TOPOLOGICAL RECURSION FOR GAUSSIAN MEANS AND COHOMOLOGICAL FIELD THEORIES

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We introduce explicit relations between genus-filtrated s-loop means of the Gaussian matrix model and terms of the genus expansion of the Kontsevich–Penner matrix model (KPMM), which is the generating function for volumes of discretized (open) moduli spaces $M_{g,s}^{\text{disc}}$ (discrete volumes). Using these relations, we express Gaussian means in all orders of the genus expansion as polynomials in special times weighted by ancestor invariants of an underlying cohomological field theory. We translate the topological recursion of the Gaussian model into recurrence relations for the coefficients of this expansion, which allows proving that they are integers and positive. We find the coefficients in the first subleading order for $M_{g,1}$ for all $g$ in three ways: using the refined Harer–Zagier recursion, using the Givental-type decomposition of the KPMM, and counting diagrams explicitly.

Keywords: chord diagram, Givental decomposition, Kontsevich–Penner matrix model, discrete volume, moduli space, Deligne–Mumford compactification

1. Introduction

Multitrace means $\left(\prod_{i=1}^{s} \text{tr} H^{k_i}\right)^{\text{conn}}$ of the Gaussian unitary ensemble have long presented significant interest for investigations. First, Harer and Zagier obtained [1] a linear recurrence relation for the genus-filtrated one-trace means, which allows obtaining answers for very high genera (unattainable by other methods). Although an exact $s$-fold integral representation for $s$-trace means applicable for all $N$ was obtained by Brezin and Hikami [2] using the replica method improved in [3], the problem of constructing an effective genus expansion based on these formulas remains open. The interest in multitrace means revived after the appearance of the methods of topological recursion [4], [5] and quantum curves [6]–[8]. As shown in our first paper [9], Gaussian means via the so-called Kontsevich–Penner matrix model (KPMM) [10], [11] are directly related to discrete volumes of open moduli spaces and simultaneously to generating functions of ancestor invariants of a cohomological field theory [12].

The KPMM arises from explicit combinatorial calculations. It has been known since [13] and [14] that the KPMM is equivalent to the Hermitian matrix model with a potential whose times (coupling constants)
are related to the external-matrix eigenvalues of the KPMM via a Miwa-type transformation and whose matrix size is the coefficient of the logarithmic term. It was confirmed in [9] that the KPMM is an integral of the resolvents of the Gaussian matrix model. The resolvents containing the multitrace Gaussian means are naturally described as meromorphic (multi)differentials with zero residues over a rational Riemann surface, known as the spectral curve, and their primitives are hence meromorphic functions on the spectral curve. These primitives are conjecturally related to the so-called quantum curve (this was proved in the Gaussian case [7]; also see [15]), which is a linear differential equation that is a noncommutative quantization of the spectral curve. The spectral and quantum curves are related: the wave function arising from the spectral curve is a specialization of the KPMM free energy, which satisfies the second-order differential equation that is the quantum curve.

The geometric content of the KPMM is also quite rich: its free energy was related to structures of discretized moduli spaces in [16], and it was recently identified (see [17] and [18]) with the generating function for discrete volumes \( N_{g,s}(P_1,\ldots,P_s) \), which are quasipolynomials introduced in [19] that count integer points in the interiors \( \mathcal{M}_{g,s} \) of moduli spaces of Riemann surfaces of genus \( g \) with \( s \geq 0 \) holes with fixed perimeters \( P_j \in \mathbb{Z}_+, \ j = 1,\ldots,s \), of holes in the Strebel uniformization. Moreover, it was shown in [20] that the special times \( T_{2n}^{\pm} \) that are discrete Laplace transforms of monomials \( P_I^{2k} \), this model admits a decomposition into two Kontsevich models related by a Bogoliubov canonical transformation, which was the first example of the Givental-type decomposition formulas [21]. We use the approach in [20] to represent the free-energy expansion terms \( F_{g,s} \) of the KPMM as finite sums over graphs whose nodes are terms in the expansion of the free energy of the Kontsevich matrix model: internal edges correspond to quadratic terms in the canonical transformation operator, external half edges (dilaton leaves) correspond to the constant shifts of the higher times, and external legs (ordinary leaves) carry the times \( T_{2n}^{\pm} \). This graph representation provides another proof that \( N_{g,s}(P_1,\ldots,P_s) \) is a quasipolynomial.

It is known from [22] and [23] that the terms of the topological recursion [4], [24]–[26] based on a certain spectral curve satisfying a compatibility condition (relating the \( w_{0,1} \) and \( w_{0,2} \) invariants) describe ancestor invariants of a cohomological field theory (CohFT) or, equivalently, a Frobenius manifold.

A fundamental family of Frobenius manifolds described by Dubrovin [27] comprises Hurwitz spaces. For \( \mu = (\mu_1,\ldots,\mu_n) \), the Hurwitz space \( H_{\mu} \) consists of homotopy classes of genus-\( g \) branched covers of the sphere with \( n \) labeled points over \( \infty \) of a ramification profile \( (\mu_1,\ldots,\mu_n) \) and a simple ramification over \( \mathbb{P}^1 - \infty \). It has the dimension \( |\mu| + n + 2g - 2 \), where \( |\mu| = \mu_1 + \cdots + \mu_n \).

The two-dimensional Hurwitz–Frobenius manifold \( H_{(0,1)} \) consists of double branched covers of the sphere with two branch points and no ramification at infinity. Its free energy is

\[
F_0(t_{0,1},t_{0,0}) = \frac{1}{2}t_{0,0}^2t_{0,1} + \frac{1}{2}t_{0,1}^2 \log t_{0,1} - \frac{3}{4}t_{0,1}^2
\]

with the Euler vector field

\[
E = t_{0,0} \frac{\partial}{\partial t_{0,0}} + 2t_{0,1} \frac{\partial}{\partial t_{0,1}}.
\]

We note that expression (1.1) appears as a standard term (the perturbative part) in the expansion of any matrix model when \( t_{0,1} \) is identified with the normalized number of eigenvalues and \( t_{0,0} \) is identified with the first time. We then obtain

\[
\log \int \prod_{i=1}^{t_{0,1}} dx_i \prod_{i<j} (x_i - x_j)^2 \exp \left[ -N \sum_{i=1}^{t_{0,1}} \left( \frac{1}{2}x_i^2 - t_{0,0}x_i \right) \right] =
N^2 F_0(t_{0,1},t_{0,0}) + \sum_{g=1}^{\infty} N^{2-2g} F_g(t_{0,1}),
\]

as the quantum curve. The spectral and quantum curves are related: the wave function arising from the spectral curve is a specialization of the KPMM free energy, which satisfies the second-order differential equation that is the quantum curve.
where the leading term of the $1/N$-expansion of the free energy of the above Gaussian matrix model is exactly (1.1).

In [9], we related the discrete volumes to the Gaussian means $W_s^{(g)}(x_1, \ldots, x_s)$ and used the CohFT description further relating the discrete volumes to ancestor invariants of a CohFT. These ancestor invariants are already evaluated in terms of the closed moduli spaces $\mathcal{M}_{g,s}$ compactified by Deligne and Mumford.

This paper is organized as follows. In Sec. 2, we establish the equivalence between the Gaussian means (the correlation functions) and the terms in the expansion of the KPMM free energy.

In Sec. 3, we describe the results in [16], [19], and [20] for open discrete moduli spaces, which we used to relate the above Gaussian means and the discrete volumes purely combinatorially in [9]. The quantum curve can then be obtained as a specialization of the KPMM to the case of unit-size matrices. We describe the Givental-type decomposition formulas for the KPMM obtained in [20], representing them in terms of graph expansions for the free energy terms. This graph representation also implies that the discrete volumes are quasipolynomials and provides a link to a CohFT.

In Sec. 4, we identify the terms in the expansion of Gaussian means with the ancestor invariants of a CohFT using the results in [28] and [29]. The decomposition thus obtained has a canonical Givental form. The coefficients of this decomposition (or Laplace transforms of the quasipolynomials $N_{g,s}(P_1, \ldots, P_s)$) are the special coefficients $\hat{b}_{\vec{k}, \vec{\beta}}^{(g)}$, which in a sense “most economically” represent the genus-filtrated $s$-loop means $W_s^{(g)}(x_1, \ldots, x_s)$ and are linear combinations of the CohFT ancestor invariants of neighbouring levels.

In Sec. 5, we develop the topological recursion for Gaussian means, present the general recurrence relations for $\hat{b}_{\vec{k}, \vec{\beta}}^{(g)}$, and prove that in the admissibility domain, all these coefficients are positive integers.

In Sec. 6, we concentrate on the case of a one-loop mean. We find the first subleading coefficient $b_{g-2}^{(g)}$ in three ways: using the modified Harer–Zagier (HZ) recurrence relation, using the graph description of Givental-type decomposition in Sec. 3, and counting diagrams explicitly.

### 2. The effective matrix model for the multiloop Gaussian means

We consider a sum of connected chord diagrams based on $s$ backbones (or loop insertions) carrying the variables $u_i, i = 1, \ldots, s$. We first provide an effective matrix model description for all genus-$g$ contributions in terms of shapes, the connected fat graphs of genus $g$ with $s$ faces and with vertices of arbitrary order greater than or equal to three. From the Euler characteristic formula, for a fixed $g$ and $s$, only a finite number of such fat graphs exist, and we let $\Gamma_{g,s}$ denote this finite set. The set $\Gamma_{g,s}$ labels cells in the canonical Strebel–Penner ideal cell decomposition of the moduli space $\mathcal{M}_{g,s}$. In accordance with [29], $\Gamma_{g,s}$ is in bijection with circular chord diagrams that are also shapes in the terminology in [29], i.e., chord diagrams that are seeds and have no one-chords.

The correlation functions (or means) are given by the integrals

$$\left\langle \prod_{i=1}^{s} \left( \text{tr } H^{k_i} \right) \right\rangle = \int_{H \in \mathcal{H}_N} \left( \prod_{i=1}^{s} \text{tr } H^{k_i} \right) e^{-\frac{N}{2} \text{tr } H^2} DH, \quad (2.1)$$

where $\mathcal{H}_N$ is the set of Hermitian $N \times N$ matrices. By Wick’s theorem, any correlation function (2.1) can be represented as the sum over all possible (complete) pairings between matrix elements $M_{ij}$, where the pairings are two-point correlation functions $\langle H_{i,j} H_{k,l} \rangle = \delta_{ij} \delta_{jk}/4$. These pairing are customarily represented by edges: double lines of indices. The corresponding index lines run along faces of fat graphs containing an ordered set of $s$ vertices with the valences $k_i, i = 1, \ldots, s$, and $\sum_{i=1}^{s} k_i/2$ edges. For each vertex, we fix a cyclic order of edges incident to this vertex. Moreover, we also have a first incident edge given for each
vertex. We let \( \hat{\Gamma}(k_1, \ldots, k_s) \) denote this set of fat graphs. The sum in (2.1) then becomes

\[
\sum_{\gamma \in \hat{\Gamma}(k_1, \ldots, k_s)} N^b(\gamma) - \sum_{i=1}^s k_i / 2,
\]

where \( b(\gamma) \) is the number of boundary components of \( \gamma \).

Let \( \hat{\Gamma}(k_1, \ldots, k_s)^c \) be the subset of \( \hat{\Gamma}(k_1, \ldots, k_s) \) that consists of all the connected fat graphs and \( \langle \prod_{i=1}^s (\text{tr} H^{k_i}) \rangle_{\text{conn}} \) be the part of the sum comprising only connected diagrams. The connected correlation functions then admit the \( 1/N \)-expansion

\[
N^{s-2} \langle \prod_{i=1}^s (\text{tr} H^{k_i}) \rangle_{\text{conn}} = \sum_{g=0}^\infty N^{-2g} \langle \prod_{i=1}^s (\text{tr} H^{k_i}) \rangle_{g}^{\text{conn}},
\]

where the term with

\[
\langle \prod_{i=1}^s (\text{tr} H^{k_i}) \rangle_{g}^{\text{conn}} = |\hat{\Gamma}_g(k_1, \ldots, k_s)^c|
\]

(2.2)

corresponds to the set \( \hat{\Gamma}_g(k_1, \ldots, k_s)^c \) of connected fat graphs of genus \( g \) with ciliated vertices.

For nonciliated vertices, we then have the formula

\[
(-1)^s \langle \prod_{i=1}^s \text{tr} \log(1 - u_i H) \rangle_{g}^{\text{conn}} = \sum_{\{k_1, \ldots, k_s\} \in \mathbb{Z}_+^s} \prod_{i=1}^s \left( u_i^{k_i} / k_i \right) \langle \prod_{i=1}^s (\text{tr} H^{k_i}) \rangle_{g}^{\text{conn}} =
\]

\[
= \sum_{\{k_1, \ldots, k_s\} \in \mathbb{Z}_+^s} \sum_{\gamma \in \Gamma_g(k_1, \ldots, k_s)^c} \frac{1}{|\text{Aut}(\gamma)|} \prod_{i=1}^s u_i^{k_i},
\]

(2.3)

where \( \Gamma_g(k_1, \ldots, k_s)^c \) is the set of connected fat graphs of genus \( g \) with \( s \) nonciliated ordered vertices with the valences \( k_1, \ldots, k_s \) and \( \text{Aut}(\gamma) \) is the automorphism group of the fat graph \( \gamma \) with ordered vertices. We pass from expressions with nonciliated vertices to those with ciliated vertices (or chord diagrams) by differentiation:

\[
\langle \prod_{i=1}^s \text{tr} \log(1 - u_i H) \rangle_{g}^{\text{conn}} = \sum_{\{k_1, \ldots, k_s\} \in \mathbb{Z}_+^s} \prod_{i=1}^s \left( u_i^{k_i} \text{tr} H^{k_i} \right)_{g}^{\text{conn}} =
\]

\[
= (-1)^s \left[ \prod_{i=1}^s \left( u_i \frac{\partial}{\partial u_i} \right) \right] \langle \prod_{i=1}^s \text{tr} \log(1 - u_i H) \rangle_{g}^{\text{conn}}.
\]

(2.4)

Formula (2.3) with (2.4) taken into account yields

\[
\left[ \prod_{i=1}^s \left( u_i \frac{\partial}{\partial u_i} \right) \right] \langle \prod_{i=1}^s \text{tr} \log(1 - u_i H) \rangle_{g}^{\text{conn}} = \sum_{\gamma \in \hat{\Gamma}_g} N^{s-2g} \prod_{i=1}^s u_i^{k_i}.
\]

2.1. Summing planar subgraphs: Formulating the matrix model. We first partially resum over planar subgraphs in (2.3). A planar chord diagram on an interval is a rainbow diagram (see examples in Fig. 1). Rainbow diagrams with a given number of chords are labeled with the Catalan numbers whose generating function is

\[
f(u_i) := \frac{1 - \sqrt{1 - 4u_i^2}}{2u_i^2},
\]

(2.5)
Fig. 1. Summing rainbow diagrams of chords (dashed lines) for a single backbone (solid line): the result is the new (thickened) edge of the backbone.

Fig. 2. Resumming over ladder diagrams: the thickened edges associated with the selected ladder, which becomes an edge of a new fat graph, are darkened; the crosshatched domains become the respective three- and four-valent vertices of the new fat graph representing a shape.

and we can therefore replace the original edge of a chord diagram in the effective theory with a thickened edge carrying the factor \( f(u_i) \) and prohibit the existence of “bubbles” or rainbow subgraphs.

We now turn to summing ladder-type diagrams, where a “rung” of the ladder joins two cycles that carry (either distinct or coinciding) indices \( i \) and \( j \) (see an example in Fig. 2). Each ladder contains at least one rung, which is a chord carrying the factor \( u_i u_j \). We obtain an effective fat graph with new edges and vertices by blowing up cycles of thickened backbone edges until they are joined pairwise along rungs (each containing at least one rung). Disjoint parts of these cycles then constitute loops of lengths \( 2r_k \geq 6 \) alternatively bounded by \( r_k \) rungs (the chords) and \( r_k \) thickened edges of circular backbones. These loops then become vertices of the respective orders \( r_k \geq 3 \) of the new fat graph.

Introducing \( e^{\lambda_i} = (1 + \sqrt{1 - 4u_i^2})/2u_i \) or \( u_i = 1/(e^{\lambda_i} + e^{-\lambda_i}) \), we obtain a sum

\[
\sum_{k=1}^{\infty} (u_i u_j f(u_i) f(u_j))^k = \frac{1}{(u_i f(u_i) u_j f(u_j))^{-1} - 1} = \frac{1}{e^{\lambda_i + \lambda_j} - 1}
\]

for each ladder subgraph. We thus obtain an effective description.

**Theorem 1** [9]. The genus-\( g \) term of the (nonciliated) \( s \)-backbone connected diagrams is given by the following (finite!) sum over fat-graph shapes \( \Gamma_{g,s} \) of genus \( g \) with \( s \) faces whose vertices have valences of at least three:

\[
\left< \prod_{i=1}^{s} \text{tr} \log(e^{\lambda_i} + e^{-\lambda_i} - H) \right>_{\text{conn}}^{g,s} = \sum_{\gamma \in \Gamma_{g,s}} \frac{1}{|\text{Aut}(\gamma)|} \prod_{\text{edges}} \frac{1}{e^{\lambda_1^{(+)\gamma} + \lambda_1^{(-)\gamma}} - 1} = F_{s}^{(g)}(\lambda_1, \ldots, \lambda_s),
\]

where \((+)\) and \((-)\) denote the two sides (faces) of the edge \( e \). The quantity \( F_{s}^{(g)}(\lambda_1, \ldots, \lambda_s) \) in the right-hand side is the term in the diagram expansion of the free energy of the KPMM [10] described by the
normalized integral over Hermitian \(N \times N\) matrices \(X\)

\[
Z[\Lambda] := \exp \left[ \sum_{g,s} N^{2-2g} \left( \frac{\alpha}{2} \right)^{2-2g-s} F_s^{(g)}(\lambda) \right] = \\
\frac{\int DX \exp[-\alpha N \text{tr}(\Lambda X X/4 + \log(1 - X)/2 + X/2)]}{\int DX \exp[-\alpha N \text{tr}(\Lambda X X/4 - X^2/4)]}.
\]  

(2.8)

Here, the sum ranges all stable curves \((2g + s > 2)\) and \(\Lambda = \text{diag}(e^{\lambda_1}, \ldots, e^{\lambda_N})\).

Differentiating relation (2.7) with respect to \(\lambda_i\) in the right-hand side, we obtain the standard loop means or (connected) correlation functions \(W_s^{(g)}(x_1, \ldots, x_s)\), \(x_i = e^{\lambda_i} + e^{-\lambda_i}\), of the Gaussian matrix model with the standard topological recurrence relations [4], [24]. We hence obtain an exact relation between resolvents and terms of the expansion of the KPMM free energy:

\[
W_s^{(g)}(e^{\lambda_1} + e^{-\lambda_1}, \ldots, e^{\lambda_s} + e^{-\lambda_s}) = \prod_{i=1}^{s} \left[ \frac{1}{e^{\lambda_i} - e^{-\lambda_i}} \frac{\partial}{\partial \lambda_i} \right] F_s^{(g)}(\lambda_1, \ldots, \lambda_s).
\]  

(2.9)

The quantities \(W_s^{(g)}(x_1, \ldots, x_s)\) here have the standard topological recursion [4], [26] for the spectral curve \(x = e^\lambda + e^{-\lambda}, y = (e^\lambda - e^{-\lambda})/2\).

3. Kontsevich–Penner matrix model and discrete moduli spaces

3.1. The Kontsevich matrix model. We now turn to the cell decomposition of moduli spaces of Riemann surfaces of genus \(g\) with \(s > 0\) marked points independently proved by Harer [30] using Strebel differentials [31] and by Penner [32], [33] using hyperbolic geometry. This cell decomposition theorem states that strata in the cell decomposition of the direct product \(\mathcal{M}_{g,s} \times \mathbb{R}^*_+\) of the open moduli space and the \(s\)-dimensional space of strictly positive perimeters of holes are in one-to-one correspondence with fat graphs of genus \(g\) with \(s\) faces (these are the shapes in Sec. 2) whose edges are endowed with strictly positive numbers \(l_i \in \mathbb{R}^+_0\). The perimeters \(P_I, I = 1, \ldots, s\), are the sums of \(l_i\) taken (with multiplicities) over edges incident to the corresponding face (boundary component or hole). It is hence natural to call them the lengths of the corresponding edges.

The fundamental theorem by Kontsevich [34] establishes the relation between the intersection indices

\[
\langle \tau_{d_1} \cdots \tau_{d_s} \rangle_g := \int d\psi \prod_{I=1}^{s} \psi_I^{d_i}
\]

and the Kontsevich matrix-model integral. Here, \(\psi_I\) is a \(\psi\)-class or Chern class associated with the \(I\)th marked point, and integrals of these classes (intersection indices) are independent of the actual values of \(P_I\), being purely cohomological objects. Multiplying each \(\psi_I^{d_i}\) by \(P_I^{2d_i}\) and taking the Laplace transform with respect to all \(P_I\), we obtain

\[
\int_0^\infty dP_1 \cdots dP_s \exp \left[ -\sum_{I} P_I \lambda_I \right] \int \prod_{I=1}^{s} P_I^{2d_i} \psi_I^{d_i} = \langle \tau_{d_1} \cdots \tau_{d_s} \rangle_g \prod_{I=1}^{s} \prod_{I=1}^{s} \frac{(2d_i)!}{\lambda_i^{2d_i+1}}.
\]  

(3.1)

Using the explicit representation of \(\psi\)-classes in [34], we can represent the left-hand side of (3.1) as a sum over three-valent fat graphs with the weights \(1/(\lambda_{I_1} + \lambda_{I_2})\) on edges, where \(I_1\) and \(I_2\) are indices of two (possibly coinciding) cycles incident to a given edge. A factor \(2^{|L| - |V|}\) also appears (where \(|V|\) and \(|L|\) are
the cardinalities of the respective sets of vertices and edges). The generating function is then the famous Kontsevich matrix model

$$\exp\left[\sum_{g=0}^{\infty} \sum_{s=1}^{\infty} N^{2-2g-2s} \alpha_{2-2g-s} \xi_k(\{\xi_k\}) \right] := \frac{\int DX \exp[-\alpha N \text{tr}(X^2 \Lambda/2 + X^3/6)]}{\int DX \exp[-\alpha N \text{tr}(X^2 \Lambda/2)]},$$

(3.2)

where

$$\xi_k := \frac{1}{N} \sum_{i=1}^{N} \frac{(2k)!}{\lambda_i^{2k+1}} = \frac{1}{N} \sum_{i=1}^{N} \int_0^{\infty} dP_i P_i^{2k} e^{-\lambda_i P_i}$$

(3.3)

are the times of the Kontsevich matrix model.

### 3.2. Open discrete moduli spaces and KPMM.

As proposed in [16], we set all the lengths of edges of the Penner–Strebel graphs to be nonnegative integers $l_i \in \mathbb{Z}_+$, $i = 1, \ldots, |L| \leq 6g - 6 + 3s$. Instead of integrating over $\mathcal{M}_{g,s}$, we sum over integer points inside $\mathcal{M}_{g,s}$.

Because the length $l_i$ of each edge appears exactly twice in the sum $\sum_{i=1}^{s} P_i$, this sum is always a positive even number, and we must take this restriction into account when taking the discrete Laplace transforms with the measure $\exp[-\sum_{i=1}^{s} \lambda_i P_i]$. By analogy with the continuous Laplace transformation in the Kontsevich model, we introduce the new times

$$T_{2k}^\pm(\lambda_I) := \frac{\partial^{2k}}{\partial \lambda^{2k}_I} \frac{1}{e^{\lambda_I} - 1} = \sum_{P_i=1}^\infty (\pm 1)^{P_i} P_i^{2k} e^{-\lambda_i P_i}$$

(3.4)

as discrete Laplace transforms; the above $\mathbb{Z}_2$ restrictions ensure the existence of two sets of times.

Following [19], we thus define the discrete volumes $N_{g,s}(P_1, \ldots, P_s)$, which are weighted counts of the integer points inside $\mathcal{M}_{g,s}^\text{disc} \times \mathbb{Z}_2^s$ for fixed positive integers $P_i, I = 1, \ldots, s$, which are the perimeters of the holes (cycles). These discrete volumes are equal (modulo the standard factors of volumes of automorphism groups) to the numbers of all fat graphs with vertices of valences $\geq 3$ and with positive integer lengths of edges subject to the restriction that the lengths of all cycles (the perimeters) are fixed. Using the identity

$$\sum_{i=1}^{s} \lambda_i P_i = \sum_{e \in L} l_e (\lambda_{i_1^{(e)}} + \lambda_{i_2^{(e)}}),$$

where $l_e$ is the length of the $e$th edge and $i_1^{(e)}$ and $i_2^{(e)}$ are the indices of two (possibly coinciding) cycles incident to the $e$th edge, we obtain

$$\sum_{\{P_i\} \in \mathbb{Z}_2^s} N_{g,s}(P_1, \ldots, P_s) \exp\left[-\sum_{i=1}^{s} P_i \lambda_i \right] = \sum_{g,s, P_i} \frac{1}{|\text{Aut } \Gamma_{g,s}|} \prod_{e=1}^{|L|} \frac{1}{e^{\lambda_{i_1^{(e)}} + \lambda_{i_2^{(e)}}} - 1}. $$

(3.5)

In (3.5), we recognize the genus expansion of KPMM (2.8). We hence have the following lemma.

**Lemma 1** [9]. The generating function for the Laplace-transformed discrete volumes $N_{g,s}(P_1, \ldots, P_s)$ is KPMM (2.8). Correspondence (3.5) is given by the formula

$$\exp\left[\sum_{g,s, P_i} N^{2-2g} \alpha_{2-2g-s} N_{g,s}(P_1, \ldots, P_s) \right] \exp\left[-\sum_{i=1}^{s} P_i \lambda_i \right] =$$

$$= \frac{\int DX \exp[-\alpha N \text{tr}(AXAX/2 + \log(1 - X) + X)]}{\int DX \exp[-\alpha N \text{tr}(AXA/2X - X^2/2)]},$$

(3.6)

where the sum ranges all stable curves with $2g - 2 + s > 0$ and strictly positive perimeters $P_i$.  

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Remark 1. Formula (3.6) is applicable for all values of \( N \) and \( \lambda \). Specializing it to the case \( N = 1 \) (where we have just an ordinary integral instead of a matrix integral) and setting \( \lambda = \lambda, \alpha = 1/\hbar \), and \( x = e^\lambda + e^{-\lambda} \), we obtain

\[
\exp\left[ \sum_{g,s,P_I \in \mathbb{Z}_+^s} h^{2g+s-2} N_{g,s}(P_1^2, \ldots, P_s^2) \exp\left[ -\lambda \sum_{I=1}^s P_I \right] \right] = \sqrt{\frac{1 - e^{-2\lambda}}{\pi \hbar}} \exp\left[ (2\hbar)^{-1} e^{2\lambda} + \hbar^{-1} \lambda F(\hbar, x) \right],
\]

where the function

\[
F(\hbar, x) := \int_{-\infty}^{\infty} dt e^{-(1/\hbar)(t^2/2+xt+\log t)}
\]

satisfies the second-order differential equation

\[
\left[ \hbar^2 \frac{\partial^2}{\partial x^2} + x \hbar \frac{\partial}{\partial x} + (1 - \hbar) \right] F(\hbar, x) = 0.
\]

We thus reproduce the equation for the quantum curve in [8].

We note that the discrete volumes are quasipolynomials: their coefficients depend on the mutual parities of the \( P_I \), and we present one more proof of this below (see Corollary 1). Because generating function (2.8) is related by (2.9) to the standard \( s \)-loop Gaussian means \( W_s^{(g)} \), we have the following lemma.

**Lemma 2** [9]. The correlation functions \( W_s^{(g)}(x_1, \ldots, x_s) \) of the Gaussian matrix model subject to the standard topological recursion based on the spectral curve \( x = e^\lambda + e^{-\lambda}, \ y = (e^\lambda - e^{-\lambda})/2 \) are related to the discrete volumes by the explicit relation

\[
W_s^{(g)}(e^{\lambda_1} + e^{-\lambda_1}, \ldots, e^{\lambda_s} + e^{-\lambda_s}) = \prod_{I=1}^s \left[ \frac{1}{e^{\lambda_I} - e^{-\lambda_I}} \sum_{P_I = 1}^\infty P_I e^{-P_I \lambda_I} \right] N_{g,s}(P_1, \ldots, P_s). \tag{3.7}
\]

Matrix model (2.8) has many remarkable properties. In addition to being the generating function for the discrete volumes related to Gaussian means, it is also equivalent [13], [35] to the Hermitian matrix model with the potential determined by the Miwa change of variables \( t_k = (1/k) \tr(e^{\lambda} + e^{-\lambda})^{-k} + \delta_{k,2}/2 \), it is the generating function for the number of clean Belyi functions or the corresponding Grothendieck dessins d’enfant [36] (also see [37]) and, finally, in the special times \( T_k^\pm, r = 0, 1, \ldots \), given by (3.4), is equal to the product of two Kontsevich matrix models [34] intertwined by a canonical transformation of the variables. We now turn to this last property.

**Lemma 3** [20]. The partition function \( Z[\Lambda] \) given by (2.8) expressed in the times \( T_k^\pm(\lambda) \) given by (3.4) depends only on the even times \( T_{2k}^\pm(\lambda) \) and satisfies the exact relation

\[
Z[\Lambda] = e^{\mathcal{F}_K \{T_{2n}^+\}} = e^{C(\alpha N) e^{-N^2 A e^{\mathcal{F}_K \{T_{2n}^+\}} + \mathcal{F}_K \{T_{2n}^-\}}}, \tag{3.8}
\]

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where $\mathcal{F}_K([T_{2n}^\pm])$ is the free energy of KPMM (3.2), $T_{2n}^\pm$ given by (3.4) are therefore the times of the KdV hierarchies, and $A$ is the canonical transformation operator

$$A = \sum_{m,n=0}^{\infty} \frac{B_{2(n+m+1)}}{4(n+m+1)(2n+1)!(2m+1)!} \times$$

$$\left\{ \frac{\partial^2}{\partial T_{2n}^+ \partial T_{2m}^+} + \frac{\partial^2}{\partial T_{2n}^- \partial T_{2m}^-} + 2(2^{2(n+m+1)} - 1) \frac{\partial^2}{\partial T_{2n}^+ \partial T_{2m}^-} \right\} +$$

$$\sum_{n=2}^{\infty} \alpha N^2 \frac{2^{2n-1}}{(2n+1)!} \left( \frac{\partial}{\partial T_{2n}^+} + \frac{\partial}{\partial T_{2n}^-} \right) \right\}$$

(3.9)

Here, $C(\alpha N)$ is a function depending only on $\alpha N$ and ensuring $\mathcal{F}_{KP}[([T_{2n}^\pm])] = 0$ for $T_{2n}^\pm \equiv 0$, and $B_{2k}$ are the Bernoulli numbers generated by

$$\frac{t}{e^t - 1} = \sum_{m=0}^{\infty} \frac{B_m t^m}{m!}.$$

From this canonical transformation, we immediately obtain the (ordinary) graph representation for the term $\mathcal{F}_g,s_{1}([T_{2n}^\pm])$ of the expansion of

$$\mathcal{F}_{KP}([T_{2n}^\pm]) = \sum_{g,s} N^{2-2g} \alpha^{2-2g-s} \mathcal{F}_g,s_{1}([T_{2n}^\pm]).$$

**Lemma 4** [9], [20]. We can represent a term $\mathcal{F}_g,s_{1}([T_{2n}^\pm])$ of the genus expansion of KPMM (2.8) as a sum of a finite set of graphs $G_{g,s}$ described below; each graph contributes the factor also described below divided by the order of the automorphism group of the graph.

- Each node (vertex) $v_i, i = 1, \ldots, q$, of a graph $G_{g,s}$ has a mark + or −, the genus $g_i \geq 0$, and $s_i$ endpoints of edges incident to it ($2g_i - 2 + s_i > 0$, i.e., all nodes are stable). Each endpoint of an edge has a nonnegative integer $k_{r,i}^\pm$, $r = 1, \ldots, s_i$; these integers are subject to restriction that $\sum_{r=1}^{s_i} k_{r,i}^\pm = 3g_i - 3 + s_i$, where the superscript $+ or -$ is determined by the mark on the vertex.

- Edges can be external legs (ordinary leaves) with $k_{r,i}^\pm \geq 0$ (we let $a_i \geq 0$ denote the number of such legs incident to the $i$th vertex), half-edges (dilaton leaves) with $k_{r,i}^\pm \geq 2$ (we let $b_i \geq 0$ denote the number of such legs incident to the $i$th node), or internal edges incident to either two different nodes or the same node (their two endpoints in general have different numbers $k_{r,1,i}^\pm$ and $k_{r,2,i}^\pm$; we let $l_i$ denote the number of internal edge endpoints incident to the $i$th node).

- Each node contributes the Kontsevich intersection index $\langle \tau_{k_{r,1,i}^\pm} \cdots \tau_{k_{r,2,i}^\pm} \rangle_{g_i}$.

- Each internal edge with endpoint markings $(k_{r,1,i}^+, k_{r,2,i}^+)$ or $(k_{r,1,i}^-, k_{r,2,i}^-)$ (two endpoints of such an edge can be incident to the same node) contribute the factor

$$- \frac{B_{2(k_{r,1,i}^+ + k_{r,2,i}^+ + 1)}}{2(k_{r,1,i}^+ + k_{r,2,i}^+ + 1)(2k_{r,1,i}^+ + 1)!(2k_{r,2,i}^+ + 1)!} \times$$

and each internal edge with endpoint markings $(k_{r,1,i}^+, k_{r,2,i}^-)$ (two endpoints of such an edge can be incident only to distinct nodes having different markings $+$ and $-$) contributes the factor

$$- \frac{B_{2(k_{r,1,i}^+ + k_{r,2,i}^+ + 1)}}{2(k_{r,1,i}^+ + k_{r,2,i}^- + 1)(2k_{r,1,i}^+ + 1)!(2k_{r,2,i}^- + 1)!} \times$$

$$+ \frac{2^{2(k_{r,1,i}^+ + k_{r,2,i}^+ + 1)} - 1}{2(k_{r,1,i}^+ + k_{r,2,i}^- + 1)(2k_{r,1,i}^+ + 1)!(2k_{r,2,i}^- + 1)!}.$$
Each half-edge with the marking $r^\pm \geq 2$ contributes the factor $-2^{r^\pm - 1}/(2^{r^\pm} + 1)!$.

Each external leg with the marking $k^\pm_{r,i}$ contributes the corresponding time $T^\pm_{2k^\pm_{r,i}}$.

We have $\sum_{i=1}^{g} (g_i + l_i/2 - 1) + 1 = g$ (the total genus $g$ is equal to the sum of internal genera plus the number of loops in the graph).

We have $\sum_{i=1}^{a} a_i = s$ (the total number of external legs is fixed and equal to $s$).

From the above formulas, we obtain

$$\sum_{j=1}^{s} k^\text{ext}_{j} = 3g - 3 + s - \sum_{j=1}^{|L|} (1 + k^\text{int}_{j,1} + + k^\text{int}_{j,2}) - \sum_{j=1}^{|B|} (k^\text{half}_{j} - 1), \tag{3.10}$$

where, disregarding the node labels, $k^\text{ext}_{j} \geq 0$ are indices of the external edges, $k^\text{int}_{j,1} \geq 0$ and $k^\text{int}_{j,2} \geq 0$ are indices of endpoints of the internal edges, $k^\text{half}_{j} \geq 2$ are indices of half-edges, and $|L|$ and $|B|$ are the cardinalities of the respective sets of internal edges and half-edges of the graph.

The proof of this assertion is one more application of Wick’s theorem, now in the form of an exponential of linear-quadratic differential operator (3.9); the typical form of a term in such a sum is shown in Fig. 3.

This lemma immediately implies the following corollary.

**Corollary 1.** The quantities $F_{g,s}(\{T^\pm_{2n}\})$ are polynomials such that for each monomial $T^+_{2n_1} \cdots T^-_{2n_s}$, we have $\sum_{i=1}^{s} n_i \leq 3g - 3 + s$, and the highest term with $\sum_{i=1}^{s} n_i = 3g - 3 + s$ is

$$\langle \tau_{n_1} \cdots \tau_{n_s} \rangle_g \left( \prod_{i=1}^{s} T^+_{2n_i} + \prod_{i=1}^{s} T^-_{2n_i} \right).$$

This also implies that all discrete volumes $N_{g,s}(P_1, \ldots, P_s)$ are $\mathbb{Z}_2$-quasipolynomials in $P^2_I$.

**Proof.** The discrete volumes $N_{g,s}(P_1, \ldots, P_s)$ depend only on even powers of $P_I$ because $F_{g,s}$ depend only on even times $T^\pm_{2n}$. The quasipolynomiality follows immediately because $F_{g,s}$ are polynomials in $T^+_I$ and $T^-_{2n}$.

**Remark 2.** We note that the quadratic part of differential operator (3.9) has an alternating structure because the Bernoulli numbers $B_{2n}$ are positive for odd $n$ and negative for even $n$,

$$B_{2n} = (-1)^{n+1} \frac{2(2n)!}{(2\pi)^{2n}} \left[ 1 + \frac{1}{2^{2n}} + \frac{1}{3^{2n}} + \ldots \right].$$
3.3. Times for the multiresolvents. We first consider $N_{g,1}(P)$, which are polynomials of degree $3g - 2$ in $P^2$, are nonzero only for even $P$, and must vanish for all $P = 2, \ldots, 4g - 2$ (because the minimum number of edges of the genus-$g$ shape with one face is $2g$, and the minimum nonzero $P$ is therefore $4g$). We hence find that $N_{g,1}(P)$ has the form $\prod_{k=1}^{2g-1} (P^2 - (2k)^2) \text{Pol}_{2g-1}(P^2)$ for even $P$ (where $\text{Pol}_{2g-1}(x)$ is a polynomial of degree $g - 1$) and $N_{g,1}(P)$ vanishes for odd $P$. Hence, its Laplace transform in formula (3.7) is

$$W_1^{(g)}(e^\lambda + e^{-\lambda}) = \frac{-1}{e^\lambda - e^{-\lambda}} \sum_{i=0}^{g-1} \frac{b_i^{(g)}}{(e^\lambda - e^{-\lambda})^{4g+2i}} \left( \frac{\partial^2}{\partial \lambda^2} - (2k)^2 \right) \frac{\partial}{\partial \lambda} \frac{1}{e^{2\lambda} - 1}$$  \hspace{1cm} (3.11)

for some coefficients $b_i^{(g)}$. Using the relation

$$-\frac{\partial}{\partial \lambda} \frac{1}{e^{2\lambda} - 1} = \frac{2}{(e^\lambda - e^{-\lambda})^2}$$

and the formula

$$\left( \frac{\partial^2}{\partial \lambda^2} - m^2 \right) \frac{1}{(e^\lambda - e^{-\lambda})^m} = \frac{4m(m+1)}{(e^\lambda - e^{-\lambda})^{m+2}}, \quad m \geq 1, \quad (3.12)$$

we obtain the general representation for the one-loop mean,

$$W_1^{(g)}(e^\lambda + e^{-\lambda}) = \frac{1}{e^\lambda - e^{-\lambda}} \sum_{i=0}^{g-1} \frac{b_i^{(g)}}{(e^\lambda - e^{-\lambda})^{4g+2i}} \left( \frac{\partial^2}{\partial \lambda^2} - (2k)^2 \right) \frac{1}{e^{2\lambda} - 1} \sum_{i=0}^{g-1} \frac{b_i^{(g)}}{(e^\lambda - e^{-\lambda})^{2i}}. \quad (3.13)$$

In [9], we found the transition formulas between $b_i^{(g)}$ and the coefficients $P_{g,i}$ in [38]: because $b_i^{(g)}$ is an integer, $P_{g,i}$ is an integer, and vice versa. But the conjecture in [38] that $P_{g,s}$ is positive requires additional study.

We now consider the general $s$-resolvent case. By (2.9), the (stable) loop means (with $2g + s - 2 \geq 1$) are polynomials $W_s^{(g)}(e^{\lambda_1} + e^{-\lambda_1}, \ldots, e^{\lambda_s} + e^{-\lambda_s}) = F_{g,s}\{\{t_{2n_j+1}^{\pm}(\lambda_j)\}\}$ in times obtained by the substitution

$$T_{2d} \rightarrow t_{2d+1}^{\pm}(\lambda_j) = \frac{1}{e^{\lambda_j} - e^{-\lambda_j}} \left( \frac{\partial}{\partial \lambda_j} \right)^{2d+1} \frac{1}{e^{\lambda_j} \pm 1}. \quad (3.14)$$

All the times $t_{2d+1}^{\pm}(\lambda)$ are strictly skew-symmetric under the change of variables $\lambda \rightarrow -\lambda$.

We have the relations

$$t_{2d+1}^{-}(\lambda) + t_{2d+1}^{+}(\lambda) = \frac{1}{e^\lambda - e^{-\lambda}} \left( \frac{\partial}{\partial \lambda} \right)^{2d+1} \frac{1}{2} \sum_{j=1}^{d+1} q_{j,d} \frac{1}{(e^\lambda - e^{-\lambda})^{2j+1}}, \quad (3.15)$$

and

$$t_{2d+1}^{-}(\lambda) - t_{2d+1}^{+}(\lambda) = \frac{1}{e^\lambda - e^{-\lambda}} \left( \frac{\partial}{\partial \lambda} \right)^{2d+1} \frac{2}{e^\lambda - e^{-\lambda}} \sum_{j=1}^{d+1} \tilde{q}_{j,d} \frac{e^\lambda + e^{-\lambda}}{(e^\lambda - e^{-\lambda})^{2j+1}} \quad (3.16)$$

with some integer coefficients $q_{j,d}$ and $\tilde{q}_{j,d}$, where (3.16) follows from $1/(e^\lambda - 1) + 1/(e^\lambda + 1) = 2/(e^\lambda - e^{-\lambda})$ and from another useful representation

$$\frac{1}{e^\lambda - e^{-\lambda}} \frac{\partial}{\partial \lambda} \prod_{k=1}^{d} \left( \frac{\partial^2}{\partial \lambda^2} - (2k - 1)^2 \right) \frac{2}{e^\lambda - e^{-\lambda}} \frac{\partial}{\partial \lambda} \frac{2^{2d+1}(2d)!}{(e^\lambda - e^{-\lambda})^{2d+1}} = -2^{2d+1}(2d+1)! \frac{e^\lambda + e^{-\lambda}}{(e^\lambda - e^{-\lambda})^{2d+3}}. \quad (3.17)$$
Applying (3.11), (3.15), and (3.16), we can now expand $F_{g,s}(\{t z_{2n_{j}+1}^\pm(\lambda_{j})\})$ equivalently in the variables
\[
s_{k,\beta}(\lambda) := \frac{(e^\lambda + e^{-\lambda})^\beta}{(e^\lambda - e^{-\lambda})^{2k+\beta}}, \quad k = 0, \ldots, 3g + s - 3, \quad \beta = 0, 1.
\] (3.18)

In Sec. 4, we demonstrate that the coefficients of these expansions are related to the ancestor invariants of a CohFT.

We now present the general structure of the multiloop means.

**Lemma 5.** The general expression for a stable $(2g+s-3\geq 0)$ loop mean $W_{s}^{(g)}(e^{\lambda_{1}}+e^{-\lambda_{1}}, \ldots, e^{\lambda_{s}}+e^{-\lambda_{s}})$ in terms of the variables $s_{k,\beta}(\lambda)$ given by (3.18) is
\[
W_{s}^{(g)}(e^{\lambda_{1}}+e^{-\lambda_{1}}, \ldots, e^{\lambda_{s}}+e^{-\lambda_{s}}) = \sum_{k,\beta} \hat{W}_{k,\beta}^{(g)}(\lambda) s_{k,\beta}(\lambda),
\] (3.19)

where $k_{j}$ and $\beta_{j}$ are subject to the restrictions
\[
2g - 1 + \frac{1}{2} \sum_{j=1}^{s} \beta_{j} \leq \sum_{j=1}^{s} k_{j} \leq 3g + s - 3, \quad \sum_{j=1}^{s} \beta_{j} = 0 \pmod{2}.
\] (3.20)

The two nonstable loop means are
\[
W_{1}^{(0)}(e^{\lambda} + e^{-\lambda}) = e^{-\lambda},
\] (3.21)
\[
W_{2}^{(0)}(e^{\lambda_{1}} + e^{-\lambda_{1}}, e^{\lambda_{2}} + e^{-\lambda_{2}}) = \prod_{i=1,2} \prod_{j=1,2} \frac{1}{e^{\lambda_{i}} - e^{-\lambda_{j}}},
\] (3.22)

We prove restrictions (3.20) using two considerations: first, if we scale $\lambda_{j} \rightarrow \infty$ uniformly for all $j$, $\lambda_{j} \rightarrow \lambda_{j} + R$, then every edge contributes a factor $e^{-2R}$ plus $s$ factors due to the derivatives. The minimum number of edges (for a shape with one vertex) is $2g + s - 1$, and the minimum factor appearing is hence $e^{(-4g-3s+2)R}$, while $s_{k,\beta}(\lambda)$ scale as $e^{(-3-2k+\beta)R}$, which results in the lower estimate. The upper estimate arises from the pole behavior at $\lambda_{j} = 0$. On one hand, $s_{k,\beta}(\lambda) \sim \lambda^{-2k-3}$ as $\lambda \rightarrow 0$ irrespective of $\beta$; on the other hand, from the relation to the Kontsevich model, we can conclude that the pole structure of the derivatives of the Kontsevich KdV times is $t_{d_{j}}(\lambda_{j}) \sim \lambda_{j}^{-2d_{j}-3}$ with $\sum_{j} d_{j} \leq 3g + s - 3$ and therefore $\sum_{j} d_{j} = \sum_{j} k_{j}$, which leads to the upper estimate. That the sum of the $\beta_{j}$ factors is even follows from the symmetry of the total expression with respect to the total change of the times $T^{\pm} \rightarrow T^{\mp}$. Under this change, the variables $s_{k,\beta}(\lambda)$ behave as $s_{k,\beta}(\lambda) \rightarrow (-1)^{\beta} s_{k,\beta}(\lambda)$, and the sum of the beta factors must therefore be even.

In Sec. 5, we use the topological recursion to prove that all coefficients $\hat{W}_{k,\beta}^{(g)}$ admitted by (3.20) are positive integers (see Theorem 7).

**4. Cohomological field theory from discrete volumes**

We now describe a CohFT associated with the discrete volumes. A $d$-dimensional Frobenius manifold structure is equivalent to a CohFT for a $d$-dimensional vector space $H$ with a basis $\{e_{a}\}$ and a metric $\eta$. We show that the quasipolynomial discrete volumes are equivalent to the correlation functions of the CohFT associated with the Hurwitz Frobenius manifold $H_{0,(1,1)}$ described in Sec. 1. In this section, we give two approaches to the genus-0 case: the first approach is constructive, and the second generalizes to all genera. It also follows from the constructive approach that a homogenous CohFT arises in this case. The primary correlation functions of this CohFT turn out to be virtual Euler characteristics $\chi(M_{g,n})$ of moduli spaces.
4.1. Cohomological field theories. Given a complex vector space $H$ equipped with a complex metric $\eta$, a CohFT is a sequence of $S_s$-equivariant linear maps

$$I_{g,s}: H^{\otimes s} \to H^*(\overline{M}_{g,s})$$

that satisfy the following compatibility conditions with respect to inclusion of strata. Any partition into two disjoint subsets $I \sqcup J = \{1, \ldots, s\}$ defines a map $\phi_I: \overline{M}_{g, |I|+1} \times \overline{M}_{g, |J|+1} \to \overline{M}_{g,s}$ such that

$$\phi_I^* I_{g,s}(v_1 \otimes \cdots \otimes v_s) = I_{g, |I|+1} \otimes I_{g, |J|+1} \left( \bigotimes_{i \in I} v_i \otimes \Delta \otimes \bigotimes_{j \in J} v_j \right),$$

where $\Delta = \sum_{\alpha, \beta} \eta^{\alpha\beta} e_\alpha \otimes e_\beta$ with respect to a basis $\{e_\alpha\}$ of $H$. The map $\psi: \overline{M}_{g-1,s+2} \to \overline{M}_{g,s}$ induces

$$\psi^* I_{g,s}(v_1 \otimes \cdots \otimes v_s) = I_{g-1,s+2}(v_1 \otimes \cdots \otimes v_s \otimes \Delta).$$

The three-point function $I_{0,3}$ together with the metric $\eta$ induces a product $\bullet$ on $H$,

$$u \bullet v = \sum_{\alpha, \beta} I_{0,3}(u \otimes v \otimes e_\alpha) \eta^{\alpha\beta} e_\beta,$$

where $I_{0,3}$ takes its values in $\mathbb{C}$. A vector $e_0$ satisfying

$$I_{0,3}(v_1 \otimes v_2 \otimes e_0) = \eta(v_1 \otimes v_2) \quad \forall v_1, v_2 \in H,$$

is the identity element for the product on $H$.

An extra condition satisfied by both the CohFT under consideration and Gromov–Witten invariants pertains to the forgetful map for $s \geq 3$:

$$I_{g,s+1}(v_1 \otimes \cdots \otimes v_s \otimes e_0) = \pi^* I_{g,s}(v_1 \otimes \cdots \otimes v_s). \quad (4.1)$$

4.2. Quasipolynomials and ancestor invariants. The discrete volumes $N_{g,s}(P_1, \ldots, P_s)$ are even quasipolynomials (mod 2), i.e., they are even polynomials on each coset of $2\mathbb{Z}^s \subset \mathbb{Z}^s$. We define a basis of even quasipolynomials (mod 2) induced (via tensor product) from the single-variable basis $p_{k,\alpha}(b)$ for $k = 0, 1, 2, \ldots$ and $\alpha = 0, 1$:

$$p_{0,0}(b) = \begin{cases} 1, & b \text{ even}, \\ 0, & b \text{ odd}, \end{cases} \quad p_{0,1}(b) = \begin{cases} 0, & b \text{ even}, \\ 1, & b \text{ odd}, \end{cases}$$

$$p_{k+1,\alpha}(b) = \sum_{m=0}^b mp_{k,\alpha}(m), \quad k \geq 0.$$

Then

$$p_{k,\alpha}(b) = \frac{p_{0,k+\alpha}(b)}{4^k k!} \prod_{-k \leq m \leq k \atop m = k+\alpha \text{ (mod 2)}} (b^2 - m^2), \quad (4.2)$$

where in the second subscript we mean $k + \alpha$ (mod 2).

We set $\vec{k} = (k_1, \ldots, k_s)$ and $\vec{\alpha} = (\alpha_1, \ldots, \alpha_s)$. 

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Theorem 2. We have
\[ N_{g,s}(P_1, \ldots, P_s) = \sum_{k,\alpha} c^g_{k,\alpha} \prod_{i=1}^s p_{k_i,\alpha_i}(P_i), \]
where the coefficients are ancestor invariants,
\[ c^g_{k,\alpha} = \int_{\overline{M}_{g,s}} \mathcal{I}_{g,s}(e_{\alpha'_1} \otimes \cdots \otimes e_{\alpha'_s}) \prod_{i=1}^s \psi^{k_i}_{i}. \quad (4.3) \]

Proof. The proof is an application of [23], where theories with spectral curves satisfying special conditions were identified with semisimple CohFTs. The outcome of applying [23] is nonconstructive, and we therefore prove the genus-0 case differently, providing an explicit realization of the CohFT.

4.3. A homogeneous CohFT in genus 0. The primary correlators of a CohFT are
\[ Y_{g,s} := \int_{\overline{M}_{g,s}} \mathcal{I}_{g,s} : H^\otimes s \to \mathbb{C}, \]
and we assemble them into the generating function
\[ F(t_0, \ldots, t_{D-1}) = \sum N^{2-2g} \frac{1}{s!} Y_{g,s} = \sum N^{2-2g} F_g, \]
where \((t_0, \ldots, t_{D-1})\) in \(H^*\) is the basis dual to \(\{e_0, \ldots, e_{D-1}\}\). The genus 0 part \(F_0\) is the prepotential of the CohFT.

Theorem 3 (Manin, Theorem III.4.3 in [12]). A genus-0 CohFT can be uniquely reconstructed from abstract correlation functions.

The Deligne–Mumford compactification \(\overline{M}_{g,s}\) has a natural stratification indexed by dual graphs. The dual graph of \(\Sigma \in \overline{M}_{g,s}\) has vertices corresponding to the irreducible components of \(\Sigma\) with specified genera, edges corresponding to the nodes (cusps) of \(\Sigma\), and a tail, an edge with an open end (no vertex), corresponding to each labeled point of \(\Sigma\). If \(\Gamma\) is a dual graph of type \((g,s)\), then the collection of curves \(D_\Gamma\) whose associated dual graph is \(\Gamma\) forms a stratum of \(\overline{M}_{g,s}\). The closure \(\overline{D}_\Gamma = \bigcup_{\Gamma' \subset \Gamma} D_{\Gamma'}\), where the partial ordering is given by edge contraction, represents an element of \(H^*(\overline{M}_{g,s})\). Keel [39] proved that \(H^*(\overline{M}_{0,s})\) is generated by \(\overline{D}_\Gamma\) and derived all relations.

The proof of Theorem 3 uses the formula
\[ \int_{\overline{D}_\Gamma} I_{0,s}(v_1 \otimes \cdots \otimes v_s) = \bigotimes_{v \in \mathcal{V}_\Gamma} Y_{0,\{v\}} \left( \bigotimes_{i=1}^s v_i \otimes \Delta^\otimes |E_\Gamma| \right), \]
which defines the evaluation of a cohomology class on boundary strata tautologically from the definition of a CohFT. Because \(H^*(\overline{M}_{0,s})\) is generated by its boundary strata, and relations in \(H^*(\overline{M}_{0,s})\) agree with the relations satisfied by abstract correlation functions, this suffices for proving the theorem.

In particular, we have the primary invariants
\[ Y_{0,3}(e_0 \otimes e_0 \otimes e_1) = 1 = Y_{0,3}(e_1 \otimes e_1 \otimes e_1), \]
\[ Y_{0,s}(e_0 \otimes \text{any combination}) = 0, \quad s > 3, \quad (4.4) \]
\[ Y_{0,s}(e_1^\otimes s) = N_{0,s}(0, \ldots, 0) = \chi(\mathcal{M}_{0,s}), \quad s > 3, \]

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which define a genus-0 CohFT.

A CohFT is conformal if its prepotential is quasihomogeneous with respect to the Euler vector field:

\[ E \cdot F_0 = (3 - d)F_0 + Q(t), \]  

(4.5)

where \( Q \) is a quadratic polynomial in \( t = (t_0, \ldots, t_{D-1}) \). Using the genus-0 reconstruction in Theorem 3, Manin [12] proved that a conformal CohFT induces the following pushforward condition on the genus-0 CohFT.

Let \( \xi \) be any vector field on \( H \) treated as a manifold with coordinates \( t_0, \ldots, t_{D-1} \in H^* \). The Lie derivative with respect to \( \xi \) of the CohFT correlation functions \( I_{g,s} \) induces a natural action

\[
(\xi \cdot I)_{g,s}(v_1 \otimes \cdots \otimes v_s) = \text{deg} I_{g,s}(v_1 \otimes \cdots \otimes v_s) - \sum_{j=1}^{s} I_{g,s}(v_1 \otimes \cdots \otimes [\xi, v_j] \otimes \cdots \otimes v_s) +
\]

\[
+ \pi_s I_{g,s+1}(v_1 \otimes \cdots \otimes v_s \otimes \xi),
\]

where \( \pi : \overline{\mathcal{M}}_{g,s+1} \to \overline{\mathcal{M}}_{g,s} \) is the forgetful map, \( I_{g,s} \) are \((H^*(\overline{\mathcal{M}}_{0,s})\text{-valued}) \) tensors on \( H \), and the vector field \( \xi \) acts infinitesimally on \( I_{g,s} \).

A CohFT is homogeneous of weight \( d \) if

\[
(E \cdot I)_{g,s} = ((g - 1)d + s)I_{g,s}.
\]  

(4.6)

If a prepotential satisfies homogeneity condition (4.5), then the proof of Theorem 3 implies that the corresponding genus-0 CohFT is homogeneous. The Lie derivative of the bivector \( \Delta \) dual to the metric \( \eta \) on \( H \) can be calculated in flat coordinates as

\[
\mathcal{L}_E \cdot \Delta = \mathcal{L}_E \cdot \eta^{ij}e_i \otimes e_j = \eta^{ij}([E, e_i] \otimes e_j + e_i \otimes [E, e_j]) = (d - 2)\eta^{ij}e_i \otimes e_j = (d - 2)\Delta,
\]

where we use a choice of flat coordinates [40] in which \( \eta = \delta_{i,D-1-i} \) and \( E = \sum_i(\alpha_i t_i + \beta_i)(\partial/\partial t_i) \), where \( \alpha_i + \alpha_{D-1-i} = 2 - d \).

4.4. Proof of Theorem 2 in genus 0. We can now prove the genus-0 case of Theorem 2. For this, we produce a prepotential from the primary (constant) terms of \( N_{0,s}(P_1, \ldots, P_s) \), which uniquely (and constructively) determines a genus-0 CohFT. Moreover, the quasihomogeneity of the prepotential implies a homogeneous CohFT. The higher coefficients of \( N_{0,s}(P_1, \ldots, P_s) \) satisfy a homogeneity condition that makes them the correlation functions of the homogeneous CohFT.

The prepotential

\[
F_0 = \sum_{s \geq 3} \frac{1}{s!} N_{0,s}(0)t_1^s = \frac{1}{2} t_1^2 t_1 + \sum_{s \geq 3} \frac{1}{s!} \log(1 + t_1) - \frac{3}{4} t_1^2
\]  

(4.7)

assembled from \( N_{0,s}(0) = (-1)^s(s - 3)! \) is quasihomogeneous with respect to the Euler vector field \( E = t_0(\partial/\partial t_0) + 2(1 + t_1)(\partial/\partial t_1) \):

\[
E \cdot F_0 = 4F_0 + t_1^2 + t_0^2.
\]

This ensures that the genus-0 CohFT \( I_{0,s} \) produced from Theorem 3 satisfies

\[
\pi_s I_{g,s+1}(e_S \otimes e_1) = \frac{1}{2} \left( 1 - g + s - \text{deg} - \sum \alpha_k \right) I_{g,s}(e_S),
\]  

(4.8)

where \( \alpha_0 = 1 \) and \( \alpha_1 = 2 \) are the coefficients of \( E \) and \( e_S = e_{i_1} \otimes \cdots \otimes e_{i_s} \). The CohFT also satisfies pullback condition (4.1).
**Theorem 4** [41]. A semisimple homogenous CohFT with a flat identity is uniquely and explicitly reconstructible from genus-0 data.

Therefore, given the genus-0 primary invariants \( N_{0,s}(\vec{0}) \), there is a unique homogenous CohFT with a flat identity. Below, we show that its correlation functions agree with the coefficients of \( N_{g,s}(P_1, \ldots, P_s) \).

Pushforward relation (4.8) expressed in terms of correlators is [9]

\[
\int_{\mathcal{M}_{g,s+1}} I_{g,s+1}(e_S \otimes e_1) \prod_{i=1}^{s} \psi_i^{k_i} = \left( \sum_{i=1}^{s} \frac{k_i}{2} + \chi_{g,s} \right) \int_{\mathcal{M}_{g,s}} I_{g,s}(e_S) \prod_{i=1}^{s} \psi_i^{k_i} + \\
+ \sum_{j=1}^{s} \int_{\mathcal{M}_{g,s}} I_{g,s}(e_S \setminus \{j\} \otimes e_j^*) \prod_{i=1}^{s} \psi_i^{k_i - \delta_{ij}}.
\]

The condition \( E \cdot F_0 = 4F_0 + t_1^2 + t_0^2 \) on \( N_{0,s}(\vec{0}) \) is a specialization to \( g = 0 \) and \( P_1 = 0 \) of the divisor equation [17]

\[
N_{g,s+1}(0, P_1, \ldots, P_s) = \sum_{j=1}^{s} \sum_{k=1}^{P_j-1} k N_{g,s}(P_1, \ldots, P_s)|_{P_j=k} + \\
+ \left( \frac{1}{2} \sum_{j=1}^{s} P_j + \chi_{g,s} \right) N_{g,s}(P_1, \ldots, P_s).
\]

The flat identity pullback condition is known as the string equation on correlators for \( 2g - 2 + s > 0 \),

\[
\int_{\mathcal{M}_{g,s+1}} I_{g,s+1}(v_1 \otimes \cdots \otimes v_s \otimes e_0) \prod_{i=1}^{s} \psi_i^{k_i} = \sum_{j=1}^{s} \int_{\mathcal{M}_{g,s}} I_{g,s}(v_1 \otimes \cdots \otimes v_s) \prod_{i=1}^{s} \psi_i^{k_i - \delta_{ij}},
\]

and agrees with the recurrence relation [17]

\[
N_{g,s+1}(1, P_1, \ldots, P_s) = \sum_{j=1}^{s} \sum_{k=1}^{P_j} k N_{g,s}(P_1, \ldots, P_s)|_{P_j=k}.
\]

In particular, this proves the genus-0 case of Theorem 2 because recurrence relations (4.9) and (4.10) uniquely determine the correlation functions of \( I_{0,s} \) and \( N_{0,s}(P_1, \ldots, P_s) \).

This constructive proof explicitly describes the genus-0 classes \( I_{0,s}(e_S) \in H^*(\overline{M}_{0,s}) \):

\[
\int_{\overline{M}_{0,s}} I_{0,s}(e_S) = \begin{cases} 
\chi(\overline{M}_{0,s}), & e_S = e_1^{\otimes s}, \\
0, & \text{otherwise}.
\end{cases}
\]

4.5. General proof of Theorem 2 using the method of Dunin-Barkowski, Orantin, Shadrin, and Spitz. We establish the correspondence between correlation functions of the CohFT and discrete volumes in higher genera using the results in [23], where it was shown that for spectral curves satisfying a compatibility condition, the Givental reconstruction of higher-genus correlation functions can be formulated in terms of graphs, and the same graphs can be used to calculate the topological recursion.

In [23], a local spectral curve \((\Sigma, B, x, y)\) was associated with any semisimple CohFT using Eynard’s technique [22]. The Givental R-matrix yields the bidifferential \( B \) on the spectral curve

\[
\sum_{p,q} \tilde{B}_{p,q} z^p w^q = \frac{1}{z + w} \left[ \delta^{ij} - \sum_{k=1}^{N} R^i_k(-z) R^j_k(-w) \right], \tag{4.11}
\]
where \( B_{i,j} \) are coefficients of an asymptotic expansion of the Laplace transform of the regular part of the Bergmann bidifferential \( B \) expressed in terms of the local coordinates \( s_i = \sqrt{x - x(a_i)} \), where \( dx(a_i) = 0 \). The \( R \)-matrix together with the transition matrix \( \Psi \) from a flat basis to a normalized canonical basis expresses the meromorphic differential \( y \, dx \) in terms of \( s_i \). In particular, this implies compatibility condition (4.13) between the differential \( y \, dx \) and the bidifferential \( B \).

We can apply the method in [23] in either direction, beginning with a semisimple CohFT or with a spectral curve. Prepotential \( F_0 \) (4.7) yields a semisimple CohFT, thus generating the \( R \)-matrix and the transition matrix \( \Psi \) and hence the spectral curve. But having a candidate for the spectral curve, we can start with the spectral curve and use the method in [23] to obtain the coefficients of \( N_{g,s}(P_1, \ldots, P_s) \) as ancestor invariants of a CohFT. Because it agrees with the above CohFT in genus 0, by uniqueness, it is the same CohFT produced by Teleman’s theorem.

The spectral curves for the discrete volumes and Gromov–Witten invariants of \( \mathbb{P}^1 \) are similar,

\[
\begin{align*}
\text{discrete volumes} & \quad x = z + \frac{1}{z}, \quad y = z, \quad B = \frac{dz \, dz'}{(z - z')^2}, \\
\text{Gromov–Witten invariants} & \quad x = z + \frac{1}{z}, \quad y = \log z, \quad B = \frac{dz \, dz'}{(z - z')^2},
\end{align*}
\]

and because \( x \) and \( B \) uniquely determine the \( R \)-matrix, it is the same for both curves. The \( R \)-matrix for the Gromov–Witten invariants of \( \mathbb{P}^1 \) is [23]

\[
R(u) = \sum_{k=0}^{\infty} R_k u^k, \quad R_k = \frac{(2k - 1)!!(2k - 3)!!}{2^{4k} k!} \left( \frac{1}{2ki} \left( -1 \right)^{k+1} \frac{2k+1}{(-1)^{k+1}} \right).
\]

The results in [23] can be applied to those spectral curves for which the Laplace transform of \( y \, dx \) is related to this \( R \)-matrix (which is essentially the Laplace transform of the regular part of the bidifferential).

For local coordinates \( s_i, \ i = 1, 2 \), near \( x = \pm 2 \) given by \( x = s_i^2 \pm 2 \), we have

\[
\begin{align*}
y &= 1 + s_1 + \frac{1}{2} s_1^2 + \sum_{k=1}^{\infty} (-1)^{k-1} \frac{(2k - 3)!!}{2^{4k} k!} s_1^{2k+1}, \\
y &= -1 + is_2 + \frac{1}{2} s_2^2 - i \sum_{k=1}^{\infty} \frac{(2k - 3)!!}{2^{4k} k!} s_2^{2k+1},
\end{align*}
\]

and we hence obtain

\[
\begin{align*}
(y \, dx)_1 &= \sqrt{\frac{u}{2\sqrt{\pi}}} \int_{\gamma_1} e^{-u(x-2)} y \, dx \sim \sum_{k=0}^{\infty} (-1)^{k-1} \frac{(2k + 1)!!(2k - 3)!!}{2^{4k+1} k!} u^{-(k+1)}, \\
(y \, dx)_2 &= \sqrt{\frac{u}{2\sqrt{\pi}}} \int_{\gamma_2} e^{-u(x+2)} y \, dx \sim -i \sum_{k=0}^{\infty} \frac{(2k + 1)!!(2k - 3)!!}{2^{4k+1} k!} u^{-(k+1)},
\end{align*}
\]

where \((-1)!! = 1, (-3)!! = -1\), and we let \( \sim \) denote the Poincaré asymptotic correspondence in the parameter \( u \).

The compatibility condition between the differential \( y \, dx \) and the bidifferential \( B \) is

\[
\frac{1}{\sqrt{2}} \left( 1 + i \right) \cdot \frac{1}{\sqrt{2}} R(u) = \left( (y \, dx)_1 \quad (y \, dx)_2 \right), \quad (4.13)
\]

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which uses the first row of the transition matrix $\Psi = (1/\sqrt{2})(1_{1 \times 4} - i_{1 \times 4})$. A direct verification indicates that it is satisfied for $x = z + 1/z$, $y = z$, and $B = dz\,dz'/(z - z')^2$.

In this case, the calculations in [23] gives the choice of times

$$\xi_0^0 = \frac{1}{2} \left( \frac{1}{1 - z} - \frac{1}{1 + z} \right), \quad \xi_1^0 = \frac{1}{2} \left( \frac{1}{1 - z} + \frac{1}{1 + z} \right), \quad \xi_i^k = \sum_k p_{k, i} z^k$$

and the main result

$$W_s^{(g)}(x_1, \ldots, x_s) = \sum_{\vec{k}, \vec{\alpha}} c_{\vec{k}, \vec{\alpha}} \prod_{i=1}^s \xi_{k_i, \alpha_i},$$

where the coefficients are ancestor invariants (4.3). As noted above, the CohFT produced this way necessarily coincides with the homogeneous CohFT produced by Teleman’s theorem because they both use the Givental reconstruction and the same initial data.

4.6. Ancestor invariants and Gaussian means. Lemma 2 and formulas (3.11) and (3.17) express the loop means directly in terms of the ancestor invariants.

**Theorem 5.** We have the explicit relation between ancestor invariants (4.3) of a CohFT and the Gaussian means

$$W_s^{(g)}(e^{\lambda_1} + e^{-\lambda_1}, \ldots, e^{\lambda_s} + e^{-\lambda_s}) = \sum_{\vec{k}, \vec{\alpha}} c_{\vec{k}, \vec{\alpha}} \prod_{j=1}^s \hat{p}_{k_j, \alpha_j}(\lambda_j),$$

where

$$\hat{p}_{k, \alpha}(\lambda) = \begin{cases} 2^1 - 2r(2r + 1)s_{r, 0}(\lambda), & k = 2r, \alpha = 0, \\ 2^{-2r}(2r + 1)s_{r, 1}(\lambda), & k = 2r, \alpha = 1, \\ 2^{-2r + 2}2r(2r + 1)s_{r, 1}(\lambda), & k = 2r - 1, \alpha = 0, \\ 2^{-2r - 1}s_{r, 0}(\lambda), & k = 2r + 1, \alpha = 1, \end{cases}$$

and $s_{r, \beta}(\lambda), \beta = 0, 1$, are defined in (3.18).

**Example 1.** The topological (degree-0) part of the CohFT is

$$I_{g,s}(e_{\alpha_1} \otimes \cdots \otimes e_{\alpha_s}) = \epsilon(\vec{\alpha})2^g + \text{higher degree terms},$$

where $\epsilon(\vec{\alpha}) \equiv \sum_{i=1}^s \alpha_i \pmod{2}$ is 0 or 1. This explains the asymptotic behavior of the topological invariants $W_s^{(g)}$ at their poles.

**Example 2.** If $\{e_0, e_1\}$ is a basis of $H$ corresponding to flat coordinates, then

$$\int_{\mathcal{M}_{g,s}} I_{g,s}(e_1^\otimes s) = \chi(\mathcal{M}_{g,s}).$$

This uses the fact that $N_{g,s}(0, 0, \ldots, 0) = \chi(\mathcal{M}_{g,s})$ and

$$p_{k, \alpha}(0) = \begin{cases} 1, & (k, \alpha) = (0, 0), \\ 0, & \text{otherwise.} \end{cases}$$
We thus identify the coefficients $b_{k,g}^0$ of expansions (3.19) with (linear combinations) of the ancestor invariants $c_{k,1}^{g}$ using identification (4.16). We have

$$b_{r,0}^0 = 2^{1-2r}(2r + 1)c_{2r,0}^g + 2^{-1-2r}c_{2r+1,1}^g,$$

$$b_{r,1}^0 = 2^{-2r}(2r + 1)c_{2r+1,1}^g + 2^{2-2r}2r(2r + 1)c_{2r-1,0}^g,$$

for $s = 1$, and for general $s$, we have up to $2^s$ terms $c_{k,1}^{g}$ with all admissible substitutions $(k_1, 1) \leftrightarrow (k_1-1, 0)$.

5. The topological recursion

In this section, we present the main ingredients of the topological recursion method developed in [4], [5], [24], [25]. In parallel, we adapt the general construction to the Gaussian means $W_s^{(g)}(x_1, \ldots, x_s)$.

1. The input is a spectral curve $\Sigma_{x,y} = 0$ with two meromorphic differentials $dx$ and $dy$ on this curve. The zeros of $dx$ are the branch points. For the Gaussian means, this curve is the sphere $yx - y^2 = 1$, and we use the convenient local coordinates

$$x = e^\lambda + e^{-\lambda}, \quad y = e^\lambda, \quad dx = (e^\lambda - e^{-\lambda}) d\lambda.$$  \hfill (5.1)

In the Gaussian mean case, we consider the covering of this sphere by two maps: $y = e^\lambda$ and $\overline{y} = e^{-\lambda}$. The sphere is represented as a cylinder obtained from the strip $\text{Im } \lambda \in [0, 2\pi]$ by identifying points $(x, 0)$ of the real line $\text{Im } \lambda = 0$ with the points $(x, 2i\pi)$ of the line $\text{Im } \lambda = 2\pi$. We have two branch points $\lambda = 0, i\pi$.

2. The next ingredient is the Bergmann 2-differential $B(p, q)$, which is a symmetric differential with zero $A$-cycles (which are absent in the genus-0 case considered here) and with a double pole at coinciding $p$ and $q$. We also need its antiderivative $E(p, q)$, which is a 1-differential in $p$ and a function of $q$ defined as $\int_q^0 B(p, \bullet)$. In the Gaussian mean case, we have

$$B(p, q) = \frac{de^\lambda de^\mu}{(e^\lambda - e^\mu)^2}, \quad E(p, q) = \frac{de^\lambda}{e^\lambda - e^\mu}, \quad p = e^\lambda, \quad q = e^\mu.$$  \hfill (5.2)

3. We define the recursion kernel $K(p, q)$ as a $(1, -1)$-differential

$$K(p, q) = E(p, q) \frac{1}{(y(q) - \overline{y}(q))dx}.$$  

For Gaussian means, we have

$$K(p, q) = \frac{de^\lambda}{e^\lambda - e^\mu (e^\mu - e^{-\mu})^2d\mu}, \quad p = e^\lambda, \quad q = e^\mu,$$  \hfill (5.3)

where one factor $e^\mu - e^{-\mu}$ in the denominator comes from the difference $y(q) - \overline{y}(q)$ and the other comes from $dx$. 1703
4. We introduce the correlation functions \( W_s^{(g)}(p_1, \ldots, p_s) \) as symmetric \( s \)-differentials defined recursively. We choose one of the variables \( p_r \) as a root. Then

\[
W_{3}^{(0)}(p_1, p_2, p_3) = \sum_{\text{Res } dx=0} K(p_1, q)[B(p_2, q) + B(\overline{p}_2, q)][B(p_3, q) + B(\overline{p}_3, q)],
\]

\[
W_{1}^{(1)}(p_1) = \sum_{\text{Res } dx=0} K(p_1, q)B(q, \overline{q}),
\]

\[
W^{(g)}(p_1, p_2, \ldots, p_s) = \sum_{\text{Res } dx=0} K(p_1, q)\left[ \sum_{k=2}^{s} \left[ B(p_k, q) + B(\overline{p}_k, q) \right] W_{s-1}^{(g)}(q, p_2, \ldots, \hat{p}_k, \ldots, p_s) + \right.
\]

\[
\left. + W_{s+1}^{(g-1)}(q, p_2, \ldots, p_s) + \sum' \right|_{l \cup J = \{p_2, \ldots, p_s\}} W_{|I|+1}^{(g_1)}(q, \{p_i\}_{i \in I}) W_{|J|+1}^{(g_2)}(q, \{p_j\}_{j \in J}) \right].
\]

where the right-hand side is manifestly symmetric with respect to all \( p_2, \ldots, p_s \) but not with respect to \( p_1 \) and \( \sum' \) means that we take only stable terms (those with \( 2g - 2 + s > 0 \)) explicitly segregating the only nonstable contribution (the term with \( [B(p_k, q) + B(\overline{p}_k, q)] \)). The hat over a symbol indicates its omission from the list of arguments, and in the last term, we sum over all partitions of the set of arguments \( \{p_2, \ldots, p_s\} \) into two nonintersecting subsets \( I \) and \( J \). We show the recurrence relation schematically in Fig. 4.

Using (5.6), we construct all higher \( W_{s}^{(g)} \) from \( W_{3}^{(0)}(p_1, p_2, p_3) \) and \( W_{1}^{(1)}(p_1) \).

The lemma in [4] states that although recurrence relations (5.6) are not explicitly symmetric with respect to permutations of all \( p_1, \ldots, p_s \), the whole sum in the right-hand side is in fact symmetric.

5.1. The topological recursion for the Gaussian means. In any local theory satisfying the topological recursion, all stable \( W_{s}^{(g)}(x_1, \ldots, x_s) \) have singularities only at the branch points. In the Gaussian case, we therefore conclude that the only singularities in the right-hand side of (5.6) in addition to high-order poles at the branch points (for \( W_{s}^{(g)} \), the highest possible order of a pole is \( 6g + 2s - 3 \)) are simple poles at \( q = p_1 \) arising from \( K(p_1, q) \) and double poles at \( q = p_k \) and \( q = \overline{p}_k \) arising from \( [B(p_k, q) + B(\overline{p}_k, q)] \). We can therefore integrate over \( q \) in the right-hand side by evaluating residues at these points, not at the branch points, which drastically simplifies actual calculations.

**Definition 1.** For each (stable) pair \((g, s)\), we introduce the set of admissible pairs of Young tableaux \((D_1, D_2)\) (as usual, we let \( l(D) \) and \( |D| \) denote the respective length (the number of columns) and volume
(the number of boxes) of a Young tableau $D$)

$$Y_s^{(g)} = \{(D_0, D_1)\}: s = l(D_0) + l(D_1), \quad D_1 \in 2\mathbb{Z}_{\geq 0},$$

$$(5.7)$$

$$2g - 1 + s + \frac{l(D_1)}{2} \leq |D_0| + |D_1| \leq 3g + 2s - 3.$$ 

For each pair $(\tilde{k}, \tilde{\beta})$, we define the pair $(D_0, D_1) \in Y_s^{(g)}$ as follows. We split the set of indices $\tilde{k}$ into two disjoint subsets, one with $\beta_i = 0$ and the other with $\beta_i = 1$, and we arrange the $k_i$ inside each subset in a nonincreasing order, thus obtaining two Young tableaux $D_0$ and $D_1$ with columns of heights $k_i$.

The Young tableau $D_0$ contains $l(D_0)$ columns of positive integer heights $t_r$ such that $t_r > t_s$ for $r < s$, and we let $d_r$ denote the number of columns of the same height $t_r$. Hence, $|D_0| = \sum_r d_r t_r$. Correspondingly, $D_1$ contains an even number $l(D_1)$ of columns of heights $r_j$, $r_j > r_i$ for $j < i$, and we let $k_j$ denote the number of columns of the same height $r_j$. Hence, $|D_1| = \sum_j k_j r_j$.

**Definition 2.** We define a homomorphism

$$\mathcal{F}: \mathbb{Z}[Y_s^{(g)}] \rightarrow R[e^{\lambda_1}, \ldots, e^{\lambda_s}]$$

to the space $R$ of rational functions of $e^{\lambda_i}$, $i = 1, \ldots, s$, regarded as formal complex variables obtained by mapping each pair $(D_0, D_1) \in Y_s^{(g)}$ to the function of $\lambda_i$, $i = 1, \ldots, s$, defined by

$$(5.8)$$

$$\mathcal{F}(D_0, D_1) := \sum_{L_1 \cup L_2 \cup \cdots \cup R_1 \cup R_2 \cup \cdots = \{1, \ldots, s\}} \prod_i \left( \prod_{\alpha \in L_i} s_{t_i-1,0}(\lambda_\alpha) \right) \prod_j \left( \prod_{\gamma \in R_j} s_{r_j-1,1}(\lambda_\gamma) \right),$$

where the sum ranges all partitions of the set $\{1, \ldots, s\}$ into disjoint subsets $R_i$ and $L_j$ with the respective cardinalities $d_i$ and $k_j$ and where $s_{k,\beta}(\lambda) := (e^\lambda + e^{-\lambda})^\beta (e^\lambda - e^{-\lambda})^{-2k-3}$.

**Theorem 6.** The Gaussian means $W_s^{(g)}(x_1, \ldots, x_s)$ are the coefficients of the differentials

$$W_s^{(g)}(p_1, p_2, \ldots, p_s) = W_s^{(g)}(x_1, \ldots, x_s) \, dx_1 \cdots dx_s,$$

where

$$W_s^{(g)}(x_1, \ldots, x_s) = W_s^{(g)}(e^{\lambda_1} + e^{-\lambda_1}, \ldots, e^{\lambda_s} + e^{-\lambda_s}) =$$

$$\sum_{(D_0, D_1) \in Y_s^{(g)}} \hat{b}_{D_0, D_1}^{(g)} \mathcal{F}(D_0, D_1),$$

and we label the expansion coefficients with $\hat{b}_{D_0, D_1}^{(g)}$ by the above two Young tableaux $D_0$ and $D_1$.

**Proof.** Recurrence relations described by (5.6) (or graphically in Fig. 4) are governed by two operations on the basic functions $s_{k,0}(\lambda)$ and $s_{k,1}(\lambda)$: “product” and “coproduct.” In Sec. 5.2, we present these operations right on the level of Young tableaux.

The product operation occurs in the second and third terms in (5.6). We define it on the set of basis functions and continue by bilinearity to products of these functions constituting $W_s^{(g)}(e^{\lambda_1} + e^{-\lambda_1}, \ldots, e^{\lambda_s} + e^{-\lambda_s})$. This operation produces a linear combination of basis functions $s_{k,\beta}(\lambda)$ from two basis functions $s_{k_1,\beta_1}(\lambda)$ and $s_{k_2,\beta_2}(\lambda)$, is denoted by a standard “pairing” symbol, and is given by the integral

$$(5.9)$$

$$\text{“product”: } s_{k_1,\beta_1}(\lambda)s_{k_2,\beta_2}(\lambda) := \sum_{\text{Res } dx_0 = 0} K(p_1, q)s_{k_1,\beta_1}(\lambda q)s_{k_2,\beta_2}(\lambda q) \, dx_q^2,$$
Recalling that \( dx_q = (e^{\lambda_1} - e^{-\lambda_1}) d\lambda_q \) and that instead of evaluating this integral by residues at the branch points, we can evaluate it at its only simple pole \( p_1 = q \) outside the branch points, taking explicit form (3.18) of the basic vectors into account, we obtain

\[
\sum_{\text{Res } dx_q = 0} K(p_1, q) s_{k_1,\beta_1}(\lambda_q) s_{k_2,\beta_2}(\lambda_q) dx_q^2 = - \text{Res}_{p_1=q} K(p_1, q) s_{k_1,\beta_1}(\lambda_q) s_{k_2,\beta_2}(\lambda_q) dx_q^2
\]

\[
= \frac{(e^{\lambda_1} + e^{-\lambda_1})^{\beta_1+\beta_2}}{(e^{\lambda_1} - e^{-\lambda_1})^2} d\lambda_1 = \frac{(e^{\lambda_1} + e^{-\lambda_1})^{\beta_1+\beta_2}}{(e^{\lambda_1} - e^{-\lambda_1})^2} dx_1.
\]

Hence, recalling that \((e^\mu + e^{-\mu})^2 = (e^\mu - e^{-\mu})^2 + 4\), we obtain the product operation rule

\[
s_{k_1,\beta_1}(\lambda) s_{k_2,\beta_2}(\lambda) = \begin{cases} 
  s_{k_1+k_2+2,\beta_1+\beta_2}(\lambda_1), & \beta_1 + \beta_2 < 2, \\
  s_{k_1+k_2+1,0}(\lambda_1) + 4s_{k_1+k_2+2,0}(\lambda_1), & \beta_1 = \beta_2 = 1.
\end{cases}
\]

(5.10)

The second operation we need is the “coproduct” operation, which we encounter in the first term in the right-hand side of (5.6). This operation produces a term bilinear in \( s_{k_1,\beta_1}(\lambda_1) \) and \( s_{k_2,\beta_2}(\lambda_p) \) from a basis function \( s_{k,\beta}(\lambda) \), is denoted by an uparrow beside the symbol of this function, is to be continued by linearity to products of these basic functions, and is given by the integral

\[
\text{“coproduct”}: \quad (s_{k,\beta}(\lambda))^\uparrow := \sum_{\text{Res } dx_q = 0} K(p_1, q)[B(p_k, q) + B(\overline{p}_k, q)] s_{k,\beta}(\lambda_q) dx_q,
\]

(5.11)

where we can again integrate by residues at \( q = p_1 \) and \( q = p_k \) (for the term with \( B(p_k, q) \)) and at \( q = p_1 \) and \( q = \overline{p}_k \) (for the term with \( B(\overline{p}_k, q) \)). The calculations involve combinatorics of the geometric progression type but are otherwise straightforward. The two cases \( \beta = 0 \) and \( \beta = 1 \) are rather different, and two integrations hence yield two cases of the “coproduct” operation:

\[
(s_{k,0}(\lambda))^\uparrow = \sum_{\text{Res } dx_q = 0} K(p_1, q)[B(p_k, q) + B(\overline{p}_k, q)] s_{k,1}(\lambda_q) dx_q =
\]

\[
= \sum_{m=0}^k (2 + 2k - 2m)s_{m,0}(\lambda_1)s_{k-m,0}(\lambda_p) dx_1 dx_p +
\]

\[
+ \sum_{m=0}^{k+1} (3 + 2k - 2m)[4s_{m,0}(\lambda_1)s_{k+1-m,0}(\lambda_p) + s_{m,1}(\lambda_1)s_{k+1-m,1}(\lambda_p)] dx_1 dx_p
\]

(5.12)

and

\[
(s_{k,1}(\lambda))^\uparrow = \sum_{\text{Res } dx_q = 0} K(p_1, q)[B(p_k, q) + B(\overline{p}_k, q)] s_{k,1}(\lambda_q) dx_q =
\]

\[
= \sum_{m=0}^k (2 + 2k - 2m)s_{m,1}(\lambda_1)s_{k-m,0}(\lambda_p) dx_1 dx_p +
\]

\[
+ \sum_{m=0}^k (1 + 2k - 2m)s_{m,0}(\lambda_1)s_{k-m,1}(\lambda_p) dx_1 dx_p +
\]

\[
+ \sum_{m=0}^{k+1} 4(3 + 2k - 2m)[s_{m,0}(\lambda_1)s_{k+1-m,1}(\lambda_p) + s_{m,1}(\lambda_1)s_{k+1-m,0}(\lambda_p)] dx_1 dx_p.
\]

(5.13)
The two correlation functions we need to start the recursive procedure are

\[ W_3^{(0)}(x_1, x_2, x_3) = 4s_{0,0}(\lambda_1)s_{0,0}(\lambda_2)s_{0,0}(\lambda_3) \, dx_1 \, dx_2 \, dx_3 + \]
\[ + [s_{0,1}(\lambda_1)s_{0,1}(\lambda_2)s_{0,0}(\lambda_3) + s_{0,1}(\lambda_1)s_{0,0}(\lambda_2)s_{0,1}(\lambda_3) + \]
\[ + s_{0,0}(\lambda_1)s_{0,1}(\lambda_2)s_{0,1}(\lambda_3)] \, dx_1 \, dx_2 \, dx_3 \quad (5.14) \]

and

\[ W_1^{(1)}(x) = s_{1,0}(\lambda) \, dx. \quad (5.15) \]

### 5.2. Recurrence relations determining $\hat{b}^{(g)}_{D_0, D_1}$.

We now present recurrence relations for the coefficients $\hat{b}^{(g)}_{D_0, D_1}$ exclusively in terms of operations on the sets of Young tableaux. We consider the $\mathbb{Z}_{\geq 0}$ module $\mathbb{Z}_{\geq 0}[Y_s^{(g)}]$ of functions on $Y_s^{(g)}$ with nonnegative integer coefficients.

The main result in this section is the following theorem.

**Theorem 7.** All the coefficients $\hat{b}^{(g)}_{D_0, D_1}$ of expansions of the loop means in Theorem 6 in the range determined by relations (5.7) are positive integers, i.e., we have unique $W_s^{(g)} \in \mathbb{Z}_{\geq 0}[Y_s^{(g)}]$ such that

\[ F(W_s^{(g)}) = W_s^{(g)}(e^{\lambda_1} + e^{-\lambda_1}, \ldots, e^{\lambda_r} + e^{-\lambda_r}). \]

The element $W_s^{(g)} \in \mathbb{Z}_{\geq 0}[Y_s^{(g)}]$ is determined by the recursion based on the coproduct and product operations described below.

**Proof.** The proof is constructive and pertains to lifting the product and coproduct operations to the sets of Young tableaux. We must first define an auxiliary space of Young tableaux.

**Definition 3.** We define the set of pairs of Young tableaux $\overline{Y}_s^{(g)}$ whose elements are those of $Y_s^{(g)}$ with the following extra data. We assign the label 1 to exactly one column in every pair $(D_0, D_1) \in Y_s^{(g)}$ (this pertains to symmetrization with respect to all arguments expect $\lambda_1$ in (5.8)). The elements $(D_0, D_1)$ of $\overline{Y}_s^{(g)}$ with the same tableaux $(D_0, D_1)$ but with assignments of the label 1 to columns of different “sorts” (differing in height and/or “color”: we color $D_0$ white and $D_1$ gray) are regarded as different. We consider the $\mathbb{Z}_{\geq 0}$-module $\mathbb{Z}_{\geq 0}[\overline{Y}_s^{(g)}]$ of functions on $\overline{Y}_s^{(g)}$ with nonnegative integer coefficients.

**Definition 4.** We define the embedding $S$ of $\mathbb{Z}_{\geq 0}[Y_s^{(g)}]$ into $\mathbb{Z}_{\geq 0}[\overline{Y}_s^{(g)}]$ generated by the following linear map. The image $S(D_0, D_1)$ is the sum of elements $(D_0, D_1)$ (with unit coefficients) based on the same pair of Young tableaux $(D_0, D_1)$ and having labels 1 of all possible sorts (see examples below). Under the homomorphism $F$, such sums become expressions totally symmetric in all their $\lambda$ arguments.

The product and coproduct operations map $\mathbb{Z}_{\geq 0}[Y_s^{(g)}]$ to $\mathbb{Z}_{\geq 0}[\overline{Y}_s^{(g)}]$. We use the convenient graphical form of representing these operations (as usual, if one of the tableaux is empty, then we omit it to save space). We let the first (white) tableau denote $D_0$ and the second (gray) tableau denote $D_1$. For example, $W_3^{(0)} = 4 \square \square + 1 \square \square \square$ and $W_1^{(1)} = 1 \overline{\square}$, and the coefficients of pairs of “white” and “gray” tableaux are precisely $\hat{b}^{(g)}_{D_0, D_1}$.

The coproduct operation is $\text{CP} : \mathbb{Z}_{\geq 0}[Y_s^{(g)}] \to \mathbb{Z}_{\geq 0}[\overline{Y}_{s+1}^{(g)}]$. This operation produces two columns (labeled 1 and $p$) from one column of each sort in $(D_0, D_1) \in Y_s^{(g)}$ according to the following rules:
• We apply the coproduct operation to exactly one column of each sort (the sorts differ in column height and color). This operation is based on relations (5.12) and (5.13) and is

\[
\left(\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
1
\end{array}
\end{array}
\end{array}
\end{array}
\end{array}
\right)^k + 1 = \sum_{m=0}^k 2(k - m + 1) + m + 1 \left(\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
1
\end{array}
\end{array}
\end{array}
\end{array}
\right)^k + 1
\]

+ \sum_{m=0}^{k+1} \left[ 4m + 1 \left(\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
1
\end{array}
\end{array}
\end{array}
\end{array}
\right)^k + 1
\]

+ \sum_{m=0}^{k+1} \left. \left[ m + 1 \left(\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
1
\end{array}
\end{array}
\end{array}
\end{array}
\right)^k + 1 \right] \right]
\]

• We absorb the thus obtained columns labeled 1 and \( p \) into a pair of Young tableaux (the other columns remain unaltered). Among the remaining columns, we have \( k \geq 0 \) columns of the same sort as the column labeled \( p \). We then multiply the resulting tableaux by \( k + 1 \), subsequently erasing the label \( p \) but retaining the label 1.

Applying the coproduct operation, we thus obtain a linear combination of pairs of Young tableaux with positive integer coefficients. Exactly one column in each pair is labeled 1.

**Example 3.** We first calculate \( W_4^{(0)} \). Because product operations do not contribute in this case, the answer is obtained with the action of the coproduct operation on \( W_3^{(0)} \). For elements of this Young tableau, we have

\[
\left(\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
1
\end{array}
\end{array}
\end{array}
\end{array}
\right)^k = 2 \left(\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
1
\end{array}
\end{array}
\end{array}
\right)^k + \left(\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
1
\end{array}
\end{array}
\end{array}
\right)^k + \left(\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
1
\end{array}
\end{array}
\end{array}
\right)^k + \left(\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
1
\end{array}
\end{array}
\end{array}
\right)^k,
\]

\[
\left(\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
1
\end{array}
\end{array}
\end{array}
\end{array}
\right)^k = 2 \left(\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
1
\end{array}
\end{array}
\end{array}
\right)^k + \left(\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
1
\end{array}
\end{array}
\end{array}
\end{array}
\right)^k + \left(\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
1
\end{array}
\end{array}
\end{array}
\end{array}
\right)^k + \left(\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
1
\end{array}
\end{array}
\end{array}
\end{array}
\right)^k.
\]

Therefore, for elements of \( W_3^{(0)} \), we obtain (we explicitly segregate the multipliers appearing due to the symmetrization with respect to \( p \))

\[
\left( W_3^{(0)} \right)^k = 4 \left(\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
1
\end{array}
\end{array}
\end{array}
\end{array}
\right)^k + \left(\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
1
\end{array}
\end{array}
\end{array}
\end{array}
\right)^k + \left(\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
1
\end{array}
\end{array}
\end{array}
\end{array}
\right)^k =
\]

\[
= 8 \cdot 3 \left(\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
1
\end{array}
\end{array}
\end{array}
\end{array}
\right)^k + 16 \cdot 3 \left(\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
1
\end{array}
\end{array}
\end{array}
\end{array}
\right)^k + 48 \left(\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
1
\end{array}
\end{array}
\end{array}
\end{array}
\right)^k + 12 \left(\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
1
\end{array}
\end{array}
\end{array}
\end{array}
\right)^k + 4 \left(\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
1
\end{array}
\end{array}
\end{array}
\end{array}
\right)^k +
\]

\[
+ 2 \left(\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
1
\end{array}
\end{array}
\end{array}
\end{array}
\right)^k + 4 \left(\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
1
\end{array}
\end{array}
\end{array}
\end{array}
\right)^k + 12 \left(\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
1
\end{array}
\end{array}
\end{array}
\end{array}
\end{array}
\right)^k + 3 \left(\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
1
\end{array}
\end{array}
\end{array}
\end{array}
\end{array}
\right)^k +
\]

\[
+ 2 \cdot 2 \left(\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
1
\end{array}
\end{array}
\end{array}
\end{array}
\right)^k + 4 \cdot 2 \left(\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
1
\end{array}
\end{array}
\end{array}
\end{array}
\end{array}
\right)^k + 12 \left(\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
1
\end{array}
\end{array}
\end{array}
\end{array}
\end{array}
\right)^k + 12 \left(\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
1
\end{array}
\end{array}
\end{array}
\end{array}
\end{array}
\right)^k =
\]

\[
= 24 \left(\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
1
\end{array}
\end{array}
\end{array}
\end{array}
\right)^k + 48 \left(\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
1
\end{array}
\end{array}
\end{array}
\end{array}
\end{array}
\right)^k + 12 \left(\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
1
\end{array}
\end{array}
\end{array}
\end{array}
\end{array}
\right)^k +
\]

\[
+ 12 \left(\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
1
\end{array}
\end{array}
\end{array}
\end{array}
\end{array}
\right)^k + 3 \left(\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
1
\end{array}
\end{array}
\end{array}
\end{array}
\end{array}
\right)^k + 4 \left(\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
1
\end{array}
\end{array}
\end{array}
\end{array}
\end{array}
\right)^k.
\]
It can be seen that we automatically obtain symmetrized expressions with respect to $p_1$: every term in brackets contains exactly one appearance of the label 1 for every sort of columns and hence belongs to the image of the map $S$. After the map $F$, the total answer is therefore totally symmetric in all its arguments and is

$$W_4^{(0)} = 24 \begin{array}{c}
\end{array} + 48 \begin{array}{c}
\end{array} + 12 \begin{array}{c}
\end{array} + 12 \begin{array}{c}
\end{array} + 3 \begin{array}{c}
\end{array} + 4 \begin{array}{c}
\end{array}.$$  \hfill (5.19)

We thus obtain an expression in the module $Z_{\geq 0}[Y_4^{(0)}]$. The coefficients in this expression are exactly $\hat{b}_{D_0, D_1}$ in the planar four-backbone case.

The product operation produces one column with the label 1 (or a linear combination of such columns) from two columns according to the rules

$$n_1 \begin{array}{c}
\end{array} + n_2 \begin{array}{c}
\end{array} = \begin{array}{c}
\end{array}, \quad n_1 \begin{array}{c}
\end{array} + n_2 + 1 \begin{array}{c}
\end{array} = \begin{array}{c}
\end{array}, \quad n_1 \begin{array}{c}
\end{array} + n_2 \begin{array}{c}
\end{array} + 1 \begin{array}{c}
\end{array} = \begin{array}{c}
\end{array}.$$  \hfill (5.20)

We have two cases.

1. The first case is where we do the product inside the same pair of diagrams $(D_0, D_1)$,

   $$\text{PI: } Z_{\geq 0}[Y_s^{(g)}] \rightarrow Z_{\geq 0}[\tilde{Y}_{s-1}^{(g+1)}].$$

   In this case, we must take all possible (pairwise) products between different types of columns (one product operation for each pair of types) and also products inside the same type (if we have more than one column of the same type in $D_0$ or in $D_1$). The following factors arise additionally:

   - We have a factor 2 if we take the product between different types of columns.
   - We have an additional factor 2 if we take a product in a term of $W_s^{(g)}$ with $s > 2$, i.e., if the result of PI acting on $(D_0, D_1)$ contains more than one column.

2. The second case is where we take the product of two different pairs of Young tableaux $(D_0, D_1) \in Y_{s_1}^{(g_1)}$ and $(D_0', D_1') \in Y_{s_2}^{(g_2)}$,

   $$\text{PII: } Z_{\geq 0}[Y_{s_1}^{(g_1)}] \times Z_{\geq 0}[Y_{s_2}^{(g_2)}] \rightarrow Z_{\geq 0}[\tilde{Y}_{s_1+s_2-1}^{(g_1+g_2)}].$$

   In this case, we must take all possible products between all column types in the first pair and in the second pair (one product for each pair of types from different pairs of tableaux) unless $(D_0, D_1) = (D_0', D_1')$, in which case we take only every type of pairing between entries of the tableaux $(D_0, D_1)$ into account only once. We then take the union of two of the above pairs of Young tableaux; as a result, we obtain a linear combination of Young tableaux of the form

$$\begin{array}{ccc}
d_1 & d_2 & \cdots \\
d'_1 & d'_2 & \cdots \\
\end{array} \begin{array}{ccc}
d_1 & d_2 & \cdots \\
d'_1 & d'_2 & \cdots \\
\end{array} \begin{array}{ccc}
t_1 & t_2 & \cdots \\
t'_1 & t'_2 & \cdots \\
\end{array},$$  \hfill (5.21)

where we have exactly one column labeled 1 and every term has $d_j$ columns from $D_0$, $d'_j$ columns from $D'_0$ and, correspondingly, $t_i$ columns from $D_1$ and $t'_i$ columns from $D'_1$. 1709
We obtain the following combinatorial factors:

- We multiply the obtained Young tableaux by the product of binomial coefficients

\[
\prod_{j=1}^{k} \frac{d_j + d_j'}{d_j} \prod_{i=1}^{r} \frac{t_i + t'_i}{t_i}.
\]

- We multiply by a factor of two if we take the product between different types of columns and/or if we take a product between two different tableaux (i.e., if \(D_0 \neq D'_0\) and/or \(D_1 \neq D'_1\)); in other words, the only situation where we do not have this factor occurs if we take the product between two equal Young tableaux, \(D_0 = D'_0\) and \(D_1 = D'_1\), and we take a product of terms of the same type in these two tableaux.

- We multiply by an additional factor of two if the result of product of Young tableaux \((D_0, D_1)\) and \((D'_0, D'_1)\) contains more than one column, i.e., if \(s_1 = l(D_0) + l(D_1) > 1\) and/or \(s_2 = l(D'_0) + l(D'_1) > 1\) (in other words, the only situation where we do not have this factor is \(s_1 = s_2 = 1\)).

**Example 4.** We next calculate \(W_2^{(1)}\) (in the third line, we explicitly indicate the combinatorial factors due to the product process):

\[
W_2^{(1)} = (W_1^{(1)})^\dagger + W_3^{(0)} = (\square)\uparrow + 4 \begin{array}{c} \square \end{array} + \begin{array}{c} \square \end{array} + \begin{array}{c} \square \end{array} + \begin{array}{c} \square \end{array} =
\]

\[
= 4 \begin{array}{c} 1 \end{array} + 12 \begin{array}{c} 1 \end{array} + 2 \begin{array}{c} 1 \end{array} + 4 \begin{array}{c} 1 \end{array} + 20 \begin{array}{c} 1 \end{array} + 5 \begin{array}{c} 1 \end{array} + 3 \begin{array}{c} 1 \end{array} + 1 \begin{array}{c} 1 \end{array} +
\]

\[
+ 4 \cdot 2 \begin{array}{c} 1 \end{array} + 1 \cdot 4 \begin{array}{c} 1 \end{array} + 4 \cdot 2 \begin{array}{c} 1 \end{array} + 1 \cdot 2 \begin{array}{c} 1 \end{array} =
\]

\[
= 4 \left(\begin{array}{c} 1 \\ 1 \end{array}\right) + 12 \left(\begin{array}{c} 1 \\ 1 \end{array}\right) + 20 \left(\begin{array}{c} 1 \\ 1 \end{array}\right) + 5 \left(\begin{array}{c} 1 \\ 1 \end{array}\right) + 3 \left(\begin{array}{c} 1 \\ 1 \end{array}\right) =
\]

\[
= 4 \begin{array}{c} 1 \end{array} + 12 \begin{array}{c} 1 \end{array} + 20 \begin{array}{c} 1 \end{array} + 5 \begin{array}{c} 1 \end{array} + 3 \begin{array}{c} 1 \end{array}.
\]

With this expression and \(W_2^{(1)}\), we can now calculate \(W_1^{(2)}\):

\[
W_1^{(2)} = W_2^{(1)} + W_1^{(1)} \times W_1^{(1)} =
\]

\[
= 4 \begin{array}{c} 1 \end{array} + 12 \begin{array}{c} 1 \end{array} + 20 \begin{array}{c} 1 \end{array} + 5 \begin{array}{c} 1 \end{array} + 3 \begin{array}{c} 1 \end{array} + \begin{array}{c} \times \end{array} =
\]

\[
= 4 \cdot 2 \begin{array}{c} 1 \end{array} + 12 \begin{array}{c} 1 \end{array} + 20 \begin{array}{c} 1 \end{array} + 5 \cdot 2 \left(\begin{array}{c} 1 \\ 1 \end{array}\right) + 3 \left(\begin{array}{c} 1 \\ 1 \end{array}\right) + 1 \cdot 1 \begin{array}{c} 1 \end{array} =
\]

\[
= 3 \cdot 5 \cdot 7 \begin{array}{c} 1 \end{array} + 3 \cdot 7 \begin{array}{c} 1 \end{array}.
\]

The same answer follows from the Harer–Zagier recurrence relation: \(b_1^{(2)} = 3 \cdot 5 \cdot 7\) and \(b_0^{(2)} = 3 \cdot 7\).
Example 5. The first example in which all three of the above operations contribute is calculating \( W^{(1)}_3 \):

\[
W^{(1)}_3 = \left( W^{(1)}_2 \right) + W^{(0)}_2 \times W^{(1)}_1.
\]  

(5.24)

Here, the first term (with coproduct) contains 67 summands, the second contains 21 summands, and the third contains only three summands. Summing, we again obtain a result that is totally symmetric in all \( p_i \) including \( p_1 \). This answer has the form

\[
W^{(1)}_3 = 24 + 192 + 240 + 288 + 480 + 560 + 30 + \\
+ 18 + 24 + 120 + 72 + 140 + 120 + \\
+ 120 + 30 + 140.
\]  

(5.25)

A second example is \( W^{(0)}_5 \):

\[
W^{(0)}_5 = \left( W^{(0)}_4 \right) \times W^{(0)}_3.
\]  

(5.26)

The first term contains 85 summands, and the second term contains seven summands, which we present in order to clarify the count of symmetry coefficients (for brevity, we omit unit binomial coefficients):

\[
W^{(0)}_3 \times W^{(0)}_3 = 16 + 4 + 4 + 16 + \\
+ 4 + 4 + 4 + 16 + \\
= 16 \cdot 2 \cdot \left( \frac{4}{2} \right) + 4 \cdot 2 \cdot \left( \frac{3}{1} \right) + 4 \cdot 2 \cdot \left( \frac{3}{1} \right) + \\
+ 2 \cdot \left( \frac{4}{2} \right) + 2 \cdot \left( \frac{3}{1} \right) + 2 \cdot \left( \frac{2}{1} \right) + \left( \frac{2}{1} \right).
\]

Here, because we apply the product operation to identical objects, we take each type of product into account only once, but if we take a product of different entries or different types of columns inside the same entry, we must multiply by two. One factor of two is always present because the result contains more than one column.

The sum in (5.26) is totally symmetric in all \( p_i \) including \( p_1 \), and we obtain the answer

\[
W^{(0)}_5 = 192 + 768 + 1152 + 960 + 144 + \\
+ 240 + 288 + 288 + 144 + 288 + \\
+ 240 + 24 + 72 + 60 + 18 + \\
+ 72 + 12 + 60.
\]  

(5.27)
It is obvious that all the coefficients in the coproduct and product relations are positive integers, and the result is therefore always an integer and positive. Moreover, it follows from the lemma in [4] that the result of jointly applying the coproduct and product operations is automatically symmetric with respect to permutations of all arguments including $p_1$; therefore, the result of jointly applying the operations CP, PI, and PII is in the image $S(Y_s^{(g)})$ in $Y_s^{(g)}$, which we can naturally identify with the module $Y_s^{(g)}$ itself:

$$\text{CP}(W_{s-1}^{(g)}) + \text{PI}(W_{s+1}^{(g-1)}) + \sum_{g_1+g_2=g \atop s_1+s_2=s+1} \text{PII}(W_{s_1}^{(g_1)} \times W_{s_2}^{(g_2)}) = W_s^{(g)} \in \mathbb{Z}_{\geq 0}[Y_s^{(g)}].$$

This concludes the proof of the theorem.

6. The one-backbone case

6.1. The Harer–Zagier recurrence relation and representation in terms of graphs in Sec. 3.

In the case of chord diagrams with one backbone, we have representation (3.13) and the alternative representation

$$W_1^{(g)}(e^\lambda + e^{-\lambda}) = \sum_{r=0}^{3g-2} (-1)^r \frac{\kappa_{g,1,r}}{2d-r(d-r)!} e^{\lambda} - e^{-\lambda} \left( \frac{\partial}{\partial \lambda} \right)^{2d-2r+1} \frac{2}{e^{2\lambda} - 1}, \quad d = 3g - 2, \quad (6.1)$$

where it follows from reasonings related to the stratification of closed moduli spaces that $\kappa_{g,1,r}$ are conjecturally positive rational numbers, $\kappa_{g,1,0} = \langle \tau_{3g-2} \rangle_g$.

For the coefficients $b_k^{(g)}$ in decomposition (3.13) based on the Harer–Zagier recurrence relation [1], we obtained the following recurrence relation (independently found in [42]).

**Proposition 1** [9]. The coefficients $b_k^{(g)}$ in decomposition (3.13) satisfy the three-term recurrence relation

$$(4g + 2k + 6)b_k^{(g+1)} = (4g + 2k + 1)(4g + 2k + 3)(4g + 2k + 2)b_k^{(g)} + 4(4g + 2k - 1)b_k^{(g-1)}.$$ 

All these coefficients are positive integers.\(^1\)

In [9], we used recurrence relation (6.2) to develop a method that allows determining $b_{g-1-k}^{(g)}$ for any fixed $k \geq 0$ and for all $g$. For example, we have just two-term relations for the boundary coefficients

$$(4g + 6)b_0^{(g+1)} = (4g - 1)(4g + 3)(4g + 2)b_0^{(g)},$$

$$(6g + 6)b_1^{(g+1)} = 4(6g + 1)(6g + 3)(6g - 1)b_1^{(g-1)},$$

which immediately give

$$b_{g-1}^{(g)} = \frac{2^{g-1} (6g - 3)!!}{3^g g!}, \quad b_0^{(g)} = \frac{(4g)!}{8^g g! (2g + 1)!!}.$$ 

Substituting $b_{g-1}^{(g)}$ in (6.1) and evaluating the leading term ($r = 0$), we obtain the highest Kontsevich coefficient $\kappa_{g,1,0} = \langle \tau_{3g-2} \rangle_g = 1/2^{3g} 3^g g!$.

\(^1\)Of course, the property of positivity and integrality of the coefficients $b_k^{(g)}$ is a particular case of the general statement in Theorem 7.
Solving recurrence relation (6.2) for the first subleading term, we obtain
\[ b_{g-2}^{(g)} = \frac{1}{5} \frac{2g-2}{3g-2} \frac{(6g-5)!!}{(g-2)!} \quad \text{or} \quad \kappa_{g,1,1} = \frac{1}{5} [12g^2 - 7g + 5] \kappa_{g,1,0}, \quad g \geq 2. \quad (6.4) \]

For the next term, we have
\[ b_{g-3}^{(g)} = \frac{(2g-1) 2^{g-3}}{5^2 3^{g-3} (g-3)!} \frac{(6g-7)!!}{(g-3)!!} - \frac{7}{10} \frac{2^{g-3}}{(3g-2)!!}, \quad (3g-2)!! = \prod_{k=3}^{g} (3k-2), \quad (6.5) \]

and so on. The complete multistep procedure was described in [9].

We can alternatively derive \( b_{g-2}^{(g)} \) from the graph representation in Lemma 4. For this, it suffices to take only the part with the times \( T_{2k}^+ \). The highest term for genus \( g \) is \( \langle \tau_{3g-2} \rangle g T_{6g-4}^+ \).

According to Lemma 4, the first-order correction (or the coefficient of \( T_{6g-6}^+ \)) comes from only two terms: from the graph with one vertex and one internal edge with the endpoint markings \((0,0)\) and from the graph with one vertex and one half-edge with the marking 2 (see Fig. 5). The corresponding coefficient is then
\[ \frac{B_2}{4} \langle \tau_{3g-3} \rangle g_{g-1} + \frac{3}{5} \langle \tau_{3g-3} \tau_2 \rangle g, \quad (6.6) \]

and we need only know the corresponding intersection indices. While \( \langle \tau_{3g-3} \tau_0 \rangle g_{g-1} = \langle \tau_{3g-5} \rangle g_{g-1} \), we calculated the intersection index \( \langle \tau_{3g-3} \tau_2 \rangle g \) in [9] using the Virasoro conditions for the Kontsevich matrix model, and the result is
\[ \langle \tau_2 \tau_{3g-3} \rangle g = \frac{1}{5} [12g(g - 1) + 5] \langle \tau_{3g-2} \rangle g, \quad g \geq 2. \quad (6.7) \]

Using formula (5.7) and the fact that \( B_2 = 1/24 \), we find that the coefficient of \( T_{6g-6}^+ \) is \( [12g^2 - 7g + 5]/5 \) in full agreement with (6.4).

Below, we present the third approach for calculating the same quantity using explicit diagram counting.

### 6.2. The 2-cycles and the recurrence relation for \( b_{g-1}^{(g)} \) and \( b_{g-2}^{(g)} \).

#### 6.2.1. Contracting edges in genus-\( g \) graphs.

We now find \( b_{g-2}^{(g)} \) using explicit fat graph counting. For this, we consider the set of shapes with one boundary component and one marked edge. We let \( \Gamma^{(g)} \) denote the sets of combinatorial types of the corresponding genus-\( g \) shapes and \( V^{(g)} \) denote cardinalities of these sets.

We first consider the procedure of edge contraction in genus-\( g \) graphs. We let \( \Gamma^{(g)}_{g;3-2} \) denote the set of genus-\( g \) shapes with a marked edge with all vertices having valence three and with \( q \) 2-cycles (all these 2-cycles are of the form as in the rightmost diagram in Fig. 6). We let \( V^{(g)}_{g;3-3} \) denote the number of such diagrams. We let \( \Gamma^{(g)}_{4,3-3}, \Gamma^{(g)}_{4,4,3-3}, \) and \( \Gamma^{(g)}_{5,3-3} \) denote the respective sets of genus-\( g \) shapes with the marked edge and with one four-valent vertex, two four-valent vertices, and one five-valent vertex and with all other vertices having valence three. The numbers of the corresponding shapes are \( V^{(g)}_{4,3-3}, V^{(g)}_{4,4,3-3}, \) and \( V^{(g)}_{5,3-3}. \)
We now consider the contraction process. We never contract the marked edge corresponding to the ends of the backbone and can contract any other edge in any graph in $\Gamma^{(g)}_{q;3-3}$ (there are $6g - 4$ contractible edges in total) every time obtaining a graph in $\Gamma^{(g)}_{4,3-3}$. Vice versa, every graph in $\Gamma^{(g)}_{4,3-3}$ can be obtained from two graphs in $\Gamma^{(g)}_{q;3-3}$. We therefore have the equality

$$
(6g - 4) \sum_{q=0}^{\max} V^{(g)}_{q;3-3} = 2V^{(g)}_{4,3-3} = (6g - 4)b^{(g)}_{g-1}.
$$

A more interesting situation occurs when we want to contract two edges. We have three possible outcomes:

1. When we contract two disjoint edges, we obtain a graph in $V^{(g)}_{4,4,3-3}$.
2. When we contract two edges with incidence one, we obtain a graph in $V^{(g)}_{5,3-3}$.
3. We are not allowed to contract two edges with incidence two (which therefore constitute a 2-loop).

We begin with the first variant. The total number of disjoint pairs of edges is

$$
\frac{1}{2}(6g - 4)(6g - 5) - \# \text{ of incident pairs of edges}.
$$

The number of edges of incidence one and two can be easily counted: this is three times the number of vertices minus four because of the marked edge minus twice the number of 2-loops in a graph in $V^{(g)}_{q;3-3}$, i.e., $3(4g - 2) - 4 - 2q$. The number of pairs of incidence two is obviously $q$. The total number of nonincident pairs can then be easily counted: $(3g - 4)(6g - 5) + q$. We note that from each such pair, we produce a graph in $V^{(g)}_{4,4,3-3}$, and each graph in $V^{(g)}_{4,4,3-3}$ can be produced in exactly four ways from the graphs in $V^{(g)}_{q;3-3}$ with some $q$ (it might be the same graph in $V^{(g)}_{q;3-3}$ that produces a graph in $V^{(g)}_{4,4,3-3}$, and we then count this case with the corresponding multiplicity). The resulting relation is

$$
\sum_{q=0}^{\max} [(3g - 4)(6g - 5) + q] V^{(g)}_{q;3-3} = 4V^{(g)}_{4,4,3-3}.
$$

Analogously, each graph in $V^{(g)}_{5,3-3}$ can be obtained by contracting two edges with incidence one in exactly five ways from graphs in $V^{(g)}_{q;3-3}$, i.e., we obtain

$$
\sum_{q=0}^{\max} [12g - 10 - 2q] V^{(g)}_{q;3-3} = 5V^{(g)}_{5,3-3}.
$$

The total number of diagrams with $6g - 6$ nonmarked edges is precisely the sum of $V^{(g)}_{4,4,3-3}$ and $V^{(g)}_{5,3-3}$ and is given by a combination of $b$ factors. We therefore obtain

$$
V^{(g)}_{4,4,3-3} + V^{(g)}_{5,3-3} = b^{(g)}_{g-2} + \frac{(3g - 2)(3g - 3)}{2} b^{(g)}_{g-1}.
$$

We have the three above equations for three unknowns $V^{(g)}_{4,4,3-3}$, $V^{(g)}_{5,3-3}$, and $\sum_{q=1}^{\max} q V^{(g)}_{q;3-3}$. The solution is

$$
V^{(g)}_{4,4,3-3} = \frac{1}{4} \left[ (3g - 4)(6g - 5)b^{(g)}_{g-1} + gb^{(g)}_{g-1} - \frac{20}{3} b^{(g)}_{g-2} \right],
$$

$$
V^{(g)}_{5,3-3} = 2(g - 1)b^{(g)}_{g-1} + \frac{8}{3} b^{(g)}_{g-2},
$$

$$
\sum_{q} q V^{(g)}_{q;3-3} = gb^{(g)}_{g-1} - \frac{20}{3} b^{(g)}_{g-2}.
$$
Fig. 6. The procedure of gluing handles into two sides of two arbitrary edges of a three-valent graph $\Gamma_{3-3}$, which increase the genus by one: we can regard it as blowing up handles from a pair of punctures.

We note that there is another particular combination of $V$ that produces an interesting relation:

$$\begin{align*}
(2!)^2 V^{(g)}_{4,4,3-3} + 5V^{(g)}_{5,3-3} = (6g - 5)(6g - 6)V^{(g)}_{3-3},
V^{(g)}_{3-3} &= \max_{q=0} \sum q V^{(g)}_{q;3-3},
\end{align*}$$

(6.16)

(We have verified the validity of this relation for $g = 3$ using the data in [43].)

6.2.2. Blowing up process, $g \to g + 1$. We now consider the “inverse” process shown in Fig. 6, which allows blowing up a handle from a pair of marked sides of edges in a graph in $\Gamma^{(g)}_{q;3-3}$; the number of 2-cycles is irrelevant here. At the first stage, we allow “bubbling” of two sides of edges as in the middle diagram in Fig. 6; we must now also allow this bubbling on the marked edge. We must also represent this marked edge as a subdiagram containing three edges joined at a single vertex: two edges are incident to the rest of the diagram (their ends are the ends of the marked edge and are therefore always different), and the third edge is the tail. We can consider bubbling process sequentially. In the first stage, we have $2(6g - 1)$ possibilities for setting a bubble on an edge side, while in the second stage, we already have $2(6g + 1)$ such possibilities because we increased the total number of edges by two in the first process. Therefore, the total number of possibilities is

$$\frac{1}{2} 2^2 (6g - 1)(6g + 1) = 2(6g - 1)(6g + 1).$$

Every time we blow up two bubbles in a graph in $\Gamma^{(g)}_{3-3}$, we obtain a graph in $\Gamma^{(g+1)}_{q;3-3}$ with $q \neq 0$. Vice versa, every graph in $\Gamma^{(g+1)}_{q;3-3}$ with nonzero $q$ can be obtained in exactly $q$ ways from graphs in $\Gamma^{(g)}_{3-3}$. (We note that the number of 2-cycles does not necessarily increase in this process: if we blow up a bubble on an edge in a 2-cycle in the initial graph, we destroy this 2-cycle and can therefore, in principle, even reduce the number of 2-cycles in this process, but every time we obtain a graph of genus $g + 1$ with at least one 2-cycle.)

We thus obtain the relation

$$2(6g - 1)(6g + 1)V^{(g)}_{3-3} = \sum_{q=0}^{\max} q V^{(g+1)}_{q;3-3},$$

(6.17)

whence substituting the result in (6.15) and recalling that $V^{(g)}_{3-3}$ is merely $b^{(g)}_{g-1}$, we obtain the new relation for the $b$ values

$$2(6g - 1)(6g + 1)b^{(g)}_{g-1} = (g + 1)b^{(g+1)}_{g} - \frac{20}{3} b^{(g+1)}_{g-1}.$$  

(6.18)

From this relation, we immediately obtain

$$b^{(g)}_{g-2} = \frac{3}{10} \frac{g(g - 1)}{2g - 1} b^{(g)}_{g-1},$$

(6.19)

which coincides with (6.4).
7. Conclusion

Using the topological recursion method to construct generating functions for CohFTs occupies an important place in contemporary mathematical physics (see, e.g., [44], where all invariants of the descendant fields were recently constructed for all orders of the genus expansion of equivariant Gromov–Witten invariants of the projective line $\mathbb{P}^1$ using the topological recursion method). In this respect, it seems interesting to investigate the possibility of applying Givental-type decompositions in the quantum spectral curve approach.

REFERENCES

1. J. Harer and D. Zagier, Invent. Math., 85, 457–485 (1986).
2. É. Brezin and S. Hikami, Commun. Math. Phys., 283, 507–521 (2008); arXiv:0708.2210v1 [hep-th] (2007).
3. A. Morozov and Sh. Shakirov, “From Brezin–Hikami to Harer–Zagier formulas for Gaussian correlators,” arXiv:1007.4100v1 [hep-th] (2010).
4. L. Chekhov and B. Eynard, JHEP, 0603, 014 (2006); arXiv:hep-th/0504116 (2005).
5. B. Eynard and N. Orantin, Commun. Number Theory Phys., 1, 347–452 (2007).
6. S. Gukov and P. Su/suppresslkowski, JHEP, 1202, 070 (2012); arXiv:1108.0002 (2011).
7. M. Mulase and P. Sukowski, “Spectral curves and the Schrödinger equations for the Eynard–Orantin recursion,” arXiv:1210.3006 (2012).
8. O. Dumitrescu and M. Mulase, “Quantization of spectral curves for meromorphic Higgs bundles through topological recursion,” arXiv:1411.1023 (2014).
9. J. E. Andersen, L. O. Chekhov, P. Norbury, and R. C. Penner, “Models of discretized moduli spaces, cohomological field theories, and Gaussian means,” arXiv:1501.05867v1 (2015).
10. L. Chekhov and Yu. Makeenko, Modern Phys. Lett. A, 7, 1223–1236 (1992); arXiv:hep-th/9201033v1 (1992).
11. J. Ambjørn, L. Chekhov, C. F. Kristjansen, and Yu. Makeenko, Nucl. Phys. B, 404, 127–172 (1993); Erratum, 449, 681 (1995); arXiv:hep-th/9302014v1 (1993).
12. Y. Manin, Frobenius Manifolds, Quantum Cohomology, and Moduli Spaces (AMS Colloq. Publ., Vol. 47), Amer. Math. Soc., Providence, R. I. (1999).
13. L. Chekhov and Yu. Makeenko, Phys. Lett. B, 278, 271–278 (1992); arXiv:hep-th/920206v1 (1992).
14. A. Marshakov, A. Mironov, and A. Morozov, Phys. Lett. B, 265, 99–107 (1991).
15. P. Norbury, “Quantum curves and topological recursion,” arXiv:1502.04394v1 [math-ph] (2015).
16. L. Chekhov, J. Geom. Phys., 12, 153–164 (1993); arXiv:hep-th/9205106v1 (1992).
17. P. Norbury, Trans. Amer. Math. Soc., 365, 1687–1709 (2013).
18. M. Mulase and M. Penkava, Adv. Math., 230, 1322–1339 (2012); arXiv:1009.2135v2 [math.AG] (2010).
19. P. Norbury, Math. Res. Lett., 17, 467–481 (2010).
20. L. Chekhov, Acta Appl. Math., 48, 33–90 (1997); arXiv:hep-th/9509001v1 (1995).
21. A. B. Givental, Moscow Math. J., 1, 551–568 (2001).
22. B. Eynard, Commun. Number Theory Phys., 8, 541–588 (2014); arXiv:1110.2949v1 [math-ph] (2011).
23. P. Dunin-Barkowski, N. Orantin, S. Shadrin, and L. Spitz, Commun. Math. Phys., 328, 669–700 (2014); arXiv:1211.4021v1 [math-ph] (2012).
24. B. Eynard, JHEP, 0411, 031 (2004); arXiv:hep-th/0407261v1 (2004).
25. L. Chekhov, B. Eynard, and N. Orantin, JHEP, 0612, 053 (2006); arXiv:math-ph/0603003v2 (2006).
26. A. Alexandrov, A. Mironov, and A. Morozov, Internat. J. Mod. Phys. A, 19, 4127–4165 (2004); arXiv:hep-th/0310113v1 (2003).
27. B. Dubrovin, “Geometry of 2D topological field theories,” in: Integrable Systems and Quantum groups (Lect. Notes Math., Vol. 1620, M. Francaviglia and S. Greco, eds.), Springer, Berlin (1996), pp. 120–348; arXiv:hep-th/9407018v1 (1994).
28. P. Dunin-Barkowski, P. Norbury, N. Orantin, P. Popolitov, and S. Shadrin, “Superpotentials and the Eynard–Orantin topological recursion,” (in preparation).
29. J. E. Andersen, R. C. Penner, C. M. Reidys, and M. S. Waterman, *J. Math. Biol.*, 67, 1261–1278 (2013).
30. J. Harer, *Invent. Math.*, 84, 157–176 (1986).
31. K. Strebel, *Quadratic Differentials* (Ergebn. Math. ihrer Grenzg., Vol. 3), Springer, Berlin (1984).
32. R. C. Penner, *Commun. Math. Phys.*, 113, 299–339 (1987).
33. R. C. Penner, *J. Differential Geom.*, 27, 35–53 (1988).
34. M. L. Kontsevich, *Commun. Math. Phys.*, 147, 1–23 (1992).
35. S. Kharchev, A. Marshakov, A. Mironov, A. Morozov, and A. Zabrodin, *Phys. Lett. B*, 275, 311–314 (1992); arXiv:hep-th/9111037v1 (1991).
36. J. Ambjørn and L. Chekhov, *Ann. Inst. Henri Poincaré D*, 1, 337–361 (2014); arXiv:1404.4240v2 [math.AG] (2014).
37. A. Alexandrov, A. Mironov, A. Morozov, and S. Natanzon, *JHEP*, 1411, 080 (2014); arXiv:1405.1395v3 [hep-th] (2014).
38. J. E. Andersen, L. O. Chekhov, C. M. Reidys, R. C. Penner, and P. Sułkowski, *Nucl. Phys. B*, 866, 414–443 (2013); arXiv:1205.0658v1 [hep-th] (2012).
39. S. Keel, *Trans. Amer. Math. Soc.*, 330, 545–574 (1992).
40. B. Dubrovin, “Geometry of 2D topological field theories,” in: *Integrable Systems and Quantum Groups* (Lect. Notes Math., Vol. 1620, M. Francaviglia and S. Greco, eds.), Springer, Berlin (1996), pp. 120–348.
41. C. Teleman, *Invent. Math.*, 188, 525–588 (2012).
42. U. Haagerup and S. Thorbjørnsen, *Infin. Dimens. Anal. Quantum Probab. Relat. Top.*, 15, 1250003 (2012).
43. R. J. Milgram and R. C. Penner, “Riemann’s moduli space and the symmetric group,” in: *Mapping Class Groups and Moduli Spaces of Riemann Surfaces* (Contemp. Math., Vol. 150, C.-F. Bödigheimer and R. M. Hain, eds.), Amer. Math. Soc., Providence, R. I. (1993), pp. 247–290.
44. B. Fang, C.-C. M. Liu, and Z. Zong, “The Eynard–Orantin recursion and equivariant mirror symmetry for the projective line,” arXiv:1411.3557v2 [math.AG] (2014).