The Impact of Exponential Utility Costs in Bottleneck Routing Games

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Abstract—We study bottleneck routing games where the social cost is determined by the worst congestion on any edge in the network. Bottleneck games have been studied in the literature by having the player’s utility costs to be determined by the worst congested edge in their paths. However, the Nash equilibria of such games are inefficient since the price of anarchy can be very high with respect to the parameters of the game. In order to obtain smaller price of anarchy we explore exponential bottleneck games where the utility costs of the players are exponential functions on the congestion of the edges in their paths. We find that exponential bottleneck games are very efficient giving a poly-log bound on the price of anarchy: $O(\log L \cdot \log |E|)$, where $L$ is the largest path length in the players strategy sets and $E$ is the set of edges in the graph.

I. INTRODUCTION

Motivated by the selfish behavior of entities in communication networks, we study routing games in general networks where each packet’s path is controlled independently by a selfish player. We consider noncooperative games with $N$ players, where each player has a pure strategy profile from which it selfishly selects a single path from a source node to a destination node such that the selected path minimizes the player’s utility cost function (such games are also known as atomic or unsplittable-flow games). We focus on bottleneck games where the objective for the social outcome is to minimize $C$, the maximum congestion on any edge in the network. Typically, the congestion on an edge is a non-decreasing function on the number of paths that use the edge; here, we consider the congestion to be simply the number of paths that use the edge.

Bottleneck congestion games have been studied in the literature [1]–[4] where each player’s utility cost is the worst congestion on its path edges. In particular, player $i$ has utility cost function $\max_{e \in p_i} C_e$, where $p_i$ is the path of the player and $C_e$ denotes the congestion of edge $e$. In [1] the authors observe that bottleneck games are important in networks for various practical reasons. In wireless networks the maximum congested edge is related to the lifetime of the network since the nodes adjacent to high congestion edges transmit large number of packets which results to higher energy utilization. Thus, minimizing the maximum edge congestion immediately translates to longer network lifetime. High congestion edges also result to congestion hot-spots in the network which may slow down the performance of the whole network. Hot spots may also increase the vulnerability of the network to malicious attacks which aim to increase the congestion of edges in the hope to bring down the network or degrade its performance. Thus, minimizing the maximum congested edge results to hot-spot avoidance and also to more secure networks.

Bottleneck games are also important from a theoretical point of view since the maximum edge congestion is immediately related to the optimal packet scheduling. In a seminal result, Leighton et al. [5] showed that there exist packet scheduling algorithms that can deliver the packets along their chosen paths in time very close to $C + D$, where $D$ is the maximum chosen path length. This work on packet scheduling has been extended in [6]–[9]. When $C \gg D$, the congestion becomes the dominant factor in the packet scheduling performance. Thus, smaller $C$ immediately implies faster delivery time for the packets in the network.

A natural problem that arises concerns the effect of the players’ selfishness on the welfare of the whole network measured with the social cost $C$. We examine the consequence of the selfish behavior in pure Nash equilibria which are stable states of the game in which no player can unilaterally improve her situation. We quantify the effect of selfishness with the price of anarchy ($\text{PoA}$) [10], [11], which expresses how much larger is the worst social cost in a Nash equilibrium compared to the social cost in the optimal coordinated solution. The price of anarchy provides a measure for estimating how closely do Nash equilibria of bottleneck routing games approximate the optimal $C^*$ of the respective routing optimization problem.

Ideally, the price of anarchy should be small. However, the findings in the literature show that bottleneck games are not efficient, namely, the price of anarchy may be large. In [1] it is shown that if the edge-congestion function is bounded by some polynomial with degree $d$ (with respect to the packets that use the edge) then $\text{PoA} = O(|E|^d)$, where $E$ is the set of edges in the graph. In [2] the authors consider the case $d = 1$ and they show that $\text{PoA} = O(L + \log |V|)$, where $L$ is the maximum path length in the players strategies and $V$ is the set of nodes. This bound is asymptotically tight since there are game instances with $\text{PoA} = \Omega(L)$. Note that $L \leq |E|$, and
further $L$ may be significantly smaller than $|E|$. However, $L$
 can still be proportional to the size of the graph, and thus the
price of anarchy can be large.

A. Contributions

In this work we focus on exploring alternative utility cost
functions for the players that have better impact on the social
cost $C$. We introduce exponential bottleneck games where
the player utilities are exponential functions on the congestion
of the edges of the paths. In particular, the player utility cost
function for player $i$ is:

$$
\sum_{e \in p_i} 2^{C_e},
$$

where $p_i$ is the player’s chosen path. Note that the new utility
cost is a sum of exponential terms on the congestion of the
edges in the path (instead of the max that we described earlier).
Using the new utility cost functions we show that exponential
games have always Nash equilibria which can be obtained by
best response dynamics. The main result is that the price of
anarchy is poly-log:

$$
P oA = O(\log L \cdot \log |E|),
$$

where $L$ is the maximum path length in the players strategy set
and $E$ is the set of edges in the graph. This price of anarchy
bound is a significant improvement over the price of anarchy
from the regular utility cost functions described earlier.

Exponential cost functions are legitimate metrics for the
utility costs of players since they reflect the performance of the
chosen paths according to congestion. Each player is motivated
to select a path with lower utility cost since it will provide
a better quality path with lower congestion that can affect
positively the player’s performance. As we discuss in Section V
the reason that we use exponential cost functions instead of
polynomial ones is that low degree polynomials give high price
of anarchy.

B. Related Work

Congestion games were introduced and studied in [12], [13].
Koutsoupias and Papadimitriou [10] introduced the notion of
price of anarchy in the specific parallel link networks model in
which they provide the bound $P oA = 3/2$. Since then, many
routing and congestion game models have been studied which
are distinguished by the network topology, cost functions, type
of traffic (atomic or splittable), and kind of equilibria (pure or
mixed). Roughgarden and Tardos [14] provided the first result
for splittable flows in general networks in which they showed that
$P oA \leq 4/3$ for each $p$ player cost which reflects to the sum of
congestions of the edges of a path. Pure equilibria with atomic
flow have been studied in [2], [13], [15]–[17] (work fits into this
category), and with splittable flow in [14], [18]–[20].
Mixed equilibria with atomic flow have been studied in [10],
[11], [21]–[29], and with splittable flow in [30], [31].

Most of the work in the literature uses a cost metric measured
as the sum of congestions of all the edges of the player’s path
[14], [15], [17], [19], [20], [26]. Our work differs from these
approaches since we adopt the exponential metric for player
cost. The vast majority of the work on routing games has been
performed for parallel link networks, with only a few exceptions
on general network topologies [2], [15], [18]–[30], which we
consider here.

Our work is close to [2], where the authors consider the
player cost $C_i$ and social cost $C$. They prove that the price of
stability is 1. They show that the price of anarchy is bounded
by $O(L + \log n)$, where $L$ is the maximum allowed path length.
They also prove that $\kappa \leq P oA \leq c(n^2 + \log^2 n)$, where $\kappa$
is the size of the largest edge-simple cycle in the graph and $c$
is a constant. Some of the techniques that we use in our proofs
(for example expansion) were introduced in [2]. Another related
result for general networks which has a brief discussion of the
convergence of maximum player cost ($C_i$) games is [16] where
the authors focus on parallel link networks, but also give some
results for general topologies on convergence to equilibria.

Bottleneck congestion games have been studied in [1], where
the authors consider the maximum congestion metric in general
networks with splittable and atomic flow (but without consider-
ning path lengths). They prove the existence and non-uniqueness
of equilibria in both the splittable and atomic flow models. They
show that finding the best Nash equilibrium that minimizes the
social cost is a NP-hard problem. Further, they show that the
price of anarchy may be unbounded for specific edge congestion
functions (these are functions of the number of paths that use
the edge). If the edge congestion function is polynomial with
degree $p$ then they bound the price of anarchy with $O(m^p)$,
where $m$ is the number of edges in the graph. In the splittable
case they show that if the users always follow paths with low
congestion then the equilibrium achieves optimal social cost.

Outline of Paper

In Section II we give basic definitions. In section III we show
that exponential bottleneck games have always Nash equilibria.
We study the price of anarchy in Section IV. We finish with
conclusions and future work in Section V.

II. Definitions

A. Path Routings

Consider an arbitrary graph $G = (V, E)$ with nodes $V$ and
edges $E$. Let $\Pi = \{\pi_1, \ldots, \pi_N\}$ be a set of packets such that
each $\pi_i$ has a source $u_i$ and destination $v_i$. A routing $p =\
p_i[p_1, p_2, \cdots, p_N]$ is a collection of paths, where $p_i$ is a path for
packet $\pi_i$ from $u_i$ to $v_i$. We will denote by $E(p_i)$ the set of
edges in path $p_i$. Consider a particular routing $p$. The edge-
congestion of an edge $e$, denoted $C_e$, is the number of paths in
$p$ that use edge $e$. For any set of edges $A \subseteq E$, we will denote
by $C_A = \max_{e \in A} C_e$. For any path $q$, the path-congestion is
$C_q = C_{E(q)}$. For any set of edges $A \subseteq E$, we will also use the notation
$C_A = C_{E_A}$. The network congestion is $C = C_E$, which is the
maximum edge-congestion over all edges in $E$.

We continue with definitions of exponential functions on
congestion. Consider a routing $p$. For any edge $e$, we will
denote $\tilde{C}_e = 2^{C_e}$. For any set of edges $A \subseteq E$, we will denote
$\tilde{C}_A = \sum_{e \in A} \tilde{C}_e$. For any path $q$, we will denote $\tilde{C}_q = \tilde{C}_{E(q)}$. 
For any path \( p_i \in \mathcal{P} \) we will denote \( \tilde{C}_i = \tilde{C}_{p_i} \). We denote the length (number of edges) of any path \( q \) as \(|q|\). Whenever necessary we will append \((p)\) in the above definitions to signify the dependance on routing \( p \). For example, we will write \( C(p) \) instead of \( C \).

### B. Routing Games

A routing game in graph \( G \) is a tuple \( \mathcal{R} = (G, \mathcal{N}, \mathcal{P}) \), where \( \mathcal{N} = \{1, 2, \ldots, N\} \) is the set of players such that each player \( i \) corresponds to a packet \( \pi_i \) with source \( u_i \) and destination \( v_i \), and \( \mathcal{P} \) are the strategies of the players. We will use the notation \( p_i \) to denote player \( i \) and its respective packet. In the set \( \mathcal{P} = \bigcup_{i \in \mathcal{N}} \mathcal{P}_i \) the subset \( \mathcal{P}_i \) denotes the strategy set of player \( i \), which a collection of available paths in \( G \) for player \( i \) from \( u_i \) to \( v_i \). Any path \( p \in \mathcal{P}_i \) is a pure strategy available to player \( i \). A pure strategy profile is any routing \( p = [p_1, p_2, \ldots, p_N] \), where \( p_i \in \mathcal{P}_i \). The longest path length in \( \mathcal{P} \) is denoted \( L(\mathcal{P}) = \max_{p \in \mathcal{P}} |p| \). (When the context is clear we will simply write \( L \)).

For game \( R \) and routing \( p \), the social cost (or global cost) is a function of routing \( p \), and it is denoted \( SC(p) \). The player or local cost is also a function on \( p \) denoted \( pc_i(p) \). We use the standard notation \( p_i \) to refer to the collection of paths \( \{p_{i1}, \ldots, p_{iM}\} \), and \( p_i \); \( \cdot \) as an alternative notation for \( p \) which emphasizes the dependence on \( p_i \). Player \( \pi_i \) is locally optimal (or stable) in routing \( p \) if \( pc_i(p) \leq pc_i(p') \) for all paths \( p' \in \mathcal{P}_i \). A greedy move by a player \( \pi_i \) is any change of its path from \( p_i \) to \( p' \), which improves the player’s cost, that is, \( pc_i(p) > pc_i(p') \). Best response dynamics are sequences of greedy moves by players.

A routing \( p \) is in a Nash Equilibrium (we say \( p \) is a Nash-routing) if every player is locally optimal. Nash-routings quantify the notion of a stable selfish outcome. In the games that we study there could exist multiple Nash-routings. A routing \( p^* \) is an optimal pure strategy profile if it has minimum attainable social cost: for any other pure strategy profile \( p \), \( SC(p^*) \leq SC(p) \).

We quantify the quality of the Nash-routings with the price of anarchy \( (PoA) \) (sometimes referred to as the coordination ratio) and the price of stability \( (PoS) \). Let \( \mathcal{P} \) denote the set of distinct Nash-routings, and let \( SC^* \) denote the social cost of an optimal routing \( p^* \). Then,

\[
PoA = \sup_{p \in \mathcal{P}} \frac{SC(p)}{SC^*}, \quad PoS = \inf_{p \in \mathcal{P}} \frac{SC(p)}{SC^*}.
\]

### III. Exponential Bottleneck Games and their Stability

Let \( \mathcal{R} = (G, \mathcal{N}, \mathcal{P}) \) be a routing game such that for any routing \( p \) the social cost function is \( SC = C \), and the player cost function is \( pc_i = \tilde{C}_i \). We refer to such routing games as exponential bottleneck games.

We show that exponential games have always Nash-routings. We also show that there are instances of exponential games that have multiple Nash-routings. The existence of Nash routings relies on finding an appropriate potential function that provides an ordering of the routings. Given an arbitrary initial state a greedy move of a player can only give a new routing with smaller order. Thus, best response dynamics (repeated greedy moves) converge to a routing where no player can improve further, namely, they converge to a Nash-routing. The potential function that we will use is: \( f(p) = \tilde{C}_E(p) \). We show that any greedy move gives a new routing with lower potential.

**Lemma 3.1:** If in routing \( p \) a player \( \pi_i \) performs a greedy move, then the resulting routing \( p' \) has \( \tilde{C}_E(p') > \tilde{C}_E(p) \).

**Proof:** Suppose that player \( \pi_i \) has path \( p_i \in p \) and switches to path \( p'_i \in p' \). Then, \( \tilde{C}_{p_i}(p) > \tilde{C}_{p'_i}(p') \). Let \( A = E(p_i) \setminus E(p'_i) \) and \( B = E(p'_i) \setminus E(p_i) \). It has to be that \( \tilde{C}_A(p) > \tilde{C}_B(p') \) since \( \pi_i \)'s cost decreases. Further, \( \tilde{C}_B(p') = 2\tilde{C}_B(p) \) and \( \tilde{C}_A(p) = 2\tilde{C}_A(p') \), since the presence or absence of player’s path in the edges \( A \) and \( B \) alters their total cost by a factor of 2. Let \( H = E \setminus \{A \cup B\} \). We have that \( \tilde{C}_H(p) = \tilde{C}_H(p') \), since \( \pi_i \) does not affect those edges. Since \( E = H \cup A \cup B \) and \( H, A, B \) are disjoint, we have that

\[
\tilde{C}_E(p) = \tilde{C}_H(p) + \tilde{C}_A(p) + \tilde{C}_B(p) = \tilde{C}_H(p') + 2\tilde{C}_A(p') + \frac{\tilde{C}_B(p')}{2} = \tilde{C}_H(p') + \tilde{C}_A(p') + \tilde{C}_B(p') + \left(\frac{\tilde{C}_A(p') - \frac{\tilde{C}_B(p')}{2}}{2}\right) = \tilde{C}_E(p') + \left(\frac{\tilde{C}_A(p') - \frac{\tilde{C}_B(p')}{2}}{2}\right).
\]

Since \( \tilde{C}_A(p') = 2\tilde{C}_A(p') \) and \( \tilde{C}_A(p) > \tilde{C}_B(p') \), we have that \( 2\tilde{C}_A(p') > \tilde{C}_B(p') \), or equivalently

\[
\frac{\tilde{C}_A(p') - \frac{\tilde{C}_B(p')}{2}}{2} > 0.
\]

Therefore, \( \tilde{C}_E(p') > \tilde{C}_E(p) \), as needed.

Since the result of the potential function cannot be smaller than zero, Lemma 3.1 implies that best response dynamics converge to Nash-routings. Thus, we have: \( \tilde{C}_E \)

\[Fig. 1. \quad An exponential game instance with multiple Nash-routings\]

We continue to show that there are exponential games with multiple Nash-routings. Consider the example of Figure 1. There are three players \( \pi_1, \pi_2, \pi_3 \) with respective sources \( u_1, u_2, u_3 \) and destinations \( v_1, v_2, v_3 \). The strategy set of each player are all feasible paths from their source to destination. In
the left part of Figure [1] is a Nash-routing \( p = [p_1, p_2, p_3] \) with social cost \( SC(p) = 2 \) and respective player costs \( pc_1(p) = 4, pc_2(p) = 8, \) and \( pc_3(p) = 6. \) On the right part of the same figure is another Nash-routing \( p' = [p'_1, p'_2, p'_3] \) with social cost \( SC(p') = 1 \) and respective player costs \( pc_1(p') = 2, pc_2(p') = 6, \) and \( pc_3(p') = 6. \) Thus, we can make the following observation:

**Observation 3.3:** There exist exponential game instances with multiple Nash-routings.

### IV. Price of Anarchy

We bound the price of anarchy in exponential bottleneck games. Consider an exponential bottleneck routing game \( R = (G, N, \mathcal{P}). \) Let \( p = [p_1, \ldots, p_N] \) be an arbitrary Nash-routing with social cost \( C \); from Theorem 3.2 we know that \( p \) exists. Let \( p^* = [p^*_1, \ldots, p^*_N] \) represent the routing with optimal social cost \( C^*. \) Let \( L \) be the maximum path length in the players strategy sets and \( L^* \leq L \) be the longest path in \( p^*. \) Denote \( l^* = \log L^*. \)

We will obtain an upper bound on the price of anarchy \( PoA = C/C^* \) by finding a lower bound on the number of players as well as the number of edges in the \( p. \) The proof relies on the notion of self-sufficient sets:

**Definition 4.1 (Self-sufficient set):** Consider an arbitrary set of players \( S \) in Nash-routing \( p \) in game \( R. \) We label the equilibrium of \( S \) as self-sufficient if, after removing the paths of all players \( \notin S \) from \( p, \) for every \( \pi_i \in S, \) the cost \( C_{p_i^*} \) remains at least \( pc_i(p). \) Thus \( \pi_i \) cannot switch to path \( p_i^* \) only because of other players in \( S. \)

If a set \( S \) is not self-sufficient, then additional players \( S' \) must be present to guarantee the Nash-routing. Thus, we define the notion of support sets:

**Definition 4.2 (Support set):** If \( S \) is not a self-sufficient set, then there is a set of players \( S', \) where \( S \cap S' = \emptyset, \) such that the paths of the players in \( S' \cup S \) guarantee that \( C_{p_i^*} \) remains at least \( pc_i(p) \) even if all the other players \( \notin S' \cup S \) are removed from the game.

The players in \( S' \) may not be self-sufficient either. This process is repeated until a self-sufficient set is found. Our goal is to find a lower bound on a self-sufficient set players. We start with a small set of players based on \( C \) and the optimal congestion value \( C^* \), prove they are not self-sufficient and consider a sequence of expansions that will eventually lead to a self-sufficient set. We find the minimum number of these expansions to terminate the process and thus find the minimum number of players (and edges) needed to support a maximum equilibrium congestion of \( C. \) For a given graph \( G \) and players/edges this gives us an upper bound on \( C \) relative to \( C^*. \)

Initially assume \( C^* = 1, \) i.e every player in the optimally congested network has a unique optimal path to its destination of length at most \( L^*. \) For the game \( \hat{G} \) we will consider sets of players in stages, depending on their costs in \( \hat{G}. \) Let \( S^i \) denote the set of players in stage \( i \), \( 1 \leq i \leq \hat{C} \) with player costs \( C : 2^{\hat{C}+1} \leq \hat{C} \leq 2^{\hat{C}+1}. \) Consider an arbitrary player \( \pi \) in stage \( i. \) We let \( P^* \) denote its optimal path and \( \Phi(P^*) \) the minimum cost of path \( P^* \) in \( \hat{G}. \) Since \( II \) is in equilibrium, we must have \( \Phi(P^*) > 2^{C+1}. \)

We formally define expansion chains as follows: In stage \( i, 1 \leq i \leq \hat{C} - 1, \) let \( A^{(i)} \) denote the set of all players occupying exactly one edge of congestion \( \hat{C} - i + 1 \), let \( B^{(i)} \) denote the set of all players whose maximum edge congestion \( C' \) satisfies \( \hat{C} - i \geq C' > \hat{C} - i + 1 \) and finally let \( D^{(i)} = S^{(i)} = A^{(i)} \) or \( B^{(i)}. \)

For \( i > 1, \) a level \( i \) expansion chain consists of a single chain of nodes \( r \rightarrow X_{i+1}(r) \rightarrow X_{i+2}(r) \rightarrow \ldots, \) where the root node \( r \) represents the players of \( \{B^{(i)}, D^{(i)}\}. \) Thus there are two possible expansion chains rooted at level \( i, \) except for level 1, where \( A^{(1)} \) can also be the root node for a third expansion chain. The rest of the chain consists of a sequence of nodes such that node \( X_{i+k}(r) \) represents the support set of players of node \( X_{i+k-1}(r). \)

We first show below, a sufficient condition on \( \hat{C} \) for expansion chains to exist at any stage. For technical reasons, we will use \( l^*_1 = \log_2 (L^* - 1). \)

**Lemma 4.3:** Given a non-empty player set \( X^i \in \{A^{(i)}, B^{(i)}, D^{(i)}\}, \) either there exists an expansion chain rooted at \( X^i \) or the players of \( X^i \) are on the expansion chain of other players for all stages \( i: 1 \leq i \leq \hat{C} - l^*_i - 11. \)

**Proof:** To prove the existence of expansion chains at any stage \( i, \) we need to show that the set of players \( X^{(i)} \) is not self-sufficient. Consider each of the possible elements of \( X^{(i)} \) separately. First consider the set \( D^{(i)}. \) Clearly, \( D^{(i)}'s \) equilibrium is not self-sufficient since the maximum congestion experienced by players in \( D^{(i)} \) is \( \hat{C} - i - l^*_i - 2 \) and the maximum cost of an optimal path composed exclusively of edges from \( D^{(i)} \) is \( (L^* - 1 + 1) \cdot 2^{\hat{C} - i - l^*_i - 2} = 2^{\hat{C} - i - 2} + 2^{\hat{C} - i - 1} - 2, \) which is strictly less than the minimum required cost of an optimal path \( \Phi(P^*) + 1. \)

Next consider the set \( B^{(i)}. \) Assume for purposes of contradiction that \( B^{(i)} \) is self-sufficient, i.e there are a sufficient number of edges composed exclusively of players in \( B^{(i)} \) that are also on all the optimal paths of \( B^{(i)} \) and each optimal path has cost at least \( \Phi(P^*) + 1. \) Let \( B^{(j)} \) denote the edges of congestion \( \hat{C} - i - j \) composed exclusively of players in \( B^{(i)}, \) where \( 0 \leq j \leq \hat{C} - i - 1. \) Note that a single player in \( B^{(i)} \) may have several edges across different \( B^{(j)}'s. \) Each edge of \( B^{(j)} \) contributes \( 2^{\hat{C} - i - j} \) to the total cost of each of the \( \hat{C} - i - j \) players on the edge. Since the total cost of each player in \( B^{(i)} \) is bounded by \( 2^{\hat{C}+1}, \) we must have

\[
\sum_{j=0}^{\hat{C}-i-1} |B^{(j)}| (\hat{C} - i - j) 2^{\hat{C} - i - j} \leq |B^{(i)}| 2^{\hat{C}+1} \leq \sum_{j=0}^{l^*_i + 2} |B^{(j)}| \left( \frac{\hat{C} - i - j}{2^{l^*_i + 1}} \right) \leq |B^{(i)}| \tag{1}
\]

Since \( B^{(i)} \) is in equilibrium, each of the \( |B^{(j)}| \) optimal paths has cost \( > \Phi(P^*) \). For \( j \geq 1, \) each edge \( e \in B^{(j)} \) on an optimal path \( P^*_e \) contributes \( \Phi(P^*)/2^{l^*_i + 1} \) towards the cost of this path. (Each edge in \( B^{(i)} \) contributes \( \Phi(P^*) ). \) Now using the fact that
\(B^{(i)}\)'s equilibrium is self-contained, we must have
\[
\sum_{e \in B_{0}^{(i)}} \Phi(P^{*}) + \sum_{j=1}^{\hat{C}-i-1} \sum_{e \in B_{j}^{(i)}} \frac{\Phi(P^{*})}{2^{j-1}} > \sum_{P^{*}_{opt}} \Phi(P^{*})
\]
\[
\equiv |B_{0}^{(i)}| + \sum_{j=1}^{\hat{C}-i-1} |B_{j}^{(i)}| > |B^{(i)}| \tag{2}
\]

We note the following: edges of congestion \(\leq \hat{C} - i - l^{*} - 3\) must account for less than half the cost of any optimal path on which they are present. The maximum contribution of such edges over \(L^{*}-1\) edges of the optimal path is \(\Phi(P^{*})/2\), implying that there must be one edge of higher congestion \((\geq \hat{C} - i - l^{*} - 2)\) that contributes more than half of the required total cost \(\geq \Phi(P^{*})+1\). Thus we must have
\[
\sum_{j=i+1}^{\hat{C}-i-3} |B_{j}^{(i)}| < \frac{|B_{0}^{(i)}| + \sum_{j=1}^{i+2} |B_{j}^{(i)}|}{2^{j-1}} \tag{3}
\]
and therefore Eq. 2 becomes
\[
|B_{0}^{(i)}| + \sum_{j=1}^{i+2} |B_{j}^{(i)}| > |B^{(i)}|/2 \tag{4}
\]
Comparing Eq. 4 with Eq. 1 we get
\[
2|B_{0}^{(i)}| + \sum_{j=1}^{i+2} |B_{j}^{(i)}| \geq \sum_{j=0}^{i+2} |B_{j}^{(i)}| (\hat{C} - i - j)
\]
or simplifying
\[
\sum_{j=0}^{i+2} |B_{j}^{(i)}| \cdot (8 + j - (\hat{C} - i)) \geq 0 \tag{5}
\]
Since \(|B_{j}^{(i)}| > 0\) for at least some \(j : 1 \leq j \leq i + 2\), Eq. 5 is impossible for \((\hat{C} - i) > i + 10\), which contradicts the assumption that \(B^{(i)}\) is self-sufficient.

Finally for the case of players from \(A^{(i)}\), each subset of \(\hat{C} - i + 1\) players shares an edge. Thus the maximum number of optimal edges available from within the set is \(|A^{(i)}|/(\hat{C}-i+1)\). Since this is much less than the number of optimal paths \(|A^{(i)}|\), players in \(A^{(i)}\) are also not self-sufficient.

Concluding, none of the player sets \(A^{(i)}, B^{(i)}, D^{(i)}\) are self-sufficient and hence either these players are on the expansion chains of some other players or there are expansion chains rooted at these players in stage \(i : 1 \leq i \leq \hat{C} - l^{*} - 11\).

The above lemma guarantees the existence of at least one expansion chain rooted at stage 1 when \(C = O(l^{*})\). We now want to find the minimum number of edges required to support the game with equilibrium cost \(2^{C}\). This corresponds to finding the smallest expansion chain rooted at stage 1. By our definition, an expansion chain consists of new players occupying the expansion edges of players on the previous levels. It would seem that chains should consist of type \(B\) players since they occupy multiple edges and thus fewer players are required. However as the lemma below shows it is players of type \(A\) that minimize the expansion edges.

Consider an arbitrary player \(\pi\) of type \(B\) occupying edges \(E = \{e_{1}, e_{2}, \ldots, e_{k}\}\) of non-increasing congestion \(c_{1} \geq c_{2} \geq \ldots \geq c_{k}\) that are optimal edges (expansion edges) of other players, where we assume maximum congestion \(c_{1} \geq 2\). We want to answer the following question: Is there an alternate equilibrium/game containing player(s) with the same total equilibrium cost as \(\pi\), but requiring fewer edges to support this equilibrium cost. Note that when comparing these two games, the actual routing paths (i.e. source-destination) do not have to be the same. All we need to show is the existence of an alternate game (even with different source-destination pairs for the players) that has the same equilibrium cost.

In particular, consider an alternate game \(G^{'}\) in which \(\pi\) is replaced by a set \(P = \{\pi_{1}, \pi_{2}, \ldots, \pi_{k}\}\) of type \(A\) players occupying single edges of congestion \(c_{1}, c_{2}, \ldots, c_{k}\), where \(\pi\) and the set \(P\) are also in equilibrium in their respective games. The equilibrium cost of \(P\) and set \(P\) is the same \((\sum_{j=1}^{k} 2^{c_{j}})\) as they are occupying edges of the same congestion. Since both \(A\) in game \(G\) and the set of players \(P\) are in equilibrium and occupying expansion edges of other players in their respective games, \(C^{*} = 1\) implies they must have their own expansion edges in their respective games. Suppose we can show that the number of expansion edges required by the \(k\) players in \(P\) is at most those required by the single player of type \(B\). Since \(\pi\) is an arbitrary type \(B\) player, this argument applied recursively implies that all expansion edges in the game \(G\) should be occupied by type \(A\) players to minimize the total number of expansion edges. Thus we will have shown that any equilibrium with cost \(\hat{C}\) can be supported with fewer total players if they are of type \(A\) than if they are of type \(B\). Let \(\pi^{*}\) and \(P^{*}\) denote the expansion edges of \(\pi\) and the set \(P\) respectively.

**Lemma 4.4:** \(|P^{*}| \leq |\pi^{*}|\) for arbitrary players \(\pi\) and set \(P\) with the same equilibrium cost.

**Proof:** We prove this by strong induction on the length of player \(\pi\)'s path. For the basis, assume player \(\pi\) is on path \((e_{1}, e_{2})\) of length 2 in \(G\), with edges of congestion \(c_{1}\) and \(c_{2}\) respectively, where \(c_{1} \geq c_{2}\). Simultaneously consider two players \(\pi_{1}\) and \(\pi_{2}\) on single edges in game \(G^{'}\) with respective costs \(2^{c_{1}}\) and \(2^{c_{2}}\). We need to show that every possible optimal path (i.e. expansion edges) for \(\pi\) in \(G\) has two equivalent optimal paths (of the same or lower total cost) for the two players \(\pi_{1}\) and \(\pi_{2}\) in \(G^{'}\).

Suppose the optimal path of \(\pi\) is \(\pi^{*} = (e_{1}^{*}, e_{2}^{*}, \ldots, e_{m}^{*})\) in non-increasing order of congestion \(c_{1}^{*} \geq c_{2}^{*} \geq \ldots \geq c_{m}^{*}\). Consider two cases:

**Case 1:** \(c_{1}^{*} < c_{1}^{*}\): Since \(\pi\) is in equilibrium, \(\sum_{i=1}^{m} 2^{c_{i}^{*}} \geq (2^{c_{1}} + 2^{c_{2}})/2\). Since \(c_{1}^{*} < c_{1}\), there exists \(c_{j}^{*}\) such that \(\sum_{i=1}^{j} 2^{c_{i}^{*}} = 2^{c_{1}^{*}}\). Hence optimal path \(\pi^{*}\) can be partitioned into two paths, \(\pi_{1}^{*} = (e_{1}^{*}, \ldots, e_{j}^{*})\) and \(\pi_{2}^{*} = (e_{j+1}^{*}, \ldots, e_{m}^{*})\) with costs \(C(\pi_{1}^{*}) = 2^{c_{1}^{*}} - 1\) and \(C(\pi_{2}^{*}) \geq 2^{c_{1}^{*}} - 1\). Thus the edges of \(\pi_{1}^{*}\) and \(\pi_{2}^{*}\) can serve as expansion edges for \(\pi_{1}\) and \(\pi_{2}\) in alternate game \(G^{'}\) with appropriate endpoints, specifically, the endpoints of \(\pi_{1}^{*}\) and \(\pi_{2}^{*}\) will be the same as the endpoints
of edge $e_1$ and $e_2$ in $G'$. Hence $|P^*| = |\pi^*|$ in this case as desired.

Case 2 $c^*_i \geq c_1$: There are at least $c_1 \geq 2$ players on player $\pi$’s optimal path with costs $\geq 2^{c_1}$. Since $C^* = 1$, these players must have independent optimal paths of cost $\geq 2^{c_1}-1$. Hence at least $c_1 \geq 2$ such optimal paths are needed to support $\pi$ in game $G$. In contrast, in game $G'$, the two players $\pi_1$ and $\pi_2$ can be supported by two edges of congestion $c_1-1$ and $c_2-1$, respectively. Hence $|P^*| = 2 \leq |\pi^*|$ in this case as well.

For the inductive hypothesis assume $|P^*| \leq |\pi^*|$ for all paths up to length $k > 2$. Consider player $\pi$ occupying edges of non-increasing congestion $c_1, \ldots, c_{k+1}$ in $G$ whose optimal path has edges of non-increasing congestion $c_1, \ldots, c_m$. As before consider two cases, Case 1 $c_1^* < c_1$: let $j_1$ and $j_2$ be the indices such that 1) $\sum_{i=1}^{j_1} 2^{c_i} = (2^{c_1} + 2^{c_2})/2$, and 2) $\sum_{j_2}^{j_1} 2^{c_i} = (\sum_{i=1}^{k+1} 2^{c_i})/2$. Note that since $c_1^* < c_1$, indices $j_1$ and $j_2$ exist with $j_1 < j_2 \leq m$. Instead of player $\pi$ consider two new players $P_1$ and $P_2$, where $P_1$ occupies two edges of congestion $c_1$ and $c_2$ and $P_2$ occupies edges of congestion $c_3, c_4, \ldots, c_{k+1}$. From above, $j_2$ edges are required to satisfy $P_1$ and $P_2$ and $|\pi^*| = m \geq j_2$. Players $P_1$ and $P_2$ have path lengths $< k$ and thus by the inductive hypothesis, the number of expansion edges $P^*$ required to support $P_1$ and $P_2$ assuming they were replaced by type $A$ players satisfies $|P^*| \leq j_2 \leq |\pi^*|$ as desired.

Case 2 $c^*_i \geq c_1$: First assume $m \geq 2$. Let $j$ be the largest index such that $\sum_{i=1}^{j} 2^{c_i} \leq 2^{c_1}+1$. Clearly $j$ exists since $c_1^* \geq c_1$. Now instead of player $\pi$, consider two players $P_1$ and $P_2$ with $P_1$ occupying edges of congestion $c_1, c_2, \ldots, c_j$ and $P_2$ occupying edges of congestion $c_{j+1}, \ldots, c_m$, respectively. The edge of congestion $c_1^*$ can satisfy $P_1$ while the remaining edges of the optimal path $\pi^*$ can satisfy $P_2$. As in the previous case, players $P_1$ and $P_2$ have path lengths $< k$ and thus by the inductive hypothesis, the number of expansion edges $P^*$ required to support $P_1$ and $P_2$ assuming they were replaced by type $A$ players satisfies $|P^*| \leq |\pi^*|$ as desired. The case when $m = 1$ is omitted for brevity.

As a consequence of lemma 4.3 we have

Lemma 4.5: For $C > i+1$, the expansion chain rooted in stage 1 and occupying the minimum number of edges consists only of players of type $A$ (other than the root).

Next we derive the size of the smallest network required to support an equilibrium congestion of $C$. Without loss of generality, we assume there exists at least one type $A$ player in stage 1, i.e a single edge of congestion $C$ and derive the minimum chain rooted at $A^{(i)}$. From lemma 4.5 there exists an expansion chain rooted at $A^{(i)}$ with only type $A$ players. Among all such expansion chains, the one with the minimum number of players (equivalently edges, since each type $A$ player occupies a single edge) is defined below.

Theorem 4.6: $E_{C_{\text{min}}}$, the expansion chain with minimum number of edges that supports a self-sufficient equilibrium rooted at $A^{(i)}$ is defined by $E_{C_{\text{min}}} : A^{(i)} \rightarrow A^{(i+2)} \rightarrow A^{(2i+3)} \rightarrow \ldots \rightarrow A^{(C-1)}$. Every player in $E_{C_{\text{min}}}$ has an optimal path whose length is the maximum allowed $L^*$. The depth of chain $E_{C_{\text{min}}}$ is $O(C/l^*)$.

For technical reasons, we don’t terminate $EC_{\text{min}}$ with players from $A^{(C)}$ i.e single edges of congestion 1. Such a network can be shown to be unstable (i.e no equilibrium exists). Rather, the optimal paths of players from $A^{(C-1)}$ (i.e with player cost 4) are of length 2 with congestion 0 in $G$. This does not affect our count of the total number of edges required to derive the PoA below. We need a lower bound on the number of edges to derive an upper bound on the PoA, so (under)counting $E_{C_{\text{min}}}$ only up to stage $A^{(C-1)}$ is acceptable for our purposes.

To prove this theorem, we need a couple of technical lemmas which determine the minimum rate of expansion of an expansion chain. We describe these lemmas using the preliminary setup below. Let $\pi$ denote the set of $C-i+1$ players occupying a single edge in $A^{(i)}$, for some $i \geq 1$. Let $\pi_m \in \pi$ denote an arbitrary player with $\pi_m = (e_1, e_2, \ldots, e_k)$ denoting $\pi_m$’s optimal path, where $k \leq L^*$. For the moment, assume all edges on $\pi_m^*$ have the same congestion $c$. We first note that the largest stage from which type $A$ players can support $\pi_m$ is $i + l^* + 1$ since the player cost is $PC_m = 2^{C-m+1}$ and we must have $k \cdot 2^c \geq 2^{C-m}$. Using $k \leq L^*$, we must have congestion $c \geq C - i - l^*$ and the largest stage where this is possible is stage $i + l^* + 1$. Next consider the two (partial) expansion chains $EC_1: \pi \rightarrow A^{i+l^*+1}$ and $EC_2: \pi \rightarrow A^{i+l^*+1}$, where $1 \leq j \leq l^*$. We evaluate both chains at stage $i + l^* - 1$.

Let $|EC_1|$ and $|EC_2|$ denote the number of edges in the respective chains. Then we have,

Lemma 4.7: $|EC_1| \leq |EC_2|$, i.e expanding directly to the $l^* + 1$th succeeding stage is cheaper than expanding via an intermediate stage.

Proof: First consider $EC_1$. Since $|\pi| = C - i + 1$ and $C^* = 1$, there are $C - i + 1$ optimal paths at the first expansion stage of $EC_1$. Each optimal path length is the longest allowed i.e $2^i$. Clearly $C - i - l^*$ players on each edge of such path are enough to support the equilibrium cost of $\pi$. Thus the total of expansion edges in $EC_1$ is $(C - i + 1)2^{l^*}$.

For $EC_2$, again there are $C - i + 1$ optimal paths at the first expansion stage. However each edge of each optimal path now has congestion $C - i - l^* + 1$. Each optimal path must have length $l \geq 2^{i+1}$, since $l \cdot 2^{C-i-j+1} \geq 2^{C-i}$. Thus the total number of edges at this stage of $EC_2$ is at least $(C - i + 1)2^{i+1}$ while the total number of players is at least $(C - i - 1)(C - i - j + 1)2^{i+1}$. Each of these players has its own optimal path, with each edge on a path having congestion $C - i - l^*$, by definition of $EC_2$. The cost of each optimal path must be at least $2^{C-i-l^*}$ and so the length $l$ of each such path is at least $2^r - 1$ and the total number of players is at least $(C - i - 1)(C - i - j + 1)2^{i+1}$. Adding the edges in both stages and simplifying, we get the overall number of edges required to support the equilibrium of $\pi$ in $EC_2$ as

$$|EC_1| = (C - i + 1)2^{l^*}$$

Using the fact that $\hat{C} \geq i + j + 1$ by definition of expansion, we can see that the number of edges in $EC_2$ is at least as much as $|EC_1| = (C - i + 1)2^{l^*}$. 

Now consider the two (partial) expansion chains $EC_3 : \pi \rightarrow A^{(i+1+t^*)} \rightarrow A^{(i+1+t^*+k)}$ and $EC_4 : \pi \rightarrow A^{(i+1+t^*-j)} \rightarrow A^{(i+1+t^*+k)}$ where $1 \leq j \leq t^*$ and $j + k \leq t^* + 1$. (Note that the condition on $j + k$ is because one cannot directly expand beyond $t^* + 1$ stages due to the maximum optimal path length constraint). Then we have

**Lemma 4.8:** $|EC_3| \leq |EC_4|$. Expanding to larger stages (i.e. any stage after $i + t^* + 1$) is cheaper via stage $i + t^* + 1$ than via any intermediate stage before it. Equivalently (since larger stages imply expansion edges with lower congestions), when starting from stage $i$ it is cheapest to expand via the lowest possible congested edges which are in stage $i + t^* + 1$.

Due to space constraints, we skip the proof which counts edges similar to the previous lemma. The proof of Theorem 4.6 follows from lemmas 4.7 and 4.8 using the fact that starting from any stage $i$, the minimum cost expansion arises by selecting players from stage $i + t^* + 1$ to occupy expansion edges, with all optimal path lengths being the maximum possible $L^*$. Due to space constraints, we omit a formal proof by induction for showing that the number of expansion edges is minimized when all edges on an optimal path have the same congestion.

$EC_{\min}$ defined in Theorem 4.6 is also the minimum sized chain when the root players are from $B(1)$ or $D(1)$ although the number of edges required in the supporting graph is slightly different as we see later. In these cases, all stages (other than the root) in the minimum expansion chain consist of type $A$ players by lemma 4.5 and the proof of Theorem 4.6 is immediately applicable in choosing the specific indices of the expansion stages required to support the equilibrium). As we will show later, the PoA is maximized when the chain is rooted at $A(1)$.

**Theorem 4.9:** When $C^* = 1$, the upper bound $\kappa$ on the Price of Anarchy PoA of game $G$ is given by the minimum of $1$ $\kappa = O(1)$ or $2$ $\kappa\left(\log(\kappa L^*)\right) \leq \log L^* \cdot \log |E|$

**Proof:** To obtain an upper bound on the PoA, we want to find the smallest graph that can support an equilibrium cost of $2^\hat{C}$. Since the optimal path length $L^*$ can range from $O(1)$ to $O(|E|)$, we evaluate smallness both in terms of path length and number of edges.

Clearly, in the case when there is no expansion in $G$, the Price of Anarchy is $O(\log L^*)$, since by lemma 4.3, $C \leq t^* + 11$ and the PoA $C/C^* = O(\log L^*)$. Consider the case when there is expansion in the network i.e $\hat{C} >> \log L^*$. To bound the PoA, we will compute the number of edges in the minimum sized expansion chain. First assume there exists a single edge of congestion $\hat{C}$ (labeled as player set $\pi$) and exactly one expansion chain $EC_{\min} : \pi \rightarrow A^{(i+t^*)+2} \rightarrow A^{(2t^*+3)} \rightarrow \ldots$ in the graph i.e the only players in the graph are those required to be on the expansion edges of $EC_{\min}$. Using the standard notion of depth, the node corresponding to the player set $A^{(i+k(\hat{C}+1))}$ on $EC_{\min}$ is defined to be at depth $k$, with the root node at depth 0. At a given depth $k$, we define the following notations: Let $E_k$ denote the total number of expansion edges at depth $k$ (i.e the edges on comprising the optimal paths of players at depth $k - 1$), $P_k$ denote the minimum number of players who require players from $p_{k+1}$ on their optimal paths and $C_k$ denote the congestion on any expansion edge.

At depth 0, we have $E_0 = 1$ (a single edge $e$ of congestion $C_0 = \hat{C}$) Note that $P_0 = \hat{C} - 1$. Even though we have $\hat{C}$ players, one of these players might have its optimal path coincident with edge $e$. However for all $k > 1$, $P_k = E_kC_k$ since all the edges in $E_k$ are already optimal edges of players from $P_{k-1}$. We also have $C_k = \hat{C} - kl^* - k$ (by definition of type $A$ congestion), and finally $E_k = P_{k-1}L^*$, since every packet in $P_{k-1}$ has its own optimal path ($C^* = 1$) and every optimal path on $EC_{\min}$ is of length $L^*$. Putting these together, we obtain a recursive definition of $E_k = (L^*)^kP_{k-1}\Pi_{l=1}^{k-1}C_{l-1}$. We terminate our evaluation of the expansion chain when expansion edges have a congestion of $2$, i.e $\hat{C} - kl^* - k = 2$ which implies a depth of $d = (\hat{C} - 2)/(l^* + 1)$.

For technical reasons, we don’t terminate the chain with players from $A(C^*)$ i.e single edges of congestion 1. Such a network can be shown to be unstable (i.e no equilibrium exists. Rather, the optimal paths of players from $A(C^* - 1)$ (i.e. with player cost 4) are of length 2 with congestion 0 in $\hat{G}$. This does not significantly affect our count of the total number of edges required to derive the PoA below.

Thus the total number of edges in $EC_{\min}$ is bounded by

$$|EC_{\min}| = 1 + (\hat{C} - 1)
\begin{align*}
n & \times (L^* + (L^*)^2(\hat{C} - l^* - 1) + \\
& \ldots + (L^*)^d_{l=1}((\hat{C} - l^* - t) \\
& \leq \kappa(\log(\kappa L^*)) - 1) \leq \log L^* \cdot \log |E|
\end{align*}$$

Hence the PoA is bounded by a polylog function of $\log |E|$ in the worst case.

Can we get a larger upper bound on the PoA if the expansion chain is rooted at $B(1)/D(1)$ instead of $A(1)$? To examine this, let $\hat{C} - q$ be the largest congestion in $\hat{G}$, $q > 0$. We need $2^q$ such edges in order to satisfy the maximum player cost of $2^\hat{C}$. All these edges can be used as expansion edges for other players. From the analysis in Theorem 4.9 we note that expansion between stages occurs at a factorial rate. Thus using these $2^q$ edges as high up in the chain as possible (thereby reducing the need for new expansion edges) will minimize the expansion rate. The best choice for $q$ then is $l^*$. In this case, we have a single player $\pi_m$ in equilibrium in $\hat{G}$, occupying $L^*$ edges of congestion $\hat{C} - l^*$. These $L^*$ edges are also the optimal edges of $\pi_m$, i.e its equilibrium and optimal paths are identical. Hence the first stage of expansion in this chain is for the $L^*(\hat{C} - l^* - 1)$ players on the $L^*$ edges of $\pi_m$. From this point on the minimum sized chain for this graph is identical to the minimum sized chain $EC_{\min}$ defined above. The total number of edges in this chain can be computed in a manner similar to above. While the number of edges is smaller than $EC_{\min}$, it can be shown that the PoA is also smaller.
\[ \hat{C} - l^* \text{. Hence the upper bound on the } PoA \text{ is obtained using an expansion chain rooted at } A^{(1)}. \]

So far we have assumed the optimal bottleneck congestion \( C^* = 1 \) in our derivations. We now show that increasing \( C^* \) decreases the PoA and hence the previous derivation is the upper bound. We first evaluate the impact of expansion chains. Having \( C^* > 1 \) implies that more players can share expansion edges and thus the rate of expansion as well as the depth of an expansion chain (if it exists) should decrease. We first show that expansion chains exist even for arbitrary \( C^* = M \).

**Lemma 4.10:** Given a non-empty player set \( X^{(i)} \in \{A^{(i)}, B^{(i)}, D^{(i)}\} \), either there exists an expansion chain rooted at \( X^{(i)} \) or the players of \( X^{(i)} \) are on the expansion chain of other players for all stages \( i : \hat{C} - i > 8M + l^*_i + 2 \).

**Proof:** We provide a brief outline of the proof. First consider the case of players from \( A^{(i)} \). As before, the maximum number of optimal edges available from within the set is \(|A^{(i)}|/(\hat{C} - i + 1)\). However each group of \( M \) players could have their optimal paths (of length one) on one such edge. Thus the number of distinct optimal paths (edges) required is only \(|A^{(i)}|/M\). If \(|A^{(i)}|/M \leq |A^{(i)}|/(\hat{C} - i + 1)\) or equivalently \( \hat{C} - i \leq M - 1 \), then the players in \( A^{(i)} \) are in a self-sustained equilibrium. This is not true for the given value of \( i \) in the lemma and hence there must be an expansion chain rooted at \( A^{(i)} \). Similarly for the case of players from \( B^{(i)} \), the main modification from \(4.3\) is in Eq. \(2\) which now becomes

\[ |B^{(i)}| \geq |B^{(i)}|/M \]

(10)

for making \( B^{(i)} \) self-sustained since the set of \( B^{(i)} \) players only need \(|B^{(i)}|/M\) optimal paths. Following the same derivation as in lemma \(4.3\), Eq. \(5\) becomes

\[ \sum_{j = 1}^{l^*_i + 2} |B^{(i)}| / 2j + 1 \cdot (8M + j - (\hat{C} - i)) \geq 0 \]

(11)

For the given values of \( i \) and \( 1 \leq j \leq l^*_i + 2 \), this is impossible and hence \( B^{(i)} \) must participate in an expansion chain. The arguments for \( D^{(i)} \) are similar to lemma \(4.4\). □

Similarly Lemmas \(4.4\) and \(4.5\) can be suitably modified and the minimum sized chain in this case has the same structure as defined in Theorem \(4.6\). Analogous to the \( C^* = 1 \) case, the maximum PoA occurs when \( EC_{min} \) is rooted at \( A^{(1)} \). We calculate this PoA with \( C^* = M \), below.

**Theorem 4.11:** When \( C^* = M \), the upper bound \( \kappa \) on the Price of Anarchy PoA of game \( G \) is given by the minimum of 1) \( \kappa = O(\log L^*/M) \) or 2) \( \kappa(\log(L^*\kappa)) \leq \frac{l^* \log |E|}{M} \).

**Proof:** Suppose \( \hat{C} \) is such that there is no expansion in \( G \). This implies that \( \hat{C} \leq 8M + l^*_1 + 3 \). The PoA is \( \hat{C}/C^* \) which can be seen to be \( O(\log L^*/M) \). Conversely, if there is expansion we have the following: At depth 0, \( E_0 = 1 \), \( C_0 = \hat{C} \) and \( P_0 = \hat{C} - M \) since upto \( M \) players may have this edge as their optimal. As before \( C_k = \hat{C} - k l^* - k \) and \( P_k = E_k C_k \).

However, now \( E_k \) the number of expansion edges at depth \( k \) becomes \( E_k = P_{k-1} L^*/M \) since upto \( M \) players can share the same optimal path. Using a similar derivation as before we get, \( E_k = ((L^*)^k/M^{k-1}) \cdot ((C/M) - 1) \prod_{t=1}^{k-1} C_t \) which after some algebraic manipulation leads to

\[ |E_{C_{min}}| \geq \left( \frac{\hat{C}^C \sqrt{C(L^*)^C}}{M^C e^C} \right)^{l^*} \]

(12)

Substituting \( \kappa = \hat{C}/M \) and simplifying, we get \( l^* \log |E| \geq \hat{C}(\log \kappa + l^* - 1) \) which leads to

\[ \kappa(\log(L^*\kappa)) \leq \frac{l^* \log |E|}{M} \]

(13)

It can be seen that the PoA decreases with increasing optimal congestion \( M \).

**V. Conclusions**

In this work we have considered exponential bottleneck games with player utility costs that are exponential functions on the congestion of the edges of the players paths. The social cost is \( C \), the maximum congestion on any edge in the graph. We show that the price of anarchy is poly-log with respect to the size of the game parameters: \( O(\log L \cdot \log |E|) \), where \( L \) is the largest path length in the players strategy sets, and \( E \) is the set of edges in the graph.

![Fig. 2. High price of anarchy with social cost k](image-url)
the players have source \( u \) and destination \( v \) which are connected by edge \( e \). The graph consists of \( k-1 \) edge-disjoint paths from \( u \) to \( v \) each of length \( k \). There is a Nash equilibrium, depicted in the top of Figure 2, where every player chooses to use a path of length 1 on edge \( e \). This is an equilibrium because the cost of each player is \( k \), while the cost of every alternative path is also \( k \). Since the congestion of edge \( e \) is \( k \) the social cost is \( k \). The optimal solution for the same routing problem is depicted in the bottom of Figure 2, where every player uses a edge-disjoint path and thus the maximum congestion on any edge is 1. Therefore, the price of anarchy is at least \( k \). Since we can choose \( k = \Theta(\sqrt{n}) \), where \( n \) is the number of nodes in the graph, the price of anarchy is \( \Omega(\sqrt{n}) \).

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