Singular Lagrangians and the Dirac–Bergmann Algorithm in Classical Mechanics

J. David Brown

Department of Physics, North Carolina State University, Raleigh, NC 27695

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Abstract

Textbook treatments of classical mechanics typically assume that the Lagrangian is nonsingular; that is, the matrix of second derivatives of the Lagrangian with respect to the velocities is invertible. This assumption insures that (i) Lagrange’s equations can be solved for the accelerations as functions of coordinates and velocities, and (ii) the definitions of the conjugate momenta can be inverted to solve for the velocities as functions of coordinates and momenta. This assumption, however, is unnecessarily restrictive—there are interesting classical dynamical systems with singular Lagrangians. The algorithm for analyzing such systems was developed by Dirac and Bergmann in the 1950’s. After a brief review of the Dirac–Bergmann algorithm, several examples are presented using familiar components: point masses connected by massless springs, rods, cords and pulleys.
I. INTRODUCTION

The central focus of any advanced book on classical mechanics is the Lagrangian formulation of dynamics. With few exceptions these books assume that the Lagrangian \( L(q, \dot{q}) \) is nonsingular. That is, the matrix

\[
L_{ij} \equiv \frac{\partial^2 L}{\partial \dot{q}_i \partial \dot{q}_j}
\]

of second derivatives of \( L \) with respect to the velocities \( \dot{q}_i \equiv dq_i/dt \) is invertible. If \( L_{ij} \) is not invertible, we cannot solve Lagrange’s equations for the accelerations as functions of the coordinates and velocities.

The starting point for the Hamiltonian formulation of mechanics is the Lagrangian. If \( L_{ij} \) is not invertible, the definitions of momenta in terms of coordinates and velocities cannot be inverted for the velocities as functions of coordinates and momenta. The Hamiltonian theory cannot be constructed in the usual way.

The purpose of this article is to point out that there are, in fact, physically interesting classical dynamical systems with singular Lagrangians. The formalism for treating such systems was developed in the 1950’s by Dirac\(^{1-5}\) and by Bergmann and collaborators\(^{6-11} \) following earlier work by Rosenfeld\(^{12} \). This formalism is referred to as the Dirac–Bergmann algorithm. To be more precise, Dirac and Bergmann, and also Rosenfeld\(^{13}\) showed that a singular Lagrangian system can be placed in the form of a “constrained Hamiltonian system” in which the evolution is constrained to a subspace of phase space. The singular nature of the system is most clearly exhibited in Hamiltonian form.

The primary motivation for Dirac and Bergmann was to understand the structure of field theories such as electromagnetism and general relativity. These theories, as well as Yang–Mills theory and string theory, are gauge theories—they contain degrees of freedom that do not alter the physical state of the system. Gauge theories are described by singular Lagrangians, but (as will be seen in the examples) not all singular Lagrangian systems are gauge theories.

In Section II we outline the Dirac–Bergmann algorithm, the steps for converting a singular Lagrangian system into a constrained Hamiltonian system. The detailed reasoning is spelled out in numerous books\(^{14-21} \) and review articles\(^{13,22,23} \). Section III contains a number of physical examples of singular systems constructed from familiar elements found in textbook classical mechanics problems: point masses connected by massless springs, rods, cords
and pulleys.

In Sec. III A we consider a “compound spring” obtained by welding a spring with stiffness $k_1$ and relaxed length $\ell_1$ to a second spring with stiffness $k_2$ and relaxed length $\ell_2$. The system of Sec. III B consists of a pendulum attached to two spring. In Sec. III C we analyze three masses moving on a circular ring and connected by springs. None of these first three examples contain any gauge freedom.

In Sec. III D we consider four masses fixed at the centers of freely extensible rods. The ends of the rods are connected, and the connection points (the “corners”) slide freely on the vertical posts. We consider two versions of this system: one with springs extending from the corners to the ceiling, the other with springs extending from the masses to the ceiling. In both cases the system is described by a singular Lagrangian, but the placement of the springs plays an important role which is clearly revealed in the Hamiltonian formulation. When the springs are attached to the masses, the system contains gauge freedom. When the springs are attached to the corners, there is no gauge freedom.

The system discussed in Sec. III E consists of a single loop of cord weaving between three pairs of pulleys. The lower pulley of each pair is fixed, while the upper pulley is attached to a mass and a spring. This system is a gauge theory, and can be generalized to any number of pairs of pulleys.

Section IV lists two more problems described by singular Lagrangians, without solutions. These problems are left as exercises for the reader.

The common element in each of these singular systems is the presence of degrees of freedom with no inertial response. Consider, for example, a two–particle system with coordinates $x_1$, $x_2$, and Lagrangian $L = m_1 \dot{x}_1^2/2 + m_2 \dot{x}_2^2/2 - V(x_1, x_2)$. The matrix of second derivatives of $L$ is nonsingular since $L_{ij}$ is diagonal with entries $m_1$ and $m_2$. Now set the mass $m_1$ to zero so that $L$ becomes singular. We can now vary the coordinate $x_1$ without any inertial response—changing $x_1$ does not cause any mass in the system to move. This observation provides an intuitive test for singular systems in classical mechanics. Imagine fixing the location of each mass, and ask: Is the system rigid? If not, then there are degrees of freedom with no inertial response. The Lagrangian for such a system is singular.

Section V contains concluding remarks.
II. DIRAC–BERGMANN ALGORITHM

Consider a system with \( \bar{N} \) generalized coordinates \( q_i \), where \( i = 1, \ldots, \bar{N} \). The velocities are denoted by \( \dot{q}_i \). The Lagrangian \( L(q, \dot{q}) \) is assumed to be singular, so the rank of the matrix \( L_{ij} \) (the number of linearly independent rows or columns) is less than \( \bar{N} \), say, \( \bar{M} \). We assume that the rank \( \bar{M} \) is constant throughout phase space. The following is a short summary of the Dirac–Bergmann algorithm for converting this system into constrained Hamiltonian form. This summary is not intended as a substitute for the more thorough treatments given elsewhere.\(^{13-23}\)

- Compute the conjugate momenta \( p_i = \partial L/\partial \dot{q}_i \). Since the Lagrangian is singular, these relations cannot be inverted for the velocities as functions of coordinates and momenta. This implies the existence of \( \bar{N} - \bar{M} \) relations among the coordinates and momenta.\(^{25}\) These relations are the primary constraints, denoted \( \phi_a(q, p) = 0 \), with the index \( a \) ranging from 1 to \( \bar{N} - \bar{M} \).

- Define the canonical Hamiltonian \( H_C \) by writing \( p_i \dot{q}_i - L(q, \dot{q}) \) in terms of the \( q \)'s and \( p \)'s. It can be shown that this is always possible. Note that \( H_C(q, p) \) is not unique, because one can always use the constraints \( \phi_a(q, p) = 0 \) to write some of the canonical variables in terms of others.

- Define the primary Hamiltonian \( H_P \) by adding the primary constraints with Lagrange multipliers to the canonical Hamiltonian. That is, \( H_P = H_C + \lambda^a \phi_a \), where \( \lambda^a \) are the Lagrange multipliers.

- Impose the conditions \( [\phi_a, H_P] = 0 \), referred to as “consistency conditions,” where \( [\cdot, \cdot] \) is the Poisson bracket. These conditions assure that the primary constraints are preserved under time evolution. The consistency conditions (one for each value of the index \( a \)) will reduce to a combination of (i) identities when the primary constraints \( \phi_a(q, p) = 0 \) hold; (ii) restrictions on the Lagrange multipliers; and/or (iii) restrictions on the \( q \)'s and \( p \)'s. The restrictions on the Lagrange multipliers express some \( \lambda \)'s in terms of \( q \)'s, \( p \)'s, and the remaining \( \lambda \)'s. Restrictions on the \( q \)'s and \( p \)'s are secondary constraints, which we write as \( \psi_m(q, p) = 0 \).

- The consistency conditions must be applied to the secondary constraints to insure
their preservation in time: \([\psi_m, H_P] = 0\). This can yield identities, further restrictions on the Lagrange multipliers, and/or tertiary constraints, which are further restrictions on the \(q\)’s and \(p\)’s. We continue to apply the consistency conditions to identify higher–order constraints and restrictions on the Lagrange multipliers. The process naturally stops when the consistency conditions have been applied to all constraints. We extend the range of the index \(m\) and let \(\psi_m(q, p)\) denote all of the secondary, tertiary, and higher–order constraints.

- The total Hamiltonian \(H_T\) is obtained from the primary Hamiltonian \(H_P\) by incorporating the restrictions on Lagrange multipliers. In the most general case, a subset of the Lagrange multipliers will remain free.

- The primary, secondary, tertiary, etc. constraints \(\phi_a\) and \(\psi_m\) are separated into first and second class. First class constraints have the property that their Poisson bracket with all constraints vanish when the constraints hold. Second class constraints have nonvanishing Poisson bracket with at least one other constraint. Let \(\mathcal{C}^{(fc)}_\alpha\) denote the set of first class constraints, and \(\mathcal{C}^{(sc)}_\mu\) denote the set of second class constraints.

- A subset of first class constraints can be constructed from the primary constraints \(\phi_a(q, p)\). These are the primary first class constraints which we denote \(\mathcal{C}^{(pfc)}_A\). It can be shown that the total Hamiltonian can be written as \(H_T = H_{fc} + \Lambda^A \mathcal{C}^{(pfc)}_A\), where the Lagrange multipliers \(\Lambda^A\) are free and the first class Hamiltonian \(H_{fc}\) has vanishing Poisson bracket with all of the constraints (when the constraints hold).

Before continuing, a few comments are in order. The equations of motion generated by the total Hamiltonian \(H_T\) through the Poisson bracket are equivalent to Lagrange’s equations for the original Lagrangian system. Since the Lagrange multipliers \(\Lambda^A\) are completely arbitrary, the phase space transformations generated by the primary first class constraints \(\mathcal{C}^{(pfc)}_A\) do not change the physical state of the system. We refer to such transformations as gauge transformations. Therefore, primary first class constraints generate gauge transformations.

The Dirac conjecture\textsuperscript{14} says that all first class constraints \(\mathcal{C}^{(fc)}_\alpha\) generate gauge transformations. Counterexamples to this conjecture have been described in the literature by a number of researchers.\textsuperscript{18,31–37} Other researchers have argued against these counterexamples,
citing subtleties in the way that the constraints are written. \textsuperscript{19,38} The Dirac conjecture is often taken as an assumption. \textsuperscript{20}

Here is the next step in the algorithm:

- Assuming the Dirac conjecture holds, each of the first class constraints has the status of a gauge generator. These constraints can be treated on an equal footing by constructing the \textit{extended Hamiltonian} \( H_E = H_{fc} + \Lambda^\alpha \mathcal{C}_\alpha^{(fc)} \). This is the sum of the first class Hamiltonian \( H_{fc} \) and a linear combination of all first class constraints with unrestricted Lagrange multipliers \( \Lambda^\alpha \). The equations of motion defined by the extended Hamiltonian are not strictly equivalent to the original Lagrangian equations of motion. Nevertheless, the theories agree for the evolution of physical variables (variables that are invariant under gauge transformations.)

Phase space functions \( F \) are evolved in time with either the extended Hamiltonian, \( \dot{F} = [F, H_E] \), or the total Hamiltonian, \( \dot{F} = [F, H_T] \). Physical trajectories are those that lie in the subspace of phase space where the constraints hold.

The constraint relations can be used freely after computing Poisson brackets, but not before. For example, the constraints can be used to alter the equations of motion \( \dot{F} = [F, H_E] \) (or \( \dot{F} = [F, H_T] \)) but not the functions that appears in the Poisson bracket.

We now have options. One option:

- Eliminate the second class constraints leaving the gauge freedom generated by the first class constraints intact. We do this by replacing the Poisson bracket with the Dirac bracket, defined as follows. Let \( \mathcal{M}_{\mu\nu} = [\mathcal{C}_\mu^{(sc)}, \mathcal{C}_\nu^{(sc)}] \) denote the matrix of Poisson brackets among the second class constraints, and let \( \mathcal{M}^{\mu\nu} \) denote its inverse. The Dirac bracket is

\[
[F, G]^* = [F, G] - [F, \mathcal{C}_\mu^{(sc)}] \mathcal{M}^{\mu\nu} [\mathcal{C}_\nu^{(sc)}, G] \tag{2}
\]

where \( F \) and \( G \) are phase space functions. (Summation over repeated indices is implied.)

Like the Poisson bracket, the Dirac bracket is antisymmetric and obeys the Jacobi identity. It also satisfies \( [F, \mathcal{C}_\mu^{(sc)}]^* = 0 \) for any phase space function \( F \). This allows us to use the second class constraints to simplify \( F \) and \( G \) before computing the bracket \( [F, G]^* \).
Because the Poisson bracket of the extended Hamiltonian with a second class constraint will vanish when the constraints hold, it follows that 
$[F, H_E]^*$ equals $[F, H_E]$ when the constraints hold. (Likewise for the total Hamiltonian $H_T$.) Thus, the equations of motion can be defined using either the Dirac bracket or the Poisson bracket.

• We can now eliminate a subset of phase space variables by imposing the second class constraints $C^{(sc)}_\mu = 0$ and using the Dirac bracket. In particular, we can use $C^{(sc)}_\mu = 0$ to eliminate variables from the extended Hamiltonian (or total Hamiltonian), resulting in a partially reduced Hamiltonian $H_{PR}$. Time evolution becomes $\dot{F} = [F, H_{PR}]^*$.

A second option:

• Eliminate both first and second class constraints by imposing gauge conditions. Canonical gauge conditions, like constraints, are restrictions on the phase space variables. Let us denote the constraints and gauge conditions, combined, by $C^{(\text{all})}_M$. A good set of gauge conditions will have the property that $C^{(\text{all})}_M$ are second class. That is, the matrix of Poisson brackets $M_{MN} = [C^{(\text{all})}_M, C^{(\text{all})}_N]$ is invertible. Let $M^{MN}$ denote the inverse and define the Dirac bracket by

$$[F, G]^* = [F, G] - [F, C^{(\text{all})}_M] M^{MN} [C^{(\text{all})}_N, G] ,$$

where summations over $M$ and $N$ are implied.

• Now use the constraints and gauge conditions $C^{(\text{all})}_M = 0$ to eliminate a subset of phase space variables. We can eliminate variables from the extended Hamiltonian (or total Hamiltonian), resulting in a fully reduced Hamiltonian $H_{FR}$. Time evolution becomes $\dot{F} = [F, H_{FR}]^*$.

In the next section we apply the Dirac–Bergmann algorithm to analyze problems in classical mechanics that are described by singular Lagrangians.

III. EXAMPLES

A. Compound spring

Form a “compound spring” by welding two springs together, as shown in Fig. 1. One end
FIG. 1. The compound spring. The mass moves vertically, with gravity acting in the downward direction. The generalized coordinates are the lengths of the two springs.

of the compound spring is attached to the ceiling, and a mass \( m \) hangs from the other end. Let \( x_1 \) and \( x_2 \) denote the lengths of the two springs, so the distance between the ceiling and the mass is \( x_1 + x_2 \). The Lagrangian for this system is

\[
L = \frac{m}{2}(\dot{x}_1 + \dot{x}_2)^2 + mg(x_1 + x_2) - \frac{k_1}{2}(x_1 - \ell_1)^2 - \frac{k_2}{2}(x_2 - \ell_2)^2. 
\]  

(4)

The matrix of second derivatives of \( L \) with respect to the velocities \( \dot{x}_i \),

\[
L_{ij} = \begin{pmatrix} m & m \\ m & m \end{pmatrix},
\]  

(5)

is singular with rank 1. The momenta are

\[
p_1 = \frac{\partial L}{\partial \dot{x}_1} = m(\dot{x}_1 + \dot{x}_2),
\]  

(6a)

\[
p_2 = \frac{\partial L}{\partial \dot{x}_2} = m(\dot{x}_1 + \dot{x}_2),
\]  

(6b)

and we can identify the primary constraint

\[
\phi \equiv p_2 - p_1
\]  

(7)

by inspection.

Next, construct the canonical Hamiltonian by writing \( p_i \dot{x}_i - L \) (a sum over the repeated index \( i \) is implied) in terms of \( p \)'s and \( x \)'s:

\[
H_C(x,p) = \frac{1}{2m}p_1p_2 - mg(x_1 + x_2) + \frac{k_1}{2}(x_1 - \ell_1)^2 + \frac{k_2}{2}(x_2 - \ell_2)^2.
\]  

(8)

The leading term \( p_1p_2/(2m) \) can be written in other ways, such as \( p_1^2/(2m) \) or \( (p_1^2 + p_2^2)/(4m) \), by invoking the constraint \( \phi = 0 \).
The primary Hamiltonian is \( H_P = H_C + \lambda \phi \). The consistency condition \( [\phi, H_P] = 0 \) yields
the secondary constraint
\[
\psi = k_1(x_1 - \ell_1) - k_2(x_2 - \ell_2) ,
\]
and the condition \( [\psi, H_P] = 0 \) restricts the Lagrange multiplier to
\[
\lambda = \frac{k_1 p_2 - k_2 p_1}{2m(k_1 + k_2)} .
\]
The application of consistency conditions is now complete.

The secondary constraint \( \psi = 0 \) has a direct physical interpretation via Newton’s third
law. It tells us that the force \( k_1(x_1 - \ell_1) \) that spring 1 exerts on spring 2 is equal but opposite
to the force \( k_2(\ell_2 - x_2) \) that spring 2 exerts on spring 1.

The total Hamiltonian is obtained by using the result (10) for \( \lambda \) in the primary Hamilto-
nian:
\[
H_T = \frac{1}{2m} p_1 p_2 + \frac{k_1 p_2 - k_2 p_1}{2m(k_1 + k_2)} (p_2 - p_1) - mg(x_1 + x_2) + \frac{k_1}{2} (x_1 - \ell_1)^2 + \frac{k_2}{2} (x_2 - \ell_2)^2 .
\]
The two constraints are second class, since \( [\phi, \psi] = k_1 + k_2 \) is nonzero. There are no first
class constraints, so the system has no gauge freedom and the total Hamiltonian, first class
Hamiltonian, and extended Hamiltonian coincide: \( H_T = H_{fc} = H_E \).

Let \( C^{(sc)}_\mu = \{ \phi, \psi \} \) denote the set of second class constraints. The matrix \( \mathcal{M}_{\mu\nu} = [C^{(sc)}_\mu, C^{(sc)}_\nu] \) is invertible with inverse
\[
\mathcal{M}^{\mu\nu} = \frac{1}{k_1 + k_2} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} .
\]

We now construct the Dirac bracket as in Eq. (2). The nonzero brackets among the phase
space variables are
\[
[x_1, p_1]^* = [x_1, p_2]^* = k_2/(k_1 + k_2) ,
\]
\[
[x_2, p_1]^* = [x_2, p_2]^* = k_1/(k_1 + k_2) .
\]

We can use the constraints to eliminate two of the phase space variables. For example,
solving \( \phi = \psi = 0 \) for \( x_2 \) and \( p_2 \), we find
\[
x_2 = \ell_2 + \frac{k_1}{k_2} (x_1 - \ell_1) ,
\]
\[
p_2 = p_1 ,
\]
and the Hamiltonian reduces to
\[ H_R = \frac{p_1^2}{2m} - \frac{mg}{k_2} [(k_1 + k_2)x_1 + k_2\ell_2 - k_1\ell_1] + \frac{1}{2}(k_1 + k_2^2/k_2)(x_1 - \ell_1)^2 . \] (15)

Note that in the absence of first class constraints, the partially and fully reduced Hamiltonians coincide. Here we use $H_R$ to denote this reduced Hamiltonian.

The time evolution of any function of the phase space variables $x_1, p_1, x_2, p_2$ can be obtained from $H_R$ and the Dirac bracket. In particular we have
\[ \dot{x}_1 = [x_1, H_R]^* = \frac{k_2}{m(k_1 + k_2)}p_1 , \] (16a)
\[ \dot{p}_1 = [p_1, H_R]^* = -k_1(x_1 - \ell_1) + mg , \] (16b)

which form a closed set of differential equations for $x_1$ and $p_1$ with general solution
\[ x_1(t) = A \cos(\omega t) + B \sin(\omega t) + \ell_1 + mg/k_1 , \] (17a)
\[ p_1(t) = \frac{k_1}{\omega} (B \cos(\omega t) - A \sin(\omega t)) . \] (17b)

Here, $A$ and $B$ are constants and the angular frequency is defined by
\[ \omega = \sqrt{k_1k_2/(m(k_1 + k_2))} . \] (18)

Given $x_1(t)$, we can determine $x_2$ as a function of time from the result (14a). The position of the mass below the ceiling, $x_1(t) + x_2(t)$, then follows. We find that the mass executes simple harmonic motion about its equilibrium position $\ell_1 + \ell_2 + mg(k_1 + k_2)/(k_1k_2)$ with angular frequency $\omega$.

**B. Pendulum and two springs**

Figure 2 shows a pendulum of length $\ell$ hanging from two springs. Each spring has stiffness $k$, and for simplicity we take the relaxed length of each spring to be zero. The generalized coordinates are the Cartesian coordinates $x$ and $y$ of the point where the springs attach to the pendulum, and the angle $\theta$ of the pendulum rod. (The Cartesian coordinate origin is midway between the points where the springs attach to the ceiling. The angle $\theta$ is measured from the negative $y$–axis.)
FIG. 2. A pendulum hanging from two springs. The springs are attached to the ceiling at the points $x = \pm d, y = 0$.

The kinetic energy for this system is $T = (m/2)(\dot{X}^2 + \dot{Y}^2)$, where $X = x + \ell \sin \theta$ and $Y = y - \ell \cos \theta$ are the Cartesian coordinates of the mass $m$. The Lagrangian is

$$L = \frac{m}{2}(\dot{x}^2 + \dot{y}^2 + \ell^2 \dot{\theta}^2) + m\ell(\dot{x} \cos \theta + \dot{y} \sin \theta)\dot{\theta} - mg(y - \ell \cos \theta) - k(x^2 + y^2 + d^2), \quad (19)$$

and the matrix of second derivatives of $L$ with respect to the velocities $\dot{x}, \dot{y}, \dot{\theta}$ is

$$L_{ij} = \begin{pmatrix} m & 0 & m\ell \cos \theta \\ 0 & m & m\ell \sin \theta \\ m\ell \cos \theta & m\ell \sin \theta & m\ell^2 \end{pmatrix}. \quad (20)$$

This matrix is singular with rank 2.

The momenta for this system are

$$p_x = m\dot{x} + m\ell \dot{\theta} \cos \theta, \quad (21a)$$

$$p_y = m\dot{y} + m\ell \dot{\theta} \sin \theta, \quad (21b)$$

$$p_\theta = m\ell(\dot{x} \cos \theta + \dot{y} \sin \theta) + m\ell^2 \dot{\theta}. \quad (21c)$$

Since the Lagrangian is quadratic in the velocities, the constraint can be constructed as $\phi = V^i(p_i - L_i)$, where the vector $V^i = (-\ell \cos \theta, -\ell \sin \theta, 1)$ spans the null space of $L_{ij}$. (Details are given at the end of this paper.) In this case $L_i = 0$ and the primary constraint is

$$\phi = -\ell p_x \cos \theta - \ell p_y \sin \theta + p_\theta. \quad (22)$$
The canonical Hamiltonian can be written as

\[ H_C = \frac{1}{2m}(p_x^2 + p_y^2) + k(x^2 + y^2 + d^2) + mg(y - \ell \cos \theta) , \tag{23} \]

and the primary Hamiltonian is \( H_P = H_C + \lambda \phi \).

The consistency condition \([\phi, H_P] = 0\) yields the secondary constraint

\[ \psi = 2k\ell (x \cos \theta + y \sin \theta) , \tag{24} \]

which gives \( \tan \theta = -x/y \). This tells us that the angle of the force exerted by the springs on the massless connection point must coincide with the angle of the pendulum rod. This is a consequence of Newton’s third law—the forces exerted by the springs on the rod must be equal in magnitude but opposite in direction to the force that the rod exerts on the springs—and the fact that the rod can only exert a force along its own direction, at angle \( \theta \).

The condition \([\psi, H_P] = 0\) determines the Lagrange multiplier to be

\[ \lambda = \frac{p_x \cos \theta + p_y \sin \theta}{m(\ell + x \sin \theta - y \cos \theta)} , \tag{25} \]

and

\[ H_T = \frac{1}{2m}(p_x^2 + p_y^2) + k(x^2 + y^2 + d^2) + mg(y - \ell \cos \theta) \]

\[ + \frac{p_x \cos \theta + p_y \sin \theta}{m(\ell + x \sin \theta - y \cos \theta)}(-\ell p_x \cos \theta - \ell p_y \sin \theta + p_\theta) \tag{26} \]

is the total Hamiltonian.

Note that the denominator in Eq. (25) is the coefficient of \( \lambda \) in \([\psi, H_P]\). This coefficient vanishes when \( x = -\ell \sin \theta, y = \ell \cos \theta \). At these points in phase space the Lagrange multiplier is not determined. This is not a shortcoming of the Dirac–Bergmann formalism, rather, it is a property of the physical system defined by the Lagrangian (19). When \( x = -\ell \sin \theta \) and \( y = \ell \cos \theta \), the mass \( m \) is at the origin and the pendulum rod can rotate without any inertial resistance, and without any change in the potential energy. Thus, at these points in phase space, the system exhibits a gauge–like freedom in which multiple configurations are physically indistinguishable. We can avoid this complication by assuming the mass stays below the ceiling, so that \( Y = y - \ell \cos \theta \) is always negative.

With this assumption the constraints are second class: \([\phi, \psi] = 2k\ell(\ell + x \sin \theta - y \cos \theta) \neq 0\). There is no gauge freedom, so \( H_E = H_{fc} = H_T \). We can construct the Dirac bracket from
Eq. (2) with
\[ M_{\mu \nu} = \frac{1}{2k\ell(x \sin \theta - y \cos \theta)} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \] (27)

The constraints \( \phi = \psi = 0 \) imply
\[ \theta = -\arctan(x/y), \] (28a)
\[ p_\theta = \frac{\ell}{\sqrt{x^2 + y^2}} (xp_y - yp_x), \] (28b)

and we can use these results to reduce the Hamiltonian:
\[ H_R = \frac{1}{2m} (p_x^2 + p_y^2) + mg(y + \ell/\sqrt{x^2 + y^2}) + k(x^2 + y^2 + d^2). \] (29)

The nonzero Dirac brackets among the remaining variables are
\[ [x, p_x]^* = \frac{r + \ell x^2/r^2}{r + \ell}, \] (30a)
\[ [x, p_y]^* = [y, p_x]^* = \frac{\ell xy/r^2}{r + \ell}, \] (30b)
\[ [y, p_y]^* = \frac{r + \ell y^2/r^2}{r + \ell}, \] (30c)

where \( r \equiv \sqrt{x^2 + y^2} \). The equations of motion are
\[ \dot{x} = [x, H_R]^* = \frac{\ell x(xp_x + yp_y) + px^3}{mr^2(r + \ell)}, \] (31a)
\[ \dot{y} = [y, H_R]^* = \frac{\ell y(xp_x + yp_y) + py^3}{mr^2(r + \ell)}, \] (31b)
\[ \dot{p}_x = [p_x, H_R]^* = -2kx, \] (31c)
\[ \dot{p}_y = [p_y, H_R]^* = -mg - 2ky. \] (31d)

These can be solved numerically in a straightforward fashion. The angle \( \theta(t) \) and its conjugate \( p_\theta(t) \) follow from Eqs. (28).

C. Masses, springs and ring

Three identical masses slide without friction on a ring of radius \( R \). The masses are connected by springs, as shown in Fig. 3. Each spring has stiffness \( k \) and for simplicity we set the relaxed lengths equal to zero. The generalized coordinates for this system are the
angles $\theta_1$, $\theta_2$, $\theta_3$ of the three masses and the Cartesian coordinates $x$ and $y$ of the point where the springs connect. (The origin is at the center of the ring. All angles are measured with respect to the $x$–axis.)

The Lagrangian for this system is

$$L = \frac{mR^2}{2}(\dot{\theta}_1^2 + \dot{\theta}_2^2 + \dot{\theta}_3^2) - V(\theta, x, y)$$  \hspace{1cm} (32)

with potential energy

$$V(\theta, x, y) = \frac{k}{2} \left\{ (x - R \cos \theta_1)^2 + (y - R \sin \theta_1)^2 + (x - R \cos \theta_2)^2 + (y - R \sin \theta_2)^2 
+ (x - R \cos \theta_3)^2 + (y - R \sin \theta_3)^2 \right\}.$$ \hspace{1cm} (33)

The conjugate momenta are

$$p_i \equiv \frac{\partial L}{\partial \dot{\theta}_i} = mR^2 \dot{\theta}_i, \hspace{1cm} (34a)$$
$$p_x \equiv \frac{\partial L}{\partial \dot{x}} = 0, \hspace{1cm} (34b)$$
$$p_y \equiv \frac{\partial L}{\partial \dot{y}} = 0, \hspace{1cm} (34c)$$

where $i = 1, 2, \text{and} \ 3$. We have two primary constraints,

$$\phi_1 = p_x, \hspace{1cm} (35a)$$
$$\phi_2 = p_y, \hspace{1cm} (35b)$$

and the primary Hamiltonian is

$$H_P = \frac{1}{2mR^2}(p_1^2 + p_2^2 + p_3^2) + V(\theta, x, y) + \lambda_1 p_x + \lambda_2 p_y.$$ \hspace{1cm} (36)
The consistency conditions $[\phi_a, H_P] = 0$ lead to the secondary constraints

$$
\psi_1 = kR(\cos \theta_1 + \cos \theta_2 + \cos \theta_3) - 3kx ,
$$

$$
\psi_2 = kR(\sin \theta_1 + \sin \theta_2 + \sin \theta_3) - 3ky ,
$$

and the conditions $[\psi_m, H_P] = 0$ yield restrictions on the Lagrange multipliers:

$$
\lambda_1 = -\frac{1}{3mR} (p_1 \sin \theta_1 + p_2 \sin \theta_2 + p_3 \sin \theta_3) ,
$$

$$
\lambda_2 = \frac{1}{3mR} (p_1 \cos \theta_1 + p_2 \cos \theta_2 + p_3 \cos \theta_3) .
$$

The secondary constraints tell us that the three springs connect at the point given by the average location of the three masses. This is required for the forces $\vec{F}_i = -k(x - R \cos \theta_i, y - R \sin \theta_i)$ that the springs exert on the massless connection point to sum to zero.

Inserting the results for the Lagrange multipliers into the primary Hamiltonian, we find the total Hamiltonian

$$
H_T = \frac{1}{2mR^2} (p_1^2 + p_2^2 + p_3^2) + V(\theta, x, y) - \frac{p_x}{3mR} (p_1 \sin \theta_1 + p_2 \sin \theta_2 + p_3 \sin \theta_3) \\
+ \frac{p_y}{3mR} (p_1 \cos \theta_1 + p_2 \cos \theta_2 + p_3 \cos \theta_3) .
$$

The constraints $C^{(sc)}_\mu = \{\phi_1, \phi_2, \psi_1, \psi_2\}$ are second class, with Poisson bracket

$$
\mathcal{M}_{\mu\nu} = [C^{(sc)}_\mu, C^{(sc)}_\nu] = \begin{pmatrix} 0 & 0 & 3k & 0 \\ 0 & 0 & 0 & 3k \\ -3k & 0 & 0 & 0 \\ 0 & -3k & 0 & 0 \end{pmatrix} .
$$

We now construct the Dirac bracket as defined in Eq. (2). The brackets among the phase space variables include

$$
[\theta_i, p_j]^* = \delta_{ij} ,
$$

$$
[x, p_i]^* = -\frac{R}{3} \sin \theta_i ,
$$

$$
[y, p_i]^* = \frac{R}{3} \cos \theta_i ,
$$

where $i$ and $j$ range over 1, 2, 3. The remaining Dirac brackets vanish.
We can use the constraints to eliminate four of the phase space variables; the natural choice is $x, y, p_x$ and $p_y$. From $C^{(sc)}_\mu = 0$ we find $p_x = p_y = 0$ and

\[
x = \frac{R}{3} (\cos \theta_1 + \cos \theta_2 + \cos \theta_3) ,
\]
\[
y = \frac{R}{3} (\sin \theta_1 + \sin \theta_2 + \sin \theta_3) .
\]

The reduced Hamiltonian is

\[
H_R = \frac{1}{2mR^2} (p_1^2 + p_2^2 + p_3^2) - \frac{kR^2}{3} (\cos(\theta_1 - \theta_2) + \cos(\theta_2 - \theta_3) + \cos(\theta_3 - \theta_1)) + kR^2
\]

From here we obtain the equations of motion for the angles,

\[
\dot{\theta}_i = [\theta_i, H_R]^* = p_i/(mR^2) ,
\]
and their conjugate momenta,

\[
\dot{p}_1 = [p_1, H_R]^* = \frac{kR^2}{3} (\sin(\theta_2 - \theta_1) + \sin(\theta_3 - \theta_1)) ,
\]
\[
\dot{p}_2 = [p_3, H_R]^* = \frac{kR^2}{3} (\sin(\theta_3 - \theta_2) + \sin(\theta_1 - \theta_2)) ,
\]
\[
\dot{p}_3 = [p_3, H_R]^* = \frac{kR^2}{3} (\sin(\theta_1 - \theta_3) + \sin(\theta_2 - \theta_3)) .
\]

It is straightforward to solve these equations numerically. Equations (42) then determine the coordinates $x, y$ as functions of time.

### D. Masses, Rods and Springs

The system shown in Fig. 4 consists of four masses fixed at the midpoints of four massless, freely extensible rods. A freely extensible rod is rigid in transverse directions but does not have a fixed length. That is, the rods can expand or contract as needed to span the distance between the vertical posts. Figure 4 shows two versions of this system. In the left figure, springs are attached to the connection points between the rods (the “corners” with coordinates $y_1$ through $y_4$). In the right figure the springs are attached to the masses.

The Lagrangian for this system is

\[
L = \frac{m}{2} \left[ \left( \frac{\dot{y}_1 + \dot{y}_2}{2} \right)^2 + \left( \frac{\dot{y}_2 + \dot{y}_3}{2} \right)^2 + \left( \frac{\dot{y}_3 + \dot{y}_4}{2} \right)^2 + \left( \frac{\dot{y}_4 + \dot{y}_1}{2} \right)^2 \right] - V(y) ,
\]

FIG. 4. Four massless, freely extensible rods are connected to each other at the corner posts. The rods slide freely along the posts. In the left figure springs are attached to the corners. In the right figure springs are attached to the masses.

with potential energy

\[ V(y) = mg \left[ y_1 + y_2 + y_3 + y_4 \right] + \frac{k}{2} \left[ (a - y_1)^2 + (a - y_2)^2 + (a - y_3)^2 + (a - y_4)^2 \right] \tag{47} \]

when the springs are attached to the corners, and

\[ V(y) = mg \left[ y_1 + y_2 + y_3 + y_4 \right] + \frac{k}{8} \left[ (2a - y_1 - y_2)^2 + (2a - y_2 - y_3)^2 + (2a - y_3 - y_4)^2 + (2a - y_4 - y_1)^2 \right] \tag{48} \]

when the springs are attached to the masses. Here, \( k \) denotes the spring constant and \( a = h - \ell \), where \( h \) is the height of the ceiling and \( \ell \) is the relaxed length of each spring.

The momenta are

\[ p_1 = \frac{m}{4} (\dot{y}_1 + 2\dot{y}_2 + \dot{y}_3) , \tag{49a} \]
\[ p_2 = \frac{m}{4} (\dot{y}_1 + 2\dot{y}_2 + \dot{y}_3) , \tag{49b} \]
\[ p_3 = \frac{m}{4} (\dot{y}_2 + 2\dot{y}_3 + \dot{y}_4) , \tag{49c} \]
\[ p_4 = \frac{m}{4} (\dot{y}_3 + 2\dot{y}_4 + \dot{y}_1) , \tag{49d} \]

which yield the single primary constraint

\[ \phi = p_1 - p_2 + p_3 - p_4 . \tag{50} \]
The canonical Hamiltonian is given by
\[ H_C = \frac{1}{2m} \left[ \frac{5}{2}(p_1^2 + p_2^2 + p_3^2 + p_4^2) - 2(p_1p_2 + p_2p_3 + p_3p_4 + p_4p_1) + p_1p_3 + p_2p_4 \right] + V(y) , \] (51)
and the primary Hamiltonian is \( H_P = H_C + \lambda \phi \).

Focus on the case shown on the left of Fig. 4 with springs attached to the corners and the potential energy of Eq. (47). The consistency conditions yield a secondary constraint
\[ \psi = -k(y_1 - y_2 + y_3 - y_4) , \] (52)
and the restriction
\[ \lambda = -\frac{5}{4m}(p_1 - p_2 + p_3 - p_4) \] (53)
on the Lagrange multiplier. The Lagrange multiplier can be simplified to \( \lambda = 0 \) using the constraint \( \phi = 0 \).

The secondary constraint is interesting because it places a restriction on the configuration of the system that might not be obvious. One can imagine moving any one of the corners, by hand, independently of the others. The mechanical arrangement of rods and posts do not place any restrictions on the \( y \) values. But as a dynamical system, the \( y \)'s must obey \( y_1 - y_2 + y_3 - y_4 = 0 \). This is because the \( y \)'s can be changed while keeping the masses in place. For example, if \( y_1 \) and \( y_3 \) are increased by some amount \( \delta y \), while \( y_2 \) and \( y_4 \) are decreased by the same amount \( \delta y \), the masses remain unmoved. There is no inertial resistance to this type of motion. As a result, the springs can instantly “snap” the corners into the preferred configuration satisfying \( y_1 - y_2 + y_3 - y_4 = 0 \). This is the configuration that minimizes the potential (47) while keeping the locations of the masses fixed.

In this example the total Hamiltonian \( H_T \) coincides with the canonical Hamiltonian \( H_C \). The constraints \( \phi, \psi \) are second class, so we can use them to eliminate two of the phase space variables. For example, let
\[ y_4 = y_1 - y_2 + y_3 , \] (54a)
\[ p_4 = p_1 - p_2 + p_3 . \] (54b)

Then the reduced Hamiltonian becomes
\[ H_R = \frac{1}{2m} \left[ 3(p_1^2 + p_2^2 + p_3^2) - 4(p_1p_2 + p_2p_3 + 2p_1p_3) + 2gm(y_1 + y_3) \right. \\
+ k \left[ y_1^2 + y_2^2 + y_3^2 - y_1y_2 + y_1y_3 - y_2y_3 - 2a(y_1 + y_3 - a) \right] . \] (55)
The equations of motion are obtained from $H_R$ and the Dirac bracket. Writing these as a system of second order equations for the coordinates, we find

\begin{align}
\ddot{y}_1 &= -g - \frac{k}{2m}(3y_1 - y_3 - 2a), \\
\ddot{y}_2 &= -g - \frac{k}{2m}(4y_2 - y_1 - y_3 - 2a), \\
\ddot{y}_3 &= -g - \frac{k}{2m}(3y_3 - y_1 - 2a).
\end{align}

(56a)

(56b)

(56c)

This simple system of linear equations can be solved analytically for $y_1$, $y_2$ and $y_3$ as functions of time $t$. The remaining variable $y_4(t)$ is determined from Eq. (54a).

The general solution is a linear combination of three harmonic modes: (i) the rods remain horizontal ($y_1 = y_2 = y_3 = y_4$) and oscillate up and down with frequency $\sqrt{k/m}$; (ii) the rod between $y_1$ and $y_2$ and the rod between $y_3$ and $y_4$ remain horizontal as they oscillate $180^\circ$ out of phase with frequency $\sqrt{2k/m}$; (iii) the rod between $y_2$ and $y_3$ and the rod between $y_4$ and $y_1$ remain horizontal as they oscillate $180^\circ$ out of phase with frequency $\sqrt{2k/m}$.

Now specialize to the system shown on the right side of Fig. 4, with springs attached to the masses and the potential energy of Eq. (48). In this case the time derivative of $\phi$ vanishes, $[\phi, H_P] = 0$, so there are no secondary constraints and the Lagrange multiplier $\lambda$ is unrestricted. The total Hamiltonian $H_T$ coincides with the primary Hamiltonian $H_P$.

Since $\phi$ is the only constraint, it is necessarily first class and it generates a gauge transformation. Explicitly, let $G = \epsilon \phi$ where $\epsilon$ is an arbitrary function of time. The gauge transformation of any phase space function $F(q, p)$ is determined by the Poisson bracket of $F$ with the gauge generator $G$; that is, $\delta F = [F, G]$. For the phase space coordinates, we have $\delta y_1 = \epsilon$, $\delta y_2 = -\epsilon$, $\delta y_3 = \epsilon$, $\delta y_4 = -\epsilon$ and $\delta p_i = 0$ (with $i = 1, \ldots, 4$). This describes a change in the $q$'s and $p$'s for which the masses don’t move and the spring lengths don’t change. In other words, the physical state of the system is unchanged.

Observables are phase space functions that are gauge invariant. Observables include the locations of the masses, namely, $x_{12} \equiv (y_1 + y_2)/2$, $x_{23} \equiv (y_2 + y_3)/2$, $x_{34} \equiv (y_3 + y_4)/2$ and $x_{41} \equiv (y_4 + y_1)/2$, and the momenta $p_i$. We can identify the physical meaning of the $p$'s by computing the time derivatives of the masses’ locations using the total Hamiltonian. This shows that $p_i = mv_i$, where $v_i$ is the average of the velocities of the two masses adjacent to the corner $y_i$.

From this simple example we see that each physical state of the system is described by a
curve in phase space, a curve defined by the transformation \( \delta F = \epsilon [F, \phi] \). These curves are called “gauge orbits”.

We can select a single point on each gauge orbit to represent the physical state of the system. We do this by choosing a gauge condition \( \chi(q, p) = 0 \) such that \( \chi \) and \( \phi \), together, form a set of second class constraints. For example, we can require the rod between \( y_1 \) and \( y_2 \) to remain horizontal by choosing

\[
\chi = y_2 - y_1. \tag{57}
\]

Since \( [\chi, \phi] \neq 0 \), the set \( \mathcal{C}^{(all)} = \{ \chi, \phi \} \) is indeed second class. The Dirac bracket is constructed as in Eq. (3), and we can use the constraints to eliminate two of the variables, say, \( y_1 \) and \( p_1 \). The fully reduced Hamiltonian is

\[
H_{FR} = \frac{1}{2m} [3(p_2^2 + p_4^2) + 4(p_3^2 - p_2p_3 - p_4p_3) + 2p_2p_4] + V(y)|_{y_1 = y_2} \tag{58}
\]

where \( V(y)|_{y_1 = y_2} \) is the potential energy of Eq. (48) evaluated at \( y_1 = y_2 \). The nonzero Dirac brackets among the remaining variables are

\[
[y_3, p_3]^* = [y_4, p_4]^* = 1, \tag{59a}
\]

\[
[y_2, p_2]^* = [y_4, p_2]^* = -[y_4, p_2]^* = 1/2, \tag{59b}
\]

and the equations of motion \( \dot{F} = [F, H_{FR}]^* \) are

\[
\dot{y}_2 = \frac{1}{2m} (3p_2 - 2p_3 + p_4), \tag{60a}
\]

\[
\dot{y}_3 = \frac{1}{2m} (-p_2 + 6p_3 - 3p_4), \tag{60b}
\]

\[
\dot{y}_4 = \frac{1}{2m} (-p_2 - 2p_3 + 5p_4), \tag{60c}
\]

\[
\dot{p}_2 = ak - mg - \frac{k}{4} (3y_2 + y_3), \tag{60d}
\]

\[
\dot{p}_3 = ak - mg - \frac{k}{4} (y_2 + 2y_3 + y_4), \tag{60e}
\]

\[
\dot{p}_4 = ak - mg - \frac{k}{4} (y_2 + y_3 + 2y_4). \tag{60f}
\]

These results imply \( \ddot{y}_i = ak/m - g - (k/m)y_i \), where \( i = 2, 3, 4 \). Thus, each of the three corners \( y_2, y_3, \) and \( y_4 \) independently execute simple harmonic motion with frequency \( \sqrt{k/m} \) about their equilibrium positions \( a - mg/k \). The corner \( y_1 \) moves in sync with \( y_2 \), due to the gauge condition \( \chi = y_2 - y_1 = 0 \).
There are other interesting gauge choices. For example, we can freeze the corner \( y_4 \) by letting
\[
\chi = y_4 - a + mg/k .
\]
(61)

After constructing the Dirac bracket, eliminating the variables \( x_4 \) and \( p_4 \) and reducing the Hamiltonian, we find the equations \( \ddot{y}_i = ak/m - g - (k/m)y_i \) for \( i = 1, 2, 3 \). The corners \( y_1, y_2, \) and \( y_3 \) independently execute simple harmonic motion with frequency \( \sqrt{k/m} \), while \( y_4 \) remains fixed.

The evolution of the observables is, of course, independent of the gauge choice. In particular, the motions of the masses are the same whether we choose \((57)\) or \((61)\). It is not difficult to see that each of the four masses executes simple harmonic motion with frequency \( \sqrt{k/m} \). However, the amplitudes and phases are not independent—they must satisfy \( x_{12} + x_{34} = x_{23} + x_{41} \). This relationship follows from the definitions \( x_{12} \equiv (y_1 + y_2)/2, \ etc. \)

E. Pairs of pulleys

Figure 5 shows three pairs of massless pulleys. Each pair consists of a fixed lower pulley and an upper pulley attached to a mass and a spring. A cord runs over and under the pulleys, as shown, with the left end attached to the right end. That is, the left and right sides of the figure are “periodically identified” so that the cord forms a single continuous loop. (This can be achieved in three dimensions by attaching the springs and the bottom pulleys to circular supports.) Note that we can construct such a system using any number of pairs of pulleys. We will focus on the version with three pairs.

The coordinates for this system are the angles \( \alpha_1, \alpha_2, \alpha_3 \) of the lower, fixed pulleys. Consider the height of mass \( m_2 \). If the angle \( \alpha_1 \) increases by \( \delta \alpha_1 \), the height of \( m_2 \) increases by \( R \delta \alpha_1/2 \) where \( R \) is the radius of the lower pulley. If the angle \( \alpha_2 \) increases by \( \delta \alpha_2 \), the height of \( m_2 \) decreases by \( R \delta \alpha_2/2 \). Thus we see that the height of \( m_2 \) is \( h_2 = R(\alpha_1 - \alpha_2)/2 + c \), where \( c \) is a constant. Likewise, the height of \( m_1 \) is \( h_1 = R(\alpha_3 - \alpha_1)/2 + c \) and the height of \( m_3 \) is \( h_3 = R(\alpha_2 - \alpha_3)/2 + c \).

Let \( m_1 = m_2 = m_3 \) and use the common notation \( m \) for each mass. The kinetic energy for this system is \( T = (m/2)(\dot{h}_1^2 + \dot{h}_2^2 + \dot{h}_3^2) \), or
\[
T = \frac{mR^2}{8} \left[ (\dot{\alpha}_1 - \dot{\alpha}_2)^2 + (\dot{\alpha}_2 - \dot{\alpha}_3)^2 + (\dot{\alpha}_3 - \dot{\alpha}_1)^2 \right] .
\]
(62)
The gravitational potential energy $mg(h_1 + h_2 + h_3)$ is simply a constant. The spring potential energy is $(k/2)[(a - h_1)^2 + (a - h_2)^2 + (a - h_3)^2]$, where the constant $a$ depends on the height of the ceiling and the relaxed length of each spring. To within an additive constant, the total potential energy is

$$V(\alpha) = \frac{kR^2}{8} \left[ (\alpha_1 - \alpha_2)^2 + (\alpha_2 - \alpha_3)^2 + (\alpha_3 - \alpha_1)^2 \right].$$

(63)

As usual the Lagrangian is $L = T - V$.

The momenta for this system are

$$p_1 = \frac{mR^2}{4}(2\dot{\alpha}_1 - \dot{\alpha}_2 - \dot{\alpha}_3),$$

(64a)

$$p_2 = \frac{mR^2}{4}(2\dot{\alpha}_2 - \dot{\alpha}_3 - \dot{\alpha}_1),$$

(64b)

$$p_3 = \frac{mR^2}{4}(2\dot{\alpha}_3 - \dot{\alpha}_1 - \dot{\alpha}_2).$$

(64c)

The matrix of second derivatives $\partial^2 L/\partial \dot{\alpha}_i \partial \dot{\alpha}_j$ has rank 2 and there is one primary constraint:

$$\phi = p_1 + p_2 + p_3.$$  

(65)

The canonical Hamiltonian can be written as

$$H_C = \frac{4}{3mR^2}(p_1^2 + p_1p_2 + p_2^2) + V(\alpha).$$

(66)
where $V(\alpha)$ is given in Eq. (63). The primary Hamiltonian is $H_P = H_C + \lambda \phi$.

The Poisson bracket $[\phi, H_P]$ vanishes identically, so the consistency condition does not lead to any further constraints and does not restrict the Lagrange multiplier. The primary constraint $\phi$ is first class. We see that this is a gauge theory with gauge generator $G = \epsilon \phi$.

Under a gauge transformation the phase space coordinates transform as

$$\delta \alpha_i = [\alpha_i, G] = \epsilon,$$  \hspace{1cm} (67a)
$$\delta p_i = [p_i, G] = 0,$$  \hspace{1cm} (67b)

for $i = 1, 2, 3$. Physically, the gauge freedom arises because the pulleys can rotate by equal amounts ($\delta \alpha_1 = \delta \alpha_2 = \delta \alpha_3$), causing the cord to cycle through the system while leaving each mass fixed in place. The observables for this system include the differences, $\alpha_3 - \alpha_1$, $\alpha_1 - \alpha_2$, $\alpha_2 - \alpha_3$, which are proportional to the heights $h_1$, $h_2$, $h_3$ of the masses. The observables also include the momenta $p_i$. In physical terms, these are given by

$$p_1 = mR(v_1 - v_3)/2,$$  \hspace{1cm} (68a)
$$p_2 = mR(v_2 - v_1)/2,$$  \hspace{1cm} (68b)
$$p_3 = mR(v_3 - v_2)/2,$$  \hspace{1cm} (68c)

where $v_1$, $v_2$, and $v_3$ are the velocities of the three masses.

Let us fix the gauge with the condition $\chi = 0$ where

$$\chi = \alpha_1 + \alpha_2 + \alpha_3.$$  \hspace{1cm} (69)

The set $\{\chi, \phi\}$ is second class, and the nonzero Dirac brackets are $[\alpha_i, p_j]^* = 2/3$ for $i = j$ and $[\alpha_i, p_j]^* = -1/3$ for $i \neq j$. We can use $\chi = 0$ and $\phi = 0$ to eliminate two of the variables, say, $\alpha_3$ and $p_1$. Then the fully reduced Hamiltonian is

$$H_{FR} = \frac{4}{3mR^2} (p_2^2 + p_2 p_3 + p_3^2) + \frac{3kR^2}{4} (\alpha_1^2 + \alpha_1 \alpha_2 + \alpha_2^2).$$  \hspace{1cm} (70)

The equations of motion for the remaining variables are

$$\dot{\alpha}_1 = -\frac{4}{3mR^2} (p_2 + p_3),$$  \hspace{1cm} (71a)
$$\dot{\alpha}_2 = \frac{4}{3mR^2} p_2,$$  \hspace{1cm} (71b)
$$\dot{p}_2 = -\frac{3kR^2}{4} \alpha_2,$$  \hspace{1cm} (71c)
$$\dot{p}_3 = \frac{3kR^2}{4} (\alpha_1 + \alpha_2).$$  \hspace{1cm} (71d)
The resulting second order equations for the angles are $\ddot{\alpha}_1 = -(k/m)\alpha_1$ and $\ddot{\alpha}_2 = -(k/m)\alpha_2$. Thus, the pulleys $\alpha_1$ and $\alpha_2$ execute independent simple harmonic motion with angular frequency $\sqrt{k/m}$. The third angle is determined from the gauge condition as $\alpha_3 = -\alpha_1 - \alpha_2$.

The observables for this system include the heights $h_i$ of the masses. Each mass executes simple harmonic motion with frequency $\sqrt{k/m}$, subject to the restriction $h_1 + h_2 + h_3 = \text{const.}$ The restriction follows from the relations $h_1 = R(\alpha_3 - \alpha_1)/2 + c$, etc.

IV. EXERCISES FOR THE READER

The following problems are left as exercises for the reader. Solutions can be found in the supplementary material.42

1. A pendulum of mass $m$ and length $\ell$ hangs from the ceiling, as shown in Fig. 6. Two massless springs are attached to the ceiling, a distance $D$ apart, with spring #1 wound around the pendulum rod. The two springs are attached to each other. Let each spring have stiffness $k$ and a relaxed length of zero. Use the angle of the pendulum and the length of spring #1 as generalized coordinates.

2. Two massless, frictionless pulleys are arranged as shown in Fig. 7. The axis of the upper pulley is fixed, while the lower pulley is free to move vertically. The mass $m$ is also restricted to move vertically. Note the direction in which the cords are wound around the pulleys. Use the orientation angles $\alpha_1$ and $\alpha_2$ as generalized coordinates.

FIG. 6. A pendulum with two springs. One spring is wound around the pendulum rod.

FIG. 7. Two massless, frictionless pulleys are arranged as shown...
FIG. 7. A mass $m$ hanging from a series of pulleys. The radii of the two pulleys can be different.

V. CONCLUSIONS

Classical mechanics textbooks typically avoid singular Lagrangians by fiat. For many popular books, such as Classical Mechanics by Goldstein and Mechanics by Landau and Lifschitz, this is understandable since these were written before the work of Dirac and Bergmann. But today we need not limit our attention to nonsingular systems. The Dirac–Bergmann algorithm is a natural extension of the standard Lagrangian and Hamiltonian formalism, and is not overly difficult to apply. It allows us to analyze interesting singular systems and creates a closer link to modern field theories with gauge freedom.

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This is the view taken by Dirac. Alternatively, some authors define a gauge transformation as an invariance of the Lagrangian. With this definition the primary first class constraints, by themselves, do not always generate gauge transformations. Rather, gauge transformations can be associated with particular linear combinations of primary and secondary first class constraints.

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40 Here we assume $y < 0$ and $-\pi/2 \leq \theta \leq \pi$, which implies $Y = y - \ell \cos \theta < 0$.

41 A freely extensible rod can be approximated by a series of fixed–length rods and cylinders, where the rods slide through the cylinders without friction. A vertical guide can be used to keep the mass at the midpoint between the vertical posts.

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