I-Semiprime Submodules

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Abstract
Let $R$ be a commutative ring with identity and $I$ a fixed ideal of $R$ and $M$ be an unitary $R$-module. We say that a proper submodule $N$ of $M$ is $I$-semi prime submodule if $a \in R$, $x \in M$ with $a^2x \in N - IN$ implies that $ax \in N$. In this paper, we investigate some properties of this class of submodules. Also, some characterizations of $I$-semiprime submodules will be given, and we show that under some assumptions $I$-semiprime submodules and semiprime submodules are coincided.

Keywords: Prime submodules, weakly semiprime submodules, semiprime submodules, $I$-semiprime submodules.
1. Main result

**Definition (1.1):**
(i) Let I be an ideal of R and M an R-module. A proper submodule N of M is called an I-semiprime submodule of M, if $a^2x \in N - IN$ for all $a \in R, x \in M$ implies that $x \in N$.

(ii) An ideal A is called I-semiprime ideal iff for every $a \in R$ and any ideal I, $(a^2) \subseteq A - IA$ implies $(a) \subseteq A$.

Now, it is clear that every semiprime submodule N of M is an I-semiprime submodule of M. But the converse need not be true. For example, consider Z-module $M = Z_{24}$ and N = (8). Then if $I = [N: M]N = [(8); Z_{24}] = (8)$. So N is an I-semiprime submodule of M. But N is not semiprime in M, since $2^2 \cdot (2) = 8 \notin N$, but 2.2 $\notin N$.

**Proposition (1.2):**
1. Let N, K are two submodules of an R-module M. If $N \subseteq K$ and N is I-semiprime submodule of M, then N is I-semiprime submodule of K.

2. If $I_1 \subseteq I_2$. Then if N is $I_1$-semiprime implies N is $I_2$-semi prime.

3. If N is semi prime then N is I-semiprime.

**Proof:** 1, 2 and 3 are trivial.

The following theorem gives a useful characterization for I-semiprime submodules.

**Theorem (1.3):** Let N a proper submodule of an R-module M. Then N is I-semiprime submodule of M in M if and only if for any ideal A of R and submodule K of M such that $A^2K \subseteq N - IN$, we have AK $\subseteq N$.

**Proof:** Suppose that N is I-semi prime submodule of M, and $A^2K \subseteq N - IN$ for A is an ideal of R and submodule K of M. If $AK \subseteq N$, so there exist $x \in K$ and $a \in A$ such that $ax \notin N$. Now, $a^2x \in A^2K \subseteq N - IN$. We claim that $a^2x \in IN$, because if $a^2x \notin IN$, we get $ax \notin N$ which is a contradiction. Thus $a^2x \in IN$. Since $a^2K \subseteq N - IN$, there exists $m \in K$ such that $a^2m \in a^2K \subseteq N - IN$. This implies $am \in N$. On the other hand $a^2x + a^2m = a^2(x + m) \in N - IN$. This implies $a(x + m) \in N$; that is $ax + am \in N$. But $am \in N$, so $ax \in N$ which is a contradiction. Therefore $AK \subseteq N$.

Conversely suppose that $a^2m \in N - IN$ for $a \in R$ and $m \in M$. Then $(a^2)(m) \subseteq N - IN$. So by assumption, $(a)(m) \subseteq N$. Therefore $am \in N$. Thus N is an I-semiprime submodule of M.

**Corollary (1.4):** Let N a proper submodule of an R-module M. Then N is I-semiprime submodule in M if and only if for any ideal A of R such that $A^2M \subseteq N - IN$, we have AM $\subseteq N$.

**Remark (1.5):** If I-semiprime submodule of an R-module M, then it is not necessarily that $[N: M] I$-semiprime ideal, for example: Suppose that $N = (0)$ of the Z-module $Z_{4}$, then N is I-semiprime. But $[N: M] = (0); Z_{4} = 4Z$ is not an I-semiprime ideal of Z where $I = [N: M]$, since $2^2 \in [N: M] - I[N: M]$, but $2 \notin (0); Z_{4} = 4Z$.

Now, we give characterizations of I-semiprime submodule. But first, we need the following definitions.

[Recall that an R-module M is called a multiplication module if every submodule N of M has the form IM for some ideal I of R, [5]. And an R-module M is called faithful if it has zero annihilator, [6].]

**Theorem (1.6):** Let N a proper submodule of a finitely generated faithful multiplication R-module M with $I[N: M] = [IN: M]$. If N is I-semiprime submodule in M if and only if $I[IN: M]$ is an I-semi prime ideal of R.

**Proof:** Suppose that N is I-semi prime submodule in M. Let $a \in R$ with $a^2 \in [N: M] - I[N: M]$. Then $a^2M \subseteq N$. If $a^2M \subseteq IN$. Then $a^2 \in [IN: M] = I[N: M]$ which is contradiction. Assume $a^2M \subseteq N - IN$. Then $a^2M \subseteq N - IN$. But N is I-semi prime submodule. So $am \subseteq N$, thus $a \in [N: M]$. Hence $[N: M]$ is an I-semi prime ideal of R.

**Remark (1.7):** Let N a proper submodule of a finitely generated faithful multiplication R-module M with $I[N: M] = [IN: M]$. If N is I-semi prime submodule in M if and only if $I[IN: M]$ is an I-semi prime ideal of R.
primary submodule if \( rx \in N - IN \) for all \( r \in R, x \in M \) implies that either \( r \in \sqrt{[N:M]} \) or \( x \in N \), [8]. And recall that an ideal \( I \) is called radical if \( I = \sqrt{I} \), [9].

By using these concepts we can give the following proposition.

**Proposition (1.7):** Let \( N \) a proper submodule of an \( R \)-module \( M \). If \( N \) is \( I \)-prime then \( N \) is \( I \)-semiprime.

**Proof:** Let \( N \) is \( I \)-prime submodule of an \( R \)-module \( M \). Assume that \( a^2m \in N - IN \), where \( a \in R, m \in M \). Since \( a^2m = a(am) \in N - IN \) and \( N \) is \( I \)-prime submodule of \( M \), then either \( am \in N \) or \( a \in [N:M] \). In any case, we have \( am \in N \). Therefore \( N \) is \( I \)-semi prime submodule of \( M \).

**Proposition(1.8):** Let \( N \) a proper submodule of an \( R \)-module \( M \) such that \([N:M]\) is radical ideal. If \( N \) is \( I \)-primary submodule in \( M \), then \( N \) is an \( I \)-prime (and hence \( I \)-semi prime) submodule of \( M \).

**Proof:** Let \( N \) is \( I \)-primary submodule and \([N:M]\) is radical ideal. Assume that \( a^2m \in N - IN \), where \( a \in R, m \in M, m \notin N \). Since \( N \) is \( I \)-primary submodule of \( M \) and \( m \notin N \), then \( a \in \sqrt{[N:M]} \).

But \([N:M]\) is radical, so \( a \in [N:M] \). Therefore \( N \) is \( I \)-prime (and hence \( I \)-semi prime) submodule of \( M \).

From proposition (1.8) we get the following:

**Corollary (1.9):** Let \( N \) a proper submodule of an \( R \)-module \( M \) such that \([N:M]\) is semi prime ideal of \( R \). If \( N \) is \( I \)-primary submodule in \( M \), then \( N \) is an \( I \)-prime (and hence \( I \)-semi prime) submodule of \( M \).

**Proposition (1.10):** Let \( M \) be an \( R \)-module. Let \( N \) be an \( I \)-semiprime submodule of \( M \). If \((r + [N:M])^2m \notin IN \) for all \( r \in R - [N:M] \), then \( N \) is a semiprime submodule of \( M \).

**Proof:** Suppose that \((r + [N:M])^2m \notin IN \), we show that \( N \) is a semiprime. Let \( a \in R \) and \( m \in M \) such that \( a^2m \in N \). If \( a^2m \notin IN \), then \( N \) is a semiprime gives \( am \in N \). So assume that \( a^2m \in IN \). First suppose that \( a^2 \notin IN \), say \( a^2m \notin IN \) where \( n \in N \). Then \( a^2(m+n) \in N - IN \), so \( a(m+n) \in N \). Hence \( am \in N \). So we can assume that \( a^2 \subseteq IN \). Next, suppose that \( (a+b)^2m \notin IN \) for some \( b \in [N:M] \). Therefore \( (a+b)^2m \notin IN \) and \( (a+b)m \in N \). Hence \( am \in N \). So we can assume that \((a + [N:M])^2m \subseteq IN \). Since \((a + [N:M])^2m \subseteq IN \) there exists \( r \in [N:M] \) and \( x \in N \) such that \((a + r)^2x \notin IN \). Then \((a + r)^2(m + x) \in N - IN \). So \((a + r)(m + x) \in N \). Hence \( am \in N \). So \( N \) is a semiprime submodule of \( M \).

**Proposition (1.11):** Let \( M \) be an \( R \)-module. Let \( N \) be an \( I \)-semiprime submodule of \( M \). If \((r^2)^2N \notin IN \) for some \( r \in [N:M] \), then \( N \) is a semiprime submodule of \( M \).

**Proof:** Let \( a \in R \) and \( m \in M \) such that \( a^2m \in N \). Suppose \( a^2N \subseteq IN \). If \( a^2m \notin IN \), then \( a^2m \in N - IN \), and \( N \) is an \( I \)-semiprime gives \( am \in N \). Suppose that \( r^2m \notin IN \). Therefore \((a+r)^2m = (a^2 + r^2)m \notin IN \) and \( (a+r)m \in N \). So \( am \in N \). Now, we can assume that \( r^2m \in IN \). But \((r^2)^2N \subseteq N \), so there exists \( x \in N \) such that \( r^2x \notin IN \). Then \((a+r)^2(m + x) = (a^2 + r^2)(m + x) \in N - IN \) and \( (a+r)(m+x) \in N \). So \( am \in N \). Then \( N \) is a semiprime submodule of \( M \).

Recall that a proper submodule \( N \) of \( M \) is called an irreducible submodule if for each \( K \), \( K \) be two submodules of \( M \) such that \( L \cap K = N \), then either \( L = N \) or \( K = N \).

**Theorem (1.12):** Let \( N \) be an irreducible submodule of an \( R \)-module \( M \). Then \( N \) is an \( I \)-prime if and only if \( I \)-semiprime submodule of \( M \).

**Proof:** Suppose that \( N \) is \( I \)-prime submodule irreducible submodule in \( M \). Assume that \( N \) is not \( I \)-prime, so there exists \( a \in R; a \notin [N:M]; m \notin N \) such that \( am \in N - IN \). Since \( a \notin [N:M] \), so there exists \( x \in M \) such that \( ax \notin N \). Claim that \( L \cap K = N \) where \( K = N + (ax) \), \( L = N + (m) \). Now, let \( b \in L \cap K \), so \( b \in N + (ax) \), and \( b \in N + (m) \), then there exists \( n, w \in N \) and \( r, s \in R \) such that \( b = w + sax \in N + r \), then \( sax-n+w \in N \) and \( s \in N - IN \). Therefore \( sa^2x \in N - IN \). But \( N \) is \( I \)-prime, then \( sax \notin N \) and so \( b \notin N \). Thus, \( L \cap K \subseteq N \) and it is clear that \( N \subseteq L \cap K \). Therefore the claim \( L \cap K = N \) is true. But \( N \) is an irreducible submodule of \( M \).

**Theorem (1.13):** Let \( N \) a proper submodule of a faithful multiplication \( R \)-module \( M \) and \( A \) be a finitely generated faithful multiplication ideal of \( R \). Then \( N \) is \( I \)-semiprime submodule in \( AM \) if and only if \( [N:A] \) is an \( I \)-semiprime in \( M \).

**Proof:** Suppose that \( N \) is \( I \)-semiprime submodule in \( AM \). Let \( a \in R \) and \( m \in M \) such that \( \alpha^2m \in [N:A] - [N:A] \). Then \( \alpha^2Am \subseteq N - IN \). If \( \alpha^2Am \notin IN \), so by [8, lemma 2.15]
Proposition (1.16): Let $M$ be a $R$-module and let $N$ be a proper submodule of $M$. Then $N$ is $I$-semiprime in $M$ if and only if $N/I\cap N$ is $I$-semiprime submodule of $M/I\cap N$.

Proof: Suppose that $N$ is $I$-semiprime in $M$. Let $a \in R$, $x \in M$ such that $0 \neq a^2x + IN = a^2(x + IN) \in N/I\cap N$. Since $N$ is $I$-semiprime submodule of $M$, so $a \in N$. Therefore $a(x + IN) \in N/I\cap N$. Hence $N/I\cap N$ is $0$-semiprime submodule of $M/I\cap N$.

Conversely suppose that $(N/I)\cap N$ is $0$-nearly prime in $M/I\cap N$. Let $a \in R$, $x \in M$ such that $a^2x \in N - I\cap N$. So $0 \neq a^2(x + IN) \in N/I\cap N$. But $N/I\cap N$ is $0$-semiprime in $M/I\cap N$. Thus $a(x + IN) \in N/I\cap N$. Hence, $ax \in N$. Therefore $N$ is $I$-semiprime submodule of $M$.
Proof. Let \( a^2m \in N_1 - IN_1 \) where \( a \in R, m \in M_1 \). Then \( a^2(m_1, 0) \in (N_1 \oplus N_2) - I(N_1 \oplus N_2) \). Since \( (N_1 \oplus N_2) \) is an \( I \)-semi prime, then \( a(m_1, 0) \in N_1 \oplus N_2 \) and so \( am_1 \in N_1 \). Hence \( N_1 \) is \( I \)-semi prime in \( M_1 \).

similarly \( N_2 \) is an \( I \)-semi prime in \( M_2 \).

In what follows give some of characterizations for \( I \)-semi prime submodules.

Theorem (1.18): Let \( N \) be a proper submodule of \( M \), then the following are equivalent:

1. \( N \) is an \( I \)-semi prime submodule of \( M \).
2. For \( r \in R, [N:M (r^2)] = [IN:M (r^2)] \cup [N:M (r)] \).
3. For \( r \in R, [N:M (r^2)] = [IN:M (r^2)] \) or \([N:M (r^2)] = [N:M (r)] \).

Proof: (1) \( \rightarrow \) (2): Suppose that \( N \) is an \( I \)-semi prime submodule of \( M \). Let \( r \in R, m \in [N:M (r^2)] \). So \( r^2m \in N \). If \( r^2m \notin IN \), then \( rm \in N \), because \( N \) is an \( I \)-semi prime submodule of \( M \). If \( r^2m \in IN \), then \( m \in [IN:M (r^2)] \). Hence \( [N:M (r^2)] \subseteq [IN:M (r)] \cup [N:M (r)] \). Since \( IN \subseteq N \), so \( [IN:M (r^2)] \cup [N:M (r)] \subseteq [N:M (r^2)] \). Therefore \( [N:M (r^2)] = [IN:M (r^2)] \cup [N:M (r)] \).

(2) \( \rightarrow \) (3): It is clear because \([N:r] \) is a submodule of \( M \).

(3) \( \rightarrow \) (1): Let \( r \in R \) and \( m \in M \) such that \( r^2m \in N - IN \). Then \( m \in [IN:M (r^2)] \) and \( m \notin [IN:M (r^2)] \). Then by assumption, \( m \in [N:M (r)] \). Therefore \( m \in [N:M (r)] \). Thus \( N \) is an \( I \)-semi prime submodule of \( M \).

Proposition (1.19): Let \( N \) be a proper submodule of \( M \). If \( N \) is an \( I \)-semi prime submodule of \( M \), then for all \( m \in M - N \),

\[ \sqrt{[N:m]} = \sqrt{[IN:m]} \cup \sqrt{[N:m]} \]

Proof: Suppose that \( N \) is an \( I \)-semi prime submodule of \( M \). Let \( m \in M - N \) and \( r \in \sqrt{[N:m]} \). Hence \( [N:M (r^2)] \subseteq [IN:M (r)] \cup [N:M (r)] \). Since \( IN \subseteq N \), so \( [N:M (r^2)] \cup [N:M (r)] \subseteq [N:M (r^2)] \). Therefore \( \sqrt{[N:m]} = \sqrt{[IN:m]} \cup [N:m] \).

Theorem (1.20):

Let \( R = R_1 \times R_2 \) and \( M = M_1 \times M_2 \) with \( (r_1, r_2)(m_1, m_2) = (r_1m_1, r_2m_2) \) be an \( R \)-module, where \( r_i \in R_i, m_i \in M_i \). Then we have:

1. If \( N_1 \) is an \( I_1 \)-semi prime submodule of \( M_1 \) such that \( IN_1 \times M_2 \subseteq \cap \{N_1 \times M_2 | \} \), then \( N_1 \times M_2 \) is an \( I \)-semi prime submodule of \( M \).

2. If \( N_2 \) is an \( I_2 \)-semi prime submodule of \( M_2 \) such that \( IN_2 \times M_1 \subseteq \cap \{N_2 \times M_1 | \} \), then \( N_1 \times M_2 \) is an \( I \)-semi prime submodule of \( M \).

Proof: Because the prove of (1) and (2) are similar, so we only prove (1). Hence suppose that \( N_1 \) is an \( I_1 \)-semi prime submodule of \( M_1 \) and \( (a, b) \in R \times R \) and \( (m_1, m_2) \in M \) with \( (a, b)^2(m_1, m_2) = (a^2m_1, b^2m_2) \in N_1 \times M_2 - I(N_1 \times M_2) \). and \( N_1 \times M_2 - I(N_1 \times M_2) \subseteq N_1 \times M_2 - IN_1 \times M_2 = (N_1 - IN_1) \times M_2 \). We have \( a^2m_1 \in N_1 - IN_1 \) but \( N_1 \) is \( I \)-semi prime submodule of \( M_1 \). Then \( am_1 \in N_1 \). This give \( (a, b)(m_1, m_2) \in N_1 \times M_2 \). Hence \( N_1 \times M_2 \) is an \( I \)-semi prime submodule of \( M_1 \times M_2 \).

Proposition (1.21): Let \( R = R_1 \times R_2, M_1 \) be an \( R \)-module \( (i = 1, 2) \) with \( M = M_1 \times M_2 \). Let \( I_i \) and \( I_2 \) be ideals of \( R_i \) and \( R_2 \) respectively with \( I = I_1 \times I_2 \). Then all the following types are \( I \)-semi prime submodule of \( M_1 \times M_2 \):

1. \( N_1 \times M_2 \) where \( N_1 \) is an \( I_1 \)-semi prime submodule of \( M_1 \) and \( I_2 \times M_2 = M_2 \).

2. \( N_1 \times N_2 \) where \( N_2 \) is an \( I_2 \)-semi prime submodule of \( M_2 \) and \( I_1 \times M_1 = M_1 \).

Proof: 1. Suppose that \( N_1 \) is an \( I_1 \)-semi prime submodule of \( M_1 \) and \( I_2 \times M_2 = M_2 \). Let \( (a, b) \in R \) and \( (m_1, m_2) \in M \) such that \( (a^2, b^2)(m_1, m_2) = (a^2m_1, b^2m_2) \in N_1 \times M_2 - I(N_1 \times M_2) = N_1 \times M_2 - (I_1 \times M_2)(N_1 \times M_2) = (N_1 \times M_2 - I(N_1 \times M_2)(I_1 \times M_2) = (N_1 - IN_1) \times M_2 \). Then \( a^2m_1 \in N_1 - IN_1 \) and \( N_1 \) is \( I \)-semi prime submodule of \( M_1 \), so \( am_1 \in N_1 \). Therefore \( (a, b)(m_1, m_2) \in N_1 \times M_2 \). So \( N_1 \times M_2 \) is an \( I \)-semi prime submodule of \( M_1 \times M_2 \).

2. The proof is similar to part (1).
Remark (1.22): Let \( R = R_1 \times R_2 \). Let \( M_i \) be an \( R_i \)-module \((i=1,2)\) with \( M = M_1 \times M_2 \). Let \( I_1 \) and \( I_2 \) be ideals of \( R_1 \) and \( R_2 \) respectively with \( I = I_1 \times I_2 \). Then all the following types are \( I \)-semiprime submodule of \( M_1 \times M_2 \).

1- \( N_1 \times N_2 \) where \( N_i \) is a proper submodule of \( M_i \) with \( I_i N_i = N_i \) for \( i = 1, 2 \).

2- \( N_1 \times M_2 \) where \( N_1 \) is a prime submodule of \( M_1 \).

3- \( M_1 \times N_2 \) where \( N_2 \) is a prime submodule of \( M_2 \).

Proof. 1. Since \( I_1 N_2 = N_1 \) and \( I_2 N_2 = N_2 \). Then \( I_1 N_1 \times I_2 N_2 = (I_1 \times I_2) (N_1 \times N_2) = I (N_1 \times N_2) = N_1 \times N_2 \). So \( N_1 \times N_2 \) is a prime submodule of \( M_1 \times M_2 \). Thus there is nothing to prove.

2. Let \( N_1 \) be a prime submodule of \( M_1 \). Then \( N_1 \times M_2 \) is a prime submodule of \( M_1 \times M_2 \) \cite{11} and hence \( I \)-prime (\( I \)-semiprime) submodule of \( M_1 \times M_2 \) by (1.6).

3. The proof is similar to the part (2).

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