Explicit minimal embedded resolutions of divisors on models of the projective line

Andrew Obus* and Padmavathi Srinivasan

Abstract

Let $K$ be a discretely valued field with ring of integers $O_K$ with perfect residue field. Let $K(x)$ be the rational function field in one variable. Let $\mathbb{P}^1_{O_K}$ be the standard smooth model of $\mathbb{P}^1_K$ with coordinate $x$. Let $f(x) \in O_K[x]$ be a squarefree polynomial with corresponding divisor of zeroes $\text{div}_0(f)$ on $\mathbb{P}^1_{O_K}$. We give an explicit description of the minimal embedded resolution $Y$ of the pair $(\mathbb{P}^1_{O_K}, \text{div}_0(f))$ by using Mac Lane’s theory to write down the discrete valuations on $K(x)$ corresponding to the irreducible components of the special fiber of $Y$.

Keywords: Mac Lane valuation, Embedded resolution, Regular model

Mathematics Subject Classification: Primary: 14B05, 14J17; Secondary: 13F30, 14H25

1 Introduction

Let $K$ be a discretely valued field with ring of integers $O_K$ with perfect residue field. Let $K(x)$ be the rational function field in one variable. Let $\mathbb{P}^1_{O_K}$ be the standard smooth model of $\mathbb{P}^1_K$ with coordinate $x$. Let $f(x) \in O_K[x]$ be a squarefree polynomial with corresponding divisor of zeroes $\text{div}_0(f)$ on $\mathbb{P}^1_{O_K}$. A minimal embedded resolution of the pair $(\mathbb{P}^1_{O_K}, \text{div}_0(f))$ is a regular model $Y$ of $\mathbb{P}^1_{O_K}$ with a birational morphism $\pi: Y \to \mathbb{P}^1_{O_K}$ such that the strict transform of $\text{div}_0(f)$ is regular, and such that any other modification $\pi': Y' \to \mathbb{P}^1_{O_K}$ with $Y'$ regular and the strict transform of $\text{div}_0(f)$ regular factors uniquely as $Y' \to Y \xrightarrow{\pi} \mathbb{P}^1_{O_K}$. The main result of this paper is the following theorem (See Theorem 4.3 for a more precise statement, with notation as defined in Notation 3.9. Also see the last paragraph of Sect. 1.1.)

**Theorem 1.1** Let $f \in O_K[x]$ be a squarefree polynomial. There is an explicit description of the minimal embedded resolution $Y$ of the pair $(\mathbb{P}^1_{O_K}, \text{div}_0(f))$ when $\deg(f) \geq 2$. More specifically, we write down the discrete valuations on $K(x)$ corresponding to the irreducible components of the special fiber of $Y$.

1Note that such a resolution exists only when $f$ is squarefree.

2When $\deg(f) = 1$, the divisor $\text{div}_0(f)$ is already regular on the standard model $\mathbb{P}^1_{O_K}$. © The Author(s), under exclusive licence to Springer Nature Switzerland AG 2022.
Remark 1.2 This minimal embedded resolution is a key technical input to [8], where it is used to help prove a conductor-discriminant inequality for hyperelliptic curves in residue characteristic \( \neq 2 \), as we now describe.

It is well-known that an algorithm for strong embedded resolution of singularities in dimension \( n - 1 \) gives rise to an algorithm for resolution of singularities in dimension \( n \). The motivation for the current paper is to explicitly understand regular models of cyclic covers of \( \mathbb{P}^1_K \) branched at \( \text{div}_0(f) \) by explicitly constructing embedded resolutions of pairs \((\mathbb{P}^1_{O_K}, \text{div}_0(f))\) first. The eventual goal of these constructions is to give an upper bound on the number of components in the exceptional fiber of such a resolution; see [8] for an application to proving conductor-discriminant inequalities for degree 2 covers of \( \mathbb{P}^1_K \), and forthcoming work of the authors for higher degree cyclic covers. We do so by capitalizing on the recent revival in [9, 10] of explicit descriptions of normal and regular models of \( \mathbb{P}^1_K \), using descriptions of valuations of \( K(x) \) (now called “Mac Lane valuations”) going back to Mac Lane [6].

In [10, Proposition 3.4], Rüth shows that normal models of \( \mathbb{P}^1_K \) are in bijection with non-empty finite collections of discrete valuations on \( K(x) \) (extending the given valuation on \( K \)) whose residue fields have transcendence degree 1 over the residue field of \( K \). Over algebraically closed fields, it is known that analogous valuations with value group \( \mathbb{Q} \) on the rational function field can be constructed from supremum norms on non-archimedean disks. Over non-algebraically closed discretely valued fields, Rüth ([10, Proposition 4.56], restated in Proposition 2.3) shows that there is a similar description of valuations in terms of “diskoids”, which are Galois stable collections of non-archimedean disks defined over the algebraic closure. In fact, he shows that these diskoids can be explicitly described by giving a certain sequence of polynomials \( \phi_i \) in \( K[x] \) of increasing degree (whose roots correspond to the “centers” of a nested sequence of diskoids) and a corresponding sequence of rational numbers \( \lambda_i \) (“radii” of the diskoids) – such a description goes back to Mac Lane [6] from 1936. These \( \phi_i \) can be thought of as successive lower degree approximations to the roots of a polynomial \( f \in O_K[x] \), and each rational number \( \lambda_i \) is simply \( \nu_K(\phi_i(\alpha)) \) for any root \( \alpha \) of \( f \) (Corollary 2.7). Using successive \( \phi_i\)-adic expansions, one can easily compute the valuation of any given polynomial from this description, by a procedure analogous to the computation of the Gauss valuation; see the discussion surrounding (2.1). Mac Lane valuations have been implemented in Sage in [11]. In [9, Theorem 7.8] (restated here in Proposition 3.10), the authors describe the minimal regular resolution of a model of \( \mathbb{P}^1_K \) with irreducible special fiber corresponding to a valuation \( \nu \), using the same polynomials \( \phi_i \) that show up in the description of \( \nu \), and natural Farey paths between successive \( \lambda_i \).

The bulk of the paper is devoted to proving Theorem 4.3, which is Theorem 1.1 in the case where \( f \) is monic and irreducible and the residue field \( k \) is algebraically closed. The general result can easily be derived from Theorem 4.3; see Remarks 5.11 and 5.12. So for the rest of the introduction, assume \( f \) is monic and irreducible and \( k \) is algebraically closed. To each such \( f \in O_K[x] \), there is a canonical diskoid centered about the roots of \( f \) giving rise to a valuation \( \nu_f \) on \( K(x) \) (Sect. 4.1). By Rüth’s correspondence, this valuation \( \nu_f \) corresponds to a normal model of \( \mathbb{P}^1_K \) with irreducible special fiber, which we will call the \( \nu_f\)-model. By \( \nu_f\)-component, we mean the strict transform of the special fiber of the \( \nu_f\)-model in any model that dominates it. In what follows, we use \( \text{div}_0(f) \) to mean the zero divisor of \( f \) on any model of \( \mathbb{P}^1_K \); the model will be clear from context.
Concurrent to our work in [7] (an earlier version of [8]), in [3, Theorem 3.16], the authors also noted that \( \text{div}_0(f) \) is a normal crossings divisor on the minimal regular resolution \( Y^\text{reg}_v \) of the \( v_f \)-model \( Y_v \), which implies that the minimal regular model \( Y^\text{reg}_{v_f,0} \) dominating \( Y^\text{reg}_v \) and \( \mathbb{P}^1_{\mathcal{O}_K} \) is an embedded resolution of \( (\mathbb{P}^1_{\mathcal{O}_K}, \text{div}_0(f)) \). However, \( Y^\text{reg}_{v_f,0} \) is never the minimal embedded resolution of the pair \( (\mathbb{P}^1_{\mathcal{O}_K}, \text{div}_0(f)) \). In fact, for the applications to regular models of hyperelliptic curves, we are sometimes forced to work with (regular) contractions of \( Y^\text{reg}_{v_f,0} \) where the strict transform of \( \text{div}_0(f) \) is also regular.

Determining whether the horizontal part of \( \text{div}_0(f) \) remains regular on these contractions can be challenging because it might specialize to a node.

The two main insights of this paper are the following. First, using the machinery of Mac Lane valuations, it is possible to explicitly modify \( f \) to write down a rational function \( g \) that cuts out the unique irreducible horizontal divisor in \( \text{div}_0(f) \) on natural contractions of \( Y^\text{reg}_{v_f,0} \). Note that checking regularity of \( \text{div}_0(g) \) at its unique closed point \( y \) is equivalent to checking whether \( g \) is in the square of the maximal ideal at \( y \). This is hard to check directly since this local ring is 2-dimensional. The second main insight is to use the \( \psi_n \)-adic expansion of \( f \) to write down an analogous explicit decomposition \( g = \sum_i g_i \). The terms \( g_i \) in this decomposition vanish along vertical components through the closed point \( y \) (a computation back in a 1-dimensional local ring), even though \( g \) itself does not, and we can exploit the orders of vanishing to determine when \( g \) is in the square of the maximal ideal. It turns out that the Mac Lane descriptions of vertical components are tailor-made for computing orders of vanishing of functions along these components!

Our main theorem shows that quite often it is possible to contract entire tails in the dual graph of \( Y^\text{reg}_{v_f,0} \) and in fact, the minimal embedded resolution we are after is the minimal regular resolution of one of two neighbouring components of the \( v_f \)-component in the dual graph of \( Y^\text{reg}_{v_f,0} \). We do not see any way to deduce our main theorem directly from [3, Theorem 3.16].

1.1 Outline of the paper

In Sect. 2, we introduce Mac Lane valuations. As we have mentioned, a normal model of \( \mathbb{P}^1_K \) corresponds to a finite set of Mac Lane valuations, one valuation for each irreducible component of the special fiber. Mac Lane valuations are also in one-to-one correspondence with diskoids, which are Galois orbits of rigid-analytic disks in \( \mathbb{P}^1_K \). We will use the diskoid perspective often, and it is introduced in in Sect. 2.2.

In Sect. 3, we prove several results about the correspondence between Mac Lane valuations and normal models of \( \mathbb{P}^1_K \). For instance, if \( \mathcal{Y} \) is a normal model of \( \mathbb{P}^1_K \) with special fiber consisting of several irreducible components, each corresponding to a Mac Lane valuation, results in Sect. 3 can be used to determine which irreducible component a point of \( \mathbb{P}^1_K \) specializes to. After this, we cite a result (Proposition 3.10) from [9] giving an explicit criterion for when a normal model of \( \mathbb{P}^1_K \) is regular. More specifically, using that Mac Lane valuations correspond to normal models of \( \mathbb{P}^1_K \) with irreducible special fiber, Proposition 3.10 takes a Mac Lane valuation as input and gives the minimal regular resolution of the corresponding normal model as output (as a finite set of Mac Lane valuations, of course)!

---

[1] Here \( \psi_n \) is the last polynomial that shows up in the Mac Lane description of \( v_f \).
In Sect. 4, we first define the canonical valuation \( v_f \) associated to a polynomial \( f \). The minimal embedded resolution of the pair \((\mathbb{P}^1_{\mathcal{O}_K}, \text{div}_0(f))\) is a certain contraction of \( Y_{v_f,0}^{\text{reg}} \). So we are led to an analysis of regularity of the strict transform of \( \text{div}_0(f) \), which we will henceforth call \( D \). We first define three types of regular models of \( \mathbb{P}^1_K \) that can arise as contractions of \( Y_{v_f,0}^{\text{reg}} \). Viewing these contractions as a sequence of closed point blow-downs, a short argument shows that if we want the blow-down to stay regular and dominate \( \mathbb{P}^1_{\mathcal{O}_K} \), there is a unique component that can be blown down at every stage (for instance, the \( v_f \)-component is the only \(-1\)-component that can be blown down in the model \( Y_{v_f,0}^{\text{reg}} \) by the minimality of the construction of \( Y_{v_f,0}^{\text{reg}} \)). As we proceed through this natural sequence of blow-downs, we first go through a sequence of models we call “Type I” models. If \( D \) stays regular on all Type I regular blow-downs of \( Y_{v_f,0}^{\text{reg}} \), we then move on to the “Type II” models. We continue contracting in this way, and after the Type II models, naturally comes the unique “Type III” model. (See Definition 4.8.)

In Sect. 5, we run thus argument. The crux is to show that \( D \) is not regular on the unique Type III model (Proposition 5.7), and we use this to show that the minimal embedded resolution of \( D \) must be a special Type I or a Type II model (Corollary 5.8). We then show that if \( D \) is regular on a Type I or Type II model, then the model must include a component corresponding to one of two additional canonical valuations attached to the polynomial \( f \), denoted \( v'_f, v''_f \) (Proposition 5.9) – these turn out to be neighbouring valuations to \( v_f \) in the dual graph of \( Y_{v_f,0}^{\text{reg}} \). The technical lemmas needed for these regularity arguments use an analysis of valuations of individual terms in the \( \varphi_n \)-adic expansion of \( f \) along vertical components of these models (Lemma 5.6 for the unique Type III model, and Lemma 5.5 for Type I and Type II models). The Mac Lane machinery for describing these vertical components is perfectly equipped for carrying out such calculations. Finally, in Theorem 4.3, we show that the minimal embedded resolution of the pair \((\mathbb{P}^1_{\mathcal{O}_K}, \text{div}_0(f))\) is the minimal regular model dominating \( \mathbb{P}^1_{\mathcal{O}_K} \) and either the \( v'_f \)-model or the \( v''_f \)-model.

**Notation and conventions**

Throughout, \( K \) is a Henselian field with respect to a discrete valuation \( v_K \). In much of the paper (Sect. 2.3, Sect. 3, Sect. 4, and all of Sect. 5 until the very end) we will further assume that the residue field \( k \) of \( K \) is algebraically closed, but this will be noted specifically and is not a running assumption for the paper. We denote an algebraic closure of \( K \) by \( \overline{K} \). We fix a uniformizer \( \pi_K \) of \( v_K \) and normalize \( v_K \) so that \( v_K(\pi_K) = 1 \). Note that the valuation \( v_K \) uniquely extends to a valuation on \( \overline{K} \), which we also call \( v_K \).

For an integral \( K \)-scheme or \( \mathcal{O}_K \)-scheme \( S \), we denote the corresponding function field by \( K(S) \). If \( \mathcal{Y} \to \mathcal{O}_K \) is an arithmetic surface, an irreducible codimension 1 subscheme of \( \mathcal{Y} \) is called vertical if it lies in a fiber of \( \mathcal{Y} \to \mathcal{O}_K \), and horizontal otherwise. Let \( f \in K(\mathcal{Y}) \). We denote the divisor of zeroes of \( f \) by \( \text{div}_0(f) \). For any discrete valuation \( v \), we denote the corresponding value group by \( \Gamma_v \). If \( P \) is a closed point on \( \mathcal{Y} \), we denote the corresponding local ring by \( \mathcal{O}_{\mathcal{Y},P} \) and maximal ideal by \( m_{\mathcal{Y},P} \).

Throughout this paper, we fix a system of homogeneous coordinates \( \mathbb{P}^1_K = \text{Proj} K[x_0, x_1] \), and \( x := x_1/x_0 \) and \( \mathbb{P}^1_{\mathcal{O}_K} := \text{Proj} \mathcal{O}_K[x_0, x_1] \).
All minimal polynomials are assumed to be monic. When we refer to the *denominator* of a rational number, we mean the positive denominator when the rational number is expressed as a reduced fraction.

**2 Mac Lane valuations**

**2.1 Definitions and facts**

We recall the theory of inductive valuations, which was first developed by Mac Lane in [6]. We also use the more recent [10] as a reference. Inductive valuations give us an explicit way to talk about normal models of $\mathbb{P}^1$.

Define a *geometric valuation* of $K(x)$ to be a discrete valuation that restricts to $v_k$ on $K$ and whose residue field is a finitely generated extension of $k$ with transcendence degree 1. We place a partial order $\preceq$ on valuations by defining $v \preceq w$ if $v(f) \leq w(f)$ for all $f \in K[x]$. Let $v_0$ be the *Gauss valuation* on $K(x)$. This is defined on $K[x]$ by $v_0(a_0 + a_1x + \cdots + a_nx^n) = \min_{0 \leq i \leq n} v_k(a_i)$, and then extended to $K(x)$.

We consider geometric valuations $v$ such that $v \preceq v_0$. By the triangle inequality, these are precisely those geometric valuations for which $v(x) \geq 0$. This entails no loss of generality, since $x$ can always be replaced by $x^{-1}$. We would like an explicit formula for describing geometric valuations, similar to the formula above for the Gauss valuation, and this is achieved by the so-called inductive valuations or Mac Lane valuations. Observe that the Gauss valuation is described using the $x$-adic expansion of a polynomial. The idea of a Mac Lane valuation is to “declare” certain polynomials $\phi_i$ to have higher valuation than expected, and then to compute the valuation recursively using $\phi_i$-adic expansions.

More specifically, if $v$ is a geometric valuation such that $v \preceq v_0$, the concept of a *key polynomial* over $v$ is defined in [6, Definition 4.1] (or [10, Definition 4.7]). Key polynomials are monic polynomials in $O_k[x]$ — we do not give a definition, which would require more terminology than we need to develop, but see Lemmas 2.1 and 2.9 below for the most useful properties. If $\phi \in O_k[x]$ is a key polynomial over $v$, then for $\lambda > v(\phi)$, we define an augmented valuation $v' = [v, v'(\phi) = \lambda]$ on $K[x]$ by

$$v'(a_0 + a_1\phi + \cdots + a_r\phi^r) = \min_{0 \leq i \leq r} v(a_i) + i\lambda$$

(2.1)

whenever the $a_i \in K[x]$ are polynomials with degree less than $\deg(\phi)$. We should think of this as a “base $\phi$ expansion”, and of $v'(f)$ as being the minimum valuation of a term in the base $\phi$ expansion of $f$ when the valuation of $\phi$ is declared to be $\lambda$. By [6, Theorems 4.2, 5.1] (see also [10, Lemmas 4.11, 4.17]), $v'$ is in fact a discrete valuation. In fact, the key polynomials are more or less the polynomials $\phi$ for which the construction above yields a discrete valuation for $\lambda > v(\phi)$. The valuation $v'$ extends to $K(x)$.

We extend this notation to write Mac Lane valuations in the following form:

$$[v_0, v_1(\phi_1(x)) = \lambda_1, \ldots, v_n(\phi_n(x)) = \lambda_n].$$

Here each $\phi_i(x) \in O_k[x]$ is a key polynomial over $v_{i-1}$, we have that $\deg(\phi_{i-1}(x)) \mid \deg(\phi_i(x))$, and each $\lambda_i$ satisfies $\lambda_i > v_{i-1}(\phi_i(x))$. By abuse of notation, we refer to such a valuation as $v_n$ (if we have not given it another name), and we identify $v_i$ with $[v_0, v_1(\phi_1(x)) = \lambda_1, \ldots, v_i(\phi_i(x)) = \lambda_i]$ for each $i \leq n$. The valuation $v_i$ is called a truncation of $v_n$. One sees without much difficulty that $v_n(\phi_i) = \lambda_i$ for all $i$ between 1 and $n$.

It turns out that the set of Mac Lane valuations on $K(x)$ exactly coincides with the set of geometric valuations $v$ with $v \preceq v_0$ ([2, Corollary 7.4] and [6, Theorem 8.1], or [10, ...])
Theorem 4.31]). Furthermore, every Mac Lane valuation is equal to one where the degrees of the \( \varphi_i \) are strictly increasing ( [6, Lemma 15.1] or [10, Remark 4.16]), so we may and do assume this to be the case for the rest of the paper. This has the consequence that the number \( n \) is well-defined. We call \( n \) the inductive valuation length of \( v \). In fact, by [6, Lemma 15.3] (or [10, Lemma 4.33]), the degrees of the \( \lambda_i \) are invariants of \( v \), once we require that they be strictly increasing. If \( f \) is a key polynomial over \( v = [v_0, v_1(\varphi_1) = \lambda_1, \ldots, v_n(\varphi_n) = \lambda_n] \) and either \( \deg(f) > \deg(\varphi_n) \) or \( v = v_0 \), we call \( f \) a proper key polynomial over \( v \). By our convention, each \( \varphi_i \) is a proper key polynomial over \( v_{i-1} \).

We collect some basic results on Mac Lane valuations and key polynomials that will be used repeatedly.

**Lemma 2.1** Suppose \( f \) is a proper key polynomial over \( v = [v_0, v_1(\varphi_1) = \lambda_1, \ldots, v_n(\varphi_n) = \lambda_n] \) with \( n \geq 1 \). If \( f = \varphi_n^e + a_{e-1}\varphi_n^{e-1} + \cdots + a_0 \) is the \( \varphi_n \)-adic expansion of \( f \), then \( v_n(a_0) = v_n(\varphi_n^e) = e\lambda_n \), and \( v_n(a_i\varphi_n^j) \geq e\lambda_n \) for all \( i \in \{1, \ldots, e-1\} \). In particular, \( v_n(f) = e\lambda_n \).

**Proof** This follows from [6, Theorem 9.4] (or [10, Lemma 4.19(ii), (iii)]).

**Example 2.2** If \( K = \text{Frac}(\mathcal{O}_K[\varphi]/(\varphi^3)) \), then the polynomial \( f(x) = x^3 - 9 \) is a proper key polynomial over \( [v_0, v_1(x) = 2/3] \). In accordance with Lemma 2.1, we have \( v_1(f) = v_1(9) = v_1(x^3) = 3 \cdot 2/3 = 2 \). If we extend \( v_1 \) to a valuation \( [v_0, v_1(x) = 2/3, v_2(f(x)) = \lambda_2] \) with \( \lambda_2 > 2 \), then the valuation \( v_2 \) notices “cancellation” in \( x^3 - 9 \) that \( v_1 \) does not.

### 2.2 Mac Lane valuations and diskoids

Given \( \varphi \in \mathcal{O}_K[x] \) monic, irreducible and \( \lambda \in \mathbb{Q}_{\geq 0} \), we define the diskoid \( D(\varphi, \lambda) \) with “center” \( \varphi \) and radius \( \lambda \) to be \( D(\varphi, \lambda) := \{ \alpha \in \bar{K} \mid v_K(\varphi(\alpha)) \geq \lambda \} \) (we only treat diskoids with non-negative, finite radius in the sense of [10, Definition 4.40]). By [10, Lemma 4.43], a diskoid is a union of a disk with all of its \( \text{Gal}(\bar{K}/K) \)-conjugates. Such a diskoid is said to be defined over \( K \), since \( \varphi \in \mathcal{O}_K[x] \). Notice that the larger \( \lambda \) is, the smaller the diskoid is. We now state the fundamental correspondence between Mac Lane valuations and diskoids.

**Proposition 2.3** (cf. [10, Theorem 4.56], see also [9, Proposition 5.4]) There is a bijection from the set of diskoids to the set of Mac Lane valuations that sends a diskoid \( D \) to the valuation \( v_D \) defined by \( v_D(f) = \inf_{\alpha \in D} v_K(\varphi(\alpha)) \). The inverse sends a Mac Lane valuation \( v = [v_0, \ldots, v_n(\varphi_n) = \lambda_n] \) to the diskoid \( D_v \) defined by \( D_v = D(\varphi_n, \lambda_n) \). Alternatively,

\[
D_v = \{ \alpha \in \bar{K} \mid v_K(f(\alpha)) \geq v(f) \forall f \in K[x] \},
\]

is a presentation of \( D_v \) independent of the description of \( v \) as a Mac Lane valuation.

Lastly, if \( D \) and \( D' \) are diskoids, then \( D \subseteq D' \) if and only if \( v_D \geq v_{D'} \). If \( v \) and \( v' \) are Mac Lane valuations, then \( v \geq v' \) if and only if \( D_v \subseteq D_{v'} \).

The following proposition is crucial for our method.

**Proposition 2.4** Let \( \alpha \in \mathcal{O}_K \) and let \( f \in K[x] \) be the minimal polynomial for \( \alpha \). Then there exists a unique Mac Lane valuation \( v_f = [v_0, \ldots, v_n(\varphi_n) = \lambda_n] \) over which \( f \) is a proper key polynomial.
Proof Consider the unique valuation \( v_L \) on \( L := K[x]/(f) \) extending \( v_K \). This lifts to a discrete pseudovaluation on \( K[x] \) in the language of \([10, \S 4.6]\) (a valuation which can take the value \( \infty \) on an ideal, in this case \((f)\)). By \([10, \text{Corollary 4.67}]\), it can be written as a so-called “infinite inductive valuation” \([v_0, \ldots, v_n(\psi_n) = \lambda_n] \), with \( f \) a proper key polynomial over \( v_f := [v_0, \ldots, v_n(\psi_n) = \lambda_n] \). This shows the existence of \( v_f \).

If \( f \) is a proper key polynomial over some other valuation \( v \), then for sufficiently large \( \lambda \), one can construct inductive valuations \( v' = [v_f, v'/(f) = \lambda] \) and \( v' = [v, v'(f) = \lambda] \). By Proposition 2.3, these inductive valuations correspond to the same diskoid, and are thus the same. Applying the “only if” direction of \([10, \text{Theorem 4.33}] \) (or \([6, \text{Theorem 15.3}] \)) to \( v_f \) and \( v' \), and then the “if” direction of the same theorem to \( v_f \) and \( v \) shows that \( v_f = v \). \( \square \)

To close out Sect. 2.2, we prove several results linking Mac Lane valuations evaluated at a polynomial to the valuation of that polynomial at a particular point.

**Definition 2.5** \(([10, \text{Definition 4.4, Lemma 4.24}]\)) If \( v = [v_0, v_1(\psi_1) = \lambda_1, \ldots, v_n(\psi_n) = \lambda_n] \) is a Mac Lane valuation and \( f \in K[x] \), then a \( v \)-reciprocal of \( f \) is a polynomial \( f' \in K[x] \) such that \( v(ff' - 1) > 0 \) and \( v(f') = v_{n-1}(f') = -v(f) \).

By \([6, \text{Lemma 9.1}] \) (or \([10, \text{Lemma 4.24}] \)), any \( f \in K[x] \) with \( v(f) = v_{n-1}(f) \) has a \( v \)-reciprocal. In this case, it is clear from Definition 2.5 that \( f \) and \( f' \) being \( v \)-reciprocals is a symmetric relation.

**Proposition 2.6** Suppose \( v = [v_0, v_1(\psi_1) = \lambda_1, \ldots, v_n(\psi_n) = \lambda_n] \) is a Mac Lane valuation, \( \alpha \in D(\psi_n, \lambda_n) \), and \( g \in K[x] \) such that \( v(g) = v_{n-1}(g) \). Then \( v_K(g(\alpha)) = v(g) \).

**Proof** Let \( D := D(\psi_n, \lambda_n) \) be the diskoid corresponding to \( v \) and let \( D' := D(g, v_K(g(\alpha))) \) with corresponding valuation \( v' \). These two diskoids share the common element \( \alpha \). By \([10, \text{Lemma 4.44}] \), either \( D \subseteq D' \) or \( D' \subseteq D \), and then Proposition 2.3 shows that either \( v' \leq v \) or \( v \leq v' \).

Since \( \alpha \in D \), by Proposition 2.3 we have \( v_K(g(\alpha)) = v(g) \). Suppose \( v_K(g(\alpha)) > v(g) \). Since \( v'(g) = v_K(g(\alpha)) \) by definition, we have \( v(g) < v'(g) \). Since either \( v' \leq v \) or \( v \leq v' \), it follows that \( v \leq v' \). Let \( g' \in K[x] \) be a \( v \)-reciprocal of \( g \), i.e., \( gg' = 1 + h \) with \( v(h) > 0 \) (\( g' \) exists because \( v(g) = v_{n-1}(g) \)). Since \( v \leq v' \), we have \( 0 < v(h) \leq v'(h) \). In particular, \( v'(gg') = v(gg') = 0 \), so \( v'(g') = -v'(g) < -v(g) = v(g') \). But this contradicts \( v \leq v' \). \( \square \)

**Corollary 2.7** If \( f \) is a key polynomial over \( v = [v_0, v_1(\psi_1) = \lambda_1, \ldots, v_n(\psi_n) = \lambda_n] \) with root \( \alpha \in K \), then \( v_K(g(\alpha)) = v(g) \) for all \( g \in O_K[x] \) of degree less than \( \deg(f) \). In particular, \( v_K(g(\alpha)) = \lambda_i \) for all \( 1 \leq i \leq n \).

**Proof** Consider a Mac Lane valuation \( w_f = [w_0, w_1(\psi_1) = \lambda_1, \ldots, w_n(\psi_n) = \lambda_n, v_{n+1}(f) = \lambda_{n+1}] \), with \( \lambda_{n+1} \) large. Then \( v_{n+1}(g) = v_n(g) \) and \( \alpha \in D(f, \lambda_{n+1}) \), so the corollary follows from Proposition 2.6. \( \square \)

### 2.3 Ramification of Mac Lane valuations

For Sect. 2.3, we assume that the residue field \( k \) of \( K \) is algebraically closed.

If \( v \) and \( w \) are two Mac Lane valuations such that the value group \( \Gamma_w \) contains the value group \( \Gamma_v \), we write \( e(w/v) \) for the ramification index \([\Gamma_w : \Gamma_v] \).

Remark 2.8 Observe that if \([v_0, v_1(\varphi_1) = \lambda_1, \ldots, v_n(\varphi_n) = \lambda_n]\) is a Mac Lane valuation, where each \(\lambda_i = b_i/c_i\) in lowest terms, then the ramification index \(e(\varphi_i/\varphi_0)\) equals \(\text{lcm}(c_1, \ldots, c_n)\). Consequently, \(e(\varphi_i/\varphi_j) = \text{lcm}(c_1, \ldots, c_i)/\text{lcm}(c_1, \ldots, c_j)\) for \(i \geq j\).

Lemma 2.9 Suppose \(f\) is a proper key polynomial over \(v = [v_0, v_1(\varphi_1) = \lambda_1, \ldots, v_n(\varphi_n) = \lambda_n]\).

(i) If \(n = 0\), then \(f\) is linear. If \(n \geq 1\), then \(\varphi_1\) is linear. Every monic linear polynomial in \(O_K[x]\) is a key polynomial over \(v_0\).

(ii) If \(n \geq 1\), then \(\deg(f)/\deg(\varphi_n) = e(\nu_n/\nu_{n-1})\).

Proof Part (i) follows from [9, Remark 5.2(i)] for \(n = 0\), and then for general \(n \geq 1\) by applying the \(n = 0\) case to \(\varphi_1\) and \(v_0\). Part (ii) follows from [6, Theorem 12.1] (one can also use the second equation of [10, Lemma 4.30], where \(\text{lcm}(c_1, \ldots, c_j)\) for all \(i\). Let \(N_n = \text{lcm}(c_{n-1}, c_i)\) if \(n > 1\), and let \(N_n = 1\) if \(n = 1\). Then \(N_n = e(\nu_{n-1}/\nu_0) = \deg(\varphi_n)\), and thus \(\Gamma_{\nu_{n-1}} = (1/N_n)\mathbb{Z} = (1/\deg(\varphi_n))\mathbb{Z}\).

Proof That \(\deg(\varphi_1) = 1\) is Lemma 2.9(ii), which proves the corollary when \(n = 1\). By Remark 2.8, \(e(\nu_{j+1}/\nu_j) \text{lcm}(c_1, \ldots, c_j) = \text{lcm}(c_1, \ldots, c_{j+1})\). The rest of the corollary follows from Lemma 2.9(ii) and induction.

Remark 2.10 The assumption \(k\) algebraically closed is required above to apply [9, Remark 5.2(ii)] and to assume \(F_m = F_{m-1} = k\) in [10, Lemma 4.30].

Corollary 2.11 Let \(v = [v_0, v_1(\varphi_1) = \lambda_1, \ldots, v_n(\varphi_n) = \lambda_n]\) be a Mac Lane valuation of inductive valuation length \(n \geq 1\). Write \(\lambda_i = b_i/c_i\) in lowest terms for all \(i\). Let \(N_n = \text{lcm}(c_{n-1}, c_i)\) if \(n > 1\), and let \(N_n = 1\) if \(n = 1\). Then \(N_n = e(\nu_{n-1}/\nu_0) = \deg(\varphi_n)\), and thus \(\Gamma_{\nu_{n-1}} = (1/N_n)\mathbb{Z} = (1/\deg(\varphi_n))\mathbb{Z}\).

Proof If \(\lambda_i \in \Gamma_{\nu_{i-1}}\), then \(e(\nu_i/\nu_{i-1}) = 1\). If \(i = n\), applying Lemma 2.9(ii) to \(\nu_n\), contradicts the fact that \(\deg(f) > \deg(\varphi_n)\). For \(i < n\), applying Lemma 2.9(ii) to \(\nu_i\) contradicts the fact that \(\deg(\varphi_{i+1}) > \deg(\varphi_i)\).

Lemma 2.12 Let \([v_0, v_1(\varphi_1) = \lambda_1, \ldots, v_n(\varphi_n) = \lambda_n]\) be a valuation over which \(f\) is a proper key polynomial. Then for \(1 \leq i \leq n\), we have \(\lambda_i \notin \Gamma_{\nu_{i-1}} = (1/N_i)\mathbb{Z}\).

Proof If \(\lambda_i \in \Gamma_{\nu_{i-1}}\), then \(e(\nu_i/\nu_{i-1}) = 1\). If \(i = n\), applying Lemma 2.9(ii) to \(\nu_n\), contradicts the fact that \(\deg(f) > \deg(\varphi_n)\). For \(i < n\), applying Lemma 2.9(ii) to \(\nu_i\) contradicts the fact that \(\deg(\varphi_{i+1}) > \deg(\varphi_i)\).

3 Mac Lane valuations, normal models and regular resolutions

In Sect. 3.1, we prove results on the specialization of horizontal divisors, expressed in terms of Mac Lane valuations. In Sect. 3.2 we recall a result from [9], giving a criterion in terms of Mac Lane valuations for when a model of \(\mathbb{P}^1_k\) is regular. Lastly, in Sect. 3.3, we discuss valuations that are in a geometric sense “nearby” to a given Mac Lane valuation in a regular model of \(\mathbb{P}^1_k\). These valuations will play a crucial role throughout the rest of the paper.

A normal model of \(\mathbb{P}^1_k\) is a flat, normal, proper \(O_K\)-curve with generic fiber isomorphic to \(\mathbb{P}^1_k\). By [10, Corollary 3.18], normal models \(\mathcal{Y}\) of \(\mathbb{P}^1_k\) are in one-to-one correspondence with non-empty finite collections of geometric valuations, by sending \(\mathcal{Y}\) to the collection of geometric valuations corresponding to the local rings at the generic points of the irreducible components of the special fiber of \(\mathcal{Y}\). Via this correspondence, the multiplicity
of an irreducible component of the special fiber of a normal model \(\mathcal{Y}\) of \(\mathbb{P}^1_\mathbb{K}\) corresponding to a Mac Lane valuation \(v\) equals \(e(v/v_0)\).

We say that a normal model of \(\mathbb{P}^1_\mathbb{K}\) includes a Mac Lane valuation \(v\) if a component of the special fiber corresponds to \(v\). If \(\mathcal{Y}\) includes \(v\), we call the corresponding irreducible component of its special fiber the \(v\)-component of the special fiber of \(\mathcal{Y}\) (or simply the \(v\)-component of \(\mathcal{Y}\), even though it is not an irreducible component of \(\mathcal{Y}\)). If \(\mathcal{S}\) is a finite set of Mac Lane valuations, then the \(S\)-model of \(\mathbb{P}^1_\mathbb{K}\) is the normal model including exactly the valuations in \(\mathcal{S}\). If \(\mathcal{S} = \{v\}\), we simply say the \(v\)-model instead of the \(\{v\}\)-model. Recall that we fixed a coordinate \(x\) on \(\mathbb{P}^1_\mathbb{K}\), that is, a rational function \(x\) on \(\mathbb{P}^1_\mathbb{K}\) such that \(K(\mathbb{P}^1_\mathbb{K}) = K(x)\).

For the remainder of Sect. 3, we assume the residue field \(k\) of \(K\) is algebraically closed, but the statements above about the correspondence between normal models and collections of geometric valuations are true without this assumption.

### 3.1 Specialization of horizontal divisors

Each \(\alpha \in \mathbb{K} \cup \{\infty\}\) corresponds to a point of \(\mathbb{P}^1(\mathbb{K})\) given by \(x = \alpha\), which lies over a unique closed point of \(\mathbb{P}^1_{\mathbb{K}}\). If \(\mathcal{Y}\) is a normal model of \(\mathbb{P}^1_\mathbb{K}\), the closure of this point in \(\mathcal{Y}\) is a subscheme that we call \(D_\alpha\); note that \(D_\alpha\) is a horizontal divisor (the model will be clear from context, so we omit it to lighten the notation).

If \(v\) is a Mac Lane valuation, then the reduced special fiber of the \(v\)-model of \(\mathbb{P}^1_\mathbb{K}\) is isomorphic to \(\mathbb{P}^1_\mathbb{K}\) (see, e.g., [9, Lemma 7.1]). It will be useful to have an explicit coordinate on this special fiber (that is, a rational function \(y\) such that the function field of the special fiber is \(k(y)\)).

**Lemma 3.1** Let \(v = [v_0, v_1(\varphi_1) = \lambda_1, \ldots, v_n(\varphi_n) = \lambda_n]\) be a Mac Lane valuation, and let \(e = e(v_n/v_{n-1})\). There exists a monomial \(t\) in \(\varphi_1, \ldots, \varphi_{n-1}\) such that \(v(t\varphi^n_\ell) = 0\), and for any such \(t\), the restriction of \(t\varphi^n_\ell\) to the reduced special fiber of the \(v\)-model of \(\mathbb{P}^1_\mathbb{K}\) is a coordinate on the \(v\)-component that vanishes at the specialization of \(\varphi_n = 0\).

**Proof** Let \(\mathcal{O} \subseteq K[x]\) be the subring of elements \(f\) such that \(v(f) \geq 0\), and let \(\mathcal{O}^+ = \{g \in \mathcal{O} | v(g) > 0\}\). By [6, Theorem 12.1] (or [10, Lemma 4.29] and the discussion before that lemma), \(\mathcal{O}/\mathcal{O}^+ \cong k[y]\), where \(y\) is the image of \(\varphi^n_\ell\) in \(\mathcal{O}/\mathcal{O}^+\), for any \(t \in K[x]\) with \(v(t\varphi^n_\ell) = 0\) and \(v(t) = v_{n-1}(t)\) (in the notation of [10], the example used is \(t = (S')^t\)). Since \(v(\varphi^n_\ell) \in \Gamma_{v_{n-1}},\) we can take \(t\) to be a monomial in \(\varphi_1, \ldots, \varphi_{n-1}\). Since \(\text{Spec} \mathcal{O}\) is an affine open of the \(v\)-model with reduced special fiber \(\text{Spec} \mathcal{O}/\mathcal{O}^+ \cong \text{Spec} k[y] \cong \mathbb{A}^1_k \subseteq \mathbb{P}^1_k\), we have that \(y\) is a coordinate on the reduced special fiber of the \(v\)-model of \(\mathbb{P}^1_k\).

**Proposition 3.2** Let \(v = [v_0, v_1(\varphi_1) = \lambda_1, \ldots, v_n(\varphi_n) = \lambda_n]\) be a Mac Lane valuation and let \(\mathcal{Y}\) be the \(v\)-model of \(\mathbb{P}^1_\mathbb{K}\). As \(\alpha\) ranges over \(\mathbb{K}\), all \(D_\alpha\) with \(v_K(\varphi_n(\alpha)) > \lambda_n\) meet on the special fiber, all \(D_\alpha\) with \(v_K(\varphi_n(\alpha)) < \lambda_n\) meet at a different point on the special fiber, and no \(D_\alpha\) with \(v_K(\varphi_n(\alpha)) = \lambda_n\) meets any \(D_\beta\) with \(v_K(\varphi_n(\beta)) = \lambda_n\).

**Proof** Let \(\mathcal{Y}\) be the \(v\)-model of \(\mathbb{P}^1_\mathbb{K}\). Using the coordinate \(y = t\varphi^n_\ell\) from Lemma 3.1 on the reduced special fiber of \(\mathcal{Y}\), we will show that all \(\alpha \in \mathbb{K}\) with \(v_K(\varphi_n(\alpha)) < \lambda_n\) specialize to \(y = \infty\), all \(\alpha \in \mathbb{K}\) with \(v_K(\varphi_n(\alpha)) > \lambda_n\) specialize to \(y = 0\) and all \(\alpha \in \mathbb{K}\) with \(v_K(\varphi_n(\alpha)) = \lambda_n\) specialize to some point \(y = a\) with \(a \notin \{0, \infty\}\). We now work out the details.
Let \( O \subseteq K[x] \) be the subring of elements \( f \) such that \( v(f) \geq 0 \), and let \( O^+ \) be the ideal of elements \( g \) where \( v(g) > 0 \). Suppose \( \alpha \in D(\psi_n, \lambda_n) \). Proposition 2.3 shows that \( v_K(g(\alpha)) > 0 \) for \( g \in O^+ \), thus evaluating \( y \) at \( \alpha \) gives a well-defined element of \( k \).

Furthermore, \( y = y(\alpha) \) is precisely the point where \( D_\alpha \) meets the special fiber of \( \mathcal{Y} \). We now compute:

\[
y(\alpha) = 0 \iff v_K(t(\alpha)\psi_n(\alpha)^s) > 0 \\
\iff v_K(t(\alpha)\psi_n(\alpha)^s) > v(t(\psi_n^s)) \\
\iff v_K(\psi_n(\alpha)) > \lambda_n \quad \text{(} \therefore v_K(t(\alpha)) = v(t) \text{)}.
\]

This shows that all \( D_\alpha \) for which \( v_K(\psi_n(\alpha)) > \lambda_n \) intersect on the special fiber at the point \( y = 0 \), but none of them intersect any \( D_\beta \) for which \( v_K(\psi_n(\beta)) = \lambda_n \). All such \( D_\beta \) intersect the reduced special fiber \( \Lambda_\alpha^1 \cong \text{Spec } k[y] \) of \( \text{Spec } O \) at some point where \( y \neq 0 \).

Now let \( \alpha \notin D(\psi_n, \lambda_n) \). We will show that \( D_\alpha \cap (\text{Spec } O)_s \) is empty by contradiction. Suppose not. Let \( P \in D_\alpha \cap (\text{Spec } O)_s \) be a closed point of \( \text{Spec } O \). We have a well-defined element \( g(P) \in k \) for every \( g \in O \) coming from evaluating \( g \) at \( P \). Since \( P \) is the closed point of \( D_\alpha \cong \text{Spec } A \) with \( A \subseteq O_K(\alpha) \), it follows that \( g(\alpha) \in O_K(\alpha) \) and furthermore, \( g(P) = g(\alpha) \mod m_{O_K(\alpha)} \). We will now construct a \( g \in O \) with \( v_K(g(\alpha)) < 0 \), which is a contradiction. Let \( b \) be such that \( bv(\psi_n) \in \mathbb{Z}_{>0} \), and let \( g := \psi_n^b/n_{K(\psi_n)} \). Then \( v(g) = 0 \) so \( g \in O \), but

\[
v_K(g(\alpha)) = b(v_K(\psi_n(\alpha)) - v(\psi_n)) < 0.
\]

Thus \( D_\alpha \) does not intersect the special fiber of \( \text{Spec } O \), so \( D_\alpha \) specializes to a point of \( \mathcal{Y}_s \setminus (\text{Spec } O)_s \), which is the “point at infinity” where \( y = \infty \) on the reduced special fiber of \( \mathcal{Y} \). This finishes the proof.

\[\square\]

**Corollary 3.3** Let \( v = [v_0, v_1(\psi_1) = \lambda_1, \ldots, v_n(\psi_n) = \lambda_n] \) be a Mac Lane valuation and let \( \mathcal{Y} \) be a normal model of \( \mathbb{P}^1_k \) including \( v \). If \( \alpha, \beta \in K \) are such that \( v_K(\psi_n(\beta)) \leq \lambda_n \leq v_K(\psi_n(\alpha)) \) and \( v_K(\psi_n(\beta)) \neq v_K(\psi_n(\alpha)) \), then \( D_\alpha \) and \( D_\beta \) do not meet on the special fiber of \( \mathcal{Y} \).

**Proof** Immediate from Proposition 3.2. \[\square\]

**Corollary 3.4** Let \( v = [v_0, v_1(\psi_1) = \lambda_1, \ldots, v_n(\psi_n) = \lambda_n] \) and \( v' = [v_0, v_1(\psi_1) = \lambda_1, \ldots, v'_n(\psi_n) = \lambda'_n] \) be Mac Lane valuations with \( \lambda'_n < \lambda_n \). Let \( \mathcal{Y} \) be a model of \( \mathbb{P}^1_k \) including \( v \) and \( v' \)-components intersect, say at a point \( z \). Then \( D_\alpha \) meets \( z \) if and only if \( \lambda'_n < v_K(\psi_n(\alpha)) < \lambda_n \).

**Proof** We may assume \( \mathcal{Y} \) is the \( \{v, v'\}\)-model \( \mathbb{P}^1_k \). Let \( \mathcal{Y} \) and \( \mathcal{Y}' \) be the \( v \)- and \( v' \)-components of \( \mathcal{Y} \), respectively, so that \( z = \mathcal{Y} \cap \mathcal{Y}' \). First suppose \( \lambda'_n < v_K(\psi_n(\alpha)) < \lambda_n \). If \( D_\alpha \) meets a point of \( \mathcal{Y} \setminus \mathcal{Y}' \), then by Proposition 3.2 applied to the blow down of \( \mathcal{Y}' \subseteq \mathcal{Y} \) (i.e., the \( v \)-model of \( \mathbb{P}^1_k \)), all \( D_\alpha \) outside of \( D(\psi_n, \lambda_n) \) intersect this point on \( \mathcal{Y} \subseteq \mathcal{Y}' \). So if we blow down \( \mathcal{Y} \subseteq \mathcal{Y}' \), then all \( D_\alpha \) for \( \alpha \notin D(\psi_n, \lambda_n) \) specialize to the same point. Since we can find \( \alpha_1, \alpha_2 \in K \setminus D(\psi_n, \lambda_n) \) with \( v_K(\psi_n(\alpha_1)) = \lambda'_n \) and \( \lambda'_n < v_K(\psi_n(\alpha_2)) < \lambda_n \), the previous line contradicts Proposition 3.2 applied to the \( v' \)-model of \( \mathbb{P}^1_k \). The same argument applied to the blow down of \( \mathcal{Y}' \) (i.e., the \( v \)-model of \( \mathbb{P}^1_k \)) yields a contradiction if \( D_\alpha \) intersects a point of \( \mathcal{Y} \setminus \mathcal{Y}' \). So \( D_\alpha \) meets the intersection point \( z \) of the two irreducible components of the special fiber.
Now, suppose \( v_K(\varphi_n(\alpha)) \leq \lambda'_n \). Fix \( \beta \in \overline{K} \) such that \( \lambda'_n < v_K(\varphi_n(\beta)) < \lambda_n \). Corollary 3.3 shows that \( D_\alpha \) and \( D_\beta \) do not meet on the \( v' \)-model of \( \mathbb{P}_K^1 \), and thus not on \( \mathcal{Y} \) either. In particular, since \( D_\beta \) meets \( z \) by the previous paragraph, \( D_\alpha \) does not. A similar proof works if \( v_K(\varphi_n(\alpha)) \geq \lambda_n \) using the \( v \)-model instead of the \( v' \)-model. This completes the proof of the corollary. \( \square \)

### 3.2 Resolution of singularities on normal models of \( \mathbb{P}_K^1 \)

Let \( \mathcal{Y} \) be a normal model of \( \mathbb{P}_K^1 \). A minimal regular resolution of \( \mathcal{Y} \) is a (proper) regular model \( \mathcal{Z} \) of \( \mathbb{P}_K^1 \) with a surjective, birational morphism \( \pi : \mathcal{Z} \to \mathcal{Y} \) such that the special fiber of \( \mathcal{Z} \) contains no \(-1\)-components (\cite[Definition 2.2.1]{1}). Such minimal regular resolutions exist and are unique, e.g., by \cite[Theorem 2.2.2]{1}.

In the remainder of Sect. 3.2, we recall a fundamental result from \cite{9} (which requires \( k \) algebraically closed), expressing minimal regular resolutions of models of \( \mathbb{P}_K^1 \) with irreducible special fiber in terms of Mac Lane valuations.

#### 3.2.1 Shortest \( N \)-paths

We start by recalling the notion of shortest \( N \)-path, introduced in \cite{9}.

**Definition 3.5** Let \( N \) be a natural number, and let \( a > a' \geq 0 \) be rational numbers. An \( N \)-path from \( a \) to \( a' \) is a decreasing sequence \( a = b_0/c_0 > b_1/c_1 > \cdots > b_r/c_r = a' \) of rational numbers in lowest terms such that

\[
\frac{b_i}{c_i} - \frac{b_{i+1}}{c_{i+1}} = \frac{N}{\text{lcm}(N, c_i) \cdot \text{lcm}(N, c_{i+1})}
\]

for \( 0 \leq i \leq r - 1 \). If, in addition, no proper subsequence of \( b_0/c_0 > \cdots > b_r/c_r \) containing \( b_0/c_0 \) and \( b_r/c_r \) is an \( N \)-path, then the sequence is called the shortest \( N \)-path from \( a \) to \( a' \).

**Remark 3.6** By \cite[Proposition A.14]{9}, the shortest \( N \)-path from \( a' \) to \( a \) exists and is unique.

**Remark 3.7** Observe that two successive entries \( b_i/c_i > b_{i+1}/c_{i+1} \) of a shortest 1-path satisfy \( b_i/c_i - b_{i+1}/c_{i+1} = 1/(c_i c_{i+1}) \).

**Example 3.8** The sequence \( 1 > 1/2 > 2/5 > 3/8 > 1/3 > 0 \) is a concatenation of the shortest 1-path from 1 to 3/8 with the shortest 1-path from 3/8 to 0. Note that the denominators increase until 3/8 and then decrease afterwards.

#### 3.2.2 Regular resolutions

The following proposition expresses minimal regular resolutions in terms of Mac Lane valuations and shortest \( N \)-paths. We fix the following notation.

**Notation 3.9** If \( v \) is a Mac Lane valuation, then \( \mathcal{Y}_v \) is the \( v \)-model of \( \mathbb{P}_K^1 \), and \( \mathcal{Y}_v^{\text{reg}} \) is its minimal regular resolution. Furthermore, \( \mathcal{Y}_{v, 0} \) is the \( \{v, 0\} \)-model of \( \mathbb{P}_K^1 \), and \( \mathcal{Y}_{v, 0}^{\text{reg}} \to \mathcal{Y}_{v, 0} \) is its minimal regular resolution. Observe that if \( \mathcal{X} \) is the \( v_0 \)-model of \( \mathbb{P}_K^1 \), then contracting the \( v \)-component of \( \mathcal{Y}_{v, 0} \) yields a canonical map \( \mathcal{Y}_{v, 0}^{\text{reg}} \to \mathcal{X} \) factoring through \( \mathcal{Y}_{v, 0} \).

**Proposition 3.10** (\cite[Theorem 7.8]{9}) Let \( v = \{ v_0, v_1(\varphi_1) = \lambda_1, \ldots, v_n(\varphi_n) = \lambda_n \} \). For each \( i \), write \( \lambda_i = b_i/c_i \) in lowest terms, and let \( N_i = \text{lcm}(c_i, c_{i+1}) = \deg(\varphi_i) \) (Corollary 2.11).

Set \( \lambda_0 = \lfloor \lambda_1 \rfloor \), as well as \( N_0 = N_1 = 1 \) and \( e(v_0/v_{-1}) = 1 \). Then the minimal regular
Corollary 3.13 With the notation of Proposition 3.10, the valuations included in $\mathcal{Y}_v^{\text{reg}}$ are the valuations included in $\mathcal{Y}_v^{\text{reg}}_{\nu,0}$ as well as $v_0$ and the valuations $[v_0, v_1(\psi_1) = \lambda]$ for $\lambda \in \{1, 2, \ldots, \lambda_0 - 1\}$. Equivalently, the valuations included in $\mathcal{Y}_v^{\text{reg}}$ are exactly the valuations we would get from Proposition 3.10 if we changed our convention from $\lambda_0 = \lfloor \lambda_1 \rfloor$ to $\lambda_0 = 0$.

Proof If $\lambda_0 = 0$, then $\mathcal{Y}_v^{\text{reg}}$ includes $v_0$, so $\mathcal{Y}_v^{\text{reg}} = \mathcal{Y}_v^{\text{reg}}_{\nu,0}$. If $\lambda_0 \geq 1$, then if $Z$ is the normal model of $\mathbb{P}^1_K$ including the valuations included in $\mathcal{Y}$ as well as $v_0$, then there may be a
singularity where the components corresponding to \(v_0\) and \([v_0, v_1(\varphi_1) = \lambda_0]\) cross. Since \(v_0\) and \([v_0, v_1(\varphi_1) = 0]\) are the same valuation, and since \(\lambda_0 > \lambda_0 - 1 > \cdots > 1 > 0\) is the shortest 1-path from \(\lambda_0\) to 0, [9, Corollary 7.5] shows that resolving this singularity yields exactly the description of \(\mathcal{Y}_{v_0^{\text{reg}}}^{\text{reg}}\) in the statement of the corollary. The equivalent description is clear, since \(\lambda_1 < 1\) is equivalent to \(\lambda_0 = 0\).

**Proposition 3.14** Let \(v = [v_0, v_1(\varphi_1) = \lambda_1, \ldots, v_n(\varphi_n) = \lambda_n]\) be a Mac Lane valuation. Let \(\mathcal{Y}_{v_0^{\text{reg}}}^{\text{reg}}\) be the minimal regular resolution of the \([v, v_0]\)-model of \(\mathbb{P}_K^1\). If \(w\) is a valuation included in \(\mathcal{Y}_{v_0^{\text{reg}}}^{\text{reg}}\), then \(e(w/v_0) \leq e(v/v_0)\), and furthermore, if \(e(w/v_0) = e(v/v_0)\), then \(w \leq v\).

**Proof** For a contradiction, choose a valuation \(w\) such that \(e(w/v_0)\) is maximal among those \(w\) violating the proposition, and among these choose \(w\) such that \(w\) is maximal under \(\leq\).

First observe that, since \(e(v_i/v_0) \leq e(v_n/v_0)\) for all \(i \leq n\) and \(v_i \leq v_n\), we may assume

\[
w \neq v_i \quad \text{for any } i.
\]

(3.1)

Let \(c_w\) be the self-intersection number of the \(w\)-component of \(\mathcal{Y}_{v_0^{\text{reg}}}^{\text{reg}}\). Since \(\mathcal{Y}_{v_0^{\text{reg}}}^{\text{reg}}\) is the minimal regular resolution of the \(\mathcal{Y}_{v, v_0}\)-model, and since \(w \notin \{v, v_0\}\) by (3.1), we have \(c_w \neq -1\), thus \(c_w \leq -2\). By standard intersection theory on regular arithmetic surfaces (e.g., [9, (3.4)]), we have

\[-c_w e(w/v_0) = \sum_{w'} e(w'/v_0),\]

where the sum is taken over all \(w'\) such that the \(w'\)-component intersects the \(w\)-component. Since \(w \neq v_i\) for any \(i\) by (3.1), Figure 1 shows that there are at most two such \(w'\).

Since \(-c_w \geq 2\) and by assumption \(e(w/v_0) \geq e(w'/v_0)\) for all \(w'\) in the sum, we find that there are exactly two \(w'\) and \(e(w'/v_0) = e(w/v_0)\) for each of them. By Remark 3.12, one of the \(w'\) satisfies \(w \prec w'\). Since \(w\) is maximal under \(\prec\), we conclude that \(w'\) does not violate the proposition. But \(w \prec w'\) and \(e(w/v_0) = e(w'/v_0)\) imply that \(w\) does not violate the proposition either, a contradiction.

\[\square\]

### 3.3 Valuations related to a given Mac Lane valuation

Let \(v = [v_0, v_1(\varphi_1) = \lambda_1, \ldots, v_n(\varphi_n) = \lambda_n]\). Recall that \(c_i\) is the denominator of \(\lambda_i\), when written in lowest terms. Let \(N_n = \text{lcm}(c_1, \ldots, c_{n-1}) = \text{deg}(\varphi_n)\) (Corollary 2.11). We assume that \(n \geq 1\) and \(\lambda_n \notin \Gamma_{v_0} = (1/N_n)\mathbb{Z}\).

Let \(\mathcal{Y}_v\) be the \(v\)-model of \(\mathbb{P}_K^1\), and let \(\mathcal{Y}_{v_0}^{\text{reg}}\) be its minimal regular resolution. By Proposition 3.10, the following Mac Lane valuations are included in \(\mathcal{Y}_{v_0}^{\text{reg}}\):

- \(v' : = v_{n-1}\lambda' = [v_0, v_1(\varphi_1) = \lambda_1, \ldots, v_n(\varphi_n) = \lambda_n]\),
- \(v'' : = v_{n-1}\lambda'' = [v_0, v_1(\varphi_1) = \lambda_1, \ldots, v_{n-1}(\varphi_{n-1}) = \lambda_{n-1}, v_n(\varphi_n) = \lambda'_n]\),

where \(\lambda'\) is the entry directly following \(\lambda_n\) in the shortest \(N_n\)-path from \(\lambda_n\) to \(e(v_{n-1}/v_0)\), and \(\lambda''\) is the entry directly preceding \(\lambda_n\) in the shortest \(N_n\)-path from \([N_n\lambda_n]/N_n\) to \(\lambda_n\). The valuation \(v'\) (resp. \(v''\)) is called the successor (resp. precursor) valuation to \(v\).

In the description of the minimal embedded resolution of an irreducible horizontal divisor, we have to analyze various contractions of the model described in Proposition 3.10.
Referring to certain special valuations appearing in Proposition 3.10. Let us write

\[ v^* := v_{n-1}^\lambda, \quad v^* := v_n^\lambda \]

where \( v^* \) (resp. \( v^{**} \)) can represent any of the valuations amongst the \( v_{n-1}^\lambda \) (resp. the \( v_n^\lambda \)) from Proposition 3.10. Later in Definition 4.10, we will specialize to specific choices of \( v^* \) and \( v^{**} \) depending on which regular model is being considered. In the remainder of this section, we establish some inequalities bounding \( \lambda^*, \lambda^{**} \) that are valid for all choices of \( v^*, v^{**} \). These will then be used in the proof of Lemma 5.5, which is the key technical input for the main theorem of the paper.

Since \( \lambda_n \not\in (1/N_n)\mathbb{Z} \), we have \( [N_n\lambda_n] \leq N_n\lambda' < N_n\lambda < N_n\lambda'' \leq [N_n\lambda_n] \), the first inequality coming from [9, Corollaries A.7, A.11]. Write \( \hat{\lambda}_n \) (resp. \( \hat{\lambda}', \hat{\lambda}'', \hat{\lambda}^*, \hat{\lambda}^{**} \)) for \( N_n\lambda_n - [N_n\lambda_n] \) (resp. \( N_n\lambda' - [N_n\lambda_n] \), \( N_n\lambda'' - [N_n\lambda_n] \), \( N_n\lambda^* - [N_n\lambda_n] \), \( N_n\lambda^{**} - [N_n\lambda_n] \)). Then we obtain

\[ 0 \leq \hat{\lambda}' < \hat{\lambda}_n < \hat{\lambda}'' \leq 1. \]

**Proposition 3.15** Let \( e, e', e'', e^* \), and \( e^{**} \) be the denominators of \( \hat{\lambda}_n, \hat{\lambda}', \hat{\lambda}'', \hat{\lambda}^*, \) and \( \hat{\lambda}^{**} \), respectively.

(i) The number \( \hat{\lambda}' \) immediately follows \( \hat{\lambda}_n \) in the shortest 1-path from \( \hat{\lambda}_n \) to 0.
(ii) The number \( \hat{\lambda}'' \) immediately precedes \( \hat{\lambda}_n \) in the shortest 1-path from 1 to \( \hat{\lambda}_n \).
(iii) The number \( \hat{\lambda}^* \) is on the shortest 1-path from \( \hat{\lambda}_n \) to 0.
(iv) The number \( \hat{\lambda}^{**} \) is on the shortest 1-path from 1 to \( \hat{\lambda}_n \).

**Proof** By [9, Lemma A.7], \( N_n\lambda' \) immediately follows \( N_n\lambda_n \) in the shortest 1-path from \( N_n\lambda_n \) to \( N_ne(v_{n-1})\lambda_{n-1} \), and thus in the shortest 1-path from \( N_n\lambda_n \) to \( [N_n\lambda_n] \) by [9, Lemma A.11]. Since translating by an integer preserves shortest 1-paths, subtracting \( [N_n\lambda_n] \) from all entries of these paths yields part (i). Part (ii) follows similarly, using that \( N_n\lambda'' \) immediately precedes \( N_n\lambda_n \) in the shortest \( N_n \)-path from \( [N_n\lambda_n] \) to \( N_n\lambda_n \). The proofs of parts (iii) and (iv) are essentially the same as the proofs of parts (i) and (ii), respectively.

**Example 3.16** If \( \hat{\lambda}_n = 3/8 \), we would have \( \hat{\lambda}' = 1/3 \) and \( \hat{\lambda}'' = 2/5 \) (cf. Example 3.8). We could take \( \hat{\lambda}^* \) to be 1/3 or 0, and we could take \( \hat{\lambda}^{**} \) to be 2/5, 1/2, or 1.

**Corollary 3.17** Let \( e, e', e'', e^* \), and \( e^{**} \) be as in Proposition 3.15. Then

(i) \( \lambda_n - \lambda' = 1/(N_nee') \).
(ii) \( \lambda'' - \lambda_n = 1/(N_nee'') \).
(iii) \( \lambda_n - \lambda^* \geq 1/(N_nee^*) \), with equality if and only if \( \lambda^* = \lambda' \).
(iv) \( \lambda^{**} - \lambda_n \geq 1/(N_nee^{**}) \), with equality if and only if \( \lambda^{**} = \lambda'' \).

**Proof** By Proposition 3.15(i) and the definition of 1-path, \( \hat{\lambda}_n - \hat{\lambda}' = 1/(ee') \), from which part (i) follows. Part (ii) follows similarly, using Proposition 3.15(ii). To prove part (iii), note that Proposition 3.15(iii) shows that \( \hat{\lambda}^* \) is on the shortest 1-path from \( \hat{\lambda}_n \) to 0, but that \( \hat{\lambda}^* \) does not directly follow \( \hat{\lambda}_n \) on this path unless \( \lambda^* = \lambda' \). The definition of shortest
A. Obus, P. Srinivasan
Res. Number Theory

1-paths shows that \( \hat{\lambda}_n - \hat{\lambda} = 1/ee^* \) if and only if \( \lambda^* = \lambda' \). Since \( \hat{\lambda}_n - \hat{\lambda} \) is a multiple of \( 1/ee^* \) by common denominators, part (iii) follows. The proof of part (iv) is exactly the same, using \( \lambda^{**}, \lambda'', \) and Proposition 3.15(iv) instead of \( \lambda^*, \lambda', \) and Proposition 3.15(iii). \( \square \)

**Lemma 3.18** Let \( v, v', v'', v^* \) be as above. If \( e, e', e'', e^* \) are defined as in Proposition 3.15, then \( e = e(v/v_{n-1}), e' = e(v'/v_{n-1}), e'' = e(v''/v_{n-1}), \) and \( e^{**} = e(v^{**}/v_{n-1}) \).

**Proof** By construction, \( e \) is the denominator of \( N_n \lambda_n \) (and similarly for \( e', e'', e^* \), and \( e^{**} \)). By [9, Lemma 5.3(ii)], \( e(v/v_0) = \lcm(N_n, c_n) \), where \( c_n \) is the denominator of \( \lambda_n \). By [9, Lemma A.6], this is equal to \( N_n e \). Since \( N_n = e(v_{n-1}/v_0) \), we have \( e = e(v/v_{n-1}) \). This proves the lemma for \( e \), and the proofs for \( e', e'', e^* \), and \( e^{**} \) are identical. \( \square \)

4 Some regular models of \( \mathbb{P}^1 \) attached to a polynomial

Throughout Sect. 4, we assume that the residue field \( k \) of \( K \) is algebraically closed.

Let \( \alpha \in \mathcal{O}_K \) such that \( v_K(\alpha) > 0 \) and the minimal polynomial \( f(x) \in K[x] \) of \( \alpha \) has degree at least 2. In this section, we first define a canonical Mac Lane valuation \( v_f \) attached to \( f \). We then define certain natural contractions of the minimal regular resolution of the \( v_f \)-model, called “Type I”, “Type II”, or “Type III” models. These are candidate models for the horizontal divisor \( D_\alpha \) to be regular on. We prove some technical results about these three kinds of models. These results will then be used in the next section to show that the minimal regular model on which \( D_\alpha \) is regular is a special kind of Type I or Type II model.

4.1 The Mac Lane valuation associated to a polynomial

Write

\[ v_f = [v_0, v_1(\phi_1) = \lambda_1, \ldots, v_n(\phi_n) = \lambda_n] \]

for the unique Mac Lane valuation on \( K(x) \) over which \( f \) is a proper key polynomial (Proposition 2.4(iv)). As usual, write \( v_0, v_1, \ldots, v_n = v_f \) for the intermediate valuations. For \( 1 \leq i \leq n \), write \( \lambda_i = b_i/c_i \) in lowest terms. Let \( N_i = \lcm(c_1, \ldots, c_{i-1}) = \deg(\phi_i) \) (Corollary 2.11). Furthermore, pick once and for all a root \( \alpha \) of \( f \).

**Remark 4.1** If the roots of \( f \) generate a tame extension, it is easy to read off the polynomials \( \psi_i \) and integers \( \lambda_i \) from the truncations of Newton-Puiseux expansions of the roots of \( f \) with respect to some choice of uniformizer \( t \), as we now explain. Using Proposition 2.4(iii), we see that we can take \( \psi_i \) to be the minimal polynomials of the truncations of the Newton-Puiseux expansions just before there is a jump in the lcm of the denominators of the exponents in the expansion. If \( \alpha \) is a root of \( f \), then Corollary 2.7 shows that \( \lambda_i = v_K(\psi_i(\alpha)) = \sum_{\psi_i(\beta) = 0} v_K(\alpha - \beta) \). If \( \deg(\phi_i) = m \), then the Galois group of the splitting field of the tame extension generated by the roots of \( \phi_i \) is generated by the automorphism \( t^{1/m} \mapsto \zeta_m t^{1/m} \) for a primitive \( m \)th root of unity \( \zeta_m \). Since the induced \( \mathbb{Z}/m\mathbb{Z} \)-action on the roots of \( \phi_i \) is transitive, a direct computation then shows that for each root \( \beta \) of \( \phi_i \), the quantity \( v_K(\alpha - \beta) \) is equal to one of the \( \lambda_i \) where the lcm of the denominators of the exponents jumps. (This is the content of [12, Lemma 8.13] using the language of characteristic/jump exponents.)
For example, let $K = \mathbb{C}(t)$ and let $f$ be the minimal polynomial of $2t - t^{5/2} + t^{8/3} - 3t^{7/2} + t^{23/6}$. Then $v_f$ has the form

$$v_f = [v_0, v_1(\varphi_1) = \lambda_1, v_2(\varphi_2) = \lambda_2],$$

and we can take $\varphi_1 = x - 2t$ and $\varphi_2$ to be the minimal polynomial of $2t - t^{5/2}$, with $\lambda_1 = 5/2$ and $\lambda_2 = 5/2 + 8/3$. This example also shows that $\deg(\varphi_i)$ and the invariants $\lambda_i$ contain the same information as the characteristic exponents of the Newton-Puiseux expansion of a root of $f$ as in [12, Example 8.13] in the tame case.

For the rest of this section we will use the following notation.

**Notation 4.2** Lemma 2.12 implies that we are in the situation of Sect. 3.3. Like in Sect. 3.3, let

- $v_f' = [v_0, v_1(\varphi_1) = \lambda_1, \ldots, v_{n-1}(\varphi_{n-1}) = \lambda_{n-1}, v_n(\varphi_n) = \lambda']$
- $v_f'' = [v_0, v_1(\varphi_1) = \lambda_1, \ldots, v_{n-1}(\varphi_{n-1}) = \lambda_{n-1}, v_n(\varphi_n) = \lambda'']$

be the successor and precursor valuations to $v_f$, respectively.

For simplicity, we write $e = e(v_f'/v_{n-1})$, $e' = e(v_f'/v_{n-1})$, and $e'' = e(v_f''/v_{n-1})$. This is consistent with the notation in Lemma 3.18 and Proposition 3.15. We record for later usage that $e = \deg(f)/\deg(\varphi_n)$ by Lemma 2.9(ii). With this notation, we are ready to state the main result of this paper. We postpone the proof to Section 5.

**Theorem 4.3** Let $f \in \mathcal{O}_K[x]$ be a monic irreducible polynomial of degree $\geq 2$, and let $\mathcal{X}$ be the $v_0$-model of $\mathbb{P}_K^1$. Let $v_f$ be the unique Mac Lane valuation over which $f$ is a key polynomial, and let $v_f'$ and $v_f''$ be the valuations defined in Notation 4.2. For any Mac Lane valuation $v$, let $\mathcal{Y}_v^\text{reg}$ be defined as in Notation 3.9.

(i) If $e(v_f'/v_0) \leq e(v_f''/v_0)$, then the minimal embedded resolution of $(\mathcal{X}, \text{div}_0(f))$ is $c : \mathcal{Y}_{v_f'}^\text{reg} \to \mathcal{X}$, where $c$ is the canonical contraction from Notation 3.9.

(ii) If $e(v_f'/v_0) > e(v_f''/v_0)$, then the minimal embedded resolution of $(\mathcal{X}, \text{div}_0(f))$ is $c : \mathcal{Y}_{v_f''}^\text{reg} \to \mathcal{X}$, where $c$ is the canonical contraction from Notation 3.9.

We now give two basic examples illustrating Theorem 4.3.

**Example 4.4** If $f = x^8 - x^{3/2}$, then $v_f = [v_0, v_1(x) = 3/8]$. As in Example 3.8, the shortest 1-path from 1 to 3/8 is given by 1 \ensuremath{\rightarrow} 1/2 \ensuremath{\rightarrow} 3/8 and the shortest 1-path from 3/8 to 0 is given by 3/8 \ensuremath{\rightarrow} 1/3 \ensuremath{\rightarrow} 0, yielding $v_f' = [v_0, v_1(x) = 1/3]$ and $v_f'' = [v_0, v_1(x) = 1/2]$. Since $e(v_f'/v_0) = 3$ and $e(v_f''/v_0) = 2$, part (ii) of Theorem 4.3 applies, and the minimal embedded resolution of $(\mathcal{X}, \text{div}_0(f))$ is $\mathcal{Y}_{v_f''}^\text{reg} \to \mathcal{X}$.

**Remark 4.5** In Example 4.4, applying Proposition 3.10 shows that $\mathcal{Y}_{v_f'}^\text{reg}$ includes the valuations $v_{\lambda} := [v_0, v_1(x) = \lambda]$, for $\lambda \in \{0, 1/2, 1\}$. In particular, there exist $\lambda$ both greater than and less than 3/8 for which $v_{\lambda}$ is included. By Corollary 3.4, this implies that $D_{\alpha}$, for $\alpha$ a root of $f$, specializes to the intersection of two components (the ones corresponding to $\lambda = 0$ and $\lambda = 1/2$). This property makes $\mathcal{Y}_{v_{1/2}}^\text{reg}$ a prototype for what we will call a “Type I model” in the sequel. In particular, $v_{1/2}$ is one of the $v_{1,\lambda}$ and $v_0$ is one of the $w_{0,\lambda}$; see Definition 4.8(i).
Example 4.6 If $f$ is Eisenstein, then $v_f = [v_0, v_1(x) = 1/\deg(f)]$. The shortest 1-path from $1/\deg(f)$ to 0 is given by $1/\deg(f) > 0$ and the shortest 1-path from 1 to $1/\deg(f)$ is given by $1 > 1/2 > \cdots > 1/(\deg(f) - 1) > 1/\deg(f)$. So $v'_f = [v_0, v_1(x) = 0] = v_0$, and $v''_f = [v_0, v_1(x) = 1/(\deg(f) - 1)]$. Since $e(v'_f/v_0) = 1$ and $e(v''_f/v_0) = \deg(f) - 1$, part (i) of Theorem 4.3 applies, and the minimal embedded resolution of $(\mathcal{X}, \text{div}_0(f))$ is $\mathcal{Y}_{v'_f,0} \to \mathcal{X}$.

But $\mathcal{Y}_{v'_f,0} = \mathcal{Y}_{v_0,0} = \mathcal{X}$, so this recovers the easy-to-verify fact that if $\alpha$ is a root of $f$, then $D_\alpha$ is regular on the $v_0$-model $\mathcal{X}$ of $\mathbb{P}^1_K$.

Remark 4.7 In Example 4.6, since $\mathcal{Y}_{v'_f,0} = \mathcal{X}$ has irreducible special fiber, $D_\alpha$, for $\alpha$ a root of $f$, specializes to only one irreducible component. This property makes $\mathcal{Y}_{v'_f,0}$ a prototype for what we will call a "Type II model" in the sequel. In particular, $v_0$ is one of the $w_{0,\lambda}$; see Definition 4.8(ii).

4.2 The model $\mathcal{Y}_{v'_f}$ and its contractions

As we start contracting components in $\mathcal{Y}_{v'_f}$ to identify the minimal embedded resolution of the the pair $(\mathbb{P}^1_K, \text{div}_0(f))$, we go through an intermediate sequence of regular models of $\mathbb{P}^1_K$ that naturally breaks up into three types (Definition 4.8), based on the specialization behaviour of $D_\alpha$ (Proposition 4.15). To understand whether $D_\alpha$ is regular on these contractions, we also need to understand where some closely related divisors specialize on each of these three types of models (Corollary 4.16). The goal of the rest of the subsection is to prove Proposition 4.17, which lets us write down an explicit function that cuts out the divisor $D_\alpha$ on each of these three types of models – the forms of the explicit functions look different in each of these three cases, hence the subdivision. In Sect. 5 we will finally use these explicit functions to understand the regularity of $D_\alpha$ on each of these three types of models. We will show that the minimal embedded resolution has to be one of the Type I or Type II models, and $D_\alpha$ is not regular on the unique Type III model.

We use the notation of Proposition 3.10 and Figure 1.

Definition 4.8 Fix $f$ as in this section. The Type I, II, and III models below implicitly depend on $f$.

- A Type I model of $\mathbb{P}^1_K$ is any regular contraction of $\mathcal{Y}_{v'_f}$ that includes at least one of the $v_{n,\lambda}$ and one of the $w_{n-1,\lambda}$, but does not include $v_f$.
- A Type II model of $\mathbb{P}^1_K$ is any regular contraction of $\mathcal{Y}_{v'_f}$ that does not include $v_f$ or any $v_{n,\lambda}$, but does include at least one of the $w_{n-1,\lambda}$.
- Assuming that $\mathcal{Y}_{v'_f}$ includes at least one valuation other than $v_f$, the $v_{n,\lambda}$, and the $w_{n-1,\lambda}$, we define the Type III model of $\mathbb{P}^1_K$ to be the model where the $v_f$-component is contracted, as well as all the $v_{n,\lambda}$ and the $w_{n-1,\lambda}$.

Remark 4.9 Since $v_{n-1}$ is one of the $w_{n-1,\lambda}$, one sees that the Type III model is the contraction of the $v_{n-1}$-component in $\mathcal{Y}_{v_{n-1}}$.

Definition 4.10 Given a Type I or Type II model $\mathcal{Y}$, define

$$v_f^* = [v_{n-1}, v_f^*(\varphi_n) = \lambda^*] = w_{n-1,\lambda^*},$$

where $\lambda^*$ is maximal such that $w_{n-1,\lambda^*}$ is included in $\mathcal{Y}$.
• Given a Type I model \( \mathcal{Y} \), define
\[
v_f^{**} = [v_{n-1}, v_f^{**}(\varphi_n)] = v_{n, \lambda^{**}},
\]
where \( \lambda^{**} \) is minimal such that \( v_{n, \lambda^{**}} \) is included in \( \mathcal{Y} \).

• If \( v_f^* \) (resp. \( v_f^{**} \)) is defined, define \( e^* \) (resp. \( e^{**} \)) to be the denominator of \( N_n\lambda^* \) (resp. \( N_n\lambda^{**} \)). Note that this notation is consistent with that of Proposition 3.15.

• Given a Type III model \( \mathcal{Y} \), define \( v_{n-1}' \) and \( v_{n-1}'' \) to be the successor and precursor valuations to \( v_{n-1} \), respectively.

\textbf{Remark 4.11} Note that the \( v_f^* \) and \( v_f^{**} \)-components of \( \mathbb{P}^1_K \) intersect using Proposition 3.10 and Remark 3.12.

\textbf{Remark 4.12} By Lemma 3.18, \( e^* = e(v_f^*/v_{n-1}) \) and \( e^{**} = e(v_f^{**}/v_{n-1}) \).

\subsection*{4.2.1 Specializations of horizontal divisors}

\textbf{Lemma 4.13} On the model \( \mathcal{Y}_{\mathcal{Y}}^{\text{reg}} \) the only component of the special fiber that \( D_\alpha \) meets is the \( v_f \)-component.

\textbf{Proof} The multiplicity of the \( v_f \)-component of \( \mathcal{Y}_{\mathcal{Y}}^{\text{reg}} \) in the special fiber is \( e(v_f/v_{n-1})e(v_{n-1}/v_0) \). But \( e(v_f/v_{n-1}) = \deg(f)/\deg(\varphi_n) \) by Lemma 2.9(ii) and \( e(v_{n-1}/v_0) = \deg(\varphi_n) \) by Corollary 2.11. So the multiplicity is equal to \( \deg(f) \).

By Proposition 2.7, \( v_f(\varphi_n(\alpha)) = \lambda_n \). So by [9, Lemma 7.3(iii)] and Proposition 3.2, \( D_\alpha \) intersects a regular point \( z \) on the \( v_f \)-model of \( \mathbb{P}^1_K \), which is also a smooth point of the reduced special fiber by [9, Lemma 7.1]. By the previous line, we conclude that the point \( z \) is not part of the base locus of the projection \( \mathcal{Y}_{\mathcal{Y}}^{\text{reg}} \to \mathcal{Y}_{\mathcal{Y}} \), and this proves the lemma. \( \square \)

\textbf{Lemma 4.14} Let \( y \) be a point on the \( v_f \)-component of \( \mathcal{Y}_{\mathcal{Y}}^{\text{reg}} \).

(i) Suppose \( \mathcal{Y} \) is a Type I model, and \( \tau: \mathcal{Y}_{\mathcal{Y}}^{\text{reg}} \to \mathcal{Y} \) is the standard contraction. Then \( \tau(y) \) lies on the intersection of the \( v_f^* \) and \( v_f^{**} \)-components of \( \mathcal{Y} \).

(ii) Suppose \( \mathcal{Y} \) is a Type II model, and \( \tau: \mathcal{Y}_{\mathcal{Y}}^{\text{reg}} \to \mathcal{Y} \) is the standard contraction. Then \( \tau(y) \) lies only on the \( v_f^* \)-component of \( \mathcal{Y} \).

(iii) Suppose \( \mathcal{Y} \) is the Type III model, and \( \tau: \mathcal{Y}_{\mathcal{Y}}^{\text{reg}} \to \mathcal{Y} \) is the standard contraction. Then \( \tau(y) \) lies on the intersection of the \( v_{n-1}' \) and \( v_{n-1}'' \)-components of \( \mathcal{Y} \).

\textbf{Proof} This follows from Remark 3.12 and Figure 1. \( \square \)

\textbf{Proposition 4.15} Let \( \alpha, f, v_f, v_f^*, v_f^{**}, v_{n-1}' \) and \( v_{n-1}'' \) be as in this section.

(i) If \( \mathcal{Y} \) is a Type I model of \( \mathbb{P}^1_K \), then the divisor \( D_\alpha \) on \( \mathcal{Y} \) meets the intersection of the two components of the special fiber corresponding to \( v_f^* \) and \( v_f^{**} \).

(ii) If \( \mathcal{Y} \) is a Type II model of \( \mathbb{P}^1_K \), then the divisor \( D_\alpha \) on \( \mathcal{Y} \) intersects only the \( v_f^* \)-component of the special fiber.

(iii) If \( \mathcal{Y} \) is the Type III model of \( \mathbb{P}^1_K \), then the divisor \( D_\alpha \) on \( \mathcal{Y} \) meets the intersection of the two components of the special fiber corresponding to \( v_{n-1}' \) and \( v_{n-1}'' \).

\textbf{Proof} By Lemma 4.13, \( D_\alpha \) meets the special fiber of \( \mathcal{Y}_{\mathcal{Y}}^{\text{reg}} \) only on the \( v_f \)-component. Parts (i), (ii), and (iii) of the proposition now follow from the respective parts of Lemma 4.14. \( \square \)

\textbf{Corollary 4.16} Let \( \mathcal{Y} \) be a Type I or Type II model of \( \mathbb{P}^1_K \). Let \( \alpha_n \) be a root of \( \varphi_n \).
(i) Suppose $\beta \in \overline{K}$ has degree less than $\deg(\varphi_n)$ over $K$. Then $D_\alpha$ and $D_\beta$ do not meet on the special fiber of $\mathcal{Y}$.

(ii) If $\mathcal{Y}$ is Type I, then $D_\alpha$ and $D_{\alpha e}$ do not meet on the special fiber of $\mathcal{Y}$.

(iii) If $\mathcal{Y}$ is Type II or Type III, then $D_\alpha$ and $D_{\alpha e}$ meet on the special fiber of $\mathcal{Y}$.

**Proof** By Proposition 4.15, $D_\alpha$ specializes to the $v_f^+$-component of the special fiber of $\mathcal{Y}$. By Corollary 2.11 and Lemma 3.18, the multiplicity of this component is $N_\alpha e^* = \deg(\varphi_n) e^* \geq \deg(\varphi_n)$. So by [5, Lemma 5.1(a)], $D_\beta$ does not specialize to this component. This proves part (i).

To prove part (ii), assume $\mathcal{Y}$ is Type I. Note that $\alpha_n$ is a root of $\varphi_n$, we have $v_K(\varphi_n(\alpha_n)) = \infty$, which does not lie between $\lambda^*$ and $\lambda^{**}$. As a consequence, Corollary 3.4 and Proposition 4.15(i) show that $D_\alpha$ does not meet $D_{\alpha e}$ on the special fiber of $\mathcal{Y}$.

To prove part (iii), it suffices to assume $\mathcal{Y}$ is Type II, since a Type III model is a contraction of a Type II model. Since both $v_K(\varphi_n(\alpha_n)) = \lambda_n$ and $v_K(\varphi_n(\alpha_n)) = \infty$ are greater than $\lambda^*$, Proposition 3.2 shows that they meet on the special fiber of the $v_f^+$-model of $\mathbb{P}_K^1$. This point is not a base point of the contraction $\mathcal{Y} \to \mathcal{Y}_{v_f^+}$, because that would violate Proposition 4.15(ii). Thus, $D_\alpha$ and $D_{\alpha e}$ meet on $\mathcal{Y}$. \hfill \Box

4.2.2. The final result of this section, Proposition 4.17, shows how to appropriately modify the function $f$ to make a function that precisely cuts out the divisor $D_\alpha$ on each of the three types of models –

**Proposition 4.17** Let $\mathcal{Y}$ be a Type I, Type II, or Type III model of $\mathbb{P}_K^1$, and let $v_f^+$ and $v_f^{**}$ be defined accordingly.

(i) If $\mathcal{Y}$ is Type I, the quantity $b := e(\lambda_n - \lambda^*)/(\lambda^{**} - \lambda^*)$ is an integer. Furthermore, there exists a monomial $s$ in $\varphi_1, \ldots, \varphi_{n-1}$ over $K$ such that the divisor $D_\alpha$ is locally cut out by $sf/\varphi_n^b$.

(ii) If $\mathcal{Y}$ is Type II, there exists a monomial $t$ in $\varphi_1, \ldots, \varphi_{n-1}$ such that the divisor $D_\alpha$ is locally cut out by $sf$, where $s = t^e$.

(iii) If $\mathcal{Y}$ is Type III, then there exists $s \in K(x)$ such that the divisor $D_\alpha$ is locally cut out by $sf$, and such that the support of $s$ is locally (near $D_\alpha$) contained in the special fiber of $\mathcal{Y}$.

**Remark 4.18** Since $\varphi_1, \ldots, \varphi_{n-1}$ all have degree lower than $\deg(\varphi_n)$, Corollary 4.16(i) shows that the support of $s$ is locally (near $D_\alpha$) contained in the special fiber of $\mathcal{Y}$ in parts (i) and (ii), as well as part (iii).

To prove Proposition 4.17, we first need to compute the orders of vanishing of various auxiliary functions that will be used to modify the function $f$ along vertical components of Type I, II, and III models. This is accomplished in Lemma 4.19. The proof also needs two other short lemmas (Lemma 4.20 and Lemma 4.21).

**Lemma 4.19** Let $f = \varphi_n^e + a_{e-1}\varphi_n^{e-1} + \cdots + a_0$ be the $\varphi_n$-adic expansion of $f$. Let $\mathcal{Y}$ be a Type I or Type II model of $\mathbb{P}_K^1$, and let $v_f^+$ and $v_f^{**}$ be defined accordingly. Let $a_{e} = 1$.

(i) We have $v_f^+(f) = v_f^+(\varphi_n^e) = e\lambda^*$.

(ii) We have $v_f^+(a_i\varphi_n^e) > e\lambda^*$ for $0 \leq i \leq e - 1$.

(iii) In the case of a Type I model, we have $v_f^{**}(f) = v_f^{**}(a_0) = e\lambda_n$. 


(iv) In the case of a Type I model, we have \( v_{\sharp}^* (a_i \varphi_n^i) > e \lambda_n \) for \( 1 \leq i \leq e \).

**Proof** By Lemma 2.1, \( \varphi_n^e \) is a term in the \( \varphi_n \)-adic expansion of \( f \) with minimal \( v_f \)-valuation. It is also the term whose valuation is decreased the most when \( v_f \) is replaced with \( v_f^* \). Thus \( \varphi_n^e \) is the unique term in the \( \varphi_n \)-adic expansion of \( f \) with minimal \( v_f^* \)-valuation. Since \( v_f^* (\varphi_n) = \lambda^* \) by definition, this proves parts (i) and (ii).

Similarly, by Lemma 2.1, \( a_0 \) is a term in the \( \varphi_n \)-adic expansion of \( f \) with minimal \( v_f \)-valuation. It is also the term whose valuation is increased the least when \( v_f \) is replaced by \( v_f^{**} \). Thus \( a_0 \) is the unique term in the \( \varphi_n \)-adic expansion of \( f \) with minimal \( v_f^{**} \)-valuation. Since \( v_f^{**} (a_0) = v_f (a_0) = e \lambda_n \) (Lemma 2.1), this proves parts (iii) and (iv).

**Lemma 4.20** On a Type I model \( \mathcal{Y} \), we have \( \lambda^{**} - \lambda^* = 1/(N_n e^{\lambda^*}) \).

**Proof** Since \( \mathcal{Y} \) is regular and the \( v_f^* \)- and \( v_f^{**} \)-components intersect, [9, Corollary 7.6] (with \( \mathcal{X} = \mathcal{X}' \) there) shows that \( \lambda^{**} > \lambda^* \) is the shortest \( N_n \)-path. By [9, Corollary A.7], \( \lambda^{**} > \lambda^* \) is a shortest 1-path, where \( \lambda^* \) and \( \lambda^{**} \) are as in Proposition 3.15. By the definition of a 1-path, \( \lambda^{**} - \lambda^* = 1/(e^{\lambda^*}) \), so \( \lambda^{**} - \lambda^* = 1/(N_n e^{\lambda^*}) \).

**Lemma 4.21** On a Type II model \( \mathcal{Y} \), we have \( \Gamma_{v_f} = \Gamma_{v_n-1} \).

**Proof** If \( \mathcal{Y} \) is a Type II model, then it dominates \( \mathcal{Y}_{v_f} \), and thus includes all the valuations therein. On the other hand, by the definition of a Type II model, \( \mathcal{Y} \) does not include any valuation of the form \( [v_0, v_1(\varphi_1) = \lambda_1, \ldots, v_{n-1}(\varphi_{n-1}) = \lambda_{n-1}, v_n(\varphi_n) = \lambda] \) with \( \lambda > \lambda^* \).

Applying Proposition 3.10 to \( v_f^* \), this forces the \( \beta_n \) referred to in the first bullet point of Proposition 3.10 to equal \( \lambda^* \). So \( \lambda^* \in (1/N_n)\mathbb{Z} = \Gamma_{v_n-1} \). Since \( \Gamma_{v_f} = [v_{n-1}, v_1(\varphi_1) = \lambda^*] \), it follows that \( \lambda^* \) together with \( \Gamma_{v_n-1} \) generates \( \Gamma_{v_f} \). Combining the previous two sentences, we get \( \Gamma_{v_f} = \Gamma_{v_n-1} \).

**Proof of Proposition 4.17** To prove the first assertion of part (i), note that \( \lambda^{**} - \lambda^* = 1/(N_n e^{\lambda^*}) \) by Lemma 4.20. So \( b = N_n e^{\lambda^*}(\lambda_n - \lambda^*) \). Since the denominator of \( \lambda_n \) divides \( e(v_f/v_0) = N_n e \) and that of \( \lambda^* \) divides \( e(v_f^*/v_0) = N_n e^{\lambda^*} \), we have that \( b \) is an integer, and is in fact divisible by \( e^{\lambda^*} \).

Now, assume we are on a Type I model and let \( y \) be the point where \( D_o \) meets the special fiber of \( \mathcal{Y} \), i.e., the specialization of \( f(x) = 0 \). The function \( f \) in general does not locally cut out \( D_o \) at \( y \), because div(\( f \)) might also include vertical components passing through \( y \). By Proposition 4.15(i), \( z \) is the intersection of the \( v_f^* \) and \( v_f^{**} \)-components of the special fiber. By Corollary 4.16(ii), (i), the specialization of \( \varphi_i = 0 \) is not \( y \) for any \( 1 \leq i \leq n \). So to finish the proof of part (i), it suffices to construct \( s \) as in the proposition such that \( v_f^*(sf/\varphi_n) = v_f^{**}(sf/\varphi_n) = 0 \).

By Lemma 4.19(i), we have \( v_f^*(f/\varphi_n) = (e-b)\lambda^* \). Likewise, by Lemma 4.19(iii), we have \( v_f^{**}(f/\varphi_n) = e \lambda_n - b \lambda^{**} \). Since \( e^{\lambda^*} \mid b \), and the denominators of \( \lambda_n \) and \( \lambda^{**} \) are \( N_n e \) and \( N_n e^{\lambda^*} \) respectively, \( e \lambda_n - b \lambda^{**} \in \Gamma_{v_n-1} = (1/N_n)\mathbb{Z} \). This means that there exists \( s \) in the proposition such that \( v_f^*(sf/\varphi_n) = 0 \). Since \( v_f^*(sf/\varphi_n) = 0 \) is reduced to showing that \( (e-b)\lambda^* = e \lambda_n - b \lambda^{**} \). But this is immediate upon plugging in the definition of \( b \).

Now we prove part (ii). Let \( y \) be as in part (i). By Proposition 4.15(ii), \( y \) lies on a unique component of the special fiber, namely the \( v_f^* \)-component. Furthermore, since the value group of \( v_f^* \) is \( \Gamma_{v_n-1} \) (Lemma 4.21), we have that \( v_f^*(\varphi_n) = \lambda^* \in \Gamma_{v_n-1} \). Thus we can find \( t \)
as in the proposition such that \( v^*_\tau(t) = -\lambda^* \). By Lemma 4.19(i), \( v^*_\tau(tf) = v^*_\tau(sf) = 0 \). By Corollary 4.16(i), the specialization of \( \psi_i = 0 \) is not \( y \) for any \( 1 \leq i \leq n - 1 \). So \( sf \) cuts out \( D_\alpha \), proving part (ii).

For part (iii), note that \( \mathcal{Y} \) is regular, and is thus a local UFD. Since \( \text{div}(f) \) and \( D_\alpha \) agree on the generic fiber in a neighborhood of \( D_\alpha \), there exists \( s \in K(\mathcal{X}) \) with the desired property.

\[ \square \]

5 Minimal embedded resolution

In this section, we prove our main result, Theorem 4.3, which explicitly gives the minimal embedded resolution of \( (\mathcal{X}, \text{div}_0(f)) \), where \( \mathcal{X} \) is the \( v_0 \)-model of \( \mathbb{P}^1_K \) and \( f \in \mathcal{O}_K[x] \) is a monic polynomial of degree at least 2. We begin in Sect. 5.1 with some general results on regularity, and then return to Mac Lane valuations and models of \( \mathbb{P}^1_K \) for the proof in Sect. 5.2. The main technical lemma that makes everything work is Lemma 5.5, which depends heavily on the work in Sect. 4.

Throughout Sect. 5, with the exception of Remark 5.12 and the conclusion of the paper immediately following it, we assume that the residue field \( k \) of \( K \) is algebraically closed.

5.1 Generalities on regular models

Lemma 5.1 If \( \mathcal{X} \) is a regular model of \( \mathbb{P}^1_K \) and \( D \) is a reduced, effective, regular divisor on \( \mathcal{X} \) and if \( f : \mathcal{X}' \to \mathcal{X} \) is a modification, then the strict transform \( D' \) of \( D \) in \( \mathcal{X}' \) is regular.

Proof Since \( \mathcal{X} \) is normal, \( f \) is an isomorphism above points of codimension 1, thus over the generic point of each component of \( D \). So \( D' \to D \) is proper and birational. Since \( \dim(D') = \dim(D) = 1 \), \( D' \to D \) is finite as well, and thus it is an isomorphism, proving the lemma.

The following proposition is well-known, but we were unable to find an exact reference. We state it only in the generality we need.

Proposition 5.2 If \( \mathcal{X} \) is a regular model of \( \mathbb{P}^1_K \) and \( D \) is an integral horizontal divisor on \( \mathcal{X} \), then there is a unique minimal modification \( \mathcal{X}' \to \mathcal{X} \) such that \( \mathcal{X}' \) is regular and the strict transform of \( D \) is regular.

Proof By [4, Theorem 9.2.26], there exists some modification \( \mathcal{Y} \to \mathcal{X} \) with \( \mathcal{Y} \) regular under which the total transform of \( D \) has normal crossings, and in particular, the strict transform of \( D \) is thus regular. We now prove that a minimal such \( \mathcal{Y} \) is unique. By [4, Theorem 9.2.2], the morphism \( \mathcal{Y} \to \mathcal{X} \) is a finite sequence of blowups at reduced closed points.

We now prove the proposition by induction on the minimum number \( n \) of blowups of \( \mathcal{X} \) at closed points required to make the strict transform of \( D \) regular. The case \( n = 0 \) is trivial. If not, since blowups in centers outside \( \text{Supp}(D) \) do not affect \( D \), any minimal sequence of blowups making the strict transform of \( D \) regular begins with blowing up the (unique) intersection point \( x \) of \( D \) with the special fiber of \( \mathcal{X} \). Replacing \( \mathcal{X} \) with its blowup at \( x \), and noting that the strict transform of \( D \) is still integral on this blowup and then applying induction completes the proof. 

\[ \square \]
Lemma 5.3  Let \( \mathcal{Y} \) be a regular snc-model of a smooth curve \( Y \) over \( K \), and let \( y \in \mathcal{Y} \) be a closed point. Let \( \text{div}(f), \text{div}(g) \) be the divisors in \( \text{Spec} \, \hat{O}_{\mathcal{Y}, y} \) of functions \( f, g \in \hat{O}_{\mathcal{Y}, y} \) respectively.

(i) Suppose \( \text{div}(f) \) is of the form \( \sum_{i=1}^{r} c_i D_i \) for some integers \( c_i \geq 0 \) where the \( D_i \) are Weil prime divisors. If \( \sum_i c_i \geq 2 \), then \( f \in m_{\mathcal{Y}, y}^2 \).

(ii) Suppose \( \text{div}(f) = D \) where \( D \) is a Weil prime divisor corresponding to one of the irreducible components of the special fiber of \( \mathcal{Y} \) passing through \( y \). Then \( f \in m_{\mathcal{Y}, y} \cap m_{\mathcal{Y}, y}^2 \).

(iii) If \( \text{div}(f) = D \) and \( \text{div}(g) = E \), where \( D \) and \( E \) are Weil prime divisors corresponding to two different components of the special fiber of \( \mathcal{Y} \) passing through \( y \), then the images of \( f \) and \( g \) are linearly independent in \( m_{\mathcal{Y}, y}/m_{\mathcal{Y}, y}^2 \).

Proof  First note that the regular local ring \( \hat{O}_{\mathcal{Y}, y} \) is a UFD and thus every height one prime ideal is principal. Thus in the situation of part (i), \( f = w \prod_i f_i^{c_i} \), where \( w \) is a unit and \( \text{div}(f_i) = D_i \). Since the \( f_i \) lie in the maximal ideal, this proves (i).

In fact, by [1, Lemma 2.3.2 and its proof], we can write

\[
\hat{O}_{\mathcal{Y}, y} \cong \hat{O}_K [y_1, y_2]/(y_1^m \cdots y_r^m - u \pi_K),
\]

with \( r \in \{1, 2\} \). The irreducible components of the special fiber passing through \( y \) are cut out by \( y_1 \) if \( r = 1 \) and, by \( y_1 \) and \( y_2 \) if \( r = 2 \). So in the situation of part (ii), we have \( f = w y_1 \) or \( f = w y_2 \), with \( w \) a unit. Since \( m_{\mathcal{Y}, y} = (y_1, y_2) \), this proves part (ii). In the situation of part (iii), we have \( r = 2 \), and the result follows from the fact that the images of \( w_1 y_1 \) and \( w_2 y_2 \) are linearly independent in \( (y_1, y_2)/(y_1, y_2)^2 \).

Proposition 5.4  Let \( \mathcal{Y} \) be a regular model of \( \mathbb{P}^1_K \), and let \( y \) be the point where \( D_{\alpha} \) intersects the special fiber. Let \( g \in \hat{O}_{\mathcal{Y}, y} \) be such that \( \text{div}(g) = D_{\alpha} \) on \( \text{Spec} \, \hat{O}_{\mathcal{Y}, y} \). Then \( D_{\alpha} \) is regular if and only if \( g \notin m_{\mathcal{Y}, y}^2 \).

Proof  This is [4, Corollary 4.2.12].

5.2 Non-archimedean analysis of valuations in an expansion

Maintain our notation from Sect. 4. In particular, for the remainder of the paper, \( f \in \hat{O}_K [x] \) is monic and irreducible of degree at least 2, \( \alpha \) is a root of \( f \), and on any regular model of \( \mathbb{P}^1_K \), the divisor \( D_{\alpha} \) is the horizontal divisor corresponding to \( \alpha \) as in Sect. 3.1. As in Sect. 4, we use the notation \( \nu_f \) for the unique Mac Lane valuation over which \( f \) is a proper key polynomial. We also use the valuations \( \nu_f', \nu_f'' \) from Notation 4.2, and we use the concept of Type I/II/III models associated to \( f \) from Definition 4.8, which give rise to valuations \( \nu_f^*, \nu_f^{**}, \nu_{f,-1}^* \) and \( \nu_{f,-1}^{**} \) as in Definition 4.10. As in Sect. 3.1, we write \( e = e(\nu_f/\nu_{n-1}) \), \( e' = e(\nu_f'/\nu_{n-1}) \), \( e'' = e(\nu_f''/\nu_{n-1}) \), and, when there is a Type I/II model in play, \( e^* = e(\nu_f^*/\nu_{n-1}) \) and \( e^{**} = e(\nu_f^{**}/\nu_{n-1}) \).

We decompose the function cutting out the unique horizontal divisor of \( D_{\alpha} \) using the \( \nu_{\alpha, n} \)-adic expansion of \( f \), and analyze which of the terms in the decomposition are in \( m_{\mathcal{Y}, y}^2 \) for Type I/II models \( \mathcal{Y} \). This will be the key technical input for analyzing regularity of \( D_{\alpha} \) on these models in the next section.

Lemma 5.5  Let \( \mathcal{Y} \) be a Type I or Type II model of \( \mathbb{P}^1_K \), and let \( y \in \mathcal{Y} \) be the point where \( D_{\alpha} \) meets the special fiber of \( \mathcal{Y} \). Let \( s \) be as in Proposition 4.17(i), (ii), let \( b \) be as in Proposition 4.17(i) if \( \mathcal{Y} \) is Type I and let \( b = 0 \) if \( \mathcal{Y} \) is Type II. If \( f = \nu_n^e + a_{e-1} \nu_n^{e-1} + \cdots + a_0 \) is
the $\varphi_n$-adic expansion of $f$, then we can write

$$\frac{sf}{\varphi_n^b} = s\varphi_n^{e-b} + sa_{e-1}\varphi_n^{e-1-b} + \cdots + sa_0\varphi_n^{-b}. \tag{5.1}$$

Then,

1. All terms $sa_i\varphi_n^{i-b}$ of \((5.1)\) for $1 \leq i \leq e - 1$ are in $m_{2,n}^2$.
2. We have $sa_0\varphi_n^{-b}$ in $m_{2,n}^2$ if and only if $v_f^* \neq v_f^{-}$.
3. We have $sa_i\varphi_n^{i-b}$ in $m_{2,n}^2$ if and only if $\mathcal{Y}$ is Type II or $v_f^{**} \neq v_f^*$.
4. Suppose $\mathcal{Y}$ is Type I. If $v_f^* = v_f^{-}$ and $v_f^{**} = v_f^*$, then $sa_i\varphi_n^{i-b}$ and $sa_0\varphi_n^{-b}$ generate linearly independent elements of $m_{2,Y}/m_{2,Y}^2$.

**Proof** Let $y$ be the point where $D_\alpha$ intersects the special fiber of $\mathcal{Y}$. Recall from Proposition 4.17 that $sf/\varphi_n^b$ cuts out $D_\alpha$. By Remark 4.18, the horizontal part of $\text{div}(s)$ does not contain $y$. The same is true for all of the $\text{div}(a_i)$, since the $a_i$ have degree less than $\varphi_n$ by definition. Furthermore, Corollary 4.16(ii) shows that the same is true for the horizontal part of $\text{div}(\varphi_n)$ if $\mathcal{Y}$ is Type I.

By Proposition 4.15, $y$ is the intersection of the $v_f^*$- and $v_f^{**}$-components if $\mathcal{Y}$ is Type I, and $y$ lies on only the $v_f^*$-component of $\mathcal{Y}$ is Type II. Write $D^*$ and $D^{**}$ for the prime divisors corresponding to the $v_f^*$- and $v_f^{**}$-components, respectively.

We now prove part (i). Assume $1 \leq i \leq e - 1$. By Lemma 4.19(i), (ii), $v_f^*(f) < v_f^*(a_i\varphi_n^b)$, and since the divisor of $sf/\varphi_n^b$ is horizontal by construction, so $0 = v_f^*(sf/\varphi_n^b) < v_f^*(sa_i\varphi_n^{i-b})$. Thus $D^*$ lies in the support of $sa_i\varphi_n^{i-b}$. If $\mathcal{Y}$ is Type I, the same is true for $D^{**}$ using Lemma 4.19(iii), (iv). Since no horizontal component of $\text{div}(sa_i\varphi_n^{i-b})$ passes through $y$, we have that $sa_i\varphi_n^{i-b} \in \mathcal{O}_{\mathcal{Y}, y}$ and thus, Lemma 5.3(i) shows that $sa_i\varphi_n^{i-b} \in m_{2,Y}^2$. On the other hand, if $\mathcal{Y}$ is Type II, then Corollary 4.16(iii) shows that the horizontal part of $\text{div}(\varphi_n)$ does not pass through $y$. In this case, Lemma 5.3(ii) shows that $sa_i\varphi_n^{i} \in m_{2,Y}^2$. This concludes the proof of part (i).

For part (ii), Lemma 4.19(i), (ii) show as above that $D^*$ is in the support of $\text{div}(sa_0\varphi_n^{-b})$. If $\mathcal{Y}$ is Type I, then Lemma 4.19(iii) shows that $v_f^{**}(f) = v_f^{**}(a_0)$, so $0 = v_f^{**}(sf/\varphi_n^b) = v_f^{**}(sa_0\varphi_n^{-b})$, meaning that $D^{**}$ is not in the support of $\text{div}(sa_0\varphi_n^{-b})$. Observe further that the horizontal support of $\text{div}(sa_0\varphi_n^{-b})$ does not pass through $y$, regardless of whether $\mathcal{Y}$ is Type I or Type II. This means that we have $sa_0\varphi_n^{-b} \in \mathcal{O}_{\mathcal{Y}, y}$ and by Lemma 5.3(i), we thus have $sa_0\varphi_n^{-b} \in m_{2,Y}^2$ if and only if the multiplicity of $D^*$ in $\text{div}(sa_0\varphi_n^{-b})$ is at least 2.

By Corollary 2.11 and Lemma 3.18, the multiplicity of $D^*$ in the special fiber is $N_n e^*$, so its multiplicity in $\text{div}(sa_0\varphi_n^{-b})$ is $N_n e^* v_f^*(sa_0\varphi_n^{-b})$. Since $v_f^*(sf/\varphi_n^b) = 0, v_f^*(a_0) = v_f^{**}(a_0) = e\lambda_n$ (Lemma 4.19(iii)), and $v_f^*(f) = e\lambda^*$ (Lemma 4.19(i)), we have

$$N_n e^* v_f^*(sa_0\varphi_n^{-b}) = N_n e^* v_f^*(a_0/f) = N_n e^*(e\lambda_n - e\lambda^*) \geq 1.$$

where the inequality follows from Corollary 3.17(iii), and equality holds if and only if $v_f^* = v_f^{-}$. So the multiplicity of $D^*$ in $\text{div}(sa_0\varphi_n^{-b})$ is at least 2 if and only if $v_f^* \neq v_f^{-}$, finishing part (ii).
For part (iii), first suppose \( \mathcal{Y} \) is Type I. Then Lemma 4.19(i), (iii), (iv) show using similar reasoning to part (ii) that \( D^{**} \) is in the support of \( \text{div}(\psi_n^{e-b}) \) but \( D^* \) is not. This proves the first assertion of part (iii). Since the horizontal part of \( \text{div}(\psi_n^{e-b}) \) does not pass through \( y \), the same reasoning as in part (ii) reduces us to showing that the multiplicity of \( D^{**} \) in \( \text{div}(\psi_n^{e-b}) \) is at least 2 if and only if \( v_y^{**} \neq v_y^{'} \).

By Corollary 2.11 and Lemma 3.18, the multiplicity of \( D^{**} \) in the special fiber is \( N_n e^{**} \), so its multiplicity in \( \text{div}(\psi_n^{e-b}) \) is \( N_n e^{**} v_y^{**}(\psi_n^{e-b}) \). Since \( v_y^{**}(sf/\psi_n^{b}) = 0 \) and \( v_y^{**}(f) = e\lambda_n \) (Lemma 4.19(i)), we have

\[
N_n e^{**} v_y^{**}(\psi_n^{e-b}) = N_n e^{**} v_y^{**}(\psi_n^{b}/f) = N_n e^{**} (e\lambda^{**} - e\lambda_n) \geq 1,
\]

where the inequality follows from Corollary 3.17(iv), and equality holds if and only if \( v_y^{**} = v_y^{'} \). So the multiplicity of \( D^{**} \) in \( \text{div}(\psi_n^{e-b}) \) is at least 2 if and only if \( v_y^{**} \neq v_y^{'} \), proving part (iii) in this case.

Now suppose \( \mathcal{Y} \) is Type II. Then \( \psi_n^{e-b} = \psi_n^{e} \), and by Corollary 4.16(iii), the horizontal part of \( \text{div}(\psi_n^{e}) \) does meet \( y \). By Proposition 4.17, \( s \) can be taken to be an eth power in \( K[x] \). Since \( e \geq 2 \), we have \( \psi_n^{e} \in \mathfrak{m}^e \subseteq \mathfrak{m}^2 \), finishing the proof of part (iii).

Lastly, by the proofs of parts (ii) and (iii), if \( v^{'} = v^{'} \) and \( v^{**} = v^{''} \), then \( \text{div}(\psi_n^{e-b}) = D^{**} \) and \( \text{div}(sa\psi_n^{e-b}) = D^* \) in Spec \( \hat{O}_{\mathcal{Y},y} \). Applying Lemma 5.3(iii) completes the proof of part (iv).

\[ \square \]

**Lemma 5.6** Assume the Type III model \( \mathcal{Y} \) of \( \mathbb{P}^1_k \) exists. Let \( s \) be as in Proposition 4.17(ii), and write \( sf = \psi_n^{e} + sae_{-1}\psi_n^{e-1} + \cdots + sa \) for the product of \( s \) with the \( \varphi_n \)-adic expansion of \( f \). Then

\( \begin{align*}
(\text{i}) & \quad v^{'}_{n-1}(\psi_n^{e}) = v^{''}_{n-1}(\psi_n^{e}) = 0, \\
(\text{ii}) & \quad v^{'}_{n-1}(sa\psi_n^{l}) > 0 \quad \text{and} \quad v^{''}_{n-1}(sa\psi_n^{l}) > 0 \quad \text{for} \quad 0 \leq i < e.
\end{align*} \)

**Proof** By Proposition 4.15(iii), the divisor \( D_n \) (which is locally the same as \( \text{div}(sf) \)) meets the intersection of the \( v^{'}_{n-1} \) and \( v^{''}_{n-1} \)-components of the special fiber of \( \mathcal{Y} \). Thus \( v^{'}_{n-1}(sf) = v^{''}_{n-1}(sf) = 0 \). So it suffices to show that, for \( 0 \leq i < e \), both \( v^{'}_{n-1}(\psi_n^{e}) < v^{''}_{n-1}(sa\psi_n^{l}) \) and \( v^{''}_{n-1}(\psi_n^{e}) < v^{''}_{n-1}(sa\psi_n^{l}) \), or equivalently, that

\[
v^{'}_{n-1}(\psi_n^{e}) < v^{''}_{n-1}(a_i\psi_n^{l}) \quad \text{and} \quad v^{''}_{n-1}(\psi_n^{e}) < v^{''}_{n-1}(a_i\psi_n^{l}). \tag{5.2}
\]

Fix \( i \) such that \( 0 \leq i < e \). We first claim that

\[
v^{'}_{n-1}(\psi_n^{e}) < v^{''}_{n-1}(a_i\psi_n^{l}). \tag{5.3}
\]

By Lemma 2.1, \( v_y(\psi_n^{e}) \leq v_y(a_i\psi_n^{l}) \). Since \( \deg(a_i) < \deg(\psi_n) \), we have \( v_{n-1}(a_i) = v_y(a_i) \). On the other hand, applying Lemma 2.1 to \( \varphi_n \) and \( v_{n-1} \) for the equality below, we have

\[
v^{'}_{n-1}(\varphi_n) = e_{n-1}\lambda_{n-1} < v_y(\varphi_n), \tag{5.4}
\]

where \( e_{n-1} = \deg(\varphi_n)/\deg(\varphi_{n-1}) \). Write \( \delta = v_y(\varphi_n) - v_{n-1}(\varphi_n) \). Since \( e > i \), we have

\[
v^{'}_{n-1}(\varphi_n) = v_y(\varphi_n) - e\delta < v_y(\varphi_n) - i\delta \leq v_y(a_i\varphi_n^{l}) - i\delta = v_{n-1}(a_i\varphi_n^{l}), \tag{5.5}
\]

proving (5.3).
Now, write $\varphi_n = \varphi_{n-1}^{e_n-1} + b_{n-1}\varphi_{n-1}^{e_n-1} - \cdots + b_0$ for the $\varphi_{n-1}$-adic expansion of $\varphi_n$, and recall from Lemma 2.1 that

$$v_{n-1}(\varphi_n) = v_{n-1}(\varphi_{n-1}^{e_n-1}) = v_{n-1}(b_0).$$

Furthermore, the term whose valuation decreases the most upon replacing $v_{n-1}$ with $v_{n-1}'$ is $\varphi_{n-1}^{e_n-1}$ and the term whose valuation increases the least upon replacing $v_{n-1}$ with $v_{n-1}''$ is $b_0$ (since it does not increase at all), Thus,

$$v_{n-1}'(\varphi_n) = v_{n-1}'(\varphi_{n-1}^{e_n-1})$$

and

$$v_{n-1}''(\varphi_n) = v_{n-1}''(b_0).$$

Let $c$ be the degree of $\varphi_{n-1}$ in the $\varphi_{n-1}$-adic expansion of $a_i\varphi_n$, and note that

$$c < e_{n-1}e,$$

since $\deg(a_i\varphi_n^i) < \deg(\varphi_n^e)$. Then, we have

$$v_{n-1}'(\varphi_n^e) = v_{n-1}'(\varphi_{n-1}^{e_n-1}) = v_{n-1}(\varphi_{n-1}^{e_n-1} - e_{n-1}e(\lambda_{n-1} - \lambda_{n-1}') \leq v_{n-1}(a_i\varphi_n^i)$$

and

$$v_{n-1}''(\varphi_n^e) = v_{n-1}''(b_0) = v_{n-1}(b_0) \leq v_{n-1}(a_i\varphi_n^i) \leq v_{n-1}(a_i\varphi_n^i).$$

This proves (5.2), and thus the lemma. □

5.3 The minimal embedded resolution

In this subsection, we prove Theorem 4.3, and then extend that result to a proof of Theorem 1.1.

Proposition 5.7 If $\mathcal{Y}$ is the Type III model of $\mathbb{P}^1_k$, then $D_a$ is not regular on $\mathcal{Y}$.

Proof By Proposition 4.15(iii), $D_a$ meets the intersection $y$ of the $v_{n-1}'$- and $v_{n-1}''$- components of the special fiber of $\mathcal{Y}$. Let $D'$ and $D''$ be the respective corresponding Weil prime divisors on $\mathcal{Y}$.

Let $s$ be as in Lemma 5.6. Write $f = \varphi_n^e + ae_{e-1}\varphi_n^{e-1} + \cdots + a_0$, and set $ae = 1$. By Proposition 4.17(iii), $sf$ cuts out $D_a$ locally, so by Proposition 5.4, it suffices to show that $s\varphi_n^i \in m^2_{\lambda,\mathcal{Y}}$, for $0 \leq i \leq e$. By Lemma 5.6, neither $D'$ nor $D''$ appears with a negative coefficient in any div($s\varphi_n^i$).

Recall that in Spec $\mathcal{O}_{\mathcal{Y},y}$, the support of $s$ is contained in the special fiber and, by Corollary 4.16(iii), $y$ is in the support of the horizontal part $D_{a_n}$ of div($\varphi_n$). Since $e \geq 2$, the divisor of $s\varphi_n^i$ is at least $eD_{a_n} \geq 2D_{a_n}$. By Lemma 5.3(i), $s\varphi_n^i \in m^2_{\lambda,\mathcal{Y}}$. If $0 \leq i \leq e - 1$, Lemma 5.6(ii) shows that both $D'$ and $D''$ lie in the support of div($s\varphi_n^i$). We again use Lemma 5.3(i) to conclude that $s\varphi_n^i \in m^2_{\lambda,\mathcal{Y}}$. □

Corollary 5.8 If $\mathcal{Y}$ is a non-trivial regular contraction of $\mathcal{Y}_{\mathcal{Y}}^{\text{reg}}$ on which $D_a$ is regular, then $\mathcal{Y}$ is Type I or Type II.

Proof Suppose $\mathcal{Y}$ is a non-trivial regular contraction of $\mathcal{Y}_{\mathcal{Y}}^{\text{reg}}$ that is not Type I or Type II. Then in the language of Proposition 3.10 and Corollary 3.13 applied to $\mathcal{Y}$, the model $\mathcal{Y}$ includes none of the $w_{n-1,\lambda}$ or $v_{n,\lambda}$. Thus $\mathcal{Y}$ is dominated by the unique Type III model $\mathcal{Z}$ of $\mathbb{P}^1_k$, given that $\mathcal{Z}$ includes exactly those valuations in $\mathcal{Y}_{\mathcal{Y}}^{\text{reg}}$ that are not among the $w_{n-1,\lambda}$ or $v_{n,\lambda}$. By Proposition 5.7, $D_a$ is not regular on $\mathcal{Z}$. By Lemma 5.1, $D_a$ is therefore not regular on any regular contraction of $\mathcal{Z}$, which finishes the proof. □
Proof By Corollary 5.8, we may assume that $\mathcal{Y}$ is either Type I or Type II. We show that if $\mathcal{Y}$ is Type I (resp. Type II), then $D_\alpha$ is regular on $\mathcal{Y}$ if and only if $\mathcal{Y}$ includes $v'_f$ or $v''_f$. This yields the proposition.

Let $y \in \mathcal{Y}$ be the point where $D_\alpha$ meets the special fiber, and let $s$, $b$, and the $a_i$ be as in Lemma 5.5. By Propositions 4.17 and 5.4, $D_\alpha$ being regular is equivalent to $sf/\psi_n^b \notin m_{\mathcal{Y},y}^2$. By Lemma 5.5(i), this is equivalent to $s\psi_n^{-b} + sa_0\psi_n^{-b} \notin m_{\mathcal{Y},y}^2$. By Lemma 5.5(ii), (iii), $s\psi_n^{-b} + sa_0\psi_n^{-b} \notin m_{\mathcal{Y},y}^2$ implies either $v^*_f = v'_f$, or $\mathcal{Y}$ is Type I and $v^{**}_f = v''_f$. If $\mathcal{Y}$ is Type II, the reverse implication also follows from Lemma 5.5(iii), (iii), and if $\mathcal{Y}$ is Type I, the reverse implication follows from Lemma 5.5(iv). We have shown that $D_\alpha$ is regular if and only if $v^*_f = v'_f$ or $\mathcal{Y}$ is Type I and $v^{**}_f = v''_f$. By the definition of $v^*_f$ and Type I/II models, $v^*_f = v'_f$ is equivalent to $v'_f$ being included in $\mathcal{Y}$. Likewise, if $\mathcal{Y}$ is Type I, then $v^{**}_f = v''_f$ is equivalent to $v''_f$ including $v''_f$. This finishes the proof.

Since $\mathcal{Y}^\text{reg}_{v'_f,0}$ is a blowup of $\mathcal{Y}^\text{reg}_{v'_f}$ (and similarly for $\mathcal{Y}^\text{reg}_{v''_f,0}$), the following corollary is immediate.

Corollary 5.10 The divisor $D_\alpha$ is regular on $\mathcal{Y}^\text{reg}_{v'_f}, \mathcal{Y}^\text{reg}_{v''_f}, \mathcal{Y}^\text{reg}_{v'_f,0},$ and on $\mathcal{Y}^\text{reg}_{v''_f,0}$.

We now have the main result of the paper.

Proof of Theorem 4.3 By Corollary 5.10, both $\mathcal{Y}^\text{reg}_{v'_f,0} \rightarrow \mathcal{X}$ and $\mathcal{Y}^\text{reg}_{v''_f,0} \rightarrow \mathcal{X}$ are embedded resolutions of $\text{div}_0(f)$. Since $\mathcal{Y}^\text{reg}_{v'_f,0}$ and $\mathcal{Y}^\text{reg}_{v''_f,0}$ are both contractions of $\mathcal{Y}^\text{reg}_{v'_f,0}$, the minimal embedded resolution is as well. By Corollary 5.8, the minimal embedded resolution includes either $v'_f$ or $v''_f$. It obviously includes $v_0$ as well, so it is either $\mathcal{Y}^\text{reg}_{v'_f,0}$ or $\mathcal{Y}^\text{reg}_{v''_f,0}$. In particular, one of these models dominates the other, and the dominated one is the minimal embedded resolution.

Suppose $e(v'_f/v_0) \leq e(v''_f/v_0)$ as in part (i). Since $v'_f \prec v''_f$, Proposition 3.14 applied to $v'_f$ shows that $v''_f$ is not included in $\mathcal{Y}^\text{reg}_{v'_f,0}$, which shows that $\mathcal{Y}^\text{reg}_{v'_f,0}$ is the dominated one, thus proving the theorem. If $e(v'_f/v_0) > e(v''_f/v_0)$ as in part (ii), then the same proposition applied to $v''_f$ shows that $v'_f$ is not included in $\mathcal{Y}^\text{reg}_{v''_f,0}$, showing that $\mathcal{Y}^\text{reg}_{v''_f,0}$ is the dominated one, again proving the theorem.

Remark 5.11 Given Theorem 4.3 and Remark 5.12, and assuming $k$ is algebraically closed, one can construct a minimal embedded resolution of $(\mathbb{P}^1_{\mathcal{O}_K}, \text{div}_0(f))$ for arbitrary square-free $f \in \mathcal{O}_K[x]$ as follows.

First, one can always make a change of variables by taking some $\gamma \in \text{PGL}_2(\mathcal{O}_K)$ such that the zeroes of the rational function $f(\gamma x)$ all lie in $\mathcal{O}_K$. Replacing $f$ by the numerator of $f(\gamma x)$, we may thus assume that all roots of $f$ lie in $\mathcal{O}_K[x]$. Letting $\pi_K$ be a uniformizer of $K$, we then have the irreducible factorization $f = \pi_K^bf_1 \cdots f_r \in \mathcal{O}_K[x]$, where all $f_i$ monic and distinct, $\pi_K$ is a uniformizer of $K$, and $b \in \{0, 1\}$, since $f$ is squarefree. Let $\mathcal{Y}$ be the minimal embedded resolution of $(\mathbb{P}^1_{\mathcal{O}_K}, \text{div}_0(f))$, and let $\mathcal{Y}'$ be the minimal normal model of $\mathbb{P}^1_{K}$ dominating all $\mathcal{Y}_i$. Then $\mathcal{Y}'$ is regular (see, e.g., [8, Lemma 5.3]), and the minimal embedded resolution of $(\mathbb{P}^1_{\mathcal{O}_K}, \text{div}_0(f))$ is the minimal blowup $\mathcal{Y} \rightarrow \mathcal{Y}'$ separating the strict transforms of $\text{div}_0(\pi_K)$ and the $\text{div}_0(f_i)$ on $\mathcal{Y}'$. Thus, neither the irreducibility nor
the monicity of \( f \) is a serious condition, but the statement of Theorem 4.3 is much cleaner when they are in place.

**Remark 5.12** Regular resolutions satisfy étale descent. That is, if \( L/K \) is an unramified, algebraic field extension and \( f \in O_K[x] \) is a monic irreducible polynomial, then \( \mathcal{Y} \) is an embedded resolution of \( (\mathbb{P}^1_{O_K}, \text{div}_0(f)) \) if and only if \( Z := \mathcal{Y} \times_{O_K} O_L \) is an embedded resolution of \( (\mathbb{P}^1_{O_L}, \text{div}_0(f)) \), in which case we have \( Z \cong (\mathcal{Y} \times_{O_K} O_L)/\text{Gal}(L/K) \). Moreover, the geometric valuations corresponding to the irreducible components of \( \mathcal{Y} \) are obtained by restricting the Mac Lane valuations included in \( Z \) to \( K(x) \).

**Proof of Theorem 1.1** Suppose \( K \) is a complete discrete valuation field with perfect residue field \( k \), and that \( f \in K[x] \). If \( K^{ur} \) is the completion of the maximal unramified extension of \( K \), then Theorem 4.3 and Remark 5.11 allow us to construct the minimal regular resolution \( Z \) of \( (\mathbb{P}^1_{O_{K^{ur}}}, \text{div}_0(f)) \). To explicitly present the minimal regular resolution of \( (\mathbb{P}^1_{O_{K^{ur}}}, \text{div}_0(f)) \) as a collection of geometric valuations, simply let \( \mathcal{Y} \) be the normal model of \( \mathbb{P}^1_K \) corresponding to the set of restrictions of all valuations included in \( Z \) to \( K(x) \). This completes the proof of Theorem 1.1.

**Acknowledgements**

The authors would like to acknowledge the hospitality of the Mathematisches Forschungsinstitut Oberwolfach, where they participated in the “Research in Pairs” program that was integral to the writing of this paper. They would also like to thank Dino Lorenzini for useful conversations, and the referees for their thoughtful comments and suggestions to improve the exposition.

**Data availability Statement** Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

**Author details**

1 Baruch College, 1 Bernard Baruch Way, New York, NY 10010, USA; 2 University of Georgia, 452 Boyd Graduate Studies, 1023 D. W. Brooks Drive, Athens, GA 30602, USA.

Received: 16 August 2021 Accepted: 7 March 2022 Published online: 13 April 2022

**References**

1. Conrad, B., Edixhoven, B., Stein, W. J. (p) has connected fibers. Doc. Math. 8, 331–408 (2003)
2. Fernández, J., Guàrdia, J., Montes, J., Nart, E.: Residual ideals of MacLane valuations. J. Algebra 427, 30–75 (2015)
3. Kunzweiler, S., Wewers, S.: Integral differential forms for superelliptic curves (2020). arXiv:2003.12357
4. Liu, Q.: Algebraic geometry and arithmetic curves. In: Oxford Graduate Texts in Mathematics, vol. 6, Oxford University Press, Oxford, Translated from the French by Reinie Erné, Oxford Science Publications (2002)
5. Liu, Q., Lorenzini, D.: Models of curves and finite covers. Compos. Math. 118(1), 61–102 (1999). https://doi.org/10.1023/A:1001417251990
6. MacLane, S.: A construction for absolute values in polynomial rings. Trans. Am. Math. Soc. 40(3), 363–395 (1936)
7. Obus, A., Srinivasan, P.: Conductor-discriminant inequality for hyperelliptic curves in odd residue characteristic (2019). arXiv:1910.02589v1
8. Obus, A., Srinivasan, P.: Conductor-discriminant inequality for hyperelliptic curves in odd residue characteristic (2019). arXiv:1910.02589v2
9. Obus, A., Wewers, S.: Explicit resolution of weak wild arithmetic surface singularities. J. Algebraic Geom. 29(4), 691–728 (2020)
10. Rüth, J.: Models of curves and valuations, Ph.D. Thesis, Universität Ulm (2014). https://doi.org/10.18725/OPARU-3275. https://oparu.uni-ulm.de/xmlui/handle/123456789/3302
11. Rüth, J.: A framework for discrete valuations in Sage. https://trac.sagemath.org/ticket/21869
12. Srinivasan, P.: Conductors and minimal discriminants of hyperelliptic curves: a comparison in the tame case (2019). arXiv:1910.08228
13. The Stacks Project Authors, The Stacks Project. https://stacks.math.columbia.edu

**Publisher’s Note**

Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.