STREAMING CORESETS FOR SYMMETRIC TENSOR FACTORIZATION

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ABSTRACT

Factorizing tensors has recently become an important optimization module in a number of machine learning pipelines, especially in latent variable models. We show how to do this efficiently in the streaming setting. Given a set of \( n \) vectors, each in \( \mathbb{R}^d \), we present algorithms to select a sublinear number of these vectors as coreset, while guaranteeing that the CP decomposition of the \( p \)-moment tensor of the coreset approximates the corresponding decomposition of the \( p \)-moment tensor computed from the full data. We introduce two novel algorithmic techniques: online filtering and kernelization. Using these two, we present four algorithms that achieve different tradeoffs of coreset size, update time and working space, beating or matching various state of the art algorithms. In case of matrices (2-ordered tensor) our online row sampling algorithm guarantees \((1 \pm \epsilon)\) relative error spectral approximation. We show applications of our algorithms in learning single topic modeling.

Keywords Tensor · Factorization · Subspace Embedding · Online · Streaming · Lp

1 Introduction

Much of the data that is consumed in data mining and machine learning applications arrives in a streaming manner. The data is conventionally treated as a matrix, with a row representing a single data point and the columns its corresponding features. Since the matrix is typically large, it is advantageous to be able to store only a small number of rows and still preserve some of its “useful” properties. One such abstract property that has proven useful in a number of different settings, such as solving regression, finding various factorizations, is subspace preservation. Given a matrix \( A \in \mathbb{R}^{n \times d} \), an \( m \times d \) matrix \( C \) is subspace preserving for the \( \ell_2 \) norm if, \( \forall x \in \mathbb{R}^d \),

\[
\left| \sum_{\tilde{a}_j^T \in C} (\tilde{a}_j^T x)^2 - \sum_{i \in [n]} (a_i^T x)^2 \right| \leq \epsilon \cdot \sum_{i \in [n]} (a_i^T x)^2
\]

We typically desire \( m \ll n \) and \( \tilde{a}_j \)'s represent the subsampled and rescaled rows from \( A \). Such a sample \( C \) is often referred to as a coreset. This property has been used to obtain approximate solutions to many problems such as regression, low rank approximation etc [8], while having \( m \) to be at most \( O(d^2) \). Such property has been defined for other \( \ell_p \) norms too [37, 42, 45].

Matrices are ubiquitous and depending on the application one can assume that the data is coming from a generative model, i.e. there is some distribution from which every incoming row is sampled and given to user and the task is to estimate the hidden variables of this generative model. A much used technique to learn these variables is by using a low rank representation data(matrix). A low rank representation (even in the \( \ell_2 \) norm) of a matrix is not,
however, unique (since $AB = (AQ)(Q'B)$ for any orthonormal $Q$). This often creates an issue in using the low rank representation to estimate the hidden variables. This is one of the reasons to look at higher order moments of the data i.e. tensors. Tensors are formed by outer product of data vectors, i.e. for a dataset $A \in \mathbb{R}^{n \times d}$ one can use a $p$ order tensor $T \in \mathbb{R}^{d \times \ldots \times d}$ as $T = \sum_{i=1}^{n} a_i \otimes p$, where $p$ is set by user depending on the number of latent variables one is expecting in the generative model [46]. The decomposition of such a tensor is unique under a mild assumption [2]. Factorization of tensors into its constituent elements has found uses in many machine learning applications such as topic modeling [3], various latent variable models [20, 23, 24], training neural networks [36] etc.

For a $p$-order moment tensor $T = \sum_{i} a_i \otimes p$ created using the set of vectors $\{a_i\}$ and for $x \in \mathbb{R}^{d}$ one of the important property one needs to preserve is $\overline{T}(x, \ldots, x) = \sum_{i} (a_i^T x)^p$. This operation is also called tensor contraction [15]. Now if we wish to “approximate” it using only a subset of the rows in $A$, the above property for $\ell_p$ norm subspace preservation does not suffice.

For tensor factorization, which is performed using power iteration, a coreset $C \subseteq \{a_i\}$, in order to give a guaranteed approximation to the tensor factorization, needs to satisfy the following natural extension of the $\ell_2$ subspace preservation condition:

$$\sum_{a_i \in A} (a_i^T x^p) \approx \sum_{a_j \in C} (\tilde{a}_j^T x^p)$$

Ensuring this tensor contraction property enables one to approximate the CP decomposition of $T$ using only the vectors $\tilde{a}_j$’s via power iteration method [3]. A related notion is that of $\ell_p$ subspace embedding where we need that $C$ satisfies the following, $\forall x \in \mathbb{R}^d$

$$\sum_{a_i \in A} |a_i^T x|^p \approx \sum_{a_j \in C} |\tilde{a}_j^T x|^p$$

The two properties are the same for even $p$, as both LHS and RHS is just sum of non negative terms. But they slightly differ for odd values of $p$.

In this work, we show that it is possible to create coresets for the above property in streaming and restricted streaming (online) setting. In restricted streaming setting an incoming point, when it arrives, is either chosen in the set or discarded forever. We consider the following formalization of the above two properties. Given a query space of vectors $Q \subseteq \mathbb{R}^d$ and $\epsilon > 0$, we aim to choose a set $C$ which contains sampled and rescaled rows from $A$ to ensure that $\forall x \in Q$ with probability at least 0.9, the following properties hold,

$$|\sum_{a_j \in C} (\tilde{a}_j^T x)^p - \sum_{i \in [n]} (a_i^T x)^p| \leq \epsilon \cdot \sum_{i \in [n]} |a_i^T x|^p$$  \hspace{1cm} (1)

$$|\sum_{a_j \in C} |\tilde{a}_j^T x|^p - \sum_{i \in [n]} |a_i^T x|^p| \leq \epsilon \cdot \sum_{i \in [n]} |a_i^T x|^p$$  \hspace{1cm} (2)

Note that neither property follows from the other. For even values of $p$, the above properties are identical and imply a relative error approximation as well. For odd values of $p$, the $\ell_p$ subspace embedding as equation (2) gives a relative error approximation but the tensor contraction as equation (1) implies an additive error approximation, as LHS terms are not sum of absolutes. It can become relative error under non-negativity constraints on $a_i$ and $x$. This happens, for instance, for the important use case of topic modeling, where $p = 3$ typically.

**Our Contributions:** We give a method to sample rows in streaming manner for a $p$ order tensor in order to estimate its latent factors. For a given matrix $A \in \mathbb{R}^{n \times d}$, a $k$-dimensional query space $Q \subseteq \mathbb{R}^{k \times d}$, some integer $k \geq 2$ and $\epsilon > 0$,

- We give an algorithm (LineFilter) that is able to select rows, it takes $O(d^2)$ update time, and returns a sample of size $O(n^{1-2/p}d/pk(1+1\log \|A\| - d^{-1} \min_i \log \|a_i\|))$ such that the set of selected rows forms a coreset having the guarantees stated in equations (1) and (2) (Theorem 4.2). It is a streaming algorithm but also works well in the restricted streaming (online) setting.
- We improve the sampling complexity of our coreset to $O(d^{p/2}k(\log n)^{10\epsilon^{-5}})$ by a streaming algorithm (LineFilter+StreamingLW) with amortized update time $O(d^2)$ (Theorem 4.4). It requires slightly higher working space $O(d^{p/2}k(\log n)^{11\epsilon^{-5}})$.
- We present a kernelization technique which, for any vector $v$, creates two vectors $\tilde{v}$ and $\hat{v}$ such that for any $x, y \in \mathbb{R}^d$,

$$|x^T y|^p = |\tilde{x}^T \tilde{y}| \cdot |\hat{x}^T \hat{y}| = |\tilde{x}^T |\tilde{y}|^{2p/(p+1)}$$
We use the following notation throughout the paper. A scalar is denoted by a lower case letter, e.g. \( p \) while a vector is denoted by a boldface lower case letter, e.g. \( \mathbf{a} \). By default all vectors are considered as column vectors unless specified otherwise. Matrices and sets are denoted by boldface upper case letters, e.g. \( \mathbf{A} \). Specifically, \( \mathbf{A} \) denotes an \( n \times d \) matrix with set of rows \( \{ \mathbf{a}_i \} \) and, in the streaming setting, \( \mathbf{A}_i \) represents the matrix formed by the first \( i \) rows of \( \mathbf{A} \) that have arrived. We will interchangeably refer to the set \( \{ \mathbf{a}_i \} \) as the input set of vectors as well as the rows of the matrix \( \mathbf{A} \). A tensor is denoted by a bold calligraphic letter e.g. \( \mathcal{T} \). Given a set of \( d \)–dimensional vectors \( \mathbf{a}_1, \ldots, \mathbf{a}_n \), from which a \( p \)-order symmetric tensor \( \mathcal{T} \) is obtained as \( \mathcal{T} = \sum_{i=1}^n \mathbf{a}_i \otimes_p \) i.e. the sum of the \( p \)-th order outer product of each of the vectors. It is easy to see that a symmetric tensor \( \mathcal{T} \) satisfies the following: \( \forall i_1, i_2, \ldots, i_p: \mathcal{T}_{i_1, i_2, \ldots, i_p} = \mathcal{T}_{i_2, i_1, \ldots, i_p} = \cdots = \mathcal{T}_{i_p, i_1, \ldots, i_{p-1}} \), i.e. same value for all possible permutations of \( (i_1, i_2, \ldots, i_p) \). We define the scalar quantity, also known as tensor contraction, as \( \mathcal{T}(\mathbf{x}, \ldots, \mathbf{x}) = \sum_{i=1}^n (\mathbf{a}_i^T \mathbf{x})^p \), where \( \mathbf{x} \in \mathbb{R}^d \). There are three widely used tensor decomposition techniques known as CANDECOMP/PARAFAC(CP), Tucker and Tensor Train decomposition. Our work focuses on CP decomposition.

### Table 1: Table Comparing Existing Work and Current Contributions.

| Algorithm | Sample Size \( O(\cdot) \) | Update time \( O(\cdot) \) | Working space \( O(\cdot) \) |
|-----------|-----------------------------|-----------------------------|-----------------------------|
| StreamingWCB \[47\] | \( d^p k e^{-2} \) | \( d^p p \log d \) | \( d^p k e^{-2} \) |
| StreamingLW \[49\] | \( d^{p/2} k e^{-2} \) | \( d^{p/2} \) | \( d^{p/2} k e^{-2} \) |
| StreamingFC \[48\] | \( d^{p/2} k e^{-2} \) | \( d \) | \( d^{p/2} k e^{-2} \) |
| LineFilter (Theorem 4.2) | \( n^{1 - 1/p} d k e^{-2} \) | \( d^2 \) | \( d^2 \) |
| LineFilter+StreamingLW (Theorem 4.4) | \( d^{p/2} k e^{-2} \) | \( d^2 \) | \( d^{p/2} k e^{-2} \) |
| KernelFilter (Theorem 4.6) (even \( p \)) | \( d^{p/2} k e^{-2} \) | \( d^p \) | \( d^p \) |
| KernelFilter (Theorem 4.7) (odd \( p \)) | \( n^{(p-1)/(p+1)} d^{p/2} k e^{-2} \) | \( d^{p+1} \) | \( d^{p+1} \) |
| LineFilter+KernelFilter (Theorem 4.8) (even \( p \)) | \( d^{p/2} k e^{-2} \) | \( d^2 \) | \( d^p \) |
| LineFilter+KernelFilter (Theorem 4.8) (odd \( p \)) | \( n^{(p-2)/(p+2)} d^{p/2} k e^{-2} \) | \( d^{p+1} \) | \( d^{p+1} \) |

Using this technique, we give an algorithm (KernelFilter) which takes \( O(n d^p) \) time and samples \( O(\frac{d^{p/2} k}{\epsilon^2}(1 + p(\log \| \mathbf{A} \| - d^{-p/2} \min_i \log \| \mathbf{a}_i \|))) \) vectors to create a coreset having the same guarantee as \( (1) \) and \( (2) \) (Theorem 4.6) for even value \( p \). For odd value \( p \) it takes \( O(n d^{p+1}) \) time and samples \( O(\frac{n^{(p-1)/(p+1)} d^{p/2} k}{\epsilon^2}(1 + (p + 1)(\log \| \mathbf{A} \| - d^{-p/2} \min_i \log \| \mathbf{a}_i \|)))^{p/(p+1)} \) vectors to create a coreset having the same guarantee as \( (1) \) and \( (2) \) (Theorem 4.7). Both update time and working space of the algorithm for even \( p \) is \( O(d^p) \) and for odd \( p \) it is \( O(d^{p+1}) \). It is a streaming algorithm but also works well in the restricted streaming (online) setting.

- We combine both online algorithms and propose another algorithm (LineFilter+KernelFilter) which has \( O(d^2) \) amortized update time and returns \( O(\frac{d^{p/2} k}{\epsilon^2}(1 + p(\log \| \mathbf{A} \| - d^{-p/2} \min_i \log \| \mathbf{a}_i \|))) \) for even \( p \) and \( O(\frac{n^{(p-1)/(p+2)} d^{p/2} k}{\epsilon^2}(1 + (p + 1)(\log \| \mathbf{A} \| - d^{-[p/2]} \min_i \log \| \mathbf{a}_i \|)))^{p/(p+1)} \) vectors for odd \( p \) as coreset with same guarantees as equation \( (1) \) and \( (2) \) (Theorem 4.6). Although it uses \( O(d^{p+1/2}) \) working space.

- For the \( p = 2 \) case, both LineFilter and KernelFilter translate to an online algorithm for sampling rows of the matrix \( \mathbf{A} \), while guaranteeing a relative error spectral approximation (Theorem 4.9). This is an improvement (albeit marginal) over the online row sampling result by [6]. The additional benefit of this new online algorithm over [6] is that it does not need the knowledge of \( \sigma_{\min}(\mathbf{A}) \) to get a relative error approximation.

The rest of this paper is organized as follows: In section 2 we look at some preliminaries for tensors and coresets. We also describe the notation used throughout the paper. Section 3 discusses related work. In section 4 we state all the four streaming algorithms we propose. We also show how our problem of preserving tensor contraction relates to preserving low subspace embedding. In section 5 we describe the guarantees given by our algorithm and some proofs. In section 6 we describe how our algorithm can be used in case of streaming single topic modeling. We give empirical result that compare our sampling scheme with other schemes.

## 2 Preliminaries

We use the following notation throughout the paper. A scalar is denoted by a lower case letter, e.g. \( p \) while a vector is denoted by a boldface lower case letter, e.g. \( \mathbf{a} \). By default all vectors are considered as column vectors unless specified otherwise. Matrices and sets are denoted by boldface upper case letters, e.g. \( \mathbf{A} \). Specifically, \( \mathbf{A} \) denotes an \( n \times d \) matrix with set of rows \( \{ \mathbf{a}_i \} \) and, in the streaming setting, \( \mathbf{A}_i \) represents the matrix formed by the first \( i \) rows of \( \mathbf{A} \) that have arrived. We will interchangeably refer to the set \( \{ \mathbf{a}_i \} \) as the input set of vectors as well as the rows of the matrix \( \mathbf{A} \). A tensor is denoted by a bold calligraphic letter e.g. \( \mathcal{T} \). Given a set of \( d \)–dimensional vectors \( \mathbf{a}_1, \ldots, \mathbf{a}_n \), from which a \( p \)-order symmetric tensor \( \mathcal{T} \) is obtained as \( \mathcal{T} = \sum_{i=1}^n \mathbf{a}_i \otimes_p \) i.e. the sum of the \( p \)-th order outer product of each of the vectors. It is easy to see that a symmetric tensor \( \mathcal{T} \) satisfies the following: \( \forall i_1, i_2, \ldots, i_p: \mathcal{T}_{i_1, i_2, \ldots, i_p} = \mathcal{T}_{i_2, i_1, \ldots, i_p} = \cdots = \mathcal{T}_{i_p, i_1, \ldots, i_{p-1}} \), i.e. same value for all possible permutations of \( (i_1, i_2, \ldots, i_p) \). We define the scalar quantity, also known as tensor contraction, as \( \mathcal{T}(\mathbf{x}, \ldots, \mathbf{x}) = \sum_{i=1}^n (\mathbf{a}_i^T \mathbf{x})^p \), where \( \mathbf{x} \in \mathbb{R}^d \). There are three widely used tensor decomposition techniques known as CANDECOMP/PARAFAC(CP), Tucker and Tensor Train decomposition. Our work focuses on CP decomposition.
We denote 2-norm for a vector $x$ as $\|x\|$, and any $p$-norm, for $p \neq 2$ as $\|x\|_p$. We denote the 2-norm or spectral norm of a matrix $A$ by $\|A\|$. 

**Coresets.** A coreset is a small summary of data which can give provable guarantees for a particular optimization problem. Formally, given a set $X \subseteq \mathbb{R}^d$, set of queries $Q$ and a nonnegative cost function $f_q(x)$ with parameter $q \in Q$ and data point $x \in X$, a set of subsampled and appropriately reweighted points $C$ is called a coreset if $\forall q \in Q$, $|\sum_{x \in X} f_q(x) - \sum_{x \in C} f_q(x)| \leq \epsilon \sum_{x \in X} f_q(x)$ for some $\epsilon > 0$.

To guarantee the above approximation one can define a set of scores, termed as sensitivities [32] corresponding to each point. This can be used to create coresets via importance sampling. The sensitivity of a point $x$ is defined as $s_x = \sup_{q \in Q} \frac{|\sum_{x \in X} f_q(x)|}{\sum_{x \in X} f_q(x)}$. In [32] authors show that using any upper bounds to the sensitivity scores, we can create a probability distribution, which can be used to sample a coreset. The size of the coreset depends on the sum of these upper bounds and the dimension of the query space.

We use the following two inequality to state our guarantees.

**Theorem 2.1. (Bernstein [10])** Let the scalar random variables $x_1, x_2, \ldots, x_n$ be independent that satisfy $\forall i \in [n], |x_i - \mathbb{E}[x_i]| \leq b$. Let $X = \sum_i x_i$ and let $\sigma^2 = \sum_i \sigma_i^2$ be the variance of $X$, where $\sigma_i^2$ is the variance of $x_i$. Then for any $t > 0$,

$$\Pr(X > \mathbb{E}[X] + t) \leq \exp\left(-\frac{t^2}{2\sigma^2 + bt/3}\right)$$

**Theorem 2.2. (Matrix Bernstein [43])** Let $X_1, \ldots, X_n$ are independent $d \times d$ random matrices such that $\forall i \in [n], \|X_i\| - \mathbb{E}[\|X_i\|] \leq b$ and $\text{var}(\|X_i\|) \leq \sigma^2$ where $X = \sum_{i=1}^n X_i$, then for some $t > 0$,

$$\Pr(\|X\| - \mathbb{E}[\|X\|] \geq t) \leq d \exp\left(-\frac{\delta^2}{2\sigma^2 + bt/3}\right)$$

3 Related Work

Coresets are small summaries of data which can be used as a proxy to the original data with provable guarantees. The term was first introduced in [33] where they used coresets for the shape fitting problem. Coresets for clustering problem were described in [34]. In [31] authors gave a generalized framework to construct coresets based on importance sampling using sensitivity scores introduced in [32]. Interested reader can check [3, 9, 11]. Various online sampling schemes for spectral approximation are discussed in [6, 13].

Tensor decomposition is unique under minimal assumptions [2]. Therefore it has become very popular in various latent variable modeling applications [20, 23, 3], neural networks [36] etc. However in general (i.e. without any assumption) most of the tensor problems including tensor decomposition are NP-hard [5]. There has been much work on fast tensor decomposition techniques. Various tensor sketching methods for tensor operations are discussed in [14, 12, 15]. The area of online tensor power iterations has also been explored in [4, 7]. Various heuristics for tensor sketching as well as RandNLA techniques [8] over matricized tensors for estimating low rank tensor approximation have been studied in [10].

In the online setting, for a matrix $A \in \mathbb{R}^{n \times d}$ where rows are coming in streaming manner, the guarantee achieved by [6] while preserving additive error spectral approximation with sample size $O(d(\log d)(\log \epsilon \|A\|^2/\delta))$. $\|A x\|^2 - \|C x\|^2 \leq \epsilon \|A x\|^2 + \delta, \forall x \in \mathbb{R}^d$.

The problem of $\ell_p$ subspace embedding has been explored in both offline [33, 34, 52, 55] and streaming setting [40]. As any offline algorithm to construct coresets can be used as streaming algorithm [54], we use the known offline algorithms and summarize their results in streaming version in table 1. The algorithm in [41] samples $O(n^{1-2/p}\text{poly}(d))$ rows and gives poly($d$) error relative subspace embedding but in $O(\text{nnz}(A))$ time. For streaming $\ell_p$ subspace embedding [40], give a one pass deterministic algorithm for $\ell_p$ subspace embedding for $1 \leq p \leq \infty$. For some constant $\gamma \in (0, 1)$ the algorithm takes $O(n^\gamma d^\gamma)$ space and $O(n^{\gamma}d^2 + n^\gamma d^3 \log n)$ update time to return a $1/d^{O(1/\gamma)}$ error relative subspace embedding for any $\ell_p$ norm.

4 Algorithms and Guarantees

In this section we discuss our two major contributions. We first introduce the two algorithmic modules–LineFilter and KernelFilter. LineFilter, on arrival of each row, simply decides whether to sample it or
not. The probability of sampling is computed based on the stream seen till now, where as in KernelFilter, for every incoming row \(a_i\), the decision of sampling it, depends on two rows \(\hat{a}_i\) and \(\hat{a}_i\) we create from \(a_i\) such that: for any vector \(x\), there is a similar transformation \((\hat{x}\) and \(\hat{x}\)) and we get, \(|a_i^T x|^p = |\hat{a}_i^T x| \cdot |\hat{a}_i^T x| = |\hat{a}_i^T x|^{2p/(p+1)}\). We call it kernelization. Here we mention both \(\hat{a}_i\) and \(\hat{a}_i\) to simplify our analysis, even though the objective function can also be defined using just the \(\hat{a}_i\) and its counter part of \(x\).

Note that both LineFilter and KernelFilter are restricted streaming algorithms in the sense that each row is selected / processed only when it arrives. This online nature of the two algorithms allows us to use these as modules in order to create the following algorithms:

1. LineFilter+StreamingLW: The output streams of LineFilter is fed to a StreamingLW, which is a merge-and-reduce based streaming algorithm based on Lewis Weights. Here StreamingLW outputs the final coreset.
2. LineFilter+KernelFilter: The output of LineFilter is first kernelized. Which is for final sampling by KernelFilter. It is essentially a version of LineFilter with different \(p\).

The algorithms compute a score for every incoming row and based on the score the sampling probability of the row is decided. The score depends on the incoming row (say \(x_i\)) and some prior knowledge (say \(M\)) of the data which we have already seen. Here, we define \(M = X_{i-1}^T X_{i-1}\) and \(Q\) is an orthonormal column basis where \(X_{i-1}\) represents the matrix with rows \(\{x_1, \ldots, x_{i-1}\}\). Now we present the method which is called by both LineFilter and KernelFilter for computing sampling probability.

**Algorithm 1 OnlineScore\((x_i, M, Q, p)\)**

```python
def OnlineScore(x_i, M, Q, p):
    if x_i in column space(Q):
        M_f = M_T - M_T x_i x_i^T M_T
    else:
        M = M + x_i x_i^T; M_f = M_T
        Q = orthonormal column basis(M)

    M = M + x_i x_i^T; M_f = M_T
    M = M + x_i x_i^T; M_f = M_T
    Q = orthonormal column basis(M)

    return c_i, M_f
```

Here if the incoming row \(x_i\) lies in the subspace spanned by \(Q\) (i.e. if \(|Qx_i| = |x_i|\)), then the algorithm takes \(O(m^3)\) time else it takes \(O(m^3)\) where \(x_i \in \mathbb{R}^m\). Here we have used a modified version of Sherman Morrison formula to compute \((X_{i-1}^T x_i)^T = (X_{i-1}^T x_i + x_i x_i^T)^T = (M + x_i x_i^T)^T\). Note that in our setup \(M\) need not be full rank, so we use the formula \((X_i X_i)^T = M_T - M_T x_i x_i^T M_T + x_i x_i^T M_T\). In the following lemma we prove this formula.

**Lemma 4.1.** Given a rank-\(k\) positive semi-definite matrix \(M \in \mathbb{R}^{d \times d}\) and a vector \(x\) such that it completely lies in the column space of \(M\). Then we have,

\[
(M + xx^T)^T = M_T - \frac{M_T xx^T M_T}{1 + x^T M_T x}
\]

**Proof.** The proof is in the similar spirit to lemma 5.3 Consider \([\mathbf{V}, \Sigma, \mathbf{V}] = \text{SVD}(M)\) and since \(x\) lies completely in the column space of \(M\), hence \(\exists y \in \mathbb{R}^k\) such that \(\mathbf{V} y = x\). Note that \(\mathbf{V} \in \mathbb{R}^{d \times k}\).

\[
(M + xx^T)^T = (\mathbf{V} \Sigma \mathbf{V}^T + \mathbf{V} y y^T \mathbf{V}^T)^T = \mathbf{V} (\Sigma + yy^T)^{-1} \mathbf{V}^T = \mathbf{V} \left(\Sigma^{-1} - \frac{\Sigma^{-1} yy^T \Sigma^{-1}}{1 + y^T \Sigma^{-1} y}\right) \mathbf{V} = M_T - \frac{M_T xx^T M_T}{1 + x M_T x}
\]

The third equality is by Sherman Morrison Formula.

**4.1 LineFilter**

Here we present our first streaming algorithm which ensures equations (1) and (2). The algorithm can also be used in restricted streaming (online) setting where for every incoming row we get only one chance to decide whether to sample it or not. Due to its nature of filtering out rows we call it LineFilter algorithm. The algorithm tries to reduce the
Given LineFilter equation (1). We create have access to the entire data. A use sensitivity based framework to decide our sampling probability where we know sensitivity scores are well defined the difference between the original and sampled term, through Bernstein inequality [10] and try to reduce it. Here we preserve $\ell_p$ than that of $\ell_2$. Here we present a streaming algorithm which returns a coreset for the same problem with its coreset much smaller.

4.2 StreamingLW

First we want to point out that our coresets for tensor contraction i.e. equation (1) also gives a coreset which is sublinear to input size but gives little benefits by taking very less working space and computation time, which are independent of $p$ i.e. order of the tensor. However LineFilter gives a coreset which is sublinear to input size but as $p$ increases the factor $n^{1-2/p}$ tends to $n$. Hence for higher $p$ the coresets might be as big as the entire dataset. We discuss the proof of the theorem along with its supporting lemma in section 5.

4.2 LineFilter+StreamingLW

Here we present a streaming algorithm which returns a coreset for the same problem with its coreset much smaller than that of LineFilter. First we want to point out that our coresets for tensor contraction i.e. equation (1) also preserve $\ell_p$ subspace embedding i.e. equation (2). For simplicity we show this relation in the offline setting, where we have access to the entire data $A$. For a matrix $A \in \mathbb{R}^{n \times d}$, we intend to preserve the tensor contraction property in equation (1). We create a coreset $C$ by sampling original row vectors $a_i$ with appropriate scaling. We analyze the variance of the difference between the original and sampled term, through Bernstein inequality [10] and try to reduce it. Here we use sensitivity based framework to decide our sampling probability where we know sensitivity scores are well defined for positive cost function [32]. Now with the tensor contraction the problem is that for odd $p$ and for some $x$, the cost $(a_i^T x)^p$ could be negative, for some $i \in [n]$. So for every row $i$ we define the sensitivity score as follows,

$$s_i = \sup_x \frac{|a_i^T x|^p}{\sum_{j=1}^n |a_j^T x|^p}$$

Here by sampling enough number of rows based on above defined sensitivity scores also preserve $\sum_{i=1}^n |a_i^T x|^p = \|Ax\|_p$ [32]. The sampled rows creates a coreset $C$ which is $\ell_p$ subspace embedding, i.e. $\forall x, \|Ax\|_p = \|Cx\|_p \leq \epsilon \|Ax\|_p$. We define and discuss the online version of these scores in section 5 which also preserve tensor contraction. It is not difficult to show that the offline scores defined above also preserve tensor contraction. Sampling based methods used in [37, 42, 45] to get a coreset for $\ell_p$ subspace embedding also preserve tensor contraction. This is because all these sampling based methods try reducing the variance of the difference between original and expected term.

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**Algorithm 2 LineFilter**

**Require**: Streaming rows $a_i^T, i = 1, \ldots, n, p \geq 2, r > 1$

**Ensure**: Coreset $C$ satisfying eqn (1) and (2) w.h.p.

1. Set $M = O^{d \times d}$, $L = 0$, $C = \emptyset$
2. $Q = \text{orthonormal column basis of } M$
3. $\text{while current row } a_i^T \text{ is not the last row do}$
   - $\hat{e}_i = \text{OnlineScore}(a_i, M, Q, p)$
   - $l_i = \min\{\hat{e}_i^{p/2 - 1}(\hat{e}_i)^{p/2}, 1\}; L = L + l_i; p_i = \min\{r L_i / L, 1\}$
4. $\text{Sample } a_i / \sqrt{p_i} \text{ in } C \text{ with probability } p_i$
5. Return $C$

Every time a row $a_i \in \mathbb{R}^d$ comes, the LineFilter calls the function $\text{OnlineScore}(i)$ which returns a score $\hat{e}_i$. Then LineFilter computes $l_i$, which is an upper bound of the sensitivity score. Based on $l_i$ the row’s sampling probability is decided. We formally define and discuss sensitivity scores of our problem in section 5. Now we present the streaming algorithm LineFilter takes just $O(nd^2)$. Now we summarize the guarantees of the LineFilter in the following theorem.

**Theorem 4.2.** Given $A \in \mathbb{R}^{n \times d}$ whose rows are coming in streaming manner, LineFilter selects a set $C$ of size $O(2^{1 - 2/p}d (1 + \log \|A\| - d^{-1} \min \log \|a_i\|))$ using both working space and update time $O(d^2)$. Suppose $Q$ is a fixed $k$-dimensional subspace, then with probability at least 0.9, for $\epsilon > 0$, $\forall x \in \mathbb{R}$, the set $C$ satisfies both tensor contraction and $\ell_p$ subspace embedding as in equations (1) and (2) respectively.

It is worth noting that LineFilter benefits by taking very less working space and computation time, which are independent of $p$ i.e. order of the tensor. However LineFilter gives a coreset which is sublinear to input size but as $p$ increases the factor $n^{1-2/p}$ tends to $n$. Hence for higher $p$ the coresets might be as big as the entire dataset. We discuss the proof of the theorem along with its supporting lemma in section 5.
As we know that any offline coreset algorithm can be made a streaming algorithm using merge and reduce method [34]. For \( p \geq 2 \) the sampling complexity of [42] is best among all other methods we mentioned. Hence here we use Lewis Weights sampling [42] as the offline method along with merge and reduce to convert it into a streaming algorithm which we call StreamingLW. The following lemma summarizes the guarantee one gets from StreamingLW.

**Lemma 4.3.** Given a set of \( n \) streaming vectors \( \{a_i\} \), and \( Q \), a fixed \( k \)-dimensional subspace, the StreamingLW returns \( C \) such that, with probability 0.9 and \( \epsilon > 0 \), \( \forall x \in \mathbb{R}^d \), it satisfies tensor contraction and \( \ell_p \) subspace embedding as in equations (1) and (2) respectively.

It requires \( O(dp/2) \) amortized update time and uses \( O(dp/2\epsilon^{-5}\log^{11}n) \) working space to return a coreset \( C \) of size \( O(dp/2\epsilon^{-5}\log^{10}n) \).

**Proof.** Here the data is coming in streaming sense and it is feed to the StreamingLW for \( \ell_p \) subspace embedding. We use merge and reduce from [34] for streaming data. From [42] we know that for a set \( P \) of size \( n \) takes \( O(ndp/2) \) time to return a coreset \( Q \) of size \( O(dp/2(\log d)\epsilon^{-5}) \). Note that for the StreamingLW in section 7 of [34] we set \( M = O(dp/2(\log d)\epsilon^{-5}) \). The method returns \( Q_i \) as the \((1 + \delta_i) \) coreset for the partition \( P_i \) where \(|P_j|\) is either \( 2^iM \) or 0, here \( \rho_j = \epsilon/(c(j + 1)^2) \) such that \( 1 + \delta_i = \prod_{j=0}^i(1 + \rho_j) \leq 1 + \epsilon/2, \forall j \in [\log n] \). Thus we have \(|Q_i|\) is \( O(dp/2(\log d)(i + 1)^{10}\epsilon^{-5}) \). In StreamingLW the method reduce sees at max \( \log n \) many coresets at any point of time. Hence the total working space is \( O(dp/2(\log^{11}n)(\log d)\epsilon^{-5}) \). Now the amortized time spent per update is,

\[
\sum_{i=1}^{\lceil\log(n/M)\rceil} \frac{1}{2^iM}(Q_i|dP/2) = \sum_{i=1}^{\lceil\log(n/M)\rceil} \frac{1}{2^iM}(M(i + 1)^4dP/2) \leq dp/2
\]

So the finally the algorithm return \( Q \) as the final coreset of \( O(dp/2(\log^{10}n)(\log d)\epsilon^{-5}) \) rows and uses \( O(dp/2) \) amortized update time. \( \square \)

Note that in this case both update time and working space has dominating term as functions of \( d \) and \( p \). The coreset size also has major contributing factor which is \( \epsilon^{-5} \).

Now we propose our second algorithm where we feed the output of LineFilter to StreamingLW method. Here every incoming row is fed to LineFilter, which quickly computes a sampling probability and based on which the row gets sampled. Now if it gets sampled then we pass it to the StreamingLW method, which returns a coreset. The entire algorithm gets an improved amortized update time compared to StreamingLW and improved sampling complexity compared to LineFilter. We call this algorithm LineFilter + StreamingLW and summarize the guarantees in the following theorem.

**Theorem 4.4.** Consider \( A \in \mathbb{R}^{n \times d} \) whose rows are given to LineFilter + StreamingLW in streaming manner. It requires \( O(d^2) \) amortized update time and uses \((1 - 2/p)^{11}dp/2\epsilon^{-5}\log^{11}n\) working space to return a coreset \( C \) of size \((1 - 2/p)^{10}dp/2\epsilon^{-5}\log^{10}n\) such that with at least 0.9 probability, \( C \) satisfies both tensor contraction and \( \ell_p \) subspace embedding as in equations (1) and (2) respectively.

**Proof.** Here the data is coming in streaming sense. Now LineFilter filters out the rows with small sensitivity scores and only the sampled rows (high sensitivity score) are feed to StreamingLW. We know that for a set \( P \) of size \( n \) takes \( O(ndp/2) \) time to return a coreset \( Q \) of size \( O(dp/2(\log d)\epsilon^{-5}) \) by [42]. But here the LineFilter ensures that StreamingLW only gets \( O(n^{1-2/p}d) \), hence the amortized update time is same as that of LineFilter, i.e. \( O(d^2) \). Now similar to the above proof [42] by the StreamingLW from section 7 of [34] we set \( M = O(dp/2(\log d)\epsilon^{-5}) \). The method returns \( Q_i \) as the \((1 + \delta_i) \) coreset for the partition \( P_i \) where \(|P_j|\) is either \( 2^iM \) or 0, here \( \rho_j = \epsilon/(c(j + 1)^2) \) such that \( 1 + \delta_i = \prod_{j=0}^i(1 + \rho_j) \leq 1 + \epsilon/2, \forall j \in [\log n] \). Thus we have \(|Q_i|\) is \( O(dp/2(\log d)(i + 1)^{10}\epsilon^{-5}) \). Hence the total working space is \( O((1 - 2/p)^{11}dp/2(\log^{11}n)(\log d)\epsilon^{-5}) \). So finally LineFilter + StreamingLW returns a coreset \( Q \) of \( O((1 - 2/p)^{10}dp/2(\log^{10}n)(\log d)\epsilon^{-5}) \) rows. \( \square \)

This is an improved streaming algorithm which gives the same guarantee as lemma 4.3 but using very less amortized update time. Hence asymptotically we get an improvement in the overall run time of the algorithm and yet get a coreset better than LineFilter. It is important to note that we could improve the run time of the streaming result because our LineFilter can be used in online setting, which returns a sub-linear size coreset (i.e. \( o(n) \)) and its update time is less than the amortized update time of StreamingLW. Note that LineFilter + StreamingLW is a streaming algorithm, whereas LineFilter or the next algorithm that we propose, works even in restricted streaming setting.
4.3 KernelFilter

Now we discuss our second streaming algorithm for the tensor contraction guarantee as equation (1). First we give a reduction from $p$-order to $q$-order where $q \leq 2$.

**Lemma 4.5.** For a vector $x \in \mathbb{R}^d$ it can be transformed to $\langle x \rangle$ and $\hat{x}$ such that for any two $d$-dimensional vectors $x$ and $y$ with their similar transformations we get,

$$
|x^T y|^p = |\langle x \rangle^T \hat{y}| \cdot |\hat{x}^T y| = \begin{cases} 
|x^T y|^2 & \text{if } p \text{ even} \\
|x^T y|^{2p/(p+1)} & \text{if } p \text{ odd}
\end{cases}
$$

**Proof.** The term $|x^T y|^p = |x^T y|^{p/2} |x^T y|^{p/2}$. We define $|x^T y|^{p/2} = |\langle x \rangle^T \hat{y}|$ and $|x^T y|^{p/2} = |\hat{x}^T y|$. For even valued $p$ we know $[p/2] = [p/2]$, so we write it as $|x^T y|^{p/2} = |\langle x \rangle^T \hat{y}|$. So we get $|x^T y|^{p/2} = (\langle x \otimes y \rangle)^{p/2} = |\hat{x}^T y|^2$. Here the vector $\hat{x} = \text{vec}(x \otimes y) \in \mathbb{R}^{p/2}$ and similarly $\hat{y}$ is also defined. Now for odd value of $p$ we have $\hat{x} = \text{vec}(x \otimes (y^{(p-1)/2})) \in \mathbb{R}^{(p-1)/2}$ and $\hat{y} = \text{vec}(x \otimes (y^{(p+1)/2})) \in \mathbb{R}^{(p+1)/2}$. Similarly $\hat{y}$ and $\hat{y}$ are defined for odd value of $p$. So we get $|x^T y|^{p/2} = |(x \otimes (y^{(p-1)/2}), y^{(p+1)/2})| \cdot |(x \otimes (y^{(p+1)/2}), y^{(p+1)/2})| = |\hat{x}^T \hat{y}| \cdot |x^T y|$. Further note that $|\hat{x}^T \hat{y}| = |\hat{x}^T \hat{y}|^{(p-1)/(p+1)}$, hence we get $|x^T y|^p = |x^T y|^{2p/(p+1)}$.

Now we discuss our second streaming algorithm for the tensor contraction guarantee as equation (1). First we give a streaming algorithm which is in the same spirit of LineFilter, which computes the sampling probability based on $\hat{a}_i$ and the counterparts of the previously seen rows. As the vector $\hat{a}_i$ only depend on $a_i$, this algorithm can also be used in online setting. Since we give a sampling based coreset, it retains the structure of the input data. So one need not convert $x$ into its corresponding higher dimensional vector, instead one can use the same $x$ on the sampled coreset to compute the desired operation. We call it KernelFilter and give it as algorithm.

**Algorithm 3 KernelFilter**

**Require:** Streaming rows $a_1, a_2, \ldots, a_n$, $r > 1$, $p \geq 2$

**Ensure:** Coreset $C$ satisfying eqn (1) and (2) w.h.p.

$M = \text{vec}(a_i \otimes \hat{a}_i)$, $M = 0^{d/(p+1) \times d/(p+1)}$

$L = 0$, $C = \emptyset$

$Q = $ orthonormal column basis of $M$

$Q = $ orthonormal column basis of $M$

$\varepsilon \leq n \text{ do}$

$\hat{a}_i = \text{vec}(a_i \otimes \hat{a}_i)$

$[\hat{e}_i, M, Q] = \text{OnlineScore}(\hat{a}_i, M, Q, p)$

$l_i = (\hat{e}_i)^2$

if $p/2 = 0$ then

$l_i = (l_i)^2$

else

$l_i = (l_i)^{2p/(p+1)}$

end if

$L = L + l_i; p_i = \min \{ r l_i / L, 1 \}$

Sample $a_i / \sqrt{p_i}$ in $C$ with probability $p_i$

end while

We summarize the guarantees of KernelFilter in the following two theorems.

**Theorem 4.6.** Given a matrix $A \in \mathbb{R}^{n \times d}$ whose rows are coming one at a time and an even value $p$, the KernelFilter selects a set $C$ of size $O(\frac{d^{p/2}}{\varepsilon^2} (1 + p(\log \|A\| - d - p/2 \min_i \log \|a_i\|)))$ with working space and update time $O(d^p)$. Suppose $Q$ is a fixed $k$-dimensional subspace, then with probability atleast $0.9$, with $\varepsilon > 0$, $\forall x \in Q$ we have $C$ satisfying both tensor contraction and $l_p$ subspace embedding as in equations (1) and (2) respectively.

**Theorem 4.7.** Given a matrix $A \in \mathbb{R}^{n \times d}$ whose rows are coming one at a time and an odd integer $p$, $p \geq 3$, the algorithm KernelFilter selects a set $C$ of size $O(\frac{n^{p/(p+1)} d^{p/2}}{\varepsilon^2} (1 + (p+1)(\log \|A\| - d - p/2 \min_i \log \|a_i\|))^{p/(p+1)})$
with working space and update time \(O(d^{p+1})\). Suppose \(Q\) is a fixed \(k\)-dimensional subspace, then with probability atleast 0.9, with \(\epsilon > 0, \forall x \in Q\) we have \(C\) satisfying both tensor contraction and \(\ell_p\) subspace embedding as in equations \(1\) and \(2\) respectively.

The working space and the computation time of the above algorithm are functions of \(d\) and \(p\). But compared to LineFilter, the KernelFilter returns an asymptotically smaller coreset. This is because the \(l_i\) gives a tighter upper bound of the online sensitivity score compared to what LineFilter gives. We discuss the proof of the above two theorems along with its supporting lemmas in section 5.

4.4 LineFilter+KernelFilter

Here we briefly sketch our final algorithm which achieves both the best sampling complexity as well as update time. We use our LineFilter along with KernelFilter to give a streaming algorithm that benefits both in space and time. For every incoming row LineFilter quickly decides its sampling probability and samples according to it and pass it to KernelFilter. We summarize the guarantee of LineFilter+KernelFilter by the following two lemmas.

**Theorem 4.8.** Consider \(A \in \mathbb{R}^{n \times d}\) whose rows are coming one at a time. The algorithm LineFilter+KernelFilter takes \(O(d^2)\) amortized update time and uses \(O(d^{p+1})\) working space to return \(C\) such that with at least 0.9 probability, \(\epsilon > 0, \forall x \in Q\), \(C\) satisfies both tensor contraction and \(\ell_p\) subspace embedding as equations \(1\) and \(2\) respectively. The size of \(C\) is as follows for integer \(p \geq 2\):

- \(p\) even: \(O(\frac{d^{p/2}k}{\epsilon^2}(1 + p(\log \|A\| - d^{-p/2}\min_i \log \|a_i\|)))\)
- \(p\) odd: \(O(\frac{(p-2)(p+1)d^{p/2}k}{\epsilon^2}(1 + (p + 1)(\log \|A\| - d^{-p/2}\min_i \log \|a_i\|)))^{p/(p+1)}\)

**Proof.** Here as LineFilter!'s

We note that the dependence on \(n\) is at most \(n^{0.1}\) and is always \(o(n^{1/(p+3)})\). Further unlike LineFilter the factor of \(n\) gradually reduces with increase in \(p\). Since the LineFilter only chooses a sublinear number of samples, which are further passed to KernelFilter, hence the amortized update time of LineFilter+KernelFilter is same as the update time of LineFilter.

4.5 \(p = 2\) case

In case of matrix i.e. \(p = 2\) the LineFilter and KernelFilter are just the same. This is because for every incoming row \(a_i\), the kernelization returns the same row as it is. Hence KernelFilter’s sampling process is exactly same as LineFilter. While we use the sensitivity framework, for \(p = 2\) our proofs are novel in the following sense:

1. When creating the sensitivity scores in the online setting, we also do not need to use a regularization term as \(6\), instead relying on a novel analysis when the matrix is rank deficient. Hence we get a relative error bound without making the number of samples depend on the smallest non zero singular value (which \(6\) need for online row sampling for matrices).
2. We do not need to use a martingale based argument, since the sampling probability of a row does not depend on the previous samples.

Our algorithm gives a coreset which preserves relative error approximation (i.e. subspace embedding). Note that lemma 3.5 of \(6\) can be used to achieve the same but it requires the knowledge of \(\sigma_{\min}(A)\)(smallest singular value of \(A\)). There we need \(\delta = \epsilon \sigma_{\min}(A)\) which gives sampling complexity as \(O(d(\log d)(\log \kappa(A))/\epsilon^2)\). Our algorithm gives relative error approximation even when \(\kappa(A) = 1\), which is not clear in \(6\).

**Corollary 4.1.** Given a matrix \(A \in \mathbb{R}^{n \times d}\) with rows coming one at a time, for \(p = 2\) the algorithm takes \(O(d^2)\) update time and samples \(O(d^2(1 + \log \|A\| - d^{-1}\min_i \log \|a_i\|)))\) rows and preserves the following with probability at least 0.9, \(\forall x \in \mathbb{R}^d (1 - \epsilon)\|Ax\|^2 \leq \|Cx\|^2 \leq (1 + \epsilon)\|Ax\|^2\).

Just by using Matrix Bernstein inequality \(39\) we can slightly improve the sampling complexity from \(O(d^2)\) to \(O(d\log d)\). For simplicity we modify the sampling probability to \(p_i = \min\{r_i, 1\}\) and get the following guarantee.

**Theorem 4.9.** The above modified algorithm samples \(O(d\log d)\) \((1 + \log \|A\| - d^{-1}\min_i \log \|a_i\|)))\) rows and preserves the following with probability at least 0.9, \(\forall x \in \mathbb{R}^d (1 - \epsilon)\|Ax\|^2 \leq \|Cx\|^2 \leq (1 + \epsilon)\|Ax\|^2\)
Proof. We prove this lemma in 2 parts. First we show that sampling $a_i$ with probability $p_i = \min\{r\hat{l}_i, 1\}$ where $\hat{l}_i = \min\{(1 + ||a_i^T(A^T)^T||) a_i, 1\}$ preserves $||C^T C|| \leq (1 + \varepsilon) ||A^T A||$, for $\forall x \in \mathbb{R}^d$. Next we give the bound on expected sample size.

We define, $u_i = (A^T A)^{-1/2} a_i$ and we define a random matrix $X_i$ corresponding to each streaming row as,

$$X_i = \begin{cases} (1/p_i - 1)u_i u_i^T & \text{if } a_i \text{ is sampled in } \hat{A} \\ -u_i u_i^T & \text{else} \end{cases}$$

Now we have,

$$\hat{l}_i = a_i^T (A_{i-1}^T A_{i-1} + a_i a_i^T)^T a_i \geq a_i^T (A^T A)^T a_i = u_i^T u_i$$

For $p_i \geq \min\{ru_i^T u_i, 1\}$, if $p_i = 1$, then $||X_i|| = 0$, else $p_i = ru_i^T u_i < 1$. If the row is not sampled then we get $u_i^T u_i = ||X_i|| \leq 1/r$. Next when the row is sampled we have $||X_i|| - E[||X_i||] \leq ||u_i u_i^T||/p_i \leq 1/r$. Next we bound $E[X_i^2]$.

$$E[X_i^2] = p_i(1/p_i - 1)^2 ||u_i u_i^T||^2 + (1 - p_i)(u_i u_i^T)^2 \leq ||u_i u_i^T||^2/p_i \leq (u_i u_i^T)^2/r$$

Let $X = \sum_{i=1}^n X_i$, then variance of $||X||$

$$\text{var}(||X||) = \frac{\sum_{i=1}^n \text{var}(X_i)}{\sum_{i=1}^n E[|X_i|^2]} \leq \frac{n}{\sum_{j=1}^n u_j u_j^T/r} \leq 1/r$$

Next by applying matrix Bernstein theorem [2, 2] with appropriate $r$ we get,

$$\text{Pr}(||X|| \geq \varepsilon) \leq d \exp \left( \frac{-\varepsilon^2 / 2}{1/r + \varepsilon/(3r)} \right) \leq 0.1$$

This implies that the modified algorithm preserves spectral approximation with high probability, i.e. $||C^T C|| \leq (1 + \varepsilon) ||A^T A||$.

Now once we have the above result using lemma 4 of [19] we conclude that the expected number of samples are $O(\sum_{i=1}^n \hat{l}_i (\log d)/\varepsilon^2)$. Now from lemma [5, 3] we know that for $p = 2$, $\sum_{i=1}^n \hat{l}_i$ is $O(d(1 + \log ||A|| - \min_i \log ||a_i||))$. Finally to get $\text{Pr}(||Cx|| - ||Ax|| \geq \varepsilon) \leq 0.1$ the algorithm samples $O(\frac{d\log d}{\varepsilon^2}(1 + \log ||A|| - \min_i \log ||a_i||))$ rows. \qed

5 Proofs

In this section we prove our main theorems. While doing so whenever needed we also state and prove the supporting lemmas for them.

5.1 LineFilter

Here we give sketch of the proof for theorem [2, 2]. For ease of notation, the rows are considered numbered according to their order of arrival. The supporting lemmas are for the online setting which also work for the streaming case. We show that it is possible to generalize the notion of sensitivity for the online setting as well as give an upper bound to it. We define the online sensitivity of any $i^{th}$ row as: $\sup_{x \in \mathcal{Q}} \frac{||a_i^T x||^p}{\sum_{j=1}^n ||a_j^T x||^p} = \sup_{y \in \mathcal{Q}'} \frac{||u_i^T y||^p}{\sum_{j=1}^n ||u_j^T y||^p}$, where $\mathcal{Q}' = \{y \mid y = \Sigma V^T x, x \in \mathcal{Q}\}$, $\text{svd}(A) = U \Sigma V^T$ and $\mathcal{Q}$ is the query space. Notice that the denominator now contains a sum only over the rows that have arrived. We note that while online sampling results often need the use of martingales as an analysis tool, e.g. [6], in our setting, the sampling probability of each row does depend on the previous rows, but not on whether they were sampled or not. Hence the sampling decision of each row is independent. So, application of Bernstein’s inequality [2, 1] suffices.

We first show that the sampling of the incoming rows using $p_i$’s defined in LineFilter, are upper bounds to the online sensitivity scores.

Lemma 5.1. Consider $A \in \mathbb{R}^{n \times d}$, whose rows are provided in an online manner to LineFilter. Let $\hat{l}_i = \min\{p_i^{-1} - 1, (a_i^T M^T a_i)^{p/2}, 1\}$, and $M$ is a $d \times d$ matrix maintained by the algorithm. Then $\forall i \in [n]$, $\hat{l}_i$ satisfies

$$\hat{l}_i \geq \sup_{x} \frac{||a_i^T x||^p}{\sum_{j=1}^n ||a_j^T x||^p}$$

Proof. We define the online sensitivity scores $\hat{s}_i$ for each point $i$ as follows,
\[ \tilde{s}_i = \sup_x \frac{|a_i^T x|^p}{\sum_{j=1}^k |a_j^T x|^p} = \sup_y \frac{|a_i^T y|^p}{\sum_{j=1}^k |a_j^T y|^p} \]

Here \( y = \Sigma V^T x \) where \([U, \Sigma, V] = \text{svd}(A)\) and \( \tilde{u}_i^T \) is the \( i^{th} \) row of \( U \). Now at this \( i^{th} \) step we define \([U_i, \Sigma_i, V_i] = \text{svd}(A_i)\). So we rewrite the above optimization function as follows with \( y = \Sigma_i V_i^T x \) and \( \tilde{u}_i^T \) is the \( i^{th} \) row of \( U_i, \)

\[ \tilde{s}_i = \sup_x \frac{|a_i^T x|^p}{\sum_{j=1}^k |a_j^T x|^p} = \sup_y \frac{|a_i^T y|^p}{\|U_i y\|^p} = \sup_y \frac{|a_i^T y|^p}{\|U_i y + \sum_{j=1}^k |a_j^T y|^p} \]

Let there be an \( x^* \) which maximizes \( \tilde{s}_i \). Corresponding to it we have \( y^* = \Sigma V_i^T x^* \). For a fixed \( x \), let \( f(x) = \frac{|a_i^T x|^p}{\sum_{j=1}^k |a_j^T x|^p} \) and correspondingly we have \( g(y) = \frac{|a_i^T y|^p}{\|U_i y\|^p} \). By assumption we have \( f(x^*) \geq f(x), \forall x \). We prove by contradiction that its corresponding \( g(y^*) \geq g(y), \forall y \), where \( y = \Sigma V_i^T x \). Let \( \exists y' \) such that \( g(y') \geq g(y^*) \). Then we get \( x' = V_i \Sigma^{-1} y' \) for which \( f(x') \geq f(x^*) \). This contradicts our assumption, unless \( x' = x^* \). To maximize the score, \( x \) is chosen from the row space of \( A_i \). Now without loss of generality we assume that \( \|y\| = 1 \) and we know that if \( x \) is in the row space of \( A_i \), then \( y \) is in the row space of \( U_i \). Hence we get \( \|U_i y\| = \|y\| = 1 \).

We break denominator into sum of numerator and the rest, i.e. \( \|U_i y\|^p = |\tilde{u}_i^T y|^p + \sum_{j=1}^{i-1} |\tilde{u}_j^T y|^p \). Consider the denominator term which is \( \sum_{j=1}^{i-1} |\tilde{u}_j^T y|^p \geq \left( \sum_{j=1}^{i-1} |\tilde{u}_j^T y|^2 \right)^{\frac{p}{2}} \cdot f(n) \). From this we estimate \( f(n) \) as follows,

\[
\begin{align*}
\sum_{j=1}^{i-1} |\tilde{u}_j^T y|^2 &= \left( \sum_{j=1}^{i-1} |\tilde{u}_j^T y|^2 \right) \\
&\leq \left( \sum_{j=1}^{i-1} |\tilde{u}_j^T y|^2 \right)^{p/2} \left( \sum_{j=1}^{i-1} (p/(p-2)) \right)^{1-2/p} \\
&\leq \left( \sum_{j=1}^{i-1} |\tilde{u}_j^T y|^p \right)^{2/p} \cdot (i)^{-1/2p} \tag{5}
\end{align*}
\]

Here equation (4) is by holder’s inequality, where \( 2/p + 1 - 2/p = 1 \). So we rewrite the above term as \( \left( \sum_{j=1}^{i-1} |\tilde{u}_j^T y|^p \right)^{2/p} (i)^{-1/2p} \geq \sum_{j=1}^{i-1} |\tilde{u}_j^T y|^2 = 1 - |\tilde{u}_i^T y|^2 \). Now substituting this in equation (5) we get,

\[
\begin{align*}
\left( \sum_{j=1}^{i-1} |\tilde{u}_j^T y|^p \right)^{2/p} &\geq \left( \frac{1}{i} \right)^{1-2/p} (1 - |\tilde{u}_i^T y|^2) \\
\sum_{j=1}^{i-1} |\tilde{u}_j^T y|^p &\geq \left( \frac{1}{i} \right)^{p/2-1} (1 - |\tilde{u}_i^T y|^2)^{p/2}
\end{align*}
\]

So we get \( \tilde{s}_i \leq \sup_y \frac{|\tilde{u}_i^T y|^p}{\|U_i y + \sum_{j=1}^{i-1} |\tilde{u}_j^T y|^p \cdot (i)^{-1/2p}} \). Note that this function increases with value of \( |\tilde{u}_i^T y| \), which is maximum when \( y = \frac{\tilde{u}_i}{\|\tilde{u}_i\|} \).

\[
\hat{s}_i \leq \frac{\|\tilde{u}_i\|^p}{\|\tilde{u}_i\|^p + (1/i)^{p/2-1} (1 - \|\tilde{u}_i\|^2)^{p/2}}
\]

We know that a function \( \frac{a}{a+b} \leq \min\{1, a/b\} \), so we get \( \hat{\tilde{s}}_i = \min\{1, i^{p/2-1} \|\tilde{u}_i\|^p \} \). Note that \( \hat{\tilde{s}}_i = i^{p/2-1} \|\tilde{u}_i\|^p \) when \( \|\tilde{u}_i\|^p < (1/i)^{p/2-1} \).

Although the \( \hat{s}_i \)'s are computed very quickly but the algorithm gives a loose upper bound due to the additional factor of \( i^{p/2-1} \) in the definition of \( \hat{s}_i \). As \( i \) increases it also increases. Now with these upper bounds we get the following.

**Lemma 5.2.** Let \( r \) provided to \texttt{LineFilter} be \( O(ker^{-2} \sum_{j=1}^n \|\tilde{l}_j\|) \). Let \texttt{LineFilter} returns a coreset \( C \). Then with probability at least 0.9, \( \forall x \in Q \), \( C \) satisfies the tensor contraction as in equation (1) and \( \ell_p \) subspace embedding as in equation (4).
Proof. For simplicity we prove this lemma at the last timestamp $n$. But it can also be proved for any timestamp $t_i$ which is why the LineFilter can also be used in restricted streaming (online) setting. Now for a fixed $x \in \mathbb{R}^d$ and its corresponding $y$, we define a random variables as follows, i.e. based on the choice of LineFilter.

$$w_i = \begin{cases} \frac{1}{p_i} (u_i^T y)^p & \text{with probability } p_i \\ 0 & \text{with probability } (1 - p_i) \end{cases}$$

Where $u_i^T$ is the $i^{th}$ row of $U$ for $[U, \Sigma, V] = \text{svd}(A)$ and $y = \Sigma V^T x$. Here we get $E[w_i] = (u_i^T y)^p$. In our online algorithm we have defined $p_i = \min \{ r \hat{l}_i / \sum_{j=1}^i \hat{l}_j, 1 \}$. When $p_i$ is not 1, we have $p_i = r \hat{l}_i / \sum_{j=1}^i \hat{l}_j \geq r (u_i^T y)^p / \sum_{j=1}^i (u_j^T y)^p$ for $i$.

Now to apply Bernstein 2.1 we bound the term $|w_i - E[w_i]| \leq b$. Consider the two possible cases,

**Case 1:** When $w_i$ is non zero, then $|w_i - E[w_i]| \leq |w_i| \leq \frac{(\sum_{j=1}^n \hat{l}_j) \sum_{j=1}^n |u_j^T y|^p}{r (u_i^T y)^p}$. Note for $p_i = 1$, $|w_i - E[w_i]| = 0$.

**Case 2** When $w_i$ is 0 then $p_i < 1$. So we have $1 > \sum_{j=1}^i \hat{l}_j \geq \frac{(\sum_{j=1}^n \hat{l}_j) \sum_{j=1}^n |u_j^T y|^p}{r (u_i^T y)^p}$. So $|w_i - E[w_i]| = |E[w_i]| = \frac{|u_i^T y|^p}{\sum_{j=1}^n (u_j^T y)^p}$.

So $b = \frac{(\sum_{j=1}^n \hat{l}_j) \sum_{j=1}^n |u_j^T y|^p}{r}$. Next we bound the variance of $W = \sum_{i=1}^n w_i$. Let $\sigma^2 = \text{var} \{ \sum_{i=1}^n w_i \} = \sum_{i=1}^n \sigma_i^2$, where $\sigma_i^2 = \text{var}(w_i)$ and we get

$$\sigma^2 = \sum_{i=1}^n E[w_i^2] - (E[w_i])^2 \leq \frac{\sum_{i=1}^n |u_i^T y|^2 p_i}{\sum_{i=1}^n p_i} \leq \frac{\sum_{i=1}^n |u_i^T y|^{2p} (\sum_{k=1}^n \hat{l}_k (\sum_{j=1}^n |u_j^T y|^p)^2)}{(\sum_{j=1}^n \hat{l}_j (2 + \epsilon/3))} \leq \sum_{k=1}^n \hat{l}_k$$

Note that $||Uy||_p^p = \sum_{j=1}^n |u_j^T y|^p$, now setting $t = \epsilon \sum_{j=1}^n |u_j^T y|^p$, let

$$P = \Pr \left( |W - \sum_{j=1}^n (u_j^T y)^p| \geq \epsilon \sum_{j=1}^n |u_j^T y|^p \right)$$

Then by applying Bernstein 2.1 we get,

$$P \leq \exp \left( \frac{(\epsilon \sum_{j=1}^n |u_j^T y|^p)^2}{2 \sigma^2 + b t / 3} \right) \leq \exp \left( \frac{-r \epsilon^2 (||Uy||_p^p)^2}{(||Uy||_p^p)^2 \sum_{j=1}^n \hat{l}_j (2 + \epsilon/3)} \right) = \exp \left( \frac{-r \epsilon^2}{(2 + \epsilon/3) \sum_{j=1}^n \hat{l}_j} \right)$$

Now to ensure that the above probability at most 0.1, $\forall x \in Q$ we use $\epsilon$-net argument where we take a union bound over $(2/\epsilon)^k$, $x$ from the net. Note that for our purpose 1/2-net also suffices. Hence with the union bound over all $x$ in 1/2-net we need to set $r = O \left( k \sum_{j=1}^n \hat{l}_j \right) / \epsilon^2$.

Now to ensure the guarantee for $\ell_p$ subspace embedding one can define the random variable $w_i$ as follows and follow the above proof.

$$w_i = \begin{cases} \frac{1}{p_i} (u_i^T y)^p & \text{with probability } p_i \\ 0 & \text{with probability } (1 - p_i) \end{cases}$$

Again by setting the $r = O \left( k \sum_{j=1}^n \hat{l}_j \right)$ one can get

$$P = \Pr \left( |W - ||A x||_p^p| \geq \epsilon ||A x||_p^p \right) \leq 0.1$$

Finally get the guarantee of $\ell_p$ subspace embedding as in equation 3 using the same sampling complexity.

Now in order to bound the number of samples, we need a bound on the quantity $\sum_{j=1}^n \hat{l}_j$ which we demonstrate in the following lemma.

**Lemma 5.3.** The $\hat{l}_i$ in LineFilter algorithm which satisfies lemma 5.1 and lemma 5.2 has $\sum_{i=1}^n \hat{l}_i = O(n^{1-2/p}(d + d \log ||A|| - \min_i \log ||a_i||))$. 

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Proof. Recall that $A_i$ denotes the $i \times d$ matrix of the first $i$ vectors. LinFilter maintains the covariance matrix $M$. At the $(i - 1)^{th}$ step we have $M = A_{i-1}^T A_{i-1}$. This is then used to define the score $\hat{e}_i$ for the next step $i$, as $\hat{e}_i = \text{a}_i^T (A_i^T A_i)\text{a}_i$ and $\text{a}_i^T$ is the $i^{th}$ row of $A_i$. The scores $\hat{e}_i$ are also called online leverage scores. We first give a bound on $\sum_{i=1}^n \hat{e}_i$. A similar bound is given in the online matrix row sampling by [6], albeit for a regularized version of the scores $\hat{e}_i$. As the rows are coming, the rank of $M$ increases from 1 to at most $d$. We say that the algorithm is in phase-$k$ if the rank of $M$ equals $k$. For each phase $k \in [1, d-1]$, let $i_k$ denote the index where row $\text{a}_{i_k}$ caused a phase-change in $M$ i.e. rank of $(A_{i_k}^T A_{i_k})$ is $k$, while rank of $(A_{i_k}^T A_{i_k})$ is $k$. For each $i_k$, the leverage score $\hat{e}_{i_k} = 1$, since row $\text{a}_{i_k}$ does not lie in the space of rows $\{a_1, \ldots, a_{i_k-1}\}$. There are at most $d$ such indices.

We now bound the $\sum_{i \in [i_k, i_k+1]} \hat{e}_i$. Suppose the thin-SVD $(A_{i_k}^T A_{i_k}) = \mathbf{V} \mathbf{\Sigma}_{i_k} \mathbf{V}^T$, all entries in $\mathbf{\Sigma}_{i_k}$ being positive. Furthermore, for any $i$ in this phase, i.e. for $i \in [i_k, i_k+1]$, $\mathbf{V}$ forms the basis of the row space of $A_i$. Define $\mathbf{X}_i = \mathbf{V}^T (A_i^T A_i) \mathbf{V}$ and $\text{a}_i = \mathbf{V}b_i$. Notice that each $\mathbf{X}_i \in \mathbb{R}^{k \times k}$, and $\mathbf{X}_{i_k} = \mathbf{\Sigma}_{i_k}$. Also, $\mathbf{X}_{i_k}$ is positive definite and hence full rank for each $i \in [i_k, i_k+1]$. We also have $\mathbf{X}_i = \mathbf{X}_{i-1} + \text{a}_i b_i^T$.

So we have, $\hat{e}_i = \text{a}_i^T (A_i^T A_i)\text{a}_i = \text{b}_i^T \mathbf{V}^T (\mathbf{X}_{i-1} + \text{a}_i \text{b}_i \mathbf{V}^T) \mathbf{V} b_i = \text{b}_i^T (\mathbf{X}_{i-1} + \text{a}_i \text{b}_i) \mathbf{V} b_i = \text{b}_i^T (\mathbf{X}_{i-1} + \text{a}_i \text{b}_i) \mathbf{V} b_i$ where the last equality uses the invertibility of the matrix. Now using matrix determinant lemma [35] on $\det(\mathbf{X}_{i-1} + \text{a}_i \text{b}_i)$ we get,

$$\det(\mathbf{X}_{i-1} + \text{a}_i \text{b}_i) = \det(\mathbf{X}_{i-1})(1 + \text{b}_i^T \mathbf{X}_{i-1}^{-1} \text{b}_i)$$

$$\exp(\hat{e}_i / 2) \leq \frac{\det(\mathbf{X}_{i-1} + \text{a}_i \text{b}_i)}{\det(\mathbf{X}_{i-1})}$$

Inequality (i) follows as $\mathbf{X}_{i-1}^{-1} - (\mathbf{X}_{i-1} + \text{a}_i \text{b}_i)^{-1} \succeq 0$ (i.e. p.s.d.). Inequality (ii) follows from the fact that $1 + x \geq \exp(x/2)$ for $x \leq 1$ and by definition $\hat{e}_i \leq 1$. Now since $\hat{e}_{i_k} = 1$, we analyze

$$\prod_{i \in [i_k+1, i_{k+1}]} \exp(\hat{e}_i / 2) \leq \prod_{i \in [i_k+1, i_{k+1}]} \frac{\det(\mathbf{X}_i)}{\det(\mathbf{X}_{i_k+1})} \leq \frac{\det(\mathbf{X}_{i_k+1})}{\det(\mathbf{X}_{i_k+1})}$$

Taking the product over all the phases $\exp\left(\sum_{i \in [1, d-1]} \hat{e}_i / 2\right)$ gets,

$$\exp\left(\sum_{i \in [1, d-1]} \hat{e}_i / 2\right) = \exp((d - 1)/2) \prod_{k \in [1, d-1]} \exp(\hat{e}_i / 2)$$

$$= \exp((d - 1)/2) \prod_{k \in [1, d-1]} \frac{\det(\mathbf{X}_{i_k+1})}{\det(\mathbf{X}_{i_k+1})}$$

$$= \exp((d - 1)/2) \frac{\det(\mathbf{X}_{i_k+1})}{\det(\mathbf{X}_{i_k+1})} \prod_{k \in [2, d-1]} \frac{\det(\mathbf{X}_{i_k+1})}{\det(\mathbf{X}_{i_k+1})}$$

Because we know that $(A_{i_{k+1}}^T A_{i_{k+1}}) \succeq (A_{i_{k+1}}^T A_{i_{k+1}})$ so we get, $\det(A_{i_{k+1}}^T A_{i_{k+1}}) \succeq \det(A_{i_{k+1}}^T A_{i_{k+1}})$. We get $\exp((d - 1)/2)$, as there are $d - 1$ many $i$ such that $\hat{e}_i = 1$. Hence,

$$\exp\left(\sum_{i \in [1, d-1]} \hat{e}_i / 2\right) \leq \exp((d - 1)/2) \frac{\det(A_{i_d}^T A_{i_d})}{\det(A_{i_{d+1}}^T A_{i_{d+1}})} \frac{\det(A_{i_{d+1}}^T A_{i_{d+1}})}{\det(A_{i_{d+1}}^T A_{i_{d+1}})}$$

Furthermore, we know $\hat{e}_{i_d} = 1$ so for $i \in [i_d, n]$, the matrix $M$ is full rank. We follow the same argument as above, and obtain

$$\exp\left(\sum_{i \in [i_d, n]} \hat{e}_i / 2\right) \leq \exp((d - 1)/2) \frac{\det(A_{i_d}^T A_{i_d})}{\det(A_{i_{d+1}}^T A_{i_{d+1}})} \frac{\det(A_{i_{d+1}}^T A_{i_{d+1}})}{\det(A_{i_{d+1}}^T A_{i_{d+1}})}$$

Let $a_1$ be the first incoming row. Now multiplying the above two expressions and taking logarithm of both sides, and accounting for the indices $i_k$ for $k \in [2, d]$,

$$\sum_{i \leq n} \hat{e}_i \leq d/2 + 2d \log \|A\| - 2 \log \|a_1\| \leq d/2 + 2d \log \|A\| - \min_i 2 \log \|a_i\|$$
Now, we give a bound on $\sum_{i=1}^{n} \tilde{t}_i$ where $\tilde{t}_i = \min\{1, t_i^{p/2-1} \tilde{e}_i^{p/2}\} \leq \min\{1, n_i^{p/2-1} \tilde{e}_i^{p/2}\}$. We consider two cases. When $\tilde{e}_i^{p/2} \geq n_i^{-1/p}$ then $\tilde{t}_i = 1$, this implies that $\tilde{e}_i \geq n_i^{2/p-1}$. But we know $\sum_{i=1}^{n} \tilde{e}_i \leq O(d + d \log \|A\| - \min_i \log \|a_i\|)$ and hence there are at-most $O(n_i^{-2/p}(d + d \log \|A\| - \min_i \log \|a_i\|))$ indices such $\tilde{t}_i = 1$. Now for the case where $\tilde{e}_i^{p/2} < n_i^{-1/p}$, we get $\tilde{e}_i^{p/2} = \sum_{i=1}^{n} n_i^{p/2-1} \tilde{e}_i^{p/2} = \sum_{i=1}^{n} n_i^{p/2-1} \tilde{e}_i \leq \sum_{i=1}^{n} n_i^{1-2/p} \tilde{e}_i$ is $O(n_i^{-1/p}(d + d \log \|A\| - \min_i \log \|a_i\|))$.

With lemmas 5.1, 5.2 and 5.3, we prove that the guarantee in theorem 4.2 is achieved by LineFilter. The bound on space is evident from the fact that we are maintaining the matrix $M$ in the algorithm which uses $O(d^2)$ space and returns a coreset of size $O(\frac{\alpha_{\circ}^{1-2/p}}{\epsilon} (1 + \log \|A\| - d^{-1/2} \min_i \log \|a_i\|))$.

### 5.2 KernelFilter

In this section we give sketch of the proof of theorem 4.6 and theorem 4.7. We use sensitivity based framework to decide sampling probability of each incoming row. The novelty in this algorithm is by reducing the $p$ order operation to a $q$ order operation where $q \leq 2$. Now we give bound on sensitivity score of every incoming row.

**Lemma 5.4.** Consider $A \in \mathbb{R}^{n \times d}$ rows are coming in online manner to KernelFilter. The term $\tilde{t}_i$ defined in the algorithm upper bounds the online sensitivity score, i.e. $\forall i \in [n], \tilde{t}_i \geq \sup_x \frac{|a_i^T x|^p}{\|A_i x\|^p} = \tilde{s}_i$.

**Proof.** For even value $p$ using kernel trick we change the $p$ order operation to a $2$ order operation. Now $\tilde{t}_i$ are nothing but online leverage scores for the matrix $\hat{A}$ where each row $\hat{A}_i = \text{vec}(a_i^T \odot \gamma^2)$. Now since $\|A_i x\|_p^p = \|\hat{A}_i x\|_2^p$ hence $\forall i \in [n], \tilde{t}_i \geq \sup_x \frac{|a_i^T x|^p}{\|A_i x\|^p}$ defines the $\tilde{t}_i$ in KernelFilter for even $p$ value upper bounds the desired sensitivity score $\tilde{s}_i$.

The lemma is non trivial for odd $p$ value. Using lemma 4.5 we define the online sensitivity scores $\tilde{s}_i$ for each point $i$ as follows,

$$\tilde{s}_i = \sup_{\{x : \|x\| = 1\}} \frac{|a_i^T x|^p}{\|A_i x\|^p}$$

$$\tilde{s}_i = \sup_{\{x : \|x\| = 1\}} \frac{|\hat{A}_i x_{2p/(p+1)}|^p}{\|\hat{A}_i x\|_2^p} = \sup_{\{x : \|x\| = 1\}} \frac{|\hat{A}_i x_{2p/(p+1)}|^p}{\|\hat{A}_i x\|_2^p}$$

$$\tilde{s}_i = \sup_{\{y : \|y\| = 1\}} \frac{|\hat{U}_i y_{2p/(p+1)}|^p}{\|\hat{U}_i y\|_2^p} \leq \sup_{\{y : \|y\| = 1\}} \frac{|\hat{U}_i y_{2p/(p+1)}|^p}{\|\hat{U}_i y\|_2^p}$$

$$\tilde{s}_i = \sup_{\{y : \|y\| = 1\}} \frac{|\hat{U}_i y_{2p/(p+1)}|^p}{\|\hat{U}_i y\|_2^p} = \frac{|\hat{U}_i y_{2p/(p+1)}|^p}{\|\hat{U}_i y\|_2^p}$$

$$\tilde{s}_i \leq \frac{|\hat{U}_i y_{2p/(p+1)}|^p}{\|\hat{U}_i y\|_2^p}$$

The equality (i) is by lemma 4.5. Now let $[\hat{U}_i, \hat{\Sigma}_i, \hat{V}_i] = \text{svd}(\hat{A}_i)$, then $\hat{A}_i^T x = \hat{U}_i y$ where $\hat{U}_i$ is the $i^{th}$ row of $\hat{U}_i$ and $y = \hat{\Sigma}_i \hat{V}_i x$. We substitute these to get the equality (ii). Note that without loss of generality we can consider $\|y\| = 1$. The next inequality is because $\|\hat{U}_i y\|_2^p \geq \|\hat{U}_i y\|_2^p$ and finally we get $\tilde{s}_i \leq \tilde{t}_i$ as defined in KernelFilter for odd $p$ value.

Unlike LineFilter, the KernelFilter does not use any additional factor of $i$. Hence it gives tighter upper bounds to the sensitivity scores compared to lemma 5.1. It will be evident when we sum these upper bounds. Next we show with these $\tilde{t}_i$, we can claim the following.

**Lemma 5.5.** In the KernelFilter let $r$ is set as $O(k \sum_{i=1}^{n} \tilde{t}_i / \epsilon^2)$ then for some fixed $k$-dimensional subspace $Q$ the set $C$ with probability 0.9 $\forall x \in Q$ satisfies tensor contraction as in equation 1 and $\ell_p$ subspace embedding as in equation 2.

**Proof.** For simplicity we prove this lemma at the last timestamp $n$. But it can also be proved for any timestamp $t_i$, which is why the KernelFilter can also be used in restricted streaming (online) setting. Also as we showed the LineFilter for the tensor contraction operation. Here we show this for $\ell_p$ subspace embedding. Now for some fixed $x \in \mathbb{R}^d$ our algorithm takes the following random variable for every row $i$.

$$w_i = \begin{cases} (1/p_i - 1) |a_i^T x|^p & \text{w.p. } p_i \\ - |a_i^T x|^p & \text{w.p. } (1 - p_i) \end{cases}$$
Now to show the concentration of the expected term we will apply Bernstein’s inequality \(\text{2.1}\) on \(W = \sum_{i=1}^n w_i\). For this first we bound \(|w_i - E[w_i]| = |w_i|\) as \(E[w_i] = 0\) and then we give a bound on \(\text{var}(W)\). Here \(p_i = \min\{1, r l_i / \sum_{j=1}^i l_j\}\).

Now for the \(i\)th timestep if \(p_i = 1\) then \(|w_i| = 0\), else if \(p_i < 1\) and \(\text{KernelFilter}\) samples the row then \(|w_i| \leq |\alpha_i T x|^p / p_1 = |\alpha_i T x|^p \sum_{j=1}^i l_j / (r l_i) \leq \|Ax\|_p^p |\alpha_i T x|^p \sum_{j=1}^i l_j / (r |\alpha_i T x|^p) \leq \|Ax\|_p^p \sum_{j=1}^i l_j / r\). Next when \(\text{KernelFilter}\) does not sample means that \(p_i < 1\), then \(1 > r l_i / \sum_{j=1}^i l_j \geq r |\alpha_i T x|^p / (\|Ax\|_p^p \sum_{j=1}^i l_j)\). Finally we get \(|\alpha_i T x|^p \leq \|Ax\|_p^p \sum_{j=1}^i l_j / r\). So for each \(i\) we get \(|w_i| \leq \|Ax\|_p^p \sum_{j=1}^i l_j / r\).

Next we bound \(\sigma^2 = \text{var}(W) = \sum_{i=1}^n \text{var}(w_i) = \sum_{i=1}^n E[w_i^2]\) as follows,

\[
\sigma^2 = \sum_{i=1}^n E[w_i^2] = \sum_{i=1}^n |\alpha_i T x|^2 p / p_i = \sum_{i=1}^n |\alpha_i T x|^2 p \sum_{j=1}^i \tilde{l}_j / (r \tilde{l}_i) \leq \|Ax\|_p^p \sum_{i=1}^n |\alpha_i T x|^2 p \sum_{j=1}^i \tilde{l}_j / (r |\alpha_i T x|^p) \leq \|Ax\|_p^p \sum_{j=1}^i \tilde{l}_j / r
\]

Now we can apply Bernstein’s inequality on the sum of random variables to bound the event \(P: \text{Pr}(\|W\| \geq \epsilon \|Ax\|_p^p)\). Here we have \(b = \|Ax\|_p^p / r, \sigma^2 = \|Ax\|_p^p / r\) and we set \(t = \epsilon \|Ax\|_p^p\), then we get

\[
P \leq \exp \left( - \frac{\epsilon^2 \|Ax\|_p^p}{2 \|Ax\|_p^p \sum_{j=1}^n \tilde{l}_j / r + \epsilon \|Ax\|_p^p \sum_{j=1}^n \tilde{l}_j / 3r} \right) = \exp \left( - \frac{\epsilon^2 \|Ax\|_p^p}{2 + \epsilon / 3) \|Ax\|_p^p \sum_{j=1}^n \tilde{l}_j} \right)
\]

Now to ensure that the above probability at most 0.1, \(\forall x \in \mathbb{Q}\) we use \(\epsilon\)-net argument where we take a union bound over \((2/\epsilon)^k, x\) from the net. Note that for our purpose \(1/2\)-net also suffices. Hence with the union bound over all \(x\) in \(1/2\)-net we need to set \(r = O\left(\frac{k \sum_{j=1}^n \tilde{l}_j}{x^2}\right)\).

Now to ensure the guarantee for tensor contraction as equation \(\text{1}\) one can define the random variable \(w_i\) as follows and follow the above proof.

\[
w_i = \begin{cases} 
(1/p_i - 1) |\alpha_i T x|^p & \text{w.p. } p_i \\
-|\alpha_i T x|^p & \text{w.p. } (1 - p_i)
\end{cases}
\]

Again by setting the \(r = O\left(\frac{k \sum_{j=1}^n \tilde{l}_j}{x^2}\right)\) one can get

\[
P = \text{Pr}(\|W - \sum_{j=1}^n |\alpha_i T x|^p\| \geq \epsilon \sum_{j=1}^n |\alpha_i T x|^p) \leq 0.1
\]

Finally get the guarantee of tensor contraction as in equation \(\text{2}\) using the same sampling complexity. \(\square\)

Now in order to bound the number of samples, we need a bound on the quantity \(\sum_{j=1}^n \tilde{l}_j\) which we demonstrate in the following lemma.

**Lemma 5.6.** The \(\tilde{l}_j\)’s used in \(\text{KernelFilter}\) which satisfy lemma \(\text{5.4}\) and \(\text{5.5}\) has \(\sum_{i=1}^n \tilde{l}_i\) as

- \(p\) even: \(O(d^{p/2}(1 + p(\log \|A\| - d^{-[p/2]} \min_i \log \|\alpha_i\|)))\)
- \(p\) odd: \(O(n^{1/(p+1)} \cdot d^{p/2}(1 + p(1 + 1(\log \|A\| - d^{-[p/2]} \min_i \log \|\alpha_i\|))^{p/(p+1)}))\)

**Proof.** Let \(\hat{c}_i = \|\hat{u}_i\|\). Now for even \(p\) case \(\sum_{i=1}^n \hat{c}_i = \sum_{i=1}^n c_i^2\). From lemma \(\text{5.3}\) we get \(\sum_{i=1}^n c_i^2\) is \(O(d^{p/2}(1 + \log \|\hat{A}\| - d^{-[p/2]} \min_i \log \|\hat{u}_i\|))\). Now with \([\mathbf{u}, \mathbf{\Sigma}, \mathbf{V}] = \text{svd}(A)\) we have \(\hat{u}^T = \text{vec}(\alpha_i T \phi \phi / \hat{A}) = \text{vec}(\mathbf{u}_j^T \Sigma \mathbf{V}^T)^{p/2}\). So we get \(\|\hat{A}\| \leq \sigma_1^{1/2}\). Hence \(\sum_{i=1}^n \hat{c}_i = O(d^{p/2}(1 + p(\log \|A\| - d^{-[p/2]} \min_i \log \|\alpha_i\|))^{p/(p+1)}\).

Now for the odd \(p\) case \(\sum_{i=1}^n \hat{c}_i = \sum_{i=1}^n c_i^{2p/(p+1)}\). From lemma \(\text{5.3}\) we get \(\sum_{i=1}^n c_i^{2p/(p+1)}\) is \(O(d^{p/2}(1 + \log \|\hat{A}\| - d^{-[p/2]} \min_i \log \|\hat{u}_i\|))\). Now with \([\mathbf{u}, \mathbf{\Sigma}, \mathbf{V}] = \text{svd}(A)\) we have \(\hat{u}^T = \text{vec}(\alpha_i T \phi \phi / \hat{A}) = \text{vec}(\mathbf{u}_j^T \Sigma \mathbf{V}^T)^{p/2}\). So we get \(\|\hat{A}\| \leq \sigma_1^{(p+1)/2}\). Hence \(\sum_{i=1}^n c_i^{2p/(p+1)}\) is \(O(d^{p/2}(1 + p(\log \|A\| - d^{-[p/2]} \min_i \log \|\alpha_i\|))^{p/(p+1)}\). Now let \(\hat{c}\) is a vector with each index \(\hat{c}_i\) is defined as above. Then for odd \(p\) case \(\sum_{i=1}^n \hat{c}_i = \|\hat{c}\|^{2p/(p+1)} \leq n^{1/(p+1)} \|\hat{c}\|^{2p/(p+1)}\) which is \(O(n^{1/(p+1)} \cdot d^{p/2}(1 + p(1 + 1(\log \|A\| - d^{-[p/2]} \min_i \log \|\alpha_i\|))^{p/(p+1)}))\). \(\square\)
The proof of the above lemma is similar to that of lemma 5.3. It implies that the lemma 5.4 gives tighter sensitivity bounds compared to lemma 5.1 as the factor of p reduces as p increases. Now with lemmas 5.4, 5.5 and 5.6 we prove that the guarantee in theorem 5.6 and theorem 5.7 is achieved by LineFilter+KernelFilter. The working space bound of $O(d^p)$ is evident from the fact that the algorithm is maintaining a $d^{p/2} \times d^{p/2}$ matrix for even p and for odd p it maintains $d^{p/2} \times d^{p/2}$ matrix working space of $O(d^{p+1})$ is needed.

6 Applications

Here we show how our methods can also be used for learning latent variable model using tensor factorization. We use a corollary which summarizes the guarantees we get on latent variables by learning them using tensor factorization on our coreset. We discuss it in the appendix A.6.

**Streaming Single Topic Model:** Here we empirically show how sampling using LineFilter+KernelFilter can preserve tensor contraction as in equation (1). This can be used in single topic modeling where documents are coming in streaming manner. We compare our method with 2 other sampling schemes, namely – uniform and online leverage scores which we call LineFilter(2).

We use a subset of 20Newsgroups data (pre-processed). We took a subset of 10k documents and considered the 100 most frequent words. We normalized each document vector, such that its $\ell_1$ norm is 1 and created a matrix $A \in \mathbb{R}^{10K \times 100}$. We feed its row one at a time to LineFilter+KernelFilter with $p = 3$, which returns a coreset $C$. We run tensor based single topic modeling [3] on $A$ and $C$, to return 12 top topic distributions from both. We take the best matching between empirical and estimated topics based on $\ell_1$ distance and compute the average $\ell_1$ difference between them. We run this entire method 5 times and report the median of the $\ell_1$ differences. Here the coreset sizes are over expectation.

| SAMPLE | UNIFORM | LINEFILTER(2) | LINEFILTER + KERNELFILTER |
|--------|---------|---------------|--------------------------|
| 50     | 0.5725  | 0.6903        | 0.5299                   |
| 100    | 0.5093  | 0.6385        | 0.4379                   |
| 200    | 0.4687  | 0.5548        | 0.3231                   |
| 500    | 0.3777  | 0.3992        | 0.2173                   |
| 1000   | 0.2548  | 0.2318        | 0.1292                   |

From the table it can be seen, that our algorithm LineFilter+KernelFilter performs better or at par with both uniform and LineFilter(2) thus supporting our theoretical claims.

6.1 Latent Variable Modeling

Under the assumption that the data is generated by some generative model such as Gaussian Mixture model, Topic model, Hidden Markov model etc, one can represent the data in terms of higher order (say 3) moments as $T_3$ to realize the latent variables [3]. Next the tensor is reduced to an orthogonally decomposable tensor by multiplying a matrix called whitening matrix ($M \in \mathbb{R}^{d \times k}$), such that $W^T M_2 W = I_k$. Here $k$ is the number of number of latent variables we are interested and $M_2 \in \mathbb{R}^{d \times d}$ is the 2nd order moment. Now the reduced tensor $\tilde{T}_r = \tilde{T}_3(W, W, W)$ is a $k \times k \times k$ orthogonally decomposable tensor. Now by running robust tensor power iteration (RTPI) on $\tilde{T}_r$ we get the eigenvalue/eigenvector pair on which upon applying inverse whitening we get the estimated latent factors and its corresponding weights [3].

Note that we give guarantee over the $d \times d \times d$ tensor where as the main theorem 5.3 [3] has conditioned over the smaller orthogonally reducible tensor $\tilde{T}_r \in \mathbb{R}^{k \times k \times k}$. Now rephrasing this main theorem 5.1 of [3] we get that the $\|M_3 - \tilde{T}_3\| \leq \varepsilon \|W\|^{-3}$ where $M_3$ is the true 3rd order tensor with no noise and $\tilde{T}_3$ is the empirical tensor that we get from the dataset. Now we state the guarantees that one gets by applying the RTPI on our sampled data.

**Corollary 6.1.** For a dataset $A \in \mathbb{R}^{n \times d}$ with rows coming in streaming fashion and the algorithm LineFilter+KernelFilter guarantees [1] such that for all unit vector $x \in Q$, it ensures $\sum_{i \leq n} | x^T x |^d \leq \varepsilon \|W\|^{-3}$. Then applying the RTPI on the sampled coreset $C$ returns $k$ eigenpairs $\{\lambda_i, v_i\}$ of the reduced (orthogonally decomposable) tensor, ensures that for all $i \in [k],$

$$\|v_{\pi(i)} - v_i\| \leq 8\varepsilon/\lambda_i \quad |\lambda_{\pi(i)} - \lambda_i| \leq 5\varepsilon$$
Here precisely we have \( Q \) as the column space of the \( W^T \), where \( W \) is the whitening matrix as defined above.

### 6.1.1 Tensor Contraction

Now we show empirically that how coreset from LineFilter can preserve tensor contraction. We compare our method with 2 other sampling schemes, namely – uniform and LineFilter(2). Here LineFilter(2) has sampling probabilities \( p_i = \min\{r\|u_i\|^2, 1\} \), We call it leverage as it is very much related to it. In this section

**Dataset:** We generated a dataset with 200k rows in \( \mathbb{R}^{30} \). Each coordinate of row vector was uniformly generated entry between 0 and 1. Further each vector was normalized to have \( \ell_2 \) norm as 1. Hence we had a matrix of size 200x \times 30 but we ensured that it had rank 12. Furthermore 99.99% of the rows in the matrix spanned only an 8-dimensional subspace in \( \mathbb{R}^{30} \) and its orthogonal 4 dimensional subspace was spanned by the remaining 0.01% of the rows. We simulated these rows to come in online fashion and applied the 3 sampling strategies. We generated 3-mode tensors \( T \) using the sampled rows and tensor \( T \) using the entire dataset.

**Uniform:** Here we sample rows uniformly at random from the dataset. It means that every row has a chance of getting sampled with a probability of 1/\( n \). Intuitively it is highly unlikely to pick rows from the subspace with few rows. Hence the required property might not be preserved for \( x \) coming from that particular row space.

**LineFilter(2):** Here we sample rows based online leverage scores \( c_i = a^T_i(A_i A_i^T)^{-1} a_i \). We define a sampling probability for an incoming row \( i \) as \( p_i = c_i/\sum_{j=1}^{k} c_j \). Rows with high leverage scores have higher chance of getting sampled. Though leverage score sampling preserved rank of the the data, but it is not known to preserve higher order moments of the data.

**LineFilter:** Sample rows of \( A \) as per LineFilter.

The following tables compare the error \( |T(x,x,x) - \tilde{T}(x,x,x)\|/(|T(x,x,x)|) \) values between three sampling schemes mentioned above. Here \( T(x, x, x) = \sum_{i=1}^{k} (a_i^T x)^3 \). In table (3), \( Q \) is set of right singular vectors of \( A \) corresponding to the 5 smallest singular values. The table reports \( \sum_{x \in [Q]} T(x,x,x) - \sum_{x \in [Q]} \tilde{T}(x,x,x) \|/(\sum_{x \in [Q]} T(x,x,x)) \). The table (4) reports for \( x \) as the right singular vector of the smallest singular value of \( A \). \( \tilde{T} \) is the tensor we get from different sampling techniques. For each sampling technique and each sample size we ran 5 random experiments and report the mean of the experiments. Here as a result of the sampling techniques, the sample sizes mentioned are in expectation and not exact. We set the sampling parameter in each method to get a similar expected sample size. Here we choose this \( x \) because this direction captures the worst direction, as in the direction which has highest variance in the sampled data. Intuitively this is because that the direction corresponding to the smallest singular value will have very few representative rows and it will have very less chance to get sampled.

| Table 3: Error with \( x \in [Q] \) |
|---|---|---|---|
| SAMPLE | UNIFORM | LINEFILTER(2) | LINEFILTER |
| 200 | 1.0195 | **0.3641** | 0.4814 |
| 250 | 1.0126 | 0.7021 | **0.4555** |
| 300 | 1.0650 | **0.3697** | 0.4307 |
| 350 | 1.0095 | 0.5355 | **0.3948** |

| Table 4: Error with \( x \) as right singular vector of the smallest singular value |
|---|---|---|---|
| SAMPLE | UNIFORM | LINEFILTER(2) | LINEFILTER |
| 200 | 1.0000 | **0.5909** | 1.0437 |
| 250 | 1.0000 | 0.6781 | **0.6737** |
| 300 | 1.0000 | 0.9135 | **0.6598** |
| 350 | 1.0000 | 0.7431 | **0.4575** |

It can be seen from the tables, that our algorithm beats uniform sampling and performs better or at par with leverage score sampling thus supporting our theoretical claims. There are no theoretical claims for leverage score sampling for tensor factorization.
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