TIGHT CONVERGENCE RATES OF THE GRADIENT METHOD ON HYPOCOCONVEX FUNCTIONS

TEODOR ROTARU¹², FRANÇOIS GLINEUR ², AND PANAGIOTIS PATRINOS¹

ABSTRACT. We perform the first tight convergence analysis of the gradient method with fixed step sizes applied to the class of smooth hypoconvex (weakly-convex) functions, i.e., smooth nonconvex functions whose curvature belongs to the interval $[\mu, L]$ with $\mu < 0$. The results fill the gap between convergence rates for smooth nonconvex and smooth convex functions.

The convergence rates were identified using the performance estimation framework adapted to hypoconvex functions. We provide mathematical proofs for a large range of step sizes and prove that the convergence rates are tight when all step sizes are smaller or equal to $1/L$. Finally, we identify the optimal constant step size that minimizes the worst-case of the gradient method applied to hypoconvex functions.

1. Introduction

Finding tight convergence rates for optimization methods on different classes of functions is a problem that receives an increasing attention. While most of the work targets convex functions, not as much is known about the guarantees in the nonconvex context. However, the latter is addressed with a higher interest in the recent years due to large-scale optimization problems or deep learning. For this reason, we aim to provide some insights about a specific subclass of nonconvex functions – the smooth hypoconvex ones. This class can be studied using its key property of curvature subtraction from a convex function. Thus, we analyze the hypoconvex functions with the canonical optimization algorithm, i.e., the gradient method, and provide tight convergence results for a large range of steps.

1.1. Tight convergence analysis. Suppose that one solves an optimization problem through an iterative method and is interested in a convergence measure after applying an arbitrary number of steps. Drori and Teboulle tackle this problem by introducing in [6] a novel tool for the worst-case analysis of first-order methods – the Performance Estimation Problem (PEP). They model the problem of finding the worst behavior as an optimization problem, and solve a convex relaxation of it with semidefinite programming.

Intuitively, a distance to the solution after performing $N$ steps of the optimization method $\mathcal{M}$ is maximized with respect to a class of functions $\mathcal{F}$. One must provide a characterization of the solution and an initial condition that ensure boundedness of the problem. A general form of the PEP problem is the following optimization program:

$$\begin{align*}
\text{maximize} & \quad \mathcal{P}(f, x_0) \\
\text{subject to} & \quad f \in \mathcal{F} \\
& \quad \text{Optimality conditions on } x, \\
& \quad \text{Initial conditions on } x_0
\end{align*}$$

(1.1)

¹Department of Electrical Engineering (ESAT-STADIUS) – KU Leuven, Kasteelpark Arenberg 10, 3001 Leuven, Belgium
²Department of Mathematical Engineering (ICTEAM-INMA) – UCLouvain, Avenue Georges Lemaître 4, 1348 Louvain-la-Neuve, Belgium
E-mail addresses: teodor.rotaru@kuleuven.be, francois.glineur@uclouvain.be, panos.patrinos@esat.kuleuven.be.

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where $\mathcal{P}$ is a performance measure and the decision variables are the function $f$ from a generic class of functions $\mathcal{F}$ and the initial point $x_0$.

The problem (1.1) is infinite dimensional because the maximization is over a class of functions. Therefore, it has to be reformulated as a finite dimensional problem. A rigorous discretization is done by using necessary and sufficient interpolating conditions of functions from $\mathcal{F}$. These conditions were first established in the context of PEP by A. Taylor, F. Glineur, J. Hendrickx in [12].

Worst-case convergence rates of first-order methods applied to smooth convex functions have been extensively analyzed with PEP. For instance, the gradient method with step sizes lower than $\frac{1}{L}$, where $L$ is the Lipschitz constant, applied to convex functions was studied in [6], while the strongly-convex functions were analyzed in [12] for fixed step sizes and in [3] with line-search. An efficiency analysis of first-order methods on smooth functions under the PEP framework is given in [8].

1.2. Work on smooth nonconvex functions. Exact convergence rates for the larger class of nonconvex functions have not been studied so deeply. A central reason is that one cannot guarantee the convergence to a global minimum point, hence the convergence analysis is made with respect to finding a stationary point. In particular, this is not sufficient to be a local minimum. The performance of algorithms for smooth nonconvex functions is measured in terms of the minimum gradient norm over the iterations.

Within the PEP framework for tight analysis, the gradient method for nonconvex smooth functions was studied by Taylor in [10, page 190] for the particular step size of $\frac{1}{L}$. The result was extended by Drori and Shamir in [5, Corollary 1] for step sizes lower than $\frac{1}{L}$ and then improved by Abbaszadeheinekasti et. al. in [1] for even larger step sizes, i.e., up to $\frac{\sqrt{L}}{L}$. In this paper, we make a further extension to embed hypocoercive functions and closely follow the PEP formulation from [1].

1.3. Contributions. The study is performed under the following Assumptions:

Assumption 1.1. The functions we analyze are smooth, with a curvature belonging to the interval $[\mu, L]$, with $L > 0$ and $\mu \leq 0$, and have a global minimum.

Assumption 1.2. We analyze the iterations of the gradient method with fixed but variable step sizes $\frac{1}{h_i}$, where $i$ denotes the iteration number and $h_i$ belongs to $(0, 2)$.

Assumption 1.3. The gradient method converges to a stationary point.

The choice of step sizes belonging to $(0, 2)$ is because the gradient method converges for this interval (for example, see [9, Section 2.1.5]).

Using the technique of performance estimation, the following results are proved:

(i) The first upper bounds of the convergence rates for the gradient method on hypocoercive functions for step sizes in $(0, \tilde{h})$, where $\tilde{h} \in [\frac{3}{4}, 2)$ is a threshold with an analytical expression depending on the ratio between $\mu$ and $L$. (Theorem 4.1)

(ii) The tightness of the upper bounds for step sizes $h_i \in (0, 1)$. (Proposition 4.3)

(iii) The tight convergence rate for the gradient method on convex functions with step sizes $h_i \in (0, \frac{1}{L})$. (Proposition 4.4)

From this findings, one can deduce:

(i) The optimal constant step size recommendation with respect to $\mu$ and $L$ such that the worst-case upper bound is minimized. (Proposition 4.7)

(ii) The existence of three worst-case regimes with respect to intervals of a constant step size $h$:

- $h \in (0, 1]$,
- $h \in (1, \tilde{h}]$ and
- $h \in (\tilde{h}, 2)$.

Following extensive numerical simulations, we conjecture:

(i) The upper bound for the gradient method on convex functions with constant step size $h \in \left(\frac{3}{4}, 2\right)$. (Conjecture 4.6)

(ii) A partial description of the upper bound of the third regime corresponding to $h \in (\tilde{h}, 2)$. (Conjecture 4.8)

The work relies on Theorem 3.1, which provides the interpolation conditions for smooth hypocoercive functions, together with a characterization of the global minimum of such functions.
Structure of the paper. In Section 2 we introduce notations and definitions. Section 3 is dedicated to the formulation of performance estimation problem for the gradient method applied to hypoconvex functions. In Section 4 we present the results on the exact convergence rates and, as a direct application, we recommend the optimal constant step size that minimizes the rate.

2. Definitions and setup

Consider a $d$-dimensional function $f : \mathbb{R}^d \to \mathbb{R}$ and the unconstrained optimization problem

$$\min_{x \in \mathbb{R}^d} f(x),$$

where $f \in C^1(\mathbb{R}^d)$ is smooth.

2.1. Gradient method. Assume one computes $N$ steps of the gradient method generating the iterations $x_i$, starting from the initial point $x_0$,

$$x_{i+1} = x_i - \frac{1}{L} g_i, \quad \forall i = 0, \ldots, N-1 \quad (2.1)$$

where $g_i := \nabla f(x_i)$ is the gradient of $f$ with respect to $x_i$. Any optimal solution, the global minimum value and the optimal gradient are denoted by $x^*, f^* := f(x^*)$ and $g^* := \nabla f(x^*) = 0$, respectively. Let $\Delta > 0$; then for all starting points $x_0$ it is assumed that

$$f(x_0) - f_i \leq \Delta \quad (2.2)$$

Black-box model. An index set is denoted by $I$. For all $i \in I$ it is assumed to have access to a first-order oracle that provides the triplets $T := \{(x_i, g_i, f_i)\}_{i \in I} \subseteq \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}$, i.e., the points, the gradients and the function values.

2.2. Smooth hypoconvex functions. Let the following known result on smooth convex functions:

**Proposition 2.1.** If $f : \mathbb{R}^d \to \mathbb{R}$ is smooth and convex, then

$$f(x) \geq f(y) + \langle \nabla f(y), x - y \rangle, \quad \forall x, y \in \mathbb{R}^d.$$

The maximum and minimum curvature of a function are defined as following.

**Definition 2.2.** Let $L > 0$. A function $f : \mathbb{R}^d \to \mathbb{R}$ has a maximum curvature $L$ if and only if the function $g : \mathbb{R}^d \to \mathbb{R}$, $g := \frac{1}{2} ||x||^2 - f$, is convex.

**Definition 2.3.** Let $\mu > -\infty$. A function $f : \mathbb{R}^d \to \mathbb{R}$ has a minimum curvature $\mu$ if and only if the function $g : \mathbb{R}^d \to \mathbb{R}$, $g := f - \frac{\mu}{2} ||x||^2$, is convex.

Depending on the sign of $\mu$, $f$ is (i) hypoconvex for $\mu < 0$, (ii) convex for $\mu = 0$ or (iii) strongly-convex for $\mu > 0$.

**Definition 2.4.** By $\mathcal{F}_{\mu,L}(\mathbb{R}^d)$ we denote the class of $d$-dimensional smooth functions whose curvature belongs to the interval $[\mu, L]$, where $L > 0$ and $\mu \in (-\infty, L]$.

The analysis in this paper is restricted to $\mu \leq 0$. To simplify the notation, we use $\mathcal{F}_{\mu,L}$ and implicitly consider a $d$-dimensional real function. The functions $f \in \mathcal{F}_{\mu,L}$ are characterized by the following lemma:

**Lemma 2.5 (Quadratic bounds).** If $f \in \mathcal{F}_{\mu,L}$ with $L > 0$ and $\mu \in (-\infty, L]$, then $\forall x, y \in \mathbb{R}^d$ the following quadratic upper and lower bounds state:

$$\frac{\mu}{2} ||x - y||^2 \leq f(x) - f(y) - \langle \nabla f(y), x - y \rangle \leq \frac{L}{2} ||x - y||^2 \quad (2.3)$$

**Proof.** The bounds result by inserting the Proposition 2.1 in the definitions of upper curvature (2.2) and lower curvature (2.3), respectively. \hfill \Box

In the convergence rates’ derivations, the ratio $\kappa := \frac{L}{\mu}$ is employed. In the smooth strongly-convex context, $\kappa$ is referred as the inverse condition number.

**Remark 2.6.** For $f \in \mathcal{F}_{\mu,L}$, the Lipschitz constant is defined as $L := \max\{\mu, L\}$, while the hypoconvexity constant as $\bar{\mu} := \max\{-L, \mu\}$. Within this paper, the analysis is performed only with respect to the curvature bounds $\mu$ and $L$, thus allowing $\mu < -L$. 
Definition 2.7 ([12], Definition 2). A set of triplets \( \mathcal{T} := \{(x_i, g_i, f_i)\}_{i \in I} \), with \( x_i, g_i \in \mathbb{R}^d, f_i \in \mathbb{R} \), is called \( \mathcal{T}_{\mu, L} \)-interpolable if and only if there exists a function \( f \in \mathcal{T}_{\mu, L}(\mathbb{R}^d) \) such that \( \nabla f(x_i) = g_i \) and \( f(x_i) = f_i \) for all \( i \in I \).

3. Performance estimation

The objective is to find the worst-case convergence rate when applying \( N \) steps of the gradient method on smooth hypoconvex functions. To this end, in (3.1) is instantiated the general concept of a PEP (1.1), using the minimum gradient norm over the iterations as a performance measure and the distance between function values (2.2) as an initial condition.

\[
\begin{align*}
\text{maximize} & \quad \min_{0 \leq \|g\| \leq N} \|g\|^2 \\
\text{subject to} & \quad f \in \mathcal{T}_{\mu, L} \\
& \quad x_{i+1} = x_i - \frac{g_i}{\mu}, \quad i \in \{0, \ldots, N - 1\} \\
& \quad f(x) \geq f_i, \quad \forall x \in \mathbb{R}^d \\
& \quad f_0 - f_* \leq \Delta
\end{align*}
\]

(3.1)

The decision variables of problem (3.1) are \( f \) and \( x_0 \). Because the maximum is taken over the entire class of functions \( f \in \mathcal{T}_{\mu, L} \), we deal with an infinite-dimensional optimization problem, with an infinite number of constraints. In order to solve it, one must relax it and show that the relaxed formulation leads to the same optimal solution. This step is usual in the PEP methodology and is done by restricting the functions to the iterations and use specific interpolation conditions.

3.1. Finite-dimensional formulation of the PEP. A finite-dimensional formulation of (3.1) is defined with the help of Theorem 3.1. This provides necessary and sufficient interpolation conditions of smooth hypoconvex functions \( f \in \mathcal{T}_{\mu, L} \) and a characterization of the global minimum \( f_* \).

**Theorem 3.1.** Let \( \mathcal{T} = \{(x_i, g_i, f_i)\}_{i \in I} \), \( L > 0 \) and \( \mu \in (0, L] \). \( \mathcal{T} \) is \( \mathcal{T}_{\mu, L} \)-interpolable if and only if for every pair of indices \((i, j)\), with \( i, j \in I \):

\[
f_i - f_j - \langle g_j, x_j - x_i \rangle \geq \frac{1}{2(1 - \frac{\mu}{L})} \left( \|g_i - g_j\|^2 + \mu \|x_i - x_j\|^2 - 2\frac{\mu}{L} \langle g_j - g_i, x_j - x_i \rangle \right)
\]

(3.2)

Moreover, among the set of interpolating functions, there exists one such that its global minimum \( f_* \) is characterized by

\[
f_* = \min_{x \in \mathbb{R}^d} f(x) = \min_{i \in I} \{f_i - \frac{\mu}{L} \|g\|^2\}
\]

(3.3)

Let \( i_* \in \text{arg min}_{i \in I} \{f_i - \frac{\mu}{L} \|g\|^2\} \); then a global minimizer of this function is given by \( x_* = x_{i_*} - \frac{\mu}{L} g_{i_*} \).

Based on Theorem 3.1, the infinite-dimensional PEP (3.1) is transformed into the following finite-dimensional PEP, where \( I = \{0, \ldots, N, \ast\} \).

\[
\begin{align*}
\text{maximize} & \quad \min_{0 \leq \|g\| \leq N} \|g\|^2 \\
\text{subject to} & \quad \{(x_i, g_i, f_i)\}_{i \in I} \text{ satisfy (3.2)} \\
& \quad x_{i+1} = x_i - \frac{g_i}{\mu}, \quad i \in \{0, \ldots, N - 1\} \\
& \quad f_i - \frac{\mu}{L} \|g\|^2 - f_* \geq 0, \quad i \in \{0, \ldots, N\} \\
& \quad f_0 - f_* \leq \Delta
\end{align*}
\]

(3.4)

With regard to its application to our problem, we comment on Theorem 3.1 in the next two paragraphs. The two parts of the theorem are extensions of previous results based on which we construct our proof in Appendix A.

**Interpolation conditions.** The first part of Theorem 3.1 is a direct extension of [12, Theorem 4] and includes smooth hypoconvex functions besides smooth strongly-convex functions. The only difference is allowing negative curvatures due to \( \mu < 0 \). A graphical interpretation of the interpolating function for the particular case \( \mu = -L \) is given in [10, page 71].
Characterization of the optimal point. An optimality condition of the type \( f_i - f_* \geq 0 \) would not guarantee the existence of a function with a global minimum \( f_* \). The second part of Theorem 3.1 provides such guarantees and leads to an exact formulation, i.e., problem (3.4). The result is given in [5, Theorem 7] for the smooth case \( (\mu = -L) \) and exploited in [1] to obtain tightness guarantees of the convergence rates for the gradient method. Theorem 3.1 includes general curvatures, belonging to the range \( [\mu, L] \), with \( L > 0 \). This extension was suggested in [4, Remark 2.1] for strongly-convex functions, which are similar in some sense to the hypoconvex ones.

Proposition 3.2 shows that the problems (3.1) and (3.4) share the same optimal values. In particular, a worst-case function \( f_* \), solution of the infinite dimensional problem (3.4), can be obtained from the solution \( \mathcal{T} = [(\tilde{x}, \tilde{g}, \tilde{f})]_{\alpha f} \) of (3.4) through the necessary and sufficient interpolation conditions from Theorem 3.1.

Proposition 3.2. The solutions of the optimization problems (3.1) and (3.4) are the same.

Proof. Since problem (3.4) is a relaxation of problem (3.1), it is sufficient to show that for any feasible solution of (3.4), \( \mathcal{T} = [(\tilde{x}, \tilde{g}, \tilde{f})]_{\alpha f} \), there exists a smooth hypoconvex function \( f \in \mathcal{F}_{\mu,L} \), with \( L > 0, \mu \in (-\infty,0] \), such that \( f(\tilde{x}) = f_* \), \( \nabla f(\tilde{x}) = g_* \), \( \forall i \in \mathcal{I} \) and \( \min_{x \in \mathcal{X}} f(x) \geq f_* \). Since \( \mathcal{T} \) is a feasible point of problem (3.4), the assumptions of Theorem 3.1 are valid, hence there exists such a function. \( \square \)

3.2. Solving the PEP. Using an auxiliary variable \( l \), it is obtained the nonconvex quadratic problem with quadratic constraints

\[
\begin{align*}
\text{maximize} \quad & l \\
\text{subject to} \quad & \{(x_i, g_i, f_i)\}_{i \in \mathcal{I}} \text{ satisfy (3.2)} \\
& x_{i+1} = x_i - \frac{1}{L} g_i, \quad i \in \{0, \ldots, N-1\} \\
& f_i - \frac{1}{2L} \|g_i\|^2 - f_* \geq 0, \quad i \in \{0, \ldots, N\} \\
& f_* - f_0 - \Delta \geq 0 \\
& \|g_i\|^2 - l \geq 0, \quad 0 \leq i \leq N.
\end{align*}
\]

(3.5)

on which the mathematical proof of the convergence rate is built up.

The program (3.5) is intractable and we relax it following the same steps as in [12, Sections 3.2-3.3]. We obtain a tractable convex reformulation using a Gram matrix to describe the iterates \( x_i \) and the gradients \( g_i \). Let

\[
P := \begin{bmatrix} g_0 & g_1 & \cdots & g_N & x_0 \end{bmatrix}
\]

and the symmetric \((N + 2) \times (N + 2)\) Gram matrix \( G = P^T P \). The iterates of the gradient method \( x_i \) can be expressed using the gradients and the initial value \( x_0 \):

\[
x_i = x_0 + \frac{1}{2} \sum_{k=0}^{i-1} h_k g_k
\]

Therefore, one can replace the iterations \( x_i \) in the interpolation conditions and reformulate all the constraints from (3.5) in terms of entries of \( G \), along with function values \( f_i \). Then the following tractable SDP is obtained:

\[
\begin{align*}
\text{maximize} \quad & l \\
\text{subject to} \quad & f_i - f_j + \text{tr}(A_{ij} G) \geq 0, \quad i \neq j \\
& f_i - \frac{1}{2L} G_{ii} - f_* \geq 0, \quad i \in \{0, \ldots, N\} \\
& f_* - f_0 - \Delta \geq 0 \\
& G_{ii} - l \geq 0, \quad 0 \leq i \leq N \\
& G \succeq 0
\end{align*}
\]

(3.6)
where the \( A_{ij} \) matrices are formed according to the interpolation conditions. A detailed explanation of computing \( A_{ij} \) and obtaining (3.6) is given in [12, Section 3.3]. The relaxation is exact for \( d \geq N + 2 \) (the large-scale setting), while for smaller dimensions one can introduce a nonconvex rank constraint; for more details, see [12, Theorem 5].

**Software tools.** One can solve (3.6) using an SDP solver. Alternatively, problem (3.4) can be directly solved using the Matlab toolbox PESTO [11] or the Python toolbox PEPit [7].

### 4. Worst-Case Convergence Rate

In this section we state our results, the focal point being Theorem 4.1 on the worst-case convergence rates.

The result was inferred by solving a large number of PEPs (3.4) for multiple setups of the parameters. We exploited the homogeneity conditions with respect to \( L \) and \( \Delta \) from [12, Section 3.5] and fixed \( L = \Delta = 1 \). In this way, it was enough to consider only two unknown parameters: the step sizes \( h_i \) and lower curvatures \( \mu \). The obtained analytical expressions for the convergence rates were easily extended to arbitrary positive \( L \) and \( \Delta \) using the homogeneity conditions.

However, even though the numerical solution of PEP helped us derive the analytical expression, its proof is independent.

**Theorem 4.1.** Let \( f \in \mathcal{F}_{L,\Delta}(\mathbb{R}^d) \) be a smooth hypoconvex function, with \( L > 0 \) and \( \mu \in (-\infty,0) \), and \( \kappa := \frac{2}{\mu} \). Let \( \bar{h}(\kappa) := \frac{1\kappa - 1}{L + \kappa} \) and \( N \) iterations of the gradient method (2.1) with \( h_i \in (0, \bar{h}(\kappa)] \), \( i \in \{0, \ldots, N - 1\} \), generating the sequence \( x_0, \ldots, x_N \) starting from \( x_0 \). Assume that \( f \) has a global minimum \( f^* \) and \( f(x_0) - f^* \leq \Delta \), where \( \Delta > 0 \). Then

\[
\min_{0 \leq i \leq N} \|\nabla f(x_i)\|^2 \leq \frac{2L(f(x_0) - f^*)}{1 + \sum_{i=0}^{N-1} \left(2h_i - h_i^2 + \frac{\kappa}{1+\kappa}\right)} \tag{4.1}
\]

In particular, if \( h_i \in (0, 1] \), \( \forall i \in \{0, \ldots, N - 1\} \),

\[
\min_{0 \leq i \leq N} \|\nabla f(x_i)\|^2 \leq \frac{2L(f(x_0) - f^*)}{1 + \sum_{i=0}^{N-1} \left(2h_i - h_i^2 + \frac{\kappa}{1+\kappa}\right)} \tag{4.2}
\]

while if \( h_i \in [1, \bar{h}(\kappa)] \), \( \forall i \in \{0, \ldots, N - 1\} \),

\[
\min_{0 \leq i \leq N} \|\nabla f(x_i)\|^2 \leq \frac{2L(f(x_0) - f^*)}{1 + \sum_{i=0}^{N-1} h_i(2-h_i^2+\kappa h_i) \frac{\kappa}{2(1+\kappa)h_i}} \tag{4.3}
\]

To give an intuition of the bounds, we plot in Figure 1 the general inner term from the sum in the denominator of (4.1)

\[
s_i(h_i, \kappa) := 2h_i - h_i^2 \frac{\kappa}{2 \min \left(\frac{1}{h_i}, \frac{1}{\kappa}\right) - (1 + \kappa)} \tag{4.4}
\]

In Appendix B we prove Theorem 4.1 by the standard technique in the PEP framework: for every feasible solution of problem (3.5) the decision variable \( I \) is upper-bounded by the right-hand-side of (4.1). Further, we discuss some particular cases.

**The smooth nonconvex case.** For \( \mu = -L \) it is recovered the result from [1, Theorem 2]: for all \( h_i \in (0, \sqrt{\kappa}] \)

\[
\min_{0 \leq i \leq N} \|\nabla f(x_i)\|^2 \leq \frac{2L(f(x_0) - f^*)}{1 + \sum_{i=0}^{N-1} \left(2h_i - \frac{\kappa}{2} \max(1, h_i)\right)}
\]
Convergence rate of the gradient method on hypoconvex functions

![Graph](image)

**Figure 1.** Dependence on the step size $h_i$ and the ratio $\kappa$ of the general term (4.4) from the sum in the denominator of the upper bound (4.1). For every $\kappa$, the step sizes belong to a sub-interval of $(0,2)$. The step size $h = 1$ marks the transition between the two regimes.

**Descent lemma based result.** In Corollary 4.2 is showed that assuming only knowledge of the upper curvature of a function (i.e., letting $\mu \to -\infty$), the upper bound is obtained by applying the descent lemma.

**Corollary 4.2.** Let $f$ be a function for which only its upper curvature $L > 0$ is known. Then the upper bound (4.5), derived by Nesterov in [9, page 28, formula (1.2.15)] based on descent lemma, is tight.

\[
\min_{0 \leq i \leq N} \|\nabla f(x_i)\|^2 \leq \frac{2L(f(x_0) - f_*)}{1 + \sum_{i=0}^{N-1} (2h_i - h_i^2)}, \quad \forall h_i \in (0,2) \tag{4.5}
\]

**Proof.** Due to the arbitrary lower curvature of $f$, the bound is obtained by taking the limit $\kappa \to -\infty$ in Theorem 4.1. \qed

4.1. **Tightness of the results.** A natural question after finding the upper bound (4.1) concerns its tightness. For $h_i \in (0,1]$, the bound’s exactness is proved by constructing a one-dimensional worst-case function example inspired from [1, Proposition 4].

**Proposition 4.3.** If $h_i \in (0,1], i \in \{0, \ldots, N - 1\}$, the optimal value of (3.1) is

\[
\frac{2L(f(x_0) - f_*)}{1 + \sum_{i=0}^{N-1} (2h_i - h_i^2)}
\]

**Proof.** Given the upper bound from (4.2), it is enough to find a worst-case function $f \in \mathcal{F}_{\mu,L}$ for which this bound is reached. Let $U$ be the square root of the optimal value,

\[
U = \sqrt{\frac{2L \Delta}{1 + \sum_{i=0}^{N-1} (2h_i - h_i^2)}}
\]

For all $i = 0, \ldots, N$, let

\[
x_i = U + U \sum_{j=0}^{i-1} h_j \\

f_i = \Delta - \frac{U^2}{L} \sum_{j=0}^{i-1} h_j \left(1 - \frac{h_j}{2(1 - \kappa)}\right) \tag{4.6}
\]

\[
g_i = U
\]
and define the points

\[ \bar{x}_i := x_i - \frac{k}{1 - k} \frac{h_i}{L} U \in [x_{i+1}, x_i], \quad i \in [0, \ldots, N - 1]. \]

Then a worst-case function is the following piecewise quadratic function \( f : \mathbb{R} \to \mathbb{R} \):

\[
f(x) = \begin{cases} 
\frac{\mu}{2} x^2 & x \in (-\infty, x_1] \\
\frac{\mu}{2} (x - x_{i+1})^2 + U(x - x_{i+1}) + f_{i+1} & x \in [x_{i+1}, \bar{x}_i] \\
\frac{\mu}{2} (x - x_i)^2 + U(x - x_i) + f_i & x \in [\bar{x}_i, x_i] \\
\frac{\mu}{2} (x - x_0)^2 + U(x - x_0) + f_0 & x \in [x_0, \infty) 
\end{cases}
\]

By construction, \( f(x_i) = f_i \) and \( \nabla f(x_i) = g_i \). The curvature is alternating between \( \mu \) and \( L \), having the iterates \( x_i \) and the points \( \bar{x}_i \) as inflection points. The optimal solution is \( (x_*, f_*) = (0, 0) \). One can directly check that the function \( f \) satisfies the interpolation conditions (3.2) from Theorem 3.1. Therefore, the function \( f \) is the optimal of the initial PEP (3.1), hence the bound from (4.2) is exact.

\[ \square \]

**Larger step sizes.** We do not provide a worst-case function example for \( h \in [1, \bar{h}(\kappa)] \). However, the primal solution of PEP can be seen as a (numerical) proof of the lower bound. More precisely, for every solution of the optimization problem \( T = (x_0, \kappa, f_i) \) of (4.2), there exists an interpolating function \( f \) because of Proposition 3.2. From the numerical simulations we observed that worst-case candidate functions are \((N+1)\)-dimensional, where \( N \) is the number of iterations.

### 4.2. The convex case.

The class of convex functions can be seen as a particular class of hypoconvex functions with \( \mu = 0 \). For performance measures in terms of the gradient norm, the problem of finding exact worst-convergence rates for the gradient method on smooth convex functions was little explored. In particular, in [8, Theorem 5.1] the exact rate for the constant step size \( h = 1 \) is determined. With the help of Theorem 4.1, we extend the upper bounds for step sizes \( h_i \in (0, \frac{1}{2}) \).

The rate from Proposition 4.4 is similar to the one conjectured in [6, Conjecture 3.1] for step sizes \( h_i \leq 1 \), where the distance to the optimal value \( f(x_*) - f \), is measured, instead of the gradient norm.

**Proposition 4.4 (Exact worst-case rate for convex functions).** Let \( f \in F_{h_i} \) be a smooth convex function and \( f_* \) its global minimum. Let \( x_0 \) be such that \( f(x_0) - f_* \leq \Delta \), with \( \Delta > 0 \) and consider \( N \) iterations of the gradient method (2.1) with \( h_i \in (0, \frac{1}{2}) \). Then

\[
\min_{0 \leq i \leq N} \| \nabla f(x_i) \|^2 \leq \frac{2L(f(x_0) - f_*)}{1 + 2 \sum_{i=0}^{N-1} h_i}
\]

and there exists a function \( f \) for which the bound is tight.
Proof. The upper bound results by taking the limit $\kappa \to 0$ in Theorem 4.1. Alternatively, one can directly plug $\kappa = 0$ in the proof of Theorem 4.1 (see B).

A worst-case smooth convex function is obtained from Proposition (4.3) for $\kappa = 0$:

$$f(x) = \begin{cases} \frac{1}{2}x^2 & x \in (-\infty, x_i] \\ U(x - x_{i+1}) + f_{i+1} & x \in [x_{i+1}, x_i] \\ \frac{1}{2}(x - x_0)^2 + U(x - x_0) + f_0 & x \in [x_0, \infty) \end{cases}$$

where $U$ is the positive square root of the right-hand-side of (4.7), and $x_i$ and $f_i$ are computed by setting $\kappa = 0$ in (4.6).

If all step-sizes are equal, then in Proposition 4.5 we have a particular result on the $N$-th gradient norm.

**Proposition 4.5** (Exact worst-case rate for convex functions with constant step-size). Let $f \in \mathcal{F}_{0,L}$ be a smooth convex function and $f_i$ its global minimum. Let $x_0$ be such that $f(x_0) - f_i \leq \Delta$ with $\Delta > 0$ and consider $N$ iterations of the gradient method (2.1) with constant step-size $h \in (0, \frac{\kappa}{2})$. Then

$$\|\nabla f(x_N)\| \leq \frac{2L(f(x_0) - f_i)}{1 + 2Nh}$$

and there exists a function $f$ for which the bound is tight.

Proof. When a constant step-size is used, the gradient norm is non-increasing for smooth convex functions, therefore the minimum corresponds to the last iteration. To demonstrate this statement, one can particularize the analysis from [2, Theorem 1] to the gradient method, and obtain that the sequence $\|x_i - x_{i+1}\|^2$ is monotonically non-increasing. The rest of the proof is a corollary of Proposition 4.4 for $h_i = h$.

Following extensive numerical simulations, we conjecture the following upper bound for step sizes larger than $\frac{\kappa}{2}$.

**Conjecture 4.6.** Let $f \in \mathcal{F}_{0,L}$ be a smooth convex function and $f_i$ its global minimum. Let $x_0$ be such that $f(x_0) - f_i \leq \Delta$, with $\Delta > 0$, and consider $N$ iterations of the gradient method with a constant step size $h \in (\frac{\kappa}{2}, 2)$. Then

$$\|\nabla f(x_N)\|^2 \leq 2L(f(x_0) - f_i) \max \left\{ (1 - h)^{2N}, \frac{1}{1 + 2Nh} \right\}$$

4.3. **Application: the optimal step size.** A direct benefit of the upper bounds from Theorem 4.1 is the deduction of the optimal constant step size that minimizes the worst-case convergence rate of the gradient method applied to hypoconvex functions. A better recommendation than the smooth nonconvex case is obtained by exploiting the curvature properties.

**Proposition 4.7.** Let $f \in \mathcal{F}_{\mu,L}$ be a smooth hypoconvex function with $L > 0$ and $\mu < 0$, and $\kappa = \frac{\mu}{2} < 0$. Then the optimal step size $h_*$ for the gradient method with respect to the worst-case convergence rate from Theorem 4.1 is

$$h_*(\kappa) = \begin{cases} h_{opt} & \kappa \leq \bar{\kappa} \\ \bar{h}(\kappa) & \bar{\kappa} < \kappa < 0 \end{cases}$$

where $h_{opt}$ is the unique solution in $[1, \bar{h}(\kappa)]$ of

$$-\kappa(1 + \kappa)h^3 + [3\kappa + (1 + \kappa)^2]h^2 - 4(1 + \kappa)h + 4 = 0$$

and $\bar{\kappa} = \frac{-0.5 \pm \sqrt{0.25 + 4}}{2} = -0.1001$.

Proof. Minimizing the upper bound from Theorem 4.1 is equivalent with maximizing the general term $s(\kappa)$ (4.4)) from the sum in the denominator:

$$h_* = \arg\max_{h \in (0, \min(\bar{h}(\kappa), 1))} 2h - h^2 \frac{-\kappa}{-(1 + \kappa) + 2 \min\left(1, \frac{1}{h}\right)}$$

(4.9)
4.1 We split the analysis on the intervals 
shows $\tilde{h}(\kappa) = 2/\sqrt{3}$ recommended in [1] for smooth nonconvex functions (in red) and the optimal step size recommendation from Proposition 4.7 (in blue), in term of of the constant (4.11).

We split the analysis on the intervals $h \in (0, 1]$ and $h \in [1, \tilde{h}(\kappa)]$:

$$c_1 := \max_{h \in (0, 1)} 2h - h^2 \frac{-\kappa}{1 - \kappa} = 2 - \frac{-\kappa}{1 - \kappa}$$

$$c_2 := \max_{h \in [1, \tilde{h}(\kappa)]} 2h - h^3 \frac{-\kappa}{2 - (1 + \kappa)h} \geq 2 - \frac{\kappa}{2 - (1 + \kappa)} = c_1$$

Hence, $c_2 \geq c_1$ and $h_1$ belongs to $[1, \tilde{h}(\kappa)]$. We denote by $c(h)$ the objective function from

$$h_1 = \arg \max_{1 \leq h \leq \tilde{h}(\kappa)} 2h + \frac{\kappa h^3}{2 - h(1 + \kappa)} \quad (4.10)$$

One can check that $c(h)$ is strictly concave on $[1, 2)$ and its first derivative is

$$c'(h) = -\kappa(1 + \kappa)h^3 + \left[3(1 + \kappa) + (1 + \kappa)^2\right]h^2 - 4(1 + \kappa)h + 4$$

By solving the optimization problem (4.10) we get the optimal step-size expression (4.8).

To the limit $\kappa \to -\infty$, from (4.10) we get the “classical optimal step size” $h = 1$. Likewise, for $\kappa = -1$ we obtain the optimal step size for smooth nonconvex function given in [1, Theorem 3], $h = \frac{2}{\sqrt{3}}$.

One may be interested in the actual benefit of using our recommendation. To answer this question, we focus on constant step sizes $h$, and slightly weaken the upper bound on (4.3) by neglecting the 1 in the denominator, obtaining $\bar{c}(f(x_0) - f_*) \leq \tilde{w}(\kappa, h)$ with $\tilde{w}(\kappa, h)$ defined as

$$\tilde{w}(\kappa, h) := \frac{2(1 - h) + 2(1 - kh)}{h(2 - h)(2 - kh)} \quad (4.11)$$

Figure 3 shows $\tilde{w}$ for special choices of $h$: (i) the “classical optimal step” $h = 1$, (ii) the optimal step size for smooth functions $h_1^* = \frac{2}{\sqrt{3}}$ and (iii) the optimal step size from Proposition 4.7. When the nonconvexity decreases, the optimal step size from (4.10) provides a larger improvement in comparison to the step $h_1^*$, that does not take a benefit from the curvature. To the limit $\kappa \to \infty$, the classical optimal step size $h = 1$ is recovered.

In Figure 4 we plot the optimal constant step size $h_1(\kappa)$ that minimizes the upper bound of (4.1) and the threshold $\tilde{h}(\kappa)$ of the rate from Theorem 4.1. The non-smooth part that appears for $\kappa > \kappa$ is due to the threshold $\tilde{h}(\kappa)$. For this close-to-convex case it is suggested that the optimal step size belongs to the range of large steps, $h > \tilde{h}(\kappa)$, which is not covered by Theorem 4.1.
The upper threshold. The worst-case convergence rate from Theorem 4.1 is valid for step sizes up to \( \tilde{h}(\kappa) \). This limit is derived from the proof of the upper bound (see Appendix B), in particular from the non-negativity of some dual multipliers. For \( \kappa \to -\infty, \tilde{h} \to 2 \), i.e., the entire range of step sizes is covered, as in Corollary 4.2. For \( \kappa = -1 \), it is recovered the upper limit from \([1, \text{Theorem } 2], \tilde{h} = \sqrt{3}\), while for \( \kappa = 0 \) the limit \( \tilde{h} = \frac{2}{3} \) from Proposition 4.4.

4.4. On the regime with larger step sizes. The last part of the paper is dedicated to the insights about the third regime, with steps \( h \in (\tilde{h}(\kappa), 2) \). Using the numerical results from the solution of the problem (3.4) for multiple setups, we conjecture the following expression of the worst-case convergence rate for constant step sizes.

**Conjecture 4.8.** Let \( f \in \mathcal{F}_{\mu,L} \) be a smooth hypoconvex function, \( f_* \) its global minimum and \( x_0 \) the starting point such that \( f(x_0) - f_* \leq \Delta \), with \( \Delta > 0 \). Consider \( N \) iterations of the gradient method (2.1) with a constant step \( h \in (\tilde{h}(\kappa), 2) \). Then

\[
\min_{\theta \in [0, N]} \|\nabla f(x_\theta)\|^2 \leq 2L(f(x_0) - f_*) \max \left\{ (1 - h)^{2N}, \frac{1}{q + N \frac{\mu + 0.2 - k h}{2 - h(1 + \kappa)}} \right\}
\]

where for \( q = q(L, \kappa, h) \) we lack an analytical expression.

From the upper bound expression, one can infer there exists \( N_0 \) such that, when performing \( N > N_0 \) iterations, the first \( N_0 \) steps belong to the first regime and the next ones to the second regime. For the latter, the fraction multiplying \( N \) is the same as in (4.3).

**Optimal step-size.** In the asymptotic behavior, i.e., for a large number of iterations \( N \), the term \( q \) becomes negligible. Hence, minimizing the upper bound leads to the optimization problem (4.9), but for the entire range \( h \in [0, 2) \). Therefore, we propose Conjecture 4.9 where the threshold \( \tilde{k} \) from Proposition 4.7 disappears and the recommendation covers all regimes.

**Conjecture 4.9.** Let \( f \in \mathcal{F}_{\mu,L} \) be a smooth hypoconvex function with \( L > 0 \) and \( \mu < 0 \), and \( \kappa = \frac{L}{\mu} < 0 \). Then the optimal step size \( h_* \) that minimizes the asymptotic worst-case convergence rate of the gradient method applied to hypoconvex functions is the unique solution in \([1, 2)\) of

\[
-\kappa(1 + \kappa)h^3 + \left[ 3\kappa + (1 + \kappa)^2 \right]h^2 - 4(1 + \kappa)h + 4 = 0
\]

**Proof.** Similar to the proof in Proposition 4.7, we have to solve the maximization problem

\[
h_* = \arg \max_{0 < h < 2} 2h + \frac{\kappa h^3}{2 - h(1 + \kappa)}
\]
The objective function $c(h) = 2h + \frac{3}{2(1 + h)}$ is strictly concave on $[1, 2)$ and its first derivative is
\[
c'(h) = -\kappa(1 + \kappa)h^3 + \left[3\kappa + (1 + \kappa)^2\right]h^2 - 4(1 + \kappa)h + 4 \tag{1}
\]
Because $c'(1) = 1 > 0$ and $c'(2) = 4\kappa(1 - \kappa) < 0$, there exists an unique solution $h_* \in [1, 2)$ of $c'(h) = 0$.

Figure 4 shows this conjectured extension of the optimal step-size $h_*(\kappa)$ in black dashed line. To the limit $\kappa \to 0$, the optimal step-size for the convex case is obtained as the upper bound of the convergence interval.

5. Conclusion

We performed the first worst-case analysis using the performance estimation technique on smooth hypoconvex functions. We introduced Theorem 3.1 for interpolation with functions from this class and characterization of the optimal point; this theorem extends previous results in the literature. We selected the gradient method and identified three regimes with respect to the range of step sizes. For two of them we provided a proof of the upper bound $h \leq \bar{h}(\kappa)$, and in addition show tightness for short steps ($h \leq 1$). As a particular case of Theorem 4.1, we proved tight convergence rates on convex functions for the steps range $h \in [0, \bar{h}]$. As a direct application of our results, we offered recommendations for the optimal constant step size. Concerning the third regime with $h \in (\bar{h}(\kappa), 2)$, we conjectured a partial analytical rate which is useful in the asymptotic regime to recommend the optimal step size.

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Appendix A. Proof of interpolation conditions for smooth hypoconvex functions

The proof of Theorem 3.1 is based on the same steps as the one of [12, Theorem 4], which is more detailed. The following results from [12] are employed.

**Theorem A.1** ([12], Theorem 1). (Convex interpolation) The set \( \{ (x, g, f) \}_{i \in I} \) is \( T_{0, \infty} \)-interpolable if and only if
\[
 f_i - f_j - (g_i, x_i - x_j) \geq 0 \quad i, j \in I.
\]

**Lemma A.2.** Consider a set \( \{ (x, g, f) \}_{i \in I} \). For any constants \( \mu, L \) with \( \mu \in ( - \infty, L ] \), \( 0 < L \leq \infty \), the following propositions are equivalent:

(i) \( \{ (x, g, f) \}_{i \in I} \) if \( T_{\mu, L} \)-interpolable,

(ii) \( \{ (x, g, f) \}_{i \in I} \) if \( T_{\mu, L} \)-interpolable.

**Proof.** This lemma is a direct extension of [12, Lemma 1] in the sense of including the hypoconvex functions, i.e., \( \mu < 0 \). The idea is to use minimal curvature subtraction and write Lemma 2.5 for \( h(x) := f(x) - \frac{\mu}{2} \| x \|^2 \). Then \( \nabla h(x) = \nabla f(x) - \mu x \) and from (2.3) we have the following equivalent inequalities:
\[
\frac{\mu}{2} \| x - y \|^2 \leq h(x) - h(y) - \langle \nabla h(y), x - y \rangle \leq \frac{\mu}{2} \| x - y \|^2.
\]

Then \( \{ (x, g, f) \}_{i \in I} \) is \( T_{\mu, L} \)-interpolable if and only if \( \{ (x, g, f) \}_{i \in I} \) is \( T_{0, L} \)-interpolable. \( \square \)

**Lemma A.3** ([12], Lemma 2). Consider a set \( \{ (x, g, f) \}_{i \in I} \). The following propositions are equivalent \( \forall L : 0 < L \leq + \infty \):

(i) \( \{ (x, g, f) \}_{i \in I} \) is \( T_{0, L} \)-interpolable,

(ii) \( \{ (g, x, (g, x)) \} \) is \( T_{0, L} \)-interpolable.

We proceed now to the proof of Theorem 3.1.

**Proof of Theorem 3.1.** We divide it in two parts: (i) proving the interpolation conditions and (ii) proving the characterization of the global minimum \( f, (3.3) \).

(i) Proof of the interpolation conditions. Taylor et. al. state in [12, Theorem 4] the interpolation conditions of smooth strongly-convex functions \( f \in T_{\mu, L} \), with \( \mu > 0 \). The inequalities from Lemma 2.5 are valid for any finite \( \mu \leq L \), therefore we have the same equivalences from [12, Theorem 4]:

(a) \( \{ (x, g, f) \}_{i \in I} \) is \( T_{\mu, L} \)-interpolable,

(b) \( \{ (x, g, f) \}_{i \in I} \) is \( T_{0, L} \)-interpolable,

(c) \( \{ (g, x, (g, x)) \} \) is \( T_{0, L} \)-interpolable.

(ii) Proof of the characterization of an optimal point. We follow the steps as in the proof of [5, Theorem 7] for smooth nonconvex functions. Consider the function
\[
 Z(y) := \min_{x \in X} \left\{ \frac{L}{2} \| y - \sum_{i \in I} \alpha_i \langle x_i - \frac{1}{L}(g_i - \mu x_i), y \rangle \|_2^2 + \sum_{i \in I} \alpha_i \langle f_i - \frac{\mu}{2} \| x_i \|^2 - \frac{1}{2L} \| g_i - \mu x_i \|^2 \rangle \right\}
\]
Lipschitz gradient and satisfies
\[ \Delta \] for the second inequality:
\[ \alpha \]

We proceed to define the function
\[ W \]

Using algebraic manipulations, we can express \( \hat{W} \)
\[ W \in \mathcal{F}_{\mu, L} \]

We proceed to define the function
\[ \hat{W}(y) := Z(y) + \frac{\mu}{2} \| y \|^2, \]

Using algebraic manipulations, we can express \( \hat{W} \) as
\[ \hat{W}(y) = \min_{\alpha \in \Delta^J} \left\{ \frac{\mu}{2} \| y - \sum_{i \in J} \alpha_i (x_i - \frac{1}{f_i} g_i) \|^2 + \frac{\mu}{2} \| \sum_{i \in J} \alpha_i (x_i - \frac{1}{f_i} g_i) \|^2 + \sum_{i \in J} \alpha_i \left( f_i - \frac{\mu}{2} \| g_i \|^2 - \frac{\mu}{2} \| x_i - \frac{1}{f_i} g_i \|^2 \right) \} \] (A.1)

Further, we lower bound the first squared norm by 0 and we use the convexity of the squared norm for the second inequality:
\[ \hat{W}(y) \geq \min_{\alpha \in \Delta^J} \left\{ \frac{\mu}{2} \| y \|^2 + \sum_{i \in J} \alpha_i \left( f_i - \frac{\mu}{2} \| g_i \|^2 - \frac{\mu}{2} \| x_i - \frac{1}{f_i} g_i \|^2 \right) \} \]
\[ \geq \min_{\alpha \in \Delta^J} \left\{ \sum_{i \in J} \alpha_i \left( f_i - \frac{\mu}{2} \| g_i \|^2 \right) \} \] (A.2)

For the upper bound, in (A.1) we take \( y := x_i - \frac{1}{f_i} g_i \), and \( \alpha = e_i \), the \( i \)-th unit vector:
\[ \hat{W}(x_i - \frac{1}{f_i} g_i) \leq \frac{\mu}{2} \| y \|^2 + f_i - \frac{\mu}{2} \| g_i \|^2 - \frac{\mu}{2} \| x_i - \frac{1}{f_i} g_i \|^2 = f_i - \frac{\mu}{2} \| g_i \|^2 \] (A.3)

Therefore, from (A.2) and (A.3) it follows that \( \hat{W}(x_i - \frac{1}{f_i} g_i) = f_i - \frac{\mu}{2} \| g_i \|^2 \), which ends the proof. □
Appendix B. Proof of the convergence rate

This section proves the central result of the paper, the convergence rate for smooth hypoconvex functions, i.e., Theorem 4.1.

Proof. We consider a feasible point \((i, x_i, \{(g_i, f_i)\}_{i \in I})\) of problem (3.5)\(^{9}\):

\[
\text{maximize} \quad l
\]

subject to

\[
\begin{aligned}
& f_i - f_j - (g_i, x_i - x_j) \geq \frac{1}{N_i} \left( \frac{1}{2} \|g_i - g_j\|^2 + \kappa L \|x_i - x_j\|^2 - 2\alpha (g_i - g_j, x_j - x_i) \right) \forall i, j \in I \\
x_{i+1} = x_i - \frac{1}{\kappa} h g_i, \quad i \in [0, \ldots, N - 1] \\
f_i - f - \frac{1}{\kappa} \|g\|^2 - f_c \geq 0, \quad i \in [0, \ldots, N] \\
f_i - f_0 - \Delta \geq 0
\end{aligned}
\]

and show that the right-hand-side of (4.1),

\[
U = \frac{2L\Delta}{1 + \sum_{i=0}^{N-1} \frac{2h_i - \frac{\alpha}{\kappa} \|g_i\|^2}{\min \left( \frac{2}{\kappa}, 1 \right)^{|i+1|}}}
\]

is an upper bound of \(l\).

The dual values were identified using the PESTO toolbox \(^{11}\) and only the active constraints were selected for the proof, i.e., the ones with non-zero multipliers. The following inequalities are active:

(i) Interpolability conditions – only for adjacent pairs \((i, i+1)\) and \((i+1, i)\)

1. \((i, i+1)\) with the dual multipliers \(\alpha_i\):

\[
f_i - f_{i+1} + \frac{h_i}{1 - \alpha_i} (g_i, g_{i+1}) - \frac{1}{2(1 - \alpha_i)} \|g_i - g_{i+1}\|^2 \geq 0
\]

2. \((i+1, i)\) with the dual multipliers \(\alpha_i = B\); not active for \(h_i \in (0, 1]\)

\[
f_{i+1} - f_i + \frac{h_{i+1}}{1 - \alpha_{i+1}} (g_{i+1}, g_i) - \frac{1}{2(1 - \alpha_{i+1})} \|g_{i+1} - g_i\|^2 \geq 0
\]

(ii) Optimality conditions – only for \(i = N\), with the dual multiplier \(B\)

\[
f_N - f_N - \frac{1}{\kappa} \|g_N\|^2 \geq 0
\]

(iii) Initial condition – with the dual multiplier \(B\)

\[
f_0 - f_0 + \Delta \geq 0
\]

(iv) Performance measure – with dual multipliers \(\sigma_i\)

\[
\|g_i\|^2 - l \geq 0, \quad 0 \leq i \leq N
\]

We found \(B := \frac{N}{\kappa}\). For \(h_i \in (0, 1]\), the dual multipliers are:

\[
\begin{align*}
\alpha_i &= B, \quad i = 0, \ldots, N - 1 \\
\sigma_0 &= \frac{h_0 B}{L} \left[ 1 - \frac{1 - h_0}{2(1 - \alpha_0)} \right] \\
\sigma_i &= \frac{h_i B}{L} \left[ 1 - \frac{1 - h_i}{2(1 - \alpha_i)} + \frac{h_i B}{L} \frac{1}{2(1 - \kappa)} \right], \quad i = 1, \ldots, N - 1 \\
\sigma_N &= 1 - \sum_{j=0}^{N-1} \sigma_j = \frac{B}{2L} + \frac{h_{i+1} B}{L} \frac{1}{2(1 - \kappa)} \\
\end{align*}
\]

For \(h_i \in [1, \tilde{h}(\kappa)]\), the non-negative dual multipliers are:

\[
\begin{align*}
\alpha_i &= B \frac{1 - h_i}{2(1 - \alpha_i \kappa)}, \quad i = 0, \ldots, N - 1 \\
\sigma_0 &= \frac{h_0 B}{L} \left[ 1 - \frac{1 - h_0}{2(1 - \alpha_0 \kappa)} \right] \\
\sigma_i &= \frac{h_i B}{L} \left[ 1 - \frac{1 - h_i}{2(1 - \alpha_i \kappa)} + \frac{h_i B}{L} \frac{1}{2(1 - \kappa \alpha_i \kappa)} \right], \quad i = 1, \ldots, N - 1
\end{align*}
\]
\[
\sigma_N = 1 - \sum_{i=0}^{N-1} \sigma_i = \frac{B}{2} + \frac{k_N - 1}{2} \frac{1}{[2(1+\kappa)k_{N-1}]} \tag{B.8}
\]

The non-negativity of the multipliers leads to the threshold \( \tilde{h}(\kappa) \). Because \( h_i > 0, 2 - (1 + \kappa)h_i > 0 \) (from \( \kappa \leq 0 \) and \( h_i < 2 \)) and \( B > 0 \), these three non-negativity conditions for \( h_i \in [1, \tilde{h}(\kappa)] \) can be rewritten as:

\[
\begin{align*}
2 - (1 + \kappa)h_i \frac{2\alpha_i}{2(1 + \kappa)^2} & = 1 - \kappa h_i \geq 0 \\
2 - (1 + \kappa)h_i \frac{2\alpha_i}{2(1 + \kappa)^2} & = \kappa h_i^2 - 2(1 + \kappa)h_0 + 3 \geq 0 \\
2 - (1 + \kappa)h_i \frac{2\alpha_i}{2(1 + \kappa)^2} & = \kappa h_i^2 - 2(1 + \kappa)h_i + 3 + \frac{h_0 - (1 + \kappa)h_i}{2(1 + \kappa)} \geq 0
\end{align*}
\]

The first inequality is valid because \( \kappa < 0 \). The fraction from the last inequality is positive, but can be arbitrarily small because of \( h_{i-1} \) and hence the worst-case to analyze is the same as for \( \sigma_0 \). Therefore, a sufficient condition is

\[\kappa h_i^2 - 2(1 + \kappa)h_i + 3 \geq 0,
\]

leading to the condition \( h \leq \tilde{h}(\kappa) = \frac{1 + 3 + \sqrt{1 + 3 + 4\kappa}}{2\kappa} \).

Following the idea of \([1, \text{Theorem 2}],\) showing \( I \leq U \) is equivalent with proving that adding non-negative terms to \( I - U \) keeps the difference non-positive. More specifically,

\[
I - U + \sum_{i=0}^{N-1} \sigma_i (\|g_i\|^2 - l) + B (f_i - f_0 + \Delta) + B (f_N - f_{N-1} - \frac{1}{\kappa}\|g_N\|^2)
\]

\[
+ \sum_{i=0}^{N-1} \alpha_i (f_i - f_{i+1} + \frac{h_i}{1 - \kappa^2} (g_{i+1}, g_{i+1}) - \frac{1}{2(1 + \kappa^2)} \|g_i - g_{i+1}\|^2 + \frac{h_i (1 - \kappa^2)}{2(1 + \kappa^2)} \|g_i\|^2)
\]

\[
+ \sum_{i=0}^{N-1} (\alpha_i - B) (f_{i+1} - f_i + \frac{h_i}{1 - \kappa^2} (g_{i+1}, g_{i+1}) - \frac{1}{2(1 + \kappa^2)} \|g_i - g_{i+1}\|^2 + \frac{h_i (1 - \kappa^2)}{2(1 + \kappa^2)} \|g_i\|^2) \leq 0
\]

The proof is based on basic algebraic manipulations. The main idea is to form squares in the left hand side of the inequality. First, we apply some straightforward simplifications, extract the term corresponding to \( i = N \) and group the terms with respect to the gradients.

\[
\sum_{i=0}^{N-1} \frac{1}{2} \left( - (1 + \kappa)\alpha_i + \kappa B \right) (g_{i+1}, g_{i+1}) - \frac{2\alpha_i - B}{\kappa} \|g_i - g_{i+1}\|^2 + \left( -\frac{(1 + \kappa)\alpha_i}{\kappa} + B + \frac{1 + \kappa^2}{\kappa} \sigma_i \right) \|g_i\|^2 + (1 - \kappa)(\sigma_N - B) \frac{1}{\kappa} \|g_N\|^2 \leq 0
\]

Further, we multiply the inequality by 2 and form the squares in every inner term of the sum:

\[
\frac{1}{\kappa} \left[ -2(1 + \kappa)\alpha_i - \kappa B \right] (g_{i+1}, g_{i+1}) - \frac{2\alpha_i - B}{\kappa} \|g_i - g_{i+1}\|^2 + \left[ -2\alpha_i - B \right] h_i + 2(1 + \kappa)\alpha_i - B + \frac{1 + \kappa^2}{\kappa} \sigma_i \|g_i\|^2
\]

\[
\leq \frac{1}{\kappa} \left[ -(2\alpha_i - B) h_i + 2(1 + \kappa)\alpha_i - B + \frac{2(1 + \kappa^2)}{\kappa} \sigma_i \right] (g_{i+1}, g_{i+1}) - \left[ (1 + \kappa)\alpha_i - B \right] \|g_i\|^2
\]

\[
\leq \frac{1}{\kappa} \left[ (1 + \kappa)\alpha_i - B \right] \|g_{i+1}\|^2
\]  \tag{B.9}
Then we replace the identity (B.9) in the inequality and adjust the indices of the gradients:

\[
\sum_{i=0}^{N-1} \left( -\frac{2\alpha_i - B}{\sigma} + \frac{\alpha_i}{\sigma} \right) \|g_i - g_{i+1}\|^2 + 2(1 - \alpha) \|\sigma_i - B\|_N^2 + \sum_{i=0}^{N-1} \frac{1}{\sigma} \left[ -(2\alpha_i - B)\sigma_i + 2[(1 + \alpha)\alpha_i - B] + \frac{2(1 + \alpha)\alpha_i}{\sigma} \right] - [(1 + \alpha)\alpha_i - B]\|g_i\|^2 - \sum_{i=1}^{N} \frac{h_i}{\sigma} [(1 + \alpha)\alpha_i - B]\|g_i\|^2 \leq 0
\]

For simplicity, we divide the inequality by \( \frac{\sigma}{\sigma_N} > 0 \) and define the scalars \( T_1, T_2, T_3 \) and \( T_4 \) that multiply the squared gradient norms. Showing that these scalars are non-positive will finish the proof.

\[
\sum_{i=0}^{N-1} \left( -\frac{2\alpha_i}{\sigma} + \frac{\alpha_i}{\sigma} \right) \|g_i - g_{i+1}\|^2 + \sum_{i=1}^{N} \frac{h_i}{\sigma} [(1 + \alpha)\alpha_i - B]\|g_i\|^2 \\
T_1 := \sum_{i=1}^{N} \frac{h_i}{\sigma} [(1 + \alpha)\alpha_i - B]\|g_i\|^2 + \sum_{i=1}^{N} \left[ -(2\alpha_i - B)\sigma_i + 2[(1 + \alpha)\alpha_i - B] + \frac{2(1 + \alpha)\alpha_i}{\sigma} \right] - [(1 + \alpha)\alpha_i - B]\|g_i\|^2 - \sum_{i=1}^{N} \frac{h_i}{\sigma} [(1 + \alpha)\alpha_i - B]\|g_i\|^2 \\
T_2 := \sum_{i=1}^{N} \left[ -(2\alpha_i - B)\sigma_i + 2[(1 + \alpha)\alpha_i - B] + \frac{2(1 + \alpha)\alpha_i}{\sigma} \right] - [(1 + \alpha)\alpha_i - B]\|g_i\|^2 \\
T_3 := \sum_{i=0}^{N-1} \left( -\frac{2\alpha_i - B}{\sigma} + \frac{\alpha_i}{\sigma} \right) \|g_i - g_{i+1}\|^2 + \sum_{i=1}^{N} \frac{h_i}{\sigma} [(1 + \alpha)\alpha_i - B]\|g_i\|^2 \\
T_4 := \sum_{i=0}^{N-1} \left( -\frac{2\alpha_i - B}{\sigma} + \frac{\alpha_i}{\sigma} \right) \|g_i - g_{i+1}\|^2
\]

For every coefficient \( T_i \) we study the two possible cases: \( 0 < h_i \leq 1 \) and \( 1 \leq h_i \leq \bar{h}(\alpha) \), respectively.

(i) \( T_1 = -\frac{2\alpha_i}{\sigma} + \frac{\alpha_i}{\sigma} \) and hence:

\[
T_1(h_i \leq 1) = -1 + h_i \leq 0
\]

1. For \( h_i \leq 1, \frac{\sigma_i}{\sigma_N} = 1 \), and hence:

\[
T_1(h_i \geq 1) = -\frac{2\sigma_i}{\sigma} - 1 + h_i (1 + \alpha)\frac{\sigma_i}{\sigma} - \frac{\alpha_i}{\sigma} = 0.
\]

Hence \( T_1 \leq 0, \forall h_i \in (0, \bar{h}(\alpha)) \).

(ii) \( T_2 = h_i \left[ -(2\alpha_i - B)\sigma_i + 2[(1 + \alpha)\alpha_i - B] + \frac{2(1 + \alpha)\alpha_i}{\sigma} \right] - [(1 + \alpha)\alpha_i - B]\|g_i\|^2 - h_{i-1} [(1 + \alpha)\alpha_i - B]\|g_i\|^2 \)

In the analysis of \( T_2 \), there are four possible cases due to the two choices of \( h_i \) and \( h_{i+1} \), i.e., \( h_{i+1} \leq 1 \). The identity (B.11) is derived by just replacing the expressions of \( \alpha_i \) and \( B \) for every possible case.

\[
D = \max \left\{ h_i + h_{i+1}, \frac{2h_i h_{i+1} - (1 + h_i h_{i+1})}{(2 - (1 + h_i h_{i+1}))} \right\}
\]

1. For \( h_i \leq 1, \frac{\sigma_i}{\sigma_N} = 1 \) and from (B.11) we get \( D = h_i + h_{i+1} \).

\[
T_2(h_i \leq 1) = h_i + h_{i+1} - h_i - h_{i+1} (1 + \alpha)\frac{\sigma_i}{\sigma} - \frac{\alpha_i}{\sigma} = h_i (1 + (1 + \alpha)\frac{\sigma_i}{\sigma}) - h_{i+1} (1 + \alpha)\frac{\sigma_i}{\sigma}.
\]

We next proceed to the two subcases:

(i) If \( h_{i+1} \leq 1, \frac{\sigma_{i+1}}{\sigma_N} = 1 \) and

\[
T_2(h_i \leq 1, h_{i+1} \leq 1) = 0
\]
(ii) If \( h_{l-1} \geq 1 \), then \( \frac{n_{l-1}}{B} = \frac{1 - \frac{1}{x_{h_{l-1}}}}{2 - (1 + \kappa) x_{h_{l-1}}} \). Then
\[
T_2(h_l \leq 1, h_{l-1} \geq 1) = h_{l-1}(1 + \kappa) \frac{1 - \frac{1}{x_{h_{l-1}}}}{2 - (1 + \kappa) x_{h_{l-1}}} \leq 0
\]

2. For \( h_l \geq 1 \), \( \frac{n_l}{B} = \frac{1 - x_{h_l}}{2 - (1 + \kappa) x_{h_l}} \) and from (B.11) we get \( D = \frac{2(1 - x_{h_l} h_{l-1} - (1 + \kappa) h_{l-1})}{(2 - (1 + \kappa) x_{h_l}) (2 - (1 + \kappa) h_{l-1})} \). After direct algebraic manipulations, one obtains
\[
T_2(h_l \geq 1) = h_{l-1} \left[ \frac{1 + \kappa}{2 - (1 + \kappa) x_{h_{l-1}}} - ((1 + \kappa) \frac{n_{l-1}}{B} - \kappa) \right]
\]
We next proceed to the two subcases:

(i) If \( h_{l-1} \leq 1 \), then \( \frac{n_{l-1}}{B} = 1 \), hence
\[
T_2(h_l \geq 1, h_{l-1} \leq 1) = (1 + \kappa) h_{l-1} \frac{1 - x_{h_l}}{2 - (1 + \kappa) x_{h_l}} \leq 0
\]

(ii) If \( h_{l-1} \geq 1 \), then \( \frac{n_{l-1}}{B} = \frac{1 - \frac{1}{x_{h_{l-1}}}}{2 - (1 + \kappa) x_{h_{l-1}}} \), and by direct computations we obtain
\[
T_2(h_l \geq 1, h_{l-1} \geq 1) = 0
\]

Hence, \( T_2 \leq 0 \), \( \forall h_{l} \in (0, \bar{h}(\kappa)] \).

(iii) \( T_3 = 2(1 - \kappa) \left[ \frac{\sigma_{h_{l-1}}}{B} - \frac{1}{2} \right] - h_{N-1} \left[ (1 + \kappa) \frac{n_{N-1}}{B} - \kappa \right] \)

1. \( h_{N-1} \leq 1 \): The following expressions for \( \sigma_{N} \) and \( \sigma_{N-1} \) state:
\[
\frac{2 \sigma_{N}}{B} = 1 + \frac{n_{N-1}}{1 + \kappa}, \quad \frac{n_{N-1}}{B} = 1
\]
By replacing them in the expression of \( T_3 \), we obtain
\[
T_3(h_{N-1} \leq 1) = 0
\]

2. \( h_{N-1} \geq 1 \): The following expressions for \( \sigma_{N} \) and \( \sigma_{N-1} \) state:
\[
\frac{2 \sigma_{N}}{B} = 1 + h_{N-1} \frac{1}{2 - (1 + \kappa) x_{h_{N-1}}}, \quad \frac{n_{N-1}}{B} = \frac{1 - \frac{1}{x_{h_{N-1}}}}{2 - (1 + \kappa) x_{h_{N-1}}}
\]
By replacing them in the expression of \( T_3 \) we get
\[
T_3(h_{N-1} \geq 1) = 0
\]

Hence, \( T_3 = 0 \), \( \forall h_{l} \in (0, \bar{h}(\kappa)] \).

(iv) \( T_4 = \left[ - (2 \frac{\sigma_{h_{l-1}}}{B} - 1) x_{h_{l-1}} + 2 \left[ (1 + \kappa) \frac{n_{h_{l-1}}}{B} - 1 \right] + \frac{2 \sigma_{h_{l-1}}}{B} - \sigma_{h_{l-1}} \right] - \left[ (1 + \kappa) \frac{n_{h_{l-1}}}{B} - \kappa \right] \)

1. For \( h_l \leq 1 \), the following expressions for \( \sigma_{h_{l}} \) and \( \sigma_{h_{l}} \) state:
\[
\frac{\sigma_{h_{l}}}{B} = 1 - \frac{1 - \frac{1}{x_{h_{l}}}}{2 - (1 + \kappa) x_{h_{l}}}, \quad \frac{n_{h_{l}}}{B} = 1
\]
We replace them in the expression of \( T_4 \) and obtain
\[
T_4(h_l \leq 1) = 0
\]

2. For \( h_l \geq 1 \), the following expressions for \( \sigma_{h_{l}} \) and \( \sigma_{h_{l}} \) state:
\[
\frac{\sigma_{h_{l}}}{B} = 1 - \frac{1 - \frac{1}{x_{h_{l}}}}{2 - (1 + \kappa) x_{h_{l}}}, \quad \frac{n_{h_{l}}}{B} = \frac{1 - \frac{1}{x_{h_{l}}}}{2 - (1 + \kappa) x_{h_{l}}}
\]
We replace them in the expression of \( T_4 \) and after direct computations obtain:
\[
T_4(h_l \geq 1) = 0
\]

Hence, \( T_4 = 0 \), \( \forall h_{l} \in (0, \bar{h}(\kappa)] \).

To conclude, all terms from (B.10) are non-positive:

- \( T_1 \) if \( 1 \leq h_l \leq \frac{1 + \kappa \nu - \kappa \nu}{1 + \kappa \nu} \)
- \( T_2 \) if \( 1 - h_l (1 - h_{l-1}) \geq 0 \)
- \( T_3 = 0 \) otherwise
- \( T_4 = 0 \)
• $T_A = 0.$

Therefore, (B.10) is proved through a series of equivalent inequalities, hence $l \leq U.$ 

□