Test vectors for non-Archimedean Godement–Jacquet zeta integrals

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Abstract

Given an induced representation of Langlands type $(\pi, V_\pi)$ of $GL_n(F)$ with $F$ non-Archimedean, we show that there exist explicit choices of matrix coefficient $\beta$ and Schwartz–Bruhat function $\Phi$ for which the Godement–Jacquet zeta integral $Z(s, \beta, \Phi)$ attains the $L$-function $L(s, \pi)$.

1. Introduction

Let $F$ be a non-Archimedean local field with ring of integers $\mathcal{O}$, maximal ideal $p$, and uniformiser $\varpi$, so that $\varpi \mathcal{O} = p$ and $\mathcal{O}/p \cong \mathbb{F}_q$ for some prime power $q$. We normalise the absolute value $|\cdot|$ on $F$ such that $|\varpi| = q^{-1}$.

Let $(\pi, V_\pi)$ be a generic irreducible admissible smooth representation of $GL_n(F)$, where $F$ is a non-Archimedean local field. Given a matrix coefficient $\beta(g) = \langle \pi(g) \cdot v_1, \tilde{v}_2 \rangle$ of $\pi$, where $v_1 \in V_\pi$ and $v_2 \in \tilde{V}_\pi$, and given a Schwartz–Bruhat function $\Phi \in \mathcal{S}(\text{Mat}_{n \times n}(F))$, we define the Godement–Jacquet zeta integral \[ Z(s, \beta, \Phi) := \int_{GL_n(F)} \beta(g)\Phi(g)|\det g|^{s + \frac{n-1}{2}} \, dg, \] which is absolutely convergent for $\Re(s)$ sufficiently large. The test vector problem for Godement–Jacquet zeta integrals is the following.

**Test Vector Problem.** Given a generic irreducible admissible smooth representation $(\pi, V_\pi)$ of $GL_n(F)$, determine the existence of $K$-finite vectors $v_1 \in V_\pi$, $\tilde{v}_2 \in \tilde{V}_\pi$, and a Schwartz–Bruhat function $\Phi \in \mathcal{S}(\text{Mat}_{n \times n}(F))$ such that

\[ Z(s, \beta, \Phi) = L(s, \pi). \]

The Archimedean analogue of this problem has been resolved for $F = \mathbb{C}$ by Ishii \[4\] and for $F = \mathbb{R}$ by Lin \[12\]. For non-Archimedean $F$, the spherical case is resolved in \[3, \text{Lemma 6.10}\]: one takes $v_1$ and $v_2$ to be spherical vectors and

\[ \Phi(x) = \begin{cases} 1 & \text{if } x \in \text{Mat}_{n \times n}(\mathcal{O}), \\ 0 & \text{otherwise.} \end{cases} \]

We solve the ramified case of this problem.

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\[1\] The author has been unable to verify certain aspects of \[12\]. In particular, the functions constructed in \[12, (6.5)\] and \[6.7\] are defined only on the maximal compact subgroup $K = O(n)$ of $GL_n(\mathbb{R})$. For these functions to be elements of certain induced representations of $GL_n(\mathbb{R})$, they must transform under the action of diagonal matrices $a = \text{diag}(a_1, \ldots, a_n) \in A_n(\mathbb{R})$ in a specified manner, and this action does not seem to be compatible with the definitions \[12, (6.5)\] and \[6.7\] when $k \in K$ is taken to be a diagonal orthogonal matrix.

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Theorem 1.2. Let $(\pi, V_\pi)$ be a generic irreducible admissible smooth representation of $\text{GL}_n(F)$ of conductor exponent $c(\pi) > 0$. Let $\beta(g)$ denote the matrix coefficient $\langle \pi(g) \cdot v^0, v^0 \rangle$, where $v^0 \in V_\pi$ is the newform of $\pi$ normalised such that $\beta(1) = 1$. Define the Schwartz–Bruhat function $\Phi \in \mathcal{S}(\text{Mat}_{n \times n}(F))$ by

$$
\Phi(x) := \begin{cases} 
\omega_\pi^{-1}(x_{n,n}) 
& \text{if } x \in \text{Mat}_{n \times n}(O) \text{ with } x_{n,1}, \ldots, x_{n,n-1} \in \mathfrak{p}^{\xi(\pi)} \text{ and } x_{n,n} \in O^\times, \\
\text{vol}(K_0(\mathfrak{p}^{\xi(\pi)})) 
& \text{otherwise}, \\
0 
& \text{otherwise},
\end{cases}
$$

(1.3)

where $\omega_\pi$ denotes the central character of $\pi$ and the congruence subgroup $K_0(\mathfrak{p}^{\xi(\pi)})$ is as in (3.1). Then for $\Re(s)$ sufficiently large,

$$
Z(s, \beta, \Phi) = L(s, \pi).
$$

2. Induced representations of Langlands type

Rather than working with generic irreducible admissible smooth representations, we will work in the more general setting of induced representations of Langlands type; see [2, Section 1.5] for further details.

Given representations $\pi_1, \ldots, \pi_r$ of $\text{GL}_{n_1}(F), \ldots, \text{GL}_{n_r}(F)$, where $n_1 + \cdots + n_r = n$, we form the representation $\pi_1 \boxtimes \cdots \boxtimes \pi_r$ of $\text{M}_F(F)$, where $\boxtimes$ denotes the outer tensor product and $\text{M}_F(F)$ denote the block-diagonal Levi subgroup of the standard parabolic subgroup $P(F) = P(n_1, \ldots, n_r)(F)$ of $\text{GL}_n(F)$. We then extend this representation trivially to a representation of $P(F)$. By normalised parabolic induction, we obtain an induced representation $\pi$ of $\text{GL}_n(F)$,

$$
\pi = \bigoplus_{j=1}^{r} \pi_j := \text{Ind}_{P(F)}^{\text{GL}_n(F)} \bigotimes_{j=1}^{r} \pi_j.
$$

When $\pi_1, \ldots, \pi_r$ are irreducible and essentially square-integrable, $\pi_1 \boxtimes \cdots \boxtimes \pi_r$ is said to be an induced representation of Whittaker type; such a representation is admissible and smooth. Moreover, if each $\pi_j$ is of the form $\sigma_j \mid \det |^s |, \text{ where } \sigma_j \text{ is irreducible, unitary, and square-integrable, and } \Re(t_{1}) \geq \cdots \geq \Re(t_{r}), \text{ then } \pi \text{ is said to be an induced representation of Langlands type. Every irreducible admissible smooth representation } \pi \text{ of } \text{GL}_n(F) \text{ is isomorphic to the unique irreducible quotient of some induced representation of Langlands type. If } \pi \text{ is also generic, then it is isomorphic to some (necessarily irreducible) induced representation of Langlands type.}

An induced representation of Langlands type $(\pi, V_\pi)$ is isomorphic to its Whittaker model $W(\pi, \psi)$, the image of $V_\pi$ under the map $v \mapsto \Lambda(\pi(\cdot) \cdot v)$, where $\Lambda : V_\pi \to \mathbb{C}$ is the unique (up to scalar multiplication) nontrivial Whittaker functional associated to an additive character $\psi$ of $F$. This is a continuous linear functional that satisfies

$$
\Lambda(\pi(u) \cdot v) = \psi_n(u) \Lambda(v)
$$

for all $v \in V_\pi$ and $u \in N_n(F)$, where $N_n(F)$ denotes the unipotent radical of the standard minimal parabolic subgroup and $\psi_n(u) := \psi(u_{1,2} + u_{2,3} + \cdots + u_{n-1,n})$.

An induced representation of Langlands type $\pi$ is said to be spherical if it has a $K$-fixed vector, where $K := \text{GL}_n(O)$. Such a spherical representation $\pi$ must be a principal series representation of the form $| \cdot |^{s_1} \boxtimes \cdots \boxtimes | \cdot |^{s_n}$; furthermore, the subspace of $K$-fixed vectors must be one dimensional. This $K$-fixed vector, unique up to scalar multiplication, is called the spherical vector of $\pi$. In the induced model of $\pi$, the normalised spherical vector is the unique
smooth right $K$-invariant function $f^\circ : \text{GL}_n(F) \to \mathbb{C}$ satisfying

$$f^\circ(uag) = f^\circ(g)\delta_n^{1/2}(a)\prod_{i=1}^n |a_i|^{t_i},$$

for all $u \in N_n(F)$, $a = \text{diag}(a_1, \ldots, a_n) \in A_n(F) \cong F^n$, the subgroup of diagonal matrices, and $g \in \text{GL}_n(F)$, where $\delta_n(a) := \prod_{i=1}^n |a_i|^{n-2i+1}$ denotes the modulus character of the standard minimal parabolic subgroup, and normalised such that

$$f^\circ(1_n) = \prod_{i=1}^{n-1} \prod_{j>i+1} \zeta_F(1 + t_i - t_j), \quad \zeta_F(s) := \frac{1}{1 - q^{-s}}.$$  

The normalised spherical Whittaker function $W^\circ$ in the Whittaker model $\mathcal{W}(\pi, \psi)$ is given by the analytic continuation of the Jacquet integral

$$W^\circ(g) := \int_{N_n(F)} f^\circ(w_nug)\overline{\psi_n(u)}\, du,$$

where $w_n = \text{antidiag}(1, \ldots, 1)$ is the long Weyl element. The Jacquet integral is absolutely convergent if $\Re(t_1) > \cdots > \Re(t_n)$ \cite[Section 3]{10} and extends holomorphically as a function of the complex variables $t_1, \ldots, t_n$ \cite[1]{1}. The Haar measure on $N_n(F)$ is $du = \prod_{j=1}^{n-1} \prod_{\ell=j+1}^n du_{j,\ell}$, where for $u_{j,\ell} \in F$, $du_{j,\ell}$ is the additive Haar measure on $F$ normalised to give $\mathcal{O}$ volume 1. With this normalisation of Haar measures and with $\psi$ an unramified additive character of $F$, the normalised spherical vector $W^\circ \in \mathcal{W}(\pi, \psi)$ satisfies $W^\circ(1_n) = 1$.

3. The newform

For each nonnegative integer $m$, we define the congruence subgroup $K_0(p^m)$ of $K$ by

$$K_0(p^m) := \{ k \in K : k_{n,1}, \ldots, k_{n,n-1} \in p^m \}. \quad (3.1)$$

**Theorem 3.2** \cite[Théorème (5)]{8}. Let $(\pi, V_\pi)$ be an induced representation of Langlands type of $\text{GL}_n(F)$. Then either $\pi$ is spherical, so that

$$V_\pi := \{ v \in V_\pi : \pi(k) \cdot v = v \text{ for all } k \in K \}$$

is one dimensional, or $\pi$ is ramified, in which case $V_\pi^K$ is trivial and there exists a minimal positive integer $m = c(\pi)$ for which the vector subspace

$$V_\pi^{K_0(p^m)} := \{ v \in V_\pi : \pi(k) \cdot v = \omega_\pi(k_{n,n})v \text{ for all } k \in K_0(p^m) \}$$

is nontrivial; moreover, $V_\pi^{K_0(p^{c(\pi)})}$ is one dimensional.

**Definition 3.3.** The vector $v^\circ \in V_\pi^{K_0(p^{c(\pi)})}$, unique up to scalar multiplication, is called the newform of $\pi$. The nonnegative integer $c(\pi)$ is called the conductor exponent of $\pi$, where we set $c(\pi) = 0$ if $\pi$ is spherical.

For each $m$, we may view $V_\pi^{K_0(p^m)}$ as the image of the projection map $\Pi^m : V_\pi \to V_\pi$ given by

$$\Pi^m(v) := \int_K \xi^m(k)\pi(k) \cdot v \, dk, \quad (3.4)$$

$$\xi^m(k) := \begin{cases} 
\omega_\pi^{-1}(k_{n,n}) & \text{if } m > 0 \text{ and } k \in K_0(p^m), \\
\text{vol}(K_0(p^m)) & \text{if } m = 0 \text{ and } k \in K, \\
0 & \text{otherwise}. 
\end{cases} \quad (3.5)$$
Here \( dk \) is the Haar measure on the compact group \( K \) normalised to give \( K \) volume 1. In particular, for any \( v \in V_\pi \), we have that
\[
\Pi(\pi)(v) = \langle v, \tilde{v}^\circ \rangle v^\circ, \tag{3.6}
\]
where \( v^\circ \in V_\pi^{K_0(\mathfrak{p}(\pi))} \) and \( \tilde{v}^\circ \in V_\pi^{K_0(\mathfrak{p}(\pi))} \) are normalised such that \( \langle v^\circ, \tilde{v}^\circ \rangle = 1 \).

We write \( W^\circ \) for the newform in the Whittaker model \( W(\pi, \psi) \) normalised such that \( W^\circ(1_n) = 1 \), where \( \psi \) is an unramified additive character; we also normalise \( v^\circ \in V_\pi \) and the Whittaker functional \( \Lambda \) such that \( \Lambda(v^\circ) = W^\circ(1_n) = 1 \). Note that if \( \pi \) is spherical, then the newform in the Whittaker model is precisely the normalised spherical Whittaker function.

A key property of \( W^\circ \) is the fact that it is a test vector for certain Rankin–Selberg integrals.

**Theorem 3.7** (Jacquet–Piatetski-Shapiro–Shalika [8, Théorème (4)], Jacquet [7], Matringe [13, Corollary 3.3]). Let \( \pi \) be an induced representation of Langlands type, and let \( W^\circ \in W(\pi, \psi) \) denote the newform in the Whittaker model. Then for any spherical representation of Langlands type \( \pi' \) of \( \text{GL}_{n-1}(F) \) with normalised spherical Whittaker function \( W'^\circ \in W(\pi', \overline{\psi}) \), the \( \text{GL}_n \times \text{GL}_{n-1} \) Rankin–Selberg integral
\[
\Psi(s, W^\circ, W'^\circ) := \int_{N_{n-1}(F) \backslash \text{GL}_{n-1}(F)} W^\circ(g) W'^\circ(g) \det g^s \frac{dg}{|\det g|} \tag{3.8}
\]
is equal to the Rankin–Selberg L-function \( L(s, \pi \times \pi') \).

Here the Haar measure on \( \text{GL}_n(F) \) is that induced from the Iwasawa decomposition \( \text{GL}_n(F) = N_n(F) A_n(F) K \), namely \( dg = du \delta_n^{-1}(a) d^s a \, dk \), where \( d^s a = \prod_{i=1}^n d^s a_i \) with the multiplicative Haar measure on \( F^\times \) given by \( d^s a_i = \zeta_F(1)|a_i|^{-1} \, da_i \).

**Theorem 3.9** [11, Theorem 2.1.1]. Let \( \pi \) be an induced representation of Langlands type, and let \( W^\circ \in W(\pi, \psi) \) denote the newform in the Whittaker model. Then for any spherical representation of Langlands type \( \pi' \) of \( \text{GL}_{n}(F) \) with normalised spherical Whittaker function \( W'^\circ \in W(\pi', \overline{\psi}) \), the \( \text{GL}_n \times \text{GL}_n \) Rankin–Selberg integral
\[
\Psi(s, W^\circ, W'^\circ, \Phi^\circ) := \int_{N_n(F) \backslash \text{GL}_n(F)} W^\circ(g) W'^\circ(g) \Phi(e_n g) \det g^s \frac{dg}{|\det g|} \tag{3.10}
\]
is equal to the Rankin–Selberg L-function \( L(s, \pi \times \pi') \), where \( e_n := (0, \ldots, 0, 1) \in \text{Mat}_{1 \times n}(F) \) and \( \Phi^\circ \in \mathcal{F}(\text{Mat}_{1 \times n}(F)) \) is given by
\[
\Phi^\circ(x_1, \ldots, x_n) := \begin{cases} 
\frac{\omega_\pi^{-1}(x_n)}{\text{vol}(K_0(\mathfrak{p}(\pi)))} & \text{if } c(\pi) > 0, x_1, \ldots, x_{n-1} \in \mathfrak{p}(\pi), \text{ and } x_n \in \mathcal{O}^\times, \\
1 & \text{if } c(\pi) = 0 \text{ and } x_1, \ldots, x_n \in \mathcal{O}, \\
0 & \text{otherwise.}
\end{cases}
\]

4. A propagation formula

We now present a propagation formula for spherical Whittaker functions. This is a recursive formula for a \( \text{GL}_n(F) \) Whittaker function in terms of a \( \text{GL}_{n-1}(F) \) Whittaker function.

**Lemma 4.1.** Let \( \pi' = [\cdot | \cdot] \quad \boxplus \cdots \boxplus [\cdot | \cdot]^{\prime} \) be a spherical representation of Langlands type of \( \text{GL}_n(F) \). Then the normalised spherical Whittaker function \( W'^\circ \in W(\pi', \overline{\psi}) \)
satisfies

\[ W^{\pi}(g) = |\det g|^{t'_1 + \frac{n-1}{2}} \int_{\text{GL}_{n-1}(F)} W^{\pi}_0(h) |\det h|^{-t'_1 - \frac{n}{2}} \times \int_{\text{Mat}(n-1) \times 1(F)} \Phi'(h^{-1} (1_{n-1} v) g) \psi(e_{n-1} v) dv dh, \]

(4.2)

where \( W^{\pi}_0 \in \mathcal{W}(\pi'_0, \overline{\psi}) \) is the normalised spherical Whittaker function of the spherical representation of Langlands type \( \pi'_0 := | \cdot |_{1}^{t'_0} \mathbb{T} \cdots \mathbb{T} | \cdot |_{1}^{t'_{n}} \) of \( \text{GL}_{n-1}(F) \) and \( \Phi' \in \mathcal{S}(\text{Mat}(n-1) \times 1(F)) \) is the Schwartz–Bruhat function

\[ \Phi'(x) := \begin{cases} 1 & \text{if } x \in \text{Mat}(n-1) \times 1(\mathcal{O}), \\ 0 & \text{otherwise}. \end{cases} \]

**Proof.** Let \( f^{\pi} \) be the normalised spherical vector in the induced model of \( \pi' \), so that

\[ f^{\pi}(1_n) = \prod_{i=1}^{n-1} \prod_{j=i+1}^{n} \zeta_F(1 + t'_i - t'_j), \]

(4.3)

\[ f^{\pi}(uag) = f^{\pi}(g) \delta^{1/2}_{n}(a) \prod_{i=1}^{n} |a_i|^{t'_i}, \]

(4.4)

\[ f^{\pi}(gk) = f^{\pi}(g) \]

(4.5)

for all \( u \in N_n(F), a = \text{diag}(a_1, \ldots, a_n) \in A_n(F), g \in \text{GL}_n(F), \) and \( k \in K \). We claim that \( f^{\pi} \) is also given by the Godement section

\[ f^{\pi}(g) := |\det g|^{t'_1 + \frac{n-1}{2}} \int_{\text{GL}_{n-1}(F)} \Phi'(h^{-1} (0 1_{n-1}) g) f^{\pi}_0(h) |\det h|^{-t'_1 - \frac{n}{2}} dh. \]

(4.6)

Here \( f^{\pi}_0 \) is the normalised spherical vector in the induced model of \( \pi'_0 \), so that

\[ f^{\pi}_0(1_{n-1}) = \prod_{i=2}^{n-1} \prod_{j=i+1}^{n} \zeta_F(1 + t'_i - t'_j), \]

(4.7)

\[ f^{\pi}_0(u'a'h) = f^{\pi}_0(h) \delta^{1/2}_{n-1}(a') \prod_{i=2}^{n} |a'_i|^{t'_i}, \]

(4.8)

\[ f^{\pi}_0(hk') = f^{\pi}_0(h) \]

(4.9)

for all \( u' \in N_{n-1}(F), a' = \text{diag}(a'_2, \ldots, a'_n) \in A_{n-1}(F), h \in \text{GL}_{n-1}(F), \) and \( k' \in \text{GL}_{n-1}(\mathcal{O}) \).

We then insert the identity (4.6) into the Jacquet integral

\[ W^{\pi}(g) := \int_{N_n(F)} f^{\pi}(w_n u g) \psi_n(u) du, \]

write \( w_n = \begin{pmatrix} 0 & 1 \\ -u_n & 0 \end{pmatrix} \) and \( u' = \begin{pmatrix} u' & 0 \\ 0 & 1 \end{pmatrix} \) for \( u' \in N_{n-1}(F) \) and \( v \in \text{Mat}(n-1) \times 1(F) \), and make the change of variables \( h \mapsto w_n^{-1} u' a'h \) to obtain the identity (4.2).

So it remains to show that \( f^{\pi} \) is indeed given by (4.6). We first show that this is an element of the induced model of \( \pi' \), just as in [6, Proposition 7.1]. We replace \( g \) with \( \begin{pmatrix} 0 & 0 \\ v_0 & a_0 \end{pmatrix} g \), where \( v \in \text{Mat}_{1 \times (n-1)}(F), u' \in N_{n-1}(F), a_1 \in F^\times, \) and \( a' \in A_{n-1}(F) \). Upon making the change of variables \( h \mapsto u' a'h \) and using (4.8), we see that (4.4) is satisfied. Next, we check that \( f^{\pi} \) given by (4.6) satisfies (4.5), which follows easily from the fact that \( \Phi'(xk) = \Phi'(x) \) for all
$x \in \text{Mat}_{(n-1) \times n}(F)$ and $k \in K$. Finally, we confirm the normalisation (4.3). To see this, we use the Iwasawa decomposition $h = u'a'k'$ in (4.6), in which case the Haar measure is $dh = \delta_{n-1}^{-1}(a') \, du' \, d^\times a' \, dk'$. The integral over $\text{GL}_{n-1}(O) \supset k'$ is trivial. We then make the change of variables $u' \mapsto u'^{-1}$, $a' \mapsto a'^{-1}$, so that

$$f^\circ_0(1_n) = f^\circ_0(1_{n-1}) \int_{\text{GL}_{n-1}(F)} \Phi'(0 \, a'u') \prod_{i=2}^n |a'_i|^{(-t'_i/2 - \delta_{n-1})} |\det a'|^{t'_1 + \frac{n-1}{2}} \, d^\times a' \, du',$$

recalling (4.8). Writing $du' = \prod_{j=2}^{n-1} \prod_{\ell = j+1}^{n} du'_{j,\ell} \, d^\times a'_{j}$ and $d^\times a' = \prod_{i=2}^n d^\times a'_i$ and making the change of variables $u'_{j,\ell} \mapsto a'^{-1}_{j} u'_{j,\ell}$, this becomes

$$f^\circ_0(1_{n-1}) \prod_{j=2}^{n-1} \prod_{\ell = j+1}^{n} \int_{\mathcal{O} \setminus \{0\}} \prod_{i=2}^n |a'_i|^{1+t'_i - t'_i} \, d^\times a'_i.$$

The integral over $\mathcal{O} \ni u'_{j,\ell}$ is 1, while the integral over $\mathcal{O} \setminus \{0\} \ni a'_i$ is $\zeta_F(1+ t'_1 - t'_i)$. Recalling the normalisation (4.7) of $f^\circ_0(1_{n-1})$, we see that (4.3) is indeed satisfied.

5. Proof of Theorem 1.2

Proof of Theorem 1.2. Let $\pi$ be a ramified induced representation of Langlands type of $\text{GL}_n(F)$, so that $c(\pi) > 0$, and let $\pi' = \left| \cdot \right|^{t'_1} \boxtimes \cdots \boxtimes \left| \cdot \right|^{t'_n}$ be an arbitrary spherical representation of Langlands type of $\text{GL}_n(F)$. We insert the identity (4.2) for the normalised spherical Whittaker function $W^\circ_0 \in W(\pi', \pi)$ into the $\text{GL}_n \times \text{GL}_n$ Rankin–Selberg integral (3.10). Just as in [6, Equation (8.1)], we fold the integration over $N_{n-1}(F) \setminus N_n(F) \cong \text{Mat}_{(n-1) \times 1}(F)$ in $v$ and make the change of variables $g \mapsto (\begin{smallmatrix} 0 & v \\ h & 0 \end{smallmatrix})$. In this way, we find that $\Psi(s, W^\circ, W^\circ, \Phi^\circ)$ is equal to

$$\int_{N_{n-1}(F) \setminus \text{GL}_{n-1}(F)} W^\circ_0(h) |\det h|^{-\frac{s}{2}} \int_{\text{GL}_{n}(F)} W^\circ \left( \begin{smallmatrix} h & 0 \\ 0 & 1 \end{smallmatrix} \right) g \Phi(g) |\det g|^{s+t'_1 + \frac{n-1}{2}} \, dg \, dh,$$

with $\Phi(x) := \Phi^\circ(e_n x) \Phi'(0 \, 1 \, 0 \, x)$ as in (1.3).

We claim that

$$\Phi(g) = \int_K \xi^{c(\pi)}(k) \Phi(k^{-1} g) \, dk,$$

with $\xi^{c(\pi)}$ as in (3.5). Indeed, $\xi^{c(\pi)}(k)$ vanishes unless $k \in K_0(p^{c(\pi)})$, in which case $\Phi(k^{-1} g)$ vanishes unless $g \in \text{Mat}_{n \times n}(O)$ with $g_{1,1}, \ldots, g_{n,n-1} \in p^{c(\pi)}$ and $g_{n,n} \in O^\times$. Then as $k^{-1} \in K_0(p^{c(\pi)})$, it is easily checked that

$$\omega^{-1}_\pi(e_n k^{-1} g' e_n) = \omega_\pi(k_{n,n}) \omega^{-1}_\pi(g_{n,n}),$$

using the fact that $e_n k^{-1} g' e_n - e_n k^{-1} e_n g_{n,n} \in p^{c(\pi)}$, $e_n k^{-1} e_n k_{n,n} - 1 \in p^{c(\pi)}$, and $c(\omega_\pi) \leq c(\pi)$. Thus (5.2) follows.

We insert (5.2) into (5.1) and make the change of variables $g \mapsto kg$, so that the integral over $K \ni k$ is

$$\int_K W^\circ \left( \begin{smallmatrix} h & 0 \\ 0 & 1 \end{smallmatrix} \right) kg \xi^{c(\pi)}(k) \, dk = \Lambda \left( \pi \left( \begin{smallmatrix} h & 0 \\ 0 & 1 \end{smallmatrix} \right) \cdot \int_K \xi^{c(\pi)}(k) \pi(k) \cdot (\pi(g) \cdot v^\circ) \, dk \right).$$
We note that
\[ \int_K \xi^{(\pi)}(k) \pi(k) \cdot (\pi(g) \cdot v^\circ) \, dk = \Pi^{(\pi)}(\pi(g) \cdot v^\circ) = \beta(g)v^\circ, \]
where \( \beta(g) := \langle \pi(g) \cdot v^\circ, \tilde{v}^\circ \rangle \), recalling (3.4) and (3.6), so that
\[ \int_K W^\circ \left( \begin{array}{cc} h & 0 \\ 0 & 1 \end{array} \right) k g \xi^{(\pi)}(k) \, dk = \beta(g)W^\circ \left( \begin{array}{cc} h & 0 \\ 0 & 1 \end{array} \right). \] (5.3)

Combining (5.1) with (5.2) and (5.3), we find that
\[ \Psi(s, W^\circ, W'^\circ, \Phi^\circ) = Z(s + t'_1, \beta, \Phi) \Psi(s, W^\circ, W'^\circ_0), \]
recalling the definitions (1.1) of the Godement–Jacquet zeta integral and (3.8) of the \( GL_n \times GL_{n-1} \) Rankin–Selberg integral. From Theorems 3.9 and 3.7,
\[ \Psi(s, W^\circ, W'^\circ, \Phi^\circ) = L(s, \pi \times \pi'), \quad \Psi(s, W^\circ, W'^\circ_0) = L(s, \pi \times \pi'_0). \]
Moreover, [9, (9.5) Theorem] implies
\[ L(s, \pi \times \pi') = L(s, \pi \times | \cdot |^t s_1) L(s, \pi \times \pi'_0) = L(s + t'_1, \pi) L(s, \pi \times \pi'_0). \]
Since \( L(s, \pi \times \pi'_0) \) is not uniformly zero, we conclude that
\[ Z(s + t'_1, \beta, \Phi) = L(s + t'_1, \pi). \] \( \square \)

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