The bi-Hamiltonian structure and new solutions of KdV6 equation

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Abstract. We show that the KdV6 equation recently studied in [1,2] is equivalent to the Rosochatius deformation of KdV equation with self-consistent sources (RD-KdVESCS) recently presented in [9]. The $t$-type bi-Hamiltonian formalism of KdV6 equation (RD-KdVESCS) is constructed by taking $x$ as evolution parameter. Some new solutions of KdV6 equation, such as soliton, positon and negaton solution, are presented.

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Key words: KdV6 equation, Rosochatious deformation of KdV equation with self-consistent source, bi-Hamiltonian structure, positon solution, negaton solution.

1 Introduction

Recently, the 5 authors of [1] applied the Painlevé analysis to the class of sixth-order nonlinear wave equations, and found 4 cases that pass the Painlevé test. Three of those cases correspond to previously known integrable equations, whereas the fourth one turns out to be new:

$$\left( \partial_x^2 + 8u_x \partial_x + 4u_{xx} \right) \left( u_t + u_{xxx} + 6u_x^2 \right) = 0. \quad (1)$$

This equation, as it stands, does not belong to any recognizable theory. In the variables $v = u_x, w = u_t + u_{xxx} + 6u_x^2, (1)$ is converted to

$$v_t + v_{xxx} + 12vv_x - w_x = 0, \quad (2a)$$
$$w_{xxx} + 8vw_x + 4wv_x = 0, \quad (2b)$$

which is referred as KdV6 equation in [1] and regarded as a nonholonomic deformation of the KdV equation. The authors of [1] found Lax pair and an auto-Bäcklund transformation for (2). They claimed that (2) is different from the KdV equation with self-consistent sources (KdVESCS) and reported that they were unable to find higher symmetries and asked if higher conserved densities and a Hamiltonian formalism exist for (2).

In [2], Kupershmidt described (2) as a nonholonomic perturbations of bi-Hamiltonian systems. By rescaling $v$ and $t$ in (2), one gets

$$u_t = 6uu_x + u_{xxx} - w_x, \quad (3a)$$
$$w_{xxx} + 4uw_x + 2wu_x = 0, \quad (3b)$$

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which can be converted into a nonholonomic perturbations of bi-Hamiltonian systems [2]

\[ u_t = B^1\left(\frac{\delta H_{n+1}}{\delta u}\right) - B^1(w) = B^2\left(\frac{\delta H_n}{\delta u}\right) - B^1(w), \]
\[ B^2(w) = 0, \]

where

\[ B^1 = \partial = \partial_x, \quad B^2 = \partial^3 + 2(u\partial + \partial u) \]

are the two standard Hamiltonian operators of the KdV hierarchy, \( n = 2 \), and

\[ H_1 = u, \quad H_2 = \frac{u^2}{2}, \quad H_3 = \frac{u^3}{3} - \frac{u_x^2}{2}, \ldots \]

Then the author in [2] believed that he could prove the integrability of KdV6 equation by constructing the infinite commuting hierarchy KdV_\(n\) [4] with a common infinite set of conserved densities. Some solutions for [2] were obtained in [1, 3].

The soliton equations with self-consistent sources (SESCS) have attracted much attention (see [4]-[7]) and have important physical applications, for example, the KdV equation with self-consistent sources (KdVESCS) describes the interaction of long and short capillary-gravity waves [4]. The Rosochatius deformation of finite-dimensional integrable Hamiltonian system (FDIHS) also has important physical application, for example, the Garner-Rosochatius system can be used to solve the multicomponent coupled nonlinear Schrödinger equation [8]. We generalized the Rosochatius deformation from FDIHS to SESCS and presented many Rosochatius deformations of SESCS (RD-SESCS) in [9], such as RD-KdVESCS which stationary reduction gives rise to the well-known generalized Henon-Heiles system [10].

In this paper, we would like to answer the questions mentioned in [1]. We will first show that (3) is equivalent to the Rosochatius deformation of KdVESCS (RD-KdVESCS) presented in [9]. It is known [11, 12] that some soliton equations have both \( x \)- and \( t \)-type Hamiltonian formulation. However the Hamiltonian formulation for KdV6 equation (RD-KdVESCS) can not be written in usual way. We will formulate it as an infinite-dimensional integrable bi-Hamiltonian system with a \( t \)-type Hamiltonian operator by taking \( t \) as the 'spatial' variable and \( x \) as the evolution parameter as in the case of KdVESCS [13,14]. Since the KdV6 equation can be regarded as the KdV equation with no-homogeneous term and \( w \) is related to the square of eigenfunction, we may apply the method of variant of constant to find some new solutions of KdV6 equation starting from the known solutions of KdV equation.

The present paper is organized as follows. We will first convert the KdV6 equation into RD-KdVESCS, and present extension of KdV6 equation in section 2. In section 3, we will describe RD-KdVESCS (KdV6 equation) and RD-mKdVESCS as a \( t \)-type Hamiltonian system by taking \( x \) as the evolution parameter, respectively. Then following the procedure given in [11]-[14] by means of the \( t \)-type Miura transformation relating these two Hamiltonian systems, we will construct the second \( t \)-type Hamiltonian structure for KdV6 equation (RD-KdVESCS) from the first Hamiltonian structure of RD-mKdVESCS,
and present infinite chain of local commuting vector fields for KdV6 equation. Finally in section 4, starting from the solutions of KdV, we obtain many new solutions of KdV6 equation, such as soliton, positon and negaton solution.

## 2 KdV6 equation is equivalent to RD-KdV ESCS

By rescaling \( u \) and \( t \) and using the Galilean invariance of KdV equation, KdV6 equation (3) can be rewritten as

\[
\begin{align*}
    u_t &= \frac{1}{4} (u_{xxx} + 6uu_x) - w_x, \quad (6a) \\
    w_{xxx} + 4(u - \lambda_1)w_x + 2uw_x &= 0 \quad (6b)
\end{align*}
\]

where \( \lambda_1 \) is a parameter.

Set

\[ w = \varphi^2, \]

then (6b) yields

\[
    w_{xxx} + 4(u - \lambda_1)w_x + 2uw_x = 2\varphi[\varphi_{xx} + (u - \lambda_1)\varphi]_x + 6\varphi_x[\varphi_{xx} + (u - \lambda_1)\varphi] = 0,
\]

which immediately gives rise to

\[
    \varphi_{xx} + (u - \lambda_1)\varphi = \frac{\mu}{\varphi^3},
\]

where \( \mu \) is an integrable constant. So KdV6 equation (6) is equivalent to

\[
\begin{align*}
    u_t &= \frac{1}{4} (u_{xxx} + 6uu_x) - (\varphi^2)_x, \quad (8a) \\
    \varphi_{xx} + (u - \lambda_1)\varphi &= \frac{\mu}{\varphi^3}, \quad (8b)
\end{align*}
\]

which is just the RD-KdV ESCS presented in [9]. The Lax pair for (8) reads [9]

\[
\begin{align*}
    \left( \begin{array}{c} \psi_1 \\ \psi_2 \end{array} \right)_x &= U \left( \begin{array}{c} \psi_1 \\ \psi_2 \end{array} \right), \quad U = \left( \begin{array}{cc} 0 & 1 \\ \lambda - u & 0 \end{array} \right) \quad (9a) \\
    \left( \begin{array}{c} \psi_1 \\ \psi_2 \end{array} \right)_t &= N \left( \begin{array}{c} \psi_1 \\ \psi_2 \end{array} \right), \quad N = \left( \begin{array}{cc} \lambda^2 - \frac{u}{2} \lambda - \frac{u_x}{4} - \frac{v^2}{2} + \frac{1}{2} \varphi^2 & \frac{u}{4} \\ -\frac{u}{4} & \frac{1}{2} \lambda - \lambda_1 \end{array} \right) - \frac{1}{2} \lambda - \lambda_1 \left( \begin{array}{cc} \varphi_x & -\varphi^2 \\ \varphi^2 & \frac{u}{\varphi} - \varphi_x \end{array} \right). \quad (9b)
\end{align*}
\]

More generally, the multi-component extension of KdV6 equation is given by

\[
\begin{align*}
    u_t &= \frac{1}{4} (u_{xxx} + 6uu_x) - \sum_{j=1}^{N} w_{jx}, \quad (10a) \\
    w_{jxx} + 4(u - \lambda_j)w_{jx} + 2uw_j &= 0, \quad j = 1, 2, \cdots, N. \quad (10b)
\end{align*}
\]
Under the transformation \( w_j = \varphi_j^2 \), (10) can be converted into the following RD-KdVESCS

\[
\begin{align*}
  u_t &= \frac{1}{4}(u_{xxx} + 6uu_x) - \sum_{j=1}^{N}(\varphi_j^2)_x, \\
  \varphi_{jxx} + (u - \lambda_j)\varphi_j &= \frac{\mu_j}{\varphi_j}, \quad j = 1, 2, \ldots, N.
\end{align*}
\] (11a)

The Lax pair for (11) is given by (9a) with

\[
N = \left( \begin{array}{cc}
-\lambda^2 - \frac{u}{2}\lambda - \frac{u_{xx}}{4} - \frac{u^2}{4} + \frac{1}{2} \sum_{j=1}^{N} \varphi_j^2 & -\lambda + \frac{u}{2} \\
-\frac{u}{4} & -\frac{u}{4} + \frac{1}{2} \sum_{j=1}^{N} \frac{1}{\lambda - \lambda_j} \left( \varphi_j^2 - \varphi_j^3 + \frac{\mu_j}{\varphi_j} - \varphi_j^2 \varphi_{jx} \right)
\end{array} \right).
\] (12)

### 3 Bi-Hamiltonian structure of KdV6 equation

In this section, we will follow the method in [11]-[14] to construct the bi-Hamiltonian formalism for KdV6 equation (RD-KdVESCS). First we will present the \( t \)-type Hamiltonian formalism for RD-KdVESCS and RD-mKdVESCS. For the RD-KdVESCS \( \mathcal{S} \), set

\[
\frac{1}{8} u_{xx} + \frac{3}{4} u^2 - \varphi^2 = c, \quad q_t = c_x,
\]

\( q = u, \ p = \frac{1}{8} u_x, \ Q = \varphi, \ P = \varphi_x, \ R = (Q,q,P,p,c)^T, \) (13)

then \( \mathcal{S} \) becomes \( x \)-evolution equations and can be written as a \( t \)-type Hamiltonian system

\[
R_x = \left( \begin{array}{cc}
P \\
-8p \\
(\lambda_1 - q)Q + \frac{\mu}{Q} \\
\frac{3}{8}q^2 - \frac{1}{2}Q^2 - \frac{1}{2}c \\
q_t
\end{array} \right) = K_1 = \Pi_0 \nabla H_1,
\] (14a)

where \( \nabla \) means variational derivative, \( \nabla H = (\frac{\delta H}{\delta Q}, \frac{\delta H}{\delta q}, \frac{\delta H}{\delta P}, \frac{\delta H}{\delta p}, \frac{\delta H}{\delta c})^T \), and the \( t \)-type Poisson operator \( \Pi_0 \) and conserved density \( H_1 \) are given by

\[
\Pi_0 = \left( \begin{array}{cccc}
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
-1 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 2q_t
\end{array} \right),
\] (14b)

\[
H_1 = \frac{1}{2} P^2 - 4p^2 - \frac{1}{2} \lambda_1 Q^2 + \frac{1}{2} q Q^2 - \frac{1}{8} q^3 + \frac{1}{2} cq + \frac{1}{2} \mu.
\] (14c)
The Rosochatius deformation of mKdV equation with self-consistent source (RD-mKdVESCS)
is defined as [9]
\begin{equation}
v_t = \frac{1}{4}(v_{xxx} - 6v_x^2v_x) + \frac{1}{2}(\Phi_1 \Phi_2)_x, \quad (15a)
\end{equation}
\begin{equation}
\Phi_{1x} = v \Phi_1 + \lambda_1 \Phi_2, \quad \Phi_{2x} = \Phi_1 - v \Phi_2 + \frac{\mu}{\lambda_1 \Phi_1^3}.
\end{equation}

Let
\begin{equation}
\frac{1}{4}(v_{xx} - 2v^3) + \frac{1}{2} \varphi_1 \varphi_2 = -\bar{c}, \quad v_t = -\bar{c},
\end{equation}
\begin{equation}
\bar{q} = v, \quad \bar{p} = \frac{1}{2}v_x, \quad \bar{Q} = \varphi_1, \quad \bar{P} = \varphi_2, \quad \bar{R} = (\bar{Q}, \bar{q}, \bar{P}, \bar{p}, \bar{c})^T,
\end{equation}
then RD-mKdVESCS [15] can be written as a \( t \)-type Hamiltonian system
\begin{equation}
\bar{R}_x = \begin{pmatrix}
\bar{q}\bar{Q} + \lambda_1 \bar{P} \\
\bar{Q} - \bar{q}\bar{P} + \frac{\mu}{\lambda_1 \bar{Q}^3} \\
\bar{q}^3 - \bar{Q}\bar{P} - 2\bar{c} \\
-\bar{c}_t
\end{pmatrix} = \bar{K}_1 = \bar{\Pi}_0 \nabla \bar{H}_1,
\end{equation}
where \( \nabla \bar{H} = \left( \frac{\delta \bar{H}}{\delta \bar{Q}}, \frac{\delta \bar{H}}{\delta \bar{q}}, \frac{\delta \bar{H}}{\delta \bar{P}}, \frac{\delta \bar{H}}{\delta \bar{p}}, \frac{\delta \bar{H}}{\delta \bar{c}} \right)^T \), and the \( t \)-type Poisson operator \( \bar{\Pi}_0 \)
and conserved density \( \bar{H}_1 \) are given by
\begin{equation}
\bar{\Pi}_0 = \begin{pmatrix}
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
-1 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -\frac{1}{2} \partial_t
\end{pmatrix},
\end{equation}
\begin{equation}
\bar{H}_1 = \bar{q}\bar{P}\bar{Q} + \frac{1}{2} \lambda_1 \bar{P}^2 + \bar{p}^2 - \frac{1}{2} \bar{Q}^2 - \frac{1}{4} \bar{q}^4 + 2\bar{c}\bar{q} + \frac{1}{2} \frac{\mu}{\lambda_1 \bar{Q}^2}.
\end{equation}

The Miura map relating systems [14] to [17], ie \( R = M(\bar{R}) \), is given by
\begin{equation}
M : \quad Q = \bar{Q},
q = -\bar{q}^2 - 2\bar{p},
P = \lambda_1 \bar{P} + \bar{q}\bar{Q},
p = \frac{1}{4} \bar{q}^3 - \frac{1}{2} \bar{c} - \frac{1}{4} \bar{Q}\bar{P} + \frac{1}{2} \bar{q}\bar{p},
c = \bar{H}_1 - \bar{q}_t = \bar{H}_1 - \bar{q}_t = \frac{1}{2} \bar{Q}^2 - \frac{1}{4} \bar{q}^4 + \bar{p}^2 + 2\bar{c}\bar{q} + \frac{1}{2} \lambda_1 \bar{P}^2 + \bar{q}\bar{Q}\bar{P} + \frac{1}{2} \frac{\mu}{\lambda_1 \bar{Q}^2} - \bar{q}_t
\end{equation}
which can be proved through direct calculations.

Denote
\begin{equation}
M' \equiv \frac{DR}{DR^T}
\end{equation}
where \( \frac{\partial M}{\partial R} \) is the Jacobi matrix consisting of Frechêt derivative of \( M \), \( M' \) denotes adjoint of \( M' \). According to the standard procedure [11]–[14], applying the map \( M \) to the first Hamiltonian structure of RD-mKdVESCS (17), we can generate the second Hamiltonian structure of the RD-KdVESCS (14),

\[
\Pi_1 = M' \Pi_0 M'^* = \begin{pmatrix}
0 & 0 & \lambda_1 & -\frac{1}{4}P & P \\
0 & 0 & 2Q & -\frac{1}{2}q & -8p + 2\partial_t \\
-\lambda_1 & -2Q & 0 & -\frac{1}{2}P & (\lambda_1 - q)Q + \frac{\mu}{Q} \\
\frac{1}{4}Q & \frac{1}{2}q & -\frac{2}{3}q^2 & -\frac{1}{3}c & \frac{3}{8}q^2 - \frac{1}{2}c - \frac{1}{2}q^2 \\
-\frac{1}{P} & 8p + 2\partial_t & (q - \lambda_1)Q - \frac{\mu}{Q} & -\frac{2}{3}q^2 + \frac{1}{2}c + \frac{1}{2}Q^2 & q\partial_t + \partial_t q
\end{pmatrix}
\]  

(19)

and the bi-Hamiltonian structure for KdV6 equation or RD-KdVESCS (8) under the transformation (7) and (13) is given by

\[
R_x = \Pi_0 \frac{\delta H_1}{\delta R} = \Pi_1 \frac{\delta H_0}{\delta R}, \quad H_0 = c.
\]  

(20)

Since the \( t \)-type Possion operator \( \Pi_0 \) is invertible, we can immediately construct a recursion operator

\[
\Phi = \Pi_1 (\Pi_0)^{-1}
\]

(21)

which has the hereditary property [11]. Applying \( \Phi \) to the vector field \( K_1 \) (14), we can generate the hierarchy of Hamiltonian commuting vector fields (symmetries)

\[
K_n = \Phi^{n-1} K_1,
\]  

(22)

and obtain a hierarchy of infinite-dimensional integrable bi-Hamiltonian systems

\[
R_x = \Phi^{n-1} K_1 = K_n = \Pi_0 \nabla H_n = \Pi_1 \nabla H_{n-1},
\]  

(23)

for example

\[
K_2 = \begin{pmatrix}
\lambda_1 P + 2PQ + \frac{1}{4}qP \\
2QP + q_t \\
-2P + \lambda_1^2 Q - \frac{1}{4} \lambda_1 q Q + \frac{1}{4} q^2 Q - Q^3 - cQ + \frac{\lambda_1 \mu}{Q^2} + \frac{\mu}{2Q^2} \\
-\frac{1}{4} P^2 + p_t + \frac{1}{4} qQ^2 - Q_1 Q^2 - \frac{1}{4} \lambda_1 Q^2 - \frac{1}{4} \mu \\
2QP + Q_t + c_t
\end{pmatrix}
\]

with the conserved functional densities \( H_2 \) given by

\[
H_2 = \frac{1}{4} Q^4 + 2PQ - q^2 Q^2 + \frac{1}{4} P^2 q + \frac{1}{4} \lambda_1 Q^2 - \frac{1}{4} \lambda_1 Q^2 + \frac{1}{2} \lambda_1 P^2 + \frac{\mu q}{4Q^2} + \frac{\mu \lambda_1}{2Q^2} - q p_t + \frac{1}{2} cQ^2 + \frac{1}{4} c^2.
\]
4 The new solutions of KdV6

The KdV equation is
\[ u_t = \frac{1}{4}(u_{xxx} + 6uu_x). \] (24)

The lax pair is
\[ \psi_{xx} + u\psi = \lambda\psi, \] (25a)
\[ \psi_t = \frac{-u_x}{4} \psi + \left(\frac{u}{2} + \lambda\right)\psi_x. \] (25b)

Let the functions \( \phi_1, \phi_2, \ldots, \phi_n \) be \( n \) different solutions of the system (25) with the corresponding \( \lambda = \lambda_1, \lambda_2, \ldots, \lambda_n \). We construct two Wronskian determinants from these functions:
\[ W_1 = W(\phi_1, \phi_2, \ldots, \phi_n), \] (26a)
\[ W_2 = W(\phi_1, \phi_2, \ldots, \phi_n, \lambda), \] (26b)

where \( m_i \geq 0 \) are given numbers and \( \phi_j^{(n)} := \partial^n\phi_j(x, \lambda)|_{\lambda=\lambda_j} \). The generalized Darboux transformation of equation (24) and system (25) is given by [15]
\[ \bar{u} = u + 2\partial_x^2 \ln W_1, \] (27a)
\[ \bar{\psi} = \frac{W_2}{W_1}, \] (27b)

namely, system (25) is covariant with respect to the action of (27). For any initial solution of (24), \( \bar{u} \) and \( \bar{\psi} \) are new solution of (24) and (25). Now we will take \( u = 0 \) in what follows.

4.1 soliton solution

In (26), let \( n = 1, m_1 = 0, \lambda = \frac{k_2^2}{4}, \lambda_1 = \frac{k_1^2}{4} \) and take
\[ \phi_1(x, t, k) = \cosh \Theta, \quad \psi_1(x, t, k) = \sinh \Theta, \] (28a)
\[ \Theta = \frac{k}{2}(x + \frac{1}{4}k^2t) + \alpha, \quad \Theta_1 = \frac{k_1}{2}(x + \frac{1}{4}k_1^2t) + \alpha \] (28b)

where \( \alpha \) is an arbitrary constant. By using (26) and (27), we obtain the single-soliton solution and the corresponding eigenfunction with \( k = k_1 \) for the KdV equation (24)
\[ \bar{u} = \frac{k_1^2}{2} \text{sech}^2 \Theta_1, \] (29)
\[ \bar{\psi}_1(x, t, k_1) = \frac{\beta k_1}{2} \text{sech} \Theta_1, \] (30)

where \( \beta \) is an arbitrary constant as well.

Since KdV6 equation (9) can be considered to be KdV equation (24) with non-homogeneous terms and \( w \) is related to the square of eigenfunction by (7), we may apply
the method of variation of constant to find the solution of Eq. (6) by using the solution \( \tilde{u} \) of Eq. (24) and corresponding eigenfunction \( \tilde{\psi}_1 \). Taking \( \alpha \) and \( \beta \) in (28b) and (30) to be time-dependent functions \( \alpha(t) \) and \( \beta(t) \) and using (7), and requiring that

\[
 u = \frac{k_1^2}{2} \text{sech}^2 \Theta_1, \tag{31a}
\]

\[
 w = \tilde{\psi}_1^2(x, t, k_1) = \frac{\beta(t)k_1^2}{4} \text{sech}^2 \Theta_1, \tag{31b}
\]

\[
 \Theta_1 = \frac{k_1}{2}(x + \frac{1}{4}k_1^2t) + \alpha(t), \tag{31c}
\]

satisfy the Eq. (6). We find that \( \alpha(t) \) can be an arbitrary function of \( t \) and

\[
 \beta(t)^2 = -4\alpha'(t)k_1. \tag{32}
\]

So the single-soliton solution of KdV6 equation (6) is given by

\[
 u = \frac{k_1^2}{2} \text{sech}^2 \Theta_1, \tag{33a}
\]

\[
 w = -\alpha'(t)k_1 \text{sech}^2 \Theta_1. \tag{33b}
\]

Its shape is shown in figure 1. Notice that \( \Theta_1 \) contains an arbitrary t-function \( \alpha(t) \). This implies that the insertion of sources into KdV equation may cause the variation of the speed of the soliton solution. So the dynamics of solution of KdV6 equation turns out to be much richer than that of solution of KdV equation.

![Figure 1. The shape of single soliton solution for u and w when \( \alpha(t) = -2t, k_1 = 1, t = 3 \).](image)

### 4.2 The first and second order of positon solution

In (20), set \( n = 1, m_1 = 1, \lambda = -\frac{k^2}{4}, \lambda_1 = -\frac{k_1^2}{4} \) and take

\[
 \phi_1(x, t, k) = \sin \Theta, \quad \psi_1(x, t, k) = \cos \Theta, \tag{34a}
\]

\[
 \Theta = \frac{k}{2}(x + x_1(k) - \frac{1}{4}k^2t) - \frac{1}{8}(k - k_1)\alpha, \tag{34b}
\]
where \(x_1(k)\) is a function that is analytic in the vicinity of the point \(k\) and has real Taylor expansion coefficients. By using (26), (27) and (34), we obtain first order of the one-position solution and the corresponding eigenfunction with \(k = k_1\) for the KdV equation (24) [15]

\[
\begin{align*}
\bar{u} &= -\frac{16k_1^2 \sin \Theta_1 (8 \sin \Theta_1 + k_1 \gamma \cos \Theta_1)}{(4 \sin 2\Theta_1 + k_1 \gamma)^2}, \\
\bar{\psi}_1(x, t, k_1) &= -\frac{4 \beta k_1^2 \sin \Theta_1}{4 \sin 2\Theta_1 + k_1 \gamma}, \\
\Theta_1 &= \frac{k_1}{2} (x + x_1(k_1) - \frac{1}{4} k_1^2 t), \\
\gamma &= -8 \partial_k \Theta|_{k=k_1} = 3k_1^2 t - 4(x + x_2(k_1)) + \alpha, \ x_2(k_1) = [x_1 + 4k \partial_k x_1(k)]_{k=k_1}
\end{align*}
\]

(35a)

(35b)

(35c)

(35d)

where \(\alpha, \beta\) are arbitrary constants. Similarly, by using (7) and the method of variation of constant we present first order of the one-position solution for the KdV6 equation (6)

\[
\begin{align*}
u &= -16k_1^2 \sin \Theta_1 (8 \sin \Theta_1 + k_1 \gamma \cos \Theta_1), \\
w &= -\frac{16k_1^2 \alpha'(t) \sin^2 \Theta_1}{(4 \sin 2\Theta_1 + k_1 \gamma)^2}, \\
\bar{\gamma} &= 3k_1^2 t - 4(x + x_2(k_1)) + \alpha(t).
\end{align*}
\]

(36a)

(36b)

(36c)

(36d)

(36) implies that for fixed \(t\) and \(x \rightarrow \pm \infty\), we have the asymptotic estimate

\[
\begin{align*}
u &= \frac{2k_1}{x} \sin 2\Theta_1 [1 + O(x^{-1})], \\
w &= \frac{k_1 \alpha'(t)}{x^2} \sin^2 \Theta_1 [1 + O(x^{-1})].
\end{align*}
\]

(37a)

(37b)

If \(x\) is fixed and \(t \rightarrow \pm \infty\), the solution has the asymptotic behavior

\[
\begin{align*}
u &= -\frac{8 \sin 2\Theta_1}{3k_1 t} [1 + O(t^{-1})], \\
w &= -\frac{16 \alpha'(t)}{9k_1^4 t^2} \sin^2 \Theta_1 [1 + O(t^{-1})].
\end{align*}
\]

(38a)

(38b)

A positon solution as a function of \(x\), \(u\), and \(w\) have a second-order pole. This pole is situated at the point \(x = x_0(t)\) which oscillates around the point \(x_{as}(t)\) with the amplitude \(\frac{1}{k_1}\), where \(x_{as}(t) = -\frac{3}{2}k_1^2 t - x_2(k_1) - \frac{\alpha(t)}{4}\). The exact position of this pole can be determined by solving the following equation with \(\delta = k_1 \bar{\gamma}\),

\[
\delta = -4 \sin \frac{1}{8} \left[ \delta - 4k_1^3 t + 4k_1 (x_1 - x_2) - k_1 \alpha(t) \right].
\]

So the positon solution of KdV6 equation (6) is long-range analogue of soliton and is slowly decreasing, oscillating solution. The shape and motion of the single positon is shown in figure 2.
Figure 2. The shape and motion of one-positon solution for $u$ and $w$ when $\alpha(t) = -2t$, $x_1(k_1) = 2k_1$, $k_1 = 1$.

In order to find the second order of one-positon solution, we take $\Theta$ in (34a) to be

$$\Theta = \frac{k}{2}(x + x_1(k) - \frac{1}{4}k^2 t) - \frac{1}{8}(k - k_1)^2 \alpha, \quad (39)$$

we have

$$W_1 = W(\phi_1, \partial_k \phi_1, \partial^2_k \phi_1)|_{k=k_1} = \frac{1}{128}\{ -32 \sin^2 \Theta_1 \cos \Theta_1 + k_1^2 \gamma^2 \cos \Theta_1 + [12k_1^2 \nu$$

$$- 4k_1(4x + 4x_1(k_1) + k_1 \alpha - 2k_1^2 x''_1(k_1)) \sin \Theta_1 \}, \quad (40a)$$

$$W_2 = W(\phi_1, \partial_k \phi_1, \partial^2_k \phi_1, \psi_1)|_{k=k_1} = -\frac{1}{64}k_1^3(4\sin 2\Theta_1 + k_1 \gamma), \quad (40b)$$

$$\Theta_1 = \frac{k_1}{2}(x + x_1(k_1) - \frac{1}{4}k^2 t), \quad \gamma = -8\partial_k \Theta|_{k=k_1} = 3k_1^2 t - 4(x + x_2(k_1)), \quad (40c)$$

$$\nu = -4\partial^2_k \Theta|_{k=k_1} = 3k_1 t - 4\partial_k x_2(k)|_{k=k_1} + \alpha. \quad (40d)$$

By the similar process, by taking $\alpha = \alpha(t)$ in (40d) we obtain the second-order of the one-positon solution from (7) and (27)

$$u = 2\partial^2_t \ln W_1, \quad (41a)$$

$$w = -\frac{2\alpha'(t)}{k_1^3}(\frac{W_2}{W_1})^2. \quad (41b)$$

Its shape is shown in figure 3.

Figure 3. The shape of second-order positon solution for $u$ and $w$ when $x_1(k_1) = 2k_1$, $\alpha(t) = -2t^2$, $k_1 = 1$, $t = 2$. 
4.3 The first and second order of negaton solution

In (26), set \( n = 1, m_1 = 1, \lambda = \frac{k^2}{4}, \lambda_1 = \frac{k_1^2}{4} \) and take

\[
\phi_1(x, t, k) = \sinh \Theta, \quad \psi_1(x, t, k) = \cosh \Theta, \tag{42a}
\]

\[
\Theta = \frac{k}{2}(x + x_1(k) + \frac{1}{4}k^2t) + \frac{1}{8}(k - k_1)\alpha. \tag{42b}
\]

We obtain the first order of one-negaton solution and the corresponding eigenfunction with \( k = k_1 \) for the KdV equation (24)

\[
\bar{u} = \frac{-16k_1^2 \sinh \Theta_1(8\sinh \Theta_1 - k_1\gamma \cosh \Theta_1)}{(4\sinh 2\Theta_1 - k_1\gamma)^2}, \tag{43a}
\]

\[
\bar{\psi}_1(x, t, k_1) = -\frac{4\beta k_1^2 \sinh \Theta_1}{4\sinh 2\Theta_1 - k_1\gamma}, \tag{43b}
\]

\[
\Theta_1 = \frac{k_1}{2}(x + x_1(k_1) + \frac{1}{4}k_1^2t), \tag{43c}
\]

\[
\gamma = 8\partial_k \Theta|_{k=k_1} = 3k_1^2 t + 4(x + x_2(k_1)) + \alpha, \tag{43d}
\]

where \( \alpha, \beta \) are arbitrary constants.

\[
\begin{array}{ccc}
\text{t=-15} & \text{t=2} & \text{t=20} \\
\text{w} & \text{w} & \text{w} \\
\text{t=-10} & \text{t=2} & \text{t=10} \\
\end{array}
\]

Figure 4. The shape and motion of one-negaton solution for \( u \) and \( w \) when \( \alpha(t) = -2t, \ x_1(k_1) = 2k_1, \ k_1 = 1. \)

Similarly, by using (7) and the method of variation of constant we present the first order of one-negaton solution for the KdV6 equation (6).
\[ u = \frac{-16k_1^2 \sinh \Theta_1 (8 \sinh \Theta_1 - k_1 \bar{\gamma} \cosh \Theta_1)}{(4 \sinh 2 \Theta_1 - k_1 \bar{\gamma})^2} \]  
\[ w = \frac{-16k_1^2 \alpha'(t) \sinh \Theta_1}{(4 \sinh 2 \Theta_1 - k_1 \bar{\gamma})^2} \]  
\[ \bar{\gamma} = 3k_1^2 t + 4(x + x_2(k_1)) + \alpha(t). \]  
\( (44a) \) \( (44b) \) \( (44c) \)

Similarly, negaton solution of (4) have second-order pole. The shape and motion of the negaton is shown in figure 4.

Now we take \( \Theta \) in (42a) to be

\[ \Theta = k_2 \left( x + x_1(k) + \frac{1}{4} k_1^2 t + \frac{1}{8} (k - k_1)^2 \alpha \right), \]  
\( (45) \)

we have

\[ W_1 = W(\phi_1, \partial_k \phi_1, \partial_k^2 \phi_1)_{k=k_1} = \frac{1}{128} \{32 \sinh^2 \Theta_1 \cosh \Theta_1 - k_1^2 \bar{\gamma}^2 \cosh \Theta_1 + [12k_1^2 \nu \\
- 4k_1(-4x - 4x_1(k_1) + k_1\alpha + 2k_1^2 x_1''(k_1))] \sinh \Theta_1 \}, \]  
\( (46a) \)

\[ W_2 = W(\phi_1, \partial_k \phi_1, \partial_k^2 \phi_1, \psi_1)_{k=k_1} = \frac{1}{64} k_1^3 (-4 \sin 2 \Theta_1 + k_1 \bar{\gamma}), \]  
\( (46b) \)

\[ \Theta_1 = \frac{k_1^2}{2} (x + x_1(k_1) + \frac{1}{4} k_1^2 t), \quad \bar{\gamma} = 8 \partial_k \Theta |_{k=k_1} = 3k_1^2 t + 4(x + x_2(k_1)), \]  
\( (46c) \)

\[ \nu = 4 \partial_k^2 \Theta |_{k=k_1} = 3k_1 t + 4 \partial_k x_2(k)|_{k=k_1} + \alpha. \]  
\( (46d) \)

Similarly, we obtain the second-order of negaton solution from (7) and (27)

\[ u = 2 \partial_x^2 \ln W_1, \]  
\( (47a) \)

\[ w = \frac{2 \alpha'(t)}{k_1^3 \left( \frac{W_2}{W_1} \right)^2}. \]  
\( (47b) \)

Its shape is shown in figure 5.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure5.png}
\caption{The shape of second-order negaton solution for \( u \) and \( w \) when \( x_1(k_1) = 2k_1, \ \alpha(t) = 2t^2, \ k_1 = 1, \ t = 5. \)}
\end{figure}

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