Universal $R$-matrix for esoteric quantum group

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Abstract

The universal $R$-matrix for a class of esoteric (non-standard) quantum groups $\mathcal{U}_q(gl(2N + 1))$ is constructed as a twisting of the universal $R$-matrix $R_S$ of the Drinfeld-Jimbo quantum algebras. The main part of the twisting element $\mathcal{F}$ is chosen to be the canonical element of appropriate pair of separated Hopf subalgebras (quantized Borel’s $B(N) \subset \mathcal{U}_q(gl(2N + 1))$), providing the factorization property of $\mathcal{F}$. As a result, the esoteric quantum group generators can be expressed in terms of the Drinfeld-Jimbo ones.
1 Introduction

Quasitriangular Hopf algebras (quantizations of universal enveloping Lie algebras $U(g)$) and quantum groups (deformations of algebra of functions on Lie groups $Fun(G)$) are the subject of active research during the last decade originated in famous Drinfeld’s report [1]. There are different starting points of the quantum group theory: generators and defining relations [1], $R$-matrix or solution to the Yang-Baxter equation (FRT approach) [3], deformation quantization or star product [4]. Although up to now there is no complete transformation theory of quantum groups, particular studies of twistings are of great importance [5].

In this paper a twisting element $F$ is constructed giving the esoteric quantum group of [6, 7] as a deformation of the Drinfeld-Jimbo quantum group (the standard one) and the corresponding universal $R$-matrix from that of quantum $U_q(gl(2N + 1))$. The question of relation between $R_{FG}$ and $R_S$ via twisting was already discussed in Refs. [8, 9, 10] within the FRT-approach and in terms of matrix 2-cocycles $\chi$ on quantum groups, which are the images of “universal” twisting elements in the fundamental representation of the quantum algebra: $\chi(T^i_j, T^k_l) = F^{ik}_{jl}$. Below we are dealing with a quasitriangular Hopf algebra $A$ only fixing appropriate Hopf subalgebras in it. The latter ones can be considered as mutually dual and the twisting cocycle $F$ is given by the corresponding canonical element $\sum e_i \otimes e^i \in B(N) \otimes B(N)^{\text{op}}_*$ (the subscript “op” means the opposite multiplication).

The Letter is organized as follows. After reminding briefly the basic material on twisting of Hopf algebras (Sec.2), we construct the universal twist for the $U_q(gl(3))$ case, when the Cremmer-Gervais $R$-matrix (found in [11] for $gl(N)$) coincides with the esoteric one. It is shown that knowing the universal twist one can express the FRT-approach generators in terms of the original quantum algebra generators. Sec.4 is devoted to esoteric $U_q(gl(2N + 1))$ for general $N$. The Letter is concluded by outlining few possible applications of twisting element.
2 Twisting of Hopf algebras

A Hopf algebra $\mathcal{A}(m, \Delta, \varepsilon, S)$ with multiplication $m: \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$, coproduct $\Delta: \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$, counit $\varepsilon: \mathcal{A} \to C$, and antipode $S: \mathcal{A} \to \mathcal{A}$ (see definitions in Refs. [1, 3, 12]) can be transformed [13] with an invertible element $F \in \mathcal{A} \otimes \mathcal{A}$, $F = \sum f_i^{(1)} \otimes f_i^{(2)}$, into a twisted one $\mathcal{A}_t(m, \Delta_t, \varepsilon, S_t)$. This Hopf algebra $\mathcal{A}_t$ has the same multiplication and counit maps but the twisted coproduct and antipode

$$\Delta_t(a) = F \Delta_t(a) F^{-1}, \quad S_t(a) = v S_t(a) v^{-1}, \quad v = \sum f_i^{(1)} S(f_i^{(2)}), \quad a \in \mathcal{A}.$$ 

Sometimes it appears to be useful to combine twist with a homomorphism of $\mathcal{A}$. The twisting element has to satisfy the identities

$$(\varepsilon \otimes \text{id})(F) = (\text{id} \otimes \varepsilon)(F) = 1, \quad (1)$$

$${\mathcal{F}_{12}(\Delta \otimes \text{id})(F) = \mathcal{F}_{23}(\text{id} \otimes \Delta)(F), \quad (2)}$$

where the first one is just a normalizing condition and follows from the second relation modulo a non-zero scalar factor.

A quasitriangular Hopf algebra $\mathcal{A}(m, \Delta, \varepsilon, S, \mathcal{R})$ has additionally an element $\mathcal{R} \in \mathcal{A} \otimes \mathcal{A}$ (a universal $R$-matrix) satisfying [1]

$$(\Delta \otimes \text{id})(\mathcal{R}) = \mathcal{R}_{13} \mathcal{R}_{23}, \quad (\text{id} \otimes \Delta)(\mathcal{R}) = \mathcal{R}_{13} \mathcal{R}_{12}. \quad (3)$$

The coproduct $\Delta$ and its opposite $\Delta^{op}$ are related by the similarity transformation (twisting) with $\mathcal{R}$

$$\Delta^{op}(a) = \mathcal{R} \Delta(a) \mathcal{R}^{-1}, \quad a \in \mathcal{A}.$$ 

A twisted quasitriangular quantum algebra $\mathcal{A}_t(m, \Delta_t, \varepsilon, S_t, \mathcal{R}_t)$ has the twisted universal $R$-matrix

$$\mathcal{R}_t = \sigma(F) \mathcal{R} F^{-1}, \quad (4)$$

where $\sigma$ means permutation of the tensor factors: $\sigma(f \otimes g) = (g \otimes f)$. 

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Although, in principle, the possibility to quantize an arbitrary Lie bialgebra has been proved [14], an explicit formulation of Hopf operations remains a nontrivial task. In particular, the knowledge of explicit form of the twisted cocycle is a rare case even for classical universal enveloping algebras, despite of advanced Drinfeld’s theory [15]. Most of such explicitly known twisting elements have the factorization property with respect to comultiplication (cf.(3))

\[(\Delta \otimes id)(F) = F_{23}F_{13} \quad \text{or} \quad (\Delta \otimes id)(F) = F_{13}F_{23}\]

and similar involving \((id \otimes \Delta)\). To satisfy the twist equation, these identities are combined with additional requirements \(F_{12}F_{23} = F_{23}F_{12}\) or the Yang-Baxter equation on \(F\) \[16, 17]\.

In particular, a twisting element can be used to construct a nontrivial tensor product of Hopf algebras \(A\) and \(B\) \[16, 18\]. Given an element \(F \in A \otimes B\), \(F = \sum a_i \otimes b_i\) such that

\[\((\Delta_A \otimes id)(F) = F_{23}F_{13}, \quad (id \otimes \Delta_B)(F) = F_{12}F_{13}\),\]

one can define the twisted tensor product \(A \overset{\tilde{F}}{\otimes} B\) coinciding with \(A \otimes B\) as an algebra and endowed with a new coproduct

\[\Delta_t(a \otimes b) = F_{14}(id \otimes \sigma \otimes id)(\Delta_A(a) \otimes \Delta_B(b))F_{14}^{-1},\]

where \(F_{14} = \sum a_i \otimes 1 \otimes 1 \otimes b_i \in (A \otimes B) \otimes (A \otimes B)\) and the antipode

\[S(a \otimes b) = F^{-1}S_A(a) \otimes S_B(b)F.\]

Taking \(B = A^\ast_{op}\) (the dual with the opposite multiplication) and the canonical element \(F = \sum e_i \otimes e^i\) as a twisting cocycle, one gets \[16\] the dual to the Drinfeld quantum double

\[D(A^\ast) = (A \overset{\tilde{F}}{\otimes} A^\ast_{op})^*\].

Let us note that the element \(F\) can be replaced by \((id \otimes \varphi)(F)\) where \(\varphi\) is a Hopf automorphism of \(B\). Such a modification may be nontrivial if \(A \otimes B\) is embedded into a larger Hopf algebra \(H\) and \(\varphi\) is not extended to an automorphism of entire \(H\). A twisting element of this kind with appropriate Hopf algebras \(A, A^\ast_{op} \subset U_q(sl(2N + 1))\) will be constructed in the next sections.
3 The Cremmer-Gervais universal $R$-matrix for $gl(3)$

From the study of the quantum $sl(N)$ Toda field theory \[11\] the Cremmer-Gervais solution to the Yang-Baxter equation $R_{CG}$ was obtained, which was different from the standard one

$$R_S = q \sum_i e_{ii} \otimes e_{ii} + \sum_{i \neq j} e_{ii} \otimes e_{jj} + \omega \sum_{i < j} e_{ij} \otimes e_{ji}, \quad q = e^\gamma, \quad \omega = q - q^{-1}. \quad (6)$$

The $R$-matrix $R_{CG}$ for $N = 3$ coincides with the Fronsdal-Galindo $R$-matrix of esoteric quantum groups \[3, 4\]

$$R_{CG} = R_S + (p - 1) (e_{11} \otimes e_{22} + e_{22} \otimes e_{33})$$

$$+ (p^{-1} - 1) (e_{22} \otimes e_{11} + e_{33} \otimes e_{22}) + (p^2/q - 1)e_{11} \otimes e_{33}$$

$$+ (q/p^2 - 1)e_{33} \otimes e_{11} + q\nu (e_{32} \otimes e_{12} - p^2/q^2 e_{12} \otimes e_{32}). \quad (7)$$

Here $p$ and $\nu$ are two additional independent deformation parameters.

The quasitriangular Hopf algebra $U_q(sl(3))$ can be defined by the two triples of generators $\{h_i, e_i, f_i\}, i = 1, 2$, subjected to the relations \[1, 2\]

$$q^{h_i} e_j = q^{a_{ij}} e_j q^{h_i}, \quad q^{h_i} f_j = q^{-a_{ij}} f_j q^{h_i}, \quad [e_i, f_j] = \delta_{ij} \frac{q^{h_i} - q^{-h_i}}{q - q^{-1}},$$

$$e_k^2 e_l - (q + q^{-1}) e_k e_l e_k + e_l e_k^2, \quad f_k^2 f_l - (q + q^{-1}) f_k f_l f_k + f_l f_k^2, \quad k \neq l. \quad (8)$$

where the Cartan matrix elements are $a_{ii} = 2, a_{ii+1} = a_{i+1i} = -1$. The coproduct on these generators reads

$$\Delta(h_i) = h_i \otimes 1 + 1 \otimes h_i, \quad \Delta(e_i) = e_i \otimes q^{h_i} + 1 \otimes e_i, \quad \Delta(f_i) = f_i \otimes 1 + q^{-h_i} \otimes f_i. \quad (9)$$

Having introduced elements corresponding to the composite root

$$e_{13} = e_1 e_2 - q e_2 e_1, \quad f_{13} = f_2 f_1 - q^{-1} f_1 f_2,$$

one gets the universal $R$-matrix in the factorized form \[19, 20, 21, 22, 23\]:

$$R_S = q^{t_0} \exp_{q^{-2}}(\omega e_2 \otimes f_2) \exp_{q^{-2}}(\omega e_{13} \otimes f_{13}) \exp_{q^{-2}}(\omega e_1 \otimes f_1), \quad (10)$$
where \( t_0 = \sum_{ij} (a^{-1})_{ij} h_i \otimes h_j \) is the canonical element of the Cartan subalgebra \( \mathcal{H} \otimes \mathcal{H} \) and the q-exponential is

\[
\exp_q(x) = \sum_{n=0}^{\infty} \frac{x^n}{[n; q]!} = \left\{ \prod_{k=0}^{\infty} (1 - (1 - q)xq^k) \right\}^{-1}, \quad [n; q]! = \frac{q^n - 1}{q - 1}. \tag{11}
\]

The new matrix elements of \( R_{CG} \) in Eq. (7) correspond to contributions of \( e_1 \otimes f_2 \) and \( f_2 \otimes e_1 \) in the fundamental representation. Although commuting with each other, the elements \( e_1 \) and \( f_2 \) do not generate independent Hopf subalgebras because \( h_1 \) does not commute with \( f_2 \) nor does \( h_2 \) with \( e_1 \). To overcome this obstacle let us extend \( U_q(\mathfrak{sl}(3)) \) with the central element

\[
C = e_{11} + e_{22} + e_{33}, \quad h_1 = e_{11} - e_{22}, \quad h_2 = e_{22} - e_{33},
\]

and perform a diagonal twist to separate the above mentioned Hopf subalgebras

\[
F^{(1)} = \exp\left(\frac{\gamma}{2}[e_{11} \wedge e_{22} + e_{11} \wedge e_{33} + e_{22} \wedge e_{33}]\right), \quad q = e^\gamma.
\]

The twisted coproducts \( \Delta_t = F \Delta F^{-1} \) of the generators \( \tilde{e}_1 = e_1 q^{\frac{1}{2}(e_{11} + e_{22})} \) and \( \tilde{f}_2 = f_1 q^{-\frac{1}{2}(e_{22} + e_{33})} \) do not contain common elements:

\[
\Delta_t(\tilde{e}_1) = \tilde{e}_1 \otimes q^{2e_{11}} + 1 \otimes \tilde{e}_1, \quad \Delta_t(\tilde{f}_2) = \tilde{f}_2 \otimes 1 + q^{2e_{33}} \otimes \tilde{f}_2.
\]

The Hopf subalgebra \( B(1)_- \) generated by \( \{e_{33}, \tilde{f}_2\} \) appears to be dual but with opposite product \( B(1)^*_o \) to the Hopf subalgebra \( B(1) \) spanned by \( \{e_{11}, \tilde{e}_1\} \). So, the corresponding canonical element is

\[
F^{(2)} = \exp_q^2 (\mu \tilde{e}_1 \otimes \tilde{f}_2) q^{2e_{11} \otimes e_{33}} \tag{12}
\]

with independent parameter \( \mu \). This element can be used for further twisting already twisted \( U_q(\mathfrak{gl}(3)) \). With extra diagonal twist depending on \( (e_{11} - e_{33}) \otimes C \) we get three parameter universal \( R \)-matrix, reduced to \( (\mathfrak{f}) \) in the fundamental representation.

Let us briefly discuss the relations among the FRT-generators of the Drinfeld-Jimbo (standard) quantum algebra and the twisted one. Taking the first factor of \( \mathcal{A} \otimes \mathcal{A} \) in the fundamental representation, we get three \( 3 \times 3 \) matrices

\[
F_{21} = (\rho \otimes \text{id}) \sigma(F), \quad L_S^{(+)} = (\rho \otimes \text{id}) R_S, \quad F_{12} = (\rho \otimes \text{id}) F,
\]
entries of which are expressed in terms of the standard generators \([8]\). Multiplying these matrices one gets the \(L\)-matrix of the FRT-approach

\[ L_{\text{CG}}^{(+)} = F_{21} L_{S}^{(+)} F_{12}^{-1} \]

entries of which are generators of the esoteric quantum algebra, adding the same formulas for

\[ L_{\text{CG}}^{(-)} = (\rho \otimes \text{id}) \sigma (R_{\text{CG}}^{-1}) = F_{21} L_{S}^{(-)} F_{12}^{-1} \]

### 4 Universal R-matrix for esoteric quantum algebra

The \(U_q(sl(2N+1))\) analogue of the \(R\)-matrix \([7]\) \([6,7]\) has quite a few non-zero entries, so we will not write it here. This \(R\)-matrix \(R_{FG}\) can be obtained as the reduction to the fundamental representation of our final result – universal \(R_{FG}\). Instead we start with the Drinfeld-Jimbo quantum algebra. The quasitriangular Hopf algebra \(U_q(sl(2N+1))\) is generated by \(2N\) triples \(\{h_i, e_i, f_i\}\) satisfying relations \([8]\) and having coproducts \([9]\) with \(i = 1, 2, \ldots, 2N\) and three-diagonal Cartan matrix \(\{a_{ij}\}\): \(a_{ii} = 2\), \(a_{i,i+1} = a_{i+1,i} = -1\). Serre relations read

\[ e_\ell^2 e_l = (q + q^{-1}) e_k e_l e_k + e_l e_k^2, \quad l = k \pm 1, \]

for two neighbouring root vectors and \(e_k e_l = e_l e_k\) for distant ones. Similar equalities hold for \(f_i\) generators. The ordered product of \(q\)-exponentials in the universal \(R\)-matrix includes factors corresponding to all positive roots \([19, 20, 21, 22, 23]\):

\[ R = q^{t_0} \prod_{\alpha \in \Phi^+} \exp_q^{-2}(\omega e_\alpha \otimes f_\alpha). \]

Composite root vectors are defined according to

\[ e_{\alpha + \beta} = e_\alpha e_\beta - q e_\beta e_\alpha, \quad f_{\alpha + \beta} = f_\beta f_\alpha - q^{-1} f_\alpha f_\beta, \quad \alpha < \beta . \]

Following the procedure of the preceding section we fix two Hopf subalgebras \(B(N) = \{h_j, e_j; j = i = 1, 2, \ldots, N\}\) and \(B_-(N) = \{h_k, f_k; k = i = N+1, N+2, \ldots, 2N\}\). They
have elements \(h_N\) and \(h_{N+1}\) commuting nontrivially with \(f_{N+1}\) and \(e_N\). Extending \(\mathcal{U}_q(sl(2N+1))\) by the central element \(C = \sum_{i=1}^{2N+1} e_{ii}\) and performing the "diagonal twist" with

\[
\mathcal{F}^{(1)} = \exp\left(\frac{\gamma}{2} H_{N+1} \wedge Z_{N+1}\right), \quad H_{N+1} = \sum_{i=1}^{N+1} e_{ii}, \quad Z_{N+1} = \sum_{k=N+1}^{2N+1} e_{kk},
\]

one gets two independent Hopf subalgebras of \(\mathcal{U}_q(gl(2N+1))\). Their independence can be easily seen from the coproduct on new generators

\[
e_i \rightarrow e_i, \quad e_{i < N} \rightarrow e_N e_i^{2N+1}, \quad f_{N+1} \rightarrow f_{N+1} e^{-2Z_{N+1}}, \quad f_i \rightarrow f_i, \quad i > N + 1,
\]

\[
\Delta_t(e_{N}) = e_N \otimes q^{e_{NN}+H_N} + 1 \otimes e_N, \quad H_N = \sum_{i=1}^{N} e_{ii},
\]

\[
\Delta_t(f_{N+1}) = f_{N+1} \otimes 1 + q^{e_{NN+2}+Z_N} \otimes f_{N+1}, \quad Z_N = \sum_{k=N+2}^{2N+1} e_{kk},
\]

Let us introduce "primed" notations \(i' = 2N+2-i\) and \(\alpha'_j = \alpha_{2N+1-j}, \quad i, j = 1, \ldots, N\), corresponding to reflection of the Dynkin diagram for \(sl(2N+1)\). Now we can identify the Hopf subalgebra \(\mathcal{B}(N)_- = \{e_{kk}, f_{k-1}; k = N + 2, \ldots, 2N + 1\}\) with \(\mathcal{B}(N)_{op}^*\), the dual to \(\mathcal{B}(N) = \{e_{ii}, e_i; i = 1, \ldots, N\}\) having the opposite product. The non-vanishing matrix elements of the pairing between the generators (\(\alpha\) is a simple positive root) are

\[
< e_{ii}, e_{i'i'} >= 1, \quad < e_{\alpha}, f_{\alpha'} >= 1.
\]

Corresponding canonical element is given by the ordered product

\[
\mathcal{F}^{(2)} = \prod_{\alpha \in \Phi^+} \exp_{q^2}(\mu_\alpha e_{\alpha} \otimes g_{\alpha'}) q^{t_0 + H_N \otimes Z_N},
\]

where \(\Phi^+\) is the set of all positive roots of \(sl(N+1)\) and \(t_0 = \sum e_{ii} \otimes e_{i'i'}\). Element \(g_{\alpha'}\) if \(\alpha\) is a simple root and is defined by \(g_{\alpha' + \beta'} = g_{\beta'} g_{\alpha'} - q^{-1} g_{\alpha'} g_{\beta'}\) for the case of composite roots. Because of the inverted ordering in primed roots compared to non-primed ones, \(g_{\alpha'} = f_{\alpha'}\) only for the simple roots. The element \(\mathcal{F}^{(2)}\) involves \(N(N+1)/2\) parameters \(\mu_\alpha\) among which those \(N\) corresponding to the simple roots.
are independent and $\mu_{\alpha+\beta} = \mu_\alpha \mu_\beta$. Actually, $\mathcal{F}^{(2)}$ is not exactly the canonical element but related to it by the transformation of the kind just described (cf. the remark at the end of Section 2). We deliberately make no difference between them so as to simplify the presentation.

Due to the definition of the canonical element it satisfies the factorization properties (6). Hence one can take it as a twisting element $\mathcal{F}^{(2)}$. There is still a part of the centrally extended Cartan subalgebra $\{e_{ii}, i = 1, 2, \ldots, 2N + 1\}$ which is invariant with respect to the composition of two twists $\mathcal{F}^{(1)}$ and $\mathcal{F}^{(2)}$. For $N = 1$ its dimension is two: $\sum_{i=1}^{3} \alpha_i, ([e_{ii}, e_{12}] \wedge e_{23} + e_{12} \wedge [e_{ii}, e_{23}]) = 0$ means $2\alpha_2 = \alpha_1 + \alpha_3$; in the general case that subalgebra is formed by $\{e_{ii} - e_{i'i'}, C; i \leq N\}$ ($C$ is the central element). Hence there is a possibility for additional diagonal twisting with more parameters in resulting quantum algebra:

$$\mathcal{F}^{(3)} = \exp(A^{ik}(e_{ii} - e_{i'i'}) \wedge (e_{kk} - e_{k'k'}) + B^i(e_{ii} - e_{i'i'}) \wedge C)).$$

Finally we arrive to the following

**Proposition.** The esoteric quantum algebra $\mathcal{U}_{FG}(gl(2N + 1))$ defined by the $R$-matrix of the type (4), (5) is a twisting of the quasitriangular Hopf algebra $\mathcal{U}_q(gl(2N + 1))$ with the twisting element $\mathcal{F} = \mathcal{F}^{(3)}\mathcal{F}^{(2)}\mathcal{F}^{(1)}$ where $\mathcal{F}^{(i)}$ are given by expressions (13), (14), (15), and the universal $R$-matrix

$$\mathcal{R}_{FG} = \mathcal{F}_2\mathcal{R}_S\mathcal{F}^{-1}.$$ 

That way twisted esoteric quantum algebra $\mathcal{U}_q(gl(2N + 1))$ and its universal $R$-matrix have $(N + 1)(N + 2)/2$ parameters, which is in accordance with [10].

**5 Conclusion**

The explicit expression of the twist $\mathcal{F}$ has been obtained due to appropriate choice of initial diagonal twist providing two independent Hopf subalgebras. We hope that
a similar procedure could clarify interrelation between the Cremmer-Gervais quantum algebra and $\mathcal{U}_q(gl(N))$ for $N > 3$.

There are various possibilities to use the universal/algebraic twist:

(i) relations among the FTR-approach generators of the twisted and original quantum algebras;

(ii) evaluation of the Clebsch-Gordan coefficients (CGC) of the twisted algebra in terms of the original CGC and the matrix $F = (\rho_\lambda \otimes \rho_\mu)\mathcal{F}$ in the tensor product of the irreducible representations $V_\lambda \otimes V_\mu$;

(iii) explicit construction according to the quantum inverse scattering method of new integrable models corresponding to twisted $R$-matrices in various irreducible representations (cf.\cite{24}).

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