PROJECTIVE SURFACES WITH MANY NODES

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Abstract. We prove that a smooth projective complex surface \( X \), not necessarily minimal, contains \( h^{1,1}(X) - 1 \) disjoint \((-2)\)-curves if and only if \( X \) is isomorphic to the minimal rational ruled surface \( F_2 \) or \( \mathbb{P}^2 \) or a fake projective plane.

We also describe smooth projective complex surfaces \( X \) with \( h^{1,1}(X) - 2 \) disjoint \((-2)\)-curves.

1. Introduction

Throughout this paper, we work over the field \( \mathbb{C} \) of complex numbers.

A smooth rational curve on a surface with self-intersection \(-2\) is called a \((-2)\)-curve or a nodal curve as it may be contracted to give a nodal singularity (conical double point). For a smooth surface \( X \), we denote by \( \mu(X) \) the maximum of the cardinality of a set of disjoint \((-2)\)-curves on \( X \). Hodge index theorem implies that

\[ \mu(X) \leq \rho(X) - 1 \leq h^{1,1}(X) - 1, \]

in particular, \( X \) contains at most \( h^{1,1}(X) - 1 \) disjoint nodal curves, where \( \rho(X) \) denotes the Picard number and \( h^{1,1}(X) \) the \((1,1)\)-th Hodge number of \( X \).

A result of I. Dolgachev, M. Mendes Lopes, and R. Pardini gives a classification of smooth projective complex surfaces \( X \) with \( q(X) = p_g(X) = 0 \) containing \( \rho(X) - 1 \) disjoint nodal curves ([6], Theorem 3.3 and Proposition 4.1).

Theorem 1.1. ([6]) Let \( X \) be a smooth projective surface, not necessarily minimal, with \( q(X) = p_g(X) = 0 \). Then \( \mu(X) = h^{1,1}(X) - 1 \) if and only if \( X \) is isomorphic to the minimal rational ruled surface \( F_2 \) or the complex projective plane \( \mathbb{P}^2 \) or a fake projective plane.

Note that \( \rho(X) = h^{1,1}(X) \) for a smooth projective surface \( X \) with \( p_g(X) = 0 \). The Hirzebruch surface \( F_2 \) contains one nodal curve, while \( \mathbb{P}^2 \) or a fake projective plane contains none. The latter two cases were not mentioned in [6], as the authors focused on the case with \( \mu(X) > 0 \).

A \( \mathbb{Q} \)-homology projective plane is a normal projective surface having the same \( \mathbb{Q} \)-homology groups as \( \mathbb{P}^2 \). If a \( \mathbb{Q} \)-homology projective plane has rational singularities only, then both the surface and its resolution have \( p_g = q = 0 \). Theorem 1.1 also gives the following classification of \( \mathbb{Q} \)-homology projective planes with nodes only.

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Corollary 1.2. Let $S$ be a $\mathbb{Q}$-homology projective plane. Assume that all singularities of $S$ are nodes. Then $S$ is isomorphic to $\mathbb{P}^2$ or a fake projective plane or a cone in $\mathbb{P}^3$ over a conic curve.

In this paper we first show that the condition "$q(X) = p_g(X) = 0$" in Theorem 1.1 is not necessary.

Theorem 1.3. Let $X$ be a smooth projective surface, not necessarily minimal. Then $\mu(X) = h^{1,1}(X) - 1$ if and only if $X$ is isomorphic to $F_2$ or $\mathbb{P}^2$ or a fake projective plane.

Next, we describe smooth projective complex surfaces $X$ with $\mu(X) = h^{1,1}(X) - 2$.

Theorem 1.4. Let $X$ be a smooth projective surface, not necessarily minimal. Assume that $\mu(X) = h^{1,1}(X) - 2$. Then $X$ belongs to one of the following cases:

1. nef $K_X$:
   - (a) a bi-elliptic surface, i.e. a minimal surface of Kodaira dimension $0$ with $q = 1$, $p_g = 0$, $h^{1,1} = 2$;
   - (b) a minimal surface of Kodaira dimension $1$ with $q = 1$, $p_g = 0$, $h^{1,1} = 2$;
   - (c) an Enriques surface with $8$ disjoint nodal curves;
   - (d) a minimal surface of Kodaira dimension $1$ with $q = p_g = 0$ whose elliptic fibration has $2$ reducible fibres of type $I_0^*$ whose end components give the $8$ disjoint nodal curves;
   - (e) a ball quotient with $q = 0$, $p_g = 1$, i.e. a minimal surface of general type with $q = 0$, $p_g = 1$, $h^{1,1} = 2$;
   - (f) a minimal surface of general type with $q = p_g = 0$, $K^2 = 1, 2, 4, 6, 7, 8$ containing $8 - K^2$ disjoint nodal curves;

2. non-nef $K_X$:
   - (a) the blowup of a fake projective plane at one point or at two infinitely near points;
   - (b) a relatively minimal irrational ruled surface; or its blow-up at two infinitely near points on each of $k \geq 1$ fibres so that each of the $k$ fibres becomes a string of $3$ rational curves $(-2)\cdots(-1)(-2)$;
   - (c) a rational ruled surface $F_e$, $e \neq 2$;
   - (d) the blowup of $F_e$ at two infinitely near points on each of $k \geq 1$ fibres so that each of the $k$ fibres becomes a string of $3$ rational curves $(-2)\cdots(-1)\cdots(-2)$;
   - (e) the blowup of $F_2$ at two infinitely near points away from the negative section so that one fibre becomes a string of $3$ rational curves $(-1)\cdots(-2)\cdots(-1)$;
   - (f) the blowup of $F_2$ at one point away from the negative section; or equivalently the blowup of $F_1$ at one point on the negative section.

We remark that all cases of Theorem 1.4 are supported by an example except the case (1-f) with $K^2 = 1$, or 7.

For the case (1-b), such surfaces can be obtained by taking a quotient $(E \times C)/G$ of the product of an elliptic curve $E$ and a hyperelliptic curve $C$ of genus $g(C) \geq 2$ by a group $G$ of order 2 acting on $E$ as a translation by a point of order 2 and on $C$ as the hyperelliptic involution. (If $g(C) = 1$ we get a bi-elliptic surface.)

For (1-c), such Enriques surfaces were completely classified in [13]. See also [11] and [9] for explicit examples.
For (1-d), the Jacobian fibration of such a surface is a rational elliptic surface $Y$ with two singular fibres of type $I_0^*$. (The Jacobian fibration of an elliptic fibration has singular fibres of the same type as the original fibration (cf. [5]).) In other words, such surfaces are torsors of $Y$, i.e., can be obtained by performing logarithmic transformations on $Y$. If the orders of logarithmic transformations are $(2, 2)$, then the resulting surface is an Enriques surface belonging to the case (1-c). Such a rational elliptic surface $Y$ can be constructed in many ways, e.g., by blowing up the base points of a specific cubic pencil on $\mathbb{P}^2$ ([5]) or by taking a minimal resolution of a $\mathbb{Z}/2$-quotient of the product of an elliptic curve $E$ and $\mathbb{P}^1$ where the group acts on $E$ as the inversion and on $\mathbb{P}^1$ as an involution. It is easy to see that any such rational elliptic surface $Y$ is a special case of (2-d). (Consider a free pencil $|N_1 + 2C + N_2|$ on $Y$ where $C$ is a section meeting two simple components $N_1$ and $N_2$ of the two reducible fibres.)

For the case (1-f) with $K^2 = 2, 4, 6$, examples can be found in [2]. See Remark 4.2.

**Notation**

$\mathbb{F}_e := \text{Proj}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(e))$, $e \geq 0$, a rational ruled surface (Hirzebruch surface)

$\mathbb{P}^n$: the complex projective $n$-space

$\rho(Y)$: the Picard number of a variety $Y$

$K_Y$: the canonical class of $Y$

$b_i(Y)$: the $i$-th Betti number of $Y$

$e(Y)$: the topological Euler number of $Y$

$c_i(X)$: the $i$-th Chern class of $X$

$e_{\text{orb}}(S)$: the orbifold Euler number of a surface $S$ with quotient singularities only

$h^{i,j}(X)$: the $(i, j)$-th Hodge number of a smooth variety $X$

$q(X) := \dim H^1(X, \mathcal{O}_X)$ the irregularity of a smooth surface $X$

$p_g(X) := \dim H^2(X, \mathcal{O}_X)$ the geometric genus of a smooth surface $X$

$|G|$: the order of a finite group $G$

$(-m)$-curve: a smooth rational curve on a surface with self-intersection $-m$

$\mu(Z)$: the maximum of the cardinality of a set of disjoint $(-2)$-curves on a smooth surface $Z$

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2. **The orbifold Bogomolov-Miyaoka-Yau inequality**

A singularity $p$ of a normal surface $S$ is called a quotient singularity if the germ at $p$ is analytically isomorphic to the germ of $\mathbb{C}^2/G_p$ at the image of the origin $O \in \mathbb{C}^2$ for some nontrivial finite subgroup $G_p$ of $GL_2(\mathbb{C})$ not containing quasi-reflections. Brieskorn classified all such finite subgroups of $GL(2, \mathbb{C})$ in [4]. We call $G_p$ the local fundamental group of the singularity $p$.

Let $S$ be a normal projective surface with quotient singularities and $f : S' \to S$ be a minimal resolution of $S$. For each quotient singular point $p \in S$, there is a string of smooth rational curves $E_j$ such that

$$f^{-1}(p) = \bigcup_{j=1}^{l} E_j.$$
It is well-known that quotient singularities are log-terminal singularities. Thus one can compare canonical classes $K_{S'}$ and $K_S$, and write

$$K_{S'} \equiv \sum_{\text{num}} f^* K_S - \sum D_p,$$

where $D_p = \sum_{j=1}^{l} (a_j E_j)$ is an effective $\mathbb{Q}$-divisor supported on $f^{-1}(p) = \bigcup_{j=1}^{l} E_j$ and $0 \leq a_j < 1$. It implies that

$$K^2_S = K^2_{S'} - \sum_{p \in \text{Sing}(S)} D^2_p.$$  

The coefficients $a_1, \ldots, a_l$ of $D_p$ are uniquely determined by the system of linear equations

$$D_p \cdot E_i = -K_{S'} \cdot E_i = 2 + E^2_i \quad (1 \leq i \leq l).$$

In particular, $D_p = 0$ if and only if $p$ is a rational double point.

Also we recall the orbifold Euler characteristic

$$e_{\text{orb}}(S) := e(S) - \sum_{p \in \text{Sing}(S)} \left(1 - \frac{1}{|G_p|}\right)$$

where $e(S)$ is the topological Euler number of $S$, and $|G_p|$ the order of the local fundamental group $G_p$ of $p$.

The following theorem is called the orbifold Bogomolov-Miyaoka-Yau inequality.

**Theorem 2.1** ([15], [14], [10], [12]). Let $S$ be a normal projective surface with quotient singularities such that $K_S$ is nef. Then

$$K^2_S \leq 3e_{\text{orb}}(S).$$

In particular,

$$0 \leq e_{\text{orb}}(S).$$

The second (weaker) inequality holds true even if $-K_S$ is nef.

**Theorem 2.2** ([8]). Let $S$ be a normal projective surface with quotient singularities such that $-K_S$ is nef. Then

$$0 \leq e_{\text{orb}}(S).$$

The following corollary is well-known (e.g. [7], Corollary 3.4) and immediately follows from Theorems 2.1 and 2.2.

**Corollary 2.3.** A $\mathbb{Q}$-homology projective plane with quotient singularities only has at most 5 singular points.

### 3. Proof of Theorem 1.3

The “if”-part is trivial.

Let $X$ be a smooth projective surface, not necessarily minimal, with $h^{1,1}(X) - 1$ disjoint nodal curves. We shall show that $X$ is isomorphic to $\mathbb{F}_2$ or $\mathbb{P}^2$ or a fake projective plane. Let

$$f : X \to S$$

be the contraction morphism of the $h^{1,1}(X) - 1$ disjoint nodal curves.

Note first that $\rho(S) = 1$. Thus $K_S$ is nef or anti-ample.
Assume that $K_S$ is nef. Then we can apply the orbifold Bogomolov-Miyaoka-Yau inequality (Theorem 2.1). Note that $K_S^2 = K_X^2$. From Noether formula

$$K_X^2 = 12\{1 - q(X) + p_g(X)\} - \{2 - 4q(X) + h^{1,1}(X) + 2p_g(X)\}.$$ 

Also we have

$$c_{orb}(S) = e(S) - \frac{h^{1,1}(X) - 1}{2}.$$ 

Since $e(S) = e(X) - (h^{1,1}(X) - 1) = 3 - 4q(X) + 2p_g(X)$, Theorem 2.1 implies that

$$12(1 - q + p_g) - (2 - 4q + h^{1,1} + 2p_g) \leq 3\left(3 - 4q + 2p_g - \frac{h^{1,1} - 1}{2}\right),$$

i.e.,

$$4q + 4p_g + \frac{h^{1,1}}{2} \leq \frac{1}{2},$$

hence

$$q(X) = p_g(X) = 0, \quad h^{1,1}(X) = 1.$$ 

In particular, $b_2(X) = 1$, $e(X) = 3$, $K_X^2 = 9$. Note that $K_X = f^*K_S$ is nef, hence $X$ is not rational. Thus, by classification theory of complex surfaces (see [3]), $X$ must be a fake projective plane, i.e. a smooth surface of general type with $q = p_g = 0$, $K^2 = 9$.

Assume that $-K_S$ is ample. Then $-K_X = -f^*K_S$ is nef and non-zero, hence $X$ has Kodaira dimension $\kappa(X) = -\infty$. Suppose $X$ is not rational. Then there is a morphism $g : X \to C$ onto a curve of genus $\geq 1$, with general fibres isomorphic to $\mathbf{P}^1$. Since a curve of genus $\geq 1$ cannot be covered by a rational curve, we see that all nodal curves of $X$ are contained in a union of fibres. This implies that $S$ has Picard number $\geq 2$, a contradiction. Thus $X$ is rational. Now by Theorem 3.3 of [6], $X \cong \mathbf{F}_2$ or $\mathbf{P}^2$.

Here we give an alternative proof. Since we assume that $X$ is rational, $S$ is a $\mathbb{Q}$-homology projective plane with nodes only. Let $k$ be the number of nodes on $S$. Then $k \leq 5$ by Corollary 2.3. Note that $b_2(X) = 1 + k$, so $K_X^2 = 9 - k$. Let $L$ be the sublattice of the cohomology lattice of $X$ generated by the class of $K_X$ and the classes of the $k$ nodal curves. Then $L$ is of finite index in the cohomology lattice that is unimodular, hence $|\det(L)|$ is a square integer. Note that $|\det(L)| = (9 - k)2^k$. If $k \leq 5$, then it is a square integer only if $k = 0$ or 1. If $k = 0$, then $X \cong \mathbf{P}^2$. If $k = 1$, then $K_X^2 = 8$ and $\rho(X) = 2$, hence $X \cong \mathbf{F}_2$.

This completes the proof of Theorem 1.2.

Remark 3.1. Proposition 4.1 of [6] was also proved by using the orbifold Bogomolov-Miyaoka-Yau inequality. Our proof is just a slight refinement of their argument.

4. PROOF OF THEOREM 1.4

For a smooth surface $Z$, we denote by $\mu(Z)$ the maximum of the cardinality of a set of disjoint $(-2)$-curves of $Z$. The following useful lemma is due to M. Mendes Lopes and R. Pardini.

Lemma 4.1. Let $X$ be a smooth surface with Kodaira dimension $\kappa(X) \geq 0$. Let $\phi : X \to Y$ be the map to the minimal model, and let $r := \rho(X) - \rho(Y)$. Then

$$\mu(X) \leq \mu(Y) + \frac{r}{2}.$$
Proof. The proof is essentially contained in the proof of Proposition 4.1 of [6].

Use induction on $r$.

When $r = 0$, it is trivial.

Assume $r > 0$. Write

$$K_X = \phi^* K_Y + E$$

and let $C_1, \ldots, C_{\mu(X)}$ be disjoint $(-2)$-curves on $X$. For each $i$ there are 2 possibilities:

1. $C_i$ is exceptional for $f$, hence $(\phi^* K_Y) C_i = 0$.
2. $C_i$ is not exceptional for $\phi$. Then since $K_X C_i = 0$ and $K_Y$ is nef, we see that $(\phi^* K_Y) C_i = 0$, $EC_i = 0$, hence $C_i$ is disjoint from the support of $E$.

Let $E_1$ be an irreducible $(-1)$-curve of $X$ and let $X_1$ be the surface obtained by blowing down $E_1$. If $E_1$ does not intersect any of the $C_i$’s, then the $C_i$’s give $\mu(X)$ disjoint $(-2)$-curves on $X_1$, hence $\mu(X) \leq \mu(X_1)$ and the statement follows by induction, i.e.,

$$\mu(X) \leq \mu(X_1) \leq \mu(Y) + \frac{r - 1}{2}.$$

So assume that $E_1 C_1 > 0$. By the above remark, this implies that $C_1$ is exceptional for $\phi$. In particular, we have $C_1 E_1 = 1$. Notice that $E_1 C_i = 0$ for every $i > 1$. Indeed, if, say, $E_1 C_2 = 1$, then the images of $C_1$ and $C_2$ in $X_1$ are $(-1)$-curves that intersect, contradicting the assumption that $\kappa(X_1) \geq 0$. Hence the image of $C_1$ in $X_1$ is a $(-1)$-curve that can be contracted to get a surface $X_2$ with $\mu(X) - 1$ disjoint $(-2)$-curves, and again we get the result by induction, i.e.,

$$\mu(X) - 1 \leq \mu(X_2) \leq \mu(Y) + \frac{r - 2}{2}. \qed$$

Now we prove Theorem 1.4. Let

$$f : X \rightarrow S$$

be the contraction morphism of the disjoint nodal curves $C_1, \ldots, C_{\mu(X)}$, where $\mu(X) = h^{1,1}(X) - 2$.

Note first that $K_X = \phi^* K_S$. Thus $K_X$ is nef if and only if $K_S$ is nef.

Assume that $K_X$ is nef.

Then we again apply the orbifold Bogomolov-Miyaoka-Yau inequality (Theorem 2.1). In this case we have

$$K_S^2 = K_X^2 = 12\{1 - q(X) + p_g(X)\} - \{2 - 4q(X) + h^{1,1}(X) + 2p_g(X)\},$$

$$e_{\text{orb}}(S) = e(S) - h^{1,1}(X) - \frac{2}{2} = 4 - 4q(X) + 2p_g(X) - \frac{h^{1,1}(X) - 2}{2}. \text{Thus Theorem 2.1 implies that}$$

$$12(1 - q + p_g) - (2 - 4q + h^{1,1} + 2p_g) \leq 3\left(4 - 4q + 2p_g - \frac{h^{1,1} - 2}{2}\right),$$

and hence,

$$4q + 4p_g + \frac{h^{1,1}}{2} \leq 5.$$ 

This inequality has the following solutions:

- (i) $q(X) = 1, p_g(X) = 0, h^{1,1}(X) = 2$;
- (ii) $q(X) = 0, p_g(X) = 1, h^{1,1}(X) = 2$;
• (iii) \( g(X) = p_g(X) = 0, 2 \leq h^{1,1}(X) \leq 10. \)

Assume the case (i). In this case, \( e(X) = 0 \) and \( K_X^2 = 0. \) Since \( X \) is a minimal surface of Kodaira dimension \( \kappa(X) \geq 0, \) the classification theory of complex surfaces (cf. [3]) shows that \( X \) belongs to the case (1-a) or (1-b).

Assume the case (ii). In this case, \( e(X) = 6 \) and \( K_X^2 = 18. \) Hence, \( X \) is of general type. Since \( 3e(X) = K_X^2, \) it is a ball quotient. This gives the case (1-e).

Assume the case (iii). In this case, \( e(X) = h^{1,1}(X) + 2 \) and \( K_X^2 = 10 - h^{1,1}(X). \)

If \( h^{1,1}(X) = 10, \) then \( e(X) = 12 \) and \( K_X^2 = 0. \) Since \( X \) is a minimal surface of Kodaira dimension \( \kappa(X) \geq 0, X \) is an Enriques surface or a minimal surface of Kodaira dimension \( \kappa(X) = 1. \) This gives the case (1-c) and (1-d). In the latter case, a fibre of the elliptic fibration of \( X \) is a rational multiple of \( K_X, \) hence the 8 nodal curves must be contained in fibres of the elliptic fibration. By the formula for computing the topological Euler number of a fibration (cf. [3], Chap. III) this is possible only if the reducible fibres are two fibres of type \( I_0^5 \) and the eight nodal curves are the end-components of these fibres.

If \( 2 \leq h^{1,1}(X) \leq 9, \) then \( 8 \geq K_X^2 \geq 1. \) Since \( X \) is a minimal surface of Kodaira dimension \( \kappa(X) \geq 0, X \) is of general type. By Theorem [13] any minimal surface of general type with \( p_g(X) = 0 \) and \( K_X^2 = 8 \) cannot contain a \((-2)\)-curve, hence belongs to the case (1-f).

We claim that \( K_X^2 \neq 3, 5. \) This can be proved by a lattice theoretic argument. Let \( L \) be the cohomology lattice \( H^2(X, \mathbb{Z})/(\text{torsion}), \) which is an odd unimodular lattice of signature \((1, h^{1,1}(X) - 1). \) Let \( M \) be the sublattice of \( L \) generated by the classes of the nodal curves \( C_1, \ldots, C_{\mu(X)} \) where \( \mu(X) = h^{1,1}(X) - 2. \) Consider the homomorphism of quadratic forms of finite abelian groups

\[ \tau : M/2M \to L/2L. \]

Note that \( M/2M \cong (\mathbb{Z}/2\mathbb{Z})^{\mu(X)} \) is totally isotropic, and \( L/2L \cong (\mathbb{Z}/2\mathbb{Z})^{\mu(X)+2}. \) Assume that \( K_X^2 = 3. \) Then \( \mu(X) = 5, \) so the kernel \( \ker(\tau) \) must have length \( \geq 2. \)

If \( \sum_{j=1}^{4} C_{i_j} \mod 2M \in \ker(\tau), \) then \( \sum_{j=1}^{4} C_{i_j} = 2D + \text{torsion} \) for some divisor \( D. \)

Since \( D \cdot K_X = 0, D^2 \) is an even integer. This implies that \( k \) is a multiple of 4. This means that any non-trivial element of \( \ker(\tau) \) is a sum of 4 members of \( C_1, \ldots, C_5. \)

But this is impossible since \( \ker(\tau) \) has length \( \geq 2. \) Assume that \( K_X^2 = 5. \) Then \( \mu(X) = 3, \) so the kernel \( \ker(\tau) \) must have length \( \geq 1. \) But no linear combination of \( C_1, C_2, C_3 \) gives a non-trivial element of \( \ker(\tau). \) This gives the case (1-f).

Assume that \( K_X \) is not nef and \( \kappa(X) \geq 0. \)

In this case \( X \) is not minimal. Consider the map \( \phi : X \to Y \) to the minimal model, and let

\[ r := \rho(X) - \rho(Y). \]

By Lemma 4.1,

\[ h^{1,1}(Y) + r - 2 = h^{1,1}(X) - 2 = \mu(X) \leq \mu(Y) + \frac{r}{2} \leq h^{1,1}(Y) - 1 + \frac{r}{2}, \]

hence

\[ r \leq 2. \]

If \( r = 1, \) then the above inequality shows that

\[ h^{1,1}(Y) - 1 = \mu(X) \leq \mu(Y) + \frac{1}{2}. \]
hence $\mu(Y) = h^{1,1}(Y) - 1$. So by Theorem 1.3 $Y$ is a fake projective plane and
$\mu(X) = h^{1,1}(Y) - 1 = 0$.
If $r = 2$, then the above inequality shows that

$$h^{1,1}(Y) = \mu(X) \leq \mu(Y) + 1 \leq h^{1,1}(Y),$$

hence $\mu(Y) = h^{1,1}(Y) - 1$. So by Theorem 1.3 $Y$ is a fake projective plane and
$\mu(X) = h^{1,1}(Y) = 1$. This gives the case (2-a).

Assume that $\kappa(X) = -\infty$ and $X$ is irrational.
In this case $X$ is an irrational ruled surface. If $X$ is relatively minimal, then
$\mu(X) = h^{1,1}(X) - 2 = 0$. Assume that $\mu(X) = h^{1,1}(X) - 2 > 0$. The (-2)-curves
must be contained in the union of fibers of the Albanese fibration $a_X$ on $X$. Let
$\phi : X \to Y$ be the map to a relatively minimal irrational ruled surface. Then

$$\mu(X) = h^{1,1}(X) - 2 = h^{1,1}(Y) + \rho(X) - \rho(Y) - 2 = \rho(X) - \rho(Y),$$

i.e. the number of disjoint nodal curves on $X$ is the same as the number of blowups from $Y$ to $X$. This is possible only if the number of nodal curves contained in each reducible fibre of $a_X$ is the same as the number of blowups on the corresponding fibre of $Y$. The only possibility is that each reducible fibre of $a_X$ is a string of three smooth rational curves $(-2) - (-1) - (-2)$ obtained by blowing up twice. This gives the case (2-b).

Assume that $\kappa(X) = -\infty$ and $X$ is rational.
This case has been classified in [1], Theorem 3.3 and Remark 3. This gives the cases (2-c), (2-d), (2-e), (2-f).

This completes the proof of Theorem 1.4.

Remark 4.2. (1) There are examples of the case (1-f) with $K^2 = 6, 4, 2$, as given in [2]. In the paper they give a complete classification of the surfaces $Y$ occurring as the minimal resolution of a surface $Z := (C_1 \times C_2)/G$, where $G$ is a finite group with an unmixed action on a product of smooth projective curves $C_1 \times C_2$ of respective genera $\geq 2$, and such that (i) $Z$ has only rational double points as singularities, (ii) $q(Y) = p_g(Y) = 0$. In particular they show that $Z$ has only nodes as singularities, and the number of nodes is even and equal to $t := 8 - K_Z^2$ (see Corollary 5.3, ibid).
Furthermore, they give examples with $t = 2, 4, 6$. The case $t = 0$, i.e., $G$ acts freely on $C_1 \times C_2$, was completely classified in [1].

The cases $t = 6, 4$ can also be obtained as the quotient of a minimal surface of general type with $K^2 = 8$ and $p_g = 0$ by an action of $(\mathbb{Z}/2\mathbb{Z})^2$, or by an action of $\mathbb{Z}/2\mathbb{Z}$, where each non-trivial involution has isolated fixed points only. This was confirmed by Ingrid Bauer.

(2) We do not know the existence of the case (1-f) with $K^2 = 1$, i.e. a Godeaux surface with 7 disjoint nodal curves. However, there is a possible construction of such an example. If one can find a minimal surface of general type with $K^2 = 8$ and $p_g = 0$ admitting an action of $(\mathbb{Z}/2\mathbb{Z})^3$, each of the 7 involutions having isolated fixed points only, then the quotient has the minimal resolution with $K^2 = 1, p_g = 0$, and 7 disjoint nodal curves.

(3) We do not know the existence of the case (1-f) with $K^2 = 7$.

Remark 4.3. The case (2-a) gives counterexamples to Proposition 4.1 of [6]. Indeed the authors, though their proof was correct, overlooked the case of fake projective planes for the minimal case, and consequently the case of blowups of fake projective
planes for the non-minimal case as they used induction on the number of blowups from the minimal model. Thus the first statement of their proposition holds true except for the case where $Y$ is a fake projective plane, and the second statement except for the case where $Y$ is the blowup of a fake projective plane at one point or at two infinitely near points.

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