On Fourier Coefficients of the Symmetric Square L-Function at Piatetski-Shapiro Prime Twins

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Abstract: Let \( \mathbb{P}_c(x) = \{ p \leq x | p, [p^r] \text{ are primes} \}, c \in \mathbb{R}^+ \setminus \mathbb{N} \) and \( \lambda_{sym^2f}(n) \) be the \( n \)-th Fourier coefficient associated with the symmetric square \( L \)-function \( L(s, sym^2f) \). For any \( \lambda > 0 \), we prove that the mean value of \( \lambda_{sym^2f}(n) \) over \( \mathbb{P}_c(x) \) is \( \ll x \log^{-\lambda-2} x \) for almost all \( c \in (\epsilon, (\sqrt{3}+3)/8 - \epsilon) \) in the sense of Lebesgue measure. Furthermore, it holds for all \( c \in (0, 1) \) under the Riemann Hypothesis. Furthermore, we obtain that asymptotic formula for \( \lambda_{sym^2f}^2(n) \) over \( \mathbb{P}_c(x) \) is \( \sum_{p | q \text{ prime} \leq x, d | [p]} \lambda_{sym^2f}^2(p) = \frac{x}{c \log^2 x} (1 + o(1)) \), for almost all \( c \in (\epsilon, (\sqrt{3}+3)/8 - \epsilon) \), where \( \lambda_{sym^2f}(n) \) is the normalized \( n \)-th Fourier coefficient associated with a holomorphic cusp form \( f \) for the full modular group.

Keywords: primes; Fourier coefficient; symmetric square \( L \)-function

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1. Introduction

Let \( k \) be an even positive integer, \( f \) be a holomorphic cusp form of weight \( k \) for the full modular group and \( \lambda_f(n) \) be the normalized \( n \)-th Fourier coefficient of \( f \), i.e.,

\[
f(z) = \sum_{n=1}^{\infty} \lambda_f(n) n^{\frac{k-1}{2}} e(nz).
\]

If we assume that \( f \) is an eigenform of all the Hecke operators, then \( f \) can be normalized such that \( \lambda_f(1) = 1 \) and \( \lambda_f(n) \) is real. We define the Hecke \( L \)-function associated to \( f \) for \( \Re s > 1 \) by

\[
L(s, f) = \sum_{n=1}^{\infty} \frac{\lambda_f(n)}{n^s} = \prod_p (1 - \lambda_f(p)p^{-s} + p^{-2s})^{-1}.
\]

For any prime \( p \) and all integers \( \nu \geq 0 \), we have

\[
\lambda_f(p^\nu) = \alpha_f(p)^\nu + \alpha_f(p)^{\nu-1}\beta_f(p) + \cdots + \beta_f(p)^{\nu},
\]

where \( \alpha_f(p), \beta_f(p) \) are the local parameters of \( L(s, f) \) at prime \( p \), satisfying

\[
\alpha_f(p) + \beta_f(p) = \lambda_f(p), \quad \alpha_f(p)\beta_f(p) = 1.
\]

Then we have

\[
\lambda_f^2(p) = 1 + \lambda_f(p^2).
\]

Deligne [1,2] proved Ramanujan–Petersson conjecture, i.e.,

\[
|\lambda_f(n)| \leq d(n) \ll n^{\epsilon}
\]

for all \( n \geq 1 \), where \( d(n) = \sum_{d|n} 1 \).
In order to detect the sign changes of $\lambda_f(n)$, many authors have studied the mean value of $\lambda_f(n)$ and obtained some good results. For example, see [1,3–22]. In addition, the sums of $\lambda_f(n)$ over primes have also been studied. It is known that (see for example Section 5.6 of Iwaniec and Kowalski [23]) there exists a constant $C > 0$ such that

$$\sum_{p \leq N} \lambda_f(p) \ll_f N \exp(-C \sqrt{\log N}). \quad (1)$$

The upper bound of (1) may reach $N^{1/2+\varepsilon}$, assuming the Riemann Hypothesis. Furthermore, we can establish that

$$\sum_{p \leq N} \lambda_f^2(p) \sim \frac{N}{\log N}, \quad \text{as } N \to \infty,$$

by using the analytic properties of the Rankin–Selberg $L$-function $L(s, f \otimes \bar{f})$.

Another interesting question considered by many authors is the mean value of $\sum_{n \leq x} \lambda_f^2([n^g])$ over certain sets of primes. For example, Baier and Zhao [24] studied the distribution of $\lambda_f(n)$ at Piatetski–Shapiro primes by considering estimates of exponential sum involving Hecke eigenvalue. Moreover, they conjectured that, for $1 < c < 1 + \theta$ with some suitable $\theta > 0$, there exists a constant $c_f > 0$ such that

$$\sum_{n \leq N} \lambda_{f}^2([n^g]) \sim c_f \frac{N}{c \log N}, \quad \text{as } N \to \infty.$$

Furthermore, we can define Piatetski–Shapiro prime twins if $p, q$ are primes and $q = [p^c]$, for any fixed $c \in \mathbb{R}^+ \setminus \mathbb{N}$. Balog [25] and Dufner [26] proved that

$$\sum_{\substack{p, q \text{ prime} \atop p \leq x, q \leq [p^c]}} \frac{1}{c \log^2 x} (1 + o(1)), \quad x \to \infty, \quad (2)$$

for almost all $c \in (0,1)$ in the sense of Lebesgue measure. Furthermore, assuming the Riemann Hypothesis of automorphic $L$-function $L(s, f)$ is true, they found that (2) holds for all $c \in (0,1)$. Furthermore, Zhang and Zhai [27] studied the mean value of $\lambda_f(n)$ over Piatetski–Shapiro prime twins.

Motivated by the above results, we are interested in the distribution of $\lambda_f^2(n)$ at Piatetski–Shapiro prime twins. For the form $f$, we know the $L(s, \text{sym}^2 f)$ is an $L$-function for some $GL(3, \mathbb{Z})$ automorphic representation, which is often called the symmetric-square lift of $f$. The $n$-th Fourier coefficient of $L(s, \text{sym}^2 f)$ satisfies

$$\lambda_{\text{sym}^2 f}(n) = \sum_{m^2 = n} \lambda_f(m^2).$$

The symmetric square $L$-function associated to $f$ is defined by

$$L(s, \text{sym}^2 f) = \prod_p \prod_{0 < j \leq 2} \left(1 - \frac{\alpha_f(p)^{2-j} \beta_f(p)^j}{p^s}\right)^{-1} = \sum_{n=1}^{\infty} \frac{\lambda_{\text{sym}^2 f}(n)}{n^s}$$

in the half-plane $\Re s > 1$. Then, for all $n \geq 1$, $\lambda_{\text{sym}^2 f}(n)$ is also multiplicative, real and

$$|\lambda_{\text{sym}^2 f}(n)| \leq d_3(n),$$

where $d_3(n) = \sum_{m|n} d(m)$. Furthermore, for all primes $p$, we have

$$\lambda_{\text{sym}^2 f}(p) = \lambda_f(p^2).$$
Many authors studied the mean value of \( \lambda_{sym}f(n) \). For example, see References [28–36]. In this paper, we consider the mean value of Fourier coefficients of symmetric square \( L \)-function over Piatetski–Shapiro prime twins and obtain the following results, which imply a result on the distribution of \( \lambda_f^2(n) \) at Piatetski–Shapiro prime twins.

**Theorem 1.** For almost all \( c \in \left( \varepsilon, (\sqrt{5} + 3)/8 - \varepsilon \right) \) and any \( A > 0 \), we have

\[
\sum_{p \leq x} \lambda_{sym}^2(f)(p) \ll \frac{x}{\log^{A+2} x}.
\]

**Theorem 2.** Assuming the Riemann Hypothesis of symmetric square \( L \)-function is true, (3) holds for all \( c \in (0, 1) \).

**Corollary 1.** For almost all \( c \in \left( \varepsilon, (\sqrt{5} + 3)/8 - \varepsilon \right) \), we have

\[
\sum_{p \leq x} \lambda_f^2(p) = \frac{x}{c \log^2 x} (1 + o(1)), \ x \to \infty.
\]

**Proof.** The result follows easily from Theorem 1 and (2), if we notice that \( \lambda_f^2(p) = 1 + \lambda_f(p^2) \). \( \square \)

**Corollary 2.** Assuming the Riemann Hypothesis of symmetric square \( L \)-function is true, (4) holds for all \( c \in (0, 1) \).

**Proof.** The result follows from Theorem 2 and (2). \( \square \)

**Notation.** Throughout the paper, \( \varepsilon \) always denotes a sufficiently small positive constant. Let \( \delta_1(\varepsilon) \) be sufficiently small and depend on \( \varepsilon \). We write \( f(x) \ll g(x) \), or \( f(x) = O(g(x)) \), to mean that \( |f(x)| \leq C g(x) \). Let \( \rho = \sigma + i \eta \) be the nontrivial zero of the symmetric square \( L \)-function \( L(s, sym^2 f) \). As usual, \( \Lambda(n) \) is the von Mangoldt function.

2. Auxiliary Lemmas

**Lemma 1.** Let \( \eta \) run through a countable set of reals, with \( c(\eta) \) arbitrary complex such that \( S(t) = \sum_{\eta} c(\eta)e(\eta t) \) is absolutely convergent. Let \( \alpha \in \mathbb{R}, \Theta \in (0, 1), \delta = \Theta/\alpha \). Then

\[
\int_{-\alpha}^{\alpha} |S(t)|^2 dt \ll \Theta \int_{-\infty}^{\infty} \left| \delta^{-1} \sum_{t \leq \eta \leq t + \delta} c(\eta) \right|^2 dt.
\]

**Proof.** This lemma is Lemma 1 of Gallagher [37]. \( \square \)

**Lemma 2.** Let \( T \geq T_0 \), where \( T_0 \) is a sufficiently large real number. The

\[
\int_T^{2T} |L(\alpha + it, sym^2 f)|^2 dt \ll \begin{cases} (10 + k + T)(\log(10 + k + T))^{17}, & \text{if } 2/3 \leq \alpha \leq 1, \\ (10 + k + T)^{3(1-\alpha)}(\log(10 + k + T))^{17}, & \text{if } 1/3 \leq \alpha \leq 2/3, \\ (10 + k + T)^{3(1-2\alpha)+1}(\log(10 + k + T))^{17}, & \text{if } 0 \leq \alpha \leq 1/3. \end{cases}
\]

**Proof.** This lemma is Lemma 3.1 of Lü [33]. \( \square \)

**Lemma 3.** For \( 1/2 \leq \sigma \leq 1 \), \( T \geq 3 \), define

\[
N(\sigma, T) = \# \{ \eta = \beta + i \omega : L(\eta, sym^2 f) = 0, \sigma \leq \beta \leq 1, |\omega| \leq T \}.
\]
Then we have
\[ N(\sigma, T) \ll T^{\frac{5(1-\sigma)}{3-2\sigma} + \epsilon}, \text{ if } 1/2 \leq \sigma \leq 3/4, \]
and
\[ N(\sigma, T) \ll T^{3(1-\sigma) + \epsilon}, \text{ if } 3/4 < \sigma \leq 1. \]

**Proof.** From Lemma 2, we have
\[ \int_T^{2T} |L(1/2 + it, \text{sym}^2 f)|^2 dt \ll T^{3/2 + \epsilon}. \tag{5} \]
This combined with Theorem 1.1, 1.2 of Ye and Zhang [38] and [39] gives this lemma. □

**Lemma 4.** For any \( u \geq 2, T \geq 2 \), we have
\[ \sum_{n < u} \Lambda(n) \lambda_{\text{sym}^2 f}(n) = - \sum_{\rho = \sigma + i \eta, |\eta| \leq T} \frac{\rho^u}{\rho} + O \left( \log^3 u + u \log^5 (uT) \right). \]

**Proof.** See, for example, Iwaniec and Kowalski [23]. For convenience of calculation, we reduce the summation range of \( n \) from \( n \leq u \) to \( n < u \). The contribution of \( n = u \) is \( O(\log^3 u) \), in view of \( |\lambda_{\text{sym}^2 f}(p^h)| \leq d_3(p^h) \ll \log^2 u \).

**Lemma 5.** Let \( L(s, f) \) be an L-function of degree \( k \) such that Rankin–Selberg convolutions \( L(s, f \otimes f) \) and \( L(s, f \otimes \overline{f}) \) exist, and the latter has a simple pole at \( s = \alpha + it = 1 \) while the former is entire if \( f \neq \overline{f} \). Suppose that the ramified primes \( |\alpha_f(p)| \leq p/2 \). There exists an absolute constant \( c > 0 \) such that \( L(s, f) \) has no zeros in the region
\[ \alpha \geq 1 - \frac{c}{k^4 \log(q(f) |t| + 3)^3}, \]
where \( q(f, s) \) is the analytic conductor and \( q(f) = q(f, 0) \).

**Proof.** This lemma is Theorem 5.10 of Iwaniec and Kowalski [23]. □

**Lemma 6.** Let \( x, T \in \mathbb{R}^+, x \to \infty \) and
\[ S := \sum_{\rho = \sigma + i \eta, |\eta| \leq T} x^\sigma, \]
we have
\[ S \ll \begin{cases} x^{1/2} T^{5/4 + \epsilon}, & \text{if } T \geq x^{4/5}; \\ x^{3/2} T^{5/2 + \epsilon} e^{-\sqrt{3} \log M \log T}, & \text{if } x^{9/20} \leq T < x^{4/5}; \\ x^{3/4} T^{5/6 + \epsilon} + x^{1-\beta} T^{3\beta + \epsilon}, & \text{if } T < x^{9/20}. \end{cases} \]

**Proof.** Note that
\[ S = \int_{1/2}^{1-\beta} x^\sigma d_\sigma N(\sigma, T), \]
where $\beta = \beta(T)$ is the width of the zero-free region of the symmetric square $L$-function in Lemma 5. Using integration by parts and Lemma 3, we obtain

\begin{equation}
S \ll T^{1/2} + \int_{1/2}^{3/4} \exp(f_1(\sigma))d\sigma + \int_{3/4}^{1-\beta} \exp(f_2(\sigma))d\sigma
\end{equation}

where $\ll$ is fulfilled apart from a fixed number of log-factors,

\[ f_1(\sigma) = \frac{5(1 - \sigma)}{3 - 2\sigma} \log T + \sigma \log x + \varepsilon \log T \]

and

\[ f_2(\sigma) = 3(1 - \sigma) \log T + \sigma \log x + \varepsilon \log T. \]

We estimate $S_1$ first. The first and second derivatives of $f_1(\sigma)$ are

\[ f_1'(\sigma) = -\frac{5 \log T}{(3 - 2\sigma)^2} + \log x \]

and

\[ f_1''(\sigma) = -\frac{20 \log T}{(3 - 2\sigma)^3} < 0, \text{ if } \sigma < 3/2. \]

Then $f_1(\sigma)$ is a concave function as $\sigma < 3/2$. Let $f_1'(\sigma_1) = 0$, we have $\sigma_1 = 1 - \frac{1}{2} \sqrt{\frac{3 \log T}{\log x}}$.

So we have the following three cases.

**Case 1.** When $T \geq x^{4/5}$. In this case, we have $\sigma_1 \leq 1/2$ and $f_1(\sigma)$ is monotonically decreasing in $[1/2, 3/4]$. Hence

\[ \max_{\sigma \in [1/2, 3/4]} f_1(\sigma) = f_1(1/2) = (5 \log T)/4 + (\log x)/2 + \varepsilon \log T. \]

**Case 2.** When $x^{9/20} \leq T < x^{4/5}$. In this case, the function $f_1(\sigma)$ takes the extreme value at $\sigma_1$, which gives

\[ \max_{\sigma \in [1/2, 3/4]} f_1(\sigma) = f_1(\sigma_1) = (5 \log T)/2 + (3 \log x)/2 - \sqrt{5 \log T \log x} + \varepsilon \log T. \]

**Case 3.** When $T < x^{9/20}$. In this case, we have $\sigma_1 > 3/4$ and $f_1(\sigma)$ is monotonically increasing in $[1/2, 3/4]$. Hence,

\[ \max_{\sigma \in [1/2, 3/4]} f_1(\sigma) = f_1(3/4) = (5 \log T)/6 + (3 \log x)/4 + \varepsilon \log T. \]

Combining all the above cases, we have

\begin{equation}
S_1 = \int_{3/2}^{3/4} \exp(f_1(\sigma))d\sigma \ll \begin{cases} 
T^{5/4 + \varepsilon}, & \text{if } T \geq x^{4/5}; \\
T^{5/2 + \varepsilon} e^{-\sqrt{5 \log x} \log T}, & \text{if } x^{9/20} \leq T < x^{4/5}; \\
T^{3/4 + \varepsilon}, & \text{if } T < x^{9/20}. 
\end{cases}
\end{equation}

Next, we need to estimate $S_2$. It is easy to see that $f_2(\sigma)$ is linear function in $[3/4, 1 - \beta]$, hence

\[ f_2(\sigma) \ll \max((f_2(3/4), f_2(1 - \beta)) = \max((3 \log T)/4 + (3 \log x)/4, 3\beta \log T + (1 - \beta) \log x) + \varepsilon \log T. \]
We split the summation range of $q$ which proves this lemma. □

**Lemma 7.** Let $x_0 := 2$, $x_{0+1} := x_0 + x_0 \log^{-2} x_0$. Let $\gamma_0$ be a constant and

$$D(\gamma_0, x_0, T) = \int_{\gamma_0}^{\gamma_0+\log^{-2} x_0} \sum_{x_0^{1/3} < m \leq x_0^{1/2}} \Lambda(m) \sum_{\substack{p = m \leq q \leq \gamma_0 \leq \frac{T}{2} \leq |\gamma| \leq T}} \left( m + 1 \right)^{\gamma_0} - m^{\gamma_0} \rho \left| dx \right|.$$  

If $T > x_0^{1/\gamma_0-\delta}$, we have

$$D(\gamma_0, x_0, T) \ll x_0^{2/\gamma_0-3} \left( \sum_{\substack{p = m \leq q \leq \gamma_0 \leq \frac{T}{2} \leq |\gamma| \leq T}} x_0^c \right)^2.$$  

(9)

If $T \leq x_0^{1/\gamma_0-\delta}$, we have

$$D(\gamma_0, x_0, T) \ll x_0^{2/\gamma_0-3} \left( \sum_{\substack{p = m \leq q \leq \gamma_0 \leq \frac{T}{2} \leq |\gamma| \leq T}} x_0^c \right)^2 \left( T x_0^{-1/\gamma_0} \right)^2.$$  

(10)

**Proof.** This lemma follows from (13) and (16) of Dufner [26]. □

**3. Proof of Theorem 2**

In this section, we write $\gamma = 1/c$. Then we have

$$\Pi_{2,c}(x) := \sum_{\substack{p \text{ prime} \leq q \leq x^{1/4-c} \rho}} \lambda_{sym^2 f}(p) = \sum_{\substack{p \text{ prime} \leq q \leq x^{1/4-c} \rho}} \lambda_{sym^2 f}(p) = \sum_{q \leq x^c} \sum_{q^c \leq p \leq (q+1)^c} \lambda_{sym^2 f}(p).$$

We split the summation range of $q$ into two parts: $q \leq x^c - 1$ and $x^c - 1 < q \leq x^c$ to get

$$\Pi_{2,c}(x) = \sum_{q \leq x^c - 1} \lambda_{sym^2 f}(p) + \sum_{x^c - 1 < q \leq x^c} \sum_{q^c \leq p \leq (q+1)^c} \lambda_{sym^2 f}(p)$$

$$= \sum_{q \leq x^c} \sum_{q^c \leq p \leq (q+1)^c} \lambda_{sym^2 f}(p) - \sum_{x^c - 1 < q \leq x^c} \sum_{x^c - 1 < q \leq x^c \rho \leq (q+1)^c} \lambda_{sym^2 f}(p)$$

(11)

:= $E_1 - E_2$.

For $E_2$, there is at most one prime $q_x$ satisfying $x^c - 1 < q_x \leq x^c$, hence

$$E_2 \ll \sum_{q_x^c \leq p \leq (q_x+1)^c} \left| \lambda_{sym^2 f}(p) \right| \ll (q_x + 1)^c - q_x^c \ll x^{1-c},$$

(12)

if we notice that $\left| \lambda_{sym^2 f}(p) \right| \leq d_2(p) \ll 1$.

For $E_1$, by the definition of the von Mangoldt function, we have

$$E_1 = \sum_{q \leq x^c} \sum_{q^c \leq n \leq (q+1)^c} \frac{\Lambda(n) \lambda_{sym^2 f}(n)}{\log n} + O \left( \sum_{q \leq x^c} \sum_{n = q^c \leq (q+1)^c} \frac{\lambda_{sym^2 f}(p^n) \log p}{\log p^n} \right).$$

(13)
The error term of the above formula contributes
\[
\ll \sum_{q \leq x} \sum_{q^h \leq \rho < (q+1)\gamma} |\lambda_{\text{sym}^2 f}(p^h)| \\
= \sum_{q \leq x} \sum_{q^h \leq \rho < (q+1)\gamma} |\lambda_{\text{sym}^2 f}(p^h)| + \sum_{q \leq x} \sum_{q^h \leq \rho < (q+1)\gamma} |\lambda_{\text{sym}^2 f}(p^h)| \\
\ll \sum_{q \leq x} \sum_{q^h \leq \rho^2 < (q+1)\gamma} 1 + \sum_{q \leq x} \sum_{q^h \leq \rho < (q+1)\gamma} \log^2 x \\
\ll \sum_{q \leq x} \left( (q + 1)^{\gamma/2} - q^{\gamma/2} \right) + \sum_{q \leq x} \left( (q + 1)^{\gamma/3} - q^{\gamma/3} \right) \log^3 x \\
\ll \sum_{q \leq x} q^{\gamma/2 - 1} + \sum_{q \leq x} q^{\gamma/3 - 1} \log^3 x \ll x^{1/2}.
\]

We use the same method to deal with the main term of (13) and get
\[
E_1 = \sum_{2 \leq m \leq x} \frac{\Lambda(m)}{\log m} \sum_{m^\gamma \leq \rho < (m+1)^\gamma} \frac{\Lambda(n)\lambda_{\text{sym}^2 f}(n)}{\log n} + O(\lambda^{1-c/2}) + O(\lambda^{1/2}),
\]
where the first $O$-term comes from
\[
\sum_{m=\rho^h \leq x} \frac{\Lambda(m)}{\log m} \sum_{m^\gamma \leq \rho < (m+1)^\gamma} \frac{\Lambda(n)\lambda_{\text{sym}^2 f}(n)}{\log n} \\
= \sum_{m=\rho^h \leq x} \frac{\log q}{\log q^h} \sum_{m^\gamma \leq \rho < (m+1)^\gamma} \frac{\lambda_{\text{sym}^2 f}(p^r) \log p}{\log p^r} \\
\ll \sum_{m=\rho^h \leq x} \sum_{m^\gamma \leq \rho < (m+1)^\gamma} |\lambda_{\text{sym}^2 f}(p^r)| \ll x^{1-c/2}.
\]

Let
\[
W(x) := \sum_{2 \leq m \leq x} \frac{\Lambda(m)}{\log m} \sum_{m^\gamma \leq \rho < (m+1)^\gamma} \frac{\Lambda(n)\lambda_{\text{sym}^2 f}(n)}{\log n}.
\]

By the range of $n$ and Taylor’s formula, we have
\[
\log n = \gamma \log m + O(m^{-1})
\]
and
\[
\frac{1}{\log n} = \frac{1}{\gamma \log m} + O\left( \frac{1}{m \log^2 m} \right).
\]

Then
\[
W(x) = \gamma^{-1} \sum_{2 \leq m \leq x} \frac{\Lambda(m)}{\log m} \sum_{m^\gamma \leq \rho < (m+1)^\gamma} \Lambda(n)\lambda_{\text{sym}^2 f}(n) + O(x^{1-c} \log^3 x),
\]
(15)
where the $O$-term comes from

$$
\sum_{2 \leq m \leq x^c} \frac{\Lambda(m)}{m \log^3 m} \sum_{m^\gamma \leq n \leq (m+1)^\gamma} \Lambda(n) \lambda_{\text{sym}}^2 f(n)
$$

$$
\ll \log^3 x \sum_{2 \leq m \leq x^c} m^{\gamma-2} \log^{-3} m
$$

$$
\ll x^{1-c} \log^3 x \sum_{2 \leq m \leq x^c} \frac{1}{m \log m}
$$

$$
\ll x^{1-c} \log^3 x.
$$

From (14), (15) and partial summation, we obtain

$$
E_1 \ll \log^{-2} x \cdot \max_{x_0 \leq x} \left| \Psi_{2,c}(x_0) \right| + x^{1-c} \log^3 x + x^{1-c/2} + x^{1/2},
$$

where

$$
\Psi_{2,c}(x) = \sum_{m \leq x^c} \frac{\Lambda(m)}{m} \sum_{m^\gamma \leq n \leq (m+1)^\gamma} \Lambda(n) \lambda_{\text{sym}}^2 f(n).
$$

To get (3), it suffices to prove that

$$
\left| \Psi_{2,c}(x) \right| \ll x \log^{-A} x, \quad \text{for any } A > 0.
$$

The inequality (17) will be proved from a variant of (17) for short intervals. Let $x_0 := 2$, $x_{v+1} := x_v + x_v \log^{-2} x_v$, we can see that the inequality

$$
\left| \Delta \Psi_{2,c}(x_v) \right| = \left| \sum_{x_0^c < m \leq x_{v+1}^c} \frac{\Lambda(m)}{m} \sum_{m^\gamma \leq n \leq (m+1)^\gamma} \Lambda(n) \lambda_{\text{sym}}^2 f(n) \right| \ll x_v \log^{-A-2} x_v
$$

implies (17). We use Lemma 4 and get

$$
\sum_{m^\gamma \leq n \leq (m+1)^\gamma} \Lambda(n) \lambda_{\text{sym}}^2 f(n) = - \sum_{\rho = \sigma + i \eta \atop \rho \leq T_v} \frac{(m+1)^\gamma \rho - m^\gamma \rho}{\rho} + O \left( \log m + \frac{(m+1)^\gamma \log^3 (mT)}{T} \right).
$$

Taking $T_v = x_v \log^{-2A-3} x_v$, then we obtain

$$
\left| \Delta \Psi_{2,c}(x_v) \right| \ll \left| \Phi(\gamma, x_v, T_v) \right| + x_v^c \log^{2A+4} x_v,
$$

where

$$
\Phi(\gamma, x_v, T_v) = \sum_{x_0^c < m \leq x_{v+1}^c} \frac{\Lambda(m)}{m} \sum_{\rho = \sigma + i \eta \atop \rho \leq T_v} \frac{(m+1)^\gamma \rho - m^\gamma \rho}{\rho}.
$$

Under the Riemann Hypothesis, we have

$$
\Phi(\gamma, x_v, T_v) = \sum_{x_0^c < m \leq x_{v+1}^c} \frac{\Lambda(m)}{m} \int_{m^\gamma}^{(m+1)^\gamma} \sum_{\rho = \sigma + i \eta \atop \rho \leq T_v} \frac{1}{t} \log^1 dt
$$

$$
= \sum_{x_0^c < m \leq x_{v+1}^c} \frac{\Lambda(m)}{m} \int_{m^\gamma}^{(m+1)^\gamma} \sum_{\rho = \sigma + i \eta \atop \rho \leq T_v} e^{(-1/2+i\eta) \log^1} dt.
$$
Making the change of variables $u = \log t$, we deduce that

$$\Phi(\gamma, x_\nu, T_\nu) = \sum_{x_\nu^0 < m \leq x_\nu^1} \Lambda(m) \int_{\gamma \log m}^{\gamma \log (m+1)} e^{u/2} \sum_{\rho = \sigma + \mathrm{i} \eta} \frac{d\rho}{\pi} \, du.$$ 

Using the Cauchy–Schwarz inequality twice, we get

$$|\Phi(\gamma, x_\nu, T_\nu)|^2 \ll \sum_{x_\nu^0 < m \leq x_\nu^1} \Lambda^2(m) \sum_{x_\nu^0 < m \leq x_\nu^1} \int_{\gamma \log m}^{\gamma \log (m+1)} e^{u/2} \sum_{\rho = \sigma + \mathrm{i} \eta} \frac{d\rho}{\pi} \, du \leq \sum_{x_\nu^0 < m \leq x_\nu^1} \Lambda^2(m) \int_{\gamma \log m}^{\gamma \log (m+1)} \sum_{\rho = \sigma + \mathrm{i} \eta} \frac{d\rho}{\pi} \, du$$

$$\ll x_\nu \log^{-1} x_\nu \cdot \int_{\log x_\nu}^{\gamma + \log x_\nu} \sum_{\rho = \sigma + \mathrm{i} \eta} \frac{d\rho}{\pi} \, du. \quad (22)$$

Making the change of variables $t = u(2\pi) - (\gamma + \log x_{\nu+1} + \log x_\nu)(4\pi)^{-1}$, we deduce that the last integral in (22) can be written as

$$2\pi \int_{-a}^{a} \left| \sum_{|\eta| \leq T_\nu} c(\eta) e(\eta t) \right|^2 \, dt, \quad (23)$$

where $a = (\gamma + \log x_{\nu+1} - \log x_\nu)(4\pi)^{-1}$ and

$$c(\eta) = \begin{cases} e^{i\eta(\gamma + \log x_{\nu+1} + \log x_\nu)/2}, & \text{if } |\eta| \leq T_\nu; \\ 0, & \text{otherwise}. \end{cases}$$

Applying Lemma 1 with $\Theta = 1/2$ and $\delta = \Theta/a \sim 2\pi/\gamma$ to estimate the integral in (23), we have

$$\ll \int_{-a}^{a} \left| \sum_{|\eta| \leq T_\nu} c(\eta) \right|^2 \, dt$$

$$= \int_{-\infty}^{\infty} \delta^{-1} \sum_{\eta_1 \eta_2 \leq T_\nu} e^{i(\eta_1 - \eta_2)(\gamma + \log x_{\nu+1} + \log x_\nu)/2} \, d\eta_1 d\eta_2$$

$$\ll \delta^{-2} \cdot T_\nu \ll x_\nu \log^{-2A-3} x_\nu. \quad (24)$$

From (20), (22) and (24), we get

$$|\Delta_\Psi_{2c}(x_\nu)| \ll x_\nu \log^{-A-2} x_\nu + x_\nu^0 \log^{2A+4} x_\nu.$$

This combined with (16) and (17) gives

$$E_1 \ll x \log^{-A-2} x. \quad (25)$$

Theorem 2 follows from (11), (12) and (25).
4. Proof of Theorem 1

To prove Theorem 1, we need Lemma 6 and Lemma 7. Furthermore, we only need to estimate \( \Phi(\gamma, x, T) \) unconditionally. Note that

\[
\Phi(\gamma, x, T) \ll \log x \cdot \max_{1 \leq T \leq T_0} |\Phi_1(\gamma, x, T)|,
\]

where

\[
\Phi_1(\gamma, x, T) = \sum_{\sqrt{x} \leq m \leq \sqrt{x} + 1} \Lambda(m) \sum_{\rho - e^{1/6} \leq T \leq \rho \leq T + e^{1/6}} \frac{(m + 1)^\rho - m^\rho}{\rho}.
\]

We consider the following integral mean value of \( \Phi_1(\gamma, x, T) \),

\[
D(\gamma_0, x, T) = \int_{\gamma_0}^{\gamma_0 + \log^2 x} |\Phi_1(\gamma, x, T)|^2 d\gamma
\]

with \( \gamma_0 \in \left[\frac{1}{\log T}, \frac{1}{\log^2 x}\right] \).

From Lemma 6, we know that the upper bound of \( S \) depends on the range of \( T \), so we have the following three cases.

Case 1. When \( T \geq x_0^{4/5} \).

In this case, we have \( x_0^{1/70 - \delta_1(\epsilon)} < T \) and use (9) to get

\[
D(\gamma_0, x, T) \ll x_0^{2/70} T^{-3} \left( x_0^{1/2} \gamma_0^{5/4 + \epsilon} \right)^2 = x_0^{1+2/70} T^{-1-2+2\epsilon}
\]

\[
\ll x_0^{1+3/2\gamma_0+2\log \gamma_0+1/2-2\epsilon} \delta_1(\epsilon).
\]

Note that if \( 1/\gamma_0 \leq \left(\sqrt{5} + 3\right)/8 - \epsilon \), we have

\[
D(\gamma_0, x, T) \ll x_0^{2-\epsilon}.
\]

Case 2. When \( x_0^{9/20} \leq T < x_0^{4/5} \).

In case of \( x_0^{1/70 - \delta_1(\epsilon)} < T \), we use (9) and get

\[
D(\gamma_0, x, T) \ll x_0^{2/70} T^{-3} \left( x_0^{3/2} \gamma_0^{5/2 + \epsilon} - \sqrt{\log x} - \log T \right)^2
\]

\[
= x_0^{3+2/70} \gamma_0^{2+2\epsilon} - 2\sqrt{\log x} - \log T
\]

\[
= e^{\left(2+2\epsilon\right) \left(\sqrt{\frac{\log T}{\log x}}\right)^2 - 2\sqrt{\log T} + \frac{3+2\epsilon}{\gamma_0}}
\]

\[
= (2+2\epsilon) \left(\frac{\log T}{\log x}\right)^2 - 2\log T + \frac{3+2\epsilon}{\gamma_0},
\]

For convenience, we take \( t = \sqrt{\frac{\log T}{\log x}} \) and consider a quadratic function \( g(t) = (2 + 2\epsilon)t^2 - 2\sqrt{5}t + 3 + 2/\gamma_0 \).

If \( x_0^{1/70 - \delta_1(\epsilon)} \leq x_0^{9/20} \leq T \), we have \( t \in \left[\sqrt{9/20}, \sqrt{47/5}\right] \) and \( g(t) \) is monotonically decreasing in this interval. Hence,

\[
g(t) \leq g\left(\sqrt{9/20}\right) = 9(1 + \epsilon)/10 + 2/\gamma_0 \leq 9/5 + 9/10\epsilon + 2\delta_1(\epsilon)
\]

with \( 1/\gamma_0 \leq 9/20 + \delta_1(\epsilon) \). Therefore,

\[
D(\gamma_0, x, T) \ll x_0^{2-\epsilon}.
\]
If \( x_0^{9/20} < x_0^{1/\gamma_0 - \delta_1(\epsilon)} < T < x_0^{4/5} \), we get analogously \( t \in \left( \sqrt{1/\gamma_0 - \delta_1(\epsilon)} , \sqrt{4/5} \right) \)
and
\[
g(t) < g \left( \sqrt{1/\gamma_0 - \delta_1(\epsilon)} \right) = 3 + 4/\gamma_0 - 2\sqrt{5(1/\gamma_0 - \delta_1(\epsilon)) + 2\epsilon/\gamma_0 - (2 + 2\epsilon)\delta_1(\epsilon)}
\leq 3 + 4/\gamma_0 - 2\sqrt{5/\sqrt{\gamma_0}} + 2\epsilon/\gamma_0 + \left( \sqrt{10/\gamma_0 - 2 - 2\epsilon} \right) \delta_1(\epsilon)
\leq 2 - \epsilon,
\]
where we choose \( \delta_1(\epsilon) \) sufficiently small and use the elementary inequality
\[
\sqrt{1-x} \geq 1 - x/\sqrt{2}, \text{ for } x \in [0,1/2].
\]
Therefore,
\[
D(\gamma_0, x_0, T) \ll x_0^{2-\epsilon}.
\]
In case of \( T \leq x_0^{1/\gamma_0 - \delta_1(\epsilon)} \), we use (10) and get
\[
D(\gamma_0, x_0, T) \ll x_0^{3/2} + x_0^2 T^2 + 2\epsilon T^2 \left( T x_0^{1/\gamma_0} \right)^2
= x_0^{3/2} T^2 + 2\epsilon T^2 \left( T x_0^{1/\gamma_0} \right)^2
= x_0^{3/2} T^2 + 2\epsilon T^2 \left( T x_0^{1/\gamma_0} \right)^2
\]
so we consider a new quadratic function \( g_1(t) = (4 + 2\epsilon) t^2 - 2\sqrt{5} t + 3 \).
If \( x_0^{9/20} \leq T \leq x_0^{1/\gamma_0 - \delta_1(\epsilon)} < x_0^{4/5} \), we get \( t \in \left[ \sqrt{9/20}, \sqrt{1/\gamma_0 - \delta_1(\epsilon)} \right] \)
and
\[
g_1(t) \leq g \left( 1/\gamma_0 - \delta_1(\epsilon) \right) = 4 \left( 1/\gamma_0 - \delta_1(\epsilon) \right) - 2\sqrt{5} \cdot \sqrt{1/\gamma_0 - \delta_1(\epsilon)} + 3 + 2\epsilon/\gamma_0 - 2\epsilon \delta_1(\epsilon)
\leq 4/\gamma_0 - 2\sqrt{5/\sqrt{\gamma_0}} + 3 + 2\epsilon/\gamma_0 + \left( \sqrt{10/\gamma_0 - 4 - 2\epsilon} \right) \delta_1(\epsilon)
\leq 2 - \epsilon.
\]
Therefore,
\[
D(\gamma_0, x_0, T) \ll x_0^{2-\epsilon}.
\]

**Case 3.** When \( T < x_0^{9/20} \).
If \( x_0^{1/\gamma_0 - \delta_1(\epsilon)} < T < x_0^{9/20} \), we use (9) to get
\[
x_0^{2/\gamma_0} T^{-3} \left( x_0^{3/4} T^{5/6 + \epsilon} \right)^2 \ll x_0^{2-\epsilon}
\]
and
\[
x_0^{2/\gamma_0} T^{-3} \left( x_0^{1-\beta} T^{3\beta + \epsilon} \right)^2 \ll x_0^{2-\epsilon}.
\]
Therefore,
\[
D(\gamma_0, x_0, T) \ll x_0^{2/\gamma_0} T^{-3} \left( x_0^{3/4} T^{5/6 + \epsilon} + x_0^{1-\beta} T^{3\beta + \epsilon} \right)^2 \ll x_0^{2-\epsilon}.
\]
If \( T \leq x_0^{1/\gamma_0 - \delta_1(\epsilon)} < x_0^{9/20} \) or \( T < x_0^{9/20} \leq x_0^{1/\gamma_0 - \delta_1(\epsilon)} \), we use (10) to get
\[
x_0^{2/\gamma_0} T^{-3} \left( x_0^{3/4} T^{5/6 + \epsilon} \right)^2 \left( T x_0^{1/\gamma_0} \right)^2 \ll x_0^{2-\epsilon}
\]
and
\[
x_0^{2/\gamma_0} T^{-3} \left( x_0^{1-\beta} T^{3\beta + \epsilon} \right)^2 \left( T x_0^{1/\gamma_0} \right)^2 \ll x_0^{2-\epsilon}.
\]
Therefore, 
\[ D(\gamma_0, x_\nu, T) \ll x_\nu^{2/\nu - 1/T - 3} \left( x_\nu^{3/4} T^{5/6 + \epsilon} + x_\nu^{1 - \beta} T^{3\beta + \epsilon} \right)^2 \left( T x_\nu^{-1/\nu} \right)^2 \ll x_\nu^{2 - \epsilon}. \]

Combining all the above cases, we obtain 
\[ D(\gamma_0, x_\nu, T) = \int_{\gamma_0}^{\gamma_0 + \log^{-2} x_\nu} |\Phi_1(\gamma, x_\nu, T)|^2 d\gamma \ll x_\nu^{2 - \epsilon}. \]

Inequality (20) gives us 
\[ \int_{\gamma_0}^{\gamma_0 + \log^{-2} x_\nu} |\Psi_{2, c}(x)|^2 d\gamma \ll \log^4 x \cdot \int_{\gamma_0}^{\gamma_0 + \log^{-2} x_\nu} |\Phi(\gamma, x_\nu, T)|^2 d\gamma + x_\nu^{2c} \log^{4A + 10} x \]
\[ \ll \log^6 x \cdot \max_{1 \leq t \leq T_0} \int_{\gamma_0}^{\gamma_0 + \log^{-2} x_\nu} |\Phi_1(\gamma, x_\nu, T)|^2 d\gamma + x_\nu^{2c} \log^{4A + 10} x \]
\[ \ll x^{2 - \epsilon}. \]

Let \( \gamma_0 := \frac{1}{(\sqrt{3} + 3/8 - \epsilon)} \gamma_{i+1} := \gamma_i + \frac{1}{\log^2 x_\nu} \) and \( A_i := \{ \gamma \in [\gamma_i, \gamma_{i+1}] : |\Psi_{2, c}(x)| > x \log^{-A} x \} \). Then by Tschebyschev’s inequality, we obtain 
\[ \mu(A_i) \ll x^{-2} \log^2 A x \cdot \int_{\gamma_i}^{\gamma_{i+1}} |\Psi_{2, c}(x)|^2 d\gamma \]
\[ \ll x^{-\epsilon} \log^2 A x \ll x^{-\epsilon}. \]

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