Spectral asymptotic and positivity for singular
Dirichlet-to-Neumann operators

Ali BenAmor∗

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Abstract

In the framework of Hilbert spaces we shall give necessary and sufficient conditions to define a Dirichlet-to-Neumann operator via Dirichlet principle. For singular Dirichlet-to-Neumann operators we will establish Laurent expansion near singularities as well as Mittag–Leffler expansion for the related quadratic form. The established results will be exploited to solve definitively the problem of positivity of the related semigroup in the $L^2$ setting. The obtained results are supported by some examples on Lipschitz domains. Among other results, we shall demonstrate that regularity of the boundary may affect positivity and derive Mittag-Leffler expansion for the eigenvalues of singular Dirichlet-to-Neumann operators.

Key words: Dirichlet-to-Neumann operator, spectral asymptotic, positivity preserving.

1 Introduction

Singular D-to-N operators are one parameter family of operators which have standing singularities at eigenvalues of some Dirichlet operator. As illustration, consider the following example. Let $\mathcal{E}$ be the quadratic form associated with Neumann’s Laplacian on the unit disc $\mathbb{D}$ in $\mathbb{R}^2$:

$$D(\mathcal{E}) = H^1(\mathbb{D}), \quad \mathcal{E}[u] = \int_{\mathbb{D}} |\nabla u|^2 \, dx, \quad \forall \, u \in H^1(\mathbb{D}).$$

Let $dS$ be the surface measure on $\Gamma$. Set $L^2(\Gamma) := L^2(\Gamma, dS)$ and let $J$ be the operator ‘trace on the boundary’

$$J : H^1(\mathbb{D}) \to L^2(\Gamma), \quad u \mapsto u|_{\Gamma}.$$ 

It is well known that $J$ is bounded and

$$\text{ran } J = H^{1/2}(\Gamma), \quad \ker J = H^1_0(\mathbb{D}).$$

∗Department of technology and engineering for transport, High School for Transport and Logistics, University of Sousse, Tunisia. E-mail: ali.benamor@ipeit.rnu.tn
Let $-\Delta_D$ be the Dirichlet Laplacian on $\mathbb{D}$. It is nothing else but the selfadjoint operator associated to the quadratic form $\mathcal{E}|_{\ker J}$.

Let $\lambda \in \mathbb{R}$ such that $\lambda$ is not an eigenvalue of $-\Delta_D$, $\psi \in H^{1/2}(\Gamma)$ and $u_\lambda \in H^1(\mathbb{D})$ be the unique solution of the boundary value problem

$$\begin{cases}
-\Delta u_\lambda - \lambda u_\lambda = 0, & \text{in } \mathbb{D} \\
 u_\lambda = \psi, & \text{on } \Gamma
\end{cases}$$

Define the form $\mathcal{E}_\lambda$ as follows:

$$D(\mathcal{E}_\lambda) = H^{1/2}(\Gamma), \quad \mathcal{E}_\lambda[\psi] := \mathcal{E}[u_\lambda] = \int_{\mathbb{D}} |\nabla u_\lambda|^2 dx - \lambda \int_{\mathbb{D}} u_\lambda^2 dx \quad \forall \psi \in H^{1/2}(\Gamma).$$

Then $\mathcal{E}_\lambda$ is a closed lower semibounded and densely defined quadratic form in $L^2(\Gamma)$ (see [AM12]). Let us denote by $\tilde{L}_\lambda$ the selfadjoint operator associated to $\mathcal{E}_\lambda$ via Kato’s representation theorem. The family $\tilde{L}_\lambda$ is a typical family of singular Dirichlet-to-Neumann (D-to-N for short) operators with singularities being the Dirichlet eigenvalues. In fact, in case $\lambda$ is a Dirichlet eigenvalue then the above mentioned boundary value problem is not uniquely solvable. It has as many linearly independent solutions as the multiplicity of $\lambda$. Hence $\mathcal{E}_\lambda$ is not well defined and one can not define a quadratic form by the described procedure. This explains the connotation 'singular'.

The Dirichlet eigenvalues are also singularities for the quadratic form $\mathcal{E}_\lambda$, the resolvent, the semigroup and the eigenvalues of $\tilde{L}_\lambda$.

Let us observe that for $\lambda \leq 0$, according to Dirichlet principle, it holds

$$\mathcal{E}_\lambda[\psi] = \inf\{ \int_{\mathbb{D}} |\nabla u|^2 dx - \lambda \int_{\mathbb{D}} u^2 dx : u \in H^1(\mathbb{D}), J u = \psi \}.$$ 

Now some natural questions arise: How do spectral objects of $L_\lambda$, behave near these singularities? How do these singularities influence positivity or sub-Markov property for the related semigroup?

In this paper we shall first, put the above procedure in the abstract setting of Hilbert spaces in order to construct D-to-N operators in Hilbert spaces. Concretely, let $\mathcal{H}, \mathcal{H}_{aux}$ be Hilbert spaces, $\mathcal{E}$ a lower semibounded quadratic form with domain $\mathcal{D} \subset \mathcal{H}$ and $J$ a linear operator $J : \text{dom } J \subset \mathcal{D} \rightarrow \mathcal{H}_{aux}$ with dense range. We shall give necessary and sufficient conditions ensuring Dirichlet principle to hold. Precisely we aim for finding necessary and sufficient conditions so that

$$\mathcal{E}[Ju] := \inf\{ \mathcal{E}(v, v) : v \in S(u), Jv = Ju \} = \min\{ \mathcal{E}(v, v) : v \in S(u), Jv = Ju \}, \quad \forall u \in \mathcal{D}, \quad (1.1)$$

with unique minimizer, where $S(u)$ is a linear manifold (to be determined) depending on the vector $u$. It turns out that (1.1) holds if and only if $\text{dom } J$ is the direct sum of the kernel of $J$ and some specific subspace. Then we shall give necessary and sufficient conditions for the quadratic form $\mathcal{E}$ defined via Dirichlet principle, to be lower semibounded and closed in $\mathcal{H}_{aux}$. The obtained form is commonly named the trace form of $\mathcal{E}$ and the
related operator is commonly named the Dirichlet-to-Neumann operator.

Once the construction has been done we consider a positive form $E$ and $J$ such that $\ker J$ is dense in $H_{aux}$ and the form $E_D := E|_{\ker J}$ is closed and has discrete spectrum. We construct the singular form $\hat{E}_\lambda$, trace of the form $E - \lambda$, where $\lambda \in \mathbb{R}$ is not an eigenvalue of $E_D$. Our major contribution in this respect is to write a representation formula for $\hat{E}_\lambda$ (Theorem 3.1). The formula involves $\hat{E}_0$, some Dirichlet operator and an abstract Poisson kernel operator. It plays a central role in the development of the paper.

Then we shall turn our attention to analyze some properties of the singular D-to-N operator $\hat{L}_\lambda$ associated with $\hat{E}_\lambda$. Extending $\hat{E}_\lambda$ to complex values $z$, we shall show that $\hat{E}_z$ is meromorphic with simple poles coinciding with the eigenvalues of $E_D$. At this stage our main contribution is to establish Laurent and Mittag–Leffler expansions for $\hat{E}_z$ near each singularity (Theorem 4.1 and Theorem 4.2).

As a byproduct, we shall determine the exact rate of growth for $|\hat{E}_z[\psi]|$ as $z$ approaches any singularity. The main input for proving the mentioned results is the representation formula of $\hat{E}_z$.

In case $\hat{L}_\lambda$ has compact resolvent, pushing our analysis forward we shall examine the behavior of the eigenvalues near the singularities.

As applications we shall consider the special case of $L^2$ spaces. Our major contribution in this framework is to utilize the obtained asymptotic from former sections to establish necessary and sufficient conditions ensuring positivity preservation property of the obtained semigroup near singularities (see Theorem 5.1). These conditions involve some abstract Poisson kernel operator and the eigenfunctions of the singularities. Thereby we completely solve the problem of positivity preservation in a general framework. Here the main ingredient is the Mittag–Leffler expansion for the trace form together with Beurling–Deny criterion.

Finally, we analyze the singular D-to-N operator related to Neumann’s Laplacian on Euclidean Lipschitz domains. Here we shall be able to write Mittag–Leffler expansion of $\hat{E}_\lambda$ with coefficients depending only on Dirichlet eigenvalues and boundary integrals of the normal derivatives of the related eigenfunctions. Moreover, in some cases we shall write Mittag–Leffler expansion for the eigenvalues of the singular D-to-N operator. The expansion involves the Dirichlet eigenvalues solely!

Besides, we shall demonstrate that regularity of the boundary may affect positivity as well as multiplicities of Dirichlet eigenvalues. Whereas for negative $\lambda$ the semigroup is even ultracontractive.

We quote that construction of D-to-N operators in the setting of Hilbert spaces via Dirichlet principle was already performed by many authors [AtE12, Pos16, BBST19], following different approaches and under more restrictive assumptions. Some spectral properties for the D-to-N operator on Lipschitz domains were established in [AM12, BtE15]. Analysis of positivity preservation for intervals and the unit disc was elaborated in [Dan14].

However, as long as we know, there is no systematic studies neither concerning spectral asymptotic near singularities nor concerning positivity property of the semigroup related to singular D-to-N operators.
2 D-to-N operators via Dirichlet principle

Let \( \mathcal{H}, \mathcal{H}_{\text{aux}} \) be two Hilbert spaces. Let \( (\cdot, \cdot), (\cdot, \cdot)_{\text{aux}} \) denote the scalar products on \( \mathcal{H} \) and \( \mathcal{H}_{\text{aux}} \), respectively and \( \| \cdot \|, \| \cdot \|_{\text{aux}} \) be the corresponding norms.

We shall use the connotation 'form' for any sesquilinear symmetric form as for the related quadratic form.

Let \( \mathcal{E} \) be a lower semibounded form with domain \( D \subseteq \mathcal{H} \). For \( u \in D \) we abbreviate \( \mathcal{E}[u] := \mathcal{E}(u, u) \) and for every \( \lambda \in \mathbb{R} \) we set

\[
\mathcal{E}_\lambda, \; \text{dom} \mathcal{E}_\lambda := D, \; \mathcal{E}_\lambda[u] := \mathcal{E}[u] - \lambda \|u\|^2.
\]

Assume we are given a linear operator \( J: \text{dom} J \subseteq D \rightarrow \mathcal{H}_{\text{aux}} \) with dense range.

2.1 The positive case

Assume that \( \mathcal{E} \) is positive.

Being inspired by [BBST19, Theorem 2.1, Lemma 2.8] let us define \( \breve{\mathcal{E}} \) with domain in \( \mathcal{H}_{\text{aux}} \) as follows:

\[
\text{dom} \breve{\mathcal{E}} := \text{ran} J, \; \breve{\mathcal{E}}[Ju] := \inf \{ \mathcal{E}[v] : v \in \text{dom} J, \; Jv = Ju \}.
\]

We aim to find necessary and sufficient conditions for the Dirichlet principle to occur. Namely, conditions ensuring

\[
\inf \{ \mathcal{E}[v] : v \in \text{dom} J, \; Jv = Ju \} = \min \{ \mathcal{E}[v] : v \in \text{dom} J, \; Jv = Ju \},
\]

with a unique minimizer.

To achieve our goal we introduce the linear subspace defined by

\[
\mathcal{H}^J_{\text{har}} := \{ u \in \text{dom} J, \; \mathcal{E}(u, v) = 0, \; \forall v \in \ker J \},
\]

and for every \( u \in \text{dom} J \) we designate by \( C_u \) the linear manifold

\[
C_u := u + \ker J.
\]

The subscript 'har' stands for 'harmonic' as indicated by the first example.

Let us observe that the infimum can be written as

\[
\inf \{ \mathcal{E}[v] : v \in C_u \}.
\]

Let us first solve the uniqueness problem.

Theorem 2.1. Assume that for each \( u \in \text{dom} J \) the infimum is attained at some \( Pu \).

Then

1. \( Pu \) should satisfies

\[
Pu \in C_u, \; \mathcal{E}(Pu, v) = 0, \; \forall v \in \ker J.
\]

In particular, \( Pu \in \mathcal{H}^J_{\text{har}} \) and

\[
\mathcal{E}[Ju] = \inf \{ \mathcal{E}[v] : v \in \mathcal{H}^J_{\text{har}} \cap C_u \}.
\]
2. Uniqueness. Pu is unique if and only if

\[ \mathcal{H}_{\text{har}}^J \cap \ker J = \{ 0 \}. \]  

(2.3)

Proof. Let \( u \in \text{dom } J \). Assume that the infimum is attained at some \( Pu \). From the definition \( Pu \in C_u \).

Now let \( v \in \ker J \). Then for any \( t > 0 \), \( v_t := tv + Pu \in C_u \). Hence \( \mathcal{E}[v_t] \geq \mathcal{E}[Pu] \). An elementary computation leads to

\[ \mathcal{E}[v_t] = t^2 \mathcal{E}[v] + 2t \text{Re } \mathcal{E}(v, Pu) + \mathcal{E}[Pu] \geq \mathcal{E}[Pu]. \]

Dividing by \( t \) and letting \( t \downarrow 0 \) yields \( \text{Re } \mathcal{E}(v, Pu) \geq 0 \).

Similarly, changing \( v_t \) by \( w_t := itv + Pu \) we obtain \( \text{Im } \mathcal{E}(v, Pu) = 0 \). Thus \( \mathcal{E}(v, Pu) \geq 0 \) and \( Pu \in \mathcal{H}_{\text{har}}^J \).

As \( Pu \in \mathcal{H}_{\text{har}}^J \cap C_u \subset C_u \) we achieve

\[ \mathcal{E}[Pu] \leq \inf \{ \mathcal{E}[v], \ v \in \mathcal{H}_{\text{har}}^J \cap C_u \} \leq \mathcal{E}[Pu], \]

which ends the proof of the first assertion.

Uniqueness: Assume that \( Pu \) is unique for every \( u \in \text{dom } J \). In particular for \( u \in \ker J \) we get \( \mathcal{E}[Ju] = 0 \) and by uniqueness \( Pu = 0 \). Now let \( v \in \mathcal{H}_{\text{har}}^J \cap \ker J \). Then \( \mathcal{E}[v] = 0 \).

By uniqueness, once again, we obtain \( v = Pu = 0 \). Thereby \( \mathcal{H}_{\text{har}}^J \cap \ker J = \{ 0 \} \).

Conversely assume that \( \mathcal{H}_{\text{har}}^J \cap \ker J = \{ 0 \} \) and that for some \( u \in \text{dom } J \) the infimum is attained at \( Pu, Pu' \). Then \( Pu - Pu' \in \mathcal{H}_{\text{har}}^J \cap \ker J = \{ 0 \} \) and then \( Pu = Pu' \).

\[ \square \]

Remark 2.1. 1. The \( Pu \) might be non-unique. It is for instance the case if \( 0 \) is an eigenvalue of \( \mathcal{E} \).

2. Under assumption of Theorem 2.1 the map

\[ \text{dom } J \to \mathcal{H}_{\text{har}}^J, \ u \mapsto Pu, \]

is linear.

From now on we maintain the assumption (2.3).

The converse of the latter theorem solves the problem \( \inf = \min \).

Theorem 2.2. Assume that

\[ \mathcal{H}_{\text{har}}^J \cap C_u \neq \emptyset, \ \forall u \in \text{dom } J. \]  

(2.4)

Let \( Pu \) be any element from \( \mathcal{H}_{\text{har}}^J \cap C_u \). Then

\[ \inf \{ \mathcal{E}[v], \ v \in C_u \} = \mathcal{E}[Pu]. \]  

(2.5)
Proof. For each \( v \in \text{dom} \ J \), we set \( P_v \) any element from \( \mathcal{H}_\text{har}^J \cap C_v \). From the very definition we get
\[
\inf \{ \mathcal{E}[w], \ w \in C_u \} \leq \mathcal{E}[P_u], \ \forall u \in \text{dom} \ J.
\]
On the other hand we have \( v - P_v \in \text{ker} \ J \). Since in particular \( P_v \in \mathcal{H}_\text{har}^J \) we get \( \mathcal{E}(P_v, P_v - v) = 0 \) and then
\[
\mathcal{E}[P_v] = \mathcal{E}(P_v, v), \ \forall v \in \text{dom} \ J.
\]
On the other hand the positivity of \( \mathcal{E} \) together with the latter identity lead to
\[
0 \leq \mathcal{E}[P_v - v] = \mathcal{E}[P_v] - 2\mathcal{E}(P_v, v) + \mathcal{E}[v] = -\mathcal{E}[P_v] + \mathcal{E}[v]. \quad (2.6)
\]
Hence we achieve
\[
\mathcal{E}[v] \geq \mathcal{E}[P_v], \ \forall v \in \text{dom} \ J. \quad (2.7)
\]
Let now \( u \in \text{dom} \ J \) and \( v \in C_u \). Then, as \( P_v \in C_v \) we get \( JP_v = Jv = Ju \). Thereby \( P_v \in C_u \) and \( Pu - P_v \in \text{ker} \ J \cap \mathcal{H}_\text{har}^J \). Hence \( \mathcal{E}[Pu - P_v] = 0 \). Set \( w = Pu - P_v \). Since \( w \in \text{ker} \ J \) and \( P_v \in \mathcal{H}_\text{har}^J \) we obtain \( \mathcal{E}(w, P_v) = 0 \). A straightforward computation leads to
\[
\mathcal{E}[Pu] = \mathcal{E}[w + P_v] = \mathcal{E}[w] + 2\mathcal{E}(w, P_v) + \mathcal{E}[P_v] = \mathcal{E}[P_v]. \quad (2.8)
\]
Finally putting all together we achieve
\[
\inf \{ \mathcal{E}[v], \ v \in C_u \} \geq \inf \{ \mathcal{E}[P_v], \ v \in C_u \} = \mathcal{E}[P_u],
\]
and hence
\[
\inf \{ \mathcal{E}[v], \ v \in C_u \} = \mathcal{E}[P_u].
\]
which completes the proof.

\[ \square \]

**Remark 2.2.** One may think that condition (2.4) is equivalent to \( \mathcal{H}_\text{har}^J \cap \text{ker} \ J \neq \emptyset \).

However, it is not true. Indeed, let \( E_0 \) be the smallest eigenvalue of the Dirichlet Laplacian on the unit ball. Let
\[
\text{dom} \mathcal{E} = H^1(B), \ \mathcal{E}[v] = \int_B |\nabla v|^2 \, dx - E_0 \int_B v^2 \, dx.
\]
Then \( \mathcal{H}_\text{har}^J \cap \text{ker} \ J \) is the linear span of the eigenfunction of \( E_0 \), say \( u_0 \). Let \( u \in H^1(B) \setminus H^1_0(B) \). Then any \( v \in \mathcal{H}_\text{har}^J \) should be proportional to \( u_0 \). On the other hand any \( v \in \mathcal{H}_\text{har}^J \) can not be in \( H^1_0(B) \), which is a contradiction. Hence \( \mathcal{H}_\text{har}^J \cap C_u = \emptyset \) whereas \( \mathcal{H}_\text{har}^J \cap \text{ker} \ J \neq \emptyset \).

Combining both theorems we obtain:

**Corollary 2.1.** The infimum is attained and is unique for every \( u \in \text{dom} \ J \) if and only if \( \mathcal{H}_\text{har}^J \cap C_u \) is a singleton.
The latter theorems together with the corollary have very far-reaching consequences.

**Theorem 2.3.**  
1. The infimum is attained and is unique for every $u \in \text{dom } J$, if and only if
   \[ \text{dom } J = \mathcal{H}^J_{\text{har}} \oplus \ker J. \]  
   (2.9)

2. Assume that the decomposition (2.9) holds. Let $u \in \text{dom } J$ and $Pu \in \mathcal{H}^J_{\text{har}}$ be the component of $u$ corresponding to the direct sum. Then
   \[ \inf \{ \mathcal{E}[v] : v \in C_u \} = \mathcal{E}[Pu]. \]  
   (2.10)

**Proof.** Assume that for every $u \in \text{dom } J$ the infimum is attained at some $Pu$ and is unique. Then according to Theorem 2.3 $Pu \in \mathcal{H}^J_{\text{har}} \cap C_u$. Moreover $u$ can be written, in a unique manner, as $u = Pu + u - Pu$ with $Pu \in \mathcal{H}^J_{\text{har}}$ and $u - Pu \in \ker J$. The converse: If (2.9) is fulfilled, mimicking the proof of Theorem 2.2 one shows that
   \[ \inf \{ \mathcal{E}[v] : v \in C_u \} = \mathcal{E}[Pu], \]
and the infimum is attained at the sole element $Pu$. \hfill \Box

Once we have solved the problem of existence and uniqueness of the minimizer, we are in a comfortable situation to discuss the closedness $\hat{\mathcal{E}}$. We quote that under condition of Theorem 2.3 we have
   \[ \text{dom } \hat{\mathcal{E}} = \text{ran } J, \ \hat{\mathcal{E}}[Ju] = \mathcal{E}[Pu], \ \forall u \in \text{dom } J. \]  

Here is an improvement of [BBST19, Lemma 3.4].

We set $\mathcal{E}^J$ the form defined by
   \[ \text{dom } \mathcal{E}^J = \text{dom } J, \ \mathcal{E}^J[u] := \mathcal{E}[u] + \|Ju\|^2_{\text{aux}}. \]

Observe that if the direct sum decomposition (2.9) is fulfilled then $\mathcal{E}^J$ defines a scalar product on $\mathcal{H}^J_{\text{har}}$.

**Theorem 2.4.** Suppose that condition (2.9) is fulfilled. Then $\hat{\mathcal{E}}$ is closed if and only if $(\mathcal{H}^J_{\text{har}}, \mathcal{E}^J)$ is a Hilbert space.

**Proof.** Assume that $\hat{\mathcal{E}}$ is closed. Let $(u_n)$ be a $\mathcal{E}^J$-Cauchy sequence in $\mathcal{H}^J_{\text{har}}$. Owing to the fact that $Pu_n = u_n$ for every integer $n$, we get
   \[ \hat{\mathcal{E}}[Ju_n - Ju_m] = \mathcal{E}[u_n - u_m] \to 0, \ \text{and } \|Ju_n - Ju_m\|_{\text{aux}} \to 0. \]

Thereby $(Ju_n)$ is $\hat{\mathcal{E}}$-Cauchy and by closedness of $\hat{\mathcal{E}}$ there is $u \in \text{dom } J$ such that
   \[ \lim_{n \to \infty} \hat{\mathcal{E}}[Ju_n - Ju] + \|Ju_n - Ju\|_{\text{aux}} = 0. \]

Recalling that $\hat{\mathcal{E}}[Ju_n - Ju] = \mathcal{E}[u_n - Pu]$ and $Ju = JPu$ and that $Pu \in \mathcal{H}^J_{\text{har}}$ we conclude that $(u_n)$ is $\mathcal{E}^J$-convergent to $Pu$ and hence converges in $(\mathcal{H}^J_{\text{har}}, \mathcal{E}^J)$.

Conversely, assume that $(\mathcal{H}^J_{\text{har}}, \mathcal{E}^J)$ is a Hilbert space. Let $(Ju_n)$ and $v \in \mathcal{H}^J_{\text{aux}}$ be such that $Ju_n \to v$ and $(Ju_n)$ is $\hat{\mathcal{E}}$-Cauchy. Then $(Pu_n)$ is a Cauchy sequence in $(\mathcal{H}^J_{\text{har}}, \mathcal{E}^J)$. Thereby there is $u \in \mathcal{H}^J_{\text{har}}$ such that $\mathcal{E}^J[Ju_n - u] = \mathcal{E}^J[Pu_n - Pu] \to 0$. Thus $v = Ju$ and
   \[ \hat{\mathcal{E}}[Ju_n - Ju] = \mathcal{E}[Pu_n - Pu] \to 0, \] showing that $\hat{\mathcal{E}}$ is closed. \hfill \Box
2.2 The general case

Here we no longer assume positivity of $\mathcal{E}$, but lower semi-boundedness: there is a real $c \geq 0$ such that

$$\mathcal{E}[u] \geq -c\|u\|^2, \ \forall \ u \in \text{dom} \mathcal{E}. $$

One is tempted to define $\check{\mathcal{E}}$ via formula (2.1). However, one is faced to a new problem with lower semi-boundedness of $\check{\mathcal{E}}$. If $\check{\mathcal{E}}$ were defined by (2.1) and if it were lower semibounded then there should be a real constant $c' \geq 0$ such that

$$\check{\mathcal{E}}[Ju] \geq -c'\|Ju\|_{\text{aux}}^2, \ \forall \ u \in \text{dom} \ J. $$

This leads, in particular to

$$\mathcal{E}[v] \geq 0, \ \forall \ v \in \ker J. $$

Unfortunately this is a strong restriction.

Being inspired by Theorem 2.1, we define

$$\check{\mathcal{E}}[Ju] := \inf \{\mathcal{E}[v], \ v \in \mathcal{H}^J_{\text{har}} \cap C_u\}, \ \forall \ u \in \text{dom} \ J. \tag{2.11}$$

We first note that, thanks to Theorem 2.1, this definition coincides with the former one for positive forms.

We first solve the problem of lower semi-boundedness.

**Lemma 2.1.** The functional $\check{\mathcal{E}}$ is lower semibounded if and only if there is a real $c' \geq 0$ such that

$$\mathcal{E}[v] \geq -c'\|Jv\|_{\text{aux}}^2, \ \forall \ v \in \mathcal{H}^J_{\text{har}}. \tag{2.12}$$

**Proof.** Obviously if $\check{\mathcal{E}}$ is lower semibounded then inequality (2.12) holds true. Conversely, if (2.12) holds true, then for any $u \in \text{dom} \ J$, $v \in \mathcal{H}^J_{\text{har}} \cap C_u$ we obtain

$$\mathcal{E}[v] \geq -c'\|Jv\|_{\text{aux}}^2 = -c'\|Ju\|_{\text{aux}}^2. $$

Taking the infimum over $\mathcal{H}^J_{\text{har}} \cap C_u$ we deduce that $\check{\mathcal{E}}$ is lower semibounded. \qed

Concerning uniqueness of infimum, Theorem 2.1 still holds in this general framework. Under condition (2.12), Theorem 2.2 still holds true as well. For, one has to change $\mathcal{E}$ by $\mathcal{E} + c'J$. Moreover, Theorem 2.4. still holds in this general framework.

**Theorem 2.5.**

1. Assume that (2.9) holds. Then For every $u \in \text{dom} \ J$ there is a unique $Pu \in \mathcal{H}^J_{\text{har}} \cap C_u$ such that $\check{\mathcal{E}}[Ju] = \mathcal{E}[Pu]$.

2. Assume that (2.9) and (2.12) hold. Then the form $\check{\mathcal{E}}$ is closed if and only if $(\mathcal{H}^J_{\text{har}}, \mathcal{E}^{(1+c')J})$ is a Hilbert space.

The proof runs exactly as the positive case, so we omit it.
3 The singular D-to-N operator

Let $\mathcal{E}$ be a positive form and $\lambda \in \mathbb{R}$. Set

$$\mathcal{E}_\lambda := \mathcal{E} - \lambda.$$ 

We introduce the quadratic form

$$\mathcal{E}_D : \text{dom}\, \mathcal{E}_D = \ker J, \quad \mathcal{E}_D[u] = \mathcal{E}[u].$$

The subscript D stands for 'Dirichlet'.

Let us stress that the positivity assumption for $\mathcal{E}$ is not crucial. For, if $\mathcal{E}$ is lower semi-bounded one can shift it to get a positive form.

We assume that $\ker J$ is dense in $H$ and $\mathcal{E}_D$ is closed. Let $L_D$ be the positive selfadjoint operator related to $\mathcal{E}_D$. We suppose that $L_D$ has compact resolvent and designate by $\sigma_e(L_D)$ the set of eigenvalues of $L_D$.

In order do construct $\mathcal{E}_\lambda$ via the Dirichlet principle, we should have, among other conditions

$$\mathcal{H}^J_{\text{har}}(\lambda) \oplus \ker J = \text{dom} J,$$

where $\mathcal{H}^J_{\text{har}}(\lambda) = \{u \in \text{dom} J, \mathcal{E}_\lambda(u, v) = 0, \forall v \in \ker J \}$. The condition $\mathcal{H}^J_{\text{har}}(\lambda) \cap \ker J = \{0\}$ forces $\lambda$ not to be an eigenvalue of $L_D$. Hence from know on we assume

$$\lambda \in \mathbb{R} \setminus \sigma_e(L_D). \quad (3.1)$$

Lemma 3.1. The following two conditions are equivalent:

1. $\text{dom} J = \mathcal{H}^J_{\text{har}} \oplus \ker J$.

2. $\text{dom} J = \mathcal{H}^J_{\text{har}}(\lambda) \oplus \ker J$, for every $\lambda \in \mathbb{R} \setminus \sigma_e(L_D)$.

Proof. The implication 2 $\Rightarrow$ 1 is obvious. Let us prove the reversed implication.

Suppose $\mathcal{H}^J_{\text{har}} \oplus \ker J = \text{dom} J$. Let $u \in \text{dom} J$. We already know from Theorem 2.1 that if $\mathcal{H}^J_{\text{har}} \cap \ker J = \{0\}$ then $\mathcal{H}^J_{\text{har}} \cap C_u = \{Pu\}$. Let $\lambda \in \mathbb{R} \setminus \sigma_e(L_D)$. Set

$$w_\lambda := \lambda(L_D - \lambda)^{-1}Pu, \quad u_\lambda := Pu + w_\lambda.$$ 

Then $w_\lambda \in \ker J$. On the one hand $Ju_\lambda = JPu = Ju$, yielding thereby $u - u_\lambda \in \ker J$. On the other one, a straightforward computation shows that $u_\lambda \in \mathcal{H}^J_{\text{har}}(\lambda)$. Hence $u = (u - u_\lambda) + u_\lambda$ is the sum of an element from $\ker J$ and an element from $\mathcal{H}^J_{\text{har}}(\lambda)$. Let us prove uniqueness of the latter decomposition. Let $u \in \mathcal{H}^J_{\text{har}}(\lambda) \cap \ker J$. Then $\mathcal{E}(u, v) = \lambda(u, v)$ for any $v \in \ker J$. As $u \in \ker J$, if $u \neq 0$ then $\lambda$ is an eigenvalue of $L_D$, which is a contradiction and the proof is finished.

As we aim for defining $\mathcal{E}_\lambda$ via Dirichlet principle, we adopt from now on, the following assumption:

$$\text{dom} J = \mathcal{H}^J_{\text{har}} \oplus \ker J. \quad (3.2)$$
Under assumption (3.2), according to the latter lemma together with Theorem 2.5, we are able to define \( \tilde{E}_\lambda \) via the Dirichlet principle:

\[
\text{dom } \tilde{E}_\lambda = \text{ran } J, \quad \tilde{E}_\lambda [Ju] = \inf \{ E_\lambda [v], \ v \in \mathcal{H}^J_{\text{har}}(\lambda) \cap C_u \}
\]

\[
= \min \{ E_\lambda [v], \ v \in \mathcal{H}^J_{\text{har}}(\lambda) \cap C_u \} = E_\lambda [P_\lambda u],
\]

for any \( \lambda \in \mathbb{R} \setminus \sigma_e(L_D) \). Here \( P_\lambda u \) is the component of \( u \) from \( \mathcal{H}^J_{\text{har}}(\lambda) \) corresponding to the direct sum decomposition \( \text{dom } J = \mathcal{H}^J_{\text{har}}(\lambda) \oplus \ker J \). 

For \( \lambda = 0 \) we shall denote \( \tilde{E}_0 \) simply by \( \tilde{E} \) and \( P_0 \) by \( P \).

Let us define the abstract Poisson kernel operator, \( \Pi \) as follows:

\[
\Pi : \text{dom } \Pi = \text{ran } J \subset \mathcal{H} \rightarrow \mathcal{H}^J_{\text{har}}, \quad \text{such that } \Pi J = P.
\] (3.3)

Observe that assumption (3.2) ensures that \( \Pi \) is well defined.

**Lemma 3.2.** The operator \( \Pi \) is an isometric isomorphism from the normed space \( (\text{ran } J, \tilde{E}^{-1}) \) into the normed space \( (\mathcal{H}^J_{\text{har}}, E_J) \). Moreover, it holds

\[
\Pi = (J|_{\mathcal{H}^J_{\text{har}}})^{-1}.
\]

We are in position now to establish a representation formula for \( \tilde{E}_\lambda \) which will play a decisive role for investigating its properties.

**Theorem 3.1** (A representation formula). Let \( u \in \text{dom } J \) and \( \psi = Ju \). Then

\[
E_\lambda [\psi] = \tilde{E} [\psi] - \lambda (L_D (L_D - \lambda)^{-1} \Pi \psi, \Pi \psi).
\] (3.5)

**Proof.** Let \( u, \psi \) be as in the theorem. Set

\[
K := L_D^{-1}, \ u_\lambda := P_\lambda u, \ \psi := \Pi \psi \text{ and } w_\lambda := \lambda (1 - \lambda K)^{-1} K v = \lambda K (1 - \lambda K)^{-1} v.
\]

Then \( v \in \mathcal{H}^J_{\text{har}}, \ Jv = \psi = Ju \) and \( w_\lambda \in \ker J \). Let \( w \in \ker J \). Then

\[
E_\lambda (v + w_\lambda, w) = E_\lambda (v, w) + E_\lambda (w_\lambda, w) = -\lambda (v, w) + \lambda E_{D,\lambda}((L_D - \lambda)^{-1} v, w)
\]

\[
= -\lambda (v, w) + \lambda (v, w) = 0.
\] (3.6)
Hence \( u_\lambda = v + w_\lambda \).

By definition of \( \mathcal{E}_\lambda \) we have \( \mathcal{E}_\lambda[\psi] = \mathcal{E}_\lambda[u_\lambda] \). Observing that \( v \) and \( w_\lambda \) are \( \mathcal{E} \)-orthogonal, we accordingly obtain:

\[
\mathcal{E}_\lambda[\psi] = \mathcal{E}[v + w_\lambda] - \lambda\|v + w_\lambda\|^2 = \mathcal{E}[v] + \mathcal{E}[w_\lambda] - \lambda\|v + w_\lambda\|^2
\]

As \( w_\lambda = \lambda(L_D - \lambda)^{-1}v \) we get

\[
\mathcal{E}_\lambda[w_\lambda] = \lambda(v, w_\lambda).
\]

In particular, \( (v, w_\lambda) \) is real. Thus

\[
\mathcal{E}_\lambda[\psi] = \mathcal{E}[\psi] - \lambda\|v\|^2 - \lambda(v, w_\lambda).
\]

Having the formulae of \( v \) and \( w_\lambda \) in mind we achieve

\[
\mathcal{E}_\lambda[\psi] = \mathcal{E}[\psi] - \lambda\|\Pi\psi\|^2 - \lambda^2((1 - \lambda K)^{-1}K\Pi\psi, \Pi\psi)
\]

and the proof is finished.

\[ \square \]

**Remark 3.1.** Formula \((3.5)\) highlights the connection between, Dirichlet Laplacian, Dirichlet principle, Poisson kernel, and D-to-N operator. Furthermore it highlights the singular part of \( \mathcal{E}_\lambda \).

For \( \mathcal{E}_\lambda \) to define a lower semibounded closed form for any \( \lambda \in \mathbb{R} \setminus \sigma(L_D) \), according to Theorem \((2.5)\) we have to impose further restrictions. Hence from know on we shall assume, unless otherwise stated, that: for any \( \lambda \in \mathbb{R} \setminus \sigma_e(L_D) \) there is \( c_\lambda > 0 \) such that

\[
\mathcal{E}_\lambda[u] \geq -c_\lambda\|J u\|^2_{\text{aux}}, \quad \forall u \in \mathcal{H}^J_{\text{har}}(\lambda) \tag{3.9}
\]

and

\[
(\mathcal{H}^J_{\text{har}}(\lambda), \mathcal{E}_\lambda^{1+c_\lambda J}) \text{ is a Hilbert space.} \tag{3.10}
\]

On the light of Theorem \((2.5)\) under assumptions \((3.9)-(3.10)\) together with Lemma \((3.1)\) the form \( \hat{\mathcal{E}}_\lambda \) is lower semibounded densely defined and closed.

Henceforth, we designate by \( \tilde{L}_\lambda \) the selfadjoint operator related to \( \mathcal{E}_\lambda \) via Kato representation theorem. For \( \lambda = 0 \), the operator \( \tilde{L}_0 \) will be denoted simply by \( \tilde{L} \).

At this stage we would like to emphasize that similar construction for \( \hat{\mathcal{E}}_\lambda \) was developed in \([\text{Pos16}]\) via the concept of ’boundary pairs’. However, under the additional stronger assumptions that \( \mathcal{E} \) is closed, \( J : (\mathcal{D}, \mathcal{E}_1^{1/2}) \rightarrow \mathcal{H}_{\text{aux}} \) is bounded and the boundary pair is elliptically regular.

### 4 The asymptotic

In order to perform asymptotic in the complex plane we shall first extend the trace from to complex numbers by extending formula \((3.5)\). Precisely, for every \( z \in \mathbb{C} \setminus \sigma_e(L_D) \) we define

\[
\hat{\mathcal{E}}_z[\psi] := \hat{\mathcal{E}}[\psi] - z(L_D(L_D - z)^{-1}\Pi\psi, \Pi\psi), \quad \forall \psi \in \text{ran } J. \tag{4.1}
\]
Formula (4.1) shows that the mapping

\[ z \mapsto \hat{\mathcal{E}}_z[\psi], \]

is meromorphic with poles the eigenvalues of \( L_D \). Owing to selfadjointness of \( L_D \) they are all simple poles.

From now on we designate by \( E \) any eigenvalue of \( L_D \) and \( P_E \) its associated eigenprojection.

### 4.1 Laurent and Mittag–Leffler expansions for the form

**Theorem 4.1** (Laurent expansion). Let \( E \) be an eigenvalue of \( L_D \) and \( P_E \) be its associated eigenprojection. Let \( C_E \) be a positively oriented small circle around \( E \). Set

\[
A_0 = \frac{1}{2i\pi} \int_{C_E} (z - E)^{-1}(L_D - z)^{-1} \, dz \quad \text{and} \quad r_E = \|A_0\|.
\]

Then for every \( \psi \in \text{ran} \, J \) it holds

\[
\hat{\mathcal{E}}_z[\psi] = \hat{\mathcal{E}}[\psi] + \frac{z}{z - E}\|L_D^{1/2}P_E\Pi\psi\|^2 - z \sum_{k=0}^{\infty} (z - E)^k\|L_D^{1/2}A_0^{k+1}\Pi\|2, \quad 0 < |z - E| < r_E,
\]

where the series is absolutely convergent.

**Proof.** From the standard theory of meromorphic operator valued functions and since \( E \) is a simple pole for \( (L_D - z)^{-1} \), the following Laurent expansion holds true

\[
(L_D - z)^{-1} = \frac{A_{-1}}{z - E} + \sum_{k=0}^{\infty} (z - E)^kA_k, \quad 0 < |z - E| < r_E \quad \text{uniformly},
\]

where \( A_k = A_0^{k+1} \) and \( A_{-1} = -P_E \). Finally making use of the representation formula (4.1) for \( \hat{\mathcal{E}}_z \) together with continuity os the scalar product, we get the desired expansion. \(\square\)

**Corollary 4.1.** The following asymptotic behavior is true:

\[
\lim_{z \to E} ((z - E)\hat{\mathcal{E}}_z[\psi]) = E\|L_D^{1/2}P_E\Pi\psi\|^2, \quad \forall \psi \in \text{ran} \, J.
\]

It follows in particular,

1. \( \lim_{\lambda \uparrow E} \hat{\mathcal{E}}_\lambda[\psi] = -\infty \) for all \( \psi \in \text{ran} \, J \).
2. \( \lim_{\lambda \downarrow E} \hat{\mathcal{E}}_\lambda[\psi] = \infty \) for all \( \psi \in \text{ran} \, J \).
3. \( |\hat{\mathcal{E}}_z[\psi]| \) grows as fast as \( E|z - E|^{-1}\|L_D^{1/2}P_E\Pi\psi\|^2 \) when approaching the singularity \( E \).
The corollary derives directly from Theorem 4.1, so we omit its proof. We proceed now to establish Mittag-Leffler expansion for \( \hat{\mathcal{E}}_z \).

Let \( E_0 \leq E_1 \leq \cdots \leq E_k \cdots \) be the increasing arrangement for the eigenvalues of \( L_D \) where each \( E_k \) is repeated as many times as its multiplicity. Let \( (u_k) \) be the corresponding orthonormal basis of eigenfunctions.

**Theorem 4.2** (Mittag-Leffler expansion). Let \( z \in \mathbb{C} \setminus \sigma_e(L_D) \) and \( \psi \in \text{ran} J \). Then

\[
\hat{\mathcal{E}}_z[\psi] = \hat{\mathcal{E}}[\psi] + z \sum_{k=0}^{\infty} \frac{E_k}{E_k - z} |(\Pi \psi, u_k)|^2 ,
\]

(4.3)

where the series converges absolutely.

**Proof.** By the spectral theorem we get, for every \( u \in \mathcal{H} \)

\[
L_D(L_D - z)^{-1} u = \sum_{k=0}^{\infty} \frac{E_k}{E_k - z} (u, u_k) u_k .
\]

Making use of the representation formula (4.1) together with the continuity of the scalar product we obtain the sought formula.

**Remark 4.1.**

1. The connotation 'Mittag–Leffler expansion' is justified by the fact that the expansion can be written in the form

\[
\hat{\mathcal{E}}_z[\psi] = \hat{\mathcal{E}}[\psi] + z \sum_{k=0}^{\infty} |(\Pi \psi, u_k)|^2 \left( \sum_{k=0}^{\infty} \frac{E_k^2}{|E_k|^2 + E_k(z + E_k)} \right) |(\Pi \psi, u_k)|^2.
\]

This is plainly the Mittag-Leffler expansion for \( \hat{\mathcal{E}}_z \).

2. For later use, we emphasize that the expansion can also be written in an other form. For, let \( m_k \) be the multiplicity of \( E_k \) and \( (u_{1k}, \ldots, u_{mk}) \) be an orthonormal eigenbasis for \( E_k \). Then according to the expansion (4.3) we have

\[
\hat{\mathcal{E}}_z[\psi] = \hat{\mathcal{E}}[\psi] + z \sum_{k=0}^{\infty} \frac{E_k}{E_k - E_k} \sum_{l=1}^{m_k} |(\Pi \psi, u_{lk})|^2 .
\]

(4.4)

**4.2 The eigenvalues near the poles**

Assume that \( \hat{L}_\lambda \) has compact resolvent for some (and hence every) \( \lambda \in \mathbb{R} \setminus \sigma_e(L_D) \). Let us turn our attention to study properties of eigenvalues of \( \hat{L}_\lambda \) near the singularities. To that and we shall establish a monotony property for \( \hat{\mathcal{E}}_\lambda \).

**Lemma 4.1.** For every fixed \( \psi \in \text{ran} J \), the mapping \( \lambda \mapsto \hat{\mathcal{E}}_\lambda[\psi] \) is strictly decreasing on each interval of \( \mathbb{R} \setminus \sigma_e(L_D) \).
Proof. According to formula (3.1), the form $\tilde{\mathcal{E}}_\lambda[\psi]$ is $\lambda$-differentiable. Moreover, making use of the first resolvent formula, we obtain
\[
\frac{d}{d\lambda} \tilde{\mathcal{E}}_\lambda[\psi] = -(L_D(L_D - \lambda)^{-2}\Pi\psi, \Pi\psi) = -\|L^{1/2}_D(L_D - \lambda)^{-1}\Pi\psi\|^2 \leq 0, \ \forall \lambda \in \mathbb{R} \setminus \sigma_e(L),
\]
which was to be proved.

Theorem 4.3. Let $\tilde{E}(\lambda)$ be an eigenvalue of $\tilde{\mathcal{E}}_\lambda$.

1. The mapping 
\[
\mathbb{R} \setminus \sigma_e(L_D) \to \mathbb{R}, \ \lambda \mapsto \tilde{E}(\lambda),
\]
is strictly decreasing on each interval of $\mathbb{R} \setminus \sigma(L_D)$.

2. Let $E$ be any eigenvalue of $L_D$ which is a singularity for $\tilde{E}(\lambda)$. Then
\[
\lim_{\lambda \uparrow E} \tilde{E}(\lambda) = -\infty, \ \lim_{\lambda \downarrow E} \tilde{E}(\lambda) = \infty.
\]

Proof. The first assertion is consequence of the min-max principle for successive eigenvalues together with Lemma 4.1. The second assertion follows from monotony of $\tilde{E}(\lambda)$ and the fact that $E$ is a singularity for $\tilde{E}(\lambda)$.

Remark 4.2. We stress that not every eigenvalue of $L_D$ is a singularity for $\tilde{E}(\lambda)$. Concrete examples for this fact can be found in [Dan14] or in the examples analyzed at the end of the current paper.

5 Positivity preservation

In this section we assume that $\mathcal{H} = L^2(X, m)$ and $\mathcal{H}_{aux} = L^2(X, \mu)$ (real Hilbert spaces), where $(X, m), (X, \mu)$ are $\sigma$-finite measure spaces and $m, \mu$ are positive measures on some $\sigma$-algebras of $X$.

We maintain the assumption that $\mathcal{E}$ is positive together with assumptions ensuring closedness of $\tilde{\mathcal{E}}_\lambda$ from the latter section. Furthermore, we assume that the form $\mathcal{E}$ is closed and is a semi-Dirichlet form, i.e. its related semigroups
\[
e^{-tL}, \ t > 0 \text{ is positivity preserving.}
\]
Equivalently,
\[
u \in \text{dom } \mathcal{E} \Rightarrow |u| \in \text{dom } \mathcal{E} \text{ and } \mathcal{E}[|u|] \leq \mathcal{E}[u],
\]
or (Beurling–Deny criterion)
\[
u \in \text{dom } \mathcal{E} \Rightarrow u^\pm \in \text{dom } \mathcal{E} \text{ and } \mathcal{E}(u^+, u^-) \leq 0.
\]
Thereby the form $\mathcal{E}_D$ is a semi-Dirichlet form as well and hence its related semigroup, $e^{-tL_D}, \ t > 0 \text{ is also positivity preserving.}$

Obviously $\mathcal{E}_\lambda$ is a semi-Dirichlet form for every $\lambda \in \mathbb{R}$.  

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Let \( T_t(\lambda) := e^{-tL_\lambda}, \ t > 0 \) be the semigroup related to the form \( \mathcal{E}_\lambda \). We shall exploit the already established asymptotic to discuss into which extent the positivity preservation property is inherited by the semigroup of the D-to-N operator, \( T_t(\lambda) \). It is expected that positivity property will depend on \( \lambda \).

We shall use the abbreviation p.p. to mean 'positivity preserving'.

At this stage we mention that some partial results concerning positivity in one and two dimensions can be found in [Dan14] and for bounded Lipschitz domain in [AM12].

It is not possible to go ahead without some additional assumptions on the map \( J \). Henceforth we assume that \( \text{dom} \ J = \text{dom} \ E = D \) furthermore \( u \in D \Rightarrow |Ju| = J|u| \).

Let us first investigate positivity of \( \Pi \).

**Lemma 5.1.** Let \( \psi \in \text{ran} J \) be positive. Then \( \Pi \psi \) is positive as well.

**Proof.** Let \( \psi \in \text{ran} J \) be positive and \( u \in D \) such that \( Ju = \psi \). By assumption (5.2) we get \( Ju = J|u| \geq 0 \). Thus we may and shall assume that \( u \geq 0 \). By assumption (5.2) once again we obtain \( J(|Pu|) = |JPu| = Ju \). Owing to Dirichlet principle we get \( \mathcal{E}[Ju] = \mathcal{E}[Pu] \leq \mathcal{E}[|Pu|] \). On the other hand as \( \mathcal{E} \) is a semi-Dirichlet form we get \( \mathcal{E}[|Pu|] \leq \mathcal{E}[Pu] \). Hence from uniqueness we derive \( Pu = |Pu| \geq 0 \). Thus \( \Pi \psi = Pu \geq 0 \).

**Lemma 5.2.** \( T_1(0) \) is p.p.

**Proof.** We shall prove that \( \mathcal{E} \) is a semi-Dirichlet form. Let \( u \in D \). Then by Dirichlet principle we have

\[
\mathcal{E}[|Ju|] = \mathcal{E}[J|u|] = \inf \{ \mathcal{E}[v], \ v \in D, \ Jv = J|u| \}.
\]

Now if \( v \) is such that \( Jv = J|u| \) then, by assumption (5.2) we get \( J|v| = |Jv| = J|u| = Jv \). Hence making use of the semi-Dirichlet property for \( \mathcal{E} \) we achieve

\[
\mathcal{E}[|Ju|] \leq \inf \{ \mathcal{E}[|v|], \ v \in D, Jv = Ju \} \\
\leq \inf \{ \mathcal{E}[v], \ v \in D, Jv = Ju \} = \mathcal{E}[Ju],
\]

which completes the proof.

**Proposition 5.1.** Let \( E_0 \) be the smallest eigenvalue of \( L_D \). Then for every \( \lambda < E_0 \), the semigroup \( T_t(\lambda) \) is p.p.

**Proof.**

1. **Step 1:** \( \lambda \leq 0 \). Then \( \mathcal{E}_\lambda \) is a positive semi-Dirichlet form. Hence Lemma 5.2 applied to \( \mathcal{E}_\lambda \) instead of \( \mathcal{E} \) yields the p.p. for \( T_t(\lambda) \).

2. **Step 2:** \( 0 < \lambda < E_0 \). Here we use Beurling–Deny criterion together with the representation formula. By polarization we get

\[
\mathcal{E}_\lambda(\psi^+,\psi^-) = \mathcal{E}(\psi^+,\psi^-) - \lambda(L_D(L_D - \lambda)^{-1}\Pi\psi^+\Pi\psi^-) \\
= \mathcal{E}(\psi^+,\psi^-) - \lambda(\Pi\psi^+\Pi\psi^-) - \lambda^2((L_D - \lambda)^{-1}\Pi\psi^+\Pi\psi^-) \quad (5.3)
\]
According to Lemma 5.2, the first term is negative, whereas the second term is negative owing to Lemma 5.1. As \( \lambda < E_0 \), we have

\[
(L_D - \lambda)^{-1} = \int_0^\infty e^{-\lambda t} T_i u dt.
\]

As \( T_i \) is p.p. we conclude that \((L_D - \lambda)^{-1}\) is p.p. as well. Hence the third term is also negative, leading to \( \hat{\mathcal{E}}(\psi^+, \psi^-) \leq 0 \) and the proof is finished.

We proceed now to analyze positivity of \( \hat{T}_t(\lambda) \) for \( \lambda > E_0 \). To achieve our purpose we shall utilize Mittag–Leffler expansion for \( \hat{\mathcal{E}}_\lambda \).

**Theorem 5.1.** Let \( \lambda > E_0 \) and \( E \) be an eigenvalue of \( L_D \) with multiplicity \( m \). Let \((v_1, \ldots, v_m)\) be an orthonormal basis for \( \ker(L_D - E) \). Then \( \hat{T}_t(\lambda), t > 0 \) is p.p. on a left (resp. right) neighborhood of \( E \) if and only if one of the following equivalent conditions is fulfilled

1. \[
\sum_{k=1}^m (\Pi \psi^+, v_k) \cdot (\Pi \psi^-, v_k) \geq 0, \quad (\text{resp.} \leq 0), \quad \forall \psi \in \text{ran} J.
\]

2. \[
(L_D P_E \Pi \psi^+, \Pi \psi^-) \geq 0, \quad (\text{resp.} \leq 0), \quad \forall \psi \in \text{ran} J.
\]

**Proof.** We shall use Beurling–Deny criterion. Let \( \psi \in \text{ran} J \). By polarization and according to formula (4.2) we obtain

\[
\hat{\mathcal{E}}_\lambda(\psi^+, \psi^-) = \hat{\mathcal{E}}(\psi^+, \psi^-) + \lambda \sum_{k=0}^\infty \frac{E_k}{\lambda - E_k} (\Pi \psi^+, u_k)(\Pi \psi^-, u_k).
\]

(5.4)

Thereby the leading term in the expansion near \( E \) is

\[
\frac{\lambda E}{\lambda - E} \sum_{k=1}^m (\Pi \psi^+, v_k) \cdot (\Pi \psi^-, v_k).
\]

Accordingly, \( \hat{\mathcal{E}}(\psi^+, \psi^-) \) is negative in a left (resp. right) neighborhood of \( E \) if and only if

\[
\sum_{k=1}^m (\Pi \psi^+, v_k) \cdot (\Pi \psi^-, v_k) \geq 0, \quad (\text{resp.} \leq 0).
\]

Now the selfadjointness of \( L_D \) yields \( \text{ran} P_E = \ker(L_D - E) \). Hence \( P_E = \sum_{k=1}^m (\cdot, v_k) v_k \) and

\[
(L_D P_E \Pi \psi^+, \Pi \psi^-) = E \sum_{k=1}^m (\Pi \psi^+, v_k)(\Pi \psi^-, v_k).
\]

Thus both conditions of the theorem are equivalent and by Beurling–Deny criterion they are both equivalent to positivity to the left (resp. to the right).
Remark 5.1. The latter theorem yields the following observations for the semigroup $\tilde{T}_t(\lambda)$ with $\lambda > E_0$:

1. The semigroup $\tilde{T}_t(\lambda)$ can not be simultaneously p.p. on both sides of any singularity.
2. If it is p.p. on one side then it is necessary non p.p. on the other one.
3. It might be non p.p. on both sides of some singularities.

Theorem 5.1 leads immediately to the following conclusions:

Corollary 5.1. Under assumptions of Theorem 5.1, suppose there are $\psi_1, \psi_2 \in \text{ran } J$ such that the sums

$$\sum_{k=1}^{m} (\Pi \psi_1^+, u_k) \cdot (\Pi \psi_1^-, u_k) \quad \text{and} \quad \sum_{k=1}^{m} (\Pi \psi_2^+, u_k) \cdot (\Pi \psi_2^-, u_k),$$

have opposite signs. Then $\tilde{T}_t(\lambda)$ is not p.p. in any neighborhood of $E$.

Corollary 5.2. Assume that $E$ is simple and that the associated eigenfunction, $u_E$ can be chosen to have constant sign. Then $\tilde{T}_t(\lambda)$ is p.p. to the left of $E$ whereas it is non p.p. to the right of $E$.

We close this section with some topological properties concerning $\tilde{E}_\lambda$ and the set of $\lambda$ where $\tilde{T}_t(\lambda)$ is positive.

For the concept of Mosco convergence we refer the reader to [Mos94].

Lemma 5.3. Let $\lambda \in \mathbb{R} \setminus \sigma_e(L_D)$ and $(\lambda_n) \subset \mathbb{R} \setminus \sigma_e(L_D)$ such that $\lambda_n \to \lambda$. Then the sequence $\tilde{E}_{\lambda_n}$ converges to $\tilde{E}_{\lambda}$ in the sense of Mosco.

Proof. We shall show that left and right Mosco limits of $\tilde{E}_{\lambda_n}$ both coincide with $\tilde{E}_{\lambda}$. As the topology of Mosco convergence is metrizable we get the result.

Assume that $\lambda_n \uparrow \lambda$. Then by Lemma 5.1 $\tilde{E}_{\lambda_n}[\psi] \downarrow \tilde{E}_\lambda[\psi]$. As Mosco convergence is equivalent to strong resolvent convergence, according to Kato monotone convergence theorem, [Kat95] Theorem 3.11, p.459] of quadratic forms, we obtain $\tilde{E}_{\lambda_n} \to \tilde{E}_\lambda$ in the sense of Mosco.

Now assume that $\lambda_n \downarrow \lambda$. By the same arguments as before together with [Kat95] Theorem 3.13a, p.461] we conclude that $\tilde{E}_{\lambda_n} \to \tilde{E}_\lambda$ in the sense of Mosco. \(\square\)

Proposition 5.2. The set

$$\mathcal{P} := \{ \lambda \in \mathbb{R} \setminus \sigma_e(L_D): \tilde{T}_t(\lambda) \text{ is p.p.} \}$$

is closed in $\mathbb{R} \setminus \sigma(L_D)$.

Proof. Let $\lambda \in \mathbb{R} \setminus \sigma_e(L_D)$ and $(\lambda_n) \subset \mathcal{P}$ such that $\lambda_n \to \lambda$. According to Lemma 5.3 together with the fact Mosco convergence yields strong convergence of the related semigroups, we obtain, for any $0 \leq \psi \in L^2(X, \mu)$:

$$0 \leq \tilde{T}_t(\lambda_n) \psi \to \tilde{T}_t(\lambda) \psi \text{ in } L^2(X, \mu),$$

and then $\tilde{T}_t(\lambda) \psi \geq 0$, yielding positivity of $\tilde{T}_t(\lambda)$.

\(\square\)
6 Examples

6.1 The singular D-to-N operator on Lipschitz domains

In this section we shall use the elaborated theory from former sections to study D-to-N operators related to Neumann Laplacian on Lipschitz domains.

Let \( \Omega \subset \mathbb{R}^d \) be a nonempty open bounded connected subset with Lipschitz boundary \( \Gamma \) and \( \mathcal{E} \) the gradient Dirichlet form on \( H^1(\Omega) \):

\[
D(\mathcal{E}) = H^1(\Omega), \quad \mathcal{E}[u] = \int_{\Omega} |\nabla u|^2 \, dx, \quad \forall u \in H^1(\Omega).
\]

(6.1)

It is well known that the quadratic form \( \mathcal{E} \) is closed and densely defined in \( L^2 := L^2(\Omega, dx) \). Moreover, \( \mathcal{E} \) is a Dirichlet form, i.e.,

\[
u \in H^1(\Omega) \Rightarrow \nu_{0,1} := (\nu \vee 0) \wedge 1 \in H^1(\Omega) \text{ and } \mathcal{E}[\nu_{0,1}] \leq \mathcal{E}[\nu].
\]

The positive selfadjoint operator associated to \( \mathcal{E} \), which we denote by \( L \) is commonly named the Neumann Laplacian on \( \Omega \). As \( \mathcal{E} \) is a Dirichlet form its related semigroup \( e^{-tL} \), \( t > 0 \) is Markovian and hence is p.p.

Let \( dS \) be the surface measure on \( \Gamma \) (the \((d-1)-\text{Hausdorff measure of } \Gamma \)). Set \( L^2(\Gamma) := L^2(\Gamma, dS) \) and let \( J \) be the operator ‘trace to the boundary’

\[
J : H^1(\Omega) \to L^2(\Gamma), \quad J \mapsto u|_{\Gamma}.
\]

Then \( J \) is bounded (see Proposition 6.2). Moreover it is well known that

\[
\ker J = H^1_0(\Omega) \quad \text{and} \quad \text{ran } J = H^{1/2}(\Gamma),
\]

and \( J \) has dense range. Thus \( \mathcal{E}_D \) is the closed quadratic from associated with the Dirichlet Laplacian on \( \Omega \) and is a Dirichlet form as well. Owing to boundedness of \( \Omega \) its well known that \( L_D \) has compact resolvent with simple smallest eigenvalue \( E_0 > 0 \).

In this case we have

\[
\mathcal{H}^l_{\text{har}}(\lambda) = \{u \in H^1(\Omega), \, -\Delta u - \lambda u = 0 \text{ on } \Omega\}.
\]

According to [FMM98, Theorem 10.1], for any \( \psi \in H^{1/2}(\Gamma) \) there is a unique \( u \in H^1(\Omega) \) such that

\[
\begin{cases}
-\Delta u = 0, & \text{in } \Omega \\
u = \psi, & \text{on } \Gamma
\end{cases}
\]

All these considerations lead to the decomposition \( H^1(\Omega) = H^1_0(\Omega) \oplus \mathcal{H}^l_{\text{har}} \).

Thus all conditions are fulfilled to define \( \mathcal{E}_\lambda \) via Dirichlet principle. In fact, let \( \lambda \in \mathbb{R} \setminus \sigma_e(L_D) \) and \( \psi \in H^{1/2}(\Gamma) \). Then \( P_\lambda u \) is the unique element from \( H^1(\Omega) \) which solves the boundary value problem

\[
\begin{cases}
-\Delta P_\lambda u - \lambda P_\lambda u = 0, & \text{in } \Omega \\
P_\lambda u = \psi, & \text{on } \Gamma
\end{cases}
\]
and
\[ D(\tilde{\mathcal{E}}_{\lambda}) = H^{1/2}(\Gamma), \quad \tilde{\mathcal{E}}_{\lambda}[\psi] = \mathcal{E}_{\lambda}[P_{\lambda}u] = \int_{\Omega} |\nabla P_{\lambda}u|^2 \, dx - \lambda \int_{\Omega} (P_{\lambda}u)^2 \, dx, \quad \forall \psi \in H^{1/2}(\Gamma). \]

We proceed to show that \( \tilde{\mathcal{E}}_{\lambda} \) is lower semibounded.

**Lemma 6.1.** There is a finite constant \( c > 0 \) such that
\[
\int_{\Omega} (\Pi \psi)^2 \, dx \leq c \int_{\Gamma} \psi^2 \, d\Gamma, \quad \forall \psi \in H^{1/2}(\Gamma). \tag{6.2}
\]

**Proof.** Let \( \psi \in H^{1/2}(\Gamma) \). Let \( G \) be the fundamental solution of the Laplacian on \( \mathbb{R}^d \). According to [FMM98, Identity 10.5], \( \Pi \psi \) is given by

\[
\Pi \psi(x) = \int_{\Gamma} G(x - y) S^{-1}(\psi(y)) \, dS, \quad x \in \Omega, \tag{6.3}
\]

where the operator \( S \) is as defined in [FMM98, p.10]. A routine computation leads to

\[
c_1 := \sup_{x \in \Omega} \int_{\Gamma} G(x - y) \, dS < \infty, \quad c_2 := \sup_{y \in \Gamma} \int_{\Omega} G(x - y) \, dx < \infty.
\]

Thus by Hölder inequality we obtain

\[
\int_{\Omega} (\Pi \psi)^2 \, dx \leq c_1 c_2 \int_{\Gamma} (S^{-1}(\psi(y)))^2 \, dS.
\]

According to [FMM98, Theorem 8.1], the operator \( S^{-1} \) operates on the whole space \( H^{1/2}(\Gamma) \). In particular, \( \int_{\Gamma} (S^{-1}(\psi(y)))^2 \, dS < \infty \), which completes the proof. \( \square \)

**Lemma 6.2.** Let \( \lambda \in \mathbb{R} \setminus \sigma_e(L_D) \). Then there is a finite constant \( c = c(\lambda) > 0 \) such that
\[
\tilde{\mathcal{E}}_{\lambda}[\psi] \geq -c \int_{\Gamma} \psi^2 \, dS, \quad \forall \psi \in H^{1/2}(\Gamma). \tag{6.4}
\]

**Proof.** According to the representation formula from Theorem 3.1 we have
\[
\tilde{\mathcal{E}}_{\lambda}[\psi] = \tilde{\mathcal{E}}[\psi] - \lambda (L_D (L_D - \lambda)^{-1} \Pi \psi, \Pi \psi)_{L^2(\Omega)}, \quad \forall \psi \in H^{1/2}(\Gamma).
\]

Hence the inequality of the lemma is automatically satisfied for \( \lambda \leq 0 \).

Let \( \lambda > 0 \). Let us quote that \( \mathcal{E} \geq 0 \). Therefore, owing to the representation formula together with Lemma 6.1 and Cauchy-Schwartz inequality we obtain
\[
\tilde{\mathcal{E}}_{\lambda}[\psi] \geq -\lambda \|L_D (L_D - \lambda)^{-1} \Pi \psi\|_{L^2(\Omega)}^2 \geq -c \|\psi\|_{L^2(\Gamma)}^2, \quad \forall \psi \in H^{1/2}(\Gamma), \tag{6.4}
\]

which was to be proved. \( \square \)
Proposition 6.1. The form \( \mathcal{E}_\lambda \) is closed.

Proof. Step 1: \( \lambda = 0 \). We shall use Theorem 2.4. Let \((u_n) \subset \mathcal{H}_{\text{har}}^J \) be \( \mathcal{E}^J \)-Cauchy. As \( \mathcal{E} \) is positive we get \( \|Ju_n - Ju_m\|_{L^2(\Gamma)} \to 0 \) and hence \( \mathcal{E}[u_n - u_m] \to 0 \). Recalling that \( \Pi Ju_n = u_n \), making use of Lemma (6.1) we achieve \( \|u_n - u_m\|_{L^2(\Omega)} \to 0 \). Thus there is \( u \in H^1(\Omega) \) such that \( u_n \rightharpoonup u \) in \( H^1(\Omega) \). Moreover, for any \( v \in C^\infty_c(\Omega) \) we get

\[ \mathcal{E}(u_n, v) \to \mathcal{E}(u, v). \]

As \((u_n) \subset \mathcal{H}_{\text{har}}^J \) we obtain that \( u \in \mathcal{H}_{\text{har}}^J \) as well and then \((\mathcal{H}_{\text{har}}^J, \mathcal{E}^J)\) is a Hilbert space. According to Theorem 2.4, \( \mathcal{E} \) is closed.

Stem 2: General \( \lambda \). We use the representation formula together with Theorem 2.5. Let \((u_n) \subset \mathcal{H}_{\text{har}}^J(\lambda)\) be \( \mathcal{E}_\lambda^{(1+cJ)} \)-Cauchy. Then \((Ju_n)\) is a Cauchy sequence in \( L^2(\Gamma) \) and \( \mathcal{E}_\lambda[u_n - u_m] \to 0 \). By Lemma 6.1 we get that \( \Pi Ju_n = Pu_n \) is a Cauchy sequence in \( L^2(\Omega) \). Now the representation formula yields

\[ \mathcal{E}_\lambda[u_n - u_m] = \tilde{\mathcal{E}}_\lambda[Ju_n - Ju_m] = \tilde{\mathcal{E}}[Ju_n - Ju_m] - \lambda(L_D(L_D - \lambda)^{-1}\Pi Ju_n - u_m, \Pi Ju_n - u_m). \]

Hence \((Ju_n)\) is \( \tilde{\mathcal{E}}_1 \)-Cauchy. By the first step, there is \( u \in \mathcal{H}_{\text{har}}^J \) such that

\[ \mathcal{E}[Pu_n - u] + \|Ju_n - Ju\|^2_{L^2(\Gamma)} \to 0. \]

Additional use of the representation formula together with Lemma 6.1 lead to

\[ \mathcal{E}_\lambda[u_n - u_\lambda] = \tilde{\mathcal{E}}_\lambda[Ju_n - Ju] = \mathcal{E}[Pu_n - u] \]

\[ - \lambda(L_D(L_D - \lambda)^{-1}\Pi Ju_n - u, \Pi Ju_n - u) \to 0. \quad (6.5) \]

As \( u_\lambda \in \mathcal{H}_{\text{har}}^J(\lambda) \) and \( Ju_\lambda = Ju \), we get \( \mathcal{E}_\lambda[u_n - u_\lambda] + (1+c)\|Ju_n - Ju_\lambda\|^2_{L^2(\Gamma)} \to 0 \). Thereby \((\mathcal{H}_{\text{har}}^J(\lambda), \mathcal{E}^{(1+cJ)})\) is a Hilbert space and \( \tilde{\mathcal{E}}_\lambda \) is closed.

Here \( \tilde{L}_\lambda \) is the D-to-N operator with respect to the boundary \( \Gamma \).

We close this subsection with a compactness result which was already proved in [AM12, Theorem 3.1]. Here we give a new proof.

Proposition 6.2. It holds

1. The operator \( J \) is compact.

2. For every \( \lambda \in \mathbb{R} \setminus \sigma_\epsilon(L_D) \) the operator \( \tilde{L}_\lambda \) has compact resolvent.

Proof. According to [JW84, Example 3, p.30] the measure \( dS \) is a \((d-1)\)-measure, i.e.: for some \( c_1, c_2 \),

\[ c_1r^{d-1} \leq \int_{B(x,r)} dS \leq c_2r^{d-1}, \forall x \in \Gamma, \ 0 < r \leq 1. \]

By [BA07, Lemma 6.1] there is finite constant \( c \) such that for \( d \geq 3, \ 2 < p \leq \frac{2(d-2)}{d-1} \) and for \( d = 2, \ p \geq 2 \) the following inequality is true

\[ (\int_\Gamma |u|^p dS)^{2/p} \leq c(\int_{\mathbb{R}^d} |\nabla u|^2 dx + \int_{\mathbb{R}^d} u^2 dx), \forall u \in H^1(\mathbb{R}^d). \quad (6.6) \]
On the other hand, according to [Ste70, Theorem 5, p. 181] there is bounded linear extension operator for $H^1(\Omega)$ into $H^1(\mathbb{R}^d)$. Thus the latter inequality holds on $H^1(\Omega)$ which in turn, according to [BA07, Theorem 7.1] yields compactness of $J$.

By [BBST19, Theorem 2.10] compactness of $J$ yields compactness of the resolvent of $\hat{L}$. Let $(\psi_k) \subset \text{ran } J$ be such that $\sup_k (\hat{E}_\lambda[\psi_k] + (1 + c)\|\psi_k\|_{L^2(\Gamma)}^2) < \infty$, $c = c(\lambda)$ is a lower bound for $\hat{E}_\lambda$. By the latter formula together with inequality (6.1) we conclude that $\sup_k (\hat{E}_\lambda[\psi_k] + \|\psi_k\|_{L^2(\Gamma)}^2) < \infty$. As $(\hat{L} + 1)^{-1}$ is compact there is a subsequence $(\psi_{k_j})$ and $\psi \in L^2(\Gamma)$ such that $\psi_{k_j} \to \psi$ in $L^2(\Gamma)$. This means that the embedding $(\text{ran } J, \hat{E}_\lambda + 1 + c) \to L^2(\Gamma)$ is compact which is in turn equivalent to compactness of $(\hat{L}_\lambda + 1 + c)^{-1}$.

\[ \square \]

### 6.2 Asymptotic and positivity

Regarding positivity property for $\hat{T}_t(\lambda)$ the problem is completely solved for the unit disc. Whereas for bounded Lipschitz domains it is proved in [AM12] that $\hat{T}_t(\lambda)$ is p.p. for $\lambda < E_0$.

Having the theoretical results from the former sections in hands we shall show that the latter property holds in general.

**Proposition 6.3.** The following assertions are true.

1. For every $\lambda \leq 0$, the semigroup $\hat{T}_t(\lambda)$ is sub-Markovian, i.e. $\hat{T}_t(\lambda)1 \leq 1$ for any $t > 0$.

2. For every $\lambda < 0$, the semigroup $\hat{T}_t(\lambda)$ is ultracontractive:

   \[ \hat{T}_t(\lambda) : L^2(\Gamma) \to L^\infty(\Gamma) \text{ is bounded, } \forall t > 0. \]

3. For every $\lambda < E_0$, the semigroup $\hat{T}_t(\lambda)$ is p.p.

**Proof.** As for $\lambda \leq 0$ the form $\mathcal{E}_\lambda$ is a Dirichlet form, the first assertion follows from [BBST19] and the fact that Dirichlet forms have sub-Markovian semigroups.

Observe that for $\lambda < 0$ the scalar products $\mathcal{E}_\lambda$ are equivalent on $H^1(\Omega)$. Thus, using inequality (6.6) together with Dirichlet principle we obtain a Sobolev type inequality: for some finite constant $c = c(\lambda)$ we have

\[ (\int \Gamma |\psi|^p dS)^{2/p} \leq c \mathcal{E}_\lambda[\psi], \forall \psi \in H^{1/2}(\Gamma). \quad (6.7) \]

It is well known that (see [Dav89, p.75]) Sobolev type inequality with $p > 2$ together with Dirichlet property for $\mathcal{E}_\lambda$ lead to ultracontractivity. The third assertion follows from Lemma 5.2. \[ \square \]

Now we proceed to establish necessary and sufficient conditions for p.p. near Dirichlet eigenvalues. These conditions will be enlightened by the asymptotic of $\mathcal{E}_\lambda$ in this special situation. In particular we shall show that near Dirichlet eigenvalues positivity depends
solely on the behavior of either the Dirichlet eigenfunctions or their normal derivatives. Let us first write the representation formula of $\tilde{E}_z$ for this particular case. To that end, we denote by $G_\Omega$ the Green kernel of the Dirichlet Laplacian on $\Omega$ and

$$K := L^2 \to L^2, \quad Ku = \int_\Omega G_\Omega(\cdot, y)u(y)\,dy.$$ 

Let $z \in \mathbb{R} \setminus \sigma_e(L_\Omega)$ and $\psi \in H^{1/2}$ be given. Consider $v, w_z$ solutions of

$$\begin{cases} -\Delta v = 0, & \text{in } \Omega, \\ v = \psi, & \text{on } \Gamma \end{cases} \quad (6.8)$$

and

$$\begin{cases} -\Delta w_z - zw_z = zv, & \text{in } \Omega, \\ w_z = 0, & \text{on } \Gamma \end{cases} \quad (6.9)$$

Then

$$v = \Pi \psi, \quad w_z - zKw_z = zKv, \quad \text{and } P_zu = v + w_z. \quad (6.10)$$

Since $z \notin \sigma_e(-\Delta_\Omega)$, the operator $(1 - zK)$ is invertible and

$$w_z = z(1 - zK)^{-1}Kv = zK(1 - zK)^{-1}v. \quad (6.11)$$

Thus in this situation the representation formula takes the form

$$\tilde{E}_z[\psi] = \tilde{E}[\psi] - z \int_\Omega \Pi \psi(1 - zK)^{-1}\Pi \psi \,dx. \quad (6.12)$$

For bounded Lipschitz domains, we shall show that the Mittag–Leffler expansion for the trace form has a simpler expression involving the trace form at $z = 0$, the eigenvalues and the normal derivatives of the eigenfunctions of the the Dirichlet Laplacian.

Henceforth, we designate by $\nu$ the outward normal unit vector on $\Gamma$.

We recall that according to [JK95, Theorems 1.1-1.3], if $u \in H^1(\Omega)$ and $\Delta u \in L^2(\Omega)$ then $u \in H^{3/2}(\Omega)$. By [JW84, Theorem 1, p.8] we obtain $u|_\Gamma \in H^1(\Gamma)$. Thus the uniqueness part of [PMM98, Theorem 10.1] leads to $\frac{\partial u}{\partial \nu}|_\Gamma \in L^2(\Gamma)$.

Moreover, owing to [Tar07, Lemma 14.4] the following version of Green’s formula occurs

$$\int_\Omega (-\Delta u)v\,dx = \int_\Omega \nabla u \nabla v\,dx - \int_\Gamma \frac{\partial u}{\partial \nu}v\,dS, \, \forall u \in H^{3/2}(\Omega), \, v \in H^1(\Omega).$$

**Proposition 6.4.** Let $z \in \mathbb{C} \setminus \sigma_e(L_\Omega)$. Then for any $\psi \in H^{1/2}(\Gamma)$ it holds

$$\tilde{E}_z[\psi] = \tilde{E}[\psi] + \sum_{k=0}^\infty \frac{z}{E_k(z - E_k)} \left( \int_\Gamma \frac{\partial u_k}{\partial \nu} \psi \,dS \right)^2.$$
Proof. We claim that for all \( \psi \in H^{1/2}(\Gamma) \) and \( k \) we have
\[
(\Pi \psi, u_k) = -\frac{1}{E_k} \int_\Gamma \frac{\partial u_k}{\partial \nu} \psi \, dS.
\] (6.13)

Once identity (6.13) has been proved, the result would follow from Theorem 4.2. Let us prove (6.13). From the above discussion we have \( \Pi \psi, u_k \in H^{3/2}(\Omega) \). Thus utilizing Green formula we obtain
\[
\int_\Omega (-\Delta u_k) \cdot \Pi \psi \, dx = E_k \int_\Omega u_k \Pi \psi \, dx = \int_\Omega \nabla u_k \cdot \nabla \Pi \psi \, dx - \int_\Gamma \frac{\partial u_k}{\partial \nu} \psi \, dS
\]
\[
= \int_\Omega (-\Delta \Pi \psi) \cdot u_k \, dx + \int_\Gamma \frac{\partial \Pi \psi}{\partial \nu} u_k \, dS - \int_\Gamma \frac{\partial u_k}{\partial \nu} \psi \, dS
\]
\[
= -\int_\Gamma \frac{\partial u_k}{\partial \nu} \psi \, dS.
\]

This leads to
\[
(\Pi \psi, u_k) = -E_k^{-1} \int_\Gamma \frac{\partial u_k}{\partial \nu} \psi \, dS,
\]
and the claim is proved.

\[\square\]

**Theorem 6.1.** Let \( \lambda > E_0 \) and \( E \) be an eigenvalue of the Dirichlet Laplacian with multiplicity \( m \). Let \( (v_1, \cdots, v_m) \) be an orthonormal basis for \( \ker(L_D - E) \). Then \( \hat{T}_t(\lambda), t > 0 \) is p.p. on a left (resp. right) neighborhood of \( E \) if and only if one of the following conditions is fulfilled:

1. \[
\sum_{k=1}^{m} (\Pi \psi^+, v_k) \cdot (\Pi \psi^-, v_k) \geq 0, \quad (\text{resp.} \leq 0), \forall \psi \in L^2(\Gamma).
\]

2. \[
\sum_{k=1}^{m} \left( \int_\Gamma \frac{\partial v_k}{\partial \nu} \psi^+ \, dS \right) \cdot \left( \int_\Gamma \frac{\partial v_k}{\partial \nu} \psi^- \, dS \right) \geq 0, \quad (\text{resp.} \leq 0), \forall \psi \in L^2(\Gamma).
\]

It follows, in particular that if there is \( \psi_1, \psi_2 \in L^2(\Gamma) \) are such that the above corresponding sums have opposite signs then \( \hat{T}_t(\lambda) \) is not p.p. in any neighborhood of \( E \).

**Proof.** Let \( \lambda, E \) and \( (v_1, \cdots, v_m) \) be as in the theorem and \( \psi \in H^{1/2}(\Gamma) \). From Green formula we get
\[
(\Pi \psi^+, v_k) \cdot (\Pi \psi^-, v_k) = \frac{1}{E^2} \left( \int_\Gamma \frac{\partial v_k}{\partial \nu} \psi^+ \, dS \right) \cdot \left( \int_\Gamma \frac{\partial v_k}{\partial \nu} \psi^- \, dS \right).
\]

Hence both sums appearing in the statement of the theorem are equal up to the positive factor \( 1/E^2 \). By Theorem 5.1 positivity holds if and only if one of the equivalent conditions 1–2 hold for every \( \psi \in H^{1/2}(\Gamma) \). Finally, the continuity of the scalar product together with the fact that \( H^{1/2}(\Gamma) \) is dense in \( L^2(\Gamma) \) gives the result, which completes the proof. \[\square\]
Corollary 6.1. Let $E$ be a simple eigenvalue of $L_D$ with associated normalized eigenfunction $u_E$. Assume that $u_E$ or $\frac{\partial u_E}{\partial 
u}$ can be chosen to have constant sign. Then $\tilde{T}_1(\lambda)$ is not p.p. in any right neighborhood of $E$ whereas it is p.p in $\{\lambda: -r_E < \lambda - E < 0\}$.

Proof. If $u_E$ has a constant sign, owing to positivity of $\Pi$ we get $(\Pi \psi^+, u_E) \cdot (\Pi \psi^-, u_E) \geq 0$. Whereas if $\frac{\partial u_E}{\partial 
u}$ has constant sign we obtain $(\int_{\Gamma} \frac{\partial u_E}{\partial 
u} \psi^+ \, dS) \cdot (\int_{\Gamma} \frac{\partial u_E}{\partial 
u} \psi^- \, dS) \geq 0$, and the result follows from Theorem 6.1.

6.3 The D-to-N on the unit disc revisited

For the unit disc the eigenvalues of the Dirichlet Laplacian are $j_{k,l}^2$, $k, l \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$ where $j_{k,l}$ are the positive zeros of the Bessel functions $J_k$. They are either simple or double eigenvalues. The set of simple eigenvalues consists of $\{j_{0,l}^2, l \in \mathbb{N}_0\}$ with associated normalized eigenfunctions

$$u_{0,l}(r) := c_l J_0(j_{0,l}r), \quad c_l = \frac{\sqrt{2}}{|J_0'(j_{0,l})|}.$$

The set of double eigenvalues consists of $\{j_{k,l}^2, k \in \mathbb{N}, l \in \mathbb{N}_0\}$ with associated normalized eigenfunctions

$$u_{k,l}(r) := c_{k,l} J_k(j_{k,l}r) \cos(k\theta), \quad v_{k,l}(r) := c_{k,l} J_k(j_{k,l}r) \sin(k\theta), \quad c_{k,l} = \frac{\sqrt{2}}{\sqrt{\pi} |J'_k(j_{k,l})|}. \quad (6.14)$$

Let us mention that for $v_{k,l}$ the integer $k$ runs $\mathbb{N}$.

In order to compute Mittag–Leffler expansion we have to compute the normal derivatives of the eigenfunctions. Obviously if $E$ is an eigenvalue of the Dirichlet Laplacian on the unit disc with associated normalized eigenfunction $u_E$ then

$$\frac{\partial u_E}{\partial 
u} \bigg|_{r=1} = c\sqrt{E} J_0'(\sqrt{E}).$$

Thus

$$\frac{\partial u_{0,l}}{\partial 
u} = c_l j_{0,l} J'_0(j_{0,l}), \quad \frac{\partial u_{k,l}}{\partial 
u} = c_{k,l} j_{k,l} J'_k(j_{k,l}) \cos(k\theta), \quad \frac{\partial v_{k,l}}{\partial 
u} = c_{k,l} j_{k,l} J'_k(j_{k,l}) \sin(k\theta). \quad (6.15)$$

Proposition 6.5. Let $\psi \in H^{1/2}(\Gamma)$. Then

$$\tilde{\mathcal{E}}_z[\psi] = \tilde{\mathcal{E}}[\psi] + \sum_{l=0}^{\infty} \frac{2z}{(z - j_{0,l}^2)} \left(\int_0^{2\pi} \psi(\theta) \, d\theta\right)^2 + \sum_{k=1, l=0}^{\infty} \frac{2z}{\pi(z - j_{k,l}^2)} \left(\int_0^{2\pi} \cos(k\theta) \psi(\theta) \, d\theta\right)^2$$

$$+ \sum_{k=1, l=0}^{\infty} \frac{2z}{\pi(z - j_{k,l}^2)} \left(\int_0^{2\pi} \sin(k\theta) \psi(\theta) \, d\theta\right)^2.$$
Proof. Let \( \psi \in H^{1/2}(\Gamma) \). Making use of Mittag-Leffler expansion from Proposition \( \text{(6.4)} \) together with the normal derivatives \( \text{(6.15)} \) we obtain

\[
\hat{E}_z[\psi] = \hat{E}[\psi] + z \sum_l \frac{1}{J_{0,l}(z - j_{0,l}^2)} (c_l j_{0,l} J'_0(j_{0,l}))^2 \left( \int_0^{2\pi} \psi(\theta) d\theta \right)^2 \\
+ z \sum_{k=1, l=0}^\infty \frac{1}{j_{k,l}^2 (z - j_{k,l}^2)} \left( c_{k,l} j_{k,l} J'_k(j_{k,l}) \right)^2 \left( \int_0^{2\pi} \cos(k\theta) \psi(\theta) d\theta \right)^2 \\
+ z \sum_{k=1, l=0}^\infty \frac{1}{j_{k,l}^2 (z - j_{k,l}^2)} \left( c_{k,l} j_{k,l} J'_k(j_{k,l}) \right)^2 \left( \int_0^{2\pi} \sin(k\theta) \psi(\theta) d\theta \right)^2 \tag{6.16}
\]

Taking the the expression of \( c_l \) and \( c_{k,l} \) from \( \text{(6.15)} \) into account leads to the sought formula. \( \square \)

The eigenvalues as well as the eigenfunctions of \( \tilde{L}_z \) were computed in \[Dan14\]. The eigenvalues are

\[
\hat{E}_k(z) = k - \sqrt{\lambda} J_{k+1}(\sqrt{z}) \quad k \in \mathbb{N}_0,
\]

with normalized eigenfunctions \( \psi_k \):

\[
\frac{1}{\sqrt{2\pi}}, \quad \frac{1}{\sqrt{\pi}} \cos(k\theta), \quad \frac{1}{\sqrt{\pi}} \sin(k\theta), \quad k \in \mathbb{N}.
\]

We quote that \( \hat{E}_0(\lambda) \) is the sole simple eigenvalue whereas all others are double eigenvalues. Moreover the eigenfunctions are \( z \)-independent.

Let us quote that one can rediscover the values of the eigenfunctions and the eigenvalues of \( \hat{L}_z \) on the light of formula \( \text{(6.16)} \).

We shall use Mittag–Leffler expansion for \( \hat{E}_z \) from Proposition \( \text{(6.5)} \) to write Mittag–Leffler expansion for \( \hat{E}_k(z) \).

**Proposition 6.6.** Let \( \hat{E}_k(z) \) be an eigenvalue of \( \tilde{L}_\lambda \) with eigenfunction \( \psi_k \). Then

\[
\hat{E}_k(z) = k + \sum_{l=0}^\infty \frac{2z}{z - j_{k,l}^2}. \tag{6.17}
\]

We mention that formula \( \text{(6.17)} \) was established in \[Dan14\], using properties of Bessel functions. We shall give an other proof.

**Proof.** In case \( k = 0 \), then \( \psi_0 = \frac{1}{\sqrt{2\pi}} \). Hence by formula \( \text{(6.16)} \) we get

\[
\hat{E}_0(z) = \sum_{l=0}^\infty \frac{2z}{(z - j_{0,l}^2)}.
\]

In case \( k \neq 0 \), then either \( \psi_k(\theta) = \frac{1}{\sqrt{\pi}} \cos(k\theta) \) or \( \psi_k(\theta) = \frac{1}{\sqrt{\pi}} \sin(k\theta) \). Thus in both cases we have \( \int_0^{2\pi} \psi_k d\theta = 0 \). Whereas in case \( \psi_k(\theta) = \frac{1}{\sqrt{\pi}} \cos(k\theta) \) we get

\[
\int_0^{2\pi} \cos(n\theta) \psi_k(\theta) d\theta = \sqrt{\pi} \delta_{n,k}, \quad \int_0^{2\pi} \sin(n\theta) \psi_k(\theta) d\theta = 0.
\]
Similar formulae hold in case \( \psi_k(\theta) = \frac{1}{\sqrt{\pi}} \sin(k\theta) \). Thus in both case we get

\[
\tilde{E}_k(z) = \tilde{\mathcal{E}}[\psi_k] + \sum_{l=1}^{\infty} \frac{2z}{z - j_{k,l}^2} = k + \sum_{l=1}^{\infty} \frac{2z}{z - j_{k,l}^2},
\]

which completes the proof.

We turn our attention to analyze positivity of \( \tilde{T}_t(\lambda) \) for \( \lambda > E_0 \).

**Proposition 6.7.** Let \( E \) be a simple Dirichlet eigenvalue of \( L_D \). Then the semigroup is p.p. on every small left neighborhood of \( E \) whereas it is non-p.p. on any right neighborhood of \( E \).

**Proof.** For the unit disc if \( E \) is a simple eigenvalue then the associated normalized eigenfunction is radially symmetric and is of the type

\[
u_E(r) = cJ_0(\sqrt{E}r).
\]

Plainly

\[
\frac{\partial u_E}{\partial \nu} = \frac{\partial u_E}{\partial r} \bigg|_{r=1} = c\sqrt{E}J_0'(\sqrt{E}) \neq 0,
\]

because \( J_0 \) has only simple positive zeros. Hence \( \frac{\partial u_E(r)}{\partial \nu} \) has constant sign and on the light of Corollary 6.1, we get the result.

The latter proposition was established in [Dan14, Theorem 1.1], however with a different proof.

Let us now turn our attention to double eigenvalues. Let \( E \) be a double eigenvalue for \( L_D \) and \( (u_1, u_2) \) be an orthonormal basis for \( \ker(L_D - E) \). Then

\[
u_1(r, \theta) = c_1J_m(\sqrt{E}r) \cos(m\theta), \quad \nu_2(r, \theta) = c_2J_m(\sqrt{E}r) \sin(m\theta).
\]

Hence

\[
\frac{\partial u_1}{\partial \nu} = \frac{\partial u_1}{\partial r} \bigg|_{r=1} = -mc_1\sqrt{E}J_m'(\sqrt{E}) \sin(m\theta), \quad \frac{\partial u_2}{\partial \nu} = \frac{\partial u_2}{\partial r} \bigg|_{r=1} = mc_2\sqrt{E}J_m'(\sqrt{E}) \cos(m\theta).
\]

They both change signs many times. Thus we are led to use the last assertion of Theorem 6.1 to handle the question.

The following result was already proved in [Dan14]. We shall prove it by using our method.

**Proposition 6.8.** The semigroup \( \tilde{T}_t(\lambda) \) is non p.p. on any neighborhood of any double eigenvalue.

**Proof.** Take \( \psi_k = \frac{\partial u_k}{\partial \nu} \), \( k = 1, 2 \). Then

\[
\int_{\Gamma} \nu_k \nu_k^+ dS = \int_{\Gamma} (\nu_k^+)^2 dS > 0, \quad \int_{\Gamma} \nu_k \nu_k^- dS = -\int_{\Gamma} (\nu_k^-)^2 dS < 0.
\]
On the other hand, owing to the $L^2(\Gamma)$ orthogonality of $\psi_1, \psi_2$ we get
\[
\int_{\Gamma} \psi_2 \psi_1^+ dS = \int_{\Gamma} \psi_2 \psi_1^- dS.
\]
A straightforward computation leads to (up to a constant)
\[
\int_{\Gamma} \psi_2 \psi_1^+ dS = \int_{0}^{2\pi} \cos(m\theta) \sin^+(m\theta) d\theta = \frac{1}{m} \int_{0}^{2m\pi} \cos(\theta) \sin^+(\theta) d\theta
\]
\[
= \frac{1}{m} \sum_{k=0}^{2m-1} \int_{k}^{(k+1)\pi} \cos(\theta) \sin^+(\theta) d\theta = \frac{1}{m} \sum_{k=0}^{m-1} \int_{2k\pi}^{(2k+1)\pi} \cos(\theta) \sin(\theta) d\theta = 0.
\]
Putting all together we obtain
\[
\sum_{k=1}^{2} \left( \int_{\Gamma} \frac{\partial u_k}{\partial \nu} \psi_1^+ dS \right) \cdot \left( \int_{\Gamma} \frac{\partial u_k}{\partial \nu} \psi_1^- dS \right) < 0.
\]
Thus, according to Theorem 6.1, $\hat{T}_t(\lambda)$ is non p.p. on a left neighborhood of $E$. To show that $\hat{T}_t(\lambda)$ is non p.p. on a right neighborhood of $E$ we take $\psi_m(\theta) = \cos(m\theta)$. Set
\[
I_1^+ = \int_{\Gamma} \cos((m+1)\theta) \psi_1^+(\theta) d\theta, \quad I_2^+ = \int_{\Gamma} \sin((m+1)\theta) \psi_1^+(\theta) d\theta.
\]
A lengthy computation leads to
\[
I_1^+ = \frac{2m + 3}{2(2m+1)} \sin(\frac{\pi}{2m}) - \frac{1}{2}.
\]
On can easily check that $I_1^+ \neq 0$. By orthogonality consideration we obtain $I_1^- = I_1^+$ and hence $I_1^- I_1^+ > 0$.
Similarly we get $I_2^+ = I_2^-$ and then $I_2^+ I_2^- \geq 0$. Summarizing we get
\[
\sum_{k=1}^{2} \left( \int_{\Gamma} \frac{\partial u_k}{\partial \nu} \psi_{m-1}^+ dS \right) \cdot \left( \int_{\Gamma} \frac{\partial u_k}{\partial \nu} \psi_{m-1}^- dS \right) > 0.
\]
Once again, according to Theorem 6.1, $\hat{T}_t(\lambda)$ is non p.p. on a right neighborhood of $E$.

### 6.4 The D-to-N on the square

Compared to the unit disc we shall see that for the square the picture changes drastically regarding positivity. In fact, we shall show that p.p. fails to hold true in any neighborhood of any Dirichlet eigenvalue except the smallest one. Therefore regularity of the boundary affect positivity of the semigroup.

We consider the square $\Omega = (0, 1) \times (0, 1)$. It is known that the eigenvalues of the Dirichlet Laplacian on the the square are
\[
E_{m,n} = \pi^2(m^2 + n^2), \quad m, n \in \mathbb{N},
\]
with associated normalized eigenfunctions

\[ u_{m,n}(x, y) = 2 \sin(m\pi x) \sin(n\pi y). \]

The eigenvalues are either simple or double eigenvalues. Moreover, the eigenfunction associated to the smallest eigenvalue \( E_0 := E_{1,1} \) can be chosen to be positive. The normal derivatives of \( u_{m,n} \) are, respectively

\[
\frac{\partial u_{m,n}}{\partial \nu} = 2\pi \cdot \begin{cases} 
-n \sin(m\pi x) & \text{on } (0,1) \times \{0\} \\
(-1)^m m \sin(n\pi y) & \text{on } \{1\} \times (0,1) \\
n(-1)^n \sin(m\pi x) & \text{on } (0,1) \times \{1\} \\
-m \sin(n\pi y) & \text{on } \{0\} \times (0,1)
\end{cases}.
\]

They are in \( L^2(\Gamma) \) and all change sign except for \( m = n = 1 \). For simple eigenvalues \( E_{m,m} \), \( m \geq 2 \) the normal derivatives are

\[
\frac{\partial u_{m,m}}{\partial \nu} = 2m\pi \cdot \begin{cases} 
-\sin(m\pi x) & \text{on } (0,1) \times \{0\} \\
(-1)^m m \sin(m\pi y) & \text{on } \{1\} \times (0,1) \\
(-1)^m \sin(m\pi x) & \text{on } (0,1) \times \{1\} \\
-m \sin(m\pi y) & \text{on } \{0\} \times (0,1)
\end{cases}.
\]

We know from Proposition 5.1 that \( \tilde{T}_t(\lambda) \) is p.p. to the left \( E_{1,1} \). Thus by Theorem 6.1 it is non p.p. on a left neighborhood of \( E_{1,1} \).

**Proposition 6.9.** Let \( E_{m,n} > E_0 \) be an eigenvalue of the Dirichlet Laplacian on the square. Then \( \tilde{T}_t(\lambda) \) is non p.p. in any neighborhood of \( E_{m,n} \).

**Proof.** Let \( E_{m,m} \) be a simple eigenvalue of \( L_D \) with \( m \geq 2 \). For any \( \psi \in L^2(\Gamma) \), we set

\[
I^\pm(\psi) = \int_{\Gamma} \frac{\partial u_{m,m}}{\partial \nu} \psi^\pm \, dS.
\]

First case: \( m \) is even. For the choice \( \psi = \frac{\partial u_{m,m}}{\partial \nu} \) owing to the fact that the normal derivative changes sign we get

\[
I^+(\psi)I^-(\psi) < 0. \quad (6.18)
\]

Let \( \psi \in L^2(\Gamma) \). An elementary computation shows that for even \( m \) we have

\[
I^\pm(\psi) = \int_{\Gamma} \frac{\partial u_{m,m}}{\partial \nu} \psi^\pm \, dS = 2m\pi \int_0^1 \sin(m\pi x)(\psi^+(1, x) - \psi^-(x, 1)) \, dx \\
+ 2m\pi \int_0^1 \sin(m\pi x)(\psi^+(0, x) - \psi^-(x, 0)) \, dx.
\]

We consider the case \( m = 2 \), the proof general even \( m \) is similar. Let us choose now

\[
\psi(x, 0) = \cos(\pi x), \quad \psi(1, x) = x, \quad \psi(x, 1) = \cos(\pi x), \quad \psi(0, x) = x.
\]
Then with this choice of $\psi$ we obtain (up to a factor)

$$I^+(\psi) = \int_0^{1/2} \sin(2\pi x)(x - \cos(\pi x)) \, dx + \int_{1/2}^1 x \sin(2\pi x) \, dx$$

$$\quad + \int_0^{1/2} \sin(2\pi x)(x - \cos(\pi x)) \, dx + \int_{1/2}^1 x \sin(2\pi x) \, dx$$

$$= 2 \int_0^1 x \sin(2\pi x) \, dx - 2 \int_0^{1/2} \sin(2\pi x) \cos(\pi x) \, dx$$

$$= \frac{1}{2\pi} - \frac{4}{3\pi}.$$  

Whereas

$$I^-(\psi) = 2 \int_{1/2}^1 \sin(2\pi x) \cos(\pi x) \, dx = -\frac{4}{3\pi}.$$  

Thus

$$I^+(\psi)I^-(\psi) > 0 \quad (6.19)$$

Inequalities (6.18) in conjunction with Theorem 6.1 yield that $\tilde{T}_i(\lambda)$ is non p.p. in any neighborhood of $E_{m,m}$.

**Second case:** $m$ is odd. Choosing $\psi$ as in the first step we get $I^+(\psi)I^-(\psi) < 0$.

When choosing $\psi(x,0) = 1 = \psi(1,x)$, $\psi(x,1) = \psi(0,x) = -1$,

we get $I^+(\psi) = I^-(\psi) = 4$ and then $I^+(\psi)I^-(\psi) > 0$. Finally an application of Theorem 6.1 yields the first assertion.

Now let $E_{m,n}$ be a double eigenvalue with eigenfunctions $v_1, v_2$. As $\frac{\partial v_1}{\partial \nu}$ changes sign, taking $\psi = \frac{\partial v_1}{\partial \nu}$ we obtain $\int_\Gamma \psi_1^+ \psi_1^+ \, dS > 0$ and $\int_\Gamma \psi_1^- \psi_1^- \, dS < 0$.

Let us emphasize that $\frac{\partial v_1}{\partial \nu}$, $\frac{\partial v_2}{\partial \nu}$ are $L^2(\Gamma)$-orthogonal. Thereby, $\int_\Gamma \frac{\partial v_1}{\partial \nu} \psi_1^+ \, dS = \int_\Gamma \frac{\partial v_2}{\partial \nu} \psi_1^- \, dS$.

Furthermore an elementary computations leads to $\int_\Gamma \frac{\partial v_1}{\partial \nu} \psi_1^+ \, dS = 0$.

Owing to these considerations we achieve

$$2 \sum_{k=1}^2 \left( \int_\Gamma \frac{\partial v_k}{\partial \nu} \psi_1^+ \, dS \right) \cdot \left( \int_\Gamma \frac{\partial v_k}{\partial \nu} \psi_1^- \, dS \right) < 0.$$  

Once again, by Theorem 6.1 we conclude that $\tilde{T}_i(\lambda)$ is non p.p. to the left of $E_{m,n}$.

To prove non positivity to the right we follow the strategy we used for the unit disc. Let us choose

$$\psi_{m,n} = \frac{\partial u_{m+1,n+1}}{\partial \nu}.$$  

Set

$$I_{1}^\pm = \int_\Gamma \frac{\partial u_{m,n}}{\partial \nu} \psi_{m,n}^\pm \, dS, \quad I_{2}^\pm = \int_\Gamma \frac{\partial u_{m,n}}{\partial \nu} \psi_{m,n}^\pm \, dS.$$
Orthogonality leads to $I_{1,2}^+ = I_{1,2}^-$. Moreover, an elementary computation leads to

$$I_1^+ = 2n(n + 1) \int_0^1 \sin(m\pi x) \sin^+((m + 1)\pi x) \, dx$$

$$- 2m(m + 1) \int_0^1 \sin(n\pi x) \sin^+((n + 1)\pi x) \, dx$$

$$\quad = \frac{mn(n + 1)(2m - 1)}{(2m + 1)\pi} \sin\left(\frac{\pi}{m + 1}\right) - \frac{mn(m + 1)(2n - 1)}{(2n + 1)\pi} \sin\left(\frac{\pi}{n + 1}\right).$$

On the other hand the functions

$$[1, \infty) \to \mathbb{R}, \ x \mapsto \frac{2x - 1}{(2x + 1)(x + 1)} \quad \text{and} \quad x \mapsto \sin\left(\frac{\pi}{x + 1}\right),$$

are both positive and strictly decreasing. Hence $I_1^+ \neq 0$. Putting all together we achieve $I_1^+ I_1^- + I_2^+ I_2^- > 0$. According to Theorem 6.1, $\tilde{T}_t(\lambda)$ is non p.p. to the right of $E_{m,n}$ and the proof is finished.

6.5 The D-to-N on the unit ball

We denote by $B$ the unit ball in $\mathbb{R}^3$. The eigenvalues of the Dirichlet Laplacian on $B$ are squares of the positive zeros of the modified spherical Bessel functions of the first kind

$$j_n(z) := \sqrt{\frac{\pi}{2z}} J_{n+1/2}(z).$$

They coincide with squares of positive zeros of $J_{n+1/2}$ and shall be enumerated

$$j_{nk}^2, \ k, n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\},$$

with respective multiplicities $2n + 1$ and normalized eigenfunctions

$$u_{nkl} := c_{nkl} j_n(j_{nk} r) Y_{nl}(\theta, \varphi), \ |l| \leq n; \ c_{nkl} = \frac{\sqrt{2}}{|j_n'(j_{nk})|}.$$

where $Y_{nl}$ are the normalized spherical harmonics.

The respective normal derivatives are:

$$\frac{\partial u_{nkl}}{\partial \nu} = \frac{\partial u_{nkl}}{\partial r} \bigg|_{r=1} = c_{nkl} j_n(j_{nk}) Y_{nl}(\theta, \varphi).$$

**Proposition 6.10.** Let $z \in \mathbb{C} \setminus \sigma_e(L_D)$ and $\psi \in H^{1/2}(\Gamma)$. Then

$$\tilde{\mathcal{E}}_z[\psi] = \tilde{\mathcal{E}}[\psi] + \sum_{k=0}^{\infty} \sum_{n=0}^{\infty} \sum_{l=-n}^{n} \frac{2z}{(z - j_{nk}^2)} \left| \int_{\Gamma} Y_{nl} \psi \, dS \right|^2.$$ (6.20)
Proof. According to Proposition 6.4 we have

$$\tilde{\mathcal{E}}_z[\psi] = \tilde{\mathcal{E}}[\psi] + z \sum_{k=0}^{\infty} \sum_{n=0}^{\infty} \sum_{l=-n}^{n} \frac{1}{j_{nk}^2 (z - j_{nk}^2)} \left| \int_{\Gamma} \frac{\partial u_{nk}}{\partial \nu} \psi dS \right|^2.$$

Inserting the expression of the normal derivatives of the Dirichlet eigenfunctions in the latter identity gives the result. \( \square \)

We shall use the series expansion (6.20) to derive Mittag–Leffler expansion for the eigenvalues of \( \tilde{L}_z \).

**Theorem 6.2.** For any \( z \in \mathbb{C} \setminus \sigma_e(L_D) \) it holds:

1. The spherical harmonics are the eigenfunctions of \( \tilde{L}_z \).

2. For each \( n \in \mathbb{N}_0 \) the spherical harmonics \( Y_{nl} \), \( |l| \leq n \) are normalized eigenfunctions of an eigenvalue \( \tilde{E}_n(z) \) of \( \tilde{L}_z \). Moreover,

$$\tilde{E}_n(z) = n + \sum_{k=0}^{\infty} \frac{2z}{z - j_{nk}^2}. \quad (6.21)$$

Proof. We first compute the eigenfunctions and the eigenvalues of \( \tilde{\mathcal{E}} \). Making use of spherical coordinates we get that for any \( Y_{nl} \) the solution of

$$\begin{cases} 
-\Delta u = 0, & \text{in } B, \\
u = Y_{nl}, & \text{on } \Gamma
\end{cases} \quad (6.22)$$

is given by \( r^n Y_{nl} \). Hence \( \Pi(Y_{nl}) = r^n Y_{nl} \) and \( \tilde{\mathcal{E}}(Y_{nl}, \psi) = \mathcal{E}(\Pi Y_{nl}, \Pi \psi) = n \int_{\Gamma} Y_{nl} \psi dS \), by Green’s formula, for any \( \psi \in H^{1/2}(\Gamma) \). This shows that \( n \) is an eigenvalue of \( \tilde{\mathcal{E}} \) of order \( 2n + 1 \) with normalized eigenfunction \( Y_{nl}, |l| \leq n \).

Let \( \psi \in H^{1/2}(\Gamma) \), by polarization and since the \( Y_{nl} \) are an orthonormal basis for \( L^2(\Gamma) \) we get

$$\tilde{\mathcal{E}}_z(Y_{mp}, \psi) = \tilde{\mathcal{E}}(Y_{mp}, \psi) + \sum_{k=0}^{\infty} \sum_{n=0}^{\infty} \sum_{l=-n}^{n} \frac{2z}{z - j_{nk}^2} \left( \int_{\Gamma} Y_{nl} \psi dS \right) \left( \int_{\Gamma} Y_{nl} \bar{Y}_{mp} dS \right)$$

$$= m + \sum_{k=0}^{\infty} \frac{2z}{z - j_{mk}^2} \left( \int_{\Gamma} Y_{mp} \psi dS \right)$$

For every \( p \in \mathbb{N}_0 \) and every \( |m| \leq p \), \( Y_{mp} \) is an eigenfunction of \( \tilde{\mathcal{E}}_z \) associated to the eigenvalue

$$m + \sum_{k=0}^{\infty} \frac{2z}{z - j_{mk}^2},$$

which is of order \( 2m + 1 \). \( \square \)
Remark 6.1. On the light of Proposition 6.5 and Theorem 6.2 it seems that formula (6.21) should be true for every dimension $d \geq 2$.

Let us now investigate positivity of $\tilde{T}_t(\lambda)$, $\lambda > E_0$. We first record that owing to our general result form Corollary 6.1 the semigroup $\tilde{T}_t(\lambda)$ is p.p. to the left of $E_0 = j_{00}^2$ and non p.p. on a right neighborhood of it.

**Proposition 6.11.** Let $j_{nk}^2$ be an eigenvalue of $L_D$. Then

1. The semigroup $\tilde{T}_t(\lambda)$ is p.p. on a left neighborhood of $j_{0k}$ whereas it is non p.p. on a right neighborhood of $j_{0k}$.

2. The semigroup $\tilde{T}_t(\lambda)$ is non p.p. in any neighborhood of $j_{nk}^2$ for $n \geq 1$.

**Proof.** For $n = 0$ the normal derivatives of the eigenfunctions are constant. Hence by Corollary 6.1 we get assertion 1.

In what follows $c$ designates a nonzero generic constant which may differ from line to line. Assume now that $n \geq 1$. Then the normal derivatives of the real eigenfunctions of $j_{nk}^2$ are

$$v_l := cP_n^l(\cos \theta) \cos(l\varphi), \quad l = 0, \cdots, n,$$

$$w_l := cP_n^l(\cos \theta) \sin(l\varphi), \quad l = 1, \cdots, n.$$

We recall that

$$P_n^l(\cos \theta) = c_n^l |\sin(\theta)|^n, \quad w_n = P_n^n(\theta) \sin(n\varphi).$$

Let us choose $\psi = w_n$. Then

$$\int_{\Gamma} w_n w_n^+ dS > 0, \quad \int_{\Gamma} w_n w_n^- dS < 0.$$

Let $l = 1, \cdots, n - 1$. Then $\int_{\Gamma} v_l w_n dS = \int_{\Gamma} w_l w_n dS = 0$. Thus

$$\int_{\Gamma} v_l w_n^+ dS = \int_{\Gamma} v_l w_n^- dS, \quad \int_{\Gamma} w_l w_n^- dS = \int_{\Gamma} w_l w_n^+ dS.$$

Owing to the orthogonality of Legendre polynomials we obtain, for any $l = 1, \cdots, n - 1$

$$\int_{\Gamma} v_l w_n^+ dS = \int_0^\pi P_n^l(\cos \theta) P_n^n(\cos \theta) \sin(\theta) d\theta \cdot \int_0^{2\pi} \cos(l\varphi) \sin^+(n\varphi) d\varphi$$

$$= 0 = \int_{\Gamma} w_l w_n^+ dS.$$

Consequently, according to Theorem 6.1 we conclude that $\tilde{T}_t(\lambda)$ is non p.p. on left neighborhoods of $j_{nk}^2$.

Let us show non positivity to the right of $j_{nk}^2$.

We recall that

$$P_n^l(\cos \theta) = c_n^l |\sin(\theta)|^n, \quad w_n = P_n^n(\theta) \sin(n\varphi).$$

Arguing as in the former examples it suffices to find a function $\psi$ which is $L^2(\Gamma)$ orthogonal to all $v_l, w_l$ and such that $\int_{\Gamma} w_n \psi^+ dS \neq 0$. 

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Assume that $n$ is even. Let us choose $\psi(\varphi) = \sin((n + 1)\varphi)$. By a straightforward computation we obtain (up to a constant)
\[
\int_{\Gamma} w_n \psi^+ dS = \int_0^\pi \sin(n\varphi) \sin^+((n + 1)\varphi) \, d\varphi + \int_0^\pi \sin(n\varphi) \sin^-((n + 1)\varphi) \, d\varphi
\]
\[
= 2 \int_0^\pi \sin(n\varphi) \sin^+((n + 1)\varphi) \, d\varphi = \frac{n(2n - 1)}{2(2n + 1)} \sin\left(\frac{\pi}{n + 1}\right) \neq 0.
\]
For odd $n$ we choose $\psi(\varphi) = \sin((n + 2)\varphi)$ and obtain $\int_{\Gamma} w_n \psi^+ dS \neq 0$.

Once again according to Theorem 6.1 the semigroup $\bar{T}_t(\lambda)$ is non p.p. on right neighborhoods of $j_{nk}^2$.

\[
\square
\]

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