A GENERAL DOUBLE INEQUALITY RELATED TO OPERATOR MEANS AND POSITIVE LINEAR MAPS

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Abstract. Let \( A, B \in \mathcal{B}(\mathcal{H}) \) be such that \( 0 < b_1 I \leq A \leq a_1 I \) and \( 0 < b_2 I \leq B \leq a_2 I \) for some scalars \( 0 < b_i < a_i, \ i = 1, 2 \) and \( \Phi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{K}) \) be a positive linear map. We show that for any operator mean \( \sigma \) with the representing function \( f \), the double inequality

\[
\omega^{1-\alpha}(\Phi(A) \#_{\alpha} \Phi(B)) \leq (\omega\Phi(A)) \nabla_{\alpha} \Phi(B) \leq \frac{\alpha}{\mu} \Phi(A \sigma B)
\]

holds, where \( \mu = \frac{a_1 b_1 (f(b_2 a_1^{-1}) - f(a_2 b_1^{-1}))}{b_1 a_1 - a_1 a_2} \), \( \nu = \frac{a_1 a_2 f(b_2 a_1^{-1}) - b_1 b_2 f(a_2 b_1^{-1})}{a_1 a_2 - b_1 b_2} \), \( \omega = \frac{\alpha \nu}{(1-\alpha)\mu} \) and \( \#_{\alpha} \) (\( \nabla_{\alpha} \), resp.) is the weighted geometric (arithmetic, resp.) mean for \( \alpha \in (0, 1) \).

As applications, we present several generalized operator inequalities including Diaz–Metcalf and reverse Ando type inequalities. We also give some related inequalities involving Hadamard product and operator means.

1. Introduction

In what follows, \( \mathcal{B}(\mathcal{H}) \) denotes the \( C^* \)-algebra of all bounded linear operators acting on a Hilbert space \( (\mathcal{H}, \langle \cdot, \cdot \rangle) \) and \( I \) stands for the identity operator. A selfadjoint operator \( A \in \mathcal{B}(\mathcal{H}) \) is said to be positive (strictly positive, resp.) if \( \langle A \xi, \xi \rangle \geq 0 \) (\( \langle A \xi, \xi \rangle > 0, \xi \neq 0 \), resp.) for all \( \xi \in \mathcal{H} \) and we write \( A \geq 0 \) (\( A > 0 \), resp.). For selfadjoint operators \( A, B \in \mathcal{B}(\mathcal{H}) \), by \( A \geq B \) (\( A > B \), resp.) we mean \( A - B \geq 0 \) (\( A - B > 0 \), resp.). A linear map \( \Phi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{K}) \) between \( C^* \)-algebras is called positive (strictly positive, resp.) if it maps positive (strictly positive, resp.) operators into positive (strictly positive, resp.) operators and is said to be unital if it maps identity operator to identity operator in the corresponding \( C^* \)-algebra.

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By an operator monotone function, we mean a continuous real-valued function $f$ defined on an interval $J$ such that $A \geq B$ implies $f(A) \geq f(B)$ for all self adjoint operators $A, B$ with spectra in $J$. Some structure theorems on operator monotone functions can be found in [2].

The axiomatic theory for operator means for pairs of positive operators have been developed by Kubo and Ando [6]. A binary operation $\sigma$ defined on the set of strictly positive operators is called an operator mean provided that

(i) $I \sigma I = I$;
(ii) $C^*(A \sigma B)C \leq (C^*AC)\sigma(C^*BC)$;
(iii) $A_n \downarrow A$ and $B_n \downarrow B$ imply $(A_n \sigma B_n) \downarrow A \sigma B$, where $A_n \downarrow A$ means that $A_1 \geq A_2 \geq \cdots$ and $A_n \to A$ as $n \to \infty$ in the strong operator topology;
(iv) $A \leq B$ and $C \leq D$ imply that $A \sigma C \leq B \sigma D$.

There exists an affine order isomorphism between the class of operator means and the class of positive operator monotone functions $f$ defined on $(0, \infty)$ with $f(1) = 1$ via $f(t)I = I \sigma (tI)$ ($t > 0$). In addition, $A \sigma B = A^{1/2}f(A^{-1/2}BA^{-1/2})A^{1/2}$ for all strictly positive operators $A, B$. The operator monotone function $f$ is called the representing function of $\sigma$. Using a limit argument by $A_\epsilon = A + \epsilon I$, one can extend the definition of $A \sigma B$ to positive operators as well. The operator means corresponding to operator monotone functions $(1 - \alpha) + \alpha t$ and $t^\alpha$ with $0 \leq \alpha \leq 1$ are called weighted arithmetic and weighted geometric means and are denoted by $\nabla_\alpha$ and $\#_\alpha$, respectively. In particular $\nabla_{1/2}$ and $\#_{1/2}$ or simply written as $\nabla$ and $\#$ are called arithmetic and geometric mean, respectively.

In this paper we establish a general double inequality involving operator means and positive linear maps, which unifies and includes the recent results of [7, 8] concerning Diaz–Metcalf and reverse Ando type inequalities. We also give some related inequalities involving Hadamard product and operator means.

2. Main Result

We start this section with our main result. It extends the reverse Ando’s inequality presented in [8] and some Diaz–Metcalf type inequalities in [7] as we see in the sequel.
Theorem 2.1. Let $A, B \in \mathbb{B} (\mathcal{H})$ be such that $0 < b_1 I \leq A \leq a_1 I$ and $0 < b_2 I \leq B \leq a_2 I$ for some scalars $0 < b_i < a_i, \ i = 1, 2$ and $\Phi : \mathbb{B} (\mathcal{H}) \to \mathbb{B} (\mathcal{H})$ be a positive linear map. Then for any operator mean $\sigma$ with the representing function $f$, the following double inequality holds:

$$
\omega^{1-\alpha}(\Phi(A)\#_{\alpha}\Phi(B)) \leq (\omega\Phi(A))\nabla_{\alpha}\Phi(B) \leq \frac{\alpha}{\mu}\Phi(A\sigma B),
$$

where $\mu = \frac{a_1 b_1 (f(b_2 a_1^{-1}) - f(a_2 b_1^{-1}))}{b_1 b_2 - a_1 a_2}$, $\nu = \frac{a_1 a_2 f(b_2 a_1^{-1}) - b_1 b_2 f(a_2 b_1^{-1})}{a_1 a_2 - b_1 b_2}$, $\omega = \frac{\alpha \nu}{(1-\alpha)\mu}$ and $\alpha \in (0, 1)$.

Proof. The conditions $0 < b_1 I \leq A \leq a_1 I$ and $0 < b_2 I \leq B \leq a_2 I$ implies that $0 < b_2 a_1^{-1} A \leq b_2 I \leq B \leq a_2 I \leq a_2 b_1^{-1} A$. Consequently, $0 < b_2 a_1^{-1} < a_2 b_1^{-1}$. The function $f : (0, \infty) \to (0, \infty)$ being operator monotone is strictly increasing and concave [2, Corollary 1.12]. Therefore $x^{-1} f(x)$ is operator monotone decreasing (see [2, Corollary 1.14]). This implies that $\mu$ and $\nu$ are positive. In fact,

$$
\mu = \frac{1}{b_2 a_1^{-1} - b_2 a_1^{-1}} (f(b_2 a_1^{-1}) - f(a_2 b_1^{-1})) > 0
$$

and

$$
\nu = \frac{a_2 b_2}{a_1 a_2 - b_1 b_2} (b_2 a_1^{-1} f(b_2 a_1^{-1}) - a_2 b_1^{-1} f(a_2 b_1^{-1})) > 0.
$$

The first inequality in (2.1) follows on using the weighted arithmetic-geometric mean inequality $\omega^{1-\alpha}(X\#_{\alpha}Y) = (\omega X)\#_{\alpha}Y \leq (\omega X)\nabla_{\alpha}Y$.

For the second inequality in (2.1), consider $\mu t + \nu$. Note that $f(t)$ and the line $\mu t + \nu$ intersect at the points $(a_2 b_1^{-1}, f(a_2 b_1^{-1}))$ and $(b_2 a_1^{-1}, f(b_2 a_1^{-1}))$. Thus, since $f(t)$ is concave [2, Corollary 1.12], we see that

$$
\mu t + \nu \leq f(t)
$$

for all $t \in [b_2 a_1^{-1}, a_2 b_1^{-1}]$. So, \[\alpha t + (1-\alpha) \left( \frac{\alpha \nu}{(1-\alpha)\mu} \right) \leq \frac{\alpha}{\mu} f(t)\]

for all $\alpha \in (0, 1)$. Hence, since $b_2 a_1^{-1} I \leq A^{-1/2} B A^{-1/2} \leq a_2 b_1^{-1} I$, we obtain

$$
\alpha A^{-1/2} B A^{-1/2} + (1-\alpha) \left( \frac{\alpha \nu}{(1-\alpha)\mu} \right) I \leq \frac{\alpha}{\mu} f(A^{-1/2} B A^{-1/2}),
$$
which implies
\[
\alpha B + (1 - \alpha) \left[ \frac{\alpha \nu}{(1 - \alpha) \mu} \right] A \leq \frac{\alpha}{\mu} A \sigma B.
\] (2.3)

Since \( \Phi \) is positive and linear, (2.3) yields
\[
(\omega \Phi(A)) \nabla_\alpha \Phi(B) \leq \frac{\alpha}{\mu} \Phi(A \sigma B).
\]

\[\square\]

Remark 2.2. The condition \( 0 < b_i < a_i, \ i = 1, 2 \) in Theorem 2.1 can be replaced by either \( 0 < b_1 < a_1 \) and \( 0 < b_2 \leq a_2 \) or \( 0 < b_1 \leq a_1 \) and \( 0 < b_2 < a_2 \).

3. Applications

Now we present several applications of our main Theorem 2.1.

3.1. Diaz–Metcalf type inequalities. Taking \( \alpha = 1/2 \) and \( \sigma = \# \), so that \( f(t) = \sqrt{t} \), in the second inequality of (2.1), we get the following Diaz–Metcalf type inequality of the second type:

**Corollary 3.1.** [7, Theorem 2.1] Let \( A, B \in \mathbb{B}(\mathcal{H}) \) be positive invertible operators such that \( m_2^2 I \leq A \leq M_2^2 I \) and \( m_2^2 I \leq B \leq M_2^2 I \) for some positive real numbers \( m < M \) and \( m_2 < M_2 \) and \( \Phi : \mathbb{B}(\mathcal{H}) \to \mathbb{B}(\mathcal{K}) \) be a positive linear map. Then
\[
\frac{M_2 m_2}{M_1 m_1} \Phi(A) + \Phi(B) \leq \left( \frac{M_2}{m_1} + \frac{m_2}{M_1} \right) \Phi(A \# B).
\]

If \( m^2 A \leq B \leq M^2 A \) for some positive real numbers \( m < M \), then by considering \( \Psi(C) = \Phi(A^{1/2} C A^{1/2}) \) and noting that \( m^2 I \leq A^{-1/2} B A^{-1/2} \leq M^2 I \) and \( 1I \leq I \leq 1I \) we obtain the following inequality:
\[
Mm \Psi(I) + \Psi(A^{-1/2} B A^{-1/2}) \leq (M + m) \Psi(I \# A^{-1/2} B A^{-1/2}).
\]

Therefore we reach to the following Diaz–Metcalf type inequality of the first type:

**Corollary 3.2.** [7, Theorem 2.1] Let \( A, B \in \mathbb{B}(\mathcal{H}) \) be positive invertible operators such that \( m^2 A \leq B \leq M^2 A \) for some positive real numbers \( m < M \) and \( \Phi : \mathbb{B}(\mathcal{H}) \to \mathbb{B}(\mathcal{K}) \) be a positive linear map. Then
\[
Mm \Phi(A) + \Phi(B) \leq (M + m) \Phi(A \# B).
\]
3.2. Inequalities complementary to Ando’s inequality. Ando’s inequality [1] states that if $A, B \in \mathbb{B}(\mathcal{K})$ are positive operators, $\alpha \in [0, 1]$ and $\Phi$ is a positive linear map, then

$$\Phi(A\#_\alpha B) \leq \Phi(A)\#_\alpha \Phi(B).$$

The following result is an additive reverse of the second type of this inequality:

**Corollary 3.3.** Let $A, B, \Phi$ and $\mu, \nu$ and $\alpha$ be as in Theorem 2.1. Then

$$\Phi(A)\#_\alpha \Phi(B) - \Phi(A\#_\alpha B) \leq \left(1 - \alpha\right)\left(\mu\alpha^{-1}\right)^{\frac{1}{\alpha - 1}} - \nu) \Phi(A).$$

(3.1)

In particular, when $\alpha = 1/2$.

$$\Phi(A)\#\Phi(B) - \Phi(A\#B) \leq \left(\frac{1}{4\mu} - \nu\right) \min\{a_1, a_2\} I$$

(3.2)

whenever $\Phi(I) \leq I$.

**Proof.** From inequality (2.2) we get

$$(1 - \alpha)(\mu\alpha^{-1})^{\frac{1}{\alpha - 1}} + \alpha(\mu\alpha^{-1})t \leq f(t) - (\nu + (\alpha - 1)(\mu\alpha^{-1})^{\frac{1}{\alpha - 1}}).$$

As in the proof of Theorem 2.1 the above inequality yields

$$(1 - \alpha)(\mu\alpha^{-1})^{\frac{1}{\alpha - 1}}\Phi(A) + \alpha(\mu\alpha^{-1})\Phi(B) \leq \Phi(A\sigma B) - (\nu + (\alpha - 1)(\mu\alpha^{-1})^{\frac{1}{\alpha - 1}})\Phi(A).$$

Now on using weighted arithmetic-geometric mean inequality on the left hand side of the above inequality we get

$$\Phi(A)\#_\alpha \Phi(B) \leq (1 - \alpha)\left((\mu\alpha^{-1})^{\frac{1}{\alpha - 1}}\right)^{\alpha} \Phi(A) + \alpha \left((\mu\alpha^{-1})^{\frac{1}{\alpha - 1}}\right)^{\alpha - 1} \Phi(B) \leq \Phi(A\sigma B) - (\nu + (\alpha - 1)(\mu\alpha^{-1})^{\frac{1}{\alpha - 1}})\Phi(A).$$

Taking $\sigma = \#_\alpha$ and using the given conditions on $A, B$ we get the desired inequality (3.1). Inequality (3.2) immediately follows from (3.1) for $\alpha = 1/2$ because both $\Phi(A)\#\Phi(B)$ and $\Phi(A\#B)$ are symmetric with respect to $A$ and $B$. \qed

Using the technique in Subsection 3.1 we can get the following reverse Ando inequality of the second type:
Corollary 3.4. [8, Theorem 1] Let $A, B \in \mathcal{B}(\mathcal{H})$ be positive invertible operators such that $mA \leq B \leq MA$ for some positive real numbers $m < M$ and $\Phi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ be a positive linear map. Then for $\alpha \in (0, 1)$,

$$\Phi(A\#_\alpha B) - \Phi(A\#_\alpha B) \leq \left[ (1 - \alpha) \left( \frac{M^\alpha - m^\alpha}{\alpha(M - m)} \right)^{\alpha^{-1}} - \frac{Mm^\alpha - mM^\alpha}{M - m} \right] \Phi(A).$$

3.3. Shisha–Mond and Kalmkin–McLenaghan inequalities. Now, we present a Kalmkin–McLenaghan type inequality due to Seo \[8\]:

Theorem 3.5. Let $A, B, \Phi$ and $\mu, \omega$ be as in Theorem 2.1. Then

$$\Phi(A\sigma B)^{-1/2}\Phi(B)\Phi(A\sigma B)^{-1/2} - \Phi(A\sigma B)^{1/2}\Phi(A)^{-1}\Phi(A\sigma B)^{1/2} \leq \left( \mu^{-1} - 2\sqrt{\omega} \right) I.$$

Proof. Now from (2.1) for $\alpha = 1/2$ we get

$$\Phi(A\sigma B)^{-1/2}\Phi(B)\Phi(A\sigma B)^{-1/2} + \omega\Phi(A\sigma B)^{-1/2}\Phi(A)\Phi(A\sigma B)^{-1/2} \leq \mu^{-1} I,$$

whence

$$\Phi(A\sigma B)^{-1/2}\Phi(B)\Phi(A\sigma B)^{-1/2} - \Phi(A\sigma B)^{1/2}\Phi(A)^{-1}\Phi(A\sigma B)^{1/2} \leq \left( \mu^{-1} - 2\sqrt{\omega} \right) I$$

$$\leq \mu^{-1} I - \omega\Phi(A\sigma B)^{-1/2}\Phi(A)\Phi(A\sigma B)^{-1/2} - \Phi(A\sigma B)^{1/2}\Phi(A)^{-1}\Phi(A\sigma B)^{1/2}$$

$$\leq \left( \mu^{-1} - 2\sqrt{\omega} \right) I$$

$$- \left( \sqrt{\omega} \left( \Phi(A\sigma B)^{-1/2}\Phi(A)\Phi(A\sigma B)^{-1/2} \right)^{1/2} - \left( \Phi(A\sigma B)^{1/2}\Phi(A)^{-1}\Phi(A\sigma B)^{1/2} \right)^{1/2} \right)^2$$

$$\leq \left( \mu^{-1} - 2\sqrt{\omega} \right) I,$$

which is a generalized operator Shisha–Mond inequality. \hfill \Box

If $\sigma$ is taken to be $\#$, we get the following Operator Shisha–Mond inequality:

Corollary 3.6. [7, Theorem 2.1] Let $A, B \in \mathcal{B}(\mathcal{H})$ be positive invertible operators such that $m^2A \leq B \leq M^2A$ for some positive real numbers $m < M$ and $\Phi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ be a positive linear map. Then

$$\Phi(A^*B)^{1/2}\Phi(B)\Phi(A^*B)^{1/2} - \Phi(A^*B)^{1/2}\Phi(A)^{-1}\Phi(A^*B)^{1/2} \leq (\sqrt{M} - \sqrt{m})^2 I.$$
Using the technique in Subsection 3.1 we can get the following Kalmkin–McLenaghan inequality:

**Corollary 3.7.** [7, Theorem 2.1] Let $A, B \in \mathbb{B}(\mathcal{H})$ be positive invertible operators such that $m_1^2 I \leq A \leq M_1^2 I$ and $m_2^2 I \leq B \leq M_2^2 I$ for some positive real numbers $m_1 < M_1$ and $m_2 < M_2$ and $\Phi : \mathbb{B}(\mathcal{H}) \to \mathbb{B}(\mathcal{K})$ be a positive linear map. Then

$$
\Phi(A^\frac{1}{2}B)^\frac{1}{2}\Phi(B)\Phi(A^\frac{1}{2}B)^\frac{1}{2} - \Phi(A^\frac{1}{2}B)^\frac{1}{2}\Phi(A)^{-1}\Phi(A^\frac{1}{2}B)^\frac{1}{2} \leq \left(\sqrt{\frac{M_2}{m_1}} - \sqrt{\frac{m_2}{M_1}}\right)^2 I.
$$

### 3.4. Ozeki–Izumino–Mori–Seo type inequalities.

Now we present a generalized operator Ozeki–Izumino–Mori–Seo type inequality. To achieve it, we need the following lemma, which is helpful in proving many operator inequalities.

**Lemma 3.8.** Let $\Phi$ be a unital positive linear map on $\mathbb{B}(\mathcal{H})$, $A \in \mathbb{B}(\mathcal{H})$ is selfadjoint with $bI \leq A \leq aI$. Then

$$
\Phi(A^2) - \Phi(A)^2 \leq \frac{1}{4}(a - b)^2 I.
$$

*Proof.* The proof is an easy consequence of both facts

$$
\Phi(A^2) - \Phi(A)^2 = \Phi(|A - \alpha I|^2) - |\Phi(A - \alpha I)|^2 \quad (\alpha \in \mathbb{R})
$$

and

$$
\frac{1}{4}(a - b)^2 I - \left|A - \frac{a + b}{2} I\right|^2 = (aI - A)(A - bI) \geq 0.
$$

□

**Theorem 3.9.** Suppose that $\Phi : \mathbb{B}(\mathcal{H}) \to \mathbb{B}(\mathcal{K})$ is a strictly positive linear map with $\Phi(I) \leq I$. Assume that $A, B \in \mathbb{B}(\mathcal{H})$ are such that $0 < b_1 I \leq A \leq a_1 I$ and $0 < b_2 I \leq B \leq a_2 I$. Then

$$
\Phi(A)^{1/2} \Phi(|A^{-1/2}(A\sigma B)|^2) \Phi(A)^{1/2} - |\Phi(A)^{-1/2}\Phi(A\sigma B)\Phi(A)^{1/2}|^2 
\leq \frac{a_1^2}{4}(f(a_2b_1^{-1}) - f(b_2a_1^{-1}))^2 I.
$$
Proof. Consider the strictly positive linear map $\Psi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ defined by

$$\Psi(C) = \Phi(A)^{-1/2} \Phi(A^{1/2}CA^{1/2}) \Phi(A)^{-1/2}.$$ 

Clearly $\Psi(I) = I$. Utilizing Lemma 3.8, we obtain

$$\Psi(C^2) - \Psi(C)^2 \leq \frac{1}{4}(a - b)^2 I$$

for all $C$ with $0 < bI \leq C \leq aI$. Put $C := f(A^{-1/2}BA^{-1/2})$. The given conditions on $A, B$ imply that

$$b_2 a_1^{-1} I \leq A^{-1/2}BA^{-1/2} \leq a_2 b_1^{-1} I.$$ 

The operator monotonicity of $f$ then yields

$$0 < bI = f(b_2 a_1^{-1}) I \leq C = f(A^{-1/2}BA^{-1/2}) \leq f(a_2 b_1^{-1}) I = aI.$$ 

Thus inequality (3.3) gives rise to

$$\Phi(A)^{-1/2} \Phi(|A^{-1/2}(A\sigma B)|^2) \Phi(A)^{-1/2} - \left(\Phi(A)^{-1/2} \Phi(A\sigma B) \Phi(A)^{-1/2}\right)^2 \leq \frac{1}{4}(f(a_2 b_1^{-1}) - f(b_2 a_1^{-1}))^2 I.$$ 

Now on pre and post multiplying by $\Phi(A)$, we obtain

$$\Phi(A)^{1/2} \Phi(|A^{-1/2}(A\sigma B)|^2) \Phi(A)^{1/2} - \left|\Phi(A)^{-1/2} \Phi(A\sigma B) \Phi(A)^{1/2}\right|^2 \leq \frac{a_1^2}{4}(f(a_2 b_1^{-1}) - f(b_2 a_1^{-1}))^2 I.$$ 

□

Corollary 3.10. Let $A, B \in \mathcal{B}(\mathcal{H})$ be such that $0 < b_1 I \leq A \leq a_1 I$ and $0 < b_2 I \leq B \leq a_2 I$, respectively. Then for any unit vector $x \in \mathcal{H}$,

$$\langle Ax, x \rangle \langle |A^{-1/2}(A\sigma B)|^2 x, x \rangle - \langle A\sigma Bx, x \rangle^2 \leq \frac{a_1^2}{4}(f(a_2 b_1^{-1}) - f(b_2 a_1^{-1}))^2.$$ 

Proof. The corollary follows on considering the unital positive linear map $\Phi$ on $\mathcal{B}(\mathcal{H})$ given by $\Phi(C) = \langle Cx, x \rangle$. □
Corollary 3.11. [4, Theorem 4.5] Let $A, B \in \mathcal{B}(\mathcal{H})$ be such that $0 < b_1 I \leq A \leq a_1 I$ and $0 < b_2 I \leq B \leq a_2 I$, respectively. Then for any unit vector $x \in \mathcal{H}$,

$$\langle Ax, x \rangle \langle Bx, x \rangle - \langle A \# Bx, x \rangle^2 \leq \left( \frac{\sqrt{a_1 a_2} - \sqrt{b_1 b_2}}{2} \right)^2 \min\{a_1 b_1^{-1}, a_2 b_2^{-1}\}.$$  

Proof. Replacing $\sigma$ by $\#$ in Corollary 3.10, we get the desired result. \qed

3.5. Greub–Rheinboldt type inequality. Greub–Rheinboldt [3] showed that if $A \in \mathcal{B}(\mathcal{H})$ be such that and $0 < m I \leq A \leq M I$, then

$$\langle Ax, x \rangle \langle A^{-1}x, x \rangle \leq \frac{(M + m)^2}{4mM} \quad (x \in \mathcal{H}, \|x\| = 1).$$

The first consequence of our main result is a generalized Greub-Rheinboldt inequality.

Corollary 3.12. If $A \in \mathcal{B}(\mathcal{H})$ be such that $0 < m I \leq A \leq M I$, then for any $0 < \alpha < 1$, any operator mean $\sigma$ and any positive linear map $\Phi$ the following Greub–Rheinboldt type inequality is valid:

$$\Phi(A) \#_{\alpha} \Phi(A^{-1}) \leq \frac{\alpha}{\mu \omega^{1-\alpha}} \Phi(A \sigma A^{-1}),$$

where $\mu$ and $\omega$ are determined as in Theorem 2.1 with $a_1 = M, b_1 = m, a_2 = m^{-1}$ and $b_2 = M^{-1}$.

Proof. Using (2.1) with $B = A^{-1}$ we get the desired inequality. \qed

4. Inequalities involving Hadamard product and operator means

In this section we present several inequalities involving Hadamard product and operator means.

If $U$ is the isometry of $H$ into $H \otimes H$ given by $U e_n = e_n \otimes e_n$, where $\{e_n\}$ is a fixed orthonormal basis of $H$, then the Hadamard product $A \circ B$ of (bounded) operators $A$ and $B$ on $H$ for $\{e_n\}$ is expressed by

$$A \circ B = U^* (A \otimes B) U.$$

A real valued continuous function $f$ is called supermultiplicative (submultiplicative, resp.) if $f(xy) \geq f(x)f(y)$ ($f(xy) \leq f(x)f(y)$, resp.).
Theorem 4.1. Let $A, B, C, D \in \mathbb{B}(\mathcal{H})$ be such that $b_1 I \leq A \leq a_1 I$, $b_2 I \leq B \leq a_2 I$, $b_3 I \leq C \leq a_3 I$ and $b_4 I \leq D \leq a_4 I$ for some scalars $0 < b_i < a_i$, $i = 1, \ldots, 4$. Then for any operator mean $\sigma$, whose representing function $f$ is submultiplicative, the following generalized inequalities hold for $\alpha \in (0, 1)$:

(i) $(\omega(A \circ B)) \nabla_\alpha (C \circ D) \leq \frac{\alpha}{\mu} ((A \sigma C) \circ (B \sigma D))$.

(ii) $\omega^{1-\alpha} ((A \circ B) \#_\alpha (C \circ D)) \leq \frac{\alpha}{\mu} ((A \sigma C) \circ (B \sigma D))$.

(iii) $(A \circ B) \#_\alpha (C \circ D) - ((A \sigma C) \circ (B \sigma D)) \leq (\frac{\alpha}{\mu} \omega^{\alpha-1} - 1) a_1 a_2 f(a_3 a_4 b_1^{-1} b_2^{-1}) I$.

In particular,

$(A \circ B) \#(C \circ D) - ((A \# C) \circ (B \# D))$

$\leq \left\{ \frac{1}{2\mu \sqrt{\omega}} - 1 \right\} \min \left\{ a_1 a_2 f(a_3 a_4 b_1^{-1} b_2^{-1}), a_3 a_4 (a_1^{-1} a_2^{-1} b_3 b_4^{1/2}) \right\} I$.

(iv) $(A \circ B) \#_\alpha (C \circ D) - ((A \#_\alpha C) \circ (B \#_\alpha D)) \leq \left( (1 - \alpha)(\mu \alpha^{-1})^{\frac{\alpha}{\alpha-1}} - \nu \right) a_1 a_2 I$.

In particular,

$(A \circ B) \#(C \circ D) - ((A \# C) \circ (B \# D)) \leq \left\{ \frac{1}{4\mu} - \nu \right\} \min \{ a_1 a_2, a_3 a_4 \} I$.

(v)

$\left( (A \sigma C) \circ (B \sigma D) \right)^{-1/2} (C \circ D) (\nabla_\alpha (A \sigma C) \circ (B \sigma D))^{-1/2}$

$- (\nabla_\alpha (A \sigma C) \circ (B \sigma D))^{-1/2} (A \circ B)^{-1} (\nabla_\alpha (A \sigma C) \circ (B \sigma D))^{-1/2} \leq \left( \mu^{-1} - 2\sqrt{\omega} \right) I$,

where $\omega = \frac{a(b_1 b_2 b_3 f(a_3 a_4 b_1^{-1} b_2^{-1}) - a_1 a_2 a_4 f(b_3 b_4 a_1^{-1} a_2^{-1}))}{(1-\alpha)a_1 a_2 b_1 b_2 (f(b_3 b_4 a_1^{-1} a_2^{-1}) - f(a_3 a_4 b_1^{-1} b_2^{-1}))}$, $\mu = \frac{a_1 a_2 b_1 b_2 (f(b_3 b_4 a_1^{-1} a_2^{-1}) - f(a_3 a_4 b_1^{-1} b_2^{-1}))}{b_1 b_2 b_4 - a_1 a_2 a_4}$ and $\nu = \frac{(1-\alpha) \omega \mu}{\alpha}$.

Proof. We have

$(A \otimes B)^{1/2} (f(X) \otimes f(Y))(A \otimes B)^{1/2} \geq (A \otimes B)^{1/2} f(X \otimes Y)(A \otimes B)^{1/2}$.

Taking $X = A^{-1/2} C A^{-1/2}$ and $Y = B^{-1/2} D B^{-1/2}$, we obtain

$(A \sigma C) \otimes (B \sigma D) \geq (A \otimes B) \sigma (C \otimes D)$. (4.1)
Also, simple arguments leads to \( a_1^{-1}a_2^{-1}b_3b_4A \otimes B \leq C \otimes D \leq b_1^{-1}b_2^{-1}a_3a_4A \otimes B \), so on replacing \( A \) by \( A \otimes B \) and \( B \) by \( C \otimes D \) in Theorem 2.1 and taking \( \Phi(Y) = U^*YU \), where \( U \) an isometry satisfying \( U^*(A \otimes B)U = A \circ B \), and using (4.1) we get (i) and (ii). The rest of the inequalities can be proved similarly. \( \square \)

**Theorem 4.2.** Let \( A, B, C, D, \sigma \) and \( f \) as in Theorem 4.1. Then, the following generalized inequalities hold:

(i) \((A \circ B)\sigma(C \circ D) \leq \frac{1}{\omega}(AσC) \circ (BσD))\)

(ii) \((A \circ B)\sigma(B \circ D) - ((AσC) \circ (BσD)) \leq -g(t_0)(A \circ B)\),

where \( \omega = \frac{a_1a_2b_1b_2(f(a_3a_4b_1^{-1}b_3^{-1})-f(a_1^{-1}a_2^{-1}b_3b_4))}{a_1a_2a_4^{-1}b_1^2b_3^2b_4^2} \)

\( \mu = \frac{a_1a_2b_1b_2(f(a_3a_4b_1^{-1}b_3^{-1})-f(a_1^{-1}a_2^{-1}b_3b_4))}{a_1a_2a_4^{-1}b_1^2b_3^2b_4^2} \)

\( \nu = \frac{a_1a_2a_4^{-1}b_1^{-1}b_3b_4}{a_1a_2a_4^{-1}b_1^2b_3^2b_4^2} \), and \( g(t) = \mu t + \nu - f(t) \) for \( c \) and \( t_0 \) some fixed points in \((a_1^{-1}a_2^{-1}b_3b_4, a_3a_4b_1^{-1}b_2^{-1})\).

**Proof.** It is known [5] that if \( A, B > 0 \) be such that \( aA \geq B \geq bA \) for some scalars \( a \geq b > 0 \) and \( \Phi \) is a positive linear map, then for any connection \( \sigma \),

\[ \Phi(A)\sigma\Phi(B) \leq \frac{1}{\omega}\Phi(A\sigma B) \quad (4.2) \]

and

\[ \Phi(A)\sigma\Phi(B) - \Phi(A\sigma B) \leq -g(t_0)\Phi(A) \quad (4.3) \]

where \( \omega = \frac{f(a)-f(b)}{(a-b)f(c)} \) for some fixed \( c \in (b, a) \) and \( g(t) = \mu t + \nu - f(t) \), \( t_0 \) a fixed point in \((b, a)\) with \(-g(t_0) \geq 0 \), \( \mu = \frac{f(a)-f(b)}{a-b}, \nu = \frac{af(b)-bf(a)}{a-b} \) and \( f(t) \) is the representing function of \( \sigma \).

As in Theorem 4.1, replacing \( A \) by \( A \otimes B \) and \( B \) by \( C \otimes D \), in (4.2) and (4.3) and using inequality (4.1), we get

\[ \Phi(A \otimes B)\sigma\Phi(C \otimes D) \leq \frac{1}{\omega}\Phi((A\sigma C) \otimes (B\sigma D)) \]

and

\[ \Phi(A \otimes B)\sigma\Phi(C \otimes D) - \Phi((A\sigma C) \otimes (B\sigma D)) \leq -g(t_0)(A \otimes B) \].
Once again (i) and (ii) can be deduced by considering $\Phi(Y) = U^*YU$, where $U$ an isometry satisfying $U^*(A \otimes B)U = A \circ B$. □

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