Hawking radiation of the Vaidya–Bonner–de Sitter black hole

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New Journal of Physics 9 (2007) 252
Received 9 March 2007
Published 2 August 2007
Online at http://www.njp.org/
doi:10.1088/1367-2630/9/8/252

Abstract. Considering the unfixed background space-time and the self-gravitational interaction, we view the Hawking radiation of the Vaidya–Bonner–de Sitter black hole by the Hamilton–Jacobi method and the radial geodesic method. The result shows the tunneling rate is related not only to the change of Bekenstein–Hawking entropy but also to the integral of the black hole mass and charge, which does not satisfy the unitary theory and is not in accordance with the known result.

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1. Introduction

Hawking radiation [1] is interpreted as a quantum-tunneling process triggered by vacuum fluctuations near the horizon. When the real tunneling procedure is taken into account, the derived radiation spectrum deviates from the purely thermal one and the tunneling rate is related to the change of Bekenstein–Hawking entropy. This was mentioned in recent studies [2]–[11], where the background space-time and the self-gravitational interaction were considered.
However, these studies are mainly limited to the Hawking radiation of static and stationary black holes. For the case of non-stationary black holes, these were not thoroughly researched [12, 13]. As we know, due to absorption and evaporation, the mass and other parameters of black holes are not fixed, and should change with time. Therefore, generally speaking, black holes are non-stationary and stationary black holes are only ideal models. So the study of non-stationary black holes is rather meaningful in the cognition and exploration on black holes.

In this paper, considering the unfixed background space-time and the self-gravitational interaction, we view the Hawking radiation of the non-stationary Vaidya–Bonner–de Sitter black hole. The tunneling rate of the radiation particle is related to two parts: one part is the change of Bekenstein–Hawking entropy; another part is the integral of the black hole mass and charge. This result does not satisfy the unitary theory and is not in accordance with Parikh and Wilczek’s [2] result. The remainder of this paper is organized as follows. The Hawking radiation of the Vaidya–Bonner–de Sitter black hole is viewed by the Hamilton–Jacobi method and the radial geodesic method in sections 2 and 3, respectively. Section 4 contains some discussion and the conclusion.

2. Hamilton–Jacobi method

The line element of the non-stationary Vaidya–Bonner–de Sitter black hole represented in Eddington–Finkelstein coordinates is given by [14]

$$ds^2 = - \left(1 - \frac{2M(v)}{r} + \frac{Q^2(v)}{r^2} - \frac{1}{3} \lambda r^2 \right) dv^2 + 2dvdr + r^2(d\theta^2 + \sin^2 \theta d\phi^2),$$

with the electromagnetic potential

$$A_{\mu} = \left(\frac{Q(v)}{r}, 0, 0, 0\right).$$

This line element is different from that of a stationary sphere-symmetrically charged black hole (such as the Ressiner–Nordström black hole) and there is no coordinate that is singular in the line element and the mass $M(v)$ and charge $Q(v)$ of the black hole change with time. The entropy is $S = \pi r_h^2$, and the event horizon $r_h$ satisfies the following equation as

$$1 - 2\dot{r}_h - \frac{2M(v)}{r} + \frac{Q^2(v)}{r^2} - \frac{1}{3} \lambda r^2 = 0.$$

In which $\dot{r}_h = \partial r / \partial v$ denotes the event horizon changes with time. Due to this change, it is not convenient for us to discuss the Hawking radiation of the black hole. A helpful treatment should be in a motion coordinate system. Besides, the horizons (obtained from $g^{\mu\nu} \partial_\mu f \partial_\nu f = 0$) and the time-like limit surfaces (obtained from $g_{00} = 0$) are not coincident with each other, which causes some difficulties in calculating the tunneling rate. So we introduce the following coordinate transformation

$$R = r - r_h, \quad (dR = dr - \dot{r}_h dv).$$

Then new line element takes on the form as

$$ds^2 = - \left(1 - 2\dot{r}_h - \frac{2M(v)}{r} + \frac{Q^2(v)}{r^2} - \frac{1}{3} \lambda r^2 \right) dv^2 + 2dvR + r^2(d\theta^2 + \sin^2 \theta d\phi^2).$$
In this line element, the metrics satisfy Landau’s condition of the coordinate clock synchronization. This provides the advantageous condition to discuss the Hawking radiation. The horizons coincide with the time-like limit surfaces, which imply the geometrical optics limit can be relied upon here. Using the WKB approximation, we can get the relationship between the tunneling rate and the imaginary part of radiation particle’s action as \( \Gamma \sim \exp(-2\text{Im} I) \). In the investigation of the Hawking radiation, the crux is to derive the action. There are two methods to derive it, namely the radial geodesic method and the Hamilton–Jacobi method. The radial geodesic method was developed by Parikh and Wilczek [2] and was mainly adopted to study the Hawking radiation of static and stationary black holes. In this method, to get the action, one should introduce the Painlevé coordinate and explore the motion equations of radiation particles. Moreover, massless and massive particles should be differentiated for their motion equations are different. The Hamilton–Jacobi method was extensively applied to the non-thermal radiation of black holes in 1990s [15, 16] and has attracted people’s attention again recently [17]–[20]. Since the derivation of the action is dependent on the Hamilton–Jacobi equation, one can avoid performing the Painlevé coordinate transformation and exploring the motion equations. The virtue of this method consists in that one can get not only the action of particles from the stationary black holes but also that from non-stationary black holes by solving Hamilton–Jacobi equations. Considering the properties of the non-stationary Vaidya–Bonner–de Sitter space-time, we adopt the Hamilton–Jacobi method to explore the action here. The radiation particle’s action satisfies the relativistic Hamilton–Jacobi equation

\[
g^{\mu \nu} (\partial_\mu I + q A_\mu)(\partial_\nu I + q A_\nu) + u^2 = 0, \tag{6}\]

where \( u \) and \( q \) are the mass and charge of the particle, respectively. Substituting the inverse metric tensors \( g^{\mu \nu} \) derived from the line element (5) into the Hamilton–Jacobi equation yields

\[
2(\partial_\nu I + q A_\nu) \partial_R I + \left(1 - 2 \dot{r}_h - \frac{2M(v)}{r} + \frac{Q(v)}{r^2} - \frac{1}{3} \lambda r^2 \right) (\partial_R I)^2 \\
+ r^{-2}(\partial_\theta I)^2 + (r \sin \theta)^{-2}(\partial_\varphi I)^2 + u^2 = 0. \tag{7}\]

It is difficult to get the expression of the action from equation (7) for it is a function of \( v, R, \theta \) and \( \varphi \). Considering the properties of the Vaidya–Bonner–de Sitter space-time, we carry out the separation of variables as

\[
I = W(v, R) + \Theta(\theta) + Y(\varphi), \tag{8}\]

and let \( \partial_\nu I = \partial_\nu W(v, R) = -\omega \). So equation (7) becomes

\[
-2(\omega - q A_\nu)\partial_R W(v, R) + \left(1 - 2 \dot{r}_h - \frac{2M(v)}{r} + \frac{Q^2(v)}{r^2} - \frac{1}{3} \lambda r^2 \right) [\partial_R W(v, R)]^2 \\
+ r^{-2}[\partial_\theta \Theta(\theta)]^2 + (r \sin \theta)^{-2}[\partial_\varphi Y(\varphi)]^2 + u^2 = 0, \tag{9}\]

which is an equation of \( W(v, R), \Theta(\theta) \) and \( Y(\varphi) \), but our real interest is in the expression of \( W(v, R) \) for the imaginary part can be produced from it. Solving it, there is

\[
W(v, R) = \int \frac{(\omega - q A_\nu)^2 - (1 - 2 \dot{r}_h - [2M(v)/r] + [Q^2(v)/r^2] - (1/3) \lambda r^2)H}{1 - 2 \dot{r}_h - [2M(v)/r] + [Q^2(v)/r^2] - (1/3) \lambda r^2} \, dR, \tag{10}\]

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The solution is singular at the event horizon. Near the event horizon, we have

\[ 1 - 2\dot{r}_h - \frac{2M(v)}{r} + \frac{Q^2(v)}{r^2} - \frac{1}{3}\lambda r^2 = \left( \frac{M(v)}{r_h^2} - \frac{Q^2(v)}{r_h^3} - \frac{1}{3}\lambda r_h \right) (r - r_h) + \cdots. \]  

(12)

Finishing the integral and substituting the integral result into equation (8), we can get the imaginary part of the action as

\[ \text{Im} I = \frac{\pi (\omega - q A_h)}{[M(v)/r_h^2] - [Q^2(v)/r_h^3] - (1/3)\lambda r_h}, \]  

(13)

where \( A_h = Q(v)/r_h \) expresses the electromagnetic potential at the event horizon. Using the WKB approximation, we can get the tunneling rate of the radiation particle. However, it is easy to find the radiation spectrum is the only leading term, which is not in accordance with recent research. The reason is that the unfixed background space-time and the self-gravitational interaction were not taken into account. Now let us incorporate these and move onto discussing the Hawking radiation of this black hole.

Considering the unfixed background space-time, we fix the total energy and charge of the black hole to fluctuate. When the particle with energy \( \omega \) and charge \( q \) tunnels across the event horizon, the mass and charge of the black hole will change into \( M(v) - \omega \) and \( Q(v) - q \). We hypothesize that the particle tunnels across the horizon only in a radial direction. In this paper, we discuss the Hawking radiation in the following case, namely there are no other particles to be absorbed and emitted by the black hole during this time \( \Delta v \) that the particle tunnels across the potential barrier. Since the fact that the particle tunnels across the barrier is an instantaneous process, the time \( \Delta v \) is infinity short. Therefore, the case that it is too late for other particles to be absorbed or emitted by the black hole in this time \( \Delta v \) is possible, and we can study the Hawking radiation of the non-stationary Vaidya–Bonner–de Sitter black hole follows the case of stationary black holes. Taking the self-gravitational interaction into account, we get the imaginary part of the actual action as

\[ \text{Im} I = \pi \int_{(0,0)}^{(\omega,q)} \frac{d\omega' - A_h' dq'}{M(v) - \omega'/r_h^2 - [(Q(v) - q')^2/r_h^3] - (1/3)\lambda r_h}, \]

(14)

where

\[ M' = M(v) - \omega', \quad Q' = Q(v) - q', \quad \dot{r}_h' = \frac{\partial r_h}{\partial v}, \]

(15)

and \( \dot{r}_h' \) satisfies the horizon equation. Due to \( \dot{r}_h' \) is a function of \( M(v) \), \( Q(v) \) and \( r_h' \), it is difficult to get the result from equation (14) by a direct integral. From the horizon equation

\[ 1 - 2\dot{r}_h - 2M(v)/r + Q^2(v)/r^2 - (1/3)\lambda r^2 = 0 \]

and the black hole entropy \( S = \pi r_h^2 \), we have

\[ dS = 2\pi \frac{dM - (Q/r_h) dQ + r_h ((\partial r_h/\partial M) dM + (\partial r_h/\partial Q) dQ)}{(M/r_h^2) - (Q^2/r_h^3) - (1/3)\lambda r_h}. \]

(16)
So the imaginary part is

\[
\text{Im} I = -\frac{\Delta S_{\text{BH}}}{2} + \pi \int_{(M(v), Q(v))} \left( \frac{\partial T_1/\partial M'}{(M'/r_h^2) - (Q^2/r_h^3) - (1/3)\lambda r_h'} \right) \frac{r_h'}{r_h^2} \left( \partial r_h'/\partial M' \right) \text{d}M' + \left( \partial r_h'/\partial Q' \right) \text{d}Q',
\]

where \( \Delta S_{\text{BH}} = S_{\text{BH}}[M(v) - \omega, Q(v) - q] - S_{\text{BH}}[M(v), Q(v)] \) is the change of Bekenstein–Hawking entropy before and after the particle tunnels out of the event horizon. And then the tunneling rate is obtained as

\[
\Gamma \sim \exp(-2\text{Im} I) = \exp \left[ \frac{\Delta S_{\text{BH}}}{2} - 2\pi \int_{(M(v), Q(v))} \left( \frac{\partial T_1/\partial M'}{(M'/r_h^2) - (Q^2/r_h^3) - (1/3)\lambda r_h'} \right) \frac{r_h'}{r_h^2} \left( \partial r_h'/\partial M' \right) \text{d}M' + \left( \partial r_h'/\partial Q' \right) \text{d}Q' \right].
\]

Clearly, the tunneling rate is related to two parts: one part is the change of Bekenstein–Hawking entropy; another part is the integral of the black hole mass and charge, which is in accordance with Parikh and Wilczek’s result and does not satisfy the unitary theory.

In the following, we discuss the Hawking radiation of the black hole at the cosmological horizon. Replace \( r_h \) by \( r_c \) in equations (3)–(5), (7), (9), (10), (12) and (13). Considering the unfixed background space-time and self-gravitational interaction, when the particle with energy \( \omega \) and charge \( q \) tunnels into the cosmological horizon, the imaginary part of the radiation particle’s action should be written as

\[
\text{Im} I = \pi \int_{(0,0)}^{(\omega, q)} \frac{\text{d}\omega'}{[M(v) + \omega'/r_c^2] - [(Q(v) + q')^2/r_c^4] - (1/3)\lambda r_c^3} - \pi \int_{(M(v), Q(v))} \frac{\text{d}M'' - (Q''/r_c^2) \text{d}Q''}{(M''/r_c^2) - (Q''^2/r_c^3) - (1/3)\lambda r_c^3},
\]

where \( M'' = M(v) + \omega', Q'' = Q(v) + q', \dot{r}_c'' = \partial r_c'/\partial v \), and \( r_c'' \) satisfy the horizon equation. Finishing the integral, we get

\[
\text{Im} I = -\frac{\Delta S'_{\text{BH}}}{2} + \pi \int_{(M(v), Q(v))} \frac{r_c''((\partial r_c''/\partial M'') \text{d}M' + (\partial r_c''/\partial Q'') \text{d}Q'')}{(M''/r_c^2) - (Q''^2/r_c^3) - (1/3)\lambda r_c^3},
\]

where \( \Delta S'_{\text{BH}} = S_{\text{BH}}[M(v) + \omega, Q(v) + q] - S_{\text{BH}}[M(v), Q(v)] \) is the change of Bekenstein–Hawking entropy before and after the particle tunnels into the cosmological horizon. So the tunneling rate is

\[
\Gamma \sim \exp(-2\text{Im} I) = \exp \left[ \frac{\Delta S'_{\text{BH}}}{2} - 2\pi \int_{(M(v), Q(v))} \frac{r_c''((\partial r_c''/\partial M'') \text{d}M' + (\partial r_c''/\partial Q'') \text{d}Q'')}{(M''/r_c^2) - (Q''^2/r_c^3) - (1/3)\lambda r_c^3} \right].
\]

The tunneling rate is related not only to the change of Bekenstein–Hawking entropy but also to the integral of the black hole mass and charge, which is in accordance with equation (18).

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3. Radial geodesic method

In this section, we adopt the radial geodesic method to review the Hawking radiation of the Vaidya–Bonner–de Sitter black hole at the event horizon. We know the black hole is charged, and the corresponding radiation particles should be also charged, therefore, the effect of the electromagnetic field has to be considered with the particle emission. We first explore the motion equation of the particle. According to [8], the radial geodesic of the charged particle, which is different from that of the uncharged massless particle, is the phase velocity of the particle. From the line element (5), the phase velocity can be given as

\[ r_p = \frac{-1}{2} \frac{g_{00}}{g_{01}} = \frac{1}{2} \left( 1 - 2\tilde{r}_h - \frac{2M(v)}{r} + \frac{Q^2(v)}{r^2} - \frac{1}{3}\lambda r^2 \right). \]  

(22)

In the following, we discuss the Hawking radiation of the black hole given in the case of section 2. Considering the unfixed background space-time, when the particle with energy \( \omega \) and charge \( q \) tunnels out from the event horizon, the mass and charge of the black hole should be replaced by \( M(v) - \omega \) and \( Q(v) - q \) in equation (22). We find the horizon is still coincident with the time-like limit surfaces after the particle emission, and the geometrical optics limit can be relied upon here. In the semi-classical limit, the imaginary part of the particle action is expressed as

\[ \text{Im} S = \int_{r_i}^{r_f} L \, dt = \text{Im} \int_{r_i}^{r_f} (L - \Pi_{A\mu} \dot{A}_\mu) \, dt = \text{Im} \left[ \int_{r_i}^{r_f} \int_0^{P_r} \int_0^{P_{A\mu}} \int_0^{P_\mu} dP_r' \, dP_r \right. \\
\left. \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad \quad \left. \frac{A_{\mu} P_\mu}{A_{\mu} (A_{\mu}, r, P_r)} \right] \], \]  

(23)

where \( r_i \) and \( r_f \) denote the locations of the event horizon before and after the particle goes missing and the distance depends on the energy and charge of the outgoing particle. \( (P_r, P_{A\mu}) \) are canonical momentums conjugated to coordinates \( (r, A_\mu) \), respectively. For proceeding with an explicit calculation, we adopt Hamilton canonical equation

\[ \dot{r} = \frac{\partial H}{\partial P_r} \bigg|_{(r, A_\mu, P_{A\mu})} = d[M(v) - \omega]; \]  

\[ \dot{A}_\mu = \frac{\partial H}{\partial P_{A\mu}} \bigg|_{(A_\mu, r, P_r)} = A_\mu d[Q(v) - q']; \]  

(24)

where \( A_\mu = (Q(v) - q'/r) \). Substituting equations (22) and (24) into (23), we have

\[ \text{Im} S = \text{Im} \left[ \int_{r_i}^{r_f} \int_{(M(v), Q(v))}^{(M(v) - \omega, Q(v) - q)} \frac{dr}{F_r} \left[ d[M(v) - \omega'] - A_\mu d[Q(v) - q'] \right] \right] \\
= 2\text{Im} \int_{r_i}^{r_f} \int_{(M(v), Q(v))}^{(M(v) - \omega, Q(v) - q)} \frac{d[M(v) - \omega'] - [Q(v) - q'/r]d[Q(v) - q']}{1 - 2\tilde{r}_h - [2M(v) - \omega']/r + [(Q(v) - q')^2/r^2] - (1/3)\lambda r^2} \, dr \]

\[ = -\pi \int_{(M(v), Q(v))}^{(M(v) - \omega, Q(v) - q)} \frac{d[M(v) - \omega'] - [Q(v) - q'/r]d[Q(v) - q']}{[M(v) - \omega'/r_h^2] - [(Q(v) - q')^2/r_h^2] - (1/3)\lambda r_h^2}, \]  

(25)
which is in accordance with equation (14). Further calculating, we can also get the same result, namely the tunneling rate of the particle is

$$\Gamma \sim \exp(-2\text{Im} I) = \exp\left[\Delta S_{\text{BH}} - 2\pi \int_{(M(v), Q(v))}^{(M(v) - \omega, Q(v) - q)} \frac{r_h'((\partial \dot{r}_h^i/\partial M')dM' + (\partial \dot{r}_h^i/\partial Q')dQ')}{(M'/r_h^2) - (Q^2/r_h^3) - (1/3)\lambda r_h^3}\right],$$

(26)

in which $M' = M(v) - \omega'$, $Q' = Q(v) - q'$, $\dot{r}_h' = \partial \dot{r}_h^i/\partial v$. The result is consistent with equations (18) and (21).

4. Discussion and conclusion

From equations (18), (21) and (26), we know the tunneling rate is related to two parts: one part is the change of Bekenstein–Hawking entropy; another part is related to the integral of the black hole mass and charge. This result is not in accordance with Parikh and Wilczek’s result.

In fact, Chowdhury’s work [21] also disagrees with Parikh and Wilczek’s result. It showed that $\exp(-2\text{Im}(\int Pdr))$ is not invariant under canonical transformations in general. Further study in [22], which showed the temperature of the Hawking radiation should be twice as large as originally calculated. On the basis of Akhmedov and Chowdhury’s works [21, 22], Mitra [23] further studied the Hawking radiation of stationary black holes, and this work is in accordance with Parikh and Wilczek’s opinion. In this paper, we researched the Hawking radiation of the non-stationary black hole, which is different from that of [21–23] and their works limited to stationary black holes. Our result in this paper can be reduced to the case of the stationary black holes and is rather meaningful.

When $\lambda = 0$, we can get the tunneling rate of the radiation particle from equation (18) as

$$\Gamma \sim \exp(-2\text{Im} I) = \exp\left[\Delta S_{\text{BH}} - 2\pi \int_{(M(v), Q(v))}^{(M(v) - \omega, Q(v) - q)} \frac{r_h'((\partial \dot{r}_h^i/\partial M')dM' + (\partial \dot{r}_h^i/\partial Q')dQ')}{(M'/r_h^2) - (Q^2/r_h^3) - (1/3)\lambda r_h^3}\right].$$

(27)

It corresponds to that of the Vaidya–Bonner black hole. When $\dot{r}_h = \text{const}$, the tunneling of the radiation particle is reduced to

$$\Gamma \sim \exp(-2\text{Im} I) = \exp(\Delta S_{\text{BH}}),$$

(28)

which is the known result (Parikh and Wilczek’s result).

In this paper, using the Hamilton–Jacobi method and the radial geodesic method, we discussed the Hawking radiation of the Vaidya–Bonner–de Sitter black hole by considering the unfixed background space-time and the self-gravitational interaction. The result does not satisfy the unitary theory and is not in accordance with Parikh and Wilczek’s result.

Acknowledgments

This work is supported by National Natural Science Foundation of China (grant no 10347008) and Sichuan Province Foundation for Fundamental Research of Projects (grant no 05JY029-092).
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