Conditional Risk Minimization for Stochastic Processes

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Abstract

We study the task of learning from non-i.i.d. data. In particular, we aim at learning predictors that minimize the conditional risk for a stochastic process, i.e. the expected loss of the predictor on the next point conditioned on the set of training samples observed so far. For non-i.i.d. data, the training set contains information about the upcoming samples, so learning with respect to the conditional distribution can be expected to yield better predictors than one obtains from the classical setting of minimizing the marginal risk. Our main contribution is a practical estimator for the conditional risk based on the theory of non-parametric time-series prediction, and a finite sample concentration bound that establishes uniform convergence of the estimator to the true conditional risk under certain regularity assumptions on the process.

1 Introduction

One of the cornerstone assumptions in the analysis of machine learning algorithms is that the training examples are independently and identically distributed (i.i.d.). Interestingly, this assumption is hardly ever fulfilled in practice: dependencies between examples exist even in the famous textbook example of classifying e-mails into spam or ham. For example, when multiple emails are exchanged with the same writer, the contents of later emails depend on the contents of earlier ones. For this reason, there is growing interest in the development of algorithms that learn from dependent data and still offer generalization guarantees similar to the i.i.d. situation.

In this work, we are interested in learning algorithms for stochastic processes, i.e. data sources with samples arriving in a sequential manner. Traditionally, the generalization performance of learning algorithms for stochastic processes is phrased in terms of the marginal risk: the expected loss for a new data point that is sampled with respect to the underlying marginal distribution, regardless of which samples have been observed before. In this work, we instead are interested in the conditional risk, i.e. the expectation over the loss taken with respect to the conditional distribution of the next sample given the samples observed so far. For i.i.d. data both notions of risk coincide. For dependent data, however, they can differ drastically and the conditional risk is the more promising quantity for sequential prediction tasks. Imagine, for example, a self-driving car. At any point of time it makes its next decision, e.g. determines if there is a pedestrian in front of the vehicle based on the image from camera. A typical machine learning approach (based on the marginal risk) would use a single classifier that works well on average. However, choosing different classifiers at each step such that they are adapted to work well in the current conditions (based on the conditional risk) is clearly beneficial in this case.

There are two main challenges when trying to learn predictors of low conditional risk. First, the conditional distribution typically changes in each step, so we are trying to learn a moving target. Second, we cannot make use of out-of-the-box empirical risk minimization, since that would just lead to predictors of low marginal risk.
Our main contributions in this work are the following:

- a non-parametric **empirical estimator of the conditional risk** with finite history,
- a proof of **consistency** under mild assumptions on the process
- a **finite sample concentration bound** that, under certain technical assumptions, guarantees and quantifies the uniform convergence of the above estimator to the true conditional risk.

Our results provide the necessary tools to theoretically justify and practically perform empirical risk minimization with respect to the conditional distribution of a stochastic process. To our knowledge, our work is the first one providing a consistent algorithm for this problem.

## 2 Risk minimization

We study the problem of **risk minimization**: having observed a sequence of examples, $S = (z_i)_{i=1}^N$ from a stochastic process, our goal is to select a predictor, $h$, of minimal risk from a fixed hypothesis set $\mathcal{H}$. The risk is defined as the expected loss for the next observation with respect to a given loss function $\ell : \mathcal{H} \times Z \to [0,1]$. For example, in a classification setting, one would have $Z = \mathcal{X} \times \mathcal{Y}$, where $\mathcal{X}$ are inputs and $\mathcal{Y}$ are class labels, and solve the task of identifying a predictor, $h : \mathcal{X} \to \mathcal{Y}$, that minimizes the expected $0/1$-loss, $\ell(h, z) = \mathbb{I}[h(x) \neq y]$, for $z = (x, y)$.

Different distribution lead to different definitions of risk. The simplest possibility is the **marginal risk**,

$$R_{\text{mar}}(h) = \mathbb{E}[\ell(h, z_{N+1})],$$

which has two desirable properties: it is in fact independent of the actual value of $N$ (for the type of the processes that we consider), and under weak conditions on the process it can be estimated by a simple average of the losses over the training set, i.e. the **empirical risk**,

$$\hat{R}_{\text{mar}}(h) = \frac{1}{N} \sum_{i=1}^N \ell(h, z_i).$$

On the downside, the minimizer of the marginal risk might have low prediction performance on the actual sequence because it tries to generalize across all possible histories, while what we care about in the end is the prediction only for one observed realization. For an i.i.d. process this would not matter, since any future sample would be independent of the past observations anyway. For a dependent process, however, the sequence $z_{N}^{N}$ (where $z_{i}^{j}$ is a shorthand notation for $(z_{i}, \ldots, z_{j})$) might carry valuable information about the distribution of $z_{N+1}$, see Section 5 for an example and numerical simulations.

In this work we study the **conditional risk** for a finite history of length $d$,

$$R(h, \bar{z}) = \mathbb{E}\left[\ell(h, z_{N+1}) | z_{N-d+1}^{N} = \bar{z}\right],$$

for any $\bar{z} \in Z^d$. Our goal is to identify a predictor of minimal conditional risk in the hypothesis set, i.e. to solve the following optimization problem

$$\min_{h \in \mathcal{H}} R(h, z_{N-d+1}^{N}).$$

Note that on a practical level this is a more challenging problem than marginal risk minimization: the conditional risk depends on the history, $z_{N-d+1}^{N}$, so different predictors will be optimal for different histories and time steps. However, this comes with the benefit that the resulting predictor is tuned for the actually observed history.

For better understanding let us consider a related problem of time series prediction that can be formulated as a conditional risk minimization. One can consider constant predictors and the loss should measure the distance of the prediction from the next value of the process, e.g. for a square loss this would mean minimizing $\mathbb{E}\left[(h - z_{N+1})^2 | z_{N-d+1}^{N}\right]$ over $h \in Z$. Notice that there is a
big difference from the standard time series approaches to prediction, where one minimizes a fixed (not changing with time) measure of risk over predictors that can take a finite history into account to make their predictions, which can be written as $\mathbb{E} \left[ (h(z_{N-d+1}^N) - z_{N+1})^2 \right]$ with $h: \mathbb{Z}^d \rightarrow \mathbb{Z}$. In this way we choose a fixed function $h$ based on the data and use it henceforth. In our case, at each step we try to find a (simpler) predictor that minimizes the risk for this particular step, meaning that we can perform optimization over less complex predictors, but we need to recompute it at every step.

3 Related work

While statistical learning theory was first formulated for the i.i.d. setting (Vapnik and Chervonenkis, 1971), it was soon recognized that extension to non-i.i.d. situations, in particular many classes of stochastic processes, were possible and useful. As in the i.i.d. case, the core of such results is typically formed by a combination of a capacity bound on the class of considered predictors and a law of large numbers argument, that ensures that the empirical average of function values converges to a desired expected value. Combining both, one obtains, for example, that empirical risk minimization (ERM) is a successful learning strategy.

Most existing results study the situation of stationary stochastic processes, for which the definition of a marginal risk makes sense. The consistency of ERM or similar principles can then be established under certain conditions on the dependence structure, for example for processes that are $\alpha$-, $\beta$- or $\phi$-mixing (Yu, 1994; Karandikar and Vidyasagar, 2002; Steinwart and Christmann, 2009; Zou et al., 2009), exchangeable, or conditionally i.i.d. (Berti and Rigo, 1997; Pestov, 2010). Asymptotic or distribution dependent results were furthermore obtained even for processes that are just ergodic (Adams and Nobel, 2010), and Markov chains with countably infinite state-space (Gamarnik, 2003).

All of the above works aim to study the minimizer of a long-term or marginal risk. Actually minimizing the conditional risk has not received much attention in the literature, even though some conditional notions of risk were noticed and discussed. The most popular objective is the conditional risk based on the full history, that is $\mathbb{E} [\ell(h, z_{N+1}) | z_N^s]$. For example, Pestov (2010) and Shalizi and Kontorovitch (2013) argue in favor of minimizing this conditional risk, but focus on exchangeable processes, for which the unweighted average over the training samples can be used for this purpose.

The following two papers focus on the variants of the conditioning, but consider the situations where the objective is close to the marginal risk. Kuznetsov and Mohri (2014) look at the minimization of $\mathbb{E} [\ell(h, z_{N+1}) | z_N^s]$ with integer gap $s$ for the non-stationary processes, but the convergence of their bound requires $s$ growing as the amount of data grows. Mohri and Rostamizadeh (2013) discuss the conditional risk based on the full history in the context of generalization guarantees for stable algorithms. Their proofs require an assumption that one can freely remove points from the conditioning set without changing the distribution. In our notation this assumption means $\mathbb{E} [\ell(h, z_{N+1}) | z_N^N] = \mathbb{E} [\ell(h, z_{N+1}) | z_N^{N-s}]$ for integer $s$’s. This again allows the conditional risk to be approximated by the marginal one by separating the point in the loss from the history by an arbitrarily large gap. In both cases, this makes the problem much easier for mixing processes, since for big values of $s$, $z_{N+s}$ is almost independent of $z_N^N$. In contrast to these two works, in our setting the conditional risk is indeed different from the marginal one (see Figure 1 for an example).

Agarwal and Duchi (2013) extend online-to-batch conversion to mixing processes. The authors construct an estimator for the marginal risk, and then show that it can also be used for an average over $m$ future conditional risks. In our notation this corresponds to $\frac{1}{m} \sum_{i=1}^m \mathbb{E} [\ell(h, z_{N+i}) | z_N^N]$. Their results are based on the idea of separating the point in the loss and the history by a large enough gap. Similarly to the above papers, the convergence only holds for $m \rightarrow \infty$, where the average conditional risk converges to the marginal one, while our setting corresponds the case without any gap, $m = 1$, with conditioning on a finite history.

1 Apparently, the need for this assumption was realized only after publication of the JMLR paper of the same title. Our discussion is based on the PDF version of the manuscript from the author homepage, dated 10/10/13.
Wintenberger (2014) introduces a novel online-to-batch conversion technique to show bounds in the regret framework for the cumulative conditional risk, defined as a sum of conditional risks, \( \sum_{i=1}^{N} \mathbb{E}[\ell(h, z_{i+1}) | z_{1}^{i}] \). Our setting is a harder version of this problem, when we need to minimize each summand separately, not only the whole sum.

The work of Kuznetsov and Mohri (2015) is the most related to ours. They aim to minimize the conditional risk based on the full history by using a weighted empirical average, in the spirit of our estimator. They provide a generalization bound for fixed, non-random weights and, based on their results, they derive a heuristic procedure for finding the weights without guarantees of convergence. The main difference of our work is that we provide a data-dependent way to choose weights with the proof of the convergence.

We are not aware of any work that provides a convergent algorithm for the conditional risk based on the full or finite history. In our work we focus on the finite history and in Section 6 we discuss the relation between the two notions.

On a technical level, our work is related to the task of one-step ahead prediction of time series (Modha and Masry 1996, 1998; Meir 2000; Alquier and Wintenberger 2012). The goal of these methods is to reason about the next step of a process, though not in order to choose a hypothesis of minimal risk, but to predict the value for the next observation itself. Our work on empirical conditional risk is inspired by this school of thought, in particular on kernel-based nonparametric sequential prediction (Biau et al., 2010).

4 Results

In this section we present our results together with the assumptions needed for the proofs.

When we want to perform the optimization (4) in practice, we do not have access to the conditional distribution of \( z_{N+1} \). Thus, we take the standard route and aim at minimizing an empirical estimator, \( \hat{R} \), of the conditional risk.

Our first contribution is the definition of a suitable conditional risk estimator when the process takes values in \( Z \subseteq \mathbb{R}^k \). The estimator is based on the notion of a smoothing kernel.

**Definition 1.** A function \( K : \mathbb{R}^{kd} \rightarrow \mathbb{R}_+ \) is called a smoothing kernel if it satisfies

1. \( \int_{\mathbb{R}^{kd}} K(\bar{z}) d\bar{z} = 1 \),
2. \( K \) is bounded by \( K_1 \),
3. \( \forall i, j = 1, \ldots, kd : \int_{\mathbb{R}^{kd}} \bar{z}_i K(\bar{z}) d\bar{z} = 0 \) and \( \int_{\mathbb{R}^{kd}} \bar{z}_i \bar{z}_j K(\bar{z}) d\bar{z} \leq K_2 \),
4. \( K \) is Lipschitz continuous of order \( \gamma \) with Lipschitz constant \( L \).

A typical example is a squared exponential kernel, \( e^{-\gamma \| \bar{z} \|^2} \), but many other choices are possible. We can now define our estimator.

**Definition 2.** For a smoothing kernel function, \( K \), and a bandwidth, \( b > 0 \), we define the empirical conditional risk estimator

\[
\hat{R}(h, \bar{z}) = \frac{\hat{q}(h, \bar{z})}{\hat{p}(\bar{z})},
\]

for

\[
\hat{q}(h, \bar{z}) = \frac{1}{nb^d} \sum_{i \in I} \ell(h, z_{i+1}) K( (\bar{z} - z_{i-d+1}^{i}) / b )
\]

[2]Here and later in this section, as well as in Section 4, we use kernel only in the sense of kernel-based non-parametric density estimation, not in the sense of positive definite kernel functions from kernel methods.
and

\[ \hat{\beta}(\varepsilon) = \frac{1}{nb^d} \sum_{i \in I} K((\varepsilon - z_{i-d+1}^i)/b), \tag{8} \]

where \( I = \{d, d+1, \ldots, N-1\} \) is the index set of samples used and \( n = |I| \).

In words, the estimator is a weighted average loss over the training set, where the weight of each sample, \( z_{i+1} \), is proportional to how similar its history, \( z_{i-d+1} \), is to the target history, \( \varepsilon \). Similar kernel-based non-parametric estimators have been used successfully in time series prediction (Györfi et al., 2002). Note, however, that risk estimation might be an easier task than that, especially for processes of complex objects, since we do not have to predict the actual values of \( z_{N+1} \), but only the loss it causes for a hypothesis \( h \). In the self-driving car example, compare the pixel-wise prediction of the next image to the prediction of the loss that our classifier causes.

Our main result in this work is the proof that minimizing the above empirical conditional risk is a successful learning strategy for finding a minimizer of the conditional risk. The following well-known result (Vapnik, 1998) shows that it suffices to focus on uniform deviations of the estimator from the actual risk.

**Lemma 1.**

\[ R(h_S, z_{N-d+1}^N) - \inf_{h \in H} R(h, z_{N-d+1}^N) \leq 2 \sup_{h \in H} \left| R(h, z_{N-d+1}^N) - \hat{R}(h, z_{N-d+1}^N) \right|. \tag{9} \]

As our first result we show the convergence of such uniform deviations to zero. But before we can make the formal statement, we need to introduce the technical assumptions and a few definitions.

We assume that we observe data from a stationary \( \beta \)-mixing stochastic process \( \{z_i\}_{i=1}^{\infty} \) taking values in \( \mathcal{Z} = [0,1]^k \). Stationarity means that for all \( m \geq 1 \) the vector \( z_0^m \) has the same distribution as \( z_{s+1}^m \) for all \( s \geq 0 \). In order to quantify the dependence between the past and the future of the process, we consider mixing coefficients.

**Definition 3.** Let \( \sigma(z_i^j) \) be a sigma algebra generated by \( z_i^j \). Then the \( j \)-th \( \beta \)-mixing coefficient is

\[ \beta(j) = \sup_{i \in \mathbb{N}} \mathbb{E} \sup_{A \in \sigma(z_{i+j}^\infty)} |\mathbb{P}[A|z_i^j] - \mathbb{P}[A]|. \tag{10} \]

A process is called \( \beta \)-mixing if \( \beta(j) \to 0 \) as \( j \to \infty \). We call a process an exponentially \( \beta \)-mixing if \( \beta(j) \leq c_1 e^{-c_2 j} \) for some \( c_1, c_2 > 0 \). On a high level, a process is mixing if the head of the process, \( z_1^j \), and the tail of the process, \( z_{i+j}^\infty \), become as close to independent from each other as wanted when they are separated by a large enough gap.

Many classical stochastic processes are \( \beta \)-mixing, see (Bradley, 2005) for a detailed survey. For example, many finite-state Markov and hidden Markov models as well as autoregressive moving average (ARMA) processes fulfill \( \beta(j) \to 0 \) at an exponential rate (Athreya and Pantula, 1986), while certain diffusion processes are \( \beta \)-mixing at least with polynomial rates (Chen et al., 2010). Clearly, i.i.d. processes are \( \beta \)-mixing with \( \beta(j) = 0 \) for all \( j \).

To control the complexity of the hypothesis space we use covering numbers.

**Definition 4.** A set, \( V \), of \( \mathbb{R} \)-valued functions is a \( \theta \)-cover (with respect to the \( \ell_p \)-norm) of \( \mathcal{F} \subset \{ f : \mathcal{X} \to \mathbb{R} \} \) on a sample \( x_1, \ldots, x_n \) if

\[ \forall f \in \mathcal{F} \ \exists v \in V \ \left( \frac{1}{n} \sum_{i=1}^{n} |f(x_i) - v(x_i)|^p \right)^{1/p} \leq \theta. \tag{11} \]

The \( \theta \)-covering number of a function class \( \mathcal{F} \) on a given sample \( x_1, \ldots, x_n \) is

\[ N_p(\theta, \mathcal{F}, x^n) = \min \{ |V| : V \text{ is an } \theta \text{-cover w.r.t. } \ell_p \text{-norm of } \mathcal{F} \text{ on } x^n \}. \tag{12} \]

The maximal \( \theta \)-covering number of a function class \( \mathcal{F} \) is

\[ N_p(\theta, \mathcal{F}, n) = \sup_{x^n \in \mathcal{X}^n} N_p(\theta, \mathcal{F}, x^n). \tag{13} \]
Theorem 1. Assume that

- there exist $D_0, D_1 > 0$ such that $D_0 \lambda(B) \leq \mathbb{P} \left[ \mathbf{z}^i_{t-d+1} \in B \right] \leq D_1 \lambda(B)$ for $B \in \mathcal{B}(\mathbb{R}^d)$, where $\lambda$ is the Lebesgue measure on $\mathbb{R}^k$;
- for every hypothesis $h \in \mathcal{H}$, the conditional risk $R(h, z)$ is $L_R$-Lipschitz continuous in $z$.

Then

$$\sup_{h \in \mathcal{H}} \left| R(h, \mathbf{z}^N_{t-d+1}) - \hat{R}(h, \mathbf{z}^N_{t-d+1}) \right| \to 0 \text{ in probability} \quad (14)$$

if $b \to 0$ as $n \to \infty$ slowly enough (depending on the covering number of the hypothesis set and the mixing rate of the process). The same statement holds almost surely for an exponentially $\beta$-mixing process.

We will refer to the first assumption of Theorem 1 as smoothness and to the second one as robustness. The need for these assumptions is due to the fact that we use nonparametric estimates for the conditional risk, because as it is shown in [Györfi and Lugosi, 1992] the ergodicity (which is implied by mixing) itself is not enough to show even $L_1$-consistency of kernel density estimators. Because of this, additional assumption are required. Smoothness means that the marginal distribution of the process and the Lebesgue measure are mutually absolute continuous. This assumption, for example, satisfied for processes with a density, which is bounded from below away from 0. Caires and Ferreira (2005) argue that it implies some kind of recurrence of the process, that is that for almost every point in the support, the process visits its neighborhood infinitely often. The usage of local averaging estimation implicitly assumes the continuity of the underlying function, that is the robustness assumption. The proof would work with weaker, but more technical assumption, however, we stick to this, more natural one.

As a second result we establish the convergence rate of the estimator.

Theorem 2. Assume that

- there exist $D_0, D_1 > 0$ such that $D_0 \lambda(B) \leq \mathbb{P} \left[ \mathbf{z}^i_{t-d+1} \in B \right] \leq D_1 \lambda(B)$ for $B \in \mathcal{B}(\mathbb{Z}^d)$, where $\lambda$ is a Lebesgue measure on $\mathbb{R}^k$;
- the random vector $\mathbf{z}^i_{t-d+1}$ has a density $p$. Also, $q(h, z) = R(h, z)p(z)$ and $p(z)$ are both twice continuously differentiable in $z$ with second derivatives bounded by $D_2$;
- loss function $\ell$ is $L_H$-Lipschitz in the first argument.

Then the following holds for $t_1 = \frac{1}{2}(t_0 - K_2 D_2 d^2 b^2)$, $t_2 = t_1 b^d/(64 K_1 L_H)$, $t_3 = \left( \frac{3b}{64 d^2 t_1} \right)^{\frac{1}{2}}$ and any $\mu, a > 0$ such that $4\mu ad = N$:

$$\mathbb{P} \left[ \sup_{h, z} \left| R(h, z) - \hat{R}(h, z) \right| > t \right] \leq 32 \left( \frac{\sqrt{kd} t_3}{2} \right)^{kd} N_1(t_2, \mathcal{H}, n) e^{-\mu t_2^2 b^d/(2048 K_1^2)} + 4 \left( \frac{\sqrt{kd} t_3}{2} \right)^{kd} (\mu - 1) \beta(2ad). \quad (15)$$

$$\hat{R}(h, \mathbf{z}^N_{t-d+1}) \approx R(h, \mathbf{z}^N_{t-d+1}) \quad (16)$$

The first assumption in Theorem 2 is the smoothness condition of Theorem 1. The second one is a stricter version of robustness needed to quantify the convergence rate, we will refer to it as strong robustness. The last assumption is a standard way to relate the covering numbers of the induced space $\{\ell(h, \cdot) : h \in \mathcal{H}\}$ to $\mathcal{H}$ (such losses are sometimes called admissible).

As an example, let us consider a case of an exponentially $\beta$-mixing process, e.g. when $\beta(j) \leq e^{-j}$. For $\mathcal{H}$ with a finite fat-shattering dimension, $N_1(t_2, \mathcal{H}, n) = \text{poly}(n) = \text{poly}(N)$, and if we choose $\mu \approx N^{2/3}$, $2ad \approx N^{1/3}$ and $b = N^{-1/\beta}$, then the bound of Theorem 2 is approximately $\text{poly}(N)(e^{-c_1 N^{1/3}} + e^{-c_2 N^{1/3}})$ for a fixed $t$.

Before we present the proofs, we introduce some auxiliary results.
We start by splitting the index set \( \mathcal{I} \) of \( \mathbf{I} \) and the bound of the lemma is trivial in this case. Let \( y \) be some variable from \( \mathcal{J} \) above step, there is no function that takes variables from different blocks. Let \( \mathcal{J} \) differ from \( \mathcal{J} \) above, since here we split the variables themselves. Thanks to the Lemma 2 (Yu (1994), Corollary 2.7), For a \( \beta \)-mixing sequence of random variables \( \mathbf{z}_1^n \), let \( \mathbf{z}_1^n \) be its \((\mu,a)\)-independent block copy. Then for any measurable function \( g \) defined on every second block of \( \mathbf{z}_1^n \) of length \( a \) and bounded by \( B \), it holds

\[
\mathbb{E} [g(\mathbf{z}_1^n)] - \mathbb{E} [g(\mathbf{z}_1^n)] \leq B(\mu - 1)\beta(a). \tag{17}
\]

Note that an application of this lemma and the proof of Lemma 3 does not require \( 2\mu a = n \) to hold exactly. It is possible to work with \( \mu \) and \( a \) such that \( 2\mu a < n \) by putting all the remaining points into the last block. However, for the convenience of notations we will write equality.

The proof of Theorems 1 and 2 relies on Lemma 3, a concentration inequality that bounds the uniform deviations of functions on blocks of a \( \beta \)-mixing stochastic process. This lemma uses a popular independent block technique. However, it is not a direct application of the previous results (e.g. (Mohri and Rostamizadeh, 2009)) as a careful decomposition is required to deal with the fact the summands are defined on overlapping sets of variables.

**Lemma 3.** Let \( \mathcal{F} = \{ f : \mathcal{Z}_d \to [0,B] \} \) be a class of functions defined on blocks of \( d+1 \) variables. For any integers \( \mu, a \), such that \( 4\mu ad = N \), let \( \mathbf{z}_1^n \) be an \((\mu,2ad)\)-independent block copy of \( \mathbf{z}_1^n \). Then

\[
\mathbb{P} \left[ \sup_{f \in \mathcal{F}} \left| \frac{1}{N} \sum_{i \in \mathcal{I}} f(z_{i+d-1}) - \mathbb{E} [f(z_{i+d-1})] \right| > t / 4 \right] \leq 32B^2 \mathbb{E} \left[ \mathcal{N}_1 (1/64, \mathcal{F}, \{ z_{i+1}^{d+1}, i \in \mathcal{I} \}) \right] e^{-\mu t^2/(2048B^2)} + 4(\mu - 1)\beta(2ad). \tag{19}
\]

**Proof.** We start by splitting the index set \( \mathcal{I} = \{ d, d+1, \ldots, N-1 \} \) into two sets \( \mathcal{I}_1 \) and \( \mathcal{I}_2 \), such that \( \mathcal{I}_1 = \{ i \in \mathcal{I} : \lfloor \frac{i}{2} \rfloor \text{ is odd} \} \) and \( \mathcal{I}_2 = \{ i \in \mathcal{I} : \lfloor \frac{i}{2} \rfloor \text{ is even} \} \). Then, recalling that \( n = N - d - 1 \) and, hence, \( \frac{1}{n} = \frac{N - d - 1}{N} \leq \frac{2}{N} \) for \( N > 2(d+1), \)

\[
\mathbb{P} \left[ \sup_{f \in \mathcal{F}} \left| \frac{1}{N} \sum_{i \in \mathcal{I}_1} f(z_{i+d-1}) - \mathbb{E} [f(z_{i+d-1})] \right| > t / 4 \right] \leq 32B^2 \mathbb{E} \left[ \mathcal{N}_1 (1/64, \mathcal{F}, \{ z_{i+1}^{d+1}, i \in \mathcal{I}_1 \}) \right] e^{-\mu t^2/(2048B^2)} + 4(\mu - 1)\beta(2ad). \tag{20}
\]

Both summands can be bounded in the same way, so we focus on the first one. We are going to use the independent block technique due to Yu (1994). For this we choose \( \mu \) and \( a \) such that \( 4\mu ad = N \). We will split the sample \( \mathbf{z}_1^n \) into 2\( \mu \) blocks of 2\( ad \) consecutive points (Note that is a different splitting than the one above, since here we split the variables themselves). Thanks to the above step, there is no function that takes variables from different blocks. Let \( \mathcal{J} = \{ y_1, \ldots, y_n \} \), where \( y_k = ((2k - 1)2ad + 1, \ldots, (2k - 1)2ad + 2ad) \) and \( \mathcal{J} = \{ \bar{y}_1, \ldots, \bar{y}_n \} \), where \( \bar{y}_k = ((2k - 1)2ad + 1, \ldots, (2k - 1)2ad + 2ad) \). In this way, the blocks within \( \mathcal{J} \) and \( \mathcal{J} \) are separated by gaps of \( 2ad \) points. Note that thanks to the first splitting, we can rewrite

\[
\frac{1}{N} \sum_{i \in \mathcal{I}_1} f(z_{i+d-1}) = \frac{1}{N} \sum_{i \in \mathcal{I}_1 \cap \mathcal{J}^c} f(z_{i+d-1}) + \frac{1}{N} \sum_{i \in \mathcal{I}_1 \cap \mathcal{J}} f(z_{i+d-1}) \tag{22}
\]

\[
= \frac{1}{2\mu} \sum_{j=1}^{\mu} \frac{1}{2ad} \sum_{i \in \mathcal{I}_1 \cap \mathcal{J}^c} f(z_{i+d-1}) + \frac{1}{2\mu} \sum_{j=1}^{\mu} \frac{1}{2ad} \sum_{i \in \mathcal{I}_1 \cap \mathcal{J}} f(z_{i+d-1}) \tag{23}
\]

\[
= \frac{1}{2\mu} \sum_{j=1}^{\mu} \frac{1}{2ad} \sum_{i \in \mathcal{I}_1 \cap \mathcal{J}^c} f(z_{i+d-1}) + \frac{1}{2\mu} \sum_{j=1}^{\mu} \frac{1}{2ad} \sum_{i \in \mathcal{I}_1 \cap \mathcal{J}} f(z_{i+d-1}). \tag{24}
\]
where we defined composite hypotheses $F_f(y) = \frac{1}{2ad} \sum_{i \in I_1 \cap y} f(z_i^{i+1})$ with $z_y = \{z_i\}_{i \in y}$.

Then
\[
P \left[ \sup_{f \in \mathcal{F}} \left| \frac{1}{N} \sum_{i \in I_1} f(z_i^{i+1}) - \mathbb{E} [f(z_i^{i+1})] \right| > t/4 \right]
\leq 2P \left[ \sup_{f \in \mathcal{F}} \left| \frac{1}{\mu} \sum_{j=1}^{\mu} F_f(z_{y_j}) - \mathbb{E} [F_f(z_{y_j})] \right| > t/4 \right].
\]

(25)

Let $z_{y_1}, \ldots, z_{y_\mu}$ be the random variables having the same marginal distributions as $z_y$‘s, but drawn independently from each other. Using the fact that probability of an event is an expectation of the indicator of the same event, we can apply Lemma 2 to obtain
\[
P \left[ \sup_{f \in \mathcal{F}} \left| \frac{1}{\mu} \sum_{j=1}^{\mu} F_f(z_{y_j}) - \mathbb{E} [F_f(z_{y_j})] \right| > t/4 \right]
\leq \sum_{j=1}^{\mu} \left( \mathbb{E} \left[ F_f(z_{y_j}) \right] - \mathbb{E} [F_f(z_{y_j})] \right) + (\mu - 1)\beta(2ad).
\]

(26)

The first term can be bounded using the standard techniques for uniform laws of large numbers for i.i.d. random variables. We are going to use the bound in terms of covering numbers. Following the standard proof, e.g., (Györfi et al. [2002] Theorem 9.1), we obtain
\[
P \left[ \sup_{f \in \mathcal{F}} \left| \frac{1}{\mu} \sum_{j=1}^{\mu} F_f(z_{y_j}) - \mathbb{E} [F_f(z_{y_j})] \right| > t/4 \right]
\leq 8e^{-\mu t^2/(2048b^2)}.
\]

(27)

where $F(f)$ is a class of composite hypotheses. The only thing that is left is to connect the covering number of $F(f)$ to the covering number of $\mathcal{F}$. This follows from the fact that for any $f, g \in \mathcal{F}$ and any fixed blocks $y_1, \ldots, y_\mu$ on $z^n$:
\[
\frac{1}{\mu} \sum_{j=1}^{\mu} |F_f(y_j) - F_g(y_j)| \leq \frac{2}{n} \sum_{j=1}^{\mu} \sum_{i \in I_1 \cap y_j} \left| f(z_i^{i+1}) - g(z_i^{i+1}) \right| .
\]

(28)

\[
\leq \frac{2}{n} \sum_{i \in I_1} \left| f(z_i^{i+1}) - g(z_i^{i+1}) \right| .
\]

(29)

\[
\leq \frac{2}{n} \sum_{i \in I_1} \left| f(z_i^{i+1}) - g(z_i^{i+1}) \right| .
\]

(30)

\[
\leq \frac{2}{n} \sum_{i \in I_1} \left| f(z_i^{i+1}) - g(z_i^{i+1}) \right| .
\]

(31)

\[
\leq \frac{2}{n} \sum_{i \in I_1} \left| f(z_i^{i+1}) - g(z_i^{i+1}) \right| .
\]

(32)

\[
\leq \frac{2}{n} \sum_{i \in I_1} \left| f(z_i^{i+1}) - g(z_i^{i+1}) \right| .
\]

(33)

\[
\leq \frac{2}{n} \sum_{i \in I_1} \left| f(z_i^{i+1}) - g(z_i^{i+1}) \right| .
\]

(34)

\[
\leq \frac{2}{n} \sum_{i \in I_1} \left| f(z_i^{i+1}) - g(z_i^{i+1}) \right| .
\]

(35)

\[
\leq \frac{2}{n} \sum_{i \in I_1} \left| f(z_i^{i+1}) - g(z_i^{i+1}) \right| .
\]

(36)

\[
\leq \frac{2}{n} \sum_{i \in I_1} \left| f(z_i^{i+1}) - g(z_i^{i+1}) \right| .
\]

(37)

\[
\leq \frac{K_1}{nb^2} \sum_{i \in I_1} \left| f(h, z_i^{i+1}) - g(h, z_i^{i+1}) \right| .
\]

(38)
Lemma 4 shows how to express the concentration of the following decomposition
\[ \mathcal{N}_1(\varepsilon, \mathcal{F}, \{ \tilde{z}_{i+1}^t, i \in I \}) \leq \mathcal{N}_1(\varepsilon b^d / K_1, \mathcal{L}(\mathcal{H}), \{ \tilde{z}_{i+1}, i \in I \}), \]
where \( \mathcal{L}(\mathcal{H}) = \{ f(z) = \ell(h, z) : h \in \mathcal{H} \} \). Second, by the Lipschitz property of the loss function we get:
\[ \mathcal{N}_1(\varepsilon b^d / K_1, \mathcal{L}(\mathcal{H}), \{ \tilde{z}_{i+1}, i \in I \}) \leq \mathcal{N}_1(\varepsilon b^d / (K_1 L_H), \mathcal{H}, \{ \tilde{z}_{i+1}, i \in I \}). \]

Now we are ready to prove Theorem 1.

The proof is based on the argument of Collomb [1984] with appropriate modifications to achieve the uniform convergence over hypotheses.

**Theorem 1**. We start by reducing the problem to the supremum over the histories. Then we make the following decomposition
\[ \sup_{h, \tilde{z}} \left| R(h, \tilde{z}) - \tilde{R}(h, \tilde{z}) \right| \leq \frac{1}{\inf \mathbb{E}[\hat{p}(\tilde{z})]} (T_1 + T_2 + T_3), \]
where
\[ T_1 = \sup_{h, \tilde{z}} \left| \tilde{R}(h, \tilde{z})(\mathbb{E}[\hat{p}(\tilde{z})] - \hat{p}(\tilde{z})) \right|, \]
\[ T_2 = \sup_{h, \tilde{z}} \left| \hat{q}(h, \tilde{z}) - \mathbb{E}[\hat{q}(h, \tilde{z})] \right|, \]
\[ T_3 = \sup_{h, \tilde{z}} \left| \mathbb{E}[\hat{q}(h, \tilde{z})] - R(h, \tilde{z}) \mathbb{E} [\hat{p}(\tilde{z})] \right|. \]

First, note that by the smoothness assumption \( \mathbb{E}[\hat{p}(\tilde{z})] \) ≥ \( D_0 \int K(\tilde{z}) d\tilde{u} = D_0 \). A minor modification of Lemma 5 from Collomb [1984] coupled with robustness gives us the convergence of \( T_3 \).

Next, note that, by Lemma 5, \( \mathbb{P}[T_i > t] \leq N \mathbb{P}\left( \tilde{T}_i > t \right) \) for \( i = 1, 2 \), where
\[ \tilde{T}_1 = \sup_{h} \left| \mathbb{E}[\hat{p}(\tilde{z})] - \hat{p}(\tilde{z}) \right|, \]
\[ \tilde{T}_2 = \sup_{h} \left| \hat{q}(h, \tilde{z}) - \mathbb{E}[\hat{q}(h, \tilde{z})] \right|. \]

and \( N \) is a covering of a \( kd \)-dimensional hypercube with an appropriate ball width. Now, by Lemma 5 we get a bound on \( \tilde{T}_1 \) and \( \tilde{T}_2 \) and obtain the convergence in probability.

For exponentially β-mixing processes, the same Lemma gives an exponential bound on \( \tilde{T}_1 \) and \( \tilde{T}_2 \) and hence we get the almost sure convergence of \( T_1 \) and \( T_2 \) to 0 (by the Borel-Cantelli lemma).

The proof of the convergence rate requires a bit different decomposition than in Theorem 1 and more delicate treatment of the terms using the additional assumptions. We introduce two further lemmas: Lemma 4 shows how to express the concentration of \( \tilde{R}(h, \tilde{z}) \) in terms of the concentration of \( \hat{q}(h, \tilde{z}) \) and \( \hat{p}(\tilde{z}) \). Lemma 4 uses covers to eliminate the supremum over \( \tilde{z} \).

**Lemma 4.** Assume smoothness and strong robustness. Then, for \( R = R(h, \tilde{z}) \), \( \tilde{R} = \tilde{R}(h, \tilde{z}) \), \( \hat{q} = \hat{q}(h, \tilde{z}) \), \( \hat{p} = \hat{p}(\tilde{z}) \), and with \( t_1 = \frac{1}{2}(tD_0 - K_3 D_2 b^2) \),
\[ \mathbb{P}\left[ \sup_{h, \tilde{z}} |\tilde{R} - R| > t \right] \leq \mathbb{P}\left[ \sup_{h, \tilde{z}} \left| \hat{q} - \mathbb{E}[\hat{q}] \right| > t_1 \right] + \mathbb{P}\left[ \sup_{\tilde{z}} \left| \hat{p} - \mathbb{E}[\hat{p}] \right| > t_1 \right]. \]

**Proof.** We start with the following decomposition
\[ \tilde{R}(h, \tilde{z}) - R(h, \tilde{z}) = \frac{\tilde{R}(h, \tilde{z}) (\mathbb{E}[\hat{p}(\tilde{z})] - \hat{p}(\tilde{z}))}{\mathbb{E}[\hat{p}(\tilde{z})]} + \frac{\hat{q}(h, \tilde{z}) - q(h, \tilde{z})}{\mathbb{E}[\hat{p}(\tilde{z})]} \]
\[ + \frac{R(h, \tilde{z}) (\hat{p}(\tilde{z}) - \mathbb{E}[\hat{p}(\tilde{z})])}{\mathbb{E}[\hat{p}(\tilde{z})]} . \]

Using the fact that smoothness implies \( \mathbb{E}[\hat{p}(\tilde{z})] \geq D_0 \int K(\tilde{z}) d\tilde{z} = D_0 \) and that \( \tilde{R}(h, \tilde{z}) \) and \( R(h, \tilde{z}) \) are both upper bounded by 1, we can bound the left hand side of (45) by
\[ \mathbb{P}\left[ \sup_{h, \tilde{z}} |\hat{q}(h, \tilde{z}) - q(h, \tilde{z})| > \frac{1}{2} tD_0 \right] \]
\[ + \mathbb{P}\left[ \sup_{\tilde{z}} |\hat{p}(\tilde{z}) - \mathbb{E}[\hat{p}(\tilde{z})]| + \sup_{\tilde{z}} |p(\tilde{z}) - \mathbb{E}[p(\tilde{z})]| > \frac{1}{2} tD_0 \right]. \]
Let us consider a fixed $\tau$. We denote by $V$ the closest $	au$-covering number of $kd$-dimensional hypercube (in $\ell_2$-norm) with
\[
\tau = \left(\frac{b^{d+1}L}{3L}\right)^{\frac{1}{2}},
\]
and, invoking the assumptions on the kernel,
\[
\left|q(h, \bar{z}) - \mathbb{E}[\hat{q}(h, \bar{z})]\right| \leq \frac{1}{2} D_{2K_2d^2b^2}.
\]

Lemma 5. Let $N_\tau$ be an $\tau$-covering number of a $kd$-dimensional hypercube (in $\ell_2$-norm) with
\[
\tau = \left(\frac{b^{d+1}L}{3L}\right)^{\frac{1}{2}},
\]
then
\[
P\left[\sup_{h, \bar{z}} |\hat{q}(h, \bar{z}) - \mathbb{E}[\hat{q}(h, \bar{z})]| > t_1\right] \leq N_\tau \mathbb{P}\left[\sup_{h} |\hat{q}(h, z) - \mathbb{E}[\hat{q}(h, z)]| > \frac{t_1}{3}\right],
\]
\[
P\left[\sup_{\bar{z}} |\hat{p}(\bar{z}) - \mathbb{E}[\hat{p}(\bar{z})]| > t_1\right] \leq N_\tau \mathbb{P}\left[|\hat{p}(\bar{z}) - \mathbb{E}[\hat{p}(\bar{z})]| > \frac{t_1}{3}\right].
\]

Proof. We again prove the statement only for $q$, while the argument for $p$ goes along the same lines. Let us consider $V$, a fixed $\tau$-covering of $Z^d$ with $\tau$ to be set later. We denote by $v(\bar{z})$ the closest element of the covering to $\bar{z}$. Then we have
\[
\sup_{h, \bar{z}} |\hat{q}(h, \bar{z}) - \mathbb{E}[\hat{q}(h, \bar{z})]| \leq \sup_{h, \bar{z}} |\hat{q}(h, \bar{z}) - \hat{q}(h, v(\bar{z}))| + \sup_{h, \bar{z}} |\hat{q}(h, v(\bar{z})) - \mathbb{E}[\hat{q}(h, v(\bar{z}))]| + \sup_{h, \bar{z}} |\mathbb{E}[\hat{q}(h, v(\bar{z}))] - \mathbb{E}[\hat{q}(h, \bar{z})]|.
\]
Using the Lipschitz property of the kernel, we can bound $|\hat{q}(h, \bar{z}) - \hat{q}(h, v(\bar{z}))|$ by
\[
\frac{1}{n b^{d+1}} \sum_{i \in I} \ell(h, z_{i+1}) \left(\mathcal{K}(\bar{z} - z_{i-1-d+1})/b - \mathcal{K}(v(\bar{z}) - z_{i-1-d+1})/b]\right) \leq \frac{L \tau^\gamma}{n b^{d-1}} \sum_{i \in I} \ell(h, z_{i+1}) \left\|\bar{z} - v(\bar{z})\right\|_2^\gamma \leq \frac{L \tau^\gamma}{n b^{d-1}}.
\]
Now, setting $\tau = \left(\frac{b^{d+1}L}{3L}\right)^{\frac{1}{2}}$, we can ensure that
\[
\sup_{h, \bar{z}} |\hat{q}(h, \bar{z}) - \mathbb{E}[\hat{q}(h, \bar{z})]| \leq \sup_{h, v \in V} |\hat{q}(h, v) - \mathbb{E}[\hat{q}(h, v)]| + \frac{2}{3} t_1,
\]
which lead us to the statement of the lemma.

Theorem. The proof is a combination of Lemmas 4 and 5 followed by Corollary 1. To obtain the final result we also bound the $\tau$-covering number of $kd$-dimensional hypercube by $\left(\frac{\sqrt{kd}}{2\tau}\right)^{kd}$. Note that, technically, to have the concentration for $\hat{p}$ we do not need to take the last step in Lemma 3 with coverings, however, to unify the final statement, we include the covering term in the bound.
Figure 1: Illustration of the data distributions in Section 5 by their $y$-expectations. (a): limit distribution of the hidden Markov chain. (b)–(d): conditional distributions of the $(n+1)$-th sample with history lengths 1, 2 and 4. (e): conditional distribution of the $(n+1)$-th sample with full history.

5 Simulations

This section illustrates the problem setting and the proposed estimator of the conditional risk in a synthetic setting that is easy to analyze but difficult to learn for other techniques. Our goal is to highlight the differences between marginal and conditional risks and between conditioning on the finite history and full history.

Let $\mathcal{Z} = \mathcal{X} \times \mathcal{Y}$ with $\mathcal{X} = [0, 10] \times [0, 10] \in \mathbb{R}^2$ and $\mathcal{Y} = \{\pm 1\}$. We generate data using a time-homogeneous hidden Markov process with four latent states (Figure 1a). Each state, $i$, is associated with an emission probability distribution, $\mu_i(x, y)$, that is uniform in $x$ and deterministic in $y$, with $y = \text{sign}(f_i(x))$ for an affine function $f_i$. At any step, $i$, we observe a sample $z_i = (x_i, y_i)$ from the distribution associated with the current latent state $s_i$. Figure 1b depicts the situation for $N = 1000$.

Drawn without order, the empirical distribution of the samples $z_1^N$ resembles the limit distribution of the hidden Markov chain (Figure 1d), which is also the marginal distribution of the next sample, $p(z_{N+1})$. Using the dependencies in the sequence, however, we can obtain a more informed estimate, $p(z_{N+1} | z_1^N)$ or its finite history counterparts, $p(z_{N+1} | z_{N-d}^N)$ for any history length $d$.

Figures 1c to 1h visualize these distribution: already with a short history length, the conditional distribution is easier learnable (has a lower Bayes risk) than the marginal one. Thus, identifying a good classifier for the conditional distribution at each step, i.e. minimizing the conditional risk, can lead to an overall lower error rate than finding a single predictor that is optimal for the limit distribution, i.e. minimizing the marginal risk.

We illustrate the proposed kernel-based estimator of the conditional risk, for $d = 4$ and a stratified set kernel, $K(S, \bar{S}) = \frac{1}{2|S_+||S_-|} \sum_{(i, j) \in S_+ \times S_-} k(x_i, \bar{x}_j) + \frac{1}{2|S_+||S_-|} \sum_{(i, j) \in S_- \times S_+} k(x_i, \bar{x}_j)$, for $S = \{(x_i, y_i)\}_{i=1}^d$ and $\bar{S} = \{\bar{x}_i, \bar{y}_i\}_{i=1}^d$, where $S_+/S_-$ are the sets of positive/negative examples in $S$ (and analogously for $\bar{S}$). As base kernel, $k(x, \bar{x})$ we use the squared exponential kernel. Figure 1c depicts the same samples, $z_1^N$, as Figure 1a, but each point $z_i$ is drawn at a size proportionally to its contribution to the conditional risk estimate at time $N = 1001$ for $d = 4$, i.e. its kernel weight $K(z_{j-d}^j, z_N^N)$. The points of the history $z_{N-3}^N$ are marked with crosses. One can see that the
resulting distribution indeed resembles the conditional distribution with history length 4 (Figure 1g), which is also close to the full conditional distribution (Figure 1h).

Figure 2 shows a numeric evaluation over 100 randomly generated Markov chains. In each case, we learn a linear predictor by approximately minimizing the estimator of the conditional risk from 5000 samples of the process. Then, we measure its quality using 5000 samples from the conditional distribution, \( p(z_{5001} | z_{5000}) \). To learn the predictor, we use squared-loss as a convex surrogate for 0/1-loss in the conditional risk estimator. The resulting expression is convex in the weight vector of the predictor and can be minimized by solving a weighted least squares problem, where the weight of each sample is proportional to the kernel similarity between its history and the most recent \( d \) elements of the sequence.

The results show that learning by conditional risk minimization (solid line) consistently outperforms empirical risk minimization (dashed line) for a wide range of kernel parameters (bandwidth of the Gaussian base kernel). Interestingly, even \( d = 1 \) is often enough to achieve a noticeable improvement. As an additional baseline, we add a sliding-window method that performs empirical risk minimization, but only on the samples in the history (dotted line). As expected, this heuristic works better than the plain ERM for long enough histories, but it does not achieve the same prediction quality as the conditional risk minimizer.

6 Discussions

As we noticed in the introduction, our definition of the conditional risk differs from the one usually considered in the literature because we condition on the last \( d \) examples instead of the whole sample. The relation between the two notions of the conditional risk was thoroughly discussed in (Caires and Ferreira, 2005) and, in short, if one is interested in the full history, then it is still possible to find a minimizer of this risk based on the risk with finite history for a wide range of processes called approximately Markov (for Markov processes the two notions coincide for the correct value of \( d \)). A stationary Gaussian process is an example of such a process.

This relation also highlights a different problem - which value of \( d \) to choose in practice. This is a hard question for theory, since it requires to take into account computational costs: for bigger values of \( d \) it may take more time to compute the weights. In addition, one may like to increase \( d \) with the amount of data and while there are partial justifications of this approach (Schäfer, 2002), its consequences are unclear. The current practical solution is to use cross-validation over feasible choices.
Another important parameter we need to choose is the bandwidth $b$. Its choice is a known problem for the kernel estimators themselves. The usual solution is cross-validation, see [Györfi et al., 1989] for more details. The proof of validity for such procedure in our setting is possible future work.

With all its benefits, the conditional risk minimization comes with the downside that it requires an intensive computation at each time step. Usually, the amount of computations for a single kernel is linear in $d$, meaning that it takes $O(dN)$ to compute the weights, and if $d$ is proportional to $N$, it may become quadratic. Afterwards, one also has to perform the optimization itself. While reducing the optimization costs is an algorithm-dependent problem, it may be possible to optimize computation of weights by some iterative procedure exploiting the fact that the finite history $\hat{z}_{N-d+1}$ changes only by one point at each step.

Our approach inherits the main problem of nonparametric methods – the curse of dimensionality. For high-dimensional spaces this makes it even harder to consider longer history because of the computational consideration above. We believe that in practice this effect can be mitigated by an appropriate choice of the kernel function.

7 Conclusion and possible extensions

In this paper we introduced an empirical estimator of the conditional risk for vector-valued stochastic processes and we proved concentration bounds showing that the estimator converges uniformly to the true risk for large classes at an exponential rate, if the process is $\beta$-mixing with sufficiently fast rates.

It is possible to generalize our results in several ways. The stationarity assumption is not essential in our proofs and can be relaxed to conditionally stationary processes from [Caires and Ferreira, 2005]. In this case smoothness and robustness will have to be assumed for each time step.

The assumption of the support to be a hypercube was also made for convenience and it can easily be any compact set. It also can be relaxed by different means. For example, we could make additional assumptions on the moments of $q$ and $p$ or restrict the supremum over histories to the set $\{z : \|z\| \leq c_n\}$, with $c_n$ increasing as $n$ grows like it was done in [Hansen, 2008]. This would lead to a slower convergence rate, though. Another option is to assume that the distribution of $z_{d+1}$ is tight, meaning that for every $\varepsilon > 0$ there is a compact set $K_\varepsilon$ such that the probability mass assigned to $K_\varepsilon^c$ is less than $\varepsilon$. This approach would require the knowledge of behavior of the covering numbers of the sets $K_\varepsilon$. While all these extension may allow to include more “real life” distributions, it is more of a technical contribution.

While $\beta$-mixing assumption covers a wide range of stochastic processes considered in the literature, there are other dependency measures that may be more suitable for concrete situations. It should be possible to extend our results to these cases as long as a particular dependency measure allows for uniform convergence of the empirical averages.

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