Quasi-Periodic Solutions of Heun’s Equation

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Abstract: By exploiting a recently developed connection between Heun’s differential equation and the generalized associated Lamé equation, we not only recover the well known periodic solutions, but also obtain a large class of new, quasi-periodic solutions of Heun’s equation. Each of the quasi-periodic solutions is doubly degenerate.
Heun’s equation, a second order linear differential equation with four regular singular points has been extensively discussed in the mathematics literature [1, 2, 3]. In recent years, this equation has also appeared in a number of physical problems, like quasi-exactly solvable systems [4], sphaleron stability [5], Calogero-Sutherland models [6], higher dimensional correlated systems [7], Kerr-de Sitter black holes [8], and finite lattice Bethe ansatz systems [9].

The so-called periodic (also termed as polynomial) solutions of Heun’s equation have been well studied. However, as emphasized in ref. [1], much less attention has been devoted to the quasi-periodic (also termed as non-polynomial) solutions. In this letter we obtain a large class of (mostly new) quasi-periodic solutions. We shall show that each such solution is doubly degenerate.

The canonical form of Heun’s equation is given by [1]

\[
\frac{d^2}{dx^2} + \left( \frac{\gamma}{x} + \frac{\delta}{x-1} + \frac{\epsilon}{x-c} \right) \frac{d}{dx} + \frac{\alpha \beta x - q}{x(x-1)(x-c)} \right] G(x) = 0 ,
\]

(1)

where \(\alpha, \beta, \gamma, \delta, \epsilon, q, c\) are parameters. The parameters are not all independent; the constraint relation is

\[
\gamma + \delta + \epsilon = \alpha + \beta + 1.
\]

(2)

The four regular singular points of the differential equation are located at \(x = 0, 1, c \neq 0, 1\), and the point at infinity.

Let us make a change of independent variables using the transformation \(x = \text{sn}^2(y, m)\) [10], where \(m \equiv 1/c\). Then Heun’s equation takes the form [1]

\[
F''(y) + \left[ (1 - 2\epsilon)m \frac{\text{sn}(y, m)\text{cn}(y, m)}{\text{dn}(y, m)} + (1 - 2\delta)\frac{\text{sn}(y, m)\text{dn}(y, m)}{\text{cn}(y, m)} + (2\gamma - 1)\frac{\text{cn}(y, m)\text{dn}(y, m)}{\text{sn}(y, m)} \right] F'(y) \\
- [4mq - 4\alpha \beta \text{sn}^2(y, m)] F(y) = 0,
\]

(3)

where \(G(x) \equiv F(y)\). The periodic solutions of eq. (3) correspond to the polynomial solutions of eq. (1) while the quasi-periodic solutions of this equation correspond to the non-polynomial solutions of eq. (1).

Before describing the quasi-periodic solutions of eq. (3), it may be worthwhile explaining how we arrived at such solutions. A few years ago, we studied in some detail the Schrödinger equation for the associated Lamé (AL) potentials [11, 12]

\[
V(y, m) = a(a+1)\text{sn}^2(y, m) + b(b+1)\text{sn}^2(y + K(m), m) = a(a+1)\text{sn}^2(y, m) + b(b+1)m \frac{\text{cn}^2(y, m)}{\text{dn}^2(y, m)},
\]

(4)
which after a transformation can be shown to be a special case of eq. (3). In particular, the band edges and the mid-band states of several of these potentials were studied, when $a$ and $b$ were related in various specific ways. Further, when $a, b$ are both integers, we showed that these potentials had the special feature of possessing only a finite number of band gaps. This study has been extended to the generalized associated Lamé (GAL) potentials [13, 14]

\[ V(y, m) = a(a + 1)\text{sn}^2(y, m) + b(b + 1)\text{sn}^2(y + K(m), m) + f(f + 1)\text{sn}^2(y + K'(m), m) + g(g + 1)\text{sn}^2(y + iK'(m), m) \]

\[ = a(a + 1)\text{cn}^2(y, m) + f(f + 1)\text{dn}^2(y, m) + g(g + 1)\text{sn}^2(y, m) \]

(5)

The point to note is that after a transformation, the Schrödinger equation for the potential given in eq. (5) is identical in form to eq. (3). In particular, let us start from the Schrödinger equation

\[ -\frac{d^2\psi(y)}{dy^2} + V(y, m)\psi(y) = E\psi(y) , \]

(6)

with $V(y, m)$ given by eq. (5). On substituting

\[ \psi(y) = \text{dn}^{-b}(y)\text{cn}^{-f}(y)\text{sn}^{-g}(y)\phi(y) , \]

(7)

one can show that $\phi(y)$ satisfies Heun’s equation (3). More precisely, $\phi(y)$ satisfies the differential equation

\[ \phi''(y) + 2\left[ mb\frac{\text{sn}(y, m)\text{cn}(y, m)}{\text{dn}(y, m)} + f\frac{\text{sn}(y, m)\text{dn}(y, m)}{\text{cn}(y, m)} - g\frac{\text{cn}(y, m)\text{dn}(y, m)}{\text{sn}(y, m)} \right] \phi'(y) - [R - Q\text{sn}^2(y, m)]\phi(y) = 0 , \]

(8)

where

\[ R = -E + m(g + b)^2 + (f + g)^2 , \quad Q = (b + f + g)(b + f + g - 1) - a(a + 1) . \]

(9)

Thus, once we obtain solutions of the Schrödinger equation for the GAL potential (3), then we can immediately write the solutions of eq. (3) and hence that of the original Heun’s eq. (1) with the identification

\[ \gamma = \frac{1}{2} - g , \quad \delta = \frac{1}{2} - f , \quad \epsilon = \frac{1}{2} - b , \quad \alpha + \beta = \frac{1}{2} - (b + f + g) , \]

\[ 4\alpha\beta = Q , \quad 4mq = R , \quad F(y) \equiv \phi(y) . \]

(10)

A few comments are in order.
1. The Schrödinger eq. (6) for the GAL potential (5) is invariant under \( y \rightarrow y + K(m) \) provided \( a \) and \( b \) are interchanged and so also are \( f \) and \( g \). Hence, the eigenvalues \( E \) are invariant under \( a \leftrightarrow b \) and \( f \leftrightarrow g \) while the corresponding eigenfunctions are related to each other by the translation \( y \rightarrow y + K(m) \). Likewise, the eigenvalues \( E \) are also invariant under \( a \leftrightarrow f \) and \( b \leftrightarrow g \), while the corresponding eigenfunctions are related to each other by the translation \( y \rightarrow y + K(m) + iK'(m) \). Similarly, the eigenvalues \( E \) are invariant under the transformation \( a \leftrightarrow g \) and \( b \leftrightarrow f \), while the corresponding eigenfunctions are related to each other by the translation \( y \rightarrow y + iK'(m) \).

2. The Schrödinger eq. (6) for the GAL potential (5) is also invariant under \( a \rightarrow -a - 1 \) and/or \( b \rightarrow -b - 1 \) and/or \( f \rightarrow -f - 1 \) and/or \( g \rightarrow -g - 1 \). As a result the eigenvalues \( E \) as well as the corresponding eigenfunctions are invariant under one or several of these transformations.

3. It may be noted that, except for the invariance under \( a \rightarrow -a - 1 \), eq. (3) is not invariant under any of the above transformations. However, the connection between the GAL problem and eq. (3) and the invariances of the GAL equation, can be exploited to obtain several more solutions of Heun’s equation. For example, from eq. (9), it follows that if under any of the above transformations, if \( b_1, f_1, g_1 \) change to \( b_2, f_2, g_2 \) and the energies \( E \) remain invariant, then the corresponding eigenvalues \( R = 4mq \) of Heun’s eq. (3) are related by

\[
R_1 - m(b_1 + 1 + g_1)^2 - m(f_1 + g_1)^2 = R_2 - m(b_2 + g_2)^2 - (f_2 + g_2)^2.
\] (11)

Our strategy is now clear. We shall first obtain solutions of the Schrödinger eq. (6) for the GAL potential (5) and then using the connections given in eqs. (10) and (11) and the symmetries of the GAL equation, we shall obtain a host of solutions of Heun’s equation. This strategy is demonstrated below by discussing one example in detail. As mentioned previously, the focus here is on the quasi-periodic solutions of the GAL equation.

We now show that when either \( a + b + f + g \) and/or \( a - b - f - g \) is an arbitrary half-integer (but not an integer), then one can obtain doubly degenerate eigenstates of the GAL eq. (6) which correspond to the mid-band states (rather than band-edges) of this periodic problem. In the special case when \( a + b + f + g \) and/or \( a - b - f - g \) is an integer, then one obtains two distinct (non-degenerate) eigenstates which
correspond to the band edge eigenstates. It turns out that depending on whether $b$ or $f$ or $g$ is half-integral (while the other two are integral), we need to use a different ansatz. Let us consider all three cases one by one.

**Case 1: $b$ half-integral**

We start from eq. (8) and substitute the ansatz

$$\phi(y) = [\text{cn}(y,m) + i\text{sn}(y,m)]^t Z(y),$$

where $t$ is any real number. It follows that $Z(y)$ satisfies the equation

$$Z''(y) + [2it\text{dn}(y,m) + 2mb\frac{\text{sn}(y,m)\text{cn}(y,m)}{\text{dn}(y,m)} - 2g\frac{\text{cn}(y,m)\text{dn}(y,m)}{\text{sn}(y,m)} + 2f\frac{\text{dn}(y,m)\text{sn}(y,m)}{\text{cn}(y,m)}]Z'(y)$$

$$+[-(R + t^2) + (Q + t^2)\text{msn}^2(y,m) - 2itg\frac{\text{cn}(y,m)}{\text{sn}(y,m)} + 2itf(1 - m)\frac{\text{sn}(y,m)}{\text{cn}(y,m)}]Z(y) = 0,$$

where $R$ and $Q$ are as given by eq. (9). Not surprisingly, $Z(y) = \text{constant}$ is a solution with energy $E = (4t^2 + m)/4$ provided $f = g = 0$, $b = 1/2$, $a = t - 1/2$ (i.e. $b + f + g = 1/2$).

One can build solutions for higher half-integer values of $b + f + g$ from here. In particular, if $b + f + g = 2M + \frac{1}{2}$, we choose the ansatz ($M = 0, 1, 2, ...$)

$$Z(y) = \sum_{k=0}^{M} A_k \text{sn}^{2k}(y,m) + \text{cn}(y,m)\text{sn}(y,m) \sum_{k=0}^{M-1} B_k \text{sn}^{2k}(y,m),$$

while if $b + f + g = 2M + 3/2$, then we consider the ansatz ($M = 0, 1, 2, ...$)

$$Z(y) = \text{cn}(y,m) \sum_{k=0}^{M} A_k \text{sn}^{2k}(y,m) + \text{sn}(y,m) \sum_{k=0}^{M} B_k \text{sn}^{2k}(y,m).$$

Substitution into eq. (13) followed by lengthy but straightforward algebra yields analytic expressions for the energy eigenvalues for arbitrary $M$ and $b = 1/2, 3/2$. In particular, for $b = 1/2$, we find

$$b = 1/2, \ f + g = N, \ g = p, \ a = t - 1/2, \ E = [t^2 + m(g + b)^2],$$

where both $f, g$ are nonnegative integers satisfying $f + g = N$ with $p, N = 0, 1, 2, ...$. Since $p \leq N$, it follows that for a given $N$, there are $N + 1$ different solutions. Similarly, when $b = 3/2$, we get $g = p, f = N - p, a = t - 1/2$, and

$$E = [1 + t^2 + m(g + b)^2] - m(2g + 1) \pm \sqrt{(2g + 1)^2m^2 + 4m(N + 1)(f - g) + 4(1 - m)t^2}$$
where, \( f \) and \( g \) are again nonnegative integers. Note that, in this case too, for a given \( N \), there are \( N + 1 \) different solutions. In all cases, the corresponding eigenfunctions have the form given in eqs. (14) and (15). For small values of \( N \), the explicit coefficients \( A_k, B_k \) appearing in the eigenfunction expressions can be easily written down.

We can immediately write down the solutions of Heun’s eq. (3) by making use of eq. (10). In particular, we find the following two classes of solutions:

\[
\gamma = \frac{1}{2} - p, \quad \delta = \frac{1}{2} - N + p, \quad \epsilon = 0, \quad \alpha = -\frac{N - t}{2}, \quad \beta = -\frac{N + t}{2}, \quad 4mq = N^2 - t^2; \quad (18)
\]

\[
\gamma = \frac{1}{2} - p, \quad \delta = \frac{1}{2} - N + p, \quad \epsilon = -1, \quad \alpha = -\frac{N + 1 - t}{2}, \quad \beta = -\frac{N + 1 + t}{2}, \quad 4mq = N^2 - t^2 - 1 + (2p + 1) \pm \sqrt{(2p + 1)^2m^2 + 4m(N + 1)(N - 2p) + 4(1 - m)t^2}. \quad (19)
\]

The corresponding eigenfunctions are generically as given by eqs. (12), (14) and (15) with \( F(y) \equiv \phi(y) \).

As an illustration, for \( N = 1 \), the eigenfunction is \( Z(y) = A\text{cn}(y, m) + B\text{sn}(y, m) \) with \( B/A = it \) for \( f = 1, g = 0, b = 1/2 \), while for \( g = 1, f = 0, b = 1/2 \), one gets \( B/A = i \).

Several remarks are in order at this stage.

1. Since the Schrödinger eq. (3) for the GAL potential (5) as well as Heun’s eq. (3) are invariant under \( y \rightarrow y + 2K(m) \) and \( y \rightarrow y + 2iK'(m) \), hence \( F(y) \), \( F(y + 2K(m)) \) and \( F(y + 2iK'(m)) \) are all eigenfunctions of Heun’s eq. (3) with the same eigenvalue. As a consequence, \( F(y) = [\text{cn}(y, m) - i\text{sn}(y, m)] Z(y) \) is also an eigenfunction of Heun’s eq. (3) with the same eigenvalue. Thus for any nonintegral \( t \), each level is doubly degenerate. The same remark also applies to the other two solutions (when \( f \) or \( g \) are half integral) discussed below.

2. For the special case of \( f = g = 0 \) and \( t = 1/2 \), the results (as expected) are identical to those already obtained in ref. [12].

3. For integral \( t \), both \( a, b \) are half integral (while \( f, g \) are integral) and each of these solutions reduces to two nondegenerate periodic solutions of the GAL equation.
4. We can generate new solutions of Heun’s eq. (3) by considering the transformations $y \to y + K(m), y \to y + iK'(m), y \to y + K(m) + iK'(m)$ and the fact that the eigenvalues are invariant provided we interchange $a, b, f, g$ appropriately as explained above. Obviously, each of these new solutions is also doubly degenerate. In particular, by starting from solution (18) and using eq. (11), the three sets of new solutions of Heun’s eq. (3) are

$$
\gamma = \frac{1}{2} - N + p, \quad \delta = \frac{1}{2} - p, \quad \epsilon = 1 - t, \quad \alpha = -\frac{N + t - 2}{2}, \\
\beta = -\frac{N + t}{2}, \quad 4mq = N^2 - t^2 + m(N + t)[N + t - 2p - 1], \quad F \equiv F(y + K(m)). \quad (20)
$$

$$
\gamma = 1 - t, \quad \delta = 0, \quad \epsilon = \frac{1}{2} - N + p, \quad \alpha = -\frac{N + t - p - 1}{2}, \\
\beta = -\frac{N + t}{2}, \quad 4mq = m(N + t)[N + t - 2p - 1], \quad F \equiv F(y + iK'(m)). \quad (21)
$$

$$
\gamma = 0, \quad \delta = 1 - t, \quad \epsilon = \frac{1}{2} - p, \quad \alpha = -\frac{p + t - 1}{2}, \\
\beta = -\frac{p + t}{2}, \quad 4mq = 0, \quad F \equiv F(y + K(m) + iK'(m)). \quad (22)
$$

Similarly, by starting from solution (19) with $b = 3/2$, three sets of new solutions can be immediately written down.

5. For low values of $N$, the corresponding eigenfunctions can be explicitly shown. For example, for $g = 0, f = 1$, i.e. $p = 0, N = 1$, the eigenfunction corresponding to solution (18) is given by

$$
F(y) = [cn(y, m) + isn(y, m)]^t[cn(y, m) + itsn(y, m)], \quad (23)
$$

while the eigenfunctions corresponding to solutions (20) to (22) are given by

$$
F \equiv F(y + K(m)) \propto [cn(y, m) + i\sqrt{1 - m}sn(y, m)]^t[tcn(y, m) + i\sqrt{1 - m}sn(y, m)][dn(y, m)]^{-(1 + t)}, \quad (24)
$$

$$
F \equiv F(y + iK'(m)) \propto [1 - dn(y, m)]^t[t - dn(y, m)][sn(y, m)]^{-(1 + t)}, \quad (25)
$$

$$
F \equiv F(y + K(m) + iK'(m)) \propto [dn(y, m) - \sqrt{1 - m}]^t[tdn(y, m) - \sqrt{1 - m}][cn(y, m)]^{-(1 + t)}. \quad (26)
$$
6. We can generate even more new solutions of Heun’s eq. (3) by considering the transformations 
\( b \to -b - 1, f \to -f - 1 \) and \( g \to -g - 1 \) either singly or in various combinations. As an illustration, 
by starting from solution (18) and using eq. (11), we get the following seven sets of new solutions.

\[
\gamma = \frac{1}{2} - p, \quad \delta = \frac{1}{2} + p - N, \quad \epsilon = 2, \quad \alpha = -\frac{N - t - 2}{2}, \\
\beta = -\frac{N + t - 2}{2}, \quad 4m_q = N^2 - t^2 - 2m(2p - 1), \quad F \equiv \frac{F(y)}{dn^2(y)}. \tag{27}
\]

\[
\gamma = \frac{1}{2} - p, \quad \delta = \frac{3}{2} + N - p, \quad \epsilon = 0, \quad \alpha = \frac{N + t + 1 - 2p}{2}, \\
\beta = -\frac{N + 1 - t - 2p}{2}, \quad 4m_q = N^2 - t^2 - (2p - 1)(2N - 2p + 1), \quad F \equiv \frac{F(y)}{cn^2N - 2p + 1(y)}. \tag{28}
\]

\[
\gamma = \frac{3}{2} + p, \quad \delta = \frac{1}{2} + p - N, \quad \epsilon = 0, \quad \alpha = \frac{2p + t + 1 - N}{2}, \\
\beta = -\frac{2p + 1 - t - N}{2}, \quad 4m_q = N^2 - t^2 - (2p + 1)(2N - 2p - 1), \quad F \equiv \frac{F(y)}{dn^2(y)cn^{2N - 2p + 1}(y)}. \tag{29}
\]

\[
\gamma = \frac{1}{2} - p, \quad \delta = \frac{3}{2} + N - p, \quad \epsilon = 2, \quad \alpha = \frac{N + t + 3 - 2p}{2}, \quad \beta = -\frac{N + 3 - t - 2p}{2}, \\
4m_q = N^2 - t^2 - 2m(2p - 1) - (2p - 1)(2N - 2p + 1), \quad F \equiv \frac{F(y)}{dn^2(y)cn^{2N - 2p + 1}(y)}. \tag{30}
\]

\[
\gamma = \frac{3}{2} + p, \quad \delta = \frac{1}{2} + p - N, \quad \epsilon = 2, \quad \alpha = \frac{2p + t + 3 - N}{2}, \quad \beta = -\frac{2p + 3 - t - N}{2}, \\
4m_q = N^2 - t^2 + 2m(2p + 3) - (2p + 1)(2N - 2p - 1), \quad F \equiv \frac{F(y)}{dn^2(y)sn^{2p + 1}(y)}. \tag{31}
\]

\[
\gamma = \frac{3}{2} + p, \quad \delta = \frac{3}{2} + N - p, \quad \epsilon = 0, \quad \alpha = \frac{N + t + 2}{2}, \quad \beta = -\frac{N + 2 - t}{2}, \\
4m_q = N^2 - t^2 + 4(N + 1), \quad F \equiv \frac{F(y)}{cn^{2N - 2p + 1}(y)sn^{2p + 1}(y)}. \tag{32}
\]

\[
\gamma = \frac{3}{2} + p, \quad \delta = \frac{3}{2} + N - p, \quad \epsilon = 2, \quad \alpha = \frac{N + t + 4}{2}, \quad \beta = -\frac{N + 4 - t}{2}, \\
4m_q = N^2 - t^2 + 2m(2p + 3) + 4(N + 1), \quad F \equiv \frac{F(y)}{dn^2(y)cn^{2N - 2p + 1}(y)sn^{2p + 1}(y)}. \tag{33}
\]

7. Starting from these seven solutions, we can generate further new solutions by considering the trans-
formations \( y \to y + K(m), y \to y + iK'(m), y \to y + K(m) + iK'(m) \) and interchanging \( a, b, f, g \) 
appropriately. One can show that in this way, by starting from the solution (18) with \( b = 1/2 \) one 
has 20 independent solutions.
8. There is one more remarkable symmetry associated with eqs. (14) and (15). Note that eq. (15) is invariant under $t \to -t$ followed by $i \to -i$. Under this transformation, ansatz (14) becomes

$$\phi(y) = [\text{cn}(y, m) - i\text{sn}(y, m)]^{-t}Z(y),$$

and hence it follows that solutions with the ansatz (14) and (34) as well as with the ansatz $\phi(y) = [\text{cn}(y, m) - i\text{sn}(y, m)]^tZ(y)$ are degenerate in energy (i.e. have the same value of $R$). This then gives us 12 additional solutions.

9. Thus starting with the solution (18), for $b = 1/2$, we can generate 32 new sets of quasi-periodic solutions of Heun’s equation. In particular, 8 of these solutions have $\gamma = 1/2 - p, \gamma = 1/2 + p - N, \gamma = 1/2 + p, \epsilon = 1 \pm t$, another 8 have $\gamma = 0/2, \delta = 1 \pm t, \epsilon = 1/2 - p, \delta = 1/2 - p, \epsilon = 1 \pm t, another 8 have \gamma = 1 \pm t, \delta = 0, \epsilon = 1 \pm t, another 8 have \gamma = 0/2, \delta = 1 \pm t, \epsilon = 1 \pm t, while 4mq can be computed by using eqs. (10) and (18).

10. Proceeding in the same way, by starting from the solution (19) with $b = 3/2$, we also generate 32 independent solutions in each of which 4mq takes two possible values. The corresponding values of $\gamma, \delta, \epsilon$ are the same as in the $b = 1/2$ case except that the values 0, 2 are now replaced everywhere by $-1, 3$ respectively.

Case 2: $f$ half-integral

We start from eq. (35) and substitute the ansatz

$$\phi(y) = [\text{dn}(y, m) + i\text{sn}(y, m)]^tZ(y),$$

where $t$ is any real number and $k = \sqrt{m}$. It follows that $Z(y)$ satisfies the equation

$$Z''(y) + [2ikt\text{cn}(y, m) + 2mb\frac{\text{sn}(y, m)\text{cn}(y, m)}{\text{dn}(y, m)} - 2g\frac{\text{cn}(y, m)\text{dn}(y, m)}{\text{sn}(y, m)} + 2f\frac{\text{dn}(y, m)\text{sn}(y, m)}{\text{cn}(y, m)}]Z'(y) + \left[-(R + mt^2) + (Q - t^2)\text{msn}^2(y, m) - 2ikt\frac{\text{dn}(y, m)}{\text{sn}(y, m)} - 2ikt(1 - m)\frac{\text{sn}(y, m)}{\text{dn}(y, m)} + ikt(2b + 2f + 2g - 1)\text{sn}(y, m)\text{dn}(y, m)\right]Z(y) = 0,$$
where \( R \) and \( Q \) are as given by eq. (9). Not surprisingly, \( Z(y) = \text{constant} \) is a solution with energy \( E = \frac{(4mt^2 + 1)}{4} \) provided \( b = g = 0, \ f = 1/2, \ a = t - 1/2 \) (i.e. \( b + f + g = 1/2 \)).

One can build solutions for higher values of \( b + f + g \) from here. In particular, for \( b + f + g = 2M + 1/2 \), we consider the ansatz \((M = 0, 1, 2, ...)\)

\[
Z(y) = \sum_{k=0}^{M} A_k \text{sn}^{2k}(y, m) + \text{sn}(y, m) \sum_{k=0}^{M-1} B_k \text{sn}^{2k}(y, m),
\]

while if \( b + f + g = 2M + 3/2 \) we take the ansatz \((M = 0, 1, 2, ...)\)

\[
Z(y) = \text{dn}(y, m) \sum_{k=0}^{M} A_k \text{sn}^{2k}(y, m) + \text{sn}(y, m) \sum_{k=0}^{M} B_k \text{sn}^{2k}(y, m).
\]

Substitution into eq. (36) yields analytic expressions for the energy eigenvalues and eigenfunctions for arbitrary \( M \) for \( f = 1/2 \) and \( f = 3/2 \). In particular, for \( f = 1/2 \), we find that

\[
f = 1/2, \ g = p, \ b + g = N, \ a = t - 1/2, \ E = \lfloor mt^2 + (g + f)^2 \rfloor,
\]

where both \( b, g \) are nonnegative integers satisfying \( b + g = N \) with \( N = 0, 1, 2, ... \).

Similarly, when \( f = 3/2, a = t - 1/2, g = p, b + g = N \) we find that

\[
E = [(1 + t^2)m + (g + f)^2] - (2g + 1) \pm \sqrt{(2g + 1)^2 + 4m(N + 1)(b - g) - 4m(1 - m)t^2}
\]

where, \( b \) and \( g \) are again nonnegative integers. In all these cases, the corresponding eigenfunctions have the form as given above in eqs. (37) and (38). For small values of \( N \), the explicit coefficients \( A_k, B_k \) in the eigenfunction expressions can be easily written down.

Using eq. (9), we can write down the corresponding solutions of Heun’s eq. (3). They are given by

\[
\gamma = \frac{1}{2} - p, \ \delta = 0, \ \epsilon = \frac{1}{2} - N + p, \ \alpha = -\frac{N - t}{2}, \ \beta = -\frac{N + t}{2}, \ 4mq = m(N^2 - t^2).
\]

\[
\gamma = \frac{1}{2} - p, \ \delta = -1, \ \epsilon = \frac{1}{2} - N + p, \ \alpha = -\frac{N + 1 - t}{2}, \ \beta = -\frac{N + 1 + t}{2},
\]

\[
4mq = m(N^2 - t^2 - 1) + (2p + 1) \pm \sqrt{(2p + 1)^2 + 4m(N + 1)(N - 2p) - 4m(1 - m)t^2}.
\]

Corresponding to each of these two solutions, we can again write down 32 independent sets of solutions as shown above in Case 1 when \( b \) was half-integral.
Case 3: $g$ half-integral

We start from eq. (8) and substitute the ansatz
\[
\phi(y) = [\text{dn}(y, m) + k\text{cn}(y, m)]^t Z(y),
\]
where $t$ is any real number. It then follows that $Z(y)$ satisfies the equation
\[
Z''(y) + \left[ -2ktsn(y, m) + \frac{2mb}{\text{dn}(y, m)} - 2g \frac{\text{cn}(y, m)\text{dn}(y, m)}{\text{sn}(y, m)} + 2f \frac{\text{dn}(y, m)\text{sn}(y, m)}{\text{cn}(y, m)} \right] Z'(y)
\]
\[
+ \left[ -R + (Q + t^2)\text{msn}^2(y, m) - 2kt \frac{\text{cn}(y, m)}{\text{dn}(y, m)} - 2kt \frac{\text{dn}(y, m)}{\text{cn}(y, m)} \right] Z(y) = 0,
\]
where $R$ and $Q$ are as given by eq. (9). Clearly, $Z(y) = \text{constant}$ is a solution with energy $E = (1 + m)/4$ provided $b = f = 0$, $g = 1/2$, $a = t - 1/2$ (i.e. $b + f + g = 1/2$).

One can build solutions for higher values of $b + f + g$ from here. In particular, when $b + f + g = 2M + 1/2$, we consider the ansatz ($M = 0, 1, 2, \ldots$)
\[
Z(y) = \sum_{k=0}^{M} A_k \text{sn}^{2k}(y, m) + \text{cn}(y, m)\text{dn}(y, m) \sum_{k=0}^{M-1} B_k \text{sn}^{2k}(y, m),
\]
while if $b + f + g = 2M + 3/2$ then we consider the ansatz ($M = 0, 1, 2, \ldots$)
\[
Z(y) = \text{cn}(y, m) \sum_{k=0}^{M} A_k \text{sn}^{2k}(y, m) + \text{dn}(y, m) \sum_{k=0}^{M} B_k \text{sn}^{2k}(y, m).
\]
Substitution into eq. (44) leads to analytic expressions for the energy eigenvalues and eigenfunctions for arbitrary $M$ when $b = 1/2, 3/2$. In particular, for $b = 1/2$, we find that
\[
g = 1/2, \ b + f = N, \ a = t - 1/2, \ E = [(f + g)^2 + m(g + b)^2],
\]
where both $b, f$ are nonnegative integers satisfying $b + f = N$ with $N = 0, 1, 2, \ldots$. Similarly, when $g = 3/2, a = t - 1/2, b + f = N$ we obtain
\[
E = (f + g)^2 + m(g + b)^2 - [1 + 2f + (2b + 1)m] \pm \sqrt{(1 - m)[(2f + 1)^2 - (2b + 1)^2m] + 4mt^2}.
\]
In all these cases, the corresponding eigenfunctions have the form given above in eqs. (45) and (46).
We now write down the solutions of Heun’s eq. (3) corresponding to solutions (47) and (48):

\[
\gamma = 0, \quad \delta = \frac{1}{2} - N + p, \quad \epsilon = \frac{1}{2} - p, \quad \alpha = -\frac{N - t}{2}, \quad \beta = -\frac{N + t}{2}, \quad 4mq = 0; \quad (49)
\]

\[
\gamma = -1, \quad \delta = \frac{1}{2} - N + p, \quad \epsilon = \frac{1}{2} - p, \quad \alpha = -\frac{N + 1 - t}{2}, \quad \beta = -\frac{N + 1 + t}{2},
\]

\[
4mq = 2(N - p) + (2p + 1)m \pm \sqrt{(1 - m)[(2N - 2p + 1)^2 - (2p + 1)^2m] + 4mt^2}. \quad (50)
\]

Corresponding to each of these two solutions, we can again write down 32 independent sets of solutions as done above when \( b \) was half-integral.

Summarizing, in this letter we have obtained new 192 sets of quasi-periodic solutions of Heun’s eq. (3), each of which is doubly degenerate. In each set, solutions exist with polynomials of arbitrary order \( N \), and for each \( N \), there are \( N + 1 \) distinct solutions.

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References

[1] For an excellent up to date mathematical summary of Heun’s equation, see A. Ronveaux (ed.), *Heun’s Differential Equation* (Oxford University Press, 1995).

[2] R.S. Maier, arXiv:math.CA/0408317.

[3] A. Erdélyi et al. (ed.), *Higher Transcendental Functions* (Bateman Manuscript Project) Vol. III (McGraw-Hill, 1955).

[4] N.H. Christ and T.D. Lee, Phys. Rev. **D12** (1975) 1606; D.P. Jatkar, C.N. Kumar and A. Khare, Phys. Lett. **A142** (1989) 200; A. Khare and B.P. Mandal, Phys. Lett. **A239** (1998) 197.

[5] Y. Brihaye, S. Giller, P. Kosinski and J. Kunz, Phys. Lett. **B293** (1992) 383; S. Briabant and Y. Brihaye, Jour. Math. Phys. **34** (1994) 2107; Y. Brihaye, S. Giller and P. Kosinski, Jour. Phys. **A28** (1995) 421.
[6] K. Takemura, Comm. Math. Phys. 235 (2003) 467; Jour. Nonlinear Math. Phys. 11 (2004) 21; arXiv:math.CA/0406141.

[7] R.K. Bhaduri, A. Khare, J. Law, M.V.N. Murthy and D. Sen, Jour. Phys. A30 (1997) 2557.

[8] M. Suzuki, E. Takasugi and H. Umetsu, Prog. Theor. Phys. 100 (1998) 491.

[9] P. Dorey, J. Suzuki and R. Tateo, Jour. Phys. A37 (2004) 2047.

[10] For the properties of Jacobi elliptic functions, see, for example, M. Abramowitz and I. Stegun, Handbook of Mathematical Functions (Dover, 1964); I. S. Gradshteyn and I. M. Ryzhik, Table of Integrals, Series and Products (Academic Press, 2000).

[11] A. Khare and U. Sukhatme, Jour. Math. Phys. 40 (1999) 5473.

[12] A. Khare and U. Sukhatme, Jour. Math. Phys. 42 (2001) 5652.

[13] A. Treibich and J.-L. Verdier, C.R. Acad. Sci. Paris 311 (1990) 51.

[14] A. Khare and U. Sukhatme, arXiv:math-ph/0505027.

[15] For a recent attempt, see for example, N. Gurappa and P.K. Panigrahi, Jour. Phys. A37 (2004) L605.