ON THE SPECTRAL ESTIMATES FOR THE
SCHRÖDINGER TYPE OPERATORS:
THE CASE OF SMALL LOCAL DIMENSION

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In memory of M.Sh. Birman, a Scientist and a Person

Abstract. The behavior of the discrete spectrum of the Schrödinger
operator $-\Delta - V$, in quite a general setting, to a large extent is de-
termined by the behavior of the corresponding heat kernel $P(t; x, y)$
as $t \to 0$ and $t \to \infty$. If this behavior is powerlike, i.e.,
\[ \|P(t; \cdot, \cdot)\|_{L^\infty} = O(t^{-\delta/2}), \; t \to 0; \|P(t; \cdot, \cdot)\|_{L^\infty} = O(t^{-D/2}), \; t \to \infty, \]
then it is natural to call the exponents $\delta, D$ ‘the local dimension’
and ‘the dimension at infinity’, respectively. The character of spec-
tral estimates depends on the relation between these dimensions.
We analyze the case where $\delta < D$, insufficiently studied before.
Our applications concern both combinatorial and metric graphs.

1. Introduction

One of the most influential papers by M.Sh.Birman has been [2]
(1961). The approach developed there, under the name ‘The Birman-
Schwinger Principle’, has been the source of inspiration and one of th e
main tools in the spectral analysis for Schrödinger type operators.

In [13] this tool was applied to eigenvalue estimates for such operators
in a very general setting, and it turned out that these estimates depend
essentially on two numerical characteristics of the operator, $\delta$ and $D$,
that can be called the local dimension and the dimension at infinity. For
the standard Schrödinger operator on $\mathbb{R}^d$, these characteristics coincide
with the dimension; in general, $\delta \neq D$.

In the survey paper [14] various relations between dimensions were
discussed and the main attention was given there to the effects appear-
ing when $\delta \geq D$. The simplest example with $\delta < D$ is given by the
lattice $\mathbb{Z}^d$ (i.e., by the discrete Schrödinger operator); here $D = d$ and
$\delta = 0$. In this case some peculiarities in the spectral distribution were

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discovered in [15]. In particular, the large coupling constant eigenvalue estimates, that are order sharp in \( \mathbb{R}^d \), are not sharp in \( \mathbb{Z}^d \) any more. Deeper deliberations on the effects found in [15] have lead the authors to understanding that these peculiarities are common to all situations when \( \delta < D \). In the present paper we consider the cases of rather general combinatorial graphs, where \( \delta = 0 \), and quantum (metric) graphs, where \( \delta > 1 \) is arbitrarily close to 1; we restrict ourselves to the situation where \( D > 2 \). We find a range of spectral estimates for Schrödinger type operators on such graphs. For quantum graphs, it turns out that such estimates are determined by the corresponding estimates for the associated combinatorial graph, which is rather unexpected since the quantum graph contains much more ‘flesh’. This phenomenon is supported by Theorem 4.1, where the relations between dimensions of these graphs are established. We find also criteria for the Birman-Schwinger operator to belong to various Schatten, or ‘weak’ Schatten classes, and conditions for the validity of a Weyl type eigenvalue asymptotics. The main results in this direction are Theorems 5.3 and 5.5.

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2. OPERATORS ON GENERAL MEASURE SPACES

2.1. Dimensions of a semigroup. Let \((X, \sigma)\) be a measure space with sigma-finite measure. We denote \( L^q(X) = L^q(X, \sigma) \) and \( \| \cdot \|_q = \| \cdot \|_{L^q(X)} \). Often we drop the symbol \( X \) in our notation. Let \( \mathcal{A} \) be a non-negative self-adjoint operator in \( L^2(X) \) and \( P(t) = \exp(-\mathcal{A}t) \) be the corresponding semigroup. We assume that for any \( t > 0 \) the operator \( P(t) \) is positivity preserving and is bounded as acting from \( L^1 \) to \( L^\infty \). It is well known that under these assumptions \( P(t) \) is an integral operator whose kernel \( P(t; x, y) \) (heat kernel) is well-defined for \( t > 0 \) as a function in \( L^\infty(X \times X) \) (see [13] for details). We denote

\[
M_\mathcal{A}(t) = \| P(t; \cdot, \cdot) \|_{L^\infty(X \times X)}.
\]

The above described class of operators \( \mathcal{A} \) (we use the notation \( \mathcal{P} \) for it) includes the Laplacian, both in its continuous and discrete versions, and also many other important operators; see e.g. [13, 10]. For simplicity, we use below the term ‘Laplacian’ for any operator \( \mathcal{A} \in \mathcal{P} \).

The function \( M_\mathcal{A}(t) \) is non-increasing. Its main characteristics are the behavior as \( t \to 0 \) and as \( t \to \infty \). In this paper we always assume
that there are two non-negative exponents $\delta, D$, such that
\begin{equation}
M_A(t) = O(t^{-\delta/2}), \quad t \to 0; \quad M_A(t) = O(t^{-D/2}), \quad t \to \infty,
\end{equation}
and moreover,
\begin{equation}
D \geq \delta, \quad D > 2.
\end{equation}

Denote by $a$ the quadratic form of $A$. It follows from (2.1), (2.2) that $M_A(t) \leq Ct^{-D/2}$, with some $C$, for all $t \in (0, \infty)$, and hence, by the Varopoulos theory, see [17], the last inequality is equivalent to the ‘embedding theorem’
\begin{equation}
\|u\|_p^2 \leq Ca[u], \quad p = p(D) = 2D/(D - 2), \quad \forall u \in \text{Dom}(a).
\end{equation}

2.2. General eigenvalue estimates. Let $V \geq 0$ be a measurable function on $X$. Under some additional assumptions, the operator $A - V$ defined via its quadratic form $a[u] - \int_X V|u|^2d\sigma$ is self-adjoint, with negative spectrum consisting of a finite number of eigenvalues of finite multiplicities. Below $N_-(A - V)$ stands for the total multiplicity of the negative spectrum of $A - V$. It is proved in [13] that the number $N_-(A - V)$ can be conveniently estimated in terms of the function $M_A(t)$. This estimate is an abstract version of Lieb’s approach to the proof of the Rozenblum – Lieb – Cwikel (RLC) estimate.

The estimates for $N_-(A - V)$ depend on which one of the exponents $\delta, D$ in (2.1) is bigger than the other one. In particular, the following version of RLC estimate is valid (see, e.g., [14], Remark 1 in section 3.2, see also [13] and especially section 3.1 there – with $d$ replaced by $D$ – for the details of the proof). We write it down for the operator with a large parameter $\alpha > 0$ (the coupling constant) incorporated.

**Theorem 2.1.** Suppose that $D \geq \delta, \quad D > 2$. Then for any $V \in L^{D/2}(X)$ and any $\alpha > 0$ the operator $A - \alpha V$ is well-defined, its negative spectrum is finite, and the following estimate is satisfied:
\begin{equation}
N_-(A - \alpha V) \leq C \alpha^{D/2} \int_X V^{D/2}d\sigma, \quad C = C(X, D).
\end{equation}

It is convenient to formulate the estimates of this type in terms of the corresponding Birman – Schwinger operator $B_V$. Recall (see, e.g., [2, 5]) that $B_V$ is the operator in $H_a$ generated by the quadratic form
\begin{equation}
b_V[u] = \int_X V|u|^2d\sigma.
\end{equation}
Due to the inequality (2.3), for $V \in L^{D/2}$ this operator is well defined. The Rayleigh quotient for $B_V$ is

$$b_V[u]/a[u], \quad u \in H_a;$$

its eigenvalue counting function is denoted by $n(s, B_V)$.

The next result is an equivalent reformulation of Theorem 2.1.

**Theorem 2.2.** Under the assumptions of Theorem 2.1, $B_V \in \Sigma^{D/2}$, and

$$\|B_V\|_{\Sigma^{D/2}} \leq C \|V\|_{L^{D/2}}. \quad (2.6)$$

We recall that $\Sigma_p$ stands for the class of all compact operators with the powerlike estimate for the $s$-numbers, $s_n(T) = O(n^{-1/p})$, see [4], §11.6. Similar classes with $o$ in place of $O$ are denoted by $\Sigma_p^{(0)}$, they are closed in $\Sigma_p$. Below we also use the standard Neumann – Schatten classes $\mathcal{S}_p$, $0 < p \leq \infty$.

Suppose now that $\delta < D$. Then (2.1) implies $M_A(t) = O(t^{-q})$, $t \to 0, \infty$, with any $q \in [\delta/2, D/2]$. It follows that an estimate similar to (2.6) but with any such exponent $q$ instead of $D/2$ (and with a constant depending on $q$) is also valid:

$$\|B_V\|_{\Sigma_q} \leq C \|V\|_{L^q}, \quad \delta/2 \leq q \leq D/2, \quad q > 1. \quad (2.7)$$

In the case of the Euclidean Laplacian on $\mathbb{R}^d$, $d \geq 3$ (here $D = \delta = d$) the estimate (2.6) is known to be sharp, in the sense that for $V \neq 0$ the operator $B_V$ cannot belong to any class, smaller than $\Sigma_{d/2}$. It was shown in [15] that for the lattice Laplacian the situation is different: $V \in \ell^{D/2}(\mathbb{Z}^d)$, $d \geq 3$, yields $B_V \in \Sigma_{d/2}^{(0)}$ (or, in other terms, $N_-(A - \alpha V) = o(\alpha^{d/2})$). Our next result shows that a similar fact takes place in the general case, provided that $D > \delta$.

**Theorem 2.3.** Suppose that in the assumptions of Theorem 2.1 we have $D > \delta$. Then $B_V \in \Sigma_{D/2}^{(0)}$.

**Proof.** Fix a number $q \in (\max(\delta/2, 1), D/2)$. Functions $V \in L^{D/2} \cap L^{\delta/2}$ belong to $L^q$ and form a dense subset in $L^{D/2}$. For such functions $V$, the estimate (2.7) implies $B_V \in \Sigma_q \subset \Sigma_{D/2}^{(0)}$. By continuity (see [4], Theorem 11.6.7), this inclusion carries over to any $V \in L^{D/2}$. \(\square\)

The following result shows that the estimate (2.7) with any $q < D/2$ is not sharp, in the sense that the class of admissible potentials can be considerably widened, and the operators $B_V$ for $V \in L^q$ belong actually to the class $\mathcal{S}_q$ which is smaller than $\Sigma_q$. Below $L^q_w$ stands for the weak $L^q$-space (see, e.g., [1]).
Theorem 2.4. Suppose that in the assumptions of Theorem 2.1 we have $D > \delta$. Then for any $q \in (\max(\delta/2, 1), D/2)$

$$V \in L^q_w \implies B_V \in \Sigma_q; \quad V \in L^q \implies B_V \in \mathcal{S}_q,$$

with the estimates

$$\|B_V\|_{\Sigma_q} \leq C\|V\|_{L^q}^q; \quad \|B_V\|_{\mathcal{S}_q} \leq C\|V\|_{L^q}^q.$$

Equivalently,

$$\|B_V\|_{\Sigma_q} \leq C\|V\|_{L^q}^q; \quad \|B_V\|_{\mathcal{S}_q} \leq C\|V\|_{L^q}^q.$$

The result follows from (2.7) by the real interpolation (see [1]).

3. Combinatorial graphs

In the rest of the paper we consider operators on graphs. In this section we treat combinatorial graphs (notation $G$), and in the next two sections we discuss metric graphs (notation $\Gamma$). In both cases we always assume that the graph is connected, has an infinite number of vertices, and has no loops, vertices with degree one, or multiple edges (see, e.g., [9] for the main notions of the graph theory). We denote the set of edges by $\mathcal{E}$ and the set of vertices by $\mathcal{V}$. The notation $v \sim v'$ means that the vertices $v, v'$ are connected by an edge that we sometimes denote by $(v, v')$. We always suppose that degrees of all vertices are finite:

$$\deg(v) = \#\{v' \in \mathcal{V}, v' \sim v\} < \infty, \forall v \in \mathcal{V}.$$

With each edge $e \in \mathcal{E}$ we associate a weight $g_e > 0$. We need such ‘weighted graphs’ when dealing with metric graphs in Sect. 4, 5.

On the set $\mathcal{V}$ we consider the counting measure $\sigma : \sigma(v) = 1$ for any $v \in \mathcal{V}$. The basic Hilbert space in this section is $\ell^2(\mathcal{V}) = L^2(\mathcal{V}, \sigma)$; we write also $\ell^q = L^q(\mathcal{V}, \sigma), \ell^q_w = L^q_w(\mathcal{V}, \sigma)$. The quadratic form

$$a_G[f] = \sum_{e \in \mathcal{E} : e = (v, v')} g_e |f(v) - f(v')|^2,$$

with the domain $f \in \ell^2(\mathcal{V}), a_G[f] < \infty$, defines in $\ell^2(\mathcal{V})$ a nonnegative self-adjoint operator, $A = A_G = -\Delta_G$. In particular, if the weights $g_e$ and the degrees $\deg(v)$ are uniformly bounded, the operator $A$ is bounded. Due to the inclusions $\ell^1(\mathcal{V}) \subset \ell^2(\mathcal{V}) \subset \ell^\infty(\mathcal{V})$, the operators $\exp(-At)$ are bounded as acting from $\ell^1$ to $\ell^\infty$, so that $M_A(t) \leq C$. This means that $\delta = 0$. Our main assumption (cf. (2.1), (2.2)) is that

$$M_{A_G}(t) = O(t^{-D/2}), \ t \to \infty, \ \text{with some } D > 2.$$
The corresponding inequality (2.3):

\[ \|f\|_2^2 \leq C a_G[f], \quad p = p(D) = 2D(D - 2)^{-1} \]

is certainly satisfied for all \( f \) with finite support. Hence, the closure \( \mathcal{H}(G) \) of the set of all such functions in the metric \( a_G[f] \) is embedded in \( \ell^p \). In notation of section 2, \( \mathcal{H}(G) \) plays the role of the space \( H_a \).

In what follows, \( \mathcal{H}_\text{fin}(G) \) stands for the set of all finitely supported functions in \( \mathcal{H}(G) \), considered as a linear subspace in \( \mathcal{H}(G) \).

There are many geometric and analytic criteria for the relation (3.2) to hold. Without going into details, we refer to [6], [8], [16], where such criteria are presented. An example of graph satisfying (3.2) is the integer lattice \( \mathbb{Z}^d \); for each edge \( e \) we take \( g_e = 1 \). Here \( D = d \), which can be checked by a direct computation of the heat kernel. This case was the object of our study in the paper [15]. Here we extend some of its results to general graphs.

The following result, in its main part, is just a special case of Theorems 2.2 - 2.4. The only essential novelty is that the condition \( q > 1 \) that implicitly appears in Theorem 2.4 is no longer necessary. Note that now \( b_V \) takes the form

\[ b_V[f] = \sum_{v \in V} V(v) |f(v)|^2. \]

**Theorem 3.1.** Let (3.2) be satisfied.

1° Suppose \( V \in \ell^{D/2}(\mathcal{V}) \). Then

\[ \|B_V\|_{\Sigma_{D/2}} \leq C \|V\|_{\ell^{D/2}}, \]

and in addition, \( B_V \in \Sigma^{(0)}_{D/2} \).

2° If \( V \in \ell^{q}(\mathcal{V}) \) (or \( V \in \ell^{q}(\mathcal{G}) \)) for some \( q \in (0, D/2) \) then \( B_V \in \Sigma_q \) (resp., \( B_V \in \mathcal{S}_q \)), and the estimates (2.8) hold true.

The proof is the same as for \( G = \mathbb{Z}^d \), see [15], and we skip it.

In contrast to the general situation of section 2, for graphs it is also possible to obtain a lower bound for \( n(s, B_V) \) in terms of the distribution function for \( V \), i.e.,

\[ \nu(\tau, V) = \#E(\tau, V); \quad E(\tau, V) = \{ v \in \mathcal{V} : V(v) > \tau \}, \quad \tau > 0. \]

This estimate does not require any assumptions about \( V \). We need, however, two additional assumptions about the graph \( G \): the weights \( g_e \) should be uniformly bounded:

\[ g_e \leq g_0, \]

\[ \text{From now on, we formulate the results in the terms of the Birman-Schwinger operator only.} \]
and the degrees of the vertices should be uniformly bounded:

\( \deg(v) = \# \{ v' \in V, v' \sim v \} \leq d. \)

**Theorem 3.2.** Let (3.6), (3.7) be satisfied for a graph \( G \). Then for any \( V \geq 0 \)

\( n(s, B_V) \geq (d + 1)^{-1} \nu(g_0(d + 1)s, V), \)

and therefore, for any \( q > 0 \),

\( \| B_V \|_{\Sigma_q} \geq c \| V \|_{\ell_q}, \| B_V \|_{L_q} \geq c \| V \|_{\ell_q}; \ c = c(g, d, g_0) > 0. \)

**Proof.** It is well known that the set \( V \) can be broken into the union of no more than \( d + 1 \) disjoint subsets \( V_j \), so that no pair of vertices in the same subset is connected by an edge. Therefore, for any fixed \( \tau > 0 \) the set \( E(\tau, V) = \{ v \in V : V(v) > \tau \} \) splits into the union of no more than \( d + 1 \) disjoint subsets \( \Omega_j = E(\tau, V) \cap V_j \). For at least one of them, say \( \Omega_1 \), we have \( \# \Omega_1 \geq (d + 1)^{-1} \nu(\tau, V) \). Now, consider the subspace \( L \subset \mathcal{K}(G) \) generated by the functions \( f_{v'}(v) = \delta_{v,v'}, v' \in \Omega_1 \).

These functions are mutually orthogonal both in the metric (3.1) and with respect to the quadratic form \( b_V \) in (3.4). So, for any \( f(v) = \sum_{v' \in \Omega_1} c_{v'} \delta_{v,v'} \in L \) we have

\( a_G[f] = \sum_{v \in \Omega_1} |c_v|^2 \sum_{e \ni v} g_e \leq g_0(d + 1) \sum_{v \in \Omega_1} |c_v|^2, \)

while \( b_V[f] = \sum_{v \in \Omega_1} |c_v|^2 V(v) \geq \tau \sum_{v \in \Omega_1} |c_v|^2 \). So we have constructed the subspace \( L \), \( \dim L \geq (d + 1)^{-1} \nu(\tau, V) \), on which

\( b_V[f] \geq \tau g_0^{-1}(d + 1)^{-1} a_G[f]. \)

This implies (3.8) by the variational principle. The estimates (3.9) follow from (3.8) in a standard way. \( \square \)

4. **Metric graphs: upper estimates**

4.1. **The Laplacian and the decomposition of the space.** Each edge \( e \) of a metric graph \( \Gamma \) is considered as a line segment of the length \( l_e > 0 \). With \( \Gamma \) we associate the combinatorial graph \( G = G(\Gamma) \), with the same set of vertices \( V \), the same set of edges \( \mathcal{E} \), and the same connection relations. To any edge \( e \) of \( G(\Gamma) \) we assign the weight \( g_e = l_e^{-1} \). If \( v \in V \), then \( S(v) \) stands for its *star*: \( S(v) = \cup_{e \ni v} e \). If \( e = (v, v') \), then we define \( S(e) = S(v) \cup S(v') \).

The Lebesgue measure on the edges induces a measure on \( \Gamma \), and our basic Hilbert space is \( L^2(\Gamma) \) with respect to this measure.
On the space $H^1(\Gamma)$ of continuous functions $\varphi$ on $\Gamma$, such that $\varphi \in H^1(e)$ on each edge and $\int_{\Gamma}(|\varphi'|^2 + |\varphi|^2)dy < \infty$, we consider the quadratic form

$$a_\Gamma[\varphi] := \int_{\Gamma} |\varphi'(y)|^2 dy.$$  

(4.1)

The Laplacian $A_\Gamma$ in $L^2(\Gamma)$ is determined by this quadratic form. It acts as $-\frac{d^2}{dy^2}$ on each edge. Its domain, $\text{Dom}(A_\Gamma)$, consists of all functions $\varphi$ belonging to $H^2$ on each edge, continuous at all vertices, satisfying Kirchhoff conditions, and such that

$$\sum_{e \in \mathcal{E}} \|\varphi\|^2_{H^2(e)} < \infty.$$  

First we consider the relations between the exponents $\delta$, $D$ for the semi-groups generated by the operators $A_\Gamma$ in $L^2(\Gamma)$ and $A_G(\Gamma)$ in $\ell^2(G(\Gamma))$.

To this end, let us consider two pre-Hilbert spaces that are linear subspaces in the space $H^1_{\text{comp}}(\Gamma)$ of all compactly supported functions from $H^1(\Gamma)$. One of them, $H^1_{\text{comp,pl}}(\Gamma)$, is formed by functions linear on each edge; the subscript pl stands for ‘piecewise-linear’. Any function $\varphi \in H^1_{\text{comp,pl}}(\Gamma)$ is determined by its values $\varphi(v)$ at the vertices. Given a sequence $f = \{f(v)\}$, $v \in \mathcal{V}$ with finite support, we denote by $Jf$ the (unique) function in $H^1_{\text{comp,pl}}(\Gamma)$, such that $(Jf)(v) = f(v)$, $\forall v \in \mathcal{V}$. The mapping $J$ defines an isometry between the pre-Hilbert spaces $H^1_{\text{comp,pl}}(\Gamma)$ equipped with the metric $a_\Gamma$, and $\mathcal{H}_{\text{fin}}(G)$, equipped with the metric $a_G$ defined in (3.1). By means of this isometry we identify these pre-Hilbert spaces.

Another subspace is $H^1_{\text{comp,D}}$ consisting of all functions $\varphi \in H^1_{\text{comp}}(\Gamma)$, such that $\varphi(v) = 0$ for all $v \in \mathcal{V}$. It is clear that

$$H^1_{\text{comp}} = H^1_{\text{comp,pl}} \oplus H^1_{\text{comp,D}}$$  

(4.2)

(the orthogonal decomposition in the metric $a_\Gamma$). We will denote by $\varphi_{\text{pl}}$ and $\varphi_{\mathcal{D}}$ the components of a given element $\varphi$ with respect to this decomposition.

4.2. **Dimensions of the metric graph.** The following theorem shows that the local dimension of the Laplacian on $\Gamma$ is any number $\delta > 1$; it can be chosen arbitrarily close to 1 (that supports the intuitive understanding of the local dimension), while the dimension at infinity is the same as it is for $G$. We believe that the estimate (4.3) below is satisfied with $\delta = 1$. However, a weaker result that we prove in Theorem 4.1, 1°, is sufficient for our main conclusions on the spectral estimates.
Theorem 4.1. 1° For any $\delta > 1$,
\begin{equation}
M_{A_{\Gamma}}(t) \leq C(\delta) t^{-\frac{\delta}{4}}, \quad t \in (0, 1).
\end{equation}
2° If the lengths of the edges are uniformly bounded,
\begin{equation}
l_e \leq l_+, \quad \forall e \in E,
\end{equation}
and $M_{A_{\Gamma}}(t) = O(t^{-\frac{D}{4}})$, $D > 2$, as $t \to \infty$, then also $M_{A_{\Gamma}}(t) = O(t^{-\frac{D}{4}})$.

Proof. 1° It suffices to prove (4.3) for $\delta \in (1, 2)$. One can find $s > 0$, such that for any point $z \in \Gamma$ there exists a simple path $S(z)$ in $\Gamma$ containing $z$ and having length $s$. To show this, fix a vertex $v_0 \in V$ and set $s$ as the minimal length of edges containing $v_0$. For $z \in S(v_0)$ the statement is obvious, for any other $z \in \Gamma$, take as $S(z)$ a segment with length $s$, containing $z$, of an arbitrary simple path connecting $z$ with $v_0$. Further on we treat such a path $S(z)$ as an interval.

For a fixed $z \in \Gamma$, consider the operator $T_z$ mapping a function $\varphi$ on $\Gamma$ to its restriction to $S(z)$. The operator $T_z$ is obviously bounded as acting from $H^1(\Gamma) = \text{Dom}(A^\frac{4}{2})$ to $H^1(S(z))$ and from $L^2(\Gamma)$ to $L^2(S(z))$, with norms not greater than 1. By interpolation, $T_z$ is bounded as acting from $\text{Dom}(A^\frac{\delta}{4})$ to $H^{\frac{\delta}{2}}(S(z))$, for any $\delta \in (0, 2)$, again with norm not greater than 1. For $\delta > 1$ the space $H^{\frac{\delta}{2}}(S(z))$ is embedded in $C(S(z))$, with the same norm of the embedding operator for all $z$. We use the fact that $z$ is arbitrary to conclude that the operator $(I + A_{\Gamma})^{-\frac{\delta}{4}}$ is bounded as acting from $L^2(\Gamma)$ to $L^\infty(\Gamma)$. Next, we have
\begin{equation}
\exp(-tA_{\Gamma}) = t^{-\frac{\delta}{4}}(I + A_{\Gamma})^{-\frac{\delta}{4}} \left[(t(I + A_{\Gamma}))^\frac{\delta}{4} \exp(-tA_{\Gamma})\right].
\end{equation}

By the spectral theorem, the operator in brackets is bounded in $L^2(\Gamma)$ uniformly in $t \in (0, 1)$ and, therefore,
\begin{equation}
\| \exp(-tA_{\Gamma}) \|_{L^2(\Gamma) \to L^\infty(\Gamma)} = O(t^{-\frac{\delta}{4}}), \quad t \in (0, 1).
\end{equation}
This estimate, together with its dual, imply (4.3); see [13], section 2.1, for details and further references.

2° By Theorem II.3.1 in [17], it is sufficient to prove that the Sobolev inequality
\begin{equation}
\|\varphi\|_{L^p}^2 \leq C(p)a_{\Gamma}[\varphi], \quad p = p(D) = 2D/(D - 2),
\end{equation}
holds for any $\varphi \in H^1_{\text{comp}}(\Gamma)$. Since the decomposition (4.2) is orthogonal, it is sufficient to establish (4.5) separately for the components $\varphi_{pl}$ and $\varphi_{p}$. For the term $\varphi_{pl} = Jf$, its norm in $L^p(\Gamma)$ is majorized by the norm of $f$ in $\ell^p(G)$, so the Sobolev inequality for $\varphi_{pl}$ follows from
the corresponding inequality for $f$. For $\varphi_D$, due to (4.4), the required Sobolev inequality holds on each edge, with a common constant, and the summation gives (4.5). $\square$

So, the general results of Section 2 apply to the metric graphs. However, the analysis carried out below gives somewhat more complete and detailed picture.

4.3. Birman-Schwinger operators. As always, we suppose that the combinatorial graph $G$ satisfies (3.2). Hence, by Theorem 4.1 and (2.3), the space $H^1 = H^1(\Gamma)$, defined as the closure of $H^1_{\text{comp}}(\Gamma)$ in the metric $a_\Gamma$, is a Hilbert space of functions, embedded in $L^p(\Gamma)$.

Taking closure of both terms in the decomposition (4.2) in the same metric, we obtain the Hilbert spaces $H^1_{pl}$ and $H^1_D$, so that

$$H^1_D(\Gamma) = \bigoplus_{e \in E} H^1(\epsilon)$$

and

$$H^1(\Gamma) = H^1_{pl} \oplus H^1_D.$$ 

(4.6)

The isometry $J$ extends to the isometry of $\mathcal{H}(G)$ onto $H^1_{pl}$. The quadratic form (2.5) in our case is

$$b_V[\varphi] = \int_{\Gamma} V(y)|\varphi(y)|^2dy.$$ 

(4.7)

In general, the decomposition (4.6) does not reduce the corresponding operator $B_V$. Still, we introduce the operators $B_{V,pl}$ and $B_{V,D}$, acting in $H^1_{pl}$ and $H^1_D$ respectively and generated by the quadratic form (4.7) restricted to the corresponding subspace. The spectral estimates for $B_V$ easily reduce to the ones for these two operators. Indeed, it is clear that $B_V$ is bounded (compact) if and only if these two operators possess this property. Moreover, due to the inequality

$$b_V[\varphi] \leq 2(b_{V,pl}[\varphi_{pl}] + b_V[\varphi_D]),$$

we have (in the case of compactness)

$$\max\{n(s, B_{pl}), n(s, B_D)\} \leq n(s, B_V) \leq n(s/2, B_{pl}) + n(s/2, B_D).$$ 

(4.8)

Now we are ready to proceed to the upper spectral estimates for the operators $B_{V,D}$ and $B_{V,pl}$. They are given in Lemmas 4.2, 4.3. The resulting estimates for our original operator $B_V$ will be formulated in the next section 5. The lower estimates, showing that the result is sharp, are also derived in section 5.
The structure of the operator $B_{V,D}$ is simple:

\begin{equation}
B_{V,D} = \sum_{e \in E} B_{V,e,D}
\end{equation}

where $B_{V,e,D}$ stands for the operator in $H^{1,0}(e)$, generated by the quadratic form similar to (4.7), with the integration over the edge $e$.

Consider now the quadratic form (4.7) for $\varphi \in \mathcal{H}^{1}_{pl}(\Gamma)$. Let $f = \{f(v)\}$ be the restriction of $\varphi$ onto $V$, i.e., $f(v) = \varphi(v), \ \forall v \in V$. Then

\begin{equation}
b_{V}[\varphi_{pl}] = b_{V}[Jf] = \sum_{e \in E} \int_{e} V(y)|(Jf)(y)|^{2} dy.
\end{equation}

The corresponding operator on $H^{1}_{pl}$ is $B_{V,pl}$. Consider also the operator $\hat{B}_{V,pl}$ in $\mathcal{H}^{1}(G)$, generated by the quadratic form $\hat{b}_{V}[f] \equiv b_{V}[Jf]$. It is clear that the operators $B_{V,pl}$ and $\hat{B}_{V,pl}$ are unitarily equivalent.

4.4. Operator $B_{V,D}$. The orthogonal decomposition (4.9) reduces the study of the spectrum of $B_{V,D}$ to the same problem for a family of finite intervals, and thus makes the task elementary. In what follows we always assume that the condition (4.4) is satisfied.

We associate with $V$ the sequence

\begin{equation}
\eta_{V} = \{\eta_{V}(e)\}, \quad \eta_{V}(e) = l_{e} \int_{e} V dy, \ e \in E.
\end{equation}

It is well known (see, e.g., the estimate (4.8) and Theorem 4.6 in [3], where one has to take $l = m = 1$), that

\begin{equation}
n(\lambda, B_{V,e,D}) \leq C\lambda^{-1/2}\sqrt{\eta_{V}(e)}, \quad \forall \lambda > 0.
\end{equation}

and

\begin{equation}
\lambda^{1/2}n(\lambda, B_{V,e,D}) \rightarrow \frac{1}{\pi} \int_{e} \sqrt{V} dx, \quad \lambda \rightarrow 0.
\end{equation}

Let $\nu(s, \eta_{V}) = \#\{e : \eta_{V}(e) > s\}, \ s > 0$, be the distribution function for the sequence (4.11). We say that $\eta_{V} \rightarrow 0$ if $\nu(s, \eta_{V}) < \infty$ for any $s > 0$.

The next statement follows from (4.9), due to (4.12), (4.13).

**Lemma 4.2.** 1° If $\eta_{V} \in \ell^{\infty}$ then the operator $B_{V,D}$ is bounded and

\begin{equation}
\|B_{V,D}\| \leq C\|\eta_{V}\|_{\ell^{\infty}}, \quad C > 0.
\end{equation}

If $\eta_{V} \rightarrow 0$ then the operator $B_{V,D}$ is compact.

2° For any $q \in (\frac{1}{2}, \infty)$,

\begin{equation}
\|B_{V,D}\|_{\ell^{q}} \leq C(q)\|\eta_{V}\|_{\ell^{q}}, \quad \|B_{V,D}\|_{\ell^{q}} \leq C(q)\|\eta_{V}\|_{\ell^{q}}.
\end{equation}
3° Let $\eta_V \in \ell^{1/2}$. Then
$$\|B_{V,D}\|_{1/2} \leq C \sum_{e \in \ell} \sqrt{\eta_V(e)}$$
and
$$\lambda^{1/2} n(\lambda, B_{V,D}) \to \frac{1}{\pi} \int_{\Gamma} \sqrt{V} dy, \quad \lambda \to 0.$$  

**Proof.** The reasoning is rather standard, see, e.g., [12], and we prove only the statement 2° for the classes $\Sigma_q$. If $\eta_V \in \ell_q^w$, then, after an appropriate enumeration of edges, $e_j$, we have:
$$\eta_V(e_j) \leq M j^{-1/q}.$$  
Hence, by (4.12),
$$n(\lambda, B_{V,e_j,D}) \leq CM^{1/2} \lambda^{-1/2} j^{-1/2q}.$$  
In particular, $n(\lambda, B_{V,e_j,D}) = 0$ if $j > C^2qM^q\lambda^{-q}$. Therefore,
$$n(\lambda, B_{V,D}) = \sum_e n(\lambda, B_{V,e,D}) \leq CM^{1/2} \lambda^{-1/2} \sum_{j \leq C^2qM^q\lambda^{-q}} j^{-1/2q}$$
and, since $2q > 1$,
$$n(\lambda, B_{V,D}) \leq C'M^q\lambda^{-q},$$
whence the result. □

4.5. **Operator $B_{V,pl}$.** We compare our operator $B_{V,pl}$ (or, equivalently, $\hat{B}_{V,pl}$) with the operator $B_{\kappa V}$, where the discrete potential $\kappa V = \{\kappa_V(v)\}$ is chosen in a special way:

$$\kappa_V(v) = \int_{S(v)} V dy = \sum_{e \ni v} l_e^{-1} \eta_V(e), \quad \forall v \in V.$$  

Let us return to the quadratic form in (4.10). Choose an edge $e = (v, v')$. Identifying $e$ with the interval $(0, l_e)$, we have, for $f \in \mathcal{H}(G)$,
$$\int_e V(y)(Jf)(y)^2 dy = l_e^{-2} \int_0^{l_e} V(y)|f(v)(l_e - y) + f(v')y|^2 dy \leq \max\{|f(v)|^2, |f(v')|^2\} \int_e V(y) dy.$$  
Summing up the integrals over all $e \in \ell$, we see that
$$\hat{b}_V[f] \leq b_{\kappa_V}[f].$$  
This leads us to the following result.
Lemma 4.3. Suppose the operator $B_{\kappa V}$ on the combinatorial graph $G(\Gamma)$ is bounded, compact, or lies in one of the classes $\mathcal{E}_q$, $\Sigma_q$, $q > 0$. Then the same is true for the operator $B_{V,pl}$, and the estimate

$$\|B_{V,pl}\| \leq \|B_{\kappa V}\|$$

holds in the (quasi)-norm of the corresponding class.

5. Metric graphs: estimates in Schatten classes and asymptotics

5.1. Lower bounds. We start by a simple graph-theoretic lemma.

Lemma 5.1. Let condition (3.7) be satisfied for $G(\Gamma)$. Then the set $\mathcal{E}$ can be split into no more than $2d^2 + 1$ subsets $\mathcal{E}_j$, such that $S(e) \cap S(e') = \emptyset$ for any $e \neq e'$ in the same $\mathcal{E}_j$.

Proof. Order the edges in $\mathcal{E}$ in an arbitrary way. We must color the edges in $2d^2 + 1$ colors so that the stars of the edges of the same color are disjoint. Suppose that we have already colored all edges $e_k$, $k < n$. The star of $e_n$ can have common edges with no more than $2d^2$ stars of the previously colored edges. So we apply the unused color to $e_n$. □

Now we are in a position to give some lower estimates for our original operator $B_V$. Similarly to Theorem 3.2, they require additional conditions: namely, we assume that the edge lengths $l_e$ are bounded and separated from zero:

$$(5.1) \quad 0 < l_- \leq l_e \leq l_+, \quad \forall e \in \mathcal{E},$$

and also that (3.7) is satisfied. Note that the right inequality in (5.1) coincides with (4.4), and that the left inequality implies the condition (3.6) for the combinatorial graph $G(\Gamma)$. We also note that for the graphs satisfying (5.1) the (quasi-)norms of the sequences $\eta_V, \kappa_V$ in any space $\ell^q, \ell^q_w$ are equivalent to each other.

Lemma 5.2. Suppose the conditions (3.7) and (5.1) are satisfied. Then

1° If the operator $B_V$ is bounded, then $\eta_V \in \ell_\infty$ and

$$\|B_V\| \geq c\|\eta_V\|_\infty.$$  

2° If $B_V$ is compact, then the sequence $\eta_V$ tends to zero. Moreover, there exist constants $c', c''$ depending on $d$ and on $l_+/l_-$, such that

$$(5.2) \quad n(s, B_V) \geq c'\nu(c''s, \eta_V), \quad \forall s > 0.$$  

Proof. The reasoning follows the scheme repeatedly used in the literature, see, e.g., [5, 12, 15]. Let $e = (v, v')$ be an edge. Take a function $\varphi_e \in H^1_{pl}(\Gamma)$, such that $\varphi_e(y) = 1$ for $y \in e$ and $\varphi_e(y) = 0$ for $y \not\in e$. Then $\varphi_e \in \mathcal{E}$ and

$$\|\varphi_e\|_{\ell^q_w} \leq c_{\varphi_e} \|\varphi_e\|_{\ell^q}.$$
outside the set $S(e) = S(v) \cup S(v')$. Such function is unique, and
\[ \int_{\Gamma} |\varphi'_e|^2 dy \leq 2(d-1)l^{-1}. \] Moreover, \[ \int_{\Gamma} V|\varphi_e|^2 dy \geq l^+_1 \eta_V(e), \] and hence,
\[ \int_{\Gamma} V|\varphi_e|^2 dy \geq (2d - 2)^{-1} l^{-1}_+ \eta_V(e). \]

The statement 1° immediately follows.

Further, if for two edges $e_1, e_2$ the sets $S(e_1), S(e_2)$ are disjoint, then
the corresponding functions $\varphi_{e_1}, \varphi_{e_2}$ are orthogonal both in $H^1(\Gamma)$ and
in the space $L^2$ with the weight $V$. Let us say that a subset $F \subset E$ is
nice, if the sets $S(e), e \in F$, are mutually disjoint. By restricting the
quadratic form $b_V$ onto the linear span of the functions $\varphi_e, e \in F$, we
obtain an operator whose eigenvalues, up to the ordering, are exactly
the numbers in the left-hand side of (5.3). Therefore, the numbers
$\eta_V(e)$, listed in the decreasing order, do not exceed the eigenvalues
$\lambda_n(B_V)$ (or even, $\lambda_n(B_{V,pl})$), multiplied by $2(d-1)l_l^{-1}$. By Lemma 5.1
the set $E$ can be split into no more than $2d^2 + 1$ nice subsets, and this
leads to the statement 2°.

Note that a similar lower estimate for the operator $B_{V,D}$ (in place of
$B_V$) does not hold.

5.2. The operator $B_V$. Now we easily obtain the following result
for our original operator $B_V$ in the space $H^1(\Gamma)$, generated by the
quadratic form (4.7). We compare it with the ‘discrete’ operator $B_{\mathbf{x}_V}$
in the space $H(G), G = G(\Gamma)$, where the discrete potential $\mathbf{x}_V$ is
defined by (4.16).

Theorem 5.3. Let $\Gamma$ be a metric graph satisfying the conditions (3.7)
and (5.1). Suppose also that $D > 2$. Then the operator $B_V$ is bounded
(resp., compact, lies in one of the classes $\mathfrak{S}_q, \mathfrak{S}_{q,w}$ with $q > 1/2$) if
and only if the following two conditions are satisfied.

1° The operator $B_{\mathbf{x}_V}$ belongs to the corresponding class;
2° The sequence $\mathbf{x}_V$ belongs to $\ell^\infty$, resp., its subspace of sequences
tending to zero, $\ell^q$, or $\ell^q_w$.

If $q < D/2$, the condition 1° follows from 2° and thus, can be removed.

Note that the last statement follows from the upper estimates for
the combinatorial graphs, see Theorem 3.1, 2°.

Remark 5.4. In particular, $B_V \in \mathfrak{S}_q$ if, and under assumptions (5.1)
and (3.7), only if $\int_{\Gamma} V dy < \infty$.

Proof. Part ‘if’ follows from Lemmas 4.2 and 4.3, due to the second
inequality in (4.8).
Part 'only if': if $B_V$ possesses one of the properties mentioned in the assumption, then the same is true for the operator $B_K$ due to the first inequality in (4.8). The condition $2^*$ is fulfilled by Lemma 5.2.

This result shows that the spectral properties of the operator $B_V$ are basically determined by the ones for its discrete analogue. It is worth noting that the result for $q > D/2$ should be considered as 'conditional': indeed, our results for general combinatorial graphs (Theorem 3.1) concern only the case $q \leq D/2$. For more advanced results, one needs an additional information about the structure of $G$. For the important case $G = \mathbb{Z}^d$, $d \geq 3$, such results were obtained in [15].

In particular, a construction of 'sparse potentials' was suggested there, that allows one to construct a discrete potential producing an operator $B_V$ with an arbitrary prescribed asymptotic behavior of the spectrum. In this connection, we note that this construction extends to arbitrary combinatorial graphs with $D > 2$. This material will be presented elsewhere.

Theorem 5.3 does not include the borderline case $q = 1/2$. For this case, a simple sufficient condition for $B_V \in \mathcal{S}_{1/2}$, can be given.

**Theorem 5.5.** Let $\eta_V \in \ell^{1/2}$. Then

\begin{equation}
\|B_V\|_{\Sigma_{1/2}} \leq C \|\eta_V\|_{\ell^{1/2}},
\end{equation}

or, equivalently,

\[ N_-(A - \alpha V) \leq C \alpha^{1/2} \sum_{e \in E} \eta_V(e)^{1/2}. \]

The Weyl-type asymptotic formula

\begin{equation}
N_-(A - \alpha V) \sim \frac{\alpha^{1/2}}{\pi} \int_{\Gamma} \sqrt{V} \, dy
\end{equation}

is valid.

**Proof.** By Lemma 4.2, the estimate (5.4) and the asymptotics (5.5) hold for the operator $B_V;D$. Since $K_V$ lies in $\ell^{1/2}$ together with $\eta_V$, the operator $B_V;pl$ belongs to $\mathcal{S}_{1/2}$, and thus to $\Sigma_{1/2}^{(0)}$, with the corresponding estimate for its quasi-norm. Hence, it does not contribute to the asymptotics of order $1/2$. Both statements of Theorem follow from this fact.

A necessary and sufficient condition for the validity of (5.4) and (5.5) can be obtained by analogy with [11]. We do not present it here.
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