Notes on Chern-Simons Theory in the Temporal Gauge

Andrey Smirnov

Abstract

We analyze the perturbative series expansion of vacuum expectation values (vevs) for Wilson loop operators in Chern-Simons (CS) gauge theory in the temporal gauge \( A_0 = 0 \). Following J. Labastida and E. Pérez we introduce the notion of the kernels of knot polynomial invariants - the (non-invariant) vevs of the Wilson loops, arising from CS theory in the temporal gauge. A method for exact calculation of the kernels of knot polynomial invariants is presented.

Contents

1 Introduction

2 Chern-Simons theory in the temporal gauge

2.1 The propagator of the gauge fields in the temporal gauge

2.2 The Abelian case as a basic example

3 The Non-Abelian Case

3.1 Labastida-Pérez formula

3.2 Two-Component Links

3.3 \( I_{R_1, R_2}(c_1, c_2) \) case

3.4 \( I_{R_1, R_2}(c_1, c_2) \) case

3.5 \( I_{m_1, m_2}(c_1 c_2) \) case

4 The intersection point operator method

4.1 The intersection point operator

4.2 The vev of the trefoil knot in \( GL(2) \) case

5 Conclusion

1 Introduction

In 1988, Edward Witten showed the connection between Chern-Simons field theory and the theory of knots in three-dimensional space [1]. Since that time the knot theory has been intensively studied using the standard quantum field theory methods. Different approaches inherent in quantum field theory established important connections between different types of knot invariants and provided a lot of new constructions for them.

The CS theory has been studied in different ways. Using series of non-perturbative methods Edward Witten in his original paper proved the equivalence of vacuum expectation values for Wilson lines operators and known polynomial knot invariants [2]-[6]. Perturbative studies performed in series of paper [7]-[15] established the connection between the coefficients of perturbative series expansion of vevs for Wilson loops and finite type (Vassiliev) knot invariants [17]-[18].

Perturbative series expansion has been studied for different gauge fixing. The study of CS theory in the covariant Landau gauge performed in [7][8] showed the equivalence of coefficients for the perturbative series expansion of the vacuum expectation values for Wilson loop operators in Chern-Simons theory in the temporal gauge, providing a method for exact calculation of the kernels of knot polynomial invariants.
and the integral representation for invariants by Bott and Taubes [19]. The Kontsevich integral representation for Vassiliev invariants turned out to correspond to the coefficients of vevs of Wilson lines operators in the light-cone gauge [14, 20].

In this paper we concentrate our attention on the study of CS theory in the temporal gauge. Different aspects of this gauge were discussed in [10, 13, 16]. In the temporal gauge $A_0 = 0$ the cubic term of the CS action disappears and the main ingredient for perturbative calculations is the gauge propagator. As it was noted in [13] the calculation of the gauge propagator in non-covariant gauges is plagued with ambiguities which should be solved by demanding additional properties for the correlation functions of gauge fields. As it was shown in [14] in non-covariant gauges we need to introduce some additional term to the propagator in order to obtain the knot invariants. Unfortunately, at present the correction term for the propagator in the temporal gauge is not known yet. In this paper we work with the propagator (9) without the correction term what leads to non-invariant quantities for vevs of the Wilson loop operators. On the other hand as it was argued in [13] the propagator (6) contains enough information about a knot to reconstruct the full knot invariants in any order. Following the definitions of [13] we call these non-invariant vevs of Wilson loops the kernels of polynomial invariants.

The aim of this paper is to present the method for calculation of the kernels of the knot polynomial invariants in a very simple way. We show that there exists an operator $X : R_1 \otimes R_2 \rightarrow R_1 \otimes R_2$, where $R_1$ and $R_2$ are some representations of gauge group of CS action, such that the kernel of polynomial invariant for a knot with $m$ intersection points can be obtained by appropriate contraction of indexes for $m$ copies of the operator $X$.

The structure of the article is as follows: in section 2 we discuss the main ingredients of perturbative calculations in CS theory for the temporal gauge and explain in details the calculation of vevs for Wilson loops in the simplest abelian case. In section 3 we discuss the geometrical Labastida-Pérez formula for perturbative series expansion in non-abelian case and present some results of explicit calculations with this formula for the gauge group $GL(N)$. In section 4 we introduce a notion of the intersection operator, we calculate explicitly this operator for the case of $GL(2)$ gauge group. As an example, using this operator we calculate the exact answer for kernel of polynomial invariant for the trefoil knot in $GL(2)$ case.

## 2 Chern-Simons theory in the temporal gauge

### 2.1 The propagator of the gauge fields in the temporal gauge

Let $G$ be a semi-simple Lie group and $A = A^a_{\mu}(x)dx^\mu F^a$ be a $G$-connection on $\mathbb{R}^3$, where $F^a$ are the generators of Lie algebra of $G$ in the fundamental representation. The Chern-Simons field theory is defined by the following action:

$$S[A] = \int_{\mathbb{R}^3} tr \left( A \wedge dA + g \frac{2}{3} A \wedge A \wedge A \right)$$

(1)

where $tr$ denotes the trace over the fundamental representation of $G$ and $g$ is some parameter.

The CS theory is a particular example of a topological field theory [13], that implies that all the observables in this theory are some topological invariants. In this way, the CS theory is a natural tool for study of three-dimensional topology, for example the topology of three-dimensional knots. Indeed, let us consider the following gauge invariant operator associated with a knot $c \in \mathbb{R}^3$:

$$W_R(c) = tr \left( P \exp \left( g \oint c A^a_{\mu}(x) R^a dx^\mu \right) \right),$$

(2)

this operator is just a Wilson loop associated with the knot $c$ or a trace of the holonomy of $A$ along the path $c$. The index $R$ means that one-form $A$ takes values in representation $R$ of $G$. The natural class of the knot invariants provided by the CS theory is the vacuum expectation values of these operators:

$$< W_R(c) >= \int DA \exp(-S[A]) W_R(c)$$

(3)

In [1] E.Witten using non-perturbative methods showed that these invariants are in fact the well known polynomial knot invariants with the argument $t = \exp(g^2)$. To compute [23] using the standard perturbative methods (with respect to the parameter $g$) we need to chose a gauge fixing condition to make the associated functional integral well defined. Different choices of a gauge fixing lead to different descriptions of the same polynomial
invariants [13 14 7]. The aim of this paper is to study the structure of the perturbative expansion of these polynomials in the so called temporal gauge. In the temporal gauge the condition imposed on the field \( A \) is:

\[
A_0 = 0
\]

In this case the cubic term of the action (11) disappears and we arrive to a free field theory with the following action:

\[
S[A]|_{A_0=0} = \int_{\mathbb{R}^3} d^3 x \, A_\mu^a \left( \delta^{ab} \, \epsilon^{\mu 0 \nu} \frac{\partial}{\partial x_0} \right) A_\nu^b, \quad \delta^{ab} = \text{tr}(F^a F^b) \tag{4}
\]

We have the following equation for the Green function:

\[
\delta^{ab} \, \epsilon^{\mu 0 \nu} \partial_0 \Delta^{b c, \nu \eta}(x) = \delta^{ac} \, \delta^{\mu \eta} \delta(x)
\]

for the solution of this equation we have (for careful derivation of this formula see [13]):

\[
\Delta^{b c, \nu \eta}(x) = - \epsilon^{\mu 0 \eta} \delta^{b c} \delta(x_1) \delta(x_2) \frac{1}{2} \text{sign}(x_0) + f^{b c \nu \eta}(x_1, x_2), \tag{5}
\]

where \( f^{b c \nu \eta}(x_1, x_2) \) is an arbitrary tensor that does not depend on \( x_0 \). Particular choice of \( f^{b c \nu \eta}(x_1, x_2) \) depends on a prescription, some examples of different prescriptions can be found in [13, 16]. In this work we will restrict our attention to the case \( f^{b c \nu \eta}(x_1, x_2) = 0 \). In this case one obtains the following components of the propagator:

\[
< A^a_\mu(x), A^b_\nu(y) > = 0, \mu = 0, 1, 2; \quad < A^a_\mu(x), A^b_\nu(y) > = \epsilon^{\mu \nu} \delta^{ab} \delta(x_1 - y_1) \delta(x_2 - y_2) \frac{1}{2} \text{sign}(x_0 - y_0), \quad \mu, \nu = 1, 2 \tag{6}
\]

As we will see below, perturbation theory based on the propagator (6) leads to the quantities for \( < W_R(c) > \) that do not coincide with the known knot invariants obtained using non-perturbative methods and depend on a projection of the knot to the two-dimensional \( (x_1, x_2) \) plane. As it was conjectured in [13 14] this deviation arises because we do not take into account the prescription depending term in (3).

Despite the vacuum expectation values of Wilson lines in this description are not knot invariants, they contain a lot of information about knots and as it was conjectured in [13] they can be used for a reconstruction of the full invariants. Following the notation of [13] we will denote these quantities by \( < \hat{W}_R(c) > \) and call them kernels of polynomial invariants. The aim of this paper is to present the method that allows us to find exact expressions for the kernels of the polynomial invariants for every knot and for every representation in a very simple and elegant way.

### 2.2 The Abelian case as a basic example

Let us analyze in details the structure of the perturbative series expansion with propagator (6) in the Abelian case. In this case for the propagator we have:

\[
\Pi_{\mu \nu}(x-y) = < A_\mu(x), A_\nu(y) > = \frac{1}{2} \epsilon^{\mu \nu} \delta(x_1 - y_1) \delta(x_2 - y_2) \text{sign}(x_0 - y_0), \quad \mu, \nu = 1, 2, \quad A_0 = 0 \tag{7}
\]

The Wilson lines are presented by a ”simple” exponents:

\[
W(c) = \exp \left( g \oint_c A_\mu(x) \, dx^\mu \right) \tag{8}
\]

We omit the label of representation because in the Abelian case the Wilson loop operators for different representations differ by a constant. For the perturbative expansion we get:

\[
< \hat{W}(c) > = < \sum_{n=0}^{\infty} \frac{1}{n!} \left( g \oint_c A_\mu(x) \, dx^\mu \right)^n > = \sum_{n=0}^{\infty} \frac{(2n-1)!!}{2n!} g^{2n} \left( \int_c \int_c dx_\mu \, dx_\nu < A_\mu(x), A_\nu(x) > \right)^{2n}
\]

and after the summation we obtain:

\[
< \hat{W}(c) > = \exp (g^2 L_c) \tag{9}
\]

where:

\[
L_c = \frac{1}{2} \int_c \int_c dx_\mu \, dy_\nu < A_\mu(x), A_\nu(y) > = \frac{1}{2} \int_c \int_c dx^\mu \, dy^\nu \frac{1}{2} \epsilon^{\mu \nu} \delta(x_1 - y_1) \delta(x_2 - y_2) \text{sign}(x_0 - y_0)
\]
Let us parametrize the knot by a parameter $t$ running from 0 to 1, then we can rewrite the last integral in the following form:

$$L_c = \frac{1}{4} \int_0^1 \int_0^1 dt_1 dt_2 \left( \frac{dx_1}{dt_1} \frac{dy_2}{dt_2} - \frac{dx_2}{dt_1} \frac{dy_1}{dt_2} \right) \delta(x_1(t_1) - y_1(t_2)) \delta(x_2(t_1) - y_2(t_2)) \text{sign}(x_0(t_1) - y_0(t_2))$$

To perform the integration we need to solve the following equations:

$$\begin{align*}
    x_1(t_1) - y_1(t_2) &= 0 \\
    x_2(t_1) - y_2(t_2) &= 0
\end{align*}\quad(10)$$

The solutions of these equations are the self-intersection points of two dimensional curve $(x_1(t), x_2(t))$ which is the projection of the knot $c$ on the plane $(x_1, x_2)$. Let us denote by $t_1^k < t_2^k$ the values of the parameter $t$ in the intersection points, then the two-dimensional delta function in the integral can be represented in the form:

$$\delta(x_1(t_1) - y_1(t_2)) \delta(x_2(t_1) - y_2(t_2)) = \sum_k \left( \delta(t_1 - t_1^k) \delta(t_2 - t_2^k) + \delta(t_1 - t_2^k) \delta(t_2 - t_1^k) \right)$$

Substituting this expression into (10) and integrating over $t_1$ and $t_2$ we arrive to the following simple expression:

$$L_c = \sum_k \epsilon_k\quad(11)$$

where the quantities $\epsilon_k$ are the "sings" of the intersection points. They can take values $\pm 1$ and are defined in the following way:

$$\epsilon_k = \frac{\frac{dx_1}{dt_1}(t_1^k)}{|\frac{dx_1}{dt_1}(t_1^k)|} \frac{\frac{dy_2}{dt_2}(t_2^k)}{|\frac{dy_2}{dt_2}(t_2^k)|} \text{sign}(x_0(t_1^k) - y_0(t_2^k))\quad(12)$$

Note that the expectation value (9) for the knot can be expressed in terms of product over the intersection points:

$$\langle W(c) \rangle = \prod_k \exp(g^2 \epsilon_k)\quad(13)$$

We see that the answer for vev in the abelian case has a form of a product over the intersection points of some quantity $\exp(g^2 \epsilon_k)$ that depends only on the intersection point. As we will see below in the non-abelian case this formula generalizes to the contraction of some tensors corresponding to intersection points.

## 3 The Non-Abelian Case

### 3.1 Labastida-Pérez formula

In the case of non-Abelian gauge group the Wilson line operator is more complicated:

$$W_R(c) = \text{tr}P \exp \left( g \oint_c A_\mu(x) dx^\mu \right)$$

The vacuum expectation value has the form:

$$\langle W(c) \rangle = \sum_{m=0}^{\infty} I^R_m(c)g^{2m}$$

where the expansion are presented in terms of ordered multidimensional integrals:

$$I_m^R(c) = \text{tr}(R^{a_1} R^{a_2} ... R^{a_m}) \int_0^1 dx_{\mu_1} \int_0^{x_{\mu_1}} dx_{\mu_2} ... \int_0^{x_{\mu_{m-1}}} dx_{\mu_m} \langle A_{\mu_1}^{a_1} A_{\mu_2}^{a_2} ... A_{\mu_m}^{a_m} \rangle$$
where $R^a$ are the generators of the gauge group in a representation $R$. Using Wick theorem, and the following facts:

$$\int_0^1 dx_{\mu_1} \int_0^{x_{\mu_1}} dx_{\mu_2} \cdots \int_0^{x_{\mu_{m-1}}} dx_{\mu_m} = \int dx_1 \cdots dx_m \prod_{k=1}^{m-1} \theta(x_k - x_{k+1}),$$

$$< A^a_\mu(x), A^b_\mu(y) >= 0, \mu = 0, 1, 2;$$

$$< A^a_\mu(x), A^b_\nu(y) > = \frac{1}{2} \epsilon^{\mu \nu \rho} \delta^{bc} \delta(x_1 - y_1) \delta(x_2 - y_2) \text{sign}(x_0 - y_0), \mu, \nu = 1, 2$$

we arrive to the following formula for $I^R_m$ (J.M.F.Labastida and E.Pérez [13]):

$$I^R_m(c) = \sum_{i_1 < i_2 < \cdots < i_m} \epsilon_{i_1} \epsilon_{i_2} \cdots \epsilon_{i_m} D(i_1, i_2, \ldots, i_m) + \frac{1}{(t!)^2} \sum_{j \notin S_2, \in \sigma,_{1, \ldots, m-2}} \epsilon_j^2 \epsilon_{i_1} \cdots \epsilon_{i_{m-2}} D(j, \sigma, i_1, \ldots, i_{m-2}) +$$

$$\frac{1}{(t!)^2} \sum_{j \notin S_r, \in \sigma,_{1, \ldots, m-r}} \epsilon_j^r \epsilon_{i_1} \cdots \epsilon_{i_{m-r}} D(j, \sigma, i_1, \ldots, i_{m-r}) + \sum_{\sigma \in S_m} \epsilon_j^m D(j, \sigma)$$

(14)

The first term in this big sum comes from the contribution in which all the propagators are attached to different crossings. The second when two propagators are attached to the same crossing and rest to different crossings and so on. The factors $D(j, \sigma, i_1, \ldots, i_{m-r})$ are group factors and they can be computed in the following way: we attach to every crossing $i_k$ a group generator and $r$ generators to the crossing $j$. Then travelling along the knot from some base point we multiply this generators in the order that they encounter. When we arrive to the crossing $j$ first time one encounters product of $r$ group generators and the second time the product is rearranged in accordance with the permutation $\sigma \in S_r$. After returning to the base point we should take a trace of obtained group factor.

For example let us calculate the group factor $D(3, \sigma, 1, 5)$, where $\sigma \in S_2$ (we consider the group $S_2$ as a permutaions of two-element set $\{b, c\}$) for the knot projection represented in fig. 1. According to the receipt, we should attach one generator $R^a$ to point 1, one generator $R^d$ to point 5, and the product of two generators $R^b R^c$ to point 3. Running along the knot projection from the base point $p$ we encounter the chosen points in the following order: 1, 3, 5, 1, 5, 3, then we get the following product of the generators attached to the points: $R^a \cdot R^b R^c \cdot R^d \cdot R^a \cdot R^d \cdot R^a R^b R^c$, where we rearranged the product of generators corresponding to the point 3 according to permutation $\sigma$ when we arrived to the point 3 for the second time. Finally, taking the trace we have:

$$D(3, \sigma, 1, 5) = tr(R^a R^b R^c R^d R^a R^d R^a R^b R^c)$$

![Figure 1](image-url)
Using modern computational tools such as Maple or Mathematica and formula \(14\) one can perform computations in perturbation theory for the first several orders. In the table below some exact results of these calculus for \(gl(N)\)-case in fundamental representation \(F\) are presented:

| Knot | \(I_1^F\) | \(I_2^F\) | \(I_3^F\) | \(I_4^F\) |
|------|-------------|-------------|-------------|-------------|
| \(3_1\) | \(3N^2\) | \(3/4 N(5 + N^2)\) | \(1/12 N^2(53 + N^2)\) | \(\frac{1}{192} N(284 + 363 N^2 + N^4)\) |
| \(4_1\) | 0 | \(3N(N - 1)(N + 1)\) | 0 | \(\frac{7}{144} N(N - 1)(N + 1)(5 N^2 + 46)\) |
| \(5_1\) | \(5N^2\) | \(5/4 N(9 + N^2)\) | \(\frac{5}{36} N^2(149 + N^2)\) | \(\frac{7}{360} N(1712 + 1287 N^2 + N^4)\) |
| \(5_2\) | \(5N^2\) | \(1/4 N (33 + 17 N^2)\) | \(\frac{5}{36} N^2 (131 + 19 N^2)\) | \(\frac{1}{10} N (4096 + 10263 N^2 + 641 N^4)\) |
| \(6_1\) | \(2N^2\) | \(1/2 N (-13 + 17 N^2)\) | \(1/9 N^2 (-47 + 59 N^2)\) | \(\frac{1}{192} N (-2394 + 1193 N^2 + 1393 N^4)\) |

Where we have chosen the two-dimensional projections of the knots as in the fig. 2

![Figure 2](image)

In the Abelian case \(N = 1\) we get the following results:

\[
< \hat{W}_F(3_1) > = \sum_{m=0}^{\infty} I_m^F(3_1) g^{2m} = 1 + 3 g^2 + 9/2 g^4 + 9/2 g^6 + 27/8 g^8 + 81/32 g^{10} + \ldots = e^3 g^2
\]

\[
< \hat{W}_F(4_1) > = \sum_{m=0}^{\infty} I_m(4_1) g^{2m} = 1 + 0 g^2 + 0 g^4 + 0 g^6 + 0 g^8 + 0 g^{10} + \ldots = 1
\]

\[
< \hat{W}(5_1) > = \sum_{m=0}^{\infty} I_m^F(5_1) g^{2m} = 1 + 5 g^2 + 25/2 g^4 + 125/6 g^6 + 625/24 g^8 + 625/24 g^{10} + \ldots = e^5 g^2
\]

\[
< \hat{W}_F(5_2) > = \sum_{m=0}^{\infty} I_m^F(5_2) g^{2m} = 1 + 5 g^2 + 25/2 g^4 + 125/6 g^6 + 625/24 g^8 + 625/24 g^{10} + \ldots = e^5 g^2
\]

\[
< \hat{W}_F(6_1) > = \sum_{m=0}^{\infty} I_m^F(5_3) g^{2m} = 1 + 2 g^2 + 2 g^4 + 4/3 g^6 + 2/3 g^8 + 4/15 g^{10} + \ldots = e^9 g^2
\]
We see that in full agreement with (13) the vevs of the Wilson loops in this cases are just products of simple exponent operators over intersection points. In the first non-abelian case $N = 2$ we get:

$$< \hat{W}_F(3_1) > = \sum_{m=0}^{\infty} I_m^F(3_1) g^{2m} = 2 + 12 g^2 + \frac{27}{2} g^4 + 19 g^6 + \frac{73}{4} g^8 + \frac{279}{20} g^{10} + ... = ?$$

$$< \hat{W}_F(4_1) > = \sum_{m=0}^{\infty} I_m^F(4_1) g^{2m} = 2 + 18 g^4 + \frac{9}{2} g^8 + ... = ?$$

$$< \hat{W}_F(5_1) > = \sum_{m=0}^{\infty} I_m^F(5_1) g^{2m} = 2 + 20 g^2 + \frac{65}{2} g^4 + 85 g^6 + \frac{955}{8} g^8 + \frac{271}{6} g^{10} + ... = ?$$

$$< \hat{W}_F(5_2) > = \sum_{m=0}^{\infty} I_m^F(5_2) g^{2m} = 2 + 20 g^2 + \frac{101}{2} g^4 + 115 g^6 + \frac{1539}{8} g^8 + \frac{1535}{6} g^{10} + ... = ?$$

$$< \hat{W}_F(6_1) > = \sum_{m=0}^{\infty} I_m^F(6_1) g^{2m} = 2 + 8 g^2 + 55 g^4 + 84 g^6 + \frac{4111}{24} g^8 + \frac{2912}{15} g^{10} + ... = ?$$

In this case it is difficult to find any regularity in the coefficients of the expansions and sum them to the exact expressions of vevs for kernels of associated Wilson loops operators. It indicates that for these vevs in non-abelian case has much more complicated structure. In section 4 we present the example of such vev for the simplest trefoil knot $3_1$ in the case $N = 2$.

### 3.2 Two-Component Links

In order to derive the notion of intersection point operator we need to find a contribution to vev of Wilson loop coming from a single two-dimensional intersection point as in fig.3. For our analysis it is much more convenient to assume that the pathes $i j$ and $k m$ belong to different contours.

![Figure 3](image)

In this connection let us temporary proceed to the consideration of two-component links. More precisely, we are interested in the following quantity:

$$I_{R_1 R_2}(c_1, c_2) = \frac{< \hat{W}_{R_1}(c_1), \hat{W}_{R_2}(c_2) >}{< W_{R_1}(c_1) > < W_{R_2}(c_2) >} = \sum_{k=0}^{\infty} I_{R_1 R_2}^k(c_1, c_2) g^{2k} = 1 + \sum_{k=1}^{\infty} I_{R_1 R_2}^k(c_1, c_2) g^{2k} \tag{15}$$

where $c_1$ and $c_2$ are two contours in $\mathbb{R}^3$ and $W_{R_i}(c_i)$ are associated Wilson lines operators in representation $R_i$ of the gauge group:

$$W_{R_i}(c_i) = tr P \exp \left( g \oint_{c_i} A_\mu dx^\mu \right) \tag{16}$$

In (15) we divided vev $< \hat{W}_{R_1}(c_1), \hat{W}_{R_2}(c_2) >$ by the product of vevs $< \hat{W}_{R_1}(c_1) > < \hat{W}_{R_2}(c_2) >$, which means that in perturbation series expansion we will only take into account the terms with different ends of the propagators attached to the different contours, in other words, we do not take into consideration the self-intersection points of the contours $c_1$ and $c_2$.

### 3.3 $I_{1 R_1 R_2}(c_1, c_2)$ case

In the order $g^2$ we have the following integral:

$$I_{1 R_1 R_2}^k(c_1, c_2) = g^2 tr(R_1^a) tr(R_2^b) \oint_{c_1} dx^\mu \oint_{c_2} dy^\nu < A^a_\mu(x), A^b_\nu(y) >= 0$$
3.4 \( I_{2}^{R_{1} R_{2}}(c_1, c_2) \) Case

In the order \( g^4 \) we have two contributions:

\[
I_{2}^{R_{1} R_{2}}(c_1, c_2) = M_1 + M_2
\]

where

\[
M_1 = tr(R_{1}^{a_1} R_{2}^{a_2}) \int dx_{1}^{\mu_1} dy_{1}^{\nu_1} dy_{2}^{\nu_2} < A_{\mu_1}^{a_1}(x_1), A_{\nu_1}^{b_1}(y_1) > < A_{\mu_2}^{a_2}(x_2), A_{\nu_2}^{b_2}(y_2) >
\]

\[
M_2 = tr(R_{1}^{a_1} R_{2}^{a_2}) \int dx_{2}^{\mu_2} dy_{1}^{\nu_1} dy_{2}^{\nu_2} < A_{\mu_1}^{a_1}(x_1), A_{\nu_1}^{b_1}(y_1) > < A_{\mu_2}^{a_2}(x_2), A_{\nu_2}^{b_2}(y_2) >
\]

Introducing the parameterizations \( t \) and \( s \) for contours \( c_1 \) and \( c_2 \) respectively we get:

\[
G(t, s) = \epsilon_{\mu \nu} \frac{dx^\mu}{dt}(t) \frac{dy^\nu}{ds}(s) \delta(x_1(t) - y_1(s)) \delta(x_2(t) - y_2(s)) \frac{1}{2} \text{sign}(x_0(t) - y_0(s)) = \sum K_k \delta(t - t_k) \delta(s - s_k)
\]

Here \( \epsilon_k = \pm 1 \) and sum runs over the intersection points of two contours lying in the plane \( (x_1, x_2) \) which are the projections of three dimensional contours on the plane. Using this notation we can rewrite the integrals for \( M_1 \) and \( M_2 \) in the form:

\[
M_1 = tr(R_{1}^{a_1} R_{2}^{a_2}) \int dt_1 \int dt_2 \int ds_1 \int ds_2 G(t_1, s_1) G(t_2, s_2)
\]

\[
M_2 = tr(R_{1}^{a_1} R_{2}^{a_2}) \int dt_1 \int dt_2 \int ds_1 \int ds_2 G(t_1, s_1) G(t_2, s_1)
\]

and using the following fact:

\[
\int dt_1 \int dt_2 \int ds_1 \int ds_2 = \int dt_1 \int dt_2 \int ds_1 \int ds_2 \theta(t_1 - t_2) \theta(s_1 - s_2)
\]

we have:

\[
M_1 = tr(R_{1}^{a_1} R_{2}^{a_2}) \sum_{k_1 k_2} \epsilon_k \epsilon_k \theta(t_{k_1} - t_{k_2}) \theta(s_{k_1} - s_{k_2})
\]

\[
M_2 = tr(R_{1}^{a_1} R_{2}^{a_2}) \sum_{k_1 k_2} \epsilon_k \epsilon_k \theta(t_{k_1} - t_{k_2}) \theta(s_{k_2} - s_{k_1})
\]

and finally we arrive to the following expression:

\[
I_{2}^{R_{1} R_{2}}(c_1, c_2) = M_1 + M_2 = \sum_{\sigma \in S_2} tr(R_{1}^{a_1} R_{2}^{a_2}) tr(R_{2}^{a_{\sigma(1)}} R_{2}^{a_{\sigma(2)}}) \sum_{k_1 k_2} \epsilon_k \epsilon_k \theta(t_{k_1} - t_{k_2}) \theta(s_{k_{\sigma(1)}} - s_{k_{\sigma(2)}})
\]

where \( S_2 \) is the permutation group of two elements.

3.5 \( I_{m}^{R_{1} R_{2}}(c_1 c_2) \) Case

The last formula (24) has obvious straightforward generalization for \( I_{m}^{R_{1} R_{2}}(c_1 c_2) \):

\[
I_{m}^{R_{1} R_{2}}(c_1 c_2) = \sum_{\sigma \in S_m} M_{\sigma}
\]

where \( S_m \) is the permutation group for a set with \( m \) elements.

\[
M_{\sigma} = tr(R_{1}^{a_1} ... R_{k}^{a_m}) tr(R_{2}^{a_{\sigma(1)}} ... R_{2}^{a_{\sigma(m)}}) \sum_{k_1 k_2 ... k_m} \epsilon_{k_1} \epsilon_{k_2} ... \epsilon_{k_m} \left( \prod_{i=1}^{m-1} \theta(t_{k_{i}} - t_{k_{i+1}}) \right) \left( \prod_{j=1}^{m-1} \theta(s_{k_{\sigma(i)}} - s_{k_{\sigma(i+1)}}) \right)
\]
Let us note that in the Abelian case \( M_\sigma \) does not contain non-commuting lie-algebra structures and the sum of \( M_\sigma \) is expressed through the linking number of two contours:

\[
\sum_{\sigma \in S_m} M_\sigma = \sum_{\sigma \in S_m} \left( \sum_{k_1 k_2 \ldots k_m} \epsilon_{k_1} \epsilon_{k_2} \ldots \epsilon_{k_m} \left( \prod_{i=1}^{m-1} \theta(t_{k_i} - t_{k_{i+1}}) \right) \left( \prod_{j=1}^{m-1} \theta(s_{\sigma(j)} - s_{\sigma(j+1)}) \right) \right) = \frac{L_{12}}{m!} \tag{27}
\]

where the quantity \( L_{12} \) is just a sum of crossing signs over two dimensional intersection points:

\[
L_{12} = \sum_k \epsilon_k
\]

The definition of \( L_{12} \) coincides precisely with definition of linking number of two knots. The linking number of two two-component link known to be the topological invariant of the link, and we arrive to the following result: in abelian case the vev \( \langle 15 \rangle \) is just the exponent of linking number of the link:

\[
I(c_1, c_2) = \sum_{m=0}^\infty g^{2m} \frac{L_{12}^m}{m!} = \exp(g^2 L_{12})
\]

4 The intersection point operator method

4.1 The intersection point operator

Using the following property of the tensor product of operators:

\[
tr(R_1^{a_1} R_2^{a_2} \ldots R_1^{a_m})tr(R_2^{\sigma(1)} R_2^{\sigma(2)} \ldots R_2^{\sigma(m)}) = tr(R_1^{a_1} R_1^{a_2} \ldots R_1^{a_m} \otimes R_2^{\sigma(1)} R_2^{\sigma(2)} \ldots R_2^{\sigma(m)})
\]

we can rewrite expression for \( I(c_1, c_2) \) in the form:

\[
M_\sigma = tr(\hat{M}_\sigma)
\]

\[
\hat{M}_\sigma = R_1^{a_1} \ldots R_1^{a_m} \otimes R_2^{\sigma(1)} \ldots R_2^{\sigma(m)} \sum_{k_1 k_2 \ldots k_m} \epsilon_{k_1} \epsilon_{k_2} \ldots \epsilon_{k_m} \left( \prod_{i=1}^{m-1} \theta(t_{k_i} - t_{k_{i+1}}) \right) \left( \prod_{j=1}^{m-1} \theta(s_{\sigma(j)} - s_{\sigma(j+1)}) \right) = \frac{L_{12}}{m!} \tag{28}
\]

Let us consider instead of the sums (15) and (25) the following operator:

\[
M = \sum_{m=0}^\infty g^{2m} \sum_{\sigma \in S_m} R_1^{a_1} \ldots R_1^{a_m} \otimes R_2^{\sigma(1)} \ldots R_2^{\sigma(m)} \sum_{k_1 k_2 \ldots k_m} \epsilon_{k_1} \epsilon_{k_2} \ldots \epsilon_{k_m} \left( \prod_{i=1}^{m-1} \theta(t_{k_i} - t_{k_{i+1}}) \right) \theta \left(s_{\sigma(j)} - s_{\sigma(j+1)}\right) = 1 + g^2 R_1^{a_1} \otimes R_2^{a_2} \sum_{k_1} \epsilon_{k_1} + g^4 \sum_{\sigma \in S_2} R_1^{a_1} R_2^{a_2} \otimes R_2^{\sigma(1)} R_2^{\sigma(2)} \left( \prod_{k_1} \theta(t_{k_1} - t_{k_2}) \theta(s_{\sigma(1)} - s_{\sigma(2)}) \right) + \ldots \tag{29}
\]

and calculate the contribution coming from one point. To find this contribution we should to assume \( k_1 = k_2 = \ldots = k_m \) in the \( g^{2m} \)-order term of \( M \) expansion and take into account the following fact:

\[
\left( \prod_{k_1}^{m-1} \theta(t_{k_1} - t_{k_{i+1}}) \right) |_{k_1=k_2=\ldots=k_m} = \frac{1}{m!}, \quad \left( \prod_{k_1}^{m-1} \theta(s_{\sigma(j)} - s_{\sigma(j+1)}) \right) |_{k_1=k_2=\ldots=k_m} = \frac{1}{m!}
\]

Let us denote:

\[
\{ R_1^{a_1} R_1^{a_2} \ldots R_1^{a_m} \} = \frac{1}{m!} \sum_{\sigma \in S_m} R_2^{\sigma(1)} R_2^{\sigma(2)} \ldots R_2^{\sigma(m)}
\]

then for a single point contribution we get:

\[
X(\epsilon g^2) = \sum_{k=0}^\infty \epsilon^k g^{2k} \frac{1}{k!} \{ R_1^{a_1} R_1^{a_2} \ldots R_1^{a_k} \} \otimes \{ R_2^{a_1} R_2^{a_2} \ldots R_2^{a_k} \}
\]

One can treat \( X(\pm g^2) \) as an operator acting in the space \( R_1 \otimes R_2 \). It is convenient to consider the operator \( X \) as a tensor with four indexes \( X_{i j k m} \), so that each index is associated with a leg of intersection vertex
The main feature of this operator is that the two adjacent point contribution can be simply expressed as the corresponding product of two such operators fig.5.

4.2 The vev of the trefoil knot in GL(2) case

As a more explicit example let us consider how the intersection point operator (30) can be used for calculation of the kernels of polynomial invariants. Let us summate the series (30) for operators $F^a$ in fundamental representation of $gl(2)$. In this case we have:

$$X(g^2) = X(g^2) = \sum_{k=0}^{\infty} \frac{g^{2k}}{k!} (F^a_1 F^a_2 ... F^a_k)^{\otimes 2} = A(g^2) \hat{I}d + B(g^2) \hat{P}$$

(31)

where $\hat{I}d$, $\hat{P}$ are the identity and interchange operators in $\mathbb{C}^2 \otimes \mathbb{C}^2$ and:

$$A(g^2) = 1/6 \epsilon g^2 + 1/6 \epsilon g^2 g^2 + 5/6 - 1/3 g^2,$$

$$B(g^2) = 2/3 \epsilon g^2 - 2/3 + 1/6 g^2 + 1/6 \epsilon g^2 g^2$$

$$\hat{I}d = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad \hat{P} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

To find the kernel of polynomial invariant, for example of the trefoil knot, we need to chose some projection of the knot to two-dimensional plane. For instance as in the fig.6. Then, we attach to every intersection point of obtained two-dimensional curve the tensor $X(\epsilon g^2)$, where $\epsilon$ is the sign of the intersection point defined by (12). Finally, to find the kernel of polynomial invariant in this case, we contract the indexes of the tensors in accordance with two dimensional diagram fig.6:

$$\langle \hat{W}(3_1) \rangle = \sum_{i, j, k, m, l} X^{ij}_{km} X^{km}_{sl} X^{ls}_{jk} =$$
Figure 6: Two dimensional projection of the trefoil knot and contraction of intersection operators.

\[
2 \left( \frac{5}{6} e^g + 1 \right) + 1 \left( 1 - 1 + 1 \right) + \\
+ 6 \left( \frac{1}{6} + 1 \right) \left( \frac{2}{3} - 2 + 1 \right) + 1 \left( 1 + 1 \right) + \\
+ 2 \left( 2 - 2 + 1 \right) + 1 \left( 1 + 1 \right) \\
\]

\[ (32) \]

The expansion of \( < \hat{W}(3_1) > \) in \( g^2 \) coincide precisely with the result obtained in section 3.1 for the trefoil knot \( 3_1 \) in the \( N = 2 \) case by means of Labastida-Pérez formula:

\[
< \hat{W}(3_1) > = 2 + 12 g^2 + 27 g^4 + 19 g^6 + 73 g^8 + 279 g^{10} + O(g^{12})
\]

Therefore, we observed an interesting property of CS theory in the temporal gauge with the propagator (6): the vevs of Wilson loops are factorized into the product of the intersection point operators corresponding to the crossings of two-dimensional projection of the knot. We note that in this case the vevs of Wilson loops are not knot invariants and depends on the projection chosen, moreover, the kernels of the polynomial invariants are not polynomials in \( e^g \) anymore (as it could be seen from (32)).

Of course, without the prescription depending term in the propagator (5) the theory is incomplete, and appropriate choice for this term is needed. Nevertheless, the first term of the propagator (5) that we used in this work, contains only information about crossings, and the second, prescription depending term, does not depend on the crossing sings, as it does not depend on \( x_0 \). This leads us to the conjecture, that in the presence of the prescription depending term, the property of vevs to be factorized into the product of some tensors corresponding to crossings, should be conserved.

5 Conclusion

There are a lot of the combinatorial constructions for the knot polynomial invariants in terms of regular two-dimensional projections. For example the Jones polynomial arising from the braid group representations \[9\] or the Kauffman construction of Jones polynomials in terms of \( R \)-matrices and the ”creation-annihilation” operators \[5\]. All this constructions provide some tensors corresponding to intersection points, and some additional tensors corresponding to free lines, like the Kauffman ”creation” and ”annihilation” operators which correspond to critical points of the knots in the Morse representation. The natural way for deriving these representations form CS theory is to use the temporal gauge fixing as the perturbation theory in this gauge depends only on two-dimensional representation of the knot. The CS theory in the temporal gauge with propagator (6) contains only information about crossings, and operators corresponding to the Kauffman creation annihilation operators can not be derived by means of this propagator. In this way, we should conclude, that the prescription depending term in (5) plays a crucial role in the construction of correct perturbation theory for CS in the temporal gauge.

To find exact expression for this term we need some additional physical restrictions on the form of the propagator. As an example of such a restriction we can demand that the propagator gives the perturbative series expansion for Wilson lines is in agreement with some general properties of CS theory, for example the factorization theorem \[12\]-[14]. The examples of restrictions on prescription depending term arising from factorization theorem can be found in \[13\]. Another example of restriction gives consideration of the unknot vevs for a projection without intersection points. In this case the first term of the propagator (5) does not play any
role and the perturbation series expansion contains only multidimensional integrals on products of prescription depending terms. We should demand that this series expansion coincides with the known series for vevs of unknot. All this will be considered elsewhere.

Acknowledgments

The author is grateful to A.Morozov for fruitful discussions and interest to this work. The work was partly supported by RFBR grant 09-02-00393, RFBR grant 06-01-92054-KEn, RFBR-CNRS grant 09-01-93106, and grant for support of scientific schools NSh-3036.2008.2. The work was also supported in part by the "Dynasty" Foundation.

References

[1] E.Witten, Commun.Math.Phys. 121, 351-399 (1989)
[2] V.F.R. Jones, Bull. AMS 12 (1985) 103; Ann. of Math. 126 (1987) 335.
[3] P.Freyd, D.Yetter, J.Hoste, W.B.R.Lickorish, K.Millet, and A.Ocneanu, Bull. AMS 12 (1985) 239.
[4] Y.Akutsu and M. Wadati, J.Phys.Soc.Jap. 56 (1987) 839 and 3039
[5] L.H.Kauffman, Note di Matematica vol. 9 17-32 (1989)
[6] L.H.Kauffman, Trans, Am. Math. Soc. 318 (1990) 417
[7] D.Bar-Natan, Ph.D. Thesis, Princeton Univercity (1991).
[8] E.Guadagnini, M.Martellini and Mitchev, Phys.Lett.B227 (1989) 111; B228 (1989) 489; Nucl.Phys. B330 (1990) 575.
[9] E.Guadagnini, CERN-TH 5827-90 (1990)
[10] A. Morozov, A. Rosly, early 90's, unpublished
[11] M.Alvarez, and J.M.F.Labastida, Nucl. Phys. B395,(1993) 198; B433 (1995) 555.
[12] J.M.F.Labastida and E.Pérez, Nucl.Phys. B527, (1998) 499;
[13] J.M.F.Labastida and E.Pérez, CERN-TH/98-193, US-FT-11/98 (1998) 499;
[14] J.M.F.Labastida and E.Pérez, Journ.Math.Phys. 39, (1998) 5183
[15] T.Dimofte, S. Gukov, J. Lenells, D.Zagier, arXiv: hep-th 0903.2472v1 (2009)
[16] A.S.Cattaneo, P.Cotta-Ramusino, J.Frohlich and M. Martellini, Jour.Math.Phys. 36 (1995) 6137.
[17] V.A. Vassiliev, Amer. Math. Soc., Providence, RI, (1990), 23
[18] V.A.Vassiliev, Translation of Mathematical Monographs, vol.98, AMS, (1992)
[19] R.Bott and C.Taubes, Jour.Math.Phys. 35 (1994) 5247.
[20] M.Kontsevich, Advances in Soviet Math. 16, part 2 (1993) 137
[21] I.A. Dynnikov, Funkts. Anal. Prilozh. vol. 33, 25-37, (1999)