CIRCLE-VALUED MORSE THEORY FOR COMPLEX HYPERPLANE ARRANGEMENTS

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ABSTRACT. Let $\mathcal{A}$ be an essential complex hyperplane arrangement in $\mathbb{C}^n$, and $H$ denote the union of the hyperplanes. We develop the real-valued and circle-valued Morse theory on the space $M = \mathbb{C}^n \setminus H$ and prove, in particular, that $M$ has the homotopy type of a space obtained from a manifold $V$, fibered over a circle, by attaching to it $|\chi(M)|$ cells of dimension $n$. We compute the Novikov homology $\hat{H}_*(M, \xi)$ for a large class of homomorphisms $\xi: \pi_1(M) \to \mathbb{R}$.

1. INTRODUCTION

Let $f$ be a holomorphic Morse function without zeros on a complex analytic manifold. It gives rise to a real-valued Morse function $z \mapsto |f(z)|$ and a circle-valued Morse function $z \mapsto f(z)/|f(z)|$. These two functions can be used to study the topology of the underlying manifold. There are, however, numerous technical problems, and this approach works only in some rare particular cases. This paper is about one of such cases, namely, the case of the complement to a complex hyperplane arrangement in $\mathbb{C}^n$.

Let $\xi_i : \mathbb{C}^n \to \mathbb{C}$ be non-constant affine functions ($1 \leq i \leq m$); put $H_i = \text{Ker} \xi_i$. Denote by $\mathcal{A}$ the hyperplane arrangement $\{H_1, \ldots, H_m\}$ and put

$$H = \bigcup_i H_i, \quad M(\mathcal{A}) = \mathbb{C}^n \setminus H.$$ 

We will abbreviate $M(\mathcal{A})$ to $M$. The rank of $\mathcal{A}$ is the maximal codimension of a non-empty intersection of some subfamily of $\mathcal{A}$. We say that $\mathcal{A}$ is essential if $\text{rk} L = n$. Assume that $\mathcal{A}$ is essential. We prove that $M$ has the homotopy type of a space obtained from a finite $n$-dimensional CW complex fibered over a circle, by attaching $|\chi(M)|$ cells of dimension $n$, and apply these results to the computation of the Novikov homology of $M$.

The homology of rank one local systems over $M$ has been extensively studied. It was shown that this homology vanishes except in the case of dimension $n$ for generic local systems (see [2], [5]).
These homology groups play an important role in the theory of hypergeometric integrals and have been investigated in relation with the cohomology of the twisted de Rham cohomology of logarithmic forms (see [3], [4]).

**Definition 1.1.** A homomorphism \( \xi : \pi_1(M) \to \mathbb{R} \) is called positive if its value on the positive meridian of each hyperplane \( H_i \) is strictly positive.

We show that the structure of the Novikov homology \( \hat{H}_*(X, \xi) \) for positive homomorphisms \( \xi \) is similar to that of the generic local coefficient homology, namely, it vanishes in all degrees except \( n \).

## 2. Main results

Let \( \mathcal{A} \) be an essential arrangement. Let \( \alpha = (\alpha_1, \ldots, \alpha_m) \) be a string of complex numbers. P. Orlik and H. Terao [8] proved that for \( \alpha \) outside a closed algebraic subset of \( \mathbb{C}^m \) the multivalued holomorphic function \( \Phi_\alpha = \xi_1^{\alpha_1} \cdot \xi_2^{\alpha_2} \cdot \ldots \cdot \xi_m^{\alpha_m} \) has only non-degenerate critical points (see the works of K. Aomoto [2], and A. Varchenko [12] for partial results in this direction).

In this paper we work only with \( \alpha \in \mathbb{R}^m \). It follows from the Orlik-Terao theorem that there is an open dense subset \( W \subset \mathbb{R}^m \) such that for \( \alpha \in W \) the function \( \Phi_\alpha \) has only non-degenerate critical points. Consider a real-valued \( C^\infty \) function

\[
f_\alpha(z) = \prod_i |\xi_i(z)|^{\alpha_i}, \quad f_\alpha : \mathbb{C}^n \setminus H \to \mathbb{R}.
\]

**Lemma 2.1.** Let \( \alpha \in W \). Then \( f_\alpha \) is a Morse function. The index of every critical point of \( f_\alpha \) equals \( n \).

**Proof.** Let \( \omega_\alpha = \sum_i \alpha_i \xi_i \). Then \( f_\alpha(z) = C \cdot \exp(\text{Re} \int_{z_0}^z \omega_\alpha) \), therefore, \( \log f_\alpha(z) \) locally is the real part of a holomorphic Morse function. In general, if \( h \) is a holomorphic Morse function on an open subset of \( \mathbb{C}^n \), then \( \text{Re} h \) is a real-valued Morse function, and the index of every critical point of \( \text{Re} h \) equals \( n \). Our assertion follows. \( \square \)

Let \( \varepsilon > 0 \) and put

\[
(1) \quad V = \{ z \in \mathbb{C}^n \mid f_\alpha(z) = \varepsilon \}, \quad N = \{ z \in \mathbb{C}^n \mid f_\alpha(z) \geq \varepsilon \}.
\]

**Definition 2.2.** A vector \( \alpha \in \mathbb{R}^m \) is called positive if \( \alpha_i > 0 \) for all \( i \). The set of all positive vectors is denoted by \( \mathbb{R}^m_+ \). The rank of the vector \( \alpha \in \mathbb{R}^m \) is the dimension of the \( \mathbb{Q} \)-vector space generated by the components of \( \alpha \) in \( \mathbb{R} \).
Recall that we denote $\mathbb{C}^n \setminus H$ by $M$.

**Theorem 2.3.** Let $\alpha$ be any positive vector. Then for every $\varepsilon > 0$ small enough

1) The inclusion $N \subset M$ is a homotopy equivalence. The space $V = \partial N$ is a $C^\infty$ manifold of dimension $2n - 1$.

2) The space $N$ has the homotopy type of the space $V$ with $|\chi(M)|$ cells of dimension $n$ attached.

3) If $\alpha$ has rank 1, then $V$ is fibered over a circle and the fiber has the homotopy type of a finite CW-complex of dimension $n - 1$.

To state the next theorem we recall the definition of the Novikov homology. Let $G$ be a group, and $\xi : G \to \mathbb{R}$ a homomorphism. Put $G_C = \{ g \in G \mid \xi(g) \geq C \}$. The Novikov completion $\hat{\Lambda}_\xi$ of the group ring $\Lambda = \mathbb{Z}G$ with respect to the homomorphism $\xi : G \to \mathbb{R}$ is defined as follows (see the thesis of J.-Cl. Sikorav [11]):

$$\hat{\Lambda}_\xi = \left\{ \lambda = \sum_{g \in G} n_g g \mid \text{where } n_g \in \mathbb{Z} \text{ and } \supp \lambda \cap G_C \text{ is finite for every } C \right\}.$$

Let $X$ be a connected topological space and denote $\pi_1(X)$ by $G$. Let $\xi : G \to \mathbb{R}$ be a homomorphism. The Novikov homology $\hat{H}_*(X, \xi)$ is by definition the homology of the chain complex

$$\hat{\mathcal{S}}_*(\tilde{X}) = \hat{\Lambda}_\xi \otimes \mathcal{S}_*(\tilde{X})$$

where $\mathcal{S}_*(\tilde{X})$ is the singular chain complex of the universal covering of $X$.

**Theorem 2.4.** For any positive homomorphism $\xi : \pi_1(M) \to \mathbb{R}$ the Novikov homology $\hat{H}_k(M, \xi)$ vanishes for $k \neq n$ and is a free $\hat{\Lambda}_\xi$-module of rank $|\chi(M)|$ if $k = n$.

3. THE GRADIENT FIELD IN THE NEIGHBOURHOOD OF $H$

In this section we develop the main technical tools of the paper. The results of this section are valid for every arrangement, essential or not. Let

$$v_\alpha(z) = \frac{\nabla f_\alpha(z)}{f_\alpha(z)}.$$

Denote by $u_j$ the gradient of the function $z \mapsto |\xi_j(z)|$. Then

$$v_\alpha(z) = \sum_{j=1}^m \alpha_j \frac{u_j(z)}{|\xi_j(z)|}.$$
For a linear form $\beta : \mathbb{C}^n \to \mathbb{C}$, $\beta(z) = a_1z_1 + \ldots + a_nz_n$ the gradient of the function $|\beta(z)|$ is easy to compute, namely

$$\text{grad} |\beta(z)| = \frac{\beta(z)}{|\beta(z)|} \cdot (\bar{a}_1, \bar{a}_2, \ldots, \bar{a}_n)$$

(it follows, in particular, that the norm of this gradient is constant).

**Lemma 3.1.** Assume that the intersection of all hyperplanes of $A$ is non-empty. Let $\Gamma \subset \mathbb{R}^m_+$ be a compact subset. Then there is $K > 0$ such that

$$||v_\alpha(z)|| \geq K \sum_i \frac{1}{|\xi_i(z)|}$$

for every $z \in \mathbb{C}^n \setminus H$ and every $\alpha \in \Gamma$.

**Proof.** We can assume that the intersection of the hyperplanes contains 0, that is, the arrangement is central. Furthermore, it suffices to prove the Lemma for the case when

$$\bigcap_j \ker \xi_j = \{0\}.$$ 

Indeed, let $L = \bigcap_j \ker \xi_j$. Then it follows from (2) that both sides of our inequality (3) are invariant with respect to translations by vectors in $L$, and it is sufficient to prove the formula for the vector field $v_\alpha|_L$.

Furthermore, the both sides of the inequality are homogeneous of degree $-1$, and it is sufficient to prove the inequality for $z \in \Sigma \setminus H$, where $\Sigma$ stands for the sphere of radius 1 and center 0.

We will proceed by induction on $m$. The case $m = 1$ being obvious, we will assume that $m > 1$. Choose some $\kappa > 0$, and for $i \neq j$ let

$$U_{i,j} = \left\{ z \in \Sigma \mid |\xi_i(z)| < \kappa |\xi_j(z)| \right\}.$$ 

These are open sets and it follows from the condition (4) that their union $U = \bigcup_{i,j} U_{i,j}$ covers the set $H \cap \Sigma$. We will now prove (3) for $z \in U_{i,j} \setminus H$. To simplify the notation let us assume $i = 1, j = m$. Put

$$A_m(z) = v_\alpha(z), \quad A_{m-1}(z) = \sum_{j=1}^{m-1} \alpha_j \frac{u_j(z)}{|\xi_j(z)|},$$ 

$$B_m(z) = \sum_{i=1}^m \frac{1}{|\xi_i(z)|}, \quad B_{m-1}(z) = \sum_{i=1}^{m-1} \frac{1}{|\xi_i(z)|}.$$
By the induction assumption we have $||A_{m-1}(z)|| \geq DB_{m-1}(z)$, where $D$ is some positive constant. An easy computation shows that for $z \in U_{i,j}$ we have

$$||A_m(z)|| \geq (D - \kappa \alpha_m K_m - \kappa D)B_m(z)$$

where $K_m = ||u_m(z)||$. Choosing $\kappa$ sufficiently small, we conclude that for $z \in U_{i,j}$ we have

$$||A_m(z)|| \geq (D' - \kappa \alpha_m K_m - \kappa D)B_m(z)$$

where $K_m = ||u_m(z)||$. Choosing $\kappa$ sufficiently small, we conclude that $v_\alpha(z) \neq 0$ for $z \in \Sigma \setminus U$. This is in turn obvious since

$$f_\alpha(\mu z) = \mu^{\alpha_1 + \cdots + \alpha_m} f_\alpha(z) \quad \text{for} \quad \mu \in \mathbb{R}_+,$$

therefore, $f_\alpha'(z) \neq 0$ for every $z \notin H$, since all $\alpha_i$ are positive. The proof of Lemma 3.1 is now over.

Proposition 3.2. Let $\Gamma \subset \mathbb{R}^m_+$ be a compact subset. There is an open neighbourhood $U$ of $H$, and numbers $A, B > 0$ such that

1) For some $\delta > 0$ the set $H(\delta)$ is in $U$.

2) For every $z \in U \setminus H$ and every $\alpha, \beta \in \Gamma$ we have

$$||v_\alpha(z)|| \geq A,$$

$$||v_\alpha(z) - v_\beta(z)|| \leq B \cdot \max_i |\alpha_i - \beta_i| \cdot ||v_\beta(z)||.$$

Proof. For a multi-index $I = (i_1, \ldots, i_s)$ let us denote by $H_I$ the intersection of the hyperplanes $H_{i_1}, \ldots, H_{i_s}$. Proceeding by induction on $\dim H_I$ we will construct for every $I$ with $H_I \neq \emptyset$ a neighbourhood $U_I$ of the subset $H_I$ such that the properties 1) and 2) of the Proposition hold if we replace in the formulae $H$ by $H_I$ and $U$ by $U_I$. Assume that this is done for every $H_J$ with $\dim H_J \leq k - 1$; put

$$U_{k-1} = \bigcup_{\dim H_J \leq k-1} U_J.$$

Let $I$ be a multi-index with $\dim H_I = k$. We will construct the neighbourhood $U_I$. We will assume that $k > 0$. The proof of our assertion for the case $\dim H_I = 0$ (the initial step of the induction) is similar and will be omitted.

We can assume that the multi-index $I$ includes all the values of $j$ such that $H_I \subset H_j$. To simplify the notation let us assume that
Let $\mu > 0$, and consider the subset $U'_\mu = H_1(\mu) \setminus U_{k-1}$. For $z \in U'_\mu$ the second term of (8) is bounded uniformly with respect to $\alpha \in \Gamma$ if $\mu > 0$ is sufficiently small. As for the first term, its norm converges to $\infty$ when $d(z, H_I) \to 0$ as it follows from Lemma 3.1, applied to the arrangement $\{H_1, \ldots, H_r\}$. An easy computation shows now that for every $\mu > 0$ sufficiently small the inequalities (5) and (6) hold for $z \in U'_\mu$ and every $\alpha, \beta \in \Gamma$. Put $U_I = U'_\mu \cup U_{k-1}$. The properties 1) and 2) for $H_I$ and $U_I$ are now easy to deduce.

The neighbourhood

$$U_{n-1} = \bigcup_{\dim H_J \leq n-1} U_J.$$ 

satisfy the properties required in the statement of our Proposition.

We will also use the normalized gradient

$$w_\alpha(z) = \frac{\nabla f_\alpha(z)}{||\nabla f_\alpha(z)||}.$$ 

Choose a neighbourhood $U$ of $H$ so that the conclusion of Proposition 3.2 holds.

**Proposition 3.3.** Let $\alpha \in \mathbb{R}^m_+$. There is $D > 0$ such that

$$\langle v_\alpha(z), w_\beta(z) \rangle \geq D$$

for every $z \in U \setminus H$ and every positive vector $\beta$ with $\max_i |\alpha_i - \beta_i|$ sufficiently small.

**Proof.** For $z \in U \setminus H$ we have

$$\left| \langle v_\alpha(z) - v_\beta(z), w_\beta(z) \rangle \right| \leq B \cdot \max_i |\alpha_i - \beta_i| \cdot ||v_\beta(z)||.$$ 

On the other hand $\langle v_\beta(z), w_\beta(z) \rangle = ||v_\beta(z)||$ (since these vector fields are collinear and $||w_\beta(z)|| = 1$), therefore,

$$\langle v_\alpha(z), w_\beta(z) \rangle \geq (1 - B \max_i |\alpha_i - \beta_i|) \cdot ||v_\beta(z)||.$$ 

If $\alpha - \beta$ is small enough, then the right-hand side of the above inequality is greater than a positive constant again by Proposition 3.2.

□
4. The homotopy type of $M$

In this section we prove the first two assertions of Theorem 2.3. We fix a positive vector $\alpha$. Choose a neighbourhood $U$ of $H$ so that the conclusion of Proposition 3.2 holds. Observe that for $\varepsilon > 0$ small enough the set $f^{-1}_\alpha([0, \varepsilon])$ is in $U$, therefore, $\varepsilon$ is a regular level of $f_\alpha$, and $V$ is a submanifold of $M$ of dimension $2n - 1$. This proves the second part of the assertion 1).

To prove the first part we use the shift along the flow lines of $w_\alpha$ to construct the deformation retraction of $M$ onto $N = f^{-1}_\alpha(\varepsilon, \infty)$. If $\varepsilon > 0$ is sufficiently small, then $M \setminus N \subset U$, and for every integral curve $\gamma(t)$ of $w_\alpha$ starting at a point $x \in M \setminus N$ we have

$$\frac{d}{dt}f_\alpha(\gamma(t)) = \langle \text{grad} f_\alpha(\gamma(t)), \gamma'(t) \rangle = f_\alpha(\gamma(t)) \cdot ||v_\alpha(\gamma(t))|| \geq Af_\alpha(x)$$

(for every $t$ such that $\gamma(t)$ is in the set $U$). Therefore, this trajectory will reach $f^{-1}_\alpha(\varepsilon)$, and our deformation retraction is well-defined.

Moving forward to the assertion 2), let us first outline the proof. Choose $\beta \in \mathbb{R}^m_+$ so that $f_\beta : M \to \mathbb{R}$ is a Morse function, and $\beta$ is sufficiently close to $\alpha$ so that the property (6) holds. We are going to apply the Morse theory to the restriction of $f_\beta$ to the manifold $N$ with boundary $V$. Our setting differs from the classical one (see [6]), essentially in 2 points: A) the manifold $N$ is not compact, B) the function $f_\beta$ is not constant on the boundary of $N$. The main technical tool to deal with these issues is Proposition 3.2, which describes the behaviour of $\text{grad} f_\beta$ in a neighbourhood of $H$. Using the classical Morse-theoretic schema (cf [6], §3), we choose a suitable gradient-like vector field $u_\beta$ for $f_\beta$, and use the gradient descent along $(-u_\beta)$-trajectories to describe the changement of the homotopy type of sublevel sets while crossing critical values. It is technically convenient to replace the couple $(N, V)$ by a suitable thickening. Let $0 < \varepsilon_0 < \varepsilon$ and put

$$N' = \{z \mid f_\alpha(z) \geq \varepsilon_0\}, \quad L = \{z \mid \varepsilon_0 \leq f_\alpha(z) \leq \varepsilon\}.$$

**Lemma 4.1.** The inclusion of pairs $(N, V) \subset (N', L)$ is a homotopy equivalence.

**Proof.** The shift along the trajectories of the vector field $w_\alpha$ determines a deformation retraction of $(N', L)$ onto $(N, V)$.

We proceed to the construction of a gradient-like vector field for $f_\beta$. We assume that $\beta$ is sufficiently close to $\alpha$ so that the property (9) in Proposition 3.3 holds. For each critical point $c$ of $f_\beta$ choose a neighbourhood $R_c$ of $c$ such that $\overline{R_c} \cap \overline{U} = \emptyset$; let $R$ denote the union
of all $R_c$. Put
\[ K = \{ z \mid f_\alpha(z) \leq \varepsilon \}. \]

Using a partition of unity, it is easy to construct a $C^\infty$ vector field $u_\beta$ on $\mathbb{C}^n$ such that

1) $u_\beta | N$ is a gradient-like vector field for $f_\beta$.
2) $\text{supp} (w_\beta - u_\beta) \subset R \cup K$.
3) $u_\beta | (K - L) = 0$.
4) For $z \in L$ we have $u_\beta(z) = h(f_\beta(z))w_\beta(z)$ where $h$ is a $C^\infty$ function with the following properties:

\[ h(t) = 0 \quad \text{if} \quad t \leq \varepsilon_0, \quad h(t) = 1 \quad \text{if} \quad t \geq \varepsilon, \]
\[ h'(t) > 0 \quad \text{for every} \quad \varepsilon_0 < t < \varepsilon. \]

Observe that the vector field $u_\beta$ is bounded, therefore, its integral curves are defined on the whole of $\mathbb{R}$.

Now we can investigate the sublevel sets of the function $f_\beta$. For $b \in \mathbb{R}$ put
\[ Y_b = \{ z \mid f_\beta(z) \leq b \text{ and } f_\alpha(z) \geq \varepsilon_0 \}. \]

It follows from the construction of the vector field $u_\beta$, that $Y_b$ is $(-u_\beta)$-invariant, that is, every $(-u_\beta)$-trajectory $\gamma(z, t)$ starting at a point $z \in Y_b$ remains in $Y_b$ for all $t \geq 0$. Furthermore, it follows from Proposition 3.3 that $L$ is also $(-u_\beta)$-invariant. Therefore, $Y_b \cup L$ is $(-u_\beta)$-invariant.

\textbf{Proposition 4.2.} Let $0 \leq a < b$ and assume that $f_\beta$ does not have critical points in $Y_b \setminus Y_a$. Then

1) There is $C > 0$ such that for every $z \in Y_b \setminus (Y_a \cup L)$ we have
\[ f'_\beta(z)(u_\beta(z)) \geq C. \]

2) There is $T > 0$ such that $\gamma(z, T) \in \text{Int} L \cup \text{Int} Y_a$ for every $z \in Y_b$.

3) The inclusion
\[ (Y_a \cup L, L) \subset (Y_b \cup L, L) \]

is a homotopy equivalence.

\textbf{Proof.} 1) Consider two cases: A) $z \notin U$, and B) $z \in U$. For the case A) observe that the set $Y_b \setminus (\text{Int} Y_a \cup U)$ is compact (since the arrangement $\mathcal{A}$ is essential), the function $f_\beta$ has no critical points in it and $u_\beta$ is a gradient-like vector field for $f_\beta$. Thus the property (10) is true for $z \in Y_b \setminus (\text{Int} Y_a \cup U)$. 

For the case B) observe that when \( z \in (Y_b \cap U) \setminus (Y_a \cup L) \) we have \( u_\beta(z) = w_\beta(z) \) and
\[
f'_\beta(z)(u_\beta(z)) = \langle v_\beta(z)f_\beta(z), w_\beta(z) \rangle \geq a||v_\beta(z)|| \geq aA
\]
(see Proposition 3.2).

2) Let \( T > \frac{b-a}{C} \), and assume that \( \gamma(z, T) \notin \text{Int } L \cup \text{Int } Y_a \). The set \( Y_a \) is obviously invariant with respect to the shift along \((-u_\beta)\)-trajectories. The same is true for \( L \), as it follows from (9). Therefore, \( \gamma(z, \tau) \notin \text{Int } L \cup \text{Int } Y_a \) for all \( \tau \in [0, T] \). By the first assertion of the Lemma we have
\[
f_b(\gamma(z, T)) < f_b(\gamma(z, 0)) - C \cdot \frac{b-a}{C} \leq a
\]
which contradicts the assumption \( \gamma(z, T) \notin \text{Int } Y_a \).

3) Define a homotopy
\[H_t : Y_b \cup L \to Y_b \cup L, \quad t \in [0, T], \quad H_t(z) = \gamma(z, t).\]
The sets \( Y_b, L, Y_a \) are \((-u_\beta)\)-invariant and it follows from the second assertion of the Lemma that \( H_T(z) \in Y_a \cup L \) for every \( z \in Y_b \cup L \). Applying the next Lemma, we accomplish the proof of the Proposition.

**Lemma 4.3.** Let \((X, A) \subset (Y, B)\) be pairs of topological spaces, and \( H_t : Y \to Y \) be a homotopy, where \( t \in [0, T] \) such that
1) \( X, A, B \) are invariant under \( H_t \) for each \( t \).
2) \( H_0 = \text{Id}, \quad H_T(Y) \subset X, \quad H_T(B) \subset A. \)
Then the inclusion \((X, A) \hookrightarrow (Y, B)\) is a homotopy equivalence.

*Proof.* The proof repeats the arguments of [9], Lemma 1.8, page 171.

**Corollary 4.4.** Assume that \( f_\beta \) does not have critical points in \( Y_a \). Then the inclusion \( L \subset Y_a \cup L \) is a homotopy equivalence.

Now let \( c \) be a critical value of \( f_\beta \); we will describe the homotopy type of the pair \((Y_{c+\delta} \cup L, L)\) in terms of \((Y_{c-\delta} \cup L, L)\). Denote by \( p_1, ..., p_k \) the critical points on the level \( c \) and let \( H_1, ..., H_k \) be the corresponding \((-u_\beta)\)-invariant handles around the critical points; denote the union \( \cup_i H_i \) by \( H \). If \( \delta > 0 \) is sufficiently small, then we can choose the handles in such a way that they do not intersect \( L \), and, therefore, the set
\[
L \cup Y_{c-\delta} \cup H
\]
is homeomorphic to the result of attaching of $k$ handles of index $n$ to $L \cup Y_{c-\delta}$.

**Lemma 4.5.** 
1) There is $C > 0$ such that for every 
   
   $z \in Y_{c+\delta} \setminus (Y_{c-\delta} \cup H \cup L)$ 

   we have $f'_\beta(z)(u_\beta(z)) \geq C$.

2) There is $T > 0$ such that 
   
   $\gamma(z, T) \in \text{Int} (Y_{c-\delta} \cup H \cup L)$ 

   for every $z \in Y_{c+\delta}$.

3) The inclusion 
   
   $(Y_{c-\delta} \cup H \cup L, L) \subset (Y_{c+\delta} \cup L, L)$ 

   is a homotopy equivalence.

**Proof.** The proof repeats the corresponding steps of the proof of Proposition 4.2 and will be omitted. □

The point 2) of Theorem 2.3 follows by the usual inductive procedure of Morse theory. Namely, we recover the homotopy type of a manifold by successive attaching of handles corresponding to the critical points of a Morse function on it.

5. The Novikov Homology of $V$ and $M$

In this section we prove the assertion 3) of Theorem 2.3 and Theorem 2.4.

Returning to the space $M = \mathbb{C}^n \setminus H$, observe that $H_1(M, \mathbb{Z})$ is a free abelian group of rank $m$ generated by the meridians of the hyperplanes $H_i$. The elements of the dual basis in the group $\text{Hom}(\pi_1(M), \mathbb{Z})$ will be denoted by $\theta_i$ (where $1 \leq i \leq m$). For $\alpha \in \mathbb{R}^m$ denote by $\bar{\alpha} : \pi_1(M) \to \mathbb{R}$ the homomorphism $\sum_i \alpha_i \theta_i$. It is clear that $\alpha \in \mathbb{R}^m$ is positive if and only if $\bar{\alpha}$ is a positive homomorphism. The composition 

$$\pi_1(N) \cong \pi_1(M) \overset{\overline{\alpha}}{\to} \mathbb{R},$$

where the first homomorphism is induced by the inclusion $N \subset M$, will be denoted by the same symbol $\overline{\alpha}$ by an abuse of notation. We will restrict ourselves to the case $n \geq 2$, where $V$ is connected. The proof in the case $n = 1$ is easy and will be omitted. The composition 

$$\pi_1(V) \to \pi_1(M) \overset{\overline{\alpha}}{\to} \mathbb{R},$$
where the first homomorphism is induced by the inclusion \( V \subset M \), will be denoted by \( \alpha \). Recall the holomorphic 1-form

\[
\omega_\alpha = \frac{d\Phi_\alpha}{\Phi_\alpha} = \sum_j \alpha_j \frac{d\xi_j}{\xi_j}
\]

and denote its real and imaginary parts by \( R \) and \( I \) respectively. Then

\[
R = df_\alpha f_\alpha.
\]

Denote by \( \hat{\alpha} \) the element of the group \( H^1(M, \mathbb{R}) \) corresponding to \( \alpha \) under the canonical isomorphism

\[
\text{Hom}(\pi_1(M), \mathbb{R}) \approx H^1(M, \mathbb{R}).
\]

Then the cohomology class of \( I \) equals \( 2\pi \hat{\alpha} \). Let \( \iota_\alpha \) be the vector field dual to \( I \). Since \( \omega_\alpha \) is a holomorphic form, we have

\[
\text{grad} f_\alpha(z) = -\sqrt{-1} \cdot \iota_\alpha(z).
\]

If \( \varepsilon > 0 \) is small enough so that \( V \) is contained in the neighbourhood \( U \) from Proposition 3.2 we have \( ||\iota_\alpha(z)|| \geq A \). Observe that \( \iota_\alpha(z) \) is orthogonal to \( \text{grad} f_\alpha(z) \) and, therefore, tangent to \( V = f_\alpha^{-1}(\varepsilon) \). We deduce that the restriction to \( V \) of the 1-form \( \mathfrak{I} \) does not vanish, and, moreover, the norm of its dual vector field is bounded from below by a strictly positive constant.

**Proposition 5.1.** The Novikov homology \( \hat{H}_k(V, \alpha) \) vanishes for all \( k \).

**Proof.** Let \( p : \tilde{V} \rightarrow V \) be the universal covering of \( V \). The closed 1-form \( p^*(\mathfrak{I}) \) is cohomologous to zero; let \( p^*(\mathfrak{I}) = dF \), where

\[
F = \text{arg}(\xi_1^{\alpha_1} \cdot \xi_2^{\alpha_2} \cdot ... \cdot \xi_m^{\alpha_m})
\]

is a real-valued function on \( \tilde{V} \) without critical points. Denote by \( B_n \) the subset \( F^{-1}([\varepsilon, -\infty, -n]) \). The singular chain complexes \( C_*(n) = \mathcal{S}_*(\tilde{V}, B_n) \) form an inverse system, and the Novikov homology \( \hat{H}_*(V, \alpha) \) is isomorphic to the homology of its inverse limit (see [11]). For every \( k \) we have an exact sequence

\[
\lim^1 H_{k+1}(C_*(n)) \rightarrow H_k(\lim_{\leftarrow} C_*(n)) \rightarrow \lim_{\rightarrow} H_k(C_*(n)).
\]

The lift of the vector field \( \iota_\alpha \) to \( \tilde{V} \) will be denoted by the same letter \( \iota_\alpha \). The standard argument using the shift diffeomorphism along the trajectories of \( -\iota_\alpha \) shows that \( H_k(C_*(n)) = 0 \) for every \( k \); our Proposition follows. \( \square \)

Consider now the case when \( \alpha \) is of rank one, that is, all \( \alpha_i \) are rational multiples of one real number. In this case the differential
Proposition 5.2. The map \( g \) is a fibration of \( V \) over \( S^1 \).

Proof. The map \( g \) does not have critical points. Consider the vector field

\[
y_\alpha(z) = \frac{\iota_\alpha(z)}{||\iota_\alpha(z)||^2}.
\]

For \( x \in V \) denote by \( \gamma(x, t; y_\alpha) \) the \( y_\alpha \)-trajectory starting at \( x \). Since the norm of \( \iota_\alpha(z) \) is bounded away from zero in \( V \), the trajectory is defined on the whole of \( \mathbb{R} \). We have also

\[
d\left( g(\gamma(x, t; y_\alpha)) \right) = 1.
\]

Pick any \( b \in \mathbb{R} \) and let \( \bar{b} \) be the image of \( b \) under the projection \( \mathbb{R} \to \mathbb{R}/a \mathbb{R} \approx S^1 \); let \( c \in S^1 \) be the image of \( b + a/2 \). Let \( V_0 = g^{-1}(\bar{b}) \).

It is easy to check that the map

\[
(x, t) \mapsto \gamma(x, t; y_\alpha)
\]

is a diffeomorphism

\[
V_0 \times ] - a/2, a/2[ \approx g^{-1}(S^1 \setminus \{c\})
\]

compactible with projections. Therefore, \( g \) is a locally trivial fibration.

It is clear that any fiber of \( g \) is locally the set of zeros of a holomorphic function, therefore, it is a closed complex analytic submanifold of \( \mathbb{C}^n \) and has a homotopy type of a CW-complex of dimension \( \leq n - 1 \) (see [1]). Moreover, it is not difficult to prove that for any \( a \in \mathbb{C}^n \) the distance function \( d(z) = ||z - a||^2 \) restricted to \( X \) has no critical points outside a ball \( D(a, R) \) of sufficiently large radius \( R \). Therefore, the Morse theory applied to the function \( d|X \), implies that \( X \) has the homotopy type of a finite CW-complex of dimension \( \leq n - 1 \). A similar argument shows that the manifold \( V \) itself is homotopy equivalent to a finite CW complex. The proof of Theorem 2.3 is now complete.

Proof of Theorem 2.4 Let \( \Lambda \) be the group ring of the fundamental group of \( N \). Let \( \xi : \pi_1(M) \to \mathbb{R} \) be a positive homomorphism, then \( \xi = \overline{\alpha} \) where \( \alpha \in \mathbb{R}^m \) is a positive vector. Let \( \hat{\Lambda}_{\overline{\alpha}} \) denote the Novikov completion of \( \Lambda \) with respect to \( \overline{\alpha} \). Denote by \( q : \tilde{N} \to N \) the universal covering of \( N \). Put \( \tilde{V} = q^{-1}(V) \).

We have the short exact sequence of free \( \hat{\Lambda}_{\overline{\alpha}} \)-complexes

\[
0 \to \hat{\Lambda}_{\overline{\alpha}} \otimes \mathcal{S}_*(\tilde{V}) \to \hat{\Lambda}_{\overline{\alpha}} \otimes \mathcal{S}_*(\tilde{N}) \to \hat{\Lambda}_{\overline{\alpha}} \otimes \mathcal{S}_*(\tilde{N}, \tilde{V}) \to 0
\]
Observe that the inclusion $V \subset N$ induces a surjective homomorphism of fundamental groups. Therefore, the space $\bar{V}$ is connected and the covering $\bar{V} \rightarrow V$ is a quotient of the universal covering $\tilde{V} \rightarrow V$, so that

$$H_*(\Lambda_\alpha \otimes S_*(\bar{V})) = H_*(\Lambda_\alpha \otimes (\tilde{\mathcal{L}}_\alpha \otimes S_*(\bar{V}))),$$

where $\mathcal{L}$ is the group ring of the fundamental group of $V$, and $\tilde{\mathcal{L}}_\alpha$ is the Novikov completion of $\mathcal{L}$ with respect to $\alpha$. By Proposition 5.1 the Novikov homology of $V$ vanishes. Consider the long exact sequence of homology modules, derived from the short exact sequence (12). Since $H_*(\Lambda_\alpha \otimes S_*(\bar{V})) = 0$, we deduce the isomorphisms

$$\widehat{H}_*(N, \bar{\alpha}) \approx H_*(\Lambda_\alpha \otimes S_*(\bar{N}, \bar{V})).$$

Observe now that the homology of the couple $(\bar{N}, \bar{V})$ is a free module over $\Lambda$ of rank $|\chi(M)|$, concentrated in degree $n$. Theorem 2.4 follows.

6. NON-ESSENTIAL ARRANGEMENTS

Let us consider the case of non-essential arrangements $\mathcal{A}$. Assume that $\text{rk } \mathcal{A} = l < n$. The function $f_\alpha$ is not a Morse function in this case. However, the analog of Theorem 2.3 is easily obtained by reduction to the case of essential arrangements.

Denote by $\pi : \mathbb{C}^l \oplus \mathbb{C}^k \rightarrow \mathbb{C}^l$ the projection onto the first direct summand. Let $\mathcal{A}$ be a hyperplane arrangement in $\mathbb{C}^l$, defined by affine functions $\xi_i : \mathbb{C}^l \rightarrow \mathbb{C}$. The functions $\xi_i \circ \pi$ determine a hyperplane arrangement in $\mathbb{C}^{l+k}$ which will be called $k$-suspension of $\mathcal{A}$. It is not difficult to prove the next proposition.

**Proposition 6.1.** Let $\mathcal{A}$ be a hyperplane arrangement in $\mathbb{C}^n$ of rank $l$. The $\mathcal{A}$ is linearly isomorphic to the $(n - l)$-suspension of an essential hyperplane arrangement $\mathcal{A}_0$ in $\mathbb{C}^l$.

The complement $M(\mathcal{A})$ is diffeomorphic to $M(\mathcal{A}_0) \times \mathbb{C}^{n-l}$. Denote $M(\mathcal{A})$ by $M$ for brevity. We obtain the following generalizations of theorems 2.3 and 2.4.

**Theorem 6.2.** There is an $l - 1$-dimensional manifold $Y$, fibered over a circle, such that $M$ is homotopy equivalent to the result of attaching to $Y$ of $|\chi(M)|$ cells of dimension $l$. 
Theorem 6.3. For every positive homomorphism $\xi : \pi_1(M) \to \mathbb{R}$ the Novikov homology $\hat{H}_k(M, \xi)$ vanishes for every $k \neq l$ and is a free module of rank $|\chi(M)|$ for $k = l$.

Remark 6.4. Theorem 2.3 leads to a quick proof of the well-known fact that the space $M$ is homotopy equivalent to a finite CW-complex.

Remark 6.5. A. Suciu communicated to us that our results are related to the recent work [10] of S. Papadima and A. Suciu. Let $A$ denote the Orlik-Solomon algebra for the arrangement $\mathcal{A}$, which is isomorphic to the cohomology ring of $M(\mathcal{A})$. We denote by $A^j$ the degree $j$ part of $A$. Now the Aomoto complex is the cochain complex $A^* = \oplus_{j \geq 0} A^j$ whose coboundary operator is defined to be the multiplication by $a \in A^1$. The corresponding cohomology is denoted by $\hat{H}^*(A, a)$. Let us call an element $a$ resonant in degree $j$ if the cohomology $H^j(A, a)$ does not vanish. Let $b \in \mathbb{R}^m$; recall from Section 5 the corresponding homomorphism $\tilde{b} : \pi_1(M) \to \mathbb{R}$ and the cohomology class $\hat{b} = \sum_i b_i \theta_i \in H^1(M, \mathbb{R})$.

Proposition 16.7 of the paper [10] says, that if the Novikov homology $\hat{H}_j(M(\mathcal{A}), \tilde{b})$ equals 0 for $j \leq q$, then $\hat{b}$ is non-resonant in degrees $j \leq q$. Thus our Theorem 2.4 implies that every positive vector $b$ the corresponding cohomology class $\hat{b}$ is non-resonant in degrees strictly less than the rank of the arrangement. This also follows from a result due to S. Yuzvinsky (see [13], [14]).

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