Error Estimates for Measurements of Cosmic Shear

Dipak Munshi and Peter Coles

1Max-Planck-Institut fur Astrophysik, Karl-Schwarzschild-Str.1, D-85740, Garching, Germany
2School of Physics & Astronomy, University of Nottingham, University Park, Nottingham, NG7 2RD, United Kingdom

ABSTRACT
In the very near future, weak lensing surveys will map the projected density of the universe in an unbiased way over large regions of the sky. In order to interpret the results of studies it is helpful to develop an understanding of the errors associated with quantities extracted from the observations. In a generalization of one of our earlier works, we present estimators of the cumulants and cumulant correlators of the weak lensing convergence field, and compute the variance associated with these estimators. By restricting ourselves to so-called compensated filters we are able to derive quite simple expressions for the errors on these estimates. We also separate contributions from cosmic variance, shot noise and intrinsic ellipticity of the source galaxies.

Key words: Cosmology: theory – large-scale structure of the Universe – Methods: analytical – Methods: statistical

1 INTRODUCTION
The recent detections of weak gravitational lensing of background galaxy images by large-scale structure (Bacon et al. 2000; van Waerbeke et al. 2000) provide an added impetus to the development of statistical methods for handling high-quality data from weak lensing surveys which should provide us with valuable information about the mass distribution in the universe (Mellier 1999; Bernardeau & Schneider 2001). The particular benefit of weak lensing surveys is that they permit us to probe the distribution of underlying mass in a fashion that does not depend on an understanding of the relationship between galaxies and the distribution of dark matter.

Following the directions set in earlier work by Gunn (1967), Blandford et al. (1991), Miralda-Escudé (1991) and Kaiser (1992), most current progress in weak lensing can broadly be divided into two categories. Villumsen (1996), Stebbins (1996), Bernardeau et al. (1997) and Kaiser (1987) have focussed on the linear and quasi-linear regime by assuming a large smoothing angle. Several other authors have developed computational techniques to simulate weak lensing catalogs: numerical simulations of weak lensing typically employ N-body simulations, through which ray tracing experiments are conducted (Schneider & Weiss 1988; Jaroszyn’ski et al. 1990; Lee & Paczyn’ski 1990; Jaroszyn’ski 1991; Babul & Lee 1991; Bartelmann & Schneider 1991: Blandford et al. 1991). Building on the earlier work of Wambsganns et al. (1995, 1997, 1998) the most detailed numerical studies of lensing have been made by Wambsganns, Cen & Ostriker (1998). Other recent studies using ray-tracing experiments have been conducted by Premadi, Martel & Matzner (1998), van Waerbeke, Bernardeau & Mellier (1998), Bartelmann et al. (1998), Couchman, Barber & Thomas (1998) and Jain, Seljak & White (2000). While the former, perturbative analyses can provide valuable information at large smoothing angle, this approach can not be used to study lensing on small angular scales, because the perturbation series involved start to diverge.

More recent studies (Hui 1999; Munshi & Jain 2000, 2001; Munshi & Coles 1999, 2000; Valageas 1999a,b) have demonstrated that, in the highly non-linear regime, it is possible to combine the well-motivated hierarchical ansatz (Davis & Peebles 1977; Peebles 1980; Fry 1984; Fry & Peebles 1978; Szapudi & Szalay 1993, 1997; Colombi et al. 1997; Scoccimarro & Frieman 1998; Scoccimarro et al. 1998; Balbin & Schaeffer 1989; Bernardeau & Schaeffer 1992; Bernardeau & Schaeffer 1999) with the scaling relation for evolution of two-point correlation functions (Hamilton et al. 1991; Nityananda & Padmanabhan 1994; Jain, Mo & White 1995; Padmanabhan et al. 1996; Peacock & Dodds 1996) to make very accurate predictions of the statistics of the lensing convergence field for very small smoothing angular scales. This approach offers an analytic route to the study of small scale lensing to complement the numerical approaches mentioned above.

* e-mail: mdipak@mpa-garching.mpg.de
† e-mail: Peter.Coles@nottingham.ac.uk
The hierarchical ansatz has been used to show that lower-order moments such as cumulants and cumulant correlators can be modelled very accurately. It has also been found that the probability distribution function (PDF) and the bias associated with hot spots in convergence maps can also be predicted very accurately using this formalism. Higher order moments are more sensitive to the tail of the distribution function they represent, and are consequently more sensitive to measurement errors arising from finite size of the sample. Although there have been many detailed studies to quantify measurement errors for moments of the density field (Colombi et al. 1995; Colombi et al. 1996; Szapudi & Colombi 1996; Hui & Gaztanaga 1998) similar studies for weak lensing surveys are still lacking. Schneider et al. (1998) proposed different estimators for extracting the variance from convergence maps and the errors associated with them. In this paper we extend such results to incorporate all higher order cumulants and their two-point counterparts the cumulant correlators (Szapudi & Szalay 1997) which were used in the context of lensing by Munshi & Coles (2000). We study the contribution to the error involved in using these estimators by computing their variance. We list contributions from different sources (including the discrete nature of the source distribution, the intrinsic ellipticities associated with source galaxies, and the finite size of the catalogue). In previous studies of error estimations the higher order cumulants were assumed to be zero, as there has not been until recently an analytic prediction for the hierarchical parameters $S_N$ in the highly non-linear regime. Combining our result with recent analytical prediction for $S_N$ parameters for small smoothing angles will provide an accurate way to compute estimation errors and hence actual possibility of measuring these quantities from observational data.

The layout of the paper is follows. In Section 2 we introduce the estimators, and in Section 3 we explain the different types of averaging involved in computing the dispersion and mean of these estimators. In Section 4 we develop a diagrammatic formalism to compute the mean and the dispersion and derive very general expression for scatter in estimates of the cumulant correlators of arbitrary order for an arbitrary number of points. In Section 5 we discuss the importance of our results in a general cosmological context. We have presented the detailed expressions for specific lower-order moments in an appendix for easy reference.

## 2 Estimators for Cumulants and Cumulant Correlators

The statistics most frequently used to quantify the nature of clustering from galaxy catalogs are the moments of various orders. These are useful both to quantify the nature of non-Gaussianity and also to constrain the nature of initial conditions. A particularly useful way of combining moments is in the form of cumulants, which have been used to quantify both galaxy clustering and lensing surveys. Unlike the cumulants derived from galaxy catalogs, cumulants of lensing fluctuations can also differentiate between different cosmological models. These will be the most useful statistical descriptors for future weak lensing surveys.

Schneider et al. (1998) have proposed the use of aperture mass statistics based on the use of a compensated filter function $U$ to smooth the weak lensing convergence field $\kappa$ defined over a circular patch of sky with a radius $\theta_0$:

$$M_{ap}(\theta_0) = \int d\theta U(\theta)\kappa(\theta).$$  \hspace{1cm} (1)

This particular filter function has many useful properties which allows us write $M_{ap}$ in terms of the measured tangential component of the shear $\gamma_t$ inside a circle of radius $\theta$ on the sky:

$$M_{ap}(\theta_0) = \int d\theta Q(\theta)\gamma_t(\theta),$$  \hspace{1cm} (2)

where $Q(\theta_0)$ and $U(\theta)$ are related by

$$Q(\theta_0) = \frac{2}{\theta_0^2} \int_0^{\theta_0} d\theta' U(\theta') - U(\theta_0)$$  \hspace{1cm} (3)

(Schneider et al. 1998). We will use the second definition of $M_{ap}$ in our analysis. The analysis can be extended to any other specific form of window function, such as a top hat, although it will no longer possible to directly relate the smoothed convergence fields with the galaxy shear $\gamma_t$ measured observationally.

We begin by defining an estimator for the cumulants which is a natural generalization of the lower-order estimators used by Schneider et al. (1998):

$$M_N = \frac{(\pi \theta_0^2)^N}{(n)_N} \left[ \sum_{(i_1, \ldots, i_N)} Q_{i_1} \ldots Q_N \epsilon_{i_1} \ldots \epsilon_{i_N} \right],$$  \hspace{1cm} (4)

where $n$ is the number of galaxies in the patch of size $\pi \theta_0^2$, the $\epsilon_i$ are individual image ellipticities, and $N$ is the order of the cumulant. The function $Q$ and its relation to the compensated filter function $U$ are defined above. In equation (4) we use the notation

$$(n)_N \equiv n(n-1) \ldots (n-N+1) = \frac{n!}{N!}.$$  \hspace{1cm} (5)
We propose a family of new estimators for cumulant correlators. Cumulant correlators were introduced in the context of galaxy surveys by Szapudi & Szalay (1997). The new estimators are defined as:

\[ M_{N_1;N_2} = \frac{(\pi\theta_0^2)^{(N_1+N_2)}}{(n_1)!N_1-1(n_2)!N_2-1} \left[ \sum_{i_1=1}^{n_1} \sum_{j_1=1}^{n_2} Q_{i_1} \epsilon_{i_1} \cdots Q_{i_{N_1}} \epsilon_{i_{N_1}} Q_{j_1} \epsilon_{j_1} \cdots Q_{j_{N_2}} \epsilon_{j_{N_2}} \right]. \]  

(6)

This approach can in principle be extended to s-point cumulant correlators which are defined over s different patches of the sky where measurements have been conducted.

\[ M_{N_1 \ldots N_s} = \frac{(\pi\theta_0^2)^{(N_1+N_2+\ldots+N_s)}}{(n_1)!N_1-1(n_2)!N_2-1 \ldots (n_s)!N_s-1} \left[ \sum_{(i_1, \ldots, i_{N_1})} \sum_{(j_1, \ldots, j_{N_2})} \cdots \sum_{(m_1, \ldots, m_{N_s})} \prod_{p=1}^{N_1} Q_{i_p} \epsilon_{i_p} \prod_{q=1}^{N_2} \epsilon_{j_q} Q_{j_q} \cdots \prod_{m=1}^{N_s} \epsilon_{m} Q_{l_m} \right]. \]  

(7)

For detailed description of these quantities see Munshi et al. (2000) context of galaxy surveys and Munshi & Coles (2000) for their weak lensing counterparts. It is well-known that these quantities carry more information then their one point counterparts the cumulants.

In order to be useful, the signal-to-noise ratio involved in measurements of these quantities should be high. Our main aim in this paper is to develop analytical results which take into account contributions from various sources of error (or “noise”). These include the distribution of intrinsic ellipticities for the lensed galaxies, shot-noise resulting from the discrete nature of galaxy distributions, and the finite size of the catalogues. While the last two contributions also arise during the analysis of projected galaxy catalogs, the intrinsic ellipticity distribution is a source of uncertainty unique to weak lensing surveys.

Finite catalogue size has two principal effects. The first is that a finite volume (or projected area) can not reveal information about structures on a scale larger than the sample. In cosmic microwave background studies this is sometimes, though not entirely accurately, referred to as “cosmic variance”. We can generally neglect this first aspect of finite volume by using compensated filters. The other main effect is more difficult and involves the effect of sample boundaries or edges imposed by sample geometry, particularly if it involves complicated masks. We shall generally refer to this second source of error as “finite volume”.

We will show that the different contributions to measurement errors of cumulant correlators are in general factorizable and can be separated into “pure” terms and hybrid errors associated with measurements of one-point cumulants. Our results are quite general and are valid for arbitrary order and for arbitrary number of smoothed patches. Using the rules we have developed in this paper it will also be possible to compute the higher order moments of errors associated with different estimators. For related discussions, see Szapudi & Colombi (1996) and Szapudi, Colombi & Bernardeau (1999).

3 DIFFERENT TYPES OF AVERAGING

The total shear \( \epsilon \) can decomposed into the part which is due to the intrinsic source ellipticity \( \epsilon^{(s)} \) and the ellipticity introduced by distortion of images due to weak lensing, which we will denote by \( \gamma \). To proceed further we need to consider the effects of three different types of averaging process. First, there is the average over positions of source galaxies within the patches which are used to compute the cumulant correlators. Let us denote this operator, operating on an arbitrary statistic \( M \), by \( P(M) \) where

\[ P(M) = \prod_{i=1}^{n} \int \frac{d^2\theta_i}{\pi\theta_0^2} \; M. \]  

(8)

Second, we have the average over the distribution of intrinsic ellipticities. This particular source of noise is generally assumed to be Gaussian. It is further assumed that the ellipticities of neighbouring galaxies (in projection) do not correlate with each other. For a given estimator say \( M \) we will denote this average by \( G(M) \). It will only operate on the intrinsic ellipticity variables i.e. \( \epsilon_{ij} \). So we can write:

\[ G(\epsilon_{ij}) = \delta_{ij} \frac{\sigma^2}{2}, \quad G(\epsilon_{i1}\epsilon_{j1}) = 0, \quad G(\epsilon_{i1}\epsilon_{j1}\epsilon_{i2}) = \delta_{ij}\delta_{kl} \frac{\sigma^2}{2} + \text{cycl.perm.}, \quad G(\gamma\gamma \ldots) = \gamma\gamma \ldots \]  

(9)

Finally, there is the ensemble average over different realizations of the sky. The ensemble averaging is commonly denoted by \( \langle \ldots \rangle \). Putting these averages together we can write the expectation value of a given statistic \( M \) for a particular patch on the sky as

\[ E(M) = \langle G(P(M)) \rangle. \]  

(10)

For a more detailed description of these operators and their commuting properties see Schneider et al. (1998).
4 MEAN AND DISPERSION OF ESTIMATORS

The basic formalism we will adopt in our analysis of the bias and dispersion of these estimators is similar to that developed by Schneider et al. (1998). Throughout the following analysis we will focus exclusively upon compensated filters. It was shown by previous studies that such filters are local in nature, meaning that correlations between neighboring cells will be quite negligible. As we shall see, our analytical results for the appropriate error terms for the one-point cumulants contain only higher-order one-point cumulants, and not two-point cumulants. Although our results will be quite accurate for the specific case of compensated filter functions, for other smoothing windows it is necessary to consider the correlation between different smoothing windows. This introduces a bias in our estimators in addition to the scatter we are studying here: for a thorough discussion of the ramifications of this, see Bernardeau et al. (1997).

We will consider several patches of the sky where measurements of shear are performed, and we will do an averaging over galaxy positions within each of these patches, intrinsic galaxy ellipticity and finally an ensemble averaging over all sky-positions. These results then will be generalized to the case when simultaneous measurements are carried over many s-tuples of patches for correlator of order $s$.

To compute the mean or dispersion of these estimators we use a diagrammatic technique, which will simplify the computation and will allow us to write a very general expression for the dispersion associated with them.

4.1 Rules

To compute the dispersion we have to consider an identical copy of the same patch. After expanding the multinomial expression that results from this splitting into intrinsic and lensing-induced shear we can express the statistic as a sum of various terms which are just products of various combination of powers of $\epsilon^s$ and $\gamma$ represented as a diagram as shown in Figure 1. To compute the third order moment of errors it is necessary to consider three copies of the same patch, and so on. The action of the various operators as discussed above will result in pairing of these stochastic variables which can be computed by following the rules listed below.

The different pairings of $\epsilon$ will be considered between copies of the same patch. The different types of pairing and the rules for dealing with them are:

- **$\gamma$ pairing:** each of these terms will contribute one $M_s^2$ term which will denote the contribution to the error the from discrete nature of the galaxy distribution (denoted by solid lines in Figure 1). Here $M_s$ is defined by
  \[
  M_s^2 = \pi \theta_0^2 \int_{\Omega} d^2 \theta \; Q^2(\theta) \gamma_s^2(\theta) \tag{11}
  \]

- **$\epsilon$ pairing:** each of these pairings will contribute a $M_s^2$ term, which arises from the intrinsic ellipticity of galaxies (denoted by dashed lines in Figure 1).
  \[
  M_s^2 = \pi \theta_0^2 \int_{\Omega} d^2 \theta \; Q^2(\theta). \tag{12}
  \]

The intrinsic ellipticity distribution is assumed to be Gaussian and uncorrelated with other patches, as mentioned above.

- **No pairing of $\gamma$ or $\epsilon$** which will take contribution from correlation term between different patches (denoted by black dots in Figure 1). The amplitude associated with these terms are simply $M_{ap}$ which we have defined before.

Total number of pairs which can be made out of $n$ objects is $n!$: out of these pairs some of them will involve $\gamma$ pairing and some of them will correspond to $\epsilon$ pairing.
given above we can finally write down the expression for the di-

\[ \sigma \]

The second term represents only \( \gamma \) from terms in which the \( \gamma \) are unbiased estimators of multi-point moments.

As we discussed above, for the computation of variances we have to consider two copies of the same patch. Following the rules

\[ E(P(A(M_N))) = \left\langle \prod_{i=1}^{N} \int_{\Omega} d^{2} \theta_{i} Q(\theta_{i}) \gamma_{i}(\theta_{i}) \right\rangle = \langle M_{ap}^{N} \rangle \]

and similarly for cumulant correlators,

\[ E(P(A(M_{N_{1},N_{2},...}))) = \left\langle \prod_{i=1}^{N_{1}} \int_{\Omega_{1}} d^{2} \theta_{i} Q(\theta_{i}) \gamma_{i}(\theta_{i}) \prod_{j=1}^{N_{2}} \int_{\Omega_{2}} d^{2} \theta_{j} Q(\theta_{j}) \gamma_{j}(\theta_{j}) \right\rangle = \langle M_{ap}^{N_{1}}(\gamma_{1})M_{ap}^{N_{2}}(\gamma_{2}) \rangle , \]

which shows that they are unbiased estimators of multi-point moments.

4.3 Dispersion

As we discussed above, for the computation of variances we have to consider two copies of the same patch. Following the rules
given above we can finally write down the expression for the dispersion of an \( N \)-th order cumulant as

\[ \sigma^{2}(M_{N}) = (n_{2N}) (M_{ap}^{2N}) + (N!) \sum_{p=1}^{N} (N_{r}) (n_{2N-r}) (M_{ap}^{2(N-r)}) (M_{ap}^{2p}) \]

\[ + (N!) \sum_{r=1}^{N} (N_{r}) (n_{2N-r}) (M_{ap}^{2(N-r)}) (\sigma_{\gamma}^{2})^{r} \]

\[ + (N!) \sum_{2 \leq p + r \leq N} (N_{r}) (n_{2N-p-r}) (M_{ap}^{2(N-r)}) (M_{ap}^{2(N-p-r)}) (M_{ap}^{2p}) (\sigma_{\gamma}^{2})^{r}. \]

The symbol \( N_{r} \) is used to denote the number of combinations of \( p \) objects taken from \( N \). The first term here represents

the case when all points are \( \gamma \) in both patches and there is no pairing of these points within the copies of the same patch.

The second term represents only \( \gamma \) pairing between pairs of \( \gamma \) from two different copies of the same patch. The third term

represents the \( \epsilon \) pairing within the copies of the same patch and the last terms is a mixture when some of the couplings are

\( \epsilon \) coupling and some of them are \( \gamma \) coupling.

A similar analysis can be performed for \( s \)-point smoothed cumulant correlators of order \( N_{1} + \ldots + N_{s} \). We find

\[ \sigma^{2}(M_{N_{1},N_{2},...N_{s}}) = \prod_{i=1}^{s} ((n_{i})_{2N}) (M_{ap}^{2(N_{i})})^{N_{i}} + (N_{i}!) \sum_{p=1}^{N_{i}} (N_{i}) (n_{i})_{2N_{i}-p}) (M_{ap}^{2(N_{i})}) (M_{ap}^{2(N_{i}-p)}) (M_{ap}^{2p}) \]
4.4 Errors associated with \( S_N \) parameters

For practical purposes the normalised cumulants and cumulant correlators generally involve

\[
S_N = \frac{\langle M_{ap}^N \rangle}{\langle M_{ap}^2 \rangle^{N-1}}.
\]

These quantities are of great importance as they can quantify the non-Gaussianity in the weak lensing field and also can be used to study the effects of background cosmology on weak lensing statistics. Although the \( S_N \) parameters for weak lensing surveys have already been measured using top-hat filters, there has been not work in this direction for computation of \( S_N \) parameters using compensated filters for high \( N \).

Our error expression can be written in a compact form if we use the parameters \( \Sigma_N \) parameters which can be related to the \( S_N \) parameters by following expressions. These definition of \( \Sigma_N \) also include the disconnected contributions.

\[
\begin{align*}
\langle M_{ap}^4 \rangle &= S_4 \langle M_{ap}^2 \rangle^3 + 3 \langle M_{ap}^2 \rangle^2 = \Sigma_4 (\langle M_{ap}^2 \rangle)^{4/2} \\
\langle M_{ap}^5 \rangle &= S_5 \langle M_{ap}^2 \rangle^4 + 10 S_3 \langle M_{ap}^2 \rangle^3 = \Sigma_5 (\langle M_{ap}^2 \rangle)^{5/2} \\
\langle M_{ap}^6 \rangle &= S_6 \langle M_{ap}^2 \rangle^5 + 15 S_4 \langle M_{ap}^2 \rangle^4 + 10 S_3^2 \langle M_{ap} \rangle^6 = \Sigma_6 (\langle M_{ap} \rangle)^{6/2}
\end{align*}
\]

Using the expressions derived in previous sections and in the limit when number of source galaxies are much higher compared to unity we can write down the following expressions for variance in \( M_3 \) and \( M_2 \).

\[
\sigma^2(M_3) = \left[ \Sigma_6 - \Sigma_3 + \frac{9 \Sigma_4}{\rho} \right] \langle M_{ap}^2 \rangle^{3/2} + \frac{18}{\rho^2} + \frac{6}{\rho^3} \langle M_{ap}^2 \rangle^{3/2}
\]
Figure 4. Estimation error for computation of $\langle M^2_{ap} \rangle$ as a function of smoothing angle $\theta_0$. A survey area of $1^\circ$ is assumed for left panel and a survey area of $8^\circ$ is assumed in the right panel. Different curves denote different background cosmologies. Line style is the same as that of Figure 2. While measurement errors are dominated by the shot noise (which are induced due to intrinsic ellipticity distribution of the galaxy) and the Poisson shot noise for smaller smoothing angles. In large smoothing angles it is more dominated by finite size of the survey area. Estimation errors for the variance depends on the higher order moments. In particular we found that the normalized cumulant $S_4$ appears. Unfortunately there are no theoretical estimates for computation of $S_4$ using compensated filters. Earlier results in the literature used $S_4 = 0$, which we have used here. Clearly the fractional errors are quite small even for moderate survey size and for large sky coverage power spectrum estimation can be carried out with very high precision. All our calculations are done under the hypothesis that the correlation length scale associated with compensated filter is quite small compared to length-scales associated with field of view. This was demonstrated by Schneider et al. (1998). For top-hat filters correlations between neighboring cells will have to be taken into account. In our computation the intrinsic ellipticity of the source galaxy $\sigma_e$ is set to be equal to 2.

Figure 5. Estimation error for computation of $S_3$. We have assumed that all sources are at a fixed red-shift. A survey area of radius $1^\circ$ (left panel) and $8^\circ$ (right panel) are assumed. Number density of source galaxies $n_{\text{survey}}$ in the left panel is 30 per arcminute² in the right panel it is fixed at 60 per arcminute². Error is plotted as a function of the smoothing angle $\theta_0$. Different curves denote different background cosmologies. Line style is same as in previous figures. While the errors are mostly dominated by shot noise at a very small resolution angle, finite size of the survey are more pronounced for larger smoothing angles. However for smoothing angular scales of few arc minutes the fractional errors are quite small for reasonable sky coverage. This is more interesting as non-Gaussian estimators such as $S_3$ can be used to distinguish between different cosmologies. Notice that fractional errors do not change much while we change the background cosmology. For computation of the fractional errors in $S_3$ we have used the analytical results for compensated filters by Bernardeau & Valageas (2000) which are in agreement with numerical computations by Reblinsky et al. (1999). Although these results are for smoothing radius $\theta_0 = 4'$, the $S_N$ parameters are known not to change much at very small angular scales where we can assume them approximately constant for computations of error bars.
Various estimators have already been proposed for the computation of the variance and skewness of cosmic shear smoothed one- and two-point probability distribution functions. These quantities are widely used to quantify the statistical nature of clustering of the mass distribution in the study of galaxy surveys. In this context, estimators of these statistics are prone to error from finite catalogue size and Poisson (discreteness) effects. The application of similar methods to weak lensing studies is clearly appropriate, but introduces an additional source of error. This paper allows for these additional error terms.

The behaviour of $M_{ap}$ for various cosmologies is shown in Figure 2. The function $\rho$ is plotted in Figure 3. Using the results we have obtained, the scatter associated with the $S_N$ parameters can now be written in terms of these quantities:

$$\frac{\sigma(S_N)}{S_N} = \frac{\sigma(M_{ap}^N)}{(M_{ap}^N)} + (N - 1)\frac{\sigma(M_{ap}^2)}{(M_{ap}^2)}$$  

(21)

For multiple fields covering the whole field of view factors of $\sqrt{N_f}$ (where $N_f$ is the number of cells covering the field) will have to be incorporated (see Schneider et al. 1998). For example, the variance is a factor $N_f$ smaller. This, of course, is only the case for compensated filters like those we are studying here. Effect of over-sampling has not been studied in the context of weak lensing and needs a dedicated analysis.

Final results for the scatter in variance and $S_3$ are plotted in Figure 4 and Figure 5 respectively. For specific case studies we have assumed two different hypothetical survey strategies. In one case we assume that the radius of the survey is $1^\circ$ in the second case it is fixed at $8^\circ$. The number density of source galaxies in the first case is assumed to be $30 \text{arcmin}^{-2}$ and twice that value in the second case. The parameter $\sigma_r$ which characterizes the intrinsic ellipticity distribution of galaxies is fixed at .2 in both cases. While the first case is representative of various ground based surveys with less exposure time which are currently in progress, the second case is more realistic and hope to be achieved by future weak lensing survey with long exposure time probably from outside earths atmosphere. In both cases the errors are minimum for arcminute scale. Our study shows that skewness parameters $S_3$ can be measured very accurately from second type of surveys although signals might be detectable even in first case with considerable uncertainty resulting in large error bars. This underlines the importance of future weak lensing surveys from cosmological perspective.

Direct determination of higher order $S_N$ parameters for the case of weak lensing surveys has so far only been reported by Bernardeau et al. (2002). Higher moments become more difficult to measure because better statistics are needed. As the expression for the error in such quantities contain $S_N$ parameters of higher order we need to make certain order-of-magnitude approximations to make progress in estimating errors. Schneider et al. (1998) assumed that $\Sigma_4 = 3$. In addition to that we have assumed that $\Sigma_6 = \Sigma_2^2$ for error expression in $S_3$. A more accurate error estimation will have to wait till we can compute these quantities analytically. However any analytical results will have to be checked with numerical result for which we need to have a better understanding of their error properties. Our results are only a first step in this cycle and will have to be refined by future studies.

We stress again that our derivation is only exact for the case of compensated filters when neighboring smoothed regions are not correlated. For the case of other filter functions such as the top-hat filter a more general treatment is necessary. We plan to present such results elsewhere. We have also assumed that the source redshifts are determined with high accuracy. Lack of knowledge of source redshifts is a potential source of additional error which too needs to be taken into account for more realistic estimation of cosmological parameters from future weak lensing surveys.

5 DISCUSSION

It has been the purpose of this paper to estimate the errors involved with the extraction of statistical information from weak lensing surveys. We have focussed on the cumulants and cumulants correlators, which are normalized moments of the smoothed one- and two-point probability distribution functions. These quantities are widely used to quantify the statistical nature of clustering of the mass distribution in the study of galaxy surveys. In this context, estimators of these statistics are prone to error from finite catalogue size and Poisson (discreteness) effects. The application of similar methods to weak lensing studies is clearly appropriate, but introduces an additional source of error. This paper allows for these additional error terms.

Various estimators have already been proposed for the computation of the variance and skewness of cosmic shear smoothed with a particular window function. We have also generalized these suggestions to statistics of arbitrary order via a generalized estimator which estimates the statistics of smoothed convergence field in an unbiased way. We have also proposed a new set of estimators which are useful for measuring cumulant correlators, and which are natural generalizations of their one-point counterparts. We have shown that our estimators constitute a family of unbiased estimators for cumulant correlators.

We have also computed the dispersion of these estimators, which is essential to determine the signal-to-noise ratio associated with them. We have found compact expressions for the dispersions for arbitrary order and also for arbitrary number of points. We have been also able to separate contributions from different sources of noises such as the finite size of the galaxy catalog, finite width of ellipticity distribution of source galaxies and Poisson noise due to finite number of source galaxies in the field of view. Our results are valid for both large and small smoothing angles. In case of large smoothing
angles, where perturbative calculations are still valid, one need to use the quasi-linear values of $S_N$ and $C_{pq}$ parameters associated with the expressions for finite volume corrections. For smaller smoothing angles we have to replace these numbers by forms suitable for the highly non-linear regime, which have already been computed by several authors recently based on the hierarchical ansatz (Hui 1999; Munshi & Jain 2000, 2001; Munshi 2000, Valageas 2000a,b).

The effect of source clustering, which we have ignored, will also introduce corrections terms in the measurement of lower order moments. We have also ignored the effect due to lens coupling. Some of these issues have been studied by Bernardeau et al. (1997) and Schneider et al. (1998) with the result that such corrections seem to be negligible, at least in the quasi-linear regime. Studying the effects of source clustering using ray-tracing experiments in the highly non-linear regime is difficult because most such simulations propagate light rays backward: the source position is consequently left arbitrary and determined only by lensing due to the intervening mass.

The validity of the Born approximation, which underpins lensing calculations, has also been studied in the quasi-linear regime. It has been shown that corrections arising from higher-order terms in the photon propagation equation are negligible in quasi-linear regime. Similar conclusions have also been found to be valid in the highly non-linear regime by comparing ray-tracing simulation against analytical results using hierarchical ansatz. (Hui 1999; Munshi & Jain 2000, 2001; Munshi 2000; Valageas 2000a,b).

We have also assumed that the galaxy intrinsic ellipticities are not correlated but it may be possible that the galaxies are not randomly oriented and there may be a coherent alignment due to the geometry of the large-scale structure in which they are embedded (e.g. Heavens, Refregier & Heymans 2000). So far however no convincing observations of nearby structures have indicated that such an alignment exists (e.g. Mellier 1999) although several attempts have been made to unearth one.

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APPENDIX A: EXPLICIT EXPRESSIONS FOR ERROR TERMS

A1 Cumulants

Error terms for the one-point smoothed cumulants involve three contributions. For example, for the lowest order cumulant,

$$\sigma^2(M_1) = \frac{1}{n} \left[ n(n-1)\langle M_{ap}^2 \rangle + n\langle M_{ap}^2 \rangle + nM_g^2 \left( \frac{\sigma^2}{2} \right) \right] - \langle M_{ap} \rangle^2. \quad (A1)$$

The first of these contributions derives from finite volume effects included in the higher order moments such as $M_{ap}$. For example the dispersion in variance depends on the fourth moment, and the dispersion in skewness will similarly depend on the sixth order moment. The terms denoted by $M_g^2$ are related to the fact that we have a finite number of galaxies: this will vanish if we take the limiting case of infinite number of galaxies. The terms with $M_g^2$ are due to the finite width of the intrinsic galactic ellipticity distribution and would vanish in the limit in which there are an infinite number of galaxies in the survey.

For the higher-order cumulants there are differences. Unlike the expression for the dispersion of $M_1$, we will have mixed terms in the expression for the dispersion in $M_2$ which are denoted by various products of $M_s$, $M_{ap}$ and $M_g$. The following expression was derived by Schneider et al. (1998):

$$\sigma^2(M_2) = \frac{1}{n^2(n-1)^2} \left\{ n(n-1)\ldots(n-3)\langle M_{ap}^4 \rangle + 4n(n-1)(n-2)\langle M_{ap}^2 M_{ap}^2 \rangle + 2n(n-1)\langle (M_g^2)^3 \rangle \right\}$$

$$+ \left\{ 4n(n-1)(n-2)M_g^2 \langle M_{ap}^2 \rangle + 4n(n-1)(n-2)M_g^3 \langle M_g \rangle \right\} \left( \frac{\sigma^2}{2} \right)^2 + 2n(n-1)M_g^2 \left( \frac{\sigma^2}{2} \right)^2 - \langle M_{ap}^2 \rangle^2. \quad (A2)$$

At third order we get the following expression for dispersion of $M_3$:

$$\sigma^2(M_3) = \frac{1}{n^3(n-1)^3(n-2)^2} \left\{ n(n-1)\ldots(n-5)\langle M_{ap}^6 \rangle + 9n(n-1)\ldots(n-4)\langle M_{ap}^2 M_g^4 \rangle \right\}.$$
error associated with its measurement can be expressed as

\[ +18n(n-1)\ldots(n-3)\langle M^6_{ap} \rangle^2 \]  
\[ + \left\{ 9n(n-1)(n-2) M^2_g \langle (M^2_s)^2 \rangle + 18n(n-1)\ldots(n-3) M^2_g \langle M^2_{ap} \rangle S + 9n(n-1)\ldots(n-4) M^2_g \langle M^2_{ap} \rangle \right\} \left( \frac{\sigma^2}{2} \right) \] 
\[ + \left\{ 18n(n-1)(n-2) M^2_g \langle M^2_s \rangle + 18n(n-1)\ldots(n-3) M^2_g \langle M^2_{ap} \rangle \right\} \left( \frac{\sigma^2}{2} \right)^2 \] 
\[ + 6n(n-1)(n-2) M^2_g \left( \frac{\sigma^2}{2} \right)^3 \} - \langle M^4_{ap} \rangle^2 \]  

(A3)

Notice that we can write (Fry 1984) \( \langle M^6_{ap} \rangle = S_0 \langle M^2_{ap} \rangle^3 + 15 S_4 \langle M^4_{ap} \rangle^2 + 10 S^2_6 \langle M^2_{ap} \rangle^4 + 15 \langle M^2_{ap} \rangle^3 \). For top-hat window functions we have analytic expressions for all lower order \( S_N \) parameters which can be used to estimate the effects of finite volume in the highly non-linear regime. For other window functions there is no such analytical expressions.

Generally speaking, for large values of \( n \), only the dominant contributions are considered and the rest are neglected. On the smaller angular scales the finite width of the galaxy ellipticity distribution dominates, and for very large smoothing angles it is the finite volume effect that dominates. Hence one can often neglect the terms containing \( M_{ap} \) altogether.

At fourth order we get

\[ \sigma^2(M_4) = \frac{1}{n^4(n-1)^4(n-2)^2(n-3)^2} \left\{ n(n-1)\ldots(n-7)\langle M^6_{ap} \rangle + 16n(n-1)\ldots(n-6)\langle M^6_{ap} M^2_{ap} \rangle + \ldots + 144n(n-1)(n-2)\ldots(n-5)\langle M^6_{ap} \rangle \right\} \left( \frac{\sigma^2}{2} \right) \] 
\[ + \left\{ 144n(n-1)(n-2)\ldots(n-3) \langle M^2_{ap} \rangle^2 \langle (M^2_s)^2 \rangle \right\} \left( \frac{\sigma^2}{2} \right)^2 \] 
\[ + 24n(n-1)(n-2)(n-3) M^2_g \left( \frac{\sigma^2}{2} \right)^3 \} - \langle M^4_{ap} \rangle^2 \]  

(A4)

In the statistical study of large scale distribution of galaxies generally cumulants are normalized by dividing them with suitable power of two-point cumulant or by the variance, for example in the construction of the \( S_N \) parameters. Since both the numerator and denominator are both affected by errors we have discussed above they will introduce a ratio bias as discussed by Hui & Gaztanaga (1999).

Finally in this section we mention that the above results do not depend on the scale of non-linearity probed by weak lensing, but the appropriate values of \( S_N \) must be used. For example, when large smoothing angles are considered we should use the quasi-linear values of \( S_N \) parameters of the convergence map.

A2 Cumulant Correlators

To compute the cumulant correlators we consider two patches in the sky in the direction of \( \gamma_1 \) and \( \gamma_2 \). The results are very similar to the case of one-point smoothed cumulants. As before the errors associated with measurements can be differentiated in three types, i.e. the finite volume effects, errors due to intrinsic ellipticity of the source galaxies and errors associated with finite number of galaxies. The lowest order cumulant correlator is of course the smoothed two-point correlation function. The error associated with its measurement can be expressed as

\[ \sigma^2(M_{11}) = \frac{1}{n^2 n^2} \left\{ n_1(n_1-1)n_2(n_2-1)\langle M^2_{ap} \rangle^2 \langle (M^2_{ap})^2 \rangle + n_1 n_2(n_2-1)\langle M^2_{ap} \rangle \langle (M^2_{ap})^2 \rangle + \ldots \right\} \] 
\[ + \left\{ n_1 n_2(n_2-1)\langle M^2_g \rangle \langle (M^2_{ap})^2 \rangle + n_1 n_2(n_2-1)\langle M^2_g \rangle \langle (M^2_{ap})^2 \rangle + n_1 n_2\langle M^2_g \rangle \langle (M^2_{ap})^2 \rangle \right\} \left( \frac{\sigma^2}{2} \right) \] 
\[ + n_1 n_2 M^2_g \left( \frac{\sigma^2}{2} \right)^3 \} - \langle M^4_{ap} \rangle \langle M^4_{ap} \rangle. \]  

(A5)

We have taken account that the number of galaxies in two different patches can be different. Notice that error now contains terms which are mainly the correlation of measurement errors in two different patch.
Terms of higher order begin with $M_{21}$, which can be given in the form

$$
\sigma^2(M_{21}) = \frac{1}{\pi^2 n_2^2 (n_2 - 1)^2} \left\{ 4n_1(n_1 - 1)(n_1 - 2)n_2(n_2 - 1)(M_2^2(\gamma_1)M_{ap}^2(\gamma_1)M_{ap}^2(\gamma_2)) + n_1(n_1 - 1)(n_1 - 2)(n_1 - 3)n_2(n_2 - 1)(M_2^2(\gamma_1)M_{ap}^2(\gamma_2)) + 2n_1(n_1 - 1)n_2(M_2(\gamma_1)^2M_{ap}(\gamma_2)) \right\}
$$

$$
+ \left\{ 4n_1(n_1 - 1)(n_1 - 2)n_2(n_2 - 1)M_2^2(\gamma_1)(M_{ap}^2(\gamma_1)M_{ap}^2(\gamma_2)) + n_1(n_1 - 1)n_2(n_2 - 1)M_2^2(\gamma_1)(M_2^2(\gamma_1)M_{ap}^2(\gamma_2)) + 4n_1(n_1 - 1)n_2M_2^2(\gamma_1)(M_{ap}^2(\gamma_1)M_2^2(\gamma_2)) \right\} \left( \frac{\sigma^2}{2} \right)
$$

$$
+ \left\{ n_1(n_1 - 1)(n_1 - 2)(n_1 - 3)n_2M_2^2(\gamma_2)(M_{ap}^2(\gamma_1)) + 4n_1(n_1 - 1)(n_1 - 2)n_2M_2^2(\gamma_2)(M_2^2(\gamma_2)M_{ap}^2(\gamma_1)) + 2n_1(n_1 - 1)n_2M_2^2(\gamma_2)(M_{ap}^2(\gamma_1)) \right\} \left( \frac{\sigma^2}{2} \right)^2
$$

$$
+ 4n_1(n_1 - 1)n_2(M_2^2(\gamma_1))^2(M_{ap}^2(\gamma_2)) \left( \frac{\sigma^2}{2} \right)^3 - (M_{ap}^2(\gamma_1)M_{ap}^2(\gamma_2))^2.
$$

(A6)

These expressions are cumbersome, but worth listing because these are the cumulant correlators most likely to be measurable. In the above expressions we used

$$
M_2^2(\gamma_i) = \frac{\pi}{\Omega_i} \int d^2 \theta \ Q^2(\theta) \gamma_i^2(\theta)
$$

$$
M_2^2(\gamma_i) = \frac{\pi}{\Omega_i} \int d^2 \theta \ Q^2(\theta).
$$

$$
M_{ap}^2(\gamma_i) = \frac{\pi}{\Omega_i} \int d^2 \theta \ Q(\theta) \gamma_i(\theta).
$$

(A7)

In deriving the above expressions for measurement errors in two-point cumulant correlators we have again assumed that the intrinsic ellipticities of galaxies do not cross correlate among different patches. At increasingly higher orders the expressions for error contribution becomes more complicated, although as explained above for large number of galaxies we can take the $n_1 \rightarrow \infty$ and $n_2 \rightarrow \infty$ limit which will simplify these expressions.

As in the case of cumulants normalized cumulant correlators can be derived by dividing cumulant correlators by suitable powers of variance within these patches and the correlation among these patches (see Munshi & Coles 2000). Normalized cumulant correlators are also denoted by $C_{pq}$ and in addition to the errors we have already discussed we will have ratio bias too.

The formalism we have developed above can also be extended to compute the skewness associated with these estimators. Statistical studies for galaxy distribution have already shown that estimated values of these moments are more likely to have a lower value than its mean (Szapudi & Colombi 1996). This will mean that the probability distribution of these estimators are skewed. The skewness associated with these estimators can also be computed using the procedure outlined above and we hope to present a detailed analysis elsewhere; see Szapudi et al. (2000) for comments in a similar vein.

The results presented above are valid for only one particular patch of the sky for computation of cumulants and a single pair of patches for measurements of cumulant correlators. However it is straightforward to generalize the above results for large number of patches or pairs of patches which will help to increase the signal to noise ratio (see Schneider et al. 1998).

Detailed expressions for many-point cumulant correlators can also be obtained using the formalism developed here. The general expression we have presented in the text are valid for an arbitrary number of points and do not depend directly on the filter function used. In deriving the above results we have assumed that the two patches have same size and hence the same variance, but they might contain different number of galaxies, i.e. $n_1$ and $n_2$. It is not difficult to generalize the results to the case when the variance in these two patches are also different. We aim to present numerical results for specific filters which arise from above results and their comparison against simulated noisy convergence maps in the near future.