Topological Pressure of Free Semigroup Actions for Non-Compact Sets and Bowen’s Equation, I

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Abstract
The applicability of Bowen’s equation to an arbitrary subset \( Z \) of a compact metric space has been well-studied by Climenhaga (Ergod Theory Dyn Syst 31(4):1163–1182, 2011). This paper aims to generalize the main results obtained by Climenhaga to free semigroup actions. To this end, we adopt the Carathéodory–Pesin structure (C–P structure) and introduce the notions of the topological pressure and lower and upper capacity topological pressures of a free semigroup action for an arbitrary subset. Some properties of these notions follow and we obtain the following three main results. First, by Bowen’s equation, we characterize the Hausdorff dimension of an arbitrary subset, where the points of the subset have the positive lower Lyapunov exponents and satisfy a tempered contraction condition. Second, we give an estimation of the topological pressure of a free semigroup action on an arbitrary subset. Finally, we analyze the relationship between the upper capacity topological pressure of a skew-product transformation and a free semigroup action in an arbitrary subset.

Keywords C–P structure · Free semigroup actions · Topological pressure · Skew-product · Hausdorff dimension · Bowen’s equation

Mathematics Subject Classification 37B40 · 37C45 · 37C85

1 Introduction
As an application of topological pressure, Bowen’s equation plays an important role in dynamical systems. Bowen [7] first studied the connection between topological pressure and Hausdorff dimension. He proved that, for certain compact sets \( J \subset C \) which arise as invariant sets of fractional linear transformations \( f \) of the Riemann sphere, the Hausdorff dimension \( t = \dim_H J \) is the unique root of the equation (namely, Bowen’s equation)
\[ P_J(-t \varphi) = 0, \quad (1.1) \]

where \( P_J \) is the topological pressure of the map \( f : J \to J \), and \( \varphi(z) = \log |f'(z)| \). Ruelle [31] showed that (1.1) yields the Hausdorff dimension of \( J \) as \( f \) is a \( C^{1+\epsilon} \) conformal map on a Riemann manifold and \( J \) is a repeller. Gatzouras and Peres [17] further extended the result to the case in which \( f \) is \( C^1 \). When \( X \) is a metric space (not necessarily a manifold), Rugh [32] redefined the conformal map and confirmed that the result remains valid. By the Bowen’s equation, there exists extensive literature that studied the settings of Julia sets and non-conformal repellers (see, for example, [1,9,15,22,23,28,29,34–36]).

For a given map \( f \), we conclude the above results derive the Hausdorff dimension of some dynamically significant set \( J \) through Bowen’s equation. This leads to an important question of how to find the Hausdorff dimension of subset \( Z \subset J \).

For certain subsets, results in this direction are given by the multifractal analysis (see, for example, [2,26,40]).

By Carathéodory structure, Pesin [24] (or see [25]) gave a new characterization of topological pressure for non-compact sets, which we referred to as the Carathéodory–Pesin structure or C–P structure for short. This extended the earlier definition of topological entropy for non-compact sets by Bowen [6]. Using the C–P structure introduced in [24], Barreira and Schmeling [3] introduced the notion of the \( u \)-dimension \( \text{dim}_u Z \) for positive functions \( u \), and showed that \( \text{dim}_u Z \) is the unique number \( t \) such that \( P_Z( -tu ) = 0 \). They also confirmed that for a subset \( Z \) of a conformal repeller \( J \), where \( u = \log \| Df \| > 0 \), \( \text{dim}_u Z = \text{dim}_H Z \) holds. This implies that the Hausdorff dimension of any subset \( Z \subset J \) can be given by Bowen’s equation, regardless of compact or invariant \( Z \). Moreover, Climenhaga [14] examined that the applicability of Bowen’s equation to an arbitrary subset \( Z \) of a compact metric space. The paper needs positive lower Lyapunov exponents on \( Z \) with a tempered contraction condition only, and generalizes the results of the uniformly expanding case.

**Definition 1.1** Let \( X \) be a compact metric space with metric \( d \) and \( f : X \to X \) a continuous map. We say that \( f : X \to X \) is **conformal** with factor \( a(x) \) if for every \( x \in X \), we have

\[
a(x) = \lim_{y \to x} \frac{d(f(x), f(y))}{d(x, y)},
\]

where \( a : X \to [0, \infty) \) is continuous.

Denote the Birkhoff sums of \( \log a \) by

\[
\lambda_n(x) = \frac{1}{n} S_n(\log a)(x) = \frac{1}{n} \sum_{k=0}^{n-1} \log a(f^k(x)),
\]

the lower and upper limits of this sequence are then the **lower** and **upper Lyapunov exponent**, respectively:

\[
\underline{\lambda}(x) = \liminf_{n \to \infty} \lambda_n(x), \quad \overline{\lambda}(x) = \limsup_{n \to \infty} \lambda_n(x).
\]

If the limit exists, then their common value is the **Lyapunov exponent**:

\[
\lambda(x) = \lim_{n \to \infty} \lambda_n(x).
\]

We say that a point \( x \in X \) satisfies **tempered contraction condition** ([14]), if

\[
\inf_{n \in \mathbb{N}, \ 0 \leq k \leq n} \{ S_{n-k} \log a(f^k(x)) + n\varepsilon \} > -\infty \quad \text{for every} \ \varepsilon > 0. \quad (1.2)
\]
Denote \( B \) by the set of all points in \( X \) which satisfy (1.2). Given \( E \subset \mathbb{R} \) and let \( A(E) \) be the set of points along whose orbits all the asymptotic exponential expansion rates of the map \( f \) lie in \( E \):

\[
A(E) = \{ x \in X : [\lambda(x), \overline{\lambda}(x)] \subseteq E \},
\]

and \( A(\alpha) = A([\alpha]) \).

More precisely, Climenhaga [14] proved the following theorem ([13, Theorem 2.4]):

**Theorem 1.1** Let \( X \) be a compact metric space and \( f : X \to X \) be continuous and conformal with factor \( a(x) \). Suppose that \( f \) has no critical points and no singularities—that is, that \( 0 < a(x) < \infty \) for all \( x \in X \). Consider \( Z \subset A((0, \infty)) \cap B \). Then the Hausdorff dimension of \( Z \) is given by

\[
dim H Z = t^* = \sup \{ t \geq 0 : P_Z(f, -t \log a) > 0 \}
\]

\[
= \inf \{ t \geq 0 : P_Z(f, -t \log a) \leq 0 \}.
\]

Furthermore, if \( Z \subset A((\alpha, \infty)) \cap B \) for some \( \alpha > 0 \), then \( t^* \) is the unique root of Bowen’s equation

\[
P_Z(f, -t \log a) = 0.
\]

Finally, if \( Z \subset A(\alpha) \) for some \( \alpha > 0 \), then \( P_Z(f, -t \log a) = h_Z(f) - t \alpha \), and hence

\[
dim H Z = \frac{1}{\alpha} h_Z(f).
\]

Here \( P_Z(f, -t \log a) \) denotes the topological pressure of \( f \) with respect to \(-t \log a\) on \( Z \), \( h_Z(f) \) is the topological entropy on \( Z \) (see [24]) and \( \dim H(Z) \) is the Hausdorff dimension of \( Z \).

The above result focuses on a single map. In the past two decades, researchers are growing to study the case of free semigroup actions. Related studies include [4,8,10–12,18–20,30,37,38]. Naturally, we wonder if the result of Theorem 1.1 remains valid in the case of free semigroup actions. To answer this question and generalize the result of Climenhaga [14], we introduce the notions of new topological pressure by using C–P structure.

This paper is organized as follows. In Sect. 2, we give our main results. In Sect. 3, we give some preliminaries. In Sect. 4, by using the C–P structure we give the new definitions of the topological pressure and lower and upper capacity topological pressures of free semigroup actions. Several of their properties are provided. In Sect. 5, we give other two definitions of the topological pressure and lower and upper capacity topological pressures of free semigroup actions on arbitrary subset of \( X \), which are both equivalent to the definitions in Sect. 4. In Sects. 6, 7 and 8, we give the proofs of the main results, respectively.

## 2 Statement of Main Results

Let \((X, d)\) be a compact metric space, denote \( G \) the free semigroup with \( m \) generators \( f_0, f_1, \ldots, f_{m-1} \) acting on \( X \) and let \( f_i : X \to X \) be continuous and conformal with factor \( a_i(x) \) (\( i = 0, 1, \ldots, m - 1 \)). Let \( \Phi = \{ \log a_0, \log a_1, \ldots, \log a_{m-1} \} \). For \( w = i_1 i_2 \cdots i_n \in F_m^+ \), where \( F_m^+ \) denotes the set of all finite words of symbols \( 0, 1, \ldots, m - 1 \), denote

\[
S_w \Phi(x) := \log a_{i_1}(x) + \log a_{i_2}(f_{i_1}(x)) + \cdots + \log a_{i_n}(f_{i_{n-1}i_{n-2} \cdots i_1}(x))
\]
and
\[ \lambda_w(x) = \frac{1}{|w|} S_w \Phi(x), \]
where
\[ a_i(x) = \lim_{y \to x} \frac{d(f_i(x), f_i(y))}{d(x, y)}, \quad i = 0, 1, \ldots, m - 1, \]
and
\[ f_{i_0 \cdots i_{n-2} \cdots i_1} := f_{i_0} \circ f_{i_{n-2}} \circ \cdots \circ f_{i_1}. \]
For any \( \omega = (i_0, i_1, \ldots) \in \Sigma_m^+, \) where \( \Sigma_m^+ \) denotes the one-side symbol space, denote \( \omega|_{[0,n-1]} := i_0 i_1 \cdots i_{n-1} \) and set
\[ \underline{\lambda}_\omega(x) = \liminf_{n \to \infty} \lambda_{\omega|_{[0,n-1]}}(x), \]
and
\[ \overline{\lambda}_\omega(x) = \limsup_{n \to \infty} \lambda_{\omega|_{[0,n-1]}}(x), \]
then we call \( \underline{\lambda}_\omega(x) \) and \( \overline{\lambda}_\omega(x) \) the lower and upper Lyapunov exponents of the free semigroup \( G \) relative to \( \omega \in \Sigma_m^+ \) at \( x \). If the two are equal (i.e., if the limit exists), then their common value is the Lyapunov exponent relative to \( \omega \in \Sigma_m^+ \) at \( x \):
\[ \lambda_\omega(x) = \lim_{n \to \infty} \lambda_{\omega|_{[0,n-1]}}(x). \]

Our first main result relates the Hausdorff dimension of \( Z \) to the topological pressure of \( \Phi \) on \( Z \), provided every point in \( Z \) has \( \underline{\lambda}_\omega(x) > 0 \) and satisfies the following so-called tempered contraction condition:
\[ \inf_{w \in F_m^+, \|w\| \leq \|w\|} \{ S_w \Phi(x) - S_w' \Phi(x) + \|w\| + |w| \varepsilon \} > -\infty, \text{ for any } \varepsilon > 0. \tag{2.1} \]
Denote \( B \) as the set of all points in \( X \) which satisfy (2.1).

**Remark 2.1**

1. Proposition 6.2 shows that if for any the \( \omega \in \Sigma_m^+ \), the Lyapunov exponent of \( x \) exists and is positive, that is, if \( \underline{\lambda}_\omega(x) = \overline{\lambda}_\omega(x) > 0 \) for every \( \omega \in \Sigma_m^+ \), then \( x \) satisfies (2.1).
2. We observe that if \( a_i(x) \geq 1 \) for any \( x \in X \) and \( i = 0, 1, \ldots, m - 1 \), then \( B = X \).
3. We say that \( x \) has bounded contraction if \( \inf \{ S_w \Phi(x) - S_{w'} \Phi(x) : w \in F_m^+, \|w\| \leq \|w\| \} > -\infty \). Any point which has bounded contraction satisfies (2.1).

Given \( E \subset \mathbb{R} \) and let \( \mathcal{A}(E) \) be the set of points along whose orbits all the asymptotic exponential expansion rates of the \( G \) relative to \( \omega \in \Sigma_m^+ \) lie in \( E \):
\[ \mathcal{A}(E) = \{ x \in X : [\underline{\lambda}_\omega(x), \overline{\lambda}_\omega(x)] \subset E, \omega \in \Sigma_m^+ \}. \]

In particular, \( \mathcal{A}((0, \infty)) \) is the set of all points for which \( \underline{\lambda}_\omega(x) > 0 \) for every \( \omega \in \Sigma_m^+ \) and \( \mathcal{A}(\emptyset) = \mathcal{A}(\{0\}) \). The first main result deals with subsets \( Z \subset X \) that lie in both \( \mathcal{A}((0, \infty)) \) and \( B \).

**Example 2.1** Let \( X = S^1 \) be the unit circle, \( f_0 : X \to X, f_0(x) = x^2 \) and \( f_1 : X \to X, f_1(x) = x^3 \). Then for any \( \omega \in \Sigma_2^+ \) and \( x \in X, [\underline{\lambda}_\omega(x), \overline{\lambda}_\omega(x)] \subset [\log 2, \log 3] \). Moreover, for any \( x \in X, x \) satisfies tempered contraction condition. Thus, \( \mathcal{A}((0, \infty)) = B = X \).
Theorem 2.2 Let \((X, d)\) be a compact metric space, \(G\) be the free semigroup with \(m\) generators \(f_0, f_1, \ldots, f_{m-1}\) acting on \(X\) and let \(f_i : X \to X\) be continuous and conformal with factor \(a_i(x)\). Assume \(f_i\) has no critical points and singularities, that is, \(0 < a_i(x) < \infty\) for all \(x \in X\) and \(i \in \{0, 1, 2, \ldots, m-1\}\). Consider \(Z \subset \mathcal{A}((0, \infty)) \cap \mathcal{B}\) and \(\Phi = \{\log a_0, \log a_1, \ldots, \log a_{m-1}\}\). Then the Hausdorff dimension of \(Z\) is given by

\[
dim_H Z = t^* = \sup \{t \geq 0 : P_Z(G, -t\Phi) > 0\} = \inf \{t \geq 0 : P_Z(G, -t\Phi) \leq 0\}.
\]

Furthermore, if \(Z \subset \mathcal{A}((\alpha, \infty)) \cap \mathcal{B}\) for some \(\alpha > 0\), then \(t^*\) is the unique root of Bowen’s equation

\[
P_Z(G, -t\Phi) = 0.
\]

Finally, if \(Z \subset \mathcal{A}(\alpha)\) for some \(\alpha > 0\), then \(P_Z(G, -t\Phi) = h_Z(G) - t\alpha\), and hence

\[
dim_H Z = \frac{1}{\alpha} h_Z(G).
\]

Here \(h_Z(G)\) is the topological entropy on \(Z\) defined by Ju et al in [18] and \(P_Z(G, -t\Phi)\) denotes the topological pressure of \(G\) with respect to \(-t\Phi\) on \(Z\) (see Sect. 4).

Remark 2.2 When \(m = 1\), i.e., \(G = \{f\}\), \(\Phi = \{\log a\}\), the result coincides with [14].

Next, we give our second main result. Considering a compact metric space \((X, d)\), a free semigroup \(G = \{f_0, f_1, \ldots, f_{m-1}\}\) acting on \(X\), where \(f_i(i = 0, 1, \ldots, m-1)\) is continuous transformation from \(X\) to itself and \(\varphi_0, \varphi_1, \ldots, \varphi_{m-1} \in C(X, \mathbb{R})\), where \(C(X, \mathbb{R})\) denotes the Banach algebra of real-valued continuous functions of \(X\) equipped with the supremum norm. Denote \(\Phi = \{\varphi_0, \varphi_1, \ldots, \varphi_{m-1}\}\). For \(w = i_1i_2\cdots i_n \in F_m^+\), \(B_w(x, r)\) is the \((w, r)\)-Bowen ball at \(x\) and denote

\[
S_w \Phi(x) := \varphi_1(x) + \varphi_2(f_1(x)) + \cdots + \varphi_n(f_{i_{n-1}i_{n-2}\cdots i_1}(x)).
\]

Let \(\mu\) be a Borel probability measure on \(X\) and \(w \in F_m^+\). Denote

\[
P(G, \Phi, x) := \lim_{r \to 0} \lim_{n \to \infty} \frac{1}{n} \max \{\log \mu(B_w(x, r)) - S_w \Phi(x)\},
\]

and

\[
\overline{P}(G, \Phi, x) := \lim_{r \to 0} \lim_{n \to \infty} \frac{1}{n} \min \{\log \mu(B_w(x, r)) - S_w \Phi(x)\}.
\]

Now we give two estimations about topological pressure on \(Z \subset X\).

Theorem 2.3 Let \(\mu\) denote a Borel probability measure on \(X\), \(Z\) be a Borel subset of \(X\) and \(s \in \mathbb{R}\).

1. If \(P(G, \Phi, x) \geq s\) for all \(x \in Z\) and \(\mu(Z) > 0\) then \(P_Z(G, \Phi) \geq s\).
2. If \(\overline{P}(G, \Phi, x) \leq s\) for all \(x \in Z\) then \(P_Z(G, \Phi) \leq s\).

Remark 2.3 When \(\varphi_0 = \varphi_1 = \cdots = \varphi_{m-1} = 0\), it coincides with the result in [18]. Moreover, if \(m = 1\), then the above theorem coincides with the main results that Ma and Wen proved in [21]. A similar result of topological entropy were obtained by Biś for pseudogroups in [5].
Finally, the third result describes the relationship between the upper capacity topological pressure of a free semigroup action and the upper capacity topological pressure of a skew-product transformation. Let $X$ be a compact metric space with metric $d$ and suppose a free semigroup $G = \{f_0, f_1, \ldots, f_{m-1}\}$ with $m$ generators acting on $X$, $\varphi \in C(X, \mathbb{R})$, where $f_0, f_1, \ldots, f_{m-1}$ are continuous transformations from $X$ to itself. Let $F : \Sigma_m \times X \to \Sigma_m \times X$ be a skew-product transformation and $g : \Sigma_m \times X \to \mathbb{R}$ be defined by the formula $g(\omega, x) = c + \varphi(x)$, where $\Sigma_m$ denotes the two-side symbol space and $c$ is a constant. $\overline{CP}_{\Sigma_m \times Z}(F, g)$ denotes the upper capacity topological pressure of $F$ with respect to $g$ on $\Sigma_m \times Z$ (see [24]), $\overline{CP}_{Z}(G, \varphi)$ the upper capacity topological pressure of $G$ with respect to $\varphi$ on $Z$ (see Sect. 4), $P_{\Sigma_m \times X}(F, g)$ the topological pressure of $F$ with respect to $g$ on $\Sigma_m \times X$ (see [39]), $P_{X}(G, \varphi)$ the topological pressure defined by Lin et al in [19], $h_{\Sigma_m \times X}(F)$ the topological entropy (see [39]) and $h_{X}(G)$ the topological entropy of $G = \{f_0, f_1, \ldots, f_{m-1}\}$ (see [8]). Then we have the following theorem:

**Theorem 2.4** For any set $Z \subset X$, we have

$$\overline{CP}_{\Sigma_m \times Z}(F, g) = \log m + \overline{CP}_{Z}(G, \varphi) + c.$$  

**Remark 2.4**

(1) If $g(\omega, x) = \varphi(x) \equiv 0$, it is $\overline{CP}_{\Sigma_m \times Z}(F) = \log m + \overline{CP}_{Z}(G)$, which coincides with the result in [18]. When $Z = X$, we have $\overline{CP}_{\Sigma_m \times X}(F, g) = P_{\Sigma_m \times X}(F, g)$, $\overline{CP}_{X}(G, \varphi) = P_{X}(G, \varphi)$, then we can get $P_{\Sigma_m \times X}(F, g) = \log m + P_{X}(G, \varphi) + c$, which has been proved by Lin et al in [19]. Further, if $g(\omega, x) = \varphi(x) \equiv 0$, it is easy to get $h_{\Sigma_m \times X}(F) = \log m + h_{X}(G)$, which has been proved by Bufetov in [8].

(2) This result also retains valid for one-side shift, i.e.

$$\overline{CP}_{\Sigma_m \times Z}(F, g) = \log m + \overline{CP}_{Z}(G, \varphi) + c.$$  

**3 Preliminaries**

**3.1 Carathéodory–Pesin structure**

Let $X$ and $S$ be arbitrary sets and $\mathcal{F} = \{U_s : s \in S\}$ a collection of subsets in $X$. From Pesin [24], we assume that there exist two functions $\eta, \psi : S \to \mathbb{R}^+$ satisfying the following conditions:

1. there exists $s_0 \in S$ such that $U_{s_0} = \emptyset$; if $U_s = \emptyset$ then $\eta(s) = \psi(s) = 0$; if $U_s \neq \emptyset$ then $\eta(s) > 0$ and $\psi(s) > 0$;
2. for any $\delta > 0$ one can find $\varepsilon > 0$ such that $\eta(s) \leq \delta$ for any $s \in S$ with $\psi(s) \leq \varepsilon$;
3. for any $\varepsilon > 0$ there exists a finite or countable subcollection $\mathcal{G} \subset S$ which covers $X$ (i.e., $\bigcup_{s \in \mathcal{G}} U_s \supset X$) and $\psi(G) := \sup\{\psi(s) : s \in \mathcal{G}\} \leq \varepsilon$.

Let $\xi : S \to \mathbb{R}^+$ be a function, we say that the set $S$, collection of subsets $\mathcal{F}$, and the set functions $\xi, \eta, \psi$ satisfying conditions (1), (2) and (3), form the Carathéodory–Pesin structure or C–P structure $\tau$ on $X$ and write $\tau = (S, \mathcal{F}, \xi, \eta, \psi)$.

Given a subset $Z$ of $X$, $\alpha \in \mathbb{R}$, and $\varepsilon > 0$, we define

$$M(Z, \alpha, \varepsilon) := \inf_{\mathcal{G}} \left\{ \sum_{s \in \mathcal{G}} \xi(s) \eta(s)^{\alpha} \right\},$$

where the infimum is taken over all finite or countable subcollections $\mathcal{G} \subset S$ covering $Z$ with $\psi(\mathcal{G}) \leq \varepsilon$. By condition (3) the function $M(Z, \alpha, \varepsilon)$ is correctly defined. It is non-decreasing as $\varepsilon$ decreases. Therefore, the following limit exists:
\[ m(Z, \alpha) = \lim_{\varepsilon \to 0} M(Z, \alpha, \varepsilon). \]

It was shown in [24] that there exists a critical value \( \alpha_C \in [-\infty, \infty] \) such that
\[
m(Z, \alpha) = 0, \quad \alpha > \alpha_C, \\
m(Z, \alpha) = \infty, \quad \alpha < \alpha_C.
\]

The number \( \alpha_C \) is called the Carathéodory–Pesin dimension of the set \( Z \).

Now we assume that the following condition holds:
\[
(3') \text{ there exists } \varepsilon > 0 \text{ such that for any } 0 < \varepsilon \leq \varepsilon \text{ there exists a finite or countable subcollection } \mathcal{G} \subset S \text{ covering } X \text{ such that } \psi(s) = \varepsilon \text{ for any } s \in \mathcal{G}.
\]

Given \( \alpha \in \mathbb{R} \) and \( \varepsilon > 0 \), for any subset \( Z \subset X \), define
\[
R(Z, \alpha, \varepsilon) = \inf_{\mathcal{G}} \left\{ \sum_{s \in \mathcal{G}} \xi(s) \eta(s)^{\alpha} \right\},
\]
where the infimum is taken over all finite or countable subcollections \( \mathcal{G} \subset S \) covering \( Z \) such that \( \psi(s) = \varepsilon \) for any \( s \in \mathcal{G} \). Set
\[
\underline{r}(Z, \alpha) = \liminf_{\varepsilon \to 0} R(Z, \alpha, \varepsilon), \quad \overline{r}(Z, \alpha) = \limsup_{\varepsilon \to 0} R(Z, \alpha, \varepsilon).
\]

It was shown in [24] that there exist \( \underline{\alpha}_C, \overline{\alpha}_C \in \mathbb{R} \) such that
\[
\underline{r}(Z, \alpha) = \infty, \quad \alpha < \underline{\alpha}_C, \quad \overline{r}(Z, \alpha) = 0, \quad \alpha > \overline{\alpha}_C, \\
\overline{r}(Z, \alpha) = \infty, \quad \alpha < \overline{\alpha}_C, \quad \underline{r}(Z, \alpha) = 0, \quad \alpha > \underline{\alpha}_C.
\]

The numbers \( \underline{\alpha}_C \) and \( \overline{\alpha}_C \) are called the lower and upper Carathéodory–Pesin capacities of the set \( Z \) respectively.

For any \( \varepsilon > 0 \) and subset \( Z \subset X \), put
\[
\Lambda(Z, \varepsilon) := \inf_{\mathcal{G}} \left\{ \sum_{s \in \mathcal{G}} \xi(s) \right\},
\]
where the infimum is taken over all finite or countable subcollections \( \mathcal{G} \subset S \) covering \( Z \) such that \( \psi(s) = \varepsilon \) for any \( s \in \mathcal{G} \).

Assume that the function \( \eta \) satisfies the following condition:
\[
(4) \eta(s_1) = \eta(s_2) \text{ for any } s_1, s_2 \in S \text{ for which } \psi(s_1) = \psi(s_2).
\]

It was shown in [24] that if the function \( \eta \) satisfies condition (4) then for any subset \( Z \subset X \),
\[
\underline{\alpha}_C = \liminf_{\varepsilon \to 0} \frac{\log \Lambda(Z, \varepsilon)}{\log (1/\eta(\varepsilon))}, \quad \overline{\alpha}_C = \limsup_{\varepsilon \to 0} \frac{\log \Lambda(Z, \varepsilon)}{\log (1/\eta(\varepsilon))}.
\]

### 3.2 Words and Sequences

Let \( F_m^+ \) be the set of all finite words of symbols \( 0, 1, \ldots, m - 1 \). For any \( w \in F_m^+ \), \( |w| \) stands for the length of \( w \), that is, the digits of symbols in \( w \). Denote \( F_m^+(n) = \{ w \in F_m^+ : |w| = n, \ n \in \mathbb{N} \} \). Obviously, \( F_m^+ \) with respect to the law of composition is a free semigroup with \( m \) generators. We write \( w' \leq w \) if there exists a word \( w'' \in F_m^+ \) such that \( w = w''w' \). For \( w = i_1 \cdots i_k \in F_m^+ \), denote \( \overline{w} = i_k \cdots i_1 \).
Let $\Sigma_m$ be the set of all two-side infinite sequences of symbols 0, 1, \ldots, $m - 1$, that is,

$$\Sigma_m = \{ \omega = (\ldots, i_{-1}, i_0, i_1, \ldots) : i_j = 0, 1, \ldots, m - 1 \text{ for all integer } j \}.$$

The metric on $\Sigma_m$ is defined by

$$d'(\omega, \omega') = 1/2^k, \text{ where } k = \inf \{ n : i_n \neq i'_n \}.$$

Obviously, $\Sigma_m$ is compact with respect to this metric. The Bernoulli shift $\sigma_m : \Sigma_m \to \Sigma_m$ is a homeomorphism of $\Sigma_m$ given by the formula

$$(\sigma_m \omega)_k = i_{k+1}.$$

Suppose that $\omega \in \Sigma_m$, $w \in F_m^+$, $a, b$ are integers, and $a \leq b$. We write $\omega|_{[a,b]} = w$ if $w = i_0 i_{a+1} \cdots i_{b-1} i_b$.

Let $\Sigma_m^+$ be the set of all one-side infinite sequences of symbols 0, 1, \ldots, $m - 1$,

$$\Sigma_m^+ = \{ \omega = (i_0, i_1, \ldots) : i_j = 0, 1, \ldots, m - 1 \text{ for all integer } j \}.$$

### 3.3 Hausdorff Dimension

Let $(X, d)$ be a metric space and $\mathcal{D}(Z, r)$ denote the collection of countable open covers $\{U_i\}_{i=1}^{\infty}$ of $Z$ for which $\text{diam} U_i < r$ for all $i$. Given a subset $Z \subset X$, $t \geq 0$, define

$$m_H(Z, t, r) = \inf_{\mathcal{D}(Z, r)} \left\{ \sum_{U_i \in \mathcal{D}(Z, r)} (\text{diam}(U_i))^t \right\}.$$

When $r$ decreases, $m_H(Z, t, r)$ increases. Therefore there exists the limit

$$m_H(Z, t) = \lim_{r \to 0} m_H(Z, t, r)$$

which is called the $t$-dimensional Hausdorff measure of $Z$. The number

$$\dim_H(Z) = \inf \{ t > 0 : m_H(Z, t) = 0 \} = \sup \{ t > 0 : m_H(Z, t) = \infty \}$$

is called the Hausdorff dimension of $Z$.

One may equivalently define Hausdorff dimension of using covers by open balls rather than arbitrary open sets. Let $\mathcal{D}^b(Z, r)$ denote the collection of countable open balls covers $\{B(x_i, r_i)\}_{i=1}^{\infty}$ of with $x_i \in Z$ and $r_i < r$ for all $i$, and then define

$$m_H^b(Z, t, r) = \inf_{\mathcal{D}^b(Z, r)} \left\{ \sum_{B(x_i, r_i) \in \mathcal{D}^b(Z, r)} (\text{diam} B(x_i, r_i))^t \right\}.$$

Finally, define $m_H^b(Z, t)$ and $\dim_H^b Z$ by the same procedure as above. Moreover, from [13,16] we know $\dim_H^b Z = \dim_H Z$.

### 4 Topological Pressure and Lower and Upper Capacity Topological Pressures of a Free Semigroup Action and Their Properties

In this section, we introduce the definitions of topological pressure and lower and upper capacity topological pressures of a free semigroup action by using C–P structure and provide some properties of them.
4.1 Topological Pressure and Lower and Upper Capacity Topological Pressures

Let $X$ be a compact metric space with metric $d$, $f_0$, $f_1$, $\ldots$, $f_{m-1}$ continuous transformations from $X$ to itself. Suppose that a free semigroup $G = \{f_0, f_1, \ldots, f_{m-1}\}$ with $m$ generators acting on $X$. Given $\varphi_0, \varphi_1, \ldots, \varphi_{m-1} \in C(X, \mathbb{R})$, denote $\Phi = \{\varphi_0, \varphi_1, \ldots, \varphi_{m-1}\}$. Let $w = i_1 i_2 \ldots i_n \in F_m^+$, where $i_j = 0, 1, \ldots, m-1$ for all $j = 1, \ldots, n$ and $f_w = f_{i_1} \circ f_{i_2} \circ \cdots \circ f_{i_n}$. Obviously, $f_{ww'} = f_w f_{w'}$. For $w = i_1 i_2 \ldots i_n \in F_m^+$, denote

$$S_w \Phi (x) := \varphi_{i_1}(x) + \varphi_{i_2}(f_{i_1}(x)) + \cdots + \varphi_{i_n}(f_{i_1} \circ f_{i_2} \circ \cdots \circ f_{i_n}(x)).$$

Considering a finite open cover $\mathcal{U}$ of $X$, let

$$S_{n+1}(\mathcal{U}) = \{U = (U_0, U_1, \ldots, U_n) : U \in \mathcal{U}^{n+1}\},$$

where $\mathcal{U}^{n+1} = \prod_{i=1}^{n+1} \mathcal{U}$ and $n \geq 0$. For any string $U \in S_{n+1}(\mathcal{U})$, define the length of $U$ as $m(U) := n + 1$. We put $S = S(\mathcal{U}) = \bigcup_{n \geq 1} S_n(\mathcal{U})$. For any $\omega = (i_1, i_2, \ldots, i_n, \ldots) \in \Sigma_m^+$, $n \geq 1$, and a given string $U = (U_0, U_1, \ldots, U_n) \in S_{n+1}(\mathcal{U})$, we associate the set

$$X_\omega(U) = \{x \in X : x \in U_0, f_{i_1} \circ \cdots \circ f_{i_n}(x) \in U_j, j = 1, 2, \ldots, n\}.$$

If $w_U = \omega|_{[0,n]-1} = i_1 i_2 \cdots i_n \in F_m^+$, we also denote $X_\omega(U)$ by $X_{w_U}(U)$ for the sake of convenience. Define

$$\mathcal{F} = \{X_\omega(U) : U \in S(\mathcal{U}) \text{ and } \omega \in \Sigma_m^+\},$$

and three functions $\xi, \eta, \psi : S \to \mathbb{R}^+$ as follows

$$\xi(U) = \exp \left( \sup_{x \in X_{w_U}(U)} S_{w_U} \Phi(x) \right),$$

$$\eta(U) = \exp(-m(U)), \quad \psi(U) = m(U)^{-1}.$$

It is easy to verify that the set $S$, collection of subsets $\mathcal{F}$, and the functions $\xi, \eta,$ and $\psi$ satisfy the conditions (1), (2) and (3) in Sect. 3.1 and hence they determine a C–P structure $\tau = (S, \mathcal{F}, \xi, \eta, \psi)$ on $X$.

Given $w \in F_m^+$, $|w| = N$, $Z \subset X$ and $\alpha \in \mathbb{R}$, we define

$$M_w(Z, \alpha, \Phi, \mathcal{U}, N) = \inf_{G_w} \left\{ \sum_{U \in G_w} \xi(U) \eta(U)^\alpha \right\},$$

$$= \inf_{G_w} \left\{ \sum_{U \in G_w} \exp \left( -\alpha m(U) + \sup_{x \in X_{w_U}(U)} S_{w_U} \Phi(x) \right) \right\},$$

where the infimum is taken over all finite or countable collections of strings $G_w \subset S(\mathcal{U})$ such that $m(U) \geq N + 1$ for all $U \in G_w$ and $G_w$ covers $Z$ (i.e., for any $U \in G_w$, there exists $w_U \in F_m^+$ such that $\overline{w} \leq \overline{w_U}$ and $\bigcup_{U \in G_w} X_{w_U}(U) \supset Z$).

Let

$$M(Z, \alpha, \Phi, \mathcal{U}, N) = \frac{1}{m^N} \sum_{|w| = N} M_w(Z, \alpha, \Phi, \mathcal{U}, N).$$
We can easily verify that the function \( M(Z, \alpha, \Phi, U, N) \) is non-decreasing as \( N \) increases. Therefore there exists the limit

\[
m(Z, \alpha, \Phi, U) = \lim_{N \to \infty} M(Z, \alpha, \Phi, U, N).
\]

Furthermore, given \( w \in F_m^+ \) and \(|w| = N\), by the condition (3') in Sect. 3.1, we can define

\[
R_w(Z, \alpha, \Phi, U, N) := \inf_{G_w} \left\{ \sum_{U \in G_w} \exp \left( -\alpha(N+1) + \sup_{x \in X_w(U)} S_w \Phi(x) \right) \right\}
\]

where \( \Lambda_w(Z, \Phi, U, N) = \inf_{G_w} \left\{ \sum_{U \in G_w} \exp \left( \sup_{x \in X_w(U)} S_w \Phi(x) \right) \right\} \), the infimum is taken over all finite or countable collections of strings \( G_w \subset S(U) \) such that \( m(U) = N + 1 \) for all \( U \in G_w \) and \( G_w \) covers \( Z \) (i.e., for any \( U \in G_w \), there exists \( w_U \in F_m^+ \) such that \( w_U = w \) and \( \bigcup_{U \in G_w} X_{w(U)}(U) \supset Z \)).

Let

\[
R(Z, \alpha, \Phi, U, N) = \frac{1}{m^N} \sum_{|w| = N} R_w(Z, \alpha, \Phi, U, N),
\]

and

\[
\Lambda(Z, \Phi, U, N) = \frac{1}{m^N} \sum_{|w| = N} \Lambda_w(Z, \Phi, U, N).
\]

It is easy to see that

\[
R(Z, \alpha, \Phi, U, N) = \Lambda(Z, \Phi, U, N) \exp(-\alpha(N+1)).
\]

Set

\[
\underline{r}(Z, \alpha, \Phi, U) = \liminf_{N \to \infty} R(Z, \alpha, \Phi, U, N),
\]

\[
\overline{r}(Z, \alpha, \Phi, U) = \limsup_{N \to \infty} R(Z, \alpha, \Phi, U, N).
\]

The C–P structure \( \tau \) generates the Carathéodory–Pesin dimension of \( Z \) and the lower and upper Carathéodory–Pesin capacities of \( Z \) with respect to \( G \). We denote them by \( P_Z(G, \Phi, U), \overline{CP}_Z(G, \Phi, U) \) and \( \underline{CP}_Z(G, \Phi, U) \) respectively. We have

\[
P_Z(G, \Phi, U) = \inf \{ \alpha : m(Z, \alpha, \Phi, U) = 0 \} = \sup \{ \alpha : m(Z, \alpha, \Phi, U) = \infty \},
\]

\[
\overline{CP}_Z(G, \Phi, U) = \inf \{ \alpha : \underline{r}(Z, \alpha, \Phi, U) = 0 \} = \sup \{ \alpha : \underline{r}(Z, \alpha, \Phi, U) = \infty \},
\]

and

\[
\underline{CP}_Z(G, \Phi, U) = \inf \{ \alpha : \overline{r}(Z, \alpha, \Phi, U) = 0 \} = \sup \{ \alpha : \overline{r}(Z, \alpha, \Phi, U) = \infty \}.
\]
Theorem 4.1 For any set $Z \subset X$, the following limits exist:
\[
P_Z(G, \Phi) = \lim_{|U| \to 0} P_Z(G, \Phi, U),
\]
\[
CP_Z(G, \Phi) = \lim_{|U| \to 0} CP_Z(G, \Phi, U),
\]
\[
\overline{CP}_Z(G, \Phi) = \lim_{|U| \to 0} \overline{CP}_Z(G, \Phi, U).
\]

Proof We use the analogous method as that of [24]. Let $V$ be a finite open cover of $X$ with diameter smaller than the Lebesgue number of $U$. Then each element $V \in V$ is contained in some element $U(V) \in U$. For any $V = (V_0, V_1, \ldots, V_k) \in S_{k+1}(V) \subset S(V)$, we associate the string $U(V) = (U(V_0), U(V_1), \ldots, U(V_k)) \in S_{k+1}(U) \subset S(U)$. Let $w = i_1 i_2 \ldots i_N \in \mathcal{F}_m$. If $G_w \subset S(V)$ covers $Z$, for each $V \in G_w$, $m(V) \geq N + 1$ and there is $w_V \in \mathcal{F}_m$ such that $\bar{w} \leq \bar{w}_V$. We denote the word that corresponds to $U(V)$ by $w_{U(V)}$ such that $w_{U(V)} = w_V$, then $U(G_w) = \{U(V) : V \in G_w\} \subset S(U)$ also covers $Z$. Let
\[
\gamma = \gamma(U) := \sup\{|\varphi_j(x) - \varphi_j(y)| : x, y \in U \text{ for some } U \in U \text{ and } j = 0, 1, \ldots, m - 1\}.
\]
It follows that
\[
\sup_{x \in x_{w_{U(V)}(U(V))}} S_{w_{U(V)}(U(V))}\Phi(x) \leq \sup_{y \in x_{w_V}(V)} S_{w_V}\Phi(y) + \gamma m(V).
\]

Note that $m(V) = m(U(V))$. Then for any $\alpha \in \mathbb{R}$ and $N > 0$, one can easily see that
\[
M_w(Z, \alpha, \Phi, U, N) \leq M_w(Z, \alpha - \gamma, \Phi, \mathcal{V}, N).
\]

Then
\[
M(Z, \alpha, \Phi, U, N) \leq M(Z, \alpha - \gamma, \Phi, \mathcal{V}, N).
\]
Moreover,
\[
m(Z, \alpha, \Phi, U) \leq m(Z, \alpha - \gamma, \Phi, \mathcal{V}).
\]
This implies that
\[
P_Z(G, \Phi, U) - \gamma \leq P_Z(G, \Phi, \mathcal{V}).
\]
Since $X$ is compact, it has finite open covers of arbitrarily small diameter. Therefore,
\[
P_Z(G, \Phi, U) - \gamma \leq \liminf_{|\mathcal{V}| \to 0} P_Z(G, \Phi, \mathcal{V}).
\]
If $|U| \to 0$, then $\gamma \to 0$ and hence
\[
\limsup_{|U| \to 0} P_Z(G, \Phi, U) \leq \liminf_{|\mathcal{V}| \to 0} P_Z(G, \Phi, \mathcal{V}).
\]
This implies the existence of the first limit. The existence of the two other limits can be proved in a similar fashion.

Remark 4.1

(1) It is easy to see that $P_Z(G, \Phi) \leq CP_Z(G, \Phi) \leq \overline{CP}_Z(G, \Phi)$.  

\[ Springer\]
(2) If \( \psi_i(x) = 0, i = 0, 1, \ldots, m - 1 \), we can obtain \( P_Z(G, 0) = h(Z) \), where \( h(Z) \) is the topological entropy on \( Z \) in [18]. Similarly, the lower and upper capacity topological pressures are the corresponding the lower and upper capacity topological entropies in [18].

(3) When \( m = 1 \), i.e., \( G = \{f\} \) and \( \Phi = \{\psi\} \), we obtain \( P_Z(G, \Phi) = P_Z(\psi), CP_Z(G, \Phi) = CP_Z(\psi), \) and \( \overline{CP}_Z(\psi) \), for any set \( Z \subset X \), where \( P_Z(\psi), CP_Z(\psi) \) and \( \overline{CP}_Z(\psi) \) are the topological pressure and lower and upper capacity topological pressures in [24]. That is to say, this definition agrees with Pesin’s [24]. Moreover, if \( Z = X \), then \( P_X(f, \psi) = CP_X(f, \psi) = \overline{CP}_X(f, \psi) = P(f, \psi) \), which is equivalent to the classical topological pressure in [39].

### 4.2 Properties of Topological Pressure and Lower and Upper Capacity Topological Pressures

Using the basic properties of the Carathéodory–Pesin dimension [24] and definitions, we get the following basic properties of topological pressure and lower and upper capacity topological pressures of free semigroup actions.

#### Proposition 4.1

1. \( P_\emptyset(G, \Phi) \leq 0 \).
2. \( P_Z(G, \Phi) \leq P_{Z_1}(G, \Phi) \) if \( Z_1 \subset Z_2 \).
3. \( P_Z(G, \Phi) = \sup_{i \geq 1} P_{Z_i}(G, \Phi) \) where \( Z = \bigcup_{i \geq 1} Z_i \) and \( Z_i \subset X, i = 1, 2, \ldots \).

#### Proposition 4.2

1. \( CP_\emptyset(G, \Phi) \leq 0, \overline{CP}_\emptyset(G, \Phi) \leq 0 \).
2. \( CP_{Z_1}(G, \Phi) \leq CP_{Z_2}(G, \Phi) \) and \( \overline{CP}_{Z_1}(G, \Phi) \leq \overline{CP}_{Z_2}(G, \Phi) \) if \( Z_1 \subset Z_2 \).
3. \( CP_Z(G, \Phi) \geq \sup_{i \geq 1} CP_{Z_i}(G, \Phi) \) and \( \overline{CP}_Z(G, \Phi) \geq \sup_{i \geq 1} \overline{CP}_{Z_i}(G, \Phi) \), where \( Z = \bigcup_{i \geq 1} Z_i \) and \( Z_i \subset X, i = 1, 2, \ldots \).
4. If \( g : X \to X \) is a homeomorphism which commutes with \( G \) (i.e., \( f_i \circ g = g \circ f_i \) for all \( f_i \in \{f_0, f_1, \ldots, f_{m-1}\} \)), then
   \[
   \begin{align*}
   P_Z(G, \Phi) &= P_g(Z)(G, \Phi \circ g^{-1}), \\
   CP_Z(G, \Phi) &= CP_g(Z)(G, \Phi \circ g^{-1}), \\
   \overline{CP}_Z(G, \Phi) &= \overline{CP}_g(Z)(G, \Phi \circ g^{-1}),
   \end{align*}
   \]
   where \( \Phi \circ g^{-1} = \{\psi_0 \circ g^{-1}, \psi_1 \circ g^{-1}, \ldots, \psi_{m-1} \circ g^{-1}\} \).

Obviously, the functions \( \eta \) and \( \psi \) satisfy condition (4) in Sect. 3.1. Therefore, similar to the Theorem 2.2 in [24], we obtain the following lemma.

#### Lemma 4.2

For any open cover \( \mathcal{U} \) of \( X \) and any set \( Z \subset X \), we have
\[
\begin{align*}
CP_Z(G, \Phi, \mathcal{U}) &= \liminf_{N \to \infty} \frac{\log \Lambda(Z, \Phi, \mathcal{U}, N)}{N}, \\
\overline{CP}_Z(G, \Phi, \mathcal{U}) &= \limsup_{N \to \infty} \frac{\log \Lambda(Z, \Phi, \mathcal{U}, N)}{N}.
\end{align*}
\]
**Proof** We will prove the first equality; the second one can be proved in a similar fashion. Put
\[
\alpha = \text{CP}_Z(G, \Phi, U), \quad \beta = \liminf_{N \to \infty} \frac{\log \Lambda(Z, \Phi, U, N)}{N}.
\]
Given \(\gamma > 0\), choose a sequence \(N_i \to \infty\) such that
\[
0 = r(Z, \alpha + \gamma, \Phi, U) = \lim_{i \to \infty} R(Z, \alpha + \gamma, \Phi, U, N_i).
\]
It follows that \(R(Z, \alpha + \gamma, \Phi, U, N_i) \leq 1\) for all sufficiently large \(i\). Therefore, for such \(i\)
\[
\Lambda(Z, \Phi, U, N_i) \exp(-\alpha \gamma(N_i + 1)) \leq 1. \tag{4.1}
\]
Moreover,
\[
\alpha + \gamma \geq \frac{\log \Lambda(Z, \Phi, U, N_i)}{N_i + 1}.
\]
Therefore,
\[
\alpha + \gamma \geq \liminf_{N \to \infty} \frac{\log \Lambda(Z, \Phi, U, N)}{N}.
\]
Hence,
\[
\alpha \geq \beta - \gamma. \tag{4.2}
\]
Let us now choose a sequence \(N'_i\) such that
\[
\beta = \lim_{i \to \infty} \frac{\log \Lambda(Z, \Phi, U, N'_i)}{N'_i}.
\]
We have that
\[
\lim_{i \to \infty} R(Z, \alpha - \gamma, \Phi, U, N'_i) \geq r(Z, \alpha - \gamma, \Phi, U) = \infty.
\]
This implies that \(R(Z, \alpha - \gamma, \Phi, U, N'_i) \geq 1\) for all sufficiently large \(i\). Therefore, for such \(i\)
\[
\Lambda(Z, \Phi, U, N'_i) \exp(-(\alpha - \gamma)(N'_i + 1)) \geq 1
\]
and hence
\[
\alpha - \gamma \leq \frac{\log \Lambda(Z, \Phi, U, N'_i)}{N'_i + 1}.
\]
Taking the limit as \(i \to \infty\) we obtain that
\[
\alpha - \gamma \leq \liminf_{N \to \infty} \frac{\log \Lambda(Z, \Phi, U, N)}{N} = \beta,
\]
and consequently,
\[
\alpha \leq \beta + \gamma. \tag{4.3}
\]
Since \(\gamma\) can be chosen arbitrarily small, the inequalities (4.2) and (4.3) imply that \(\alpha = \beta\). □
Remark 4.2 By the Theorem 4.1 and Lemma 4.2, we can obtain
\[
\begin{align*}
CP_Z(G, \Phi) &= \lim_{|U| \to 0} \liminf_{N \to +\infty} \frac{\log \Lambda(Z, \Phi, U, N)}{N}, \\
\overline{CP}_Z(G, \Phi) &= \lim_{|U| \to 0} \limsup_{N \to +\infty} \frac{\log \Lambda(Z, \Phi, U, N)}{N}.
\end{align*}
\]

Theorem 4.3 If \( \Phi = \{\varphi_0, \varphi_1, \ldots, \varphi_{m-1}\} \) and \( \Psi = \{\psi_0, \psi_1, \ldots, \psi_{m-1}\} \), we have
\[
\begin{align*}
|P_Z(G, \Phi) - P_Z(G, \Psi)| &\leq \max_{0 \leq i \leq m-1} \|\varphi_i - \psi_i\|, \\
|\overline{CP}_Z(G, \Phi) - \overline{CP}_Z(G, \Psi)| &\leq \max_{0 \leq i \leq m-1} \|\varphi_i - \psi_i\|, \\
|\underline{CP}_Z(G, \Phi) - \underline{CP}_Z(G, \Psi)| &\leq \max_{0 \leq i \leq m-1} \|\varphi_i - \psi_i\|,
\end{align*}
\]
where \( \| \cdot \| \) denotes the supremum norm in the space of continuous functions on \( X \).

Proof Notice that for any \( w \in F^+_m \),
\[
\sup_{x \in X} S_w \Phi(x) - \sup_{x \in X} S_w \Psi(x) \leq \sup_{x \in X} |S_w \Phi(x) - S_w \Psi(x)| \leq w \max_{0 \leq i \leq m-1} \|\varphi_i - \psi_i\|.
\]
Hence
\[
M_w(Z, \alpha, \Phi, U, N) \leq M_w(Z, \alpha - \max_{0 \leq i \leq m-1} \|\varphi_i - \psi_i\|, \Psi, U, N).
\]
Similarly, we have
\[
M_w(Z, \alpha, \Phi, U, N) \geq M_w(Z, \alpha + \max_{0 \leq i \leq m-1} \|\varphi_i - \psi_i\|, \Psi, U, N).
\]
Therefore,
\[
M(Z, \alpha + \max_{0 \leq i \leq m-1} \|\varphi_i - \psi_i\|, \Psi, U, N) \leq M(Z, \alpha, \Phi, U, N)
\leq M(Z, \alpha - \max_{0 \leq i \leq m-1} \|\varphi_i - \psi_i\|, \Psi, U, N).
\]
Taking limit \( N \to \infty \) yields
\[
m(Z, \alpha + \max_{0 \leq i \leq m-1} \|\varphi_i - \psi_i\|, \Psi, U) \leq m(Z, \alpha, \Phi, U)
\leq m(Z, \alpha - \max_{0 \leq i \leq m-1} \|\varphi_i - \psi_i\|, \Psi, U).
\]
Thus
\[
P_Z(G, \Psi, U) - \max_{0 \leq i \leq m-1} \|\varphi_i - \psi_i\| \leq P_Z(G, \Phi, U) \leq P_Z(G, \Psi, U) + \max_{0 \leq i \leq m-1} \|\varphi_i - \psi_i\|.
\]
Let \( |U| \to 0 \), and we obtain
\[
P_Z(G, \Psi) - \max_{0 \leq i \leq m-1} \|\varphi_i - \psi_i\| \leq P_Z(G, \Phi) \leq P_Z(G, \Psi) + \max_{0 \leq i \leq m-1} \|\varphi_i - \psi_i\|,
\]
which establishes the first inequality. The proof of the two other inequalities is similar. \( \square \)

For a free semigroup \( G \) with \( m \) generators acting on \( X \), denoting the maps corresponding to the generators by \( f_0, f_1, \ldots, f_{m-1} \), a set \( Z \subset X \) is called \( G \)-invariant if \( f_i^{-1}(Z) = Z \) for all \( f_i \in G \). For invariant sets, similar to the lower and upper capacity topological pressures of a single map [24], we have the following theorems.

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Theorem 4.4 For any $G$-invariant set $Z \subset X$, we have

$$\mathcal{CP}_Z(G, \Phi) = \mathcal{CP}_Z(G, \Phi).$$

Moreover, for any open cover $\mathcal{U}$ of $X$, we have

$$\mathcal{CP}_Z(G, \Phi, \mathcal{U}) = \mathcal{CP}_Z(G, \Phi, \mathcal{U}).$$

Proof Let $Z \subset X$ be a $G$-invariant set. For any $w^{(1)}$, $w^{(2)} \in F^+_m$ where $|w^{(1)}| = p$ and $|w^{(2)}| = q$, we choose two collections of strings $\mathcal{G}_w^{(1)} \subset S_{p+1}(\mathcal{U})$ and $\mathcal{G}_w^{(2)} \subset S_{q+1}(\mathcal{U})$ which cover $Z$. Supposing that $U = (U_0, U_1, \ldots, U_p) \in \mathcal{G}_w^{(1)}$ and $V = (V_0, V_1, \ldots, V_q) \in G_w^{(2)}$, we define

$$UV = (U_0, U_1, \ldots, U_p, V_0, V_1, \ldots, V_q).$$

For a fixed $i \in \{0, 1, \ldots, m - 1\}$, we consider

$$\mathcal{G}_w := \{UV : U \in \mathcal{G}_w^{(1)}, V \in \mathcal{G}_w^{(2)}\} \subset S_{p+q+2}(\mathcal{U}),$$

where $w = w^{(1)}i w^{(2)}$. Then

$$X_w(UV) = X_w^{(1)}(U) \cap (f_i \circ f_{w^{(1)}})^{-1}(X_w^{(2)}(V)),$$

and $m(UV) = m(U) + m(V)$. Since $Z$ is a $G$-invariant set, the collection of strings $\mathcal{G}_w$ also covers $Z$. By the definition of $\Lambda_w(Z, \Phi, \mathcal{U}, p + q + 1)$, we have

$$\Lambda_w(Z, \Phi, \mathcal{U}, p + q + 1) \leq \sum_{UV \in \mathcal{G}_w} \exp \left( \sup_{x \in X_w(UV)} S_w \Phi(x) \right)$$

$$\leq c \times \left\{ \sum_{U \in \mathcal{G}_w^{(1)}} \exp \left( \sup_{x \in X_w^{(1)}(U)} S_w^{(1)} \Phi(x) \right) \right\} \times \left\{ \sum_{V \in \mathcal{G}_w^{(2)}} \exp \left( \sup_{x \in X_w^{(2)}(V)} S_w^{(2)} \Phi(x) \right) \right\},$$

where $c = \max_{0 \leq i \leq m - 1} e^{\|\psi_i\|}$, which implies

$$\frac{1}{m} \sum_{i=0}^{m-1} \Lambda_w(Z, \Phi, \mathcal{U}, p + q + 1) \leq c \times \Lambda_w^{(1)}(Z, \Phi, \mathcal{U}, p) \times \Lambda_w^{(2)}(Z, \Phi, \mathcal{U}, q).$$

It follows that

$$\Lambda(Z, \Phi, \mathcal{U}, p + q + 1) \leq c \times \Lambda(Z, \Phi, \mathcal{U}, p) \times \Lambda(Z, \Phi, \mathcal{U}, q).$$

Let $a_p = \log \Lambda(Z, \Phi, \mathcal{U}, p)$. Note that $\Lambda(Z, \Phi, \mathcal{U}, p) \geq e^{-p \max_{0 \leq i \leq m - 1} \|\psi_i\|}$. Therefore, $\inf_{p>1} \frac{a_{p-1}}{p} > -\infty$. The desired result is now a direct consequence of Remark 4.2 and the following Lemma 4.5.

Lemma 4.5 Let $\{a_p\} = 1, 2, \ldots$ be a sequence of numbers satisfying $\inf_{p>1} \frac{a_{p-1}}{p} > -\infty$ and $a_{p+q+1} \leq c' + a_p + a_q$ for all $p, q > 1$ where $c' > 0$ is a constant. Then the limit $\lim_{p \to \infty} \frac{a_p}{p}$ exists and coincides with $\inf_{p>1} \left( \frac{a_{p-1}}{p} + \frac{c'}{p} \right)$.

Proof The proof is similar to the Theorem 4.9 in [39], so we omit the proof. \hfill \square
Remark 4.3 Indeed, when \( Z = X \) and \( \Phi = \{ \varphi \} \), i.e., \( \varphi_0 = \varphi_1 = \cdots = \varphi_{m-1} = \varphi \), it is easy to get \( CP_Z(G, \Phi) = CP_X(G, \Phi) = P_X(G, \varphi) \), where \( P_X(G, \varphi) \) is denoted by the topological pressure of free semigroup actions on \( X \) in [19].

Example 4.6 Let \( X = [0, 1] \), \( G \) is generated by \( f_0 \) and \( f_1 \), where \( f_0 \) and \( f_1 \) are defined by

\[
f_0(x) = \begin{cases} 
-2x + \frac{1}{3} & x \in [0, \frac{1}{6}) \\
2x - \frac{1}{3} & x \in (\frac{1}{6}, \frac{1}{3}) \\
3x - \frac{2}{3} & x \in (\frac{1}{3}, \frac{4}{9}) \\
-3x + 2 & x \in (\frac{4}{9}, \frac{5}{9}) \\
3x - \frac{4}{3} & x \in (\frac{5}{9}, \frac{2}{3}) \\
x - \frac{2}{3} & x \in (\frac{2}{3}, \frac{5}{6}) \\
-2x + \frac{8}{3} & x \in [\frac{5}{6}, 1]
\end{cases}
\]

\[
f_1(x) = \begin{cases} 
-\frac{1}{3} \sin(3\pi x) + \frac{1}{3} & x \in [0, \frac{1}{3}) \\
5x - \frac{4}{3} & x \in (\frac{1}{3}, \frac{6}{15}) \\
-5x + \frac{8}{3} & x \in [\frac{6}{15}, \frac{7}{15}) \\
5x - 2 & x \in (\frac{7}{15}, \frac{8}{15}) \\
-5x + \frac{10}{3} & x \in (\frac{8}{15}, \frac{9}{15}) \\
5x - \frac{8}{3} & x \in (\frac{9}{15}, \frac{10}{15}) \\
\frac{1}{3} \sin(3\pi x) + \frac{2}{3} & x \in [\frac{10}{15}, 1]
\end{cases}
\]

The images of functions \( f_0 \) and \( f_1 \) are shown in Fig. 1.

Choose \( Z = (\frac{1}{3}, \frac{2}{3}) \), then \( Z \) is \( G \)-invariant, namely, \( f_0^{-1}(Z) = Z \) and \( f_1^{-1}(Z) = Z \). Thus, for any \( \Phi = \{ \varphi_0, \varphi_1 \} \), where \( \varphi_0, \varphi_1 \in C(X, \mathbb{R}) \), we have

\[
CP_Z(G, \Phi) = CP_Z(G, \Phi).
\]

Next, we discuss the relationship between the topological pressure and upper capacity topological pressure of a free semigroup action generated by \( G \) on \( Z \) when \( Z \) is a compact \( G \)-invariant set. Given a compact \( G \)-invariant set \( Z \subset X \) and an open cover \( \mathcal{U} \) of \( X \), we choose any \( \alpha > P_Z(G, \mathcal{U}) \), then

\[
m(Z, \alpha, \Phi, \mathcal{U}) = \lim_{N \to \infty} M(Z, \alpha, \Phi, \mathcal{U}, N) = 0.
\]

Since \( M(Z, \alpha, \Phi, \mathcal{U}, N) \) is non-decreasing as \( N \) increases and non-negative, it follows that \( M(Z, \alpha, \Phi, \mathcal{U}, N) = 0 \) for any \( N \). Therefore, for any \( w \in F_{m+}^{+} \) and \( |w| = N \), we have \( M_w(Z, \alpha, \Phi, \mathcal{U}, N) = 0 \). Let \( c := \max_{0 \leq i \leq m-1} e^{\|\varphi_i\|} \), \( 0 < p < 1 \), such that \( cp < 1 \). For \( M_w(Z, \alpha, \Phi, \mathcal{U}, 2) = 0 \), there exists \( A_w \subset \hat{S}(\mathcal{U}) \) such that \( A_w \) covers \( Z \) (i.e., for any \( U \in A_w \), there exists \( w_U \in F_{m}^{+} \) such that \( |w_U| = m(U) - 1 \), \( \bar{w} \leq \bar{w_U} \) and \( \bigcup_{U \in A_w} X_{w_U}(U) \supset Z \) and

![Fig. 1](image-url) Left: the graph of \( f_0 \). Right: the graph of \( f_1 \)

\[\text{Springer}\]
For any $U$ of $X$, we have

$$w(1)jw(2)j...jn \in A_{\alpha } \cdot \Phi  \cdot \Gamma 1w(1)jw(2)j...jn \subseteq w(1)jw(2)j...jn \subseteq A_{\alpha } \cdot \Phi  \cdot \Gamma 1w(1)jw(2)j...jn.$$  

Therefore, for any $\omega \in \Sigma _{m}^{+}$, there exists $\Gamma _{\omega }$ covering $Z$ and $Q(Z, \alpha , \Phi , \Gamma _{\omega }) < \infty$. Put

$$\mathcal{F} = \{ \Gamma _{\omega } : \omega \in \Sigma _{m}^{+} \}.$$  

Similar to [18], the following condition is given and the following Theorem 4.8 holds under this condition.

**Condition 4.7** For any $N > 0$ and any $w = i_{1}i_{2}...i_{N} \in F_{m}^{+}$, there exists $\Gamma _{\omega } \in \mathcal{F}$ such that for any $U \in \Gamma _{\omega }$, $\overline{w} \leq \overline{w}U$ and $N + 1 \leq m(U) \leq N + K$, where $wU$ is the word corresponds to $U$ and $K$ is given by (4.5).

**Theorem 4.8** Under the condition 4.7, for any compact $G$-invariant set $Z \subset X$, we have

$$P_{Z}(G, \Phi ) = CP_{Z}(G, \Phi ) = C\overline{P}_{Z}(G, \Phi ).$$  

Moreover, for any open cover $\mathcal{U}$ of $X$, we have

$$P_{Z}(G, \Phi , \mathcal{U}) = CP_{Z}(G, \Phi , \mathcal{U}) = C\overline{P}_{Z}(G, \Phi , \mathcal{U}).$$  

**Proof** Under the condition 4.7, for any $N > 0$ and any $w = i_{1}i_{2}...i_{N} \in F_{m}^{+}$, there is $\Gamma _{\omega } \in \mathcal{F}$ covering $Z$ such that for any $U \in \Gamma _{\omega }$, the word corresponds to $U$ is $wU$ and $\overline{w} \leq \overline{w}U$. Then for any $x \in Z$, there exists a string $U = (U_{0}, U_{1}, ..., U_{N}, ..., U_{N+p}) \in \Gamma _{\omega }$ such that
Let \( x \in X \) be a point in the phase space, where \( 0 \leq P < K \). Let \( U^* = (U_0, U_1, \ldots, U_N) \). Then \( X_{U^*} \subset X \) is compact. Using \( \Gamma_{\omega}^* \) denotes the collection of all substrings constructed above and let

\[
\varepsilon(|U|) := \sup \{|\varphi_i(x) - \varphi_j(y)| : x, y \in U, \forall U \in \mathcal{U}, 0 \leq i \leq m - 1\}.
\]

Because of \( \varphi_i \in C(X, \mathbb{R}) \), \( \varphi_i \) is uniformly continuous for \( 0 \leq i \leq m - 1 \). Then \( \varepsilon(|U|) \) is finite and \( \lim_{|U| \to 0} \varepsilon(|U|) = 0 \).

\[
\sup_{y \in X_{U^*}} S_y \Phi(y) \leq \sup_{x \in X_{U^*}} S_x \Phi(x) + K \cdot \max_{0 \leq i \leq m - 1} \|\varphi_i\| + N \varepsilon(|U|).
\]

Therefore,

\[
\sum_{U^* \in \Gamma_{\omega}^*} \exp \left( \sup_{y \in X_{U^*}} S_y \Phi(y) \right) \leq \max\{1, e^{\alpha K} \cdot e^{K \cdot \max_{0 \leq i \leq m - 1} \|\varphi_i\|} \cdot Q(Z, \alpha, \Phi, \Gamma_{\omega}) \}
\]

It follows that

\[
e^{-(\alpha + \varepsilon(|U|))N} A_{U^*}(Z, \Phi, \mathcal{U}, N) < \infty.
\]

Moreover,

\[
e^{-(\alpha + \varepsilon(|U|))N} A(Z, \Phi, \mathcal{U}, N) < \infty.
\]

By Lemma 4.2

\[
\alpha + \varepsilon(|U|) > \overline{CP}_Z(G, \Phi, \mathcal{U}),
\]

that is,

\[
\alpha > \overline{CP}_Z(G, \Phi, \mathcal{U}) - \varepsilon(|U|).
\]

Letting \( |U| \to 0 \), we get \( P_Z(G, \Phi) > \overline{CP}_Z(G, \Phi) \). \( \square \)

### 5 Two Equivalent Definitions of Topological Pressure in the Present Paper

Let \( (X, d) \) be a compact metric space. Now, we describe two other approaches to redefine the topological pressure and lower and upper capacity topological pressures of \( G = \{f_0, \ldots, f_{m-1}\} \) on any subset of \( X \) and \( \Phi = \{\varphi_0, \varphi_1, \ldots, \varphi_{m-1}\} \), where \( f_i(i = 0, 1, \ldots, m-1) \) is continuous and \( \varphi_0, \varphi_1, \ldots, \varphi_{m-1} \in C(X, \mathbb{R}) \).

For each \( w \in F_m^+ \), a new metric \( d_w \) on \( X \) (named Bowen metric) is given by

\[
d_w(x_1, x_2) = \max_{w' \leq w} d(f_{w'}(x_1), f_{w'}(x_2)).
\]

Clearly, if \( \overrightarrow{w} \leq \overrightarrow{w'} \), then \( d_{w'}(x_1, x_2) \leq d_w(x_1, x_2) \) for all \( x_1, x_2 \in X \).
5.1 Definition using Bowen Balls

Fix a number $\delta > 0$. Given $w \in F^+_m$ and a point $x \in X$, define the $(w, \delta)$-Bowen ball at $x$ by

$$B_w(x, \delta) = \{y \in X : d(f_w'(x), f_w'(y)) < \delta, \text{ for } w' \leq w\}.$$ 

Put $S = X \times F^+_m$. We define the collection of subsets

$$\mathcal{F} = \{B_w(x, \delta) : x \in X, w \in F^+_m\},$$

and three functions $\xi, \eta, \psi : S \to \mathbb{R}$ as follows

$$\xi(x, w) = \exp \left( \sup_{y \in B_w(x, \delta)} S_w \Phi(y) \right),$$

$$\eta(x, w) = \exp \left( -(|w| + 1) \right), \quad \psi(x, w) = (|w| + 1)^{-1}.$$

One can easily verify that the collection of subsets $\mathcal{F}$ and the functions $\xi, \eta, \psi$ satisfy conditions (1), (2), (3) and (3') in Sect. 3.1. Therefore, they determine a C–P structure $\tau = (S, \mathcal{F}, \xi, \eta, \psi)$ on $X$.

Given $w \in F^+_m$, $|w| = N$, $Z \subset X$ and $\alpha \in \mathbb{R}$, we define

$$M_w(Z, \alpha, \Phi, \delta, N) := \inf_{G_w} \left\{ \sum_{B_w(x, \delta) \in G_w} \xi(x, |w|) \eta(x, |w|)^{\alpha} \right\}$$

$$= \inf_{G_w} \left\{ \sum_{B_w(x, \delta) \in G_w} \exp \left( -\alpha \cdot (|w| + 1) + \sup_{y \in B_w(x, \delta)} S_w \Phi(y) \right) \right\},$$

where the infimum is taken over all finite or countable subcollections $G_w \subset \mathcal{F}$ covering $Z$ (i.e., for any $B_{w'}(x, \delta) \in G_w$, $w' \leq w$ and $\bigcup_{B_{w'}(x, \delta) \in G_w} B_{w'}(x, \delta) \supset Z$).

Let

$$M(Z, \alpha, \Phi, \delta, N) = \frac{1}{m^N} \sum_{|w|=N} M_w(Z, \alpha, \Phi, \delta, N).$$

We can easily verify that the function $M(Z, \alpha, \Phi, \delta, N)$ is non-decreasing as $N$ increases. Therefore, there exists the limit

$$\overline{m}(Z, \alpha, \Phi, \delta) = \lim_{N \to \infty} M(Z, \alpha, \Phi, \delta, N).$$

Furthermore, by the condition (3') in Sect. 3.1, we can define

$$\overline{R}_w(Z, \alpha, \Phi, \delta, N) = \inf_{G_w} \left\{ \sum_{B_w(x, \delta) \in G_w} \exp \left( -\alpha \cdot (N + 1) + \sup_{y \in B_w(x, \delta)} S_w \Phi(y) \right) \right\}$$

$$= \overline{\Lambda}_w(Z, \Phi, \delta, N) \exp(-\alpha \cdot (N + 1)),$$

where $\overline{\Lambda}_w(Z, \Phi, \delta, N) = \inf_{G_w} \left\{ \sum_{B_w(x, \delta) \in G_w} \exp \left( \sup_{y \in B_w(x, \delta)} S_w \Phi(y) \right) \right\}$, the infimum is taken over all finite or countable subcollections $G_w \subset \mathcal{F}$ covering $Z$ and the words corresponding to every ball in $G_w$ are all equal.
Let
\[
R(Z, \alpha, \Phi, \delta, N) = \frac{1}{m^N} \sum_{|w|=N} R_w(Z, \alpha, \Phi, \delta, N),
\]
\[
\Lambda(Z, \Phi, \delta, N) = \frac{1}{m^N} \sum_{|w|=N} \Lambda_w(Z, \Phi, \delta, N).
\]

We set
\[
r(Z, \alpha, \Phi, \delta) = \liminf_{N \to \infty} R(Z, \alpha, \Phi, \delta, N),
\]
\[
r(Z, \alpha, \Phi, \delta) = \limsup_{N \to \infty} R(Z, \alpha, \Phi, \delta, N).
\]

The C–P structure \(\tau\) generates the Carathéodory–Pesin dimension of \(Z\) and the lower and upper Carathéodory–Pesin capacities of \(Z\) with respect to \(G\). We denote them by
\[
P_Z(G, \Phi, \delta) = \inf\{\alpha : m(Z, \alpha, \Phi, \delta) = 0\} = \sup\{\alpha : m(Z, \alpha, \Phi, \delta) = \infty\},
\]
\[
CP_Z(G, \Phi, \delta) = \inf\{\alpha : r(Z, \alpha, \Phi, \delta) = 0\} = \sup\{\alpha : r(Z, \alpha, \Phi, \delta) = \infty\},
\]
\[
CP_Z(G, \Phi, \delta) = \inf\{\alpha : r(Z, \alpha, \Phi, \delta) = 0\} = \sup\{\alpha : r(Z, \alpha, \Phi, \delta) = \infty\}.
\]

**Theorem 5.1** For any set \(Z \subset X\), the following limits exist:

\[
P_Z(G, \Phi) = \lim_{\delta \to 0} P_Z(G, \Phi, \delta),
\]
\[
CP_Z(G, \Phi) = \lim_{\delta \to 0} CP_Z(G, \Phi, \delta),
\]
\[
CP_Z(G, \Phi) = \lim_{\delta \to 0} CP_Z(G, \Phi, \delta).
\]

**Proof** Let \(U\) be a finite open cover of \(X\), and \(\delta(U)\) is the Lebesgue number of \(U\). It is easy to see that for any \(x \in X\), if \(x \in X_\omega(U)\) for some \(U \in S_{k+1}(U)\) and some \(\omega \in \Sigma_m^+\) then
\[
B_{\omega[0,k-1]}(x, \frac{1}{2}\delta(U)) \subset X_\omega(U) \subset B_{\omega[0,k-1]}(x, 2\delta(U)).
\]

It follows from Theorem 4.1 that
\[
P_Z(G, \Phi) = \lim_{\delta \to 0} P_Z(G, \Phi, \delta),
\]
\[
CP_Z(G, \Phi) = \lim_{\delta \to 0} CP_Z(G, \Phi, \delta),
\]
\[
CP_Z(G, \Phi) = \lim_{\delta \to 0} CP_Z(G, \Phi, \delta).
\]

\[\square\]

**Remark 5.1** Similar to Remark 4.2, we have

\[
CP_Z(G, \Phi) = \lim_{\delta \to 0} \liminf_{N \to \infty} \frac{\log \Lambda(Z, \Phi, \delta, N)}{N},
\]
\[
CP_Z(G, \Phi) = \lim_{\delta \to 0} \limsup_{N \to \infty} \frac{\log \Lambda(Z, \Phi, \delta, N)}{N}.
\]
5.2 Definition Using the Center of Bowen’s Ball

Put \( \mathcal{S} = X \times F_m^+ \), define the collection of subsets

\[ \mathcal{F} = \{ B_w(x, \delta) : x \in X, w \in F_m^+ \} \]

We redefine three functions \( \xi, \eta, \psi : \mathcal{S} \rightarrow \mathbb{R} \) as follows

\[
\begin{align*}
\xi(x, w) &= \exp \left( S_w \Phi_1(x) \right), \\
\eta(x, w) &= \exp \left( -|w| \right), \\
\psi(x, w) &= |w| - 1.
\end{align*}
\]

One can easily verify that the collection of subsets \( \mathcal{F} \) and the functions \( \xi, \eta, \psi \) satisfy conditions (1), (2), (3) and \((3')\) in Sect. 3.1. Therefore, they determine a C–P structure \( \tau = (\mathcal{S}, \mathcal{F}, \xi, \eta, \psi) \) on \( X \). Given \( w \in F_m^+ \), \( |w| = N \), \( Z \subset X \) and \( \alpha \in \mathbb{R} \), we define

\[
M_w'(Z, \alpha, \Phi, \delta, N) := \inf_{\mathcal{G}_w} \left\{ \sum_{B_w(x, \delta) \in \mathcal{G}_w} \xi(x, |w|) \eta(x, |w|)^\alpha \right\}
\]

\[
= \inf_{\mathcal{G}_w} \left\{ \sum_{B_w(x, \delta) \in \mathcal{G}_w} \exp \left( -\alpha \cdot |w| + S_w \Phi_1(x) \right) \right\},
\]

where the infimum is taken over all finite or countable subcollections \( \mathcal{G}_w \subset \mathcal{F} \) covering \( Z \) (i.e., for any \( B_{w'}(x, \delta) \in \mathcal{G}_w, \ w \leq w', \ x \in Z \) and \( \bigcup_{B_{w'}(x, \delta) \in \mathcal{G}_w} B_{w'}(x, \delta) \supset Z \)).

Let

\[
M'(Z, \alpha, \Phi, \delta, N) = \frac{1}{mN} \sum_{|w| = N} M_w'(Z, \alpha, \Phi, \delta, N).
\]

We can easily verify that the function \( M'(Z, \alpha, \Phi, \delta, N) \) is non-decreasing as \( N \) increases. Therefore, there exists the limit

\[
m'(Z, \alpha, \Phi, \delta) = \lim_{N \rightarrow \infty} M'(Z, \alpha, \Phi, \delta, N).
\]

Furthermore, by the condition \((3')\) in Sect. 3.1, we can define

\[
R_w'(Z, \alpha, \Phi, \delta, N) = \inf_{\mathcal{G}_w} \left\{ \sum_{B_w(x, \delta) \in \mathcal{G}_w} \exp \left( -\alpha \cdot N + S_w \Phi_1(x) \right) \right\}
\]

\[
= \Lambda_w'(Z, \Phi, \delta, N) \exp(-\alpha \cdot N),
\]

where \( \Lambda_w'(Z, \Phi, \delta, N) = \inf_{\mathcal{G}_w} \left\{ \sum_{B_w(x, \delta) \in \mathcal{G}_w} \exp \left( S_w \Phi_1(x) \right) \right\} \), the infimum is taken over all finite or countable subcollections \( \mathcal{G}_w = \{ B_w(x, \delta) : x \in Z \} \subset \mathcal{F} \) covering \( Z \) and the words corresponding to every ball in \( \mathcal{G}_w \) are all equal.

Let

\[
R'(Z, \alpha, \Phi, \delta, N) = \frac{1}{mN} \sum_{|w| = N} R_w'(Z, \alpha, \Phi, \delta, N),
\]

\[
\Lambda'(Z, \Phi, \delta, N) = \frac{1}{mN} \sum_{|w| = N} \Lambda_w'(Z, \Phi, \delta, N).
\]
We set

\[ r'(Z, \alpha, \Phi, \delta) = \liminf_{N \to \infty} R'(Z, \alpha, \Phi, \delta, N), \]

\[ \overline{r}'(Z, \alpha, \Phi, \delta) = \limsup_{N \to \infty} R'(Z, \alpha, \Phi, \delta, N). \]

The C–P structure \( \tau \) generates the Carathéodory–Pesin dimension of \( Z \) and the lower and upper Carathéodory–Pesin capacities of \( Z \) with respect to \( G \). We denote them by \( P_Z'(G, \Phi, \delta), \overline{C_P'}(G, \Phi, \delta) \), and \( \overline{C_P'}(G, \Phi, \delta) \) respectively. We have that

\[ P_Z'(G, \Phi, \delta) = \inf \{ \alpha : m'(Z, \alpha, \Phi, \delta) = 0 \} = \sup \{ \alpha : m'(Z, \alpha, \Phi, \delta) = \infty \}, \]

\[ \overline{C_P'}(G, \Phi, \delta) = \inf \{ \alpha : r'(Z, \alpha, \Phi, \delta) = 0 \} = \sup \{ \alpha : r'(Z, \alpha, \Phi, \delta) = \infty \}, \]

\[ \overline{C_P'}(G, \Phi, \delta) = \inf \{ \alpha : \overline{r}'(Z, \alpha, \Phi, \delta) = 0 \} = \sup \{ \alpha : \overline{r}'(Z, \alpha, \Phi, \delta) = \infty \}. \]

**Theorem 5.2** For any set \( Z \subset X \), the following limits exist:

\[ P_Z(G, \Phi) = \lim_{\delta \to 0} P_Z'(G, \Phi, \delta), \]

\[ \overline{C_P}(G, \Phi) = \lim_{\delta \to 0} \overline{C_P'}(G, \Phi, \delta), \]

\[ \overline{C_P}(G, \Phi) = \lim_{\delta \to 0} \overline{C_P'}(G, \Phi, \delta). \]

**Proof** We use the analogous method as that of [13]. On the one hand, given \( \delta > 0 \), let

\[ \epsilon(\delta) := \sup \{ |\varphi_i(x) - \varphi_i(y)| : d(x, y) < \delta, i = 0, 1, \ldots, m-1 \}. \]

and observe that since \( \varphi_i \in C(X, \mathbb{R}) \) and \( X \) is compact, \( \varphi_i \) is in fact uniformly continuous, hence \( \epsilon(\delta) \) is finite and \( \lim_{\delta \to 0} \epsilon(\delta) = 0 \). Furthermore, given \( x \in X, w \in F_m^+ \) and \( |w| = N \), then for any \( y \in B_w(x, \delta) \), we have

\[ |\varphi_i(f_w'(x)) - \varphi_i(f_w'(y))| \leq \epsilon(\delta) \text{ for any } w' \leq w, i = 0, 1, \ldots, m-1. \]

Thus

\[ |S_w \Phi(x) - S_w \Phi(y)| \leq |w|\epsilon(\delta). \]

Now fix \( \delta > 0 \), choose a finite open cover \( U \) of \( X \) with \( |U| < \delta \) and let \( \gamma(U) \) be the Lebesgue number of \( U \). Let \( G_w = \{ B_w'(x, \frac{1}{2}\gamma(U)) : x \in Z, w \leq w' \} \) be a open cover of \( Z \), then for each \( B_w'(x, \frac{1}{2}\gamma(U)) \in G_w \) there exists \( U \in S_{|w'|+1}(U) \) such that \( B_w'(x, \frac{1}{2}\gamma(U)) \subset X_{w'}(U) \). Set

\[ G'_w = \{ U : B_w'(x, \frac{1}{2}\gamma(U)) \subset X_{w'}(U) \} \]

and then

\[ M_w(Z, \alpha, \Phi, U, N) \leq \sum_{U \in G'_w} \exp \left( -\alpha m(U) + \sup_{y \in X_{w'}(U)} S_w \Phi(y) \right) \]

\[ \leq \sum_{B_w'(x, \frac{1}{2}\gamma(U)) \in G_w} \exp \left( -\alpha m(U) + S_w \Phi(x) + |w'|\epsilon(\delta) \right) \]

\[ \leq e^{-\alpha} \cdot \sum_{B_w'(x, \frac{1}{2}\gamma(U)) \in G_w} \exp \left( -|w'|(|\alpha - \epsilon(\delta)) + S_w \Phi(x) \right). \]
Moreover, we can get
\[ M_w(Z, \alpha, \Phi, \mathcal{U}, N) \leq e^{-\alpha} \cdot M'_w(Z, \alpha - \varepsilon(\delta), \Phi, \frac{1}{2} \gamma(\mathcal{U}), N), \]
which implies
\[ M(Z, \alpha, \Phi, \mathcal{U}, N) \leq e^{-\alpha} \cdot M'(Z, \alpha - \varepsilon(\delta), \Phi, \frac{1}{2} \gamma(\mathcal{U}), N). \]
Taking the limit \( N \to \infty \) yields
\[ m(Z, \alpha, \Phi, \mathcal{U}) \leq e^{-\alpha} \cdot m'(Z, \alpha - \varepsilon(\delta), \Phi, \frac{1}{2} \gamma(\mathcal{U})). \]
Therefore
\[ P_Z(G, \Phi, \mathcal{U}) \leq P'_Z(G, \Phi, \frac{1}{2} \gamma(\mathcal{U})), \]
and as \( \delta \to 0 \), that is, \( \varepsilon(\delta) \to 0 \), \( |\mathcal{U}| \to 0 \), we obtain
\[ P_Z(G, \Phi) \leq \liminf_{\gamma(\mathcal{U}) \to 0} P'_Z(G, \Phi, \frac{1}{2} \gamma(\mathcal{U})). \]
On the other hand, fix a cover \( \mathcal{U} \) of \( X \) with \( |\mathcal{U}| < \delta \). Given \( w \in F^+_w \), \( |w| = N \) and \( \mathcal{G}_w \subset S(\mathcal{U}) \) covering \( Z \), we may assume without loss of generality that for every \( U \in \mathcal{G}_w \), we have \( X_{wU}(U) \cap Z \neq \emptyset \). Then for each \( U \in \mathcal{G}_w \), we can choose \( x \in X_{wU}(U) \cap Z \). We observe
\[ X_{wU}(U) \subset B_{wU}(x, \delta). \]
Using \( F_w \) denotes the collection of all \( (wU, \delta) \)-Bowen ball \( B_{wU}(x, \delta) \) constructed above and then
\[ M_w(Z, \alpha, \Phi, \mathcal{U}, N) = \inf_{\mathcal{G}_w} \left\{ \sum_{U \in \mathcal{G}_w} \exp \left( -\alpha m(U) + \sup_{y \in X_{wU}(U)} S_{wU}(y) \right) \right\} \]
\[ \geq e^{-\alpha} \cdot \inf_{F_w} \left\{ \sum_{B_{wU}(x, \delta) \in F_w} \exp \left( -\alpha |wU| + S_{wU}(x) \right) \right\} \]
\[ \geq e^{-\alpha} \cdot M'_w(Z, \alpha, \Phi, \delta, N). \]
It follows
\[ M(Z, \alpha, \Phi, \mathcal{U}, N) \geq e^{-\alpha} \cdot M'(Z, \alpha, \Phi, \delta, N). \]
Hence
\[ m(Z, \alpha, \Phi, \mathcal{U}) \geq e^{-\alpha} \cdot m'(Z, \alpha, \Phi, \delta). \]
Therefore
\[ P_Z(G, \Phi, \mathcal{U}) \geq P'_Z(G, \Phi, \delta), \]
and taking the limit as \( \delta \to 0 \) gives
\[ P_Z(G, \Phi) \geq \limsup_{\delta \to 0} P'_Z(G, \Phi, \delta), \]
which completes the proof of the first. The existence of the two other limits can be proved in a similar way.

\[\square\]

**Remark 5.2**

1. When \(m = 1\), i.e., \(G = \{f\}, \Phi = \{\varphi\}\), then \(P_Z(G, \Phi)\) coincides with the topological pressure using the centre of Bowen ball which is defined by Climenhaga [14] for every \(Z \subset X\).

2. Similar to Remark 4.2, we have

\[
\begin{align*}
CP_Z(G, \Phi) &= \lim_{\delta \to 0} \liminf_{N \to \infty} \frac{\log \Lambda'(Z, \Phi, \delta, N)}{N}, \\
\overline{CP}_Z(G, \Phi) &= \lim_{\delta \to 0} \limsup_{N \to \infty} \frac{\log \Lambda'(Z, \Phi, \delta, N)}{N}.
\end{align*}
\]

**6 The Proof of Theorem 2.2**

Let \((X, d)\) be a compact metric space and \(G = \{f_0, f_1, \ldots, f_{m-1}\}\) be a free semigroup with \(m\) generators acting on \(X\), where \(f_0, f_1, \ldots, f_{m-1}\) are continuous transformations from \(X\) to itself. In this section, as the application of the topological pressure, we give the connection between topological pressure and Hausdorff dimension on some \(Z\) in the form of Bowen’s equation, which extends the results of Climenhaga [14]. Before proving the Theorem 2.2, we give some relevant results.

**Proposition 6.1** Let \(f_i : X \to X\) be as in Theorem 2.2. Fix \(0 < \alpha \leq \beta < \infty\) and \(Z \subset A([\alpha, \beta])\), then

1. for any \(t \in \mathbb{R}\) and \(h > 0\), we have

\[
P_Z(G, -t\Phi) - \beta h \leq P_Z(G, -(t + h)\Phi) \leq P_Z(G, -t\Phi) - \alpha h.
\]

2. The equation \(P_Z(G, -t\Phi) = 0\) has a unique root \(t^*\) and

\[
\frac{h_Z(G)}{\beta} \leq t^* \leq \frac{h_Z(G)}{\alpha}.
\]

3. If \(\alpha = \beta\), then \(t^* = \frac{h_Z(G)}{\alpha}\).

Here \(h_Z(G)\) is the topological entropy in Ju et al [18], \(\Phi = \{\log a_0, \log a_1, \ldots, \log a_{m-1}\}\) and \(t \Phi = \{t \cdot \log a_0, t \cdot \log a_1, \ldots, t \cdot \log a_{m-1}\}\).

**Proof** (1) Given arbitrary \(\varepsilon > 0\) and \(k \geq 1\), let

\[
Z_k = \left\{ x \in Z : \frac{1}{|w|}S_w\Phi(x) \in (\alpha - \varepsilon, \beta + \varepsilon), \text{ for any } |w| \geq k \right\}.
\]
and notice that $Z = \bigcup_{k=1}^{\infty} Z_k$. Now fix $t \in \mathbb{R}$, $h > 0$, $w = i_1i_2 \cdots i_N \in F_m^+$ and $N \geq k$. It follows that for any $\delta > 0$, $s \in \mathbb{R}$,

$$M_w'(Z_k, s, -(t + h)\Phi, \delta, N)$$

$$\leq \inf_{G_w} \left\{ \frac{1}{\exp \left( -s \cdot |w'| - (t + h)S_w'\Phi(x) \right)} \right\}$$

$$\leq \inf_{G_w} \left\{ \frac{1}{\exp \left( -s \cdot |w'| - tS_w'\Phi(x) - h|w'|(\alpha - \varepsilon) \right)} \right\}$$

$$= \inf_{G_w} \left\{ \frac{1}{\exp \left( -(s + h(\alpha - \varepsilon)) \cdot |w'| - tS_w'\Phi(x) \right)} \right\}$$

$$= M_w'(Z_k, s + h(\alpha - \varepsilon), -t\Phi, \delta, N),$$

where $G_w = \{ B_{w'}(x, \delta) : \bar{w} \leq w', x \in Z_k \} \subset F$ covers $Z_k$. It follows that

$$M'(Z_k, s, -(t + h)\Phi, \delta, N) \leq M'(Z_k, s + h(\alpha - \varepsilon), -t\Phi, \delta, N)$$

and then

$$m'(Z_k, s, -(t + h)\Phi, \delta) \leq m'(Z_k, s + h(\alpha - \varepsilon), -t\Phi, \delta).$$

Then we have

$$P_{Z_k}'(G, -(t + h)\Phi, \delta) \leq P_{Z_k}'(G, -t\Phi, \delta) - h(\alpha - \varepsilon).$$

Letting $\delta \to 0$, it follows that

$$P_{Z_k}(G, -(t + h)\Phi) \leq P_{Z_k}(G, -t\Phi) - h(\alpha - \varepsilon).$$

Using the Proposition 4.1 and taking the supremum over all $k \geq 1$, we can get

$$P_{Z}(G, -(t + h)\Phi) \leq P_{Z}(G, -t\Phi) - h(\alpha - \varepsilon).$$

Since $\varepsilon > 0$ is arbitrary, this establishes the right of inequality (6.1).

Using the similar computation, we obtain

$$M_w'(Z_k, s, -(t + h)\Phi, \delta, N) \geq M_w'(Z_k, s + h(\beta + \varepsilon), -t\Phi, \delta, N).$$

It follows that

$$M'(Z_k, s, -(t + h)\Phi, \delta, N) \geq M'(Z_k, s + h(\beta + \varepsilon), -t\Phi, \delta, N),$$

and then

$$m'(Z_k, s, -(t + h)\Phi, \delta) \geq m'(Z_k, s + h(\beta + \varepsilon), -t\Phi, \delta).$$

Hence,

$$P_{Z_k}'(G, -(t + h)\Phi, \delta) \geq P_{Z_k}'(G, -t\Phi, \delta) - h(\beta + \varepsilon).$$

Letting $\delta \to 0$, it follows that

$$P_{Z_k}'(G, -(t + h)\Phi) \geq P_{Z_k}'(G, -t\Phi) - h(\beta + \varepsilon).$$
Take the supremum over all \( k \geq 1 \), and by the Proposition 4.1 we can get
\[
P_Z(G, -(t + h)\Phi) \geq P_Z(G, -t\Phi) - h(\beta + \varepsilon).
\]
Since \( \varepsilon > 0 \) is arbitrary, this establishes the left of inequality (6.1). This complete the proof of the inequality (6.1).

(2) We observe that the map \( t \mapsto P_Z(G, -t\Phi) \) is continuous and strictly decreasing. First applied the left of (6.1) with \( t = 0 \) and \( h = \frac{h_Z(G)}{\beta} \), we have
\[
P_Z(G, -\frac{h_Z(G)}{\beta}\Phi) \geq P_Z(G, 0) - h_Z(G) = 0;
\]
and second applied the right of (6.1) with \( t = 0 \) and \( h = \frac{h_Z(G)}{\alpha} \), we have
\[
P_Z(G, -\frac{h_Z(G)}{\alpha}\Phi) \leq P_Z(G, 0) - h_Z(G) = 0.
\]
Thus it get the desired result by Intermediate Value Theorem.

(3) It follows from (2) immediately. \( \square \)

Similar to [14], we also have the following proposition. The proof of this proposition is similar to that of [14]. Therefore, we omit the proof.

**Proposition 6.2** Let \( f_i : X \longrightarrow X \) be as in Theorem 2.2 and suppose that for any \( \omega \in \Sigma_m^+ \), \( \lambda_\omega(x) \) exists and is positive. Then \( x \in B \).

Before proving Theorem 2.2, similar to [14], we give the following two lemmas.

**Lemma 6.1** Let \( f_i : X \longrightarrow X \) be as in Theorem 2.2. Then given any \( x \in B \) and \( \varepsilon > 0 \), there exist \( \delta_0 = \delta_0(\varepsilon) > 0 \) and \( \eta = \eta(x, \varepsilon) > 0 \) such that for each \( w \in F_m^+ \) and \( 0 < \delta < \delta_0 \),
\[
B(x, \eta \delta e^{-[w]|(\lambda_\omega(x)+\varepsilon)|}) \subset B_w(x, \delta) \subset B(x, \delta e^{-[w]|(\lambda_\omega(x)−\varepsilon)|}). \tag{6.2}
\]

**Proof** Since \( f_i \) is conformal with factor \( a_i(x) > 0 \), for each \( i \in \{0, 1, 2, \ldots, m-1\} \) we have
\[
\lim_{y \to x} \frac{d(f_i(x), f_i(y))}{d(x, y)} = a_i(x).
\]
Since \( a_i(x) > 0 \) everywhere, we can take logarithms and get
\[
\lim_{y \to x} (\log d(f_i(x), f_i(y)) - \log d(x, y)) = \log a_i(x).
\]
It can be extended the continuous function \( \zeta_i : X \times X \longrightarrow \mathbb{R} \)
\[
\zeta_i(x, y) = \begin{cases} 
\log d(f_i(x), f_i(y)) - \log d(x, y) & x \neq y, \\
\log a_i(x) & x = y.
\end{cases}
\]
Because \( X \times X \) is compact, \( \zeta_i \) is uniformly continuous, \( i \in \{0, 1, \ldots, m-1\} \). Hence for \( \varepsilon > 0 \), there exists \( \delta_0 = \delta_0(\varepsilon) > 0 \) such that for every \( 0 < \delta < \delta_0 \) and \( (x, y), (x', y') \in X \times X \) with
\[
(d \times d)((x, y), (x', y')) := d(x, x') + d(y, y') < \delta,
\]
we have \( |\zeta_i(x, y) - \zeta_i(x', y')| < \varepsilon \). In particular, for \( x, y \in X \) with \( d(x, y) < \delta \), we have \( (d \times d)((x, y), (x, x)) < \delta \). Therefore,
\[
|\log d(f_i(x), f_i(y)) - \log d(x, y) - \log a_i(x)| = |\zeta_i(x, y) - \zeta_i(x, x)| < \varepsilon,
\]
that is,
\[
\log d(f_i(x), f_i(y)) - \log a_i(x) - \varepsilon < \log d(x, y) < \log d(f_i(x), f_i(y)) - \log a_i(x) + \varepsilon,
\]
and taking exponentials, we obtain
\[
d(f_i(x), f_i(y)) e^{-(\log a_i(x) - \varepsilon)} < d(x, y) < d(f_i(x), f_i(y)) e^{-(\log a_i(x) - \varepsilon)},
\]
whenever the middle quantity is less than \(\delta\). Given \(w = i_1 i_2 \cdots i_n \in F_m^+\). Now we show the second half of (6.2). Let \(\Phi = \{\log a_0, \log a_1, \ldots, \log a_{m-1}\}\). Suppose \(y \in B_w(x, \delta)\), then \(d(f_w'(x), f_w'(y)) < \delta\) for all \(w' \leq w\). Then repeated application of the second inequality in (6.3) yields
\[
d(x, y) < d(f_i(x), f_i(y)) e^{-(\log a_i(x)) - \varepsilon}
\]
\[
< d(f_{i_1 i_2}(x), f_{i_1 i_2}(y)) e^{-(\log a_{i_1}(f_i(x)) - \varepsilon)} e^{-(\log a_i(x) - \varepsilon)}
\]
\[
< \cdots
\]
\[
< d(f_{w'}(x), f_{w'}(y)) e^{-(\log a_{w'}(x) - \varepsilon)}
\]
\[
< \delta e^{-|w'|(\log a_{w'}(x) - \varepsilon)}.
\]

Then
\[
B_w(x, \delta) \subset B(x, \delta e^{-|w'|(\log a_{w'}(x) - \varepsilon)}).
\]

Now we prove the first inclusion in (6.2). We observe that if \(d(x, y) < \delta\), then by the first inequality in (6.3) we get
\[
d(f_i(x), f_i(y)) < d(x, y) e^{\log a_i(x) + \varepsilon}.
\]
Then if \(d(x, y) < \delta e^{-(\log a_i(x) + \varepsilon)}\), we have \(d(f_i(x), f_i(y)) < \delta\) and so
\[
d(f_{i_1}(x), f_{i_1}(y)) < d(f_i(x), f_i(y)) e^{\log a_{i_1}(f_i(x)) + \varepsilon}
\]
\[
< d(x, y) e^{\log a_{i_1}(f_i(x)) + \varepsilon} e^{\log a_i(x) + \varepsilon}.
\]
Continuing in this method, we can obtain that if
\[
d(x, y) < \delta e^{-|w'|(\log a_{w'}(x) + \varepsilon)}
\]
for each \(w' \leq w\), we have \(d(f_{w'}(x), f_{w'}(y)) < \delta\), and hence \(y \in B_w(x, \delta)\). Therefore
\[
B(x, \delta \min_{\overline{w'} \leq \overline{w}} e^{-|w'|(\log a_{w'}(x) + \varepsilon)}) \subset B_w(x, \delta),
\]
which is almost what we wanted. If the minimum was always achieved at \(w' = w\), we would be done; however, this may not be the case. Therefore, now we find what \(\eta\) should be for any \(\overline{w'} \leq \overline{w}\), and we observe that
\[
\frac{e^{-|w|(\log a_{w}(x) + 2\varepsilon)}}{e^{-|w'|(\log a_{w'}(x) + \varepsilon)}} = \frac{e^{-S_w \Phi(x) - 2|w|\varepsilon}}{e^{-S_{w'} \Phi(x) - |w'|\varepsilon}}
\]
\[
= e^{-|S_w \Phi(x) - S_{w'} \Phi(x)| - 2|w|\varepsilon - |w'|\varepsilon}
\]
\[
\leq e^{-|S_w \Phi(x) - S_{w'} \Phi(x)| + |w'|\varepsilon}.
\]
Since \(x\) satisfies the tempered contraction condition, there exists \(\eta = \eta(x, \varepsilon) > 0\) such that
\[
\log \eta < S_w \Phi(x) - S_{w'} \Phi(x) + |w|\varepsilon
\]
\[\text{ Springer}\]
for arbitrary \( w \in F_m^+ \), \( \overline{w'} \leq \overline{w} \), and hence

\[
e^{-\left( S_w \Phi(x) - S_{w'} \Phi(x) + |w| \varepsilon \right)} \leq \frac{1}{\eta}.
\]

Then for such \( w, w' \), we have

\[
\eta e^{-|w| (\lambda_w(x) + 2\varepsilon)} < e^{-|w'| (\lambda_{w'}(x) + \varepsilon)}.
\]

and combining with (6.4)

\[
B(x, \delta \eta e^{-|w| (\lambda_w(x) + 2\varepsilon)}) \subset B_w(x, \delta).
\]

Taking \( \delta_0 = \delta_0(\varepsilon/2) \) gives that desired result. \( \Box \)

**Remark 6.1**

(1) If \( x \) has bounded contraction, then \( \eta = \eta(x) \) may be chosen independently of \( \varepsilon \).

(2) Furthermore, if \( a_i(x) \geq 1, \ i \in \{0, 1, \ldots, m - 1\} \) for any \( x \in X \), then \( \eta = 1 \) suffices.

**Lemma 6.2** Let \( f_i : X \to X \) satisfy the conditions of Theorem 2.2 and fix \( Z \subset A((\alpha, \infty)) \cap B \), where \( 0 < \alpha < \infty \). Let \( t^* \) be the unique real number with \( P_Z(G, -t^* \Phi) = 0 \), whose existence and uniqueness are guaranteed by Proposition 6.1, where \( \Phi = \{\log a_0, \log a_1, \ldots, \log a_{m - 1}\} \). Then \( \dim_H Z = t^* \).

**Proof** First we prove \( \dim_H Z \leq t^* \). Given \( k \geq 1 \), and let

\[
Z_k = \{x \in Z : \lambda_w(x) > \alpha \ \text{for all} \ |w| \geq k \}
\]

and then \( Z = \bigcup_{k=1}^{\infty} Z_k \). Consider \( t > t^* \), and naturally \( P_Z(G, -t \Phi) < 0 \). Thus there exists \( \varepsilon > 0 \) with \( P_Z(G, -t \Phi) < -t \varepsilon \) and by Lemma 6.1 there exists \( \delta_0 = \delta_0(\varepsilon) \) such that for all \( x \in Z_k \), \( 0 < \delta < \delta_0 \), fixing \( w \in F_m^+ \) and \( |w| \geq k \), we have

\[
diam B_w(x, \delta) \leq 2\delta e^{-|w| (\lambda_w(x) - \varepsilon)} \leq 2\delta e^{-|w| (\alpha - \varepsilon)}. \tag{6.5}
\]

Given \( N \geq k \) and \( 0 < \delta < \delta_0 \), we have

\[
M'_w(Z_k, -t \varepsilon, -t \Phi, \delta, N) = \inf_{G_w} \left\{ \sum_{B_{w'}(x, \delta) \in G_w} \exp \left( - (t \varepsilon) \cdot |w'| - t S_{w'} \Phi(x) \right) \right\}
\]

\[
= \inf_{G_w} \left\{ \sum_{B_{w'}(x, \delta) \in G_w} \exp \left( - t \cdot |w'| (\lambda_{w'}(x) - \varepsilon) \right) \right\}
\]

\[
\geq \inf_{G_w} \left\{ \sum_{B_{w'}(x, \delta) \in G_w} \left( \frac{1}{2\delta} \right)^{diam B_{w'}(x, \delta)} \right\}
\]

\[
\geq \inf_{D(Z_k, 2\delta e^{-N(\alpha - \varepsilon)})} \left\{ \sum_{U_i \in D(Z_k, 2\delta e^{-N(\alpha - \varepsilon)})} \left( \frac{1}{2\delta} \right)^{diam U_i} \right\}
\]

\[
= (2\delta)^{-t} m_H(Z_k, t, 2\delta e^{-N(\alpha - \varepsilon)}),
\]
where \( G_w = \{ B_u(x, \delta) : \overline{w} \leq w, x \in Z_k \} \subset \mathcal{F} \) covers \( Z_k \) and \( \mathcal{D}(Z_k, 2\delta e^{-N(\alpha-\epsilon)}) \) denotes the collection of open covers \( \{U_i\} \) of \( Z_k \) for which \( \text{diam} U_i < 2\delta e^{-N(\alpha-\epsilon)} \) for all \( i \).

It follows that
\[
M'(Z_k, -t\varepsilon, -t\Phi, \delta, N) = \frac{1}{mN} \sum_{|w|=N} M'_w(Z_k, -t\varepsilon, -t\Phi, \delta, N)
\]
\[
\geq (2\delta)^{-t}m_H(Z_k, t, 2\delta e^{-N(\alpha-\epsilon)}).
\]

Taking the limit as \( N \to \infty \) gives
\[
m'(Z_k, -t\varepsilon, -t\Phi, \delta) \geq (2\delta)^{-t}m_H(Z_k, t).
\]
(6.6)

Thus
\[
-t\varepsilon > P_Z(G, -t\Phi) \geq P_{Z_k}(G, -t\Phi) = \lim_{\delta \to 0} P^\prime_{Z_k}(G, -t\Phi, \delta),
\]
and for sufficiently small \( \delta > 0 \), we have \( -t\varepsilon > P^\prime_{Z_k}(G, -t\Phi, \delta) \). Hence \( H^t(Z_k) = 0 \) by (6.6), which implies \( \dim H(Z_k) \leq t \). Then taking the union over all \( k \) gives \( \dim H(Z) \leq t \) for all \( t > t^* \). Therefore \( \dim H(Z) \leq t^* \).

Now we prove the other inequality, \( \dim H(Z) \geq t^* \). Fix \( t < t^* \), and then we show that \( \dim H(Z) \geq t \). Suppose \( t > 0 \), and by the Proposition 6.1 \( t^* \) is the unique real number such that \( P_Z(G, -t^*\Phi) = 0 \). Since the pressure function \( P_Z(G, -t\Phi) \) is decreasing, we have \( P_Z(G, -t\Phi) > 0 \). Hence we can choose \( \varepsilon > 0 \) such that
\[
P_Z(G, -t\Phi) > t\varepsilon > 0.
\]

Set \( \delta_0 = \delta_0(\varepsilon) \) be as in Lemma 6.1. Given \( k \in \mathbb{N} \), \( k \geq 1 \), consider the set
\[
Z_k = \{ x \in Z : (6.2) \text{ holds with } \eta = e^{-k} \text{ for all } w \in F_m^n \text{ and } 0 < \delta < \delta_0 \}.
\]

Obviously, \( Z = \bigcup_{k=1}^{\infty} Z_k \) and hence \( P_Z(G, -t\Phi) = \sup_{k \geq 1} P_{Z_k}(G, -t\Phi) \). Then there exists \( k \in \mathbb{N} \) with \( t\varepsilon < P_{Z_k}(G, -t\Phi) \), and we fix \( 0 < \delta < \delta_0 \) such that
\[
t\varepsilon < P^\prime_{Z_k}(G, -t\Phi, \delta).
\]
(6.7)

Let \( \beta = \max_i \{ \sup_{x \in \mathcal{X}} \log a_i(x) \} < \infty \). For any \( w = i_1i_2 \cdots i_n \in F_m^n, x \in \mathcal{X} \), denote
\[
s_w(x) = e^{-k\delta e^{-|w|((\lambda_w(x)+\varepsilon))}},
\]
and notice that for any \( n_{i+1} \in \{0, 1, \ldots, m-1\} \)
\[
\frac{s_w(x)}{s_{w_{i+1}}(x)} = \frac{e^{-S_w(x)-|w|\varepsilon}}{e^{-S_{w_{i+1}}(x)-(|w|+1)\varepsilon}} = e^{\log a_{n+1}(f_{i+1}(x)+\varepsilon)} \leq e^{\beta\varepsilon}.
\]

Moreover, given \( x \in Z_k \) and \( r > 0 \) enough small and for any \( \omega = i_1i_2 \cdots \in \Sigma_m^+ \), there exists \( n = n(x, r, \omega) \in \mathbb{N} \) such that for \( w = i_1i_2 \cdots i_n = \omega|_{0, n-1} \in F_m^n \), we have
\[
s_w(x)e^{-(\beta+\varepsilon)} \leq s_{w_{i+1}}(x) \leq r \leq s_w(x) = e^{-k\delta e^{-|w|((\lambda_w(x)+\varepsilon))}}.
\]
(6.8)

Moreover, for every \( w^\prime \in F_m^n \) and \( x \in \mathcal{X} \), we have \( \lambda_w^\prime(x) \leq \beta \) and thus \( s_{w^\prime}(x) \geq \delta e^{-\frac{k+|w^\prime|((\beta+\varepsilon))}{2}} \). It yields from (6.8) that for \( w = i_1i_2 \cdots i_n = \omega|_{0, n-1} \), where \( n = n(x, r, \omega) \), we have
\[
\delta e^{-\left(k+(n+1)(\beta+\varepsilon)\right)} \leq r.
\]
and thus
\[ n \geq -\log r + \log \delta - k \frac{\beta + \varepsilon}{\beta + \varepsilon} - 1. \]

Writing \( N := N(r, \delta) = -\log r + \log \delta - k \frac{\beta + \varepsilon}{\beta + \varepsilon} - 1 \) and observe that for every fixed \( \delta > 0 \), we have \( \lim_{r \to 0} N(r, \delta) = \infty \).

By Lemma 6.1 we have for \( w = i_1 i_2 \cdots i_n = \omega||0,n-1|, \ n = n(x, r, \omega) \in \mathbb{N}, \)
\[ B(x, r) \subset B_w(x, \delta). \]

Then given any \( \{B(x_i, r_i)\} \) with \( Z_k \subset \bigcup B(x_i, r_i) \), we can get \( Z_k \subset \bigcup B_{w_i}(x_i, \delta) \), where \( w_i = \omega|_{[0,n_i-1]}, \ n_i = n_i(x_i, r_i, \omega) \) satisfies (6.8).

Thus for all \( r > 0, \ 0 < \delta < \delta_0 \) and fixed \( w = \omega|_{[0,N-1]} \in \mathcal{F}^+_\mathbb{N}(N) \), by applying (6.8) we can get
\[
m^b_H(Z_k, t, r) = \inf_{\mathcal{D}^b(Z_k, r)} \left\{ \sum_{B(x_i, r_i) \in \mathcal{D}^b(Z_k, r)} (2r_i)^t \right\}
\geq \inf_{\mathcal{G}''_w} \left\{ \sum_{B_{w_i}(x_i, \delta) \in \mathcal{G}''_w} \left( 2e^{-\beta r_i} s_{w_i}(x_i) \right)^t \right\}
= (2\delta)^t e^{-t(k+\beta+\varepsilon)} \inf_{\mathcal{G}''_w} \left\{ \sum_{B_{w_i}(x_i, \delta) \in \mathcal{G}''_w} e^{-|w_i|t S_{w_i}(x_i) + r} \right\}
= (2\delta)^t e^{-t(k+\beta+\varepsilon)} \inf_{\mathcal{G}''_w} \left\{ \sum_{B_{w_i}(x_i, \delta) \in \mathcal{G}''_w} \exp \left( -|w_i|t \varepsilon - t S_{w_i}(x_i) \right) \right\}
\geq (2\delta)^t e^{-t(k+\beta+\varepsilon)} M'(Z_k, t, \varepsilon, -t \Phi, \delta, N),
\]

where \( \mathcal{D}^b(Z_k, r) \) denotes the collection of countable open balls covers \( \{B(x_i, r_i) : x_i \in Z_k\}_{i=1}^{\infty} \) of \( Z_k \) with \( r_i < r \) for all \( i \) and \( \mathcal{G}''_w \) denotes the collection of countable Bowen balls covers \( \{B_{w_i}(x_i, \delta) : x_i \in Z_k\}_{i=1}^{\infty} \) of \( Z_k \) for which \( \overline{w_i} \leq \overline{w_i}, \ w = \omega|_{[0,N-1]}, \ w_i = \omega|_{[0,n_i-1]} \) for all \( i \).

It follows that
\[ m^b_H(Z_k, t, r) \geq (2\delta)^t e^{-t(k+\beta+\varepsilon)} M'(Z_k, t, \varepsilon, -t \Phi, \delta, N). \]

Letting \( r \to 0 \), we can obtain the quantity on the right goes to \( \infty \) by (6.7), and thus we have \( m^b_H(Z_k, t) = \infty \). Therefore,
\[ \dim_H Z \geq \dim_H Z_k \geq t, \]
and since \( t < t^* \) was arbitrary, this establishes the Lemma.

\[ \square \]

**Proof of Theorem 2.2** Consider a decreasing sequence of positive numbers \( \alpha_k \) which converge to 0, and set \( Z_k \subset \mathcal{A}(\alpha_k, \infty) \cap Z \), so that Lemma 6.2 applies to \( Z_k \) and we have \( Z = \bigcup_{k=1}^{\infty} Z_k \). Let \( t_k \) be the unique real number with
\[ P_{Z_k}(G, -t_k \Phi) = 0 \]

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Given that $t_0$ is a unique root of Bowen's equation, we have
\[
\dim_H Z_k = t_k.
\]
Denote $t^* = \sup_k t_k$, then $\dim_H Z = t^*$. Then it remains to prove that
\[
t^* = \sup\{t \geq 0 : P_Z(G, -t\Phi) > 0\}. \tag{6.9}
\]
Given $t \geq 0$, we have
\[
P_Z(G, -t\Phi) = \sup_k P_{Z_k}(G, -t\Phi).
\]

First for any $t < t^*$, there exists $t_k$ with $t < t_k$ and hence $P_{Z_k}(G, -t\Phi) > 0$. Then $t \in \{t \geq 0 : P_Z(G, -t\Phi) > 0\}$ and thus $t \leq \sup\{t \geq 0 : P_Z(G, -t\Phi) > 0\}$. Therefore, $t^* \leq \sup\{t \geq 0 : P_Z(G, -t\Phi) > 0\}$.

Next for arbitrary $t < \sup\{t \geq 0 : P_Z(G, -t\Phi) > 0\}$, there exists $t_j > t$ with $P_{Z_k}(G, -t_j\Phi) > 0$. Then there exists $Z_k$ such that $P_{Z_k}(G, -t_j\Phi) > 0$ and thus $t_j < t_k$. It follows that $t < t_j < t_k < t^*$. So $\sup\{t \geq 0 : P_Z(G, -t\Phi) > 0\} \leq t^*$. This establishes (6.9).

Finally, it follows from (6.9) and continuity of $t \mapsto P_Z(G, -t\Phi)$ that $P_Z(G, -t^*\Phi) = 0$. If $Z \subset \mathcal{A}(\alpha, \infty)$ for some $\alpha > 0$, then Proposition 6.1 guarantees that $t^*$ is in fact the unique root of Bowen's equation.

**Example 6.3** Let $X = [0, 1]$ be a closed interval on $\mathbb{R}$. Assume $G$ is a free semigroup generated by a series of the Manneville-Pomeau maps (for the study of Manneville-Pomeau map, see, for example [27,33]), that is,
\[
f_i : X \to X, f_i(x) = x + x^{1+s_i} \mod 1, i = 0, 1, \ldots, m - 1
\]
and $0 < s_0 < s_1 < \cdots < s_{m-1} < 1$. Then $f_i$ $(i = 0, 1, \ldots, m - 1)$ is continuous conformal and has no critical points and singularities, and $a_i(x) \geq 1$ for any $x \in X$. The Manneville-Pomeau map is non-uniformly hyperbolic transformation having the most benign type of non-hyperbolicity: an indifferent fixed point at 0, i.e., $f_i(0) = 0$, and $a_i(0) = 1$, and exhibits intermittent behavior. Moreover, we can verify $X, G = \{f_0, f_1, \ldots, f_{m-1}\}$ and $\Phi = \{\log a_0, \log a_1, \ldots, \log a_{m-1}\}$ satisfy the conditions of Theorem 2.2. What’s more, for any $\omega \in \Sigma^*_m$
\[
\overline{\lambda}_\omega(0) = \underline{\lambda}_\omega(0) = \lambda_\omega(0) = 0
\]
and
\[
\underline{\lambda}_\omega(x) > 0, \text{ for any } x \in X - \{0\}.
\]
Thus $B = X$ and $\mathcal{A}((0, \infty)) = X - \{0\}$. Therefore, for any $Z \subset \mathcal{A}((0, \infty))$, by the Theorem 2.2 we get
\[
\dim_H Z = t^* = \sup\{t \geq 0 : P_Z(G, -t\Phi) > 0\}
\]
\[= \inf\{t \geq 0 : P_Z(G, -t\Phi) \leq 0\}.\]

If $Z \subset \mathcal{A}(\alpha, \infty) \cap B$ for some $\alpha > 0$, then the Hausdorff dimension $\dim_H Z$ of $Z$ is the unique root of Bowen’s equation
\[
P_Z(G, -t\Phi) = 0.
Moreover, if $Z \subset A(\alpha)$ for some $\alpha > 0$, then
\[
\dim_H Z = \frac{1}{\alpha} h(Z(G)),
\]
where $h(Z(G))$ is the topological entropy on $Z$ defined by Ju et al in [18].

7 The Proof of Theorem 2.3

**Proof of Theorem 2.3** (1) Fix an $\varepsilon > 0$ and for each $k \geq 1$. Consider the set
\[
Z_k = \left\{ x \in Z : \liminf_{n \to \infty} - \frac{1}{n} \max_{|w|=n} \{ \log \mu(B_u(x, r)) - S_u(\Phi(x)) \} > s - \varepsilon \text{ for all } r \in (0, 1/k) \right\}.
\]
Since $P(G, \Phi, x) \geq s$ for all $x \in Z$, the sequence $\{Z_k\}_{k=1}^{\infty}$ increases to $Z$. So by the continuity of the measure, we have
\[
\lim_{k \to \infty} \mu(Z_k) = \mu(Z) > 0.
\]
Then select an integer $k_0 \geq 1$ with $\mu(Z_{k_0}) > \frac{1}{2} \mu(Z)$. For each $N \geq 1$, put
\[
Z_{k_0, N} = \left\{ x \in Z_{k_0} : - \frac{1}{n} \max_{|w|=n} \{ \log \mu(B_u(x, r)) - S_u(\Phi(x)) \} > s - \varepsilon \text{ for all } n \geq N \text{ and } r \in (0, 1/k_0) \right\}.
\]
Since the sequence $\{Z_{k_0, N}\}_{N=1}^{\infty}$ increases to $Z_{k_0}$, we can pick a $N^* \geq 1$ such that $\mu(Z_{k_0, N^*}) > \frac{1}{2} \mu(Z_{k_0})$. Write $Z^* = Z_{k_0, N^*}$ and $r^* = \frac{1}{k_0}$. Then $\mu(Z^*) > 0$, for all $x \in Z^*$, $0 < r \leq r^*$ and $n \geq N^*$,
\[
\max_{|w|=n} \{ \log \mu(B_u(x, r)) - S_u(\Phi(x)) \} < -(s - \varepsilon) \cdot n.
\]
(7.1)
For any $N \geq N^*$, $w \in F_m^+$ and $|w| = N$, take a cover of $Z^*$
\[
\mathcal{F}_w = \left\{ B_{\omega_i|[0,n_i-1]}(y_i, r/2) : \omega_i \in \Sigma_m^+ \text{ and } \omega_i|[0,N-1] = w \right\}
\]
such that
\[
Z^* \cap B_{\omega_i|[0,n_i-1]}(y_i, r/2) \neq \emptyset, \quad n_i \geq N \text{ for all } i \geq 1 \text{ and } 0 < r \leq r^*.
\]
For each $i$, there exists an $x_i \in Z^* \cap B_{\omega_i|[0,n_i-1]}(y_i, r/2)$. By the triangle inequality
\[
B_{\omega_i|[0,n_i-1]}(y_i, r/2) \subset B_{\omega_i|[0,n_i-1]}(x_i, r).
\]
In combination with (7.1), we can get
\[
\sum_{i \geq 1} \exp \left( -(s - \varepsilon) \cdot n_i + S_u(\Phi(x_i)) \right) \geq \sum_{i \geq 1} \mu(B_{\omega_i|[0,n_i-1]}(x_i, r)) \geq \mu(Z^*) > 0.
\]
Therefore, $M_n'(Z^*, s - \varepsilon, \Phi, r, N) \geq \mu(Z^*) > 0$ for all $N \geq N^*$, and we obtain
\[
M'(Z^*, s - \varepsilon, \Phi, r, N) = \frac{1}{m^N} \sum_{|w|=N} M_n'(Z^*, s - \varepsilon, \Phi, r, N) \geq \mu(Z^*) > 0,
\]
and consequently

\[ m'(Z^*, s - \varepsilon, \Phi, r) = \lim_{N \to \infty} M'(Z^*, s - \varepsilon, \Phi, r, N) > 0, \]

which in turn implies that \( P_{Z^*}(G, \Phi, r) \geq s - \varepsilon. \) Then we have \( P_{Z^*}(G, \Phi) \geq s - \varepsilon \) by letting \( r \to 0. \) It follows that \( P_Z(G, \Phi) \geq P_{Z^*}(G, \Phi) \geq s - \varepsilon \) and hence \( P_Z(G, \Phi) \geq s \) since \( \varepsilon > 0 \) is arbitrary.

(2) In order to prove the second result, we need to use the following Lemma.

**Lemma 7.1** ([18,21]). Let \( r > 0 \) and \( B(r) = \{B_w(x, r) : x \in X, w \in F_m^+ \}. \) For any family \( \mathcal{F} \subset B(r), \) there exists a (not necessarily countable) subfamily \( \mathcal{G} \subset \mathcal{F} \) consisting of disjoint balls such that

\[ \bigcup_{B \in \mathcal{G}} B \subset \bigcup_{B_w(x,r) \in \mathcal{G}} B_w(x, 3r). \]

Since \( \overline{P}(G, \Phi, x) \leq s \) for all \( x \in Z, \) then for any \( \omega \in \Sigma_m^+ \) and \( x \in Z, \)

\[ \lim_{r \to 0} \liminf_{n \to \infty} -\frac{1}{n} \log \mu(B_{\omega|\{0, n-1\}}(x, r)) - S_{\omega|\{0, n-1\}}(x) \leq \overline{P}(G, \Phi, x) \leq s. \]

For any \( N \geq 1 \) and \( w = i_1 i_2 \ldots i_N \in F_m^+. \) Fix \( \varepsilon > 0, \) and put

\[ Z_k = \left\{ x \in Z : \liminf_{n \to \infty} -\frac{1}{n} \log \mu(B_{\omega|\{0, n-1\}}(x, r)) - S_{\omega|\{0, n-1\}}(x) < s + \varepsilon \right\}, \]

for all \( r \in (0, 1/k), \omega \in \Sigma_m^+ \) and \( \omega|\{0, n-1\} = w \),

then we have \( Z = \bigcup_{k \geq 1} Z_k. \) Now fix \( k \geq 1 \) and \( 0 < r < \frac{1}{3k}. \) For each \( x \in Z_k, \) we take \( \omega_x \in \Sigma_m^+ \) such that \( \omega_x|\{0, N-1\} = w, \) there exists a strictly increasing sequence \( \{n_j(x)\}_{j=1}^\infty \) such that

\[ \log \mu(B_{\omega_x|\{0, n_j(x)-1\}}(x, r)) - S_{\omega_x|\{0, n_j(x)-1\}}(x) \geq -(s + \varepsilon) \cdot n_j(x) \]

for all \( j \geq 1. \) So the set \( Z_k \) is contained in the union of the sets in the family

\[ \mathcal{F}_w = \left\{ B_{\omega_x|\{0, n_j(x)-1\}}(x, r) : x \in Z_k, \omega_x \in \Sigma_m^+, \omega_x|\{0, N-1\} = w \text{ and } n_j(x) \geq N \right\}. \]

By Lemma 7.1, there exists a sub family \( \mathcal{G}_w = \{B_{\omega_x|\{0, n_j-1\}}(x_i, r)\}_{i \in I} \subset \mathcal{F}_w \) consisting of disjoint balls such that for all \( i \in I, \)

\[ Z_k \subset \bigcup_{i \in I} B_{\omega_x|\{0, n_j-1\}}(x_i, 3r), \]

and

\[ \mu(B_{\omega_x|\{0, n_j-1\}}(x_i, r)) \geq \exp \left( - (s + \varepsilon) \cdot n_i + S_{\omega_x|\{0, n_j-1\}}(x_i) \right). \]

The index set \( I \) is at most countable since \( \mu \) is a probability measure and \( \mathcal{G}_w \) is a disjointed family of sets, each of which has positive \( \mu \)-measure. Therefore,

\[ M'_w(Z_k, s + \varepsilon, \Phi, 3r, N) \leq \sum_{i \in I} \exp \left( - (s + \varepsilon) \cdot n_i + S_{\omega_x|\{0, n_j-1\}}(x_i) \right) \]

\[ \leq \sum_{i \in I} \mu(B_{\omega_x|\{0, n_j-1\}}(x_i, r)) \leq 1. \]

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where the disjointness of \( \left\{ B_{\omega_j(0, n_j - 1)}(x_i, r) \right\}_{i \in I} \) is used in the last inequality. It follows that

\[
M'(Z_k, s + \varepsilon, \Phi, 3r, N) = \frac{1}{m^N} \sum_{|w| = N} M'_w(Z_k, s + \varepsilon, \Phi, 3r, N) \leq 1
\]

and consequently

\[
m'(Z_k, s + \varepsilon, \Phi, 3r) = \lim_{N \to \infty} M'(Z_k, s + \varepsilon, \Phi, 3r, N) \leq 1,
\]

which in turn implies that \( P'_{Z_k}(G, \Phi, 3r) \leq s + \varepsilon \) for any \( 0 < r < \frac{1}{3k} \). Letting \( r \to 0 \) yields

\[
P_{Z_k}(G, \Phi) \leq s + \varepsilon
\]

for any \( k \geq 1 \).

By Proposition 4.1(3),

\[
P_Z(G, \Phi) = P_{\cup_{k=1}^\infty Z_k}(G, \Phi) = \sup_{k \geq 1} \{ P_{Z_k}(G, \Phi) \} \leq s + \varepsilon.
\]

Since \( \varepsilon > 0 \) is arbitrary, then we can get \( P_Z(G, \Phi) \leq s \).

\[\square\]

**Corollary 7.2** Let \( \mu \) denote a Borel probability measure on \( X \), \( Z \) be a Borel subset of \( X \) and \( w \in F_m^+ \), \( |w| = n \). Consider the following quantities:

\[
\overline{P} := \sup_{x \in Z} \lim_{\delta \to 0} \liminf_{n \to \infty} -\frac{1}{n} \inf_{|w| = n} \{ \log \mu(B_w(x, \delta)) - S_w \Phi(x) \},
\]

\[
\underline{P} := \inf_{x \in Z} \lim_{\delta \to 0} \limsup_{n \to \infty} -\frac{1}{n} \sup_{|w| = n} \{ \log \mu(B_w(x, \delta)) - S_w \Phi(x) \}.
\]

Then \( P_Z(G, \Phi) \leq \overline{P} \). If in addition \( \mu(Z) > 0 \), we have \( P_Z(G, \Phi) \geq \underline{P} \).

**Proof**

(1) It is easy to verify that \( \overline{P}(G, \Phi, x) \leq \overline{P} \) for any \( x \in Z \). Hence, by Theorem 2.2(2), we can get \( P_Z(G, \Phi) \leq \overline{P} \).

(2) It is obvious to get \( P(G, \Phi, x) \geq \underline{P} \) for any \( x \in Z \). In addition \( \mu(Z) > 0 \), then we have \( P_Z(G, \Phi) \geq \underline{P} \) by Theorem 2.2(1).

\[\square\]

**Remark 7.1** When \( m = 1 \), i.e., \( G = \{ f \} \), \( \Phi = \{ \varphi \} \), the Corollary 7.2 coincides with the results that Climenhaga proved in [13].

### 8 The Proof of Theorem 2.4

Let \( X \) be a compact metric space with metric \( d \), suppose a free semigroup \( G = \{ f_0, f_1, \ldots, f_{m-1} \} \) with \( m \) generators act on \( X \), the generators are continuous transformations from \( X \) to itself and \( \Phi = \{ \varphi \} \), i.e., \( \varphi_0 = \varphi_1 = \cdots = \varphi_{m-1} = \varphi \in C(X, \mathbb{R}) \). For \( w = i_1 i_2 \cdots i_n \in F_m^+ \), denote \( (S_w \Phi)(x) := (S_w \varphi)(x) \).

For any subset \( Z \subset X \), \( w \in F_m^+ \) and \( \varepsilon > 0 \), a subset \( E \subset X \) is said to be a \( (w, \varepsilon, Z, f_0, \ldots, f_{m-1}) \)-spanning set of \( Z \), if for any \( x \in Z \), there exists \( y \in E \) such that \( d_w(x, y) < \varepsilon \). A subset \( F \subset Z \) is said to be a \( (w, \varepsilon, Z, f_0, \ldots, f_{m-1}) \)-separated set of \( Z \), if \( x, y \in F \), \( x \neq y \) implies \( d_w(x, y) \geq \varepsilon \).
For any $w \in F_m^+, |w| = n$ and $\varepsilon > 0$, put

$$Q_w(Z, G, \varphi, \varepsilon, n) = \inf \left\{ \sum_{x \in E} e^{(S_w \varphi)(x)} : E \text{ is a } (w, \varepsilon, Z, f_0, \ldots, f_{m-1})\text{-spanning set of } Z \right\}$$

and

$$Q(Z, G, \varphi, \varepsilon, n) = \frac{1}{m^n} \sum_{|w|=n} Q_w(Z, G, \varphi, \varepsilon, n).$$

Set

$$Q(Z, G, \varphi, \varepsilon) = \limsup_{n \to \infty} \frac{1}{n} \log Q(Z, G, \varphi, \varepsilon, n).$$

Similarly, for any $w \in F_m^+, |w| = n$ and $\varepsilon > 0$, set

$$P_w(Z, G, \varphi, \varepsilon, n) = \sup \left\{ \sum_{x \in F} e^{(S_w \varphi)(x)} : F \text{ is a } (w, \varepsilon, Z, f_0, \ldots, f_{m-1})\text{-separated set of } Z \right\}$$

and

$$P(Z, G, \varphi, \varepsilon, n) = \frac{1}{m^n} \sum_{|w|=n} P_w(Z, G, \varphi, \varepsilon, n).$$

Let

$$P(Z, G, \varphi, \varepsilon) = \limsup_{n \to \infty} \frac{1}{n} \log P(Z, G, \varphi, \varepsilon, n).$$

For any $w \in F_m^+, |w| = n$, since

$$\Lambda_w'(Z, \varphi, \varepsilon, n) = \inf_{\varphi_w} \left\{ \sum_{B_w(x, \varepsilon) \in \mathcal{G}_w} \exp \left( (S_w \varphi)(x) \right) \right\} = Q_w(Z, G, \varphi, \varepsilon, n)$$

then $\Lambda'(Z, \varphi, \varepsilon, n) = Q(Z, G, \varphi, \varepsilon, n)$. Therefore, by Remark 5.2(2) we can obtain

$$\overline{CP}_Z(G, \varphi) = \lim_{\varepsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log Q(Z, G, \varphi, \varepsilon, n).$$

If $\delta = \sup\{|\varphi(x) - \varphi(y)| : d(x, y) \leq \frac{\varepsilon}{2}\}$, similar to the Lin et al. ([18, Remark 3.4(5)]), we have

$$Q(Z, G, \varphi, \varepsilon, n) \leq P(Z, G, \varphi, \varepsilon, n) \leq e^{n\delta} Q(Z, G, \varphi, \frac{\varepsilon}{2}, n).$$

Moreover, we have

$$\overline{CP}_Z(G, \varphi) = \lim_{\varepsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log P(Z, G, \varphi, \varepsilon, n).$$

We assign the following skew-product transformation. Its base is $\Sigma_m$, its fiber is $X$, and the maps $F : \Sigma_m \times X \to \Sigma_m \times X$ and $g : \Sigma_m \times X \to \mathbb{R}$ are defined by the formula

$$F(\omega, x) = (\sigma_m \omega, f_{\omega_0}(x))$$
and
\[ g(\omega, x) = c + \varphi(x), \]
where \( \omega = (\ldots, \omega_{-1}, \omega_0, \omega_1, \ldots) \in \Sigma_m \) and \( c \) is a constant number. Here \( f_{\omega_0} \) stands for \( f_0 \) if \( \omega_0 = 0 \), and for \( f_1 \) if \( \omega_0 = 1 \), and so on and \( \varphi \in C(X, \mathbb{R}) \). Obviously, \( g \in C(\Sigma_m \times X, \mathbb{R}) \).

Then
\[ F^n(\omega, x) = (\sigma^n \omega, f_{\omega_{n-1}} f_{\omega_{n-2}} \cdots f_{\omega_0}(x)) \]
Furthermore, \( S_n g(\omega, x) = nc + \sum_{[\omega[0,n-1]]} \varphi(x) \). Let \( \omega = (\ldots, \omega_{-1}, \omega_0, \omega_1, \ldots) \in \Sigma_m \), and the metric \( D \) on \( \Sigma_m \times X \) is defined as
\[ D((\omega, x), (\omega', x')) = \max \left( d'(\omega, \omega'), d(x, x') \right) \]

For \( F : \Sigma_m \times X \to \Sigma_m \times X \), we can get a C–P structure on \( \Sigma_m \times X \). For any set \( Z \subset X \), from Pesin [24], we denote \( CP_{\Sigma_m \times X}(F, g) \) the upper capacity topological pressure of \( F \) on the set \( \Sigma_m \times Z \). Our purpose is to find the relationship between the upper capacity topological pressure \( CP_{\Sigma_m \times X}(F, g) \) of the skew-product transformation \( F \) and the upper capacity topological pressure \( CP_Z(G, \varphi) \) of a free semigroup action \( G = \{f_0, \ldots, f_{m-1}\} \) on \( Z \).

To prove this theorem, we give the following two lemmas. The proof of these two lemmas is similar to that of Lin et al [19]. Therefore, we omit the proof.

**Lemma 8.1** For any subset \( Z \) of \( X \), \( n \geq 1 \) and \( 0 < \varepsilon < \frac{1}{2} \), we have
\[ P(\Sigma_m \times Z, F, g, \varepsilon, n) \geq e^{nc} m^n P(Z, G, \varphi, \varepsilon, n). \]

**Lemma 8.2** For any subset \( Z \) of \( X \), \( n \geq 1 \) and \( \varepsilon > 0 \), we have
\[ Q(\Sigma_m \times Z, F, g, \varepsilon, n) \leq K(\varepsilon)e^{nc} m^n Q(Z, G, \varphi, \varepsilon, n), \]
where \( K(\varepsilon) \) is a positive constant that depends only on \( \varepsilon \).

**Proof of Theorem 2.4** From Lemma 8.1 we have for any subset \( Z \) of \( X \),
\[ P(\Sigma_m \times Z, F, g, \varepsilon, n) \geq e^{nc} m^n P(Z, G, \varphi, \varepsilon, n), \]
whence, taking logarithms and limits, we obtain that
\[ CP_{\Sigma_m \times Z}(F, g) \geq \log m + CP_Z(G, \varphi) + c. \]
In the same way, from Lemma 8.2, we have
\[ Q(\Sigma_m \times Z, F, g, \varepsilon, n) \leq K(\varepsilon)e^{nc} m^n Q(Z, G, \varphi, \varepsilon, n), \]
whence
\[ CP_{\Sigma_m \times Z}(F, g) \leq \log m + CP_Z(G, \varphi) + c. \]
Thus the proof is complete.

**Problem 8.3** For any subset \( Z \subset X \), is it true that \( P_{\Sigma_m \times Z}(F, g) = \log m + P_Z(G, \varphi) + c \)?

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References

1. Ban, J., Cao, Y., Hu, H.: The dimensions of a non-conformal repeller and an average conformal repeller. Trans. Am. Math. Soc. 362(2), 727–751 (2010)
2. Barreira, L., Pesin, Y., Schmeling, J.: On a general concept of multifractality: multifractal spectra for dimensions, entropies, and Lyapunov exponents. Multifractal Rigidity, Chaos 7(1), 27–38 (1997)
3. Barreira, L., Schmeling, J.: Sets of “non-binary” points have full topological entropy and full Hausdorff dimension. Israel J. Math. 116, 29–70 (2000)
4. Biš, A.: Entropies of a semigroup of maps. Discrete Contin. Dyn. Syst. Ser. A 11, 639–648 (2004)
5. Biš, A.: An analogue of the variational principle for group and pseudogroup actions. Ann. Inst. Fourier (Grenoble) 63(3), 839–863 (2013)
6. Bowen, R.: Topological entropy for noncompact sets. Trans. Am. Math. Soc. 184, 125–136 (1973)
7. Bowen, R.: Hausdorff dimension of quasicircles. Inst. Hautes tudes Sci. Publ. Math. No. 50, 11–25 (1979)
8. Bufetov, A.: Topological entropy of free semigroup actions and skew-product transformations. J. Dyn. Control Syst 5(1), 137–143 (1999)
9. Cao, Y., Pesin, Y., Zhao, Y.: Dimension estimates for non-conformal repellers and continuity of subadditive topological pressure. Geom. Funct. Anal. 29(5), 1325–1368 (2019)
10. Carvalho, M., Rodrigues, F., Varandas, P.: Semigroup actions of expanding maps. J. Stat. Phys. 166(1), 114–136 (2017)
11. Carvalho, M., Rodrigues, F., Varandas, P.: A variational principle for free semigroup actions. Adv. Math. 334, 450–487 (2018)
12. Carvalho, M., Rodrigues, F., Varandas, P.: Quantitative recurrence for free semigroup actions. Nonlinearity 31(3), 864–886 (2018)
13. Climenhaga, V.: Thermodynamic formalism and multifractal analysis for general topological dynamical systems, Thesis (Ph.D.), The Pennsylvania State University, p 131 (2010)
14. Climenhaga, V.: Bowen’s equation in the non-uniform setting. Ergod. Theory Dyn. Syst. 31(4), 1163–1182 (2011)
15. Denker, M., Urbaniński, M.: Ergodic theory of equilibrium states for rational maps. Nonlinearity 4(1), 103–134 (1991)
16. Falconer, K.J.: Fractal Geometry: Mathematical Foundations and Applications. Wiley, Chichester (1990)
17. Gatzouras, D., Peres, Y.: Invariant measures of full dimension for some expanding maps. Ergod. Theory Dyn. Syst. 17(1), 147–167 (1997)
18. Ju, Y., Ma, D., Wang, Y.: Topological entropy of free semigroup actions for noncompact sets. Discrete Contin. Dyn. Syst. 39(2), 995–1017 (2019)
19. Lin, X., Ma, D., Wang, Y.: On the measure-theoretic entropy and topological pressure of free semigroup actions. Ergod. Theory Dyn. Syst. 38(2), 686–716 (2018)
20. Ma, D., Wu, M.: Topological pressure and topological entropy of a semigroup of maps. Discrete Contin. Dyn. Syst. 31, 545–557 (2011)
21. Ma, J.H., Wen, Z.Y.: A Billingsley type theorem for Bowen entropy. C. R. Math. Acad. Sci. Paris 346, 503–507 (2008)
22. Mayer, V., Urbaniński, M.: Geometric thermodynamic formalism and real analyticity for meromorphic functions of finite order. Ergod. Theory Dyn. Syst. 28(3), 915–946 (2008)
23. Mayer, V., Urbaniński, M.: Thermodynamical formalism and multifractal analysis for meromorphic functions of finite order. Mem. Am. Math. Soc. 203(954), (2010)
24. Pesin, Y.: Dimension Theory in Dynamical Systems. The University of Chicago Press, Chicago (1997)
25. Pesin, Y., Pitskel, B.S.: Topological pressure and the variational principle for noncompact sets. Funct. Anal. Appl. 18(4), 50–63 (1984)
26. Pesin, Y., Weiss, H.: The multifractal analysis of Gibbs measures: motivation, mathematical foundation, and examples. Chaos 7(1), 89–106 (1997)
27. Pomeau, Y., Manneville, P.: Intermittent transition to turbulence in dissipative dynamical systems. Comm. Math. Phys. 74(2), 189–197 (1980)
28. Przytycki, F., Rivera-Letelier, J., Smirnov, S.: Equivalence and topological invariance of conditions for non-uniform hyperbolicity in the iteration of rational maps. Invent. Math. 151(1), 29–63 (2003)
29. Przytycki, F., Rivera-Letelier, J., Smirnov, S.: Equality of pressures for rational functions. Ergod. Theory Dyn. Syst. 24(3), 891–914 (2004)
30. Rodrigues, F.B., Varandas, P.: Specification and thermodynamical properties of semigroup actions. J. Math. Phys. 57(5), 052704, 27 (2016)
31. Ruelle, D.: Repellers for real analytic maps. Ergod. Theory Dyn. Syst. 2(1), 99–107 (1982)
32. Rugh, H. H.: On the dimensions of conformal repellers. Randomness and parameter dependency. Ann. Math. (2) 168(3), 695–748 (2008)
33. Takens, F., Verbitskiy, E.: On the variational principle for the topological entropy of certain non-compact sets. Ergod. Theory Dyn. Syst. 23(1), 317–348 (2003)
34. Urbanski, M.: On the Hausdorff dimension of a Julia set with a rationally indifferent periodic point. Studia Math. 97(3), 167–188 (1991)
35. Urbanski, M.: Parabolic Cantor sets. Fund. Math. 151(3), 241–277 (1996)
36. Urbanski, M., Zdunik, A.: Real analyticity of Hausdorff dimension of finer Julia sets of exponential family. Ergod. Theory Dyn. Syst. 24(1), 279–315 (2004)
37. Wang, Y., Ma, D.: On the topological entropy of a semigroup of continuous maps. J. Math. Anal. Appl. 427(2), 1084–1100 (2015)
38. Wang, Y., Ma, D., Lin, X.: On the topological entropy of free semigroup actions. J. Math. Anal. Appl. 435(2), 1573–1590 (2016)
39. Walters, P.: An introduction to ergodic theory. Springer, New York, Heidelberg, Berlin (1982)
40. Weiss, H.: The Lyapunov spectrum for conformal expanding maps and axiom-A surface diffeomorphisms. J. Stat. Phys. 95(3–4), 615–632 (1999)

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