New complex variables for equations of ideal barotropic fluid

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Abstract

We propose new construction of dependent variables for equations of an ideal barotropic fluid. This construction is based on a direct generalization of the known connection between Schroedinger equation and a system of Euler-type equations. The system for two complex-valued functions is derived that is equivalent to Euler equations. Possible advantages of the proposed formulation are discussed.

1 Introduction

When solving a partial problem of fluid dynamics or exploring general properties of governing equations one often use different choice of the dependent variables. Introduction of a stream function is common practice for two-dimensional problems. For a general case of a 3D time-dependent flow one can use a vector potential, a pair of stream functions (for incompressible case), Clebsch potentials and etc. Clebsch potentials are mainly used with intention to exploit preferences of Lagrange description of a fluid motion. The new representation is based on the use of multi-valued potentials and Euler approach. The paper is composed as follows. In the second section we analyze Madelung transformation that connects a generic Schroedinger equation with a system of Euler-type equations. Some generalization will be made for the case of potential flows of a barotropic fluid. In the next section the generalization of Madelung transformation for a general vector field
will be derived, that leads to the system of equations (8) for two complex-valued functions with arbitrary potentials. In the fourth section we use this arbitrariness and propose the choice of potentials, that make the system equivalent to Euler equations for an ideal barotropic fluid. To substantiate this we will derive Euler equations from the system (8). In the last section we discuss possible preferences of new choice of dependent variables and their relation to vortices.

2 Madelung transformation

Since pioneer work by E. Madelung \[1\] physical literature contains many examples of connection between Schrödinger equation of quantum mechanics and fluid dynamics. Typical exposition of this connection is the substitution

\[ \psi = \sqrt{\rho} e^{i\varphi} \]

that leads to

\[ \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \nabla \varphi) = 0 \] \hspace{1cm} (2)

\[ \frac{\partial \varphi}{\partial t} + \frac{(\nabla \varphi)^2}{2} = -V + \frac{\beta}{2} \frac{\Delta \sqrt{\rho}}{\sqrt{\rho}} \] \hspace{1cm} (3)

This trick looks slightly mystical for novice. Some historical notes and elucidation can be found in \[3\]. More clear is back substitution. Following Madelung \[2\], let’s linearize equation (2) using substitution

\[ \rho = \psi \overline{\psi}, \quad \varphi = -\frac{i\beta}{2} \ln \left( \frac{\psi}{\overline{\psi}} \right) \] \hspace{1cm} (4)

were \( \beta \) has dimension of kinematical viscosity. After simple algebra one can obtain

\[ \overline{\psi} \left( \frac{\partial \psi}{\partial t} - \frac{i\beta}{2} \Delta \psi \right) + \psi \left( \frac{\partial \overline{\psi}}{\partial t} + \frac{i\beta}{2} \Delta \overline{\psi} \right) = 0 \]

Choice

\[ \frac{\partial \psi}{\partial t} - \frac{i\beta}{2} \Delta \psi = iV \psi \]

leads to Schrödinger equation. Here \( V \) is a real-valued function of a time, coordinates and/or \( \psi \). We can conclude that this equation leads to conservation of probability, but dynamics is completely defined by potential \( V \).
Now from hydrodynamical viewpoint let’s summarize restrictions that were implicitly used in this derivation. First, interpreting $\rho$ as density of some fluid with an arbitrary equation of state, we see that fluid flow is supposed to be potential. Second, we use dimensional constant $\beta$.

To describe an ideal fluid, we can to overcome the second restriction using a non-dimensional form of equation (2) ($\beta = 1$) and the potential

$$V = \Pi (\rho) + \frac{1}{2} \frac{\Delta \sqrt{\rho}}{\sqrt{\rho}}$$

This choice give Cauche-Lagrange equation for barotropic fluid

$$\frac{\partial \varphi}{\partial t} + \frac{(\nabla \varphi)^2}{2} = -\Pi$$

but leads to

$$i \frac{\partial \psi}{\partial t} = -\frac{1}{4} \left( \Delta \psi - \frac{\psi}{\psi} \Delta \psi \right) + \left[ -\frac{1}{8} \left( \nabla \ln \left( \frac{\psi}{\psi} \right) \right)^2 + \Pi \left( \psi \bar{\psi} \right) \right] \psi$$

that differs from Schrödinger equation. This form of equation of an ideal barotropic fluid seems to be unknown.

### 3 Generalization of Madelung transformation

We consider a direct generalization of the previous scheme for the case of two complex-valued functions and introduce definitions

$$\rho = \rho_1 + \rho_2, \quad J = \rho \mathbf{V} = \rho_1 \nabla \varphi_1 + \rho_2 \nabla \varphi_2$$

(5)

Obviously, permutation of indexes should not have any physical consequence. For velocity and vorticity we obtain

$$\mathbf{V} = \frac{\rho_1}{\rho} \nabla \varphi_1 + \frac{\rho_2}{\rho} \nabla \varphi_2, \quad \nabla \times \mathbf{V} = \frac{\rho_1 \rho_2}{\rho^2} \nabla \ln \left( \frac{\rho_1}{\rho_2} \right) \times \nabla (\varphi_1 - \varphi_2)$$

(6)

The requirement of possibility to represent a vector field with a non-zero total helicity

$$H = \int \frac{\rho_1 \rho_2}{\rho^2} \ln \left( \frac{\rho_1}{\rho_2} \right) (\nabla \varphi_1 \times \nabla \varphi_2) \cdot d\mathbf{S} \neq 0$$

3
implies a multi-valuedness of potentials \( \tilde{\Pi} \) (here integral should be taken over some closed surface). That is admissible due to usage of the complex-valued variables.

Linearizing

\[
\frac{\partial \rho}{\partial t} + \nabla \cdot J = 0
\]

(7)

after some algebra we obtain

\[
\overline{\psi}_1 \left( \frac{\partial \psi_1}{\partial t} - \frac{i}{2} \Delta \psi_1 \right) + \psi_1 \left( \frac{\partial \overline{\psi}_1}{\partial t} + \frac{i}{2} \Delta \overline{\psi}_1 \right)
\]

\[+
\overline{\psi}_2 \left( \frac{\partial \psi_2}{\partial t} - \frac{i}{2} \Delta \psi_2 \right) + \psi_2 \left( \frac{\partial \overline{\psi}_2}{\partial t} + \frac{i}{2} \Delta \overline{\psi}_2 \right) = 0
\]

By inspection one can show that choice

\[
\frac{\partial \psi_k}{\partial t} - \frac{i}{2} \Delta \psi_k = U_k \psi_k
\]

with

\[
U_1 = \frac{\rho_2}{2\rho} I - iV_1, \quad U_2 = -\frac{\rho_1}{2\rho} I - iV_2
\]

where \( I, V_1, V_2 \) are real-valued functions of time, coordinates and/or \( \psi_k \) solve this equation. We obtain the following system of equations

\[
i \frac{\partial \psi_1}{\partial t} = -\frac{\Delta \psi_1}{2} + \left( \frac{\rho_2}{2\rho} iI + V_1 \right) \psi_1, \quad i \frac{\partial \psi_2}{\partial t} = -\frac{\Delta \psi_2}{2} + \left( -\frac{\rho_1}{2\rho} iI + V_2 \right) \psi_2
\]

(8)

Substitutions \( \psi_k = \sqrt{\rho_k} \exp(i\varphi) \) give the equivalent system

\[
\frac{\partial \rho_k}{\partial t} + \nabla \cdot (\rho_k \nabla \varphi_k) = (\text{(-1)} \frac{k-1}{k} \frac{\rho_1 \rho_2}{\rho} I, \quad \frac{\partial \varphi_k}{\partial t} + \frac{(\nabla \varphi_k)^2}{2} = -V_k + \frac{1}{2} \frac{\Delta \sqrt{\rho_k}}{\sqrt{\rho_k}}
\]

(9)

Equation (7) follows from the first two equations of this system.

4 New form of Euler equations

To apply the derived system to description of an ideal barotropic flow we need a proper choice of the potentials \( I, V_1, V_2 \). By inspection it was found that

\[
V_1 = \Pi (\rho) - \frac{\rho_2^2}{2\rho^2} w^2 + \frac{1}{2} \frac{\Delta \sqrt{\rho_1}}{\sqrt{\rho_1}}
\]

(10)
\[ V_2 = \Pi (\rho) - \frac{\rho_1^2}{2\rho^2} w^2 + \frac{1}{2} \Delta \sqrt{\rho_2} \]

\[ I = \nabla \cdot w + \frac{w}{\rho} \cdot \left( \frac{\rho_2}{\rho_1} \nabla \rho_1 + \frac{\rho_1}{\rho_2} \nabla \rho_2 \right) \]

(11)

(12)

make system equivalent to Euler equations. Here \( w = \nabla (\varphi_1 - \varphi_2) \). The invariance of systems (8),(9) with respect to both Galilei group and indexes permutation can be directly checked.

Substitution of (10-12) into (9) give

\[ \frac{\partial \rho_1}{\partial t} + \nabla \cdot (\rho_1 \nabla \varphi_1) = \frac{\rho_1 \rho_2}{\rho} I, \quad \frac{\partial \rho_2}{\partial t} + \nabla \cdot (\rho_2 \nabla \varphi_2) = -\frac{\rho_1 \rho_2}{\rho} I, \]

(13)

(14)

(15)

From equations (13) follows (7).

Now we start derivation of equation for flux \( J \). First, multiplying (14),(15) by \( \rho_k \) respectively, summing and taking gradient of result, then adding to obtained equation (13), multiplied by \( \nabla \varphi_k \) respectively, one can obtain

\[ \frac{\partial J}{\partial t} + \nabla \left( \frac{J_1^2}{2\rho_1} + \frac{J_2^2}{2\rho_2} \right) + \left[ \nabla \rho_1 \frac{\partial \varphi_1}{\partial t} + \nabla \rho_2 \frac{\partial \varphi_2}{\partial t} \right] \]

\[ + \left( \frac{\nabla \cdot j_1}{\rho_1} + \frac{\nabla \cdot j_2}{\rho_2} - \frac{\rho_1 \rho_2}{\rho} I \right) w = -\nabla \left( \rho \Pi - \frac{\rho_1 \rho_2}{\rho} \frac{w^2}{2} \right) \]

where \( j_k = \rho_k \nabla \varphi_k \). Using identities

\[ \frac{J^2}{2\rho} = \frac{j_1^2}{2\rho_1} + \frac{j_2^2}{2\rho_2} - \frac{\rho_1 \rho_2}{\rho} \frac{w^2}{2} \]

\[ \frac{J \nabla \cdot J}{\rho} = \frac{j_1 \nabla \cdot j_1}{\rho_1} + \frac{j_2 \nabla \cdot j_2}{\rho_2} - \frac{\rho_1 \rho_2}{\rho} \left( \frac{\nabla \cdot j_1}{\rho_1} - \frac{\nabla \cdot j_2}{\rho_2} \right) w \]

after some algebra one can obtain

\[ \frac{\partial J}{\partial t} + \nabla \left( \frac{J^2}{2\rho} \right) - \frac{J^2 \nabla \rho}{2\rho} + \frac{J \nabla \cdot J}{\rho} \]
Algebraic transformations of terms in square braces with account for first identity and (14),(15) lead to equation

\[
\frac{\partial J}{\partial t} + \nabla \left( J \cdot \frac{\nabla \rho}{2\rho} \right) - J \cdot \nabla \rho + J \times \nabla \times V = -\rho \nabla \Pi
\]

Using definition of velocity and equations (14),(15) after direct algebra one can show that terms in square braces give Lamb vector

\[
V \times \nabla \times V = \frac{\rho_1 \rho_2}{\rho} \left[ \nabla \ln \left( \frac{\rho_1}{\rho_2} \right) \left( \frac{\partial \varphi_1}{\partial t} - \frac{\partial \varphi_2}{\partial t} \right) + \left( \frac{\nabla \cdot j_1}{\rho_1} - \frac{\nabla \cdot j_2}{\rho_2} - I \right) \right] w = -\rho \nabla \Pi
\]

We obtain the equation

\[
\frac{\partial J}{\partial t} + \nabla \left( J \cdot \frac{J}{2\rho} \right) - J \cdot \nabla \rho + J \times \nabla \times V + V \nabla \cdot J = -\rho \nabla \Pi \quad (16)
\]

To make last step in derivation one should use continuity equation to obtain from (16) Euler equation in Gromeka-Lamb form

\[
\frac{\partial \rho}{\partial t} + \nabla \cdot J = 0, \quad \frac{\partial V}{\partial t} + \nabla \left( \frac{V \cdot V}{2} \right) - V \times \nabla \times V = -\nabla \Pi \quad (17)
\]

The result is as follows: System (8) is equivalent to system of Euler equation (17).

5 Discussion

First of all, the attractive feature of (8) is the homogeneity both dependent variables and equations in contrast to the non-homogeneity of velocity/density and form of equations in (18). This property can be used both numerically and analytically. Homogeneity and elimination of the convective
derivative can substantially simplify numerical algorithm. As far as multi-valuedness is concerned, the possibility of use multi-valued potentials was clearly demonstrated in [3]. In analytical way the aforementioned property can simplify proof of existence and uniqueness theorems. Also application of geometrical methods to partial differential equations (8) is looking quite natural.

This formulation of Euler equation can have another interesting property. Zeroes of solution of nonlinear Schroedinger equation correspond to a vortex axes (topological defects) [3]. At a moment the condition $\psi = 0$ defines two surfaces, and their intersection defines a space curve (possibly, disconnected). Note similarity with definition of a vortex as zero of an analitical complex-valued function in two-dimensional hydrodynamic of ideal incompressible fluid. If the system (8) inherits this property from its prototype (1) the known problem of a vortex definition [3] can be solved in general case.

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