A TEMPERLEY-LIEB ANALOGUE FOR THE BMW ALGEBRA

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To Toshiaki Shoji on his 60th birthday

Abstract. The Temperley-Lieb algebra may be thought of as a quotient of the Hecke algebra of type A, acting on tensor space as the commutant of the usual action of quantum $\mathfrak{sl}_2$ on $(\mathbb{C}(q)^2)^n$. We define and study a quotient of the Birman-Wenzl-Murakami algebra, which plays an analogous role for the 3-dimensional representation of quantum $\mathfrak{sl}_2$. In the course of the discussion we prove some general results about the radical of a cellular algebra, which may be of independent interest.

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1. Introduction

Let \( \mathfrak{g} \) be a finite dimensional simple complex Lie algebra, \( \mathcal{U}(\mathfrak{g}) \) its universal enveloping algebra, and \( \mathcal{U}_q = \mathcal{U}_q(\mathfrak{g}) \) its Drinfeld-Jimbo quantisation, the latter being an algebra over the function field \( K := \mathbb{C}(q^\mathbb{Z}) \), \( q \) an indeterminate. As explained in [LZ, §6], the finite dimensional \( \mathfrak{g} \)-modules correspond bijectively to the “type \((1, 1, \ldots, 1)\) modules” of \( \mathcal{U}_q \), with corresponding modules having the same character (which is an element of the weight lattice of \( \mathfrak{g} \)).

Let \( V \) be an irreducible finite dimensional \( \mathfrak{g} \)-module, and \( V_q \) its \( q \)-analogue (the corresponding \( \mathcal{U}_q \)-module). It is known that there is an action of the \( r \)-string braid group \( B_r \) on the tensor space \( V_q^\otimes r \) which commutes with the action of \( \mathcal{U}_q \), and in [LZ] a sufficient condition was given in order that \( B_r \) span \( \text{End}_{\mathcal{U}_q} V_q^\otimes r \). This condition, that \( V \) be “strongly multiplicity free” (see [LZ, §3]) was shown to be satisfied when \( V \) is any irreducible module for \( \mathfrak{sl}_2 \).

When \( V = V(1) \), the (natural) two-dimensional \( \mathfrak{sl}_2 \)-module, it is known that the two braid generators satisfy a quadratic relation, and together with the quantum analogue of the relation which expresses the vanishing of alternating tensors of rank \( \geq 3 \) these relations give a presentation of the Temperley-Lieb algebra (see §2 below and [GL03]). In this work we study the algebra which occurs when we start with the three-dimensional \( \mathfrak{sl}_2 \)-module \( V(2) \). It follows from our earlier work that this algebra is a quotient of the BMW (Birman-Wenzl-Murakami) algebra, and we give a presentation for a quotient of the latter, whose semisimple quotient specialises generically to the endomorphism algebra of tensor space. One of the major differences between our case and the classical Temperley-Lieb case is that neither the BMW algebra we start with nor its Brauer specialisation at \( q = 1 \) is semisimple.

This work makes extensive use of the cellular structure of the BMW algebra and its “classical” specialisation, the Brauer algebra. We use specialisation arguments to relate the quantum and classical \((q = 1)\) situations. Because of this, and also because we have in mind applications of this work to cases where the modules concerned may not be semisimple, we shall work in integral lattices for the modules we encounter, and with integral forms of the endomorphism algebras. Such constructions are closely related to the “Lusztig form” of the irreducible \( \mathcal{U}_q \)-modules.

In §5 we prove some general results concerning the radical of a cellular algebra. These characterise it quite explicitly, and give a general criterion for an ideal to contain the radical. These results may have some interest, independently of the rest of this work.

2. Dimensions

Let \( \mathfrak{g} = \mathfrak{sl}_2 \) and write \( V(d) \) for the \((d + 1)\)-dimensional irreducible representation of \( \mathfrak{g} \) on homogeneous polynomials of degree \( d \) in two variables, say \( x \) and \( y \). The standard generators \( e, f, h \) of \( \mathfrak{sl}_2 \) act as \( x \frac{\partial}{\partial y}, y \frac{\partial}{\partial x}, x \frac{\partial}{\partial x} - y \frac{\partial}{\partial y} \) respectively.
2.1. Dimension of the endomorphism algebra. We start by giving a (well known) recursive formula for \( \dim_{\mathbb{C}} \text{End}_{\mathfrak{g}} V(d)^{\otimes r} = \dim_{\mathbb{C}(z)} \text{End}_{\mathfrak{gl}_2} V(d)^{\otimes r} \). From the classical Clebsch-Gordan formula, we have for \( n \geq m \),

\[
V(n) \otimes V(m) \cong V(n + m) \oplus V(n + m - 2) \oplus \cdots \oplus V(n - m).
\]

Define the coefficients \( m^r_n(i) \) by

\[
V(n)^{\otimes r} \cong \bigoplus_{i \equiv rn(\mod 2)} m^r_n(i) V(i),
\]

and the dimension

\[
d(n, r) := \dim \text{End}_{\mathfrak{g}}(V(n)^{\otimes r}).
\]

Then \( d(n, r) = \sum_{i \equiv rn(\mod 2)} m^r_n(i)^2 \). Since all the modules \( V(n) \) are self dual, it also follows from Schur’s lemma that \((V(n) \otimes V(n), V(n) \otimes V(n)) = (V(n)^{\otimes 2r}, V(0)) \otimes \mathfrak{g}\), where \(( , )_\mathfrak{g}\) denotes multiplicity. Thus

\[
d(n, r) = \sum_{i \equiv rn(\mod 2)} m^r_n(i)^2 = m^r_{n}(0).
\]

Using (2.1), it is a straightforward combinatorial exercise to prove the following recursive formula. Let \( x \) be an indeterminate and write \([n]_x = \frac{x^n - x^{-n}}{x - x^{-1}} \in \mathbb{Z}[x \pm 1]\) for the “\( x \)-analogue" of \( n \in \mathbb{Z} \). Then define the integers \( a^r_n(k) \) by \([n + 1]^r_x \sum_k a^r_n(k) x^k \), where the sum is over \( k \) such that \(-nr \leq k \leq nr \) and \( k \equiv nr \mod 2 \). Finally, define the integers \( b^r_n(k) \) \((-nr \leq k \leq nr \) \) by downward recursion on \( k \) as follows.

Set \( b^r_n(nr) = a^r_n(nr) = 1 \), and then \( \sum_{i \geq k} b^r_n(i) = a^r_n(k) \). Equivalently, \( b^r_n(k) = a^r_n(k) - a^r_n(k + 1) \).

**Proposition 2.1.** We have, for all \( n, r \) and \( k \),

\[
m^r_n(k) = b^r_n(k).
\]

2.2. The case \( n = 1 \). In this case one verifies easily that

\[
a^r_1(k) = \begin{cases} 
\left(\frac{r}{k}\right) & \text{if } k \equiv r \pmod{2} \\
0 & \text{otherwise}
\end{cases}
\]

It is then straightforward to compute that

\[
m^r_1(k) = \left(\frac{r}{r+k}\right) \frac{2(k + 1)}{r + k + 2}.
\]

In particular,

\[
d(1, r) = \dim \text{End}_{\mathfrak{sl}_2} V(1)^{\otimes r} = m^r_1(0) = \frac{1}{r + 1} \left(\begin{array}{c} 2r \\ r \end{array}\right).
\]
2.3. The case $n = 2$. With the above notation, we have

$$a_2^r(2\ell) = \sum_{k \geq \frac{L + r}{2}}^r \binom{r!}{2k - (\ell + r)!}((\ell + r - k)!)(r - k)!$$

In this case, we have from Proposition 2.1 that $m_2^r(2\ell) = a_2^r(2\ell) - a_2^r(2\ell + 2).$ This relation easily yields the following formula for $d(2, r).$

$$d(2, r) = \dim \text{End}_{U_q} V(2)_{\otimes r} = m_2^r(0)$$

$$(2.5) = \binom{2r}{r} + \sum_{p=0}^{r-1} \binom{2r}{2p} \binom{3p - 2r + 1}{p + 1}.$$

Thus for $r = 1, 2, 3, 4, 5$ the respective dimensions are 1, 3, 15, 91, and 603.

3. Some generators and relations for $\text{End}_{U_q}(V(n)_{\otimes r})$

In this section, we review the results of [LZ] which pertain to the structure of the endomorphism algebras we wish to study.

3.1. The general case. Recall that with $\mathfrak{g}$ and $U_q$ as above, given any $U_q$-module $V_q$, there is an operator $\hat{R} \in \text{End}_{U_q}(V_q \otimes V_q)$, known as an “$R$-matrix” (see [LZ §6.2]). Denote by $R_i$ the element $id_{V_q}^{\otimes i-1} \otimes \hat{R} \otimes id_{V_q}^{\otimes r-i-1}$ of $\text{End}_{U_q} V_{\otimes r}$ ($i = 1, \ldots, r - 1$). It is well known that the $R_i$ satisfy the braid relations:

$$(3.1) \quad R_i R_j = R_j R_i \text{ if } |i - j| \geq 2 \quad R_i R_{i+1} R_i = R_{i+1} R_i R_{i+1} \text{ for } 1 \leq i \leq r - 1.$$

Moreover if $V_q$ is strongly multiplicity free (for the definition see [LZ §7]), it follows from [LZ Theorem 7.5] that the endomorphisms $R_i$ generate $\text{End}_{U_q} (V_q^{\otimes r})$. Assume henceforth that $V_q$ is strongly multiplicity free. The following facts may be found in [LZ §§3, 7].

Firstly, $V_q = L_{\lambda_0}$, the unique irreducible module for $U_q$ with highest weight $\lambda_0$, and $V_q \otimes V_q$ is multiplicity free as $U_q$-module. Write

$$V_q \otimes V_q \cong \bigoplus_{\mu \in \mathcal{P}(\lambda_0)} L_{\mu},$$

where $\mathcal{P}(\lambda_0)$ is the relevant set of dominant weights of $\mathfrak{g}$, and $L_{\mu}$ is the irreducible $U_q$-module with highest weight $\mu$. Let $P(\mu)$ be the projection : $V_q \otimes V_q \longrightarrow L_{\mu}$. These projections clearly span $\text{End}_{U_q} (V_q \otimes V_q)$, and we have (see [LZ (6.10)])

$$(3.2) \quad \hat{R} = \sum_{\mu \in \mathcal{P}(\lambda_0)} \varepsilon(\mu) q^{(\chi_\mu(C) - 2\chi_{\lambda_0}(C))} P(\mu),$$

where $C \in U(\mathfrak{g})$ is the classical quadratic Casimir element, $\chi_\lambda(C)$ is the scalar through which $C$ acts on the (classical) irreducible $U(\mathfrak{g})$-module with highest weight $\lambda$, and $\varepsilon(\mu)$ is the sign occurring in the action of the interchange $s$ on the classical limit $V \otimes V$ of $V_q \otimes V_q$ as $q \rightarrow 1$. 
3.2. Relations for the case $g = \mathfrak{sl}_2$. It was proved in [LZ] that all irreducible modules for $\mathcal{U}_q(\mathfrak{sl}_2)$ are strongly multiplicity free. In this subsection, we make explicit the relations above when $g = \mathfrak{sl}_2$ and $V_q = V(n)_q$. These statements are all well known. As above, we think of $V(n)$ as the space $\mathbb{C}[x,y]_n$ of homogeneous polynomials of degree $n$. This has highest weight $n$, with $h = x \frac{\partial}{\partial x} - y \frac{\partial}{\partial y}$ acting on $x^n$ as highest weight vector. We have seen that

$$V(n) \otimes V(n) = \oplus_{\ell=0}^n V(2\ell).$$

It is easy to compute the highest weight vectors in the summands of (3.3), which leads to

3.1. *The endomorphism* $s : v \otimes w \mapsto w \otimes v$ of $V(n) \otimes V(n)$ *acts on* $V(2\ell)$ *as* $(-1)^{n+\ell}$. *That is, in the notation of [3.1] $\epsilon(2\ell) = (-1)^{d+\ell}$."

Next, if $\theta = x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y}$ is the Euler form, the Casimir $C$ is given by $C = \theta + \frac{1}{2} \theta^2$. It follows that

3.2. *The Casimir* $C$ *acts on* $V(n)$ *as multiplication by* $\chi_n(C) = \frac{1}{2} n(n + 2)$.

Applying the statements in §3.1 here, we obtain

**Proposition 3.3.** Let $V$ be the irreducible $\mathfrak{sl}_2$ module $V(n)_q$, with $V(n)_q$ its quantum analogue. Then $V(n)_q \otimes V(n)_q = \oplus_{\ell=0}^n V(2\ell)_q$, and if $\hat{R}$ is the $R$-matrix acting on $V(n)_q \otimes V(n)_q$, then

$$\hat{R} = \sum_{\ell=0}^n (-1)^{n+\ell} q^{\frac{1}{2}((\ell(2\ell+2)-n(n+2))} P(2\ell),$$

where $P(2\ell)$ is the projection to the component $V(2\ell)_q$.

Now let $E_q(n,r) := \text{End}_{\mathcal{U}_q}(V(n)_q \otimes r)_q$, and let $R_i \in E_q(n,r)$ be the endomorphism defined above ($i = 1, \ldots, r-1$). We have seen that the $R_i$ generate $E_q(n,r)$, and that they satisfy the relations (3.1). From the relation (3.4), we deduce that for all $i$,

$$\prod_{\ell=0}^n \left( R_i - (-1)^{n+\ell} q^{\frac{1}{2}(\ell(2\ell+2)-n(n+2))} \right) = 0.$$ 

Writing $T_i := (-1)^n q^{\frac{1}{2}n(n+2)} R_i$, the above relation simplifies to

$$\prod_{\ell=0}^n (T_i - (-1)^{\ell} q^{\ell(\ell+1)}) = 0.$$ 

Now the relations (3.1) and (3.3) do not provide a presentation for $E_q(n,r)$, and it is one of our objectives to determine further relations among the $R_i$. These will suffice to present $E_q(n,r)$ as an associative algebra generated by the $R_i$ only in some special cases.
3.3. Basic facts about $\mathcal{U}_q(\mathfrak{sl}_2)$. It will be convenient to establish notation for the discussion below by recalling the following basic facts. The algebra $\mathcal{U}_q: = \mathcal{U}_q(\mathfrak{sl}_2)$ has generators $e, f$ and $k^{\pm 1}$, with relations $kek^{-1} = q^2e$, $kfk^{-1} = q^{-2}f$, and $ef - fe = \frac{k-q^{-1}}{q-q^{-1}}$. The weight lattice $P$ is identified with $\mathbb{Z}$, and for $\lambda, \mu \in P$, $(\lambda, \mu) = \frac{1}{2} \lambda \mu$. In general, if $M$ is a $\mathcal{U}_q$-module with weights $\lambda_1, \ldots, \lambda_d$ $(d = \dim M)$, then the quantum dimension of (i.e. quantum trace of the identity on) $M$ is $\dim_q M = \sum_{i=1}^d q^{-2(\rho, \lambda_i)}$, where $2\rho$ is the sum of the positive roots, in this case 2. Hence $\dim_q V(n) = q^n + q^{n-2} + \cdots + q^{-n} = [n + 1]_q$. The comultiplication is given by $\Delta(e) = e \otimes k + 1 \otimes e$, $\Delta(f) = f \otimes 1 + k^{-1} \otimes f$, $\Delta(k) = k \otimes k$.

3.4. The case $n = 1$: the Temperley-Lieb algebra. In this case the relation (3.5) reads

\[(R_i + q^{\frac{1}{2}})(R_i - q^{\frac{1}{2}}) = 0.\]

Renormalising by setting $T_i = q^{\frac{1}{2}}R_i$ $(i = 1, 2, \ldots, r - 1)$ we obtain

\[(T_i + q^{-1})(T_i - q) = 0.\]

Now it is well known that the associative $\mathbb{C}(q)$-algebra with generators $T_1, \ldots, T_{r-1}$ and relations (3.1) and (3.5) is the Hecke algebra $H_r(q)$ of type $A_{r-1}$ with parameter $q$. The algebra $H_r(q)$ has a $\mathbb{C}(q)$-basis $\{T_w \mid w \in \text{Sym}_r\}$, and we may therefore speak of the action of $T_w(= q^{\ell(w)}R_w)$ (where $\ell(w)$ is the usual length function in $\text{Sym}_r$) on $V^{\otimes r}$.

Evidently we have a surjection $\phi : H_r(q) \rightarrow E_q(1, r)$, and we shall determine $\ker \phi$. The next statement is just the quantum analogue of the fact that there are no non-zero alternating tensors in $V^{\otimes 3}$ if $V$ is 2-dimensional.

Lemma 3.4. Let $V = V(1)_q$ be the two-dimensional irreducible $\mathcal{U}_q(\mathfrak{sl}_2)$-module. In the notation above, let $E(\varepsilon) = \sum_{w \in \text{Sym}_3} (-q)^{-\ell(w)}T_w \in H_3(q)$. Then $E(\varepsilon)$ acts as zero on $V^{\otimes 3}$.

Proof. Let $v_1 \in V$ have weight 1, and take $v_2 = fv_1$ as the complementary basis element, which has weight $-1$. We have $V \otimes V \cong L_0 \oplus L_1$, where $L_0$ is the trivial $\mathcal{U}_q$ module, and $L_1$ is the irreducible $\mathcal{U}_q$-module of dimension 3. By computing the action of $\Delta(e)$ and $\Delta(f)$, one sees easily that $L_0$ is spanned by $qv_1 \otimes v_2 - v_2 \otimes v_1$, and $L_1$ has basis $v_1 \otimes v_1$, $v_2 \otimes v_2$, and $v_2 \otimes v_1 + q^{-1}v_1 \otimes v_2$.

Now $T = q^{\frac{1}{2}}R$ acts on $L_0$ as $-q^{-1}$ and on $L_1$ as $q$. If $P(0)$, $P(1)$ are the projections of $V \otimes V$ onto $L_0$, $L_1$ respectively, this implies that $P(0) = -\frac{1}{q+q^{-1}}(T - q)$, and $P(1) = \frac{1}{q+q^{-1}}(T + q^{-1})$. As above, write $P_i(j)$ for the projection of $V^{\otimes r}$ obtained by applying $P(j)$ to the $(i, i + 1)$ factors of $V^{\otimes r}$ $(i = 1, \ldots, r - 1; \ j = 0, 1)$. Then $P_i(1) = -\frac{1}{q+q^{-1}}(T_i + q^{-1})$, etc.

Next observe that since $(T_i + q^{-1})E(\varepsilon) = E(\varepsilon)(T_i + q^{-1}) = 0$ for $i = 1, 2$, we have $P_i(1)E(\varepsilon) = E(\varepsilon)P_i(1) = 0$ for $i = 1, 2$. Since $P_i(0) + P_i(1) = \text{id}_{V^{\otimes 3}}$, it follows that $E(\varepsilon)V^{\otimes 3} \subseteq P_1(0)V^{\otimes 3} \cap P_2(0)V^{\otimes 3}$ \[= L_0 \otimes V \cap V \otimes L_0.\]

Now $L_0 \otimes V$ and $V \otimes L_0$ are two irreducible submodules of $V^{\otimes 3}$. Hence they either coincide or have zero intersection. But $L_0 \otimes V$ has basis $\{(qv_1 \otimes v_2 - v_2 \otimes v_1) \otimes v_i \mid i = 1, 2\}$. Hence $qv_1 \otimes v_1 \otimes v_2 - v_1 \otimes v_2 \otimes v_1$ is in $V \otimes L_0$, but not in $L_0 \otimes V$. 


It follows that $E(\varepsilon)V^{\otimes 3} = 0$. \hfill \Box

This enables us to prove

**Theorem 3.5.** Let $V$ be the two-dimensional irreducible representation of $\mathcal{U}_q(\mathfrak{sl}_2)$. For each integer $r \geq 2$, let $E_q(1, r) = \text{End}_{L_q(V)}(V^{\otimes r})$. Then $E_q(1, r)$ is isomorphic to the Temperley-Lieb algebra $TL_r(q)$ (cf. [GL04, p. 144]).

**Proof.** We have seen above that $E_q(1, r)$ is generated as $\mathcal{K}$-algebra by the endomorphisms $T_1, \ldots, T_{r-1}$. By (3.7), they generate a quotient of the Hecke algebra $H_r(q)$ and by Lemma 3.4 that quotient is actually a quotient of $H_r(q)/I$, where $I$ is the ideal generated by the element $E(\varepsilon)$ defined above. But up to the automorphism $T_i \mapsto T_i' = -T_i + q - q^{-1}$, this is precisely the idempotent $E_1$ of [GL04, 4.9)]. It follows (see, e.g. [GL04, 4.17]) that the quotient $H_r(q)/I$ is isomorphic to $TL_r(q)$.

But this latter algebra is well known (cf. [GL96] or [GL03]) to have dimension $\frac{1}{r+1}\binom{2r}{r}$, which by (2.3) above is the dimension of $E_q(1, r)$. The theorem follows. \hfill \Box

It follows that in this case, the endomorphism algebra has a well understood cellular structure (see [GL96]).

4. The case $n = 2$: action of the BMW algebra

In this section we take $V$ to be $V_q(2)$, the three-dimensional irreducible module for $\mathcal{U}_q(\mathfrak{sl}_2)$. In accord with the notation of the last section, we write $E_q(2, r) := \text{End}_{L_q(V)}(V_q(2)^{\otimes r})$.

4.1. The setup, and some relations. In this situation, $V \otimes V \cong V_0 \oplus V_1 \oplus V_2$, where $V_0$ is the trivial module, and $V_1, V_2$ are respectively the three and five dimensional irreducible modules. As above, we therefore have operators $R_i, P_i(j)$ ($i = 1, \ldots, r-1; j = 0, 1, 2$) on $V^{\otimes r}$, where $P_i(j)$ is the projection $V \otimes V \rightarrow V_j$, applied to the $(i, i+1)$ factors of $V^{\otimes r}$, appropriately tensored with the identity endomorphism of $V$.

The $R_i$ here satisfy the braid relations (as they always do), and the cubic relation

$$(R_i - q^{-4})(R_i + q^{-2})(R_i - q^2) = 0.$$  

Now if $L$ is any strongly multiplicity free $\mathcal{U}_q$-module such that the trivial module $L_0$ is a summand of $L \otimes L$, and $f \in \text{End}_{L_q}(L \otimes L)$, then writing $P_i(0)$ for the projection $L \otimes L \rightarrow L_0$, applied to the $(i, i+1)$ components of $L^{\otimes r}$, we have

$$P_i(0)f_{i\pm 1}P_i(0) = \frac{1}{(\dim_q(L))^2}\tau_{q, L \otimes L}(f)P_i(0),$$

where $\tau_{q, M}$ denotes the quantum trace of an endomorphism of the $\mathcal{U}_q$-module $M$, and $f_i$ is $f$ applied to the $(i, i+1)$ components on $L^{\otimes r}$.

To apply (4.2) to the case when $f = \tilde{R}$, we shall use

$$\tau_{q, V(n)^{\otimes 2}}(\tilde{R}) = q^{\frac{1}{2}n(n+2)}[n+1]_q$$

and $f_{i\pm 1}$ is $f$ applied to the $(i, i+1)$ components on $L^{\otimes r}$.

To apply (4.2) to the case when $f = \tilde{R}$, we shall use

$$\tau_{q, V(n)^{\otimes 2}}(\tilde{R}) = q^{\frac{1}{2}n(n+2)}[n+1]_q$$

This may be proved in several different ways, including the use of the explicit expression given in Proposition 3.3 for $\tilde{R}$. 

Applying (4.2) to the cases \( f = R \) and \( f = P(0) \) in turn, we obtain for our case \( L = V_q(2) \),
\[
P_i(0)P_{i \pm 1}(0) = q^4[3]_{q^{-1}}P_i(0) \quad \text{for } i = 1, \ldots, r.
\]
and
\[
P_i(0)P_{i \pm 1}(j)P_i(0) = [2j + 1]_{q^3}P_i(0) \quad \text{for } i = 1, \ldots, r \quad \text{and } j = 0, 1, 2.
\]

In the equations (4.4), (4.5), the applicable range of values for \( i \) is understood to be such that \( P_{k(i)}(j) \) and \( R_{k(i)} \) makes sense for the relevant functions \( k(i) \) of \( i \).

Since \( R \) acts on \( V_0, V_1 \) and \( V_2 \) respectively as \( q^{-4}, -q^{-2} \) and \( q^2 \), we also have
\[
P_i(0) = \frac{q^8(R_i + q^{-2})(R_i - q^2)}{(1 + q^2)(1 - q^6)}.
\]

4.2. The BMW algebra. We recall some basic facts concerning the BMW algebra, suitably adapted to our context. Let \( K = \mathbb{C}(q^{1/2}) \) as above, and let \( \mathcal{A} \) be the ring \( \mathbb{C}[y^\pm 1, z] \), where \( y, z \) are indeterminates.

The BMW algebra \( \text{BMW}_{r}(y, z) \) over \( \mathcal{A} \) is the associative \( \mathcal{A} \)-algebra with generators \( g_1, \ldots, g_{r-1} \) and \( e_1, \ldots, e_{r-1} \), subject to the following relations:

The braid relations for the \( g_i \):
\[
g_i g_j = g_j g_i \quad \text{if } |i - j| \geq 2
\]
\[
g_i g_{i+1} g_i = g_{i+1} g_i g_{i+1} \quad \text{for } 1 \leq i \leq r - 1;
\]
The Kauffman skein relations:
\[
g_i - g_i^{-1} = z(1 - e_i) \quad \text{for all } i;
\]
The de-looping relations:
\[
g_i e_i = e_i g_i = y e_i;
\]
\[
e_i g_i^{\pm 1} e_i = y^{\mp 1} e_i;
\]
\[
e_i g_i^{\pm 1} e_i = y^{\mp 1} e_i.
\]
The next four relations are easy consequences of the previous three.

\[
ee_i e_i^{\pm 1} e_i = e_i;
\]
\[
(g_i - y)(g_i^2 - zg_i - 1) = 0;
\]
\[
ze_i^2 = (z + y^{-1} - y)e_i;
\]
\[
y e_i = g_i^2 - zg_i - 1.
\]

It is easy to show that \( \text{BMW}_{r}(y, z) \) may be defined using the relations (4.7), (4.9), (4.11) and (4.13) instead of (4.7), (4.8) and (4.9), i.e. that (4.8) is a consequence of (4.11) and (4.13).

We shall require a particular specialisation of \( \text{BMW}_{r}(y, z) \) to a subring \( \mathcal{A}_q \) of \( K \), which is defined as follows. Let \( \mathcal{S} \) be the multiplicative subset of \( \mathbb{C}[q, q^{-1}] \) generated by \( [2]_q, [3]_q \) and \( [3]_q - 1 \). Let \( \mathcal{A}_q := \mathbb{C}[q, q^{-1}] \mathcal{S} := \mathbb{C}[q, q^{-1}, [2]_q^{-1}, [3]_q^{-1}, (q^2 + q^{-2})^{-1}] \) be the localisation of \( \mathbb{C}[q, q^{-1}] \) at \( \mathcal{S} \).

Now let \( \psi : \mathbb{C}[y^\pm 1, z] \rightarrow \mathcal{A}_q \) be the homomorphism defined by \( y \mapsto q^{-4} \), \( z \mapsto q^2 - q^{-2} \). Then \( \psi \) makes \( \mathcal{A}_q \) into an \( \mathcal{A} \)-module, and the specialisation \( \text{BMW}_{r}(q) := \mathcal{A}_q \otimes_{\mathcal{A}} \text{BMW}_{r}(y, z) \) is the \( \mathcal{A}_q \)-algebra with generators which we denote, by abuse
of notation, $g_i^{±1}, e_i$ ($i = 1, \ldots, r - 1$) and relations (4.14) below, with the relations (4.15) being consequences of (4.14).

\begin{align*}
g_i g_j & = g_j g_i \text{ if } |i - j| \geq 2 \\
g_i g_{i+1} g_i & = g_{i+1} g_i g_{i+1} \text{ for } 1 \leq i \leq r - 1 \\
g_i - g_i^{-1} & = (q^2 - q^{-2})(1 - e_i) \text{ for all } i \\
g_i e_i & = e_i g_i = q^{-4} e_i \\
e_i g_i^{±1} e_i & = q^{±4} e_i \\
e_i g_i^{±1} e_i & = q^{±4} e_i.
\end{align*}

(4.14)

\begin{align*}
e_i e_i^{±1} e_i & = e_i \\
g_i - q^2)(g_i + q^{-2}) & = -q^{-4}(q^2 - q^{-2})e_i \\
(g_i - q^{-4})(g_i - q^2)(g_i + q^{-2}) & = 0 \\
e_i^2 & = (q^2 + 1 + q^{-2})e_i.
\end{align*}

(4.15)

We shall be concerned with the following two specialisations of $BMW_r(q)$.

**Definition 4.1.** Let $\phi_q : A_q \longrightarrow K = \mathbb{C}(q^2)$ be the inclusion map, and let $\phi_1 : BMW_r(q) \longrightarrow \mathbb{C}$ be the $\mathbb{C}$-algebra homomorphism defined by $q \mapsto 1$. Define the specialisations $BMW_r(K) := K \otimes_{\phi_q} BMW_r(q)$, and $BMW_r(1) := \mathbb{C} \otimes_{\phi_1} BMW_r(q)$.

The next statement is straightforward.

**Lemma 4.2.** (1) The algebra $BMW_r(q)$ may be regarded as an $A_q$-module in the $K$-algebra $BMW_r(K)$.

(2) The specialisation $BMW_r(1)$ is isomorphic to the Brauer algebra $B_r(3)$ over $\mathbb{C}$.

(3) Let $I$ be the two-sided ideal of $BMW_r(q)$ generated by $e_1, \ldots, e_{r-1}$. There is a surjection $BMW_r(q) \rightarrow H_r(q^2)$ of $A_q$-algebras with kernel $I$, where $H_r(q^2)$ is the Hecke algebra discussed above (3.4).

**Proof.** Note for the first two statements, that by [X] Theorem 3.11 and its proof, $BMW_r(y, z)$ has a basis of “r-tangles” $\{T_d\}$, where $d$ runs over the Brauer $r$-diagrams, which form a basis of $B_r(\delta)$ over any ring. The same thing applies to $BMW_r(y, z)$; thus $BMW_r(q)$ may be thought of as the subring of $BMW_r(K)$ consisting of the $A_q$-linear combinations of the $T_d$, while $B_r(3)$ is realised as the set of $\mathbb{C}$-linear combinations of the diagrams $d$. \qed

Note that in view of the third relation in (4.15), the element $\frac{e_i}{[3]_q}$ of $BMW_r(q)$ is an idempotent. Moreover it follows from (4.13) or (4.15) that

\begin{equation}
\frac{e_i}{[3]_q} = \frac{(g_i - q^2)(g_i + q^{-2})}{(q^{-4} - q^2)(q^{-4} + q^{-2})}.
\end{equation}

(4.16)
Taking into account the cubic relation (4.11), or its specialisation in (4.15), we also have the idempotents $d_i$ and $c_i$ in $\text{BMW}_r(K)$, where
\begin{align*}
    d_i &= \frac{(g_i - q^2)(g_i - q^{-4})}{(q^{-2} + q^2)(q^{-2} + q^{-4})}, \\
    c_i &= \frac{(g_i + q^{-2})(g_i - q^{-4})}{(q^2 + q^{-2})(q^2 - q^{-4})}. \\
\end{align*}

**Lemma 4.3.** If $\text{BMW}_r(q)$ is thought of as an $A_q$-submodule of $\text{BMW}_r(K)$ as in Lemma 4.2(1), then the idempotents $e_i[3]_q^{-1}$, $d_i$ and $c_i$ all lie in $\text{BMW}_r(q)$.

**Proof.** It is evident that $e_i[3]_q^{-1} \in \text{BMW}_r(q)$. Since $(q^{-2} + q^2)(q^{-2} + q^{-4})$ is invertible in $A_q$, clearly $d_i \in \text{BMW}_r(q)$. But it is easily verified that $e_i[3]_q^{-1} + d_i + c_i = 1$, whence the result. \hfill \Box

The relevance of the above for the study of endomorphisms is evident from the next result.

**Theorem 4.4.** With the above notation, there is a surjection $\eta_q$ from the algebra $\text{BMW}_r(K) \to E_q(2, r)$ which takes $e_i$ to $[3]_q P_i(0)$ and $g_i$ to $R_i$.

**Proof.** In view of the above discussion, it remains only to show that the endomorphisms $\eta_q(g_i) = R_i$ and $\eta_q(e_i) = [3]_q P_i(0)$ satisfy the relations (4.7), (4.9), (4.11) and (4.13) for the appropriate $y$ and $z$. Now the braid relations (4.7) are always satisfied by the $R_i$; further, (4.11) with the $R_i$ replacing the $g_i$ is just (4.1). Now a simple calculation shows that in our specialisation, $z^{-1}(z + y^{-1} - y) = [3]_q := \delta$. It follows that (4.13) may be written
\begin{align*}
    \delta^{-1} e_i &= \frac{1}{\delta y z} (g_i - q^2)(g_i + q^{-2}) \\
    &= \frac{-1}{[3]_q q^{-4}(q^2 - q^{-2})} (g_i - q^2)(g_i + q^{-2}) \\
    &= \frac{-1}{(q - q^{-1})q^4} (g_i - q^2)(g_i + q^{-2}) \\
    &= \frac{q^8}{(1 + q^2)(1 - q^6)} (g_i - q^2)(g_i + q^{-2}). \\
\end{align*}
Thus (4.13) with $\delta P_i(0)$ replacing $e_i$ is just (4.6). Finally, the first delooping relation follows immediately from (4.11) and (4.13), while the other two follow from (4.4). \hfill \Box

We wish to illuminate which relations are necessary in addition to those which define $\text{BMW}_r(K)$, to obtain $E_q(2, r)$, i.e. we wish to study $\text{Ker}(\eta_q)$. Note that the specialisation of Lemma 4.2(2) is the classical limit as $q \to 1$ of $\text{BMW}_r(q)$, and that this is just the Brauer algebra with parameter $\delta_{q^{-1}} = 3$ in accord with [LZ, §3]. We shall study $\text{Ker}(\eta_q)$ by first examining the classical case, and then use specialisation arguments. The cellular structure of the algebras involved will play an important role in what follows.
4.3. **Tensor notation and quantum action.** In this subsection we establish notation for basis elements of tensor powers, which is convenient for computation of the actions we consider. Since the $\mathfrak{sl}_2$-module $V(2)$ is the classical limit at $q \to 1$ of $V_q(2)$, we do this for the quantum case, and later obtain the classical one by putting $q = 1$.

Maintaining the notation of section 3.3 and proceeding as in the proof of Lemma 3.4, let $v_{-1} \in V_q(2)$ be a basis element of the $-2$ weight space, and let $e, f, k$ be the generators of $\mathcal{U}_q$ referred to in §3.3. Then $v_0 := ev_{-1}$ and $v_1 := ev_0$ have weights 0, 2 respectively, and $\{v_0, v_{\pm 1}\}$ is a basis of $V_q(2)$. Moreover it is easily verified that $fv_1 = (q + q^{-1})v_0$ and $fv_0 = (q + q^{-1})v_{-1}$. The tensor power $V_q(2)^{\otimes r}$ has a basis consisting of elements $v_{i_1, i_2, \ldots, i_r} := v_{i_1} \otimes v_{i_2} \otimes \cdots \otimes v_{i_r}$. Note that $v_{i_1, i_2, \ldots, i_r}$ is a weight element of weight $2(i_1 + \cdots + i_r)$ for the action of $U_q(\mathfrak{sl}_2)$.

Now $V_q(2)^{\otimes 2}$ has a canonical decomposition $V_q(2)^{\otimes 2} = L(0)_q \oplus L(2)_q \oplus L(4)_q$, where $L(i)_q$ is isomorphic to $V(i)_q$ for all $i$. We shall give bases of the three components, which consist of weight vectors.

**Lemma 4.5.** The three components of $V_q(2)^{\otimes 2}$ have bases as follows.

1. $L(0)_q$: $v_{-1, -1} - q^2v_{0,0} + q^2v_{1, -1}$.
2. $L(2)_q$: $v_{0,1} - q^2v_{1,0}; v_{-1, 1} - v_{0, -1} + (1 - q^2)v_{0, 0}; v_{-1, 0} - q^2v_{0, -1}$.
3. $L(4)_q$: $v_{1, 1}; v_{0, 1} + q^{-2}v_{1, 0}; v_{-1, 1} + (1 + q^{-2})v_{0, 0} + q^{-4}v_{1, -1}; v_{-1, 0} + q^{-2}v_{0, -1}; v_{-1, -1}$.

The corresponding statement for the classical case is obtained by putting $q = 1$ above.

The $R$-matrix $R = R_1$ acts on the three components above via the scalars $q^{-4}, -q^{-2}$ and $q^2$ respectively.

The proof is a routine calculation, which makes use of the fact that several of the basis elements above are characterised by the fact that they are annihilated by $e$ and/or $f$.

It is useful to record the action of the endomorphism $e_1$ of $V_q(2)^{\otimes 2}$ (see Theorem 4.4) on the basis elements $v_{i,j}$.

**Lemma 4.6.** The endomorphism $e_1$ of $V_q(2)^{\otimes 2}$ acts as follows. Let $w_0 = -q^2v_{0,0} + q^2v_{1, -1} + v_{-1, 1} \in L(0)_q$. Then

$$e_1v_{i,j} = \begin{cases} w_0, & \text{if } (i, j) = (1, -1), \\ q^{-2}w_0, & \text{if } (i, j) = (-1, 1), \\ -q^{-2}w_0, & \text{if } (i, j) = (0, 0), \\ 0, & \text{if } i + j \neq 0. \end{cases}$$

**Proof.** Since $e_1$ acts as 0 on $L(2)_q$ and $L(4)_q$, and as $[3]_q$ on $L(0)_q$, this follows from an easy computation with the bases in Lemma 4.5.

The next result will be used in the next section.

**Lemma 4.7.** The $\mathcal{U}_q(\mathfrak{sl}_2)$-homomorphism $e_2 : L(2)_q \otimes L(2)_q \longrightarrow V_q(2) \otimes L(0)_q \otimes V_q(2)$ which is obtained by restricting $e_2$ to $L(2)_q \otimes L(2)_q \subset V_q(2)^{\otimes 4}$, is an isomorphism.

**Proof.** Let $u_i$ be the weight vector of weight $2i$ of $L(2)_q$ which is given in Lemma 4.5 ($i = 0, \pm 1$). Then $L(2)_q \otimes L(2)_q$ has basis $\{u_{i,j} := u_i \otimes u_j \mid i, j = 0, \pm 1\}$. Similarly, $V_q(2) \otimes L(0)_q \otimes V_q(2)$ has basis $\{x_{i,j} := v_i \otimes w_0 \otimes v_j \mid i, j = 0, \pm 1\}$. 


Now from Lemma 4.6, $e_2v_{i,j,a,b} = 0$ unless $j + a = 0$. This fact may be used to easily compute $e_2u_{i,j}$ in terms of $x_{a,b}$. The resulting $9 \times 9$ matrix of the linear map $e_2$ is then readily seen to have determinant $\pm q^{-4}(q^2 + q^{-2} - 1)(q^4 + 1 - q^{-2} + q^{-4})$, which is non-zero, whence the result. \qed

**Corollary 4.8.** The statement in Lemma 4.7 remains true in the classical case ($q = 1$).

This is clear since the determinant in the proof of Lemma 4.7 does not vanish at $q = 1$.

## 5. The Radical of a Cellular Algebra

In the next section we shall meet some cellular algebras which are not semisimple. This section is devoted to proving some general results about such algebras, which we use below. In this section only, we take $B = B(\Lambda, M, C, *)$ to be any cellular algebra over a field $F$, and prove some general results concerning its radical $R$. These results may be of some interest independently of the rest of this work. We assume that the reader has some acquaintance with the general theory of cellular algebras (see [GL96, §§1-3]); notation will be as is standard in cellular theory. In particular, for any element $\lambda \in \Lambda$, the corresponding cell module will be denoted by $W(\lambda)$ and its radical with respect to the canonical invariant bilinear form $\phi_\lambda$ by $R(\lambda)$. Since there is no essential loss of generality, we shall assume for ease of exposition, that $B$ is quasi-hereditary.

Let $\lambda \in \Lambda$, and write $W(\lambda)^*$ for the dual of $W(\lambda)$; this is naturally a right $B$-module, and we have a vector space monomorphism (cf. [GL96, (2.2)(i)]) $C^\lambda : W(\lambda) \otimes_F W(\lambda)^* \to B$ defined by

\[
C^\lambda(C_S \otimes C_T) = C^\lambda_{S,T} \quad \text{for } S, T \in M(\lambda). \tag{5.1}
\]

Denote the image of $C^\lambda$ by $B(\{\lambda\})$. This is a subspace of $B$, isomorphic as $(B, B)$-bimodule to $B(\leq \lambda)/B(< \lambda)$ (see [loc. cit.]GL96), and we have a vector space isomorphism

\[
B \sim \bigoplus_{\lambda \in \Lambda} B(\{\lambda\}). \tag{5.2}
\]

Note that $W(\lambda)$ and $W(\lambda)^*$ are equal as sets. We shall therefore differentiate between them only when actions are relevant.

Now let $\pi : B \to B := B/R$ be the natural map from $B$ to its largest semisimple quotient. Then $\overline{B} \cong \bigoplus_{\lambda \in \Lambda} \overline{B}(\lambda)$, where $\overline{B}(\lambda) \cong M(\lambda)(F) \cong \text{End}_F(L(\lambda))$. Thus $\pi$ may be written $\pi = \bigoplus_{\lambda \in \Lambda} \pi_\lambda$, where $\pi_\lambda : B \to \text{End}_F(L(\lambda))$ is the representation of $B$ on $L(\lambda)$. We collect some elementary observations in the next Lemma.

**Lemma 5.1.**

(1) The restriction $\pi_\lambda : B(\{\lambda\}) \to \text{End}_F(L(\lambda))$ is a surjective linear map for each $\lambda \in \Lambda$.

(2) Let $A$ be any semisimple $F$-algebra and let $\sigma : B \to A$ be a surjective homomorphism. Then $\sigma$ factors through $\pi$ as shown.
The restriction $\pi_\lambda$ of $\pi$ to $B(\lambda)$ is either zero or an isomorphism.

(4) Denote by $\Lambda^0$ the set $\{\lambda \in \Lambda \mid \pi_\lambda \text{ is non-zero}\}$. Let $J = \text{Ker}(\sigma)$. Then $\pi_\lambda(J) = 0$ if and only if $\lambda \in \Lambda^0$.

(5) The radical $\mathcal{R}$ is the set of elements of $B$ which act as zero on each irreducible module $L(\lambda)$.

Proof. The statement (1) follows from the cyclic nature of the cell modules ([GL96 (2.6)(i)]). The only other statement deserving of comment is (4), which follows immediately from the observation that $A \cong B/J \cong \overline{B}/\overline{J}$, where $\overline{J} = J/R$ is a two-sided ideal of the semisimple algebra $\overline{B}$. The ideal $\overline{J}$ therefore acts trivially in precisely those irreducible representations of $B$ which “survive” in the quotient, and non-trivially in the others. Note that (5) follows immediately from (4).

Corollary 5.2. Let $\eta : B \rightarrow \text{End}_F(W)$ be a representation of $B$ in the semisimple $B$-module $W$, and write $E = \text{Im}(\eta)$. Let $N = \text{Ker}(\eta : B \rightarrow E)$. Then $E \cong B/N \cong \oplus_{\lambda \in \Lambda^0} B(\lambda)$, where $\Lambda^0$ is the set of $\lambda \in \Lambda$ such that $L(\lambda)$ is a direct summand of $W$, regarded as an $E$-module. Moreover $\Lambda^0$ is characterised as the set of $\lambda \in \Lambda$ such that $N$ acts trivially on $L(\lambda)$.

To study the action of ideals on the $L(\lambda)$ we shall require the following results.

Lemma 5.3. Assume that $B$ is quasi-hereditary; i.e. that for all $\lambda \in \Lambda$, $\phi_\lambda \neq 0$.

1. In the notation of (5.1), if $x, y$ and $z$ are elements of $W(\lambda)$, then $C^\lambda(x \otimes y)z = \phi_\lambda(y, z)x$.

2. If $x$ or $y$ is in $R(\lambda)$, then $\pi_\lambda(C^\lambda(x \otimes y)) = 0$.

3. The radical $\mathcal{R}$ of $B$ has a filtration $(\mathcal{R}(\lambda) = \mathcal{R} \cap B(\leq \lambda))$ by two sided ideals such that there is an isomorphism of $(B, B)$-bimodules

$$\mathcal{R}(\lambda)/\mathcal{R}(< \lambda) \cong W(\lambda) \otimes R(\lambda)^* + R(\lambda) \otimes W(\lambda)^* \subset W(\lambda) \otimes W(\lambda)^*.$$ 

Proof. (1) is just [GL96 (2.4)(iii)]. To see (2), note that if $y \in R(\lambda)$, then from (1), $C^\lambda(x \otimes y)z = \phi_\lambda(y, z)x = 0$ for all $z \in W(\lambda)$. If $x \in R(\lambda)$, then again by (1), $C^\lambda(x \otimes y)z \in R(\lambda)$, whence $C^\lambda(x \otimes y)$ acts as 0 on $L(\lambda) = W(\lambda)/R(\lambda)$.

Now suppose $b \in C^\lambda(W(\lambda) \otimes R(\lambda) + R(\lambda) \otimes W(\lambda))$. Then by (2), $\pi_\lambda(b) = 0$. We shall prove

$$\exists \text{ elements } b_\lambda \in B(\{\lambda\'} \quad (\lambda' < \lambda) \text{ such that } b + \sum_{\lambda' < \lambda} b_\lambda \in \mathcal{R}. \quad (5.4)$$

We do this recursively as follows. Suppose we have a subset $\Gamma \subseteq \Lambda$ and an element $\sum_{\gamma \in \Gamma} b_\gamma \in \sum_{\gamma \in \Gamma} B(\{\gamma\})$ such that for any $\beta \in \Lambda$ which satisfies $\beta \geq \gamma$ for some $\gamma \in \Gamma$, we have $\pi_\beta(\sum_{\gamma \in \Gamma} b_\gamma) = 0$. We show that if there is $\mu \in \Lambda$ such that $\pi_\mu(\sum_{\gamma \in \Gamma} b_\gamma) \neq 0$, then we may increase $\Gamma$ to obtain another set with the same
properties. For this, take \( \mu \in \Lambda \) such that \( \pi_\mu(\sum_{\gamma \in \Gamma} b_\gamma) \neq 0 \), and maximal with respect to this property. Note that since \( \pi_\mu(b_\gamma) \neq 0 \) implies that \( \gamma \geq \beta \), we have \( \mu \leq \gamma \) for some element \( \gamma \in \Gamma \). By Lemma 5.1(1), there is an element \( b_\mu \in B(\{\mu\}) \) such that \( \pi_\mu(\sum_{\gamma \in \Gamma} b_\gamma) = \pi_\mu(-b_\mu) \). Let \( \Gamma' = \Gamma \cup \{\mu\} \). If \( \beta \geq \gamma' \) for some \( \gamma' \in \Gamma \), we show that \( \pi_\beta(\sum_{\gamma' \in \Gamma'} b_{\gamma'}) = 0 \).

There are two cases. If \( \gamma' \in \Gamma \), then \( \pi_\beta(\sum_{\gamma \in \Gamma} b_\gamma) = 0 \). If \( \pi_\beta(b_\mu) \neq 0 \), then \( \beta \leq \mu \) and so \( \gamma' \leq \beta \leq \mu \), whence by hypothesis \( \pi_\mu(\sum_{\gamma \in \Gamma} b_\gamma) = 0 \), a contradiction. Hence \( \pi_\beta(b_\mu) = 0 \), which proves the assertion in this case.

The remaining possibility is that \( \gamma' = \mu \). In this case, since \( \pi_\mu(\sum_{\gamma' \in \Gamma'} b_{\gamma'}) = 0 \) by construction, we may suppose \( \beta > \mu \). But then by the maximal nature of \( \mu \), \( \pi_\beta(\sum_{\gamma \in \Gamma} b_\gamma) = 0 \). Moreover since \( \beta > \mu \), \( \pi_\beta(b_\mu) = 0 \). Hence \( \Gamma' \) and \( \sum_{\gamma' \in \Gamma'} b_{\gamma'} \) have the same property as \( \Gamma \) and \( \sum_{\gamma \in \Gamma} b_\gamma \). Note that \( \Gamma' \) is obtained from \( \Gamma \) by adding an element \( \mu \) such that \( \mu \leq \gamma \) for some \( \gamma \in \Gamma \).

Now to prove the assertion (5.4), start with \( \Gamma = \{\lambda\} \) and \( b_\lambda = b_1 \). The argument above shows that we may repeatedly add elements \( \mu < \lambda \) to \( \Gamma \), with corresponding \( \pi_\mu(b_\mu) = 0 \), eventually coming to a set \( \Gamma_{\max} \) such that \( \sum_{\mu \in \Gamma_{\max}} b_\mu \) acts trivially on each \( L(\beta) \) (\( \beta \in \Lambda \)).

This completes the proof of (5.4), and hence of (3).

The arguments used in the proof of the above Lemma may be applied to yield the following result, in which we use the standard notation of [GL96] for cellular theory.

**Theorem 5.4.** Let \( B = (\Lambda, M, C, \ast) \) be a cellular algebra over a field \( \mathbb{F} \), and assume that \( B \) is quasi-hereditary, i.e. that the invariant form \( \phi_\lambda \) on each cell module is non-zero. For \( \lambda \in \Lambda \), denote by \( W(\lambda) \) and \( R(\lambda) \) respectively the corresponding cell module and its radical.

1. Let \( \lambda \in \Lambda \) and take any elements \( x \in W(\lambda), \ z \in R(\lambda) \). Then there exist elements \( r(x, z) \in C^\lambda(x \otimes z) + B(\langle < \lambda \rangle) \) and \( r(z, x) \in C^\lambda(z \otimes x) + B(\langle < \lambda \rangle) \), both in \( \mathcal{R} \), the radical of \( B \).
2. Let \( X \) be a subset of \( B \) such that for all \( \lambda \in \Lambda \), \( x \in W(\lambda) \) and \( z \in R(\lambda) \), \( X \) contains elements \( r(x, z) \) and \( r(z, x) \) as in (1). Then the linear subspace of \( B \) spanned by \( X \) contains \( \mathcal{R} \).
3. Suppose \( J \) is a two-sided ideal of \( B \) such that \( J^\ast = J \). Let \( \Lambda^0 := \{\lambda \in \Lambda \mid JL(\lambda) = 0\} \). Then \( J \supseteq \mathcal{R} \) if and only if, for all \( \lambda \in \Lambda^0 \), \( R(\lambda) \subseteq JW(\lambda) \).

**Proof.** The argument given in the proof of Lemma 5.3(3) proves the statement (1).

For each \( \lambda \in \Lambda \), let \( w_\lambda = \dim W(\lambda), \ r_\lambda = \dim R(\lambda), \) and \( l_\lambda = \dim L(\lambda) = \dim (W(\lambda)/R(\lambda)) = w_\lambda - r_\lambda \). Then

\[
\dim_\mathbb{F}(\mathcal{R}) = \dim_\mathbb{F}(B) - \sum_{\lambda \in \Lambda} l_\lambda^2
= \sum_{\lambda \in \Lambda} w_\lambda^2 - \sum_{\lambda \in \Lambda} l_\lambda^2
= \sum_{\lambda \in \Lambda} r_\lambda(w_\lambda + l_\lambda).
\]
But it is evident from an easy induction in the poset \( \Lambda \) that the dimension of the space spanned by the elements \( r(x,z) \) and \( r(z,x) \) is at least equal to

\[
\sum_{\lambda \in \Lambda} (2w_{\lambda}r_{\lambda} - r_{\lambda}^2) \\
= \sum_{\lambda \in \Lambda} r_{\lambda}(2w_{\lambda} - r_{\lambda}) \\
= \sum_{\lambda \in \Lambda} r_{\lambda}(w_{\lambda} + l_{\lambda}).
\]

Comparing with \( \dim(\mathcal{R}) \), we obtain the statement (2).

We now turn to (3). We begin by showing

(5.5) \( J \supseteq \mathcal{R} \iff R(\lambda) \subseteq JW(\lambda) \) for all \( \lambda \in \Lambda \).

First assume \( J \supseteq \mathcal{R} \) and suppose \( z \in R(\lambda) \); then take \( x, y \in W(\lambda) \), such that \( \phi_{\lambda}(x,y) \neq 0 \). Since \( \mathcal{R} \), and therefore \( J \), contains an element \( r(z,x) \) of the form above, we have \( JW(\lambda) \ni r(z,x)y = \phi_{\lambda}(x,y)z \). Hence \( R(\lambda) \subseteq JW(\lambda) \) for each \( \lambda \in \Lambda \).

Conversely, suppose \( JW(\lambda) \supseteq R(\lambda) \) for each \( \lambda \in \Lambda \). Since \( J^* = J \), to show that \( J \supseteq \mathcal{R} \), it will suffice to show that for any \( \lambda \in \Lambda \), and \( x \in W(\lambda), z \in R(\lambda) \), there is an element \( r(z,x) \in J \), of the form above. Now by hypothesis, \( z \in JW(\lambda) \); hence \( C^\lambda(z \otimes x) = C^\lambda(ay \otimes x) \in aC^\lambda(y \otimes x) + B(<\lambda), \) for some \( a \in J \) and \( y \in W(\lambda) \). Hence there is an element \( a_1 = C^\lambda(z \otimes x) + b \in J \) where \( b \in B(<\lambda) \). If \( a_1 \not\in \mathcal{R} \), then there is an element \( \lambda' < \lambda \) such that \( a_1L(\lambda') \neq 0 \), since \( a_1L(\mu) = 0 \) for all \( \mu \neq \lambda \).

By the cyclic nature of cell modules, if \( JL(\mu) \neq 0 \), then \( JW(\mu) = W(\mu) \). Thus for any two elements \( p, q \in W(\mu) \), since \( p \in JW(\mu) \), the argument above shows that \( C^\mu(p \otimes q) + b' \in J \) for some \( b' \in B(<\mu) \). It follows that the argument in the proof of Lemma 5.3 may be applied to show that \( a_1 \) may be recursively modified by elements of \( J \), to yield an element \( a_0 = r(z,x) \in J \cap \mathcal{R} \) as required.

The statement (5.5) now follows from (2). To deduce (3), observe that if \( \lambda \in \Lambda \setminus \Lambda^0 \), then since \( JL(\lambda) \neq 0 \), there are elements \( a \in J \) and \( x \in W(\lambda) \) such that \( ax \not\in R(\lambda) \).

But then by Lemma 5.3(1), \( W(\lambda) = B \cdot ax \subseteq J \cdot x \subseteq JW(\lambda) \), whence \( JW(\lambda) \supseteq R(\lambda) \) always holds \emph{a fortiori} for \( \lambda \in \Lambda \setminus \Lambda^0 \). In view of (5.5) this completes the proof of (3). \( \square \)

**Corollary 5.5.** Let notation be as in Theorem 5.4. Assume that for all \( \lambda \in \Lambda \) such that \( JL(\lambda) = 0 \), \( R(\lambda) \) is either zero or irreducible. Assume further that for any \( \lambda \in \Lambda \) such that \( JL(\lambda) = 0 \) and \( R(\lambda) \neq 0 \), \( JW(\lambda) \neq 0 \). Then \( J \) contains the radical \( \mathcal{R} \) of \( B \).

**Proof.** It follows from Theorem 5.4 that it suffices to show that for any \( \lambda \) such that \( JL(\lambda) = 0 \), \( JW(\lambda) = R(\lambda) \). But by hypothesis, if \( R(\lambda) \neq 0 \) for some such \( \lambda \), \( JW(\lambda) \) is a non-zero submodule of \( R(\lambda) \). By irreducibility, it follows that \( JW(\lambda) = R(\lambda) \), whence the result. \( \square \)

### 6. The classical 3-dimensional case

**6.1. The setup.** Let \( V = V(2) \), the classical three-dimensional irreducible representation of \( \mathfrak{sl}_2(\mathbb{C}) \). In this section we shall construct a quotient of the Brauer
algebra $B_r(3)$ which is defined by adding a single relation to the defining relations of $B_r(3)$, and which maps surjectively onto $\text{End}_{S_t}(V^\otimes r)$. For small $r$ we are able to show that our quotient is isomorphic to the endomorphism algebra. We shall make extensive use of the cellular structure of $B_r(3)$, as outlined in [GL96, §4]. In analogy with the Temperley-Lieb case above, where the case $r = 3$ (i.e. $V(1)^{\otimes 3}$) was critical, we start with the case $r = 4$, i.e. $V(2)^{\otimes 4}$.

Recall that given a commutative ring $A$, the Brauer algebra $B_r(\delta)$ over $A$ may be defined as follows. It has generators $\{s_1, \ldots, s_{r-1}; e_1, \ldots, e_{r-1}\}$, with relations $s_i^2 = 1$, $e_i^2 = \delta e_i$, $s_i e_i = e_i s_i = e_i$ for all $i$, $s_i s_j = s_j s_i$, $s_i e_j = e_j s_i$, $e_i e_j = e_j e_i$ if $|i-j| \geq 2$, and $s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$, $e_i e_{i+1} e_i = e_i$ and $s_i e_{i+1} e_i = s_{i+1} e_i$, $e_{i+1} e_i s_{i+1} = e_{i+1} s_i$ for all applicable $i$. We shall assume the reader is familiar with the diagrammatic representation of a basis of $B_r(\delta)$, and how basis elements are multiplied by concatenation of diagrams. In particular, the group ring $A\text{Sym}_r$ is the subalgebra of $B_r(\delta)$ spanned by the diagrams with $r$ “through strings”, and the algebra contains elements $w \in \text{Sym}_r$ which are appropriate products of the $s_i$.

In this section we take $A = \mathbb{C}$ and $\delta = 3$. The algebra $B_r(3)$ acts on $V^\otimes r$ as follows. We take the same basis $\{v_i \mid i = 0, \pm 1\}$ as in [GL96, 3.3]. Then $s_i$ acts by interchanging the $i$th and $(i+1)$st factors in the tensor $v_j \otimes \cdots \otimes v_j$. We define $w_0 \in V \otimes V$ as the specialisation at $q = 1$ of the element $w_0$ of [GL96, 3.3] i.e. $w_0 = v_{1,1} + v_{1,-1} - v_{0,0}$. Then the action of $e_1$ is obtained by putting $q = 1$ in Lemma 4.6 i.e.

$$e_1 v_{i,j} = \begin{cases} w_0, & \text{if } (i,j) = (1,-1) \text{ or } (-1,1), \\ -w_0, & \text{if } (i,j) = (0,0), \\ 0, & \text{if } i + j \neq 0. \end{cases}$$

The element $e_i$ acts on the $i, i+1$ components similarly. In addition to the elements $s_i$ and $e_i$, it will be useful to define the endomorphisms $e_{i,j} := (1,i)(2,i+1)e_1(1,i)(2,i+1)$, where we use the usual cycle notation for permutations in $B_r(\delta)$. The endomorphism $e_{i,j}$ acts on the $i$th and $j$th components of $V^\otimes r$ as $e_1$, and leaves the other components unchanged.

6.2. Cellular structure. The Brauer algebra $B_r(\delta)$ was proved in [GL96, §4] to have a cellular structure. This facilitates discussion of its representation theory. We begin by reviewing briefly the cells and cell modules for $B_r(\delta)$. Our notation here differs slightly from that in loc. cit.

Given an integer $r \in \mathbb{Z}_{\geq 0}$, define $T(r) := \{t \in \mathbb{Z} \mid 0 \leq t \leq r, \text{ and } r - t \in 2\mathbb{Z}\}$. For $t \in \mathbb{Z}_{\geq 0}$, let $\mathcal{P}(t)$ denote the set of partitions of $t$. Define $\Lambda(r) := \Pi_{t \in T(r)} \mathcal{P}(t)$. This set is partially ordered by stipulating that $\lambda \geq \lambda'$ if $|\lambda| > |\lambda'|$ or $|\lambda| = |\lambda'|$ and $\lambda > \lambda'$ in the dominance order on partitions of $|\lambda|$. For any partition $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_p)$, denote by $|\lambda|$ the sum $\sum \lambda_i$ of its parts. Given $\lambda \in \Lambda(r)$, the corresponding set $M(\lambda)$ (cf. [GL96, (1.1) and §4]) is the set of pairs $(S, \tau)$, where $S$ is an involution with $|\lambda|$ fixed points in $\text{Sym}_r$ and $\tau$ is a standard tableau of shape $\lambda$.

If $(S, \tau)$ and $(S', \tau')$ are two elements of $M(\lambda)$, the basis element of $B_r(\delta)$ is, in the notation of [GL96, (4.10)],

$$C^\lambda_{(S,\tau),(S',\tau')} := \sum_{w \in \text{Sym}_{|\lambda|}} p_{\tau,\tau'}(w)[S, S', w],$$
where \( w(\tau, \tau') \) is the element of \( \text{Sym}_{|\lambda|} \) corresponding to \( \tau, \tau' \) under the Robinson-Schensted correspondence, and for \( u \in \text{Sym}_{|\lambda|} \), \( c_u = \sum_{w \in \text{Sym}_{|\lambda|}} p_u(w)w \) is the corresponding Kazhdan-Lusztig basis element of \( \mathbb{Z}\text{Sym}_{|\lambda|} \). The cardinality \( |M(\lambda)| \) is easily computed. Let \( k(\lambda) = \frac{r-|\lambda|}{2} \), and let \( d_\lambda \) be the dimension of the representation (Specht module) of the symmetric group \( \text{Sym}_{|\lambda|} \) corresponding to \( \lambda \). For any integer \( t \geq 0 \) denote by \( t!! \) the product of the odd positive integers \( 2i + 1 \leq t \). Then we have, for any \( \lambda \in \Lambda(r) \),

\[
|\lambda| = \left( \frac{r}{|\lambda|} \right) (2k(\lambda))!!d_\lambda := w_\lambda.
\]

Now assume that the ground ring \( A \) is a field. We recall some facts from cellular theory.

6.1. Maintain the notation above.

1. For each \( \lambda \in \Lambda(r) \), there is a left \( B_r(\delta) \)-module \( W(\lambda) \), of dimension \( w_\lambda \) over \( A \).
2. The module \( W(\lambda) \) has a bilinear form \( \phi_\lambda : W(\lambda) \times W(\lambda) \to A \), which is invariant under the \( B_r(\delta) \)-action.
3. Let \( R(\lambda) \) be the radical of the form \( \phi_\lambda \). Then \( L(\lambda) := W(\lambda)/R(\lambda) \) is either an irreducible \( B_r(\delta) \)-module, or is zero. The non-zero \( L(\lambda) \) are pairwise non-isomorphic, and all irreducible \( B_r(\delta) \)-modules arise in this way.
4. All composition factors \( L(\mu) \) of \( W(\lambda) \) satisfy \( \mu \geq \lambda \).
5. \( B_r(\delta) \) is semisimple if and only if each form \( \phi_\lambda \) is non-degenerate. Equivalently, the \( W_\lambda \) form a complete set of representatives of the isomorphism classes of simple \( B_r(\delta) \)-modules.

We shall make use of the following facts.

Proposition 6.2. Take \( A = \mathbb{C} \) and assume that \( \delta \neq 0 \).

1. The algebra \( B_r(\delta) \) is quasi-hereditary; that is, each form \( \phi_\lambda \) in the assertion 6.1(2) is non-zero.
2. The algebra \( B_r(3) \) is semisimple if and only if \( r \leq 4 \).

Proof. The statement (1) is immediate from [GL96, Corollary (4.14)], while (2) follows from [RS, Theorem 2.3].

6.3. The case \( r = 4 \). It is clear from dimension considerations that when \( r \leq 3 \), the surjection \( \eta : B_r(3) \to \text{End}_{sl_2} V(2)^{\otimes r} \) (and its quantum analogue) is an isomorphism. The case \( r = 4 \) is therefore critical. In this subsection we shall treat the classical case when \( r = 4 \).

In terms of 6.1, we now take \( r = 4 \) and \( \delta = 3 \). Our purpose is to identify the kernel of the surjection \( \eta : B_4(3) \to \text{End}_{sl_2(\mathbb{C})} V^{\otimes 4} \). Define the element \( \Phi \in B_4(3) \) by

\[
\Phi = F_eF - F - \frac{1}{4} Fe_1e_4F,
\]

where notation is as in 6.1

The next statement summarises some of the properties of \( \Phi \).

Proposition 6.3. Let \( F, \Phi \in B_4(3) \) be the elements defined in (6.2). Then
(1) $e_i \Phi = 0$ for $i = 1, 2, 3$.
(2) $\Phi^2 = -4\Phi$.
(3) $\Phi$ acts as 0 on $V^{\otimes 4}$. That is, $\Phi \in \text{Ker}(\eta)$.

Proof. First note that

$$e_2 F e_2 = e_2 + e_2 e_{1,4}.$$  \hfill (6.3)

To see this, observe that $e_2 F e_2 = e_2(1 - s_1 - s_3 + s_1 s_3) e_2 = e_2^2 - e_2 s_1 e_2 - e_2 s_3 e_2 + e_2 s_1 s_3 e_2 = 3e_2 - 2e_2 + e_2 e_{1,4}$.

To prove (1), note that it is trivial that $e_i F = 0$ for $i = 1, 3$, and hence that $e_i \Phi = 0$ for $i = 1, 3$. But the relation $e_2 \Phi = 0$ now follows easily from (6.3) and the fact that $e_2^2 = 3e_{1,4}$.

It follows from (1) that $e_2 F \Phi = 0$, since $F \Phi = 4\Phi$ (recall $F^2 = 4F$). Hence $\Phi^2 = -F \Phi = -4\Phi$ which proves (2).

For (3), note that $F$ maps $V^{\otimes 4}$ onto $L(2) \otimes L(2)$, and hence that $\Phi(V^{\otimes 4}) \subseteq L(2) \otimes L(2)$. But by Corollary 4.8, $e_2$ acts injectively on $L(2) \otimes L(2)$, whence it follows from the fact that $e_2 \Phi = 0$, just proved, that $\Phi(V^{\otimes 4}) = 0$. \hfill $\square$

**Theorem 6.4.** The kernel of $\eta : B_4(3) \rightarrow \text{End}_{S_4} V(2)^{\otimes 4}$ is generated by the element $\Phi$ above.

Proof. The set $\Lambda(4)$ has 8 elements, ordered as follows:

$$(4) > (3, 1) > (2^2) > (2, 1^2) > (1^4) > (2) > (1^2) > (0).$$

The dimensions of the corresponding cell modules, which by the assertion 6.1(5) and Proposition 6.2(2) are simple in this case, are given respectively by

$$1, 3, 2, 3, 1, 6, 6, 3.$$

Now since $B_4(3)$ is semisimple, it is isomorphic to a sum $\oplus_{j=1}^{5} M(j)$ of 2-sided ideals, which are isomorphic to matrix algebras of size given in the list above (thus, e.g., dim $M(1) = 1$, while dim $M(7) = 36$). Moreover the 2-sided ideal $\mathcal{I}$ of $B_4(3)$ which is generated by the $e_i$ is cellular, and is the sum of the matrix algebras $M(j)$ for $j \geq 6$.

Note that $B_4(3)/\mathcal{I} \cong \text{CSym}_4$. Let $\mathcal{P}$ be the 2-sided ideal of $B_4(3)$ generated by $\Phi$. Then $\mathcal{P} + \mathcal{I} \ni F$, and $\frac{1}{4} F$ is an idempotent in $\text{CSym}_4$ which generates a left ideal on which $\text{Sym}_4$ acts as $\text{Ind}_K^{\text{Sym}_4}(\varepsilon)$, where $K$ is the subgroup of $\text{Sym}_4$ generated by $s_1, s_3$ and $\varepsilon$ denotes the alternating representation. But it is easily verified that $\text{Ind}_K^{\text{Sym}_4}(\varepsilon)$ is isomorphic to the sum of the irreducible representations of $\text{Sym}_4$ which correspond to the partitions $(2^2), (2, 1^2)$ and $(1^4)$, each one occurring with multiplicity one. Here we use the standard parametrisation in which the irreducible complex representations of $\text{Sym}_n$ correspond to partitions of $n$, the trivial representation corresponding to the partition $(n)$.

It follows that the 2-sided ideal of $\text{CSym}_4$ generated by $F$ is the image of $\oplus_{j=3}^{5} M(j)$ under the surjection $B_4(3)/\mathcal{I} \rightarrow \text{CSym}_4$. It follows that $\mathcal{I} + \mathcal{P} = \oplus_{j\geq 3} M_j$, whence $\dim(\mathcal{I} + \mathcal{P}) = \dim \mathcal{I} + 14$.

But using the dimension formula for $\dim \text{End}_{S_4} V^{\otimes 4}$ in (2.5), the kernel $N$ of $\eta$ has dimension 14 in this case. Since $\mathcal{P} \subseteq N$, it follows that

$$\dim(\mathcal{I} + \mathcal{P}) \leq \dim(\mathcal{I} + N) \leq \dim \mathcal{I} + \dim \mathcal{P} \leq \dim \mathcal{I} + \dim N = \dim \mathcal{I} + 14,$$
with equality if and only if $I \cap N = 0$ and $P = N$.

Since we have proved equality, the theorem follows. \hfill $\Box$

6.4. **The case** $r = 5$. This is the first case where $B := B_r(3)$ is not semisimple. We shall analyse this case to show how our methods yield non-trivial information on the algebras, such as the dimension of the radical. For this subsection only, we denote $B_5(3)$ by $B$.

In this case the cells are again totally ordered; we write them as follows.

\begin{equation}
(5) > (4,1) > (3,2) > (3,1^2) > (2^2,1) > (2,1^3) > (1^5)
\end{equation}

(6.4) $> (3) > (2,1) > (1^3) > (1)$.

If $W(\lambda)$ denotes the cell module corresponding to $\lambda$, the dimensions of the $W(\lambda)$ above are respectively given by:

\begin{align*}
1, 4, 5, 6, 5, 4, 1, 10, 20, 10, 15.
\end{align*}

Recall that $L(\lambda)$ is the irreducible head of $W(\lambda)$ for $\lambda \in \Lambda(5)$; write $l_\lambda := \dim L(\lambda)$. These integers are the dimensions of the simple $B$-modules.

We define the following 2-sided ideals of $B$. Let $\mathcal{R}$ be the radical of $B_5(3)$, $I = B/\langle 3 \rangle$ the ideal generated by the $e_i$, $P$ the ideal generated by $\Phi$, and $N$ the kernel of $\eta : B \rightarrow E := \text{End}_{sl_5}(V^\otimes 5)$.

We shall prove

**Theorem 6.5.** Let $B = B_5(3)$ as above, let $\mathcal{R}$ be its radical, and maintain the above notation.

1. The cell modules of $B$ are all simple except for those corresponding to the partitions $(2,1)$ and $(1^3)$, whose simple heads have dimension $15,6$ respectively.
2. The composition factors of $W(1^3)$ are $L(1^3)$ and $L(2,1^3)$.
3. The composition factors of $W(2,1)$ are $L(2,1)$ and $L(2^2,1)$.
4. The radical of $B$ has dimension $239$.
5. The kernel $N$ of $\eta : B \rightarrow E$ is generated by $\Phi$ modulo the radical. That is, in the notation above, $N = P + \mathcal{R}$.

**Proof.** It is easily verified that $E \cong M_1(\mathbb{C}) \oplus M_4(\mathbb{C}) \oplus M_{10}(\mathbb{C}) \oplus M_{15}(\mathbb{C}) \oplus M_5(\mathbb{C}) \oplus M_6(\mathbb{C})$, where $M_n(\mathbb{C})$ denotes the algebra of matrices of size $n$ over $\mathbb{C}$. Further, $\eta$ induces a surjection $\overline{\eta} : B/\mathcal{R} \cong \oplus_{\lambda \in \Lambda} M_{i_\lambda}(\mathbb{C}) \rightarrow E$. We shall determine which of the simple components $M_{i_\lambda}(\mathbb{C})$ are in the kernel of $\overline{\eta}$.

Observe that since $B/I \cong \mathbb{C}(\text{Sym}_5)$, the cell modules $W(\lambda)$ ($|\lambda| = 5$) are all irreducible, and clearly $\mathcal{R} \subseteq I \cap N$.

Now the element $F = (1 - s_1)(1 - s_3)$ generates the 2-sided ideal of $\mathbb{C}(\text{Sym}_5)$ which corresponds to the irreducible representations of $\text{Sym}_5$ which are constituents of $\text{Ind}^\mathbb{C}(\varepsilon)$, where $K$ is the subgroup generated by $s_1$ and $s_3$. An easy computation shows that these representations are precisely those which correspond to the partitions $\lambda$ with $|\lambda| = 5$ and $\lambda \neq (5)$, $(4,1)$. Let $\Lambda^1 = \{ \lambda \in \Lambda \mid |\lambda| = 5, \lambda \neq (5), (4,1) \}$, and write $\Lambda^0 := \Lambda \setminus \Lambda^1$. It follows from the above that $N$ acts non-trivially on the simple modules $W(\lambda)$ for $\lambda \in \Lambda^1$ (since $\Phi \in N$ does), and hence that $\text{Ker}(\overline{\eta}) \supseteq \oplus_{\lambda \in \Lambda^1} M_{i_\lambda}(\mathbb{C})$. Using the number and dimensions of the matrix components of $E$, it follows, by comparing the sizes of the matrix algebras on both sides,
that $W(1)$ is simple, one of the 10 dimensional cell modules is simple, the other has head of dimension 6, and $W(2, 1)$ has head of dimension 15.

Now the Gram matrix associated with the bilinear form $\phi_{(1^2)}$ on $W(1^3)$ is given by

$$
\begin{bmatrix}
3 & 1 & -1 & 1 & 1 & -1 & 1 & 0 & 0 & 0 \\
1 & 3 & 1 & -1 & 1 & 0 & 0 & -1 & 1 & 0 \\
-1 & 1 & 3 & 1 & 0 & 1 & 0 & -1 & 0 & 1 \\
1 & -1 & 1 & 3 & 0 & 0 & 1 & 0 & -1 & 1 \\
1 & 1 & 0 & 0 & 3 & 1 & -1 & 1 & -1 & 0 \\
-1 & 0 & 1 & 0 & 1 & 3 & 1 & 1 & 0 & -1 \\
1 & 0 & 0 & 1 & -1 & 1 & 3 & 0 & 1 & -1 \\
0 & -1 & -1 & 0 & 1 & 1 & 0 & 3 & 1 & 1 \\
0 & 1 & 0 & -1 & -1 & 0 & 1 & 1 & 3 & 1 \\
0 & 0 & 1 & 1 & 0 & -1 & -1 & 1 & 1 & 3
\end{bmatrix}
$$

Since this has rank 6, $W(1^3)$ is reducible. To understand the composition factors, note that the cell corresponding to the partition $1^3$ contains just the longest element $w_0$ of Sym$_3$. The Kazhdan-Lusztig basis element $c_{w_0} = \sum_{w \in \text{Sym}_3} \varepsilon(w)w$, and from this one sees easily that the element $\sum_{w \in (s_1, s_2, s_3)} \varepsilon(w)w$ of $B$ acts trivially on $W(1^3)$, whence it follows that $W(1^3)$ has no submodule isomorphic to $L(5)$ or $L(4, 1)$. Similarly, since $E(5) = \sum_{w \in \text{Sym}_3} \varepsilon(w)w$ also acts trivially on $W(1^3)$ (note that $E(5)e_i = 0$ for all $i$), $W(1^3)$ has no submodule isomorphic to $L(1^5)$. It follows that $R(1^3) \cong L(4, 1)$, proving (2).

Now consider $W(2, 1)$. The corresponding cell of Sym$_3$ contains the elements $r_1, r_2, r_1r_2$ and $r_2r_1$, where the simple generators of Sym$_3$ are denoted $r_1, r_2$ to avoid confusion with $s_1, s_2 \in B$. The corresponding Kazhdan-Lusztig basis elements are $1 - r_1, 1 - r_2, (1 - r_1)(1 - r_2)$ and $(1 - r_2)(1 - r_1)$. In analogy with the previous case, one now verifies easily that $\sum_{w \in (s_1, s_2, s_3)} w$ and $E(5)$ act trivially on $W(2, 1)$ whence the latter cell module has no simple submodule isomorphic to $L(5), L(4, 1), L(3, 2)$ or $L(1^5)$. It follows by dimension that $R(3, 2) \cong L(2^2, 1)$, completing the proof of (1), (2) and (3).

Clearly $\dim \mathcal{R} = \dim B - \dim \overline{B} = 20^2 + 10^2 - (15^2 + 6^2) = 239$, which proves (4).

To prove (5) observe that since $F \in (\mathcal{P} + \mathcal{I})$, the argument concerning induced representations above shows that $B/(\mathcal{P} + \mathcal{I}) \cong M_1(C) \oplus M_4(C)$. Hence

$$\dim(\overline{P} + \mathcal{I}) \leq \dim(\overline{N} + \mathcal{I}) \leq \dim \overline{N} + \dim \mathcal{I},$$

with equality if and only if $\overline{N} \cap \mathcal{I} = 0$ and $\overline{P} + \mathcal{R} = \overline{N}$. But $\dim \overline{P} + \mathcal{I} = \dim \overline{B} - 17$, and by the above argument, this is equal to $\dim \overline{N} + \dim \mathcal{I}$, whence $\overline{P} + \mathcal{R} = \overline{N}$, i.e. $\overline{N} = \Phi + \mathcal{R}$.

Note that it is possible that the methods of [HW] could be applied to give alternative proofs of some parts of Theorem 6.5.

6.5. The general classical case. Our objective in this section is to check some cases of our main conjecture below, and reduce it to a specific question about the action of $\Phi$ on certain cell modules of $B_3(3)$. To do this we shall utilise the general results of §6.4 above about the radical of a cellular algebra.

In view of the results of the last subsection we make the
6.6. Conjecture. Let \( B = B_r(3) \), \( E = \text{End}_{\mathfrak{sl}_2(\mathbb{C})}(V(2)^{\otimes r}) \) and \( \eta : B \to E \) the natural surjection discussed above. The kernel \( N \) of \( \eta \) is generated by the element \( \Phi = F e_2 F - F - \frac{1}{4} F e_2 e_{14} F \in B \).

To make use of the theory in the last section, we shall develop more detail concerning the cellular structure of \( B_r \). We maintain the notation above. In particular \( R \) denotes the radical of \( B \), \( \mathcal{I} \) is the two-sided ideal generated by the \( e_i \) and \( \mathcal{P} \) denotes the ideal generated by \( \Phi \).

We start with the following elementary observation.

Lemma 6.7. Let \( t \geq 4 \) be an integer and consider the symmetric group \( \text{Sym}_t \) generated by simple transpositions \( s_1, \ldots, s_{t-1} \). The two-sided ideal of \( \mathbb{C}\text{Sym}_t \) generated by \( F = (1 - s_1)(1 - s_3) \) is the sum of the minimal ideals corresponding to all partitions with at least 4 boxes in the first two columns. This is an easy exercise, which may be proved by induction on \( t \).

Theorem 6.8. Let \( \eta : B_r(3) \to E := \text{End}_{\mathfrak{sl}_2(\mathbb{C})}(V(2)^{\otimes r}) \) be the surjection discussed above, and let \( N = \text{Ker}(\eta) \). Define \( \Lambda^0 \subseteq \Lambda \) by \( \Lambda^0 = \{(t), (t-1, 1), t^3 \mid 0 \leq t \leq r; \ t \equiv r(\text{mod } 2)\} \), and let \( \Lambda^1 := \Lambda \setminus \Lambda^0 \). Let \( \Phi \) be the element of \( B = B_r \) defined above. Then

1. For \( \lambda \in \Lambda^1 \), there is an element \( x_\lambda \in L(\lambda) \) such that \( \Phi x_\lambda \neq 0 \).
2. \( N \) acts trivially on \( L(\lambda) \) if and only if \( \lambda \in \Lambda^0 \).
3. \( E \cong \bigoplus_{\lambda \in \Lambda^0} \overline{\Phi}(\lambda) \).
4. If \( \mathcal{P} \) denotes the ideal of \( B \) generated by \( \Phi \), we have \( \mathcal{P} + \mathcal{R} = N \), where \( \mathcal{R} \) is the radical of \( B \).

Proof. Take \( \lambda \in \Lambda^1 \). If \( t = |\lambda| \geq 4 \), consider the subalgebra of \( B \) generated by the elements \( \{s_ie_{i+1}e_{i+3} \ldots e_{i+2k-1} \mid 1 \leq i \leq t-1\} \), where \( r = t + 2k \). This is isomorphic to \( \mathbb{C}\text{Sym}_t \), and \( \Phi \) acts on the corresponding cell modules as \( -F \). The statement (1) is now clear for this case, given Lemma 6.7.

Now suppose \( t = |\lambda| \leq 3 \). Now in analogy to the above argument, we consider the “leftmost” part of the diagrams, completed with \( e_5e_7 \ldots e_6e_8 \ldots \) on the right according as \( t \) is odd or even. The cases \( r = 4, 5 \), which are known by §§6.3.6.4 produce, when appropriately completed, elements \( x_\lambda \in L(\lambda) \) as required. This proves (1).

To see (2), observe that (1) shows that \( N \) acts non-trivially on the simple modules \( L(\lambda) \) for \( \lambda \in \Lambda^1 \), and so the set of \( \lambda \) such that \( N \) acts trivially in \( L(\lambda) \) is contained in \( \Lambda^0 \). But \( |\Lambda^0| = r + 1 \), and it is easy to see that \( V^{\otimes r} \) has \( r + 1 \) distinct simple components (as \( \mathfrak{sl}_2 \)-module). It follows that \( N \) acts trivially in at least \( r + 1 \) of the simple modules \( L(\lambda) \), and (2) is immediate (cf. [5.2]), as is (3).

Since \( \Phi \in \mathcal{P} + \mathcal{R} \), the latter is a two-sided ideal of \( B \) which acts non-trivially on \( L(\lambda) \) for \( \lambda \in \Lambda^1 \). But \( \Phi \in N \), so that \( \mathcal{P} + \mathcal{R} \) acts trivially on \( L(\lambda) \) for \( \lambda \in \Lambda^0 \). The statement (4) follows.

Combined with the results of §5 Theorem 6.8 leads to the following criterion for the truth of Conjecture 6.6.

Corollary 6.9. The conjecture 6.6 is equivalent to the following statement. For each \( \lambda \in \Lambda^0 \) (as above) the \( B_r \)-submodule of \( W(\lambda) \) generated by \( \Phi W(\lambda) \) contains \( R(\lambda) \).
Proof. It follows from Theorem 6.8(4) that the conjecture is equivalent to the statement that $\mathcal{P} \supseteq \mathcal{R}$. By Theorem 5.4(3) this is equivalent to the stated criterion. \(\square\)

Next we show that the Conjecture is true for $r = 5$.

**Proposition 6.10.** If $r = 5$, $\mathcal{P} = \langle \Phi \rangle$ contains the radical $\mathcal{R}$ of $B$. Hence by 6.9 the conjecture is true for $r = 5$.

**Proof.** In view of Theorem 6.9, it suffices to show that $\mathcal{P}W(\lambda) = R(\lambda)$ for $\lambda = (1^3)$ and $\lambda = (2, 1)$. But again by Theorem 6.9 we have the situation of Corollary 5.5 here, whence it suffices to show simply that $\Phi$ acts non-trivially on the cell modules $W(1^3)$ and $W(2, 1)$. This will require two calculations, which we now proceed to outline.

**The case $W(1^3).** In this case $M(\lambda) = \{(ij), \tau \}$ where $(ij)$ is a transposition in $\text{Sym}_5$ and $\tau$ is the unique standard tableau of shape $1^3$. Thus we may write a basis for $W(1^3)$ as $\{C_{ij} \mid 1 \leq i < j \leq 5\}$. Recalling that the Kazhdan-Lusztig basis element of $\mathbb{C}\text{Sym}_3$ corresponding to $(\tau, \tau)$ is $E(3) := \sum_{w \in \text{Sym}_3} \varepsilon(w)w$, the following facts are easily verified using the diagrammatic representation of $B_5$.

$$s_1C_{45} = -C_{45}; \quad s_3C_{45} = C_{35}; \quad FC_{45} = 2(C_{45} - C_{35});$$

$$e_2C_{45} = 0; \quad e_4C_{35} = -C_{23}; \quad e_{14}C_{23} = 0.$$  

Using these equations one calculates in straightforward fashion that

$$\Phi C_{45} = 2(C_{23} - C_{13} - C_{24} + C_{14} - C_{45} + C_{35}) \neq 0.$$  

**The case $W(2, 1).** In this case $M(\lambda) = \{(ij), \tau_k \}$ where $(ij)$ is a transposition in $\text{Sym}_5$ and $\tau_k$ is one of the two standard tableau of shape $(2, 1)$. Explicitly,

$$\tau_1 = \begin{pmatrix} 1 & 3 \\ 2 \end{pmatrix}, \quad \tau_2 = \begin{pmatrix} 1 & 2 \\ 3 \end{pmatrix}.$$  

Thus we may write a basis for $W(2, 1)$ as $\{C_{ij, \tau_k} \mid 1 \leq i < j \leq 5, \quad k = 1, 2\}$. In this case we need to recall that the Kazhdan-Lusztig cell representation of $\mathbb{C}\text{Sym}_3$ which corresponds to $(2, 1)$ may be thought of as having basis $\{c_{\tau_1}, c_{\tau_2}\}$ and action by $\text{Sym}_3 = \langle \tau_1, \tau_2 \rangle$ given by

$$r_1c_{\tau_1} = -c_{\tau_1}; \quad r_1c_{\tau_2} = c_{\tau_2} - c_{\tau_1}; \quad r_2c_{\tau_1} = c_{\tau_1} - c_{\tau_2}; \quad r_2c_{\tau_2} = -c_{\tau_2}.$$  

With these facts one verifies easily the following facts

$$s_1C_{45, \tau_1} = -C_{45, \tau_1}; \quad s_3C_{45, \tau_1} = C_{35, \tau_1}; \quad FC_{45, \tau_1} = 2(C_{45, \tau_1} - C_{35, \tau_1}).$$  

Further,

$$e_2C_{45, \tau_1} = 0; \quad e_4C_{35, \tau_1} = C_{23, \tau_1} - C_{23, \tau_2};$$

$$e_2FC_{45, \tau_1} = 2(C_{23, \tau_2} - C_{23, \tau_1}); \quad e_{14}C_{23, \tau_k} = 0 \text{ for } k = 1, 2.$$  

Using these equations, it is straightforward to calculate that

$$\Phi C_{45, \tau_1} = 2(C_{23, \tau_2} - C_{13, \tau_2} - C_{24, \tau_2} + C_{14, \tau_2} - C_{23, \tau_1} + C_{13, \tau_1} + C_{24, \tau_1} - C_{14, \tau_1} - C_{45, \tau_1} + C_{35, \tau_1}) \neq 0.$$  

This completes the proof of the Proposition. \(\square\)

A computer calculation has been done to verify the case $r = 6$.  

Theorem 6.11. Let $\eta$ be the surjective homomorphism from $B_r := B_r(3)$ to $E_r := \text{End}_{\mathbb{C}[q]}(V(2)^{\otimes r})$, and let $\Phi \in B_r$ be the element defined above. Then for $r \leq 6$ $\Phi$ generates the kernel of $\eta$.

Proof. We have proved the result for $r \leq 5$. The case $r = 6$ was checked by a computer calculation, which verified that $\dim(\Phi)$ is correct in that case. Since we know that $\Phi \in \text{Ker}(\eta)$, the result follows. \qed

We are grateful to Derek Holt for doing this computation for us using the Magma computational algebra package, with an implementation of noncommutative Gröbner basis due to Allan Steel.

7. The quantum case

In this section, we develop the theme of §6.2 and consider the BMW algebra $BMW_r(q)$ over $A_q$, and its specialisation $BMW_r(\mathcal{K})$. The results of the last section on the Brauer algebra all generalise to the present case, and we deduce some new ones through the technique of specialisation. One of the key observations is that $BMW_r(q)$ has the $\mathbb{C}$ algebra $BMW_r(1) \cong B_r(3)$ (cf. Lemma 6.2) as a specialisation.

7.1. Specialisation and cell modules. In analogy with the case of the Brauer algebra, of which it is a deformation, $BMW_r(q)$ has a cellular structure [X Theorem 3.11] and is also quasi hereditary [X Theorem 4.3]. For each partition $\lambda \in \Lambda(r)$, there is therefore a cell module $W_q(\lambda)$ of dimension $w_\lambda$ for $BMW_r(q)$. Each cell module has a non-zero irreducible head $L_q(\lambda)$, and these irreducibles form a complete set of representatives of the isomorphism classes of simple $BMW_r(q)$-modules. Furthermore, $BMW_r(q)$ is semisimple if and only if all the cell modules are simple (see [X §3]).

Recall that $BMW_r(q)$ is the $A_q$-algebra defined by the presentation (4.14), where $A_q$ is the localisation of $\mathbb{C}[q^{\pm 1}]$ at the multiplicative subset $S$ generated by $[2]_q, [3]_q$ and $[3]_q - 1$. By Lemma 6.2 $BMW_r(q)$ may be thought of as an integral form of $BMW_r(\mathcal{K})$.

One may identify $BMW_r(q)$ with the $A_q$-algebra generated by $(r, r)$-tangle diagrams, which satisfy the usual relations (cf. e.g., [X Definition 2.5]). For each Brauer $r$-diagram $T$ [GL96, §4], it is explained in [X p. 285] how to construct an $(r, r)$-tangle diagram $T_q$ by lifting each intersection in $T$ to an appropriate crossing. The tangle diagrams obtained this way form a basis of $BMW_r(q)$, which we shall denote by $T_q$.

The cell modules $W_q(\lambda)$ of $BMW_r(q)$ are parametrised by partitions $\lambda \in \Lambda(r)$. They may also be described diagramatically, in a similar way to the cell modules of the Brauer algebra $B_r(3)$ (cf. §6.2). We proceed to give this description. Let $t \in T(r)$; that is, $0 \leq t \leq r$ and $r - t \in 2\mathbb{Z}$. For a partition $\lambda$ of $t$, we take $M(\lambda)$ to be the set defined in §6.2 for the Brauer algebra, viz $M(\lambda)$ is the set of pairs $(S, \tau)$ where $S$ is an involution in $\text{Sym}_t$ with $|\lambda| = t$ fixed points, and $\tau$ is a standard tableau of shape $\lambda$. In analogy with §6.2 if $(S, \tau)$ and $(S', \tau')$ are two elements of $M(\lambda)$, we obtain the (cellular) basis element $C^\lambda_{(S, \tau), (S', \tau')}(q)$ of $BMW_r(q)$ by

$$C^\lambda_{(S, \tau), (S', \tau')}(q) = \sum_{w \in \text{Sym}_t} P_{w(\tau, \tau'), w}(q)[S, S', w]_q,$$

where $P_{w(\tau, \tau'), w}(q)$ is the monomial basis element of $BMW_r(q)$. The relation (7.1) provides a way to compute $C^\lambda_{(S, \tau), (S', \tau')}(q)$ explicitly.
where $C_v = \sum_{w \in \text{Sym}_v} P_{v,w}(q)T_w$ is the Kazhdan-Lusztig basis element of the Hecke algebra $H_1(q^2)$, $[S, S', w]_q$ is the element of the basis $T_q$ (i.e. dangle) corresponding to the Brauer diagram $[S, S', w]$, and all other notation is as in [6.2]. Note that $P_{v,w}(q) \in \mathbb{Z}[q^{\pm 1}] \subset A_q$, so that $C_{(S, r)}^{(S', t)}(q) \in \text{BMW}_r(q)$.

Now for each element $(S, \tau) \in M(\lambda)$, the arguments leading to [X Cor. 3.13] describe how to associate to $(S, \tau)$ an $(r, t)$ dangle which we denote by $(S, \tau)_q$. These form an $\mathcal{A}_q$-basis of $W_q(\lambda)$, with the action of $\text{BMW}_r(q)$ given by concatenation, using the relations in [X Def. 2.5] and the action of the Hecke algebra $H_1(q^2)$ on its cell modules (which have basis $\{\tau\}$). The next statement is a general result about cellular algebras, adapted to our situation.

**Proposition 7.1.** Let $\phi : \mathcal{A}_q \rightarrow R$ be a homomorphism of commutative rings with 1, and denote by $\text{BMW}_r^\phi$ the specialisation $R \otimes_\phi \text{BMW}_r(q)$. Then

1. There is a natural bijection between the $\mathcal{A}_q$-basis $\{(S, \tau)_q\}$ of $W_q(\lambda)$ and an $R$-basis of the specialised cell module $W^\phi(\lambda)$.
2. If $a \in \text{BMW}_r(q)$, the matrix of $1 \otimes a \in \text{BMW}_r^\phi$ with respect to the basis in (1) is obtained by applying $\phi$ to the entries of the matrix of $a$.
3. The Gram matrix of the canonical form on $W^\phi(\lambda)$ is obtained from that of $W_q(\lambda)$ by applying $\phi$ to the entries of the latter.
4. If $W^\phi(\lambda)$ is simple, so is $W_q(\lambda)$.
5. We have $\text{rank}_{\mathcal{A}_q} L_q(\lambda) \geq \text{rank}_R L^\phi(\lambda)$, where $L_q(\lambda)$ is the simple head of $W_q(\lambda)$, etc.
6. For any pair $\mu \geq \lambda \in \Lambda(r)$, the multiplicity $[W_q(\lambda) : L_q(\mu)] \leq [W^\phi(\lambda) : L^\phi(\mu)]$.

**Proof.** The bijection of (1) arises from any $\mathcal{A}_q$-basis $\{\beta\}$ of $W_q(\lambda)$, by taking $\beta \mapsto 1 \otimes \beta$. Given this, the assertion (2) is clear, as is (3). If $\Delta(\lambda)$ is the determinant of the Gram matrix of $W_q(\lambda)$ (i.e. the discriminant), the discriminant $\Delta^\phi(\lambda)$ of $W^\phi(\lambda)$ is given by $\Delta^\phi(\lambda) = \phi(\Delta(\lambda))$. If $W^\phi(\lambda)$ is simple, then $\Delta^\phi(\lambda) \neq 0$, whence $\Delta_q(\lambda) \neq 0$. This implies that if $\phi_\ast$ is the inclusion of $\mathcal{A}_q$ in $\mathcal{K}$, then $W^\phi_q(\lambda)(= W_\mathcal{K}(\lambda))$ is simple as $\text{BMW}_r(\mathcal{K})$ module. It follows that $W_q(\lambda)$ has no non-trivial $\text{BMW}_r(q)$-submodules, whence (4). Finally, note that $\text{rank}_{\mathcal{A}_q}(L_q(\lambda))$ equals the rank of the Gram matrix of the form. Since this cannot increase on specialisation, (5) follows. To see (6), observe that any composition series of $W_q(\lambda)$ specialises (under the functor $R \otimes_\phi -$) to a chain of submodules of $W^\phi(\lambda)$. But by (3), the specialisation of $L_q(\mu)$ has $L^\phi(\mu)$ as a subquotient, from which (6) follows.

---

### 7.2. An element of the quantum kernel

We next consider some elements of $\text{BMW}_r(q)$ which will play an important role in the remainder of this work, and which will be used to define the Temperley-Lieb analogue of the title. Let $f_i = -g_i - (1 - q^{-2})e_i + q^2$, and set

$$F_q = f_1f_3.$$  

We also define $e_{14} = g_3^{-1}g_1e_2g_1^{-1}g_3$ and $e_{1234} = e_2g_1g_3^{-1}g_2g_1^{-1}g_3$.

The next two results are quantum analogues of Proposition 6.3.
Lemma 7.2. The following identities hold in $\text{BMW}_4(q)$ (and hence in $\text{BMW}_r(q)$).

\begin{align*}
(7.3) \quad f_i &= \frac{(g_i - q^2)(g_i - q^{-4})}{q^{-2} + q^{-4}} \\
(7.4) \quad e_i f_i &= 0, \quad f_i^2 = (q^2 + q^{-2})f_i, \quad i = 1, 2, 3, \\
(7.5) \quad e_2 F_q e_2 &= ae_2 - de_{1234} + ae_{14}, \\
(7.6) \quad e_2 F_q e_2 e_{14} &= e_{14} e_2 F_q e_2 = (q^2 + q^{-2})^2 e_2 e_{14}, \\
(7.7) \quad e_2 F_q e_{1234} &= e_{1234} F_q e_2 = -de_2 + ae_{1234} + q^{-4}ae_{14},
\end{align*}

where

\[ a = 1 + (1 - q^{-2})^2, \quad \tilde{a} = 1 + (1 - q^2)^2, \quad d = (q - q^{-1})^2 = q^2(a - 1) = q^{-2}(\tilde{a} - 1). \]

The first relation follows easily from the relations (4.15), and the others are straightforward consequences. Note that $F_q$ potent of Lemma 4.2. Alternatively, one may use the representation of elements of the BMW algebra by tangle diagrams, and the multiplication by composition of diagrams, to verify the above statements.

Define the following element of $\text{BMW}_4(q)$:

\begin{equation}
(7.8) \quad \Phi_q = a F_q e_2 F_q - b F_q - c F_q e_{14} F_q + d F_q e_{1234} F_q,
\end{equation}

where

\[ b = 1 + (1 - q^2)^2 + (1 - q^{-2})^2, \]

\[ c = \frac{1 + (2 + q^{-2})(1 - q^{-2})^2 + (1 + q^2)(1 - q^2)^4}{([3]_q - 1)^2}. \]

Proposition 7.3. The elements $F_q, \Phi_q$ have the following properties:

1. $F_q^2 = (q^2 + q^{-2})^2 F_q$.
2. $e_i \Phi_q = \Phi_q e_i = 0$ for $i = 1, 2, 3$.
3. $\Phi_q^2 = - (q^2 + q^{-2})^2 (1 + (1 - q^2)^2 + (1 - q^{-2})^2) \Phi_q$.
4. $\Phi_q$ acts as $0$ on $V_q^{\otimes 4}$.

Proof. Part (1) immediately follows from the second relation in (7.4).

The fact that $e_1 \Phi_q = e_3 \Phi_q = 0$ follows from the first relation of (7.4) in Lemma 7.2.

Now

\[ e_2 \Phi_q = ae_2 F_q e_2 F_q - be_2 F_q - ce_2 F_q e_{14} F_q + de_2 F_q e_{1234} F_q. \]

Using the relations (7.3), (7.5) and (7.7), we readily obtain $e_2 \Phi_q = 0$. It can be similarly shown that $\Phi_q e_i = 0$ for $i = 1, 2, 3$. Thus by part (2), we see $\Phi_q F_q e_2 = 0$, and therefore that $\Phi_q^2 = -b \Phi_q F_q = -(q^2 + q^{-2})^2 b \Phi_q$.

The proof of part (4) proceeds in much the same way as in the classical case. Note that $\Phi_q(V_q^{\otimes 4}) \subset L(2)_q \otimes L(2)_q$. Thus by Lemma 7.1, $\Phi_q(V_q^{\otimes 4}) \cong e_2 \Phi_q(V_q^{\otimes 4})$ as $\mathcal{U}_q(\mathfrak{sl}_2)$-modules. Since $e_2 \Phi_q = 0$ by part (2), the proof is complete. \qed

7.3. A regular form of quantum $\mathfrak{sl}_2$. In this subsection we consider the quantised universal enveloping algebra of $\mathfrak{sl}_2$ over the ring $\mathcal{A}_q$ and its representations. By “regular form” we shall understand an $\mathcal{A}_q$-lattice in a $\mathcal{K}$-representation of $\mathcal{U}_q(\mathfrak{sl}_2)$. Denote by $\mathcal{U}_q$ the $\mathcal{A}_q$-algebra generated by $e, f, k^\pm 1$ and $h := \frac{k - k^{-1}}{q - q^{-1}}$, subject to the usual relations, and call it the regular form of $\mathcal{U}_q(\mathfrak{sl}_2)$. Recall (Definition 4.1)
that we have homomorphisms $\phi_1$ and $\phi_q$ from $A_q$ to $C, K$ respectively; the resulting specialisation $C \otimes_{\phi_1} U_{A_q}$ at $\phi_1$ is isomorphic to the universal enveloping algebra of $U(sl_2)$ of $sl_2$ with

$$
1 \otimes e \mapsto \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad 1 \otimes f \mapsto \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad 1 \otimes h \mapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad 1 \otimes k \mapsto \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.
$$

The $A_q$-span $V_q^{reg}(2)$ of the vectors $v_0, v_{\pm 1}$ (see Section 4.3) forms a $U_{A_q}$-module, which is an $A_q$-lattice in $V_q(2)$. Denote the $r$-th tensor power of $V_q^{reg}(2)$ over $A_q$ by $V_q^{reg}(2)^{\otimes r}$; this has an $A_q$-basis consisting of the elements $v_{i_1,i_2,\ldots,i_r}$. Then

$$
\mathcal{K} \otimes_{\phi_q} V_q^{reg}(2)^{\otimes r} = V_q(2)^{\otimes r}, \quad \text{as } U_q(sl_2)-\text{module};
$$

$$
\mathbb{C} \otimes_{\phi_q} V_q^{reg}(2)^{\otimes r} \cong V(2)^{\otimes r}, \quad \text{as } sl_2-\text{module}.
$$

Remark 7.4. The vectors $1 \otimes v_{i_1,i_2,\ldots,i_r}$ form a basis for $\mathbb{C} \otimes_{A_q} V_q^{reg}(2)^{\otimes r}$. It follows that $1 \otimes v \in \mathbb{C} \otimes_{\phi_q} V_q^{reg}(2)^{\otimes r}$ is zero if and only if $v \in (q-1)V_q^{reg}(2)^{\otimes r}$.

Denote by $E_q^{reg}(r)$ the $A_q$ algebra $End_{U_q}(V_q^{reg}(2)^{\otimes r})$. Recall (Theorem 4.4) that we have the surjection $\eta_q : BMW_r(K) \longrightarrow E_q(r) = K \otimes_{\phi_q} E_q(r)$. The next result shows that $\eta_q$ preserves the $A_q$-structures.

**Lemma 7.5.** We have $\eta_q(BMW_r(q)) \subseteq E_q^{reg}(r)$. In particular, if $[3]_q^{-1}e_i, d_i$ and $c_i$ are the idempotents of Theorem 4.4, then $\eta_q(g_i), \eta_q(\frac{\eta_q(g_i)}{[3]_q}), \eta_q(d_i)$ and $\eta_q(c_i)$ belong to $E_q^{reg}(r)$ for all $i$.

**Proof.** The formulæ in Lemma 4.6 show explicitly that $\eta_q(d_i) \in E_q^{reg}(r)$. A similar computation shows that $\eta_q(e_i) \in E_q^{reg}(r)$, as follows. Evidently it suffices to treat the case $i = 1$. Write $\eta_q(d_1)(v_{k,l}) := x_{k,l}$; clearly we need only show that $x_{k,l} \in V_q^{reg}(2)^{\otimes 2}$ for $k, l = 0, \pm 1$. But one verifies easily that the following explicit formulæ describe the action of $d_i$.

Write $u_{-1} = q^{-2}v_{-1,1} = v_0, u_0 = -q^{-2}v_{-1,1} + (1-q^{-2})v_{0,0} + q^{-2}v_{1,-1}$ and $u_1 = q^{-2}v_{0,1} - v_0, 0$, and observe that $u_i \in V_q^{reg}(2)^{\otimes 2}$ for $i = 0, \pm 1$. Then $x_{1,1} = x_{-1,-1} = 0, x_{0,1} = \frac{1}{1-q^{-2}q^{u_1}}u_1, x_{0,0} = -q^{-2}v_{1,0}, x_{0,-1} = \frac{1}{1-q^{-2}q^{u_0}}u_0, x_{-1,-1} = -x_{-1,1} = \frac{1}{1-q^{-2}}u_0, x_{0,1} = \frac{1}{1-q^{-2}}u_1, x_{0,-1} = \frac{1}{1-q^{-2}}u_{0,1}, x_{-1,0} = -q^{-2}x_{0,-1}$.

This shows that $\eta_q(d_i) \in E_q^{reg}(r)$, and since $\eta_q(\frac{\eta_q(g_i)}{[3]_q}) + \eta_q(d_i) + \eta_q(c_i) = 1$, it follows that $\eta_q(c_i) \in E_q^{reg}(r)$. But $\eta_q(g_i) = q^{-4}\eta_q(\frac{\eta_q(g_i)}{[3]_q}) - q^{-2}\eta_q(d_i) + q^2\eta_q(c_i)$, whence the result.

As a $U_q(sl_2)$-module, $V_q(2)^{\otimes r} \cong \mathcal{K} \otimes_{\phi_q} V_q^{reg}(2)^{\otimes r}$ is the direct sum of isotypic components $I_q(2l)$, where every irreducible $U_q(sl_2)$-submodule of $I_q(2l)$ has highest weight $2l$. It follows from the $U_q(sl_2)$ case of Theorem 8.5 in [LZ] that $I_q(2l)$ is an irreducible $U_q(sl_2) \otimes_K BMW_r(K)$-submodule of $V_q(2)^{\otimes r}$.

**Lemma 7.6.**

1. $I_q^{reg}(2l) := I_q(2l) \cap V_q^{reg}(2)^{\otimes r}$ is a $BMW_r(q) \otimes_{A_q} U_{A_q}$-submodule of $V_q^{reg}(2)^{\otimes r}$.

2. The specialisation $I(2l) := C \otimes_{\phi_l} I_q^{reg}(2l)$ of $I_q^{reg}(2l)$ is isomorphic as a $U(sl_2)$-module to the isotypic component of $V(2)^{\otimes r}$ with highest weight $2l$.

3. $I(2l)$ is an irreducible $B_r(3) \otimes U(sl_2)$-module of $V(2)^{\otimes r}$.

**Proof.** By Lemma 7.3, $I_q^{reg}(2l)$ is stable under the action of $BMW_r(q)$. Since it is evidently a $U_{A_q}$-module and the $U_{A_q}$ action commutes with the action of $BMW_r(q)$, part (1) follows.
Thus, the specialisation argument of Proposition 7.1(5) shows that \( \dim \sum \) components with different highest weights. Thus the map \( W \) has the same dimension as that of the irreducible BMW\((\Lambda)\)-module, which we refer to as \( L_q^{BMW}(2l) \). Similarly, \( I(2l) \) is the direct sum of \( \dim C(V(2l)) \) copies of an irreducible \( B_r(3) \)-module \( L^{Br}(2l) \). Recall that both \( \dim K(V(2l)) \) and \( \dim C(V(2l)) \) are equal to \( 2l + 1 \).

**Lemma 7.7.** With notation as above, the irreducible \( BMW_r(q) \)-module \( L_q^{BMW}(2l) \) has the same dimension as that of the irreducible \( B_r(3) \)-module \( L^{Br}(2l) \).

**Proof.** If \( l \neq l' \), \( I(2l) \) and \( I(2l') \) intersect trivially since they are isotypical components with different highest weights. Thus \( \sum \dim I_q(2l) = 3^r = \sum \dim I(2l) \). But the specialisation argument of Proposition 7.1(5) shows that \( \dim K I_q(2l) \geq \dim C I(2l) \), whence

\[
\dim K I_q(2l) = \dim C I(2l).
\]

Thus

\[
\dim K L^{BMW}_q(2l) = \frac{\dim K I_q(2l)}{2l + 1} = \frac{\dim C I(2l)}{2l + 1} = \dim C L^{Br}(2l).
\]

Denote by \( \mathcal{R}(\Lambda) \) the radical of the BMW algebra \( BMW_\Lambda(\Lambda) \) and let \( BMW_\Lambda(\Lambda) = BMW_r(\Lambda) / \mathcal{R}(\Lambda) \) be its natural semisimple quotient. Then as explained in [5], \( BMW_\Lambda(\Lambda) = \bigoplus_{\lambda \in \Lambda} B_\lambda(\Lambda) \) with \( B_\lambda(\Lambda) \cong \text{End}_\Lambda(L_\Lambda(\lambda)) \), where \( L_\Lambda(\lambda) \) is the simple head of the cell module \( W_\Lambda(\lambda) \). As in Lemma 5.1, the surjective algebra homomorphism \( \eta : BMW_r(\Lambda) \to \text{End}_{BMW_r(\Lambda)}(V_q^{\otimes r}) \) induces a surjection \( \overline{\eta} : BMW_r(\Lambda) \to \text{End}_{BMW_r(\Lambda)}(V_q^{\otimes r}) \).

Similarly, let \( \overline{B}_{\Lambda}(3) \) denote the largest semi-simple quotient of the Brauer algebra. Then \( \overline{B}_{\Lambda}(3) = \bigoplus_{\lambda \in \Lambda} \overline{B}_\lambda(\Lambda) \) with \( \overline{B}_\lambda(\Lambda) \cong \text{End}_\Lambda(L(\lambda)) \), where \( L(\lambda) \) is the simple head of the cell module \( W(\lambda) \). Let \( \overline{\eta} : \overline{B}_{\Lambda}(3) \to \text{End}_{BMW_r(\Lambda)}(V_q^{\otimes r}) \) be the surjection induced by the map \( \eta : B_{\Lambda}(3) \to \text{End}_{BMW_r(\Lambda)}(V_q^{\otimes r}) \).

Recall that in analogy with (5.2), we have

\[
BMW_r(q) = \bigoplus_{\lambda \in \Lambda} A_q C_{S,T}^\lambda, \quad \text{where} \quad B_{A_q}(\lambda) = \sum_{S,T \in M(\lambda)} A_q C_{S,T}^\lambda.
\]

Taking appropriate tensor products with \( \mathcal{K} \) and \( \mathbb{C} \) respectively, and writing \( B_C(\lambda) \) for what was denoted \( B(\lambda) \) in (5.16), we obtain

\[
BMW_r(\mathcal{K}) = \bigoplus_{\lambda \in \Lambda} B_\mathcal{K}(\lambda), \quad \text{where} \quad B_\mathcal{K}(\lambda) = \sum_{S,T \in M(\lambda)} \mathcal{K} C_{S,T}^\lambda, \quad \text{and}
\]

\[
BMW_r(\mathbb{C}) = \bigoplus_{\lambda \in \Lambda} B_\mathbb{C}(\lambda), \quad \text{where} \quad B_\mathbb{C}(\lambda) = \sum_{S,T \in M(\lambda)} \mathbb{C} C_{S,T}^\lambda.
\]
Proposition 7.8. Maintain the above notation. Then \( \overline{\eta_q(B_q(\lambda))} \neq 0 \) if and only if \( \overline{\eta(B(\lambda))} \neq 0 \). For such \( \lambda \), we have \( \dim_K \overline{B_q(\lambda)} = \dim_K B(\lambda) \).

Proof. Let \( I^q_{reg}(\lambda) := B_{A_q}(\{\lambda\})V_{reg}(2)_{\otimes r} \), where \( B_{A_q}(\{\lambda\}) \) is defined by equation (7.10). Set \( I_q(\lambda) := K \otimes \phi I^q_{reg}(\lambda) \) and \( I(\lambda) := C \otimes \phi I^q_{reg}(\lambda) \). Then

\[
I(\lambda) = B_C(\{\lambda\})(C \otimes \phi) V_{reg}(2)_{\otimes r}.
\]

Note that \( I_q(\lambda) \) is a \( U_q(\mathfrak{sl}_2) \)-isotypic component of \( V_q(2)_{\otimes r} \), and \( I(\lambda) \) is isomorphic to an \( \mathfrak{sl}_2 \)-isotypic component of \( V(2)_{\otimes r} \). The \( U_q(\mathfrak{sl}_2) \)-highest weight of \( I_q(\lambda) \) is equal to the \( \mathfrak{sl}_2 \)-highest weight of \( I(\lambda) \).

If \( \overline{\eta_q(B_q(\lambda))} = 0 \), then \( I_q(\lambda) = 0 \). In this case, \( I(\lambda) = 0 \) and this is equivalent to \( \overline{\eta(B(\lambda))} = 0 \). If \( \overline{\eta_q(B_q(\lambda))} \neq 0 \), then \( I_q(\lambda) \neq 0 \), and it follows from Remark 7.4 that \( I(\lambda) \neq 0 \). Therefore \( \overline{\eta(B(\lambda))} \neq 0 \).

With the first statement of the Proposition established, the second follows immediately from Lemma 7.7. \( \square \)

Recall that \( \Lambda^0 \) denotes the set of all partitions with 3 or fewer boxes in the first two columns, and \( \Lambda^1 = \Lambda(r) \setminus \Lambda^0 \). We have the following analogue of Theorem 6.8.

Theorem 7.9. Let \( N_K \) be the kernel of the surjective map \( \eta_q : BMW_r(\mathcal{K}) \to \text{End}_{U_q(\mathfrak{sl}_2)}(V_q(2)_{\otimes r}) \). Denote by \( \mathcal{P}_K \) the two-sided ideal of \( BMW_r(\mathcal{K}) \) generated by \( \Phi_q \), and by \( \mathcal{R}_K \) the radical of \( BMW_r(\mathcal{K}) \).

1. \( \text{End}_{U_q(\mathfrak{sl}_2)}(V_q(2)_{\otimes r}) \cong \bigoplus_{\lambda \in \Lambda^0} \overline{B_q(\lambda)} \).
2. If \( \lambda \in \Lambda^1 \), then \( \Phi_q(I_q(\lambda)) \neq 0 \).
3. \( N_K \) acts trivially on \( L_q(\lambda) \) if and only if \( \lambda \in \Lambda^0 \).
4. \( \mathcal{P}_K + \mathcal{R}_K = N_K \).

Proof. Part (1) is an easy corollary of Proposition 7.8 in view of Theorem 6.8(3).

For any \( \lambda \in \Lambda^1 \), \( B_r(3) \Phi W(\lambda) = W(\lambda) \) by Theorem 6.8(1) and the cyclic property of \( W(\lambda) \). Since \( B_r(3) \Phi W(\lambda) \cong C \otimes \phi \Phi BMW_r(\mathcal{K}) \Phi W_{A_q}(\lambda) \) and \( W(\lambda) \cong C \otimes \phi \Phi W_{A_q}(\lambda) \), it follows that \( BMW_r(\mathcal{K}) \Phi W_{A_q}(\lambda) = W_{A_q}(\lambda) \) since \( \mathcal{K} \otimes \mathcal{A}_q W_{A_q}(\lambda) \) and \( W(\lambda) \) have the same dimensions. This implies part (2).

The proof of parts (3) and (4) is essentially the same as that of Theorem 6.8(2), (4), and will be omitted. \( \square \)

Remark 7.10. (1) Although Theorem 7.9 has been stated over \( \mathcal{K} \), it is clear that the statements (1)-(4) hold integrally, i.e., if we replace \( \mathcal{K} \) by \( A_q \) and all \( \mathcal{K} \) vector spaces by the corresponding free \( A_q \)-modules.

(2) Note that (2) and (3) of Theorem 7.9 imply that \( \Phi_q(I_q(\lambda)) = 0 \) if and only if \( \lambda \in \Lambda^0 \). This is because \( \Phi_q \in N_q \) implies (by (3)) that \( \Phi_q \) acts trivially on \( L_q(\lambda) \) for \( \lambda \in \Lambda^0 \), while (3) shows that \( \Phi_q(I_q(\lambda)) \neq 0 \) for \( \lambda \in \Lambda \setminus \Lambda^0 \).

The final result of this section is that to determine whether \( \Phi_q \) generates \( N_q \), it suffices to check the classical case.

Proposition 7.11. With notation as in Theorem 7.9, if \( \langle \Phi \rangle = \mathcal{P} \) contains \( \mathcal{R} \), the radical of \( B_r(3) \), then \( \langle \Phi_q \rangle_{BMW_r(\mathcal{K})} = \mathcal{P}_K \) contains the radical \( \mathcal{R}_K \) of \( BMW_r(\mathcal{K}) \).

Proof. We have already noted that by [11, Theorem 3.11], \( BMW_r(\mathcal{K}) \) is cellular, with the canonical anti-involution being defined by \( g_i^* = g_i \) and \( e_i^* = e_i \). It follows that
\[ \Phi^*_q = \Phi_q, \text{ and hence that } \mathcal{P}_\mathcal{K} \text{ is a self-dual two sided ideal of } \text{BMW}_r(\mathcal{K}). \]

Hence we may apply Theorem 7.9 to deduce that \( \mathcal{P}_\mathcal{K} \supseteq \mathcal{R}_\mathcal{K} \) if and only if \( \mathcal{P}_\mathcal{K} W_\mathcal{K}(\lambda) = \mathcal{R}_\mathcal{K}(\lambda) \) for each \( \lambda \in \Lambda^0 \), where \( \Lambda^0 \) is as in Theorem 7.9.

Now we are given that \( \mathcal{P} \supseteq \mathcal{R} \), whence \( \mathcal{P} W(\lambda) = R(\lambda) \), where \( W(\lambda) = W_\mathcal{C}(\lambda) \), etc. Write \( \mathcal{P}_q := (\Phi_q)_{\text{BMW}_r(q)} \).

Let \( \lambda \in \Lambda^0 \) and consider the \( \mathcal{A}_q \)-submodule \( \mathcal{P}_q W_{\mathcal{A}_q}(\lambda) \) of \( \mathcal{R}_{\mathcal{A}_q}(\lambda) \). For any element \( r \in \mathcal{R}_{\mathcal{A}_q}(\lambda) \), since \( 1 \otimes \phi_i, r \in 1 \otimes \phi_i \), \( \mathcal{P}_q W_{\mathcal{A}_q}(\lambda) \), there exist elements \( r_0 \in \mathcal{P}_q W_{\mathcal{A}_q}(\lambda) \) and \( w = (q - 1)r_1 \in (q - 1)W_{\mathcal{A}_q}(\lambda) \) such that \( r = r_0 + w \). But \( \mathcal{P}_q W_{\mathcal{A}_q}(\lambda) \subseteq \mathcal{R}_{\mathcal{A}_q}(\lambda) \) by Theorem 7.9 (3), whence \( w \in \mathcal{R}_{\mathcal{A}_q}(\lambda) \), and so evidently \( r_1 \in \mathcal{R}_{\mathcal{A}_q}(\lambda) \), since \( r_1 \) has zero inner product with \( W_{\mathcal{A}_q}(\lambda) \).

It follows that multiplication by \( q - 1 \) is an invertible endomorphism of the quotient \( \mathcal{R}_{\mathcal{A}_q}(\lambda)/\mathcal{P}_q W_{\mathcal{A}_q}(\lambda) \), whence the latter is an \( \mathcal{A}_q \)-torsion module. Hence \( \mathcal{K} \otimes_{\mathcal{A}_q} (\mathcal{R}_{\mathcal{A}_q}(\lambda)/\mathcal{P}_q W_{\mathcal{A}_q}(\lambda)) = 0 \), i.e. \( \mathcal{R}_\mathcal{K}(\lambda) = \mathcal{P}_\mathcal{K} W_\mathcal{K}(\lambda) \).

Corollary 7.12. \( \begin{align*} 
(1) & \text{ With the above notation, if } \Phi \text{ generates } \mathcal{N} \text{ then } \Phi_q \text{ generates } \mathcal{N}_\mathcal{K}. \\
(2) & \text{ If } r \leq 6, \text{ then } \Phi_q \text{ generates } \mathcal{N}_\mathcal{K} \text{ as an ideal of } \text{BMW}_r(\mathcal{K}). 
\end{align*} \)

The first statement is evident from Proposition 7.11 while the second follows from the first, together with Theorem 6.11.

We end this section by noting that our results imply the quantum analogues of the results of §6.

Corollary 7.13. \( \begin{align*} 
(1) & \text{ The algebra } \text{BMW}_4(\mathcal{K}) \text{ is semisimple.} \\
(2) & \text{ The cell modules of } \text{BMW}_5(\mathcal{K}) \text{ are all simple except for those corresponding to the partitions } (2, 1) \text{ and } (1^3), \text{ whose simple heads have dimensions } 15, 6 \text{ respectively.} \\
(3) & \text{ The cell modules } W_\mathcal{K}(1^3) \text{ and } W_\mathcal{K}(2, 1) \text{ have two composition factors each.} \\
(4) & \text{ The radical of } \text{BMW}_5(\mathcal{K}) \text{ has dimension } 239. 
\end{align*} \)

Proof. All statements are easy consequences of Proposition 7.1. For example, it follows from loc. cit.(5) and (6), that if \( W^\phi(\lambda) \) has just two composition factors, then either \( W^\phi(\lambda) \) is irreducible, or it also has two composition factors, whose dimensions are the same as those of \( W^\phi(\lambda) \). This implies the statements (3), (4) and (5) above. \( \square \)

8. A BMW-analogue of the Temperley-Lieb algebra

Although implicit above, we complete this work with an explicit definition of our analogue of the Temperley-Lieb algebra, together with some of its properties, as well as some questions about it.

Definition 8.1. Let \( \mathcal{A}_q \) be the ring \( \mathbb{C}[q^{\pm 1}, [3]_q^{-1}, (q + q^{-1})^{-1}, (q^2 + q^{-2})^{-1}] \). The \( \mathcal{A}_q \)-algebra \( \mathcal{P}_r(q) \) has generators \( \{g_i^{\pm 1}, e_i \mid 1 = 1, \ldots, r - 1 \} \) and relations given by (4.14) together with \( \Phi_q = 0 \), where \( \Phi_q \) is the word in the generators defined in (4.13).

We reproduce the relations here for convenience.
let $\phi \in \mathbb{C}$, and let $C$.

where $V$ is a module for $P$.

Some open problems.

8.2. Some open problems. We finish with some problems relating to $P_r(q)$.

(1) Determine whether $P_r(q)$ is generically semisimple, in particular whether $P_r(K)$ is semisimple. By Proposition 7.11 this is true provided that $P_r(C)$ is semisimple. The latter algebra has been shown (Proposition 6.10) to be semisimple for $r \leq 5$ and the case $r = 6$ has been verified by computer.

(2) A question equivalent to (1) is to determine whether $P_r(K)$ has dimension given by the formula (2.5). More explicitly, we know that

$$\dim_K P_r(K) \geq \binom{2r}{r} + \sum_{p=0}^{r-1} \binom{2r}{2p} \binom{2p}{p} \frac{3p - 2r + 1}{p + 1}.$$
with equality if and only if the ideal of $BMW_r(K)$ which is generated by $\Phi_q$ contains the radical $\mathcal{R}(K)$ of $BMW_r(K)$.

We therefore ask whether equality holds in (8.2).

(3) Is $P_r(q)$ free as $A_q$-module?

(4) Determine whether $P_r(q)$ has a natural cellular structure.

(5) Generalise the program of this work to higher dimensional representations of quantum $\mathfrak{sl}_2$.

Finally, we note that an affirmative answer to Conjecture [6.6] implies an affirmative answer to both (1) and (2) above.

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