On the Kontsevich \( \star \)-Product Associativity Mechanism

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Abstract—The deformation quantization by Kontsevich is a way to construct an associative noncommutative star-product \( \star = \times + \hbar \{ \}_\hbar + \mathcal{O}(\hbar) \) in the algebra of formal power series in \( \hbar \) on a given finite-dimensional affine manifold: here \( \times \) is the usual multiplication, \( \{ \}_\hbar \neq 0 \) is the Poisson bracket, and \( \hbar \) is the deformation parameter. The product \( \star \) is assembled at all powers \( \hbar^{k \geq 0} \) via summation over a certain set of weighted graphs with \( k + 2 \) vertices; for each \( k > 0 \), every such graph connects the two co-multiples of \( \star \) using \( k \) copies of \( \{ \}_\hbar \). Cattaneo and Felder interpreted these topological portraits as genuine Feynman diagrams in the Ikeda–Izawa model for quantum gravity. By expanding the star-product up to \( \hbar^3 \), i.e., with respect to graphs with at most five vertices but possibly containing loops, we illustrate the mechanism \( \text{Assoc} = \star \) (Poisson) that converts the Jacobi identity for the bracket \( \{ \}_\hbar \) into the associativity of \( \star \).

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Denote by \( \times \) the multiplication in the commutative associative unital algebra \( C^\infty(\mathbb{R}^n) \) of scalar functions on a smooth \( n \)-dimensional real manifold \( \mathbb{R}^n \). Suppose first that a noncommutative deformation \( \star = \times + \hbar \{ \}_\hbar + \mathcal{O}(\hbar) \) of \( \times \) is still unital \( f \star 1 = 1 \star f \) and associative, \( (f \star g) \star h = f \star (g \star h) \) for \( f, g, h \in C^\infty(\mathbb{R}^n)[[\hbar]] \). By taking \( 3! = 6 \) copies of the associativity equation for the star-product \( \star \), we infer that the skew-symmetric part of the leading deformation term, \( \{ f, g \}_\star := \frac{1}{\hbar} (f \star g - g \star f) \big|_{\hbar = 0} \), is a Poisson bracket\(^3\).

Now the other way round: can the multiplication \( \times \) on a Poisson manifold \( \mathbb{R}^n \) be deformed using the bracket \( \{ \}_\hbar \) such that the \( \{ [\hbar]\}\)\([-\text{linear}\] star-product \( \star = \times + \hbar \{ \}_\hbar + \mathcal{O}(\hbar) \) stays associative? Kontsevich proved [1] that on finite-dimensional affine\(^4\) Poisson manifolds this is always possible: from \( \{ \}_\hbar \), one obtains the bi-differential terms \( B_k(\cdot, \cdot) \) at all powers \( \hbar^{k \geq 0} \) in the formal series for \( \star \). This associative unital \( \star \)-product was constructed in [1] using a pictorial language: the operators \( B_k = \sum_{|\Gamma|} w(\Gamma) \times B_k^\Gamma(\cdot, \cdot) \) are encoded by the weighted oriented graphs \( \Gamma \) with \( k + 2 \) vertices and \( 2k \) edges but without tadpoles or multiple edges; in every such \( \Gamma \), there are \( k \) internal vertices (each of them is a tail for two edges) and \( 2 \) sinks (no issued edges). The Poisson bracket \( \{ \}_\hbar \) with coefficients \( \mathcal{P}^\hbar(u) \) at \( u \in \mathbb{R}^n \) provides the “building block” \( \mathcal{A} = \sum_{i,j=1}^n \frac{\partial f}{\partial x_i} \frac{\partial f}{\partial x_j} \) in which \( \sum_{i,j=1}^n \) is implicit and the vertex contains \( \mathcal{P}^\hbar(u) \). To indicate the ordering of indexes in \( \mathcal{P}^\hbar(u) \), the out-going edges are

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\(^3\) The left-hand side of the Jacobi identity \( \sum_{ij} \{ f, g \} \star h = 0 \) is an obstruction to the associativity of the star-product: whenever the Jacobi identity is violated, one cannot have that \( (f \star g) \star h = f \star (g \star h) \).

\(^4\) On affine manifolds \( \mathbb{R}^n \), the only shape of coordinate changes is \( \bar{u} = A \cdot u + \bar{c} \). Yet no loss of generality occurs if the space \( \mathbb{R}^n \) is the fibre in an affine bundle \( \pi \) of physical fields \( u = \phi(x) \) over the space-time \( M^{\infty} \times \chi \); the Jacobians \( \partial u/\partial x = \pi(x) \) are then constant over \( \mathbb{R}^n \). (The arguments of \( \star \) are local functionals of sections, \( \varphi \in \Gamma(\pi) \to \mathbb{C} \); the \( \star \)-product is marked by the variational Poisson brackets \( \{ \}_\hbar \) on the jet space \( J^\infty(\pi) \).) The deformation quantization from [1] is lifted to the gauge field set-up in [2].
ordered by Left $\prec$ Right. The edges carry the derivatives $\partial_i \equiv \partial/\partial u^i$ and $\partial_j \equiv \partial/\partial u^j$, respectively. Every such derivation acts on the content of the vertex at the arrowhead via the Leibniz rule (and it does so independently from the other in-coming arrows, if any).

The weights $w(\Gamma) \in \mathbb{R}$ of such graphs $\Gamma$ are given by the integrals over configuration spaces of $k$ distinct points in the hyperbolic plane $\mathbb{H}^2$ (e.g., in its upper half-plane model).

The associativity postulate for $\star$ yields the infinite system of quadratic algebraic equations for the weights $w(\Gamma)$ of graphs. Kontsevich shows [1] that the left-hand side $\text{Jac}_{\rho}(\cdot, \cdot, \cdot) := \sum_{\sigma} \{ \rho \}^\sigma \cdot \nabla^\rho \cdot \nabla^\sigma$ of the Jacobi identity for $\{ \rho \}$ is the only obstruction to the balance $\text{Assoc}(f, g, h) := (f \star g) \star h - f \star (g \star h) = 0$ at all powers $h^k$ of the deformation parameter at once. The core question that we address in this note is how the mechanism $\text{Assoc} = \bigcirc$ (Poisson) works explicitly, making the star-product $\star = \bigcirc + h \{ \rho \} + \sigma(h)$ associative by virtue of Jacobi identity for the Poisson bracket $\{ \rho \}$. Expanding the Kontsevich $\star$-product in $h$ up to $\sigma(h^3)$ and with respect to all the graphs $\Gamma_i$ such that $w(\Gamma_i) \neq 0$, we obtain

$$\begin{align*}
\star \star \star &= \star \star + \frac{h^1}{6!} \left( \begin{array}{c}
\star \star \star \\
\star \star \star \\
\star \star \star
\end{array} \right) + \frac{h^2}{225} \left( \begin{array}{c}
\star \star \star \\
\star \star \star \\
\star \star \star
\end{array} \right) + \frac{h^3}{3} \left( \begin{array}{c}
\star \star \star \\
\star \star \star \\
\star \star \star
\end{array} \right) + \frac{h^4}{6} \left( \begin{array}{c}
\star \star \star \\
\star \star \star \\
\star \star \star
\end{array} \right) + \left( \begin{array}{c}
\star \star \star \\
\star \star \star \\
\star \star \star
\end{array} \right)
\end{align*}$$


5 For example, $\{ f, g \}_\rho(u) = f - \frac{i}{\hbar} \frac{\partial}{\partial u} f (u) \frac{\partial}{\partial u} g (u)$, see (1) above.

6 Willwacher and Felder (2010) conjecture that the weights can be irrational numbers for some graphs.

7 The wedge factors within the integrand in the formula for $w(\Gamma)$ are copies of the kernel of the singular linear integral operator $(d * d)^{-1}$ in the hyperbolic geometry of $\mathbb{H}^2$, see [3]. Cattaneo and Felder also showed that the $\star$-product of two functions $f, g \in C^\infty(N^n \to \mathbb{C})$ amounts to the Feynman path integral calculation of the correlation function, $(f \star g)(x) = \int_{x(0)=x} D\mathcal{D}[\phi(x(0))] \times g(\phi(x)) \times \exp \left( \frac{i}{\hbar} S(\mathcal{D}[\phi], \eta) \right)$, in the Ikeda–Izawa topological open string model on a disk $D = \mathbb{H}^2$ with boundary $\partial D \ni 0, 1, \infty$; here $\mathcal{D} \ni D \to T^* D \otimes T^* (T^* N^n)$ All details and further references are found in [3, 4]; still let us remember that within the Ikeda–Izawa model, the perturbative expansion in $\hbar$ run, in particular, over the graphs with tadpoles (which must be regularized by hand) but at the same time, those path integral calculations reproduce only the weighted oriented graphs without ”eyes” (e.g., as in $\longrightarrow), \multimap$, see Eq. (1) above). Because, to the best of our knowledge, the eye-containing graphs $\Gamma_i$ such that $w(\Gamma_i) \neq 0$ cannot at all once be eliminated from the star-product $\star$ via gauge transformations of its arguments and of its output, see Remark 1 on p. 4 and [11], many graphs in the original construction of $\star$ were not recovered in [3]. Hence there is an open problem to extend or modify the Ikeda–Izawa Poisson $\sigma$-model such that in the new set-up, the correlation functions would expand with respect to all the Kontsevich graphs $\Gamma_i$ with $w(\Gamma_i) \neq 0$.

8 That system solution is not claimed unique: one is provided by the Kontsevich integrals. Number-theoretic properties of those weights were explored by Kontsevich in the context of motives and by Willwacher–Felder in the context of Riemann $\zeta$-function.

9 Ensuring the associativity $\text{Assoc}(f, g, h) = 0$, the tri-vector $\text{Jac}_{\rho}(\cdot, \cdot, \cdot)$ is not necessarily (indeed, far not always!) evaluated at the three arguments $f, g, h$ of the associator for $\star$.

10 Balancing the associativity of a star-product order-by-order up to $\sigma(h^3)$, Penkava and Vanhaecke (1998) derived a set of weights for the $(k + 2)$-vertex Kontsevich graphs without loops. Yet no loops are destroyed in either of the copies of $\star$ when the composition $\star \circ \star$ is taken; the associativity of loopless star-products is only a part of the full claim for $\star$. So, we integrate over the configuration spaces of $k \geq 3$ points in $\mathbb{H}^2$ for all the Kontsevich graphs (e.g., with loops).
In every composition \( \star \circ \star \) the sums of graphs act on sums of graphs by linearity; each in-coming edge acts via the Leibniz rule (see above). The mechanism for \( \text{Assoc} (f,g,h) \) to vanish is two-step: first, the sums in \( \star \circ \star \) are reduced using the antisymmetry of the Poisson bi-vector \( \mathcal{P} \). The output is then reduced modulo the (consequences of) Jacobi identity\(^{11}\),

\[
\text{Jac}_\mathcal{P}(f,g,h) = \begin{array}{c}
\begin{array}{c}
\text{L}_R \\
\text{R}_L \\
\text{R}_R \\
\text{L}_L
\end{array}
\end{array} - \begin{array}{c}
\begin{array}{c}
\text{L}_R \\
\text{R}_L \\
\text{R}_R \\
\text{L}_L
\end{array}
\end{array} - \begin{array}{c}
\begin{array}{c}
\text{L}_R \\
\text{R}_L \\
\text{R}_R \\
\text{L}_L
\end{array}
\end{array} = 0
\tag{2}
\]

For \( \star \) given by (1), the associator contains 6 terms at \( \hbar \), 38 terms \(~\hbar^2\), and 218 terms \(~\hbar^3\). After the use of \( \mathcal{P}^{ij} = -\mathcal{P}^{ji} \), we infer that \( \text{Assoc} (f,g,h) \) starts at \( \hbar^2 \) with 2/3 times (2). Next, there are 39 terms at \( \hbar^3 \); we now examine how their sum \( A \) vanishes by virtue of (2) and its differential consequences\(^{12}\). Of them, three which are the easiest to recognize are\(^{13}\)

\[
\text{Jac}_\mathcal{P}(f,g,h) = \begin{array}{c}
\begin{array}{c}
\text{L}_R \\
\text{R}_L \\
\text{R}_R \\
\text{L}_L
\end{array}
\end{array} - \begin{array}{c}
\begin{array}{c}
\text{L}_R \\
\text{R}_L \\
\text{R}_R \\
\text{L}_L
\end{array}
\end{array} - \begin{array}{c}
\begin{array}{c}
\text{L}_R \\
\text{R}_L \\
\text{R}_R \\
\text{L}_L
\end{array}
\end{array} = 0
\tag{3}
\]

Working out the Leibniz rule in (4), we collect the graphs according to the number of derivatives falling on each of \((f,g,h)\). The edge \( \mathcal{P} \) provides the differential orders\(^{14}\) \((3,1,1)\), \((2,2,1)\), \((2,1,2)\), and \((2,1,1)\) twice. Likewise, we see \((1,1,1)\) in (2) and \((2,2,1)\) in (3).

**Lemma.** A tri-differential operator \( \sum_{|\alpha|=k}\partial^\alpha I_{\alpha} \otimes \partial^\alpha J \otimes \partial^\alpha K \) vanishes identically iff all its homogeneous components vanish: \( c^{IJK} = 0 \) for every triple \((I,J,K)\) of multi-indices; here \( \partial^\alpha L = \partial^\alpha_1 \circ \cdots \partial^\alpha_n \) for a multi-index \( L = (\alpha_1,\ldots,\alpha_n) \). Moreover, the sums \( \sum_{|\alpha|=k}\partial^\alpha I_{\alpha} \otimes \partial^\alpha J \otimes \partial^\alpha K \) are then zero for all \((i,j,k)\); in a vanishing sum \( X \) of graphs, we denote by \( X_{ijk} \) its vanishing restriction\(^{15}\) to a fixed differential order \((i,j,k)\).

The Poisson bi-vector components \( \mathcal{P}^{ij} \) can also serve as arguments of the Jacobiator:\(^{16}\)

\[
I_{j} := \partial^i \left( \text{Jac}_\mathcal{P}(\mathcal{P}^{ij}, g, h) \right) \partial^j f
\]

\[
\partial^i \left( \begin{array}{c}
\begin{array}{c}
\text{L}_R \\
\text{R}_L \\
\text{R}_R \\
\text{L}_L
\end{array}
\end{array} - \begin{array}{c}
\begin{array}{c}
\text{L}_R \\
\text{R}_L \\
\text{R}_R \\
\text{L}_L
\end{array}
\end{array} - \begin{array}{c}
\begin{array}{c}
\text{L}_R \\
\text{R}_L \\
\text{R}_R \\
\text{L}_L
\end{array}
\end{array} \right) = 0.
\]

\(^{11}\)By default, the \( L \prec R \) edge ordering equals the left \( \prec \) right direction in which edges start on these pages.

\(^{12}\)Within the variational geometry of Poisson field models (cf. [2]), a tiny leak of the associativity for \( \star \) may occur, if it does at all, only at orders \( \hbar^3 \) because at most one arrow falls on \( \text{Assoc} (f,g,h) = \sigma(\hbar^3) \). But unlike the always vanishing first variation of the homologically trivial functional \( \text{Jac} (\star, \cdot, \cdot) \equiv 0 \), its higher-order variations can be nonzero.

\(^{13}\)We use the Einstein summation convention; a sum over all indices is also implicit in the graph notation.

\(^{14}\)In fact, the double edge to \( f \) contributes with zero at \((3,1,1)\) due to the skew-symmetry \( \mathcal{P}^{ij} = -\mathcal{P}^{ji} \).

\(^{15}\)For example, relation (3) is the consequence of (4) at order \((2,2,1)\); restriction of (4) to \((2,1,1)\) yields

\[
0 = \begin{array}{c}
\begin{array}{c}
\text{L}_R \\
\text{R}_L \\
\text{R}_R \\
\text{L}_L
\end{array}
\end{array} - \begin{array}{c}
\begin{array}{c}
\text{L}_R \\
\text{R}_L \\
\text{R}_R \\
\text{L}_L
\end{array}
\end{array} - \begin{array}{c}
\begin{array}{c}
\text{L}_R \\
\text{R}_L \\
\text{R}_R \\
\text{L}_L
\end{array}
\end{array}
\tag{4}
\]

Similarly, we have \( S_{g} := \mathcal{P}^{ij}\partial^i \text{Jac}_\mathcal{P}(f,\partial^i g, h) = 0 \) and \( S_{h} := \mathcal{P}^{ij}\partial^i \text{Jac}_\mathcal{P}(f,g,\partial^i h) = 0 \).

\(^{16}\)The three tadpoles produce \( \text{Jac}_\mathcal{P}(\partial^i \mathcal{P}^{ij}, g, h) \partial^i f = 0 \), which plays its rôle in \( A_{111} \) (see the claim below).
Likewise, \( I_g := \partial_i (\text{Jac}_g(f, \mathcal{P}^g, h)) \partial_j g = 0 \) and \( I_h := \partial_i (\text{Jac}_h(f, g, \mathcal{P}^g)) \partial_j h = 0 \). It is the expansion of \( I_f, I_g, I_h \) via the Leibniz rule that produces the graphs with “eyes”. It also yields an order \((1,1,1)\) differential operator on \((f, g, h)\) which cannot be obtained from (4).

Claim. The sum \( A \) of 39 terms at \( h^3 \) in \( \text{Assoc}(f, g, h) \) vanishes by virtue of restriction of \( S_f, S_g, S_h \) and \( I_f, I_g, I_h \) to the orders \((i, j, k)\) that are present in \( A \).

Indeed, we have\(^{17} \) \( A_{221}^{[3]} = \frac{2}{3} (S_f)_{221} \), \( A_{222}^{[3]} = \frac{2}{3} (S_g)_{222} \), and \( A_{212}^{[3]} = \frac{2}{3} (S_h)_{212} \), see (3). Finally, we deduce that

\[
A_{111}^{[8]} = \frac{1}{6} (I_f - I_h)_{111}, \quad A_{112}^{[9]} = \left( \frac{1}{6} I_f + \frac{1}{6} I_g - \frac{1}{3} S_h \right)_{112}, \\
A_{212}^{[4]} = \frac{1}{3} (I_f - I_h)_{212}, \quad A_{211}^{[9]} = \left( \frac{1}{3} I_f - \frac{1}{6} I_g - \frac{1}{6} S_h \right)_{211}.
\]

The total number of terms which we thus eliminate equals \((3 + 3 + 3) + 8 + 9 + 4 + 9 = 39\).

Remark 1. The deformation quantization is a gauge theory: each argument \(\bullet\) of \(\star\) marks its gauge class \([\bullet]\) under the linear maps

\[
t: \bullet \mapsto [\bullet] = \bullet + h^3 \left( I^{(0)} \partial_i \partial_j \left( \mathcal{P}^g \right)^{\bullet} + I^{(1)} \partial_i \partial_j \partial_\bullet \right) + h^4 (0)
\]

where the constants \( I^{(n)} \in \mathbb{R} \) can be arbitrary\(^{18} \) and \( h \) is formally invertible over \( \mathbb{R} \). In turn, the star–products are gauged\(^{19} \) by using \( t: f \star g := t^\dagger (t(f) \star t(g)) \). This degree of freedom extends the uniqueness problem for Kontsevich’s solution \( \bullet \) of \(\text{Assoc}(f, g, h) = 0 \). Namely, not the exact balance of power series but an equivalence \([\bullet] = \) of gauge classes (up to unrelated transformations at all steps) can be sought in \([f] \star [g] \star [h] = [f] \star [g] \star [h] \). \([f] \star [g] \star [h] \).

Remark 2. Each graph \( \Gamma \) in (1) encodes the polydifferential operator of scalar arguments in a coordinate–free way. The Jacobians \( \partial u / \partial \tilde{u} \) of affine mappings appear on the edges but then they join the content \( \mathcal{P}^g \) of internal vertices at the arrowtails,\(^{20} \) forming \( \mathcal{P}^{\alpha \beta} \) from \( \mathcal{P}^g \). Independent from \( u \in \mathbb{R}^n \), these Jacobians stay invisible to all in-coming arrows (if any). So, the operator given by a graph \( \Gamma \) with \( \tilde{P}(u) \) in its vertices is equal to the one for \( \tilde{P}(\tilde{u}(u)) \) there. This reasoning works for the variational Poisson brackets \( \{ , \}_\phi \) on \( F^n(\pi) \) for affine bundles \( \pi \) with fibre \( N^n \) over points \( x \in M^m \), see [2]. The graphs \( \Gamma \) then yield local variational polydifferential operators yet the pictorial language of [1] is the same\(^{21} \).

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\(^{17}\)By using the symbol \([m] \) we indicate the number \( m \) of terms that are eliminated at each step.

\(^{18}\)The view [3] on -products as -expansions of path integrals shows that the graphs \( \Gamma \) in (1) are genuine Feynman diagrams for the channel marked by \( \mathcal{P} \). The weights \( w(\Gamma) \) integrate over the energy of each intermediate vertex. Quite naturally, a particle \( \bullet \) shares its energy–mass with the interaction carriers \( \mathcal{P} \) as it gets coated by them. But no object \( \bullet \) can spend more energy on growing its gauge tail than the amount it actually has; hence every set \([\bullet]\) is bounded in the space of parameters \( I \).

\(^{19}\)For example, the loop graph at \( h^2/6 \) in (1) is gauged out by \( t(\bullet) = \bullet + h^2 \left( \frac{1}{2} \mathcal{P}^{\alpha \beta} \right) \), see [1] for further details.

\(^{20}\)E.g., \( \partial_\alpha \mathcal{P}^{\alpha \beta} \partial_\beta = \partial_i \times \frac{\partial u^i}{\partial \alpha} \mathcal{P}^{\alpha \beta} \bigg|_{u(x)} \partial_\beta \frac{\partial u^i}{\partial \beta} \partial_j = \partial_j \mathcal{P}^{\alpha \beta} \bigg|_{u(x)} \partial_\beta \) so that \( (f, g) \mathcal{P}(u) = (f, g) \mathcal{P}(u) \).

\(^{21}\)A sought–for extension of the Ikeda–Izawa topological open string geometry—namely, its lift from the affine Poisson manifolds \( (N^n, [\bullet], [\bullet]) \) in [3, 4] to the variational set-up \( (F^n(\pi), [\bullet], [\bullet]) \) of jet spaces in [2] is a mechanism to quantize Poisson field models. This will be the object of another paper.
REFERENCES

1. M. Kontsevich, “Deformation quantization of Poisson manifolds. I,” Lett. Math. Phys. 66, 157–216 (2003), arXiv:q-alg/9709040.

2. A. V. Kiselev, “Deformation approach to quantisation of field models,” Preprint IHÉS/M/15/13 (IHÉS, Bures-sur-Yvette, 2015), pp. 1–37.

3. A. S. Cattaneo and G. Felder, “A path integral approach to the Kontsevich quantization formula,” Comm. Math. Phys. 212, 591–611 (2000), arXiv:q-alg/9902090.

4. N. Ikeda, “Two-dimensional gravity and nonlinear gauge theory,” Ann. Phys. (N.Y.) 235, 435–464 (1994), arXiv:hep-th/9312059.