A Social Welfare Optimal Sequential Allocation Procedure*

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Abstract

We consider a simple sequential allocation procedure for sharing indivisible items between agents in which agents take turns to pick items. Supposing additive utilities and independence between the agents, we show that the expected utility of each agent is computable in polynomial time. Using this result, we prove that the expected utilitarian social welfare is maximized when agents take alternate turns. We also argue that this mechanism remains optimal when agents behave strategically.

1 Introduction

There exist a variety of mechanisms to share indivisible goods between agents without side payments [Brams and Fishburn, 2000; Herreiner and Puppe, 2002; Brams et al., 2003; Brams and Kaplan, 2004; Brams et al., 2012]. One of the simplest is to let the agents take turns to pick an item. This mechanism is parameterized by a policy, the order in which agents take turns. In the alternating policy, agents take turns in a fixed order, whereas in the balanced alternating policy, the order of agents reverses each round. Bouveret and Lang (2011) study a simple model of this mechanism in which agents have strict preferences over items, and utilities are additive. They conjecture that the alternating policy maximizes the expected utilitarian social welfare for an arbitrary utility function, and prove it does so asymptotically. We close both conjectures. Surprisingly, we prove that the expected utility of each agent can be computed in polynomial time for any policy and utility function. Using this result, we prove that the alternating policy maximizes the expected utilitarian social welfare for any number of items and any linear utility function including Borda. Our results provide some justification for a mechanism in use in school playgrounds around the world.

2 Notation

Each of the \( n \) agents has a total preference order over the \( p \) items. A profile is an \( n \)-tuple of such orders. Agents have the same utility function. An item ranked in \( k \)th position has a utility \( g(k) \). For Borda utilities, \( g(k) = p - k + 1 \). The utility of a set of items is the sum of their individual utilities. Preference orders are independent and drawn uniformly at random from all \( p! \) possibilities (full independence). Agents take turns to pick items according to a policy, a sequence \( \pi = \pi_1 \ldots \pi_p \in \{1, 2, \ldots, n\}^p \). At the \( k \)-th step, agent \( \pi_k \) chooses one item from the remaining set. Without loss of generality, we suppose \( \pi_1 = 1 \). For profile \( R \), \( u_R(i, \pi) \) denotes the utility gained by agent \( i \) supposing every agent always chooses the item remaining that they rank highest. We write \( \pi_\pi(\pi) \) for the expectation of \( u_R(i, \pi) \) over all possible profiles. We take an utilitarian standpoint, measuring social welfare by the utility of each agent can be computed in just \( \frac{(k-1)}{2k} \) for \( k \geq 3 \) and \( \gamma_k = \gamma_k/k \) for all \( k \) and expect empty sums are zero and empty products are one.

3 Computing the Expected Social Welfare

Bouveret and Lang 2011 conjectured that it is NP-hard to compute the expected social welfare of a given policy as such a calculation depends on the super-exponential number of possible profiles. However, as we show now, the expected utility of each agent can be computed in just \( O(np^2) \) time for an arbitrary utility function, and \( O(np) \) time for Borda utilities. We begin with this last case, and then extend the results to the general case. Let \( P^n_p \) denote the set of all policies of length \( p \) for \( n \) agents. For \( p \geq 2 \), we define an operator \( P^n_p \rightarrow P^n_{p-1} \) mapping \( \pi \mapsto \tilde{\pi} \), by deleting the first entry. More precisely, \( \tilde{\pi}_i = \pi_{i+1} \) for \( i \in \{1, \ldots, p-1\} \). For example, \( \pi = 1211 \) and \( \tilde{\pi} = 211 \).

Lemma 1. For Borda scoring, \( n \geq 2 \) agents, \( p \geq 2 \) items and \( \pi \in P^n_p \) with \( \pi_1 = 1 \), we have:

\[
\gamma_1(\pi) = p + \gamma_1(\tilde{\pi}) \quad \gamma_i(\pi) = \frac{p+1}{p} \gamma_i(\tilde{\pi}), \quad i \neq 1
\]
and these values can be computed in $O(np)$ time.

**Proof.** Agent 1 picks her first item, giving her a utility of $p$. After that, from her perspective, it’s the standard game on $p - 1$ items with policy $\tilde{\pi}$, so she expects to get an utility of $\tilde{\pi}(\tilde{\pi})$. This proves the first equation. For the other agents, it is more involved. Let $i \in \{2, \ldots, n\}$ be a fixed agent. For $q \in \{1, \ldots, p\}$, let $a_i(q, \pi)$ denote the probability that under policy $\pi$ agent $i$ gets the item with utility $q$. Note that this probability does not depend on the utility function but only on the ranking: it is the probability that agent $i$ gets the item of rank $p - q + 1$ in her preference order. By the definition of expectation,

$$\bar{w}_i(\pi) = \sum_{q=1}^{p} a_i(q, \pi)q. \quad (1)$$

There are three possible outcomes of the first move of agent $1$ with respect to the item that has utility $q$ for agent $i$. With probability $(q - 1)/p$, agent 1 has picked an item with utility less than $q$ (for agent $i$), with probability $(p - q)/p$, agent 1 has picked an item with utility more than $q$, and with probability $1/p$ it was the item of utility equal to $q$. In the first case there are only $q - 2$ items of utility less than $q$ left, hence the probability for agent $i$ to get the item of utility $q$ is $a_i(q - 1, \tilde{\pi})$. In the second case there are still $q - 1$ items of value less than $q$, hence the probability to get the item of utility $q$ is $a_i(q, \tilde{\pi})$. In the third case, the probability to get the item of utility is zero, and together we obtain

$$a(q, \pi) = \frac{q - 1}{p} a_i(q - 1, \tilde{\pi}) + \frac{p - q}{p} a_i(q, \tilde{\pi}). \quad (2)$$

Substituting this into (1) yields

$$\bar{w}_i(\pi) = \sum_{q=1}^{p} \left[ \frac{q - 1}{p} a_i(q - 1, \tilde{\pi}) + \frac{p - q}{p} a_i(q, \tilde{\pi}) \right] q$$

$$= \sum_{q=1}^{p} \frac{(q - 1)q}{p} a_i(q - 1, \tilde{\pi}) + \sum_{q=1}^{p} \frac{(p - q)q}{p} a_i(q, \tilde{\pi})$$

In the first sum we substitute $q' = q - 1$ and this yields

$$\bar{w}_i(\pi) = \sum_{q'=0}^{p-1} \frac{q'}{p} \cdot a_i(q', \tilde{\pi}) \cdot (q' + 1) + \sum_{q=1}^{p} \frac{p - q}{p} a_i(q, \tilde{\pi}) \cdot q$$

The first term in the first sum and the last term in the second sum are equal to zero, so they can be omitted and we obtain

$$\bar{w}_2(\pi) = \sum_{q'=1}^{p-1} \frac{q'}{p} \cdot a_i(q', \tilde{\pi}) \cdot (q' + 1) + \sum_{q=1}^{p} \frac{p - q}{p} a_i(q, \tilde{\pi}) \cdot q$$

$$= \sum_{q=1}^{p-1} a_i(q, \tilde{\pi}) \left[ q \cdot (q + 1) + \frac{p - q}{p} \cdot q \right]$$

$$= \frac{p + 1}{p} \sum_{q=1}^{p-1} a_i(q, \tilde{\pi}) \cdot q = \frac{p + 1}{p} \bar{w}_2(\pi).$$

The time complexity follows immediately from the recursions.

| $\pi^1$ | $\pi^2$ | $\pi^3$ | $\pi^4$ | $\pi^5$ | $\pi^6$ |
|---------|---------|---------|---------|---------|---------|
| $\pi_1(\pi)$ | $\pi_2(\pi)$ | $\pi_3(\pi)$ | $\pi_4(\pi)$ | $\pi_5(\pi)$ | $\pi_6(\pi)$ |
| 1 2 3 4 5 6 | 1 2 3 4 5 6 | 1 2 3 4 5 6 | 1 2 3 4 5 6 | 1 2 3 4 5 6 | 1 2 3 4 5 6 |
| 2 1 3 4 5 6 | 2 1 3 4 5 6 | 2 1 3 4 5 6 | 2 1 3 4 5 6 | 2 1 3 4 5 6 | 2 1 3 4 5 6 |
| 3 1 2 4 5 6 | 3 1 2 4 5 6 | 3 1 2 4 5 6 | 3 1 2 4 5 6 | 3 1 2 4 5 6 | 3 1 2 4 5 6 |
| 4 1 2 3 5 6 | 4 1 2 3 5 6 | 4 1 2 3 5 6 | 4 1 2 3 5 6 | 4 1 2 3 5 6 | 4 1 2 3 5 6 |
| 5 1 2 3 4 6 | 5 1 2 3 4 6 | 5 1 2 3 4 6 | 5 1 2 3 4 6 | 5 1 2 3 4 6 | 5 1 2 3 4 6 |
| 6 1 2 3 4 5 | 6 1 2 3 4 5 | 6 1 2 3 4 5 | 6 1 2 3 4 5 | 6 1 2 3 4 5 | 6 1 2 3 4 5 |

**Table 1:** Expected utilities and expected utilitarian social welfare for $\pi^1 = 121212$ and $\pi^2 = 111222$

**Example 1.** Consider two agents with Borda utilities and the policies $\pi^1 = 121212$ and $\pi^2 = 111222$. We compute expected utilities and expected social welfare for each of them using Lemma 1. Table 1 shows results up to two decimal places. Note that expected values computed in all examples in the paper coincide with the results obtained by the brute-force search algorithm from [Bouveret and Lang, 2011].

Due to the linearity of Borda scoring the probabilities $a_i(q, \pi)$ in the proof of Lemma 1 cancel, and this will allow us to solve recursions explicitly and to prove our main result about the optimal policy for Borda scoring in Section 5.

In the general case, we can still compute the expected utilities $\tilde{w}_i(i, \pi)$, and thus $\tilde{w}_p(\pi)$, but we need the probabilities $a_i(q, \pi)$ from the proof of Lemma 1: $a_i(q, \pi)$ is the probability that under policy $\pi$, agent $i$ gets the item ranked at position $p - q + 1$ in her preference order. Computing these probabilities using (2) adds a factor of $p$ to the runtime.

**Lemma 2.** For $n \geq 2$ agents, $p \geq 2$ items, a policy $\pi \in P^n_p$ and an arbitrary scoring function $g$, the expected utility for agent $i$ is

$$\bar{w}_i(\pi) = \sum_{q=1}^{p} a_i(q, \pi)g(q)$$

and can be computed in $O(np^3)$ time.

Lemma 2 allows us to resolve an open question from [Bouveret and Lang, 2011].

**Corollary 1.** For $n$ agents and an arbitrary scoring utility function $g$, the expected utility of each agent, as well as the expected utilitarian social welfare can be computed in polynomial time.

For some special policies, the recursions in Lemma 1 can be solved explicitly. A particularly interesting policy is the strictly alternating one (denoted ALT POLICY) $\pi = 123\ldots n123\ldots n123\ldots n\ldots$.

**Proposition 1.** Let $\pi$ be the strictly alternating policy of length $p$ starting with 1. The expected utilities and utilitarian social welfare for two agents and Borda scoring are

$$\tilde{w}(\pi) = \frac{1}{3}[(2p - 1)(p + 1) + \gamma_{p+1}]$$

$$(\bar{w}_1(\pi), \bar{w}_2(\pi)) = \left\{ \begin{array}{ll} \left( \frac{p(p+1)}{3}, \frac{p^2-1}{3} \right) + \frac{1}{3}\gamma_{p+1} & \text{if } p \text{ is even,} \\
\left( \frac{p(p+1)}{3} + \frac{1}{3}\gamma_{p+1}, \frac{p^2-1}{3} \right) & \text{if } p \text{ is odd.} \end{array} \right.$$

For $n$ agents the expectations are, for all $i \in \{1, \ldots, n\}$,

$$\bar{w}_i(\pi) = \frac{p^2}{n+1} + O(p), \quad \tilde{w}(\pi) = \frac{np^2}{n+1} + O(p).$$
Proof: A proof for the two agents case can be done by straightforward induction using the recursions from Lemma 1. For $n$ agents we outline the proof. The full proof is given in Appendix A in the online version [Kalinowski et al., 2013a]. First, we solve the recursions for the expected utility of agent $i$ when the number of items is $p = i - 1 \pmod{n}$. Using these values, we approximate the expected utility for the remaining combinations of $p$ and $i$.

Example 2. Consider the policy $\pi^1$ from Example 1. Using Proposition 1, we get: $\overline{\pi_1}(\pi) = \frac{6(6+1)}{3} = 14$, $\overline{\pi_2}(\pi) = \frac{6^2-1}{1} + \frac{1}{\gamma_{6+1}} = 11.67 + 2.19/3 = 12.4$ and $\overline{\pi}(\pi) = 26.4$. These values coincide with results in Table 1.

For a fixed number $n$ of agents, we write the number of items as $p = kn + r$ with $0 \leq r < n$. We call a policy $\pi = \pi_1 \ldots \pi_p$ balanced if $\{\pi_{in+1}, \ldots, \pi_{in+n}\} = \{1, 2, \ldots, n\}$ for all $i \in \{0, \ldots, k - 1\}$ and $\{\pi_{kn+1}, \ldots, \pi_{kn+r}\} = r$. For any balanced policy, the expected utility of any agent lies between that of agents 1 and $n$ in the alternating policy. Thus every agent has expected utility $p^2/(n + 1) + O(p)$.

Proposition 3. For $n$ agents, and Borda utilities, the expected utilitarian social welfare of Random is $\frac{n(p+1)}{2}$. Proof. As the probability of each agent obtaining the $i$th item is $1/n$, the expected utilitarian social welfare is $n \sum_{i=1}^{p} \frac{i}{n} = \frac{p(p+1)}{2}$.

Table 2 summarizes the expected utilitarian social welfare for different mechanisms. Clearly, BestPref is an upper bound on the expected utilitarian social welfare for any allocation mechanism. As in [Bouveret and Lang, 2011], we define asymptotic optimality of a sequence of policies $(\pi^{(p)})_{p=1,2,\ldots}$ where $\pi^{(p)}$ is a policy for $p$ items by

$$\lim_{p \to \infty} \frac{\overline{\text{SW}}(\pi^{(p)})}{\overline{\text{SW}}(\pi)} = 1.$$ 

As can be seen from the table, ALTPolicy is an asymptotically optimal policy. By the observation in the end of the previous section the same is true for any balanced policy, and this implies Proposition 5 in [Bouveret and Lang, 2011]. However, the proof in [Bouveret and Lang, 2011] is incorrect as it implies that the expected utility is $p^2/n + O(1)$ for every agent which contradicts our upper bound for BestPref. See Appendix C for a detailed discussion of the gaps in the proof. Of course, for any given $p$ and preference orderings, ALTPolicy may not give the maximal utilitarian social welfare possible.

Example 3. Consider two agents and six items with the following preferences: $1 \succ 2 \succ 3 \succ 4 \succ 5 \succ 6$ and $1 \succ 6 \succ 2 \succ 3 \succ 4 \succ 5$. The ALTPolicy policy gives items $\{1, 2, 4\}$ and $\{6, 3, 5\}$ to agents 1 and 2 respectively. Hence, the total welfare is $(6 + 5 + 3) + (5 + 3 + 1) = 23$. Consider a policy $\pi = 121111$ which gives the following items to agents: $\{1, 2, 3, 4, 5\}$ and $\{6\}$. The total welfare is now $(6 + 5 + 4 + 3 + 2) + (6) = 25$. The Random mechanism gives the worst expected utilitarian social welfare among the three mechanisms. Moreover, as $n$ increases the expected utilitarian social welfare produced by Random declines compared with the other two mechanisms: $\lim_{p \to \infty} \frac{\overline{\text{SW}}(\text{ALTPolicy}^{(p)})}{\overline{\text{SW}}(\text{BestPref}^{(p)})} = \frac{n+1}{2n}$.

With two agents, the expected loss using ALTPolicy compared to BestPref (which requires full revelation of the

| ALTPolicy | BestPref | RANDOM |
|-----------|----------|-------|
| $\frac{p^2}{2} + O(p)$ | $\frac{p+1}{2} + O(1)$ | $\frac{p}{2}$ |

\[\frac{(p-1)!}{2} \geq \frac{(p+1)/2 - i(i+1)/2}{2}.\] So the expected utilitarian social welfare is

\[
\frac{1}{p!} \sum_{i=1}^{p} (p-1)! \left[ \frac{i^2}{2} + \frac{1}{2} p(p+1) - \frac{1}{2} i(i+1) \right]
\]

\[
= \frac{1}{2p} \sum_{i=1}^{p} [p(p+1) + i^2 - i] = \frac{(p+1)(4p-1)}{6}.
\]

As can be seen from the table, ALTPolicy is an asymptotically optimal policy. By the observation in the end of the previous section the same is true for any balanced policy, and this implies Proposition 5 in [Bouveret and Lang, 2011]. However, the proof in [Bouveret and Lang, 2011] is incorrect as it implies that the expected utility is $p^2/n + O(1)$ for every agent which contradicts our upper bound for BestPref. See Appendix C for a detailed discussion of the gaps in the proof. Of course, for any given $p$ and preference orderings, ALTPolicy may not give the maximal utilitarian social welfare possible.
Proposition 4. For two agents and any $\varepsilon > 0$, with probability at least $1 - \varepsilon$, ALTPOLICY is a $(1 - 1/6\varepsilon)$-approximation of the optimal expected utilitarian social welfare.

Proof. Let the random variables $z_1$ and $z_2$ denote the utilitarian social welfare for ALTPOLICY and BESTPREF. Then $z_2 - z_1 \geq 0$ and for the expectations we have $\mathbb{E}(z_1) = 2p^2 + p - 1 $ and $\mathbb{E}(z_2) = 4p^2 + 3p - 1$.

So $\mathbb{E}(z_2 - z_1) < p/6$, and by Markov’s inequality

$$P \left( z_2 - z_1 \geq \frac{p}{6\varepsilon} \right) \leq \frac{p/6}{p/(6\varepsilon)} = \varepsilon.$$

Writing it multiplicatively, with probability at least $1 - \varepsilon$,

$$\frac{z_1}{z_2} > \frac{p^2/2 - p/(6\varepsilon)}{p^2/2} = 1 - \frac{1}{3p\varepsilon}.$$ 

A similar result holds for more than two agents.

Proposition 5. For $n$ agents, there exists a constant $C$ such that for every $\varepsilon > 0$ with probability at least $1 - \varepsilon$ ALTPOLICY is a $(1 - C\varepsilon)$-approximation of the optimal expected utilitarian social welfare.

5 Optimality of the Alternating Policy

We now consider the problem of finding the policy that maximizes the expected utilitarian social welfare for Borda utilities and the full independence assumption.

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Recall that we consider that this problem is NP-hard. We close this problem, by proving that ALTPOLICY is in fact the optimal policy for any given $p$ with two agents.

Theorem 1. The expected utilitarian social welfare is maximized by the alternating policy for two agents supposing Borda utilities and the full independence assumption.

Note that by linearity of expectation this implies optimality of the alternating policy for every linear scoring function $g(k) = \alpha k + \beta$ with $\alpha, \beta \in \mathbb{R}, \alpha \leq 0$. In particular, the result also holds for quasi-indifferent scoring where $g(k) = N + (p - k + 1)$ for large $N$.

In the following let $\pi^*_p$ always be the alternating policy of length $p$. Also recall that due to symmetry we can only consider policies that starts with 1, e.g. policy 212 is equivalent to 121. To prove Theorem 1 we need to prove that for any policy $\pi$ of length $p$ the expected utilitarian social welfare is smaller or equal to the expected utilitarian social welfare of $\pi^*_p$. That is, $\text{dsw}_\pi = \text{sw}(\pi) - \text{sw}(\pi^*_p) \leq 0$. We proceed in two steps. First, we describe $\text{dsw}_\pi$ recursively, by representing the policy $\pi$ in terms of its deviations from $\pi^*_p$. Second, given the recursive description of $\text{dsw}_\pi$, we prove by induction that this difference is never positive (Proposition 6). The proof is not trivial as the natural inductive approach to derive $\text{dsw}_\pi \leq 0$ from $\text{dsw}_\pi \leq 0$ does not go through. Hence, we will prove a stronger result in Theorem 2 that implies Proposition 6 and Theorem 1.
Example 5. The hypothesis is too weak as the following example shows.

We might try to prove Proposition 6 in another direction. We describe this strengthening first and then outline the induction argument. The technical details of the proof are presented in the online Appendix D.

To remedy this problem we would like to strengthen the proposition, for example by proving \( a + b \leq f(a, b) \) for all \((a, b) \in \bigcup_k A_k \) where \( f \) is some function with \( f(x, y) \leq 0 \) for all \((x, y)\). The difficulty of finding such a function is indicated by Figure 3 showing the set \( A_{10} \). Different markers distinguish the points arising from \( A_9 \) by following \( \pi^* \) from those deviating from \( \pi^* \).

The key idea of our proof is to strengthen Proposition 6 in another direction. We describe this strengthening first and then outline the induction argument. The technical details of the proof are presented in the online Appendix D.

Consider a policy \( \pi \) that is represented by a node \( n_\pi \) at level \( k \) in the policy tree. Instead of requiring the inequality \( a + b \leq 0 \) only for the point \((a, b) \in A_k \) that corresponds to policy \( \pi \), we also require it for (i) all policies that lay on the path that follow only the right branches from \( n_\pi \) and (ii) all policies that lay on the path that starts from \( n_\pi \) by following the left branch once and then only follow the right branches. To formalize this idea, for \( k \geq 1 \) we define functions \( f_k, g_k : \mathbb{R}^2 \to \mathbb{R}^2 \) by

\[
f_k(x, y) = (x + \delta_{k+1}, \frac{k+2}{k+1}(y - \delta_{k+1})^2)
\]

and

\[
g_k(x, y) = (y, \frac{k+2}{k+1}(x + \delta_{k+1}^2 - \delta_{k+1})^2)
\]

Note that \( A_{k+1} \) is \( f_k(A_k) \cup g_k(A_k) \) for all \( k \geq 1 \), as \( f_k \) encodes the case when we follow the left branch and \( g_k \) the right branch. Figure 2 illustrates this correspondence. We also consider iterated compositions of these functions. For every \( k \geq 1 \) let \( G_{k0} \) denote the identity on \( \mathbb{R}^2 \), i.e. \( G_{k0}(x, y) = (x, y) \), and for \( m \geq 1 \) let \( G_{km} \) denote the function

\[
G_{km} = g_{k+m-1} \circ g_{k+m-2} \circ \cdots \circ g_k
\]

Applying \( G_{km} \) to the point \((a, b) \in A_k \) corresponding to \( \pi \in P_k \) gives the point \((a', b') \in A_{k+m} \) which corresponds
to the policy \( \pi' \in \mathcal{P}_{k+m} \) that is obtained from \( \pi \) by following the right branch \( m \) times. For all \( k \geq 1 \) and \( m \geq 1 \), we define the function \( F_{km} = G_{k+1,m-1} \circ f_k \). \( F_{km} \) corresponds to starting in level \( k \), following the first left branch and then \( m - 1 \) right branches. For \( (x, y) \in A_k \), \( G_{km}(x, y) \in A_{k+m} \) for \( m \geq 0 \) and \( F_{km}(x, y) \in A_{k+m} \) for \( m \geq 1 \). Proposition 6 is a consequence of the following theorem.

**Theorem 2.** Let \( A_1 = \{(0, 0)\} \) and

\[
A_{k+1} = f_k(A_k) \cup g_k(A_k) = \left\{ \left( y, \frac{k+2}{k+1} x \right) \right\} \cup \left\{ \left( x + \delta_{k+1}, \frac{k+2}{k+1} y - \delta_{k+1} \right) \right\}
\]

for \( k \geq 1 \) where \( (x, y) \in A_k \), \( \delta_k = \frac{1}{3} (k + (-1)^k \gamma_k) \). Then for every \( k \geq 1 \) and every \( (x, y) \in A_k \), the following statements are true.

1. For all \( m \geq 0 \), if \( (x', y') = G_{km}(x, y) \) then \( x' + y' \leq 0 \).
2. For all \( m \geq 1 \), if \( (x', y') = F_{km}(x, y) \) then \( x' + y' \leq 0 \).

**Proof sketch.** We provide the full proof in the Appendix C. We give a high-level overview. We start with a few technical lemmas to derive an explicit description of functions \( (x', y') = F_{km}(x, y) \) and \( (x'', y'') = G_{km}(x, y) \). This gives us explicit expressions for the sums \( x' + y' \) and \( x'' + y'' \) in terms of \( x \) and \( y \). Then we proceed through the induction proof. We summarize the induction step here. Suppose the statements of the theorem are already proved for all sets \( A_l \) with \( l < k \). Let \( (x, y) \) be an arbitrary element of \( A_k \). Suppose \( (x, y) = f_{k-1}(\bar{x}, \bar{y}) \) for some \( (\bar{x}, \bar{y}) \in A_{k-1} \) (the case \( (x, y) = g_{k-1}(\bar{x}, \bar{y}) \) is similar). Figure 4 shows \( (\bar{x}, \bar{y}) \) and \( (x, y) \) that is obtained from \( (\bar{x}, \bar{y}) \) by following the left branch. By the induction hypothesis, \( x' + y' \leq 0 \) whenever \( (x', y') \in \{ G_{k-1,m}(\bar{x}, \bar{y}), F_{k-1,m}(\bar{x}, \bar{y}) \} \). The corresponding nodes are highlighted in gray in Figure 4. To complete the induction step we need to show \( x' + y' \leq 0 \) for \( (x', y') \in \{ G_{km}(x, y), F_{km}(x, y) \} \). The corresponding nodes are indicated by dashed circles. The result for \( (x', y') = G_{km}(x, y) \) (gray and dashed) follows immediately as \( (x', y') = G_{km}(x, y) = F_{k-1,m+1}(\bar{x}, \bar{y}) \). For \( (x', y') = F_{km}(x, y) \) we first express \( x' + y' \) in terms of \( x \) and \( y \). Then, by induction \( x'' + y'' \leq 0 \) for \( (x'', y'') = G_{k-1,m-1}(\bar{x}, \bar{y}) \). Inverting the representation of \( x'' + y'' \) in terms of \( \bar{x} \) and \( \bar{y} \), we derive a bound \( \bar{x} + \bar{y} \leq -c(m) \leq 0 \), depending on \( m \), and this stronger bound is used to prove \( x' + y' \leq 0 \) for \( (x', y') = F_{km}(x, y) \).

The extension of Theorem 2 to \( n \) agents is not straightforward. Firstly, it requires deriving exact recursions for the expected utility for an arbitrary \( p \). This is not trivial, as Proposition 1 only provides asymptotics. An easier extension might be to other utility functions. The alternating policy is not optimal for all scoring functions. For example, it is not optimal for the \( k \)-approval scoring function which has \( g(i) = 1 \) for \( i \leq k \) and 0 otherwise. However, we conjecture that ALTPOLICY is optimal for all convex scoring functions (which includes lexicographical scoring).

6 Strategic Behaviour

So far, we have supposed agents sincerely pick the most valuable item left. However, agents can sometimes improve their utility by picking less valuable items. To understand such strategic behaviour, we view this as a finite repeated game with perfect information. [Kohler and Chandrasekaran, 1971] proves that we can compute the subgame perfect Nash equilibrium for the alternating policy with two agents by simply reversing the policy and the preferences and playing the game backwards. More recently, [Kalinowski et al., 2012] prove this holds for any policy with two agents.

We will exploit such reversal symmetry. We say that a policy \( \pi \) is reversal symmetric if and only the reversal of \( \pi \), after interchanging the agents if necessary, equals \( \pi \). The policies 1212 and 1221 are reversal symmetric, but 1121 is not. The next result follows quickly by expanding and rearranging expressions for the expected utilitarian social welfare using the fact that we can compute strategic play by simply reversing the policy and profile and supposing truthful behaviour.

**Theorem 3.** For two agents and any utility function, any reversal symmetric policy that maximizes the expected utilitarian social welfare for truthful behaviour also maximizes the expected utilitarian social welfare for strategic behaviour.

As the alternating policy is reversal symmetric, it follows that the alternating policy is also optimal for strategic behaviour. Unfortunately, the generalisation of these results to more than two agents is complex. Indeed, for an unbounded number of agents, computing the subgame perfect Nash equilibrium becomes PSPACE-hard [Kalinowski et al., 2013b].

7 Conclusions

Supposing additive utilities, and full independence between agents, we have shown that we can compute the expected utility of a sequential allocation procedure in polynomial time for any utility function. Using this result, we have proven that the expected utilitarian social welfare for Borda utilities is maximized by the alternating policy in which two agents pick items in a fixed order. We have argued that this mechanism remains optimal when agents behave strategically. There remain open several important questions. For example, is the alternating policy optimal for more than two agents? What happens with non-additive utilities?
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