On “Regular Landsberg metrics are always Berwald” by Z. I. Szabó

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In his paper [2], Z. I. Szabó claimed (Theorem 3.1) that all sufficiently smooth Landsberg Finsler metrics are Berwald; this claim solves the long-standing “unicorn” problem. Unfortunately, as I explain below, the proof of the statement has a gap.

Following [2], let us consider a smooth $n$–dimensional manifold $M$ with a proper Finsler metric $F : TM \to \mathbb{R}$. The second differential of $\frac{1}{2} F_{|x}^2$ will be denoted by $g = g_{(x,y)}$ and should be viewed as a Riemannian metric on the punctured tangent space $T_x M - \{0\}$.

For a smooth curve $c(t)$ connecting two points $a, b \in M$, we denote by

$$\tau : T_a M \to T_b M, \quad \tau(\underbrace{a}_{\in T_a M}, \underbrace{y_a}_{\in T_a M}) = (b, \phi(y_a))$$

the Berwald parallel transport along the curve $c$. Following [1], Z. I. Szabó considers the following Riemannian metric $g$ on $M$ canonically constructed by $F$ by the formula

$$g_{(x)}(\xi, \eta) := \int_{\substack{y_x \in T_x M \\widetilde{F}(x,y_x) \leq 1}} g_{(x,y_x)}(\xi, \eta) d\mu_{(x,y_x)}$$

where $\xi, \eta \in T_x M$ are two arbitrary vectors, and the volume form $d\mu$ on $T_x M$ is given by $d\mu_{(x,y_x)} := \sqrt{\text{det}(g_{(x,y_x)})} dy_1 \wedge \cdots \wedge dy^n$.

Z. I. Szabó claims that if the Finsler metric $F$ is Landsberg, the Berwald parallel transport preserves the Riemannian metric $g$. According to the definitions in Section
2 of [2], this claim means that for every \( \xi, \eta, \nu \in T_aM \)
\[
    g_{(a)}(\xi, \eta) = g_{(b)}(d_\nu \phi(\xi), d_\nu \phi(\eta)).
\]

(2)

This claim is crucial for the proof; the remaining part of the proof is made of relatively simple standard arguments, and is correct. The claim itself is explained very briefly; basically Z. I. Szabó writes that, for Landsberg metrics, the unite ball \( \{ y_x \in T_xM \mid F(x, y_x) \leq 1 \} \), the volume form \( d\mu \), and the metric \( g_{(x,y_x)} \) are preserved by the parallel transport, and, therefore, the metric \( g \) given by (1) must be preserved as well.

Indeed, for Landsberg metrics, the unite ball and the volume form \( d\mu \) are preserved by the parallel transport. Unfortunately, it seems that the metric \( g \) is preserved in a slightly different way one needs to prove the claim.

More precisely, plugging (1) in (2), we obtain
\[
    \int_{y_a \in T_aM \atop F(a, y_a) \leq 1} g_{(a, y_a)}(\xi, \eta) d\mu_{(a, y_a)} = \int_{y_b \in T_bM \atop F(b, y_b) \leq 1} g_{(b, y_b)}(d_\nu \phi(\xi), d_\nu \phi(\eta)) d\mu_{(b, y_b)}.
\]

(3)

As it is explained for example in Section 2 of [2], for every Finsler metric, the parallel transport preserves the unite ball:
\[
    \phi(\{ y_a \in T_aM \mid F(a, y_a) \leq 1 \}) = \{ y_b \in T_bM \mid F(b, y_b) \leq 1 \}.
\]

(4)

The condition that \( F \) is Landsberg implies \( \phi_* d\mu_{(a, y_a)} = d\mu_{(b, \phi(y_a))} \). Thus, Szabó’s claim is trivially true if at every \( y_a \in T_aM \)
\[
    g_{(a, y_a)}(\xi, \eta) = g_{(b, \phi(y_a))}(d_\nu \phi(\xi), d_\nu \phi(\eta)).
\]

(5)

But the condition that the metric is Landsberg means that
\[
    g_{(a, y_a)}(\xi, \eta) = g_{(b, \phi(y_a))}(d_{y_a} \phi(\xi), d_{y_a} \phi(\eta))
\]

only, i.e., (3) coincides with the definition of the Landsberg metric at the only point \( y_a = \nu \in T_aM \).

Since no explanation why (3) holds is given in the paper, I tend to suppose that Z. I. Szabó oversaw the difference between the formulas (5) and (6); anyway, at the present
point, the proof of Theorem 3.1 in [2] is not complete. Unfortunately, I could not get any explanation from Z. I. Szabó by email.

The unicorn problem remains open until somebody closes the gap, or presents another proof, or proves the existence of a counterexample; at the present point I can do neither of these.

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References

[1] Z. I. Szabó: *Berwald metrics constructed by Chevalley’s polynomials*, arXiv:math.DG/0601522 (2006)

[2] Z. I. Szabó: *Regular Landsberg metrics are always Berwald*, Ann. Glob. Anal. Geom. 2008

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