Subword complexity of the Fibonacci-Thue-Morse sequence: the proof of Dekking’s conjecture

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Abstract

Recently F. M. Dekking conjectured the form of the subword complexity function for the Fibonacci-Thue-Morse sequence. In this note we prove his conjecture by purely computational means, using the free software Walnut.

1 Introduction

Recall that the Fibonacci numbers $F_n$ are defined by $F_0 = 0$, $F_1 = 1$, and $F_n = F_{n-1} + F_{n-2}$.

In this note we will use the Fibonacci (or “Zeckendorf”) representation of natural numbers [6, 9]. This is a map from $\mathbb{N}$ to the set of binary strings, written $(n)_F$, such that

(a) If $(n)_F = w$, and $t = |w|$, then $n = \sum_{1 \leq i \leq t} w[i]F_{t+2-i}$,

(b) $w[1]$ (if it exists) is 1;

(c) $w$ does not have two consecutive 1’s.

The inverse map is written $[w]_F$. Thus, Fibonacci representation essentially writes a natural number as a sum of Fibonacci numbers, with no two consecutive Fibonacci numbers appearing in the sum.

The so-called Fibonacci-Thue-Morse sequence $(ftm[n])_{n \geq 0}$ is defined to be the sum, taken modulo 2, of the bits of the Fibonacci representation of $n$. See, for example, [1, Examples 7.8.2, 7.8.4] and [4], where this sequence is discussed. It is sequence A095076 in the On-Line Encyclopedia of Integer Sequences (OEIS). The first few terms of the sequence are as follows:

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The sequence \( \text{ftm} \) is Fibonacci-automatic. This means there is a deterministic finite automaton with output (DFAO) that, on input the Fibonacci representation of \( n \), computes \( \text{ftm}[n] \). The state name \( q \) and the corresponding output \( a \) is written as \( q/a \), and the output of the machine is the output of the last state reached after following the transitions corresponding to the input, starting from the initial state \( q_0 \). In fact, it is easy to see that this DFAO has 4 states, as follows:

![Diagram of Fibonacci automaton for the sequence \( \text{ftm} \)](image)

Each state is labeled with the syntax “name/output”.

The subword complexity function \( \rho_s(n) \) of a sequence \( s \) is defined to be the number of distinct length-\( n \) contiguous blocks appearing in \( s \). Recently Dekking \cite{Dekking} gave a conjecture on the form of \( \rho_{\text{ftm}}(n) \). In this note we prove this conjecture.

## 2 Dekking’s conjecture

One form of Dekking’s conjecture involves the first difference \( d(n) := \rho_{\text{ftm}}(n + 1) - \rho_{\text{ftm}}(n) \) of the subword complexity function. It is well-known, and easy to see, that \( d(n) \) counts the number of right-special factors, where a factor \( x \) is right-special if both \( x0 \) and \( x1 \) are factors of \( \text{ftm} \).

**Conjecture 1** (Dekking).

\[
(d(n))_{n \geq 0} = 1, 2, 4, 6, 10, 6, 6, 8, 8, 8, 8, 6, 6, 6, 8, 8, 8, 6, 6, 6, \ldots
\]
\[
= 1, 2, 4, 6, 10, 6^2, 8^1, 6^1, 8^2, 6^4, 8^3, 6^4, 8^5, 6^9, 8^8, 6^{12}, 8^{13} \ldots
\]
\[
= (1, 2, 4, 6, 10) \prod_{i \geq 2} 6^{F_i + (-1)^i} 8^{F_i}.
\]

Here the exponents on the numbers denote repetition factors, and the \( \prod \) symbol denotes concatenation.
3 Outline of the proof

We can prove Dekking’s conjecture via purely computational means, using a decision procedure implemented in the free software Walnut [7].

There are five steps in the proof.

1. We create a first-order logical formula specifying that the length-\(n\) factor \(\text{ftm}[i..i+n-1]\) beginning at position \(i\) is a right-special factor.

2. Using the ideas in [8], we can create a deterministic finite automaton (DFA) accepting the Fibonacci representation of the pairs \((i,n)\) such that
   (a) \(\text{ftm}[i..i+n-1]\) is a right-special factor; and
   (b) \(\text{ftm}[i..i+n-1]\) is the earliest occurrence of that factor in \(\text{ftm}\).

3. Using the ideas in [3], we can read off, from the automaton in the previous step, a linear representation for \(d(n)\). This consists of vectors \(v, w\) and a matrix-valued morphism \(\gamma\) such that \(d(n) = v\gamma((n)_F)w\) for all \(n \geq 0\). The rank of the linear representation is the size of \(v\).

4. Using the so-called “semigroup trick” [3], we can prove that the semigroup generated by the two matrices \(\gamma(0), \gamma(1)\) is finite. Furthermore, we can create a Fibonacci-DFAO computing \(d(n)\).

5. This DFAO can be minimized using the standard methods [5], and Dekking’s result can be read off the result.

4 Details of the proof

Now we provide a few more details about each step.

1. The logical formula is created in several steps. First, we need a formula asserting that the factor \(\text{ftm}[i..i+n-1]\) is equal to the factor \(\text{ftm}[j..j+n-1]\). One obvious way to write this would be
   \[
   \forall t \ (t < n) \implies \text{ftm}[i + t] = \text{ftm}[j + t].
   \] (1)

However, for computational purposes, it turns out to be more efficient to formulate this assertion somewhat differently:

   \[
   \text{ftmfactoreq}(i, n) := \forall t, u \ (t \geq i \land t < i + n \land i + u = t + j) \implies \text{ftm}[t] = \text{ftm}[u].
   \]

It is easy to see this is logically equivalent to (1).

Next, we create a formula asserting that \(\text{ftm}[i..i+n-1]\) is right-special:

   \[
   \text{isftmrs}(i, n) := \exists j, k \ \text{ftmfactoreq}(i, j, n) \land \text{ftmfactoreq}(i, k, n) \land \text{ftm}[j+n] \neq \text{ftm}[k+n].
   \]
Finally, we create a formula asserting that \( \text{ftm}[i..i + n − 1] \) is right-special and also is the first occurrence of that factor:

\[
\text{ftmrsn}(i, n) := \text{isftmr}(i, n) \land \forall j \ (j < i) \implies \neg \text{ftmfactoreq}(i, j, n).
\]

2. We can translate the first-order formulas above into automata using the Walnut system. The created automata recognize the Fibonacci representations of the values of the free variables in the formulas that make the formula true. We do this with the following Walnut commands, which are more or less verbatim translations of the predicates above.

```python
def ftmfactoreq "?msd_fib At,u (t>=i & t<i+n & i+u=t+j) => FTM[t]=FTM[u]":
def isftmrs "?msd_fib Ej,k $ftmfactoreq(i,j,n) & $ftmfactoreq(i,k,n)
  & FTM[j+n] != FTM[k+n]":
def ftmrsn "?msd_fib $isftmrs(i,n) & Aj (j<i) => ~$ftmfactoreq(i,j,n)":
```

The automata that are created by Walnut are DFA’s (deterministic finite automata). The DFA for \text{ftmfactoreq} has 91 states, the DFA for \text{isftmrs} has 45 states, and the DFA for \text{ftmrsn} has 46 states.

3. We can create the linear representation with the Walnut command

```python
eval ftmatrix n "?msd_fib $ftmrsn(i,n)"
```

This gives us a linear representation \((v, \gamma, w)\) of rank 46, which is stored in the file \text{lftmatrix.mpl}.

For technical reasons, this representation now needs to be adjusted by replacing \(v\) with \(v\gamma(000)\).

4. The “semigroup trick” gives a DFAO with 75 states.

5. After minimization, the Fibonacci-DFAO produced is below.

![Figure 2: Fibonacci-DFAO computing \(d(n)\)](image)
5 Deducing Dekking’s result from the automaton

Theorem 2. We claim that

\[ d(n) = 8 \text{ for } F_{2m+1} + 2 \leq n \leq F_{2m+1} + F_{2m-2} + 1 \]
\[ d(n) = 6 \text{ for } F_{2m+1} + F_{2m-2} + 2 \leq n \leq F_{2m+2} \]
\[ d(n) = 8 \text{ for } F_{2m+2} + 1 \leq n \leq F_{2m+2} + F_{2m-1} \]
\[ d(n) = 6 \text{ for } F_{2m+2} + F_{2m-1} + 1 \leq n \leq F_{2m+3} + 1 \]

for \( m \geq 2 \).

Proof. We can prove this with Walnut, too. The first step is to take the DFAO given in Figure 2, and add it to Walnut’s Word Automata Library. We’ll use the name FTMD for this DFAO. It is given in Appendix A.

Next, we will create individual formulas for each of the intervals given in the statement of the theorem; we’ll call them interval1, ..., interval4. In order to specify them we need ways to specify that \( x \) is a Fibonacci number, an odd-indexed Fibonacci number, an even-indexed Fibonacci number, or that two numbers represent consecutive Fibonacci numbers. These are done, respectively, with the formulas fibo, oddfib, evenfib, and consecfib. The first three of these use the standard formalism of regular expressions [5, §2.5]; for example, evenfib specifies that the Fibonacci representation must consist of 1 followed by an even number of 0’s, which corresponds to \( F_{2n} \) for some \( n \). The last is more complicated; basically, it asserts that \( t \) and \( u \) are both Fibonacci numbers, and there is no Fibonacci number strictly between \( t \) and \( u \). We also need a formula specifying that \( x \) is of the form \( F_n + F_{n-3} \); this is done with a regular expression in fib1001. This gives the following formulas.

\[
\text{reg fibo msd_fib } "0*10*":
\]
\[
\text{reg oddfib msd_fib } "0*10(00)*":
\]
\[
\text{reg evenfib msd_fib } "0*1(00)*":
\]
\[
\text{def consecfib } "(?msd_fib (t<u) \& $oddfib(t) \& $evenfib(u) \& $consecfib(t,u) \& $fib1001(y) \& (t<y) \& (y<=u) \& (y+2<=n) \& (n<y+1))":
\]
\[
\text{reg fib1001 msd_fib } "0*10010*":
\]

From these we can create formulas for our four kinds of intervals:

\[
\text{def interval1 } "(?msd_fib Et,u,y $oddfib(t) \& $evenfib(u) \& $consecfib(t,u) \& $fib1001(y) \& (t<y) \& (y<=u) \& (t+2<=n) \& (n<=y+1))":
\]
\[
\text{def interval2 } "(?msd_fib Et,u,y $oddfib(t) \& $evenfib(u) \& $consecfib(t,u) \& $fib1001(y) \& (t<y) \& (y<=u) \& (y+2<=n) \& (n<=u))":
\]
\[
\text{def interval3 } "(?msd_fib Ew,x,z $evenfib(x) \& $oddfib(z) \& $consecfib(x,z) \& $fib1001(w) \& (x<w) \& (w<=z) \& (x+1<=n) \& (n<=w))":
\]
\[
\text{def interval4 } "(?msd_fib Ew,x,z $evenfib(x) \& $oddfib(z) \& $consecfib(x,z) \& $fib1001(w) \& (x<w) \& (w<=z) \& (w+1<=n) \& (n<=z+1))":
\]

As a double-check, we can now verify with Walnut that our four defined intervals cover all integers \( n \geq 7 \) with the formula all,
def all "?msd_fib $interval1(n) | $interval2(n) | $interval3(n) | $interval4(n)":
and the resulting automaton accepts all $n \geq 7$.

Finally, we perform four tests, one for each interval, checking each of the assertions in the statement of the theorem. Each of these tests returns true, so the theorem is proven.

eval test1 "?msd_fib An $interval1(n) \Rightarrow \text{FTMD}[n]=@8":
eval test2 "?msd_fib An $interval2(n) \Rightarrow \text{FTMD}[n]=@6":
eval test3 "?msd_fib An $interval3(n) \Rightarrow \text{FTMD}[n]=@8":
eval test4 "?msd_fib An $interval4(n) \Rightarrow \text{FTMD}[n]=@6":

Corollary 3. Dekking’s conjecture is correct.

6 Going further

It should be possible to prove similar results for the analogues of the Fibonacci-Thue-Morse sequence for other kinds of numeration systems, such as those built from the recurrence $E_n = kE_{n-1} + E_{n-2}$ for an integer $k \geq 3$.

References

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Appendix A  Walnut file FTMD.txt

```
msd_fib
0 1
0 -> 0
1 -> 1
1 2
0 -> 2
2 4
0 -> 3
1 -> 4
3 6
0 -> 5
1 -> 6
4 10
0 -> 7
5 6
0 -> 8
1 -> 9
6 6
0 -> 10
7 8
0 -> 11
1 -> 12
8 6
0 -> 5
1 -> 13
9 8
0 -> 14
10 8
0 -> 15
1 -> 16
11 6
0 -> 11
1 -> 12
12 6
0 -> 11
13 6
0 -> 14
14 8
0 -> 14
1 -> 9
15 8
0 -> 10
1 -> 12
16 8
0 -> 11
```