Abstract

Weighting methods that adjust for observed covariates, such as inverse probability weighting, are widely used for causal inference and estimation with incomplete outcome data. These methods are appealingly interpretable: we observe an outcome of interest on one ‘observation’ sample and wish to know its average over another ‘target’ sample, so we weight the observation sample so it looks like the target and calculate the weighted average outcome. In this paper, we discuss a minimax linear estimator for this retargeted mean estimation problem. It is a weighting estimator that optimizes for similarity, as stochastic processes acting on a class of smooth functions, between the empirical measure of the covariates on the weighted observation sample and that on the target sample. This approach generalizes methods that optimize for similarity of the covariate means or marginal distributions, which correspond to taking this class of functions to be linear or additive respectively. Focusing on the case that this class is the unit ball of a reproducing kernel Hilbert space, we show that the minimax linear estimator is semiparametrically efficient under weak conditions; establish bounds attesting to the estimator’s good finite sample properties; and observe promising performance on simulated data throughout a wide range of sample sizes, noise levels, and levels of overlap between the covariate distributions for the observation and target samples.

Keywords: causal inference, covariate balance, kernel methods, minimax estimation, semiparametric efficiency.

1 Introduction

In this paper, we focus on an estimand that arises frequently in causal inference. We consider an observational study in which we observe for each unit a covariate vector $X_i$, a categorical treatment status $W_i \in 0 \ldots C$, and an outcome
\( Y_i = Y_i^{(W_i)} \in \mathbb{R} \), and assume that as a function of \((X_i, W_i)\), we can calculate indicators \( T_i = T(X_i, W_i) \) that mark units as members of a target group of scientific interest. Our goal will be to estimate the average, over this target group, of the potential outcome \( Y_i^{(0)} \) that they would have been experienced had they received the treatment of interest \( W_i = 0 \).

The well-known problem of estimating a mean outcome when some outcomes are missing is such a problem. In that case, we observe the outcome of interest if we observe an outcome at all and our target group is the entire population we sample from, i.e. we have \( W_i = 0 \) iff we actually observe the outcome \( Y_i \) and \( T_i = 1 \) for all \( i \). However, the flexibility afforded us in this framework to define our target group introduces little additional complexity and can be valuable. For example, if we are wondering whether to recommend a change to treatment \( W_i = 0 \) for those who are above a given age and currently taking another treatment \( W_i = 1 \), it is natural to estimate the average outcome we’d expect to see for that specific group if that recommendation were followed, which we can do by defining \( T_i \) in terms of both \( X_i \) and \( W_i \).

The mean with outcomes missing is identifiable when missingness arises from a strongly ignorable mechanism (see e.g. Rosenbaum and Rubin, 1983). For our more general problem, we make the following assumptions generalizing those that comprise strong ignorability.

**Assumption 1 (Overlap).** The covariate distribution on the target population is dominated by that of the treatment \( W_i = 0 \) population, i.e.

\[
P[X_i \in \cdot \mid T_i = 1] \ll P[X_i \in \cdot \mid W_i = 0].
\]

**Assumption 2 (Unconfoundedness).** Conditional on the covariates, the potential outcome mean is the same for units in the treatment and target groups, i.e.

\[
E[Y_i^{(0)} \mid X_i, W_i = 0] = E[Y_i^{(0)} \mid X_i, T_i = 1].
\]

Under these assumptions, our causal estimand \( E[Y_i^{(0)} \mid T_i = 1] \) is identified as a linear functional of the regression of the observed outcome on covariates and treatment, i.e. it may be written

\[
\psi^c(m) = E[m(X_i, 0) \mid T_i = 1] \quad \text{where} \quad m(x, w) = E[Y_i \mid X_i = x, W_i = w].
\]

We will consider a simple linear estimator of the form \( \hat{\psi}^c = n_T^{-1} \sum_{i=1}^n 1\{W_i = 0\} \hat{\gamma}_i Y_i \) for \( n_T = \sum_{i=1}^n T_i \). In many applications, estimators of this form are desirable for their ease of interpretation: we observe our potential outcome of interest \( Y_i^{(0)} \) on the subsample receiving the treatment \( W_i = 0 \), so we weight that subsample so that it looks like the target subsample and calculate the weighted average outcome. To simplify our discussion, we will not focus directly on the problem of estimating \( \psi^c(m) \), instead devoting our attention to the problem of estimating \( \psi(m) = E[T_i m(X_i, 0)] \) by an estimator of the form \( \hat{\psi} = n^{-1} \sum_{i=1}^n 1\{W_i = 0\} \hat{\gamma}_i Y_i \). The estimator \( \hat{\psi}^c = (n/n_T) \hat{\psi} \) will then be an estimator of the desired form for \( \psi^c(m) \), as \( n/n_T \to (E[T_i])^{-1} = \psi^c(m)/\psi(m) \).
Our estimator $\hat{\psi}_{ML}$ will be a minimax linear estimator of a sample-average version of our estimand, $\hat{\psi}(m) = n^{-1} \sum_{i=1}^{n} T_i m(X_i, 0)$, conditional on the study design $(X_i, W_i)_{i \leq n}$. We choose the weights that result in the best estimate of $\hat{\psi}(m)$ of the form $n^{-1} \sum_{i=1}^{n} \gamma_i Y_i$ in the worst case over regression functions $m(\cdot, 0)$ in an absolutely convex class $\mathcal{F}$ and over conditional variance functions $\text{var}[Y_i \mid X_i = x, W_i = w]$ bounded by a constant $\sigma^2$. This defines our weights $\hat{\gamma} \in \mathbb{R}^n$ as the solution to the convex optimization problem below.

$$
\hat{\gamma} = \arg \min_{\gamma} I_F^2(\gamma) + \frac{\sigma^2}{n^2} \|\gamma\|^2 \quad \text{where} \quad I_F(\gamma) = \sup_{f \in \mathcal{F}} \frac{1}{n} \sum_{i=1}^{n} [1(\{W_i = 0\}) \gamma_i - T_i] f(X_i).
$$

(2)

Here $I_F$ measures how well the $\gamma$-weighted average of a function $f(x)$ observed on the treatment samples matches the average over the target that is our estimand for all functions $f \in \mathcal{F}$. As a result the minimax linear weights enforce the sample balance condition $n^{-1} \sum_{i=1}^{n} 1(\{W_i = 0\}) \gamma_i f(X_i) \approx n^{-1} \sum_{i=1}^{n} T_i f(X_i)$ uniformly over $\mathcal{F}$. Furthermore, as long as this class $\mathcal{F}$ is sufficiently expressible, they will converge in empirical mean square to our functional’s Riesz representer, the unique square integrable function satisfying $E[\gamma \psi(X_i, W_i) f(X_i, W_i)] = \psi(f)$ for all square integrable functions $f(x, w)$,

$$
\gamma_{\psi}(x, w) = 1_{\{w = 0\}} g_{\psi}(x) \quad \text{for} \quad g_{\psi}(x) = \frac{P(T_i = 1 \mid X_i = x)}{P(W_i = 0 \mid X_i = x)}
$$

(3)

whenever this function is bounded (Hirshberg and Wager, 2018).\(^1\) It is conventional to call weights like $\gamma_{\psi}(X_i, W_i)$ inverse probability weights, as they essentially invert the probabilistic mechanism that assigns units to our treatment and target groups to ensure unbiasedness of the infeasible estimator $\psi_\star = n^{-1} \sum_{i=1}^{n} \gamma_{\psi}(X_i, W_i) Y_i$ (see e.g. Hernán and Robins, 2015).

Much has already been said in favor of minimax linear estimators. They perform well in practice in a variety of applications including the missing outcomes problem discussed above (Armstrong and Kolesář, 2018; Imbens and Wager, 2017; Kallus, 2016; Wang and Zubizarreta, 2017; Zubizarreta, 2015). And working in a fixed-design regression setting with independent observations $Y_i = m(Z_i) + \varepsilon_i$ of a function $m(\cdot)$ at fixed points $Z_1 \ldots Z_n$, Donoho (1994) and related papers (Armstrong and Kolesář, 2018; Cai and Low, 2003; Donoho and Liu, 1991; Ibragimov and Khas’minskii, 1985; Johnstone, 2015; Juditsky and Nemirovski, 2009) have established a number of desirable theoretical properties. Among them is that if the regression function $m(\cdot)$ is in a convex set $\mathcal{F}$ and the noise $\varepsilon_i$ is Gaussian, the minimax risk over linear estimators of a linear functional $\psi(m)$ will exceed the minimax risk over all estimators by no more than 25%. This holds on any fixed sample $Z_1 \ldots Z_n$ of any size.

However, this result and many of the others mentioned characterize the behavior of the estimator only when nature saddles us with the least favorable regression function $m_\star \in \mathcal{F}$, the regression function $m \in \mathcal{F}$ that makes estimation

\(^1\)A class $\mathcal{F}$ is sufficiently expressive in this sense if its span is dense in the space $L_2(P)$ of square integrable functions.
of \( \psi(m) \) most difficult. This leaves open the possibility that the minimax linear estimator estimator may be outperformed substantially in the more typical case that \( m \) is not least favorable.\(^2\) Furthermore, as the least favorable function \( m_\star \) in this fixed-design setting is chosen as a function of the design \( Z_1 \ldots Z_n \) to maximize risk, these results are pessimistic even relative to random-design minimax results in which the least favorable function is chosen as a function of the distribution of that design.

In the random-design setting we consider here, these fixed-design results hold for our estimator \( \hat{\psi}_{ML} \) conditional on \( Z_i = (X_i, W_i) \). However, they leave us with a limited understanding of how its behavior depends on the distribution of our observations \( (Y_i, X_i, W_i) \). In random design, the difficulty of estimating \( \psi(m) \) for \( m(x, w) = E[Y_i \mid X_i = x, W_i = w] \) is determined essentially by the regularity of functions \( g_\psi \) and \( m(\cdot, 0) \) that summarize the conditional distributions of \( W_i \) and \( Y_i \) respectively given covariates \( X_i \) (Robins et al., 2009), whereas the fixed-design results depend on that of \( m \) alone.\(^3\) Thus, with the goal of better understanding the behavior of this estimator \( \hat{\psi}_{ML} \), we focus in this paper on bounding its error \( \hat{\psi}_{ML} - \psi(m) \) as a function of the distribution of our observations and in particular as a function of \( g_\psi \) and \( m(\cdot, 0) \).

Focusing on the case that the function class \( \mathcal{F} \) over which \( \psi_{ML} \) is minimax is the unit ball of a reproducing kernel Hilbert space (RKHS), we show several desirable random-design properties for \( \hat{\psi}_{ML} \): (i) asymptotic efficiency under no smoothness assumptions on \( g_\psi \) and relatively weak assumptions on \( m(\cdot, 0) \) and (ii) finite-sample bounds on the estimator’s conditional bias that decay rapidly as a function of the smoothness of both \( g_\psi \) and \( m(\cdot, 0) \).\(^4\) In addition, we demonstrate the estimator’s performance on simulated data for a wide range of sample sizes, noise levels, and levels of overlap between the covariate distributions for the observation and target samples.

\(^2\)Armstrong and Kolesár (2018, Corollary 3.2) also characterize behavior at the constant function \( f(x) = 0 \), which is perhaps too easy to be typical.

\(^3\)A well-known reflection of this phenomenon is that when the treatment assignment mechanism is known, as it is in an experiment, the inverse probability weighting estimator \( \hat{\psi}^* \) based on the true inverse probability weighting function \( g_\psi \) is \( n^{-1/2} \) consistent without any regularity assumptions on \( m(\cdot, 0) \). More quantitatively, the minimax rate for estimating \( \psi(m) \) is determined as much by the regularity of the inverse probability weighting function \( g_\psi \) as it is by that of the regression function \( m(\cdot, 0) \), and there is a regular \( n^{-1/2} \) rate estimator when \( g_\psi \) and \( m(\cdot, 0) \) are Hörder-smooth functions on \( \mathbb{R}^d \) of order \( \beta_\theta \) and \( \beta_m \) if and only if \( \beta_m + \beta_\theta \geq d/2 \) (Robins et al., 2009).

\(^4\)We defer to a later paper consideration of the case that \( m(\cdot, 0) \) is not smooth enough to justify efficiency on its own. In this case, efficiency can be shown for \( \hat{\psi}_{ML} \) by applying our argument in Section 3.1 below to a smoother approximation \( \tilde{m} \) to \( m(\cdot, 0) \), and bounding the degree to which the inaccuracy of this approximation impacts our estimator’s error. As the difference \( \tilde{m} - m(\cdot, 0) \) impacts our estimator through an inner product with \( g_\psi \), a smooth \( g_\psi \) will be approximately orthogonal to the largely nonsmooth difference \( \tilde{m} - m(\cdot, 0) \) and therefore allows for smoothness of \( g_\psi \) to compensate for lack of smoothness of \( m(\cdot, 0) \).
1.1 Understanding the Estimator

As a first step toward understanding the behavior of our estimator, we decompose its error into a bias term and a noise term. We will consider estimation of the sample-average version of our estimand, \( \hat{\psi}(m) := n^{-1} \sum_{i=1}^{n} T_i m(X_i, 0) \), as the behavior of the difference \( \psi(m) - \hat{\psi}(m) \) is out of our hands. We write

\[
\hat{\psi}_{ML} - \hat{\psi}(m) = \frac{1}{n} \sum_{i=1}^{n} 1_{\{W_i = 0\}} \hat{\gamma}_i Y_i - T_i m(X_i, 0)
\]

\[
= \frac{1}{n} \sum_{i=1}^{n} \left[ 1_{\{W_i = 0\}} \hat{\gamma}_i - T_i \right] m(X_i, 0) + 1_{\{W_i = 0\}} \hat{\gamma}_i \varepsilon_i, \quad \varepsilon_i = Y_i - m(X_i, W_i).
\]

(4)

It is clear from this expression that what we are minimizing in (2) to define our weights is, in fact, the mean squared error conditional on \( (X_i, W_i)_{i \leq n} \). Our bias term is \( E[\hat{\psi}_{ML} | X, W] - \hat{\psi}(m) \) and our noise term is \( \hat{\psi}_{ML} - E[\hat{\psi}_{ML} | X, W] \).

Supposing that \( m(., 0) \) is really in the class \( \mathcal{F} \) that our estimator is minimax over, our bias term is bounded by \( I_F(\hat{\gamma}) \). This allows us to bound our bias term using a simple argument (see e.g. Kallus, 2016). We use the property that our regression adjustment. The bias term of the linear estimator with the infeasible weights \( \gamma^\star = \gamma_\psi(X, W) \). Rearranging this condition yields the bound

\[
I_F^2(\hat{\gamma}) \leq I_F^2(\gamma^\star) + \frac{\sigma^2}{n^2} \sum_{i=1}^{n} (\gamma^\star_i)^2 - \hat{\gamma}_i^2.
\]

(5)

As a result, we have \( I_F(\hat{\gamma}) \leq I_F(\gamma^\star) + \sigma \| \gamma_\psi \|_{\infty} n^{-1/2} \). Furthermore, \( I_F(\gamma^\star) \) is the supremum of the empirical process \( n^{-1} \sum_{i=1}^{n} \delta_{X_i, W_i} \), indexed by the class of functions \( \mathcal{H} = \{ [\gamma_\psi(x, w) - T(x, w)] f(x) : f \in \mathcal{F} \} \), and for the same reason that weighting by the Riesz representer \( \gamma_\psi \) results in unbiased estimation, each function in this class has mean zero. Using well known tools from Empirical Process Theory, this supremum can be shown to concentrate at \( n^{-1/2} \)-rate on a quantity comparable to the Rademacher complexity \( R_n(\mathcal{F}) \) of the set of outcome models \( \mathcal{F} \). Consequently, this argument shows that our estimator will be consistent at \( n^{-1/2} \) rate when the class \( \mathcal{F} \) is small enough that \( R_n(\mathcal{F}) = O(n^{-1/2}) \).

However, this is essentially the limit of this argument’s power in this context. While a variant of this argument can be used to show that the bias term of a regression-augmented weighting estimator like the one discussed in Hirshberg and Wager (2018) is \( o_p(n^{-1/2}) \), it cannot be used for the same purpose without regression adjustment. The bias term of the linear estimator with the infeasible weights \( \gamma^\star, n^{-1} \sum_{i=1}^{n} [\gamma_\psi(X_i, W_i) - T_i] m(X_i, 0) \), has mean zero but standard deviation on the order of \( n^{-1/2} \). As a result, the simple argument above, which relies solely on the characterization that our estimator performs as well as this.

\footnote{The relevant tools are the symmetrization technique and the Ledoux-Talagrand Contraction Lemma (see e.g. Giné and Nickl, 2015, Theorems 3.1.21 and 3.2.1).}
infeasible estimator, cannot be used to show that our estimator’s bias term is negligible.

Using more refined arguments, many methods like this one have been shown to control the bias term at $o_p(n^{-1/2})$ rate and as a consequence achieve semi-parametric efficiency. These methods include Empirical Balancing Calibration Weighting (Chan et al., 2015), the Covariate Balancing Propensity Score (Fan et al., 2016), and the Minimal Approximately Balancing Weights (Wang and Zubizarreta, 2017). They differ from ours primarily in that they optimize for some desirable property of the weights subject to bounds on the maximal conditional bias $I_{F_n}$ over a finite-dimensional sieve $F_n$, e.g. polynomials of order $K_n \to \infty$, where we do so for a fixed class $F$. All of these arguments rely on the phenomenon that there are weights that achieve better control on $I_{F_n}$ than the inverse probability weights $\gamma$ do for sufficiently small classes $F_n$. Clearly this is the case for classes $F_n$ of dimension no larger than $n$, as in that case the condition $I_{F_n}(\gamma) = 0$ is a solvable set of linear equations. This phenomenon is quite robust. When $m(\cdot,0)$ and $g_\psi$ are sufficiently smooth, even methods that estimate inverse propensity weights $\hat{\gamma}$ by maximum likelihood within some appropriate sequence of finite-dimensional model classes $G_n$ have been shown to achieve better control on the bias term and therefore semiparametric efficiency (Hirano et al., 2003).

But these approaches do not line up well with the minimax framework we’ve discussed. When working with a finite-dimensional sieve $F_n$, the maximal conditional bias $I_{F_n}(\hat{\gamma})$ over $F_n$ will generally not directly bound the conditional bias at $m(\cdot,0)$. Thus, the decay of the bias in these sieve-based methods is limited by the the error in approximating the regression function $m(\cdot,0)$ and the inverse probability weights $g_\psi$ by functions in the finite-dimensional sieve class $F_n$, which may decay substantially slower in moderate sample sizes than in the asymptotic regime. In particular, the error in approximating a Sobolev-smooth function of $x \in \mathbb{R}^d$ by a function in any $K$-dimensional class can decay as slowly as a polynomial in $1/\log(K)$ in the preasymptotic regime $K \leq 2^d$ (Kuhn et al., 2016).

What we show here is that the use of finite dimensional sieves is not necessary to establish negligible bias — in fact, we can get this by solving the minimax problem (2) over a moderately large infinite-dimensional class of $F$ of possible regression functions. We will focus on the case that $F$ is the unit ball of a Reproducing Kernel Hilbert Space (RKHS), e.g. the Sobolev space $H^k$ of square-integrable functions with square integrable weak partial derivatives of order up to $k > d/2$.

At the heart of our argument will be a characterization of our estimator $\hat{\psi}_{ML}$ as the average, over our target subsample, of a kernel-ridge-regression estimate $\hat{m}(\cdot)$ of the regression function $m(\cdot,0)$ based on the subsample $\{i : W_i = 0\}$ of units receiving the treatment of interest. Specifically,
Lemma 1. If \( F \) is the unit ball of an RKHS with norm \( \| \cdot \| \), then

\[
\frac{1}{n} \sum_{i=1}^{n} 1_{\{W_i = 0\}} \hat{\gamma}_i Y_i = \frac{1}{n} \sum_{i=1}^{n} T_i \hat{m}(X_i) \quad \text{where} \quad \hat{\gamma} = \arg \min_{\gamma} I_F(\gamma) + \frac{\sigma^2}{n^2} \| \gamma \|^2, \quad I_F(\gamma) = \sup_{f \in F} \frac{1}{n} \sum_{i=1}^{n} [1_{\{W_i = 0\}} \gamma_i - T_i] f(X_i); \quad \hat{m} = \arg \min_{m} \frac{1}{n_Z} \sum_{i: W_i = 0} (Y_i - m(X_i))^2 + \frac{\sigma^2}{n_Z} \| m \|^2 \quad \text{where} \quad n_Z = |\{i : W_i = 0\}|.
\]

We use this duality result, proven in Appendix E, to characterize the bias term of our estimator as the bias of this ridge regression estimator, conditional on the design \((X_i, W_i)_{i \leq n}\), averaged over the target subsample. As the weight \( \sigma^2/n_Z \) of the penalty term in our ridge regression is small, \( \hat{m} \) will be fairly unbiased estimator of a regression function \( m \) in our RKHS. Furthermore, what bias there is will be attenuated by averaging over the target sample to a degree that depends on the smoothness of ratio of covariate densities within these two groups, \( P[X_i = x \mid T_i = 1]/P[X_i = x \mid W_i = 0] \propto g_\psi(x) \). Using this argument to characterize our estimator’s bias term and a variant of Theorem 2 from Hirshberg and Wager (2018) to characterize our noise term, we establish finite sample bounds on the error of our estimator.

As a consequence, we obtain simple conditions sufficient for our estimator to be semiparametrically efficient: in essence, our RKHS must be a space of sufficiently smooth functions and our regression function \( m(\cdot, 0) \) must be one of them. Smoothness of \( m(\cdot, 0) \) in excess of the level required to be in the RKHS will then improve the higher order terms in our bound. We will state this result after introducing the necessary definitions in Section 2.

1.2 Related Work

Our approach is motivated as a minimax linear estimator, a type of estimator that has been widely studied in the past several decades (Armstrong and Kolešar, 2018; Cai and Low, 2003; Donoho, 1994; Donoho and Liu, 1991; Ibragimov and Khas’minskii, 1985; Imbens and Wager, 2017; Johnstone, 2015; Juditsky and Nemirovski, 2009; Kallus, 2016; Zubizarreta, 2015). However, while the estimator we study is a minimax linear estimator in fixed design, we focus on its random design behavior in the tradition of covariate balancing or calibration estimators in causal inference (Athey et al., 2016; Chan et al., 2015; Chen et al., 2008; Graham et al., 2012, 2016; Hainmueller, 2012; Hirshberg and Wager, 2018; Imai and Ratkovic, 2014; Kallus, 2016; Wang and Zubizarreta, 2017; Wong and Chan, 2017; Zhao, 2016; Zubizarreta, 2015). Recently Kallus (2016) and Wong and Chan (2017) have also worked in an RKHS setting and characterized the behavior the same estimator and a similar one respectively, establishing consistency at \( O_p(n^{-1/2}) \) rate using essentially the aforementioned argument based on
Our primary contribution relative to this work is a sharper characterization of the estimator’s asymptotic and finite sample behavior.

The analysis of the estimator hinges largely on its equivalent characterization as a plug-in estimator $\psi(\hat{m})$ using kernel ridge regression estimate of the regression function $m(\cdot, 0)$. In this way, we extend the work of (Hsu et al., 2012) on random design ridge regression in Hilbert spaces by focusing on bias in what is essentially a generalization task: the problem of predicting outcomes on target population different from that from which the training sample for the regression was drawn. Recently Newey and Robins (2018) analyzed the behavior of a plug-in $\psi(\hat{m})$ estimator using finite-dimensional sieve least squares estimator $\hat{m}$ in a setting generalizing ours. Making use of orthogonality between factors in the error of their estimator, they established surprising efficiency properties for such an estimator: it is first-order semiparametrically efficient under weakest possible smoothness assumptions on the regression function and furthermore has fast decay in its higher order error terms. Using some similar arguments, we show efficiency for our method under weak assumptions and characterize our estimator’s higher order error terms. Our focus differs largely in that we establish these results as a consequence of finite sample error bounds, avoiding where possible large constant factors and terms with poor preasymptotic decay.

2 Background Information on Reproducing Kernel Hilbert Spaces

Let $\mathcal{X}$ be a compact metric space. A Reproducing Kernel Hilbert Space $\mathcal{H}_K$ of functions on $\mathcal{X}$ is a complete normed vector space with its norm $\|\cdot\|_K$ induced by an inner product $\langle \cdot, \cdot \rangle$ in the sense that $\|f\|^2 = \langle f, f \rangle$ and on which point evaluation is continuous in the sense that for all $x \in \mathcal{X}$ there is a constant $C_x$ such that $f(x) \leq C_x \|f\|$. By the Riesz representation theorem (see e.g. Peypouquet, 2015, Theorem 1.4.1), this implies that each $x \in \mathcal{X}$ corresponds to a unique element $K_x$ in the RKHS such that $f(x) = \langle K_x, f \rangle$. We call the function $K(x, y) = (K_x, K_y)$ the kernel associated $\mathcal{H}_K$. If the kernel is continuous, it is bounded as a consequence of the the compactness of $\mathcal{X} \times \mathcal{X}$, and furthermore $\|\cdot\|_\infty \leq M_K \|\cdot\|$ for the finite constant $M_K = \sup_x K(x, x)$, as by Cauchy-Schwartz $\|f\|_\infty ^2 = \sup_x \langle K_x, f \rangle \leq \sqrt{\sup_x \langle K_x, K_x \rangle} \langle f, f \rangle$.

Given any finite measure $\nu$ with support equal to $\mathcal{X}$, we can completely characterize an RKHS $\mathcal{H}_K$ with a continuous kernel $K$ in terms of the spectral decomposition of a compact positive integral operator

$$L_{K, \nu}(f)(x) = \int K(x, x') f(x') d\nu(x')$$

mapping the space of square integrable functions $L_2(\nu)$ to itself (see e.g. Cucker and Zhou, 2007, Chapter 4). Its eigenfunctions $\{\phi_j\}_{j \in \mathbb{N}}$ form an orthonormal basis for $L_2(\nu)$ and its scaled eigenfunctions $\sqrt{\lambda_j} \phi_j$, where $\lambda_j$ is the eigenvalue corresponding to $\phi_j$, form an orthonormal basis for $\mathcal{H}_K$. One useful consequence
is the square root of our integral operator, the operator $L_{K,\nu}^{1/2}$ mapping $\sum_j f_j \phi_j$ to $\sum_j f_j \sqrt{\nu_j} \phi_j$, maps an orthonormal basis of $L_2(\nu)$ to an orthonormal basis of our $\mathcal{H}_K$, so (i) $\mathcal{H}_K$ is the image of the square integrable functions $L_2(\nu)$ under $L_{K,\nu}^{1/2}$ and (ii) the RKHS inner product satisfies $\langle L_{1/2}^{1/2} f, L_{1/2}^{1/2} g \rangle = \langle f, g \rangle_{L_2(\nu)}\) where the $\langle f, g \rangle_{L_2(\nu)} = \int f(x) g(x) d\nu$ is the standard inner product on $L_2(\nu)$.

Generalizing (i), we can think of the space of square integrable functions $L_2(\nu)$ and our RKHS $\mathcal{H}_K$ as elements of a continuum of spaces, the images $L_{K,\nu}^\kappa((L_2(\nu))$ of $L_2(\nu)$ under powers of $L_{K,\nu}$. Corresponding to these spaces, we define the family of norms $\| f \|_{L_{K,\nu}^\kappa} = \| L_{K,\nu}^{\kappa/2} f \|_{L_2(\nu)}$, with $\| f \| = \| f \|_{L_{K,\nu}^{1/2}}$. This exponent $\kappa$ will be a useful quantitative notion of smoothness.

One familiar scale of spaces like this is the scale of Sobolev spaces $H^s$ of $s$-times weakly differentiable periodic functions on the unit cube endowed with Lebesgue measure $\mu$ (see e.g. Kühn et al., 2014). These spaces have the characterization $H^s = \{ \sum_{k \in \mathbb{Z}^d} f_k (1 + \|k\|^2_2)^{-s/2} e^{2\pi i k \cdot \cdot} : \sum_{k \in \mathbb{Z}^d} |f_k|^2 \leq 1 \}$, with the fourier basis functions as eigenfunctions irrespective of $s$. It is clear from this that if $K_\kappa$ is the kernel of $H^s$, then for all $s'$, $H^{s'}$ is the image of $L_2(\mu)$ under $L_{K_\kappa,\mu}^{s'/s}$ or equivalently the image of $H^s$ under $L_{K_\kappa,\mu}^{(s'/s-1)/2}$.

While our space $\mathcal{H}_K$ itself is not defined with reference to any particular measure $\nu$, many of the the objects discussed above are. One useful relation between operators $L_{K,\nu}$ and $L_{K,\nu'}$ defined in terms of different measures is that for all $\phi$,

$$\langle \phi, L_{K,\nu'} \phi \rangle_{L_2(\nu')} = \int K(x,y) \phi(x) \phi(y) \frac{d\nu'}{d\nu}(x) \frac{d\nu'}{d\nu}(y) \int \phi(x) d\nu(x) d\nu(y)$$

As mentioned in Bach (2017), this identity and the extremal characterization of the eigenvalues $\lambda_{j,\nu}$ and $\lambda_{j,\nu'}$ offered by the Courant-Fischer minimax theorem (see e.g. Horn et al., 1990) imply that $\lambda_{j,\nu} \leq \| d\nu'/d\nu \|_\infty^2 \lambda_{j,\nu'}$ for all $j$. As our finite-sample bounds will depend on the eigenvalues of integral operators defined in terms of the unknown distribution of our data, this phenomenon makes our bounds much less opaque than they otherwise would be. In particular, under weak assumptions the relevant eigenvalues decay at the same rate as those of the operator $L_{K,\mu}$ for Lebesgue measure $\mu$, which can often be calculated straightforwardly. This approach will be used in the proof of Lemma 4.

3 Main Results

Having reviewed these properties, we are prepared to state and prove our results. We’ll start with the asymptotic results.

**Setting** We observe $(X_i, W_i, Y_i)_{i \leq n}$ iid from a distribution $P$ with $m(x, w) = \mathbb{E}[Y_i \mid X_i = x, W_i = w]$ and $v(x, w) = \text{var}[Y_i \mid X_i = x, W_i = w]$. For some

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binary function $T(x, w)$, we define $T_i = T(W_i, X_i)$, and we will assume that $T$ is chosen so that the target and treatment groups overlap in the sense that $g_\psi(x) < \infty P - a.s.$ for $g_\psi$ defined in (3). We consider the estimands $\psi^c(m)$ and $\psi(m)$ defined in the introduction as the sample variant of $\psi(m)$, $\hat{\psi}(m) = n^{-1} \sum_{i=1}^{n} T(X_i, W_i)m(X_i, 0)$. We write $P_Z$ for the distribution of $X_i$ conditional on $W_i = 0$ and assume that its support is a compact metric space $\mathcal{X}$, working with an RKHS $\mathcal{H}_K$ of functions on $\mathcal{X}$ with kernel $K$ and norm $\| \cdot \|$. We will also write $P_T$ for the distribution of $X_i$ conditional on $T_i = 1$.

**Smoothness assumptions** For spaces of functions on subsets of $\mathbb{R}^d$, we measure smoothness by one of two standards, (i) the maximal Sobolev norm $\sup_t \| f \|_{L^1} \| f \|_{H^s}$ of an element of the unit ball of $\mathcal{H}_K$ or (ii) the H"older norm $\| K \|_{C^{2\sigma}}$ of the kernel $K$. We define these norms in Section B of the Appendix.

**Assumption 3.** The unit ball of our RKHS is contained in a ball in the Sobolev space $H^s$, i.e. $\sup_{\| f \|_{H^s} \leq 1} \| f \|_{H^s} < \infty$, for $s > d$.

**Assumption 4.** The kernel $K$ of our space satisfies the H"older-type smoothness condition $\| K \|_{C^{2\sigma}} < \infty$ for noninteger $s > d/2$.

Assumption 4 implies containment of the unit ball of $\mathcal{H}_K$ in a ball in a Hölder space, i.e. $\sup_{\| f \|_{H^s} \leq 1} \| f \|_{C^{2\sigma}} < \infty$ (Cucker and Zhou, 2007, Theorem 5.5.).

**Theorem 2.** In the setting described above, let $\mathcal{H}_K$ be dense in $L_2(P_Z)$ and satisfy Assumption 3 or 4, and let $\| m(\cdot, 0) \|_{\mathcal{H}_K} < \infty$ and either

$$\| m(\cdot, 0) \|_{L^{1/2, \infty}_{\mathcal{K}, P_Z}} < \infty \quad \text{or} \quad \| g_\psi \|_{L^{1/2, \infty}_{\mathcal{K}, P_Z}} < \infty \quad \text{hold for any} \quad \epsilon > 0. \quad (10)$$

Then for any $\sigma > 0$, the estimator $\hat{\psi}_{ML} = n^{-1} \sum_{i=1}^{n} \hat{\gamma}_i Y_i$ with weights $\hat{\gamma}$ defined in (7) is a semiparametrically efficient estimator of $\psi(m)$, with the asymptotic characterization

$$\hat{\psi}_{ML} - \psi(m) = n^{-1} \sum_{i=1}^{n} \nu(X_i, W_i, Y_i) + o_p(n^{-1/2}) \quad \text{where} \quad \nu(x, w, y) = T(x, w)m(x, 0) - \psi(m) + \gamma_\psi(x, w)(y - m(x, 0)). \quad (11)$$

Furthermore, $\hat{\psi}_{ML}^c = (n_T/n)^{-1} \hat{\psi}_{ML}$ is a semiparametrically efficient estimator of $\psi^c(m)$, with the asymptotic characterization

$$\hat{\psi}_{ML}^c - \psi^c(m) = n^{-1} \sum_{i=1}^{n} p_T^{-1} \nu(X_i, W_i, Y_i) + o_p(n^{-1/2}). \quad (12)$$

Thus, $n^{1/2}(\hat{\psi}_{ML} - \psi(m))$ and $n^{1/2}(\hat{\psi}_{ML}^c - \psi^c(m))$ are asymptotically normal with mean zero and variances $V = E \nu(X_i, W_i, Z_i)^2$ and $V' = p_T^{-2}V$ respectively, and given a consistent estimate $\hat{V}$ of $V$, $\hat{\psi}_{ML} \pm z_{\alpha/2} \sqrt{\hat{V}/n^{1/2}}$ and $\hat{\psi}_{ML}^c \pm z_{\alpha/2} (n_T/n)^{-1} \sqrt{\hat{V}' n^{-1/2}}$ are asymptotically valid confidence intervals of level $1 - \alpha$ for $\psi(m)$ and $\psi^c(m)$ respectively. But it is not clear from these asymptotic characterizations (11) and (12) that in any finite sample the $o_p(n^{-1/2})$
remainder term will be small enough that these confidence intervals will be approximately valid. To inform about the magnitude of this remainder, we will now state a nonasymptotic characterization of our estimator’s error.

**Theorem 3.** In the setting described above, consider the estimator \( \hat{\psi}_{ML} = n^{-1} \sum_{i=1}^{n} 1_{(W_i,=0)} \hat{\gamma}_i Y_i \) with weights \( \hat{\gamma} \) defined in (7). Let the decreasing sequences of eigenvalues \( \lambda_{j,T} \) and \( \lambda_{j,Z} \) of \( L_{K,P_T} \) and \( L_{K,P_Z} \) respectively satisfy the bounds \( \lambda_{j,T} \leq C_{\lambda,T} j^{-\alpha}, \lambda_{j,Z} \leq C_{\lambda,Z} j^{-\alpha} \) and the eigenfunctions \( \phi_j \) of \( L_{K,P_Z} \) satisfy the bound \( \| \phi_j \|_{\infty} \leq C_{\phi} \lambda_j^{-\beta/2} \) with \( \alpha > 1 \) and \( \alpha(1-\beta) > 1 \). Define \( \lambda = \sigma^2/n, p_Z = P\{ W_i = 0 \}, p_T = P\{ T_i = 1 \} \). For any \( \eta > 0, \delta \in (0,1) \) and any \( \kappa_m \in [1/2,1], \kappa_g \in [0,1/2] \) such that \( \| m(\cdot,0) \|_{L_{K,P_Z}^{\kappa_m}} < \infty \) and \( \| g_{\psi} \|_{L_{K,P_Z}^{\kappa_g}} < \infty \), with probability \( 1 - 3\delta \),

\[
E[|\hat{\psi}_{ML} - \psi(m)|] \leq \zeta^{-1} p_Z \| m(\cdot,0) \|_{L_{K,P_Z}^{\kappa_m}} + c_1 \lambda^{\kappa_m+\kappa_g} \| g_{\psi} \|_{L_2(P_Z)} + c_2(\eta) \lambda^{\kappa_m+1/2 - 1/\alpha} n^{-1/2} \sqrt{p_Z} \| g_{\psi} \|_{\infty} \log(2\delta^{-1}) + c_3 \lambda^{\kappa_m} n^{-1/2} \sqrt{p_Z} \| g_{\psi} \|_{\infty} \log(2\delta^{-1}) + c_4(\eta) \lambda^{\kappa_m-1/2} n^{-1} M_K \log(2\delta^{-1}) \quad (13)
\]

\[
\zeta = \max \left\{ 0, 1 - \frac{8c_2(\eta) \lambda^{(1-1/\alpha)\eta} \log(4\delta^{-1}n^2)}{np_Z - \sqrt{2np_Z} \log(\delta^{-1})} - \sqrt{16c_2(\eta) \lambda^{-1/\alpha+\eta} \log(4\delta^{-1}n^2)} \right\}
\]

and with probability \( 1 - \exp\{-c_d(\eta_{\eta}) n^{r^2/M_{F_2}} \} - 4\delta \),

\[
\left| \hat{\psi}_{ML} - E[\hat{\psi}_{ML} | X,W] - n^{-1} \sum_{i=1}^{n} \eta_{\psi}(X_i,W_i)(Y_i - m(X_i,0)) \right| \leq \delta^{-1/2} \| \psi \|_{\infty} n^{-1/2} \min\{a, b\}^{1/2};
\]

\[
a = s \left( c_u n^{-1/2} + c_u,1 n^{-1} \right) + \bar{R} + \lambda,
\]

\[
b = 2 \max \left\{ s^2 r^2, \frac{\bar{R} + \lambda}{\eta_{Q} - 2s^{-1} \eta_{C}} \right\};
\]

\[
s = 2\eta_{C} \lambda^{-1/2} \eta_{C}^{-1/2};
\]

\[
r = \max \left\{ 7c_{Q} n^{-\frac{1}{2}} \lambda_{1/2}^{1/2}, \lambda_{1/2}^{1/2} \eta_{Q}^{-1/2} \right\} ;
\]

\[
\bar{R} = c_{1,R} \lambda^{2s} + c_{2,R} n^{-1/2} \lambda^{s};
\]

when \( \lambda \leq c_{1,R}^{(1-2s)} \) and \( \log(\delta^{-1}) \leq \sqrt{12} p_{Z}^{-\eta} (n/M_{F_2})^{1/(\alpha+1)} \).

\[
(14)
\]

Here the constants \( c_\cdot, M_\cdot \) may be functions of \( \delta \) but not of \( n \) or \( \lambda \). They are defined in Appendix A.

Via the triangle inequality, the sum of these two bounds is a bound on the magnitude of the remainder, i.e. the deviation of our estimator from our
idealized asymptotic characterization $\psi(m) + n^{-1/2} \sum_{i=1}^{n} \ell(X_i, W_i, Y_i)$ in (11), as

$$\hat{\psi}_{ML} - \tilde{\psi}(m) - n^{-1} \sum_{i=1}^{n} \gamma_{\psi}(X_i, W_i)(Y_i - m(X_i, 0)) = \hat{\psi}_{ML} - \psi(m) - n^{-1} \sum_{i=1}^{n} \ell(X_i, W_i, Y_i).$$

Thus, the claim (11) made by Theorem 2 holds if the two bounds (13) and (14) are $o(n^{-1/2})$ for all $\delta > 0$. To prove Theorem 2, it suffices to establish bounds on the eigenvalues of $L_{K,P,E}$ and $L_{K,P,T}$ and the supremum norm of the eigenfunctions of the former.

The behavior of these eigenvalues and eigenfunctions can be characterized in terms of the measures of the smoothness of the space $\mathcal{H}_K$ that we discussed above. We prove the following lemma in Appendix B and using it prove Theorem 2 in Appendix C.

**Lemma 4.** Let $\mathcal{H}_K$ be an RKHS of functions on a compact set $X \subseteq \mathbb{R}^d$ and $\nu$ be a measure on $X$ that is strongly equivalent to Lebesgue measure $\mu$ in the sense that $\eta \leq d\nu/d\mu \leq \eta^{-1}$ for some $\eta > 0$. Then the decreasing sequence of eigenvalues $\lambda_j$ and the corresponding eigenfunctions $\phi_j$ of $L_{K,\nu}$ satisfy $\lambda_j = O(j^{-\alpha})$ and $\|\phi_j\|_\infty = O(\lambda_j^{-\beta/2})$ with (i) $\alpha = 2s/d$ and $\beta = d/(2s)$ if $\sup_{\|f\|_{\mathcal{H}_K} \leq 1} \|f\|_{H^s} < \infty$ and (ii) $\alpha = (2s + d)/d$ and $\beta = d/(2s + d)$ if $\|K\|_{C^{2s}} < \infty$ and $s$ is not an integer.

We close the section with a few remarks.

**Remark 1.** Some estimators of $\psi^c(m)$ are translation invariant in the sense that estimates based on observations $Y_i$ and translated versions $Y_i' = Y_i + t$ differ by exactly $t$. The estimator $\hat{\psi}_{ML}^c$ that we discuss here is not. This is a consequence of the regularization implicit in our choice of weights. In the averaged ridge regression interpretation of our estimator (6), the penalty $\|\hat{\psi}\|_{\mathcal{H}_K}^2$ that we use when we estimate $m(\cdot, 0)$ penalizes deviation of our estimator from the constant function $f(x) = 0$, even if that deviation takes the form of a constant translation. As penalties on translations are light for most reasonable RKHS norms, this is not generally a problem if $E[Y_i | W_i = 0]$ is not too large. However, modifying our estimator so that it is translation invariant makes it somewhat more foolproof. A simple way to do this is to use the estimator $\hat{\psi}_{ML}^c = (n_T/n)\bar{Y}_0 + n^{-1} \sum_{i:W_i=0} \gamma_i(Y_i - \bar{Y}_0)$ where $\bar{Y}_0 = n^{-1} \sum_{i:W_i=0} Y_i$. This is a very simple augmented minimax linear estimator (Hirshberg and Wager, 2018) incorporating a constant estimate $\bar{Y}_0$ of $m(\cdot, 0)$. See Kallus (2016, Section 4.5) for an alternative approach to translation invariance and its generalizations that substitutes a translation invariant seminorm for the norm $\|\cdot\|_{H_K}$.

**Remark 2.** Our first-order asymptotic result, Theorem 2, requires no assumptions on the inverse propensity weighting function $g_\psi$ beyond its boundedness. Our assumption that it is is bounded is a strict overlap assumption in the sense of D’Amour et al. (2017), which ensures that our target population and the population that receives our treatment of interest are sufficiently similar that the rate at which $\psi(m)$ can be estimated is not impacted by identification issues.
(see e.g. Khan and Tamer, 2010). It does require smoothness of the regression function \( m(\cdot, 0) \). In particular, it requires that \( m(\cdot, 0) \) is in the RKHS \( \mathcal{H}_K \) and that \( \mathcal{H}_K \) satisfies Assumption 3 or 4.

Under Assumption 3, the condition \( m(\cdot, 0) \in \mathcal{H}_K \) implies Sobolev-type smoothness of \( m(\cdot, 0) \) of order \( s > d \). This seems to be twice as strong as should be necessary. Efficient estimation of \( \psi(m) \) is possible so long as \( m(\cdot, 0) \) is H"older-smooth of order \( s > d/2 \) (Robins et al., 2009), and the linear plug-in estimator of Newey and Robins (2018) is efficient in this case. Furthermore, Sobolev-type smoothness of order \( s > d/2 \) is sufficient to show that our estimator is consistent at \( O_p(n^{-1/2}) \) rate by the simple argument based on (5) discussed in Section 1.1. Thus, it would be strange if twice this level of smoothness were required for efficiency of our estimator. This seemingly excessive level of smoothness is needed only to ensure an adequately slow rate of growth for the eigenfunctions of \( L_{K,Z} \) in supremum norm, a significant challenge in the characterization of the performance of RKHS methods (see discussion in Zhou, 2002). In some cases, it is clear that we do not need this degree of smoothness. In particular if \( \mathcal{H}_K \) is the Sobolev space of periodic functions on the unit cube in \( \mathbb{R}^d \) and \( P_Z \) is uniform measure on this cube, the eigenfunctions of \( L_{K,Z} \) will be the Fourier basis functions, which are bounded in supremum norm.

The implication of Assumption 4 that \( \|m(\cdot, 0)\|_{C^s} < \infty \) for \( s > d/2 \) is closer to what we expect. This is the aforementioned minimal level of H"older-type smoothness required for efficient estimation of \( \psi(m) \). However, insofar as the finiteness of \( \|m(\cdot, 0)\|_{C^s} \) is implied by and not equivalent to our assumptions, this should not be taken as a claim that we establish efficiency under weakest-possible conditions.

**Remark 3.** While the first order asymptotic behavior of our estimator is essentially not impacted by the smoothness of \( g_\psi \) when \( m(\cdot, 0) \in \mathcal{H}_K \), the higher order remainder terms are strongly affected. In particular, lack of smoothness can limit the rate of convergence of our weights \( \hat{\gamma} \) to \( \gamma_\psi \), with the consequence that our bound (14) decays no faster than \( n^{-1/2} \lambda^{\kappa_g} \) where \( \kappa_g \) is the largest \( \kappa \leq 1/2 \) such that \( \|g_\psi\|_{L_{K,Z}} < \infty \).

**Remark 4.** In Theorem 2, we take \( \lambda = \sigma^2/n \) for constant \( \sigma \). This choice is the natural one in our minimax approach, as \( \sigma_n \to \infty \) or \( \sigma_n \to 0 \) would yield minimax estimators in settings in which the noise level was either increasing or decreasing with sample size. In addition, it is a robust choice when \( m(0, \cdot) \in \mathcal{H}_K \), as it results in first-order asymptotic efficiency under the minimal smoothness assumptions we consider here.

However, other perspectives on our estimator motivate the use of \( \lambda \) asymptotically larger than \( 1/n \). Interpreting our estimator as an averaged ridge regression estimator (8) or as an inverse probability weighting estimator with inverse probability weights estimated by least squares (26), the choice \( \lambda \approx 1/n \) results in unusually weak regularization. As discussed in Hirshberg and Wager (2018, Appendix B.3), by taking \( \lambda \gg 1/n \), it is possible to get faster convergence of the weights \( \hat{\gamma} \) to the Riesz representer \( \gamma_\psi \), with the optimal tuning typically being \( \lambda = n^{-1/2} r_n \) where \( r_n \) is the minimax rate for estimating \( \gamma_\psi \).
While estimating nuisance parameters like \( m(\cdot, 0) \) and \( g_\psi \) at the optimal rate is not always desirable in semiparametric problems like this one (see Hirshberg and Wager, 2018, Section 1.4), these tuning choices are, in fact, frequently justified by our bounds. Our proof of Theorem 2 in Appendix C shows that our estimator is efficient for \( \lambda \) satisfying the rate bounds \( \frac{1}{n} \lesssim \lambda \ll \frac{1}{n} \left[ \frac{1}{2} (\kappa_m + \kappa_g) \right] \) in terms of a sort of smoothness exponent \( \kappa_m \in [1/2, 1] \) for \( m(\cdot, 0) \), the largest \( \kappa \) in that range such that \( \| m(\cdot, 0) \|_{L^{\kappa, P_Z}} < \infty \), and an analogous quantity \( \kappa_g \in [0, 1/2] \) for \( g_\psi \).

Furthermore, our bounds suggest that this sort of tuning may lead to faster decay of the remainder terms. While the idea of choosing \( \lambda \) on the basis of these parameters \( \kappa_m, \kappa_g \) to minimize these bounds is impractical in that we don’t know these parameters and not necessarily optimal insofar as these bounds are not necessarily sharp, it does provide some potentially useful intuition. The tuning parameters \( \lambda \) that result in the smallest remainder bound are typically between our robust choice \( \lambda \approx \frac{1}{n} \) and the optimal choice for estimating \( \gamma_\psi \), \( \lambda \approx \frac{n - 1/2}{r} \). Thus, we should not necessarily expect optimal performance either from tuning approaches that assume \( \sigma^2 = n\lambda \) on the basis of a fixed-design minimaxity argument nor from approaches that tune \( \lambda \) for estimation of \( \gamma_\psi \) by cross-validation.

### 3.1 Proof Sketch

In this section, we sketch a proof of (13) from Theorem 3. A detailed proof, as well as a proof of (14) based on that Theorem 2 in Hirshberg and Wager (2018), we defer to Section D in the Appendix.

Our approach works with the averaged ridge regression interpretation our estimator. In (4) above, we decomposed the error \( \psi_{ML} - \hat{\psi}(m) \) of our estimator, written in weighting form (7), into its design-conditional bias and its variation around it. Consider the same decomposition of our estimator expressed in averaged ridge regression form (8).

\[
\hat{\psi}_{ML} - \hat{\psi}(m) = \frac{1}{n} \sum_{i=1}^{n} T_i [E[\hat{m}(X_i) \mid X, W] - m(X_i, 0)] + \frac{1}{n} \sum_{i=1}^{n} T_i [\hat{m}(X_i) - E[\hat{m}(X_i) \mid X, W]].
\]

The quantity \( E[\hat{\psi}_{ML} \mid X, W] - \hat{\psi}(m) \) that we are bounding is a sample average \( n^{-1} \sum_{i=1}^{n} T_i b(X_i) \) of the conditional bias function \( b(x) = E[\hat{m}(x) \mid X, W] - m(x, 0) \) of our regression estimator, which can be written explicitly in the form \( b = -\lambda \hat{L} + \lambda I \) where \( \hat{L} \) is a sample-based approximation to the smoothing operator \( L_{K,P_Z} \) associated with our RKHS. We will write this average as the sum of the corresponding population average and its deviation.

\( ^6 \) Here \( a_n \lesssim b_n \) and \( a_n \ll b_n \) have their conventional meanings \( a_n/b_n \not\to \infty \) and \( a_n/b_n \to 0 \).
We have products discussed above in Section 2. For any approximation $\tilde{g}$ from it, i.e., norms, then bound such deviations uniformly over our bias function and bound both terms.

We do a change of measure to rewrite it as an inequality (see e.g. Bartlett et al., 2005, Theorem 2.1). This results in the second through fourth terms in (13).

To bound the deviation term, we will show that with high probability our bias function $b$ is in a class $B$ of functions with bounded RKHS and $L_2(P_Z)$ norms, then bound such deviations uniformly over $b \in B$ using Talagrand’s inequality (see e.g. Bartlett et al., 2005, Theorem 2.1). This results in the second through fourth terms in (13).

To show that the population average is bounded by the first term in (13), we do a change of measure to rewrite it as an $L_2(P_Z)$ inner product between our bias function $b$ and the inverse probability weighting function $g_\psi$, using in the last step the self-adjointness of the operator $L_{K,P_Z}$. Then via Cauchy-Schwartz we have the bound

$$\int T(x,w)b(x)dP \leq p_Z\|g_\psi - \tilde{g}\|_{L_2(P_Z)}\|b\|_{L_2(P_Z)} + p_Z\|\tilde{g}\|\|L_{K,P_Z}b\|.$$ (18)

Thus, to complete our argument it suffices to bound $\|b\|$, $\|b\|_{L_2(P_Z)}$, and $\|L_{K,P_Z}b\| = \lambda\|L_{K,P_Z}[L + \lambda I]^{-1}m(\cdot,0)\|$ and then to optimally choose an approximation $\tilde{g}$ to $g_\psi$ to minimize (18). For the former task, we follow an argument of Hsu et al. (2012) based on the operator-norm convergence of $L_{K,P_Z}^{-1/2}L_{K,P_Z}^{-1/2}$ to the identity. For the latter task, we use a simple corollary of Cucker and Zhou (2007, Theorem 4.1), which establishes that if $\|g\|_{L_2(\nu)} < \infty$,

$$\inf_{g' : \|g'\|_2 \leq R} \|g' - g\|_{L_2(\nu)} \leq \left(2\|g\|_{L_2(\nu)}\right)^{1/2 - \nu} R^{1/2 - \nu}.$$ (19)
The reasoning behind this two-term bound based on the approximation $\tilde{g}$ merits some explanation. Suppose that instead of ridge regression, we estimated $m(\cdot, 0)$ using least squares regression in the span of the first $K \leq n$ eigenvectors $\phi_1 \ldots \phi_K$ of $L_{K,P_x}$. Then the bias function $b$ would be orthogonal to any function $\tilde{g}$ in this span, and while $\langle g_\psi - \tilde{g}, b \rangle_{L_2(P_x)} = \langle g_\psi, b \rangle_{L_2(P_x)}$, we are better off applying Cauchy-Schwartz to the former, and when we do, it is best to take $\tilde{g}$ to be the best approximation to $g_\psi$ in that span (see e.g. Newey and Robins, 2018, Proof of Lemma A5). In essence, insofar as the function $g_\psi$ is smooth, the nonsmooth components of the function $b$ we are averaging against it are irrelevant. When we do ridge regression, we do not get strict orthogonality because our estimator does shrinkage along all eigenvectors of $L \approx L_{K,P_x}$. However, much of this bias is along the higher order eigenvectors of $L$. When we do ridge regression, we do not get strict orthogonality because our estimator does shrinkage along all eigenvectors of $L_{K,P_x}$ and therefore approximately orthogonal to functions in $H_K$. By working with the smooth approximation $\tilde{g}$ to $g_\psi$, our bound exploits this.

4 Empirical Performance

We evaluate the performance of our estimator on the famous example of Kang and Schafer (2007) and a more recent example of Hainmueller (2012). In these examples, we estimate a mean outcome when some outcomes are missing by a strongly ignorable mechanism (Rosenbaum and Rubin, 1983), an instance of the estimand strongly ignorable mechanism (Rosenbaum and Rubin, 1983), an instance of the examples, we estimate a mean outcome when some outcomes are missing by a strong ignorability mechanism (Rosenbaum and Rubin, 1983), an instance of the estimand strongly ignorable mechanism (Rosenbaum and Rubin, 1983), an instance of the

We will look at, in addition to root mean squared error and bias, the width and coverage of 95% confidence intervals of the form $\hat{\psi} \pm z_{0.025} \hat{V}^{1/2}/n^{1/2}$, where

$$
\hat{V} = n^{-1} \sum_{i: T_i=1} (\hat{m}(X_i) - \hat{\psi})^2 + n^{-1} \sum_{i: W_i=0} \hat{\gamma}_i^2 (Y_i - \hat{m}(X_i))^2.
$$

(20)
Here $\gamma_i$ are the weights used in the given estimator\(^7\) and $\hat{m}$ is an OLS estimate of $m(x, 0)$ based on the sample receiving treatment $W_i = 0$. $\hat{V}$ is based on a variant of the asymptotic characterization (11) with the limit of $\gamma_i$ substituted for $\gamma_0(X_i, W_i)$, which will hold for these estimators so long as the conditional bias $E[\hat{\psi} | X, W] - \psi(m) = o_p(n^{-1/2})$.

The Kang and Schafer example was designed to illustrate that methods using estimated inverse propensity weights can be unstable. Our observations $X_i \in \mathbb{R}^4, W_i \in \{0, 1\}, Y_i \in \mathbb{R}$ are defined in terms of a latent vector of standard normal random variables $Z_i \in \mathbb{R}^4$: we have $X_{i1} = \exp(Z_{i1}/2), X_{i2} = Z_{i2}/(1 + \exp(Z_{i1}) + 10), X_{i3} = (Z_{i1}Z_{i3}/25 + .06)^3, X_{i4} = (Z_{i2} + Z_{i4} + 20)^2$; $P\{W_i = 0 | Z_i\} = \logit^{-1}(-Z_{i1} + 0.5Z_{i2} - 0.25Z_{i3} - 0.1Z_{i4})$; and $Y_i = 210 + 27.4Z_{i1} + 13.7(Z_{i2} + Z_{i3} + Z_{i4}) + \sigma_x\epsilon_i$ for standard normal $\epsilon_i$ when $W_i = 0$. In this example, the instability of the IPW and AIPW estimators persists even into large sample sizes, while the OLS estimator performs extremely well even in small samples. These phenomena are explained in detail by a comment on Kang and Schafer (2007) by Robins et al. (2007). In summary, there are regions of poor overlap. We show here that our estimator $\hat{\psi}_{ML}$, while not reliant on the linearity of $m(x, 0)$, also performs very well in all sample sizes. Furthermore, when the sample size is small and noise level $\sigma_x$ is large, the inclusion of regularization in our estimator’s implicit estimate of $\hat{m}$ helps us — in these settings, $\hat{\psi}_{ML}$ and $\hat{\psi}_{ML_t}$ with larger values of the tuning parameter $\sigma$ outperform OLS.

In the example of Hainmuller that we consider, we observe $X_i \in \mathbb{R}^6$ with $X_{i1}, \ldots, X_{i3}$ jointly normal with mean zero and covariance matrix $\Sigma$ defined in Appendix G and $X_{i4} \sim \text{Uniform}([-3, 3]), X_{i5} \sim \chi^2_1$, and $X_{i6} \sim \text{Bernoulli}(1/2)$ independent and independent of $X_{i1}, \ldots, X_{i3}$; missingness follows a probit model $P\{W_i = 0 | X_i\} = \psi(\eta^{-1}[X_{i1} + 2X_{i2} - 2X_{i3} - X_{i4} - 0.5X_{i5} + X_{i6}])$; and outcomes follow a quadratic model $Y_i = (X_{i1} + X_{i2} + X_{i6})^2 + \sigma_x\epsilon_i$. In this example, there is better overlap than in that of Kang & Schafer and the logit missigness model used in the IPW and AIPW methods is barely misspecified, so the IPW and AIPW perform well in moderate and large samples. On the other hand, misspecification of the linear outcome model relied upon by OLS causes substantial bias in all sample sizes. Our estimators $\hat{\psi}_{ML}$ and $\hat{\psi}_{ML_t}$ perform well in all sample sizes, as they do in several variations on this example discussed in Appendix G.

\(^7\)While the OLS estimator is not typically considered a weighting estimator, it is linear in $Y$ and can therefore be expressed in that form. Lemma 1 shows that it is, in fact, a limiting ($\sigma \to 0$) case of our estimator $\hat{\psi}_{ML}$ in which we work with the RKHS of linear functions $f(x) = f^T x$ with the Euclidean inner product $\langle f(x), g(x) \rangle = f^T g$. 

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Figure 1: Boxplots of our estimates over 1000 replications with $\sigma_x = 50$ at sample sizes 50, 200, 1000 in the example of Kang and Schafer (2007). The grey horizontal line indicates the value of estimand. As the IPW and AIPW estimators were very variable, some larger estimates are cut off to allow some detail to be visible in the plot.
Figure 2: Root mean squared error (rmse), bias, and confidence interval half-width and coverage over 1000 replications in the example of Kang and Schafer (2007). Here we take the tuning parameter \( \sigma \) to be 0.1 in the estimators ML and MLt. The notation MLt 10\( \sigma \) and 100\( \sigma \) indicates the substitution of 1 and 10 respectively.
| $n$ | 50  | 200  | 1000 | 4000 | 50  | 200  | 1000 | 4000 |
|-----|-----|------|------|------|-----|------|------|------|
| MSE | IPW | 1.74 | 1.84 | 0.77 | 0.34 | 6.9  | 4.11 | 2.45 | 1.15 |
|     | AIPW | 0.50 | 0.8 | 0.08 | -0.06 | 0.89 | 0.97 | 0.99 | 1    |
|     | OLS | -0.48 | -0.28 | -0.11 | -0.09 | 0.84 | 0.95 | 0.99 | 0.99 |
|     | ML | -1.28 | -1.07 | -1.13 | -1.13 | 0.84 | 0.88 | 0.72 | 0.23 |
|     | MLt | -2.44 | -1.91 | -0.58 | -0.53 | 0.7 | 0.79 | 0.9 | 0.97 |
| Coverage | IPW | 2.43 | 1.24 | 0.71 | 0.41 | 4.41 | 2.91 | 1.47 | 1.02 |
|     | AIPW | 2.21 | 1.96 | 1.91 | 0.65 | 3.86 | 2.42 | 1.42 | 0.83 |
|     | OLS | 1.78 | 1.78 | 0.83 | 0.71 | 3.44 | 2.76 | 0.82 | 0.46 |
|     | ML | 1.40 | 1.40 | 1.10 | 0.72 | 2.41 | 2.41 | 0.37 | 0.24 |
|     | MLt | 2.23 | 1.70 | 1.70 | 0.71 | 3.07 | 2.10 | 0.91 | 0.91 |
|     | AIPW | 0.03 | 0.18 | 0.08 | 0.05 | 0.96 | 0.98 | 0.99 | 0.99 |
|     | OLS | 0.71 | 0.99 | 1.29 | 1.94 | 1.06 | 1.39 | 0.65 | 0.35 |
|     | ML | 0.82 | 0.7 | 0.54 | 0.58 | 0.86 | 0.98 | 0.99 | 0.99 |
|     | MLt | 0.81 | 0.53 | 0.55 | 0.58 | 0.86 | 0.98 | 0.99 | 0.99 |
| Bias | IPW | 1.70 | 1.70 | 0.81 | 0.61 | 2.87 | 2.79 | 0.91 | 0.82 |
|     | AIPW | 2.21 | 1.96 | 1.91 | 0.65 | 3.86 | 2.42 | 1.42 | 0.83 |
|     | OLS | 2 | 2 | 1 | 1 | 2 | 2 | 1 | 1 |
|     | ML | 1.40 | 1.40 | 1.10 | 0.72 | 2.41 | 2.41 | 0.37 | 0.24 |
|     | MLt | 2.23 | 1.70 | 1.70 | 0.71 | 3.07 | 2.10 | 0.91 | 0.91 |
|     | AIPW | 1.70 | 1.70 | 0.81 | 0.61 | 2.87 | 2.79 | 0.91 | 0.82 |
|     | OLS | 2 | 2 | 1 | 1 | 2 | 2 | 1 | 1 |
|     | ML | 1.40 | 1.40 | 1.10 | 0.72 | 2.41 | 2.41 | 0.37 | 0.24 |
|     | MLt | 2.23 | 1.70 | 1.70 | 0.71 | 3.07 | 2.10 | 0.91 | 0.91 |

Figure 3: Root mean squared error (rmse), bias, and confidence interval half-width and coverage over 1000 replicates of Outcome Design 3 from Hainmueller (2012). Here we take the tuning parameter $\sigma$ to be 0.1 in the estimators ML and MLt. The notation MLt 100 signifies the substitution of 1 and 10 respectively.
5 Application: the LaLonde Study

We apply our method to estimate the impact of the National Supported Work (NSW) Demonstration, a labor training program, on post-intervention income levels. In this study, participants were randomly selected for admission to the program, so experimental estimates of a treatment effect are available. As a result, it has been used to test methods for estimation of treatment effects in observational studies. Attempts have been made to use larger nonexperimental control groups to replicate the experimental estimate, but this has proven challenging for many of the methods considered. This problem was famously discussed in LaLonde (1986) and later in Dehejia and Wahba (1999).

We follow Dehejia and Wahba (1999) in working with a subset of the male participants in the experimental sample in which pre-intervention income history is available for at least two years. The latter restriction allows us to adjust for 4 continuous covariates and 4 binary ones: two years of pre-intervention income, age (in years), education (in years of schooling), and indicators for attainment of a high-school diploma, marriage status (married/unmarried), identification as black, and identification as hispanic. The former is in recognition of both substantially different eligibility criteria and realization of the intervention for men and women (see LaLonde, 1986). In this subset, the experimental treatment and control subsamples have 185 and 260 units respectively.

In this context, the primary causal estimand that has been discussed is the average treatment effect on the treated, $\tau_T = E[Y_i(1) \mid W_i = 1] - E[Y_i(0) \mid W_i = 1]$. In the experimental sample, randomization ensures that $E[Y_i(0) \mid W_i = 1] = E[Y_i(0) \mid W_i = 0] = E[Y_i \mid W_i = 0]$, and a simple simple difference-in-means estimate $T_i = \frac{1}{n_T} \sum_{i: W_i = 1} Y_i - \frac{1}{n_Z} \sum_{i: W_i = 0} Y_i$ for $n_T = \sum_{i=1}^n 1_{W_i = 1}$ and $n_Z = \sum_{i=1}^n 1_{W_i = 0}$ yields a 95% confidence interval of $1794 \pm 1315$. In our attempt to replicate this estimate this using a nonexperimental control group, we observe that under our identification assumptions, $E[Y_i(0) \mid W_i = 1] = \psi^c(m)$ where $\psi^c$ is defined as in (1) for $T_i = 1_{W_i = 1}$. For the treatment effect $\tau_T$, we use the point estimator $\hat{\tau}_T = \frac{1}{n_T} \sum_{i: W_i = 1} Y_i - m_{ML}^{c}$, taking the parameter $\sigma$ in (7) to be 0.1 and using the Matérn kernel with $\nu = 3/2$ when calculating $\hat{\psi}_{ML}^{c} = (n/n_T)\hat{\psi}_{ML}$. Around it, we give 95% confidence intervals $\hat{\tau}_T \pm z_{0.025}\hat{V}_{1/2}/n_T^{1/2}$ based on the variance estimator

$$\hat{V} = \hat{V}_1 + \hat{V}_2 \quad \text{for}$$

$$\hat{V}_1 = n_T^{-1} \sum_{i: W_i = 1} Y_i^2 - \left( n_T^{-1} \sum_{i: W_i = 1} Y_i \right)^2;$$

$$\hat{V}_2 = n_T^{-1} \sum_{i: W_i = 1} \left( \hat{m}(X_i) - \hat{\psi}_{ML}^{c} \right)^2 + n_T^{-1} \sum_{i: W_i = 0} \hat{\gamma}^2_i (Y_i - \hat{m}(X_i))^2;$$

in which we use an auxiliary ordinary least squares estimator $\hat{m}$ of $m(X_i, 0)$.

We consider the use of non-experimental control samples constructed by LaLonde from the Population Survey of Income Dynamics (PSID-1) and the
Current Population Survey (CPS-1) and as a small subset of the latter called CPS-3 chosen to have characteristics like the experimental sample. This data is made available with and summarized in Dehejia and Wahba (1999). Our point estimates vary substantially depending on the control group used. We estimate 95% confidence intervals of $525 \pm 2684$, $1233 \pm 2733$, $1783 \pm 1652$, and $770 \pm 1785$ using the additional control units from the CPS-3 sample, the PSID-1 sample, the CPS-1 sample, and the PSID-1 and CPS-1 samples combined. This may be suggestive of a problem, perhaps caused by adjusting for a fairly limited set of covariates, but the standard error of our estimators is sufficiently large that differences between these estimates could simply be explained by random variation. Thus, the experimental data provide little evidence that can be used to validate or invalidate our approach. The same qualitative behavior is observed in the results of Dehejia and Wahba (1999).

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incomplete outcome data. *Journal of the American Statistical Association*, 110(511):910–922, 2015.
A Constants used in the statement of Theorem 3

A.0.1 Constants used in Equation 13

\[ c_1 = 2q(\kappa_m)^{2\eta - 1}q(\kappa_m + 1/2)^{2\eta}q(\kappa_g); \]
\[ c_2(\eta) = (1 + \eta)q(\kappa_m - 1/2)^{-1}\sqrt{8\|g_\omega\|_\infty^{-1/\alpha}(p_TC_{\lambda,T})^{1/\alpha}}/(1 - 1/\alpha); \]
\[ c_3 = \sqrt{2}q(\kappa_m)^{-1}; \]
\[ c_4(\eta) = 2q(\kappa_m - 1/2)^{-1}(1/3 + 1/\eta); \]
\[ q(x) = \left(\frac{x}{1-x}\right)^{1-x} + \left(\frac{1-x}{1-x}\right)^{-x}; \]
\[ c_\zeta = C_\phi C_{\lambda,Z}^{1/\alpha}\left((\frac{\beta}{1-\beta})^{1/\alpha+\beta}+\frac{\alpha}{(1+\alpha\beta)(\alpha-(1+\alpha\beta))}\right). \]

A.0.2 Constants used in Equation 14

\[ c_p(\eta_Q) = \frac{(1-\eta_Q)^2}{2(1+\eta_Q)(21-11\eta_Q)} \approx .02; \]
\[ \eta_Q = (61 - 8\sqrt{39})/49 \approx .23 \]
\[ \eta_C = (c_{1,C} + c_{2,C}n^{-\frac{1}{\alpha+1}} + c_{3,C}n^{-\frac{1}{\alpha+1}})/(7C_Q)^{1+1/\alpha}; \]
\[ M_{\lambda^*} = M_K + \|g_\omega\|_\infty; \]
\[ M_{\lambda^*} = \|g_\omega\|_\infty(M_K + \|g_\omega\|_\infty); \]
\[ c_{u,1} = (1 + \eta)^{2(1/2)^2/2} \max\{1, \|g_\omega\|_\infty\} \left(\|g_\omega\|_{L_2(p_\zeta)} + C_{\lambda,Z}^{1/2}(1-\alpha)^{-1/2}\right); \]
\[ + 2^{1/2}p_{\zeta}^{1/2} \max\{1, \|g_\omega\|_\infty\} \left(\lambda_1, z + \|g_\omega\|_{L_2(p_\zeta)}\right) \sqrt{\log(2\delta^{-1})}; \]
\[ c_{u,2} = 2M_{\lambda^*}(1/3 + 1/\eta) \log(2\delta^{-1}); \]
\[ c_Q = \{2p_\zeta^2M_{\lambda^*}^2[1 + (p_\zeta C_{\lambda,Z})^{1/\alpha}(1 - 1/\alpha)]\}^{1/2}; \]
\[ c_{1,C} = 2(1 + \eta)(\|g_\omega\|_\infty \{3[1 + (p_\zeta C_{\lambda,Z})^{1/\alpha}(1 - 1/\alpha)]\}^{1/2}; \]
\[ c_{2,C} = \max\{1, \|g_\omega\|_\infty\} \sqrt{2\log(2\delta^{-1})}; \]
\[ c_{3,C} = 2M_{\lambda^*}\left(\frac{1}{3} + \frac{1}{\eta}\right) \log(2\delta^{-1}); \]
\[ c_{1,R}, c_2,R = \|g_\omega\|_{L_2(p_\zeta)}^2, 0 \text{ if } \kappa_g = 1/2; \]
\[ c_{1,R}, c_2,R = (x + x^{-1})^2, (x + x^{-1}) \|g_\omega\|_\infty \text{ otherwise} \]
\[ \text{with } x = 4(2\delta^{-1}p_\zeta)^{1/2-n}\|g_\omega\|_{L_2(p_\zeta)}. \]
B Smoothness, Eigenvalues, and Eigenfunctions

The conditions of Lemma 4 are defined in terms of the Hölder norm \( \| \cdot \|_{C^\alpha} \) and the Sobolev norm \( \| \cdot \|_{H^s} \). We define these norms, then prove the lemma.

\[
\|f\|_{C^\alpha} = \sum_{\beta \in \mathbb{N}^d: \|\beta\|_1 \leq [s]} \|D^\beta f\|_\infty + \sum_{\beta \in \mathbb{N}^d: \|\beta\|_1 = [s]} \sup_{x, x' \in \mathbb{R}^d} \frac{|D^\beta f(x) - D^\beta f(x')|}{\|x - x'\|_{\ell_2^d}^{s-[s]}};
\]

\[
\|f\|_{H^s} = \sum_{\beta \in \mathbb{N}^d: \|\beta\|_1 \leq [s]} \|D^\beta f\|_{L_2(\mu)} + \sum_{\beta \in \mathbb{N}^d: \|\beta\|_1 = [s]} \left[ \int \frac{|D^\beta f(x) - D^\beta f(x')|^2}{\|x - x'\|_{\ell_2^d}^{2(s-[s]) + d}} \, d\mu(x) \, d\mu(x') \right]^{1/2};
\]

\[D^\beta f = \frac{\partial^{\beta_1}}{\partial x_1} \cdots \frac{\partial^{\beta_d}}{\partial x_d} f.\]

Here \( \mu \) is Lebesgue measure.

**Proof of Lemma 4.** In this proof, we will use the Gagliardo-Nirenberg inequality (Hajaiej et al., 2010, Theorem 1.2), in particular the bounds

\[
\|f\|_\infty \lesssim \|f\|_{C^\alpha}^{1-\theta} \|f\|_{L_2(\mu)}^\theta, \quad \theta = 1 - d/(2s+d);
\]

\[
\|f\|_\infty \lesssim \|f\|_{H^\mu}^{1-\theta} \|f\|_{L_2(\mu)}^\theta, \quad \theta = 1 - d/(2s).
\]

In case (ii), Kühn (1987, Theorem 4) established the claimed eigenvalue bound with \( \alpha = 2s/d + 1 \) under a condition \( \sup_{x \in \mathcal{X}} \| K(x, \cdot) \|_{C^\alpha} < \infty \) weaker than our assumption on \( K \). In addition, Cucker and Zhou (2007, Theorem 5.5) established the bound \( \sup_{f: \|f\|_{\mathcal{H}_K} \leq 1} \|f\|_{C^\alpha} < \infty \). This, in combination with the Gagliardo-Nirenberg inequality, imply that \( \|f\|_\infty \lesssim \|f\|_{\mathcal{H}_K}^{d/(2s+d)} \|f\|_{L_2(\mu)}^{1-d/(2s+d)} \).

As \( \|\phi_j\|_{\mathcal{H}_K} = \lambda_j^{-1/2} \) and \( \|\phi\|_{L_2(\mu)} \leq \eta^{-1} \|\phi\|_{L_2(\nu)} = \eta^{-1} \), we have \( \|\phi_j\|_\infty \lesssim \lambda_j^{-d/[2(2s+d)]} \) as claimed.

In case (i), we use the bound \( a_j(\mathcal{B}_{H^\mu}^s) \lesssim j^{-s/d} \) (see e.g. Kühn et al., 2014) where

\[a_j(\mathcal{F}) = \inf_{\{A \leq j\}} \sup_{f \in \mathcal{F}} \|f - Af\|_{L_2(\mu)} \text{ and } \mathcal{B}_{H^\mu}^s = \{f: \|f\|_{H^\mu} \leq 1\}.\]

Observe that \( a_j \) has the homogeneity property \( a_j(s\mathcal{F}) = sa_j(\mathcal{F}) \) and the increasing property \( A \subseteq B \implies a_j(A) \leq a_j(B) \). As our assumption \( \sup_{\|f\|_{C^\alpha} \leq 1} \|f\|_{H^\mu} < \infty \) implies that the unit ball \( \mathcal{B}_K \) of our RKHS is in \( s\mathcal{B}_{H^\mu}^s \) for some \( s \), we have \( a_j(\mathcal{B}_K) \lesssim j^{-s/d} \) as well. This is helpful because \( a_j(\mathcal{B}_K) = \lambda_j^{1/2} \) where \( \lambda_j \) is the \( j \)th eigenfunction of the integral operator \( L_{K,\mu} \). To see this, observe that if the range of \( A \) does not contain the span of the first \( j-1 \) eigenfunctions \( \phi_1 \cdots \phi_{j-1} \), there is a function \( f = \sum_{k=1}^{j-1} f_k \lambda_k^{1/2} \phi_k \) in \( \mathcal{B}_K \) with \( \|f - Af\|_{L_2(\mu)} = \|f\|_{L_2(\mu)} = ... \).
$|\sum_{k=1}^{n-1} f_k^2 \lambda_k|^{1/2} \geq \lambda_j^{1/2}$, whereas if it is the identity restricted to that span, we have $\|f - A f\|_{L_2(\nu)} \leq \lambda_j^{1/2}$ whenever $f \in B_K$. Thus, our saddle point is attained with $A$ equal to this restricted identity and $f = \phi_j$. This implies that $\lambda_j \lesssim j^{-2s/d}$, and as discussed in our review of RKHSes, strong equivalence of $\mu$ and $\nu$ implies the same rate for the eigenvalues $\lambda_{j,\nu}$ of $L_{K,\nu}$.

To bound the eigenfunctions, recall that $\|f\|_{H_\kappa} \lesssim \|f\|_{\mathcal{H}K}$. This and the Gagliardo-Nirenberg inequality imply the bound $\|f\|_\infty \lesssim \|f\|_{H_\kappa}^{d/(2s)} \|f\|_{L_2(\mu)}^{1-d/(2s)}$. As $\|\phi_j\|_{H_\kappa} = \lambda_j^{-1/2}$ and $\|\phi\|_{L_2(\mu)} \leq \eta^{-1} \|\phi\|_{L_2(\nu)} = \eta^{-1}$, we have $\|\phi_j\|_\infty \lesssim \lambda_j^{-d/(4s)}$ as claimed.

\[\square\]

### C Asymptotics

**Proof of Theorem 2.** It suffices to establish the characterization (11), as this implies that $\hat{\psi}$ is efficient (Hirshberg and Wagner, 2018, Proposition 3).

First consider our bound (13) on our bias term. Lemma 4 implies that the eigenvalue and eigenfunction bounds assumed in Theorem 3 are satisfied with $\alpha > 1$ and $\alpha(1 - \beta) > 1$ under either Assumption 3 or Assumption 4. $\zeta^{-1}$ is bounded if $\lambda^{-1/(\alpha + \beta)} \log(n)/n \to 0$ or equivalently if $[\log(n)/n]^{1/(1/\alpha + \beta)} \ll \lambda$. When this happens, our bound term is $o(n^{-1/2})$ as long as $\lambda^{\kappa_m + \kappa_p}$ is, i.e. if $\lambda \ll n^{-1/[2(\kappa_m + \kappa_p)]]}$.

Now consider the bound (14) on the deviation of our noise term from our desired asymptotic characterization. This bound will be negligible if the factor $a$ goes to zero. This will happen if (i) $R \to 0$, (ii) $s \ll \sqrt{n}$ and therefore the first term in $a$ goes to zero. Referring to the first claim of Lemma 12, we have (i) given $\lambda \to 0$ and our assumption that $\mathcal{H}_K$ is dense in $L_2(P_Z)$. Unpacking (ii), we are assured that the second term in $s$ is $o(\sqrt{n})$ given (i) if $\lambda \gtrsim n^{-1}$ and the first term in $s$ is $o(\sqrt{n})$ if $\lambda^{-1} n^{-(1/2 + 1/(1 + \alpha))}$ is or equivalently $n^{-(1/2 + 1/(1 + \alpha))} \ll \lambda$.

Collecting all of our conditions, we have efficiency if $\mathcal{H}_K$ is dense, $n^{-1} \lesssim \lambda$, and

$$\max\{[\log(n)/n]^{1/(1/\alpha + \beta)}, n^{-1/(2 + 1/(1 + \alpha))}\} \ll \lambda \ll n^{-1/[2(\kappa_m + \kappa_p)]}.$$ 

The condition $n^{-1} \lesssim \lambda$ implies all of our lower bounds, as our condition $\alpha(1 - \beta) > 1$ is equivalent to $1/\alpha + \beta < 1$ and therefore $[\log(n)/n]^{1/(1/\alpha + \beta)} \ll n^{-1}$ and our assumption $\alpha > 1$ implies that $n^{-(1/2 + 1/(1 + \alpha))} \ll n^{-1}$. Thus, if $\mathcal{H}_K$ is dense, we have efficiency for $\lambda$ satisfying $n^{-1} \lesssim \lambda \ll n^{-1/[2(\kappa_m + \kappa_p)]}$. Our results for $\hat{\psi}^c = (n_T/n)^{-1} \hat{\psi}$ then follow straightforwardly. Its characterization (12) follows from (11) and the convergence of $n_T/n$ to its mean $p_T$. And efficiency follows as well, as it is clear that dividing the efficient and therefore regular asymptotically linear estimator $\hat{\psi}$ by $n_T/n$ to yield $\hat{\psi}^c$ gives us another regular asymptotically linear estimator, and all regular asymptotically linear estimators are efficient in problems like this one, in which the space of models we allow is nonparametric (see e.g. Newey, 1994, Theorem 2.1).

\[\square\]
D Proof of Theorem 3

D.1 The bias term bound (13)

In this section, we prove the bound (13) by filling in the details from our sketch in Section 3.1. We will begin by proving the lemma below, then show that it implies the bound (13).

Lemma 5. In the setting described in Section 3, consider the estimator \( \hat{\psi}_{ML} = n^{-1} \sum_{i=1}^{n} 1_{(W_i = 0)} \gamma_i Y_i \) with weights \( \gamma \) defined in (7). Define \( \lambda = \sigma^2 / n, L = L_{K, P_Z}, L_\lambda = L + \lambda I, P_Z = P\{W_i = 0\}, \) and \( P_T = P\{T_i = 1\} \).

For any \( \eta > 0, \delta \in (0,1), \kappa_m \in [1/2,1], \kappa_g \in [0,1/2], \) on an event of probability \( 1 - 3\delta, \)

\[
\begin{align*}
|E[\hat{\psi}_{ML} | X, W] - \hat{\psi}(m)|/\left(\zeta^{-1}p_Z\|m(\cdot,0)\|_{L_{K,P_Z}}\right) &\leq \\
&= c_1 \lambda^{\kappa_m + \kappa_g} \|\psi\|_{L_\lambda(P_Z)} \\
&+ c_2(\eta) \lambda^{\kappa_m - 1/2} R_{\alpha}(T(x, w)b(x) : \|b\| \leq 1, \|b\|_{L_\alpha(P_Z)} \leq c'_2 \lambda^{1/2}) \\
&+ c_3 \lambda^{\kappa_m - 2} \sqrt{p_Z} \|\psi\|_\infty \log(2\delta^{-1}) \\
&+ c_4(\eta) \lambda^{\kappa_m - 2} n^{-1} M_K \log(2\delta^{-1});
\end{align*}
\]

\[
\zeta = \max\left\{0, 1 - \frac{8U^2 \log(4\delta^{-1} n_\delta^2)}{n_\delta} - \sqrt{\frac{16U^2 \log(4\delta^{-1} n_\delta^2)}{n_\delta}}\right\} ;
\]

\[
U = \text{ess sup}_{X \sim P_Z} \left\|L_{\lambda}^{-1/2} K_X \right\| ;
\]

\[
n_\delta = np_Z - \sqrt{2np_Z \log(\delta^{-1})} ,
\]

\[
c_1 = 2q(\kappa_m)^{2\kappa_m - 1} q(\kappa + 1/2 + 2\kappa) q(2\kappa_g); \\
c_2(\eta) = 2(1 + \eta) q(\kappa_m - 1/2)^{-1}; \\
c'_2 = q(\kappa_m - 1/2) / q(\kappa_m); \\
c_3 = \sqrt{2} q(\kappa_m)^{-1}; \\
c_4(\eta) = 2q(\kappa_m - 1/2)^{-1} (1/3 + 1/\eta); \\
q(x) = \left(\frac{x}{1-x}\right)^{1-x} + \left(\frac{x}{1-x}\right)^{-x} .
\]

D.1.1 Characterizing the bias function \( b. \)

The optimization problem (8) defining \( \hat{m} \) has an explicit solution (see e.g. Hsu et al., 2012)

\[
\hat{m} = \left[ \frac{1}{n_Z} \sum_{i: W_i = 0} K_{X_i} \otimes K_{X_i} + \lambda I \right]^{-1} \frac{1}{n_Z} \sum_{i: W_i = 0} K_{X_i} Y_i ,
\]

written in terms of \( \lambda = \sigma^2 / n_Z \) and the rank-one operator \( K_z \otimes K_z \) defined by \( [K_z \otimes K_z] f = K_z(f, K_z) \). Then because for the \( W_i = 0 \) units \( E[Y_i | X, W = \)
\( m(X, 0) = \langle K_{X_x}, m(\cdot, 0) \rangle \), we have \( K_X, E[Y_i \mid X, W] = K_X, \langle K_{X_x}, m(\cdot, 0) \rangle = [K_{X_x} \otimes K_{X_x}] m(\cdot, 0) \). In terms of the operator \( \hat{L} = n_Z^{-1} \sum_{i:W_i=0} K_{X_x} \otimes K_{X_x} \), we may write

\[
E[\hat{m} \mid X, W] = \left[ \hat{L} + \lambda I \right]^{-1} \hat{L} m(\cdot, 0) = \left( I - \lambda \left[ \hat{L} + \lambda I \right]^{-1} \right) m(\cdot, 0)
\]

and therefore

\[
b = -\lambda \left[ \hat{L} + \lambda I \right]^{-1} m(\cdot, 0).
\]

The operator \( \hat{L} \) is an empirical version of our integral operator \( L := L_{K,P_Z} \). Our characterization of \( b \) will rely on the convergence of \( LL^{-1} \) to the its mean \( I \) in the operator norm \( ||A|| = \sup_{||f|| \leq 1} ||Af|| \). Using the shorthand \( \hat{L}_\lambda := \hat{L} + \lambda I \) and \( L_\lambda := L + \lambda I \) for the regularized versions of these operators, we may write

\[
b = -\lambda \left[ \hat{L}_\lambda^{-1} \hat{L}_\lambda \right] \left[ \hat{L}_\lambda^{-1} m(\cdot, 0) \right] = -\lambda \left[ \hat{L}_\lambda^{-1} L_\lambda \right] \left[ L_\lambda^{-1} L^{\kappa^{-1/2}} \left[ L^{1/2} - \kappa^{-1} m(\cdot, 0) \right] \right].
\]

Then bounding operator/vector products in terms of the operator norm,

\[
||b|| \leq \lambda ||\hat{L}_\lambda^{-1} L_\lambda|| \left( ||L_\lambda^{-1} L^{\kappa^{-1/2}}|| \right) \left( ||L^{1/2} - \kappa^{-1} m(\cdot, 0) || \right),
\]

and analogously, because \( ||b||_{L_2(P_Z)} \leq ||L^{1/2}b|| \) and \( ||AB|| \leq ||A|| ||B|| \),

\[
||b||_{L_2(P_Z)} \leq \lambda \left[ ||\hat{L}_\lambda^{-1} L_\lambda^{-1}|| \right] \left( ||L_\lambda^{-1} L^{\kappa}|| \right) \left( ||L^{1/2} - \kappa^{-1} m(\cdot, 0) || \right).
\]

These two bounds form the basis for our characterization of \( b \) as an element of the set \( B = \{ b : ||b|| \leq s, ||b||_{L_2(P_Z)} \leq r \} \).

To bound the leading operator norm factor \( \|\hat{L}_\lambda^{-1} L_\lambda\| = \|L^{1/2} \hat{L}_\lambda^{-1} L_\lambda^{1/2}\| \), we use an argument of Hsu, Kakade, and Zhang (2012, Lemmas 25 and 26). Observe first that

\[
\left\| L^{1/2} \hat{L}_\lambda^{-1} L_\lambda^{1/2} \right\| \leq (1 - \|\Delta\|)^{-1} \text{ where } \Delta = L^{-1/2}(\hat{L}_\lambda - L_\lambda)L^{-1/2}.
\]

Here \( \Delta \) is a centered mean of iid rank-one operators,

\[
\Delta = n_Z^{-1} \sum_{i:W_i=0} \hat{X}_i \otimes \hat{X}_i - E_{X \sim P_Z} \hat{X}_i \otimes \hat{X}_i \text{ where } \hat{X}_i = L^{-1/2} K_{X_x},
\]

and it satisfies the condition \( \|E_{X \sim P_Z} \hat{X}_i \otimes \hat{X}_i\| = \|L^{-1/2} LL^{-1/2}\| \leq 1 \). We bound \( ||\Delta|| ||L^{1/2} - \kappa^{-1} m(\cdot, 0) || \) using a concentration inequality for such averages (Oliveira et al., 2010, Lemma 1). If \( ||\hat{X}_i|| \leq U \) almost surely, with probability \( 1 - \delta \),

\[
\left\| n_Z^{-1} \sum_{i:W_i=0} \hat{X}_i \otimes \hat{X}_i - E_{X \sim P_Z} \hat{X}_i \otimes \hat{X}_i \right\| < \frac{8U^2 \log(4\delta^{-1}n_Z^2)}{n_Z} + \sqrt{\frac{16U^2 \log(4\delta^{-1}n_Z^2)}{n_Z}}
\]

Consequently, on an event of probability \( 1 - \delta \),

\[
\left\| L^{1/2} \hat{L}_\lambda^{-1} L_\lambda^{1/2} \right\| \leq \max \left\{ 0, 1 - \frac{8U^2 \log(4\delta^{-1}n_Z^2)}{n_Z} - \sqrt{\frac{16U^2 \log(4\delta^{-1}n_Z^2)}{n_Z}} \right\}^{-1}.
\]
As long as $U$, which take to be the sharp upper bound $U = \text{ess sup}_X \| L^{-1/2}_\lambda K_X \|$, is small relative to $\sqrt{n(Z)}$, this factor will be essentially one.

To eliminate this bound’s dependence on $n(Z)$, observe that $\log((4\delta^{-1})/x)$ is an increasing function, so our bound will remain valid if we substitute an upper bound on $n(Z)$. Furthermore, in terms of $p_Z = P\{W_i = 0\}$, $n(Z) \geq n(1 - \epsilon)p_Z$ with probability $1 - \exp(-nc^2p_Z/2)$ by the lower tail of the multiplicative Chernoff bound (see e.g. Mitzenmacher and Upfal, 2005, Theorem 4.5). For $\epsilon = [2\log(\delta^{-1})/(np_Z)]^{1/2}$, this is probability $1 - \delta$, and we have $(1 - \epsilon)n = np_Z - \sqrt{2np_Z \log(\delta^{-1})}$. Therefore by the union bound, on an event of probability $1 - 2\delta$,

$$\| L^{1/2}_\lambda L^{-1}_\lambda L^{1/2}_\lambda \| \leq \zeta^{-1} \quad \text{for} \quad \zeta = \max\left\{ 0, 1 - \frac{8U^2\log(4\delta^{-1}n_\delta^2)}{n_\delta} - \sqrt{16U^2\log(4\delta^{-1}n_\delta^2)} \right\};$$

$$n_\delta = np_Z - \sqrt{2np_Z \log(\delta^{-1})}. \quad (22)$$

We can bound the other operator norm factors, which take the form $\| L^{-1}_\lambda L^{\alpha} \|$, using the following lemma. It is proven via straightforward calculation in Section E.

**Lemma 6.** Let $L$ be a compact operator on an RKHS. Then if $\alpha \in [0, 1],$$$
\| L^{-1}_\lambda L^{\alpha} \| \leq q(\alpha)^{-1}\lambda^{\alpha-1} \quad \text{where} \quad q(x) = \left(\frac{x}{1-x}\right)^{1-x} + \left(\frac{x}{1-x}\right)^{-x}.
$$

Finally, we write $\| L^{1/2-\kappa_m} m(\cdot, 0) \|$ in the equivalent form $\| m(\cdot, 0) \|_{L^2_{K,P(Z)}}$. Thus, on the aforementioned event of probability $1 - 2\delta$,

$$\| b \| \leq s := \lambda^{\kappa_m-1/2} q(\kappa_m - 1/2)^{-1} \zeta^{-1} \| m(\cdot, 0) \|_{L^2_{K,P(Z)}}$$

$$\| b \|_{L^2(P_Z)} \leq r := s\lambda^{1/2} q(\kappa_m - 1/2)/q(\kappa_m) \quad (23)$$

We are now ready to complete the proof of our lemma above by (i) bounding our sketch’s bound (18) on the population average of $b$ and (ii) bounding the deviation of the sample average we characterize in the lemma from that population average.

**D.1.2 Bounding the population average**

In this section, we bound the right side of (18),

$$p_Z^{-1} \int T(x, u)b(x)dP \leq \| g\Phi - \hat{g} \|_{L^2(P_Z)} \| b \|_{L^2(P_Z)} + \| \hat{g} \| \| Lb \|.
$$

Hsu, Kakade, and Zhang (2012) complete this argument by invoking a similar inequality. Theirs involves a log factor involving a parameter of $L_\lambda$, whereas the one we use here involves a factor of $\log(n)$ in its place.
Our tools from the previous section provide all we need to bound \( \| Lb \|. \) On the probability \( 1 - 2\delta \) event we work on,

\[
\| Lb \| = \lambda \| \tilde{L}^{-1}_\lambda m(\cdot, 0) \|
\leq \lambda \| \tilde{L}^{-1}_\lambda (L^{\kappa_m - 1/2} L^{1/2 - \kappa_m}) m(\cdot, 0) \|
\leq \lambda \| L^{\kappa_m + 1/2} L^{1/2 - \kappa_m} \| \| L^{1/2 - \kappa_m} m(\cdot, 0) \|
\leq \lambda^{\kappa_m + 1/2} q(\kappa_m + 1/2)^{-1} \zeta^{-1} \| m(\cdot, 0) \|_{L^2_{K, v}}
\]

and therefore

\[
p_Z^{-1} \int T(x, w)b(x)dP(x, w) \leq s \left( \| g_\psi - \tilde{g} \|_{L^2_{2(\nu)}} + t \| \tilde{g} \| \right)
\text{ where } t = \lambda^{1/2} q(\kappa_m)/q(\kappa_m + 1/2).
\]

We complete this part of our argument by minimizing over approximations to \( g_\psi \). To do this, we apply the following corollary of Cucker and Zhou (2007, Theorem 4.1), which we prove by a simple calculation in Section E.

**Lemma 7.** If \( \| g \|_{L^2_{K, \nu}} < \infty \) for \( \kappa \in [0, 1/2] \),

\[
\inf_{g'} \left\{ \| g' - g \|_{L^2_{2(\nu)}} + t \| g' \| \right\} \leq 2^{2\kappa} Z^2 \| \tilde{g} \|_{L^2_{2(\nu)}} q(2\kappa)
\text{ where } q(x) = \left( \frac{x}{1 - x} \right)^{1-x} + \left( \frac{x}{1 - x} \right)^{-x}.
\]

As our bound above is \( s \) times the quantity bounded in this lemma for \( t = \lambda^{1/2} C_{\kappa_m + 1/2}/C_{\kappa_m} \), we have

\[
p_Z^{-1} \int T(x, w)b(x)dP \leq Z^{-1} 2^{2\kappa} [q(\kappa_m)/q(\kappa_m + 1/2)]^{2\kappa} \| g_\psi \|_{L^2_{2(\nu)} q(2\kappa)}
\]

\[
= 2\zeta^{-1} \lambda^{\kappa_m + \kappa_0} \| m(\cdot, 0) \|_{L^2_{K, v}} \| g_\psi \|_{L^2_{2(\nu)} q(\kappa_m)}^{2\kappa_0} q(\kappa_m + 1/2)^{2\kappa_0} q(2\kappa)
\]

on an event on probability \( 1 - 2\delta \).

**D.1.3 Bounding the deviation term**

In this section, we bound the deviation term \( n^{-1} \sum_{i=1}^n T_i b(X_i) - \int T(x, w)b(x)dP \) from (16). Our approach will be to bound this expression uniformly over uniformly over \( b' \in B \), i.e. to bound supremum of the mean-zero empirical process

\[
\sup_{b' \in B} n^{-1} \sum_{i=1}^n T_i b'(X_i) - E T_i b'(X_i).
\]

To do this, we use a form of Talarand’s inequality (Bartlett et al., 2005, Theorem 2.1). On an event of probability \( 1 - \delta \),

\[
\sup_{b' \in B} \left| n^{-1} \sum_{i=1}^n T_i b'(X_i) - E T(x, w)b'(x) \right| \leq t_\eta \text{ for all } \eta > 0;
\]

\[
t_\eta = 2(1 + \eta) R_n \{ T(x, w)b'(x) : b' \in B \} + \sqrt{2V \log(2\delta^{-1}) \over n} + 2U \left( 1 \over 3 + 1 \over \eta \right) \log(2\delta^{-1});
\]

(24)
where $V \geq \sup_{b \in B} E[T_i b(X_i)^2]$, $U \geq \sup_{b \in B} \|T_i b(X_i)\|_{\infty}$. A calculation analogous to (17) shows that $E[T_i b(X_i)^2] = p_Z \int b'(x)g_\nu(x)dP_Z$, and via Hölder’s inequality we may take $V = s^2 p_Z \|g_\nu\|_{\infty}$. And as the unit ball of our RKHS is bounded by $M_K$, we may take $U = rM_K$.

Thus, on the intersection of this event and the event of probability $1 - 2\delta$ on which the results of the previous section hold, which has probability at least $1 - 3\delta$, we have the bound $|n^{-1} \sum_{i=1}^n T_i b(X_i) - \int T(x, w)b(x)dP| \leq t_\eta$ as well.

In the simplified form that appears in the statement of Lemma 5, we pull out a common factor of $s$, writing

$$t_\eta/s = 2(1 + \eta)Rn \{T(x, w)b'(x) : \|b'\| \leq 1, \|b'\|_{L_2(P_T)} \leq r/s\}$$

$$+ (r/s) \sqrt{2p_Z \|g_\nu\|_{\infty} \log(2\delta^{-1})}$$

$$+ 2M_K \left(\frac{1}{3} + \frac{1}{\eta}\right) \log(2\delta^{-1})$$

and write $r/s$ and $s$ explicitly.

**D.1.4 Proving (13) from Lemma 5**

To prove (13) from Lemma 5, we substitute upper bounds for a few quantities in (21). To establish these bounds, we use the lemmas stated below, which are proven in Appendix E.

Lemma 8 implies that that our expression for $\zeta$ in terms of $\alpha, \beta, n$ in (13) bounds the corresponding quantity $\zeta$ in (21).

To bound the second term in (21), we will use Lemma 9, a generalization of Mendelson’s bound on the local Rademacher complexity of the unit ball in an RKHS (Mendelson, 2002). This term is a multiple of the Rademacher complexity of the set $\{T_i b(x) : \|b\| \leq 1, \|b\|_{L_2(P_Z)} \leq c_2 \lambda^{1/2}\}$, and as established above, this $\|b\|_{L_2(P_Z)}$ bound on $b'$ implies that $\|T(x, w)b'(x)\|_{L_2(P)} \leq t$ for $t = c_2 \lambda^{1/2} \sqrt{p_Z \|g_\nu\|_{\infty}}$. Thus, it suffices to bound the Rademacher complexity $R$ of the set $\{T_i b(x) : \|b\| \leq 1, E(T_i b(X_i))^2 \leq t^2\}$, and we apply Lemma 9 with $g = 0$, $Z_i = T_i$, and an iid Rademacher sequence $\sigma_1 \ldots \sigma_n$ independent of $(X_i, W_i)_{i \leq n}$. This yields the bound $R^2 \leq (2/n) \sum_{j=1}^\infty \lambda_j \wedge t^2$ in terms of the eigenvalues $\lambda_j$ of the integral operator $L_{K,\nu}$ associated with the measure $\nu = p_T \cdot P_T$, a scaled version of the distribution of the covariate $X_i$ on the target population. Thus, $\lambda_j = p_T \lambda_j, T$ for eigenvalues $\lambda_j, T$ of $L_{K,\nu}$, and our bound may be rewritten in the form $R^2 \leq (2/n) \sum_{j=1}^\infty (p_T \lambda_j) \wedge t^2$ and bounded using Lemma 10 to complete our proof.

**Lemma 8.** Let $\mathcal{H}_K$ be an RKHS of functions on a compact set $\mathcal{X}$, Let $\nu$ be a finite measure with support equal to $\mathcal{X}$, define $[L_{K,\nu}]f(x) = \int K(x, t)f(t)d\nu(t)$, and let $(\lambda_j, \phi_j)_{j \in \mathbb{N}}$ be its eigenvalues and eigenfunctions scaled so that $\|\phi_j\|_{L_2(\nu)} = 1$, and assume that $\lambda_j \leq C_j \lambda^{-\alpha}$ and that $\|\phi_j\|_{L_\infty(\nu)} \leq C \lambda_j^{-\beta/2}$ with $\alpha(1 - \beta) >$
2. Then, \[
\operatorname{ess} \sup_{X \sim \nu} \left\| (L_{K,\nu} + \lambda I)^{-1/2} K_{x} \right\| \leq C \lambda^{-(1/\alpha + \beta)/2} \quad \text{where}
\]
\[
C = C_{\rho} C_{\lambda}^{1/(2\alpha)} \left[ \left( \frac{\beta}{1 - \beta} \right)^{1/\alpha + \beta} + \frac{\alpha}{(1 + \alpha \beta)(\alpha - (1 + \alpha \beta))} \right]^{1/2}.
\]

**Lemma 9.** Let \( H_{K} \) be an RKHS of functions on a compact set \( \mathcal{X} \), let \((X_1, Z_1) \ldots (X_n, Z_n) \) \( \text{iid} \) \( \nu_{x,z} \) where the marginal \( \nu_x \) on \( X_1 \) has support equal to \( \mathcal{X} \), and let \( s_z(x) = E[Z_i^2 | X_i = z] \) satisfy \( s_z(x) > 0 \) a.e. \( \nu_x \). Define the measure \( \nu \) by \( d\nu = s_z d\nu_x \) and let \( \{\lambda_j : j \in 1 \ldots \infty\} \) be the eigenvalues of \( [L_{K,\nu} f](x) = \int K(x, t) f(t) d\nu(t) \) in decreasing order. For a \( \nu \)-square-integrable function \( g \), define the set \( B^* = \{f - sg : \|f\|_{H_{K}} \leq 1, s \in [0,1]\} \). In terms of an identically distributed sequence \( \sigma_1 \ldots \sigma_n \) satisfying \( E[\sigma_i \sigma_j | X_1, Z_1 \ldots X_n, Z_n] = 0 \) for \( i \neq j \), the local multiplier complexity

\[
M_n\{z f(x) : f \in B^*, \ E(Z_i f(X_i))^2 \leq t^2\} := \sup_{f \in B^*} \sup_{E(Z_i f(X_i))^2 \leq t^2} \left| n^{-1} \sum_{i=1}^{n} \sigma_i Z_i f(X_i) \right|
\]
is bounded by

\[
3^{1/2} \left\| E[\sigma_i^2 | X_i, Z_i] \right\|_{L_{\infty}(\nu_{x,z})} n^{-1/2} \sqrt{\sum_{j=0}^{\infty} \lambda_j \wedge t^2} \quad \text{where} \quad \lambda_0 = (1 + \sqrt{\lambda_1})^2 \left\| g \right\|_{L_{2}(\nu)}^2.
\]

If \( g = 0 \), we may take \( \lambda_0 = 0 \) and the leading constant to be \( 2^{1/2} \). For \( t = \infty \), we have the tighter bound

\[
2^{1/2} \left\| E[\sigma_i^2 | X_i, Z_i] \right\|_{L_{\infty}(\nu_{x,z})} n^{-1/2} \left( \|g\|_{L_{2}(\nu)}^2 + \sqrt{\sum_{j=0}^{\infty} \lambda_j} \right).
\]

If \( g = 0 \), we may take the leading constant to be 1.

**Lemma 10.** If \( \lambda_j \leq C n^{-\alpha} \) for \( \alpha > 1 \), \( \sum_{j=1}^{\infty} \lambda_j \wedge t^2 \leq C^{1/\alpha}(1 - 1/\alpha)^{-1} t^{2(1 - 1/\alpha)} \).

### D.2 Proof of the noise term bound (14)

In this section, we prove the bound (14) from Theorem 3. This is a slight variation on the bound from Theorem 2 of Hirshberg and Wager (2018). We will work with a characterization of the noise term from (4),

\[
\hat{\psi}_{ML} - E[\hat{\psi}_{ML} | X, W] = 1_{\{W_i = 0\}} \hat{\gamma} \varepsilon_i \quad \text{where} \quad \varepsilon_i = Y_i - m(X_i, W_i).
\]

We will show convergence of this quantity to the iid sum \( n^{-1} \sum_{i=1}^{n} \gamma_\psi(X_i, W_i) \varepsilon_i \) by showing convergence of \( 1_{\{W_i = 0\}} \hat{\gamma} \) to \( \gamma_\psi(X_i, W_i) \). To do this, we use Lemma 8
of Hirshberg and Wager (2018). This suffices, as in Hirshberg and Wager (2018, Appendix A.4) it is shown that if \( \gamma_1 \ldots \gamma_n \) satisfy the bound \( n^{-1} \sum_{i=1}^{n} (\gamma_i - \hat{\gamma}_i(X_i, W_i))^2 \leq a \wedge b \) with probability \( 1 - \delta' \), then the bound (14) we aim to prove holds with probability \( 1 - \delta - \delta' \).

In order to apply Lemma 8 of Hirshberg and Wager (2018) in this setting, we must establish that the weights \( \hat{\gamma} \) that we discuss here are an instance of the weights \( \hat{\gamma} \) discussed in that paper. We prove this proposition in Appendix E.

**Proposition 11.** Let \( h(x, w, f) = T(x, w)f(x, 0) \), let \( \mathcal{B} \) be the unit ball of a reflexive space of functions on a set \( \mathcal{X} \), and let \( \mathcal{B}^C \) be the unit ball for the cartesian product of \( C+1 \) copies of this space considered as functions \( f(x, w) \) on \( (\mathcal{X}, \{0 \ldots C\}) \). Then the primal problem

\[
\ell_{n, \mathcal{B}^C}(\gamma) = \ell^2_{h, \mathcal{B}^C}(\gamma) + \frac{\sigma^2}{n^2} \| \gamma \|^2, \quad I_{h, \mathcal{F}} = \sup_{f \in \mathcal{F}} \frac{1}{n} \sum_{i=1}^{n} [h(X_i, W_i, f) - \gamma_i f(X_i, W_i)],
\]

(25)

has a unique minimum at \( \hat{\gamma} \) satisfying \( \hat{\gamma}_i = 0 \) for \( W_i \neq 0 \). Furthermore, the dual

\[
M_{n, \mathcal{B}^C}(\gamma) = -\frac{\sigma^2}{n} \| g \|^2_{\mathcal{B}^C} - \frac{1}{n} \sum_{i=1}^{n} g(X_i, W_i)^2 + \frac{2}{n} \sum_{i=1}^{n} h(X_i, W_i, g),
\]

(26)

has a possibly nonunique maximum, and for any \( \hat{\gamma} \) at which its maximum is attained, \( \hat{\gamma}_i = \hat{\gamma}(X_i, W_i) \) and \( \hat{\gamma}(s, w) = 0 \) for \( w \neq 0 \).

Here we take \( \mathcal{B} \) to be the unit ball of our RKHS \( \mathcal{H}_K \). Subject to the constraint that \( \hat{\gamma}_i = 0 \) if \( W_i \neq 0 \), a constraint that is satisfied by the solution of (25), this problem reduces to the problem (7) that defines our weights. Thus, our weights solve it. Having established this, we may now show convergence of \( \hat{\gamma} \) to \( \gamma \).

As we know that both \( \hat{\gamma} \) and \( \gamma_\psi \) satisfy the property \( g(\cdot, w) = 0 \) for \( w \neq 0 \), we apply Lemma 8 of Hirshberg and Wager (2018) with \( \hat{g} \) satisfying this property and with \( \mathcal{F} = \mathcal{F} = \{ f : \| f(\cdot, 0) \|_{\mathcal{H}_K} \leq 1, f(\cdot, w) = 0 \) for \( w \neq 0 \}. \) The resulting bound will be stated in terms of a few properties of the sets \( \mathcal{F}^*(t) = \{ f - s\gamma_\psi : f \in \mathcal{F}, s \in [0, 1], \| f - \gamma_\psi \|_{L_2(\mathcal{P})} \leq t \} \) and \( \mathcal{H}^*(t) = \{ T(x, w)f(x, 0) - \gamma_\psi(x, w)f(x, w) : f \in \mathcal{F}^*(t) \}. \) The relevant properties are, in terms of a convenient choice of constant \( \eta_Q = (61 - 8\sqrt{39})/49 \approx .23 \) and arbitrary \( \eta_C > 0, \)

\[
r_Q(\eta_Q) = 7 \inf \{ r > 0 : R_n(\mathcal{F}^*(r)) \leq r^2/(2M_{\mathcal{F}}) \} \quad \text{and} \quad r_C(\eta_C, \delta) = \inf \{ r > 0 : u(\mathcal{H}^*(r), \delta) \leq \eta_C r^2 \}
\]

where

\[
u(\mathcal{H}, \delta) = \min_{\eta < 0} 2(1 + \eta)R_n(\mathcal{H}) + \sigma(\mathcal{H}) \sqrt{\frac{2 \log(2\delta^{-1})}{n}} + 2M_H \left( \frac{1}{3} + \frac{1}{\eta} \right) \log(2\delta^{-1}) \frac{1}{n};
\]

\[
M_{\mathcal{G}} = \sup_{g \in \mathcal{G}} \| g \|_{\infty}; \quad \sigma(\mathcal{G}) = \sup_{g \in \mathcal{G}} \| g \|_{L_2(\mathcal{P})}.
\]

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It will also be stated in terms of a measure of the approximability of the Riesz representer $\gamma_\psi$ by a function $\tilde{g}$ with $\tilde{g}(\cdot, 0) \in \mathcal{H}_K$, specifically a bound $\bar{R}$ satisfying with probability $1 - \delta$

$$\bar{R} \geq \frac{1}{n} \sum_{i=1}^n [(\tilde{g} - \gamma_\psi)(X_i, W_i)]^2 - \frac{2}{n} \sum_{i=1}^n (T_i - \gamma_\psi(X_i, W_i))(\tilde{g} - \gamma_\psi)(X_i, W_i) + \frac{\sigma^2 \|\tilde{g}(\cdot, 0)\|_{\mathcal{H}_K}^2}{n}. \quad (27)$$

In terms of these quantities, this lemma yields the bound $n^{-1} \sum_{i=1}^n (\hat{g}(X_i, W_i) - \gamma_\psi(X_i, W_i))^2 \leq a \wedge b$ with probability $1 - \exp\{-c_p(\eta Q)n r_Q(\eta Q)^2/M_{\mathcal{F}}^2\} - 4\delta$ where

$$a = su(\mathcal{H}^*, \delta) + \bar{R};$$

$$b = 2s_Q^2 + 2 - \frac{\bar{R} + s_Q^2/n}{\eta Q - 2s_Q^{-1} \eta C} + \frac{44M_{\mathcal{F}}^2 s_Q^2 \log(\delta^{-1})}{n};$$

$$s = 1 + \left[2\eta C \sigma^{-2} n r^2 + \sigma^{-1} n^{1/2} \bar{R}^{1/2}\right];$$

$$r = r_Q(\eta Q) \vee r_C(\eta C, \delta) \vee \sigma n^{-1/2} \eta_C^{-1/2};$$

$$c_p(\eta Q) = \frac{(1 - \eta Q)^2}{2(1 + \eta Q)(21 - 11\eta_Q)} \approx .02$$

To complete our proof, it suffices to bound these quantities. We do this below in Section F, bounding $u(\mathcal{H}^*, \delta)$, $r_Q(\eta_Q)$, and $r_C(\eta C, \delta)$ using Lemmas 9 and 10.

The lemma stated below, which is proven in Section E, characterizes $\bar{R}$ when we take $\lambda = \sigma^2/n$.

**Lemma 12.** Suppose that we observe $X_1, W_1, \ldots, X_n, W_n$ iid and let $P_Z$ be the conditional distribution of $X_i$ given $W_i = 0$ and have support equal to a compact set $\mathcal{X}$. Let $\mathcal{H}_K$ be an RKHS of functions of $\mathcal{X}$ and $\gamma_\psi(x, w) = 1_{\{w=0\}} g_\omega(x)$ be the Riesz representer for the functional $f(x, w) \rightarrow E T(x, w)f(x, 0)$ and for an approximation $\tilde{g}(x, w) = 1_{\{w=0\}} \tilde{g}(x, w)$, define

$$\bar{R}_{\lambda, \tilde{g}} = \frac{1}{n} \sum_{i=1}^n (\tilde{g}(X_i, W_i) - \gamma_\psi(X_i, W_i))^2 - \frac{2}{n} \sum_{i=1}^n (T_i - \gamma_\psi(X_i, W_i))(\tilde{g}(X_i, W_i) - \gamma_\psi(X_i, W_i)) + \lambda \|\tilde{g}\|_{\mathcal{H}_K}^2.$$

1. If $\|g_\omega\|_{L_2(P_Z)} < \infty$, $\mathcal{H}_K$ is dense in $L_2(P_Z)$, and $\lambda \rightarrow 0$, then $\gamma_\psi$ has a sequence of approximations $\tilde{g}_n(x, w) = 1_{\{w=0\}} \tilde{g}_n(x)$ such that $\bar{R}_{\lambda_n, \tilde{g}_n} = o_p(1)$.

2. Furthermore, if $\|g_\omega\|_{L_{K, P_Z}} < \infty$ for $\kappa \in (0, 1/2)$, $\gamma_\psi$ has an approximation $\tilde{g}(x, w) = 1_{\{w=0\}} \tilde{g}(x)$ such that with probability $1 - \delta$

$$\bar{R}_{\lambda, \tilde{g}} \leq \alpha^2 \kappa^2 + 2a_\kappa \|g_\psi\|_{L_{K, P_Z}}^{-1/2}\lambda^\kappa \quad \text{where}$$

$$\alpha = x + x^{-1} \quad \text{where} \quad x = 4(25^{-1} P_Z) \tau^{-1/2}\kappa \|g_\psi\|_{L_{K, P_Z}}^2$$

where $P_Z = P\{W_i = 0\}$.

---

9Here rather than the general definition of $u(\cdot, \delta)$ given in Hirshberg and Wager (2018), we use a specific instance based on a convenient form of Talagrand’s inequality (Bartlett et al., 2005, Theorem 2.1).
Proof of Lemma 1. To simplify our notation, we’ll use $Z_i$ as a shorthand for $1_{\{W_i=0\}}$. Our weighting problem (7) is

$$\frac{\sigma^2}{n^2} \sum_{i:Z_i=1} \gamma_i^2 + \sup_{f:\|f\|\leq 1} \left[ \frac{1}{n} \sum_i (T_i - Z_i \gamma_i) (K_{X_i,f}) \right]^2$$

$$= \frac{\sigma^2}{n^2} \sum_{i:Z_i=1} \gamma_i^2 + \left( \frac{1}{n} \sum_i (T_i - Z_i \gamma_i) K_{X_i}, \frac{1}{n} \sum_j (T_j - Z_j \gamma_j) K_{X_j} \right)$$

$$= \frac{\sigma^2}{n^2} \sum_{i:Z_i=1} \gamma_i^2 + \frac{1}{n^2} \sum_{i,j} (T_i - Z_i \gamma_i)(T_j - Z_j \gamma_j) K(X_i,X_j)$$

$$= \frac{1}{n^2} \left[ \sigma^2 \gamma^T \gamma + \gamma^T K_{Z,T} 1 - 2 \gamma^T K_{Z,Z} \gamma + \gamma^T (K_{Z,Z} + \sigma^2 I) \gamma \right]$$

where $K$ is the Gram matrix ($K_{i,j} = K(X_i,X_j)$), $1$ is a vector of $|\{i : T_i = 1\}|$ ones, and subscripting by $Z$ or $T$ takes the rows of columns corresponding to units in those groups. At the minimum over $\gamma$, the derivative with respect to $\gamma$ will be zero, so our weights solve $(K_{Z,Z} + \sigma^2 I) \gamma = K_{Z,T} 1$, and the weighted average of treatment outcomes is

$$n^{-1} \sum_{i=1}^n Z_i \gamma_i Y_i = n^{-1} Y_{Z,T} \gamma = n^{-1} Y_{Z,T} (K_{Z,Z} + \sigma^2 I)^{-1} K_{Z,T} 1. \quad (29)$$

Now consider ridge regression on the treated units. We estimate $\hat{m}$ solving

$$\min_{f} \sum_{i:Z_i=1} (Y_i - \langle K_{X_i}, f \rangle)^2 + \sigma^2 \|f\|^2.$$

We can write it equivalently in constrained form,

$$\min_{r,f} \sum_{i:Z_i=1} r_i^2 + \sigma^2 \|f\|^2 \quad \text{where} \quad r_i = \langle K_{X_i}, f \rangle - Y_i.$$

This problem is solved by a saddle point of the Lagrangian (Peypouquet, 2015, Theorem 3.6.8),

$$L((r,f),\lambda) = \sum_{i:Z_i=1} r_i^2 + \sigma^2 \|f\|^2 + 2 \sum_{i:Z_i=1} \lambda_i (\langle K_{X_i}, f \rangle - Y_i - r_i).$$
For given $\lambda$, we can minimize over $(r, f)$ explicitly, solving the conditions $r_i - \lambda_i = 0$ and $\sigma^2 f + \sum_{i:Z_i=1} \lambda_i K_{X_i} = 0$ that arise from setting the derivatives with respect to $r_i$ and $f$ to zero. Substituting the optimal values $\hat{r}_i = \lambda_i$ and $\hat{f} = -\sigma^{-2} \sum_{i:Z_i=1} \lambda_i K_{X_i}$,

$$L((\hat{r}, \hat{f}), \lambda) = \sum_{i \in Z} \lambda_i^2 + \sigma^{-2} \left( \sum_{i:Z_i=1} \lambda_i K_{X_i} + \sum_{j:Z_j=1} \lambda_j K_{X_j} \right)$$

$$+ 2 \sum_{i:Z_i=1} \lambda_i \left[ -\sigma^{-2} \left( \sum_{j:Z_j=1} \lambda_j K_{X_j}, K_{X_i} \right) - Y_i - \lambda_i \right]$$

$$= \sum_{i:Z_i=1} \lambda_i^2 + \sigma^{-2} \sum_{i:Z_i=1} \lambda_i \lambda_j K(X_i, X_j)$$

$$- 2\sigma^{-2} \sum_{i:Z_i=1} \lambda_i \lambda_j K(X_j, X_i) - 2 \sum_{i:Z_i=1} (\lambda_i Y_i + \lambda_i^2)$$

$$= -\lambda^T \lambda - 2\lambda^T Y_Z - \sigma^{-2} \lambda^T K_{Z,Z} \lambda$$

$$= -2\lambda^T Y_Z - \lambda^T \left( \sigma^{-2} K_{Z,Z} + I \right) \lambda.$$

This is maximized at $\hat{\lambda} = - \left( \sigma^{-2} K_{Z,Z} + I \right)^{-1} Y_Z = -\sigma^2 \left( K_{Z,Z} + \sigma^2 I \right)^{-1} Y_Z$.

Thus, we have a saddle at $((\hat{r}, \hat{f}), \hat{\lambda})$ and the function $\hat{m}$ solving our problem is $\hat{f}$. Substituting in $\hat{\lambda}$ into our expression for $\hat{f}$ above,

$$(K_x, \hat{f}) = \left[ -\sigma^{-2} \sum_{i:Z_i=1} \left[ -\sigma^2 \left( K_{Z,Z} + \sigma^2 I \right)^{-1} Y_Z \right] K_{X_i}, K_x \right]$$

$$= \sum_{i:Z_i=1} Y_Z^T \left( K_{Z,Z} + \sigma^2 I \right)^{-1} K(X_i, x).$$

Therefore our ridge regression prediction $\hat{f}$, averaged over our target sample, is

$$\left( n^{-1} \sum_{j:Y_j=1} K_{X_j}, \hat{f} \right) = n^{-1} \sum_{j:Y_j=1} \sum_{i:Z_i=1} Y_Z^T \left( K_{Z,Z} + \sigma^2 I \right)^{-1} K(X_i, X_j).$$

$$= n^{-1} Y_Z^T \left( K_{Z,Z} + \sigma^2 I \right)^{-1} K_{Z,T} 1.$$

This is the weighted average of treatment outcomes using our minimax weights, completing our proof.

Proof of Lemma 6. The eigenvalues of the product have the form $\sigma^\alpha / (\sigma + \lambda) = 1/(\sigma^{1-\alpha} + \lambda \sigma^{-\alpha})$ where $\sigma$ is an eigenvalue of $L$. We maximize this expression over all $\sigma$. The derivative of the denominator is

$$(1 - \alpha)\sigma^{-\alpha} - \lambda \alpha \sigma^{-(1+\alpha)} = \sigma^{-(1+\alpha)}[(1 - \alpha)\sigma - \lambda \alpha]$$

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and is zero at \( \sigma = \lambda \alpha/(1 - \alpha) \). To check that this is a minimum of the denominator, we check positivity of the second derivative at this point. The second derivative of the denominator is
\[
-\alpha(1 - \alpha)\sigma^{-(1-\alpha)} + \lambda \alpha (1+\alpha) \sigma^{-(2+\alpha)} = \alpha \sigma^{-(2+\alpha)}[-(1 - \alpha)\sigma + \lambda(1 + \alpha)]
\]
and at \( \sigma = \lambda \alpha/(1 - \alpha) \) this takes the value \( \alpha \sigma^{-(2+\alpha)} \lambda > 0 \). Evaluating the expression \( 1/(\sigma^{1-\alpha} + \lambda \sigma^{-\alpha}) \) at \( \sigma = \lambda \alpha/(1 - \alpha) \) yields our bound. \( \square \)

**Proof of Lemma 7.** Reparameterizing Cucker and Zhou (2007, Theorem 4.1) in terms of \( \kappa = \theta/(4 + 2\theta) \) gives the bound
\[
\inf_{\|g'\| \leq R} \|g' - g\| \leq (2\|g\|_{L_2(\nu)})^{1-2\kappa} R^{-\frac{2\kappa}{1-2\kappa}}.
\]
If this infimum is attained at some \( g' \), then
\[
\|g' - g\|_{L_2(\nu)} + t\|g'\| \leq (2\|g\|_{L_2(\nu)})^{1-2\kappa} R^{-\frac{2\kappa}{1-2\kappa}} + tR,
\]
and the infimum over \( g' \) of \( \|g' - g\|_{L_2(\nu)} + t\|g'\| \) can be no larger than the minimum of the right side above over \( R \). This bound is convex and differentiable in \( R \), so it is minimized at a zero of its derivative, which occurs where
\[
t = (2\|g\|_{L_2(\nu)})^{\frac{1}{1-2\kappa}} \frac{2\kappa}{1 - 2\kappa} R^{-1 + \frac{2\kappa}{1-2\kappa}} \text{ i.e. } R = t^{1-2\kappa} 2\|g\|_{L_2(\nu)} \left( \frac{2\kappa}{1 - 2\kappa} \right)^{1-2\kappa}.
\]
Evaluating our bound at this value of \( R \) yields our bound. \( \square \)

**Proof of Lemma 8.** Expanding \( K_x \) in the orthonormal basis \( (\lambda_j^{1/2} \phi_j)_{j \in \mathbb{N}} \) of \( \mathcal{H}_K \), we have
\[
K_x = \sum_j \langle K_x, \lambda_j^{1/2} \phi_j \rangle \lambda_j^{1/2} \phi_j = \sum_j \lambda_j^{1/2} \phi_j(x) \lambda_j^{1/2} \phi_j
\]
and consequently
\[
\left\| (L_{K, \nu} + \lambda I)^{-1/2} K_x \right\|^2 = \left\| \sum_j \lambda_j^{1/2} \phi_j(x) \lambda_j^{1/2} \phi_j \right\|^2 = \sum_j \frac{\lambda_j \phi_j(x)^2}{\lambda_j + \lambda} \leq C^2_{\phi} \sum_j \frac{\lambda_j^{1-\beta}}{\lambda_j + \lambda}.
\]
Here the last step holds \( \nu \)-almost-everywhere by our assumption \( \|\phi_j\|_{L_\infty(\nu)} \leq C_{\phi} \lambda_j^{-\beta/2} \).

The function \( t^{1-\beta}/(t + \lambda) \) is increasing for \( 0 \leq t < \lambda(1 - \beta)/\beta \) — the sign of its derivative is that of \( (1 - \beta)t^{-\beta}(t + \lambda) - t^{1-\beta}((1 - \beta)\lambda - \beta t) \). Thus, we may substitute our bound \( C_{\lambda_j^{-\alpha}} \) for eigenvalues \( \lambda_j \) smaller than this threshold. Ordering the eigenvalues \( \lambda_j \) so they are decreasing and taking \( J = \max\{j \in \mathbb{N} : \lambda_j \geq \lambda(1 - \beta)/\beta \} \), we bound the sum (30) above by
\[
C^2_{\phi} \sum_{j \leq J} \frac{\lambda_j^{1-\beta}}{\lambda_j + \lambda} + C^2_{\phi} \sum_{j > J} \frac{(C_{\lambda_j^{-\alpha}})^{1-\beta}}{C_{\lambda_j^{-\alpha}} + \lambda}.
\]
Furthermore, as (i) for all $J$ terms in the first sum here, $\lambda_j^{1-\beta}/(\lambda_j + \lambda) \leq \lambda_j^{1-\beta} \leq \lambda_j^{1-\beta}$ and (ii) $\lambda_j^{1-\beta}/\beta \leq \lambda_j \leq C \lambda J^{-\alpha}$ and therefore $J \leq [C \beta/(\lambda j^{(1-\beta)}))]^{1/\alpha}$, we may bound it by
\[
C_\phi^2 \left[ \frac{C \beta}{\lambda(1-\beta)} \right]^{1/\alpha} \left[ \frac{1}{\beta} \right]^{1/\alpha} \leq \lambda - \beta \leq \left[ \frac{C \beta}{\lambda(1-\beta)} \right]^{1/\alpha} \lambda^{-1/(\alpha+\beta)}.
\]
In addition, we may bound the second sum here by an integral,
\[
C_\phi^2 \int_0^\infty \left( \frac{C^{-1/\alpha} t}{C^{-1/\alpha} t + \lambda} \right)^{-\alpha(1-\beta)} dt = C_\phi^2 C_{\lambda}^{1/\alpha} \int_0^\infty \frac{s^{-\alpha(1-\beta)}}{s^{-\alpha} + \lambda} ds,
\]
decompose that integral into two pieces,
\[
\int_0^{\lambda^{-1/\alpha}} \frac{s^{-\alpha(1-\beta)}}{s^{-\alpha} + \lambda} ds + \int_\lambda^{\lambda^{-1/\alpha}} \frac{s^{-\alpha(1-\beta)}}{s^{-\alpha} + \lambda} ds,
\]
and bound each piece as follows:
\[
\int_0^{\lambda^{-1/\alpha}} \frac{s^{-\alpha(1-\beta)}}{s^{-\alpha} + \lambda} ds \leq \int_0^{\lambda^{-1/\alpha}} s^{-\alpha\beta} ds \leq \frac{1}{1 + \beta \lambda^{-1/(\alpha+\beta)}}
\]
\[
\int_\lambda^{\lambda^{-1/\alpha}} \frac{s^{-\alpha(1-\beta)}}{s^{-\alpha} + \lambda} ds \leq \lambda^{-1} \int_0^\infty \frac{s^{-\alpha(1-\beta)}}{s^{-\alpha} + \lambda} ds
\]
\[
= \lambda^{-1} \left[ 1 - \alpha(1-\beta) \right] \left[ 0 - \left( \lambda^{-1/\alpha} \right)^{1-\alpha(1-\beta)} \right]
\]
\[
= \lambda^{-1/(\alpha+\beta)} \alpha(1-\beta) - 1.
\]
To guarantee that the latter integral converges, we use our assumption $\alpha(1-\beta) > 1$.

Putting everything together, (31) and therefore (30) is bounded by
\[
C_\phi^2 C_{\lambda}^{1/\alpha} \left[ \left( \frac{\beta}{1-\beta} \right)^{1/\alpha+\beta} + \frac{1}{1 + \alpha \beta} + \frac{1}{\alpha(1-\beta) - 1} \right] \lambda^{-1/(\alpha+\beta)}
\]
\[
= C_\phi^2 C_{\lambda}^{1/\alpha} \left[ \left( \frac{\beta}{1-\beta} \right)^{1/\alpha+\beta} \frac{\alpha}{(1 + \alpha \beta)(\alpha - (1 + \alpha \beta))} \right] \lambda^{-1/(\alpha+\beta)}.
\]
Thus, the square root of this quantity bounds $\| [L_{K,\nu} + \lambda I]^{-1/2} K_x \| \nu$-a.e. as claimed.

**Proof of Lemma 9.** The unit ball of $H_K$ can be characterized as $\{ \sum_{j=1}^\infty f_j^{1/2} \phi_j : \sum_{j=1}^\infty f_j^2 \leq 1 \}$ where $\phi_j$ are eigenfunctions of $L_{K,\nu}$ that form an orthonormal
basis of $L_2(\nu)$. Let $\phi_0 = g/\|g\|_{L_2(\nu)}$ and $\hat{f}_j = \phi_j - \langle \phi_j, \phi_0 \rangle_{L_2(\nu)} \phi_0$ for $j \geq 1$. Any function in our set $B^*$ can be written in the form

$$f - sg = \sum_{j=1}^{\infty} f_j \lambda_j^{1/2} \hat{f}_j + \left[ \sum_{j=1}^{\infty} f_j \lambda_j^{1/2} \langle \phi_j, \phi_0 \rangle_{L_2(\nu)} - s \|g\|_{L_2(\nu)} \right] \phi_0$$

for $\sum_{j=1}^{\infty} f_j^2 \leq 1$, $s \in [0, 1]$. By Cauchy-Schwartz, the bracketed term is bounded by $\lambda_0^{1/2} = \sqrt{\sum_{j=1}^{\infty} \lambda_j \langle \phi_j, \phi_0 \rangle_{L_2(\nu)}^2 + \|g\|_{L_2(\nu)}^2}$. Thus, $B^*$ is contained in the set $B'$ of functions of the form

$$f = f_0 \lambda_0^{1/2} \phi_0 + \sum_{j=1}^{\infty} f_j \lambda_j^{1/2} \hat{f}_j$$

for $\sum_{j=1}^{\infty} f_j^2 \leq 1, f_0^2 \leq 1$.

Define the rescaled basis functions $\hat{\phi}_j = \phi_j / \|\phi_j\|_{L_2(\nu)}$ and $\tilde{\lambda}_j = \lambda_j \|\phi_j\|_{L_2(\nu)}^2 = \lambda_j (1 - \langle \phi_j, \phi_0 \rangle_{L_2(\nu)}^2)$ for $j \geq 1$ and let $\hat{\phi}_0 = \phi_0$ and $\tilde{\lambda}_0 = \lambda_0$. Equivalently, we may say that $B'$ is the set of functions

$$f = \sum_{j=0}^{\infty} f_j \tilde{\lambda}_j^{1/2} \hat{\phi}_j$$

for $\sum_{j=1}^{\infty} f_j^2 \leq 1, f_0^2 \leq 1$.

From this point we imitate the proof of Mendelson (2002, Theorem 41). If $f$ is a function of the form above, $Z_i f(X_i)$ satisfies

$$E(Z_i f(X_i))^2 = \int f(x)^2 s_x \text{d} \nu_x = \int f(x)^2 \text{d} \nu(x) = \sum_{j=1}^{\infty} f_j^2 \tilde{\lambda}_j.$$

Therefore if $f(x)$ is in the set $B'_i = \{ f \in B' : E(Z_i f(X_i))^2 \leq t^2 \}$, it has coefficients that satisfy

$$f_0^2 \leq 1, \sum_{j=1}^{\infty} f_j^2 \leq 1, \sum_{j=0}^{\infty} f_j^2 \tilde{\lambda}_j / t^2 \leq 1.$$

Now consider the set $\mathcal{E}_i$ of functions $f$ with coefficients satisfying $\sum_{j=0}^{\infty} f_j^2 (1 \lor \tilde{\lambda}_j / t^2) \leq 1$. As $\sum_{j=0}^{\infty} f_j^2 (1 \lor \tilde{\lambda}_j / t^2) \leq f_0^2 + \sum_{j=1}^{\infty} f_j^2 + \sum_{j=0}^{\infty} f_j^2 \tilde{\lambda}_j / t^2 \leq 3$ for all functions in $B'_i$, $\sqrt{3} \mathcal{E}_i$ contains $B'_i$ and therefore also $B'_i = \{ f \in B' : E(Z_i f(X_i))^2 \leq t^2 \}$. Thus, we will use $\sqrt{3} M_n \{ z f(x) : f \in \mathcal{E}_i \}$ to bound $M_n \{ z f(x) : f \in B'_i \}$.

In the case that $t = \infty$, we can improve this constant $\sqrt{3}$ to $\sqrt{2}$. $\mathcal{E}_\infty$ is the set of functions $f$ with coefficients satisfying $\sum_{j=0}^{\infty} f_j^2 \leq 1$, and as $f_0^2 + \sum_{j=1}^{\infty} f_j^2 \leq 2$ for $f \in B'$, $\sqrt{2} \mathcal{E}_\infty \supset B' \supset B^*$.

We will bound $M_{n,2} \{ z f(x) : f \in \mathcal{E}_i \} = \| \text{sup}_{f \in \mathcal{E}_i} n^{-1} \sum_{i=1}^{n} s_i Z_i f(X_i) \|^{1/2}$, as by Jensen’s inequality this quantity bounds $M_n \{ z f(x) : f \in \mathcal{E}_i \}$ itself. Writ-
ing $e_0,e_1, \ldots$ for the standard basis for $\ell_2$, we have

$$M_{n,2}\{zf(x) : f \in \mathcal{E}_t\}^2 = E \sup_{f \in \mathcal{E}_t} \left( \sum_{j=0}^{\infty} f_j e_j, n^{-1} \sum_{i=1}^{\infty} \sum_{j=0}^{\infty} \sigma_i Z_i \hat{\lambda}_j^{1/2} \hat{\phi}_j(X_i)e_j \right)_{\ell_2}^2$$

$$= E \sup_{f \in \mathcal{E}_t} \left( \sum_{j=0}^{\infty} f_j \sqrt{1 + \hat{\lambda}_j/t^2} e_j, n^{-1} \sum_{i=1}^{\infty} \sum_{j=0}^{\infty} \sigma_i Z_i \sqrt{1 + \hat{\lambda}_j/t^2} \hat{\phi}_j(X_i)e_j \right)_{\ell_2}^2$$

$$\leq E \left| n^{-1} \sum_{i=1}^{\infty} \sum_{j=0}^{\infty} \sigma_i \sqrt{\hat{\lambda}_j + t^2} \hat{\phi}_j(X_i)e_j \right|_{\ell_2}^2$$

$$= n^{-2} \sum_{i=1}^{\infty} \sum_{j=0}^{\infty} (\hat{\lambda}_j + t^2) E[\sigma_i^2 \hat{X}_i^2 \hat{\phi}_j(X_i)^2]$$

$$= n^{-1} \sum_{j=0}^{\infty} \left( \hat{\lambda}_j + t^2 \right) E[\sigma_i^2 | X_i, Z_i] E[\hat{X}_i^2 \hat{\phi}_j(X_i)^2]$$

$$\leq n^{-1} \sum_{j=0}^{\infty} \left( \hat{\lambda}_j + t^2 \right) \|E[\sigma_i^2 | X_i, Z_i]\|_{L_\infty(\nu, x, z)} \|Z_i^2 \hat{\phi}_j(X_i)^2\|_{L_1(\nu, x, z)}$$

$$= \left( n^{-1} \sum_{j=0}^{\infty} \hat{\lambda}_j + t^2 \right) \|E[\sigma_i^2 | X_i, Z_i]\|_{L_\infty(\nu, x, z)}.$$ 

The inequalities above are via Cauchy-Schwarz and Hölder’s inequality respectively.

All that remains now is to simplify this bound so we need not discuss $\hat{\lambda}_j$. Recall that $\hat{\lambda}_0 = \left( \sqrt{\sum_{j=1}^{\infty} \lambda_j (\phi_j, \phi_0)_{L_2(\nu)}^2 + \|g\|_{L_2(\nu)}^2} \right)^2$ and that $\hat{\lambda}_j = \lambda_j (1 - (\phi_j, \phi_0)_{L_2(\nu)}^2)$ for $j \geq 1$. Typically when we consider the local Rademacher complexity, i.e. the case $t < \infty$, we take $t$ small, so we will have $t$ in our sum rather than the large $\hat{\lambda}_0$ term. Thus, in that case we will not be able to cancel terms appearing in $\lambda_0$ and the rest of the sum. To bound $\lambda_0$, we apply Hölder’s inequality to the term inside the square root in our expression for $\hat{\lambda}_0$, yielding $\hat{\lambda}_0 \leq \lambda_0' = [(\lambda_1^{1/2} + 1)\|g\|_{L_2(\nu)}^2]^2$. Typically $\hat{\lambda}_0 > t$, so the looseness of this bound will be irrelevant. For the other terms, we use the simple bound $\hat{\lambda}_j \leq \lambda_j$ for $j \geq 1$. Thus, letting $\lambda_j' = \lambda_j$, we have our claimed bound $M_n\{zf(x) : f \in \mathcal{B}_t, E(Z_i f(X_i))^2 \leq t^2\} \leq \sqrt{3} M_{n,2}\{zf(x) : f \in \mathcal{E}_t\} \leq \|E[\sigma_i^2 | X_i, Z_i]\|_{L_\infty(\nu, x, z)}^{1/2} \cdot 3^{1/2} 2^{-1/2} \sum_{j=0}^{\infty} \lambda_j' \cdot t$.

In the case $t = \infty$, terms from $\lambda_0$ and the other terms in our sum will cancel,
giving us a better bound.

\[
\sum_{j=0}^{\infty} \tilde{\lambda_j} = \sum_{j=1}^{\infty} \lambda_j \langle \phi_j, \phi_0 \rangle_{L_2(\nu)}^2 + 2\|g\|_{L_2(\nu)} \sqrt{\sum_{j=1}^{\infty} \lambda_j \langle \phi_j, \phi_0 \rangle_{L_2(\nu)}^2} \\
+ \|g\|_{L_2(\nu)} + \sum_{j=1}^{\infty} \lambda_j \left(1 - \langle \phi_j, \phi_0 \rangle_{L_2(\nu)}^2\right) \\
= 2\|g\|_{L_2(\nu)} \sqrt{\sum_{j=1}^{\infty} \lambda_j \langle \phi_j, \phi_0 \rangle_{L_2(\nu)}^2 + \|g\|_{L_2(\nu)} + \sum_{j=1}^{\infty} \lambda_j} \\
\leq 2\|g\|_{L_2(\nu)} \sqrt{\sum_{j=1}^{\infty} \lambda_j + \|g\|_{L_2(\nu)} + \sum_{j=1}^{\infty} \lambda_j} \\
= \left(\|g\|_{L_2(\nu)} + \sqrt{\sum_{j=1}^{\infty} \lambda_j}\right)^2.
\]

Therefore \(M_n\{z f(x) : x \in B^*\} \leq 2^{1/2} M_n,2(\mathcal{E}_\infty) \leq 2^{1/2}\|E[\sigma_t^2 \mid X_t, Z_t]\|_{L_\infty(\nu_{x,z})}^{1/2} n^{-1/2}(\|g\|_{L_2(\nu)} + \sqrt{\sum_{j=1}^{\infty} \lambda_j})\) as claimed.

**Proof of Lemma 10.** Let \(J = \max\{j : \lambda_j \geq t^2\}\). \(J\) satisfies \(t^2 \leq \lambda_J \leq CJ^{-\alpha}\) and therefore \(J \leq (C/t^2)^{1/\alpha}\). Therefore,

\[
\sum_{j=1}^{\infty} \lambda_j \wedge t^2 = Jt^2 + \sum_{j=J+1}^{\infty} \lambda_j \\
\leq (C/t^2)^{1/\alpha} t^2 + \int_{(C/t^2)^{1/\alpha}}^{\infty} Cs^{-\alpha} ds \\
= C^{1/\alpha} t^{2(1-1/\alpha)} + [C/(\alpha - 1)][(C/t^2)^{1/\alpha}]^{1-\alpha} \\
= C^{1/\alpha}[\alpha/(\alpha - 1)] t^{2(1-1/\alpha)}.
\]

**Proof of Proposition 11.** Consider the decomposition of the expression maximized in \(I_h,B^C\)

\[
\frac{1}{n} \sum_{i=1}^{n} [h(X_i, W_i, f) - \gamma_i f(X_i, W_i)] = \frac{1}{n} \sum_{i=1}^{n} [T(X_i, W_i) - 1_{\{W_i = 0\}} \gamma_i] f(x, 0) + \frac{1}{n} \sum_{i : W_i \neq 0} \gamma_i f(x, w).
\]

If there were nonzero weights \(\gamma_i\) in the second sum, functions \(f(\cdot, 1) \ldots f(\cdot, C)\) could be chosen from the (symmetric) unit ball \(B\) that make the second term match the first in sign. It follows that the weights \(\gamma'_i = 1_{\{W_i = 0\}} \gamma_i\) satisfy
\[ I^2_{\delta,BC}(\gamma') \leq I^2_{\delta,BC}(\gamma) \] and, unless \( \gamma' = \gamma \), \( \|\gamma'\|^2 < \|\gamma\|^2 \). Therefore it suffices to optimize over weights of the form \( 1_{\{W_i = 1\}} \gamma_i \).

The space normed by \( BC \) that we consider is reflexive, as cartesian products of reflexive spaces are reflexive. Then Hirshberg and Wagner (2018, Lemma 5) establishes that \( M_{\delta,BC} \) has a maximum at some possibly nonunique function \( \hat{g} \) and that the weights satisfy \( \gamma_i = \hat{g}(X_i, W_i) \) for all such functions. To establish our last claimed property, observe that taking \( g(\cdot, w) \neq 0 \) for \( w \neq 0 \) decreases the second term of \( M_{\delta,BC} \) without increasing any other term.

\[ \square \]

**Proof of Lemma 12.** Let \( Z_i = 1_{\{W_i = 0\}} \). As \( \hat{g}(w, x) \) takes the form \( 1_{\{w = 0\}} g'(x) \) and \( \gamma_0(w, x) \) the form \( 1_{\{w = 0\}} g(x) \), we can rewrite the quantity we are bounding as

\[
\frac{1}{n} \sum_{i=1}^{n} Z_i (g'(X_i) - g(X_i))^2 - \frac{2}{n} \sum_{i=1}^{n} (T_i - Z_i g(X_i)) Z_i (g'(X_i) - g(X_i)) + \lambda g'_{\|H_K|}^2.
\]

The middle term is centered, so we bound it using Chebyshev’s inequality. With probability greater than \( 1 - \delta/2 \),

\[
\left| \frac{1}{n} \sum_{i=1}^{n} (T_i - Z_i g(X_i)) Z_i (g'(X_i) - g(X_i)) \right| < (n\delta/2)^{-1/2} E[(T_i - Z_i g(X_i))^2 Z_i (g'(X_i) - g(X_i))^2]^{1/2}
\]

\[
\leq (n\delta/2)^{-1/2} \|g\|_{L_\infty(P_Z)} E[Z_i (g'(X_i) - g(X_i))^2]^{1/2}
\]

\[
= (n\delta/2)^{-1/2} \|g\|_{L_\infty(P_Z)} E[Z_i]^{1/2} \|g' - g\|_{L_2(P_Z)}.
\]

We use Markov’s inequality for the first term. With probability greater than \( 1 - \delta/2 \),

\[
\frac{1}{n} \sum_{i=1}^{n} Z_i (g'(X_i) - g(X_i))^2 \leq (\delta/2)^{-1} E[Z_i (g'(X_i) - g(X_i))^2] = (\delta/2)^{-1} E[Z_i] \|g' - g\|_{L_2(P_Z)}^2.
\]

Thus, with probability \( 1 - \delta \), we have the bound

\[
2\delta^{-1} p Z \|g' - g\|_{L_2(P_Z)}^2 + 2^{3/2}(\delta^{-1} p Z)^{1/2} n^{-1/2} \|g\|_{L_\infty(P_Z)} \|g' - g\|_{L_2(P_Z)} + \lambda \|g'\|_{H_K}^2.
\]

(32)

Our first claim follows by observing that because \( H_K \) is dense in \( L_2(P_Z) \), there exists a sequence \( g'_j \) satisfying \( \|g'_j - g\|_{L_2(P_Z)} \to 0 \) as \( j \to \infty \) and \( \|g'_j\|_{H_K} < \infty \), as so long as \( \lambda_n \to 0 \), we can take a subsequence \( g'_n = g_{j_n} \) such that \( \lambda_n \|g_n\|_{H_K} \to 0 \).

Our second claim is a consequence of Cucker and Zhou (2007, Theorem 4.1), which establishes that if \( \|\cdot\|_{L_{k,\nu}} < \infty \),

\[
\inf_{g' : \|g'\|_{H_K} \leq R} \|g' - g\|_{L_2(P_Z)} \leq \left( 2 \|g\|_{L_2(P_Z)} \right)^{\frac{1}{\nu}} r^{-\frac{2}{1-2\nu}}.
\]

(33)

Thus, (32) is bounded by \( aR^{-2\theta} + bR^{-\theta} + \lambda R^2 \) for \( \theta = 2\kappa/(1 - 2\kappa) \), \( a = 2\delta^{-1} p Z \left( 2 \|g\|_{L_2(P_Z)} \right)^{\frac{1}{\nu}} r^{-\frac{2}{1-2\nu}} \), and \( b = 2^{3/2}\delta^{-1/2} p Z^{1/2} n^{-1/2} \|g\|_{L_\infty(P_Z)} (2 \|g\|_{L_2(P_Z)})^{\frac{1}{\nu}} \).
To bound the minimum of this expression, we first complete the square, writing it in the form $a(R^\theta + b/(2a))^2 - b^2/(4a) + \lambda R^2$. As the second term is constant, we will minimize a bound on the sum of the first and last term. This bound, the square of $a^{1/2}(R^\theta + b/(2a)) + \lambda^{1/2}R$, arises from the elementary inequality $a^2 + b^2 \leq (a + b)^2$ for $a, b > 0$. As it is convex and differentiable in $R$, it is minimized where its derivative is zero, at $R_* = (\lambda/a)^{-1/[2(1+\theta)]}g^{1/(1+\theta)}$. Thus, the infimum $(\ast)$ from (33) satisfies the bound

$$(\ast) \leq \left[ a^{1/2}(R_*^\theta + b/(2a)) + \lambda^{1/2}R_* \right]^2 - b^2/(4a) = x^2 + (b/a^{1/2})x \quad \text{for} \quad x = a^{1/2}R_*^\theta + \lambda^{1/2}R_* = x \left( a^{\theta/2} + a^{-\theta/2} \right) = \lambda \left( a^{1/2-\nu} + a^{\nu-1/2} \right) \quad \text{where}$$

$a^{1/2-\nu} = 4(2\delta^{-1}pZ)^{1/2-\nu} \|g\|^2_{L^2(pZ)}$

$b/a^{1/2} = 2\|g\|_{\infty} n^{-1/2}$.

This is our claimed bound.

\[\Box\]

## F Calculations used in the proof of (14)

In this section, we establish bounds on the quantities appearing in the bound (28). Because $f(\cdot, w) = 0$ when $w \neq 0$ for all $f \in \mathcal{F}$,

\[\mathcal{F}^*(t) = \left\{ 1_{\{w=0\}}f'(x) : f' \in \mathcal{B}^*, \|1_{\{w=0\}}f'\|_{L^2(p)} \leq t \right\};\]

\[\mathcal{H}^*(t) = \left\{ \|T(x, 0) - g_\psi(x)\|1_{\{w=0\}}f'(x) : 1_{\{w=0\}}f'(x) \in \mathcal{B}^*, \|1_{\{w=0\}}f'\|_{L^2(p)} \leq t \right\};\]

\[\mathcal{B}^* = \left\{ f - s g_\psi(x) : \|f\| \leq 1, s \in [0, 1] \right\}.\]

Via the triangle inequality, $M_{\mathcal{F}^*} \leq M_K + \|g_\psi\|_{\infty}$ and $M_{\mathcal{H}^*} \leq \|g_\psi\|_{\infty}(M_K + \|g_\psi\|_{\infty})$.

### F.1 Bounding non-aggregate terms

In this section, we bound the terms that appear in our bound: $\hat{R}$, $r_Q(\eta_C)$, $r_C(\eta_C)$, and $u(\mathcal{H}^*, \delta)$.

#### F.1.1 Bounding $\hat{R}$.

In the event that $\|g_\psi\|_{\mathcal{H}^*_K} < \infty$, we may take $\bar{g} = \gamma_\psi$, in which case the condition (27) is satisfied deterministically with $\hat{R} = (\sigma^2/n)\|g_\psi\|^2_{\mathcal{H}^*_K}$. If instead we have $\|g_\psi\|_{L^2_{\mathcal{H}^*_K}^{\nu_1}} < \infty$ for $\nu_1 \in (0, 1)$, we use the second claim of Lemma 12.

#### F.1.2 Bounding $r_Q(\eta_Q)$.

$r_Q(\eta_Q)$ is a fixed point of the local Rademacher complexity of the class $\mathcal{F}^*$. In the terms of Lemma 9, $R_n(\mathcal{F}^*(t)) = M_n\{zf(x) : f \in \mathcal{B}^*, E(Z_i f(X_i))^2 \leq t^2\}$ for...
Consequently, \( t \) independent of \( \nu \) apply Lemma 9 with noting that case \( \nu \). Our approach will be to find a simple function \( r \) for \( t = 2(1 + \| \psi \| \_{\infty}) \) for \( \| \psi \| \_{\infty} \leq (3/n) \sum_{j=0}^{\infty} (pZ \lambda_j, Z) \). Via Lemma 10, the assumptions of Theorem 3 guarantee that \( \sum_{j=1}^{\infty} \lambda_j, Z \) and \( t \leq (pZ C_\lambda, Z) \) for \( \lambda_0, Z = pZ (1 + \sqrt{pZ \lambda_1, Z})^2 \| g_\psi \|_{L_2(pZ)}^2 \). Via Lemma 10, the assumptions of Theorem 3 guarantee that \( \sum_{j=1}^{\infty} \lambda_j, Z \) and \( t \leq (pZ C_\lambda, Z) \) for \( \lambda_0, Z = pZ (1 + \sqrt{pZ \lambda_1, Z})^2 \| g_\psi \|_{L_2(pZ)}^2 \). Consequently, \( R(\mathcal{F}^* (t)) \leq C' n^{-1/2} t^{1-1/\alpha} \) where \( C' = \{3[1 + (pZ C_\lambda, Z)]^{1/\alpha}(1 - 1/\alpha)\}^{1/2} \). To bound \( r_Q (\eta Q) \), we take 7 times the solution to fixed point equation \( C' n^{-1/2} t^{1-1/\alpha} = t^2 / (2M_{\mathcal{F}^*}) \), which is \( t = (2M_{\mathcal{F}^*} C' n^{-1/2})^{1/(1+1/\alpha)} = c_Q \). For \( \eta \), we can solve the equation \( C' n^{-1/2} t^{1-1/\alpha} = t^2 / (2M_{\mathcal{F}^*}) \), which is \( t = (2M_{\mathcal{F}^*} C' n^{-1/2})^{1/(1+1/\alpha)} = c_Q \). For \( \eta \), we can solve the equation \( C' n^{-1/2} t^{1-1/\alpha} = t^2 / (2M_{\mathcal{F}^*}) \), which is \( t = (2M_{\mathcal{F}^*} C' n^{-1/2})^{1/(1+1/\alpha)} = c_Q \). Recalling our recent assumption that \( t \leq \min \{1, \lambda_0, Z \} \), this means that we have the bound

\[
r_Q (\eta Q) \leq 7C_Q n^{-1/2} (1+1/\alpha) \tag{35}
\]

if it is no larger than \( 7 \min \{1, \lambda_0, Z \}^{1/2} \) and therefore if it is no larger than \( 7P_{\mathcal{Z}}^{1/2} \| g_\psi \|_{L_2(pZ)} \).

### F.1.3 Bounding \( r_C(\eta C) \)

Our approach will be to find a simple function \( u' (\cdot) \) for which we can solve the fixed point equation \( u' (t) = \eta_c t^2 \) and which, at that fixed point \( t \), we have \( \eta_c t^2 = u' (t) \geq u(\mathcal{H}^* (t), \delta) \) and therefore \( t \geq r_C(\eta C) \). Recall that

\[
u = 2(1 + \eta) R^*_n (\mathcal{H}^* (t)) + \sigma(\mathcal{H}^* (t)) \sqrt{2 \log (2t^{-1}) \over n} + 2 M_{\mathcal{H}^*} \left( \frac{1}{3} + \frac{1}{\eta} \right) \log (2t^{-1}) \over n}.
\]

Our first step will be to establish that for all \( \eta > 0 \), when \( t \leq P_{\mathcal{Z}}^{1/2} \| g_\psi \|_{L_2(pZ)} \),

\[
u = 2(1 + \eta) R^*_n (\mathcal{H}^* (t)) + \sigma(\mathcal{H}^* (t)) \sqrt{2 \log (2t^{-1}) \over n} + 2 M_{\mathcal{H}^*} \left( \frac{1}{3} + \frac{1}{\eta} \right) \log (2t^{-1}) \over n}.
\]

Here we include the third term in \( u(\mathcal{H}^* (t), \delta) \) as-is and include a bound on the second using \( \sigma(\mathcal{H}^* (t)) \leq \| T(x, 0) - g_\psi (x) \|_{\infty} \leq \max \{1, \| g_\psi (x) \|_{\infty} \} \), so what remains to do is show that \( C_1 n^{-1/2} t^{1-1/\alpha} \) bounds \( R_n (\mathcal{H}^* (t)) \). To do this, we apply Lemma 9 with \( g = g_\psi \), \( Z_i = 1_{W_i = 0} \) and \( \sigma_i = \sigma_i (T(X_i, W_i) - g_\psi (X_i, W_i)) \) for an iid Rademacher sequence \( \sigma_i \ldots \sigma_n \) independent of \( (X_i, W_i) \). Then, noting that \( \| \sigma_i^2 \|_{\infty} \leq \| (T(x, w) - g_\psi (x, w)) \|_{\infty} \leq \| g_\psi \|_{\infty}^2 \) and that as in the previous case \( \nu = pZ \cdot P_{\mathcal{Z}} \) and therefore \( \lambda_1 = pZ \lambda_1, Z \), we have the bound \( R_n (\mathcal{H}^* (t)) \leq \| g_\psi \|_{\infty} (3/n) \sum_{j=0}^{\infty} (pZ \lambda_j, Z) \wedge t^2 \) where \( \lambda_0, Z = pZ (1 + \sqrt{pZ \lambda_1, Z}) \| g_\psi \|_{L_2(pZ)} \).
Using Lemma 10 to bound this, we have \( R_a(H^*(t)) \leq C_1 n^{-1/2} t^{1-1/\alpha} \) when \( t \leq p_1^{1/2} |g_\psi|_{L_2(p_2)} \).

Having established the validity of our bound (36), we now define something that will act as a bound on it: \( u_a(t) = an^{-1/2} t^{1-1/\alpha} \), a multiple of its asymptotically dominant term. We will solve the fixed point equation \( u'_a(t) = \eta_C t^2 \) and then select \( a \) so that \( u_a(t_a) \) upper bounds the right side above at \( t = t_a \), ensuring that \( t_a \geq r_C(\eta_C) \) as desired. The solution to this fixed point equation is \( t_a = (an^{-1/2}/\eta_C)^{1/(1+1/\alpha)} \). When our condition \( t_a \leq p_1^{1/2} |g_\psi|_{L_2(p_2)} \) for the validity of our bound on \( R_a(F^*(t)) \) is satisfied, clearly \( t_a \) at \( a = C_1 \) bounds the first term in (36). To incorporate the other terms as well, we will bound their ratios with \( u_a(t_a) \), too. For all \( a \geq 1 \), the ratio of the second and third terms in (36) and \( u_a(t_a) \) are respectively

\[
C_2 a^{-1} t_a^{1/\alpha} = C_2 \eta_C^{-1/(\alpha+1)} a^{1/(1+1/\alpha)} n^{-1/(2(\alpha+1))} \leq C_2 \eta_C^{-1/(\alpha+1)} a^{-1/2(\alpha+1)}; \\
C_3 n^{-1/2} a^{-1} t_a^{1/\alpha-1} = C_3 n^{-1/2} a^{-1} (an^{-1/2}/\eta_C)^{(1-\alpha)/(1+\alpha)} \leq C_3 \eta_C^{(\alpha-1)/(\alpha+1)} n^{-1/(\alpha+1)}.
\]

It follows that \( t_a \geq r_C(\eta_C) \) if (i) \( a \) is no smaller than the sum of our three ratio bounds and also no smaller than one, the latter being required for the validity of our second and third ratio bounds and (ii) \( t_a \) is no larger than \( p_1^{1/2} |g_\psi|_{L_2(p_2)} \), required for the validity of our local Rademacher complexity bound. Thus, in terms of \( C_1, C_2, C_3 \) defined in (36), we have

\[
r_C(\eta_C) \leq \max \left\{ \eta_C^{-1}, C_1 \eta_C^{-1}, C_2 \eta_C^{-1/(\alpha+1)} a^{1/(\alpha+1)} n^{-1/2(\alpha+1)} + C_3 n^{-1/(\alpha+1)} \right\} \frac{1}{n^{1/2(\alpha+1)}} \\
(37)
\]

so long as the entire bound is no larger than \( p_1^{1/2} |g_\psi|_{L_2(p_2)} \). Here the expression within the maximum plays the role of \( a \eta_C^{-1} \), and we take this maximum to ensure that our condition \( a \geq 1 \) holds.

We will use a variant of this bound with simpler dependence on \( \eta_C \);

\[
r_C(\eta_C) \leq \left[ \eta_C^{-1} \left( C_1 + C_2 n^{-1/2(\alpha+1)} + C_3 n^{-1/(\alpha+1)} \right) \right] \frac{1}{n^{1/2(\alpha+1)}}, \\
(38)
\]

valid when \( 2^{\alpha+1} \leq \eta_C \leq \left[ (C_2/(2C_3)) n^{1/2} \right]^{\frac{1}{\alpha+1}} \), the parenthesized term exceeds 1, and the entire bound is no larger than \( p_1^{1/2} |g_\psi|_{L_2(p_2)} \). To show that this is a valid upper bound, we will show that the bracketed expression in (38) exceeds the right branch of the maximum in (37). The difference between these two expressions is

\[
C_2 n^{-1/2(\alpha+1)} \left( \eta_C^{-1} - \eta_C^{-1/(\alpha+1)} \right) + C_3 n^{-1/(\alpha+1)} \left( \eta_C^{-1} - \eta_C^{-1/(\alpha+1)} \right),
\]

which is positive when the ratio

\[
C_2 n^{-1/2(\alpha+1)} \left( \eta_C^{-1} - \eta_C^{-1/(\alpha+1)} \right) / C_3 n^{-1/(\alpha+1)} \left( \eta_C^{-(\alpha+1)/\alpha} - \eta_C^{-1} \right)
\]

is larger than 1. The solution to this fixed point equation is \( t_a = (an^{-1/2}/\eta_C)^{1/(1+1/\alpha)} \). Thus, in the right branch of the maximum in (37), the difference between these expressions is

\[
C_2 n^{-1/2(\alpha+1)} \left( \eta_C^{-1} - \eta_C^{-1/(\alpha+1)} \right) + C_3 n^{-1/(\alpha+1)} \left( \eta_C^{-1} - \eta_C^{-1/(\alpha+1)} \right) > 1,
\]

which shows that the right branch of the maximum in (37) is exceeded by the bracketed expression in (38) for the validity of our bound on \( r_C(\eta_C) \).
exceeds one. This ratio is bounded above
\[
(C_2/C_3) n^{\frac{1}{\alpha+1}} \eta C^{-\frac{\alpha+1}{\alpha+2}} \left( 1 - \eta C^{-\frac{1}{\alpha+1}} \right),
\]
To complete our argument, observe that our lower bound \(2^{\alpha+1}\) on \(\eta C\) implies that the parenthesized factor is at least \(1/2\), and consequently that our upper bound on \(\eta C\) implies that the quantity above is at least one as required.

**F.1.4 Bounding \(u(H^*, \delta)\)**

To bound \(u(H^*, \delta)\), we bound \(R_n(H^*)\) and \(\sigma(H^*)\). We bound the latter using Hölder’s inequality and the triangle inequality,

\[
\sigma(H^*) \leq \max\{1, \|g_\psi\|_\infty\} \sqrt{\sup_{\|f\| \leq 1} \mathbb{E} \left[ |f(X_1) + g_\psi(X_1)|^2 \right]} = \max\{1, \|g_\psi\|_\infty\} \sqrt{\sup_{\|f\| \leq 1} p_Z \mathbb{E}|f(X_1) + g_\psi(X_1)|^2 \mid W_i = 1]}
\]

\[
\leq p_Z^{1/2} \max\{1, \|g_\psi\|_\infty\} \left( \sup_{\|f\| \leq 1} \|f\|_{L_2(p_Z)} + \|g_\psi\|_{L_2(p_Z)} \right)
\]

\[
= p_Z^{1/2} \max\{1, \|g_\psi\|_\infty\} \left( \lambda_j^{1/2} + \|g_\psi\|_{L_2(p_Z)} \right)
\]

The identity \(\sup_{\|f\| \leq 1} \|f\|_{L_2(p_Z)} = \lambda_{1, Z}\) used in the last step follows from the representation of this unit ball as the set \(\{\sum_{j=1}^{\infty} f_j \lambda_j^{1/2} \psi_j(x) : \sum_{j=1}^{\infty} f_j^2 \leq 1\}\) in terms of \(L_2(p_Z)\)-orthonormal eigenfunctions \(\psi_j\).

We bound the Rademacher complexity using Lemma 9 with \(t = \infty\), \(g = g_\psi\), \(Z = 1_{\{W_i = 0\}}\) and \(\sigma = \sigma(T(X_i, W_i) - g_\psi(X_i, W_i))\) for an iid Rademacher sequence \(\sigma'_1 \ldots \sigma'_{\nu}\) independent of \((X_i, W_i)_{i \leq n}\). Then, noting that \(\|\sigma_t^2\|_\infty = \|(T(x, w) - g_\psi)^2\|_\infty \leq \|g_\psi\|_2^2\) and that as in the previous case \(\nu = p_Z \cdot P_Z\) and therefore \(\lambda_j = p_Z \lambda_j, Z\), we have the bound

\[
R_n(H^*) \leq 2^{1/2} \|g_\psi\|_\infty n^{-1/2} \left( p_Z^{1/2} \|g_\psi\|_{L_2(p_Z)} + \sqrt{\sum_{j=1}^{\infty} p_Z \lambda_j, Z} \right)
\]

\[
\leq 2^{1/2} p_Z^{1/2} \|g_\psi\|_\infty n^{-1/2} \left( \|g_\psi\|_{L_2(p_Z)} + \sqrt{C_{\lambda, Z} \int_{s=1}^{\infty} s^{-\alpha}} \right)
\]

\[
= 2^{1/2} p_Z^{1/2} \|g_\psi\|_\infty \left( \|g_\psi\|_{L_2(p_Z)} + C_{\lambda, Z}^{1/2}(\alpha - 1)^{-1/2} \right)n^{-1/2}.
\]
and therefore

\[ u(\mathcal{H}^*, \delta) \leq \min_{\eta > 0} c_{u,1, \eta} n^{-1/2} + c_{u,2, \eta} n^{-1}; \]  

(39)

\[ c_{u,1, \eta} = (1 + \eta)2^{3/2} p_Z^{1/2} \max\{1, \|g_\psi\|_\infty\} \left( \|g_\psi\|_{L2(p_Z)} + C_{1/2}(1 - \alpha)^{-1/2} \right) \]

\[ + 2^{1/2} p_Z^{1/2} \max\{1, \|g_\psi\|_\infty\} \left( \lambda_{1,2} + \|g_\psi\|_{L2(p_Z)} \right) \sqrt{\log(2\delta^{-1})}; \]

\[ c_{u,2, \eta} = 2M_{\mathcal{H}^*}(1 + 1/\eta) \log(2\delta^{-1}). \]

F.2 Aggregating terms

In order to simplify our statement of (14) as much as possible, we equate our bounds (35) and (38) on \( r_Q(\eta_Q) \) and \( r_C(\eta_C) \) by setting

\[ \eta_C = \left( c_1 + c_2 n^{-\frac{1}{2(\alpha + 1)}} + c_3 n^{-\frac{1}{\alpha + 1}} \right) / (7c_Q)^{1+1/\alpha}. \]  

(40)

Having chosen this value of \( \eta_C \), \( r = r_Q(\eta_Q) \vee r_C(\eta_C) \vee n^{-1/2}\sigma^{-1}\eta_Q^{-1/2} \) satisfies the bound \( r \leq 7c_Q n^{-\frac{1}{2(\alpha + 1)}} \vee n^{-1/2}\sigma^{-1}\eta_Q^{-1/2} \). This is usually optimal. Among choices of \( \eta_C \), this one results in the sharpest bound (28) except in the case that \( b < a \) and \( b \) is equal to the second of the three expressions of which it is the maximum.

These choices yield the following bound, which is expressed in terms of \( \lambda = \sigma^2/n \) and approximately constant factors \( c_s \) defined in Appendix A. With probability \( 1 - \exp\{-c_p(\eta_Q)nr^2/M_{\mathcal{F}_s}^2 \} - 4\delta \),

\[ \left| \hat{\psi}_{ML} - E[\hat{\psi}_{ML} | X, W] \right| = n^{-1} \sum_{i=1}^n \gamma_\psi(X_i, W_i)(Y_i - m(X_i, 0)) \leq n^{-1/2}(a \wedge b)1/2\|v\|_\infty \delta^{-1/2}; \]

\[ a = s \left( c_{u,1} n^{-1/2} + c_{u,2} n^{-1} \right) + \bar{R} + \lambda; \]

\[ b = 2s^2 r^2 \vee 2 \frac{\bar{R} + \lambda}{\eta_Q - 2s^{-1}\eta_C} \vee \frac{44M_{\mathcal{F}_s}^2 s^2 \log(\delta^{-1})}{n}; \]

\[ s = 1 \vee \left[ 2\eta_{\mathcal{C}} \lambda^{-1} r^2 + \lambda^{-1/2} \bar{R}^{1/2} \right]; \]

\[ r = 7c_Q n^{-\frac{1}{2(\alpha + 1)}} \vee \lambda^{1/2}\eta_Q^{-1/2}; \]

\[ \bar{R} = c_{1,R} \lambda^{2s} + c_{2,R} n^{-1/2}\lambda^{s}. \]  

(41)

To complete our proof, we will simplify this bound under mild restrictions on the range of \( \lambda \) and \( \delta \). First we simplify \( s \), dropping the maximum with 1 by introducing the restriction \( \lambda \leq c_{1,R}^{(1/2-2s)}/R \). Under that restriction, the other term in the maximum is at least as large as \( \lambda^{-1/2}\bar{R}^{1/2} \geq c_{1,R}^{1/2} \lambda^{s} = 1 \).

Our final simplification is to drop the final term of the bound \( b \). It is smaller than the first unless \( \delta \) is extremely small, i.e.

\[ 2s^2 r^2 \geq \frac{44M_{\mathcal{F}_s}^2 s^2 \log(\delta^{-1})}{n} \iff nr^2/(22M_{\mathcal{F}_s}^2) \geq \log(\delta^{-1}), \]  

(49)
where

\[
\frac{nr^2}{44M_F^2} \geq \frac{49c_{\delta}^2 n^{1-\frac{1}{\alpha+1}}}{44M_F^2},
\]

\[
= \left[ 12p_Z M_F^{-2/\alpha} \right]^{\alpha/(\alpha+1)} n^{1/(\alpha+1)}
\]

\[
\geq \sqrt{12} \frac{p_Z M_F^{1-2/\alpha}}{M_F^{1/\alpha}} n^{1/(\alpha+1)}
\]

Above in the second to last step, we lower bound the bracketed term by one, lower bound 49/44 by one, and combine factors of \(M_F\). In the last step, we use our knowledge that \(\alpha > 1\) and therefore \(\alpha/(\alpha+1) \in (1/2, 1)\) to lower bound the constant factors. Thus, we may drop the final term so long as we impose the restriction \(\log(\delta^{-1}) \leq \sqrt{12} \frac{p_Z (n/M_F^2)}{M_F^{1/\alpha}}\).

### G Additional Simulation Results

In this section, we include simulation results for several variations of the example of Hainmueller (2012) discussed in Section 4. The variation included in Section 4 used outcome design 3 from Hainmueller (2012). Here we show results for outcome designs 1 and 2, in which we have \(Y_i = X_{i1} + X_{i2} + X_{i3} - X_{i4} + X_{i5} + X_{i6} + \sigma \varepsilon_i\) and \(Y_i = X_{i1} + X_{i2} + 0.2X_{i3}X_{i4} - \sqrt{X_{i5}} + \sigma \varepsilon_i\) respectively. As these outcomes models are more linear than outcome design 3, results in these variations are more favorable to OLS. The covariance matrix of \(X_{i1} \ldots X_{i3}\) used in all variations is

\[
\Sigma := \begin{pmatrix}
2 & 1 & -1 \\
1 & 1 & -0.5 \\
-1 & -0.5 & 1
\end{pmatrix}.
\]
| $n$ | 50 | 200 | 1000 | 4000 | 50 | 200 | 1000 | 4000 |
|-----|----|-----|------|------|----|-----|------|------|
| $\sigma = 0$ (no overlap) | | | | | | | | |
| IPW | 0.74 | 0.26 | 0.12 | 0.06 | 1 | 0.05 | 0.25 | 0.14 |
| AIPW | 0.69 | 0.15 | 0.08 | 0.04 | 0.88 | 0.94 | 0.97 | 0.95 |
| OLS | -0.03 | 0.01 | 0.00 | 0.00 | 0.94 | 0.96 | 0.97 | 0.96 |
| ML | -0.38 | -0.22 | -0.14 | -0.05 | 0.97 | 0.88 | 0.88 | 0.88 |
| MLt | 0.55 | 0.24 | 0.11 | 0.06 | 0.9 | 0.48 | 0.22 | 0.12 |
| MLt 10σ | 0.57 | 0.25 | 0.14 | 0.07 | 0.86 | 0.45 | 0.21 | 0.11 |
| MLt 100σ | 0.62 | 0.26 | 0.16 | 0.09 | 0.84 | 0.54 | 0.06 | 0.06 |
| $\sigma = 10$ | | | | | | | | |
| IPW | 4.44 | 1.18 | 0.53 | 0.26 | 6.15 | 3.74 | 1.42 | 0.71 |
| AIPW | 4.26 | 1.14 | 0.53 | 0.26 | 6.13 | 3.74 | 1.42 | 0.71 |
| OLS | 9.27 | 2.74 | 1.32 | 0.66 | 5.88 | 0.87 | 0.11 | 0.06 |
| ML | -0.35 | -0.12 | -0.06 | -0.02 | 0.98 | 0.99 | 1 | 1 |
| MLt | 0.04 | 0 | -0.04 | -0.02 | 0.93 | 0.99 | 1 | 1 |
| MLt 10σ | 0.05 | -0.01 | -0.07 | -0.04 | 0.85 | 0.98 | 1 | 1 |
| MLt 100σ | 0.12 | -0.01 | -0.17 | -0.08 | 0.86 | 0.97 | 1 | 1 |
| $\sigma = 100$ | | | | | | | | |
| IPW | 2.16 | 0.99 | 0.47 | 0.24 | 1.8 | 0.56 | 0.36 | 0.15 |
| AIPW | 2.14 | 0.99 | 0.47 | 0.24 | 1.8 | 0.56 | 0.36 | 0.15 |
| OLS | 0.47 | 0.27 | 0.13 | 0.06 | 0.95 | 0.96 | 0.98 | 0.98 |
| ML | -0.21 | -0.12 | -0.06 | -0.03 | 0.94 | 0.95 | 0.96 | 0.96 |
| MLt | 0.05 | 0 | -0.05 | -0.02 | 0.94 | 0.95 | 0.96 | 0.96 |
| MLt 10σ | 0.07 | -0.03 | -0.12 | -0.06 | 0.93 | 0.95 | 0.96 | 0.96 |
| MLt 100σ | 0.19 | -0.08 | -0.27 | -0.13 | 0.92 | 0.94 | 0.96 | 0.96 |

Figure 4: Root mean squared error (rmse), bias, and confidence interval half-width and coverage over 1000 replications of Outcome Design 1 from Hainmueller (2012). Here we take the tuning parameter $\sigma$ to be 0.1 in the estimators ML and MLt. The notation MLt 10σ and 100σ indicates the substitution of 1 and 10 respectively.
| \( n \) | 50  | 200  | 1000  | 4000  | 50  | 200  | 1000  | 4000  |
|-----|-----|------|-------|-------|-----|------|-------|-------|
| IPW | 0.46 | 0.28 | 0.12 | 0.06  | 0.0  | 0.48 | 0.24  | 0.11  |
| AIPW| 0.13 | 0.03 | 0.02 | 0.01  | 0.04 | 0.92 | 0.92  | 0.94  |
| OLS | 0.82 | 0.44 | 0.23 | 0.11  | 0.04 | 0.95 | 0.98  | 0.99  |
| ML  | 0.41 | 0.21 | 0.10 | 0.05  | 0.08 | 0.98 | 0.97  | 0.95  |
| MLs | 0.37 | 0.21 | 0.10 | 0.09  | 0.68 | 0.66 | 0.67  | 0.67  |
| MLt 10 | 0.57 | 0.43 | 0.21 | 0.11  | 0.8  | 0.41 | 0.21  | 0.11  |
| MLt 100 | 0.85 | 0.54 | 0.31 | 0.15  | 0.95 | 0.41 | 0.21  | 0.11  |
| MLt 1000 | 0.96 | 0.84 | 0.72 | 0.65  | 0.74 | 0.35 | 0.16  | 0.08  |
| IPW  | 2.86 | 1.17 | 0.52 | 0.24  | 0.95 | 0.92 | 0.97  | 0.97  |
| AIPW | 0.03 | 0.04 | 0.02 | 0.02  | 0.94 | 0.99 | 0.99  | 0.99  |
| OLS  | 0.99 | 0.56 | 0.28 | 0.15  | 0.96 | 0.96 | 0.96  | 0.96  |
| ML  | 0.61 | 0.36 | 0.19 | 0.14  | 0.66 | 0.99 | 0.99  | 0.99  |
| MLs | 0.61 | 0.36 | 0.21 | 0.14  | 0.66 | 0.99 | 0.99  | 0.99  |
| MLt 10 | 0.75 | 0.43 | 0.21 | 0.11  | 0.8  | 0.41 | 0.21  | 0.11  |
| MLt 100 | 0.94 | 0.68 | 0.42 | 0.31  | 0.95 | 0.41 | 0.21  | 0.11  |
| MLt 1000 | 1.18 | 0.71 | 0.42 | 0.31  | 0.95 | 0.41 | 0.21  | 0.11  |
| IPW  | 3.1 | 1.4 | 0.52 | 0.25  | 0.95 | 0.92 | 0.97  | 0.97  |
| AIPW | 0.03 | 0.04 | 0.02 | 0.02  | 0.94 | 0.99 | 0.99  | 0.99  |
| OLS  | 0.99 | 0.56 | 0.28 | 0.15  | 0.96 | 0.96 | 0.96  | 0.96  |
| ML  | 0.61 | 0.36 | 0.19 | 0.14  | 0.66 | 0.99 | 0.99  | 0.99  |
| MLs | 0.61 | 0.36 | 0.21 | 0.14  | 0.66 | 0.99 | 0.99  | 0.99  |
| MLt 10 | 0.75 | 0.43 | 0.21 | 0.11  | 0.8  | 0.41 | 0.21  | 0.11  |
| MLt 100 | 0.94 | 0.68 | 0.42 | 0.31  | 0.95 | 0.41 | 0.21  | 0.11  |
| MLt 1000 | 1.18 | 0.71 | 0.42 | 0.31  | 0.95 | 0.41 | 0.21  | 0.11  |

Figure 5: Root mean squared error (rmse), bias, and confidence interval half-width and coverage over 1000 replications of Outcome Design 2 from Hainmueller (2012). Here we take the tuning parameter \( \sigma \) to be 0.1 in the estimators ML and MLt. The notation MLt 10\( \sigma \) and 100\( \sigma \) indicates the substitution of 1 and 10 respectively.