A KUROSH TYPE THEOREM FOR TYPE $\text{II}_1$ FACTORS

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Abstract. We prove a Kurosh type theorem for free-product type $\text{II}_1$ factors. In particular, if $M = LF_r \otimes \mathcal{R}$, then the free-product type $\text{II}_1$ factors $M * \ldots * M$ are all prime and pairwise non-isomorphic. We also study the case of crossed product type $\text{II}_1$ factors. This paper is a continuation of our previous papers [Oz2] [OP], where the structure of (tensor products of) word hyperbolic group type $\text{II}_1$ factors was studied.

1. Introduction

The classification of type $\text{II}_1$ factors (of discrete groups) was initiated by Murray and von Neumann [MvN] who distinguished the hyperfinite type $\text{II}_1$ factor $\mathcal{R}$ from the group factor $LF_r$ of the free group $F_r$ on $r \geq 2$ generators. Thirty years later, Connes [Co2] proved uniqueness of the injective type $\text{II}_1$ factor. Thus, the group factor $L\Gamma$ of an ICC amenable group $\Gamma$ is isomorphic to the hyperfinite type $\text{II}_1$ factor $\mathcal{R}$. On the other hand, the isomorphism problem of free group factors remains open. To solve this problem, Voiculescu invented free probability theory, which led to a number of deep results on the structure of free group factors (cf. the survey paper [Vo2]). Apart from these results and results of Connes [Co1] and Cowling and Haagerup [CH], the classification of type $\text{II}_1$ factors has been vague by and large. Recently, however, a breakthrough came when Popa [Po2] [Po4] found that unitary conjugacy results can be deduced from existence of finite-dimensional bimodules and obtained quite precise classification theorems for certain classes of type $\text{II}_1$ factors. On the other hand, a $C^*$-algebraic method [Oz2] was proved to be useful in study of type $\text{II}_1$ factors. These methods in combination yielded some prime factorization results in [OP].

This paper is a continuation of [Oz2] and [OP], where the structure of (tensor products of) word hyperbolic group type $\text{II}_1$ factors was studied. In this paper, we will study the structure of free-products and crossed products of certain type $\text{II}_1$ factors. A crucial ingredient of the argument is a computation of the kernels of certain morphisms on $C^*$-algebras. The idea of exploiting a ‘boundary’ to compute...
such kernels is due to Skandalis [Sk] and developed by Higson and Guentner [HG]. We will take advantage of this idea. We denote by $\mathcal{S}$ the class of countable discrete groups $\Gamma$ such that the left and right translation action of $\Gamma \times \Gamma$ on the Stone-Čech remainder $\partial^\beta \Gamma = \beta \Gamma \setminus \Gamma$ is amenable (see Section 4 for details). The class $\mathcal{S}$ was suggested by Skandalis and it contains all subgroups of word hyperbolic groups and discrete subgroups of connected simple Lie groups of rank one [HG] [Sk]. The class $\mathcal{S}$ also contains a group with an infinite amenable normal subgroup (cf. Corollary 4.5). The main result of [Oz2] was solidity of the group factor $L\Gamma$ of a group $\Gamma$ in $\mathcal{S}$. The $q$-Gaussian von Neumann algebras (for certain values of $q$) are other examples of solid factors. Indeed, solidity for certain values of $q$ was proved by Shlyakhtenko [Sh] while factoriality for all values of $q$ was proved by Ricard [Ri]. The main results of [OP] were unique prime factorization and rigidity of their tensor products.

The main result of this paper is a Kurosh type theorem for a free-product of certain type II$_1$ factors. Although the theorem is not as precise as the original Kurosh theorem in group theory, it implies, for instance, that the iterated free-product type II$_1$ factors

$$L\mathcal{F}_\infty \ast (L\mathcal{F}_\infty \otimes \mathcal{R})^n, \quad n = 1, 2, \ldots$$

are mutually non-isomorphic. This is a contrast to Dykema’s theorem (Theorem 3.5 in [Dy1]) that $L\mathcal{F}_\infty \ast (L\mathcal{F}_\infty \otimes \mathcal{L}_\infty [0, 1])^n$ are all isomorphic. There is an obvious similarity between these results and the isomorphism problem of free group factors. In fact, according to Dykema and Rădulescu [DR], isomorphism of free group factors would imply that $\mathcal{M}_1 \ast \mathcal{M}_2 = \mathcal{M}_1 \ast \mathcal{M}_2 \ast L\mathcal{F}_\infty$ for any type II$_1$ factors $\mathcal{M}_1$ and $\mathcal{M}_2$. The proof of the above theorem consists of an adaptation for free-product of the method developed in [Oz2] and [OP] (which we will review in Section 2) and Popa’s work [Po1] on normalizers in a free-product.

**Definition 1.1.** A type II$_1$ factor $\mathcal{M}$ is prime if $\mathcal{M} \neq \mathcal{M}_1 \otimes \mathcal{M}_2$ for any type II$_1$ factors $\mathcal{M}_1$ and $\mathcal{M}_2$. A (finite) von Neumann algebra $\mathcal{M}$ is solid if for any diffuse von Neumann subalgebra $\mathcal{A}$, the relative commutant $\mathcal{A}' \cap \mathcal{M}$ is injective. A (finite) von Neumann algebra is semisolid if for any type II$_1$ von Neumann subalgebra $\mathcal{Q}$, the relative commutant $\mathcal{Q}' \cap \mathcal{M}$ is injective. A von Neumann algebra is semiexact if it contains a ultraweakly dense exact $C^*$-algebra.

There are obvious implications; solid $\Rightarrow$ semisolid $\Rightarrow$ prime for a non-injective type II$_1$ factor. We will see none of these implications is reversible. For a technical reason, the results in this paper are valid only for semiexact von Neumann algebras. There are plenty of semiexact von Neumann algebras. A discrete group $\Gamma$ is exact if and only if its reduced group $C^*$-algebra $C^*_\text{r} \Gamma$ is an exact $C^*$-algebra (cf. [KW]). Thus, by definition, the group von Neumann algebra $L\Gamma$ of a discrete exact group $\Gamma$ is semiexact. The above mentioned Kurosh type theorem implies that every free-product of semiexact type II$_1$ factors is prime. This gives an example of prime type II$_1$ factors which are not semisolid.
We also deal with crossed product and prove that the group-measure space von Neumann algebra $\Gamma \times L^\infty[0, 1]$ of a measure-preserving action of a group $\Gamma$ on the standard probability space $[0, 1]$ is semisolid provided that the group $\Gamma$ is in $\mathcal{S}$. This generalizes Adams’ theorem [Ad] that a measurable orbit equivalence relation of a non-amenable hyperbolic group is indecomposable. This also gives an example of semisolid type II$_1$ factors with the property $(\Gamma)$. Note that a type II$_1$ factor with the property $(\Gamma)$ cannot be solid by Proposition 7 in [Oz2].

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2. Conventions and Preliminary Background

For a discrete group $\Gamma$, we denote by $\lambda$ (resp. $\rho$) the left (resp. right) regular representation of $\Gamma$ on $\ell^2\Gamma$. The reduced group $C^*$-algebra $C^*_\lambda \Gamma$ (resp. $C^*_\rho \Gamma$) is the $C^*$-subalgebra in $\mathbb{B}(\ell^2\Gamma)$ which is generated by $\lambda(\Gamma)$ (resp. $\rho(\Gamma)$) and the group von Neumann algebra $L\Gamma$ is the von Neumann algebra generated by $\lambda(\Gamma)$. We denote by $C^*_{\lambda, \rho} \Gamma$ the $C^*$-subalgebra in $\mathbb{B}(\ell^2\Gamma)$ which is generated by $C^*_{\lambda} \Gamma$ and $C^*_{\rho} \Gamma$. Given a finite von Neumann algebra $\mathcal{M}$, we assume that there is a distinguished faithful normal trace $\tau$ on $\mathcal{M}$ and the trace on its von Neumann subalgebra $\mathcal{N} \subset \mathcal{M}$ is the restriction of $\tau$ on $\mathcal{N}$. So, we will write $L^2\mathcal{M}$ without specifying the trace $\tau$. We denote by $\hat{a}$ the vector in $L^2\mathcal{M}$ corresponding to $a \in \mathcal{M}$. The Hilbert space $L^2\mathcal{M}$ is an $\mathcal{M}$-$\mathcal{M}$ bimodule with $a \widehat{xb} = \hat{axb}$ for $a, b, x \in \mathcal{M}$. The canonical conjugation $J_\mathcal{M}$ on $L^2\mathcal{M}$ is given by $J_\mathcal{M}a = a^*$ for $a \in \mathcal{M}$. We simply denote $J_\mathcal{M}$ by $J$ if there are no confusions. We will use the same notations for a $C^*$-algebra with a faithful trace. When dealing with $C^*$-algebras, the symbol $\otimes$ means the algebraic tensor product while $\otimes_{\text{min}}$ means the minimal (spatial) tensor product. The spatial tensor product of von Neumann algebras is denoted by $\hat{\otimes}$ and the Hilbert space tensor product of Hilbert spaces is simply denoted by $\otimes$. The term ‘ucp’ is an abbreviation for ‘unital completely positive’. All representations and homomorphisms are assumed to be self-adjoint and non-degenerate. All groups denoted by $\Gamma$ and $\Delta$ are assumed to be countable and discrete. All von Neumann algebras are assumed to have separable predual.

We review the method developed in [Oz2] and [OP]. It tells when von Neumann subalgebra $\mathcal{Q}$ in a finite von Neumann algebra $\mathcal{M}$ has the relative commutant $\mathcal{Q}' \cap \mathcal{M}$ that is not injective.
Let $\mathcal{M} \subset \mathbb{B}(L^2\mathcal{M})$ be a finite von Neumann algebra and $\mathcal{Q} \subset \mathcal{M}$ be an injective von Neumann subalgebra. Then, there exists a conditional expectation $\Psi_\mathcal{Q}$ from $\mathbb{B}(L^2\mathcal{M})$ onto $\mathcal{Q}'$ which is proper, i.e.,

$$\forall x \in \mathbb{B}(L^2\mathcal{M}) \quad \Psi_\mathcal{Q}(x) \in \text{conv}^{\text{uw}}\{uxu^*: u \in \mathcal{U}(\mathcal{Q})\}.$$ 

It follows that $\Psi_\mathcal{Q}|_{\mathcal{M}'} = \text{id}_{\mathcal{M}'}$ and that $\Psi_\mathcal{Q}$ is a trace preserving conditional expectation from $\mathcal{M}$ onto $\mathcal{Q}' \cap \mathcal{M}$, which coincides with the unique trace preserving conditional expectation $E_{\mathcal{Q}'}|_{\mathcal{M}}$ from $\mathcal{M}$ onto $\mathcal{Q}' \cap \mathcal{M}$. Since $\Psi_\mathcal{Q}|_{\mathcal{M}'} = \text{id}_{\mathcal{M}'}$, the ucp map $\Psi_\mathcal{Q}$ is an $\mathcal{M}'$-bimodule map;

$$\forall x, y \in \mathcal{M}', \forall b \in \mathbb{B}(L^2\mathcal{M}) \quad \Psi_\mathcal{Q}(xby) = x\Psi_\mathcal{Q}(b)y.$$ 

In particular, we have

$$\forall a \in \mathcal{M}, \forall x \in \mathcal{M}' \quad \Psi_\mathcal{Q}(ax) = E_{\mathcal{Q}'}|_{\mathcal{M}}(a)x.$$ 

Now Lemma 5 in [Oz2] can be interpreted as follows;

**Lemma 2.1 ([Oz2]).** Let $\mathcal{M} \subset \mathbb{B}(L^2\mathcal{M})$ be a finite von Neumann algebra with an injective von Neumann subalgebra $\mathcal{Q} \subset \mathcal{M}$. If there exist unital ultraweakly dense $C^*$-subalgebras $B \subset \mathcal{M}$ and $C \subset \mathcal{M}'$ with $B$ exact such that the ucp map

$$\tilde{\Psi}_\mathcal{Q}: B \otimes C \ni \sum_{k=1}^n a_k \otimes x_k \mapsto \Psi_\mathcal{Q}(\sum_{k=1}^n a_k x_k) \in \mathbb{B}(\mathcal{H})$$

is continuous w.r.t. the minimal tensor norm on $B \otimes C$, then the relative commutant $\mathcal{Q}' \cap \mathcal{M}$ is injective.

As in [OP], in an actual application, non-injectivity of $\mathcal{Q}' \cap \mathcal{M}$ forces

$$\mathbb{K}(\mathcal{K}) \otimes_{\text{min}} \mathbb{B}(L^2\mathcal{N}) \not\subset \ker \Psi_\mathcal{Q}$$

for some von Neumann subalgebra $\mathcal{N} \subset \mathcal{M}$ and a Hilbert space $\mathcal{K}$ such that $L^2\mathcal{M} = \mathcal{K} \otimes L^2\mathcal{N}$ as a right $\mathcal{N}$ module. If this is the case, we may find a finite rank projection $p$ on $\mathcal{K}$ with $b = \Psi_\mathcal{Q}(p \otimes 1_\mathcal{N}) \neq 0$. Since $\Psi_\mathcal{Q}$ is proper, $b$ commutes with the right $\mathcal{N}$ action, or equivalently $b \in (\mathbb{B}(\mathcal{K}) \otimes \mathcal{N}) \cap \mathcal{Q}'$. By Proposition 1.3.2 in [Po2], we have $(\text{Tr} \otimes \tau_\mathcal{N})(b) \leq \text{Tr}(p) < \infty$. Thus, there is a non-zero spectral projection $e$ of $b$ with $(\text{Tr} \otimes \tau_\mathcal{N})(e) < \infty$. It follows that $\mathcal{H} = eL^2\mathcal{M}$ is a $\mathcal{Q} \cap \mathcal{N}$-sub-bimodule of $L^2\mathcal{M}$ with $\dim_{\mathcal{N}} \mathcal{H} < \infty$. (Strictly speaking, $\mathcal{H}$ is a $\mathcal{Q} \cap \mathcal{N}'$ bimodule where $e' = J_\mathcal{M}eJ_\mathcal{M}$.) If in addition $\mathcal{M}$ and $\mathcal{N}$ are factors, then we can apply the following Lemma 5 in [Po3] and Proposition 12 in [OP].

**Lemma 2.2 ([OP] [Po3]).** Let $\mathcal{N}$ and $\mathcal{Q}$ be subfactors in a type $\text{II}_1$ factor $\mathcal{M}$. If there exists a non-zero $Q, Q'$ sub-bimodule $\mathcal{H} \subset L^2\mathcal{M}$ with $\dim_{\mathcal{N}} \mathcal{H} < \infty$, then there exist projections $e \in \text{Proj}(\mathcal{N})$ and $q \in \text{Proj}(\mathcal{Q})$, a non-zero partial isometry $v \in \mathcal{M}$ and a homomorphism $\theta: q\mathcal{Q}q \to e\mathcal{N}e'$ such that

$$vv^* \in (q\mathcal{Q}q)' \cap q\mathcal{M}q, \quad v^*v \in \theta(q\mathcal{Q}q)' \cap \mathcal{eM}e \text{ and } xv = v\theta(x) \text{ for } x \in q\mathcal{Q}q.$$
We need two more lemmas. The first is about normalizers in a free-product due to Popa. The following is formally stronger than stated in Theorem 6.1 and Corollary 4.3 in [Po1], but their proofs are same (along Proposition 4.1 in [Po1]).

Lemma 2.3 ([Po1]). Let \( \mathcal{N}_1 \subset \mathcal{M}_1 \) and \( \mathcal{M}_2 \) be finite von Neumann algebras and let \( \mathcal{M} = \mathcal{M}_1 \ast \mathcal{M}_2 \) be their free-product. If \( \mathcal{N}_1 \) is diffuse and a unitary operator \( u \in \mathcal{M} \) satisfies \( u^* \mathcal{N}_i u \subset \mathcal{M}_i \) for some \( i \), then \( i = 1 \) and \( u \in \mathcal{M}_1 \).

The last lemma in this section is about nuclearity of reduced free-product \( C^* \)-algebras. We recall that a ucp map \( \varphi : A \to B \) is said to be nuclear if there exist nets of ucp maps \( \beta^\lambda : A \to \mathbb{M}_{n(\lambda)}(\mathbb{C}) \) and \( \alpha^\lambda : \mathbb{M}_{n(\lambda)}(\mathbb{C}) \to B \) such that \( \alpha^\lambda \circ \beta^\lambda \to \varphi \) in the point-norm topology. If \( \varphi : A \to B \) is a nuclear ucp map and \( B \subset \mathbb{B}(\mathcal{H}) \), then the ucp map
\[
\varphi \times \text{id}_{B'} : A \otimes B' \ni \sum a_k \otimes x_k \mapsto \sum \varphi(a_k)x_k \in \mathbb{B}(\mathcal{H})
\]
is continuous w.r.t. the minimal tensor norm on \( A \otimes B' \).

Lemma 2.4. Let \( B_i \subset \mathbb{B}(\mathcal{H}_i) \) be a \( C^* \)-subalgebra with a \( B_i \)-cyclic unit vector \( \xi_i \in \mathcal{H}_i \). We denote by \( \omega_i \) the vector state corresponding to \( \xi_i \). If both \( B_i \) are exact, then the inclusion map of \( (B_1, \omega_1) * (B_2, \omega_2) \) into \( (\mathbb{B}(\mathcal{H}_1), \omega_1) * (\mathbb{B}(\mathcal{H}_2), \omega_2) \) is nuclear.

Proof. It suffices to show that the inclusion map is approximated by ucp maps which factor through nuclear \( C^* \)-algebras. We first note that if \( B \subset \mathbb{B}(\mathcal{H}) \) is exact, then so is \( B + \mathbb{K}(\mathcal{H}) \). (This is well-known and the proof is involved. Indeed, by Kirchberg’s theorem [Ki1], \( B \) is locally reflexive and the quotient \( C := B/(B \cap \mathbb{K}(\mathcal{H})) \) is exact. Moreover, the short exact sequence
\[
0 \to \mathbb{K}(\mathcal{H}) \to B + \mathbb{K}(\mathcal{H}) \to C \to 0
\]
has ucp local splittings by the Effros-Haagerup lifting theorem [EH]. Now, the exactness of \( B + \mathbb{K}(\mathcal{H}) \) follows from that of \( \mathbb{K}(\mathcal{H}) \) and \( C \) by the 3-by-3 lemma.) By replacing \( B_i \) with \( B_i + \mathbb{K}(\mathcal{H}_i) \), we may assume that \( \mathbb{K}(\mathcal{H}_i) \subset B_i \). Since \( B_i \) is exact, there are a net of finite dimensional subspaces \( \mathcal{K}_i^\lambda \subset \mathcal{H}_i \) with \( \xi_i \in \mathcal{K}_i^\lambda \) with the corresponding compression \( \beta_i^\lambda : B_i \to \mathbb{B}(\mathcal{K}_i^\lambda) \), and a net of ucp maps \( \alpha_i^\lambda : \mathbb{B}(\mathcal{K}_i^\lambda) \to \mathbb{B}(\mathcal{H}_i) \) such that the net \( \alpha_i^\lambda \circ \beta_i^\lambda \) converges pointwise to the inclusion \( B_i \hookrightarrow \mathbb{B}(\mathcal{H}_i) \).

Since the rank-one projection \( p_i \) corresponding to \( \xi_i \) is in \( B_i \), we have \( \lim_\lambda \alpha_i^\lambda(p_i) = p_i \). Thus, by perturbing \( \alpha_i^\lambda \), we may assume that \( \alpha_i^\lambda(p_i) = p_i \) for all \( \lambda \). It follows from Choda-Blanchard-Dykema’s theorem [BD] that
\[
\beta_1^\lambda \ast \beta_2^\lambda : (B_1, \omega_1) \ast (B_2, \omega_2) \to (\mathbb{B}(\mathcal{K}_1^\lambda), \omega_1) \ast (\mathbb{B}(\mathcal{K}_2^\lambda), \omega_2)
\]
and
\[
\alpha_1^\lambda \ast \alpha_2^\lambda : (\mathbb{B}(\mathcal{K}_1^\lambda), \omega_1) \ast (\mathbb{B}(\mathcal{K}_2^\lambda), \omega_2) \to (\mathbb{B}(\mathcal{H}_1), \omega_1) \ast (\mathbb{B}(\mathcal{H}_2), \omega_2)
\]
are ucp maps such that the net \( (\alpha_1^\lambda \ast \alpha_2^\lambda) \circ (\beta_1^\lambda \ast \beta_2^\lambda) \) converges pointwise to the inclusion map \( (B_1, \omega_1) \ast (B_2, \omega_2) \hookrightarrow (\mathbb{B}(\mathcal{H}_1), \omega_1) \ast (\mathbb{B}(\mathcal{H}_2), \omega_2) \). Since the \( C^* \)-algebras \( (\mathbb{B}(\mathcal{K}_1^\lambda), \omega_1) \ast (\mathbb{B}(\mathcal{K}_2^\lambda), \omega_2) \) are nuclear (see e.g., [Ki2] or [Oz4]), we are done. \( \square \)
3. Free-Product

We recall the reduced free-product construction (in the tracial setting). Let $B_i$, $i \in \{1, 2\}$ be a $C^*$-algebra with a faithful trace $\tau_i$. Let $B_i$ act on the GNS-Hilbert space $\mathcal{H}_i = L^2(B_i, \tau_i)$. We denote by $\hat{a} \in \mathcal{H}_i$ the vector associated with $a \in B_i$ and denote by $\xi_i = \hat{1} \in \mathcal{H}_i$ the cyclic separating trace vector for $B_i$. The canonical conjugation $J_i$ on $\mathcal{H}_i$ is given by $J_i \hat{a} = \hat{a}^*$. Let $B^0_i = \ker \tau_i \subset B_i$ and let $\mathcal{H}^0_i = \mathcal{H}_i \ominus \mathbb{C} \xi_i$ be the closure of $B^0_i$ in $\mathcal{H}_i$. Then the free-product Hilbert space is

$$\mathcal{H} = \mathbb{C} \xi \oplus \bigoplus_{n \geq 1} \mathcal{H}^0_{i_1} \otimes \mathcal{H}^0_{i_2} \otimes \cdots \otimes \mathcal{H}^0_{i_n}.$$  

We shall describe the left action $\lambda_i : \mathbb{B}(\mathcal{H}_i) \to \mathbb{B}(\mathcal{H})$. It is convenient to introduce a subspace $\mathcal{H}(i) \subset \mathcal{H}$ which is the closed span of $\mathbb{C} \xi_i$ and those direct summands with $i \neq i_1$ in the above representation. Then, there is a canonical unitary operator

$$U_i : \mathcal{H} \to \mathcal{H}(i) \otimes \mathcal{H}(i)$$

which identify $\mathcal{H}(i) \cong \mathbb{C} \xi_i \otimes \mathcal{H}(i)$ and $\mathcal{H}(i)^\perp \cong \mathcal{H}_i^0 \otimes \mathcal{H}(i)$. We define

$$\lambda_i : \mathbb{B}(\mathcal{H}_i) \ni a \mapsto U_i^*(a \otimes 1_{\mathcal{H}(i)})U_i \in \mathbb{B}(\mathcal{H}).$$

The reduced free-product $C^*$-algebra $(B, \tau) = (B_1, \tau_1) \ast (B_2, \tau_2)$ is the $C^*$-algebra $B$ in $\mathbb{B}(\mathcal{H})$ generated by $\lambda_1(B_1)$ and $\lambda_2(B_2)$ with the distinguished trace $\tau(\cdot) = (\cdot, \xi, \xi)$ on $B$. We will omit $\lambda_i$ when there are no confusions. The vector $\xi \in \mathcal{H}$ is a cyclic separating trace vector for $B$ and the corresponding conjugation operator $J$ is given by

$$J(\hat{a}_1 \otimes \cdots \otimes \hat{a}_n) = \hat{a}^*_n \otimes \cdots \otimes \hat{a}^*_1 = (J_{i_n} \hat{a}_n) \otimes \cdots \otimes (J_{i_1} \hat{a}_1)$$

for $a_k \in B^0_{i_k}$ with $i_1 \neq i_2, \ldots, i_{n-1} \neq i_n$. In particular, $J\mathcal{H}(i) \subset \mathcal{H}$ is the closed linear span of $\mathbb{C} \xi_i$ and $\hat{a}_i \otimes \cdots \otimes \hat{a}_i$ with $i_n \neq i$. Let $V_i : \mathcal{H} \to J\mathcal{H}(i) \otimes \mathcal{H}_i$ be the unitary operator which identifies $J\mathcal{H}(i) \cong J\mathcal{H}(i) \otimes \mathbb{C} \xi_i$ and $(J\mathcal{H}(i))^\perp \cong J\mathcal{H}(i) \otimes \mathcal{H}^0_i$. We note that $V_i$ intertwines the right actions of $\mathcal{M}_i$. Moreover, the following is true.

**Lemma 3.1.** We have

$$\lambda_i(a) = JV_i^*(J_{J\mathcal{H}(i)} \otimes J_i a J_i)V_i J$$

$$= V_i^*(P_{\xi} \otimes a + \lambda_i(a)_{J_{J\mathcal{H}(i)} \otimes \mathbb{C} \xi} \otimes 1_{\mathcal{H}_i}) V_i$$

$$= V_j^*(\lambda_i(a)_{J\mathcal{H}(i)} \otimes 1_{\mathcal{H}_i}) V_j$$

for every $a \in \mathbb{B}(\mathcal{H}_i)$ and $j \neq i$, where $P_{\xi}$ is the orthogonal projection onto $\mathbb{C} \xi$.

**Proof.** We first observe that the unitary operator

$$(J_{J\mathcal{H}(i)} \otimes J_i)V_i JU_i^* : \mathcal{H}_i \otimes \mathcal{H}(i) \to \mathcal{H}(i) \otimes \mathcal{H}_i$$

is nothing but the flip of the tensor components $\mathcal{H}_i$ and $\mathcal{H}(i)$. Hence, we have $JV_i^*(J \otimes J_i)(1 \otimes a)(J \otimes J_i)V_i J = U_i^*(a \otimes 1)U_i = \lambda_i(x)$ and the first equation follows.
We denote by $P_i$ the orthogonal projection from $\mathcal{H}$ onto $\mathbb{C}\xi \oplus \mathcal{H}_i^0$. We note that $P_i$ commutes with $\lambda_i(a)$. Since $U_iV_i^*(\xi \otimes \xi) = \xi \otimes \xi$ for every $\xi \in \mathcal{H}_i$, we have
\[
V_i\lambda_i(a)P_iV_i^*(\xi \otimes \xi) = V_iU_i^*(a\xi \otimes \xi) = \xi \otimes a\xi = (P_i \otimes a)(\xi \otimes \xi)
\]
for every $\xi \in \mathcal{H}_i$. Hence, we have $\lambda_i(a)P_i = V_i^*(P_i \otimes a)V_i$. We observe that
\[
(U_i \otimes H_i)V_i(1 - P_i) = (1_{H_i} \otimes V_i)U_i(1 - P_i)
\]
as a partial isometry from $(1 - P_i)\mathcal{H}$ onto $\mathcal{H}_i \otimes (\mathcal{H}(i) \cap J\mathcal{H}(i) \ominus \mathbb{C}\xi) \otimes \mathcal{H}_i$. It follows that
\[
V_i^*(\lambda_i(a) \otimes 1)V_i(1 - P_i) = V_i^*(U_i^* \otimes 1)(a \otimes 1 \otimes 1)(U_i \otimes 1)V_i(1 - P_i)
\]
\[= V_i^*(U_i^* \otimes 1)(a \otimes 1 \otimes 1)(1 \otimes V_i)U_i(1 - P_i)
\]
\[= V_i^*(U_i^* \otimes 1)(1 \otimes V_i)U_i(1 - P_i)\lambda_i(a)(1 - P_i)
\]
\[= \lambda_i(a)(1 - P_i).
\]
Since $Q_i := V_i(1 - P_i)V_i^*$ is the projection onto $(J\mathcal{H}(i) \ominus \mathbb{C}\xi) \otimes \mathcal{H}_i$, we have
\[
\lambda_i(a) = \lambda_i(a)P_i + \lambda_i(a)(1 - P_i) = V_i^*((P_i \otimes a) + (\lambda_i(a) \otimes 1)Q_i)V_i
\]
as we claimed. Finally, if $i \neq j$, then $(U_i \otimes 1_{H_i})V_j = (1_{H_i} \otimes V_j)U_i$ and we have
\[
V_j^*(\lambda_i(a)(J\mathcal{H}(i) \otimes 1)V_j = U_j^*(1 \otimes V_j^*)(a \otimes 1 \otimes 1)(1 \otimes V_j)U_i = \lambda_i(a).
\]
\[
\square
\]
We denote $C_i = JB_iJ$, $C = JB$ and $D_i = V_i^*(\mathbb{B}(J\mathcal{H}(i)) \ominus \mathbb{B}(\mathcal{H}_i))V_i$ for simplicity. The above lemma implies that $\lambda_i(a)D_j \subset D_j$ for any $i, j \in \{1, 2\}$ and $a \in \mathbb{B}(\mathcal{H}_i)$.

**Proposition 3.2.** Let $(B, \tau) = (B_1, \tau_1) \ast (B_2, \tau_2)$, $\mathcal{H} = L^2B$ and $C = JB$ be as above, and let $\Psi: \mathbb{B}(\mathcal{H}) \rightarrow \mathbb{B}(\mathcal{H})$ be a $C$-bimodule ucp map. If $B_i$ are both exact and $D_i \subset \ker \Psi$ for both $i \in \{1, 2\}$, then the ucp map
\[
\tilde{\Psi}: B \otimes C \ni \sum_{k=1}^n a_k \otimes x_k \mapsto \Psi(\sum_{k=1}^n a_k x_k) \in \mathbb{B}(\mathcal{H})
\]
is continuous w.r.t. the minimal tensor norm on $B \otimes C$.

**Proof.** For simplicity, we let $\tilde{B}_1 = \lambda(\mathbb{B}(\mathcal{H}_i))$ and $\tilde{B} = C^*(\tilde{B}_1, \tilde{B}_2)$. We claim that $[\tilde{B}, C] \subset \ker \Psi$. Since $\Psi$ is a $C$-bimodule map and the closed linear span of $\bigcup_{j=1,2} C[\tilde{B}, C_j]C$ contains $[\tilde{B}, C]$, it suffices to show $[\tilde{B}, C_j] \subset \ker \Psi$ for each $j \in \{1, 2\}$. But since $\tilde{B}D_k \tilde{B} \subset D_k$ by Lemma 3.1 and the closed linear span of $\bigcup_{i=1,2} \tilde{B}[\tilde{B}_i, C_j]\tilde{B}$ contains $[\tilde{B}, C_j]$, it suffices to show $[\tilde{B}_i, C_j] \subset \bigcup_{k=1,2} D_k$ for each $i, j \in \{1, 2\}$. Now, it is not hard to see from Lemma 3.1 that $[\tilde{B}_i, C_j] = \{0\}$ for $i \neq j$ and that
\[
[\lambda_i(a), J\lambda_i(b)J] = V_i^*(\mathbb{C}P_\xi \otimes [a, J,bJ_i])V_i \in D_i
\]
for every $a \in \mathbb{B}(\mathcal{H}_i)$ and $b \in B_i$.

Since $[\tilde{B}, C] \subset \ker \Psi$ and $\Psi$ is a $C$-bimodule map, we have $\Psi(\tilde{B}) \subset C'$. Lemma 2.4 implies that the inclusion map $\iota : B \hookrightarrow \tilde{B}$ is nuclear and so is the ucp map $\Psi_i = \Psi \circ \iota : B \to C'$. Therefore, the product ucp map

$$\tilde{\Psi} = \Psi_i \times \text{id}_C : B \otimes C \to \mathbb{B}(\mathcal{H})$$

is continuous w.r.t. the minimal tensor norm. □

**Theorem 3.3.** Let $\mathcal{M}_1$ and $\mathcal{M}_2$ be semiexact finite factors and let $\mathcal{M} = \mathcal{M}_1 \ast \mathcal{M}_2$ be their free-product. If $\mathcal{Q} \subset \mathcal{M}$ is an injective type $\Pi_1$ subfactor whose relative commutant $\mathcal{Q}' \cap \mathcal{M}$ is a non-injective factor, then there exist $i \in \{1, 2\}$ and a unitary operator $u \in \mathcal{M}_i$ such that $u^* \mathcal{Q} u \subset \mathcal{M}_i$ in $\mathcal{M}$.

**Proof.** We follow the notations used above. Let $B_i \subset \mathcal{M}_i$ be ultraweakly dense exact $C^*$-algebras, $(B, \tau) = (B_1, \tau_1) \ast (B_2, \tau_2)$ be their free-product and let $C = \mathcal{J}B_1J$.

Then, $B$ is ultraweakly dense in $\mathcal{M}$ and is exact by Dykema’s theorem [Dy2]. It follows Lemma 2.1 that the $C$-bimodule ucp map

$$\tilde{\Psi} : B \otimes C \ni \sum_{k=1}^n a_k \otimes x_k \mapsto \Psi_i(\sum_{k=1}^n a_kx_k) \in \mathbb{B}(L^2\mathcal{M})$$

cannot be continuous w.r.t. the minimal tensor norm. But by Proposition 3.2, this implies that $D_i \not\subset \ker \tilde{\Psi}_i$ for some $i \in \{1, 2\}$. Now the following discussion following Lemma 2.1 applies (for $\mathcal{N} = \mathcal{M}_i$) and yields a $\mathcal{Q}$-$\mathcal{M}_i$ sub-bimodule $\mathcal{K}$ in $L^2\mathcal{M}$ with $\dim \mathcal{K}_{\mathcal{M}_i} < \infty$.

It follows Lemma 2.2 that there exist projections $e \in \text{Proj}(\mathcal{M}_i)$ and $q \in \text{Proj}(\mathcal{Q})$, a non-zero partial isometry $v \in \mathcal{M}$ and a homomorphism $\theta : q\mathcal{Q}q \to e\mathcal{M}_ie$ such that $vv^* \in (q\mathcal{Q}q)' \cap q\mathcal{M}_q$, $v^*v \in \theta(q\mathcal{Q}q)' \cap e\mathcal{M}e$ and $xv = \theta(x)$ for $x \in q\mathcal{Q}q$. Since $(q\mathcal{Q}q)' \cap q\mathcal{M}_q = q(\mathcal{Q}' \cap \mathcal{M})q$, we have $vv^* = qq'$ for some $q' \in \text{Proj}(\mathcal{Q}' \cap \mathcal{M})$. By restricting $\theta$ and $v$ if necessary, we may assume $\tau(q) = 1/m$ and $\tau(q') = 1/n$ for some $m, n \in \mathbb{N}$. Let $u_1, \ldots, u_m \in \mathcal{Q}$ (resp. $u'_1, \ldots, u'_n \in \mathcal{Q}' \cap \mathcal{M}$) be partial isometries such that $u_j^*u_j = q$ and $\sum_{j=1}^m u_ju_j^* = 1$ (resp. $(u'_k)^*u'_k = q'$ and $\sum_{k=1}^n u_k^*u_k = 1$).

By Lemma 2.2, $v^*v \in e\mathcal{M}_ie$. We note that $\tau(v^*v) = \tau(qq') = (mn)^{-1}$. Let $w_{j,k}$ be partial isometries in $\mathcal{M}_i$ such that $w_{j,k}w_{j,k}^* = v^*v$ and $\sum_{j=1}^m \sum_{k=1}^n w_{j,k}^*w_{j,k} = 1$. Then $u = \sum_{j,k} u_j^*v_i^*u_{j,k}$ is the desired unitary operator. Indeed, $u^*xu = \sum_{j,k} w_{j,k}^* \theta(u_j^*xu_{j,k})w_{j,k} \in \mathcal{M}_i$ for $x \in \mathcal{Q}$. □

The Kurosh subgroup theorem states that if $\Lambda$ is a subgroup in a free product $\Gamma_1 \ast \Gamma_2$, then $\Lambda$ is freely generated by a free subgroup in $\Gamma_1 \ast \Gamma_2$ and/or conjugates of subgroups in $\Gamma_1$ and/or $\Gamma_2$. In particular, if $\Lambda \leq \Gamma_1 \ast \Gamma_2$ is a freely-indecomposable non-infinite-cyclic subgroup, then $\Lambda$ is conjugated to a subgroup in $\Gamma_1$ or $\Gamma_2$. The following corollary is an analogue of this for type $\Pi_1$ factors.
Corollary 3.4. Let $M_1$ and $M_2$ be semiexact finite factors and let $M = M_1 \ast M_2$ be their free-product. If $\mathcal{N} \subset M$ is a non-prime non-injective subfactor whose relative commutant $\mathcal{N}' \cap M$ is a factor, then there exist $i \in \{1, 2\}$ and a unitary operator $u \in M$ such that $u^* \mathcal{N}' u \subset M_i$ in $M$. In particular, $M$ is prime unless one of $M_i$ is trivial or both $M_i$ are isomorphic to $\mathbb{M}_2(\mathbb{C})$.

Proof. Since $\mathcal{N}$ is non-prime non-injective, there are II$_1$-factors $\mathcal{N}_1$ and $\mathcal{N}_2$ with $\mathcal{N}_2$ non-injective such that $\mathcal{N} = \mathcal{N}_1 \otimes \mathcal{N}_2$. By Proposition 13 in [OP], there is an injective type II$_1$ subfactor $Q \subset \mathcal{N}_1$ with $Q' \cap M = \mathcal{N}_1' \cap M$. Since $\mathcal{N}_2 \subset \mathcal{N}_1' \cap M$, the relative commutant $\mathcal{N}_1' \cap M$ is non-injective and the center of $\mathcal{N}_1' \cap M$ is contained in (the center of) $\mathcal{N}' \cap M$. Hence the factoriality of $\mathcal{N}' \cap M$ implies that $Q' \cap M$ is a non-injective factor. Now it follows from Theorem 3.3 that there exist $i \in \{1, 2\}$ and a unitary operator $u \in M$ such that $u^*Qu \subset M_i$. If $v \in U(u^*\mathcal{N}_2u)$, then $v$ commutes with $u^*Qu$ and hence $v \in M_i$ by Lemma 2.3. It follows that $u^*\mathcal{N}_2u \subset M_i$. The same argument applies to $u^*\mathcal{N}_2u$ instead of $u^*Qu$ and we have $u^*\mathcal{N}_1u \subset M_i$. Consequently, we have $u^*\mathcal{N}u \subset M_i$. We note that $M$ is a non-injective factor unless one of $M_i$ is trivial or both $M_i$ are isomorphic to $\mathbb{M}_2(\mathbb{C})$ [Dy1]. Since $M$ is not unitarily conjugated into $M_i$, it cannot be non-prime. \(\square\)

One of the consequences of the Kurosh subgroup theorem is the isomorphism theorem that if $\Gamma_0$ and $\Lambda_0$ are free groups and $\Gamma_1, \ldots, \Gamma_n$ and $\Lambda_1, \ldots, \Lambda_m$ are freely-indecomposable non-infinite-cyclic groups with $\Gamma = \ast_{i=0}^n \Gamma_i = \ast_{j=0}^m \Lambda_j$, then $\Gamma_0 = \Lambda_0$, $n = m$ and, modulo permutation of indices, $\Gamma_i$ and $\Lambda_i$ are conjugated in $\Gamma$ for every $i \geq 1$. The following is an analogue of this for type II$_1$ factors.

Corollary 3.5. Let $M_0, \ldots, M_n$ and $\mathcal{N}_0, \ldots, \mathcal{N}_m$ be semiexact finite factors such that $M_0$ and $\mathcal{N}_0$ are semisolid (possibly one-dimensional) and that $M_1, \ldots, M_n$ and $\mathcal{N}_1, \ldots, \mathcal{N}_m$ are non-prime non-injective. If $M = \ast_{i=0}^n M_i = \ast_{j=0}^m \mathcal{N}_j$, then $n = m$ and, modulo permutation of indices, $M_i$ and $\mathcal{N}_i$ are unitarily conjugated in $M$ for every $i \geq 1$.

Proof. Without loss of generality, we assume $m \geq n$ (and we no longer need the assumption that $\mathcal{N}_0$ is semisolid). Let $M = \ast_{i=0}^n M_i = \ast_{j=0}^m \mathcal{N}_j$. By Corollary 3.4, there exist maps $i: \{1, \ldots, m\} \rightarrow \{1, \ldots, n\}$ and $j: \{1, \ldots, n\} \rightarrow \{0, \ldots, m\}$ and unitaries $u_1, \ldots, u_m, v_1, \ldots, v_n$ in $M$ such that $u_i^* \mathcal{N}_j u_j \subset M_{i(j)}$ for every $j \geq 1$ and $v_i^* M_i v_i \subset \mathcal{N}_{j(i)}$ for every $i \geq 1$. It follows that $v_{i(j)}^* u_j^* \mathcal{N}_j u_j v_{i(j)} \subset \mathcal{N}_{j(i(j))}$ for every $j \geq 1$. But by Lemma 2.3, this implies that $j(i(j)) = j$ and $u_j v_{i(j)} \in \mathcal{N}_j$. Therefore, the map $i$ is a bijection and the above inclusion maps are all isomorphisms. Hence, we have $u_j^* \mathcal{N}_j u_j = M_{i(j)}$ for every $j \geq 1$. \(\square\)

4. Crossed-Product

We recall the crossed product construction. Let $A$ be a $C^*$-algebra with a $\Gamma$-action $\alpha: \Gamma \rightarrow \text{Aut}(A)$. A covariant representation of $(\Gamma, A)$ is a pair $(\sigma, \pi)$ of
representations of $\Gamma$ and $A$ respectively on a Hilbert space $H$ such that
\[ \forall s \in \Gamma, \forall a \in A \quad \text{Ad}_{\sigma(s)}(\pi(a)) = \sigma(s)\pi(a)\sigma(s)^{-1} = \pi(\alpha_s(a)). \]
The full crossed product $C^*$-algebra $C^*(\Gamma, A)$ is the universal $C^*$-algebra generated by $\sigma(\Gamma)$ and $\pi(A)$ under the covariance condition. To define the reduced crossed product, we fix a faithful representation $\pi : A \to B(H)$. Define a new representation $\tilde{\pi} : A \to B(\ell_2\Gamma \otimes H)$ by
\[ \tilde{\pi}(a)(\delta_t \otimes \zeta) = \delta_t \otimes \pi(\alpha_t^{-1}(a))\zeta \quad \text{for } t \in \Gamma \text{ and } \zeta \in H. \]
It is convenient to introduce orthogonal projections $e(t)$ of $\ell_2\Gamma$ onto $\mathbb{C}\delta_t$ so that $\tilde{\pi}(a) = \sum_{t \in \Gamma} e(t) \otimes \pi(\alpha_t^{-1}(a))$, where the sum converges strongly. It is easily verified that $(\lambda \otimes 1, \tilde{\pi})$ is a covariant representation of $(\Gamma, A)$. The reduced crossed product $C^*$-algebra $C^*_{\text{red}}(\Gamma, A)$ is the $C^*$-subalgebra in $B(\ell_2\Gamma \otimes H)$ generated by $(\lambda \otimes 1)(\Gamma)$ and $\tilde{\pi}(A)$. We note that $C^*_{\text{red}}(\Gamma, A)$ does not depends on the choice of a faithful representation $\pi$. By definition, $C^*_{\text{red}}(\Gamma, A)$ is canonically isomorphic to a quotient of $C^*(\Gamma, A)$. If $A$ is an exact $C^*$-algebra and $\Gamma$ is an exact group, then $C^*_{\text{red}}(\Gamma, A)$ is exact also (cf. [KW]).

A compact $\Gamma$-space is a compact topological space $X$ together with a continuous action of $\Gamma$ on it. Recall that we say a compact $\Gamma$-space $X$ is amenable (or, the $\Gamma$-action on $X$ is amenable) if there exists a sequence of continuous $\mu_n : X \to \text{Prob}(\Gamma)$ such that
\[ \lim_{n \to \infty} \sup_{x \in X} \|s.\mu^n_x - \mu^n_{s.x}\| = 0 \]
for every $s \in \Gamma$, where $\text{Prob}(\Gamma) = \{ \mu \in \ell_1\Gamma : \mu \geq 0, \|\mu\| = 1 \}$ and $(s.\mu)(t) = \mu(s^{-1}t)$ for $\mu \in \text{Prob}(\Gamma)$ and $s \in \Gamma$. We note that the Stone-Čech compactification $\beta\Gamma$ with the left translation action of $\Gamma$ is amenable iff $\Gamma$ is exact [AD] [GK] [Oz1]. We assume that the reader is familiar with basic facts on amenability for group actions. We refer [AD] [AR] for detail.

Let $X$ be a compact $\Gamma$-space. A unital $\Gamma$-$C(X)$-$C^*$-algebra is a unital $C^*$-algebra $A$ such that

1. $A$ contains $C(X)$ in its center,
2. there is a $\Gamma$-action $\alpha : \Gamma \to \text{Aut}(A),$
3. $(\alpha_s(f))(x) = f(s^{-1}x)$ for every $s \in \Gamma$, $f \in C(X)$ and $x \in X.$

If the compact $\Gamma$-space $X$ is amenable, then we have $C^*_{\text{red}}(\Gamma, A) = C^*(\Gamma, A)$ canonically for every $\Gamma$-$C(X)$-$C^*$-algebra $A$.

We first prove a general result on amenability of a group action. Suppose that $\Gamma$ acts on a set $K$. The $\Gamma$-action extends to a continuous action on the Stone-Čech compactification $\beta K$ of $K$ and then restricts to a continuous action on the Stone-Čech remainder $\partial^3 K = \beta K \setminus K$. We are interested when the compact $\Gamma$-space $\partial^3 K$ is amenable. It is well-known that the unitary representation $\sigma_K : \Gamma \to B(\ell_2K)$, given by $\sigma_K(s)\delta_x = \delta_{s.x}$, is weakly contained in the left regular representation $\lambda$ iff the isotropy subgroups are all amenable.
Proposition 4.1. Let $\Gamma$ be an exact group, $K$ be a countable set on which $\Gamma$ acts and $\partial^3K$ be the Stone-Cech remainder of $K$. The following are equivalent.

1. The compact $\Gamma$-space $\partial^3K$ is amenable.
2. There exists a map $\mu: K \to \text{Prob}(\Gamma)$ such that $\lim_{s \to \infty} \left\| s.\mu_x - \mu_{s,x} \right\| = 0$ for every $s \in \Gamma$.
3. There exists a ucp map $\varphi: C^*_\lambda(\Gamma) \to \mathbb{B}(\ell_2K)$ such that $\varphi(\lambda(s)) - \sigma_K(s) \in \mathbb{K}(\ell_2K)$ for every $s \in \Gamma$.

Proof. 1$\Rightarrow$3. We note that $C(\partial^3K) \cong \ell_\infty K/c_0K \subset \mathbb{B}(\ell_2K)/\mathbb{K}(\ell_2K)$. If the compact $\Gamma$-space $\partial^3K$ is amenable, then the $C^*$-algebra $C^*_\lambda(\Gamma, C(X))$ is nuclear and the natural homomorphism

$$C^*_\lambda \Gamma \subset C^*_\lambda(\Gamma, C(X)) \cong C^*(\Gamma, C(X)) \to \mathbb{B}(\ell_2K)/\mathbb{K}(\ell_2K)$$

is continuous. Moreover, it has a ucp lifting $\varphi$ by the Choi-Sub injion lifting theorem.

3$\Rightarrow$2. By a generalized Weyl-von Neumann theorem (Theorem II.5.3 in [Da]), there exists an isometry $V: \ell_2K \to \ell_2\Gamma$ such that $V^*\lambda(s)V - \varphi(\lambda(s)) \in \mathbb{K}(\ell_2\Gamma)$ for every $s \in \Gamma$. For every $x \in K$, we set $x = |V\delta_x|^2 \in \text{Prob}(\Gamma)$. Then, we have

$$\left\| s.\mu_x - \mu_{s,x} \right\|_1 \leq \left\| V\delta_x + V\delta_{s,x} \right\|_2 \left\| V\delta_x - V\delta_{s,x} \right\|_2 \leq 2(2 - 2\Re(V^*\lambda(s)V\delta_x, \delta_{s,x}))^{1/2}.$$ 

Since $V^*\lambda(s)V - \sigma_K(s) \in \mathbb{K}(\ell_2\Gamma)$ for every $s \in \Gamma$, we are done.

2$\Rightarrow$1. Since the state space $S$ of $\ell_\infty K$ is compact, the map $\mu: K \to \text{Prob}(\Gamma)$ extends to a continuous map $\tilde{\mu}$ from $\beta K$ into $S$. By the condition 2, the map $\tilde{\mu}$ is $\Gamma$-equivariant on $\partial^3K$. Hence, the amenability of $\partial^3K$ follows from that of $S$. However, since the state space $S$ is amenable iff the underlying space $\beta \Gamma$ is amenable, the amenability of $S$ follows from the exactness of $\Gamma$. \hfill \Box

Consider the $\Gamma \times \Gamma$-action on $\Gamma$ given by the left and right translations. Let $\mathcal{S}$ be the class of countable discrete groups $\Gamma$ such that the compact $\Gamma \times \Gamma$-space $\partial^3\Gamma$ is amenable. The class $\mathcal{S}$ contains all word hyperbolic groups (and more generally groups which are hyperbolic relative to a family of amenable subgroups) [HG][Oz3][Sk]. Moreover, every subgroups of a group in $\mathcal{S}$ is again in $\mathcal{S}$ and $\mathcal{S}$ is closed under free-product (with finite amalgamation).

Although it is irrelevant to the rest of paper, we make the following observation. A group $\Gamma$ is said to be inner-amenable if there exists a state $m$ on $\ell_\infty K$ with $c_0 K \subset \ker m$ such that $m(\sigma_s(f)) = m(f)$ for every $s \in \Gamma$ and $f \in \ell_\infty K$, where $\sigma_s(f)(t) = f(sts)$. In other words, $\Gamma$ is inner-amenable if $\partial^3\Gamma$ carries a probability measure which is invariant under the conjugation action of $\Gamma$. (There is another definition of inner-amenable that requires only $m(\delta_s) = 0$ instead of $c_0 K \subset \ker m$. But, they coincide if the group in consideration is ICC.) If $\Gamma \in \mathcal{S}$, then the conjugation action of $\Gamma$ on $\partial^3\Gamma$ is amenable since it is the restriction of the $\Gamma \times \Gamma$-action to its diagonal subgroup $\Gamma$. It follows that every inner-amenable group in $\mathcal{S}$ is amenable.
This was first proved in [dH]. It is also not difficult to prove that every torsion-free non-amenability group in $S$ is ICC.

We look at the crossed product construction more carefully in the tracial setting. Let $A$ be a $C^*$-algebra with a faithful trace $\tau$ and let $\alpha: \Gamma \rightarrow Aut(A, \tau)$ be a trace preserving action. If $\pi_\tau: A \rightarrow B(L^2A)$ is the GNS-representation of $(A, \tau)$, then $C^*_{red}(\Gamma, A)$ is the $C^*$-subalgebra in $B(\ell_2 \Gamma \otimes L^2 A)$ generated by $(\lambda \otimes 1)(\Gamma)$ and $\pi_\tau(A)$. The vector $\xi = \delta_e \otimes \hat{1}_A$ is a cyclic separating trace vector for $C^*_{red}(\Gamma, A)$ with the canonical conjugation $J$ on $\ell_2 \Gamma \otimes L^2 A$ given by $Jx\xi = x^*\xi$ for $x \in C^*_{red}(\Gamma, A)$. Let $u$ be the representation of $\Gamma$ on $L^2 A$ given by $u(s)\hat{a} = \alpha_s(a)$ for $s \in \Gamma$ and $a \in A$.

A simple calculation shows that

$$J(\lambda \otimes 1)(s)J = (\rho \otimes u)(s)$$

for $s \in \Gamma$ and $J\pi_\tau(a)J = 1 \otimes \pi_\tau^2(a)$ for $a \in A$,

where $\pi_\tau^2(a) = J_A\pi_\tau(a)J_A$. For simplicity, denote by $C^*(B, C)$ the $C^*$-subalgebra in $B(\ell_2 \Gamma \otimes L^2 A)$ generated by $B = C^*_{red}(\Gamma, A)$ and $C = JBJ$.

Let $I_1 = K(\ell_2 \Gamma) \otimes_{\min} B(L^2A)$. We note that $I_1$ is the hereditary $C^*$-subalgebra in $B(\ell_2 \Gamma \otimes L^2 A)$. It is not hard to see that both $B$ and $C$ are in the multiplier of $I_1$ and hence $I = I_1 \cap C^*(B, C)$ is an ideal in $C^*(B, C)$.

**Proposition 4.2.** Let $\alpha: \Gamma \rightarrow Aut(A, \tau)$ and $I \subset C^*(B, C)$ be as above. Suppose that $A'' \subset B(L^2 A)$ is injective and $\Gamma \in S$. Then, the homomorphism

$$\nu: B \otimes C \ni \sum_{k=1}^n a_k \otimes x_k \mapsto \sum_{k=1}^n a_k x_k + I \in C^*(B, C)/I$$

is continuous w.r.t. the minimal tensor norm on $B \otimes C$.

**Proof.** Consider the $C^*$-algebra $D_1 = C^*(B, C, \ell_\infty \Gamma \otimes \mathbb{C}1)$ generated by $C^*(B, C)$ and $\ell_\infty \Gamma \otimes 1$. Since $D_1$ is in the multiplier of $I_1$, there is a natural inclusion

$$C^*(B, C)/I \hookrightarrow (D_1 + I_1)/I_1 = E_1.$$

Let $D \subset D_1$ be the $C^*$-algebra generated by $\hat{\pi}(A)$, $J\hat{\pi}(A)J$ and $\ell_\infty \Gamma \otimes \mathbb{C}1$. Then, it is not hard to see that $E = (D + I_1)/I_1 \subset E_1$ is a $(\Gamma \times \Gamma)$-$\partial\Gamma$-$C^*$-algebra with the commuting $\Gamma$-actions $Ad_{\lambda \otimes 1}$ and $Ad_{\rho \otimes u}$. Since $\Gamma \in S$, the canonical homomorphism from $C^*(\Gamma \times \Gamma, E)$ onto $E_1$ factors through $C^*_{red}(\Gamma \times \Gamma, E)$. Since $\hat{\pi}(A)^n$ is injective, the natural homomorphism $\hat{\pi}(A) \otimes J\hat{\pi}(A)J \rightarrow E$ is continuous w.r.t. the minimal tensor norm. Therefore, the homomorphism $\nu$ is continuous on $B \otimes_{\min} C \cong C^*_{red}(\Gamma \times \Gamma, \hat{\pi}(A) \otimes_{\min} J\hat{\pi}(A)J)$. \qed

Let

$$K_1 = \{ x \in \mathbb{B}(\ell_2 \Gamma \otimes L^2 A) : (\omega \otimes \text{id})(x^*x + xx^*) \in \mathbb{K}(L^2 A) \ \forall \omega \in \mathbb{B}(\ell_2 \Gamma)^+ \}.$$

We note that $K_1$ is the hereditary $C^*$-subalgebra in $\mathbb{B}(\ell_2 \Gamma \otimes L^2 A)$ generated by $\ell_\infty(\Gamma, \mathbb{K}(L^2 A))$. It is not hard to see that both $B$ and $C$ are in the multiplier of $K_1$ and hence $K = K_1 \cap C^*(B, C)$ is an ideal in $C^*(B, C)$. 
Proposition 4.3. Let $\alpha: \Gamma \to \text{Aut}(A, \tau)$ and $K \subset C^*(B, C)$ be as above. Suppose that $A'' \subset \mathcal{B}(L^2A)$ is injective and there exists a unital completely positive map $\theta: \mathcal{B}(L^2A \otimes \ell_2\Gamma) \to \mathcal{B}(L^2A)$ such that the elements $\theta((u \otimes \rho)(s)) - u(s), \theta(\pi_\tau(a) \otimes 1) - \pi_\tau(a)$ and $\theta(\pi_\tau(a) \otimes 1) - \pi_\tau(a)$ are all in $\mathcal{K}(L^2A)$ for every $s \in \Gamma$ and $a \in A$. Then, the homomorphism

$$
\mu: B \otimes C \ni \sum_{k=1}^n a_k \otimes x_k \mapsto \sum_{k=1}^n a_k x_k + K \in C^*(B, C)/K
$$

is continuous w.r.t. the minimal tensor norm on $B \otimes C$.

Proof. Since $A'' \subset \mathcal{B}(L^2A)$ is injective, there is a canonical homomorphism

$$
\Phi: (\mathcal{B}(\ell_2\Gamma) \otimes A'') \otimes_{\text{min}} (A' \otimes \mathcal{B}(\ell_2\Gamma)) \to \mathcal{B}(\ell_2\Gamma \otimes L^2A \otimes \ell_2\Gamma)
$$

given by $\Phi(x \otimes 1 \otimes 1) = x \otimes 1$ for $x \in \mathcal{B}(\ell_2\Gamma) \otimes A''$ and $\Phi(1 \otimes 1 \otimes y) = 1 \otimes y$ for $y \in A' \otimes \mathcal{B}(\ell_2\Gamma)$. Let $U_0: \ell_2\Gamma \otimes L^2A \to L^2A \otimes \ell_2\Gamma$ be the unitary operator given by $U_0(\delta_t \otimes \xi) \mapsto u(t)\xi \otimes \delta_t$ so that the elements

$$
\text{Ad}_{U_0}((\rho \otimes u)(s)) = (1 \otimes \rho)(s), \quad \text{Ad}_{U_0}(1 \otimes \pi_\tau^e(a)) = \sum_{t \in \Gamma} \pi_\tau^e(\alpha_t(a)) \otimes e(t)
$$

are in $A' \otimes \mathcal{B}(\ell_2\Gamma)$. Let $U_1 = \sum_{t \in \Gamma} (\rho \otimes u)(t) \otimes e(t) \in B' \otimes \mathcal{B}(\ell_2\Gamma)$ be the unitary operator on $\ell_2\Gamma \otimes L^2A \otimes \ell_2\Gamma$ so that

$$
\text{Ad}_{U_1}((1 \otimes 1 \otimes \rho)(s)) = (\rho \otimes u \otimes \rho)(s), \quad \text{Ad}_{U_1(1 \otimes U_0)}(1 \otimes 1 \otimes \pi_\tau^e(a)) = 1 \otimes \pi_\tau^e(a) \otimes 1.
$$

It follows that for the homomorphism

$$
\tilde{\Phi} = \text{Ad}_{U_1} \Phi \text{Ad}_{1 \otimes U_0}: B \otimes_{\text{min}} C \to \mathcal{B}(\ell_2\Gamma \otimes L^2A \otimes \ell_2\Gamma),
$$

we have

$$
\tilde{\Phi}((\lambda \otimes 1 \otimes 1 \otimes 1)(s)) = \lambda(s) \otimes 1 \otimes 1,
$$
$$
\tilde{\Phi}(\pi_\tau(a) \otimes 1 \otimes 1) = \pi_\tau(a) \otimes 1,
$$
$$
\tilde{\Phi}((1 \otimes 1 \otimes \rho \otimes u)(s)) = (\rho \otimes u \otimes \rho)(s),
$$
$$
\tilde{\Phi}(1 \otimes 1 \otimes 1 \otimes \pi_\tau^e(a)) = 1 \otimes \pi_\tau^e(a) \otimes 1.
$$

It follows that for the unital completely positive map

$$
\tilde{\theta} = \text{id} \otimes \theta: \mathcal{B}(\ell_2\Gamma \otimes L^2A \otimes \ell_2\Gamma) \to \mathcal{B}(\ell_2\Gamma \otimes L^2A),
$$
the elements

\[ \tilde{\Phi}((\lambda \otimes 1 \otimes 1 \otimes 1)(s)) - \lambda(s) \otimes 1 = 0, \]
\[ \tilde{\Phi}(\tilde{\pi}_r(a) \otimes 1 \otimes 1) - \tilde{\pi}_r(a) = \sum_{t \in \Gamma} e(t) \otimes (\theta(\pi_r(\alpha_t^{-1}(a)) \otimes 1) - \pi_r(\alpha_t^{-1}(a))), \]
\[ \tilde{\Phi}((1 \otimes 1 \otimes \rho \otimes u)(s)) - (\rho \otimes u)(s) = \rho(s) \otimes (\theta((u \otimes \rho)(s)) - u(s)), \]
\[ \tilde{\Phi}(1 \otimes 1 \otimes 1 \otimes \pi^\circ_r(a)) - 1 \otimes \pi^\circ_r(a) = 1 \otimes (\theta(\pi^\circ_r(a) \otimes 1) - \pi^\circ_r(a)) \]

are all in \( K_1 \) for every \( s \in \Gamma \) and \( a \in A \). Since \( (C^*(B, C) + K_1)/K_1 = C^*(B, C)/K \) canonically, the completely positive map \( \tilde{\Phi} : B \otimes_{\min} C \rightarrow C^*(B, C) + K_1 \) passes to the homomorphism \( \mu \).

Let us recall the Bernoulli shift (or the wreath product) of \( \Delta \) by \( \Gamma \) is the group \( \Gamma \ltimes \Delta_\Gamma \), where \( \Delta_\Gamma = \bigoplus_{\Gamma} \Delta \) is the direct sum of \( \Delta \)'s indexed by \( \Gamma \) and \( \Gamma \) acts on \( \Delta_\Gamma \) by left translation. We view an element \( x \in \Delta_\Gamma \) as a function \( x : \Gamma \rightarrow \Delta \) with finite support \( \{ t \in \Gamma : x(s) \neq e_\Delta \} =: \text{supp} \ x \).

**Proposition 4.4.** Let \( \Gamma \) and \( \Delta \) be groups with \( \Delta \) amenable. Then, the Bernoulli shift action \( \Gamma \) on \( A = C^*_\Delta(\Delta_\Gamma) \) satisfies the assumptions in Proposition 4.3.

**Proof.** We note that \( L^2 A \) is canonically isomorphic to \( \ell_2 \Delta_\Gamma \). We fix a proper length function \( l_\Gamma \) on \( \Gamma \), i.e., \( l_\Gamma \) is a non-negative function on \( \Gamma \) such that (i) \( l_\Gamma(s) = 0 \) iff \( s = e \), (ii) \( l_\Gamma(st) \leq l_\Gamma(s) + l_\Gamma(t) \) for \( s, t \in \Gamma \), and (iii) the set \( \{ s \in \Gamma : l_\Gamma(s) \leq R \} \) is finite for every \( R > 0 \). Likewise \( l_\Delta \). For \( y \in \Delta_\Gamma \) and \( t \in \Gamma \), we set \( w(y, t) = l_\Gamma(t) + l_\Delta(y(t)) \) if \( t \in \text{supp} \ y \) and \( w(y, t) = 0 \) otherwise. Further, set \( w(y) = \sum_{t \in \Gamma} w(y, t) \) and \( n(y) = |\text{supp} \ y| \). It follows that \( n(y)/w(y) \rightarrow 0 \) as \( y \rightarrow \infty \). Define \( \xi : \Delta_\Gamma \rightarrow \ell_2 \Gamma \) by \( \xi_y(t) = (w(y, t)/w(y))^{1/2} \). Since \( |w(y, t) - w(s(y), st)| \leq l_\Gamma(s) \) for \( s \in \Gamma \) and \( t \in \text{supp} y \), we have for every \( s \in \Gamma \) that

\[
\|\lambda(s)\xi_y - \xi_{s(y)}\|_2 \leq \|\lambda(s)\xi_y\|^2 - \xi_{s(y)}^2 \|
\]

\[
= \sum_{t \in \Gamma} \frac{|w(y, t)/w(y) - w(s(y), st)/w(s(y))|}{w(y)}
\]

\[
= \left( \sum_{t \in \Gamma} |w(y, t) - w(s(y), st)|/w(y) \right) + |w(s(y)) - w(y)|/w(y)
\]

\[
\leq 2l_\Gamma(s)n(y)/w(y) \rightarrow 0 \text{ as } y \rightarrow \infty.
\]

Moreover for each \( x \in \Delta_\Gamma \), we have \( \|\xi_{xy} - \xi_y\|_2 \leq 2w(x)/w(y) \rightarrow 0 \) and \( \|\xi_{yx} - \xi_y\|_2 \leq 2w(x)/w(y) \rightarrow 0 \) as \( y \rightarrow \infty \).

Let \( V : \ell_2 \Delta_\Gamma \rightarrow \ell_2 \Delta_\Gamma \otimes \ell_2 \Gamma \) be the isometry given by \( V\delta_y = \delta_y \otimes \xi_y \). Then the unital completely positive map \( \theta : \mathbb{B}(\ell_2 \Delta_\Gamma \otimes \ell_2 \Gamma) \rightarrow \mathbb{B}(\ell_2 \Delta_\Gamma) \), given by \( \theta(z) = V^* z V \),
satisfies the assumption of Proposition 4.3 (with \( \rho \) replaced with \( \lambda \)). Indeed,
\[
V^*(u \otimes \lambda)(s)V\delta_y = \langle \lambda(s)\xi_y, \xi_{s(y)} \rangle \delta_{s(y)},
V^*(\lambda(x) \otimes 1)V\delta_y = \langle \xi_y, \xi_{xy} \rangle \delta_{xy},
V^*(\rho(x) \otimes 1)V\delta_y = \langle \xi_y, \xi_{yx^{-1}} \rangle \delta_{yx^{-1}}
\]
for every \( s \in \Gamma \) and \( x \in \Delta_\Gamma \). \( \square \)

The group \( \Gamma \) acts on \( \Delta_\Gamma \) by Bernoulli shift and \( \Delta_\Gamma \times \Delta_\Gamma \) acts on \( \Delta_\Gamma \) by left and right translations. These actions induce an action of \( \Lambda := \Gamma \rtimes (\Delta_\Gamma \times \Delta_\Gamma) \) on the Stone-Čech remainder \( \partial^\beta \Delta_\Gamma \). We observe from the above proof that this action is amenable provided that \( \Gamma \) is exact. Indeed, the map \( \xi_2 : \Delta_\Gamma \to \ell^1(\Gamma) \) gives rise to a continuous map from the Stone-Čech compactification \( \beta_\partial \Delta_\Gamma \) into the state space of \( \ell^\infty(\Gamma) \) whose restriction to \( \partial^\beta \Delta_\Gamma \) is \( \Lambda \)-equivariant (where \( \Delta_\Gamma \times \Delta_\Gamma \) acts trivially on \( \ell^\infty(\Gamma) \)). Since \( \Delta_\Gamma \) is amenable there exists a \( \Lambda \)-equivariant conditional expectation from \( \ell^\infty(\Lambda) \) onto \( \ell^\infty(\Gamma) \). Composing these maps, we obtain a \( \Lambda \)-equivariant continuous map from \( \partial^\beta \Delta_\Gamma \) into the state space of \( \ell^\infty(\Lambda) \). Therefore, the amenability of \( \partial^\beta \Delta_\Gamma \) follows from that of the state space of \( \ell^\infty(\Lambda) \), which is amenable when \( \Gamma \) (and hence \( \Lambda \)) is exact.

**Corollary 4.5.** If \( \Gamma \in \mathcal{S} \) and \( \Delta \) is amenable, then the Bernoulli shift \( \Gamma \ltimes \Delta_\Gamma \) is in \( \mathcal{S} \).

**Proof.** By Propositions 4.2, 4.3 and 4.4 (and their proof), the homomorphism
\[
B \otimes C \ni \sum_{k=1}^n a_k \otimes x_k \mapsto \sum_{k=1}^n a_k x_k + (I \cap K) \in C^*(B, C)/(I \cap K)
\]
is continuous w.r.t. the minimal norm and has a ucp lifting. Since \( I_1 \cap K_1 = \mathbb{K}(\ell_2(\Gamma) \otimes \ell_2(\Delta_\Gamma)) \), the claim follows from Proposition 4.1. \( \square \)

If \( A = A \) is a von Neumann algebra, then the crossed product von Neumann algebra is \( \Gamma \ltimes A \subset C^*_{\text{red}}(\Gamma, A)'' \subset \mathcal{B}(\ell^2(\Gamma) \otimes L^2(A)) \). The following are the main results of this section.

**Theorem 4.6.** Let \( (\mathcal{A}, \tau) \) be a commutative von Neumann algebra with a faithful trace \( \tau \) and a trace preserving action \( \alpha : \Gamma \to \text{Aut}(\mathcal{A}, \tau) \) and let \( \mathcal{M} = \Gamma \ltimes \mathcal{A} \) be its crossed product (which may not be a factor). If \( \Gamma \in \mathcal{S} \), then \( \mathcal{M} \) is semisolid. In particular, any non-injective subfactor of \( \mathcal{M} \) is prime.

**Proof.** We follow the notations used above. In particular, \( B = C^*_{\text{red}}(\Gamma, A) \) and \( C = JBJ \). Let \( \mathcal{Q} \subset \mathcal{M} \) be a type II\(_1\) von Neumann subalgebra. Passing to a subalgebra if necessary, we may assume \( \mathcal{Q} \) is injective. For a proof by contradiction, suppose that \( \mathcal{Q}' \cap \mathcal{M} \) is not injective. It follows from Lemma 2.1 that the ucp map
\[
\tilde{\Psi}_\mathcal{Q} : B \otimes C \ni \sum_{k=1}^n a_k \otimes x_k \mapsto \Psi_\mathcal{Q}(\sum_{k=1}^n a_k x_k) \in \mathcal{B}(L^2(\mathcal{M}))
\]
cannot be continuous w.r.t. the minimal tensor norm. But, by Proposition 4.2 this implies that $\mathcal{K}(\ell_2\Gamma) \otimes B(L^2\mathcal{A}) \not\subset \Psi_Q$. Now the discussion following Lemma 2.1 applies (for $\mathcal{N} = \mathcal{A}$) and yields a non-zero $Q\mathcal{A}$ bimodule $\mathcal{H}$ with $\dim \mathcal{H}_\mathcal{A} < \infty$. This is absurd since $\mathcal{A}$ is commutative while $Q$ is of type $\Pi_1$.

**Theorem 4.7.** Let $\Gamma$ be an exact group and $\mathcal{R}$ be the hyperfinite type $\Pi_1$ factor. Consider the Bernoulli product $\mathcal{M} = \Gamma \times \bigotimes_\Gamma \mathcal{R}$. Then, for any diffuse von Neumann subalgebra $Q \subset \bigotimes_\Gamma \mathcal{R}$, the relative commutant $\mathcal{Q}' \cap \mathcal{M}$ is injective.

**Proof.** Let $\Delta$ be an amenable ICC group so that $L\Delta = \mathcal{R}$ and let $\mathcal{A} = C^*_\Delta \Delta \Gamma$. Let a diffuse von Neumann subalgebra $Q \subset \bigotimes_\Gamma \mathcal{R} = A''$ be given. We will follow the proof of Theorem 6 in [Oz2]. Passing to a von Neumann subalgebra if necessary, we may assume that $Q$ is generated by a single unitary operator $v \in A''$ with $\lim_k v^k = 0$ ultraweakly. Fix a non-trivial ultrafilter $\omega$ and consider the proper conditional expectation $\Psi_Q$ from $B(\ell_2\Gamma \otimes L^2\mathcal{A})$ onto $\tilde{\pi}_v(Q)'$ given by

$$\Psi_Q(x) = \text{weak}^*\text{-lim}_{n \to \omega} \sum_{k=1}^n \text{Ad}(\phi_v)(x).$$

Then, for any $x = \sum_{t \in \Gamma} e(t) \otimes x(t) \in \ell_\infty(\Gamma, \mathcal{K}(L^2\mathcal{A}))$, we have

$$\Psi_Q(x) = \text{weak}^*\text{-lim}_{n \to \omega} \sum_{t \in \Gamma} e(t) \otimes u(t)^* (\sum_{k=1}^n \text{Ad}(\phi_v)(u(t)x(t)u(t)^*))u(t) = 0$$

since each $u(t)x(t)u(t)^*$ is a compact operator. In particular, $\Psi_Q(K) = 0$. It follows from Lemma 2.1 and Proposition 4.3 that the relative commutant $Q' \cap \mathcal{M}$ is injective. \hfill \square

**Remark 4.8.** A similar proof applies to $\mathcal{M} = \Gamma \times \mathcal{R}$, where $\Gamma \in \mathcal{S}$ and $\mathcal{R}$ is the hyperfinite type $\Pi_1$ factor. (Even though the $C^*$-algebra $C^*_\text{red}(\Gamma, \mathcal{R})$ is not exact, Lemma 2.1 is applicable since the pair $C^*_\text{red}(\Gamma, \mathcal{R}) \subset \mathcal{M}$ satisfies local reflexivity.) It follows that if $Q \subset \mathcal{M}$ is a non-injective subfactor, then the relative commutant $Q' \cap \mathcal{M}$ is injective. In particular, any non-McDuff subfactor of $\mathcal{M}$ is prime. This result applies to the factors constructed in [NPS].

Consider an essentially-free measure-preserving ergodic action of a non-amenable hyperbolic group $\Gamma$ on the standard probability space $[0, 1]$ and let

$$\mathcal{A} = L^\infty[0, 1] \subset \Gamma \times L^\infty[0, 1] = \mathcal{M}.$$

Then, Adams’ theorem [Ad] says $(\mathcal{A} \subset \mathcal{M}) \neq (\mathcal{A}_1 \subset \mathcal{M}_1) \otimes (\mathcal{A}_2 \subset \mathcal{M}_2)$ for any type $\Pi_1$ factors $\mathcal{M}_i$ with Cartan subalgebras $\mathcal{A}_i \subset \mathcal{M}_i$. Combined with Popa’s theorem [Po2], this implies that $\mathcal{M} \neq \mathcal{M}_1 \otimes \mathcal{M}_2$ for any HT type $\Pi_1$ factors $\mathcal{M}_i$. Theorem 4.6 generalizes these to that $\mathcal{M} \neq \mathcal{M}_1 \otimes \mathcal{M}_2$ for any type $\Pi_1$ factors $\mathcal{M}_i$.

By considering an ergodic but not strongly-ergodic action of $\Gamma \in \mathcal{S}$, we obtain a semisolid (and hence prime) type $\Pi_1$ factor with the property $(\Gamma)$. 
Voiculescu proved that the free group factors do not have regular diffuse injective subalgebras. In particular, do not have Cartan subalgebras. The Bernoulli shift of by is an example of a solid factor with a Cartan subalgebra.

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