ON THE GLOBAL 2-HOLONOMY FOR A 2-CONNECTION ON A 2-BUNDLE

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Abstract. A crossed module constitutes a strict 2-groupoid $G$ and a $G$-valued 2-cocycle on a manifold defines a 2-bundle. A 2-connection on this 2-bundle is given by a Lie algebra $g$ valued 1-form $A$ and a Lie algebra $h$ valued 2-form $B$ over each coordinate chart together with 2-gauge transformations between them, which satisfy the compatibility condition. Locally, the path-ordered integral of $A$ gives us the local 1-holonomy, and the surface-ordered integral of $(A, B)$ gives us the local 2-holonomy. The transformation of local 2-holonomies from one coordinate chart to another is provided by the transition 2-arrow, which is constructed from a 2-gauge transformation. We can use the transition 2-arrows and the 2-arrows provided by the $G$-valued 2-cocycle to glue such local 2-holonomies together to get a global one, which is well defined.

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1. Introduction

Higher gauge theory is a generalization of gauge theory that describes the dynamics of higher dimensional extended objects. See e.g. [3] [4] [9] [17] for 2-gauge theory and [14] [18] [23] for 3-gauge theory. It involves higher algebraic structures and higher geometrical structures in...
mathematics; higher groups, higher bundles (gerbes) and higher connections, etc. (cf. e.g. [11][12][13][14][15] and references therein). An important physical quantity in 2-gauge theory is the Wilson surface [8][17]. This is a 2-dimensional generalization of Wilson loop or holonomy in differential geometry. We will discuss the global 2-holonomy for a 2-connection on a 2-bundle.

Let us recall definitions of 2-bundles and 2-connections. Suppose that \((G, H, \alpha, \triangleright)\) is a crossed module, where \(\alpha : H \rightarrow G\) is a homomorphism of Lie groups and \(\triangleright\) is a smooth left action of \(G\) on \(H\) by automorphisms. Similarly, \((g, h, \alpha, \triangleright)\) is a differential crossed module, where \(\alpha : \mathfrak{h} \rightarrow \mathfrak{g}\) is a homomorphism of Lie algebras and \(\triangleright\) is a smooth left action of \(\mathfrak{g}\) on \(\mathfrak{h}\) by automorphisms. A local 2-connection over an open set \(U\) is given by a \(g\)-valued 1-form \(A\) and a \(h\)-valued 2-form \(B\) over \(U\) such that

\[
dA + A \wedge A = \alpha(B).
\]

A 2-gauge-transformation from a local 2-connection \((A, B)\) to another one \((A', B')\) is given by a \(G\)-valued function \(g\) and a \(h\)-valued 1-form \(\varphi\) such that

\[
g \triangleright A' = -\alpha(\varphi) + A + dg \cdot g^{-1},
\]

\[
g \triangleright B' = B - d\varphi - A \triangleright \varphi + \varphi \wedge \varphi.
\]

Given a crossed module \((G, H, \alpha, \triangleright)\), there exists an associated strict 2-groupoid denoted by \(G\). A 2-bundle over a manifold \(M\) is given by a nonabelian \(G\)-valued 2-cocycle on \(M\). This is a collection of \((U_i, \rho_{ij}, f_{ijk})\), where \(\{U_i\}_{i \in I}\) is an open cover of the manifold \(M\), \(\rho_{ij} : U_i \cap U_j \rightarrow G\) and \(f_{ijk} : U_i \cap U_j \cap U_k \rightarrow H\) are smooth maps satisfying

\[
\alpha\left(f_{ijk}^{-1}\right) \rho_{ij} \rho_{jk} = \rho_{ik},
\]

and the 2-cocycle condition

\[
g_{ij} \triangleright f_{jkl} f_{jkl} = f_{ijk} f_{ikl}.
\]

A 2-connection on this 2-bundle over \(M\) is given by a collection of local 2-connections \((A_i, B_i)\) over each coordinate chart \(U_i\), together with a 2-gauge transformation \((\rho_{ij}, a_{ij})\) over each intersection \(U_i \cap U_j\) from the local 2-connection \((A_i, B_i)\) to another one \((A_j, B_j)\). They satisfy the following compatibility condition:

\[
a_{ij} + g_{ij} \triangleright a_{jk} = f_{ijk} a_{ik} f_{ijk}^{-1} + A_i \triangleright f_{ijk} f_{ijk}^{-1} + df_{ijk} f_{ijk}^{-1},
\]

over each triple intersection \(U_i \cap U_j \cap U_k\). Note that minus signs in (1.2) become plus if \(\varphi\) is replaced by \(-\varphi\). See also Remark 2.1 and 4.4 for this form of 2-gauge-transformations and the compatibility condition.

Given a \(g\)-valued 1-form \(A\) on an open set \(U\), the 1-holonomy \(F_A(\rho)\) along a Lipschitzian path \(\rho : [a, b] \rightarrow U\) is well defined. It is given by the path-ordered integral. More precisely, \(F_A(\rho)\) is the unique solution to the ODE

\[
\frac{d}{dt} F_A(\rho_{[a,t]}) = F_A(\rho_{[a,t]}) \rho^* A_t \left( \frac{\partial}{\partial t} \right)
\]

with the initial condition \(F_A(\rho_{[a,t]})|_{t=a} = 1_G\), where \(\rho_{[a,t]}\) is the restriction of the curve \(\rho\) to \([a, t]\) and \(\rho^* A_t\) is the value of the pull back \(g\)-valued 1-form \(\rho^* A\) at \(t \in [a, b]\). Moreover, we can
integrate the 2-connection \((A_i, B_i)\) along a surface \(\gamma : [0, 1]^2 \rightarrow U_i\) to get a 2-arrow in \(\mathcal{G}\), called the \textit{local 2-holonomy}. It is a surface-ordered integral. If we denote the boundary of \(\gamma\) as follows

\[
\begin{array}{c}
\gamma^u \\
\gamma^l \\
\gamma^r \\
\gamma^d
\end{array}
\]

(1.7)

the local 2-holonomy is a 2-arrow in \(\mathcal{G}\):

\[
\begin{array}{c}
F_{A_i}(\gamma^u) \\
F_{A_i}(\gamma^l) \\
F_{A_i}(\gamma^r) \\
F_{A_i}(\gamma^d)
\end{array}
\]

(1.8)

It was proved by Schreiber and Waldorf \cite{20} that there exists a bijection between 2-connections on the trivial 2-bundle and 2-functors (play the role of 2-holonomy):

\[
\{\text{smooth 2-functors } \mathcal{P}_2(M) \rightarrow \mathcal{G}\} \cong \{A \in \Lambda^1(M, g), B \in \Lambda^1(M, h); dA + A \wedge A = \alpha(B)\},
\]

where \(\mathcal{P}_2(M)\) is path 2-groupoid of manifold \(M\). The local 1- and 2-holonomies are well defined.

See also Martins-Picken \cite{12} for the theory of local 1- and 2-holonomies. The problem is how to define the global 2-holonomy for a 2-connection on a nontrivial 2-bundle. This is known for Abelian 2-bundles by Mackaay-Picken \cite{11}. Schreiber and Waldorf \cite{21} proved the equivalence of several 2-categories associated a 2-connection to show the existence of the transport functor, which plays the role of global 2-holonomy. Parzygnat \cite{16} studied its generalization, explicit computations and application to magnetic monopoles. On the other hand, Martins and Picken \cite{13} introduced the notion of parallel transport by using the language of double groupoids. They also give the method of glueing local 2-holonomies to get a global one for the cubical version. This is a cubical description, rather than a simplicial description (see also Soncini-Zucchini \cite{22} for this approach). A cubical 2-bundle does not seem to be a direct generalization of the ordinary case of (principal) bundles and connections. Recently Arias Abad and Schätz \cite{2} compared these two approaches locally. In this paper we will give an elementary approach to this problem, including an algorithm to calculate the global 2-holonomy.

As a model, let us consider first how to glue local 1-holonomies to get a global one. Recall that a 1-\textit{connection} on \(M\) is given by a collection of local 1-forms \(A_i\) over coordinate charts \(U_i\), together with transition functions \(g_{ij}\) on \(U_i \cap U_j\), which satisfy the 1-cocycle condition. They satisfy the following \textit{compatibility condition}:

\[
A_i = g_{ij}^{-1} A_j g_{ij} + g_{ij}^{-1} dg_{ij}
\]
over $U_i \cap U_j$. Let $\rho : [0,1] \to M$ be a loop, i.e., $\rho(0) = \rho(1)$. We divide the interval $[0,1]$ into several subintervals $I_i := [t_i, t_{i+1}]$, $i = 1, \ldots, N$, such that the image $\rho(I_i)$ is contained in a coordinate chart denoted by $U_i$. We have local 1-holonomies $F_{A_i}(\rho_{I_i})$. We glue $F_{A_{i-1}}(\rho_{I_{i-1}})$ with $F_{A_i}(\rho_{I_i})$ by the gauge transformation $g_{(i-1)}(x)$ at point $x = \rho(t_i)$ to get the following path:

\begin{equation}
F_{A_i}(\rho_{I_i})
\end{equation}

The composition of elements of $G$ along this path is the global 1-holonomy of the connection along the loop $\rho$. Its conjugacy class is independent of the choice of the open sets $U_i$ containing the paths $\rho(I_i)$. This is because that if we use $U'_\varphi$ and $A'_\varphi$ instead of $U_i$ and $A_i$, respectively, we have the following commutative diagram

\begin{equation}
F_{A_{i-1}}(\rho_{I_{i-1}})
\end{equation}

where $x = \rho(t_i)$, $y = \rho(t_{i+1})$. Here the 1-cocycle condition implies the commutativity of two triangles, and $g_{\varphi_i}$ as a gauge transformation provides the commutative quadrilateral. The wavy path is what we obtain when $U_i$ and $A_i$ are replaced by $U'_{\varphi}$ and $A'_{\varphi}$, respectively. So the 1-arrows represented by the wavy and dotted paths coincide. When $U_i = U_N$ is replaced by $U'_{\varphi}$, we get the conjugacy of the global 1-holonomy by the element $g_{\varphi'_{1}}(\rho(0))$. 

To construct the the global 2-holonomy, we consider a surface given by the union of the mapping $\gamma$ in (1.7) and a mapping $\tilde{\gamma} : [0, 1]^2 \rightarrow U_j$, such that the left path $\tilde{\gamma}^l$ above coincides with the right path $\gamma^r$ in (1.7). Then we also have the following 2-arrow

$$F_{A_j}(\tilde{\gamma}^l) \rightarrow F_{A_j}(\gamma^r)$$

(1.11)

in $G$ by the surface-ordered integration of the 2-connection $(A_j, B_j)$ over $U_j$. The path $\tilde{\gamma}^l$ coincides with $\gamma^r$, but the local 1-holonomy $F_{A_j}(\tilde{\gamma}^l)$ in (1.11) is usually different from $F_{A_i}(\gamma^r)$ in (1.8). So we can not glue the 2-arrows in (1.8) and (1.11) directly. But we can integrate the 2-gauge transformation $(g_{ij}, a_{ij})$ along the path $\rho = \gamma^r = \tilde{\gamma}^l$ in $U_i \cap U_j$ to get the 2-arrow

$$\psi_{ij}(\rho)$$

(1.12)

in the 2-groupoid $G$. We call this 2-arrow (1.12) the transition 2-arrow along the path $\rho$. It can be used to connect two arrows (1.8) and (1.11) to get
Now consider 4 adjacent rectangles $\gamma^{(a)} : [0, 1]^2 \rightarrow U_\alpha$, $\alpha = i, j, k, l$,

\[
\begin{array}{cccc}
\gamma(i) & \gamma(k) \\
\gamma(j) & \gamma(l)
\end{array}
\]

(1.13)

in four different coordinate charts. We can connecting the local 2-holonomies by using the transition 2-arrows along their common boundaries to get the following diagram:

(1.14)

We add the following 2-arrow in $\mathcal{G}$ in the central rectangle:

(1.15)
where \( f_{ikj}(y_2) \) and \( f_{ij}(y_2) \) are provided by the \( G \)-valued 2-cocycle of the 2-bundle. Note that diagrams (1.14), (1.15) are similar to figure 3 in [13], p. 3358, for the cubical 2-holonomy, where the 2-arrow in the central rectangle in (1.14) is provided directly by the definition of 2-cubical bundles. It is not a composition.

Now fix coordinate charts \( \{U_i\} \) of \( M \). Let \( \gamma : [0,1]^2 \to M \) be a Lipschitzian mapping. To define the global 2-holonomy, we divide the square \([0,1]^2\) into the union of small rectangles \( \Box_{ij} := [t_a, t_{a+1}] \times [s_b, s_{b+1}], \ a = 0, \ldots, N, \ b = 0, \ldots, M, \) where \( 0 = t_0 < t_1 < \cdots < t_N = 1, \ 0 = s_0 < s_1 < \cdots < s_M = 1 \). We choose the rectangles sufficiently small so that \( \gamma(\Box_{ab}) \) is contained in some coordinate chart \( U_i \) for each small rectangle \( \Box_{ab} \). We also require \( \gamma(\Box_{ab}) \) and \( \gamma(\Box_{aM}) \) are in the same coordinate chart for each \( a \). For any two adjacent rectangles whose images under \( \gamma \) are contained in two different coordinate charts, we use the transition 2-arrow along their common path to glue these two local 2-holonomies (the transition 2-arrow is the identity when they are in the same coordinate chart). In this construction, there exist an extra rectangle for any 4 adjacent rectangles as in (1.14). We use the 2-arrows provided by the \( G \)-valued 2-cocycle as in (1.15) to fill them. The resulting 2-arrow is denoted by \( \text{Hol}(\gamma) \) and its \( H \)-element is denoted by \( \text{Hol}_\gamma \). We will assume \( \gamma \) to be a loop in the loop space \( \mathcal{LM} \), i.e., \( \gamma(0, \cdot) \equiv \gamma(1, \cdot), \gamma(\cdot, 0) \equiv \gamma(\cdot, 1) \). Denote \( H/ \sim \) by \( H/[G,H] \), where \( h \sim h' \) when \( h = g \circ h' \) for some \( g \in G \). In fact, \( H/[G,H] \) is commutative (cf. [21], Lemma 5.8).

**Theorem 1.1.** For a loop \( \gamma \) in the loop space \( \mathcal{LM} \), the global 2-holonomy \( \text{Hol}_\gamma \) constructed above, as an element of \( H/[G,H] \), is well-defined. In particular when \( \gamma \) is a sphere, \( \text{Hol}_\gamma \) is in \( \ker \alpha \).

See theorem 4.15 of [21] for the existence theorem of the transport functor, and [13] [16] for the cubical version. When \( \gamma \) is a sphere, \( \gamma(\cdot, 0) \equiv \gamma(\cdot, 1) \equiv * \) is a fixed point. So if we write \( \text{Hol}(\gamma) \) as the 2-arrow \( (g, \text{Hol}_\gamma) \) in \( \mathcal{G} \) for some \( g \in G \), its target is also \( g \). This implies that \( \alpha(\text{Hol}_\gamma) = 1_H \).

To show the well-definedness of \( \text{Hol}_\gamma \), we have to prove that it is independent of the choice of the coordinate charts \( \{U_i\} \), division of the square \([0,1]^2\) into the union of small rectangles \( \Box_{ab} \), the choice of the coordinate chart \( U_i \) for each rectangle \( \Box_{ab} \) such that \( \gamma(\Box_{ab}) \subset U_i \) and reparametrization of the loop \( \gamma \) in the loop space \( \mathcal{LM} \).

In Section 2, we recall definitions of a crossed module, a differential crossed module, a strict 2-category and the construction of the strict 2-groupoid \( \mathcal{G} \) associated to a crossed module. In Section 3 and 4, we develop the theory of path-ordered and surface-ordered integrals. We use the method in [20] (and similarly that in [12]), where the authors only consider the local 2-holonomies for bigons. A **bigon** is a mapping \( \gamma : [0,1]^2 \to M \) such that its left and right boundaries degenerate to two points. In our case, after division of the mapping \( \gamma : [0,1]^2 \to U \), we have to consider general Lipschitzian mappings \( \Box_{ab} \to U \). In Section 3, we discuss the local 1-holonomy along the loop as the boundary of a mapping \( \gamma : [0,1]^2 \to U \) and obtain its differentiation in terms of 1-curvatures. We also give the transformation law of local 1-holonomies under a 2-gauge transformation. In Section 4, we construct the local 2-holonomy along a mapping and give the transformation law of local 2-holonomies under a 2-gauge transformation, which is a commutative cube. We also introduce the transition 2-arrow along a path in the intersection \( U_i \cap U_j \), which is constructed from a 2-gauge-transformation \((g_{ij}, a_{ij})\). The compatibility cylinder of three transition 2-arrows along a path in the triple intersection \( U_i \cap U_j \cap U_k \) is commutative.
The $G$-valued 2-cocyle condition gives us a commutative tetrahedron. The commutative cubes, the compatibility cylinders and the 2-cocyle tetrahedra are used in the last section to show the well-definedness of the global 2-holonomy. From 3-cells (5.6)-(5.9) as a 3-dimensional version of (1.10), it is quite intuitionistic to see that the global 2-holonomy is independent of the choice of the coordinate chart $U_i$ for each rectangle $\square_{ab}$ such that $\gamma(\square_{ab}) \subset U_i$.

2. (Differential) crossed modules and 2-categories

2.1. Crossed modules and differential crossed modules. A crossed module $(G,H,\alpha,\triangleright)$ of Lie groups is given by a Lie group map $\alpha : H \to G$ together with a smooth left action $\triangleright$ of $G$ on $H$ by automorphisms, such that:

(1) for each $g \in G$ and $h \in H$, we have
\[
\alpha(g \triangleright h) = g\alpha(h)g^{-1};
\]
(2) for any $f,h \in H$, we have
\[
\alpha(f) \triangleright h = fhf^{-1}.
\]

Here the smooth left action $\triangleright$ of $G$ on $H$ by automorphisms means that we have
\[
(gg') \triangleright h = g \triangleright (g' \triangleright h) \quad \text{and} \quad g \triangleright (hh') = g \triangleright h \cdot g \triangleright h',
\]
for any $g,g' \in G$, $h,h' \in H$. In particular, we have
\[
g \triangleright 1_H = 1_H, \quad (g \triangleright h)^{-1} = g \triangleright (h^{-1}).
\]

A differential crossed module is given by Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$ and a homomorphism of Lie algebras $\alpha_* : \mathfrak{h} \to \mathfrak{g}$, together with a smooth left action $\triangleright$ of $\mathfrak{g}$ on $\mathfrak{h}$ by automorphisms, such that:

(1) for any $x \in \mathfrak{g}$, $u \in \mathfrak{h}$, we have $\alpha_*(x \triangleright u) = [x,\alpha_*(u)]$;
(2) for any $v,u \in \mathfrak{h}$, we have $\alpha_*(v) \triangleright u = [v,u]$.

Here the smooth left action $\triangleright$ of $\mathfrak{g}$ on $\mathfrak{h}$ by automorphisms means that for any $x,y \in \mathfrak{g}$, $u,v \in \mathfrak{h}$, we have
\[
x \triangleright [u,v] = [x \triangleright u,v] + [u,x \triangleright v] \quad \text{and} \quad [x,y] \triangleright u = x \triangleright (y \triangleright u) - y \triangleright (x \triangleright u).
\]

Without loss of generality, we assume that groups $G$ and $H$ are matrix groups. In this case, a product of group elements is realized as a product of matrices. Moreover, their Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$ also consist of matrices. The smooth left action $\triangleright$ of $G$ on $H$ induces an action of $G$ on $\mathfrak{h}$ and an action of $\mathfrak{g}$ on $H$ by
\[
g \triangleright y = \frac{d}{dt}|_{t=0} \exp(ty), \quad x \triangleright h = \frac{d}{dt}|_{t=0} \exp(tx) \triangleright h,
\]
where $y \in \mathfrak{h}$, $x \in \mathfrak{g}$, respectively. And $\alpha_*(x) = \frac{d}{dt}|_{t=0} \alpha(\exp(tx))$. By abuse of nations, we will also denote $\alpha_*$ by $\alpha$. In particular, for any $x \in \mathfrak{g}$, it follows from (2.4) that
\[
x \triangleright 1_H = 0.
\]

Let $G \ltimes H$ be the wreath product of groups $G$ and $H$ given by the action $\triangleright$, i.e.
\[
(g_1,h_1) \cdot (g_2,h_2) := (g_1g_2,g_1 \triangleright h_2 \cdot h_1).
\]

This product is associative since we have
\[
[(g_1,h_1) \cdot (g_2,h_2)] \cdot (g_3,h_3) = (g_1g_2g_3,g_1g_2 \triangleright h_3 \cdot g_1 \triangleright h_2 \cdot h_1) = (g_1,h_1) \cdot [(g_2,h_2) \cdot (g_3,h_3)],
\]
by using \((2.9)\), and
\[(g,h)^{-1} = (g^{-1},g \triangleright h^{-1}) .\]
Set \(g_j = \exp(sX), h_j = \exp(sX)\) in \((2.7)\), \(j = 1,2\), where \(X \in \mathfrak{g}, Y \in \mathfrak{h}\). Then differentiate it with respect to \(s\) at \(s = 0\) to get
\[(2.10) \quad (X,Y) \cdot (g,h) = (Xg,X \triangleright h + hY), \quad (g,h) \cdot (X,Y) = (gX,g \triangleright Y \cdot h).\]
Similarly, we have
\[(2.11) \quad (X,Y) \cdot (X',Y') = (XX',X \triangleright Y' + Y'Y),\]
which provides the wreath product \(\mathfrak{g} \ltimes \mathfrak{h}\) the structure of a Lie algebra.

**Lemma 2.1.** For any \((g,h) \in G \ltimes H\) and \((X,Y) \in \mathfrak{g} \ltimes \mathfrak{h}\), we have
\[(2.12) \quad Ad_{(g,h)}(X,Y) = \left( Ad_gX, (Ad_gX) \triangleright h^{-1} \cdot h + Ad_{h^{-1}}(g \triangleright Y) \right).\]

**Proof.** Note that by using the multiplication law \((2.7)-(2.9)\), we have
\[\text{Ad}_{(g,h)}(\exp(sX),\exp(sY)) = (g,h)(\exp(sX),\exp(sY))(g^{-1},g \triangleright h^{-1}) = (g\exp(sX)g^{-1},(g\exp(sX)g^{-1}) \triangleright h^{-1} \cdot g \triangleright \exp(sY) \cdot h).\]
Then take derivatives with respect to \(s\) at \(s = 0\) to get \((2.12)\). \(\square\)

### 2.2. Strict 2-categories

A 2-category is a category enriched over the category of all small categories. In particular, a strict 2-category \(\mathcal{C}\) consists of collections \(\mathcal{C}_0\) of objects, \(\mathcal{C}_1\) of arrows, and \(\mathcal{C}_2\) of 2-arrows, together with
- functions \(s_n,t_n : \mathcal{C}_i \to \mathcal{C}_n\) for all \(0 \leq n < i \leq 2\), called the \(n\)-source and \(n\)-target,
- functions \(\#_n : \mathcal{C}_{n+1} \times \mathcal{C}_{n+1} \to \mathcal{C}_{n+1}\), \(n = 0,1\), called the \((vertical) n\)-composition,
- a function \(\#_0 : \mathcal{C}_2 \times \mathcal{C}_2 \to \mathcal{C}_2\), called the \((horizontal) 0\)-composition,
- a function \(1_s : C_i \to C_{i+1}\), \(i = 0,1\), called the identity.

Two arrows \(\gamma\) and \(\gamma'\) are called \(n\)-composable if the \(n\)-target of \(\gamma\) coincides with the \(n\)-source of \(\gamma'\). For example, two 2-arrows \(\phi\) and \(\psi\) are called \(1\)-composable if the 1-target of \(\phi\) coincides with the 1-source of \(\psi\). In this case, their vertical composition \(\phi \#_1 \psi\) is \(x \begin{array}{c} \phi \phantom{y} \\ \psi \end{array} y\), where \(A = s_1(\phi), B = t_1(\phi) = s_1(\psi), C = t_1(\psi), x = s_0(\phi) = s_0(\psi), \text{ etc.}\)

Two 2-arrows \(\phi\) and \(\psi\) are called \(\text{(horizontally) 0-composable}\) if the 0-target of \(\phi\) coincides with the 0-source of \(\psi\). In this case, their horizontal composition \(\phi \#_0 \psi\) is \(x \begin{array}{c} \phi \\ \psi \end{array} y \begin{array}{c} \phi \\ \psi \end{array} z\). In particular, when \(\phi = 1_A\), we call \(1_A \#_0 \psi\) whiskering from left by the 1-arrow \(A\), and denote it by \(A \#_0 \psi\):
\[x \begin{array}{c} A \phantom{y} \\ B \end{array} y \begin{array}{c} \psi \\ C \end{array} \begin{array}{c} \phi \\ \psi \end{array} z\]. Similarly, we define whiskering from right by a 1-arrow.
The identities satisfy
\[(2.13) \quad 1_x\#_0 A = A = A\#_1 1_y, \quad 1_A\#_1 \phi = \phi = \phi\#_1 1_B,\]
for any 1-arrow \( A : x \rightarrow y \) and any 2-arrow \( \phi : A \Rightarrow B \). The composition \( \#_p \) satisfies the associativity
\[(2.14) \quad (\phi\#_p \psi)\#_p \omega = \phi\#_p (\psi\#_p \omega),\]
if they are \( p \)-composable, for \( p = 0 \) or 1.

The horizontal composition satisfies the interchange law:
\[(2.15) \quad (A\#_0 \psi)\#_1 (\phi\#_0 D) = \phi\#_0 (\psi\#_0 B)\#_1 (C\#_0 \psi),\]
\[\begin{array}{ccc}
\xymatrix{
& A \\
\downarrow^\phi & & \downarrow^\psi \\
\downarrow^\gamma & B \ar[u] & C \\
\downarrow^\alpha (h^{-1})g & \psi \downarrow & \psi \downarrow \\
& D
}\end{array}
\begin{array}{ccc}
\xymatrix{
& A \\
\downarrow^\phi & & \downarrow^\psi \\
\downarrow^\gamma & B \ar[u] & C \\
\downarrow^\alpha (h^{-1})g & \psi \downarrow & \psi \downarrow \\
& D
}\end{array}
\]

namely, the vertical composition of the left two 2-arrows coincides with the vertical composition of the right two 2-arrows. They are both equal to the horizontal composition \( \phi\#_0 \psi \). The interchange law allows us to change the order of compositions of 2-arrows, up to whiskerings.

The interchange law (2.15) is a special case of the following more general compatibility condition for different compositions. If \((\beta, \beta'), (\gamma, \gamma') \in C_k \times C_k \) are \( p \)-composable and \((\beta, \gamma), (\beta', \gamma') \in C_k \times C_k \) are \( q \)-composable, \( p, q = 0, 1 \), then we have
\[(2.16) \quad (\beta\#_p \beta')\#_q (\gamma\#_p \gamma') = (\beta\#_q \gamma)\#_p (\beta'\#_q \gamma'),\]
\[\begin{array}{ccc}
\xymatrix{
& \beta \\
\downarrow^\gamma & \downarrow^\beta & \downarrow^\gamma \\
& \beta
}\end{array}
\begin{array}{ccc}
\xymatrix{
& \beta \\
\downarrow^\gamma & \downarrow^\beta & \downarrow^\gamma \\
& \beta
}\end{array}
\]
Here \( p = 0, q = 1 \) in the right diagram. The first identity of the interchange law (2.15) is exactly the condition (2.16) with \( p = 0, q = 1, \beta = 1_A, \beta' = \psi, \gamma = \phi, \gamma' = 1_D \), by using the property (2.13) for identities. It is similar for the second identity in (2.15), (2.13), (2.14) and (2.16) are the axioms that a strict 2-category should satisfy.

A 1-arrow \( A : x \rightarrow y \) is called invertible, if there exists another 1-arrow \( B : y \rightarrow x \) such that \( 1_x = A\#_0 B \) and \( B\#_0 A = 1_y \). A strict 2-category in which every 1-arrow is invertible is called a strict 2-groupoid. A 2-arrow \( \varphi : A \Rightarrow B \) is called invertible if there exists another 2-arrow \( \psi : B \Rightarrow A \) such that \( \psi\#_1 \varphi = 1_B \) and \( \varphi\#_1 \psi = 1_A \). \( \psi \) is uniquely determined and called the inverse of \( \varphi \).

2.3. The strict 2-groupoid \( G \) associated to a crossed module.

Proposition 2.1. A crossed module \((G, H, \alpha, \triangleright)\) constitutes a strict 2-groupoid with only one object \( \bullet \), 1-arrows given by elements of \( G \) and 2-arrows given by elements \((g, h) \in G \times H\)
\[\begin{array}{ccc}
\xymatrix{
& \alpha(h^{-1})g \\
\downarrow^h & & \\
& \alpha(h^{-1})g
}\end{array}
\]
We denote this strict 2-groupoid by $G$. Any two 1-arrows $g : \bullet \rightarrow \bullet$ and $g' : \bullet \rightarrow \bullet$ are 0-composable and $g \#_0 g' = gg'$. The 1-source of 2-arrow $(g, h)$ is $g$, while its 1-target is $\alpha(h^{-1})g$.

The vertical composition of two 2-arrows $(g, h)$ and $(g', h')$ is

$$
(2.17) \quad (g, h) \#_1 (g', h') := (g, hh')
$$

if they are 1-composable, i.e., $g' = \alpha(h^{-1})g$. This composition is well defined since their targets are equal, i.e. $\alpha(h'^{-1})\alpha(h^{-1})g = \alpha(hh'^{-1})g$. The horizontal composition is

$$
(2.18) \quad (g, h) \#_0 (g', h') := (gg', g \triangleright h' \cdot h)
$$

This is exactly the multiplication of the wreath product $G \ltimes H$ in (2.7). So it satisfies the associativity (2.14) by (2.8). Note that for any two 2-arrows, their horizontally composition always exists. When $h = 1_H$ or $h' = 1_H$ in (2.18), we have 2-arrows

$$
(2.19) \quad (gg', g \triangleright h') : \bullet \quad g \quad g' \quad \bullet
$$

respectively. They are whiskering from left or right by a 1-arrow, respectively. From above we see that whiskering from right by a 1-arrow is always trivial in $G$. We have identities $1_1 = 1_G, 1_g = (g, 1_H)$. The horizontal composition satisfies the interchange law:

$$
(2.20) \quad (gg', g \triangleright h' \cdot h) = (gg', h \cdot [\alpha(h^{-1})g] \triangleright h').
$$

This is because

$$
g \triangleright h' \cdot h = hAd_{h^{-1}}(g \triangleright h') = h \cdot \alpha(h^{-1}) \triangleright (g \triangleright h') = h \cdot [\alpha(h^{-1})g] \triangleright h',
$$

by (2.2) and left action $\triangleright$ of $G$ on $H$.

It is easy to check that $G$ satisfies axioms (2.13) (2.14) and (2.16). So it is a strict 2-category. Moreover, it is a strict 2-groupoid.

**Remark 2.1.** Proposition [2.1] is well known. But here we write compositions of 1- or 2-arrows in the natural order, which is different from that in [12] [20] [21]. It has the advantage that the order of a product of group elements is the same as that of corresponding arrows appear in the diagram. But this makes our formulae of 2-gauge-transformations in (1.2) and the compatibility conditions (1.7) a little bit different from the standard ones.

The condition (1.3) in the definition of a nonabelian $G$-valued 2-cocycle is equivalent to say that $f_{ijk}$ defines a 2-arrow

$$
(g_{ij}g_{jk}, f_{ijk}) :
$$
in $G$, while the 2-cocycle condition (1.4) is equivalent to commutativity of the following tetrahedron:

\[ \begin{align*}
\sum_{a,b} K^a X_a \text{ and elements } X_a \text{'s of } \mathfrak{t} \text{. Since } \mathfrak{t} \text{ is assumed to be a matrix Lie algebra, we have } [X, X'] = XX' - X'X \text{ for any } X, X' \in \mathfrak{t}.
\end{align*} \]

For $K = \sum_a K^a X_a, M = \sum_b M^b X_b \in \Lambda^2(U, \mathfrak{g}),$ define

\[ (2.22) \]

\[ K \wedge M : = \sum_{a,b} K^a \wedge M^b X_a X_b, \quad dK = \sum_a dK^a X_a, \]

and for $\Psi = \sum_b \Psi^b Y_b \in \Lambda^s(U, \mathfrak{h}),$ define

\[ (2.23) \]

\[ K \triangleright \Psi := \sum_{a,b} K^a \wedge \Psi^b X_a \triangleright Y_b. \]

The 1-curvature 2-form and 2-curvature 3-form are defined as

\[ \Omega^A := dA + A \wedge A, \quad \Omega_2^{(A,B)} := dB + A \triangleright B, \]

respectively. Under the 2-gauge transformation (1.2), these curvatures transform as follows:

\[ \Omega^A' - \alpha(B') = g^{-1} \triangleright (\Omega^A - \alpha(B)) , \quad \Omega_2^{(A',B')} = g^{-1} \triangleright \Omega_2^{(A,B)} + [\Omega^A' - \alpha(B')] \triangleright \varphi, \]

(cf. [3, 23]). The fake 1-curvature is $\Omega^A - \alpha(B)$. We only consider 2-connections with vanishing fake 1-curvatures, i.e. (1.3) holds. In this case the 2-curvature 3-form is covariant under 2-gauge transformations (1.2).

3. The Local 1-Holonomy

3.1. The local 1-holonomy along a loop and its variation. By the definition of 1-holonomy in (1.6), it is easy to see that

\[ (3.1) \]

\[ F_A(\rho \# \tilde{\rho}) = F_A(\rho) F_A(\tilde{\rho}), \]
where \( \# \) is the composition of two paths. We use the natural order, i.e. we write \( \rho \# \tilde{\rho} \) if the endpoint of \( \rho \) coincides with the starting point of \( \tilde{\rho} \).

Now consider a surface given by a Lipschitzian mapping \( \gamma : [0,1]^2 \rightarrow U \). We denote by \( \gamma_{[t_1,t_2],s} \) the curve given by the mapping \( \gamma \) restricted to the horizontal interval \([t_1, t_2] \times \{s\}\), and denote by \( \gamma_{t,[s_1,s_2]} \) the curve given by the mapping \( \gamma \) restricted to the vertical interval \([t] \times [s_1, s_2]\). Also denote by \( \gamma_{t,s} \) the point \( \gamma(t,s) \). In the following we will also use the notations

\[
\gamma_{t:s}^\pm := \gamma_{[0,t]:0}^\pm \# \gamma_{[0,t]:s},
\]

for the lower and upper boundaries of the surface \( \gamma \) restricted to \([0, t] \times [0, s]\), respectively.

The 1-holonomy along the loop as the boundary of the surface \( \gamma : [0, t] \times [s_0, s] \rightarrow U \) is

\[
u_{A,s_0}(t,s) := F_A(\gamma_{[s_0,s],s}) \cdot F_A(\gamma_{[t,s],s})^{-1} \cdot F_A(\gamma_{[0,t]:s})^{-1},
\]

for \( s \geq s_0 \). When \( s_0 = 0 \), denote

\[
u_{A}(s,t) := u_{A,0}(s,t) = F_A(\gamma_{t:s}^-) F_A(\gamma_{t:s}^+)^{-1}.
\]

From the above diagram (3.3), \( u_A(t,s) \) is the composition of 1-holonomies of two loops. Namely,

\[
u_{A}(t,s) = Ad_{F_A(\gamma_{[s_0,s],s})} u_{A,s_0}(t,s) \cdot u_A(t,s_0).
\]

The following proposition tells us how the 1-holonomy \( u_{A,s_0}(t,s) \) changes as \( s \) increase for fixed \( t \) (cf. lemma B. 1 of [19]).

**Proposition 3.1.** \( u_{A,s_0} \) satisfies the following ODE of second order:

\[
\frac{\partial^2 u_{A,s_0}}{\partial t \partial s} \bigg|_{(t,s_0)} = Ad_{F_A(\gamma_{[0,t],s_0})} \gamma^* \Omega^A_{(t,s_0)} \left( \frac{\partial}{\partial t}, \frac{\partial}{\partial s} \right).
\]

**Proof.** Differentiate (3.3) with respect to \( s \) to get

\[
\frac{\partial}{\partial s} u_{A,s_0}(s,t) = F_A(\gamma_{[s_0,s],s}) \left[ \gamma^* A(0,s) \left( \frac{\partial}{\partial s} \right) F_A(\gamma_{[t,s],s}) + \frac{\partial}{\partial s} F_A(\gamma_{[0,t],s}) \right]
\]

\[-F_A(\gamma_{[0,t],t}) \cdot \gamma^* A(t,s) \left( \frac{\partial}{\partial s} \right) F_A(\gamma_{[t,s],s})^{-1} F_A(\gamma_{[0,t],s})^{-1},
\]

where the notation \( \gamma^* \) is as in [19].
by using the ODE (1.6). Note that by definition, we have

\[ FA(\gamma_{t;[s_0, s]}) \big|_{s=s_0} = 1G, \quad \frac{\partial}{\partial t} FA(\gamma_{t;[s_0, s]}) \big|_{s=s_0} = 0. \]

Then differentiate the above identity with respect to \( \alpha \)

\[ (3.10) \]

Then, it is easy to see that

\[ (3.9) \]

corresponding \( h \) by applying \( \alpha \) with \( \Box \). The result is proved.

Differentiate both sides of (3.5) with respect to \( s \) if we use the notation

\[ (3.8) \]

\[ (3.7) \]

\[ \gamma^* \cdot \gamma A_{(t, s_0)}(\partial_s F) \gamma^* \Omega^A_{(t, s_0)} \left( \frac{\partial}{\partial t}, \frac{\partial}{\partial s} \right). \]

The result is proved.

The proposition implies that

\[ \frac{\partial}{\partial s} u_{A, s_0} \bigg|_{(t, s_0)} = -\int_0^t Ad_{FA}(\gamma_{(0, \tau; s_0)}^* \gamma \Omega^A_{(\tau; s_0)}) \left( \frac{\partial}{\partial \tau}, \frac{\partial}{\partial s} \right) d\tau. \]

Differentiate both sides of (3.5) with respect to \( s \), then take \( s_0 = s \) and use the above formula to get

\[ (3.7) \]

\[ \frac{\partial}{\partial s} u_A(t, s) = -\mathcal{A}_t(s) u_A(t, s), \]

with

\[ (3.8) \]

\[ \mathcal{A}_t(s) := \int_0^t Ad_{FA}(\gamma_{(\tau; s)}^* \gamma \Omega^A_{(\tau; s)}) \left( \frac{\partial}{\partial \tau}, \frac{\partial}{\partial s} \right) d\tau, \]

if we use the notation \( \gamma_{(\tau; s)} \) in (3.2) and \( Ad_{FA}(\gamma_{(0, \tau; s)}) \gamma \Omega^A_{(\tau; s)} = Ad_{FA}(\gamma_{(\tau; s)}) \). Now define a corresponding \( h \)-valued 1-form

\[ (3.9) \]

\[ \mathcal{B}_t(s) := \int_0^t FA(\gamma_{(\tau; s)}^* \gamma B_{(\tau; s)}) \left( \frac{\partial}{\partial \tau}, \frac{\partial}{\partial s} \right) d\tau. \]

Then, it is easy to see that

\[ (3.10) \]

\[ \alpha(\mathcal{B}_t(s)) = \mathcal{A}_t(s), \]

by applying \( \alpha \) to (3.9) and using (1.1), (2.1).
3.2. The transformation law of local 1-holonomies under a 2-gauge transformation. Suppose that \( \rho : [a, b] \rightarrow U \) be a Lipschitzian curve. Let \((A, B)\) and \((A', B')\) be two local 2-connection over \( U \) such that \((g, \varphi)\) is a 2-gauge-transformation \((1.2)\) from \((A, B)\) to \((A', B')\). To construct the 2-arrow relating 1-holonomies \( F_A(\rho) \) and \( F_{A'}(\rho) \), we define an \( H \)-valued function \( h(\rho_{[a,b]}) \) satisfying the following ODE

\[
\frac{d}{dt} h(\rho_{[a,t]}) = F_A(\rho_{[a,t]}) \triangleright \rho^* \frac{\partial}{\partial t} \cdot h(\rho_{[a,t]})
\]

with initial value \( 1_H \). Then \( (F_A(\rho_{[a,t]}), h(\rho_{[a,t]})) \) is a 2-arrow in \( \mathcal{G} \) by the following proposition. We call it the 2-gauge-transformation along the curve \( \rho_{[a,t]} \) associated to the 2-gauge-transformation \((1.2)\) (cf. the pseudonatural transformation in \([21]\)).

**Proposition 3.2.** Suppose that \((g, \varphi)\) is 2-gauge-transformation \((1.2)\) from \((A, B)\) to \((A', B')\). Then \( h(\rho_{[a,t]}) \) satisfies the target-matching condition

\[
\alpha \left( h(\rho_{[a,t]})^{-1} \right) F_A(\rho_{[a,t]}) g(\rho(t)) = g(\rho(a)) F_{A'}(\rho_{[a,t]}),
\]

and satisfies the following composition formula

\[
h(\rho_{[a,t+t']}) = F_A(\rho_{[a,t]}) \triangleright h(\rho_{[t,t+t']}) \cdot h(\rho_{[a,t]}),
\]

which corresponds to the diagram

\[
\begin{array}{c}
\begin{array}{cc}
\bullet & \bullet \\
F_A(\rho_{[a,t]}) & F_A(\rho_{[t,t+t']}) \\
\end{array}
\end{array}
\]

\[
\begin{array}{c}
\begin{array}{cc}
\rho(\rho(a)) & \rho(\rho(t)) \\
\bullet & \bullet \\
F_{A'}(\rho_{[a,t]}) & F_{A'}(\rho_{[t,t+t']}) \\
\end{array}
\end{array}
\]

\[
\begin{array}{c}
\begin{array}{cc}
\cdot & \cdot \\
\rho(\rho(a)) & \rho(\rho(t)) \\
\end{array}
\end{array}
\]

\[
\begin{array}{c}
\begin{array}{cc}
\cdot & \cdot \\
\rho(\rho(a)) & \rho(\rho(t)) \\
\end{array}
\end{array}
\]

\[
\begin{array}{c}
\begin{array}{cc}
\cdot & \cdot \\
\rho(\rho(a)) & \rho(\rho(t)) \\
\end{array}
\end{array}
\]

**Proof.** Set

\[
\beta(t) := \frac{g_a^{-1} \alpha (h_t^{-1})}{F_A(t) g_t},
\]

where \( h_t = h(\rho_{[a,t]}), F_A(t) := F_A(\rho_{[a,t]}) \) and \( g_t = g(\rho(t)) \). Differentiating it with respect to \( t \), we get

\[
\beta'(t) = -g_a^{-1} \alpha (h_t^{-1}) \alpha \left( \frac{dh_t}{dt} \right) \alpha(h_t^{-1}) F_A(t) g_t + g_a^{-1} \alpha (h_t^{-1}) F_A(t) \rho^* A_t \left( \frac{\partial}{\partial t} \right) g_t
\]

\[
+ g_a^{-1} \alpha (h_t^{-1}) F_A(t) \frac{dg_t}{dt}
\]

\[
= \alpha \left( g_t^{-1} \triangleright \rho^* \varphi_t \left( \frac{\partial}{\partial t} \right) \right) + g_t^{-1} \rho^* A_t \left( \frac{\partial}{\partial t} \right) g_t + g_t^{-1} dg_t \left( \frac{\partial}{\partial t} \right)
\]

\[
= \beta(t) \rho^* A' \left( \frac{\partial}{\partial t} \right)
\]
by the 2-gauge-transformation (1.2) at the point \( \rho(t) \), and
\[
\alpha \left( \frac{d h_t}{d t} \right) \alpha \left( h_t^{-1} \right) F_A(t) g_t = \alpha \left( F_A(t) \triangleright \rho^* \varphi_t \left( \frac{\partial}{\partial t} \right) \cdot h_t \right) \alpha \left( h_t^{-1} \right) F_A(t) g_t
\]
\[
= F_A(t) \alpha \left( \rho^* \varphi_t \left( \frac{\partial}{\partial t} \right) \right) g_t = F_A(t) g_t \alpha \left( g_t^{-1} \triangleright \rho^* \varphi_t \left( \frac{\partial}{\partial t} \right) \right),
\]
by using the ODE (3.11) satisfied by \( h_t \) and (2.1). And \( \beta(a) = 1_G \). So \( \beta(t) \) and \( F_A(\rho(a,t)) \) satisfy the same ODE with the same initial condition. They must be identical. (3.12) is proved.

To show (3.13), set
\[
\sigma(\tau) := F_A(\rho(a,t)) \triangleright h(\rho_{[t,t+\tau]} \cdot h(\rho(a,t)).
\]
Then \( \sigma(0) = h(\rho_{[a,t]}) \) and
\[
\frac{d}{d\tau} \sigma(\tau) = F_A(\rho_{[a,t]} \triangleright h(\rho_{[t,t+\tau]} \cdot h(\rho_{[a,t]}))
\]
\[
= F_A(\rho_{[a,t+\tau]} \triangleright \rho^* \varphi_{t+\tau} \left( \frac{\partial}{\partial \tau} \right)) \sigma(\tau),
\]
by using (3.1) and (3.11). So \( \sigma(\tau) \) and \( h(\rho_{[a,t+\tau]} \cdot h(\rho_{[a,t]}) \) satisfy the same ODE with the same initial condition. They must be identical. (3.14) is proved. \( \square \)

**Remark 3.1.** (1) Differentiating (3.13) with respect to \( \tau \) at \( \tau = 0 \), we get (3.11). Here \( \frac{d}{d\tau} \bigg|_{\tau=0} h(\rho_{[t,t+\tau]} \cdot h(\rho_{[a,t]})) = \rho^* \varphi_t \left( \frac{\partial}{\partial \tau} \right) \). On the other hand, differentiating (3.12) with respect to \( t \) at \( t = a \), we get the first formula of the 2-gauge-transformation (1.2).

(2) By the natural order of compositions, the Lie algebra element in ODE (1.1) for the local 1-holonomy and that in ODE (3.1) for the local 2-holonomy are on the right of products, but the Lie algebra element in ODE (3.11) for \( h \) is on the left of a product. This is because that the horizontal composition (2.13) (i.e. the wreath product) change the order of H-elements.

4. The local 2-holonomy

**4.1. The local 2-holonomy: the surface-ordered integral.** Given a 2-connection \( (A,B) \) over an open set \( U \), to construct the local 2-holonomy along a Lipschitzian mapping \( \gamma : [0,1]^2 \rightarrow U \), we define an \( H \)-valued function \( H_{A,B}(t,s) \) satisfying the ODE

\[
(4.1) \quad \frac{d}{ds} H_{A,B}(t,s) = H_{A,B}(t,s) \mathcal{B}(s)
\]
for fixed \( t \), with the initial condition \( H_{A,B}(t,0) \equiv 1_H \), where \( \mathcal{B}(s) \) is the \( h \)-valued function given by (3.9). Denote \( \text{Hol}(\gamma|_{[0,t] \times [0,s]} : (F_A(\gamma_{t,s}^+) , H_{A,B}(t,s)) \), which is called the local 2-holonomy along the mapping \( \gamma|_{[0,t] \times [0,s]} \).

**Lemma 4.1.** (1) \( (F_A(\gamma_{t,s}^+) , H_{A,B}(t,s)) \) is a 2-arrow with target \( F_A(\gamma_{t,s}^-) \) in \( G \). Namely the \( H \)-element \( H_{A,B}(t,s) \) satisfies the target-matching condition

\[
(4.2) \quad \alpha(H_{A,B}(t,s)^{-1}) F_A(\gamma_{t,s}^+) = F_A(\gamma_{t,s}^-).
\]

(2) \( H_{A,B}(t,s) \) satisfies the following composition formulae:

\[
(4.3) \quad H_{A,B}(t + t', s) = F_A(\gamma|_{[0,t']}) \triangleright \hat{H}_{A,B}(t', s) \cdot H_{A,B}(t,s)
\]
which corresponds to the diagram

\[ \begin{array}{c}
\text{Diagram 1} \\
\end{array} \]

where \( \tilde{H}_{A,B} \) is the \( H \)-element of the local 2-holonomy associated to the mapping \( \tilde{\gamma}(\cdot, \cdot) = \gamma(t+\cdot, \cdot) \) for fixed \( t \); and

\[ H_{A,B}(t, s + s') = H_{A,B}(t, s) \cdot F_A(\gamma_{0,0,s}) \triangleright \tilde{H}_{A,B}(t, s') \]

which corresponds to the diagram

\[ \begin{array}{c}
\text{Diagram 2} \\
\end{array} \]

where \( \tilde{H}_{A,B} \) is the \( H \)-element of the local 2-holonomy associated to the mapping \( \tilde{\gamma}(\cdot, \cdot) = \gamma(\cdot, s+\cdot) \) for fixed \( s \).

**Proof.**

(1) It is sufficient to show that \( \alpha(H_{A,B}(t, s)^{-1}) = u_A(t, s) \). By (4.1), we have

\[ \frac{d}{ds}H_{A,B}(t, s)^{-1} = -\mathcal{A}_t(s)H_{A,B}(t, s)^{-1}. \]

So \( \alpha(H_{A,B}(s)^{-1}) \) satisfies the ODE

\[ \frac{d}{ds}\alpha(H_{A,B}(t, s)^{-1}) = -\mathcal{A}_t(s)\alpha(H_{A,B}(t, s)^{-1)), \]

with \( H_{A,B}(t, 0)^{-1} = 1_H. \) Comparing it with (4.7), we see that \( \alpha(H_{A,B}(t, s)^{-1}) \) and \( u_A(t, s) \) satisfy the same ODE with the same initial condition. So they must be identical.

(2) We denote by the right hand sight of (4.8) as \( \beta(s) \). Then,

\[ \begin{align*}
\beta'(s) &= F_A(\gamma_{0,0,s}) \triangleright \tilde{H}_{A,B}(t', s) \int_0^{t'} F_A(\gamma_{t,0,s} \# \gamma_{t, t+\tau,s}) \triangleright \gamma^* B(t + \tau, s) \left( \frac{\partial}{\partial \tau} \frac{\partial}{\partial s} \right) d\tau \\
&\quad \cdot H_{A,B}(t, s) + \beta(s) \int_0^t F_A(\gamma_{t,s} \# \gamma_{t, t+\tau,s}) \triangleright \gamma^* B(\tau, s) \left( \frac{\partial}{\partial \tau} \frac{\partial}{\partial s} \right) d\tau =: I_1 + I_2,
\end{align*} \]
and

\[
I_1 = \beta(s) A d_{H_{A,B}(t,s)}^{-1} \int_0^t F_A \left( \gamma_{[0,t];0} \# \gamma_{[0,s]} \# \gamma_{[t,t+\tau];s} \right) \triangleright \gamma^* B_{(t+\tau,s)} \left( \frac{\partial}{\partial \tau}, \frac{\partial}{\partial s} \right) d\tau
\]

\[
= \beta(s) \int_0^t \left[ \alpha \left( H_{A,B}(t,s)^{-1} \right) F_A \left( \gamma_{t,s}^0 \right) \cdot F_A \left( \gamma_{[t,t+\tau];s} \right) \right] \triangleright \gamma^* B_{(t+\tau,s)} \left( \frac{\partial}{\partial \tau}, \frac{\partial}{\partial s} \right) d\tau
\]

\[
= \beta(s) \int_t^{t+\nu} F_A \left( \gamma_{t,s}^0 \right) \triangleright \gamma^* B_{(t,s)} \left( \frac{\partial}{\partial \kappa}, \frac{\partial}{\partial s} \right) d\kappa,
\]

by using target-matching condition (4.2) for \( H_{A,B} \). Thus the sum of \( I_1 \) and \( I_2 \) is exactly \( \beta(s) \circ \gamma \). The result follows. The proof of (4.1) is similar. \( \square \)

**Remark 4.1.** Differentiating (4.2) with respect to \( t \) and \( s \) at \( t = s = 0 \), we get the vanishing (4.1) of the fake 1-curvature.

### 4.2. The transformation law of local 2-holonomies under a 2-gauge transformation.

**Proposition 4.1.** Under the 2-gauge-transformation \((g, \varphi)\) from a 2-connection \((A, B)\) to another one \((A', B')\) in (1.3), the \( H \)-elements of the local 2-holonomies satisfy the following the transformation law:

\[
g(\gamma_{0,0}) \triangleright H_{A',B'}(t,s) = h(\gamma_{t,s}^+)^{-1} H_{A,B}(t,s) h(\gamma_{t,s}^-),
\]

i.e., the following cube

\[
\begin{array}{c}
\begin{array}{c}
F_A(\gamma_{[0,t];0}) \\
F_A(\gamma_{[0,s]}) \\
\end{array} \\
\begin{array}{c}
H_{A,B}(t,s) \\
H_{A',B'}(t,s) \\
\end{array} \\
\begin{array}{c}
F_A(\gamma_{[0,t];s}) \\
F_A(\gamma_{[0,s]}) \\
\end{array}
\end{array}
\]

is commutative, where \( g_0 := g(\gamma_{0,0}), F_a := F_A(\gamma_{t,[0,s]}), h_a := h(\gamma_{t,[0,s]}), h_b := h(\gamma_{0,[0,s]}), \) and \( F_c := F_A(\gamma_{[0,t];s}) \). The front face represents the 2-arrow given by \( h(\gamma_{[0,t];s}) \).

**Remark 4.2.** (1) By the composition formula in (2.15), we have

\[
\begin{align*}
& h(\gamma_{t,s}^+) = F_A(\gamma_{[0,t];0}) \triangleright h(\gamma_{t,[0,s]}) \cdot h(\gamma_{0,[t];0}), \\
& h(\gamma_{t,s}) = F_A(\gamma_{[0,t];s}) \triangleright h(\gamma_{[0,t];s}) \cdot h(\gamma_{0,[t];s}),
\end{align*}
\]
correspond to the following diagrams:

respective. For example, \( h(\gamma_{t,s}^+) \) is the \( H \)-element of the composition of the following two arrows:

The left one is whiskered from left by 1-arrow \( F_A(\gamma_{[0,t],0}) \), corresponding to the wavy path, while the right one is trivially whiskered from right by a 1-arrow.

(2) Differentiating \( h(\gamma_{t,s}^+)g(\gamma_{0,0}) \triangleright H_{A',B'}(t,s) = H_{A,B}(t,s)h(\gamma_{t,s}^-) \) in \([4.7]\) with respect to \( t \) and \( s \) at \( t = s = 0 \), we get the second formula of the 2-gauge-transformation \([4.2]\) (cf. subsection 3.3.2 of \([20]\)).

To prove Proposition \([4.1]\) set

(4.8) \[ F(s) := h(\gamma_{t,s}^+)^{-1}H_{A,B}(t,s)h(\gamma_{t,s}^-). \]

To show \( F(s) = g(\gamma_{0,0}) \triangleright h_{A',B'}(s) \), it is sufficient to check that they satisfy the same ODE with the same initial condition. To find the ODE satisfied by \( F(s) \), we take derivatives with respect to \( s \) on both sides of \([4.8]\). So we have to know two derivatives \( \frac{d}{ds}h(\gamma_{t,s}^+) \). To simplify it, we rewrite \( F(s) \) in the following form:

(4.9) \[ F(s) = H_{A,B}(t,s)F_s \quad \text{with} \quad F_s = h(\gamma_{t,s}^-) Ad_{\{H_{A,B}(s)h(\gamma_{t,s}^-)\}^{-1}}(h(\gamma_{t,s}^+)^{-1}). \]

Note that by the target-matching conditions \([3.12]\) and \([4.2]\) for \( h(\gamma_{t,s}^-) \) and \( H_{A,B}(t,s) \), respectively, we see that

(4.10) \[ \alpha(H_{A,B}(t,s)h(\gamma_{t,s}^-)) = F_A(\gamma_{t,s}^-) \cdot g(\gamma_{t,s}) \cdot F_{A'}(\gamma_{t,s}^-)^{-1}g(\gamma_{0,0})^{-1} = \tilde{g}_{t,s}^{-1}. \]

where

(4.11) \[ \tilde{g}_{t,s} = g(\gamma_{0,0}) \cdot F_A(\gamma_{t,s}^-) \cdot g(\gamma_{t,s}^-)^{-1} \cdot F_A(\gamma_{t,s}^-)^{-1} \]
corresponds to the dotted loop in the following cube:

```
\[ \begin{array}{c}
\cdots \quad F_A(\gamma_{[0,t],s})^{-1} \\
\quad \\ \\
\quad \\ \\
\quad \\ \\
\cdots \quad F_A(\gamma_{[t,0],s})^{-1} \\
\end{array} \]
```

So we only need to find the derivative of the term \( F_s \). This term has a good geometric interpretation in terms of the 1-holonomy of the \( g \ltimes h \)-valued connection

\[(4.12) \quad \mathfrak{A} = (A, \varphi).\]

See Lemma 3.19 in [20] for this method. As before, let \( u_\mathfrak{A}(t, s) \) be the 1-holonomy for the loop as the boundary of the image of the rectangle \([0, t] \times [0, s]\) under the mapping \( \gamma \), with respect to the \( g \ltimes h \)-valued 1-form \( \mathfrak{A} \). Write

\[(4.13) \quad u_\mathfrak{A}(t, s) = \left( g^\tau_1(s), h^\tau_1(s) \right).\]

**Lemma 4.2.** We have \( h^\tau_1(s) = F_s \) with \( F_s \) given by (4.9).

**Proof.** Recall that for a Lipschitzian curve \( \rho: [a, b] \rightarrow U \) and the \( g \ltimes h \)-valued 1-form \( \mathfrak{A} \) on \( U \), \( F_\mathfrak{A}(\rho) \) is the 1-holonomy satisfying

\[
\frac{d}{dt}F_\mathfrak{A}(\rho_{[a,t]}) = F_\mathfrak{A}(\rho_{[a,t]}) \rho^* \mathfrak{A}_\tau \left( \frac{\partial}{\partial \tau} \right).
\]

If we write \( F_\mathfrak{A}(\rho_{[a,t]}) := \left( g(\tau), h(\tau) \right) \), then this ODE can be written as

\[
\begin{cases} 
\frac{d}{d\tau}g(\tau) = g(\tau)\rho^* A_\tau \left( \frac{\partial}{\partial \tau} \right), \\
\frac{d}{d\tau}h(\tau) = g(\tau) \triangleright \rho^* \varphi_\tau \left( \frac{\partial}{\partial \tau} \right) \cdot h(\tau),
\end{cases}
\]

by using (2.10). By comparing ODE’s in (4.14) with (3.11) and (1.6), we see that \( g(\tau) = F_A(\rho_{[a,t]}), h(\tau) = h(\rho_{[a,t]}), \) i.e.,

\[(4.15) \quad F_\mathfrak{A}(\rho) = (F_A(\rho), h(\rho)).\]

Apply (4.15) and the composition formula (3.11) of 1-holonomies to the boundary of the square \([0, t] \times [0, s]\) to get

\[(4.16) \quad \left( g^\tau(s), h^\tau_1(s) \right) = u_\mathfrak{A}(t, s) = \left( F_A(\gamma_{t,s}^-), h(\gamma_{t,s}^-) \right) \left( F_A(\gamma_{t,s}^+), h(\gamma_{t,s}^+) \right)^{-1}.\]

Consequently, by the multiplication law (2.7) and (2.11) of \( G \ltimes H \) and the interchange law (2.20), we see that \( h^\tau_1(s) \) as the \( H \)-element of \( u_\mathfrak{A}(t, s) \) is equal to

\[(4.17) \quad h^\tau_1(s) = h(\gamma^-_{t,s}) \cdot \left[ \alpha \left( h(\gamma^-_{t,s}) \right)^{-1} F_A(\gamma^-_{t,s}) F_A(\gamma^+_{t,s}) \right] \triangleright h(\gamma^+_{t,s})^{-1} = h(\gamma_{t,s}^-) \cdot g_{t,s} \triangleright h(\gamma^+_{t,s})^{-1},\]
where \( \tilde{g}_{t,s} \) is given by (4.11). Then the result follows from (4.9)–(4.10) and the formula of \( h_t^1(s) \) in (4.17).

**Remark 4.3.** In definition (4.8), \( F(s) \) is the \( H \)-element of the vertical composition of three 3-arrows in the cube (4.6). Here we reinterpret the part \( F_s \) as the \( H \)-element of the horizontal composition

\[
(F_A(\gamma_{t,s}^-), h(\gamma_{t,s}^-)) \#_0 (F_A(\gamma_{t,s}^+), h(\gamma_{t,s}^+))^{-1},
\]

e.g., the horizontal composition of 2-arrows corresponding to the left, front, right and back face in the cube (4.6).

**Proof of Proposition 4.1.** Now we can write

\[
F(s) = H_{A,B}(t,s)h_t^1(s)
\]

by (4.9) and Lemma 4.2. We need to find the ODE satisfied by \( h_t^1(s) \). Note that by (3.7)–(3.8), we see that \( u_\alpha(t,s) = (g_t^\dagger(s), h_t^1(s)) \) satisfies the ODE

\[
\frac{d}{ds} u_\alpha(t,s) = -\mathcal{D}_t u_\alpha(t,s), \quad \text{with} \quad \mathcal{D}_t(s) = \int_0^t Ad_{F_A(\gamma_{t,s})} \gamma^* \Omega^A(t,s) \left( \frac{\partial}{\partial \tau}, \frac{\partial}{\partial s} \right) d\tau,
\]

where \( \Omega^A \) is the curvature of the \( g \times h \)-valued connection \( \mathfrak{A} \), i.e.

\[
\Omega^A = d\mathfrak{A} + \mathfrak{A} \wedge \mathfrak{A} = (dA, d\varphi) + (A, \varphi) \wedge (A, \varphi) = (dA + A \wedge A, d\varphi + A \triangleright \varphi - \varphi \wedge \varphi),
\]

by using (2.11) and the definition of wedges in (2.22)–(2.23). Then we can write

\[
\Omega^A = (\alpha(B), Y) \quad \text{with} \quad Y_p = B_p - g_p \triangleright B_p,
\]

at a point \( p \), by the 2-gauge-transformations (1.2).

If we write \( \mathcal{D}_t(s) := (\mathcal{D}_t^g(s), \mathcal{D}_t^h(s)) \in \mathfrak{g} \times \mathfrak{h} \), (4.20) implies that

\[
\frac{d}{ds} \left( g_t^\dagger(s), h_t^1(s) \right) = - (\mathcal{D}_t^g(s), \mathcal{D}_t^h(s))(g_t^\dagger(s), h_t^1(s)) = - (\mathcal{D}_t^g(s)g_t^\dagger(s), \mathcal{D}_t^g(s) \triangleright h_t^1(s) + h_t^1(s)\mathcal{D}_t^h(s))
\]

by using (2.10), i.e. we have

\[
\frac{d}{ds} g_t^\dagger(s) = -\mathcal{D}_t^g(s)g_t^\dagger(s),
\]

\[
\frac{d}{ds} h_t^1(s) = -\mathcal{D}_t^g(s) \triangleright h_t^1(s) - h_t^1(s)\mathcal{D}_t^h(s).
\]

The second equation is ODE for \( h_t^1(s) \) if we know \( \mathcal{D}_t \). To calculate \( \mathcal{D}_t \), note that

\[
F_A(\gamma_{t,s}) = (F_A(\gamma_{t,s}^-), h(\gamma_{t,s}^-))
\]

by (4.15) and that it follows from Lemma 2.1 that for any \( G \times H \)-valued function \( (\tilde{g}, \tilde{h}) \),

\[
Ad_{(\tilde{g}, \tilde{h})} \Omega^A = Ad_{(\tilde{g}, \tilde{h})}(\alpha(B), Y) = \left( Ad_{\tilde{g}} \alpha(B), \alpha(\tilde{g} \triangleright B) \triangleright \tilde{h}^{-1} \cdot \tilde{h} + Ad_{\tilde{h}^{-1}}(\tilde{g} \triangleright Y) \right)
\]

\[
= (\alpha(\tilde{g} \triangleright B), \tilde{g} \triangleright (B - \tilde{h}^{-1} \cdot \tilde{g} \triangleright B \cdot \tilde{h} + Ad_{\tilde{h}^{-1}}(\tilde{g} \triangleright Y))
\]

\[
= (\alpha(\tilde{g} \triangleright B), \tilde{g} \triangleright B + Ad_{\tilde{h}^{-1}}(\tilde{g} \triangleright (Y - B)))]
\]

\[
= (\alpha(\tilde{g} \triangleright B), \tilde{g} \triangleright B - Ad_{\tilde{h}^{-1}}(\tilde{g} \triangleright g_p) \triangleright B_p)
\]

\[
(4.24)
\]
at point \( p = \gamma_{\tau,s} \), by 2-gauge-transformations (1.2), (1.21) and
\[
\alpha(\tilde{g} \triangleright B) \triangleright \tilde{h}^{-1} = \tilde{g} \triangleright B \cdot \tilde{h}^{-1} - \tilde{h}^{-1} \cdot \tilde{g} \triangleright B.
\]
Apply (4.23)-(4.24) to \( \mathcal{D}_t \) in (4.20) to get
\[
\mathcal{D}_t(s) = \int_0^t \text{Ad}(F_\alpha(\gamma_{\tau,s}) \triangleright h(\gamma_{\tau,s}))^* \Omega^e_{(\tau,s)} \left( \frac{\partial}{\partial \tau} \cdot \frac{\partial}{\partial s} \right) d\tau
\]
\[
= \int_0^t \left( \alpha(F_\alpha(\gamma_{\tau,s}) \triangleright \gamma^* B), F_\alpha(\gamma_{\tau,s}) \triangleright \gamma^* B \right.
\]
\[
- \text{Ad}_{h(\gamma_{\tau,s})}^{-1} \left[ (F_\alpha(\gamma_{\tau,s}) g(\gamma_{\tau,s}) \triangleright \gamma^* B') \right] d\tau,
\]
(here 2-forms \( \gamma^* B \) and \( \gamma^* B' \) take value at \((\frac{\partial}{\partial \tau}, \frac{\partial}{\partial s})\)). Consequently, we see that
\[
\mathcal{D}_t^e(s) = \int_0^t \alpha \left( F_\alpha(\gamma_{\tau,s}) \triangleright \gamma^* B \right) \left( \frac{\partial}{\partial \tau} \cdot \frac{\partial}{\partial s} \right) d\tau = \alpha(\mathcal{B}_t(s)) = \mathcal{B}_t(s),
\]
\[
\mathcal{D}_t^a(s) = \mathcal{B}_t(s) - \int_0^t \text{Ad}_{h(\gamma_{\tau,s})}^{-1} \left[ (F_\alpha(\gamma_{\tau,s}) g(\gamma_{\tau,s}) \triangleright \gamma^* B') \right] d\tau.
\]
Now apply (4.20) to (4.22) to get the ODE satisfied by \( h^i_1(s) \):
\[
\frac{dh^i_1(s)}{ds} = -\mathcal{B}_t(s) \triangleright h^i_1(s) - h^i_1(s)
\]
\[
\cdot \left\{ \mathcal{B}_t(s) - \int_0^t \text{Ad}_{h(\gamma_{\tau,s})}^{-1} \left[ (F_\alpha(\gamma_{\tau,s}) g(\gamma_{\tau,s}) \triangleright \gamma^* B') \right] d\tau \right\}.
\]
This integrant can be simplified to be
\[
\text{Ad}_{h(\gamma_{\tau,s})}^{-1} [(F_\alpha(\gamma_{\tau,s}) g(\gamma_{\tau,s}) \triangleright \gamma^* B')_{(\tau,s)}] = [\alpha(h(\gamma_{\tau,s})^{-1}) F_\alpha(\gamma_{\tau,s}) g(\gamma_{\tau,s}) \triangleright \gamma^* B']_{(\tau,s)}
\]
\[
= [g(\gamma_{0,0}) F_\alpha'(\gamma_{\tau,s})] \triangleright \gamma^* B'_{(\tau,s)};
\]
by the target-matching condition. At last differentiate (4.19) with respect to \( s \) and use the ODE (4.27) satisfied by \( h^i_1(s) \) and (4.28) to get
\[
\frac{d}{ds} F(s) = H_{A,B}(t,s) \mathcal{B}_t(s) h^i_1(s) - H_{A,B}(t,s) \alpha(\mathcal{B}_t(s)) \triangleright h^i_1(s)
\]
\[
- H_{A,B}(t,s) h^i_1(s) \left\{ \mathcal{B}_t(s) - \int_0^t \left[ g(\gamma_{0,0}) F_\alpha'(\gamma_{\tau,s}) \triangleright \gamma^* B' \left( \frac{\partial}{\partial \tau} \cdot \frac{\partial}{\partial s} \right) \right] d\tau \right\}
\]
\[
= H_{A,B}(t,s) h^i_1(s) \cdot g(\gamma_{0,0}) \triangleright \mathcal{B}_t(s) = F(s) \cdot g(\gamma_{0,0}) \triangleright \mathcal{B}_t(s)
\]
by
\[ \alpha(\mathcal{B}(s)) \triangleright h_t^1(s) = \mathcal{B}(s) h_t^1(s) - h_t^1(s) \mathcal{B}(s), \]
\[ \mathcal{B}(s) = \int_0^s F_A'(\gamma_{\tau,s}) \triangleright \gamma^* B'_{\tau,s} \left( \frac{\partial}{\partial \tau}, \frac{\partial}{\partial s} \right) d\tau. \]

Now \( F(s) \) and \( g(\gamma_{0:0}) \triangleright H_{A',B'}(s) \) satisfy the same ODE with the same initial condition. So they must be identical. \( \Box \)

4.3. The compatibility cylinder of transition 2-arrows. For a Lipschitzian curve \( \rho : [a, b] \rightarrow U_i \cap U_j \), define \( \psi_{ij}(\rho_{[a,b]}) \) to be the \( H \)-element of the 2-gauge-transformation along the curve \( \rho \) (with \( \varphi \) replaced by \( a_{ij} \) in (3.11)),

constructed from the the 2-gauge-transformation \( (g_{ij}, a_{ij}) \). Namely, it is the unique solution to the ODE
\[ \frac{d}{dt} \psi_{ij}(\rho_{[a,t]}) = F_A_i(\rho_{[a,t]}) \triangleright \gamma^* a_{ij} \left( \frac{\partial}{\partial t} \right) \psi_{ij}(\rho_{[a,t]}), \]
with initial condition \( 1_H \). We call \( \Psi_{ij}(\rho) := (F_A_i(\rho) g_{ij}(\rho(b)), \psi_{ij}(\rho)) \) the transition 2-arrow along the path \( \rho \).

**Proposition 4.2.** Let \( \rho \) be as above, \( x = \rho(a) \) and \( y = \rho(t) \). If the 2-gauge-transformation \( (g_{ij}, a_{ij}) \) satisfies the compatibility condition (1.5), then \( \psi_{ij}(\rho) \) satisfies
\[ g_{ij}(x) \triangleright \psi_{jk}(\rho) = \psi_{ij}^{-1}(\rho) \cdot F_A_i(\rho) \triangleright f_{ijk}(y) \psi_{ik}(\rho) f_{ijk}^{-1}(x). \]
i.e., the following cylinder

(4.31)

The cylinder \( C_{ijk} \)
is commutative. The front face represents the transition 2-arrow given by \( \psi_{ik}(\rho) \).
Remark 4.4. Follows.

Then,

\[ \mu(t) := \psi_{ij}^{-1}(t) \cdot g_i(t) \triangleright f_{ijk}(t) \cdot \psi_k(t) \cdot f_{ij}^{-1}(x). \]

Proof. Denote \( \psi_{ij}(t) := \psi_{ij}(\rho_{[a,b]}), g_i(t) := F_{Ai}(\rho_{[a,b]}), g_{ij}(t) := g_{ij}(\rho(t)) \) and \( y := \rho(t) \). Set

\[
\mu(t) := \psi_{ij}^{-1}(t) \cdot g_i(t) \triangleright f_{ijk}(t) \cdot \psi_k(t) \cdot f_{ij}^{-1}(x).
\]

Then,

\[
\mu'(t) = \psi_{ij}^{-1}(t)g_i(t) \triangleright \left[ - \gamma^*a_{ij} \left( \frac{\partial}{\partial t} \right) f_{ijk}(t) + \gamma^*A_i \left( \frac{\partial}{\partial t} \right) f_{ijk}(t) + f_{ijk}'(t) \right. \\
\left. + f_{ijk}(t)\gamma^*a_{ik} \left( \frac{\partial}{\partial t} \right) \psi_k(t)f_{ij}^{-1}(x) \right]
\]

by using the equation (4.29) satisfied by \( \psi_{ij}(t) \), the compatibility condition (1.5) and the target-matching condition (3.12). This is the same ODE satisfied by \( g_{ij}(x) \triangleright \psi_k(t) \cdot f_{ij}^{-1}(x) \). The result follows.

Remark 4.4. (1) Differentiating (4.30) with respect to \( t \) at \( t = a \), we get the compatibility condition (1.5).

(2) The gauge transformation (1.4), if \( \varphi \) is replaced by \(-\varphi\), coincides with that in proposition 3.10 of [20], but with primed and unprimed terms interchanged.

(3) The union of any 3 compatibility cylinders \( C_{ijk}, C_{jkl} \) and \( C_{ijl} \) as in (4.31) over the intersection \( U_i \cap U_j \cap U_k \cap U_l \) gives us the 4-th compatibility cylinder \( C_{ikl} \) by their commutativity and commutative tetrahedra (5.3). Hence, the 4 compatibility conditions (4.30) over this intersection are consistent, and so are their differentiations (1.5).

5. The global 2-holonomy

5.1. The invariance of the global 2-holonomies under the change of coordinate charts.

The 2-cocycle condition (1.4) implies that

\[
\int_{ikj} f_{ij}^{-1} f_{ij}^{-1} = f_{ik}^{-1} f_{ikj}
\]

by permutation \((i, j, k, l) \rightarrow (l, i, k, j)\), which corresponds to the following diagrams:

\[
\text{(5.2)}
\]
namely, the following tetrahedron is commutative.

\[
\begin{array}{c}
\text{(5.3)} \\
\end{array}
\]

The tetrahedron \( T_{ijk}^b \)

 Fix a rectangle \( \square_{ab} \) such that \( \gamma(\square_{ab}) \subset U_k \). Suppose that the image \( \gamma(\square_{ab}) \) is also contained in the coordinate chart \( U_q \). Let us show that the global 2-holonomy is invariant if we use the 2-connection \((A_q, B_q)\) on \( U_q \) instead of the 2-connection \((A_k, B_k)\) over \( U_k \), when calculating the local 2-holonomy for \( \gamma|_{\square_{ab}} \). Now consider 9 adjacent rectangles in the above. By our construction, the corresponding 2-holonomy is represented by the following diagram:

\[
\begin{array}{c}
\text{(5.4)} \\
\end{array}
\]
where \( H_\alpha := H_{A_{\alpha}, B_{\alpha}}(\gamma^{(\alpha)}) \). We do not draw the \( H \)-elements of corresponding to \( \gamma^{(l')}, \gamma^{(k')}, \gamma^{(p')} \). Now consider the eight rectangles adjacent to \( H_k \) in \((5.1)-(5.2)\). We apply the 2-cocycle condition \((5.1)-(5.2)\) to change two rectangles (corresponding to the dotted ones in the following diagram) to get the following diagram (we denote \( \gamma^j := \gamma^{(k)j}, \# = u, d, l, r \)):

\[
\begin{array}{c}
g_{P_{ik}}(\gamma) & F_{A_{ik}}(\gamma^u) & g_{P_{ik}'}(\gamma^u) \\
g_{P_{kj}}(\gamma) & F_{A_{kj}}(\gamma^d) & F_{A_{kj}'}(\gamma^d) \\
F_{A_{ik}}(\gamma^u) & F_{A_{kj}}(\gamma^d) & H_k \\
g_{P_{ij}}(\gamma) & F_{A_{ij}}(\gamma^d) & F_{A_{ij}'}(\gamma^d) \\
g_{P_{jm}}(\gamma) & F_{A_{jm}}(\gamma^u) & g_{P_{jm}'}(\gamma^u) \\
\end{array}
\]

\((5.5)\)

If we use the local 2-connection \((A_p, B_q)\) over the coordinate chart \( U_q \) instead of the local 2-connection \((A_k, B_k)\) over the coordinate chart \( U_k \), we claim that the 2-holonomies are the same.

\((5.6)\)
Namely in (5.6) the 2-arrow in \( \mathcal{G} \) represented by the bottom 2-cells (i.e. diagram (5.5)) is the same as the 2-arrow represented by the upper 2-cells, which is the same as the diagram (5.5) with subscript \( k \) replaced by \( q \). In (5.6) there is only one cube

\[
\begin{array}{c}
\text{(5.7)}
\end{array}
\]

which is the commutative cube (4.6) of the 2-gauge transformation from the local 2-holonomy \( H_{A_q,B_q} \) to \( H_{A_k,B_k} \) by Proposition 4.1, where

\[
g_2 = g_{qk}(y_2), \quad g_3 = g_{qk}(y_3), \quad g'_2 = g_{qk}(z_2), \quad g'_3 = g_{qk}(z_3),
\]

and the front face represents the 2-arrows given by \( \psi_{qk}(\gamma^d) \). There are four compatibility cylinders in (5.6)

\[
\begin{array}{c}
\text{(5.8)}
\end{array}
\]

which are commutative by Proposition 4.2. Here the front face of the first cylinder represents the 2-arrow given by \( \psi_{qj}(\gamma^d) \) and the front triangle of the second cylinder represents the 2-arrow...
given by \( f_{qkp}^{-1}(y_3) \). There are four 2-cocycle tetrahedra in (5.6)

![Diagram](image_url)

which are commutative by the 2-cocycle condition (5.3) at points \( y_2, y_3, z_2, z_3 \), respectively (the front triangle of the second tetrahedron represents the 2-arrows given by \( f_{qkp}^{-1}(y_3) \)). The commutativity of a cube, a cylinder or a tetrahedron means that the bottom 2-arrow is equal to the composition of the remaining 2-arrows. By (5.7)–(5.9), it is easy to see that 2-arrows represented by vertical 2-cells in (5.6) appear twice and in reverse directions, and so they are cancelled. Hence, 2-arrows represented by the upper and bottom 2-cells in (5.6) must coincide.

If a rectangle \( \square_{a0} \) is contained in \( U_k \), which is adjacent to the upper boundary of \([0,1]^2\), and the local 2-connection \( (A_k, B_k) \) over the open set \( U_k \) is replaced by the local 2-connection \( (A_q, B_q) \) over the open set \( U_q \), we have the following commutative 3-cells:

![Diagram](image_url)

In this case, we have an extra 2-arrow:

![Diagram](image_url)
whose $H$-element is denoted by $h_0$. Meanwhile, $\Box_{aM}$ is in the same open set $U_k$, which is adjacent to the lower boundary of $[0,1]^2$. When the local 2-connection $A_k$, $B_k$ over the open set $U_k$ is replaced by the local 2-connection $A_q$, $B_q$ over the open set $U_q$, we have the following 3-cells:

$$\gamma$$

(5.12)

with an extra 2-arrow represented by the front 2-cells, which is the inverse of the 2-arrow in (5.11). Its $H$-element is $h_0^{-1}$. Thus after $U_k$ replaced by $U_q$, $\text{Hol}_\gamma$ is changed to

$$g_1 \triangleright h_0 \cdot \text{Hol}_\gamma \cdot g_2 \triangleright h_0^{-1} \sim h_0 \cdot \text{Hol}_\gamma \cdot h_0^{-1} = \text{Ad}_{h_0} \text{Hol}_\gamma = \alpha (h_0) \triangleright \text{Hol}_\gamma \sim \text{Hol}_\gamma$$

in $H/[G,H]$, for some $g_1, g_2 \in G$. Here $g_1 \triangleright$s represent whiskering by some 1-arrows.

If a mapping $\gamma : \Box_{ab} \longrightarrow U_\alpha$ is divided into four 4 adjacent rectangles $\gamma^{(i)}, \gamma^{(j)}, \gamma^{(k)}$ and $\gamma^{(l)}$ as in (1.13). We have a local 2-holonomy associated to each small rectangle in $U_\alpha$. The local 2-holonomy $\text{Hol}(\gamma|_{\Box_{ab}})$ is the composition of four local 2-holonomies $\text{Hol}(\gamma^{(\alpha)})$’s by using composition formulae (4.3)-(4.4) in Lemma 4.3. So $\text{Hol}_\gamma$ is invariant under the refinement of a division. For any two different divisions of the square $[0,1]^2$, we can refine them to get a common refinement. Therefore $\text{Hol}_\gamma$ is independent of the division we choose.

If we choose another coordinate charts $\{U'_i\}$, then $\{U_i \cup U'_i\}$ are also coordinates charts. By the above result, the global 2-holonomy constructed by coordinate charts $\{U_i\}$ is the same as that by $\{U_i \cup U'_i\}$. So it is the same as that constructed by $\{U'_i\}$.

5.2. **The independence of reparametrization.** We sketch the proof. A loop $\gamma$ in the loop space $LM$ is given by a family of loops $\gamma_s : [0,1] \rightarrow M$ with $\gamma_s(0) = \gamma_s(1)$, for $s \in [0,1]$, and $\gamma_0 \equiv \gamma_1$. A reparametrization of such a loop is given by a mapping

$$\Xi : [0,1]^2 \rightarrow [0,1]^2, \quad (t', s') \mapsto (\alpha(t', s'), \beta(s'))$$

We must have

$$\frac{d\beta}{ds}(s') > 0, \quad \frac{\partial \alpha}{\partial t'}(t', s') > 0,$$

since $\Xi$ must map a loop to a loop. Here we assume first that the starting points of loops $\gamma_s$ are fixed for each $s$. Namely, $\Xi$ maps the left and right boundaries of $[0,1]^2$ to themselves.

Let $[0,1]^2$ be divided into rectangles $\Box_{ab}$’s. The pull back quadrilateral $\Box_{ab} : = \Xi^* \Box_{ab}$ may have curved left and right boundaries, but its upper and lower boundaries must be straight.

(5.14)
where $\square_1 := \tilde{\square}_{ab}$ and $\square_2 := \tilde{\square}_{(a+1)b}$. Denote the composition $\tilde{\gamma} := \gamma \circ \Xi$. If $[0,1]^2$ is divided into sufficiently small rectangles $\square_{ab}$’s, we can assume the left and right boundaries of $\tilde{\square}_{ab}$ are describe by functions $t' = \kappa_j(s'), j = 1, 2$, such that $\kappa_j$ is monotonic function of $s'$ by \((5.13)\). Then we have

$$\tilde{\square}_{ab} := \{(s', t'); s' \in (s_a', s_b'), t' \in (\kappa_1(s'), \kappa_2(s'))\}.$$ 

For $\gamma|\square_1 : \square_1 \to U_i$, we have the $H$-element of local 2-holonomy $H_{A,B}^\gamma|\square_1 (s), s \in (s_a, s_b)$, satisfying ODE \((4.1)\). Define $H_{A,B}(s') = H_{A,B}^\gamma|\square_1 (\beta(s')), s' \in (s_a', s_b')$. Then it directly follows from the ODE satisfied by $H_{A,B}^\gamma|\square_1 (s)$ that

$$\frac{d}{ds'} \tilde{H}_{A,B}(s') = \tilde{H}_{A,B}(s')\tilde{\mathcal{B}}(s'), \tag{5.15}$$

by changing variables, where

$$\tilde{\mathcal{B}}(s') := \int_{\kappa_1(s')}^{\kappa_2(s')} F_A(\gamma_{s',t'}; s') \mathcal{B}(\gamma_{s',t'}; s') \left( \frac{\partial}{\partial \tau}, \frac{\partial}{\partial s'} \right) d\tau', \tag{5.16}$$

$\gamma_{s',t'}$ is defined similarly, and the pull back of 1-holonomy is well defined. Namely, we have

$$\frac{d}{dt'} F_A(\gamma_{[s_1(t'),t']}; s') = F_A(\gamma_{[s_1(t'),t']}; s') \gamma^* A \left( \frac{\partial}{\partial t'} \right),$$

and

$$\frac{d}{ds'} F_A(\gamma_{s'}^l) = F_A(\gamma_{s'}^l) \gamma^* A (X_{s'}),$$

where $\gamma_{s'}^l$ is the restriction of $\gamma$ to the curved left boundary $\partial_s \square_1$ of the quadrilateral $\square_1$, and $X_{s'} = \kappa_1(s') \partial_{t'} + \partial_s$ is its tangential vector. The above equations imply that

$$H_{A,B}^\gamma|\square_1 (s') = \tilde{H}_{A,B}(s').$$

So it is sufficient to show that we can use the pull back quadrilaterals $\tilde{\square}_{ab}$’s instead of rectangles to calculate the global 2-holonomy of $\tilde{\gamma}$.

Suppose that the images of $\tilde{\gamma}$ over $\square_1$ and $\square_2$ are in the same coordinate chart $U_i$. Exactly as Lemma \((4.1)\) by using the ODE \((5.15)-(5.16)\) satisfied by $H_{A,B}$, we can prove the curved quadrilateral version of composition formulae for local 2-holonomies, similar to \((4.3)\).

\begin{equation}
\text{ Hol} \left( \tilde{\gamma}_1 \right) \#_1 \text{ Hol} \left( \tilde{\gamma}_2 \right) = \text{ Hol} \left( \tilde{\gamma}_1 \right), \tag{5.17}
\end{equation}

namely, we have

$$\text{ Hol} \left( \tilde{\gamma}_1 \square_1 \right) \#_1 \text{ Hol} \left( \tilde{\gamma}_1 \square_1 \right) = \text{ Hol} \left( \tilde{\gamma}_1 \square_1 \right),$$

$$\text{ Hol} \left( \tilde{\gamma}_2 \square_2 \right) \#_1 \text{ Hol} \left( \tilde{\gamma}_2 \right) = \text{ Hol} \left( \tilde{\gamma}_2 \right) \#_1 \text{ Hol} \left( \tilde{\gamma}_2 \right). \tag{5.18}$$

Here we omit the whiskering parts. Thus we can use $\tilde{\square}_1$ and $\tilde{\square}_1 \cup \tilde{\square}_2$ to calculate 2-holonomy, whose common boundary is straight.
Now suppose the images of of $\tilde{\gamma}$ over $\tilde{\square}_1$ and $\tilde{\square}_2$ are in different coordinate charts $U_i$ and $U_j$, respectively. We have to add a transition 2-arrow $\Psi_{ij}(\partial_t \tilde{\square}_1)$. Note that the transition 2-arrow along the interval $\partial_t \tilde{\square}_1$ in (5.14) under the map $\gamma$ satisfies

$$\frac{d}{ds}\psi_{ij}(s) = F_A(\gamma'(s)) \triangleright \gamma^*a_{ij}(\frac{\partial}{\partial s})\psi_{ij}(s),$$

where $\gamma'(s)$ is the restriction of $\gamma$ to the right boundary $\partial_t \tilde{\square}_1$ of $\square_1$. By pulling back $\Xi$, we get $\tilde{\psi}_{ij}(s') := \psi_{ij}(\beta(s'))$ satisfying

$$\frac{d}{ds}\tilde{\psi}_{ij}(s') = F_A(\tilde{\gamma}'(s')) \triangleright \tilde{\gamma}^*a_{ij}(Y_{s'}) \cdot \tilde{\psi}_{ij}(s')$$

where $\tilde{\gamma}'(s')$ is the restriction of $\tilde{\gamma}$ to the curved right boundary $\partial_t \tilde{\square}_1$ of $\tilde{\square}_1$, and $Y_{s'} = k'(s')\partial_{s'} + \partial_{\nu'}$ is its tangential vector. Let $\Psi_{ij}(\partial_t \tilde{\square}_1)$ be the 2-arrows given by $\tilde{\psi}_{ij}(\beta(s'))$.

We claim that

$$\Psi_{ij}^{-1}(\partial_t \tilde{\square}_1) \#_1\text{Hol} (\tilde{\gamma}_{ij}|\tilde{\square}_1) = \text{Hol} (\tilde{\gamma}_{ij}|\tilde{\square}_1') \#_1\Psi_{ij}^{-1}(\partial_t \tilde{\square}_1') \#_1\text{Hol} (\tilde{\gamma}_{ij}|\tilde{\square}_1'),$$

i.e. the composition of left two 2-arrows in (5.20) is equal to the composition of the following three 2-arrows

$$\Psi_{ij}^{-1}(\partial_t \tilde{\square}_1) \#_1\text{Hol} (\tilde{\gamma}_{ij}|\tilde{\square}_1') = \text{Hol} (\tilde{\gamma}_{ij}|\tilde{\square}_1') \#_1\Psi_{ij}^{-1}(\partial_t \tilde{\square}_1') \#_1\text{Hol} (\tilde{\gamma}_{ij}|\tilde{\square}_1'),$$

where $h'' := \text{Hol} (\tilde{\gamma}_{ij}|\tilde{\square}_1')$. The transition 2-arrow along $\partial_t \tilde{\square}_1$ in (5.20) is replaced by the transition 2-arrow along the straight interval $\partial_t \tilde{\square}_1'$ in (5.22). To prove this claim, we divide $\tilde{\square}_1'$ in (5.17) repeatedly to get the diagram

$$\text{Hol} (\tilde{\gamma}_{ij}|\tilde{\square}_1') \#_1\Psi_{ij}^{-1}(\partial_t \tilde{\square}_1) \#_1\text{Hol} (\tilde{\gamma}_{ij}|\tilde{\square}_1')$$

where $\tilde{\square}_1^{(4)}$ is the part between the dotted path and the left boundary $\partial_t \tilde{\square}_2$. To prove the claim (5.21), note that we can use (5.6) to replace the local 2-holonomy of small rectangles in $\tilde{\square}_1^{(3)}$ for 2-connection over $U_i$ instead of 2-connection over $U_j$. So we have

$$\text{RHS of (5.21)} = \text{Hol} (\tilde{\gamma}_{ij}|\tilde{\square}_1^{(4)}) \#_1\Psi_{ij}^{-1}(\partial_t \tilde{\square}_1^{(4)}) \#_1\text{Hol} (\tilde{\gamma}_{ij}|\tilde{\square}_1^{(3)}) \#_1\text{Hol} (\tilde{\gamma}_{ij}|\tilde{\square}_1')$$
corresponding to the diagram

Note that the dotted path $\partial_t \bar{\square}_1^{(4)}$ in (5.23) converges to the curved path $\partial_t \bar{\square}_2$ if we divide $\bar{\square}_1^n$ repeatedly in (5.17). So the transition 2-arrow $\Psi_{ij} \left( \partial_t \bar{\square}_1^{(4)} \right)$ converges to $\Psi_{ij} \left( \partial_t \bar{\square}_2 \right)$, meanwhile $\text{Hol} \left( \bar{\gamma}_j | \bar{\square}_1^{(4)} \right)$ converges to the identity. So the left-hand side of (5.21) converges to the left-hand side of (5.24). The claim is proved. In summary, in our algorithm to calculate the global 2-holonomy of the mapping $\bar{\gamma}$, we can use the pull back quadrilaterals $\bar{\square}_{ab}$’s instead of rectangles, and consequently, $\text{Hol} \left( \bar{\gamma} \right) = \text{Hol} \left( \gamma \right)$.

If $\Xi$ does not fix the starting points of loops $\gamma$, then $\Xi : [0, 1]^2 \to \square \cup \square''$ in the following diagram:

$\text{Hol} \left( \bar{\gamma} | \square'' \right)$ can be replaced by $\text{Hol} \left( \bar{\gamma} | \square \right)$ in the expression of $\text{Hol} \left( \bar{\gamma} \right)$ by conjugacy. We omit the details.

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