DERIVATIONS ON THE ALGEBRA OF MULTIPLE HARMONIC $q$-SERIES AND THEIR APPLICATIONS

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Abstract. We introduce derivations on the algebra of multiple harmonic $q$-series and show that they generate linear relations among the $q$-series which contain the derivation relations for a $q$-analogue of multiple zeta values due to Bradley. As a byproduct we obtain Ohno-type relations for finite multiple harmonic $q$-series at a root of unity.

1. Introduction

In this article we introduce derivations on the algebra of multiple harmonic $q$-series and show that they generate linear relations among the $q$-series, which are a slight generalization of Bradley’s result in [3]. As a byproduct we obtain Ohno-type relations for finite multiple harmonic $q$-series at a root of unity introduced by Bachmann, Tasaka and the author in [2].

For a tuple $k = (k_1, \ldots, k_r)$ of positive integers with $k_1 \geq 2$, the multiple zeta value (MZV) $\zeta(k)$ is defined by

$$\zeta(k) = \sum_{m_1 > \cdots > m_r \geq 1} \frac{1}{m_1^{k_1} \cdots m_r^{k_r}}.$$  

In [6] Ihara, Kaneko and Zagier introduce derivations on the algebra of MZVs and described linear relations among MZVs by using them (see also [5]). The precise statement is as follows. Let $h = Q(x, y)$ be the non-commutative polynomial ring, and set $z_k = x^{k-1}y$ for $k \geq 1$. Denote by $h^0$ the $Q$-submodule of $h$ spanned by $1$ and the monomials $z_{k_1} \cdots z_{k_r}$ with $k_1 \geq 2$. Then the $Q$-linear map $Z : h^0 \to \mathbb{R}$ is uniquely defined by $Z(1) = 1$ and $Z(z_{k_1} \cdots z_{k_r}) = \zeta(k_1, \ldots, k_r)$. Now define the $Q$-linear derivation $\partial_n (n \geq 1)$ on $h$ by $\partial_n(x) = -\partial_n(y) = x(x + y)^{n-1}y$. Then it holds that $Z(\partial_n(w)) = 0$ for any $n \geq 1$ and $w \in h^0$. This gives linear relations among MZVs, which are called the derivation relations.

In [3] Bradley proves that the derivation relations hold also for a $q$-analogue of MZVs defined by

$$\zeta_q(k) = \sum_{m_1 > \cdots > m_r \geq 1} \frac{q^{(k_1-1)m_1 + \cdots + (k_r-1)m_r}}{[m_1]^{k_1} \cdots [m_r]^{k_r}},$$

where $q$ is a parameter satisfying $|q| < 1$ and $[m] = (1 - q^m)/(1 - q)$. The $q$-analogue model (1.1) is often called the Bradley-Zhao model, which is a special value of the
q-zeta function studied by Kaneko, Kurokawa and Wakayama \cite{8} in the case of $r = 1$ and Zhao \cite{15} for $r \geq 2$. In \cite{3} Bradley establishes that the model (1.1) satisfies Ohno’s relation \cite{10} for MZVs in the same form, and then the derivation relations are obtained as a corollary.

In this article we extend the derivation relations to a more general class of multiple harmonic \( q \)-series. Denote by \( Z_q \) the \( \mathbb{Q} \)-vector space spanned by the \( q \)-series of the following form

\[
(1 - q)^s \sum_{m_1 > \cdots > m_r \geq 1} \frac{q^{l_1 m_1 + \cdots + l_r m_r}}{[m_1]^{k_1} \cdots [m_r]^{k_r}},
\]

where \( s \in \mathbb{Z} \), \( l_1 \geq 1 \) and \( k_j \geq 1 \), \( k_j \geq l_j \geq 0 \) for any \( j \). Note that the space \( Z_q \) contains the Bradley-Zhao model and other various \( q \)-analogue models due to, for example, Schlesinger \cite{14}, Ohno-Okuda-Zudilin \cite{11, 17}, Okounkov \cite{12} and Bachmann-Kühn \cite{1}.

The proof of our derivation relations runs almost parallel to that in \cite{6}. We define a subalgebra \( \widehat{\mathfrak{H}}^0 \) of a non-commutative polynomial ring with two indeterminates and the map \( Z_q \) which sends an element of \( \widehat{\mathfrak{H}}^0 \) to \( Z_q \). As for the algebra \( \mathfrak{h}^0 \) of MZVs we can define two commutative products called the stuffle product and the shuffle product, and establish the double shuffle relations. Then we define the derivations \( \partial_n \) \( (n \geq 1) \) on the algebra \( \widehat{\mathfrak{H}}^0 \) which relate the two products, and we obtain the derivation relations on \( Z_q \) from the double shuffle relations.

In this paper, as a byproduct of the above construction, we also prove Ohno-type relations for finite multiple harmonic \( q \)-series at a root of unity Let \( n \) be a positive integer and \( \zeta_n \) a primitive \( n \)-th root of unity. Set

\[
z_n(k; \zeta_n) = \sum_{n > m_1 > \cdots > m_r \geq 1} \frac{q^{(k_1-1)m_1 + \cdots + (k_r-1)m_r}}{[m_1]^{k_1} \cdots [m_r]^{k_r}} |_{q=\zeta_n}.
\]

In \cite{2} Bachmann, Tasaka and the author found that the value \( z_n(k; \zeta_n) \) reproduces the finite multiple zeta values (FMZVs) and the symmetric multiple zeta values (SMZVs) introduced by Kaneko and Zagier \cite{9} through an algebraic operation and an analytic one, respectively. Kaneko and Zagier conjecture that there exists a \( \mathbb{Q} \)-algebra isomorphism which sends FMZVs to SMZVs (see Section \ref{sec:4.1} for the precise statement). The above-mentioned property of \( z_n(k; \zeta_n) \) offers an explanation, though not a proof, for the conjecture.

In \cite{13} Oyama proves the Ohno-type relations for FMZVs and SMZVs by making use of the double shuffle relations among them and the derivations on the algebra \( \mathfrak{h} \). In a similar manner we can prove Ohno-type relations for \( z_n(k; \zeta_n) \) by using the derivations defined in this paper. Our relations turn into the relations due to Oyama through the algebraic and analytic operation. Moreover, we also obtain Ohno-type relations for the cyclotomic analogue of FMZVs introduced in \cite{2}.

The paper is organized as follows. In Section \ref{sec:2} we formulate the double shuffle relations for multiple harmonic \( q \)-series. We prove the derivation relations on the
space $\mathcal{Z}_q$ in Section 3. In Section 4 we show the Ohno-type relations for finite multiple harmonic $q$-series at a root of unity $z_n(k; \zeta_n)$.

2. Double shuffle relations for multiple harmonic $q$-series

2.1. Preliminaries. Let $\mathbb{N}$ be the set of positive integers. We introduce the letter $\mathbb{T}$ and set $\hat{\mathbb{N}} = \{\mathbb{T}\} \sqcup \mathbb{N} = \{1, 2, \ldots\}$. Throughout this paper we call an ordered set $k = (k_1, \ldots, k_r)$ of elements of $\hat{\mathbb{N}}$ an index. An empty set $\emptyset$ is regarded as an index with $r = 0$. We denote by $\hat{I}$ the set of indices:

$$\hat{I} = \{(k_1, \ldots, k_r) \mid r \geq 0 \text{ and } k_1, \ldots, k_r \in \hat{\mathbb{N}}\}.$$

We will use the subsets $I, \hat{I}_0$ and $I_0$ of $\hat{I}$ defined by

$$I = \{(k_1, \ldots, k_r) \in \hat{I} \mid r \geq 0 \text{ and } k_j \neq \mathbb{T} \text{ for any } j\},$$

$$\hat{I}_0 = \{(k_1, \ldots, k_r) \in \hat{I} \mid r \geq 0 \text{ and } k_1 \neq 1\},$$

$$I_0 = I \cap \hat{I}_0.$$

Note that the empty index $\emptyset$ belongs to all the three sets above.

Fix a parameter $q \in \mathbb{C}$. For $m \geq 1$ we set

$$F_1(m) = \frac{q^m}{[m]}, \quad F_k(m) = \frac{q^{(k-1)m}}{[m]^k} \quad (k \geq 1),$$

where $[m] = (1 - q^m)/(1 - q)$ is the $q$-integer.

Now assume that $|q| < 1$. For a non-empty index $k = (k_1, \ldots, k_r)$ which belongs to $\hat{I}_0$, we set

$$\zeta_q(k) = \sum_{m_1 > \cdots > m_r \geq 1}^{r} \prod_{j=1}^{r} F_{k_j}(m_j).$$

(2.1)

It converges absolutely because $|I_k(m)| \leq |1 - q||q|^m/(1 - |q|)$ for any $k \in \hat{\mathbb{N}} \setminus \{1\}$ and $m \geq 1$. If $k \in I_0$, it is often called the Bradley-Zhao model of a $q$-analogue of multiple zeta values [3, 15]. Hereafter we consider a more general class of $q$-series of the form (2.1) containing the factor $F_1(m) = q^m/[m]$. By definition we set $\zeta_q(\emptyset) = 1$.

For a non-empty index $k = (k_1, \ldots, k_r) \in \hat{I}$ we define the multiple polylogarithm of one variable $L_k(t)$ by

$$L_k(t) = \sum_{m_1 > \cdots > m_r \geq 1}^{r} t^{m_k} \prod_{j=1}^{r} F_{k_j}(m_j).$$

We set $L_\emptyset(t) = 1$. Note that $L_k(1) = \zeta_q(k)$ if $k \in \hat{I}_0$.

Let $\hbar$ be a formal variable and set $\mathcal{C} = \mathbb{Q}[\hbar, \hbar^{-1}]$. Denote by $\mathfrak{H}$ the non-commutative polynomial ring over $\mathcal{C}$ with two indeterminates $a$ and $b$. Set

$$e_\mathbb{T} = ab, \quad e_k = a^{k-1}(a + \hbar)b \quad (k \geq 1).$$
We denote by $\mathfrak{H}$ the subalgebra freely generated by the set $\{e_k\}_{k \in \mathbb{N}}$. Hereafter we will also use the element $e_\pi (n \in \mathbb{N})$ of $\mathfrak{H}$ defined by

$$e_\pi = a^n b = \sum_{j=2}^{n} (-h)^{n-j} e_j + (-h)^{n-1} e_\pi.$$  

For a non-empty index $k = (k_1, \ldots, k_r)$ we define $e_k = e_{k_1} \cdots e_{k_r}$. For the empty index we set $e_\emptyset = 1$.

We denote by $\mathfrak{H}_1$ the $C$-submodule of $\mathfrak{H}$ spanned by the monomials $e_k$ with $k \in \mathfrak{I}_0$. We endow $\mathbb{C}$ with $C$-module structure such that $\hbar$ acts as multiplication by $1 - q$, and define the $C$-linear map $Z_q : \mathfrak{H}_1 \to \mathbb{C}$ by

$$Z_q(e_k) = \zeta_q(k)$$

for any $k \in \mathfrak{I}_0$. Since

$$\frac{q_{km}}{[m]^k} = \sum_{j=1}^{k-l} \binom{k-l}{j-1} (1-q)^{k-l-j} F_{l+j}(m) \quad (k > l \geq 0),$$

$$\frac{q_{lm}}{[m]^l} = \sum_{j=2}^{k} (q-1)^{k-j} F_j(m) + (q-1)^{k-1} F_{k}(m)$$

for $k \geq 1$ and $m \geq 1$, we see that the image $Z_q(\mathfrak{H}_1)$ is equal to the $\mathbb{Q}$-vector space $\mathcal{Z}_q$ spanned by the $q$-series of the form (1.2).

2.2. **Double shuffle relations.** The double shuffle relations for multiple harmonic $q$-series $\zeta_q(k)$ are described in terms of two commutative and associative multiplication called the **shuffle product** $*_q$ and the **shuffle product** $\pi_q$. Here we recall their definitions and basic properties. See, e.g., [4, 16] for details.

Denote by $\mathfrak{I}$ the $C$-submodule of $\mathfrak{H}$ spanned by the set $\{e_k\}_{k \in \mathfrak{I}_0}$. For $m \geq 1$ it holds that

$$F_{\pi}(m)^2 = F_2(m) - (1-q) F_{\pi}(m), \quad F_{\pi}(m) F_k(m) = F_{k+1}(m) \quad (k \geq 1),$$

$$F_k(m) F_l(m) = F_{k+l}(m) + (1-q) F_{k+l-1}(m) \quad (k, l \geq 1).$$

Motivated by the above properties we define the $C$-bilinear symmetric map $\circ_q : \mathfrak{I} \times \mathfrak{I} \to \mathfrak{I}$ by

$$e_\pi \circ_q e_\pi = e_2 - \hbar e_\pi, \quad e_\pi \circ_q e_k e_l = e_k e_l + \hbar e_{k+l-1}$$

for $k, l \geq 1$. Note that $e_\pi \circ_q e_\pi = e_{\pi+1}$ for $n \geq 1$, where $e_\pi$ is defined by (2.2).

The shuffle product $*_{q}$ is the $C$-bilinear binary operation on $\mathfrak{H}_1$ uniquely determined by

$$1 *_{q} w = w *_{q} 1 = 1,$$

$$(e_k w) *_{q} (e_l w') = e_k (w *_{q} e_l w') + e_l (e_k w *_{q} w') + (e_k \circ_q e_l)(w *_{q} w')$$

for $k, l \geq 1$. Here we recall their definitions and basic properties. See, e.g., [4, 16] for details.
for \( w, w' \in \hat{\mathcal{H}}^1 \) and \( k, l \in \hat{\mathbb{N}} \). Note that the subalgebra \( \hat{\mathcal{H}}^0 \) is closed under the stuffle product \(*_q\).

**Proposition 2.1.** For \( w, w' \in \hat{\mathcal{H}}^0 \) it holds that
\[
Z_q(w*_q w') = Z_q(w)Z_q(w').
\]

Let \( D_r = \{ t \in \mathbb{C} \mid |t| < r \} \) be the open disc with radius \( r \). Denote by \( \mathcal{H} \) the \( \mathbb{C} \)-vector space of holomorphic functions \( f \) on the unit disk \( D_1 \) satisfying the following condition:
\[
0 < \forall r < 1, \exists C > 0, \forall j \in \mathbb{N}, \forall t \in \overline{D_r} : |f(q^j t)| \leq C|q|^j.
\]

We define the \( \mathbb{Q} \)-linear action of \( \hat{\mathcal{H}} \) on \( \mathcal{H} \) by \((hf)(t) = (1 - q)f(t)\) and
\[
(af)(t) = (1 - q)\sum_{j=1}^{\infty} f(q^j t), \quad (bf)(t) = \frac{t}{1-t}f(t).
\]

Note that \( tf(t)/(1-t) \in \mathcal{H} \) for any bounded holomorphic function \( f \) on \( D_1 \). Hence, for any index \( k = (k_1, \ldots, k_r) \), the polylogarithm \( L_k \) is written as
\[
L_k = e_{k_1} \cdots e_{k_r}(1),
\]
where \( 1 \) is the constant function \( 1(t) \equiv 1 \). We define the \( \mathcal{C} \)-linear map \( L : \hat{\mathcal{H}}^1 \to \mathcal{F} + \mathbb{C} 1 \), \( w \mapsto L_w = L_k \) for any index \( k \).

For \( f, g \in \mathcal{F} \) it holds that
\[
(af)(ag) = a((af)g + f(af) + hf g), \quad (bf)g = f(bg) = b(fg).
\]

Motivated by the above properties we define the shuffle product \( \mathfrak{m}_q \) as the \( \mathcal{C} \)-bilinear map \( \mathfrak{m}_q : \hat{\mathcal{H}} \times \hat{\mathcal{H}} \to \hat{\mathcal{H}} \) by
\[
1 \mathfrak{m}_q w = w \mathfrak{m}_q 1 = w, \quad (aw) \mathfrak{m}_q (aw') = a((aw) \mathfrak{m}_q w' + w \mathfrak{m}_q (aw') + h w \mathfrak{m}_q w'), \quad (bw) \mathfrak{m}_q w' = w \mathfrak{m}_q (bw') = b(w \mathfrak{m}_q w')
\]
for \( w, w' \in \hat{\mathcal{H}} \).

**Proposition 2.2.**

(i) For \( w, w' \in \hat{\mathcal{H}}^1 \) it holds that \( L_{w \mathfrak{m}_q w'} = L_w L_{w'} \).

(ii) The subalgebra \( \hat{\mathcal{H}}^0 \) is closed under the shuffle product \( \mathfrak{m}_q \).

(iii) For \( w, w' \in \hat{\mathcal{H}}^0 \) it holds that \( Z_q(w \mathfrak{m}_q w') = Z_q(w)Z_q(w') \).

Now we can state the double shuffle relations for multiple harmonic \( q \)-series, which follow from Proposition 2.1 and Proposition 2.2 (iii).

**Theorem 2.3.** For any \( w, w' \in \hat{\mathcal{H}}^0 \) it holds that
\[
Z_q(w*_q w' - w \mathfrak{m}_q w') = 0.
\]
3. Derivation relations

3.1. Algebraic formulas of formal power series. Denote by $\mathcal{S}[[X]]$ the non-commutative ring of formal power series in $X$ whose coefficients belong to $\mathcal{S}$. We extend the shuffle product on $\mathcal{S}$ to $\mathcal{S}[[X]]$ by $\mathcal{C}[[X]]$-linearity. Then we can define the logarithm and the exponential with respect to the shuffle product by

$$
\log_{\mathcal{S}_q} (1 + f(X)) = \frac{\sum_{n=1}^{\infty} (-1)^{n-1}}{n} (f(X))^{m_q n}, \quad \exp_{\mathcal{S}_q} (f(X)) = \frac{\sum_{n=0}^{\infty} 1}{n!} (f(X))^{m_q n}
$$

(3.1)

for $f(X) \in X\mathcal{S}[[X]]$, where $f(X)^{m_q 1} = f(X)$ and $f(X)^{m_q (n+1)} = f(X)\mathcal{S}_q (f(X)^{m_q n})$ for $n \geq 1$. They are inverse to each other.

Similarly we extend the stuffle product to the ring $\mathcal{S}_1[[X]]$ of formal power series over $\mathcal{S}_1$, and define the logarithm $\log_{\mathcal{S}_q} (1 + f(X))$ and the exponential $\exp_{\mathcal{S}_q} (f(X))$ for $f(X) \in X\mathcal{S}_1[[X]]$ by (3.1) with $\mathcal{S}_q$ replaced by $\mathcal{S}_q$.

Now set

$$
\psi(X) = \frac{1}{\hbar} a \log (1 + \hbar bX) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \hbar^{n-1} a b^n X^n,
$$

(3.2)

$$
\phi(X) = (\log (1 + aX)) b = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} a^n b X^n.
$$

(3.3)

Note that they belong to the ideal $X\mathcal{S}_1[[X]]$ because $\hbar^{n-1} a b^n = e_\mathcal{T} (e_1 - e_\mathcal{T})^{n-1}$ and $a^n b = e_\pi$ for $n \geq 1$. We will make use of the following formulas.

**Proposition 3.1.** It holds that

$$
\psi(X) = \log_{\mathcal{S}_q} \left( \frac{1}{1 - e_\mathcal{T} X} \right),
$$

(3.4)

$$
\phi(X) = \log_{\mathcal{S}_q} \left( \frac{1}{1 - e_\mathcal{T} X} \right).
$$

(3.5)

**Proof.** First we prove (3.4). Since the both sides belong to $X\mathcal{S}_1[[X]]$, it suffices to show that the derivatives with respect to $X$ are equal, that is,

$$
\frac{d}{dX} \left( \frac{1}{1 - e_\mathcal{T} X} \right) = \frac{1}{1 - e_\mathcal{T} X} \mathcal{S}_q \left( \frac{ab}{1 + \hbar b X} \right).
$$

(3.6)

Rewrite the right hand side as follows.

$$
\left( 1 + ab X \frac{1}{1 - e_\mathcal{T} X} \right) \mathcal{S}_q \left( \frac{ab}{1 + \hbar b X} \right)
= ab \frac{1}{1 + \hbar b X} + ab X \left\{ \frac{1}{1 - e_\mathcal{T} X} \mathcal{S}_q \left( \frac{ab}{1 + \hbar b X} \right) + \frac{1}{1 + \hbar b X} (a + \hbar) b \frac{1}{1 - e_\mathcal{T} X} \right\}.
$$
Thus we find that
\[
(1 - abX) \left\{ \frac{1}{1 - e_T X} \right\}^\infty_q \left( \frac{ab^{1 + h} X}{1 + h X} \right) = ab \frac{1}{1 + h X} \left\{ 1 + (a + h) X \frac{1}{1 - e_T X} \right\}
\]
\[
= \frac{e_T}{1 - e_T X}.
\]
This completes the proof of (3.6).

The formula (3.5) is proved in much the same way. It suffices to prove that
\[
\frac{d}{dX} \left( \frac{1}{1 - e_T X} \right) = \frac{1}{1 - e_T X} * q \frac{d}{dX} \phi(X).
\]
The right hand side is equal to
\[
\sum_{n=1}^\infty (-X)^{n-1} \frac{1}{1 - e_T X} * q \epsilon^\pi.
\]
For \( n \geq 1 \) it holds that
\[
(3.7) \quad \frac{1}{1 - e_T X} * q \epsilon^\pi = \left( 1 + e_T X \frac{1}{1 - e_T X} \right) * q \epsilon^\pi
\]
\[
= \epsilon^\pi + X \left\{ e_T \left( \frac{1}{1 - e_T X} * q \epsilon^\pi \right) + \epsilon^\pi \frac{1}{1 - e_T X} + (e_T \circ_q \epsilon^\pi) \frac{1}{1 - e_T X} \right\}.
\]
Since \( e_T \circ_q \epsilon^\pi = e^\pi \) we see that
\[
(1 - e_T X) \left( \frac{1}{1 - e_T X} * q \epsilon^\pi \right) = \epsilon^\pi + (X \epsilon^\pi \epsilon^\pi + \epsilon^\pi) \frac{1}{1 - e_T X}
\]
\[
= (\epsilon^\pi + X \epsilon^\pi) \frac{1}{1 - e_T X}.
\]
Therefore
\[
\frac{1}{1 - e_T X} * q \frac{d}{dX} \phi(X) = \frac{1}{1 - e_T X} \left( \sum_{n=1}^\infty (-X)^{n-1} (\epsilon^\pi + X \epsilon^\pi) \right) \frac{1}{1 - e_T X}
\]
\[
= \frac{e_T}{(1 - e_T X)^2} = \frac{d}{dX} \left( \frac{1}{1 - e_T X} \right).
\]

3.2. Derivations. Here we define three derivations \( \delta_n, d_n \) and \( \partial_n (n \geq 1) \) on \( \mathcal{J} \), and prove a relation among them. They are regarded as counterparts of the derivations on the algebra of multiple zeta values due to Ihara, Kaneko and Zagier [6].

**Definition 3.2.** Let \( \delta_n : \mathcal{J} \to \mathcal{J} (n \geq 1) \) be the \( \mathcal{C} \)-linear derivation uniquely determined by
\[
\delta_n(a) = 0, \quad \delta_n(b) = \frac{(-1)^n-1}{n} (b + 1) a^n b.
\]
We define the $\mathcal{C}$-algebra homomorphism $\Phi_X : \mathfrak{H} \to \mathfrak{H}[[X]]$ by

$$\Phi_X = \exp \left( \sum_{n=1}^{\infty} X^n \delta_n \right).$$

We prove another representation of $\Phi_X$ on the subalgebra $\hat{\mathfrak{H}}^1$.

**Lemma 3.3.** Set

$$D_X^* = \sum_{n=1}^{\infty} X^n \delta_n. \quad (3.8)$$

For any $w \in \hat{\mathfrak{H}}^1$ it holds that

$$D_X^*(w) = \phi(X) *_q w - \phi(X)w,$$

where $\phi(X) \in X\hat{\mathfrak{H}}^1[[X]]$ is given by (3.3).

**Proof.** Define the map $\delta'_n : \hat{\mathfrak{H}}^1 \to \hat{\mathfrak{H}}^1$ by

$$\delta'_n(w) = \frac{(-1)^{n-1}}{n} (e_{P^q} w - e_{P}w).$$

By direct calculation we see that it is a $\mathcal{C}$-linear derivation and $\delta'_n(e_k) = \delta_n(e_k)$ for $k \in \hat{\mathbb{N}}$. Hence $\delta'_n = \delta_n$ on the subalgebra $\hat{\mathfrak{H}}^1$. Using $\phi(X) = \sum_{n=1}^{\infty} (-1)^{n-1} e_{P} X^n / n$, we get the desired formula. $\square$

**Proposition 3.4.** For $w \in \hat{\mathfrak{H}}^1$, it holds that

$$\Phi_X(w) = (1 - e_{P^q} X) \left( \frac{1}{1 - e_{P^q} X} *_q w \right).$$

**Proof.** Denote by $\varphi_X$ the $\mathcal{C}$-linear map given in the right hand side above, that is,

$$\varphi_X(w) = (1 - e_{P^q} X) \left( \frac{1}{1 - e_{P^q} X} *_q w \right).$$

By a similar calculation to that in the proof of Proposition 3.1 (see (3.7)), we find that

$$\varphi_X(e_k) = (e_k + (e_{P^q} e_k) X) \frac{1}{1 - e_{P^q} X}$$

and $\varphi_X(e_k w) = \varphi_X(e_k) \varphi_X(w)$ for $k \in \hat{\mathbb{N}}$ and $w \in \hat{\mathfrak{H}}^1$. Hence $\varphi_X(w w') = \varphi_X(w) \varphi_X(w')$ for any $w, w' \in \hat{\mathfrak{H}}^1$, and it suffices to show $\Phi_X(e_k) = \varphi_X(e_k)$ for any $k \in \hat{\mathbb{N}}$.

Let $D_X^* : \mathfrak{H}[[X]] \to \mathfrak{H}[[X]]$ be the operator defined by (3.3), where $\delta_n$ is extended to $\mathfrak{H}[[X]]$ by $\mathcal{C}[[X]]$-linearity. Then $D_X^*(a) = 0$ and, by induction on $s \geq 1$, we see that

$$(D_X^*)^s(b) = (b + 1) (\phi(X))^{*s},$$
where \( \phi(X) \) is given by (3.3). From the definition of \( \Phi_X \) and (3.5), we find that

\[
\Phi_X(a) = a, \quad \Phi_X(b) = (b + 1) \exp_*(\phi(X)) - 1 = (1 + aX)b \frac{1}{1 - abX}.
\]

Using the above formulas, we see that \( \Phi_X(e_k) \) is equal to the right hand side of (3.9) for any \( k \in \mathbb{N} \).

Next we consider derivations associated with the shuffle product \( \mathfrak{m}_q \).

**Definition 3.5.** Let \( d_n : \mathfrak{H}[[X]] \to \mathfrak{H}[[X]] \) be the \( \mathcal{C}[[X]] \)-linear derivation defined by

\[
d_n(w) = \frac{(-h)^{n-1}}{n} \{ (ab^n) \mathfrak{m}_q w - ab^n w \}.
\]

We define the \( \mathcal{C}[[X]] \)-algebra homomorphism \( \Psi_X : \mathfrak{H}[[X]] \to \mathfrak{H}[[X]] \) by

\[
\Psi_X = \exp \left( \sum_{n=1}^{\infty} X^n d_n \right).
\]

As we will see below the operator \( \Psi_X \) also has another representation, but this time on \( \mathfrak{H}[[X]] \).

**Lemma 3.6.** Set

\[
D_X^w = \sum_{n=1}^{\infty} X^n d_n.
\]

Define \( \rho_s(X) \in \mathfrak{H}^{1[[X]]} \) (\( s \geq 1 \)) by the recurrence relation

\[
\rho_1(X) = 1,
\]

\[
\rho_{s+1}(X) = (\psi(X) + \log(1 + hbX)) \rho_s(X) + \psi(X) \mathfrak{m}_q \rho_s(X) \quad (s \geq 1),
\]

where \( \psi(X) \) is given by (3.2).

(i) For \( s \geq 1 \) and \( w \in \mathfrak{H}[[X]] \) it holds that

\[
(D_X^w)^s(w) = (\psi(X) \rho_s(X)) \mathfrak{m}_q w - \psi(X) (\rho_s(X) \mathfrak{m}_q w).
\]

(ii) For \( s \geq 1 \) it holds that \( \psi(X) \rho_s(X) = (\psi(X))^{\mathfrak{m}_q^s} \).

**Proof.** By induction on \( s \).

**Proposition 3.7.** For \( w \in \mathfrak{H}[[X]] \), it holds that

\[
\Psi_X(w) = (1 - eT X) \left( \frac{1}{1 - eT X \mathfrak{m}_q w} \right).
\]

**Proof.** Let \( \varphi_X \) be the \( \mathcal{C}[[X]] \)-linear map on \( \mathfrak{H}[[X]] \) defined by

\[
\varphi_X(w) = (1 - eT X) \left( \frac{1}{1 - eT X \mathfrak{m}_q w} \right).
\]
A similar calculation to that in the proof of Proposition 3.1 shows that
\[ \varphi_X(a) = a(1 + \hbar bX) \frac{1}{1 - abX}, \quad \varphi_X(b) = (1 - abX)b \frac{1}{1 - abX}, \]
and that \( \varphi_X \) is a \( \mathbb{C}[[X]] \)-algebra homomorphism. Therefore it suffices to show that \( \varphi_X(u) = \Psi_X(u) \) for \( u \in \{a, b\} \).

First we calculate \( \Psi_X(a) \). From Lemma 3.6 and the definition of \( \Psi_X \) we see that
\[ \Psi_X(a) = a + \sum_{s=1}^{\infty} \frac{1}{s!} (a + h) \psi(X) \rho_s(X) = a + (a + h) \left( \exp_{a} (\psi(X)) - 1 \right). \]
Because of (3.4) it is equal to
\[ a + (a + h) \left( \frac{1}{1 - abX} - 1 \right) = a(1 + \hbar bX) \frac{1}{1 - abX} = \varphi_X(a). \]

Next we calculate \( \Psi_X(b) \). Using Proposition 3.1 and (3.4) we see that
\[ \Psi_X(b) = b \exp_{a} (\psi(X)) - a \eta(X) = b \frac{1}{1 - abX} - ab \eta(X), \]
where \( \eta(X) \) is given by
\[ \eta(X) = \frac{1}{h} \log (1 + \hbar bX) \sum_{s=1}^{\infty} \frac{1}{s!} \rho_s(X). \]
Since
\[ a \eta(X) = \psi(X) \sum_{s=1}^{\infty} \frac{1}{s!} \rho_s(X) = \exp_{a} (\psi(X)) - 1 = abX \frac{1}{1 - abX} \]
and the map \( w \mapsto aw \) of left multiplication by \( a \) is injective on \( \mathfrak{H}[[X]] \), we find that \( \eta(X) = bX(1 - abX)^{-1} \). Thus we see that
\[ \Psi_X(b) = b \frac{1}{1 - abX} - ab^2 X \frac{1}{1 - abX} = (1 - abX)b \frac{1}{1 - abX} = \varphi_X(b). \]
This completes the proof.

Lastly we define the derivations \( \partial_n \) on \( \mathfrak{H} \) which relate \( \delta_n \) and \( d_n \).

**Definition 3.8.** For \( n \geq 1 \) we define the \( \mathcal{C} \)-linear derivation \( \partial_n : \mathfrak{H} \to \mathfrak{H} \) by
\[ \partial_n(a) = \frac{(-1)^{n}}{n} a \{ a(b + 1) + \hbar b \}^{n-1} (a + h)b, \]
\[ \partial_n(b) = \frac{(-1)^{n-1}}{n} a \{ (b + 1)a + \hbar b \}^{n-1} (b + 1)b. \]
We also define the \( \mathcal{C} \)-algebra homomorphism \( \Delta_X : \mathfrak{H} \to \mathfrak{H}[[X]] \) by
\[ \Delta_X = \exp \left( \sum_{n=1}^{\infty} X^n \partial_n \right). \]
Remark 3.9. Using
\[ \{a(b + 1) + hb\}(a + h) = (a + h)\{(b + 1)a + hb\}, \]
\[ \{(b + 1)a + hb\}(b + 1) = (b + 1)\{a(b + 1) + hb\}, \]
we see that
\[
\begin{align*}
\partial_n(a) &= (-1)^n a(a + h) \{(b + 1)a + hb\}^{n-1} b, \\
\partial_n(b) &= (-1)^{n-1} a(b + 1) \{a(b + 1) + hb\}^{n-1} b.
\end{align*}
\]

(3.14) \hspace{1cm} (3.15)

Theorem 3.10. It holds that \( \Phi_X = \Psi_X \Delta_X \) on \( \hat{\mathcal{H}} \).

Proof. It suffices to show that \( \Phi_X(u) = \Psi_X(\Delta_X(u)) \) for \( u \in \{a, b\} \). For that purpose we extend the map \( \Delta_X \) to the quotient field of \( \mathcal{H} \) by \( \Delta_X(w^{-1}) = -w^{-1}\Delta_X(w)w^{-1} \) for \( w \in \mathcal{H} \setminus \{0\} \), and calculate \( \Delta_X(a) \) and \( \Delta_X(b) \). Set \( z = a(b + 1) + hb \). From (3.13) and (3.14), we see that
\[
\partial_n(z) = \partial_n(z + h) = \partial_n((a + h)(b + 1)) = 0.
\]
Hence \( \Delta_X(z) = z \). Moreover, using (3.12) we find that
\[
\partial_n(a^{-1} - z^{-1}) = \partial_n(a^{-1}) = \frac{(-1)^{n-1} z^n (a^{-1} - z^{-1})}{n},
\]
for \( n \geq 1 \). Therefore
\[
\Delta_X(a^{-1} - z^{-1}) = (\exp(\log(1 + zX))) (a^{-1} - z^{-1}) = (1 + zX)(a^{-1} - z^{-1}).
\]
Since \( \Delta_X(z^{-1}) = z^{-1} \) we get
\[
\Delta_X(a) = (\Delta_X(a^{-1}))^{-1} = a \frac{1}{1 + (a + h)bX}.
\]
(3.16)

Combining (3.16) and \( \Delta_X(z) = z \), we find that
\[
\Delta_X(b) = \{1 + (a + (a + h)b)X\} b \frac{1}{1 + hbX}.
\]
(3.17)

From (3.16), (3.17) and (3.11) with \( \varphi_X \) replaced by \( \Psi_X \), we see that \( \Psi_X(\Delta_X(u)) \) is equal to \( \Phi_X(u) \) given by (3.10) for \( u \in \{a, b\} \). \( \square \)

As a corollary of Proposition 3.4, Proposition 3.7 and Theorem 3.10, we obtain the following relation, which plays a crucial role in the rest of this paper.

Corollary 3.11. For \( w \in \hat{\mathcal{H}} \), it holds that
\[
\frac{1}{1 - e^X} w = \frac{1}{1 - e^X} \mathbb{q} \Delta_X(w).
\]
(3.18)
3.3. Derivation relations for multiple harmonic q-series. Making use of (3.18) we prove the derivation relations for multiple harmonic q-series.

**Lemma 3.12.** The $\mathcal{C}$-subalgebra $\hat{\mathcal{H}}^0$ is invariant under the derivation $\partial_n \ (n \geq 1)$.

**Proof.** It suffices to prove that $\partial_n(e_k) \in \hat{\mathcal{H}}^0$ for $n \geq 1$ and $k \in \hat{\mathbb{N}}$. Note that the operator of left multiplication by $a$ leaves $\hat{\mathcal{H}}^0$ invariant and the formula (3.12) implies that the element

$$\partial_n(a) = \frac{(-1)^n}{n} a(a + e_1)^{n-1} e_1$$

belongs to $\hat{\mathcal{H}}^0$. Hence

$$\partial_n(e_1) = \partial_n(z - a) = -\partial_n(a) \in \hat{\mathcal{H}}^0,$$

where $z = a(b + 1) + \hbar b$. From it we see that $\partial_n(e_k)$ belongs to $\hat{\mathcal{H}}^0$ for $k \geq 2$ by induction on $k$ using

$$\partial_n(e_1 - e_T) = \hbar \partial_n(b) = \frac{(-1)^{n-1}}{n} (a + e_T)(a + e_1)^{n-1}(e_1 - e_T),$$

which belongs to $\hat{\mathcal{H}}^0$, and hence also does $\partial_n(e_T)$. \qed

Now we are in a position to prove the derivation relations.

**Theorem 3.13.** For any $n \geq 1$ and $w \in \hat{\mathcal{H}}^0$, it holds that $Z_q(\partial_n(w)) = 0$.

**Proof.** Corollary 3.11 implies that

$$e^n_T * w - e^n_T \Pi_q w = \sum_{r=1}^n \frac{1}{r!} \sum_{\substack{m \geq 0, j_1, \ldots, j_r \geq 1 \\text{mand} \ j_1 + \cdots + j_r = n}} e^m_T \Pi_q (\partial_{j_1} \cdots \partial_{j_r}(w))$$

for $n \geq 1$ and $w \in \hat{\mathcal{H}}^0$. Using the double shuffle relations (Theorem 2.3) we see that

$$\sum_{r=1}^n \frac{1}{r!} \sum_{\substack{m \geq 0, j_1, \ldots, j_r \geq 1 \\text{mand} \ j_1 + \cdots + j_r = n}} Z_q(e^m_T) Z_q(\partial_{j_1} \cdots \partial_{j_r}(w)) = 0.$$

Now the induction on $n$ implies that $Z_q(\partial_n(w)) = 0$ for any $w \in \hat{\mathcal{H}}^0$. \qed

Let us compare our derivation relations (Theorem 3.13) with those for multiple zeta values. Let $\mathfrak{h} = \mathbb{Q}(x, y)$ be the non-commutative polynomial ring with indeterminates $x$ and $y$. Set $z_k = x^{k-1} y \ (k \geq 1)$ and $\mathfrak{h}^0 = \mathbb{Q} + x \mathfrak{h}$. An index which belongs
to $I_0$ is said to be admissible. For an admissible index $\mathbf{k} = (k_1, \ldots, k_r)$, the multiple zeta value (MZV) $\zeta(\mathbf{k})$ is defined by

$$\zeta(\mathbf{k}) = \sum_{m_1 > \cdots > m_r \geq 1} \frac{1}{m_1^{k_1} \cdots m_r^{k_r}}.$$ 

Then the $\bQ$-linear map $Z : \mathfrak{h}^0 \to \bR$ is uniquely determined by $Z(1) = 1$ and $Z(z_{k_1} \cdots z_{k_r}) = \zeta(k_1, \ldots, k_r)$ for any admissible index $(k_1, \ldots, k_r)$. Now define the $\bQ$-linear derivation $\tilde{\partial}_n$ on $\mathfrak{h}$ by

$$\tilde{\partial}_n(x) = x(x+y)^{n-1}y, \quad \tilde{\partial}_n(y) = -(x+y)^{n-1}y.$$ 

Then it holds that

$$Z(\tilde{\partial}_n(w)) = 0 \quad (n \geq 1, w \in \mathfrak{h}^0),$$

which is called the derivation relation for MZVs [5, 6].

Set $\mathfrak{h}^1 = \bQ + y \mathfrak{h}_y$. It is a $\bQ$-algebra freely generated by the set $\{z_k\}_{k \geq 1}$. Hence it is embedded into $\hat{\mathfrak{h}}^0$ through the $\bQ$-algebra homomorphism $\iota : \mathfrak{h}^1 \to \hat{\mathfrak{h}}^0$ defined by $\iota(z_k) = e_k$ ($k \geq 1$). From (3.19), (3.20) and (3.21) we see that

$$\frac{(-1)^n}{n} \iota \tilde{\partial}_n = \partial_n \iota$$

on $\mathfrak{h}^1$. Therefore Theorem 3.13 implies that

$$Z_q(\iota(\tilde{\partial}_n(w))) = 0 \quad (n \geq 1, w \in \mathfrak{h}^0).$$

For an admissible index $(k_1, \ldots, k_r)$, the $q$-series

$$Z_q(\iota(z_{k_1} \cdots z_{k_r})) = \sum_{m_1 > \cdots > m_r \geq 1} \frac{q^{(k_1-1)m_1 + \cdots + (k_r-1)m_r}}{[m_1]^{k_1} \cdots [m_r]^{k_r}}$$

is nothing but the Bradley-Zhao model of a $q$-analogue of MZVs. Thus we obtain another proof for the following theorem due to Bradley [3].

**Corollary 3.14.** The Bradley-Zhao model of a $q$-analogue of multiple zeta values satisfies the derivation relations for multiple zeta values in the same form.

### 4. Ohno-type relations

#### 4.1. Finite multiple harmonic $q$-series at a root of unity.

In [2] Bachmann, Tasaka and the author introduce finite multiple harmonic $q$-series at a root of unity and find a connection to finite multiple zeta values (FMZVs) and symmetric multiple zeta values (SMZVs). Here we briefly recall the results in [2].

Suppose that $n \geq 2$ and $\zeta_n$ is a primitive $n$-th root of unity. For an index $\mathbf{k} = (k_1, \ldots, k_r)$ which belongs to $I$, we set

$$z_n(\mathbf{k}; \zeta_n) = \sum_{n > m_1 > \cdots > m_r \geq 1} \prod_{j=1}^r F_{k_j}(m_j)|_{q=\zeta_n}$$

where $F_{k_j}(m_j)$ denotes the $k_j$-th power sum of $m_j$. Then

$$Z_q(z_n(\mathbf{k}; \zeta_n)) = 0 \quad (n \geq 1, \mathbf{k} \in I).$$

This is called the finite Ohno-type relation for MZVs.
and call it the finite multiple harmonic \( q \)-series at a root of unity. By definition we set \( z_n(k) = 0 \) if \( r \geq n \). Note that, if \( n \) is a prime \( p \), \( z_p(k; \zeta_p) \) belongs to the integer ring \( \mathbb{Z}[\zeta_p] \) because \( [m]_{\zeta=\zeta_p} \) is a cyclotomic unit for \( 0 < m < p \).

Now we recall the definition of FMZVs. Set

\[
\mathcal{A} = \prod_{p \text{ prime}} \mathbb{F}_p / \bigoplus_{p \text{ prime}} \mathbb{F}_p.
\]

It is endowed with a \( \mathbb{Q} \)-algebra structure by diagonal multiplication. An element of \( \mathcal{A} \) is represented by a sequence \((a_p)_p\) of elements of \( \mathbb{F}_p \), and two elements \((a_p)_p\) and \((b_p)_p\) of \( \mathcal{A} \) are equal if \( a_p = b_p \) for all but a finite number of primes \( p \).

Let \( k = (k_1, \ldots, k_r) \) be an index which belongs to \( I \). The FMZV \( \zeta_{\mathcal{A}}(k) \) is the element of \( \mathcal{A} \) defined by

\[
\zeta_{\mathcal{A}}(k) = \left( \sum_{p > m_1 > \cdots > m_r \geq 1} \frac{1}{m_1^{k_1} \cdots m_r^{k_r}} \mod p \right).
\]

Next we recall the definition of SMZVs. We define the stuffle product \( * \) on \( \mathfrak{h}^1 = \mathbb{Q} + \mathfrak{h}y \) by

\[
1 \ast w = w \ast 1 = w,
\]

\[
(z_kw) \ast (z_lw') = z_k(w \ast z_lw') + z_l(z_kw \ast w') + z_{k+l}(w \ast w')
\]

for \( w, w' \in \mathfrak{h}^1 \) and \( k, l \geq 1 \). We denote by \( \mathfrak{h}^1 \) the commutative \( \mathbb{Q} \)-algebra \( \mathfrak{h}^1 \) equipped with the multiplication \( * \). Let \( \mathcal{Z} = \sum_{k \in I} \mathbb{Q} \zeta(k) \) the \( \mathbb{Q} \)-vector space spanned by all the MZVs. Then there exists a unique \( \mathbb{Q} \)-algebra homomorphism

\[
R : \mathfrak{h}^1 \longrightarrow \mathcal{Z}[T]
\]

such that \( R(e_1) = T \) and \( R(e_k) = \zeta(k) \) for any admissible index \( k \).

For an index \( k \) which belongs to \( I \) we set \( R_k(T) = R(e_k) \). Then the SMZV is defined by

\[
\zeta_S(k_1, \ldots, k_r) = \sum_{i=1}^r (-1)^{k_1+\cdots+k_i} R_{k_1,k_i-1,\ldots,k_1}(T)R_{k_{i+1},\ldots,k_{r-1},k_r}(T).
\]

The right hand side does not depend on \( T \) and belongs to \( \mathcal{Z} \).

In [9] Kaneko and Zagier conjecture that there exists the \( \mathbb{Q} \)-algebra homomorphism

\[
\varphi : \mathcal{A} \longrightarrow \mathcal{Z}/\zeta(2)\mathcal{Z}
\]

such that \( \varphi(\zeta_{\mathcal{A}}(k)) \equiv \zeta_S(k) \mod \zeta(2)\mathcal{Z} \) for any \( k \in I \).

**Theorem 4.1.** [2]

(i) For a prime \( p \), we denote by \( \mathfrak{p}_p \) the ideal of \( \mathbb{Z}[\zeta_p] \) generated by \( 1 - \zeta_p \), and identify \( \mathbb{Z}[\zeta_p]/\mathfrak{p}_p \) with the finite field \( \mathbb{F}_p \). Then, for any \( k \in I \), it holds that

\[
(z_p(k; \zeta_p) \mod \mathfrak{p}_p)_p = \zeta_{\mathcal{A}}(k)
\]
in \( A \).

(ii) For any \( k \in I \), the limit

\[
\xi(k) = \lim_{n \to \infty} z_n(k; e^{2\pi i/n})
\]

converges and it holds that \( \text{Im} \xi(k) \in \pi \mathbb{Z} \) and

\[
\text{Re} \xi(k) \equiv \zeta_S(k) \mod \zeta(2) \mathbb{Z}.
\]

4.2. Double shuffle relations. Here we state the double shuffle relations for the finite multiple harmonic \( q \)-series at a root of unity. The proof is similar to that for FMZVs (see [7]).

In this subsection we fix a primitive \( n \)-th root of unity \( \zeta_n \) for \( n \geq 2 \). We denote by the same letter \( z_n \) the \( \mathbb{C} \)-linear map \( z_n : \hat{H}_1 \to \mathbb{C} \) uniquely determined by \( z_n(e_k) = z_n(k; \zeta_n) \) for any \( k \in I \), where the \( \mathbb{C} \)-module structure of \( \mathbb{C} \) is defined by \( hc = (1-\zeta_n)c \) for \( c \in \mathbb{C} \).

**Theorem 4.2.**

(i) For \( n \geq 2 \) and \( w, w' \in \hat{H}_1 \), it holds that

\[
z_n(w \ast_q w') = z_n(w)z_n(w').
\]

(ii) Let \( \psi : \hat{H}_1 \to \hat{H}_1 \) be the \( \mathbb{C} \)-algebra anti-involution defined by

\[
\psi(e_1) = -e_1, \quad \psi(e_1) = -e_1, \quad \psi(e_k) = (-1)^k \sum_{j=2}^k \binom{k-2}{j-2} h^{k-j} e_j \ (k \geq 2).
\]

Then it holds that

\[
z_n(w \ast_q w') = z_n(\psi(w)w')
\]

for \( n \geq 2 \) and \( w, w' \in \hat{H}_1 \).

**Proof.** The proof for (i) is similar to that of Proposition 2.1. Here we prove (ii).

For the time being we assume that \( |q| < 1 \). For \( m \geq 1 \) we define the \( \mathbb{C} \)-linear map \( A_m : \hat{H}_1 \to \mathbb{C} \) by

\[
A_m(1) = 1, \quad A_m(e_k) = \sum_{m=m_1 > m_2 > \cdots > m_r > 0} \prod_{j=1}^r F_{k_j}(m_j)
\]

for \( k = (k_1, \ldots, k_r) \in \hat{I} \). Then we have

\[
L_w(t) = \sum_{m=1}^{\infty} t^m A_m(w) \quad (w \in \hat{H}_1).
\]

Proposition 2.2 implies that

\[
A_m(w_1 \ast_q w_2) = \sum_{\alpha+\beta=m, \alpha, \beta \geq 0} A_\alpha(w_1) A_\beta(w_2)
\]
for \( m \geq 1 \) and \( w_1, w_2 \in \mathcal{F}_q^1 \). Note that it is an equality of rational functions in \( q \) without poles at \( n \)-th roots of unity for \( n > m \). Hence we can set \( q = \zeta_n \).

Now let us prove (1.2). We may assume that \( w = e_k \) and \( w' = e_1 \) for some indices \( k = (k_1, \ldots, k_r) \) and \( l = (l_1, \ldots, l_s) \). From the previous observation we see that

\[
zn(e_k \, \overline{w} \, e_1) = \sum_{n > m > 0} A_m(e_k \, \overline{w} \, e_1)\big|_{q = \zeta_n} = \sum_{n > m > 0} \sum_{\alpha + \beta = m} A_\alpha(e_k) A_\beta(e_1)\big|_{q = \zeta_n}
\]

\[
= \sum_{n > \alpha > \beta \geq 0} A_{n - \alpha}(e_k) A_\beta(e_1)\big|_{q = \zeta_n}.
\]

When \( q = \zeta_n \) it holds that

\[
F_\tau(n - m) = -F_1(m), \quad F_1(n - m) = -F_\tau(m),
\]

\[
F_k(n - m) = (-1)^k \sum_{j=2}^{k} (1 - \zeta_n)^{k-j} \binom{k-2}{j-2} F_j(m)
\]

for \( n > m > 0 \) and \( k \geq 2 \). By changing the summation variable \( m_j \) to \( n - m_{r+1-j} \) in \( A_{n - \alpha}(e_k) \) and using the above formulas, we see that

\[
\sum_{n > \alpha > \beta \geq 0} A_{n - \alpha}(e_k) A_\beta(e_1)\big|_{q = \zeta_n} = z_n(\psi(e_k)e_1).
\]

This completes the proof. \( \square \)

4.3. **Ohno-type relations.** In [13] Oyama proves linear relations for FMZVs and SMZVs of a similar form to Ohno’s relations for MZVs [10]. Here we prove their \( q \)-analogue for the finite multiple harmonic \( q \)-series at a root of unity.

Denote by \( \mathcal{H}_Q^1 \) the subalgebra of \( \mathcal{F}_Q^1 \) generated by the set \( \{e_k\}_{k \geq 1} \) over \( Q \). Set \( e_0 = a \). Then we can identify \( \mathcal{H}_Q^1 \) with the non-commutative polynomial ring \( Q\langle e_0, e_1 \rangle \).

Note that \( e_k = e_0^{k-1} e_1 \) for \( k \geq 1 \).

Let \( \tau \) be the \( Q \)-linear involution on \( \mathcal{H}_Q^1 \) defined by \( \tau(e_0) = e_1 \) and \( \tau(e_1) = e_0 \). Suppose that \( k \in I \setminus \{\emptyset\} \). The monomial \( e_k \) is uniquely written in the form \( e_k = we_1 \) for some word \( w \) in \( \{e_k\}_{k \geq 1} \). Then we define the **Hoffman dual** \( k^\tau \) by the relation \( \tau(w)e_1 = e_k^\tau \). For example, if \( k = (2, 3, 1) \), we have \( e_k = (e_0 e_1 e_0^2 e_1) e_1 \) and hence \( e_k^\tau = \tau(e_0 e_1 e_0^2 e_1) e_1 = e_1 e_0 e_0^2 e_0 e_1 = e_1 e_2 e_1 e_2 \). Therefore \( (2, 3, 1)^\tau = (1, 2, 1, 2) \).

For a tuple of non-negative integers \( e = (e_1, \ldots, e_r) \) we define the **depth** \( \text{dep}(e) \) and the **weight** \( \text{wt}(e) \) by \( \text{dep}(e) = r \) and \( \text{wt}(e) = \sum_{j=1}^{r} e_j \), respectively.

The Ohno-type relations are given as follows.
Theorem 4.3. Suppose that $k \in I \setminus \{\emptyset\}$ and $\text{dep}(k) = r$. Set $s = \text{dep}(k^\vee) = \text{wt}(k) - r + 1$. For $m \geq 0$ and $n \geq r + m + 1$, it holds that

$$\sum_{e \in (\mathbb{Z}_{\geq 0})^s \atop \text{wt}(e) = m} z_n((k^\vee + e)^\vee; \zeta_n) = \sum_{l=0}^{m} \frac{1}{n} \binom{n}{m - l + 1} (1 - \zeta_n)^{m-l} \sum_{e' \in (\mathbb{Z}_{\geq 0})^r \atop \text{wt}(e') = l} z_n(k + e'; \zeta_n).$$

(4.3)

Proof. The proof is similar to that given in [13]. From (3.16) and (3.17), we see that $\Delta_X(e_k) = \left( e_0 \frac{1}{1 + e_1 X} \right)^{k-1} e_1 \left( 1 + e_0 X \frac{1}{1 + e_1 X} \right)$ $(k \geq 1)$.

(4.4)

For $k = (k_1, \ldots, k_r) \in I \setminus \{\emptyset\}$ and $s \geq 0$, we define $a_s(k)$ by

$$a_s(k) = \sum \prod_{1 \leq i \leq r} \left[ \left( \prod_{1 \leq l \leq k_i} \left( e_0 e_1^{j_{l(i)}} \right) \right) e_1 \right],$$

where the sum in the right hand side is over the set

$$\{(j_{l(i)})_{1 \leq l \leq r} \in (\mathbb{Z}_{\geq 0})^{\text{wt}(k)} \mid \sum_{i=1}^{r} \sum_{l=1}^{k_i} j_{l(i)} = s\}$$

and $\prod_{1 \leq i \leq k} Y_i$ stands for the ordered product $Y_1 \cdots Y_k$. We also set

$$A_{k,s,p} = \sum_{\lambda_1, \ldots, \lambda_r \in \{0,1\} \atop \lambda_1 + \cdots + \lambda_r = p} a_s(k_1 + \lambda_1 - 1, \ldots, k_r + \lambda_r - 1)$$

for $k = (k_1, \ldots, k_r) \in I \setminus \{\emptyset\}$ and $s, p \geq 0$. Then the equality (4.4) implies that

$$\Delta_X(e_k) = \sum_{p,s \geq 0} (-1)^s X^{p+s} A_{k,s,p}.$$

From the above formula and Corollary 3.11, we find that

$$e_{T}^m * q e_k = \sum_{l,s,p \geq 0 \atop l+s+p=m} (-1)^s e_{T}^l \zeta_{l} A_{k,s,p}$$

for $m \geq 0$. Hence Theorem 4.2 implies that

$$(-1)^m z_n(e_{T}^m) z_n(e_k) = \sum_{p=0}^{\min\{m,r\}} (-1)^p \sum_{l,s \geq 0 \atop l+s = m-p} z_n(e_{T}^l A_{k,s,p}).$$
for $m \geq 0$ and $n \geq 1$. Now use the following formula [13, Lemma 2.4]

$$\sum_{l,s \geq 0} e_{l,s}^p A_{k,s,p} = \sum_{\lambda \in \{0,1\}^r} \sum_{\nu \in (\mathbb{Z}_{\geq 0})^d} \sum_{\mu \in (\mathbb{Z}_{\geq 0})^d} e_{((k+\lambda)^\nu + e)^\nu},$$

where $r = \text{dep}(k)$ and $d = \text{dep}((k + \lambda)^\nu)$. Then we obtain

$$(4.5) \quad (-1)^m z_n(\{T\}^m) z_n(k) = \sum_{p=0}^{\min\{m,r\}} (-1)^p \sum_{\lambda \in \{0,1\}^r} \sum_{\nu \in (\mathbb{Z}_{\geq 0})^d} \sum_{\mu \in (\mathbb{Z}_{\geq 0})^d} z_n((k + \lambda)^\nu + e)^\nu)$$

for $m \geq 0$.

Now we take $k \in I \setminus \{0\}$, and set $r = \text{dep}(k)$ and $s = \text{dep}(k^\nu)$. We show that

$$(4.6) \quad \sum_{\nu \in (\mathbb{Z}_{\geq 0})^r} z_n((k^\nu + e)^\nu) = \sum_{l=0}^{m} (-1)^{m-l} z_n(\{T\}^{m-l}) \sum_{\nu \in (\mathbb{Z}_{\geq 0})^r} \sum_{\mu \in (\mathbb{Z}_{\geq 0})^d} z_n(k + \lambda + \mu).$$

for $0 \leq m \leq n - r - 1$ by induction on $m$. If $m = 0$ it follows from $(k^\nu)^\nu = k$. Suppose that $m \geq 1$. Separate the right hand side of (4.5) into two parts with $p = 0$ and $p \geq 1$. Then, from the induction hypothesis, we see that

$$(4.7) \quad \sum_{\nu \in (\mathbb{Z}_{\geq 0})^r} z_n((k^\nu + e)^\nu) = (-1)^m z_n(\{T\}^m) z_n(k)$$

$$+ \sum_{l=1}^{m} (-1)^{m-l} z_n(\{T\}^{m-l}) \sum_{p=1}^{l} (-1)^{p-1} \sum_{\lambda \in \{0,1\}^r} \sum_{\nu \in (\mathbb{Z}_{\geq 0})^r} \sum_{\mu \in (\mathbb{Z}_{\geq 0})^d} z_n(k + \lambda + \mu).$$

For $1 \leq p \leq l$ we define the map

$$\nu_p : \{\lambda \in \{0,1\}^r \mid \text{wt}(\lambda) = p\} \times \{\mu \in (\mathbb{Z}_{\geq 0})^r \mid \text{wt}(\mu) = l - p\}$$

$$\rightarrow \{e \in (\mathbb{Z}_{\geq 0})^r \mid \text{wt}(e) = l\}$$

by $\nu_p(\lambda, \mu) = \lambda + \mu$. Then it holds that

$$\sum_{p=1}^{l} (-1)^{p-1} \sum_{\lambda \in \{0,1\}^r} \sum_{\nu \in (\mathbb{Z}_{\geq 0})^r} \sum_{\mu \in (\mathbb{Z}_{\geq 0})^d} z_n(k + \lambda + \mu) = \sum_{e \in (\mathbb{Z}_{\geq 0})^r} \sum_{p=1}^{l} (-1)^{p-1} |\nu_p^{-1}(e)| \sum_{\lambda \in \{0,1\}^r} w_{\lambda}(\lambda) = \sum_{e \in (\mathbb{Z}_{\geq 0})^r} \sum_{p=1}^{l} (-1)^{p-1} |\nu_p^{-1}(e)|.$$

For a tuple $e = (e_1, \ldots, e_r)$ of non-negative integers, set

$$\text{supp}(e) = \{j \in \{1,2,\ldots,r\} \mid e_j \geq 1\}.$$

Then we see that

$$\nu_p^{-1}(e) = \{(\lambda, e - \lambda) \mid \lambda \in \{0,1\}^r, \text{wt}(\lambda) = p, \text{supp}(\lambda) \subset \text{supp}(e)\}.$$
Hence
\[ \sum_{p=1}^{l} (-1)^{p-1} |\nu_p^{-1}(e)| = \sum_{p=1}^{l} (-1)^{p-1} \left( \left| \text{supp}(e) \right| \right) = 1 \]
for \( e \in (\mathbb{Z}_0)^r \) satisfying \( \text{wt}(e) = l \). Therefore the right hand side of (4.7) is equal to that of (1.6).

To derive the desired equality from (4.6), we should prove that
\[ z_n(\{1\}^r; \zeta_n) = \frac{(-1)^r n}{n+1} (1 - \zeta_n)^r \]
for \( n > r \geq 0 \). We obtain it by calculating the generating function
\[ \sum_{r=0}^{n-1} z_n(\{1\}^r; \zeta_n)T^r = \prod_{m=1}^{n-1} \left( 1 + \frac{q^m}{m} T \right) \bigg|_{q = \zeta_n} = \prod_{m=1}^{n-1} \frac{1 - \zeta_m^n (1 - (1 - \zeta_n)T)}{1 - \zeta_m^n} \]
\[ = \frac{1 - (1 - (1 - \zeta_n)T)^n}{n(1 - \zeta_n)T} = \sum_{r=0}^{n-1} \frac{(-1)^r}{n} \left( \frac{n}{r+1} \right) (1 - \zeta_n)^rT^r. \]
This completes the proof. \( \square \)

As a corollary of Theorem 4.3 we reproduce the Ohno-type relations for FMZVs and SMZVs.

**Corollary 4.4.** [13] Suppose that \( k \in I \setminus \{\emptyset\} \). Set \( r = \text{dep}(k) \) and \( s = \text{dep}(k^\vee) \). Then it holds that
\[ \sum_{e \in (\mathbb{Z}_0)^r \atop \text{wt}(e) = m} \zeta_F((k^\vee + e)^\vee) = \sum_{e' \in (\mathbb{Z}_0)^r \atop \text{wt}(e') = m} \zeta_F(k + e') \]
for \( m \geq 0 \) and \( F = A \) or \( S \).

**Proof.** First we consider the case of FMZVs. If \( n \) is a prime \( p \), then \( \frac{1}{p} \left( m, \frac{p}{m-l+1} \right) \) is an integer for \( m \geq l \geq 0 \). Therefore, taking modulo \( (1 - \zeta_p) \) of both sides of (4.3), we obtain the desired relation for FMZVs.

To consider the case of SMZVs, we set \( \zeta_n = e^{2\pi i/n} \) in (4.3) and calculate the limit as \( n \to \infty \). Using Stirling’s formula, we see that
\[ \frac{1}{n} \left( \frac{n}{m-l+1} \right) (1 - e^{2\pi i/n})^{m-l} \to (-2\pi i)^{m-l} \quad (n \to \infty) \]
for \( m \geq l \geq 0 \). Hence it holds that
\[ \sum_{e \in (\mathbb{Z}_0)^r \atop \text{wt}(e) = m} \xi((k^\vee + e)^\vee) = \sum_{l=0}^{m} (-2\pi i)^{m-l} \sum_{e' \in (\mathbb{Z}_0)^r \atop \text{wt}(e') = m} \xi(k + e'), \]
where \( \xi(k) \) is defined by (4.1). Taking the real parts modulo \( \zeta(2)\mathbb{Z} = \pi^2 \mathbb{Z} \), we obtain the desired equality for SMZVs. \( \square \)
Finally we prove Ohno-type relations for the cyclotomic analogue of FMZVs introduced in [2]. As an analogue of $\mathcal{A}$ we define

$$
\mathcal{A}^{\text{cyc}} = \left( \prod_{p:\text{prime}} \mathbb{Z}[\zeta_p]/(p) \right) / \left( \bigoplus_{p:\text{prime}} \mathbb{Z}[\zeta_p]/(p) \right).
$$

It also carries the $\mathbb{Q}$-algebra structure. The cyclotomic analogue of FMZV is defined by

$$
Z^{\text{cyc}}(k) = (z_p(k; \zeta_p) \mod (p))_p \in \mathcal{A}^{\text{cyc}}
$$

for $k \in I$.

To write down the Ohno-type relation we need the $\mathbb{Q}$-linear map $L$ defined as follows. We define the stuffle product $*$ on $H_1^Q$ by

$$
1 \ast w = w \ast 1 = 1,
$$

$$
(e_k w) \ast (e_l w') = e_k (w \ast e_l w') + e_l (e_k w \ast w') + e_{k+l} (w \ast w')
$$

for $w, w' \in H_1^Q$ and $k, l \geq 1$. We define the $\mathbb{Q}$-linear map $L : H_1^Q \to H_1^Q$ by

$$
L(e_k) = -\frac{1}{2\text{dep}(k) + 1} e_1 * e_k
$$

for $k \in I$.

Let $\mathbb{Q}I$ be the $\mathbb{Q}$-vector space with the basis $I$. By abuse of notation we denote by the same letter $L$ the $\mathbb{Q}$-linear transformation on $\mathbb{Q}I$ defined by $L(k) = \sum_{k'} a_{k,k'} k'$, where $a_{k,k'}$ is a rational number determined by $L(e_k) = \sum_{k'} a_{k,k'} e_{k'}$. We also extend the map $Z^{\text{cyc}} : I \to \mathcal{A}^{\text{cyc}}$ to $\mathbb{Q}I$ by $\mathbb{Q}$-linearity.

Set $\varpi = (1 - \zeta_p)_p \in \mathcal{A}^{\text{cyc}}$. Then it holds that

$$
\varpi Z^{\text{cyc}}(k) = Z^{\text{cyc}}(L(k))
$$

(see [2]).

**Corollary 4.5.** For $k \in I \setminus \{\emptyset\}$ and $m \geq 0$, it holds that

$$
\sum_{e \in (\mathbb{Z}_{>0})^s, \text{wt}(e)=m} Z^{\text{cyc}}((k^r + e)^r) = \sum_{l=0}^m (-1)^{m-l} \frac{1}{m-l+1} \sum_{e' \in (\mathbb{Z}_{>0})^r, \text{wt}(e')=l} Z^{\text{cyc}}(L^{m-l}(k + e')),
$$

where $r = \text{dep}(k)$ and $s = \text{dep}(k^r)$.

**Proof.** It follows from Theorem 4.3 (4.8) and

$$
\frac{1}{p} \left( \frac{p}{m-l+1} \right) \equiv \frac{(-1)^{m-l}}{m-l+1} \mod p \quad (m \geq l \geq 0)
$$

for any prime $p$ such that $p \geq r + m + 1$.  \qed
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