On the Converse of Talagrand’s Influence Inequality

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Abstract

In [Tal94], Talagrand showed a generalization of the celebrated KKL theorem. In this work, we prove that the converse of this generalization also holds. Namely, for any sequence of numbers $0 < a_1, a_2, \ldots, a_n \leq 1$ such that $\sum_{j=1}^n a_j/(1 - \log a_j) \geq C$ for some constant $C > 0$, it is possible to find a roughly balanced Boolean function $f$ such that $\text{Inf}_j[f] < a_j$ for every $1 \leq j \leq n$.

1 Introduction

In their seminal paper [KKL88], Kahn, Kalai and Linial showed that for any Boolean function $f$ there exists a coordinate $1 \leq j \leq n$ such that $\text{Inf}_j[f] \geq c \cdot \frac{\log n}{n} \cdot \text{Var}[f]$, where $c > 0$ is a universal constant. This result, followed by the generalizations of Bourgain et al. [BKK+92], Talagrand [Tal94] and Friedgut [Fri98], was a milestone for numerous results in different areas in computer science and mathematics such as hardness of approximation [DS02, CKK+06, KR08], distributed computing [BL90], communication complexity [Raz95], metric embeddings [KR09, DKSV06], learning theory [DS08, OW09], random $k$-SAT [Fri99], random graphs [FK90] and extremal combinatorics [OW09].

Talagrand’s paper “On Russo’s approximate zero-one law” [Tal94], generalized KKL’s result and stated that for every Boolean function $f$,

$$\sum_{j=1}^n \frac{\text{Inf}_j[f]}{1 - \log \text{Inf}_j[f]} \geq K \cdot \text{Var}[f],$$

where $K > 0$ is a universal constant. We refer to this sum as Talagrand sum.

We study whether the converse Talagrand’s theorem holds. Namely, given a sequence of numbers, $0 < a_1, a_2, \ldots, a_n \leq 1$ whose Talagrand sum is greater than a constant $C > 0$, can one find a roughly balanced Boolean function $f$ such that $\text{Inf}_j[f] < a_j$ for all $1 \leq j \leq n$. We show that this is true (up to a constant) not only for balanced functions, but also for unbalanced functions.

2 Main Result

Let $n$ be a positive integer (which is henceforth fixed). A Boolean function on $n$ variables is a function $f : \{0,1\}^n \to \{0,1\}$. Let $x$ be a uniformly chosen element in $\{0,1\}^n$. For $1 \leq j \leq n$, denote by $x^j$ the vector $x$ with the $j$-th coordinate flipped, i.e., $x^j \overset{\text{def}}{=} (x_1, \ldots, x_{j-1}, 1 - x_j, x_{j+1}, \ldots, x_n)$.

We define the influence of the $j$-th variable on $f$ to be $\text{Inf}_j[f] \overset{\text{def}}{=} \Pr[f(x) \neq f(x^j)]$.  

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That is, $\text{Inf}_j[f]$ is the probability that flipping the $j$-th bit affects the outcome of the function. We say that a Boolean function $f$ is balanced if $\Pr[f(x) = 0] = 1/2$. Throughout this paper, $\ln$ denotes the natural logarithm and $\log$ denotes the base-2 logarithm.

**Theorem 1.** Let $0 < a_1, a_2, \ldots, a_n \leq 1$ and denote

$$\alpha \overset{\text{def}}{=} \sum_{j=1}^{n} \frac{a_j}{1 - \log a_j} - 2 \ln 2.$$ 

Then, for any $0 < \mu \leq 1 - \exp(-\alpha/8)$, there exists a Boolean function $f$ on $n$ variables such that

$$\mu \leq \mathbb{E}[f(x)] \leq \frac{3}{4}\mu + \frac{1}{4}$$

and

$$\text{Inf}_j[f] < a_j \quad \text{for all} \ 1 \leq j \leq n.$$ 

**Proof:** Following Ben-Or and Linial’s example of tribes [BL90], which minimizes the influence of each variable, we wish to construct a function in a similar manner by aggregating variables $x_j$ into tribes of various sizes according to the desired bound $a_j$ on their influence.

We may assume without loss of generality that $a_1 \geq a_2 \geq \cdots \geq a_n$. Define an integer $m \geq 0$ and a sequence of integers $k_0, k_1, \ldots, k_m, k_{m+1}$ by

$$k_0 \overset{\text{def}}{=} 0,$$

$$k_i \overset{\text{def}}{=} \min\{k \geq 1 : a_{k_0 + \cdots + k_{i-1} + k} > 2^{1-k}\}, \quad 1 \leq i \leq m,$$

$$k_{m+1} \overset{\text{def}}{=} n - (k_0 + \cdots + k_m) + 1,$$

where $m$ is defined by the condition $\{k \geq 1 : a_{k_0 + \cdots + k_m + k} > 2^{1-k}\} = \emptyset$. Note that $2 \leq k_1 \leq k_2 \leq \cdots \leq k_m$. The integers $k_1, \ldots, k_m$ represent the sizes of the tribes. We shall show that there exists an integer $0 \leq m_* \leq m$ such that the function $f : \{0,1\}^n \to \{0,1\}$ defined by

$$f(x_1, \ldots, x_n) \overset{\text{def}}{=} \bigvee_{i=1}^{m_*} \bigwedge_{j=1}^{k_i} x_{k_1 + \cdots + k_{i-1} + j}$$

(1)

satisfies the conclusion of the theorem. We first show the following.

**Claim 2.**

$$\sum_{i=1}^{m} 2^{-k_i} \geq \frac{\alpha}{8}.$$

**Proof:** For $0 \leq i \leq m$, denote $s_i \overset{\text{def}}{=} k_0 + \cdots + k_i$. Note that, by the definition of $k_i$, we have

$$a_{s_i + j} \leq 2^{1-j}, \quad 0 \leq i \leq m, \quad 1 \leq j \leq k_{i+1} - 1$$

and, in particular,

$$a_{s_i} \leq a_{s_{i-1}} = a_{s_{i-1} + k_{i-1}} \leq 2^{2-k_i}, \quad 1 \leq i \leq m.$$
Thus, with this choice of $\alpha$, the claim follows by the definition of $\alpha$.

We construct the function as described in (4), where we keep adding tribes until we reach a point at which the expected value becomes at least $\mu$. We denote this tribe by $m_\ast + 1$, so that at the end of the process we have $m_\ast$ tribes. Precisely, $m_\ast$ is defined by

$$m_\ast \overset{\text{def}}{=} \min \left\{ 1 \leq r \leq m : \prod_{i=1}^{r} (1 - 2^{-k_i}) \leq 1 - \mu \right\}.$$ 

Indeed, this is well-defined, since by Claim 3.

$$\prod_{i=1}^{m} (1 - 2^{-k_i}) \leq \exp \left( - \sum_{i=1}^{m} 2^{-k_i} \right) \leq \exp \left( - \frac{\alpha}{8} \right) \leq 1 - \mu.$$ 

Thus, with this choice of $m_\ast$, we have

$$\mathbb{E}[f(x)] = \Pr[f(x) = 1] = 1 - \Pr[f(x) = 0] = 1 - \prod_{i=1}^{m_\ast} (1 - 2^{-k_i}) \geq \mu.$$ 

We now show that the expectation of the function $f$ constructed above is less than $\frac{3}{4} \mu + \frac{1}{4}$. Indeed, by the minimality of $m_\ast$, and since $k_i \geq 2$ for all $1 \leq i \leq m$, we have

$$\mathbb{E}[f(x)] = 1 - \prod_{i=1}^{m_\ast} (1 - 2^{-k_i}) < 1 - (1 - \mu)(1 - 2^{-k_{m_\ast}}) \leq 1 - \frac{3}{4} (1 - \mu) = \frac{3}{4} \mu + \frac{1}{4}.$$ 

It remains to check that the influence of the $j$-th variable on $f$ is bounded above by $a_j$. Indeed, if $s_{i-1} < j \leq s_i$ for some $1 \leq i \leq m_\ast$, then

$$\text{Inf}_j[f] = 2^{1-k_i} \prod_{\ell=1}^{m_\ast} (1 - 2^{-k_{\ell}}) \leq 2^{1-k_i} < a_{s_i} \leq a_j.$$ 

Since $\text{Inf}_j[f] = 0$ for all $s_{m_\ast} < j \leq n$, the proof is complete.

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