The boundedness of some singular integral operators on weighted Hardy spaces associated with Schrödinger operators

Hua Wang *
School of Mathematical Sciences, Peking University, Beijing 100871, China

Abstract

Let \( L = -\Delta + V \) be a Schrödinger operator acting on \( L^2(\mathbb{R}^n) \), \( n \geq 1 \), where \( V \neq 0 \) is a nonnegative locally integrable function on \( \mathbb{R}^n \). In this paper, we first define molecules for weighted Hardy spaces \( H^p_L(w)(0 < p \leq 1) \) associated to \( L \) and establish their molecular characterizations. Then by using the atomic decomposition and molecular characterization of \( H^p_L(w) \), we will show that the imaginary power \( L^{i\gamma} \) is bounded on \( H^p_L(w) \) for \( \frac{n}{n+1} < p \leq 1 \), and the fractional integral operator \( L^{-\alpha/2} \) is bounded from \( H^p_L(w) \) to \( H^{p/p}_{w^{1/p}} \), where \( 0 < \alpha < \min\{n/2, 1\} \), \( n/(n+1) < p \leq n/(n+\alpha) \) and \( 1/q = 1/p - \alpha/n \).

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1 Introduction

Let \( n \geq 1 \) and \( V \) be a nonnegative locally integrable function defined on \( \mathbb{R}^n \), not identically zero. We define the form \( Q \) by

\[
Q(u, v) = \int_{\mathbb{R}^n} \nabla u \cdot \nabla v \, dx + \int_{\mathbb{R}^n} Vuv \, dx
\]

with domain \( \mathcal{D}(Q) = \mathcal{V} \times \mathcal{V} \) where

\[
\mathcal{V} = \{ u \in L^2(\mathbb{R}^n) : \frac{\partial u}{\partial x_k} \in L^2(\mathbb{R}^n) \text{ for } k = 1, \ldots, n \text{ and } \sqrt{V}u \in L^2(\mathbb{R}^n) \}.
\]

*E-mail address: wanghua@pku.edu.cn.
It is well known that this symmetric form is closed. Note also that it was shown by Simon [15] that this form coincides with the minimal closure of the form given by the same expression but defined on \( C_0^\infty (\mathbb{R}^n) \) (the space of \( C^\infty \) functions with compact supports). In other words, \( C_0^\infty (\mathbb{R}^n) \) is a core of the form \( Q \).

Let us denote by \( L \) the self-adjoint operator associated with \( Q \). The domain of \( L \) is given by

\[
D(L) = \{ u \in D(Q) : \exists v \in L^2 \text{ such that } Q(u, \varphi) = \int_{\mathbb{R}^n} v \varphi \, dx, \forall \varphi \in D(Q) \}.
\]

Formally, we write \( L = -\Delta + V \) as a Schrödinger operator with potential \( V \). Let \( \{ e^{-tL} \}_{t > 0} \) be the semigroup of linear operators generated by \( -L \) and \( p_t(x, y) \) be their kernels. Since \( V \) is nonnegative, the Feynman-Kac formula implies that

\[
0 \leq p_t(x, y) \leq \frac{1}{(4\pi t)^{n/2}} e^{-\frac{|x-y|^2}{4t}}
\]

for all \( t > 0 \) and \( x, y \in \mathbb{R}^n \).

Since the Schrödinger operator \( L \) is a self-adjoint positive definite operator acting on \( L^2(\mathbb{R}^n) \), then \( L \) admits the following spectral resolution

\[
L = \int_0^\infty \lambda \, dE_L(\lambda),
\]

where the \( E_L(\lambda) \) are spectral projectors. For any \( \gamma \in \mathbb{R} \), we shall define the imaginary power \( L^{i\gamma} \) associated to \( L \) by the formula

\[
L^{i\gamma} = \int_0^\infty \lambda^{i\gamma} \, dE_L(\lambda).
\]

By the functional calculus for \( L \), we can also define the operator \( L^{i\gamma} \) as follows

\[
L^{i\gamma}(f)(x) = \frac{1}{\Gamma(-i\gamma)} \int_0^\infty t^{-i\gamma - 1} e^{-tL}(f)(x) \, dt.
\]

By spectral theory \( \| L^{i\gamma} \|_{L^2 \rightarrow L^2} = 1 \) for all \( \gamma \in \mathbb{R} \). Moreover, it was proved by Shen [12] that \( L^{i\gamma} \) is a Calderón-Zygmund operator provided that \( V \in RH_{n/2} \) (Reverse Hölder class). We refer the readers to [6,7,14] for related results concerning the imaginary powers of self-adjoint operators.

For any \( 0 < \alpha < n \), the fractional integrals \( L^{-\alpha/2} \) associated to \( L \) is defined by

\[
L^{-\alpha/2}(f)(x) = \frac{1}{\Gamma(\alpha/2)} \int_0^\infty t^{\alpha/2 - 1} e^{-tL}(f)(x) \, dt.
\]
Since the kernel \( p_t(x, y) \) of \( \{e^{-tL}\} \) satisfies the Gaussian upper bound (1.1), then it is easy to check that \( |L^{-\alpha/2}(f)(x)| \leq C I_\alpha(|f|)(x) \) for all \( x \in \mathbb{R}^n \), where \( I_\alpha \) denotes the classical fractional integral operator (see [17])
\[
I_\alpha(f)(x) = \frac{\Gamma\left(\frac{n-\alpha}{2}\right)}{2^n \pi^{\frac{n+\alpha}{2}}} \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}} dy.
\]

Hence, by using the \( L^p-L^q \) boundedness of \( I_\alpha \) (see [17]), we have
\[
\|L^{-\alpha/2}(f)\|_{L^q} \leq C \|I_\alpha(f)\|_{L^q} \leq C \|f\|_{L^p},
\]
where \( 1 < p < n/\alpha \) and \( 1/q = 1/p - \alpha/n \). For more information about the fractional integrals \( L^{-\alpha/2} \) associated to a general class of operators, we refer the readers to [3,9,20].

In [16], Song and Yan introduced the weighted Hardy spaces \( H^1_L(w) \) associated to \( L \) in terms of the area integral function and established their atomic decomposition theory. They also showed that the Riesz transform \( \nabla L^{-1/2} \) is bounded on \( L^p(w) \) for \( 1 < p < 2 \), and bounded from \( H^1_L(w) \) to the classical weighted Hardy space \( H^1(w) \) (see [4,18]).

Recently, in [19], we defined the weighted Hardy spaces \( H^p_L(w) \) associated to \( L \) for \( 0 < p < 1 \) and gave their atomic decompositions. We also obtained that \( \nabla L^{-1/2} \) is bounded from \( H^p_L(w) \) to the classical weighted Hardy space \( H^p(w) \) (see also [4,18]) for \( n/(n+1) < p < 1 \). In this article, we first define weighted molecules for the weighted Hardy spaces \( H^p_L(w) \) associated to \( L \) and then establish their molecular characterizations. As an application of the molecular characterization combining with the atomic decomposition of \( H^p_L(w) \), we shall obtain some estimates of \( L^{i\gamma} \) and \( L^{-\alpha/2} \) on \( H^p_L(w) \) for \( n/(n+1) < p \). Our main results are stated as follows.

**Theorem 1.1.** Let \( L = -\Delta + V, n/(n+1) < p \leq 1 \) and \( w \in A_1 \cap RH_{(2/p)'} \). Then for any \( \gamma \in \mathbb{R} \), the imaginary power \( L^{i\gamma} \) is bounded from \( H^p_L(w) \) to the weighted Lebesgue space \( L^p(w) \).

**Theorem 1.2.** Let \( L = -\Delta + V, n/(n+1) < p \leq 1 \) and \( w \in A_1 \cap RH_{(2/p)'} \). Then for any \( \gamma \in \mathbb{R} \), the imaginary power \( L^{i\gamma} \) is bounded on \( H^p_L(w) \).

**Theorem 1.3.** Suppose that \( L = -\Delta + V \). Let \( 0 < \alpha < n/2, n/(n+1) < p \leq 1, 1/q = 1/p - \alpha/n \) and \( w \in A_1 \cap RH_{(2/p)'} \). Then the fractional integral operator \( L^{-\alpha/2} \) is bounded from \( H^p_L(w) \) to \( L^q(w^{d/p}) \).

**Theorem 1.4.** Suppose that \( L = -\Delta + V \). Let \( 0 < \alpha < \min\{n/2, 1\}, n/(n+1) < p \leq n/(n+\alpha), 1/q = 1/p - \alpha/n \) and \( w \in A_1 \cap RH_{(2/p)'} \). Then the fractional integral operator \( L^{-\alpha/2} \) is bounded from \( H^p_L(w) \) to \( H^q_L(w^{d/p}) \).
2 Notations and preliminaries

First, let us recall some standard definitions and notations. The classical $A_p$ weight theory was first introduced by Muckenhoupt in the study of weighted $L^p$ boundedness of Hardy-Littlewood maximal functions in [10]. A weight $w$ is a locally integrable function on $\mathbb{R}^n$ which takes values in $(0, \infty)$ almost everywhere, $B = B(x_0, r_B)$ denotes the ball with the center $x_0$ and radius $r_B$. We say that $w \in A_1$, if

$$\frac{1}{|B|} \int_B w(x) \, dx \leq C \cdot \text{ess inf}_{x \in B} w(x)$$

for every ball $B \subseteq \mathbb{R}^n$.

where $C$ is a positive constant which is independent of $B$.

A weight function $w$ is said to belong to the reverse Hölder class $RH_r$ if there exist two constants $r > 1$ and $C > 0$ such that the following reverse Hölder inequality holds

$$\left( \frac{1}{|B|} \int_B w(x)^r \, dx \right)^{1/r} \leq C \left( \frac{1}{|B|} \int_B w(x) \, dx \right)$$

for every ball $B \subseteq \mathbb{R}^n$.

Given a ball $B$ and $\lambda > 0$, $\lambda B$ denotes the ball with the same center as $B$ whose radius is $\lambda$ times that of $B$. For a given weight function $w$, we denote the Lebesgue measure of $B$ by $|B|$ and the weighted measure of $B$ by $w(B)$, where $w(B) = \int_B w(x) \, dx$.

We give the following results which will be often used in the sequel.

**Lemma 2.1** ([5]). Let $w \in A_1$. Then, for any ball $B$, there exists an absolute constant $C$ such that

$$w(2B) \leq C \, w(B).$$

In general, for any $\lambda > 1$, we have

$$w(\lambda B) \leq C \cdot \lambda^n w(B),$$

where $C$ does not depend on $B$ nor on $\lambda$.

**Lemma 2.2** ([5]). Let $w \in A_1$. Then there exists a constant $C > 0$ such that

$$C \cdot \frac{|E|}{|B|} \leq \frac{w(E)}{w(B)}$$

for any measurable subset $E$ of a ball $B$. 


Given a Muckenhoupt’s weight function $w$ on $\mathbb{R}^n$, for $0 < p < \infty$, we denote by $L^p(w)$ the space of all functions satisfying

$$
\|f\|_{L^p(w)} = \left( \int_{\mathbb{R}^n} |f(x)|^p w(x) \, dx \right)^{1/p} < \infty.
$$

Throughout this article, we will use $C$ to denote a positive constant, which is independent of the main parameters and not necessarily the same at each occurrence. By $A \sim B$, we mean that there exists a constant $C > 1$ such that $\frac{1}{C} \leq A/B \leq C$. Moreover, we denote the conjugate exponent of $s > 1$ by $s' = s/(s-1)$.

3 Atomic decomposition and molecular characterization of weighted Hardy spaces

Let $L = -\Delta + V$. For any $t > 0$, we define $P_t = e^{-tL}$ and

$$
Q_{t,k} = (-t)^k \frac{d^k P_s}{ds^k} \bigg|_{s=t} = (tL)^k e^{-tL}, \quad k = 1, 2, \ldots.
$$

We denote simply by $Q_t$ when $k = 1$. First note that Gaussian upper bounds carry over from heat kernels to their time derivatives.

Lemma 3.1 ([2,11]). For every $k = 1, 2, \ldots$, there exist two positive constants $C_k$ and $c_k$ such that the kernel $p_{t,k}(x,y)$ of the operator $Q_{t,k}$ satisfies

$$
|p_{t,k}(x,y)| \leq \frac{C_k}{(4\pi t)^{n/2}} e^{-\frac{|x-y|^2}{4t}}
$$

for all $t > 0$ and almost all $x, y \in \mathbb{R}^n$.

Set

$$
H^2(\mathbb{R}^n) = \overline{R(L)} = \{Lu \in L^2(\mathbb{R}^n) : u \in L^2(\mathbb{R}^n)\},
$$

where $\overline{R(L)}$ stands for the range of $L$. We also set

$$
\Gamma(x) = \{(y,t) \in \mathbb{R}^{n+1}_+ : |x-y| < t\}.
$$

For a given function $f \in L^2(\mathbb{R}^n)$, we consider the area integral function associated to Schrödinger operator $L$(see [1,8])

$$
S_L(f)(x) = \left( \int_{\Gamma(x)} |Q_{t,k}(f,y)|^2 \, dydt \right)^{1/2}, \quad x \in \mathbb{R}^n.
$$
Similarly, in [19], we introduced the notion of weighted atoms for \(L^p\) and proved their atomic characterizations.

**Definition 3.2** ([16]). Let \(M \in \mathbb{N}\). A function \(a(x) \in L^2(\mathbb{R}^n)\) is called a \((1,2,M)\)-atom with respect to \(w\) (or a \(w\)-(1,2,M)-atom) if there exist a ball \(B = B(x_0,r_B)\) and a function \(b \in \mathcal{D}(L^M)\) such that

(a) \(a = L^M b\);

(b) \(\text{supp} L^k b \subseteq B, \quad k = 0, 1, \ldots, M\);

(c) \(\| (r_B^2 L^k b) \|_{L^2(B)} \leq r_B^{2M} |B|^{1/2} w(B)^{-1}, \quad k = 0, 1, \ldots, M\).

**Theorem 3.3** ([16]). Let \(M \in \mathbb{N}\) and \(w \in A_1 \cap RH_2\). If \(f \in H^1_L(w)\), then there exist a family of \(w\)-(1,2,M)-atoms \(\{a_j\}\) and a sequence of real numbers \(\{\lambda_j\}\) with \(\sum_j |\lambda_j| \leq C\|f\|_{H^1_L(w)}\) such that \(f\) can be represented in the form \(f(x) = \sum_j \lambda_j a_j(x)\), and the sum converges both in the sense of \(L^2(\mathbb{R}^n)\)-norm and \(H^1_L(w)\)-norm.

Similarly, in [19], we introduced the notion of weighted atoms for \(H^p_L(w)(0 < p < 1)\) and proved their atomic characterizations.

**Definition 3.4** ([19]). Let \(M \in \mathbb{N}\) and \(0 < p < 1\). A function \(a(x) \in L^2(\mathbb{R}^n)\) is called a \((p,2,M)\)-atom with respect to \(w\) (or a \(w\)-(p,2,M)-atom) if there exist a ball \(B = B(x_0,r_B)\) and a function \(b \in \mathcal{D}(L^M)\) such that

(a’) \(a = L^M b\);

(b’) \(\text{supp} L^k b \subseteq B, \quad k = 0, 1, \ldots, M\);

(c’) \(\| (r_B^2 L^k b) \|_{L^2(B)} \leq r_B^{2M} |B|^{1/2} w(B)^{-1/p}, \quad k = 0, 1, \ldots, M\).

**Theorem 3.5** ([19]). Let \(M \in \mathbb{N}\), \(0 < p < 1\) and \(w \in A_1 \cap RH(2/p)'\). If \(f \in H^p_L(w)\), then there exist a family of \(w\)-(p,2,M)-atoms \(\{a_j\}\) and a sequence of real numbers \(\{\lambda_j\}\) with \(\sum_j |\lambda_j|^p \leq C\|f\|_{H^p_L(w)}^p\) such that \(f\) can be represented in the form \(f(x) = \sum_j \lambda_j a_j(x)\), and the sum converges both in the sense of \(L^2(\mathbb{R}^n)\)-norm and \(H^p_L(w)\)-norm.

For every bounded Borel function \(F : [0, \infty) \to \mathbb{C}\), we define the operator \(F(L) : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)\) by the following formula

\[ F(L) = \int_0^\infty F(\lambda) \, dE_L(\lambda), \]

where \(E_L(\lambda)\) is the distribution function of \(L\).
where $E_L(\lambda)$ is the spectral decomposition of $L$. Therefore, the operator $\cos(t\sqrt{L})$ is well-defined on $L^2(\mathbb{R}^n)$. Moreover, it follows from [13] that there exists a constant $c_0$ such that the Schwartz kernel $K_{\cos(t\sqrt{L})}(x,y)$ of $\cos(t\sqrt{L})$ has support contained in $\{(x,y) \in \mathbb{R}^n \times \mathbb{R}^n : |x-y| \leq c_0 t\}$. By the functional calculus for $L$ and Fourier inversion formula, whenever $F$ is an even bounded Borel function with $F \in L^1(\mathbb{R})$, we can write $F(\sqrt{L})$ in terms of $\cos(t\sqrt{L})$; precisely

$$F(\sqrt{L}) = (2\pi)^{-1} \int_{-\infty}^{\infty} \hat{F}(t) \cos(t\sqrt{L}) \, dt,$$

which gives

$$K_{F(\sqrt{L})}(x,y) = (2\pi)^{-1} \int_{|t| \geq c_0^{-1}|x-y|} \hat{F}(t) K_{\cos(t\sqrt{L})}(x,y) \, dt.$$

**Lemma 3.6** ([8]). Let $\varphi \in C_0^\infty(\mathbb{R})$ be even and $\text{supp} \varphi \subseteq [-c_0^{-1},c_0^{-1}]$. Let $\Phi$ denote the Fourier transform of $\varphi$. Then for each $j = 0, 1, \ldots$, and for all $t > 0$, the Schwartz kernel $K_{(t^2L)^j \Phi(t\sqrt{L})}(x,y)$ of $(t^2L)^j \Phi(t\sqrt{L})$ satisfies

$$\text{supp} K_{(t^2L)^j \Phi(t\sqrt{L})} \subseteq \{(x,y) \in \mathbb{R}^n \times \mathbb{R}^n : |x-y| \leq t\}.$$

For a given $s > 0$, we set

$$\mathcal{F}(s) = \left\{ \psi : \mathbb{C} \to \mathbb{C} \text{ measurable}, |\psi(z)| \leq C \frac{|z|^s}{1 + |z|^{2s}} \right\}.$$

Then for any nonzero function $\psi \in \mathcal{F}(s)$, we have the following estimate (see [16])

$$\left( \int_0^\infty \| \psi(t\sqrt{L})f \|_{L^2(\mathbb{R}^n)}^2 \frac{dt}{t} \right)^{1/2} \leq C \| f \|_{L^2(\mathbb{R}^n)}. \quad (3.1)$$

We are now going to define the weighted molecules corresponding to the weighted atoms mentioned above.

**Definition 3.7.** Let $\varepsilon > 0$, $M \in \mathbb{N}$ and $0 < p \leq 1$. A function $m(x) \in L^2(\mathbb{R}^n)$ is called a $w-(p,2,M,\varepsilon)$-molecule associated to $L$ if there exist a ball $B = B(x_0,r_B)$ and a function $b \in \mathcal{D}(L^M)$ such that

(A) $m = L^M b$; 
(B) $\| (r_B^2 L)^k b \|_{L^2(2B)} \leq r_B^{2M} |B|^{1/2} w(B)^{-1/p}$, $k = 0, 1, \ldots, M$; 
(C) $\| (r_B^2 L)^k b \|_{L^2(2^{j+1}B \setminus 2jB)} \leq 2^{-j \varepsilon} r_B^{2M} |2^j B|^{1/2} w(2^j B)^{-1/p}$, $k = 0, 1, \ldots, M$, $j = 1, 2, \ldots$. 

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Note that for every $w$-$(p, 2, M)$-atom $a$, it is a $w$-$(p, 2, M, \varepsilon)$-molecule for all $\varepsilon > 0$. Then we are able to establish the following molecular characterization for the weighted Hardy spaces $H^p_L(w)(0 < p \leq 1)$ associated to $L$.

**Theorem 3.8.** Let $\varepsilon > 0$, $M \in \mathbb{N}$, $0 < p \leq 1$ and $w \in A_1 \cap RH_{(2/p)}^\infty$.

(i) If $f \in H^p_L(w)$, then there exist a family of $w$-$(p, 2, M, \varepsilon)$-molecules $\{m_j\}$ and a sequence of real numbers $\{\lambda_j\}$ with $\sum_j |\lambda_j|^p \leq C\|f\|^p_{H^p_L(w)}$ such that $f(x) = \sum_j \lambda_j m_j(x)$, and the sum converges both in the sense of $L^2(\mathbb{R}^n)$-norm and $H^p_L(w)$-norm.

(ii) Assume that $M > \frac{4}{p}(\frac{1}{p} - \frac{1}{2})$. Then every $w$-$(p, 2, M, \varepsilon)$-molecule $m$ is in $H^p_L(w)$. Moreover, there exists a constant $C > 0$ independent of $m$ such that $\|m\|_{H^p_L(w)} \leq C$.

**Proof.** (i) is a straightforward consequence of Theorems 3.3 and 3.5.

(ii) We follow the same constructions as in [8]. Suppose that $m$ is a $w$-$(p, 2, M, \varepsilon)$-molecule associated to a ball $B = B(x_0, r_B)$. Let $\varphi \in C_0^\infty(\mathbb{R})$ be even with $\text{supp} \varphi \subseteq [-2c_0, 2c_0]$ and let $\Phi$ denote the Fourier transform of $\varphi$. We set $\Psi(x) = x^2 \Phi(x), x \in \mathbb{R}$. By the $L^2$-functional calculus of $L$, for every $m \in L^2(\mathbb{R}^n)$, we can establish the following version of the Calderón reproducing formula

$$m(x) = c_\psi \int_0^\infty (t^2 L)^M \Psi(t \sqrt{L})(m)(x) \frac{dt}{t}, \quad (3.2)$$

where the above equality holds in the sense of $L^2(\mathbb{R}^n)$-norm. Set $U_0(B) = 2B$, $U_j(B) = 2^{j+1}B \setminus 2^jB, j = 1, 2, \ldots, \text{then we can decompose}$

$$\mathbb{R}^n \times (0, \infty) = \left( \bigcup_{j=0}^\infty U_j(B) \times (0, 2^j r_B) \right) \cup \left( \bigcup_{j=1}^\infty 2^j B \times (2^{j-1} r_B, 2^j r_B) \right).$$

Hence, by the formula (3.2), we are able to write

$$m(x) = c_\psi \sum_{j=0}^\infty \int_0^{2^j r_B} (t^2 L)^M \Psi(t \sqrt{L})(m \chi_{U_j(B)})(x) \frac{dt}{t}$$

$$+ c_\psi \sum_{j=1}^\infty \int_{2^{j-1} r_B}^{2^j r_B} (t^2 L)^M \Psi(t \sqrt{L})(m \chi_{2^j B})(x) \frac{dt}{t}$$

$$= \sum_{j=0}^\infty m_j^{(1)}(x) + \sum_{j=1}^\infty m_j^{(2)}(x).$$
Let us first estimate the terms \( \{m_j^{(1)}\}_{j=0}^{\infty} \). We will show that each \( m_j^{(1)} \) is a multiple of \( w-(p, 2, M) \)-atom with a sequence of coefficients in \( L^p \). Indeed, for every \( j = 0, 1, 2, \ldots, \) one can write

\[
m_j^{(1)}(x) = L^M b_j(x),
\]

where

\[
b_j(x) = c_\psi \int_0^{2^{j+1}r_B} t^{2M} \Psi^2(t \sqrt{L})(m \chi_{U_j(B)})(x) \frac{dt}{t}.
\]

By Lemma 3.6, we can easily conclude that for every \( k = 0, 1, \ldots, M, \) \( \text{supp}(L^k b_j) \subseteq 2^{j+1} B \). Since

\[
\left\| (2^{j+1} r_B) L^k b_j \right\|_{L^2(2^{j+1} B)} = \sup_{\|h\|_{L^2(2^{j+1} B)} \leq 1} \left| \int_{2^{j+1} B} [(2^{j+1} r_B) L^k b_j(x)h(x)] dx \right|.
\]

Then it follows from Hölder’s inequality and the estimate (3.1) that

\[
\left| \int_{2^{j+1} B} [(2^{j+1} r_B) L^k b_j(x)h(x)] dx \right|
\]

\[
= c_\psi (2^{j+1} r_B) 2^k \left| \int_0^{2^{j+1} r_B} \int_{2^{j+1} B} t^{2M} L^k \Psi(t \sqrt{L})(m \chi_{U_j(B)})(y) \Psi(t \sqrt{L})(h)(y) \frac{dy dt}{t} \right|
\]

\[
\leq c_\psi (2^{j+1} r_B) 2^k (2^j r_B)^{2M-2k} \left| \int_0^{2^{j+1} r_B} \int_{2^{j+1} B} (t^2 L)^k \Psi(t \sqrt{L})(m \chi_{U_j(B)})(y) \Psi(t \sqrt{L})(h)(y) \frac{dy dt}{t} \right|
\]

\[
\leq c_\psi (2^{j+1} r_B)^{2M} \left( \int_0^\infty \| (t^2 L)^k \Psi(t \sqrt{L})(m \chi_{U_j(B)}) \|_{L^2(\mathbb{R}^n)}^2 \frac{dt}{t} \right)^{1/2}
\]

\[
\times \left( \int_0^\infty \| \Psi(t \sqrt{L})(h \chi_{2^{j+1} B}) \|_{L^2(\mathbb{R}^n)}^2 \frac{dt}{t} \right)^{1/2}
\]

\[
\leq c_\psi (2^{j+1} r_B)^{2M} \| m \chi_{U_j(B)} \|_{L^2(\mathbb{R}^n)} \cdot \| h \chi_{2^{j+1} B} \|_{L^2(\mathbb{R}^n)}
\]

\[
\leq C \cdot 2^{-j^p} (2^j r_B)^{2M} |2^j+1 B|^{1/2} w(2^j+1 B)^{-1/p}.
\]

Hence

\[
\left\| (2^{j+1} r_B) L^k b_j \right\|_{L^2(2^{j+1} B)} \leq C \cdot 2^{-j^p} (2^j r_B)^{2M} |2^j+1 B|^{1/2} w(2^j+1 B)^{-1/p},
\]

which implies our desired result. Next we consider the terms \( \{m_j^{(2)}\}_{j=1}^\infty \).
For every $j = 1, 2, \ldots$, we write
\[
m_j^{(2)}(x) = c_\psi \int_{2j-1}^{2j} (t^2 L)^M \Psi^2(t \sqrt{L})(m(x)) \frac{dt}{t} - c_\psi \int_{2j-1}^{2j} (t^2 L)^M \Psi^2(t \sqrt{L})(m(x))(21) \frac{dt}{t} \]
\[
= m_j^{(21)}(x) - m_j^{(22)}(x).
\]
To deal with the term $m_j^{(21)}$, we recall that $m = L^M b$ for some $b \in \mathcal{D}(L^M)$.
Then we have
\[
m_j^{(21)}(x) = c_\psi \int_{2j-1}^{2j} (t^2 L)^M \Psi^2(t \sqrt{L})(L^M b)(x) \frac{dt}{t}
\]
\[
= L^M b_j^{(21)}(x),
\]
where
\[
b_j^{(21)}(x) = c_\psi \int_{2j-1}^{2j} (t^2 L)^M \Psi^2(t \sqrt{L})(b(x)) \frac{dt}{t}.
\]
Since $b(x) = b(x) \chi_{2j B}(x) + \sum_{i=1}^{\infty} b(x) \chi_{U_i(B)}(x)$. Then we can further write
\[
b_j^{(21)}(x) = b_{1,j}^{(21)}(x) + \sum_{l=j}^{\infty} b_{l,j}^{(21)}(x),
\]
where
\[
b_{1,j}^{(21)}(x) = c_\psi \int_{2j-1}^{2j} (t^2 L)^M \Psi^2(t \sqrt{L})(b(x) \chi_{2j B})(x) \frac{dt}{t}
\]
and
\[
b_{l,j}^{(21)}(x) = c_\psi \int_{2j-1}^{2j} (t^2 L)^M \Psi^2(t \sqrt{L})(b(x) \chi_{U_l(B)})(x) \frac{dt}{t}.
\]
By using Lemma 3.6 again, we have $\text{supp} \ (L^k b_{1,j}^{(21)}) \subseteq 2^j B$ and $\text{supp} \ (L^k b_{l,j}^{(21)}) \subseteq 2^{l+1} B$ for every $k = 0, 1, \ldots, M$. Moreover, it follows from Minkowski’s integral inequality that
\[
\left\| \left[ (t^2 R)^2 L \right]^k b_{1,j}^{(21)} \right\|_{L^2(2j B)}
\]
\[
= c_\psi (2^j R)^{2k} \left\| \int_{2j-1}^{2j} (t^2 L)^{M+k} \Psi^2(t \sqrt{L})(b(x) \chi_{2j B}) \frac{dt}{t} \right\|_{L^2(2j B)}
\]
\[
\leq c_\psi (2^j R)^{2k} \int_{2j-1}^{2j} \left\| (t^2 L)^{M+k} \Psi^2(t \sqrt{L})(b(x) \chi_{2j B}) \right\|_{L^2(2j B)} \frac{dt}{10^{2k+1}}
\]
\[
\leq C \left\| b(x) \chi_{2j B} \right\|_{L^2(2j B)}
\]
\[
\begin{align*}
&\leq C \sum_{l=0}^{j-1} \| b \chi_{U_l(B)} \|_{L^2(2^l B)} \\
&\leq C \sum_{l=0}^{j-1} 2^{-l} r_{2^l B} M |2^l B|^{-1/2} w(2^l B)^{-1/p}.
\end{align*}
\]

When \(0 \leq l \leq j-1\), then \(2^l B \subseteq 2^j B\). By using Lemma 2.2, we can get
\[
\frac{w(2^l B)}{w(2^j B)} \geq C \cdot \frac{|2^l B|}{|2^j B|}.
\]

Consequently
\[
\left\| \left[ (2^j r_B)^2 L \right]^k b_{1,j}^{(21)} \right\|_{L^2(2^j B)}
\leq C \cdot 2^{-j(2M-n(1/p-1/2))} \cdot (2^j r_B)^{2M} |2^j B|^{-1/2} w(2^j B)^{-1/p} \sum_{l=0}^{\infty} \frac{1}{2^{l+1}} \cdot \frac{1}{2^{l(n/p-n/2)}}
\leq C \cdot 2^{-j(2M-n(1/p-1/2))} \cdot (2^j r_B)^{2M} |2^j B|^{-1/2} w(2^j B)^{-1/p}.
\]

On the other hand
\[
\left\| \left[ (2^{l+1} r_B)^2 L \right]^k b_{l+1,j}^{(21)} \right\|_{L^2(2^{l+1} B)}
= c_{\psi}(2^{l+1} r_B)^{2k} \left\| \int_{2^{l-1} r_B}^{2^{l+1} r_B} t^{2M} L^{M+k} \Psi^2(t \sqrt{L}) (b \chi_{U_l(B)}) \frac{dt}{t} \right\|_{L^2(2^{l+1} B)}
\leq c_{\psi}(2^{l+1} r_B)^{2k} \left\| (t^2 L)^{M+k} \Psi^2(t \sqrt{L}) (b \chi_{U_l(B)}) \right\|_{L^2(2^{l+1} B)} \frac{dt}{t^{2k+1}}
\leq C(2^{l+1} r_B)^{2k} \| b \chi_{U_l(B)} \|_{L^2(2^{l+1} B)} \cdot \frac{1}{(2^j r_B)^{2k}}
\leq C \cdot 2^{-l} (2^{l+1} r_B)^{2M} |2^{l+1} B|^{-1/2} w(2^{l+1} B)^{-1/p}.
\]

Observe that \(2M > n(1/p-1/2)\). Thus, from the above discussions, we have proved that each \(m_j^{(21)}\) is a multiple of \(w\cdot (p, 2, M)\)-atom with a sequence of coefficients in \(l^p\). Finally, we estimate the terms \(\{m_j^{(22)}\}_{j=1}^{\infty}\). For every \(j = 1, 2, \ldots\), we decompose \(m_j^{(22)}\) as follows
\[
m_j^{(22)}(x) = c_{\psi} \sum_{l=j}^{\infty} \int_{2^{l-1} r_B}^{2^{l+1} r_B} (t^2 L)^{M} \Psi^2(t \sqrt{L}) (m \chi_{U_l(B)})(x) \frac{dt}{t}
= \sum_{l=j}^{\infty} L^M b_{l,j}^{(22)}(x),
\]
where
\[ b^{(22)}_{ij}(x) = c_\psi \int_{2^{j-1}r_B}^{2^jr_B} t^{2M} \Psi^2(t \sqrt{L})(m \chi_{U_i(B)})(x) \frac{dt}{t}. \]

It follows immediately from Lemma 3.6 that \( \text{supp}(L^k b^{(22)}_{ij}) \subseteq 2^{i+1}B \) for every \( k = 1, 2, \ldots, M \) and \( l \geq j \). Moreover
\[
\left\| (2^{2r_B} L)^k b^{(22)}_{ij} \right\|_{L^2(2^{i+1}B)} = c_\psi (2^{2r_B})^k \left\| \int_{2^{j-1}r_B}^{2^jr_B} t^{2M} L^k \Psi^2(t \sqrt{L})(m \chi_{U_i(B)}) \frac{dt}{t} \right\|_{L^2(2^{i+1}B)} \\
\leq c_\psi (2^{2r_B})^{2k} (2^{2r_B})^{2M-2k} \int_{2^{j-1}r_B}^{2^jr_B} \left\| (t^2 L)^k \Psi^2(t \sqrt{L})(m \chi_{U_i(B)}) \right\|_{L^2(2^{i+1}B)} \frac{dt}{t} \\
\leq C \cdot 2^{-\varepsilon} (2^{2r_B})^{2M} |2^{i+1}B|^{1/2} w(2^{i+1}B)^{-1/p}.
\]

Therefore, we have showed that each \( m^{(22)}_j \) is also a multiple of \( w\)-\((p, 2, M)\)-atom with a sequence of coefficients in \( l^p \). This completes the proof of Theorem 3.8. \( \square \)

4  Proofs of Theorems 1.1 and 1.2

**Proof of Theorem 1.1.** For any \( \gamma \in \mathbb{R} \), since the operator \( L^{i\gamma} \) is linear and bounded on \( L^2(\mathbb{R}^n) \), then by Theorems 3.3 and 3.5, it is enough to show that for any \( w\)-\((p, 2, M)\)-atom \( a \), \( M \in \mathbb{N} \), there exists a constant \( C > 0 \) independent of \( a \) such that \( \|L^{i\gamma}(a)\|_{L^p(w)} \leq C \). Let \( a \) be a \( w\)-\((p, 2, M)\)-atom with \( \text{supp} a \subseteq B = B(x_0, r_B) \), \( \|a\|_{L^2(B)} \leq |B|^{1/2} w(B)^{-1/p} \). We write
\[
\|L^{i\gamma}(a)\|_{L^p(w)}^p = \int_{2B} |L^{i\gamma}(a)(x)|^p w(x) \, dx + \int_{(2B)^c} |L^{i\gamma}(a)(x)|^p w(x) \, dx \\
= I_1 + I_2.
\]

Set \( s = 2/p > 1 \). Note that \( w \in RH_s' \), then it follows from Hölder’s inequality, the \( L^2 \) boundedness of \( L^{i\gamma} \) and Lemma 2.1 that
\[
I_1 \leq \left( \int_{2B} |L^{i\gamma}(a)(x)|^2 \, dx \right)^{p/2} \left( \int_{2B} w(x)^s \, dx \right)^{1/s'} \\
\leq C \|a\|_{L^2(B)}^p \cdot \frac{w(2B)}{|2B|^{1/s}} \\
\leq C.
\]
On the other hand, by using Hölder’s inequality and the fact that \( w \in RH_{s'} \), we can get

\[
I_2 = \sum_{k=1}^{\infty} \int_{2^{k+1}B \setminus 2^kB} \left| L^\gamma(a)(x) \right|^p w(x) \, dx
\]

\[
\leq C \sum_{k=1}^{\infty} \left( \int_{2^{k+1}B \setminus 2^kB} \left| L^\gamma(a)(x) \right|^2 \, dx \right)^{p/2} \cdot \frac{w(2^{k+1}B)}{|2^{k+1}B|^{1/s}}. \tag{4.2}
\]

For any \( x \in 2^{k+1}B \setminus 2^kB, \ k = 1, 2, \ldots \), by the expression (1.2), we can write

\[
\left| L^\gamma(a)(x) \right| \leq C \int_0^\infty e^{-tL}a(x) \frac{dt}{t} \\
\leq C \int_0^{r_B^2} e^{-tL}a(x) \frac{dt}{t} + C \int_r^{\infty} e^{-tL}a(x) \frac{dt}{t} \\
= I + II.
\]

For the term I, we observe that when \( x \in 2^{k+1}B \setminus 2^kB, \ y \in B \), then \( |x - y| \geq 2^{k-1}r_B \). Hence, by using Hölder’s inequality and the estimate (1.1), we deduce

\[
\left| e^{-tL}a(x) \right| \leq C \cdot \frac{t^{1/2}}{(2^{k-1}r_B)^n+1} \int_B |a(y)| \, dy \\
\leq C \cdot \frac{t^{1/2}}{(2^{k}r_B)^n+1} \|a\|_{L^2(\mathbb{R}^n)} |B|^{1/2} \\
\leq C \cdot w(B)^{-1/p} \cdot \frac{t^{1/2}}{2^{k(n+1)} \cdot r_B}. \tag{4.3}
\]

So we have

\[
I \leq C \cdot \frac{1}{2^{k(n+1)}w(B)^{1/p}} \cdot \frac{1}{r_B} \int_0^{r_B^2} \frac{dt}{\sqrt{t}} \\
\leq C \cdot \frac{1}{2^{k(n+1)}w(B)^{1/p}}.
\]

We now turn to estimate the other term II. In this case, since there exists a function \( b \in \mathcal{D}(L^M) \) such that \( a = L^Mb \) and \( \|b\|_{L^2(B)} \leq r_B^{2M} |B|^{1/2} w(B)^{-1/p}, \)
then it follows from Hölder’s inequality and Lemma 3.1 that
\[
|e^{-tL}a(x)| = |(tL)^{\frac{1}{M}} e^{-tL}b(x)| \cdot \frac{1}{t^M}
\]
\[
\leq C \cdot \frac{1}{(2k-1)^{n+1}} \int_B |b(y)|\, dy \cdot \frac{1}{t^{M-1/2}}
\]
\[
\leq C \cdot \frac{1}{(2k)^{n+1}} \|b\|_{L^2(\mathbb{R}^n)} |B|^{1/2} \cdot \frac{1}{t^{M-1/2}}
\]
\[
\leq C \cdot \frac{r_B^{2M-1}}{2^{k(n+1)}w(B)^{1/p}} \cdot \frac{1}{t^{M-1/2}}.
\]

(4.4)

Consequently
\[
\Pi \leq C \cdot \frac{1}{2^{k(n+1)}w(B)^{1/p}} \cdot r_B^{2M-1} \int_{r_B}^{\infty} \frac{dt}{t^{M+1/2}}
\]
\[
\leq C \cdot \frac{1}{2^{k(n+1)}w(B)^{1/p}},
\]

where in the last inequality we have used the fact that \(M \geq 1\). Therefore, by combining the above estimates for \(I\) and \(\Pi\), we obtain
\[
|L^{i\gamma}(a)(x)| \leq C \cdot \frac{1}{2^{k(n+1)}w(B)^{1/p}}, \text{ when } x \in 2^{k+1}B \setminus 2^k B.
\]

(4.5)

Substituting the above inequality (4.5) into (4.2) and using Lemma 2.1, then we have
\[
I_2 \leq C \sum_{k=1}^{\infty} \frac{1}{2^{kp(n+1)}w(B)} \cdot w(2^{k+1}B)
\]
\[
\leq C \sum_{k=1}^{\infty} \frac{1}{2^{kp(n+1)-n}}
\]
\[
\leq C,
\]

(4.6)

where the last series is convergent since \(p > n/(n+1)\). Summarizing the estimates (4.1) and (4.6) derived above, we complete the proof of Theorem 1.1.

**Proof of Theorem 1.2.** Since the operator \(L^{i\gamma}\) is linear and bounded on \(L^2(\mathbb{R}^n)\), then by using Theorems 3.3, 3.5 and 3.8, it suffices to verify that for every \(w\)-\((p, 2, 2M)\)-atom \(a\), the function \(m = L^{i\gamma}(a)\) is a multiple of \(w\)-\((p, 2, M, \varepsilon)\)-molecule for some \(\varepsilon > 0\), and the multiple constant is independent of
a. Let \( a \) be a \( w-(p, 2, 2M) \)-atom with \( \text{supp} \ a \subseteq B = B(x_0, r_B) \). Then by definition, there exists a function \( \tilde{b} \in \mathcal{D}(L^{2M}) \) such that \( a = L^{2M}(\tilde{b}) \) and \( \| (r_B^2 L)^k \tilde{b} \|_{L^2(B)} \leq r_B^{4M} |B|^{1/2} w(B)^{-1/p} \), \( k = 0, \ldots, 2M \). We set \( \tilde{b} = L^{i\gamma} (L^M \tilde{b}) \), then we have \( m = L^M (\tilde{b}) \). Obviously, \( m(x) \in L^2(\mathbb{R}^n) \). Moreover, for \( k = 0, 1, \ldots, M \), we can deduce

\[
\| (r_B^2 L)^k \tilde{b} \|_{L^2(2B)} = \frac{1}{r_B^{2M}} \| L^{i\gamma} [(r_B^2 L)^{M+k} b] \|_{L^2(2B)} \\
\leq C \cdot \frac{1}{r_B} \| (r_B^2 L)^{M+k} b \|_{L^2(B)} \\
\leq C \cdot r_B^{2M} |B|^{1/2} w(B)^{-1/p}.
\]

It remains to estimate \( \| (r_B^2 L)^k \tilde{b} \|_{L^2(2^{j+1}B \setminus 2^j B)} \) for \( k = 0, 1, \ldots, M, \ j = 1, 2, \ldots \). We write

\[
\| (r_B^2 L)^k \tilde{b}(x) \| = |L^{i\gamma} [(r_B^2 L)^k L^M b](x)| \\
\leq C \int_0^{r_B^2} e^{-t\|L \|} |r_B L^{M+k} b|(x) \frac{dt}{t} + C \int_{r_B^2}^\infty e^{-t\|L \|} |r_B L^{M+k} b|(x) \frac{dt}{t} \\
= I' + II' .
\]

As mentioned in the proof of Theorem 1.1, we know that when \( x \in 2^{j+1}B \setminus 2^j B \), \( y \in B \), then \( |x-y| \geq 2^{j-1} r_B \), \( j = 1, 2, \ldots \). It follows from Hölder’s inequality and the estimate (1.1) that

\[
I' \leq C \int_0^{r_B^2} \frac{t^{1/2}}{(2^{j-1} r_B)^{n+1}} \| r_B L^{M+k} b \|_{L^2(\mathbb{R}^n)} |B|^{1/2} \frac{dt}{t} \\
\leq C \cdot \frac{1}{(2^{j} r_B)^{n+1}} \left( \frac{1}{r_B^{2M}} \cdot r_B^{4M} |B|^{1/2} w(B)^{-1/p} \right) |B|^{1/2} \int_0^{r_B^2} \frac{dt}{\sqrt{t}} \\
\leq C \cdot \frac{1}{2^{j(n+1)}} \cdot r_B^{2M} w(B)^{-1/p} .
\]

Since \( B \subseteq 2^j B \), \( j = 1, 2, \ldots \), then by using Lemma 2.2, we can get

\[
\frac{w(B)}{w(2^j B)} \geq C \cdot \frac{|B|}{|2^j B|} .
\]

Hence

\[
I' \leq C \cdot \frac{1}{2^{j(n+1)-n/p}} \cdot r_B^{2M} w(2^j B)^{-1/p} , \quad \text{when } x \in 2^{j+1}B \setminus 2^j B .
\]

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Applying Hölder’s inequality and Lemma 3.1, we obtain

\[ II' \leq C \cdot r_B^{2k} \int_{r_B^2}^{\infty} (tL)^{M+k}e^{-tL}(b)(x) \frac{dt}{e^{tL+1}} \]
\[ \leq C \cdot r_B^{2k} \int_{r_B^2}^{\infty} \frac{1}{(2^{j-1}r_B)^{n+1}} \|b\|_{L^2(\mathbb{R}^n)} |B|^{1/2} \frac{dt}{e^{tL+1}} \]
\[ \leq C \cdot \frac{1}{(2^{j}r_B)^{n+1}} \left(r_B^{4M+2k}|w(B)|^{-1/p} \right) \int_{r_B^2}^{\infty} \frac{dt}{e^{tL+1}} \]
\[ \leq C \cdot \frac{1}{2^{j(n+1)}} \cdot r_B^{2M} w(B)^{-1/p}. \]

(4.9)

It follows immediately from the above inequality (4.8) that

\[ II' \leq C \cdot \frac{1}{2^{j[(n+1)-n/p]}} \cdot r_B^{2M} w(2^j B)^{-1/p}, \quad \text{when } x \in 2^{j+1} B \setminus 2^j B. \]

Combining the above estimates for I’ and II’, we thus obtain

\[ \| (r_B^2 L) ^{k} b \|_{L^2(2^{j+1} B \setminus 2^j B)} \leq C \cdot \frac{1}{2^{j[(n+1)-n/p]}} \cdot r_B^{2M} |2^j B|^{1/2} w(2^j B)^{-1/p}. \]

Observe that \( p > n/(n+1) \). If we set \( \epsilon = (n+1) - n/p \), then we have \( \epsilon > 0 \). Therefore, we have proved that the function \( m = L^{\gamma}(a) \) is a multiple of \( w^{-}(p, 2, M, \epsilon) \)-molecule. This completes the proof of Theorem 1.2. \( \square \)

5 Proofs of Theorems 1.3 and 1.4

Proof of Theorem 1.3. As in the proof of Theorem 1.1, it suffices to prove that for every \( w^{-}(p, 2, M) \)-atom \( a \), \( M > (3n)/4 \), there exists a constant \( C > 0 \) independent of \( a \) such that \( \|L^{-\alpha/2}(a)\|_{L^q(w^{q/p})} \leq C \). We write

\[ \|L^{-\alpha/2}(a)\|_{L^q(w^{q/p})}^q = \int_{2B} |L^{-\alpha/2}(a)(x)|^q w(x)^{q/p} \, dx + \int_{(2^j B)^c} |L^{-\alpha/2}(a)(x)|^q w(x)^{q/p} \, dx \]
\[ = J_1 + J_2. \]

First note that \( 0 < \alpha < n/2 \), \( 1/q = 1/p - \alpha/2 \), then we are able to choose a number \( \mu > q \) such that \( 1/\mu = 1/2 - \alpha/2 \). Set \( s = 2/p \), then by a simple calculation, we can easily see that \( (q/p) \cdot (\mu/q)' = s' \) and \( 1 - q/\mu = q/(ps') \). Applying Hölder’s inequality, the \( L^2-L^\mu \) boundedness of \( L^{-\alpha/2} \), Lemma 2.1
and \( w \in RH_{s'} \), we can get
\[
J_1 \leq \left( \int_{2B} |L^{-\alpha/2}(a)(x)|^{q \frac{p}{q'}} dx \right)^{q/p} \left( \int_{2B} w(x)^{q \frac{p}{q'}} dx \right)^{1/q}
\]
\[
= \left( \int_{2B} |L^{-\alpha/2}(a)(x)|^\mu dx \right)^{q/\mu} \left( \int_{2B} w(x)^\mu dx \right)^{q/(p\mu)}
\]
\[
\leq C \| a \|_{L^2(\mathbb{R}^n)}^q \left( \frac{w(2B)}{|2B|^{1/s}} \right)^{q/p}.
\]
(5.1)

We now turn to deal with \( J_2 \). Using the condition \( w \in RH_{s'} \) and Hölder’s inequality, we obtain
\[
J_2 = \sum_{k=1}^{\infty} \int_{2^{k+1}B \setminus 2^kB} |L^{-\alpha/2}(a)(x)|^q w(x)^{q/p} dx
\]
\[
\leq C \sum_{k=1}^{\infty} \left( \int_{2^{k+1}B \setminus 2^kB} |L^{-\alpha/2}(a)(x)|^\mu dx \right)^{q/\mu} \left( \frac{w(2^{k+1}B)}{|2^{k+1}B|^{1/s}} \right)^{q/p}.
\]
(5.2)

For any \( x \in 2^{k+1}B \setminus 2^kB, k = 1, 2, \ldots \), by the expression (1.3), we can write
\[
|L^{-\alpha/2}(a)(x)| \leq C \int_0^\infty e^{-tL}(a)(x) \frac{dt}{t^{1-\alpha/2}}
\]
\[
\leq C \int_0^{r_B^2} e^{-tL}(a)(x) \frac{dt}{t^{1-\alpha/2}} + C \int_{r_B^2}^\infty e^{-tL}(a)(x) \frac{dt}{t^{1-\alpha/2}}
\]
\[
= III + IV.
\]

For the term III, it follows immediately from (4.3) that
\[
III \leq C \cdot \frac{1}{2^{k(n+1)}w(B)^{1/p}} \cdot \frac{1}{r_B} \int_{r_B^2}^{\infty} dt \frac{1}{t^{1-\alpha/2}}
\]
\[
\leq C \cdot \frac{r_B^{n\alpha}}{2^{k(n+1)}w(B)^{1/p}}.
\]

For the other term IV, by the previous estimate (4.4), we thus have
\[
IV \leq C \cdot \frac{1}{2^{k(n+1)}w(B)^{1/p}} \cdot r_B^{2M-1} \int_{r_B^2}^{\infty} dt \frac{1}{t^{M+1/2-\alpha/2}}
\]
\[
\leq C \cdot \frac{r_B^{\alpha}}{2^{k(n+1)}w(B)^{1/p}},
\]
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where the last inequality holds since $M > (3n)/4 > 1/2 + \alpha/2$. Combining the above estimates for III and IV, we obtain

$$|L^{-\alpha/2}(a)(x)| \leq C \cdot \frac{r_B^n}{2^{k(n+1)}w(B)^{1/p}}, \quad \text{when } x \in 2^{k+1}B \setminus 2^k B. \quad (5.3)$$

Substituting the above inequality (5.3) into (5.2) and using Lemma 2.1, we can get

$$J_2 \leq C \sum_{k=1}^{\infty} 2^{k+1}B \cdot \left( \frac{r_B^n}{2^{k(n+1)}w(B)^{1/p}} \right)^q \cdot \left( \frac{w(2^{k+1}B)}{|2^{k+1}B|^{1/s}} \right)^q \cdot \left( \frac{w(B)}{|B|^{1/p}} \right)^{1/p} \leq C,$$

where in the last inequality we have used the fact that $q > p > n/(n+1)$.

Therefore, by combining the above inequality (5.4) with (5.1), we conclude the proof of Theorem 1.3.

Proof of Theorem 1.4. As in the proof of Theorem 1.2, it is enough to show that for every $w$-atom $a$, the function $m = L^{-\alpha/2}(a)$ is a multiple of $w$-molecule for some $\epsilon > 0$, and the multiple constant is independent of $a$. Let $a$ be a $w$-atom with $supp a \subseteq B = B(x_0, r_B)$, and $a = L^{2M}(b)$, where $M > (3n)/4 > \max\{\frac{n}{2} - \frac{1}{2}, \frac{1}{2} + \frac{n}{2}\}$, $b \in D(L^{2M})$. Set $\tilde{b} = L^{-\alpha/2}(L^2M)b$, then we have $m = L^M(\tilde{b})$. It is easy to check that $m(x) \in L^2(\mathbb{R}^n)$. As before, since $0 < \alpha < n/2$, then we may choose a number $\mu > 2$ such that $1/\mu = 1/2 - \alpha/n$. For $k = 0, 1, \ldots, M$, by using Hölder’s inequality, the $L^2-L^\mu$ boundedness of $L^{-\alpha/2}$, we obtain

$$\| (r_B^nL)^k \tilde{b} \|_{L^2(2B)} \leq \frac{1}{r_B^n} \| (r_B^nL)^{M+k}b \|_{L^{p}(2B)} |2B|^{1/2-1/\mu} \leq C \cdot \frac{1}{r_B^n} \| (r_B^nL)^{M+k}b \|_{L^2(2B)} |B|^{1/2-1/\mu} \leq C \cdot r_B^{2M} |B|^{1/2+\alpha/n} w(B)^{-1/p}. \quad (5.5)$$

Note that $1/q = 1/p - \alpha/n$, then a straightforward computation yields that $q/p < (2/p)'$ whenever $0 < \alpha < n/2$. By our assumption $w \in RH_{(2/p)'}$, then we have $w \in RH_{q/p}$. Consequently

$$w^{q/p}(B)^{p/q} \leq C \cdot \frac{w(B)}{|B|^{1-p/q}}.$$
which implies
\[ w(B)^{-1/p} \leq C \cdot |B|^{1/q-1/p} w^{q/p}(B)^{-1/q}. \]  
(5.6)

Substituting the above inequality (5.6) into (5.5), we can get
\[ \| (r_B^2 L)^k \tilde{b} \|_{L^2(2B)} \leq C \cdot r_B^{2M} |B|^{1/2} w^{q/p}(B)^{-1/q}. \]  
(5.7)

It remains to estimate \( \| (r_B^2 L)^k \tilde{b} \|_{L^2(2^j+1B\setminus 2^j B)} \) for \( k = 0, 1, \ldots, M, \) \( j = 1, 2, \ldots \). We write
\[ |(r_B^2 L)^k \tilde{b}(x)| = |L^{-\alpha/2} [(r_B^2 L)^k L^M b](x)| \leq C \int_0^{r_B^2} e^{-tL} [r_B^{2k} L^{M+k} b](x) \frac{dt}{t^{1-\alpha/2}} + C \int_{r_B^2}^{\infty} e^{-tL} [r_B^{2k} L^{M+k} b](x) \frac{dt}{t^{1-\alpha/2}} = \text{III}'+\text{IV}'. \]

For the term III', by using the same arguments as in the proof of (4.7), we have
\[ \text{III}' \leq C \cdot 1 \frac{1}{2^{j(n+1)}} \cdot r_B^{2M} w(B)^{-1/p} \frac{1}{r_B} \int_0^{r_B^2} \frac{dt}{t^{1/2-\alpha/2}} \leq C \cdot 1 \frac{1}{2^{j(n+1)}} \cdot r_B^{2M+\alpha} w(B)^{-1/p}. \]

For the term IV', we follow the same arguments as that of (4.9) and then obtain
\[ \text{IV}' \leq C \cdot 1 \frac{1}{2^{j(n+1)}} \cdot w(B)^{-1/p} \frac{1}{r_B} \int_{r_B^2}^{\infty} \frac{dt}{t^{M+k+1/2-\alpha/2}} \leq C \cdot 1 \frac{1}{2^{j(n+1)}} \cdot r_B^{2M+\alpha} w(B)^{-1/p}. \]

Combining the above estimates for III' and IV', we can get
\[ |(r_B^2 L)^k \tilde{b}(x)| \leq C \cdot 1 \frac{1}{2^{j(n+1)}} \cdot r_B^{2M+\alpha} w(B)^{-1/p}, \quad \text{when} \ x \in 2^{j+1}B \setminus 2^j B. \]

Since \( w \in A_1 \), then it follows from the previous inequality (4.8) that
\[ |(r_B^2 L)^k \tilde{b}(x)| \leq C \cdot \frac{1}{2^{j(n+1)-n/2}} \cdot r_B^{2M+\alpha} w(2^j B)^{-1/p}, \quad \text{when} \ x \in 2^{j+1}B \setminus 2^j B. \]

Similar to the proof of (5.6), we can also show that
\[ w(2^j B)^{-1/p} \leq C \cdot |2^j B|^{1/q-1/p} w^{q/p}(2^j B)^{-1/q}. \]
Hence
\[
|(r^2 B L)^k \tilde{b}(x)| \leq C \cdot \frac{1}{2^{j((n+1)-n/q)}} \cdot r^{2M} B^{q/p} (2^j B)^{-1/q}, \quad \text{when } x \in 2^{j+1} B \setminus 2^j B.
\]

Therefore
\[
\| (r^2 B L)^k \tilde{b} \|_{L^2(2^{j+1} B \setminus 2^j B)} \leq C \cdot \frac{1}{2^{j((n+1)-n/q)}} \cdot r^{2M} B^{1/2} |2^j B|^{1/2} w^{q/p} (2^j B)^{-1/q}.
\]

Observe that 1 ≥ q > p > n/(n + 1). If we set \( \varepsilon = (n + 1) - n/q \), then \( \varepsilon > 0 \). Summarizing the estimates (5.7) and (5.8) derived above, we finally conclude the proof of Theorem 1.4.

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