Distributed Compression of Graphical Data*

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Abstract

In contrast to time series, graphical data is data indexed by the vertices and edges of a graph. Modern applications such as the internet, social networks, genomics and proteomics generate graphical data, often at large scale. The large scale argues for the need to compress such data for storage and subsequent processing. Since this data might have several components available in different locations, it is also important to study distributed compression of graphical data. In this paper, we derive a rate region for this problem which is a counterpart of the Slepian–Wolf theorem. We characterize the rate region when the statistical description of the distributed graphical data can be modeled as being one of two types – as a member of a sequence of marked sparse Erdős–Rényi ensembles or as a member of a sequence of marked configuration model ensembles. Our results are in terms of a generalization of the notion of entropy introduced by Bordenave and Caputo in the study of local weak limits of sparse graphs. Furthermore, we give a generalization of this result for Erdős–Rényi and configuration model ensembles with more than two sources.

1 Introduction

Nowadays, storing and processing data that in its native form is indexed by combinatorial objects other than just linearly ordered time or multidimensional arrays is of great importance in many applications such as the internet, social networks and biology. For instance, a social network could be presented as a graph where each vertex models an individual and each edge stands for a friendship. Also, vertices and edges can carry marks, e.g. the mark of a vertex might describe some characteristics of the individual represented by the vertex, and the mark of an edge might describe some property of the nature of the interaction between the two individuals whose friendship is represented by the edge. The overall graphical data is then comprised of both the structure of the underlying graph and the data indexed by the graph, i.e. the vertex and edge marks. Due to the sheer amount of such data in many applications, the question of how to compress it for efficient storage has drawn attention, see e.g. [BV04], [CS12], [Abb16], [MTS16], [BV17], [DA20].

As the data is not always available in one location, it is also important to consider distributed compression of graphical data. This latter question is the focus of this paper. Traditionally, when dealing with time series, distributed lossless compression is modeled using two (or more) possibly dependent jointly stationary and ergodic processes representing the components of the data at the individual locations. In this case, the rate region, which characterizes how efficiently the data can be compressed, is given by the Slepian–Wolf theorem [CT12]. We adopt an analogous framework, namely

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that two jointly defined marked random graphs on the same vertex set are presented to two encoders, one to each encoder. Each encoder is then required to individually compress its data such that a third party, having access to the two compressed representations, can recover both marked graph realizations with a vanishing probability of error in the asymptotic limit of the size of the data.

We characterize the compression rate region for two scenarios, namely, a sequence of marked sparse Erdős–Rényi ensembles and a sequence of marked configuration model ensembles. We employ the framework of local weak convergence, also called the objective method, as a counterpart for marked graphs of the notion of stochastic processes for time series [BS01, AS04, AL07]. Our characterization of the rate region is best understood in terms of a generalization of a measure of entropy introduced by Bordenave and Caputo [BC15], which we call the marked BC entropy [DA19]. It turns out that, for the sequences of ensembles we study in this paper and even more generally, as proved in [DA19], this notion of entropy captures the per–vertex growth rate of the portion of the Shannon entropy of the graphical data that is over and above an entropy of connectivity which is controlled entirely by the average degree of the graph ensemble and not the detailed statistics of the graphical data. Indeed, to the highest order, the marked BC entropy captures the part of the overall entropy that truly depends on the empirical characteristics of the graphical data and not just on the underlying connectivity structure of the graph. This motivates the marked BC entropy as a natural measure governing the asymptotic compression bounds, since it is sensitive to the details of the statistics of the ensembles and scales linearly with the number of vertices of the underlying graph. Moreover, we generalize the two graphical source result to the case where there are more than two graphical sources.

The paper is organized as follows. In Section 2 we introduce the notation and formally state the problem. Sections 3 and 4 give a brief introduction to the concept of local weak convergence and to the marked BC entropy, mostly specialized for the examples we study. Finally, in Section 5, we characterize the rate region for distributed lossless compression in the scenarios we present in Section 2, i.e. graphical data analogs of the Slepian–Wolf theorem in these scenarios. Also, in Section 5.5, we generalize this result to the case where there are more than two graphical sources.

We close this section by introducing some of the main notational conventions used in this paper. The set of natural numbers is denoted by $\mathbb{N}$ and the set of real numbers is denoted by $\mathbb{R}$. For $n \in \mathbb{N}$, $\{n\}$ denotes the set $\{1, 2, \ldots, n\}$. For a probability distribution $P$ on a finite set, $H(P)$ denotes its Shannon entropy. Also, for a random variable $X$ taking values in a finite set, we denote by $H(X)$ its Shannon entropy. We write $=: $ for equality by definition. For a positive integer $N$ and a sequence of positive integers $\{a_i\}_{1 \leq i \leq k}$ such that $\sum_{i=1}^{k} a_i \leq N$, we define

$$\binom{N}{\{a_i\}_{1 \leq i \leq k}} := \frac{N!}{a_1! \cdots a_k!(N-a_1-\cdots-a_k)!}.$$ 

For sequences of real numbers $a_n$ and $b_n$, defined for all sufficiently large values of $n$, we write $a_n = O(b_n)$ if, for some constant $C \geq 0$, we have $|a_n| \leq C|b_n|$ for $n$ large enough. We write $a_n = o(b_n)$ if $a_n/b_n \to 0$ as $n \to \infty$. We denote by $I[A]$ the indicator of the event $A$. For a probability distribution $P$, $X \sim P$ denotes that the random variable $X$ has law $P$. Throughout the paper logarithms are to the natural base.

2 Problem Statement

Let $\Xi$ and $\Theta$ be finite sets. A marked graph with edge mark set $\Xi$ and vertex mark set $\Theta$ is a graph where each edge carries a mark in $\Xi$ and each vertex carries a mark in $\Theta$. All graphs encountered in this paper are assumed to be simple, i.e. without multiple edges or self loops, unless otherwise stated. Also, we assume that all edge and vertex mark sets are finite. For two vertices $v$ and $w$ in a graph $G$, $v \sim_G w$ denotes that $v$ and $w$ are adjacent in $G$. We denote the set of vertices in $G$ by $V(G)$. A finite
sequence of nonnegative integers \((d(1), \ldots, d(n))\) is said to be graphic if there is a simple graph on \(n\) vertices with vertex \(i\) having degree \(d(i)\) for \(1 \leq i \leq n\). A simple characterization of graphic sequences is provided by the well-known theorem of Erdös and Gallai [Cho86, EG60].

Let \(G\) be a marked graph on a finite vertex set with edges and vertices carrying marks in the sets \(\Xi\) and \(\Theta\), respectively. We denote the edge mark count vector of \(G\) by \(\vec{m}_G = \{m_G(x)\}_{x \in \Xi}\), where \(m_G(x)\) is the number of edges in \(G\) carrying mark \(x\). We denote the vertex mark count vector of \(G\) by \(\vec{d}_G = \{d_G(\theta)\}_{\theta \in \Theta}\), where \(d_G(\theta)\) denotes the number of vertices in \(G\) carrying mark \(\theta\). Additionally, for a graph \(G\) on the vertex set \([n]\), we denote the degree sequence of \(G\) by \(\vec{d}_G = \{d_G(1), \ldots, d_G(n)\}\), where \(d_G(i)\) denotes the degree of vertex \(i\). For a degree sequence \(\vec{d} = (d(1), \ldots, d(n))\) and a nonnegative integer \(k\), we define
\[
c_k(\vec{d}) := |\{1 \leq i \leq n : d(i) = k\}|.
\] (1)

Also, for two degree sequences \(\vec{d} = (d(1), \ldots, d(n))\) and \(\vec{d}' = (d'(1), \ldots, d'(n))\), and two nonnegative integers \(k\) and \(l\), we define
\[
c_{k,l}(\vec{d}, \vec{d}') := |\{1 \leq i \leq n : d(i) = k, d'(i) = l\}|.
\] (2)

Given a degree sequence \(\vec{d} = (d(1), \ldots, d(n))\), we let \(G^{(n)}_{\vec{d}}\) denote the set of simple unmarked graphs \(G\) on the vertex set \([n]\) such that \(d_G(i) = d(i)\) for \(1 \leq i \leq n\).

When discussing distributed compression of graphical data with two sources, we assume that \(\Xi_1\) and \(\Xi_2\) are two fixed and finite sets of edge marks and \(\Theta_1\) and \(\Theta_2\) are two fixed and finite sets of vertex marks. For \(i \in \{1, 2\}\) and \(n \in \mathbb{N}\), let \(G^{(n)}_1\) denote the set of marked graphs on the vertex set \([n]\) with edge and vertex mark sets \(\Xi_i\) and \(\Theta_i\) respectively. For two graphs \(G_1 \in G^{(n)}_1\) and \(G_2 \in G^{(n)}_2\), \(G_1 \oplus G_2\) denotes the superposition of \(G_1\) and \(G_2\) which is a marked graph defined as follows: a vertex \(1 \leq v \leq n\) in \(G_1 \oplus G_2\) carries the mark \((\theta_1, \theta_2)\) where \(\theta_i\) is the mark of \(v\) in \(G_i\). Furthermore, we place an edge in \(G_1 \oplus G_2\) between vertices \(v\) and \(w\) if there is an edge between them in at least one of \(G_1\) or \(G_2\), and mark this edge \((x_1, x_2)\), where, for \(1 \leq i \leq 2\), \(x_i\) is the mark of the edge \((v, w)\) in \(G_i\) if it exists and \(x_i\) otherwise. Here \(x_1\) and \(x_2\) are auxiliary marks not present in \(\Xi_1 \cup \Xi_2\). Note that \(G_1 \oplus G_2\) is a marked graph with edge and vertex mark sets \(\Xi_{1,2} := (\Xi_1 \cup \{x_1\}) \times (\Xi_2 \cup \{x_2\}) \setminus \{(x_1, x_2)\}\) and \(\Theta_{1,2} := \Theta_1 \times \Theta_2\), respectively. We use the terminology jointly marked graph to refer to a marked graph with edge and vertex mark sets \(\Xi_{1,2}\) and \(\Theta_{1,2}\) respectively. With this, let \(G^{(n)}_{1,2}\) denote the set of jointly marked graphs on the vertex set \([n]\). Moreover, for \(i \in \{1, 2\}\), we say that a graph is in the \(i\)-th domain if its edge and vertex marks come from \(\Xi_i\) and \(\Theta_i\) respectively. For a jointly marked graph \(G_{1,2}\) and \(1 \leq i \leq 2\), the \(i\)-th marginal of \(G_{1,2}\), denoted by \(G_i\), is the marked graph in the \(i\)-th domain obtained by projecting all vertex and edge marks onto \(\Xi_i\) and \(\Theta_i\), respectively, followed by removing edges with mark \(x_i\). Note that any jointly marked graph \(G_{1,2}\) is uniquely determined by its marginals \(G_1\) and \(G_2\), because \(G_{1,2} = G_1 \oplus G_2\). Given an edge mark count vector \(\vec{m} = \{m(x)\}_{x \in \Xi_{1,2}}\), for \(x_1 \in \Xi_1 \cup \{x_1\}\) and \(x_2 \in \Xi_2 \cup \{x_2\}\), with an abuse of notation we define
\[
m(x_1) := \sum_{(x_1',x_2') \in \Xi_{1,2}} m((x_1', x_2')) = \sum_{(x_1',x_2') \in \Xi_{1,2}} m((x_1', x_2')).
\] (3)

Likewise, given a vertex mark count vector \(\vec{u} = \{u(\theta)\}_{\theta \in \Theta_{1,2}}\), we define, for \(\theta_1 \in \Theta_1\) and \(\theta_2 \in \Theta_2\),
\[
u(\theta_1) := \sum_{\theta_2' \in \Theta_2} u(\theta_1, \theta_2'), \quad \nu(\theta_2) := \sum_{\theta_1' \in \Theta_1} u(\theta_1', \theta_2).
\] (4)

Assume that we have a sequence of random marked graphs \(G^{(n)}_{1,2} \in G^{(n)}_{1,2}\), defined for all \(n\) sufficiently large, drawn for each \(n\) according to some ensemble distribution on \(G^{(n)}_{1,2}\). Additionally, assume that
there are two encoders who want to compress realizations of such jointly marked graphs in a distributed fashion. Namely, the $i$-th encoder, $1 \leq i \leq 2$, has only access to the $i$-th marginal $G_i^{(n)}$. We assume that the distribution of $G_{1,2}^{(n)}$ is known.

**Definition 1.** A sequence of $(n, L_1^{(n)}, L_2^{(n)})$ codes is a sequence of triples $(f_1^{(n)}, f_2^{(n)}, g^{(n)})$, defined for all sufficiently large $n$, such that

$$f_i^{(n)} : G_i^{(n)} \rightarrow [L_i^{(n)}], \quad i \in \{1, 2\},$$

and

$$g^{(n)} : [L_1^{(n)}] \times [L_2^{(n)}] \rightarrow G_{1,2}^{(n)}.$$

The probability of error for this code corresponding to the ensemble of $G_{1,2}^{(n)}$, which is denoted by $P_e^{(n)}$, is defined as

$$P_e^{(n)} := \mathbb{P} \left( g^{(n)}(f_1^{(n)}(G_1^{(n)}), f_2^{(n)}(G_2^{(n)})) \neq G_{1,2}^{(n)} \right).$$

Now we define our achievability criterion.

**Definition 2.** A rate tuple $(\alpha_1, R_1, \alpha_2, R_2) \in \mathbb{R}^4$ is said to be achievable for distributed compression of the sequence of random graphs $G_{1,2}^{(n)} \in G_{1,2}^{(n)}$ if there is a sequence of $(n, L_1^{(n)}, L_2^{(n)})$ codes such that

$$\limsup_{n \rightarrow \infty} \frac{\log L_i^{(n)} - (\alpha_i n \log n + R_i n)}{n} \leq 0, \quad i \in \{1, 2\},$$

and also $P_e^{(n)} \rightarrow 0$. The rate region $\mathcal{R} \subset \mathbb{R}^4$ is defined as follows: for fixed $\alpha_1$ and $\alpha_2$, if there are sequences $R_1^{(m)}$ and $R_2^{(m)}$ with limit points $R_1$ and $R_2$ in $\mathbb{R}$, respectively, such that for each $m$ the rate tuple $(\alpha_1, R_1^{(m)}, \alpha_2, R_2^{(m)})$ is achievable, then we include $(\alpha_1, R_1, \alpha_2, R_2)$ in the set $\mathcal{R}$.

In this paper, we characterize the above rate region for the following two sequences of ensembles:

**A sequence of Erdős–Rényi ensembles:** Assume that nonnegative real numbers $\vec{p} = \{p_x\}_{x \in \Xi_{1,2}}$ together with a probability distribution $\vec{q} = \{q_{\theta}\}_{\theta \in \Theta_{1,2}}$ are given such that, for all $x_1 \in \Xi_1$ and $x_2 \in \Xi_2$, we have

$$\sum_{(x'_1, x'_2) \in \Xi_{1,2}} p(x'_1, x'_2) > 0 \quad \text{and} \quad \sum_{x'_1 = x_1} p(x'_1, x_2) > 0,$$

and, for all $(\theta_1, \theta_2) \in \Theta_{1,2}$, we have

$$\sum_{\theta'_2 \in \Theta_2} q(\theta_1, \theta'_2) > 0 \quad \text{and} \quad \sum_{\theta'_1 \in \Theta_1} q(\theta'_1, \theta_2) > 0.$$

For $n \in \mathbb{N}$ large enough, we define the probability distribution $G(n; \vec{p}, \vec{q})$ on $G_{1,2}^{(n)}$ as follows: for each pair of vertices $1 \leq i < j \leq n$, the edge $(i, j)$ is present in the graph and has mark $x \in \Xi_{1,2}$ with probability $p_x / n$, and is not present with probability $1 - \sum_{x \in \Xi_{1,2}} p_x / n$. Furthermore, each vertex in the graph is given a mark $\theta \in \Theta_{1,2}$ with probability $q_\theta$. The choice of edge and vertex marks is done independently.

The conditions in (7) and the conditions for $x_i \in \Xi_i$, $i = 1, 2$, in (6) are required only to ensure that the sets of vertex marks and edge marks are chosen to be as small as possible, and these conditions could be relaxed if desired.
A sequence of configuration model ensembles: Fix $\Delta \in \mathbb{N}$. Suppose that a probability distribution $\vec{r} = \{r_k\}_{k=0}^\Delta$ supported on the set $\{0, \ldots, \Delta\}$ is given, such that $r_0 < 1$. Moreover, assume that probability distributions $\vec{\gamma} = \{\gamma_x\}_{x \in \Xi_{1,2}}$ and $\vec{q} = \{q_\theta\}_{\theta \in \Theta_{1,2}}$ on the sets $\Xi_{1,2}$ and $\Theta_{1,2}$, respectively, are given. We assume that, for all $x_1 \in \Xi_1 \cup \{\circ_1\}$ and $x_2 \in \Xi_2 \cup \{\circ_2\}$, we have
\begin{equation}
\sum_{(x'_1, x'_2) \in \Xi_{1,2}} \gamma(x'_1, x'_2) > 0 \quad \text{and} \quad \sum_{(x'_1, x'_2) \in \Xi_{1,2}, \ x'_1 = x_1} \gamma(x'_1, x'_2) > 0,
\end{equation}
and, for all $(\theta_1, \theta_2) \in \Theta_{1,2}$, we have
\begin{equation}
\sum_{\theta'_1 \in \Theta_1} q(\theta_1, \theta'_1) > 0 \quad \text{and} \quad \sum_{\theta'_1 \in \Theta_1} q(\theta'_1, \theta_2) > 0.
\end{equation}
Furthermore, for each $n$, the degree sequence $\vec{d}^{(n)} = \{d^{(n)}(1), \ldots, d^{(n)}(n)\}$ is given such that, for all $1 \leq i \leq n$, we have $d^{(n)}(i) \leq \Delta$ and also $\sum_{i=1}^n d^{(n)}(i)$ is even. Let $m_n := (\sum_{i=1}^n d^{(n)}(i))/2$. Additionally, if, for $0 \leq k \leq \Delta$, $c_k(\vec{d}^{(n)})$ denotes the number of $1 \leq i \leq n$ such that $d^{(n)}(i) = k$, we assume that, for some constant $K > 0$, we have
\begin{equation}
\sum_{k=0}^\Delta |c_k(\vec{d}^{(n)}) - n r_k| \leq K n^{1/2}.
\end{equation}
Now, for fixed $\vec{r}$, $\vec{\gamma}$ and $\vec{q}$ as above, and a sequence $\vec{d}^{(n)}$ satisfying (10), we define the law $\vec{G}(n; \vec{d}^{(n)}, \vec{\gamma}, \vec{q}, \vec{r})$ on $\mathcal{G}_1(n)$, for $n \in \mathbb{N}$ large enough, as follows. First, we pick an unmarked graph on the vertex set $[n]$ uniformly at random among the set of graphs $\mathcal{G}$ with maximum degree $\Delta$ such that for each $0 \leq k \leq \Delta$, $c_k(\vec{d}^{(n)}) = c_k(\vec{d}^{(n)})$. Then, we assign i.i.d. marks with law $\vec{\gamma}$ on the edges and i.i.d. marks with law $\vec{q}$ on the vertices.

The conditions in (9) and the conditions for $x_i \in \Xi_i$, $i = 1, 2$, in (8) are required only to ensure that the sets of vertex marks and edge marks are chosen to be as small as possible, and these conditions could be relaxed if desired. However, the conditions in (8) for $x_i = \circ_i$, $i = 1, 2$, are essential, as will be pointed out at the appropriate point in the proofs, since they ensure that neither of the two underlying unmarked graphs is a subgraph of the other.

As we will discuss in Section 3 below, the sequence of Erdős–Rényi ensembles defined above converges in the local weak sense to a marked Poisson Galton Watson tree. Moreover, the sequence of configuration model ensembles converges in the same sense to a marked Galton Watson process with degree distribution $\vec{r}$. In Section 5, we will characterize the achievable rate regions for lossless distributed compression of graphical data modeled as coming from one of the two sequences of ensembles above in terms of these limiting objects for the above two sequences of ensembles respectively. The formulation of this result will be in terms of a measure of entropy, namely the marked BC entropy, discussed in Section 4 below.

Remark 1. It should be pointed out that a rate region in the sense of Definition 2 need not be a topologically closed set, in contrast to what one is used to in the discussion of the Slepian-Wolf region in the traditional case. Further, while $\alpha_1$ and $\alpha_2$ can be restricted to being nonnegative, $R_1$ and $R_2$ should be thought of as real numbers. Indeed, the rate regions for the two sequences of ensembles considered in this paper, which are characterized in Theorem 3, are not topologically closed sets. The correct way to think of such a rate region is in terms of the subsets of $(R_1, R_2) \in \mathbb{R}_2$, parametrized by $(\alpha_1, \alpha_2) \in \mathbb{R}_2$.

\[1\]The fact that each degree is bounded by $\Delta$, $r_0 < 1$ and the sum of degrees is even implies that $\vec{d}^{(n)}$ is a graphic sequence for $n \in \mathbb{N}$ large enough. This is, for instance, a consequence of Theorem 4.5 in [BC15].
for which \((\alpha_1, R_1, \alpha_2, R_2)\) lies in the rate region, and each such subset is topologically closed as a subset of \(\mathbb{R}^2\). Further, for any \((\alpha_1, R_1, \alpha_2, R_2)\) in the rate region, if \(\alpha'_1 > \alpha_1\) then \((\alpha'_1, R'_1, \alpha_2, R_2)\) lies in the rate region for all \(R'_1 \in \mathbb{R}\), and a similar statement holds if one replaces the index 1 by the index 2.

3 The Framework of Local Weak Convergence

In this section, we discuss the framework of local weak convergence mainly in the context of the Erdős–Rényi and configuration model ensembles discussed in Section 2. For a general discussion, the reader is referred to [BS01, AS04, AL07].

Let \(\Xi\) and \(\Theta\) be fixed finite sets. A rooted marked graph is a marked graph \(G\) with edge and vertex mark sets \(\Xi\) and \(\Theta\) respectively, together with a distinguished vertex \(o\). We denote such a rooted marked graph by \((G, o)\). For a rooted marked graph \((G, o)\) and a nonnegative integer \(h \geq 0\), \((G, o)_h\) denotes the \(h\) neighborhood of \(o\), i.e. the subgraph consisting of vertices with distance no more than \(h\) from \(o\). Note that \((G, o)_h\) is connected, by definition. Two rooted marked graphs \((G_1, o_1)\) and \((G_2, o_2)\) are said to be isomorphic if there is a vertex bijection between the connected components of the roots in the two graphs that maps \(o_1\) to \(o_2\), preserves adjacencies, and also preserves edge and vertex marks. With this, we denote the isomorphism class corresponding to a rooted marked graph \((G, o)\) by \([G, o]\). We use \([G, o]\) as a shorthand for \([[(G, o)_h]\).

Let \(\mathcal{G}_*(\Xi, \Theta)\) denote the set of isomorphism classes \([G, o]\) of rooted marked graphs on a countable vertex set with edge and vertex marks coming from the sets \(\Xi\) and \(\Theta\), respectively. It can be shown that \(\mathcal{G}_*(\Xi, \Theta)\) can be metrized as a Polish space, i.e. a complete separable metric space [AL07]. In order to do this, we employ the metric on \(\mathcal{G}_*(\Xi, \Theta)\) denoted by \(d_*\), defined as follows: given \([G, o]\) and \([G', o']\) in \(\mathcal{G}_*(\Xi, \Theta)\), let \(\hat{h}\) be the supremum over all nonnegative integers \(h \geq 0\) such that \((G, o)_h \equiv (G', o')_h\), where \((G, o)\) and \((G', o')\) are arbitrary members in the isomorphism classes \([G, o]\) and \([G', o']\) respectively. If there is no such \(h\) (which can only happen if the mark of \(o\) and \(o'\) in \(G\) and \(G'\), respectively, are not the same), we define \(\hat{h} = 0\). With this, \(d_*([G, o], [G', o'])\) is defined to be \(1/(1 + \hat{h})\). One can check that \(d_*\) is a metric; in particular, it satisfies the triangle inequality. Let \(\mathcal{T}_*(\Xi, \Theta)\) denote the subset of \(\mathcal{G}_*(\Xi, \Theta)\) comprised of the isomorphism classes \([G, o]\) arising from some \((G, o)\) where the graph underlying \(G\) is a tree.

We write \(\mathcal{P}(\mathcal{G}_*(\Xi, \Theta))\) for the set of probability distributions on \(\mathcal{G}_*(\Xi, \Theta)\) when it is viewed as a complete separable metric space with its Borel \(\sigma\)-algebra. Given \(\mu \in \mathcal{P}(\mathcal{G}_*(\Xi, \Theta))\), let \(\deg_\mu\) denote the expected degree at the root in \(\mu\). For \(x \in \Xi\) let \(\deg_x(\mu)\) denote the expected number of edges in \(\mu\) connected to the root which carry mark \(x\), and define \(\deg(\mu) := \{\deg_x(\mu)\}_{x \in \Xi}\). For \(\theta \in \Theta\), let \(\Pi_\theta(\mu)\) denote the probability that the mark at the root in \(\mu\) is \(\theta\), and let \(\Pi(\mu) := \{\Pi_\theta(\mu)\}_{\theta \in \Theta}\).

For a finite marked graph \(G\) and a vertex \(v\) in \(G\), let \(G(v)\) denote the connected component of \(v\). With this, if \(v\) is a vertex chosen uniformly at random in \(G\), we define \(U(G)\) be the law of \([G(v), v]\), which is a probability distribution on \(\mathcal{G}_*(\Xi, \Theta)\). If \(G^{(n)}\) denotes the set of marked graphs on the vertex set \([n]\) with edge and vertex mark sets \(\Xi\) and \(\Theta\) respectively, then a sequence of graphs \(G^{(n)} \in \mathcal{G}^{(n)}\) is said to converge in the local weak sense if the sequence of probability distributions \(U(G^{(n)})\) converges weakly in the usual sense [Bil13] as probability distributions on \(\mathcal{G}_*(\Xi, \Theta)\). We now describe what this notion means in more detail in the context of the two sequences of ensembles that are studied in this paper.

Let \(G_{1,2}^{(n)}\) be a random jointly marked graph with law \(G(n; \bar{p}, \bar{q})\) and let \(v_n\) be a vertex chosen uniformly at random in the set \([n]\). A simple Poisson approximation implies that \(D_2(v_n)\), the number of edges adjacent to \(v_n\) with mark \(x \in \Xi_{1,2}\), converges in distribution to a Poisson random variable with

\(^2\)As all elements in an isomorphism class are isomorphic, the definition is invariant under the choice of the representatives.
the argument above, one can see that, almost surely, 
\( \mu \) as a certain stationarity condition at the limit, called 
root is chosen with distribution \( \vec{q} \)
random tree constructed as follows. First, we generate the degree of the root
choose its vertex mark according to the distribution \( \vec{q} \)
with depth
similar argument can be repeated for any other vertex in the neighborhood of
let
\( G \)
for each offspring, i.e. vertex
\( x \)
goes to infinity. Moreover,
\( \Theta \) be the set of isomorphism classes \( [G, o, v] \) where \( G \) is a marked connected graph with
mean \( p_\ast \), as \( n \) goes to infinity. Moreover, \( \{ D_x(v_n) \}_{x \in \Xi_{1,2}} \) are asymptotically mutually independent. A similar argument can be repeated for any other vertex in the neighborhood of \( v_n \). Also, it can be shown that the probability of having cycles of any fixed length converges to zero. In fact, the isomorphism class of \( (G^{(n)}_{1,2}, v_n)_h \) converges in distribution to that of a rooted marked Poisson Galton Watson tree with depth \( h \).

More precisely, let \( (T^{ER}_{1,2}_i, o) \) be a rooted jointly marked tree defined as follows. First, the mark of
\( \mu^{ER}_{1,2} \) is the local weak limit
of the sequence \( G^{(n)}_{1,2} \), where the term
“local” is meant to indicate that we require the convergence in distribution of the isomorphism class
of each fixed depth neighborhood of a typical vertex (i.e. a vertex chosen uniformly at random).

With this, we say that, almost surely, \( \mu^{ER}_{1,2} \) is the local weak limit of the sequence \( G^{(n)}_{1,2} \). Similarly to
the argument above, one can see that, almost surely, \( \mu^{ER}_{i} \) is the local weak limit of the sequence \( G^{(n)}_{i} \).

A similar picture also holds for the configuration model. Let \( (T^{CM}_{1,2}, o) \) be a rooted jointly marked random tree constructed as follows. First, we generate the degree of the root \( o \) with law \( \bar{r} \). Then, for each offspring \( w \) of \( o \), we independently generate the offspring count of \( w \) with law \( r^\prime = \{ r^\prime_k \}_{k=0}^{\Delta-1} \)
defined as
\[
r^\prime_k = \frac{(k+1)r_{k+1}}{\mathbb{E}[X]}, \quad 0 \leq k \leq \Delta - 1,
\]
where \( X \) has law \( \bar{r} \). We continue this process recursively, i.e. for each vertex other than the root,
we independently generate its offspring count with law \( r^\prime \). The distribution \( r^\prime \) is called the size-biased
distribution, and takes into account the fact that each vertex other than the root has an extra edge
by virtue of its being defined via an edge to an earlier defined vertex, and hence its degree should be
biased in order to get the correct degree distribution \( \bar{r} \). Then, for each vertex and edge existing in
the graph \( T^{CM}_{1,2} \), we generate marks independently with laws \( \bar{q} \) and \( \bar{r} \), respectively. Let \( \mu^{CM}_{1,2} \) be the
law of \( [T^{CM}_{1,2}, o] \). Moreover, for \( 1 \leq i \leq 2 \), let \( \mu^{CM}_i \) be the law of \( [T^{CM}_{i}, o] \). It can be shown that if
\( G^{(n)}_{1,2} \) has law \( G(n; \vec{d}^{(n)}, \vec{q}, \vec{r}) \), with these random graphs being constructed independently on a joint
probability space, then, almost surely, \( \mu^{CM}_{1,2} \) is the local weak limit of \( G^{(n)}_{1,2} \), and \( \mu^{CM}_i \) is the local weak
limit of \( G^{(n)}_{i} \), for \( 1 \leq i \leq 2 \). \( \mu^{CM}_i \) depends on the choice of the underlying parameters \( (\vec{r}, \vec{q}, \vec{r}) \), but we
suppress this from the notation, for readability.

A probability distribution on \( \mathcal{G}_*(\Xi, \Theta) \) is called sofic if it is the local weak limit of a sequence of finite
simple marked graphs. Not all probability distributions on \( \mathcal{G}_*(\Xi, \Theta) \) are sofic. In fact, the condition
that all vertices have the same chance of being chosen as the root for a finite graph manifests itself
as a certain stationarity condition at the limit, called unimodularity [AL07]. To define unimodularity,
let \( \mathcal{G}_*(\Xi, \Theta) \) be the set of isomorphism classes \( [G, o, v] \) where \( G \) is a marked connected graph with
two distinguished vertices \( o \) and \( v \) in \( V(G) \) (ordered, but not necessarily distinct). Here, isomorphism is defined by an adjacency preserving vertex bijection which preserves vertex and edge marks, and also maps the two distinguished vertices of one object to the respective ones of the other. A measure \( \mu \in \mathcal{P}(\mathcal{G}_* (\Xi, \Theta)) \) is said to be unimodular if, for all measurable functions \( f : \mathcal{G}_* (\Xi, \Theta) \rightarrow \mathbb{R}_+ \), we have

\[
\int \sum_{v \in V(G)} f([G, o, v]) d\mu([G, o]) = \int \sum_{v \in V(G)} f([G, v, o]) d\mu([G, o]).
\]  

(11)

Here the summation is taken over all vertices \( v \) which are in the same connected component of \( G \) as \( o \).

It can be seen that it suffices to check the above condition for a function \( f \) such that \( f([G, o, v]) = 0 \) unless \( v \sim_G o \). This is called involutive invariance [AL07]. Let \( \mathcal{P}_u(\mathcal{G}_* (\Xi, \Theta)) \) denote the set of unimodular probability measures on \( \mathcal{G}_* (\Xi, \Theta) \). Also, since \( \mathcal{T}_*(\Xi, \Theta) \subset \mathcal{G}_*(\Xi, \Theta) \), we can define the set of unimodular probability measures on \( \mathcal{T}_*(\Xi, \Theta) \) and denote it by \( \mathcal{P}_u(\mathcal{T}_*(\Xi, \Theta)) \). A sofic probability measure is unimodular. Whether the other direction also holds is unknown.

## 4 The BC Entropy

In this section, we discuss a notion of entropy for probability distributions on the space \( \mathcal{G}_*(\Xi, \Theta) \) of isomorphism classes of rooted marked graphs with edge and vertex mark sets \( \Xi \) and \( \Theta \) respectively. This is a generalization to the marked framework of the notion of entropy introduced by Bordenave and Caputo in [BC15], who considered the unmarked case. This generalization is due to us, and the reader is referred to [DA19] for more details. To distinguish it from the Shannon entropy, we call this notion of entropy the marked BC entropy. In fact, the discussion in [DA19] is for a more general setting in which each edge is allowed to carry two directional marks, one towards each of its endpoints. The setup in this paper, where an edge is allowed to carry only one mark, can be considered as a special case where the two directional marks have the same value. In the following, we give the definition of the marked BC entropy from [DA19], restricted to the setting in this paper where each edge is allowed to carry only one mark.

The following general lemma, whose proof is straightforward using Stirling’s approximation, is often used in this paper. See Appendix A for a proof.

**Lemma 1.** Let \( k \in \mathbb{N} \). Let \( a_n \) and \( b_1^n, \ldots, b_k^n \) be sequences of integers, defined for all sufficiently large \( n \).

1. Assume that \( a_n = \sum_{i=1}^k b_i^n \) for all \( n \). If \( a_n/n \rightarrow 0 \) and, for each \( 1 \leq i \leq k \), \( b_i^n/n \rightarrow b_i \geq 0 \), we have

\[
\lim_{n \rightarrow \infty} \frac{1}{n} \log \left( \frac{a_n}{\left( b_i^n \right)_{1 \leq i \leq k}} \right) = aH \left( \left\{ \frac{b_i}{a} \right\}_{1 \leq i \leq k} \right).
\]

2. Assume that \( a_n \geq \sum_{i=1}^k b_i^n \) for all \( n \). If \( a_n/n \rightarrow 0 \) and \( b_i^n/n \rightarrow b_i \geq 0 \), we have

\[
\lim_{n \rightarrow \infty} \frac{\log \left( \frac{a_n}{\left( b_i^n \right)_{1 \leq i \leq k}} \right) - \left( \sum_{i=1}^k b_i^n \right) \log n}{n} = \sum_{i=1}^k s(2b_i),
\]

where \( s(x) \) is defined to be \( \frac{x}{2} - \frac{x}{2} \log x \) for \( x > 0 \) and \( 0 \) if \( x = 0 \).
Throughout the discussion in this section, up to the definition of BC entropy in Definition 5, we assume that the edge and vertex mark sets, Ξ and Θ respectively, are fixed and finite. For edge and vertex mark count vectors \( \mathbf{m} = \{m(x)\}_{x \in \Xi} \) and \( \mathbf{u} = \{u(\theta)\}_{\theta \in \Theta} \), respectively, define \( \|\mathbf{m}\|_1 := \sum_{x \in \Xi} m(x) \) and \( \|\mathbf{u}\|_1 := \sum_{\theta \in \Theta} u(\theta) \).

Given \( n \in \mathbb{N} \), together with edge and vertex mark count vectors \( \mathbf{m} = \{m(x)\}_{x \in \Xi} \) and \( \mathbf{u} = \{u(\theta)\}_{\theta \in \Theta} \) respectively, let \( G_{\mathbf{m}, \mathbf{u}}^{(n)} \) denote the set of marked graphs \( G \) on the vertex set \( \{1, \ldots, n\} \) such that \( \mathbf{m}_G = \mathbf{m} \) and \( \mathbf{u}_G = \mathbf{u} \). Note that \( G_{\mathbf{m}, \mathbf{u}}^{(n)} \) is empty unless \( \|\mathbf{u}\|_1 = n \) and \( \|\mathbf{m}\|_1 \leq \binom{n}{2} \).

We define an average degree vector to be a vector of nonnegative reals \( \overline{\mathbf{d}} := \{d_x\}_{x \in \Xi} \) such that \( \sum_{x \in \Xi} d_x > 0 \).

**Definition 3.** Given an average degree vector \( \overline{\mathbf{d}} \) and a probability distribution \( Q = \{q_\theta\}_{\theta \in \Theta} \), we say that a sequence \( (\mathbf{m}^{(n)}, \mathbf{u}^{(n)}) \), comprised of edge mark count vectors and vertex mark count vectors \( \mathbf{m}^{(n)} \) and \( \mathbf{u}^{(n)} \) respectively, is adapted to \( (\overline{\mathbf{d}}, Q) \), if the following conditions hold:

1. For each \( n \), we have \( \|\mathbf{m}^{(n)}\|_1 \leq \binom{n}{2} \) and \( \|\mathbf{u}^{(n)}\|_1 = n \);
2. For \( x \in \Xi \), we have \( m^{(n)}(x)/n \to d_x/2 \);
3. For \( \theta \in \Theta \), we have \( u^{(n)}(\theta)/n \to q_\theta \);
4. For \( x \in \Xi \), \( d_x = 0 \) implies \( m^{(n)}(x) = 0 \) for all \( n \);
5. For \( \theta \in \Theta \), \( q_\theta = 0 \) implies \( u^{(n)}(\theta) = 0 \) for all \( n \).

If \( \mathbf{m}^{(n)} \) and \( \mathbf{u}^{(n)} \) are sequences such that \( (\mathbf{m}^{(n)}, \mathbf{u}^{(n)}) \) is adapted to \( (\overline{\mathbf{d}}, Q) \) then one can show as a simple consequence of Lemma 1 that

\[
\log |G_{\mathbf{m}^{(n)}, \mathbf{u}^{(n)}}^{(n)}| = \|\mathbf{m}^{(n)}\|_1 \log n + nH(Q) + n \sum_{x \in \Xi} s(d_x) + o(n). \tag{12}
\]

See Appendix B for a proof. To simplify the notation, we may write \( s(\overline{\mathbf{d}}) \) for \( \sum_{x \in \Xi} s(d_x) \).

To give the definition of the marked BC entropy, we first define the upper and the lower marked BC entropy.

**Definition 4.** Assume \( \mu \in \mathcal{P}(\mathcal{G}(\overline{\mathbf{d}}, \Xi, \Theta)) \) is given, with \( 0 < \deg(\mu) < \infty \). For \( \epsilon > 0 \), and edge and vertex mark count vectors \( \overline{\mathbf{m}} \) and \( \overline{\mathbf{u}} \) respectively, define

\[
G_{\overline{\mathbf{m}}, \overline{\mathbf{u}}}^{(\epsilon)}(\mu, \epsilon) := \{ G \in G_{\overline{\mathbf{m}}, \overline{\mathbf{u}}}^{(\epsilon)} : d_{LP}(U(G), \mu) < \epsilon \}.
\]

Here, \( d_{LP} \) denotes the Levy–Prokhorov distance [Bill13]. Fix an average degree vector \( \overline{\mathbf{d}} \) and a probability distribution \( Q = \{q_\theta\}_{\theta \in \Theta} \), and also fix sequences of edge and vertex mark count vectors \( \mathbf{m}^{(n)} \) and \( \mathbf{u}^{(n)} \) respectively such that \( (\mathbf{m}^{(n)}, \mathbf{u}^{(n)}) \) is adapted to \( (\overline{\mathbf{d}}, Q) \). With these, define

\[
\sum_{\overline{\mathbf{d}}, Q}^U(\mu, \epsilon)|(\mathbf{m}^{(n)}, \mathbf{u}^{(n)})| := \limsup_{n \to \infty} \frac{\log |G_{\overline{\mathbf{m}}^{(n)}, \overline{\mathbf{u}}^{(n)}}^{(n)}(\mu, \epsilon)| - \|\mathbf{m}^{(n)}\|_1 \log n}{n},
\]

which we call the \( \epsilon \)-upper marked BC entropy. Since this is increasing in \( \epsilon \), we can define the upper marked BC entropy as

\[
\sum_{\overline{\mathbf{d}}, Q}^U(\mu)|(\mathbf{m}^{(n)}, \mathbf{u}^{(n)})| := \lim_{\epsilon \downarrow 0} \sum_{\overline{\mathbf{d}}, Q}^U(\mu, \epsilon)|(\mathbf{m}^{(n)}, \mathbf{u}^{(n)})|.
\]
We may define the ε–lower marked BC entropy $\Sigma_{\vec{d},Q}(\mu, \epsilon)|_{(\vec{m}(n), \vec{\omega}(n))}$ similarly as

$$\Sigma_{\vec{d},Q}(\mu, \epsilon)|_{(\vec{m}(n), \vec{\omega}(n))} := \liminf_{n \to \infty} \frac{1}{n} \log \|G(n)_{\vec{m}(n),\vec{\omega}(n)}(\mu, \epsilon) - \|\vec{m}(n)\|_1 \log n.$$ 

Since this is increasing in $\epsilon$, we can define the lower marked BC entropy $\Sigma_{\vec{d},Q}(\mu)|_{(\vec{m}(n), \vec{\omega}(n))}$ as

$$\Sigma_{\vec{d},Q}(\mu)|_{(\vec{m}(n), \vec{\omega}(n))} := \lim_{\epsilon \downarrow 0} \Sigma_{\vec{d},Q}(\mu, \epsilon)|_{(\vec{m}(n), \vec{\omega}(n))}.$$ 

Now, we state the following properties of the upper and lower marked BC entropy, which will lead to the definition of the marked BC entropy. The reader is referred to [DA19] for a proof and more details.

**Theorem 1** (Theorem 1 in [DA19]). Let an average degree vector $\vec{d} = \{d_x\}_{x \in \Xi}$ and a probability distribution $Q = \{q_\theta\}_{\theta \in \Theta}$ be given. Suppose $\mu \in \mathcal{P}(\mathcal{G}_s(\Xi, \Theta))$ with $0 < \deg(\mu) < \infty$ satisfies any one of the following conditions:

1. $\mu$ is not unimodular;
2. $\mu$ is not supported on $T_s(\Xi, \Theta)$;
3. $\deg_{\vec{d},x}(\mu) \neq d_x$ for some $x \in \Xi$, or $\Pi_\theta(\mu) \neq q_\theta$ for some $\theta \in \Theta$.

Then, for any choice of the sequences $\vec{m}(n)$ and $\vec{\omega}(n)$ such that $(\vec{m}(n), \vec{\omega}(n))$ is adapted to $(\vec{d}, Q)$, we have $\Sigma_{\vec{d},Q}(\mu)|_{(\vec{m}(n), \vec{\omega}(n))} = -\infty$.

A consequence of Theorem 1 is that the only case of interest in the discussion of marked BC entropy is when $\mu \in \mathcal{P}_u(T_s(\Xi, \Theta))$, $\vec{d} = \vec{\deg}(\mu)$, $Q = \vec{\Pi}(\mu)$, and the sequences $\vec{m}(n)$ and $\vec{\omega}(n)$ are such that $(\vec{m}(n), \vec{\omega}(n))$ is adapted to $(\vec{\deg}(\mu), \vec{\Pi}(\mu))$. In particular, the only upper and lower marked BC entropies of interest are $\Sigma_{\vec{\deg}(\mu),\vec{\Pi}(\mu)}(\mu)|_{(\vec{m}(n), \vec{\omega}(n))}$ and $\Sigma_{\vec{\deg}(\mu),\vec{\Pi}(\mu)}(\mu)|_{(\vec{m}(n), \vec{\omega}(n))}$ respectively.

The following theorem establishes that the upper and lower marked BC entropies do not depend on the choice of the defining pair of sequences $(\vec{m}(n), \vec{\omega}(n))$. Further, this theorem establishes that the upper marked BC entropy is always equal to the lower marked BC entropy. The reader is referred to [DA19] for a proof and more details.

**Theorem 2** (Theorem 2 in [DA19]). Assume that an average degree vector $\vec{d} = \{d_x\}_{x \in \Xi}$ together with a probability distribution $Q = \{q_\theta\}_{\theta \in \Theta}$ are given. For any $\mu \in \mathcal{P}(\mathcal{G}_s(\Xi, \Theta))$ such that $0 < \deg(\mu) < \infty$, we have

1. The values of $\Sigma_{\vec{d},Q}(\mu)|_{(\vec{m}(n), \vec{\omega}(n))}$ and $\Sigma_{\vec{d},Q}(\mu)|_{(\vec{m}(n), \vec{\omega}(n))}$ are invariant under the specific choice of the sequences $\vec{m}(n)$ and $\vec{\omega}(n)$ such that $(\vec{m}(n), \vec{\omega}(n))$ is adapted to $(\vec{d}, Q)$. With this, we may simplify the notation and unambiguously write $\Sigma_{\vec{d},Q}(\mu)$ and $\Sigma_{\vec{d},Q}(\mu)$.
2. $\Sigma_{\vec{d},Q}(\mu) = \Sigma_{\vec{d},Q}(\mu)$. We may therefore unambiguously write $\Sigma_{\vec{d},Q}(\mu)$ for this common value, and call it the marked BC entropy of $\mu \in \mathcal{P}(\mathcal{G}_s(\Xi, \Theta))$ for the average degree vector $\vec{d}$ and a probability distribution $Q = \{q_\theta\}_{\theta \in \Theta}$. Moreover, $\Sigma_{\vec{d},Q}(\mu) \in [-\infty, s(\vec{d}) + H(Q)]$.

From Theorem 1 we conclude that unless $\vec{d} = \vec{\deg}(\mu)$, $Q = \vec{\Pi}(\mu)$, and $\mu$ is a unimodular measure on $T_s(\Xi, \Theta)$, we have $\Sigma_{\vec{d},Q}(\mu) = -\infty$. In view of this, for $\mu \in \mathcal{P}(\mathcal{G}_s(\Xi, \Theta))$ with $0 < \deg(\mu) < \infty$, we write $\Sigma(\mu)$ for $\Sigma_{\vec{d}(\mu),\vec{\Pi}(\mu)}(\mu)$. Likewise, we may write $\Sigma(\mu)$ and $\Sigma(\mu)$ for $\Sigma_{\vec{\deg}(\mu),\vec{\Pi}(\mu)}(\mu)$ and $\Sigma_{\vec{\deg}(\mu),\vec{\Pi}(\mu)}(\mu)$, respectively. These are both equal to $\Sigma(\mu)$ by part 2 of the theorem. Note that, unless $\mu \in \mathcal{P}_u(T_s(\Xi, \Theta))$, we have $\Sigma(\mu) = \Sigma(\mu) = \Sigma(\mu) = -\infty$.

We are now in a position to define the marked BC entropy.
We next connect the asymptotic behavior of the entropy of the ensembles defined in Section B14a that we have for details). straightforward, and is therefore omitted.

In Appendix H, we have provided the details of calculating the marked BC entropy for several examples. We next connect the asymptotic behavior of the entropy of the ensembles defined in Section 2 to the marked BC entropy of their local weak limits. We first consider a sequence of Erdős–Rényi ensembles. Let \( n \in \mathbb{N} \) be large enough, and assume that \( G(1) \) has law \( \mathcal{G}(n; \overline{p}, \overline{q}) \). Let \( d_{1,2}^{ER} := \text{deg}(\mu_{1,2}^{ER}) = \sum_{x \in \Xi_{1,2}} p_x \). For \( x_i \in \Xi_i \) and \( \theta_i \in \Theta_i \), \( 1 \leq i \leq 2 \), let

\[
\begin{align*}
p_{x_1} &:= \sum_{x_1' \in \Xi_1 \cup \{\varnothing_2\}} p(x_1, x_1'), \\
p_{x_2} &:= \sum_{x_1' \in \Xi_1 \cup \{\varnothing_1\}} p(x_1', x_2), \\
q_{\theta_1} &:= \sum_{\theta_1' \in \Theta_1} q(\theta_1, \theta_1'), \\
q_{\theta_2} &:= \sum_{\theta_2' \in \Theta_2} q(\theta_2, \theta_2').
\end{align*}
\]

For \( 1 \leq i \leq 2 \), let \( d_i^{ER} := \text{deg}(\mu_i^{ER}) = \sum_{x_i \in \Xi_i} p_{x_i} \). If \( Q = (Q_1, Q_2) \) has law \( \overline{q} \), it can be verified by using Lemma 1 in a manner similar to the proof of (12) in Appendix B that we have

\[
\begin{align*}
H(G_1^{(n)}) &= \frac{d_1^{ER}}{2} n \log n + n \left( H(Q) + \sum_{x_1 \in \Xi_1} s(p_{x_1}) \right) + o(n), \\
H(G_2^{(n)}) &= \frac{d_2^{ER}}{2} n \log n + n \left( H(Q_1) + \sum_{x_1 \in \Xi_1} s(p_{x_1}) \right) + o(n), \\
H(G_2^{(n)}) &= \frac{d_2^{ER}}{2} n \log n + n \left( H(Q_2) + \sum_{x_2 \in \Xi_2} s(p_{x_2}) \right) + o(n).
\end{align*}
\]

Using Theorem 3 in [DA19], it can be seen that the coefficients of \( n \) in equations (14a)–(14c) are \( \Sigma(\mu_{1,2}^{ER}) \), \( \Sigma(\mu_1^{ER}) \) and \( \Sigma(\mu_2^{ER}) \), respectively (see Appendix H for details).

Before discussing configuration model ensembles, we state two lemmas, which are used at several points. The proof of the following Lemma 2 straightforward, and is therefore omitted.

**Lemma 2.** Let \( \Delta \in \mathbb{N} \). Let \( Y \) be a random variable taking values in \( \{0, 1, \ldots, \Delta\} \), and let \( 0 \leq \epsilon \leq 1 \). Let \( \{V_i\}_{i \geq 1} \) be a sequence of i.i.d. Bernoulli random variables with \( \mathbb{P}(V_i = 1) = \epsilon \), and let \( Y_1 := \sum_{i=1}^{V_i} V_i \), where \( Y_1 = 0 \) when \( V_i = 0 \). Then, we have

\[
H(Y_1, Y - Y_1) = H(Y_1, Y) = H(Y) + \mathbb{E}[Y] H(V_1) - \mathbb{E} \left[ \log \frac{Y}{Y_1} \right].
\]

The proof of the following Lemma 3 is given in Appendix C.

**Lemma 3.** Let \( \Delta \in \mathbb{N} \). Let \( Y \) be a random variable taking values in \( \{0, 1, \ldots, \Delta\} \), such that \( d := \mathbb{E}[Y] > 0 \). For all \( n \in \mathbb{N} \) large enough, let \( \overline{a}^{(n)} := (a^{(n)}(1), \ldots, a^{(n)}(n)) \) be a degree sequence of length \( n \) with entries bounded by \( \Delta \) such that \( b_n := \sum_{i=1}^{n} a^{(n)}(i) \) is even and, for \( 0 \leq k \leq \Delta \), we have \( c_k(\overline{a}^{(n)})/n \to \mathbb{P}(Y = k) \). Then, we have

\[
\lim_{n \to \infty} \frac{\log |G^{(n)}_{\overline{a}^{(n)}}| - \frac{b_n}{2} \log n}{n} = -s(d) - \mathbb{E} \left[ \log Y \right],
\]

where we recall that \( G^{(n)}_{\overline{a}^{(n)}} \) denotes the set of simple unmarked graphs \( G \) on the vertex set \([n]\) such that \( d_G(i) = a^{(n)}(i) \) for \( 1 \leq i \leq n \).
Now, we are ready to state our main result, which is to characterize the rate region in Definition 2.

Consider now a sequence of configuration model ensembles. Namely, for all \( n \in \mathbb{N} \) large enough, let \( G_{1,2}^{(n)} \) be distributed according to \( \mathcal{G}(n; \bar{d}^{(n)}, \bar{\gamma}, \bar{q}, \bar{r}) \). Let \( X \) be a random variable with law \( \bar{r} \) and \( \Gamma^k = (\Gamma_1^k, \Gamma_2^k), 1 \leq k \leq \Delta \), an i.i.d. sequence distributed according to \( \bar{\gamma} \). With this, let

\[
X_1 := \sum_{k=1}^{X} \mathbb{1} [\Gamma_1^k \neq o_1], \quad X_2 := \sum_{k=1}^{X} \mathbb{1} [\Gamma_2^k \neq o_2],
\]

where \( X_1 = X_2 = 0 \) if \( X = 0 \). Then, if \( d_1^{CM} := \deg(\mu_{1,2}^{CM}) \) and, for \( 1 \leq i \leq 2 \), \( d_i^{CM} := \deg(\mu_i^{CM}) \), it can be seen that

\[
\begin{align*}
H(G_{1,2}^{(n)}) &= \frac{d_1^{CM}}{2} n \log n + n \left(-s(d_1^{CM}) + H(X) - \mathbb{E} \log X! \right) \\
&\quad + H(Q) + \frac{d_1^{CM}}{2} H(\Gamma) + o(n), \quad (16a) \\
H(G_{1}^{(n)}) &= \frac{d_1^{CM}}{2} n \log n + n \left(-s(d_1^{CM}) + H(X_1) - \mathbb{E} \log X_1! \right) \\
&\quad + H(Q_1) + \frac{d_1^{CM}}{2} H(\Gamma_1 | \Gamma_1 \neq o_1) + o(n), \quad (16b) \\
H(G_{2}^{(n)}) &= \frac{d_2^{CM}}{2} n \log n + n \left(-s(d_2^{CM}) + H(X_2) - \mathbb{E} \log X_2! \right) \\
&\quad + H(Q_2) + \frac{d_2^{CM}}{2} H(\Gamma_2 | \Gamma_2 \neq o_2) + o(n), \quad (16c)
\end{align*}
\]

where \( \Gamma \) is distributed according to \( \bar{\gamma} \). Also, using Theorem 3 in [DA19], it can be seen that the coefficients of \( n \) in equations (16a)–(16c) are \( \Sigma(\mu_{1,2}^{CM}), \Sigma(\mu_1^{CM}) \) and \( \Sigma(\mu_2^{CM}) \), respectively (see Appendix H for details). The proof of equations (16a)–(16c), which is given in Appendix D, and depends on both Lemma 2 and Lemma 3.

If \( \mu_{1,2} \) is any one of the two distributions \( \mu_{1,2}^{R} \) or \( \mu_{1,2}^{CM} \), and \( \mu_1 \) and \( \mu_2 \) are its marginals, we define the conditional marked BC entropies as \( \Sigma(\mu_2 | \mu_1) := \Sigma(\mu_{1,2}) - \Sigma(\mu_1) \) and \( \Sigma(\mu_1 | \mu_2) := \Sigma(\mu_{1,2}) - \Sigma(\mu_2) \).

## 5 Main Results

Now, we are ready to state our main result, which is to characterize the rate region in Definition 2 for a sequence of Erdős–Rényi ensembles and a sequence of configuration model ensembles. In the following, for pairs of reals \( (\alpha, R) \) and \( (\alpha', R') \), we write \( (\alpha, R) \succ (\alpha', R') \) if either \( \alpha > \alpha' \), or \( \alpha = \alpha' \) and \( R > R' \). We also write \( (\alpha, R) \succeq (\alpha', R') \) if either \( (\alpha, R) \succ (\alpha', R') \) or \( (\alpha, R) = (\alpha', R') \).

**Theorem 3.** Assume \( \mu_{1,2} \) is a member of either of the two families of distributions \( \mu_{1,2}^{R} \) (parametrized by \( \bar{p}, \bar{q} \)) or \( \mu_{1,2}^{CM} \) (parametrized by \( \bar{\gamma}, \bar{q}, \bar{r} \)) defined in Section 3. Then, if \( \mathcal{R} \) is the rate region for the sequence of ensembles corresponding to \( \mu_{1,2} \), as defined in Section 2, a rate tuple \( (\alpha_1, R_1, \alpha_2, R_2) \in \mathcal{R} \) if and only if

\[
(\alpha_1, R_1) \succeq \left( (d_1 - d_2)/2, \Sigma(\mu_1 | \mu_2) \right), \quad (17a)
\]
\[ (\alpha_2, R_2) \geq \left( (d_{1,2} - d_1)/2, \Sigma(\mu_2|\mu_1) \right), \] 
(17b)

\[ (\alpha_1 + \alpha_2, R_1 + R_2) \geq \left( (d_{1,2}/2, \Sigma(\mu_{1,2})) \right), \] 
(17c)

where \( d_{1,2} := \text{deg}(\mu_{1,2}), d_1 := \text{deg}(\mu_1) \) and \( d_2 := \text{deg}(\mu_2) \).

We prove the achievability for the Erdős–Rényi case and the configuration model case in Sections 5.1 and 5.2, respectively. Subsequently, we prove the converses for the two cases in Sections 5.3 and 5.4, respectively.

**Remark 3.** Although our achievability analysis shares some well-known concepts with the classical Slepian–Wolf, such as the random binning method, there are several factors that make the analysis for graphical data much more challenging compared to the classical results for time series. For one thing, as we saw in Section 4, our entropy analysis is up to the first two leading terms, one which scales like \( n \log n \) and the other which scales like \( n \). This is reflected in the statement of the above Theorem 3 as the appearance of two rate parameters \( \alpha \) and \( R \) for each source. On the other hand, the classical operational meaning of the conditional Shannon entropy does not easily extend to similar operational meanings for the conditional marked BC entropy. More precisely, in the classical setting of two i.i.d. sources \( X \) and \( Y \) with a joint distribution \( p_{X,Y} \), roughly speaking, any typical sequence \( (x_1, \ldots, x_n) \) has approximately the same number of conditional typical sequences \( (y_1, \ldots, y_n) \), and the number of such conditional typical sequences is asymptotically related to the conditional Shannon entropy \( H(Y|X) \). However, it turns out that a similar property does not necessarily hold in our setting for sparse graphical data. See Appendix I for details. This in part makes our analysis more complicated compared to the classical setting as we need to carefully control the number of conditional typical graphs. This requires separate treatment for the Erdős–Rényi and the configuration model ensembles, as is discussed in Sections 5.1 and 5.2 below, respectively.

**Remark 4.** Recall from Section 4 that the coefficient of \( n \) in the ensemble entropies of the Erdős–Rényi and the configuration model ensembles and their marginals are equal to the marked BC entropy of their corresponding local weak limits. This is a key reason why the rate region in Theorem 3 above is characterized in terms of the marked BC entropy. The reason why the ensemble entropies and the marked BC entropies match is that the Erdős–Rényi and the configuration model ensembles are almost uniform over the typical graphs with respect to their corresponding local weak limits. For the Erdős–Rényi case, it is well known that the Erdős–Rényi ensemble is close in distribution to a distribution on the set of graphs with a typical number of edges. For the configuration model case, we learn from the techniques used in [BC15] and [DA19] to prove the properties of the BC entropy that the configuration model ensemble covers the set of typical graphs roughly uniformly in an asymptotic sense. In other words, the fact that the ensemble entropies and the marked BC entropy match is not a coincidence.

As is the case for the classical Slepian–Wolf theorem, one can generalize the above result to more than two sources. The definition of the rate region as well as its characterization can be naturally extended to this case. In Section 5.5 below, we generalize the Erdős–Rényi and configuration model ensembles to more than two sources, define the corresponding Slepian-Wolf rate region, and characterize the rate region for each of these cases in Theorem 4. The proof structure is similar to that for the scenario with two sources, and is highlighted in Appendix G.

### 5.1 Proof of Achievability for the Erdős–Rényi case

Here we show that a rate tuple \((\alpha_1, R_1, \alpha_2, R_2)\) is achievable for the Erdős–Rényi ensemble if it satisfies the following

\[ (\alpha_1, R_1) \succ (d_1^{\text{ER}} - d_2^{\text{ER}})/2, \Sigma(\mu_1^{\text{ER}}|\mu_2^{\text{ER}})), \] 
(18a)
Note that if a rate tuple \((\alpha_1, R_1, \alpha_2, R_2)\) satisfies the weak inequalities (17a)–(17c) then, for any \(\epsilon > 0\), \((\alpha'_1, R'_1 + \epsilon, \alpha'_2, R'_2 + \epsilon)\) satisfies the strict inequalities (18a)–(18c). As we show below, this implies that \((\alpha'_1, R'_1 + \epsilon, \alpha'_2, R'_2 + \epsilon)\) is achievable. Hence, after sending \(\epsilon \to 0\), we get \((\alpha'_1, R'_1, \alpha'_2, R'_2) \in \mathcal{R}\).

We show that any \((\alpha_1, R_1, \alpha_2, R_2)\) satisfying (18a)–(18c) is achievable by employing a random binning method. More precisely, for \(i \in \{1, 2\}\), we set \(L_i^{(n)} = \lfloor \exp(\alpha_i n \log n + R_i n) \rfloor\) and for each \(G_i \in \mathcal{G}_i^{(n)}\), we assign \(f_i^{(n)}(G_i)\) uniformly at random in the set \([L_i^{(n)}]\) and independent of everything else.

To describe our decoding scheme, we first need to set up some notation. Let \(\mathcal{M}^{(n)}\) denote the set of edge count vectors \(\tilde{m} = \{m(x)\}_{x \in \Xi_{1,2}}\) such that

\[
\sum_{x \in \Xi_{1,2}} |m(x) - np_x/2| \leq n^{2/3}.
\]

Moreover, let \(\mathcal{U}^{(n)}\) denote the set of vertex mark count vectors \(\tilde{u} = \{u(\theta)\}_{\theta \in \Theta_{1,2}}\) such that

\[
\sum_{\theta \in \Theta_{1,2}} |u(\theta) - nq_\theta| \leq n^{2/3}.
\]

Furthermore, we define \(\mathcal{G}_{\tilde{p}, \tilde{q}}^{(n)}\) to be the set of graphs \(H_{1,2}^{(n)} \in \mathcal{G}_{1,2}^{(n)}\) such that \(\tilde{m} H_{1,2}^{(n)} \in \mathcal{M}^{(n)}\) and \(\tilde{u} H_{1,2}^{(n)} \in \mathcal{U}^{(n)}\). Upon receiving \((i, j) \in [L_1^{(n)}] \times [L_2^{(n)}]\), we form the set of graphs \(H_{1,2}^{(n)} \in \mathcal{G}_{\tilde{p}, \tilde{q}}^{(n)}\) such that \(f_i^{(n)}(H_{1,2}^{(n)}) = i\) and \(f_j^{(n)}(H_{1,2}^{(n)}) = j\), where \(H_{1,2}^{(n)}\) and \(H_{2,1}^{(n)}\) are the marginals of \(H_{1,2}^{(n)}\). If this set has only one element, we output this element as the decoded graph; otherwise, we report an error.

In what follows, assume that \(G_{1,2}^{(n)}\) is a random graph with law \(\mathcal{G}(n; \tilde{p}, \tilde{q})\). We consider the following four error events corresponding to the above scheme:

\[
\mathcal{E}_1^{(n)} := \{G_{1,2}^{(n)} \notin \mathcal{G}_{\tilde{p}, \tilde{q}}^{(n)}\},
\]

\[
\mathcal{E}_2^{(n)} := \{\exists H_{1,2}^{(n)} \in \mathcal{G}_{\tilde{p}, \tilde{q}}^{(n)}: H_{1,2}^{(n)} \neq G_{1,2}^{(n)}, H_{2,1}^{(n)} \neq G_{2,1}^{(n)}, f_i^{(n)}(H_{1,2}^{(n)}) = f_i^{(n)}(G_{1,2}^{(n)}), i \in \{1, 2\}\},
\]

\[
\mathcal{E}_3^{(n)} := \{\exists H_{2,1}^{(n)} \notin \mathcal{G}_{\tilde{p}, \tilde{q}}^{(n)}: H_{2,1}^{(n)} \neq G_{2,1}^{(n)}, f_j^{(n)}(H_{2,1}^{(n)}) = f_j^{(n)}(G_{1,2}^{(n)})\},
\]

\[
\mathcal{E}_4^{(n)} := \{\exists H_{1,2}^{(n)} \neq G_{1,2}^{(n)}: H_{1,2}^{(n)} \neq G_{2,1}^{(n)}, f_i^{(n)}(H_{1,2}^{(n)}) = f_i^{(n)}(G_{1,2}^{(n)})\}.
\]

Note that outside the above four events the decoder successfully decodes the input graph \(G_{1,2}^{(n)}\).

Using Chebyshev’s inequality, for some \(\kappa > 0\) we have \(\mathbb{P}(\mathcal{E}_1^{(n)}) \leq \kappa n^{-1/3}\), which converges to zero as \(n\) goes to infinity. Moreover, using the union bound, we have

\[
\mathbb{P}(\mathcal{E}_2^{(n)}) \leq \frac{|\mathcal{G}_{\tilde{p}, \tilde{q}}^{(n)}|}{L_1^{(n)} L_2^{(n)}}.
\]

Note that, for each graph \(H_{1,2}^{(n)} \in \mathcal{G}_{\tilde{p}, \tilde{q}}^{(n)}\), the mark count vectors \(\tilde{m} H_{1,2}^{(n)}\) and \(\tilde{u} H_{1,2}^{(n)}\) are in the sets \(\mathcal{M}^{(n)}\) and \(\mathcal{U}^{(n)}\) respectively. Additionally, we have \(|\mathcal{M}^{(n)}| \leq (2n^{2/3} + 1)|\Xi_{1,2}|\) and \(|\mathcal{U}^{(n)}| \leq (2n^{2/3} + 1)|\Theta_{1,2}|\). Therefore,

\[
|\mathcal{G}_{\tilde{p}, \tilde{q}}^{(n)}| \leq (2n^{2/3} + 1)(|\Xi_{1,2}| + |\Theta_{1,2}|) \max_{\tilde{m} \in \mathcal{M}^{(n)}} A_1(\tilde{m}, \tilde{u}),
\]
where

\[ A_1(\tilde{m}, \tilde{u}) := \left( \frac{n}{\{u(\theta)\}_{\theta \in \Theta_{1,2}}} \right) \left( \frac{m(x)}{x \in \Xi_{1,2}} \right). \]

Now, let \( m^{(n)} \) and \( u^{(n)} \) be sequences in \( \mathcal{M}^{(n)} \) and \( \mathcal{U}^{(n)} \), respectively. Then, for all \( x \in \Xi_{1,2} \) and \( \theta \in \Theta_{1,2} \), we have \( m^{(n)}(x)/n \to p_x/2 \) and \( u^{(n)}(\theta)/n \to q_\theta \). Thereby, using Lemma 1, we have

\[
\lim_{n \to \infty} \log A_1(\tilde{m}^{(n)}, \tilde{u}^{(n)}) - \left( \sum_{x \in \Xi_{1,2}} m^{(n)}(x) \right) \log n = H(\tilde{q}) + \sum_{x \in \Xi_{1,2}} s(p_x) = \Sigma(\mu_{1,2}).
\]

Substituting this into (20) and using the fact that \( \sum |m^{(n)}(x) - np_x/2| \leq n^{2/3} \), we have

\[
\limsup_{n \to \infty} \frac{\log |G_{\tilde{p}, q}^{(n)}| - n \frac{d_{ER}^{(n)}}{2} \log n}{n} \leq \Sigma(\mu_{1,2}^{ER}). \tag{21}
\]

Substituting this into (19), we have

\[
\limsup_{n \to \infty} \frac{1}{n} \log P\left( E_2^{(n)} \right) \leq \limsup_{n \to \infty} \frac{\log |G_{\tilde{p}, q}^{(n)}| - n \frac{d_{ER}^{(n)}}{2} \log n - n \Sigma(\mu_{1,2}^{ER})}{n} + \limsup_{n \to \infty} \frac{n \alpha_2}{n} \log n + n(R_1 + R_2) - \log L_1^{(n)} L_2^{(n)}.
\]

The first term is nonpositive due to (21), the second term is strictly negative due to the assumption (18c), and the third term is nonpositive due to our choice of \( L_1^{(n)} \) and \( L_2^{(n)} \). Consequently, the RHS is strictly negative, which implies that \( P(E_2^{(n)}) \to 0 \).

Now, we show that \( P(E_3^{(n)}) \cap E_1^{(n)} \) vanishes. In order to do so, for \( H_1^{(n)} \in G_1^{(n)} \), define \( S_2^{(n)}(H_1^{(n)}) := \{ H_2^{(n)} \in G_2^{(n)} : H_1^{(n)} \oplus H_2^{(n)} \in \tilde{G}_{\tilde{p}, \tilde{q}}^{(n)} \} \). Using the union bound, we have

\[
P\left( E_3^{(n)} \setminus E_1^{(n)} \right) \leq \sum_{H_1^{(n)} \in G_1^{(n)}} P(G_1^{(n)} = H_1^{(n)}) \frac{|S_2^{(n)}(H_1^{(n)})|}{L_2^{(n)}} \leq \frac{1}{L_2^{(n)}} \max_{H_1^{(n)} \in G_1^{(n)}} |S_2^{(n)}(H_1^{(n)})|.
\]

It can be shown that (See Appendix E)

\[
\limsup_{n \to \infty} \frac{\max_{H_1^{(n)} \in G_1^{(n)}} \log |S_2^{(n)}(H_1^{(n)})| - n \frac{d_{1,2}^{ER} + d_{1,2}^{ER}}{2} \log n}{n} \leq \Sigma(\mu_2^{ER} | \mu_1^{ER}), \tag{23}
\]
where $H_1^{(n)}$ is the first marginal of $H_1^{(n)}$. Substituting this in (22), we get
\[
\limsup \frac{1}{n} \log \mathbb{P} \left( \mathcal{E}_3^{(n)} \setminus \mathcal{E}_1^{(n)} \right) \leq \limsup \frac{n \mu_{ER}^{2} - \mu_{ER}^{3} \log n + n \Sigma \mu_{ER}^{2} | \mu_{ER}^{1} | - \log L_2^{(n)}}{n} \\
\leq \limsup \frac{n \mu_{ER}^{2} - \mu_{ER}^{3} \log n + n \Sigma \mu_{ER}^{2} | \mu_{ER}^{1} | - R_2}{n} \\
+ \limsup \frac{n \alpha_2 \log n + n R_2 - \log L_2^{(n)}}{n}
\] (24)

Note that the first term is strictly negative due to the assumption (18b), while the second term is nonpositive due to our way of choosing $L_2^{(n)}$. This means that $\mathbb{P}(\mathcal{E}_3^{(n)} \setminus \mathcal{E}_1^{(n)})$ goes to zero as $n$ goes to infinity. Similarly, $\mathbb{P}(\mathcal{E}_4^{(n)} \setminus \mathcal{E}_1^{(n)})$ converges to zero as $n \to \infty$. This means that there exists a sequence of deterministic codebooks with vanishing probability of error, which completes the proof of achievability.

5.2 Proof of Achievability for the Configuration model

Our achievability proof for this case is very similar in nature to that for the Erdős–Rényi case, with the modifications discussed below.

Let $\mathcal{D}^{(n)}$ be the set of degree sequences $\vec{d}$ with entries bounded by $\Delta$ such that $c_k(\vec{d}) = c_k(\vec{d}^{(n)})$ for all $0 \leq k \leq \Delta$. Moreover, redefine $\mathcal{M}^{(n)}$ to be the set of mark count vectors $\vec{m}$ such that $\sum_{x \in \mathcal{E}_{1,2}} m(x) = m_n$ and $\sum_{x \in \mathcal{E}_{1,2}} |m(x) - m_n \Delta| \leq n^{2/3}$, where we recall that $m_n = (\sum_{i=1}^{n} d^{(n)}(i))/2$.

We use the same definition for $\mathcal{U}^{(n)}$ as in the previous section, i.e. the set of vertex mark count vectors $\vec{u}$ such that $\sum_{\theta \in \Theta_{1,2}} |u(\theta) - nq(\theta)| \leq n^{2/3}$.

In what follows, let $X$ be a random variable with law $\vec{r}$, $X_1$ and $X_2$ defined as in (15), and $\Gamma = (\Gamma_1, \Gamma_2)$ a random variable with law $\vec{\gamma}$.

We define $\mathcal{W}^{(n)}$ to be the set of graphs $H_{1,2}^{(n)} \in \mathcal{G}_{1,2}^{(n)}$ such that: (i) $\vec{d}_{H_{1,2}^{(n)}}^{(n)} \in \mathcal{D}^{(n)}$, (ii) $\vec{m}_{H_{1,2}^{(n)}}^{(n)} \in \mathcal{M}^{(n)}$, (iii) $\vec{u}_{H_{1,2}^{(n)}}^{(n)} \in \mathcal{U}^{(n)}$, (iv) for all $0 \leq l \leq k \leq \Delta$, recalling the notation in (2), we have
\[
|c_{k,l}(\vec{d}_{G_{1,2}^{(n)}}^{(n)}, \vec{d}_{G_{1,2}^{(n)}}^{(n)}) - n \mathbb{P}(X = k, X_1 = l)| \leq n^{2/3},
\] (25)
and (v) for all $0 \leq l \leq k \leq \Delta$ we have
\[
|c_{k,l}(\vec{d}_{G_{1,2}^{(n)}}^{(n)}, \vec{d}_{G_{1,2}^{(n)}}^{(n)}) - n \mathbb{P}(X = k, X_2 = l)| \leq n^{2/3}.
\] (26)

We employ a similar random binning framework as in Section 5.1. For decoding, upon receiving a pair $(i, j)$, we form the set of graphs $H_{1,2}^{(n)} \in \mathcal{W}^{(n)}$ such that $f_1^{(n)}(H_{1,2}^{(n)}) = i$ and $f_2^{(n)}(H_{1,2}^{(n)}) = j$. If this set has only one element, we output it as the source graph; otherwise, we output an indication of error. In order to prove the achievability, we consider the four error events $\mathcal{E}_1^{(n)}$, $1 \leq i \leq 4$, defined exactly like those in the previous section, with $G_{i,j}^{(n)}$ being replaced with $\mathcal{W}^{(n)}$.

It can be shown that if $G_{1,2}^{(n)} \sim \mathcal{G}(n; \vec{d}^{(n)}, \vec{q}, \vec{\gamma}, \vec{\gamma})$, the probability of $G_{1,2}^{(n)} \in \mathcal{W}^{(n)}$ goes to one as $n$ goes to infinity (see Lemma 4 in Appendix D). Therefore, $\mathbb{P}(\mathcal{E}_1^{(n)})$ goes to zero as $n \to \infty$.

To show that $\mathbb{P}(\mathcal{E}_2^{(n)})$ vanishes, similar to the analysis in Section 5.1, we find an asymptotic upper bound for $\log |\mathcal{W}^{(n)}|$. By only considering the conditions (i), (ii) and (iii) in the definition of $\mathcal{W}^{(n)}$,
we have
\[
\log |\mathcal{W}(n)| \leq \log \left( \sum_{k=0}^{n} \binom{n}{k} \right) + \log |\mathcal{G}(n)| + \log \left( \frac{2n^{2/3} + 1}{\Theta_1} \max_{m \in M(n)} \left( \left\{ m(n) \right\}_{x \in \Xi_1, 2} \right) \right) + \log \left( \frac{2n^{2/3} + 1}{\Theta_1, 2} \max_{\bar{a} \in U(n)} \left( \frac{n}{n} \right) \right).
\]  
(27)
By assumption, we have \( \alpha_0 < 1 \), hence \( d_{1.2}^{CM} > 0 \). The condition (10) together with Lemma 3 in Appendix D then implies that
\[
\lim_{n \to \infty} \frac{\log |\mathcal{G}(n)| - n d_{1.2}^{CM} \log n}{n} = \lim_{n \to \infty} \frac{\log |\mathcal{G}(n)| - m_n \log n}{n} + \lim_{n \to \infty} \frac{m_n - n d_{1.2}^{CM}}{2} \log n
\]
(28)
\[
= -s(d_{1.2}^{CM}) - \mathbb{E} \left[ \log X! \right],
\]
where on the second line we have used the bound \( |m_n - n d_{1.2}^{CM}/2| \leq K \Delta n^{1/2} \) which is implied by (10). Using this together with Lemma 1 for the other terms in (27), we have
\[
\limsup_{n \to \infty} \frac{\log |\mathcal{W}(n)| - n d_{1.2}^{CM} \log n}{n} \leq -s(d_{1.2}^{CM}) + H(X) + \frac{d_{1.2}^{CM}}{2} H(\Gamma) + H(Q) - \mathbb{E} \left[ \log X! \right] = \Sigma(\mu_1^{CM}),
\]
where \( \Gamma \) and \( Q \) are random variables with law \( \tilde{\gamma} \) and \( \tilde{q} \), respectively.

Now, in order to show that \( \mathbb{P}(\mathcal{E}_3^{(n)} \setminus \mathcal{C}_1^{(n)}) \) vanishes, we prove a counterpart for (23). For \( H_1^{(n)} \in \mathcal{G}_1^{(n)} \), we define \( S_2^{(n)}(H_1^{(n)}) \) to be the set of graphs \( H_2^{(n)} \in \mathcal{G}_2^{(n)} \) such that \( H_1^{(n)} \oplus H_2^{(n)} \in \mathcal{W}(n) \). Then, it can be shown (see Appendix F) that
\[
\limsup_{n \to \infty} \frac{\max_{H_1^{(n)} \in \mathcal{W}(n)}}{n} \log |S_2^{(n)}(H_1^{(n)})| - n \frac{d_{1.2}^{CM} - d_1^{CM}}{2} \log n \leq \Sigma(\mu_2^{CM}/\mu_1^{CM}).
\]  
(29)
Then, similar to (24), this shows that \( \mathbb{P}(\mathcal{E}_3^{(n)} \setminus \mathcal{C}_1^{(n)}) \) vanishes as \( n \to \infty \). Similarly, \( \mathbb{P}(\mathcal{E}_4^{(n)} \setminus \mathcal{C}_1^{(n)}) \) vanishes as \( n \to \infty \). This completes the proof of achievability.

5.3 Proof of the Converse for the Erdős–Rényi case

In this section, we show that every rate tuple \( (\alpha_1, R_1, \alpha_2, R_2) \in \mathcal{R} \) for the Erdős–Rényi scenario must satisfy the conditions (17a)–(17c). By definition, for a rate tuple \( (\alpha_1, R_1, \alpha_2, R_2) \in \mathcal{R} \), there exist sequences \( R_1^{(m)} \) and \( R_2^{(m)} \) such that for each \( m, (\alpha_1, R_1^{(m)}, \alpha_2, R_2^{(m)}) \) is achievable and, besides, we have \( R_1^{(m)} \to R_1 \) and \( R_2^{(m)} \to R_2 \). If we show that \( (\alpha_1, R_1^{(m)}, \alpha_2, R_2^{(m)}) \) satisfies (17a)–(17c) for each \( m \), it is easy to see that \( (\alpha_1, R_1, \alpha_2, R_2) \) must also satisfy the same inequalities. Therefore, it suffices to show that any achievable rate tuple satisfies (17a)–(17c).

For this, take an achievable rate tuple \( (\alpha_1, R_1, \alpha_2, R_2) \) together with a corresponding sequence of \( \langle n, L_1^{(n)}, L_2^{(n)} \rangle \) codes \( \langle f_1^{(n)}, f_2^{(n)}, g^{(n)} \rangle \). By definition, we have
\[
\limsup_{n \to \infty} \frac{\log L_i^{(n)} - \alpha_i n \log n + R_i n}{n} \leq 0 \quad i \in \{1, 2\},
\]  
(30)
and also the error probability $P_e^{(n)}$ goes to zero as $n$ goes to infinity. Now, we define the set $\mathcal{A}^{(n)} \subseteq \mathcal{G}^{(n)}_{1,2}$ as

$$
\mathcal{A}^{(n)} := \mathcal{G}^{(n)}_{p,q} \cap \{ H_{1,2}^{(n)} \in \mathcal{G}^{(n)}_{1,2} : g^{(n)}(f_1^{(n)}(H_{1,2}^{(n)}), f_2^{(n)}(H_{1,2}^{(n)})) = H_{1,2}^{(n)} \},
$$

where $\mathcal{G}^{(n)}_{p,q}$ was defined in Section 5.1. In fact, $\mathcal{A}^{(n)}$ is the set of “typical” graphs with respect to the Erdős–Rényi model that are successfully decoded by the code $(f_1^{(n)}, f_2^{(n)}, g^{(n)})$. In the following, let $G_{1,2}^{(n)} \sim \mathcal{G}^{(n)}_{p,q}$ be distributed according to the Erdős–Rényi model. Moreover, let $P_{ER}^{(n)}$ be the law of $G_{1,2}^{(n)}$, i.e. for $H_{1,2}^{(n)} \in \mathcal{G}^{(n)}_{1,2}$, $P_{ER}^{(n)}(H_{1,2}^{(n)}) := \mathbb{P}(G_{1,2}^{(n)} = H_{1,2}^{(n)})$. With this, we define a random variable $\tilde{G}_{1,2}^{(n)}$ whose distribution is the conditional distribution of $G_{1,2}^{(n)}$, conditioned on lying in $\mathcal{A}^{(n)}$, i.e.

$$
\mathbb{P}\left(\tilde{G}_{1,2}^{(n)} = H_{1,2}^{(n)} \right) = \begin{cases} 
P_{ER}^{(n)}(H_{1,2}^{(n)})/\pi_n & H_{1,2}^{(n)} \in \mathcal{A}^{(n)}, \\
0 & \text{otherwise.} \end{cases}
$$

(32)

where $\pi_n := \mathbb{P}\left(\tilde{G}_{1,2}^{(n)} \in \mathcal{A}^{(n)}\right)$ is the normalizing factor. Note that, since $P_e^{(n)} \rightarrow 0$ as $n \rightarrow \infty$ and $P(G_{1,2}^{(n)} \in \mathcal{A}^{(n)}) \rightarrow 1$ as $n \rightarrow \infty$, we have $\pi_n > 0$ for all sufficiently large $n$, and in fact $\pi_n \rightarrow 1$ as $n \rightarrow \infty$. Additionally, let $\tilde{P}_{ER}^{(n)}$ be the law of $\tilde{G}_{1,2}^{(n)}$. If, for $i \in \{1, 2\}$, $\tilde{M}_i^{(n)}$ denotes $f_i^{(n)}(\tilde{G}_{1,2}^{(n)})$, we have

$$
\log L_1^{(n)} + \log L_2^{(n)} \geq H(\tilde{M}_1^{(n)}) + H(\tilde{M}_2^{(n)}) \geq H(\tilde{M}_1^{(n)}, \tilde{M}_2^{(n)}) = H(\tilde{G}_{1,2}^{(n)}),
$$

(33)

where the last equality follows from the fact that, by definition, $\tilde{G}_{1,2}^{(n)}$ takes values among the graphs that are successfully decoded, and hence is uniquely identified given $\tilde{M}_1^{(n)}$ and $\tilde{M}_2^{(n)}$.

Now, we find a lower bound for $H(\tilde{G}_{1,2}^{(n)})$. For doing so, note that for $H_{1,2}^{(n)} \in \mathcal{G}^{(n)}_{1,2}$ and $n$ large enough, we have

$$
- \log P_{ER}^{(n)}(H_{1,2}^{(n)}) = - \sum_{x \in \mathcal{E}_{1,2}} m_{H_{1,2}^{(n)}}(x) \log \frac{p_x}{n} - \left( \begin{bmatrix} n \\ 2 \end{bmatrix} - \sum_{x \in \mathcal{E}_{1,2}} m_{H_{1,2}^{(n)}}(x) \right) \log \left( 1 - \frac{\sum_{x \in \mathcal{E}_{1,2}} p_x}{n} \right) - \sum_{\theta \in \mathcal{E}_{1,2}} u_{H_{1,2}^{(n)}}(\theta) \log q_\theta.
$$

(34)

On the other hand, due to the definition of $\mathcal{G}^{(n)}_{p,q}$, if $H_{1,2}^{(n)} \in \mathcal{G}^{(n)}_{p,q}$ then, for all $x \in \mathcal{E}_{1,2}$ and $\theta \in \mathcal{E}_{1,2}$, we have

$$
n \frac{p_x}{2} - n^{2/3} \leq m_{H_{1,2}^{(n)}}(x) \leq \frac{p_x}{2} + n^{2/3}, \quad \text{and} \quad n q_\theta - n^{2/3} \leq u_{H_{1,2}^{(n)}}(\theta) \leq n q_\theta + n^{2/3}.
$$

Substituting these in (34) and using the inequality $\log(1 - x) \leq -x$ which holds for $x \in (0, 1)$, for $n$ large enough, we have

$$
- \log P_{ER}^{(n)}(H_{1,2}^{(n)}) \geq \sum_{x \in \mathcal{E}_{1,2}} \left( \frac{p_x}{2} - n^{2/3} \right) \left( \log n - \log p_x \right) + \left( \begin{bmatrix} n \\ 2 \end{bmatrix} - \sum_{x \in \mathcal{E}_{1,2}} \left( \frac{p_x}{2} + n^{2/3} \right) \right) - \sum_{\theta \in \mathcal{E}_{1,2}} (n q_\theta - n^{2/3}) \log q_\theta.
$$

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Using \( \sum_{x \in \Xi_1, 2} p_x = d_{\text{ER}}^{1, 2} \) and simplifying the above, we realize that there exists a constant \( c > 0 \) that does not depend on \( n \) or \( H_{1, 2}^{(n)} \), such that, for all \( H_{1, 2}^{(n)} \in \mathcal{G}_{p, q}^{(n)} \) and thus, in particular, for all \( H_{1, 2}^{(n)} \in \mathcal{A}^{(n)} \), we have

\[
- \log P_{\text{ER}}(H_{1, 2}^{(n)}) \geq n \frac{d_{\text{ER}}^{1, 2}}{2} \log n - n \sum_{x \in \Xi_{1, 2}} \frac{p_x}{2} \log p_x + n \sum_{x \in \Xi_{1, 2}} \frac{p_x}{2} - n \sum_{\theta \in \Theta_{1, 2}} q_{\theta} \log q_{\theta} - cn^{2/3} \log n
\]

\[
= n \frac{d_{\text{ER}}^{1, 2}}{2} \log n + n \sum_{\theta \in \Theta_{1, 2}} P_{\text{ER}}^{(n)}(H_{1, 2}^{(n)}) \log P_{\text{ER}}^{(n)}(H_{1, 2}^{(n)}).
\]

Now, if \( \tilde{G}_{1, 2}^{(n)} \) is the random variable defined in (32), we have

\[
H(\tilde{G}_{1, 2}^{(n)}) = - \sum_{H_{1, 2}^{(n)} \in \mathcal{A}^{(n)}} P_{\text{ER}}^{(n)}(H_{1, 2}^{(n)}) \log P_{\text{ER}}^{(n)}(H_{1, 2}^{(n)})
\]

\[
= \log \pi_n - \frac{1}{\pi_n} \sum_{H_{1, 2}^{(n)} \in \mathcal{A}^{(n)}} P_{\text{ER}}^{(n)}(H_{1, 2}^{(n)}) \log P_{\text{ER}}^{(n)}(H_{1, 2}^{(n)}).
\]

Note that since the probability of error of the above code vanishes, i.e. \( P_{\text{c}}^{(n)} \to 0 \) and \( \mathbb{P}(\tilde{G}_{1, 2}^{(n)} \in \mathcal{G}_{p, q}^{(n)}) \to 1 \), we have \( \pi_n \to 1 \) as \( n \to \infty \). On the other hand, with probability one, we have \( \tilde{G}_{1, 2}^{(n)} \in \mathcal{G}_{p, q}^{(n)} \). Also, by the definition of \( \pi_n \), we have \( \sum_{H_{1, 2}^{(n)} \in \mathcal{A}^{(n)}} P_{\text{ER}}^{(n)}(H_{1, 2}^{(n)}) = \pi_n \). Thereby, employing the bound (35), we have

\[
\lim_{n \to \infty} \frac{H(\tilde{G}_{1, 2}^{(n)}) - n \frac{d_{\text{ER}}^{1, 2}}{2} \log n}{n} \geq \Sigma(\mu_{1, 2}^{\text{ER}}).
\]

(36)

Now, using the assumption (30) together with the bound (33), we have

\[
0 \geq \limsup_{n \to \infty} \frac{\log L_1^{(n)} + \log L_2^{(n)} - (\alpha_1 + \alpha_2)n \log n - n(R_1 + R_2)}{n} \geq \liminf_{n \to \infty} \frac{H(\tilde{G}_{1, 2}^{(n)}) - n \frac{d_{\text{ER}}^{1, 2}}{2} \log n - n \Sigma(\mu_{1, 2}^{\text{ER}})}{n} + \liminf_{n \to \infty} \frac{n \frac{d_{\text{ER}}^{1, 2}}{2} \log n + n \Sigma(\mu_{1, 2}^{\text{ER}}) - (\alpha_1 + \alpha_2)n \log n - n(R_1 + R_2)}{n}.
\]

(37)

The first term is nonnegative due to (36). Consequently,

\[
0 \geq \liminf_{n \to \infty} \frac{n \left(\frac{d_{\text{ER}}^{1, 2}}{2} - \alpha_1 - \alpha_2\right)}{n} \log n + n(\Sigma(\mu_{1, 2}^{\text{ER}}) - R_1 - R_2).
\]

(38)

Note that this is impossible unless \( \alpha_1 + \alpha_2 \geq d_{\text{ER}}^{1, 2}/2 \). Furthermore, if \( \alpha_1 + \alpha_2 = d_{\text{ER}}^{1, 2} \), it must be the case that \( R_1 + R_2 \geq \Sigma(\mu_{1, 2}^{\text{ER}}) \). But this is precisely (17c) for \( \mu_{1, 2} = \mu_{1, 2}^{\text{ER}} \).

Now, we turn to showing (17a). We have

\[
\log L_1^{(n)} \geq H(\tilde{M}_1^{(n)}) \geq H(\tilde{M}_1^{(n)} | \tilde{M}_2^{(n)})
\]

\[
= H(\tilde{G}_{1, 2}^{(n)}, \tilde{M}_1^{(n)} | \tilde{M}_2^{(n)}) - H(\tilde{G}_{1, 2}^{(n)} | \tilde{M}_2^{(n)})
\]

\[
\overset{(a)}{=} H(\tilde{G}_{1, 2}^{(n)} | \tilde{M}_2^{(n)})
\]

\[
\overset{(b)}{\geq} H(\tilde{G}_{1, 2}^{(n)} | \tilde{G}_{2}^{(n)})
\]

\[
= H(\tilde{G}_{1, 2}^{(n)}) - H(\tilde{G}_{2}^{(n)}).
\]

(39)
where (a) uses the facts that $\tilde{G}_{1}^{(n)}$ is a function of $\tilde{G}_{1}^{(n)}$ and also, since $\tilde{G}_{1,2}^{(n)} \in A^{(n)}$, given $\tilde{M}_{1}^{(n)}$ and $\tilde{M}_{2}^{(n)}$ we can unambiguously determine $\tilde{G}_{1,2}^{(n)}$ and hence $\tilde{G}_{1,2}^{(n)}$. Also, (b) uses data processing inequality.

Now, we find an upper bound for $H(\tilde{G}_{2}^{(n)})$. Note that since $\tilde{G}_{1,2}^{(n)} \in A^{(n)}$ with probability one, we have

$$H(\tilde{G}_{2}^{(n)}) \leq \log |A_{2}^{(n)}|,$$

where

$$A_{2}^{(n)} := \{ H_{2}^{(n)} \in G_{2}^{(n)} : H_{1}^{(n)} + H_{2}^{(n)} \in A^{(n)} \text{ for some } H_{1}^{(n)} \in G_{1}^{(n)} \}.$$

Now, take $H_{2}^{(n)} \in A_{2}^{(n)}$ and let $H_{1}^{(n)} \in G_{1}^{(n)}$ be such that $H_{1}^{(n)} := H_{1}^{(n)} + H_{2}^{(n)} \in A^{(n)}$. Since $A^{(n)} \subseteq G_{p,q}$, by definition we have that, for all $x \in \Xi_{1,2}$ and all $\theta \in \Theta_{1,2}$,

$$\sum_{x \in \Xi_{1,2}} |m_{H_{1,2}^{(n)}}(x) - np_{x}/2| \leq n^{2/3}$$

and

$$\sum_{\theta \in \Theta_{1,2}} |u_{H_{1,2}^{(n)}}(\theta) - nq_{\theta}| \leq n^{2/3}.$$

Moreover, for $x_{2} \in \Xi_{2}$ and $\theta_{2} \in \Theta_{2}$ we have $m_{H_{2}^{(n)}}(x_{2}) = \sum_{x_{1} \in \Xi_{1} \cup \{0_{1}\}} m_{H_{1,2}^{(n)}}((x_{1}, x_{2}))$ and $u_{H_{2}^{(n)}}(\theta_{2}) = \sum_{\theta_{1} \in \Theta_{1}} m_{H_{1,2}^{(n)}}((\theta_{1}, \theta_{2}))$. Using this in the above and using the triangle inequality, we realize that for $H_{2}^{(n)} \in A_{2}^{(n)}$ we have $\tilde{m}_{H_{2}^{(n)}} \in M_{2}^{(n)}$ and $\tilde{u}_{H_{2}^{(n)}} \in U_{2}^{(n)}$, where $M_{2}^{(n)}$ is the set of edge mark count vectors $\tilde{m}$ such that $\sum_{x_{2} \in \Xi_{2}} |m(x_{2}) - np_{x}_{2}/2| \leq n^{2/3}$ and $U_{2}^{(n)}$ is the set of vertex mark count vectors $\tilde{u}$ such that $\sum_{\theta_{2} \in \Theta_{2}} |u(\theta_{2}) - nq_{\theta_{2}}| \leq n^{2/3}$. Consequently, we have

$$|A_{2}^{(n)}| \leq (2n^{2/3} + 1)^{(|\Xi_{2}| + |\Theta_{2}|)} \left( \max_{\tilde{m} \in M_{2}^{(n)}} \left( \left\{ m(x_{2}) \right\}_{x_{2} \in \Xi_{2}} \right) \right) \left( \max_{\tilde{u} \in U_{2}^{(n)}} \left( \left\{ u(\theta_{2}) \right\}_{\theta_{2} \in \Theta_{2}} \right) \right).$$

Using Lemma 1 and the definition of $M_{2}^{(n)}$ and $U_{2}^{(n)}$ above, with $Q = (Q_{1}, Q_{2}) \sim \tilde{q}$, an argument similar to the one that was used to establish (21) implies that

$$\limsup_{n \to \infty} \frac{\log |A_{2}^{(n)}| - n^{d_{ER}^{2}}} {n} \leq H(Q_{2}) + \sum_{x_{2} \in \Xi_{2}} s(p_{x_{2}}) = \Sigma(\mu_{2}^{ER}).$$

Substituting this into (40), we get

$$\limsup_{n \to \infty} \frac{\log H(\tilde{G}_{2}^{(n)}) - n^{d_{ER}^{2}}} {n} \leq \Sigma(\mu_{2}^{ER}).$$

Using this together with (36) and substituting into (39) we get

$$\liminf_{n \to \infty} \frac{\log L_{1}^{(n)} - n^{d_{ER}^{1} - d_{ER}^{2}}} {n} \geq \Sigma(\mu_{1}^{ER}) - \Sigma(\mu_{2}^{ER}) = \Sigma(\mu_{1}^{ER} | \mu_{2}^{ER}).$$

Using a similar method as in (37) and (38), this implies (17a). The proof of (17b) is similar. This completes the proof of the converse for the Erdős–Rényi case.

### 5.4 Proof of the Converse for the Configuration Model

The proof of the converse for the configuration model is similar to that for the Erdős–Rényi model presented in the previous section. Take an achievable rate tuple $(a_{1}, R_{1}, a_{2}, R_{2})$ together with a
sequence of \( (n, L_1^{(n)}, L_2^{(n)}) \) codes \( (f_1^{(n)}, f_2^{(n)}, g^{(n)}) \) achieving this rate tuple. Moreover, redefine the set \( \mathcal{A}^{(n)} \) to be
\[
\mathcal{A}^{(n)} := \mathcal{W}^{(n)} \cap \{ H_{1,2}^{(n)} \in G_1^{(n)} : g^{(n)}(f_1^{(n)}(H_1^{(n)}), f_2^{(n)}(H_2^{(n)})) = H_{1,2}^{(n)} \},
\]
where the set \( \mathcal{W}^{(n)} \) was defined in Section 5.2. Now, let \( G_{1,2}^{(n)} \sim \mathcal{G}(n; \bar{d}^{(n)}, \bar{\gamma}, \bar{q}, \bar{r}) \) be distributed according to the configuration model ensemble, and let \( \tilde{G}_{1,2}^{(n)} \in \mathcal{A}^{(n)} \) have the distribution obtained from that of \( G_{1,2}^{(n)} \) by conditioning on it lying in the set \( \mathcal{A}^{(n)} \). Note that the normalizing constant \( \pi_n := \mathbb{P}(G_{1,2}^{(n)} \in \mathcal{A}^{(n)}) \) goes to 1 as \( n \to \infty \) since \( \mathbb{P}(G_{1,2}^{(n)} \in \mathcal{W}^{(n)}) \to 1 \) and the error probability of the code, \( P_e^{(n)} \), vanishes. Moreover, let \( P_{CM}^{(n)} \) and \( \tilde{P}_{CM}^{(n)} \) be the laws of \( G_{1,2}^{(n)} \) and \( \tilde{G}_{1,2}^{(n)} \), respectively. In the following, we show that
\[
\liminf_{n \to \infty} \frac{H(\tilde{G}_{1,2}^{(n)}) - n \frac{d_{CM}}{2} \log n}{n} \geq \Sigma(\mu_{1,2}^{CM}),
\]
and
\[
\limsup_{n \to \infty} \frac{H(\tilde{G}_{1,2}^{(n)}) - n \frac{d_{CM}}{2} \log n}{n} \leq \Sigma(\mu_{2}^{CM}).
\]
The rest of the proof is then identical to that of the previous section, so we only focus on proving the statements in (42) and (43).

For (42), note that for \( H_{1,2}^{(n)} \in G_{1,2}^{(n)} \) such that \( \log H_{1,2}^{(n)} \in D^{(n)} \), where \( D^{(n)} \) was defined in Section 5.2, we have
\[
- \log P_{CM}^{(n)}(H_{1,2}^{(n)}) = \log \left( \prod_{k=0}^{n} c_k(\bar{d}^{(n)}) \right) + \log |g_{\bar{d}^{(n)}}^{(n)}| - \sum_{x \in \Xi_{1,2}} m_{H_{1,2}^{(n)}}(x) \log \gamma_x - \sum_{\theta \in \Theta_{1,2}} u_{H_{1,2}^{(n)}}(\theta) \log q_\theta.
\]
Now, if \( H_{1,2}^{(n)} \in \mathcal{W}^{(n)} \), using the definition of \( \mathcal{W}^{(n)} \) we realize that there exists a constant \( c > 0 \) such that
\[
- \log P_{CM}^{(n)}(H_{1,2}^{(n)}) \geq \log \left( \prod_{k=0}^{n} c_k(\bar{d}^{(n)}) \right) + \log |g_{\bar{d}^{(n)}}^{(n)}| - \sum_{x \in \Xi_{1,2}} m_n \gamma_x \log \log q_\theta - \sum_{\theta \in \Theta_{1,2}} n q_\theta \log q_\theta - c n^{2/3} =: K_n.
\]
Note that the right hand side is a constant independent of \( H_{1,2}^{(n)} \) and is denoted by \( K_n \). Since \( \tilde{G}_{1,2}^{(n)} \) falls in \( \mathcal{W}^{(n)} \) with probability one, this means that \( H(\tilde{G}_{1,2}^{(n)}) \geq \log \pi_n + K_n \). But \( \pi_n \to 1 \) as \( n \to \infty \). Therefore, using the assumption (10) together with (28) from Section 5.2 and also the fact that \( m_n/n \to d_{1,2}^{CM}/2 \), we realize that
\[
\liminf_{n \to \infty} \frac{H(\tilde{G}_{1,2}^{(n)}) - n \frac{d_{CM}}{2} \log n}{n} \geq H(X) - s(d_{1,2}^{CM}) - \mathbb{E} \log X] + \frac{d_{CM}}{2} H(\Gamma) + H(Q),
\]
where \( X \sim \tilde{\Gamma} \), \( \Gamma \sim \bar{\gamma} \) and \( Q \sim \bar{q} \). Note that the right hand side is precisely \( \Sigma(\mu_{1,2}^{CM}) \). Hence we have proved (42).

In order to show (43), note that \( H(\tilde{G}_{2}^{(n)}) \leq \log |A_{2}^{(n)}| \) where \( A_{2}^{(n)} \) consists of graphs \( H_{2}^{(n)} \in G_{2}^{(n)} \) such that, for some \( H_{1}^{(n)} \in G_{1}^{(n)} \), we have \( H_{1}^{(n)} \oplus H_{2}^{(n)} \in A_{2}^{(n)} \). Since \( A_{2}^{(n)} \subseteq \mathcal{W}^{(n)} \), we have for all \( H_{2}^{(n)} \in A_{2}^{(n)} \) that
\[
\sum_{x \in \Xi_2} |m_{H_{2}^{(n)}}(x) - m_n \gamma_{x_2}| \leq n^{2/3} \quad \text{and} \quad \sum_{\theta_2 \in \Theta_2} |u_{H_{2}^{(n)}}(\theta_2) - n q_{\theta_2}| \leq n^{2/3}.
\]
On the other hand, the condition (26) implies that $\bar{d} \in \mathcal{D}_2^{(n)}$ where $\mathcal{D}_2^{(n)}$ denotes the set of degree sequences $\bar{d}$ of size $n$ with elements bounded by $\Delta$ such that

$$|c_k(\bar{d}) - n \mathbb{P}(X_2 = k)| \leq (\Delta + 1)n^{2/3}, \quad \forall 0 \leq k \leq \Delta,$$  \hspace{1cm} (45)

where $X_2$ is the random variable defined in (15). Consequently, we have

$$\log |A_2^{(n)}| \leq \log |\mathcal{D}_2^{(n)}| + \max_{\bar{d} \in \mathcal{D}_2^{(n)}} \log |G^{(n)}_{\bar{d}}| + \max_{H_2^{(n)} \in A_2^{(n)}} \log \left( \sum_{x_2 \in \Xi_2} \frac{m_{H_2^{(n)}}(x_2)}{\left\{ m_{H_2^{(n)}}(x_2) \right\}_{x_2 \in \Xi_2}} \right)$$

$$+ \max_{H_2^{(n)} \in A_2^{(n)}} \log \left( \{ u_{H_2^{(n)}}(\vartheta_2) \}_{\vartheta_2 \in \Theta_2} \right).$$  \hspace{1cm} (46)

Note that (45) implies that $|\mathcal{D}_2^{(n)}| \leq (2(\Delta + 1)n^{2/3} + 1)^{\Delta + 1} \max_{\bar{d} \in \mathcal{D}_2^{(n)}} \left( \frac{n}{\Delta + 1} \right)$). Therefore, Lemma 1 implies that

$$\limsup_{n \to \infty} \frac{1}{n} \log |\mathcal{D}_2^{(n)}| \leq H(X_2).$$  \hspace{1cm} (47)

On the other hand, the assumptions $r_0 < 1$ and (8) imply that $d^{CM}_2 > 0$. Hence, using Lemma 3, we have

$$\limsup_{n \to \infty} \frac{1}{n} \max_{d \in \mathcal{D}_2^{(n)}} \log |G^{(n)}_{\bar{d}}| - n d^{CM}_2 \log n \leq -s(d^{CM}_2) - \mathbb{E} \left[ \log X_2 \right].$$  \hspace{1cm} (48)

Moreover, if $H_2^{(n)}$ is a sequence in $A_2^{(n)}$, from (44), for all $x_2 \in \Xi_2$, we have

$$\lim_{n \to \infty} \frac{m_{H_2^{(n)}}(x_2)}{\sum_{x_2' \in \Xi_2} m_{H_2^{(n)}}(x_2')} = \gamma_{x_2} = \mathbb{P}(\Gamma_2 = x_2|\Gamma_2 \neq o_2),$$

where $\Gamma = (\Gamma_1, \Gamma_2)$ has law $\gamma$. Additionally, we have

$$\lim_{n \to \infty} \frac{1}{n} \sum_{x_2 \in \Xi_2} m_{H_2^{(n)}}(x_2) = \frac{d^{CM}_2}{2}.$$

Thereby, from Lemma 1, we have

$$\limsup_{n \to \infty} \frac{1}{n} \max_{H_2^{(n)} \in A_2^{(n)}} \log \left( \sum_{x_2 \in \Xi_2} \frac{m_{H_2^{(n)}}(x_2)}{\left\{ m_{H_2^{(n)}}(x_2) \right\}_{x_2 \in \Xi_2}} \right) \leq \frac{d^{CM}_2}{2} - H(\Gamma_2|\Gamma_2 \neq o_2).$$  \hspace{1cm} (49)

Finally, as we have $u_{H_2^{(n)}}(\vartheta_2)/n \to q_{\vartheta_2}$ for all $\vartheta_2 \in \Theta_2$, another usage of Lemma 1 implies that

$$\limsup_{n \to \infty} \frac{1}{n} \max_{H_2^{(n)} \in A_2^{(n)}} \log \left( \{ u_{H_2^{(n)}}(\vartheta_2) \}_{\vartheta_2 \in \Theta_2} \right) \leq H(Q_2),$$  \hspace{1cm} (50)

where $Q = (Q_1, Q_2)$ has law $\tilde{q}$. Now, combining (47), (48), (49) and (50) and substituting into (46), and also using the bound $H(\hat{G}_2^{(n)}) \leq \log |A_2^{(n)}|$, we realize that

$$\limsup_{n \to \infty} \frac{H(\hat{G}_2^{(n)}) - n d^{CM}_2 \log n}{n} \leq H(X_2) - s(d^{CM}_2) - \mathbb{E} \left[ \log X_2 \right] + \frac{d^{CM}_2}{2} - (\frac{d^{CM}_2}{2} + H(Q_2)).$$

But the right hand side is precisely $\Sigma(\mu_2^{CM})$. This completes the proof of (43). As was mentioned before, the rest of the proof is identical to that in the previous section.
5.5 Generalization to more than two sources

Assume we have \( k \geq 2 \) sources of graphical data. For \( 1 \leq i \leq k \), let \( \Theta_i \) and \( \Xi_i \) denote the vertex and edge mark sets for the \( i \)th domain. For \( i \in [k] \) and \( n \in \mathbb{N} \), \( \mathcal{G}_i^{(n)} \) denotes the set of marked graphs on the vertex set \([n] \) with vertex and edge marks coming from \( \Theta_i \) and \( \Xi_i \), respectively. Given \( A \subseteq [k] \) nonempty and for \( G_i \in \mathcal{G}_i^{(n)} \), \( i \in A \), we define \( \bigoplus_{i \in A} G_i \) to be the superposition of graphs in \( A \), which is a single marked graph on the vertex set \([n] \) such that a vertex \( v \in [n] \) carries a vertex mark \( (\theta_i : i \in A) \in \Theta_A := \prod_{i \in A} \Theta_i \) such that \( \theta_i \) is the mark of \( v \) in \( G_i \). Moreover, an edge between vertices \( v \) and \( w \) exists in \( \bigoplus_{i \in A} G_i \) if such an edge exists in at least one of the graphs \( G_i, i \in A \). If this is the case, the mark of this edge is defined to be \( (x_i : i \in A) \), where for \( i \in A \), \( x_i \) is the mark of the edge \((v, w)\) in \( G_i \) if such an edge exists in \( G_i \). Otherwise, we set \( x_i = \circ_i \), where \( \circ_i \) for \( i \in [k] \) is an auxiliary mark not present in \( \Xi_i \). For nonempty \( A \subseteq [k] \), we denote \( (\circ_i : i \in A) \) by \( \circ_A \). Note that with \( \Xi_A := (\prod_{i \in A} (\Xi_i \cup \{\circ_i\})) \setminus \{\circ_A\}, \bigoplus_{i \in A} G_i \) is a marked graph with vertex and edge mark sets \( \Theta_A \) and \( \Xi_A \), respectively. Let \( \mathcal{G}_A^{(n)} \) denote the set of marked graphs in domain \( A \), which is the set of marked graphs on the vertex set \([n] \) together with vertex and edge mark sets \( \Theta_A \) and \( \Xi_A \), respectively. Given \( G \in \mathcal{G}_A^{(n)} \) and \( A \subseteq [k] \), we can naturally define the projection of \( G \) onto domain \( A \) by projecting all vertex and edge marks onto \( \Theta_A \) and \( \Xi_A \), respectively, followed by removing edges with mark \( \circ_A \). It can be checked that the resulting graph, denoted by \( G_A \), lies in domain \( A \), i.e. \( G_A \in \mathcal{G}_A^{(n)} \).

A sequence of \( \langle n, L_i^{(n)} : i \in [k] \rangle \) codes is defined as a sequence of tuples \( \langle (f_i^{(n)} : i \in [k]), g^{(n)} \rangle \) such that \( f_i^{(n)} : \mathcal{G}_i^{(n)} \to [L_i^{(n)}] \) for \( i \in A \) are encoding functions, and \( g^{(n)} : \prod_{i \in [k]} [L_i^{(n)}] \to \mathcal{G}_A^{(n)} \) is the corresponding decoding function. Given a sequence of ensembles \( \mathcal{G}_i^{(n)} \) on \( \mathcal{G}_A^{(n)} \), the probability of error \( P_e^{(n)} \) is defined to be the probability that \( g^{(n)}(f_i^{(n)}(G_i^{(n)} : i \in [k])) \neq G_A^{(n)} \).

We say that a rate tuple \( (\alpha_i, R_i) : i \in [k] \) is achievable for the distributed compression of the sequence of random graphs \( G_i^{(n)} \in \mathcal{G}_i^{(n)} \) if there is a sequence of \( \langle n, L_i^{(n)} : i \in [k] \rangle \) codes such that for \( i \in [k] \),

\[
\limsup_{n \to \infty} \frac{\log L_i^{(n)} - (\alpha_i n \log n + R_i n)}{n} \leq 0,
\]

and also \( P_e^{(n)} \to 0 \). We say that \( (\alpha_i, R_i) : i \in [k] \) lies in the rate region \( \mathcal{R} \) if there exist sequences \( R_i^{(m)} \) for \( i \in [k] \) such that \( R_i^{(m)} \to R_i \) as \( m \to \infty \) and, for each \( m \), \( (\alpha_i, R_i^{(m)}) : i \in [k] \) is achievable.

We can naturally generalize the Erdős–Rényi and the configuration model ensembles of Section 2 to the above setting.

**A sequence of Erdős–Rényi ensembles:** Given a sequence of nonnegative real numbers \( \vec{p} = \{p_x\}_{x \in \Xi[k]} \) and a probability distribution \( \vec{q} = \{q_{\theta}\}_{\theta \in \Theta[k]} \), assume that for all \( i \in [k] \) and \( x_i \in \Xi_i \) we have

\[
\sum_{(x'_j : j \in [k]) \in \Xi[k]: x_i' = x_i} p_{(x'_j : j \in [k])} > 0. \tag{51}
\]

Moreover, assume that for all \( i \in [k] \) and all \( \theta_i \in \Theta_i \) we have

\[
\sum_{(\theta'_j : j \in [k]) \in \Theta[k]: \theta_i' = \theta_i} q_{(\theta'_j : j \in [k])} > 0. \tag{52}
\]

For \( n \in \mathbb{N} \) large enough, the probability distribution \( \mathcal{G}(n; \vec{p}, \vec{q}) \) on \( \mathcal{G}_A^{(n)} \) is defined as follows: for each pair of vertices \( 1 \leq i < j \leq n \), the edge \((i, j)\) exists and has a mark \( x \in \Xi[k] \) with probability \( p_x / n \), and is not present with probability \( 1 - \sum_{x \in \Xi[k]} p_x / n \). Moreover, each vertex is independently given a mark \( \theta \in \Theta[k] \) with probability \( q_{\theta} \). The choices of edge and vertex marks are done independently.
The conditions in (51) and (52) are required only to ensure that the sets of vertex marks and edge marks are chosen to be as small as possible, and these conditions could be relaxed if desired.

A sequence of configuration model ensembles: Similar to the configuration model ensemble for two sources as we defined in Section 2, assume that $\Delta \in \mathbb{N}$ and a probability distribution $\vec{r} = \{r_i\}_{i=0}^\Delta$ is given such that $r_0 < 1$. Moreover, for each $n$, the degree sequence $d(n) = \{d^{(n)}(1), \ldots, d^{(n)}(n)\}$ is given such that for $i \in [n]$, $d^{(n)}(i) \leq \Delta$, $\sum_{i=1}^n d^{(n)}(i)$ is even, and (10) is satisfied. Additionally, assume that probability distributions $\vec{\gamma} = \{\gamma_x\}_{x \in \Xi[n]}$ and $\vec{q} = \{q_{\theta}\}_{\theta \in \Theta_i}$ are given such that for all $i \in [k]$ and $x_i \in \Xi_i$ we have

$$\sum_{(x'_j, j \in [k]) \in \Xi_A[x'_i = x_i]} \gamma(x'_j, j \in [k]) > 0,$$

and for all $A \subseteq [k]$ nonempty, $A \neq [k]$, we have

$$\sum_{(x'_j, j \in [k]) \in \Xi_A, (x'_j, j \in A) = \varnothing_A} \gamma(x'_j, j \in [k]) > 0.$$

We also assume that for all $i \in [k]$ and $\theta_i \in \Theta_i$ we have

$$\sum_{(\theta'_j, j \in [k]) \in \Theta_A, \theta'_j = \theta_i} q(\theta'_j, j \in [k]) > 0.$$  

With these, for $n$ large enough, we define the probability distribution $\mathcal{G}(n; \vec{d}^{(n)}, \vec{\gamma}, \vec{q}, \vec{r})$ on $\mathcal{G}[n]$ as follows. Similar to the ensemble for two sources, we pick an unmarked graph on the vertex set $[n]$ uniformly at random among the set of graphs with maximum degree $\Delta$ such that for $0 \leq k \leq \Delta$, $c_k(d_G) = c_k(\vec{d}(n))$. Then, we assign i.i.d. marks with law $\vec{\gamma}$ on the edges and i.i.d. marks with law $\vec{q}$ on the vertices.

The conditions in (53) and (55) are required only to ensure that the sets of vertex marks and edge marks are chosen to be as small as possible, and these conditions could be relaxed if desired. However, the conditions in (54) are essential, since they ensure that for all $A \subseteq [k]$ nonempty, $A \neq [k]$, the underlying unmarked graph of the projection of the overall graph onto domain $A$ is not a subgraph of the underlying unmarked graph of the projection onto domain $A^c$.

Similar to our discussion in Section 3, it can be seen that the local weak limit of the sequence of Erdős–Rényi ensembles above is a marked Poisson Galton–Watson tree, which we denote by $\mu_{[k]}^{\text{ER}}$. Likewise, the local weak limit of the sequence of configuration model ensembles above is a marked Galton–Watson tree with degree distribution $\vec{r}$, which we denote by $\mu_{[k]}^{\text{CM}}$. For $A \subseteq [k]$ nonempty, we denote the projection of $\mu_{[k]}^{\text{ER}}$ and $\mu_{[k]}^{\text{CM}}$ to domain $A$ by $\mu_A^{\text{ER}}$ and $\mu_A^{\text{CM}}$, respectively. For nonempty $A \subseteq [k]$, $A \neq [k]$, we define $\Sigma(\mu_A^{\text{ER}} | \mu_{[k]}^{\text{ER}})$ to be $\Sigma(\mu_A^{\text{ER}}) - \Sigma(\mu_{[k]}^{\text{ER}})$. We similarly define $\Sigma(\mu_A^{\text{CM}} | \mu_{[k]}^{\text{CM}})$.

We are now ready to characterize the rate region for the multi-source scenarios above in the following Theorem 4. This is a generalization of Theorem 3, and its proof is similar to that of Theorem 3. We highlight the proof of Theorem 4 in Appendix G.

**Theorem 4.** Assume $\mu_{[k]}$ is either of the two distributions $\mu_{[k]}^{\text{ER}}$ or $\mu_{[k]}^{\text{CM}}$ defined above. Then, if $\mathcal{R}$ is the rate region for the sequence of ensembles corresponding to $\mu_{[k]}$, as defined above, a rate tuple $((\alpha_i, R_i) : i \in [k]) \in \mathcal{R}$ if and only if for every nonempty $A \subseteq [k]$, $A \neq [k]$, we have

$$\sum_{i \in A} \alpha_i, \sum_{i \in A} R_i \geq \left((d_{[k]} - d_{A^c})/2, \Sigma(\mu_A | \mu_{[k]})\right),$$

where $d_{[k]}$ is the degree in $\mu_{[k]}$.
\[
\left( \sum_{i \in [k]} \alpha_i, \sum_{i \in [k]} R_i \right) \succeq \left( \frac{d_{[k]}}{2}, \Sigma(\mu_{[k]}) \right),
\]
where \( d_{[k]} = \text{deg}(\mu_{[k]}) \) and \( d_{Ac} = \text{deg}(\mu_{Ac}) \).

6 Conclusion

We gave a counterpart of the Slepian–Wolf Theorem for distributed compression of graphical data, employing the framework of local weak convergence. We derived the rate region for two families of sequences of graph ensembles, namely sequences of Erdős–Rényi ensembles having a local weak limit and sequences of configuration model ensembles having a local weak limit. Furthermore, we gave a generalization of this result for Erdős–Rényi and configuration model ensembles with more than two sources.

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A Proof of Lemma 1

Throughout this section, we treat $0 \log 0$ as equal to 0. Consider the first part of Lemma 1. Since $a_n/n \to a > 0$ as $n \to \infty$, using Stirling’s approximation we have $\log a_n! = a_n \log a_n - a_n + o(n)$. Similarly, from the assumption that $b_i^n/n \to b_i \geq 0$ as $n \to \infty$ for $1 \leq i \leq k$, we have $\log b_i^n! = b_i^n \log b_i^n - b_i^n + o(n)$, which holds irrespective of whether $b_i > 0$ or $b_i = 0$. Hence we have

$$\log \left( \sum_{1 \leq i \leq k} b_i^n \right) = a_n \log a_n - a_n - \sum_{i=1}^k b_i^n \log b_i^n + \sum_{i=1}^k b_i^n + o(n)$$

$$= a_n \log \frac{a_n}{n} - \sum_{i=1}^k \frac{b_i^n}{n} \log \frac{b_i^n}{n} + o(n),$$

where we have used $a_n = \sum_{i=1}^k b_i^n$. This gives

$$\lim_{n \to \infty} \frac{1}{n} \log \left( \sum_{1 \leq i \leq k} b_i^n \right) = a \log a - \sum_{i=1}^k b_i \log b_i$$
where we have used

Therefore, we have

Using \((b_i^n)\) and since \(b_i^n \to b_i \geq 0\) as \(n \to \infty\) for \(1 \leq i \leq k\), we have \(\log(b_i^n) = b_i^n \log b_i^n - b_i^n + o(n)\), which holds irrespective of whether \(b_i > 0\) or \(b_i = 0\). Moreover, with \(b_n := \sum_{i=1}^k b_i^n\), we have \(\log(a_n - b_n)! = (a_n - b_n) \log(a_n - b_n) - (a_n - b_n) + o(n)\).

Next, consider the second part of Lemma 1. Since \(a_n / \binom{n}{2} \to 1\) as \(n \to \infty\), using Stirling’s approximation we have \(\log(a_n! - a_n log a_n - a_n + o(n))\). As noted earlier, since \(b_i^n / n \to b_i \geq 0\) as \(n \to \infty\) for \(1 \leq i \leq k\), we have \(\log(b_i^n) = b_i^n \log b_i^n - b_i^n + o(n)\), which holds irrespective of whether \(b_i > 0\) or \(b_i = 0\). Moreover, with \(b_n := \sum_{i=1}^k b_i^n\), we have \(\log(a_n - b_n)! = (a_n - b_n) \log(a_n - b_n) - (a_n - b_n) + o(n)\).

Therefore, we have

\[
\log \left( \frac{a_n}{\{b_i^n\}_{1 \leq i \leq k}} \right) = a_n \log a_n - a_n - \sum_{i=1}^k b_i^n \log b_i^n + \sum_{i=1}^k b_i^n - (a_n - b_n) \log(a_n - b_n) + (a_n - b_n) + o(n)
\]

\[
= a_n \log a_n - \sum_{i=1}^k b_i^n \log \left( \frac{a_n}{b_i^n} \right) - (a_n - b_n) \log \left( \frac{a_n - b_n}{n} \right) + o(n),
\]

where we have used \(a_n = b_n + (a_n - b_n)\). This gives

\[
\frac{1}{n} \log \left( \frac{a_n}{\{b_i^n\}_{1 \leq i \leq k}} \right) = \frac{a_n}{n} \log a_n - \sum_{i=1}^k b_i^n \log \left( \frac{a_n - b_n}{n} \right) = \frac{a_n}{n} \log a_n - \sum_{i=1}^k b_i^n \log \left( \frac{a_n - b_n}{n} \right) + o(1)
\]

\[
= -\frac{a_n}{n} \log \left( 1 - \frac{b_n}{a_n} \right) + \frac{b_n}{n} \log \left( \frac{a_n - b_n}{a_n} \right) + \frac{b_n}{n} \log \left( \frac{a_n - b_n}{a_n} \right) - \sum_{i=1}^k \frac{b_i^n}{n} \log \left( \frac{a_n - b_n}{n} \right) + o(1).
\]

Since \(b_n/a_n \to 0\) as \(n \to \infty\), we write \(\log(1 - b_n/a_n) = -b_n/a_n + O(b_n^2/a_n^2)\). Consequently, we have

\[
-\frac{a_n}{n} \log \left( 1 - \frac{b_n}{a_n} \right) = \frac{b_n}{a_n} + O\left( \frac{b_n^2}{na_n} \right),
\]

and since \(\frac{b_n^2}{na_n} \to 0\) we have

\[
\lim_{n \to \infty} -\frac{a_n}{n} \log \left( 1 - \frac{b_n}{a_n} \right) = b.
\]

Further, since \((a_n - b_n)/(n^2/2) \to 1\) we have

\[
\lim_{n \to \infty} \frac{b_n}{n} \log \left( \frac{a_n - b_n}{n^2/2} \right) = 0.
\]

Using (57) and (58) in (56), we get

\[
\lim_{n \to \infty} \frac{\log \left( \frac{a_n}{\{b_i^n\}_{1 \leq i \leq k}} \right) - b_n \log n}{n} = b - b \log 2 - \sum_{i=1}^k b_i \log b_i
\]

\[
= \sum_{i=1}^k s(2b_i),
\]

which completes the proof.
B Calculations for Deriving (12)

Note that we have

\[
|G_{\tilde{m}(n), \tilde{u}(n)}^{(n)}| = \frac{n!}{\prod_{\theta \in \Theta} u^{(n)}(\theta)!} \times \frac{n(n-1)!}{\prod_{x \in \Xi} m^{(n)}(x)! \times \left(\frac{n(n-1)}{2} - \|\tilde{m}^{(n)}\|_1\right)!}. \tag{59}
\]

Since \(u^{(n)}(\theta)/n \to q_\theta\) for all \(\theta \in \Theta\), from part 1 of Lemma 1 we have

\[
\log \frac{n!}{\prod_{\theta \in \Theta} u^{(n)}(\theta)!} = nH(Q) + o(n). \tag{60}
\]

Moreover, since for all \(x \in \Xi\) we have \(m^{(n)}(x)/n \to d_x/2 < \infty\), from part 2 of Lemma 1 we have

\[
\log \frac{n(n-1)!}{\prod_{x \in \Xi} m^{(n)}(x)! \times \left(\frac{n(n-1)}{2} - \|\tilde{m}^{(n)}\|_1\right)!} = \|\tilde{m}^{(n)}\|_1 \log n + n \sum_x s(d_x) + o(n). \tag{61}
\]

Using (60) and (61) in (59), we get

\[
\log |G_{\tilde{m}(n), \tilde{u}(n)}^{(n)}| = \|\tilde{m}^{(n)}\|_1 \log n + nH(Q) + n \sum_x s(d_x) + o(n),
\]

which is precisely what was stated in (12).

C Proof of Lemma 3

The assumptions of the lemma imply that \(b_n/n \to d/2 > 0\) and, in particular, \(b_n \to \infty\) as \(n \to \infty\). Therefore, Theorem 4.6 in [McK85] implies that

\[
\lim_{n \to \infty} \frac{|G_{\tilde{a}(n)}^{(n)}|}{\alpha_n \prod_{i=1}^{b_n-1} a^{(n)}(i)!} = 1,
\]

where

\[
\alpha_n := \exp \left(-\lambda_n - \lambda_n^2\right), \quad \lambda_n := \frac{1}{2b_n} \sum_{i=1}^{n} a^{(n)}(i)(a^{(n)}(i) - 1),
\]

and

\[
(b_n - 1)! := (b_n - 1) \times (b_n - 3) \times \cdots \times 1 = \frac{b_n!}{2^{b_n/2}(b_n/2)!}.
\]

Under the assumptions of the lemma, we have \(b_n/n \to d/2\) as \(n \to \infty\). Therefore, using Stirling’s approximation, we have \(\log(b_n - 1)! = \frac{b_n}{2} \log n - n s(d) + o(n)\). Moreover, since \(c_k(\tilde{a}^{(n)})/n \to P(Y = k)\) as \(n \to \infty\) for all \(0 < k \leq \Delta\), we have

\[
\frac{1}{n} \log \prod_{i=1}^{n} a^{(n)}(i)! = \frac{1}{n} \sum_{k=0}^{\Delta} c_k(\tilde{a}^{(n)}) \log k! = \mathbb{E} [\log Y!] + o(1).
\]

On the other hand, we have

\[
\lim_{n \to \infty} \lambda_n = \lim_{n \to \infty} \frac{1}{2b_n/n} \sum_{i=1}^{n} a^{(n)}(i)(a^{(n)}(i) - 1)
\]

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\[ = \frac{1}{d} \lim_{n \to \infty} \frac{\Delta}{n} \sum_{k=1}^{c_k(d(n))} k(k-1) \]

\[ = \frac{1}{d} E[Y(Y-1)] := \lambda. \]

This implies that, as \( n \to \infty \), \( \alpha_n \to \alpha := \exp(-\lambda - \lambda^2) > 0 \). Therefore, \( \frac{1}{n} \log \alpha_n \to 0 \) as \( n \to \infty \). Putting these together, we get the desired result.

**D Asymptotic behavior of the entropy of the configuration model**

Here, we prove (16a)–(16c). Let \( X \) be a random variable with law \( \theta \), and let \( X_1 \) and \( X_2 \) be defined as in (15). Let \( \Gamma = (\Gamma_1, \Gamma_2) \) and \( Q = (Q_1, Q_2) \) denote random variables with laws \( \gamma \) and \( \theta \), respectively. Let \( \beta_i := \mathbb{P}(\Gamma_i \neq \omega_1) \) and let \( \Gamma_1 \) be a random variable on \( \Xi \) with the law of \( \Gamma_1 \) conditioned on \( \Gamma_1 \neq \omega_1 \).

As in Section 5.2, we let \( D(n) \) denote the set of degree sequences \( d = (d(1), \ldots, d(n)) \) with entries bounded by \( \Delta \) such that \( c_k(d) \) for all \( 0 \leq k \leq \Delta \). Let \( F_{1,2}^{(n)} \) be a simple unmarked graph chosen uniformly at random from the set \( \{ \mathbb{P}_{\Gamma_1} \} \), where we recall that \( \mathbb{P}_{\Gamma_1} \) denotes the set of degree sequences \( d \) on the vertex set \( [n] \) such that \( d_G(i) = d(i) \) for \( 1 \leq i \leq n \). By definition, \( G_{1,2}^{(n)} \sim G(n; d(n), \gamma, \theta) \) is obtained from \( F_{1,2}^{(n)} \) by adding independent edge and vertex marks according to the laws of \( \gamma \) and \( \theta \), respectively. If we first create \( G_{1,2}^{(n)} \) from \( F_{1,2}^{(n)} \), and then drop the edges with the first domain mark \( \omega_1 \), if \( F_{1}^{(n)} \) denotes the unmarked version of the resulting marked graph, then \( F_{1}^{(n)} \) is effectively obtained from \( F_{1,2}^{(n)} \) by independently removing each edge with probability \( 1 - \beta_1 \). Also, the corresponding first domain marked graph, i.e. \( G_{1}^{(n)} \), obtained from \( G_{1,2}^{(n)} \) in this way is effectively obtained from \( F_{1}^{(n)} \) by adding independent vertex and edge marks to \( F_{1}^{(n)} \) with the laws of \( Q_1 \) and \( \Gamma_1 \), respectively. With this viewpoint, we may consider \( G_{1,2}^{(n)}, F_{1,2}^{(n)}, G_{1}^{(n)} \) and \( F_{1}^{(n)} \) as being defined on a joint probability space.

As in Section 5.2, we let \( W(n) \) denote the set of graphs \( H_{1,2}^{(n)} \in \mathcal{M}^{(n)} \) such that: (i) \( \overrightarrow{d}_{H_{1,2}^{(n)}} \in D^{(n)} \), (ii) \( m_{H_{1,2}^{(n)}} \in \mathcal{M}^{(n)} \), (iii) \( m_{H_{1,2}^{(n)}} \in U^{(n)} \), (iv) for all \( 0 \leq l \leq k \leq \Delta \), recalling the notation of (2), we have

\[
|c_{k,l}(\overrightarrow{d}_{H_{1,2}^{(n)}}, \overrightarrow{d}_{H_{1,2}^{(n)}}) - nP(X = k, X_1 = l)| \leq n^{2/3},
\]

and (v) for all \( 0 \leq l \leq k \leq \Delta \) we have

\[
|c_{k,l}(\overrightarrow{d}_{H_{1,2}^{(n)}}, \overrightarrow{d}_{H_{1,2}^{(n)}}) - nP(X = k, X_2 = l)| \leq n^{2/3}.
\]

Here, as in Section 5.2, \( M^{(n)} \) denotes the set of mark count vectors \( m \) such that \( \sum_{x \in \Xi_{1,2}} m(x) = m_n \) and \( \sum_{x \in \Xi_{1,2}} |m(x) - m_n \gamma| \leq n^{2/3} \), where we recall that \( m_n := (\sum_{i=1}^{n} d^{(n)}(i))/2 \), while, as in Section 5.2, \( U^{(n)} \) denotes the set of vertex mark count vectors \( u \) such that \( \sum_{\theta \in \Theta_{1,2}} |u(\theta) - nu_{\theta}| \leq n^{2/3} \).

We can now prove the following lemma.

**Lemma 4**. If \( G_{1,2}^{(n)} \sim G(n; \overrightarrow{d}^{(n)}, \gamma, \theta) \), we have \( \mathbb{P}(G_{1,2}^{(n)} \notin W^{(n)}) \leq \kappa n^{-1/3} \) for some constant \( \kappa > 0 \).

**Proof.** Condition (i) in the definition of \( W^{(n)} \) holds for every realization of \( G_{1,2}^{(n)} \). Chebyshev’s inequality implies that conditions (ii) and (iii) hold with probability at least \( 1 - \kappa_1 n^{-1/3} \), for some \( \kappa_1 > 0 \). To
show (iv), fix $0 \leq l \leq k \leq \Delta$ and, for $1 \leq i \leq n$, let $Y_i$ be the indicator of the event that $d_{G_1^{(n)}}(i) = k$ and $d_{G_1^{(n)}}(i) = l$. With $Y := \sum_{i=1}^{n} Y_i$, we have $c_{k,l}(\overrightarrow{d_{G_1^{(n)}}}, \overrightarrow{d_{G_1^{(n)}}}) = Y$. Note that an edge of $G_1^{(n)}$ exists in $G_1^{(n)}$ if its mark is not of the form $(a_1, x_2)$, which happens with probability $\beta_1$. Therefore,

$$E[Y_i|F_{1,2}^{(n)}] = \mathbb{1} \left[ d_{G_{1,2}^{(n)}}(i) = k \right] \left( \frac{d_{G_{1,2}^{(n)}}(i)}{l} \right) \beta_1^l (1 - \beta_1)^{k-l}. $$

Consequently,

$$E[Y|F_{1,2}^{(n)}] = c_k(d^n) \left( \frac{k}{l} \right) \beta_1^l (1 - \beta_1)^{k-l}. $$

Since this is a constant, it is also equal to $E[Y]$. Now, if $s_{k,l} := \mathbb{P}(X = k, X_1 = l)$, we have $s_{k,l} = r_k(\frac{l}{k}) \beta_1^l (1 - \beta_1)^{k-l}$. Hence the assumption in (10) implies that

$$|E[Y] - ns_{k,l}| \leq Kn^{1/2} \left( \frac{k}{l} \right) \beta_1^l (1 - \beta_1)^{k-l}. \quad (62)$$

Furthermore, since edge marks are chosen independently conditioned on $F_{1,2}^{(n)}$, if $i$ and $j$ are nonadjacent vertices in $F_{1,2}^{(n)}$, then $Y_i$ and $Y_j$ are conditionally independent, conditioned on $F_{1,2}^{(n)}$. As a result, if $\mathcal{I}$ denotes the set of $(i, j)$ with $1 \leq i \neq j \leq n$ such that $i$ and $j$ are not adjacent in $F_{1,2}^{(n)}$, we have

$$E[Y^2|F_{1,2}^{(n)}] = \sum_{i=1}^{n} E[Y_i^2|F_{1,2}^{(n)}] + \sum_{1 \leq i \neq j \leq n} E[Y_i Y_j|F_{1,2}^{(n)}] \leq n + \sum_{(i,j) \notin \mathcal{I}} E[Y_i Y_j|F_{1,2}^{(n)}] + \sum_{(i,j) \in \mathcal{I}} E[Y_i Y_j|F_{1,2}^{(n)}] \leq n + 2m_n + \sum_{(i,j) \in \mathcal{I}} E[Y_i Y_j|F_{1,2}^{(n)}] \leq n + 2m_n + \sum_{(i,j) \in \mathcal{I}} E[Y_i|F_{1,2}^{(n)}] E[Y_j|F_{1,2}^{(n)}] \leq n + 2m_n + \sum_{1 \leq i \neq j \leq n} E[Y_i|F_{1,2}^{(n)}] E[Y_j|F_{1,2}^{(n)}] \leq n + 2m_n + \mathbb{E}\left[Y|F_{1,2}^{(n)}\right]^2,$$

where (a) uses the fact that, conditioned on $F_{1,2}^{(n)}$, the random variables $Y_i$ and $Y_j$ are conditionally independent for $(i, j) \in \mathcal{I}$. From (10), we have $|m_n - nd_{G_{1,2}^{(n)}}/2| \leq \kappa_2 K n^{1/2}$, where $\kappa_2 := (\Delta + 1)/2$ and $d_{G_{1,2}^{(n)}} := \deg(\mu_{G_{1,2}^{(n)}}) = \sum_{k=0}^{\Delta} k r_k$. As a consequence of the above discussion, we have $\text{Var}(Y|F_{1,2}^{(n)}) \leq \kappa_3 n$ for some $\kappa_3 > 0$. On the other hand, as we saw above, $\mathbb{E}\left[Y|F_{1,2}^{(n)}\right] = \mathbb{E}[Y]$. Therefore, using the law of total variance, we have $\text{Var}(Y) \leq \kappa_3 n$. This, together with (62) and Chebyshev’s inequality, implies that the condition (iv) holds with probability at least $1 - \kappa_4 n^{-1/3}$, for some $\kappa_4 > 0$. Similarly, the same statement holds for condition (v).

Let $B_{1,2}^{(n)}$ be the set of pairs of degree sequences $\tilde{d}$ and $\tilde{\delta}$ with $n$ elements bounded by $\Delta$ such that for all $0 \leq k, l \leq \Delta$, $|c_{k,l}(\tilde{d}, \tilde{\delta}) - n \mathbb{P}(X_1 = k, X - X_1 = l)| \leq n^{2/3}$. Moreover, let $B_{1}^{(n)}$ be the set of $\tilde{d}$
such that for some \( \delta \), we have \((\vec{d}, \delta) \in B_{1,2}^{(n)}\). For \( \vec{d} \in B_{1}^{(n)} \), let \( B_{2|1}(\vec{d}) \) be the set of degree sequences \( \bar{\delta} \) such that \((\vec{d}, \bar{\delta}) \in B_{1,2}^{(n)}\).

In order to show (16a), note that since \( G_{1,2}^{(n)} \) is formed by adding independent vertex and edge marks to \( F_{1,2}^{(n)} \), we have

\[
H(G_{1,2}^{(n)}) = \log |D^{(n)}| + \log |G_{d,n}^{(n)}| + m_n H(\Gamma) + n H(Q).
\]

From (10), we have \(|m_n - nd_{1,2}^{CM}/2| \leq \frac{(\Delta + 1)K}{2} n^{1/2} \). Moreover, we have \( E[X] > 0 \). Consequently, using Lemma 3 and the fact that \( \frac{1}{n} \log |D^{(n)}| \to H(X) \), we get (16a).

We now turn to showing (16b). Since the expected number of the edges in \( F_{1}^{(n)} \) is \( nd_{1}^{CM}/2 \), we have

\[
H(G_{1}^{(n)}) = H(F_{1}^{(n)}) + n \frac{d_{1}^{CM}}{2} H(\Gamma_{1} | \Gamma_{1} \neq \circ_{1}) + n H(Q_{1}).
\]

With this, we focus on \( H(F_{1}^{(n)}) \). With \( E_{n} \) being the indicator of the event that \( G_{1,2}^{(n)} \notin W^{(n)} \), we have

\[
H(F_{1}^{(n)} | E_{n} = 1) \leq H(F_{1}^{(n)}) \leq \log |D^{(n)}| + \log |G_{d,n}^{(n)}| + m_{n} \log 2 \\
\leq H(G_{1}^{(n)}) + m_{n} \log 2 \\
\leq \kappa' n \log n,
\]

where in the last line, \( \kappa' > 0 \) is obtained from (16a). Putting this together with Lemma 4, we have

\[
H(F_{1}^{(n)} | E_{n} = 1) P(E_{n} = 1) \leq \kappa' n \log n \kappa n^{-1/3}.
\]

Note that the right hand side of (65) above is \( o(n) \). On the other hand, by the definition of \( W^{(n)} \), if \( E = 0 \), we have \( d_{\vec{d}}^{cm} F_{1}^{(n)} \in B_{1}^{(n)} \). Therefore, \( H(F_{1}^{(n)} | E_{n} = 0) \leq \log |B_{1}^{(n)}| + \max_{\vec{d} \in B_{1}^{(n)}} \log |G_{d}^{(n)}| \). The assumption \( n_{0} < 1 \) together with (8) imply that \( d_{1}^{CM} > 0 \). Additionally note that, by definition, for \( \vec{d} \in B_{1}^{(n)} \), we have \( |e_{k}(\vec{d}) - n P(X_{1} = k)| \leq n^{2/3} \) for all \( 0 \leq k \leq \Delta \). Thereby, we have

\[
\limsup_{n \to \infty} \frac{\log |B_{1}^{(n)}|}{n} \leq H(X_{1}).
\]

Putting the above together with Lemma 3 and (65), we have

\[
\limsup_{n \to \infty} \frac{H(F_{1}^{(n)}) - n d_{1}^{CM} \log n}{n} \leq \limsup_{n \to \infty} \frac{\log |B_{1}^{(n)}| + \max_{\vec{d} \in B_{1}^{(n)}} \log |G_{d}^{(n)}| - n d_{1}^{CM} \log n}{n} \\
\leq \limsup_{n \to \infty} \frac{\log |B_{1}^{(n)}| + \max_{\vec{d} \in B_{1}^{(n)}} \log |G_{d}^{(n)}| - \sum_{i=1}^{\Delta} n i d_{i}^{CM} \log n}{n} \\
+ \max_{\vec{d} \in B_{1}^{(n)}} \frac{\sum_{i=1}^{\Delta} i d_{i}^{CM}}{n} \log n - n d_{1}^{CM} \log n \\
\leq -s(d_{1}^{CM}) + H(X_{1}) - E[\log X_{1}] \, ,
\]

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where in the last line, have used the fact that due to the definition of $B_1^{(n)}$, for $\tilde{d} \in B_1^{(n)}$, we have $|\sum_{i=1}^n d(i) - d_1^{CM}| \leq \Delta n^{2/3} = o(n/\log n)$. Now, let $\tilde{F}_1^{(n)}$ be the unmarked graph consisting of the edges removed from $F_1^{(n)}$ to obtain $F_1^{(n)}$, and note that

$$H(F_1^{(n)}) = H(F_1^{(n)}, \tilde{F}_1^{(n)}) - H(\tilde{F}_1^{(n)}|F_1^{(n)}) = H(F_1^{(n)}|F_1^{(n)}) + m_nH(\beta_1) - H(\tilde{F}_1^{(n)}|F_1^{(n)}).$$

Furthermore, conditioned on $E_n = 0$, we have $\tilde{d}_i \in B_2(n)(\tilde{d}_i^{(n)})$. Moreover, the assumption (\ref{eq:Gamma}), for $x_1 = \gamma_1$, together with $r_0 < 1$, implies that $d_1^G - d_1^{CM} > 0$. Hence, using a similar method to that used in proving (\ref{eq:CM1}), we have

$$\limsup_{n \to \infty} \frac{H(\tilde{F}_1^{(n)}|F_1^{(n)}) - n \frac{d_1^{CM} - d_1^G}{2} \log n}{\log n} \leq -s(d_1^{CM} - d_1^G) + H(X - X_1|X_1) - \log(X - X_1)].$$

To see this, with $E_n$ as defined previously, we may write

$$H(\tilde{F}_1^{(n)}|F_1^{(n)}) \leq 1 + H(\tilde{F}_1^{(n)}|F_1^{(n)}, E_n = 0) = H(\tilde{F}_1^{(n)}|F_1^{(n)}, E_n = 1) = \log(B_2(n)(\tilde{d}_i^{(n)})) + \max_{\delta \in B_2(n)(\tilde{d}_i^{(n)})} \log |G_\delta^{(n)}|.

Note that, conditioned on $E_n = 0$, we have $\tilde{d}_i \in B_1^{(n)}$. Hence, we have $\log(B_2(n)(\tilde{d}_i^{(n)})) = nH(X - X_1|X_1) + o(n)$. Furthermore, using Lemma 3 and the fact that for $\tilde{d} \in B_2(n)(\tilde{d}_i^{(n)})$, we have $|\sum_{i=1}^n \delta(i) - (d_1^{CM} - d_1^{CM})/2| \leq \Delta n^{2/3} = o(n/\log n)$, we have

$$\log |G_\delta^{(n)}| = n(-s(d_1^{CM} - d_1^{CM}) - \log(X - X_1)] + \frac{d_1^{CM} - d_1^{CM}}{2} \log n + o(n).

Putting the above together, we arrive at (\ref{eq:CM1}).

On the other hand, using the definition of $F_{1,2}$, we have $H(F_{1,2}^{(n)}) = \log |\mathcal{D}| + \log |G_\delta^{(n)}|$. Employing Lemma 3 and using the assumption (\ref{eq:CM2}), we have

$$\lim_{n \to \infty} \frac{\log |G_\delta^{(n)}| - n \frac{d_1^{CM} - d_1^{CM}}{2} \log n}{n} = -s(d_1^{CM} - \log X)].$$

Furthermore, we have $\np \log |\mathcal{D}| \to H(X)$. Therefore, we have

$$\lim_{n \to \infty} \frac{H(F_{1,2}^{(n)} - n \frac{d_1^{CM} - d_1^{CM}}{2} \log n}{n} = -s(d_1^{CM} + H(X) - \log X)].$$
Using (68) and (69) back in (67), followed by a simplification using Lemma 2, we get

\[
\liminf_{n \to \infty} \frac{H(F_1^{(n)}) - n \frac{d_{1,2}^{CM}}{2} \log n}{n} = \liminf_{n \to \infty} \frac{H(F_1^{(n)}) - n \frac{d_{1,2}^{CM}}{2} \log n + m_n \beta_1 - H(F_1^{(n)}) + n \frac{d_{1,2}^{CM} - d_1^{CM}}{2}}{n} \\
\geq \liminf_{n \to \infty} \frac{H(F_1^{(n)}) - n \frac{d_{1,2}^{CM}}{2} \log n + \frac{d_{1,2}^{CM}}{2} H(\beta_1)}{n} - \limsup_{n \to \infty} \frac{H(F_1^{(n)}) - n \frac{d_{1,2}^{CM} - d_1^{CM}}{2} \log n}{n} \\
\geq -s(d_{1,2}^{CM}) + H(X) - E[\log X] + \frac{d_{1,2}^{CM}}{2} H(\beta_1) \\
\geq -s(d_{1,2}^{CM}) + H(X) - E[\log X] + \frac{d_{1,2}^{CM}}{2} H(\beta_1) + H(X - X_1 | X_1) - E[\log(X - X_1)!!] \\
= H(X) + d_{1,2}^{CM} H(\beta_1) - E \left[ \log \left( \frac{X}{X_1} \right) \right] - E[\log X_1] \\
- \frac{d_{1,2}^{CM}}{2} H(\beta_1) - s(d_{1,2}^{CM}) + s(d_{1,2}^{CM} - d_1^{CM}) - H(X - X_1 | X_1) \\
= H(X_1, X - X_1) - H(X - X_1 | X_1) - E[\log X_1] \\
- s(d_{1,2}^{CM}) + s(d_{1,2}^{CM} - d_1^{CM}) - \frac{d_{1,2}^{CM}}{2} H(\beta_1) \\
= H(X_1) - E[\log X_1] - s(d_{1,2}^{CM}) + s(d_{1,2}^{CM} - d_1^{CM}) - \frac{d_{1,2}^{CM}}{2} H(\beta_1),
\]

where in (a), we have used Lemma 2. Since \( \beta_1 = d_{1,2}^{CM} / d_{1,2}^{CM} \), we may write

\[
-s(d_{1,2}^{CM}) + s(d_{1,2}^{CM} - d_1^{CM}) - \frac{d_{1,2}^{CM}}{2} H(\beta_1) = \frac{d_{1,2}^{CM}}{2} \log d_{1,2}^{CM} - \frac{d_{1,2}^{CM}}{2} + \frac{d_{1,2}^{CM}}{2} - \frac{d_1^{CM}}{2} \\
- \frac{d_{1,2}^{CM}}{2} \log (d_{1,2}^{CM} - d_1^{CM}) + \frac{d_1^{CM}}{2} \log (d_{1,2}^{CM} - d_1^{CM}) \\
+ \frac{d_1^{CM}}{2} \log d_1^{CM} - \frac{d_1^{CM}}{2} \log d_{1,2}^{CM} + \frac{d_{1,2}^{CM}}{2} \log (d_{1,2}^{CM} - d_1^{CM}) \\
- \frac{d_{1,2}^{CM}}{2} \log (d_{1,2}^{CM} - d_1^{CM}) - \frac{d_{1,2}^{CM}}{2} \log d_{1,2}^{CM} + \frac{d_1^{CM}}{2} \log d_{1,2}^{CM} \\
= -s(d_1^{CM}) + s(d_{1,2}^{CM} - d_1^{CM}).
\]

Substituting this into (70), we arrive at

\[
\liminf_{n \to \infty} \frac{H(F_1^{(n)}) - n \frac{d_1^{CM}}{2} \log n}{n} \geq -s(d_1^{CM}) + H(X_1) - E[\log X_1].
\]

This, together with (66) and (63), completes the proof of (16b). The proof of (16c) is similar.
E Bounding $|S_2^{(n)}(H_1^{(n)})|$ for $H_{1,2}^{(n)} \in \mathcal{G}_{1,2}^{(n)}$ in the Erdős–Rényi case

Note that for $H_{1,2}^{(n)} \in \mathcal{G}_{1,2}^{(n)}$ and $G_2^{(n)} \in \mathcal{G}_2^{(n)}$, if $H_{1,2}^{(n)} \oplus G_2^{(n)} \in \mathcal{G}_{1,2}^{(n)}$, we have $\tilde{m}_{H_{1,2}^{(n)} \oplus G_2^{(n)}} \in \mathcal{M}^{(n)}$ and $
abla H_{1,2}^{(n)} \oplus G_2^{(n)} \in \mathcal{U}^{(n)}$. On the other hand, for fixed $\tilde{m} \in \mathcal{M}^{(n)}$ and $\nabla \in \mathcal{U}^{(n)}$, the number of $G_2^{(n)}$ such that $\tilde{m}_{H_{1,2}^{(n)} \oplus G_2^{(n)}} = \tilde{m}$ and $\nabla_{H_{1,2}^{(n)} \oplus G_2^{(n)}} = \nabla$ is at most

$$A_2(\tilde{m}, \nabla) := \left( \prod_{x_1 \in \Xi_1} \left( m(x_1) \right) \right) \times \left( \prod_{x_2 \in \Xi_2 \cup \{02\}} \left( m(x_1) \right) \right) \times \left( \prod_{\theta_1 \in \Theta_1} \left( u(\theta_1) \right) \right) \times \left( \prod_{\theta_2 \in \Theta_2} \left( u(\theta_2) \right) \right),$$

where we have used the notational conventions in (3) and (4). Consequently, we have

$$\max_{H_{1,2}^{(n)} \in \mathcal{G}_{1,2}^{(n)}} |S_2^{(n)}(H_1^{(n)})| \leq |\mathcal{M}^{(n)}| |\mathcal{U}^{(n)}| \max_{\tilde{m} \in \mathcal{M}^{(n)}} A_2(\tilde{m}, \nabla) \leq (2n^{2/3} + 1)(|\Xi_2| + |\Theta_1, 2|) \max_{\tilde{m} \in \mathcal{M}^{(n)}} A_2(\tilde{m}, \nabla). \quad (72)$$

Now, if $\tilde{m}^{(n)}$ and $\nabla^{(n)}$ are sequences in $\mathcal{M}^{(n)}$ and $\mathcal{U}^{(n)}$, respectively, then for all $x \in \Xi_{1,2}$ we have $m^{(n)}(x)/n \rightarrow p_x/2$. Furthermore, for all $x_1 \in \Xi_1$ and $\theta_1 \in \Theta_1$, we have $m^{(n)}(x_1)/n \rightarrow p_x/2$ and $u^{(n)}(\theta_1)/n \rightarrow q_{\theta_1}$. As a result, using Lemma 1, for any such sequences $\tilde{m}^{(n)}$ and $\nabla^{(n)}$, with $Q = (Q_1, Q_2)$ having law $\tilde{q}$, we have

$$\lim_{n \to \infty} \frac{\log A_2(\tilde{m}^{(n)}, \nabla^{(n)}) - (\sum_{x_2 \in \Xi_2} m^{(n)}(x_1, x_2)) \log n}{n} \quad = \sum_{x_2 \in \Xi_2} s(p_{01, x_2}) + \sum_{x_1 \in \Xi_1} \frac{p_{x_1} H \left( \left\{ \frac{p(x_1, x_2)}{p_x} \right\}_{x_2 \in \Xi_2 \cup \{02\}} \right)}{2}$$

$$+ \sum_{\theta_1 \in \Theta_1} q_{\theta_1} H \left( \left\{ \frac{q_{\theta_1, \theta_2}}{q_{\theta_1}} \right\}_{\theta_2 \in \Theta_2} \right)$$

$$= H(Q_2 | Q_1) + \sum_{x \in \Xi_{1,2}} s(p_x) - \sum_{x_1 \in \Xi_1} s(p_{x_1})$$

$$= \Sigma(\mu_2^{\text{ER}} | \mu_1^{\text{ER}}),$$

where the second equality follows by rearranging the terms and using the definition of $s(.)$. Using the fact that $m^{(n)}(x_1, x_2) - np_{x_1, x_2}/2 \leq n^{2/3}$, we have

$$\lim_{n \to \infty} \frac{\log A_2(\tilde{m}^{(n)}, \nabla^{(n)}) - n \frac{d_{\text{ER}}(\mu_2, \mu_1)}{2} \log n}{n} = \Sigma(\mu_2^{\text{ER}} | \mu_1^{\text{ER}}).$$

This together with (72) implies (23).

F Bounding $|S_2^{(n)}(H_1^{(n)})|$ for $H_{1,2}^{(n)} \in \mathcal{W}^{(n)}$ in the configuration model

Here, we find an upper bound for $\max_{H_{1,2}^{(n)} \in \mathcal{W}^{(n)}} |S_2^{(n)}(H_1^{(n)})|$, where $\mathcal{W}^{(n)}$ is defined in Section 5.2, and use it to show (29). Take $H_{1,2}^{(n)} \in \mathcal{W}^{(n)}$ and assume $\tilde{H}_2^{(n)} \in S_2^{(n)}(H_1^{(n)})$. With $\tilde{H}_1^{(n)} := H_1^{(n)} \oplus \tilde{H}_2^{(n)},$
let \( \overline{H}_1^{(n)} \) be the subgraph of \( \hat{H}_1^{(n)} \) consisting of the edges not present in \( H_1^{(n)} \). Employing the notation of Appendix D, we have \( \overrightarrow{dg}_{\hat{H}_1^{(n)}} \in B_{2|1}^{(n)}(\overrightarrow{dg}_{H_1^{(n)}}) \), which follows from the definition of the set \( \mathcal{W}^{(n)} \). Therefore, we can think of \( \hat{H}_1^{(n)} \) as being constructed from \( H_1^{(n)} \) by adding a graph to \( H_1^{(n)} \) with degree sequence \( \overrightarrow{dg}_{\hat{H}_1^{(n)}} \in B_{2|1}^{(n)}(\overrightarrow{dg}_{H_1^{(n)}}) \), marking its edges, adding second domain marks to edges in \( H_1^{(n)} \), and also adding second domain marks to vertices. Motivated by this, we have

\[
\max_{H_1^{(n)} \in \mathcal{W}^{(n)}} \log |S_2^{(n)}(H_1^{(n)})| \leq \max_{H_1^{(n)} \in \mathcal{W}^{(n)}} \log |B_{2|1}^{(n)}(\overrightarrow{dg}_{H_1^{(n)}})| + \max_{H_1^{(n)} \in \mathcal{W}^{(n)}, \delta \in B_{2|1}^{(n)}(\overrightarrow{dg}_{H_1^{(n)}})} \log |G_{\theta}^{(n)}|
\]

\[
+ \max_{u \in M^{(n)}} \log \left| \prod_{\theta_1 \in \Theta_1} \left( \frac{\log |G_{\theta_1}^{(n)}| - \sum_{i=1}^{n} \overline{\delta}_i \log n}{n} \right) \right|
\]

(73)

We establish an upper bound for each term. The definition of \( B_{2|1}^{(n)} \) implies that

\[
\lim_{n \to \infty} \frac{1}{n} \max_{H_1^{(n)} \in \mathcal{W}^{(n)}} \log |B_{2|1}^{(n)}(\overrightarrow{dg}_{H_1^{(n)}})| = H(X - X_1|X_1),
\]

(74)

where \((X, X_1)\) are defined as in Section 5.2. Note that the assumption (8), for \( x_1 = 0_1 \), together with \( r_0 < 1 \), implies that \( d_{1,2}^{CM} - d_1^{CM} > 0 \). On the other hand, we have

\[
\limsup_{n \to \infty} \max_{H_1^{(n)} \in \mathcal{W}^{(n)}} \frac{\log |G_{\theta}^{(n)}| - \sum_{i=1}^{n} \overline{\delta}_i \log n}{n} \leq \limsup_{n \to \infty} \max_{H_1^{(n)} \in \mathcal{W}^{(n)}, \delta \in B_{2|1}^{(n)}(\overrightarrow{dg}_{H_1^{(n)}})} \frac{\log |G_{\theta}^{(n)}| - \sum_{i=1}^{n} \delta_i \log n}{n}
\]

\[
+ \limsup_{n \to \infty} \max_{H_1^{(n)} \in \mathcal{W}^{(n)}, \delta \in B_{2|1}^{(n)}(\overrightarrow{dg}_{H_1^{(n)}})} \frac{1}{n} \left( \sum_{i=1}^{n} \overline{\delta}_i \log n - n \frac{d_{1,2}^{CM} - d_1^{CM}}{2} \log n \right).
\]

(75)

By definition, for \( \overline{\delta} = (\overline{\delta}_1, \ldots, \overline{\delta}_n) \in B_{2|1}^{(n)}(\overrightarrow{dg}_{H_1^{(n)}}) \), we have

\[
\left| \left( \sum_{i=1}^{n} \overline{\delta}_i \right) - n(d_{1,2}^{CM} - d_1^{CM}) \right| = \left| \left( \sum_{k=0}^{\Delta} kc_k(\overline{\delta}) \right) - nE[X - X_1] \right|
\]

\[
\leq \sum_{k=0}^{\Delta} k |c_k(\overline{\delta}) - nP(X - X_1 = k)|
\]

\[
\leq \sum_{k=0}^{\Delta} k \sum_{j=0}^{\Delta} |c_{j,k}(\overrightarrow{dg}_{H_1^{(n)}}, \overline{\delta}) - nP(X_1 = j, X - X_1 = k)|
\]

\[
\leq \Delta n^{2/3}.
\]
This implies that the second term in the right hand side of (75) vanishes. Therefore, Lemma 3 implies that
\[
\limsup_{n \to \infty} \max_{\mathcal{M}(n)} \frac{\log |\gamma(n)| - n \frac{d_{CM}^1}{2} \log n}{n} \leq -s(d_{1,2}^CM - d_{1}^CM) - \mathbb{E} [\log(X - X_1)!].
\]

Furthermore, if \(\tilde{m}^{(n)}\) is a sequence in \(\mathcal{M}^{(n)}\), by definition we have \(\sum_{x \in \Xi_1} |m^{(n)}(x) - m_n \gamma_x| \leq n^{2/3}\). Therefore, we have
\[
\lim_{n \to \infty} \frac{m_n - \sum_{x \in \Xi_1} m^{(n)}(x)}{n} = \frac{d_{CM}^1}{2} \left( 1 - \sum_{x \in \Xi_1} \gamma_x \right),
\]
where \(\gamma_x\) for \(x_1 \in \Xi_1\) is defined to be \(\sum_{x, x_2 \in \Xi_2} \gamma(x_1, x_2)\). Similarly, for \(x_2 \in \Xi_2\), we have
\[
\lim_{n \to \infty} \frac{m^{(n)}(\{x_1, x_2\})}{n} = \frac{\gamma(x_1, x_2)}{1 - \sum_{x \in \Xi_1} \gamma_x} = \frac{\gamma(x_1, x_2)}{\sum_{x' \in \Xi_2} \gamma(x_1, x')}.\]

Consequently, using Lemma 1, we have
\[
\lim_{n \to \infty} \frac{1}{n} \log \left( \frac{m_n - \sum_{x \in \Xi_1} m^{(n)}(x)}{n} \right) = \frac{d_{CM}^1}{2} \left( 1 - \sum_{x \in \Xi_1} \gamma_x \right) H \left( \left\{ \frac{\gamma(x_1, x_2)}{\sum_{x' \in \Xi_2} \gamma(x_1, x')} \right\} \right)_{x_1, x_2} (77)
\]
\[
= \frac{d_{CM}^1}{2} P(\Gamma_1 = x_1) H(\Gamma_2 | \Gamma_1 = x_1).
\]

Here, \(\Gamma = (\Gamma_1, \Gamma_2)\) has law \(\gamma\). On the other hand, for \(x_1 \in \Xi_1\) and \(x_2 \in \Xi_2\), we have \(m^{(n)}(x_1)/n \to d_{CM}^1 \gamma_x\), and \(m^{(n)}(x_1, x_2)/m^{(n)}(x_1) \to \gamma(x_1, x_2)/\gamma_x\). Consequently, another use of Lemma 1 implies that for all \(x_1 \in \Xi_1\), we have
\[
\lim_{n \to \infty} \frac{1}{n} \log \left( \frac{m^{(n)}(x_1)}{m((x_1, x_2))} \right) = \frac{d_{CM}^1}{2} \gamma_x H \left( \left\{ \frac{\gamma(x_1, x_2)}{\gamma_x} \right\} \right)_{x_2} (78)
\]
\[
= \frac{d_{CM}^1}{2} P(\Gamma_1 = x_1) H(\Gamma_2 | \Gamma_1 = x_1).
\]

Putting together (77) and (78), we realize that
\[
\lim_{n \to \infty} \max_{\tilde{m} \in \mathcal{M}^{(n)}} \log \left( \frac{m_n - \sum_{x \in \Xi_1} m^{(n)}(x)}{n} \prod_{x_1 \in \Xi_1} \left( \frac{m(x_1)}{m((x_1, x_2))} \right)_{x_2 \in \Xi_2} \right) = \frac{d_{CM}^1}{2} P(\Gamma_1 = x_1) H(\Gamma_2 | \Gamma_1 = x_1) (79)
\]
\[
= \frac{d_{CM}^1}{2} H(\Gamma_2 | \Gamma_1).
\]

Using a similar technique, if \(\tilde{u}^{(n)}\) is a sequence in \(\mathcal{U}^{(n)}\), for all \(\theta_1 \in \Theta_1\) and \(\theta_2 \in \Theta_2\), we have \(\tilde{u}^{(n)}(\theta_1) \to q_{\theta_1}\) and \(\tilde{u}^{(n)}(\theta_1, \theta_2) \to \frac{q_{\theta_1} q_{\theta_2}}{q_{\theta_1}}\). Thereby, using Lemma 1, for all \(\theta_1 \in \Theta_1\), we have
\[
\lim_{n \to \infty} \frac{1}{n} \log \left( \frac{\tilde{u}^{(n)}(\theta_1)}{\{u^{(n)}((\theta_1, \theta_2))}_{\theta_2 \in \Theta_2} \right) = q_{\theta_1} H \left( \left\{ \frac{q(\theta_1, \theta_2)}{q_{\theta_1}} \right\} \right)_{\theta_2 \in \Theta_2} = P(\Gamma_1 = \theta_1) H(\Gamma_2 | \Gamma_1 = \theta_1),
\]
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where \( Q = (Q_1, Q_2) \) has law \( \tilde{q} \). Consequently, we have

\[
\lim_{n \to \infty} \frac{1}{n} \max_{u \in U(n)} \log \prod_{\theta_1 \in \Theta_1} \left( \frac{u(\theta_1)}{\{u((\theta_1, \theta_2))\}_{\theta_2 \in \Theta_2}} \right) = \sum_{\theta_1 \in \Theta_1} P(Q_1 = \theta_1) H(Q_2 | Q_1 = \theta_1) = H(Q_2 | Q_1). \quad (80)
\]

Putting (74), (76), (79), and (80) back into (73), we get

\[
\lim_{n \to \infty} \frac{\max_{H_1^{(n)} \in \mathcal{W}(n)} \log |S_2^{(n)}(H_1^{(n)})| - n \frac{d_{CM}^2 - d_{1,2}^2}{2}}{\log n} = -s(d_{1,2}^2 - d_{CM}^2) + H(X - X_1 | X_1) - \mathbb{E} [\log(X - X_1)!] + \frac{d_{1,2}^2}{2} H(\Gamma_2 | \Gamma_1) + H(Q_2 | Q_1).
\]

Using Lemma 2 and rearranging, this is precisely equal to \( \Sigma(\mu_2^CM | \mu_1^CM) \), which completes the proof of (29).

**G Proof of Theorem 4: generalization to multiple sources**

The proof of Theorem 4 is similar to that of Theorem 3 which was given in Section 5.

It is easy to verify that if \( G_{[k]}^{(n)} \) is distributed according to either the multi-source Erdős–Rényi ensembles or the multi-source configuration model ensembles discussed in Section 5.5, then given any nonempty \( A \subset [k], A \neq [k] \), the joint distribution of \((G_A^{(n)}, G_{A^c}^{(n)})\) is similar to that of a two-source ensemble as in Section 2 with the following mark sets:

\[
\begin{align*}
\tilde{\Xi}_1 &:= \left\{ x_A \in \Xi_A : \sum_{(x'_j : j \in [k]) : (x'_j : j \in A)} p(x'_j : j \in [k]) > 0 \right\} \\
\tilde{\Xi}_2 &:= \left\{ x_{A^c} \in \Xi_{A^c} : \sum_{(x'_j : j \in [k]) : (x'_j : j \in A^c)} p(x'_j : j \in [k]) > 0 \right\} \\
\tilde{\Xi}_{1,2} &:= \Xi_{[k]} \\
\hat{\Theta}_1 &:= \left\{ \theta_A \in \Theta_A : \sum_{(\theta'_j : j \in [k]) : (\theta'_j : j \in A)} q(\theta'_j : j \in [k]) > 0 \right\} \\
\hat{\Theta}_2 &:= \left\{ \theta_{A^c} \in \Theta_{A^c} : \sum_{(\theta'_j : j \in [k]) : (\theta'_j : j \in A^c)} q(\theta'_j : j \in [k]) > 0 \right\} \\
\hat{\Theta}_{1,2} &:= \Theta_{[k]}
\end{align*}
\]

Moreover, we set \( \tilde{\omega}_1 := \circ_A \) and \( \tilde{\omega}_2 := \circ_{A^c} \). To establish the analogy, for the Erdős–Rényi ensemble, we define \( \tilde{\rho} = \{ \tilde{\rho}_x \}_{x \in \tilde{\Xi}_{1,2}} \) such that \( \tilde{\rho}_x = p_x \) for \( x \in \tilde{\Xi}_{1,2} \). Furthermore, we define \( \tilde{\zeta} = \{ \tilde{\zeta}_\theta \}_{\theta \in \hat{\Theta}_{1,2}} \) such that \( \tilde{\zeta}_\theta = q_\theta \) for \( \theta \in \hat{\Theta}_{1,2} \). Similarly, for the configuration model ensemble, we let \( \tilde{\gamma} = \{ \tilde{\gamma}_x \}_{x \in \tilde{\Xi}_{1,2}} \) such that \( \tilde{\gamma}_x = \gamma_x \) for \( x \in \tilde{\Xi}_{1,2} \), and define \( \tilde{\zeta} = \{ \tilde{\zeta}_\theta \}_{\theta \in \hat{\Theta}_{1,2}} \) where \( \tilde{\zeta}_\theta = q_\theta \) for \( \theta \in \hat{\Theta}_{1,2} \). It can be easily verified that (6) and (7) follow from the assumptions (51) and (52). Likewise, (8) and (9) follow from (53), (54) and (55).
Using this observation together with (14a)–(14c), we realize that for the multi-source Erdős–Rényi ensemble and nonempty $A \subseteq [k]$, we have

$$H(G^{(n)}_A) = \frac{d^{ER}_A}{2} n \log n + n \left( H(Q_A) + \sum_{x \in \Xi_A} s(p_x) \right) + o(n),$$

(81)

where $d^{ER}_A = \deg(\mu^{ER}_A)$, and with $Q = (Q_i : i \in [k])$ having law $\bar{q}$, we let $Q_A := (Q_i : i \in A)$. In fact, the coefficient of $n$ in the above expression is $\Sigma(\mu^{ER}_A)$. Similarly, the above observation together with (16a)–(16c) establishes that for the multi-source configuration model ensemble and for nonempty $A \subseteq [k]$,

$$H(G^{(n)}_A) = \frac{d^{CM}_A}{2} n \log n + n \left( -s(d^{CM}_A) + H(X_A) - \mathbb{E}[\log X_A] \right) + H(Q_A) + \frac{d^{CM}_A}{2} H(\Gamma_A | \Gamma_A \neq \phi_A) + o(n)$$

where $d^{CM}_A := \deg(\mu^{CM}_A)$. In the above expression, with $X \sim \bar{r}$ and $\Gamma^i = (\Gamma^i_j : j \in [k])$ for $1 \leq i \leq \Delta$ which are i.i.d. with law $\bar{r}$, we define $X_A := \sum_{i \in A} \mathbb{I} [\Gamma^i_j \neq o_j$ for some $j \in A]$. Here, if $X = 0$, then $X_A := 0$. Moreover, $Q = (Q_i : i \in [k])$ has law $\bar{q}$ and $Q_A := (Q_i : i \in A)$. Furthermore, $\Gamma = (\Gamma_i : i \in [k])$ has law $\bar{r}$ and $\Gamma_A := (\Gamma_i : i \in A)$. It can be seen that the coefficient of $n$ in the above expression is $\Sigma(\mu^{CM}_A)$.

G.1 Proof of converse

Observe that for both the Erdős–Rényi and the configuration model ensembles, for $A \subset [k]$ nonempty, $A \neq [k]$, even if all the encoders in the set $A$ as well as all the encoders in the set $A^c$ can cooperate, since the distribution of $(G^{(n)}_A, G^{(n)}_{A^c})$ is identical to a two-source ensemble as was discussed above, using the converse result corresponding to the two-source case (i.e. Sections 5.3 and 5.4), with $\alpha_B := \sum_{i \in B} \alpha_i$ and $R_B := \sum_{i \in B} R_i$ for $B \subset [k]$, for $(\alpha_i, R_i) : i \in [k]) \in \mathcal{R}$, we must have

$$(\alpha_A, R_A) \succeq (\max(k) - d_{A^c})/2, \Sigma(\mu_A | \mu_{A^c})$$

$$(\alpha_{A^c}, R_{A^c}) \succeq (\max(k) - d_A)/2, \Sigma(\mu_{A^c} | \mu_A)$$

$$(\alpha(k), R(k)) \succeq (d_{k}/2, \Sigma(\mu_{k}))$$

Here, $\mu$ denotes $\mu^{ER}$ or $\mu^{CM}$, depending on the ensemble. Repeating this for all nonempty $A \subset [k]$, $A \neq [k]$, recovers all the necessary inequalities and completes the converse proof.

G.2 Proof of achievability for the Erdős–Rényi ensemble

Similar to Section 5.1, we employ a random binning codebook construction with $L^{(n)}_i = \lfloor \exp(\alpha_i n \log n + R_in) \rfloor$ for $i \in [k]$. More precisely, For $i \in [k]$ and $H^{(n)}_i \in \mathcal{G}^{(n)}_i$, we generate $f^{(n)}_i(H^{(n)}_i)$ uniformly in $[L^{(n)}_{i}]$. The choice of $f^{(n)}_i(H^{(n)}_i)$ is made independently for each $H^{(n)}_i \in \mathcal{G}^{(n)}_i$ and also for each domain $i \in [k]$. To explain the decoding procedure, similar to Section 5.1, let $\mathcal{M}^{(n)}$ be the set of $\bar{m} = \{m(x)\}_{x \in [k]}$ such that $\sum_{x \in [k]} |m(x) - np_x/2| \leq n^{2/3}$. Furthermore, let $\mathcal{U}^{(n)}$ be the set of $\bar{u} = \{u(\theta)\}_{\theta \in \Theta_{[k]}}$ such that $\sum_{\theta \in \Theta_{[k]}} |u(\theta) - np(\theta)| \leq n^{2/3}$. With these, let $\mathcal{G}^{(n)}_{\bar{u}, \bar{m}}$ be the set of $H^{(n)}_{[k]} \in \mathcal{G}^{(n)}_{[k]}$ such that $\bar{m}^{H^{(n)}_{[k]}} \in \mathcal{M}^{(n)}$ and $\bar{u}^{H^{(n)}_{[k]}} \in \mathcal{U}^{(n)}$. At the receiver, upon receiving bin indices $i_j$, $1 \leq j \leq k$, we form the set of $H^{(n)}_{[k]} \in \mathcal{G}^{(n)}_{\bar{u}, \bar{m}}$ such that $f^{(n)}_j(H^{(n)}_{j}) = i_j$ for $j \in [k]$. If there is only one graph in this
set, the decoder outputs that graph; otherwise, it reports an error. It can be easily seen that the error
events are as follows:
\[ \mathcal{E}_1^{(n)} = \{ G_{[k]}^{(n)} \notin G_{[k]}^{(n)} \}, \]
and, for each nonempty \( A \subset [k] \),
\[ \mathcal{E}_A^{(n)} = \{ \exists H_{[k]}^{(n)} \in \mathcal{G}_{[k]}^{(n)} : H_i^{(n)} = G_i^{(n)} \text{ for } i \notin A, \]
\[ H_i^{(n)} \neq G_i^{(n)}, f_i^{(n)}(H_i^{(n)}) = f_i^{(n)}(G_i^{(n)}) \text{ for } i \in A \} \]

For nonempty \( A \subset [k] \) and \( H_A^{(n)} \in \mathcal{G}_{[k]}^{(n)} \), we denote \( (f_i^{(n)}(H_i^{(n)}) : i \in A) \) by \( f_A^{(n)}(H_A^{(n)}) \). Note that we may treat \( f_A^{(n)}(H_A^{(n)}) \) as an integer in the range \( \prod_{i \in A} L_i^{(n)} \approx |\exp(\alpha_A n \log n + R_A n)| \).

Recall that \( \alpha_A = \sum_{i \in A} \alpha_i \) and \( R_A = \sum_{i \in A} R_i \). Observe that due to our random binning procedure, \( f_A^{(n)}(H_A^{(n)}) \)
is uniformly distributed in the range \( \prod_{i \in A} L_i^{(n)} \). Moreover, for \( H_A^{(n)} \) such that \( H_i^{(n)} \neq G_i^{(n)} \) for \( i \in A, f_A^{(n)}(H_A^{(n)}) \) is independent from \( f_A^{(n)}(G_A^{(n)}) \). Thereby, for nonempty \( A \subset [k], A \neq [k] \), using the previously discussed fact that \( \{G_A^{(n)}, G_A^{(n)}\} \) is distributed according to a two-source ensemble, and using the analysis of Section 5.1, we realize that the probabilities of the error events \( \mathcal{E}_A^{(n)} \), \( \mathcal{E}_{[k]}^{(n)} \), \( \mathcal{E}_{[k]}^{(n)} \), and \( \mathcal{E}_1^{(n)} \) vanish as \( n \to \infty \) given that \( (\alpha_A, R_A) \geq ((d_{[k]} - d_A)/2, \Sigma(\mu_{ER}^A \mu_{ER}^A)), (\alpha_{A'}, R_{A'}) \geq ((d_{[k]} - d_A)/2, \Sigma(\mu_{ER}^A \mu_{ER}^A)), (\alpha_{[k]}, R_{[k]}) \geq (d_{[k]}/2, \Sigma(\mu_{ER}^A)). \) Repeating this argument for all nonempty \( A \subset [k], A \neq [k] \), we realize that the probabilities of all error events vanish, which completes the proof of achievability.

### G.3 Proof of achievability for the configuration model ensemble

We again employ a random binning procedure as in the above, where, for \( i \in [k] \) and \( H_i^{(n)} \in \mathcal{G}_{[k]}^{(n)} \), we choose \( f_i^{(n)}(H_i^{(n)}) \) uniformly in the set \([L_i^{(n)}] \) with \( L_i^{(n)} = |\exp(\alpha_i n \log n + R_i n)| \). To explain the decoding procedure, similar to the setup in Section 5.2, we define \( \mathcal{D}^{(n)} \) be the set of degree sequences \( \bar{d} \) such that \( c_i(\bar{d}) = c_i(\bar{d}^{(n)}) \) for all \( 0 \leq i \leq \Delta \). Moreover, let \( \mathcal{M}^{(n)} \) be the set of \( \bar{m} = (m(x) : x \in \Xi_{[k]}) \) such that \( \sum_{x \in \Xi_{[k]}} m(x) = m_n, \) where \( m_n := \langle \sum_{i=1}^n d_i^{(n)}(i) \rangle / 2, \) and \( \sum_{x \in \Xi_{[k]}} |m(x) - m_n \gamma x| \leq n^{2/3} \). Also, let \( \mathcal{U}^{(n)} \) be the set of \( \bar{u} = (u(\theta) : \theta \in \Theta_{[k]}) \) such that \( \sum_{\theta \in \Theta_{[k]}} |u(\theta) - n\phi| \leq n^{2/3} \). Let the random variables \( X \) and \( X_A \) for \( A \subset [k] \) nonempty be defined as above, i.e. \( X \sim \bar{P} \) and with \( \Gamma^i = (\Gamma^i_j : j \in [k]) \) for \( 1 \leq i \leq \Delta \) being i.i.d. with law \( \mathcal{P} \), we define \( X_A := \sum_{i=1}^N \mathbb{1}[\Gamma^i_j = c_j \text{ for some } j \in A] \) if \( X > 0, \) and \( X_A := 0 \) if \( X = 0 \). With this, let \( \mathcal{W}^{(n)} \) be the set of \( H_{[k]}^{(n)} \in \mathcal{G}_{[k]}^{(n)} \) such that \( (i) \bar{d}_{H_{[k]}^{(n)}} \in \mathcal{D}^{(n)}, (ii) \bar{m}_{H_{[k]}^{(n)}} \in \mathcal{M}^{(n)}, (iii) \bar{u}_{H_{[k]}^{(n)}} \in \mathcal{U}^{(n)}, \) and \( (iv) \) for all \( A \subset [k] \) nonempty and \( 0 \leq j \leq i \leq \Delta, \) we have
\[ |c_{i,j}(\bar{d}_{H_{[k]}^{(n)}}^{-}, \bar{d}_{H_{[k]}^{(n)}}^{+}) - n \mathbb{P}(X = i, X_A = j)| \leq n^{2/3}. \]

At the decoder, upon receiving \( \bar{r}_j : 1 \leq j \leq k \), we form the set of graphs \( H_{[k]}^{(n)} \in \mathcal{W}^{(n)} \) such that \( f_i^{(n)}(H_i^{(n)}) = i_j \) for \( 1 \leq j \leq k \). If there is only one graph in this set, the decoder outputs this graph; otherwise, it reports an error. It can be easily seen that the error events are as follows:
\[ \mathcal{E}_1^{(n)} = \{ \bar{G}_{[k]}^{(n)} \notin \mathcal{W}^{(n)} \}, \]
and for nonempty \( A \subset [k], \)
\[ \mathcal{E}_A^{(n)} = \{ \exists H_{[k]}^{(n)} \in \mathcal{W}^{(n)} : H_i^{(n)} = G_i^{(n)} \text{ for } i \notin A \} \]
which we call $\xi$. Similar to the above discussion in Section 5.2, we realize that the probabilities of all error events vanish, which completes the proof of achievability.

H Some Examples of Calculating the marked BC Entropy

In this appendix, we provide some examples of calculating the marked BC entropy defined in Section 4. To simplify the discussion, we focus on a special yet rich class of probability distributions on $G_\#(\Xi, \Theta)$ which we call depth-1 unimodular Galton-Watson trees defined as follows. The reader is referred to [DA19] for more details for the general setting.

We first need to make some definitions. As in Section 2, let $\Xi$ and $\Theta$ be finite sets of edge marks and vertex marks, respectively. Given a marked graph $G$ and two adjacent vertices $v$ and $w$ in $G$, let $\xi_G(v, w) = \xi_G(w, v) \in \Xi$ be the mark on the edge connecting $v$ to $w$. Moreover, we denote the mark of a vertex $v$ in $G$ by $\tau_G(v)$. For a rooted marked graph $(G, o)$, $\theta, \theta' \in \Theta$, and $x \in \Xi$, let

$$E(\theta, x, \theta')(G, o) := \{|v \sim_G o : \tau_G(o) = \theta, \tau_G(v) = \theta', \xi_G(o, v) = x\}|.$$ 

For $[G, o] \in G_\#(\Xi, \Theta)$, we write $E(\theta, x, \theta')(G, o)$ for $E(\theta, x, \theta')(G, o)$ when $(G, o)$ is an arbitrary member of the isomorphism class $[G, o]$. It is easy to verify to see that this definition does not depend on the choice of the representative in the isomorphism class. Given a probability distribution $P \in \mathcal{P}(G_\#(\Xi, \Theta))$, for $\theta, \theta' \in \Theta$ and $x \in \Xi$, we define

$$e_P(\theta, x, \theta') := \mathbb{E}_P[E(\theta, x, \theta')(G, o)],$$

where the expectation is with respect to $[G, o]$ with distribution $P$.

Recall that $T_\#(\Xi, \Theta)$ denotes the subset of $G_\#(\Xi, \Theta)$ which consists of the isomorphism classes $[G, o]$ arising from some $\{G, o\}$ where the graph underlying $G$ is a tree. Let $T_\#(\Xi, \Theta)$ be the subset of $T_\#(\Xi, \Theta)$ consisting of the isomorphism classes $[T, o] \in T_\#(\Xi, \Theta)$ where $[T, o]$ has depth at most one, i.e. all the vertices in $T$ have distance at most one from the root node $o$. This includes an isolated root with degree zero.

**Definition 6.** A probability distribution $P \in \mathcal{P}(T_\#(\Xi, \Theta))$ is called admissible if $\mathbb{E}_P[\deg_T(o)] < \infty$ and $e_P(\theta, x, \theta') = e_P(\theta', x, \theta)$ for all $\theta, \theta' \in \Theta$ and $x \in \Xi$.

It can be shown that for a unimodular $\mu \in \mathcal{P}_u(T_\#(\Xi, \Theta))$ with $\deg(\mu) < \infty$, $\mu_1$ which is defined to be the law of $[T, o]$ when $[T, o]$ has law $\mu$, is admissible [DA19, Lemma 1]. This in particular highlights the importance of the concept of admissibility. Below, given an admissible $P \in \mathcal{P}(T_\#(\Xi, \Theta))$, we define a unimodular measure in $\mathcal{P}_u(T_\#(\Xi, \Theta))$ which is called the marked unimodular Galton-Watson tree with depth-1 neighborhood distribution $P$, and is denoted by $\text{UGWT}_1(P).$ For $\theta, \theta' \in \Theta$ and $x \in \Xi$ such that $e_P(\theta, x, \theta') > 0$, we define $\tilde{P}_{\theta', x, \theta} \in \mathcal{P}(T_\#(\Xi, \Theta))$ via

$$\tilde{P}_{\theta', x, \theta}([T, o]) := \frac{1}{e_P(\theta', x, \theta)} [\tau_T(o) = \theta'] P([\tilde{T}, o]) E(\theta', x, \theta)(\tilde{T}, o),$$ \hspace{1cm} (82)

This discussion can be made more general to include any depth, see [DA19] for more details.
where \([\tilde{T}, o] \in T_1^*(\Xi, \Theta)\) is obtained from \([T, o]\) by adding an edge to the root \(o\) in \([T, o]\) which has edge mark \(x\), and the vertex mark of the endpoint of this edge other than \(o\) is \(\theta\). It is straightforward to verify that \(\hat{\mathcal{P}}_{\theta,x,\theta}\) is a probability distribution.

With this, for an admissible \(P\) as above, we define \(\text{UGWT}_1(P)\) to be the law of \([T, o]\) when \((T, o)\) is the random rooted marked tree constructed as follows. First, we sample the 1 neighborhood of the root, i.e. \((T, o)_1\), according to \(P\). Then, for each offspring \(v \sim_{T} o\) of the root, we sample \([\tilde{T}, \tilde{o}]\) according to the law \(\hat{\mathcal{P}}_{\theta,v,\theta}\) where \(\theta = \tau_T(o), \theta' = \tau_T(v)\), and \(x = \xi_T(o, v)\). Note that by definition we have \(\tau_T(\tilde{o}) = \theta' = \tau_T(v)\). This means that we can add \((T, o)\) as a subtree below node \(v\). We repeat this process independently for each \(v \sim_{T} o\). At this point, \((T, o)\) has depth at most 2. Subsequently, we follow the same procedure for vertices at depth 2, 3, and so on inductively to construct \((T, o)\). Finally, we define \(\text{UGWT}_1(P)\) to be the law of \([T, o]\).

For \(P \in \mathcal{P}(\mathcal{T}_1^*(\Xi, \Theta))\) admissible such that \(d := \mathbb{E}_P[\text{deg}_T(o)] > 0\), let \(\pi_P\) denote the probability distribution on \(\Theta \times \Xi \times \Theta\) defined as

\[
\pi_P(\theta, x, \theta') := \frac{e_P(\theta, x, \theta')}{d}.
\]

(83)

Since for \([T, o] \in \mathcal{T}_s^*(\Xi, \Theta)\) we have \(\text{deg}_T(o) = \sum_{\theta, x, \theta'} E(\theta, x, \theta')(\{T, o\})\), we have \(d = \sum_{\theta, x, \theta'} e_P(\theta, x, \theta')\) and \(\pi_P\) is indeed a probability distribution.

For \(P \in \mathcal{P}(\mathcal{T}_1^*(\Xi, \Theta))\) admissible such that \(H(P) < \infty\) and \(d := \mathbb{E}_P[\text{deg}_T(o)] > 0\), define

\[
J(P) := -s(d) + H(P) - \frac{d}{2} H(\pi_P) - \sum_{\theta, x, \theta'} \mathbb{E}_P[\log E(\theta, x, \theta')(\{T, o\})!],
\]

(84)

where \(s(d) := \frac{d}{2} - \frac{1}{2} \log d\). Note that since \(0 < d < \infty\), \(s(d)\) is finite. On the other hand, by assumption we have \(H(P) < \infty\). Also, \(H(\pi_P) \geq 0\) and \(\mathbb{E}_P[\log E(\theta, x, \theta')(\{T, o\})!] \geq 0\) for all \(\theta, \theta' \in \Theta\) and \(x \in \Xi\). Therefore, \(J(P)\) is well defined and is in the range \([-\infty, \infty)\).

**Definition 7.** We say that a probability distribution \(P \in \mathcal{P}(\mathcal{T}_1^*(\Xi, \Theta))\) is strongly admissible if \(P\) is admissible, \(H(P) < \infty\), and \(\mathbb{E}_P[\text{deg}_T(o) \log \text{deg}_T(o)] < \infty\).

The following result gives a recipe for calculating the marked BC entropy of \(\text{UGWT}_1(P)\) when \(P\) is strongly admissible. Theorem 5 below is a direct consequence of Theorem 3 and Proposition 5 in [DA19].

**Theorem 5.** Let \(P \in \mathcal{P}(\mathcal{T}_1^*(\Xi, \Theta))\) be strongly admissible. Then, with \(\mu := \text{UGWT}_1(P)\), we have

\[
\Sigma(\mu) = J(P).
\]

Now, we apply this result to several examples, namely the local weak limits of the sequences of Erdős–Rényi and the configuration model ensembles defined in Section 2, as well as a marked \(d\)–regular distribution which will be useful for our discussion in Appendix I.

**H.1 Local Weak Limit of the Sequence of Erdős–Rényi Ensembles**

Recall from Section 3 that the local weak limit of the sequence of Erdős–Rényi ensembles defined in Section 2 is \(\mu_{1,2}^\text{ER}\). Recalling the definition of \(\mu_{1,2}^\text{ER}\) from Section 3, since the procedure of generating the depth–1 neighborhood of the root is the same as that for each offspring, we have \(\mu_{1,2}^\text{ER} = \text{UGWT}_1(P_{1,2})\) where \(P_{1,2} \in \mathcal{P}(\mathcal{T}_1^*(\Xi_{1,2}, \Theta_{1,2}))\) is defined as follows. The root is randomly assigned a mark in \(\Theta_{1,2}\) with distribution \(\tilde{q}\). For \(x \in \Xi_{1,2}\), we independently generate \(D_x\) with law Poisson\(\{p_x\}\) and add \(D_x\) many edges with mark \(x\) to the root. Then the vertex mark of each offspring of the root is independently
assigned with distribution \( \tilde{q} \). Using the thinning property of the Poisson distribution, the number of edges connected to the root with edge mark \( x \) and vertex mark \( \theta' \) at the endpoint other than the root has law \( \text{Poisson}(p_x q \theta') \), independent for \( x \in \Xi_{1,2} \) and \( \theta' \in \Theta_{1,2} \). Since the mark at the root is \( \theta \) with probability \( q_0 \), for \( \theta, \theta' \in \Theta_{1,2} \) and \( x \in \Xi_{1,2} \), we have

\[
\Pr(E(\theta, x, \theta')(T, o) = k) = q_0 \Pr(\Lambda_{x, \theta'} = k) + (1 - q_0) \mathbb{1}[k = 0],
\]

where the probability on the left hand side is with respect to \( P_{1,2} \), and \( \Lambda_{x, \theta'} \) denotes the number of edges connected to the root with edge mark \( x \) and vertex mark \( \theta' \) at the endpoint other than the root. From the above discussion, \( \Lambda_{x, \theta'} \) is a Poisson\((p_x q \theta')\) random variable. Furthermore, \( \Lambda_{x, \theta'} \) are independent for \( x \in \Xi_{1,2} \) and \( \theta' \in \Theta_{1,2} \). As a result, we have

\[
e_{P_{1,2}}(\theta, x, \theta') = q_0 p_x q \theta',
\]

and

\[
\pi_{P_{1,2}}(\theta, x, \theta') = \frac{e_{P_{1,2}}(\theta, x, \theta')}{d^\text{ER}_{1,2}} = \frac{q_0 p_x q \theta'}{d^\text{ER}_{1,2}},
\]

where \( d^\text{ER}_{1,2} = \sum_{x \in \Xi_{1,2}} p_x \) is the expected degree at the root in \( P_{1,2} \). Note that an object \([T, o] \in \mathcal{T}^*_1(\Xi_{1,2}, \Theta_{1,2})\) is uniquely determined by knowing the mark at the root as well as the number of edges connected to the root with mark \( x \) and the vertex mark \( \theta' \) at the endpoint other than the root for each \( x \in \Xi_{1,2} \) and \( \theta' \in \Theta_{1,2} \). Since \( \Lambda_{x, \theta'} \) are independent for \( x \in \Xi_{1,2} \) and \( \theta' \in \Theta_{1,2} \), and they are all independent from the vertex mark at the root, we have

\[
H(P_{1,2}) = H(Q) + \sum_{x \in \Xi_{1,2}} \sum_{\theta' \in \Theta_{1,2}} H(\Lambda_{x, \theta'}),
\]

where \( Q \) has law \( \tilde{q} \) and \( \Lambda_{x, \theta'} \) is a Poisson\((p_x q \theta')\) random variable as was defined above. Further simplifying this expression using the identities \( \sum_{x \in \Xi_{1,2}} p_x = d^\text{ER}_{1,2} \) and \( \sum_{\theta' \in \Theta_{1,2}} q \theta' = 1 \), we get

\[
H(P_{1,2}) = H(Q) + \sum_{x \in \Xi_{1,2}} p_x - \sum_{x \in \Xi_{1,2}} p_x \log p_x - \left( \sum_{x \in \Xi_{1,2}} p_x \right) \left( \sum_{\theta' \in \Theta_{1,2}} q \theta' \log q \theta' \right) + \sum_{x \in \Xi_{1,2}, \theta' \in \Theta_{1,2}} \mathbb{E}[\log \Lambda_{x, \theta'}]
\]

\[
= H(Q) + d^\text{ER}_{1,2} - \sum_{x \in \Xi_{1,2}} p_x \log p_x + d^\text{ER}_{1,2} H(Q) + \sum_{x \in \Xi_{1,2}, \theta' \in \Theta_{1,2}} \mathbb{E}[\log \Lambda_{x, \theta'}].
\]

On the other hand, using (86), we have

\[
H(\pi_{P_{1,2}}) = \sum_{\theta, \theta' \in \Theta_{1,2}, x \in \Xi_{1,2}} \frac{q_0 p_x q \theta'}{d^\text{ER}_{1,2}} \log \frac{d^\text{ER}_{1,2}}{q_0 p_x q \theta'}
\]

\[
= \log d^\text{ER}_{1,2} - \sum_{x \in \Xi_{1,2}, \theta' \in \Theta_{1,2}} \frac{p_x q \theta'}{d^\text{ER}_{1,2}} \sum_{\theta \in \Theta_{1,2}} q \theta \log q \theta
\]

\[
- \sum_{x \in \Xi_{1,2}, \theta \in \Theta_{1,2}} \frac{p_x q \theta}{d^\text{ER}_{1,2}} \sum_{\theta' \in \Theta_{1,2}} q \theta' \log q \theta'
\]

\[
- \frac{1}{d^\text{ER}_{1,2}} \sum_{\theta, \theta' \in \Theta_{1,2}} q \theta \sum_{x \in \Xi_{1,2}} p_x \log p_x
\]

\[
= \log d^\text{ER}_{1,2} + 2H(Q) - \frac{1}{d^\text{ER}_{1,2}} \sum_{x \in \Xi_{1,2}} p_x \log p_x.
\]
Furthermore, using (85), we have
\[
\sum_{\theta, x, \theta'} \mathbb{E}_{P_{1,2}} [\log E(\theta, x, \theta')([T, o])!] = \sum_{\theta, x, \theta'} q_\theta \mathbb{E} [\log \Lambda_{x, \theta'}!] = \sum_{x, \theta'} \mathbb{E} [\log \Lambda_{x, \theta'}!].
\] (89)

Substituting (87), (88), and (89) into (84) and simplifying, we get
\[
J(P_{1,2}) = -s(d_{1,2}^{ER}) + H(P_{1,2}) - \frac{d_{1,2}^{ER}}{2} H(\pi_{P_{1,2}}) - \sum_{\theta, x, \theta'} \mathbb{E}_{P_{1,2}} [\log E(\theta, x, \theta')([T, o])!]
\]
\[
= -\frac{d_{1,2}^{ER}}{2} + \frac{d_{1,2}^{ER}}{2} \log d_{1,2}^{ER} + H(Q) + d_{1,2}^{ER} - \sum_{x \in \Xi_{1,2}} p_x \log p_x + d_{1,2}^{ER} H(Q) + \sum_{x \in \Xi_{1,2}, \theta' \in \Theta_{1,2}} \mathbb{E} [\log \Lambda_{x, \theta'}!]
\]
\[
- \frac{d_{1,2}^{ER}}{2} \left( \log d_{1,2}^{ER} + 2H(Q) - \frac{1}{d_{1,2}^{ER}} \sum_{x \in \Xi_{1,2}} p_x \log p_x \right)
\]
\[
- \sum_{x \in \Xi_{1,2}, \theta' \in \Theta_{1,2}} \mathbb{E} [\log \Lambda_{x, \theta'}!]
\]
\[
= \frac{d_{1,2}^{ER}}{2} + H(Q) - \sum_{x \in \Xi_{1,2}} \frac{p_x}{2} \log p_x
\]
\[
= H(Q) + \sum_{x \in \Xi_{1,2}} \left( \frac{p_x}{2} - \frac{p_x}{2} \log p_x \right)
\]
\[
= H(Q) + \sum_{x \in \Xi_{1,2}} s(p_x).
\]

It is easy to verify that $P_{1,2}$ is strongly admissible. Consequently, using Theorem 5, we get
\[
\Sigma(m_{1,2}^{ER}) = H(Q) + \sum_{x \in \Xi_{1,2}} s(p_x).
\] (90)

Using similar arguments for the marginals $\mu_1^{ER}$ and $\mu_2^{ER}$, if $Q = (Q_1, Q_2)$ has law $\bar{q}$, we realize that
\[
\Sigma(m_1^{ER}) = H(Q_1) + \sum_{x \in \Xi_1} s(p_{x_1}),
\]

and
\[
\Sigma(m_2^{ER}) = H(Q_2) + \sum_{x \in \Xi_2} s(p_{x_2}).
\]

**H.2 Local Weak Limit of the Sequence of Configuration Model Ensembles**

Recall from Section 3 that the local weak limit of the sequence of configuration model ensembles defined in Section 2 is $\mu_{1,2}^{CM}$. In this section, we calculate the marked BC entropy of $\mu_{1,2}^{CM}$, i.e. the quantity $\Sigma(\mu_{1,2}^{CM})$, as well the marked BC entropy of the marginals $\mu_1^{CM}$ and $\mu_2^{CM}$. We do this by using Theorem 5 discussed above. At the end of this section, we justify the result through an intuitive argument. It is easy to see that $\mu_{1,2}^{CM} = \text{UGWT}_{1}(P_{1,2})$ where $P_{1,2} \in \mathcal{P}(\Xi_{1,2}, \Theta_{1,2})$ is defined as follows. The degree of the root is $X$ which has law $\bar{r}$, the root and each of its offsprings are independently assigned a vertex mark with law $\bar{q}$, and each edge is independently assigned an edge mark with law $\bar{\gamma}$. With $[T, o]$ with...
law $P_{1,2}$, let $Q$ denote the vertex mark at the root. Furthermore, for $x \in \Xi_{1,2}$ and $\theta' \in \Theta_{1,2}$, let $\Lambda_{x,\theta'}$ be the number of edges connected to the root with edge mark $x$ which have a vertex mark $\theta'$ at the endpoint other than the root. Observe that for $\theta, \theta' \in \Theta_{1,2}$ and $x \in \Xi_{1,2}$, we have
\[
\mathbb{P}(E(\theta, x, \theta')(T, o) = k) = q_0 \mathbb{P}(\Lambda_{x, \theta'} = k) + (1 - q_0) \mathbb{I}[k = 0],
\] (91)
where the probability on the left hand side is with respect to $P_{1,2}$. Note that, conditioned on $X$, \(\{\Lambda_{x, \theta'}\}_{x \in \Xi_{1,2}, \theta' \in \Theta_{1,2}}\) have a multinomial distribution with parameters \(\{\gamma_x q_{\theta'}\}_{x \in \Xi_{1,2}, \theta' \in \Theta_{1,2}}\). As a result, for $x \in \Xi_{1,2}$ and $\theta, \theta' \in \Theta_{1,2}$, we have
\[
e_{P_{1,2}}(\theta, x, \theta') = d^\text{CM}_{1,2} q_0 \gamma_x q_{\theta'},
\]
and
\[
\pi_{P_{1,2}}(\theta, x, \theta') = q_0 \gamma_x q_{\theta'}.
\]
(92)
On the other hand, note that there is a one to one correspondence between $[T, o]$ with law $P_{1,2}$ and the collection of random variables $(X, Q, \{\Lambda_{x, \theta'}\}_{x \in \Xi_{1,2}, \theta' \in \Theta_{1,2}})$. As a result, we have
\[
H(P_{1,2}) = H(X, Q, \{\Lambda_{x, \theta'}\}_{x \in \Xi_{1,2}, \theta' \in \Theta_{1,2}}) = H(X) + H(Q|X) + H(\{\Lambda_{x, \theta'}\}_{x \in \Xi_{1,2}, \theta' \in \Theta_{1,2}}|X, Q)
\]
(93)
where the second line uses the fact that $Q$ is independent from everything else. Recall that, conditioned on $X$, \(\{\Lambda_{x, \theta'}\}_{x \in \Xi_{1,2}, \theta' \in \Theta_{1,2}}\) has a multinomial distribution with parameters \(\{\gamma_x q_{\theta'}\}_{x \in \Xi_{1,2}, \theta' \in \Theta_{1,2}}\).

\[
H(\{\Lambda_{x, \theta'}\}_{x \in \Xi_{1,2}, \theta' \in \Theta_{1,2}}|X) = -\mathbb{E} \left[ \log \mathbb{P}(\{\Lambda_{x, \theta'}\}_{x \in \Xi_{1,2}, \theta' \in \Theta_{1,2}}|X) \right]
\]
\[
= -\mathbb{E} \left[ \log \left( \prod_{x \in \Xi_{1,2}, \theta' \in \Theta_{1,2}} (\gamma_x q_{\theta'})^{\Lambda_{x, \theta'}} | X \right) \right] = -\mathbb{E} \left[ \log |X| \right] + \sum_{x \in \Xi_{1,2}, \theta' \in \Theta_{1,2}} (\mathbb{E} \left[ \log \Lambda_{x, \theta'} \right] - \mathbb{E} \left[ \Lambda_{x, \theta'} \right] \log(\gamma_x q_{\theta'}))
\]
\[
= -\mathbb{E} \left[ \log |X| \right] + \sum_{x \in \Xi_{1,2}, \theta' \in \Theta_{1,2}} (\mathbb{E} \left[ \log \Lambda_{x, \theta'} \right] - d^\text{CM}_{1,2} \gamma_x q_{\theta'} \log(\gamma_x q_{\theta'}))
\]
\[
= -\mathbb{E} \left[ \log |X| \right] + d^\text{CM}_{1,2} (H(\Gamma) + H(Q)) + \sum_{x \in \Xi_{1,2}, \theta' \in \Theta_{1,2}} \mathbb{E} \left[ \log \Lambda_{x, \theta'} \right] \cdot
\]
(94)
Using this in (93), we get
\[
H(P_{1,2}) = H(X) + H(Q) - \mathbb{E} \left[ \log |X| \right] + d^\text{CM}_{1,2} (H(\Gamma) + H(Q)) + \sum_{x \in \Xi_{1,2}, \theta' \in \Theta_{1,2}} \mathbb{E} \left[ \log \Lambda_{x, \theta'} \right] \cdot
\]
(95)
Additionally, using (92), we have
\[
H(\pi_{P_{1,2}}) = 2H(Q) + H(\Gamma).
\]
(96)
Furthermore, from (91), we have
\[
\sum_{\theta, x, \theta'} \mathbb{E} \left[ \log E(\theta, x, \theta')(T, o) | [T, o] \right] = \sum_{\theta, x, \theta'} q_0 \mathbb{E} \left[ \log \Lambda_{x, \theta'} \right] = \sum_{x, \theta'} \mathbb{E} \left[ \log \Lambda_{x, \theta'} \right] \cdot
\]
(97)
Substituting (95), (96), and (97) in (84), we get

\[
J(P_{1,2}) = -s(d_{1,2}^{\mathrm{CM}}) + H(X) + H(Q) - \mathbb{E} \left[ \log X! \right] + d_{1,2}^{\mathrm{CM}}(H(\Gamma) + H(Q)) + \sum_{x,\theta'} \mathbb{E} \left[ \log \Lambda_{x,\theta'} \right] \\
- \frac{d_{1,2}^{\mathrm{CM}}}{2} (2H(Q) + H(\Gamma)) - \sum_{x,\theta'} \mathbb{E} \left[ \log \Lambda_{x,\theta'} \right] \\
= -s(d_{1,2}^{\mathrm{CM}}) + H(X) - \mathbb{E} \left[ \log X! \right] + H(Q) + \frac{d_{1,2}^{\mathrm{CM}}}{2} H(\Gamma).
\]

It is easy to verify that \( P_{1,2} \) is strongly admissible. As a result, Theorem 5 implies that

\[
\Sigma(\mu_{1,2}^{\mathrm{CM}}) = -s(d_{1,2}^{\mathrm{CM}}) + H(X) - \mathbb{E} \left[ \log X! \right] + H(Q) + \frac{d_{1,2}^{\mathrm{CM}}}{2} H(\Gamma). \tag{98}
\]

Observe that if \( \Gamma = (\Gamma_1, \Gamma_2) \) has law \( \bar{\gamma} \), and with \( X_1 \) and \( X_2 \) defined in (15), we have \( \mu_{1,2}^{\mathrm{CM}} = \mathrm{UGWT}_{1}(P_1) \) where in \( P_1 \), the root has degree \( X_1 \), each vertex is independently assigned a mark whose distribution is the same as that of \( Q_1 \), and each edge has an independent edge mark whose distribution is the same as that of \( \Gamma_1 \) conditioned on \( \Gamma_1 \neq \emptyset \). As a result, using similar calculations as above, we get

\[
\Sigma(\mu_{1,2}^{\mathrm{CM}}) = -s(d_{1,2}^{\mathrm{CM}}) + H(X_1) - \mathbb{E} \left[ \log X_1! \right] + H(Q_1) + \frac{d_{1,2}^{\mathrm{CM}}}{2} H(\Gamma_1|\Gamma_1 \neq \emptyset). \tag{99}
\]

Similarly, we have

\[
\Sigma(\mu_{2,2}^{\mathrm{CM}}) = -s(d_{2,2}^{\mathrm{CM}}) + H(X_2) - \mathbb{E} \left[ \log X_2! \right] + H(Q_2) + \frac{d_{2,2}^{\mathrm{CM}}}{2} H(\Gamma_2|\Gamma_2 \neq \emptyset). \tag{100}
\]

To understand the result in (98), note that the set of typical graphs with respect to \( \mu_{1,2}^{\mathrm{CM}} \) on the vertex set \( \{1, \ldots, n\} \) is roughly the set of those graphs whose degree sequence \( \vec{d} = (d_1, \ldots, d_n) \) has an empirical distribution which is close to \( \vec{r} \) (the degree distribution at the root in \( \mu_{1,2}^{\mathrm{CM}} \)), and where the empirical distributions of the vertex and edge marks are close to \( \bar{q} \) and \( \bar{\gamma} \) respectively. The number of degree sequences \( \vec{d} \) whose empirical distribution is close to \( \vec{r} \) is asymptotically close to \( \exp(nH(X)) \) where \( X \) is a random variable with law \( \vec{r} \). On the other hand, from Theorem 2.16 in [Bol98], given such a typical degree sequence \( \vec{d} \), the number of unmarked graphs with degree sequence \( \vec{d} \) is asymptotically

\[
\exp(-\lambda/2 - \lambda^2/4) \frac{(2m)!}{m!2^m \prod_{i=1}^{m} d_i!} =: G^{(n)}(\vec{d}),
\]

where \( m = \left( \sum_{i=1}^{n} d_i / 2 \right) \) and \( \lambda = \frac{1}{m} \sum_{i=1}^{n} \left( d_i / 2 \right) \). Since \( \vec{d} \) is a typical sequence, we have \( m = nd_{1,2}^{\mathrm{CM}} / 2 + o(n) \). Therefore, using Stirling’s approximation, it is straightforward to see that

\[
\log G^{(n)}(\vec{d}) = m \log n + n(s(d_{1,2}^{\mathrm{CM}}) - \mathbb{E} \left[ \log X! \right]) + o(n),
\]

where \( X \) is a random variable with law \( \vec{r} \). So far, we have justified the role of the terms \(-s(d_{1,2}^{\mathrm{CM}}) + H(X) - \mathbb{E} \left[ \log X! \right]\) in (98). The \( H(Q) \) term correspond to vertex marks, and the term \( \frac{d_{1,2}^{\mathrm{CM}}}{2} H(\Gamma) \) corresponds to edge marks, since there are \( d_{1,2}^{\mathrm{CM}} / 2 \) many edges per vertex on average in a typical graph. Similar arguments can be used to justify (99) and (100).
H.3 Alternating Red-Blue Regular Rooted Tree

Let the vertex mark sets for the first and the second domains be \( \Theta_1 = \{ \text{black} \} \) and \( \Theta_2 = \{ \text{blue}, \text{red} \} \), respectively. Moreover, let the edge mark sets for the first and the second domains be \( \Xi_1 = \Xi_2 = \{ | \} \).

Furthermore, as in Section 2, let
\[
\Theta_{1,2} = \Theta_1 \times \Theta_2 = \{(\text{black}, \text{blue}), (\text{black}, \text{red})\}
\]
\[
\Xi_{1,2} = ((\Xi_1 \cup \{o_1\}) \times (\Xi_2 \cup \{o_2\})) \setminus \{(o_1, o_2)\}
\]
\[
= \{(|, |), (o_1, |), (|, o_2)\}.
\]

For the sake of simplicity, we may identify \( \Theta_{1,2} \) with \( \{ \text{blue}, \text{red} \} \). Fix an integer \( d \geq 3 \) and let \( \mu_{1,2} \in \mathcal{P}(\mathcal{T}_d(\Xi_{1,2}, \Theta_{1,2})) \) be defined as follows. Let \( [T_d, o] \) be the isomorphism class of a rooted \( d \) regular unmarked trees. Furthermore, we define \( [T_d^\text{blue}, o] \in \mathcal{T}_d(\Xi_{1,2}, \Theta_{1,2}) \) by adding marks to vertices and edges in \( [T_d, o] \) as follows. We give the vertex mark \text{blue} to the root \( o \), all the vertices with an odd distance from the root receive mark \text{red}, and all the vertices with an even distance from the root receive mark \text{blue}. Additionally, all the edges in \( [T_d^\text{blue}, o] \) have mark \( (|, |) \) and all the vertices with an even distance from the root receive mark \text{red}, and all the vertices with an odd distance from the root receive mark \text{blue}. Lastly, we define \( [T_d^\text{red}, o] \) by interchanging vertex marks \text{blue} and \text{red}. With this, we define \( \mu_{1,2} \in \mathcal{P}_u(\mathcal{T}_d(\Xi_{1,2}, \Theta_{1,2})) \) such that it assigns probability \( 1/2 \) to \( [T_d^\text{blue}, o] \) and probability \( 1/2 \) to \( [T_d^\text{red}, o] \). Observe that the marginal distribution \( \mu_1 \in \mathcal{P}_u(\mathcal{T}_d(\Xi_1, \Theta_1)) \) is effectively a point mass on a \( d \) regular tree (recall that since \( |\Xi_1| = |\Theta_1| = 1 \), the first domain is effectively unmarked).

Now we focus on calculating \( \Sigma(\mu_{1,2}) \). Let \( P_{1,2} \in \mathcal{P}(\mathcal{T}_d^1(\Xi_{1,2}, \Theta_{1,2})) \) be defined as follows. \( P_{1,2} \) assigns probability \( 1/2 \) to the element in \( \mathcal{T}_d^1(\Xi_{1,2}, \Theta_{1,2}) \) where the root has vertex mark \text{red}, the root has \( d \) children each with vertex mark \text{blue}, and all the edges have edge mark \( (|, |) \). Moreover, \( P_{1,2} \) assigns probability \( 1/2 \) to a similar element in \( \mathcal{T}_d^1(\Xi_{1,2}, \Theta_{1,2}) \) with the only difference that the role of vertex marks \text{blue} and \text{red} are interchanged. It is easy to verify that \( P_{1,2} \) is strongly admissible, and \( \mu_{1,2} = \text{UGWT}_1(P_{1,2}) \). Therefore, we may use Theorem 5 to calculate \( \Sigma(\mu_{1,2}) \). Indeed, we have
\[
H(P_{1,2}) = \log 2. \tag{101}
\]
Furthermore, we have
\[
e_{P_{1,2}}(\text{blue}, (|, |), \text{red}) = e_{P_{1,2}}(\text{red}, (|, |), \text{blue}) = \frac{d}{2}. \tag{102}
\]
This implies
\[
\pi_{P_{1,2}}(\text{blue}, (|, |)) = \pi_{P_{1,2}}(\text{red}, (|, |), \text{blue}) = \frac{1}{2}. \tag{103}
\]
and
\[
H(\pi_{P_{1,2}}) = \log 2. \tag{104}
\]
On the other hand,
\[
\sum_{\theta, x, \theta'} \mathbb{E}_{P_{1,2}}(\log E(\theta, x, \theta')([T, o])!] = \mathbb{E}_{P_{1,2}}(\log E(\text{blue}, (|, |), \text{red})([T, o])]
\]
\[
+ \mathbb{E}_{P_{1,2}}(\log E(\text{red}, (|, |), \text{blue})([T, o])]
\]
\[
= \frac{1}{2} \log(d!) + \frac{1}{2} \log(d!)
\]
\[
= \log(d!). \tag{105}
\]
Substituting (101), (104), and (102) into (84), we get
\[
J(P_{1,2}) = -s(d) + \log 2 - \frac{d}{2} \log 2 - \log(d!).
\]
It is easy to verify that $P_{1,2}$ is strongly admissible. Thereby, simplifying and using Theorem 5, we get
\[
\Sigma(\mu_{1,2}) = \frac{d}{2} + \frac{d}{2} \log \frac{d}{2} + \log 2 - \log(d!).
\] (106)

It is straightforward to see that using similar calculations, we get
\[
\Sigma(\mu_2) = \Sigma(\mu_{1,2}) = \frac{d}{2} + \frac{d}{2} \log \frac{d}{2} + \log 2 - \log(d!).
\] (107)

Finally, to calculate $\Sigma(\mu_1)$, we define $P_1 \in P(T_{\frac{d}{2}}(\Xi_1, \Theta_1))$ to be the point mass on a root with $d$ children. It is easy to verify that $\mu_1 = UGWT_1(P_1)$. Also, $H(P_1) = 0$. Moreover, $\pi_{P_1}(\text{black}, [1], \text{black}) = 1$ which means that $H(\pi_{P_1}) = 0$. Additionally, we have
\[
\sum_{\theta, x, \theta'} \mathbb{E}_{P_1} [\log E(\theta, x, \theta')([T, o])!] = \mathbb{E}_{P_1} [\log E(\text{black}, [1], \text{black})([T, o])!] = \log(d!).
\]

Hence, using Theorem 5, we get
\[
\Sigma(\mu_1) = -s(d) - \log(d!) = \frac{d}{2} + \frac{d}{2} \log d - \log(d!).
\] (108)

Now, we provide an intuitive explanation for the entropy formulas derived above. We begin with $\Sigma(\mu_1)$ in (108). Roughly speaking, the set of typical graphs with respect to $\mu_1$ on the vertex set $\{1, \ldots, n\}$ is the set of labeled unmarked $d$-regular graphs. Using Theorem 2.16 in [Bol98], the number of such graphs is asymptotically equal to
\[
\exp(-(d - 1)/2 - (d - 1)^2/4) \frac{(nd)!}{(nd/2)!2^{nd/2}(d!)^n} =: \text{Reg}_{n,d}.
\]

Using Stirling’s approximation, it is easy to verify that
\[
\log \text{Reg}_{n,d} = \frac{nd}{2} \log n + n(-s(d) - \log(d!)) + o(n).
\]

Note that $nd/2$ is the number of edges in a $d$-regular graph, and the coefficient of $n$ in this expression is equal to $\Sigma(\mu_1)$ as was demonstrated in (108). Note that since all the edges in $\mu_{1,2}$ also appear in $\mu_2$, and the vertex and edge marks in $\mu_{1,2}$ can be recovered from those in $\mu_2$, we have $\Sigma(\mu_{1,2}) = \Sigma(\mu_2)$ (as was stated in (107) above). Observe that roughly speaking, due to the alternating red–blue vertex marks in $\mu_2$, and the fact that the root mark is red with probability 1/2 and blue with probability 1/2, the set of $\mu_2$ typical graphs is more or less the set of $d$-regular graphs which have a red–blue vertex marking such that almost half of the vertices are red and the rest half are blue, most of the red vertices have all of their $d$ neighbors marked as blue, and most of the blue vertices have all of their $d$ neighbors marked as red. In other words, the set of $\mu_2$ typical graphs is more or less the set of bipartite $d$–regular graphs where one partition has $n/2$ red vertices, and the other partition has $n/2$ blue vertices. Given such a marked graph, we construct an unmarked bipartite graph by relabeling the vertices with mark red to $\{1, \ldots, n/2\}$, preserving their order, and relabeling the vertices with mark blue to $\{n/2 + 1, \ldots, n\}$, also preserving their order. From [BBK72], the number of $d$–regular bipartite graphs on the vertex set $\{1, \ldots, n\}$ with vertices $\{1, \ldots, n/2\}$ in one partition and vertices $\{n/2 + 1, \ldots, n\}$ in the second partition is asymptotically equal to
\[
\exp(-(d - 1)^2/2) \frac{(nd/2)!}{(d!)^n} =: \text{Bip}_{n,d}.
\]
Because of the above relabeling of vertices, given such an unmarked bipartite graph, there are \( \binom{n}{n/2} \) marked bipartite graphs as above. As a result, using Stirling’s approximation, the logarithm of the number of \( \mu_2 \) typical graphs is asymptotically
\[
\log \text{Bip}_{n,d} + \log \left( \frac{n}{n/2} \right) = \frac{dn}{2} \log n + n \left( \frac{d}{2} \log \frac{d}{2} - \frac{d}{2} - \log(d!) + \log 2 \right) + o(n).
\]
Note that \( nd/2 \) is the number of edges in a \( d \)–regular graph, and the coefficient of \( n \) in the above expression is precisely \( \Sigma(\mu_2) = \Sigma(\mu_{1,2}) \) as in (107).

I Counterexample for the Constancy of the Size of the Set of Conditional Typical Graphs

In this section, we study the asymptotic size of joint, marginal, and conditional typical graphs for the example of Appendix H.3 and we observe a behavior which is fundamentally different from what we expect from classical information theory, namely the constancy of the size of conditional typical sequences in classical information theory. This kind of behavior in part makes our analysis more complicated compared to the classical setting as we need to carefully control the number of jointly typical graphs.

Fix an integer \( d \geq 3 \) and let \( \mu_{1,2} \in \mathcal{P}_n(\mathcal{T}_1(\Theta_{1,2})) \) be the alternating red–blue \( d \)–regular random rooted tree explained in Section H.3 of Appendix H. In order to study the joint, marginal, and conditional typical graphs for this example, fix sequences \( \bar{m}^{(n)} = \{m_i^{(n)}(x)\}_{x \in \Xi_{1,2}} \) and \( \bar{u}^{(n)} = \{u_i^{(n)}(\theta)\}_{\theta \in \Theta_{1,2}} \) for \( n \geq 1 \) of edge mark and vertex mark count vectors adapted to \( (\deg(\mu_{1,2}), \bar{H}(\mu_{1,2})) \) in the sense of Definition 3. Note that since \( \deg_{(1,2)}(\mu_{1,2}) = \deg_{(o_1,1)}(\mu_{1,2}) = 0 \), condition 4 in Definition 3 implies that
\[
m^{(n)}(1, o_2) = m^{(n)}(o_1, 1) = 0 \quad \forall n.
\]
Motivated by this, the only nonzero element in \( \bar{m}^{(n)} \) is \( m^{(n)}(1, 1) \) which is the total number of edges. Therefore, we define \( m^{(n)} := m^{(n)}(1, 1) \), and to simplify the notation we write \( m^{(n)} \) instead of \( \bar{m}^{(n)} \).

Note that (109) in particular implies that for every marked graph in the joint domain \( \mathcal{G}^{(n)}_{m^{(n)}, m^{(n)}}(\mu_{1,2}) \), all the edges appear in both marginals. Following the convention in (4), we define the marginal vertex mark count vectors \( \bar{u}_1^{(n)} = (u_1^{(n)}(\theta_1))_{\theta_1 \in \Theta_{1,2}} \) and \( \bar{u}_2^{(n)} = (u_2^{(n)}(\theta_2))_{\theta_2 \in \Theta_{1,2}} \). Note that since \( \Theta_1 = \{\text{black}\} \), graphs on the first domain are effectively unmarked. Hence, we may simply write \( \mathcal{G}^{(n)}_{m^{(n)}, m^{(n)}}(\mu_{1,2}) \) instead of \( \mathcal{G}^{(n)}_{m^{(n)}, \bar{u}_1^{(n)}}(\mu_{1,2}) \) and \( \mathcal{G}^{(n)}_{m^{(n)}, \bar{u}_2^{(n)}}(\mu_{1,2}) \), respectively.

Given \( 0 < \epsilon < \epsilon' \) and \( G_1 \in \mathcal{G}^{(n)}_{m^{(n)}, \bar{u}_1^{(n)}}(\mu_1, \epsilon) \), we define the conditional typical set as
\[
\mathcal{G}^{(n)}_{m^{(n)}, \bar{u}_1^{(n)}}(\mu_2, \epsilon'|G_1) := \{ G_2 \in \mathcal{G}^{(n)}_{m^{(n)}, \bar{u}_1^{(n)}} : G_1 \oplus G_2 \in \mathcal{G}^{(n)}_{m^{(n)}, \bar{u}_1^{(n)}}(\mu_{1,2}, \epsilon') \}.
\]
In words, this is the set of graphs on the second domain which are jointly typical with \( G_1 \). Note that, due to (109), each graph in \( \mathcal{G}^{(n)}_{m^{(n)}, \bar{u}_1^{(n)}}(\mu_2, \epsilon'|G_1) \) has the same set of edges as in \( G_1 \), and only has vertex marks added to \( G_1 \). Extrapolating the results from classical information theory, we might expect that for each \( G_1 \in \mathcal{G}^{(n)}_{m^{(n)}, \bar{u}_1^{(n)}}(\mu_1, \epsilon) \), the set \( \mathcal{G}^{(n)}_{m^{(n)}, \bar{u}_1^{(n)}}(\mu_2, \epsilon'|G_1) \) has roughly the same size, and this size is dependent on the conditional marked BC entropy \( \Sigma(\mu_2|\mu_1) \). However, as we will see below, this is not true.
Proposition 1. For the above example, there exists $\epsilon_0 > 0$ such that for all $0 < \epsilon < \epsilon' < \epsilon_0$, for $n$ large enough, the set 

$$A_{n, \epsilon, \epsilon'} := \{ G_1 \in \mathcal{G}^{(n)}_{m(n)}(\mu_1, \epsilon) : \mathcal{G}^{(n)}_{m(n), \overline{u}_2}(\mu_2, \epsilon'|G_1) \text{ is empty} \},$$

is not empty.

Note that if we take a graph $G_{1,2}$ which is $\mu_{1,2}$-typical, i.e. if $G_1 \in \mathcal{G}^{(n)}_{m(n), \overline{u}_2}(\mu_{1,2}, \epsilon)$, then it is easy to verify that the marginal graph $G_1$ is $\mu_1$-typical, i.e. $G_1 \in \mathcal{G}^{(n)}_{m(n)}(\mu_1, \epsilon)$, and also by definition $\mathcal{G}^{(n)}_{m(n), \overline{u}_2}(\mu_2, \epsilon'|G_1)$ is not empty. This behavior is fundamentally different from what we know from classical information theory, where roughly speaking, all marginal typical sequences have a nonempty set of conditional typical sequences with an asymptotically constant size.

Before proving this result, we intuitively discuss why it holds. Roughly speaking, the set of $\mu_1$-typical graphs is less or more the set of almost $d$–regular graphs. On the other hand, $\mu_{1,2}$–regular graphs in addition to being almost $d$–regular, should also have a vertex marking which results in an almost bipartite partitioning. Therefore, only those $\mu_1$–regular graphs which also have at least one such almost bipartite marking can have conditional typical graphs on the second domain. But since not all $d$–regular graphs have such a bipartite partitioning, there are $\mu_1$ typical graphs for which their corresponding conditional typical set is empty.

Proof of Proposition 1. Observe that for two distinct $G_1$ and $G'_1$ in $\mathcal{G}^{(n)}_{m(n)}(\mu_1, \epsilon)$, the sets $\mathcal{G}^{(n)}_{m(n), \overline{u}_2}(\mu_2, \epsilon'|G_1)$ and $\mathcal{G}^{(n)}_{m(n), \overline{u}_2}(\mu_2, \epsilon'|G'_1)$ are distinct. To see this, assume that $G_2 \in \mathcal{G}^{(n)}_{m(n), \overline{u}_2}(\mu_2, \epsilon'|G_1) \cap \mathcal{G}^{(n)}_{m(n), \overline{u}_2}(\mu_2, \epsilon'|G'_1)$ and note that due to (109), the set of edges in $G_2$ is identical to those of $G_1$ and $G'_1$. But each edge and vertex in $G_1$ and $G'_1$ can have only one possible mark. This implies that $G_1 = G'_1$ which is a contradiction. This implies that

$$\sum_{G_1 \in \mathcal{G}^{(n)}_{m(n)}(\mu_1, \epsilon)} |\mathcal{G}^{(n)}_{m(n), \overline{u}_2}(\mu_2, \epsilon'|G_1)| \leq |\mathcal{G}^{(n)}_{m(n), \overline{u}_2}(\mu_2, \epsilon')|.$$  \hfill (111)

On the other hand,

$$\sum_{G_1 \in \mathcal{G}^{(n)}_{m(n)}(\mu_1, \epsilon)} |\mathcal{G}^{(n)}_{m(n), \overline{u}_2}(\mu_2, \epsilon'|G_1)| \geq |\mathcal{G}^{(n)}_{m(n), \overline{u}_2}(\mu_2, \epsilon'|G_1) \setminus A_{n, \epsilon, \epsilon'}|$$

$$= |\mathcal{G}^{(n)}_{m(n)}(\mu_1, \epsilon)| - |A_{n, \epsilon, \epsilon'}|.$$

Comparing this with (111), we get

$$|\mathcal{G}^{(n)}_{m(n)}(\mu_1, \epsilon)| - |A_{n, \epsilon, \epsilon'}| \leq |\mathcal{G}^{(n)}_{m(n), \overline{u}_2}(\mu_2, \epsilon')|.$$  \hfill (112)

Using the definition of the marked BC entropy (Definition 4 in Section 4) and Theorem 2, we have

$$\lim_{\epsilon \downarrow 0} \lim_{n \to \infty} \sup \frac{\log |\mathcal{G}^{(n)}_{m(n), \overline{u}_2}(\mu_2, \epsilon')| - m(n) \log n}{n} = \mathcal{\Sigma}(\mu_2) = \Sigma(\mu_2),$$

and

$$\lim_{\epsilon \downarrow 0} \lim_{n \to \infty} \inf \frac{\log |\mathcal{G}^{(n)}_{m(n)}(\mu_1, \epsilon)| - m(n) \log n}{n} = \mathcal{\Sigma}(\mu_1) = \Sigma(\mu_1),$$

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But from the calculations in Section H.3 in Appendix H, we have

\[ \Sigma(\mu_1) = -\frac{d}{2} + \frac{d}{2} \log d - \log(d!) > -\frac{d}{2} + \frac{d}{2} \log \frac{d}{2} + \log 2 - \log(d!) = \Sigma(\mu_2), \]

where the inequality holds since \( d \geq 3 \) by assumption. This means that there exists \( \epsilon_0 > 0 \) such that for all \( 0 < \epsilon < \epsilon' < \epsilon_0 \), when \( n \) is large, we have

\[ |G_{m^{(n)}}(\mu_1, \epsilon)| > |G_{m^{(n)}}(\mu_2, \epsilon')|. \]

Comparing this with (112), we realize that for this \( \epsilon_0 \), for all \( 0 < \epsilon < \epsilon' < \epsilon_0 \), when \( n \) is large, \( A_{n, \epsilon, \epsilon'} \) is not empty. This is precisely what we wanted to prove. \( \square \)