Non-perturbative improvement of composite operators with Wilson fermions

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ABSTRACT

We propose a method to improve lattice operators composed of Wilson fermions which allows the removal of all corrections of $O(a)$, including those proportional to the quark mass, leaving only errors of $O(a^2)$. The method exploits the fact that chiral symmetry is restored at short distances. By imposing this requirement on correlation functions of improved lattice operators at short distances, the coefficients which appear in these operators can be determined. The method is an extension of the improvement program of the ALPHA collaboration, which, up to now, has only been applicable in the chiral limit. The extension to quarks with non-zero masses is particularly important for applications in heavy quark physics.
1 Introduction

An important source of errors in present lattice computations is the use of a finite lattice spacing, $a$. A systematic method for reducing discretization errors order by order in $a$ was proposed by Symanzik [1] and developed in ref. [2]. It consists of modifying the action and operators by “irrelevant” terms, chosen in such a way that the convergence to the continuum limit is accelerated. For Wilson fermions, in the first implementations of this procedure, the improvement coefficients were computed in perturbation theory, leaving errors of $O(a g_0^2)$ in physical quantities [3, 4], where $g_0$ is the bare coupling constant. It turns out, however, that, for lattice spacings used in present simulations, one-loop perturbation theory is not sufficiently accurate for some quantities [3, 4], even when “tadpole resummed” [7]. Recently, a method for determining the improvement coefficients beyond perturbation theory has been proposed and implemented, thus achieving full $O(a)$ improvement [8, 9]. This method allows one to obtain the improved action and the improved vector currents, but can only be applied to other composite operators, including the axial currents, in the chiral limit. We explain the reasons for this limitation in section 3. It is particularly significant for applications involving heavy quarks, where $O(a m)$ errors may be large [5]. Important examples in the phenomenology of $B$-mesons include the calculations of the leptonic decay constant $f_B$, the form factors of semileptonic $B$ decays, the $B$-parameters and the amplitudes of the radiative decay $B \to K^*\gamma$.

In this paper we suggest a new method, based on the short-distance behaviour of gauge-invariant correlation functions, which overcomes this limitation. The basic idea is to exploit the restoration of chiral symmetry at short distances in the renormalized theory. This is achieved by constructing finite improved operators by matching lattice and continuum correlators at short distances. The matching is implemented by requiring the vanishing of chirality violating form factors in the correlation functions. This procedure is justified by the following two observations. Firstly, at short distances, renormalized perturbation theory becomes chirally invariant (explicit chiral symmetry breaking effects induced by the regularization are reabsorbed by imposing the validity of the Ward Identities of chiral symmetry, while violations from the non-vanishing of quark masses disappear at short distances). Secondly, contributions due to the spontaneous breaking of chiral symmetry, which are absent in perturbation theory, vanish in this region. So both effects decrease as we go to short distances. In all the cases of interest, including the matrix elements of quark bilinear operators considered in this letter, the terms which violate chirality vanish either quadratically with the quark mass, i.e., as $m^2|x|^2$, where $|x|$ is the Euclidean distance, or linearly as $m \Lambda_{\text{QCD}}^3 |x|^4$. On the other hand the lattice artefacts vanish as $ma$. It is the different short-distance behaviour of the $O(ma)$ terms which allows us to eliminate them. Note that in this paper we are proposing a method for the determination of the coefficients of the $O(ma)$ terms. For this reason the quark mass is a free parameter, which can be chosen to optimise the procedure and does not necessarily coincide with any physical quark mass. For example we may envisage using a mass of the order of a few hundred MeV to extract the coefficients, and use these coefficients in studies of heavy quark physics.

The reason why mass corrections, other than the lattice artefacts, vanish either as $m^2|x|^2$ or as $m \Lambda_{\text{QCD}}^3 |x|^4$, can be easily understood. The chiral structure of QCD is such that

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1 By full $O(a)$ improvement we mean that the remaining discretization errors are of $O(a^2)$.

2 By $m$ we mean some choice of physical quark mass. We do not need to pick a specific definition in this paper.
contributions to gauge invariant correlation functions which arise in perturbation theory must contain an even number of mass insertions. For this reason the short distance perturbative corrections vanish at least as $m^2|x|^2$. On the other hand contributions due to spontaneous symmetry breaking require the insertion of the mass operator, $m\bar{q}q$. By dimensional counting this implies that the corrections are of the form $mA_{QCD}^3|x|^4$.

Before presenting the technical details of the method, we explain what we mean by “short distances” throughout this paper. In the continuum we simply mean that $|x|/\Lambda_{QCD} \ll 1$, where $|x|$ is the Euclidean distance. On the lattice, in addition, we require that $|x|/a$ is sufficiently large so as to avoid lattice artefacts ($a$ is the lattice spacing). The existence of a window, such that $a \ll |x| \ll \Lambda_{QCD}^{-1}$, requires that we have a sufficiently small lattice spacing.

To implement a full $O(a)$ improvement away from the chiral limit, further conditions should be satisfied. Indeed, it is necessary to vary the lattice spacing $a$, distance $|x|$ and quark mass $m$ in such a way that the chirality violating terms of $O(m^2|x|^2)$ and $O(mA_{QCD}^3|x|^4)$ are small enough to isolate the $O(ma)$ artefacts in the correlation functions. We postpone the detailed description of these conditions until subsection 4.1, where a specific example is discussed. For the remainder of the paper we will assume that $|x|$ and $m$ are chosen so that the terms of $O(m^2|x|^2)$ are much larger than those of $O(mA_{QCD}^3|x|^4)$ (this condition is straightforward to satisfy provided the lattice spacing is sufficiently small).

We wish to stress that the requirement of the existence of the window $a \ll |x| \ll \Lambda_{QCD}^{-1}$ is common to all renormalization schemes implemented using computations at a single value of $\beta (=6/g_0^2)$. In principle the constraint $|x| \ll \Lambda_{QCD}^{-1}$ can be avoided by using a sequence of lattices of decreasing $a$, and correspondingly of decreasing physical volumes. The idea is to determine the normalization constants by imposing renormalization conditions on a lattice with a very small spacing. On such a fine-grained lattice, very small distances $|x|$ are possible (so the renormalization scale, in momentum space, can be chosen to be very large) but the small physical size precludes the calculation of hadronic correlation functions. It is therefore necessary to determine the equivalent normalization constants for the larger, but coarser, lattice, on which physical matrix elements are computed. This is achieved by increasing the lattice spacing $a$ and matching the normalization constants at fixed $|x|$, and then by increasing $|x|$ and matching at fixed $a$. This procedure is repeated until the required value of $a$ is reached. This method has been proposed within the context of the Schrödinger functional approach in ref. [8], but can also be applied here, and is illustrated with an example in the appendix.

In ref. [10] a method of non-perturbative renormalization was proposed, based on the imposition of renormalization conditions on Green functions of operators between quark and gluon states in a fixed gauge. It is possible to generalize this procedure to implement improvement, exploiting the chiral behaviour of Green functions at large momenta, but this adds a number of theoretical complications. In a fixed gauge the basis of higher dimensional operators which must be added to the bare lattice ones is larger than with the gauge invariant method proposed here, particularly when computing off-shell Green functions. This basis is restricted by BRST-invariance rather than gauge invariance, and includes operators which vanish by the equations of motion, but which are not themselves BRST-invariant [11]. A detailed description of the procedure applied to quark bilinear operators will be presented in ref. [11], together with an exploratory computation of all the subtraction coefficients.

The use of chiral Ward identities to impose constraints on the normalization constants of op-
operators is a well established technique [12]. In particular, for Wilson and tree-level improved theories, they provide the normalizations of the vector and axial-vector currents, the ratio of the normalizations of the scalar and pseudoscalar densities and the values of the finite coefficients describing the mixing of lattice operators of different chiralities (this mixing is present due to the breaking of chirality by the Wilson term). Chiral Ward identities also provide important constraints for the determination of the operators necessary for full $O(a)$ improvement [8, 9] and, where possible, we propose to exploit them to determine some of the coefficients. As explained in some detail in section 3, however, it is not possible to determine all the improvement coefficients by the use of the Ward identities only.

One can also attempt to construct an improved action and operators by plotting the behaviour of the matrix elements of these operators as a function of the lattice spacing, and fixing the improvement coefficients by requiring that there are no linear terms in $a$ (including possible logarithmic factors). Such a procedure is impracticable, however, at least without a considerable amount of additional input based on the techniques described in refs. [8, 9] and in this letter.

The plan of the remainder of this letter is as follows. In the next section we briefly review the construction of the improved action and the determination of the mass dependence of the bare coupling constant in the unquenched theory. In section 3 we explain why the chiral Ward identities cannot be used to determine all the coefficients needed to construct improved lattice composite operators. The description of our method, and its application in the construction of improved lattice quark bilinear operators for quarks with non-zero masses is presented in section 4. Finally section 5 contains our conclusions and a discussion of future prospects.

2 Improvement of the action

The Wilson fermion action ($S_{\text{Wilson}}$) is improved by the addition of the SW operator [3],

$$S = S_{\text{gauge}} + S_{\text{Wilson}} + a \int d^4x \, O_{SW}$$

where $S_{\text{gauge}}$ is the gauge action and

$$O_{SW} = -\frac{i}{4} c_{SW} \sum_{\mu \nu} \bar{q} \sigma_{\mu \nu} F_{\mu \nu} q,$$

with $\sigma_{\mu \nu} = (i/2)[\gamma_{\mu}, \gamma_{\nu}]$. Here and in the following we use continuum notation to refer to lattice quantities. A method for the non-perturbative determination of $c_{SW}$ was presented and exploited in refs. [8, 9].

In order that physical quantities should approach their continuum values with deviations of only $O(a^2)$, we need, in addition to choosing the correct value of $c_{SW}$, to adjust the bare gauge coupling $g_0$ and the bare mass $m_0$ in a way which depends on the renormalized quark mass $m$. To see this, consider the renormalized coupling constant $g(\mu^2)$ defined, for example, as the three gluon vertex (in some fixed gauge) at large virtualities characterized by the scale $\mu^2$. The corresponding renormalization constant $Z_g$, defined by

$$g(\mu^2) = Z_g g_0(a),$$

(3)
contains terms of $O(ma)$ from diagrams which include quark loops, which must therefore be removed for improvement of unquenched lattice QCD. Following ref. [8] we absorb this linear term into a redefinition of the bare coupling constant $\tilde{g}_0$,

$$\tilde{g}_0^2 = g_0^2 (1 + b_g ma) ,$$

so that $Z_g$ has no remaining linear term in $ma$.

Thus, for each value of the bare quark mass, we need to determine $\tilde{g}_0$. For simplicity we take the quarks to be degenerate, so that there is only one mass parameter $m_0$; the generalization of the following discussion to the non-degenerate case is straightforward. We can imagine, at least in principle, proceeding as follows:

- Choose a value of $g_0$, compute a short-distance physical quantity $Q$ at several values of the quark mass at this value of $g_0$ and extrapolate the results to the chiral limit. An example of a suitable quantity is the derivative of the potential between a static quark and antiquark ($V'(r)$) at a separation of a fixed number of lattice spacings ($r = na$), satisfying $r \Lambda_{QCD} \ll 1$ and $rm \ll 1$. We denote the result by $Q(n, g_0, 0)$, where the final argument represents the fact that this value corresponds to $m = 0$.

- The quantity $Q$ must be such that its linear dependence on the mass is entirely due to lattice artefacts, and that these $O(ma)$ effects can be removed by fixing $c_{SW}$ and $b_g$ only. For physical quantities, such as decay constants and form factors, which are related to matrix elements of composite operators, there are further mass dependent terms of $O(ma)$ in their renormalization constants. It is therefore not possible in general to use these quantities to determine $b_g$ (with the exception of the matrix elements of the vector current, see subsection [12] below).

- We now choose a value of $m_0$ and recompute $Q$ at different values of the bare coupling until we obtain the same result, i.e. until we find a coupling $\tilde{g}_0$ such that $Q(n, \tilde{g}_0, m) = Q(n, g_0, 0)$. In this way we obtain $\tilde{g}_0$ for each $m_0$. As discussed in the introduction the result is independent of $n$, up to quadratic corrections of order $(mr)^2$.

- Finally, we need to determine the values of the lattice spacing corresponding to $\tilde{g}_0(a)$ and of the bare quark mass ($m_0$) of the physical quark. This can be done by simulating with a series of values of $\{m_0, \tilde{g}_0(m_0)\}$ and requiring that the computed results for two physical quantities, for example the masses of two hadrons, agree with their experimental values.

The technique described above needs considerable computing resources, since it requires new simulations at a number of different values of $\beta$ in order to map out the behaviour of $\tilde{g}_0$ with $m_0$. At this stage we are not very concerned about this, since it will be some time before an accurate determination of $b_g$ is actually required. It will only be needed when precise unquenched computations become possible ($b_g$ can be taken to be zero in quenched calculations). Even then, however, the effects of improvement (beyond e.g. tree-level) will be most significant for heavy quarks, for which the effects of quark loops are expected to be strongly suppressed. We also note that in first order perturbation theory $b_g$ is small, $b_g \simeq 0.012 N_f g_0^2$, where $N_f$ is the number of quark flavours [13] [4]. Thus, for the purposes of

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3 See ref. [8] for a method based on the use of Schrödinger Functionals.

4 The one-loop results for other improvement coefficients can be found in ref. [13].
this letter, we simply note that a non-perturbative computation of $b_g$ is possible in principle and postpone any detailed consideration of the most practical way of determining it. We focus instead on the more immediate problem of determining improved composite operators, thus, in particular, removing the $O(ma)$ effects due to heavy valence-quarks.

For the remainder of this letter we assume that $c_{SW}$ is known, and that, as the bare quark mass $m_0$ is varied, the bare coupling constant has been readjusted to the corresponding value $\tilde{g}_0(m_0)$.

### 3 Chiral Ward identities

Before describing our method for the construction of improved lattice composite operators, we explain why it is not possible to use chiral Ward identities to obtain all the necessary coefficients. As an example, consider a lattice quark bilinear operator $\bar{q}\Gamma q$, where $\Gamma$ is one of the Dirac matrices, and we have suppressed the flavour indices. The improved lattice operator takes the form

$$\hat{O}_\Gamma = Z_\Gamma (\bar{q}\Gamma q + ac_\Gamma O_{4,\Gamma}) ,$$

where the $O_{4,\Gamma}$ are possible dimension 4 operators with the same quantum numbers as $O_\Gamma$ (which will be exhibited explicitly in the following section; there are no such operators for the scalar or pseudoscalar densities). The operators on the right-hand side of eq.(5) are the bare lattice operators. $\hat{O}$ is constructed so that its matrix elements between physical states have discretization errors only of $O(a^2)$. Following ref. [8], we parametrize the mass dependence in the overall renormalization constants by

$$Z_\Gamma(m) = Z_\Gamma(0)(1 + b_\Gamma ma) .$$

The point which we wish to stress here is that the Ward identities do not determine the $b_\Gamma$’s. In the remainder of this section we briefly explain the reasons for this limitation and in the following one we propose a new method for the determination of the $b_\Gamma$’s.

Consider the following continuum Ward identity

$$\left\langle \left\{ \int_R d^4x \left[ 2mP(x) - \partial_\mu A_\mu(x) \right] O(y) - \delta O(y) \right\} O_1(z_1) O_2(z_2) \cdots \right\rangle = 0 ,$$

where $P$ and $A_\mu$ are a pseudoscalar density and axial current respectively, and $O, O_1, O_2$ etc. are composite operators, which, for the purposes of this letter, we can assume to be gauge invariant. We are free to make any convenient choice of the operators $\{O_1, O_2, \cdots\}$. For simplicity of presentation we suppress the flavour indices on all the operators. The integral is over a region $R$ which includes the point $y$, but excludes the points $z_i$. $\delta O$ is the axial variation of $O$. The improved lattice version of the operator $2mP(x) - \partial_\mu A_\mu(x)$ is obtained (up to an overall normalization) by requiring it to satisfy Ward identities of the form (up to terms of $O(a^2)$)

$$\left\langle \left[ 2mP(x) - \partial_\mu A_\mu(x) \right] O_1(z_1) O_2(z_2) \cdots \right\rangle = 0 ,$$

where $x$ is separated from the other points $z_i$. It might therefore be hoped that by subsequently imposing the identity in eq.(5) it would be possible to determine the improved forms of $O$ and $\delta O$ (up to an overall normalization factor). This is not so, however, since in order to satisfy this identity up to terms of $O(a^2)$, using lattice operators, we must add irrelevant
operators to $P(x)$ and $O(y)$ which “vanish” by the equations of motion, and which therefore do not contribute to physical matrix elements. For example, the improved form of $O_\Gamma$ in off-shell matrix elements takes the form
\[
\hat{O}_\Gamma = Z_\Gamma \left( q\Gamma q + a c'_\Gamma q \left( \Gamma(\overrightarrow{\not{D}} + m_0) + (-\overrightarrow{\not{D}} + m_0)\Gamma \right) q + ac_\Gamma O_{4,\Gamma} \right),
\]
where $\overrightarrow{\not{D}} + m_0$ is the lattice version of the fermion matrix appearing in eq.(1). The terms proportional to the new coefficients $c'_\Gamma$ contribute to the identity in eq.(7) in the region where $P(x)$ comes into contact with $O(y)$ \[\text{[\text{1}]}\]. Thus, because of these “contact terms”, there are extra coefficients, the $c'_\Gamma$’s, to determine, and in fact it is not possible to obtain the $c'_\Gamma$’s and the $b_\Gamma$’s separately using the Ward identities (and so it is not possible to obtain the $b_\Gamma$’s which are needed for the evaluation of physical matrix elements).

We now justify this claim in more detail. Using the expressions (3) for the operators $P$, $A_\mu$, $O$ and $\delta O$ in eq.(7), we find that the term containing the rotated lattice operator $\delta O$ is
\[
- Z_A Z_O \left[ \frac{Z_{\delta O}}{Z_A Z_O} - 2\rho a(c'_p + c'_O) \right] \langle \delta O(y) O_1(z_1) O_2(z_2) \cdots \rangle,
\]
where $\rho$ is computed directly from the lattice correlation functions in eq.(8):
\[
2\rho = \frac{\langle \partial_\mu A_\mu(x) O_1(z_1) O_2(z_2) \cdots \rangle}{\langle P(x) O_1(z_1) O_2(z_2) \cdots \rangle}.
\]

In eq.(11), $x$ is separated from the $\{z_i\}$, and $\rho$ is independent of the choice of the operators $\{O_i\}$ and of all the coordinates. Since the term proportional to $\rho$ in eq.(10) is multiplied by a factor of $a$, it is sufficient to take any convenient, not necessarily improved, lattice pseudoscalar density $P$ or axial current $A_\mu$ in eq.(11) \[\text{[\text{1}]}\]. From eq. (10) we see that the $b_\Gamma$’s and the $c'_\Gamma$’s appear in the combination:
\[
\frac{Z_{\delta O}(0)}{Z_A(0) Z_O(0)} \left\{ 1 + (b_{\delta O} - b_O - b_A) ma \right\} - 2\rho a(c'_p + c'_O),
\]
where the $b_\Gamma$’s are defined in eq.(3). For each bilinear $O$, the $O(a)$ part of the Ward Identity in eq.(4) allows one to obtain the combination of $O(a)$ that appear in eq.(12). Thus one only obtains a linear combination of the $b_\Gamma$’s and the $c'_\Gamma$’s, and even by considering all possible bilinears it is not possible to determine any of the $b_\Gamma$’s separately.

Despite this shortcoming, chiral Ward identities should be part of any improvement program; they provide a determination of the $c_\Gamma$’s as well as many checks of the numerical results for the remaining improvement coefficients.

4 Improvement of bilinear quark operators

In this section we propose a new method for the determination of the improvement coefficients in general, and the $b_\Gamma$’s in particular. We illustrate the method, by discussing the construction of improved quark bilinear operators. We consider only flavour non-singlet bilinears, but, unless specifically required for the discussion, we drop the flavour indices.

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5 Note that the integral of $\partial_\mu A_\mu$ in eq. (6) gives a surface integral, and so produces no contact terms.

6 In this letter we have not needed to define an improved quark mass. However, comparing equations (8) and (11), leads us to write $\rho = mZ_P/Z_A$. By computing $\rho$ using improved operators $A_\mu$ and $P$, whose construction is explained in section 4, the corresponding mass $m$ is indeed a renormalized improved quantity.
4.1 Pseudoscalar and scalar densities

The renormalized, improved pseudoscalar density has the general form

\[ \hat{P}(x) \equiv Z_P P(x), \quad (13) \]

where the renormalization constant \( Z_P \equiv Z_P(m) = Z_P(0)(1 + b_P ma + \cdots) \) is, in general, a function of the quark mass and \( P(x) = \overline{\psi}(x)\gamma_5 q(x) \) is the bare lattice pseudoscalar density. We start by constructing the two-point correlation function \( \hat{G}_{PP}(x) = \langle \hat{P}(x)\hat{P}(0) \rangle \) and by studying it in the limit \( |x| \to 0 \). By \( "|x| \to 0" \) we mean “short distances” in the sense explained in the Introduction. In this region, we expect on the one hand to be able to use the operator product expansion, keeping only the leading, most singular term, and yet on the other hand to neglect contributions from contact terms.

At any given value of \( m \), provided \( ma \ll 1 \), we determine \( Z_P \) by imposing, at short distances, the following renormalization condition

\[ \hat{G}_{PP}(x) = Z_P^2(m)\langle P(x)P(0) \rangle = \langle \hat{P}(x)\hat{P}(0) \rangle_{\text{cont}} \text{ as } |x| \to 0. \quad (14) \]

\( G_{PP}(x) \equiv \langle P(x)P(0) \rangle \) is the bare lattice pseudoscalar-pseudoscalar correlation function, computed by numerical simulation at the quark mass \( m \). In eq.\((14)\) \( \langle \hat{P}(x)\hat{P}(0) \rangle_{\text{cont}} \) is the pseudoscalar-pseudoscalar correlation function computed in perturbation theory in some continuum renormalization scheme, e.g. in the \( \overline{MS} \) or the \( RI \) schemes \([10]\) at a renormalization scale \( \mu \). A convenient choice is \( \mu \sim 1/|x| \). At short distances \( \langle \hat{P}(x)P(0) \rangle_{\text{cont}} \) is independent of the quark mass, up to subasymptotic corrections vanishing as \( m^2 |x|^2 \). This shows that we can determine the \( O(am) \) correction to \( Z_P(m) \) using only gauge invariant quantities.

An alternative method to determine \( Z_P \) is the following. We define, at short distances,

\[ R_P = (1 + b_P ma)^2 = \frac{G_{PP}(x)|_{m=0}}{G_{PP}(x)|_m}, \quad (15) \]

where \( G_{PP}(x)|_{m=0} \) is the bare lattice correlator \( G_{PP} \) extrapolated to the chiral limit. Thus a computation of the ratio \( R_P \) provides us with a determination of \( b_P \). In addition, at short distances \( R_P \) is independent of \( x \), which provides us with a set of consistency conditions.

The leading physical chirality-violating correction to \( R_P(x) \) is of the form (up to logarithmic corrections) \( m^2 |x|^2 \). In order to be able to extract \( b_P \) we require \( m^2 |x|^2 \ll ma \), which at fixed \( a \) imposes constraints on the allowed values of \( m \) and \( |x| \). In practice, for values of the lattice spacing currently used in simulations \( (a^{-1} = O(2-4 \text{ GeV})) \), these conditions can be implemented without difficulty \([11]\). In addition, one can estimate the \( O(m^2 |x|^2) \) terms by studying the short-distance dependence of \( R_P(x) \) on \( x \) at fixed \( m \), and try in this way to improve the precision in the determination of \( b_P \). These considerations also apply to all the bilinears considered below.

The above is a good example illustrating the necessity of knowing \( b_g \) (or equivalently, of knowing \( g_0(a) \) for each value of the quark mass). The numerator and denominator on the right-hand side of eq.\((15)\) have to be evaluated at the same values of the coordinates \( x \) (in

\[^7\] At short distances, the terms of \( O(m^3 \Lambda_{QCD}^3 |x|^4) \) are naturally negligible compared to those of \( O(m^2 |x|^2) \) provided that the mass \( m \) is not much smaller than \( \Lambda_{QCD}^3 \).
physical units). As explained in section 2, in order to satisfy this requirement (up to $O(a^2)$ corrections), the bare coupling constant has to be varied as a function of the mass as in eq. (4).

Having determined the mass dependence of $Z_P(m)$, $Z_P(0)$ can be obtained for example, by imposing renormalization conditions on the pseudoscalar density in the chiral limit using the Schrödinger functional techniques [3] or by determining $Z_P(m)$ on quark states, as suggested in ref. [10], and then by extrapolating the results for $Z_P(m)$ to the chiral limit. In the latter case we can ignore completely the extra terms coming from operators which vanish by the equation of motions, or from non-gauge invariant operators, since the systematic error due to these terms disappears as $m \to 0$ [11].

The determination of the renormalization constant of the scalar density, $Z_S$, proceeds in exactly the same way. The ratio of the renormalization constants $Z_S(0)/Z_P(0)$, which is a finite quantity, can be determined in the standard way using chiral Ward Identities [12].

### 4.2 The Vector Currents

The improved vector current has the general form

$$\hat{V}_\mu(x) = Z_V V_\mu(x) ,$$  

(16)

where

$$V_\mu(x) = \bar{q}(x)\gamma_\mu q(x) + ac_V \sum_\nu i\partial_\nu [\bar{q}(x)\sigma_{\mu\nu}q(x)] ,$$  

(17)

and the operators on the right-hand side of (17) are the bare lattice ones. In this case, improvement requires the determination of two constants ($Z_V$ and $c_V$), one more than for the pseudoscalar and scalar densities. The necessity of the term proportional to $c_V$ for improvement has been pointed out and discussed in refs. [8, 15]. For any choice of the quark mass, $Z_V \equiv Z_V(m) = Z_V(0)(1 + b_V am)$ can be immediately fixed by requiring that the forward matrix element of the current is correctly normalized, i.e. that

$$\langle p|\hat{V}_\mu|p\rangle = Z_V(m)\langle p|V_\mu|p\rangle = 2p_\mu .$$  

(18)

This does not require the knowledge of $c_V$ since the forward matrix element of $\partial_\nu [\bar{q}(x)\sigma_{\mu\nu}q(x)]$ is zero. We then determine $c_V$ by studying the short-distance behaviour of the correlation function

$$\hat{G}_{VV}(x) = \langle 0|\hat{V}_\mu(0)\hat{V}_\nu(0)|0\rangle .$$  

(19)

c_V is to be chosen to remove the residual dependence of $\hat{G}_{VV}(x)$ on $ma$ as $|x| \to 0$.

It is also possible to exploit the chiral Ward identities (in the massless limit) to determine $c_V$ [8]. For example, having determined $Z_V$ from the normalization of the charge operator, and anticipating that we can obtain the improved axial current as described in subsection 4.3 below, the Ward identity (see eq. (7))

$$f_{abc} \sum_{\vec{x}} \langle \left( \hat{A}_0^a(t_2, \vec{x}) - \hat{A}_0^a(t_1, \vec{x}) \right) \hat{A}_b^b(y) V_{i,\text{source}}^c(0) \rangle = i N_f \langle \hat{V}_i^c(y) V_{i,\text{source}}^c(0) \rangle$$  

(20)

This is the method currently being used by the authors of ref. [16].
provides an equation for $c_V$. In eq. (20) $V^c_{i,\text{source}}(0)$ is any vector current (not necessarily improved), we have assumed that $0 < t_1 < y_0 < t_2$, and a sum over the spatial index $i$ is implied. It has been convenient to exhibit the flavour indices $\{a, b, c, \cdots\}$ explicitly. By varying the parameters in eq. (20) ($t_{1,2}, y$ and the form of $V^c_{i,\text{source}}(0)$) we obtain a series of consistency conditions.

We stress that when using Ward identities, such as that in eq. (20), we do not require “short distances”. Indeed for $|y| \to 0$ in eq. (20), the contribution from the term containing $c_V$ vanishes (i.e. is of $O(a^2)$), and hence cannot be determined. To see this, note that this term is proportional to

$$ac_V \langle \bar{q}(y)\gamma_\mu \tau^c q(y) V^c_{i,\text{source}}(0) \rangle ,$$

which vanishes at short distances by chirality (on the lattice the correlation function in eq. (21) is of $O(a)$ because of the Wilson term, leading to a contribution of $O(a^2)$). Thus, in order to use the Ward identity to determine $c_V$ we need to work at distances $y \gtrsim 1/\Lambda_{\text{QCD}}$, where the non-perturbative chiral symmetry breaking terms make the correlation function in eq. (21) non-vanishing.

If the Ward identities are used to determine $c_V$, then rather than using the independence of $\hat{G}_{VV}(x)$ of the quark mass to determine $c_V$, we can use it instead to determine $\tilde{g}_0(m) (b_g)$. Thus, we see that there are many possibilities to (over-)determine the set of parameters \{\ $b_g, Z_V(0), b_V, c_V$\}.

### 4.3 The Axial Currents

The improved form of the axial current is

$$\hat{A}_\mu(x) = Z_A A_\mu(x)$$

where

$$A_\mu(x) = \bar{q}(x)\gamma_\mu \gamma_5 q(x) + ac_A \partial_\mu [\bar{q}(x)\gamma_5 q(x)] ,$$

and the operators on the right-hand side of (23) are bare lattice ones. As in the case of the vector current, there are two constants ($Z_A$ and $c_A$) to determine, but now there is no normalization condition which would allow us to determine $Z_A$ without knowledge of $c_A$.

We start therefore with the determination of $c_A$. Following [8, 9] we exploit the fact that $\partial_\mu \hat{A}_\mu(x)$ is proportional to $P(x)$, and compute

$$R_A(x, c_A) = \frac{\langle \partial_\mu \hat{A}_\mu(x) P(0) \rangle}{\langle P(x) P(0) \rangle} .$$

$c_A$ is determined, together with $c_{SW}$, by requiring that $R_A$ is independent of $x$ (up to terms of $O(a^2)$ which we are neglecting). $R_A$ is also independent of the form of the pseudoscalar operator at the origin (e.g. $P(0)$ could be replaced by some extended operator in both the numerator and denominator of eq. (24); see also the discussion around eqs. (11) and (20)). This provides further constraints which can be exploited in the simultaneous determination of $c_A$ and $c_{SW}$ [8].

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9 We use a notation in which, for example, $P^a(x) = \bar{\psi}(x)\gamma^5 \tau^a \psi(x)$, with the generators of the flavour symmetry normalised to $\text{Tr}(\tau^a \tau^b) = \frac{1}{2}\delta^{ab}$. We present the results for an $SU(N_f)$ flavour symmetry, and $f^{abc}$ are the corresponding structure constants, $[\tau^a, \tau^b] = if^{abc}\tau^c$. 
The determination of $Z_A$ can now be achieved in a number of ways. For example, one can start by calculating $Z_A$ in the chiral limit by imposing a chiral Ward identity [9], using the improved operators $\hat{V}$ and $\hat{A}$. We will see below that in this case we do not need to know the values of $Z_V$ and $c_V$. For example, consider the identity

$\sum_{\vec{x},\vec{y}} \langle P^c(z) \left( \hat{A}_0^n(t_2, \vec{x}) - \hat{A}_0^n(t_1, \vec{x}) \right) \hat{A}_0^b(\vec{y}) P^d(0) \rangle = i N_f \sum_{\vec{y}} \langle P^c(z) \hat{V}_0^c(\vec{y}) P^d(0) \rangle$ ,

where $0 < t_1 < y_0 < t_2 < z_0$ and again it has been convenient to exhibit the flavour indices \{a, b, c, \ldots\} explicitly. Any convenient form for the lattice pseudoscalar densities, $P$, can be used in eq.\(\text{(25)}\); indeed the pseudoscalar densities could also be replaced by other operators. Both sides of eq.\(\text{(25)}\) can be simplified. Current conservation implies that the left-hand side is independent of $t_{1,2}$ and $y_0$ (provided that the above time-ordering is preserved) and hence that the two terms are equal (note that as we are summing over $\vec{x}$ and $\vec{y}$, we can, for example, change variables $\vec{x} \rightarrow \vec{y}$ in the second term). Combining this with the normalization of the charge operator on the right-hand side we obtain

$\sum_{\vec{x},\vec{y}} \langle P^c(z) \hat{A}_0^a(x) \hat{A}_0^b(y) P^d(0) \rangle = \frac{N_f^2}{2} \langle P^c(z) P^c(0) \rangle$ ,

where $0 < y_0 < x_0 < z_0$. For each set of values of \{x_0, y_0, z_0\} satisfying this ordering, and for each choice of $\vec{x}$, eq.\(\text{(25)}\) is an equation for the one unknown $Z_A(0)$ ($c_A$ having been determined as described above). Finally, the values of $Z_A(m) = Z_A(0)(1 + b_A ma)$ away from the chiral limit can be obtained from the condition that at short distances the correlation function $\langle \hat{A}_\mu(x)\hat{A}_\nu(0) \rangle$ is independent of the quark mass,

$\langle \hat{A}_\mu(x)\hat{A}_\nu(0) \rangle |_{m} = \langle \hat{A}_\mu(x)\hat{A}_\nu(0) \rangle |_{m=0} . \tag{27}$

For any value of the quark mass, eq.\(\text{(27)}\) represents an equation for $b_A$. With this method, no perturbative calculations, either on the lattice or in the continuum are necessary.

Alternatively one can calculate the correlation function $\langle A_\mu(x)A_\nu(0) \rangle$ at short distances in continuum perturbation theory (here $A_\mu$ is the continuum axial current and, at short distances, the correlation function is independent of the mass) and determine $Z_A(m)$ by imposing that $\langle \hat{A}_\mu(x)\hat{A}_\nu(0) \rangle$ is equal to this perturbative result. Finally, as done for the pseudoscalar density, we can compute $Z_A$ on quark states, for different masses, and obtain $Z_A(0)$ by extrapolating the result to the chiral limit. This gives the correct result provided the renormalization scale is large enough and the renormalization conditions are compatible with the Ward identities for the axial current [10].

### 4.4 The Tensor bilinears

Finally, we briefly sketch the corresponding analysis for the tensor operator. The improved tensor operator has the form

$\hat{T}_{\mu\nu}(x) = Z_T T_{\mu\nu}(x)$ \tag{28}

where

$T_{\mu\nu}(x) = i \vec{q}(x)\sigma_{\mu\nu}q(x) + ac_T(\partial_\mu V_\nu - \partial_\nu V_\mu)$ , \tag{29}
and the operators on the right-hand side of (29) are bare lattice ones. Since we are neglecting terms of $O(a^2)$, we do not need to include the $O(a)$ terms proportional to $c_V$ in the expression for $V_\mu$ given in eq.(17).

Again we have two constants ($Z_T$ and $c_T$) to determine and, as in the case of the axial current, we start with the determination of $c_T$. This can be achieved by noting that the correlation function $\langle \hat{V}_\rho(0) \rangle$ is zero at short distances. This is because the tensor and vector currents have different chiralities, and chirality is a good symmetry at short distances. The vanishing of this correlation function represents an equation for $c_T$.

The coefficient $c_T$ can also be determined using Ward Identities in the chiral limit. For example, consider the identity

$$\sum_{\vec{x}} \sum_{\mu\nu} \epsilon_{\mu\nu\rho\sigma} \langle \hat{A}_0^a(t_2, \vec{x}) - \hat{A}_0^a(t_1, \vec{x}) \rangle \hat{T}_{\mu\nu}^b(y) \hat{T}_{\rho\sigma, \text{source}}^c(0) = 2d_{abc} \langle \hat{T}_{\rho\sigma}^c(y) \hat{T}_{\rho\sigma, \text{source}}^c(0) \rangle,$$

(30)
in which no indices are summed unless explicitly indicated, and we have assumed $0 < t_1 < y_0 < t_2$. This identity is valid for any form of the tensor density at the origin; the simplest case is to assume that $\hat{T}_{\rho\sigma, \text{source}} = i\bar{q}\sigma_{\rho\sigma}q$. The set of equations (30) for different choices of $\rho$, $\sigma$, $y$ and $\hat{T}_{\rho\sigma, \text{source}}$, allow us to determine $c_T$ as well as providing many consistency checks. Note that $Z_T$ appears as an overall factor on both sides of the equation, and so cannot be obtained from eq.(30).

The overall normalization constant $Z_T$ can then be determined, for example, by computing the correlation function $\langle \hat{T}_{\rho\sigma}^c(y) \hat{T}_{\rho\sigma}^c(0) \rangle$ at short distances in perturbation theory (in some renormalization scheme) and on the lattice and imposing that they are equal. $Z_T(m)$ is now the only unknown in this equation and can therefore be obtained.

$Z_T$ can also be obtained in the chiral limit by imposing a normalization condition on the matrix element of the tensor operator between quark states. The mass dependence of $Z_T(m)$ is then determined in the standard way, by requiring that the correlation function $\langle \hat{T}_{\rho\sigma}^c(y) \hat{T}_{\rho\sigma}^c(0) \rangle$ is independent of the quark mass at short distances.

## 5 Conclusions

In this letter we have proposed a method for the construction of improved lattice composite operators, containing quarks with non-zero masses. The remaining discretization errors in matrix elements of these operators are of $O(a^2)$. The method builds on the improvement program of the ALPHA collaboration and extends it away from the chiral limit. This is of particular importance in the study of the decays of hadrons containing a heavy quark. A detailed numerical study is now necessary to determine the precision with which the coefficients can be obtained; the results will be presented in a future paper.

Our proposed method is based on the observation that chiral symmetry is restored at short distances. By imposing this requirement on correlation functions of composite operators, we can determine the coefficients which are needed to construct the improved operators. For purposes of illustration, in this letter we have studied the application of our method to quark bilinear operators; the approach can, however, also be generalized to deal with more complicated composite operators, such as four-quark operators relevant for studies of the effective weak Hamiltonian.
We are also carrying out a pilot study of the determination of the normalization coefficients by imposing normalization conditions between quark and gluon states in a fixed gauge. Again the key ingredient in the method is the restoration of chiral symmetry at large momenta or short-distances. The basis of higher dimensional operators which must be considered (and hence the number of coefficients) is larger in this case, and the method will be presented in detail in ref. [11], together with the results of the preliminary numerical study.

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Appendix

In this appendix we sketch the procedure which allows, in principle, the evaluation of $Z_p$ away from the chiral limit, in a way which obviates the need for the condition $|x| \ll \Lambda_{\text{QCD}}^{-1}$. Following ref. [3], we start with a fine lattice with a spacing which is so small that one can perturbatively match the lattice results onto a continuum renormalization scheme with negligible errors from higher order corrections, and, after successive iterations, we end with a course lattice which is sufficiently large to allow the computation of hadronic correlation functions. For simplicity we present the discussion in the quenched case only, which obviates the need to determine $b_g$. The generalization to the unquenched case introduces further technical complications, but the basic strategy is the same. The steps are the following [4]:

i) On a lattice with a very small spacing, $a_1$, and bare coupling $g_1$, we use our method to determine $Z_p(g_1, \mu, m_1)$ for a given value of the quark mass $m_1$, and for a renormalization scale $\Lambda_{\text{QCD}} \ll \mu \ll a_1^{-1}$. This condition is readily satisfied since we are working on a very fine lattice. We do this using two-point functions at distances $|x_1| \sim 1/\mu$.

ii) We evaluate $R_p(x_2, g_1, m_1)$ as defined in eq. (15) and a second physical quantity $Q(x_2, g_1, m_1)$ for which the $O(a)$ effects can be removed by adjusting $c_{\text{SW}}$ only (see section 2). We choose $|x_2|$ to be larger than $|x_1|$, e.g. a simple choice is $|x_2| = 2|x_1|$.

iii) The next step is to obtain the same physical quark mass on both lattices. With a coupling constant $g_2$, corresponding to a lattice spacing $a_2$ larger than $a_1$, e.g. $a_2 = 2a_1$, we adjust the mass of the quark until $Q(x_2, g_1, m_1) = Q(x_2, g_2, m_2)$ ($x_2$ is the same in physical units on both lattices). The calibration of the lattice spacing is performed, as usual, in the chiral limit, using, for example, $Q(x_2, g_1, 0) = Q(x_2, g_2, 0)$.

In this appendix, we exhibit explicitly the relevant arguments of $Z_p$, $G_{PP}$, $R_p$ etc. where needed. Some of these arguments were implicit in the earlier sections.
iv) We find the new value of $b_P$ by imposing the condition $R_P(x_2, g_2, m_2) = R_P(x_2, g_1, m_1)$.

v) At this point we only require $Z_P$ in the chiral limit. This is obtained by imposing the condition $Z_P(g_2, \mu, 0)G_{PP}(x, g_2, 0) = Z_P(g_1, \mu, 0)G_{PP}(x, g_1, 0)$.

By iterating the steps ii) – v) we can reach a large lattice (in physical units).

For the other bilinears, one first determines the subtraction constants, $c_V$, $c_A$ and $c_T$ in the chiral limit, and then use the above procedure to obtain the overall renormalization constants.

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