Energy and Mass Generation

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Abstract

Modifications in the energy momentum dispersion laws due to a noncommutative geometry, have been considered in recent years. We examine the oscillations of extended objects in this perspective and find that there is now a "generation" of energy.

1 Introduction

Many modern approaches, particularly in Quantum Gravity introduce a minimum spacetime scale. It is well known that the introduction of such a fundamental minimum length in the universe leads to a noncommutative geometry. On the other hand there has been much discussion about how this could avert the infinities which plague Quantum Field Theory (Cf.ref.[1] and several other references therein). As pointed out by Snyder many years ago [2], there are now new commutation relations which replace the usual Quantum Mechanical relations. These are

\[ [x, y] = (\hbar^2/\hbar) L_z, \quad [t, x] = (\hbar^2/\hbar c) M_x, \text{ etc.} \]

\[ [x, p_x] = \hbar [1 + (l/\hbar)^2 p_x^2]; \tag{1} \]

where \( a \) is the fundamental length.

It was noted by the author that these new commutation relations lead to a modified energy momentum relation viz., [1 3 4]

\[ E^2 = p^2 + m^2 + \alpha l^2 p^4 \tag{2} \]
giving the so called Snyder-Sidharth Hamiltonian \[5\], \(\alpha\) being a scalar, and \(l\) the fundamental length, in units \(c = 1 = \hbar\). This leads to a modification of the usual Klein-Gordon equation, which now becomes,

\[
(D + l^2 \nabla^4 - m^2)\psi = 0
\]  

(3)

where \(D\) denotes the usual D’Alembertian.

It is well known that the usual Klein-Gordon equation, without the extra term in \[2\] describes a super position of normal mode harmonic oscillators \[6\]

Let us first start with the Klein-Gordon equation itself

\[
(D + m^2)\psi(x) = 0
\]  

(4)

Following a well known procedure, a Fourier integral over elementary plane waves is then taken to give

\[
\psi(x, t) = \int \frac{d^3 k}{\sqrt{(2\pi)^3 2\omega_k}} \left[ a(k)e^{ik.x - \omega_k t} + a^\dagger(k)e^{-ik.x + \omega_k t} \right]
\]

\[
= \int d^3 k \left[ a(k)f_k(x) + a^\dagger(k)f^*_k(x) \right]
\]  

(5)

where

\[
\omega_k = \pm \sqrt{k^2 + m^2} \quad \text{and} \quad f_k(x) = \frac{1}{\sqrt{(2\pi)^3 2\omega_k}} e^{-ik.x}
\]

An inversion then gives

\[
\int f^*_k(x, t)\psi(x, t)d^3 x = \frac{1}{2\omega_k} \left[ a(k) + a^\dagger(-k)e^{2\omega_k t} \right]
\]

\[
\int f^*_k(x, t)\dot{\psi}(x, t)d^3 x = -\frac{i}{2} \left[ a(k) - a^\dagger(-k)e^{2\omega_k t} \right]
\]  

(6)

Finally the canonical Quantum Mechanical commutation relations for the wave function (now operator) \(\phi\) and the associated momentum leads to (Cf.ref.\[6\] for details)

\[
[a(k), a^\dagger(k')] = \int d^3 x d^3 y [f^*_k(x, t)\partial_0 \psi(x, t), f_k(y, t)\partial_0 \psi(y, t)]
\]

\[
= +i \int d^3 x f^*_k(x, t)\partial_0 f_k(y, t) = \delta^3(k - k')
\]
Similarly

\[ [a(k), a(k')] = (i)^2 \int d^3x d^3y [f_k^*(x, t) \partial_0 \psi(x, t), f_{k'}^*(x, t) = 0 \]

and

\[ [a^\dagger(k), a^\dagger(k')] = 0 \tag{7} \]

This finally gives the Hamiltonian in terms of the creation and annihilation operators \( a \) and \( a^\dagger \) viz.,

\[ H = 1/2 \int d^3k \omega_k [a^\dagger(k)a(k) + a(k)a^\dagger(k)] \tag{8} \]

Discretizing the above, replacing the integrals in momentum space by summations and the Dirac-Delta function by the Kronecker Delta, we get, as is well known,

\[ H = \sum_k H_k = \sum_k 1/2 \omega_k (a_k^\dagger a_k + a_k a_k^\dagger) \quad a_k = \sqrt{\Delta V_k} a(k) \tag{9} \]

with

\[ [a_k, a_{k'}^\dagger] = \delta_{kk'} \quad [a_k, a_{k'}] = [a_k^\dagger, a_{k'}^\dagger] = 0 \tag{10} \]

In this second quantized picture, these lead to the various particle solutions given by

\[ H_k \Phi_k(n_k) = \omega_k(n_k + 1/2) \Phi_k(n_k) \tag{11} \]

2 The New Scenario

Let us see, how all this gets modified due to the new considerations and in particular the new commutation relations \(1\). We first observe that if instead of the usual energy momentum relation, we use \(2\) above, then this would lead to a modification of the frequency given in \(5\). That is we would have

\[ \omega_k = \sqrt{k^2 + m^2 + \alpha l^2 k^4} \equiv \omega_k' \tag{12} \]

(Remembering that we are in natural units). This would mean that there would be a shift in the energy of the individual oscillators (Cf. also \(7, 8\)). But also, there are other important changes.
To see this, let us consider the above in greater detail. As is well known for the individual Quantum Harmonic oscillators in (9) or (11) we have [6]

\[ a = \lambda_1 q + i\lambda_2 p \]
\[ a^\dagger = \lambda_1 q - i\lambda_2 p \] (13)

where \( \lambda_1 = \left( \frac{m\omega}{2} \right)^{\frac{1}{2}} \) and \( \lambda_2 = \left( \frac{1}{2m\omega} \right)^{\frac{1}{2}} \), \( q \) replaces \( x \) and we have dropped the subscript \( k \) for \( \omega \). This leads to, in the old case the Hamiltonian

\[ H_0 = \omega(a^\dagger a + \frac{1}{2}) \] (14)

In our case however there is the additional term in (2). To see this in greater detail we note that owing to the commutation relations (1) we have

\[ [q, p] = i(1 + l^2p^2) \] (15)

and so,

\[ [a, a^\dagger] = (1 + l^2p^2) \] (16)

Whence we have

\[ H = \frac{m\omega}{2} [aa^\dagger + a^\dagger a] \]
\[ = \frac{m\omega}{2} [2a^\dagger a + (1 + l^2p^2)] \]
\[ = m\omega[a^\dagger a + \frac{1}{2}(1 + l^2p^2)] \] (17)

Finally we have

\[ H = m\omega[(a^\dagger a + \frac{1}{2}) + \frac{l^2}{2}p^2] \]
\[ = H'_0 + \frac{m\omega}{2}l^2p^2 \] (18)

where \( H'_0 \) is given by

\[ H'_0 \equiv m\omega(a^\dagger a + \frac{1}{2}) \] (19)

that is, it would be the Hamiltonian if \( l^2 \) were 0. Using (13) it is easy to see that \( p^2 \) is given by

\[ p^2 = \left( \frac{m\omega}{2} \right)(a - a^\dagger)^2 \] (20)
Using (20) in (18), we get finally
\[ H = H'_0 + l^2 \frac{m \omega}{2} - \left( \frac{m \omega}{2} l \right)^2 (a^2 + a^\dagger) \]  
(21)

Equation (21) for the Quantum Mechanical Harmonic Oscillator already shows the effect of the extra term in the SS-Hamiltonian (2). This is an additional energy that appears due to the commutation relations (1).

But it also shows that apart from the shift in energy (or mass) for the individual oscillators, reflecting (2), the eigen states of the oscillators get mixed up due to the presence of the squares of the creation and annihilation operators. The eigen states would be combinations of states like \( \Phi(n - 2), \Phi(n + 2) \) in addition to \( \Phi(n) \), of the usual theory as in (11). In other words we have to consider a system of oscillators. This is due to the fact that a point is being replaced by an extension.

Indeed in String Theory itself, as we know we have a similar situation [1] and [9]-[12]. We have
\[ \rho \dddot{y} - T \dot{y}'' = 0, \]  
(22)
\[ \omega = \frac{\pi}{2l} \sqrt{\frac{T}{\rho}}, \]  
(23)
\[ T = \frac{mc^2}{l}; \quad \rho = \frac{m}{l}, \]  
(24)
\[ \sqrt{T/\rho} = c, \]  
(25)

\( T \) being the tension of the string, \( l \) its length and \( \rho \) the line density and \( \omega \) in (23) the frequency. The identification (23), (24) gives (25), where \( c \) is the velocity of light, and (22) then goes over to the usual d’Alembertian or massless Klein-Gordon equation and an expansion in normal modes (which are uncoupled).

Further, if the above string is quantized canonically, we get
\[ \langle \Delta x^2 \rangle \sim l^2. \]  
(26)

Thus the string can be considered as an infinite collection of harmonic oscillators [10]. Further we can see, using equations (23) and (24) and the fact that
\[ \hbar \omega = mc^2 \]
and that the extension \( l \) is of the order of the Compton wavelength in (26), a circumstance that was called one of the miracles of the string theory by
Veneziano [13, 14, 15, 16, 17].
The point is that a non zero extension for the system of oscillators would imply the commutation relations (1). This again implies an extra energy or mass on the one hand, and a coupled system of oscillators on the other, rather than a superposition of normal modes as we would otherwise have. Thus a consideration of a system of coherent oscillators would be a better starting point, if we consider the Planck scale, as in string theory and other approaches, rather than spacetime points of the usual theory. This circumstance has been overlooked.

In summary there is now an extra energy or mass that appears due to (1) and (2), on the one hand. On the other, we have to consider a system of (coupled) oscillators.

3 A system of Oscillators

We know that String Theory, Loop Quantum Gravity, the author’s own approach of Planck oscillators, and a few other approaches start from the Planck scale.

We will consider the problem from a different point of view, which also enables an elegant extension to the case of the entire Universe itself [18, 19, 20, 21]. Let us consider an array of $N$ particles, spaced a distance $\Delta x$ apart, which behave like oscillators that are connected by springs. This is because of the results in the last section that we have to consider coupled oscillators. The starting point for such a programme can be found in [22]. We then have

$$ r = \sqrt{N\Delta x^2} $$

(27)

$$ ka^2 \equiv k\Delta x^2 = \frac{1}{2}k_BT $$

(28)

where $k_B$ is the Boltzmann constant, $T$ the temperature, $r$ the extent and $k$ is the spring constant given by

$$ \omega_0^2 = \frac{k}{m} $$

(29)

$$ \omega = \left( \frac{k}{m a^2} \right)^{\frac{1}{2}} \frac{1}{r} = \omega_0 \frac{a}{r} $$

(30)
We now identify the particles with Planck masses and set $\Delta x \equiv a = l_P$, the Planck length. It may be immediately observed that use of (29) and (28) gives $k_B T \sim m_P c^2$, which of course agrees with the temperature of a black hole of Planck mass. Indeed, Rosen [27] had shown that a Planck mass particle at the Planck scale can be considered to be a Universe in itself a Schwarzschild Black Hole of radius equalling the Planck length. We also use the fact deduced earlier to that a typical elementary particle like the pion can be considered to be the result of $n \sim 10^{40}$ Planck masses. Using this in (27), we get $r \sim l$, the pion Compton wavelength as required. Whence the pion mass is given by

$$m = m_P / \sqrt{n}$$

Further, in this latter case, using (27) and the fact that $N = n \sim 10^{40}$, and (28), i.e. $k_B T = kl^2 / N$ and (29) and (30), we get for a pion, remembering that $m_p^2 / n = m^2$,

$$k_B T = \frac{m^3 c A l^2}{\hbar^2} = mc^2,$$

which of course is the well known formula for the Hagedorn temperature for elementary particles like pions [28]. In other words, this confirms the earlier conclusions that we can treat an elementary particle as a series of some $10^{40}$ Planck mass oscillators.

However it must be observed from (30) and (29), that while the Planck mass gives the highest energy state, an elementary particle like the pion is in the lowest energy state. This explains why we encounter elementary particles, rather than Planck mass particles in nature. Infact as already noted [20], a Planck mass particle decays via the Bekenstein radiation within a Planck time $\sim 10^{-42} secs$. On the other hand, the lifetime of an elementary particle would be very much higher.

In any case the efficacy of our above oscillator model can be seen by the fact that we recover correctly the masses and Compton scales in the order of magnitude sense and also get the correct Bekenstein and Hagedorn formulas as seen above, and further we even get the correct estimate of the mass and size of the Universe itself, as will be seen below.

Using the fact that the Universe consists of $N \sim 10^{80}$ elementary particles like the pions, the question is, can we think of the Universe as a collection of $nN$ or $10^{120}$ Planck mass oscillators? This is what we will now show. Infact if we use equation (27) with $\bar{N} \sim 10^{120}$,

$$\bar{N} \sim 10^{120},$$
we can see that the extent is given by \( r \sim 10^{28} \text{cms} \) which is of the order of the diameter of the Universe itself. We shall shortly justify the value for \( \bar{N} \).

Next using (30) we get

\[
\hbar \omega_0^{(\text{min})} \left( \frac{l_P}{10^{28}} \right)^{-1} \approx m_P c^2 \times 10^{60} \approx M c^2
\]

which gives the correct mass \( M \), of the Universe which in contrast to the earlier pion case, is the highest energy state while the Planck oscillators individually are this time the lowest in this description. In other words the Universe itself can be considered to be described in terms of normal modes of Planck scale oscillators.

More generally, if an arbitrary mass \( M \), as in (31), is given in terms of \( \bar{N} \) Planck oscillators, in the above model, then we have from (31) and (27):

\[
M = \sqrt{\bar{N} m_P} \quad \text{and} \quad R = \sqrt{\bar{N} l_P},
\]

where \( R \) is the radius of the object. Using the fact that \( l_P \) is the Schwarzchild radius of the mass \( m_P \), this gives immediately,

\[
R = 2GM/c^2
\]

a relation that has been deduced alternatively. In other words, such an object, the Universe included as a special case, shows up as a Black Hole, a super massive Black Hole in this case.

4 Discussion

The interesting question is, from where does the extra energy come? We must remember that in the above considerations we have an a priori dark energy background, and it is this energy that manifests itself as the extra energy or mass or frequency.

This can be illustrated by considering the background electromagnetic field, or the Zero Point Field as a collection of ground state oscillators (Cf.ref.[1] and references therein). It is known that the probability amplitude is

\[
\psi(x) = \left( \frac{m\omega}{\pi\hbar} \right)^{1/4} e^{-\left(m\omega/2\hbar\right)x^2}
\]
for displacement by the distance \( x \) from its position of classical equilibrium. So the oscillator fluctuates over an interval

\[
\Delta x \sim (\hbar/m\omega)^{1/2}
\]

The background electromagnetic field is an infinite collection of independent oscillators, with amplitudes \( X_1, X_2 \) etc. The probability for the various oscillators to have amplitudes \( X_1, X_2 \) and so on is the product of individual oscillator amplitudes:

\[
\psi(X_1, X_2, \cdots) = \exp[-(X_1^2 + X_2^2 + \cdots)]
\]

wherein there would be a suitable normalization factor. This expression gives the probability amplitude \( \psi \) for a configuration \( B(x, y, z) \) of the magnetic field that is described by the Fourier coefficients \( X_1, X_2, \cdots \) or directly in terms of the magnetic field configuration itself by, as we saw,

\[
\psi(B(x, y, z)) = P \exp \left( -\int \int \frac{B(x_1) \cdot B(x_2)}{16\pi^3 h c r_{12}^3} d^3 x_1 d^3 x_2 \right).
\]

\( P \) being a normalization factor. At this stage, we are thinking in terms of energy without differentiation, that is, without considering Electromagnetism or Gravitation etc as separate. Let us consider a configuration where the magnetic field is everywhere zero except in a region of dimension \( l \), where it is of the order of \( \sim \Delta B \). The probability amplitude for this configuration would be proportional to

\[
\exp[-((\Delta B)^2 l^4 / \hbar c)]
\]

So the energy of fluctuation in a region of length \( l \) is given by finally, the density [29][19]

\[
B^2 \sim \frac{\hbar c}{l^4}
\]

So the energy content in a region of volume \( l^3 \) is given by

\[
\beta^2 \sim \frac{\hbar c}{l}
\]

Equation (32) can be written as

\[
\text{Energy} \sim m_e c^2 \sim \frac{\hbar c}{l}
\]
Equation (33) shows that $l$ is the Compton wavelength of an elementary particle, which thus arises naturally. On the other hand if in (33), we consider the energy of a Planck mass rather than that of an elementary particle, we will get the Planck length.

For another perspective, it is interesting to derive a model based on the theory of phonons which are quanta of sound waves in a macroscopic body [30]. Phonons are a mathematical analogue of the quanta of the electromagnetic field, which are the photons, that emerge when this field is expressed as a sum of Harmonic oscillators. This situation is carried over to the theory of solids which are made up of atoms that are arranged in a crystal lattice and can be approximated by a sum of Harmonic oscillators representing the normal modes of lattice oscillations. In this theory, as is well known the phonons have a maximum frequency $\omega_m$ which is given by

$$\omega_m = c \left( \frac{6\pi^2}{v} \right)^{1/3}$$  \hspace{1cm} (34)

In (34) $c$ represents the velocity of sound in the specific case of photons, while $v = V/N$, where $V$ denotes the volume and $N$ the number of atoms. In this model we write

$$l \equiv \left( \frac{4}{3}\pi v \right)^{1/3}$$

$l$ being the inter particle distance. Thus (34) now becomes

$$\omega_m = c/l$$  \hspace{1cm} (35)

Let us now liberate the above analysis from the immediate scenario of atoms at lattice points and quantized sound waves due to the Harmonic oscillations and look upon it as a general set of Harmonic oscillators as above. Then we can see that (35) and (32) are identical as

$$\omega = \frac{mc^2}{\hbar}$$  \hspace{1cm} (36)

Using (36), we can once again recover both the Planck length and an elementary particle Compton wavelength. On the other hand it has been shown that starting with the background Zero Point Field the Quantum Mechanical commutation relations in $JXJ$ yield the Quantum Mechanical spin at the Compton wavelength [11]. It is to be noted that this Quantum Mechanical
spin feature is absent at the Planck length.
We finally comment the following. If in the considerations of Section 2 we take the particle to have negligible mass or vanishing mass as in the case of radiation, then this new physics as embodied in (18) leads to an increased frequency of the radiation, or an apparent increase in its velocity leading to different wavelengths travelling with different speeds. The effect is very minute and can be observed only for very high frequency radiation like Gamma Rays from Gamma Ray Bursts. Already there are claims that such lags in arrival times of the Gamma Rays have indeed been found [31].

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