CORDES CHARACTERIZATION FOR PSEUDODIFFERENTIAL OPERATORS WITH SYMBOLS VALUED ON A NONCOMMUTATIVE C*-ALGEBRA

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Abstract. Given a separable unital C*-algebra $A$ with norm $\| \cdot \|$, let $E$ denote the Banach-space completion of the $A$-valued Schwartz space on $\mathbb{R}^n$ with norm $\| f \|_2 = \|(f, f)\|^{1/2}$, $(f, g) = \int f(x)^* g(x) \, dx$. The assignment of the pseudodifferential operator $B = b(x, D)$ with $A$-valued symbol $b(x, \xi)$ to each smooth function with bounded derivatives $b \in B^A(\mathbb{R}^{2n})$ defines an injective mapping $O$, from $B^A(\mathbb{R}^{2n})$ to the set $\mathcal{H}$ of all operators with smooth orbit under the canonical action of the Heisenberg group on the algebra of all adjointable operators on the Hilbert module $E$. It is known that $O$ is surjective if $A$ is commutative. In this paper, we show that, if $O$ is surjective for $A$, then it is also surjective for $M_k(A)$.

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1. Introduction

Let $A$ be a separable C*-algebra with norm $\| \cdot \|$ and unit $1$, and let $S^A(\mathbb{R}^n)$ denote the set of all $A$-valued smooth (Schwartz) functions on $\mathbb{R}^n$ which, together with all their derivatives, are bounded by arbitrary negative powers of $|x|$, $x \in \mathbb{R}^n$. We equip it with the $A$-valued inner-product

$$\langle f, g \rangle = \int f(x)^* g(x) \, dx,$$

which induces the norm $\| f \|_2 = \|(f, f)\|^{1/2}$, and denote by $E$ its Banach-space completion with this norm. The inner product $\langle \cdot, \cdot \rangle$ turns $E$ into a Hilbert module $[4]$. The set of all (bounded) adjointable operators on $E$ is denoted $B^*(E)$.

Let $B^A(\mathbb{R}^{2n})$ denote the set of all smooth bounded functions from $\mathbb{R}^{2n}$ to $A$ whose derivatives of arbitrary order are also bounded. For each $b$ in $B^A(\mathbb{R}^{2n})$, a linear mapping from $S^A(\mathbb{R}^n)$ to itself is defined by the formula

$$(Bu)(x) = \frac{1}{(2\pi)^{n/2}} \int e^{ix \cdot \xi} b(x, \xi) \hat{u}(\xi) \, d\xi,$$

where $\hat{u}$ denotes the Fourier transform,

$$\hat{u}(\xi) = (2\pi)^{-n/2} \int e^{-iy \cdot \xi} u(y) \, dy.$$

The operator $B := b(x, D)$ extends to an element of $B^*(E)$ whose norm satisfies the following estimate. There exists $K > 0$ depending only on $n$ such that

$$\|B\| \leq K \sup \{ \| \partial^\alpha_{\xi} \partial^\beta_x b(x, \xi) \| ; (x, \xi) \in \mathbb{R}^{2n} \text{ and } \alpha, \beta \leq (1, \cdots, 1) \}.$$

This generalization of the Calderón-Vaillancourt Theorem $[1]$ was proven by Merklen $[8,9]$, see also $[3,10,11]$. 

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The estimate \(\|z\|_2 \geq 2\|z\|_1\) implies that the mapping
\[
\mathbb{R}^{2n} \ni (z, \zeta) \mapsto B_{z,\zeta} = T_{-z}M_z \widetilde{M} \widetilde{T} z \in \mathcal{B}^*(E)
\]
is smooth (i.e., \(C^\infty\) with respect to the norm topology), where \(T_z\) and \(M_z\) are defined by \(T_z u(x) = u(x - z)\) and \(M_z u(x) = e^{i\zeta \cdot x} u(x)\), \(u \in \mathcal{S}(\mathbb{R}^n)\). That follows just as in the scalar case \([3\text{, Chapter 8}]\).

**Definition 1.** We call Heisenberg smooth an operator \(B \in \mathcal{B}^*(E)\) for which the mapping \(O_A\) is smooth, and denote by \(\mathcal{H}\) the set of all such operators.

The elements of \(\mathcal{H}\) are the smooth vectors for the canonical action of the Heisenberg group on \(\mathcal{B}^*(E)\).

We therefore have a mapping
\[
O_A : \mathcal{B}^A(\mathbb{R}^{2n}) \longrightarrow \mathcal{H}
\]
\(b \mapsto b(x, D)\).

It is a standard result that, in the scalar case \((\mathcal{A} = \mathbb{C})\), \(O_A\) is injective. For general \(\mathcal{A}\), injectiveness follows from the scalar case by a duality argument. Cordes \([2]\) proved that \(O_A\) is surjective in the scalar case. We have shown \([7]\) that this also happens if \(\mathcal{A}\) is unital and commutative.

In this paper, we show that, if \(O_A\) is surjective, then \(O_{M_k(\mathcal{A})}\) is also surjective. We show that by first noticing that the Hilbert module \(E_k\) for the matrix case is a Banach-space direct sum of \(k^2\) copies of \(E\). Then it follows that a bounded operator on \(E_k\) (regarded only as a Banach space) is smooth under the action of the Heisenberg group if and only if it is a matrix whose entries are operators on \(E\) which are also smooth under the Heisenberg group. When we impose that such a matrix be an adjointable \(M_k(\mathcal{A})\)-module homomorphism, then we get precisely the pseudodifferential operators of the form \((\mathbb{I})\).

Given a skew-symmetric \(n \times n\) matrix \(J\) and \(F \in \mathcal{B}^A(\mathbb{R}^n)\), let us denote by \(L_F\) the pseudodifferential operator \(a(x, D) \in \mathcal{B}^*(E)\) with symbol \(a(x, \xi) = F(x - J\xi)\). Let us further denote by \(R_F\) the pseudodifferential operator with symbol \(b(x, \xi) = F(x + J\xi)\) defined similarly as in \((\mathbb{I})\), except that \(b(x, \xi)\) multiplies \(\hat{u}(\xi)\) on the right. At the end of Chapter 4 in \([10]\), Rieffel made a conjecture that may be rephrased as follows: any \(B \in \mathcal{H}\) which commutes with every \(R_G, G \in \mathcal{B}^A(\mathbb{R}^n)\), is of the form \(B = L_F\) for some \(F \in \mathcal{B}^A(\mathbb{R}^n)\).

Using Cordes’ characterization of the Heisenberg-smooth operators in the scalar case, we have shown \([3]\) that Rieffel’s conjecture is true when \(\mathcal{A} = \mathbb{C}\). The second author \([8\text{, Theorem 3.5}]\) proved further that Rieffel’s conjecture is true for any separable \(C^*-\)algebra \(\mathcal{A}\) for which the operator \(O_A\) is a bijection.

The assumption of separability of \(\mathcal{A}\) is needed to justify several results about vector-valued integration \([9\text{, Apêndice}]\).

2. **Adjointable operators**

Let us denote by \(E_k\) the Hilbert module obtained using the procedure described in the first paragraph of this paper with \(\mathcal{A}\) replaced by \(M_k(\mathcal{A})\), the \(C^*-\)algebra of \(k\)-by-\(k\) matrices with entries in \(\mathcal{A}\).

Using that the norm \(\|a_{i,j}\|_{\mathcal{B}^*(E)} := \max\{|a_{i,j}|; 1 \leq i, j \leq k\}\) is equivalent to the \(C^*-\)norm \(\|\cdot\|\) of \(M_k(\mathcal{A})\), one easily proves that a given function
\[
f = ((f_{ij}))_{1 \leq i, j \leq k} : \mathbb{R}^n \to M_k(\mathcal{A})
\]
Proposition 1. For each \((l, m)\), \(1 \leq l, m \leq k\), the maps
\[
P_{lm} : S^{M_k(A)}(\mathbb{R}^n) \ni (f_{ij})_{1 \leq i, j \leq k} \mapsto f_{lm} \in S^A(\mathbb{R}^n)
\]
and
\[
I_{lm} : S^A(\mathbb{R}^n) \ni f \mapsto ((\delta_{il}\delta_{jm}f))_{1 \leq i, j \leq k} \in S^{M_k(A)}(\mathbb{R}^n)
\]
\((\delta_{pq} = 1 \text{ if } p = q \text{ and } \delta_{pq} = 0 \text{ if } p \neq q)\) extend continuously to
\[
P_{lm} : E_k \longrightarrow E \text{ and } I_{lm} : E \longrightarrow E_k.
\]
Moreover, \(\|P_{lm}\| = 1\) and \(I_{lm}\) is an isometry.

Proof: For each \(f \in S^A(\mathbb{R}^n)\) and each \((i, j)\), we have:
\[
\left( \int I_{lm}(f)(x)^* I_{lm}(f)(x) \, dx \right)_{ij} = \delta_{im}\delta_{jm} \int f(x)^* f(x) \, dx;
\]
and, then,
\[
\|I_{lm}(f)\|^2 = \|((\delta_{il}\delta_{jm}1))_{1 \leq i, j \leq k} \cdot \|f\|^2 = \|f\|^2.
\]
This shows that \(I_{lm}\) is an isometry.

Given \(f = ((f_{ij}))_{1 \leq i, j \leq k} \in S^{M_k(A)}(\mathbb{R}^n)\), we have:
\[
\|P_{ml}(f)\|^2 = \left\| \int f_{ml}(x)^* f_{ml}(x) \, dx \right\| \leq \left\| \sum_{j=1}^k \int f_{ij}(x)^* f_{ij}(x) \, dx \right\|
\]
(we have used that \(\|a\| \leq \|a + b\|\) if \(a\) and \(b\) are two positive elements of any \(C^*\)-algebra, and that the integral of a positive valued function is also positive). The right-hand side of the previous inequality equals
\[
\left\| \left( \int f(x)^* f(x) \, dx \right)_{il} \right\| \leq \left\| \int f(x)^* f(x) \, dx \right\|_{\infty} \leq \left\| \int f(x)^* f(x) \, dx \right\|.
\]
This shows that \(\|P_{ml}\| \leq 1\). The equality holds because, for any \(g \in S^A(\mathbb{R}^n)\),
\(P_{ml}(I_{ml}(g)) = g\) and \(\|g\|_2 = \|I_{ml}(g)\|_2\).

\[
E_k \ni f \mapsto (P_{ij}(f))_{1 \leq i, j \leq k} \in \bigoplus_{1 \leq i, j \leq k} E
\]
is a Banach space isomorphism. The right action of \(M_k(A)\) on \(E_k\) is then given by matrix multiplication, while the \(M_k(A)\)-valued inner-product on \(E_k\) is given by:
\[
\langle (f_{ij})_{1 \leq i, j \leq k}, (g_{ij})_{1 \leq i, j \leq k} \rangle = \left( \sum_{i=1}^k (f_{ii}, g_{ii}) \right)_{1 \leq i, j \leq k}.
\]

Proof: Using that \(P_{lm}I_{ml}\) equals the identity on \(E\) for every \((l, m)\), that \(P_{lm}I_{pq} = 0\) if \(l \neq p\) or \(m \neq q\) and that \(\sum_{lm} I_{lm}P_{lm}\) equals the identity on \(E_k\), it follows that
\[
\bigoplus_{i,j=1}^k E \ni (f_{ij})_{1 \leq i, j \leq k} \mapsto \sum_{l,m=1}^k I_{lm}(f_{lm}) \in E_k
\]
is an inverse for the map defined in (5). The statements about the action of \(M_k(A)\) and about the inner-product follow by density, since they hold on \(S^{M_k(A)}(\mathbb{R}^n)\). □
Let $\mathcal{L}(E_k)$ denote the algebra of bounded operators on the Banach space $E_k$. In order to describe which elements of $\mathcal{L}(E_k)$ belong to $\mathcal{B}^*(E_k)$ (i.e., which of them are adjointable $M_k(\mathcal{A})$-module homomorphisms), it is convenient to define an isomorphism

$$
\bigoplus_{i,j=1}^{k} E \cong \bigoplus_{p=1}^{k^2} E,
$$

using the bijection $\phi : \{1, \cdots, k\} \times \{1, \cdots, k\} \rightarrow \{1, 2, \cdots, k^2\}$ defined by listing the pairs $(l, m)$ column after column,

$$
\phi(1,1) = 1, \cdots, \phi(k,1) = k, \phi(1,2) = k+1, \cdots, \phi(k,2) = 2k, \\
\cdots, \phi(1,k) = k^2-(k-1), \cdots, \phi(k,k) = k^2.
$$

The composition of the two isomorphisms defined in (5) and (7) induces the isomorphism

$$
E_k \cong \bigoplus_{p=1}^{k^2} E,
$$

which, by its turn, induces

$$
\mathcal{L}(E_k) \ni T \mapsto \left((P_p T I_q)\right)_{1 \leq p,q \leq k^2} \in M_{k^2}(\mathcal{L}(E_k)).
$$

Here, abusing notation, we have written $P_p$ and $I_q$ where we really meant $P_{\phi^{-1}(p)}$ and $I_{\phi^{-1}(q)}$.

The following theorem is purely algebraic and could be stated for general rings and modules.

**Theorem 1.** Using the isomorphism (7) as an identification, a given

$$
T = \left((T_{pq})\right)_{1 \leq p,q \leq k^2} \in \mathcal{L}(E_k)
$$

is a (right) $M_k(\mathcal{A})$-module homomorphism if and only if

$$
T = \begin{bmatrix}
\tilde{T} & 0 & \cdots & 0 \\
0 & \tilde{T} & \cdots & 0 \\
\vdots & \ddots & \ddots & \vdots \\
0 & 0 & \cdots & \tilde{T}
\end{bmatrix},
$$

where $\tilde{T}$ is a $k$-by-$k$ matrix of bounded (right) $\mathcal{A}$-module homomorphisms and 0 denotes the $k$-by-$k$ zero block.

**Proof:** Given $T = \left((T_{pq})\right)_{1 \leq p,q \leq k^2} \in \mathcal{L}(E_k)$, each $T_{pq} = P_p T I_q$ is obviously bounded. If $T$ is an $M_k(\mathcal{A})$-module homomorphism, then $T_{pq}$ is an $\mathcal{A}$-module homomorphism since, for every $a \in \mathcal{A}$ and $f \in E$, we have

$$
I_q(fa) = I_q(f)
$$

$$
\begin{bmatrix}
a & 0 & \cdots & 0 \\
0 & a & \cdots & 0 \\
\vdots & \ddots & \ddots & \vdots \\
0 & 0 & \cdots & a
\end{bmatrix}.$$
Given an integer $l$, $1 \leq l \leq k^2$, let $l_1$ and $l_2$ be the integers defined by $0 \leq l_1 \leq k - 1$, $1 \leq l_2 \leq k$ and $l = kl_1 + l_2$. The product of two matrices can then be expressed by

$$
\begin{bmatrix}
a_1 & a_{1+k} & \cdots & a_{1+(k-1)k} \\
a_2 & a_{2+k} & \cdots & a_{2+(k-1)k} \\
\vdots & \vdots & & \vdots \\
a_k & a_{k+k} & \cdots & a_{k+(k-1)k}
\end{bmatrix}
\begin{bmatrix}
b_1 & b_{1+k} & \cdots & b_{1+(k-1)k} \\
b_2 & b_{2+k} & \cdots & b_{2+(k-1)k} \\
\vdots & \vdots & & \vdots \\
b_k & b_{k+k} & \cdots & b_{k+(k-1)k}
\end{bmatrix}
= 
\begin{bmatrix}
c_1 & c_{1+k} & \cdots & c_{1+(k-1)k} \\
c_2 & c_{2+k} & \cdots & c_{2+(k-1)k} \\
\vdots & \vdots & & \vdots \\
c_k & c_{k+k} & \cdots & c_{k+(k-1)k}
\end{bmatrix}
$$

with

$$c_l = \sum_{j=1}^{k} a_{l_2+k(j-1)} b_j + kl_1.$$

This formula holds if the two matrices that we multiply belong to $M_k(A)$ or if the left one is an element of $E_k$ regarded as a matrix by $[5]$. With this notation, if a given $T = ((T_{pq}))_{1 \leq p,q \leq k^2} \in \mathcal{L}(E_k)$ is an $M_k(A)$-module homomorphism, then, for every $(a_l)_{1 \leq l \leq k^2} \in E_k$ (we now refer to the isomorphism $[5]$), and for every

$$
\begin{bmatrix}
b_1 & b_{1+k} & \cdots & b_{1+(k-1)k} \\
b_2 & b_{2+k} & \cdots & b_{2+(k-1)k} \\
\vdots & \vdots & & \vdots \\
b_k & b_{k+k} & \cdots & b_{k+(k-1)k}
\end{bmatrix}
\in M_k(A),
$$

we have, for every $1 \leq p \leq k^2$,

$$
\sum_{l=1}^{k^2} \sum_{j=1}^{k} T_{pl} a_{l_2+k(j-1)} b_j + kl_1 = \sum_{j=1}^{k} \sum_{l=1}^{k^2} T_{p_2+k(j-1), l} a_l b_{kp_1+j}.
$$

We now apply this equality, for each $a \in E$ and each $(q,r)$, $1 \leq q,r \leq k^2$, to $a_l = \delta_{ql} a$ and $b_l = \delta_{rl} 1$. The only nonvanishing term in the left side sum will satisfy $l_2 + k(j-1) = q$ (hence $j - 1 = q_1$ and $l_2 = q_2$) and $kl_1 + j = r$ (hence $l_1 = r_1$ and $j = r_2$); hence $l = kr_1 + q_2$ and $j = q_1 + 1 = r_2$. The only nonvanishing term in the right side sum will satisfy $l = q$ and $kp_1 + j = r$ (hence $p_1 = r_1$ and $j = r_2$). Equation (11) then becomes

$$T_{p,kr_1+q_2} \delta_{q_1+1,r_2} = T_{p_2+k(r_2-1), q} \delta_{p_1,r_1}.$$

We can now see that, if $q_1 + 1 \neq r_2$ and $p_1 = r_1$, then $T_{p_2+k(r_2-1),q} = 0$. This proves that, for each $(p,q)$, $T_{pq} = 0$ unless $p = p_2 + qk_1$. In other terms, if $p_1 \neq q_1$, then $T_{p,q} = 0$. Therefore, all blocks outside the diagonal in (10) indeed vanish.

Finally, letting $q_1 + 1 = r_2$ and $p_1 = r_1$, we get $T_{p, kp_1+q_2} = T_{p_2+kq_1,q}$, or $T_{kp_1+p_2,kp_1+q_2} = T_{kq_1+p_2,kq_1+q_2}$, proving that all blocks along the diagonal in (10) are indeed equal.
We have proven that any module homomorphism on $E_k$ is of the form \((10)\). To see that the converse is also true, we only need to remark that, under the description of $E_k$ given by Proposition \(2\), the action of a $T$ as in \((10)\) on $E_k$ is given by left multiplication by $T$.

\[ \text{Theorem 2. A given } T = ((T_{pq}))_{1 \leq p,q \leq k^2} \in \mathcal{L}(E_k) \text{ belongs to } B^*(E_k) \text{ if and only if it is of the form } \text{(11)}, \text{ with } \tilde{T} \in M_k(B^*(E)). \]

\[ \begin{align*}
\langle Tf, g \rangle_{ij} &= \sum_{l,m=1}^{k} \langle \tilde{T}_{lm} f_{mi}, g_{lj} \rangle \text{ and } \sum_{l,m=1}^{k} \langle f_{mi}, \tilde{S}_{lm} g_{lj} \rangle = \langle f, Sg \rangle_{ij}.
\end{align*} \]

From this, it follows that, if each $\tilde{T}_{ij}$ is adjointable and if $\tilde{S}_{ij} = T_{ji}^*$ for all $(i,j)$, then $S$ is the adjoint of $T$.

Conversely, suppose that $T$ is adjointable and that its adjoint is $S$. The equality of the two sums in \((12)\) for particular choices of $f$ and $g$ will imply that each $\tilde{T}_{ij}$ is adjointable. \(\square\)

3. **Heisenberg-smooth adjointable operators**

The mapping \((3)\) may be defined for any $B$ in $\mathcal{L}(E)$ or in $\mathcal{L}(E_k)$. It thus makes sense to talk about Heisenberg-smooth operators in $\mathcal{L}(E)$ or in $\mathcal{L}(E_k)$. Given $B = ((B_{pq}))_{1 \leq p,q \leq k^2} \in \mathcal{L}(E_k)$, we have

\[ B_{\zeta} = ((B_{pq}_\zeta))_{1 \leq p,q \leq k^2} = ((P_p B_{\zeta} I_q))_{1 \leq p,q \leq k^2} \]

(it is enough to check these equalities on the dense subset of Schwartz functions). We then get:

\[ \text{Proposition 3. A given } B = ((B_{pq}))_{1 \leq p,q \leq k^2} \in \mathcal{L}(E_k) \text{ is Heisenberg-smooth if and only if each } B_{pq} \in \mathcal{L}(E) \text{ is Heisenberg-smooth.} \]

This leads to our main result:

\[ \text{Theorem 3. If the unital separable C*-algebra } A \text{ is such that the map } O_A \text{ defined in } \text{(2)} \text{ is a bijection, then the map } O_{M_k(A)} \text{ is also a bijection.} \]

\[ \text{Proof: Given any Heisenberg-smooth operator } B \in B^*(E_k), \text{ we have to show that it is of the form } B = b(x,D) \text{ for some } b \in B^{M_k(A)}(\mathbb{R}^{2n}). \text{ Let } ((B_{pq}))_{1 \leq p,q \leq k^2} \text{ be the matrix that corresponds to } B \text{ by the isomorphism } \text{(9). By Proposition } 3 \text{ and by the assumption that } O_A \text{ is surjective, each } B_{pq} \text{ is a pseudodifferential operator of the type defined in } \text{(11). By Theorem } 1 \text{ } B \text{ is of the form } \text{(10). That is, there exist } b_{ij} \in B^A(\mathbb{R}^{2n}), 1 \leq i,j \leq k, \text{ such that, with} \]

\[ \tilde{T} = \begin{bmatrix}
 b_{11}(x,D) & b_{12}(x,D) & \cdots & b_{1k}(x,D) \\
 b_{21}(x,D) & b_{22}(x,D) & \cdots & b_{2k}(x,D) \\
 \vdots & \vdots & \ddots & \vdots \\
 b_{k1}(x,D) & b_{k2}(x,D) & \cdots & b_{kk}(x,D)
\end{bmatrix}, \]

\[ \]
we have
\[
B = \begin{bmatrix}
\tilde{T} & 0 & \cdots & 0 \\
0 & \tilde{T} & \cdots & 0 \\
& \ddots & \ddots & \ddots \\
& & 0 & 0 & \cdots & \tilde{T}
\end{bmatrix}.
\]
This implies that $B$ and $b(x, D)$ are equal, if $b \in \mathcal{B}^{M_k(A)}(\mathbb{R}^{2n}) = M_k(\mathcal{B}^A(\mathbb{R}^{2n}))$ is given by $b = ((b_{ij}))_{1 \leq i, j \leq k}$. Indeed, the equality of the two operators can be easily verified on $S^{M_k(A)}(\mathbb{R}^n)$.

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\section*{References}

[1] A. P. Calderón & R. Vaillancourt, On the boundedness of pseudo-differential operators; J. Math. Soc. Japan 23 (1971), 374-378.
[2] H. O. Cordes, On pseudodifferential operators and smoothness of special Lie-group representations, Manuscripta Math. 28 (1979), 51-69.
[3] H. O. Cordes, The technique of Pseudodifferential Operators, London Mathematical Society Lecture Note Series 202, Cambridge University Press, Cambridge, 1995.
[4] C. Lance, Hilbert $C^*$-modules - A toolkit for operator algebraists, London Mathematical Society Lecture Note Series 210, Cambridge University Press, Cambridge, 1995.
[5] I. L. Hwang, The $L^2$-boundedness of pseudodifferential operators, Trans. Amer. Math. Soc. 302-1 (1987), 55-76.
[6] S. T. Melo & M. I. Merklen, On a conjectured Beals-Cordes-type characterization, Proc. Amer. Math. Soc. 130-7 (2002), 1997-2000.
[7] S. T. Melo & M. I. Merklen, Pseudodifferential operators with $C^*$-algebra-valued symbol: Abstract characterizations, Proc. Amer. Math. Soc. 136-1 (2008), 219-227.
[8] M. I. Merklen, Boundedness of Pseudodifferential Operators of $C^*$-Algebra-Valued Symbol, Proc. Roy. Soc. Edinburgh Sect. A 135-6 (2005), 1279-1286.
[9] M. I. Merklen, Resultados motivados por uma caracterização de operadores pseudodiferenciais conjecturada por Rieffel, Tese de Doutorado, Universidade de São Paulo, 2002, http://arxiv.org/abs/math.OA/0309464.
[10] M. Rieffel, Deformation Quantization for Actions of $\mathbb{R}^d$, Memoirs of the American Mathematical Society 506, 1993.
[11] J. Seiler, Continuity of edge and corner pseudodifferential operators, Math. Nachr. 205 (1999), 163-182.