DUAL DIVERGENCES ESTIMATION FOR CENSORED SURVIVAL DATA

MOHAMED CHERFI

ABSTRACT. This paper is devoted to robust estimation based on dual divergences estimators for parametric models in the framework of right censored data. We give limit laws of the proposed estimators and examine their asymptotic properties through a simulation study.

Key words and phrases: Robust estimation; Minimum divergence estimators; Kaplan-Meier estimator; M-estimators.

AMS Subject Classification: 62N01; 62N02.

1. INTRODUCTION

In engineering and biomedical sciences, parametric models are frequently used in analyzing survival data. This analysis is often complicated by the presence of right censoring. Typically right censored data arise in medical studies when patients cannot be followed to the event of interest.

A common parametric method of estimation is the maximum likelihood approach which is efficient if the specified parametric model is valid. However, in many situations in practice, there is no certainty that the data come from a specified parametric model and may, in fact, come from some neighborhood of the model. Likelihood based estimation procedures can lead to poor results when the underlying model is misspecified or contaminated. In such instances, the maximum likelihood is not robust against data or model inadequacies and the need for robust statistical techniques for estimation and testing has been stressed by many authors, we may refer to Huber (1981), Hampel et al. (1986), Maronna et al. (2006) and the references therein.

In this paper, we consider parametric estimation for right censored data with and without contamination, and try to balance the dual aims of robustness and efficiency using minimum divergence estimators.
Keziou (2003) and Broniatowski and Keziou (2009) introduced the class of dual divergences estimators for general parametric models, the procedure being based on the optimization of a new dual form of a divergence and includes the maximum likelihood as a benchmark. Toma and Broniatowski (2010) have proved that this class contains robust and efficient estimators and proposed robust test statistics based on divergences estimators.

A major advantage of the method is that it does not require additional accessories such as kernel density estimation or other forms of nonparametric smoothing to produce nonparametric density estimates of the true underlying density function. The plug-in of the empirical distribution function is sufficient for the purpose of estimating the divergence in the case of i.i.d. data. For the right-censoring scenario, one can replace the empirical distribution function with the corresponding estimate of the cumulative distribution function based on the Kaplan-Meier estimate Kaplan and Meier (1958). Thus in this situation one can also estimate the divergence measure without having to take recourse to nonparametric smoothing techniques in contrast with existing method, see Yang (1991), Ying (1992) that need a nonparametric estimate of the true density function. Another feature of the proposed method is its flexibility, that is it leads to a wide class of $M$-estimators indexed by the divergence function and by some instrumental value of the parameter, called here escort parameter. Relevant choices induce efficiency and robustness properties of the proposed estimators.

The paper is organized as follows. In Section 2, we present the class of dual divergences estimators in the censored case. Asymptotic properties of the proposed estimators are derived in Section 3. We give a brief discussion on the choice of the escort parameter in Section 4. In Section 5, we present Monte Carlo simulation studies to show the performance of the proposed estimators from both robustness and small sample accuracy points of view. Proofs are deferred to the Appendix.

2. Dual divergences for censored data

The class of dual divergences estimators has been recently introduced by Keziou (2003), Broniatowski and Keziou (2009). In the following, we shortly recall their context and definition.
Recall that the $\phi$-divergence between a bounded signed measure $Q$ and a probability $P$ on $\mathcal{D}$, when $Q$ is absolutely continuous with respect to $P$, is defined by

$$D_\phi(Q, P) := \int_{\mathcal{D}} \phi \left( \frac{dQ}{dP}(x) \right) dP(x),$$

where $\phi$ is a convex function from $]-\infty, \infty[ \to [0, \infty]$ with $\phi(1) = 0$.

Well-known examples of divergences are the Kullback-Leibler, modified Kullback-Leibler, $\chi^2$, modified $\chi^2$ and Hellinger divergences, they are obtained respectively for $\phi(x) = x \log x - x + 1$, $\phi(x) = -\log x + x - 1$, $\phi(x) = \frac{1}{2}(x-1)^2$, $\phi(x) = \frac{1}{2}\frac{(x-1)^2}{x}$ and $\phi(x) = 2(\sqrt{x} - 1)^2$. All these divergences belong to the class of the so called “power divergences” introduced in Cressie and Read (1984) (see also Liese and Vajda (1987) chapter 2). They are defined through the class of convex functions $x \in [0, +\infty[ \to \phi_{\gamma}(x) := \frac{x^\gamma - \gamma x + \gamma - 1}{\gamma(\gamma - 1)}$ (2.1) if $\gamma \in \mathbb{R} \setminus \{0, 1\}$, $\phi_0(x) := -\log x + x - 1$ and $\phi_1(x) := x \log x - x + 1$. (For all $\gamma \in \mathbb{R}$, we define $\phi_{\gamma}(0) := \lim_{x \downarrow 0} \phi_{\gamma}(x)$). So, the $KL$-divergence is associated to $\phi_1$, the $KL_m$ to $\phi_0$, the $\chi^2$ to $\phi_2$, the $\chi_m^2$ to $\phi_{-1}$ and the Hellinger distance to $\phi_{1/2}$. We refer to Liese and Vajda (1987) for an overview on the origin of the concept of divergences in statistics.

Let $X_1, \ldots, X_n$ be an i.i.d. sample with p.m. $P_{\theta_0}$. Consider the problem of estimating the population parameters of interest $\theta_0$, when the underlying identifiable model is given by $\{P_{\theta} : \theta \in \Theta\}$ with $\Theta$ a subset of $\mathbb{R}^d$.

Let $\phi$ be a function of class $C^2$, strictly convex and satisfies

$$\int \left| \phi' \left( \frac{p_\theta(x)}{p_\alpha(x)} \right) \right| dP_\theta(x) < \infty. \quad (2.2)$$

By Lemma 3.2 in Broniatowski and Keziou (2006), if the function $\phi$ satisfies: There exists $0 < \eta < 1$ such that for all $c$ in $[1 - \eta, 1 + \eta]$, we can find numbers $c_1$, $c_2$, $c_3$ such that

$$\phi(cx) \leq c_1 \phi(x) + c_2 |x| + c_3, \text{ for all } x, \quad (2.3)$$

then the assumption (2.2) is satisfied whenever $D_\phi(P_\theta, P_\alpha)$ is finite. From now on, $\mathcal{U}$ will be the set of $\theta$ and $\alpha$ such that $D_\phi(P_\theta, P_\alpha) < \infty$. Note that all the real convex functions $\phi_{\gamma}$ pertaining to the class of power divergences defined in (2.1) satisfy the condition (2.3). Take for example the exponential distribution with density $p_\theta(x) = \theta e^{-\theta x}$ for $x \geq 0$ and $\theta > 0$, then $\mathcal{U} := \{\alpha, \theta > 0 : \gamma\theta + (1 - \gamma)\alpha > 0\}$. 


Under (2.2), using Fenchel duality technique, the divergence $D_\phi(\theta, \theta_0)$ can be represented as resulting from an optimization procedure, this elegant result was proven in Keziou (2003), Liese and Vajda (2006) and Broniatowski and Keziou (2009). Broniatowski and Keziou (2006) called it the dual form of a divergence, due to its connection with convex analysis.

Under the above conditions, the $\phi$-divergence:

$$D_\phi(P_\theta, P_{\theta_0}) = \int \phi\left(\frac{p_\theta(x)}{p_{\theta_0}(x)}\right) dP_{\theta_0},$$

can be represented as the following form:

$$D_\phi(P_\theta, P_{\theta_0}) = \sup_{\alpha \in \mathcal{U}} \int h(\theta, \alpha) dP_{\theta_0}, \quad (2.4)$$

where $h(\theta, \alpha) : x \mapsto h(\theta, \alpha, x)$ and

$$h(\theta, \alpha, x) := \int \phi'(\frac{p_\theta(x)}{p_{\alpha}(x)}) dP_{\theta} - \left[\frac{p_\theta(x)}{p_{\alpha}(x)}\phi'\left(\frac{p_\theta(x)}{p_{\alpha}(x)}\right) - \phi\left(\frac{p_\theta(x)}{p_{\alpha}(x)}\right)\right]. \quad (2.5)$$

According to Liese and Vajda (2006), under the strict convexity and the differentiability of the function $\phi$, it holds

$$\phi(t) \geq \phi(s) + \phi'(s)(t - s), \quad (2.6)$$

where the equality holds only for $s = t$. Now, let $\theta$ and $\theta_0$ be fixed and put $t = p_\theta(x)/p_{\theta_0}(x)$ and $s = p_\theta(x)/p_{\alpha}(x)$ in (2.6) and (2.4) will follow by integrating with respect to $P_{\theta_0}$.

Since the supremum in (2.4) is unique and is attained in $\alpha = \theta_0$, independently upon the value of $\theta$, define the class of estimators of $\theta_0$ by

$$\hat{\alpha}_{\phi}(\theta) := \arg \sup_{\alpha \in \mathcal{U}} \int h(\theta, \alpha) dP_{\theta_0}, \quad \theta \in \Theta, \quad (2.7)$$

where $h(\theta, \alpha)$ is the function defined in (2.5). This class is called “dual $\phi$-divergence estimators” (D$\phi$DE’s).

Let us now turn to the estimation using divergences in our setting. In the case of right censored data only

$$Z = \min (X, Y) \text{ and } \delta = 1_{\{X \leq Y\}}$$

are observable. $\delta$ indicates whether $X$ has been censored or not. The variables $X_i$ are randomly generated from the true distribution $P_{\theta_0}$ which is modeled by the
parametric family \( \{P_{\theta}, \theta \in \Theta\} \). Given a set \((Z_i, \delta_i), i = 1, \ldots, n\) of independent copies of \((Z, \delta)\), it is then our goal to draw some inference on the true but unknown lifetime distribution \(P_{\theta_0}\).

Throughout the rest of the paper we will assume that the variable of interest \(X\) and the censoring variable \(Y\) are independent and \(G\) denotes the unknown distribution of censoring time \(Y\). The distribution \(F\) of the observation \(Z = \min(X, Y)\), satisfies \(1 - F = (1 - P_{\theta_0})(1 - G)\).

Kaplan and Meier (1958) developed a nonparametric estimator for the survival function which is a strongly consistent estimator of the target survival function under appropriate conditions (see Peterson (1977), Miller (1981))

\[
\hat{P}_n(x) = 1 - \prod_{i=1}^{n} \left[ 1 - \frac{\delta(i)}{n - i + 1} \right] 1_{\{Z(i) \leq x\}}
\]

where \((Z(i), \delta(i)), i = 1, \ldots, n\), are the \(n\) pairs of observations ordered over the \(Z(i)\) and \(1_A\) denotes indicator function of \(A\). If all \(\delta\)'s are equal to 1, \(\hat{P}_n\) reduces to the ordinary empirical distribution function \(P_n\).

Thus, in the right censoring context described above, we can replace \(P_n\) in (2.7) by \(\hat{P}_n(x)\) which provides a consistent estimator of the true distribution function in this context. Therefore, for the right censoring situation the “dual \(\phi\)-divergence estimators” (\(D\phi\)DE’s), is defined by replacing \(P_n\) in (2.7) by \(\hat{P}_n\), that is

\[
\hat{\alpha}_{\phi}(\theta) := \arg \sup_{\alpha \in \mathcal{U}} \int h(\theta, \alpha) d\hat{P}_n, \quad \theta \in \Theta.
\] (2.8)

Following Stute (1995), the Kaplan-Meier integral \(\int h(\theta, \alpha) d\hat{P}_n\) may be written as

\[
\sum_{i=1}^{n} W_{in} h(\theta, \alpha, Z(i))
\]

where for \(1 \leq i \leq n\)

\[
W_{in} = \frac{\delta(i)}{n - i + 1} \prod_{j=1}^{i-1} \left[ \frac{n - j}{n - j + 1} \right] \delta(j).
\]

The corresponding estimating equation for the unknown parameter is then given by

\[
\int \frac{\partial}{\partial \alpha} h(\theta, \alpha) d\hat{P}_n = 0.
\] (2.9)
Formula (2.8) defines a family of $M$-estimators for censored data indexed by the function $\phi$ specifying the divergence and by some instrumental value of the parameter $\theta$, called here escort parameter, see also Broniatowski and Vajda (2009). The choices of $\phi$ and $\theta$ represent a major feature of the estimation procedure, since they induce efficiency and robustness properties.

An $M$-estimator of $\psi$-type is the solution of the vector equation:

$$\int \psi(x; \alpha) \mathrm{d}\hat{P}_n = 0,$$

(2.10)

where the elements of $\psi(x; \alpha)$ represent the partial derivatives of $h(\theta, \alpha, x)$ with respect to the components of $\alpha$.

The first extension of $M$-estimators to censored data was noted in Reid (1981), she derived the influence function and then the asymptotic normality. Oakes (1986) considered $M$-estimators (2.10) with $\psi(x; \theta) = -\log f(x; \theta)$ and called them approximate MLEs (hereafter AMLE). Wang (1995) studied the strong consistency of $M$-estimators using the law of large numbers of the Kaplan-Meier integral developed by Stute and Wang (1993) and Stute (1995). Wang (1999) extended asymptotic results for $M$-estimators to the censored case.

The Hellinger distance have been used by Yang (1991) and Ying (1992). Estimation under misspecification have been considered by Suzukawa et al. (2001). Basu et al. (2006) developed a robust estimation, adapting the robust density power divergence methodology of Basu et al. (1998).

3. ASYMPTOTIC PROPERTIES

In this section, we establish the consistency and asymptotic normality of the class of dual divergences estimators in the right censored situation.

For a distribution $P$, let $\tau_P = \sup \{x : P(x) < 1\}$ denote the upper bound of the support of $P$.

Assume that $\theta_0$ is an interior point of $\Theta$, the convex function $\phi$ has continuous derivatives up to 4th order and the density $p_\alpha(x)$ has continuous partial derivatives up to 3th order (for all $x \lambda - a.e.$). Hereafter, $\hat{p}_\alpha$ will denotes the derivative with respect to $\alpha$ of $p_\alpha$, $\| \cdot \|$ the Euclidean norm, and, for a real valued function $g$, its
total variation or variation norm is defined as

\[ \|g\|_v = \sup_{N + 1} \sum_{j=1}^N |g(x_j) - g(x_{j-1})|, \]

where the supremum is taken over all \( N \) and over all choices of \( \{x_j\} \) such that \( -\infty = x_0 < x_1 < \ldots < x_N < x_{N+1} = +\infty \).

Let \( S \) be the \( d \times d \) matrix with entries

\[ S_{ij} = -P_{\theta_0} \frac{\partial^2}{\partial \alpha_i \partial \alpha_j} h(\theta, \theta_0). \]

We precise some notations for the asymptotic results in this section. The following quantities have been introduced in Stute (1995a) and Wang (1999).

Denote \( m(y) = p(\delta = 1|Y = y) \), decompose \( F \) into two subdistributions \( F_0, F_1 \), such that

\[ F(y) = F_0(y) + F_1(y), \]

where \( F_0(y) = P(Y \leq y, \delta = 0) = \int_{-\infty}^y (1 - m(t)) dF(t) = \int_{-\infty}^y (1 - P_{\theta_0}(t)) dG(t), \)

\[ F_1(y) = P(Y \leq y, \delta = 1) = \int_{-\infty}^y m(t) dF(t) = \int_{-\infty}^y (1 - G(t( -) )) dP_{\theta_0}, \]

and their empirical counterparts

\[ F_{jn}(y) = \frac{1}{n} \sum_{i=1}^n 1\{Z_i \leq y, \delta_i = j\}, \quad j = 0, 1. \]

Define

\[ \xi_0(x) = \exp \left\{ \int \frac{1_{(y<x)} dF_0(y)}{1 - F(y)} \right\}, \quad (3.1) \]

and, for \( i = 1, \ldots, d \),

\[ \xi_{1i}(x) = [1 - F(x)]^{-1} \int_{(x<y)} \frac{\partial}{\partial \alpha_i} h(\theta, \alpha, y) \xi_0(y) dF_1(y), \quad (3.2) \]

\[ \xi_{2i}(x) = \int \frac{\partial}{\partial \alpha_i} h(\theta, \alpha, z) \xi_0(z) C(x \land z) dF_1(z), \quad (3.3) \]

where

\[ C(x) = \int \frac{1_{(y<x)} dF_0(y)}{[1 - F(y)]^2} = \int \frac{1_{(y<x)} dG(y)}{[1 - P_{\theta_0}(y)] [1 - G(y)]^2}. \quad (3.4) \]

Let \( U(\alpha) = (U_1, \ldots, U_d)^\top \) denote the random variable defined as:

\[ U_i(\alpha) = \frac{\partial}{\partial \alpha_i} h(\theta, \alpha, Y) \xi_0(Y) \delta + \xi_{1i}(Y)(1 - \delta) - \xi_{2i}(Y), \quad i = 1 \ldots, d. \quad (3.5) \]
When $\alpha = \theta_0$,

$$U_i(\theta_0) = \frac{\partial}{\partial \alpha_i} h(\theta, \theta_0, Y) \delta + \xi_{1i}(Y)(1 - \delta) - \xi_{2i}(Y), \; i = 1 \ldots, d.$$  

Denote $V$ the $d \times d$ matrix

$$V = E \left( U(\theta_0) U(\theta_0)^\top \right). \tag{3.6}$$

3.1. **Consistency.** In Theorem 1 below, we prove that $\hat{\alpha}_\phi(\theta)$ exist and are consistent. We will consider the following conditions.

(R.0) $\tau_{P_{\theta_0}} \leq \tau_G$, where equality may hold except when $G$ is continuous at $\tau_{P_{\theta_0}}$, and, the probability mass of $P_{\theta_0}$ at $\tau_{P_{\theta_0}}$: $P_{\theta_0} \left( \tau_{P_{\theta_0}} - \tau_{P_{\theta_0}} \right) > 0$;

(R.1) There exists a neighborhood $N(\theta_0)$ of $\theta_0$ such that the first and second order partial derivatives (w.r.t $\alpha$) of $\phi' \left( p_{\theta}(x)/p_{\alpha}(x) \right) p_{\theta}(x)$ are dominated on $N(\theta_0)$ by some integrable functions. The third order partial derivatives (w.r.t $\alpha$) of $h(\theta, \alpha, x)$ are dominated on $N(\theta_0)$ by some $P_{\theta_0}$-integrable functions and the matrices $S$ and $V$ are non singular;

(R.2) $\left\| \frac{\partial}{\partial \alpha} h(\theta, \theta_0) \right\|_V < \infty$.

These conditions are mild and can be satisfied in most of circumstances. The condition (R.0) ensures that $X$ is observable on the hole of the support of $P_{\theta_0}$. Note that if $\tau_{P_{\theta_0}} > \tau_G$ holds, the $X_i$ in $[\tau_G, \infty)$ is certainly censored. In a large number of practical situations, $\tau_{P_{\theta_0}} = \tau_G = \infty$, hence the condition (R.0) is satisfied.

Condition (R.1) is about usual regularity properties of the underlying model, it guarantees that we can interchange integration and differentiation and the existence of the variance-covariance matrices, it is similar to regularity conditions used in Keziou (2003) and Broniatowski and Keziou (2009) in the uncensored case.

Condition (R.2) is needed to apply the L.I.L in the proof of Theorem 1. The requirement that $\psi(x; \alpha) := \frac{\partial}{\partial \alpha} h(\theta, \alpha)$ be of bounded variation is standard in $M$-estimation, see for instance Welsh (1989). Keep in mind the assumed regularity conditions on the criterion function, that is, $h(\theta, \alpha)$ in the present framework, to see that it holds for most regular models.

It is also noted that conditions (R.1) and (R.2) are independent of $G$.

**Theorem 1.** Let $B(\theta_0, n^{-1/3}) := \{ \theta \in \Theta, \| \theta - \theta_0 \| \leq n^{-1/3} \}$. Assume that conditions (R.0-2) hold, then as $n$ tends to infinity, with probability one, the function
\[ \alpha \mapsto \int h(\theta, \alpha) \, d\hat{P}_n \] attains its local maximum at some point \( \hat{\alpha}_\phi(\theta) \) in the interior of \( B(\theta_0, n^{-1/3}) \), which implies that the estimate \( \hat{\alpha}_\phi(\theta) \) is consistent and satisfies
\[
\int \frac{\partial}{\partial \alpha} h(\theta, \hat{\alpha}_\phi(\theta)) \, d\hat{P}_n = 0.
\]

The proof of Theorem 1 is postponed to the Appendix.

In practice, to obtain the estimate \( \hat{\alpha}_\phi(\theta) \), we use gradient descent algorithms in the optimization in (2.9). These algorithms depend on some initial parameter value of \( \alpha \). Hence, it is desirable to prove that in a neighborhood of \( \theta_0 \) there exists a maximum of \( \int h(\theta, \alpha) \, d\hat{P}_n \) which does indeed converge to \( \theta_0 \). Note that the initial parameter value may provide a local maximum (not necessarily global) of \( \int h(\theta, \alpha) \, d\hat{P}_n \). The local and global estimates coincide if the function \( \alpha \in \Theta \mapsto \int h(\theta, \alpha) \, d\hat{P}_n \) is strictly concave and \( \Theta \) is convex, see for instance Broniatowski and Keziou (2009, Remark 3.5).

The aim of Theorem 1 is not to establish the optimal rate of the estimate but merely the existence and the consistency (a.s.) of the estimate. We have considered \( n^{-1/3} \) because it works well, indeed, in Taylor expansion (A.3), in the proof, the third term of the right hand side is \( O(1) \) only for this rate, which is the major key of the demonstration, for similar arguments in the estimation of copula models see Bouzebda and Keziou (2010).

3.2. Asymptotic normality. In Theorem 2 below, we give the limit law of the estimates \( \hat{\alpha}_\phi(\theta) \) under the following conditions. From now on, \( \xrightarrow{d} \) denotes the convergence in distribution.

\begin{enumerate}
\item[(R.3)] For all \( 1 \leq i \leq d \), \( E\left[ \left( \frac{\partial}{\partial \alpha_i} h(\theta, \alpha, Y) \xi_0(Y) \delta \right)^2 \right] < \infty; \)
\item[(R.4)] For all \( 1 \leq i \leq d \), \( \int \left| \frac{\partial}{\partial \alpha_i} h(\theta, \alpha, x) \right| C^{1/2}(x) \, dP_{\theta_0} < \infty. \)
\end{enumerate}

Conditions (R.3-4) are essential for the asymptotic results of M-estimators in the censored case, see for instance Wang (1999) and Basu et al. (2006) in the case of density power divergence method.
Theorem 2. Assume that assumptions (R.0-4) hold. Then, as $n \to \infty$

$$\sqrt{n} (\hat{\alpha}_\phi(\theta) - \theta_0) \xrightarrow{d} \mathcal{N} \left(0, S^{-1} V S^{-1} \right)$$

The proof of Theorem 2 is postponed to the Appendix.

4. ADAPTIVE CHOICE OF THE ESCORT PARAMETER

Analogously as in the uncensored case, the very peculiar choice of the escort parameter defined through $\theta = \theta_0$ has same limit properties as the AMLE. The DφDE $\hat{\alpha}_\phi(\theta_0)$, in this case, has variance which indeed coincides with the AMLE for censored data. If $\theta$ is a real parameter, the asymptotic distribution of $\sqrt{n} (\hat{\alpha}_\phi(\theta) - \theta_0)$ is normal with mean zero and variance

$$\int \frac{\hat{p}_{\theta_0}^2(x)}{p_{\theta_0}(x) G(x)} \, dx - \int \frac{\hat{p}_{\theta_0}^2(x)}{\hat{P}_{\theta_0}(x) \hat{G}(x)} \, dx, \quad (4.1)$$

where $\hat{p}_{\theta}$ is the derivative with respect to $\theta$ of $p_{\theta}$ and $\hat{I}_{\theta_0}$ is the Fisher information matrix

$$\hat{I}_{\theta_0} := \int \frac{\hat{p}_{\theta_0} \hat{p}_{\theta_0}^\top}{p_{\theta_0}} \, d\lambda.$$ 

Observe that if there is no censorship, that is $G \equiv 0$, the variance of $\hat{\alpha}_\phi(\theta_0)$ is $\frac{1}{\hat{I}_{\theta_0}}$.

This result is of some relevance, since it leaves open the choice of the divergence, while keeping good asymptotic properties.

In practice, the consequence is that the escort parameter should be chosen as a the AML estimator of $\theta_0$, say $\hat{\theta}_n$, which under the model is a consistent estimate of $\theta_0$. In turn we may expect that the resulting estimator $\hat{\alpha}_\phi(\hat{\theta}_n)$ inherits both good asymptotic properties under the model, and, under contamination through a tuning of the divergence index $\gamma$.

Consider the power divergences family Cressie and Read (1984), the estimating equation (2.9) reduces to

$$- \int \left( \frac{p_\theta(x)}{p_\alpha(x)}} \right)^{\gamma - 1} \frac{\hat{p}_\alpha(x)}{p_\alpha(x)} p_\theta(x) \, dx + \frac{1}{n} \sum_{i=1}^{n} W_{in} \left( \frac{p_\theta(Z(i))}{p_\alpha(Z(i))} \right)^\gamma \frac{\hat{p}_\alpha(Z(i))}{p_\alpha(Z(i))} = 0, \quad (4.2)$$

where $W_{in}$ are the Kaplan-Meier weights. The estimate $\hat{\alpha}_\phi(\theta)$ is the solution in $\alpha$ of (4.2).
An improvement of the present estimate results in the plugging of a preliminary consistent estimate of $\theta_0$, say $\hat{\theta}_n$, as an adaptive escort parameter $\theta$ choice.

![Figure 1](image-url)

**Figure 1.** Behaviour of the ratio $\frac{p_{\hat{\theta}_n}(x)}{p_{\theta_0}(x)}$ under contamination, for a randomly generated exponential sample $\text{exp}(1)$ of size 100 with $\text{exp}(1/9)$ as censoring distribution and 20% of contamination by $\text{exp}(0.1)$.

Let $x$ be some outlier, the role of the outlier $x$ in (4.2) appears in the term

$$\left(\frac{p_{\hat{\theta}_n}(x)}{p_{\alpha}(x)}\right)^\gamma \frac{\hat{p}_\alpha(x)}{p_\alpha(x)}.$$  

(4.3)

The estimate $\hat{\alpha}_\phi(\theta)$ is robust if this term is stable. That is, if it is small when $\alpha$ is near $\theta_0$. If the escort parameter $\hat{\theta}_n$ is not a robust estimator, the ratio $\frac{p_{\hat{\theta}_n}(x)}{p_{\theta_0}(x)}$ can be very large, see Figure 1. This is due to the fact that the outlier $x$ will be more likely under $P_{\hat{\theta}_n}$, that is $\hat{\theta}_n$ will lead to an over evaluation of $p_{\hat{\theta}_n}(x)$ with respect to the expected value under $\theta_0$, say $p_{\theta_0}(x)$. To guard against such situations, compensate through the choice of $\gamma$, this requires further investigation.

One proposal for the choice of the divergence, is to look for values of the tuning parameter $\gamma$ to obtain a bounded influence function in the spirit of Toma and Broniatowski (2010), we leave this issue open for future research.
We now prove that the subsequent estimator \( \hat{\alpha}_\phi \left( \hat{\theta}_n \right) \) enjoys a limit normal law under the model, see Theorem 3 below.

Recall that, when \( \theta = \theta_0, S = -\phi''(1)I_{\theta_0} \). Also, when \( \alpha = \theta = \theta_0 \), we have

\[
U = \phi''(1)\frac{\hat{\theta}_n}{P_{\theta_0}}\xi_0(Y)\delta + \xi_1(Y)(1 - \delta) - \xi_2(Y),
\]

and the matrix \( V \) defined in (3.6) is

\[
V = E\left( UU^\top \right),
\] (4.4)

(R.5) For all \( 1 \leq i, j \leq d \), any one of the following conditions holds:

(i) \( \theta \mapsto \frac{\partial^2}{\partial \alpha_i \partial \alpha_j} h(\theta, \theta_0, x) \) is continuous at \( \theta_0 \) uniformly in \( x \);

(ii) \( \int \sup_{\{\theta : |\theta - \theta_0| \leq \rho\}} \left| \frac{\partial^2}{\partial \alpha_i \partial \alpha_j} h(\theta, \theta_0) - \frac{\partial^2}{\partial \alpha_i \partial \alpha_j} h(\theta_0, \theta_0) \right| dP_{\theta_0} = \epsilon_\rho \to 0, \) as \( \rho \to 0. \)

(iii) \( x \mapsto \frac{\partial^2}{\partial \alpha_i \partial \alpha_j} h(\theta, \theta_0, x) \) is continuous in \( x \) for \( \theta \in \text{a neighborhood of } \theta_0 \) and

\[
\lim_{\theta \to \theta_0} \left\| \frac{\partial^2}{\partial \alpha_i \partial \alpha_j} h(\theta, \theta_0, \cdot) - \frac{\partial^2}{\partial \alpha_i \partial \alpha_j} h(\theta_0, \theta_0, \cdot) \right\|_V = 0;
\]

(iv) \( \theta \mapsto \int \frac{\partial^2}{\partial \alpha_i \partial \alpha_j} h(\theta, \theta_0) dP_{\theta_0} \) is continuous at \( \theta = \theta_0, \) and

\[
x \mapsto \frac{\partial^2}{\partial \alpha_i \partial \alpha_j} h(\theta, \theta_0, x) \) is continuous in \( x \) for \( \theta \in \text{a neighborhood of } \theta_0 \)

and

\[
\lim_{\theta \to \theta_0} \left\| \frac{\partial^2}{\partial \alpha_i \partial \alpha_j} h(\theta, \theta_0, \cdot) - \frac{\partial^2}{\partial \alpha_i \partial \alpha_j} h(\theta_0, \theta_0, \cdot) \right\|_V < \infty;
\]

(v) \( \theta \mapsto \int \frac{\partial^2}{\partial \alpha_i \partial \alpha_j} h(\theta, \theta_0) dP_{\theta_0} \) is continuous at \( \theta = \theta_0, \) and

\[
\int \frac{\partial^2}{\partial \alpha_i \partial \alpha_j} h(\hat{\theta}_n, \theta_0) d\hat{P}_n \xrightarrow{P} \int \frac{\partial^2}{\partial \alpha_i \partial \alpha_j} h(\theta, \theta_0) dP_{\theta_0} < \infty,
\]

uniformly for \( \theta \in \text{a neighborhood of } \theta_0. \)

Condition (R.5) is related to Lemma 1 in Wang (1999) and ensures the convergence

\[
\int \frac{\partial^2}{\partial \alpha_i \partial \alpha_j} h(\hat{\theta}_n, \theta_0) d\hat{P}_n \xrightarrow{P} \int \frac{\partial^2}{\partial \alpha_i \partial \alpha_j} h(\theta, \theta_0) dP_{\theta_0}, 1 \leq i, j \leq d,
\]

provided that \( \int \left| \frac{\partial^2}{\partial \alpha_i \partial \alpha_j} h(\theta, \theta_0) \right| dP_{\theta_0} < \infty, 1 \leq i, j \leq d, \hat{\theta}_n \xrightarrow{P} \theta_0 \) and condition (R.0) holds.
Theorem 3. Assume that assumptions (R.0-5) hold. Then, as \( n \to \infty \)

\[
\sqrt{n} \left( \hat{\alpha} \left( \hat{\theta}_n \right) - \theta_0 \right) \xrightarrow{d} N \left( 0, \phi''^{-2}(1)I_{\theta_0}^{-1}VI_{\theta_0}^{-1} \right),
\]

where \( V \) is defined in (4.4).

The proof of Theorem 3 is postponed to the Appendix.

5. Simulation

In this section, we present results of a simulation study which was conducted to explore the properties of newly proposed dual \( \phi \)-divergence estimators (D\( \phi \)DE). These estimators are also compared with some other methods, including maximum likelihood estimator (MLE), approximate maximum likelihood estimator (AMLE) and estimators based on density power divergence method (MDPDE). Following Stute (1995), the Kaplan-Meier integral \( \int h(\theta, \alpha)d\hat{P}_n \) may be written as

\[
\sum_{i=1}^{n} W_{in} h(\theta, \alpha, Z_{(i)})
\]

where for \( 1 \leq i \leq n \)

\[
W_{in} = \frac{\delta(i)}{n - i + 1} \prod_{j=1}^{i-1} \left[ \frac{n - j}{n - j + 1} \right]^{\delta(j)}
\]

Figure 2 presents the Kaplan-Meier estimator of the survival function for a randomly generated exponential sample \( \exp(1) \) of size 100 with \( \exp(1/9) \) as censoring distribution.

In this simulation study we will use the power divergences family Cressie and Read (1984). In this case

\[
\int h(\theta, \alpha)d\hat{P}_n = \frac{1}{\gamma - 1} \int \left( \frac{p_\theta}{p_\alpha} \right)^{\gamma-1} d\theta - \frac{1}{\gamma} \int \left[ \left( \frac{p_\theta}{p_\alpha} \right)^{\gamma} - 1 \right] d\hat{P}_n - \frac{1}{\gamma - 1}.
\]

Consider the lifetime distribution to be the one parameter exponential \( \exp(\theta) \) with density \( p_\theta(x) = \theta e^{-\theta x}, \ x \geq 0. \) The MLE of \( \theta_0 \) is given by

\[
\hat{\theta}_{n,MLE} = \frac{\sum_{i=1}^{n} \delta_i}{\sum_{i=1}^{n} Z_i}, \quad (5.1)
\]
and the AMLE of Oakes (1986) is defined by
\[
\hat{\theta}_{n,\text{AMLE}} = \frac{\sum_{i=1}^{n} \delta_i}{\sum_{i=1}^{n} W_{in} Z(i)}.
\]

It follows that for \( \gamma \in \mathbb{R} \setminus \{0, 1\} \)
\[
\frac{1}{\gamma - 1} \int \left( \frac{p_{\theta}}{p_{\alpha}} \right)^{\gamma - 1} dP_{\theta} = \frac{\theta^{\gamma} \alpha^{1-\gamma}}{(\gamma - 1) [\gamma \theta + (1 - \gamma)\alpha]}.
\]
and
\[
\int h(\theta, \alpha) d\hat{P}_n = \frac{\theta^{\gamma} \alpha^{1-\gamma}}{(\gamma - 1) [\gamma \theta + (1 - \gamma)\alpha]}
- \frac{1}{\gamma} \sum_{i=1}^{n} W_{in} \left[ \left( \frac{\theta}{\alpha} \right)^{\gamma} \exp \left\{ -\gamma(\theta - \alpha)Z(i) \right\} - 1 \right].
\]

For \( \gamma = 0 \),
\[
\int h(\theta, \alpha) d\hat{P}_n = \sum_{i=1}^{n} W_{in} \left[ (\theta - \alpha)Z(i) - \log \left( \frac{\theta}{\alpha} \right) \right].
\]
Observe that this divergence leads to the AMLE, independently upon the value of \( \theta \).
For $\gamma = 1$,

$$
\int h(\theta, \alpha) d\hat{P}_n = \log \left( \frac{\theta}{\alpha} \right) - \frac{(\theta - \alpha)}{\theta} - \sum_{i=1}^{n} W_{in} \left[ \frac{\theta}{\alpha} \exp \left( -\frac{(\theta - \alpha)}{z_i} \right) - 1 \right].
$$

To make some comparisons, beside dual $\phi$-divergences estimators, we considered minimum density power divergence estimators of Basu et al. (2006), (MDPDE’s), recall that the density power divergence between $g$ and another density $f$ is

$$
d_{\beta}(g, f) = \int \left\{ f^{1+\beta}(z) - \left( 1 + \frac{1}{\beta} \right) g(z) f^{\beta}(z) + \frac{1}{\beta} g^{1+\beta}(z) \right\} dz \text{ for } \beta > 0.
$$

The values of $\gamma$ are chosen to be $-1, 0, 0.5, 1, 2$ which corresponds to the well known standard divergences: $\chi^2_m$—divergence, $KL_m$, the Hellinger distance, $KL$ and the $\chi^2$—divergence respectively. For the MDPDE’s we take the following values of $\beta$: 0.1, 0.5, 1.

A sample is generated from $\exp(1)$ and 0, 10, 25 of the observations are contaminated by $\exp(5)$ successively. We have used an exponential censoring scheme, the censoring distribution is taken to be $\exp(1/9)$, that the proportion of censoring is 10%. The $D\phi$DE’s $\hat{\alpha}_\phi(\theta)$ are calculated for samples of sizes 25, 50, 75, 100 and the hole procedure is repeated 1000 times. The value of escort parameter $\theta$ is taken to be the AMLE. We carried out Kaplan-Meier analysis with the Survival package Therneau and original R port by Thomas Lumley (2009) within the R Language R Development Core Team (2009).

Tables 1 and 2 provide the MSE of various estimates under the model, according to an an increasing proportion of censoring. As expected, when there is no contamination, MLE produces most efficient estimators. A close look at the results of the simulations show that the $D\phi$DE’s performs well under the model, when no outliers are generated. For small sample size $n = 25$ and $n = 50$, the performance of the estimator under the model is comparable to that of MDPDE’s. Indeed in terms of empirical MSE the $D\phi$DE’s with $\gamma = -1$ produces a lower MSE than the MDPDE’s for all considered values of $\beta$. As $n$ grows up, the MDPDE’s prevail.

Thus, the $D\phi$DE’s are shown to be an attractive alternative to both the AMLE and MDPDE’s in these settings.
Table 1. MSE of the estimates with 10% of censoring

| $n$    | 25  | 50  | 75  | 100 | 150 | 200 |
|--------|-----|-----|-----|-----|-----|-----|
| MLE    | 0.0572 | 0.0250 | 0.0157 | 0.0122 | 0.0079 | 0.0058 |
| $\gamma$ |
| -1     | **0.0517** | 0.0335 | 0.0188 | 0.0178 | 0.0100 | 0.0090 |
| 0      | 0.0685 | 0.0281 | 0.0166 | 0.0135 | 0.0084 | 0.0062 |
| 0.5    | 0.0727 | 0.0287 | 0.0168 | 0.0138 | 0.0085 | 0.0063 |
| 1      | 0.0824 | 0.0302 | 0.0174 | 0.0143 | 0.0086 | 0.0063 |
| 2      | 0.2533 | 0.1156 | 0.0597 | 0.0436 | 0.0151 | 0.0084 |
| $\beta$ |
| 0.1    | 0.0643 | 0.0272 | 0.0162 | 0.0131 | 0.0083 | 0.0061 |
| 0.5    | 0.0772 | 0.0368 | 0.0209 | 0.0173 | 0.0112 | 0.0083 |
| 1      | 0.1042 | 0.0506 | 0.0279 | 0.0232 | 0.0154 | 0.0108 |

Table 2. MSE of the estimates with 20% of censoring

| $n$    | 25  | 50  | 75  | 100 | 150 | 200 |
|--------|-----|-----|-----|-----|-----|-----|
| MLE    | **0.0627** | 0.0280 | 0.0174 | 0.0134 | 0.0088 | 0.0068 |
| $\gamma$ |
| -1     | 0.0655 | 0.0395 | 0.0262 | 0.0195 | 0.0154 | 0.0138 |
| 0      | 0.0892 | 0.0395 | 0.0248 | 0.0172 | 0.0113 | 0.0083 |
| 0.5    | 0.0991 | 0.0440 | 0.0273 | 0.0184 | 0.0119 | 0.0087 |
| 1      | 0.1268 | 0.0541 | 0.0336 | 0.0213 | 0.0131 | 0.0094 |
| 2      | 0.3703 | 0.2233 | 0.1919 | 0.1391 | 0.0689 | 0.0510 |
| $\beta$ |
| 0.1    | 0.0816 | 0.0362 | 0.0224 | 0.0155 | 0.0102 | 0.0075 |
| 0.5    | 0.0919 | 0.0420 | 0.0247 | 0.0171 | 0.0119 | 0.0085 |
| 1      | 0.1166 | 0.0559 | 0.0318 | 0.0218 | 0.0162 | 0.0110 |

We now turn to the comparison of these various estimators under contamination. The DφDE’s yield clearly the most robust estimate and outperform the MLE substantially. We can see from Tables 3 and 4 that the DφDE with $\gamma = -1$ has the
Table 3. MSE of the estimates with 20% of contamination–10% of censoring

|   | 25  | 50  | 75  | 100 | 150 | 200 |
|---|-----|-----|-----|-----|-----|-----|
| MLE | 0.2413 | 0.1354 | 0.0975 | 0.0916 | 0.0798 | 0.0771 |
| $\gamma$ | | | | | | |
| -1 | 0.0576 | 0.0617 | 0.0620 | 0.0626 | 0.0605 | 0.0627 |
| 0 | 0.0852 | 0.0812 | 0.0709 | 0.0710 | 0.0666 | 0.0674 |
| 0.5 | 0.0860 | 0.0820 | 0.0717 | 0.0718 | 0.0676 | 0.0683 |
| 1 | 0.0872 | 0.0826 | 0.0723 | 0.0724 | 0.0682 | 0.0689 |
| 2 | 0.0939 | 0.0843 | 0.0738 | 0.0735 | 0.0692 | 0.0697 |
| $\beta$ | | | | | | |
| 0.1 | 0.0904 | 0.0905 | 0.0829 | 0.0835 | 0.0834 | 0.0854 |
| 0.5 | 0.1134 | 0.1237 | 0.1243 | 0.1269 | 0.1369 | 0.1405 |
| 1 | 0.1231 | 0.1372 | 0.1424 | 0.1449 | 0.1524 | 0.1547 |

Table 4. MSE of the estimates with 20% of contamination–20% of censoring

|   | 25  | 50  | 75  | 100 | 150 | 200 |
|---|-----|-----|-----|-----|-----|-----|
| MLE | 0.2785 | 0.1629 | 0.1165 | 0.1081 | 0.0962 | 0.0926 |
| $\gamma$ | | | | | | |
| -1 | 0.0624 | 0.0661 | 0.0674 | 0.0684 | 0.0670 | 0.0689 |
| 0 | 0.0943 | 0.0898 | 0.0811 | 0.0796 | 0.0751 | 0.0758 |
| 0.5 | 0.0957 | 0.0914 | 0.0826 | 0.0809 | 0.0768 | 0.0774 |
| 1 | 0.0975 | 0.0928 | 0.0840 | 0.0820 | 0.0781 | 0.0784 |
| 2 | 0.1076 | 0.0971 | 0.0872 | 0.0845 | 0.0801 | 0.0801 |
| $\beta$ | | | | | | |
| 0.1 | 0.0963 | 0.0967 | 0.0891 | 0.0884 | 0.0881 | 0.0900 |
| 0.5 | 0.1127 | 0.1235 | 0.1226 | 0.1241 | 0.1335 | 0.1369 |
| 1 | 0.1225 | 0.1348 | 0.1391 | 0.1409 | 0.1503 | 0.1523 |

smallest MSE over all other $D\phi$DE’s and the MDPDE’s for all considered values of $\beta$. As $n$ increases all the $D\phi$DE’s compare favorably with MDPE for all $\beta$. 
In the case of long-tailed contamination in the form of an \( \exp(0.1) \) distribution, simulations results (not reported in this paper) emphasise that the MDPDE’s are more robust than our proposed estimators. In conclusion, without contamination the D\( \phi \)DE's express a good small sample size performance which is comparable to the AMLE and MDPDE’s. For medium and large sample sizes the MDPDE’s are preferable. Under main body contamination, the D\( \phi \)DE’s are more powerful.

6. Concluding remarks

We have introduced a new estimation procedure in parametric models in the case of right censored data. The method is based on the dual representation of \( \phi \)-divergences. The estimators are easily computed and exhibit appropriate asymptotic behaviour.

We have presented an adaptive choice of the escort parameter \( \theta \) that leads to efficient and robust estimates. It will be interesting to investigate theoretically the problem of the choice of the divergence which leads to an “optimal” estimate in terms of efficiency and robustness. One approach is to minimize an estimated asymptotic mean squared error of the estimator when it is mathematically tractable, which is not an easy task in the context of censored data and lays beyond the scope of the present work.

Appendix A. Proofs

A.1. Proof of Theorem 1. Under the assumptions (R.0), (R.1) and by applying the Strong Law of Large Numbers (SLLN) for censored data, see for instance Stute and Wang (1993), Stute (1995) and Proposition 1 in Wang (1999), we can see that

\[
\int \frac{\partial}{\partial \alpha} h(\theta, \theta_0) \, d\hat{P}_n \rightarrow \int \frac{\partial}{\partial \alpha} h(\theta, \theta_0) \, dP_{\theta_0} = 0, \quad (A.1)
\]

and

\[
\int \frac{\partial^2}{\partial \alpha \partial \alpha^\top} h(\theta, \theta_0) \, d\hat{P}_n \rightarrow \int \frac{\partial^2}{\partial \alpha \partial \alpha^\top} h(\theta, \theta_0) \, dP_{\theta_0} = -S < 0, \quad (A.2)
\]
Now, for any $\alpha = \theta_0 + un^{-1/3}$, with $\|u\| \leq 1$, consider a Taylor expansion of
\[
\int h(\theta, \alpha) \, d\hat{P}_n \text{ in } \alpha \text{ in a neighborhood of } \theta_0.
\]
Using (R.1), one finds
\[
n\int h(\theta, \alpha) \, d\hat{P}_n - n \int h(\theta, \theta_0) \, d\hat{P}_n = n^{2/3} u \int \frac{\partial}{\partial \alpha} h(\theta, \theta_0) \, d\hat{P}_n + O(1),
\]
(A.3)
uniformly in $u$ with $\|u\| \leq 1$. Observe that,
\[
\left| \int \frac{\partial}{\partial \alpha} h(\theta, \theta_0) \, d(\hat{P}_n - P_{\theta_0}) \right| = \left| \int (\hat{P}_n - P_{\theta_0}) \, d \left[ \frac{\partial}{\partial \alpha} h(\theta, \theta_0) \right] \right|
\leq \sup_x |\hat{P}_n(x) - P_{\theta_0}(x)| \int |\frac{\partial}{\partial \alpha} h(\theta, \theta_0)|.
\]

On the other hand, under condition (R.2), by the LIL of Földes and Rejtő (1981), we have
\[
\int \frac{\partial}{\partial \alpha} h(\theta, \theta_0) \, d\hat{P}_n = O \left( n^{-1/2} (\log \log n)^{1/2} \right).
\]
Therefore, using (A.1) and (A.2), we obtain for any $\alpha = \theta_0 + un^{-1/3}$, with $\|u\| = 1$,
\[
n\int h(\theta, \alpha) \, d\hat{P}_n - n \int h(\theta, \theta_0) \, d\hat{P}_n = O \left( n^{1/6} (\log \log n)^{1/2} \right) - \frac{1}{2} n^{1/3} S + O(1),
\]
Observe that the right-hand side vanishes when $\alpha = \theta_0$, and that the left-hand side, by (A.2), becomes negative for all $n$ sufficiently large. Thus, by the continuity of $\alpha \mapsto \int h(\theta, \alpha) \, d\hat{P}_n$, it holds that as $n \to \infty$, with probability one,
\[
\alpha \mapsto \int h(\theta, \alpha) \, d\hat{P}_n
\]
reaches its maximum value at some point $\hat{\alpha}_\phi(\theta)$ in the interior of $B(\theta_0, n^{-1/3})$. Therefore, the estimate $\hat{\alpha}_\phi(\theta)$ satisfies
\[
\int \frac{\partial}{\partial \alpha} h(\theta, \hat{\alpha}_\phi(\theta)) \, d\hat{P}_n = 0 \text{ and } \|\hat{\alpha}_\phi(\theta) - \theta_0\| = O(n^{-1/3}).
\]

A.2. Proof of Theorem 2. Using (R.1), simple calculus give
\[
P_{\theta_0} \frac{\partial}{\partial \alpha} h(\theta, \alpha) = 0
\]
(A.4)
and
\[
P_{\theta_0} \frac{\partial^2}{\partial \alpha \partial \alpha^\top} h(\theta, \theta_0) = - \int \phi'' \left( \frac{p_{\theta}}{p_{\theta_0}} \right) \frac{p_{\theta_0}^2}{p_{\theta_0}^*} \hat{p}_{\theta_0} \hat{p}_{\theta_0}^\top \, d\lambda =: - S.
\]
(A.5)
Observe that the matrix $S$ is symmetric and positive since the second derivative $\phi''$ is nonnegative by the convexity of $\phi$. Let $U_n(\theta_0) := \hat{P}_n \frac{\partial}{\partial \alpha} h(\theta, \theta_0)$, and use (A.4) and (R.0), (R.3) and (R.4) in connection with the Central Limit Theorem for censored data (CLT), see for instance Stute (1995a), Wang (1999) to see that
\[
\sqrt{n}U_n(\theta_0) \to \mathcal{N}(0, V).
\] (A.6)

Also, let $S_n(\theta_0) := \hat{P}_n \frac{\partial^2}{\partial \alpha \partial \alpha} h(\theta, \theta_0)$, and use (A.5) and (R.0) in connection with the SLLN to conclude that
\[
S_n(\theta_0) \to -S \ (a.s.).
\] (A.7)

Using the fact that $\hat{P}_n \frac{\partial}{\partial \alpha} h(\theta, \hat{\alpha}_\phi(\theta)) = 0$ and a Taylor expansion of $\hat{P}_n \frac{\partial}{\partial \alpha} h(\theta, \hat{\alpha}_\phi(\theta))$ in $\hat{\alpha}_\phi(\theta)$ around $\theta_0$, we obtain
\[
0 = \hat{P}_n \frac{\partial}{\partial \alpha} h(\theta, \hat{\alpha}_\phi(\theta)) = \hat{P}_n \frac{\partial}{\partial \alpha} h(\theta, \theta_0) + (\hat{\alpha}_\phi(\theta) - \theta_0)^\top \hat{P}_n \frac{\partial^2}{\partial \alpha \partial \alpha} h(\theta, \theta_0) + o_P \left( \frac{1}{\sqrt{n}} \right).
\]

Hence,
\[
\sqrt{n}(\hat{\alpha}_\phi(\theta) - \theta_0) = -S_n(\theta_0)^{-1} \sqrt{n}U_n(\theta_0) + o_P(1).
\] (A.8)

Using (A.6) and (A.7) and Slutsky Theorem, we conclude then
\[
\sqrt{n}(\hat{\alpha}_\phi(\theta) - \theta_0) \to \mathcal{N}(0, S^{-1}VS^{-1})
\] (A.9)

A.3. Proof of Theorem 3. By a Taylor expansion of $\hat{P}_n \frac{\partial}{\partial \alpha} h(\hat{\theta}_n, \hat{\alpha}_\phi(\hat{\theta}_n))$ in $\hat{\alpha}_\phi(\hat{\theta}_n)$ around $\theta_0$, we obtain
\[
0 = \hat{P}_n \frac{\partial}{\partial \alpha} h(\hat{\theta}_n, \hat{\alpha}_\phi(\theta)) = \hat{P}_n \frac{\partial}{\partial \alpha} h(\hat{\theta}_n, \theta_0) + (\hat{\alpha}_\phi(\hat{\theta}_n) - \theta_0)^\top
\]
\[
\hat{P}_n \frac{\partial^2}{\partial \alpha \partial \alpha} h(\hat{\theta}_n, \theta_0) + o_P \left( \frac{1}{\sqrt{n}} \right).
\]

Taylor expansions of $\hat{P}_n \frac{\partial}{\partial \alpha} h(\hat{\theta}_n, \hat{\alpha}_\phi(\hat{\theta}_n))$ and $\hat{P}_n \frac{\partial}{\partial \alpha} h(\hat{\theta}_n, \theta_0)$ in $\hat{\theta}_n$ around $\theta_0$, and the $\sqrt{n}$-consistency of $\hat{\theta}_n$ to $\theta_0$ yield
\[
0 = \hat{P}_n \frac{\partial}{\partial \alpha} h(\theta_0, \hat{\alpha}_\phi(\theta)) = \hat{P}_n \frac{\partial}{\partial \alpha} h(\theta_0, \theta_0) + (\hat{\alpha}_\phi(\hat{\theta}_n) - \theta_0)^\top
\]
\[
\hat{P}_n \frac{\partial^2}{\partial \alpha \partial \alpha} h(\hat{\theta}_n, \theta_0) + o_P \left( \frac{1}{\sqrt{n}} \right).
\]

Let $U_n := \hat{P}_n \frac{\partial}{\partial \alpha} h(\theta_0, \theta_0)$ and $S_n := \hat{P}_n \frac{\partial}{\partial \alpha} h(\theta_0, \theta_0)$. By the CLT
\[
\sqrt{n}U_n \to \mathcal{N}(0, V),
\] (A.10)
where \( V \) is defined in (4.4).

Use condition (R.5) and the fact that \( S = -P_{\theta_0} \frac{\partial}{\partial \alpha} h(\theta_0, \theta_0) = -\phi''(1) I_{\theta_0} \), in connection with Lemma 1 in Wang (1999) to conclude that

\[
S_n \overset{P}{\to} \phi''(1) I_{\theta_0}.
\]  

(A.11)

The theorem now follows from (A.10), (A.11) and Slutsky’s theorem. This concludes the proof.

References

Basu, A. and Lindsay, B. G. (1994). Minimum disparity estimation for continuous models: efficiency, distributions and robustness. *Ann. Inst. Statist. Math.*, 46(4), 683–705.

Basu, A., Harris, I. R., Hjort, N. L., and Jones, M. C. (1998). Robust and efficient estimation by minimising a density power divergence. *Biometrika*, 85(3), 549–559.

Basu, S., Basu, A., and Jones, M. C. (2006). Robust and efficient parametric estimation for censored survival data. *Ann. Inst. Statist. Math.*, 58(2), 341–355.

Bouzebda, S. and Keziou, A. (2010). Estimation and tests of independence in copula models via divergences. *Kybernetika*, 46(1), 178-201.

Broniatowski, M. and Keziou, A. (2006). Minimization of \( \phi \)-divergences on sets of signed measures. *Studia Sci. Math. Hungar.*, 43(4), 403–442.

Broniatowski, M. and Keziou, A. (2009). Parametric estimation and tests through divergences and the duality technique. *J. Multivariate Anal.*, 100(1), 16–36.

Broniatowski, M. and Vajda, I. (2009). Several applications of divergence criteria in continuous families. Technical Report 2257, Academy of Sciences of the Czech Republic, Institute of Information Theory and Automation.

Cressie, N. and Read, T. R. C. (1984). Multinomial goodness-of-fit tests. *J. Roy. Statist. Soc. Ser. B*, 46(3), 440–464.

Földes, A. and Rejtő, L. (1981). A LIL type result for the product limit estimator. *Z. Wahrsch. Verw. Gebiete*, 56(1), 75–86.

Hampel, F. R., Ronchetti, E. M., Rousseeuw, P. J., and Stahel, W. A. (1986). *Robust statistics*. Wiley Series in Probability and Mathematical Statistics: Probability and Mathematical Statistics. John Wiley & Sons Inc., New York. The approach based on influence functions.
Huber, P. J. (1981). *Robust statistics*. John Wiley & Sons Inc., New York. Wiley Series in Probability and Mathematical Statistics.

Jiménez, R. and Shao, Y. (2001). On robustness and efficiency of minimum divergence estimators. *Test*, **10**(2), 241–248.

Kaplan, E. L. and Meier, P. (1958). Nonparametric estimation from incomplete observations. *J. Amer. Statist. Assoc.*, **53**, 457–481.

Keziou, A. (2003). Dual representation of φ-divergences and applications. *C. R. Math. Acad. Sci. Paris*, **336**(10), 857–862.

Liese, F. and Vajda, I. (1987). *Convex statistical distances*, volume 95 of Teubner-Texte zur Mathematik [Teubner Texts in Mathematics]. BSB B. G. Teubner Verlagsgesellschaft, Leipzig. With German, French and Russian summaries.

Liese, F. and Vajda, I. (2006). On divergences and informations in statistics and information theory. *IEEE Trans. Inform. Theory*, **52**(10), 4394–4412.

Lindsay, B. G. (1994). Efficiency versus robustness: the case for minimum Hellinger distance and related methods. *Ann. Statist.*, **22**(2), 1081–1114.

Maronna, R. A., Martin, R. D., and Yohai, V. J. (2006). *Robust statistics*. Wiley Series in Probability and Statistics. John Wiley & Sons Ltd., Chichester. Theory and methods.

Miller, Jr., R. G. (1981). *Survival analysis*. John Wiley & Sons Inc., New York. With notes by Gail Gong, With problem solutions by Alvaro Munoz, Wiley Series in Probability and Mathematical Statistics.

Morales, D., Pardo, L., and Vajda, I. (1995). Asymptotic divergence of estimates of discrete distributions. *J. Statist. Plann. Inference*, **48**(3), 347–369.

Oakes, D. (1986). An approximate likelihood procedure for censored data. *Biometrics*, **42**(1), 177–182.

Peterson, Jr., A. V. (1977). Expressing the Kaplan-Meier estimator as a function of empirical subsurvival functions. *J. Amer. Statist. Assoc.*, **72**(360, part 1), 854–858.

R Development Core Team (2009). *R: A Language and Environment for Statistical Computing*. R Foundation for Statistical Computing, Vienna, Austria. ISBN 3-900051-07-0.

Reid, N. (1981). Influence functions for censored data. *Ann. Statist.*, **9**(1), 78–92.
Stute, W. (1995). The statistical analysis of Kaplan-Meier integrals. In Analysis of censored data (Pune, 1994/1995), volume 27 of IMS Lecture Notes Monogr. Ser., pages 231–254. Inst. Math. Statist., Hayward, CA.

Stute, W. (1995a). The central limit theorem under random censorship. Ann. Statist., 23(2), 422–439.

Stute, W. and Wang, J.-L. (1993). The strong law under random censorship. Ann. Statist., 21(3), 1591–1607.

Suzukawa, A., Imai, H., and Sato, Y. (2001). Kullback-Leibler information consistent estimation for censored data. Ann. Inst. Statist. Math., 53(2), 262–276.

Toma, A. and Broniatowski, M. (2010). Dual divergence estimators and tests: robustness results. Journal of Multivariate Analysis.

Wang, J.-L. (1995). M-estimators for censored data: strong consistency. Scand. J. Statist., 22(2), 197–205.

Wang, J.-L. (1999). Asymptotic properties of M-estimators based on estimating equations and censored data. Scand. J. Statist., 26(2), 297–318.

Welsh, A. H. (1989). On M-processes and M-estimation. Ann. Statist., 17(1), 337–361.

Yang, S. (1991). Minimum Hellinger distance estimation of parameter in the random censorship model. Ann. Statist., 19(2), 579–602.

Ying, Z. (1992). Minimum Hellinger-type distance estimation for censored data. Ann. Statist., 20(3), 1361–1390.