1. Introduction

Zahorski [14] and Choquet [1] (see also Tolstov [12]) proved a result characterizing curves \((f : [a, b] \to \mathbb{R}^n)\) that allow a differentiable parametrization (resp. a differentiable parametrization with almost everywhere non-zero derivative) as those curves having the \(V^*\) property (resp. which are also not constant on any interval). Fleissner and Foran [7] reproved this later (for real functions only and not considering the case of a.e. nonzero derivatives) using a different result of Tolstov. The definition of \(V^*\) is classical; see e.g. [11]. The mentioned results were generalized by L. Zajíček and the author [4] to curves with values in Banach spaces (and also metric spaces using the metric derivative instead of the usual one). Laczkovich, Preiss [9], and Lebedev [10] studied (among other things) the case of \(C^n\)-parametrizations of real-valued functions \((n \geq 2)\). For a nice survey of differentiability of real-valued functions via homeomorphisms, see [8]. L. Zajíček and the author [5] characterized the situation when a Banach space-valued curve admits a \(C^2\)-parametrization (for Banach spaces with a \(C^1\) norm) or a parametrization with finite convexity (for arbitrary Banach spaces).

Let \(X\) be a normed linear space, and \(f : [a, b] \to X\). We say that \(f\) is Lebesgue equivalent to \(g : [a, b] \to X\) provided there exists a homeomorphism \(h\) of \([a, b]\) onto itself such that \(g = f \circ h\). In the present note, we prove the following two theorems characterizing the situation when a vector-valued path allows a twice differentiable parametrization (resp. such a parametrization with almost everywhere non-zero derivative):

**Theorem 1.** Let \(X\) be a normed linear space, and \(f : [a, b] \to X\) be continuous. Then the following are equivalent.

(i) \(f\) is Lebesgue equivalent to a twice differentiable function \(g\).

(ii) \(f\) is Lebesgue equivalent to a differentiable function \(g\) whose derivative is pointwise Lipschitz.

(iii) \(f\) is \(V^*\).

**Theorem 2.** Let \(X\) be a normed linear space, and \(f : [a, b] \to X\) be continuous. Then the following are equivalent.

(i) \(f\) is Lebesgue equivalent to a twice differentiable function \(g\) with \(g'(x) \neq 0\) for a.e. \(x \in [a, b]\).

(ii) \(f\) is Lebesgue equivalent to a differentiable function \(g\) whose derivative is a pointwise Lipschitz function which is non-zero a.e. in \([a, b]\).

(iii) \(f\) is \(V^*\), and \(f\) is not constant in any interval.

As a matter of fact, a definition of a new notion of a \(V^*\) function (see Definition 5 below) involving a certain fractional variation, that was inspired by the results of Laczkovich, Preiss, and Lebedev, is necessary to achieve our goal.

The case of \(n\)-times differentiable functions for \(n \geq 3\) is more complicated even in the case \(X = \mathbb{R}\), and this case is treated in a separate paper [3] (where we also prove a version of Zahorski lemma for \(n\)-times differentiable homeomorphisms). The difficulty in the case of higher order derivatives of paths stems from the fact that although for a curve parametrized by the arc-length, the first derivative (provided it exists) is equal to the tangent (and thus has norm 1), the magnitude of higher-order derivatives is not thus simply bounded. The proof in the real-valued case of \(n \geq 3\) uses some auxiliary variations and proceeds in a rather indirect way. This is
a similar phenomenon as the case of $C^1$ parametrizations being different from the case of $C^n$ ($n > 1$) parametrizations; see e.g. [9, p. 405] (since, in some sense, the $C^1$ case corresponds to twice-differentiable function case).

2. Preliminaries

By $\lambda$ we will denote the Lebesgue measure on $\mathbb{R}$. By $X$, we will always denote a normed linear space, and by $B(x, r)$ an open ball with center $x$ and radius $r > 0$. If $X$ is separable, then it is well known that $X$ admits an equivalent Gâteaux differentiable norm (see e.g. [2]). For $f : [a, b] \to X$ we define the derivative $f'$ as usual (at the endpoints, we take the corresponding unilateral derivatives). Similarly, the second derivative $f''(x)$ of $f$ at $x$ is defined as $f''(x) := (f')(x)$. Note that the property of “being twice differentiable” is preserved under equivalent renormings of $X$.

We say that $f$ is pointwise-Lipschitz at $x \in [a, b]$ provided $\lim_{t \to 0} \frac{\|f(x+t) - f(x)\|}{|t|}$ is finite. We say that $f$ is pointwise-Lipschitz provided $f$ is pointwise-Lipschitz at each $x \in [a, b]$.

Let $f : [a, b] \to X$ be continuous, and assume that $X$ has a Gâteaux differentiable norm (there is no loss of generality in this assumption since the continuity of $f$ implies that $\text{span}(f([a, b]))$ is separable). By $K_f$ we will denote the set of points $x \in [a, b]$ such that there is no open interval $U$ containing $x$ such that $f|_U$ is either constant or admits an arc-length parametrization which is twice differentiable.

In the case of $X = \mathbb{R}$, the set $K_f$ coincides with the set of points of varying monotonicity of $f$ (see e.g. [9]). Obviously, $K_f$ is closed and $\{a, b\} \subset K_f$. We easily see that $K_f$ does not depend on the choice of the (equivalent) Gâteaux smooth norm on $X$. It is easy to see that if $f : [a, b] \to X$ is twice differentiable (and $X$ has a Gâteaux differentiable norm), then $f'(x) = 0$ for each $x \in K_f$ by the chain rule for derivatives and by the continuity of $f'$.

Let $K \subset [a, b]$ be a closed set with $a, b \in K$. We say that an interval $(c, d) \subset [a, b]$ is contiguous to $K$ in $[a, b]$ provided $c, d \in K$ and $(c, d) \cap K = \emptyset$ (i.e. it is a maximal open component of $[a, b] \setminus K$ in $[a, b]$).

By $V(f, [a, x])$ we denote the (usual) variation of $f$ on $[a, x]$. We will sometimes use the notation $v_f(x) := V(f, [a, x])$ for $x \in [a, b]$. We say that $\{y_i\}_{i=0}^N$ is a partition of $[a, b]$ provided $a = y_0 < y_1 < \cdots < y_N = b$.

We shall need the following lemma. For a proof, see e.g. [4, Lemma 2.7].

Lemma 3. Let $\{a, b\} \subset B \subset [a, b]$ be closed, and $f : [a, b] \to \mathbb{R}$ be continuous. If $\lambda(f(B)) = 0$, then we have $V(f, [a, b]) = \sum_{i \in \mathcal{I}} V(f_{|I_i}, [c_i, d_i])$, where $I_i = (c_i, d_i)$, $i \in \mathcal{I} \subset \mathbb{N}$ are all intervals contiguous to $B$ in $[a, b]$.

As in [9], for $g : [a, b] \to \mathbb{R}$, $\alpha \in (0, 1)$, and $K \subset [a, b]$, we will define $V_\alpha(g, K)$ as a supremum of sums

$$\sum_{i=1}^m |g(b_i) - g(a_i)|^\alpha,$$

where the supremum is taken over all collections $\{[a_i, b_i]\}_{i=1}^m$ of non-overlapping intervals in $[a, b]$ with $a_i, b_i \in K$ for $i = 1, \ldots, m$.

We will need the following auxiliary lemma:

Lemma 4. Let $\alpha \in (0, 1)$, $A \subset \mathbb{R}$ be bounded, $f : A \to \mathbb{R}$ be uniformly continuous with $V_\alpha(f, A) < \infty$. Then $\lambda(f(A)) = 0$. 
Proof. By [9, Theorem 2.10] it follows that $SV_\alpha(f, A) = 0$ (see [9] for the definition of $SV_\alpha$). It is easy to see that $SV_\alpha(f, A) = 0$ implies $SV_1(f, A) = 0$, and thus [9, Theorem 2.9] shows that $\lambda(f(A)) = 0$. \hfill \square

We will need the following notion which plays the rôle of $VBG_*$ for the second order differentiability.

**Definition 5.** We say that a continuous $f : [a, b] \to X$ is $VBG_\chi$ provided $f$ has bounded variation, and there exist closed sets $A_m \subset [a, b]$ ($m \in M \subset \mathbb{N}$) such that $K_f = \bigcup_{m \in M} A_m$, and $V_2(f, A_m) < \infty$ for each $m \in M$.

It is easy to see that if $f$ is $VBG_\chi$ and $g$ is Lebesgue equivalent to $f$, then $g$ is $VBG_\chi$. Also, it is easily seen that the class of $VBG_\chi$ functions does not depend on the equivalent norm of $X$.

The following example shows that we cannot equivalently replace $v_f$ by $f$ in Definition 5 (even in the case $X = \mathbb{R}$).

**Example 6.** There exists a continuous function $f : [0, 1] \to \mathbb{R}$ with bounded variation such that $f$ is not $VBG_\chi$, but there exist closed $A_m \subset K_f$ such that $K_f = \bigcup A_m$ and $V_2(f, A_m) < \infty$.

*Proof.* Let $C \subset [0, 1]$ be the standard middle-thirds Cantor set. By $\mathcal{I}_n$ we will denote the collection of all intervals contiguous to $C$ such that $\lambda(I) < 3^{-n}$ for $I \in \mathcal{I}_n$, and by $K^n_i$, where $i = 1, \ldots, 2^n$, $n \in \mathbb{N}$, denote the closed intervals at level $n + 1$ of the construction. It is easy to see that there exist open intervals $I_{nik} \subset [0, 1]$ and numbers $a_{nik} > 0$, where $n, k \in \mathbb{N}$ and $i = 1, \ldots, 2^n$, such that

(i) $I_{nik} \cap I_{n'i'k'} = \emptyset$ whenever $(n, i, k) \neq (n', i', k')$,

(ii) $\sum_{n,k\in\mathbb{N}} \sum_{i=1}^{2^n} a_{nik} < \infty$,

(iii) $\sum_{k\in\mathbb{N}} \sqrt{a_{nik}} = \infty$ whenever $n \in \mathbb{N}$ and $i = 1, \ldots, 2^n$,

(iv) $\text{card} \{ (n, i, k) : I_{nik} \subset I \} < \infty$ for all $m \in \mathbb{N}$ and $I \in \mathcal{I}_m$,

(v) if $k \neq k'$, then there exists $x \in C$ such that either $I_{nik} < x < I_{nik'}$ or $I_{nik'} < x < I_{nik}$,

(vi) $I_{nik} \subset (K^n_i \cap \bigcup \mathcal{I}_n)$ for all $i = 1, \ldots, 2^n$, $n, k \in \mathbb{N}$.

Let $I = (a, b) \subset [0, 1]$ be an open interval. We denote $l(I) = a$, $r(I) = b$, and $c(I) = \frac{a+b}{2}$. We will define $f(x) := 0$ whenever $x \not\in [0, 1] \setminus (\bigcup_{n,k\in\mathbb{N}} \bigcup_{i=1}^{2^n} I_{nik})$. For $I \in \mathcal{I}_n$, we will define $f(I) := a_{nik}$, and $f$ to be continuous and affine on $[l(I_{nik}), r(I_{nik})]$ and $[c(I_{nik}), r(I_{nik})]$.

Then $f$ is a continuous function and by (ii) it is easy to see that $V(f, [0, 1]) < \infty$. Index the countable family of closed sets

$$\{C\} \cup \{l(I_{nik}), c(I_{nik}), r(I_{nik})\} : n, k \in \mathbb{N}, i = 1, \ldots, 2^n\}
$$

as $(A_m)_{m \in \mathbb{N}}$. It is easy to see that $K_f = \bigcup_{m \in \mathbb{N}} A_m$ and $V_2(f, A_m) < \infty$ for all $m \in \mathbb{N}$ (since $f|_C \equiv 0$, and all those $A_m$ that satisfy $A_m \neq C$ are finite).

Now we will show that $f$ is not $VBG_\chi$. Suppose that $\mathcal{A}_l$ satisfy $V_2(\mathcal{A}_l, A_m) < \infty$, and $K_f = \bigcup_{m \in \mathbb{N}} A_m$. Since $C = \bigcup (C \cap A_m)$, by the Baire category theorem, there exists $m_0$ and an open interval $U$ such that $C \cap U \subset C \cap \mathcal{A}_{m_0} \cap U$ and $C \cap U \neq \emptyset$. Thus, there exists $n \in \mathbb{N}$ and $i \in \{1, \ldots, 2^n\}$ such that $K^n_i \subset U$, and conditions (v),(vi) imply that

$$V_2(f, \mathcal{A}_{m_0}) \geq \sum_{I \in \mathcal{I}_n} (V(v_f, I))^* \geq \sum_k \sqrt{a_{nik}} = \infty,$$
which contradicts the choice of the sets $\tilde{A}_m$. Thus, $f$ is not $VBG_\frac{3}{2}$.

3. LEMMATA

The following lemma is a sufficient condition for a function to be $VBG_\frac{3}{2}$.

**Lemma 7.** Let $f : [a, b] \to X$ have a pointwise-Lipschitz derivative. Then $f$ is $VBG_\frac{3}{2}$.

**Proof.** Because $f'$ is continuous on $[a, b]$ (and thus bounded), we see that $f$ is Lipschitz (and thus has finite variation). For $j \in \mathbb{N}$ define

$$D_j = \{ x \in [a, b] : \| f'(x) - f'(z) \| \leq j|x - z| \text{ for all } z \in B(x, 1/j) \}.$$  

It is easy to see that $[a, b] = \bigcup_j D_j$, and $D_j$ is closed. Let $D_j = \bigcup_{k \in \mathbb{N}} D_{jk}$ be such that each $D_{jk}$ is closed, and $\operatorname{diam}(D_{jk}) < 1/j$. We order the doubly-indexed sequence $(K_f \cap D_{jk})_{j,k}$ into a single sequence (while omitting empty sets); we will call the new sequence $A_m$ ($m \in \mathcal{M} \subset \mathbb{N}$).

It remains to show that $V_\frac{3}{2}(v_f, A_m) < \infty$, where $m \in \mathcal{M}$. Let $m \in \mathcal{M}$, and fix $j, k \in \mathbb{N}$ such that $A_m = D_{jk} \cap K_f$. Let $x < y$ be such that $x, y \in A_m$. Note that

$$|v_f(y) - v_f(x)| \leq \int_x^y \| f'(s) \| \, ds \leq j(y - x)^2. \tag{3.1}$$

Applying (3.1) to $[x, y] = [a_i, b_i]$, $i \in \{1, \ldots, N\}$, where $[a_i, b_i]$ are non-overlapping intervals with $a_i, b_i \in A_m$, we obtain

$$\sum_{i=1}^N |v_f(b_i) - v_f(a_i)|^{\frac{3}{2}} \leq \sqrt{j} \sum_{i=1}^N (b_i - a_i) \leq \sqrt{j}(b - a). \tag{3.2}$$

By taking a supremum over all sequences $\{[a_i, b_i]\}_{i=1}^N$ as above, we obtain that $V_\frac{3}{2}(v_f, A_m) < \infty$. \qed

**Lemma 8.** Let $\zeta : [\sigma, \tau] \to \mathbb{R}$ be a continuous strictly increasing Lipschitz function with $\zeta(0) = 0$, and $\lambda(F) = 0$ for some closed $F \subset [\sigma, \tau]$ with $\sigma, \tau \in F$. Then $\lambda(\sqrt{\zeta}(F)) = 0$, where $\sqrt{\zeta}(x) := \sqrt{\zeta(x)}$ for $x \in [\sigma, \tau]$.

**Proof.** Since the function $g(x) = \sqrt{x}$ on $[0, \infty)$ has property (N) (i.e. it maps zero sets onto zero sets), the conclusion easily follows. \qed

We will need the following simple lemma.

**Lemma 9.** Let $h_m : [a, b] \to [c_m, d_m]$ ($m \in \mathcal{M} \subset \mathbb{N}$) be continuous increasing functions such that $\sum_{m \in \mathcal{M}} h_m(x) < \infty$ for all $x \in [a, b]$. Let $K \subset [a, b]$ be closed and such that $\lambda(h_m(K)) = 0$ for all $m \in \mathcal{M}$. Then $h : [a, b] \to [c, d]$, defined as $h(x) := \sum_{m \in \mathcal{M}} h_m(x)$, is a continuous and increasing function (for some $c, d \in \mathbb{R}$) such that $\lambda(h(K)) = 0$.

**Proof.** The continuity and monotonicity of $h$ follows easily by the assumptions. Let $K \subset [a, b]$ be closed with $\lambda(h_m(K)) = 0$ for all $m \in \mathcal{M}$. Without any loss of generality, we can assume that $[a, b] \subset K$. Let $(c_p, d_p)$ ($p \in \mathcal{P} \subset \mathbb{N}$) be all
the intervals contiguous to $K$ in $[a,b]$. Let $\varepsilon > 0$ and find $M \in \mathbb{N}$ such that
\[ \sum_{m \in M \cap (M, \infty)} (h_m(b) - h_m(a)) < \varepsilon. \]
Then
\[ \lambda(h([a,b])) = \sum_{m \in M} (h_m(b) - h_m(a)) \leq \varepsilon + \sum_{m \in M \cap [1,M]} (h_m(b) - h_m(a)) = \varepsilon + \sum_{m \in M \cap [1,M]} \lambda(h_m(b) - h_m(a)) \]
where we used Lemma 3 to obtain the second equality. Since $\text{card} \left( h((c_p, d_p)) \cap h((c_q, d_q)) \right) \leq 1$ for $p, q \in \mathcal{P}$, $p \neq q$, we obtain the equality
\[ \lambda(h([a,b])) = \lambda(h(\bigcup_{p \in \mathcal{P}} (c_p, d_p))). \]
Since the set $h(K) \cap h(\bigcup_{p \in \mathcal{P}} (c_p, d_p))$ is countable, we get $\lambda(h(K)) = 0$. \hfill \blackqed

**Lemma 10.** Suppose that $X$ is a normed linear space with a Gâteaux smooth norm. Let $f : [a,b] \to X$ be a continuous $VBG^{1/2}$ function which is not constant on any interval. Then there exists a continuous strictly increasing $v : [a,b] \to [\alpha, \beta]$ such that $\lambda(v(K_f)) = 0$, $f \circ v^{-1}$ is twice differentiable on $[\alpha, \beta] \setminus v(K_f)$ with $(f \circ v^{-1})'(x) \neq 0$ for $x \in [\alpha, \beta] \setminus v(K_f)$, and for each $x \in K_f$ there exists $0 < C_x < \infty$ such that
\[ \|f(y) - f(z)\| \leq C_x|v(z) - v(y)|(|v(z) - v(x)| + |v(y) - v(x)|), \]
whenever $y, z \in [a, b]$, and $\text{sgn}(y-x) = \text{sgn}(z-x)$.

**Proof.** Let $A_m$ ($m \in \mathcal{M} \subset \mathbb{N}$) be as in the definition of $VBG^{1/2}_\mathcal{L}$ for $g = f \circ v^{-1}$. Note that $g$ is 1-Lipschitz, and $K_g = v_f(K_f)$. Since $f$ is $VBG^{1/2}_\mathcal{L}$, by Lemma 4 we have $\lambda(v_f(K_f)) = \lambda(v_g(K_g)) = 0$. Let $\ell = v_f(b)$. Note that because $g$ is an arc-length parametrization of $f$, we have $V(g, [c, d]) = d - c$ for all $0 \leq c < d \leq \ell$ (we will use this fact frequently without necessarily repeating it). Let $(c_p, d_p) (p \in \mathcal{P} \subset \mathbb{N})$ be all the intervals contiguous to $K_g$ in $[0, \ell]$. Since $\lambda(v_g(K_g)) = 0$, by Lemma 3 (applied to $f = v_g$) we have $V(g, [0, \ell]) = \ell = \sum_{p \in \mathcal{P}} V(g, [c_p, d_p]) = \sum_{p \in \mathcal{P}} (d_p - c_p)$, and thus $\lambda(K_g) = \ell - \lambda(\bigcup_{p \in \mathcal{P}} (c_p, d_p)) = 0$. For $m \in \mathcal{M}$ and $x \in [0, \ell]$, we define $v_m(x)$ as a supremum of the sums
\[ \sum_{i=1}^N (b_i - a_i)^{1/2}, \]
where the supremum is taken over all finite sequences $\{(a_i, b_i)\}_{i=1}^N$ of non-overlapping intervals in $[0, \ell]$ such that $a_i, b_i \in (A_m \cup \{0\}) \cap [0, x]$ for $i = 1, \ldots, N$. Similarly, we define $\hat{v}_m(x)$ for $x \in [0, \ell]$ as a supremum of the sums in (3.4), where the supremum is taken over all finite sequences $\{(a_i, b_i)\}_{i=1}^N$ of non-overlapping intervals in $[0, \ell]$ such that $a_i, b_i \in (A_m \cup \{x, \ell\}) \cap [x, \ell]$ for $i = 1, \ldots, N$. Note that $v_m$ is increasing and $\hat{v}_m$ is decreasing on $[0, \ell]$. Note that $g$ is affine on each $[c_p, d_p]$, and
\[ v_m(x) = v_m(z) + (x - z)^{1/2} \quad \text{for} \quad x \in [c_p, d_p], \]
where $z = \max(A_m) \in \{0\} \cap [0, c_p]$, and similarly for $\hat{v}_m$. Thus $v_m$ (and similarly $\hat{v}_m$) is twice (or even infinitely many times) differentiable on $[0, \ell] \setminus v_f(K_f)$ with $v'_m(x) > 0$ for all $x \in [0, \ell] \setminus v_f(K_f)$. Find $\varepsilon_m > 0$ such that
(a) if we define $w(x) := \sum_m \varepsilon_m \cdot (v_m(x) - \tilde{v}_m(x))$, then $w(0)$, and $w(\ell)$ are finite (and thus $w(x)$ is finite for all $x \in [0, \ell]$), and $w$ is continuous on $[0, \ell]$ (provided all $v_m, \tilde{v}_m$ were continuous).

(b) for all $m \in M$ and $p \in P$ with $c_p + 1/m < d_p - 1/m$ and all $x \in (c_p + 1/m, d_p - 1/m)$, we have $\varepsilon_m \max(|v_m''(x)|, -\tilde{v}'_m(x), |\tilde{v}_m''(x)|) < 2^{-m}$.

By (b), it is easy to see that $w'(x)$ exists, is positive, and $w''(x)$ exists for each $x \in [0, \ell] \setminus v_f(K_f)$. Put $v := w \circ v_f$, $\alpha = v(a)$, and $\beta = v(b)$.

To show that $v$ is strictly increasing, it is enough to show that $w$ is strictly increasing (as $v_f$ is strictly increasing by the fact that $f$ is not constant on any interval). On the other hand, to show that $w$ is strictly increasing, it is enough to show that $v_m$ is strictly increasing for each $m \in M$. Fix $m \in M$. Let $x, y \in [0, \ell]$ with $x < y$. If $x, y \in [c_p, d_p]$ for some $p \in P$, then (3.5) implies that $v_m(x) < v_m(y)$, and similarly if $x \in (c_p, d_p)$ or $y \in (c_p, d_p)$ for some $p \in P$ (resp. $p' \in P$). If $x, y \in K_f$, and $(x, y) \cap A_m = \emptyset$, then

$$v_m(t) = v_m(z) + \sqrt{t - z} \quad \text{for all } t \in [x, y],$$

where $z = \max((A_m \cup \{0\}) \cap [0, x])$, and thus $v_m(x) < v_m(y)$. Finally, if there exists $q \in A_m \cap (x, y)$, then $v_m(x) \leq v_m(q) < v_m(y)$, and thus $v_m(x) < v_m(y)$ also in this case. By a similar argument, $\tilde{v}_m$ is strictly decreasing.

For a fixed $m \in M$, we will prove that whenever $r, s \in A_m \cup \{0, \ell\}$ with $r < s$, then

$$v_m(s) - v_m(r) \leq \sum_{p \in P: (c_p, d_p) \cap [r, s] \neq \emptyset} (v_m(d_p) - v_m(c_p)).$$

A symmetrical argument then shows that

$$\tilde{v}_m(r) - \tilde{v}_m(s) \leq \sum_{p \in P: (c_p, d_p) \cap [r, s] \neq \emptyset} (\tilde{v}_m(c_p) - \tilde{v}_m(d_p)).$$

To prove (3.7), fix $\varepsilon_0 > 0$, and let $\{[a_i, b_i]\}_{i=1}^N$ be non-overlapping intervals in $[r, s]$ such that $a_i, b_i \in (A_m \cup \{r, s\}) \cap [r, s]$ for $i = 1, \ldots, N$ such that $v_m(s) = v_m(r) + \sum_{i=0}^{N-1} (b_i - a_i)^2 + \varepsilon$, for some $0 \leq \varepsilon < \varepsilon_0/2$. For $i \in \{1, \ldots, N\}$ by Lemma 3 applied to $f = g$ on $[a, b] = [a_i, b_i]$ and $B = (A_m \cup \{r, s\}) \cap [a_i, b_i]$ (note that $\lambda(g(A_m)) = 0$ since $\lambda(g(K_g)) = 0$, and thus $\lambda(g(B)) = 0$), let $(\gamma^i_1, \delta^i_2)$ ($j \in \{1, \ldots, J^i\}$) be a finite collection of intervals contiguous to $A_m \cup \{r, s\}$ in $[a_i, b_i]$ such that $(b_i - a_i) \leq \sum_{j=1}^{J^i} (\delta^i_2 - \gamma^i_1) + \left(\frac{\varepsilon_0}{4N}\right)^2$. Then

$$v_m(s) - v_m(r) \leq \sum_{i=1}^N \sum_{j=1}^{J^i} (\delta^i_2 - \gamma^i_1)^2 + \frac{\varepsilon_0}{2} + \varepsilon.$$

By Lemma 8 applied to $\zeta(x) = x - \gamma^i_1$ on $[\sigma, \tau] = [\gamma^i_1, \delta^i_2]$, $F = K_g \cap [\gamma^i_1, \delta^i_2]$, and because $v_m(x) = v_m(\gamma^i_1) + (x - \gamma^i_1)^{\frac{1}{2}}$ for $x \in [\gamma^i_1, \delta^i_2]$, we have that $\lambda(v_m(K_g \cap [\gamma^i_1, \delta^i_2])) = 0$, and by Lemma 3 applied to $f = v_m$ on $[a, b] = [\gamma^i_1, \delta^i_2]$, and $B = K_g \cap [\gamma^i_1, \delta^i_2]$, we obtain that $(\delta^i_2 - \gamma^i_1)^{\frac{1}{2}} \leq \sum_{p \in P: (c_p, d_p) \subset [\gamma^i_1, \delta^i_2]} (v_m(d_p) - v_m(c_p))$ for each $i \in \{1, \ldots, N\}$ and $j \in \{1, \ldots, J^i\}$. Combining this inequality with (3.9), we get
$v_m(s) - v_m(r) \leq \sum_{p \in \mathcal{P}} (v_m(d_p) - v_m(c_p)) + \varepsilon_0$, and by sending $\varepsilon_0 \to 0$ it follows that (3.7) holds.

To show that $v$ is continuous, it is enough to show that each $v_m$ is continuous (as this implies that $w$ is continuous by the choice of $\varepsilon_m$'s, and the continuity of $v_f$ follows from e.g. [6, §2.5.16]). Fix $m \in \mathcal{M}$. From (3.5), it follows that

$(*)$ $v_m$ is continuous from the right at all points $x \in \bigcup_{p \in \mathcal{P}} (c_p, d_p)$, and continuous from the left at all points $x \in \bigcup_{p \in \mathcal{P}} (c_p, d_p)$.

If $(x, y) \cap A_m = \emptyset$ for some $y > x$ with $y \in (0, \ell] \cap K_g$, then (3.6) implies that $v_m$ is continuous from the right at $x$. If $x \in A_m$ is a right-hand-side accumulation point of $A_m$ (i.e. $A_m \cap (x, x + \delta) \neq \emptyset$ for all $\delta > 0$), then (3.7) implies that $\lim_{y \to x+} v_m(y) = v_m(x)$, since

\[
(3.10) \quad \sum_{p \in \mathcal{P}} (v_m(d_p) - v_m(c_p)) \to 0
\]
as $y \to x+$. Now the monotonicity of $v_m$ implies that it is continuous from the right at $x$. Concerning the continuity from the left, by (*) it is enough to prove that $v_m$ is continuous from the left at all points $y \in (K_g \cap (0, \ell]) \setminus \bigcup_{p \in \mathcal{P}} (c_p, d_p)$. Fix such a point $y$. If there is an $x \in (0, y]$ such that $(x, y) \cap A_m = \emptyset$, then (3.6) implies that $v_m$ is continuous from the left at $y$. If $y$ is a left-hand-side accumulation points of $A_m$, then (3.7) together with (3.10) imply that $v_m$ is continuous from the left at $y$. A similar argument as above yields the continuity of $\tilde{v}_m$.

Now we will prove that $\lambda(v(K_f)) = 0$. Note that we already established that $\lambda(K_g) = 0$. Because $K_g = v_f(K_f)$, it is enough to prove that $\lambda(w(K_g)) = 0$. To apply Lemma 9 to $h'_s$, where $h_{2k} := h_k \cdot \varepsilon_k$, and $h_{2k+1} := -h_k \cdot \varepsilon_k$, we have to check that $\lambda(v_m(K_g)) = 0$ and $\lambda(\tilde{v}_m(K_g)) = 0$ for all $m \in \mathcal{M}$. Let $m \in \mathcal{M}$. Then (3.7) applied to $r = 0$, and $s = \ell$ shows that $v_m(0) \leq \sum_{p \in \mathcal{P}} (v_m(d_p) - v_m(c_p))$, and since $v_m(K_g) \cap v_m(\bigcup_{p \in \mathcal{P}} (c_p, d_p)) = \emptyset$, we get $\lambda(v_m(K_g)) = 0$. Similarly, we obtain $\lambda(\tilde{v}_m(K_g)) = 0$. Thus, Lemma 9 shows that $\lambda(w(K_g)) = 0$.

To prove that the second derivative of $f \circ v^{-1}$ exists and the first derivative is non-zero on $[\alpha, \beta] \setminus v(K_f)$, let $x \in [\alpha, \beta] \setminus v(K_f)$. Put $y = w^{-1}(x)$. There exists $p \in \mathcal{P}$ and $q \in \mathbb{N}$ such that $y \in (c_p, 1/q, d_p - 1/q)$. Since (by the chain rule and the smoothness of the norm on $X$) $g$ is twice differentiable on $(c_p, d_p)$ and $\|g'(x)\| = 1$ for all $x \in (c_p, d_p)$ (because $g$ is the arc-length parametrization of $f$ and $g'$ is continuous on $(c_p, d_p)$), it is enough to prove that $w'(y)$ exists, is non-zero, and $w''(y)$ exists (since then $(f \circ v^{-1})'(x) = g''(y) \cdot (w^{-1})''(x)$, and $(f \circ v^{-1})''(x) = g''(y) \cdot (w^{-1})''(x)$). But by the choice of $\varepsilon_m$ (for $m > q$), and by the properties of $v_m$, $\tilde{v}_m$ for all $m$, it is easy to see that $w'(y)$ exists, $w'(y) > 0$, and $w''(y)$ exists; the rest is a straightforward application of the “derivative of the inverse” rule.

To prove (3.3) for $f$ and $v$, by a substitution using $v_f$, it is easy to see that it is enough to establish a version of (3.3), where $f$ is replaced by $g$, and $v$ by $w$. To that end, take $m \in \mathcal{M}$ such that $x \in A_m$, and let $C_m = (\varepsilon_m)^{-2}$. Take $y, z \in [0, \ell]$. Without any loss of generality, we can assume that $x < y < z$ (if $y < x$, then a symmetric estimate using $\tilde{v}_m$ yields the conclusion). Let $0 < \varepsilon_0 < v_m(z) - v_m(x)$. Find a sequence $\{(a_i, b_i)\}_{i=1}^N$ of non-overlapping intervals with endpoints in $(A_m \cup \{x, y\}) \cap [x, y]$ with $b_i < a_{i+1}$ for $i = 1, \ldots, N - 1$, and such that $v_m(y) = \ldots,$
Lemma 11. Let $F \subset [\alpha, \beta]$ be closed, $\{\alpha, \beta\} \subset F$, and $\lambda(F) = 0$. Then there exists an (increasing) continuously differentiable homeomorphism $h$ of $[\alpha, \beta]$ onto itself such that $h'(x) = 0$ if and only if $x \in h^{-1}(F)$, $h$ is twice differentiable on $[\alpha, \beta] \setminus h^{-1}(F)$, and $h^{-1}$ is absolutely continuous.

Proof. Since we were not able to locate a reference in the literature for this exact statement, we will sketch the proof. Let $(a_i, b_i)$ (where $i \in \mathcal{I} \subset \mathbb{N}$) be all the intervals contiguous to $F$ in $[\alpha, \beta]$. For each $i \in \mathcal{I}$ find a $C^1$ function $\psi_i : (a_i, b_i) \to \mathbb{R}$ such that

- $\psi_i(x) \geq 0$ for all $x \in (a_i, b_i)$, and $\lim_{x \to a_i+} \psi_i(x) = \lim_{x \to b_i-} \psi_i(x) = \infty$,
- $m_i := \\min_{x \in (a_i, b_i)} \psi_i(x) > 0$, and if $|\mathcal{I}| = \aleph_0$, then $\lim_{i \to \infty} m_i = \infty$,
- $\sum_{i \in \mathcal{I}} \int_{a_i}^{b_i} \psi_i(t) \, dt < \infty$.

Such functions $\psi_i$ clearly exist. Define $\psi : [\alpha, \beta] \to \mathbb{R}$ as $\psi(x) := \psi_i(x)$ for $x \in (a_i, b_i)$, and $\psi(x) = 0$ for $x \in F$. It is easy to see that $\psi$ is integrable. Define $k(x) := \int_{a}^{x} \psi(t) \, dt$; then $k$ is continuous and (strictly) increasing. By integrability of $\psi$, it follows that $k$ has Luzin’s property $(N)$, and thus $k$ is absolutely continuous by the Banach-Zarecki theorem (see e.g. [13, Theorem 3]). It is easy to see that $k$ is twice differentiable on $[\alpha, \beta] \setminus F$ with $k''(x) > 0$. We also have that $k'(x) = \infty$ for $x \in F \setminus (\bigcup_{i \in \mathcal{I}} \{a_i\})$, as for $x \in F$ and $t > 0$ small enough, we have

$$k(x + t) - k(x) \geq m_j(x + t - a_j) + \sum_{(a_i, b_i) \subset [x, x+t]} m_i(b_i - a_i) \geq m_t \cdot t,$$

where $j \in \mathcal{I}$ is such that $x + t \in (a_j, b_j)$ and for $m_t := \\min \{m_k : (a_k, b_k) \cap [x, x+t] \neq \emptyset\}$ we have $\lim_{t \to 0^+} m_t = \infty$ by the choice of $\psi_i$. If $x = a_i$ for some $i \in \mathcal{I}$, then we
have \( k(x+t) - k(x) \geq t \cdot \min_{y \in [x,x+t]} \psi_i(y) \), and the minimum goes to infinity with \( t \to 0^+ \) by the choice of \( \psi_i \). By continuity and symmetry, the rest follows. Now define \( \varphi(x) := \alpha + \frac{\beta - \alpha}{k(x)} k(x) \), \( h := \varphi^{-1} \), and the lemma easily follows.

\[ \square \]

4. Proofs of the main results

Proof of Theorem 1. The implication (i) \( \implies \) (ii) is trivial. To prove that (ii) \( \implies \) (iii), let \( h \) be a homeomorphism such that \( g = f \circ h \) has pointwise-Lipschitz derivative. Then Lemma 7 implies that \( g \) is \( \text{VBG}_{\frac{1}{2}} \). By a remark following Definition 5, it follows that \( f \) is \( \text{VBG}_{\frac{1}{2}} \).

To prove that (iii) \( \implies \) (i), without any loss of generality, we can assume that the norm on \( X \) is Gâteaux differentiable (since \( \text{span}(f([a,b])) \) is separable and second order differentiability of a path does not depend on the equivalent norm on \( X \)). First, assume that \( f \) is not constant on any interval. Lemma 10 implies that there exists an increasing homeomorphism \( v : [a,b] \to [\alpha,\beta] \) such that \( f \circ v^{-1} \) is differentiable on \( [\alpha,\beta] \), twice differentiable on \( [\alpha,\beta] \setminus v(K_v) \), and \( \lambda(v(K_v)) = 0 \). Apply Lemma 11 to \( F = v(K_f) \) to obtain an (increasing) continuously differentiable homeomorphism \( h : [\alpha,\beta] \to [\alpha,\beta] \) such that \( h'(x) = 0 \) iff \( x \in h^{-1}(v(K_f)) \), and such that \( h \) is twice differentiable on \( [\alpha,\beta] \setminus h^{-1}(v(K_f)) \). Let \( g = f \circ v^{-1} \circ h \). By the chain rule for derivatives, we have that \( g \) is twice differentiable on \( [\alpha,\beta] \setminus h^{-1}(v(K_f)) \).

Let \( x \in h^{-1}(v(K_f)) \). Then by (3.3) there exists a \( C_x > 0 \) such that

\[
\frac{\|f \circ v^{-1}(y) - f \circ v^{-1}(z)\|}{|y-z|} \leq 2C_x |y - z|
\]

for \( z < y < h(x) \) or \( h(x) < y < z \) (and by continuity this holds also for \( y = h(x) \), and \( y, z \in [\alpha,\beta] \). It follows that \( (f \circ v^{-1})'(h(x)) = 0 \). Thus \( g'(x) = 0 \) by the chain rule. It also follows from (4.1) that \( (f \circ v^{-1})'(h(x)) = 0 \) is pointwise-Lipschitz at \( h(x) \) with constant \( 2C_x \). This implies that

\[
\left\| \frac{g'(x+t) - g'(x)}{t} \right\| = \left\| \frac{(f \circ v^{-1})'(h(x+t))}{t} \cdot h'(x+t) \right\|
\]

\[
= \left\| \frac{(f \circ v^{-1})'(h(x+t))}{t} - (f \circ v^{-1})'(h(x)) \right\| \cdot h'(x+t)
\]

\[
\leq 2C_x \cdot \left\| \frac{h(x+t) - h(x)}{t} \right\| \cdot h'(x+t),
\]

for all \( x + t \in [\alpha,\beta] \). The continuity of \( h' \) at \( x \) shows that \( g''(x) = 0 \). It is easy to see that \( f \) is Lebesgue equivalent to \( g \) (by composing \( v^{-1} \circ h \) with an affine change of parameter).

If \( f \) is constant on some interval, then let \( (c_i, d_i) \ (i \in I \subset \mathbb{N}) \) be the collection of all maximal open intervals such that \( f \) is constant on each \( (c_i, d_i) \). It is easy to see that we can find a continuous function \( \tilde{f} : [a,b] \to X \) such that \( f = \tilde{f} \) on \([a,b] \setminus \bigcup_i (c_i, d_i) \), \( \tilde{f} \) is affine and non-constant on \((c_i, (c_i + d_i)/2), ((c_i + d_i)/2, d_i)\), and such that \( \tilde{f} \) is \( \text{VBG}_{\frac{1}{2}} \). By the previous paragraph, there exists a homeomorphism \( h \) of \([a,b] \) onto itself such that \( \tilde{f} \circ h \) is twice differentiable. It follows that \( f \circ h \) is twice differentiable (since \( (f \circ h)'(x) = (f \circ h)''(x) = 0 \) for all \( x \in \bigcup_i (c_i, d_i) \) by the construction).
Proof of Theorem 2. The implication (i) $\implies$ (ii) is trivial. To prove that (ii) $\implies$ (iii), note that if $g'(x) \neq 0$ for a.e. $x \in [a, b]$, then $g$ is not constant in any interval. This notion is clearly stable with respect to Lebesgue equivalence. The rest follows from Theorem 1.

To prove that (iii) $\implies$ (i), we can follow the proof of the corresponding implication of Theorem 1. To see that the resulting function $g$ has non-zero derivative almost everywhere, we note that the homeomorphism $h$ obtained by applying the Lemma 11 has an absolutely continuous inverse. The rest follows easily. \(\square\)

The following example shows that even in the case of $X = \mathbb{R}$, $VBG_{1/2}$ functions do not coincide with continuous functions satisfying $V_{1/2}(f, K_f) < \infty$.

Example 12. There exists a continuous $VBG_{1/2}$ function $f : [0, 1] \to \mathbb{R}$ such that $V_{1/2}(f, K_f) = \infty$ (and thus $f$ is not Lebesgue equivalent to a $C^2$ function by [9, Remark 3.6]).

Proof. Let $a_n \in (0, 1)$ be such that $a_n \downarrow 0$. Define $f(a_{2k}) = 0$, $f(a_{2k+1}) = 1/k^2$ for $k = 1, \ldots$, and $f(0) = f(1) = 0$. Extend $f$ to be continuous and affine on the intervals $[a_{2k+1}, a_{2k}]$ and $[a_{2k+2}, a_{2k+1}]$. Then $K_f = \{0, 1\} \cup \{a_n : n \geq 2\}$ and it is easy to see that $f$ is $VBG_{1/2}$ but $V_{1/2}(f, K_f) = \infty$. \(\square\)

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*E-mail address*: duda@karlin.mff.cuni.cz

**Charles University, Department of Mathematical Analysis, Sokolovská 83, 186 75 Praha 8 - Karlín, Czech Republic**