THE POINCARÉ SERIES FOR THE ALGEBRAS OF JOINT INVARIANTS AND COVARIANTS OF $n$ LINEAR FORMS.

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Abstract. Explicit formulas for computation of the Poincaré series for the algebras of joint invariants and covariants of $n$ linear forms are found. Also, for these algebras we calculate the degrees and asymptotic behaviours of the degrees.

Keywords: classical invariant theory; invariants; Poincaré series; combinatorics

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1. Let $V_1$ be the complex vector space of linear binary forms endowed with the natural action of the special linear group $SL_2$. Consider the corresponding action of the group $SL_2$ on the algebras of polynomial functions $\mathbb{C}[nV_1]$ and $\mathbb{C}[nV_1 \oplus \mathbb{C}^2]$, where $nV_1 := \bigoplus_{i=1}^n V_1$. Denote by $\mathcal{I}_n = \mathbb{C}[nV_1]^{SL_2}$ and by $\mathcal{C}_n = \mathbb{C}[nV_1 \oplus \mathbb{C}^2]^{SL_2}$ the corresponding algebras of invariant polynomial functions. In the language of classical invariant theory the algebras $\mathcal{I}_n$ and $\mathcal{C}_n$ are called the algebra of joint invariants and the algebra of joint covariants for the $n$ linear binary forms respectively. A generating set of the algebra $\mathcal{I}_n$ was conjectured by Nowicki [1]. It had been proved later by different authors, for instance see [2], [3]. The algebras $\mathcal{C}_n$, $\mathcal{I}_n$ are affine graded algebras under the usual degree:

$$\mathcal{C}_n = (\mathcal{C}_n)_0 + (\mathcal{C}_n)_1 + \cdots + (\mathcal{C}_n)_j + \cdots, \quad \mathcal{I}_n = (\mathcal{I}_n)_0 + (\mathcal{I}_n)_1 + \cdots + (\mathcal{I}_n)_j + \cdots,$$

where each of subspaces $(\mathcal{C}_n)_j$ and $(\mathcal{I}_n)_j$ is finite-dimensional. The formal power series

$$P(\mathcal{C}_n, z) = \sum_{j=0}^{\infty} \dim(\mathcal{C}_n)_j z^j, P(\mathcal{I}_n, z) = \sum_{j=0}^{\infty} \dim(\mathcal{I}_n)_j z^j,$$

are called the Poincaré series of the algebras $\mathcal{C}_n$ and $\mathcal{I}_n$. In the paper [4] the following expressions for the Poincaré series of those algebras was derived:

$$P(\mathcal{I}_n, z) = \sum_{k=1}^{n} \frac{(-1)^{n-k}(n)_{n-k}}{(k-1)!(n-k)!} \frac{d^{k-1}}{dz^{k-1}} \left( \frac{z}{1-z^2} \right)^{2n-k-1},$$

$$P(\mathcal{C}_n, z) = \sum_{k=1}^{n} \frac{(-1)^{n-k}(n)_{n-k}}{(k-1)!(n-k)!} \frac{d^{k-1}}{dz^{k-1}} \left( \frac{1+z}{1-z^2} \right)^{2n-k-1}.$$
where \((n)_m := n(n+1) \cdots (n+m-1), (n)_0 := 1\) denotes the shifted factorial.

In the present paper those formulas are reduced to the following forms:

\[
\mathcal{P}(\mathcal{I}_n, z) = \frac{N_{n-2}(z^2)}{(1 - z^2)^{2n-3}} \quad \text{and} \quad \mathcal{P}(\mathcal{C}_n, z) = \frac{W_{n-1}(z^2) + nzN_{n-1}(z^2)}{(1 - z^2)^{2n-1}},
\]

where

\[
N_n(z) = \sum_{k=1}^{n} \frac{1}{k} \binom{n-1}{k-1} \binom{n}{k-1} z^{k-1}
\]

denotes the Narayana polynomials and the Narayana polynomials of type B respectively.

Also, the degrees of algebras \(\mathcal{I}_n, \mathcal{C}_n\) and asymptotic behaviors of the degrees are calculated using the explicit expressions for the Poincaré series.

2. Let us prove several auxiliary combinatorial identities.

**Lemma 1.** Let \(m, k, s\) be non-negative integers. The generalized Le Jen Shoo identity holds:

\[
\sum_{i=0}^{\min\{k,m\}} \binom{m}{i} \binom{m+2s}{i+s} \binom{k-i+2m+2s}{2m+2s} = \binom{m+k+s}{m+s} \binom{m+k+2s}{m+s}.
\]

**Proof.** Taking into account

\[
\binom{m}{i} = 0, \text{ for } i > m, \quad \text{and} \quad \binom{k-i+2m+2s}{k-i} = 0, \text{ for } i > k,
\]

we have

\[
\sum_{i=0}^{\infty} \binom{m}{i} \binom{m+2s}{i+s} \binom{k-i+2m+2s}{2m+2s} = \sum_{i=0}^{k} \binom{m}{i} \binom{m+2s}{i+s} \binom{k-i+2m+2s}{2m+2s} + \sum_{i=k+1}^{\infty} \binom{m}{i} \binom{m+2s}{i+s} \binom{k-i+2m+2s}{2m+2s} =
\]

\[
= \sum_{i=0}^{k} \binom{m}{i} \binom{m+2s}{i+s} \binom{k-i+2m+2s}{2m+2s} + \sum_{i=k+1}^{\infty} \binom{m}{i} \binom{m+2s}{i+s} \cdot 0 =
\]

\[
= \sum_{i=0}^{k} \binom{m}{i} \binom{m+2s}{i+s} \binom{k-i+2m+2s}{2m+2s} = \sum_{i=0}^{m} \binom{m}{i} \binom{m+2s}{i+s} \binom{k-i+2m+2s}{2m+2s} + \sum_{i=m+1}^{\infty} \binom{m}{i} \binom{m+2s}{i+s} \binom{k-i+2m+2s}{2m+2s} =
\]

\[
= \sum_{i=0}^{m} \binom{m}{i} \binom{m+2s}{i+s} \binom{k-i+2m+2s}{2m+2s} + \sum_{i=m+1}^{\infty} 0 \cdots \binom{m+2s}{i+s} \binom{k-i+2m+2s}{2m+2s} =
\]

\[
= \sum_{i=0}^{m} \binom{m+2s}{i+s} \binom{k-i+2m+2s}{2m+2s} = \sum_{i=0}^{\min\{k,m\}} \binom{m}{i} \binom{m+2s}{i+s} \binom{k-i+2m+2s}{2m+2s}.
\]
Now the statement follows immediately from following identity, see [5]:

\[
\begin{align*}
(a + c + d + e) \left( \begin{array}{cc} b + c + d + e & a + c \\ a + e & c + e \end{array} \right) &= \sum_i \left( \begin{array}{cc} a + d & b + c \\ i + d & i + c \end{array} \right) \left( \begin{array}{cc} a + b + c + d + e - i & a + b + c + d \\ a + b + c + d & a + b + c + d \end{array} \right),
\end{align*}
\]

if we set \(a = m, b = m + s, c = s, d = 0\) and \(e = k\). □

**Lemma 2.** Let \(k, n > 1\) be non-negative integers; then

\[
\sum_{i=0}^{\min\{k,n-1\}} (-1)^i \binom{n+i-1}{i} \binom{n+k-2}{k-i} \binom{n+2k-i-1}{2k} = \binom{n+k-1}{k+1} \binom{n-2+k}{k}.
\]

**Proof.** We have

\[
\begin{align*}
\sum_{i=0}^{\min\{k,n-1\}} (-1)^i \binom{n+i-1}{i} \binom{n+k-2}{k-i} \binom{n+2k-i-1}{2k} &= \sum_{i=0}^{n-1} (-1)^i \binom{n+i-1}{i} \binom{n+k-2}{k-i} \binom{n+2k-i-1}{2k} = \\
&= \sum_{i=0}^{k} (-1)^i \binom{n+i-1}{i} \binom{n+k-2}{k-i} \binom{n+2k-i-1}{2k}.
\end{align*}
\]

Note that

\[
\binom{n+i-1}{i} \binom{n+k-2}{k-i} = \frac{n+i-1}{n-1} \binom{n+k-2}{n-i-2} \binom{n+i-2}{n-2} = \frac{n+i-1}{n-1} \binom{n+k-2}{n-2} \binom{k}{i},
\]

and

\[
\frac{n-1}{k+1} \binom{n+k-1}{k} = \binom{n+k-1}{k+1}.
\]

So we prove that

\[
\sum_{i=0}^{k} (-1)^i(n-1+i) \binom{k}{i} \binom{n+2k-i-1}{2k} = \binom{n+k-1}{k+1}.
\]

Let us put \(S_1 = \sum_{i=0}^{k} (-1)^i(n-1+i) \binom{k}{i} \binom{n+2k-i-1}{2k}\). We have:

\[
S_1 = (n-1) \sum_{i=0}^{k} (-1)^i \binom{k}{i} \binom{n+2k-i-1}{2k} + \sum_{i=0}^{k} (-1)^i \binom{k}{i} \binom{n+2k-i-1}{2k}.
\]

Using the following identity, see [6], p.8

\[
\sum_i (-1)^i \binom{n-i}{p-i} \binom{p}{i} = \binom{n-p}{m},
\]
we get:

\[ S_1 = (n - 1) \binom{n + k - 1}{k} - k \sum_{i=1}^{k} (-1)^{i-1} \binom{k - 1}{i-1} \binom{n + 2k - (i - 1) - 2}{2k} = \]

\[ = (n - 1) \binom{n + k - 1}{k} - k \sum_{i=0}^{k-1} (-1)^i \binom{k - 1}{i} \binom{n + 2k - 2 - i}{n - 2 - i} = \]

\[ = (n - 1) \binom{n + k - 1}{k} - k \binom{n + k - 1}{n - 2} = \]

\[ = (n - 1) \left( \frac{(n + k - 1)!}{k!(n - 1)!} - \frac{(n + k - 1)!k}{(k + 1)!(n - 1)!} \right) = \frac{n - 1}{k + 1} \binom{n + k - 1}{n - 2} = \binom{n + k - 1}{k + 1}. \]

This concludes the proof. \( \square \)

Substituting \( m = n - 3 \) and \( s = 1 \) into Lemma 1, we obtain:

\[ \sum_i \binom{n - 3}{i} \binom{n - 1}{i + 1} \binom{2n + k - i - 4}{k - i} = \binom{n + k - 1}{n - 2} \binom{n - 2 + k}{n - 2}. \]

Multiplying both sides by \( \frac{1}{n - 1}, (n > 2) \) and using Lemma 2, we get:

\[ \sum_{i=0}^{\min\{k, n-1\}} (-1)^i \binom{n + i - 1}{i} \binom{n + k - 2}{k - i} \binom{n + 2k - i - 1}{2k} = \]

\[ = \sum_{i=0}^{\min\{k, n-3\}} \binom{n - 3}{i} \binom{n - 2}{i} \binom{2n + k - i - 4}{k - i} \frac{1}{i + 1}. \]

3. We use the derived above combinatorial identities to simplify expressions for the Poincaré series \( P(I_n, z) \) and \( P(C_n, z) \) from [1].

**Theorem 1.** The following formulas hold:

\((i)\) \[ P(I_n, z) = \sum_{k=1}^{n-2} \frac{1}{k} \binom{n - 3}{k - 1} \binom{n - 2}{k - 1} z^{2k-2} \]

\[ \frac{1}{(1 - z^2)^{2n-3}} \]

\( (ii) \) \[ P(C_n, z) = \sum_{k=0}^{n-1} \binom{n - 1}{k} z^{2k} + \sum_{k=0}^{n-2} \binom{n - 2}{k} \binom{n}{k + 1} z^{2k+1} \]

\[ \frac{1}{(1 - z^2)^{2n-1}}. \]

**Proof.** (i) Let us expand function

\[ \sum_{k=1}^{n} \frac{(-1)^{n-k}(n)_{n-k}}{(k - 1)! (n-k)!} d^{k-1} \frac{dz}{(1 - z^2)^{2n-k-1}} \left( \frac{z}{1 - z^2} \right)^{2n-k-1}, \]
into the Taylor series about \( z \). We have

\[
P(I_n, z) = \sum_{k=1}^{n} \frac{(-1)^{n-k}(n)_{n-k}}{(k-1)! (n-k)!} \frac{d^{k-1}}{dz^{k-1}} \left( z^{2n-k-1} \sum_{i=0}^{\infty} \frac{(2n-k+i-2)}{i} z^{2i} \right) =
\]

\[
= \sum_{k=1}^{n} \frac{(-1)^{n-k}(n)_{n-k}}{(k-1)! (n-k)!} \frac{d^{k-1}}{dz^{k-1}} \left( \sum_{i=0}^{\infty} \frac{(2n-k+i-2)}{i} z^{2i+2n-k-1} \right) =
\]

\[
= \sum_{k=1}^{n} \frac{(-1)^{n-k}(n)_{n-k}}{(k-1)! (n-k)!} \sum_{i=0}^{\infty} \frac{(2n-k+i-2)}{i} \frac{(2i+2n-k-1)!}{(2i+2n-2k)!} z^{2i+2n-2k}.
\]

Substituting \( j = n - k \), we have:

\[
P(I_n, z) = \sum_{j=0}^{n-1} \frac{(-1)^{j}(n)_{j}}{(n-j-1)! j!} \sum_{i=0}^{\infty} \frac{(n+j+i-2)}{i} \frac{(2i+n+j-1)!}{(2i+2j)!} z^{2i+2j} =
\]

\[
= \sum_{j=0}^{n-1} \frac{(-1)^{j}(n+j-1)!}{(n-j-1)! j!(n-1)!} \sum_{i=0}^{\infty} \frac{(n+i+j-2)}{i} \frac{(n+2i+2j-j-1)!}{(2i+2j)!} z^{2i+2j} =
\]

\[
= \sum_{j=0}^{n-1} (-1)^{j} \frac{(n+j-1)!}{j!(n-1)!} \sum_{i=0}^{\infty} \frac{(n+i+j-2)}{i} \frac{(2i+n+j-1)!}{(2i+2j)!} z^{2i+2j} =
\]

\[
= \sum_{k=0}^{\infty} \sum_{i=0}^{\min\{k,n-1\}} (-1)^{i} \binom{n+i-1}{i} \binom{n+k-2}{k-i} \binom{n+2k-i-1}{2k} z^{2k}.
\]

Using (1), we get:

\[
P(I_n, z) = \sum_{k=0}^{\infty} \sum_{i=0}^{\min\{k,n-3\}} \binom{n-3}{i} \binom{n-2}{i} \binom{2n+k-i-4}{k-i} \frac{1}{i+1} z^{2k} =
\]

\[
= \sum_{k=0}^{n-3} \binom{n-3}{k} \binom{n-2}{k} \frac{z^{2k}}{k+1} + \sum_{i=0}^{\infty} \binom{(2n-3)+i-1}{i} z^{2i}.
\]

Note that

\[
\frac{1}{(1-z^2)^{2n-3}} = \sum_{i=0}^{\infty} \binom{2n-4+i}{i} z^{2i}.
\]

This completes the proof.

(ii) Denote by

\[
A_n(z) = \sum_{k=1}^{n} \frac{(-1)^{n-k}(n)_{n-k}}{(k-1)! (n-k)!} \frac{d^{k-1}}{dz^{k-1}} \left( \frac{z^{2n-k-1}}{(1-z^2)^{2n-k}} \right),
\]
and let $B_n(z) = \mathcal{P}(C_n, z) - A_n(z)$. Reasoning as in the proof of (i), we have

$$A_n(z) = \sum_{k=0}^{\infty} \min\{k, n-1\} \sum_{i=0}^{n-k} \frac{(-1)^i}{i!} \binom{n+i-1}{i} \binom{n+k-1}{k-i} \binom{n+2k-i-1}{2k} z^{2k} = \sum_{k=0}^{\infty} \binom{n+k-1}{k} z^{2k} \sum_{i=0}^{\min\{k, n-1\}} (-1)^i \binom{k}{i} \binom{n+2k-i-1}{2k} \sum_{k=0}^{\infty} \binom{n+k-1}{k} z^{2k} - \sum_{k=0}^{\infty} \binom{n+k-1}{k} z^{2k}.$$

By using the Le Jen Shoo’s identity, we get:

$$A_n(z) = \sum_{k=0}^{\infty} \frac{(n-1)^2}{z^{2k}} \sum_{i=0}^{\infty} \binom{n-i}{i} \binom{2n+k-i-2}{k-i} z^{2k} = \sum_{k=0}^{\infty} \frac{(n-1)^2 z^{2k}}{(1-z^2)^{2n-1}}.$$

We see that

$$B_n(z) = \sum_{k=0}^{\infty} \frac{(-1)^{n-k}}{(k-1)! (n-k)!} d^{k-1} \left( \frac{z^{2n-k}}{(1-z^2)^{2n-k}} \right) = \sum_{k=0}^{\infty} \frac{(-1)^i}{i!} \binom{n+i-1}{i} \binom{n+k-1}{k-i} \binom{n+2k-i-1}{2k+1} z^{2k+1} = \sum_{k=0}^{\infty} \binom{n+k-1}{k} z^{2k+1} \sum_{i=0}^{\min\{k, n-1\}} (-1)^i \binom{k}{i} \binom{n+2k-i-1}{n-1-i} = \sum_{k=0}^{\infty} \binom{n+k-1}{n-1} \binom{n}{n-1} z^{2k+1}.$$

Using lema 1 ($m = n-2, s = 1$), we have:

$$B_n(z) = \sum_{k=0}^{\infty} \frac{(-1)^{n-k}}{(k-1)! (n-k)!} d^{k-1} \left( \frac{z^{2n-k}}{(1-z^2)^{2n-k}} \right) = \sum_{k=0}^{\infty} \binom{n+k-1}{k} z^{2k+1} \sum_{i=0}^{\min\{k, n-1\}} (-1)^i \binom{k}{i} \binom{n+2k-i-1}{n-1-i} = \sum_{k=0}^{\infty} \binom{n+k-1}{n-1} \binom{n}{n-1} z^{2k+1}.$$

Thus

$$\mathcal{P}(C_n, z) = A_n(z) + B_n(z) = \frac{\sum_{k=0}^{n-1} \binom{n-1}{k}^2 z^{2k} + \sum_{k=0}^{n-2} \binom{n-2}{k} \binom{n}{k+1} z^{2k+1}}{(1-z^2)^{2n-1}}.$$

Let us rewrite the expressions in terms of the Narayana polynomials $N_n(z)$ and the Narayana polynomials of type B $W_n(z)$ where

$$N_n(z) = \sum_{k=1}^{n} \frac{1}{k} \binom{n-1}{k-1} \binom{n}{k-1} z^{k-1} \text{ and } W_n(z) = \sum_{k=0}^{n} \binom{n}{k}^2 z^k.$$
We get

\[ P(I_n, z) = \frac{N_{n-2}(z^2)}{(1-z^2)^{2n-3}} \quad \text{and} \quad P(C_n, z) = \frac{W_{n-1}(z^2) + nzN_{n-1}(z^2)}{(1-z^2)^{2n-1}}. \]

4. The transcendence degrees over \( \mathbb{C} \) for the algebras \( I_n, C_n \) is equal to order of the pole for \( P(I_n, z), P(C_n, z) \) respectively, see [8]. Note that for all \( n N_n(1) \neq 0 \) and \( W_n(1) \neq 0 \). These arguments proves

**Theorem 2.** The following formulas hold

\begin{align*}
(i) & \quad \text{tr deg}_C I_n = 2n - 3, \\
(ii) & \quad \text{tr deg}_C C_n = 2n - 1.
\end{align*}

Let \( R = R_0 \oplus R_1 \oplus \cdots \) be a finitely generated graded complex algebra, \( R_0 = \mathbb{C} \). Denote by

\[ P(R, z) = \sum_{j=0}^{\infty} \dim R_j z^j, \]

its Poincaré series. Letting \( r \) be the transcendence degree of the quotient field of \( R \) over \( \mathbb{C} \), the number

\[ \deg(R) := \lim_{z \to 1} (1-z)^r P(R, z), \]

is called the degree of the algebra \( R \). The first two terms of the Laurent series expansion of \( P(R, z) \) at the point \( z = 1 \) have the following form

\[ P(R, z) = \frac{\deg(R)}{(1-z)^r} + \frac{\psi(R)}{(1-z)^{r-1}} + \cdots \]

The numbers \( \deg(R), \psi(R) \) are important characteristics of the algebra \( R \). For instance, if \( R \) is an algebra of invariants of a finite group \( G \) then \( \deg(R)^{-1} \) is order of the group \( G \) and \( 2 \frac{\psi(R)}{\deg(R)} \) is the number of pseudo-reflections in \( G \), see [7].

We know explicit forms for the Poincaré series for the algebras of joint invariants and covariants of \( n \) linear forms. Thus we can prove the following statement.

**Theorem 3.** The degrees of the algebras of joint invariants and covariants of \( n \) linear forms are equal to

\begin{align*}
(i) & \quad \deg(P(I_n, z)) = \frac{N_n(z^2)}{2^{2n-3}} = \frac{(2n-4)}{(n-2)2^{2n-3}}, \\
(ii) & \quad \deg(P(C_n, z)) = \frac{W_{n-1}(z^2)}{2^{2n-2}}.
\end{align*}
Proof. (i) Using Theorem 1 and Theorem 2, we have:

\[
\deg(\mathcal{I}_n) = \lim_{z \to 1} (1-z)^{2n-3} \mathcal{P}(\mathcal{I}_n, z) = \lim_{z \to 1} (1-z)^{2n-3} \frac{\sum_{k=1}^{n-2} \frac{1}{k} \binom{n-3}{k-1} \binom{n-2}{k-1} z^{2k-2}}{(1-z^2)^{2n-3}} = \frac{N_{n-2}(1)}{2^{2n-3}}
\]

Note that the number \(N_{n-2}(1)\) equal to the Catalan numbers, see [10]. It now follows that

\[
\deg(\mathcal{I}_n) = \frac{\binom{2n-4}{n-2}}{(n-1)2^{2n-3}}
\]

(ii) We have

\[
\deg(\mathcal{C}_n) = \lim_{z \to 1} (1-z)^{2n-1} \mathcal{P}(\mathcal{C}_n, z) = \lim_{z \to 1} (1-z)^{2n-1} \frac{\sum_{k=0}^{n-1} \binom{n-1}{k} z^{2k} + \sum_{k=0}^{n-2} \binom{n-2}{k} \binom{n}{k+1} z^{2k+1}}{(1-z^2)^{2n-1}} = \frac{\binom{2n-2}{n-1} + nN_{n-1}(1)}{2^{2n-1}} = \frac{\binom{2n-2}{n-1}}{2^{2n-2}}
\]

Note that asymptotically, the Catalan numbers grow as

\[
C_n = \frac{1}{n+1} \binom{2n}{n} \sim \frac{4^n}{n^{3/2} \sqrt{\pi}}.
\]

It is easy to calculate asymptotic behaviours of the degrees of the algebras \(\mathcal{I}_n\) i \(\mathcal{C}_n\):

**Corollary 1.** Asymptotic behaviours of the degrees of the algebras of joint invariants and covariants of \(n\) linear forms as \(n \to \infty\) are follows

\[
\deg(\mathcal{I}_n) \sim \frac{1}{2 \sqrt{\pi n^3}} \quad \text{and} \quad \deg(\mathcal{C}_n) \sim \frac{1}{\sqrt{\pi n}}.
\]

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