Do zealots increase or decrease the polarization in social networks?

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Zealots are the nodes in a social network who do not change their opinions under social pressure, and have been considered crucial in the study of opinion dynamics on complex networks. Here we show that whether or not their presence affects the social polarization depends sensitively on their topological characteristics, and on the initial conditions of the dynamics. To this end, we first quantify the amount of polarization when nodes of the network can exist in two opposite states. We then consider two types of initial conditions, and two types of zealot assignments. Using two different random graph models as substrates, we study polarization dynamics as the fraction of zealots is varied for all four combinations. Our results indicate that zealots are effective only when they are chosen uniformly randomly in the network, and the random initial conditions are used. Even when one of these conditions is violated, increasing zealots’ fraction does not make any substantial change in the polarization values.

I. INTRODUCTION

Social polarization is being studied at an increasing rate by researchers from various fields in recent years [1–4]. This is partly because of an availability of technological tools that make possible gathering and analyzing data about social systems on an unprecedented scale [5–6]. At the same time, it is also becoming clear that the same tools are causing increase in the polarization [7–9]. Polarization in social systems can lead to a number of undesirable effects on democratic institutions, and can lead to a biased decision making. Moreover, in a polarized society, false information can propagate relatively easily, which in turn leads to increase in the intolerance to opposing views and segregation of ideologies. Thus, it has become an indispensable necessity to gain a proper understanding of its emergence and stability.

In this paper, we study an effect of zealotry on the emergence of polarization in social networks. Zealots are the nodes in a social network who never change their opinion, and are considered highly influential in the opinion dynamics on complex networks [10–16]. To support this hypothesis, past studies about zealotry have used models with random initial conditions (RICS) where every person in the network has some opinion initially, and it is independent of the opinions of its neighbors [11–14]. In contrast, many disputes in real-social networks originate on a small subset of nodes (called seeds), and then spread. Such ‘seed initial conditions’ or SICs have recently been shown to lead to considerably different results [17]. This is probably because they effectively lead to initial conditions with correlated node states. Hence it is important to study the effect of zealots with SICs in opinion dynamics on networks. We note that the idea of propagation starting from different seeds has been considered in other situations also [18, 19].

There is another aspect to zealotry that has not been analyzed as per our knowledge: the topology based zealotry. The past studies on zealotry in complex networks have assumed that the probability for a vertex to be a zealot is independent of its network characteristics like its degree or its local clustering. As we show in the following pages, the topology based zealotry leads to considerably different outcomes than the case where zealots are randomly chosen. Thus, in this paper we have two types of initial conditions (RICS vs SICs) and two types of zealotries (Uniform vs Topology based), or four combinations in total. We find that, in only one of these cases the value of the fraction of zealots makes a significant impact on the polarization, while in the other three cases, zealots don’t affect it much.

One of the novel aspects of our work is a new method for the quantification of the network polarization when the vertex states are known. The polarization related studies so far have relied on the fractions of vertices with different opinions to quantify it. However, this misses an important information about how they are connected to each other. Our quantification takes into account both: the fractions as well as the way vertices with different opinions are connected to each other. We argue that this leads to a quantification of polarization that is consistent with our intuitive notions.

The rest of the paper is organized as follows. In Sec. [II] we briefly review a simple majority rule model introduced in [17], to be used in the rest of the paper. In Sec. [IIA] we discuss the idea of quantifying polarization in networks based on the fractions of opposite opinions and their inter-connectivity. The effect of zealots is studied in Sec. [II] with respect to RICs and SICs using the Erdős-Reényi graph as well as the configuration model with a power-law degree distribution. Finally, a degree based zealotry is introduced and studied in Sec. [IV]. We conclude in Sec. [V] with a discussion.

II. THE MODEL

Consider an undirected network with $N$ nodes and $m$ edges. In our model, every node could be in three different states: $+1$, $−1$ or $0$, where $+1$ and $−1$ represent the
opposite opinions while the 0 corresponds to the neutral point of view. At each discrete time step, the state of each node $i$ is updated according to the following majority rule:

$$x_i(t + 1) = \text{sgn}(x_i(t) + \sum_j A_{ij}x_j(t))$$

(1)

Here $x_i(t)$ represents the state of node $i$ at time $t$, and $A_{ij}$ is the $(i,j)^{th}$ element of the adjacency matrix. Also, $\text{sgn}(x)$ is the sign function which takes the value +1 if $x > 0$, the value −1 if $x < 0$ and the value 0 otherwise.

Each node $i$ in the network could be a zealot with a probability $p_z$, which means that if it has a concrete opinion (i.e. if $x_i \in \{-1, +1\}$), then it will never change its state whatever the states of its neighbors.

We study this model for two different types of initial conditions: ‘random initial conditions’ (RICs), in which every node is initially in one of the two states $\{-1, +1\}$ uniformly randomly, and the ‘seed initial conditions’ (SICs) in which all nodes are in 0 state except two nodes which have exactly opposite states. We find that almost in all cases, the network quickly stabilizes with the state of each node becoming constant in time. Depending on the initial conditions and other parameter values, this equilibrium state could be homogeneous (all the nodes in the same state) or heterogeneous.

**A. What constitutes a polarization?**

We want to quantify the asymptotic equilibrium states of a network with respect to the amount of polarization. One way to do this is to measure the fraction of nodes with each state. In our previous study [17], we used this idea to define the quantifier $R$ (called ‘the polarization index’) which assigns values between 0 to 1 to network states:

$$R = 1 - 4(n^- - 0.5)^2$$

(2)

Here, $n^-$ represents the fraction of nodes with state $−1$, and $R \in [0, 1]$. It can be easily verified that this definition assigns $R = 0$ to the homogeneous states, whereas the states with roughly equal numbers of $+1$ and $−1$ nodes are considered highly polarized, and are assigned values close to 1. A large number of studies related to binary opinion dynamics use similar quantifiers for polarization based only on the fraction of nodes in each group [11, 20–22]. We point out that the aim here is to quantify the polarization of a network as opposed to the polarization of a topic which has received an increasing amount of attention in recent years [18, 19, 21].

However, there exists another aspect to social polarization which is not captured by $R$: in a polarized social network, nodes with similar opinions are usually observed to be preferentially connected to each other. We can measure this tendency using the assortativity coefficient $r$ of the network with respect to the node states [23, 24]. If the connected nodes tend to have similar opinions, $r$ would have a positive value, whereas if the opposite is true, $r$ would have a negative value. If no such tendency exists, $r$ would be close to 0. It makes sense then to quantify the polarization using both $R$ and $r$ together. Fig. 1 shows four network states for the Zachary karate network:

1. Small $R$, small $r$: A state dominated by only one type (+1 or $−1$), and the similar nodes are not preferentially connected. This we would intuitively label as low polarization state.

2. Small $R$, large $r$: A state dominated by only one type, but the similar nodes are preferentially connected to each other forming a “fringe” group. Here we would say that this state has moderate polarization.

3. Large $R$, small $r$: Roughly equal numbers of two node type exist, but since similar nodes don’t preferentially connect to each other, no big extreme groups are formed. Thus, we would label this as a state with moderate polarization.

4. Large $R$, large $r$: A state with roughly equal numbers of the two types of node values, and connected nodes tend to have similar values. Here two extreme groups with opposite opinions are formed leading to a high polarization.

Accordingly, a simple way to assign “correct” polarization values to network states is to define the polarization index to be the product of $R$ and $r$:

$$\phi = R \times r$$

(3)

Note that the minimum value of $\phi$ is not 0 since $r$ could have a negative value if the connected nodes prefer to possess opposite opinions, although this is rare in practice, and for the dynamics we study here, only small negative values might occur. In this setting, a network state in which every node is assigned $+1$ and $−1$ randomly would have high value of $R$, but since $r$ would be close to 0, the value of $\phi$ would not be high.

The incorporation of the network structure into the quantification of polarization has been discussed in [25], but has not been actually implemented. Our approach avoids calculation of high modularity partitions [21], and also improves over the method of using community boundaries proposed in [26] because we don’t need to explicitly identify the groups of nodes to calculate the polarization. This is a great advantage since one can then quantify the network states when no obvious groups exist in the first place.

**III. EFFECT OF ZEALOTRY WITH THE TWO TYPES OF INITIAL CONDITIONS**

Now we discuss the effect of presence of zealots with the two types of initial conditions. We will denote the
FIG. 1. A chart showing four types of network states, and the amounts of polarization (low, moderate or high) that we would naturally assign them. See the text for a detailed explanation.

polarization corresponding to the SICs and RICs by $\phi_S$ and $\phi_R$ respectively. In Fig. 2 we show the histograms of $\phi$ for the co-appearance network of characters in the novel Les Miserables [27]. Plots in the figure clearly show that when SICs are used, increasing the fraction of zealots has almost no effect on the final value of the polarization. On the other hand, for RICs, a high zealot fraction quenches the polarization.

Causal reasoning behind this can be understood by the following argument. In case of RICs, when the fraction of zealots is high, opinions’ influences cannot freely propagate through the network as many paths are blocked by zealots. Hence, there are no “long range effects” since a given node can neither influence nodes that are far away from it nor gets influenced by them. This means that an initial random assignment of opinions almost “stays put” asymptotically, and since these assignments are random, the value of $r$ remains close to zero. Hence even when the numbers of the two types of nodes are comparable (implying $R \approx 1$), the polarization $\phi = R \times r$ has a small value. On the other hand, with SICs, almost all the nodes are neutral initially, and zealots start blocking the propagation only after they pick up a concrete opinion. However, by this time, the majority pressure anyway almost locks the system to a steady state, and hence the zealots can’t influence it much. We demonstrate the difference between the SICs and RICs using three different random graph models: the Erdős-Rényi graph, the configuration model with a power-law degree distribution, and the stochastic block model.

A. Erdős-Reényi network

Erdős-Reényi network or ER network is constructed by starting with $N$ isolated vertices, and then connecting every pair of vertices with probability $c/N$ where $c$ is a parameter. It is easy to see that $c$ is the value of the average degree for the constructed graph. Also, the degree values of vertices in the ER graph are Poisson distributed, and hence the vertices have degree values close to $c$. The variations of the average value of the polarization $\langle \phi \rangle$ with the zealot fraction $p_z$ for different values of $c$, are shown in Fig. 3. As can be seen from the figure, as expected, average polarization $\langle \phi \rangle$ is almost unaffected for SICs, whereas for RICs, it steadily decreases as the number of zealots is increased. Since increasing $c$ for a given size

FIG. 2. Distributions of $\phi$ with SICs and RICs for different zealot densities $p_z$ for the Les Miserables co-appearance network [27]. The average polarization $\langle \phi \rangle$ is seen to be unaffected for SICs whereas drops to 0 for RICs for high values of $p_z$.

FIG. 3. Variation of the average value of the polarization $\langle \phi \rangle$ with $p_z$ for the Erdős-Rényi graph with SICs (top panel) and RICs (bottom panel) for different values of the average degree $c$. The network size in each case is $N = 5000$, and averages are obtained using 1000 random realizations.
of the network amounts to increasing its link density, it is clear then from the same figure that the higher link density leads to lesser average polarization.

B. Configuration model with a power-law degree distribution

The ER network is homogeneous in the sense that its degree distribution is peaked around the average value. To check the effect of degree heterogeneity, we use the configuration model with a power-law degree distribution. In a nutshell, the configuration model is constructed by randomly connecting fixed number $N$ of vertices to each other such that the degree values in the constructed graph come from a prescribed degree sequence [24]. The degree sequence is usually drawn from some distribution.

Since our interest is in studying the effect of degree heterogeneity, we draw a degree sequence from a power-law distribution ($p(k) \sim k^{-\alpha}$) with a minimum degree value $k_{\text{min}}$. The scaling-index $\alpha$ determines the degree heterogeneity, smaller $\alpha$ implying higher heterogeneity. The variation of the average value of $\langle \phi \rangle$ for this case is shown in Fig. 4. Similar to the ER network case, the SICs are seen to be almost unaffected by the presence of zealots, whereas for the RICs, the average polarization eventually drops to zero.

IV. DEGREE-BASED ZEALOTRY

An interesting variation over the uniform assignment of zealotry to the vertices is to bias it towards the high degree vertices. In particular, we make zealots all the vertices with degree above a certain value. In other words, a vertex with degree $k$ is zealot if:

$$k > (1 - z)k_{\text{max}}$$  \hspace{1cm} (4)

where $k_{\text{max}}$ is the maximum degree value in the network, and $z \in [0, 1]$; if $z = 0$, there are no zealots, and if $z = 1$, every vertex is a zealot. Of course, there is no need to restrict ourselves to the zealotry according to the degree, and one can use any other topological property like the local clustering or a centrality.

Fig. 5 shows results of using the degree-based zealotry for the configuration model with the power-law degree distribution. It should be noted that, the actual fraction of zealots in the network is smaller than $z$ because the degree distribution is a power-law. Curves in the figure indicate that over a large range of $z$, the polarization doesn’t change appreciably for both SICs and RICs. However, when $z = 1$, every vertex is a zealot, and there is a sudden change in the polarization values, especially for RICs. This is understandable because for RICs, when $z = 1$, $R \approx 1$, but the randomly assigned values don’t change after the assignment leading to $r \approx 0$.

V. CONCLUSIONS

We have taken an important step in quantifying the polarization of a social network when the node states are known. Unlike the previous quantifiers which only use the fractions of nodes in each of the discrete states, our quantifier also takes into account how the nodes with different states are connected to each other. Because of
this, for example, when the traditional quantifiers suggest a high value of polarization for a given network, our quantifier shows that polarization could be much smaller because of the way the nodes are connected.

We have also shown that the effectiveness of zealots in social networks is restricted by the initial conditions of the dynamics, as well as their topological properties. To accomplish this, we have introduced a ‘topology based zealotry’, and in particular, we considered its special case in which zealotry depends on the degree. We anticipate that future investigations about the ‘topological zealotry’ will lead to many interesting insights in the opinion dynamics on networks.

The deterministic model of opinion dynamics used here of course is too simple to be of much use in practical contexts, and should only be looked at as a starting point of the investigations about ‘topological zealotry’. Our future studies aim to investigate more realistic models of opinion dynamics on static as well as time-varying networks.

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[1] C. Castellano, S. Fortunato, and V. Loreto, Rev. Modern Phys. **81**, 591 (2009).
[2] J. K. Lee, J. Choi, C. Kim, and Y. Kim, Journal of communication **64**, 702 (2014).
[3] K. Garimella, G. De Francisci Morales, A. Gionis, and M. Mathioudakis, in *Proceedings of the 2017 ACM on Web Science Conference* (ACM, 2017) pp. 263–266.
[4] D. Spohr, Business Information Review **34**, 150 (2017).
[5] A. Menon, in *Proceedings of the 2012 workshop on Management of big data systems* (ACM, 2012) pp. 31–32.
[6] T. Finin, W. Murnane, A. Karandikar, N. Keller, J. Martineau, and M. Dredze, in *Proceedings of the NAACL HLT 2010 Workshop on Creating Speech and Language Data with Amazon’s Mechanical Turk* (Association for Computational Linguistics, 2010) pp. 80–88.
[7] C. R. Sunstein, *#Republic: Divided democracy in the age of social media* (Princeton University Press, 2018).
[8] K. Garimella and I. Weber, arXiv preprint arXiv:1703.02769 (2017).
[9] J. G. Webster, Nw. Ul. Rev. **104**, 593 (2010).
[10] M. Mobilia, Phys. Rev. Lett. **91**, 028701 (2003).
[11] M. Mobilia and I. T. Georgiev, Phys. Rev. E **71**, 046102 (2005).
[12] S. Galam and F. Jacobs, Physica A: Statistical Mechanics and its Applications **381**, 306 (2007).
[13] J. Xie, S. Sreenivasan, G. Korniss, W. Zhang, C. Lim, and B. K. Szymanski, Physical Review E **84**, 011130 (2011).
[14] A. Waagen, G. Verma, K. Chan, A. Swami, and R. D’Souza, Phys. Rev. E **91**, 022811 (2015).

[15] P. P. Klamser, M. Wiedermann, J. F. Donges, and R. V. Donner, Phys. Rev. E **96**, 052315 (2017).
[16] N. Khalil, M. San Miguel, and R. Toral, Phys. Rev. E **97**, 012310 (2018).
[17] S. M. Shekatkar and S. Barve, Europhys. Lett. **122**, 38002 (2018).
[18] A. Morales, J. Borondo, J. C. Losada, and R. M. Benito, Chaos: An Interdisciplinary Journal of Nonlinear Science **25**, 033114 (2015).
[19] K. Garimella, G. D. F. Morales, A. Gionis, and M. Mathioudakis, ACM Transactions on Social Computing, 1, 3 (2018).
[20] J.-M. Esteban and D. Ray, Econometrica: Journal of the Econometric Society, 819 (1994).
[21] M. Conover, J. Ratkiewicz, M. R. Francisco, B. Gonçalves, F. Menczer, and A. Flammini, Icwsm **133**, 89 (2011).
[22] D. Guilbeault, J. Becker, and D. Centola, Proceedings of the National Academy of Sciences **115**, 9714 (2018).
[23] M. E. Newman, Phys. Rev. Lett. **89**, 208701 (2002).
[24] M. E. J. Newman, *Networks* (Oxford university press, 2018).
[25] A. Bramson, P. Grim, D. J. Singer, S. Fisher, W. Berger, G. Sack, and C. Flocken, The Journal of Mathematical Sociology **40**, 80 (2016).
[26] P. H. C. Guerra, W. Meira Jr, C. Cardie, and R. Kleinberg, in *ICWSM* (2013).
[27] D. E. Knuth, *The Stanford GraphBase: a platform for combinatorial computing* (AcM Press New York, 1993).