SOLVABILITY OF A COUPLED NONLINEAR SYSTEM OF SKOROHOD-LIKE STOCHASTIC DIFFERENTIAL EQUATIONS MODELING ACTIVE–PASSIVE PEDESTRIANS DYNAMICS THROUGH A HETEROGENEOUS DOMAIN AND FIRE

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Abstract

We study the existence and uniqueness of solutions to a coupled nonlinear system of Skorohod-like stochastic differential equations with reflecting boundary condition. The setting describes the evacuation dynamics of a mixed crowd composed of both active and passive pedestrians moving through a domain with obstacles, fire and smoke. As main working techniques, we use compactness methods and the Skorohod’s representation of solutions to SDEs posed in bounded domains. The challenge is to handle the coupling and the nonlinearities present in the model equations together with the multiplicity of the domain and the pedestrian-obstacle interaction.

Keywords: Pedestrian dynamics, Stochastic differential equations, reflecting boundary condition, Skorohod equations, heterogeneous domain, fire

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1. Introduction

In this paper, we study the solvability of a coupled nonlinear system of Skorohod-like stochastic differential equations modeling the dynamics of pedestrians through a...
heterogenous domain in the presence of fire. The standing modeling assumption is that
the crowd of pedestrians is composed of two distinct populations: an *active population*
these pedestrians are aware of the details of the environment and move towards the
exit door, and a *passive population* these pedestrians are not aware of the details of
the geometry and move randomly to explore the environment and eventually to find
the exit. All pedestrians are seen as moving point particles driven by a suitable over-
damped Langevin model, which will be described in Section 3. Our model belongs to
the class of social-velocity models for crowd dynamics. It is posed in a two dimensional
multiple connected region $D$, containing obstacles with a fixed location. Furthermore,
a stationary fire, which produces smoke, is placed within the geometry forcing the
pedestrians to choose a proper own velocity such that they evacuate. The fire is also
seen as an obstacle.

To keep a realistic picture, the overall dynamics is restricted to a bounded "perfo-
rated" domain, i.e. the obstacles are seen as impenetrable regions. The geometry
is described in Subsection 3.1 see Figure 1 to fix ideas. In this framework, we
consider reflecting boundary conditions and plan, as further research, to treat the case
of mixed reflection–flux boundary conditions so that the exits can allow for outflux.
In this framework, we focus on the interior obstacles. To achieve a correct dynamics
of dynamics of the pedestrians close to the boundary of the interior obstacles, we
choose to work with the classical Skorohod's formulation of SDEs; we refer the reader
to the textbook [26] for more details on this subject. Note that this approach is
needed especially because of the chosen dynamics for the passive particles, as the
active pedestrians are able to avoid collisions with the obstacles by using a motion
planning map (*a priori* given paths – solution to a suitable Eikonal-like equation; cf.
Appendix A).

2. Related contributions. Main questions of this research

A number of relevant results are available on the dynamics of mixed active-passive
pedestrian populations. As far as we are aware, the first questions in this context
were posed in the modeling and simulation study [27] while considering the evacuation
dynamics of a mixed active-passive pedestrian populations in a complex geometry in
the presence of a fire as well as of a slowly spreading smoke curtain. From a stochastic
processes perspective, various lattice gas models for active-passive pedestrian dynamics
have been recently explored in \[8,9\]. See also \[32\] for a result on the weak solvability of a
deterministic system of parabolic partial differential equations describing the interplay
of a mixture of fluids for active-passive populations of pedestrians.

The discussion of the active-passive pedestrian dynamics at the level of SDEs is
new and brings in at least a twofold challenge: (i) the evolution system is nonlinear
and coupled and (ii) pedestrians have to cross a domain with forbidden regions (the
obstacles). Various solution strategies have been already identified for deterministic
crowd evolution equations. We mention here the two more prominent: a granular media
approach, where collisions with obstacles are tackled with techniques of non-smooth
analysis cf. e.g. \[17\], and a reflection-of-velocities approach as it is done e.g. in \[22\].
If some level of noise affects the dynamics, then both these approaches fail to work.
On the other hand, there are several results for stochastic differential equations with
reflecting boundary conditions, one of them being the seminal contribution of Skorohod
in \[31\], where the author provided the existence and uniqueness to one dimensional
stochastic equations for diffusion processes in a bounded region. A direct approach to
the solution of the reflecting boundary conditions and reductions to the case including
nonsmooth ones are reported in \[24\]. Extending results by Tanaka, the author of \[29\]
proves the existence and uniqueness of solutions to the Skorohod equation posed in a
bounded domain in \(\mathbb{R}^d\) where a reflecting boundary condition is applied.

The main question we ask in this paper is whether we can frame our crowd dynamics
model as a well-posed system of stochastic evolution equations of Skorohod type.
Provided suitable restrictions on the geometry of the domain, on the structure of
nonlinearites as well as data and parameters, we provide in Section \[6\] a positive answer
to this question. This study opens the possibility of exploring further our system
from the numerical analysis perspective so that suitable algorithms can be designed to
produce simulations forecasting the evacuation time based on our model. A couple of
follow-up open questions are given in the conclusion; see Section \[7\].
3. Setting of the model equations

3.1. Geometry

We consider a two dimensional domain, which we refer to as \( \Lambda \). As a building geometry, parts of the domain are filled with obstacles. Their collection is denoted by \( G = \bigcup_{k=1}^{N_{\text{obs}}} G_k \), for all \( k \in \{1, \ldots, N_{\text{obs}} \in \mathbb{N} \} \). A fire \( F \) is introduced somewhere in this domain and is treated in this context as an obstacle for the motion of the crowd. Moreover, the domain has the exit denoted by \( E \). Our domain represents the environment where the crowd of pedestrians is located. The crowd tries to find the fastest way to the exit, avoiding the obstacles and the fire. Let \( D := \Lambda \setminus (G \cup E \cup F) \subset \mathbb{R}^2 \) with the boundary \( \partial D \) such that \( \partial \Lambda \cap \partial G_k = \emptyset \), \( \partial \Lambda \cap \partial G_k = \emptyset \) and \( F \cap G_k = \emptyset \), we also denote \( S = (0, T) \) for some \( T \in \mathbb{R}_+ \). Furthermore, \( N_A \) is the total number of active agents, \( N_P \) is the total number of passive particles with \( N := N_A + N_P \) and \( N_A, N_P, N \in \mathbb{N} \).
3.2. Active population

For \( i \in \{1, \ldots, N_A\} \) and \( t \in S \), let \( x_{a_i} \) denote the position of the pedestrian \( i \) belonging to the active population at time \( t \). We assume that the dynamics of active pedestrians is governed by

\[
\begin{aligned}
\frac{dx_{a_i}(t)}{dt} &= -\Upsilon\left(s(x_{a_i}(t)) \frac{\nabla \phi(x_{a_i}(t))}{|\nabla \phi(x_{a_i}(t))|} \right) \left(p_{\text{max}} - p(x_{a_i}(t), t)\right), \\
\quad x_{a_i}(0) &= x_{a_{i,0}},
\end{aligned}
\]

(1)

where \( x_{a_{i,0}} \) represents the initial configuration of active pedestrians inside \( D \). In (1), \( \nabla \phi \) is the minimal motion path of the distance between particle positions \( x_{a_i} \) and the exit location \( E \) (it solves the Eikonal-like equation). The function \( \phi(\cdot) \) encodes the familiarity with the geometry; see also [33] for a related setting. We refer to it as the motion planning map. In this context, \( p(x, t) \) is the local discomfort (a realization of the social pressure) so that

\[
p(x, t) = \mu(t) \int_{D \cap B(x, \tilde{\delta})} \sum_{j=1}^{N} \delta(y - x_{c_j}(t)) \, dy,
\]

(2)

for \( \{x_{c_j}\} := \{x_{a_i}\} \cup \{x_{b_k}\} \) for \( i \in \{1, \ldots, N_A\}, k \in \{1, \ldots, N_P\}, j \in \{1, \ldots, N_A + N_P\} \).

In [2], \( \delta \) is the Dirac (point) measure and \( B(x, \tilde{\delta}) \) is a ball center \( x \) with small enough radius \( \tilde{\delta} \) such that \( \tilde{\delta} > 0 \). Hence, the discomfort \( p(x, t) \) represents a finite measure on the bounded set \( D \cap B(x, \tilde{\delta}) \). In addition, we assume the following structural relation between the smoke extinction and the walking speed (see in [20], [28]) as a function \( \Upsilon : \mathbb{R}^2_+ \to \mathbb{R}^2 \) such that

\[
\Upsilon(x) = -\zeta x + \eta,
\]

(3)

where \( \zeta, \eta \) are given real positive numbers. The dependence of the model coefficients on the local smoke density is marked via a smooth relationship with respect to an a priori given function \( s(x, t) \) describing the distribution of smoke inside the geometry at position \( x \) and time \( t \).

3.3. Passive population

For \( k \in \{1, \ldots, N_P\} \) and \( t \in S \), let \( x_{p_k} \) denote the position of the pedestrian \( k \) belonging at time \( t \) to the passive population. The dynamics of the passive pedestrians
is described here as a system of stochastic differential equations as follows:

\[
\begin{aligned}
d\mathbf{x}_{p_k}(t) &= \sum_{j=1}^{N} \frac{x_{c_j} - x_{p_k}}{\varepsilon + |x_{c_j} - x_{p_k}|} \omega(|x_{c_j} - x_{p_k}|, s(x_{p_k}, t))dt + \beta(s(x_{p_k}, t))dB(t), \\
x_{p_k}(0) &= x_{p_k0},
\end{aligned}
\]  

(4)

where \(x_{p_k0}\) represents the initial configuration of passive pedestrians inside \(D\) and \(\varepsilon > 0\). In [4], \(\omega\) is a Morse-like potential function (see e.g. Ref. [4] for a setting where a similar potential has been used). We take \(\omega : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}^2\) to be

\[
\omega(x, y) = -\beta(y) \left( C_A e^{-x_A} + C_R e^{-x_R} \right), \quad \text{for } x, y \in \mathbb{R} \times \mathbb{R}^2
\]  

(5)

while \(C_A > 0, C_R > 0\) are the attractive and repulsive strengths and \(\ell_A > 0, \ell_R > 0\) are the respective length scales for attraction and repulsion. Moreover, the coefficient \(\beta\) is the Heaviside step function. As in Subsection 3.2, the dependence of the model coefficients on the smoke is marked via a smooth relationship with respect to an a priori given function \(s(x, t)\) describing the distribution of smoke inside the geometry at position \(x\) and time \(t\). Note that the passive pedestrians do not possess any knowledge on the geometry of the walking space. In particular, the location of the exit is unknown; see [11] for a somewhat related context.

4. Technical preliminaries and assumptions

4.1. Technical preliminaries

We recall the classical Ascoli-Arzelà Theorem:

A family of functions \(U \subset C(\bar{S}; \mathbb{R}^d)\) is relatively compact (with respect to the uniform topology) if

i. for every \(t \in \bar{S}\), the set \(\{f(t); f \in U\}\) is bounded.

ii. for every \(\varepsilon > 0\) and \(t, s \in \bar{S}\), there is \(\tilde{\delta} > 0\) such that

\[
|f(t) - f(s)| \leq \varepsilon,
\]  

(6)

whenever \(|t - s| \leq \tilde{\delta}\) for all \(f \in U\).

For a function \(f : S \to \mathbb{R}^d\), we introduce the definition of Hölder seminorms as

\[
[f]_{C^\alpha(S; \mathbb{R}^d)} = \sup_{t \neq s, t, s \in \bar{S}} \frac{|f(t) - f(s)|}{|t - s|^\alpha},
\]  

(7)
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for \( \alpha \in (0, 1) \) and the supremum norm as

\[
\|f\|_{L^\infty(S;\mathbb{R}^d)} = \text{ess sup}_{t \in S} |f(t)|.  
\]

(8)

We refer to [1] and [17] for more details on these spaces.

Using Ascoli-Arzelà Theorem starting from the facts:

i'. there is \( M_1 > 0 \) such that \( \|f\|_{L^\infty(S;\mathbb{R}^d)} \leq M_1 \) for all \( f \in U \),

ii'. for some \( \alpha \in (0, 1) \), there is an \( M_2 > 0 \) such that \( \|f\|_{C^{\alpha}(\bar{S};\mathbb{R}^d)} \leq M_2 \) for all \( f \in U \),

we infer that the set

\[
K_{M_1M_2} = \{ f \in C(\bar{S};\mathbb{R}^d) ; \|f\|_{L^\infty(S;\mathbb{R}^d)} \leq M_1, [f]_{C^{\alpha}(\bar{S};\mathbb{R}^d)} \leq M_2 \}
\]

(9)

is relatively compact in \( C(\bar{S};\mathbb{R}^d) \).

For \( \alpha \in (0, 1) \), \( T > 0 \) and \( p > 1 \), the space \( W^{\alpha,p}(S;\mathbb{R}^d) \) is defined as the set of all \( f \in L^p(S;\mathbb{R}^d) \) such that

\[
[f]_{W^{\alpha,p}(S;\mathbb{R}^d)} := \int_0^T \int_0^T \frac{|f(t) - f(s)|^p}{|t-s|^{1+\alpha p}} dt ds < \infty.
\]

This space is endowed with the norm

\[
\|f\|_{W^{\alpha,p}(S;\mathbb{R}^d)} = \|f\|_{L^p(S;\mathbb{R}^d)} + [f]_{W^{\alpha,p}(S;\mathbb{R}^d)}.
\]

Moreover, we have the following embedding

\[
W^{\alpha,p}(S;\mathbb{R}^d) \subset C^{\gamma}(\bar{S};\mathbb{R}^d) \quad \text{for } \alpha p - \gamma > 1
\]

and \( [f]_{C^{\gamma}(S;\mathbb{R}^d)} \leq C_{\gamma,\alpha,p}[f]_{W^{\alpha,p}(S;\mathbb{R}^d)} \). Relying on the Ascoli-Arzelà Theorem, we have the following situation:

ii". for some \( \alpha \in (0, 1) \) and \( p > 1 \) with \( \alpha p > 1 \), there is \( M_2 > 0 \) such that \( [f]_{W^{\alpha,p}(S;\mathbb{R}^d)} \leq M_2 \) for all \( f \in U \).

If i' and ii" hold, then the set

\[
K'_{M_1M_2} = \{ f \in C(\bar{S};\mathbb{R}^d) ; \|f\|_{L^\infty(S;\mathbb{R}^d)} \leq M_1, [f]_{W^{\alpha,p}(S;\mathbb{R}^d)} \leq M_2 \}
\]

(10)

is relatively compact in \( C(\bar{S};\mathbb{R}^d) \), if \( \alpha p > 1 \) (see e.g. [16], [9]).
4.2. Assumptions

To be successful with our analysis, we rely on the following assumptions:

(A_1) The functions $b : D \times D \to \mathbb{R}^2 \times \mathbb{R}^2$, and $\sigma : D \times D \to \mathbb{R}^{2 \times 2} \times \mathbb{R}^{2 \times 2}$ satisfy $|\sigma(x)| \leq L$, $|b(x)| \leq L$ for all $x \in D \times D$. Here $\sigma$ and $b$ incorporate the right-hand sides of the SDEs (1) and (4) in their respective dimensionless form indicated in Appendix B.

(A_2) $p_{\text{max}} = N|D|$, where $|D|$ denotes the area of $D$.

(A_3) $\Upsilon, \omega, \beta \in C^1(\mathbb{R}^2)$.

(A_4) $s \in C^1(\bar{S}; \mathbb{R}^2)$.

(A_5) $\partial D$ is $C^{2,\alpha}$ with $\alpha \in (0, 1)$, or at least satisfying the exterior sphere condition.

It is worth mentioning that assumptions (A_1) and (A_2) correspond to the modeling of the situation, while (A_3)-(A_5) are of technical nature. The latter fit to the type of solution we are searching for; clarifications in this direction are given in the next Section.

5. The Skorohod equation

5.1. Concept of solution

Take $x \in \partial D$ arbitrarily fixed. We define the set $\mathcal{N}_x$ of inward normal unit vectors at $x \in \partial D$ by

$$\mathcal{N}_x = \bigcup_{r > 0} \mathcal{N}_{x,r},$$
$$\mathcal{N}_{x,r} = \{ n \in \mathbb{R}^2 : |n| = 1, B(x - rn, r) \cap D = \emptyset \},$$

(11)

where $B(z, r) = \{ y \in \mathbb{R}^2 : |y - z| < r \}, z \in \mathbb{R}^2, r > 0$. Mind that, in general, it can happen that $\mathcal{N}_x = \emptyset$. In this case, the uniform exterior sphere condition is not satisfied (see, for instance, the examples in [6], Fig. 5 and in [7], page 4).

We complement our list of assumptions (A_1)-(A_5) with three specific conditions on the geometry of the domain $D$:

(A_6) (Uniform exterior sphere condition). There exists a constant $r_0 > 0$ such that $\mathcal{N}_x = \mathcal{N}_{x,r_0} \neq \emptyset$ for any $z \in \partial D$. 
There exits constants $\delta > 0$ and $\delta' \in [1, \infty)$ with the following property: for any $x \in \partial D$ there exists a unit vector $l_x$ such that

$$< l_x, n > \geq 1/\delta'$$

for any $n \in \bigcup_{y \in B(x, \delta) \cap \partial D} N_y$.

where $< \cdot, \cdot >$ denotes the usual inner product in $\mathbb{R}^2$.

There exist $\delta'' > 0$ and $\nu > 0$ such that for each $x_0 \in \partial D$ we can find a function $f \in C^2(\mathbb{R}^2)$ satisfying

$$< y - x, n > + \frac{1}{\nu} < \nabla f(x), n > |y - x|^2 \geq 0,$$

for any $x \in B(x_0, \delta'') \cap \partial D, y \in B(x_0, \delta'') \cap \partial \bar{D}$ and $n \in N_x$.

Let $W(\mathbb{R}^2)$ and $W(D)$ be the space of continuous paths in $\mathbb{R}^2$ and $D$, respectively. The following relation is called the Skorohod equation: Find $(\xi, \phi) \in W(\mathbb{R}^2)$ such that

$$\xi(t) = w(t) + \phi(t),$$

where $w \in W(\mathbb{R}^2)$ is given so that $w(0) \in D$. The solution of (13) is a pair $(\xi, \phi)$, which satisfies the following two conditions:

(a) $\xi \in W(D)$;

(b) $\phi \in C(\bar{S})$ with bounded variation on each finite time interval satisfying $\phi(0) = 0$ and

$$\phi(t) = \int_0^t n(y) d|\phi|_y,$$

$$|\phi|_t = \int_0^t 1_{\partial D}(\xi(y)) d|\phi|_y,$$

where

$$n(y) \in N_{\xi(y)} \text{ if } \xi(y) \in \partial D,$$

$$|\phi|_t = \text{ total variation of } \phi \text{ on } [0, t]$$

$$= \sup_{\mathcal{T} \in \mathcal{G}(0,t)} \sum_{k=1}^{n_{\mathcal{T}}} |\phi(t_k) - \phi(t_k - 1)|.$$

In (15), we denote by $\mathcal{G}([0, t])$ the family of all partitions of the interval $[0, t]$ and take a partition $\mathcal{T} = \{0 = t_0 < t_1 < \ldots < t_n = t\} \in \mathcal{G}([0, t])$. The supremum in (15) is taken over all partitions of type $0 = t_0 < t_1 < \ldots < t_n = t$. 

Let $W(D)$ and $W(\mathbb{R}^2)$ be the space of continuous paths in $\mathbb{R}^2$ and $D$, respectively. The following relation is called the Skorohod equation: Find $(\xi, \phi) \in W(\mathbb{R}^2)$ such that

$$\xi(t) = w(t) + \phi(t),$$

where $w \in W(\mathbb{R}^2)$ is given so that $w(0) \in D$. The solution of (13) is a pair $(\xi, \phi)$, which satisfies the following two conditions:

(a) $\xi \in W(D)$;

(b) $\phi \in C(\bar{S})$ with bounded variation on each finite time interval satisfying $\phi(0) = 0$ and

$$\phi(t) = \int_0^t n(y) d|\phi|_y,$$

$$|\phi|_t = \int_0^t 1_{\partial D}(\xi(y)) d|\phi|_y,$$

where

$$n(y) \in N_{\xi(y)} \text{ if } \xi(y) \in \partial D,$$

$$|\phi|_t = \text{ total variation of } \phi \text{ on } [0, t]$$

$$= \sup_{\mathcal{T} \in \mathcal{G}(0,t)} \sum_{k=1}^{n_{\mathcal{T}}} |\phi(t_k) - \phi(t_k - 1)|.$$
Conditions (a) and (b) guarantee that $\xi$ is a reflecting process on $D$.

**Theorem 5.1.** Assume conditions (a) and (b). Then for any $w \in W(\mathbb{R}^2)$ with $w(0) \in D$, there exists a unique solution $\xi(t, w)$ of the equation (13) such that $\xi(t, w)$ is continuous in $(t, w)$.

For the proof of this Theorem, we refer the reader to Theorem 4.1 in [29].

To come closer to the model equations for active-passive pedestrian dynamics described in Section 3, we introduce the mappings

$$b : D \times D \rightarrow \mathbb{R}^2 \times \mathbb{R}^2, \quad \sigma : D \rightarrow \mathbb{R}^{2 \times 2} \times \mathbb{R}^{2 \times 2}$$

and consider a Skorohod-like system on the probability space $(\Omega, \mathcal{F}, P)$

$$dX_t = b(X_t(t))dt + \sigma(X_t(t))dB(t) + d\Phi_t,$$  

(17)

or (17) can be written component-wise as

$$dX^{(I)}_t = b(X_t(t))_I dt + \sum_{J=1}^{2} \sigma_{IJ}(X_t(t))_J dB^{(J)}(t) + d\Phi^{(I)}_t,$$

for $1 \leq I \leq 4, 1 \leq J \leq 2$

with

$$X(0) = X_0 \in D,$$  

(18)

where the initial value $X_0$ is assumed to be an $\mathcal{F}_0$-measurable random variable and $B(t)$ is a 2-dimensional $\mathcal{F}_t$-Brownian motion with $B(0) = 0$. Here, $\{\mathcal{F}_t\}$ is a filtration such that $\mathcal{F}_0$ contains all $\mathbb{P}$-negligible sets and $\mathcal{F}_t = \bigcap_{\varepsilon>0}\mathcal{F}_{t+\varepsilon}$. The structure of (17) is provided in Section 6.2. Similarly to the deterministic case, we can now define the following concept of solutions to (17). More details of the structure of (17)-(18) are listed in Section 6.2.

**Definition 5.1.** A pair $(X_t, \Phi_t)$ is called solution to (17)-(18) if the following conditions hold:

(i) $X_t$ is a $D$-valued $\mathcal{F}_t$-adapted continuous process;

(ii) $\Phi(t)$ is an $\mathbb{R}^2$-valued $\mathcal{F}_t$-adapted continuous process with bounded variation
on each finite time interval such that \( \Phi(0) = 0 \) with

\[
\Phi(t) = \int_0^t n(y)d|\Phi|_y,
\]

\[
|\Phi|_t = \int_0^t 1_{\partial D}(X(y))d|\Phi|_y.
\]

(19)

(iii) \( n(s) \in N_{X(s)} \in \partial D. \)

Note that the Definition 5.1 ensures that \( X_t \) entering (17) is a reflecting process.

6. Solvability of Skorohod-like system

In this section, we establish the solvability of the Skorohod-like system by showing the existence and uniqueness of solutions in the sense of Definition 5.1 to the problem (17)–(18). It turns out that for completing the well-posedness study of our system, more work is required. We comment on this matter in Remark 6.1.

6.1. Statement of the main results

The main results of this paper are stated in Theorem 6.1 and Theorem 6.2. In the frame of this paper, the focus lies on ensuring the existence and uniqueness of Skorohod solutions to our crowd dynamics problem.

**Theorem 6.1.** Assume that \((A_1)-(A_7)\) hold. There exists at least a solution to the Skorohod-like system (17)–(18) in the sense of Definition 5.1.

**Theorem 6.2.** Assume that \((A_1)-(A_8)\) hold. There is a unique strong solution to (17)–(18).

These statements are proven in the next two subsections.
6.2. Structure of the proof of Theorem 6.1

For convenience, we rephrase the solution to the system (80) and (81) in terms of the vector $X^n_t$, $n \in \mathbb{N}$, such that

$$X^n_t := (X^n_{a_i}(t), X^n_{b_k}(t))^T, \quad i \in \{1, \ldots, N_A\}, k \in \{1, \ldots, N_P\},$$

(20)

$$F_1(X^n_t, t) := \kappa \Upsilon(S(X^n_{a_i}(t))); \quad F_2(X^n_t, t) := \kappa \sum_{j=1}^{N} \frac{X^n_{c_j}(t) - X^n_{p_k}(t)}{\epsilon + |X^n_{c_j}(t) - X^n_{p_k}(t)|} \omega(|X^n_{c_j}(t) - X^n_{p_k}(t)|, S(X^n_{p_k}(t), t)),$$

(21)

$$\tilde{\sigma}(X^n_t, t) := \kappa \beta(S(X^n_{p_k}(t), t), t).$$

(22)

Furthermore, we set

$$b(X^n_t, t) := \begin{bmatrix} F_1(X^n_t, t) \\ F_2(X^n_t, t) \end{bmatrix} \quad \text{and} \quad \sigma(X^n_t, t) := \begin{bmatrix} \mathbf{o} \\ \tilde{\beta} \end{bmatrix},$$

(23)

with

$$\mathbf{o} := \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{and} \quad \tilde{\beta} := \begin{bmatrix} \tilde{\sigma}_{11} & \tilde{\sigma}_{12} \\ \tilde{\sigma}_{21} & \tilde{\sigma}_{22} \end{bmatrix},$$

(24)

where $\tilde{\sigma}_{IJ} := (\tilde{\sigma}(X^n_{I}, t))_{I,J}$ for $1 \leq I, J \leq 2$ and the initial datum is

$$X^n(0) := X_0 := \begin{bmatrix} X_{a_i,0} \\ X_{b_k,0} \end{bmatrix}.$$

(25)

We denote by $\{\Phi^n_t\}$ the associated process of $\{X^n_t\}$ as in (17), viz.

$$\Phi^n_t := \begin{bmatrix} \Phi^n_1(t) \\ \Phi^n_2(t) \end{bmatrix}.$$  

(26)

We use the compactness method together with the continuity result of the deterministic case stated in Theorem 5.1 for proving the existence of solutions to (17)-(18). We follow the arguments by G. Da Prato and J. Zabczyk (2014) (cf. [12], Section 8.3) and a result of F. Flandoli (1995) (cf. [16]) for martingale solutions. The starting point of this argument is based on considering a sequence $\{X^n_t\}$ of solutions of the following system of Skorohod-like stochastic differential equations

$$\begin{cases} 
\frac{dX^n_t = b(X^n_t(h^n(t)))dt + \sigma(X^n_t(h^n(t)))dB(t) + d\Phi^n_t,} 
X^n(0) = X_0 \in D,
\end{cases}$$

(27)
where $X^0_n \in D$ is given, and
\begin{equation}
  h^n(0) = 0, \tag{29}
\end{equation}
\begin{equation}
  h^n(t) = (k-1)2^{-n}, \quad (k-1)2^{-n} < t \leq k2^{-n}, \quad k = 1, 2, \ldots, n \text{ and } n \geq 1. \tag{30}
\end{equation}
Moreover, by Theorem 5.1 we have a unique solution of (28). Hence, $X^t_n$ obtained for $0 \leq t \leq k2^{-n}$ and for $k2^{-n} < t \leq (k+1)2^{-n}$ is uniquely determined as solution of the following Skorohod equation
\begin{equation}
  X^t_n = X^t_n(k2^{-n}) + b(X^t_n(k2^{-n}))(t - k2^{-n}) + \sigma(X^t_n(k2^{-n}))(B(t) - B(k2^{-n})) + \Phi^n_t. \tag{31}
\end{equation}
Let us call
\begin{equation}
  Y^t_n := X_0 + \int_0^t b(X^t_n(h^n(y)))dy + \int_0^t \sigma(X^t_n(h^n(y)))dB(y). \tag{32}
\end{equation}
We define the family of laws
\begin{equation}
  \{P(Y^t_n); 0 \leq t \leq T, n \geq 1\}. \tag{33}
\end{equation}
(33) is a family of probability distributions of $Y^n_t$. Let $\mathcal{P}^n$ be the laws of $Y^n_t$.

The compactness argument proceeds as follows. We begin with $Y^n_t, n \in \mathbb{N}$, given cf. (32). The construction of $Y^n_t$ is investigated on a probability space $(\Omega, \mathcal{F}, P)$ with a filtration $\{\mathcal{F}_t\}$ and a Brownian motion $B(t)$. Next, let $\mathcal{P}^n$ be the laws of $Y^n_t$ which is defined cf. (33). Then, by using Prokhorov’s Theorem (cf. [2], Theorem 5.1), we can show that the sequence of laws $\{\mathcal{P}^n(Y^n_t)\}$ is weakly convergent as $n \to \infty$ to $\mathcal{P}(Y_t)$ in $C(\bar{S}; \mathbb{R}^2 \times \mathbb{R}^2)$. Then, by using the “Skorohod Representation Theorem” (cf. [12], Theorem 2.4), this weak convergence holds in a new probability space with a new stochastic process, for a new filtration. This leads to some arguments for weak convergence results of two stochastic processes in two different probability spaces together with the continuity result in Theorem 5.1 that we need to use to obtain the existence of our Skorohod-like system (17). Finally, we prove the uniqueness of solutions to our system.

6.3. Proof of Theorem 6.1

Let us start with handling the tightness of the laws $\{\mathcal{P}^n\}$ through the following Lemma.
Lemma 6.1. Assume that \((A_1)-(A_5)\) hold. Then, the family \(\{\mathcal{P}^n\}\) given by (33) is tight in \(C(\bar{S}, \mathbb{R}^2 \times \mathbb{R}^2)\).

Proof. To prove the wanted tightness, let us recall the following compact set in \(C(\bar{S}, \mathbb{R}^2 \times \mathbb{R}^2)\)

\[
K_{M_1M_2} = \{ f \in C(\bar{S}; \mathbb{R}^2 \times \mathbb{R}^2) : \|f\|_{L^\infty(\bar{S}; \mathbb{R}^2 \times \mathbb{R}^2)} \leq M_1, [f]_{C^\varepsilon(\bar{S}; \mathbb{R}^2 \times \mathbb{R}^2)} \leq M_2 \}.
\] (34)

Now, we show that for a given \(\varepsilon > 0\), there are \(M_1, M_2 > 0\) such that

\[
P(Y^n \in K_{M_1M_2}) \leq \varepsilon, \text{ for all } n \in \mathbb{N}.
\] (35)

This means that

\[
P(\|Y^n\|_{L^\infty(\bar{S}; \mathbb{R}^2 \times \mathbb{R}^2)} > M_1 \text{ or } [Y^n]_{C^\varepsilon(\bar{S}; \mathbb{R}^2 \times \mathbb{R}^2)} > M_2) \leq \varepsilon.
\] (36)

A sufficient condition for this to happen is

\[
P(\|Y^n\|_{L^\infty(\bar{S}; \mathbb{R}^2 \times \mathbb{R}^2)} > M_1) < \frac{\varepsilon}{2} \text{ and } P([Y^n]_{C^\varepsilon(\bar{S}; \mathbb{R}^2 \times \mathbb{R}^2)} > M_2) < \frac{\varepsilon}{2},
\] (37)

where \(Y^n\) denotes either \(Y_t\) or \(Y_r\).

We consider first \(P(\|Y^n\|_{L^\infty(\bar{S}; \mathbb{R}^2 \times \mathbb{R}^2)} > M_1) < \frac{\varepsilon}{2}\). Using Markov's inequality (see e.g. [19], Corollary 5.1), we get

\[
P(\|Y^n\|_{L^\infty(\bar{S}; \mathbb{R}^2 \times \mathbb{R}^2)} > M_1) \leq \frac{1}{M_1} \mathbb{E} \sup_{t \in \bar{S}} |Y^n_t|,
\] (38)

but

\[
\sup_{t \in \bar{S}} |Y^n_t| = \sup_{t \in \bar{S}} \left\{ X_{a,0} + \int_0^t F_1(X^n_y(h^n(y)))dy \right\}
\]
\[
, \left\{ X_{p,0} + \int_0^t F_2(X^n_y(h^n(y)))dy + \int_0^t \sigma(X^n_y(h^n(y)))dB(y) \right\}.
\] (39)

We estimate

\[
\sup_{t \in \bar{S}} |Y^n_t| = \sup_{t \in \bar{S}} \left\{ |X_{a,0}| + \int_0^t F_1(X^n_y(h^n(y)))dy \right\}
\]
\[
, \left\{ |X_{p,0}| + \int_0^t F_2(X^n_y(h^n(y)))dy + \int_0^t \sigma(X^n_y(h^n(y)))dB(y) \right\}.
\] (40)

Since \(F_1, F_2\) are bounded, then we have

\[
\int_0^T F_1(X^n_y(h^n(y)))dy \leq C \text{ and } \int_0^T F_2(X^n_y(h^n(y)))dy \leq C.
\] (41)
Taking the expectation on (40), we are led to

\[
E \left[ \sup_{t \in S} |Y^n_t| \right] \leq C + E \left[ \sup_{t \in S} \int_0^t \sigma(X^n_y(h^n(y)))dB(y) \right].
\]  

(42)

On the other hand, the Burkholder-Davis-Gundy’s inequality \( [\text{B-D-G inequality}] \) implies

\[
E \left[ \sup_{t \in S} \left| \int_0^t \sigma(X^n_y(h^n(y)))dB(y) \right| \right] \leq E \left[ \int_0^t |\sigma(X^n_y(h^n(y)))|^2 dy \right]^{1/2}.
\]  

(43)

Then, we have the following estimate

\[
E \left[ \sup_{t \in S} |Y^n_t| \right] \leq C + E \left[ \int_0^t |\sigma(X^n_y(h^n(y)))|^2 dy \right]^{1/2} \leq C
\]  

(44)

Hence, for \( \varepsilon > 0 \), we can choose \( M_1 > 0 \) such that \( P(\|Y^n_t\|_{L^\infty(\bar{S};\mathbb{R}^2 \times \mathbb{R}^2)} > M_1) < \frac{\varepsilon}{2} \).

In the sequel, we consider the second inequality \( P([Y^n]_{C^\gamma(\bar{S};\mathbb{R}^2 \times \mathbb{R}^2)} > M_2) < \frac{\varepsilon}{2} \), this reads

\[
P([Y^n]_{C^\gamma(\bar{S};\mathbb{R}^2 \times \mathbb{R}^2)} > M_2) = P \left( \sup_{t \neq r, r \in S} \left| \frac{Y^n_t - Y^n_r}{|t - r|^{\alpha}} \right| > M_2 \right) \leq \frac{\varepsilon}{2}.
\]  

(45)

Let us introduce another class of compact sets now in the Sobolev space \( W^{\alpha,p}(0,T;\mathbb{R}^2 \times \mathbb{R}^2) \) (which for suitable exponents \( \alpha p - \gamma > 1 \) lies in \( C^\gamma(\bar{S};\mathbb{R}^2 \times \mathbb{R}^2) \)). Additionally, we recall the relatively compact sets \( K^\prime_{M_1M_2} \), defined as in Section 4, such that

\[
K^\prime_{M_1M_2} = \{ f \in C(\bar{S};\mathbb{R}^2 \times \mathbb{R}^2) : \| f \|_{L^\infty(\bar{S};\mathbb{R}^2 \times \mathbb{R}^2)} \leq M_1, \| f \|_{W^{\alpha,p}(\bar{S};\mathbb{R}^2 \times \mathbb{R}^2)} \leq M_2 \}.
\]  

(46)

A sufficient condition for \( K^\prime_{M_1M_2} \) to be a relative compact underlying space is \( \alpha p > 1 \) (see e.g. \( [10, 9] \)). Having this in mind, we wish to prove that there exits \( \alpha \in (0,1) \) and \( p > 1 \) with \( \alpha p > 1 \) together with the property: given \( \varepsilon > 0 \), there is \( M_2 > 0 \) such that

\[
P([Y^n]_{W^{\alpha,p}(\bar{S};\mathbb{R}^2 \times \mathbb{R}^2)} > M_2) < \frac{\varepsilon}{2} \text{ for every } n \in \mathbb{N}.
\]  

(47)

\(^1\text{See e.g. } [21], \text{ Theorem 3.28 (The Burkholder-Davis-Gundy's inequality). Let } M \in \mathcal{M}^{c,loc} \text{ and call } M_* := \max_{0 \leq s \leq T} |M_s|. \text{ For every } m > 0, \text{ there exists universal positive constants } k_m, K_m \text{ (depending only on } m), \text{ such that the inequalities}

\[
k_m E(< M >_{\mathbb{T}}^m \leq E((M_*^2)^{2m}) \leq K_m E(< M >_{\mathbb{T}}^m)
\]

hold for every stopping time } T. \text{ Note that } \mathcal{M}^{c,loc} \text{ denotes the space of continuous local martingales and } < X > \text{ represents the quadratic variance process of } X \in \mathcal{M}^{c,loc}.\)
Using Markov’s inequality, we obtain

\[ P([Y^n]_{\mathcal{W}^{\alpha,p}(\mathbb{R}^2 \times \mathbb{R}^2)} > M_2) \leq \frac{1}{M_2} E \left[ \int_0^T \int_0^T \frac{|Y^n_t - Y^n_r|^p}{|t-r|^{1+\alpha p}} \, dt \, dr \right] \]

\[ = \frac{C}{M_2} \int_0^T \int_0^T E[|Y^n_t - Y^n_r|^p] \frac{1}{|t-r|^{1+\alpha p}} \, dt \, dr. \]  

(48)

For \( t > r \), we have

\[ Y^n_t - Y^n_r = \begin{bmatrix} \int_r^t F_1(X^n_y(h^n(y))) \, dy \\ \int_r^t F_2(X^n_y(h^n(y))) \, dy + \int_r^t \sigma(X^n_y(h^n(y))) \, d\beta(y) \end{bmatrix}. \]

(49)

Let us introduce some further notation. For a vector \( u = (u_1, u_2) \), we set \( |u| := |u_1| + |u_2| \). At this moment, we consider the following expression

\[ |Y^n_t - Y^n_r| = \left| \int_r^t F_1(X^n_y(h^n(y))) \, dy \right| + \left| \int_r^t F_2(X^n_y(h^n(y))) \, dy + \int_r^t \sigma(X^n_y(h^n(y))) \, d\beta(y) \right|. \]

(50)

Taking the modulus up to the power \( p > 1 \) together with applying Minkowski inequality, we have

\[ |Y^n_t - Y^n_r|^p = \left( \left| \int_r^t F_1(X^n_y(h^n(y))) \, dy \right|^p \right) \]

\[ + \left( \left| \int_r^t F_2(X^n_y(h^n(y))) \, dy + \int_r^t \sigma(X^n_y(h^n(y))) \, d\beta(y) \right|^p \right) \]

\[ \leq C \left( \left| \int_r^t F_1(X^n_y(h^n(y))) \, dy \right|^p \right) \]

\[ + \left( \left| \int_r^t F_2(X^n_y(h^n(y))) \, dy + \int_r^t \sigma(X^n_y(h^n(y))) \, d\beta(y) \right|^p \right) \]

\[ \leq C \left( \int_r^t |F_1(X^n_y(h^n(y)))|^p \, dy + \int_r^t |F_2(X^n_y(h^n(y)))|^p \, dy + \int_r^t |\sigma(X^n_y(h^n(y)))|^p \, d\beta(y) \right). \]

(51)

Taking the expectation on (51), we obtain the following estimate

\[ E[|Y^n_t - Y^n_r|^p] \leq C(t-r)^p + C E \left( \left| \int_r^t \sigma(X^n_y(h^n(y))) \, d\beta(y) \right|^p \right). \]

(52)
Applying the Burkholder-Davis-Gundy’s inequality to the second term of the right hand side of (52), we obtain

\[ E \left[ \left| \int_t^r \sigma(X^n_y(h^n(y)))dB(y) \right|^p \right] \leq CE \left[ \left( \int_t^r dy \right)^{p/2} \right] \leq C(t-r)^{p/2}. \tag{53} \]

On the other hand, if \( \alpha < \frac{1}{2} \), then

\[ \int_0^T \int_0^T \frac{1}{|t-r|^{1+(\alpha-\frac{1}{2})p}} dt dr < \infty. \tag{54} \]

Consequently, we can pick \( \alpha < \frac{1}{2} \). Taking now \( p > 2 \) together with the constraint \( \alpha p > 1 \), we can find \( M_2 > 0 \) such that

\[ P([Y^n_t]_{W^{\alpha,p}(S;\mathbb{R}^2 \times \mathbb{R}^2)} > M_2) < \frac{\varepsilon}{2}. \tag{55} \]

This argument completes the proof of this Lemma. \( \square \)

From Lemma \( 6.1 \), we have obtained that the sequence \( \{P^n\} \) is tight in \( C(\bar{S};\mathbb{R}^2 \times \mathbb{R}^2) \). Applying the Prokhorov’s Theorem, there are subsequences \( \{P^n_k\} \) which converge weakly to some \( P(Y_t) \) as \( n \to \infty \). For simplicity of the notation, we denote these subsequences by \( \{P^n\} \). This means that we have \( \{P^n\} \) converging weakly to some probability measure \( P \) on Borel sets in \( C(\bar{S};\mathbb{R}^2 \times \mathbb{R}^2) \).

Since we have that \( P^n(Y^n_t) \) converges weakly to \( P(Y_t) \) as \( n \to \infty \), by using the “Skorohod Representation Theorem”, there exists a probability space \((\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P})\) with the filtration \( \{\tilde{\mathcal{F}}_t\} \) and \( \tilde{Y}^n_t, \tilde{Y}_t \) belonging to \( C(\bar{S};\mathbb{R}^2 \times \mathbb{R}^2) \) with \( n \in \mathbb{N} \), such that \( P(\tilde{Y}) = P(Y) \), \( P(\tilde{Y}^n_t) = P(Y^n_t) \), and \( \tilde{Y}^n_t \to \tilde{Y}_t \) as \( n \to \infty \), \( \tilde{P}\)-a.s. Moreover, let \((\tilde{X}^n_t, \tilde{\Phi}^n_t)\) and \((\tilde{X}_t, \tilde{\Phi}_t)\) be the solutions of the Skorohod equations

\[ \tilde{X}^n_t = \tilde{Y}^n_t + \tilde{\Phi}^n_t, \]
\[ \tilde{X}_t = \tilde{Y}_t + \tilde{\Phi}_t, \tag{56} \]

respectively. Then the continuity result in Theorem \( 5.1 \) implies that the sequence \((\tilde{X}^n_t, \tilde{\Phi}^n_t)\) converges to \((\tilde{X}_t, \tilde{\Phi}_t) \in C(\bar{S};D \times D) \times C(\bar{S}) \) uniformly in \( t \in \bar{S} \), \( \tilde{P}\)-a.s as \( n \to \infty \). Hence, we still need to prove that \( \tilde{Y}^n_t \) converges to \( \tilde{Y}_t \) in some sense, where we denote

\[ \tilde{Y}^n_t := \tilde{X}_0 + \int_0^t b(\tilde{X}^n_y(h^n(y)))dy + \int_0^t \sigma(\tilde{X}^n_y(h^n(y)))d\tilde{B}(y). \tag{57} \]
and
\[ \tilde{Y}_t := \tilde{X}_0 + \int_0^t b(\tilde{X}^n_y(y))dy + \int_0^t \sigma(\tilde{X}^n_y(y))d\tilde{B}(y). \] (58)

To complete the proof of the existence of solutions to the problem (17)-(18) in the sense of Definition 5.1, we consider the following Lemma.

**Lemma 6.2.** The pair \((\tilde{X}_t, \tilde{\Phi}_t)\) \(\in C(\tilde{S}; D \times D) \times C(\tilde{S})\) cf. (56) is a solution of the Skorohod-like system
\[ \tilde{X}_t = \tilde{X}_0 + \int_0^t b(\tilde{X}_y(y))dy + \int_0^t \sigma(\tilde{X}_y(y))d\tilde{B}(y) + \tilde{\Phi}_t, \] (59)
with \(\tilde{X}_0 \in D\).

**Proof.** We consider the term \(\int_0^t \sigma(\tilde{X}^n_t(h_n(y)))d\tilde{B}(y)\) with the step process \(\sigma(\tilde{X}^n_t(h_n(y)))\).

Approximating this stochastic integral by Riemann-Stieltjes sums (see e.g. [13]), it yields
\[ \int_0^t \sigma(\tilde{X}^n_y(h_n(y)))d\tilde{B}(y) = \sum_{k=0}^{n-1} \sigma(\tilde{X}^n_t(h_n(t)))(B(t_{k+1}^n) - B(t_k^n)). \] (60)

This gives by taking the limit \(n \to \infty\) in (60)
\[ \lim_{n \to \infty} \int_0^t \sigma(\tilde{X}^n_y(h_n(y)))d\tilde{B}(y) = \lim_{n \to \infty} \sum_{k=0}^{n-1} \sigma(\tilde{X}^n_t(h_n(t)))(B(t_{k+1}^n) - B(t_k^n)) \]
\[ = \sum_{k=0}^{n-1} \sigma(\tilde{X}_t(t))(B(t_{k+1}^n) - B(t_k^n)) = \int_0^t \sigma(\tilde{X}_y(y))d\tilde{B}(y). \] (61)

By the fact that \((\tilde{X}_t^n, \tilde{\Phi}_t^n)\) converges to \((\tilde{X}_t, \tilde{\Phi}_t)\) \(\in C(\tilde{S}; D \times D) \times C(\tilde{S})\) uniformly in \(t \in [0, T]\) \(\tilde{P}\)-a.s as \(n \to \infty\) together with (61), we obtain that
\[ \tilde{X}_t^n = \tilde{X}_0 + \int_0^t b(\tilde{X}^n_y(h^n(y)))dy + \int_0^t \sigma(\tilde{X}^n_y(h^n(y)))d\tilde{B}(y) + \tilde{\Phi}_t^n. \] (62)
converges to
\[ \tilde{X}_t = \tilde{X}_0 + \int_0^t b(\tilde{X}_y(y))dy + \int_0^t \sigma(\tilde{X}_y(y))d\tilde{B}(y) + \tilde{\Phi}_t, \] \(\tilde{P}\) – a.s as \(n \to \infty\). (63)
6.3.1. Proof of Theorem 6.2

Proof. We take \( X_t, X'_t \in C(\overline{S}; D \times D) \) two solutions to (17)-(18) with the same initial values \( X(0) = X'(0) \).

Moreover, suppose that the supports of \( b \) and \( \sigma \) is included in the ball \( B(x_0, \delta) \) for some \( x_0 \in \partial D \). We use the proof idea of the Lemma 5.3 in [29]. Let us recall the assumption (A_3), where \( D \) satisfies the following condition: It exists a positive number \( \nu \) such that for each \( x_0 \in \partial D \) we can find \( f \in C^2(\mathbb{R}^2 \times \mathbb{R}^2) \) satisfying

\[
<y - x, n> + \frac{1}{\nu} < \nabla f(x), n> |y - x|^2 \geq 0.
\]

for any \( x, y \in B(x_0, \delta) \cap \partial D \) and \( n \in \mathcal{N}_x \). Then, we have

\[
< X_s - X'_s, d\Phi_s - d\Phi'_s > - \frac{1}{\nu} |X_s - X'_s|^2 < 1, d\Phi_s - d\Phi'_s >
\]

\[
= -(< X_s - X'_s, d\Phi_s > + \frac{1}{\nu} |X_s - X'_s|^2 < 1, d\Phi_s >)
\]

\[
- (< X_s - X'_s, d\Phi'_s > + \frac{1}{\nu} |X_s - X'_s|^2 < 1, d\Phi'_s >) \leq 0,
\]

where \( 1 \) is the unit vector appearing in Condition (A_7).

Using similar ideas as in [24], Proposition 4.1, we have the following estimate

\[
|X_t - X'_t|^2 \exp \left\{- \frac{1}{\nu} (\Phi(X_t) - \Phi'(X_t)) \right\} \leq
\]

\[
2 \left( \exp \left\{- \frac{1}{\nu} (\Phi(X_y) - \Phi'(X_y)) \right\} \int_0^t (b(X_y(y)) - b(X'_y(y)))dy 
\]

\[
+ \exp \left\{- \frac{1}{\nu} (\Phi(X_y) - \Phi'(X_y)) \right\} \int_0^t (\sigma(X_y(y)) - \sigma(X_y(y)))dB(y) \right)^2 
\]

\[
+ \exp \left\{- \frac{1}{\nu} (\Phi(X_y) - \Phi'(X_y)) \right\} \int_0^t \left( 2 < X_y - X'_y, l > - \frac{1}{\nu} |X_y - X'_y|^2 \right) d\Phi_y 
\]

\[
+ \exp \left\{- \frac{1}{\nu} (\Phi(X_y) - \Phi'(X_y)) \right\} \int_0^t \left( 2 < X_y - X'_y, l > - \frac{1}{\nu} |X_y - X'_y|^2 \right) d\Phi'_y 
\]

\[
2 \int_0^t \left| b(X_y(y)) - b(X'_y(y)) \right|^2 \exp \left\{- \frac{2}{\nu} (\Phi(X_y) - \Phi'(X_y)) \right\} dy 
\]

\[
+ 2 \int_0^t |\sigma(X_y(y)) - \sigma(X_y(y))|^2 \exp \left\{- \frac{2}{\nu} (\Phi(X_y) - \Phi'(X_y)) \right\} dy 
\]

\[
+ \int_0^t \left( 2 < X_y - X'_y, l > - \frac{1}{\nu} |X_y - X'_y|^2 \right) \exp \left\{- \frac{1}{\nu} (\Phi(X_y) - \Phi'(X_y)) \right\} d\Phi_y 
\]

\[
+ \int_0^t \left( 2 < X_y - X'_y, l > - \frac{1}{\nu} |X_y - X'_y|^2 \right) \exp \left\{- \frac{1}{\nu} (\Phi(X_y) - \Phi'(X_y)) \right\} d\Phi'_y.
\]
On the other hand, taking the expectation are both sides of (65) and using the Lipschitz condition to the first term of the right hand side together with (64), we are led to

\[
E \left( |X_t - X'_t|^2 \exp \left\{ -\frac{1}{\nu} (\Phi(X_t) - \Phi'(X_t)) \right\} \right) \leq C \int_0^t E \left( |X_y - X'_y|^2 \exp \left\{ -\frac{2}{\nu} (\Phi(X_y) - \Phi'(X_y)) \right\} \right) dy.
\]

(66)

This also implies that

\[
E[|X_t - X'_t|^2] \leq C \int_0^t E[|X_y - X'_y|^2] dy.
\]

(67)

Hence, \( X_t = X'_t \) for all \( t \in [0, T] \). Then, the pathwise uniqueness of solutions to (17) holds true. On the other hand, combining the Lemma 6.2 together with the fact that the pathwise uniqueness implies the uniqueness of strong solutions (see in [18], Theorem IV-1.1). Therefore, there is a unique solution \((X_t, \Phi_t) \in C(\bar{S}; D \times D) \times C(\bar{S})\) of (17).

\[\square\]

Remark 6.1. What concerns the stability with respect to data and parameters of our concept of solution to the Skorohod-like system, we can show the stability with respect to the initial data via standard arguments (see e.g. [13]). However, more work is needed, for instance in terms of energy-like estimates, to prove the structural stability with respect to the model nonlinearities and coefficients as there are no general known results in this direction. We expect however our problem to be well-posed in Hadamard’s sense.

7. Concluding remarks

In this paper, we have shown the existence and uniqueness of solutions to a system of Skorohod-like stochastic differential equations modeling the dynamics of a mixed population of active and passive pedestrians walking within a heterogeneous environment in the presence of a stationary fire. Due to the discomfort pressure term as well as to the Morse potential preventing particles (pedestrians) to overlap, our model is
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nonlinearly coupled. The main feature of the model is that the dynamics of the crowd takes place in an heterogeneous domain, i.e. obstacles hinder the motion. Hence, to allow the SDEs to account for the presence of the obstacles, we formulate our crowd dynamics scenario as a Skorohod-like system with reflecting boundary condition posed in a bounded domain in \( \mathbb{R}^2 \). Then we use compactness methods to prove the existence of solutions. The uniqueness of solutions follows by standard arguments.

There are a number of open issues that are worth to be investigated for our system:

1. To obtain a better insight on how the solution of the SDEs behave and how close is this behaviour to what is expected from standard evacuation scenarios, a convergent numerical approximation of solutions to (17)-(18) needs to be implemented. One possible route is to design an iterative weak approximation of the Skorohod system as it is done e.g. in [3], [25], and in the references cited therein.

2. We did assume that the fire is stationary, i.e. \( \partial F \) is independent on \( t \). Using the working technique from [25], we expect that it is possible to handle the case of a time-evolving fire, provided the shape of the fire \( \partial F(t) \) is sufficiently regular and it is \textit{a priori} prescribed.

3. From a mathematical point of view, the situation becomes a lot more challenging when there is a feedback mechanism between the pedestrian dynamics and the environment (fire and geometry). Empirically, such pedestrians-environment feedback was pointed out in [28]. An extension can be done in this context using the smoke observable \( s(x,t) \). As a further development of our model, we intend to incorporate the "transport" of smoke eventually via a measure-valued equation (cf. e.g. [14]), coupled with our SDEs for the pedestrian dynamics. In this case, besides the well-posedness question, it is interesting to study the large-time behavior of the system of evolution equations. Instead of a measure-valued equation for the smoke dynamics, one could also use a stochastically perturbed diffusion-transport equation. In this case, the approach from [10] is potentially applicable, provided the coupling between the SDEs for the crowd dynamics and the SPDE for the smoke evolution is done in a well-posed manner. However, in both cases, it is not yet clear cut how to couple correctly the model equations.

4. From the modeling point of view, it would be very useful to find out to which extent the motion of active particles can affect the motion of passive particles so that
the overall evacuation time is reduced. Note that our crowd dynamics context does not involve leaders, and besides the social pressure and the repelling from overlapping, there are no other imposed interactions between pedestrians. In this spirit, we are close to the setting described in [5], where active and passive particles interplay together to find exists in a maze. Further links between maze-solving strategies and our crowd dynamics scenario would need to be identified to make progress in this direction.

Appendix A. Regularized Eikonal equation for motion planning

To describe how the active population moves within $D$, we use a motion planning in terms of the solution of the following regularized Eikonal equation:

$$
\begin{cases}
-\varsigma \Delta \phi_\varsigma + |\nabla \phi_\varsigma|^2 = f^2 \quad \text{in } D, \\
\phi_\varsigma = 0 \quad \text{at } E, \\
\nabla \phi_\varsigma \cdot n = 0 \quad \text{at } \partial(\Lambda \setminus (G \cup E \cup F)),
\end{cases}
$$

(68)

where $\varsigma > 0$ given sufficiently small. In fact, $|\nabla \phi_\varsigma|$ plays the role of a priori known guidance (navigation information). Inspired very much from the implementation of video games, this is a strategy commonly used in most major crowd evacuation softwares, i.e. the map of the building to be evacuated is built-in. An alternative motion guidance strategy is suggested in [34].

We point out the existence and uniqueness of classical solutions to the problem (68) in the following Lemma.

**Lemma A.1.** Assume that $f \in C^\alpha(D)$ with $0 < \alpha < 1$. Let $D \subset \mathbb{R}^2$ be a bounded domain with $\partial D \cup \partial G \in C^{2,\alpha}$. Then the problem (68) has a unique solution $\phi_\varsigma \in C(\overline{D}) \cap C^2(D)$.

**Proof.** The idea of this proof comes from Theorem 2.1, p.10, in [30] for the case of the Dirichlet problem. In fact, the semilinear viscous problem (68) can be transformed into a linear partial differential equation via $w_a : D \rightarrow \mathbb{R}$ given by

$$
w_a(\phi_\varsigma) := \exp(-\varsigma^{-1} \phi_\varsigma) - 1,
$$

(69)

where $a = \frac{1}{\varsigma}$. Then $w_a \in C(\overline{D}) \cap C^2(D)$ becomes a solution of the following linear
partial differential equation with mixed Dirichlet-Neumann boundary conditions:

\[
\begin{cases}
-\Delta w_a + f^2 a^2 w_a + a^2 = 0 & \text{in } D, \\
w_a = 0 & \text{at } E, \\
\nabla w_a \cdot \mathbf{n} = 0 & \text{at } \partial D \cup \partial G.
\end{cases}
\]

Furthermore, there is a unique solution \( w_a \in C(\overline{D}) \cap C^2(D) \) of the problem \( \text{(70)} \) (see in Theorem 1, \[23\]). This also implies that there is a unique solution \( \phi_a \in C(\overline{D}) \cap C^2(D) \) to the problem \( \text{(68)} \).

\( \square \)

**Appendix B. Nondimensionalization**

In this section, we nondimensionalize the system \( \text{(1)}-\text{(4)} \). By this procedure, we aim to identify the relevant characteristic time and length scales involved in this crowd dynamics scenario. We let \( \hat{D} \) denote the scaled set \( \frac{x_{\text{ref}}}{\text{max}} D \). We introduce \( x_{\text{ref}}, \, x_{\text{ref}}, \, x_{\text{ref}} \) and \( t_{\text{ref}}, \, t_{\text{ref}}, \, t_{\text{ref}} \) as possible characteristic length and time scales, respectively. We choose

\[
X_{a_i}(t_{\text{ref}}) := \frac{x_{a_i}(t)}{x_{a_i}^{\text{ref}}}, \quad X_{p_k}(t_{\text{ref}}) := \frac{x_{p_k}(t)}{x_{p_k}^{\text{ref}}} \quad \text{and} \quad \tau := \frac{t}{t_{\text{ref}}} = \frac{t_{\text{ref}}}{t_{\text{ref}}} = t_{\text{ref}},
\]

where \( x_{\text{ref}} = x_{\text{ref}} = x_{\text{ref}} = t_{\text{ref}} = t_{\text{ref}} = t_{\text{ref}} \). Then, equations \( \text{(1)} \) and \( \text{(4)} \) become

\[
\begin{cases}
\frac{x_{\text{ref}}}{\text{max}} \frac{d}{dt} X_{a_i}(t_{\text{ref}}) = \Lambda_{\text{ref}} \delta_{\text{ref}}(S(x_{\text{ref}} X_{a_i}(t_{\text{ref}}))) \frac{\partial}{\partial x_{a_i}} \delta(X_{a_i}(t_{\text{ref}})) \left( \mu_{\text{ref}} \delta_x X_{a_i}(t_{\text{ref}}), t_{\text{ref}} \right), \\
X_{a_i}(0) = \frac{x_{a_i,0}}{x_{a_i}^{\text{ref}}},
\end{cases}
\]

where

\[
p(x_{a_i}(t), t) = p_{\text{ref}}(x_{\text{ref}} X_{a_i}(t_{\text{ref}}), t_{\text{ref}}) = \mu_{\text{ref}} \delta_x(t_{\text{ref}} x_{a_i}) \int_{\partial B(x_{\text{ref}} X_{a_i}, \beta_{\text{ref}})} \sum_{j=1}^{N} \delta(y_{\text{ref}} Y - x_{\text{ref}} X_{c_j}(t_{\text{ref}})) y_{\text{ref}} dY.
\]

\[
\begin{cases}
\frac{x_{\text{ref}}}{\text{max}} \frac{d}{dt} X_{p_k}(t_{\text{ref}}) = \sum_{j=1}^{N} \frac{x_{\text{ref}} X_{c_j} - x_{\text{ref}} X_{p_k}}{c_{\text{ref}} X_{c_j} - x_{\text{ref}} X_{p_k}} \omega_{\text{ref}} \tilde{\omega} \left( x_{\text{ref}} X_{c_j} - x_{\text{ref}} X_{p_k}, S(x_{\text{ref}} X_{p_k}, t_{\text{ref}}) \right) \\
\quad + \beta_{\text{ref}} \beta(S(\tilde{x}_{\text{ref}} X_{p_k}, t_{\text{ref}})) \sqrt{\frac{d\tilde{B}(t_{\text{ref}})}{d\tau}}, \\
X_{p_k}(0) = \frac{x_{p_k,0}}{x_{p_k}^{\text{ref}}},
\end{cases}
\]

\( \text{(73)} \)
\[
\omega(y, z) = \omega_{\text{ref}} \tilde{\omega}(y_{\text{ref}}, z_{\text{ref}}) = -\beta_{\text{ref}} \beta(y_{\text{ref}}) \left( C_A e^{-\frac{y_{\text{ref}}}{A}} + C_R e^{-\frac{y_{\text{ref}}}{R}} \right),
\]
(74)

\[
\beta(y) = \beta_{\text{ref}} \beta(y_{\text{ref}}) = \begin{cases} 
 1, & \text{if } y_{\text{ref}} < s_{\text{cr}}, \\
 0, & \text{if } y_{\text{ref}} \geq s_{\text{cr}}.
\end{cases}
\]
(75)

Multiplying (71) by \( \frac{t_{\text{ref}} x_{\text{ref}}}{s_{\text{ref}} x_{\text{ref}}} \), we are led to

\[
\left\{ \begin{array}{l}
\frac{d}{dt} X_{ai}(t_{\text{ref}}) = \frac{\Upsilon_{\text{ref}} t_{\text{ref}} \Phi_{\text{ref}}}{x_{\text{ref}}} \sum_{j=1}^{N} \frac{X_{ai}(t_{\text{ref}})}{x_{\text{ref}}} \tilde{\gamma}(S(x_{\text{ref}} X_{ai}(t_{\text{ref}}))) \frac{\nabla X_{ai}(x_{\text{ref}} X_{ai}(t_{\text{ref}})) - \nabla X_{ai}(x_{\text{ref}} X_{ai}(t_{\text{ref}})))}{|X_{ai}(x_{\text{ref}} X_{ai}(t_{\text{ref}})) - \nabla X_{ai}(x_{\text{ref}} X_{ai}(t_{\text{ref}}))|} \tilde{\Phi}_{\text{ref}}(S(x_{\text{ref}} X_{ai}(t_{\text{ref}}))) p_{\text{max}}, \\
-\rho_{\text{ref}} \tilde{\beta}(x_{\text{ref}} X_{ai}(t_{\text{ref}}), t_{\text{ref}})), \\
X_{ai}(0) = \frac{X_{ai}(0)}{x_{\text{ref}}}
\end{array} \right.
\]
(76)

Similarly, we obtain

\[
\left\{ \begin{array}{l}
\frac{d}{dt} X_{pk}(t_{\text{ref}}) = \omega_{\text{ref}} t_{\text{ref}} \Phi_{\text{ref}} \sum_{j=1}^{N} \frac{X_{pk}(t_{\text{ref}})}{x_{\text{ref}}} \tilde{\omega}(|x_{\text{ref}} X_{cj} - x_{\text{ref}} X_{pk}|, S(x_{\text{ref}} X_{pk}, t_{\text{ref}})) \\
+ \beta_{\text{ref}} t_{\text{ref}} \tilde{\beta}(S(x_{\text{ref}} X_{pk}, t_{\text{ref}}))) \sqrt{\frac{d\tau}{dt}} \frac{dB(t_{\text{ref}})}{\sqrt{d\tau}}, \\
X_{pk}(0) = \frac{X_{pk}(0)}{x_{\text{ref}}}
\end{array} \right.
\]
(77)

From (76) and (77) the following dimensionless numbers arise:

\[
\frac{\Upsilon_{\text{ref}} t_{\text{ref}} s_{\text{ref}} p_{\text{max}}}{x_{\text{ref}}}, \quad \frac{\Upsilon_{\text{ref}} t_{\text{ref}} s_{\text{ref}} p_{\text{ref}}}{x_{\text{ref}}}, \quad \frac{\omega_{\text{ref}} t_{\text{ref}}}{x_{\text{ref}}}, \quad \frac{\beta_{\text{ref}} t_{\text{ref}}}{x_{\text{ref}}},
\]
(78)

These dimensionless numbers indicate four different choices of the characteristic time scale \( t_{\text{ref}} \). This is due to the complexity of our system: active and passive agents interplay within the domain geometry as well as the propagation of the smoke. The choice of the corresponding time scale can be the characteristic time capturing relation between the smoke extinction, the walking speed and the discomfort level to the overall population size or the local discomfort, the one for the drift from the smoke propagation, the one for the drift produced by the action of active and passive pedestrians and the one for the amplifying factor on the noise. Therefore, in order to cover the physical relevance of the whole system, we introduce the following rate

\[
\kappa := \max \left\{ \frac{\Upsilon_{\text{ref}} t_{\text{ref}} s_{\text{ref}} p_{\text{max}}}{x_{\text{ref}}}, \frac{\Upsilon_{\text{ref}} t_{\text{ref}} s_{\text{ref}} p_{\text{ref}}}{x_{\text{ref}}}, \frac{\omega_{\text{ref}} t_{\text{ref}}}{x_{\text{ref}}}, \frac{\beta_{\text{ref}} t_{\text{ref}}}{x_{\text{ref}}} \right\}.
\]
(79)
On the other hand, a typical choice for the reference length scale is \( x_{\text{ref}} = \ell \), where \( \ell := \text{diam}(D) \). Finally, we obtain the following nondimensionalized equations

\[
\begin{align*}
\frac{d}{d\tau} X_{a_i}(\tau) &= \kappa \Upsilon(S(X_{a_i}(\tau))) \frac{\nabla X_{a_i} \phi(X_{a_i}(\tau))}{\|\nabla X_{a_i} \phi(X_{a_i}(\tau))\|} (p_{\text{max}} - p(X_{a_i}(\tau), \tau)), \\
X_{a_i}(0) &= X_{a_i,0}, \ i \in \{1, \ldots, N_A\}. 
\end{align*}
\]

(80)

\[
\begin{align*}
\frac{d}{d\tau} X_{p_k}(\tau) &= \kappa \sum_{j=1}^{N} \frac{X_{c_j} - X_{p_k}}{\|X_{c_j} - X_{p_k}\|} w(\|X_{c_j} - X_{p_k}\|, S(X_{p_k}, \tau)) + \kappa \beta(S(X_{p_k}, \tau), \tau) dB(\tau), \\
X_{p_k}(0) &= X_{p_k,0}, \ k \in \{1, \ldots, N_P\}. 
\end{align*}
\]

(81)

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