SUBFACTORS AND CONNES FUSION FOR TWISTED LOOP GROUPS

Antony Wassermann, Institut de Mathématiques de Luminy

INTRODUCTION. The study of subfactors arising from positive energy representations of loop groups was initiated with Vaughan Jones in 1989. In the 1990’s I related this to the structures arising from conformal field theory, in particular vertex algebras, their representations and intertwining operators, so-called “primary fields”. The link with subfactors was achieved by the intermediary tool of Connes fusion, a relative tensor product operation on bimodules over von Neumann algebras. With this tool, the fusion of positive energy representations of $LSU(N)$ at a fixed level could be defined analytically and explicitly computed, giving the representations the structure of a modular braided tensor category. The computation revolved around the four-point function of certain primary fields and its monodromy properties, encoded in the Knizhnik-Zamolodchikov differential equation. The computation was initially made using primary fields corresponding to arbitrary irreducible representation. In this case the corresponding smeared operators yield unbounded operators, which gives rise to certain analytic difficulties. This theory is explained in my 1994 ICM article and in Jones’ séminaire Bourbaki on this subject from 1995. Subsequently, using the fact that the smeared primary fields for the vector representation and its dual were actually bounded, I found a method for treating fusion which used only these bounded fields, published in 1998. All these computations have served as the starting point for all subsequent research in the area between operator algebras and conformal field theory. This approach has been successfully applied by my Ph.D. students to other groups and Lie algebras arising in conformal field theory, including the diffeomorphism group of the circle/Virasoro algebra (Terence Loke), the loop group of the spin group (Valerio Toledano-Laredo) and most recently Neveu–Schwarz algebra, one of the super–Virasoro algebras (Sébastien Palcoux).

Another former student Robert Verrill attempted to treat the case of the twisted loop group of $SU(2N)$, which on the surface seemed similar to these other cases but raises problems unresolved in his thesis. The same problem occurs when considering another “twisted” theory, that of the Ramond algebra, the other super–Virasoro algebra. In the language of conformal field theory, in both these cases the vertex algebra defined by the untwisted theory admits an automorphism of period two. In the loop group case, it corresponds to an outer automorphism $\tau$ of $SU(2N)$ coming from a symmetry of the Dynkin diagram which up to inner automorphisms is simply complex conjugation; for the Neveu–Schwarz algebra, it is just the $\mathbb{Z}_2$–grading of theory. Igor Frenkel and his students, Haisheng Li and Xiaoping Xu, have developed a theory of twisted representations (or modules) of vertex algebras and their intertwiners, perfectly adapted to this situation. Aspects of the theory had been discussed prior to this by Louise Dolan, Peter Goddard and Paul Mountague, in connection with the construction of the Monster vertex algebra starting from the period three “triality” outer automorphism of $SO(8)$. Based on this algebraic theory, Dan Freed, Michael Hopkins and Constantin Teleman have extended their K–theoretic explanation of fusion rules for loop groups representations to the twisted case, as is to be expected because of the appearance of twisted loop groups in boundary conformal field theory and open string theory (D–branes). In this case the outer automorphism defining the twisting can be used to modify the conjugation action of the underlying group $G$ on itself and the data is encoded in the twisted equivariant K–theory of this space, together with its structure as a module over the representation ring $R(G)$ of $G$.

On the other hand in the 1990s Adrian Ocneanu had developed a theory of modules over the modular tensor categories associated with subfactors. Using his combinatorial machinery, Ocneanu determined explicitly the possible modules for the tensor categories corresponding to $SU(N)$ for $N = 2,3,4$. These categories are the same as those originally discovered by Jones and Wenzl, coming from Temperley–Lieb algebra and more generally Hecke algebras at roots of unity; these appear as the centralizer algebras of the quantum groups of $SU(N)$ at roots of unity, the counterpart of loop groups in the theory of exactly solved models in statistical mechanics. Various authors have given an abstract formulation of Ocneanu’s ideas, in particular Vladimir Turaev and A. Kirillov, although without any specific example in mind.

In this announcement we will explain how the positive energy representations of the twisted loop group $L^*SU(2N)$ provide examples of this general theory. In this case the period two automorphism $\tau$ is most easily constructed as follows. Starting from $\mathbb{R}^N = \mathbb{R}^{4N}$ with its standard real inner product, we can identify $U(2N)$ with the subgroup of $SO(4N)$ commuting with $\rho(i)$, right multiplication by $i$. This has a period two

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automorphism \( \tau \) given conjugation by \( \rho(j) \) and with fixed point subgroup \( Sp(N) \), the orthogonal symplectic group. We will also \( \tau \) to denote conjugation by \( \rho(j) \) on \( SO(4N) \), an inner automorphism. As for loop groups, there is always an irreducible subfactor given by the failure of Haag duality and its structure can be unravelled using Connes fusion. The positive energy representations of the twisted loop group can be fused with those of the untwisted loop group to give new representations of the twisted loop group. Thus the Grothendieck group generated by the positive energy representations of the untwisted loop group form a module over \( R_\ell(SU(2N)) \), the Grothendieck ring for level \( \ell \) positive energy representations of the untwisted loop group, the so-called Verlinde algebra. The fusion rules with the fundamental representations of \( L^\ell SU(2N) \) are given by truncations of the classical tensor product rules of Sheila Sundaram for tensoring representations of \( Sp(N) \) by the fundamental representations of \( SU(N) \) (the exterior powers of the vector representation). The fusion of two positive energy representations of \( L^\ell SU(2N) \) gives a representation of \( LSU(2N) \). As predicted by the general theory, the fusion of the simplest representation with itself, corresponding to the trivial representation of \( Sp(N) \), is naturally a *-algebra within the category. Its spectral decomposition is a truncation of classical decomposition of the compact symmetric space \( SU(2N)/Sp(N) \) corresponding to the period two isomorphism, given by the classical Cartan–Helgason theorem. The principal results are described explicitly in the next section. In unpublished work, Hans Wenzl has given a construction using quantum groups at roots of unity which yields an alternative construction of this module over the tensor category, independent of conformal field theory.

The main results are proved by suitably modifying the techniques used for untwisted loop groups. So far we have only developed the unbounded approach since the primary fields creating the twisted sectors from the untwisted ones need not be bounded. (It is possible that there is a more elementary approach using only bounded operators could be found.) A technical analytic tool - the "phase theorem" - is required to pass from braiding relations between two smeared primary fields to relations between bounded intertwiners given by the phases in their polar decomposition. Because the absolute values lie in commuting von Neumann algebras, the notorious problems in manipulating formal commutation relations between unbounded operators — described in VIII.5 of Reed and Simon — can be circumvented. Many parts of the theory of Verrill, for example determination of vector primary fields between twisted modules, fit into this theory; but these computations need to be extended to other fundamental representations. The explicit construction and analytic properties of primary fields depends on the fact that the vacuum representation of \( L^\ell SO(4N) \) remains irreducible when restricted to \( L^\ell U(N) \); this is a consequence of Igor Frenkel’s twisted version of the boson–fermion correspondence. In terms of nets of von Neumann algebras, note that the local algebras defined by the twisted loop group define a double cover of the net defined by the loop group, with monodromy defined by the period two automorphism \( \tau \), regarded as an automorphism of the original net.

2. STATEMENT OF MAIN RESULTS. Let \( G = SU(2N) \) with involution \( \tau \) and set \( K = G^\tau = Sp(N) \). We define the loop group by

\[
LG = C^\infty(S^1, G)
\]

with the evident action of the rotation group \( \text{Rot} S^1 \cong \mathbb{T} \). We consider the positive energy projective representations \( H \) of \( LG \times \text{Rot} S^1 \), which are irreducible when restricted to \( LG \) are irreducible and which, when restricted to \( \text{Rot} S^1 \), have a decomposition

\[
H = \bigoplus_{n \geq 0} H(n)
\]

where

\[
R_\delta \xi = e^{i\theta \delta} \xi
\]

for \( \xi \in H(n) \) with \( \dim H(n) < \infty \), \( H(0) \neq (0) \). The projective representation \( LG \to PU(H) \) defines a central extension of \( LG \) by \( \mathbb{T} \), obtained by pulling back the central extension \( U(H) \) of \( PU(H) \). These are classified by a positive integer \( \ell \geq 1 \), called the level. Fixing a level \( \ell \), there are only finitely many irreducible positive energy representations at that level and these are classified by the zero energy space \( H(0) \). This is the subspace of \( H \) invariant under rotation, so it is invariant under the constant loops \( G \subset LG \). It is an irreducible representation of \( SU(2N) \) with signature \( f_1 \geq \cdots \geq f_{2N} \) satisfying the constraint \( f_1 - f_{2N} \leq \ell \) and it determines the isomorphism class of \( H \). (Recall that signatures that differ by a constant give the
same irreducible representation and that signatures can be identified with Young diagrams.) Let $V_f$ be the irreducible representation of $SU(2N)$ with signature $f$ and $H_f$ the positive energy irreducible representation with $H_f(0) = V_f$. In particular $V = \mathbb{C}^{2N}$ will denote the vector representation of $SU(2N)$ and $V_k = \lambda^k(V)$ the $k$th exterior power. Thus

$$V_f \otimes V_k = \bigoplus_{g < k} V_g,$$

where the signature $g$ is obtained by adding $k$ boxes to the signature $f$ so that no two lie in the same row. The rule for Connes fusion is similar

$$H_f \boxtimes H_k = \bigoplus_{g < k} H_g,$$

except only $g$ satisfying $g_1 - g_2 \leq \ell$ are allowed on the right hand side. Let $d(H_f)$ be the quantum dimension of $H_f$, uniquely specified by $d(H_0) = 1$, $d(H_f) > 0$ and

$$d(H_f)d(H_k) = \sum_{g < k} d(H_g).$$

There is an element $g_0 \in G$ such that

$$d(H_f) = \text{Tr}_{V_f}(g_0).$$

Moreover the subfactor given by the failure of Haag duality in $H_f$ is irreducible and has Jones index $d(H_f)^2$.

Likewise the twisted loop group is defined by

$$L^\tau G = \{ g \in C^\infty(\mathbb{R}, G) : f(x + 2\pi) = \tau f(x) \}.$$

with the (double cover of the) rotation group $\text{Rot}^\tau(S^1) = \mathbb{R}/4\pi\mathbb{Z}$ acting by translation. Again we consider projective representations $K$ of the semidirect product $L^\tau G \rtimes \text{Rot}^\tau(S^1)$ which are irreducible on $L^\tau G$ and which when restricted to $\text{Rot}^\tau S^1$ yield a decomposition

$$K = \bigoplus_{n \geq 0, n \in \mathbb{Z}} K(n),$$

where $R_\theta \xi = e^{in\theta} \xi$ for $\xi \in K(n)$ with $\dim K(n) < \infty$ and $K(0) \neq (0)$. The projective representation defines a central extension, classified by a level $\ell \geq 1$ and the rotation invariant subspace $K(0)$ is an irreducible representation of $K = Sp(N)$. Recall that these are classified by signatures $h_1 \geq \cdots \geq h_N \geq 0$ and in this case, at level $\ell$, must satisfy the constraint

$$h_1 + h_2 \leq \ell.$$

Let $W_h$ be the irreducible representation of $Sp(N)$ with signature $h$ and $K_h$ the positive energy irreducible representation with $K_h(0) = W_h$. The classical tensor product rule

$$V_k \otimes W_h = \bigoplus_{h < p, g < q, p + q = k} W_g$$

was determined by Sheila Sundaram (and can be verified directly using the Weyl character formula). For Connes fusion an analogous applies

$$\mathcal{H}_k \boxtimes K_h = \bigoplus_{h < p, g < q, p + q = k} K_g$$

where only $g$ with $g_1 + g_2 \leq \ell$ are allowed on the right hand side. Again there is a unique quantum dimension $d(K_h) > 0$ such that

$$\sum d(K_h)^2 = \sum d(H_f)^2,$$
and
\[ d\mathcal{H}_k d(K_h) = \sum_{h < p, f, g < f, p + q = k} d(K_g). \]

There is a constant \( C > 0 \) and an element \( k_0 \in K \) such that
\[ d(K_h) = C \cdot \text{Tr}_{W_h} k_0. \]

By Frobenius reciprocity for bimodules, \( K_h \boxtimes K_g \) is a sum of finitely many \( H_f \)'s. In particular
\[ K_0 \boxtimes K_0 = \bigoplus_{f_1 = f_2, f_3 = f_4, \ldots} \mathcal{H}_f. \]

Recall that, according to the Cartan–Helgason theorem, the spectral decomposition of \( L^2(G/K) \) as a \( G \)-module under left translation is given by
\[ L^2(G/K) = \bigoplus_{f_1 = f_2, f_3 = f_4, \ldots} V_f. \]

The subfactor defined by the failure of Haag duality in \( K_h \) is irreducible and has Jones index \( d(K_h)^2 \).

3. MODULES OVER TENSOR CATEGORIES. Given a unitary tensor category of bimodules \( X \), it is natural to look for bimodules \( Y \) such that \( Y \boxtimes Y^* \) is a finite direct sum of irreducible bimodules in the original tensor category. In that case one by decomposing all bimodules \( X \boxtimes Y \) and forming the associated Grothendieck group, one obtains a modules over the fusion ring of the original category. As an example, take the tensor category of finite–dimensional unitary representations of a finite group \( \Gamma \). Fixing a 2–cocycle \( \omega(g, h) \) of \( \Gamma \) with
\[ \omega(h, k)\omega(g, hk) = \omega(g, h)\omega(gh, k), \]
we can consider projective representations of \( \Gamma \) with cocycle \( \omega \), maps \( \pi : \Gamma \to U(W) \) such that
\[ \pi(gh) = \omega(g, h)\pi(g)\pi(h). \]

Note that the \( \omega \)-representations form a module over the ordinary representations.

In this case we can define a corresponding \( \omega \)-twisted group algebra \( C_\omega[\Gamma] \) on which \( \Gamma \) acts by conjugation by automorphisms. As a \( \Gamma \)-module we have
\[ C_\omega[\Gamma] = \bigoplus W^* \otimes W \]
summed over equivalence classes irreducible \( \omega \)-representations \( W \). There is moreover a natural coaction
\[ \delta : C_\omega[\Gamma] \to C[\Gamma] \otimes C_\omega[\Gamma], g \mapsto g \otimes g \]
where \( C[\Gamma] \) is the usual group algebra of \( \Gamma \). This coaction is ergodic in that \( \delta(a) = 1 \otimes a \) if and only if \( a \) is a scalar. Such \( \Gamma \)-algebras are all of the form \( C_{\omega_0}[\Gamma_0] \), where \( \Gamma_0 \) is a normal subgroup of \( \Gamma \) and \( \omega_0 \) is 2-cocycle of \( \Gamma_0 \); the action of \( \Gamma \) is given by \( \alpha_g(n) = \mu(g, n)gng^{-1} \), where the pair \( (\omega_0, \mu) \) defines an element of the relative group \( \Lambda(\Gamma, \Gamma_0) \) of Jones.

Analogous constructions and definitions can be made whenever one starts from a unitary tensor category, for example defined by positive energy representations of a loop group at a fixed level, reducing the classification of module categories to that of algebras in the original category admitting an ergodic coaction of the Hopf algebra \( \bigoplus X^* \boxtimes X \) of the category. However, as Wenzl has noted and as the formula for \( K_0 \boxtimes K_0 \) shows, the combinatorial structure of the subfactor for \( K_0 \) in fact approaches that of the infinite–index inclusion \( \mathbb{R}^G \subset \mathbb{R}^K \), for a minimal action of \( G \) on the hyperfinite factor \( \mathbb{R} \) (see LMS notes), a so–called “group–subgroup sufactor”. The subfactor associated with an irreducible \( \omega \)-representation \( W \) of \( \Gamma \) is just
$R^T \subset (R \otimes \text{End } V)^T$. In this sense, since $K$ is far from being a normal subgroup of $G$, the twisted loop group subfactors seem to combine features of both these types of subfactor.

4. POSITIVE ENERGY REPRESENTATIONS OF $L^* \text{SU}(2N)$. Given a complex Hilbert space $H$ and a projection $P$ on $H$, the complex canonical anticommutation relations

$$a(f)a(g)^* + a(g)^*a(f) = (f,g)I, \quad a(f)a(g) + a(g)a(f) = 0,$$

have a unique irreducible representation $F_P$ with cyclic vector $\Omega$ satisfying

$$(a(g_m)^* \cdots a(g_1)^*a(f_1) \cdots a(f_n)\Omega, \Omega) = \delta_{n,m} \det (Pf_i, g_j).$$

The space $F_P$ can be identified with fermionic Fock space and $\Omega$ is called the vacuum vector. If a unitary $u \in U(H)$ is such that $[u, P]$ is a Hilbert–Schmidt operator then there is a unitary $U \in U(F_P)$, unique up to a phase $z$ with $|z| = 1$, such that

$$Ua(f)U^* = a(uf).$$

This gives a projective unitary representation of this subgroup of $U(H)$ on $F_P$ satisfying this “quantization condition”. In particular with $H = L^2(S^1, \mathbb{C}^N)$ and $P$ the projection onto Hardy space (with values in $\mathbb{C}^N$, the natural unitary representation of $L^2(S^1) \rtimes \text{Rot } S^1$ is quantized and gives a positive energy projective representation of the subgroup $\text{LSU}(N) \rtimes \text{Rot } S^1$ at level 1. The level $\ell$ irreducible representations are obtained as summands of the $\ell$–fold tensor product. Taking $N = 1$ and $H = L^2(S^1)$, the corresponding representation of $L^\ast T$ is irreducible. This is the standard case of fermion–boson duality, so–called because a projective representation of the Abelian group $L^\ast T$, the product of $\mathbb{Z} \rtimes \mathbb{T}$ and countably many copies of $\mathbb{R} \rtimes \mathbb{R}$ can be constructed on an infinite tensor product of the corresponding Schrödinger representations (see the 2006 Cambridge lecture notes for example). More generally the representation of $L^2(\mathbb{S}^1)$ on $F_P$ is irreducible, because that of its subgroup $L^\ast U(1)^N$ is.

There is a twisted version of the above construction and a corresponding irreducibility statement, which as Igor Frenkel pointed out is somewhat easier to prove. First note that the above construction depends only on the underlying real structure of the Hilbert space $H$ and the complex structure $J = i(2P - I)$ that $P$ defines on it. Indeed $c(f) = a(f) + a(f)^*$ satisfies

$$c(f)c(g) + c(g)c(f) = 2 \text{Re}(f, g)$$

and

$$a(f) = \frac{1}{2}(c(f) - ic(Jf)).$$

Orthogonal transformations $T$ are quantized if and only if $[T, J]$ is Hilbert–Schmidt. In particular this gives a projective positive energy representation of $L^\ast SO(2N)$, irreducible since it is already irreducible on the subgroup $L^\ast U(N)$. Representations of twisted loop groups are obtained by twisting the underlying Hilbert space. The basic example is the twisted loop group $L^\ast T$ where $\tau(z) = z$. The real Hilbert space is taken to be the completion of the continuous functions $f : \mathbb{R} \to \mathbb{C}$ with $f(t + 2\pi) = f(t)$ and an appropriately defined complex structure $J$. The situation is simpler because $T^\ast = \pm 1$. The fermionic picture is described infinitesimally by operators $e_r$ with $r \in \frac{1}{2}\mathbb{Z}$ with $e^*_r = e_{-r}$ and

$$e_re_s + e_se_r = 2\delta_{r+s,0}.$$  

The infinitesimal generator $d$ for rotations satisfies $[d, e_s] = se_s$ and there is a unique irreducible representation with cyclic vector $\Omega$ satisfying $d\Omega = 0$, $e_0\Omega = 0$ and $e_r\Omega = 0$ for $r > 0$. The projective representation of $L^\ast T$ is given infinitesimally by the bosonic operators $a_r$ ($r \in 1/2 + \mathbb{Z}$) defined by

$$a_r = \frac{1}{2} \sum_{p+q=r} e_pe_q.$$
These satisfy 
\[ [a_r, e_s] = e_{r+s}, \quad [d, a_r] = ra_r \]
from which it follows that 
\[ [a_r, a_s] = r \delta_{r+s,0}. \]

On the other hand the simple combinatorial identity 
\[ \prod (1 + t^m) = \prod \frac{1 - t^{2m}}{1 - t^m} = \prod (1 - t^{2m-1})^{-1}, \]
shows that \( \Omega \) is a cyclic vector for the \( a_n \)'s. In particular \( L^T \) acts irreducibly.

To construct the level one positive energy representations of \( L^\tau SU(2N) \), note that \( \rho(j) \) lies in \( SO(4N) \) so there is an element \( X \) in the Lie algebra of \( SO(4N) \) such that \( \rho(j) = e^{2\pi X} \).

Conjugation by \( e^{tX} \) then gives an isomorphism between \( L^\tau SO(4N) \) and \( LSO(4N) \) which takes preserves positive energy representations. On the other hand \( L^\tau SU(2N) \cdot L^\tau U(1) \) is an index two subgroup of \( L^\tau U(2N) \), the restriction of \( F_{\rho} \) breaks up as the direct sum of two distinct irreducible representations of the subgroup. This gives the explicit construction of positive energy representations of \( L^\tau SU(2N) \) at level one; as Agrebaui showed in 1985, level \( \ell \) representations all arise by decomposing the \( \ell \)-fold tensor products of level one representations.

5. VERTEX ALGEBRAS, TWISTED MODULES AND INTERTWINERS. In conformal field theory, the mathematical language of vertex algebras and their representations is by now fairly well established. As introductions there are the notes of Peter Goddard on meromorphic conformal field theory and the introductory book of Victor Kac. Here we give a tailor–made approach which constructs the twisted theory in what seems to be the most direct and elementary way possible. Given a particular conformal field theory, for example the positive energy representations of a loop group at a fixed level, the vacuum representation \( H_0 \) can be given the structure of a vertex algebra. Thus to each finite energy vector \( v \in H_0 \), there is “field” given by a formal power series 
\[ V_0(a, z) = \sum \varphi(a, n) z^{-n-\delta_a} \]
where \( \varphi(a, n) : H_0(k) \to H_0(k-n) \). The field should create the state \( a \) from the vacuum:
\[ V_0(a, z)\Omega|_{z=0} = a. \]

They satisfy the following “locality relation”
\[ V_0(a, z)V_0(b, w) = V_0(b, w)V_0(a, z) \]
in sense that the matrix coefficients of each side, regarded as formal power series, define the same rational function in \( R = \mathbb{C}[z^{\pm 1}, w^{\pm 1}, (z-w)^{-1}] \). Moreover
\[ [L_{-1}, V_0(a, z)] = \frac{d}{dz} V_0(a, z), \quad [L_0, V_0(a, z)] = z \frac{dV_0(a, z)}{dz} + \delta_a V_0(a, z). \]
The following associativity relation follows automatically:
\[ V_0(a, z)V_0(b, w) = V_0(V_0(a, z-w)b, w), \]
again as formal power series developments of the same rational function in $\mathcal{R}$. For the loop group $LSU(2N)$, each element $X$ of the Lie algebra of $SU(2N)$ defines elements $X(n) = X e^{-in\theta}$ of the complexified Lie algebra of $LSU(2N)$ and hence a field $X(z) = \sum X(n)z^{-n-1}$. For different $X$ and $Y$ the fields $X(z)$ and $Y(w)$ commute in the sense of the locality relation above. The associativity relation can be used as a way of defining products $X(n) Y$ of fields, making precise the “operator product expansion” of theoretical physicists; a fundamental result of Dong implies, that if field $X$ and $Y$ are local with respect to $Z$, then so is any one of the products $X(n) Y$. This result can be used to set up inductively the correspondence between fields and states.

For any other representation $H_f$, there are similar operators $V_f(a, z)$ on $H_f$ such that

$$V_f(a, z)V_f(b, w) = V_f(b, w)V_f(a, z) = V_f(V_0(a, z-w)b, w).$$

Given a representation $H_k$ intertwiners $\varphi^k_{fg}(v, z) : H_g \to H_f \ (v \in H_k)$ should satisfy:

$$\varphi^k_{fg}(v, z)\varphi^k_g(a, w) = V_f(a, w)\varphi^k_{fg}(v, z) = \varphi^k_g(V_k(a, z-w)v, z),$$

where the intertwiners can have $\delta$’s which are rational numbers.

In the case of twisting by a period two automorphism of the loop group and hence the vertex algebra, the definition of the $V_f(a, z)$’s can still be defined, but the expansion must involve $z^{1/2}$. This is explained extensively by Li and Xu. Their approach is purely algebraic so cannot be used to deduce analytic properties of the operators $V(a, n)$ or $\varphi(v, n)$.

There is, however, a direct method of constructing these fields which also yields their analytic properties. The idea is to construct all possible fields at level one and then take tensor products. The vertex algebra of $LSO(4N)$ at level one is completely described by the fermionic/bosonic construction. Indeed for rational lattices $\Lambda \subset V$, vertex operators can be defined for every state in the bosonic Fock space $\ell^2(\Lambda) \otimes S(V)\otimes^\infty$. The simplest such operators are “current” fields

$$X(z) = \sum X(n)z^{-n-1}$$

corresponding to bosons; the other vertex operators have are essentially Fubini–Venziano vertex operators of the

$$\varphi_X(z) = z^{X(0)} \exp \int X_-(z) \exp \int X_+(z)$$

where $X_\pm(z) = \sum_{n>0} X(n)z^{-n-1}$. All other operators can be expressed in terms of these two types of operators. On the other hand the vertex operators $\varphi_X(z)$ have an important factorization property

$$\varphi_X(z) \otimes \varphi_Y(z) = \varphi_{X\otimes Y}(z).$$

Since when $\|Y\| = 1$, $\varphi_Y(z)$ yields a fermionic field, this implies analytic properties of general $\varphi_X(z)$. This means that the modes $\phi_X(n)$ map Sobolev spaces of $L_0$ into other Sobolev spaces and hence preserve the space of $C^\infty$ vectors for Rot $S^1$.

Having constructed the vertex algebra for $LSO(4N)$ at level one, the problem is then explicitly to use this to describe the twisted sector and all the intertwiners. However, Li has given a very simple description of how to pass from untwisted to twisted modules. In fact the field

$$\Delta(z) = z^{X(0)} \exp - \sum_{n>0} X(n)(-z)^n/n$$

satisfies

$$[L_{-1}, \Delta(z)] = -\frac{d\Delta}{dz} \cdot \Delta(z)V(a, z)\Delta(z)^{-1} = V(\Delta(z)a, z).$$

The fields $V(\Delta(z)a, z)$ define the fields in the twisted version of the vacuum sector. Intertwiners are then defined using the relation of Dolan, Goddard and Montague:

$$V(a, z)b = e^{zL_{-1}}V(b, -z)a.$$
At level one, this gives an explicit construction of all possible fields.

6. PRIMARY FIELDS AND BRAIDING RELATIONS. Note that intertwiners labelled by untwisted representations can only go from an untwisted or twisted representation to similar type of representation, while those labelled by a twisted representation must switch the type. From the Dolan–Goddard–Montague correspondence, it follows that to determine the dimensions of spaces of intertwiners, only those labelled by untwisted representations need be considered. From this it follows immediately that, for every twisted representation $K_h$, there is a unique intertwiner labelled by $K_h$ from the vacuum representation $\mathcal{H}_0$ to $\mathcal{K}_h$. For the representations $\mathcal{K}_h$, corresponding to the exterior powers $V_k = \lambda^k V$, we want to determine the dimension of the space of intertwiners of charge $\mathcal{K}_h$ between two twisted representations. As in the untwisted case an intertwiner of charge $\mathcal{H}_f$ is determined uniquely by $\phi(v, z)$ for $v \in \mathcal{H}_f(0) = V_f$. These fields are called primary fields. If

$$\phi(v, z) = \sum_{n \in \mathbb{Z}} \sum_{n \in \mathbb{Z}} \phi(v, n)z^{-n - \delta},$$

then the $\phi(a, z)$ defines a primary field if and only if

$$[X(m), \phi(v, n)] = \phi(Xv, m + n), [d, \phi(v, n)] = -n \phi(v, n).$$

Here $X(n) = e^{-in\theta}$ with $n$ in $\mathbb{Z}$ or $\frac{1}{2} + \mathbb{Z}$ according to whether $\tau X = X$ or $\tau X = -X$. As in the untwisted case the primary field is uniquely determined by its initial term, the restriction of $\phi(v, 0)$ to a map between $K_g(0)$ and $K_h(0)$, i.e. an element $T$ in of $\text{Hom}_K(V_f \otimes W_g, W_h)$. Thus for the exterior powers, these spaces are at most one–dimensional by Sundaram’s tensor product rule. Her rule in fact gives the exact dimension when $g$ and $h$ are permissible, i.e. $K_g$ and $K_h$ exist. For, as for untwisted representations, the condition that $T$ be the initial term of a primary field can be expressed as linear conditions given by operators from a copy of $\mathfrak{sl}_2$ given the highest/lowest weight vectors for $K$ in the -1 eigenspace of $\tau$ in $g$: for the exterior powers it is easy to check these conditions directly (compare the computations of Beauville and Faltings for the Verlinde formula).

As in the untwisted case, the braiding properties of primary fields follow by considering the reduced four-point function

$$f(z) = \sum_{i} (\phi(v_2, n_i)\phi(v_3, -n_i)v_4, v_1)z^{n_i}$$

for lowest energy vectors $v_1$. As Verrill showed, considered as a function with values in $U = \text{Hom}_K(V_2 \otimes V_3 \otimes V_4, V_1)$, this satisfies the twisted Khnizhnik–Zamolodchikov ordinary differential equation:

$$f(z) = \left( A + \frac{B}{1 - z} + \frac{C}{(1 - z)^{1/2}} \right) f(z)$$

where $A$, $B$ and $C$ are operators on $U$ together with boundary condition $D_i(z)f(z) = 0$ where the $D_i$ are analytic with values in $U^*$. Moreover reduced 2–point functions of this form exhaust all such functions. The same is true for products of primary fields with $V_2$ and $V_3$ reversed and $z$ replaced by $z^{-1}$. As in the untwisted case the monodromy of the ordinary differential equation from 0 to $\infty$ yields the braiding relation, which passing to smeared primary fields takes the form

$$\phi^g_{g_0}(a)\phi^{k^*}_{k_0}(b) = \sum_{\lambda} \lambda_0 \phi^g_{g_0}(e^{\lambda_0} \cdot a)\phi^{k^*}_{k_0}(e^{\lambda_0} \cdot a),$$

where $\lambda$, $\mu$ and $\nu$ are constants, $a$ and $b$ are supported in the upper and lower semicircle of the unit circle and $e^{\alpha}(\theta) = e^{i\alpha \theta}$. Similarly we have the Abelian braiding

$$\phi^g_{h_0}(a)\phi^{k^*}_{k_0}(b) = \varepsilon_h \phi^g_{h_0}(e^{\delta_h} \cdot b)\phi^{g^*}_{g_0}(e^{-\delta_h} \cdot a),$$

where $\varepsilon_h$ and $\delta_h$ are constants.

7. LOCAL TWISTED LOOP GROUPS. The local theory of twisted loop groups is particularly easy to deduce from the untwisted theory because of the construction by embedding in $LSO(4N)$. For an interval
Proof. Irreducibility. \( I \subset S^1 \) and \( G = SU(2N) \), the local twisted loop group \( L^*G \) is defined as the subgroup of loops such that \( f(t) = I \) for \( t \notin I \).

Generation by exponentials. Every element \( g \) of \( L^*G \) is a product of exponentials in \( L^*g \).

Proof. Note that if \( h \in L^rG \) with \( \| h - I \|_\infty < 1 \), then \( h = \exp X \) with \( X = \log I - (I - h) \) in \( L^*g \). So it suffices to find \( g_0 = I, g_1, \ldots, g_{n-1}, g_n = g \) with \( g_i \) in \( L^*G \) and \( \| g_i - g_{i-1} \|_\infty < 1 \) for \( 1 \leq i \leq n \). For then \( g_i^{-1}g_{i-1} = \exp X_i \) and \( g = \exp X_1 \cdots \exp X_n \). Since smooth loops are uniformly dense in continuous loops, it suffices to find such a chain with \( g_1, \ldots, g_{n-1} \) continuous. Now if \( g(0) = g(2\pi) \) is fixed by \( \tau \), take a path \( g' \) joining \( I \) to \( g(0) \) in \( K \). If \( g(0) \neq \tau g(2\pi) \), then we can take \( g' \) to be a path in \( G \) from \( I \) to \( g(0) \). Define a map of the boundary of the 1-simplex into \( G \) as \( g' \) on the first side emanating from a vertex, \( \tau g' \) on the second side and \( g \) on the third side, opposite the vertex. Since \( G \) is simply connected, the map can be extended to a continuous map of the 1-simplex into \( G \). Each segment parallel to the third side yields a continuous map \( h_t \) \( (0 \leq t \leq 1) \) with \( h_t(0) = h_t(2\pi) \in K \). By construction \( h_0 = I \) and \( h_1 = g \). Now take \( g_i = h_{i/n} \) for \( n \) sufficiently large.

Covering property. If \( S^1 = \bigcap_{k=1}^{g} I_k \), then \( L^*G \) is generated by the subgroups \( L^*_k G \).

Proof. By the previous result, it is enough to show that every exponential \( \exp X \) lies in the group generated by the \( L^*_k G \). Let \( \psi \in C^\infty(S^1) \) be a partition of the identity subordinate to \( (I_k) \). Then \( X = \sum \psi_k \cdot X \), a sum of commuting elements, so that

\[
\exp X = \exp \psi_1 \cdot X \cdots \exp \psi_n \cdot X.
\]

Local equivalence. Under the natural identification between \( L^rG \) and \( L^*_G \), the irreducible positive energy projective representations at level \( \ell \) become unitarily equivalent, generating hyperfinite factors of type III_1.

Proof. The automorphism \( \tau \) is the restriction of an inner automorphism \( Adx \) of \( SO(4N) \) where \( x = \rho(\ell) \), also denoted by \( \tau \). There is a natural inclusion of \( L^rG \) in \( L^rSO(4N) \). Choose a Lie algebra element \( X \) such that \( x = \exp 2\pi X \), let \( h(t) = \exp tX \). Then the map \( g \rightarrow hgh^{-1} \) gives an isomorphism between \( L^rSO(4N) \) and \( LSO(4N) \), preserving the class of positive energy representations at level \( \ell \). This yields an embedding \( \psi \) of \( L^rG \) in \( LSO(4N) \) and the level \( \ell \) representations of \( L^rG \) are obtained by decomposing the \( \ell \)-th tensor power of the vacuum representation of \( LSO(4N) \). Let \( \psi_1 \) be the natural embedding of \( LG \) in \( LSO(4N) \). We know that \( \psi \) is given by the formula \( \psi(g) = hgh^{-1} \). For the interval \( I \) we can find \( h_1 \in L_i SO(4N) \) with \( J \) a proper interval containing \( I \) and with \( h = h_1 \) on \( I \). It follows that \( \psi(g) = h_1 \psi_1(g) h_1^{-1} \) for \( g \in L^*_G \). The level \( \ell \) representations of \( LG \) and \( L^*_G \) arise by decomposing the \( \sigma = \pi_0^\ell \) of \( SO(4N) \) according to these two embeddings. Thus \( U = \sigma(h_1) \) gives a unitary intertwiner between the two representations. Let \( M = \sigma(\psi(L^*_G))'' \) and \( M_1 = \sigma(\psi(L^*_G))'' \). Let \( P \) be projections onto irreducible summands \( \sigma \circ \psi \) and \( \pi_1 \) of \( \sigma \circ \psi_1 \). Thus \( P \) lies in \( M'' \) and \( P_1 \) in \( M_1'' \). Since \( M = UM_1 U^* \), it follows that \( M_1'' = UM''U^* \), so that \( \sigma P \) lies in \( M_1'' \). Hence we can find a unitary \( V \) in \( M_1'' \) such that \( P = VPU^* V^* \). The unitary \( U \) then implements the unitary equivalence between the restrictions \( \pi \) and \( \pi_1 \) to \( L^*_G \).

von Neumann density. If \( I_1 \) and \( I_2 \) are touching intervals of the circle whose union is a proper interval \( I \) and \( \pi \) is an irreducible positive energy projective representation of \( L^rG \) on \( H \), then \( \pi(L^*_1 G \cdot L^*_2 G) \) is dense in \( \pi(L^rG) \) in \( PU(H) \) for the strong operator topology.

Proof. By local equivalence this reduces to the untwisted case, where the result is known.

Irreducibility. If \( \pi \) is an irreducible positive energy representation of \( L^rG \) on \( H \) and \( N = \pi(L^*_G)'' \) and \( M = \pi(L^*_G)'' \), where \( I^c \) is the complementary interval, then \( N'' \cap M = \emptyset \).

Proof. On removing points \( I \) and \( I^c \) can be written as the union of intervals \( I_1, I_2 \) and \( I_3, I_4 \) in anticlockwise order. It suffices to show that the von Neumann algebra generated by \( \pi(L^*_G) \) and \( \pi(L^*_G) \) is all of \( B(H) \). But this is just the von Neumann algebra generated by \( \pi(L^*_k) \) for \( 1 \leq k \leq 4 \). By two applications of the von Neumann density lemma, the von Neumann algebra generated by \( L^*_k \) with \( j \neq k \) gives the von Neumann algebra corresponding to \( L^*_k \). Two of these form an open cover of \( S^1 \) and so their von Neumann algebras generate \( B(H) \).
8. PHASE THEOREM AND CONNES FUSION. Using hermiticity, the braiding relations for smeared primary fields can be written in the form

$$a^g_0 b^g_{k0} = \sum \lambda_h b^g_{hk} a_{h0}, \quad a_{hk} b_{k0} = \varepsilon_k b_{h0} a_{g0}$$  \hfill (1)

where \(a = a^g_0\) and \(b = b_{k0}\) are called the principal parts. These two braiding relations imply the transport formula

$$ab^* b = \sum |\lambda_h| b^g_{hg} a = (b')^* b' a,$$  \hfill (2)

where

$$b' = (|\lambda_h|^{1/2} b_{h0}).$$

**Phase Theorem.** Both these sets of intertwiners can be modified so that both the principal parts are unitary operators and the non–principal parts are bounded.

The proof of this will be explained below. Granted the theorem, if \(x : \mathcal{H}_0 \to \mathcal{K}_g\) and \(y : \mathcal{H}_0 \to \mathcal{H}_k\) are arbitrary intertwiners, their non–principal parts are defined by

$$x_{ij} = a_{ij} \pi_f(a^* x), \quad y_{pq} = b_{pq} \pi_g(b^* y).$$

These still satisfy the braid relations and the transport formula. But then to compute the Connes fusion \(\mathcal{K}_g \boxtimes \mathcal{H}_k\), we note that by the transport formula

$$\|x \otimes y\|^2 = (x^* x y \Omega, \Omega) = \sum_h |\lambda_h| (a^* b^g_{hg} b_{h0} a \Omega, \Omega).$$

Hence

$$U(x \otimes y) = \oplus_h |\lambda_h|^{1/2} y_{hg} x \Omega$$

defines a unitary of \(\mathcal{K}_g \boxtimes \mathcal{H}_k\) onto \(\oplus_h \mathcal{K}_h\).

(a) The transport formula and braid relations remain true if the operator \(a\) is replaced by its phase. Let

$$b' : \mathcal{K}_g \to \oplus \mathcal{K}_h$$

be given by \(b' = (b_{h0})\). Let \(x = b^* b\) and \(y = (b')^* b'\). Thus

$$ax = ya, \quad xa^* = a^* y.$$ 

Hence, if \(X = (I + x)^{-1}\) and \(Y = (I + y)^{-1}\), then we have

$$aX = Ya.$$ 

Let

$$A = \begin{pmatrix} 0 & 0 \\ a & 0 \end{pmatrix}, \quad B = \begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix}.$$ 

Thus \(AB = BA\) on a core \(H^\infty\) for \(A, 0 \leq B \leq I\) and \(BH^\infty \subseteq D(A^* A)\). Passing to the closure of \(A\), it follows that \(AB \supseteq BA\), i.e. \(BD(A) \subseteq D(A)\) and \(AB = BA\) on \(D(A)\). But then, taking adjoints, it follows that \(BA^* \subseteq A^* B\) and hence that \(A^* AB \supseteq BA^* A\), i.e. \(BD(A^* A) \subseteq D(A^* A)\) and \(A^* AB = BA^* A\) on \(D(A^* A)\).

(This is just the statement that \(a^* a\) and \(x\) have the commuting spectral projections.) So if

$$C = A^* A = \begin{pmatrix} a^* a & 0 \\ 0 & 0 \end{pmatrix},$$

\((I + C)^{-1}\) commutes with \(B\). Similarly \((\varepsilon + C)^{-1}\), and hence its square root \((\varepsilon + C)^{-1/2}\) commutes with \(B\). Now \(AB = BA\) on \(D(A)\), so that

$$A(\varepsilon + C)^{-1/2} B = AB(\varepsilon + C)^{-1/2} = BA(\varepsilon + C)^{-1/2}$$

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on $\mathcal{D}(A)$. Since $(\varepsilon + C)^{-1/2}A$ tends in the strong operator topology to the phase of $A$, it follows that the phase of $A$ commutes with $B$.

Similarly it follows that $A_p = \pi_p((\varepsilon + a^*a)^{-1/2})$ satisfies $A_pb_{pq} = b_{pq}A_q$. Hence the intertwiners $a_{ij}\tau_j((\varepsilon + a^*a)^{-1/2})$ satisfy the braiding relations, are bounded and as $\varepsilon \to 0$ have a strong operator limit. This limit also satisfies the braiding relations, but now the principal part is the phase of $a$.

(b) The transport formula and braid relations remain true of the operator $b$ is replaced by its phase. This follows by the similar reasoning to (a), but the proof is slightly easier. Let $A_1 = a$, $A_2 = \oplus|\lambda_h|^{1/2}a_{hk}$, $B_1 = b$ and $B_2 = \oplus\varepsilon_h|\lambda_h|^{1/2}b_{hq}$. The braiding relations then take the form $A_1B_1^* = B_2^*A_2$ and $A_2B_1 = B_2A_1$. By assumption, the $A_i$'s are partial isometries. On the other hand it is easy to see that these equations are unchanged if the $B_j$'s are replaced by their phases.

(c) The transport formula and braid relations also hold with $a$ and $b$ unitary. Each set of intertwiners $(c_{ij})$ can be modified in three steps in three steps, preserving the braiding relations. Firstly $c_{ij}$ can be replaced by $\sum 2^{-n}\pi_i(g_n)c_{ij}(u_n)$ with $(g_n)$ a dense subgroup of the unitary group and $u_n$ unitaries in $\pi_0(L_1G)^\nu$ with $u_nu_n^* = \delta_{ij}I$ and $\sum u_n^*u_n = I$. Secondly passing to the phase, we many assume that $cc^* = I$. Finally, we can replace $c_{ij}$ by $c_{ij}\pi_j(u)$ where $u$ is a partial isometry in $\pi_0(L_1G)^\nu$ with $uu^* = I$ and $u^*u = c^*c$.

9. FUSION RULES. In the previous section upper bounds were obtained for fusions of twisted representations with the untwisted fundamental representations. In order to show these upper bounds are attained, we provide an algebraic model for these fusion rules. (As in the case of the Verlinde formula for fusion of untwisted representations, the fusion coefficients can also be expressed in terms of the classical tensor product coefficients by incorporating corrections given the action of a suitable affine Weyl group.)

We start by recalling that the characters of $SU(2N)$ are labelled by signatures $f_1 \geq f_2 \geq \cdots \geq f_{2N}$. On a matrix with diagonal entries $z_i$, the corresponding character is given by

$$\chi_f(z) = \frac{\det(z_j^{f_i+2N-i})}{\prod_{i<j}(z_i-z_j)}.$$ 

Diagonal matrices in $Sp(N)$ have entries $\zeta_{1}^{\pm 1}, \ldots, \zeta_{N}^{\pm 1}$ and the characters of irreducible representations are indexed by signatures $h_1 \geq \cdots \geq h_N \geq 0$, with

$$\psi_h(\zeta) = \frac{\det(\zeta^{f_i+N-i+1} - \zeta^{f_i-N+i-1})}{\prod_{i<j}((\zeta_i - \zeta_j^+)\prod_{i<j}(\zeta_i + \zeta_j^1 - \zeta_j^1))}.$$ 

Now let

$$S = \{g : g_1 \geq g_2 \geq \cdots \geq g_N \geq 0, g_i \leq \ell/2, g_i \text{ all integers or half-integers}\}.$$ 

Let $D(g) \in Sp(N)$ be the diagonal matrix with entries $\zeta_{i}^{\pm 1}$ with $\zeta_i = \exp(2\pi i(g_i + N + 1/2 - i)/(2N + \ell))$ for $g \in S$. If $h_1 + h_2 = \ell + 1$, then $\psi_h(D_g) = 0$ since the first two columns of the determinant in the numerator vanish. Let $T = \{D(g) : g \in S\}$ and let $\theta : R(Sp(N)) \rightarrow C^\nu$ denote the evaluation or restriction map. Let $S = \theta(R(Sp(N)))$ and let $\theta_h = \theta(W_h)$. The $\theta_h$'s with $h$ permissible (i.e. $h_1 + h_2 \leq \ell$) are clearly closed under multiplication by the $\theta(\chi_h)$. Moreover the fundamental representations of $Sp(N)$ are just $\chi_1, \chi_2 - \chi_1, \chi_3 - \chi_2, \ldots, \chi_{N-1} - \chi_{N-2}$ where $\chi_k$ is the character of $\Lambda^kC^\nu$. It follows that the $\theta(\psi_h)$'s generate $S$ and hence that the $Z$-linear span of the $\theta_h$'s with $h \in S$ equals $S$. The characters $\chi_k$ with $1 \leq k \leq N$ distinguish the points of $T$. Hence $S_C$ is a unital subalgebra of $C^\nu$, so must equal $C^\nu$. It follows that the $\theta_h$'s with $h$ permissible form a $Z$-basis of $S$. We claim that ker $\theta$ is the ideal in $R(Sp(N))$ generated by $\theta_h$'s with $h_1 + h_2 = \ell + 1$. Indeed if $I$ is the ideal they generate, then $R(Sp(N))/I$ is spanned by the image of the $[W_h]$'s with $h$ permissible, from the tensor product rules with the $V_h$'s. But $I \subseteq \ker \theta$ and the $\theta_h$'s are linearly independent over $Z$. Hence the images of the $[W_h]$'s give a $Z$-basis of $R(Sp(N))/I$ and therefore $I = \ker \theta$.

So far we know that

$$\mathcal{H}_k \otimes \mathcal{K}_h \leq \bigoplus_{h'p, g'q, f'p+q=k} \mathcal{K}_{g'}. \tag{*}$$

Taking quantum dimensions it follows that

$$d(\mathcal{H}_k)d(\mathcal{K}_h) \leq \bigoplus_{h'p, g'q, f'p+q=k} d(\mathcal{K}_{g'}).$$

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On the other hand $d(\mathcal{H}_k) = \chi_k(D(0))$. The quantities $\psi_h(D(0))$ are also positive and satisfy

$$\chi_k(D(0))\psi_h(D(0)) = \sum_{h' < f, g < f, p+q = k} \psi_g(D(0)).$$

Since the Perron–Frobenius eigenvalue of a matrix with non–negative entries strictly decreases on passing to a submatrix, we deduce that equality occurs in $(\ast)$ and that $d(\mathcal{K}_h) = C\psi_h(D(0))$ for some $C > 0$. It follows immediately that fusion with more general representations $H_f$ can be computed by looking at how the basis elements $\theta_h$ of $S$ transform under multiplication by $\theta(\chi_f)$. To compute $C$, we note that, from the theory of bimodules,

$$\sum_f d(H_f)^2 = \sum_h d(K_h)^2.$$

So

$$C^2 = \frac{\sum_f \chi_f(D(0))^2}{\sum_h \psi_h(D(0))^2}.$$
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