ON THE OPENNESS OF THE IDEMPOTENT BARYCENTER MAP

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ABSTRACT. We show that the openness of the idempotent barycenter map is equivalent to the openness of the map of Max-Plus convex combination. As corollary we obtain that the idempotent barycenter map is open for the spaces of idempotent measures.

1. INTRODUCTION

The notion of idempotent (Maslov) measure finds important applications in different part of mathematics, mathematical physics and economics (see the survey article [6] and the bibliography therein). Topological and categorical properties of the functor of idempotent measures were studied in [12]. There are some parallels between the theory of probability measures and idempotent measures (see e.g., [10]).

The problem of the openness of the barycentre map of probability measures was investigated in [3], [4], [2], [8] and [9]. In particular, it is proved in [8] that the barycentre map for a compact convex set in a locally convex space is open iff the map \((x, y) \mapsto 1/2(x + y)\) is open.

Zarichnyj defined in [12] the idempotent barycentre map for idempotent measures and asked the following two questions:

Question 7.2. [12] Characterize the class of max-plus convex compact spaces for which the idempotent barycenter map is open. In particular, is the latter property equivalent to the openness of the map \((x, y) \mapsto x \oplus y\)?

It is proved in [2] that the product of barycentrically open compact convex sets (i.e. compact convex sets for which the barycentre map is open) is again barycentrically open.

Question 7.3. [12] Is an analogous fact true for idempotent barycentrically open Max-Plus convex sets?

In this paper we characterize when the idempotent barycenter map is open. However we show that the openness of the idempotent barycenter map is not equivalent to the openness of the map \((x, y) \mapsto x \oplus y\). We also give an affirmative answer to the second question.

2. IDEMPOTENT MEASURES: PRELIMINARIES

In the sequel, all maps will be assumed to be continuous. Let \(X\) be a compact Hausdorff space. We shall denote the Banach space of continuous functions on \(X\) endowed with the sup-norm by \(C(X)\). For any \(c \in R\) we shall denote the constant function on \(X\) taking the value \(c\) by \(c_X\).

Let \(R_{\text{max}} = R \cup \{-\infty\}\) be the metric space endowed with the metric \(g\) defined by \(g(x, y) = |e^x - e^y|\). Following the notation of idempotent mathematics (see e.g., [7]) we use the notations \(\oplus\) and \(\odot\) in \(R\) as alternatives for max and + respectively. The convention \(-\infty \odot x = -\infty\) allows us to extend \(\odot\) and \(\oplus\) over \(R_{\text{max}}\).

Max-Plus convex sets were introduced in [3]. Let \(\tau\) be a cardinal number. Given \(x, y \in R^\tau\) and \(\lambda \in R_{\text{max}}\), we denote by \(y \oplus x\) the coordinatewise maximum of \(x\) and \(y\) and by \(\lambda \odot x\) the vector obtained from \(x\) by adding \(\lambda\) to each of its coordinates. A subset \(A\) in \(R^\tau\) is said to be Max-Plus convex if \(\lambda \odot a \oplus b \in A\) for all \(a, b \in A\) and \(\lambda \in R_{\text{max}}\) with \(\lambda \leq 0\). It is easy to check that \(A\) is Max-Plus convex iff \(\oplus_{i=1}^n \lambda_i \odot \delta_{x_i} \in A\) for all \(x_1, \ldots, x_n \in A\) and \(\lambda_1, \ldots, \lambda_n \in R_{\text{max}}\) such that \(\oplus_{i=1}^n \lambda_i \leq 0\). In the following by Max-Plus convex compactum we mean a Max-Plus convex compact subset of \(R^\tau\).

We denote by \(\odot : R \times C(X) \rightarrow C(X)\) the map acting by \((\lambda, \varphi) \mapsto \lambda \odot \varphi\), and by \(\oplus : C(X) \times C(X) \rightarrow C(X)\) the map acting by \((\psi, \varphi) \mapsto \max\{\psi, \varphi\}\).

Definition 2.1. [12] A functional \(\mu : C(X) \rightarrow R\) is called an idempotent measure (a Maslov measure) if

\[
\begin{align*}
(1) \quad & \mu(1_X) = 1; \\
(2) \quad & \mu(\lambda \odot \varphi) = \lambda \odot \mu(\varphi) \quad \text{for each} \quad \lambda \in R \quad \text{and} \quad \varphi \in C(X); \\
(3) \quad & \mu(\psi \oplus \varphi) = \mu(\psi) \oplus \mu(\varphi) \quad \text{for each} \quad \psi, \varphi \in C(X).
\end{align*}
\]

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Let $IX$ denote the set of all idempotent measures on a compactum $X$. We consider $IX$ as a subspace of $\mathbb{R}^{C(X)}$. It is shown in [12] that $IX$ is a compact Max-Plus subset of $\mathbb{R}^{C(X)}$. The construction $I$ is functorial which means that for each continuous map $f : X \to Y$ we can consider a continuous map $If : IX \to IY$ defined as follows $If(\mu)(\psi) = \mu(\psi \circ f)$ for $\mu \in IX$ and $\psi \in C(Y)$.

By $\delta_x$ we denote the Dirac measure supported by the point $x \in X$. We can consider a map $\delta X : X \to IX$ defined as $\delta X(x) = \delta_x$, $x \in X$. The map $\delta X$ is continuous, moreover it is an embedding [12]. It is also shown in [12] that the set

$$I\omega X = \{ \oplus_{i=1}^{n} \lambda_i \odot \delta_{x_i} | \lambda_i \in \mathbb{R}_{\text{max}}, i \in \{1, \ldots, n\}, \oplus_{i=1}^{n} \lambda_i = 0, x_i \in X, n \in \mathbb{N}\},$$

(i.e., the set of idempotent probability measures of finite support) is dense in $IX$.

Let $A \subset \mathbb{R}^T$ be a compact max-plus convex subset. For each $t \in T$ we put $f_t = \text{pr}_t|_A : A \to \mathbb{R}$ where $\text{pr}_t : \mathbb{R}^T \to \mathbb{R}$ is the natural projection. Given $\mu \in A$, the point $\beta_\mu(\mu) \in \mathbb{R}^T$ is defined by the conditions $\text{pr}_t(\beta_\mu(\mu)) = \mu(f_t)$ for each $t \in T$. It is shown in [12] that $\beta_\mu(\mu) \in A$ for each $\mu \in I(A)$ and the map $\beta_\mu : I(A) \to A$ is continuous. The map $\beta_\mu$ is called the idempotent barycenter map. It follows from results of [12] that for each compactum $X$ we have $\beta_{IX} \circ I(\delta X) = \text{id}_{IX}$ and for each map $f : X \to Y$ between compacta $X$ and $Y$ we have $\beta_{IY} \circ f^2 = If \circ \beta_{IX}$.

3. The openness of Max-Plus convex combination of idempotent measures

Let $X$ be a Max-Plus convex compactum. We consider a map $s_X : X \times X \times [-\infty, 0] \to X$ defined by the formula $s_X(x, y, t) = t \odot x \oplus y$. The main goal of this section is to prove that the map $s_{IX} : IX \times IX \times [-\infty, 0] \to IX$ is open for each compactum $X$. We start with a finite $X$.

**Lemma 3.1.** The map $s_{IX}$ is open for each finite compactum $X$.

**Proof.** Let $X = \{1, \ldots, n\}$. Since the functor $I$ preserves the weight [12], the compactum $IX$ is metrizable. Consider any $(\lambda, \beta, t) \in IX \times IX \times [-\infty, 0]$ and a sequence $(\alpha^j) \in IX$ converging to $t \odot \lambda \oplus \beta$. It is enough to find sequences $(\lambda^j)$, $(\beta^j)$ in $IX$ and a sequence $(t^j)$ in $[-\infty, 0]$ such that the sequence $(\lambda^j, \beta^j, t^j)$ converges to $(\lambda, \beta, t)$ and $t^j \odot \lambda^j \oplus \beta^j = \alpha^j$ for each $j \in \mathbb{N}$.

We have $\lambda = \oplus_{i=1}^{n} \lambda_i \odot \delta_i$, $\beta = \oplus_{i=1}^{n} \beta_i \odot \delta_i$ and $\alpha^j = \oplus_{i=1}^{n} \alpha_i^j \odot \delta_i$ where $\lambda_i, \beta_i, \alpha_i^j \in \mathbb{R}_{\text{max}}$ such that $\oplus_{i=1}^{n} \lambda_i = \oplus_{i=1}^{n} \beta_i = \oplus_{i=1}^{n} \alpha_i^j = 0$. Then $t \odot \lambda \oplus \beta = \oplus_{i=1}^{n} (t \odot \lambda_i \oplus \beta_i) \odot \delta_i$ and we have that the sequence $\alpha_i^j$ converges to $t \odot \lambda_i \oplus \beta_i$ for each $i \in \{1, \ldots, n\}$. We can assume (passing to a subsequence if necessary) that there exists $i_0 \in \{1, \ldots, n\}$ such that $\alpha_{i_0}^j = 0$ for each $j \in \mathbb{N}$.

Consider the case $t = 0$. We can represent $X = A \cup B \cup C$ where $A = \{i \in \{1, \ldots, n\} | \lambda_i < \beta_i\}$, $B = \{i \in \{1, \ldots, n\} | \lambda_i = \beta_i\}$ and $C = \{i \in \{1, \ldots, n\} | \lambda_i > \beta_i\}$. We can assume that $\alpha_i^j > -\frac{\lambda_i + \beta_i}{2}$ for each $j \in A \cup B$.

Consider the case $i_0 \in C$. Then $\lambda_{i_0} = \beta_{i_0} = 0$. Put

$$\lambda_i^j = \begin{cases} \lambda_i, & i \in A, \\ \alpha_i^j, & i \notin A \end{cases}$$

and

$$\beta_i^j = \begin{cases} \beta_i, & i \in B, \\ \alpha_i^j, & i \notin B \end{cases}$$

We have $\lambda_i^j \leq 0$, $\beta_i^j \leq 0$ and $\lambda_{i_0}^j = \beta_{i_0}^j = 0$. Put $\lambda^j = \oplus_{i=1}^{n} \lambda_i^j \odot \delta_i$ and $\beta^j = \oplus_{i=1}^{n} \beta_i^j \odot \delta_i$. Then the sequence $(\lambda^j, \beta^j, 0)$ converges to $(\lambda, \beta, 0)$ and $\lambda^j \oplus \beta^j = \alpha^j$ for each $j \in \mathbb{N}$.

Consider the case $i_0 \in A$. (The proof is analogous for the case $i_0 \in B$.) Put $c^j = \max\{|\alpha_i^j|, i \notin A\}$. The sequence $(c^j)$ converges to 0.

Put

$$\lambda_i^j = \begin{cases} \lambda_i, & i \in A, \\ \alpha_i - c^j, & i \notin A \end{cases}$$

and

$$\beta_i^j = \begin{cases} \beta_i, & i \in B, \\ \alpha_i^j, & i \notin B \end{cases}$$

We have $\lambda_i^j \leq 0$, $\beta_i^j \leq 0$ and $\beta_{i_0}^j = 0$. We also have $\lambda_i^j = 0$ for each $i \notin A$ such that $c^j = \alpha_i^j$. Put $\lambda^j = \oplus_{i=1}^{n} \lambda_i^j \odot \delta_i$ and $\beta^j = \oplus_{i=1}^{n} \beta_i^j \odot \delta_i$. Then the sequence $(\lambda^j, \beta^j, c^j)$ converges to $(\lambda, \beta, 0)$ and $c^j \odot \lambda^j \oplus \beta^j = \alpha^j$ for each $j \in \mathbb{N}$.

Finally consider the case $t < 0$. We have $X = A \cup B \cup C$ where $A = \{i \in \{1, \ldots, n\} | t \odot \lambda_i < \beta_i\}$, $B = \{i \in \{1, \ldots, n\} | t \odot \lambda_i = \beta_i\}$ and $C = \{i \in \{1, \ldots, n\} | t \odot \lambda_i = \beta_i\}$. We can assume that $\alpha_i^j > -\frac{\lambda_i + \beta_i}{2}$ for each $j \in A \cup B$.\]
We also have $i_0 \in A$ and $\beta_{i_0} = 0$. Put $c^j = \max\{\alpha_i^j - t - \lambda_i | i \notin A\}$ if there exists $s \in A$ such that $\lambda_s = 0$ and $c^j = \max\{\alpha_i^j - t | i \notin A\}$ otherwise. The sequence $(c_j)$ converges to $0$.

Put

$$
\lambda_i^j = \begin{cases} 
\lambda_i, & i \in A, \\
\alpha_i^j - c^j - t, & i \notin A
\end{cases}
$$

and

$$
\beta_i^j = \begin{cases} 
\beta_i, & i \in B, \\
\alpha_i^j, & i \notin B
\end{cases}
$$

We have $\lambda_i^j \leq 0$, $\beta_i^j \leq 0$ and $\beta_i^j = 0$. If $\lambda_i^j \neq 0$ for each $s \in A$, we have $\lambda_i^j = 0$ for each $i \notin A$ such that $c^j = \alpha_i^j - t$. Put $\lambda^j = \oplus_{i=1}^n \lambda_i^j \circ \delta_i$ and $\beta^j = \oplus_{i=1}^n \beta_i^j \circ \delta_i$. Then the sequence $(\lambda^j, \beta^j, t \circ c^j)$ converges to $(\lambda, \beta, t)$ and $(c^j \circ t) \circ \lambda^j \circ \beta^j = \alpha^j$ for each $j \in \mathbb{N}$.

Let

$$
X_1 \xrightarrow{p} X_2 \\
\downarrow f_1 \quad \quad \downarrow f_2 \\
Y_1 \xrightarrow{q} Y_2
$$

be a commutative diagram. The map $\chi : X_1 \rightarrow X_2 \times Y_2 Y_1 = \{(x, y) \in X_2 \times Y_1 | f_2(x) = q(y)\}$ defined by $\chi(x) = (p(x), f_1(x))$ is called a characteristic map of this diagram. The diagram is called bicommutative if the map $\chi$ is onto.

Lemma 3.2. The map $s_{IX}$ is open for each 0-dimensional compactum $X$.

Proof. Represent $X$ as the limit of an inverse system $\mathcal{C} = \{X_\alpha, p^\alpha_\beta, A\}$ consisting of finite compacta and epimorphisms. It is easy to check that $s_{IX} = \lim\{s_{I(X_\alpha)}\}$. By Proposition 2.10.9 [11] and Lemma 3.1 in order to prove that the map $s_{IX}$ is open, it is sufficient to prove that the diagram

$$
I(X_\alpha) \times I(X_\alpha) \times [-\infty, 0] \xrightarrow{I(p^\alpha_\beta)^* \times I(p^\alpha_\beta)^* \times \text{id}_{[-\infty, 0]}} I(X_\beta) \times I(X_\beta) \times [-\infty, 0] \\
\downarrow s_{I(X_\alpha)} \quad \quad \downarrow s_{I(X_\beta)} \\
I(X_\alpha) \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad I(X_\beta)
$$

is bicommutative for each $\alpha \geq \beta$.

Without loss of generality, one may assume that

$$
X_\alpha = \{x_1, \ldots, x_{n+1}\}, \quad X_\beta = \{y_1, \ldots, y_n\}
$$

(all the points are assumed to be distinct) and the map $p^\alpha_\beta$ act as follows: $p^\alpha_\beta(x_i) = y_i$ for each $i \in \{1, \ldots, n\}$ and $p^\alpha_\beta(x_{n+1}) = y_n$. Thus, given $(\nu, (\mu, \alpha, t)) \in I(X_\alpha) \times I(X_\beta) I(X_\beta) \times I(X_\beta) \times [-\infty, 0]$ one can write $\nu = \oplus_{i=1}^{n+1} \nu_i \circ \delta_i$, $\mu = \oplus_{i=1}^{n+1} \mu_i \circ \delta_i$, and $\alpha = \oplus_{i=1}^{n+1} \alpha_i \circ \delta_i$.

Since $I(p^\alpha_\beta)^* (\nu) = t \circ \mu + \alpha$, we have

$$
\nu_i = t \circ \mu_i + \alpha_i, \quad i \in \{1, \ldots, n-1\}
$$

and

$$
\nu_n + t \circ \mu_n = t \circ \mu_n + \alpha_n.
$$

Put

$$
\lambda_i = \mu_i, \quad \eta_i = \alpha_i, \quad i \in \{1, \ldots, n-1\}, \quad \lambda_n = \min\{\mu_n, \nu_n - t\}, \quad \lambda_n+1 = \min\{\mu_n, \nu_n+1 - t\}
$$

and

$$
\eta_n = \min\{\alpha_n, \nu_n\}, \quad \eta_{n+1} = \min\{\alpha_n, \nu_{n+1}\}.
$$

It is a routine checking that

$$
\lambda_n + \lambda_{n+1} = \mu_n, \quad \eta_n + \eta_{n+1} = \alpha_n
$$

and

$$
\nu_n = t \circ \lambda_n + \eta_n, \quad \nu_{n+1} = t \circ \lambda_{n+1} + \eta_{n+1}.
$$

Hence we obtain $s_{I(X_\alpha)} (\lambda, \eta, t) = \nu$ and $I(p^\alpha_\beta)^* \times I(p^\alpha_\beta)^* \times \text{id}_{[-\infty, 0]} (\lambda, \eta, t) = (\mu, \alpha, t)$ for $\lambda = \oplus_{i=1}^{n+1} \lambda_i \circ \delta_i$ and $\eta = \oplus_{i=1}^{n+1} \eta_i \circ \delta_i$. \qed
Theorem 3.3. The map $s_{1X}$ is open for each compactum $X$.

Proof. Choose a continuous onto map $f : Y \to X$ such that $Y$ is a 0-dimensional compactum and there exists a continuous $l : X \to IY$ such that $If \circ l = \delta X$. Existence of such map was proved in [12]. (It is called an idempotent Milyutin map.)

Define a map $\gamma : IX \to IY$ by the formula $\gamma = \beta_{lY} \circ II$. Then we have $If \circ \gamma = If \circ \beta_{lY} \circ II = \beta_{IX} \circ l f \circ II = \beta_{IX} \circ I(If \circ l) = \beta_{IX} \circ I(\delta X) = id_{IX}$. Since $I$ preserves surjective maps, $\gamma$ is an embedding and we can consider $IX$ as a subset of $IY$. (We identify $IX$ with $\gamma(IX)$.

Put $T = s_{1Y}^{-1}(IX)$. The map $s_{1Y}|T : T \to IX$ is open. Then we have $s_{1Y}|T = s_{1X} \circ (I f \times I f \times id_{[−\infty,0]})$ and $s_{1X}$ is open being a left divisor of the open map $s_{1Y}|T$.

4. The main results

We characterize openness of the barycenter map in this section. Since the set $I_\omega X$ is dense in $IX$, the following lemma can be obtained by direct checking for idempotent measures of finite support.

Lemma 4.1. The equality $\beta_X \circ s_{1X} = s_X \circ (\beta_X \times \beta_X \times id_{[−\infty,0]})$ holds for each Max-Plus convex compactum $X$.

Corollary 4.2. Let $X$ be a Max-Plus convex compactum, $\mu_1, \ldots, \mu_k \in IX$ and $\lambda_1, \ldots, \lambda_k \in [−\infty,0]$ be numbers such that $\max\{\lambda_1, \ldots, \lambda_k\} = 0$. Then we have $\beta_X(\oplus_{i=1}^k \lambda_i \circ \mu_i) = \oplus_{i=1}^k \lambda_i \circ \beta_X(\mu_i)$.

The notion of density for an idempotent measure was introduced in [11]. Let $\mu \in IX$. Then we can define a function $d_\mu : X \to [−\infty,0]$ by the formula $d_\mu(x) = \inf\{d(x, \varphi) \in C(X) \mid \varphi \leq 0, \varphi(x) = 0\}$, $x \in X$. The function $d_\mu$ is upper semicontinuous and is called the density of $\mu$. Conversely, each upper semicontinuous function $f : X \to [−\infty,0]$ with max $f = 0$ determines an idempotent measure $\nu_f$ by the formula $\nu_f(\varphi) = \max\{f(x) \circ \varphi(x) | x \in X\}$, for $\varphi \in C(X)$.

Lemma 4.3. Let $X$ be a Max-Plus convex compactum, $\mu \in IX$ and $U$ be an open neighborhood of $\mu$. Then there exists $\nu \in I_\omega X \cap U$ such that $\beta_X(\nu) = \beta_X(\mu)$.

Proof. By $d_\mu$ we denote the density of $\mu$. Let $U = \{U_1, \ldots, U_k\}$ be a closed Max-Plus convex cover of $X$. For $i \in \{1, \ldots, k\}$ put $s_i = \max\{d_\mu(y) \mid y \in U_i\}$ and define a function $d_i : X \to [−\infty,0]$ by the formula $d_i(x) = \begin{cases} d_\mu(x) - s_i, & x \in U_i, \\ -\infty, & x \notin U_i. \end{cases}$

It is easy to check that $d_i$ is an upper semicontinuous function with max $d_i = 0$. Denote by $\mu_i$ the idempotent measure determined by $d_i$ and put $x_i = \beta_X(\mu_i)$.

Define $\nu_U \in I_\omega X$ by the formula $\nu_U = \oplus_{i=1}^k s_i \circ \delta_{x_i}$. By Corollary 4.2 we have $\beta_X(\nu_U) = \beta_X(\oplus_{i=1}^k s_i \circ \mu_i)$. Since $\oplus_{i=1}^k s_i \circ \mu_i = \mu$, we obtain $\beta_X(\nu_U) = \beta_X(\mu)$.

Now $\{\nu_U\}$ forms a net where the set of all finite closed Max-Plus convex covers is ordered by refinement. Then $\nu_U \to \mu$.

Theorem 4.4. Let $X$ be a Max-Plus convex compactum. Then the following statements are equivalent:

1. the map $\beta_X|_{I_\omega X} : I_\omega X \to X$ is open;
2. the map $\beta_X$ is open;
3. the map $s_X$ is open.

Proof. The implication 1. $\Rightarrow$ 2. follows from Lemma 4.3.

2. $\Rightarrow$ 3. Consider any $(x, y, t) \in X \times X \times [−\infty,0]$. Let $W$ be an open neighborhood of $(x, y, t)$. We can suppose that $W = V \times U \times O$ where $V$, $U$, and $O$ are open neighborhoods of $x$, $y$, and $t$ correspondingly. Since the map $s_{1X}$ is open by Theorem 3.3, the set $s_{1X}(\beta_X^{-1}(V) \times \beta_X^{-1}(U) \times O)$ is open in $IX$. Then $\beta_X \circ s_{1X}(\beta_X^{-1}(V) \times \beta_X^{-1}(U) \times O)$ is open in $X$.

Let us show that $\beta_X \circ s_{1X}(\beta_X^{-1}(V) \times \beta_X^{-1}(U) \times O) = s_X(V \times U \times O)$. Consider any $y \in \beta_X \circ s_{1X}(\beta_X^{-1}(V) \times \beta_X^{-1}(U) \times O)$. Then there exists $(\mu, \nu, p) \in \beta_X^{-1}(V) \times \beta_X^{-1}(U) \times O$ such that $\beta_X \circ s_{1X}(\mu, \nu, p) = y$. It follows from Lemma 4.1 that $\beta_X \circ s_{1X}(\mu, \nu, p) = s_X(\beta_X(\mu), \beta_X(\nu), p$, hence $y \in s_X(V \times U \times O)$.

Now take any $z \in s_X(V \times U \times O)$. Then there exists $(r, q, p) \in V \times U \times O$ such that $z = s_X(r, q, p)$. By Lemma 4.1 we have $z = \beta_X \circ s_{1X}(\delta_{x}, \delta_{y}, p)$. Hence $z \in \beta_X \circ s_{1X}(\beta_X^{-1}(V) \times \beta_X^{-1}(U) \times O)$.

3. $\Rightarrow$ 1. Consider any $\nu = \oplus_{i=1}^k \lambda_i \circ \delta_{x_i} \in I_\omega X$. We will prove that for each net $\{x^\alpha\}$ converging to $\beta_X(\nu)$ there exists a net $\{\nu^\alpha\}$ converging to $\nu$ such that $\beta_X(\nu^\alpha) = x^\alpha$ for each $\alpha$.

We use the induction by $k$. For $k = 1$ the statement is obvious. Let us assume that we have proved the statement for each $k \leq l \geq 1$. 

Consider $k = l + 1$. Then $\nu = \sum_{i=1}^{l+1} \lambda_i \circ \delta_x$. We can assume that $\lambda_{l+1} = 0$. Put $t = \sum_{i=1}^{l} \lambda_i$ and $\nu_1 = \sum_{i=1}^{l} ((-t) \circ \lambda_i) \circ \delta_x$. We have $t \circ \nu_1 \circ \delta_{x+1} = \nu$. Hence $t \circ \beta_X(\nu_1) \circ x_{t+1} = \beta_X(\nu)$.

Consider any net $\{x^\alpha\}$ in $X$ converging to $\beta_X(\nu)$. Since the map $s_X$ is open, there exists a net $\{(y^\alpha, x^\alpha, t^\alpha)\}$ in $X \times X \times [-\infty, 0]$ converging to $(\beta_X(\nu_1), x_{t+1}, t)$ such that $t^\alpha \circ y^\alpha \circ x^\alpha_{t+1} = x^\alpha$. By the induction assumption there exists a net $\{\nu^\alpha\}$ converging to $\nu_1$ such that $\beta_X(\nu^\alpha) = y^\alpha$. Then the net $\{t^\alpha \circ \nu^\alpha \circ x^\alpha_{t+1}\}$ converges to $\nu$ and $\beta_X(t^\alpha \circ \nu^\alpha \circ x^\alpha_{t+1}) = x^\alpha$ for each $\alpha$.

Theorems 3.3 and 4.2 yield the following corollary.

**Corollary 4.5.** The map $\beta_{IX}$ is open for each compactum $X$.

Let us consider an example of a Max-Plus convex compactum $K$ such that the map $\beta_K$ is open but the map $(x, y) \mapsto x \oplus y$ is not. This example gives a negative answer to the second part of Zarichnyi Question 7.2 and demonstrate some difference between the theory of probability measures and idempotent measures. Put $K = ID$ where $D = \{0, 1\}$ is a two-point discrete compactum. Then the map $\beta_K$ is open by Corollary 4.5. Put $\nu = t \circ \delta_0 \circ \delta_1$. Then the map $\nu_{x, y} = t \circ \delta_0 \circ \delta_1$. Consider a function $\varphi \in C(D)$ defined by the formula $\varphi(i) = i$, $i \in \{0, 1\}$ and an open neighborhood $O = \{(\mu, \gamma) \in ID \times ID \mid |\mu(\varphi)| < 1/2\}$ of $(\delta_0, \delta_1)$ in $ID \times ID$. Consider any pair $(\alpha, \beta) \in ID \times ID$ such that $\alpha \circ \beta = \nu_{x, y}$ for some $i \in N$. We have $\alpha = x_0 \circ \delta_0 \circ \alpha_0 \circ \delta_1$ and $\beta = x_0 \circ \delta_0 \circ \beta_0 \circ \beta_1 = x_0 \circ \delta_0 \circ \beta_1$ for some $x_0 \circ \alpha_0 = x_0 \circ \beta_0 \circ \beta_1 = 0$. Since $x_0 \leq -1/2$, we have $x_0 = 0$ and $x_0(\varphi) \geq 1/2$. Hence $(\alpha, \beta) \notin O$ and the map $(x, y) \mapsto x \oplus y$ is not open.

Let $\{X_{\alpha}\}_{\alpha \in A}$ be a family of Max-Plus convex compacta. Then the product $X = \prod_{\alpha \in A} X_{\alpha}$ has a natural structure of Max-Plus convexity with coordinate-wise operation: $t \circ (x_{\alpha}) \oplus (y_{\alpha}) = (t \circ x_{\alpha} \oplus y_{\alpha})$ where $(x_{\alpha}), (y_{\alpha}) \in X$ and $t \in [-\infty, 0]$.

**Theorem 4.6.** Let $\{X_{\alpha}\}_{\alpha \in A}$ be a family of Max-Plus convex compacta such that all the maps $\beta_{X_{\alpha}}$ are open. Then the map $\beta_X$ is open.

**Proof.** We have by Theorem 4.1 that all the maps $s_{X_{\alpha}} : X_{\alpha} \times X_{\alpha} \times [-\infty, 0] \to X_{\alpha}$ are open. Consider the map $s_X : X \times X \times [-\infty, 0] \to X$. Take any point $(x_{\alpha}), (y_{\alpha}), t \in X \times X \times [-\infty, 0]$ and put $z = (z_{\alpha}) = s_X((x_{\alpha}), (y_{\alpha}), t)$. Let $\{z^\alpha\}$ be a net converging to $z$. Consider any $\alpha \in A$. We have that the net $\{z^\alpha\}$ converges to $z_{\alpha}$. Since the map $s_{X_{\alpha}}$ is open, there exists a net $\{(x^\alpha_{\alpha}), (y^\alpha_{\alpha}), t^\alpha\}$ converging to $(x_{\alpha}), (y_{\alpha}), t$ in $X_{\alpha} \times X_{\alpha} \times [-\infty, 0]$ such that $s_{X_{\alpha}}((x^\alpha_{\alpha}), (y^\alpha_{\alpha}), t^\alpha) = z^\alpha_{\alpha}$.

Then the net $\{(x^\alpha_{\alpha}), (y^\alpha_{\alpha}), t^\alpha\}$ converges to $(x_{\alpha}), (y_{\alpha}), t$ in $X \times X \times [-\infty, 0]$ and $s_X((x^\alpha_{\alpha}), (y^\alpha_{\alpha}), t^\alpha) = z^\alpha_{\alpha}$. Hence the map $\beta_X$ is open by Theorem 4.1.

Let $X$ be a Max-Plus convex compact. Following [5] we call a point $x \in X$ an extremal point if for each two points $y, z \in X$ and for each $t \in [-\infty, 0]$ the equality $x = t \circ y \oplus z$ implies $x \in \{y, z\}$. The set of extremal points of a Max-Plus convex compactum $X$ we denote by ext$(X)$.

**Theorem 4.7.** Let $X$ be a Max-Plus convex compactum such that the map $\beta_X$ is open. Then the set ext$(X)$ is closed in $X$.

**Proof.** Suppose the contrary. There exists a net $\{x_{\alpha}\}$ in ext$(X)$ converging to a point $x \notin ext(X)$. Then there exist $y, z \in X$ and $t \in [-\infty, 0]$ such that $x = t \circ y \oplus z$ and $x \notin \{y, z\}$. Evidently, $y \neq z$. There exist open neighborhoods $V, U$ of $y$ and $z$ correspondingly such that $x \notin Cl(V \cup U)$. We can suppose that $x_{\alpha} \notin (V \cup U)$ for each $\alpha$. Since the map $s_X$ is open, there exist $x_{\alpha}, y_{\alpha} \in U, z_{\alpha} \in U$ and $p \in [-\infty, 0]$ such that $x_{\alpha} = p \circ y_{\alpha} \circ z_{\alpha}$. Since $x_{\alpha} \in ext(X)$, we have $x_{\alpha} \in \{y_{\alpha}, z_{\alpha}\} \subset V \cup U$ and we obtain a contradiction.

An example of a compactum with the closed set of extremal points and not open barycenter map was build in [8]. We construct an idempotent counterpart. Consider a subset $Y \subset [-2, 0]^2$ defined as follows $Y = A \cup B \cup C$ where $A = \{(x, y) \in [-2, 0]^2 \mid x \in [-2, -1], y = -1\}$, $B = \{(x, y) \in [-2, 0]^2 \mid x = -1 \in [-2, -1], y \in [-2, -1]\}$ and $C = \{(x, y) \in [-1, 0]^2 \mid x = y\}$. It is easy to see that $Y$ is a Max-Plus convex compactum. Consider points $a = (-2, -1), b = (-1, -2), c = (-1, -1)$ and a sequence $(c_i)$ where $c_i = (-1 + 1, -1 + 1)$ for $i \in \mathbb{N}$. Evidently the sequence $(c_i)$ converges to $c$. Put $\nu = \delta_0 \circ \delta_0$. We have $by(\nu) = c$. It is easy to check that there is no sequence $(\nu_i)$ converging to $\nu$ and hence $by(\nu) = c_i$. Hence $by$ is not open but ext$(X) = \{a, b, (0, 0)\}$.

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