SYMMETRIES AND WAVELETS

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Abstract

Wavelet analysis of different patterns reveals some symmetries and singularities otherwise hidden in the pattern. Its general methods are briefly reviewed. Examples from turbulence, cavitation, Cherenkov gluon emission and quark-gluon jets structure in QCD are considered. Symmetries are crucial in the surrounding world and therefore lie at the very heart of any theoretical construction. Symmetries can be either global or local. The invariance under the global gauge transformation requires a definite symmetry between particles and antiparticles. More important consequences appear from the requirement of the local gauge invariance. Both in QED and QCD it defines the interaction lagrangian and, consequently, the symmetries of main equations. In classical and quantum physics the symmetries of equations and of initial (boundary) conditions determine the symmetries of the corresponding patterns observed. Thus the problem of the local analysis of these patterns should be solved.

One of the most popular methods is to use the Fourier transform. The set of functions (sine, cosine and imaginary exponents) is such that all of them are widespread along their definition axis. Thus this transform is a global one. With its help one can get the knowledge of frequencies (scales) important in the problem considered but can not learn about their locations. In other words, this transform is suited for stationary (homogeneous) processes. For non-stationary (inhomogeneous) processes one should apply a local transform. With the above set of functions, it becomes possible only if one uses the so-called windowed Fourier transform. Namely, the analyzed function is multiplied by an auxiliary function steeply decreasing (or equal to zero) outside some finite range (window) of its variables. Sometimes, this leads to uncontrollable consequences.

The only way to control such transform is to use wavelets. They allow for the direct and inverse local transform to be done. Thus one gets the local scale (frequency) characteristics (fluctuations) of the analyzed process within any required resolution. Moreover, this resolution is self-adjustable in contrast to fixed windows in the Fourier transform. It means that wavelets automatically construct the so-called Heisenberg windows adjusted to the local properties of the considered function. To resolve its fast variation with high frequencies they admit small time-window but large frequency-window while for low frequencies the wide time-window and narrow frequency-window are used, i.e., the width of the window is proportional to its mean value. That is why they are called the relative band-width filters.

Wavelets originate from the functional equation which relates the so-called scaling function with its shifted and translated version. The coefficients $h_k$ of this linear equation define in a unique way the explicit form of the wavelet which is not used directly, however. The wavelet transform
coefficients can be explicitly obtained from the iterative equations containing $h_k$. This procedure is called the fast wavelet transform. It allows for the complete decomposition of the considered process at any resolution level and does not require any integration. Computer calculations are done quickly.

For those dealing with quantum field theory, I’d like to mention also that wavelet analysis actually is very close to the renormalization group approach (see, e.g., [1]) because it uses the translated versions of the same function. These features must be further exploited.

The equation for the scaling function $\varphi(x)$ is

$$\varphi(x) = \sqrt{2} \sum_{k=0}^{2^{M-1}} h_k \varphi(2x - k)$$  \hspace{1cm} (1)

with the dyadic dilation $2$, integer translation $k$ and the coefficients $h_k$ determined from conditions of orthogonality (both mutual and to some polynomials) after one chooses their number $2M$. If the scaling function is known, one can form a ”mother wavelet” (or a basic wavelet) $\psi(x)$ according to

$$\psi(x) = \sqrt{2} \sum_{k=0}^{2^{M-1}} g_k \varphi(2x - k),$$  \hspace{1cm} (2)

where

$$g_k = (-1)^k h_{2^{M-1}-1}.$$  \hspace{1cm} (3)

The dilated and translated versions of the scaling function $\varphi$ and the ”mother wavelet” $\psi$

$$\varphi_{j,k} = 2^{j/2} \varphi(2^j x - k),$$  \hspace{1cm} (4)

$$\psi_{j,k} = 2^{j/2} \psi(2^j x - k)$$  \hspace{1cm} (5)

form the orthonormal basis.

One may decompose any function $f$ of $L^2(R)$ at any resolution level $j_n$ in a series

$$f = \sum_k s_{j_n,k} \varphi_{j_n,k} + \sum_{j \geq j_n} d_{j,k} \psi_{j,k}.$$  \hspace{1cm} (6)

The wavelet coefficients $s_{j,k}$ and $d_{j,k}$ can be calculated as

$$s_{j,k} = \int dx f(x) \varphi_{j,k}(x),$$  \hspace{1cm} (7)

$$d_{j,k} = \int dx f(x) \psi_{j,k}(x).$$  \hspace{1cm} (8)

However, in practice their values are determined from the fast wavelet transform. In general, one can get the iterative formulas of the fast wavelet transform

$$s_{j+1,k} = \sum_m h_m s_{j,2k+m};$$  \hspace{1cm} (9)

$$d_{j+1,k} = \sum_m g_m s_{j,2k+m}$$  \hspace{1cm} (10)
where
\[ s_{0,k} = \int dx f(x) \varphi(x - k). \] (11)

These equations yield fast algorithms (the so-called pyramid algorithms) for computing the wavelet coefficients, asking now just for \( O(N) \) operations to be done. Starting from \( s_{0,k} \), one computes all other coefficients provided the coefficients \( h_m, g_m \) are known. The explicit shape of the wavelet is not used. The wavelet coefficients show the strength of fluctuations of \( f \) at the location \( k \) and the scale \( j \).

The behaviour of wavelet coefficients is closely related to the structure of the local singularities of the function \( f \) and therefore can provide the knowledge which class of functions it belongs to. Also, the fractal properties of the analyzed set can be determined from the partition function \( Z_q(j) \). Namely, let us consider the sum \( Z_q \) of the \( q \)-th moments of the coefficients of the wavelet transform at various scales \( j \)
\[ Z_q(j) = \sum_k |d_{j,k}|^q, \] (12)
where the sum is over the maxima of \( |d_{j,k}| \). Then it was shown that for a fractal signal this sum should behave as
\[ Z_q(j) \propto 2^{j[\tau(q)+q/2]}, \] (13)
i.e.,
\[ \log Z_q(j) \propto j[\tau(q) + \frac{q}{2}]. \] (14)

Thus the necessary condition for a signal to possess fractal properties is the linear dependence of \( \log Z_q(j) \) on the level number \( j \). If this requirement is fulfilled the dependence of \( \tau \) on \( q \) shows whether the signal is monofractal or multifractal. Monofractal signals are characterized by a single dimension and, therefore, by a linear dependence of \( \tau \) on \( q \), whereas multifractal ones are described by a set of such dimensions, i.e., by non-linear functions \( \tau(q) \). Monofractal signals are homogeneous, in the sense that they have the same scaling properties throughout the entire signal. Multifractal signals, on the other hand, can be decomposed into many subsets characterized by different local dimensions, quantified by a weight function. The wavelet transform removes lowest polynomial trends that could cause the traditional box-counting techniques to fail in quantifying the local scaling of the signal. The function \( \tau(q) \) can be considered as a scale-independent measure of the fractal signal. It can be further related to the Renyi dimensions, Hurst and Hölder exponents (for more detail, see Ref. [5]).

The wavelet transform can be also used for representation of differential operators, for solving the differential equations etc.

Let me describe the wavelet application to the analysis of some symmetrical patterns. Many such patterns are observed, e.g., on the water surface [1] (see Figs. 1, 2). Macroscopically they are described by some order parameters. The wavelet decomposition of the observed pattern allows to ascribe the definite scale at any location within the pattern and extract wave vector information from the pattern. In particular, the regions with scales within the chosen range can be separated. E.g., the stability regions with large scales are shown in Fig. 3 in white. Thus the transition from microscopic pattern to its macroscopic description becomes feasible. The location of some defects (broken symmetry) is easily recognized as well.

Another interesting effect is the turbulence. Its wavelet analysis reveals the fractal properties of the velocity field. Closer to the topic of this conference are the similar patterns observed in jet
production in $e^+e^-$-annihilation at high energies (for the review see [5, 7]). The partonic cascade structure with "jets inside jets inside jets" leads to the mono- and multifractal distribution of the created particles within the available phase space (see [4]). The fluctuations increase in smaller phase space bins, and, correspondingly, the factorial moments of the multiplicity distributions should increase linearly (for monofractals) for smaller bins and flatten off for multifractals. This is demonstrated in Fig. 4. It is explained in QCD [8, 9, 10] as a consequence of the asymptotic freedom behavior of the coupling strength.

The fractal properties have been also discovered by the wavelet analysis in the distributions of the heartbeat intervals. One would be inclined to consider this process as a completely periodic stationary one. In fact, it has been found that it possesses the fractal properties and the distribution moments (correlations) at different scales behave in a different way for healthy and diseased patients [11, 12]. The results of the wavelet analysis were proposed as a first clinically significant measure of heart disease.

The functions somewhat similar, at first sight, to those for heartbeat intervals are obtained for the pressure variation in gas turbine compressors. However their wavelet analysis revealed absence of fractal properties with some signature of a singular behavior. More important discovery is that the behavior of the dispersion of the wavelet coefficients at some resolution level can be used as a precursor of the extremely dangerous effect [13]. It is initiated by some instability (singularity), reminds the cavitation and is called the stall. This instability results in the complete damage of the engine and an aircraft crash.

Coming back to particle interactions, let me mention the important field of wavelet application to pattern recognition in multiparticle production. One can ask what kind of patterns are formed within the available phase space by particles created in very high energy interactions. This problem becomes experimentally feasible, e.g., in AA-collisions at RHIC where more than 4000 charged particles are on average produced in a single event at c.m. energy 130 GeV/c. In a search for the ring-like structure implied by ideas on Cherenkov gluon radiation [14], the wavelet analysis of several events of central Pb-Pb collisions at l.s. energy 158 GeV/c with more than 1000 charged secondaries was done. Some events possessing such a symmetry in the long-range correlations have been found [15] (see Fig. 4). This analysis can be extended to search for other symmetry patterns as well.

In conclusion, I’d like to stress how powerful is the method of the wavelet analysis to search for symmetries as is seen already from the above examples. However, its use is at the very beginning now and further results are coming.

**Figure captions**

Fig. 1 The pattern in experimental convection (from [16]).

Fig. 2 The symmetry and imperfections in a hexagonal Benard convection cell (from [17]).

Fig. 3 The original pattern (b) is wavelet analyzed and (a) the regions of high wave vectors ($k > k_B$) are marked in white, (c) the wave-number histogram computed with the wavelet algorithm (solid line) is more precise than that with the Fourier transform (dotted line); $k_B$ is the wave number separating regions with straight rolls stable ($k > k_B$) and unstable to bending, (d)
the correlations of the pattern along various cross sections are shown to be similar (from [6]).

Fig. 4 The factorial moments $F_q$ of multiplicity distributions in $e^+e^-$-collisions at $Z^0$ energy show the mono- and multifractal behavior (linear and curved parts of the lines, correspondingly) of particles pattern in the available phase space (from [18]). Increasing $z$ corresponds to smaller bins. Experimental data are shown by dots. Different lines correspond to different calculations within some approximations of QCD.

Fig. 5 The ring-like structure of some events of Pb-Pb interaction at 158 GeV/c (from [15]) revealed by the wavelet analysis demonstrates the symmetry otherwise hidden in a huge background.

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