THE HYDRODYNAMIC LIMIT FOR LOCAL MEAN-FIELD DYNAMICS WITH UNBOUNDED SPINS

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Abstract. We consider the dynamics of a class of spin systems with unbounded spins interacting with local mean-field interactions. We prove convergence of the empirical measure to the solution of a McKean-Vlasov equation in the hydrodynamic limit and propagation of chaos. This extends earlier results of Gärtner, Comets and others for bounded spins or strict mean-field interactions.

1. Introduction and results

In this paper we consider coupled systems of $N \in \mathbb{N}$ stochastic differential equations (sde) of the form

$$d\theta_i^N(t) = -\psi' \left( \theta_i^N(t) \right) dt + \frac{1}{N^d} \sum_{j \in \mathbb{T}_N^d} J \left( \frac{j - i}{N} \right) \theta_j^N(t) dt + dB_i(t), \quad i \in \mathbb{T}_N^d,$$

(1.1)

Here we denote by $\mathbb{T}_N^d \equiv \{1, \ldots, N\}^d$ the $d$-dimensional discrete torus of side-length $N$. $\theta_i^N(t)$ take values in $\mathbb{R}$, $\psi : \mathbb{R} \to \mathbb{R}$ is a local potential that we will assume for simplicity to be a polynomial of degree $2k$ with $k \geq 2$, that is

$$\psi(\theta) = \theta^{2k} + \text{lower order terms}. \quad (1.2)$$

The interaction $J : \mathbb{T}_d \to \mathbb{R}_+$ will be assumed to be a smooth symmetric function on the $d$-dimensional unit torus $\mathbb{T}_d$. Finally, $B_i, i \in \mathbb{N}$ are iid Brownian motions.

We are interested in describing the behaviour of this system in the limit as $N \uparrow \infty$. To do so, we consider the empirical process,

$$\mu_N^t : \mathbb{R}_+ \to \mathcal{M}_1(\mathbb{T}_d \times \mathbb{R}), \quad (1.3)$$

given by

$$\mu_N^t = \frac{1}{N^d} \sum_{k \in \mathbb{T}_N^d} \delta(k/N, \theta_k^N(t)), \quad t \in \mathbb{R}_+. \quad (1.4)$$

In terms of the empirical process, the equations (1.1) can be written as

$$d\theta_i(t)^N = -\psi' \left( \theta_i(t) \right) dt + \int_{\mathbb{T}_d} \int_{\mathbb{R}} J(i/N - y) \theta_{i}^N(dy, d\theta) dt + dB_i(t), \quad i \in \mathbb{T}_N^d, \quad (1.5)$$

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Arguments developed in Section require smoothness and certain growth properties of $\psi'$ at infinity and could be readily extended to a larger class of local potentials.
Now, if $\mu^N$ converges to some measure $\mu$, then it is reasonable to expect that in the limit $N \uparrow \infty$, the $\theta_i$ will be independent diffusions and that their empirical distribution should, by the law of large numbers, converge to a measure $\mu_t(dx,d\theta) = \rho_t(x,d\theta)dx$, where, for any $x \in \mathbb{T}^d$, $\rho_t(x,d\theta)$ is the law of the diffusion

$$d\theta(t) = -\psi'(\theta(t))dt + \int_{\mathbb{T}^d} \int_{\mathbb{R}} J(x-y)\theta_i(dy,d\theta)dt + dB(t). \quad (1.6)$$

This self-consistent equation is called the McKean-Vlasov equation. The models we consider, and in fact an even richer class of models including random interactions and potentials, was studied from the point of view of large deviations by one of us [14] where also an extensive review of the history of these models is given. The main purpose of the present paper is to give a simple and transparent proof of just the law of large numbers (or hydrodynamic limit). Earlier and similar result for more restricted classes of models with strict mean-field interaction (i.e. $J$ constant) goes back to Gärtner [7] and Comets and Eisele [2], see also lecture notes [16] for a comprehensive account. A somewhat non-rigorous derivation in the local mean-field case with bounded spins was given in Katsoulakis et al. [11].

The convergence proof we present here, under assumption of sufficiently regular initial distributions - see Theorem [3.3] below, has two main steps. First, one shows that the associated local mean-field McKean-Vlasov system (L-MV), as specified in the Definition [2.6] below, has a unique solution with good regularity properties. In fact we will show that the measure $\rho_t(x,d\theta)$ is absolutely continuous with respect to Lebesgue measure and has a smooth density that is the solution of a certain partial differential equation. This will be done in Section 2 using a fixed point argument. The remainder of the proof relies on existence, unicity and regularity results for the local mean-field system of equations (L-MV). This will be done by a relative entropy estimate. In Section 3 we prove an additional propagation of chaos result that is also based on appropriate relative entropy estimates, which in their turn rely on Girsanov transforms and regularity results for solutions of (L-MV). In the concluding Section 5 we outline a proof of a large deviation principle for empirical measures $\mu^N$.

2. Local McKean-Vlasov equation

In the sequel we say that a function $f$ is smooth on the closure $\bar{D}$ of an open domain $D$ if $f$ is $C^\infty$ on $D$ with derivatives of all order having continuous extensions to $\bar{D}$.

2.1. Heat kernels for 1-dimensional diffusions. To set up the McKean-Vlasov system in a rigorous way, we consider, for smooth functions $h : \mathbb{R}_+ \to \mathbb{R}$, the sde

$$d\theta(t) = (h(t) - \psi'((\theta(t))))dt + dB(t). \quad (2.1)$$

The solution of this equation is a time-inhomogeneous Markov process whose generator is the closure of the operator

$$L_{h(t)} = \frac{1}{2} e^{2\psi(\theta)} \partial_\theta \left( e^{-2\psi(\theta)} \partial_\theta \right) + h(t)\partial_\theta = L_0 + h(t)\partial_\theta. \quad (2.2)$$

It is useful to consider $L_\theta$ as an operator on the Hilbert space $L^2\left(\mathbb{R}, e^{-2\psi}\right)$, since $L_0$ is a self-adjoint operator on this space. Below, $\langle \cdot, \cdot \rangle_\psi$, and $\| \cdot \|_{L^2(\theta)}$ denote the scalar product and the norm on $L^2\left(\mathbb{R}, e^{-2\psi}\right)$, respectively.
The formal adjoint of \( L_0 \) on \( L^2(\mathbb{R}, e^{-2\theta}) \) acts on functions \( \rho : \mathbb{R}_+ \times \mathbb{R} \) as

\[
(L_0^*\rho)_i(\theta) = (L_0\rho)_i(\theta) - h(t)e^{2\theta(\theta)}\partial_\theta(e^{-2\theta(\theta)}\rho_i(\theta)).
\] (2.3)

Condition (1.2) implies that \( L_0 \) has compact resolvent on \( L^2(\mathbb{R}, e^{-2\theta}) \). It has a smooth transition density (with respect to \( e^{-2\theta} \))

\[
q^0_\theta(\eta, \theta) = \sum_{i=1}^{\infty} e^{-\lambda_i(\theta)}\phi_i(\eta)\phi_i(\theta),
\] (2.4)

where \( \{\phi_i\}_{i \in \mathbb{N}} \) is a complete orthonormal basis of eigenfunctions of \( L_0 \) and \( \lambda_i \) are the corresponding eigenvalues. If

\[
\rho_0 \in \mathcal{D}(L_0) = \left\{ \sum_i a_i\phi_i \mid \text{ with } \sum_i \lambda_i a_i^2 < \infty \right\}
\] (2.5)

is an initial density, then \( \rho_t(\theta) = \langle \rho_0, q^0(\cdot, \theta) \rangle_\theta \) is the density at time \( t \), and it solves the Fokker-Planck equation \( \partial_t \rho_t = L_0 \rho_t \) with initial condition \( \rho_0 \).

We first show that the law of the solution of the sde (2.1) is absolutely continuous with a density that is the unique strong solution of the Fokker-Planck equation associated to the operator \( L_{h(t)}^* \). Namely, there exists a \( C^\infty((0, \infty) \times \mathbb{R}^2, \mathbb{R}) \) kernel (see (2.20) below) \( (t, \theta, \eta) \mapsto q^h_t(\theta, \eta) \), such that the following holds:

**Lemma 2.1.** Let the initial distribution of the diffusion (2.1) be absolutely continuous with respect to the measure \( e^{-2\theta(\theta)}d\theta \) with density \( \rho_0 \). Assume that \( h \) is smooth on \( \mathbb{R}_+ \). Then, for any \( t > 0 \), the distribution of \( \theta(t) \) at time \( t \) is absolutely continuous with respect to the measure \( e^{-2\theta(\theta)}d\theta \) with density \( \rho^h_t \), where,

\[
\rho^h_t(\eta) = \int_{\mathbb{R}} \rho_0(\theta)q^h_t(\theta, \eta)d\theta
\] (2.6)

is the classical solution of the Fokker-Planck equation

\[
\partial_t \rho^h_t = L_{h(t)}\rho^h_t.
\] (2.7)

**Proof.** The proof of this lemma is adapted from an argument by Rogers [15]. It is based on an application of Girsanov’s formula. Define

\[
X(t) = -\int_0^t (\psi' (B(s)) - h(s)) dB(s).
\] (2.8)

Let \( \langle X \rangle_t \) be the quadratic variation process of \( X(t) \). The results in [13, 9] imply that

\[
e^{\langle X(t) \rangle_t/2}
\] (2.9)

is a martingale. Moreover, Girsanov’s formula holds for \( \theta(t) \) in (2.1). Namely, for any bounded and continuous \( f \) on \( \mathbb{R} \),

\[
\mathbb{E}^\theta [f(\theta(t))] = \mathbb{E}^{BM}_\theta \left[ e^{\langle X(t) \rangle_t/2}f(B(t)) \right],
\] (2.10)

where \( \mathbb{P}^{BM} \) is the law of the Brownian motion starting at \( \theta \), and \( \mathbb{P}^\theta \) is the law of the time-inhomogeneous diffusion (2.1).

Set \( \tau_R = \min \{ t : |B(t)| \geq R \} \). Then (2.10) implies:

\[
\mathbb{E}^\theta [f(\theta(t))] = \lim_{R \to \infty} \mathbb{E}^{BM}_\theta \left[ e^{\langle X(t) \rangle_t/2}f(B(t)) \mathbb{1}_{\{\tau_R > t\}} \right].
\] (2.11)
Let $\psi_R$ be a sequence of smooth functions such that
\[ \psi_R(\theta) = \psi(\theta) \text{ if } |\theta| \leq R, \text{ and } \psi_R = \text{const} \text{ on both } (-\infty, -2R] \text{ and } [2R, \infty). \tag{2.12} \]

Note that on $\{ \tau_R > t \}$
\[ X(t) - \frac{1}{2}(X_t) = -\int_0^\tau (\psi_R'(B(s)) - h(s)) dB(s) - \int_0^\tau \frac{1}{2} (\psi_R''(B(s)) - h(s))^2 ds. \tag{2.13} \]

By Ito’s formula,
\[-\int_0^\tau \psi_R'(B(s))dB(s) = \psi_R(\theta) - \psi_R(B(t)) + \frac{1}{2} \int_0^\tau \psi_R''(B(s))ds. \tag{2.14} \]

By partial integration,
\[ \int_0^\tau h(s)dB(s) = h(t)B(t) - h(0)\theta - \int_0^\tau h'(s)B(s)ds. \tag{2.15} \]

In view of (2.11) we conclude that
\[ \mathbb{E}_\theta^h[f(\theta(t))] = \mathbb{E}_\theta^B \left[ e^{tF(B(s), h(s), h'(s))ds} e^{h(t)B(t) - \phi(B(t))} f(B(t)) \right], \tag{2.16} \]

where
\[ F(B, h, h') \equiv \frac{1}{2} \psi''(B) - h'B - \frac{1}{2} (\psi'(B) - h)^2. \tag{2.17} \]

By our assumptions on $\psi$ and $h$ for any $t$ there exists a finite constant $C(t, h)$ such that
\[ \max_{s \leq t, B} F(B, h(s), h'(s)) \leq C(t, h). \tag{2.18} \]

Proceeding as in [15] we infer that
\[ \mathbb{E}_\theta^h[f(\theta(t))] = e^{\phi(\theta)} \int_\mathbb{R} \mathbf{q}_t^h(\theta, \eta)e^{-2\phi(\eta)}f(\eta)d\eta \tag{2.19} \]

where the heat kernel $\mathbf{q}_t^h$ is given by
\[ \mathbf{q}_t^h(\theta, \eta) = \gamma_t(\theta, \eta)\phi_t^h(\theta, \eta), \tag{2.20} \]

with $\gamma_t(\theta, \eta) = \frac{1}{\sqrt{2\pi t}} e^{-(\eta - \theta)^2 / 2t}$, and
\[ \phi_t^h(\theta, \eta) = \mathbb{E}_{0, \eta}^B \left[ e^{h(t)\eta - h(0)\theta + \int_0^t F(B(s), h(s), h'(s))ds} \right] \tag{2.21} \]

is an exponential functional of a Brownian bridge from $\theta$ to $\eta$ in time $t$. Then (2.6) follows.

Next, recall that the Brownian bridge $B$ has the convenient representation
\[ B_s = \theta + \frac{\eta}{t}(\eta - \theta) + \frac{W_s - W_t}{t}, \quad s \in [0, t], \tag{2.22} \]

in terms of a Brownian motion $W$. Using this representation we can rewrite (2.21) as
\[ \phi_t^h(\theta, \eta) = \mathbb{E}_t^B \left[ e^{h(t)\eta - h(0)\theta + \int_0^t F(\theta + \frac{\eta}{t}(\eta - \theta) + (W_s - \frac{1}{t}W_t), h(s), h'(s))ds} \right] \tag{2.23} \]

Again, proceeding as in [15], dominated convergence arguments imply that $\phi_t^h(\theta, \eta)$ is continuously differentiable in $t$ on $(0, \infty)$ and, for every $t > 0$ it is $C^\infty$ in $\theta$ and $\eta$ on $\mathbb{R} \times \mathbb{R}$. From this the claim (2.7) of the lemma follows by a modification of standard computations employed on pp.160-161 of [15].

The proof of the preceding lemma readily yields a bound on the growth of $\rho_t$. □
Lemma 2.2. Under the conditions of Lemma 2.1 assume in addition that the initial density $\rho_0$ is bounded in $L^2(\mathbb{R}, e^{-2\phi})$, that is $||\rho_0||_{2,\phi} < \infty$. Then,

$$||\rho_t^\phi||_{2,\phi} \leq e^{\int_0^t h(s)^2 ds} ||\rho_0||_{2,\phi}^2, \quad (2.24)$$

Proof. Note that (2.18), (2.20) and (2.21) imply the following bound:

$$\sup_{\eta} ||q_t^\phi(\cdot, \eta)||_\infty + \sup_{\eta} ||\partial_\eta q_t^\phi(\cdot, \eta)||_\infty < \infty$$

for any $t > 0$. Hence, if $\rho_0$ is compactly supported, (2.6) implies that,

$$||\rho_t^\phi||_{2,\phi} + ||\partial_\eta \rho_t^\phi||_{2,\phi} < \infty,$$

(2.26)

for any $t > 0$. On the other hand, using (2.7),

$$\frac{d}{dt} ||\rho_t^\phi||_{2,\phi}^2 = -2||\partial_\eta \rho_t^\phi||_{2,\phi}^2 + 2h(t) \int_\mathbb{R} \rho_t^\phi(\eta) \partial_\eta \rho_t^\phi(\eta) e^{-2\phi(\eta)} d\eta \leq \frac{1}{2} h(t)^2 ||\rho_t^\phi||_{2,\phi}^2, \quad (2.27)$$

where the inequality follows from the elementary fact that $ab \leq \frac{a^2}{2} + b^2$, applied with $a = h(t)\rho_t^\phi(\eta)$ and $b = \partial_\eta \rho_t^\phi(\eta)$. Integrating this differential inequality yields (2.24). The general case follows by monotone convergence arguments. □

A further consequence is a Lipschitz bound on the dependence of the densities on the drift. Consider an $L^2(\mathbb{R}, e^{-2\phi})$ initial density $\rho_0$ and let $h(t)$ and $g(t)$ be two time dependent smooth drifts on $\mathbb{R}_+$.

Lemma 2.3. Define $D_t(\eta) = \rho_t^\phi(\eta) - \rho_0^\phi(\eta)$. Then the $L^2(\mathbb{R}, e^{-2\phi})$-norm of $D_t$ satisfies the following upper bound:

$$||\rho_t^\phi(\eta) - \rho_0^\phi(\eta)||_{2,\phi}^2 \leq e^{\int_0^t (g(s)^2 + h(s)^2) ds} ||\rho_0||_{2,\phi}^2 \int_0^t (g(s) - h(s))^2 ds. \quad (2.28)$$

Proof. As in the proof of Proposition 2.2, it would be enough to assume that $\rho_0$ is smooth and compactly supported. Then $||D_t||_{2,\phi}$ and $||\partial_\eta D_t||_{2,\phi}$ are finite for any $t > 0$. Hence

$$\frac{1}{2} \frac{d}{dt} ||D_t||_{2,\phi}^2 = -||\partial_\eta D_t||_{2,\phi}^2 + g(t)\langle \rho_t^\phi, \partial_\eta D_t \rangle_\phi - h(t)\langle \rho_t^\phi, \partial_\eta D_t \rangle_\phi$$

$$= -||\partial_\eta D_t||_{2,\phi}^2 + g(t)\langle D_t, \partial_\eta D_t \rangle_\phi + (g(t) - h(t))\langle \rho_t^\phi, \partial_\eta D_t \rangle_\phi \quad (2.29)$$

$$\leq \frac{g(t)^2}{2} ||D_t||_{2,\phi}^2 + \frac{(g(t) - h(t))^2}{2} ||\rho_t^\phi||_{2,\phi}^2.$$ 

Since $D_0 = 0$, the bound (2.28) follows from (2.24). □

2.2. Strong form of the local McKean-Vlasov equation. We can now formulate the McKean-Vlasov problem. To do so, we define the set $\mathcal{A}$ of admissible drift fields $h : T^d \times \mathbb{R}_+ \to \mathbb{R}$:

$$\mathcal{A} \equiv C^{0,\infty}(T^d \times \mathbb{R}_+, \mathbb{R}). \quad (2.30)$$

Next, define the set $\mathcal{B}$ of admissible density fields.

Definition 2.4. A density $\rho(x, \cdot)$ is a nice profile if it is smooth in $\eta$, and continuous in $x$ as a map from $T^d$ to $L^2(\mathbb{R}, e^{-\phi})$. In particular, nice profiles satisfy $\max_x ||\rho(x, \cdot)||_{2,\phi} < \infty$.

We denote by $\mathcal{B}$ the set of density fields $\rho : T^d \times \mathbb{R}_+ \times \mathbb{R} \mapsto \mathbb{R}$ which satisfy:

(i) $\rho_t(x, \cdot)$ is nice for any $t \in \mathbb{R}_+$. 

(ii) For any $x \in \mathbb{T}^d$, $\rho_t(x, \eta)$ are $C^{\infty, \infty}(\mathbb{R}_+ \times \mathbb{R})$ in $(t, \eta)$.

**Remark 2.5.** Continuity in $x$ in the above definition of nice profiles is redundant, and we assume it for convenience and for clarity of exposition. Theorem 2.7 and Theorem 3.3 hold if, instead of continuity, one assumes measurability and boundness - $\sup_{x \in \mathbb{T}^d} \|\rho(x, \cdot)\|_{L^\infty} < \infty$.

**Definition 2.6.** Given initial density $\rho_0(x, \theta)$ a strong solution of the McKean-Vlasov equation (1.6) is a pair $(\rho, h)$, with $\rho \in \mathcal{B}$ and $h \in \mathcal{A}$, such that

$$\forall \ x \in \mathbb{T}^d \text{ and } \forall \ t \in [0, \infty) \begin{cases} \rho_t(x, \cdot) = \rho_t^{h_t} \text{ with initial condition } \rho_0(x, \cdot) \\ h^t(\eta) = \int \int J(y - x)\rho_t(y, \eta)e^{-2\psi(\eta)}d\eta dy \end{cases} \text{ (L-MV)}$$

Above we continue to use $\rho_t^{h_t}$ for the density at time $t$ of the time inhomogeneous diffusion with generator $L_{h_t}$.

The following theorem asserts the existence and uniqueness of the McKean-Vlasov problem.

**Theorem 2.7.** Assume that $\rho_0(x, \cdot)$ is a nice initial profile. Then there exists a unique strong solution $(\rho, h)$ of the system (L-MV).

An equivalent reformulation of the theorem is to say that for any $T$ fixed a unique strong solution exists on any time interval $[0, T]$. The proof of the latter is based on contraction properties of the map $\Phi$ which we construct below.

2.3. **The map $\Phi$.** Fix an initial density $\rho_0(x, \theta)$ which satisfies the assumptions of Theorem 2.7. We define a map $\Phi : \mathcal{A} \to \mathcal{A}$ by

$$\Phi[h]^t(t) \equiv \int \int_{\mathbb{T}^d} J(y - x)\rho_t^{h_t}(y, \eta)e^{-2\psi(\eta)}d\eta dy. \quad (2.31)$$

It is useful to view this map as the composition of two maps,

$$\Phi_1 : \mathcal{A} \to \mathcal{B},$$

where, for $h \in \mathcal{A}$,

$$(\Phi_1(h))^t(t) \equiv \rho_t^{h_t}, \quad (2.33)$$

and

$$\Phi_2 : \mathcal{B} \to \mathcal{A},$$

where, for $\rho \in \mathcal{B}$,

$$(\Phi_2(\rho))^t(t) \equiv \int \int_{\mathbb{T}^d} J(y - x)\rho_t^{h_t}(y, \eta)e^{-2\psi(\eta)}d\eta dy. \quad (2.35)$$

Clearly, $\Phi(h) = \Phi_2 \circ \Phi_1(h)$. The fact that $\Phi_1$ maps $\mathcal{A}$ into $\mathcal{B}$ follows from the proof of Lemma 2.1 specifically from (2.6) and (2.20), and from Lemmas 2.2–2.3. The fact that $\Phi_2$ maps $\mathcal{B}$ into $\mathcal{A}$ follows readily from its definition and the smoothness of $J$. Therefore the composite map $\Phi$ maps $\mathcal{A}$ into $\mathcal{A}$, i.e., if $h$ is an admissible drift field then $\Phi[h]$ is also an admissible drift field.

As a first step we prove the following a priori bounds.
Lemma 2.8. Assume that \( h \in \mathcal{H} \) satisfies
\[
\|h\|_\infty \equiv \sup_{t} \max_{x} |h^i(t)| < \infty. \tag{2.36}
\]
Then,
\[
|h|_\Phi \equiv \sup_{n} \|\Phi^n[h]\|_\infty < \infty. \tag{2.37}
\]

Proof. It is enough to prove the statement of Lemma 2.8 for one-sided quantities
\[
|h|_+ \equiv \sup_{t} \max_{x} h^i(t) \quad \text{and} \quad |h|_{\Phi,+} \equiv \sup_{n} \|\Phi^n[h]\|_+. \tag{2.38}
\]
Recalling (2.6) and (2.19), we see that
\[
\Phi[h]^i(t) = \int_{\mathbb{R}^d} \int_{\mathbb{R}} J(y - x) \rho_{0}(y, \theta) e^{-\Phi^0(y, \theta)} d\theta dy. \tag{2.39}
\]
The idea behind the proof is that if \( |h|_+ \) is very large, then the strong inward drift due to the potential \( \psi \) will ensure that \( \mathbb{E}^H_n(\theta(t)) \ll |h|_+ \), which in turn implies that \( \|\Phi[h]\|_+ \) will be smaller than \( |h|_+ \). This implies that \( \|\Phi^n[h]\|_+ \) cannot grow indefinitely with \( n \), which is the assertion of the theorem.

To prove this, define
\[
\theta^*(h) = \sup \left\{ \theta : \exists y \text{ such that } \mathbb{E}^H_n(\theta(t)) \geq \theta \right\} \vee 0. \tag{2.40}
\]
Since by assumption (1.2) on \( \psi \), the derivative \( \psi'(\theta) \) tends to \( \infty \) as \( \theta \) grows, it is easy to deduce from comparison results for one dimensional diffusions (the proof of (2.43) below gives a quantitative bound along these lines) that \( \theta^*(h) \) is finite whenever \( |h|_+ < \infty \). Then, since for any \( \theta \) and \( t \), \( \mathbb{E}^H_n(\theta(t)) \leq |\theta| \vee \theta^*(h) \),
\[
|\Phi[h]|_+ \leq \theta^*(h) \hat{J}_0 + c(\rho_0), \tag{2.41}
\]
where \( \hat{J}_0 := \int_{\mathbb{R}^d} |J(x)| dx \), and
\[
c(\rho_0) = \max_{\theta} \int_{\mathbb{R}^d} \int_{\mathbb{R}} J(y - x) \rho_{0}(y, \theta) |\theta| e^{-\Phi^0(y, \theta)} d\theta dy \leq \hat{J}_0 \|\theta\|_{L^2,\Phi} \max_{x} \|\rho_{0}(x, \cdot)\|_{L^2,\Phi}. \tag{2.42}
\]
The term \( c(\rho_0) \) is just a finite constant and does not depend on \( h \), and hence is irrelevant. What we need to show is that \( \theta^*(h) \) becomes much smaller than \( |h|_+ \), as \( |h|_+ \) grows, i.e. that
\[
\lim_{H \to \infty} \sup_{|h|_+ = H} \frac{\theta^*(h)}{H} = 0. \tag{2.43}
\]
Clearly, from the discussion above, (2.43) implies that \( \{\|\Phi^n[h]\|_+\} \) is a bounded sequence, and (2.37) follows.

It remains to prove (2.43). The argument is based on the following comparison result for one-dimensional diffusions: For \( i = 1, 2 \) consider
\[
d\theta^i_t = dB(t) + b^i(t, \theta) dt. \tag{2.44}
\]
Assume that the fields \( b^i \) are continuous in \( t \) and smooth in \( \theta \) and that
\[
\forall t, \theta \quad b^1(t, \theta) \leq b^2(t, \theta) \quad \text{and} \quad \lim_{|\theta| \to \infty} \sup_{t} \text{sign}(\theta) b^2(t, \theta) < 0. \tag{2.45}
\]
Then, for any \( t \) and \( \theta \),
\[
\mathbb{E}^\Phi(\theta^1_t) \leq \mathbb{E}^\Phi(\theta^2_t). \tag{2.46}
\]
Indeed, let $\tau_n$ be the first exit time from $[-n, n]$. The second condition in (2.45) implies that $\mathbb{E}_\theta(\eta^2) = \lim_{n \to \infty} \mathbb{E}_\theta(\eta^2_{\tau_n})$. On the other hand, by the usual Yamada comparison result; see for instance Proposition 5.2.18 in [10],

$$
\mathbb{E}_\theta(\eta^2_{\tau_n}) \leq \mathbb{E}_\theta(\eta^2_{\tau_{n+1}}),
$$

(2.47)

for any $n, t$ and $\theta$. Consequently, in order to prove (2.43) we may substitute $-\psi'(\theta) + h(t)$ by a larger drift $b(t, \theta)$ which satisfies the second condition in (2.45), and, furthermore, we may choose some $\bar{\theta}$ and consider reflected diffusions on $[\bar{\theta}, \infty)$. The assumption (1.2) on $\psi$ implies that there exists a $\theta_0 \in \mathbb{R}_+$ such that $\psi$ is strictly convex on $[\theta_0, \infty)$. Consequently, if $|h| = H$, there exists $\eta \in \mathbb{R}_+$, such that for all $\theta \in [\eta, \infty)$,

$$
- \psi'(\theta) + h(t) \leq -\psi'(\theta) + H \leq -1.
$$

(2.48)

For instance (2.48) holds if we choose $\eta \sim 2\sqrt{H}$. If we denote by $\bar{\theta}_{\eta}$ the reflection at $\eta$ of the Brownian motion with unit negative drift, then

$$
\mathbb{E}_\theta(\bar{\theta}_{\eta}) \geq \mathbb{E}_\theta(\theta(t)).
$$

(2.49)

for any $|h| \leq H$ and for any $t \geq 0, \theta \geq \eta$.

The reflected diffusion $\bar{\theta}_{\eta}$ is positively recurrent and its invariant distribution has density $f_\eta(\theta) = 2e^{-2\sqrt{\theta - \eta}} \mathbb{1}_{\theta > \eta}$. Hence, by monotone coupling,

$$
\sup_{t} \mathbb{E}_\theta(\bar{\theta}_{\eta}) \leq \int_{\eta}^{\infty} \theta f_\eta(\theta)d\theta = \eta + \frac{1}{2}.
$$

(2.50)

Consequently, for any $t$ and for any $\theta > \eta$,

$$
\mathbb{E}_\theta(\bar{\theta}_{\eta}) = \mathbb{E}_\theta(\bar{\theta}_{\eta_{\tau_n}}) + \mathbb{E}_\theta(\bar{\theta}_{\eta} - \bar{\theta}_{\eta_{\tau_n}}) = \mathbb{E}_\theta(\bar{\theta}_{\eta_{\tau_n}}) + \mathbb{E}_\theta(\bar{\theta}_{\eta} - \bar{\theta}_{\eta_{\tau_n}}) \mathbb{1}_{\{\tau_n \leq t\}}
$$

$$
= (\theta - \mathbb{E}_\theta(\bar{\theta} \wedge \tau_\eta)) + \mathbb{E}_\theta(\bar{\theta}_{\eta} - \eta) \mathbb{1}_{\{\tau_n \leq t\}} \leq (\theta - \mathbb{E}_\theta(\bar{\theta} \wedge \tau_\eta)) + \frac{1}{2} \mathbb{E}_\theta(\tau_\eta \leq t).
$$

(2.51)

However, $-\mathbb{E}_\theta(\bar{\theta} \wedge \tau_\eta) + \frac{1}{2} \mathbb{E}_\theta(\tau_\eta \leq t)$ is non-positive for all $t \geq 0$ as soon as $\theta - \eta \geq K_1$ some sufficiently large constant $K_1$. In particular, if we chose $\eta \sim 2\sqrt{H}$, we conclude that there exists a finite constant $K_2$, such that $\mathbb{E}_\theta(\bar{\theta}_{\eta}) < \theta$, for all $t > 0$ and all $\theta > K_2$. The target (2.43) follows.

Differentiating both sides of (2.31) with respect to $t$ and relying on (2.7), we observe:

$$
\partial_t \Phi^{\eta+1}[h]^\ast(t) = \int_{\mathbb{T}^d} \int_{\mathbb{R}} J(y - x) (L_{\Phi^{\eta+1}[h]}(\eta) \rho_t^{\phi_{\eta+1}[h]}(\eta)) e^{-2\phi(\eta)} d\eta dy
$$

$$
= \int_{\mathbb{T}^d} \int_{\mathbb{R}} J(y - x) (\Phi^{\eta+1}[h]^\ast(t) - \psi'(\eta)) \rho_t^{\phi_{\eta+1}[h]}(\eta) e^{-2\phi(\eta)} d\eta dy.
$$

(2.52)

We can continue differentiating with respect to $t$ in (2.52). For instance,

$$
\partial_t^2 \Phi^{\eta+1}[h]^\ast(t) = \int_{\mathbb{T}^d} \int_{\mathbb{R}} J(y - x) (\partial_t \Phi^{\eta+1}[h]^\ast(t) - L_{\Phi^{\eta+1}[h]}(\psi'(\eta)) \rho_t^{\phi_{\eta+1}[h]}(\eta)) e^{-2\phi(\eta)} d\eta dy.
$$

(2.53)

By our assumption (1.2) all integrals of the form $\int_{\mathbb{R}} (\psi^{[k]}(\eta))^\ast e^{-2\phi(\eta)} d\eta$ are finite. By iteration, and in view of (2.24), we arrive to the following conclusion:
Lemma 2.9. For every $k \in \mathbb{N}$ there exists a monotone function $c_k$ on $[0, \infty)^2$, such that the following happens: If an admissible drift field $h \in \mathcal{A}$ satisfies (2.36) and hence, by Lemma 2.8 also (2.37), then, for every $T \geq 0$ and each $n \geq k$,

$$\max_{y \in T^d} \max_{\varphi \in \mathbb{B}} |\partial^k \Phi^y[h^v](s)| \leq c_k(T, |h|_\Phi).$$

(2.54)

The next lemma shows that $\Phi$ is a contraction for short times.

Lemma 2.10. Let $h, g \in \mathcal{A}$. Then, for any $t \in \mathbb{R}_+$,

$$\|\Phi[g](t) - \Phi[h](t)\|_{L^2(T^d)}^2 \leq C'_t e^{(|h|_\Phi + |g|_\Phi) t} \int_0^t \|g(s) - h(s)\|_{L^2(T^d)}^2 ds,$$

(2.55)

with

$$C'_t = \sup_{x \in T^d} \|\rho_0^x\|_{L^2}^2 \int_{T^d} \int_\mathbb{R} J(y) y^2 e^{-2|\eta|} d\eta dy.$$

(2.56)

Proof. First, for any $x$ and for any $t > 0$, using the Cauchy-Schwartz inequality, we obtain that

$$|\Phi[h]^y(t) - \Phi[g]^y(t)| \leq C_J \int_{T^d} \|\rho_0^x - \rho_t^y\|_{L^2}^2 dy,$$

(2.57)

where

$$C_J = \int_{T^d} \int_\mathbb{R} J(y) y^2 e^{-2|\eta|} d\eta dy.$$

(2.58)

By Lemma 2.3,

$$\|\rho_t^x - \rho_t^y\|_{L^2}^2 \leq e^{(x^2 + h^v(s)^2) \delta t} \|\rho_0^x\|_{L^2}^2 \int_0^t \|g^v(s) - h^v(s)\|^2 ds.$$  \hspace{1cm} (2.59)

We now use Lemma 2.8 to bound the right-hand side by

$$e^{(x^2 + h^v(s)^2) \delta t} \|\rho_0^x\|_{L^2}^2 \int_0^t (g^v(s) - h^v(s))^2 ds.$$  \hspace{1cm} (2.60)

Integrating the resulting bound over the torus yields the assertion of the lemma. \hfill $\Box$

We are now ready to prove the theorem.

Proof. (of Theorem 2.7) Let $h, g \in \mathcal{A}$ be any two initial drift fields satisfying $||h||_\infty, ||g||_\infty < \infty$. We want to show that the sequences $\Phi^y[h]$ and $\Phi^[y][g]$ converge to the same fix-point of $\Phi$. Iterating the bound in (2.55) gives

$$\sup_{s \leq t} \|\Phi^y[h](s) - \Phi^y[g](s)\|_{L^2(T^d)}^2 \leq \frac{(t C'_t e^{(|h|_\Phi + |g|_\Phi) t})^n}{n!} \sup_{s \leq t} \|h(s) - g(s)\|_{L^2(T^d)}^2.$$  \hspace{1cm} (2.61)

Therefore, if $\hat{h} = \lim_{n \to \infty} \Phi^y[h]$ exists for some $||.||_\infty$-bounded $h \in \mathcal{A}$, then $\hat{h} = \lim_{n \to \infty} \Phi^y[g]$ for any $||.||_\infty$-bounded $g \in \mathcal{A}$. That is existence would imply uniqueness.

However, plugging a $||.||_\infty$-bounded $h \in \mathcal{A}$ and $g = \Phi[h]$ into (2.61) implies that $\Phi^y[h]$ is a Cauchy sequence, and hence indeed converges to some $||.||_\infty$-bounded element $\hat{h} \in \mathcal{A}$, such that for any $t < \infty$,

$$\lim_{n \to \infty} \sup_{s \leq t} \|\Phi^y[h](s) - \hat{h}(s)\|_{L^2(T^d)}^2 ds = 0.$$  \hspace{1cm} (2.62)

Moreover, $\hat{h}$ is a fixpoint of $\Phi$. Therefore $\hat{\rho} = \Phi_1(\hat{h})$ satisfies $\hat{h} = \Phi_2(\hat{\rho})$. This is precisely (L-MV). By (2.6) and Lemma 2.9, $\hat{\rho}_1(x, \eta)$ is continuous in $x$ and smooth in $\eta$ and $t$. \hfill $\Box$
3. HYDRODYNAMIC LIMIT (HDL)

Having established the existence of a unique smooth solution to the McKean-Vlasov equations, we now prove the convergence of the empirical process of our particle system to this solution. The proof is based on entropy estimates.

Let $\rho_0(x, \theta)$ be a nice initial profile, and let $(\rho, h)$ be the unique strong solution to (L-MV) with initial condition $\rho_0$. In the sequel, elements of $\mathbb{R}^{d\theta}$ will be denoted as $\theta$. For each $N$ and $t \geq 0$ consider the following product density on $\mathbb{R}^{N\theta}$ with respect to $e^{-2\phi(\theta)}$:

$$
\rho^N_t(\theta) = \prod_{i=1}^{N^d} \rho^i \left( \frac{t}{N}, \theta^i \right).
$$

(3.1)

Another way to think about $\rho^N_t$ is as follows: For each $i = 1, \ldots, N$ define $h^i_t \equiv h^i_t$, where $h^i_t$ satisfies the second of (L-MV).

Consider

$$
d\hat{\theta}_t(t) = dB_t(t) - \psi'(\hat{\theta}_t(t))\,dt + h^i_t\,dt.
$$

(3.2)

Then $\rho^i_\theta(t)$ is the density of $\hat{\theta}_t$. (t)

Let us turn to the microscopic dynamics (1.1). Let $f^N_0(\theta)$ be the initial density of $\theta(0)$ on $\mathbb{R}^{N\theta}$ with respect to $e^{-2\phi(\theta)}$. We assume that the relative entropy

$$
\mathcal{H} \left( f^N_0 \mid \rho^N_0 \right) := \int f^N_0(\theta) \ln \left( \frac{f^N_0(\theta)}{\rho^N_0(\theta)} \right) e^{-2\phi(\theta)} \,d\theta.
$$

(3.3)

satisfies

$$
\lim_{N \to \infty} \frac{1}{N^d} \mathcal{H} \left( f^N_0 \mid \rho^N_0 \right) = 0.
$$

(C)

Our first proposition states that this property is conserved in time.

**Proposition 3.1.** Let $\mathbb{P}^N_T$ be the distribution on $\mathcal{C}([0, T], \mathbb{R}^{N\theta})$ of the diffusion process \( (1.1) \) with initial density $f^N_0$, and let $\mathbb{P}^N_T$ be the distribution of the decoupled process (3.2) with the initial density $\rho^N_0$. Then, assuming (C)

$$
\lim_{N \to \infty} \frac{1}{N^d} \mathcal{H} \left( \mathcal{P}^N_T \mid \hat{\mathcal{P}}^N_T \right) = 0,
$$

(3.4)

for any $T \geq 0$.

The proof of this proposition is given in the next subsection.

Consider now the empirical profiles $\mu^N_t$ defined in (1.3). For each $t$ the random measure $\mu^N_t$ is a probability measure on $\mathbb{R} \times T^d$. We denote the latter space as $\mathcal{M}_1(T^d \times \mathbb{R})$. It is a Polish space, and in the sequel we shall use $d_{L^p}$ for its Lévy-Prohorov type metric,

$$
d_{L^p}(\mu, \nu) := \sup_{f \in \text{Lip}(T^d \times \mathbb{R})} \left| \int f(x, \theta) \mu(\,dx, d\theta) - \int f(x, \theta) \nu(\,dx, d\theta) \right|,
$$

(3.5)

where $\text{Lip}(T^d \times \mathbb{R})$ is the set of all globally 1-Lipschitz functions $f$ with $||f||_\infty \leq 1$.

For $t \leq T$ we can think of $\mu^N_t$ in terms of marginals of the random measures on path space: For each $N$ and every $i \in T^d$, the trajectory $\theta^N_i := \theta^N_i[0, T]$ in (1.1) is a random element of $\mathcal{C}([0, T])$. Define

$$
L^N_x(dx, d\theta) = \frac{1}{N^d} \sum_i \delta_{(i/N, \theta^N_i[0,T])},
$$

(3.6)
Then, \( L_T^N \) is a random element of \( \mathbb{M}_1 \left( \mathbb{T}^d \times C \left( \left[ 0, T \right], \mathbb{R} \right) \right) \). The map

\[
L_T^N \mapsto \mu^N[0, T] := \left( \mu_t^N ; \ t \in \left[ 0, T \right] \right)
\]

is continuous from \( \mathbb{M}_1 \left( \mathbb{T}^d \times C \left( \left[ 0, T \right], \mathbb{R} \right) \right) \) to \( C \left( \left[ 0, T \right], \mathbb{M}_1(\mathbb{R} \times \mathbb{T}^d) \right) \).

With a slight abuse of notation we continue to use \( \hat{P}_T^N \) and, respectively, \( \mathbb{P}_T^N \) for the distributions of \( L_T^N \) on \( \mathbb{M}_1 \left( \mathbb{T}^d \times C \left( \left[ 0, T \right], \mathbb{R} \right) \right) \) and \( \mu^N[0, T] \) on \( C \left( \left[ 0, T \right], \mathbb{M}_1(\mathbb{R} \times \mathbb{T}^d) \right) \), whenever \( \theta \) is the decoupled diffusion \( \text{(3.2)} \) or, respectively, if \( \theta \) satisfies the local mean-field sde \( \text{(1.1)} \).

Our next proposition states an exponential concentration bound for the \textit{decoupled} measure \( \hat{P}_T^N \).

**Proposition 3.2.** Let \( \rho_0 \) be a nice initial profile. Then for any \( T < \infty \) and \( \epsilon > 0 \) there exists a positive constant \( C_T(\epsilon) > 0 \), such that

\[
\hat{P}_T^N \left( \max_{t \in [0,T]} d_{L_\rho} \left( \mu_t^N, \rho_t(x, \theta)e^{-2\rho(\theta)}d\theta \right) \geq \epsilon \right) \leq e^{-N^dC_T(\epsilon)},\tag{3.8}
\]

for all \( N \) large enough.

This proposition will be proven in Subsection 3.2 below.

We now have the tools to prove convergence to the hydrodynamic limit.

**Theorem 3.3.** Let \( \rho_0 \) be a nice initial profile. Under Assumption \( C \) the distribution \( \mathbb{P}_T^N \) of \( \mu^N[0, T] \), on \( C \left( \left[ 0, T \right], \mathbb{M}_1(\mathbb{R} \times \mathbb{T}^d) \right) \) converges to \( \delta_{\rho, \infty} \) in the following sense: For any \( \epsilon > 0 \) and \( T < \infty \),

\[
\lim_{N \to \infty} \mathbb{P}_T^N \left( \max_{t \in [0,T]} d_{L_\rho} \left( \mu_t^N, \rho_t(x, \theta)e^{-2\rho(\theta)}d\theta \right) \geq \epsilon \right) = 0.\tag{3.9}
\]

**Proof.** The entropy inequality (c.f. [17]) states that for any event \( A \) or, more generally, for any random variable \( X \),

\[
\mathbb{P}_T^N (A) \leq \frac{\log 2 + \mathcal{H} \left( \mathbb{P}_T^N \mid \hat{P}_T^N \right)}{\log \left( 1 + 1/\hat{P}_T^N(A) \right)},\tag{3.10}
\]

respectively

\[
\mathbb{E}_T^N (X) \leq \mathcal{H} \left( \mathbb{P}_T^N \mid \hat{P}_T^N \right) + \log \hat{P}_T^N (e^X).\tag{3.11}
\]

Using this with \( A = A_N \) the event considered in \( \text{(3.8)} \), and inserting the assertions of Proposition 3.1 and Proposition 3.2 into the right-hand side of this inequality immediately yields \( \text{(3.9)} \). Namely, by Proposition 3.2 \( \log \left( 1 + 1/\hat{P}_T^N(A_N) \right) \geq N^dC_T(\epsilon) \), and \( \text{(3.4)} \) applies. \( \square \)

It remains to prove Proposition 3.1 and Proposition 3.2. This is the content of Subsection 3.1 and Subsection 3.2 respectively.

### 3.1. Proof of Proposition 3.1

Set

\[
h(\theta) = \frac{1}{N^d} \sum_j J \left( \frac{j - i}{N} \right) \theta_j.\tag{3.12}
\]

By Girsanov’s formula,

\[
\mathcal{H} \left( \mathbb{P}_T^N \mid \hat{P}_T^N \right) = \mathcal{H} \left( \mathcal{L}_0^N \rho_0^N \right) + \frac{1}{2} \mathbb{E}_T^N \sum_i \int_0^T \left( h(\theta_s) - h_s \right)^2 ds.\tag{3.13}
\]
Hence,
\[
\frac{d}{dt} \mathcal{H}(\mathbb{P}_t^N \| \hat{\mathbb{P}}_t^N) = \frac{1}{2} \mathbb{E}_t^N \sum_i (h'_i(\theta) - \hat{h}_i^N)^2.
\] (3.14)

Since \((\rho, h)\) is a strong solution to (L-MV), both \(\rho\) and \(h\) are continuous in \(x\). Recall that \(J\) is also assumed to be continuous. It follows that
\[
h'_i = \left\{ \frac{1}{N^d} \sum_j J \left( \frac{j-i}{N} \right) \hat{\mathbb{P}}_t^N (\theta_j(t)) \right\} + o(1) = \hat{\mathbb{P}}_t^N (h'(\theta)) + o(1),
\] (3.15)
Theorem 2.7 the densities (with respect to \(e^{-\phi(\theta)}\)) of \(\eta_i^N\) satisfy the following property: There exists a finite constant \(C = C(t, \rho_0) < \infty\) such that
\[
\max_i \|q_i^N\|_{2,\rho} \leq C.
\] (3.21)

In such circumstances the following holds.

**Lemma 3.4.** For each \(t > 0\) there exists \(\delta > 0\) such that
\[
\lim_{N \to \infty} \frac{1}{N^d} \log \hat{\mathbb{P}}_t^N (e^{\delta N^d X_i}) = 0,
\] (3.22)
uniformly in \(t \leq T\).

The claim of Theorem 3.1 is now straightforward. It remains to prove Lemma 3.4.

**Proof of Lemma 3.4.** In view of our basic assumption (1.2), the uniform bound (3.21) implies that random variables \(\eta_i^N\) are uniformly sub-Gaussian: there exists \(\sigma = \sigma_t < \infty\), such that, for any \(\alpha \in \mathbb{R}\),
\[
\max_{i \in \mathbb{T}_N^d} \hat{\mathbb{P}}_t^N \left( e^{\alpha \eta_i^N} \right) \leq e^{\sigma \alpha^2}.
\] (3.23)

Furthermore, there exists \(\kappa_0 > 0\) and a finite convex function \(g_t\) on \((-\kappa_0, \kappa_0)\) with \(g_t(0) = 0\) such that, for any \(|\kappa| \leq \kappa_0\),
\[
\max_{i \in \mathbb{T}_N^d} \hat{\mathbb{P}}_t^N \left( e^{\kappa \eta_i^N} \right) \leq e^{\beta(\kappa)}.
\] (3.24)
Since we care only about small $\delta$ in (3.22) we can rescale both the variables $\eta_i^\varepsilon \mapsto \varepsilon \eta_i^\varepsilon$ and the kernel $K \mapsto \varepsilon K$ and assume that (3.23) holds with $\sigma_i = 1$, that (3.24) holds with $\kappa_0 = 1$ and also assume that $\max_i |K(x)| \leq 1$. Then, (3.22) is a consequence of the following, ostensibly more general, statement: Let $g$ be a finite convex function on $[-1,1]$ with $g(0) = 0$. Let $\eta_1, \eta_2, \ldots$, be independent centred random variables such that, for any $\alpha \in \mathbb{R}$ and for any $|k| \leq 1$,

$$\sup_{\eta \in \mathbb{H}} E e^{\alpha \eta} \leq e^{\alpha^2} \quad \text{and} \quad \sup_{\eta \in \mathbb{H}} E e^{\varepsilon \eta^2} \leq e^{g(\varepsilon)} \quad \text{(3.25)}$$

Finally let $K(i,j)$ be a matrix satisfying

$$\sup_{i,j \in \Omega} |K(i,j)| \leq 1. \quad \text{(3.26)}$$

Then

$$\limsup_{n \to \infty} \frac{1}{n} \log \left( \mathbb{E} e^{\varepsilon \sum_{i=1}^{n} K(i,j) \eta_i \eta_j} \right) \leq 0. \quad \text{(3.27)}$$

for all $\delta$ sufficiently small.

Indeed, define

$$a_n(\delta) = \sup_{0 \leq \gamma \leq \delta} \sup_{\eta \in \mathbb{H}} \mathbb{E} e^{\delta \sum_{i=1}^{n} K(i,j) \eta_i \eta_j}. \quad \text{(3.28)}$$

Since for any kernel $K$,

$$\mathbb{E} e^{\delta \sum_{i=1}^{n} K(i,j) \eta_i \eta_j} \leq \mathbb{E} e^{\varepsilon \sum_{i=1}^{n} K(i,j) \eta_i \eta_j \eta_i \eta_j} \left( \mathbb{E} e^{\varepsilon K(\eta_i \eta_j)^2} + \mathbb{E} e^{\varepsilon \sum_{i=1}^{n-1} K(i,j) \eta_i \eta_j} \right) \quad \text{(3.29)}$$

By Cauchy-Schwarz and (3.25),

$$\mathbb{E} \left( e^{\varepsilon \sum_{i=1}^{n} K(i,j) \eta_i \eta_j} \right) \leq e^{\frac{\varepsilon}{n^2} \sum_{i=1}^{n} K(i,j) \eta_i \eta_j} \leq e^{\frac{\varepsilon}{n^2} \sum_{i=1}^{n-1} K(i,j) \eta_i \eta_j} \left( e^{\varepsilon \sum_{i=1}^{n-1} K(i,j) \eta_i \eta_j} \right) \quad \text{(3.30)}$$

Consider the $(n-1) \times (n-1)$ kernel

$$R(i,j) = \frac{\gamma(n-1)}{n} K(i,j) + K(i,n) K(j,n) \frac{8 \gamma^2 (n-1)}{n^2}. \quad \text{(3.31)}$$

Since by assumption $\max_{i,j} |K(i,j)| \leq 1$, clearly $\max_{i,j} |R(i,j)| \leq 1$ as well, for all $\gamma$ small enough and uniformly in $n \in \mathbb{N}$. We therefore conclude that, for all sufficiently small values of $\delta$,

$$a_n(\delta) \leq a_{n-1}(\delta) e^{\frac{\gamma}{n^2} \sum_{i=1}^{n} K(i,j) \eta_i \eta_j} \leq \cdots \leq a_1(\delta) e^{\frac{\gamma}{n^2} \sum_{i=1}^{n-2} \max_{i \leq \eta \leq \delta} g(\frac{\eta}{\kappa})} \quad \text{(3.32)}$$

Since $g$ is continuous and zero at 0,

$$\lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} \max_{0 \leq \eta \leq \delta} g \left( \frac{2 \gamma}{k} \right) = 0. \quad \text{(3.33)}$$

Hence (3.27) holds. \qed

3.2. Proof of Proposition 3.2. (3.8) is a rough bound. In Section 5 below we shall discuss sharp large deviation estimates based on martingale techniques [12], see also Section 4.2.1 in [8].

If $\rho_0$ is a nice initial profile then, for any $T < \infty$, the sequence of distributions $\hat{p}_T^N$ of $L_t^\gamma$ on $\mathcal{M}_r \left( \mathbb{T}^d \times \mathcal{C}([0,T],\mathbb{R}) \right)$ and, consequently of $\mu^\gamma[0,T]$ on $\mathbb{C}([0,T],\mathcal{M}_r \times \mathbb{R}^d)$ is exponentially tight.

Indeed fix $C < \infty$ and consider the family $\mathcal{T}_{C,T}$ of one-dimensional diffusions

$$d\theta(t) = dB(t) - \psi' \left( \theta(t) \right) dt + h(t) dt, \quad \text{(3.34)}$$
with initial condition \( P(\theta(0) \in d\theta) = \rho(\theta)e^{-2\phi(\theta)} \), such that \( h \) is smooth and

\[
\max_{0 \leq t \leq T} |h_t| \leq C \quad \text{and} \quad \|\rho\|_{L^\infty} \leq C. \tag{3.35}
\]

We can parametrise elements of \( \mathcal{F}_{C,T} \) in terms of distributions \( \mathbb{P}^{h,\rho}_T \) on \( \mathbb{M}_1([0, T], \mathbb{R}) \), where \((h, \rho)\) satisfies (3.35). Indeed, if \((h, \rho)\) satisfies \( \mathbb{F}_h \)

As in the case of (3.23), under \( \hat{P}_T^{h,\rho} \) the family \( \{\mathbb{P}^{h,\rho}_T\}_{(h, \rho) \in \mathcal{F}_{C,T}} \) is uniformly sub-Gaussian, and (3.38) follows by the exponential Chebyshev inequality.

Indeed, if \( h \equiv 0 \) and the initial density \( \rho \) satisfies the second bound in (3.35), then uniform tightness follows directly from Section 8 of [1]. Cauchy-Schwarz and translation invariance of Brownian motion. The general case of \( h \) satisfying the first bound in (3.35) is then incorporated using Hölder’s inequality.

If \( \rho_0 \) is a nice initial profile, then, by Theorem [2.7] the family \( \{\mathbb{P}^{h,\rho_0(x, \cdot)}\}_{x \in \mathbb{T}^d} \) is a subset of \( \mathcal{F}_{C,T} \) for every \( T < \infty \). Since \( \mathbb{T}^d \) is compact, we can proceed as in the proof of exponential tightness for Sanov’s theorem on Polish spaces in [6].

Once exponential tightness is established, it remains to derive weak large deviation upper bounds. Let \( \rho_0 \) be a nice initial profile and let \( \rho \) be the classical solution to the local McKean-Vlasov equation \( \text{L-MV} \). We have to check that, for any \( \epsilon > 0 \), we can find \( \chi_T(\epsilon) > 0 \), such that the following holds:

Let \( \mu = \mu([0, T]) \in \mathbb{C}([0, T], \mathbb{M}_1(\mathbb{R} \times \mathbb{T}^d)) \) is such that (recall (3.5))

\[
D_T(\mu, \rho(x, \theta)e^{-2\phi(\theta)}d\theta) := \max_{\tau \in [0, T]} d_{\#} \left( \mu_\tau(x, \theta)e^{-2\phi(\theta)d\theta} \right) \geq 3\epsilon. \tag{3.37}
\]

Then,

\[
\lim_{\delta \to 0} \limsup_{N \to \infty} \frac{1}{N^d} \log \hat{P}_T^N \left( D_T(\mu, \mu^N) \leq \delta \right) \leq -\chi_T(\epsilon). \tag{3.38}
\]

Indeed, in light of all the information which we have already collected, (3.38) is just a simple concentration upper bound. If \( D_T(\mu, \rho(x, \theta)e^{-2\phi(\theta)d\theta}d\theta) \geq 3\epsilon \), then there exists \( t \in [0, T] \) and \( f \in \text{Lip}(\mathbb{T}^d \times \mathbb{R}) \), such that

\[
\left| \int f(x, \theta)\mu_t(dx, d\theta) - \int f(x, \theta)\rho_t(x, \theta)e^{-2\phi(\theta)d\theta}dx, d\theta \right| \geq 2\epsilon. \tag{3.39}
\]

Since \( \rho \) is continuous (as a strong solution to \( \text{L-MV} \)) in \( x \), we conclude that for \( \delta < \epsilon \) and \( N \) sufficiently large, the event \( \left\{ D_T(\mu, \mu^N) \leq \delta \right\} \) is included in

\[
\left\{ \left| \frac{1}{N^d} \sum_{i \in \mathbb{T}_N} \left( f \left( \frac{i}{N}, \theta_i(t) \right) - \hat{E}_T^N f \left( \frac{i}{N}, \theta_i(t) \right) \right) \right| > \epsilon \right\}, \tag{3.40}
\]

As in the case of (3.23), under \( \hat{P}_T^N \) the family of centred random variables

\[
\left\{ f \left( \frac{i}{N}, \theta_i(t) \right) - \hat{E}_T^N f \left( \frac{i}{N}, \theta_i(t) \right) \right\}_{f \in \text{Lip}(\mathbb{T}^d \times \mathbb{R}), t \in [0, T], \mu \in \mathbb{M}_1(\mathbb{R} \times \mathbb{T}^d)} \tag{3.41}
\]

is uniformly sub-Gaussian, and (3.38) follows by the exponential Chebyshev inequality.
4. Propagation of Chaos.

For the remaining two sections we shall fix a nice initial profile $\rho_0$ and assume that the initial density $f_0^N$ is in the product form, that is

$$f_0^N = \rho_0^N,$$

where $\rho_0^N$ is given by (3.1).

Given $k$ distinct points $x_1, \ldots, x_k \in \mathbb{T}^d$, let $\mathcal{P}_{T}^{N,x_1,\ldots,x_k}$ be the $\mathbb{P}_T^N$-marginal distribution on $C \left([0, T], \mathbb{R}^k\right)$ of $k$ coordinates $(\theta_1[0, T], \ldots, \theta_k[0, T])$, where, for $\ell = 1, \ldots, k$ we set $i_\ell = [N x_\ell]$.

Consider the (unique) classical solution $(\rho, h)$ to (L-MV), and let $\hat{\theta}_x, \ldots, \hat{\theta}_x$ be independent diffusions,

$$d\hat{\theta}_x(t) = (h^\ell(t) - \psi^\ell (\hat{\theta}_x(t))) dt + dB^\ell_x(t) \quad (4.1)$$

with initial densities $\rho_0(x_1, \theta) e^{-2\phi(\theta)}, \ldots, \rho_0(x_k, \theta) e^{-2\phi(\theta)}$. We use $\hat{\mathcal{P}}_{T}^{N,x_1,\ldots,x_k}$ for their product distribution on $C \left([0, T], \mathbb{R}^k\right)$.

**Theorem 4.1.** For any nice initial profile $\rho_0$, for any $k = 1, 2, \ldots$ points $x_1, \ldots, x_k \in \mathbb{T}^d$, and for any finite $T$,

$$\lim_{N \to \infty} \mathcal{H} \left( \mathcal{P}_{T}^{N,x_1,\ldots,x_k} \middle| \hat{\mathcal{P}}_{T}^{N,x_1,\ldots,x_k} \right) = 0. \quad (4.2)$$

**Proof.** Let $\hat{\mathcal{P}}_{T}^{N,x_1,\ldots,x_k}$ be the distribution of the coupled family of diffusions $\theta(t) = \{\theta_i(t)\}_{i \in \mathbb{T}^d}$ with initial product distribution $\rho_0^N$, such that the following statements hold:

a. If $i = i_j = [N x_j]$, then $\theta_i$ satisfies SDE (4.1).

b. Otherwise, $\theta_i$ satisfies (1.1).

By construction, the $\mathcal{P}_{T}^{N,x_1,\ldots,x_k}$-marginal distribution of $(\theta_1, \ldots, \theta_k)$ is exactly $\hat{\mathcal{P}}_{T}^{N,x_1,\ldots,x_k}$. Hence

$$\mathcal{H} \left( \mathcal{P}_{T}^{N,x_1,\ldots,x_k} \middle| \hat{\mathcal{P}}_{T}^{N,x_1,\ldots,x_k} \right) \leq \mathcal{H} \left( \mathcal{P}_{T}^{N} \middle| \hat{\mathcal{P}}_{T}^{N} \right). \quad (4.3)$$

We shall proceed with deriving a vanishing, as $N \to \infty$, upper bound on the latter entropy. By Girsanov’s formula,

$$\mathcal{H} \left( \mathcal{P}_{T}^{N} \middle| \hat{\mathcal{P}}_{T}^{N} \right) = \frac{1}{2} \sum_{i=1}^{k} \int_0^T \mathbb{E}_T^N \left( h^i(t) - h^i \left( \hat{\theta} \right) \right)^2 dt. \quad (4.4)$$

recall the definition (3.12) of $h^i(\hat{\theta})$.

All the above terms have the same form, so it is enough to consider the case $k = 1$. Let $x \in \mathbb{T}^d$ and $i = [N x]$. For $R > 0$, consider the cutoff, $\varphi_R(\theta)$, of $\theta$, given by

$$\varphi_R(\theta) = \theta 1_{|\theta| \leq R} + R 1_{\theta > R} - R 1_{\theta < -R}. \quad (4.5)$$

By (L-MV),

$$h^i(t) = \int_{\mathbb{T}^d} \int_{\mathbb{R}} J(y - x) \rho_i(y, \theta) \theta e^{-2\phi(\theta)} d\theta d\gamma = h^i_R(t) + g^i_R(t), \quad (4.6)$$

where

$$h^i_R(t) := \int_{\mathbb{T}^d} \int_{\mathbb{R}} J(y - x) \rho_i(y, \theta) \varphi_R(\theta) e^{-2\phi(\theta)} d\theta d\gamma. \quad (4.7)$$

Similarly,

$$h^i \left( \hat{\theta} \right) = \int_{\mathbb{T}^d} \int_{\mathbb{R}} J \left( \frac{i}{N} - y \right) \varphi_R(\theta) \mu_i^N(d\theta, d\gamma) + \frac{1}{N^d} \sum_{j \in \mathbb{T}^d} J \left( \frac{j - i}{N} \right) \left( \theta_j(t) - \varphi_R \left( \theta_j(t) \right) \right)$$

$$:= h^i_R \left( \hat{\theta} \right) + g^i_R \left( \hat{\theta} \right). \quad (4.8)$$
The function

$$f_R^*(y, \theta) = \frac{1}{R||J||_\infty (||\nabla J||_\infty + 1)} J(x - y) \varphi_R(\theta)$$

belongs to $\text{Lip} \left( \mathbb{T}^d \times \mathbb{R} \right)$. Hence, Theorem 3.3 implies that

$$\lim_{N \to \infty} \frac{1}{2} \int_0^T \mathbb{E}_T^N \left( h_R^*(t) - h_R^*(\theta) \right)^2 \, dt = 0. \quad (4.10)$$

In turn,

$$(g_R^*(t))^2 \leq \max_y J^2(y) \int_{\mathbb{T}^d} \int_{\mathbb{R}} \mathbb{1}_{||\theta|| > R} \theta^2 \rho_t(y, \theta) e^{-2\varphi_0(\theta)} \, d\theta \, dy,$$

and, consequently, by (2.24) and the uniform boundedness of $h$,

$$\lim_{R \to \infty} \int_0^T (g_R^*(t))^2 \, dt = 0. \quad (4.12)$$

Finally,

$$\mathbb{E}_T^N \left( g_R^*(\theta) \right)^2 \leq \frac{||J||^2_\infty}{N^d} \mathbb{E}_T^N \left( \sum_{j \in \mathbb{T}^d_N} (\theta_j(t) - \varphi_R(\theta_j(t)))^2 \right). \quad (4.13)$$

By the entropy inequality,

$$\mathbb{E}_T^N \left( \frac{1}{N^d} \sum_{j \in \mathbb{T}^d_N} (\theta_j(t) - \varphi_R(\theta_j(t))) \right) \leq \frac{1}{N^d} \mathbb{H} \left( \mathbb{E}_T^N \parallel \mathbb{E}_T^N \right) + \frac{1}{N^d} \sum_{j \in \mathbb{T}^d_N} \log \left( \int_{\mathbb{R}} \rho_t \left( \frac{j}{N}, \theta \right) e^{-2\varphi(\theta) + (\theta - \varphi_R(\theta))^2} \, d\theta \right). \quad (4.14)$$

By Proposition 3.1, the first term on the right hand side of (4.14) tends to zero as $N \to \infty$. On the other hand, if a density $\rho$ is such that $||\rho||_{2,\psi} < \infty$, then

$$\int_{\mathbb{R}} \rho(\theta) e^{-2\varphi(\theta) + (\theta - \varphi_R(\theta))^2} \, d\theta \leq 1 + ||\rho||_{2,\psi} \sqrt{\int_{||\theta|| > R} e^{-2\varphi(\theta) + 2\psi^2} \, d\theta}. \quad (4.15)$$

By (3.21), the norms $||\rho_t(x, \cdot)||_{2,\psi}$ are uniformly bounded in $x \in \mathbb{T}^d$ and $t \in [0, T]$. By our assumption on $\psi$,

$$\lim_{R \to \infty} \int_{||\theta|| > R} e^{-2\varphi(\theta) + 2\psi^2} \, d\theta = 0. \quad (4.16)$$

Hence,

$$\lim_{R \to \infty} \lim_{N \to \infty} \mathbb{E}_T^N \left( g_R^*(\theta) \right)^2 = 0, \quad (4.17)$$

which concludes the proof.

5. Large deviations

Large deviations for a rather general class of locally mean-field type models were investigated in [13] via a careful adaptation of ideas and techniques which were originally introduced by Dawson and Gärtner [5, 4]. It seems, however, that in a particular case we consider here, our results on the existence and uniqueness of strong solutions to the system (L-MV) and, accordingly, on hydrodynamic limits towards these strong solutions, pave the way to for a simpler and more transparent proof of the large deviation principle for the law $\mathbb{P}_T^N$ of the empirical measure $\mu_N^N = \mu_N^N[0, T]$ on $\mathbb{C} \left( [0, T], \mathbb{M}_1(\mathbb{R} \times \mathbb{T}^d) \right)$, which relies on
Lemma 5.1. This follows from (a much stronger statement of) Lemma 3.4.

5.1. **Exponential tightness.** Recall that exponential tightness for the decoupled family \( \hat{\mu}^N_t \) was already established in Subsection 3.2. Following the notation introduced in Subsection 3.1, define

\[
N^N_t = \sum_{j} \int_{0}^{t} \left( h_i(\theta_s) - h_i^t \right) dB_i(s).
\]

Then, \( e^{\eta N^N_t} - \frac{\hat{\mu}^N_t}{N^N_t} \) is a \( \hat{\mu}^N_t \)-martingale for any \( q \in \mathbb{R} \). Let \( A \subset C \left( [0, T], M_1(\mathbb{R} \times \mathbb{T}^d) \right) \) be a measurable subset. Pick positive \( q, p \) and \( r \) such that \( \frac{1}{q} + \frac{1}{p} + \frac{1}{r} = 1 \). By Girsanov’s formula, and then by Hölder’s inequality,

\[
P^N_T(A) = \hat{\mu}^N_t \left( I_A e^{\eta N^N_t} - \mathbb{E}^N_T \left( e^{\eta N^N_t} \right) \right) \leq \mathbb{E}^N_T \left( e^{\eta N^N_t} \right) \mathbb{E}^N_T \left( e^{\eta N^N_t} \right) = \mathbb{E}^N_T \left( e^{\eta N^N_t} \right).
\]

Therefore, it remains to check that there exist \( \delta > 0 \) and \( C < \infty \) such that

\[
\mathbb{E}^N_T \left( e^{\eta N^N_t} \right) \leq e^{C N^d t}.
\]

This follows from (a much stronger statement of) Lemma 3.4.

We have proved:

**Lemma 5.1.** If \( \rho_0 \) is a nice initial profile in the sense of Definition 2.4 then the law \( \mu^N_T \) of \( \mu^N \) on \( C \left( [0, T], M_1(\mathbb{R} \times \mathbb{T}^d) \right) \) is exponentially tight for any \( T \geq 0 \).

5.2. **The rate function and the result.** In order to write down an expression for the LD rate function we need to introduce some additional notation. Let \( C_{b,0}^{2,1} \left( \mathbb{R} \times \mathbb{T}^d \times \mathbb{R} \right) \) be the family of bounded and continuous (with corresponding derivatives) functions \( (t, x, \theta) \mapsto f_i(x, \theta) \). Let \( (, \cdot)_b \) be the scalar product of, depending on the context, either \( L_2 \left( \mathbb{R}, e^{-2\phi(0)}d\theta \right) \) or \( L_2 \left( \mathbb{T}^d \times \mathbb{R}, e^{-2\phi(0)}dxd\theta \right) \).

For \( r_i(x, \theta)e^{-2\phi(0)}dxd\theta \in C \left( [0, T], M_1(\mathbb{R} \times \mathbb{T}^d) \right) \) and \( f \in C_{b,0}^{2,1} \left( \mathbb{R} \times \mathbb{T}^d \times \mathbb{R} \right) \) consider

\[
\mathcal{L}_T(r | f) := \left( r_T, f_T \right)_\phi - \left( r_0, f_0 \right)_\phi - \int_{0}^{T} \left( r_t, (\partial_t + L_{\phi(t, x, \theta)} f)_\phi \right) dt.
\]

Above,

\[
L_{\phi(t, x, \theta)} = \frac{1}{2} e^{\phi} \partial_{\theta} \left( e^{-2\phi} \partial_{\theta} \right) + h(t, x, \theta) \partial_{\theta},
\]

and

\[
h'(t, x) = \int_{\mathbb{T}^d} \int_{0}^{\infty} J(y - x) \eta r_i(y, \eta) e^{-2\phi(\eta)} d\eta dy.
\]

For any \( f \in C_{b,0}^{2,1} \) we can extend \( \mathcal{L}_T \) by continuity to measures \( R \) which do not have densities with respect to \( dxd\theta \). In this way \( R \mapsto \mathcal{L}_T(R | f) \) is viewed as a continuous non-linear functional on all of \( C \left( [0, T], M_1(\mathbb{R} \times \mathbb{T}^d) \right) \).
For \( R \in C \left([0, T], \mathbb{M}_1(\mathbb{R} \times \mathbb{T}^d)\right) \) define the functional \( A_T \) via:
\[
A_T(R) = \sup_{f \in C_b^{0,1}} \left\{ \mathcal{L}_T(R | f) - \frac{1}{2} \int_0^T \left( \int_{\mathbb{R}} \mathcal{H}_t(x, \theta) R_t(\mathrm{d}x, \mathrm{d}\theta) \right) \mathrm{d}t \right\}. \tag{5.7}
\]
Since for any \( f \in C_b^{2,0,1} \) the map \( R \to \mathcal{L}_T(R | f) \) is continuous, the functional \( A_T \) is lower-semicontinuous. If \( R \neq r_t(x, \theta) e^{-2\phi(t)} \mathrm{d}\theta \mathrm{d}x \), then it is easy to check that \( A_T(R) = \infty \).

Otherwise, if \( r \) is a density of \( R \), we shall write \( A_T(r) \) instead of \( A_T(R) \).

As in [12] one concludes that if \( A_T(r) < \infty \) then there exists a drift field
\[
b_t(x, \theta) \in L_2 \left( \mathbb{R} \times \mathbb{T}^d \times \mathbb{R}, r_t(x, \theta) e^{-2\phi(t)} \mathrm{d}\theta \mathrm{d}x \right),
\]
such that
\[
\mathcal{L}_T(r | f) = \int_0^T (r_t, b_t \partial_\theta f_t) \phi \mathrm{d}t,
\]
which means that
\[
A_T(r) = \frac{1}{2} \int_0^T (r_t, b_t^2) \phi \mathrm{d}t, \text{ whenever } A_T(r) < \infty. \tag{5.9}
\]

**Remark 5.2.** Since we are working with one dimensional spins, we can always represent \( b_t = \partial_\theta g_t \), which defines \( g \) up to an addition of \( \theta \)-independent functions of \((t, x)\).

**Theorem 5.3.** Assume that \( \rho_0 \) is a nice initial profile and assume that the initial density \( f_0^N \) satisfies (D). Then the law \( \mathbb{P}_t^N \) of \( \mu^N_t \) on the space \( C \left([0, T], \mathbb{M}_1(\mathbb{R} \times \mathbb{T}^d)\right) \) satisfies a large deviation principle with rate \( N^d \) and with rate function
\[
I_T(R) = \begin{cases} 
\infty, & \text{if } R \neq r_t(x, \theta) e^{-2\phi(t)} \mathrm{d}\theta \mathrm{d}x, \\
A_T(r) \downarrow \mathcal{H} \left( r_0 \bigg| \rho_0 \right), & \text{otherwise}. 
\end{cases} \tag{5.10}
\]

### 5.3. Local mean-field systems with spatially dependent drifts.

If \( A_T(r) < \infty \), then (5.8) reads as follows: For any \( f \in C_b^{2,0,1} \left( \mathbb{R} \times \mathbb{T}^d \times \mathbb{R} \right) \),
\[
(r_t, f_t)_{\phi} - (r_0, f_0)_{\phi} - \int_0^T (r_t, \partial_t + L_{h^*(t, x)} + b_t(x, \theta) \partial_\theta f_t) \phi \mathrm{d}t = 0. \tag{5.11}
\]

Recall (5.6) how the drift \( h^* \) is related to \( r \). In this way, (5.11) is a weak form of a consistent local mean-field family of Fokker-Plank equations, and the couple \((r, h^*)\) can be interpreted as a weak solution to a local mean-field McKean-Vlasov system with an additional spatially dependent drift \( b \).

Here is the corresponding strong formulation along the lines of Definition 2.6.

**Definition 5.4.** Given a smooth space-time drift field \( b_t(x, \theta) \) and an initial density \( \rho_0(x, \theta) \), a strong solution of the McKean-Vlasov equation is a pair \((\rho, h)\), with \( \rho \in \mathcal{B} \) and \( h \in \mathcal{H} \), such that
\[
\forall x \in \mathbb{T}^d \quad \forall t \in [0, \infty) \left\{ 
\begin{array}{l}
\rho_t(x, \cdot) = \rho_t^{h^*+h^*} \quad \text{with initial condition } \rho_0(x, \cdot) \\
\text{and} \\
h^*(t) = \int \int J(y - x) \rho_t(y, \eta) e^{-2\phi(t)} \eta \mathrm{d}y \mathrm{d}\eta 
\end{array} \right. \tag{GL-MV}
\]

Above \( b^*(t, \theta) = b_t(x, \theta) \) and for any smooth field \( u(t, \theta) \) the symbol \( \rho_t^u \) stands for the density (under a tacit assumption that it is well defined) with respect to \( e^{-2\phi} \) of the one-dimensional diffusion
\[
d\theta(t) = (-\psi'(\theta(t)) + u(t, \theta(t))) \mathrm{d}t + dB(t). \tag{5.12}
\]
The proofs of Theorem 2.7 and Theorem 3.3 were based on the a priori bound (2.18). The following generalisations of these theorems are more or less straightforward:

**Theorem 5.5.** Assume that for all \( x \in \mathbb{T}^d \) the field \( b_i(x, \theta) := \partial_\theta g_i(x, \theta) \) is smooth in \((t, \theta)\). Furthermore, assume that for every \( t > 0 \)

\[
\sup_{x \in \mathbb{T}^d} \max_{b_i \leq M} \left\{ \frac{1}{2} \psi''(B) - \partial_\theta g_i(x, B) - (\psi'(B) - b_i(x, B))^2 \right\} < \infty \tag{5.13}
\]

Let \( \rho_0(x, \cdot) \) be a nice initial profile. Then, for any \( T > 0 \), there exists a unique strong solution \((\rho, h)\) of the system \((\text{GL-MV})\).

Furthermore, consider the modified system of coupled diffusions

\[
d\theta_i(t) = -\left( \psi'\left(\theta_i^N(t)\right) + b_i \left( \frac{i}{N}, \theta_i^N(t) \right) + \frac{1}{N^d} \sum_{j \in \mathbb{Z}^d} J \left( \frac{j-i}{N} \right) \theta_j^N(t) \right) dt + dB_i(t), \quad i \in \mathbb{T}^d,
\]

and let the empirical measure \( \mu_i^N \) be defined by (1.3). Let \( \mathbb{P}^N_{T} \) denote the distribution of \( \mu^N[0, T] \) under the dynamics (5.14). Then, under Assumption (12) on the product structure of the initial distribution \( f_0 \), \( \mathbb{P}^N_{T} \) converges to \( \delta_{\rho(x, \theta) e^{-2\phi(\theta)d\theta}} \) in the following sense: For any \( \epsilon > 0 \) and \( T < \infty \),

\[
\lim_{N \to \infty} \mathbb{P}^N_{T} \left( \max_{t \in [0, T]} \text{d}_{\text{LP}} \left( \mu_i^N, \rho(x, \theta) e^{-2\phi(\theta)d\theta} \right) \leq \epsilon \right) = 0. \tag{5.15}
\]

### 5.4. Scheme of the proof of the LDP Theorem 3.3

To simplify notation let us write

\[
\left\{ \mu^N \overset{\epsilon}{\sim} R \right\} = \left\{ \max_{t \in [0, T]} \text{d}_{\text{LP}} \left( \mu_i^N, R_i(x, \text{d}\theta) \text{d}x \right) \leq \epsilon \right\} \tag{5.16}
\]

in case of upper bounds, and

\[
\left\{ \mu^N \overset{\epsilon}{\preceq} R \right\} = \left\{ \max_{t \in [0, T]} \text{d}_{\text{LP}} \left( \mu_i^N, R_i(x, \text{d}\theta) \text{d}x \right) < \epsilon \right\},
\]

in the case of lower bounds.

In view of exponential tightness we need to derive asymptotic upper and lower bounds on \( \mathbb{P}^N_T (\mu^N \overset{\epsilon}{\preceq} R) \) for any \( R \in \mathbb{C} \left( [0, T], \mathcal{M}_1(\mathbb{R} \times \mathbb{T}^d) \right) \) and for small \( \epsilon > 0 \).

Proceeding as in [12, 8] the upper bound with \( A_T \) defined in (5.10) follows by Girsanov’s theorem: For any fixed \( f \in C^0_{b,1}(\mathbb{R} \times \mathbb{T}^d \times \mathbb{R}) \),

\[
\mathbb{P}^N_T \left( \mu^N \overset{\epsilon}{\preceq} R \right) \leq \mathcal{N}^f \left( L_T(R[f]) + \frac{1}{2} \int_0^T \int_{(x, \theta) \in \mathbb{T}^d} \text{r}_i(\partial_\theta f_i) \text{r}(x, \text{d}\theta) \text{d}x \text{d}t \right) \left( 1 + \mathcal{O}_f(\epsilon) \right) \mathcal{Q}^{N,f}_T (\mu^N \overset{\epsilon}{\preceq} R). \tag{5.17}
\]

Above, \( \mathcal{O}_f(\epsilon) \) is a quantity which tends to zero as \( \epsilon \to 0 \), and

\[
\frac{\text{d}\mathcal{Q}^{N,f}_T}{\text{d}\mathbb{P}^N_T} := e^{N^d \mathcal{M}_t^{N,f} - \frac{1}{2} \mathcal{M}_t^{N,f}},
\]

where \( \mathcal{M}_t^{N,f} \) is a \( \mathbb{P}^N \)-martingale,

\[
\mathcal{M}_t^{N,f} = \mu^N(t, f_t) - \mu^N(0, f_0) - \frac{1}{N} \sum_i \int_0^T \left( \partial_\theta + L_{\phi(t, (\theta, \cdot))} \right) f_i \left( \frac{i}{N}, \theta_t(s) \right) \text{d}s.
\]

Recall (3.12) and (2.2) to follow the above notation.
Optimisation over \( f \) in the first term on the right hand side of (5.17) gives \( A_T(R) \). On the other hand,
\[
\mathbb{Q}_T^{N, f}(\mu^N \overset{c.v.}{\rightarrow} R) \leq \mathbb{Q}_T^{N, f}(d_{LP}(\mu^N_0, R_0) \leq \epsilon) = \mathbb{P}_T^{N}(d_{LP}(\mu^N_0, R_0) \leq \epsilon).
\] (5.18)
The last expression is subject to stationary Sanov-type large deviations with rate function \( \mathcal{H}(R_0 | \rho_0) \).

Let us turn to the lower bound: By general methods it will follow from upper bounds if one is able to prove that there is always a unique weak solution to (5.11). Below we sketch an alternative route which is based on the approach to existence and uniqueness of strong solutions and subsequent derivation of hydrodynamic limits, as developed in Section 2-3 and formulated in Theorem 5.5

STEP 0. Recall notation (5.6). In view of the exponential tightness of the random variables \( h^N(t, x) \) by the lower semicontinuity of the functional \( \rho \mapsto h^r(t, x) \) with respect to the distance \( D_T \), we may assume that \( \sup_{t \leq T} |h^r(t, x)| \) is bounded.

STEP 1. Assume that \( r_i = r_i(x, \theta, \partial_x \psi) \) is the unique strong solution to (GL-MV) with a nice initial profile \( r_0 \) and smooth drift field \( b = \partial_x \psi \) satisfying (5.13). As in (5.17):
\[
\mathbb{P}_T^{N}(\mu^N \overset{c.v.}{\rightarrow} R) \geq \mathbb{E}^{N}[L_T(R | g) - \frac{1}{2b} \int_0^T \int_{\mathbb{R}} (\partial_\theta g_i)^2 R_i(x, \theta) dx dt + O(\epsilon)].
\] (5.19)
By our choice of \( r \), the expression \( L_T(R | g) - \frac{1}{2b} \int_0^T \int_{\mathbb{R}} (\partial_\theta g_i)^2 R_i(x, \theta) dx dt \) equals to \( A_T(r) \). Furthermore, with the notation introduced in the course of the formulation of Theorem 5.5 \( \mathbb{Q}_T^{N, g} = \mathbb{P}_T^{N, b} \). Finally, assuming that \( \mathcal{H}(r_0 | \rho_0) < \infty \),
\[
\mathbb{P}_T^{N, b}(\mu^N \overset{c.v.}{\rightarrow} R) = \mathbb{E}^{N, b}\left[ \prod_i \frac{r_0(i/N, \theta_i(0))}{\rho_0(i/N, \theta_i(0))} e^{-N \sum \log \frac{\rho_0(i/N, \theta_i(0))}{r_0(i/N, \theta_i(0))} \mu^N \overset{c.v.}{\rightarrow} R} \right].
\] (5.20)
The measure \( \mathbb{E}^{N, b}\left[ \prod_i \frac{r_0(i/N, \theta_i(0))}{\rho_0(i/N, \theta_i(0))} \right] \) is just the distribution of \( \mu^N[0, T] \) under the sde (5.14) and the product initial distribution sampled from the nice initial profile \( r_0 \). Under this measure, the law of large numbers implies that \( \lim_{N \to \infty} \frac{1}{N} \sum \log \frac{\rho_0(i/N, \theta_i(0))}{r_0(i/N, \theta_i(0))} = \mathcal{H}(r_0 | \rho_0) \).
On the other hand,
\[
\lim_{N \to \infty} \mathbb{P}_T^{N, b}\left[ \prod_i \frac{r_0(i/N, \theta_i(0))}{\rho_0(i/N, \theta_i(0))} ; \mu^N \overset{c.v.}{\rightarrow} R \right] = 1,
\] (5.21)
by (5.15).

It remains to show that strong solutions \( r \) described in STEP 1 are dense on the graph of \( A_T \). That is for any \( r \) with \( A_T(r) < \infty \) there exists a sequence \( (r^\varepsilon, b^\varepsilon) \) such that for any \( \varepsilon \) \( (r^\varepsilon, h^\varepsilon) \) is a strong solution to (GL-MV) (with smooth field \( b^\varepsilon \) satisfying (5.13)), and both \( \lim \varepsilon \rightarrow 0 \) and \( \lim A_T(r^\varepsilon) = A_T(r) \).

STEP 2. Let us go back to the linear form \( \mathcal{L}_T(r | f) \) in (5.4). It could be rewritten as
\[
\mathcal{L}_T(r | f) = \mathcal{L}_T^0(r | f) - \int_0^T (r_t, h_t \partial_\theta f) \psi dt.
\] (5.22)
Hence \( b_t = b_t^0 - h_t^\varepsilon \), where \( b_t^0 \) satisfies \( \mathcal{L}_T^0(r | f) = \int_0^T (r_t, b_t^0 \partial_\theta f) \psi dt \) for any \( f \in C^2_b(0, T) \).

Accordingly,
\[
\mathcal{A}_T(r) = \frac{1}{2} \int_0^T (r_t, (b_t^0 - h_t^\varepsilon)^2) \psi dt.
\] (5.23)
Therefore one has to show that there exists a sequence \( \{b^\varepsilon\} \) such that for any \( \varepsilon > 0 \) it complies with (5.13), and in addition, \( r^\varepsilon := p^\varepsilon \) satisfies:

\[
\forall T \quad D_T = \lim_{\varepsilon \to 0} r^\varepsilon = r \quad \text{and} \quad \lim_{\varepsilon \to 0} \int_0^T \left( r^\varepsilon_t, (b^\varepsilon_t)^2 \right)_\phi \, dt = \int_0^T \left( r_t, (b^0_t)^2 \right)_\phi \, dt \tag{5.24}
\]

Note that the densities \( r^\varepsilon(\theta, x) \) are completely decoupled, and the question is essentially about one-dimensional parabolic PDE-s in divergence form. We proceed to discuss the latter.

**STEP 3.** Let \( r_\varepsilon(\theta) \) be a density with respect to \( e^{-2\varepsilon(\theta)} d\theta \). Given \( f \in C^{2,1}(\mathbb{R} \times \mathbb{R}_+) \) we, with a slight abuse of notation, continue using

\[
\mathcal{L}^0(r \mid f) = (r_T, f_T)_\phi - (r_0, f_0)_\phi - \int_0^T (r_t, (\partial_t + L_0)f)_\phi \, dt. \tag{5.25}
\]

We have to check the following: If

\[
\mathcal{A}^0_f(r) := \sup_{f \in C^{2,1}} \left\{ \mathcal{L}^0(r \mid f) - \frac{1}{2} \int_0^T \left( r_t, (\partial_t f_t)^2 \right)_\phi \, dt \right\} < \infty, \tag{5.26}
\]

or alternatively, if \( \mathcal{L}^0(r \mid f) = \int_0^T (r_t, b_t \partial_\theta f)_\phi \, dt \) for some \( b \in L^2(\mathbb{R} \times [0, T], r e^{-2\varepsilon} d\theta dt) \), then one can find a sequence \( \{b^\varepsilon(\theta)\} \) which complies with (5.13), and in addition, the unique classical solution \( r^\varepsilon \) of

\[
\partial_t r^\varepsilon_t - L_0 r^\varepsilon_t = -e^{2\varepsilon} \partial_\theta \left( e^{-2\varepsilon} b^\varepsilon_t r^\varepsilon_t \right) \tag{5.27}
\]

with a nice initial profile \( r_0^\varepsilon \), which satisfies

\[
\forall T \quad \lim_{\varepsilon \to 0} \max_{t \leq T} d_{L^p} \left( r^\varepsilon, r \right) = 0 \quad \text{and} \quad \lim_{\varepsilon \to 0} \int_0^T \left( r^\varepsilon_t, (b^\varepsilon_t)^2 \right)_\phi \, dt = \int_0^T \left( r_t, (b^0_t)^2 \right)_\phi \, dt. \tag{5.28}
\]

**REFERENCES**

[1] Patrick Billingsley. _Convergence of probability measures_. Wiley Series in Probability and Statistics: Probability and Statistics. John Wiley & Sons, Inc., New York, second edition, 1999. A Wiley-Interscience Publication.

[2] F. Comets and Th. Eisele. Asymptotic dynamics, noncritical and critical fluctuations for a geometric long-range interacting model. _Comm. Math. Phys._, 118(4):531–567, 1988.

[3] F. Comets. Nucleation for a long range magnetic model. _Ann. Inst. H. Poincaré Probab. Statist._, 23(2):135–178, 1987.

[4] D. A. Dawson and J. Gärtner. Large deviations, free energy functional and quasi-potential for a mean-field model of interacting diffusions. _Mem. Amer. Math. Soc._, 78(398):iv+94, 1989.

[5] Donald A. Dawson and Jürgen Gärtner. Large deviations from the McKean-Vlasov limit for weakly interacting diffusions. _Stochastics_, 20(4):247–308, 1987.

[6] Amir Dembo and Ofer Zeitouni. _Large deviations techniques and applications_, volume 38 of _Stochastic Modelling and Applied Probability_. Springer-Verlag, Berlin, 2010. Corrected reprint of the second (1998) edition.

[7] Jürgen Gärtner. On the McKean-Vlasov limit for interacting diffusions. _Math. Nachr._, 137:197–248, 1988.

[8] Alice Guionnet. Large deviations and stochastic calculus for large random matrices. _Probab. Surv._, 1:72–172, 2004.

[9] Ioannis Karatzas and Johannes Ruf. Distribution of the time for explosion for one-dimensional diffusions. _Probab. Theory Related Fields_, 152(1-2):1–30, 2012.

[10] Ioannis Karatzas and Steven E. Shreve. _Brownian motion and stochastic calculus_, volume 113 of _Graduate Texts in Mathematics_. Springer-Verlag, New York, second edition, 1991.
[11] Markos A. Katsoulakis, Petr Plecháč, and Dimitrios K. Tsagkarogiannis. Mesoscopic modeling for continuous spin lattice systems: model problems and micromagnetics applications. *J. Stat. Phys.*, 119(1-2):347–389, 2005.

[12] Claude Kipnis and Stefano Olla. Large deviations from the hydrodynamical limit for a system of independent Brownian particles. *Stochastics Stochastics Rep.*, 33(1-2):17–25, 1990.

[13] Aleksandar Mijatović and Mikhail Urusov. On the martingale property of certain local martingales. *Probab. Theory Related Fields*, 152(1-2):1–30, 2012.

[14] Patrick E. Müller. Path large deviations for interacting diffusions with local mean-field interactions in random environment. *Electronic Journ. Probab.*, 22(76):1–56, 2017.

[15] L. C. G. Rogers. Smooth transition densities for one-dimensional diffusions. *Bull. London Math. Soc.*, 17(2):157–161, 1985.

[16] Alain-Sol Sznitman. Topics in propagation of chaos. In *École d’Été de Probabilités de Saint-Flour XIX—1989*, volume 1464 of *Lecture Notes in Math.*, pages 165–251. Springer, Berlin, 1991.

[17] Horng-Tzer Yau. Relative entropy and hydrodynamics of Ginzburg-Landau models. *Lett. Math. Phys.*, 22(1):63–80, 1991.

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