Constraint equations for general hypersurfaces and applications to shells

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Abstract

Hypersurfaces of arbitrary causal character embedded in a spacetime are studied with the aim of extracting necessary and sufficient free data on the submanifold suitable for reconstructing the spacetime metric and its first derivative along the hypersurface. The constraint equations for hypersurfaces of arbitrary causal character are then computed explicitly in terms of this hypersurface data, thus providing a framework capable of unifying, and extending, the standard constraint equations in the spacelike and in the characteristic cases to the general situation. This may have interesting applications in well-posedness problems more general than those already treated in the literature. As a simple application of the constraint equations for general hypersurfaces, we derive the field equations for shells of matter when no restriction whatsoever on the causal character of the shell is imposed.

1 Introduction

The Cauchy problem is a fundamental ingredient of General Relativity (and other geometric theories of gravity) as it allows one to encode the spacetime information into the geometry of suitable codimension one submanifolds and their first order variation. This geometric data allows for the reconstruction of the spacetime (more precisely the domain of dependence of the data) by solving the field equations of the theory. This, among many other reasons, makes the study of spacetime hypersurfaces an important branch of General Relativity.

Solving the evolution problem requires, in one way or another, the splitting of the spacetime in terms of a foliation by hypersurfaces. This can be done either explicitly, as in the ADM or related formalisms, or implicitly, by constructing appropriate coordinate systems in which the field equations are solved. The splitting also depends on the type of problem under consideration. For instance, in the standard Cauchy problem the splitting is performed by a family of spacelike hypersurfaces and the initial data consists of the induced metric $\gamma$ and the second fundamental form $K$. An important property of geometric theories of gravity is that they constitute a constrained system, in the sense that the initial data is subject to a set of equations called constraint equations. In the spacelike case, they take the standard form

$$2\rho \overset{\text{def}}{=} 2G_{\alpha\beta}n^\alpha n^\beta|_{\Sigma} = R(\gamma) - K_{ab}K^{ab} + K^2, \quad K \overset{\text{def}}{=} \gamma^{ab}K_{ab},$$

$$-J_a \overset{\text{def}}{=} G_{\alpha\beta}e^\alpha_a n^\beta|_{\Sigma} = D_b (K^b_a - K^b_\delta \delta_a^\delta)$$

where $D$ is the Levi-Civita covariant derivative of $(\Sigma, \gamma)$ and $R(\gamma)$ its curvature scalar. $G_{\alpha\beta}$ is the Einstein tensor of the spacetime, $n^\alpha$ is the unit normal used to define $K_{ab}$ and $e^\alpha_a$ is the push forward to $M$ of the coordinate vector $\partial_a$ in $\Sigma$. This standard splitting of spacetime into space and time (i.e. via a foliation of spacelike hypersurfaces) is not the only relevant one. Initial data can also be prescribed on a pair of null hypersurfaces with a common smooth boundary consisting on a codimension-two spacelike hypersurface. This is the characteristic initial value problem, and also gives rise to a well-posed evolution problem.
in the sense that suitable initial data (satisfying appropriate constraint equations) also define a unique spacetime solving the field equations with this initial data. The constraint equations in the characteristic case take a very different form than in the spacelike case. Most approaches require a 2+1 splitting of the null hypersurface by spacelike, codimension-one surfaces, and the constraint equations become a hierarchical set of ODE along the degeneration direction of the null hypersurface. Several forms of the constraint equations can be found e.g. in [24], [25] or [8].

The well-posedness of the characteristic initial value problem has been extended recently [6] to the limiting case where the two-surface common to the pair of null hypersurfaces degenerates to a point, i.e. when there is only one null hypersurface which is null everywhere except for a conical singularity. The initial data in such a case is similar to the characteristic one above, except for the need of a careful analysis of the conical singularity that arises, see [5], [7]. Besides the cases mentioned above, the Einstein field equations are also well-posed for the so-called Cauchy-characteristic initial value problem [30], where the initial data is prescribed on a pair of hypersurfaces, one spacelike and one null with common boundary on a codimension-two surface.

In view of the very different formulations of the constraint equations in the spacelike and in the characteristic cases, a natural question is whether there is any framework capable of dealing with both cases at once. More generally, it would be interesting to know how do the constraint equations look like for any hypersurface in the spacetime. The distinction of hypersurfaces into spacelike, null or timelike is rather artificial from a geometric perspective of submanifolds in a Lorentzian ambient spacetime. The ideal setting would be a framework where the constraint equation can be written for an arbitrary hypersurface with no restriction whatsoever in its causal character (which hence could change along the hypersurface). Besides its aesthetic appeal, having such a common framework would also be of practical interest. First of all, it would allow us to address the issue of how do the constraint equations on spacelike hypersurfaces transform smoothly into the characteristic constraint equations, which would help us clarify the very different nature of the characteristic constraint equations with respect to the spatial ones. Furthermore, whatever the final result may be, the set of variables involved in the general formulation cannot be the same as in the spacelike case (because those become singular when the hypersurface becomes null) and so the method would provide us with alternative expressions both for the standard spacelike and for the characteristic constraint equations, and this may potentially give new insights into the standard cases as well. Moreover, it is intuitively clear that the list of well-posed initial value problems discussed above should not exhaust all the possibilities. From basic causality arguments one expects that appropriate initial data prescribed on a hypersurface everywhere spacelike or null should also give rise to a well-posed initial value problem (this would correspond, in essence, to a smoothing of the characteristic or Cauchy-characteristic initial value problem discussed above). In order to start thinking about such a possibility, it is necessary to have a framework capable of dealing with the constraint equations in such a setting, and which defines the types of variables where the well-posedness problem would be addressed.

It should also be emphasized that physically relevant hypersurfaces of varying causal character are much more common than one may think a priori. A list of interesting examples can be found in the Introduction of [22] where the geometry of arbitrary hypersurfaces in the spacetime was studied. The developments of that paper were focused in generalizing the matching conditions from the case of constant causal character (well-developed both in the spacelike and timelike cases [10, 21, 27] and in the null case [3]) to the arbitrary case of varying causal character and this required a better understanding of the geometry of general hypersurfaces in a spacetime. Other relevant examples of hypersurfaces of non-constant causal character are the so-called marginally outer trapped tubes (and their close relatives, the trapping horizons [12], and dynamical/isolated horizons [3]). These are hypersurfaces foliated by codimension-two spacelike surfaces with one of its null expansions identically vanishing. A priori, these hypersurfaces may have any causal character. Under appropriate stability and energy conditions, no timelike portion may exist [11, 2] but they can still vary their causal character from spacelike to null, each case having a clear physical interpretation in terms of the energy flux that crosses the hypersurface. Marginally outer trapped tubes are physically very relevant since they are suitable quasi-local replacements for black hole event horizons, and are analyzed routinely in any numerical evolution of “black hole” mergers in any collapsing process.

The aim of this paper is to develop a consistent framework capable of describing the geometry of arbitrary
hypersurfaces in the spacetime in such a way that the constraint equations can be written down in full
generality. The starting point for the construction is based on the results of [22] (although the presentation
will be essentially self-consistent). It is clear that a fundamental ingredient of any initial value formulation
is the ability of detaching completely the hypersurface from the spacetime where it is initially sitting. This
is necessary in order to define the data at the abstract level, without the need of invoking the spacetime for
its definition. This will be the guiding principle of the derivations below. Indeed, the geometry of general
spacetime hypersurfaces will be studied in detail with the aim of extracting a set of free geometric data
living directly on the submanifold. This will allow us to define abstract data and make contact with the
spacetime construction via an appropriate notion of embedding. After this data is identified, I will derive
the constraint equations relating the hypersurface data with suitable components of the Einstein tensor of
the ambient spacetime along the hypersurface. The Einstein tensor components that can be related to the
hypersurface data are the normal-tangential component and the normal-transversal components (the precise
definitions will appear below) analogously as in the standard spacelike case. With these identities at hand,
the constraint equations will be promoted to field equations for matter-hypersurface data at the abstract
level, without the need of any embedding into a spacetime. The constraint equations will be the main result
of this paper and will open up the possibility of studying well-posedness issues (particularly in the case of
nowhere timelike initial data sets) in future developments.

As a simple application of the constraint equations for general hypersurfaces I will obtain the field
equations that need to be satisfied for shells propagating in arbitrary spacetimes, with no restriction on the
causal character of the shell. Recall that a shell arises when two spacetimes with boundary are matched
across their boundaries in such a way that a spacetime with continuous metric (in a suitable atlas) is
constructed. When the extrinsic geometry across the matching hypersurface jumps, this is interpreted as a
shell of matter-energy with support on the hypersurface. It is possible to define an energy-momentum tensor
on the shell which satisfies field equations where the sources are the jumps across the matching hypersurface of
suitable components of the spacetime energy-momentum tensor. These are the so-called Israel field equations
(also “shell equations” or “surface layer equations”) and where derived in the case of spacelike or timelike
hypersurfaces first by Lanczos [17, 18] and then put in a geometrically clear context by Israel [13]. By
performing a suitable limit of the equations (in the right variables) when the spacelike/timelike hypersurface
approaches a null limit, these equations were extended to the case of null hypersurfaces by Barrabès and
Israel [4]. The standard method to derive the Israel equations consists in using tensor distributions on the
spacetime constructed by matching two spacetimes with boundary. The energy-momentum tensor of the
matched spacetime is a distribution which, in general, has a Dirac delta part supported on the matching
hypersurface. This singular part defines the energy-momentum tensor on the shell and the contracted
(distributional) Bianchi identities lead to the shell equations. This distributional approach can in principle
also be followed in the case of matching hypersurfaces of arbitrary causal character (the distributional setting
for this case was developed in detail in [22]). Nevertheless, having the constraint equations for arbitrary
hypersurfaces at hand, the shell field equations can also be derived directly by simple subtraction of the
constraint equations at each side of the matching hypersurface. Besides its intrinsic simplicity (with no need
of using spacetime distributions and transforming the result back into hypersurface information), this has
the advantage that it works even if the hypersurface data does not come from any spacetime. This allows
us to define shells and shell equations fully independently of the existence of any spacetime where the data
is embedded.

The plan of the paper is as follows. In section 3, I will extend the results of [22] on the geometry of
general hypersurfaces. In particular, I will identify the data that allows one to reconstruct the ambient
spacetime metric along the hypersurface (this leads to the definition of hypersurface metric data). Then
I will consider the first derivatives of the spacetime metric along the hypersurface and will extract the
 corresponding free data on the hypersurface (Proposition 1). This will lead to the definition of hypersurface
data. The definition of metric hypersurface and hypersurface data have a built-in gauge freedom tied to the
choice of transversal direction used to define extrinsic properties of the hypersurface (the so-called rigging
vector). The gauge freedom on the hypersurface data at the abstract level will be studied in detail in Section
4. In section 4 I will study the Gauss and Codazzi equations of hypersurfaces in order to write down the
normal-tangential and normal-transversal components of the Einstein tensor in terms of hypersurface data.
This will be done first in terms of a natural connection [22] on the hypersurface that arises from projection along the rigging of the spacetime connection onto the submanifold. This connection, however geometrically natural in a spacetime setting, has the inconvenience that it depends on the extrinsic information of the hypersurface data. In a second step, I will rewrite the constraint equations in terms of a connection that depends solely on the metric hypersurface data and in such a way that all the dependence of the extrinsic geometry (the tensor $Y_{ab}$ introduced in the text) is fully explicit. This will be the set of equations that I will promote to constraint equations at the abstract level. In Section 5, I will obtain the shell equations mentioned above by simple subtraction of the constraint equations obtained in the previous section. During the process, a symmetric, two-covariant tensor will arise naturally from the equations. This will define the energy-momentum tensor on the shell. In terms of this tensor the shell equations take a very simple form. I will conclude with Section 6 where a brief summary of results and a discussion of future research will be given.

2 Geometry of general hypersurfaces in an $(m + 1)$-dimensional spacetime.

In this paper manifolds are always connected and paracompact. A spacetime is an $(m + 1)$-dimensional smooth oriented manifold $\mathcal{M}$ endowed with a symmetric 2-covariant tensor field $g$ of Lorentzian signature $\{-, +, \cdots, +\}$. The metric $g$ will be assumed to be $C^2$. A “hypersurface” is an embedded submanifold of codimension-one, i.e. a $n$-dimensional smooth manifold $\Sigma$ and an embedding $\Phi : \Sigma \to \mathcal{M}$, where by “embedding” we mean a smooth injective immersion which is a homomorphism between $\Sigma$ with its manifold topology and $\Phi(\Sigma)$ with its induced topology as a subset of $\mathcal{M}$. We often identify $\Sigma$ with $\Phi(\Sigma)$ when necessary.

The first fundamental form of $\Sigma$ is $\gamma \overset{\text{def}}{=} \Phi^*(g)$ of $\Sigma$. The signature of $\gamma$ at a given point $p \in \Sigma$ can be either Euclidean, Lorentzian or of type $\{0, +, +, \cdots, +\}$. In this paper we deal with arbitrary hypersurfaces and hence we will not assume that the signature of $\gamma$ remains constant in $\Sigma$. Since $\gamma$ may be degenerate at some points (or everywhere), $\Sigma$ inherits, in general, no induced metric or canonical connection from the ambient spacetime. In order to describe the intrinsic and extrinsic geometry of $\Sigma$, it is convenient to introduce an additional structure, namely a spacetime vector field along $\Sigma$ with is transverse to $\Sigma$ everywhere. Such vector field, called rigging was first introduced by Schouten [26].

Let $T\Sigma \mathcal{M}$ be the vector bundle over $\Phi(\Sigma)$ (i.e, the bundle with base $\Sigma$ and fiber at $p \in \Sigma$ the tangent space $T\Phi(p)\mathcal{M}$). Let $T\Sigma$ be the tangent bundle of $\Sigma$. Identifying $\Sigma$ with $\Phi(\Sigma)$ we can view $T\Sigma$ as a vector subbundle of $T\Sigma \mathcal{M}$. The set of smooth sections on a bundle $(E, \Omega, \pi)$ will be denoted by $\Gamma(E)$.

Definition 1 (Schouten [26]) A rigging $\ell$ is a smooth section $\ell \in \Gamma(T\Sigma \mathcal{M})$ satisfying $\ell|_p \notin T_p \Sigma$ for all $p \in \Sigma$.

An important issue concerning riggings is their existence. First of all we note that there always exists a vector bundle $T^\Sigma \Sigma$ ($T^\Sigma$ stands for transverse) over $\Sigma$ such that the vector bundle decomposition $T\Sigma \mathcal{M} = T\Sigma \oplus T^\Sigma$ holds. One way of seeing this is by selecting an arbitrary Riemannian metric $\hat{g}$ on $\mathcal{M}$ (this exists because $\mathcal{M}$ is paracompact, e.g. [19]). At a point $p \in \Sigma$ define $\hat{N}_p \Sigma$ as the vector subspace in $T_p \mathcal{M}$ consisting of vectors orthogonal to $T_p \Sigma$ with the metric $\hat{g}$. It is immediate that the collection of all $\hat{N}_p \Sigma$, $p \in \Sigma$ defines a vector bundle $\hat{N} \Sigma$ over $\Sigma$ and that $T\Sigma \mathcal{M} = T\Sigma \oplus \hat{N} \Sigma$, which proves the existence claimed. We note that this construction works in arbitrary codimension. Note, however, that the existence of the global decomposition $T\Sigma \mathcal{M} = T\Sigma \oplus T^\Sigma$ fails short of proving the existence of a rigging. For that it is necessary that a global, nowhere zero section of $T^\Sigma$ exists. The following lemma shows that this happens if and only if $\Sigma$ is orientable.

Lemma 1 Let $\Sigma$ be a hypersurface in $\mathcal{M}$. A rigging $\ell$ exists if and only if $\Sigma$ is orientable.

Proof. Select one decomposition $T\Sigma \mathcal{M} = T\Sigma \oplus T^\Sigma$. If $\Sigma$ is orientable, then there exists a smooth field of normals $n$ (i.e. a smooth field of one-forms on $\Sigma$, nowhere zero and orthogonal to all tangent vector
fields to Σ). Define ℓ|p as the unique vector satisfying ℓ|p ∈ T_p^∗Σ and n|p(ℓ|p) = 1. It is obvious that ℓ is smooth, nowhere zero and transverse to Σ everywhere, hence a rigging. For the converse, select the rigging and define, at each point p ∈ Σ the one-form n|p ∈ (T_pΣ)^⊥ ⊂ T_p^∗M (orthogonal in the sense of dual spaces) satisfying n|p(ℓ|p) = 1 (such n|p exists because ℓ|p is not tangent to Σ and hence any non-zero one-form in (T_pΣ)^⊥ when applied to ℓ|p gives a non-zero value). It is immediate to check that n : Σ → T^∗Σ, defined as n(p) def = n|p gives a smooth field of normals, and hence Σ is orientable.

In view of this lemma, we will from now one assume that all hypersurfaces are orientable unless contrarily specified.

It is clear that the choice of rigging is highly non-unique, and we will have to deal with this freedom later. Nevertheless, a rigging ℓ allows for a decomposition \( T_p^∗Σ = (ℓ|p)^⊥ ⊕ T_p^∗Σ \), where \( (ℓ|p)^⊥ \) is the vector subspace of \( T_p^∗Σ \) generated by \( ℓ|p \). Given \( V ∈ T_p^∗Σ \) we define a scalar \( V^{⊥ℓ} \) and a vector \( V^{∥ℓ} \) ∈ \( T_pΣ \) via the decomposition \( V = V^{⊥ℓ}ℓ|p + V^{∥ℓ} \). Given a section \( V ∈ Γ(TΣ^∗M) \) this decomposition defines a scalar \( V^{⊥ℓ} : Σ → ℝ \) and \( V^{∥ℓ} ∈ Γ(TΣ) \). These definitions obviously depend on the choice of rigging \( ℓ \).

The decomposition \( T_p^∗Σ = (ℓ|p)^⊥ ⊕ T_p^∗Σ \) induces a decomposition of the dual space \( T_p^∗Σ = T_p^∗Σ ⊕ N_pΣ \), where \( T_p^∗Σ = (ℓ|p)^⊥ ⊂ T_p^∗Σ \) and \( N_pΣ = (T_pΣ)^⊥ \). The latter is the normal space to Σ at p, and its elements are normal one-forms to Σ. Note that \( N_pΣ \) is independent of the rigging, while \( T_p^∗Σ \) is not. It is also clear that the collection of \( T_p^∗Σ, p ∈ Σ \), defines a vector bundle over Σ, denoted by \( T^∗Σ \). The same occurs for the collection \( N_pΣ \), which defines the normal bundle \( NΣ \). This bundle is independent of the choice of rigging. However, given \( ℓ \), we define \( n \) as the unique normal one-form \( n ∈ Γ(NΣ) \) satisfying \( n(ℓ) = 1 \). Despite the fact that \( n \) depends on \( ℓ \), we will not make this dependence explicit in the symbol in order not to make the notation cumbersome. We will do the same for several other objects defined below.

We also note that, in the same way as we have identified Σ with its image we have also identified \( TΣ \), the tangent bundle of Σ as an abstract manifold, with the vector subbundle \( TΣ ⊂ TΣ^∗M \). The precise meaning of an object in such a space will be either clear from the context, or made explicit.

In this paper we will often use index notation. To that aim, let \( \{ e_a \} \) \( a = 1, \cdots , m \) be a basis of of \( TΣ \). By definition, this means a set of \( m \) smooth sections \( e_a ∈ Γ(TΣ) \) such that for all \( p ∈ Σ \) \( \{ e_a|p \} \) is a basis of \( T_pΣ \). The set of \( m + 1 \) vectors \( \{ e_a, ℓ \} \), where \( e_a = Φ_∗(e_a) \) is clearly a basis of \( TΣ \). Its dual basis is composed by \( \{ ω^a, n \} \), where the one-forms \( \{ ω^a \} \) are defined by \( ω^a(e_b) = δ^a_b, ω^a(ℓ) = 0 \). It is clear that \( ω^a \) also depends on the choice of rigging. By construction \( \{ ω^a \} \) is a basis of \( T_p^∗Σ \). It is also immediate that the pull-back of \( ω^a \) to \( T^∗Σ \), i.e. \( ω^a = Φ_∗(ω^a) \), defines a basis of this space.

Everything we have said so far is independent of the existence of a metric g in the ambient manifold \( M \).

Assume now that \( M \) is endowed with a metric of Lorentzian signature g. We can then define the scalar \( ℓ^{(2)} = g(ℓ, ℓ) \) and the one-form \( ℓ^\ast = g(ℓ, ·) \). Pulling this back to Σ, we obtain a one-form \( ℓ^\ast = Φ_∗(ℓ^\ast) \) which can be decomposed in the basis \( \{ ω^a \} \) as \( ℓ^\ast = ℓ_aω^a \), for certain coefficients \( ℓ_a \). An alternative (and equivalent) definition of \( ℓ_a \) is \( ℓ_a = g(ℓ, e_a) \). If, as before, we denote by \( γ \), the pull-back on Σ of the ambient metric g, this tensor may be degenerate (at certain points, or nowhere, or everywhere). However, it must be the case that the square \( (m + 1) \)-matrix

\[
\begin{bmatrix}
\gamma_{ab} \\
\ell_a \\
\ell_b \\
\ell^{(2)}
\end{bmatrix}
\]

has Lorentzian signature at every point \( p ∈ Σ \) (because this is simply the matrix representation of the ambient metric g in the basis \( \{ e_a, ℓ \} \)). This suggests the following definition, where everything refers to Σ as an abstract manifold, not embedded in any ambient spacetime.

\[\text{In general, no such global basis exists, and we would need to work with bases defined on each element of a suitable open cover of } Σ. \text{ Since all the expressions below will be tensorial (unless explicitly stated), there is no loss of generality in working as if the global basis did exist. This difficulty is general to the use of index notation and it is both well-understood and harmless. An alternative is to view indices in the sense of the abstract index notation of Penrose}]}
**Definition 2** A smooth $m$-dimensional manifold $\Sigma$, a symmetric tensor $\gamma_{ab}$, a one-form $\ell_a$ and a scalar $\ell^{(2)}$ define a hypersurface metric data set provided the square $(m + 1)$-matrix

$$K \overset{\text{def}}{=} \begin{bmatrix} \gamma_{ab} & \ell_a \\ \ell_b & \ell^{(2)} \end{bmatrix}$$

has Lorentzian signature at every point $p \in \Sigma$.

**Remark 1.** Note that, by definition, $\ell_a$, $\ell^{(2)}$ cannot vanish simultaneously at any point in any hypersurface metric data.

**Remark 2.** The most interesting case for gravity arises when hypersurfaces are embedded in a spacetime, i.e. in a manifold with a metric of Lorentzian signature. Nevertheless, the signature of the ambient manifold will be used essentially nowhere below. In fact, all the developments of this paper can be generalized with very minor changes to hypersurfaces embedded in ambient manifolds endowed with a metric of arbitrary (non-degenerate) signature.

Given hypersurface metric data, we can define immediately a symmetric two-contravariant tensor $P^{ab}$, a vector $n^a$ and a scalar $n^{(2)}$ in $\Sigma$, as the unique tensors satisfying the tensor equations on $\Sigma$,

$$P^{ab}\gamma_{bc} + n^a\ell_b = \delta^a_b, \quad (3)$$

$$P^{ab}\ell_b + \ell^{(2)}n^a = 0, \quad (4)$$

$$n^a\ell_a + n^{(2)}\ell^{(2)} = 1, \quad (5)$$

$$\gamma_{ab}n^b + n^{(2)}\ell_a = 0. \quad (6)$$

Existence and uniqueness of $P^{ab}$, $n^a$ and $n^{(2)}$ is immediate by noticing that these equations can be put in matrix form as

$$\begin{bmatrix} P^{ab} & n^a \\ n^b & n^{(2)} \end{bmatrix} \overset{\text{def}}{=} \begin{bmatrix} \gamma_{ab} & \ell_a \\ \ell_b & \ell^{(2)} \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix}. \quad (7)$$

Given hypersurface metric data, we will always define $P^{ab}$, $n^a$ and $n^{(2)}$ as the solutions of (3)-(6) unless contrarily indicated. Furthermore, we define the vector field $\hat{n} \overset{\text{def}}{=} n^a\hat{e}_a \in \Gamma(T\Sigma)$.

We want to think of hypersurface metric data $\{\Sigma, \gamma, \ell, \ell^{(2)}\}$ as an abstract collection of objects, defined independently of any spacetime and any embedding. To make contact with the previous discussion, the following definition is required.

**Definition 3** A hypersurface metric data $\{\Sigma, \gamma, \ell, \ell^{(2)}\}$ is embedded in a spacetime $(M, g)$ if there exists an embedding $\Phi : \Sigma \to M$ and a rigging vector $\ell$ such that, with $\ell = g(\ell, \cdot)$,

$$\gamma = \Phi^*(g), \quad \ell = \Phi^*(\ell), \quad \ell^{(2)} = \Phi^*(g(\ell, \ell)).$$

The following lemma, gives the relationship between $n^a$, $n^{(2)}$ and $P^{ab}$ with the ambient geometry when the hypersurface metric data is embedded.

**Lemma 2** Let $\{\Sigma, \gamma, \ell, \ell^{(2)}\}$ be embedded hypersurface metric data with embedding $\Phi$, spacetime $(M, g)$ and rigging vector $\ell$. Let $\{\hat{e}_a\}$ be a basis of $T\Sigma$ and $e_a = \Phi_*(\hat{e}_a)$. Then, the vectors $n \overset{\text{def}}{=} g^{-1}(n, \cdot)$ ($g^{-1}$ is the inverse tensor of $g$) $\omega^a \overset{\text{def}}{=} g^{-1}(\omega^a, \cdot)$ and the one-forms $e_a \overset{\text{def}}{=} g(e_a, \cdot)$, $\ell = g(\ell, \cdot)$ can be decomposed as

$$n = n^a \hat{e}_a + n^{(2)}\ell, \quad (8)$$

$$\ell = \ell_a \omega^a + \ell^{(2)}n, \quad (9)$$

$$e_a = \gamma_{ab}\omega^b + \ell_a n, \quad (10)$$

$$\omega^a = P^{ab}e_b + n^a \ell. \quad (11)$$

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Proof. For (3), we only need to check that \( n \) defined by this formula is normal to \( \Sigma \) and that it satisfies \( g(n, \ell) = 1 \).
\[
\begin{align*}
g(n, e_a) &= n^b \gamma_{ab} + n^{(2)} \ell_a = 0, \quad \text{by (6)}, \\
g(n, \ell) &= n^a \ell_a + n^{(2)} \ell^{(2)} = 1, \quad \text{by (5)},
\end{align*}
\]
where we used \( g(e_a, e_b) = \gamma_{ab} \) and \( g(e_a, \ell) = \ell(e_a) = \ell_a \). Expression (9) follows from the immediate facts that \( \ell_a \omega^a + \ell^{(2)} n(e_b) = \ell_a \) and \( (\ell_a \omega^a + \ell^{(2)} n) (\ell) = \ell^{(2)} \). For (10), we should check whether \( e_a(e_b) = g(e_a, e_b) = \gamma_{ab} \) and \( e_a(\ell) = \ell(e_a) = \ell_a \). Both are immediate. For (11) we need to check \( g(\omega^a, e_b) = \delta^a_b \) and \( g(\omega^a, \ell) = 0 \). Indeed
\[
\begin{align*}
g(\omega^a, e_b) &= P^{ac} \gamma_{cb} + n^a \ell_b = \delta^a_b, \quad \text{by (8)}, \\
g(\omega^a, \ell) &= P^{ab} \ell_b + n^a \ell^{(2)} = 0, \quad \text{by (4)}.
\end{align*}
\]

A consequence of this lemma is that, for embedded hypersurface metric data, we have \( \hat{n} = n^\parallel \ell \). It also implies that, for embedded hypersurface metric data, the quantities \( P^{ab}, n^a \) and \( n^{(2)} \) can also be calculated from the expressions
\[
P^{ab} = g^{-1} (\omega^a, \omega^b), \quad n^a = g^{-1} (n, \omega^a), \quad n^{(2)} = g^{-1} (n, n).
\]
In this context, these expressions could have been obtained also from the fact that the matrix components of \( g^{-1} \) in the basis \{\( \omega^a, n \)\} is precisely
\[
\begin{bmatrix}
P^{ab} & n^a \\
n^b & n^{(2)}
\end{bmatrix}.
\]

The following simple lemma allows us to reconstruct a vector \( V^a \) from the one-form \( V_a \) \( \equiv \gamma_{ab} V^b \) and the contraction \( V^b \ell_b \), and will be used many times below.

**Lemma 3** Let \( Z_a \) and \( W \) be given. There exists a vector \( V^a \) satisfying \( V^a \ell_a = W \) and \( \gamma_{ab} V^b = Z_a \) if and only if
\[
n^b Z_b + n^{(2)} W = 0. \tag{13}
\]
Moreover, the solution is unique and reads
\[
V^a = P^{ab} Z_b + n^a W. \tag{14}
\]

**Proof.** Assume (14). Let us check that (14) solves the two equations \( V^a \ell_a = W \), \( \gamma_{ab} V^b = Z_a \). Indeed, using (3) - (5) one has
\[
\begin{align*}
\gamma_{ab} V^b &= \gamma_{ab} P^{bc} Z_c + n^b \gamma_{ab} W = (\delta^c_b - n^c \ell_a) Z_c - n^{(2)} \ell_a W = Z_a - \ell_a (n^c Z_c + n^{(2)} W) = Z_a, \tag{15} \\
V^a \ell_a &= P^{ab} Z_b \ell_a + n^a \ell_a W = -\ell^{(2)} n^b Z_b + (1 - n^{(2)} \ell^{(2)}) W = W - \ell^{(2)} (n^b Z_b + n^{(2)} W) = W. \tag{16}
\end{align*}
\]
To show necessity, let \( V^a \) solve the equations \( V^a \ell_a = W \) and \( \gamma_{ab} V^b = Z_a \). Multiplying the second by \( P^{ac} \) yields
\[
Z_a P^{ac} = V^b \gamma_{ab} P^{ac} = V^b (\delta^c_b - n^c \ell_b) = V^c - n^c W.
\]
Thus (14) is the only possible solution (by the way, this proves the claim of uniqueness). Now, the calculations (15) and (16) are still valid. The last equality in both expressions implies (13) because it cannot happen that \( \ell_a = 0 \) and \( \ell^{(2)} = 0 \) simultaneously. \( \square \)
Let us now denote the Levi-Civita covariant derivative of \((M, g)\) by \(\nabla\). Given two vectors \(X, Y \in \Gamma(T\Sigma)\) we define
\[
\nabla_X Y \overset{\text{def}}{=} (\nabla_X Y)^\ell, \\
K(X, Y) \overset{\text{def}}{=} - (\nabla_X Y)^\perp. 
\]
(17)
It is immediate to check \([22]\) that \(\overline{\nabla}\) defines a torsion-free covariant derivative on \(\Sigma\) and that \(K(X, Y)\) is a symmetric tensor. \(K\) only depends on \(n\), as the following alternative expression implies,
\[
K(X, Y) = -g(n, \nabla_X Y) = (\nabla_X n)(Y),
\]
where \(n\) in the last expression is any smooth extension of \(n\) off \(\Sigma\). This expression shows that \(K\) is the second fundamental form of \(\Sigma\) with respect to the normal \(n\). In the Riemannian case, this tensor captures the extrinsic geometry of the submanifold. However, for general hypersurfaces, the normal vector \(n\) is tangent to \(\Sigma\) at points where the hypersurface is degenerate. Thus, \(K\) gives no extrinsic information on the geometry of \(\Sigma\) at those points. In the following, we will identify a suitable tensor that will encode the information on the extrinsic geometry of \(\Sigma\). To that aim, let us introduce the Christoffel symbols of \(\Sigma\) at those points. In the following, we will identify a suitable tensor that will encode the information on the extrinsic information of the embedding. Following \([22]\) we also define a one-form \(\varphi_a\) and an endomorphism \(\Psi^a_b\) by
\[
\varphi_a = - (\nabla_{e_a} n) (\ell), \\
\Psi^a_b = \omega^b (\nabla_{e_a} \ell).
\]
(19)
(20)
Note that \(\Psi^a_b\) are simply the coefficients of \((\nabla_{e_a} \ell)_{\|}\) in the basis \(\{e_b\}\). The definitions above imply \([22]\)
\[
\nabla_{e_a} e_b = -K_{ab} \ell + \Gamma_{ba}^c e_c, \\
\nabla_{e_a} \ell = \varphi_a \ell + \Psi^b_a e_b,
\]
(21)
(22)
These equations are equivalent to the following, written in the dual basis \(\{\omega^a, n\}\) of \(\{e_a, \ell\}\)
\[
\nabla_{e_a} n = -\varphi_a n + K_{ab} \omega^b, \\
\nabla_{e_a} \omega^b = -\Psi^b_a n - \Gamma_{ca}^b \omega_c.
\]
(24)
(25)
As before, we want to identify the minimal set of quantities on \(\Sigma\) that allows us to define hypersurface data in a detached form from the spacetime and the embedding. To that aim, we will first obtain which compatibility conditions must satisfy the fields \(K_{ab}, \Gamma_{ab}^c, \varphi_a, \Psi^b_a\) when defined via \([21]-[22]\). By finding the general solution of those compatibility condition we will be able to identify the free data on the hypersurface that will encode the extrinsic information of the embedding.

The compatibility conditions arise from the fact that the connection on the ambient manifold is metric and torsion-free. Denoting by \(\nabla_a\) the direction derivative along \(e_a\), we have
\[
\nabla_a g(e_b, e_c) = e_b (\nabla_{e_a} e_c) + e_c (\nabla_{e_a} e_b) = -K_{ac} \ell_b + \Gamma_{ac}^b \ell_c - K_{ab} \ell_c + \Gamma_{ab}^c \ell_d, \\
\nabla_a e_b = \nabla_{e_a} g(e_b, \ell) = e_b (\nabla_{e_a} \ell) + \ell (\nabla_{e_a} e_b) = \varphi_a \ell_b + \Psi^c_a \gamma_{bc} - K_{ab} \ell_c + \Gamma_{ab}^c \ell_c, \\
\nabla_a \ell = 2 \ell (\nabla_{e_a} \ell) = 2 \varphi_a \ell + 2 \Psi^b_a \ell_b.
\]
Thus, the compatibility equations take the following tensorial form
\[
\nabla_a \gamma_{bc} + \ell_b K_{ac} + \ell_c K_{ab} = 0, \\
\nabla_a \ell_b - \varphi_a \ell_b + \ell_c K_{ab} - \gamma_{bc} \psi_a = 0, \\
- \frac{1}{2} \nabla_a \ell + \psi^a_b \ell_b + \ell (\varphi_a) = 0.
\]
(26)
(27)
(28)
Given hypersurface metric data, we can consider these equations as equations for the unknowns \(\{\Gamma_{ab}^c, K_{ab}, \Psi^b_a, \varphi_a\}\). The following Proposition provides the general solution in terms of a free symmetric two-covariant tensor on \(\Sigma\).
Proposition 1 Let \( \{\Sigma, \gamma, \ell, \ell^{(2)}\} \) be hypersurface metric data and \( Y_{ab} \) an arbitrary symmetric tensor on \( \Sigma \). Let \( F \) be the two-form \( F \stackrel{\text{def}}{=} \frac{1}{2} \omega \ell \) and define, in any coordinate basis,

\[
\Gamma_{ab}^c \stackrel{\text{def}}{=} \frac{1}{2} \rho_{cd} (\partial_a \gamma_{bd} + \partial_b \gamma_{ad} - \partial_d \gamma_{ab}) + n^c \left(-Y_{ab} + \frac{1}{2} (\partial_a \ell_b + \partial_b \ell_a)\right),
\]

\[
K_{ab} \stackrel{\text{def}}{=} n^{(2)} Y_{ab} + \frac{1}{2} \mathcal{L}_n \gamma_{ab} + \frac{1}{2} \left(\ell_a \partial_\ell n^{(2)} + \ell_b \partial_\ell n^{(2)}\right),
\]

\[
\varphi_a \stackrel{\text{def}}{=} \frac{1}{2} n^{(2)} \partial_a \ell^{(2)} + n^b (Y_{ab} + F_{ab}),
\]

\[
\Psi_b^a \stackrel{\text{def}}{=} P^{bc} (Y_{ac} + F_{ac}) + \frac{1}{2} n^a \partial_b \ell^{(2)},
\]

where \( \mathcal{L} \) denotes the Lie derivative. Then \( \Gamma_{ab}^c \) defines a torsion-free connection on \( \Sigma \) and \( \{\Gamma_{ab}^c, K_{ab}, \Psi_b^a, \varphi_a\} \) solves the compatibility equations \( (26)-(28) \). Conversely, any solution of these equations can be written in this form for some symmetric tensor \( Y_{ab} \). In either case, the tensor \( Y_{ab} \) satisfies the identity

\[
Y_{ab} = \frac{1}{2} (\nabla_a \ell_b + \nabla_b \ell_a) + \ell^{(2)} K_{ab}. \tag{33}
\]

Proof. We can apply Lemma 3 to equations \( (27) \) and \( (28) \) with \( V^b \rightarrow \Psi_b^a, Z_b \rightarrow \nabla_a \ell_b - \varphi_a \ell_b + \ell^{(2)} K_{ab} \) and \( W \rightarrow \frac{1}{2} \nabla_a \ell^{(2)} - \ell^{(2)} \varphi_a \). This gives, on the one hand, an explicit expression for \( \Psi_b^a \), namely

\[
\Psi_b^a = P^{bc} \nabla_a \ell_c + \ell^{(2)} P^{bc} K_{ca} + \frac{1}{2} n^b \nabla_a \ell^{(2)}, \tag{34}
\]

and, on the other, the compatibility equation \( (13) \) which reads

\[
n^b (\nabla_a \ell_b - \varphi_a \ell_b + \ell^{(2)} K_{ab}) + n^{(2)} \left(\frac{1}{2} \nabla_a \ell^{(2)} - \ell^{(2)} \varphi_a\right) = 0. \]

Recalling \( (5) \), this equation gives an explicit expression for \( \varphi_a \), namely

\[
\varphi_a = n^b (\nabla_a \ell_b + \ell^{(2)} K_{ab}) + \frac{1}{2} n^{(2)} \nabla_a \ell^{(2)}. \tag{35}
\]

Equations \( (27) \) and \( (28) \) are therefore equivalent to \( (34) \) and \( (35) \). Thus, we only need to solve \( (26) \) in terms of free data. Assume first that we are given a collection \( \{\gamma_{ab}, \ell_a, \ell^{(2)}, Y_{ab}\} \) satisfying \( (26) \). Let us define a symmetric tensor \( Y_{ab} \) by

\[
Y_{ab} \stackrel{\text{def}}{=} \frac{1}{2} (\nabla_a \ell_b + \nabla_b \ell_a) + \ell^{(2)} K_{ab}. \tag{36}
\]

We want to determine \( K_{ab} \) and \( \Gamma_{ab}^c \) in terms of \( \{\gamma_{ab}, \ell_a, \ell^{(2)}, Y_{ab}\} \). Expanding the covariant derivative in \( (26) \), we get (working in a coordinate basis)

\[
\partial_a \gamma_{bc} - \Gamma_{ba}^d \gamma_{dc} - \Gamma_{ca}^d \gamma_{bd} = -\ell_b K_{ac} - \ell_c K_{ab}. \]

Now, take the three cyclic permutations of these equations and subtract the third one to the sum of the other two. The result is

\[
\Gamma_{ab}^c \gamma_{dc} = \frac{1}{2} (\partial_a \gamma_{bc} + \partial_b \gamma_{ca} - \partial_c \gamma_{ab}) + \ell_c K_{ab} \stackrel{\text{def}}{=} Z_{ab} \tag{37}
\]

The definition of \( Y_{ab} \) \( (36) \) implies

\[
\Gamma_{ab}^c \ell_c = \frac{1}{2} (\partial_a \ell_b + \partial_b \ell_a) - Y_{ab} + \ell^{(2)} K_{ab} \stackrel{\text{def}}{=} W_{ab} \tag{38}
\]
We therefore have expressions for the contraction of $\Gamma_{ab}$ with the first fundamental form and with $\ell_c$. We can apply Lemma 3 with $Z_c \rightarrow Z_{cab}$ and $W \rightarrow W_{ab}$. This gives an explicit solution for $\Gamma_{ab}$, and a compatibility condition. The expression for $\Gamma_{ab}$, namely $\Gamma_{ab} = P^{cd} Z_{dab} + n^c W_{ab}$, gives

$$\Gamma_{ab} = \frac{1}{2} P^{cd} (\partial_a \gamma_{bd} + \partial_b \gamma_{ad} - \partial_d \gamma_{ab}) + n^c \left(-Y_{ab} + \frac{1}{2} (\partial_a \ell_b + \partial_b \ell_a)\right) + (P^{cd} \ell_d + n^c \ell^{(2)}) K_{ab}$$

$$= \frac{1}{2} P^{cd} (\partial_a \gamma_{bd} + \partial_b \gamma_{ad} - \partial_d \gamma_{ab}) + n^c \left(-Y_{ab} + \frac{1}{2} (\partial_a \ell_b + \partial_b \ell_a)\right), \tag{39}$$

where in the last equality we used (41). The compatibility condition $n^c Z_{cab} + n^{(2)} W_{ab} = 0$ reads explicitly

$$0 = \frac{1}{2} n^c (\partial_a \gamma_{bc} + \partial_b \gamma_{ac} - \partial_c \gamma_{ab}) + n^c \ell_c K_{ab} + \frac{1}{2} n^{(2)} (\partial_a \ell_b + \partial_b \ell_a) + n^{(2)} \ell^{(2)} K_{ab} - n^{(2)} Y_{ab}$$

$$= \frac{1}{2} n^c (\partial_a \gamma_{bc} + \partial_b \gamma_{ac} - \partial_c \gamma_{ab}) + \frac{1}{2} n^{(2)} (\partial_a \ell_b + \partial_b \ell_a) + K_{ab} - n^{(2)} Y_{ab}, \tag{40}$$

where in the second equality we used (55). To elaborate this expression we note the identity

$$n^c \partial_a \gamma_{bc} + n^{(2)} \partial_a \ell_b = \partial_a (n^c \gamma_{bc}) - \gamma_{bc} \partial_a n^c + n^{(2)} \partial_a \ell_b = \partial_a (-n^{(2)} \ell_b) + n^{(2)} \partial_a \ell_b - \gamma_{bc} \partial_a n^c =$$

$$= -\ell_b \partial_a n^{(2)} - \gamma_{bc} \partial_a n^c,$$

where (55) has been used. Inserting this into (40) and recalling the expression in coordinates for the Lie derivative $L_{\hat{\gamma}} Y_{ab} = n^c \partial_c \gamma_{ab} + \gamma_{ac} \partial_b n^c + \gamma_{bc} \partial_a n^c$, we find

$$K_{ab} = n^{(2)} Y_{ab} + \frac{1}{2} L_{\hat{\gamma}} \gamma_{ab} + \frac{1}{2} \left( \ell_a \partial_b n^{(2)} + \ell_b \partial_a n^{(2)} \right). \tag{41}$$

Expressions (39) and (41) give $\Gamma_{ab}$ and $K_{ab}$ explicitly in terms of $Y_{ab}$. Moreover, $Y_{ab}$ is free data because if we define $\Gamma_{ab}$ and $K_{ab}$ in terms of an arbitrary symmetric tensor $Y_{ab}$ through expressions (39)-41, then it is immediate to check that $\Gamma_{ab}$ is a torsion-free connection and, in addition, both equation (26) and expression (36) (which is now an equation) are identically satisfied with the covariant derivative $\nabla$ defined in terms of the connection $\Gamma_{ab}$. Indeed, Lemma 3 implies that expressions (39)-(41) are equivalent to (37)-(38) and, from the latter, equations (26) and (30) follow at once.

To complete the proof, we only need to notice that, irrespective of whether $Y_{ab}$ is defined by (30) or $\Gamma_{ab}^{\ell_c}$ is defined as in (40) the following holds

$$\nabla_a \ell_b = \frac{1}{2} (\nabla_a \ell_b + \nabla_b \ell_a) + \frac{1}{2} (\nabla_a \ell_b - \nabla_b \ell_a) = Y_{ab} - \ell^{(2)} K_{ab} + F_{ab} \tag{42}$$

and (34), (35) become, respectively, (32) and (31). \hfill \Box

The following corollary will be useful later

**Corollary 1** With the same hypothesis and nomenclature as in Proposition 4

$$n^b K_{ab} = n^{(2)} n^b (Y_{ab} + F_{ab}) + \frac{1}{2} (\partial_a n^{(2)} + (n^{(2)})^2 \partial_a \ell^{(2)}). \tag{43}$$

**Proof.** We only need to contract (30) with $n^b$, which in particular involves $n^b L_{\hat{\gamma}} \gamma_{ab}$. Since $L_{\hat{\gamma}} \hat{\gamma} = 0$, we have

$$n^b L_{\hat{\gamma}} \gamma_{ab} = L_{\hat{\gamma}} (n^b \gamma_{ab}) = -L_{\hat{\gamma}} (n^{(2)} \ell_a) = -(n^b \partial_b n^{(2)}) \ell_a - n^{(2)} L_{\hat{\gamma}} (\ell_a) = -(n^b \partial_b n^{(2)}) \ell_a +$$

$$+ n^{(2)} (-2n^b F_{ba} - \nabla_a (n^b \ell_b)), \tag{44}$$

where in the third equality we used the property $L_{\gamma} \alpha = i_{\gamma} d\alpha + d(i_{\gamma} \alpha)$ valid for any differential form $\alpha$. Using this expression in the contraction of $n^b$ with (30) and recalling $n^b \ell_b = 1 - n^{(2)} \ell^{(2)}$, the corollary follows directly. \hfill \Box
Following the idea of defining data on \( \Sigma \) as detached from any ambient geometry, we put forward the following definition:

**Definition 4 (Hypersurface data)** A five-tuple \( \{ \Sigma, \gamma_{ab}, \ell_a, \ell^{(2)}, Y_{ab} \} \) where \( \{ \Sigma, \gamma_{ab}, \ell_a, \ell^{(2)} \} \) is hypersurface metric data and \( Y_{ab} \) is a symmetric tensor is called hypersurface data.

Given hypersurface data, we define a torsion-free connection \( \nabla \) on \( \Sigma \) as the connection with connection coefficient symbols given by (27-32). We also define the tensors \( K_{ab}, \varphi_a, \Psi^b_a \) by the expressions (37)-(32).

As before, a notion of “embedding” for hypersurface data becomes necessary in order to link the hypersurface data with the ambient spacetime expressions (21)-(22).

**Definition 5 (Embedding of hypersurface data)** Let \( \{ \Sigma, \gamma, \ell, \ell^{(2)}, Y \} \) be hypersurface data. We will say that this data is embedded in a spacetime \( (\mathcal{M}, g) \) if there exists an embedding \( \Phi : \Sigma \rightarrow \mathcal{M} \) and a choice of rigging \( \ell \) such that, with \( \ell = g(\ell, \cdot) \),

\[
\Phi^*(g) = \gamma, \quad \Phi^*(g(\ell, \cdot)) = \ell, \quad g(\ell, \ell) = \ell^{(2)}, \quad \frac{1}{2} \Phi^*(\ell \cdot g) = Y.
\]

**Remark 3.** The last formula of the definition requires an extension of the rigging \( \ell \) off \( \Phi(\Sigma) \). The expression is, however, independent of this extension. Note also that, for embedded hypersurface data, the tensor \( Y_{ab} \) corresponds to the symmetric part of the tensor \( H_{ab} \) introduced, and extensively used, in (22) (see also (23)).

For this definition to make sense it is necessary that the covariant derivative \( \nabla \) and the tensors \( K_{ab}, \varphi_a, \Psi^b_a \) defined by Proposition 1 coincide with the corresponding tensors defined via (17), (18), (19), (20) in terms of the ambient spacetime geometry. This is taken care of in the following lemma.

**Lemma 4** Let \( \{ \Sigma, \gamma, \ell, \ell^{(2)}, Y \} \) be hypersurface data and \( \{ \Gamma^c_{ab}, K_{ab}, \varphi_a, \Psi^b_a \} \) be defined by (27-32). Assume that this hypersurface data is embedded with embedding \( \Phi \) and rigging vector \( \ell \) and let \( e_a \text{ def} \Phi_* (\hat{e}_a) \) where \( \{ \hat{e}_a \} \) be a basis of \( T\Sigma \). Then, the field equations (27) and (28) are satisfied.

**Proof.** Define (a priori new) tensors \( \tilde{K}_{ab} = K_{ba}, \tilde{\varphi}_a, \tilde{\Psi}^b_a \) and connection coefficients \( \tilde{\Gamma}^c_{ab} \) by the decompositions \( \nabla_{e_a} e_b = -\tilde{K}_{ab} \ell + \tilde{\Gamma}^c_{ba} e_c \) and \( \nabla_{e_a} \ell = \tilde{\varphi}_a \ell + \tilde{\Psi}^b_a e_b \). Then equations (29)-(32) are satisfied by this fields and hence, by Proposition 1 there exists a tensor \( \tilde{Y}_{ab} \) such that (29)-(32) are satisfied with the substitutions \( \{ \tilde{\Gamma}^c_{ab}, K_{ab}, \tilde{\varphi}_a, \tilde{\Psi}^b_a, \tilde{Y}_{ab} \} \rightarrow \{ \tilde{\Gamma}^c_{ab}, \tilde{K}_{ab}, \tilde{\varphi}_a, \tilde{\Psi}^b_a, \tilde{Y}_{ab} \} \). Thus, to prove the Lemma we only need to make sure that \( Y_{ab} = \tilde{Y}_{ab} \). Now, from the definition of embedded hypersurface data

\[
2Y_{ab} = e_a e_b (\nabla_{\mu} \ell_{\nu} + \nabla_{\nu} \ell_{\mu}) = (\nabla_{e_a} \ell)(e_b) + (\nabla_{e_b} \ell)(e_a) = \varphi_a \ell_b + \Psi^c_a \gamma_{cb} + \tilde{\varphi}_b \ell_a + \tilde{\Psi}^c_b \gamma_{ca} =
\]

\[
= \nabla_a \ell_b + \ell^{(2)} \tilde{K}_{ab} + \nabla_b \ell_a + \ell^{(2)} \tilde{K}_{ab} = \nabla_a \ell_b + \nabla_b \ell_a + 2\ell^{(2)} \tilde{K}_{ab} = 2\tilde{Y}_{ab}
\]

where in the fourth equality we have used the tilded version of (27) \( \nabla \) is the covariant derivative with connection \( \tilde{\Gamma}^c_{ab} \) and the last equality follows from (the tilded version of) Proposition 1.

Given hypersurface metric data, we have defined \( P^{ab}, n^c \) and \( n^{(2)} \) as the solutions of equations (23-30). If the data is supplemented with \( Y_{ab} \) to yield hypersurface data, then equations (20)-(28) are identically satisfied. It is clear that the fields \( P^{ab}, n^c \) and \( n^{(2)} \) will also satisfy appropriate field equations. If the hypersurface is embedded in a spacetime, the equations are easily derived by a calculation similar to derivation above leading to (26)-(28). However, as we want to work at the data level alone, we need to argue directly with the expressions on \( \Sigma \). In the following proposition we obtain a number of identities that will immediately imply the equations we are looking for. We note that the definitions of \( A_{abc}, B_{ab}, C_a \) in the proposition come directly from (26)-(28) while the definitions of \( \mathcal{D}_{abc}, \mathcal{E}_{ab}, \mathcal{F}_a \) are motivated by the spacetime calculation indicated above.
**Proposition 2** Let $n^a$, $\ell_a$, $\gamma_{ab} = \gamma_{ba}$, $\ell^{(2)}$, $P^{ab} = P^{ba}$, $n^a$, $n^{(2)}$ be arbitrary $C^1$ tensor fields on a manifold $\Sigma$ endowed with a connection $\nabla$. Let $K_{ab} = K_{ba}$, $\Psi_a^b$ and $\varphi_a$ arbitrary $C^0$ tensor fields on $\Sigma$. Define

\[
\begin{align*}
\mathcal{A}_{abc} & \overset{\text{def}}{=} \nabla_a \gamma_{bc} + \ell_b K_{ac} + \ell_c K_{ab}, \\
\mathcal{B}_{ab} & \overset{\text{def}}{=} \nabla_a \ell_b - \varphi_a \ell_b + \ell^{(2)} K_{ab} - \Psi_a^c \gamma_{bc} \\
\mathcal{C}_a & \overset{\text{def}}{=} -\frac{1}{2} \nabla_a \ell^{(2)} + \ell^{(2)} \varphi_a + \Psi_a^b \ell_b \\
\mathcal{D}_a^{bc} & \overset{\text{def}}{=} \nabla_a P^{bc} + n^b \Psi_a^c + n^c \Psi_a^b, \\
\mathcal{E}_a^b & \overset{\text{def}}{=} \nabla_a n^b + \varphi_a n^b - P^{bc} K_{ac} + n^{(2)} \Psi_a^b, \\
\mathcal{F}_a & \overset{\text{def}}{=} -\frac{1}{2} \nabla_a n^{(2)} - n^{(2)} \varphi_a + K_{ab} n^b.
\end{align*}
\]

and

\[
\begin{align*}
q & \overset{\text{def}}{=} n^a \ell_a - 1 + n^{(2)} \ell^{(2)}, \\
qu^a & \overset{\text{def}}{=} P^{ab} \ell_b + \ell^{(2)} n^a, \\
\gamma_b & \overset{\text{def}}{=} \gamma_{bc} n^c + n^{(2)} \ell_b, \\
\varphi_b & \overset{\text{def}}{=} n^c \ell^{(2)} + \Psi_a^c \varphi_a, \\
\delta_a & \overset{\text{def}}{=} P^{ac} \gamma_{cb} + n^a \ell_b - \delta_a^b.
\end{align*}
\]

Then, the following identities hold

\[
\begin{align*}
\mathcal{E}_a^b \ell_b - 2 \ell^{(2)} \mathcal{F}_a - 2 n^{(2)} \mathcal{C}_a & = \nabla_a q - K_{ac} w^c - \Psi_a^c z_a, \\
\mathcal{A}_{abc} n^c + \mathcal{E}_a^c \gamma_{bc} - 2 \mathcal{F}_a \ell_b + n^{(2)} \mathcal{B}_{ab} & = \nabla_a \varphi_b + K_{ac} (q_n^b - S_b^c) + \Psi_a \varphi_b, \\
\mathcal{D}_a^{bc} \ell_c + P^{bc} B_{ac} - 2 n^c \mathcal{C}_a + \ell^{(2)} \mathcal{E}_a^b & = \nabla_a \varphi_b + \Psi_a \varphi_b - \Phi_b \Psi_a^c - \varphi_a w_b, \\
\mathcal{D}_a^{bd} \gamma_{dc} + P^{bd} A_{adc} + \mathcal{E}_a^b \ell_c + n^b B_{ac} & = \nabla_a S_c^b + \Psi_a z_c + K_{ac} w_b, \\
\end{align*}
\]

where in the second equality we have used the definitions (43), and in the third we have used the definitions (45). The rest of expressions are more involved but can be proved similarly. \hfill \Box

The following corollary determines the equations that $P^{ab}$, $n^b$ and $n^{(2)}$ satisfy for hypersurface data.

**Corollary 2** Let $\{\Sigma, \gamma_{ab}, \ell_a, \ell^{(2)}, Y_{ab}\}$ be hypersurface data. Then, the following equations hold

\[
\begin{align*}
\nabla_a P^{bc} + n^b \Psi_a^c + n^c \Psi_a^b = 0, \\
\nabla_a n^b + \varphi_a n^b - P^{bc} K_{ac} + n^{(2)} \Psi_a^b = 0, \\
-\frac{1}{2} \nabla a n^{(2)} - n^{(2)} \varphi_a + K_{ab} n^b = 0.
\end{align*}
\]

**Proof.** Hypersurface data satisfies $q = z_a = w^a = S^a_b = 0$ and also $\mathcal{A}_{abc} = \mathcal{B}_{ab} = \mathcal{C}_a = 0$. Identities (46) then become

\[
\begin{align*}
\mathcal{E}_a^b \ell_b - 2 \ell^{(2)} \mathcal{F}_a & = 0, \\
\mathcal{E}_a^c \gamma_{bc} - 2 \mathcal{F}_a \ell_b & = 0, \\
\mathcal{D}_a^{bd} \gamma_{dc} + \mathcal{E}_a^b \ell_c & = 0.
\end{align*}
\]

Applying Lemma [3] to the first two (with $\mathcal{E}_a^b \to V^b$) yield the compatibility equation

\[
0 = \mathcal{F}_a (n^b \ell_b + n^{(2)} \ell^{(2)}) = \mathcal{F}_a
\]

\[12\]
and hence $\mathcal{E}_a^b = 0$ also. Applying now Lemma 3 to the second two gives $\mathcal{D}_a^{bc} = 0$. □

Proposition 2 has a second consequence on the relationship between $\{P^{ab}, n^b, n^{(a)}\}$ and $\{\gamma_{ab}, \ell_a, \ell^{(a)}\}$, whenever the field equations (47)-(49) hold together with the field equations (26)-(28):

**Corollary 3** Let $\gamma_{ab} = \gamma_{ba}$, $\ell_a$, $\ell^{(a)}$, $P^{ab}$, $n^b$, $n^{(a)}$ be $C^1$ fields on an $m$-dimensional connected manifold $(\Sigma, \nabla)$ satisfying the field equations (26)-(28) and (47)-(49). If the two $(m+1)$-matrices

\[
\begin{pmatrix}
\gamma_{ab} & \ell_a \\
\ell_b & \ell^{(a)}
\end{pmatrix}, \quad \begin{pmatrix}
P^{ab} & n^a \\
n^b & n^{(a)}
\end{pmatrix}
\]

are inverses from each other at one point $p \in \Sigma$, then they are inverses of each other at every point in $\Sigma$.

**Proof.** Identities (40) with $A_{abc} = B_{ab} = C_a = \mathcal{D}_a^{bc} = \mathcal{E}_a^b = \mathcal{F}_a = 0$ become a set of linear PDE for $\{q, z_a, w^a, S^a_b\}$ written in normal form. Thus, if $q = z_a = w^a = S^a_b = 0$ at one point, then they vanish everywhere on the (connected) manifold $\Sigma$.

### 3 Gauge transformations.

Up to now we have kept the rigging fixed. However, as already said, the rigging is highly non-unique. Thus, there must exist a set of transformations that keep the field equations invariant and which give essentially the same hypersurface data. This section is devoted to this issue.

Let us for the moment consider a hypersurface embedded in a spacetime. We will find how does the hypersurface data transform under an arbitrary change of rigging. Then, we will promote this transformation to gauge freedom in the hypersurface data and we will prove that the field equations are invariant under a gauge transformation.

Since a rigging is, by definition, transverse to the hypersurface any two riggings $\ell$ and $\ell'$ are related by

\[
\ell' = u (\ell + V),
\]

where $u$ is nowhere zero and $V$ is an arbitrary vector field along $\Sigma$ and tangent to $\Sigma$ everywhere. $V$ can be decomposed in the basis $\{e_a\}$ as $V = V^a e_a$. The vector $\hat{V} \overset{def}{=} V^a e_a$ is therefore a vector field of $\Sigma$.

First of all we note that the first fundamental form $\gamma_{ab}$ is independent of the choice of rigging and hence $\gamma_{ab}' = \gamma_{ab}$ (objets attached to the rigging $\ell'$ will carry a prime). Multiplying (50) by $e_a$ we obtain

\[
\ell'_a = u \left( \ell_a + V^b \gamma_{ab} \right),
\]

and squaring $\ell'$ we find

\[
\ell'^2 = g(\ell', \ell') = u^2 \left( \ell^{(a)} + 2 V^a \ell_a + V^a V^b \gamma_{ab} \right).
\]

It only remains to determine how does the tensor $Y$ change under a gauge transformation. Since we are assuming the data to be embedded, we can use $Y = \frac{1}{2} \Phi^*(\mathcal{L}_\ell g)$. We can determine $Y'$ as follows. Let $\tilde{u}$ denote any smooth extension of $u$ off $\Sigma$. Then

\[
Y' = \frac{1}{2} \Phi^*(\mathcal{L}_\ell g) = \frac{1}{2} \Phi^* \left( \mathcal{L}_{\tilde{u} (\ell + V)} g \right) = \frac{1}{2} \Phi^* \left( \tilde{u} \mathcal{L}_\ell g + d\tilde{u} \otimes \ell + \ell \otimes d\tilde{u} + \mathcal{L}_{\tilde{u} V} g \right) = u Y + \frac{1}{2} (d\ell \otimes \ell + \ell \otimes du) + \frac{1}{2} \mathcal{L}_{\tilde{u} V} \gamma,
\]

where we used the well-known properties $\Phi^*(d\tilde{u}) = d(\Phi^*(\tilde{u}))$ and $\Phi^*(\mathcal{L}_{\Phi^*(\tilde{u})} g) = \mathcal{L}_{\tilde{u} V} (\Phi^*(g))$.

The transformations above only involve a scalar function $u$ on $\Sigma$ and a vector field $\tilde{V}$ on $\Sigma$. We can therefore put forward a definition of *gauge transformation* for hypersurface data.
Definition 6 Let \( \{\gamma, \ell, \ell^{(2)}, Y\} \) be hypersurface data. Let \( u : \Sigma \to \mathbb{R} \) be a smooth scalar and \( V \in \Gamma(T\Sigma) \) a smooth vector field in \( \Sigma \). The \textit{gauge transformed} hypersurface data with gauge fields \((u, V)\) is defined as (in any basis \( \{\ell_a\} \) and with \( \tilde{V} = V^\alpha \tilde{e}_\alpha \))

\[
\begin{align*}
g'_{ab} &= \gamma_{ab}, \\
\ell'_a &= u \left( \ell_a + V^b \gamma_{ab} \right), \\
\ell^{(2)'} &= u^2 \left( \ell^{(2)} + 2V^b \ell_a + V^a V^b \gamma_{ab} \right), \\
Y_{ab}' &= u Y_{ab} + \frac{1}{2} \left( \ell_a \partial_b u + \ell_b \partial_a u \right) + \frac{1}{2} \ell_u \gamma_{ab}. \tag{52}
\end{align*}
\]

The following Lemma shows how the derived fields \( \{P^{ab}, n^a, n^{(2)}\} \) and \( \{\Gamma^c_{ab}, K_{ab}, \varphi_a, \Psi^b_a\} \) change under a gauge transformation (cf. \cite{22} and the Remark after the proof).

Lemma 5 Let \((u, V^a)\) be gauge fields. The fields \((P^{ab}, n^a, n^{(2)'}\)) corresponding to the gauge transformed data read

\[
P^{ab} = P^{ab} + V^a V^b n^{(2)} - V^a n^b - V^b n^a, \quad n^a = u^{-1} (n^a - V^a n^{(2)}) \quad n^{(2)'} = u^{-2} n^{(2)}. \tag{53}
\]

Moreover, the connection \( \Gamma^c_{ab} \) and the tensor fields \( K_{ab}, \varphi_a, \Psi^b_a \) transform according to

\[
\begin{align*}
\Gamma^c_{ab}' &= \Gamma^c_{ab} + V^c K_{ab}, \\
K_{ab}' &= u^{-1} K_{ab}, \\
\varphi_a' &= \varphi_a + u^{-2} \partial_a u - K_{ab} V^b, \\
\Psi^b_a &= u \left( \Psi^b_a + \nabla_a V^b - \varphi_a V^b + K_{ad} V^d V^b \right). \tag{54-57}
\end{align*}
\]

Proof. A straightforward calculation shows that, with the expressions \cite{53}, the primed version of \cite{53} is satisfied. Uniqueness of solutions of these equations implies the first part of the Lemma. Next we prove \cite{55}. First notice the following simple identity for the Lie derivative (a version of which was in fact already used above).

\[
A \nabla_X \gamma_{ab} = \nabla_X \gamma_{ab} - \gamma_{ac} \nabla^c \partial_b A - \gamma_{bc} \nabla^c \partial_a A. \tag{58}
\]

We now calculate \( n^{(2)'} Y' \)

\[
n^{(2)'} Y_{ab} = \frac{n^{(2)}}{u} Y_{ab} + \frac{n^{(2)'} u + n^{(2)} u}{2} \left( \ell_a \partial_b u + \ell_b \partial_a u \right) + \frac{n^{(2)'} u}{2} \ell_{u} \gamma_{ab}
\]

\[
= \frac{n^{(2)}}{u} Y_{ab} + \frac{n^{(2)'} u}{2} \left( \ell_a \partial_b u + \ell_b \partial_a u \right) + \frac{1}{2} \frac{\ell_{u} n^{(2)}}{u} \gamma_{ab} - \frac{1}{2} \left( u \gamma_{ac} V^c \partial_b n^{(2)} + u \gamma_{bc} V^c \partial_a n^{(2)} \right)
\]

\[
= \frac{n^{(2)}}{u} Y_{ab} + \frac{n^{(2)'} u}{2} \left( \ell_a \partial_b u + \ell_b \partial_a u \right) + \frac{1}{2} \frac{\ell_{u} n^{(2)}}{u} \gamma_{ab} + \frac{1}{2} \left( u \partial_a - \ell_a u \right) \partial_b n^{(2)} + \frac{1}{2} \left( u \partial_b - \ell_b u \right) \partial_a n^{(2)}
\]

where in the second equality we used \cite{55} with \( \nabla_X \) \( u \nabla_X \) and \( A \to n^{(2)'} \), and in the third equality we used the transformation law for \( \ell_a \). We now insert this into the primed version of \cite{50}. This yields

\[
K_{ab}' = \frac{n^{(2)}}{u} Y_{ab} + \frac{n^{(2)'} u}{2} \left( \ell_a \partial_b u + \ell_b \partial_a u \right) + \frac{1}{2} \left( u \partial_a \partial_b n^{(2)} + u \partial_b \partial_a n^{(2)} \right) + \frac{1}{2} \ell_{u} n^{(2)} \gamma_{ab}
\]

\[
= \frac{1}{u} \left( n^{(2)} Y_{ab} + \frac{1}{2} \left( \ell_a \partial_b n^{(2)} + \ell_b \partial_a n^{(2)} \right) + \frac{1}{2} \ell_{u} \gamma_{ab} \right) - \frac{1}{2u^2} \left( n^{(2)} \ell_a - \gamma_{ac} n^{(2)'} \right) \partial_b u - \frac{1}{2u^2} \left( n^{(2)} \ell_b - \gamma_{bc} n^{(2)'} \right) \partial_a u
\]

\[
= \frac{1}{u} K_{ab},
\]

where in the second equality we have used \( \nabla_X \) \( u \nabla_X \) and then the identity \cite{55} with \( \nabla_X \) \( u^{-1} \nabla_X \) and \( A \to u \) and in the last equality we have used \cite{9} and \cite{50}. This proves \cite{55}.
Next we address (54). $\Gamma^c_{ab}$ is uniquely defined as the unique solution of the primed versions of (37) and (38). Subtracting (37) and its primed version and using the invariance of $\gamma_{ab}$ and the transformation law for $\ell_a$ and $K_{ab}$ it follows

$$\left(\Gamma^d_{ab} - \Gamma^d_{ab}\right) \gamma_{cd} = V^d \gamma_{cd} K_{ab}$$

(59)

If we also determine $(\Gamma^d_{ab} - \Gamma^d_{ab})\ell_d$ we will be able to solve for $\Gamma^d_{ab}$. To that aim, let us start by deriving the following identity

$$\frac{1}{2} \left( \partial_a \ell'_b + \partial_b \ell'_a \right) - Y'_{ab} =$$

$$= \frac{1}{2} \left( \partial_a [u(\ell_b + \gamma_{bc} V^c)] + \partial_b [u(\ell_a + \gamma_{ac} V^c)] \right) - uY_{ab} - \frac{1}{2} \left( \ell_a \partial_b u + \ell_b \partial_a u \right) - \frac{1}{2} \ell_a\hat{\nabla} \gamma_{ab}$$

$$= u \left( \frac{1}{2} \left( \partial_a \ell_b + \partial_b \ell_a \right) - Y_{ab} \right) + \frac{1}{2} \partial_a \left( u \gamma_{bc} V^c \right) + \partial_b \left( u \gamma_{ac} V^c \right) - \ell_a\hat{\nabla} \gamma_{ab}$$

$$= u \left( \frac{1}{2} \left( \partial_a \ell_b + \partial_b \ell_a \right) - Y_{ab} \right) + \frac{1}{2} \partial_a \gamma_{bc} + \partial_b \gamma_{ac} - \partial_c \gamma_{ab} V^c.$$

Now we evaluate

$$\left(\Gamma^d_{ab} - \Gamma^d_{ab}\right) \ell_d = \frac{1}{u} \frac{1}{2} \left( \partial_a \ell'_b + \partial_b \ell'_a \right) - Y'_{ab} - \frac{1}{2} \ell'_{ab} - \frac{1}{2} \ell_{ab}$$

$$= u \left( \frac{1}{2} \left( \partial_a \ell_b + \partial_b \ell_a \right) - Y_{ab} \right) + \frac{1}{2} \partial_a \gamma_{bc} + \partial_b \gamma_{ac} - \partial_c \gamma_{ab} V^c - \ell_{cd} V^c K_{ab}$$

(60)

where in the second equality we have used the primed version of (35), equation (35) itself and the primed version of (37) and in the last equality we used the identity (60) and the transformation law for $\ell^{(2)}$. It is now immediate from this expression and (54) to conclude that (60) holds as a consequence of Lemma 8.

In order to prove the remaining transformations, we first establish the following transformation (\nabla is the covariant derivative with connection symbols $\nabla$)

$$\nabla_a \ell'_b + \ell^{(2)} K'_{ab} = \nabla_a \ell'_b - \left( \Gamma^c_{ab} - \Gamma^c_{ab} \right) \ell'_c + \ell^{(2)} K_{ab}$$

$$= \nabla_a \left( u \left[ \ell_b + \gamma_{bc} V^c \right] \right) - u V^c K_{ab} \left( \ell_c + \gamma_{cd} V^d \right) + u^{-1} \ell^{(2)} K_{ab}$$

$$= u \left( \nabla_a \ell_b + \ell^{(2)} K_{ab} \right) + \left( \ell_b + \gamma_{bc} V^c \right) \nabla_a u - u K_{ac} V^c \ell_b + u \gamma_{bc} \nabla_a V^c.$$

(61)

where in the second equality we used the gauge transformation law for $\ell_c$, $\Gamma^c_{ab}$ and $K_{ab}$ and in the third one the field equation (26) and the transformation law for $\ell^{(2)}$ was used. Proposition 1 shows that $\varphi_a$ can be written in the form

$$\varphi_a = n^b \left( \nabla_a \ell_b + \ell^{(2)} K_{ab} \right) + \frac{n^{(2)} K_{ab}}{2} \nabla_a \ell^{(2)}.$$

(62)

The transformation law (56) for $\varphi_a$ follows by a direct substitution of $n^b$, $n^{(2)}$, $\ell^{(2)}$ and (61) in the primed version of (62).

The proof of (57) is the most involved one. One option would be brute force calculation from (32). However, the gauge transformation of $P_{ab}$ is long and the calculation becomes cumbersome. Instead, we subtract the primed version of (27) and $u$ times (27) itself. This yields, after using (61) and the transformation law for $\varphi_a$,

$$\gamma_{bc} \left( \Psi^c_a - u \left[ \Psi^c_a + \nabla_a V^c - \varphi_a V^c + K_{ad} V^d V^c \right] \right) = 0.$$

(63)
This still fails short of proving \( (57) \) because \( \gamma_{bc} \) need not be invertible. To complete the argument, we compute the primed version of \( (28) \) subtracting \( u^2 \) times \( (28) \) and adding \( u^2 \) times \( (27) \) contracted with \( V^b \). A not-long calculation which uses \( (63) \) yields

\[
 u\ell_c \left( \Psi^c_{\ a} - u \left( \Psi^c_{\ a} + \nabla_a V^c - \varphi_a V^c + K_{ad} V^d V^c \right) \right) = 0. \tag{64}
\]

Lemma \( \ref{gaugetransf} \) applied to \( (63) \) and \( (64) \) establishes \( (57) \).

\( \square \)

**Remark 4.** The gauge transformation law obtained in this proposition would have been much easier to prove assuming that the hypersurface data is embedded. Indeed, in such circumstances, the spacetime definition of the connection, the second fundamental form and the tensors \( \Psi^a_{\ bc} \) and \( \varphi_a \) imply very easily \( (54) - (57) \), see \( \ref{exptensor} \). However, our definition of gauge transformation is directly at the hypersurface data level, and hence a hypersurface proof as the one above becomes necessary.

From the proof of the preceding lemma, we can infer that in order to find gauge transformation laws of contravariant tensors, it may be often convenient to study the transformation law of the one-form obtained after contraction with \( \gamma_{ab} \) and the scalar obtained by contraction with \( \ell_b \). The following proposition formalizes this observation.

**Proposition 3** Let \( Q^a = Q^a(\gamma_{bd}, \ell_b, \ell^{(2)}, Y_{bd}) \) be a vector depending on the hypersurface data and define

\[
 H^c_{\ ac} \defeq Q^c_{\ ac}, \quad R^a \defeq Q^a_{\ ac},
\]

all of them viewed as functions of the hypersurface data. Fix gauge fields \( (u, V^a) \) and let \( \hat{H}, \hat{R} \) be the functions of hypersurface data and gauge field defined by

\[
 \hat{H}_{\ ac}(\gamma_{bd}, \ell_b, \ell^{(2)}, Y_{bd}, u, V^d) \defeq H^c_{\ ac}(\gamma_{bd}, \ell_b, \ell^{(2)}, Y_{bd}, u, V^d),
\]

\[
 \hat{R}_{\ ac}(\gamma_{bd}, \ell_b, \ell^{(2)}, Y_{bd}, u, V^d) \defeq R^a(\gamma_{bd}, \ell_b, \ell^{(2)}, Y_{bd}, u, V^d),
\]

where in the right-hand side we substitute the explicit expressions \( (53) \). Then, the following two statements are equivalent.

(i) The functions \( \hat{Q}^a(\gamma_{bd}, \ell_b, \ell^{(2)}, Y_{bd}, u, V^d) \) satisfy the identities

\[
 \hat{H} = \hat{Q}^a_{\ ac},
\]

\[
 \hat{R} = u \left( \hat{Q}^a_{\ ac} + V^a \hat{H} \right). \tag{65}
\]

(ii) The functions \( \hat{Q}^a(\gamma_{bd}, \ell_b, \ell^{(2)}, Y_{bd}, u, V^d) \) satisfy \( \hat{Q}^a(\gamma_{bd}, \ell_b, \ell^{(2)}, Y_{bd}, u, V^d) = Q^a(\gamma_{bd}, \ell_b, \ell^{(2)}, Y_{bd}, u, V^d) \), where in the right-hand side the explicit expressions \( (52) \) are substituted.

**Proof.** Assume that (ii) holds. Then

\[
 \hat{H}_{\ ac} = H^c_{\ ac}(\gamma_{bd}, \ell_b, \ell^{(2)}, Y_{bd}) = Q^c_{\ ac}(\gamma_{bd}, \ell_b, \ell^{(2)}, Y_{bd})_{\ gamma_{ac}} = \hat{Q}^c_{\ ac},
\]

\[
 \hat{R} = R^a(\gamma_{bd}, \ell_b, \ell^{(2)}, Y_{bd}) = Q^a(\gamma_{bd}, \ell_b, \ell^{(2)}, Y_{bd})_{\ gamma_{ac}} = u \left( Q^a(\gamma_{bd}, \ell_b, \ell^{(2)}, Y_{bd})_{\ gamma_{ac}} + V^c Q^c_{\ ac}(\gamma_{bd}, \ell_b, \ell^{(2)}, Y_{bd})_{\ gamma_{ac}} \right) =
\]

\[
 = u \left( \hat{Q}^a_{\ ac} + V^a \hat{H} \right),
\]

which establishes the validity of (i). Conversely, assume that (i) holds. Define \( \hat{Q}^a(\gamma_{bd}, \ell_b, \ell^{(2)}, Y_{bd}, u, V^d) = Q^a(\gamma_{bd}, \ell_b, \ell^{(2)}, Y_{bd}, u, V^d) \), where in the right hand side the explicit expressions \( (52) \) are substituted. By the calculation above, this function satisfies

\[
 \hat{H} = \hat{Q}^a_{\ ac},
\]

\[
 \hat{R} = u \left( \hat{Q}^a_{\ ac} + V^c \hat{H} \right).
\]

Subtracting this to \( (63) \) and \( (66) \) yields \( \hat{Q}^a - \hat{Q}^a = 0 \) and \( u(\hat{Q}^a - \hat{Q}^a)_{\ gamma_{ac}} = 0 \). Lemma \( \ref{gaugetransf} \) implies \( \hat{Q}^a - \hat{Q}^a = 0 \) which proves (ii). \( \square \)

\[16\]
Remark 5. Obviously, this result can be applied to tensors of arbitrary rank as long as one of its indices is contravariant and all the operations and expressions involved in the lemma refer to this index, leaving all the rest of indices untouched.

As an example of the usefulness of this lemma, we obtain the transformation law of \( (n^{(2)} P_{ab} - n^a n^b) \) and of \( P_{ab} n^c - P^{ac} n^b \), which will be needed later.

Lemma 6 Under a gauge transformation with gauge fields \((u, V^n)\), the tensors \( n^{(2)} P_{ab} - n^a n^b \) and \( P_{ab} n^c - P^{ac} n^b \) transforms as

\[
\begin{align*}
n^{(2)} P_{bd} - n^b n^d = & \frac{1}{u^2} \left( n^{(2)} P_{bd} - n^b n^d \right), \\
P_{ab} n^c - P^{ac} n^b = & \frac{1}{u} \left( P_{ab} n^c - P^{ac} n^b + n^{(2)} P^{ac} - n^a n^c \right) V^b - \left( n^{(2)} P_{ab} - n^a n^b \right) V^c.
\end{align*}
\]

(67) (68)

Proof. Let \( Q^{ab} = n^{(2)} P_{ab} - n^a n^b \). According to Proposition 3 we start calculating \( H^b_{c \ell a} = Q^{ab} \gamma_{ac} = (n^{(2)} P_{ab} - n^a n^b) \gamma_{ac} = n^{(2)} \delta_c^b \) and \( R^b_{a \ell} = Q^{ab} \ell_a = -n^b \). Consequently

\[
\begin{align*}
\hat{H}^b_{c \ell a} = & \frac{n^{(2)} \delta_c^b}{u^2}, \\
\hat{R}^b_{a \ell} = & \frac{1}{u} \left( n^a - V^b n^{(2)} \right).
\end{align*}
\]

It follows from (65) and (66) that \( \hat{Q}^{ab} \) satisfies

\[
\begin{align*}
\hat{Q}^{ab} \gamma_{ac} = & \hat{H}^b_{c \ell a} = \frac{n^{(2)} \delta_c^b}{u^2}, \\
\hat{Q}^{ab} \ell_a = & \frac{\hat{R}^b_{a \ell}}{u} - V^c \hat{H}^b_{c \ell a} = -\frac{1}{u^2} n^a + \frac{1}{u^2} V^b n^{(2)} - V^c \frac{n^{(2)} \delta_c^b}{u^2} = -\frac{1}{u^2} n^a.
\end{align*}
\]

Comparing with the expressions for \( H^b_{a \ell} \) and \( R^b_{a \ell} \) above (or, alternatively, using Lemma 3), it follows immediately that \( \hat{Q}^{ab} = \frac{1}{u^2} \left( n^{(2)} P_{ab} - n^a n^b \right) \). This establishes (67). For the second statement, let \( H^b_{d c \ell} = (P_{ab} n^c - P^{ac} n^b) \gamma_{ad} = (\delta_d^b n^c - \delta_d^c n^b), \quad R^b_{a \ell} = (P_{ab} n^c - P^{ac} n^b) \ell_a = 0 \).

Equations (65) and (66) become

\[
\begin{align*}
\hat{H}^b_{d c \ell a} = & \delta_d^b n^c - \delta_d^c n^b = \frac{1}{u} \left( \delta_d^b n^c - \delta_d^c n^b \right) V^c - \delta_d^c n^b + n^{(2)} \delta_d^c V^b = \hat{Q}^{abc} \gamma_{ad}, \\
\hat{R}^b_{d c \ell a} = & 0 = u \left( \hat{Q}^{abc} \ell_a + V^a \hat{H}^b_{d c \ell a} \right) = u \hat{Q}^{abc} \ell_a + V^b n^c - V^c n^b
\end{align*}
\]

Using now Lemma 3 gives \( Q^{abc} = \frac{1}{u} \left( P_{ab} n^c - P^{ac} n^b + n^{(2)} P^{ac} - n^a n^c \right) V^b - \left( n^{(2)} P_{ab} - n^a n^b \right) V^c \), which proves (68).

\[\square\]

4 Einstein tensor on the hypersurface.

In this section we obtain the constraint equations in the case of general hypersurfaces. We start with the following lemma, which shows that the components \( G^n_{\mu, \nu, \ell}^{\rho} \) and \( G^n_{\mu, \nu, \ell, \nu}^{\rho} \) can be obtained in terms of the hypersurface data whenever this data is embedded in a spacetime (our sign conventions for the Riemann, Ricci and Einstein tensor follow [29]).
Proposition 4 Let \( \{ \Sigma, \gamma, \ell, \ell (2), Y \} \) be embedded hypersurface data with embedding \( \Phi \) and rigging \( \ell \). Let \( \{ \hat{e}_a \} \) be a basis of \( T \Sigma \) and \( e_a \overset{\text{def}}{=} \Phi_*(\hat{e}_a) \). Then
\[
G^\mu_{\nu \rho \sigma} = - R_{\alpha \beta \gamma \delta} e^\alpha_b e^\beta_c e^\gamma_d e^\delta_e n^\rho P^{bd} - \frac{1}{2} R_{\alpha \beta \gamma \delta} e^\alpha_b e^\beta_c e^\gamma_d e^\delta_e P^{bd},
\]
(69)
\[
G^\mu_{\nu \rho \sigma} = R_{\alpha \beta \gamma \delta} e^\alpha_b e^\beta_c e^\gamma_d (n^{(2)} P^{bd} - n^b n^d) + R_{\alpha \beta \gamma \delta} e^\alpha_b e^\beta_c e^\gamma_d e^\delta_e n^\rho P^{bd}.
\]
(70)
where \( R^\alpha_{\beta \gamma \delta} \) is the Riemann tensor of the ambient spacetime \( (\mathcal{M}, g) \) and \( G^\mu_{\nu} \) is the corresponding Einstein tensor.

Proof. We start by obtaining an expression for the spacetime metric \( g^{\alpha \gamma} \) in terms of the data on \( \Sigma \). Since \( \{ e_a, \ell \} \) is a basis of \( T \Sigma, \mathcal{M} \) and \( \{ \omega_a, n \} \) is the dual basis we can decompose \( g^{\alpha \gamma} \) as (see (12))
\[
g^{\alpha \gamma} \overset{\Sigma}{=} P^{ab} e^\alpha_a e^\gamma_b + n^{a} (e^\alpha_a e^\gamma_c + e^\gamma_b e^\alpha_c) + n^{(2)} \ell^\alpha \ell^\gamma.
\]
Using \( n^a e^\alpha_a = n^{a} - n^{(2)} \ell^\alpha \) the following alternative expression also holds
\[
g^{\alpha \gamma} \overset{\Sigma}{=} P^{ac} e^\alpha_a e^\gamma_c + (n^{a} \ell^\gamma + n^{(2)} \ell^\gamma) \ell^\alpha - n^{(2)} \ell^\alpha \ell^\gamma.
\]
(71)
We now calculate the curvature scalar \( R = g^{\beta \delta} R_{\beta \delta} = g^{\alpha \gamma} g^{\beta \delta} R_{\alpha \beta \gamma \delta} \). Inserting (71) and using the symmetries of the Riemann tensor yields
\[
R \overset{\Sigma}{=} R_{\alpha \beta \gamma \delta} e^\alpha_b e^\beta_c e^\gamma_d e^\delta_e P^{bd} + 4 R_{\alpha \beta \gamma \delta} e^\alpha_b e^\beta_c e^\gamma_d e^\delta_e P^{bd} - 2 n^{(2)} R_{\alpha \beta \gamma \delta} e^\alpha_b e^\beta_c e^\gamma_d e^\delta_e P^{bd} - 2 R_{\alpha \beta \gamma \delta} e^\alpha_b e^\beta_c e^\gamma_d e^\delta_e P^{bd}.
\]
(72)
We can now calculate \( G^\mu_{\nu \rho \sigma} = R_{\alpha \beta \gamma \delta} e^\alpha_b e^\beta_c e^\gamma_d e^\delta_e P^{bd} \). Inserting (71), the first term is
\[
R_{\alpha \beta \gamma \delta} e^\alpha_b e^\beta_c e^\gamma_d e^\delta_e P^{bd} = R_{\alpha \beta \gamma \delta} e^\alpha_b e^\beta_c e^\gamma_d e^\delta_e P^{bd} - R_{\alpha \beta \gamma \delta} e^\alpha_b e^\beta_c e^\gamma_d e^\delta_e P^{bd}.
\]
(73)
Subtracting (73) and \( \frac{1}{2} R \) (from (72)), it follows
\[
G^\mu_{\nu \rho \sigma} = - \frac{1}{2} R_{\alpha \beta \gamma \delta} e^\alpha_b e^\beta_c e^\gamma_d e^\delta_e P^{bd} - R_{\alpha \beta \gamma \delta} e^\alpha_b e^\beta_c e^\gamma_d e^\delta_e P^{bd} + n^{(2)} R_{\alpha \beta \gamma \delta} e^\alpha_b e^\beta_c e^\gamma_d e^\delta_e P^{bd}.
\]
Using here \( n^\gamma = n^c e^\gamma_c + n^{(2)} \ell^\gamma \) yields (69).
Regarding (70), we need to evaluate \( G^\mu_{\nu \rho \sigma} = R_{\alpha \beta \gamma \delta} e^\alpha_b e^\beta_c e^\gamma_d e^\delta_e P^{bd} \). Using (71) we get
\[
G^\mu_{\nu \rho \sigma} = R_{\alpha \beta \gamma \delta} e^\alpha_b e^\beta_c e^\gamma_d e^\delta_e P^{bd} + R_{\alpha \beta \gamma \delta} n^a \ell^\delta e^\gamma_c \ell^\gamma e^\delta_e (n^\delta - n^{(2)} \ell^\delta)\]
which becomes (70) after inserting \( n^a = n^c e^\gamma_c + n^{(2)} \ell^\gamma \).
\[
\square
\]

Remark 6. \( G^\mu_{\nu \rho \sigma} \) could also have been obtained by exploiting the gauge transformations of Section 9. Inserting the transformation (70) into (69) and using the fact that \( V^a \) is arbitrary implies the validity of (70). This method is longer but straightforward and provides a non-trivial consistency check both for the gauge transformations and for the expressions (69) and (70).

Our next aim is to write down the right-hand sides of (69) and (70) in terms of hypersurface data. We start with the following identity, which has been obtained in [22] (see formulas (12) and (13) there), and which can be proved by inserting the decomposition (21) in the Ricci identity for the Riemann tensor of the ambient spacetime.

Proposition 5 (Mars & Senovilla [22]) Let \( \{ \Sigma, \gamma, \ell, \ell (2), Y \} \) be embedded hypersurface data with embedding \( \Phi \) and rigging \( \ell \). Let \( \{ \hat{e}_a \} \) be a basis of \( T \Sigma \) and \( e_a \overset{\text{def}}{=} \Phi_*(\hat{e}_a) \). Then
\[
R^\alpha_{\beta \gamma \delta} e^\alpha_b e^\beta_c e^\gamma_d e^\delta_e = \left( \nabla^a R_{bcd} - K_{bd} \Psi^a_c + K_{bc} \Psi^a_d \right) e_a - \left( \nabla^a K_{bd} - \nabla^a K_{bd} + K_{bd} \varphi_c - K_{bd} \varphi_d \right) \ell^a.
\]
(74)
where \( R^\alpha_{\beta \gamma \delta} \) is the Riemann tensor of the ambient spacetime \( (\mathcal{M}, g) \) and \( R_{bcd} \) the curvature tensor of \( T^a_{ab} \) in the basis \( \{ \hat{e}_a \} \).
This result has the following corollary, which in particular defines two quantities denoted by (I) and (II), which will play a useful role later.

**Corollary 4** With the same hypotheses as in Proposition 3, the following identities hold

\[ R_{\alpha\beta\gamma\delta} e^\alpha_b e^\beta_c e^\gamma_d + \frac{\Sigma}{2} K_{bc} E_{d}^{(2)} + \frac{1}{2} K_{bc} \nabla_d E_{d}^{(2)} = P^{bd} (K)_{bd} - P_{bd} (K)_{bd} \]

(75)

\[ R_{\alpha\beta\gamma\delta} e^\alpha_b e^\beta_c e^\gamma_d + \frac{\Sigma}{2} K_{bc} E_{d}^{(2)} + \frac{1}{2} K_{bc} \nabla_d E_{d}^{(2)} = P^{bd} (K)_{bd} - P_{bd} (K)_{bd} \]

(76)

**Proof.** For (75), contract (74) with \( e_a^c \) and use equation (25). For (76) contract the same expression with \( e_a^c \) and use (27).

**Remark 7.** Identity (75) also appears, in a slightly modified form, in expression (17) of [22].

We are now in a position where the components \( G_{\mu} n_{\mu} e^\nu_a \) of the Einstein tensor can be obtained in terms of hypersurface data

**Theorem 1** Let \( \{ \Sigma, \gamma, \ell, e^{(2)}, Y \} \) be embedded hypersurface data with embedding \( \Phi \), rigging \( \ell \) and ambient spacetime \( (\mathcal{M}, g) \). Denote by \( G^\mu_{\nu} \) the Einstein tensor of \( (\mathcal{M}, g) \). Let \( \{ \ell \} \) be a basis of \( T \Sigma \) and \( e_a^c \) def \( \Phi \). Let \( e_a^c \) be embedded hypersurface data with embedding \( \Phi \). Then the following identities hold

\[ 2G^\mu_{\nu} n_{\mu} e^\nu_a = -P^{bd} X_{bd} - P_{bd} n_{c} \left( \ell_{b} R_{bcd} + \nabla_d (\ell^{(2)} K_{bc}) - \nabla_c (\ell^{(2)} K_{bd}) \right) - P^{ac} P^{bd} (K_{bc} Y_{da} - K_{bd} Y_{ac}) \]

(77)

\[ G^\mu_{\nu} n_{\mu} e^\nu_a = P^{bd} (K_{bd} - K_{bd}) - P^{bd} (K_{bd} - K_{bd}) \left( \nabla_d K_{bc} - \nabla_c K_{bd} \right) + \frac{1}{2} (n^{(2)} P_{bd} - n^{b} n^{d}) (K_{bd} \nabla d K_{bc} - K_{bd} \nabla c K_{bc}) + P^{bd} n_{c} [K_{bc} (Y_{da} + F_{da}) - K_{bd} (Y_{ca} + F_{ca})] - n^{b} n^{d} \ell_{a} R_{bcd} \]

(78)

**Proof.** Proposition 3 shows, in particular, that \( G^\mu_{\nu} n_{\mu} e^\nu_a = -P^{bd} X_{bd} \), where

\[ X_{bd} \defeq \left( R_{ \alpha \beta \gamma \delta } e^\alpha_b e^\beta_c e^\gamma_d + \frac{1}{2} R_{ \alpha \beta \gamma \delta } e^\alpha_b e^\beta_c e^\gamma_d P^{ac} \right) \]

So, we start finding an expression for \( X_{bd} \) in terms of hypersurface data. Using (3) and (3) in (76) yields immediately

\[ R_{ \alpha \beta \gamma \delta } e^\alpha_b e^\beta_c e^\gamma_d P^{ac} = R_{ \alpha \beta \gamma \delta } e^\alpha_b e^\beta_c e^\gamma_d + \frac{1}{2} R_{ \alpha \beta \gamma \delta } e^\alpha_b e^\beta_c e^\gamma_d P^{ac} \]

(79)

where we used \( \nabla_d K_{bc} - \nabla_c K_{bd} = Y_{ca} + F_{ca} \) (see (12)) and the fact that \( F_{ca} \) is antisymmetric. Contracting (75) with \( n^c \) and combining with (79) into \( X_{bd} \) implies easily

\[ X_{bd} = \frac{1}{2} \left( R_{bcd} + n^c \ell_{a} R_{bcd} + P^{ac} K_{bc} (Y_{da} + F_{da}) - P^{ac} K_{bd} Y_{ac} + n^c \left[ \nabla_d (\ell^{(2)} K_{bc}) - \nabla_c (\ell^{(2)} K_{bd}) \right] \right) \]

Contracting this with \( P^{bd} \). (77) follows directly after using \( P^{ac} P^{bd} K_{bc} F_{da} = 0 \), which holds because it is the contraction of a symmetric and an antisymmetric tensor.

To prove (78) we evaluate first

\[ Z_{bcd} \defeq R_{ \alpha \beta \gamma \delta } e^\alpha_b e^\beta_c e^\gamma_d n^a + n^{(2)} R_{ \alpha \beta \gamma \delta } e^\alpha_b e^\beta_c e^\gamma_d = n^a [K_{bc} (Y_{da} + F_{da}) - K_{bd} (Y_{ca} + F_{ca})] + \nabla d K_{bc} - \nabla c K_{bd} + \frac{1}{2} (K_{bc} \nabla d K_{bc} - K_{bd} \nabla c K_{bc}) \]

(80)
where we used Corollary 4 and (6), (5). Identity (78) now follows from (see (70))

\[ G^\mu_{\nu c} n_\mu e^\nu = P^{bd} Z_{bcd} - n^b n^d R_{\alpha \beta \gamma \delta} e^\alpha_\beta e^\gamma_\delta \]

after using (75) and (80). □

We have worked so far using the connection \( \Gamma_{bc} \), which arises naturally from the embedding of the hypersurface data in a spacetime. Note, however, that this connection depends on the full hypersurface data, in particular on \( Y_{ab} \). If one wishes to view the constraint equations as field equations for the hypersurface data, this mixture between the connection \( \Gamma_{bc} \) and the variable \( Y_{ab} \) obscures notably the equations. Note that this does not happen in the usual constraint equations for spacelike hypersurfaces, where the connection on \( \Sigma \) is fully independent of the extrinsic curvature. It is natural to try and do something similar in the general context and make all dependence on \( Y_{ab} \) in the constraint equations fully explicitly. We will find an example where this strategy is useful in Section 5 below.

In order to accomplish this, it is necessary to introduce a connection which does not depend on \( Y_{ab} \). In view of (29), the natural choice is the following.

**Definition 7** Let \( \{ \Sigma, \gamma, \ell^{(2)} \} \) be hypersurface metric data. We define the metric hypersurface connection \( \Gamma_{bc} \) by

\[
\Gamma_{ab}^{(c)} \overset{\text{def}}{=} \frac{1}{2} R^{cd} (\partial_a \gamma_{bd} + \partial_b \gamma_{ad} - \partial_d \gamma_{ab}) + \frac{1}{2} n^c (\partial_a \ell_b + \partial_b \ell_a). \tag{81}
\]

**Remark 8.** Despite its name, the metric hypersurface connection is, in general, not the Levi Civita connection of any metric on \( \Sigma \). Since hypersurface metric data does not define any canonical metric on \( \Sigma \), the name should not be a source of confusion.

It is clear that the metric hypersurface connection is torsion-free and that it relates to the induced connection \( \Gamma_{bc}^{(1)} \) by

\[
\Gamma_{bc}^{(1)} = \Gamma_{bc}^{(2)} - n^a Y_{ab}. \tag{82}
\]

It is well-known that two connections \( \Gamma_{bc}^{(1)} \) and \( \Gamma_{bc}^{(2)} \), whose difference tensor is \( C_{bc}^{(a)} \overset{\text{def}}{=} \Gamma_{bc}^{(1)} - \Gamma_{bc}^{(2)} \), define respective curvature tensors \( R_{bcd}^{(1)} \) and \( R_{bcd}^{(2)} \) satisfying (see e.g. formula (7.5.8) in [29])

\[
(1) R_{bcd}^{(1)} = (2) R_{bcd}^{(2)} + \nabla_c C_{bd}^{(1)} - (1) \nabla_d C_{bc}^{(1)} + C_{af}^{(1)} C_{bd}^{(1)} - C_{cf}^{(1)} C_{bd}^{(1)}, \]

where \( (1) \nabla \) is the covariant derivative of \( \Gamma_{bc}^{(1)} \). Applying this to the metric hypersurface and induced connections, we find

\[
\overline{R}_{bcd}^{f} = \overline{R}_{bcd}^{(2)} + \nabla_d (n^f Y_{bc}) - \nabla_c (n^f Y_{bd}) + n^f n^a (Y_{ca} Y_{bd} - Y_{da} Y_{bc}). \tag{83}
\]

Our aim is to rewrite the expressions in Corollary 4 in terms of the connection \( \Gamma \) and its curvature tensor. For that we need to transform covariant derivatives with respect to \( \nabla \) into covariant derivatives with \( \overline{\nabla} \). We collect the necessary expression in the following Lemma

**Lemma 7** With the definitions above, let, in addition, \( Z_{ab} \) be an arbitrary tensor. Then the following
The relation
\[ \nabla_a \ell_b = F_{ab} - \ell^{(2)} U_{ab}, \quad (84) \]
\[ \nabla_a \gamma_{bc} = -\ell_b U_{ac} - \ell_c U_{ab}, \quad (85) \]
\[ \nabla_a P^{bc} = -(n^b P^{b\ell} + n^c P^{c\ell}) F_{ab} - n^b \partial_\ell \partial_a \ell^{(2)}, \quad (86) \]
\[ \nabla_a n^b = -(n^b \partial_\ell \partial_a \ell^{(2)} - (n^{(2)} P^{b\ell} + n^b n^f) F_{ab} + P^{bf} U_{af}, \quad (87) \]
\[ \ell_a \bar{R}^a_{\ bcd} = \ell_a R^a_{\ bcd} + \nabla_c [(n \cdot \ell) Y_{bc}] - \nabla_c [(n \cdot \ell) Y_{bd}] + \]
\[ + n^f [Y_{bd} ((n \cdot \ell) Y_{cf} + F_{cf} - \ell^{(2)} U_{cf}) - Y_{bc} ((n^a \ell_b) Y_{df} + F_{df} - \ell^{(2)} U_{df})], \quad (88) \]
\[ \gamma_{af} \bar{R}^f_{\ bcd} = \gamma_{af} R^f_{\ bcd} + \ell_a \left( \nabla_c (n^{(2)} Y_{bd}) - \nabla_d (n^{(2)} Y_{bc}) \right) + Y_{bc} (U_{da} - n^{(2)} F_{da}), \quad (89) \]
\[ \nabla_d Z_{bc} - \nabla_c Z_{bd} = \hat{\nabla}_d Z_{bc} - \hat{\nabla}_c Z_{bd} + n^f (Y_{bd} Z_{fc} - Y_{bc} Z_{fd}), \quad (90) \]

where we have defined \((n \cdot \ell) \overset{\text{def}}{=} n^a \ell_a\) and \(U_{ab} \overset{\text{def}}{=} \frac{1}{2} (\ell_a \partial_\ell n^{(2)} + \ell_b \partial_\ell n^{(2)})\).

**Proof.** We start by noticing that the definition of \(U_{ab}\) allows to write (see (30))
\[ K_{ab} = n^{(2)} Y_{ab} + U_{ab}, \quad (91) \]

The relation \(\bar{F}_{ab} = \Gamma^c_{\ bc} - n^a Y_{bc}\) implies
\[ \nabla_a \ell_b = \nabla_a \ell_b - n^a Y_{ab} \ell_c = Y_{ab} + F_{ab} - \ell^{(2)} K_{ab} - (n \cdot \ell) Y_{ab} = F_{ab} - \ell^{(2)} U_{ab}, \]

where in the second equality we have used identity (12) and in the third equality we used (11) and (5). This proves (54). In order to prove (55) we use again the transformation rule for covariant derivatives
\[ \nabla_a \gamma_{bc} = \nabla_a \gamma_{bc} - n^f Y_{ab} \gamma_{fc} - n^c Y_{ac} \gamma_{bf} = -\ell_b K_{ac} - \ell_c K_{ab} + n^{(2)} \ell_a Y_{ab} + n^{(2)} \ell_b Y_{ac} = -\ell_b U_{ac} - \ell_c U_{ab}, \]

where in the third equality we used (26) and (30) and in the last one we employed (11). Expression (56) is proved similarly; first transform the covariant derivative \(\nabla\) into the covariant derivative \(\bar{\nabla}\) to obtain
\[ \nabla_a P^{bc} = \nabla_a P^{bc} + n^a Y_{af} P^{fc} + n^c Y_{af} P^{bf}, \]

Inserting equation (47) for \(\nabla_a P^{bc}\) and recalling the explicit expression for \(\Psi_a^{\ell}\) in Proposition 11 yields the result. Before proving (57) we rewrite the equation (48) for \(\nabla_a n^b\) directly in terms of hypersurface metric data. Substituting the expression for \(\varphi_a\) and \(\Psi_a^{\ell}\) given in Proposition 1 as well as \(K_{ab} = n^{(2)} Y_{ab} + U_{ab}\), into (48) implies
\[ \nabla_a n^b + n^{(2)} n^b \partial_\ell \partial_a \ell^{(2)} + n^b n^f (Y_{af} + F_{af}) + P^{bf} (n^{(2)} F_{af} - U_{af}) = 0. \]

Applying to this equation the transformation law of covariant derivatives induced by the change of connection (22) gives (77). For (88), simply multiply (54) by \(\ell_f\), use \(\ell_f \nabla_d (n^f Y_{bc}) = \nabla_d ((n \cdot \ell) Y_{bc}) - n^f Y_{bc} \nabla_d \ell_f\) (i.e. “integrate by parts”) and use (54). To address (89), contract (54) with \(\gamma_{af}\) and “integrate by parts” \(\gamma_{af}\) in the second and third terms. After using equation (55) and (6) one obtains
\[ \gamma_{af} \bar{R}^f_{\ bcd} = \gamma_{af} R^f_{\ bcd} + \nabla_c (n^{(2)} \ell_a Y_{bd}) - \nabla_d (n^{(2)} \ell_a Y_{bc}) + (n \cdot \ell) (Y_{bc} U_{da} - Y_{bd} U_{ca}) + \]
\[ + \ell_a n^f [Y_{bc} (n^{(2)} Y_{df} + U_{df}) - Y_{bd} (n^{(2)} Y_{cf} + U_{cf})]. \]

Breaking the derivative terms \(\nabla_d (n^{(2)} \ell_a Y_{bc}) = n^{(2)} Y_{bc} \nabla_d \ell_a + \ell_a \nabla_d (n^{(2)} Y_{bc})\) and using (42) yields the result. Finally, identity (90) follows directly from the transformation law
\[ \nabla_d Z_{bc} = \hat{\nabla}_d Z_{bc} + n^f (Y_{bd} Z_{fc} + Y_{cd} Z_{bf}) \]
induced by the change of connection. □

We can now rewrite Corollary 4 in terms of the connection metric hypersurface connection.

**Proposition 6** Let \( \{ \Sigma, \gamma, \ell, \ell^{(2)}, Y \} \) be embedded hypersurface data with embedding \( \Phi \) and rigging \( \ell \). Let \( \{ e_a \} \) be a coordinate basis of \( T \Sigma \) and \( e_a = \Phi_\ast (\hat{e}_a) \). Then

\[
R_{\alpha\beta\gamma\delta} e^\alpha_b e^\beta_c e^\gamma_d e^\delta_f = \ell_a R_{bc}^a + \nabla_d (\ell^{(2)} K_{bc}) - \nabla_c (\ell^{(2)} K_{bd}) - \frac{1}{2} K_{bc} \nabla_d \ell^{(2)} + \frac{1}{2} K_{bd} \nabla_c \ell^{(2)} = \frac{1}{2} K_{bd} \nabla_c \ell^{(2)} + \frac{1}{2} K_{bd} \nabla_c \ell^{(2)},
\]

where in the second equality we have used (88) and (90) applied to \( Z = \ell^{(2)} K \). Using now \( (n \cdot \ell) + n^{(2)} \ell^{(2)} = 1 \) and

\[
(n \cdot \ell) Y_{bd} + \ell^{(2)} K_{bd} = (n \cdot \ell) Y_{bd} + \ell^{(2)} n^{(2)} Y_{bd} + \ell^{(2)} U_{bd} = Y_{bd} + \ell^{(2)} U_{bd}
\]

proves (92). Identity (93) is proved by direct substitution of (89) in the left-hand side of (76) and using \( \nabla \ell_a + \ell^{(2)} K_{ca} = Y_{ca} + F_{ca} \) and applying identity (90) to \( K_{bc} = n^{(2)} Y_{bc} + U_{bc} \). □

Remark 9. Although we have chosen to state this proposition in terms of embedded hypersurface data, in fact the proof works directly at the hypersurface data level and establishes the identities

\[
(I) = \ell_a R_{bc}^a + \nabla_d (\ell^{(2)} K_{bc}) - \nabla_c (\ell^{(2)} K_{bd}) - \frac{1}{2} K_{bc} \nabla_d \ell^{(2)} + \frac{1}{2} K_{bd} \nabla_c \ell^{(2)} = \frac{1}{2} K_{bd} \nabla_c \ell^{(2)} + \frac{1}{2} K_{bd} \nabla_c \ell^{(2)}.
\]

Proof. We start with the rewriting of (I) in terms of the metric hypersurface connection. Introducing \( \ell^{(2)} \) inside the derivatives in the second and third terms yields

\[
(I) = \ell_a R_{bc}^a + \nabla_d (\ell^{(2)} K_{bc}) - \nabla_c (\ell^{(2)} K_{bd}) - \frac{1}{2} K_{bc} \nabla_d \ell^{(2)} + \frac{1}{2} K_{bd} \nabla_c \ell^{(2)} = \frac{1}{2} K_{bd} \nabla_c \ell^{(2)} + \frac{1}{2} K_{bd} \nabla_c \ell^{(2)},
\]

where in the second equality we have used (88) and (90) applied to \( Z = \ell^{(2)} K \). Using now \( (n \cdot \ell) + n^{(2)} \ell^{(2)} = 1 \) and

\[
(n \cdot \ell) Y_{bd} + \ell^{(2)} K_{bd} = (n \cdot \ell) Y_{bd} + \ell^{(2)} n^{(2)} Y_{bd} + \ell^{(2)} U_{bd} = Y_{bd} + \ell^{(2)} U_{bd}
\]

proves (92). Identity (93) is proved by direct substitution of (89) in the left-hand side of (76) and using \( \nabla \ell_a + \ell^{(2)} K_{ca} = Y_{ca} + F_{ca} \) and applying identity (90) to \( K_{bc} = n^{(2)} Y_{bc} + U_{bc} \). □

In order to rewrite the constraint equations in terms of the metric hypersurface connection the following expression are also required.

**Lemma 8** With the same definitions as in Lemma 7, let \( Z_{bc} \) be an arbitrary symmetric tensor. Then the
following expressions hold
\[
\nabla^0_c (n^{(2)} P^{bd} - n^{b} n^d) = -(n^{(2)} P^{bd} - n^{b} n^d) (2 F_{ef} n^{f} + n^{(2)} \partial_{c} \ell^{(2)}) + U_{ef} (2 n^{f} P^{bd} - P^{bf} n^{d} - P^{df} n^{b}),
\]
\[
\nabla^0_d (P^{bd} n^{c} - P^{b} c^{d}) = n^{(2)} \partial_{d} \ell^{(2)} (P^{bc} n^{d} - P^{bd} n^{c}) + F_{df} (n^{b} n^{d} P^{c} - P^{bd} n^{c} n^{f} - n^{(2)} P^{bd} P^{c} f) + \]
\[+ U_{df} (P^{bd} P^{c} f - P^{bc} P^{df}),
\]
\[
\left(\nabla^0_d Z_{bc} - \nabla^0_c Z_{bd}\right) \left( n^{(2)} P^{bd} - n^{b} n^d \right) = \nabla^0_f \left[ \left( n^{(2)} P^{bd} - n^{b} n^d \right) \left( \delta^0_d Z_{bc} - \delta^0_f Z_{bd} \right) \right] + \left( n^{(2)} P^{bd} - n^{b} n^d \right) \times \]
\[\times \left[ Z_{bc} (2 F_{ef} n^{f} + n^{(2)} \partial_{d} \ell^{(2)}) - Z_{bd} (2 F_{ef} n^{f} + n^{(2)} \partial_{c} \ell^{(2)}) \right] + \left( P^{df} n^{b} - P^{bd} n^{f} \right) \left( Z_{bc} U_{df} - 2 Z_{bd} U_{cf} \right),
\]
\[
\left(\nabla^0_d Z_{bc} - \nabla^0_c Z_{bd}\right) P^{bd} n^{c} = \nabla^0_d \left( \left( P^{bd} n^{c} - P^{bc} n^{d} \right) Z_{bc} \right) + Z_{bc} \left[ U_{df} \left( P^{bc} P^{df} - P^{bd} P^{c} f \right) + 2 F_{df} P^{bd} n^{c} n^{f} \right] \]
\[+ \left( \delta^0_{d} \partial_{d} \ell^{(2)} \right) \left( P^{bd} n^{c} - P^{bc} n^{d} \right) .
\]

**Proof.** The first identity follows by expanding the products and using (86) and (87). The term \( \partial_{c} n^{(2)} \) is dealt with using the identity
\[
\partial_{c} n^{(2)} = 2 U_{ce} n^{f} - 2 n^{(2)} F_{cf} n^{f} - \left( n^{(2)} \right)^{2} \partial_{c} \ell^{(2)},
\]
which follows immediately from \( K_{ab} = n^{(2)} Y_{ab} + U_{ab} \) and Corollary 1. Identity (85) is obtained after a straightforward calculation using equations (86) and (87). For the third expression in the lemma, write
\[
\nabla^0_d Z_{bc} - \nabla^0_c Z_{bd} \left( n^{(2)} P^{bd} - n^{b} n^d \right) = \nabla^0_f \left[ \left( n^{(2)} P^{bd} - n^{b} n^d \right) \left( \delta^0 d Z_{bc} - \delta^0 f Z_{bd} \right) \right] - \left( Z_{bc} \delta^0 d - Z_{bd} \delta^0 f \right) \nabla^0_f \left( n^{(2)} P^{bd} - n^{b} n^d \right),
\]
where in the third equality we have “integrated by parts” the factor \( n^{(2)} P^{bd} - n^{b} n^d \). Inserting now \( \nabla^0 f \) in the right hand side implies (94) after simple algebraic simplifications. For the fourth identity, a simple renaming of indices gives
\[
\nabla^0_d Z_{bc} - \nabla^0_c Z_{bd} P^{bd} n^{c} = \left( P^{bd} n^{c} - P^{bc} n^{d} \right) \nabla^0_d Z_{bc} .
\]
Integrating by parts the factor \( P^{bd} n^{c} - P^{bc} n^{d} \) and using (95) yields the result. \( \square \)

We can finally rewrite the constraint equations in terms of the metric hypersurface connection.

**Theorem 2** Let \( \{ \Sigma, \gamma, \ell^{(2)}, Y \} \) be embedded hypersurface data with embedding \( \Phi \), rigging \( \ell \) and ambient spacetime \( (M, g) \). Denote by \( G^{\mu}_{\nu} \) the Einstein tensor of \( (M, g) \). Let \( \{ e_{a} \} \) be a basis of \( T \Sigma \) and \( e_{a} = \Phi^{*} (e_{a}) \). Then the following identities hold
\[
-G^{\mu}_{\nu} n_{\mu} e^{\nu} = \frac{1}{2} R^{c}_{\ bcd} P^{bd} + \frac{1}{2} \ell a R^{a}_{\ bcd} P^{bd} n^{c} + \nabla_{d} \left( \left( P^{bd} n^{c} - P^{bc} n^{d} \right) Y_{bc} \right) + \frac{1}{2} n^{(2)} P^{bd} P^{ac} \left( Y_{bc} Y_{da} - Y_{bd} Y_{ca} \right) \]
\[+ \left( \delta^{(2)} \nabla_{d} U_{bc} + \left( U_{bc} + n^{(2)} Y_{bc} \right) \partial_{d} \ell^{(2)} + 2 Y_{bc} (F_{df} - Y_{df}) \right) n^{f} \right],
\]
\[
G^{\mu}_{\nu} n_{\mu} e^{\nu} = - \ell a R^{a}_{\ bcd} n^{b} n^{d} + \nabla_{f} \left[ \left( n^{(2)} P^{bd} - n^{b} n^d \right) \left( \delta^0 d Y_{bc} - \delta^0 f Y_{bd} \right) \right] + \left( P^{bf} - \ell (2) n^{b} n^{d} \right) \left( \nabla_{d} U_{bc} - \nabla_{f} U_{bd} \right) \]
\[+ \left( n^{(2)} P^{bd} - n^{b} n^d \right) \left[ \frac{1}{2} \left( U_{bc} + n^{(2)} Y_{bc} \right) \partial_{d} \ell^{(2)} - \frac{1}{2} \left( U_{bd} + n^{(2)} Y_{bd} \right) \partial_{c} \ell^{(2)} + \left( Y_{bc} F_{df} - Y_{bd} F_{cf} \right) \right] n^{f} \]
\[+ \left( P^{bd} n^{f} - P^{bf} n^{d} \right) Y_{bd} U_{cf} + P^{bd} n^{f} \left( U_{bc} F_{df} - U_{bd} F_{cf} \right) .
\]

**Remark 10.** At first sight these expressions look much more complicated than the corresponding expressions in Theorem 1. However, here the dependence on the extrinsic part of the data \( Y_{ab} \) is completely
explicit, while in Theorem 1 several terms (as for instance $K_{ab}$ or the connection $\nabla$) depend implicitly on this tensor.

**Proof.** According to Proposition 4 we need to compute $R_{a\beta\gamma\delta} \epsilon^{\alpha}_b \epsilon^{\beta}_c \epsilon^{\gamma}_d \epsilon^{\delta}_e P^{bd} + \frac{1}{2} R_{a\beta\gamma\delta} \epsilon^{\alpha}_b \epsilon^{\beta}_c \epsilon^{\gamma}_d \epsilon^{\delta}_e P^{ac} P^{bd}$, i.e. $(I) P^{bd} n_c + (II) P^{ac} P^{bd}$ in the notation above. For the first term, we use (92) and apply the identity (97) with $Z_{bc} = Y_{bc}$. This implies, after a number of cancellations,

\[
(I) P^{bd} n_c = \ell_a \mathring{R}^a_{bcd} P^{bd} n_c + \mathring{\nabla}_d \left[ (P^{bd} n_c - P^{bc} n^d) Y_{bc} \right] + \left( \frac{1}{2} (U_{bc} + n^{(2)} Y_{bc}) \partial_d (\ell^{(2)}) + \right.
\]
\[
+ \ell^{(2)} \nabla_d U_{bc} + Y_{bc} (F_{df} - Y_{df}) n^f \right) + P^{bd} P^{cf} Y_{bd} (U_{cf} - Y_{bf}) \right) .
\]

(100)

For the second term, we simply use $\gamma_{af} P^{ac} = \delta^c_f - n^c \ell_f$ and use the antisymmetry of $F_{ab}$ to obtain

\[
\frac{1}{2} (II) P^{ac} P^{bd} = \frac{1}{2} \mathring{R}^c_{bcd} P^{bd} - \frac{1}{2} \ell_a \mathring{R}^a_{bcd} P^{bd} n_c + P^{bd} P^{cf} \left( Y_{bd} (U_{cf} + \frac{1}{2} n^{(2)} Y_{cf}) - Y_{bd} (U_{cf} + \frac{1}{2} n^{(2)} Y_{cf}) \right)
\]
\[
- \ell^{(2)} \nabla_d U_{bc} (P^{bd} n_c - P^{bc} n^d) .
\]

(101)

Adding (100) and (101) gives (102).

From Proposition 4 in order to obtain the expression for $G_{\mu\nu} n_{\mu} e^{\nu}_c$ we need to compute $(I)(n^{(2)} P^{bd} - n^b n^d) + (II)n^{(2)} P^{bd}$. The first term follows directly from (102) and reads

\[
(I) (n^{(2)} P^{bd} - n^b n^d) = \left( n^{(2)} P^{bd} - n^b n^d \right) \left[ \ell_a \mathring{R}^a_{bcd} + \mathring{\nabla}_d Y_{bc} - \mathring{\nabla}_d Y_{bd} + \ell^{(2)} (\mathring{\nabla}_d U_{bc} - \mathring{\nabla}_d U_{bd}) + \left. \right. \right.
\]
\[
+ \frac{1}{2} (U_{bc} - n^{(2)} Y_{bc}) \partial_d (\ell^{(2)}) - \frac{1}{2} (U_{bd} - n^{(2)} Y_{bd}) \partial_c (\ell^{(2)}) \right] - n^b n^d Y_{bd} F_{cf} n^f
\]
\[
+ n^{(2)} P^{bd} n^f \left[ Y_{bd} (Y_{cf} + F_{cf}) - Y_{bd} (Y_{df} + F_{df}) \right] .
\]

(102)

On the other hand $\ell^a \gamma_{af} = -n^{(2)} \ell_f$ implies

\[
(II) n^{(2)} P^{bd} = -n^{(2)} \ell_a \mathring{R}^a_{bcd} P^{bd} + (n \cdot \ell) P^{bd} \left( \mathring{\nabla}_d U_{bc} - \mathring{\nabla}_d U_{bd} \right) + P^{bd} P^{cf} \left( Y_{bd} (U_{cf} + n^{(2)} Y_{cf}) - Y_{bd} (U_{cf} + U_{cf}) \right) .
\]

(103)

Adding the two and using $(n \cdot \ell) + \ell^{(2)} n^{(2)} = 1$ implies

\[
G_{\mu\nu} n_{\mu} e^{\nu}_c = -\ell_a \mathring{R}^a_{bcd} n^d + (n^{(2)} P^{bd} - n^b n^d) \left( \mathring{\nabla}_d Y_{bc} - \mathring{\nabla}_d Y_{bd} \right) + (P^{bd} - \ell^{(2)} n^d) \left( \mathring{\nabla}_d U_{bc} - \mathring{\nabla}_d U_{bd} \right) + \left. \right. \right.
\]
\[
+ \frac{1}{2} (n^{(2)} P^{bd} - n^b n^d) \left[ (U_{bc} - n^{(2)} Y_{bc}) \partial_d (\ell^{(2)}) - (U_{bd} - n^{(2)} Y_{bd}) \partial_c (\ell^{(2)}) \right] - n^b n^d Y_{bd} F_{cf} n^f
\]
\[
+ P^{bd} n^f \left[ Y_{bd} (Y_{cf} + F_{cf}) - Y_{bd} (Y_{df} + F_{df}) + Y_{bc} (U_{df} - n^{(2)} F_{df}) - Y_{bd} (U_{cf} - n^{(2)} F_{cf}) \right] .
\]

(104)

We use now identity (103) with $Z_{bc} = Y_{bc}$ and simplify the resulting expression to obtain (99).

□

Having obtained the constraint equations for embedded hypersurface data, we can now promote these identities to fields equations on the hypersurface data. To that aim, we introduce (still in the embedded hypersurface case) a scalar $\rho_{\ell}$ and a one-form $J_a$ on $\Sigma$ by the definitions

\[
\rho_{\ell} \overset{\text{def}}{=} -G_{\mu\nu} n_{\mu} e^{\nu}_c, \quad J_a \overset{\text{def}}{=} -G_{\mu\nu} n_{\mu} e^{\nu}_a .
\]

The sign in the definition of $\rho_{\ell}$ has been chosen so that in the spacelike case and with the standard choice of rigging $\ell^\mu = -n^\mu$ (recall that $\ell^\mu n_{\mu} = 1$ throughout this paper) $\rho_{\ell}$ coincides with the energy-density measured by the observer orthogonal to the hypersurface. By this analogy and the fact that $\rho_{\ell}$ in any case
measures the normal-transversal component of the Einstein tensor, we call $\rho_\ell$ the “energy along $\ell$”. Note, however, this name is not intended to imply any relation with a physical energy measured by any spacetime observer. Regarding $J_a$, this measures the energy flux in the case when $n$ is timelike. Again we will refer to $J_a$ as “energy flux” although in the general case $J_a$ does not correspond to physical fluxes measured by any spacetime observer. Note that both $\rho_\ell$ and $J_a$ depend on the choice of rigging. If we choose another rigging $\ell' = u(\ell + V^a e_a)$ (which implies the change $n' = \frac{1}{u} n$), this objects transform as

$$
\rho_{\ell'} = \rho_\ell + V^a J_a, \quad J'_a = \frac{1}{u} J_a
$$

We put forward the definition of matter-hypersurface data.

**Definition 8 (Matter-Hypersurface data)** An $n$-tuple $\{\Sigma, \gamma_{ab}, \ell^{(2)}, Y_{ab}, \rho_\ell, J_a\}$ formed by hypersurface data $\{\Sigma, \gamma_{ab}, \ell, (\ell^{(2)}, Y_{ab})\}$, a scalar $\rho_\ell$ and a one-form $J_a$ on $\Sigma$ is called matter-hypersurface data provided $\rho_\ell$ and $J_a$ transform as

$$
\rho_{\ell'} = \rho_\ell + V^a J_a, \quad J'_a = \frac{1}{u} J_a.
$$

under a gauge transformation defined by $(u, V^a)$ and the following constraint field equations hold.

$$
\begin{align}
\rho_\ell &= \frac{1}{2} \delta_{[a} R^{bcd]} \ell^{b} + \frac{1}{2} \delta_{a} R^{bcd} \ell^{bd} n^{c} + \delta \left( (P^{bd} n^{c} - P^{bc} n^{d}) Y_{bc} \right) + \frac{1}{2} n^{(2)} P^{bd} P^{cd} (Y_{bc} Y_{da} - Y_{bd} Y_{ca}) \\
&+ \frac{1}{2} \ell^{c} \left( n^{(2)} P^{bd} - P^{bc} n^{d} \right) \left[ \ell^{(2)} \nabla_{d} U_{bc} + (U_{bc} + n^{(2)} Y_{bc}) \partial_{d} \ell^{(2)} + 2 Y_{bc} (F_{df} - Y_{df}) n^{f} \right], \\
J_a &= \ell_{a} R_{bcd} n^{d} - \nabla_{f} \left[ (n^{(2)} P^{bd} - n^{b} n^{d}) \left( \delta_{d} Y_{bc} - \delta_{c} Y_{bd} \right) - (P^{bd} - \ell^{(2)} n^{b} n^{d}) \left( \nabla_{d} U_{bc} - \nabla_{c} U_{bd} \right) \right] \\
&- \left( n^{(2)} P^{bd} - n^{b} n^{d} \right) \ell_{a} R^{d} \partial_{d} \ell^{(2)} - \frac{1}{2} (U_{bc} + n^{(2)} Y_{bc}) \partial_{d} \ell^{(2)} + (Y_{bc} F_{df} - Y_{bd} F_{cf}) n^{f} \\
&- \left( P^{bd} n^{f} - P^{bf} n^{d} \right) Y_{bd} U_{cf} - P^{bd} n^{f} \left( U_{bc} F_{df} - U_{bd} F_{cf} \right).
\end{align}
$$

For completeness, we also add the definition of embedded matter-hypersurface data

**Definition 9 (Embedding of matter-hypersurface data)** Let $\{\Sigma, \gamma, \ell, (\ell^{(2)}, Y), \rho_\ell, J_a\}$ be matter-hypersurface data. This data will be embedded in a spacetime $(\mathcal{M}, g)$ if $\{\Sigma, \gamma, \ell, (\ell^{(2)}, Y)\}$ is embedded with embedding $\Phi$ and rigging $\ell$ and, moreover

$$
\rho_{\ell} = \Phi^{*} (G(\ell, n)), \quad J = \Phi^{*} (G(\cdot, n)),
$$

where $G$ is the 1-covariant, 1-contravariant Einstein tensor of $(\mathcal{M}, g)$ and $n$ the one-form normal to $\Sigma$ satisfying $n(\ell) = 1$.

## 5 Evolution equations for discontinuities of hypersurface data.

As discussed in the Introduction, the matching theory of two spacetimes is a useful arena to apply the results above on the constraint equations for general hypersurfaces. The ingredients for the matching are two spacetimes $(\mathcal{M}^\pm, g^\pm)$ with diffeomorphic boundaries $\partial \mathcal{M}^\pm$. Let $\Sigma$ be an abstract copy of $\partial \mathcal{M}^+$ (or $\partial \mathcal{M}^-$) and $\Phi^\pm : \Sigma \rightarrow \mathcal{M}^\pm$ be embeddings such that $\Phi^\pm (\Sigma) = \partial \mathcal{M}^\pm$. The first requirement for a successful matching is that the manifold constructed from the union of $\mathcal{M}^+$ and $\mathcal{M}^-$ with their boundaries identified via $\Phi^+ \circ (\Phi^-)^{-1}$ admits an atlas and a continuous metric $g$ such that, when restricted to $\mathcal{M}^\pm$ (the interior of each manifold with boundary) gives $g^\pm$. The necessary and sufficient conditions for this to happen were studied first by Clarke and Dray [9] in the case of boundaries with constant signature case (including null). Their result was that the necessary and sufficient condition was that the induced first fundamental form on
\[ \Sigma \] from both embeddings \( \Phi^{\pm} \) coincide. It was noticed in [22] that the arguments extend without difficulty to the case of arbitrary causal character. However, in the case when the boundary has null points, the statement in Clarke and Dray is incomplete because the equality of the first fundamental form is a necessary condition but it fails in general to be also sufficient. This was noticed in [23] where the necessary and sufficient conditions were found. With the notation introduced above we can state this theorem as follows

**Theorem 3** ([9], [22], [23]) Consider two \((m+1)\)-dimensional spacetimes \((\mathcal{M}^{\pm}, g^{\pm})\) with boundaries \(\partial \mathcal{M}^{\pm}\). They can be matched across their boundaries to produce a spacetime \((\mathcal{M}, g)\) with continuous metric (in a suitable differentiable atlas) if and only if:

(i) There exists hypersurface metric data \((\Sigma, \gamma_{ab}, \ell_{a}, \ell^{(2)})\) which can be embedded both in \((\mathcal{M}^{\pm}, g^{\pm})\) and in \((\mathcal{M}^{-}, g^{-})\) with respective embedding and riggings \(\Phi^{\pm}\) and \(\ell^{\pm}\). Moreover, the embeddings satisfy \(\Phi^{\pm}(\Sigma) = \partial \mathcal{M}^{\pm}\).

(ii) The rigging vectors \(\ell^{\pm}\) point, respectively, inside and outside of \(\mathcal{M}^{\pm}\).

It is clear that when the boundaries contain no null points, the equality of the induced metric is equivalent to items (i) and (ii) (simply chose the rigging vector to be the unit normal pointing outwards of the boundary in one spacetime and inwards in the other). When the boundary admits null points (in particular, if it is null everywhere), the existence of the rigging satisfying (i) and (ii) does not follow [23] from the equality of the first fundamental forms, so it needs to be added to the statement of the theorem.

At the hypersurface data level, the requirements of this Theorem translate into the condition that we deal with two hypersurface data \((\Sigma, \gamma_{ab}, \ell_{a}, \ell^{(2)}, Y_{ab}^{\pm})\) which differ at most on the transverse tensor \(Y_{ab}^{\pm}\), i.e. such that they define the same hypersurface metric data.

The next step in the matching procedure is to analyze whether the spacetimes match without introducing any physical energy-momentum tensor with support on the matching hypersurface. This was studied in the spacelike case by Darmois [10], Lichnerowicz [21] and O’Brien-Synge [27]. Their proposals were different but a close relationship between them could be established [5]. The Darmois matching conditions are coordinate independent and demand the coincidence of the second fundamental forms on each boundary. Given the relationship [23] between \(K_{ab}\) and \(Y_{ab}\) and using the fact that \(n^{(2)} \neq 0\) in the nowhere null case, the Darmois matching conditions are equivalent to \(Y_{ab}^{+} = Y_{ab}^{-}\) (c.f. [22]). When \([K_{ab}] = K_{ab}^{+} - K_{ab}^{-} = 0\), it follows that there is a subatlas of the matched spacetime \((\mathcal{M}, g)\) where the metric \(g\) is \(C^{1}\) [13, 5]. The Riemann tensor of \((\mathcal{M}, g)\) may be discontinuous at \(\Sigma\) but it is otherwise regular everywhere. This is why, physically, one concludes that there is no matter-energy, or gravitational field concentrated on the matching hypersurface. On the other hand, if \([K_{ab}] \neq 0\) (still in the everywhere spacelike case) then the Riemann tensor viewed as a tensor distribution in \((\mathcal{M}, g)\) (see [21, 28, 11, 22, 20] for details on how to define and use tensor distributions in this setting) has a Dirac delta function supported on \(\Sigma\). This is interpreted physically as a layer of energy and momentum concentrated on the hypersurface \(\Sigma\) (a “shell” of matter-energy). It turns out that the Dirac delta part of the Einstein tensor is (not yet leaving the nowhere null case)

\[
G^\mu_\nu = \tau^{ab} e^\mu_a e^\nu_b \delta_N \quad \text{with} \quad \tau_{ab} = - ([K_{ab}] - [K]_{\gamma_{ab}}), \quad \delta_N: \text{Dirac delta on } (\Sigma, \gamma).
\]

where the Dirac delta distribution is defined by integration with the volume form of the induced metric on the shell. The (distributional) conservation equations \(\nabla_\mu G^\mu_\nu = 0\) imply

\[
(K_{ab}^{+} + K_{ab}^{-}) \tau^{ab} = 2 [G_{\mu \nu} n^\mu n^\nu], \quad \nabla_b \tau^b_a = [J_a],
\]

which are the Israel field equations for the shell [17, 18, 13]. For the purposes of this paper, it is interesting to note that these equations can be derived directly from the constraint equations (11, 12) by simply taking the difference of both equations at each side of the matching hypersurface, and using the fact that the induced metric and the corresponding Levi-Civita connection do not jump across the shell. As mentioned in the Introduction, these field equations were extended to the null case by Barrabés and Israel [4] with an argument based on taking limits where the spacelike/timelike matching hypersurface becomes null.

The matching theory for hypersurfaces of arbitrary causal character was derived in [24], where in particular an explicit expression was obtained for the Einstein tensor distribution of the matched spacetime.
\((\mathcal{M}, g)\) in the atlas where the metric \(g\) is \(C^0\). The shell equations for matching hypersurfaces of arbitrary causal character can in principle be derived from the distributional (contracted) Bianchi identities. However, having obtained general expressions for the constraint equations of hypersurfaces of general causal character, we can also follow a different strategy. Namely, we can obtain the shell equations by simply taking differences of the constraint equations on two hypersurface data of the form \((\Sigma, \gamma_{ab}, \ell_a, \ell^{(2)}, Y^{ab}_\pm, Y^{ac}_\pm)\). This is conceptually much simpler, as there is no need to introduce spacetime distributions nor specific atlas in the matched spacetime in order to perform the calculation. In addition, it does not even need to assume that a spacetime exists. This may seem spurious, but in fact it is not. The reason is that initial data sets are, in principle, much more general than spacetimes because well-posedness is not to be expected in general if a spacetime exists. This may seem spurious, but in fact it is not. The reason is that initial data sets are, matched spacetime in order to perform the calculation. In addition, it does not even need to assume that conceptually much simpler, as there is no need to introduce spacetime distributions nor specific atlas in the \(\gamma\) when one expects a well-posedness theorem to hold (e.g. when \(\gamma_{ab}\) is of Lorentzian signature on some open subset. Even when one expects a well-posedness theorem to hold (e.g. when \(\gamma_{ab}\) is semipositive definite), the actual result is still lacking and may well be a non-trivial task to prove it. However, jump discontinuities on the data may still be considered and the field equations that they need to satisfy can be derived directly from the constraint equations. The important point is that this makes all the sense already at the initial data level, and may be useful for several things, ranging from studying shell equations on their own (i.e. detached from the spacetime) to more practical purposes like obtaining new solutions of the constraint equations from a seed solution of the constraints together with a solution of the shell equations on this metric hypersurface data background.

Let us therefore try and find the shell equations from the results in Theorem 4. Assume we are given two matter-hypersurface data \(\{\Sigma, \gamma_{ab}, \ell_a, \ell^{(2)}, Y^{ab}_\pm, Y^{ac}_\pm\}\) and let us define \(V_{ab} \equiv Y^{ab}_\pm - Y_{ab}\). Let us also define the jump in the “energy density” and in the “energy flux” as \(\rho_{\pm} \equiv \rho^+ - \rho^-\) and \(J_{\pm} \equiv J^+ - J^-\). When subtracting the constraint equations for \(\{\Sigma, \gamma_{ab}, \ell_a, \ell^{(2)}, Y^{ab}_+, Y^{ac}_+, \rho^+, J^+\}\) and for \(\{\Sigma, \gamma_{ab}, \ell_a, \ell^{(2)}, Y^{ab}_-, Y^{ac}_-, \rho^-, J^-\}\) all terms that depend exclusively on the hypersurface metric data cancel out. Furthermore, only one connection \(\nabla\) appears in all equations, since this depends only on the hypersurface metric data. Hence, subtracting the constraint equations and using the trivial identity

\[
Y_{be}Y_{da} - Y_{bd}Y_{ca} = Y_{be}V_{da} + V_{be}Y_{da} - Y_{bd}V_{ca} - V_{bd}Y_{ca},
\]

where \(\nabla_{ab} \equiv \frac{1}{2}(Y_{ab}^+ + Y_{ab}^-)\), gives

\[
[\rho_{\pm}] = \hat{\nabla}_f \left( ((P^{bf}n^c - P^{bc}n^f)V_{be}) + \frac{1}{2}n^{(2)}P^{bd}P^{ac}(Y_{be}V_{da} + V_{be}Y_{da} - Y_{bd}V_{ca} - V_{bd}Y_{ca}) \right.
\]  

\[
+ \frac{1}{2}(P^{bd}n^c - P^{bc}n^d) \left[ n^{(2)}V_{bc}\delta df^{(2)} + 2V_{bc}(F_{df} - \nabla_{df}) n^f + 2\nabla_{bc}V_{df}n^f \right],
\]

(108)

\[
[\ell_{\pm}] = -\hat{\nabla}_f \left[ n^{(2)}P^{bd} - n^{bd}n^d \right] \left[ V_{be} - V_{db} \delta f^d \right] +
\]

\[
- \left( n^{(2)}P^{bd} - n^{bd}n^d \right) \left( \frac{1}{2}n^{(2)}V_{bc}\delta df^{(2)} - \frac{1}{2}n^{(2)}V_{bd}\delta c\delta e^{(2)} + (V_{be}F_{df} - V_{db}F_{cf}) n^f \right) +
\]

\[
- (P^{bf}n^c - P^{bc}n^f)V_{bd}U_{df}.
\]

(109)

Looking at the terms involving derivatives, we realize that this expressions introduce naturally a 1–1 tensor \(\tau^{f}_{\pm}\) and vector \(T^{d}\) with the definitions

\[
\tau^{f}_{\pm} \equiv - \left( n^{(2)}P^{bd} - n^{bd}n^d \right) \left( \delta f^d V_{be} - \delta e^d V_{bd} \right),
\]

\[
T^{d} \equiv (P^{bf}n^c - P^{bc}n^f)V_{bd}.
\]

At the sight of Lemma 3 it is quite natural to enquire whether these tensors can be obtained, respectively, from contractions of a suitable 2-contravariant tensor \(\tau^{f}a\) with \(\gamma_{ac}\) and \(\ell_a\). The result is given in the following Lemma.

**Lemma 9** The tensors \(\tau^{f}_{\pm}\) and \(T^{d}\) defined above satisfy \(n^e\tau^{f}_{\pm} + n^{(2)}T^{f} \equiv 0\). Consequently, there exists a tensor \(\tau^{f}a\) such that \(\tau^{f}a\gamma_{ac} = \tau^{f}_{\pm}\) and \(\tau^{f}a\ell_a = T^{a}\). Moreover, \(\tau^{f}a\) takes the explicit form

\[
\tau^{f}a = \left( n^f p^{ac} + n^a p^{f}c \right) n^d V_{cd} - \left( n^{(2)}p^{fc}p^{ad} + p^{fa}n^n n^d \right) V_{cd} + \left( n^{(2)}p^{fa} - n^n n^f \right) p^{cd}V_{cd}.
\]

(110)
Proof. We compute
\[ \tau^f_c n^c = - (n^2) P^{bd} - n^b n^d \left( \delta^f_b V_{bc} n^c - n^f V_{bd} \right) = -n^2 P^{bf} V_{bc} n^c + n^b n^f V_{bd} = \]
\[ = -n^2 (P^{bf} n^c - P^{bc} n^f) V_{bc} = -n^2 T^f. \]

Lemma \ref{lem:existence} applied to \( Z_c \rightarrow \tau^f_c \) and \( W \rightarrow T^f \) implies immediately the existence of the tensor \( \tau^f_a \) claimed in the statement. In order to find its explicit expression we invoke again Lemma \ref{lem:existence} which gives \( \tau^f_a = P^{ac} \tau^f_c + n^a T^f \).

Expanding the right-hand side gives \( \{110\} \).

The tensor \( \tau^f_a \) has been defined by inspection of equations \( \{108\} - \{109\} \). It turns out that this allows us to rewrite not only the derivative terms of these equations in terms of \( \tau^f_a \) but in fact the full equations as well. It is straightforward to check that \( \{108\} - \{109\} \) can be rewritten in the form
\[ \nabla_a \left( \tau^{ab}_c \ell_b \right) + \tau^{ab}_c \ell_b \left( \frac{1}{2} n^{(2)} \partial_a \ell^{(2)} + F_{ad} n^d \right) - \tau^{ab} \nabla_a = [\rho \ell], \]
\[ \nabla_b \tau^b_c + \tau^b_c \left( \frac{1}{2} n^{(2)} \partial_b \ell^{(2)} + F_{bd} n^d \right) + \tau^{bd} \ell_d U_{bc} = [J_c]. \]

It is also immediate to check that, in the spacelike (or timelike case), these equations become exactly the Israel equations \( \{107\} \).

The tensor \( \tau^{ab} \) has the following properties

**Proposition 7** Let \( \tau^f_a \) be defined as in \( \{110\} \) and assume the hypersurface data is of dimension \( m \geq 2 \). Then, the following properties hold

(i) \( \tau^f_a \) is symmetric, i.e. \( \tau^{ab} = \tau^{ba} \).

(ii) At any point \( p \in \Sigma \) where \( n^{(2)}(p) \neq 0 \), \( \tau^f_a \) vanishes if and only if \( V_{ab} = 0 \), i.e. if and only if the jump in \( Y_{ab} \) vanishes. At any point where \( n^{(2)}(p) = 0 \), \( \tau^f_a \) vanishes if and only if \( V_{ab} n^b = 0 \) and \( P^{ab} V_{ab} = 0 \).

(iii) Under a gauge transformation of the data \( \{ \Sigma, \gamma_{ab}, \ell_a, \ell^{(2)}, Y^+_{ab}, Y^-_{ab} \} \) defined by \( \{ u, V^a \} \) we have \( V_{ab}' = u V_{ab} \) and
\[ \tau^f = \frac{1}{u} \tau^f_a. \] (111)

(iv) Consider “non-degenerate data in the normal gauge”, i.e. \( \{ \Sigma, \gamma_{ab}, \ell_a = 0, \ell^{(2)} = \epsilon, Y^+_{ab} = \epsilon K^+_{ab}, Y^-_{ab} = -\epsilon K^-_{ab} \} \) where \( K^a_{bc} \) is the inverse of \( \gamma^a_{bc} \).

Then
\[ \tau^{ab} = -[K^{ab}] + \gamma^{ab}[K], \]
where \( [K^{ab}] \) is the inverse of \( \gamma^{ab} \).

**Remark 11.** The condition on the dimension is necessary because in dimension \( m = 1 \) \( \{110\} \) implies \( \tau^{ab} \equiv 0 \).

Proof. Item (i) is obvious from the explicit expression \( \{110\} \). For item (ii), we notice the \( \tau^f_a \) vanishes if and only if \( \tau^f_c = 0 \) and \( T^f = 0 \). Taking the trace of the first expressions gives
\[ 0 = 0 = \tau^f = -(m - 1) (n^{(2)} P^{bd} - n^b n^d) V_{bd}. \]
Lowering the index to \( T^f \) implies
\[ 0 = T^f \gamma_a = V_{ac} n^c + \ell_a \left( n^{(2)} P^{bc} - n^b n^c \right) V_{bc} = V_{ac} n^c, \]
where in the last equality we used (113). Inserting this back into (113) gives $P^{bd}V_{bd} = 0$. Using all this in $\tau^a$ implies

$$0 = \tau^a = -n^{(2)} P^{bf} V_{bc}.$$  

So far we have found that $\tau^a$ vanishes at one point if and only if $P^{bd}V_{bd} = 0$, $n^{(2)} P^{bf} V_{bc} = 0$ and $V_a n^a = 0$. This establishes the result for points $p \in \Sigma$ where $n^{(2)}(p) = 0$. For points where $n^{(2)}(p) \neq 0$, it only remains to show that there exists no covector $S_a$ satisfying $P^{ab}S_b = 0$ and $S_a n^a = 0$ except for $S_a = 0$. Any such covector satisfies

$$\begin{bmatrix} P^{ab} & n^a \\ n^b & n^{(2)} \end{bmatrix} \begin{bmatrix} S_b \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$  

Since the square matrix is invertible $S_b = 0$ follows, as claimed.

For item (iii), the statement $V'_b = uV_{ab}$ is immediate from the definition of $V_{ab} = Y^+_a - Y^-_b$ and the Definition of gauge transformation. In order to address the gauge behaviour of $\tau^f$, Proposition 3 implies that (111) is equivalent to

$$\tau^f = \frac{1}{u} \tau^f_c,$$

$$T^f = T^f + V^a \tau^f_a.$$  

In order to prove these identities we need to compute the transformation laws of $\tau^f_c$ and of $T^f$. The first one is a simple consequence of Lemma 6 and reads

$$\tau^f_c = -\left( n^{(2)} P^{bd} - n^b n^d \right) \left( \delta^f_d V_{bc} - \delta^f_c V_{bd} \right) = -\frac{1}{u^2} \left( n^{(2)} P^{bd} - n^b n^d \right) u \left( \delta^f_d V_{bc} - \delta^f_c V_{bd} \right) = \frac{1}{u} \tau_c.$$  

For the second, we use the second part of Lemma 6 to obtain

$$T^f = \left( P^{bd} n^c - P^{bc} n^f \right) V_{bc} = \left[ P^{bf} n^c - P^{bc} n^f \right] + \left( n^{(2)} P^{bc} - n^b n^c \right) V^f - \left( n^{(2)} P^{bf} - n^b n^f \right) V^c \right] V_{bc} =$$

$$= T^f + V^a \left( n^{(2)} P^{bc} - n^b n^c \right) \left( \delta^f_a V_{bc} - \delta^f_b V_{ac} \right) = T^f + V^a \tau^a,$$

as desired. This proves claim (iii).

For claim (iv), first notice that $\ell_a = 0$ and $\ell^{(2)} = \epsilon$ imply (from (31)) that $n^{(2)} = \epsilon$, $n^a = 0$ and $P^{ab} = \gamma^{ab}$. Moreover, the expression $Y_{ab} = \epsilon K_{ab}$ is consistent with (11). Inserting this information into the general expression for $\tau^f$ yields

$$\tau^f_a = n^{(2)} \left( P^{fa} P^{cd} - P^{fc} P^{ad} \right) V_{cd} = \left( \gamma^{fa} \gamma^{cd} - \gamma^{fc} \gamma^{ad} \right) [K_{cd}] = -[K^{fa}] + \gamma^{fa} \epsilon[K].$$  

As a consequence of this proposition we find that the tensor $\tau^{ab}$ has the symmetries of an energy-momentum tensor and coincides with the standard definition of the energy-momentum tensor on the shell in the nowhere null case. Moreover, in the case of matching across null hypersurfaces, it is immediate to check that $\tau^{ab}$ coincides with the definition of energy-momentum tensor put forward by Barrabès and Israel in expression (31) in [22], where their tensor $g^{ab}$ (defined by $g^{ab} \ell_c = \delta^a_c - n^a \ell_b$) is related (in the null case) to $P^{ab}$ by $g^{ab} = P^{ab} + 2\lambda n^a n^b$, for an arbitrary $\lambda$. It can also be checked that, in the case of matching across hypersurfaces of general causal character the tensor $\tau^{ab} \epsilon^a \epsilon^b$ agrees with the Dirac delta part of the Einstein tensor of the matched spacetime $(\mathcal{M}, g)$ denoted by $\tau^{\alpha\beta}$ in [22] and given explicitly in expression (71) in that reference. All these considerations allow us to put forward the following definition, independently of whether the hypersurface data is embedded in any spacetime or not.

**Definition 10 (Shell field equations) A shell is a pair of matter-hypersurface data of the form $\{\Sigma, \gamma_{ab}, \ell_a, \ell^{(2)}, Y^+_a, \rho^+_\ell, J^+_\ell\}$. The energy-momentum tensor on the shell is the symmetric 2-covariant tensor $\tau^{ab}$ defined by

$$\tau^{ab} = \left( n^a P^{bc} + n^b P^{ac} \right) n^d V_{cd} - \left( n^{(2)} P^{ac} P^{bd} + P^{ab} n^c n^d \right) V_{cd} + \left( n^{(2)} P^{ab} - n^a n^b \right) P^{cd} V_{cd}.$$  

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where \( V_{ab} \overset{\text{def}}{=} Y_{ab}^+ - Y_{ab}^- \). The shell field equations are the pair of partial differential equations
\[
\hat{\nabla}_a \left( \tau^{ab} \ell_b \right) + \tau^{ab} \ell_b \left( \frac{1}{2} n^{(2)} \partial_a \ell^{(2)} + F_{ac} n^c \right) - \frac{1}{2} \tau^{ab} \left( Y_{ab}^+ + Y_{ab}^- \right) = [\rho \ell], \tag{115}
\]
\[
\hat{\nabla}_b \tau^b_a + \tau^b_a \left( \frac{1}{2} n^{(2)} \partial_b \ell^{(2)} + F_{bc} n^c \right) + \tau^{bc} \ell_c U_{ba} = [J_a], \tag{116}
\]
where \([\rho \ell] \overset{\text{def}}{=} \rho^+\ell - \rho^-\ell\), \([J_a] \overset{\text{def}}{=} J_a^+ - J_a^-\) and \(\hat{\nabla}\) is the metric hypersurface connection.

**Remark 12.** In the context of matching of spacetimes, it is customary to use a connection on \(\Sigma\) which is the semi-sum of the corresponding rigging connections from each side of the hypersurface. In other words, it is customary to use the connection
\[
\hat{\Gamma}_{bc}^a \overset{\text{def}}{=} \frac{1}{2} \left( \Gamma_{bc}^a + \Gamma_{bc}^- \right),
\]
where \(\Gamma_{bc}^a\) is defined by expression (29) after substitution of \(Y_{ab} \to Y_{ab}^\pm\). The relationship between \(\hat{\Gamma}_{bc}^a\) and \(\Gamma_{ab}^a\) is, obviously, \(\Gamma_{ab}^a = \hat{\Gamma}_{bc}^a + n^c Y_{bc}\). A simple calculation allows us to rewrite (115)-(116) in terms of the covariant derivative \(\hat{\nabla}\) associated to \(\hat{\Gamma}_{bc}^a\). The result (which I add for easy of comparison between our approach and the distributional approach in a matching context), is
\[
\hat{\nabla}_a \left( \tau^{ab} \ell_b \right) + \tau^{ab} \ell_b \left( \frac{1}{2} n^{(2)} \partial_a \ell^{(2)} + F_{ac} n^c + Y_{ac} n^c \right) - \frac{1}{2} \tau^{ab} Y_{ab} = [\rho \ell],
\]
\[
\hat{\nabla}_b \tau^b_a + \tau^b_a \left( \frac{1}{2} n^{(2)} \partial_b \ell^{(2)} + F_{bc} n^c + Y_{bc} n^c \right) + \tau^{bc} \ell_c \left( n^{(2)} Y_{ba} + U_{ba} \right) = [J_a].
\]

We know that both the energy-momentum tensor \(\tau^{ab}\) and \([J_a]\) have a very simple gauge behaviour. On the other hand, the gauge behaviour of the connection \(\Gamma_{bc}^a\) is complicated. It makes sense to try and rewrite the equations above in such a form that the gauge dependence becomes explicit. In order to accomplish this, it is convenient to work in coordinates. Indeed, the coordinate expression of (115) is
\[
\partial_a T^a + T^a \left( \Gamma_{ac}^{\circ} + \frac{1}{2} n^{(2)} \partial_a \ell^{(2)} + F_{ac} n^c \right) - \frac{1}{2} \tau^{ab} \left( Y_{ab}^+ + Y_{ab}^- \right) = [\rho \ell],
\]
so, we need to compute \(\Gamma_{ac}^{\circ} + \frac{1}{2} n^{(2)} \partial_a \ell^{(2)} + F_{ac} n^c\). Recalling Definition (7) we have
\[
\Gamma_{ac}^{\circ} = \frac{1}{2} P^{cd} \partial_a \gamma_{cd} + \frac{1}{2} n^c \left( \partial_a \ell_c + \partial_c \ell_a \right) \implies
\]
\[
\Gamma_{ac}^{\circ} + \frac{1}{2} n^{(2)} \partial_a \ell^{(2)} + F_{ac} n^c = \frac{1}{2} \left( P^{cd} \partial_a \gamma_{cd} + n^c \partial_a \ell_c + n^{(2)} \partial_a \ell^{(2)} \right) = \frac{1}{2} \text{tr} \left( A^{-1} \partial_a A \right),
\]
where the matrix \(A\) was introduced in Definition (9) (and we have used property (7)). We conclude therefore that
\[
\Gamma_{ac}^{\circ} + \frac{1}{2} n^{(2)} \partial_a \ell^{(2)} + F_{ac} n^c = \frac{1}{2 (\text{det} A)} \partial_a (\text{det} A)
\]
and the field equation (115) becomes
\[
\frac{1}{\sqrt{\text{det} A}} \partial_a \left( \sqrt{\text{det} A} \tau^{ab} \ell_b \right) - \frac{1}{2} \tau^{ab} \left( Y_{ab}^+ + Y_{ab}^- \right) = [\rho \ell].
\]
We can also compute the coordinate expression of equation (116):

$$
\partial_b \tau^b_{ab} - \frac{\partial}{\partial \sqrt{\det(\mathcal{A})} \tau^b_{ab}} + \frac{P_{cd} \tau^c_{cd}}{2} + \frac{n_{ab}^2}{\partial \sqrt{\det(\mathcal{A})} \tau^b_{ab}} = 0.
$$

To elaborate this further we use the explicit expression for $\Gamma^a_{bc}$ in Definition 7 and use the facts $P_{cd} \tau^b_{cd} = P_{cd} \gamma_{ac} = \tau_{bc} (\delta_a^d - n^d \gamma_{ac}) = \tau_{bc} - n^d \tau_{bc} \epsilon_c$. Hence

$$
- \frac{\partial}{\partial \sqrt{\det(\mathcal{A})} \tau^b_{ab}} + \tau_{bc} \epsilon_{U_{ba}} = \frac{1}{2} \partial^{bd} \partial_a \gamma_{ad} + \frac{\tau_{bc} \epsilon_{c}}{2} \left( \frac{1}{2} n^d (\partial_b \gamma_{da} + \partial_a \gamma_{db} - \partial_d \gamma_{ba}) + \frac{1}{2} n^{(2)} (\partial_a \epsilon_b + \partial_b \epsilon_a) + U_{ba} \right),
$$

where in the last equality we have used the definition of $U_{ba}$ (see Lemma 10) and $\gamma_{ad} n^d = -n^{(2)} \epsilon_a$. Putting things together, equation (116) becomes

$$
- \frac{\partial}{\partial \sqrt{\det(\mathcal{A})} \tau^b_{ab}} + \tau_{bc} \epsilon_{U_{ba}} = \frac{1}{2} \partial^{bd} \partial_a \gamma_{bd} = [J_a].
$$

We can summarize the result in the following Lemma

**Lemma 10** Let $\{\Sigma, \gamma_{ab}, \epsilon_{(2)}, Y_a^\pm, \rho^\pm, J^\pm\}$ be a shell. In coordinates, the shell field equations take the form

$$
\frac{1}{\sqrt{|\det(\mathcal{A})|}} \partial_a \left( \sqrt{|\det(\mathcal{A})|} \tau_{ab} \epsilon_b \right) - \frac{1}{2} \partial^{bd} \partial_a \gamma_{bd} = [J_a],
$$

where $\mathcal{A}$ is given in Definition 6.

We finish this section with an analysis of the behaviour of the shell equations under a gauge transformation. To that end let us define the following tensors

$$
B \equiv \frac{1}{\sqrt{|\det(\mathcal{A})|}} \partial_a \left( \sqrt{|\det(\mathcal{A})|} \tau_{ab} \epsilon_b \right) - \frac{1}{2} \partial^{bd} \partial_a \gamma_{bd},
$$

$$
C_a \equiv \frac{1}{\sqrt{|\det(\mathcal{A})|}} \partial_b \left( \sqrt{|\det(\mathcal{A})|} \tau_{ab} \epsilon_b \right) - \frac{1}{2} \partial^{bd} \partial_a \gamma_{bd} - [J_a].
$$

So that the shell equations are simply $B = 0$ and $C_a = 0$. The gauge behaviour of these fields is as follows

**Proposition 8** Under a gauge transformation with gauge fields $(u, V^a)$ the tensors $B$, $C_a$ defined above transform as

$$
C'_a = \frac{1}{u} C_a, \quad B' = B + V^a C_a.
$$

**Remark 13.** This gauge behaviour provides a powerful consistency check for the validity of the equations. Indeed, the validity of equation $B = 0$ for an arbitrary gauge implies immediately the validity of the equation $C_a = 0$. So, we could have concentrated on the equation for $[\rho \epsilon]$ alone and derive the equation for $[J_a]$ as a consequence of the gauge freedom, which certainly gives strong support to the validity of both equations.
Proof. We start finding the gauge behaviour of \( \det(\mathcal{A}) \), which reads explicitly

\[
\det(\mathcal{A'}) = \det \left[ \begin{array}{cc}
\gamma_{ab} & u \ell_b + V^d \gamma_{bd} \\
u \left( \ell_b + V^d \gamma_{bd} \right) & u^2 \left( \ell^2 + 2V^c \ell_c + V^d \gamma_{cd} \right)
\end{array} \right].
\]  

(118)

Multiplying the column \( b \) by \(-uV^b\) (\( b = 1, 2, 3 \)) and adding the three columns yields

\[
\left[ \begin{array}{c}
-\gamma_{aV} V^b \\
u \gamma_{aV} V^b + V^d \gamma_{bd} V^b
\end{array} \right].
\]

Adding this to the last column in the matrix (118) does not change its determinant. Hence

\[
\det(\mathcal{A'}) = \det \left[ \begin{array}{cc}
\gamma_{ab} & u \ell_a \\
u \ell_b & u^2 \ell^2(2)
\end{array} \right] = u^2 \det(\mathcal{A}),
\]

where the last equality follows, for instance, by multiplying the first three rows and first three columns by \( u \) and extracting a common factor \( u^2 \) to the matrix. Recalling the gauge transformation (114) for \( \tau^b_a \) it follows that \( \sqrt{\det(\mathcal{A})} \tau^b_a \) is gauge invariant. Since \( J^b_a = u^{-1} J_a \) and \( \gamma_{bd} \) is gauge invariant \( C_a^b = u^{-1} C_a \) follows at once.

In order to prove the second expression in (117) we recall the transformation laws \( T^a = T^a + V^b \rho^b_a \), \( [\rho^a] = [\rho] + V^a J_a \), as well as (52) for \( Y_{ab} \) to write

\[
B' = \frac{1}{u \sqrt{\det(\mathcal{A})}} \partial_a \left( u \sqrt{\det(\mathcal{A})} \left( (\tau^a \ell_b + V^b \tau^a_b) \right) \right) = \frac{1}{u} \tau^a \left( u \nabla \gamma_{ab} + \ell_a \partial_b u + \frac{1}{2} \mathcal{E}_a \gamma_{ab} \right) - [\rho] - V^a [J_a]
\]

\[
= B + V^a C_a + \frac{1}{u} \tau^a \partial_a \left( u V^b \right) + \frac{1}{2} V^a \tau^b \partial_a \gamma_{bd} - \frac{1}{2u} \tau^a \mathcal{E}_u \gamma_{ab} = B + V^a C_a,
\]

where in the second equality we have expanded the derivatives and recalled the definition of \( B \) and \( C_a \) and in the last one we have used the coordinate expression for \( \mathcal{E}_u \gamma_{ab} \).

\[\square\]

6 Conclusions

In this paper we have obtained a consistent framework to define data on \( m \)-dimensional manifolds consistent with the geometry of hypersurfaces of arbitrary causal character in \((m + 1)\)-dimensional spacetimes. We have also obtained explicitly the form of the the constraint equations (i.e. the normal-tangential and normal-transversal components of the Einstein tensor in terms of hypersurface data) when the data is embedded in a spacetime and have thus defined the constraint equations at an abstract hypersurface level. As a simple application we have derived the shell equations arising from two hypersurface data which agree except for its transverse tensor \( Y_{ab} \). In a spacetime setting, such data arises in the matching theory of two spacetimes, hence the name shell equations. These equations generalize the well-known shell equations derived in the matching of spacetimes across spacelike, timelike or null boundaries.

In a remarkable paper [15] (see also [13] [16]), the constraint equations for the normal-tangential component of the Einstein tensor in the case of null hypersurfaces was derived in full generality using an interesting geometric property of null hypersurfaces, namely that there exists an intrinsic (i.e. coordinate independent) derivative on the null hypersurface capable of evaluating the divergence of tensor densities of the form \( H^a_b \), satisfying \( H^a_b b^b = 0 \) and \( \gamma_{ac} H^c_b = \gamma_{bc} H^c_a \). This derivation depends only on the degenerate first fundamental form \( \gamma_{ab} \) of \( \Sigma \). In the results above we have dealt with data of arbitrary causal character which obviously,
must agree with the construction in [15] in the null case. In fact, as mentioned in the Introduction, there exist several distinct approaches to the initial data and constraint equations in the characteristic case. The framework above, being completely general, should be useful in trying to clarify the relationship between the various approaches to the constraint equations for characteristic initial data. This will be the subject of a future investigation.

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