INJECTIVITY OF SPHERICAL MEAN ON MÉTIVIER GROUP

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Abstract. In this article, we study the injectivity of the spherical mean for continuous functions on the Métivier group. The spherical mean is injective for \( f(z, \cdot) \in L^p(\mathbb{R}^n), 1 \leq p \leq 2 \) with tempered growth in \( z \) variable. This result is also true for a class of functions in \( L^p(\mathbb{C}^n), 1 \leq p \leq \infty \) without tempered growth. Further, we obtain a two-radii theorem for functions on the Métivier group, which are tempered in \( z \) variable and periodic in the centre variable.

1. Introduction

In integral geometry, it is an interesting question to know if the average of a continuous function \( f \) over all translates of a surface can determine \( f \). Particularly, when does the operator \( f \) into \( f * \mu_r \) turn out to be injective, where \( \mu_r \) is the normalised surface measure on \( \{ x \in \mathbb{R}^n : |x| = r \} \). In general, the answer to this is negative, since there are non-trivial bounded continuous functions, e.g. Bessel functions \( \varphi \) for which \( \varphi * \mu_r = 0 \), when \( r \) is a zero of the Bessel function. The injectivity of the spherical mean is an ever interesting question and studied by several authors including [1,5,10,11,13]. Thangavelu [10] has shown that the one radius theorem is true for \( L^p(\mathbb{R}^n) \) when \( 1 \leq p \leq 2n/(n-1) \), by exploiting the spectral decomposition of the Laplacian.

The above question (one radius theorem) was also considered for the Heisenberg group \( \mathbb{H}^n \cong \mathbb{C}^n \times \mathbb{R} \). Indeed, in [10] it is shown that if \( f \in L^p(\mathbb{H}^n), 1 \leq p < \infty \), then \( f * \mu = 0 \) implies \( f = 0 \), where \( \mu \) is a compactly supported rotation invariant probability measure with no mass at the centre. The proof of this result is based on a summability result due to Strichartz [11] for sub-Laplacian on \( \mathbb{H}^n \).

Although, in the Métivier group, denoted by \( G \cong \mathbb{C}^n \times \mathbb{R}^m \), the analogue to summability result [11] is yet to settle due to appearance of a multi-parameter singular integral due to higher dimensional centre, whose kernel need not be a Calderón-Zygmund kernel. However, we show that the mean operator \( f \) into \( f * \mu \) is injective when \( f(z, \cdot) \in L^p(\mathbb{R}^m), 1 \leq p \leq 2 \) and \( f \) is of tempered growth in \( z \) variable. This result is obtained employ the simplified \( \lambda \)-twisted spherical
mean on the Métilvier group, which we introduced in [2] and the special Hermite expansion as discussed in Section 3. Moreover, when \( \mu = \mu_e \), we prove one radius theorem for continuous functions \( f \) when \( f(z, \cdot) \in L^p(\mathbb{R}^m), 1 \leq p \leq 2 \) and \( f^\lambda(z)e^{i\lambda z \cdot x} \) is in \( L^{p'}(\mathbb{C}^n), 1 \leq p' \leq \infty \). At the end, we observed a two radii theorem for tempered continuous functions in \( z \) variable and \( 2\pi \)-periodic in \( t \) variable.

It is known that the symplectic bilinear form appears in the group action of the Métilvier group is far from \( U(n) \)-invariance, due to its higher dimensional centre, the \( \lambda \)-twisted spherical cannot be radialised as in the case of the Heisenberg group. However, it is elliptical up to a rotation [2]. We connect this elliptical mean to the twisted spherical mean of a Lie group having 3-dimensional centre, the \( \lambda \)-decomposition for studying the spherical mean in the Métilvier group. We obtain the spectral dimensional step two nilpotent Lie algebra. This fact unfolds many tools for studying the spherical mean in the Métilvier group. We obtain the spectral decomposition for \( L^2 \)-functions in terms of eigenfunctions of sub-Laplacian on this particular Lie group. This reduction eases towards proving an analogue of one radius theorem on the general Métilvier group.

2. Preliminaries and Auxiliary results

Let \( G \) be a connected, simply connected Lie group with real step two nilpotent Lie algebra \( \mathfrak{g} \). Then \( \mathfrak{g} \) has the orthogonal decomposition \( \mathfrak{g} = \mathfrak{b} \oplus \mathfrak{z} \), where \( \mathfrak{z} \) is the centre. Since \( \mathfrak{g} \) is nilpotent, the exponential map \( \exp : \mathfrak{g} \rightarrow G \) is surjective. Hence \( G \) is parameterised by \( \mathfrak{g} \) and endowed with the exponential coordinates. We identify \( X + T \in \mathfrak{b} \oplus \mathfrak{z} \) with \( \exp(X + T) \) and denote it by \((X, T) \in \mathbb{R}^d \times \mathbb{R}^m \). Since \( [\mathfrak{b}, \mathfrak{b}] \subseteq \mathfrak{z} \) and \( [\mathfrak{b}, [\mathfrak{b}, \mathfrak{b}]] = 0 \), by the Baker-Campbell-Hausdorff formula, the group law on \( G \) expressed as

\[
(X, T)(Y, S) = (X + Y, T + S + \frac{1}{2}([X, Y]),
\]

where \( X, Y \in \mathfrak{b} \) and \( T, S \in \mathfrak{z} \). For \( \omega \in \mathfrak{z}^* \), consider the skew-symmetric bilinear form \( B_\omega \) on \( \mathfrak{b} \) by \( B_\omega(X, Y) = \omega([X, Y]) \). Then \( B_\omega \) is called a non-degenerate bilinear form when \( r_\omega = \{X \in \mathfrak{b} : B_\omega(X, Y) = 0, \forall Y \in \mathfrak{b}\} \) is trivial.

We say group \( G \) is Métilvier group if \( B_\omega \) is non-degenerate for all nonzero \( \omega \in \mathfrak{z}^* \). In such cases, the dimension of \( \mathfrak{b} \) is even, say \( d = 2n \). Let \( B_1, \ldots, B_{2n} \) and \( Z_1, \ldots, Z_m \) be orthonormal bases for \( \mathfrak{b} \) and \( \mathfrak{z} \), respectively. Since \( [\mathfrak{b}, \mathfrak{b}] \subseteq \mathfrak{z} \), there exist scalars \( U_{j,l}^{(k)} \) such that

\[
[B_j, B_l] = \sum_{k=1}^{m} U_{j,l}^{(k)} Z_k, \quad \text{where} \ 1 \leq j, l \leq 2n \text{ and } 1 \leq k \leq m.
\]

For \( 1 \leq k \leq m \), define \( 2n \times 2n \) skew-symmetric matrices by \( U^{(k)} = (U_{j,l}^{(k)}) \). Then the group law for the Métilvier group can precisely be expressed as

\[
(x, t) \cdot (\xi, \tau) = \begin{pmatrix} x_i + \xi_i, \ i = 1, \ldots, 2n \\ t_j + \tau_j + \frac{1}{2}(x, U^{(j)} \xi), \ j = 1, \ldots, m \end{pmatrix},
\]
where \( x, \xi \in \mathbb{R}^{2n} \) and \( t, \tau \in \mathbb{R}^m \). Left-invariant vector fields for the Lie algebra of the M´etivier group \( G \) computed as

\[
X_j = \frac{\partial}{\partial x_j} + \frac{1}{2} \sum_{k=1}^{m} \left( \sum_{l=1}^{n} \left( x_l U_{i,j}^{(k)} + x_{n+l} U_{n+i,j}^{(k)} \right) \right) \frac{\partial}{\partial t_k},
\]

\[
X_{n+j} = \frac{\partial}{\partial x_{n+j}} + \frac{1}{2} \sum_{k=1}^{m} \left( \sum_{l=1}^{n} \left( x_l U_{l,n+j}^{(k)} + x_{n+l} U_{n+l,n+j}^{(k)} \right) \right) \frac{\partial}{\partial t_k},
\]

and \( T_k = \frac{\partial}{\partial t_k} \), where \( (x, t) = (x_1, \ldots, x_n, x_{n+1}, \ldots, x_{2n}, t_1, \ldots, t_m) \in \mathbb{R}^{2n} \times \mathbb{R}^m \), \( j = 1, \ldots, n \) and \( k = 1, \ldots, m \). As \( U^{(k)} \)'s are skew-symmetric, we obtain the following commutation relations

\[
[X_i, X_j] = \sum_{k=1}^{m} U_{i,j}^{(k)} \frac{\partial}{\partial t_k}, \quad [X_{n+i}, X_{n+j}] = \sum_{k=1}^{m} U_{n+i,n+j}^{(k)} \frac{\partial}{\partial t_k}, \quad \text{for } i, j = 1, \ldots, n.
\]

Since \( U^{(1)}, \ldots, U^{(m)} \) are linearly independent, the dimension of the space spanned by \( \{(U_{i,j}^{(1)}, \ldots, U_{i,j}^{(m)}) : i, j = 1, \ldots, n\} \) will be \( m \).

Let \( \mu_r \) be the normalised surface measure on \( \{(x, 0) : |x| = r\} \subset G \). Then the spherical mean of a function \( f \in L^1(G) \) is defined as

\[
f \ast \mu_r(x, t) = \int_{|\xi| = r} f((x, t) \cdot (-\xi, 0)) d\mu_r(\xi).
\]

Denote \( \mathbb{R}_l^l = \mathbb{R}^l \setminus \{0\} \), \( l \in \mathbb{N} \). For \( \lambda \in \mathbb{R}^m \), let \( f^\lambda(z) = \int_{\mathbb{R}^m} f(x, t) e^{i\lambda \cdot t} dt \) be the inverse Fourier transform of \( f \) in \( t \) variable, then

\[
(f \ast \mu_r)^\lambda(x) = \int_{\mathbb{R}^m} f \ast \mu_r(x, t) e^{i\lambda \cdot t} dt = \int_{|\xi| = r} f^\lambda(x - \xi) e^{\frac{i}{2} \sum_{j=1}^{m} \lambda_j (x \cdot U^{(j)} \xi)} d\mu_r(\xi).
\]

Let us define the \( \lambda \)-twisted spherical mean of \( f \in L^1(\mathbb{R}^{2n}) \) by

\[
f \times_\lambda \mu_r(x) = \int_{|\xi| = r} f(x - \xi) e^{\frac{i}{2} \sum_{j=1}^{m} \lambda_j (x \cdot U^{(j)} \xi)} d\mu_r(\xi).
\]

Then the spherical mean \( f \ast \mu_r \) on the M´etivier group \( G \) can be studied by \( \lambda \)-twisted spherical mean \( f^\lambda \times_\lambda \mu_r \) on \( \mathbb{R}^{2n} \).

For \( \lambda \in \mathbb{R}^m \), the skew-symmetric matrix \( V_\lambda = \sum_{j=1}^{m} \lambda_j U^{(j)} \) is non-singular [4]. Let \( u_1 \pm iv_1, \ldots, u_n \pm iv_n \) be the eigenvectors of \( V_\lambda \) with corresponding eigenvalues \( \pm i\mu_{\lambda,1}, \ldots, \pm i\mu_{\lambda,n} \) satisfying \( \mu_{\lambda,1} \geq \cdots \geq \mu_{\lambda,n} > 0 \). Define \( A_\lambda = (\sqrt{2} u_1, \ldots, \sqrt{2} u_n, -\sqrt{2} v_1, \ldots, -\sqrt{2} v_n) \). Then \( A_\lambda \) is an orthogonal matrix that satisfies \( V_\lambda A_\lambda = A_\lambda U_\lambda \), where

\[
U_\lambda = \begin{pmatrix} 0_n & -J_\lambda \\ J_\lambda & 0_n \end{pmatrix}
\]
with $J_\lambda = \text{diag}(\mu_{\lambda,1}, \ldots, \mu_{\lambda,n})$ and $0_n$ is zero matrix of order $n$. In view of (2.4), we have

$$
\sum_{j=1}^{m} \lambda_j \langle x, U^{(j)} \xi \rangle = \langle x, V_\lambda \xi \rangle = \langle A^\dagger_\lambda x, U_\lambda A^\dagger_\lambda \xi \rangle,
$$

where $A_\lambda A^\dagger_\lambda = I$. For $x = (x_1, \ldots, x_n, x_{n+1}, \ldots, x_{2n}) \in \mathbb{R}^{2n}$, we write $z = (z_1, \ldots, z_n) = (x_1 + ix_{n+1}, \ldots, x_n + ix_{2n})$ and say $z$ be the complexification of $x$. Thus, after complexifying (2.5), we get

$$
\sum_{j=1}^{m} \lambda_j \text{Re} \left( z \cdot U^{(j)} w \right) = \sum_{j=1}^{n} \mu_{\lambda,j} \text{Im} \left( (z_\lambda)^j \cdot (\bar{w}_\lambda)_j \right),
$$

where $z_\lambda$ and $w_\lambda$ are complexification of $A^\dagger_\lambda x$ and $A^\dagger_\lambda \xi$ respectively. The following lemma would rationalise the $\lambda$-twisted spherical mean (2.3) to a simpler mean.

**Lemma 2.1.** [2] Let $f_\lambda(x) = f(A_\lambda x)$ and $z, \tilde{z}_\lambda \in \mathbb{C}^n$ be the complexification of $x, A_\lambda x \in \mathbb{R}^{2n}$ respectively. Then $f \times \lambda \mu_r(\tilde{z}_\lambda) = f_\lambda \tilde{x}_\lambda \mu_r(z)$, where

$$
f_\lambda \tilde{x}_\lambda \mu_r(z) = \int_{|w|=r} f_\lambda(z - w) \ e^{\frac{r}{2} \sum_{j=1}^{n} \mu_{\lambda,j} \text{Im}(z_j \cdot \bar{w}_j)} \ d\mu_r(w).
$$

We say $f_\lambda \tilde{x}_\lambda \mu_r$ as modified $\lambda$-twisted spherical mean.

### 3. Twisted spherical mean and spectral decomposition

In this section, we perceive that there is a Lie group with real $3n$-dimensional step two nilpotent Lie algebra whose twisted spherical mean close with $f_\lambda \tilde{x}_\lambda \mu_r$. We look for eigenfunctions of sub-Laplacian on this particular group, and via that, obtain the spectral decomposition for $L^2$-functions. We derive some auxiliary results related to the special Hermite functions.

Consider the group $\tilde{G} \simeq \mathbb{R}^{2n} \times \mathbb{R}^n$ as $\{(x, y, t) : x, y, t \in \mathbb{R}^n\}$ equipped with the group law

$$(x, y, t) \cdot (x', y', t') = \left(x + x', y + y', t + t' + \frac{1}{2} (x' \cdot y - y' \cdot x)\right).$$

The group $\tilde{G}$ is a $3n$-dimensional Métiévier group with a basis of left-invariant vector fields

$$
X_j = \frac{\partial}{\partial x_j} + \frac{1}{2} y_j \frac{\partial}{\partial t_j}, \quad Y_j = \frac{\partial}{\partial y_j} - \frac{1}{2} x_j \frac{\partial}{\partial t_j} \quad \text{and} \quad T_j = \frac{\partial}{\partial t_j},
$$

where $j = 1, \ldots, n$. The sub-Laplacian on $\tilde{G}$ is

$$
\mathcal{L} = - \sum_{j=1}^{n} (X_j^2 + Y_j^2).
$$
For each \( \lambda' \in \mathbb{R}^n \), we can see that the operator \( \pi_{\lambda'}(x, y, t) \) acting on \( L^2(\mathbb{R}^n) \) by
\[
(3.2) \quad \pi_{\lambda'}(x, y, t) \phi(\xi) = e^{i\sum_{j=1}^{n} \lambda'_j x_j + i\sum_{j=1}^{n} \lambda'_j x_j + \frac{1}{2} \pi_j y_j} \phi(\xi + y)
\]
are all possible irreducible unitary representation of \( \tilde{G} \), where \( \phi \in L^2(\mathbb{R}^n) \). If \( \pi_{\lambda'}(z) = \pi_{\lambda'}(z, 0) \), then \( \pi_{\lambda'}(z, t) = e^{i\lambda't} \pi_{\lambda'}(z) \). Identifying \( \tilde{G} \) with \( \mathbb{C}^n \times \mathbb{R}^n \), let \( L_{\lambda'} \) be the operator defined by \( L(e^{i\lambda't} f(z)) = e^{i\lambda't} L f(z) \), where \( z = x + iy \).

Then \( L_{\lambda'} \) can precisely be expressed as
\[
(3.3) \quad L_{\lambda'} = -\Delta + \frac{1}{4} \sum_{j=1}^{n} \lambda'_j^2 |z'_j|^2 + iN_{\lambda'}, \quad \text{where} \quad N_{\lambda'} = \sum_{j=1}^{n} \lambda'_j \left( x_j \frac{\partial}{\partial y_j} - y_j \frac{\partial}{\partial x_j} \right).
\]

Let \( f \in L^1(\tilde{G}) \) and \( f^{\lambda'}(z) = \int_{\mathbb{R}^n} f(z, t) e^{i\lambda't} dt \) be the inverse Fourier transform of \( f \) in the \( t \) variable. Then, for this particular group \( \tilde{G} \) the \( \lambda' \)-twisted spherical mean can be explicitly calculated by
\[
(3.4) \quad f^{\lambda'} \times_{\lambda'} \mu_r(z) = \int_{|w| = r} f^{\lambda'}(z - w) e^{\frac{i}{2} \sum_{j=1}^{n} \lambda'_j \text{Im}(z'_j, w'_j)} d\mu_r(w).
\]

Similarly, if \( f, g \in L^1(\tilde{G}) \), then we can also define the \( \lambda' \)-twisted convolution as
\[
(3.5) \quad f^{\lambda'} \times_{\lambda'} g^{\lambda'}(z) = \int_{\mathbb{C}^n} f^{\lambda'}(z - w) g^{\lambda'}(w) e^{\frac{i}{2} \sum_{j=1}^{n} \lambda'_j \text{Im}(z'_j, w'_j)} dw.
\]

**Remark 3.1.** For any \( \lambda \in \mathbb{R}^m \), \( m \geq 2 \), the modified \( \lambda \)-twisted spherical mean \( (2.7) \) coincides with the \( \lambda' \)-twisted spherical mean \( (3.3) \), where \( \lambda' \in \mathbb{R}^n \) and each of its coordinate can be identified with the imaginary part of an eigenvalue of \( V_{\lambda} \). Therefore, studying the injectivity of spherical mean on an arbitrary Métivier group \( G \), it is enough to consider spherical mean on \( \tilde{G} \).

For \( \alpha \in \mathbb{Z}_+^n \), let \( \Phi_{\alpha}(x) = \Pi_{j=1}^{n} h_{\alpha_j}(x_j) \), where \( h_{\alpha_j} \) are normalised Hermite functions on \( \mathbb{R} \). Then \( \Phi_{\alpha} \) is an eigenfunction of Hermite operator \( H = -\Delta + |x|^2 \) with eigenvalue \( (2|\alpha| + n) \). For more details, see [12]. Moreover, for \( \lambda' = (\lambda'_1, \ldots, \lambda'_n) \in \mathbb{R}^n \setminus \{0\} \) if we define
\[
\Psi_{\alpha}^{\lambda'}(x) = \prod_{j=1}^{n} |\lambda'_j|^{rac{1}{2}} h_{\alpha_j} \left( \sqrt{|\lambda'_j|} x_j \right),
\]
then \( \Psi_{\alpha}^{\lambda'} \) are the eigenfunctions of the elliptic Hermite operator \( H_{\lambda'} = -\Delta + \sum_{j=1}^{n} (\lambda'_j x_j)^2 \) with eigenvalues \( \sum_{j=1}^{n} (2\alpha_j + 1)|\lambda'_j| \). Thus,
\[
L_{\lambda'} \left( \pi_{\lambda'}(z) \Psi_{\alpha}^{\lambda'}, \Psi_{\beta}^{\lambda'} \right) = \sum_{j=1}^{n} (2\alpha_j + 1)|\lambda'_j| \left( \pi_{\lambda'}(z) \Psi_{\alpha}^{\lambda'}, \Psi_{\beta}^{\lambda'} \right).
\]
For \( \alpha, \beta \in \mathbb{Z}_+^n \), define the function as

\[
\Psi_{\alpha\beta}^{\lambda}(z) = \left( \prod_{j=1}^{n} \sqrt{\frac{|\lambda_j|}{2\pi}} \right) \left( \pi_{\lambda'}(z) \Psi_{\alpha}^{\lambda'} \Psi_{\beta}^{\lambda'} \right).
\]

Then \( \Psi_{\alpha\beta}^{\lambda} \) are eigenfunctions of the operator \(-\Delta_z + \frac{1}{4} \sum_{j=1}^{n} \lambda_j^2 |z_j|^2\). The set \( \{ \Psi_{\alpha\beta}^{\lambda} : \alpha, \beta \in \mathbb{Z}_+^n \} \) form a complete orthonormal set for \( L^2(\mathbb{C}^n) \).

Next, we come up with some identities for \( \Psi_{\alpha\beta}^{\lambda} \) which can be derived by a suitable change of variables in the special Hermite function.

Recall that the Laguerre function \( \varphi_{\nu}^{-1} \) on \( \mathbb{C}^n \) is given by

\[
\varphi_{\nu}^{-1}(z) = L_{\nu}^{-1} \left( \frac{1}{2} |z|^2 \right) e^{-\frac{1}{4} |z|^2},
\]

where \( L_{\nu}^{-1} \) are Laguerre polynomials of type \((n - 1)\). For \( \lambda' \in \mathbb{R}_+^n \), define \( \varphi_{\nu}^{-1}(z) = \varphi_{\nu}^{-1}(\sqrt{\lambda'} z) \), where the notation \( \sqrt{\lambda'} z \) is fixed by

\[
(3.6) \quad \sqrt{\lambda'} z := \left( \sqrt{|\lambda'_1| z_1}, \ldots, \sqrt{|\lambda'_n| z_n} \right).
\]

As similar to special Hermite function, \( \Psi_{\alpha\beta}^{\lambda'} \) can be expressed in terms of Laguerre functions as

\[
(3.7) \quad \Psi_{\alpha\beta}^{\lambda'}(z) = \left( 2\pi \right)^{-\frac{n}{2}} \prod_{j=1}^{n} L_{\alpha_j}^{0} \left( \frac{1}{2} |\lambda'_j z_j|^2 \right) e^{-\frac{1}{4} |\lambda'_j z_j|^2}.
\]

Then we can derive the formula

\[
(3.8) \quad \left( \prod_{j=1}^{n} \sqrt{|\lambda'_j|} \right) \sum_{|\alpha|=k} \Psi_{\alpha\beta}^{\lambda'}(z) = \left( 2\pi \right)^{-\frac{n}{2}} \varphi_{\nu}^{-1}(z).
\]

Let \( f \in L^2(\mathbb{C}^n) \), then in view of (3.8), and the completeness of \( \Psi_{\alpha\beta}^{\lambda'} \)'s in \( L^2(\mathbb{C}^n) \), \( f \) will satisfy the identity

\[
\sum_{|\alpha|=k} \sum_{\beta} \left( f, \Psi_{\alpha\beta}^{\lambda'} \right) \Psi_{\alpha\beta}^{\lambda'} = \left( \prod_{j=1}^{n} \frac{|\lambda'_j|}{2\pi} \right) \int_{\mathbb{C}^n} f(w) \varphi_{\nu}^{-1}(z - w) e^{\frac{1}{2} \sum_{j=1}^{n} \lambda'_j \text{Im}(z_j \bar{w}_j)} dw.
\]

The right-hand side is simply \( \left( \prod_{j=1}^{n} \frac{|\lambda'_j|}{2\pi} \right) \varphi_{\nu}^{-1} \times_{\lambda'} f(z) \) and \( \varphi_{\nu}^{-1} \times_{\lambda'} f(z) = f \times_{\lambda'} \varphi_{\nu}^{-1}(z) \), we have

\[
(3.9) \quad f(z) = \left( \prod_{j=1}^{n} \frac{|\lambda'_j|}{2\pi} \right) \sum_{k=0}^{\infty} f \times_{\lambda'} \varphi_{\nu}^{-1}(z).
\]

The next proposition shows that the \( \lambda' \)-twisted spherical mean of \( \varphi_{\nu}^{-1} \) will satisfy the following functional relation.
Proposition 3.2. Denote $\varphi_{k,\lambda}^{-1}(r) = \varphi_{k,\lambda}^{-1}(w)$ for $|w| = r$. Then

$$(3.10) \quad \varphi_{k,\lambda}^{-1} \times \lambda \mu_r(z) = \left( \prod_{j=1}^n \frac{1}{\sqrt{\lambda_j}} \right) \frac{k!(n-1)!}{(k+n-1)!} \varphi_{k,\lambda}^{-1}(z) \varphi_{k,\lambda}^{-1}(r).$$

Proof. From (8, Theorem 2.1), it is known that the twisted spherical mean of $\varphi_k^{-1}$ with respect to the Heisenberg group can be written as

$$(3.11) \quad \int_{|w|=r} \varphi_k^{-1}(z-w) e^\frac{z}{w} \frac{z}{w} \sum_{i=1}^n \lambda_j \Im(z \cdot \bar{w}) d\mu_r(w) = \frac{k!(n-1)!}{(k+n-1)!} \varphi_k^{-1}(z) \varphi_k^{-1}(r).$$

Now, by a change of variable, we can rewrite

$$\varphi_{k,\lambda}^{-1} \times \lambda \mu_r(z) = \int_{|w|=r} \varphi_k^{-1}(z-w) e^\frac{z}{w} \sum_{i=1}^n \lambda_j \Im(z \cdot \bar{w}) d\mu_r(w)$$

$$(3.12) \quad = \int_{|w|=r} \varphi_k^{-1}(\sqrt{\lambda}z - \sqrt{\lambda}w) e^\frac{z}{w} \sum_{i=1}^n \lambda_j \Im(z \cdot \bar{w}) d\mu_r(w)$$

$$= \int_{|z'|=\sqrt{\lambda}w = r} \varphi_k^{-1}(z' - w) e^\frac{z}{w} \sum_{i=1}^n \lambda_j \Im(z \cdot \bar{w}) d\mu_r(w),$$

where $z' = \sqrt{\lambda}z$. By a suitable change of variable in (3.11), we can write the above equation (3.12) as

$$\int_{|z'|=\sqrt{\lambda}w = r} \varphi_k^{-1}(z' - w) e^\frac{z}{w} \sum_{i=1}^n \lambda_j \Im(z \cdot \bar{w}) d\mu_r(w) = \left( \prod_{j=1}^n \frac{1}{\sqrt{\lambda_j}} \right) \frac{k!(n-1)!}{(k+n-1)!} \varphi_k^{-1}(z') \varphi_k^{-1}(\sqrt{\lambda}w).$$

Hence the identity (3.11) is followed. \qed

Let $m = (m_1, \ldots, m_n) \in \mathbb{Z}^n$. A function $f$ on $\mathbb{C}^n$ is called $m$-homogeneous if it satisfies $f(e^{i\theta}z) = f(e^{i\theta_1}z_1, \ldots, e^{i\theta_n}z_n) = e^{im\cdot \theta} f(z)$, where $\theta = (\theta_1, \ldots, \theta_n)$. For a function $g$ on $\mathbb{C}^n$ define $m$-radialization $R_m g$ by

$$(3.13) \quad R_m g(z) = (2\pi)^{-n} \int_{0,2\pi} \int g(e^{i\theta}z) e^{-im\cdot \theta} d\theta.$$

Then $R_m f$ is $m$-homogeneous and we have

$$(3.14) \quad f(z) = \sum_m R_m f(z) e^{im\cdot \theta}.$$

The series in the right-hand side of (3.14) converges in the topology of Schwartz class function $\mathcal{S}(\mathbb{C}^n)$, see [10].

Since $\Psi_{\alpha,\beta}^{\lambda}$ is $(\beta - \alpha)$-homogeneous, we can see that

$$\left( f, \Psi_{\alpha,\beta}^{\lambda}(z) \right) = \int_{\mathbb{C}^n} f(z) \overline{\Psi_{\alpha,\beta}^{\lambda}(z)} dz$$

is nonzero only when $\beta = \alpha + m$. Thus, if $f$ is $m$-homogeneous we can write

$$(3.15) \quad f \times \lambda \varphi_{k,\lambda}^{-1} = \left( \prod_{j=1}^n \frac{\lambda_j}{2\pi} \right) \sum_{|\beta|=k} \left( f, \Psi_{\beta-m,\beta}^{\lambda} \right) \Psi_{\beta-m,\beta}^{\lambda}.$$
In [10], it has been proved that for the Heisenberg group the special Hermite series of an $m$-homogeneous function converges in the topology of $\mathcal{S}(\mathbb{C}^n)$. By imitating the prove in this case, we have the following result.

Lemma 3.3. If $f$ is a Schwartz class function and $m$-homogeneous, then the series (3.9) of $f$ converges in the topology of $\mathcal{S}(\mathbb{C}^n)$.

4. Spherical mean on the Métivier Group

This section deals with the injectivity of spherical mean $f * \mu$, where $\mu \in X_P(G)$, the space of compactly supported rotation invariant probability measure with no mass at the centre of Métivier group $G$.

Proposition 4.1. Let $1 \leq p_i \leq 2$ for $i = 1, 2$. Let $f \in C(G)$ be such that $f(z, \cdot) \in L^{p_1}(\mathbb{R}^m)$ and $f^\lambda \in L^{p_2}(\mathbb{C}^n)$ for a.e. $\lambda \in \mathbb{R}^m_+$. If $f$ satisfies $f * \mu = 0$ for some $\mu \in X_P(G)$, then $f = 0$.

Proof. For $\lambda \in \mathbb{R}^m_+$, let $f^\lambda$ and $\mu^\lambda$ be the partial Fourier transform of $f$ and $\mu$ in $t$ variable, respectively. Then applying $\lambda$-twisted convolution, we get

$$f^\lambda \times_\lambda \mu^\lambda = 0,$$

where $E = \{\mu_r\}$, $\mu_r$ is the normalised surface measure on the sphere of radius $r$ centred at the origin in $\mathbb{C}^n$, and $M$ is the measure on $E$. For more details, refer to [6]. From Lemma 2.1, we can rewrite (4.1) using the modified $\lambda$-twisted spherical mean as

$$f^\lambda \times_\lambda \mu^\lambda = \int_E (f^\lambda)_\lambda \tilde{x}_\lambda \mu_r dM,$$

where $(f^\lambda)_\lambda(x) = f^\lambda(A_\lambda x)$ defined as in Lemma 2.1. For a fixed $\lambda$, and considering Remark 3.1, we can write

$$(f^\lambda)_\lambda \tilde{x}_\lambda \mu_r = (f^\lambda)_\lambda \times_{\lambda'} \mu_r$$

for some $\lambda' \in \mathbb{R}^n_+$. In the right-hand side, $\lambda'$-twisted spherical mean is with respect to $\tilde{G}$ as defined by (3.4). By an appropriate approximation identity, we may assume that $(f^\lambda)_\lambda \in L^2(\mathbb{C}^n)$. Then applying the spectral decomposition (3.3), $(f^\lambda)_\lambda$ can be expressed in terms of $\vartheta_{k,\lambda'}^{n-1}$ as

$$(f^\lambda)_\lambda = \left(\prod_{j=1}^n \frac{\lambda_j}{2\pi}\right) \sum_{k=0}^\infty (f^\lambda)_\lambda \times_{\lambda'} \vartheta_{k,\lambda'}^{n-1},$$

where the series converges in $L^2(\mathbb{C}^n)$. Now, it is enough to prove that each spectral projection $(f^\lambda)_\lambda \times_{\lambda'} \vartheta_{k,\lambda'}^{n-1} = 0$. From (4.2) and (4.4) we have

$$\sum_{k=0}^\infty \int_E (f^\lambda)_\lambda \times_{\lambda'} \vartheta_{k,\lambda'}^{n-1} \times_{\lambda'} \mu_r dM = 0.$$
In view of Proposition (3.2), we get
\[ \sum_{k=0}^{\infty} \mu^\lambda (\vartheta_{k,\lambda}^{n-1}) (f^\lambda)_\lambda \times_{\lambda'} \vartheta_{k,\lambda'}^{n-1} = 0, \]
where
\[ \mu^\lambda (\vartheta_{k,\lambda}^{n-1}) = \int_{C^n} \vartheta_{k,\lambda}^{n-1} \mu^\lambda = \int_{E} \vartheta_{k,\lambda}^{n-1} dM. \]
Since for each \( k \in \mathbb{Z}_+ \), \( \mu^\lambda (\vartheta_{k,\lambda}^{n-1}) \) vanishes only for countable many values of \( \lambda \). Hence \( (f^\lambda)_\lambda \times_{\lambda'} \vartheta_{k,\lambda'}^{n-1} = 0 \) implies \( (f^\lambda)_\lambda \times_{\lambda'} \mu_r = 0 \) for a.e. \( \lambda \) and each \( k \). Thus, \( (f^\lambda)_\lambda = 0 \) for a.e. \( \lambda \), which concludes \( f = 0 \).

In the following result we relax the integrability condition of \( f^\lambda \) in the \( z \) variable with tempered growth with help of some approximation lemmas from Section 3.

**Theorem 4.2.** Let \( f \) be a continuous function on \( G \) with \( f(z, \cdot) \in L^p(\mathbb{R}^m) \), \( 1 \leq p \leq 2 \) and \( f^\lambda \) has tempered growth in \( \mathbb{C}^n \) for a.e. \( \lambda \in \mathbb{R}^m \). If \( f \) satisfies \( f \ast \mu = 0 \) for some \( \mu \in X^p(G) \), then \( f = 0 \).

**Proof.** Since \( f \) is integrable in the second variable, applying \( \lambda \)-twisted convolution on \( f \ast \mu = 0 \), we get \( f^\lambda \times_{\lambda'} \mu^\lambda = 0 \) for a.e. \( \lambda \). Hence we claim \( f^\lambda = 0 \) for almost all \( \lambda \). But tempered growth of \( f^\lambda \) reduces to show that
\[ \int_{C^n} f^\lambda(z)g(z)dz = 0 \]
for every \( g \) in \( \mathcal{S} (\mathbb{C}^n) \). Since \( g \) admits an \( m \)-radialization expansion, we can replace both \( g \) and \( f^\lambda \) with their \( m \)-radialization. Therefore, it is enough to consider
\[ (4.5) \quad \int_{C^n} R_m f^\lambda(z)g(z)dz = 0 \]
for all \( m \)-homogeneous \( g \in \mathcal{S} (\mathbb{C}^n) \). If we fix \( \lambda \), then there exists \( \lambda' \in \mathbb{R}^m_+ \) as in (4.3), and by Lemma 3.3 we can reciprocate \( g \) with \( g \times_{\lambda'} \vartheta_{k,\lambda'}^{n-1} \) in (4.5). Hence it is enough to examine
\[ \int_{C^n} R_m f^\lambda(z)g(z)dz = 0 \]
for every \( k \), which is equivalent to
\[ \int \vartheta_{k,\lambda'}^{n-1} \times_{\lambda'} R_m f^\lambda(z)dz = 0. \]
From (3.15) it is clear that \( \vartheta_{k,\lambda'}^{n-1} \times_{\lambda'} R_m f^\lambda = 0 \). And we also have \( \vartheta_{k,\lambda'}^{n-1} \times_{\lambda'} R_m f^\lambda = 0 \). Using Proposition 4.1 we conclude that \( \vartheta_{k,\lambda'}^{n-1} \times_{\lambda'} R_m f^\lambda = 0 \). This proves the theorem. \( \square \)
Let $\lambda \in \mathbb{R}_x$. If $f$ satisfies $f \ast \mu_r = 0$ for some $r > 0$, then $f = 0$.

Proof. Applying $\lambda$-twisted convolution on $f \ast \mu_r = 0$, we get $f^\lambda \times_\lambda \mu_r = 0$ for a.e. $\lambda$. For a fix $\lambda$, then there exists $\lambda' \in \mathbb{R}_+^n$ as in (4.3), and using Lemma 2.1 it follows that

$$(4.6) \quad (f^\lambda)_\lambda \times_{\lambda'} \mu_r = 0.$$

Further, taking the $\lambda'$-twisted convolution of equation (4.3) and $\vartheta_{\kappa,\lambda'}^{-1}$, and using Proposition 3.3 we get $\vartheta_{\kappa,\lambda'}^{-1}(r)(f^\lambda)_\lambda \times_{\lambda'} \vartheta_{\kappa,\lambda'}^{-1} = 0$ for all $\kappa \in \mathbb{Z}_+$. Since the zero sets of Laguerre polynomials are disjoint, $\vartheta_{\kappa,\lambda'}^{-1}(r) \neq 0$ for all $\kappa$ except one, say $\kappa = l$. That is, $(f^\lambda)_\lambda \times_{\lambda'} \vartheta_{\kappa,\lambda'}^{-1} = 0$ for all $\kappa \neq l$. Hence we get $(f^\lambda)_\lambda = C(f^\lambda)_\lambda \times_{\lambda'} \vartheta_{l,\lambda'}^{-1}$ for some nonzero constant $C$. Since $R_m(f^\lambda)_\lambda$ is $m$-homogeneous, in view of (3.15) we get

$$R_m(f^\lambda)_\lambda = C R_m(f^\lambda)_\lambda \times_{\lambda'} \vartheta_{l,\lambda'}^{-1} = C \left( \prod_{j=1}^n \frac{|\lambda_j'|}{2\pi} \right) \sum_{|\beta|=l} (f^\lambda)_\lambda, \Psi_{\beta-m,\beta}^\lambda \Psi_{\beta-m,\beta}^\lambda.$$

Replacing $R_m(f^\lambda)_\lambda$ with $R_m((f^\lambda)_\lambda e^{\frac{4}{l} |\lambda' z|^2})$ we have

$$R_m((f^\lambda)_\lambda e^{\frac{4}{l} |\lambda' z|^2}) = C \left( \prod_{j=1}^n \frac{|\lambda_j'|}{2\pi} \right) \sum_{|\beta|=l} (f^\lambda)_\lambda, \Psi_{\beta-m,\beta}^\lambda \Psi_{\beta-m,\beta}^\lambda e^{\frac{4}{l} |\lambda' z|^2}.$$

By the hypothesis $f^\lambda(z)e^{\frac{4}{l} |\lambda' z|^2} \in L^p(\mathbb{C}^n)$, it follows that left-hand side is in $L^p(\mathbb{C}^n)$, which makes $f = 0$ since the right-hand side is a polynomial. □

We now prove a version of the two radii theorem for the class of tempered continuous functions on the Métivier group, which are periodic in the centre variable.

Theorem 4.4. Let $f$ be a tempered continuous function in $z$ and $2\pi$-periodic in the centre variable of $G$. If $f$ satisfies $f \ast \mu_{r_i} = 0$, $i = 1, 2$, then $f = 0$ as long as

(i) $\frac{A_1}{A_2}$ is not a quotient of zeros of Laguerre polynomials $L_k^{n-1}$ for any $k$.

(ii) $\frac{A_1}{A_2}$ is not a quotient of zeros of Bessel functions $J_{n-1}$.

Proof. For $l \in \mathbb{Z}_+$, define the $l$th Fourier coefficient of $f$ by

$$f^l(z) = \int_{[0,2\pi]^m} f(z,t) e^{il t} dt.$$

It follows by Lemma 3.3 that $f^l \in L^2(\mathbb{C}^n)$. Further, taking the $l$-twisted spherical mean of $f \ast \mu_{r_i} = 0$ and uniqueness of the Fourier series, we get $f^l \times_l \mu_{r_i} = 0$.
for $i = 1, 2$. Let us fix $l \neq 0$, then using Lemma 2.1 and Equation (4.3) we can write
\[(f^l)_l \times \lambda' = 0\] for some $\lambda' \in \mathbb{R}^n_+$ and $i = 1, 2$.

where \((f^l)_l = f^l(A_ix)\) as defined in Lemma 2.1. Then using Proposition 3.2 we get
\[\vartheta_{n-1}^k(\lambda')(\varphi^l_{k,\lambda}'(z)) = 0\] for $i = 1, 2$.

Since for each $k$, either \(\vartheta_{n-1}^k(\lambda'_1) \neq 0\) or \(\vartheta_{n-1}^k(\lambda'_2) \neq 0\), we have \((f^l)_l \times \lambda' = 0\). Hence \((f^l)_l = 0\). When $l = 0$, the $l$-twisted spherical mean conditions $f * \mu_{r_i} = 0$, $i = 1, 2$ led to two radii theorem on $\mathbb{C}^n$. Hence each Fourier coefficient of $f$ is zero, and thus $f = 0$.

**Remark 4.5.** Since the twisted spherical mean of a M´etivier group is coherence with a $3n$-dimensional M´etivier group, we could show one radius theorem for $f \in L^p(G)$, $1 \leq p \leq 2$. However, when we approach as similar to the Heisenberg group [10] for $f \in L^p(G)$, $2 < p < \infty$, based on a summability result, we seek $L^p$- boundedness of a multi-parameter singular integral, whose kernel may fail to be a Calderon-Zygmund kernel.

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