On the asymptotic normality of finite population $L$-statistics

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Abstract We give sufficient conditions for the asymptotic normality of linear combinations of order statistics ($L$-statistics) in the case of simple random samples without replacement. In the first case, restrictions are imposed on the weights of $L$-statistics. The second case is on trimmed means, where we introduce a new finite population smoothness condition.

Keywords Finite population · Sampling without replacement · $L$-statistic · Trimmed mean · Hoeffding decomposition · Asymptotic normality

Mathematics Subject Classification 62E20

1 Introduction

In the case of independent and identically distributed (i.i.d.) observations, the asymptotic normality of $L$-statistics under various conditions was shown by Chernoff et al. (1967), Shorack (1972), Stigler (1973, 1974) and Mason (1981), among others. See also Serfling (1980). In the case of samples drawn without replacement, there are only a few works on the asymptotic normality of $L$-statistics, e.g., the paper of Shao (1994), where $L$-statistics under complex sampling designs are considered, and the work of Chatterjee (2011) about the case of a sample quantile.

Consider a population $\mathcal{X} = \{x_1, \ldots, x_N\}$ of size $N$ consisting of real numbers. Let $\mathcal{X} = \{X_1, \ldots, X_n\}$ be a simple random sample of size $n < N$ drawn without replacement from the population. The observations $X_1, \ldots, X_n$ are identically distributed, but they are not independent. Let $X_{1:n} \leq \cdots \leq X_{n:n}$ denote the order statistics of $\mathcal{X}$. Define the $L$-statistic
\[ L_n = L_n(\mathcal{X}) = \frac{1}{n} \sum_{j=1}^{n} c_j X_{j:n}. \]  

Here \( c_1, \ldots, c_n \) is a given sequence of real numbers called weights. Usually these weights are determined by the weight function \( J : (0, 1) \to \mathbb{R} \) as follows:

\[ c_j = J \left( \frac{j}{n+1} \right), \quad 1 \leq j \leq n. \]

Further, when we talk about the asymptotics of \( L \)-statistics, we use centered statistics \( (1) \) with \( n^{1/2} \) norming, i.e.,

\[ S_n = S_n(\mathcal{X}) = n^{1/2} (L_n - \mathbb{E} L_n). \]

Denote \( \tilde{\sigma}_n^2 = \text{Var} S_n \). We are interested in the normal approximation to the distribution function

\[ F_n(x) = \mathbb{P} \{ S_n \leq x \tilde{\sigma}_n \} . \]

Note that for correct formulations of the following asymptotic results for finite population statistics, we need to consider a sequence of populations \( \mathcal{X}_r = \{ x_{r,1}, \ldots, x_{r,N_r} \} \), with \( N_r \to \infty \) as \( r \to \infty \), and a sequence of statistics \( L_n(\mathcal{X}_r) \), based on simple random samples \( \mathcal{X}_r = \{ X_{r,1}, \ldots, X_{r,n_r} \} \) drawn without replacement from \( \mathcal{X}_r \). In order to keep the notation simple, we shall skip the subscript \( r \) in what follows.

The sample mean is the separate case of \( (1) \), where \( c_j \equiv 1, 1 \leq j \leq n \). In this case, for samples drawn without replacement, the classical result on the asymptotic normality was established by Erdős and Rényi (1959), see also Hájek (1960). Similarly as in the case of i.i.d. observations, the key asymptotic condition in Erdős and Rényi (1959) is the Lindeberg-type condition: for every \( \varepsilon > 0 \),

\[ \sigma^{-2} \mathbb{E} (X_1 - \mathbb{E} X_1)^2 \mathbb{I} \{ |X_1 - \mathbb{E} X_1| > \varepsilon \tau \sigma \} = o(1) \quad \text{as} \quad N, n \to \infty, \]

where \( \sigma^2 = \text{Var} X_1 \) and \( \tau^2 = npq \) with \( p = n/N, q = 1-p \), and \( \mathbb{I} \{ \cdot \} \) is the indicator function. Condition (2) is called the Erdős–Rényi condition. Since \( L \)-statistics can be viewed as a certain generalization of the sample mean, one can expect that conditions, sufficient for the asymptotic normality, should be similar to that used in Erdős and Rényi (1959), but with some additional restrictions to the weights \( c_1, \ldots, c_n \).

On the other hand, \( L \)-statistics is a subclass of a more general class of symmetric statistics (symmetric functions of observations). An asymptotic behaviour of the symmetric statistics differs not so much from that of the simplest linear statistic (the sample mean is an example), in the sense that, e.g., using Hoeffding’s decomposition of Bloznelis and Götze (2001), we can write

\[ S_n = U_1 + R_1, \quad \text{where} \quad U_1 = \sum_{i=1}^{n} g_1(X_i) \]
Finite population $L$-statistics

is a linear statistic with an influence function $g_1(\cdot)$, and $R_1$ is a remainder term. Then $S_n$ in (3) is asymptotically normal if its linear part $U_1$ is asymptotically normal, and the variance of $R_1$ tends to zero as the sample size

$$n_\ast := \min\{n, N - n\}$$

increases. In particular, by Bloznelis and Götze (2001), the components $U_1$ and $R_1$ are centered and uncorrelated, and (by Theorem 1 of Bloznelis and Götze (2001)) the variance of $R_1$ is bounded as follows: $\mathbb{E} R_1^2 \leq \delta_2$, where it is expected that the particular quantity $\delta_2 = o(1)$ as $n_\ast \to \infty$. We refer to (7) in Sect. 3 for the definition of $\delta_2$. In the present paper, we apply the general result on the asymptotic normality of the symmetric statistics (see Proposition 3 of Bloznelis and Götze (2001)) to the case of $L$-statistics, i.e., we replace the condition imposed on $\delta_2$ by conditions expressed in terms of the weights $c_1, \ldots, c_n$ and the population $\mathcal{X}$.

The results are presented in Sect. 2 and their proofs are given in Sect. 3.

2 Results

We assume, without loss of generality, that the values of the population $\mathcal{X}$ are arranged in the non-decreasing order, i.e., $x_1 \leq \cdots \leq x_N$. Let us use the convention \((a\wedge b) = 0\) for $a < b$. In the case of $L$-statistic (1), the function $g_1(\cdot)$ in (3) is represented by, for $1 \leq k \leq N$,

$$g_1(x_k) = -n^{-1/2} \sum_{j=1}^{n} c_j \sum_{i=1}^{N-1} \left( \mathbb{I}\{i \geq k\} - \frac{i}{N} \right) \times \frac{(i-1)(N-i-1)}{n-j} \frac{(N-2)}{n-1} (x_{i+1} - x_i),$$

see Čiginas (2012). Denote $\sigma_1^2 = \mathbb{E} g_1^2(X_1)$.

First, we consider an $L$-statistic of the general form (1), and we will require a certain smoothness of its weight function $J(\cdot)$. Recall that, if $|J(x) - J(y)| \leq B|x - y|^\delta$, for any $x, y \in (0, 1)$ and some positive real constants $B$ and $\delta$, then it is said that the function $J(\cdot)$ satisfies the Hölder condition of order $\delta$ on the interval $(0, 1)$. Reformulating Erdős–Rényi condition (2), for every $\varepsilon > 0$ we get

$$\sigma_1^{-2} \mathbb{E} g_1^2(X_1) \mathbb{I}\{|g_1(X_1)| > \varepsilon \tau \sigma_1\} = o(1) \quad \text{as} \quad n_\ast \to \infty. \quad (5)$$

Then we have the following statement.

**Theorem 1** Assume that $n_\ast \to \infty$ and $\bar{\sigma}_n \geq C_1 > 0$ for all $n_\ast$. Let $\mathbb{E}X_1^2 \leq C_2 < \infty$ and $J(\cdot)$ is bounded and satisfies the Hölder condition of order $\delta > 1/2$ on $(0, 1)$. Let (5) hold. Then $\bar{\sigma}_n^{-1} S_n$ is asymptotically standard normal.

The assumptions of Theorem 1, sufficient for the asymptotic normality of $L$-statistics, are similar to that obtained by Stigler (1974) in the i.i.d. case.
Second, we consider an important special case of (1), i.e., trimmed means. The trimmed mean is defined as follows: for any fixed numbers $0 < t_1 < t_2 < 1$,

$$M_{t_1:t_2} = ([t_2n] - [t_1n])^{-1} \sum_{j=[t_1n]+1}^{[t_2n]} X_{j:n},$$

where $[\cdot]$ is the greatest integer function. The statistic $M_{t_1:t_2}$ is represented by the weight function $J(u) = \frac{t_2 - t_1}{n-1} [t_1 < u < t_2]$. This function is bounded, but it does not satisfy the Hölder condition. Let us introduce an additional smoothness condition for the population $\mathcal{X}$. Suppose that, for some constants $C > 0$ and $1/2 < \delta \leq 1$, the inequality

$$|x_m - x_l| \leq C N^{-\delta} |m - l|$$

is satisfied for all $1 \leq l < m \leq N$.

**Theorem 2** Assume that $n_\ast \rightarrow \infty$. Let $E X_1^2 \leq C_2 < \infty$. Assume that (6) holds for some $1/2 < \delta \leq 1$, and $(1 - n/N)^{-1} n^{1/2} N^{\delta-1} \rightarrow \infty$. Then, in the case of a trimmed mean, $\tilde{\sigma}_n^{-1} S_n$ is asymptotically standard normal.

In the case of i.i.d. observations, it has been shown by Stigler (1973) that in order for the trimmed mean to be asymptotically normal, it is necessary and sufficient that the sample is trimmed at sample quantiles for which the corresponding population quantiles are uniquely defined. Thus, the conditions of Theorem 2 seem too strong. On the other hand, in finite population settings, new smoothness condition (6) has a specific interpretation. Let us take $l = 1$ and $m = N$. If the population $\mathcal{X}$ is bounded, then the condition is satisfied for $\delta = 1$. For any finite population, with the bounded variance according to Theorem 2, condition (6) is satisfied in the marginal case of $\delta = 1/2$. The latter fact follows from the Nair–Thomson inequality $|x_N - x_1| \leq \sigma \sqrt{2N}$ (see, e.g., Balakrishnan et al. (2003)). Thus, condition (6) seems very mild for small $\theta > 0$ in $\delta = 1/2 + \theta$, i.e., it holds for most of possible populations. Obviously, if we are interested in the asymptotic normality of the trimmed means, then, by the conditions of Theorem 2, for small $\theta$ we should have $n \rightarrow \infty$ quite quickly as $N \rightarrow \infty$, while in the case of $\delta = 1$ it suffices that $n \rightarrow \infty$ arbitrarily slowly with respect to the growth of the population size $N$.

3 Proofs

In the proofs of Theorems 1 and 2, we apply some tools from Čiginas (2012), where the validity of one-term Edgeworth expansion is considered in the case of $L$-statistic with a smooth weight function. In comparison with those results, the asymptotic normality in Theorem 1 requires much less smoothness from the weight function.

Following Bloznelis and Götze (2001), let $X_1, \ldots, X_N$ denote a random permutation of the ordered set $\mathcal{X} = \{x_1, \ldots, x_N\}$ which is uniformly distributed over the class of permutations. Then the first $n$ observations $X_1, \ldots, X_n$ are the simple random...
sample \( \mathbb{X} \) from \( \mathcal{X} \). Consider also the extended sample \( \mathbb{X}_2 = \{X_1, \ldots, X_{n+2}\} \), and then define the quantity \( \delta_2 \) introduced in Bloznelis and Götze (2001):

\[
\delta_2(S_n) = \mathbb{E} \left( n_\star \mathbb{D}_2 S_n \right)^2,
\]

(7)

where

\[
\mathbb{D}_2 S_n = S_n(\mathbb{X}_2 \{X_{n+1}, X_{n+2}\}) - S_n(\mathbb{X}_2 \{X_1, X_{n+2}\}) - S_n(\mathbb{X}_2 \{X_2, X_{n+1}\}) + S_n(\mathbb{X}_2 \{X_1, X_2\}).
\]

Next, for convenience, we formulate a part of Proposition 3 from Bloznelis and Götze (2001), which we apply in our proofs.

**Proposition BG** Assume that \( n_\star \to \infty \) and \( \tilde{\sigma}_n \geq C_1 > 0 \) for all \( n_\star \). Suppose that \( \delta_2 = o(1) \). Let, for every \( \varepsilon > 0 \),

\[
n_\star \mathbb{E} g_1^2(X_1 \mathbb{1}\{g_1^2(X_1) > \varepsilon\}) = o(1) \text{ as } n_\star \to \infty.
\]

(8)

Then \( \tilde{\sigma}_n^{-1} S_n \) is asymptotically standard normal.

**Proof** [Theorem 1] First, we show that \( \tilde{\sigma}_n \) is bounded as \( n_\star \to \infty \). Then condition (8) is equivalent to condition (5), see Bloznelis and Götze (2001). We show that the inequality

\[
\tilde{\sigma}_n^2 \leq \frac{n_\star}{2} \mathbb{E} (\mathbb{D}_1 S_n)^2
\]

(9)

holds. Here \( \mathbb{D}_1 S_n = S_n(\mathbb{X}_1 \{X_{n+1}\}) - S_n(\mathbb{X}_1 \{X_1\}) \), where \( \mathbb{X}_1 = \{X_1, \ldots, X_{n+1}\} \) is the extended sample. In particular, let us consider Hoeffding’s decomposition of the variance

\[
\tilde{\sigma}_n^2 = \mathbb{V} \text{a}r U_1(S_n) + \cdots + \mathbb{V} \text{a}r U_{n_\star}(S_n),
\]

(10)

as it is presented in Bloznelis and Götze (2001). Here \( U_j(S_n) \) is the \( U \)-statistic of order \( j \). From inequalities (A.22) in Bloznelis and Götze (2001), as \( i = 1 \), we have

\[
\mathbb{V} \text{a}r U_j(S_n) \leq \frac{n_\star}{2} \mathbb{V} \text{a}r U_j(\mathbb{D}_1 S_n), \quad 1 \leq j \leq n_\star,
\]

(11)

where \( \mathbb{V} \text{a}r U_j(\mathbb{D}_1 S_n) \) are the components of variance decomposition (10) for the statistic \( \mathbb{D}_1 S_n \). Next, applying (10), (11) and once again (10) to the statistic \( \mathbb{D}_1 S_n \), and noting that \( \mathbb{E} \mathbb{D}_1 S_n = 0 \), we obtain (9). Let us evaluate \( \mathbb{E} (\mathbb{D}_1 S_n)^2 \). Introduce the events \( \mathcal{R}_{1:ij} = \{R_{1:2} = i, R_{n+1:2} = j\} \), \( 1 \leq i < j \leq n + 1 \), where \( R_{1:2} < R_{n+1:2} \) denote the order statistics of the ranks \( \{R_1, R_{n+1}\} \) of \( \{X_1, X_{n+1}\} \) in the set \( \mathbb{X}_1 \). Here all the ranks \( \{R_1, \ldots, R_{n+1}\} \) of \( \mathbb{X}_1 \) are distinct if, in the case of ties on \( \mathcal{X} \), we order (select
ranks for) tied observations randomly with equal probabilities. The probabilities of
the events are

\[ p_{1;ij} := \mathbb{P}\{\mathcal{R}_{1;ij}\} = \binom{n+1}{2}^{-1}. \]

Since \( J(\cdot) \) is bounded, there exists an absolute constant \( a \) that

\[ \max_{1 \leq p \leq n} |c_p| \leq a \tag{12} \]

for all \( n \). Using Lemma 2 of Čiginas (2012) and (12), we obtain

\[
\mathbb{E}(\mathbb{D}_1S_n)^2 = \sum_{1 \leq i < j \leq n+1} \mathbb{E}\left[ \mathbb{E}(\mathbb{D}_1S_n)^2 \mid \mathcal{R}_{1;ij} \right] p_{1;ij} \\
\leq n^{-1} \sum_{1 \leq i < j \leq n+1} \mathbb{E}\left[ \sum_{p=1}^{R_{1;2}} c_p \Delta_{p:n+1} \right]^2 \mid \mathcal{R}_{1;ij} \] \tag{13}

\[ p_{1;ijlm} := \mathbb{P}\{\mathcal{B}_{1;ijlm} \mid \mathcal{R}_{1;ij}\} = \binom{l-1}{i-1} \binom{m-l-1}{j-i-1} \binom{N-m}{n+1-1} \binom{N}{n+1}. \]

For \( x_1 \leq \cdots \leq x_N \) these probabilities are the same. It follows from an argument similar to Lemma 2.1 of Balakrishnan et al. (2003). Using a generalization of the well-known Vandermonde convolution formula (with an algebraic proof which uses three polynomials instead of two), we obtain

\[
\sum_{1 \leq i < j \leq n+1} p_{1;ijlm} = \binom{N}{n+1} \sum_{s=0}^{N-1-n-s} \binom{l-1}{s} \binom{m-l-1}{t} \binom{N-m}{n-1-s-t} \]

\[ = \binom{N}{n+1}^{-1} \binom{N-2}{n-1}. \]

Then, note that

\[
\text{Var}X_1 = \frac{1}{N^2} \sum_{1 \leq l < m \leq N} (x_m - x_l)^2
\]
and continue with (13):

\[
\mathbb{E} \left( \mathbb{D}_1 S_n \right)^2 \leq a^2 n^{-1} \sum_{1 \leq i < j \leq n+1} \left[ \sum_{1 \leq l < m \leq N} (x_m - x_l)^2 \mathbb{P}(\mathcal{R}_{l:m}^{ij}) \right] p_{1:ij}^{n-1} \left( \begin{array}{c} n+1 \\ 2 \end{array} \right)^{-1} \left( \begin{array}{c} N \\ n+1 \end{array} \right)^{-1} \left( \begin{array}{c} N-2 \\ n-1 \end{array} \right) \sum_{1 \leq l < m \leq N} (x_m - x_l)^2 \\
= 2a^2 n^{-1} \frac{N}{N-1} \text{Var} X_1.
\]

Finally, from (9) we get

\[
\hat{\sigma}_n^2 \leq 2a^2 \text{Var} X_1 = O(1) \quad \text{as} \quad n_* \to \infty.
\]

Second, we show that, under the conditions of the theorem, the condition \(\delta_2(S_n) = o(1)\) as \(n_* \to \infty\) of Proposition BG is satisfied. Similarly, we introduce the events \(\mathcal{R}_{l:m}^{ij} = \{R_{l+1} = i, R_{m+1} = j\}, 1 \leq i < j \leq n+2\), where \(R_{1:4} < R_{2:4} < R_{n+1:4} < R_{n+2:4}\) denote the order statistics of the ranks \(\{R_1, R_2, R_{n+1}, R_{n+2}\}\) of \(\{X_1, X_2, X_{n+1}, X_{n+2}\}\) in the set \(X_2\). Now

\[
p_{2:ij} := \mathbb{P} \{\mathcal{R}_{l:m}^{ij}\} = \binom{i-1}{1} \binom{n+2-j}{1} / \binom{n+2}{4}.
\]

Analogously,

\[
p_{2:ilm} := \mathbb{P} \{\mathcal{B}_{l:m}^{ij} \mid \mathcal{R}_{l:m}^{ij}\} = \binom{l-1}{i-1} \binom{m-l-1}{j-i-1} \binom{N-m}{n+2-j} / \binom{N}{n+2},
\]

where the events \(\mathcal{R}_{l:m}^{ij}\) and \(\mathcal{B}_{l:m}^{ij}\) are independent. Since \(J(\cdot)\) satisfies the Hölder condition of order \(\delta > 1/2\) on \((0, 1)\), we find that

\[
|c_p - c_{p-1}| = J \left( \frac{p}{n+1} \right) - J \left( \frac{p-1}{n+1} \right) \leq B(n+1)^{-\delta}
\]

or

\[
\max_{2 \leq p \leq n} |c_p - c_{p-1}| \leq B(n+1)^{-\delta}, \quad \text{for some} \quad \delta > 1/2.
\]

Using Lemma 2 of Čiginas (2012) and (15), we obtain

\[
\delta_2(S_n) = n_*^2 \sum_{1 \leq i < j \leq n+2} \mathbb{E} \left[ \left( \mathbb{D}_2 S_n \right)^2 \right. \left. \mid \mathcal{R}_{l:m}^{ij}\right] p_{2:ij} \\
\leq n_*^2 n^{-1} \sum_{1 \leq i < j \leq n+2} \mathbb{E} \left[ \left( \sum_{p=R_{1:4}}^{R_{n+1:4}-1} (c_p - c_{p-1}) \Delta_{p:n+2} \right)^2 \right. \left. \mid \mathcal{R}_{l:m}^{ij}\right] p_{2:ij}.
\]
\[ \leq B^2 n^2 n^{-1} (n + 1)^{-2\delta} \sum_{1 \leq i < j \leq n+2} \mathbb{E} \left[ (X_{j:n+2} - X_{i:n+2})^2 \right] R_{2;ij} p_{2;ij} \]

\[ \leq B^2 n^{1-2\delta} \sum_{1 \leq l < m \leq N} \lambda_{2;lm} (x_m - x_l)^2. \]  

(16)

where \( \lambda_{2;lm} = \sum_{1 \leq i < j \leq n+2} p_{2;ij} p_{2;ijlm} \). Taking \( j = i + 1 \) and applying \( \max_{0 \leq u \leq 1} u(1 - u) \leq 1/4 \), for all \( 1 \leq i < j \leq n+2 \), we get the inequalities

\[ p_{2;ij} \leq n^2 \left( \frac{n + 2}{4} \right)^{-1} \frac{i - 1}{n} \left( 1 - \frac{i - 1}{n} \right) \leq \frac{1}{4} n^2 \left( \frac{n + 2}{4} \right)^{-1}. \]

Then, noting that, by the generalized Vandermonde convolution formula,

\[ \sum_{1 \leq i < j \leq n+2} p_{2;ijlm} = \binom{N}{n+2}^{-1} \binom{N - 2}{n}, \]

we obtain, for all \( 1 \leq l < m \leq N \),

\[ \lambda_{2;lm} \leq \frac{1}{4} n^2 \left( \frac{n + 2}{4} \right)^{-1} \left( \frac{N}{n + 2} \right) \left( \frac{N - 2}{n} \right) \leq 24N^{-2}. \]

Finally, it follows from this bound and (16) that

\[ \delta_2(S_n) \leq 24 B^2 n^{1-2\delta} \text{Var} X_1 = o(1) \text{ as } n_* \to \infty. \]

All the conditions of Proposition BG are verified. Thus, the theorem is proved. \( \square \)

**Proof** [Theorem 2] First, we show that condition \( (6) \) with \( (1 - n/N)^{-1} n^{1/2} N^{\delta-1} \to \infty \) implies \( (8) \). Noting that

\[ \sum_{j=1}^{n} (i - 1) \binom{N - i - 1}{n - j} \binom{N - 2}{n - 1}^{-1} = 1, \]

and applying (12) and (6), we get from (4) that

\[ \max_{1 \leq k \leq N} |g_1(x_k)| \leq aC n^{-1/2} N^{-\delta} \max_{1 \leq k \leq N} \sum_{i=1}^{N-1} \left| \mathbb{I} \{ i \geq k \} - \frac{i}{N} \right| = \frac{aC}{2} n^{-1/2} N^{-\delta} (N - 1). \]

Therefore, for fixed \( \varepsilon > 0 \),

\[ \mathbb{P} \{ |g_1(X_1)| > \varepsilon \} \leq \mathbb{E} \{ \max_{1 \leq k \leq N} |g_1(x_k)| > \varepsilon \} \leq \mathbb{E} \left\{ \frac{aC}{2} n^{-1/2} N^{1-\delta} > \varepsilon \right\}. \]
We obtain from here and from (3.9) of Čiginas (2012) that

\[
\frac{n_\ast a_1 g_1^2(X_1) \mathbb{I}[|g_1(X_1)| > \varepsilon]}{2} \leq \mathbb{I}\left\{ \frac{aC}{2} n^{-1/2} N^{1-\delta} > \varepsilon \right\} n_\ast a_1 g_1^2(X_1) 
\leq 4a^2 \mathbb{I}\left\{ n^{-1/2} N^{1-\delta} > \frac{2\varepsilon}{aC} \right\} \mathbb{E}|X_1|^2.
\]

Condition (8) is proved.

Second, as in the proof of Theorem 1, we verify the condition \( \delta_2(S_n) = o(1) \) as \( n_\ast \to \infty \). Write, for short, \( s = [t_1 n] + 1 \) and \( t = [t_2 n] \). Similarly, applying Lemma 2 of Čiginas (2012), we obtain

\[
\delta_2(S_n) \leq \frac{n_\ast^2}{(t-s+1)^2} \sum_{1 \leq i < j \leq n+2} p_{2;ij} \frac{A_{ij}^2}{E_{ij}}, \quad (17)
\]

where \( p_{2;ij} \) is given by (14) and

\[
A_{ij}(s, t) = \sum_{p=i}^{j-1} (\tilde{c}_p - \tilde{c}_{p-1}) \Delta_{p,n+2} \quad \text{with} \quad \tilde{c}_p = \mathbb{I}\{p \leq s \leq p \leq t\}.
\]

We can assume, without loss of generality, that \( n > (t_2 - t_1)^{-1} \). Then we have \( s < t \). It also follows from the inequality \([t_2 n] - [t_1 n] \geq t_2 n - 1 - t_1 n \) and from the same assumption that, for some constant \( C_1 > 0 \),

\[
\frac{n^2}{(t-s+1)^2} \leq \left( t_2 - t_1 - \frac{1}{n} \right)^{-2} \leq C_1. \quad (18)
\]

Let us decompose \( \mathcal{I} = \{(i, j) : 2 \leq i < j \leq n + 1\} \), for fixed \( s < t \), into mutually disjoint subsets

\[
\mathcal{I}_1 = \{(i, j) : t + 2 \leq i < j \leq n + 1\},
\mathcal{I}_2 = \{(i, j) : 2 \leq i < j \leq s\},
\mathcal{I}_3 = \{(i, j) : s + 1 \leq i < j \leq t + 1\},
\mathcal{I}_4 = \{(i, j) : s + 1 \leq i \leq t + 1, t + 2 \leq j \leq n + 1\},
\mathcal{I}_5 = \{(i, j) : 2 \leq i \leq s, s + 1 \leq j \leq t + 1\},
\mathcal{I}_6 = \{(i, j) : 2 \leq i \leq s, t + 2 \leq j \leq n + 1\},
\]

such that \( \mathcal{I} = \mathcal{I}_1 \cup \cdots \cup \mathcal{I}_6 \). Then we get

\[
A_{ij}(s, t) = \begin{cases} 
0 & \text{if } (i, j) \in \mathcal{I}_1 \cup \mathcal{I}_2 \cup \mathcal{I}_3, \\
-\tilde{c}_t \Delta_{t+1:n+2} & \text{if } (i, j) \in \mathcal{I}_4, \\
\tilde{c}_s \Delta_{s:n+2} & \text{if } (i, j) \in \mathcal{I}_5, \\
\tilde{c}_s \Delta_{s:n+2} - \tilde{c}_t \Delta_{t+1:n+2} & \text{if } (i, j) \in \mathcal{I}_6.
\end{cases}
\]
Now, by collecting the terms of the sum $\sum_{i < j}$ with the same value of $E A_{ij}^2(s, t)$ in (17), applying $E(\Delta_{t+1,n+2} - \Delta_{s,n+2})^2 \leq E \Delta_{t+1,n+2}^2 + E \Delta_{s,n+2}^2$, and then collecting terms with $E \Delta_{t+1,n+2}^2$ and $E \Delta_{s,n+2}^2$, and also invoking inequality (18), we obtain

$$\delta_2(S_n) \leq C_1 n^2 n^{-1} \binom{n + 2}{4}^{-1} \left[ \frac{(t + 1)}{2} \binom{n - t + 1}{2} \right] E \Delta_{t+1,n+2}^2$$

$$+ \binom{s}{2} \left[ (n - t + 1)^2 - \binom{n - s + 1}{2} \right] E \Delta_{s,n+2}^2 \right].$$

(19)

By applying the simple inequality $\binom{u}{v} \leq u^v / v!$, we derive

$$\left( \frac{t + 1}{2} \right) \binom{n - t + 1}{2} \leq \frac{(n + 2)^4}{4} \left[ \frac{t + 1}{n + 2} \left( 1 - \frac{t + 1}{n + 2} \right) \right]^2 \leq \frac{(n + 2)^4}{64}. \quad (20)$$

Taking $s = t$, very similarly we get

$$\left( \frac{s}{2} \right) (n - t + 1)^2 \leq \frac{(n + 1)^4}{2} \left[ \frac{t}{n + 1} \left( 1 - \frac{t}{n + 1} \right) \right]^2 \leq \frac{(n + 1)^4}{32}. \quad (21)$$

Next, it is easy to calculate (invoking Lemma 2.1 of Balakrishnan et al. (2003)) that, for $1 \leq p \leq n + 1$,

$$E \Delta_{p,n+2}^2 = \binom{N}{n+2}^{-1} \sum_{l \leq m \leq N} (l - 1) \binom{m - l - 1}{0} \binom{N - m}{n + 1 - p} (x_m - x_l)^2.$$

Then, using (6), we obtain

$$E \Delta_{p,n+2}^2 \leq \frac{C^2}{N^{2\delta}} \binom{N}{n+2}^{-1} \sum_{l \leq m \leq N} \binom{m - l - 1}{p - 1} \binom{N - m}{n + 1 - p}$$

$$= \frac{C^2}{N^{2\delta}} \frac{(N + 1)(2N - n)}{(n + 3)(n + 4)}. \quad (22)$$

Here the last equality is obtained by applying simple binomial identities $\binom{m}{k} = \binom{m - 1}{k - 1}, 1 \leq k \leq m,$

$$\sum_{j=k}^{m} \binom{j}{k} = \binom{m + 1}{k + 1}, \quad 0 \leq k \leq m,$$

and

$$\sum_{p=0}^{m} \binom{p}{j} \binom{m - p}{k - j} = \binom{m + 1}{k + 1}, \quad 0 \leq j \leq k \leq m.$$
Indeed, for instance,
\[
\sum_{1 \leq l < m \leq N} m^2 \binom{l-1}{p-1} \binom{N-m}{n+1-p} = \sum_{m=2}^{N} \left\{ \sum_{l=1}^{m-1} \binom{l-1}{p-1} \right\} m^2 \binom{N-m}{n+1-p} \\
= \sum_{m=2}^{N} m^2 \binom{m-1}{p} \binom{N-m}{n+1-p} = (p+1) \sum_{m=2}^{N} m \binom{m}{p+1} \binom{N-m}{n+1-p} \\
= (p+1)(p+2) \sum_{m=2}^{N} \binom{m+1}{p+2} \binom{N-m}{n+1-p} \\
-(p+1) \sum_{m=2}^{N} \binom{m}{p+1} \binom{N-m}{n+1-p} \\
= (p+1)(p+2) \binom{N+2}{n+4} - (p+1) \binom{N+1}{n+3},
\]
and so on. Finally, applying (20), (21) and (22), and \( n_* \leq 2n(1 - n/N) \), we continue with (19):
\[
\delta_2(S_n) \leq C_1 n_*^2 n^{-1} \left( \frac{n+2}{4} \right)^{-1} C_2^2 \frac{(N+1)(2N-n)}{(n+3)(n+4)} \left[ \frac{(n+2)^4}{64} + \frac{(n+1)^4}{32} \right] \\
\leq C_2 \left( 1 - \frac{n}{N} \right)^2 \frac{N^2(1-\delta)}{n} = o(1) \quad \text{as} \quad n_* \to \infty,
\]
for some constant \( C_2 > 0 \).

Third, one can show, with the help of (3) and (4), that \( \text{Var}U_1 \) is bounded away from zero. Therefore, the condition \( \sigma_n \geq C_1 > 0 \) of Proposition BG is satisfied in the case of the trimmed means. The theorem is proved. \( \square \)

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