ON SOME SINGULAR MEAN-FIELD GAMES

MARCO CIRANT
Dipartimento di Matematica, Università di Padova
Via Trieste 63, 35121, Padova, Italy

DIOGO A. GOMES
King Abdullah University of Science and Technology (KAUST)
CEMSE Division, Thuwal 23955-6900, Saudi Arabia

EDGARD A. PIMENTEL
Department of Mathematics, Pontifícia Universidade Católica do Rio de Janeiro
22451-900, Rio de Janeiro, Brazil

HÉCTOR SÁNCHEZ-MORGADO
Instituto de Matemáticas, Universidad Nacional Autónoma de México
Céd. México 04510, México

ABSTRACT. Here, we prove the existence of smooth solutions for mean-field games with a singular mean-field coupling; that is, a coupling in the Hamilton-Jacobi equation of the form \( g(m) = -m^{-\alpha} \) with \( \alpha > 0 \). We consider stationary and time-dependent settings. The function \( g \) is monotone, but it is not bounded from below. With the exception of the logarithmic coupling, this is the first time that MFGs whose coupling is not bounded from below is examined in the literature. This coupling arises in models where agents have a strong preference for low-density regions. Paradoxically, this causes the agents move towards low-density regions and, thus, prevents the creation of those regions. To prove the existence of smooth solutions, we consider an approximate problem for which the solutions are known. Then, we prove new a priori bounds for the solutions that show that \( \frac{1}{m} \) is bounded. Finally, using a limiting argument, we obtain the existence of solutions. The proof in the stationary case relies on a blow-up argument and in the time-dependent case on new bounds for \( m^{-1} \).

1. Introduction. Here, we examine singular second-order reduced mean-field games (MFGs) both in the stationary and time-dependent settings. Our goal is to determine conditions that ensure the existence of classical solutions of the following problems:

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Problem 1 (Stationary). Given $H : \mathbb{T}^d \times \mathbb{R}^d \to \mathbb{R}$, $H \in C^\infty$ and $\alpha > 0$, find $u, m \in C^\infty(\mathbb{T}^d)$, $m > 0$, and $\mathcal{H} \in \mathbb{R}$ that solve the (stationary) MFG:
\[
\begin{cases}
-\Delta u(x) + H(x, Du(x)) = \mathcal{H} - m^{-\alpha}(x) & \text{in } \mathbb{T}^d \\
-\Delta m(x) - \text{div}(D_p H(x, Du)m(x)) = 0 & \text{in } \mathbb{T}^d,
\end{cases}
\]
satisfying the following condition:
\[
\int_{\mathbb{T}^d} m(x)\,dx = 1. \tag{2}
\]

Problem 2 (Time-dependent). Given $H : \mathbb{T}^d \times \mathbb{R}^d \to \mathbb{R}$, $H \in C^\infty$, $\alpha > 0$, and $u^T, m^0 \in C^\infty(\mathbb{T}^d)$, $m^0 > 0$, find $u, m \in C^\infty(\mathbb{T}^d \times [0, T])$, $m > 0$, that solve the MFG:
\[
\begin{cases}
-u_t(x, t) + H(x, Du(x, t)) = \Delta u(x, t) - m^{-\alpha}(x, t) & \text{in } \mathbb{T}^d \times [0, T] \\
m_t(x, t) - \text{div}(D_p H(x, Du)m(x, t)) = \Delta m(x, t) & \text{in } \mathbb{T}^d \times [0, T],
\end{cases}
\]
and satisfy the initial-terminal boundary conditions:
\[
\begin{cases}
u(x, T) = u^T(x) & \text{in } \mathbb{T}^d \\
m(x, 0) = m^0(x) & \text{in } \mathbb{T}^d. \tag{4}
\end{cases}
\]

The MFG in Problem 2 arises as a model for the interaction of a large number of rational agents that strongly prefer empty or low-density regions. More precisely, we let $W_s$ be a $d$-dimensional Brownian motion in a filtered probability space $(\Omega, \mathcal{F}, P)$ and let $V$ be a set of admissible controls; that is, bounded progressively measurable processes on $[t, T]$ with values in $\mathbb{R}^d$. Each agent is able to forecast the distribution of all the agents for future times. Hence, knowing $m(x, s)$ for $t \leq s \leq T$, each agent seeks to optimize a stochastic control problem with dynamics
\[
\begin{align*}
dx &= \nu \, ds + \sqrt{2} \, dW_s, \\
x(t) &= x,
\end{align*}
\]
with $\nu \in V$, and cost functional
\[
J(x, t; \nu) = \mathbb{E}^{(x, t)} \left[ \int_t^T L(x(s), \nu(s)) - \frac{ds}{(m(x(s), s))^\alpha} + u^T(x(T)) \right]. \tag{6}
\]
In the above cost, the Lagrangian, $L$, is given by
\[
L(x, \nu) := \sup_{p \in \mathbb{R}^d} \left[ -p \cdot \nu - H(x, p) \right]. \tag{7}
\]
Under suitable regularity conditions for $m$, it is well known that
\[
 u(x, t) = \inf_{\nu \in V} J(x, t; \nu)
\]
solves the first equation in (3). Moreover, again under suitable regularity conditions, the optimal control is given in feedback form as
\[
\nu(s) = -D_p H(x(s), D_x u(x(s), s)).
\]
This, in turn, gives that the law of the diffusion equation in (5), $m$, solves the second equation in (3).

The stationary problem, Problem 1, has a similar interpretation; it describes the ergodic problem associated with the ergodic cost
\[
J(x; \nu) = \lim_{T \to \infty} \frac{1}{T} \mathbb{E}^{(x, 0)} \left[ \int_0^T L(x(s), \nu(s)) - \frac{ds}{(m(x(s), s))^\alpha} \right]. \tag{8}
\]
A solution \((u, \overline{H}, m)\) to Problem 1 also describes the equilibrium configuration of the system, due to the ergodic (limiting) behavior encoded in (8).

To investigate Problems 1 and 2, we introduce two auxiliary, regularized MFGs. In the stationary setting, we examine the problem:

**Problem 3** (Stationary). Given \(H : \mathbb{T}^d \times \mathbb{R}^d \to \mathbb{R}, H \in C^\infty, \alpha > 0, \epsilon > 0\) find \(u^\epsilon, m^\epsilon \in C^\infty(\mathbb{T}^d)\), with \(m^\epsilon > 0\), and \(\overline{H}\) solving the MFG:

\[
\begin{aligned}
-\Delta u^\epsilon + H(x, Du^\epsilon) &= \overline{H} - (m^\epsilon + \epsilon)^{-\alpha} \quad \text{in} \quad \mathbb{T}^d, \\
-\Delta m^\epsilon - \text{div}(D_p H(x, Du^\epsilon) m^\epsilon) &= 0 \quad \text{in} \quad \mathbb{T}^d,
\end{aligned}
\]

satisfying the following condition:

\[
\int_{\mathbb{T}^d} m^\epsilon(x)dx = 1.
\]

In the time-dependent setting, we study the following problem:

**Problem 4** (Time-dependent). Given \(H : \mathbb{T}^d \times \mathbb{R}^d \to \mathbb{R}, H \in C^\infty, \alpha > 0, \epsilon > 0, \) and \(u^T, m^0 \in C^\infty(\mathbb{T}), m^0 > 0, \) find \(u^\epsilon, m^\epsilon \in C^\infty(\mathbb{T}^d \times [0, T])\), \(m^\epsilon > 0\), that solve the MFG:

\[
\begin{aligned}
-u_t^\epsilon(x,t) + H(x, Du^\epsilon(x,t)) &= \Delta u^\epsilon(x,t) - (m^\epsilon(x,t) + \epsilon)^{-\alpha} \quad \text{in} \quad \mathbb{T}^d \times [0, T] \\
 m_t^\epsilon(x,t) - \text{div}(D_p H(x, Du^\epsilon)m(x,t)) &= \Delta m^\epsilon(x,t) \quad \text{in} \quad \mathbb{T}^d \times [0, T],
\end{aligned}
\]

where

\[
\begin{aligned}
 u^\epsilon(x, T) &= u^T(x) \quad \text{in} \quad \mathbb{T}^d \\
 m^\epsilon(x, 0) &= m^0(x) \quad \text{in} \quad \mathbb{T}^d.
\end{aligned}
\]

The existence theory for MFGs has seen significant advances in recent years. A typical MFG with a local coupling is

\[
\begin{aligned}
-u_t + \frac{|Du|^2}{2} &= \Delta u + g(m) \quad \text{in} \quad \mathbb{T}^d \\
 m_t - \text{div}(m Du) &= \Delta m.
\end{aligned}
\]

In [33, 34, 35], the authors proved several a priori estimates for weak solutions of MFGs. Because of the quadratic structure, the preceding case was one of the first MFGs to be studied. For \(g\) increasing and bounded from below, the proof of the existence of a solution was outlined in [36] and detailed in [5] using the Hopf-Cole transformation (also see [29] and [30]). One of the first examples of a singular coupling was presented in [31] to model an income function. General parabolic MFGs with a nonlocal coupling were addressed in [36] (see also [2]). The existence of weak solutions in the case of local couplings bounded by below and of polynomial growth in \(g\) was considered in [3], in the variational setting, and in [40], using general partial differential equations (PDE) methods (see [39] for the planning problem). Finally, the parabolic case with a polynomial nonlinearity \(g\) was examined in [22] for subquadratic Hamiltonians and in [23] for superquadratic Hamiltonians, see also [9]. In all these cases, the mean-field couplings were non-singular for \(m \geq 0\) and bounded from below. Explicit solutions for MFGs have been studied in [19], [12], and [18]. The study of congestion problems provides examples of singular MFGs. A model problem is

\[
\begin{aligned}
-u_t + \frac{|Du|^2}{2m^\alpha} &= \Delta u, \\
 m_t - \text{div}(m^{1-\alpha} Du) &= \Delta m
\end{aligned}
\]
with $\alpha > 0$. In the stationary case, the congestion problem was investigated in [17], and in the time-dependent case in [27] and [28] for a small terminal time, $T$. For the general case of stationary MFGs in the presence of congestion, we refer the reader to [11] and [10]. A closely related class of problems are MFGs with constraints. These were studied in a series of works in [41], [6], and [37]. An account of the regularity theory for MFGs can be found at [24]. Recently, a new class of methods based on monotone operator theory has been developed to build numerical methods in [1] and [26] to address the existence problem for a wide class of MFGs in [14], [16], and [15]. These constructions, provide however the existence of (very) weak solutions, in contrast with our results here, where our solutions are smooth.

In the problems discussed above, the Lagrangian corresponding to the Hamilton-Jacobi equation is bounded from below. Thus, it is relatively easy to get lower bounds for $u$ using the maximum principle or, alternatively, a control theory interpretation of the problem. Bounds for $u$ in $L^\infty$ are essential to prove the regularity of solutions. Thus, if the coupling is not bounded from below, the regularity problem poses substantial challenges. Prior to this paper, the only known existence results for classical solutions of MFGs with singularities that are not bounded from below were those for a logarithmic nonlinearity, $g(m) = \ln m$. This case was examined in the stationary setting in [20], [25], and [38], in the time-dependent setting in [21], and in a specific case in [29]. For weak solutions of first-order mean-field games, the results in [4] do not require lower bounds on $g(m)$ as $m \to 0$. However, the techniques used to study logarithmic nonlinearities do not apply to power-like nonlinearities as the ones in (3), except in the one-dimensional case, where the methods in [19] and [18] can be adapted accordingly.

In Section 2, we discuss the main technical assumptions under which our results are valid (Assumptions A1-A5). For example, we observe that the following Hamiltonian satisfies these assumptions:

$$H(x, p) = a(x) \left(1 + |p|^2\right)^{\frac{\gamma}{2}} + V(x),$$

where $a, V \in C^\infty(\mathbb{T}^d)$, with $a > 0$ and $\gamma$ in a suitable range to be discussed later (see Section 2). Our first main result is the following theorem:

**Theorem 1.1 (Stationary case).** Suppose Assumptions A1-A3 and A5 hold (see Section 2). Then, there exists a unique solution $(u, H, m)$ of Problem 1.

The uniqueness part of the theorem follows the classical Lasry-Lions method, so we omit it here. To handle the singular coupling in the stationary case, we study Problem 3 and use a blow-up method to obtain that $m'$ is bounded from below by a constant independent on $\epsilon$. Our method is related to the one in [8], in the setting of focusing MFG, where the coupling is a function, which is unbounded from below as $m \to \infty$: in that case, it was shown that some integrability of $m$ is enough to obtain boundedness of $m$ in $L^\infty$. Here, a similar procedure applies: the integrability of $m^{-1}$ implies bounds from above for $m^{-1}$, which allows passing to the limit in Problem 3. Second-order estimates that use the monotonicity of $-m^{-\alpha}$ give that $(m')^{-1}$ is bounded in some $L^p$ space. In Section 3.1, we establish preliminary estimates. Then, in Section 3.2, we conclude the proof of Theorem 1.1.

Our second main result concerns the time-dependent setting:

**Theorem 1.2 (Time-dependent case).** Suppose Assumptions A1-A2 and A4 hold (see Section 2). Then, there exists a unique solution $(u, m)$ to Problem 2.
As in before, the uniqueness part of the preceding theorem follows the classical Lasry-Lions method, so we omit it here. To prove Theorem 1.2, we use a limiting argument. For that, we rely on the properties of solutions to Problem 4. The existence of solutions to Problem 4 follows from standard arguments, see, for example, [22] or [23]. Here, we obtain a priori estimates for the solutions \((u^\epsilon, m^\epsilon)\) to Problem 4 that are uniform in \(\epsilon\). In Section 4.1, we prove our main estimate, which gives bounds for the coupling in Problem 4, \(- (m^\epsilon + \epsilon)^{-\alpha}\), that are uniform in \(\epsilon\). Subsequently, in Sections 4.2 and 4.3, we examine the Fokker-Planck equation and the Hamilton-Jacobi equation in (11). As remarked previously, a critical point in our estimates for the Hamilton-Jacobi equation is that the coupling in Problem 4 is not bounded from below uniformly in \(\epsilon\). Thus, to obtain bounds for \(u^\epsilon\) is \(L^\infty\), we use a delicate argument based on the nonlinear adjoint method introduced in [13] (also see [43]). Finally, in Section 4.4, we give uniform Lipschitz bounds for \(u^\epsilon\). These estimates ensure the compactness of the solution \((u^\epsilon, m^\epsilon)\). Thus, by extracting a subsequence, we have

\[(u^\epsilon, m^\epsilon) \to (u, m),\]

as \(\epsilon \to 0\), for some \((u, m)\) that solves (3)-(4) in the weak sense and inherits the regularity of \((u^\epsilon, m^\epsilon)\). In Section 4.4, we conclude the Theorem 1.2 along these lines.

2. Main assumptions. Here, we discuss the main assumptions used in this paper. Our first assumption, stated next, requires that the Hamiltonian satisfies standard growth and coercivity properties; these are similar to the ones in the literature.

A 1. The Hamiltonian \(H : T^d \times \mathbb{R}^d \to \mathbb{R}^d\) is smooth. Also, for fixed \(x, p \mapsto H(x, p)\) is a strictly convex function. In addition, there are constants, \(C_1, C_2 > 0\), such that

\[
\frac{1}{C_2} |p|^\gamma - C_1 \leq H(x, p) \leq C_1 + C_2 |p|^\gamma,
\]

and, without loss of generality, we suppose that \(H(x, p) \geq 0\). In addition, we have

\[
|D_p H(x, p)| \leq C_1 + C_2 |p|^{\gamma - 1}
\]

and

\[
|D_x H(x, p)| \leq C_1 + C_2 H(x, p).
\]

For simplicity, in the preceding assumption as in the statements of the problems, we assume that the data is smooth. Certainly this is not necessary, many of the bounds here require less regularity. However, the arguments used require certain level of differentiability of solutions. For example, the proof of Lemma 3.2 requires differentiating the Hamilton-Jacobi equation twice. Thus, stating the minimal regularity needed for our results can be fairly involved.

The next assumption requires a lower bound for the Lagrangian given by (7). We recall that if \(v = -D_p H(x, p)\), then

\[
L(x, v) = D_p H(x, p) \cdot p - H(x, p).
\]

A 2. There are constants, \(C_1, C_2 > 0\), such that

\[
D_p H(x, p) \cdot p - H(x, p) \geq \frac{H(x, p)}{C_2} - C_1.
\]

Our next assumption plays a major role in the second-order estimates for the stationary problem.
A 3. For every $\delta \in (0, 1)$, there exists $C_\delta > 0$ so that
\[ \text{Tr}(D^2_{xp}H(x,p)M) \leq \delta \text{Tr}(D^2_{pp}H(x,p)M^2) + C_\delta H(x,p), \]
for any matrix $M$.

For the time-dependent problem, our main technical condition concerns the growth of the Hamiltonian.

A 4. We assume the Hamiltonian’s growth parameter, $\gamma$, satisfies
\[ 1 < \gamma < \frac{d+2}{d+1}. \]

The preceding assumption is used together with the non-linear adjoint method to get bounds for $u$ in $L^\infty$. The main reason for this constraint is that the non-linearity is not bounded from below. Thus, proving lower bounds for $u$, which is usually an immediate corollary of the maximum principle or of the optimal control representation, becomes a non-trivial task. We do not know if this assumption is necessary for the existence of smooth solutions; the existence of weak solutions can be proved under weaker requirements, as discussed in the works mentioned in the introduction.

For the stationary problem, we need the following assumption:

A 5. We assume that $\alpha > \bar{\alpha}_{d,\gamma}$, where
\[ \bar{\alpha}_{d,\gamma} = \begin{cases} 
0 & \text{if } \gamma < \frac{d}{d-1}, \\
1 & \text{if } \gamma \geq 2 \text{ and } d = 2, \\
\max \left( \frac{\gamma}{(3-d)(d-2)}, 1 \right) & \text{if } \frac{d}{d-1} \leq \gamma < \frac{d-2}{d-3} \text{ and } d \geq 3. 
\end{cases} \]

This assumption implies the condition in Corollary 3.2. Note that $\bar{\alpha}_{N,\gamma} \to \infty$ as $\gamma \to ((d-2)/(d-3))^-$. This assumption addresses one of their key difficulties in the stationary case, how to prove lower bounds for $m^\epsilon$. In the time dependent case, these bounds are given by an iterative argument in Theorem 4.2 that ensures that the lower bounds from the initial condition propagate forward. In the next Section, we examine the stationary case and prove Theorem 1.1.

3. The stationary case. The proof of Theorem 1.1 relies on a limiting argument. In turn, it depends on a priori estimates for the solutions of Problem 3, uniform in $\epsilon$. We produce these estimates by using a blow-up method, see [8]. Preliminary results, central to our argument, are developed in what follows.

3.1. Elementary estimates. We start by proving a first-order estimate for solutions of Problem 4.

Lemma 3.1 (First-order estimates). Let $(u^\epsilon, \overline{H}, m^\epsilon)$ be a solution to (9). Suppose Assumptions A1-A2 hold. Then, there exists a constant, $C > 0$, such that
\[ \int_{\mathbb{T}^d} \frac{1}{(m^\epsilon + \epsilon)^\alpha} \, dx + \int_{\mathbb{T}^d} H(x, Du^\epsilon) \, dx + \int_{\mathbb{T}^d} H(x, Du^\epsilon) m^\epsilon \, dx + |\overline{H}^\epsilon| \leq C. \]

Proof. For ease of presentation, we drop the superscript $\epsilon$ in what follows. We integrate the Hamilton-Jacobi equation in (9) to obtain
\[ -C \leq \int_{\mathbb{T}^d} H(x, Du)dx = \overline{H} - \int_{\mathbb{T}^d} \frac{dx}{(m + \epsilon)^\alpha} \leq \overline{H}. \]
The former inequality gives a lower bound for the ergodic constant, \( \overline{H} \). Next, we fix a number \( \delta \ll 1 \) and multiply the first equation in (9) by \( m - \delta \). Then, we multiply the second equation in (9) by \(-u\) and sum them. Integrating by parts, we get

\[
\int_{\mathbb{T}^d} (H - D_p H \cdot Du) m dx - \delta \int_{\mathbb{T}^d} H dx = (1 - \delta) \overline{H} - \int_{\mathbb{T}^d} \frac{m - \delta}{(m + \epsilon)^\alpha} dx.
\]

By rearranging the terms in the previous equality, we have

\[
(1 - \delta) \overline{H} + \frac{1}{C_2} \int_{\mathbb{T}^d} |Du|^\gamma m dx + \delta \int_{\mathbb{T}^d} H dx \leq \int_{\mathbb{T}^d} \frac{m - \delta}{(m + \epsilon)^\alpha} dx + C_1.
\]

Now, we use the convexity of \( z \mapsto \frac{z^{1-\alpha}}{\alpha - 1} \) to conclude that

\[
\int_{\mathbb{T}^d} \frac{m - \delta}{(m + \epsilon)^\alpha} dx \leq \frac{(\delta + \epsilon)^{1-\alpha}}{\alpha - 1} - \frac{1}{\alpha - 1} \int_{\mathbb{T}^d} \frac{dx}{(m + \epsilon)^{\alpha-1}}.
\]

The former inequality, together with (15), implies there exists a constant, \( C > 0 \), such that

\[
\overline{H} + \int_{\mathbb{T}^d} |Du|^\gamma m dx + \int_{\mathbb{T}^d} H dx + \int_{\mathbb{T}^d} \frac{dx}{(m + \epsilon)^{\alpha-1}} dx \leq C.
\]

Combined with (14), the prior inequality concludes the proof.

Lemma 3.2 (Second-order estimates). Let \((u^\epsilon, H^\epsilon, m^\epsilon)\) be a solution to (9). Suppose Assumptions A1-A2 and A3 hold. Then,

\[
\int_{\mathbb{T}^d} \frac{|Dm|^2}{(m + \epsilon)^{\alpha+1}} dx + \int_{\mathbb{T}^d} \text{Tr} \left( D^2_{pp} H (D^2 u)^2 \right) m dx \leq C
\]

for some positive constant \( C \).

Proof. As before, we omit the superscript \( \epsilon \) throughout the proof. First, we apply the Laplace operator to the first equation in (9) to get

\[
\Delta \Delta u + \text{Tr} \left( D^2_{pp} H (D^2 u)^2 \right) + \Delta H + 2 \text{Tr} (D^2_{px} HD^2 u) + D_p HD (\Delta u) = \text{div} \left( \frac{\alpha Dm}{(m + \epsilon)^{\alpha+1}} \right).
\]

By multiplying (17) by \( m \), integrating by parts, using the second equation in (9) and Assumption A3, we obtain

\[
\alpha \int_{\mathbb{T}^d} \frac{|Dm|^2}{(m + \epsilon)^{\alpha+1}} dx + \int_{\mathbb{T}^d} \text{Tr} \left( D^2_{pp} H (D^2 u)^2 \right) m dx \leq - \int_{\mathbb{T}^d} (\Delta H + 2 \text{Tr}(D^2_{px} HD^2 u)) m dx
\]

\[
\leq \int_{\mathbb{T}^d} \left( |D^2_{xx} H| + \delta \text{Tr}(D^2_{pp} H (D^2 u)^2) + C_\delta H \right) m dx.
\]

By choosing \( \delta \in (0, 1) \) and using the estimates in Lemma 3.1, we conclude

\[
\int_{\mathbb{T}^d} \frac{|Dm|^2}{(m + \epsilon)^{\alpha+1}} dx + \int_{\mathbb{T}^d} \text{Tr} \left( D^2_{pp} H (D^2 u)^2 \right) m dx \leq C
\]

for some constant \( C > 0 \), which finishes the proof.
3.2. The blow-up and Proof of Theorem 1.1. Under suitable assumptions, the following proposition states that integrability of \((m^\epsilon + \epsilon)^{-1}\) is sufficient to obtain bounds from below for \(m^\epsilon + \epsilon\).

**Proposition 3.1.** Suppose Assumptions A1-A2 hold. Fix \(K > 0\) and assume
\[
p > \frac{\alpha d}{\gamma'}.
\] (18)

Then, there exists \(C > 0\), independent on \(\epsilon\), such that for any solution \((u^\epsilon, H^\epsilon, m^\epsilon)\) to Problem 3 satisfying
\[
\int_{\mathbb{T}^d} \frac{1}{(m^\epsilon + \epsilon)^p} \, dx \leq K,
\]
we have
\[
m^\epsilon + \epsilon \geq C
\] (19)
for all \(\epsilon\).

**Proof.** We argue by contradiction. Suppose the claim in the proposition is false. Then, there are \(\eta^\epsilon > 0, x^\epsilon \in \mathbb{T}^d\), such that
\[
0 < \eta^\epsilon := m^\epsilon(x^\epsilon) + \epsilon = \min_{\mathbb{T}^d}(m^\epsilon + \epsilon) \to 0, \quad \text{as} \quad \epsilon \to 0.
\]

For ease of presentation, we proceed with several steps.

**Step 1.** We define \(a^\epsilon := (\eta^\epsilon)^{\alpha/\gamma'}\). Next, we consider the following blow-up sequences:
\[
v^\epsilon(x) := (a^\epsilon)^{\gamma'} - 2 u^\epsilon(x^\epsilon + a^\epsilon x), \quad \mu^\epsilon(x) := \frac{m^\epsilon(x^\epsilon + a^\epsilon x) + \epsilon}{\eta^\epsilon}.
\] (20)

To simplify the notation and because no ambiguity occurs, in the remaining part of this proof, we omit the superscript \(\epsilon\) in \(a^\epsilon, \mu^\epsilon,\) and \(v^\epsilon\).

Then, \((v, \mu)\) solves
\[
\begin{cases}
-\Delta v + H^\epsilon(x,Dv) = a^{\gamma'} \bar{H} - a^{\gamma'} (\eta \mu)^{-\alpha} \\
-\Delta \mu - \text{div}(DpH^\epsilon(x,Dv) \mu) + \frac{\xi}{\eta} \text{div}(DpH^\epsilon(x,Dv)) = 0
\end{cases}
\] (21)

where \(H^\epsilon(x,p) = a^{\gamma'} H(x^\epsilon + a^\epsilon x, a^{1-\gamma'} p)\). Because \(a \to 0\), \(H^\epsilon\) satisfies Assumptions A1-A2, where \(C_1, C_2, \gamma\) are the same as for \(H\). In particular,
\[
|DpH^\epsilon(x,p)| \leq aC_1 + C_2 |p|^{\gamma - 1}.
\] (22)

Moreover, \(\mu(0) = 1\) and \(\mu \geq 1\) everywhere, so
\[
0 \leq a^{\gamma'} (\eta \mu)^{-\alpha} \leq 1
\] (23)
due to the choice of \(a\).

**Step 2.** The gradient of \(v\) is bounded in \(L^r(B_2(0))\) for any \(r > 1\). It suffices to observe that
\[
| - \Delta v + H^\epsilon(x,Dv)| \leq 1 + a^{\gamma'} |\bar{H}| \leq 2 \quad \text{in} \quad \mathbb{T}^d
\]
by (23) and (13) when \(\epsilon\) is small enough. Therefore, we apply the integral Bernstein estimates for viscous Hamilton-Jacobi-Bellman (HJB) equations to get
\[
\|Dv\|_{L^r(B_2(0))} \leq C_r.
\] (24)

For the integral Bernstein method in the special case \(H(x,p) = |p|^\gamma\), see [32]. For the general case, we argue as in [7] or [38].
Moreover, from elliptic regularity, we gather that
\[ \|D^2v\|_{L^q(B_2(0))} \leq C_q, \] (25)
for any \( q > 1 \).

**Step 3.** Now, we show that \( \mu \) is uniformly bounded from above in \( B_1(0) \). Indeed, according to (24) and (25), for any \( q > 1 \), \( B(x) = -D_pH^\epsilon(x,Dv(x)) \) satisfies
\[ \|B\|_{W^{1,q}(B_2(0))} \leq C_q. \]
Moreover, \( \mu \) is a positive solution to the following equation in divergence form
\[ -\text{div} \left( D\mu + B\mu - \frac{\epsilon}{\eta}B \right) = 0. \]

By the definition of \( \eta \), we have \( \eta \geq \epsilon \). Thus, \( 0 \leq \epsilon/\eta \leq 1 \). Hence, we apply Harnack’s inequality (see, for example, [42]) to get
\[ \max_{B_1(0)} \mu = \max_{B_1(0)} (\epsilon/\eta) \mu \leq C (\min_{B_1(0)} \mu + k) = C (\mu(0) + k) \]
for some positive constants \( C, k \) depending only on the upper bound of \( \|B\|_{L^q(B_2(0))} \).

**Step 4.** Finally, evaluating the integral of \((1/\mu)^p\) on \( T^\epsilon \) gives
\[ \int_{T^\epsilon} \left( \frac{1}{\mu} \right)^p dx \geq \delta_p > 0. \] (26)

As \( \epsilon \to 0 \), in view of the assumptions of the proposition. However, (27) contradicts (26).

First, we examine the case \( \gamma < d/(d-1) \), where no additional assumptions on \( \alpha \) are needed.

**Corollary 3.1.** Suppose Assumptions A1-A2 hold and that \( \gamma < d/(d-1) \). Then, for any solution \((u^\epsilon, H^\epsilon, m^\epsilon)\) to Problem 3, there exists \( C > 0 \), not depending on \( \epsilon \), such that
\[ m^\epsilon + \epsilon \geq C \] (28)
for all \( \epsilon \).

**Proof.** It suffices to apply Proposition 3.1 with \( p = \alpha \) and the estimate (13). Note that (18) reads \( \gamma' > d \).

Next, as an alternative to assuming \( \gamma < d/(d-1) \), we use a different proof technique that instead requires \( \alpha \) to not be too small. We do not assume, however, that \( \gamma \geq \frac{d}{d-1} \), just require a distinct condition on relating \( \alpha \) and \( \gamma \).

**Corollary 3.2.** Suppose Assumptions A1-A2 hold, \( d \geq 3 \), \( \gamma < (d-2)/(d-3) \), and
\[ \alpha > \max \left( \frac{\gamma}{(3-d) + d - 2}, \frac{1}{d-3} \right). \] (29)
Then, there exists a constant, \( C > 0 \), independent of \( \epsilon \), such that for any solution \((u^\epsilon, H^\epsilon, m^\epsilon)\) to Problem 3, we have
\[ m^\epsilon + \epsilon \geq C \] (30)
for all \( \epsilon \).
Proof. If $d > 2$, $W^{1,2}(\mathbb{T}^d)$ is continuously embedded into $L^\frac{2d}{d-2}(\mathbb{T}^d)$; then, we have

$$\left(\int_{\mathbb{T}^d} (m + \epsilon)^{(1-\alpha)\frac{d}{d-2}} \, dx\right)^\frac{d-2}{2} \leq C \left( \int_{\mathbb{T}^d} |Dm^\frac{1-\alpha}{2} |^2 \, dx + \int_{\mathbb{T}^d} (m + \epsilon)^{1-\alpha} \, dx \right).$$

The right-hand side of the previous inequality is bounded in view of (13) and (16) because

$$|Dm^\frac{1-\alpha}{2}|^2 = \left(\frac{1-\alpha}{2}\right)^2 m^{-(\alpha+1)} |Dm|^2.$$ 

Finally, because (29) guarantees (18), we apply Proposition 3.1 with $p = (\alpha - 1)\frac{d}{d-2}$.

In the case $d = 2$, arguing as in Corollary 3.2 gives that

$$\int_{\mathbb{T}^d} (m + \epsilon)^{-p} \, dx \leq C$$

for all $p > 0$, provided $\alpha > 1$ and the conditions in Proposition 3.1 hold. Finally, we prove our main result in the stationary case, Theorem 1.1. The case $d = 1$ is similar.

Proof of Theorem 1.1. Lemma 3.1 shows that $\bar{H}^\epsilon$ is uniformly bounded, thus we can extract a convergent subsequence. Depending on the values of $\gamma$ and $\alpha$, Corollaries 3.1-3.2 provide boundedness from below of $m^\epsilon + \epsilon$ independent of $\epsilon$, which implies the boundedness of $-(m^\epsilon + \epsilon)^{-\alpha}$ in $L^\infty(\mathbb{T}^d)$. Consequently, integral Bernstein estimates for viscous HJB equations (see [32, 38]) guarantee that $Du^\epsilon$ is bounded in $L^r(\mathbb{T}^d)$ for all $r$. This is enough to have the optimal drift $-D_pH(x, Du^\epsilon)$ in $L^r(\mathbb{T}^d)$ for all $r$. Hence, $m^\epsilon$ is in $C^{0,\alpha}$ by classical regularity for the Kolmogorov equation. By bootstrapping in the two equations, we have enough regularity for $u^\epsilon$ and $m^\epsilon$ to pass to the limit in (9) and obtain smooth solutions.

4. Time-dependent problem.

4.1. Estimates for the regularized MFG. Now, we consider estimates for solutions to Problem 4. We begin with an estimate that is a consequence of the optimal control representation discussed in the Introduction. Next, we use a modification of a standard technique in MFGs to prove bounds on the coupling $\frac{1}{\epsilon} (m^\epsilon + \epsilon)$ that are uniform in $\epsilon$. These estimates are essential to prove the convergence of solutions as $\epsilon \to 0$.

Proposition 4.1. Assume A1 holds. Let $(u^\epsilon, m^\epsilon)$ solve Problem 4. Then, for any solution, $\zeta : \mathbb{T}^d \times (t,T] \to \mathbb{R}$, of the heat equation

$$\zeta_t - \Delta \zeta = 0,$$ 

with $\zeta(x,t) = \zeta_0$, we have the following upper bound:

$$\int_{\mathbb{T}^d} u^\epsilon(x,t)\zeta_0(x) \, dx \leq -\int_t^T \int_{\mathbb{T}^d} \zeta(x,s) \, ds \, dx + \int_{\mathbb{T}^d} u^T(x)\zeta(x,T) \, dx.$$ 

(32)

Remark 4.1. The solution of the heat equation (31) gives the density of the solution to the stochastic differential equation (5) with $v = 0$. This choice of control is sub-optimal for the corresponding control problem and, hence, the bound in (32).
Proof. First, we multiply (31) by \( u \) and multiply the first equation in (11) by \( \zeta \). Next, we subtract these equations and integrate in \( \mathbb{T}^d \) to conclude that

\[
\frac{d}{dt} \int_{\mathbb{T}^d} u' \zeta \, dx = \int_{\mathbb{T}^d} \left( H(x, Du') + \frac{1}{(m^\epsilon + \epsilon)^\alpha} \right) \zeta \, dx.
\]

Because of A1, \( H \geq 0 \). Thus, integrating in time, we obtain the result. \( \square \)

Natural choices for \( \zeta_0 \) include the Lebesgue measure, the measure \( m^0 \), or \( \zeta_0 = \delta_{x_0} \) for \( x_0 \in \mathbb{T}^d \). The latter yields pointwise estimates.

**Corollary 4.1.** Assume A1 holds. Let \((u', m')\) solve Problem 4. Then, we have the following two estimates:

\[
\int_{\mathbb{T}^d} u'(x,0)m^0 \, dx \leq -\int_0^T \int_{\mathbb{T}^d} \frac{\mu(x,t) \, dx \, dt}{(m^\epsilon(x,t) + \epsilon)\alpha} + \int_{\mathbb{T}^d} u'(x,T)\mu(x,T) \, dx,
\]

(33)

where \( \mu(x,t) \) is the solution to the heat equation (31) with \( \mu(x,0) = m^0 \), and

\[
\int_{\mathbb{T}^d} u'(x,0) \, dx \leq -\int_0^T \int_{\mathbb{T}^d} \frac{dx \, dt}{(m^\epsilon(x,t) + \epsilon)\alpha} + \int_{\mathbb{T}^d} u'(x,T) \, dx.
\]

(34)

**Proof.** Both estimates follow from Proposition 4.1, by setting \( \zeta_0 = m^0 \) and \( \zeta_0 = 1 \).

Next, we obtain a first-order estimate for solutions to Problem 4. For \( f : \mathbb{T}^d \to \mathbb{R}^d \), we define

\[
oscf = \sup_{x \in \mathbb{T}^d} f(x) - \inf_{x \in \mathbb{T}^d} f(x).
\]

**Proposition 4.2.** Assume A1-A2 hold and \( \alpha \neq 1 \). Then, there exists a constant \( C > 0 \), such that for any solution \((u', m')\) of Problem 4, we have

\[
\int_0^T \int_{\mathbb{T}^d} cH(x, Du')m^\epsilon + \frac{m^\epsilon(x+\epsilon)\alpha - 1}{\alpha - 1} \, dx \, dt \leq CT + C \oscf u^T.
\]

(35)

**Proof.** We have

\[
\frac{d}{dt} \int_{\mathbb{T}^d} u' m^\epsilon \, dx + \int_{\mathbb{T}^d} (D_pH \cdot Du' - H)m^\epsilon \, dx = \int_{\mathbb{T}^d} \frac{m^\epsilon}{(m^\epsilon + \epsilon)\alpha} \, dx.
\]

From the preceding identity, we get

\[
\int_0^T \int_{\mathbb{T}^d} (D_pH \cdot Du' - H)m^\epsilon \, dx \, dt = \int_0^T \int_{\mathbb{T}^d} \frac{m^\epsilon}{(m^\epsilon + \epsilon)\alpha} \, dx + \int_{\mathbb{T}^d} (u'(x,0)m^\epsilon(x,0) - u'(x,T)m(x,T)) \, dx.
\]

Accordingly, using Assumption A2,

\[
c \int_0^T \int_{\mathbb{T}^d} H(x, Du')m^\epsilon \, dx \, dt \leq \int_0^T \int_{\mathbb{T}^d} \frac{m^\epsilon}{(m^\epsilon + \epsilon)\alpha} \, dx \, dt + \int_{\mathbb{T}^d} (u'(x,0)m^\epsilon(x,0) - u'(x,T)m^\epsilon(x,T)) \, dx + CT.
\]
Now, we use (33) to conclude that
\[
c \int_0^T \int_{\mathbb{T}^d} H(x, Du^\epsilon)m^\epsilon \, dx \, dt \leq CT + \int_0^T \int_{\mathbb{T}^d} u^\epsilon(x, T)(\mu(x, T) - m^\epsilon(x, T)) \, dx + \int_0^T \int_{\mathbb{T}^d} \frac{(m^\epsilon - \mu)}{(m^\epsilon + \epsilon)^\alpha} \, dx \, dt.
\]
Because both \(\mu(x, T)\) and \(m^\epsilon(x, T)\) are probability measures, we have
\[
\int_{\mathbb{T}^d} u^\epsilon(x, T)(\mu(x, T) - m^\epsilon(x, T)) \, dx \leq \text{osc} \, u^\epsilon(\cdot, T).
\]
Next, because
\[
z \mapsto z^{1-\alpha}/(\alpha - 1)
\]
is a convex function and \(\alpha \neq 1\), we have
\[
\frac{m^\epsilon - \mu}{(m^\epsilon + \epsilon)^\alpha} \leq \frac{(\mu + \epsilon)^{1-\alpha} - (m^\epsilon + \epsilon)^{1-\alpha}}{\alpha - 1}.
\]
Therefore,
\[
c \int_0^T \int_{\mathbb{T}^d} H(x, Du^\epsilon)m^\epsilon \, dx \, dt + \int_0^T \int_{\mathbb{T}^d} \frac{(m^\epsilon + \epsilon)^{1-\alpha}}{\alpha - 1} \, dx \, dt \\
\leq CT + \text{osc} \, u^\epsilon(\cdot, T) + \int_0^T \int_{\mathbb{T}^d} \frac{(\mu + \epsilon)^{1-\alpha}}{\alpha - 1} \, dx \, dt.
\]
Finally, because \(\min m^0 \leq \mu \leq \max m^0\), we have that \((\mu + \epsilon)^{1-\alpha}\) is uniformly bounded and, thus, (35) follows. \(\square\)

**Proposition 4.3.** Assume \(A1-A2\) hold. Then, there exist constants, \(c > 0\) and \(C > 0\), independent on \(\epsilon\), such that for any solution \((u^\epsilon, m^\epsilon)\) of Problem 4, we have
\[
\int_0^T \int_{\mathbb{T}^d} \frac{1}{(m^\epsilon + \epsilon)^\alpha} \, dx \, dt + c \int_0^T \int_{\mathbb{T}^d} H(x, Du^\epsilon)m^\epsilon \, dx \, dt + c \int_0^T \int_{\mathbb{T}^d} H(x, Du^\epsilon) \, dx \, dt \leq C.
\]

**Proof.** We begin by multiplying the first equation in (11) by \((m^\epsilon - m^0)\) and the second one by \((u^T - u^\epsilon)\). Next, adding the resulting expressions and integrating by parts, we obtain:
\[
-\frac{d}{dt} \int_{\mathbb{T}^d} (u^\epsilon - u^T)(m^\epsilon - m^0) \, dx + \int_{\mathbb{T}^d} H(m^\epsilon - m^0) - m^\epsilon D_p H \cdot D(u^\epsilon - u^T) \, dx \\
= \int_{\mathbb{T}^d} (u^T \Delta m^\epsilon - m^0 \Delta u^\epsilon) \, dx - \int_{\mathbb{T}^d} \frac{m^\epsilon - m^0}{(m^\epsilon + \epsilon)^\alpha} \, dx.
\]
Integrating in \([0, T]\), we have
\[
\int_0^T \int_{\mathbb{T}^d} \frac{m^0 + \epsilon}{(m^\epsilon + \epsilon)^\alpha} \, dx \, dt \\
= \int_0^T \int_{\mathbb{T}^d} (m^\epsilon + \epsilon)^{1-\alpha} + (H - D_p H \cdot Du^\epsilon)m^\epsilon - Hm^0 \, dx \, dt \\
+ \int_0^T \int_{\mathbb{T}^d} (u^T (- \text{div}(D_p Hm^\epsilon) - \Delta m) + m^0 \Delta u^\epsilon) \, dx \, dt \\
\leq C + \int_{\mathbb{T}^d} u^T(m^0 - m^\epsilon) \, dx - \int_0^T \int_{\mathbb{T}^d} cHm^\epsilon + Hm^0 + Du \cdot Dm^0 \, dx \, dt
\]
The prior inequality follows from the boundedness of
\[ \int_0^T \int_{\mathbb{R}^d} (m^\epsilon + \epsilon)^{1-\alpha} dx, \]
which holds, for 0 < \alpha \leq 1, because \( m^\epsilon \) is a probability density, and, for \alpha > 1, due to (35).

4.2. Regularity for the Fokker-Planck equation. Next, we use the structure of the Fokker-Planck equation to improve the a priori integrability for the singularity. In what follows, we set
\[ 2^* = \frac{2d}{d-2}, \]
the Sobolev conjugated exponent of 2.

**Theorem 4.1.** Assume A1-A2 hold. There exists \( C > 0 \), independent of \( \epsilon \), such that for any solution \( (u^\epsilon, m^\epsilon) \) to Problem 4, we have
\[ \left\| \frac{1}{m^\epsilon + \epsilon} \right\|_{L^\infty([0,T];L^{\frac{2(2-\gamma)}{2-\gamma}}(\mathbb{R}^d))} + \int_0^T \left\| \frac{1}{m^\epsilon + \epsilon} \right\|_{L^{\frac{2(2-\gamma)}{2-\gamma}}(\mathbb{R}^d)} \leq C, \]
uniformly in \( \epsilon \).

**Remark.** Although for \( p = \alpha \frac{2-\gamma}{2-\gamma} \), \( L^p \) is not a Banach space, the estimate in the theorem is valid without the requirement \( \alpha \frac{2-\gamma}{2-\gamma} \geq 1 \).

**Proof.** Here to simplify the notation, we drop the superscript \( \epsilon \). Fix \( \beta > 0 \), arbitrarily. From the second equation in (11), we have
\[
\frac{d}{dt} \int_{\mathbb{R}^d} \frac{dx}{(m + \epsilon)^\beta} = -\beta(\beta + 1) \left( \int_{\mathbb{R}^d} \frac{D_p H \cdot Dm}{(m + \epsilon)^{\beta+1}} dx + \int_{\mathbb{R}^d} \frac{|Dm|^2}{(m + \epsilon)^{\beta+2}} dx \right)
\leq \frac{\beta(\beta + 1)}{2} \left( \int_{\mathbb{R}^d} \frac{|D_p H|^2}{(m + \epsilon)^\beta} dx - \int_{\mathbb{R}^d} \frac{|Dm|^2}{(m + \epsilon)^{\beta+2}} dx \right)
\leq \frac{\beta(\beta + 1)}{2} \int_{\mathbb{R}^d} \frac{|D_p H|^2}{(m + \epsilon)^\beta} dx - \frac{2(\beta + 1)}{\beta} \int_{\mathbb{R}^d} |D(m + \epsilon)^{\frac{\beta}{2}}|^2 dx.
\]
(36)

Using Young’s inequality, we get
\[
\frac{\beta(\beta + 1)}{2} \int_{\mathbb{R}^d} \frac{|D_p H|^2}{(m + \epsilon)^\beta} dx \leq C_\delta \int_{\mathbb{R}^d} \frac{dx}{(m + \epsilon)^{\beta+1}} + \delta \int_{\mathbb{R}^d} \frac{dx}{(m + \epsilon)^{\beta+2\gamma/(2-\gamma)}}.
\]
(37)

Integrating (36) in time in \([0, \tau]\), with \( 0 \leq \tau \leq T \) and using (37), we obtain
\[
\int_{\mathbb{R}^d} \frac{dx}{(m(x, \tau) + \epsilon)^\beta} + \frac{2(\beta + 1)}{\beta} \tau \int_{\mathbb{R}^d} \int_0^\tau |D(m + \epsilon)^{\frac{\beta}{2}}|^2 dx dt
\leq \int_{\mathbb{R}^d} \frac{dx}{(m^0 + \epsilon)^{\beta}(x)} + C_\delta \int_0^\tau \int_{\mathbb{R}^d} |D_p H|^{\gamma/(\gamma-1)} dx dt + \delta \int_0^\tau \int_{\mathbb{R}^d} \frac{dx dt}{(m + \epsilon)^{\beta+2\gamma/(2-\gamma)}}.
\]
(38)

By choosing \( \beta = \alpha(2-\gamma)/\gamma \) and using Proposition 4.3, we get from (38) that
\[
\int_{\mathbb{R}^d} \frac{dx}{(m(x, \tau) + \epsilon)^{\alpha(2-\gamma)/\gamma}} + \tau \int_0^\tau \int_{\mathbb{R}^d} |D(m + \epsilon)^{\frac{\alpha(\gamma-2)}{2\gamma}}|^2 dx dt \leq C.
\]
(39)
Now, we observe that Sobolev’s inequality yields
\[ \| \frac{1}{m + \epsilon} \|^\beta \leq C \left( \int_{\mathbb{T}^d} \frac{dx}{(m(x, t) + \epsilon)^\beta} + \int_{\mathbb{T}^d} |D(m + \epsilon)^{-\frac{\beta}{2}}|^2 dx \right). \]
Accordingly, by setting
\[ \beta = \alpha(2 - \gamma)/\gamma \]
and using (39), we obtain that
\[ \int_0^T \| \frac{1}{m + \epsilon} \|^{\frac{\alpha(2 - \gamma)}{\gamma}} \leq C. \]

\textbf{Theorem 4.2.} Assume \textit{A1-A4} hold. Then, for any \( r \geq 1 \) there exists \( C_r > 0 \), such that for any solution \((u', m')\) of Problem 4, we have
\[ \| \frac{1}{m^\epsilon + \epsilon} \|_{L^\infty([0, T], L^r(\mathbb{T}^d))} \leq C_r, \]
uniformly in \( \epsilon \).

\textit{Proof.} Here to simplify the notation, we drop the superscript \( \epsilon \). First, we define a sequence \((\beta_n)_{n \in \mathbb{N}}\) inductively as follows:
\[ \beta_{n+1} = \frac{(2 - \gamma)}{(\gamma - 1)d} \beta_n, \]
with
\[ \beta_0 = \alpha(2 - \gamma)/\gamma, \]
Because Assumption \textit{A4} implies that
\[ \frac{(2 - \gamma)}{(\gamma - 1)d} > 1, \]
\( \beta_n \to \infty \) as \( n \to \infty \).

Next, we claim that for any \( n \in \mathbb{N} \), there exists \( C_n > 0 \), such that
\[ \int_{\mathbb{T}^d} \frac{dx}{(m(x, \tau) + \epsilon)^{\beta_n}} \leq C_n, \]
uniformly in \( \epsilon \). The proof of the claim follows by induction on \( n \). For that, set
\[ \lambda = \frac{2 - \gamma}{\gamma}. \]
Then,
\[ \frac{(d - 2)\lambda}{d\beta_{n+1}} + \frac{1 - \lambda}{\beta_n} = \frac{2 - \gamma}{\beta_{n+1}'}. \]
Therefore, using interpolation, we get
\[ \| \frac{1}{m + \epsilon} \|_{\beta_{n+1}'/(2-\gamma)} \leq \| \frac{1}{m + \epsilon} \|_{\beta_n}^{1-\lambda} \| \frac{1}{m + \epsilon} \|_{2^* \beta_{n+1}'/2}^\lambda. \]
The induction hypothesis \( \frac{1}{m + \epsilon} \leq C \) implies
\[
\int_{\mathbb{T}^d} \frac{dx}{(m + \epsilon)^{\beta_n + 1}/(2-\gamma)} = \frac{1}{m + \epsilon} \left[ \beta_n + 1 \right]^{(2-\gamma)/\gamma} 
\leq C \left[ \frac{1}{m + \epsilon} \right]^{\beta_n + 1}/(2-\gamma) \frac{1}{2^* \beta_n + 1/2} 
= C \left[ \frac{1}{m + \epsilon} \right]^{\beta_n + 1}/2^* \beta_n + 1/2.
\] (40)

A further application of Sobolev’s theorem produces
\[
\left[ \frac{1}{m + \epsilon} \right]^{\beta_n + 1}/2^* \beta_n + 1/2 \leq \hat{C} \left( \int_{\mathbb{T}^d} (m(x, t) + \epsilon)^{\beta_n + 1}/2^* \beta_n + 1/2 \right) dx dt.
\] Therefore, for each fixed \( t \), we have
\[
\int_{\mathbb{T}^d} \frac{dx}{(m + \epsilon)^{\beta_n + 1}} \leq C_\theta \int_{\mathbb{T}^d} (m + \epsilon)^{\beta_n + 1}/(2-\gamma) dx dt 
\leq C_\theta + \theta \left[ \frac{1}{m + \epsilon} \right]^{\beta_n + 1}/2^* \beta_n + 1/2,
\]
where \( C_\theta \) is a universal constant arising from Young’s inequality and that depends only on the exponent \( \frac{2-\gamma}{\gamma} \). Thus,
\[
\left[ \frac{1}{m + \epsilon} \right]^{\beta_n + 1}/2^* \beta_n + 1/2 \leq \hat{C} \int_{\mathbb{T}^d} |D(m + \epsilon)^{\beta_n + 1}/2^* \beta_n + 1/2| dx dt + \hat{C}_\theta \left[ \frac{1}{m + \epsilon} \right]^{\beta_n + 1}/2^* \beta_n + 1/2 \]. (41)

4.3. Estimates for the Hamilton-Jacobi equation in \( L^\infty(\mathbb{T}^d \times [0, T]) \). Now, we obtain uniform bounds for solutions of the Hamilton-Jacobi equation in (11).

**Proposition 4.4.** Assume A1-A2 hold and let \((u^\epsilon, m^\epsilon)\) solve Problem 4. Then,
\[
u^\epsilon(x, t) \leq \max_{y \in \mathbb{T}^d} u^T(y). \tag{42}
\]

**Proof.** The upper bound follows from the maximum principle, taking into account that \( H \geq 0 \) and that the mean-field coupling is negative. □

To investigate lower bounds for \( u^\epsilon \), we introduce the adjoint equation
\[
\rho_t - \Delta \rho - \text{div}(D_p H \rho) = 0 \quad \text{in} \quad \mathbb{T}^d \times [\tau, T], \tag{43}
\]
with initial data
\[
\rho(\cdot, \tau) = \delta_{x_0}. \tag{44}
\]
Combining the adjoint equation and the first equation in (11), we have the following representation formula for $u$:

$$u(x_0, \tau) = \int_{\mathbb{T}^d} \int_{\mathbb{T}^d} D_p H \text{Du}^\varepsilon - H \left(1 \right) \rho dx dt + \int_{\mathbb{T}^d} u(x) \rho(x, T) dx.$$ (45)

**Proposition 4.5.** Assume that A1-A2 hold. Then, for $p, q > 1$, such that

$$\frac{1}{p} + \frac{1}{q} = 1$$

and any solution $(u^\varepsilon, m^\varepsilon)$ to Problem 4, we have

$$u(x_0, \tau) \geq -C \left\| \left( \frac{1}{(m^\varepsilon + \varepsilon)^\alpha} \right) \rho \right\|_{L^1(\tau, T; L^2(T^d))},$$

where $\rho$ solves (43) with initial data (44).

**Proof.** Using A2 in (45) and combining it with A1, we get (46).

**Corollary 4.2.** Assume A1-A2 hold. Then, for $q > 1, p > 1$ such that

$$\frac{1}{p} + \frac{1}{q} = 1$$

and any solution $(u^\varepsilon, m^\varepsilon)$ to Problem 4, we have

$$\int_{\mathbb{T}^d} H \rho \leq C + \left\| \left( \frac{1}{(m^\varepsilon + \varepsilon)^\alpha} \right) \rho \right\|_{L^1(\tau, T; L^2(T^d))},$$

where $\rho$ solves (43) with initial data (44).

**Proof.** Let $p > 1$ be given by $\frac{1}{p} + \frac{1}{q} = 1$. Using A2 in (45), we get

$$c \int_{\mathbb{T}^d} H \rho \leq C + u^\varepsilon(x_0, \tau) + \left\| \left( \frac{1}{(m^\varepsilon + \varepsilon)^\alpha} \right) \rho \right\|_{L^1(\tau, T; L^2(T^d))}.$$

To end the proof, we apply Proposition 4.4.

**Proposition 4.6.** Assume that A1-A2 hold. Then, for $0 < \nu < 1$ and any solution $(u^\varepsilon, m^\varepsilon)$ to Problem 4, we have

$$\int_{\mathbb{T}^d} \int_{\mathbb{T}^d} |D\rho^\nu/2|^2 dx dt \leq C + C\|Du^\varepsilon\|^{2(\gamma - 1)}_{L^\infty(0; T; L^\infty(T^d))},$$

where $\rho$ solves (43) with initial data (44).

**Proof.** Multiply (43) by $\nu \rho^{\nu - 1}$ and integrate by parts. Then,

$$\int_{\mathbb{T}^d} (\rho^{\nu}(x, T) - \rho^{\nu}(x, \tau)) dx + \nu(\nu - 1) \int_{\mathbb{T}^d} \int_{\mathbb{T}^d} \rho^{\nu - 1} D_p H(x, Du^\varepsilon) D\rho dx dt$$

$$= \frac{4(1 - \nu)}{\nu} \int_{\mathbb{T}^d} \int_{\mathbb{T}^d} |D(\rho^{\nu/2})|^2 dx dt. \quad (48)$$

Because $\rho(\cdot, t)$ is a probability measure and $0 < \nu < 1$,

$$\int_{\mathbb{T}^d} \rho^{\nu}(x, t) dx \leq 1. \quad (49)$$

Moreover, we have the following estimate

$$|\rho^{\nu - 1} D_p H D\rho| \leq C \rho^{\nu} |D_p H|^2 + \frac{2}{\nu^2} |D(\rho^{\nu/2})|^2.$$
Accordingly, taking into account that \( \rho' \leq 1 + \rho \),
\[
\left\| \int_\tau^T \int_{T_d} \rho' \cdot H \int_{\mathbb{T}^d} \right\| dx dt \leq C \int_\tau^T \int_{T_d} |D_p H| \left( 1 + \rho \right) dx dt + \frac{2}{\nu^2} \int_\tau^T \int_{T_d} |D(\rho')|^2 dx dt.
\]
Thus,
\[
\int_\tau^T \int_{T_d} |D(\rho')|^2 dx dt \leq C \int_\tau^T \int_{T_d} |D_p H|^2 \left( 1 + \rho \right) dx dt \leq C + C \left\| Du^\varepsilon \right\|_{^2(0,T;L^\infty(T^d))}.
\]

**Lemma 4.1.** Assume A1-A2 hold. Also, suppose \( \frac{2}{2^*} < \tilde{\nu} < 1 \), \( 0 < \kappa \leq \tilde{\nu} \), and let \( q > 1 \) satisfy
\[
\frac{1}{q} = 1 - \kappa + \frac{2\kappa}{2^* \tilde{\nu}}.
\]
Then, for any solution \((u^\varepsilon, m^\varepsilon)\) to Problem 4 and any solution \( \rho \) to (43) with initial data (44), we have
\[
\left\| \rho \right\|_{L^1(\tau,T;L^q(T^d))} \leq C + C \left\| Du^\varepsilon \right\|_{^2(0,T;L^\infty(T^d \times [0,T]))}.
\]

**Proof.** Note that the expression in (50) implies \( q > 1 \) because \( 0 < \kappa < 1 \) and \( \frac{2\kappa}{2^* \tilde{\nu}} > 1 \). Using an interpolation inequality, we have
\[
\left( \int_{T_d} \rho \cdot dx \right)^\frac{1}{q} \leq \left( \int_{T_d} \rho \cdot dx \right)^{1-\kappa} \left( \int_{T_d} \rho^\frac{2^*}{2^* \tilde{\nu}} dx \right)^\frac{2^* \tilde{\nu}}{2^*}.
\]
By Sobolev’s Theorem, we have
\[
\left( \int_{T_d} \rho^\frac{2^*}{2^* \tilde{\nu}} dx \right)^\frac{2^* \tilde{\nu}}{2^*} \leq C \left( \int_{T_d} |D(\rho)|^2 dx \right)^\frac{2^* \tilde{\nu}}{2^*},
\]
and, therefore,
\[
\int_\tau^T \left( \int_{T_d} \rho \cdot dx \right)^\frac{1}{q} \leq C + C \int_\tau^T \left( \int_{T_d} |D(\rho)|^2 dx \right)^\frac{2^* \tilde{\nu}}{2^*}.
\]
Because \( \kappa \leq \tilde{\nu} \), the previous computation and Proposition 4.6 give that
\[
\left\| \rho \right\|_{L^1(\tau,T;L^q(T^d))} \leq C + C \left\| Du^\varepsilon \right\|_{^2(0,T;L^\infty(T^d \times [0,T]))},
\]
which concludes the proof.

**Corollary 4.3.** Assume A1-A2 hold. Let \( \frac{2}{2^*} < \tilde{\nu} < 1 \) and select \( 0 < \kappa \leq \tilde{\nu} \) and \( p > 1 \), such that
\[
\frac{1}{p} = \kappa - \frac{2\kappa}{2^* \tilde{\nu}}.
\]
Then, for any solution \((u^\varepsilon, m^\varepsilon)\) to Problem 4 and any solution \( \rho \) to (43) with initial data (44), we have
\[
\int_\tau^T \int_{T_d} H \rho \cdot dx dt \leq C + C \left\| \frac{1}{(m^\varepsilon + \epsilon)^p} \right\|_{^2(0,T;L^\infty(T^d))} \left( 1 + \left\| Du^\varepsilon \right\|_{^2(0,T;L^\infty(T^d \times [0,T]))} \right).
\]

**Proof.** The result follows by combining Corollary 4.2 with Lemma 4.1.
4.4. Proof of Theorem 1.2. The proof of Theorem 1.2 relies on Lipschitz bounds for the solutions of the Hamilton-Jacobi equation in (11) that are uniform in $\epsilon$. Once we establish the Lipschitz regularity, the proof follows standard arguments in parabolic regularity theory. Thus, next, we combine the estimates from the preceding sections to obtain uniform bounds for $Du^\epsilon$ in $L^\infty(T^d \times [0, T])$.

Proposition 4.7. Assume A1-A4 hold and let $(u^\epsilon, m^\epsilon)$ solve Problem 4. Then,

$$
\|Du^\epsilon\|_{L^\infty(T^d \times [0, T])} \leq C + C\|Du^\epsilon\|^{2(\gamma - 1)}_{L^\infty(T^d \times [0, T])}.
$$

Proof. For ease of notation, we omit the superscript $\epsilon$ and set $g = (m + \epsilon)^{-\alpha}$. By Theorem 4.2, $\|g\|_{L^\infty(0, T; L^p(T^d))}$ is uniformly bounded for any $p > 1$. Let $\rho$ solve (43) with initial data (44). We fix a unit vector $\xi \in \mathbb{R}^d$ and differentiate the first equation in (11) in the $\xi$ direction. Next, we multiply the resulting equation by $\rho$ and (43) by $u\xi$. By adding them and integrating by parts, we conclude that

$$
u(x_0, \tau) = \int_T^T \int_{T^d} -D\xi H\rho - g\xi \rho dx dt + \int_{T^d} (u^T)\xi \rho(x, T) dx.
$$

Take $p > d$ and define $\tilde{q}, \tilde{\nu}$ by

$$
\tilde{\nu} = \frac{1}{p} + \frac{1}{2} + \frac{1}{2}, \quad \frac{1}{q} + \frac{1}{p} = \frac{1}{2}.
$$

Thus, $\tilde{\nu} < \frac{1}{d} + \frac{1}{2} + \frac{d - 2}{2d} = 1$. Letting $\kappa = \tilde{\nu} \frac{d}{d + p}$, we obtain (51). Assumption A1 and Corollary 4.3 imply

$$
\left| \int_T^T \int_{T^d} -D\xi H\rho dx dt \right| \leq C + C\int_T^T \int_{T^d} H\rho dx dt
$$

$$
\leq C + C\|g\|_{L^\infty(0, T; L^p(T^d))}(1 + \|Du^\epsilon\|^{2(\gamma - 1)}_{L^\infty(T^d \times [0, T])}).
$$

Moreover,

$$
\left| \int_{T^d} (u^T)\xi \rho(x, T) dx \right| \leq C,
$$

because the preceding expression depends only on the terminal data and $\rho(\cdot, T)$ is a probability density. It remains to bound the term

$$
\int_T^T \int_{T^d} g\xi \rho dx dt.
$$

Integration by parts gives the following estimate:

$$
\left| \int_T^T \int_{T^d} g\xi \rho dx dt \right| \leq C\|g\|_{L^\infty(0, T; L^p(T^d))}\|\rho^{1/2}\|_{L^2(\tau, T; L^2(T^d))}\|D\rho^{\tilde{\nu}/2}\|_{L^2(T^d \times [\tau, T])}.
$$

From Proposition 4.6, it follows that

$$
\|D\rho^{\tilde{\nu}/2}\|_{L^2(T^d \times [\tau, T])} \leq C + C\|Du^{\tilde{\nu}/2(\gamma - 1)}_{L^\infty(T^d \times [0, T])}.
$$

Moreover, setting $\theta = \frac{\tilde{\nu}}{2 - \tilde{\nu}}$, we have

$$
\frac{1}{q\left(\frac{2}{2 - \tilde{\nu}}\right)} = 1 - \theta + \frac{2\theta}{2\tilde{\nu}}.
$$

(52)
Then
\[ \left( \int_{\mathbb{T}^d} \rho^{\tilde{\beta}\left(\frac{2-\alpha}{2}\right)} \right)^{\frac{1}{\tilde{\beta}\left(\frac{2-\alpha}{2}\right)}} \leq \left( \int_{\mathbb{T}^d} \rho \right)^{1-\theta} \left( \int_{\mathbb{T}^d} \rho^{\frac{\tilde{\beta}}{2}} \rho_{\varepsilon} \right)^{\frac{2}{\tilde{\beta}}}. \]

Next, Sobolev’s Theorem yields
\[ \left( \int_{\mathbb{T}^d} \rho^{\tilde{\beta}\left(\frac{2-\alpha}{2}\right)} \right)^{\frac{1}{\tilde{\beta}\left(\frac{2-\alpha}{2}\right)}} \leq C + C \left( \int_{\mathbb{T}^d} |D(\rho_{\varepsilon})|^2 \right)^{\frac{\theta}{\tilde{\beta}}}. \]

Consequently,
\[ \left( \int_{\mathbb{T}^d} \rho^{\tilde{\beta}\left(\frac{2-\alpha}{2}\right)} \right)^{\frac{2}{\tilde{\beta}}} \leq C + C \left( \int_{\mathbb{T}^d} |D(\rho_{\varepsilon})|^2 \right)^{\frac{(2-\alpha)\theta}{\tilde{\beta}}}. \]

Taking into account that \( \frac{(2-\alpha)\theta}{\tilde{\beta}} = 1 \) and using Proposition 4.6, we obtain the estimate
\[ \|\rho\|_{L^2(\mathbb{T}^d; H^2(\mathbb{T}^d))} \leq C + C\|D\rho\|_{L^\infty(\mathbb{T}^d; \mathbb{T}^d)}^{-1}. \]

The previous computation implies
\[ |u_\varepsilon(x, \tau)| \leq C + C\|g\|_{L^\infty(0, T; L^p(\mathbb{T}^d))} \left( 1 + \|D\rho\|_{L^\infty(\mathbb{T}^d; \mathbb{T}^d)}^{2(\gamma-1)} \right). \]

Accordingly, the result follows from Theorem 4.2.

\[ \square \]

**Corollary 4.4.** Assume A1-A4 hold. Then, there exists a constant, \( C > 0 \), that does not depend on \( \varepsilon \), such that for any solution \((u^\varepsilon, m^\varepsilon)\) to Problem 4, we have
\[ \|Dm^\varepsilon\|_{L^\infty(\mathbb{T}^d; \mathbb{T}^d)} \leq C. \]

Finally, we prove our main result in the time-dependent case, Theorem 1.2.

**Proof of Theorem 1.2.** Under the assumptions of the theorem, the existence of smooth solutions to Problem 4 follows from a standard argument, see, for example, [22] or [23]. In particular, strict convexity (see A1) is essential for the uniqueness of a solution by the monotonicity technique – see [35]. Since \( \|(m^\varepsilon+\varepsilon)^{-\alpha}\|_p \) is uniformly bounded for any \( p > 1 \) and \( \|Dm^\varepsilon\|_\infty \) is uniformly bounded, regularity theory applied to the first equation of (11) implies that \( \|u^\varepsilon\|_p \) and \( \|D^2u^\varepsilon\|_p \) are uniformly bounded for any \( p > 1 \). Then, Morrey’s inequality implies that for \( 0 < \beta < 1 \), \( \|u^\varepsilon\|_{C^{0,\beta}} \) is uniformly bounded. As in [22], we get that, for some \( 0 < \beta < 1 \), \( \|m^\varepsilon\|_{C^{0,\beta}} \) is uniformly bounded.

Next, using the Hopf-Cole transformation, \( v^\varepsilon = \ln(m^\varepsilon+\varepsilon) \), we have
\[ v^\varepsilon_H - D_H^p H \cdot Dv^\varepsilon - \text{div}(D_H^p H) = |Dv^\varepsilon|^2 + \Delta v^\varepsilon. \]

Consequently, as in [22], we obtain that \( v^\varepsilon \) is uniformly bounded from below and uniformly Lipschitz. Thus, \( m^\varepsilon \) is also uniformly bounded away from zero and \( (m^\varepsilon)^{-\alpha} \) is uniformly Lipschitz. Furthermore, through some subsequence, we have that \( u^\varepsilon \to u \), \( m^\varepsilon \to m \) in \( C^{0,\beta}(\mathbb{T}^d \times [0, T]) \) as \( \varepsilon \to 0 \). Hence, \( u \) is a (viscosity) solution of the first equation of (3). Furthermore, \( m \) is a weak solution of the second equation in (3). As in [22], we get uniform bounds in every Sobolev space for \((u^\varepsilon, m^\varepsilon)\). Hence, \((u, m)\) satisfies the same estimates and, thus, it is a classical solution.

\[ \square \]

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REFERENCES

[1] N. Almulla, R. Ferreira and D. Gomes, Two numerical approaches to stationary mean-field games, *Dyn. Games Appl.*, 7 (2017), 657–682.

[2] P. Cardaliaguet, *Notes on Mean-Field Games*, 2011.

[3] P. Cardaliaguet, P. J. Garber, A. Porretta and D. Tonon, Second order mean field games with degenerate diffusion and local coupling, *NoDEA Nonlinear Differential Equations Appl.*, 22 (2015), 1287–1317.

[4] P. Cardaliaguet and P. J. Graber, Mean field games systems of first order, *ESAIM Control Optim. Calc. Var.*, 21 (2015), 690–722.

[5] P. Cardaliaguet, J.-M. Lasry, P.-L. Lions and A. Porretta, Long time average of mean field games, *Netw. Heterog. Media*, 7 (2012), 279–301.

[6] P. Cardaliaguet, A. Mészáros and F. Santambrogio, First order mean field games with density constraints: Pressure equals price, *SIAM J. Control Optim.*, 54 (2016), 2672–2709.

[7] M. Cirant, Multi-population mean field games systems with Neumann boundary conditions, *J. Math. Pures Appl.* (9), 103 (2015), 1294–1315.

[8] M. Cirant, Stationary focusing mean-field games, *Comm. Partial Differential Equations*, 41 (2016), 1324–1346.

[9] M. Cirant and A. Goffi, Maximal $L^q$-regularity for parabolic Hamilton-Jacobi equations and applications to Mean Field Games, 2020, arXiv:2007.14873.

[10] D. Evangelista, R. Ferreira, D. A. Gomes, L. Nurbekyan and V. Voskanyan, First-order, stationary mean-field games with congestion, *Nonlinear Anal.*, 173 (2018), 37–74.

[11] D. Evangelista and D. A. Gomes, On the existence of solutions for stationary mean-field games with congestion, *J. Dyn. Differential Equations*, 30 (2018), 1365–1388.

[12] D. Evangelista, D. Gomes and L. Nurbekyan, Radially symmetric mean-field games with congestion, In *2017 IEEE 56th Annual Conference on Decision and Control (CDC)*, (2017), 3158–3163.

[13] L. C. Evans, Adjoint and compensated compactness methods for Hamilton-Jacobi PDE, *Arch. Ration. Mech. Anal.*, 197 (2010), 1053–1088.

[14] R. Ferreira and D. Gomes, Existence of weak solutions to stationary mean-field games through variational inequalities, *SIAM J. Math. Anal.*, 50 (2018), 5969–6006.

[15] R. Ferreira, D. Gomes and T. Tada, Existence of weak solutions to time-dependent mean-field games, arXiv preprint. arXiv:2001.03928.

[16] R. Ferreira, D. Gomes and T. Tada, Existence of weak solutions to first-order stationary mean-field games with Dirichlet conditions, *Proc. Amer. Math. Soc.*, 147 (2019), 4713–4731.

[17] D. A. Gomes and H. Mitake, Existence for stationary mean-field games with congestion and quadratic Hamiltonians, *NoDEA Nonlinear Differential Equations Appl.*, 22 (2015), 1897–1910.

[18] D. A. Gomes, L. Nurbekyan and M. Prazeres, Explicit solutions of one-dimensional, first-order, stationary mean-field games with congestion, 2016 *IEEE 55th Conference on Decision and Control (CDC)*, (2016), 4534–4539.

[19] D. A. Gomes, L. Nurbekyan and M. Prazeres, One-dimensional stationary mean-field games with local coupling, *Dyn. Games Appl.*, 8 (2018), 315–351.

[20] D. A. Gomes, S. Patrizi and V. Voskanyan, On the existence of classical solutions for stationary extended mean field games, *Nonlinear Anal.*, 99 (2014), 49–79.

[21] D. A. Gomes and E. Pimentel, Time dependent mean-field games with logarithmic nonlinearities, *SIAM J. Math. Anal.*, 47 (2015), 3798–3812.

[22] D. A. Gomes, E. A. Pimentel and H. Sánchez-Morgado, Time-dependent mean-field games in the subquadratic case, *Comm. Partial Differential Equations*, 40 (2015), 40–76.

[23] D. A. Gomes, E. Pimentel and H. Sánchez-Morgado, Time-dependent mean-field games in the superquadratic case, *ESAIM Control Optim. Calc. Var.*, 22 (2016), 562–580.

[24] D. A. Gomes, E. A. Pimentel and V. Voskanyan, *Regularity Theory for Mean-Field Game Systems*, SpringerBriefs in Mathematics. Springer, [Cham], 2016.

[25] D. Gomes and H. Sánchez-Morgado, A stochastic Evans-Aronsson problem, *Trans. Amer. Math. Soc.*, 366 (2014), 903–929.

[26] D. A. Gomes and J. Saude, Monotone numerical methods for finite-state mean-field games, arXiv preprint. arXiv:1705.00174, 2017.

[27] D. A. Gomes and V. K. Voskanyan, Short-time existence of solutions for mean-field games with congestion, *J. Lond. Math. Soc.* (2), 92 (2015), 778–799.
[28] J. Graber, Weak solutions for mean field games with congestion, Preprint. arXiv:1503.04733, 2015.

[29] O. Guéant, A reference case for mean field games models, J. Math. Pures Appl. (9), 92 (2009), 276–294.

[30] O. Guéant, Mean field games equations with quadratic Hamiltonian: A specific approach, Math. Models Methods Appl. Sci., 22 (2012), 1250022, 37 pp.

[31] O. Guéant, J.-M. Lasry and P.-L. Lions, Mean field games and applications, In Paris-Princeton Lectures on Mathematical Finance 2010, volume 2003 of Lecture Notes in Math., pages 205–266. Springer, Berlin, (2011).

[32] J.-M. Lasry and P.-L. Lions, Nonlinear elliptic equations with singular boundary conditions and stochastic control with state constraints. I. The model problem, Math. Ann., 283 (1989), 583–630.

[33] J.-M. Lasry and P.-L. Lions, Jeux à champ moyen. I. Le cas stationnaire, C. R. Math. Acad. Sci. Paris, 343 (2006), 619–625.

[34] J.-M. Lasry and P.-L. Lions, Jeux à champ moyen. II. Horizon fini et contrôle optimal, C. R. Math. Acad. Sci. Paris, 343 (2006), 679–684.

[35] J.-M. Lasry and P.-L. Lions, Mean field games, Jpn. J. Math., 2 (2007), 229–260.

[36] P. L. Lions, Collège de France course on mean-field games, 2007–2011.

[37] A. R. Mészáros and F. J. Silva, A variational approach to second order mean field games with density constraints: The stationary case, J. Math. Pures Appl. (9), 104 (2015), 1135–1159.

[38] E. A. Pimentel and V. Voskanyan, Regularity for second-order stationary mean-field games, Indiana Univ. Math. J., 66 (2017), 1–22.

[39] A. Porretta, On the planning problem for the mean field games system, Dyn. Games Appl., 4 (2014), 231–256.

[40] A. Porretta, Weak solutions to Fokker-Planck equations and mean field games, Arch. Ration. Mech. Anal., 216 (2015), 1–62.

[41] F. Santambrogio, A modest proposal for MFG with density constraints, Netw. Heterog. Media, 7 (2012), 337–347.

[42] J. Serrin, A Harnack inequality for nonlinear equations, Bull. Amer. Math. Soc., 69 (1963), 481–486.

[43] H. A. Tran, Adjoint methods for static Hamilton-Jacobi equations, Calc. Var. Partial Differential Equations, 41 (2011), 301–319.

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E-mail address: cirant@math.unipd.it
E-mail address: diogo.gomes@kaust.edu.sa
E-mail address: pimentel@puc-rio.br
E-mail address: hector@matem.unam.mx