Parameterized Bipartite Entanglement Measure

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We propose a novel parameterized entanglement measure $\alpha$-concurrence for bipartite systems. By employing positive partial transposition and realignment criteria, we derive analytical lower bounds for the $\alpha$-concurrence. Moreover, we calculate explicitly the analytic expressions of the $\alpha$-concurrence for isotropic states and Werner states.

I. INTRODUCTION

As a distinguishing feature of quantum mechanics, quantum entanglement is an important resource \cite{1,2,3} in quantum computation and quantum information processing \cite{4} such as quantum dense coding \cite{5}, clock synchronization \cite{6}, quantum teleportation \cite{7}, quantum secret sharing \cite{8} and quantum cryptography \cite{9}. One of the problems in quantum entanglement theory is to quantify the entanglement of a bipartite system. A reasonable entanglement measure has been derived by employing the positive partial transposition (PPT) and realignment criteria, we derive analytical lower bounds for the $\alpha$-concurrence. Moreover, we calculate explicitly the analytic expressions of the $\alpha$-concurrence for isotropic states and Werner states.

II. $\alpha$-CONCURRENCE

Let $\mathcal{H}_A \otimes \mathcal{H}_B$ be an arbitrary $d \times d$ dimensional bipartite Hilbert space associated with subsystems $A$ and $B$. Any pure state on the $\mathcal{H}_A \otimes \mathcal{H}_B$ can be written as in the Schmidt form,$$
|\psi\rangle = \sum_{i=1}^{r} \sqrt{\lambda_i} |a_i b_i\rangle,
$$
where $\sum_{i=1}^{r} \lambda_i = 1$ with $\lambda_i > 0$, $r$ is the Schmidt rank, $1 \leq r \leq d$. $\{ |a_i\rangle \}$ and $\{ |b_i\rangle \}$ are the local bases associated with the subsystems $A$ and $B$, respectively \cite{4}.

Definition. For any pure state $|\psi\rangle$ given in (1), the $\alpha$-concurrence is defined by
$$
C_\alpha (|\psi\rangle) = \text{Tr} \rho_A^\alpha - 1
$$
for any $0 \leq \alpha \leq 1/2$, where $\rho_A = \text{Tr}_B |\psi\rangle \langle \psi|$. From (2), for a pure state $|\psi\rangle$ given by (1) one has
$$
C_\alpha (|\psi\rangle) = \sum_{i=1}^{r} \lambda_i^\alpha - 1,
$$
where $C_\alpha (|\psi\rangle)$ satisfies $0 \leq C_\alpha (|\psi\rangle) \leq d^{1-\alpha} - 1$. It is obvious that the lower bound is attained if and only if $|\psi\rangle$ is a separable state, that is, $|\psi\rangle = |a_i\rangle |b_i\rangle$ for some $|a_i\rangle$ and $|b_i\rangle$. While the upper bound is achieved for the maximally entangled pure states $|\Psi^+\rangle = \frac{1}{\sqrt{d}} \sum_{i=1}^{d} |a_i b_i\rangle$.

For a general mixed state $\rho$ on the Hilbert space $\mathcal{H}_A \otimes \mathcal{H}_B$, the $\alpha$-concurrence is given by the convex-roof extension,
$$
C_\alpha (\rho) = \min_{\{p_i, |\psi_i\rangle\}} \sum_{i} p_i C_\alpha (|\psi_i\rangle),
$$
where the infimum is taken over all possible pure-state decompositions of $\rho = \sum_{i} p_i |\psi_i\rangle \langle \psi_i|$, with $\sum_{i} p_i = 1$ and $p_i > 0$. Before showing that $C_\alpha (\rho)$ defined in (4) is indeed a bona fide entanglement measure, we first present the following lemma, see proof in Appendix A.

Lemma 1. The function $F_\alpha (\rho) = \text{Tr} \rho^\alpha - 1$ is concave, that is,
$$
F_\alpha \left( \sum_{i} p_i \rho_i \right) \geq \sum_{i} p_i F_\alpha (\rho_i)
$$
for any $0 \leq \alpha \leq 1/2$, where $\{ p_i \}$ is a probability distribution and $\rho_i$ are density matrices. The equality holds if and only if all $p_i$ are the same for all $p_i > 0$.

By using the above Lemma 1, we have the following theorem.

**Theorem 1.** The $\alpha$-concurrence $C_{\alpha} (\rho)$ given in (4) is a well defined parameterized entanglement measure.

**Proof.** We need to verify that $C_{\alpha} (\rho)$ fulfills the following four requirements.

(E1) If $\rho$ is an entangled state, then there is at least one entangled pure state $|\psi\rangle$ in any pure state decomposition of $\rho$. Thus $C_{\alpha} (\rho) > 0$. Otherwise, $C_{\alpha} (\rho) = 0$ for separable states.

(E2) Consider $\rho = \sum_i p_i |\psi_i\rangle \langle \psi_i| \rho_2 = \sum_j q_j |\phi_j\rangle \langle \phi_j|$ be the optimal pure state decomposition of $C_{\alpha} (\rho_1) (C_{\alpha} (\rho_2))$ with $\sum_i p_i = 1 \quad (\sum_j q_j = 1)$ and $p_i > 0 \quad (q_j > 0)$. We have

$$C_{\alpha} (\rho) \leq \sum_{i=1}^k t p_i C_{\alpha} (|\psi_i\rangle) + \sum_{j=1}^l (1-t) q_j C_{\alpha} (|\phi_j\rangle)$$

$$= t C_{\alpha} (\rho_1) + (1-t) C_{\alpha} (\rho_2),$$

where the first inequality is due to that $\sum_{i=1}^k t p_i |\psi_i\rangle \langle \psi_i| + \sum_{j=1}^l (1-t) q_j |\phi_j\rangle \langle \phi_j|$ is also a pure state decomposition of $\rho$.

(E3) We adopt the approach given in [40] to show that our entanglement measure does not increase under LOCC. Denote $\tilde{\lambda}_\phi$ ($\tilde{\lambda}_\psi$) the Schur vector given by the squared Schmidt coefficients of the state $|\psi\rangle \langle \phi|$ in the decreasing order. It has been shown that the state $|\phi\rangle$ can be prepared starting from the state $|\psi\rangle$ under LOCC if and only if $\tilde{\lambda}_\psi$ is majorized by $\tilde{\lambda}_\phi$ [41], $\tilde{\lambda}_\psi \prec \tilde{\lambda}_\phi$, where the majorization means that the components $|\lambda_\psi|_i$ ($|\lambda_\phi|_i$) of $\tilde{\lambda}_\psi$ ($\tilde{\lambda}_\phi$), listed in nonincreasing order, satisfy $\sum_{i=1}^j |\lambda_\psi|_i \leq \sum_{i=1}^j |\lambda_\phi|_i$ for $1 < j \leq d$, with equality for $j = d$.

Since the entanglement cannot increase under LOCC, any entanglement measure $E$ has to satisfy that $E (|\psi\rangle \langle \phi|) \geq E (|\phi\rangle)$ whenever $\tilde{\lambda}_\psi \prec \tilde{\lambda}_\phi$. This condition, known as the Schur concavity, is satisfied if and only if $E$, given as a function of the squared Schmidt coefficients $\lambda_i$’s [42], is invariant under the permutations of any two arguments and satisfies

$$(\lambda_i - \lambda_j) \left( \frac{\partial E}{\partial \lambda_i} - \frac{\partial E}{\partial \lambda_j} \right) \leq 0$$

for any two components $\lambda_i$ and $\lambda_j$ of $\tilde{\lambda}$.

For any pure state $|\psi\rangle$ given by (1), the $\alpha$-concurrence is obviously invariant under the permutations of the Schmidt coefficients for any $0 \leq \alpha \leq 1/2$. Since

$$(\lambda_i - \lambda_j) \left( \frac{\partial C_{\alpha}}{\partial \lambda_i} - \frac{\partial C_{\alpha}}{\partial \lambda_j} \right) = \alpha (\lambda_i - \lambda_j) (\lambda_i^{\alpha-1} - \lambda_j^{\alpha-1}) \leq 0$$

for any two components $\lambda_i$ and $\lambda_j$ of the squared Schmidt coefficients of $|\psi\rangle$, we have $C_{\alpha} (|\psi\rangle) \geq C_{\alpha} (|\phi\rangle)$ for any LOCC $\Lambda$ and $0 \leq \alpha \leq 1/2$.

Next let $\rho = \sum_i p_i |\psi_i\rangle \langle \psi_i|$ be the optimal pure state decomposition of $C_{\alpha} (\rho)$ with $\sum_i p_i = 1$ and $p_i > 0$. We obtain

$$C_{\alpha} (\rho) = \sum_i p_i C_{\alpha} (|\psi_i\rangle) \geq \sum_i p_i C_{\alpha} (\Lambda |\psi_i\rangle) \geq C_{\alpha} (\Lambda \rho)$$

for any $0 \leq \alpha \leq 1/2$, where the last inequality is from the definition (4).

(E4) Let $\rho = \sum_i p_i |\psi_i\rangle \langle \psi_i|$ be the optimal pure state decomposition of $C_{\alpha} (\rho)$ with $\sum_i p_i = 1$ and $p_i > 0$. Consider stochastic LOCC protocol given by Kraus operators $A_k$ with $\sum_k A_k^\dagger A_k = I$. We have

$$C_{\alpha} (\rho) = \sum_i p_i C_{\alpha} (|\psi_i\rangle)$$

$$\geq \sum_{i,k} p_i p_{k|i} C_{\alpha} (|\psi_i^k\rangle) \geq \sum_{i,k} p_k p_{i|k} C_{\alpha} (|\psi_i^k\rangle) \geq \sum_k p_k C_{\alpha} (\rho^k),$$

where $p_{k|i} = \text{Tr} A_k |\psi_i\rangle \langle |\psi_i| A_k^\dagger$ is the probability of obtaining the outcome $k$ with $|\psi_i^k\rangle = A_k |\psi_i\rangle / \sqrt{p_{k|i}}$, and $p_k = \text{Tr} A_k \rho A_k^\dagger$ is the probability of obtaining the outcome $k$ with $\rho^k = A_k \rho A_k^\dagger / p_k$. The first inequality is due to the concavity of the Lemma 1, since $\text{Tr} B |\psi_i\rangle \langle |\psi_i| B^\dagger = \sum_{k|i} p_{k|i} \text{Tr} B |\psi_i^k\rangle \langle |\psi_i^k| B^\dagger$. The last inequality is from the definition of (4), since $\sum_{i,k} p_{k|i} |\psi_i^k\rangle \langle |\psi_i^k|$ is $\rho^k$. \hfill \Box

In [39] the parameterized entanglement monotone $q$-concurrence $C_q (|\psi\rangle)$ for any pure state $|\psi\rangle$ defined in (1) has been introduced, $C_q (|\psi\rangle) = 1 - \text{Tr} \rho^\lambda A$, where $q \geq 2$. It seems that our $\alpha$-concurrence defined by (2), $C_{\alpha} (|\psi\rangle) = \text{Tr} \rho^\lambda A_{\alpha} - 1$, is in some sense dual to the $q$-concurrence as the parameter $\alpha \in [0, 1/2]$, while the parameter $q \geq 2$. Nevertheless, these two concurrences characterize the quantum entanglement in different aspects, even though they are both derived from the Tsallis-$q$ of entanglement [38]. For large enough $q$, the $q$-concurrence $C_q (\rho)$ converges to the constant 1 for any entangled state $\rho$, while the $\alpha$-concurrence $C_{\alpha} (\rho)$ not for any $\alpha \in [0, 1/2]$. Particularly, for $\alpha = 0$, the measure $C_0 (|\psi\rangle) = r - 1$ for any pure state $|\psi\rangle$ given in (1), where $r$ is the Schmidt rank of the state $|\psi\rangle$, which is solely determined by the Schmidt rank of the bipartite pure state $|\psi\rangle$. Therefore, the $\alpha$-concernces with different $\alpha$ provide different characterizations of the feature of entanglement.
III. BOUNDS ON $\alpha$-CONCURRENCE

Owing to the optimization in the calculation of the entanglement measures, it is generally difficult to obtain analytical expressions of the entanglement measures for general mixed states. In this section, we derive analytical lower bounds for the $\alpha$-concurrence based on PPT and realignment criteria [32–36, 43].

A bipartite state can be written as $\rho = \sum_{ijkl} \rho_{ijkl} |ij⟩⟨kl|$, where the subscripts $i$ and $k$ are the row and column indices for the subsystem $A$, respectively, while $j$ and $l$ are such indices for the subsystem $B$. The PPT criterion says that if the state $\rho$ is separable, then the partial transposed matrix $\rho^T = \sum_{ijkl} \rho_{ijkl} |kl⟩⟨ij|$ with respect to the subsystem $B$ is non-negative, $\rho^T \geq 0$. While the realignment criterion says that the realigned matrix of $\rho$, $R(\rho) = \sum_{ijkl} \rho_{ijkl} |ik⟩⟨jl|$, satisfies that $\|R(\rho)\|_1 \leq 1$ if $\rho$ is separable, where $\|X\|_1$ denotes the trace norm of matrix $X$, $\|X\|_1 = \text{Tr} \sqrt{XX^T}$.

For a pure state $|ψ⟩$ given by (1), it is straightforward to obtain that [31]

$$1 \leq \|\rho^T\|_1 = \|R(\rho)\|_1 = \left(\sum_{i=1}^{r} \sqrt{λ_i}\right)^2 \leq r,$$  \hspace{1cm} (9)

where $\rho = |ψ⟩⟨ψ|$. In particular, for $\alpha = 1/2$, the 1/2-concurrence becomes $C_{1/2}(α) = \sum_{i=1}^{r} \sqrt{λ_i} - 1$. One has then

$$C_{1/2}(α) = \frac{\|\rho^T\|_1 - 1}{\sqrt{r} + 1},$$  \hspace{1cm} (10)

for any pure state $|ψ⟩$ on the $H_A \otimes H_B$.

**Theorem 2.** For any mixed state $ρ$ on the $H_A \otimes H_B$, the $α$-concurrence $C_α(ρ)$ satisfies

$$C_α(ρ) \geq \frac{d^{1-α} - 1}{d - 1} \left[\max \{\|\rho^T\|_1, \|R(ρ)\|_1 - 1\}\right].$$  \hspace{1cm} (11)

**Proof.** For a pure state $|ψ⟩$ given in (1), let us analyze the monotonicity of the following function,

$$g(α) = \sum_{i=1}^{r} λ_i^α - 1,$$  \hspace{1cm} (12)

for any $0 \leq α \leq 1/2$, where $r > 1$. The first derivative of $g(α)$ with respect to $α$ is given by

$$\frac{∂g}{∂α} = \frac{G_{ρα}}{(r^{1-α} - 1)^2},$$  \hspace{1cm} (13)

where

$$G_{ρα} = \sum_{i=1}^{r} λ_i^α \ln λ_i \left(r^{1-α} - 1\right) + \left(\sum_{i=1}^{r} λ_i^α - 1\right) r^{1-α} \ln r,$$  \hspace{1cm} (14)

Employing the Lagrange multiplies [26] under constraints $\sum_{i=1}^{r} λ_i = 1$ and $λ_i > 0$, one has that there is only one stable point $λ_i = 1/r$ for every $i = 1, \ldots, r$, for which $G_{ρα} = 0$ for any $0 \leq α \leq 1/2$. Since the second derivative at this point,

$$\frac{∂^2G_{ρα}}{∂α^2} \bigg|_{λ_i=1/r} = r^{2-α} \left\{ α (α - 1) \ln r + (2α - 1) (r^{1-α} - 1) \right\}$$

$$< 0$$  \hspace{1cm} (15)

for any $0 \leq α \leq 1/2$, the maximum extreme value point is just the maximum value point. $g(α)$ is a decreasing function for any $0 \leq α \leq 1/2$, since $∂g/∂α \leq 0$.

We have

$$C_α(|ψ⟩) \geq \frac{r^{1-α} - 1}{\sqrt{r} - 1} C_{1/2}(ψ)$$

$$\geq \frac{r^{1-α} - 1}{r - 1} (\|α\|_1 - 1)$$

$$\geq \frac{d^{1-α} - 1}{d - 1} (\|α\|_1 - 1),$$  \hspace{1cm} (16)

where $σ = |ψ⟩⟨ψ|$, the second inequality is due to (10), the last inequality is due to that $\frac{d^{1-α} - 1}{r - 1}$ is a decreasing function with respect to $r$. Assume $ρ = \sum_i p_i |ϕ_i⟩⟨ϕ_i|$ is the optimal pure state decomposition for $C_α(ρ)$. Then

$$C_α(ρ) \geq \sum_i p_i C_α(|ϕ_i⟩$$

$$\geq \frac{d^{1-α} - 1}{d - 1} \sum_i p_i (\|α_i\|_1 - 1)$$

$$\geq \frac{d^{1-α} - 1}{d - 1} (\|ρ^T\|_1 - 1),$$  \hspace{1cm} (17)

where $σ_i = |ϕ_i⟩⟨ϕ_i|$, the last inequality is due to the convex property of the trace norm and $\|ρ^T\|_1 \geq 1$ in (9).

Similar to (16) and (17), we obtain from (9) that

$$C_α(ρ) \geq \frac{d^{1-α} - 1}{d - 1} (\|R(ρ)\|_1 - 1)$$  \hspace{1cm} (18)

for any $0 \leq α \leq 1/2$.

Combining (17) and (18), we complete the proof. □

IV. $\alpha$-CONCURRENCE FOR ISOTROPIC AND WERNER STATES

In this section, we compute the $α$-concurrence for isotropic states and Werner states. Let $E$ be a convex-roof extended quantum entanglement measure. Denote $S$ the set of states and $P$ the set all pure states in $S$. Let $G$ be a compact group acting on $S$ by $(U, ρ) \mapsto UρU^†$. Assume that the measure $E$ defined on $P$ is invariant under the operations of $G$. One can define the projection $P:S \to S$ by $Pρ = \int dUUρU^†$ with the standard (normalized) Haar measure $dU$ on $G$, and the function $η$ on $PS$ by

$$η(ρ) = \min \{E(|Ψ⟩): |ψ⟩ \in P, P|ψ⟩⟨ψ| = ρ\}.$$  \hspace{1cm} (19)
Then for \( \rho \in \mathcal{PS} \), we have

\[
E(\rho) = \text{co}(\eta(\rho)),
\]

where \( \text{co}(f) \) is the convex-roof extension of a function \( f \). In other words, it is the convex hull of \( f \).

### A. Isotropic states

The isotropic states \( \rho_F \) are given by [33],

\[
\rho_F = \frac{1 - F}{d^2 - 1} (I - |\Psi\rangle\langle\Psi|) + F|\Psi\rangle\langle\Psi|,
\]

where \( |\Psi\rangle = \frac{1}{\sqrt{d}} \sum_{i=1}^{d} |ii\rangle \) and \( F = \langle\Psi|\rho_F|\Psi\rangle \). \( \rho_F \) is separable if and only if \( 0 \leq F \leq 1/d \) [24]. Inspired by the techniques adopted in [24–27], we have, see Appendix B,

\[
\eta_\alpha(\rho_F) = (Fd)^{1-\alpha} - 1
\]

for any \( 0 \leq \alpha \leq 1/2 \), where \( F > 1/d \).

Obviously, the second derivative of (22) with respect to \( F \) is non-positive for any \( 0 \leq \alpha \leq 1/2 \). Hence, \( \eta_\alpha(\rho_F) \) is concave in the whole regime \( F \in (1/d, 1) \). The \( C_\alpha(\rho_F) \) is the largest convex function that is upper bounded by \( \eta_\alpha(\rho_F) \), which is constructed in the following way. Find the line that passes through the points \( (F = 1/d, \eta_\alpha = 0) \) and \( (F = 1, \eta_\alpha = d^{1-\alpha} - 1) \) of \( \eta_\alpha(\rho_F) \) for any \( 0 \leq \alpha \leq 1/2 \). Thus, we have the following analytical formula of the \( \alpha \)-concurrence for isotropic states.

**Lemma 2.** The \( \alpha \)-concurrence for isotropic states \( \rho_F \in \mathcal{C}^d \otimes \mathcal{C}^d \ (d \geq 2) \) is given by

\[
C_\alpha(\rho_F) = \begin{cases} 
0, & F \leq 1/d, \\
\frac{d^{1-\alpha} - 1}{d-1} (dF-1), & F > 1/d,
\end{cases}
\]

where \( 0 \leq \alpha \leq 1/2 \) and \( d \geq 2 \).

Since \( \|\rho_F^d\|_1 = \|\mathcal{R}(\rho_F)\|_1 = Fd \) for \( F > 1/d \) [20, 35], surprisingly the lower bound of (11) is just exactly the (23) for every \( 0 \leq \alpha \leq 1/2 \) with \( d \geq 2 \).

The concurrence \( C(\rho_F) \) of isotropic states has been derived in [26], \( C(\rho_F) = \sqrt{\frac{2}{d(d-1)}} (dF-1) \) for any \( F > 1/d \). Fig. 1 exhibits the relations between the concurrence and the \( \alpha \)-concurrence of isotropic states for \( \alpha = 0 \) and \( 1/2 \). Especially, it shows that the \( C_0(\rho_F) = C(\rho_F) \) with \( d = 2 \), and the concurrence of isotropic states is less than the \( 0 \)-concurrence with \( d > 2 \). Moreover, we notice that the \( 1/2 \)-concurrence is bigger than the concurrence with \( d \geq 5.1508 \).

### B. Werner states

The Werner states are of the form,

\[
\rho_W = \frac{1}{2 (d+1)} \left( \sum_{k=1}^{d} |kk\rangle\langle kk| + \sum_{i<j}^{d} |\Psi_{ij}^+\rangle\langle\Psi_{ij}^+| \right) + \frac{2W}{d(d-1)} \sum_{i<j} |\Psi_{ij}^-\rangle\langle\Psi_{ij}^-|,
\]

where \( |\Psi_{ij}^\pm\rangle = (|ij\rangle \pm |ji\rangle)/\sqrt{2} \) and \( W = \text{Tr}(\rho_W \sum_{i<j} |\Psi_{ij}^-\rangle\langle\Psi_{ij}^-|) \) [27]. \( \rho_W \) is separable if and only if \( 0 \leq W \leq 1/2 \) [25, 44]. For \( W > 1/2 \), we have, see Appendix C,

\[
\eta_\alpha(\rho_W) = (2W)^{1-\alpha} - 1
\]

for any \( 0 \leq \alpha \leq 1/2 \).

It is direct to verify that the second derivative of (25) with respect to \( W \) is non-positive, namely, \( \eta_\alpha(\rho_W) \) is concave. Similar to (23), we have

**Lemma 3.** The \( \alpha \)-concurrence for Werner states \( \rho_W \in \mathcal{C}^d \otimes \mathcal{C}^d \ (d \geq 2) \) is given by

\[
C_\alpha(\rho_W) = \begin{cases} 
0, & W \leq 1/2, \\
(2^{1-\alpha} - 1) (2W - 1), & W > 1/2,
\end{cases}
\]

where \( 0 \leq \alpha \leq 1/2 \).

We remark that for \( W > 1/2 \), the lower bound of (11) for Werner states is given by

\[
C_\alpha(\rho_W) \geq \frac{2}{d(d-1)} (2W-1)
\]

Accounting to (26), we obtain

\[
(2^{1-\alpha} - 1) (2W - 1) \geq \frac{2}{d(d-1)} (2W-1)
\]
where equality holds if $d = 2$, and the inequality holds strictly for higher dimensional quantum systems.

The concurrence of Werner states has been obtained in [45], $C(\rho_W) = 2W - 1$ for $W > 1/2$. It is direct to find that $C_\alpha(\rho_W) = (2^{1-\alpha} - 1)C(\rho_W)$ for any $0 \leq \alpha \leq 1/2$. Moreover, the entanglement of formation for Werner states is given by [25], $E_F(\rho_W) = H_2[\frac{1}{2}(1 - 2\sqrt{W(1 - W)})]$. In Fig. 2 we illustrate the relations among the concurrence, entanglement of formation and $\alpha$-concurrence of Werner states for $\alpha = 0$ and $1/2$. The entanglement of formation for Werner states $E_F(\rho_W)$ is always upper bounded by the $C_0(\rho_W) = C(\rho_W)$, and larger than the $1/2$-concurrence $C_{1/2}(\rho_W)$ for $W \geq 0.6$.

V. SUMMARY

We have introduced the concept of $\alpha$-concurrence and shown that the $\alpha$-concurrence is a well defined entanglement measure. Analytical lower bounds of the $\alpha$-concurrence for general mixed states have been derived based on PPT and realignment criterion. Specifically, we have derived explicit formulae for the $\alpha$-concurrence of isotropic states and Werner states. Interestingly our lower bounds are exact for isotropic states and Werner states with $d = 2$. Our parameterized entanglement measure $\alpha$-concurrence gives a family of entanglement measures and enriches the theory of quantum entanglement, which may highlight further researches on the study of quantifying quantum entanglement and the related investigations like monogamy and polygamy relations in entanglement distribution, as well as the physical understanding of quantum correlations.

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Appendix A: Proof of Lemma 1

We only need to prove that $F_\alpha(tp + (1 - t)\sigma) \geq tF_\alpha(\rho) + (1 - t)F_\alpha(\sigma)$ for any $t \in [0, 1]$. Since for any concave function $f$, the $\text{Tr}[f(\rho)]$ is also concave [46], $f(x) = x^\alpha$ with $x \in [0, 1]$ is concave for any $0 \leq \alpha \leq 1/2$. Let $\rho' = tp + (1 - t)\sigma = \sum_j q_j |\phi_j\rangle \langle \phi_j|$, $\rho = \sum_i p_i |\psi_i\rangle \langle \psi_i|$ and $\sigma = \sum_k r_k |\xi_k\rangle \langle \xi_k|$ be corresponding eigendecompositions. We have

\[
F_\alpha(\rho') = \sum_j \{ t |\langle \phi_j|\rho|\phi_j\rangle| + (1 - t) |\langle \phi_j|\sigma|\phi_j\rangle|\}^\alpha - 1
\]

\[
\geq \sum_j \{ t |\langle \phi_j|\rho|\phi_j\rangle| + (1 - t) |\langle \phi_j|\sigma|\phi_j\rangle|\} - 1
\]

\[
= t \sum_j \left( \sum_i p_i |\langle \phi_j|\psi_i\rangle|^2 \right)^\alpha
\]

\[
+ (1 - t) \sum_j \left( \sum_k r_k |\langle \phi_j|\xi_k\rangle|^2 \right)^\alpha - 1
\]

\[
\geq t \sum_{i,j} |\langle \phi_j|\psi_i\rangle|^2 p_i^\alpha + (1 - t) \sum_{j,k} |\langle \phi_j|\xi_k\rangle|^2 r_k^\alpha - 1
\]

\[
= t \sum_i p_i^\alpha + (1 - t) \sum_k r_k^\alpha - 1
\]

\[
= tF_\alpha(\rho) + (1 - t)F_\alpha(\sigma),
\]

where the two inequalities follow from the concavity of $f$. The equality holds if $\rho$ and $\sigma$ are identical. $\square$

Appendix B: $\eta_\alpha$ for Isotropic states

Let $T_{iso}$ be the $(U \otimes U^*)$-twirling operator defined by $T_{iso}(\rho) = \int dU(U \otimes U^*) \rho(U \otimes U^*)^\dagger$, where $dU$ denotes the standard Haar measure on the group of all $d \times d$ unitary operations. Then the operator satisfies that $T_{iso}(\rho) = \rho_F(\rho)$ with $F(\rho) = |\Psi\rangle \langle \Psi|$. One has $T_{iso}(\rho_F) = \rho_F$ [25, 27, 33]. Applying $T_{iso}$ to the pure state $|\psi\rangle$ given in (1), $|\psi\rangle = \sum_i a_i \sqrt{X} U_A \otimes U_B |ii\rangle$ with $|a_i\rangle = U_A |i\rangle$ and $|b_i\rangle = U_B |i\rangle$, we have

\[
T_{iso}(|\psi\rangle \langle \psi|) = \rho_F(|\psi\rangle \langle \psi|) = \rho_F(\tilde{x}, \nu),
\]

\[
(B1)
\]
where $V = U_A^t U_B$ and

$$F (x, V) = |\langle \psi | \psi \rangle|^2 = \frac{1}{d} \sum_{i=1}^r \sqrt{\lambda_i} |V_{ii}|^2 \quad (B2)$$

with $V_{ij} = \langle i | V | j \rangle$, and $\tilde{x}$ is the Schmidt vector of (1). Then the function $\eta$ defined in (19) becomes

$$\eta_\alpha (FP) = \min_{\tilde{x}, V} \left\{ C_\alpha (\tilde{x}) : \frac{1}{d} \sum_{i=1}^r \sqrt{\lambda_i} |V_{ii}|^2 = F \right\}. \quad (B3)$$

It has been proved that the minimum above is attained for $V = I$ [27]. Therefore, we have

$$\eta_\alpha (FP) = \min_{\tilde{x}} \left\{ C_\alpha (\tilde{x}) : \frac{1}{d} \sum_{i=1}^r \sqrt{\lambda_i} = \sqrt{FD} \right\}. \quad (B4)$$

For $F \in (0, \frac{1}{2}]$, one can always chose suitable $U_A$ and $U_B$ such that $\lambda_1 = 1$, and hence $\eta_\alpha (FP) = 0$. For $F \in (\frac{1}{2}, 1]$, similar to [24, 39], by using the Lagrange multipliers [26] one can minimize (B4) subject to the constraints

$$\sum_{i=1}^r \lambda_i = 1, \quad \sum_{i=1}^r \sqrt{\lambda_i} = \sqrt{FD} \quad (B5)$$

with $FD \geq 1$. An extremum is attained when

$$\left( \sqrt{\lambda_i} \right)^{2\alpha - 1} + \lambda_1 \sqrt{\lambda_1} + \lambda_2 = 0, \quad (B6)$$

where $\lambda_1$ and $\lambda_2$ denote the Lagrange multipliers.

It is evident that $f (\sqrt{\lambda_i}) = (\sqrt{\lambda_i})^{2\alpha - 1}$ is a convex function of $\sqrt{\lambda_i}$ for any $0 \leq \alpha \leq 1/2$. Since a convex function and a linear function cross at most two points, equation (B6) has at most two possible nonzero solutions for $\sqrt{\lambda_i}$. Let $\gamma$ and $\delta$ be these two positive solutions with $\gamma > \delta$. The Schmidt vector $\tilde{x} = \{\lambda_1, \lambda_2, \ldots, \lambda_r, 0, \ldots, 0\}$ has the form,

$$\lambda_j = \begin{cases} \gamma^2, & j = 1, 2, \ldots, n, \\ \delta^2, & j = n + 1, \ldots, n + m, \\ 0, & j = n + m + 1, \ldots, d, \end{cases} \quad (B7)$$

where $r = n + m \leq d$ and $n \geq 1$. The minimization problem of (B4) has been reduced to the following minimum problem,

$$\eta_\alpha (FP) = \min_{n, m} C_{\alpha}^{nm} (F) \quad (B8)$$

with

$$C_{\alpha}^{nm} (F) = n \gamma^{2\alpha} + m \delta^{2\alpha} - 1, \quad (B9)$$

subject to the constraints

$$n \gamma^2 + m \delta^2 = 1, \quad n \gamma + m \delta = \sqrt{FD}. \quad (B10)$$

By solving Eq. (B10), we obtain

$$\gamma_{nm}^\pm (F) = \frac{n \sqrt{FD} \pm \sqrt{nm(n + m - FD)}}{n(n + m)}, \quad (B11)$$

$$\delta_{nm}^\pm (F) = \frac{m \sqrt{FD} \pm \sqrt{nm(n + m - FD)}}{m(n + m)}. \quad (B12)$$

The relation $\gamma_{nm}^\pm (F) = \delta_{nm}^\mp (F)$ suggests that we only need to consider the cases $\gamma_{nm} := \gamma_{nm}^+ (F)$ and $\delta_{nm} := \delta_{nm}^+ (F)$, which are real for $FD \leq n + m$. On the other hand, since $\delta_{nm}$ should be non-negative, we must have $FD \geq n$. Therefore, we see that $\delta_{nm} \leq \sqrt{FD}/(n + m) \leq \gamma_{nm}$, in consistent with the assumption $\gamma > \delta$. Here, $n \geq 1$ as $n = 0$ is ill defined.

We seek is the minimum of $C_{\alpha}^{nm} (F)$ over all possible $n$ and $m$, by minimizing $C_{\alpha}^{nm}$ on the parallelogram defined by $1 \leq n \leq FD$ and $FD \leq n + m \leq d$. Note that the parallelogram collapses to a line when $FD = 1$, i.e., the separable boundary. We have $\gamma_{nm} \geq \delta_{nm} \geq 0$ in the parallelogram. Moreover, $\gamma_{nm} = \delta_{nm}$ if and only if $n + m = FD$; while $\delta_{nm} = 0$ if and only if $n = FD$.

When $\alpha = 1/2$, we see from Eqs. (B9) and (B10) that Eq. (22) holds without any optimization. When $\alpha = 0$, $C_0^{nm} (F) = n + m - 1$ and Eq. (22) satisfied with the constraint conditions. From Eq.(B10) the derivatives of $\gamma_{nm}$ and $\delta_{nm}$ with respect to $n$ and $m$ are given by,

$$\frac{\partial \gamma}{\partial n} = \frac{1}{2n} \gamma - \frac{\gamma^2}{2n \gamma - \delta}, \quad \frac{\partial \delta}{\partial n} = -\frac{1}{2m} \gamma - \frac{\gamma^2}{2m \gamma - \delta},$$

$$\frac{\partial \gamma}{\partial m} = -\frac{1}{2m} \gamma - \frac{\gamma^2}{2m \gamma - \delta}, \quad \frac{\partial \delta}{\partial m} = \frac{1}{2n} \gamma - \frac{\gamma^2}{2n \gamma - \delta}. \quad (B13)$$

Hence, using Eq. (B9) we have the partial derivatives of $C_\alpha^{nm} (F)$ with respect to $n$ and $m$,

$$\frac{\partial C_\alpha^{nm}}{\partial n} = (1 - \alpha) \gamma^{2\alpha} + \alpha \gamma^2 \delta^{2\alpha - 2} - \delta^{2\alpha - 2} \gamma - \delta, \quad (B14)$$

$$\frac{\partial C_\alpha^{nm}}{\partial m} = (1 - \alpha) \delta^{2\alpha} + \alpha \delta^2 \gamma^{2\alpha - 2} - \delta^{2\alpha - 2} \gamma - \delta. \quad (B15)$$

By lengthy calculations, we have

$$\frac{\partial C_\alpha^{nm}}{\partial m} = \frac{\delta^{2\alpha + 1}}{\gamma - \delta} \left\{ (1 - \alpha) \left( \frac{\gamma}{\delta} - 1 \right) + \alpha \left( \frac{\gamma}{\delta} \right)^{2\alpha - 2} - \frac{\gamma}{\delta} \right\}. \quad (B16)$$

Denote $t = \frac{\gamma}{\delta}$. One has $t \geq 1$. Let $g (t) = (1 - \alpha) (t - 1) + \alpha t^{2\alpha - 1} - \alpha t$. We have $g (1) = 0$ and

$$\frac{\partial g}{\partial t} = (1 - 2\alpha) (1 - \alpha t^{2\alpha - 2}). \quad (B17)$$
Set \(h(t) = 1 - \alpha t^{2\alpha - 2}\) with \(h(1) = 1 - \alpha > 0\). We obtain
\[
\frac{d\eta}{dt} = -\alpha (2\alpha - 2) t^{2\alpha - 3} \geq 0. \quad \text{(B19)}
\]
From Eqs. (B19) and (B18), combining with Eq. (B17) we have
\[
\frac{\partial C_{\alpha}^{nm}}{\partial m} \geq 0. \quad \text{(B20)}
\]
Now corresponding to moving perpendicularly to and parallel to the \(n+m=\text{constant}\) boundaries of the parallelogram, we make a parameter transformation, \(u = n - m\) and \(v = n + m\). The derivative of \(C_{\alpha}^{nm}\) with respect to \(u\) is given by
\[
\frac{\partial C_{\alpha}^{nm}}{\partial u} = \frac{1}{2} \left( \frac{\partial C_{\alpha}}{\partial m} - \frac{\partial C_{\alpha}}{\partial n} \right)
= \frac{1}{2} \left\{ \gamma^{2\alpha-1} \left[ (1-\alpha) \gamma + \alpha \delta \right] - \delta^{2\alpha-1} \left[ (1-\alpha) \delta + \alpha \gamma \right] \right\}
= \frac{\gamma^{2\alpha}}{2} \left\{ 1 - \alpha + \alpha \delta \gamma \right\} \left( 1 - \alpha + \alpha \gamma \right).
\]
Set \(x = \frac{\delta}{\gamma}\) with \(0 \leq x \leq 1\). Let
\[
\eta(x) = 1 - \alpha + \alpha x - x^{2\alpha} (1 - \alpha + \alpha x^{-1}) \quad \text{(B21)}
\]
with \(\eta(1) = 0\). We have the derivative of \(\eta(x)\) respect to \(x\),
\[
\frac{d\eta}{dx} = \alpha \left\{ 1 - \frac{1}{2} x^{2\alpha-1} (1 - \alpha + \alpha x^{-1}) - x^{2\alpha-2} \right\}. \quad \text{(B22)}
\]
Again let
\[
k(x) = 2x^{2\alpha-1} (1 - \alpha + \alpha x^{-1}) - x^{2\alpha-2} \quad \text{(B23)}
\]
with \(k(1) = 1\). Then
\[
\frac{dk}{dx} = x^{2\alpha-3} l(x), \quad \text{(B24)}
\]
where
\[
l(x) = 2 (2\alpha - 1) (x - \alpha x) - (1 - \alpha) (4\alpha - 2). \quad \text{(B25)}
\]
We have \(l(1) = 0\) and
\[
\frac{dl}{dx} = 2 (2\alpha - 1) (1 - \alpha) \leq 0. \quad \text{(B26)}
\]
From Eqs. (B26) and (B24), combining Eq. (B22) we obtain \(\frac{d\eta}{dx} \geq 0\). Then for any \(x \in [0,1]\) we have \(\eta(x) \leq \eta(1) = 0\). Therefore,
\[
\frac{\partial C_{\alpha}^{nm}}{\partial u} \leq 0. \quad \text{(B27)}
\]
From Eqs. (B20) and (B27), the minimum of \(C_{\alpha}^{nm}(F)\) is obtained when \(m\) is the minimum and \(u\) is the maximum. These results imply that the minimum of \(C_{\alpha}^{nm}(F)\) occurs at the vertex of \(n = Fd\) and \(m = 0\). Specifically, since \(\gamma_{nm} = \delta_{nm}\) on the boundary \(n + m = Fd\) where Eqs. (B20) and (B27) are both hold, we have \(n' = n + m = Fd\) and \(m' = 0\). In this way, we derive an analytical expression of the function \(\eta_{\alpha}(\rho_{\mathcal{W}})\) as follows,
\[
\eta_{\alpha}(\rho_{\mathcal{W}}) = (Fd)^{1-\alpha} - 1. \quad \text{(B28)}
\]
### Appendix C: \(\eta_{\alpha}\) for Werner states
Let \(\mathcal{T}_{\mathcal{W}}(\rho) = \int dU (U \otimes U) \rho (U^{\dagger} \otimes U^{\dagger})\) be the \((U \otimes U)\)-twirling transformations [25]. Then the Werner states defined in (24) satisfy that, in analogous to the isotropic states, \(\mathcal{T}_{\mathcal{W}}(\rho) = \rho_{\mathcal{W}(\rho)}\), where \(W(\rho) = \text{Tr} \left( \rho \sum_{i<j} |\Psi_{ij}\rangle\langle\Psi_{ij}| \right)\) and \(\mathcal{T}_{\mathcal{W}}(\rho_{\mathcal{W}}) = \rho_{\mathcal{W}}\) [27, 33]. Applying \(\mathcal{T}_{\mathcal{W}}\) to the pure state \(|\psi\rangle\) defined in (1), \(|\psi\rangle = \sum_{i=1}^{r} \sqrt{\lambda_{i}} U_{A} \otimes U_{B} |i\rangle\), we have
\[
\mathcal{T}_{\mathcal{W}}(|\psi\rangle\langle\psi|) = \rho_{\mathcal{W}(\langle\psi|\psi\rangle)} = \rho_{\mathcal{W}(\tilde{\lambda}, \Lambda)} \quad \text{(C1)}
\]
where \(\Lambda = U_{A}^{\dagger} U_{B}\) and
\[
W(\tilde{\lambda}, \Lambda) = \sum_{i<j} |\langle\Psi_{ij}|\psi\rangle|^{2}
= \frac{1}{2} \sum_{i<j} |\sqrt{\lambda_{i}} \Lambda_{ji} - \sqrt{\lambda_{j}} \Lambda_{ij}|^{2} \quad \text{(C2)}
\]
with \(\Lambda_{ij} = \langle i|\Lambda|j\rangle\). Then the function \(\eta\) defined in (19) becomes
\[
\eta_{\alpha}(\rho_{\mathcal{W}}) = \min_{\tilde{\lambda}, \Lambda} \left\{ C_{\alpha} \left( \tilde{\lambda} \right) : W \left( \tilde{\lambda}, \Lambda \right) = W \right\}. \quad \text{(C3)}
\]
By \(W(\tilde{\lambda}, \Lambda) = W\) we have
\[
2W = 1 - \sum_{i=1}^{r} \lambda_{i} |\Lambda_{ii}|^{2} - 2 \sum_{i<j} \sqrt{\lambda_{i}} \lambda_{j} \text{Re} (\Lambda_{ij} \Lambda_{ji}^{*})
\leq 1 + 2 \sum_{i<j} \sqrt{\lambda_{i}} \lambda_{j} \text{Re} (\Lambda_{ij} \Lambda_{ji}^{*}) \mid
\leq 1 + 2 \sum_{i<j} \sqrt{\lambda_{i}} \lambda_{j}
= \left| \sum_{i=1}^{r} \sqrt{\lambda_{i}} \right|^{2} \quad \text{(C4)}
\]
where \(\text{Re}(z)\) is the real part of \(z\).
Note that the equalities in (C4) hold if only the two no zero components \(\Lambda_{1} = 1\) and \(\Lambda_{1} = -1\), and \(\tilde{\lambda} = (\lambda_{1}, \lambda_{2}, 0, \cdots, 0)\), which give rise to the optimal minimum of (C3) [25]. Therefore, (C3) becomes
\[
\eta_{\alpha}(\rho_{\mathcal{W}}) = \min_{\tilde{\lambda}} \left\{ C_{\alpha} \left( \tilde{\lambda} \right) : \sum_{i=1}^{2} \sqrt{\lambda_{i}}^{2} = 2W \right\}. \quad \text{(C5)}
\]
For \(W \in [0, \frac{1}{2}]\), we can always chose suitable \(U_{A}\) and \(U_{B}\) to have that \(\lambda_{1} = 1\), which results in \(\eta_{\alpha}(\rho_{\mathcal{W}}) = 0\). For \(W > 1/2\), one minimizes (C5) subject to the constraints
\[
\sum_{i=1}^{2} \lambda_{i} = 1, \quad \sum_{i=1}^{2} \sqrt{\lambda_{i}} = \sqrt{2W} \quad \text{(C6)}
\]
with \(W > 1/2\). The rest of the calculation is the similar to the one in Appendix B. We only need to set \(d = 2\) and \(F = W\). In this way, we can obtain
\[
\eta_{\alpha}(\rho_{\mathcal{W}}) = (2W)^{1-\alpha} - 1. \quad \text{(C7)}
\]
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