Anomalous Hall effect and Berry phase in two-dimensional magnetic structures

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Abstract. We discuss the topological character of the anomalous Hall effect in two different models of the electron energy spectrum. The essential property of each of these models is the nontrivial topology of electron energy bands in the momentum space. Our method is based on a mapping of the Hamiltonian, which can be effectively parameterized by the unit-vector field, onto the two-dimensional momentum space.

1. Introduction
The anomalous Hall effect (AHE) consists in appearance of a Hall voltage perpendicular to the electric current in a ferromagnet. The effect has been known for a long time but it attracted a lot of attention recently \cite{1, 2}. This is mostly related to its application in modern magnetoelectronics because the effect can be used to measure the magnitude and direction of the magnetization in layered magnetic structures \cite{3}. Several different mechanisms have been proposed so far to explain the physics of AHE \cite{1, 4, 5, 6}. However the discussions of these mechanisms are not finished yet \cite{7, 8}. One of proposed explanations – the so called intrinsic mechanism of AHE \cite{9} – is related to the topology of electron energy bands in the momentum space \cite{5, 6, 10}.

Here we discuss the topological origin of the intrinsic AHE presenting the off-diagonal conductivity in a form similar to the topological charge, which is not integer but depends continuously on the Fermi level position in metallic systems. It should be noted that we do not discuss here the effect of impurities on the intrinsic AHE. Thus, our consideration concerns the clean systems like, for example, ballistic nanostructures of characteristic size smaller than the mean free path of electrons.

The topological character and the quantization of AHE makes it of interest for possible application in quantum computations as a flux qubit. It is mostly related to the stability of such qubits in varous dephasing enviroments \cite{11}.
2. Two-dimensional electron gas with Rashba spin-orbit interaction

We consider first the model of two-dimensional electron gas (2DEG) in the $x − y$ plane under a homogeneous magnetization directed along the axis $z$, with the spin-orbit interaction described by the Rashba term [12]. The Hamiltonian of this model is

$$H = \varepsilon_k + \alpha (\sigma_x k_y - \sigma_y k_x) - M \sigma_z,$$

where $\varepsilon_k = k^2/2m$, $\sigma_\mu$ are the Pauli matrices, $\alpha$ is the coupling constant of spin-orbit interaction, $M$ is the spin splitting due to the magnetization field, and we take units with $\hbar = 1$. The energy spectrum of this Hamiltonian consists of two subbands labelled by $+\varepsilon$ and $-\varepsilon$ with dispersion relation $E_{k,\pm} = \varepsilon_k + \lambda(k)$, where we denoted $\lambda(k) = (M^2 + \alpha^2 k^2)^{1/2}$.

Let us introduce a 3D unit vector $\mathbf{n}(k)$ at each point of two-dimensional momentum plane

$$\mathbf{n}(k) = \left( \frac{\alpha k_y}{\lambda(k)}, - \frac{\alpha k_x}{\lambda(k)}, - \frac{M}{\lambda(k)} \right).$$

By using the $\mathbf{n}$ field (2), we can parameterize the manifold of $2 \times 2$ Hermitian matrices corresponding to the Hamiltonian [13]

$$H = \varepsilon_k + \lambda(k) \mathbf{\sigma} \cdot \mathbf{n}(k).$$

At $k = 0$, the vector $\mathbf{n}$ is perpendicular to the $\mathbf{k}$ plane, whereas for large $k \gg M/\alpha$, it lies in this plane and is oriented perpendicular to the momentum $\mathbf{k}$. This vector field corresponds to the topological excitation (vortex) in momentum space. The dependence $\mathbf{n}(k)$ is a mapping of the $\mathbf{k}$ plane to the unit sphere $S_2$, and the total $\mathbf{k}$ plane maps onto the lower hemisphere of $S_2$.

Due to the spin-orbit interaction, the components of velocity operators, $v_i = \partial H/\partial k_i$, are matrices in the spin space, and they can be presented using the $\mathbf{n}$ field

$$v_x = \left( \frac{\partial \lambda(k)}{\partial k_x} n_\mu + \lambda(k) \frac{\partial n_\mu}{\partial k_x} \right) \sigma_\mu, \quad v_y = \left( \frac{\partial \lambda(k)}{\partial k_y} n_\mu + \lambda(k) \frac{\partial n_\mu}{\partial k_y} \right) \sigma_\mu.$$

The off-diagonal conductivity can be calculated using the standard Kubo formula of the linear response to the external electric field [14]

$$\sigma_{xy} = e^2 \lim_{\omega \to 0} \frac{1}{\omega} \text{Tr} \int \frac{d\varepsilon}{2\pi} \frac{d^2 k}{(2\pi)^2} v_x G_k(\varepsilon + \omega) v_y G_k(\varepsilon),$$

where $G_k(\varepsilon)$ is the causal Green’s function of free electrons and the trace runs over the spin states.

In the absence of impurity scattering, the electron Green function can be found using the Hamiltonian (3)

$$G_k(\varepsilon) = \frac{\varepsilon - \varepsilon_k + \mu - \lambda(k) \mathbf{\sigma} \cdot \mathbf{n}(k)}{(\varepsilon - E_{k,+} + \mu + i\delta \text{sgn} \varepsilon)(\varepsilon - E_{k,-} + \mu + i\delta \text{sgn} \varepsilon)}$$

where $\mu$ is the chemical potential.

We consider a general case when the chemical potential $\mu$ is situated in both subbands, which corresponds to the choice $\mu > M$. When only the lower-in-energy subband $E_{k,+}$ is filled with electrons, $-M < \mu < M$, only the contribution of the filled subband, $E_{k,+}$, should be kept.

In the static limit of $\omega \to 0$, and after calculating the trace, we obtain from Eq. (5)

$$\sigma_{xy} = -\frac{e^2}{2} \int \frac{d^2 k}{(2\pi)^2} \left[ f(E_{k,+}) - f(E_{k,-}) \right] \epsilon_{\alpha\beta\gamma} n_\alpha \frac{\partial n_\beta}{\partial k_x} \frac{\partial n_\gamma}{\partial k_y},$$
where \( \epsilon_{\alpha\beta\gamma} \) is the unit antisymmetric tensor and \( f(\varepsilon) = [\exp(\varepsilon - \mu)/T + 1]^{-1} \) is the Fermi function in the limit of \( T \to 0 \) (strictly speaking, we consider limit \( T \to 0 \) and \( \omega/T \to 0 \)). The integral

\[
\Omega = \frac{1}{4\pi} \int d^2k \left[ f(E_{k,+}) - f(E_{k,-}) \right] \epsilon_{\alpha\beta\gamma} n_{\alpha} \frac{\partial n_{\beta}}{\partial k_x} \frac{\partial n_{\gamma}}{\partial k_y}
\]

(8)
is the spherical angle on \( S_2 \) enclosed by two contours \( L_+ \) and \( L_- \), where \( L_\pm \) are the mappings of the Fermi surfaces (circles) \( E_{k,+} = \mu \) and \( E_{k,-} = \mu \) to the sphere \( S_2 \), respectively.

The presence of two Fermi functions in Eq. (8) restricts the 2D momentum space to the ring. In the case when only one energy subband is filled with electrons we put \( f(E_{k,+}) = 0 \), so that the integration in (8) is over the disc of radius \( k_F \), where \( k_F \) is the Fermi momentum of electrons in the lower energy band. The integration over the whole momentum space would lead to the quantization of \( \sigma_{xy} \) because the corresponding \( \Omega \) is the half-integer topological charge. Thus, the presence of Fermi functions in (8) makes the \( \Omega \) depending continuously on the band filling. Note that for one-band filling (i.e., for \( -M < \mu < M \)) and in the limit of strong spin-orbit coupling, \( \alpha k_F \gg M \) we also obtain \( \Omega \approx 1/2 \).

The expression for off-diagonal conductivity can be also presented as the Berry phase of electrons propagating along the Fermi surface in momentum space \([5, 6, 10]\). If we start from the representation, in which the Green’s functions are diagonal, we can write Eq. (5) as

\[
\sigma_{xy} = e^2 \lim_{\omega \to 0} \frac{1}{\omega} \sum_{n \neq m} \int \frac{d\omega}{2\pi} \frac{d^2k}{(2\pi)^2} (v_x)_{nm} G_{k,mm}(\omega + \varepsilon) (v_y)_{nn} G_{k,nn}(\varepsilon),
\]

where \( m \) and \( n \) refer to the eigenstates of Hamiltonian (1), and \((v)_{nm}\) are the matrix elements of velocity in this representation. After integrating over energy, Eq. (9) can be presented in the following form \([5, 6, 10]\)

\[
\sigma_{xy} = e^2 \sum_{n} \int \frac{d^2k}{(2\pi)^2} f(E_{kn}) \left( \frac{\partial A_y(kn)}{\partial k_x} - \frac{\partial A_x(kn)}{\partial k_y} \right),
\]

(10)

where

\[
A_{\alpha}(kn) = -i \left\langle kn \left| \frac{\partial}{\partial k_{\alpha}} \right| kn \right\rangle
\]

(11)
is the gauge potential in the momentum space related to the transformation of the Hamiltonian \( H \) to the diagonal form. This transformation is local in the momentum space, leading to the nonvanishing gauge potential \( A_{\alpha}(kn) \).

The flux of curl of the gauge potential \( A(k, \pm) \) in Eq. (10) through the surface \( E_{k,\pm} = \mu \) can be presented as the circulation of vector \( A(k, \pm) \) along the circle (Fermi surface in 2D). In other words, the contribution of the filled states below the Fermi surface can be also presented by the integral of the gauge field over the Fermi surface. It was shown recently by Haldane \([15]\) that such a reduction of the integral in momentum space takes place in any dimensionality, in accordance with the Landau concept of the Fermi liquid stating that the transport properties are fully determined by the properties of electrons near the Fermi surface.

In the model of 2DEG with Rashba Hamiltonian we can calculate explicitly the eigenfunctions and we find the gauge potential

\[
A(k, \pm) = \left( \begin{array}{c} \alpha^2 k_y \\ 2\lambda(k) [M \pm \lambda(k)] \end{array} , \begin{array}{c} \alpha^2 k_x \\ 2\lambda(k) [M \pm \lambda(k)] \end{array} \right),
\]

(12)

and using (10) we can come again to the same result of Eq. (7).

It should be noted that (12) can be also found as a gauge potential corresponding to the local transformation of vector field (2) to the homogeneous field oriented along axis \( z \) in the momentum space (like in the case of local transformations in the real space \([16]\)).
3. Two-dimensional Dirac model

Here we consider the AHE in 2D model of electrons with Dirac energy spectrum described by Hamiltonian [17]

\[ H = v (k_x \sigma_x + k_y \sigma_y) + \Delta \sigma_z \]  

(13)

The non-zero mass \( \Delta \) opens a gap in the spectrum and thus splits the spectrum into two subbands with dispersions \( E_{k,\pm} = \mp E_k \), where \( E_k = (\Delta^2 + v^2 k^2)^{1/2} \). We assume that the Fermi level \( \mu \) lies above the gap. Like in the previous section we present the Hamiltonian as

\[ H = E_k \sigma \cdot \mathbf{n}(k) \]  

(14)

where the unit vector field \( \mathbf{n}(k) \) is defined by

\[ \mathbf{n}(k) = \left( \frac{vk_x}{E_k}, \frac{vk_y}{E_k}, \frac{\Delta}{E_k} \right) \]  

(15)

This field corresponds to the 2D hedgehog at large distances but for \( k \to 0 \) the vector \( \mathbf{n} \) is rotated toward axis \( z \). This field maps the momentum plane to the upper hemisphere of \( S_2 \).

Performing the calculation of AHE like in previous section we come to the same expression (7) for the off-diagonal conductivity. If the chemical potential is located within the gap then \( f(E_{k,+}) = 1 \) and \( f(E_{k,-}) = 0 \) and we get the exact quantization of the off-diagonal conductivity.

4. Conclusions

We demonstrated that the intrinsic mechanism of the AHE in two-dimensional models with complex energy spectrum can be related to the topological properties of the mapping of a parameter vector space \( \mathbf{n}(k) \) onto two-dimensional momentum space. The gauge field determining the Berry phase of electrons in momentum space corresponds to the local transformations of the \( \mathbf{n} \) field.

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