One-shot rates for entanglement manipulation under non-entangling maps

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We obtain expressions for the optimal rates of one-shot entanglement manipulation under operations which generate a negligible amount of entanglement. As the optimal rates for entanglement distillation and dilution in this paradigm, we obtain the max- and min-relative entropies of entanglement, and smoothed versions thereof. This gives a new operational meaning to these entanglement measures. Moreover, by considering the limit of many identical copies of the shared entangled state, we recover the recently found reversibility of entanglement manipulation under the class of operations which asymptotically do not generate entanglement.

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I. INTRODUCTION

In the distant laboratory paradigm of quantum information theory, a system shared by two or more parties might have correlations that cannot be described by classical shared randomness; we say a state is entangled if it contains such intrinsically quantum correlations and hence cannot be created by local operations and classical communication (LOCC). Quantum teleportation [1] shows that entanglement can actually be seen as a resource under the constraint that only LOCC operations are accessible. Indeed, one can use entanglement and LOCC to implement any operation allowed by quantum theory [1]. The development of entanglement theory is thus centered in understanding, in a quantitative manner, the interconversion of one entangled state into another by LOCC, and their use for various information-theoretical tasks [2–3].

In [4], Bennett et al proved that entanglement manipulations of bipartite pure states, in the asymptotic limit of an arbitrarily large number of copies of the state, are reversible. Given two bipartite pure states $|\psi_{AB}\rangle$ and $|\phi_{AB}\rangle$, the former can be converted into the latter by LOCC if, and only if, $E(|\psi_{AB}\rangle) \geq E(|\phi_{AB}\rangle)$, where $E$ is the von Neumann entropy of either of the two reduced density matrices of the state. For mixed bipartite states, it turns out that the situation is rather more complex. For instance, there are examples of mixed bipartite states, known as bound entangled states [5], which require a non-zero rate of pure state entanglement for their creation by LOCC in the limit of many copies, but from which no pure state entanglement can be extracted [3, 4, 5].

This inherent irreversibility in the asymptotic manipulation of entanglement led to the exploration of different scenarios for the study of entanglement, departing from the original one based on LOCC operations (see e.g. [8, 9, 10, 11, 12, 13]). The main motivation in these studies was to develop a simplified theory of entanglement manipulation, with the hope that it would also lead to new insights into the physically motivated setting of LOCC manipulations.

Recently one possible such scenario has been identified. In Refs. [14, 15, 16] the manipulation of entanglement under any operation which generates a negligible amount of entanglement, in the limit of many copies, was put forward. Remarkably, it was found that one recovers for multipartite mixed states the reversibility encountered for bipartite pure states under LOCC. In such a setting, only one measure is meaningful: the regularized relative entropy of entanglement [17, 18]; it completely specifies when a multipartite state can be converted into another by the accessible operations. This framework has also found interesting applications to the LOCC paradigm, such as a proof that the LOCC entanglement cost is strictly positive for every multipartite entangled state [19] (see [20] for a different proof), and new insights into separability criteria [21].

In this paper we analyze entanglement conversion of general multipartite states under non-entangling and approximately non-entangling operations in the single copy regime (see e.g. [22, 23, 24, 25, 26] for other studies of the single copy regime in classical and quantum information theory). We will identify the single copy cost and distillation functions under non-entangling maps with the two logarithmic robustnesses of entanglement [27, 28, 29] (one of them also referred to as the max-relative entropy of entanglement [30]), and the min-relative entropy of entanglement [30], respectively. On one hand, our findings give operational interpretation to these entanglement measures. On the other hand, they give further insight into the reversibility attained in the asymptotic regime. In fact, we will be able to recover the reversibility in the asymptotic limit by taking the appropriate limit in our finite copy formulae and using a certain extension of quantum Stein’s lemma proved in Ref. [19] (which is also the main technical tool used in [14, 15, 16]).

The paper is organized as follows. In Section II we...
introduce the necessary notation and definitions. Section III contains our main results, stated as Theorems IV, V, VI and VII. These theorems are proved in Sections IV, V, VI and VII respectively.

II. NOTATION AND DEFINITIONS

Let \( \mathcal{B}(\mathcal{H}) \) denote the algebra of linear operators acting on a finite-dimensional Hilbert space \( \mathcal{H} \), and let \( \mathcal{B}^+(\mathcal{H}) \subset \mathcal{B}(\mathcal{H}) \) denote the set of positive operators acting in \( \mathcal{H} \). Let \( \mathcal{D}(\mathcal{H}) \subset \mathcal{B}^+(\mathcal{H}) \) denote the set of states (positive operators of unit trace).

Given a multipartite Hilbert space \( \mathcal{H} = \mathcal{H}_1 \otimes \ldots \otimes \mathcal{H}_m \), we say a state \( \sigma \in \mathcal{D}(\mathcal{H}_1 \otimes \ldots \otimes \mathcal{H}_m) \) is separable if there are local states \( \sigma_j^k \in \mathcal{D}(\mathcal{H}_k) \) and a probability distribution \( \{p_j\} \) such that

\[
\sigma = \sum_j p_j \sigma_j^1 \otimes \ldots \otimes \sigma_j^m. \tag{1}
\]

We denote the set of separable states by \( \mathcal{S} \).

For given orthonormal bases \( \{\ket{i_A}^d\}_{i=1}^d \) and \( \{\ket{i_B}^d\}_{i=1}^d \) in isomorphic Hilbert spaces \( \mathcal{H}_A \) and \( \mathcal{H}_B \) of dimension \( d \), a maximally entangled state (MES) of rank \( d \) is given by

\[
\ket{\Psi_M^{AB}} = \frac{1}{\sqrt{d}} \sum_{i=1}^d \ket{i_A}^d \ket{i_B}^d.
\]

We define the fidelity of two quantum states \( \rho, \sigma \) as

\[
F(\rho, \sigma) = \left( \text{Tr} \sqrt{\sqrt{\sigma} \rho \sqrt{\sigma}} \right)^2. \tag{2}
\]

Finally, we denote the support of an operator \( X \) by \( \text{supp} (X) \). Throughout this paper we restrict our considerations to finite-dimensional Hilbert spaces, and we take the logarithm to base 2.

In [33], two generalized relative entropy quantities, referred to as the min- and max-relative entropies, were introduced. These are defined as follows.

**Definition 1** Let \( \rho \in \mathcal{D}(\mathcal{H}) \) and \( \sigma \in \mathcal{B}^+(\mathcal{H}) \) be such that \( \text{supp}(\rho) \subseteq \text{supp}(\sigma) \). Their max-relative entropy is given by

\[
D_{\text{max}}(\rho\|\sigma) := \log \min \{ \lambda : \rho \leq \lambda \sigma \}, \tag{3}
\]

while their min-relative entropy is given by

\[
D_{\text{min}}(\rho\|\sigma) := -\log \text{Tr}(\Pi_{\rho} \sigma), \tag{4}
\]

where \( \Pi_{\rho} \) denotes the projector onto \( \text{supp}(\rho) \).

As noted in [23, 33], \( D_{\text{min}}(\rho\|\sigma) \) is the relative Rényi entropy of order 0.

In [30], two entanglement measures were defined in terms of the above quantities.

**Definition 2** The max-relative entropy of entanglement of \( \rho \in \mathcal{D}(\mathcal{H}) \) is given by

\[
E_{\text{max}}(\rho) := \min_{\sigma \in \mathcal{S}} D_{\text{max}}(\rho\|\sigma), \tag{5}
\]

and its min-relative entropy of entanglement by

\[
E_{\text{min}}(\rho) := \min_{\sigma \in \mathcal{S}} D_{\text{min}}(\rho\|\sigma), \tag{6}
\]

Finally, we denote the support of an operator \( B \) on a finite-dimensional Hilbert space \( H \) in isomorphic Hilbert spaces, and we consider as follows (see also [19, 34]).

**Definition 3** The logarithmic robustness of entanglement of \( \rho \in \mathcal{D}(\mathcal{H}) \) is given by

\[
LR(\rho) := \log (1 + R(\rho)). \tag{9}
\]

We also define smoothed versions of the quantities we consider as follows (see also [19, 34]).

**Definition 4** For any \( \varepsilon > 0 \), the smooth max-relative entropy of entanglement of \( \rho \in \mathcal{D}(\mathcal{H}) \) is given by

\[
E^\varepsilon_{\text{max}}(\rho) := \min_{\bar{\rho} \in B^+(\rho)} E_{\text{max}}(\bar{\rho}), \tag{10}
\]

where \( B^+(\rho) := \{ \bar{\rho} \in \mathcal{D}(\mathcal{H}) : F(\bar{\rho}, \rho) \geq 1 - \varepsilon \} \).

The smooth logarithmic robustness of entanglement of \( \rho \in \mathcal{D}(\mathcal{H}) \) in turn is given by

\[
LR^\varepsilon(\rho) := \min_{\bar{\rho} \in B^+(\rho)} LR(\bar{\rho}). \tag{11}
\]

Finally, the smooth min-relative entropy of entanglement of \( \rho \in \mathcal{D}(\mathcal{H}) \) is defined as

\[
E^\varepsilon_{\text{min}}(\rho) := \max_{\bar{\rho} \in B^+(\rho)} \min_{\sigma \in \mathcal{S}} \left( -\log \text{Tr}(A \sigma) \right). \tag{12}
\]

We note that the definition of \( E^\varepsilon_{\text{min}}(\rho) \) which we use in this paper is different from the one introduced in [30], where the smoothing was performed over an \( \varepsilon \)-ball around the state \( \rho \), in analogy with the smooth version of \( E^\varepsilon_{\text{max}}(\rho) \) given above. The justification of this new definition arises from the fact that the expression for the min-relative entropy of entanglement of a state \( \rho \) involves
the projection onto its support, but the support of an operator in an $\varepsilon$-ball around $\rho$ may be very different from that of the support of $\rho$ itself. So, in defining $E^\varepsilon_{\min}(\rho)$, instead of replacing $\Pi$ by the projection onto the support of an operator in the $\varepsilon$-ball around $\rho$, it is meaningful to replace $\Pi$ itself by an operator $0 \leq A \leq I$, which almost projects onto the support of $\rho$.

Note also that while this new smoothing is a priori inequivalent to the one in [32], it is equivalent to the "operator-smoothing" introduced in [24], which, in addition, gives rise to a continuous family of smoothed relative Rényi entropies.

We will consider regularized versions of the min- and max-relative entropies of entanglement

\[ E^\varepsilon_{\min}(\rho) := \lim \inf_{n \to \infty} \frac{1}{n} E^\varepsilon_{\min}(\rho^\otimes n), \]
\[ E^\varepsilon_{\max}(\rho) := \lim \sup_{n \to \infty} \frac{1}{n} E^\varepsilon_{\max}(\rho^\otimes n), \]

and the quantities

\[ E_{\min}(\rho) := \lim_{\varepsilon \to 0} E^\varepsilon_{\min}(\rho) \]
\[ E_{\max}(\rho) := \lim_{\varepsilon \to 0} E^\varepsilon_{\max}(\rho) \]

(13)

In [13, 30] it was proved that $E_{\max}(\rho)$ is equal to the regularized relative entropy of entanglement [17, 18]

\[ E^\infty_R(\rho) := \frac{1}{n} E_R(\rho^\otimes n). \]

(15)

In this paper we prove that also $E_{\min}(\rho)$ is equal to $E^\infty_R(\rho)$.

We can now be more precise about the classes of maps we consider for the manipulation of entanglement, introduced in [14, 15].

Definition 5 A completely positive trace-preserving (CPTP) map $\Lambda$ is said to be a non-entangling (or separability preserving) map if $\Lambda(\sigma)$ is separable for any separable state $\sigma$. We denote the class of such maps by SEPP [27].

Definition 6 For any given $\delta > 0$ we say a map $\Lambda$ is a $\delta$-non-entangling map if $R_G(\Lambda(\sigma)) \leq \delta$ for every separable state $\sigma$. We denote the class of such maps by $\delta$-SEPP.

In the following sections we will consider entanglement manipulations under non-entangling and $\delta$-non-entangling maps. We first give the definitions of achievable and optimal rates of entanglement manipulation protocols under a general class of maps, in order to make the subsequent discussion more transparent. In the definitions we will consider maps from a multipartite state to a maximally entangled state and vice-versa. It should be understood that the two first parties share the maximally entangled state, while the quantum state of the other parties is trivial (one-dimensional).

Definition 7 A real number $R$ is said to be an $\varepsilon$-achievable one-shot dilution rate for a state $\rho$, under a class of quantum operations $\Theta$, if for some $\Lambda \in \Theta$, $F(\rho, \Lambda(\psi_M)) \geq 1 - \varepsilon$ and $\log M \leq R$. The corresponding one-shot entanglement cost is defined as

\[ E^1_{C,\Theta}(\rho) = \inf R, \]

(16)

where the infimum is taken over all $\varepsilon$-achievable rates.

We also consider a catalytic version of entanglement dilution under $\delta$-non-entangling maps.

Definition 8 A real number $R$ is said to be an $\varepsilon$-achievable one-shot catalytic dilution rate for a state $\rho$, under a class of quantum operations $\Theta$, if there exists a positive integer $K$ and a map $\Lambda \in \Theta$ such that $\Lambda(\psi_M \otimes \psi_K) = \hat{\rho} \otimes \psi_K$, with $F(\rho, \rho') \geq 1 - \varepsilon$ and $\log M \leq R$. The corresponding single shot entanglement cost is given by

\[ E^1_{C,\Theta}(\rho) = \inf R, \]

(17)

where the infimum is taken over all $\varepsilon$-achievable rates.

Finally, the next definition formalizes the notion of single-shot entanglement distillation under general classes of maps.

Definition 9 A real number $R$ is said to be an $\varepsilon$-achievable one-shot distillation rate for a state $\rho$, under a class of quantum operations $\Theta$, if for some $\Lambda \in \Theta$, $F(\psi_M, \Lambda(\rho)) \geq 1 - \varepsilon$ and $\log M \geq R$. The corresponding one-shot distillable entanglement is given by

\[ E^1_{D,\Theta}(\rho) = \sup R. \]

(18)

where the supremum is taken over all achievable rates.

In the following we shall consider $\Theta$ to be either the class of SEPP maps or the class of $\delta$-SEPP maps for a given $\delta > 0$.

III. MAIN RESULTS

The main results of our paper are given by the following four theorems. They provide operational interpretations of the smooth max- and min-relative entropies of entanglement, and the logarithmic version of the robustness of entanglement, in terms of optimal rates of one-shot entanglement manipulation protocols.

The first theorem relates the smoothed min-relative entropy of entanglement to the single-shot distillable entanglement under non-entangling maps.

\[ |E^\varepsilon_{\min}(\rho)| \leq E^1_{D,\delta\text{-SEPP}}(\rho) \leq E^\varepsilon_{\min}(\rho). \]

(19)
The following theorem relates the smoothed logarithmic robustness of entanglement to the one-shot entanglement cost under non-entangling maps.

**Theorem 2** For any state \( \rho \) and any \( \varepsilon \geq 0 \),
\[
LR^\varepsilon(\rho) \leq E_{C,\text{SEPP}}^{(1),\varepsilon}(\rho) \leq [LR^\varepsilon(\rho)].
\] (20)

We also prove an analogous theorem to the previous one, but now relating the logarithmic global robustness (alias max-relative entropy of entanglement) to the single-shot catalytic entanglement cost under \( \delta \)-non-entangling maps.

**Theorem 3** For any state \( \rho \) and any \( \delta > 0 \), there exists a positive integer \( K \) such that
\[
E^{\varepsilon}(\rho \otimes \Psi_K) - \log K - \log(1 + \delta) \leq E_{C,\delta-\text{SEPP}}^{(1),\varepsilon}(\rho) \\
\leq E_{\text{max}}^{\varepsilon}(\rho \otimes \Psi_K) - \log K.
\] (21)

Finally we show that we can partially recover the reversibility of entanglement manipulations under asymptotically non-entangling maps \([14, 27]\) from the results derived in this paper and the quantum hypothesis testing result of \([19]\).

**Theorem 4** For every state \( \rho \in \mathcal{D}(\mathcal{H}) \),
\[
E_{\text{min}}^{\varepsilon}(\rho) = E_{\text{max}}^{\varepsilon}(\rho) = E^{\varepsilon}(\rho).
\] (22)

From Theorems 1 and 3 we then find that the distillable entropy and the catalytic entanglement cost under asymptotically non-entangling maps are the same. In Refs. \([14, 27]\) one could show the same result without the need of catalysis. Here we need the extra resource of catalytic maximally entangled states because we want to ensure that already on a single-copy level, our operations only generate a negligible amount of entanglement; in Refs. \([14, 27]\), in turn, this is only the case for a large number of copies of the state.

We note that it was already proven in Refs. \([19, 30]\) that \( E_{\text{max}}^{\varepsilon}(\rho) = E_{R}^{\varepsilon}(\rho) \). Our contribution is to show that also the regularization of the smooth min-relative entropy of entanglement is equal to the regularized relative entropy of entanglement.

**IV. PROOF OF THEOREM 1**

The proof of Theorem 1 will employ the following lemma.

**Lemma 1** For any \( \Lambda \in \text{SEPP} \),
\[
E_{\text{min}}^{\varepsilon}(\rho) \geq E_{\text{min}}^{\varepsilon}(\Lambda(\rho))
\] (23)

**Proof.** Let \( 0 \leq A \leq I \) be such that \( \text{Tr}(AA(\rho)) \geq 1 - \varepsilon \) and \( E_{\text{min}}^{\varepsilon}(\Lambda(\rho)) = \min_{\sigma \in S}(\log \text{Tr}(A\sigma)) \). Setting \( \sigma_\rho \) as the optimal state in the definition of \( E_{\text{min}}^{\varepsilon}(\rho) \),
\[
E_{\text{min}}^{\varepsilon}(\rho) \geq - \log \text{Tr}(\Lambda^\dagger(A)\sigma_\rho) \\
= - \log \text{Tr}(AA(\rho)) \\
\geq \min_{\sigma \in S}(\log \text{Tr}(A\sigma)) \\
= E_{\text{min}}^{\varepsilon}(\Lambda(\rho)).
\] (24)

where \( \Lambda^\dagger \) is the adjoint map of \( \Lambda \). In the first line we used that \( 0 \leq \Lambda^\dagger(A)I \leq I \) and \( \text{Tr}(\Lambda^\dagger(A)\rho) = \text{Tr}(AA(\rho)) \geq 1 - \varepsilon \), while in the third line we use the fact that \( \Lambda(\sigma_\rho) \) is separable, since \( \Lambda \in \text{SEPP} \). \( \blacksquare \)

**Proof.** [Theorem 1] We first prove that \( E_{D,\text{SEPP}}^{(1),\varepsilon} \geq E_{\text{min}}^{\varepsilon}(\rho) \). For this it suffices to prove that any \( R \leq E_{\text{min}}^{\varepsilon}(\rho) \) is an achievable one-shot distillation rate for the state \( \rho \), under an SEPP map \( \Lambda \).

Consider the class of completely positive trace-preserving maps \( \Lambda \equiv \Lambda_\omega \) (for any operator \( 0 \leq \omega \leq I \)) whose action on a state \( \rho \) is given as follows:
\[
\Lambda(\rho) := \text{Tr}(\rho \Psi_M) + \text{Tr}((I - \rho \Psi_M)(I - \Psi_M)) = 0,
\] (25)

for any state \( \rho \in \mathcal{D}(\mathcal{H}) \). An isotropic state \( \omega \) is separable if and only if \( \text{Tr}(\omega \Psi_M) \leq 1/M \). Hence, the map \( \Lambda \) is SEPP if and only if for any separable state \( \sigma \), \( \text{Tr}(\Lambda(\sigma)\Psi_M) \leq 1/M \), or equivalently,
\[
\text{Tr}(\Lambda(\rho)) \leq 1/M.
\] (26)

We now choose \( \Lambda \) as the optimal POVM element in the definition of \( E_{\text{min}}^{\varepsilon}(\rho) \) and set \( M = 2E_{\text{min}}^{\varepsilon}(\rho) \). On one hand, as \( \text{Tr}(\rho \Psi_M) \geq 1 - \varepsilon \), we find that \( E_{\text{min}}^{\varepsilon}(\rho) \). On the other hand, by the definition of \( E_{\text{min}}^{\varepsilon}(\rho) \), we have that
\[
2^{-E_{\text{min}}^{\varepsilon}(\rho)} = \max_{\sigma \in S} \text{Tr}(A\sigma)
\] (27)

and hence \( \text{Tr}(A\sigma) \leq 1/M \) for every separable state \( \sigma \), which ensures that the map \( \Lambda \) defined by \( 25 \) is a SEPP map. Hence, \( \log M = [E_{\text{min}}^{\varepsilon}(\rho)] \) is an achievable rate and \( E_{D,\text{SEPP}}^{(1),\varepsilon} \geq [E_{\text{min}}^{\varepsilon}(\rho)] \).

We next prove the converse, namely that \( E_{D,\text{SEPP}}^{(1),\varepsilon}(\rho) \leq E_{\text{min}}^{\varepsilon}(\rho) \). Suppose \( \Lambda \) is the optimal SEPP map such that \( \text{Tr}(\Lambda(\rho)\Psi_M) \geq 1 - \varepsilon \), with \( \log M = E_{D,\text{SEPP}}^{(1),\varepsilon}(\rho) \).

By Lemma 1 we have
\[
E_{\text{min}}^{\varepsilon}(\rho) \geq E_{\text{min}}^{\varepsilon}(\Lambda(\rho)) \\
= \max_{\sigma \in S \in A \subset I \leq \varepsilon} \min_{\sigma \in S}(\log \text{Tr}(A\sigma)) \\
\geq \min_{\sigma \in S}(\log \text{Tr}(\Psi_M\sigma)) \\
\geq \log M \\
= E_{D,\text{SEPP}}^{(1),\varepsilon}(\rho),
\] (28)

where we used that \( 0 \leq \Psi_M \leq I \) and \( \text{Tr}(\Lambda(\rho)\Psi_M) \geq 1 - \varepsilon \) and that \( \text{Tr}(\Psi_M\sigma) \leq 1/M \) for every separable state \( \sigma \). \( \blacksquare \)
V. PROOF OF THEOREM \[2\]

Proof. To prove the upper bound in \[20\], consider the quantum operation \( \Lambda \) acting on a state \( \omega \) as follows:
\[
\Lambda(\omega) = \text{Tr}(\Psi_M^* \omega) \rho_e + [1 - \text{Tr}(\Psi_M^* \omega)] \pi,
\]
where \( \pi \) is the optimal state for the robustness \( R(\rho_e) \) of \( \rho_e \), which is such that \( LR^\pi(\rho) = LR(\rho_e) \).

For a separable state \( \omega \), we can rewrite eq. \[29\] as
\[
\Lambda(\omega) = q\left[\frac{\rho_e + (M - 1)\pi}{M}\right] + (1 - q)\pi,
\]
where \( q = M\text{Tr}(\Psi_M^\omega) \). For a separable state \( \omega \), \( \text{Tr}(\Psi_M^\omega) \leq 1/M \), and hence \( 0 \leq q \leq 1 \). By the convexity of the robustness we have that, for any separable state \( \omega \),
\[
R(\Lambda(\omega)) \leq qR(\sigma) + (1 - q)R(\pi),
\]
where \( \sigma := (\rho_e + (M - 1)\pi)/M \). Note that \( R(\pi) = 0 \) since \( \pi \) is separable. Furthermore, for the choice \( M = 1/R(\rho_e) \), \( \sigma \) is separable and hence \( R(\sigma) = 0 = R(\Lambda(\omega)) \), ensuring that the map \( \Lambda \) is non-entangling.

Note that \( \Lambda(\Psi_M^\omega) = \rho_e \), with the corresponding rate of \( \log M = \log(1 + R(\rho_e)) = LR^\pi(\rho) \). This then yields the upper bound in Theorem \[2\].

To prove the lower bound in \[20\], let \( \Lambda \) denote a SEPP map yielding entanglement dilution with a fidelity of at least \( 1 - \varepsilon \), for a state \( \rho_e \), i.e., \( \Lambda(\Psi_M^\omega) = \rho_e \), with \( F(\rho, \rho_e) \geq 1 - \varepsilon \), and \( \log M = E^\varepsilon_{C,G}\text{SEPP} \). The monotonicity of log robustness under SEPP maps \[15\] yields
\[
LR^\pi(\rho) \leq LR(\rho_e) = LR(\Lambda(\Psi_M^\omega)) \\
\leq LR(\Psi_M^\omega) = \log M = E^\varepsilon_{C,G}\text{SEPP}.
\]

VI. PROOF OF THEOREM \[3\]

The following lemma will be employed in the proof of Theorem \[3\].

Lemma 2 For any \( \delta > 0 \) and \( \Lambda \in \delta\text{-SEPP}, \)
\[
E^\varepsilon_{\text{max}}(\rho) \geq E^\varepsilon_{\text{max}}(\Lambda(\rho)) - \log(1 + \delta)
\]

Proof. Let \( \rho_e \) be the optimal state in the definition of \( E^\varepsilon_{\text{max}}(\rho) \), i.e., \( E^\varepsilon_{\text{max}}(\rho_e) = E^\varepsilon_{\text{max}}(\rho_e) \). By the monotonicity of the fidelity under CPTP maps we have that \( F(\Lambda(\rho), \Lambda(\rho_e)) \geq F(\rho, \rho_e) \geq 1 - \varepsilon \). Hence, using Lemma IV.1 of \[15\]
\[
E^\varepsilon_{\text{max}}(\Lambda(\rho)) \leq E^\varepsilon_{\text{max}}(\Lambda(\rho_e)) \leq E^\varepsilon_{\text{max}}(\rho_e) + \log(1 + \delta)
\]
\[
= E^\varepsilon_{\text{max}}(\rho) + \log(1 + \delta).
\]

Proof. [Theorem 3] Let us start with the achievability part, namely that for every \( \delta > 0 \) we can find a positive integer \( K \) such that \( E^\varepsilon_{C,\delta-\text{SEPP}}(\rho) \leq E^\varepsilon_{\text{max}}(\rho \otimes \Psi_K) - \log K \).

Let \( \rho_e \) be the optimal state in the definition of \( E^\varepsilon_{\text{max}}(\rho) \). Consider the family of completely positive trace-preserving maps as follows:
\[
\Lambda(\omega) = [\text{Tr}(\Psi_M \otimes \Psi_K^\omega)](\rho_e \otimes \Psi_K) \\
+ [\text{Tr}(I - \Psi_M \otimes \Psi_K^\omega)](\pi),
\]
where \( K = [1 + \delta^{-1}], M = K^{-1}2^{E^\varepsilon_{\text{max}}(\rho \otimes \Psi_K)}, \) and \( \pi \) is a state such that \( (\rho_e \otimes \Psi_K) + (MK - 1)\pi \) is an (un-normalized) separable state. Such a state \( \pi \) can always be found because \( MK \geq 1 + R(\rho_e \otimes \Psi_K) \).

We now show that with this choice of parameters the map \( \Lambda \), defined by \[34\], is \( \delta\text{-SEPP}. \) First note that since for any separable state \( \sigma \in \mathcal{B}(\mathcal{H} \otimes \mathcal{H}) \),
\[
\text{Tr}(\Psi_M \otimes \Psi_K) \sigma) \leq \frac{1}{MK},
\]
we can write
\[
\Lambda(\sigma) = p(\rho_e \otimes \Psi_K) + (1 - p)\pi,
\]
where \( p \leq \frac{1}{MK} \). This in turn can be written as
\[
\Lambda(\sigma) = q\left[\frac{(\rho_e \otimes \Psi_K) + (MK - 1)\pi}{MK}\right] + (1 - q)\pi,
\]
where \( q = pMK \). Since \( 0 \leq p \leq 1/MK \), we have that \( 0 \leq q \leq 1 \). Note that the first term in parenthesis in \[36\] is separable, due to the choice of \( p \). Using the convexity of the global robustness we then conclude that \( R_G(\Lambda(\sigma)) \leq R_G(p) \), for any separable state \( \sigma \).

Further, from the choice of \( M \) and \( K \) it follows that
\[
R_G(p) \leq \frac{1}{R_G(\rho_e \otimes \Psi_K)} \leq \frac{1}{K - 1} \leq \delta,
\]
and hence \( \Lambda \) is \( \delta\text{-SEPP} \).

Note that for \( \omega = \Psi_M \otimes \Psi_K \),
\[
\Lambda(\omega) = \Lambda(\Psi_M \otimes \Psi_K) = \rho_e \otimes \Psi_K.
\]

Hence the protocol yields a state \( \rho_e \) with \( F(\rho, \rho_e) \geq 1 - \varepsilon \) and the additional maximally entangled state \( \Psi_K \) which was employed in the start of the protocol. Its role in the protocol is to ensure that the quantum operation \( \Lambda \) is a \( \delta\text{-SEPP} \) map for any \( \delta > 0 \). Since the maximally entangled states \( \Psi_M \) and \( \Psi_K \) were employed in the protocol and \( \Psi_K \) was retrieved unchanged, the rate \( R = (\log M + \log M') - \log M = [E^\varepsilon_{\text{max}}(\rho \otimes \Psi_K)] - \log K, \) is achievable.

Next we prove the bound \( E^\varepsilon_{C,\delta-\text{SEPP}} \geq E^\varepsilon_{\text{max}}(\rho) - \log K - \log(1 + \delta) \). Let \( \Lambda \) be a \( \delta\text{-SEPP} \) map for which
\[
\Lambda(\Psi_M \otimes \Psi_K) = \rho_e \otimes \Psi_K.
\]
with $\bar{E}^{(1)}_{\rho \delta \mathrm{SEPP}} = \log M$.

Then by Lemma 2

$$E_{\max}(\rho \otimes \Psi_K) \leq E_{\max}(\rho_c \otimes \Psi_K)$$

$$= E_{\max}(\Lambda(\Psi M \otimes \Psi_K))$$

$$\leq E_{\max}(\Psi M \otimes \Psi_K) + \log(1 + \delta)$$

$$= \log M + \log K + \log(1 + \delta).$$

Hence

$$\log M \geq E_{\max}(\rho \otimes \Psi_K) - \log K - \log(1 + \delta).$$

Therefore,

$$\rho = \text{the smooth min-relative entropy of } \rho.$$
the first term does not tend to 1, since $\gamma_0 > D(\rho|\bar{\sigma})$ by assumption. Hence we obtain the bound
\begin{equation}
\text{Tr}(A_n \rho_n) < 1 - c_0,
\end{equation}
for some constant $c_0 > 0$, independent of $\varepsilon$. This clearly contradicts (48), which holds for all $n$ and any arbitrary $\varepsilon > 0$.

We also make use the following result, which appears as Proposition II.1 in [16, 19]:

**Lemma 4** [16, 18] For every $\rho \in D(\mathcal{H})$,
\begin{equation}
\lim_{n \to \infty} \min_{\sigma \in \mathcal{S}(n^{\otimes n})} \text{Tr}(\rho^{\otimes n} - 2^{\gamma n} \sigma) = \begin{cases} 0 & y \geq E^\infty_R(\rho) \\ 1 & y < E^\infty_R(\rho) \end{cases}
\end{equation}

**Proof.** (Theorem 4). That $E^\infty_R(\rho)$ was established in Ref. [30]. We hence show in this proof that $E^\infty_R(\rho)$ holds for all $\gamma$ and every arbitrary sequence of separable states $\{\sigma_n\}$.

From Lemma 3 we then find that $E^\infty_R(\rho) \leq E_{\min}(\rho)$. As $E_{\min}(\rho) \leq E_{\max}(\rho)$, we finally find that
\begin{equation}
E^\infty_R(\rho) \leq E_{\min}(\rho) \leq E_{\max}(\rho) = E^\infty_R(\rho),
\end{equation}
and so the quantities above are the same.

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[37] The acronym comes from the name separability preserving.