Stable clustering and the resolution of dissipationless cosmological N-body simulations

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\textbf{ABSTRACT}

The determination of the resolution of cosmological N-body simulations, i.e., the range of scales in which quantities measured in them represent accurately the continuum limit, is an important open question. We address it here using scale-free models, for which self-similarity provides a powerful tool to control resolution. Such models also provide a robust testing ground for the so-called stable clustering approximation, which gives simple predictions for them. Studying large N-body simulations of such models with different force smoothing, we find that these two issues are in fact very closely related: our conclusion is that the accuracy of two point statistics in the non-linear regime starts to degrade strongly around the scale at which their behaviour deviates from that predicted by the stable clustering hypothesis. Physically the association of the two scales is in fact simple to understand: stable clustering fails to be a good approximation when there are strong interactions of structures (in particular merging) and it is precisely such non-linear processes which are sensitive to fluctuations at the smaller scales affected by discretisation. Resolution may be further degraded if the short distance gravitational smoothing scale is larger than the scale to which stable clustering can propagate. We examine in detail the very different conclusions of studies by Smith et al. (2003) and Widrow et al. (2009) and find that the strong deviations from stable clustering reported by these works are the results of over-optimistic assumptions about scales resolved accurately by the measured power spectra, and the reliance on Fourier space analysis. We emphasise the much poorer resolution obtained with the power spectrum compared to the two point correlation function.

\textbf{Key words:} Cosmological structure formation, gravitational clustering, N-body simulation

\section{INTRODUCTION}

Numerical simulations using the N-body method are the primary instrument used to probe the non-linear regime of structure formation in cosmology and provide the basis for all theoretical predictions for the distribution of dark matter at the corresponding physical scales. Over the last few decades, such simulations have gained in refinement and complexity and have allowed the exploration of an ever larger range of scales (for a review see e.g. Bertschinger (1998); Springel et al. (2005); Dehnen & Reed (2011)). Nevertheless, the understanding of their precision and their convergence toward the continuum limit remains, at very least, incomplete, in particular for smaller scales (see e.g. Splinter et al. (1998); Knebe et al. (2000); Romeo et al. (2008); Joyce, Marcos & Baertschiger (2009); Power et al. (2016)). In this context “scale-free” cosmological models, in which both the expansion law and the power spectrum characterizing the initial fluctuations are simple power laws, have the advantage of relative simplicity, and they have for this reason been studied quite extensively in the literature (see e.g. Efstathiou et al. (1988); Colombi, Bouchet & Hernquist (1996); Bertschinger (1998); Jain & Bertschinger (1998); Smith et al. (2003); Knollmann, Power & Knebe (2008); Widrow et al. (2009); Orban (2013); Diemer & Kravtsov (2013)). More specifically these models provide a testing ground for the numerical method through the predicted “self-similarity” of the clustering; the temporal evolution of the clustering statistics must be equivalent to a rescaling of the distances. This fol-
allows from the fact that there is only one characteristic length scale (derived from the amplitude of the fluctuations) and one characteristic time scale in the model. Further the exact rescaling function can be determined from the evolution in the linear regime of arbitrarily small fluctuations. However, discreteness and numerical effects typically introduce additional characteristic scales (e.g., force regularization at small scales, particle density, finite box size, etc.) which lead directly to a breaking of such self-similarity. Thus the self similarity of clustering provides a potentially powerful tool to separate the scales affected by such non-physical effects from the physical results representing the continuum limit. The focus of this study is to exploit self-similar models to better understand the resolution at small scales of N-body simulations. In particular we will use simulations with a very small force smoothing which allow us to follow carefully the propagation of self-similarity to small scales in the course of a simulation.

A further motivation for studying scale-free models is that they provide a very simple analytical prediction for non-linear clustering which is the stable clustering hypothesis \citep{Davis1977, Peebles1980}. This corresponds to the assumption that once a structure is strongly non-linear it no longer evolves in physical coordinates, i.e., structures behave as though they were isolated virialized structures. While this hypothesis can be made in any cosmological model, for scale-free initial cosmologies it implies, when combined with self-similarity, that the strongly non-linear regime of the two point correlation function (and also of the power spectrum) should be a power law function of the separation, i.e. $\xi(x) \propto x^{-\gamma}$ where the exponent $\gamma_n$ is a simple function of $n$, the exponent characterizing the power law behaviour of the initial fluctuations (with power spectrum $P(k) \sim k^n$). The stable clustering hypothesis can, at best, be a good approximation because it neglects in principle the evolution of structures due to their interaction in general (and their merging in particular). It is, nevertheless, a fundamental question about non-linear clustering to understand how good an approximation stable clustering in fact provides. Indeed the assumption of the validity of this approximation at sufficiently small scales provided the basis of the assumed functional form of non-linear clustering at small scales in phenomenological approaches, like that of \cite{Hamilton1991, Peacock1996}. Hereafter PD, which were widely used to compare galaxy data to cosmological models until a few years ago.

Historically there have been numerous numerical studies of the validity of the stable clustering hypothesis in scale-free models, with, for a long time, inconclusive results. While, for example, \cite{Padmanabhan1996} and \cite{Colombi1994} reported deviations from stable clustering, \cite{Jain1997, Bertschinger1998} and \cite{Valageas2000} found results apparently in agreement with this hypothesis in the strongly non-linear regime. A subsequent larger study, by \cite{Smith2003}, reported clear deviations from the stable clustering predictions at smaller scales. These results, confirmed also by the larger study of \cite{Widrow2000}, appeared thus to unambiguously detect the inadequacy of the stable clustering hypothesis, and more specifically of the PD fits to the non-linear clustering based on it. The latter have then been superseded by fits with “halo models” which generically break stable clustering. Indeed these models are explicitly based on the assumption of smooth virialized structures which are built up through merging, which is qualitatively different from stable clustering which instead implies a hierarchy of virialized structures.

In this paper we closely re-examine the issue of the breakdown of stable clustering in scale-free cosmological models, which is, as we will see, inseparable from the issue of the resolution of N-body simulations of these models. We have been prompted to carry out the simulations and analysis reported here by results we obtained using smaller simulations, reported in a previous paper \cite{Benhaiem2013}, in which we explored clustering in scale free models in a broader class than usually considered in cosmology. The conclusions of this study appeared to be discrepant with those of \cite{Smith2003}. which, as discussed above, have been widely assumed in the literature to establish definitively clear deviations of non-linear clustering from that predicted by the stable clustering hypothesis. Indeed our conclusion — using simulations somewhat smaller than those of \cite{Smith2003}, but with higher resolution — was that the resolved (i.e. self-similar) non-linear clustering was in good agreement with the stable clustering hypothesis. Moreover, while we observed apparent deviations from the stable clustering predictions like those reported by \cite{Smith2003}, these were not in the self-similar regions. Further we have detected a clear dependence on force smoothing $\varepsilon$, by comparing simulations with different $\varepsilon$, precisely in the range of scales which has been considered by \cite{Smith2003} in their fits. This would imply that the assumptions made by \cite{Smith2003}, when obtaining their fits to the power spectrum, are strongly affected by force smoothing.

The results of \cite{Smith2003} for non self-similar cosmologies, and notably the standard ΛCDM cosmology, given in terms of the parameters of the “halo-fit” model, have been very extensively used in the literature. These have been revisited by other authors. In particular, \cite{Takahashi2012} found that the results of \cite{Smith2003} for the power spectrum at small scales (large wave-numbers) are indeed incorrect, due to the underestimation of the power generated by the effect of smoothing. \cite{Takahashi2012} have corrected the halo-fit power spectrum, and this change has then been widely adopted in the literature. While the correction of the halo-fit power spectrum was taken into account for the ΛCDM simulations, the consequences for the results of \cite{Smith2003} for scale-free simulations, and in particular for the issue of the validity of stable clustering, have not been examined in the literature other than in one other study \cite{Widrow2004}.

In the present work we thus choose our simulations to allow a detailed comparison with the results of \cite{Smith2003} for scale-free models, and to assess, in particular, the role of the force smoothing length in limiting their resolution. Specifically we present the study of six simulations, for the cases $n = -2$, $n = -1$ and $n = 0$, and for $N = 256^3$ particles (as \cite{Smith2003}) and an Einstein-de Sitter scenario.\footnote{Nevertheless it is possible, as shown in \cite{Ma2002}, to write down very specific halo models which have the exponents predicted by stable clustering at asymptotically small scales.}
ter (EdS) cosmology. For each case, we have run simulations with exactly the same initial conditions and numerical parameters, changing only the force smoothing. On the one hand we have used the same smoothing used by Smith et al. (2003), and, on the other hand, a smoothing as in Benhaiem, Joyce & Marcos (2013), smaller by a factor of six. The detailed analysis of these simulations allows us to draw clear conclusions concerning the results of Smith et al. (2003) (and also Widrow et al. (2009)) and in particular the issue of the validity of stable clustering. It also reveals that there is in fact an intimate connection between the breakdown of this same approximation and the resolution of N-body simulations. Our study also allows us to address in detail the important issue of optimal choice of smoothing in a cosmological simulation.

The paper is organized as follows. We first recall, in Sect 2 the equations of motion in an expanding universe, the self-similar evolution of scale-free models and the prediction obtained in the stable clustering hypothesis for the two point correlation function. In Sect 3 we describe the numerical simulations, and in Sect 4 we present our results. Finally in Sect 5 we summarise our main conclusions.

2 SCALE-FREE COSMOLOGIES

In a scale-free cosmology both the power spectrum, characterising initial matter density fluctuations, and the cosmological expansion are simple power law functions. In practice, the latter is usually taken to be a EdS cosmology, with the scale factor

$$a(t) = \left(\frac{t}{t_0}\right)^{2/3} \quad \text{where} \quad t_0 = \frac{1}{\sqrt{6\pi G \rho_0}}$$

(1)

and where $\rho_0$ is the mean mass density in comoving coordinates. A power spectrum $P(k) \propto k^n$ with $n \geq -3$ introduces (through its amplitude) a single length scale $L_0$. This scale can be defined to be that at which density fluctuations have, initially, a given amplitude (e.g., unity).

The gravitational evolution of such a universe can, in principle, be written in rescaled dimensionless variables (taking, e.g., $t_0$ and $L_0$ equal to unity); as a consequence any evolved quantity is a function only of these variables. In particular, any two-point statistic can be written as a function of the form $f(x/L_0, t/t_0)$. As the time $t_0$ is arbitrary, one can then infer that the temporal evolution is “self-similar”, which in this context means equivalent to a spatial rescaling. We thus have, for the two-point correlation function (CF),

$$\xi(x, a) = \xi_0 \left(\frac{x}{R_s(a)}\right)^{\gamma_{sc}}$$

(2)

where $\xi_0 = \xi(x, 1)$ and $a_0 = 1$ is an arbitrary reference scale factor. Analogously, the dimensionless power-spectrum (PS) is defined by

$$\Delta^2(k) = \frac{\pi k^3}{(2\pi)^3}P(k)$$

and, if the evolution is self-similar, we have

$$\Delta^2(k, a) = \Delta^2_0(kR_s(a))$$

(3)

where the function $R_s(a)$ is defined as the scale $L_0$, but for a scale factor $a$.

Finally the dependence of $R_s(a)$ on $a$ can be inferred from linear theory: assuming that it describes the evolution of fluctuations at large enough scales, with $\Delta^2(k, a) \propto a^2$, we can infer that

$$R_s = a^{\frac{2}{\gamma_{sc}}}.$$ (4)

One important remark on this derivation: the PS of fluctuations in a perturbed FRW cosmology cannot, strictly, be a pure power law because its integral is proportional to the one point variance of the density fluctuations which must be finite (see e.g., Gabrielli et al. (2005)). Implicitly we have thus assumed in the above derivation that there is an ultraviolet cut-off in the PS, and, crucially, that the gravitational clustering does not depend on it. Thus, in practice, only insofar as any such dependence on this scale is wiped out by the dynamics, can one expect the clustering at a given scale to be self-similar. On the basis of general theoretical arguments given originally by Zeldovich (see e.g., Peebles 1980), it is very plausible that such an assumption should be valid for exponents $n \leq 4$. In practice self-similarity has been observed numerically in models up to $n = 2$ (Benhaiem, Joyce & Marcos 2013) in the three dimensional space, and up to $n = 4$ in analogous one dimensional models (Joyce & Sicard 2014, Benhaiem, Joyce & Sicard 2013). In the N-body problem as discussed below, the ultraviolet cut-off to the power spectrum is provided by the Nyquist frequency of the initial perturbed lattice configuration: testing for self-similarity is, as we emphasize, a way to test that the evolved system does not depend on this unphysical scale.

2.1 Stable clustering prediction for two point statistics

According to the stable clustering hypothesis the statistical properties of non-linear clustering are, in physical coordinates, time independent. This is true, to a very good approximation, if matter in highly overdense regions behaves as if it were isolated from the rest of the universe. If we assume that this hypothesis is valid in a scale-free model, in which clustering develops in a self-similar way, then there is no characteristic length scale. Indeed, any characteristic scale would be proportional, in comoving coordinates, to $1/a$ because of stability, which is in contradiction with the supposed self-similar scaling. The only possibility left is then that CFs are simple power laws. For the resultant two-point CF, with a behaviour of the type $\xi(x) \sim x^{-\gamma(n)}$, it is straightforward to calculate that (Davis & Peebles 1977, Peebles 1980, Benhaiem, Joyce & Marcos 2013)

$$\gamma(n) = \frac{3(3+n)}{5+n}$$ (5)

with $0 < \gamma(n) < 3$ for $n > -3$. Thus, the larger the value of $n$, the steeper is the predicted exponent of the two-point CF.

2.2 Physical meaning of stable clustering exponent

It is instructive for what follows to recall a simple relation involving the stable clustering exponent $\gamma(n)$ which we have derived in Benhaiem, Joyce & Marcos (2013) (see also
rms density fluctuation is thus (assuming scale free initial conditions, the ratio of their initial density fluctuations is thus $L_{2}^{0}/L_{1}^{0}$ respectively, with $L_{1}^{0} \ll L_{2}^{0}$. Assuming scale free initial conditions, the ratio of their initial density fluctuations is thus $L_{2}^{0}/L_{1}^{0}$. Assuming further that they collapse and then virialize at a time which depends only on the linearly extrapolated amplitude of such an initial overdensity, we can infer that the ratio of the corresponding scale factors (at which virialization occurs)

$$\frac{a_{2}}{a_{1}} = \left(\frac{L_{2}^{0}}{L_{1}^{0}}\right)^{\frac{n+3}{2}}.$$  

(6)

If we now assume that the smaller structure, which virializes first (at scale-factor $a_{1}$), remains stable subsequently, we can infer that the relative size of the two structures at $a_{2}$ is

$$\frac{L_{1}}{L_{2}} = \frac{L_{1}^{0}}{L_{2}^{0}} \times \left(\frac{L_{1}^{0}}{L_{2}^{0}}\right)^{\frac{n+3}{2}} = \left(\frac{L_{1}^{0}}{L_{2}^{0}}\right)^{\frac{n+3}{2}}.$$  

(7)

The multiplicative factor on the right hand side of this equation is strictly less than unity, and quantities the change in the relative size of two structures between the linear and strongly non-linear regime, assuming stable clustering applies after virialization. In other words, the multiplicative factor quantifies how much a structure which virializes “shrinks” due to stable clustering relative to a structure which collapses and virializes later.

For a fixed range of initial comoving scales, the reduction in relative size increases as $\gamma_{sc}$ decreases (or correspondingly as $n$ increases). The greater is $\gamma_{sc}$ the more “concentrated” are the pre-existing virialized substructures inside a larger structure when it collapses. It is in fact precisely such a “relative shrinking” which one might expect to make stable clustering a reasonable approximation: if the substructures inside a structure are smaller (and therefore more tightly bound), the process of their disruption by tidal forces and mergers will be much slower and less efficient. Indeed, in the limit that $\gamma_{sc} \rightarrow 3$, any structure which collapses and virializes will “see” the substructures which have collapsed before it essentially as point particles, and thus stable clustering should become exact in this case. On the other hand, as $\gamma_{sc}$ decreases, we expect that the interaction between structures can lead more easily to their disruption, and in particular that mergers of substructures become much more probable.

There is a simple consequence of the relation (6) for numerical simulations. In an N-body simulation one can in principle follow the non-linear evolution of a range of comoving scales initially in the linear regime, between a minimal scale (say $L_{1}^{0}$) and a maximal scale (say $L_{2}^{0}$). In units of the initial interparticle separation, $L_{1}^{0}$ is fixed and independent of $N$, as it can be taken to correspond to a region initially enclosing some fixed minimal number of particles. The scale $L_{2}^{0}$, on the other hand, is bounded above by the box size (or some fraction of it). Thus the ratio of these scales is, to a reasonable approximation, the same for different cosmological models if one compares simulations of a given particle number $N$. It then follows from (6) that, if stable clustering is a good approximation in the non-linear regime, the range of non-linear scales ($L_{1}$ to $L_{2}$) in which it will be observed will increase monotonically with $n$. Thus not only do we expect stable clustering to be potentially a better physical approximation as $n$ increases, but we also expect that the range of scales over which it can be observed in simulations of comparable size to increase as $n$ increases. This is a behaviour we have already observed clearly in previous studies of this model in both one (Benhaiem, Joyce & Sicard 2013) and three dimensions (Benhaiem, Joyce & Marcol 2013). We will again draw particular attention to it in what follows.

3 NUMERICAL SIMULATIONS: METHODS AND RESULTS

3.1 Equations of motion

Dissipationless cosmological N-body simulations (see e.g., Bertschinger (1998), Springel et al. (2005); Dehnen & Reed (2011)) solve numerically the equations

$$\frac{d^{2}x_{i}}{dt^{2}} + 2H \frac{dx_{i}}{dt} = \frac{1}{a^{3}} F_{i},$$  

(8)

where the gravitational force is

$$F_{i} = -Gm \sum_{j \neq i} \frac{x_{i} - x_{j}}{|x_{i} - x_{j}|^{3}} W_{c}(|x_{i} - x_{j}|).$$  

(9)

In Eqs. (8) $x_{i}$ are the comoving positions of the $i = 1...N$ particles of equal mass $m$, in a cubic box of side $L$, and subject to periodic boundary conditions (the script ‘P’ indicates that the sum extends over the infinite copies); $a(t)$ is the appropriate scale factor for the cosmology considered, and $H(t) = \dot{a}/a$ is the Hubble constant. The function $W_{c}$ is a regularization of the divergence of the force at zero separation — below a characteristic scale, $\epsilon$, which is typically (but not necessarily) fixed in comoving units.

Changing the time variable to

$$\tau = \int \frac{dt}{a(t)^{3/2}},$$  

(10)

Eq. (8) can also be cast as

$$\frac{d^{2}x_{i}}{d\tau^{2}} + \Gamma \frac{dx_{i}}{d\tau} = F_{i},$$  

(11)

where, for the EdS model given by Eq. (11), we find

$$\Gamma = \frac{1}{3\Omega_{0}}.$$  

(12)

In this time variable the equations of motion are thus equivalent to those of particles in an infinite non-expanding universe subjected to a simple fluid damping.

In this representation it is clear that the scale-free property is common to a one parameter family of such models in which $\Gamma$ is constant, of which the usual EdS model is just one particular case. It is this more general class of models that, under the same hypotheses leading to self-similar evolution, also permits a simple generalization of the analytic prediction for the case of stable clustering, which we have studied in Benhaiem, Joyce & Marcos (2013). In this paper we focus on results only for the usual EdS model.

3.2 Simulation Code

We have made use of the Gadget-2 code (Springel & Volker 2005) with the modification described in detail in Benhaiem, Joyce & Sicard (2013) to simulate the equations as cast in Eqs. (11). We thus have used the static version of
the code (i.e. for a non-expanding system in periodic boundary conditions), and modified the time-integration scheme, keeping the original “Kick-Drift-Kick” structure of the leapfrog algorithm and just modifying appropriately the “Kick” and “Drift” operation to include the damping term. The structure of the code is otherwise unchanged.

Tests of this code, notably comparison between results obtained with it for the EdS case and those obtained using the standard Gadget-2 integration of this cosmology, for the same initial condition and numerical parameters, have been discussed at length in Benhaiem, Joyce & Sicard (2013). Some minor visual differences between such pairs of simulations are observed. This is to be expected as these are two different numerical integrations of the evolution of a chaotic dynamics. The statistical properties of the final configurations, which is what we measure and study in the present paper, are, however, in excellent agreement in the two simulations.

3.3 Simulation parameters

The N body method introduces in general, three unphysical length scales: the force softening scale \( \varepsilon \), the mean interparticle separation \( \Lambda = n_0^{-1/3} \) (where \( n_0 \) is the mean particle density) and the side of the periodic box \( L \). In a realistic cosmological model (e.g. \( \Lambda CDM \)) there are characteristic physical scales, and these three unphysical scales may then be specified in physical units, and can be varied independently of one another. Usually \( \varepsilon \) is referred to as the spatial resolution and given typically in kpc, \( \Lambda \) is specified by the particle mass \( (= \rho_0 \Lambda^3) \), where \( \rho_0 \) is the mean comoving mass density and given typically in solar masses, and thus referred to as the mass resolution, and finally \( L \), the box size, is given typically in Mpc. For a scale-free simulation there is no length scale other than the non-linearity scale (defined e.g. as the scale at which the mass fluctuations in a sphere are unity). The standard method of setting up initial conditions (see below) fixes this scale relative to the initial interparticle separation at the initial time, and there are thus in practice then just two dimensionless parameters: the ratio \( \varepsilon/\Lambda \), which we will refer to here simply as the resolution of the simulation, and \( N \), the particle number (and \( L = N^{1/3} \Lambda \)).

We use the version of Gadget-2 with a force smoothing parameter \( \varepsilon \) which is fixed in comoving coordinates (as is common practice in large volume cosmological simulations, and, in particular, in almost all studies of scale-free models). The force smoothing in Gadget-2 is implemented with a spline interpolation between the exact gravitational force, at 2.8\( \varepsilon \), to a vanishing force at zero separation. As anticipated above, we perform all of our simulations for two values of the smoothing length: \( \varepsilon = 0.064\Lambda \), as in Smith et al. (2003), and \( \varepsilon = 0.01\Lambda \), as we have considered in Benhaiem, Joyce & Marcos (2013). These are our “low resolution” and “high resolution” simulations respectively. We consider power law initial conditions with exponents \( n = -2, -1, 0 \). Table 1 gives the associated predicted stable clustering exponent \( \gamma_{sc} \), as well as other parameters characterising the initial amplitude and duration of the simulations which we will explain below.

All of the results reported in this paper are for simulations with \( N = 256^3 \). Results for identical simulation parameters (and values of \( \varepsilon/\Lambda \)) but with \( N = 128^3 \) and \( N = 64^3 \), have been reported in Benhaiem, Joyce & Marcos (2013). Such simulations are identical up to effects associated with the differing finite box sizes, which are expected only to become important from the time that non-linearities develop at scales approaching the box size. We have verified that this is indeed the case, and thus that all results reported here for \( N = 256^3 \) are, other than finite box size effects (which we detect as deviations at large scale from self-similarity) independent of \( N \).

One of the central points in our analysis will be the differences between these low and high resolution simulations. Indeed the question of the optimal choice of \( \varepsilon/\Lambda \) is a very important one for cosmological simulation using the N body method which has been widely discussed in the literature. While decreasing it can potentially increase the range of scales in which clustering is resolved, doing so may compromise the accuracy with which the N body system can represent the desired collisionless limit (for a discussion see e.g. Splinter et al. 1998, Knebe et al. 2000, Romeo et al. 2008; Joyce, Marcos & Baertschiger 2009). We will return to discuss this issue at some length in the final section.

For the present we underline that any non-collisional effects arising from the use of a small smoothing (and, in particular, those associated with two body collisions) should be detected as deviations from self-similarity, if they are present at a level which significantly affects the quantities we measure.

From the numerical point of view the use a smaller \( \varepsilon \) than usually employed in cosmological simulations (exemplified by the smoothing used e.g. in Smith et al. 2003) implies greater numerical cost, as smaller smoothing requires smaller time steps to integrate accurately — and maintain accurate energy conservation — on those trajectories on which the forces become dominated by a single nearby particle (for a discussion, see e.g. Knebe et al. 2000; Joyce & Sylos Labini 2013; Sylos Labini 2013). We have performed simple convergence tests as well as tests of energy conservation using the so-called Layzer-Irvine equation (see Benhaiem, Joyce & Marcos (2013); Joyce & Sylos Labini (2013); Sylos Labini (2013) for details). We have adopted finally the fiducial recommended value for the parameter controlling force accuracy, but significantly more stringent values for the time-stepping parameters as we have found these to give considerable improvement in the tests using the Layzer-Irvine equation. Finally, as underlined, we note again that self-similarity in principle robustly tests also for such non-physical effects.

To compute the power spectrum, we use the very accurate estimator of Colombi et al. (2009), and for the correlation function \( \xi(x) \), we used the so-called “full-shell” estimator which directly counts the particles in spherical shells centred on a chosen number \( N_5 \) of randomly selected particles. Our results below are obtained with \( N_5 = 10^5 \). Tests specifically we have taken ErrTolIntAccuracy=0.001, MaxRMSDisplacementFac=0.1 and MaxSizeTimestep=0.01, values which are smaller (by factors of 25 for the first, and 2.5 for the two others) than the values suggested in the Gadget-2 user guide and treated as “fiducial” in the literature (and e.g., in Smith et al. 2003). For the force accuracy we have taken ErrTolForceAcc=0.005 which is a typical fiducial value.
with a larger number and/or different distribution of the centres sampled have been performed and we found our results to be very stable. Unlike Smith et al. (2003) we do not attempt to apply any corrections to take into account discreteness effects: again, any such effects should in principle be detectable using the tests for self-similarity which we focus on here.

3.4 Initial conditions and duration of simulations

We generate our initial conditions using the code mpggrafic (Prunet et al. 2008) which employs the standard method used in cosmological simulations (see e.g. Bertschinger 1995; Joyce & Marcos 2007)): particles, initially on a simple cubic lattice, are subjected to a displacement field generated as a sum of independent Gaussian variables in reciprocal space with variance determined by the desired linear theory extrapolated amplitude by specifying the value of $\gamma$. Following standard practice, we characterize the initial and final amplitudes of the fluctuations.

$$\Delta f^2(k_N, a_0) = A_0 k_n^a,$$

which is (approximately) the normalized mass variance in a Gaussian sphere of radius $\Lambda$. In fixing the initial amplitude of our simulations as given in Table 1 we have used as guidance the previous work of Jain & Bertschinger (1998) and of Knollmann, Power & Knebe (2008) which report tests showing that self-similarity is recovered better for the cases with smaller $n$ if low amplitudes are used.

Also given in Table 1 are the final times $a_i$ considered for our analysis in each of the simulations, and the corresponding values of the linear theory extrapolated amplitude $\Delta f^2(k_0, a_f)$ at the fundamental mode of the periodic box $k_0 = 2\pi/L$. In the cases $n = -2$ and $n = -1$ our simulations thus extend to times when the normalized mass variance in a Gaussian sphere of order the size of the box is no longer much smaller than unity, and one would expect this to lead to significant finite size effects. Such effects at large scales will indeed be detected below using tests for self-similarity.

4 RESULTS

We divide our analysis into three parts. Firstly, in Sect. 4.1 we compare our results for the CF and PS in the low and high resolution simulations. We then consider, in Sect. 4.2 what conclusions concerning the validity of stable clustering may be drawn from this comparison. Finally, in Sect. 4.3 we show how, by testing for self-similarity, we can robustly determine the range of scales in which the resolved clustering is physical. This allows us to draw much stronger conclusions both about the limits on the resolution and on the validity of the predictions derived from the stable clustering hypothesis.

4.1 Resolution limits arising from force smoothing

We begin by comparing the results of our pairs of low and high resolution simulations. Shown in Fig. 1 are the CF (left panels) and the dimensionless power spectrum $\Delta^2(k)$ (right panels) for the three different values of $n$. In particular, each panel shows the results for both the low (solid blue line) and high (dashed red line) resolution simulations. The blue (red) vertical solid (dashed) line indicates the scale $2.8\pi$ in the plots of the CF, and $\pi/(2.8\pi)$ in the plots of $\Delta^2$, respectively for the low (high) resolution simulation. In these plots we have excluded the very largest scales (i.e. smallest $k$) at which the differences between the low and high resolution simulations are, as one would expect, negligibly small.

These plots show very good agreement between the CF in the high and low resolution simulations down to the smoothing scale of the latter, i.e., down to the scale at which the two body force is the same in the two simulations. This behaviour is in agreement with what one would naively expect: clustering is apparently well resolved down to the scale at which the force law deviates from the exact Newtonian force. Below this scale clustering is clearly always larger in the higher resolution simulation, so the effect of the smoothing appears to be simply to suppress clustering below the scales at which the force is smaller than its true (Newtonian) value. We underline, nevertheless, that these results alone to not allow us to conclude definitively that the clustering in either case above the smoothing scale represents accurately the physical (continuum) limit. Only the analysis using self-similarity described below will allow us to address this question.

The plots of $\Delta^2(k)$ show a distinctly different behaviour: the high and low resolution simulations strikingly differ already at a much smaller wavenumber — about a factor of four — than the naively estimated characteristic scale for the smoothing. The explanation for this behaviour is again simple: the clustering signal at a given wavenumber $k$ picks up contributions from a range of real space scales around $\pi/k$, and in particular it can receive significant contributions from scales well below $\pi/k$. Indeed, since the CF shows a clustering modified by smoothing at scales below $2.8\pi$, its Fourier transform is modified over a range of $k$ which extends well below $\pi/(2.8\pi)$. As the force smoothing is applied in real space, it is natural for it to have a localised effect on clustering in real space, while in reciprocal space its effects on clustering are “dispersed” into a broader range of wavenumbers, leading in the present case to very large effects on the PS at reciprocal space scales almost an order of magnitude.
smaller than the naively estimated scale. Again we note that, just as for the CF, this analysis does not allow us to infer the range of wavenumber in which the measured clustering in either simulation accurately represents the physical (continuum) limit.

Fig. 2 shows the ratio between the measured quantities in each pair of high and low resolution simulation (CF in left panels, $\Delta^2(k)$ in right panels): this allows a more detailed quantitative comparison of these results. The vertical line with arrow corresponds to the smoothing scale for the low resolution simulation. The conclusions drawn above concerning the differences between real and reciprocal space are very clearly visible. However, we can also note subtle differences as a function of $n$ for both quantities. In particular, the effect of the resolution on the CF shows an apparent trend as a function of $n$: for $n = -2$ the higher resolution leads to an increased clustering at all scales, starting from a scale above $2.8 \varepsilon$, while in the case $n = -1$, and more markedly in the case $n = 0$, such an increase of clustering is seen at smaller scales, but their is also a small decrease at larger scales. We will return to this detail further below, as we will see evidence that it is related to the fact that, in simulations at fixed $N$, stable clustering extends to smaller scales for larger $n$, as outlined in the discussion at the end of Sect. 2.2.

4.2 Testing the stable clustering prediction, without self-similarity

Before turning to our analysis using self-similarity, which will allow us to determine precisely which part of the clustering signals at the final time can be assumed to resolve the physical limit, let us consider the compatibility of the results above with the validity of stable clustering in the non-linear regime. As we have recalled, the stable clustering hypothesis predicts a power law behaviour of these quantities in the strongly non-linear regime, i.e., starting from scales at which structures may be expected, theoretically, to be well approximated as virialized. As we will see below (and in agreement with previous studies) such behaviour is observed starting from $\xi \sim 100$ and $\Delta^2(k) \sim 100$, consistently with what would be estimated from the naive spherical collapse model.
Further, if stable clustering is a good approximation, we would expect this always to be the case in a limited range of scale, corresponding to some finite duration after virialization. Thus we would expect that stable clustering will break down always at asymptotically small scales (compared to the non-linearity scale). In seeking to test the validity of the predictions of stable clustering one can thus envisage three possible behaviours for the CF (or PS):

- **(B1)** The CF (or PS) is in good agreement with the predicted behaviour over the full range of resolved highly non-linear scales;
- **(B2)** The CF (or PS) is consistent with the predicted behaviour over part of the range of resolved highly non-linear scales, but a break from the predicted behaviour is detected at a sufficiently small scale;
- **(B3)** The CF (or PS) is consistent nowhere with the predicted behaviour in the range of resolved highly non-linear scales.

Note that one could consistently reach different conclusions for the CF and PS. For instance (B2) could hold for the CF and (B3) for the PS: if for the former the range of scale in which the predicted power law behaviour is found is very limited, the PS may never be well approximated by the "pure" stable clustering behaviour.

Examining again the plots in Fig. 1 we observe that each of the measured CF and PS (both in the high and low resolution simulation) show, in the highly non-linear range, a behaviour which can, by eye, be fitted with a power law in a limited range. This range depends strongly both on the model (i.e. on \( n \)) and on the force smoothing. We find that it extends over a decade in the high resolution simulation for \( n = 0 \), but barely more than a factor of two for \( n = -2 \). Shown in Figs. 3-4 are the best fits to a simple power law obtained in each CF and PS in the regions which, by eye, appear to potentially admit such a fit. We recall that the predicted behaviours for stable clustering are \( \xi(r) \sim r^{-\gamma_{sc}} \) and \( \Delta^2(k) \sim k^{\gamma_{sc}} \) with \( \gamma_{sc} = 1.8 \) for \( n = 0 \), \( \gamma_{sc} = 1.5 \) for \( n = -1 \) and \( \gamma_{sc} = 1 \) for \( n = -2 \). Thus we observe that the best fit exponents for the CF in the high resolution simulations are in excellent agreement (within a few percent) with those predicted by the stable clustering hypothesis. Further, performing the same procedure on the CF measured in the low resolution simulations, we obtain exponents which are again very consistent with those predicted by stable clus-
tering, albeit within somewhat larger error bars associated with the more limited range of scale of the fits.

On the other hand, the exponents obtained in the same way for the PS are (i) in each simulation systematically smaller than those obtained with the fit to the CF, and (ii) significantly different in the low resolution and high resolution simulations. In particular, the exponents obtained from the low resolution simulation are considerably smaller than those obtained by fitting the CF for each case. We note that these latter values for the effective exponents of the PS are close to those found by Smith et al. (2003) (γ = 1.49, 1.26, 0.77 for n = −2, −1, 0 respectively). In principle, the results discussed in the previous section suggest that the latter are incorrect, as the lower fitted exponent is clearly a result of a suppression of power compared to that in the higher resolution simulation. Given, however, that we have no certainty that the higher resolution simulation is itself converged in this range, we cannot draw a definitive conclusion without using the criterion of self-similarity, as we will describe below.

In summary for the CF a good fit to the stable clustering exponent appears to be valid in some range in all simulations, and thus (B3) appears to be excluded. In the low resolution simulations the lower cut-off to this fit lies in all cases around the smoothing scale (2.8ε), and therefore at the strict lower bound of our spatial resolution. Thus, from the low resolution simulations, the conclusion we draw for all three models is (B1), assuming only that the clustering represents the physical limit at least in some part of this spatial range. For the high resolution simulations the same is not true: while for the case n = 0 the power law now extends again down to very close to the smoothing, this is definitely not the case for either n = −1 or n = −2. For n = −1 the range of the power law fit extends to a slightly smaller scale, and the CF bends away to a much shallower behaviour well above the smoothing length; for n = −2 the lower cut-off to the power law fit barely changes and the CF over the scale resolved with the smaller smoothing is everywhere well below the extrapolated power law.

Thus, for the cases n = −1 and n = −2 our high resolution simulations include a region where the CF is clearly not well described by the stable clustering prediction. If the clustering in this region is indeed resolved, i.e. representative of the continuum physical model, then our conclusion is (B2). If on the other hand this is not the case then the correct conclusion is (B1).

As we now discuss the criterion of self-similarity gives us a tool to answer the crucial question as to what the range is in which the clustering can be assumed to be physical, for both the CF and PS.

### 4.3 Self-similarity tests

To test for the self-similarity at any given time of a given quantity — here the CF or the PS — we need to simply compare it to its value at a later or an earlier time in the appropriately rescaled coordinates, or in more general at a sequence of times. For our purposes here it will be sufficient to compare the CF and the PS at the final output time (af) with the rescaled quantity at a short time before (but sufficiently long so that there is significant evolution of the clustering). We thus compare the CF (and the PS) at the final scale factor af with a slightly smaller scale factor af − n exemples choise so that the rescaling of length scales in all cases is a factor of 1.5, i.e., Rs(af) = 1.5R0(af − n exemples). This corresponds to af/af − n exemples ≈ 2.25, 1.5, 1.23 for n = 0, −1, −2 respectively.

Shown in Fig. 5 are the resulting ratios for the CF (left panels) and Δ 2(k) (right panels) for both high and low resolution simulations for each of the three models. The range of scales in which this ratio is close to unity corresponds to the range in which the clustering signal, at both af and af − n exemples, is clearly self-similar to a good approximation. Given that a fuller analysis shows that the lower cut-off to self-similarity in comoving coordinates decreases monotonically in time, we can in fact take the lower cut-off to self-similarity inferred to be that at af − n exemples, while the lower cut-off at af may be slightly smaller, and, more specifically, will be smaller by a factor af/af − n exemples if stable clustering applies at these scales.

The data in Fig. 5 show clearly that the clustering in all simulations breaks self-similarity both at large and small scales, while in an intermediate range self-similarity is obtained to a good approximation. This is very much as expected, and shows the power of the method to detect non-physical effects: at large scales self-similarity is violated due to finite box size effects, and at small scales due to effects arising from the ultraviolet cutoffs, i.e. the force smoothing and the finite particle number sampling of the continuous density field. That the two asymptotic regimes are reasonably well separated means that there is in principle no mixing of the two kinds of effects, and this is further confirmed by the fact that the deviations at large scales are completely insensitive notably to the force resolution.

We note that the differences in Δ 2(k) at small k come predominantly from the sparse mode sampling, while in the CF they arise from the sparseness noise in the estimator at very large scales. Further we observe that the deviations from self-similarity at large scales are much more marked and extend further, in particular, for the case n = −2. This is due, as has been previously documented (see in particular Jain & Bertschinger 1998), to the stronger coupling to long wavelength modes characteristics of lower n spectra, which makes the detection of self-similarity more difficult in this case.

In the behaviour at small scales (large wavenumber for PS) in Fig. 5 we observe, in contrast, an evident and very non-trivial dependence on the force smoothing. Comparing the results for the high and low resolution simulations, we see clearly, both in real space and reciprocal space, that, for n = 0, the range in which self-similarity holds to a very good approximation is extended considerably when the resolution is increases (i.e. the softening decreased). For n = −1 a similar effect is seen, albeit less pronounced, while for n = −2 the scale at which self-similarity breaks down does not appear to move significantly at all (in either space). These behaviours appear to reflect closely those we observed above for the bending in the CF and PS, and suggest that the location of this bending may be strongly correlated to the breakdown of self-similarity. Making now, for the CF and each n, a quantitative comparison between the lower cut-off to self-similarity (estimated, e.g., as that at which the ratio deviates by more than 10% from unity) and the length scale at which the bending of CF away from a power law behaviour consistent with stable clustering was observed above, we see that in all cases the former scale is in fact substantially greater.
than the latter. Further if we assume that stable clustering does indeed hold around these scales and thus infer that the scale at which self-similarity breaks at $a_f$ is indeed smaller by the factor $a_f^{-1}/a_f$, we find that, in all cases, the scale at which self-similarity breaks coincides very closely with the scale down to which the stable clustering approximation describes well the CF.

The explanation of this very strong correlation of these scales is, we believe, simply the following: the mechanism which propagates self-similarity to smaller comoving scales in the highly non-linear regime is the stability of the clustering (in the corresponding range of space and time scales). And, conversely, when the clustering is not stable — i.e., no longer well described as the stable evolution of the virialized collapsed initial over densities — there are strong deviations from self-similarity. Indeed such an association of the two scales is very plausible: the breakdown of the stable clustering hypothesis is associated with a change in the physical processes characterising the evolution of clustering. In particular it is associated with the interaction of structures and even with their merging. Both the occurrence and outcome of such processes can be expected to be sensitive to the non-physical ultraviolet scales associated with discretisation. In particular it is clear that the smallest structures which form in simulations of this kind are subject to discretisation effects which can propagate to progressively larger scales as structures merge and interact.

This hypothesis is further born out by the dependence on $n$ of the scale of the break from stable clustering: as explained in Section 2.2, we expect, in simulations that follow, as here, a comparable range of initial comoving scales from the linear into the highly non-linear regime, the ratio of the range over which stable clustering can be observed increases with $n$. This is qualitatively very clearly in line with what is observed, in the high resolution simulations (cf. Fig. 4), while it is obscured in the low resolution simulations (cf. Fig. 3) because of the larger smoothing scale.

Related to this, we remark also on another interesting feature of the results in Fig. 5 and in particular for the PS (right panels). Comparing the results for the low resolution simulation in the three cases, we observe that the wavenumber at which a clear deviation from self-similarity develops decreases as $n$ increases, i.e., for $n = 0$ this wavenumber is smallest, while for $n = -2$ it is largest. Furthermore, there
is also a marked difference in the form of the deviation from self-similarity: for \( n = -2 \) the deviation at small scales is towards a suppression of the power, while for \( n = -1 \), and more markedly for \( n = 0 \), there is a clear positive “bump”. The fact that these bumps vanish in the higher resolution simulations shows clearly that they are a result of using a smoothing which is larger than the scale to which the self-similarity can actually propagate, through stable clustering, in the absence of the smoothing. The reason is simple to understand: using such a large smoothing, the clustering mass, rather than becoming ever more concentrated at smaller scales, gets “frozen”, at a scale of order \( \varepsilon \) in direct space. In reciprocal space, which mixes over a range of small scales, this excess power is redistributed in a range of scales above \( \pi/\varepsilon \). Thus we conclude that the use of a smoothing scale larger than that to which stable clustering can actually propagate clustering in the duration of the simulation may further degrade the self-similarity of the PS in particular.

4.4 Exponents of self-similar non-linear clustering

We can check now in more detail the degree of agreement with the stable clustering predictions for the CF and PS, using the constraint imposed by self-similarity in the high resolution simulations, which clearly show self-similar behaviour in a broader range for several cases. The precise range one chooses to fit in is somewhat arbitrary, as it depends on how large a deviation from self-similarity one chooses to tolerate. We follow the procedure described in Benhaheim, Joyce & Sicard (2013), performing power law fits to both the CF and PS between an upper cut-off, chosen by eye where the functions break from a power law behaviour (corresponding in each case to \( \xi \) or \( \Delta^2(k) \) between 100 and 200), and a lower cut-off, fixed in two different ways: (i) as the scale \( x_{SS}(k_{SS}) \) at which the quantities plotted in Fig. 4 deviate by less than 10% from unity, and (ii) as the scale \( x_{SS}(k_{SS}) \) obtained by extrapolating \( x_{ss}(k_{SS}) \) assuming stable clustering to be valid between \( a_{f_-} \) and \( a_f \). We impose also the constraint that these latter scales are larger (smaller) than \( \varepsilon (5k_c) \) for the CF (PS) fits. In the case \( n = 0 \) there
Table 2. Theoretical stable clustering exponent ($\gamma_{sc} \!$) and the corresponding measured exponents for the different $n$ models, obtained by fitting the strongly non-linear CF and PS in the two range of scales described in the text: (1) indicates the more restricted range, (2) the extrapolated range. We also show for comparison the exponents reported by Smith et al. (2003).

| $n$ | $\gamma_{sc}$ | CF (1) | CF (2) | PS (1) | PS(2) | Smith et al. |
|-----|----------------|--------|--------|--------|--------|--------------|
| 0   | 1.80           | 1.87   | 1.81   | 1.76   | 1.49   |              |
| -1  | 1.50           | 1.62   | 1.56   | 1.44   | 1.39   | 1.26         |
| -2  | 1.00           | 1.08   | 1.04   | 0.90   | 0.86   | 0.77         |

is only one fit to the CF because $x_{EG} < 2.8\varepsilon$. The results of these different fits are reported in Table 2 along with the values reported by Smith et al. (2003).

By modifying marginally the fitted ranges we find best-fits with exponents varying by of order 5%. As can be seen from Table 2 our measured exponents for the CF are thus in very good agreement with the stable clustering predictions, albeit of course with a large error bar for $n = -2$ corresponding to the very limited range of the fitted region. For the PS the exponents appear to be a little lower than predicted, but the most likely explanation for this is that the results for the PS are still not converged. Indeed we observe in Fig. 3 that there is a significant deviation from unity in the range which has been treated as self-similar.

4.5 Comparison with other studies

Our results are in clear disagreement with those of Smith et al. (2003), and the explanation is clearly that their fits have been performed solely with the PS in low resolution runs, and have been slightly extended into regions in which the power is suppressed and self-similarity broken. For the case $n = 0$, we note that we could have erroneously obtained a lower exponent more consistent with the value of Smith et al. (2003) also in the high resolution simulation by fitting the PS blindly using only the chosen criterion on the breaking of self-similarity, which in fact indicates it may extend in this case right up to $k_c$.

A study of scale-free simulations has also been reported by Widrow et al. (2009) for $n \in [-2.5,-1]$, and $N$ varying from $32^3$ to as large as $1584^3$. A single value of the smoothing length in units of the interparticle distance is employed in all simulations, equal to half of that of Smith et al. (2003) (and our low resolution simulations) and thus three times larger than in our high resolution simulations. As in Smith et al. (2003) only a reciprocal space analysis of the PS is considered. Very significant discrepancies between the measured PS and those of Smith et al. (2003) are found: due to the smaller softening there is measurably more power at large $k$, in line with what we have found here. However, for what concerns the slope of the PS at the largest $k$ fitted, the results found are very consistent (for the common cases at $n \leq -1$) with those of Smith et al. (2003), and the conclusions of the paper regarding the breakdown of stable clustering concord with those of Smith et al. (2003): the “asymptotic” logarithmic slope, labelled $\mu$ of the measured PS are significantly different from that predicted by the PD fitting formulae (which is constructed to reproduce the stable clustering prediction at large $k$). We believe the analysis provided in this paper on this crucial point has the same essential shortcomings as that of Smith et al. (2003), and that this conclusion is not convincingly demonstrated. More specifically, it is based on the observed tendency (cf. the lower panels of Figure 6 (for $n = -1$) and Figure 7 (for $n = -2$) in Widrow et al. (2009)) of the measured slope $\bar{\mu}$ towards just slightly larger values than those predicted by stable clustering over last decade in $k$; for $n = -1$, $\bar{\mu}$ varies between $1.5$ and $1.8$ (where the former is the stable clustering value), and for or $n = -2$, $\bar{\mu}$ varies between $1.9$ and $2.2$ (where the stable clustering value is $-2$). In these figures it is evident that the different curves (at different times), which should overlap if there is self-similarity, in fact show a significant dispersion, and at the largest $k$ there is only data for the last time (and therefore no check of self-similarity). Given that we have seen that the PS typically begins to be visibly suppressed by softening at the corresponding wavenumbers (well below the naively estimated scale, ~ $\pi/\varepsilon$), and that such a suppression would lead also to a decrease in the fitted logarithmic slope. Only by testing these results (i) for their independence of the choice of the smoothing length, by doing higher resolution simulations as we have done, and (ii) performing the analysis of the CF in real space, could one confidently conclude that these (small) deviations from the stable clustering prediction are physical. Widrow et al. (2009) have not performed such a test, and the detailed comparison we have performed here using such a test on the results of Smith et al. (2003), which have turned out to be the result of under-estimating the effect of force smoothing on the PS, suggest that much care is needed on this point before definitive conclusions can be drawn. In short when strong conclusions are drawn from a very limited range of large $k$ closed to the inferred resolution limit in $k$, one must be very sure indeed that this resolution limit has been very robustly determined.

5 CONCLUSIONS

We have revisited the study of scale-free models, with a focus on using them as a tool to understand better what the resolution of cosmological N-body simulations truly are, i.e., how reliably such simulations can reproduce the clustering in the continuum physical limit.

5.1 Resolution in the strongly non-linear regime

Our main finding is that the measures of two point statistics in the strongly non-linear regime of our scale-free simulations, represent accurately the physical limit only in the range of scales in which stable clustering remains a good approximation. Indeed we have found a very clear and robust association between the real-space scale at which the two point CF deviates from the behaviour predicted by stable clustering and the scale at which self-similarity breaks down. We have explained that such an association is natural because the breakdown of stable clustering is indeed associated with physical processes which may intrinsically be much more sensitive to fluctuations at scales affected by the ultra-violet cut-offs — notably the grid scale and force softening — introduced by the N-body discretisation.

Let us underline, firstly, that our conclusion is not that
strongly non-linear physical clustering which is not stable cannot be resolved accurately in an N-body simulation, but just that in practice it is not accurately resolved in those we have done, nor in those of Smith et al. (2003) and of Widrow et al. (2009), which are fairly typical of current cosmological simulations. Conversely the physical processes such as merging which violate the stable clustering approximation are, in our simulations and those of Smith et al. (2003), apparently polluted by discreteness effects, and the corresponding clustering, which is measured in some cases over a significant range of scale, cannot be assumed to represent accurately the physical limit.

Secondly, we cannot and do not conclude that all N-body simulations in the literature of realistic cosmological initial conditions fail to resolve the regime in which stability of clustering is not a good approximation. We believe, however, that our results place in serious question, at least, the accuracy of all such results. As a consequence they place in doubt the accuracy in particular of popular phenomenological fits to the strongly non-linear regime based on halos models. Indeed Smith et al. (2003) is one of the reference studies in the literature for such fits (in particular the "halofit" model), and the fact that we have found its results to be not only quantitatively, but also qualitatively, incorrect for the case of scale-free initial conditions logically places in doubt the correctness of its interpretation of its simulation results for the case of spectra which are not scale free.

As we have noted in the introduction, higher resolution simulations by Takahashi et al. (2012) for these cases have in fact shown that the results of Smith et al. (2003) at small scales to be manifestly resolution dependent. Our analysis of the scale-free models leads us to the conclusion that the real limits on resolution imply that, rather than adjustment of the best fit parameters of the phenomenological halofit model, it is the correctness of fitting to any such model breaking stable clustering in the strongly non-linear regime which should be placed in question.

5.2 Real space vs. reciprocal space analysis

One important aspect of our analysis is that we studied always in parallel the two point correlation properties in both real and reciprocal space. It is very clear from our results that, to understand the issue of spatial resolution, and also indeed that of stable clustering, is it absolutely essential to consider carefully the real space quantities: the physical phenomena are expected to be characterized fundamentally by real space scales and the mixing of real space scales in reciprocal space makes it much more difficult to identify the essential dependencies. Indeed we believe that the erroneous conclusion of Smith et al. (2003) are essentially due to the use of a k space analysis only.

5.3 Choice of force smoothing

As we have noted, the question of what is the optimal smoothing for an N-body simulation of a cosmological model is an important open one, and we now summarise what conclusion we draw from our study about it.

There are two different, but related, aspects to this question of optimisation. On the one hand, there is consideration of numerical cost: the smaller the smoothing, the greater the numerical cost to integrate accurately the N body system. On the other hand, the use of a large smoothing bounds below the length scale which can be resolved, while too small a value can potentially amplify discrete effects — most evidently, two body collisions — which do not represent the physical collisionless limit. The question of its optimisation can thus be phrased as follows: how small a value of the force smoothing should be taken to maximize the range of scales over which physical clustering can be accurately simulated?

Our results show clearly that reducing force smoothing, down to the values we have considered, somewhat smaller than those typically used in cosmological simulation, never decreases the range in which non-linear clustering is self-similar (i.e. physical) to a good approximation, but can, depending on the model, increase this range. In other words in no case have we found evidence that using higher resolution produces any significant degradation of the lower resolution result, and can, on the other hand, significantly extend the range of resolved clustering (most strongly for $n = 0$). In particular we infer from this that any associated additional two body scattering does not sensibly affect the quantities we measure. This is reasonable as we have indeed, as detailed in Sect. 5.3 increased numerical accuracy specifically to ensure accurate integration of the consequent less soft two body collisions (and the rate of two body collisionality is in fact only weakly dependent on $\epsilon$, remaining finite even at $\epsilon = 0$). We have, on the other hand, found clear evidence that using a force smoothing which is larger than the scale down to which self-similarity can potentially propagate in the duration of the simulation (i.e. as seen in the higher resolution simulation) can lead to a significant degrading of the results, for the PS in particular.

In summary our results indicate that there is no apparent reason for using a finite smoothing in cosmological simulations other than a consideration of numerical cost: provided the numerical accuracy is sufficient, we have not found any evidence of adverse effects of using a small smoothing. Such effects may of course exist, and manifest themselves at yet smaller values of $\epsilon/\Lambda$, but we have not found them. We note that, for what regards two body effects, this is quite consistent with the conclusion of other detailed studies (e.g. Knebe et al. (2000); Joyce & Sylos Labini (2013)). Taking numerical cost into account, our conclusion is then that the optimal smoothing for scale-free simulations — at least for the determination of the two point statistics we have studied — is that which allows the resolution of the scale down to which self-similar clustering would propagate in the duration of the simulation if $\epsilon$ were zero. In a non scale-free simulation, the equivalent would be expected to be the scale down to which the non-linear clustering is dominated by the density fluctuations initially modelled well in the initial conditions.

In any model, if strongly non-linear clustering is stable to a good approximation, this minimal scale fixing the optimal softening can easily be estimated: it is $\sim L^0_{\Lambda}(a_v/a_f)$ where $L^0_{\Lambda}$ is the average comoving size of the first resolved non-linear structures (containing e.g. $10^5$ particles) when it virializes, at a scale factor $a_v$, and $a_f$ is the final scale factor. Let us consider just how this depends, in a given model, on the size of the simulation (i.e. $N$). In units of the interparticle separation $\Lambda$ it just decreases as the inverse of
the final scale factor (assuming fixed amplitude of power at the scale $\Lambda$), which is fixed just by $N$. Specifically, for a scale-free simulation, assuming simulations are stopped when the non-linear scale is a fixed fraction of the box size, we have $a_f \propto N^{3+\epsilon/2}$. For our $N = 256^3$ simulations we have seen (cf. Fig. 5) that the low resolution value (used also by Smith et al. (2003)) appears to be close to optimal for the case $n = -2$, but larger than the optimal value for the other two cases. In the latter cases our results do not allow us to conclude whether our high resolution values are optimal either: to do so we would need to simulate with yet smaller $\epsilon/\Lambda$ to see whether we can extend the range of measured self-similar clustering. Concerning the simulations of Widrow et al. (2009), which use an $\epsilon/\Lambda$ half that of Smith et al. (2003), and $N$ up to a factor of $4^3$ larger, the resolution appears also close to optimal for $n = -2$ but again significantly larger than optimal for $n = -1$.

5.4 Future studies

Our final conclusion from the present study is that further larger, studies of scale-free models should be undertaken to try to establish whether the breaking of stable clustering can be unambiguously detected in an N-body simulation, ideally of comparable sizes to the largest simulations currently performed in the community. As we have discussed, such simulations at larger particle number should be performed over a range of resolution (i.e. values of the parameter $\epsilon/\Lambda$) which extends to the limit in which the range of self-similar clustering observed becomes independent of its value, i.e., in the case of stable clustering a resolution high enough to follow the stable evolution of the first virialized structures through to the end of the simulation. Unless it can be shown unambiguously in scale-free models, using a combined analysis both in real and reciprocal space, that self-similarity extends into the non-linear region where the predictions of the stable clustering hypothesis are clearly wrong, we conclude that one can have little confidence that realistic cosmological simulations, where the test of self-similarity is not available, can in fact accurately trace the physical clustering into the same regime. We note that large ($N = 1024^3$) scale-free simulations with quite high resolution have in fact been performed recently by Diemer & Kravtsov (2013), but analysed only to determine the properties of halos extracted from them and without detailed consideration of tests for self-similarity (or indeed tests for stable clustering). In a forthcoming study, using simulations similar to those presented here, we will also explore in detail the clustering in scale-free simulations in terms of halo properties, and address in details the question of which of the measured properties in simulations can be shown to be self-similar and therefore physical). In particular we will aim to determine which scales are resolved within the halos, and how their properties are related to that of the CFs.

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