EFFECTS OF ANISOTROPIES IN TURBULENT MAGNETIC DIFFUSION IN MEAN-FIELD SOLAR DYNAMO MODELS

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ABSTRACT

We study how anisotropies of turbulent diffusion affect the evolution of large-scale magnetic fields and the dynamo process on the Sun. The effect of anisotropy is calculated in a mean-field magnetohydrodynamics framework assuming that triple correlations provide relaxation to the turbulent electromotive force (so-called “minimal r-approximation”). We examine two types of mean-field dynamo models: the well-known benchmark flux-transport model and a distributed-dynamo model with a subsurface rotational shear layer. For both models, we investigate effects of the double- and triple-cell meridional circulation, recently suggested by helioseismology and numerical simulations. To characterize the anisotropy effects, we introduce a parameter of anisotropy as a ratio of the radial and horizontal intensities of turbulent mixing. It is found that the anisotropy affects the distribution of magnetic fields inside the convection zone. The concentration of the magnetic flux near the bottom and top boundaries of the convection zone is greater when the anisotropy is stronger. It is shown that the critical dynamo number and the dynamo period approach to constant values for large values of the anisotropy parameter. The anisotropy reduces the overlap of toroidal magnetic fields generated in subsequent dynamo cycles, in the time-latitude “butterfly” diagram. If we assume that sunspots are formed in the vicinity of the subsurface shear layer, then the distributed dynamo model with the anisotropic diffusivity satisfies the observational constraints from helioseismology and is consistent with the value of effective turbulent diffusion estimated from the dynamics of surface magnetic fields.

Key words: convection – dynamo – magnetohydrodynamics (MHD) – Sun: activity – Sun: magnetic fields

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1. INTRODUCTION

It has long been assumed that turbulent magnetic diffusion (or eddy magnetic diffusivity) is an important part of the hydromagnetic dynamo process on the Sun (Parker 1955). It transfers the energy of magnetic fields from large to small scales and determines the characteristic spatial scale of the excited dynamo waves (the butterfly diagram) in better agreement with observations. This fact was extensively used in various solar dynamo models (Kitchatinov et al. 2000; Kitchatinov 2002; Pipin & Kosovichev 2011a; Pipin 2013).

However, properties of the anisotropic magnetic eddy diffusivity, which depends on the impact of solar rotation on convective motions, remain uncertain. Results of theoretical calculations of magnetic turbulent diffusivity coefficients strongly depend on assumed models of background turbulent flows. Mean-field magnetohydrodynamics calculations show that anisotropy of the diffusivity coefficients is strong for the regime of fast rotation, when the Coriolis number \( \Omega^* = 2\Omega_0\tau_c > 1 \); here \( \Omega_0 \) is the angular velocity, and \( \tau_c \) is a typical convective turnover time. This regime can be found in the lower part of the solar convection zone. In the upper part, \( \Omega^* \ll 1 \), and the rotationally induced anisotropy is small. Furthermore, the numerical simulations, based on a test field method (see, e.g., Käpylä et al. 2009; Brandenburg et al. 2012), confirm the analytical calculations of the rotationally induced anisotropy effects in the mean electromotive force and turbulent magnetic diffusivity. In addition, the global numerical simulations reveal a strong nonrotational anisotropy of convection in the upper part of the convection zone, where the Coriolis number is small (see, e.g., Miesch et al. 2008; Racine et al. 2011; Guerrero et al. 2013a). Similar results are suggested by the nonlocal stellar convection theory (Deng et al. 2006). At present, the origin of this anisotropy is unclear. It may be relevant to the formation of the subsurface shear layer (see, e.g., Käpylä et al. 2012; Guerrero et al. 2013b).

In theoretical dynamo calculations, this anisotropic diffusion can be taken into account by introducing an additional parameter to model the anisotropy of background turbulent flows in terms of the relative difference of root mean square velocity fluctuations of radial and horizontal flow components (Equation (A17)). Such approach has already been used in...
a mean-field model of the solar differential rotation (Kitchatinov 2004, 2011) and provided an explanation of the subsurface shear of the solar angular velocity. In our paper, we extend this idea and compute the magnetic diffusivity tensor for a range of values of the parameter of anisotropy. The calculations are performed using the so-called minimal approximation of the mean field magnetohydrodynamics (Blackman & Field 2002; Rädler et al. 2003; Brandenburg & Subramanian 2005). Given the previous results of Parker (1971) and Kitchatinov (2002), we expect that the anisotropy resulting from additional horizontal diffusion of magnetic field can change the direction of the dynamo wave propagation and increase the horizontal scale of the mean magnetic field. We find that this effect also decreases the overlap between the butterfly wings of the time-latitude diagrams of the large-scale toroidal magnetic field evolution, which improves agreement of the dynamo model with observations.

The paper is structured as follows. In the next section, we briefly outline the basic equations and assumptions. Next, we examine the simplified benchmark model suggested by Jouve et al. (2008) and then investigate the anisotropy effects in more detailed mean-field models (Pipin et al. 2013; Pipin & Kosovichev 2013b), which include the subsurface rotational shear layer and the double-cell meridional circulation suggested by recent helioseismology results (Zhao et al. 2013). Also, we present results for the triple-cell meridional circulation revealed in some recent numerical simulations (see, e.g., Käpylä et al. 2012; Guerrero et al. 2013a). In Section 3, we summarize the main results. Some mathematical details are given in the Appendix.

2. BASIC EQUATIONS

We decompose the flow, $\mathbf{U}$, and magnetic field, $\mathbf{B}$, into sums of their mean and fluctuating parts: $\mathbf{U} = \mathbf{U} + \mathbf{u}$, $\mathbf{B} = \mathbf{B} + \mathbf{b}$, where $\mathbf{U}$ and $\mathbf{B}$ represent the mean large-scale fields. Hereafter, we use small letters for the fluctuating parts and capital letters for the mean fields. The mean effect of the fluctuating turbulent flows and magnetic fields on the large-scale mean fields is described by the mean electromotive force, $\mathcal{E} = \mathbf{a} \times \mathbf{b}$, where the averaging is performed over an ensemble of the fluctuating fields. Following the two-scale approximation (Roberts & Soward 1975; Krause & Rädler 1980), we assume that the mean fields vary over much larger scales (in time and space) than do the fluctuating fields. The governing equations for the fluctuating magnetic field and velocity are written in a rotating coordinate system as follows:

$$\frac{\partial \mathbf{b}}{\partial t} = \nabla \times (\mathbf{u} \times \mathbf{B}) + \eta \nabla^2 \mathbf{b} + \mathbf{\mathcal{E}},$$

$$\frac{\partial \mathbf{u}_i}{\partial t} + 2(\Omega \times \mathbf{u})_i = -\nabla_i \left( p + \frac{\mathbf{B} \cdot \mathbf{b}}{\mu_0} \right) + \nu \Delta \mathbf{u}_i + \frac{1}{\mu_0} \nabla \cdot \mathbf{f}_i + \mathcal{F}_i,$$  \hspace{1cm} (2)

where $\mathcal{E}$ and $\mathcal{F}$ denote nonlinear contributions of the fluctuating fields, $p$ is the fluctuating pressure, $\Omega$ is the angular velocity, $\mathbf{f}$ is a random force driving the turbulence. Equations (1) and (2) are used to compute the mean electromotive force, $\mathcal{E}$. Details of the calculations are given in the Appendix.

The effect of rotational anisotropy of turbulent diffusivity results in a reduction of the isotropic part of turbulent diffusivity and an increase of the diffusivity along the rotation axis (Kitchatinov et al. 1994; Brandenburg et al. 2008). The rotational anisotropy is significant for the case of the fast rotation regime, when the typical turnover time of convective flows is of the order of magnitude of the global rotation period. In the upper part of the convection zone, where the turnover time is short, the rotational anisotropy is expected to be weak. In stratified turbulent convection, there is also nonrotational anisotropy of diffusivity that is not directly related to effect of the global rotation. It is characterized by the direction of anisotropy, $\mathbf{g}$, and the anisotropy parameter $a$ representing the ratio of the turbulent mixing along and perpendicular to $\mathbf{g}$. We expect that this nonrotational anisotropy can be significant for the slow-rotation dynamo regimes. In contrast, the effect of the global rotation decreases the nonrotational anisotropy of turbulent diffusivity (see Equations (A20)–(A23)). Thus, we take into account the influence of rotation on both the isotropic and anisotropic parts of the background turbulent flows. These are almost equally affected (reduced) as a result of rotation. Therefore, their ratio does not depend on the Coriolis number. In a simple case, when we disregard the effect of the Coriolis force, the magnetic diffusion can be written as follows:

$$\mathcal{E}^{\text{diff}} = -\eta_T \nabla \times \mathbf{B} - \frac{a}{2} \eta_T (\nabla - \mathbf{g} (\mathbf{g} \cdot \nabla)) \times \mathbf{B},$$

where the first term describes the isotropic diffusion, and the second term describes the nonrotational anisotropy, where $\mathbf{g}$ is a unit vector in the direction of anisotropy. The parameter, $a$, quantifies the level of the anisotropy (Equation (A17)). For a general case, the anisotropic part of diffusion takes into account the effect of the global rotation, and it is given by Equation (A18). The mixing length theory of stellar convection requires that $0 \leq a \leq 4$ (see Kitchatinov 2004; Rüdiger et al. 2005). Numerical simulations indicate higher values up to $a \sim 10$ (see, e.g., Figure 13 in Miesch et al. 2008).

From theoretical and numerical studies, it is expected that the nonrotational anisotropy of the background turbulent flows affects all parts of the mean electromotive force. However, because of the complexity of the full treatment, in this paper we consider corrections only to the diffusive part the electromotive force and do not discuss the inductive part.

We start with the standard mean-field induction equation in perfectly conductive media:

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{\mathcal{E}} + \mathbf{U} \times \mathbf{B}),$$

where $\mathcal{E}$ is the mean electromotive force, $\mathbf{U} = \mathbf{e}_\theta r \sin \theta \Omega (r, \theta) + \mathbf{U}^\theta (r, \theta)$ is the mean flow that includes the differential rotation, $\Omega (r, \theta)$, and meridional circulation, $\mathbf{U}^\theta (r, \theta)$; the axisymmetric magnetic field is given as follows:

$$\mathbf{B} = \mathbf{e}_\theta B + \nabla \times \frac{A \mathbf{e}_\theta}{r \sin \theta},$$

where $r$ is the radius, and $\theta$ is the polar angle, $B (r, \theta)$ is the strength of the toroidal component of magnetic field, and $A (r, \theta)$ represents vector potential of the poloidal component.

2.1. Benchmark Models Design

In this section, we examine the effect of the anisotropic mixing (hereafter referred to as “nonrotational anisotropy”) using the benchmark model presented by Jouve et al. (2008, hereafter J08). In this case, we use the simplest representation
Parameter $CS$ is the background level of the magnetic turbulent diffusivity. Thus, we have the following dynamo equations:

$$\frac{\partial A}{\partial T} = \eta \frac{\partial^2 A}{\partial x^2} + \frac{\eta}{2} \left(1 + a \frac{\sin \theta}{2x} \frac{\partial}{\partial x} - \frac{1}{2x^2} \frac{\partial^2}{\partial x^2} \frac{\partial A}{\partial x} + \frac{1}{2x} \frac{\partial}{\partial x} \frac{\partial A}{\partial \theta} + \frac{1}{2x} \frac{\partial}{\partial x} \frac{\partial A}{\partial \theta} \right)$$

and in the case of multiple circulation cells along the radius and two cells along latitude is the following:

$$\Psi = \frac{2c_0}{\pi \left(1 + \exp(-200(x-x_0))\right)} \left(1 - \frac{1}{x} \right)^{1.5} \left(\frac{\partial P_2}{\partial \theta} + m \frac{\partial P_3}{\partial \theta} \right)$$

where $P_{2,3}$ are the Legendre polynomials of $\cos \theta$, $x_0 = r_b/R$ is the inner boundary of the integration domain; parameter $m$ controls the number of cells along latitude; $c_0$ is a normalization constant; $c_1$, $c_2$ and $c_3$ control the amplitudes of flows in the case of multiple circulation cells. The velocity field is the flow given by $\mathbf{V}' = U_0 \nabla \times (c_0 \Psi)$, in the case of single-cell circulation (model C1, Equation (6), Figure 1(d)), and $\mathbf{V}' = (U_0/\bar{r}) \nabla \times (c_0 \Psi)$ for the models C2 and C3 with the multiple cells. Parameter $U_0$ is a characteristic flow speed. In cases C2 and C3, we choose $x_0 = 0.71$ to cut off penetration of the meridional circulation below $r = 0.7 R_\odot$. The stream function is similar to the one of Pipin & Kosovichev (2013b) with a modification to control the penetration of the meridional
circulation below the convection zone. This circulation pattern is illustrated in Figure 1(e). We use the same parameter value for the \( \Omega \)-effect, as in the J08, \( C_T = 1.4 \times 10^3 \). The parameters of the benchmark models are listed in the Table 1.

In summary, we conclude that models B and C1 correspond to those studied in J08. Models C2 and C3 are given for comparison. The case C2 is motivated by the recent results of helioseismology results (Zhao et al. 2013). In the set of the benchmark models, we distinguish two types of dynamo: model B represents the dynamo distributed in the solar convection zone; C1, C2, and C3 belong to the family of the flux-transport dynamo with high diffusivity. The radial profile of the latitudinal component of meridional circulation (see Figure 1(c), blue curve) and geometry of the flow Figure 1(e) are close to results determined by the helioseismology inversion. Numerical simulations of the solar convection zone often produce the multicular meridional circulation patterns, which, in some cases, have three cells stacked along the radial direction (Käpylä et al. 2012; Guerrero et al. 2013a). This circulation type is addressed in our model C3 illustrated in Figure 1(c) (red curve), and Figure 1(f). In this case, the circulation pattern only qualitatively reproduces the numerical simulations. This case helps us to highlight the important difference between the dynamo models with odd and even numbers of the circulation cells stacked along the radius.

### 2.2. Results for Benchmark Models

In the section, we investigate the dynamo instability for the benchmark models listed in the Table 1. The dynamo instability develops when the nondimensional parameter, \( C_\alpha \), which controls the magnitude of the \( \alpha \)-effect, or, parameter \( C_\gamma \), which controls the Coriolis force acting on the flux-tube rising through the solar convection zone, exceed some critical values. Figure 2(a) shows the critical threshold parameters \( C_\alpha^{(cr)} \) and \( C_\gamma^{(cr)} \) for dipole-type modes (antisymmetric relative to the equator) as a function of the anisotropy parameter, \( a \). Figure 2(b) shows the eigenfrequency, \( \omega \), of the first unstable mode. For cases B and C1 studied by Jouve et al. (2008), we get \( C_\alpha^{(cr)} = 0.408 \), \( \omega = 173 \) and \( C_\gamma^{(cr)} = 2.53 \), \( \omega = 534 \) for \( a = 0 \), in perfect agreement with J08. The main result is that the critical threshold dynamo parameters, as well as the eigenfrequency (and period) of the dynamo oscillations, vary rather little with variation of \( a \). Moreover, in cases B and C2 (even number of circulation cells), the dynamo threshold is slowly growing with increase of \( a \). In contrast, in cases C1 and C3 (odd number of cells), the dynamo threshold is slowly decreasing with increasing \( a \). As we have previously suggested (Pipin & Kosovichev 2013a), there is a similarity of the dynamo regimes operating with even or odd number of the circulation cells along the radius. Another interesting finding is that the dynamo period is growing with the increasing number of circulation cells (see also Hazra et al. 2014). However, this is not directly related to the nonrotational anisotropy effect.

Figures 2(a) and (b) show the dynamo threshold parameters for the first unstable dipole-type eigenmodes. The threshold parameters of the quadrupole-type modes vary in a similar way. For model B, the first unstable dipole mode has smaller \( C_\alpha^{(cr)} \) than the first unstable quadrupole type mode for \( a < 2 \). Both modes have similar eigenfrequencies. For model C1, the first unstable dipole-type mode is preferable in the whole range of anisotropy parameter \( a \). Meanwhile, the eigenfrequency of the first unstable quadrupole-type mode is as twice as smaller the frequency of the first unstable dipole-type mode. In model C2, the first unstable mode has the quadrupole-type symmetry for all values of \( a \). The opposite is true for model C3. Table 1 provides a summary for the parity preference.

Typical snapshots of the magnetic field distributions for models B, C1, C2, and C3, for the case of \( a = 4 \) are illustrated in

![Figure 2](https://example.com/figure2.png)

Figure 2. Panel (a) shows the dynamo instability threshold for different models. For model B, it shows the critical parameter \( C_\alpha^{(cr)} \); for models C1, C2, and C3, it shows critical parameter \( C_\gamma^{(cr)} \) (see Equation (5)); panel (b) shows the oscillation eigenfrequency (in units of the diffusive time) of the first unstable dipole- and quadrupole-like modes.

(A color version of this figure is available in the online journal.)

### Table 1

| Model | Type          | \( \alpha \)-effect | B-L term | Circulation | Parity |
|-------|---------------|----------------------|----------|-------------|--------|
|       |               |                      | c_1      | c_2         | c_3    |
| B     | Distributed   | +                    | –        | –           | –      | A      | S      |
| C1    | Flux-transport| –                    | +        | 1           | –      | A      | A      |
| C2    | –/-           | –                    | +        | 0.5         | 1.5    | S      | S      |
| C3    | –/-           | –                    | +        | 0           | 1      | 2.5    | A      | A      |
Figure 3. Snapshots of the large-scale magnetic field inside the convection zone for the benchmark dynamo models for the nonrotational anisotropy with parameter $a = 4$. The field lines show the poloidal component of the mean magnetic field, and the toroidal magnetic field is shown by the background images. Snapshots are taken for periods of polar poloidal field maxima. Panel (a) shows model B; panels (b), (c), and (d) show the same for models C1, C2, and C3.

Figure 4. Time–latitude evolution of the toroidal magnetic field at the bottom of the convection zone (contours) and the radial magnetic field at the surface (background image) for the benchmark dynamo models (Table 1): (a) model B for the anisotropy parameter $a = 4$; (b) model C1; (c) model C2; and (d) model C3.

Figure 3. The time-latitude (butterfly) diagrams of the toroidal magnetic field at $r = 0.7 R_\odot$, and the radial magnetic field at the surface are shown in Figure 4. Models C1 and C3 have some qualitative agreement with solar observations. However, the butterfly wings of the toroidal magnetic field are too wide compared to the observations. Also, for the large anisotropy parameter, $a = 4$, model C1 gives an incorrect phase relation between the maxima of the toroidal magnetic field in the equatorial region and the reversals of the radial magnetic field at the poles. Model C3 better reproduces the phase relation,
although the reversals of the polar field occur about five years earlier than the maxima of the toroidal field in the equatorial region. For the case of $a = 0$, model C3 shows a much longer polar branch of the radial magnetic field compared to the observation. Thus, including the nonrotational anisotropy of the turbulent diffusion brings this model in better agreement with the observations. In models B and C2, the toroidal field is transported to the poles at the bottom of the convection zone. In model C2, the meridional circulation moves the toroidal field toward the equator in the middle of the convection zone. As in models C1 and C3, in models B and C2 the anisotropy of turbulent diffusivity decreases the strength of the polar branch of the radial magnetic field. In addition, in model B, the poleward dynamo wave of the toroidal magnetic field becomes stationary near the surface.

Summarizing our results for the benchmark models, we conclude that the nonrotational anisotropy of the turbulent diffusivity does not significantly change the conditions of the dynamo instability. It does not significantly impact the dynamo period as well. However, it can affect the butterfly diagram, in particular, by shortening the polar branch of the radial magnetic field at the surface. Also, in models B, C2, and C3, in the upper part of the convection zone, the toroidal field migrates poleward, contrary to the observations. This migration is reversed in the dynamo model with the subsurface shear of the angular velocity, which we study in the next section.

2.3. Solar Dynamo Model with Subsurface Shear

In this section, we consider combined nonrotational and rotational anisotropy effects in the dynamo model with the subsurface shear layer, which we developed in our recent papers (Pipin & Kosovichev 2011a, 2013b; Pipin et al. 2012). The mean electromotive force is given as follows (for details, see Pipin 2008, hereafter referred to as P08):

\[ \mathcal{E}_i = (\alpha_{ij} + \gamma_{ij}^{(A)}) B_j - (\eta_{ijk} + \eta_{ijk}^{(B)}) \nabla_j B_k + \mathcal{E}_i^{(A)}, \]  

(8)

where $\mathcal{E}_i^{(A)}$ is the complete anisotropic part of magnetic diffusivity for the prescribed anisotropy of the background turbulence model. It is given by Equations (A18) and (A20). Tensor $\alpha_{ij}$ describes the $\alpha$-effect. It includes hydrodynamic ($\alpha^{(H)}_{ij}$) and magnetic ($\alpha^{(M)}_{ij}$) helicity contributions:

\[ \alpha_{ij} = C_\alpha \psi_a \sin^2 \theta \alpha^{(H)}_{ij} + \alpha^{(M)}_{ij}. \]  

(9)

We use latitudinal factor $\sin^2 \theta$ to bring the model in better agreement with observations. This correction is supported by theoretical calculations of the $\alpha$-effect for the anisotropic convective turbulent flows (Kleiner & Rogachevskii 2003). The $\alpha$-quenching function $\psi_\alpha = -3/4 \phi_6^{(a)} (\beta)$ depends on $\beta = |\mathcal{B}|/\sqrt{\mu_0 \rho u^2}$; and $\phi_6^{(a)}$ is given in P08. The magnetic helicity contribution to the $\alpha$-effect is defined as follows (P08):

\[ \alpha^{(M)}_{ij} = 2 \left( f_2^{(a)} \delta_{ij} - f_1^{(a)} \frac{\Omega_i \Omega_j}{\Omega^2} \right) \frac{\tau_c}{\mu_0 \rho u^2}. \]  

(10)

Functions $f_1^{(a)}$ and $f_2^{(a)}$ describe the effect of rotation and can be also found in P08. The evolution of magnetic helicity $\mathcal{H} = \mathbf{a} \cdot \mathbf{B}$, where $\mathbf{a}$ is the fluctuating vector-potential, and $\mathbf{B}$ is the fluctuating magnetic field, is determined from the conservation law (see Pipin 2013; Pipin et al. 2013):

\[ \frac{\partial \mathcal{H}^{(tot)}}{\partial t} = - \frac{\mathcal{H}}{\rho_2 \tau_c} - \eta \mathbf{B} \cdot \mathbf{j} - (\mathbf{U} \cdot \nabla) \mathcal{H}^{(tot)}, \]  

(11)

where $\mathcal{H}^{(tot)} = \mathbf{A} \cdot \mathbf{B} + \mathcal{H}$ is the total magnetic helicity. In the model, we assume the effective magnetic Reynolds number $\rho_2 = 10^9$.

Turbulent pumping coefficient in Equation (8), $\gamma_{ij}^{(A)}$, depends on the mean density, turbulent diffusivity stratification, and also on the Coriolis number $\Omega^* = 2\tau_c \Omega_0$, where $\tau_c$ is a typical convective turnover time, and $\Omega_0$ is the angular velocity. For a detailed formulation of $\gamma_{ij}^{(A)}$, see the above cited papers. The turbulent diffusivity is anisotropic as a result of the Coriolis force and is given by the following:

\[ \eta_{ijk} = 3 \eta_T \left\{ \left( 2 f_1^{(a)} - f_2^{(d)} \right) \delta_{ij} - 2 f_1^{(a)} \frac{\Omega_i \Omega_j}{\Omega^2} \delta_{nk} \right\}. \]  

(12)

We also include the nonlinear effects of magnetic field generation induced by the large-scale current and global rotation, which are usually called the $\Omega \times J$-effect or the $\delta$ dynamo effect (Rädler 1969). Their importance is supported by the numerical simulations (Käpylä et al. 2008; Schrinner 2011). We use the equation for $\eta_{ijk}^{(D)}$, which was suggested in P08 (also see Rogachevskii & Kleiner 2004a):

\[ \eta_{ijk}^{(D)} = 3 \eta_T C_\delta f_2^{(a)} \frac{\Omega_j}{\Omega} \left\{ \phi_2^{(a)} \delta_{ik} + \frac{\phi_2^{(a)}}{\phi_2^{(b)}} \frac{B_i B_k}{B} \right\}. \]  

(13)

where $C_\delta$ measures the strength of the $\Omega \times J$ effect, $\phi_2^{(a)} (\beta)$ are normalized expressions of magnetic quenching functions $\phi_2$, given in P08. They are defined as follows: $\phi_2^{(a)} (\beta) = (5/3)\phi_2^{(b)} (\beta)$. The functions $f_1^{(a,d)}$ in Equations (9), (12), and (13) depend on the Coriolis number. They can be found in P08 as well.

Following Pipin & Kosovichev (2011b), we use a combination of the open and closed boundary conditions at the top, controlled by a parameter $\delta = 0.99$:

\[ \delta \frac{\eta_T}{\tau_c} B + (1 - \delta) \mathcal{E}_0 = 0. \]  

(14)

This is similar to the boundary condition discussed by Kitchatinov et al. (2000). This condition results in penetration of the toroidal field to the surface, which increases the efficiency of the subsurface shear layer (Pipin & Kosovichev 2011b). For the poloidal field, we apply a condition of smooth transition from the internal poloidal field to the external potential (vacuum) field.

Summing up, the model includes magnetic field generation effects due to the differential rotation ($\Omega$-effect), turbulent kinetic helicity (the anisotropic $\alpha$-effect) and interaction of large-scale currents with the global rotation, usually called $\Omega \times J$-effect or $\delta$-effect (Rädler 1969; Käpylä et al. 2008; Schrinner 2011). For the differential rotation, we use an analytical fit to the recent helioseismology results of Howe et al. (2011; see Figure 1(c) in Pipin & Kosovichev 2013b). The subsurface rotational shear layer provides additional energy for the toroidal magnetic field generation, and also induces the equatorward drift of the toroidal magnetic field (Pipin & Kosovichev 2011b). We also take into account the turbulent transport caused by the
mean density and turbulent intensity gradients (so-called “gradient pumping”). The model includes also the magnetic helicity balance, as described by Pipin et al. (2013). The contribution of the anisotropic diffusion to the mean electromotive force is given by Equation (A20).

The profiles of the total turbulent diffusivity for $a = 0, 2,$ and 4 are shown in Figure 5(a). We see that with the anisotropy effect the total effective diffusivity is greater than $10^{12} \text{ cm}^2 \text{ s}^{-1}$ in the upper part of the convection zone. This is close to the turbulent magnetic diffusivity estimated from the sunspot decay rate (Martinez Pillet et al. 1993) and also from the cross-helicity observations (Rüdiger et al. 2011; Pipin et al. 2011).

Typical snapshots for the magnetic field and magnetic helicity distributions in the solar convection zone are shown in Figures 5(b) and (c). The toroidal magnetic field is concentrated to the bottom of the convection zone. The near surface magnetic field is weaker than the bottom field during most of the dynamo cycle.

Figure 6 shows the time-latitude diagram of the toroidal magnetic field in the subsurface shear layer ($r = 0.92 \, R_\odot$) and for the radial magnetic field at the top of the domain ($r = 0.99 \, R_\odot$). Their behavior is similar to the magnetic field evolution in solar cycles. We see that the cycle period decreases from 14 years to 10 years when $a$ increases from 0 to 4. For $a = 4$, the model has the total effective magnetic diffusivity greater than $10^{12} \text{ cm}^2 \text{ s}^{-1}$ in a large part of the convection zone. Still, the dynamo model correctly reproduces the solar cycle period and the patterns of the toroidal and poloidal magnetic field evolution.

Another interesting feature that Figure 6 demonstrates is that the overlap between the cycles decreases when the anisotropy parameter $a$ increases. We investigated this feature in detail for the distributed dynamo model without meridional circulation. To quantify the overlap between the subsequent cycles we examine the latitudinal drift of the toroidal flux maxima in the subsurface shear layer at $r = 0.92 \, R_\odot$ (see Figure 7(a)). We restrict our
consideration to the latitudinal range $0^\circ$–$40^\circ$ and compute the relative overlap between the curves that belong to subsequent cycles. Figure 7(b) shows the results for the models with $0 \leq a \leq 4$. The relative overlap time between the subsequent cycles decreases from 0.42 (5 years) to 0.25 (2.5 years) when $a$ spans from 0 to 4. The speed of the dynamo wave latitudinal migration remains almost the same for all $a$. This means that in this model, the dynamo wave is not transformed from the running to steady type with the increase of $a$.

3. DISCUSSION AND CONCLUSION

In this paper, we have examined influence of rotational and nonrotational anisotropic turbulence in the solar convection zone on magnetic diffusivity and properties of mean-field dynamo models, including the simplified benchmark model (Jouve et al. 2008) and the distributed-dynamo model with the subsurface shear layer, taking into account a detailed formulation of the mean-electromotive force (Pipin & Kosovichev 2011a, 2013b; Pipin et al. 2013). To characterized the nonrotational anisotropy, we use, as did Rüdiger et al. (2005), the anisotropy parameter $a$ derived as a relative ratio of the horizontal and vertical turbulent root mean square velocities.

The strength of the anisotropy parameter depends on theoretical models of turbulent flows. In the simple case of the mixing length theory, it is restricted to the range $a = 0$–4 ($a = 0$ corresponds to isotropic flows). The nonlocal stellar convection theory (Deng et al. 2006) suggests $a \sim 3.5$. Numerical simulations of the solar convection showed evidence for a stronger anisotropy, which may reach $a \sim 10$ (e.g., Miesch et al. 2008; Guerrero et al. 2013a). The nature of this anisotropy in the numerical simulations needs further studies to determine robustness of the current results in regimes approaching the real Sun (strong stratification and the high Reynolds number). Our results show that strong anisotropy would help the mean-field dynamo model to better reproduce the observations. It is important to note that in the simulations such anisotropic convection becomes visible even in the subsurface shear layer below $0.95 R_\odot$ where the analytical mixing-length estimations of the mean-field diffusivity coefficients show a small anisotropy. However, the current simulations still do not accurately reproduce the convection zone dynamics. Therefore, we considered the anisotropic parameter $a$ as a free parameter. The analytical calculations presented in the Appendix showed that the effect of turbulent mixing in the horizontal direction is quenched by rotation, whereas the effective structure of the given anisotropic diffusivity tensor remains the same as in the case of slow rotation.

Our study of the benchmark models suggests that the dynamo threshold parameters do not significantly change with increase of the anisotropy parameter $a$. This could be expected from the analysis given by Kitchatinov (2002). In our calculations, parameter (a) controls the excess of the magnetic diffusivity in the horizontal direction relative to the vertical direction. A local dynamo wave analysis by Kitchatinov (2002) suggests that contribution of this effect to the dynamo instability is proportional to $(k \cdot g)^2$, where $k$ is the wave vector in the direction of the dynamo wave propagation. Then, the anisotropy of the turbulent flows with $g$ perpendicular to $k$ does not affect the dynamo instability threshold and period. An interesting new finding is that the dynamo threshold can slightly decrease with the increase of $a$. This is found for the models with the odd number of the meridional circulation cell stacked along the radius. For the cases of the zero and double-cell circulation, we find the opposite behavior. We also found that the odd number of cells gives preference to dipole-like symmetry, whereas even number of cells lead to a quadrupole symmetry.

In the benchmark models, the equatorial drift of the toroidal magnetic field depends on the amplitude and type of meridional circulation. Model B (Table 1), which has no meridional circulation, does not have an equatorial branch of the toroidal magnetic field in the convection zone. This is also the result of absence of the subsurface shear layer in the simplified model. The detailed solar dynamo model, which was discussed in Section 2.3, reproduces the equatorial migration of the toroidal magnetic field in the upper part of the convection zone. This effect is induced by the dynamo wave in subsurface rotational shear layer. The effect of the subsurface shear on the large-scale dynamo was suggested earlier by Brandenburg (2005). We note a particular role of the boundary condition (Equation (14)), which allows penetration of the toroidal field to the surface, increasing the efficiency of the subsurface shear layer (Pipin & Kosovichev 2011b). Thus, the model can produce the equatorward migration of the toroidal field in the upper part of the convection zone (above $0.9 R_\odot$) even for rather small equatorial turbulent pumping, which operates in the bulk of the convection zone with amplitude less than 1 m s$^{-1}$. This is different to the model suggested by Käpylä et al. (2006), who used—in addition to the subsurface shear—strong, turbulent pumping effects in the bulk of the convection zone. For a detailed

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**Figure 7.** (a) Latitudes of maxima of the toroidal magnetic field flux in the subsurface shear layer as a function of time during a dynamo half-cycle for the model with $a = 0, 2, 4$; (b) the overlap time (relative to the cycle length) between the subsequent cycles as a function of the anisotropy parameter $a$. 
discussion of the effects of the subsurface shear layer, the turbulent pumping, and the turbulent diffusivity profiles, the reader is referred to our previous papers (Pipin & Kosovichev 2011a, 2011b).

In the flux transport models, the equatorial branch of the toroidal field is induced by the equatorial parts of the meridional circulation cells. Flux transport model C1 reproduces the results by Jouve et al. (2008) for \( a = 0 \). Among all of the flux-transport models discussed in this paper, this case is in the best agreement with observations in terms of the phase relation between the polar activity of the radial magnetic field and the equatorial activity of the toroidal magnetic field (see Figure 4(b)). Adding the nonrotational anisotropy with \( a = 4 \) breaks the agreement with observations. The more complicated patterns of magnetic field evolution are suggested by models C2 and C3. The model C2 has the equatorward migration of the toroidal magnetic field in the middle of the convection zone (see more details in Pipin & Kosovichev 2013b). The model C3 with the triple-cell circulation has, in addition, the equatorward migration near the bottom of the convection zone. The physical interpretation of these types of models suggests that the poloidal field at the surface is produced directly from the buoyant flux tubes that come from the bottom of the convection zone. In this case, the dynamo model with the double-cell meridional circulation is not of the solar type. The model with the triple-cell circulation has a qualitative agreement with observations, although the polar reversal of the radial magnetic field, \( B_r \), occurs about five years ahead of the maximum of the toroidal magnetic field, \( B_t \), in equatorial regions. This model has the dynamo period as twice as long compared with the solar cycle in the case \( a = 4 \). For the case \( a = 0 \), model C3 has a longer dynamo period of about 30 years, although the phase relation between \( B_r \) and \( B_t \) is in a better agreement with the observations. Overall, we found that the multicell meridional circulation breaks the resemblance of flux-transport models C2 and C3 with observations.

We made additional calculations for the distributed solar dynamo models similar to those discussed in Section 2.3, including the effect of multicell circulation. For the case of the double-cell circulation, we reproduced our results from the previous paper (see, Pipin & Kosovichev 2013b). We confirm the effect of the decreasing overlap between the subsequent cycles with the increasing parameter of anisotropy, \( a \). The case of the triple-cell circulation is found to be similar to model C3. However, in this case, the correct phase relation between \( B_r \) and \( B_t \) holds only for the upper part of the convection zone, where similarly to the double cell circulation model, we obtain the equatorward drift of the toroidal magnetic field (because of the equatorward meridional flow and subsurface shear layer). The results of this model are similar to the numerical simulation results reported by Käpylä et al. (2012). The differential rotation profile in their paper is different from the solar case.

With the effects of anisotropy the total magnetic diffusivity reaches values of about \( 10^{12} \) cm\(^2\) s\(^{-1}\) in the layer 0.85–0.95 \( R_\odot \), which are consistent with the estimates from measurements of the sunspot decay rate (Martínez Pillet et al. 1993) and cross-helicity observations (Rüdiger et al. 2011). We found that the anisotropy affects the overlap time between the subsequent cycles. When the anisotropy is larger, the overlap is smaller. This effect is related to stronger concentration of the toroidal magnetic field near the bottom boundary of the convection zone as a result of anisotropy and also as a consequence of a faster migration of the dynamo wave in the subsurface shear layer. In general, our model is consistent with the paradigm of solar dynamo operating in the bulk of the convection zone and shaped in the subsurface rotational shear layer.

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APPENDIX

To compute the electromotive force \( E \) for the anisotropic MHD turbulence we write Equations (1) and (2) in the Fourier space:

\[
\left( \frac{\partial}{\partial t} + \eta z^2 \right) \hat{b}_j = i z_j \int \left( \hat{u}_j (z' - q) \times B_j (q) - \hat{u}_i (z' - q) \hat{B}_j (q) \right) dq + \hat{\Theta}_j, \quad (A1)
\]

\[
\left( \frac{\partial}{\partial t} + \nu z^2 \right) \hat{u}_i = f_i + \hat{\Theta}_i - 2(\Omega z) (\hat{z} \times \hat{u}_i) + \frac{i \pi z_j}{\mu} \int \left( \hat{B}_j (z - q) B_j (q) + \hat{B}_j (z - q) \hat{B}_j (q) \right) dq, \quad (A2)
\]

where the turbulent pressure was excluded from (Equation 2) by convolution with tensor \( \pi_{ij}(z) = \delta_{ij} - \hat{z}_i \hat{z}_j \), where \( \delta_{ij} \) is the Kronecker symbol, and \( \hat{z} \) is a unit wave vector. The equations for the second-order moments that make contributions to the mean electromotive force can be found directly from Equations (A1) and (A2). We consider the high Reynolds number limit and discard the microscopic diffusion terms. Using the \( \tau \)-approximation, in which the third-order products of the fluctuating fields are approximated by the corresponding relaxation terms of the second-order contributions (see, Rädler et al. 2003; Rogachevskii & Kleewein 2003; Brandenburg & Subramanian 2005; Pipin 2008; Rogachevskii et al. 2011), we get the following:

\[
\frac{\hat{\xi}_{ij}(z, z')}{\tau^*(z)} + 2(\Omega z^2 e_{imm} z_n \hat{\gamma}_{mj}(z, z') = \right.
\]

\[
\times i z_j' \int \left\{ \hat{u}_{ij}(z, z' - q) \hat{B}_j(q) - \hat{u}_{ij}(z, z' - q) \hat{B}_j(q) \right\} dq + \frac{i \pi_{ij}}{\mu} \int \hat{m}_{ij}(z - q, z') \hat{B}_j(q) dq \]

\[
+ i \frac{z_i \pi_{ij}}{\mu} \int \hat{m}_{ij}(z - q, z') \hat{B}_j(q) dq, \quad (A3)
\]

\[
\frac{\hat{u}_{ij}(z, z')}{\tau^*(z)} = \frac{v_{ij}^{(0)}}{\tau^*(z)} - 2(\Omega z^2 e_{imm} z_n \hat{v}_{mj}(z, z') \right.
\]

\[
- 2(\Omega z^2 e_{imm} z_n \hat{v}_{im}(z, z') \right.
\]

\[
+ i \pi_{ij} \int \{ \hat{x}_{ij}(z', z - q) \hat{B}_j(q) \}
\]

\[
+ \hat{x}_{ij}(z', z - q) \hat{B}_j(q) \right\} dq + \frac{i \pi_{ij}}{\mu} \int \{ \hat{x}_{ij}(z', z - q) \hat{B}_j(q) \}
\]

\[
+ \hat{x}_{ij}(z', z - q) \hat{B}_j(q) \right\} dq, \quad (A4)
\]
\[
\frac{\hat{m}_{ij}(z,z')}{\tau^*(z)} = i\zeta_i \int \{ \hat{h}_{ij}(z' - q, z)\bar{B}_j(q) - \hat{h}_{ij}(z - q, z')\bar{B}_j(q) \} dq \\
+ i\zeta_i \int \{ \hat{h}_{ij}(z - q, z')\bar{B}_j(q) \} dq \\
+ i\zeta_i \int \{ \hat{h}_{ij}(z - q, z')\bar{B}_j(q) \} dq \\
+ \hat{n}_{ij}^{(0)}(z,z'), \quad (A5)
\]

where we introduced the ensemble averages \( \hat{v}_{ij}(z,z') = u_i(z)u_j(z') \), \( \hat{h}_{ij}(z,z') = u_i(z)b_j(z') \), \( m_{ij}(z,z') = b_i(z)b_j(z') \); the superscript \( ^{(0)} \) stands for the background state (when the mean-field is absent) of these correlations. The reader can find a comprehensive discussion of the \( \tau \)-approximation in the aforementioned papers. Furthermore, the contributions of the mean magnetic field in the equation for the turbulent stresses, \( \hat{v}_{ij} \), will be neglected because they give nonlinear terms in the cross-helicity tensor, \( \hat{h}_{ij} \).

Next, we solve Equations (A3)–(A5) in a linear approximation for the mean field \( \bar{B} \), also neglecting effects of background magnetic fluctuations in the further analysis. Thus, we obtain the following:

\[
\hat{h}_{ij}(z,z') = i\tau^*(z)\zeta_i D_{ip}(z) \\
\times \int \{ \hat{v}_{pj}(z' - q, z)\bar{B}_j(q) - \hat{v}_{pj}(z, z' - q)\bar{B}_j(q) \} dq \\
+ D_{ip}(z)\frac{i\zeta_i\tau^*(z)}{\mu} \pi_{pf} \int \hat{m}_{ij}(z - q, z')\bar{B}_j(q) dq \\
+ D_{ip}(z)\frac{i\zeta_i\tau^*(z)}{\mu} \pi_{pf} \int \hat{m}_{ij}(z - q, z')\bar{B}_j(q) dq \\
+ \hat{n}_{ij}^{(0)}(z,z'), \quad (A6)
\]

\[
\hat{v}_{ij} = D_{ijnm}(z,z') \psi_{jm}^{(0)} \\
+ D_{ijnm}(z,z') \frac{i\tau^*(z)\mu \pi_{nf}}{\zeta_i} \int \{ \hat{h}_{in}(z' - q, z)\bar{B}_j(q) - \hat{h}_{in}(z, z' - q)\bar{B}_j(q) \} dq \\
+ \hat{h}_{ij}(z,z') \frac{i\tau^*(z)\mu \pi_{nf}}{\zeta_i} \int \{ \hat{h}_{in}(z, z' - q)\bar{B}_j(q) - \hat{h}_{in}(z, z' - q)\bar{B}_j(q) \} dq \\
+ \hat{n}_{ij}(z,z') \bar{B}_j(q) dq, \quad (A7)
\]

where,

\[
D_{ip}(z) = \frac{\delta_{ip} + E_{ip}}{1 + \psi_{\Omega}}, \\
D_{ijnm}(z,z') = \frac{\delta_{ij} N + 2E_{ij}}{N^2 + 4\psi_{\Omega}^2} (\delta_{jm}(\delta_{fn} + E_{fn}) - \delta_{fm}\tilde{E}_{jm}),
\]

\[
E_{nk} = \frac{2(\Omega \cdot z)\tau z^2\zeta_p\epsilon \bar{u}_p, \quad \tilde{E}_{ml} = 2(\Omega \cdot z')\tau z^2\zeta_p\epsilon \bar{u}_p, \quad \tilde{\bar{N}} = (1 - \psi_{\Omega}^2 + \tilde{\psi}_{\Omega}), \quad \tilde{\psi}_{\Omega} = \frac{2(\Omega \cdot z)\tau^*(z)}{|z|}.
\]

To proceed further, we have to remind some conventions and notations that are widely used in the literature. The double Fourier transformation of an ensemble average of two fluctuating quantities, say \( f \) and \( g \), taken at the same time and at the different positions \( x, x' \), is given by the following:

\[
\langle f(x)g(x') \rangle = \int \int \langle \hat{f}(z)\hat{g}(z') \rangle e^{(z \cdot x + z' \cdot x')} d^3zd^3z'. \quad (A8)
\]

In the spirit of the general formalism of the two-scale approximation (Roberts & Soward 1975), we introduce two types of variables: fast and slow. They are defined by the relative, \( r = x - x' \), and mean \( R = (1/2)(x + x') \), coordinates. Then, Equation (A8) can be written in the following form:

\[
\langle f(x)g(x') \rangle = \int \int \langle \hat{f}(k + \frac{1}{2}K) \rangle e^{(K \cdot R + k \cdot r)} d^3Kd^3k. \quad (A9)
\]

where we have introduced the wave vectors \( K = (1/2)(z - z') \) and \( K = z + z' \). Then, following BS05, we define the correlation function of \( \hat{f} \) and \( \hat{g} \) obtained from (A9) by integration with respect to \( K \),

\[
\Phi(\hat{f}, \hat{g}, k, R) = \int \langle \hat{f}(k + \frac{1}{2}K) \rangle e^{(K \cdot R + k \cdot r)} d^3K d^3k. \quad (A10)
\]

For further convenience, we define the second-order correlations of velocity field and the cross-correlations of velocity and magnetic fluctuations using the following:

\[
\langle v^2 \rangle = \int \hat{u}_{ij} \cdot \hat{u}_{ij} d^3k. \quad (A11)
\]

\[
\hat{h}_{ij}(k, R) = \Phi(\hat{u}_{ij}, \hat{B}_{ij}, k, R), \quad \xi_i(R) = \langle u_iu_i \rangle = \int \hat{u}_{ij} \cdot \hat{B}_{ij} d^3k. \quad (A12)
\]

We now return to Equations (A6) and (A7). As the first step, we perform the Taylor expansion with respect to the slow variables and take the Fourier transform (A10). Details of this procedure can be found in (Brandenburg & Subramanian 2005). As a result, we get the following equations for the second moments:

\[
\hat{h}_{ij} = -i\tau^* D_{ij}(B \cdot k) \hat{v}_{ij} - \tau^* D_{ij} \hat{v}_{ij} \bar{B}_{ij,l} = \tau^{(0)}_{ijmn} \xi_{mn}^{(0)}, \quad (A13)
\]

where

\[
\tau^{(0)}_{ijmn} = \delta_{jm}\bar{h}_{jm} + \psi_{\Omega}k_{ml} \epsilon_{mn} \delta_{jm} - \delta_{mn} \hat{h}_{ij} \sqrt{M} d^3k \bar{h}_{ij} \delta_{jm} - \delta_{jm} \delta_{mn} \hat{h}_{ij},
\]

\[
M = 1 + 4\psi_{\Omega}^2.
\]

Statistical properties of the background fluctuations are described by the spectral tensor (see, e.g., Ridger et al. 2005):
The effect of the anisotropic mixing and the Coriolis force on the intensity of the velocity fluctuations in the radial and horizontal directions, and $g$ is a unit vector in the direction of anisotropy. Following the conventions given by Rüdiger et al. (2005), we write the following:

$$\{u_r^{(0)}\} = \frac{1}{3} \int \frac{E(k, R)}{4\pi k^2} \text{d}^3 k$$

(A15)

$$\langle u_h^{(0)} \rangle - 2\langle u_r^{(0)} \rangle = \frac{1}{3} \int \frac{E_1(k, R)}{4\pi k^2} \text{d}^3 k$$

(A16)

and introduce a nondimensional anisotropy parameter $a$:

$$a = \left( \frac{\langle u_h^{(0)} \rangle - 2\langle u_r^{(0)} \rangle}{\langle u_r^{(0)} \rangle} \right).$$

(A17)

Note that $a \geq -1$ because of Bochner’s theorem (see, e.g., Monin & Yaglom 1975). To integrate Equation (A13) in the $k$-space, we apply the Kolmogorov law for $E(k, R) = -\langle u^{(0)} \rangle dk/\partial r = (k/k_0)^{-q}$ and $r^* = 2\pi r/k_0$ with $q = 5/3$, $k_0 = 1/\ell_0$, $r_0 = 6/(\langle u^{(0)} \rangle)$ (see, e.g., Rädler et al. 2003; Rogachevskii & Kleerorin 2004b). We assume that $a$ is a constant over $k$. This approximation can be refined in further applications.

The isotropic part of magnetic diffusivity in rotating turbulent media was derived by Pipin (2008) and is not reproduced here. The effect of the anisotropic mixing and the Coriolis force on magnetic diffusivity is given by the following:

$$E_i^{(k)} = a \langle u^{(0)} \rangle \tau_r \left( e_{ijm}(f_3 \cdot g)^2 - f_1 \right) B_{j,m}$$

$$+ e_{ijm} B_{j,f} \left( f_3 \cdot g \right)^2 e_j e_m - f_{g} g f e_m (e \cdot g)$$

$$+ e_{ijm} (f_3 e_f e_i (e \cdot g) + f_3 g e_i (e \cdot g)) e_j e_m B_{j,l}$$

$$+ e_{ijm} (f_3 e_f e_i (e \cdot g) - f_3 g e_i (e \cdot g) - f_3 e_f e_i (e \cdot g)) B_{j,f}$$

$$+ g_j e_{jmf} (f_3 e_f e_i (e \cdot g) + f_3 g e_i (e \cdot g)) B_{m,f}$$

$$+ g_j e_{jmf} (f_3 e_f e_i (e \cdot g) + f_3 g e_i (e \cdot g)) B_{m,f}$$

$$+ g_j e_{jmf} (f_3 e_f e_i (e \cdot g) + f_3 g e_i (e \cdot g)) B_{m,f}$$

$$+ f_3 e_{f} e_m e_{mfj} (e \cdot g) B_{i,j}$$

$$+ f_2 (e_i e_l - e_i e_l) e_j e_m e_{jmf} B_{l,j} + \ldots$$

(A18)

where $e = (\Omega / \Omega^*)$ is the unit vector of angular velocity; and $f_{1-10}$ are functions of the Coriolis number, $\Omega^* = 2\Omega_0 \tau_r$. The last term in Equation (A18) means that there are additional magnetic field generation effects induced by the large-scale current and global rotation, so-called $\delta$-effect, (Rädler 1969).

For the background hydrodynamic fluctuation, we found no $\delta$-effect in the direction of large-scale magnetic field. The other terms of the $\delta$-effect can be less important for the solar type dynamo, and we skip them from our consideration.

For the case of the slow rotation, taking the Taylor expansions of $f_{1-10}$ about small $\Omega^*$, we find that $f_{1,10} = 1/6$, and the others functions are of the order of $O(\Omega^2)$. This reduces Equation (A18) to the following:

$$\mathcal{E} = -\frac{a}{2} \tau_r \nabla \times \left( \nabla \mathbf{u} \cdot \nabla \right) \mathbf{B},$$

(A19)

where $\eta_T = (\tau_c/3) \langle u^{(0)} \rangle$.

Despite a complicated form of Equation (A18), only certain combinations of $f_{1-10}$ are important in applications. For example, in the case of the spherical geometry and the axisymmetric magnetic field $\mathbf{B} = e_\varphi B + \nabla \times (A e_\varphi/\sin \vartheta)$, where $B(r, \vartheta, r)$ is the azimuthal component, $A(r, \vartheta, r)$ is proportional to the azimuthal component of the vector potential, we find the following:

$$\mathcal{E}_r = \eta_T a \left\{ \mathbf{B} \cdot \left( \Phi_1 \frac{\partial \sin \theta B}{\partial \mu} + \Phi_2 \frac{\sin \theta B}{r} \right) \right\},$$

$$\mathcal{E}_\vartheta = \eta_T a (\Phi_3 + \Phi_2 R^2) B_r,$$

$$\mathcal{E}_\varphi = \eta_T a \left\{ \Phi_1 \sin \theta \frac{\partial^2 A}{\partial \mu^2} + \Phi_1 \frac{\partial A}{r} \right\},$$

(A20)

where, $\mu = \cos \vartheta$, $\Phi_1 = f_1 + f_3 + f_5 + f_7$, $\Phi_2 = f_9 + f_6 - f_3 + f_7$, $\Phi_3 = f_{10} + f_7 - f_2 - f_6$ and

$$\Phi_1 = - \frac{1}{24\Omega^2} \left( 2 \log(1 + 4\Omega^2) + 4 \log(1 + \Omega^2) \right)$$

$$+ \left( 1 - 4\Omega^2 \right) \frac{\arctan(2\Omega^* - 2\Omega^*)}{\Omega^*}$$

$$+ \left( 4 - 4\Omega^2 \right) \frac{\arctan(\Omega^*)}{\Omega^*} - 6,$$

(A21)

$$\Phi_2 = - \frac{1}{24\Omega^2} \left( 4 \log(1 + 4\Omega^2) + 8 \log(1 + \Omega^2) \right)$$

$$+ \left( 3 - 4\Omega^2 \right) \frac{\arctan(2\Omega^*)}{\Omega^*}$$

$$+ \left( 4 - 3\Omega^2 \right) \frac{\arctan(\Omega^*)}{\Omega^*} - 18,$$

(A22)

$$\Phi_3 = \frac{1}{12\Omega^2} \left( \log(1 + 4\Omega^2) + 2 \log(1 + \Omega^2) \right)$$

$$+ \frac{\arctan(2\Omega^*)}{\Omega^*} + \frac{4 \arctan(\Omega^*)}{\Omega^*} - 6,$$

(A23)

Note that in the case of fast rotation with large Coriolis number, the only significant effect is expressed with the terms with factor $\Phi_2$. These terms appears in the spherical geometry as a result of the anisotropy. The functions $\Phi_1, \Phi_2, \Phi_3$ describe a quenching of the given anisotropy effect because $\Phi_1, \sim \Omega^{-1}$ and $\Phi_3 \sim \log(\Omega^*)/\Omega^*$ for $\Omega^* \gg 1$.
