TIME REGULARITY OF FLOWS OF NON-NEWTONIAN FLUIDS
WITH CRITICAL POWER-LAW GROWTH

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ABSTRACT. We deal with the flows of non-Newtonian fluids in three dimensional setting subjected to the homogeneous Dirichlet boundary condition. Under the natural monotonicity, coercivity and growth condition on the Cauchy stress tensor expressed by a power index $p \geq 11/5$ we establish regularity properties of a solution with respect to time variable. Consequently, we can use this better information for showing the uniqueness of the solution provided that the initial data are good enough for all power–law indexes $p \geq 11/5$. Such a result was available for $p \geq 12/5$ and therefore the paper fills the gap and extends the uniqueness result to the whole range of $p$’s for which the energy equality holds.

1. Introduction

We study the generalized Navier–Stokes system

\begin{align}
\partial_t u + (u \cdot \nabla)u - \text{div}\, \mathbf{S}(\varepsilon(u)) + \nabla \pi &= f, \\
\text{div } u &= 0
\end{align}

in $Q := (0, T) \times \Omega$ with a bounded domain $\Omega \subset \mathbb{R}^3$. Here $u : Q \to \mathbb{R}^3$ denotes the velocity field, $f : Q \to \mathbb{R}^3$ the density of the external body forces, $\pi : Q \to \mathbb{R}$ is the pressure and $\mathbf{S} : \mathbb{R}^{3 \times 3} \to \mathbb{R}^{3 \times 3}$ denotes the viscous part of the Cauchy stress. The system (1.1)–(1.2) is completed by the initial and boundary conditions

\begin{align*}
u &= 0 & \text{on } \partial \Omega \times (0, T) \\
u(0) &= u_0 & \text{in } \Omega,
\end{align*}

We consider the usual Ladyzhenskaya-type power-law fluid introduced in [14], i.e., existence of $p > 2$ such that the stress tensor is a continuous nonlinear function of the symmetric velocity gradient $\varepsilon(u)$, satisfying for all symmetric $\varepsilon, \varepsilon_1, \varepsilon_2 \in \mathbb{R}^{3 \times 3}$

\begin{align}
(\mathbf{S}(\varepsilon_1) - \mathbf{S}(\varepsilon_2)) : (\varepsilon_1 - \varepsilon_2) &\geq \begin{cases}
c(1 + |\varepsilon_1| + |\varepsilon_2|)^{p-2} |\varepsilon_1 - \varepsilon_2|^2, \\
c|\varepsilon_1 - \varepsilon_2|^2 + c|\varepsilon_1 - \varepsilon_2|^p,
\end{cases} \\
|\mathbf{S}(\varepsilon)| &\leq c(1 + |\varepsilon|^{p-1}).
\end{align}

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Without going into details, the main result of the paper can be explained as follows: the system (1.1)–(1.2) gives natural a priori estimates $u \in L^\infty(0,T; \mathcal{L}^2) \cap L^p(0,T; \mathcal{V}_p)$. If $p \geq 11/5$, it further follows that $\partial_t u \in L^p(0,T; \mathcal{V}_p')$. This means that the solution becomes an admissible test function and rigorous existence theorem can be obtained using standard compactness and monotonicity arguments. The question of uniqueness is however open in general in the above regularity class.

This “existence-uniqueness gap” is related to the fact that it is possible to test the equation by the solution, but it is not possible to do the same for the differences of solutions, in view of the nonlinear character of the problem. The key idea (due to [4]) is that in the strictly subcritical case, i.e.,

$$ p > \frac{11}{5}, $$

one obtains some room to estimate at least fractional differences of solutions. Of course the critical term here is the convective term, and due to its polynomial character, the estimates are easily computable and can be iteratively improved to the point where the obtained regularity finally implies uniqueness.

In the present paper, we extend the result in two ways. We show that the regularity is global, meaning up to the time $t = 0$, provided that $u_0 \in W^{1,p}$. Secondly, we show that the result holds even for the critical case $p = 11/5$. Here the key ingredient is a delicate estimate involving a vector-valued version of Gehring’s lemma.

The paper is organized as follows. In Section 3, we recall the appropriate function spaces, including a brief review of Nikolskii spaces. The concept of weak solution is defined, and the main result of the paper is formulated together with its corollaries. In Section 4, we establish some auxiliary estimates: we recall the standard weak-strong uniqueness, establish the initial-time regularity, and also prove the improved integrability in the critical case $p = 11/5$ based on Gehring’s’ lemma.

The iterative scheme, which results in the proof of the main theorem, is explained in the final Section 5.

2. Bibliographical overview

Although improving regularity in time of weak solutions is standard, see among others [19, Theorem III.3.5], [18, Theorem 2.7.2], [6, Section 7.1], [10, Lemma 4] there are not many works where this is done in the similar way as here. Perhaps the closest to our approach is the method from [16, Section 2], where the iterative improvement of time regularity of solutions is necessary due to terms appearing when localizing equations in time. In our article the main obstacle is the convective term. The method we use is based on the same idea of iterative improvements of time regularity as the method of [16] although the application is slightly different. It allows to handle problems connected with convective term and also with localization. In the case $p \in (11/5, 12/5]$ there appear additional difficulties that have to be overcome. In [13] this method is used to obtain full regularity of systems similar to (1.1) for $p \in [2, 4)$ if $\Omega \subset \mathbb{R}^2$. 
Improving time regularity of weak solutions is also used in [3] to compute bounds of dimension of attractor to system (1.1) if $\mathcal{S}$ has potential and $p > 12/5$. In [5] a similar iterative approach was used to derive local improvement of regularity in time for strongly nondiagonal parabolic systems of $p$-Laplace type. The main obstacle there is not a nonlinear term similar to $K_0$ below but the terms appearing due to localization in space. Technique of differences in time is used also in [7]. In [12] and [8] a similar approach is used to establish time regularity and uniqueness for the Ladyzhenskaya type fluid coupled with Cahn-Hilliard equation.

Concerning the uniqueness of solution -- already in [14], the uniqueness is established provided $p \geq 5/2$ or in case of smooth initial condition for $p \geq 12/5$. The range $p \in [11/5, 12/5)$ however remained untouched except the case of spatial periodic condition, for which one can improve even spatial regularity, see [14, 15]. Such a method is however not available for Dirichlet (or other) boundary conditions. The case of general boundary condition was firstly treated in [4], where the uniqueness in sense of trajectories was proven for $p > 11/5$. This paper therefore completes and unifies the uniqueness theory, i.e., for sufficiently regular initial condition, we have the global in time unique solution provided $p \geq 11/5$.

3. Preliminaries

3.1. Function spaces. We employ the standard Lebesgue and Sobolev spaces, pertinent to the weak formulation of our problem:

$$G = L^2(\Omega; \mathbb{R}^3) \cap \{ \text{div } u = 0, \quad u \cdot n|_{\partial \Omega} = 0 \}$$

$$V_p = W^{1,p}(\Omega; \mathbb{R}^3) \cap \{ \text{div } u = 0, \quad u|_{\partial \Omega} = 0 \}$$

Our main focus will be the time regularity of vector-valued function $u : [0, T] \to X$, where $X$ is some Banach space. The symbol $\frac{d}{dt}$ denotes the weak (distributional) derivative, and $C, C^{0,\alpha}$ are continuous and $\alpha$-Hölder continuous functions, respectively. To describe a finer scale of fractional time regularity, we will work with the so-called Nikol’skii spaces. For $u : I \to X$, where $I \subset \mathbb{R}$ is an arbitrary time interval, and $h > 0$, we set

$$I_h = \{ t \in I; \quad t + h \in I \}$$

$$\tau^h u(t) = u(t + h), \quad t \in I_h$$

$$d^h u(t) = u(t + h) - u(t), \quad t \in I_h$$

For $p \in [1, \infty]$ and $s \in (0, 1)$, the Nikol’skii space $N^{s,p}(I; X)$ is defined via the norm

$$\|u\|_{L^p(I; X)} + \sup_{h > 0} h^{-s}\|d^h u\|_{L^p(I_h; X)}$$

It is not difficult to see that for $s = 1$, the above norm is equivalent to $W^{1,p}(I; X)$. For a general $\sigma = k + s$, where $k \in \mathbb{N}$ and $s \in (0, 1)$, one defines $N^{\sigma,p}(I; X)$ as the space of functions with $(\frac{d}{dt})^j u \in L^p(I, X)$ for $j = 0, \ldots, k$ and moreover, $(\frac{d}{dt})^k u \in N^{s,p}(I; X)$.

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1Here, in the sense of trajectories means that if $u_1$ and $u_2$ coincide for all $t \in (0, t^*)$ then they coincide also for all $t \geq t^*$. 
Nikolskii spaces are one instance of fractional regularity spaces: \( N^{s,p} = B^{s,p}_\infty \), where the latter is the Besov space. The corresponding theory is treated in many books, e.g. Adams, Fournier [1] or Bennett, Sharpley [2]. Relatively elementary treatment can be found in Simon [17]. The following embeddings are standard, see e.g. [17, Corollary 26 and 33].

\[
\begin{align*}
N^{s,p}(I;X) &\hookrightarrow C^{0,\alpha}(I;X) \quad \text{if } \alpha = s - \frac{1}{p} > 0 \\
N^{s,p}(I;X) &\hookrightarrow L^q(I;X) \quad \text{if } \frac{1}{q} > \frac{1}{p} - s \geq 0
\end{align*}
\] (3.1)

Nikolskii spaces are not the best choice in view of interpolation or embedding results; note the strict condition on \( q \) in (3.1). Their relative advantage lies in simplicity of definition – we will see that it is rather straightforward to obtain estimates of \( N^{s,p} \)-norm. The following special interpolation result will be useful (see Lemma 2.3 in [4] for a simple proof).

**Lemma 3.1.** Let \( X \hookrightarrow H \), where \( H \) is a Hilbert space and \( X \) is separable and dense in \( H \). Then

\[
N^{\alpha,p}(I;X) \cap N^{\beta,p'}(I;X') \hookrightarrow N^{\alpha+\beta,2}(I;H)
\]

for any \( \alpha, \beta \geq 0 \).

### 3.2. Weak formulation and classical results.

We adopt the standard functional formulation of (1.1)–(1.2). Set

\[
\langle N(u), \psi \rangle = \int_\Omega \mathcal{S}(e(u)) : e(\psi) \, dx
\]

\[
\langle K_0(u), \psi \rangle = \int_\Omega (u \otimes u) : \nabla \psi \, dx
\]

A function \( u : [0,T] \rightarrow V_p \) will be called weak solution if it satisfies

\[
u \in L^\infty(0,T;G) \cap L^p(0,T;V_p)
\]

and the equation

\[
dt u + N(u) = K_0(u) + f \quad \text{in } V'_p
\] (3.3)

holds almost everywhere in \( I \).

The pressure is excluded from the weak formulation as usual. The critical condition \( p \geq 11/5 \) means that the derivative belongs to the corresponding dual space

\[
\frac{dt}{dt} u \in L^{p'}(0,T;V'_p).
\] (3.4)

More precisely, one has the following estimate.

**Lemma 3.2.** Let \( p \geq 11/5 \) and \( \mathcal{S} \) satisfy (1.4). Then the weak solution satisfies for almost every \( t \in (0,T) \)

\[
\| \frac{dt}{dt} u(t) \|_{V'_p} \leq C(1 + \| u(t) \|_{V_p}^{p-1} + \| f(t) \|_{V'_p})
\] (3.5)

where \( C \) possibly depends on the (essentially bounded) function \( \| u(t) \|_2 \).
Proof. Omitting the variable $t$ for simplicity, we take $\psi \in V_p$ with $\|\psi\| \leq 1$ in (3.3) and estimate

$$\left\langle \frac{d}{dt} u, \psi \right\rangle = - \langle N(u), \psi \rangle + \langle K_0(u), \psi \rangle + \langle f, \psi \rangle = D_1 + D_2 + D_3.$$ 

Clearly, it follows from (1.4) that $|D_1| + |D_3| \leq C(1 + \|u\|_{V_p}^{p-1} + \|f\|_{V'_p}).$ Using the interpolation (recall $\Omega \subset \mathbb{R}^3$)

$$\|v\|_{2p'} \leq \|v\|_{1-a}^{1-a} \|v\|_{1,p}^a \quad a = \frac{3}{5p-6}, \quad 1-a = \frac{5p-9}{5p-6},$$

which is valid all $p \geq 9/5$, we have

$$|D_2| \leq \int_{\Omega} |u|^2 |\nabla \psi| \, dx \leq \|u\|_{2p'}^2 \|\nabla \psi\|_p \leq \|u\|_{2(1-a)}^2 \|u\|_{V_p}^{2a} \|\psi\|_{V'_p} \leq C(1 + \|u\|_{V_p}^{p-1}).$$

Here we have used the Young inequality and the fact that $2a \leq p - 1$, which is just $p \geq 11/5$. □

Next, due to the monotonicity of $S$ and assumption on $p$, we have the following existence result.

**Lemma 3.3.** Let $p \geq 11/5$, $f \in L^p(0,T;V'_p)$ and $u_0 \in G$. Then there exists at least one weak solution within the class (3.2), (3.4), satisfying $u(0) = u_0$.

**Proof.** Let us outline the formal a priori estimates. Apply (3.3) to $u$. Thanks to (1.2) and the fact that $u$ vanishes on $\partial \Omega$, the convective term (the first term on the right hand side of (3.3)) disappears and we obtain the energy identity

$$(3.7) \quad \frac{1}{2} \frac{d}{dt} \|u\|_2^2 + \int_{\Omega} S(e(u)) : e(u) \, dx = \langle f, u \rangle.$$ 

In view of (1.3), Korn’s and Poincaré’s inequalities, we have

$$(3.8) \quad \int_{\Omega} S(e(u)) : e(u) \, dx \geq c \left( \|u\|_{1,2}^2 + \|u\|_{1,p}^p \right)$$

whence the estimate (3.2) follows easily. Secondly, by Lemma 3.2, one has (3.4), and $u$ is indeed an admissible test function. With a suitable approximating scheme, the above circle of reasoning can be turned into a rigorous existence theorem, employing the usual compactness and monotonicity argument to pass to the limit in nonlinear term $K_0(u)$. We omit further details, referring e.g. to [15, Chapter 5].

Note that it also follows that $u$ has a continuous representative in $C([0,T];G)$ and the initial condition $u(0) = u_0$ makes sense. □

The last classical result, we recall here (see e.g. [15]), is the “fundamental” difference inequality that can be further used for proving the weak-strong uniqueness result.
Lemma 3.4. Let \( u_1, u_2 \) be weak solutions corresponding to right hand side functions \( f_1 \) and \( f_2 \) respectively and let \( p \geq 11/5 \). Let us define

\[ p_{\text{uniq}} := \frac{2p}{2p - 3}. \]

Then

\[
\frac{d}{dt}\|u_1 - u_2\|^2 + c(\|u_1 - u_2\|_{V_2}^2 + \|u_1 - u_2\|_{V_p}^2) \\
\leq C\|u_2\|_{V_p}^{p_{\text{uniq}}}\|u_1 - u_2\|_{V_2}^2 + C\|f_1 - f_2\|_{V_p'}^2,
\]

where the constant \( C \) depends only on \( \Omega \) and \( p \).

**Proof.** Since \( p \geq 11/5 \), we can use \( u_1 - u_2 \) as a test function in (3.3) for \( u_1 \) and \( u_2 \), respectively and observe

\[
\frac{1}{2} \frac{d}{dt}\|u_1 - u_2\|^2 = \left( \frac{d}{dt}(u_1 - u_2), u_1 - u_2 \right) \\
= -\langle N(u_1) - N(u_2), u_1 - u_2 \rangle + \langle K_0(u_1) - K_0(u_2), u_1 - u_2 \rangle + \langle f_1 - f_2, u_1 - u_2 \rangle \\
=: D_1 + D_2 + D_3.
\]

By the \( p \)-ellipticity of \( N(u) \), i.e., the assumption (1.3), we have

\[
D_1 \leq -c(\|u_1 - u_2\|_{V_2}^2 + \|u_1 - u_2\|_{V_p}^2).
\]

Using the interpolation (valid for \( p \geq 3/2 \)

\[
\|v\|_{2p'} \leq C\|v\|_{2}^{\frac{2p-3}{2p'}}\|v\|_{1,2}^{\frac{1}{2p'}}
\]

integration by parts, (1.2) and the Poincaré inequality, we have

\[
D_2 = \int_{\Omega} (u_1 \otimes u_1 - u_2 \otimes u_2) \cdot \nabla (u_1 - u_2) \, dx = \int_{\Omega} (u_2 \otimes (u_1 - u_2)) \cdot \nabla (u_1 - u_2) \, dx \\
= -\int_{\Omega} (\nabla u_2 \cdot ((u_1 - u_2) \otimes (u_1 - u_2))) \, dx \leq \|u_2\|_{V_p} \|u_1 - u_2\|_{2p'}^2 \\
\leq \|u_2\|_{V_p} \|u_1 - u_2\|_{2p}^{\frac{2p-3}{2p'}} \|u_1 - u_2\|_{V_2}^2 \leq c\|u_1 - u_2\|_{V_2}^2 + C_c\|u_2\|_{V_p}^{p_{\text{uniq}}} \|u_1 - u_2\|_{2p}^2.
\]

The estimate of \( D_3 \) is straightforward. Summarizing these estimates with \( \epsilon > 0 \) small enough finishes the proof. \( \square \)

We see that if at least one weak solution belongs to \( L^{p_{\text{uniq}}}(0,T;V_p) \), one can apply the Gronwall inequality to (3.10) to conclude the continuous dependence on data and/or the uniqueness of solution. We also recall that for \( p \geq 5/2 \), one has \( p \geq p_{\text{uniq}} \), and consequently uniqueness holds true in the class of weak solutions.
3.3. Main result. Let us now formulate our main result.

**Theorem 3.1.** Let \( p \geq 11/5, f \in L^{p'}(0, T; V_p') \) and \( u_0 \in G \). In case that \( p < 5/2 \), assume moreover \( f \in N_\delta^{p'}(0, T; V_p') \) with some \( \delta > \delta_{\text{uniq}} \), where

\[
(3.11) \quad \delta_{\text{uniq}} := (p - 1) \left( \frac{5}{2p} - 1 \right).
\]

Then for an arbitrary weak solution \( u \) the following holds true:

1. For any \( t_0 \in (0, T) \) we have \( u \in L^{p_{\text{uniq}}}(t_0, T; V_p) \).
2. If \( u_0 \in V_p \), the conclusion holds for \( t_0 = 0 \) as well.

Recalling that \( p_{\text{uniq}} \) is the critical integrability condition that enables one to handle the equation for difference of two solutions (see Lemma 3.4 above) we also obtain:

**Corollary 3.1.** Let the conditions of Theorem 3.1 hold. Let \( u_1, u_2 \) be two weak solutions that coincide on some \([0, \tau], \tau > 0\). Then \( u_1 = u_2 \) on \([0, T] \). The same conclusion holds if \( u_1(0) = u_2(0) \in V_p \).

**Corollary 3.2.** Let \( p \geq 11/5 \) and \( f \in N^{1/p', p'}(0, T; V_p'), u_0 \in G, u \) be a weak solution, and \( t_0 \in (0, T) \) be arbitrary. Then

\[
u \in N^{\frac{1}{2} - \infty}(t_0, T; L^2) \cap N^{1/p, p}(t_0, T; V_p) \cap N^{1/2, 2}(t_0, T; V_2)
\]

The same conclusion holds for \( t_0 = 0 \) provided that \( u_0 \in V_p \) and \( \sup_{h \in (0, T)} \int_0^h \|f\|_{V_p'} < +\infty \).

4. Auxiliary estimates

In this section we summarize several auxiliary estimates. We establish the “initial time regularity” provided \( u_0 \) belongs to \( V_p \), see Lemma 4.1, which is a starting point of our iteration scheme. Finally, we show how the integrability of \( u \) can be improved by reverse Hölder inequality with increasing support. This allows us later to prove the main theorem also for \( p = 11/5 \).

**Definition 4.1.** Let \( f : (0, T) \to V_p' \). We call \( t \in [0, T] \) semi-Lebesgue point of \( f \) if \( \sup_{h \in (0, T-t)} \int_{t}^{t+h} \|f\|_{V_p'} \) is finite.

**Lemma 4.1.** Let \( u \) be the representative of a weak solution continuous with values in \( L^2(\Omega), p \geq 11/5 \) and \( f \in N_\delta^{p'}(0, T; V_p') \) with \( 1/p' \geq \delta > 2\tau/p' \geq 0 \). If \( t_0 \in [0, T) \) and \( u(t_0) \in V_p \) then

\[
(4.1) \quad \exists c > 0, \forall h \in (0, T - t_0) : \|u(t_0 + h) - u(t_0)\|_2^2 \leq ch^{2\tau}.
\]

If in addition \( t_0 \) is a semi-Lebesgue point of \( f \) then

\[
(4.2) \quad \exists c > 0, \forall h \in (0, T - t_0) : \|u(t_0 + h) - u(t_0)\|_2^2 \leq ch.
\]
Proof. Recall that $u$ stands for the representative continuous in $[0, T]$ with values in $L^2(\Omega)$. We can write
\[
\|u(t_0 + h) - u(t_0)\|^2_2 = (u(t_0 + h) - u(t_0), u(t_0 + h) - u(t_0))
\]
\[= (u(t_0 + h) - u(t_0), (u(t_0 + h) + u(t_0)) - 2u(t_0))
\]
\[= \|u(t_0 + h)\|^2_2 - 2\|u(t_0)\|^2_2 - 2(u(t_0 + h) - u(t_0), u(t_0))
\]
\[= I_1 + I_2.
\]
We start estimating $I_1$. Using the energy equality (3.7) and (3.8) we get
\[
I_1 = \int_{t_0}^{t_0 + h} (-\langle N(u(s)), u(s) \rangle + \langle h(s), u(s) \rangle) \, ds
\]
\[\leq \int_{t_0}^{t_0 + h} (-c(\|u(s)\|^2_{L^2} + \|u(s)\|^p_{V_p^*}) + C\|f(s)\|^p_{V_p^*}) \, ds.
\]
The term $I_2$ can be rewritten as time derivative of $u(t)$, i.e., we have
\[
I_2 = -2\int_{t_0}^{t_0 + h} \left\langle \frac{d}{dt} u(s), u(t_0) \right\rangle \, ds.
\]
Then we use (3.5) and finally Young’s inequality with $\epsilon > 0$ to obtain
\[
I_2 \leq C \int_{t_0}^{t_0 + h} (1 + \|u(s)\|^p_{V_p^*} + \|f(s)\|^p_{V_p^*}) \|u(t_0)\|_{V_p} \, ds
\]
\[\leq \int_{t_0}^{t_0 + h} \epsilon \|u(s)\|^p_{V_p^*} + C(1 + \|u(t_0)\|^p_{V_p} + \|f(s)\|^p_{V_p^*}) \, ds.
\]
If we combine the estimates of $I_1$ and $I_2$ and choose $\epsilon > 0$ sufficiently small we get
\[
\|u(t_0 + h) - u(t_0)\|^2_2
\]
\[\leq \int_{t_0}^{t_0 + h} (-\frac{c}{2}(\|u(s)\|^2_{L^2} + \|u(s)\|^p_{V_p^*}) \, ds + Ch(\int_{t_0}^{t_0 + h} \|f(s)\|^p_{V_p^*} + 1 + \|u(t_0)\|^p_{V_p}) \, ds
\]
\[\leq Ch \int_{t_0}^{t_0 + h} \|f(s)\|^p_{V_p^*} + 1 + \|u(t_0)\|^p_{V_p} \, ds = Ch(1 + \|u(t_0)\|^p_{V_p}) + C \int_{t_0}^{t_0 + h} \|f(s)\|^p_{V_p^*} \, ds.
\]
Consequently, if $t_0$ is a semi-Lebesgue point of $f : (0, T) \to V_p^*$ and $u(t_0) \in V_p$ we get (4.2).
Similarly, if $f \in N^{k,p'}(0, T; V_{p'}^*)$ with $1/p' > \delta > 2\tau/p'$ then by embedding theorem, we get that $f \in L^{p/(1-2\tau)}(0, T; V_{p'}^*)$. Thus, by Hölder’s inequality, we see that
\[
\int_{t_0}^{t_0 + h} \|f(s)\|^p_{V_p^*} \, ds \leq \left( \int_{t_0}^{t_0 + h} \|f(s)\|^\frac{p'}{V_p^*} \, ds \right)^{1-2\tau} h^{2\tau}.
\]
Hence, altogether we have
\[
\|u(t_0 + h) - u(t_0)\|^2_2 \leq C(h + h^{2\tau})
\]
and (4.1) follows. \qed
Remark 4.1. Since a weak solution has a representative continuous with values in $L^2(\Omega)$ that satisfies $u(0) = u_0$, the statement (4.2) holds provided $u_0 \in V_p$ and 0 is semi-Lebesgue point of $f$.

Remark 4.2. Note that it follows from (3.2), (3.4) and Lemma 3.1 (with $\alpha = 1/p$, $\beta = 1 + 1/p'$) that $u \in N^{1/2,2}(0,T;G)$, i.e.,

$$
\int_{t_0}^{T-h} \|d^h u\|_{2}^2 \leq C h ;
$$

in (4.2) we get this estimate, so to say, pointwise.

Lemma 4.2. Let $p \geq 11/5$, let $f \in L^{q_0}(0,T;V_{p}'_q)$ for some $q_0 > p'$. Then there is $q > p$ such that $u \in L^{q_0}_{loc}(0,T;V_p)$. Moreover, if $u_0 \in V_p$, the conclusion holds globally.

Proof. First we concentrate on the local regularity result. We show that there exist $C > 0$ such that for any $t_0 \in (0,T)$ and $h \in (0,t_0)$

$$
\left( \int_{t_0-h/2}^{t_0} \|u(t)\|_{V_p}^p \, dt \right)^{1/p} \leq C \left( 1 + \left( \int_{t_0-h}^{t_0} \|u(t)\|^{-1}_{V_p} \, dt \right)^{1/p-1} + \left( \int_{t_0-h}^{t_0} \|f(t)\|_{V_p}^{p'} \, dt \right)^{1/p} \right) .
$$

The conclusion then follows by an application of a variant of Gehring’s lemma with increasing support, see e.g. [9, Proposition V.1.1]. Here, the support grows only on one side of the interval $(t_0 - h/2,t_0)$, yet the situation can easily be accommodated according to [11, Proposition 1.3].

We fix $U \in V_p$ and test the equation (3.3) by $u(t) - U$ to obtain (using $\langle K_0(u(t)), u(t) \rangle = 0$)

$$
\frac{1}{2} \frac{d}{dt} \|u(t) - U\|_2^2 + \langle N(u(t)), u(t) \rangle = \langle K_0(u(t)) + N(u(t)), U \rangle + \langle f(t), u(t) - U \rangle .
$$

In the standard way we estimate (using the assumptions (1.3)-(1.4))

$$
\frac{d}{dt} \|u(t) - U\|_2^2 + \alpha \|u(t)\|_p^p \leq C \left( 1 + \|u(t)\|_{2p'}^2 \|U\|_{V_p} + \|u(t)\|_{V_p}^{-1} \|U\|_{V_p} \right. \\
\left. + \|f(t)\|_{V_p} (\|u(t)\|_{V_p} + \|U\|_{V_p}) \right) .
$$

Here and in what follows, $C > 0$ and $\alpha > 0$ are generic constants that may change from line to line and depend only on the data of the equation.

Invoking now the interpolation (3.6), a priori estimate (3.2) and Young’s inequality, we proceed to

$$
\begin{align*}
\|u(t)\|_{2p'}^2 \|U\|_{V_p} &\leq c \|u(t)\|_{2}^{2(5p-9)} \|u(t)\|_{V_p}^{6-6} \|U\|_{V_p} \leq C \|u(t)\|_{V_p}^{6-5} \|U\|_{V_p} \\
&\leq \epsilon \|u(t)\|_{V_p}^p + C_\epsilon (\|U\|_{V_p}^{p} + 1)
\end{align*}
$$
as by $p \geq 11/5$ we just have $6/[p(5p - 6)] + 1/p \leq 1$. Thus, we arrive at

$$
\frac{d}{dt} \|u(t) - U\|_2^2 + \alpha \|u(t)\|_{V_p}^p \leq C \left( 1 + \|U\|_{V_p}^p + \|f(t)\|_{V_p'}^p \right),
$$

which is the basis for a further investigation.

If $t_0 \in (0, T)$ and $h \in (0, t_0)$ we set

$$
U = \overline{u}_h := \int_{t_0-h}^{t_0} u(t) \, dt.
$$

We multiply (4.4) by $\xi(t) = (t - (t_0 - h))/h^2$, integrate over $t \in (t_0 - h, t_0)$ and after a simple manipulation and using the fact that $\xi(t_0 - h) = 0$, we obtain

$$
\int_{t_0-h/2}^{t_0} \|u(t)\|_{V_p}^p \, dt \leq C \left( 1 + \|\overline{u}_h\|_{V_p}^p + \int_{t_0-h}^{t_0} \frac{\|u(t) - \overline{u}_h\|_2^2}{h} + \|f(t)\|_{V_p'} \, dt \right).
$$

Observing that

$$
\|\overline{u}_h\|_{V_p}^p \leq \left( \int_{t_0-h}^{t_0} \|u(t)\|_{V_p} \, dt \right)^p
$$

it only remains to treat the second term on the right hand side of (4.5) to obtain (4.3). Towards this end, note first that the identity

$$
u(t) - \overline{u}_h = \frac{1}{h} \int_{t_0-h}^{t_0} \left( u(t) - u(s) \right) \, ds = \frac{1}{h} \int_{t_0-h}^{t_0} \int_s^t \frac{d}{d\tau} u(\tau) \, d\tau \, ds
$$

holds in $V_p'$. On the other hand $u(t) - \overline{u}_h \in V_p$ for almost all $t$ and therefore

$$
\|u(t) - \overline{u}_h\|_2^2 = \langle u(t) - \overline{u}_h, u(t) - \overline{u}_h \rangle.
$$

Consequently, we have

$$
\int_{t_0-h}^{t_0} \frac{\|u(t) - \overline{u}_h\|_2^2}{h} \, dt = \frac{1}{h^3} \int_{t_0-h}^{t_0} \int_{t_0-h}^{t_0} \int_s^t \left( \frac{d}{d\tau} u(\tau), u(t) - \overline{u}_h \right) \, d\tau \, ds \, dt.
$$
Invoking now this equation together with Lemma 3.2, we estimate the term on the right hand side further as
\[
\int_{t_0-h}^{t_0} \frac{\|u(t) - \overline{u}_h\|_2^2}{h} \, dt \leq \frac{c}{h^2} \int_{t_0-h}^{t_0} \int_{t_0-h}^{t_0} (1 + \|u(\tau)\|_{V_p}^{p-1}) (\|u(t)\|_{V_p} + \|\overline{\mathbf{u}}_h\|_{V_p}) \, d\tau \, dt
\]
\[
= c \left( 1 + \int_{t_0-h}^{t_0} \|u(\tau)\|_{V_p}^{p-1} \, d\tau \right) \int_{t_0-h}^{t_0} (\|u(t)\|_{V_p} + \|\overline{\mathbf{u}}_h\|_{V_p}) \, dt
\]
\[
\leq C \left( 1 + \left( \int_{t_0-h}^{t_0} \|u(\tau)\|_{V_p}^{p-1} \, d\tau \right) + \left( \int_{t_0-h}^{t_0} \|u(t)\|_{V_p} \, dt \right) \right),
\]
where we used the Young inequality for the last estimate. We see that (4.3) holds for \( t_0 \in (0, T) \) and \( h \in (0, t_0) \) and the local regularity result follows by Gehring’s lemma.

To prove the global improvement of regularity we extend \( u \) by 0 to \( t < 0 \) and show that if \( u_0 \in V_p \), the inequality (4.3) holds for any \( t_0 < T \) and \( h > 0 \). The situation \( t_0 \in (0, T) \), \( h \in (0, t_0) \) was treated in the previous part of the proof. Now we consider \( t_0 \in (0, T) \), \( h > t_0 \). We set \( U = u_0 \) in (4.4) to get for \( t \in (0, T) \)
\[
\frac{d}{dt} \|u(t) - u_0\|_2^2 + \alpha \|u(t)\|_{V_p}^p \leq C(1 + \|u_0\|_{V_p}^p + \|f(t)\|_{V_p}^p) \leq C(1 + \|f(t)\|_{V_p}^p).
\]
Integrating this inequality from 0 to \( t_0 \) we get
\[
\int_0^{t_0} \|u(t)\|_{V_p}^p \, dt \leq C(t_0 + \int_0^{t_0} \|f(t)\|_{V_p}^p \, dt).
\]
Further we compute
\[
\int_{t_0-h/2}^{t_0} \|u(t)\|_{V_p}^p \, dt \leq \frac{C}{h} \int_0^{t_0} \|u(t)\|_{V_p}^p \, dt \leq \frac{C}{h} (t_0 + \int_0^{t_0} \|f(t)\|_{V_p}^p \, dt)
\]
\[
\leq C(1 + \int_{t_0-h}^{t_0} \|f(t)\|_{V_p}^p \, dt).
\]
Since estimate (4.3) clearly holds also if \( t_0 < 0 \) we finally get that under the assumption \( u_0 \in V_p \) the inequality (4.3) holds for any \( t_0 < T, h > 0 \). Consequently, we get the global improvement of regularity of \( u \) by Gehring’s lemma.

5. Proof of the main theorem

This section is devoted to the proof of what we formulate as the main result: Theorem 3.1.

It seems convenient to split the idea into two auxiliary lemmas.

In Lemma 5.1, we show that the Ladyzhenskaya fluid – without the convective term – reflects the time regularity of the right-hand side in the class of Nikolskii spaces, provided the initial time regularity condition (5.2) holds. This can be seen as a generalization of a
well-known fact that the \( L^\infty(0,T; G) \cap L^p(0,T; V_p) \) norm of the solution is estimated by the \( L^{p'}(0,T; V'_p) \) norm of the right-hand side and the \( L^2 \) norm of initial condition \( u_0 \).

Lemma 5.2 then focuses on the convective term \( K_0(u) \). It shows that if \( u \in N^{r,\infty}(t_0, T; G) \cap N^{\sigma,p}(t_0, T; V_p) \), then \( K_0(u) \in N^{\delta,p'}(t_0, T; V'_p) \) for suitable \( \delta > \sigma \) depending on \( \tau \) and \( \sigma \), provided the initial time regularity at \( t = t_0 \) is satisfied, cf. (5.2). This generalizes another well-known fact, namely that \( K_0(\cdot) \) is bounded from \( L^\infty(0,T; G) \cap L^p(0,T; V_p) \) into its dual if \( p \geq 11/5 \).

**Lemma 5.1.** Let \( t_0 \in [0,T) \), \( \delta \in (0,1) \) and let \( u \in L^p(t_0, T; V_p) \) satisfy

\[
\frac{d}{dt} u + N(u) = H \quad \text{in } V'_p
\]

almost everywhere in \((t_0,T)\), where \( H \in N^{\delta,p'}(t_0, T; V'_p) \). Let us define

\[
\tau = \frac{\delta p}{2(p-1)} \quad \sigma = \frac{\delta}{p-1}
\]

and assume that \( h_0 \in (0, T - t_0) \) satisfies

\[
\|u(t_0 + h) - u(t_0)\|_2 \leq c_1 h^{2\tau} \quad \text{for } h \in (0,h_0).
\]

Then \( u \in N^{r,\infty}(t_0, T; G) \cap N^{\sigma,p}(t_0, T; V_p) \).

**Remark 5.1.** Later we will always assume that \( \sigma < 1/2 \) (and \( \tau < 1/p \)).

**Proof.** Applying \( d^h \) to (5.1) and testing the result by \( d^h u \), one obtains

\[
\frac{1}{2} \frac{d}{dt} \|d^h u\|_2^2 + \langle d^h N(u), d^h u \rangle = \langle d^h H(t), d^h u \rangle.
\]

Here

\[
\langle d^h N(u), d^h u \rangle \geq c \left( \|d^h u\|_{V'_p}^p + \|d^h u\|_{V'_2}^2 \right)
\]

in view of the \( p \)-ellipticity of \( N \), i.e., (1.3). Further, with the help of the Young inequality, we deduce that

\[
\langle d^h H(t), d^h u \rangle \leq \|d^h H(t)\|_{V'_p} \|d^h u\|_{V_p} \leq \frac{C}{2} \|d^h u\|_{V'_p}^p + C \|d^h H(t)\|_{V'_p}^{p'}
\]

and finally

\[
\sup_{t_0 \leq t \leq T-h} \|d^h u(t)\|_2^2 + c \int_{t_0}^{T-h} \left( \|d^h u\|_{V'_p}^p + \|d^h u\|_{V'_2}^2 \right) dt \leq c_1 h^{2\tau} + C \int_{t_0}^{T-h} \|d^h H(t)\|_{V'_p}^{p'} dt.
\]

The last term is estimated by \( ch^{\delta p'} = Ch^{2\tau} \) and the conclusion follows. \( \square \)

Since the embedding theorem for Nikolskii spaces is not sharp (cf. (3.1)), we will repeatedly write \( a + \epsilon \) or \( a - \epsilon \) for some number strictly larger or smaller than \( a \), respectively; the value \( \epsilon > 0 \) will be arbitrarily small and its values can change from line to line. Hence we have \( N^{s,p}(0,T) \subset L^{q \cdot \epsilon}(0,T) \), where \( 1/q = 1/p - s \) whenever \( s < 1/p \).
Lemma 5.2. Let \( p \geq 11/5 \) and \( u \in N^{\frac{1}{2} \cdot 2}(t_0, T; G) \cap N^r, \infty(t_0, T; G) \cap N^{\sigma, p}(t_0, T; V_p) \) with some \( \tau \in [0, 1/2] \) and \( \sigma \in [0, 1/p] \). Then \( K_0(u) \in N^{\delta, p'}(t_0, T; V_p') \), where

\[
\delta = \begin{cases} 
\frac{5p - 11}{5p - 6} + \frac{6\sigma}{5p - 6} + \frac{\tau(-5p + 13 - 6\sigma)}{5p - 6} & \text{if } 5p - 13 + 6\sigma < 0, \\
\frac{5p - 9}{2(5p - 6)} + \frac{3\sigma}{5p - 6} & \text{if } 5p - 13 + 6\sigma \geq 0.
\end{cases}
\]

If \( \tau = \sigma = 0 \), then \( K_0(u) \in N^{\delta, p'}(t_0, T; V_p') \) with \( \delta = \frac{5p - 11}{5p - 6} \) precisely.

Proof. Let \( \psi \in L^p(t_0, T; V_p) \) with \( \|\psi\| \leq 1 \), \( h \in (0, T - t_0) \). We set \( T_h = T - h \) and estimate

\[
\left| \int_{t_0}^{T_h} \langle d^h K_0(u), \psi \rangle \, dt \right| \leq \int_{t_0}^{T_h} \int_{\Omega} |d^h(u \otimes u)||\nabla \psi| \, dx \, dt \leq 2 \int_{t_0}^{T_h} \|d^h u\|_{L^p(V_p)} \|u\|_{L^p(V_p')} \|\nabla \psi\|_p \, dt.
\]

We use the interpolation (3.6) (and keep value for \( a \) from this) to further estimate

\[
\leq c \int_{t_0}^{T_h} \|d^h u\|_{2-a}^{1-a}\|d^h u\|_{V_p}^a \|u\|_{2-a}^{1-a}\|u\|_{V_p}^a \|\psi\|_{V_p} \, dt
\]

(5.4)

\[
\leq C \left( \int_{t_0}^{T_h} \|d^h u\|_{2-a}^{\tilde{P}(1-a)} \, dt \right)^{\frac{1}{\tilde{P}}} \left( \int_{t_0}^{T} \|d^h u\|_{V_p}^p \, dt \right)^{\frac{1}{p}} \|u\|_{L^p(0,T;G)}^a \left( \int_{t_0}^{T} \|u\|_{V_p}^p \, dt \right)^{\frac{1}{p}},
\]

where we used Hölder’s inequality with the exponents \( \tilde{P}, p/a, \infty, p_{\sigma}/a \) and \( p \). Here, \( p_{\sigma} \) is such that \( N^{\sigma, p} \subset L^{p_{\sigma}} \), i.e., it is given by

\[
\frac{1}{p_{\sigma}} = \frac{1}{p} - \sigma + \epsilon
\]

with an arbitrary small \( \epsilon > 0 \) for \( \sigma > 0 \) and \( \epsilon = 0 \) if \( \sigma = 0 \). The number \( \tilde{P} \) is computed from the Hölder’s condition, hence

\[
\frac{1}{\tilde{P}} = 1 - \frac{1}{p} - \frac{a}{p_{\sigma}} - \frac{a}{p}.
\]

Inserting the value of \( a \) from (3.6) we get

\[
\frac{1}{\tilde{P}} = \frac{5p - 9}{5p - 6} + (\sigma - \epsilon)a.
\]

Hence, using the assumptions on \( u \), we can continue in estimating of (5.4) as

\[
\leq C \left( \int_{t_0}^{T_h} \|d^h u\|_{2-a}^{\tilde{P}(1-a)} \, dt \right)^{\frac{1}{\tilde{P}}} \left( \int_{t_0}^{T} \|d^h u\|_{V_p}^p \, dt \right)^{\frac{1}{p}}.
\]

(5.5)

Finally, we distinguish two cases. If

\[
\tilde{P}(1-a) = \frac{5p - 9}{5p - 11 + 3(\sigma - \epsilon)} \leq 2 \iff -5p + 13 \leq 6 \iff \frac{5p - 13}{6} \leq \sigma - \epsilon
\]
we use the Hölder inequality on the first term to obtain
\[
\leq C \left( \int_{t_0}^{T_h} \|d^h u\|_2^2 dt \right)^{\frac{(1-a)}{2}} \left( \int_{t_0}^{T} \|d^h u\|_{V_p}^p dt \right)^{\frac{2}{p}} \leq ch^{\frac{(1-a)}{2} + a\sigma},
\]
which, after using the definition of \(a\), leads to the second part of (5.3).

In case that \(\tilde{P}(1-a) > 2\), we interpolate the first term in (5.5) into \(L^2(0,T)\) and \(L^\infty(0,T)\), which gives
\[
\leq C \left( \int_{t_0}^{T_h} \|d^h u\|_2^2 dt \right)^{\frac{1}{\tilde{P}}} \left( \int_{t_0}^{T_h} \|d^h u\|_{V_p}^p dt \right)^{\frac{2}{p}} \|d^h u\|_{L^\infty(t_0,T-h;G)} \leq ch^{\frac{1}{\tilde{P}} + a\sigma + b\tau},
\]
where for the second inequality we used the assumption on \(u\) and defined
\[
b := 1 - a - \frac{2}{\tilde{P}} = \frac{-5p + 13 + 6(\sigma - \epsilon)}{5p - 6}.
\]
This then clearly gives the first part of (5.3).

\[\square\]

\textbf{Proof of Theorem 3.1.} In case \(p \geq 5/2\), then \(p_{\text{uniq}} \leq p\) and there is nothing to prove. Hence, we will consider only the case \(11/5 \leq p < 5/2\) and prove that under the assumptions of Theorem 3.1 we have \(u \in L^{p_{\text{uniq}}}(t_0, T; V_p)\). By the embedding properties of Nikolskii spaces, it is enough to show that \(u \in N^{\sigma_{\text{uniq}} + \epsilon, p}(t_0, T; V_p)\), where
\[
\sigma_{\text{uniq}} = \frac{5}{2p} - 1.
\]
To show this property, we use Lemma 5.1. Hence, we need to check that (note that \(\delta_{\text{uniq}} = \sigma_{\text{uniq}}(p - 1)\) is defined in (3.11))
\[
K_0(u) + f \in N^{\delta_{\text{uniq}} + \epsilon, p'}(t_0, T; V'_p)
\]
and that (5.2) holds true with
\[
\tau > \frac{p\sigma_{\text{uniq}}}{2} = \frac{p'\delta_{\text{uniq}}}{2}.
\]
Note that it is the assumption of Theorem 3.1 that \(f \in N^{\delta_{\text{uniq}} + \epsilon, p'}(t_0, T; V'_p)\), so the second part of (5.6). Using the same assumption and combining it with Lemma 4.1, we also obtain the validity of (5.2) with (5.7). Thus, we just need to check the first part of (5.6), i.e., the regularity of the convective term \(K_0(u)\).

For this purpose, we use iteratively Lemmata 5.1 and 5.2. Notice that since we always will have that \(\sigma\) appearing in Lemma 5.1 fulfills \(\sigma \leq \sigma_{\text{uniq}}\) then
\[
5p - 13 + 6\sigma \leq 5p - 19 + 15/p < 0
\]
for all \(p \in [11/5, 5/2]\). Hence we shall always use the first line in (5.3). We distinguish two cases.
1. In case of \( p \in (11/5, 5/2) \) we will use an iterative scheme. If \( u \in N^{\sigma,p}(t_0,c_1 T V_p) \), we improve regularity of the convective term by Lemma 5.2 and then use Lemma 5.1 to improve the regularity of \( u \). More specifically, we obtain \( u \in N^{\tilde{\sigma},p'}(t_0,c_1 T V_p) \), where
\[
\tilde{\sigma} = \alpha + \beta \sigma, \quad \alpha = \frac{5p - 11}{(p - 1)(5p - 6)}, \quad \beta = \frac{6}{(p - 1)(5p - 6)}.
\]
Note that if \( \sigma = 0 \) we use the precise regularity of the convective term \( K_0(u) \) from Lemma 5.2. If \( \sigma > 0 \) we can assume that also \( \tau > 0 \) and we did take the last term in (5.3), namely the term with \( \tau \), into account just to avoid the presence of \( \epsilon \) in Lemma 5.2. Inequality \( p > 11/5 \) implies that \( \alpha > 0 \) and \( \beta \in (0,1) \). Hence, the mapping \( \sigma \to \tilde{\sigma} \) is a contraction on \([0,1] \). Banach contraction principle shows that starting from \( \sigma = 0 \), we can arrive arbitrarily close to the fixed point \( \sigma_{\text{max}} = \alpha/(1 - \beta) = 1/p \). But obviously \( \sigma_{\text{max}} > \sigma_{\text{uniq}} \), so the proof is concluded after finitely many steps.

2. It remains to treat the critical case \( p = 11/5 \). Observe that now one has \( \alpha = 0 \) and \( \beta = 1 \) in (5.8), so the previous iteration scheme no longer works. We modify the argument as follows: by Lemma 4.2, the solution satisfies \( u \in L^q(t_0,c_1 T V_p) \) with some \( q > p \). Following now the argument of Lemma 5.2, we apply Hölder’s inequality to (5.4) with exponents \( \tilde{P}, q/a, \infty, q/a \) and \( p \). Since \( q > p \), one has \( \tilde{P} < \infty \), and it follows that the convective term \( K_0(u) \) belongs to \( N^{\tilde{\sigma}',p'}(t_0,c_1 T V_{p'}) \) with some small positive \( \delta \).

By Lemma 5.1, the solution belongs to \( N^{\tilde{\tau},\infty}(t_0,c_1 T G) \) with some \( \tilde{\tau} > 0 \). Keeping this \( \tilde{\tau} \), and combining now Lemmas 5.1 and 5.2, while taking the last term (5.3) into account, we obtain formula for improving \( \sigma \) in the form
\[
\tilde{\sigma} = \frac{\tau(1 - 3\sigma)}{3} + \sigma = \sigma(1 - \tau) + \frac{\tau}{3}.
\]
Again the mapping \( \sigma \to \tilde{\sigma} \) is a contraction on \([0,1] \) and by iterating the procedure we can get with \( \sigma \) arbitrarily close to its fixed point \( \tilde{\sigma} = 1/3 \). Since \( 1/3 > 3/22 = \sigma_{\text{uniq}} \) for \( p = 11/5 \) we reach the value \( \sigma_{\text{uniq}} \) after finitely many iterations.

We finish the proof by final comment about \( t_0 \). Since \( u \in L^p(0,T V_p) \) initially, we may chose an arbitrary Lebesgue point of \( u(t) \) as \( t_0 \), which then can be used in Lemma 4.1. Since almost every \( t_0 \) is the Lebesgue point of \( u(t) \), we finally get the conclusion of Theorem 3.1 for all \( t_0 > 0 \). In addition, if \( u_0 \in V_r \), we may set \( t_0 := 0 \) and we again get the result of Theorem 3.1.

**Proof of Corollary 3.1.** Using Lemma 3.4, we see that it is enough to prove that \( u \in L^{p_{\text{uniq}}}(t_0,c_1 T V_p) \) for some \( t_0 \in (0,\tau) \). By Theorem 3.1, both solutions have the regularity (3.9) with some suitable \( t_0 \in (0,\tau) \), and the assertion of Corollary 3.1 follows from Lemma 3.4 and Gronwall’s lemma.

In the second part of the corollary the assumptions are chosen such that we can set \( t_0 = 0 \) by Theorem 3.1.

**Proof of Corollary 3.2.** By Theorem 3.1 and Lemma 4.1 we find \( t_0 \in (0,\tau) \) such that \( u \in L^{p_{\text{uniq}}}(t_0,c_1 T V_p) \) and moreover that (4.2) holds true. We conclude the proof by Lemma 3.4 with \( u_1 = \tau^{\beta} u \) and \( u_2 = u \) and Gronwall’s lemma.
In the second part of the corollary the assumptions are chosen such that we can set $t_0 = 0$ by Theorem 3.1.

\[ \square \]

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