Revisiting the $\mathcal{PT}$-symmetric trimer: bifurcations, ghost states and associated dynamics

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Abstract
In this paper, we revisit one of the prototypical $\mathcal{PT}$-symmetric oligomers, namely the trimer. We find all the relevant branches of ‘regular’ solutions and analyze the bifurcations and instabilities thereof. Our work generalizes the formulation that was recently proposed in the case of dimers for the so-called ‘ghost states’ of trimers, which we also identify and connect to symmetry-breaking bifurcations from the regular states. We also examine the dynamics of unstable trimers, as well as those of the ghost states in the parametric regime where the latter are found to exist. Finally, we present the current state-of-the-art for optical experiments in $\mathcal{PT}$-symmetric trimers, as well as experimental results in a gain–loss–gain three channel waveguide structure.

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(Some figures may appear in colour only in the online journal)

1. Introduction

The study of $\mathcal{PT}$-symmetry in both linear and nonlinear systems has received an increasingly large amount of attention over the past 15 years. This effort originally stemmed from the realm of quantum mechanics [1, 2], where $\mathcal{PT}$-symmetric Hamiltonians were proposed as presenting a viable alternative—also due to the fundamental nature of the corresponding parity ($\mathcal{P}$) and time-reversal ($\mathcal{T}$) symmetries—to the postulate of hermiticity. Nevertheless, and while experimental realizations in the quantum mechanical framework are still less clear, a critical observation that significantly advanced the field was made in the context of optics, both theoretically [3–5] and experimentally [6, 7]. Indeed, in this field, since losses are abundant and controllable gain is possible, an experimental synthesis of $\mathcal{PT}$-symmetric Hamiltonians was demonstrated. This opened a new chapter in the relevant investigations by enabling the
The interplay of such linear \( PT \)-symmetric Hamiltonians with the effects of nonlinearity. The latter is a feature ubiquitously present in such optical systems and a theme of particular interest and complexity in its own right. This, in turn, spearheaded additional experimental investigations in optics [8] and electrical circuit analogues of such systems [9, 10], as well as an intense effort to appreciate the impact of \( PT \)-symmetric potentials on scattering properties and transport both in the linear [11–13] and in the nonlinear [14] case. It also paved the way for numerous significant theoretical contributions on the subject. As a small sample among the many relevant topics, we highlight the facilitation of unidirectional dynamics [15], the analysis of the universality of the dynamics [16], the exploration of symmetry-breaking effects [7, 17], the study of beam switching [18], of solitons [19], of the formation of symmetric and asymmetric bright solitary waves [20, 21], of breathers [22] and their stability [23], of dark solitons [24, 25] and of vortices [24], as well as the emergence of ghost states and associated exceptional points [26–28], and the generalization of such ideas into vortex type configurations [29], \( PT \)-symmetric plaquettes [30] and higher dimensional media [31].

A theme of particular interest within these studies concerns the so-called \( PT \)-symmetric ‘oligomers’. While most of the relevant attention was focused on dimers [15, 18, 26–28] (arguably due to the corresponding experimental explorations of [6, 7, 9]), configurations with more sites, such as trimers [17, 32] and quadrtrimers [17, 30, 32, 33], have also attracted recent interest. In the present work, our aim is to revisit one of these configurations, namely the \( PT \)-symmetric trimer. Both the optical [7] and the electrical [9] implementation of the corresponding dimer strongly suggest that the experimental realization of such a trimer system may be feasible. Hence, it is particularly interesting and relevant to fully explore the \( PT \)-symmetric trimer, and provide analytical results complementing the earlier findings of [17], as well as to report on the current state-of-the-art regarding a possible experimental realization thereof in optics.

This paper will be structured as follows. In section 2, we will present the model and theoretical setup of the trimer. We will devise analytical algebraic conditions that are relevant towards identifying the full set of standing wave solutions for this configuration. Importantly, in addition to the more standard stationary solutions, we will also identify the so-called ‘ghost states’ of the model [26–28]. These are states that, remarkably, although solutions of the steady state equations, due to their complex propagation constant, are not genuine solutions of the original dynamical equations. Nevertheless, as has been argued in the case of the dimer [28], these are waveforms of potential relevance in understanding the system’s dynamics. In section 3, we will present the corresponding numerical results. In particular, we will seek both regular standing wave states and ghost states, and will build a full state diagram as a function of the gain/loss parameter \( \gamma \) of our \( PT \)-symmetric trimer. In addition to the existence properties of the obtained solutions, we will consider their stability (and potential instabilities/bifurcations) and, finally, we will examine the system’s dynamics and how the instabilities are manifested, both in the case of the ‘regular’ standing wave solutions and in that of the ghost states identified herein. In section 4 we discuss possible realizations of \( PT \)-symmetric optical systems (with a particular focus on trimers) and describe actual experimental limitations that need to be overcome. Finally, in section 5, we will summarize our findings and present our conclusions, as well as some directions for future study.

2. Model and theoretical setup

The prototypical dynamical equations for the \( PT \)-symmetric trimer model read [17]:

\[
 iu_1 = -ku_2 - |u_1|^2u_1 - i\gamma u_1
\]
Here, $u_i(t)$ ($j \in \{1, 2, 3\}$) are complex amplitudes, dots denote differentiation with respect to the variable $t$ (which is the propagation distance in the context of optics), while $k$ and $\gamma$ represent the inter-site coupling and the strength of the $\mathcal{PT}$-symmetric gain/loss parameter, respectively. In the above equations it is assumed that the first site sustains a loss at the rate $\gamma$, while the third site sustains an equal gain. The middle site suffers neither gain nor loss. Following the spirit of \cite{17,32}, we start our analysis by seeking stationary solutions of the original dynamical evolution equations (1) in the form $u_1 = a \exp(iE_1t)$, $u_2 = b \exp(iE_2t)$ and $u_3 = c \exp(iE_3t)$, where $E$ represents the nonlinear eigenvalue parameter. This way, we obtain from equations (1) the following algebraic equations:

\[ E_a = kb + |a|^2 a + iya, \]
\[ E_b = k(a + c) + |b|^2 b, \]
\[ E_c = kb + |c|^2 c - iyc. \]

(2)

Let us now use a polar representation of the three ‘sites’, namely, $a = A \exp(i\phi_a)$, $b = B \exp(i\phi_b)$, and $c = C \exp(i\phi_c)$. Then, from equations (2), one can immediately infer that $A = C$, i.e., the amplitudes of the two ‘side-sites’ of the trimer are equal. In addition, the amplitude of the central site is given, as a function of $A$, by:

\[ B^2 = \frac{E \pm \sqrt{E^2 - 8A^2(E - A^2)}}{2}. \]

(3)

In turn, the algebraic polynomial equation for the squared amplitude of $A^2 \equiv x$ is given by

\[ x[\gamma^2 + (E - x)^2]\bar{t}^2 - k^2 E[\gamma^2 + (E - x)^2] - 2k^4 x + 2k^4 E = 0. \]

(4)

Once $A$ is determined from equation (4) and subsequently $B$ from equation (3), then the two relative phases between the three sites of the trimer have to satisfy:

\[ \sin(\phi_b - \phi_a) = -\sin(\phi_b - \phi_c) = -\frac{\gamma A}{kB}, \]

(5)

\[ \cos(\phi_b - \phi_a) = \cos(\phi_b - \phi_c) = \frac{EA - A^3}{kB}. \]

(6)

The above formulation provides (via equations (3)-(4) and (5)-(6)) the full set of stationary solutions of the trimer system, for given values of the coupling strength $k$, nonlinear eigenvalue parameter $E$, and gain/loss strength $\gamma$. In what follows in our numerical section below, we will fix two of these parameters ($E$ and $k$) and vary $\gamma$ to explore the deviations from the Hamiltonian limit of $\gamma = 0$. As an important aside, let us note here that the global freedom of selecting a phase (due to the U(1) invariance of the model) can be used to choose $\phi_a = 0$. Then, it is evident that $\phi_c = -\phi_b$, which combined with the amplitude condition $A = C$ implies that $u_3 = \bar{u}_1$, where the overbar denotes complex conjugation. Clearly, this condition is in line with the demands of $\mathcal{PT}$-symmetry for our system.

As mentioned above, in addition to the regular stationary solutions for which $E$ is real, one can seek additional solutions with $E$ being complex, i.e., $E = \bar{E} \exp(i\phi_c)$. The resulting waveforms are quite special in that they are solutions of the stationary equations of motion (2), yet they are not solutions of the original dynamical evolution equations (1), because of the imaginary part of $E$. Such ‘ghost state’ solutions have recently been identified in the case of the $\mathcal{PT}$-symmetric dimer \cite{26-28} and have even been argued to play a significant role in its corresponding dynamics therein. In the present case of the trimer, to the best of our knowledge,
they have not been previously explored. Such ghost trimer states will satisfy the following
algebraic conditions:

\[
\sin \phi_a = \frac{A(B^2 + 2C^2)\gamma}{B(A^2 + B^2 + C^2)k} \\
\cos \phi_a = \frac{A(B - C)(B + C)(-A^2 + B^2 + C^2)}{B(-A^2 + B^2 - C^2)k} \\
\sin \phi_c = -\frac{(2A^2 + B^2)C\gamma}{B(A^2 + B^2 + C^2)k} \\
\cos \phi_c = \frac{(-A^2 + B^2)C(A^2 + B^2 - C^2)}{B(-A^2 + B^2 - C^2)k} \\
\sin \phi_e = \frac{(A - C)(A + C)\gamma}{(A^2 + B^2 + C^2)E} \\
\cos \phi_e = \frac{A^4 - B^4 + C^4}{(A^2 - B^2 + C^2)E}.
\]

From these equations, the amplitudes \(A, B, C\) can be algebraically identified by applying the
identity \(\sin^2 \phi + \cos^2 \phi = 1\) for each of the above angles. The relevant six algebraic equations
lead to the identification of the six unknowns, namely the three amplitudes, as well as the
phases \(\phi_a, \phi_c\) and \(\phi_e\) (for simplicity we have set \(\phi_b = 0\) hereafter, without loss of generality).
It should be noted here that should such ghost state solutions be present with
\(\phi_e \neq 0\), these
will spontaneously break the \(P\bar{T}\) symmetry, given that they will have \(A \neq C\).

Notice that for each branch of solutions that we identify in the following, we will also
examine its linear stability. This will be done through a linearization ansatz of the form
\(u_i = e^{\lambda t}[v_i + \epsilon(p_i e^{i\lambda t} + q_i e^{-i\lambda t})]\). Here the \(v_i\) for \(i = 1, 2, 3\) will denote the values of the field at the standing wave equilibria, while \(\lambda\) are the corresponding eigenvalues and \((p_i, q_i)\)
for \(i = 1, 2, 3\) denote the elements of the corresponding eigenvector which satisfies the
linearization problem at \(O(\epsilon)\); the overbar will be used to denote complex conjugation. When
the eigenvalues \(\lambda\) of the resulting 6 × 6 linearized equations have a positive real part, the
solutions will be designated as unstable (whereas otherwise they will be expected to be
dynamically stable).

We now turn to the detailed numerical analysis of the corresponding stationary, as well as
ghost branches of solutions.

3. Numerical results

Since equation (4) is a polynomial of degree 5, we expect at most five distinct real roots
(and at least one such). Indeed, for suitable choices of the free parameters \((E, k)\), we identify
five branches of stationary solutions. Figure 1 illustrates a situation with five branches under
\(E = 0.5\) and \(k = 0.1\). Two of them, denoted by blue circles and red diamonds, collide and
terminate at \(\gamma = 0.1\). The blue circles are essentially stable while the red diamonds are
unstable. Another pair of branches, namely the magenta squares and green pluses collide and
terminate at \(\gamma = 0.02\), with the magenta squares being stable and the green pluses being
unstable (i.e., both of the above collisions are examples of saddle-center bifurcations).
The black-cross branch, which is essentially unstable, persists beyond \(\gamma = 0.1\). Notice that the
Figure 1. The solution profile of equation (1) with $E = 0.5$, $k = 0.1$ and $\phi_0 = 0$. The four panels denote the solution amplitude (top left), phase differences between adjacent nodes (top right) and real and imaginary parts (second row) of eigenvalues.

Figure 2. In a way similar to that of the previous figure (i.e., with top left denoting amplitudes, top right relative phases, bottom left real and bottom right imaginary parts of the linearization eigenvalues), the four panels show the existence and stability of solutions for a trimer with parameters $E = k = 1$. There are three regular standing wave branches: the blue, the red and the black; the blue and the red are the ones disappearing hand-in-hand at $\gamma = 1.043$. Two ghost solutions are colored in magenta and green and bifurcate at the destabilization of the blue branch for $\gamma = 1.035$, while they terminate for $\gamma = 1.732$.

amplitudes of the different nodes for this branch shown in the top left panel of the figure are not constant: the upper line (standing for $B$) is slightly increasing and the lower line (standing for $A = C$) is slightly decreasing.

In the following, we focus on a typical example of the branches (both stationary and ghost ones) for a selection of the free parameters of order unity, more specifically for $E = k = 1$; cf figure 2. We identify three distinct examples of stationary states denoted by the blue-circle, red-diamond and black-cross branches. The blue-circle and red-diamond branches stem from the corresponding ‘$++0$’ and ‘$--++$’ branches, respectively, namely the second and third excited state of the Hamiltonian trimer problem of $\gamma = 0$; compare with [34]. The blue-circle branch is mostly unstable, except for a small interval of $\gamma \in [1, 1.035]$, while the red-diamond branch is chiefly stable, except for the narrow interval of values of $\gamma \in [1.035, 1.043]$. In this narrow interval, the eigenvalues of both of these branches are very close to each other.
Figure 3. The spectral planes $(\lambda_r, \lambda_i)$ of the eigenvalues $\lambda = \lambda_r + i\lambda_i$ of the solutions shown in figure 2. The first panel shows the case of $\gamma = 0.5$ where only two of the standing wave branches exist (blue circles: unstable and red diamonds: stable). The second panel for $\gamma = 1.5$ has only one standing wave (black crosses: unstable), and two asymmetric ghost states which are mirror images of each other (as are their spectra), namely magenta squares and green pluses. The third panel shows the same branches as in the top right but for $\gamma = 1.7$ close to the termination of the ghost state branches.

For the blue-circle branch, we also note that two eigenvalue pairs stemming from a complex quartet collide on the imaginary axis at $\gamma = 1$ and split as imaginary thereafter. One of these pairs exits as real for $\gamma > 1.035$, and the two branches (blue circles and red diamonds) collide shortly thereafter, i.e., at $\gamma = 1.043$.

On the other hand, figure 2 also reveals an additional branch, denoted by black crosses, that bifurcates from zero at $\gamma = 1$ and persists for all values of $\gamma$ thereafter. This is quite interesting in its own right as an observation since, as highlighted in [17], the linear critical point for the $PT$ phase transition is $\gamma = \sqrt{2} k$. Thus, this branch presents the simplest oligomer example (such ones are absent in the case of the dimer) whereby nonlinearity enables a solution family to persist past the point of the linear limit $PT$ phase transition. Additionally, it should be noted that the branch is stable for all values of $\gamma < 1.13$, but destabilizes for all larger values of $\gamma$.

In figure 2, however, in addition to the standard stationary solutions, the ghost state solutions are also shown. These are designated by the magenta squares and green pluses in the figure. These ghost solutions are also obtained for $\hat{E} = k = 1$, and importantly (and contrary to the case for the stationary states), they bear distinct amplitudes in all three sites. The two (magenta and the green) branches shown in the figure are mirror images of each other, i.e., $A, B, C$ in the magenta branch are the same as $C, B, A$ in the green branch, respectively, and their phase differences and eigenvalues are opposite to each other. Notice that as indicated above the difference in the magnitudes of $A$ and $C$ supports the fact that these branches defy the expectations of the $PT$-symmetry. Indeed, both of the branches arise through a symmetry-breaking bifurcation from the blue branch when it becomes unstable at $\gamma = 1.035$. Furthermore, it should be noted that the branches terminate at vanishing amplitude for $\gamma = 1.732$. It is interesting to point out that when performing linear stability analysis of these states, we find both of them to be unstable. Case examples of the linearization results for both the regular states and the ghost ones are shown for three different values of $\gamma$ in figure 3. For $\gamma = 0.5$, the red-diamond branch is (marginally) stable, while the blue-circle branch bears the instability that we discussed above for $\gamma < 1$. For $\gamma = 1.5$, only the black branch is present among the stationary ones and the magenta and green ghost state branches manifest their respective asymmetries with spectra that are asymmetric around the imaginary axis. This is a characteristic feature of the ghost states; see also [28, 31]. Among the two branches, the magenta is more stable and the green is highly unstable; however, even the magenta branch
is predicted to be weakly unstable with a small real positive eigenvalue. We will examine the dynamical implications of these instabilities in what follows. The last panel similarly shows the case of $\gamma = 1.7$ shortly before the disappearance of the ghost state branches.

Finally, we examine the dynamics of the different branches in figure 4. The top row panels of the figure show the evolution of the three stationary branches. Panels (a) and (b) show the blue-circle branch for $\gamma = 0.5$ and $\gamma = 1.1$, while panel (c) depicts the red-diamond standing wave branch for $\gamma = 1.1$. Notice that in the cases of (b) and (c), the corresponding branches cease to exist at $\gamma = 1.043$. Thus, in these runs, we have used the terminal point profile of the branches (at $\gamma = 1.043$) as initial data for the evolution with $\gamma = 1.1$. Importantly, we note that in the unstable evolution of cases (b) and (c), two of the sites end up growing indefinitely while the lossy site ends up decaying. On the contrary, in the case (a), only the site with gain is led to growth, while the other two are led to eventual decay. In panel (d), we show the black-cross branch, the third among the standing wave solutions identified herein for $\gamma = 1.5$. Notice that panel (d) shows a different dynamical evolution from panels (b) and (c) and is more in line with panel (a), showcasing that there are indeed two general growth scenarios: one in which the gain site ‘grabs’ along the neutral central site and leads it to indefinite growth and one in which the central site is ultimately led to decay together with the lossy site.

The four panels in the lower row show the dynamical evolution of the two ghost states (green pluses and magenta squares) for the cases of $\gamma = 1.1$ and $\gamma = 1.5$. With $E = E_r + iE_i$ being complex, the ghost state solutions under the form $u_1 = \exp(iEt)a$, $u_2 = \exp(iEt)b$ and $u_3 = \exp(iEt)c$ should evolve exponentially, as indicated by dashed lines in panels (e)–(h). In particular, the magenta-square branch with negative $E_i$ is expected to lead to growth (for all three nodes of the trimer), while the green-plus branch with positive $E_i$ is anticipated to decay (again for all nodes). The slopes of these growth/decay features are given by $-2E_i = -2E \sin \phi_c$. However, in line with their anticipated linear ‘instability’, neither of these follows exactly the dynamics anticipated above. Both of them evolve for a short period according to the expected growth or decay, and then the gain sites start to grow and the loss

Figure 4. The dynamical evolution of the amplitudes of the three sites for the solutions shown in figure 2. Notice that all solutions are shown in semilog plots. The first row shows the evolution of the three stationary branches. In (b) and (c), since these branches are absent for $\gamma = 1.1$, their profile for $\gamma = 1.043$ is initialized. The second row shows dynamics of the two ghost state solution branches. The dashed lines are the predicted dynamics of the ghost states on the basis of their growth rates (for magenta squares) or decay rates (for green pluses).
sites start to decay, regardless of the trend predicted by the form of the ghost state (discussed above). Moreover, it is relevant to note as regards the corresponding dynamics that the cases of the blue-circle and red-diamond branches of $\gamma = 1.1$ exhibit similar (asymptotic) dynamics to those of the magenta squares and of the green plus for the same parameter value; i.e., the central site is also led to growth along with the gain one. On the other hand, it is also evident that the black-cross branch for $\gamma = 1.5$ instead follows an evolution resembling the asymptotic evolution of the green-plus branch (which is different from that of the magenta squares for the latter value), i.e., here only the gain site is ultimately led to growth.

Having unveiled the existence and stability, as well as the nonlinear dynamical characteristics of the different solutions, we now turn to a discussion of the potential for experimental realization of $\mathcal{PT}$-symmetry in optical systems, especially as regards trimers, but also more generally.

4. Experimental realization of $\mathcal{PT}$ optical systems

General requirements for the realization of $\mathcal{PT}$ optical systems are the availability of adequate methods for the formation of coupled waveguide systems or arrays with the additional opportunity to spatially tailor loss and gain in the substrate. In other words, suitable fabrication conditions should allow for spatial manipulation of both real and imaginary parts of the dielectric constant.

Laser crystals and glasses are amplifying media that may provide the necessary optical gain by using different physical mechanisms. Examples are: doped bulk crystals and fibers that make use of stimulated emission to amplify a weak signal, electron–hole recombination in semiconductor optical amplifiers, parametric amplification in nonlinear crystals, or stimulated Raman scattering. Furthermore, many laser-gain materials allow for the formation of guiding index structures. However, although many amplifying materials exist, the most challenging aspect is the necessity to achieve optical gain while at the same time the real part of the refractive index should remain constant (or change only in a negligible way in order to maintain $\mathcal{PT}$-symmetry), which is difficult to achieve because of the Kramers–Kronig relation. Besides thermo-optic effects in the case of high optical powers, it turns out that other mechanisms like self- and cross-phase modulation limit the suitability of most laser-gain media for application in $\mathcal{PT}$ optics.

Another well-known amplification mechanism is optical beam coupling in photorefractive media such as photovoltaic lithium niobate (LiNbO3) crystals, which exists already at quite low optical light powers. Due to advanced waveguide formation techniques, LiNbO3 is a favorite material for use in integrated optics [35]. Aside from the diffraction of weak signal beams on recorded index gratings which leads to gain, the point symmetry 3m of LiNbO3 also enables interaction of orthogonally polarized waves [36]. A polarization grating recorded by a pump and signal beam having mutual orthogonal polarization allows for optical signal gain; the small-signal gain can reach several tens per cm for strong Fe doping [36]. At the same time, the spatially varying polarization pattern of the pump and signal beam does not induce significant phase changes for the interacting beams. Using this mechanism, the first experimental demonstration of $\mathcal{PT}$-symmetry in optics has been achieved in Fe:LiNbO3 using Ti in-diffusion to form coupled waveguide structures [7]. However, there still exist some limitations that have to be overcome in order to realize more advanced $\mathcal{PT}$ symmetric optical settings.

While spatial tailoring of optical gain may be achieved by limiting an optical pump beam to certain waveguide channels, this can be hardly done for loss (a technologically quite challenging solution consists in the formation of metallic stripes of precisely defined
width on certain channels; see [6]). Due to such difficulties, a more realistic experimental approach consists in allowing for a constant loss in all coupled channels, while this loss is overcompensated for by adjustable gain in some selected channels only.

An experimental example of a three-channel coupled waveguide structure having distributed gain and loss is shown in figure 5. While the precisely $\mathcal{PT}$-symmetric pattern of loss—neither gain nor loss—gain is not directly realizable in the above described experiments, here we focus on a somewhat different configuration featuring an alternation of gain–loss–gain which is, arguably, the nearest experimentally realizable optical variant. To achieve this, three parallel single-mode waveguide channels for a wavelength of $\lambda = 532\,\text{nm}$ are formed on a Fe-doped x-cut LiNbO$_3$ substrate by in-diffusion of a stripe-like Ti film. The propagation length is 20 mm and the linear coupling coefficient is $k \approx 0.2\,\text{mm}^{-1}$. In the sample, overall but constant loss is due to absorption of the used green light by in-diffused Fe ions. Similar to the arrangement in [7], optical gain for the extraordinarily polarized signal is achieved by pumping the sample from the top using a plane wave of ordinary polarization. An amplitude mask on top of the substrate shields the central channel ($\sharp2$), thus (equal) gain is obtained for the left ($\sharp1$) and right ($\sharp3$) channels only. The signal light is coupled from the end-facet into the central channel. As can be seen, when the pump beam is switched on at time $t_{\text{rec}} = 0$, power in the two outer channels starts to increase. Simultaneously the total power (black symbols in figure 5) increases due to the buildup of the polarization grating. Beside some asymmetries in the temporal evolution discussed below, for longer recording symmetry improves again, and the corresponding interferograms on the rhs show the development of relative phase of central and outer channels, starting from the in-phase condition at $t_{\text{rec}} = 0$ (figure 5(b)) towards a final phase difference of $\pm\pi/2$ at $t_{\text{rec}} \approx 34\,\text{min}$ (figure 5(c)). This relative phase development is in line with the earlier theoretical expectations on the basis of, for example, equations (5) and (6). This behavior is also in good agreement with simulations of this gain–loss–gain system based on the Runge–Kutta integration of the corresponding coupled-wave equations in figure 6. Of course, these runs also manifest the partial differences of this experimental realization from the genuinely $\mathcal{PT}$-symmetric case, in that ultimately all three waveguides feature growing optical power in the simulations of figure 6, a trait which is absent, for example, in figure 4 (where at least one waveguide is not growing indefinitely in power).

Obviously, some experimental problems still exist. The temporal evolution in the left and right channels is far from being perfectly symmetric, especially for intermediate recording times. In most experiments, for long recording times (i.e. high gain) output powers of the
three channels start to fluctuate slightly. Possible explanations for this behavior are large induced space-charge fields that may lead to spark plugs across the sample surface, or weak phase instabilities of the setup. A general problem is the small achievable gain, which is rarely sufficient to reach typical \( \mathcal{P} \mathcal{T} \)-symmetry-breaking thresholds in most of the fabricated samples. Gain is also limited because of the low powers of signal and pump light: a higher signal power would lead to decoupling of the excited channel due to nonlinear index changes, while a higher pump power would record a distorting phase gradient at the boundaries of the used amplitude mask (i.e. between illuminated/non-illuminated channels).

For future experiments using Fe:LiNbO\(_3\) waveguide samples, a main objective will thus be to increase optical gain by optimizing material properties. However, when doping LiNbO\(_3\) substrates with Fe using in-diffusion, the physical mechanisms that may lead to high gain when coupling orthogonally polarized waves are not yet fully understood. The influence of certain diffusion atmospheres, interference from simultaneous Ti in-diffusing used for waveguide formation, or the effect of Li out-diffusion at high temperatures and consequences on possible lattice sites of in-diffused Fe ions have not been investigated in detail. In particular, the high gain found in some Fe-doped bulk LiNbO\(_3\) crystals has not been observed in waveguide samples so far.

An alternative experimental \( \mathcal{P} \mathcal{T} \)-symmetric model system that also uses LiNbO\(_3\) with its well-developed waveguide fabrication technology as a substrate is Er doping to achieve gain in the optical communication window at 1.5 \( \mu \text{m} \). Although no detailed data on cross-phase modulation when pumped e.g. with 980 nm wavelength is available yet, data from Er-doped fiber amplifiers indicates that induced phase changes might be sufficiently small [37]. Work on such systems is currently in progress, which may perhaps allow one to avoid the described unwanted nonlinear effects that disturb the symmetry condition of the (real) refractive index in Fe-doped photorefractive LiNbO\(_3\) for higher input powers.

5. Conclusions and future challenges

In the present work, we have revisited the theme of one of the prototypical \( \mathcal{P} \mathcal{T} \)-symmetric oligomers, namely the trimer. We have illustrated the different number of branches (at least
one and at most five) of standing wave solutions that exist for this system. We have then focused on a case example of parameters of order unity and have shown that two of these standing wave branches terminate in a pairwise disappearance, while the third one in fact exists for values of the gain/loss parameter, extending past the point of the $PT$-symmetry-breaking phase transition of the linear limit occurring at $\gamma = \sqrt{2}k$.

Additionally, we have also presented the formulation of the so-called ghost states in this system and have explicitly computed them, showing how they emerge through a symmetry-breaking bifurcation (asymmetrizing the amplitudes of the two side-sites $A$ and $C$) from one of the standing wave branches. As expected, on the basis of such an effective pitchfork bifurcation, the two resulting ghost state branches are mirror images of each other (and so are their corresponding spectra). The dynamics of both the unstable stationary states and those of the ghost states revealed two possible dynamical scenarios. In one of these, the ‘neutral’ site (without gain or less) sided with the gain one, while the other corresponded to the case where it sided with the lossy site.

Additionally, we have explored the possibility of creating $PT$-symmetric systems in nonlinear optics, revealing that it is arguably simpler to create, e.g., a gain–loss–gain three-channel system, rather than the genuinely $PT$-symmetric situation where a waveguide with gain and one with loss straddle a middle one without either gain or loss. On the other hand, for this gain–loss–gain setting, we presented both physical experiments and corroborated numerical simulations revealing the partition of the fraction of optical power (initially placed at the central channel) and how it transfers more to the outer gain channels as the gain is increased beyond $\gamma/k = 1$.

There are many interesting questions that arise from the present study and are worthy of further exploration. It would be interesting to generalize our considerations herein to the case of quadrimers and to appreciate how the complexity of the relevant configurations expands, especially since in the latter case, there is generally the potential of two gain/loss parameters [33]; nevertheless, as per the above discussion on experimental possibilities in optics, the case of two waveguides with equal gain and two with equal loss would at present appear to be the most realistic one. At the same time, further experimental implementations of oligomer systems, either at the electrical circuit level, or at the optical waveguide level discussed in the last section would be particularly desirable and highly interesting for an increased understanding of the systems’ dynamics. Efforts in these directions are currently in progress and will be reported in future publications.

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