Abstract. We present a real space renormalization-group map for probabilities of random walks on a hierarchical lattice. From this, we study the asymptotic behavior of the end-to-end distance of a weakly self-avoiding random walk (SARW) that penalizes the (self-)intersection of two random walks in dimension four on the hierarchical lattice.

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1 Introduction

Self-avoiding random walk models appeared in chemical physics as models for long polymer chains. Roughly speaking, a polymer is composed of a large number of monomers which are linked together randomly but cannot overlap. This feature is modelled by a self-repulsion term. Let $A$ be a $d$-dimensional hypercubic lattice, typically the integer lattice $\mathbb{Z}^d$ (or a finite subset of $\mathbb{Z}^d$). Its elements are called sites, oriented pairs of sites are called steps. A walk $w$ on $A$ is an ordered sequence $(w(0), w(1), ..., w(k))$ of sites in $A$; $k \geq 0$, $k$ being the length of the walk [1]. Thus, a simple random walk on $A$, starting at $w(0) \in \mathbb{Z}^d$, is a stochastic process indexed by the non-negative integers.

To state a self-repulsion term, Flory [2] used the self-avoiding walk (SAW). A self-avoiding walk of length $n$ is a simple random walk which visits no site more than once. Although this simple model possesses some qualitative features of polymers, turns out to be very difficult for obtaining rigorous results. Instead of using SAW we take advantage of the several measures on random walks which favor self-avoiding walks.

In this paper we study weakly SARW (so called Domb-Joyce model or self-repellent walk). This is a measure on the set of simple walks in which self-intersections are discouraged but not forbidden. Here, double intersections of random walks are penalized by a factor $e^{-\lambda}$ (measure of (self-)intersection), $\lambda > 0$ being a small constant. This factor is needed to make the process tend to avoid itself. Thus, probabilities of the random walks are modified by a measure of (self-)intersection inside the lattice. This measure is written in terms of waiting (or local) times for the process $(t_i)$. Here, as was done in reference [5], the measure is $\sum_{i<j} t_it_j$, for all local times in the (self-)intersections of the walks on the lattice, which can be either hypercubic or hierarchical. If in this model the lattice is the hypercubic lattice, then the state space is $\mathbb{Z}^d$. To understand this model, we use a hierarchical lattice which state space is defined in next section. We want to stress that the only feature of the model that depends upon the lattice used (hypercubic
or hierarchical) is the state space in this. Thus, definitions for the weakly SARW in section 4 (interaction energy in Theorem 4) are equivalent in both cases, but with different state space in the lattice. We develop our method on a hierarchical lattice because they have the feature that the renormalization-group map is particularly simple, which is not the case on the hypercubic lattice. We believe that the results of this procedure extend to weakly SARW on the hypercubic lattice.

The hierarchical models introduced by Dyson [3] feature a simple renormalization-group transformation. This can easily be seen in related literature [4]. We would like to understand the logarithmic correction that appears in the hypercubic lattice, $d=4$, for the end-to-end distance of the weakly SARW. The real space renormalization-group map we develop here, is factorizable only in terms of a hierarchical lattice. So, we present a method in which $A$ is labeled in terms of a hierarchical metric space, from this an easy realization of the map is followed. In the integer lattice $\mathbb{Z}^d$, this is not true and technical problems arise.

We use a hierarchical lattice where the points are labeled by elements of a countable, abelian group $G$ with ultrametric $\delta$; i.e. the metric space $(G, \delta)$ is hierarchical. The hierarchical structure of this metric space induces a renormalization-group map that is “local”; i.e. instead of studying the space of random functions on the whole lattice, we can descend to the study of random functions on $L$-blocks (cosets of $G$) [5]. This simplifying feature was used by Brydges, Evans and Imbrie [5] to prove (in the $\lambda \phi^4$ Grassmann valued field representation for a weakly SARW that penalizes the (self-)intersection of two random walks) that the introduction of a sufficiently weakly self-avoidance interaction does not change the decay of the Green’s function for a particular Lévy process (continuous time random walk), [4] when $d=4$, provided the mass is introduced critically.

A rigorous proof of the end-to-end distance for the weakly SARW, $d = 4$, on the hierarchical lattice, has recently been reported [3]. This was done in the field theoretical approach by means of controlling the interacting Green’s
function and inverting the Laplace transform.

Low dimensional models are the most interesting from physical viewpoint, but rigorous results are difficult of being obtained. One major result is the proof that in high dimensions the exponents of weakly SARW take the “mean-field” value \([7]\). In this context “lace expansion” is one of the most successful tools. Contrary to what has been done from this method, in this paper we develop a map on real space which is full of the physical intuition needed for being applied on some other cases not yet solved.

Weakly SARW exhibits logarithmic corrections for physically meaningful magnitudes in the critical dimension of the model, i.e. \(d=4\). We study the end-to-end distance for weakly SARW that penalizes the double crossing of walks in \(d = 4\). A probabilistic meaning is given to the exponent of this logarithmic correction. In this paper we present an heuristic space-time renormalization-group argument to show that the end-to-end distance of a weakly self-avoiding random walk (SARW) on the hierarchical lattice, that penalizes the (self-)intersection of two walks in \(d = 4\), is asymptotic to a constant times \(T^{1/2} \log^{1/2} T\) as \(T\) tends to infinity, \(T\) being the total time for the walk. This has already been conjectured before \([8]\). This is the testing ground to check that our map reproduces previously known results with physical improved intuition. The weakly self-avoiding random walk model that penalizes the (self-)intersection of two random walks is in the same universality class as the perfect self-avoiding random walk model; therefore, the same logarithmic correction for the end-to-end distance is expected to hold for both cases regardless the details of the state space used. Real space renormalization-group methods have proved to be useful in the study of a wide class of phenomena.

Since our method is intended to provide an alternative way, full of physical intuition, to renormalize random walk models on a hierarchical lattice; we study weakly SARW in \(d = 4\) just as a testing ground. We consider our method suitable of being directly applied on kinetically growing measure models, discrete version. Kinetically growing measure models are produced
from consistent measures and are the natural framework for random walks provided these are seen like stochastic processes. Myopic self-avoiding walk (a model for adsorption of linear polymers on surfaces [9]), infinite self-avoiding walk and Laplacian (or loop-erased) random walk [13] are examples of kinetically growing measure models. These models have been studied mainly in the field theoretical framework where has been shown [10] that some contributions neglected in the derivation of the continuum problem from the discrete version of the model might be essential in determining the asymptotic behaviour of the model analytically. In fact, usually the corresponding action (in the continuum limit) is not fully renormalizable. We address this problem by presenting a real space renormalization-group suitable of been applied on discrete models. The metod can be used for obtaining both, rigorous and heuristic results for these models.

We hope this paper could be interesting for various readerships. We intend to brydge a gap between probabilistic approaches and field theoretical ones, thereby providing a new probabilistic meaning to critical and asymptotic exponents. The method we develop in this paper can be used also as an intuitive mean to search new exact results, which then remain to be proven rigorously within the framework presented here, or by other means. The renormalization scheme constructed here is not considered or known in the literature.

This paper is organized as follows; in Section 2 we present the hierarchical lattice and the Lévy process we study, on the space of simple random walks. In Section 3 we define the renormalization-group map on the hierarchical lattice and prove that two particular probabilities, functions of random walks, flow to fixed forms after applying the map. In Section 4 we apply the renormalization-group map to the weakly SARW model that penalizes the intersection of two random walks on the hierarchical lattice. In Section 5 we present an heuristic proof for the asymptotic behavior of the end-to-end distance for the weakly SARW on the hierarchical lattice, d=4. Although this is an heuristic proof, it helps in understanding the way the map can be used
and gives a new probabilistic meaning to the exponent of the logarithmic correction. Sections 3, 4 and 5 involve important results of this paper, being Theorem 4 the main result reported in the paper and an important step to obtain heuristically the end-to-end distance for the weakly SARW. Even more, we claim this Theorem to be a random walk version of the field theoretical approach in reference [5] with improved physical intuition. Finally, we summarize.

2 The hierarchical random walk.

The hierarchical lattice used in this paper was recently introduced by Brydges, Evans and Imbrie [5] [6]. Here we are presenting a slight variant of the model in reference [5].

Fix an integer $L \geq 2$. Hereafter, the points of the lattice $A$ are labeled by elements of the countable abelian group $G = \bigoplus_{k=0}^{\infty} \mathbb{Z}_{L^d}$, $d$ being the dimension of the lattice. Through the paper the abelian group $G = \bigoplus_{k=0}^{\infty} \mathbb{Z}_{L^d}$ replaces $A$.

An element $X$ in $G$ is an infinite sequence

$$X \equiv (..., X_k, ..., X_2, X_1, X_0) ; X_i \in \mathbb{Z}_{L^d} \text{ thus } X \in G = \bigoplus_{k=0}^{\infty} \mathbb{Z}_{L^d},$$

where only finitely many $X_i$ are non-zero.

Let us define subgroups

$$\{0\} = G_0 \subset G_1 \subset ... \subset G \text{ where } G_k = \{X \in G | X_i = 0, i \geq k\}, \quad (1)$$

and the norm $| \cdot |$ as

$$|X| = \begin{cases} 0 & \text{if } X = 0 \\ L^p & \text{where } p = \inf \{k | X \in G_k\} \text{ if } X \neq 0. \end{cases} \quad (2)$$

Then, the map $\delta : (X, Y) \rightarrow |X - Y|$ defines a metric on $G$. In this metric the subgroups $G_k$ are balls $|X| \leq L^k$ containing $L^{dk}$ points. Here the operation $+$ (hence - as well) is defined componentwise.
In Figure 1 we have described two examples for \( L = 2 \), a one-dimensional hierarchical lattice (Figure 1.a)) already presented in reference [5] and a two-dimensional hierarchical lattice (Figure 1.b)). In these Figures we depict \( G_k \) cosets, a way to calculate distances among points, and the concept of scales for each example.

The metric defined by eq(2) satisfies a stronger condition than the triangle inequality, namely

\[
|X + Y| \leq \text{Max}(|X|, |Y|). \tag{3}
\]

This non-archimedean property implies that every triangle is isosceles and that every point interior to a ball can be considered its center. Moreover, balls of radius \( L \) are the same as balls of diameter \( L \), and are the same as \( G_1 \) cosets. From inequality (3), it is clear that the metric introduced is an ultrametric and confers the hierarchical structure. Strictly speaking, it is only the metric space \((G, \delta)\) that is hierarchical. Here, ultrametric appears naturally as a property of polynomials. It can be shown that \( G_k \) represents polynomials of degree \( k \) on a formal basis.

Let us now introduce the Lévy process we propose in this paper. The elements of the lattice \( G \) are called sites; unoriented pairs \{\( X, Y \)\} of sites in \( G \) with \( X \neq Y \) are called bonds; oriented pairs \((X, Y)\) are called steps (or jumps) with initial site \( X \) and final site \( Y \). Let us define the Lévy process \( \{w(t)\} (\equiv \text{continuous time random walk}) \), \( w \), as an ordered sequence of sites in \( G \);

\[
(w(t_0), ..., w(t_0 + ... + t_n)), \quad w(t_0 + ... + t_i) = X_i \in G, \quad T = \sum_{i=0}^{n} t_i, \quad n \geq 0 \tag{4}
\]

where \( t_i \) is the time spent in \( X_i \in G \) (waiting time at \( X_i \)) and \( T \), fixed, is the running time for the process. For convenience we take \( X_0 = 0 \). The support of the walk \( w \) is defined by

\[
\text{supp}(w) = \{X \in G | w(t_0, ..., t_j) = X \text{ for some } j\}, \tag{5}
\]

for any \( w \). The random walk we are dealing with is not the nearest neighbor
random walk on the lattice, provided we mean neighbourhood with respect to the ultrametric distance $\delta$ previously defined.

If we compare this Lévy process with the simple random walk in the hypercubic lattice defined in the Introduction, we see that in this Section we construct a different stochastic process. Here, time is continuous and the path of the walk is given on $G$ (in the hypercubic lattice, the walk was indexed by non-negative integers, instead of waiting times, with path on $\mathbb{Z}^d$).

Besides, here we fix the running time for the process. This corresponds to walks of fixed length, feature that was not imposed on simple random walks on the hypercubic lattice.

We suppose the Lévy process in the hierarchical lattice has a probability $r dt$ ($r$ is the jumping rate) of making a step in time $(t, dt)$ and, given that it jumps, the probability of jumping from $X$ to $Y$ is $q(X,Y)$. Thus, the process, conditioned to $n$ jumps spaced by times $t_0, t_1, \ldots, t_n$, has a probability density

$$P(w) = r^n e^{-rT} \prod_{i=0}^{n-1} q(X_{i+1}, X_i), \text{ where } T = \sum_{i=0}^{n} t_i. \quad (6)$$

Define $Dw$ by

$$\int (\cdot) Dw = \sum_n \sum_{[X_i]_{i=0}^n} \int_{t_i=0}^{T} \prod_{i=0}^{n} dt_i \delta \left( \sum_{i=0}^{n} t_i - T \right) (\cdot).$$

From this and eq(6) it is straightforward to obtain

$$\int P(w) Dw = \left< \sum_{[X_i]_{i=0}^{n-1}} \prod_{i=0}^{n-1} q(X_{i+1}, X_i) \right>_{\text{Poisson}} = 1,$$

where $\prod_{i=0}^{n-1} q(X_{i+1}, X_i)$ has been normalized on $G$ and we have used

$$\int_{0}^{T} \prod_{i=0}^{n} r^n e^{-rt_i} \prod_{j=0}^{n} dt_j \delta \left( \sum_{j=0}^{n} t_j - T \right) =$$

$$r^n e^{-rT} \int_{0}^{T} \prod_{i=0}^{n} dt_i \delta \left( \sum_{i=0}^{n} t_i - T \right) = \frac{(rt^n)}{n!} e^{-rT}.$$
3 The renormalization-group map.

To start with, let us introduce a renormalization-group map on the lattice; \( R(X_i) = LX'_i \) where \( X_i \in G \) and \( LX'_i \in G' = G/G_1 \sim G \); i.e. the renormalized lattice \( G' \) is isomorphic to the original lattice \( G \).

From this renormalization-group map we construct \( R(w) = w' \), from \( w \) above as defined, to \( w' \). Here, \( w' \) is the following ordered sequence of sites in \( G' = G/G_1 \approx G \);

\[
(w'(t'_0), ..., w'(t'_0 + ... + t'_k)), \text{ where }
\]

\[
w'(t'_0, ..., t'_i) = X'_i \in G, \quad T' = \sum_{i'=0}^{k} t'_i, \quad 0 \leq k \leq n, \quad T = L^\beta T'.
\]

R maps \( w(t_0), w(t_0 + t_1), ..., w(t_0 + ... + t_n) \) to cosets \( w(t_0) + G_1, w(t_0 + t_1) + G_1, ..., w(t_0 + ... + t_n) + G_1 \). If two or more successive cosets in the image are the same, they are listed only as one site in \( w'(t'_0), ..., w'(t'_0 + ... + t'_k) \), and the times \( t'_j \) are sums of the corresponding \( t_j \) for which successive cosets are the same, rescaled by \( L^\beta \). For \( \beta = 2 \), we are dealing with normal diffusion, in case \( \beta < 2 \) with superdiffusion, and subdiffusion for \( \beta > 2 \). Additionally, the renormalization-group maps each \( G_1 \) coset to the center of the ball and rescales by \( L \). In reference [5], \( \beta \) is set to 2.

The renormalization group map, applied to functions of the hierarchical random walk, preserves locality [5]. Thus, if \( F(w) = \prod_{X_i \in G} f(w(t) = X_i) \), the effect of the renormalization-group map on \( F(w) \) can be studied as the product, for all elements of the group \( G/G_1 = G' \sim G \), of the images of the renormalization-group map of \( f(w) \) in the \( G_1 \) coset [4]. This can be seen as follows: \( \prod_{X_i \in G} \) splits into two parts; the first part is \( \prod_{X'_i \in G/G_1} \) which corresponds to \( \prod_{X'_i \in G} \). The second part is \( \prod_{X_i \in (G_1)_{X'_i}} \) and stands for the \( L \)-block (\( G_1 \) coset) of the lattice which, under the renormalization-group transformation, maps to \( LX'_i \), \( 0 \leq i' \leq k \). The geometrical interpretation of this is quite simple. The renormalization-group map applied on \( F(w) \) in \( G \) splits into that of \( f(w(t) = X_i) \) in the contracting \( G_1 \) cosets, multiplied
by the whole $G/G_1$ group. To study the renormalization of $F(w)$ in $G$ we
descend to analyze the renormalization-group action on $f(w(t) = X_i)$ in the
contracting $G_1$ coset.

In Figure 2 we present the lattice $G$. On this, walks with fixed topology in
$G/G_1$ are depicted. In this example three different types of local topologies
inside the $G_1$ cosets are illustrated. Once the renormalization-group map
is applied, the renormalized lattice and random walk are also shown. The
renormalized random walk $w'$ visits each type of $G_1$ cosets once, twice and
three times respectively. In the renormalized lattice we show the contracted
$G_1$ cosets. The particular fixed topology chosen for $w'$ is one in the class
of the simplest cases studied by our method. Here is clear what is meant
by factorization and locality of the renormalization-group map. The map
factorizes into two terms. Roughly speaking, the first term corresponds to
events inside $L$-blocks or $G_1$ cosets and the second term corresponds to events
outside the $L$-block ($G_1$ cosets); therefore in $G/G_1=G' \sim G$. Moreover, for
obtaining the flow of the interaction constant, the map descends to study
events in $G_1$ cosets; thereby preserving locality.

We can now work out probabilities at the $(p+1)^{th}$ stage in the renormal-
ization provided only that we know the probabilities at the $p^{th}$ stage. We
sum over the probabilities of all the walks $w^{(p)}$ consistent with a fixed walk
$w^{(p+1)}$ in accordance with the following;

**Definition.** Let $R(w) = w'$ be the renormalization-group map,
above as stated. Then

$$P'(w') = L^{\beta k} \int DwP(w)\chi(R(w) = w')$$

(8)

for any probability $P(w)$ where the running time for the process
$T = \sum_{i=0}^n t_i$ is fixed.

In this definition, R is a renormalization-group transformation that maps
density $P(w)$ to a new one, $P'(w')$, on rescaled coarse walks. Besides, $\chi(c)$
is the characteristic function of the condition $c$.

Let $P(w) = \prod_{X \in G} p(w(t) = X)$ then
\[ P'(w') = L^{\beta_k} \int Dw \prod_{X \in G} p(w(t) = X) \chi(R(w) = w') . \] From this and factorization properties in the hierarchical lattice follows

\[ P'(w') = \prod_{X' \in G} \left\{ L^{\beta_k} \int Dw \prod_{X \in (G_1)_{X'}} p(w(t) = X) \chi(R(w) = w') \right\} \]

\[ = \prod_{X' \in G} p'(w'(t') = X') , \]

that proves in this case the statement of preservation of locality as above given for \( F(w) \). Eq(17) and eq(21) are examples for suitable \( P(w) \).

Eq(8) corresponds precisely to

\[ P'(w') = L^{\beta_k} \sum_{[n_{i'}]_{i'=0}} \sum_{[X_i]_{i=0}} \int \prod_{i=0}^n dt_i \prod_{j'=0}^{k} \delta(\sum_{i=m_{j'-1}+1}^{m_{j'}} t_i - L^{\beta} t_{j'}) \times \]

\[ \times \prod_{j'=0}^{k} \prod_{i=m_{j'-1}+1}^{m_{j'}} \chi(X_i \in L X'_{j'}) \) P(w). \]}

Hereafter

\[ m_{j'} = \sum_{i'=0}^{j'} n_{i'} + j' \quad \text{and} \]

\[ n = \sum_{i'=0}^{k} n_{i'} + k \quad 0 \leq j' \leq k. \]

\( n_{i'} \) is the number of steps (for walks \( w \)) in the \( G_1 \) coset which, once the renormalization-group map is applied, has the image \( L X'_{j'} \). Thus, the total number of steps (for walks \( w \)) on the lattice is given by the sum of the steps within each L-block (\( G_1 \) cosets) plus \( k \) times 1 (due to the step out of the corresponding block).

**Theorem 1.** The probability in eq(6), where \( q(X_{i+1}, X_i) \) is chosen of the form \( c |X_{i+1} - X_i|^{-\alpha} \) (\( c \) is a constant fixed up to normalization and \( \alpha \) another constant), is a fixed point of the renormalization-group map \( R(w) = w' \), provided \( \beta = \alpha - d \).
\textbf{Proof.} Using the definition of the renormalization-group map on the probability given in eq(6) and doing some elementary manipulations we arrive to the following expression

$$P'(w') = L^{\beta k} r^k e^{-rT} \prod_{j'=0}^{k-1} (q(LX'_{j'} - LX'_{j'+1})L^d) \times$$

$$\prod_{j'=0}^{k} \sum_{n_{j'}} r^{n_{j'}} (q_1(L^d - 1))^{n_{j'}} \frac{(L^\beta t_{j'})^{n_{j'}}}{n_{j'}!}$$

where $\prod_{i=0}^{n-1} q(X_{i+1} - X_{i})$ has been split into two factors; the first factor corresponding to jumps from one $L$-block to another $L$-block (different $G_1$ cosets) and the second factor corresponding to jumps inside the same $L$-block or $G_1$ coset. Function $q_1$ in eq(11) is the probability of jumping to a given point within the $G_1$ coset that has the image $LX'_{j'}$ (hereafter $(G_1)_{X'_{j'}}$). There are $(L^d - 1)$ possibilities with equal probability $q_1$ and $n_{j'}$ steps.

From normalization, i.e. $\sum_{X \in G} q(X) = 1$, we get $c = \frac{L^{d-\alpha} - 1}{1 - L^{-\alpha}}$ and $q_1(L^d - 1) = 1 - L^{d-\alpha}$.

On the other hand, we know that $T = L^\beta T'$, therefore, eq (11) becomes

$$P'(w') = L^{(d+\beta-\alpha)k} r^k \prod_{j'=0}^{k-1} q(X'_{j'+1} - X'_{j'}) e^{-rL^{(d+\beta-\alpha)T'}}$$

Provided $d + \beta - \alpha = 0$, eq(12) is clearly a fixed point of the renormalization-group map $R$, i.e. $P'(w') = r^k \prod_{j'=0}^{k-1} q(X'_{j'+1} - X'_{j'}) e^{-rT'}$.

Q. E. D.

Theorem 1 corresponds to the case worked out by Brydges, Evans, and Imbrie if we choose $\beta = 2$, i.e. diffusive behavior [5].

\textbf{Theorem 2.} If the probability in eq(6) where $q(X_j, X_{j+1})$ is chosen of the form

$$q(X_j - X_{j+1}) = c \left( \frac{1}{|X_j - X_{j+1}|^\alpha} + \frac{1}{|X_j - X_{j+1}|^\gamma} \right)$$

12
(c is a constant fixed up to normalization, \(\alpha\) and \(\gamma\) are constants, \(\alpha \neq \gamma\)) then \(P(w)\) flows to the fixed point \(P'(w')\) given in theorem 1, (i.e. \(q(X_j - X_{j+1})\) is given as in theorem 1), under the renormalization-group map \(R(w) = w'\), provided \(\gamma >> \alpha\) (such that \(\log \left( \frac{L^{-\alpha} - L^{-\gamma} - 2L^{d-\gamma} - \alpha}{L^{-\alpha} - 2L^{d-\gamma}} \right) \rightarrow 0\) and \(\beta = d + \alpha\).

**Proof.** Following the same ideas as in theorem 1, from normalization, we obtain 
\[
c = \frac{(1-L^{d-\alpha})(1-L^{d-\gamma})}{(L^{d-1})(1-L^{d-\alpha} + L^{d-\gamma} - 2L^{d-\gamma} - \alpha)}
\]
and \(q_1 = c(L^{-\gamma} + L^{-\alpha})\). Then, if \(\gamma >> \alpha\), \(P'(w')\) corresponds to
\[
P'(w') = L^{(d+\beta-\alpha)k} r^k e^{-rL^\beta T'} \times
\]
\[
\prod_{j'=0}^{k-1} \frac{(L^{\alpha-d}-1)}{(1-L^{d})} |X_{j'}' - X_{j'+1}'|^{-\alpha e^{r(1-L^{d-\alpha})L^\beta T'}}
\]
If \(d + \beta - \alpha = 0\), eq(14) reduces to \(P'(w')\) as given in Theorem 1.

Q.E.D.

Theorem 2 is presented here in order to learn about the Lévy process we are studying. This case is intended to answer the question about the feasibility of introducing perturbations in the probability and the possible results to be obtained. We are looking forward to studying of asymmetric and “trapping” environments.

4 The renormalization-group map on the weakly SARW

A configurational random walk model can be defined by assigning to every n-tuple of walks \(w_1, ..., w_n\) \((n \geq 0)\) a statistical weight. For a simple random walk model, this is the product of the statistical weights for each of the n walks and can serve as a random walk representation of the Gaussian model. The best known mathematical model that involves a self-repulsion term is the self-avoiding random walk. A self-avoiding walk of length n is a simple random walk which visits no site more than once. Unfortunately, it turns out that it is extremely difficult to obtain rigorous results from this.
model for $d \leq 4$ \cite{11,12}. However, there are other ways to include self-repulsion terms in random walk models \cite{13}. These split naturally into two categories: configurational measures where random walks are weighted by the number of (self-)intersections, and kinetically growing measures where random walks are produced from consistent measures that are generated by Markovian transition probabilities on the states space of simple random walks (these measures are non-Markovian on the state space of the lattice) \cite{13}. The weakly self-avoiding random (or Domb-Joyce) model and the Edwards model are examples of the first category \cite{13}. The “true” (or “myopic”) self-avoiding walk and the Laplacian random walk are examples of the second category \cite{13}. In this paper we deal only with configurational measures.

A simple random walk model can be thought of as being endowed with a configurational measure where the weight for the self-intersections of a walk (and/or among the $n$-tuple of walks $w_1, ..., w_n$ ($n \geq 0$)) is null. Configurational measures are measures on $\Lambda_n$, the space of simple random walks of length $n$. Let $P_U(w)$ be a probability on $\Lambda_n$ given in eq(6) and $U(w)$ as the interaction energy of the walks \cite{1}. Thus, to study the effect of the renormalization-group map on $P_U(w)$ we need to follow the trajectory of $U(w)$ after applying several times the renormalization-group map.

Therefore, from the definition of the renormalization-group map in eq(8);

$$P_U'(w') = L^{\beta k} \int P_U(w) \chi(R(w) = w') Dw$$

where $Z' = Z$, it follows that

$$U'(w') = \frac{\int DwP(w)\chi(R(w) = w')U(w)}{\int DwP(w)\chi(R(w) = w')}$$

(16)

Note that eq(16) can be viewed as the expectation of $U(w)$ given that the renormalization-group map is imposed, calculated using $P(w)$ on $\Lambda_n$ defined
as in eq(6). Therefore, and hereafter, to simplify notation, we write eq(16) as \(U'(w') = <U(w)>_{w'}\).

In this Section we deal with \(U(w)\) factorizable in terms of the interaction energy with null weight for the (self-)intersection of \(n\) \((n = 2, 3, \ldots, \text{etc.})\) random walks (i.e. a simple random walk factor), and the interaction energies that weight the intersection of \(n\) random walks. Hereafter, as a hypothesis, we assume all the factors in \(U(w)\) (functions of \(w\)) as independent, simple, random variables. Thus, the conditional (given the \(R(w) = w'\) map) expectation of \(U(w)\) is the product of conditional (given the \(R(w) = w'\) map) expectation of each factor in \(U(w)\). This hypothesis follows the same spirit as in the approach used in polymer networks [14] and can be seen as a consequence of factorizability and locality of the map. See Figure 2.

To start with, we study the simple random walk model such that

\[
U(w) = \prod_{X \in G} e^{-a \sum_{j \in J_X} t_j}, \tag{17}
\]

where \(J_X = \{j \in \{0, \ldots, n\} | X_j = X\}\) for \(w(t_0 +, \ldots, +t_j) = X_j\) and \(X \in G\).

**Theorem 3.** The probability \(P_U(w)\) for the simple random walk model where \(U(w)\) is given by eq(17), is a fixed form of the renormalization-group map \(R(w) = w'\) such that, after applying the renormalization-group map, \(a' = L^\beta a\).

**Proof.** Let us split the product on sites in the lattice in \(\prod_{X \in G} e^{-a \sum_{j \in J_X} t_j}\) into two parts. The first one, i.e. \(\prod_{X' \in G/G_1}\) corresponds to \(\prod_{X' \in G}\) due to the hierarchical structure of the lattice. The second one, i.e. \(\prod_{X_i \in (G_2)_i X'_{i'}}\) stands for the \(L\)-block \((G_i, \text{coset})\) of the lattice that, under the renormalization-group transformation, maps to \(LX'_{i'}\), \((0 \leq i' \leq k)\). There are \(k\) replicas of this. If we again split \(\prod_{i=0}^{n-1} q(X_{i+1} - X_i)\) into two factors, as was done in theorem 1. We obtain

\[
U'(w') = \prod_{X'_{i'} \in G} \left\{ \prod_{j' \in J_{X'_{i'}}} e^{q_i(L^d-1)r(L^\beta t_{j'})} \right\}^{-1} \times \tag{18}
\]
\[
\left\{ \sum_{n'}^{m'} \int \prod_{i \in I_{X_i'}} dt_i \prod_{j' \in J_{X_{j'}}'} \delta\left( \sum_{i=m_{j'-1}+1}^{m_{j'}} t_i - L^\beta t_{j'}' \right) \times \sum_{X_i \in (G_1)_{X_i'}} \prod_{i=m_{j'-1}+1}^{m_{j'}} \chi(X_i \in LX_{j'}') \left( q_1(L^d - 1) r \right)^{n'} e^{-a \sum_{X_i \in G_1_{X_i'}} \sum_{j \in J_{X_i}} t_j} \right\}
\]

where we have defined, for \(X_i \in w\) and \(X_{j'}' \in w'\):

\[
I_{X_{j'}'} = \{ i \mid X_i \in LX_{j'}' \} = \bigcup_{j' \in J_{X_{j'}'}} \{ i \mid m_{j'-1} + 1 \leq i \leq m_{j'} \} = \bigcup_{j' \in J_{X_{j'}'}} \{ i \mid 0 \leq i \leq n_{j'} \}. \tag{19}
\]

Rearranging the double sum in the exponential of eq(18), we obtain

\[
U'(w') = \prod_{X_i \in G'} e^{-nL^\beta \sum_{j' \in J_{X_{j'}'}} t_{j'}'}. \tag{20}
\]

Q.E.D.

The next model we want to study is a weakly self-avoiding random walk (or Domb-Joyce model), with a configurational measure in which double intersections of walks are penalized by (roughly speaking) a factor of \(e^{-\lambda}\). As \(\lambda \to \infty\), this reduces to random walks with strict mutual avoidance. Recall that this weakly model (with \(\lambda > 0\)) and the perfect self-avoiding random walk, are in the same universality class (this implies that the critical exponents are the same). If \(\lambda = 0\), this corresponds to a simple random walk model. Next Theorem is an important result of this paper; it is a random walk version of the field theoretical approach [5] with improved physical intuition and the key stone in our method to obtain the end-to-end distance of the weakly SARW. The renormalization-group map applied on a weakly SARW involves the paramaters \(\gamma_1, \gamma_2, \beta_1, \eta, A, B, C\). These are some conditional expectations of local times for different topologies in both, \(w\) and \(w'\) random walks and are precisely defined in Figure 3.

In Figure 3, \(\gamma_1\) and \(\gamma_2\) correspond to \(O(\lambda)\) and \(O(\lambda^2)\) contributions in Figure 2.b), respectively. \(\beta_1\) corresponds to \(O(\lambda^2)\) contribution in figure 2.c)
and $\eta$ corresponds to $O(\lambda^2)$ contribution in Figure 2.d). Although $A$, $B$, and $C$ are not depicted in Figure 2, it is straightforward to figure out the corresponding pictures.

**Theorem 4.** For the weakly SARW with interaction energy

$$U(w) = \prod_{X_i \in G} e^{-\xi \sum_{i} t_i - \lambda \sum_{i<j} t_{ij} 1_{\{w(t_i) = w(t_j)\}}$$

(21)

$\xi_2 < 0$ and $\lambda > 0$ being (small) constants, the probability $P_U(w)$ flows to a fixed form after the renormalization-group transformation is applied. This fixed form is characterized by the interaction energy

$$U'(w') = \prod_{X_i' \in G} e^{-\xi' \sum_{i} t_i' - \lambda' \sum_{i<j} t_{ij}' 1_{\{w'(t_i') = w'(t_j')\}}$$

(22)

$$= \left\{ \begin{array}{l}
1 + \eta_1' \sum_{i_{\alpha_1}' < j_{\alpha_1}' < k_{\alpha_1}'} t_{i_{\alpha_1}'} t_{j_{\alpha_1}'} t_{k_{\alpha_1}'} 1_{\{w(t_{i_{\alpha_1}}') = w(t_{j_{\alpha_1}}') = w(t_{k_{\alpha_1}}')\}} + \\
\eta_2' \sum_{i_{\alpha_1}' < j_{\alpha_1}'} t_{i_{\alpha_1}'} t_{j_{\alpha_1}'} 1_{\{w(t_{i_{\alpha_1}}') = w(t_{j_{\alpha_1}}')\}} + \eta_3' \sum_{i_{\alpha_1}' \in J_{X_i'}} t_{i_{\alpha_1}'} \right\} + r'.$$

Here

\begin{align*}
\xi' & = L\beta \xi + \xi_2' \\
\xi_2' & = \gamma_1 \lambda - \gamma_2 \lambda^2 + O(\lambda^3) \\
\lambda' & = L^{(2\beta - d)} \lambda - \beta_1 \lambda^2 + O(\lambda^3) \\
\eta_1' & = \eta_1 A + \eta_2 \\
\eta_2' & = \eta_2 B + L^{(2\beta - d)} \eta_2 \\
\eta_3' & = \eta_1 C + \eta_2 \gamma_1 + L^{\beta} \eta_3 \\
r' & \sim O(\lambda^3).
\end{align*}
we rewrite eq(31) as;

\[ U'(w') = \langle U_\xi(w) \rangle_{w'} \langle U_\lambda(w) \rangle_{w'} \]

Proof. From our initial hypothesis, follows

\[ \langle U(\lambda) \rangle_{w'} = \prod_{X'_i \in G} \langle U(\lambda) \rangle_{w'}^{(G_1)_{X'_i}} \text{ where} \]

\[ \langle U(\lambda) \rangle_{w'}^{(G_1)_{X'_i}} = \left\{ \prod_{j' \in J_{X'_i}} e^{q(L^d - 1) r(L^\lambda_{j'i})} \right\}^{-1} \times \]

\[ \left\{ \sum_{X_i \in (G_1)_{X'_i}} \sum_{\tilde{n}'_i} \int \prod_{i \in I_{X'_i}} dt_i \prod_{j' \in J_{X'_i}} \delta(\sum_{i=m_{j'_{-1}}+1}^{m_{j'}} t_i - L^\lambda_{j'i}) \prod_{i=m_{j'_{-1}}+1}^{m_{j'}} \chi(X_i \in L X'_i) \right\} \]

Recall that \( w' \) is fixed; i.e. the walk, after the renormalization-group map is applied, visits each site \( X'_i \in G \) a fixed number of times \( n_{i}^{w'} \). Let us assume \( w' \) is such that \( \{ n_{i}^{w'} \} > 0 \), and at least once \( n_{i}^{w'} = 1, 2, 3 \) on \( G \). See example in Figure 2. We ask for this condition to hold, in order to learn about the flow of the interaction constant in the (self-) intersection of 2 and 3 random walks. In other words, we ask for a fixed and not totally arbitrary topology for the renormalized random walk \( w' \) on \( G \). To make this condition explicit we rewrite eq(31) as;

\[ \langle U(\lambda) \rangle_{w'} = \prod_{X'_i \in G} \prod_{n_{i}^{w'} = 0} \langle U(\lambda) \rangle_{w'}^{(G_1)_{X'_i}} \]

18
Here, $\langle U_{\lambda}(w) \rangle_{w_{n^*}}^{(G_1)_{X'}}$ is the renormalized interaction energy for all possible topologies of walks $w$ inside the $(G_1)_{X'}$ coset that, once the renormalization-group map is applied, corresponds to a fixed topology in $LX' \in G$.

Let us introduce a formal Taylor series expansion in $\lambda$, then:

$$\langle U_{\lambda}(w) \rangle_{w_{n^*}}^{(G_1)_{X'}} = \sum_{s=1}^{\infty} \frac{(-\lambda)^s}{s!} \chi_{(i_{\alpha_1}, \ldots, i_{\alpha_s}, j_{\alpha_1}, \ldots, j_{\alpha_s}) \in J_{X_i}} \left( w(t_{i_{\alpha_1}}) = w(t_{j_{\alpha_1}}) \right) \times \ldots \times w'$$

In this formal series expansion, we are writing $\langle U_{\lambda}(w) \rangle_{w_{n^*}}^{(G_1)_{X'}}$ in terms of all possible classes of topology for walks $w$ inside the $(G_1)_{X'}$ coset. Each class is an element of this Taylor series and corresponds to a fixed number $s$ of double (self)-intersections, weighted by $\lambda^s$. Here, the superscript $l.c.$ means linear contribution. We take into account only linear contributions to conditional expectations. This approach is considered to avoid double-counting sites in walks.

To start with, we analyze explicitly the case $n^*_i = 1$, (for some $0 \leq i' \leq k$) up to 2nd order in $\lambda$:

$$\langle U_{\lambda}(w) \rangle_{w_{n^*}=1}^{(G_1)_{X'}} = 1 - \lambda \times$$

$$\sum_{X_i \in (G_1)_{X'}} \sum_{i_{\alpha_1} < j_{\alpha_1}} t_{i_{\alpha_1}} t_{j_{\alpha_1}} \chi_{(w(t_{i_{\alpha_1}}) = w(t_{j_{\alpha_1}}))} \left( (i_{\alpha_1}, \ldots, i_{\alpha_s}, j_{\alpha_1}, \ldots, j_{\alpha_s}) \in J_{X_i} \right)$$

$$+ \frac{\lambda^2}{2} \sum_{X_i \in (G_1)_{X'}} \sum_{i_{\alpha_1} < i_{\alpha_2} < j_{\alpha_2}} t_{i_{\alpha_1}} t_{j_{\alpha_1}} \times \ldots \times w'$$
where $r'_1 \sim O(\lambda^3)$.

Thus, eq(35) is written as

$$\langle U_\lambda (w) \rangle_{w', n^*_p=1}^{(G_1) X'_p, l.c.} = 1 - \left( \gamma_1 \lambda - \gamma_2 \lambda^2 + O(\lambda^3) \right) \sum_{i'_{\alpha_1} \in J_{X'_p}} t'_{i'_{\alpha_1}}$$

(36)

where $\xi'_2$ is given as in eq(24),

\[
\gamma_1 = \left\langle \sum_{X_i \in (G_1) X'_p} \sum_{i'_{\alpha_1} < j_{\alpha_1}} t_{i'_{\alpha_1}} t_{j_{\alpha_1}} \mathbf{1}(w(t_{i'_{\alpha_1}}) = w(t_{j_{\alpha_1}})) \chi \left( (i_{\alpha_1}, j_{\alpha_1}) \in i'_{\alpha_1} \right) \right\rangle_{w', l.c.}^{(G_1) X'_p}
\]

(37)

\[
\gamma_2 = \frac{1}{2} \left\langle \sum_{X_i \in (G_1) X'_p} \sum_{i'_{\alpha_1} < j_{\alpha_1} < i'_{\alpha_2} < j_{\alpha_2}} t_{i'_{\alpha_1}} t_{j_{\alpha_1}} \mathbf{1}(w(t_{i'_{\alpha_1}}) = w(t_{j_{\alpha_1}})) \times \right. \]

(38)

\[
\left. \sum_{i'_{\alpha_1} < j_{\alpha_1} < i'_{\alpha_2} < j_{\alpha_2}} t_{i'_{\alpha_1}} t_{j_{\alpha_1}} \mathbf{1}(w(t_{i'_{\alpha_1}}) = w(t_{j_{\alpha_1}})) \chi \left( (i_{\alpha_1}, j_{\alpha_1}, i_{\alpha_2}, j_{\alpha_2}) \in i'_{\alpha_1} \right) \right\rangle_{w', l.c.}
\]

\[
\gamma_1 \text{ and } \gamma_2 \text{ are explained in Figure 3.}
\]

In the same spirit we study the case $n^*_p = 2$, (for some $0 \leq i' \leq k$) up to second order in $\lambda$;

$$\langle U_\lambda (w) \rangle_{w', n^*_p=2}^{(G_1) X'_p, l.c.} = 1 - \left( L^{(2\beta - d)} \lambda - \beta_1 \lambda^2 + O(\lambda^3) \right) \times$$

(39)

\[
\times \sum_{i'_{\alpha_1} < j_{\alpha_1}} t'_{i'_{\alpha_1}} t'_{j_{\alpha_1}} \mathbf{1}(w(t'_{i'_{\alpha_1}}) = w(t'_{j_{\alpha_1}}))
\]

\[
- \lambda' \sum_{j_{\alpha_1}} t'_{i'_{\alpha_1}} t'_{j_{\alpha_1}} \mathbf{1}(w(t'_{i'_{\alpha_1}}) = w(t'_{j_{\alpha_1}}))
\]

\[
\cong e^{-\xi_2 \sum_{i'_{\alpha_1} < j_{\alpha_1}} t'_{i'_{\alpha_1}}}
\]

20
where $\lambda'$ is given by eq(25) and

$$\beta_1 = \frac{1}{2} \left\langle \sum_{X_i \in (G_1)_{X_i'}} \sum_{i_{\alpha 1} < j_{\alpha 1} < i_{\alpha 2} < j_{\alpha 2}} t_{i_{\alpha 1}} t_{j_{\alpha 1}} 1_{(w(t_{i_{\alpha 1}}) = w(t_{j_{\alpha 1}}))} \times \right.$$  

$$\times t_{i_{\alpha 2}} t_{j_{\alpha 2}} 1_{(w(t_{i_{\alpha 2}}) = w(t_{j_{\alpha 2}}))} \chi \left((i_{\alpha 1}, j_{\alpha 1}, i_{\alpha 2}, j_{\alpha 2}) \in \{j_{\alpha 1} < j_{\alpha 2}\}\right) \rangle_{w'}^{(G_1)_{X_i'}, L.C.}. \quad (40)$$

$\beta_1$ is explained in Figure 3.

Finally, let us present the factor $n_{i'}^{\alpha} = 3$, (for some $0 \leq i' \leq k$), up to 2nd order in $\lambda$;

$$\langle U_{\lambda}(w) \rangle^{(G_1)_{X_i'}, L.C.}_{w'_n = 3} = (1 +$$

$$+ \eta \lambda^2 \sum_{i'_{\alpha 1} < j_{\alpha 1} < k_{\alpha 1}} t_{i'_{\alpha 1}} t_{j_{\alpha 1}} t_{k_{\alpha 1}} 1_{(w(t_{i'_{\alpha 1}}) = w(t_{j_{\alpha 1}}) = w(t_{k_{\alpha 1}}))} + r_3') \right) \rangle_{w'}^{(G_1)_{X_i'}, L.C.} \quad (41)$$

where $\eta$ is

$$\eta = \frac{1}{2} \left\langle \sum_{X_i \in (G_1)_{X_i'}} \sum_{i_{\alpha 1} < j_{\alpha 1} < i_{\alpha 2} < j_{\alpha 2}} t_{i_{\alpha 1}} t_{j_{\alpha 1}} 1_{(w(t_{i_{\alpha 1}}) = w(t_{j_{\alpha 1}}))} \times \right.$$  

$$\times t_{i_{\alpha 2}} t_{j_{\alpha 2}} 1_{(w(t_{i_{\alpha 2}}) = w(t_{j_{\alpha 2}}))} \chi \left((i_{\alpha 1}, j_{\alpha 1}, i_{\alpha 2}, j_{\alpha 2}) \in \{j_{\alpha 1} < j_{\alpha 2}\}\right) \rangle_{w'}^{(G_1)_{X_i'}, L.C.}. \quad (42)$$

$\eta$ is explained in Figure 3.

Note that in eq(41)

$$\left\langle \sum_{X_i \in (G_1)_{X_i'}} \sum_{i_{\alpha 1} < j_{\alpha 1} < i_{\alpha 2} < j_{\alpha 2}} t_{i_{\alpha 1}} t_{j_{\alpha 1}} 1_{(w(t_{i_{\alpha 1}}) = w(t_{j_{\alpha 1}}))} \chi \left((i_{\alpha 1}, j_{\alpha 1}) \in \{j_{\alpha 1} < j_{\alpha 2}\}\right) \right\rangle_{w'}^{(G_1)_{X_i'}, L.C.} = 0 \quad (43)$$

for all set of walks $\{w\} \in \Gamma_n$, being $\Gamma_n$ the space of random walks with only double (self-)intersecting walks.

Note that the case $n_{i'}^{\alpha} = 1$, for some $0 \leq i' \leq k$, shows how the weakly (self-)avoiding random walk on the $(G_1)_{X_i'}$ coset can lead to a “mass” term;
i.e. a local time contribution, after the transformation is applied. This shall be used in next section to obtain the asymptotic end-to-end distance of a weakly SARW on a hierarchical lattice, \( d = 4 \), heuristically. Similarly, the case \( n'_{ij} = 3 \), for some \( j' \neq i' \) and \( 0 \leq j' \leq k \), shows how a different realization on the \( G_1 \) cosets of the weakly SARW can render, a pair of double intersections of random walks inside the \( (G_1)_{X'_{j'}} \) coset, to the triple intersection of renormalized walks in \( X'_{j'} \) after the renormalization-group map is applied. See Figure 2.

Writing together cases \( n'_{ij} = 1, 2, 3 \) we obtain

\[
U'(w') = \prod_{X'_{i'} \in G} \left( \sum_{t'_{i'1} \in JX'_{i'}} e^{-\frac{\lambda^2}{\eta_1} \sum_{n'_{ij} \neq i'_{ij} < k'} \sum_{\{i'_{ij}, j'_{ij}, k'_{ij}\} \in JX'_{i'}} t'_{i'1} t'_{j'1} t'_{k'1} 1_{(w(t'_{i'1}) = w(t'_{j'1}) = w(t'_{k'1}))} + r'} \right) + r'.
\]

where \( r' \sim O(\lambda^3) \) bounds the cosets where \( n'_{ij} \geq 4 \), for some \( 0 \leq i' \leq k \). Note that \( \lambda^3 \) is the leading contribution to \( r' \) if \( n'_{ij} = 4 \). As \( n'_{ij} \) increases the leading contribution decreases. Here

\[
\xi' = \xi_2 + L^\beta \xi
\]

\[
\eta_1' = \eta_1^2 \quad \lambda' = L^{(2\beta - d)} \lambda - \beta' \lambda^2 + O(\lambda^3)
\]

We apply once more the renormalization-group map to eq(44). In sake of clarity let us suppress primes in eq(44), so the primed terms always correspond to the renormalized ones. From the initial hypothesis, approaching the three walks intersection factor in eq(44) to an exponential, factorize this as done
where \( r \) in eq(45), then expanding the exponential up to the first order, i.e. \( \sim (\eta \lambda^2) \); from the result in theorem 3, eq(36), eq(39) and eq(41), follows

\[
U'(w') = \prod_{X'_j \in G'} e^{-\xi' \sum_{i_{a1} \in J_{X'_j}'} t'_{i_{a1}1} - \lambda' \sum_{i_{a1} < j_{a1} < k_{a1}} \sum_{\{i'_{a1}, j'_{a1}, k'_{a1}\} \in J_{X'_j}'} t'_{i_{a1}1} t'_{j_{a1}1} t'_{k_{a1}1} 1_{(w(t_{i_{a1}}) = w(t_{j_{a1}}) = w(t_{l_{a1}}))} \times 
\]

\[
1 + \eta'_1 \sum_{i_{a1} < j_{a1} < k_{a1}} t'_{i_{a1}1} t'_{j_{a1}1} t'_{k_{a1}1} 1_{(w(t'_{i_{a1}}) = w(t'_{j_{a1}}) = w(t'_{k_{a1}}))} + \eta'_3 \sum_{i_{a1} \in J_{X'_j}'} t'_{i_{a1}1} \right) + r'.
\]

where \( r' \sim O(\lambda^3) \) and

\[
\xi_2 = \gamma_1 \lambda - \gamma_2 \lambda^2 + O(\lambda^3),
\]

\[
\xi' = L^2 \xi + \xi'_2,
\]

\[
\lambda' = L^{(2\beta - d)} \lambda - \beta_1 \lambda^2 + O(\lambda^3); \quad \eta'_1 = \eta_1 A + \eta \lambda^2
\]

\[
\eta'_2 = \eta_1 B, \quad \eta'_3 = \eta_1 C
\]

In eq(45) \( A, B \) and \( C \) are respectively given as follows;

\[
A = \left( \sum_{X_i \in (G_1)_{X_i}'} \sum_{i_{a1} < j_{a1} < k_{a1}} t_{i_{a1}1} t_{j_{a1}1} t_{k_{a1}1} 1_{(w(t_{i_{a1}}) = w(t_{j_{a1}}) = w(t_{k_{a1}}))} \times \right) \left( \sum_{X_i \in (G_1)_{X_i}'} \sum_{i_{a1} < j_{a1} < k_{a1}} t_{i_{a1}1} t_{j_{a1}1} t_{k_{a1}1} 1_{(w(t_{i_{a1}}) = w(t_{j_{a1}}) = w(t_{k_{a1}}))} \times \right)
\]

\[
\chi \left( (i'_{a1}, j'_{a1}, k'_{a1}) \in \left( i'_{a1}, j'_{a1}, k'_{a1} \right) \right) \right)^{L.c.}
\]

23
A, B and C are explained in Figure 3.

Note that, to get the result in eq(45), we have studied the trajectory, under the renormalization-group map R, of

\[ B = \left\langle \sum_{X_i \in (G_1)_{X_{i}'}} \sum_{i_{\alpha_1} < j_{\alpha_1} < k_{\alpha_1}} t_{i_{\alpha_1}} t_{j_{\alpha_1}} t_{k_{\alpha_1}} \mathbf{1}_{(w(t_{i_{\alpha_1}}) = w(t_{j_{\alpha_1}}) = w(t_{k_{\alpha_1}}))} \times \sum_{i_{\alpha_1} \in J_X_i} t_{i_{\alpha_1}} t_{j_{\alpha_1}} t_{k_{\alpha_1}} \mathbf{1}_{(w(t_{i_{\alpha_1}}) = w(t_{j_{\alpha_1}}) = w(t_{k_{\alpha_1}}))} \times \chi \left( (i_{\alpha_1}, j_{\alpha_1}, k_{\alpha_1}) \in (i'_{\alpha_1}, j'_{\alpha_1}) \right) \right\rangle_{w'} \] 

\[ C = \left\langle \sum_{X_i \in (G_1)_{X_{i}'}} \sum_{i_{\alpha_1} < j_{\alpha_1} < k_{\alpha_1}} t_{i_{\alpha_1}} t_{j_{\alpha_1}} t_{k_{\alpha_1}} \mathbf{1}_{(w(t_{i_{\alpha_1}}) = w(t_{j_{\alpha_1}}) = w(t_{k_{\alpha_1}}))} \times \sum_{i_{\alpha_1} \in J_X_i} t_{i_{\alpha_1}} t_{j_{\alpha_1}} t_{k_{\alpha_1}} \mathbf{1}_{(w(t_{i_{\alpha_1}}) = w(t_{j_{\alpha_1}}) = w(t_{k_{\alpha_1}}))} \times \chi \left( (i_{\alpha_1}, j_{\alpha_1}, k_{\alpha_1}) \in (i'_{\alpha_1}, j'_{\alpha_1}) \right) \right\rangle_{w'} \] 

When we apply the renormalization-group map to eq(49) we end up with the contributions in A, B and C. In other words, we use the same procedure above developed for the double (self-)intersection of random walks but for the triple (self-)intersection of random walks up to order $\lambda^2$, this corresponds to the first term in the corresponding Taylor series expansion.

Applying the renormalization-group map to eq(45) we obtain eq(22) (eq(44) and eq(45) are particular cases of eq(22)). Then, the proof follows from induction. We apply the renormalization-group map to $U'(w')$ (eq(22)). Recall that we suppress primes in eq(22), thus primed terms are the renormalized ones. From our original hypotheses and due to the hierarchical structure of the lattice;

\[ U'(w') = \prod_{X_i \in G} \left( \prod_{X_i \in (G_1)_{X_i'}} e^{-\xi \sum_{i_{\alpha_1} \in J_X_i} t_{i_{\alpha_1}}} \right)_{w'} \]
\[
\times \left\langle \prod_{X_i \in (G_1)_{X_i'}} e^{-\lambda \sum_{i_{\alpha_1} < j_{\alpha_1} \atop \{i_{\alpha_1}, j_{\alpha_1}\} \in J_{X_i}} t_{i_{\alpha_1}} t_{j_{\alpha_1}} t_{k_{\alpha_1}} 1(w(t_{i_{\alpha_1}}) = w(t_{j_{\alpha_1}}) = w(t_{k_{\alpha_1}}))} \right\rangle_{w'}^L_c \times
\]

\[
\langle 1 + \eta_1 \sum_{X_i \in (G_1)_{X_i'}} \sum_{i_{\alpha_1} < j_{\alpha_1} < k_{\alpha_1} \atop \{i_{\alpha_1}, j_{\alpha_1}, k_{\alpha_1}\} \in J_{X_i}} t_{i_{\alpha_1}} t_{j_{\alpha_1}} t_{k_{\alpha_1}} 1(w(t_{i_{\alpha_1}}) = w(t_{j_{\alpha_1}}) = w(t_{k_{\alpha_1}})) +
\]

\[
+ \eta_2 \sum_{X_i \in (G_1)_{X_i'}} \sum_{i_{\alpha_1} < j_{\alpha_1} \atop \{i_{\alpha_1}, j_{\alpha_1}\} \in J_{X_i}} t_{i_{\alpha_1}} t_{j_{\alpha_1}} 1(w(t_{i_{\alpha_1}}) = w(t_{j_{\alpha_1}})) +
\]

\[
+ \eta_3 \sum_{X_i \in (G_1)_{X_i'}} \sum_{i_{\alpha_1} \in J_{X_i}} t_{i_{\alpha_1}} \rangle_{w'}^L_c + r'.
\]

Assuming \(n_{*_{i'}} = 1, 2, 3\) and carrying out a Taylor series expansion up to order \(\lambda^2\), it is straightforward to prove that eq(50) leads to eq(22), provided we use the same bookkeeping device above explained. We just need to apply theorem 3, eq(36), eq(39), eq(41) and

\[
\left\langle \prod_{X_i \in (G_1)_{X_i'}} \exp(\eta_1 \sum_{i_{\alpha_1} < j_{\alpha_1} \atop \{i_{\alpha_1}, j_{\alpha_1}\} \in J_{X_i}} t_{i_{\alpha_1}} t_{j_{\alpha_1}} t_{k_{\alpha_1}} 1(w(t_{i_{\alpha_1}}) = w(t_{j_{\alpha_1}}) = w(t_{k_{\alpha_1}})) + \right\rangle_{w'}^L_c
\]

\[
+ \eta_2 \sum_{i_{\alpha_1} < j_{\alpha_1} \atop \{i_{\alpha_1}, j_{\alpha_1}\} \in J_{X_i}} t_{i_{\alpha_1}} t_{j_{\alpha_1}} 1(w(t_{i_{\alpha_1}}) = w(t_{j_{\alpha_1}})) + \eta_3 \sum_{i_{\alpha_1} \in J_{X_i}} t_{i_{\alpha_1}} \rangle_{w'}^L_c =
\]

\[
\prod_{(n_{*_{i'}})_{i'} = 0} \left\langle \prod_{X_i \in (G_1)_{X_i'}} e^{\eta_3 \sum_{i_{\alpha_1} \in J_{X_i}} t_{i_{\alpha_1}}} \right\rangle_{w'_{n_{*_{i'}}}}^L_c.
\]
\[
\begin{split}
&\times \left\langle \prod_{X_i \in (G_1)_X} e^{\eta \sum_{\text{i_1 < j_1 < k_1}} t_{i_1} t_{j_1} t_{k_1} \mathbf{1}_{(w(t_{i_1})=w(t_{j_1})=w(t_{k_1}))}} \right\rangle^L_{c}\ \\
&\times \left\langle \prod_{X_i \in (G_1)_X} e^{\eta_2 \sum_{\text{i_1 < j_1}} t_{i_1} t_{j_1} \mathbf{1}_{(w(t_{i_1})=w(t_{j_1}))}} \right\rangle^L_{c}\\
&\left\{ \right. \\
&\left. \\n&1 + A \eta_1 \sum_{\text{i_1' < j_1' < k_1'}} t_{i_1'} t_{j_1'} t_{k_1'} \mathbf{1}_{(w(t_{i_1'})=w(t_{j_1'})=w(t_{k_1'}))} + \\
&\eta'_2 \sum_{\text{i_1' < j_1'}} t_{i_1'} t_{j_1'} \mathbf{1}_{(w(t_{i_1'})=w(t_{j_1'}))} + \eta'_3 \sum_{\text{i_1' \in J_{X_1'}}} t_{i_1'} \\
&\left. \right\} + r'_3
\end{split}
\]

where \( A, \eta'_2 \) and \( \eta'_3 \) are given by Figure 3, eq(27), and eq(28); \( r' \sim O(\lambda^3) \). Note that \( A \eta_1 \) in eq(51) plus the contribution from eq(41) defines \( \eta'_1 \) as done in eq(26). Eq(51) is written up to \( O(\lambda^2) \) terms.

Q.E.D.

We claim theorem 4 is the space-time renormalization-group trajectory of the weakly SARW energy interaction studied by Brydges, Evans and Imbrie [4], provided \( \beta = 2 \) and \( d = 4 \). In reference [5] the trajectory of a \( \lambda \phi^4 \) superalgebra valued interaction was studied (this can be understood in terms of intersection of random walks due to the Mc Kane, Parisi, Sourlas Theorem [15]) using a field-theoretical version of the renormalization-group map. The field theory is defined on the same hierarchical lattice we are studying here. In this paper, we provide exact probabilistic expressions for \( \lambda' \) and \( \xi' \) (which are not given in reference [5]), these are crucial to propose an heuristic proof for the asymptotic behavior of the end-to-end distance of a weakly SARW. To do so we just need to calculate \( \gamma_1 \) from eq (37) and \( \beta_1 \) from eq(40).
Finally, we summarize important features of our method;
a) the conditional expectation of $U(w)$ can be approached in terms of the product of conditional expectations.

b) We take into account only linear contributions to conditional expectations for probabilities on $\Lambda_n$.

c) Formal Taylor series expansions are introduced.

d) We assume our model to be such that each step the renormalization-group map is applied, the number of times the renormalized walk visits any site of the lattice is 1, 2 and 3 at least once (i.e. a fixed and not totally arbitrary topology for the renormalized random walk). See Figure 2.

e) Finally, we take advantage of the hierarchical structure of the lattice. Since $G' = G/G_1 \approx G$ and the map is local, the renormalization-group transformation descends to the study of walks in the $G_1$ cosets.

From all of these, we obtain, after applying the renormalization-group map, the fixed form for a weakly SARW that penalizes, roughly speaking, the (self-)intersection of two random walks by a factor $(e^{-\lambda}, \lambda > 0$ and small). Furthermore, this fixed form is the random walk version (for $d = 4$, $\beta = 2$) of the one obtained from a field-theoretical renormalization-group map for a $\lambda \phi^4$ model recently reported by Brydges, Evans and Imbrie \[5\]. We obtain an exact probabilistic expression for the parameters that appear in the flow of the interaction factor $\lambda$ which is not given in reference \[5\]. This shall be used in next section for the heuristic study of the asymptotic behavior of the end-to-end distance for a weakly SARW model that punishes the (self-)intersection of two random walks.
5 Asymptotic end-to-end distance of a weakly SARW on a hierarchical lattice in dimension four. An heuristic example as a testing ground.

The process of renormalizing the lattice is completed by reducing all dimensions of the new lattice by a factor \( L \) each step the renormalization-group map is applied so we end up with exactly the same lattice we start with. For a diffusive simple random walk model we reduce waiting times by \( L^2 \) each step we apply the renormalization-group transformation. Moreover, when we iterate probabilities, the end-to-end distance shrinks by a factor \( L \) at each interaction, because in renormalizing the lattice we divide every length, including the end-to-end distance, by \( L \). From this viewpoint we intend to understand, heuristically, the asymptotic end-to-end distance of a weakly SARW on a hierarchical lattice in \( d = 4 \), thereby providing a new probabilistic meaning to this magnitude.

For weakly SARW, we generalize the standard scaling factor for local times of the renormalization transformation above as described by including, up to \( O(\lambda) \), the contribution of the self-repulsion term to renormalized local times. Namely, from the renormalization-group map on weakly SARW, renormalized local times are generated from the interaction. In the field theoretical approach this corresponds to generating mass. Equivalently, we can say that the interaction kills the process at a specific rate. If we take into account only \( O(\lambda) \) contributions to this and follow standard thinking, the well known asymptotic end-to-end distance for the weakly SARW in \( d = 4 \) follows. By including higher order contributions in \( \lambda \) to renormalized local times (as we have already shown this is not the case for weakly SARW on the hierarchical lattice, because these contributions are no significant), and/or different dimension for the lattice, the functional form of the end-to-end distance changes drastically. Moreover, from our method, the exponent of the logarithmic correction involved is expressed in terms of conditional expecta-
tions for random walks on the lattice, that upon calculation, give the well
known exponent. In Figure 4 the contributions to the scaling factor proposed
for the weakly SARW used to explain the asymptotic end-to-end distance in
the hierarchical lattice are depicted.

We remark that the proposition presented in this section involves heuris-
tic considerations in order to understand, from a probabilistic real-space
viewpoint, the asymptotic end-to-end distance for the weakly SARW on a
hierarchical lattice in \( d = 4 \). This has already been conjectured heuristically
before by other means. Recently a rigorous proof has been given, provided
properties of the Green function are known, in the field theoretical approach
[6]. Our proposition is anyway presented as a testing ground for our method,
and for giving probabilistic meaning to the exponent involved in the loga-
Rithmic correction. Once the method shows to be useful for explaining well
known results (at least heuristically), we shall apply this on more complicated
cases, for example kinetically growing measure model. These are renor-
malizable, in the field theoretical limit, only for particular cases. In the process
of taking the continuum limit of these discrete models some memory is lost.
Our method is suitable of being applied on the discrete models. This can be
done both, heuristically and rigorously.

A final remark before introducing the main point of this section is about
the finiteness of moments for random walks on a hierarchical lattice. The
end-to-end distance for the weakly SARW, \( d=4 \), on the hierarchical lattice,
is independent of the moment used to obtain it, as should be, provided this
is finite. Let \( \langle w^\alpha(T) \rangle \) be an \( \alpha \)-moment of the random walk, it is known that
the only finite moments for diffusive random walks on a hierarchical lattice
are \( 0 < \alpha < 2 \) [3]. This range of \( \alpha \) values is used to obtain the end-to-end
distance in the following

**Proposition.** For \( d=4 \), up to \( O(\lambda) \), the generated renormal-
ized local times (mass for the field or killing rate for the process),
from applying the renormalization-group map on the interaction,
is such that the asymptotic behavior of the end-to-end distance for a weakly SARW that penalizes the intersection of two random walks is $T^{1/2} \log^{1/8} T$ as $T$ tends to infinity.

**Proof.** After applying $(p)$ times the renormalization-group transformation on $\langle w^\alpha(T) \rangle^{1/\alpha}$ we have

$$\langle w^\alpha(T) \rangle^{1/\alpha} = \frac{\langle w^\alpha(1) \rangle^{1/\alpha(0)}}{L^p} \tag{52}$$

where we have chosen a system of units such that, for $p = 0$, $T = 1$. Hereafter, $\langle w^\alpha(1) \rangle^{1/\alpha(0)} = D$, constant. Here, we are following the standard procedure for scaling length type magnitudes (i.e. $\langle w^\alpha(T) \rangle^{1/\alpha} = \frac{\langle w^\alpha(T) \rangle^{1/\alpha}}{L}$). Moreover, by $\langle w^\alpha(1) \rangle^{1/\alpha}$ we mean $\langle w^\alpha(1) \rangle^{1/\alpha(p)}$. Since in renormalizing the lattice we divide every length, including the end-to-end distance by $L$, then, upon $p$ iterations, eq(52) follows. This is exactly what is done in scaling correlation lengths but used here on the end-to-end distance, both length type magnitudes. So eq(52) becomes

$$\langle w^\alpha(T) \rangle^{1/\alpha} = L^{-p} D \tag{53}$$

On the other hand, from what we stated in theorem 5 we know that

$$T = \frac{1}{L^{2p} \prod_{i=1}^{p} (1 + \gamma_1^* \lambda(i))} \tag{54}$$

$\gamma_1^* = \gamma_1 / L^2$ and by $T$ we mean $T(p)$. Here we have included up to $O(\lambda)$ contributions to renormalized local times for scaling the running time of the process. In this scaling factor of the renormalization transformation, the $O(\lambda)$ contribution comes from the first term in right-hand side of eq(24). See Figure 4.

From eq(54) and eq(53) follows

$$\langle w^\alpha(T) \rangle^{1/\alpha} = DT^{1/2} \left( \prod_{i=1}^{p} (1 + \gamma_1^* \lambda(i)) \right)^{1/2} \tag{55}$$

or

$$\langle w^\alpha(T) \rangle^{1/\alpha} \sim DT^{1/2} \left( e^{\gamma_1^* \sum_{i=1}^{p} \lambda(i)} \right)^{1/2}. \tag{56}$$
For $\beta = 2$, $d = 4$ and up to order $(\lambda^{(i)})^2$, follows

$$\lambda^{(i+1)} = \lambda^{(i)} - \beta_1 (\lambda^{(i)})^2. \tag{57}$$

Introducing the solution of the eq(57) recursion into eq(56) this becomes

$$\langle w^\alpha(T) \rangle^{1/\alpha} \sim DT^{1/2} e^{\frac{\gamma_1^*}{2} \ln p} \tag{58}$$

or

$$\langle w^\alpha(T) \rangle^{1/\alpha} \sim DT^{1/2} (p)^{\frac{\gamma_1^*}{2\beta_1}} \tag{59}$$

In eq(59) we have assumed $p$ to be large enough so $\lambda^{-1} << \beta_1(p)$. Taking the asymptotic limit we rewrite eq(59) as

$$\langle w^\alpha(T) \rangle^{1/\alpha} \sim DT^{1/2} \log \frac{\gamma_1^*}{2\beta_1} T^*, \tag{60}$$

which is the asymptotic behavior of the end-to-end distance.

It only remains to know the value of $(\frac{\gamma_1^*}{2\beta_1})$. Actually we can calculate $\gamma_1^*$ and $\beta_1$ from their definitions.

Let us start with $\gamma_1$, from eq(37) we obtain

$$\gamma_1 = \left\{ \prod_{i_{a_1}, j_{a_1} \in J_{X'}} \right\}^{-1} \times$$

$$\times \left( \sum_{n_{a'}} L^d \int_{i_{a'} \in J_{X'}} dt_i \prod_{i_{a'} \in J_{X'}} \delta \left( \sum_{i=m_{a_1}+1}^{m_{a_1}} t_i - L^\beta t'_{a_1} \right) \timesight.$$

$$\left. \times \left( \frac{n_{a'} + 1}{2} \right) (q_1)^{(n_{a'}-1)} r^{(n_{a'})} (L^d - 1) \times \cdots \times (L^d - (n_{a'} - 1)) \timesight.$$

$$\times \sum_{X_i \in (G_1)_X} \sum_{i_{a_1}, j_{a_1} \leq j_{a_1}} t_{i_{a_1}} t_{j_{a_1}} \mathbf{1}_{(w(t_{i_{a_1}}) = w(t_{j_{a_1}}))} \chi \left( (i_{a_1}, j_{a_1}) \in i_{a_1}' \right) \right\} \text{l.c.}$$

31
Hereafter, we assume $L^d >> n_{\nu}$, so

$$(L^d - (n_{\nu} - 1)) \sim (L^d - 1),$$

i.e. the number of points inside each $G_1$ coset is larger than the number of steps the walk $w'$ spends inside each L-block. Thus, the numerator of eq(61) can be written as

$$\left( \sum_{\nu_1' \in J_{X'}} L^\beta t_{\nu_1'} \right)^2 \prod_{\nu_1' \in J_{X'}} \int_0^1 \sum_{n_{\nu}} (n_{\nu} + 1) (n_{\nu})^2 \times$$

$$\times (n_{\nu} - 1) \frac{(r q_1 (L^d - 1) (L^\beta t_{\nu_1'} (1 - t - t^*))^{n_{\nu}}}{n_{\nu}!} dt dt^*.$$}

where we have taken $(1 - t - t^*)^{n_{\nu} - 2} \sim (1 - t - t^*)^{n_{\nu}}$. To obtain an asymptotic estimate of eq(61), we assume that the following holds

$$(n_{\nu} + 1) (n_{\nu})^2 (n_{\nu} - 1) \sim \frac{(n_{\nu})!}{(n_{\nu} - 4)!}$$

Although from this follows $n_{\nu}$ chosen to be large, we certainly assume finite local times after the renormalization-group transformation is applied.

Substituting eq(62) in eq(61), with a jumping rate $r$ such that $r q_1 (L^d - 1) \sim 1$ (as done in reference[5]) and for $\beta = 2$ we obtain, in the asymptotic limit, $\gamma_1^* \sim 8$ provided $r q_1 (L^d - 1) \sim 1$.

To calculate $\beta_1$ we use eq(40).

$$\beta_1 = \frac{1}{2} \left\{ \prod_{i_{\alpha_1}' \in J_{X'}} e^{q_1 (L^d - 1) r (L^\beta t_{i_{\alpha_1}'})} \right\}^{-1} \times$$

$$\times \left( L^d \prod_{a_{\nu}} \prod_{b_{\nu}} \int_{i_a \in I_{X'}} dt_i \prod_{i_b \in I_{X'}} dt_i \prod_{i_{\alpha_1}' \in J_{X'}} \delta \left( \sum_{i_a = m_{\alpha_1}' - 1} t_i - L^\beta t_{i_{\alpha_1}'} \right) \times$$

$$32$$
As we did for the calculation of $\gamma_1$, we assume $L^d >> n'_a$, so
\[(L^d - (n'_a - 1)) \sim (L^d - 1) \text{ and } (L^d - (n'_b - 1)) \sim (L^d - 1). \tag{64}\]
Furthermore, we choose $n'_a \sim n'_b - 1$, and approximations in eq(62) to hold for both $n'_a$ and $n'_b$ with a jumping rate $r$ such that $rq_1(L^d - 1) \sim 1$. For $\beta = 2$ in the asymptotic limit, we obtain $\beta_1 \sim 32$ provided $rq_1(L^d - 1) \sim 1$.

Finally eq(60) becomes
\[
\langle w^\alpha(T) \rangle^{1/\alpha} \sim (DT)^{1/2} \log^{1/8} T \tag{65}\]
Q.E.D.

Note that $d = 4$ is the only choice that renders eq(25) (for $\beta = 2$) to a recursion as simple as eq(57) provided that $rq_1(L^d - 1) \sim 1$.

We want to remark that the heuristic study of the asymptotic end-to-end distance of a weakly SARW on a hierarchical lattice, $d = 4$, is independent of hypothesis a) in the summary of former section. This is because we could have obtained the $O(\lambda)$ contribution to renormalized local times without introducing initial mass into the process.

6 Summary

In this paper we present a real space renormalization-group map, on the space of probabilities, to study weakly SARW that penalizes the (self-)intersection
of two random walks for a hierarchical lattice, in dimension four. This hier-
archical lattice has been labeled by elements of a countable, abelian group
$G$. For any random function $F(w)$ on $G$ of the form described in Section
3, i.e. factorizable on the lattice, (see eq(17) and eq(21) for examples of
suitable $P(w)$) we can descend from the study of the space of walks on the
whole lattice to the trajectory in the contracting $G_1$ cosets. Then we show
how the Lévy process studied in reference [5] is a particular case of the pro-
cesses that are (or flow to) fixed points of the renormalization-group map.
We apply the renormalization-group map on some random walk models with
configurational measure, working out explicitly the weakly SARW case. An
heuristic proof of the end-to-end distance for a weakly SARW on a hierarchi-
cal lattice is derived. This gives a new probabilistic meaning to the exponent
of the logarithmic correction.

In Section 4 we study a weakly SARW that penalizes the (self-)intersection
of two random walks. The weakly SARW probability studied, involves a
factor linear in local times, i.e. a random walk representation of a field-
theoretical gaussian component that adds to the corresponding term pro-
duced for the renormalization-group map applied on the weakly SARW. We
show how this probability flows to a fixed form (the random walk version
of the field-theoretical result given in reference [5]) relying on;

a) An hypothesis that assumes we can approach the expectation of the
interaction energy in terms of the product of expectations for each of its
factors, conditioned to applying the renormalization-group map.

b) The hierarchical metric space used to label the lattice that allows,
for $F(w)$ of the form described in Section 3 (i.e. eq(17) and eq(21)), the
factorization in terms of the quotient group $G/G_1$ and the image of the
renormalization-group map on the cosets $G_1$, each step the renormalization-
group is applied.

c) A class of realizations of the model such that, each step we apply
the renormalization-group transformation, the renormalized fixed walk visits
$1, 2, 3$ times different sites in the lattice $G/G_1$, at least once. Other realiza-
tions might not allow us to study the flow of the (self-)intersecting coefficients that are interesting for us.

d) A formal Taylor series expansion in $\lambda$ from which, upon renormalization, we use only the linear contributions.

Our result improves the field-theoretical approach [4] by obtaining an exact expression for the parameters that appear in the flow of $\lambda$ and $\xi$. This is a crucial feature used to obtain heuristically the asymptotic behavior of the end-to-end distance for a weakly SARW that penalizes the (self-)intersection of two random walks. Furthermore, the method here presented is full of physical intuition and suitable of being applied to discrete kinetically growing measure models.

Following standard thinking we shrink all space and time magnitudes each step the map is applied. We shrink time taking into account $O(\lambda)$ contributions to renormalized local times, generated by applying the renormalization-group map to the weakly SARW that penalizes (self-) intersections of walks. Length type magnitudes are shrunk as usual. We present this, as a possible origin for the expression $\langle w^{\alpha}(T) \rangle^{1/\alpha} \sim (DT)^{1/2} log^{1/8} T$ as $T$ tends to infinity, in $d = 4$, for a weakly SARW that penalizes the (self-)intersection of two random walks on a hierarchical lattice.

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References
[1] Aragão de Carvalho C., Caracciolo S. and Fröhlich J., *Nucl. Phys.* B 215, 209-248, 1983.
Fernández R., Fröhlich J., and Sokal A.D., “Random walks, Critical Phenomena, and Triviality in Quantum Field Theory”, Springer-Verlag, Berlin, 1992.

[2] Flory, P., *J. Chem. Phys.* 17, 303-310, 1949.

[3] Dyson, F., *Commun. Math. Phys.* 12, 91-107, 1969.

[4] Benfatto, G., Cassandro, N., Gallavotti, G., Nicolo, F., Olivieri, E., Presutti, E., Scacciatelli, E., *Commun. Math. Phys.* 59, 143-166, 1978.
Gawedzki, K., Kupiainen, A., Asymptotic Freedom Beyond Perturbation Theory in *Critical Phenomena, Random Systems, Gauge Theories*. Les Houches, eds. Osterwalder, K., Stora, R., NorthHolland, 1986.
Gawedzki, K., Kupiainen, A., *J. Stat. Phys.* 29, 683-698, 1988.

[5] Brydges D., Evans S.N., Imbrie J.Z., *Ann. Probab.* 20, 82-124, 1992.

[6] Imbrie J., Mathematical Quantum Theory Summer School, University of British Columbia, Vancouver, Canada, Summer 1993.
Imbrie J., *Centre de Recherches Mathematiques Proceedings*, Volume 7, 191-198, 1994.

[7] Brydges, D. and Spencer, T., *Commun.Math.Phys.* 97, 125-148, 1985.
Slade G., *Commun. Math. Phys.* 110, 661-683, 1987.
Slade G., *J. Phys. A; Math.Gen.* 21, L417-L420, 1988.
Madras N., *J. Stat. Phys.* 53, 689-701, 1988.
Slade G., *Ann. Probab.* 17, 91-107, 1989.
Slade G., *Lecture in Applied Mathematics* 27, 53-63, 1991.
Hara T. and Slade G., *Commun. Math. Phys.* 147, 101-136, 1992.
Madras N., Slade G., “The self-avoiding walk”, Birkhäuser, Boston, 1993.
[8] Brézin E., Le Guillou, J.C., and Zinn-Justin, J., Field theoretical approach to critical phenomena. In Domb C. and Green M.S., editors, *Phase Transitions and Critical Phenomena* (Vol. 6), Academic Press London-New York-San Francisco, 1976.

De Gennes, P.G., *Physics Lett* 38A, **339-340**, 1970.

[9] Bulgadaev S.A. and Obukhov S.P., *Phys. Lett. B* 98A, **399**, 1983.

[10] Derkachov S.E., Honkonen J. and Vasilév A.N., *J. Phys. A* 23, **2479**, 1990.

[11] Lawler G.F., *J. Phys. A* 23, **1467-1470**, 1990.

[12] Burdzy K., Lawler G.F., Polaski T., *J. Stat. Phys.* 56, **1-12**, 1989.

Alkhimov V. I., *Phys. Lett. A* 133, **15-17**, 1988.

[13] Lawler G.F., “Intersection of Random Walks”, Birkhäuser, Boston, 1991.

[14] Duplantier B., *Europhys. Lett.* 1:491, 1986.

Duplantier B., *Commun. Math. Phys.* 117, **279-329**, 1988.

Duplantier B., *J. Stat. Phys.* 54, **581-680**, 1989, and references therein.

[15] McKane, A.J., *Phys. Lett. A* 76, **22-24**, 1980.

Parisi, G., J. Phys. Lett. 43, **744-745**, 1979.

Parisi, G., J. Phys. Lett. 41, **L403-L406**, 1980.

[16] Kadanoff, L.D., *Physics* 2, **263**, 1966.

Wilson, K.G. and Kogut, J., *Phys. Rep.* 12C, **75**, 1974.

Binney, J.J., Dowrick, N.J., Fisher, A.J., and Newman, M.E.J., *The Theory of Critical Phenomena. An Introduction to the renormalization group*, Chapter 5, Oxford Science Publications, 1992.
8 Figure Captions

Figure 1: a) One dimensional and b) two dimensional hierarchical lattices. Here, \( X_k, X_{k-1}, ... \) stands for (..., 0, \( X_k, X_{k-1}, ... \)).

Figure 2: An example of locality and factorization property of the renormalization-group map for a fixed, totally arbitrary \( w' \). Contributions to formal Taylor series expansion in the interaction are also depicted.

Figure 3: Conditional expectations of local times involved in renormalized weakly SARW up to \( O(\lambda^2) \).

Figure 4: Classes of contributions to the scaling factor for the renormalized running time of the process up to \( O(\lambda) \). From the renormalization-group map transformation.
Figure 4:
| \(\gamma_1\) | is the one (self-)intersection(s) of walks \(w\) once | \(\gamma_2\) | contribution to two double (self-)intersection(s) of walks \(w\) inside the \(G_1\) coset, that after applying the renormalization twice \(\eta\) | conditional once \(\beta_1\) | the linearized applied twice \(\eta\) \(\beta_1\) | \(\beta_1\) \text{-} group map corresponds to three times times \(A\) | expectation \(LX'_i \in G \sim G/G_1\), given \(X'_i\) triple twice \(B\) | of local times \(w'\) that visits \(X'_i\) once \(C\) | given by \(X'_i\) once |

Figure 3:
For $L = 2$, $|X - Y| = (\ldots, 1, 1, 1, 1) = 2^4$; $X_i \in \mathbb{Z}_2$ and $G = \bigoplus_{k=0}^{\infty} \mathbb{Z}_2$.

1.a) $|X - Y| = L_{scale}(X, Y) = L^4$.  

1.b) $|X - Y|_H = L_{scale}(X, Y) = L^3$.  

For $L = 2$, $|X - Y| = (\ldots, 1, 2, 2) = 2^3$; $X_i \in \mathbb{Z}_4$ and $G = \bigoplus_{k=0}^{\infty} \mathbb{Z}_4$.  

Figure 1:
$G_1 \in G$ marked in Figure 2.a) like 1,2,4,5,8,9,10,11,12, 3,7 are Figure 2.b) type
6 are Figure 2.c) type

Figure 2: