $4k + 1$, $k > 0$, such that $w_{4k}(M) = 0$. Then

$$I_2 M = \hat{\chi}(M),$$

where $\hat{\chi}(M)$ is the semi-characteristic of $M$ defined in section 1.

Combining the two theorems above, we obtain

Corollary 3.7. Let $M$ be a closed spin 5-manifold such that $w_4(M) = 0$. Then we have

$$\Omega(U_M) = U_M \sim (\hat{\chi}(M) \mu)$$

where $\mu \in H^5(M; \mathbb{Z}_2)$ is the generator.

Finally we are ready for the

Proof of Theorem 1.4(a). Note first, that if $w_4(M) = 0$ it follows from corollaries 3.4 and 3.7 that

$$U_M \sim k_2(\tau_M) = U_M \sim (\hat{\chi}(M) \mu),$$

hence $k_2(\tau_M) = \hat{\chi}(M)$.

Now suppose first that $M$ satisfies conditions (i) and (ii). Let $\tau_M: M \to B_5$ be a classifying map for the tangent bundle of $M$. Condition (i) implies $\tau^*_M(k_1) = 0$ (cf. Proposition 2.9), hence by Proposition 2.9 (b) $\tau_M$ possess a lift $\tau^*_M: M \to E$. Since $k_2(\tau_M) = \hat{\chi}(M) = 0$ by condition (ii), it follows from Corollary 2.5 that $\tau_M$ lifts to $B_3$. Therefore $M$ admits an irreducible $\text{SO}(3)$-structure.
On the other hand assume that \( M \) admits an irreducible \( \text{SO}(3) \)-structure. Then by Proposition [27] we have \( w_4(M) = 0 \) and \( p_1(M) \) is divisible by five which implies condition (i). From this we have \( \tau^*_M(k_1) = 0 \) and therefore \( k_2(\tau_M) \) is defined (cf. Definition [24]). Since \( \tilde{\xi}(M) = k_2(\tau_M) \) we have to argue that \( k_2(\tau_M) = 0 \) to prove condition (ii). By assumption \( M \) admits an irreducible \( \text{SO}(3) \)-structure, hence there exists a lift of \( \tau_M \) to a map \( M \to B_3 \) for the fibration \( B \rho \) and this implies \( k_2(\tau_M) = 0 \) by Corollary [25].

In Theorem [14](a) we saw that an irreducible \( \text{SO}(3) \)-manifold \( M \) with \( w_2(M) = 0 \) possesses also a standard structure. Hence there are two different descriptions of the tangent bundle of \( M \). We close this section with a proposition which compares these two descriptions. In Lemma 3.1 of [ABBF11] the existence of an irreducible \( \text{SO}(3) \)-structure is equivalent to the existence of a 3-dimensional vector bundle \( \xi \) over \( M \) such that the tangent bundle \( \tau_M \) is isomorphic to the symmetric trace-free endomorphisms \( \text{Sym}_3(\xi) \) of \( \xi \). On the other side, since \( M \) admits also a standard \( \text{SO}(3) \)-structure, the tangent bundle is isomorphic to \( \eta \oplus \epsilon^2 \), where \( \eta \) is a rank 3 vector bundle and \( \epsilon^2 \) the trivial rank 2 vector bundle over \( M \). Finally to formulate the proposition, we have to introduce a certain kind of operation on rank 3 vector bundles over 5-manifolds:

Let \( \xi \) be a 3-dimensional vector bundle with a spin structure over \( M \). Since \( M \) is of dimension 5, we have \( \xi \in [M, \text{HP}^{\infty}] \cong [M, S^4] \). Let \( g: S^4 \to S^4 \) be a map of degree 5. Then \( g \) induces a map \( G: [M, S^4] \to [M, S^4] \), such that \( G(\xi) = g \circ \xi \). We denote by the same letter \( G \) the induced map on \([M, \text{HP}^{\infty}]\). It is clear that \( G \) depends only on the homotopy class of \( g \).

**Proposition 3.8.** Let \( M \) be a spin 5-manifold such that \( H^4(M; \mathbb{Z}) \) has no 2-torsion. Suppose furthermore that \( M \) admits an irreducible \( \text{SO}(3) \)-structure. Let \( \xi \) be 3-dimensional vector bundle with spin structure such that

\[
\tau_M \cong \text{Sym}_3(\xi).
\]

Then we have

\[
\tau_M \cong G(\xi) \oplus \epsilon^2.
\]

**Proof.** It is known that \( H^*(\text{HP}^{\infty}; \mathbb{Z}) = \mathbb{Z}[u] \) where \( u \in H^4(\text{HP}^{\infty}; \mathbb{Z}) \). Moreover we have \( G^* u = 5u \) by definition and \( p_1 = 4u \) where \( p_1 \) generates the algebra \( H^* (\text{BSO}(3); \mathbb{Z})/\text{torsion} \). Therefore by naturality we obtain \( p_1(G(\xi)) = 5p_1(\xi) = p_1(M) \) and \( w_2(G(\xi)) = 0 \). By Lemma 1 of [Tho68] we have that \( \tau_M \) and \( G(\xi) \oplus \epsilon^2 \) are stably isomorphic and by Lemma 3 of the same paper we obtain that they are isomorphic, since \( M \) admits a standard \( \text{SO}(3) \)-structure.

### 4 Proof of Theorem [14](b)

In this section we work under the assumptions of Theorem [14](b) and let \( \mathcal{P}: H^2(M; \mathbb{Z}_2) \to H^4(M; \mathbb{Z}_4) \) denote the Pontryagin square. Under these conditions the authors of [CV93] prove the following:

**Theorem 4.1.** Let \( \xi \) be a vector bundle of rank 5 over \( M \). Then \( \xi \) is uniquely determined by \( w_2(\xi), w_4(\xi) \) and \( p_1(\xi) \) such that

\[
\rho_4 p_1(\xi) = \mathcal{P} w_2(\xi) + i_4 w_4(\xi),
\]

where \( \rho_4 \) is the mod 4 reduction of an integral cohomology class and \( i_4: H^*(M, \mathbb{Z}_2) \to H^*(M, \mathbb{Z}_4) \) is the induced map from the inclusion \( \mathbb{Z}_2 \to \mathbb{Z}_4 \).
**Theorem 4.2.** For every $W \in H^2(M; \mathbb{Z}_2)$ and $P \in H^4(M; \mathbb{Z})$ there exists a 3-dimensional vector bundle $\eta$ over $M$ with $w_2(\eta) = W$ and $p_1(\eta) = P$ if and only if

$$\rho_4 P = PW$$

**Remark 4.3.** Theorem 4.2 was also proved by Woodward in [?], p. 514 and by Antieau/Williams in [?], Theorem 1. Note that the minus sign in equation (2) of [?], Theorem 1 is not correct. It should be either stated without the minus sign (which yields the same equation as in Theorem 4.2) or the authors should use the first Chern class of the complexified bundle and keep the minus sign (since the first Chern class is the negative of the first Pontryagin class). Compare also [?, p. 514] or [ˇCV93, Theorem 2] for the correct signs.

Furthermore in [ABBF11] it was proven

**Theorem 4.4.** A 5-manifold $M$ admits an irreducible $SO(3)$-structure if and only if there exists a three-dimensional oriented vector bundle $\eta$ over $M$ such that the tangent bundle is isomorphic to the bundle of symmetric trace-free endomorphisms of $\eta$.

Combining these three theorems leads us to the

**Proof of Theorem 1.4** (b). Suppose first that $p_1(M)$ is divisible by 5 and $w_4(M) = 0$. Then there is a $P \in H^4(X; \mathbb{Z})$ such that $p_1(M) = 5 \cdot P$. Moreover by Theorem 4.1 we have the relation $\rho_4 p_1(M) = P w_2(M)$. Hence with coefficients in $\mathbb{Z}_4$ we obtain

$$\rho_4 P = 5 \cdot \rho_4 P = \rho_4 5 \cdot P = \rho_4 p_1(M) = P w_2(M).$$

Hence by Theorem 4.2 there exists a vector bundle $\eta$ of rank 3 such that $p_1(\eta) = P$ and $w_2(\eta) = w_2(M)$. For the induced bundle of symmetric trace-free endomorphisms of $\eta$, $\zeta := \text{Sym}_0(\eta)$ we have

$$p_1(\zeta) = p_1(M), \quad w_2(\zeta) = w_2(M), \quad w_4(\zeta) = 0$$

(see [ABBF11, Theorem 3.2]). By Theorem 4.1 we have that $\zeta$ is isomorphic to the tangent bundle of $M$ and with Theorem 4.4 this proves that $M$ admits an irreducible $SO(3)$-structure. Now let $M$ admit an irreducible $SO(3)$-structure. Then by Proposition 2.7 we conclude $p_1(M)$ is divisible by 5 and $w_4(M) = 0$. 

**Remark 4.5.** We believe that in the general case (i.e. without the assumption on $H^4(M; \mathbb{Z})$) the theorem should be true anyway. It is also reasonable to ask if a similar approach as in [Ati70] could be used for the existence of an irreducible $SO(3)$-structure.

## 5 Examples

In [BN07] the authors classified homogeneous manifolds with irreducible $SO(3)$-structures and in [ABBF11] there were also some non-homogeneous examples mentioned as circle bundles over complex surfaces. In [ABBF11] it was shown that $SU(3)/SO(3)$ does not possess a standard $SO(3)$-structure but possesses an irreducible one with $\hat{\chi}(M) = 0$. So it was conjectured that $\hat{\chi}(M)$ is the second obstruction. However Proposition 1.8 is counterexample.
Proof of Proposition 1.8. Let \( \Sigma_d \) be the zero set of the homogeneous polynomial
\[
p_d(X_0, X_1, X_2, X_3) = X_0^d + X_1^d + X_2^d + X_3^d
\]
in \( \mathbb{C}P^3 \) for \( d \in \mathbb{N} \) and \( d \neq 0 \). \( \Sigma_d \) is a 4-dimensional submanifold which is simply connected by the Lefschetz hyperplane section theorem. Let \( u \in H^2(\mathbb{C}P^3; \mathbb{Z}) \) be such that \( H^*(\mathbb{C}P^3; \mathbb{Z}) = \mathbb{Z}[u]/(u^4) \). We denote by the same letter \( u \) the restriction of the generator of \( H^*(\mathbb{C}P^3; \mathbb{Z}) \) to \( \Sigma_d \). The following facts about \( \Sigma_d \) are well known

(a) The group \( H^2(\Sigma_d; \mathbb{Z}) \) has no torsion and its rank is equal to \( \chi(\Sigma_d) - 2 = (6 - 4d + d^2)d - 2 \),
(b) the first and second Chern classes are given by \( c_1(\Sigma) = (4 - d)u \) and \( c_2(\Sigma) = (6 - 4d + d^2)u^2 \) respectively,
(c) for the first Pontryagin class we have \( p_1(\Sigma) = (4 - d^2)u^2 \) and for the second Stiefel-Whitney class one obtains \( w_2(\Sigma) = d \cdot u \mod{2} \).

Let us consider the case \( d = 3 \). We set \( \Sigma := \Sigma_3 \) and let \( \beta \) denote the intersection form of \( \Sigma \). It is known that this space is diffeomorphic to \( \mathbb{C}P^2 \# 6 \mathbb{C}P^2 \). Furthermore let \( w \in H^2(\Sigma; \mathbb{Z}) \) such that \( w \) is orthogonal to \( u \) with respect to \( \beta \) and \( w \neq u \), i.e. \( w \sim u = 0 \). Let \( \pi: M \to \Sigma \) be the \( S^1 \)-bundle over \( \Sigma \) associated to the class \( c := u + w \). Using the Gysin-sequence for sphere bundles one obtains (see also Remark 3.3 in [ABBFT])

(a) \( k(M) = \hat{\chi}(M) = \dim_{\mathbb{R}} H^2(\Sigma; \mathbb{R}) \mod{2} \), hence \( k(M) = \hat{\chi}(M) = 1 \),
(b) \( M \) is not spin, since this is only the case if \( c \equiv w_2(\Sigma) \mod{2} \),
(c) we have \( c \sim 5u = 5u^2 + u \sim w = 5u^2 = p_1(\Sigma) \), hence by the Gysin-sequence \( \pi^*(p_1(\Sigma)) = 0 \) thus \( p_1(M) = 0 \),
(d) we also have \( w_4(M) = \pi^*(w_4(\Sigma)) \). But \( w_4(\Sigma) \equiv u^2 \mod{2} \). Again we have \( c \sim 3u = c_2(\Sigma) \) hence by naturality we obtain \( w_4(M) = \pi^*(w_4(\Sigma)) = 0 \).

This shows that \( M \) is a non spin manifold of dimension 5 with \( p_1(M) = 0 \) and \( w_4(M) = 0 \). Moreover \( H^4(M; \mathbb{Z}) \) has no 4-torsion which can be seen as follows: by the Gysin-sequence we have that \( H^4(M; \mathbb{Z}) \) is isomorphic to \( H^4(\Sigma; \mathbb{Z}) \equiv \mathbb{Z} \mod{2} \) modulo the image of the map \( H^2(\Sigma; \mathbb{Z}) \to H^4(\Sigma; \mathbb{Z}) \), \( x \mapsto c \sim x \). We can choose \( w \) in such a way that the image is \( 3\mathbb{Z} \), hence \( H^4(M; \mathbb{Z}) = \mathbb{Z}_3 \) which means, that \( M \) can not be simply connected and furthermore that \( H^4(M; \mathbb{Z}) \) has no 4-torsion. So by Theorem 1.4(b) \( M \) has to admit an irreducible \( SO(3) \)-structure with \( \hat{\chi}(M) = 1 \).

Proof of Proposition 1.9. If \( M_1 \) and \( M_2 \) are spin 5-manifolds then, the connected sum \( M_1 \# M_2 \) is again spin. The Kervaire semi-characteristic is computed by
\[
k(M_1 \# M_2) = k(M_1) + k(M_2) + 1 \mod{2}.
\]
Furthermore the first Pontryagin and the fourth Stiefel-Whitney class are additive under the operation of building connected sums. Hence we obtain
\[
k(\#_{i=1}^{2k+1} M_i) = \sum_{i=1}^{2k+1} k(M_i) + 2k \mod{2} = \sum_{i=1}^{2k+1} k(M_i) \mod{2}.
\]

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Now since $M_i$ is a spin $5$–manifold with an irreducible $\text{SO}(3)$–structure it admits a standard $\text{SO}(3)$–structure (cf. Corollary 1.6), hence by Theorem 1.3 $k(M_i) = 0$ for all $i = 1, \ldots, 2k + 1$. Thus $k(\#_{i=1}^{2k+1} M_i) = 0$ and $p_1(\#_{i=1}^{2k+1} M_i)$ is divisible by 5, so #_{i=1}^{2k+1} M_i admits an irreducible $\text{SO}(3)$–structure by Corollary 1.6.

Example 5.1. Let $N$ be a closed 3-manifold and $\Sigma$ a closed surface both orientable. We will show that connected sums of $N \times \Sigma$ admit an irreducible $\text{SO}(3)$-structure. First note that both manifolds are spin. Moreover Steenrod showed that the tangent bundle of $N$ is always parallelizable. Hence $M = N \times \Sigma$ admits an irreducible $\text{SO}(3)$-structure. By Proposition 1.9 the manifolds #_{i=1}^{2k+1}(N_i \times \Sigma_i) have an irreducible $\text{SO}(3)$-structure where $N_i$ and $\Sigma_i$ are orientable closed manifolds of dimension 3 and 2 respectively.

Example 5.2. There are two $\mathbb{S}^3$ bundles over $\mathbb{S}^2$. One is the trivial bundle covered by Example 5.1 and the other we denote by $\pi: M \to \mathbb{S}^2$. In the following we will prove that $M$ admits an irreducible $\text{SO}(3)$-structure. $M$ is simply connected with $H_2(M) = \mathbb{Z}$, $H^3(M; \mathbb{Z}) = 0$ and $w_2(M) \neq 0$. Let $\tau_M$ and $\tau_{\mathbb{S}^2}$ be the tangent bundle of $M$ and $\mathbb{S}^2$ respectively. Then $\tau_M = \eta \oplus \tau_{\mathbb{S}^2}$ with $\eta$ the vertical distribution of $\pi: M \to \mathbb{S}^2$. It follows that $0 \neq w_2(M) = w_2(\eta)$ and $p_1(M) = p_1(\eta) = 0$. Hence by Theorem 1.2 4.1 and 4.4 we have that $\tau_M \cong \text{Sym}_0(\eta)$. By Corollary 1.5(b) $M$ admits an irreducible $\text{SO}(3)$-structure.

Remark 5.3. Note that every fibre bundle $\pi: M \to \mathbb{S}^3$ is trivial if $M$ is orientable and of dimension 5, since $\pi_2$ of the diffeomorphism group of an orientable surface always vanishes.

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