HYPERBOLIC GEODESICS, KRZYZ’S CONJECTURE AND BEYOND

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Abstract. In 1968, Krzyz conjectured that for non-vanishing holomorphic functions \( f(z) = c_0 + c_1 z + \ldots \) in the unit disk with \(|f(z)| \leq 1\), we have the sharp bound \( |c_n| \leq 2/e \) for all \( n \geq 1 \), with equality only for the function \( f(z) = \exp((z^n - 1)/(z^n + 1)) \) and its rotations. This conjecture was considered by many researchers, but only partial results have been established. The desired estimate has been proved only for \( n \leq 5 \).

We provide here two different proofs of this conjecture and its generalizations based on completely different ideas.

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1. Introduction, statement of results

1.1. Conjecture. Non-vanishing holomorphic functions \( f(z) = c_0 + c_1 z + \ldots \) on the unit disk \( \Delta = \{ z : |z| < 1 \} \) (i.e., such that \( f(z) \neq 0 \) in \( \Delta \)) form the normal families admitting certain invariance properties, for example, the invariance under action of the Möbius group of conformal self-maps of \( \Delta \), complex homogeneity, etc. One of the most interesting examples of such families is the set \( B_1 \subset H^\infty \) of holomorphic maps of \( \Delta \) into the punctured disk \( \Delta^* = \Delta \setminus \{0\} \).

Compactness of \( B_1 \) in topology of locally uniform convergence on \( \Delta \) implies the existence for each \( n \geq 1 \) the extremal functions \( f_0 \) maximizing \( |c_n(f)| \) on \( B_1 \). Such functions are nonconstant and must satisfy \( |f(e^{i\theta})| = 1 \) for almost all \( \theta \in [0, 2\pi] \).

Estimating coefficients on \( B_1 \) was originated in 1940’s (see [25]). In 1968, Krzyz [18] conjectured that for all \( n \geq 1 \),

\[
|c_n| \leq 2/e, \tag{1.1}
\]

with equality only for the function \( \kappa_n(z) = \kappa(z^n) \), where

\[
\kappa(z) := \exp\left(\frac{z - 1}{z + 1}\right) = \frac{1}{e} + \frac{2}{e} z - \frac{2}{3e} z^3 + \ldots \quad \tag{1.2}
\]

and its rotations \( \epsilon_1 \kappa(\epsilon_2 z) \) with \( |\epsilon_1| = |\epsilon_2| = 1 \). Note that \( \kappa(z) \) is a holomorphic universal covering map \( \Delta \to \Delta^* \) moving 0 to \( 1/e \).

This problem has been investigated by a large number of mathematicians, however it still remained open. The estimate (1.1) was established only for some initial coefficients \( c_n \) including all \( n \leq 5 \) (see [11], [24], [28], [29], [30]). On developments related to this problem see, e.g., [2], [10], [11], [20], [21], [25], [29].

Our main goal is to prove that Krzyz’s conjecture is true for all \( n \geq 1 \):

**Theorem 1.1.** For every \( f(z) = c_0 + c_1 z + \ldots \in B_1 \) and \( n \geq 1 \), we have the sharp bound (1.1), and the equality occurs only for the function \( \kappa_n \) and its rotations.
1.2. Proofs and generalizations. We provide two completely different proofs of this theorem. The first proof relies on complex geometry of convex Banach domains and reveals geodesic features of the cover function (1.2), while the second one involves the results related to the universal Teichmüller space and extremal Beltrami coefficients following the lines originated in [14]. Both proofs shed light on the intrinsic connection between the complex geodesics and extremals of holomorphic functionals.

We also obtain some generalizations of Theorem 1.1. The arguments in the first proof of Theorem 1.1 yields in the case \( n = 2 \) the following improvement of this theorem: any functional \( J(f) = c_2 + P(c_1) \), where \( P(c_1) \) is a homogeneous polynomial of degree 2, satisfying \( |P(c_1)| < \kappa'(0) = 2/e \) for all \( f \in \mathcal{B}_1 \) distinct from \( \kappa \), satisfies

\[
\max_{\mathcal{B}_1} |J(f)| = \max_{\mathcal{B}_1} |c_2| = 2/e
\]

with the same extremal function \( \kappa_2 \) (up to rotations).

The second proof deals with more general bounded functionals \( J(f) = c_n + F(c_{m_1}, \ldots, c_{m_s}) \) on \( \mathcal{B}_1 \) where \( c_j = c_j(f); 1 \leq n, m_j \) and \( F \) is a holomorphic function of \( s \) variables in an appropriate domain of \( \mathbb{C}^s \). Assuming that this domain contains the origin \( 0 \) and that \( F, \partial F \) vanish at \( 0 \), we establish that any such a functional is sharply estimated on appropriate subsets \( E_r \subset \mathcal{B}_1 \) (with \( r \) depending on \( n \)) by

\[
\max_{E_r} |J(f)| = \max_{E_r} |c_n| = M_n r, \quad M_n = \max_{\mathcal{B}_1} |J(f)|
\]

and obtain the desired bound (1.1) in the limit as \( r \to 1 \).

2. Background: Invariant metrics on convex Banach domains

We present briefly some basic results in complex geometry of convex domains in complex Banach spaces, underlying the proofs of our main theorem.

2.1. Invariant metrics. Let \( M \) be a complex Banach manifold modeled by a Banach space \( X \). The Kobayashi metric \( d_M \) on \( M \) is the largest pseudometric \( d \) on \( M \) that does not get increased by holomorphic maps \( h : \Delta \to M \) so that for any two points \( x_1, x_2 \in M \), we have

\[
d_M(x_1, x_2) \leq \inf\{d_\Delta(0, t) : h(0) = x_1, \ h(t) = x_2\},
\]

where \( d_\Delta \) is the hyperbolic metric on the unit disk of Gaussian curvature \(-4\), hence with the differential form

\[
ds = \lambda_\Delta(z)|dz| := |dz|/(1 - |z|^2).
\]

The Carathéodory distance between \( x_1 \) and \( x_2 \) in \( M \) is

\[
c_M(x_1, x_2) = \sup d_\Delta(f(x_1), f(x_2)),
\]

where the supremum is taken over all holomorphic maps \( f : E \to \Delta \).

In the case of a bounded domain \( M \), both \( d_M \) and \( c_M \) are distances (i.e., separate the points in \( M \)). For general properties of invariant metrics we refer to [4], [12]. A remarkable fact is:

Proposition 2.1. [5], [19] If \( M \) is a convex domain in complex Banach space, then

\[
d_M(x_1, x_2) = c_M(x_1, x_2) = \inf\{d_\Delta(h^{-1}(x_1), h^{-1}(x_2)) : h \in \text{Hol}(\Delta, M)\}. \tag{2.1}
\]

Similar equality holds for the differential (infinitesimal) forms of these metrics which are defined on the tangent bundle \( TM \) of \( M \).
2.2. Complex geodesics. A holomorphic map \( h \) of the disk \( \Delta \) into a Banach manifold endowed with a pseudo-distance \( \rho \) is called complex \( \rho \)-geodesic if there exist two points \( t_1 \neq t_2 \in \Delta \) for which

\[
d_\Delta(t_1, t_2) = \rho(h(t_1), h(t_2))
\]

(one says also that their images \( h(t_1) \) and \( h(t_2) \) can be joined in \( M \) by a complex \( \rho \)-geodesic; cf. [31]). Any \( c_M \)-geodesic is also \( d_M \)-geodesic, and then the equality (2.1) holds for all points of the disk \( h(\Delta) \).

Certain conditions ensuring the existence of complex geodesics, which will be used here, are given in [3], [5].

Assume that a Banach space \( X \) has a predual space \( Y \), i.e., that \( X = Y' \) is the space of bounded linear functionals \( x(y) = \langle x, y \rangle \) on \( Y \), and consider on \( X \) the weak* topology \( \sigma(X, Y) \) which is the topology of pointwise convergence on points of \( Y \), i.e., a sequence \( \{x_n\} \subset X \) is convergent in \( \sigma(X, Y) \) to \( x \in X \) if \( x_n(y) \to x(y) \) for all \( y \in Y \).

If \( X \) has a predual \( Y \), then by the Alaoglu-Bourbaki theorem, the closure of the open unit ball \( X_1 \) of the space \( X \) in the topology \( \sigma(X, Y) \) is compact.

**Proposition 2.2.** [4], [5] Let \( M \) be a bounded convex domain in a complex Banach space \( X \) with predual \( Y \). If the closure of \( M \) is \( \sigma(X, Y) \)-compact, then every distinct pair of points in \( M \) can be joined by a complex \( c_M \)-geodesic.

This proposition also has its differential counterpart which provides that under the same assumptions, for any point \( x \in M \) and any nonzero vector \( v \in X \), there exists at least one complex geodesic \( h : \Delta \to M \) such that \( h(0) = x \) and \( h'(0) \) is collinear to \( v \) (cf. [5]).

2.3. Evaluation of holomorphic maps on geodesic disks. We shall need the following corollary of the above propositions controlling the growth of holomorphic maps with critical points on geodesic disks (cf. [17]).

**Lemma 2.3.** Let a domain \( M \) satisfy the assumptions of Proposition 2.2 and \( g \) be a holomorphic map \( M \to \Delta \) whose restriction to a geodesic disk \( h(\Delta) \subset M \), \( h(0) = 0 \), has at the origin zero of order \( m \geq 1 \), i.e.,

\[
g \circ h(t) = c_m t^m + c_{m+1} t^{m+1} + \ldots.
\]

Then the growth of \( |g| \) on this disk is estimated by

\[
|g \circ h(t)| \leq |t|^m (|t| + |c_m|)/(1 + |c_m||t|)
\]

\[= \tanh d_M \left(0, h \left( \frac{|t|^m |t| + |c_m|}{1 + |c_m||t|} \right) \right) \leq \tanh d_M(0, h(t^m)). \tag{2.2}
\]

The equality in the right inequality occurs (even for one \( t_0 \neq 0 \)) only when \( |c_m| = 1 \); then \( h(t) \) is a hyperbolic isometry of the unit disk and all terms in (2.2) are equal.

**Proof.** By Golusin’s version of Schwarz’s lemma, a holomorphic function

\[
f(t) = c_m t^m + c_{m+1} t^{m+1} + \cdots : \Delta \to \Delta \quad (c_m \neq 0, \ m \geq 1)
\]

is estimated in \( \Delta \) by

\[
|f(t)| \leq |t|^m \frac{|t| + |c_m|}{1 + |c_m||t|},
\]

and the equality occurs only for \( f_0(t) = t^m (t + c_m)/(1 + c_m^2 t) \) (see [7, Ch. 8]).

It follows from Proposition 2.1 and weak* compactness of the closure of \( M \) in \( \sigma(X, Y) \) that for any \( t_0 \neq 0 \) and \( x_0 = h(t_0) \) there exists a holomorphic map \( j : M \to \Delta \) such that

\[
d_\Delta(0, j(x_0)) = c_M(0, x_0) = d_M(0, x_0).
\]
Letting
\[ \eta(t) = |t|^m (|t| + |c_m|) / (1 + |c_m||t|), \]
one gets \( \eta(t) \leq |t| \) and
\[ |g \circ h(t_0)| \leq |j \circ h(\eta(t_0))| = \tanh d_M(0, h(\eta(t_0))) \leq \tanh d_M(0, h(t_0)), \]
which yields (2.2).

There is also a differential analog of the inequalities (2.2) which involves the infinitesimal Carathéodory and Kobayashi metrics. It will not be used here.

Lemma 2.3 straightforwardly extends to general complex Banach manifolds \( M \) having equal Carathéodory and Kobayashi distances.

### 2.4. Generalized Gaussian curvature of subharmonic metrics

The proof of Theorem 1.1 involves subharmonic conformal metrics \( \lambda(t)|dt| \) on the disk having the curvature at most \(-4\) in a somewhat generalized sense. As well-known, the Gaussian curvature of a \( C^2 \)-smooth metric \( \lambda > 0 \) is defined by
\[ k_\lambda(t) = -\frac{\Delta \log \lambda(t)}{\lambda(t)^2}, \]
where \( D \) means the Laplacian \( 4\partial^2/\partial z\partial \bar{z} \).

A metric \( \lambda(t)|dt| \) in a domain \( G \subset \mathbb{C} \) (or on a Riemann surface) has the curvature less than or equal to \( K \) in the supporting sense if for each \( K' > K \) and each \( z_0 \) with \( \lambda(z_0) > 0 \), there is a \( C^2 \)-smooth supporting metric \( \hat{\lambda} \) for \( \lambda \) at \( t_0 \) (i.e., such that \( \hat{\lambda}(t_0) = \lambda(t_0) \) and \( \hat{\lambda}(t) \leq \lambda(t) \) in a neighborhood of \( t_0 \)) with \( k_{\hat{\lambda}}(t_0) \leq K' \), or equivalently,
\[ \Delta \log \lambda \geq K\lambda^2, \] (3.3)
A metric \( \lambda \) has curvature at most \( K \) in the potential sense at \( t_0 \) if there is a disk \( U \) about \( z_0 \) in which the function
\[ \log \lambda + K \text{Pot}_U(\lambda^2), \]
where \( \text{Pot}_U \) denotes the logarithmic potential
\[ \text{Pot}_U h = \frac{1}{2\pi} \int_U h(\zeta) \log |\zeta - t| d\xi d\eta \quad (\zeta = \xi + i\eta), \]
is subharmonic. This is equivalent to \( \lambda \) to satisfy (3.3) in the sense of distributions.

One can replace above \( U \) by any open subset \( V \subset U \), because the function \( \text{Pot}_U(\lambda^2) - \text{Pot}_V(\lambda^2) \) is harmonic on \( U \).

Due to Royden [27], a conformal metric has curvature at most \( K \) in the supporting sense has curvature at most \( K \) also in the potential sense.

The following lemma concerns the circularly symmetric (radial) metrics on the disk (i.e. such that \( \lambda(t) = \lambda(|t|) \)) and is a slight improvement of the corresponding Royden’s lemma [27] to singular metrics with a prescribed singularity at the origin.

**Lemma 2.4.** [18] Let \( \lambda(|t|)|dt| \) be a circularly symmetric subharmonic metric on \( \Delta \) such that
\[ \lambda(r) = mcr^{m-1} + O(r^m) \text{ as } r \to 0 \text{ with } 0 < c \leq 1 \quad (m = 1, 2, \ldots), \] (2.4)
and this metric has curvature at most \(-4\) in the potential sense. Then
\[ \lambda(r) \geq \frac{mcr^{m-1}}{1 - c^2 r^{2m}}. \] (2.5)

Note that all metrics subject to (2.4) are dominated by \( \lambda_m(t) = m|t|^{m-1}/(1 - |t|^{2m}). \)
3. Preliminary results

We first establish some analytic and geometric facts for nonvanishing functions essentially applied in the proofs. These results have their intrinsic interest.

1. Covering maps.

Proposition 3.1. (a) Every function \( f \in B_1 \) admits factorization

\[
f(z) = \kappa \circ \hat{f}(z),
\]

where \( \hat{f} \) is a holomorphic map of the disk \( \Delta \) into itself (hence, from \( H^\infty \)) and \( \kappa \) is the function (1.2).

(b) Moreover, the map (3.1) generates an \( H^\infty \)-holomorphic map \( k : H^\infty \to B_1 \).

Proof. (a) Due to a general topological theorem, any map \( f : M \to N \), where \( M, N \) are manifolds, can be lifted to a covering manifold \( \hat{N} \) of \( N \), under appropriate relation between the fundamental group \( \pi_1(M) \) and a normal subgroup of \( \pi_1(N) \) defining the covering \( \hat{N} \) (see, e.g., [Ma]). This construction produces a map \( \hat{f} : M \to \hat{N} \) satisfying

\[
f = p \circ \hat{f},
\]

where \( p \) is a projection \( \hat{N} \to N \). The map \( \hat{f} \) is determined up to composition with the covering transformations of \( \hat{N} \) over \( N \) or equivalently, up to choosing a preimage of a fixed point \( x_0 \in \hat{N} \) in its fiber \( p^{-1}(x_0) \). For holomorphic maps and manifolds the lifted map is also holomorphic.

In our special case, \( \kappa \) is a holomorphic universal covering map \( \Delta \to \Delta_\ast = \Delta \setminus \{0\} \), and the representation (3.2) provides the equality (3.1) with the corresponding \( \hat{f} \) determined up to covering transformations of the unit disk compatible with the covering map \( \kappa \).

The assertion (b) is a consequence of a well-known property of bounded holomorphic functions in Banach spaces with sup norm given by

Lemma 3.2. Let \( E, T \) be open subsets of complex Banach spaces \( X, Y \) and \( B(E) \) be a Banach space of holomorphic functions on \( E \) with sup norm. If \( \varphi(x,t) \) is a bounded map \( E \times T \to B(E) \) such that \( t \mapsto \varphi(x,t) \) is holomorphic for each \( x \in E \), then the map \( \varphi \) is holomorphic.

Holomorphy of \( \varphi(x,t) \) in \( t \) for fixed \( x \) implies the existence of complex directional derivatives

\[
\varphi'(x,t) = \lim_{\zeta \to 0} \frac{\varphi(x,t + \zeta v) - \varphi(x,t)}{\zeta} = \frac{1}{2\pi i} \int_{|\zeta|=1} \frac{\varphi(x,t + \xi v)}{\xi^2} d\xi,
\]

while the boundedness of \( \varphi \) in sup norm provides the uniform estimate

\[
\|\varphi(x,t + cv) - \varphi(x,t) - \varphi'(x,t)cv\|_{B(E)} \leq M|c|^2,
\]

for sufficiently small \(|c|\) and \( \|v\|_Y \) (cf. [B]).

The map \( k : \hat{f} \to f \) is bounded on the ball \( H^\infty_1 \). Applying Hartog’s theorem on separate holomorphy to the sums \( g(z,t) = \hat{f}(z) + \hat{h}(z) \) of \( \hat{f} \in H^\infty_1 \), \( \hat{h} \in H_1 \) and \( t \) from a region \( B \subset \hat{C} \) so that \( g(z,t) \in H^\infty_1 \), one obtains that \( g(z,t) \) are jointly holomorphic in both variables \((z,t) \in \Delta \times B \). Thus the restriction of the map \( k_0 \) onto intersection of the ball \( H^\infty_1 \) with any complex line \( L = \{ \hat{f} + \hat{h} \} \) is \( H^\infty \)-holomorphic, and hence this map is holomorphic as the map \( H^\infty_1 \to B_1 \), which completes the proof of Lemma 3.1.

As an immediate corollary of this lemma, one gets the following known estimate, which will be used here.

Lemma 3.3. For any \( f \in B_1 \),

\[
|c_1| \leq 2/e,
\]
with equality only for the rotations \( e^{i\alpha_1} \kappa(e^{i\alpha} z) \) of \( \kappa \) (in particular, these functions maximize \( |c_1| \) among the holomorphic covering maps \( \Delta \to \Delta^* \)).

**Proof.** Given \( f \in B_1 \) distinct from \( \kappa \), one may rotate its covering map \( \hat{f} \) in (3.1) to get \( \hat{f}(0) = a \), where \( 0 < a < 1 \). By Schwarz’s lemma, \( |\hat{f}'(0)| \leq 1 - |a|^2 < 1 \); hence,

\[
|f'(0)| = |\kappa'(a)||\hat{f}'(0)| < |\kappa'(a)| = \frac{2e^{(a-1)/(a+1)}}{(a+1)^2} < \frac{2}{e};
\]

which implies (3.3).

We shall also lift the functions \( f \in B_1 \) to the universal cover of \( \Delta^* \) by the left half-plane \( \mathbb{C}_- = \{ w \in \mathbb{C} : \Re w < 0 \} \) using the map \( \kappa \circ \sigma^{-1} = \exp \), where

\[
\sigma(z) = (z - 1)/(z + 1) : \Delta \to \mathbb{C}_-.
\]

These lifts of \( f \) are reduced to choice of branches of \( \log f(z) \) determined by the values of \( \log f(0) \) in \( \mathbb{C}_- \).

2. Open domain of nonvanishing functions and its holomorphic embedding.

Consider the annuli

\[
A_r = \{ r < |z| < 1 \}, \quad 0 < r < 1,
\]

exhausting the punctured disk \( \Delta \setminus \{0\} \), and let \( B_r \) be the subset of nonvanishing functions \( f \in B_1 \) sharing the values in \( A_r \), that is,

\[
B_r = \{ f \in B_1 : f(\Delta) \subset A_r \};
\]

then \( B_r \subset B_{r'} \) if \( r < r' \). Put

\[
B_1^0 = \bigcup_r B_r;
\]

this union is located in the unit ball \( H^\infty_1 \) of the space \( H^\infty = H^\infty(\Delta) \). It will be convenient to regard the free coefficients \( c_0(f) \) as the constant elements of \( B_1 \).

The following lemma provides some needed topological properties of these sets.

**Lemma 3.4.** (a) For any \( r \in (0, 1) \), every point of \( B_r \) has a neighborhood \( U(f, \epsilon(r)) \) in \( H^\infty \), which contains only the functions belonging to some \( B_{r_*} \), where \( 0 < r_* = r_*(r) \leq r \).

(b) Each set \( B_r \) is path-wise connective in \( H^\infty_1 \).

It follows that the union \( B_1^0 \) is a domain in \( H^\infty_1 \) filled by nonvanishing functions on \( \Delta \). In particular, it contains all functions \( f \in B_1 \) which are holomorphic and nonvanishing on the closed disk \( \overline{\Delta} \).

**Proof.** To prove the assertion (a), assume the contrary, i.e., that for some \( r \) such \( r_* \) does not exist. Then there exist a function \( f_0 \in B_r \) and the sequences of functions \( f_n \in H^\infty_1 \) convergent to \( f_0 \).

\[
\lim_{n \to \infty} \|f_n - f_0\|_{H^\infty} = 0 \tag{3.4}
\]

and of points \( z_n \in \Delta \) convergent to \( z_0 \), \( |z_0| \leq 1 \), such that either \( f_n(z_n) = 0 \) \( (n = 1, 2, \ldots) \) or

\[
f_n(z_n) \neq 0, \quad \text{but} \quad \lim_{n \to \infty} f_n(z_n) = 0.
\]

The first case means that \( f_n \) vanish in \( \Delta \); in the second one, we have a sequence of nonvanishing functions \( f_n \) belonging to different sets \( B_{r_n} \), which are indexed by \( r_n \to 0 \).

If \( |z_0| < 1 \), we immediately reach a contradiction, because then the uniform convergence of \( f_n \) on compact sets in \( \Delta \) implies \( f_0(z_0) = 0 \), which is impossible.
Let $|z_0| = 1$. The values of $f_0$ on $\Delta$ must run in the annulus $A_r$, thus $\inf_{\Delta} |f_0(z)| \geq r$. Hence, for $n \geq n_0,$
$$|f_n(z_n) - f_0(z_n)| \geq ||f_0(z_n) - f_n(z_n)|| \geq \frac{r}{2},$$
and by continuity, there exists a neighborhood $\Delta(z_n, \delta_n) = \{|z - z_n| < \delta_n\}$ of $z_n$ in $\Delta$, in which $|f_n(z) - f_0(z)| > r/3$ for all $z$. This implies
$$\|f_n - f_0\|_{H_1} \geq \sup_{\Delta(z_n, \delta_n)} |f_n(z) - f_0(z)| > \frac{r}{3}.$$  
This inequality must hold for all $n \geq n_0$, contradicting (3.4). The part (a) is proved.

To show that $B_r$ is path-wise connective take its arbitrary distinct points $f_1, f_2$. Similar to (3.1) one gets
$$f_j = \chi_r \circ \tilde{f}_j, \quad j = 1, 2,$$
where $\tilde{f}_j \in H_{1r}^\infty$ and $\chi_r$ is a holomorphic universal covering map $\Delta \to A_r$. Connecting the covers $\tilde{f}_1$ and $\tilde{f}_2$ in $H_{1r}^\infty$ by the line interval $l_{1,2}(t) = t\tilde{f}_1 + (1 - t)\tilde{f}_2 \quad 0 \leq t \leq 1$, one obtains a path $\chi_r \circ l_{1,2} : [0, 1] \to B_r$ connecting $f_1$ with $f_2$. The continuity of $\chi_r \circ l_{1,2}$ in the norm of $H_1^\infty$ easily follows from the fact that the covering map $\chi_r$ is reduced to exponentiation (cf. Proposition 3.1). This completed the proof.

Observe that this lemma does not contradict to existence of sequences $\{f_n\} \in H_1^\infty$ of vanishing functions on $\Delta$ or of $f_n$ with $\lim_{n \to \infty} f_n(z_n) = 0$ convergent to $f_0 \in B_1^\infty$ only uniformly on compact sets in $\Delta$.

Note also that $B_1^\infty$ is dense in $B_1$ in the weak topology because any $f(z) = c_0 + c_1 z + \cdots \in B_1$ is approximated locally uniformly, for example, by the homotopy functions $f_r(z) = \omega^{-1}[r\omega(z)], 0 < r < 1$, with
$$\omega(z) = (z - c_0)/(1 - \tau_0 z)$$

mapping $\Delta$ holomorphically onto a subdomain $f_r(\Delta) \subset \Delta$. Hence,
$$\sup_{B_1^\infty} |c_n(f)| = \max_{B_1} |c_n(f)|,$$
and this supremum is attained only on $f_0 \in B_1$ with $\|f_0\|_{\infty} = 1$.

Now take the branch of the logarithmic function $\log w = \log |w| + i \arg w$ in the plane $C_w$ slit along the positive real semiaxes $R_+ = \{w = u + iv \in C : u > 0\}$ for which $0 < \arg w < 2\pi$ (and hence $\log(-1) = i\pi$).

Since $\Delta$ is simply connected and for every $f \in B_1^\infty$ we have $-\infty < \log |f(z)| < 0$ for all $z \in \Delta$, one can well define the composition of $f$ with the chosen branch of the logarithmic function, which generates a single valued holomorphic function
$$j_f(z) = \log f(z) : \Delta \to C_-. \quad (3.5)$$  
As was mentioned after Lemma 3.3, this means lifting $f$ to the universal cover $C_- \to \Delta \setminus \{0\}$ with the holomorphic universal covering map exp.

Every such function $j_f$ satisfies
$$\sup_{\Delta} (1 - |z|^2)^{\alpha} |\log j_f(z)| \leq \sup_{\Delta} (1 - |z|^2)^{\alpha}(\log |j_f(z)| + |\arg j_f(z)|) < \infty \quad (3.6)$$
for any $\alpha > 0$. We embed the set $jB_1^\infty$ into the Banach space $B$ of hyperbolically bounded holomorphic functions on the disk $\Delta$ with norm
$$||\psi||_B = \sup_{\Delta} (1 - |z|^2)^2|\psi(z)|.$$

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This space is dual to the space \( A_1 = A_1(\Delta) \) of integrable holomorphic functions on \( \Delta \) with \( L_1 \)-norm, and every continuous linear functional \( l_\psi \) on \( A_1 \) can be represented by

\[
l_\psi(\varphi) = (\psi, \varphi)_\Delta := \int_\Delta (1 - |z|^2)^2 \overline{\psi(z)} \varphi(z) dx dy
\]  

(3.7)

with some \( \psi \in B \), uniquely determined by \( l \) (see [3]).

We want to investigate the geometrical properties of the image \( jB_1^0 \). First of all, we have

Lemma 3.5. The functions \( jf \in jB_1^0 \) fill a convex set in \( B \).

Proof. Let \( f_1, f_2 \) be two distinct points in \( B_1^0 \); then their images \( \psi_1 = jf_1, \psi_2 = jf_2 \) are also different. The points of joining interval \( \psi_t = t\psi_1 + (1 - t)\psi_2 \) with \( 0 \leq t \leq 1 \) represent the functions \( jf_t = \log(f_1 f_2^{1-t}) \), taking again the branch of logarithm defined above. For each \( t \), the product \( f_1'(z)f_2^{1-t}(z) \neq 0 \) in \( \Delta \), and \( r < |f_1(z)| \cdot |f_2(z)|^{1-t} < 1 - r \). Hence, this interval lies entirely in \( jB_1^0 \).

Lemma 3.6. The map \( j \) is a holomorphic embedding of domain \( B_1^0 \) into the space \( B \) carrying this domain onto a holomorphic Banach manifold modeled by \( B \).

Proof. The map \( j : f \to \log f \) is one-to-one, bounded on each subset \( B_r \) and continuous on \( B_1^0 \), which follows from Lemma 3.4 and (3.6).

To check its complex holomorphy, observe that each \( f \in B_1^0 \) belongs to subsets \( B_r \) with \( r \leq r_f \) (hence \( |f(z)| \geq r_f > 0 \) in \( \Delta \)). Thus for any fixed \( h \in H^\infty \) and sufficiently small \( |t| \) (letting \( j(f) = jf \)),

\[
j(f + th) - j(f) = \log \left( 1 + \frac{h}{f} \right) = \frac{h}{f} + O(t^2),
\]

with uniformly bounded remainder for \( \|h\|_{\infty} \leq c < \infty \). This yields that the directional derivative of \( j \) at \( f \) equals \( hf/f \) and also belongs to \( B \).

In a similar way, one obtains that the inverse map \( j^{-1} : \psi \to e^{\psi} \) is holomorphic on intersections of a neighborhood of \( \psi \) in \( B \) with complex lines \( \psi + t\omega \) in \( jB_1^0 \). The lemma is proved.

Both complex structures on \( jB_1^0 \) endowed by norms on \( H^\infty \) and on \( B \) are equivalent.

3. Complex geometry of sets \( jB_1^0 \) and \( B_1^0 \).

As a subdomain of a complex manifold modeled by \( B \), the set \( jB_1^0 \) admits the invariant Kobayashi and Carathéodory metrics. Our goal is to show that the geometric features of this set are similar to bounded convex domains in Banach spaces.

Proposition 3.7. (i) The Kobayashi and Carathéodory distances on \( jB_1^0 \) are equal:

\[
d_{jB_1^0}(\psi_1, \psi_2) = c_{jB_1^0}(\psi_1, \psi_2) = \inf \{ d_{\Delta}(h^{-1}(\psi_1), h^{-1}(\psi_2)) : h \in \text{Hol}(\Delta, jB_1) \},
\]

(3.8)

and similarly for the infinitesimal forms of these metrics.

(ii) Every two points in \( jB_1^0 \) can be joined by c-geodesic (i.e., by a complex geodesic in the strongest sense).

Proof. The equality (3.8) follows from the property (ii). We establish this property in two steps.

(a) First take the c-blowing up of \( jB_1^0 \), that is, we consider the sets

\[
U_\varepsilon = \bigcup_{\psi \in jB_1^0} \{ \omega \in B : \|\omega - \psi\|_B < \varepsilon \}, \quad \varepsilon > 0.
\]

For these sets, we have

Lemma 3.8. Every set \( U_\varepsilon \) is a (bounded) convex domain in \( B \), and its weak* closure in \( \sigma(B, A_1) \) is compact.
\textbf{Proof.} The openness and connectivity of \( U_\epsilon \) are trivial. Let us check convexity. Take any two distinct points \( \omega_1, \omega_2 \) in \( U_\epsilon \) and consider the line interval

\[ \omega_t = t\omega_1 + (1-t)\omega_2, \quad 0 \leq t \leq 1, \quad (3.9) \]

joining these points. Since, by definition of \( U_\epsilon \), each point \( \omega_n (n = 1, 2) \) lies in the ball \( B(\psi_n, \epsilon) \) centered at \( \psi_n \) with radius \( \epsilon \), and the interval \( \{ \psi_t = t\psi_1 + (1-t)\psi_2 \} \) lies in \( jB_1^0 \), we have, for all \( 0 \leq t \leq 1 \),

\[ \omega_t - \psi_t = t(\omega_1 - \psi_1) + (1-t)(\omega_2 - \psi_2) \]

and

\[ \|\omega_t - \psi_t\| \leq t\|\omega_1 - \psi_1\| + (1-t)\|\omega_2 - \psi_2\| < \epsilon, \]

which shows that the interval (3.9) lies entirely in \( U_\epsilon \).

To establish \( \sigma(B, A_1) \)-compactness of the closure \( \overline{U}_\epsilon \) in \( B \), note that weak* convergence of the functions \( \omega_n \in B \) to \( \omega \) implies the uniform convergence of these functions on compact subsets of \( \Delta \). It suffices to show that for any bounded sequence \( \{ \omega_n \} \subset B \) we have the equality

\[ \lim_{n \to \infty} \int_{\Delta} \frac{(1-|\zeta|^2)^2\omega_n(z)}{\zeta - z} d\zeta d\eta = \int_{\Delta} \frac{(1-|\zeta|^2)^2\omega(z)}{\zeta - z} d\zeta d\eta, \quad z \in \Delta^c, \]

(3.10)

because the functions \( w_z(\zeta) = 1/(\zeta - z) \) span a dense subset of \( A_1(\Delta) \). But if

\[ \sup_{\Delta} (1-|\zeta|^2)^2|\omega(\zeta)| < M < \infty \quad \text{for all} \quad n, \]

the equality (3.10) is a consequence of Lebesgue’s theorem on dominant convergence. The lemma follows.

(b) We proceed to the proof of Proposition 3.7 and first establish the existence of complex geodesics in domains \( U_\epsilon, \epsilon < \epsilon_0 \). Our arguments follow [5].

Let \( \omega_1 \) and \( \omega_2 \) be distinct points in \( U_\epsilon \). By Proposition 2.1,

\[ d_{U_\epsilon}(\omega_1, \omega_2) = c_{U_\epsilon}(\omega_1, \omega_2) = \inf \{ d_\Delta(h^{-1}(\omega_1), h^{-1}(\omega_2)) : h \in \text{Hol}(\Delta, U_\epsilon) \}; \]

hence there exists the sequences \( \{ h_n \} \subset \text{Hol}(\Delta, U_\epsilon) \) and \( \{ r_n \} \), \( 0 < r_n < 1 \), such that \( h_n(0) = \omega_1 \) and \( h_n(r_n) = \omega_2 \) for all \( n \), \( \lim_{n \to \infty} r_n = r < 1 \) and \( c_{U_\epsilon}(\omega_1, \omega_2) = d_\Delta(0, r) \). Let

\[ h_n(t) = \sum_{m=0}^{\infty} a_{m,n} t^m \]

for all \( t \in \Delta \) and \( n \).

Take a ball \( B(0, R) = \{ \omega \in B : \|\omega\| < R \} \) containing \( U_\epsilon \). For any \( \omega \in B(0, R) \), the Cauchy inequalities imply \( \|a_{n,m}\|_B \leq R \) for all \( n \) and \( m \). Passing, if needed, to a subsequence of \( \{ h_n \} \), one can suppose that for a fixed \( m \), the sequence \( a_{n,m} \) is weakly* convergent to \( a_m \in B \) as \( n \to \infty \), that is

\[ \lim_{n \to \infty} (a_{n,m}, \varphi)_\Delta = (a_m, \varphi)_\Delta \quad \text{for any} \quad \varphi \in A_1. \]

Hence \( h(t) = \sum_{m=0}^{\infty} a_{m,n} t^m \) defines a holomorphic function from \( \Delta \) into \( B \). Since \( a_{n,0} = \omega_1 \) for all \( n \), we have \( h(0) = \omega_1 \).

Now, let \( \alpha, 0 < \alpha < 1, \) and \( \epsilon > 0 \) be given. Choose \( m_0 \) so that

\[ r \sum_{m=m_0}^{\infty} \alpha^m < \epsilon. \]

If \( \varphi \in A_1, \|\varphi\| = 1 \), then

\[ \sup_{|t| \leq \alpha} |(h_n(t) - h(t), \varphi)_\Delta| \leq \sum_{m=1}^{m_0-1} |(a_{n,m} - a_m, \varphi)_\Delta| + 2r \sum_{m=m_0}^{\infty} \alpha^m \]
for all \( n \), which implies that \( h_n \) is convergent to \( h \) in \( \sigma(\mathcal{B}, A_1) \) uniformly on compact subsets of \( \Delta \) as \( n \to \infty \). Since \( \overline{U}_\varepsilon \) is \( \sigma(\mathcal{B}, A_1) \) compact, \( h(\Delta) \subseteq \overline{U}_\varepsilon \), and since \( h(0) \in U_\varepsilon \), it follows that \( h(\Delta) \subseteq U_\varepsilon \). For \( r < r' < 1 \),

\[
\omega_2 = h_n(r_n) = \frac{1}{2\pi i} \int_{|t|=r} \frac{h_n(t)dt}{t-r} \to \frac{1}{2\pi i} \int_{|t|=r'} \frac{h(t)dt}{t-r} = h(r) \quad (3.11)
\]

as \( n \to \infty \). Hence,

\[
d_\Delta(0, r) = c_{U_n}(\omega, \omega_2) = c_{U_n}(h(0), h(r)),
\]

and \( h \) is a \( c \)-geodesics in \( U_n \).

There exists a holomorphic map \( g : \Delta \to U_\varepsilon \) such that for any two points \( t_1, t_2 \in \Delta \),

\[
d_\Delta(t_1, t_2) = d_{U_n}(g(t_1), g(t_2)) = c_{U_n}(g(t_1), g(t_2)), \quad (3.12)
\]

and for any pair \((t, v)\), \( t \in \Delta, \ v \in \mathbb{C} \),

\[
\mathcal{K}_{U_n}(g(t), dg(t)v) = \frac{|v|}{1 - |t|^2}, \quad (3.13)
\]

(c) Let now \( \omega_1 \) and \( \omega_2 \) be two distinct points in \( j\mathcal{B}_1^0 \). Choose a decreasing sequence \( \{\epsilon_n\} \) approaching zero and take for every \( n \) a complex geodesic \( h_n = h_{U_n} \) joining these points in \( U_n \), which was constructed in the previous step. Let \( g_n = g_{U_n} \) be the corresponding map \( \Delta \to U_n \) which provides the equalities \( (3.12), (3.13) \). Since \( d_\Delta \) is conformally invariant, one can take \( g_n \) satisfying \( g_n^{-1}(\omega_1) = 0 \), \( g_n^{-1}(\omega_2) = r_n \in (0, 1) \). Then the inequalities

\[
d_{U_m}(\omega_1, \omega_2) \leq d_{U_m}(\omega_1, \omega_2) \leq d_{j\mathcal{B}_1}(\omega_1, \omega_2) \quad \text{for} \ m \to n
\]

imply \( r_n \leq r_m \leq r_s \), where \( d_{\Delta}(0, r_s) = d_{j\mathcal{B}_1}(\omega_1, \omega_2) \). Hence, there exists \( \lim n \to \infty r_n = r' \to r_s \).

The sequence \( \{g_n\} \) is \( \sigma(\mathcal{B}, A_1) \)-compact and similar to \( (3.11) \) the weak* limit of \( g_n \) is a function \( g \in \text{Hol}(\Delta, j\mathcal{B}_1^0) \) which determines a complex geodesic for both Kobayashi and Carathéodory distances on \( j\mathcal{B}_1^0 \) joining the points \( \omega_1 \) and \( \omega_2 \) inside this set. Proposition 3.7 is proved.

An important consequence of Proposition 3.7 is that the initial domain \( \mathcal{B}_1 \) in \( H_1^\infty \) has similar complex geometric properties, since the embedding \( j \) is biholomorphic. We present it as

**Proposition 3.9.** (i) The Kobayashi and Carathéodory distances on domain \( \mathcal{B}_1 \) and the corresponding infinitesimal metrics are equal:

\[
d_{\mathcal{B}_1^0}(f_1, f_2) = c_{\mathcal{B}_1^0}(f_1, f_2) = \inf\{d_\Delta(h^{-1}(f_1), h^{-1}(f_2)) : h \in \text{Hol}(\Delta, \mathcal{B}_1)\},
\]

\[
\mathcal{K}_{\mathcal{B}_1^0}(f, v) = C_{\mathcal{B}_1^0}(f, v) \quad \text{for all} \ (f, v) \in T(\mathcal{B}_1^0). \quad (3.14)
\]

(ii) Every two points \( f_1, f_2 \) in \( \mathcal{B}_1 \) can be joined by a complex geodesic.

4. **First Proof of Theorem 1.1**

This proof involves a complex homotopy of functions \( f \in \mathcal{B}_1^0 \) and estimating the Kobayashi distance on the homotopy disks.

For any \( f \in \mathcal{B}_1^0 \), there is a complex holomorphic homotopy connecting \( f \) with \( c_0(f) \) in \( \mathcal{B}_1 \). For \( f = \kappa \circ \hat{f} \) whose the cover \( \hat{f}(z) = \hat{c}_1 z + \hat{c}_2 z^2 + \cdots \in H_1^\infty \) has zero free term, one can take

\[
\hat{f}_t(z) = \hat{f}(t z) = \hat{c}_1 t z + \cdots : \Delta \times \Delta \to \Delta, \quad (\hat{f}_0(\cdot) = 0)
\]

generating the underlying homotopy \( f_t = \kappa \circ \hat{f}_t \) in \( \mathcal{B}_1^0 \).

In the case of generic \( \hat{f}(z) = \hat{c}_0 + \hat{c}_m z + \cdots \in H_1^\infty \) (\( m \geq 1 \)), we decompose it via

\[
\hat{f} = \omega \circ \hat{g}_\alpha, \quad (4.1)
\]
where
\[ g_f(z) = (\bar{f}(z) - \hat{c}_0)/(1 - \overline{\hat{c}_0}f(z)), \]  
and set
\[ f_t(z) = \kappa \circ \omega(\hat{g}_f)(t z). \]  
The pointwise map \( t \mapsto f_t \) generates by Lemma 3.2 a holomorphic map \( \chi_f : \Delta \to H^\infty \), and the functional \( J \) is \( n \)-homogeneous with respect to this homotopy, \( J(f_t) = t^n J(f) \).

Note also that for a fixed \( \hat{c}_0 \) (regarded again as a constant function on \( \Delta \)), both maps in (4.2) are biholomorphic isometries of the ball \( H^\infty_1 \); hence
\[ \omega_s(\zeta) = \frac{\zeta \hat{g}_f/\|\hat{g}_f\|_\infty + \hat{c}_0}{1 + \overline{\hat{c}_0} \zeta \hat{g}_f/\|\hat{g}_f\|_\infty} : \Delta \to H^\infty_1 \]  
carries out the complex geodesic \( \zeta \mapsto \zeta \hat{g}_f/\|\hat{g}_f\|_\infty \) into a complex geodesic in \( H^\infty_1 \) passing through \( \hat{c}_0 \) and \( \hat{f} \), and \( \omega_s(\zeta) = \hat{f} \) at \( \zeta = \|\hat{g}_f\|_\infty \).

By Proposition 3.9, there exists for each \( f \) a complex geodesic in \( B^0_0 \) joining \( f_t \) with \( c_0(f) \); it determines a holomorphic geodesic disk isometric to the hyperbolic plane \( \mathbb{H}^2 \). We need to estimate the behavior of the distance \( d_{B^0_0}(f_t, c_0) \) for \( t \to 0 \).

**Lemma 4.1.** Let \( \hat{f} \in H^\infty_1 \) have the expansion \( \hat{f}(z) = \hat{c}_m z^m + \ldots \) with \( \hat{c}_m \neq 0 \) (\( m \geq 1 \)), and \( \|\hat{f}\| = 1 \). Then the geodesic parameter \( \zeta \) and the homotopy parameter \( t \) are related near the origin via
\[ |\zeta| = |\hat{c}_m| t^m + O(t^{|m|+1}), \quad t \to 0. \]  

**Proof.** Put \( \hat{p}_m(z) = z^m \). The homotopy disk \( \Delta(\hat{p}_m) \) of this function in \( H^\infty_1 \) is filled by the functions \( \hat{p}_{m,t}(z) = t^m z^m \) with \( |t| < 1 \), while the geodesic parameter on \( \Delta(\hat{p}_m) \) is generated by hyperbolic isometry \( \zeta \mapsto \zeta \hat{g}/\|\hat{g}\| \). So, \( \zeta = t^m \), and since
\[ \|\hat{f}_t - \hat{c}_m \hat{p}_{m,t}\|_{\mathbf{H}^\infty} = |t|^{m+1} \|\hat{c}_{m+1} + t \hat{c}_{m+2} z + \ldots\|_\infty = O(t^{m+1}), \]  
the relation (4.5) follows.

**Lemma 4.2.** For any \( f = c_0 + c_1 z + \ldots \in B^0_0 \), we have the equality
\[ d_{B^0_0}(f, c_0) = \inf\{d_{H^\infty_1}(\hat{f}, \hat{c}_0) : \kappa \circ \hat{f} = f\}; \]  

moreover, there exists a map \( \hat{f}^*(z) = c_0^* + c_1^* z + \ldots \) covering \( f \), on which the infimum in (4.6) is attained, i.e.,
\[ d_{B^0_0}(f, c_0) = d_{H^\infty_1}(\hat{f}^*, c_0^*). \]  

**Proof.** We decompose the cover \( \hat{f}(z) = \hat{c}_0 + \hat{c}_1 z + \ldots \) of \( f \) in \( H^\infty_1 \) by (4.1), (4.2), getting
\[ d_{H^\infty_1}(\hat{g}_f, 0) = d_\Delta(\|\hat{g}_f\|_\infty, 0) = \tanh^{-1}(\|\hat{f} - \hat{c}_0)/(1 - \overline{\hat{c}_0} \hat{f})\|_\infty). \]  
and then apply to \( \hat{g}_f \) the transform (4.4). This yields a complex geodesic in \( H^\infty_1 \) which connects \( \hat{c}_0 \) and \( \hat{f} \).

Now observe that the universal covering map \( \kappa_0 : \Delta \to \Delta_* \) extended by the equality (3.1) to all \( \hat{f} \in H^\infty_1 \) generates holomorphic map of the ball \( H^\infty_1 \) into domain \( B^0_0 \), which yields
\[ d_{B^0_0}(f, c_0) = d_{B^0_1}(\kappa_0 \circ \hat{f}, \kappa_0(\hat{c}_0)) \leq d_{H^\infty_1}(\hat{f}, \hat{c}_0), \]  
and
\[ d_{B^0_0}(f, c_0) \leq \inf d_{H^\infty_1}(\hat{f}, \hat{c}_0), \]  
for \( f \).
where the infimum is taken over all covers \( \hat{f} \) of \( f \).

It remains to establish that in fact one has in (4.9) the equality (so the infimum is attained). To prove this, assume to the contrary, that

\[
d_{g}^0(f, c_0) < \inf_{\hat{f}} d_{H_1^\infty}(\hat{f}, \hat{c}_0),
\]

By Proposition 3.8, there exists a complex geodesic \( h : \Delta \to B_1^0 \) joining the points \( c_0 \) and \( f \), and it follows from the above,

\[
d_{g}^0(f, c_0) = d_\Delta(\zeta_1, \zeta_2) < \tanh^{-1} \| (f - c_0)/(1 - \tau_0 f) \| \infty,
\]

where \( \zeta_1 = h^{-1}(c_0) \), \( \zeta_2 = h^{-1}(f) \). Lifting this \( h \) by (3.1) to its cover \( \hat{h} \) of the unit disk into itself, one gets the points \( \hat{h}(\zeta_1), \hat{h}(\zeta_2) \) in \( \Delta \) located in the fibers over \( c_0 \) and \( f \), respectively, and for these points

\[
d_{H_1^\infty}(\hat{h}(\zeta_1), \hat{h}(\zeta_2)) = d_\Delta(\zeta_1, \zeta_2) < \tanh^{-1} \| (f - c_0)/(1 - \tau_0 f) \| \infty,
\]

which contradicts (4.8) and completes the proof of Lemma 4.2.

Consider the functional \( J(f) = c_n \) on all \( f \in H_1^\infty \). The map \( k \) extending the function (1.2) holomorphically to \( H_1^\infty \) generates a holomorphic functional

\[
J_k = J \circ k : H_1^\infty \to \mathbb{C},
\]

with \( \max_{H_1^\infty} |J_k| = \sup_{B_1^0} |J| \). We rescale this functional taking

\[
J_k^0(\hat{f}) = \frac{J_k(f)}{C_n}, \quad C_n = \sup_{f \in B_1^0} |J(f)| = \max_{H_1^\infty} |J_k(\hat{f})|,
\]

which yields a holomorphic map of the ball \( H_1^\infty \) onto the unit disk. Similarly, let \( J^0(f) = J(f)/C_n \).

We first estimate these functionals on the set of \( f \in B_1^0 \) whose covers \( \hat{f} \) in \( H_1^\infty \) are of the form

\[
\hat{f}(z) = \hat{c}_m z^m + \hat{c}_{m+1} z^{m+1} + \ldots
\]

(4.10)

(with \( \hat{c}_0 = 0 \), \( \hat{c}_m \neq 0 \ m \geq 1 \).

Fix a small \( \varrho > 0 \) and let \( |t| < \varrho \). Take the geodesics \( h_t : \Delta \to H_1^\infty \) joining such \( \hat{f}_t \) with \( 0 \). Then

\[
J_k^0(\hat{f}_t) = \hat{\beta}_n \zeta^t + \hat{\beta}_{n+1} \zeta^{t^{m+1}} + \ldots
\]

(4.11)

(where \( \hat{\beta}_n \neq 0 \) and all \( |\hat{\beta}_j| \leq 1 \)). Applying Lemmas 2.3 and 4.1, one derives

\[
|J_k^0(\hat{f}_t)| = |J_k^0 \circ h_t(\cdot)| + O(|t|^{mn+1}) \leq |\hat{\beta}_n| |\hat{c}_m| |t|^m + O(|t|^{mn+1}), \quad t \to 0.
\]

This yields, similar to Lemma 4.2, the following upper bound for the images \( f_t = \kappa \circ \hat{f}_t \in B_1^0 \)

\[
|J^0(f_t)| \leq \inf |\hat{\beta}_n| |\hat{c}_m| |t|^m + O(|t|^{mn+1}) = \inf |\hat{\beta}_n| |\hat{c}_m| |t|^m + O(|t|^{mn+1})
\]

(4.12)

(each infimum again over \( \hat{f} \) with \( \kappa \circ \hat{f} = f \)).

Now we establish that the right-hand side of (4.12) yields simultaneously the lower asymptotic bound for \( |J^0(f_t)| \) with small \( |t| \), which means that (4.12) is in fact an asymptotic equality.

**Lemma 4.3.** For any \( f(z) = c_0 + c_m z^m + \ldots \in B_1^0 \) with \( \hat{f} \) of the form (4.10), we have

\[
|J^0(f_t)| \geq \inf |\hat{\beta}_n| |\hat{c}_m| |t|^m + O(|t|^{mn+1}),
\]

(4.13)

again taking the infimum over \( \hat{f} \) with \( \kappa \circ \hat{f} = f \).
\textbf{Proof.} Fix again a small $\varrho > 0$ and restrict $J_0^k$ to geodesic $h_t(\zeta) = h(\zeta; t)$ in $H_1^\infty$ joining 0 with $\hat{f}_t$, $|t| \leq \varrho$. The corresponding function $\hat{g}(\zeta) = J_0^k \circ h_t(\zeta)$ given by (4.11) generates a conformal metric

$$\lambda_{\hat{g}}(\zeta; t) = \frac{|\hat{g}'(\zeta; t)|}{1 - |\hat{g}(\zeta; t)|^2}$$

of Gaussian curvature $-4$ at noncritical points. The upper envelope of these metrics

$$\lambda_{j0}(\zeta; t) = \sup_{\hat{g}} \lambda_{\hat{g}}(\zeta; t)$$

(over the covers $\hat{f}$ of given $f$) is a subharmonic metric on $\Delta$ with curvature at most $-4$ in the supporting sense and hence in the potential sense. Averaging $\lambda_{j0}(\zeta; t)$ over the torus \{ $|\zeta| = |t| = r$\} yields a circularly symmetric subharmonic metric

$$\lambda_{j0}(r) = n|\hat{\beta}_n|r^n + O(r^{n+1})$$

of curvature at most $-4$ in the potential sense. Estimating this metric by Lemmas 2.4 and 4.1, one derives the following lower bound

$$\lambda_{j0}(r) \geq \frac{mn c r^{mn-1}}{1 - cr^{2mn}}, \quad (4.14)$$

where

$$c = \inf_{\hat{f}} |\hat{\beta}_n| |\hat{\epsilon}_m|$$

(this estimate depends on $\varrho$ which was fixed). On the other hand, the hyperbolic length of the interval $[0, |\hat{g}(r)|]$ equals

$$\tanh^{-1}|\hat{g}(r)| = \int_0^r \frac{d\xi}{1 - |\xi|^2} = \int_0^r \lambda_{\hat{g}}(\xi; \cdot) d\xi,$$

which implies (cf. [15])

$$\tanh^{-1}|J_0^0(f_r)| = \sup_{\hat{f}} \int_0^r \lambda_{\hat{g}}(\xi; \cdot) d\xi = \int_0^r \sup_{\hat{f}} \lambda_{\hat{g}}(\xi; \cdot) d\xi = \int_0^r \lambda_{j0}(\xi) d\xi, \quad (4.15)$$

where the second equality is obtained by taking a monotone increasing subsequence of metrics

$$\lambda_1 = \lambda_{\hat{g}}, \quad \lambda_2 = \max(\lambda_{\hat{g}_1}, \lambda_{\hat{g}_2}), \quad \lambda_3 = \max(\lambda_{\hat{g}_1}, \lambda_{\hat{g}_2}, \lambda_{\hat{g}_3}), \ldots$$

corresponding to a sequence $\{\hat{f}_j\} \subset H_1^\infty$ for which $\sup_j |J_0^0(\hat{f}_j)| = \sup_{B_0} |J_0^0(f)|$ and such that $\lim_{j \to \infty} \lambda_j = \sup_j \lambda_{\hat{g}_j}$. From (4.14) and (4.15),

$$\int_0^r \lambda_{j0}(\xi) d\xi \geq \tanh^{-1}(cr^{mn}) + O(r^{n+1}), \quad r \to 0,$$

which proves the desired estimate (4.13).

We have established that for small $r > 0$,

$$\max_{|t|=r} |J_0^0(f_t)| = r^{mn} |J_0^0(f)| = \max_{|t|=r} \inf_{\hat{f}} |\hat{\beta}_n| c m(f_t) + O(r^{n+1}). \quad (4.16)$$

One can replace in the above arguments the cover $\kappa$ by any universal covering map $\gamma^* \kappa = \kappa \circ \gamma : \Delta \to \Delta_\gamma$ with $\gamma \in \text{Mob}(\Delta)$. Fix in (3.1) $\hat{f}$ with $\hat{f}(0) = 0$ and choose $\gamma$ so that the point
\[ \gamma(0) = \kappa^{-1}(c_0) \] is placed in the closure of a fundamental triangle of a cyclic Fuchsian group \( \Gamma \) representing \( \Delta_* \) as the quotient \( \Delta/\Gamma \). Then, instead of (3.1),

\[ f(z) = (\gamma^* \kappa) \circ \hat{f}(z). \tag{4.17} \]

Applying this to generic functions \( f \in B_1^0 \) covered by arbitrary \( \hat{f}(z) = c_0 + \sum_m c_n z^m + \cdots \in H_1^\infty (m \geq 1) \), one obtains similar to Lemma 4.2 that the equalities (4.6) are extended to homotopy (4.3) as follows.

**Lemma 4.4.** For any \( \hat{f}(z) = c_0 + \sum_m c_n z^m \in H_1^\infty (c_m \neq 0, \ m \geq 1) \) and its image \( f(z) = \kappa \circ \hat{f}(z) = c_0 + \sum_m c_n z^m \in B_1^0 \),

\[ d_{B_1^0}(f_1, c_0) = \inf_{\hat{f}} d_{H_1^\infty} (\hat{f}_1, 0) = \inf \{ |c_m(\hat{f})| : \ \kappa \circ \hat{f} = f \} |t|^m + O(|t|^{m+1}), \quad t \to 0, \tag{4.18} \]

where each infimum is taken over the covers \( \hat{f} \) of \( f \) fixing the origin and attained on some \( \hat{f} \), and the estimate of the remainder is in \( H_1^\infty \)-norm. Therefore,

\[ \max_{|t|=r} |J^0(f_1)| = \max_{|t|=r} \inf_{\hat{f}} |\hat{\beta}_n||c_m|r^{mn} + O(r^{mn+1}). \tag{4.19} \]

In particular, since \( \hat{\kappa}(z) = z \),

\[ d_{B_1^0}(\kappa, c_0) = |t| + O(|t|^2), \quad t \to 0. \]

which shows that the holomorphic disk \( \Delta(\kappa) \) filled by the homotopy functions \( \kappa_t(z) = \kappa(tz), \ t \in \Delta \), is geodesic in \( B_1^0 \). Similarly, for \( \kappa_m(z) = \kappa(z^m) \),

\[ d_{B_1^0}(\kappa_m, c_0) = |t|^m + O(|t|^{m+1}), \quad t \to 0. \tag{4.20} \]

In fact, the remainder terms in the last two equalities can be omitted.

We can now complete the proof of Theorem 1.1. Let

\[ f^0(z) = c_0 + c_1 z + c_2 z^2 + \ldots \]

be an extremal function maximizing \( |J(f)| \) for \( f \in B_1^0 \); then \( |J(f^0)| = C_n \). Rotating, if needed \( f^0 \), we get \( J^0(f^0) = 1 \) and \( J^0(f^0) = r \) for the homotopy \( f^0_t(z) = f^0(rz), \ 0 < r < 1 \).

Using this homotopy, we first show that \( f^0 \) must satisfy

\[ c^0_1 = 0. \tag{4.21} \]

Indeed, assume that \( c^0_1 \neq 0 \) (hence, \( C_n > J(\kappa) \)) and apply Lemma 4.4 (with \( m = 1 \)). By (4.18),

\[ d_{B_1^0}(J^0, c_0) = |c^0_1|r + O(r^2) = \frac{|c^0_1|r}{|\gamma^* \kappa)'(0)|} + O(r^2), \quad r \to 0, \]

where \( c^0_1 \) is the first coefficient of a factorizing function \( \hat{f}^0 \) for \( f^0 \) by (4.17) and \( \gamma \) is a Möbius automorphism of \( \Delta \), on which the infima in (4.18) are attained, while by (4.19) and homogeneity of \( J \),

\[ r^n = r^n f^0(\hat{f}^0) = \inf_{\hat{f}^0} |\hat{\beta}_n||c^0_1|r^n + O(r^{n+1}). \tag{4.22} \]

This implies \( \inf_{\hat{f}^0} |\hat{\beta}_n| = 1 \) and

\[ |c^0_1| = \frac{|c^0_1|}{|\gamma^* \kappa)'(0)|} = 1. \]

By Schwarz’s lemma, the last equality can hold only when the cover \( \hat{f}^0(z) = c z, \ |c| = 1, \) and then by Lemma 3.3 \( f^0(z) = \kappa(z) \) up to rotation. This yields also

\[ |c^0_1| = |c^0_1| = |c^0_1| = 2/e, \]

violating Parseval’s equality \( \sum_{n=0}^{\infty} |c_n|^2 = 1 \) for the boundary function \( \kappa(e^{i\theta}), \ \theta \in [0, 2\pi] \) in (so \( |c_n(\kappa)| < 2/e \) for all \( n > 1 \)). This contradiction proves (4.21).
It follows that the extremals of \( J \) must be of the form
\[
f^0(z) = c_0^2 + c_0^3 z^2 + \ldots \tag{4.23}
\]
Now, if \( n = 2 \), comparison of (4.23) with Lemmas 3.3 and 4.4 (for \( m = 2 \)) and the equalities similar to (3.3), (4.22) implies that necessarily \(|c_0^2| = 1\); hence \( f^0(z) = \epsilon z^2 \). Therefore, the maximal value \(|c_0^2| \) on \( \mathcal{B}_1^0 \) must be equal to \( 2/e \) and is attained only on \( f^0(z) = \kappa(z^2) \) (up to rotations).

If \( n \geq 3 \), the same arguments as in the proof of (4.21) based on the equality (4.16) (for \( m = 2 \)) imply that also the second coefficient \( c_0^2 \) of any extremal function \( f^0 \) for \( J(f) \) must vanish; hence,
\[
f^0(z) = c_0^3 + c_0^4 z^3 + c_0^5 z^4 + \ldots \tag{4.24}
\]
In the case \( n = 3 \), the relations (4.20), (4.22), (4.24) and Lemmas 3.3 and 4.4 (for \( m = 3 \)) imply, similar to the previous case, that \( f^0(z) = \epsilon z^3 \), thus \( f^0(z) = \kappa(z^3) \) up to rotations and \( \max_{|z| = 1} |c_3| = \kappa'(0) = 2/e \).

Arguing similarly for \( n = 4, 5, \ldots \), one derives successively that for each \( n \) the extremal function \( f^0 \) must be of the form \( f^0(z) = c_0^k + c_0^{k+1} z^{k+1} + \ldots \) and coincide with \( \kappa(z^n) \) up to rotations, which implies \( \max_{|z| = 1} |c_n| = 2/e \), completing the proof of the theorem.

In the case \( n = 2 \), one can apply the above arguments to more general functional \( J(f) = c_2 + P(c_1) \) given by Proposition 1.2. Since by (1.2) \( c_2(\kappa) = 0 \), one immediately gets that any extremal \( f^0 \) of \( J \) satisfies (4.21), hence \( |J(f^0)| = |c_0^2| = 2/e \), which implies the estimate (1.4).

5. Second proof of Theorem 1.1

We first establish that \( \kappa_n(z) \) is the maximizing function in local setting compatible with Schwarz’s lemma, which provides the assertion of Theorem 1.1.

10. Consider more general bounded functionals on \( \mathcal{B}_1 \) of the form
\[
J(f) = c_n + F(c_{m_1}, \ldots, c_{m_s}) \tag{5.1}
\]
where \( c_j = c_j(f); 1 \leq n, m_j \) and \( F \) is a holomorphic function of \( s \) variables in an appropriate domain of \( \mathbb{C}^s \). We assume that this domain contains the origin \( \mathbf{0} \) and that \( F, \partial F \) vanish at \( \mathbf{0} \).

Using the factorization (3.1) and the map \( k \) generated by the function (1.2) via Proposition 3.1(b), we obtain a functional
\[
J(\kappa \circ \hat{f}) = \hat{J}(\hat{c}_1, \ldots, \hat{c}_n) \tag{5.2}
\]
on \( \hat{f}(z) = \hat{c}_0 + \hat{c}_1 z + \ldots \in H_1^\infty \), and \( \sup_{B^1_0} |J(f)| = \sup_{H_1^\infty} |\hat{J}(\hat{f})| \).

Noting that all \( \hat{f} \in H_1^\infty \) belong to the space \( \mathcal{B} \), we define for \( \varphi \in A_1(\Delta) \), \( \psi \in \mathcal{B} \) the scalar product
\[
l_\varphi(\psi) = \langle \varphi, \psi \rangle = \int_{\Delta} (1 - |z|^2)^2 \varphi(z) \overline{\psi(z)} dxdy, \tag{5.3}
\]
As was mentioned above, any linear functional on \( A_1(\Delta) \) is of such a form. Put
\[
\nu_\varphi(z) = (1 - |z|^2)^2 \overline{\psi(z)} \tag{5.4}
\]
and extend the scalar product (5.3) to all \( \mu \in L_\infty(\Delta) \) and \( \varphi \in L_1(\Delta) \) by \( \langle \varphi, \mu \rangle = \int_{\Delta} \mu \varphi dxdy \).

Then
\[
\mu - \nu_\varphi \in A_1(\Delta)^\perp = \{ \nu \in L_\infty(\Delta) : \langle \nu, \varphi \rangle = 0 \mbox{ for all } \varphi \in A_1(\Delta) \} \tag{5.5}
\]
for any \( \mu \in L_\infty(\Delta) \) extending \( l_\varphi \). In particular, this holds for the Hahn-Banach extension \( \langle \mu, \varphi \rangle \) of \( l_\varphi \) having the minimal norm.

We shall need some results from the Teichmüller space theory. Define
\[
S_F(\xi) = \hat{f}\left(\frac{1}{\xi}\right) \frac{1}{\xi^4}. \tag{5.6}
\]
These functions are holomorphic on the disk
\[ \Delta^* = \mathbb{C} \setminus \{ \Delta \} = \{ \zeta \in \mathbb{C} = \mathbb{C} \cup \{ \infty \} : |\zeta| > 1 \} \]
and belong to the space \( B(\Delta^*) \) with norm \( \| \varphi \| = \sup_{\Delta^*} |(\zeta'^2 - 1)^2| \varphi |. \) Any \( S_F \in B(\Delta^*) \) is the Schwarzian derivative
\[ S_F(\zeta) = \left( \frac{F''(\zeta)}{F'(\zeta)} \right)' - \frac{1}{2} \left( \frac{F''(\zeta)}{F'(\zeta)} \right)^2 \]
of a locally univalent in \( \Delta^* \) function
\[ w = F(\zeta) = \zeta + b_1 \zeta^{-1} + b_2 \zeta^{-2} + \ldots. \]
By the Ahlfors-Weill theorem, if \( \| S_F \| = 2k < 2 \), then \( F \) is univalent on whole disk \( \Delta^* \) and admits \( k \)-quasiconformal extension across the unit circle \( \{ |z| = 1 \} \) to \( \mathbb{C} \) with Beltrami coefficient
\[ \mu_F(\zeta) = \frac{\partial_F F/\partial_\zeta F}{- \frac{1}{2} (1 - |\zeta|^2)^2 \frac{\bar{\zeta}}{\zeta} S_F \left( \frac{1}{\zeta} \right)}. \]
Every element \( \mu \in L_\infty(\Delta) \) we consider as defined everywhere on \( \mathbb{C} \) with \( \mu(\zeta) = 0 \) for \( \zeta \in \Delta^* \).

The Schwarzians \( S_F \) of univalent functions \( F \) in \( \Delta^* \) with quasiconformal extensions to \( \mathbb{C} \) form a bounded contractible domain in the space \( B(\Delta^*) \) which models the universal Teichmüller space \( T \). Its topologies generated by the norm of \( B(\Delta^*) \) and by Teichmüller’s metric related to \( \| \mu_F \|_\infty \) are equivalent. Hence, for small \( r > 0 \),
\[ \inf \{ ||\mu_\psi|| : \|\psi\|_{B(\Delta^*)} = r \} > 0. \]
The corresponding functions \( \psi = \hat{f}(z) = S_F(1/z^4)z^4 \) with \( \|\mu_\psi\|_\infty < r \) form a (connected) domain \( U_r \) in \( H^\infty_1 \) containing the origin.

Now we can formulate our first theorem.

**Theorem 5.1.** For any functional \( J \) of type \((5.1)\), there exists a number \( r_0(J) > 0 \) such that for all \( r \leq r_0(J) \),
\[ \sup_{k(U_r)} |J(f)| = \sup_{k(U_r)} |c_n| = M_n r, \quad M_n = \max_{B_1} |J(f)|. \quad (5.7) \]

**Proof.** As was mentioned above, each function \( \psi \in B \) determines by \((5.3)\) a linear functional on \( A_1(\Delta) \). Applying to \( \psi \) the reproducing formula
\[ \psi(\zeta) = \frac{3}{\pi} \iint_{\Delta} \frac{(1 - |z|^2)^2 \psi(z)}{(1 - \zeta z)^4} dx dy, \quad \zeta \in \Delta, \quad (5.8) \]
which is valid for all \( \psi \) with \( \iint_{\Delta} (1 - |z|^2)^2 |\psi(z)| dx dy < \infty \) (see, e.g. [1]), one gets for its derivatives
\[ \psi^{(p)}(0) = \frac{3 \cdot 4 \cdot 5 \ldots (p + 4)}{\pi p!} \iint_{\Delta} (1 - |z|^2)^2 \psi(z) \zeta^p dx dy. \]
Hence the coefficients \( \hat{c}_p(\hat{f}) \) of \( \hat{f} = \psi \) are represented via
\[ \hat{c}_p = \frac{(p + 1) \ldots (p + 4)}{2 \pi} \iint_{\Delta} (1 - |z|^2)^2 \psi(z) \zeta^p dx dy, \quad (5.9) \]
or \( \hat{c}_p = M'_p(\zeta, \psi) \), \( p = 0, 1, \ldots \), with
\[ M'_p = (p + 1) \ldots (p + 4)/(2\pi). \]
Consider the bounded linear transformation
\[
\mathcal{L} : \mu \mapsto \psi(\zeta) = \frac{3}{\pi} \int_{\Delta} \frac{\mu(z)dx\,dy}{(1 - \zeta)z} : L_\infty(\Delta) \to B(\Delta).
\] (5.10)

It satisfies
\[
\langle \varphi, \mathcal{L} \mu \rangle = \int_{\Delta} \varphi \mu dx\,dy, \quad \text{for } \varphi \in A_1(\Delta),
\] (5.11)

and, similar to (5.9) the coefficients of \(\psi\) and therefore, \(\psi\) are given by
\[
\widehat{c}_p(\psi) = M'_p \int_{\Delta} \mu(z)\varphi^p dx\,dy
\] (5.12)

which represents the variation of coefficients under varying the elements \(\mu \in L_\infty(\Delta)\).

Both functionals \(J\) and \(\tilde{J}\) are extended to all such \(\psi\) as the limits of their values on \(f_r(z) = \kappa \circ \psi(rz) = c_0 + c_1rz + \cdots \in B_1^0\), \(r \neq 1\) and letting \((c_0, \ldots, c_m) = \lim_{r \to 1}(c_0r, \ldots, c_mr)\) for finite collections. Denote these extensions by \(J(\mu)\) and \(\tilde{J}(\mu)\).

Our goal is to show that for any extremal function \(\psi_0 = \mathcal{L}\mu_0\) maximizing \(J(\mu)\) on a small ball \(U_r = \{||\mu||_\infty < r\}\) (whose existence of \(f_0\) follows from compactness) defines generates a function \(f_0 = \kappa \circ f_0 \in B_1^0\). First of all, we have:

**Lemma 5.2.** For small \(r > 0\), any extremal \(\psi_0 = \mathcal{L}\mu_0\) on \(U_r\) is orthogonal to all powers \(z^p\) with \(p \neq 1, 2, \ldots, n\), i.e., for all such \(p\),
\[
\langle z^p, \psi_0 \rangle = \langle \mu_0, \varphi^p \rangle = 0,
\]

and therefore, \(\psi_0\) is a polynomial
\[
\psi_0(z) = t \sum_{j=1}^n \tilde{d}_jz^j, \quad |t| = 1.
\] (5.13)

**Proof.** By a computation, for a fixed \(r < 1\) and \(r \to 0\), one gets, using the relations (5.11) and (5.2),
\[
\max_{||\mu||_\infty \leq \tau r} \left| \tilde{J}(\tilde{f}) \right| = \max_{||\mu||_\infty \leq \tau r} \left| \frac{d}{d\tau} \sum_{j=1}^n \frac{\partial^j \tilde{J}}{\partial \varphi_n^j} \bigg|_{\varphi_n} + O_n(\tau^2) \right| = \max_{||\mu||_\infty \leq \tau r} \left| \int_{\Delta} \mu(z)\varphi_n(z) dx\,dy \right|
\] (5.14)

where \(\varphi_n\) is a polynomial of the form (5.13) whose coefficients are determined by the initial coefficients of \(\kappa\).

Now consider for any fixed \(p \neq 1, \ldots, n\) the auxiliary functional
\[
\tilde{J}_p(\mu) = \tilde{J}(\mu) + \xi \tilde{c}_p = \tilde{J}(\mu) + \xi M'_p(\mu, \varphi^p)
\]

with \(\xi \in \mathbb{C}\). Then, similar to (5.14),
\[
\max_{||\mu||_\infty \leq \tau r} \left| \tilde{J}_p(\mu) \right| = r \int_{\Delta} \left| \varphi_n(z) + \xi M'_p z^p \right| dx\,dy + O_n(\tau^2), \quad r \to 0,
\] (5.15)

and the remainder term estimate is independent of \(p\). Using the known properties of the norm
\[
h_p(\xi) = \int_{\Delta} \left| \varphi_n(z) + \xi z^p \right| dx\,dy
\]
following from the Royden\cite{26} and Earle-Kra\cite{6} lemmas, one obtains from (5.14), (5.15) that for small $\xi$ there should be
\[
\max_{\|\mu\| \leq r} |J_p(\mu)| = \max_{\|\mu\| \leq r} |\tilde{J}(\mu)| + r_o p(\xi) + O(r^2 \xi) + O(r^2),
\]
For $\|\mu\| \leq \tau r$ with fixed $r$ and $\tau \to 0$, this estimate yields
\[
\max_{\|\mu\| \leq \tau r} |J_p(\mu)| = \max_{\|\mu\| \leq \tau r} |\tilde{J}(\mu)| + \tau r_o p(\xi) + O(\tau^2 \xi) + O(\tau^2). \tag{5.16}
\]
On the other hand, as $\xi \to 0$, $\tau \to 0$,
\[
|\tilde{J}_p(\mu_0)| = |\tilde{J}(\tau_0 \mu_0)| + \Re \frac{\tilde{J}(\tau_0 \mu_0)}{|\tilde{J}(\tau_0 \mu_0)|} M_p(\xi(\tau_0 \mu_0, \nu)) + O(\tau^2 \xi^2)
\]
by suitable choices of $\xi \to 0$. Comparison with (5.16) implies the desired orthogonality $\langle \mu_0, \nu \rangle = 0$.

Substituting into (5.10) the expansion
\[
1/(1 - \bar{\tau} \xi)^4 = (1 + \bar{\tau} \xi + \bar{\tau}^2 \xi^2 + \ldots)^4 = 1 + 4 \tau \xi + \ldots,
\]
one gets after the term-wise integration the representation (5.13), completing the proof of the lemma.

We now establish that for small $r$ and some $|t| = 1$, the extremal $\mu_0$ must be of the form
\[
\mu_0(z) = rt \mu_0(z) := rt(\varphi(z))/|\varphi_n(z)|, \tag{5.17}
\]
where $\varphi_n$ is the polynomial of order $n$ given above. We suppose that (5.17) does not hold and show that this leads to a contradiction. Without loss of generality, one can take $\mu_0$ to be an extremal $L_\infty$ function arising in the Hahn-Banach extension of $l_{\psi_0}$ and satisfying (5.5). Pass to functionals
\[
J^0(f) = J(f)/M_n, \quad \tilde{J}^0(f) = J^0 \circ k(f)
\]
mapping $B^1$ onto the unit disk. The differential of $\tilde{J}^0$ at $\psi = \tilde{f} = 0$ defines a linear operator $P_n : L_\infty(\Delta) \to L_\infty(\Delta)$ acting by
\[
P_n(\mu) = \alpha_n(\mu, \varphi_n) \mu_0.
\]
Then, in view of our assumption,
\[
\{P_n(t \psi_0) : |t| < 1\} \in \{ |t| < 1\};
\]
thus by Schwarz lemma,
\[
|P_n(r \mu_0)| = \rho(r) < r. \tag{5.18}
\]
Now consider the function
\[
\omega_0 = \nu_{\psi_0} - \rho(r) \mu_0
\]
with $\nu_{\psi_0}$ defined by (5.4). We show that $\omega_0$ annihilates all functions $\varphi \in A_1(\Delta)$.

Lemma 5.2 and the mutual orthogonality of the powers $z^m, \ m \in \mathbb{Z}$, yield that $\langle \omega_0, z^p \rangle = 0$ for all $p = 0, 1, \ldots$ distinct from $n$. So we have only show that
\[
\langle \omega_0, \varphi_n \rangle = 0, \quad \varphi_n = z^n.
\]
Take the conjugate operator
\[
P_n(\varphi) = \alpha_n(\mu_0, \varphi) \varphi_n
\]
mapping $L_1(\Delta^*)$ to $L_1(\Delta^*)$. It fixes the subspace $\{t \varphi_n \ : \ t \in \mathbb{C}\}$, and $P_n(\omega_0) = 0$. Thus from (5.4) and (5.18), for some $t$,
\[
\langle \omega_0, \varphi_n \rangle = t\langle \omega_0, P_n(\varphi_n) \rangle = t\langle P_n(\omega_0), \varphi_n \rangle = 0,
\]
which means that $\omega_0 \in A_1(\Delta)$.
It is proven in the theory of extremal quasiconformal maps (see, e.g., [1]) that one of the characteristic properties of the extremal elements $\mu_0 \in L_1(\Delta)$ (the Beltrami coefficients) for functionals $l_{\psi}$ on $A_1(\Delta)$ arising by their Hahn-Banach extension to $L_1(\Delta)$ is

$$\|\mu_0\|_\infty \leq \inf\{\|\mu_0 + \omega\|_\infty : \omega \in A_1(\Delta)^\perp\}. \quad (5.19)$$

Applying (5.5) and (5.20), one obtains

$$r \leq \|\mu_0 - \omega_0\|_\infty = \|\rho_1(r)\mu_0\|_\infty = \|\rho(r)\|_\infty,$$

which contradicts (5.18) and proves (5.17).

(In fact, in our case we have more, since the Schwarzians $S_F \in \mathcal{B}(\Delta^*)$ generated by functions $f \in B^1_F$ via (5.6) satisfy $|S_f(\zeta)| = o(|\zeta|^{-1})$ as $|\zeta| \to 1$; thus their extremal $\mu_0$ are of Teichmüller form $\mu_0 = k|\varphi|/\varphi$ with $k = \text{const}$ and $\varphi \in A_1(\Delta)$; for such $\mu_0$ the inequality (5.19) is strong.)

Finally we have to prove that in (5.13)

$$\psi_0(z) = rtz^n, \quad |t| = 1.$$

This is a consequence of the extremality of $\psi_0$ which yields that the image $\psi_0(\Delta)$ must be a whole disk $\Delta_r = \{|w| < r\}$. To establish such a property of the extremal $\psi_0$, on can apply the following local existence theorem from [13] which we present here as

**Lemma 5.3.** Let $D$ be a finitely connected domain on the Riemann sphere $\hat{\mathbb{C}}$. Assume that there are a set $E$ of positive two-dimensional Lebesgue measure and a finite number of points $z_1, z_2, \ldots, z_m$ distinguished in $D$. Let $\alpha_1, \alpha_2, \ldots, \alpha_m$ be non-negative integers assigned to $z_1, z_2, \ldots, z_m$, respectively, so that $\alpha_j = 0$ if $z_j \in E$.

Then, for a sufficiently small $\varepsilon_0 > 0$ and $\varepsilon \in (0, \varepsilon_0)$, and for any given collection of numbers $w_j, s = 0, 1, \ldots, \alpha_j, j = 1, 2, \ldots, m$ which satisfy the conditions $w_0 \in D$,

$$|w_{0j} - z_j| \leq \varepsilon, \quad |w_{1j} - 1| \leq \varepsilon, \quad |w_{sj}| \leq \varepsilon (s = 0, 1, \ldots, \alpha_j, j = 1, \ldots, m),$$

there exists a quasiconformal automorphism $h_\varepsilon$ of domain $D$ which is conformal on $D \setminus E$ and satisfies

$$h_\varepsilon^{(s)}(z_j) = w_{sj} \quad \text{for all} \ s = 0, 1, \ldots, \alpha_j, \ j = 1, \ldots, m.$$

Moreover, the Beltrami coefficient $\mu_{h_\varepsilon}(z) = \partial_2 h_\varepsilon/\partial_1 h_\varepsilon$ of $h$ on $E$ satisfies $\|\mu_{h_\varepsilon}\|_\infty \leq M\varepsilon$. The constants $\varepsilon_0$ and $M$ depend only upon the sets $D, E$ and the vectors $(z_1, \ldots, z_m)$ and $(\alpha_1, \ldots, \alpha_m)$.

If the boundary $\partial D$ is Jordan or is $C^{l+\alpha}$-smooth, where $0 < \alpha < 1$ and $l \geq 1$, we can also take $z_j \in \partial D$ with $\alpha_j = 0$ or $\alpha_j \leq l$, respectively.

Now, assuming that $\psi_0(\Delta) \neq \Delta$, (and hence $\Delta_r \setminus \psi_0(\Delta)$ is an open set), one can pick a disk $E \Subset \Delta_r \setminus \psi_0(\Delta)$ and applying Lemma 5.2 vary the coefficients $\hat{c}_0, \hat{c}_1, \ldots, \hat{c}_n$ of $\psi_0$ by quasiconformal automorphisms $h_\varepsilon$ of the disk $\Delta$ conformal on $\Delta \setminus E$ so that $h_\varepsilon \circ \psi_0 \in H^1(\hat{\mathbb{C}})$ and each coefficient $\hat{c}_j(h_\varepsilon \circ \hat{f}_0)$ ranges over a small neighborhood of $\hat{c}_j(\hat{f}_0)$ for all $j = 0, 1, \ldots, n$. By appropriate choice of $h_\varepsilon$, one gets $|\hat{J}(h_\varepsilon \circ \hat{f}_0)| > |\hat{J}(\hat{f}_0)|$ (equivalently, $|J(\kappa \circ h_\varepsilon \circ \hat{f}_0)| > |J(\hat{f}_0)|$), contradicting the extremality of $f_0 = \kappa \circ \psi_0$ on the ball $\|\mu\|_\infty < r$ and on its proper convex subset consisting on $\mu$ with $\psi = \mu \in H^1(\hat{\mathbb{C}})$.

Therefore, the polynomial (5.13) must map the unit circle $\{|z| = 1\}$ onto the circle $\{|w| = r\}$ which is possible only when the $\psi_0 = rtz^p_0$, where $|t| = 1$ and $1 \leq p_0 \leq n$, in addition, $p_0$ must divide $n$. Were $p_0 < n = kp_0$, then

$$f_0(z) = \kappa(tz^{p_0}) = \frac{1}{e} + \frac{2}{e}tz^{p_0}z^{p_0} + \cdots + c_0^kz^n + \cdots$$

with $|c_0^k| \geq 2/e$, which violates Parseval’s equality $\sum_0^\infty |c_0^m t^m|^2 = r^2$ for the boundary function $\kappa(e^{ip_0}\theta)$. Hence, $\psi_0(z) = tz^n, \ |t| = 1$.

We have established the existence of $r_0 > 0$ such that, for all $r \leq r_0$, any extremal function of the rescaled functional $J^0(f)$ on $\mathcal{K}(U_r)$ is of the form $f_0 = (t_1\kappa) \circ (tz^n)$ with $|t| = |t_1| = 1$ (and
does not depend on \( r \). This sharply estimates \( J(f) \) on \( k(U_r) \) via (5.7) and completes the proof of Theorem 5.1.

2. Now, to derive the assertion of Theorem 1.1, take \( J(f) = c_n(f) \). By Theorem 5.1, we have for \( |t| = r \leq r_0 \) the sharp bound
\[
\sup_{k(t)} |c_n| = c_n(\kappa(tz^n)) = 2r/e, \quad r = |t|, \tag{5.20}
\]
and the extremal \( f_0(z) = tz^n \). This estimates yields, together with Lemma 3.3 and Schwarz’s lemma, that the equality (5.20) must hold for all \( r < 1 \) and that the functional \( j \) is a defining function of the disk \( \{ \kappa(tz^n) : |t| < 1 \} \) as a Carathéodory geodesic in \( B_1^0 \).

For any other geodesic disk \( \{ tf : |t| < 1 \} \) in \( H_1^\infty \), the previous arguments provide the strong inequality
\[
|c_n(\kappa(tf))| < |c_n(\kappa(t))|.
\]
(5.21)
In the limit as \( r \to 1 \), one obtains from (5.20) the desired bound (1.1), with equality for \( f = \kappa_n \).

It remains to show that no other extremal functions can appear in the limit case \( r = 1 \).

Let for some \( f \in B_1 \), we have the equality
\[
|c_n(f)| = 2/e = c_n(\kappa_n).
\]
(5.22)
Then all function \( f_r(z) = f(rz) \) with \( r < 1 \) belong to \( B_1 \) and
\[
|c_n(f_r)| = 2r^n/e = c_n(\kappa_n r).
\]
Fix \( r < 1 \) and represent \( f_r \) by (3.1) via \( f_r = \kappa \circ \hat{f}_r \) with corresponding \( \hat{f}_r \in H_1^\infty \). Regarding \( \hat{f}_r \) as a point of its geodesic disk \( \{ tf_r/\|f_r\|_\infty : |t| < 1 \} \) corresponding to \( t = \|f_r\|_\infty < 1 \), one obtains from (5.20) and (5.21), \( |c_n(f_r)| \leq |c_n(\kappa_n)| t \), with equality only for \( f = \kappa_n \). This yields that also (5.22) is valid only for \( f = \kappa_n \), completing the proof of Theorem 1.1.

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