Continuous-Time Mean-Variance Portfolio Selection with Constraints on Wealth and Portfolio

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Abstract

We consider continuous-time mean-variance portfolio selection with bankruptcy prohibition under convex cone portfolio constraints. This is a long-standing and difficult problem not only because of its theoretical significance, but also for its practical importance. First of all, we transform the above problem into an equivalent mean-variance problem with bankruptcy prohibition without portfolio constraints. The latter is then treated using martingale theory. Our findings indicate that we can directly present the semi-analytical expressions of the pre-committed efficient mean-variance policy without a viscosity solution technique but within a general framework of the cone portfolio constraints. The numerical simulation also sheds light on results established in this paper.

Keywords: continuous-time, mean-variance portfolio selection, bankruptcy prohibition, convex cone constraints, efficient frontier, HJB equation

1 Introduction

Since Markowitz [17] published his seminal work on the mean-variance portfolio selection sixty years ago, the mean-risk portfolio selection framework has become one of the most

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prominent topics in quantitative finance. Recently, there has been increasing interest in studying the dynamic mean-variance portfolio problem with various constraints, as well as addressing their financial applications. Typical contributions include Bielecki, Jin, Pliska and Zhou [1], Cui, Gao, Li and Li [3], Cui, Li and Li [4], Czichowsky and Schweizer [8], Heunis [11], Hu and Zhou [12], Duffie and Richardson [9], Labbé and Heunis [14], Li and Ng [15], Zhou and Li [22], and Li, Zhou and Lim [16]. The dynamic mean-variance problem can be treated in a forward-looking way by starting with the initial state. In some financial engineering problems, however, one needs to study stochastic systems with constrained conditions, such as cone-constrained policies. This naturally results in a continuous-time mean-variance portfolio selection problem with constraints for the wealth process, (see [1]), and constraints for the policies, (see [12] and [16]), for a given expected terminal target. To the best of our knowledge, despite active research efforts in this direction in recent years, there has been little progress in the continuous-time mean-variance problem with the mixed restriction of both bankruptcy prohibition and convex cone portfolio constraints. This paper aims to address this long-standing and notoriously difficult problem, not only for its theoretical significance, but also for its practical importance. New ideas, significantly different from those developed in the existing literature, establish a general theory for stochastic control problems with mixed constraints for state and control variables.

Li, Zhou and Lim [16] considered a continuous-time mean-variance problem with no-shorting constraints, while Cui, Gao, Li and Li [3] developed its counterpart in discrete-time. Bielecki, Jin, Pliska and Zhou [1] paved the way for investigating continuous-time mean-variance with bankruptcy prohibition using a martingale approach. Sun and Wang [19] introduced a market consisting of a riskless asset and one risky portfolio under constraints such as market incompleteness, no-shorting, or partial information. Labbé and Heunis [14] employed a duality method to analyze both the mean-variance portfolio selection and mean-variance hedging problems with general convex constraints. In particular, Czichowsky and Schweizer [8] further studied cone-constrained continuous-time mean-variance portfolio selection problem with the price processes being semimartingales. Cui, Gao, Li and Li [3] and Cui, Li and Li [4] derived explicit optimal semi-analytical mean-variance policies for discrete-time markets under no-shorting and convex cone constraints, respectively. Also, Föllmer and Schied [10] and Pham and Touzi [18] showed that in a constrained market, no arbitrage opportunity is equivalent to the existence of a supermartingale measure, under which the discounted wealth process of any admissible policy is a supermartingale, (see Carassus, Pham and Touzi [2] for a situation with upper bounds on proportion positions). In particular, Xu and Shreve in their two-part papers [20, 21] investigated a utility maximization problem with no-shorting constraints using duality analysis. Recently, Heunis [11] carefully considered to minimize the expected value of
a general quadratic loss function of the wealth in a more general setting where there is a specified convex constraint on the portfolio over the trading interval, together with a specified almost-sure lower-bound on the wealth at close of trade.

The existing theory and methods cannot directly handle the continuous-time mean-variance problem with the mixed restriction of both bankruptcy prohibition and convex cone portfolio constraints. Therefore, we introduce a Hamilton-Jacobi-Bellman (HJB) equation to analyze this problem. Based on our analysis, we find out that the market price of risk in policy is actually independent of the wealth process. This important finding allows us to overcome the difficulty of the original problem and also makes the similar continuous-time financial investment problem both interesting and practical. Hence, we can construct a transformation to tackle the model presented above using linear-quadratic convex optimization technique. Finally, we discuss the equivalent problem using the model of Bielecki, Jin, Pliska and Zhou \cite{1} without the viscosity solution technique developed in \cite{16}.

This paper is organized as follows. In Section 2, we formulate the continuous-time mean-variance problem with the mixed restriction of a bankruptcy prohibition and convex cone portfolio constraints. In Section 3, we transform our mean-variance problem into an equivalent mean-variance problem with a bankruptcy prohibition without convex cone portfolio constraints. We derive the pre-committed continuous-time efficient mean-variance policy for our problem using the model proposed by Bielecki, Jin, Pliska and Zhou \cite{1} in Section 4. In Section 5, we discuss properties of the continuous-time mean-variance problems with different constraints. In Section 6, we present a numerical simulation to illustrate results established in this paper. Finally, we summarize and conclude our work in Section 7.

2 Problem Formulation and Preliminaries

2.1 Notation

We make use of the following notation:

- $z^+$: the transformation of vector $z$ with every component $z_i^+ = \max\{z_i, 0\}$;
- $z^-$: the transformation of vector $z$ with every component $z_i^- = \max\{-z_i, 0\}$;
- $M'$: the transpose of any matrix or vector $M$;
- $\|M\| = \sqrt{\sum_{i,j} m_{ij}^2}$ for any matrix or vector $M = (m_{ij})$;
- $\mathbb{R}^m$: $m$ dimensional real Euclidean space;
- $\mathbb{R}_{+}^m$: the subset of $\mathbb{R}^m$ consisting of elements with nonnegative components.
The underlying uncertainty is generated on a fixed filtered complete probability space $(\Omega, \mathcal{F}, P, \{\mathcal{F}_t\}_{t \geq 0})$ on which is defined a standard $\{\mathcal{F}_t\}_{t \geq 0}$-adapted $m$-dimensional Brownian motion $W(\cdot) \equiv (W^1(\cdot), \cdots, W^m(\cdot))^\prime$. Given a probability space $(\Omega, \mathcal{F}, P)$ with a filtration $\{\mathcal{F}_t | a \leq t \leq b\}(-\infty < a < b \leq +\infty)$ and a Hilbert space $\mathcal{H}$ with the norm $\| \cdot \|_\mathcal{H}$, we can define a Banach space

$$L^2_F(a, b; \mathcal{H}) = \left\{ \varphi(\cdot) \left| \varphi(\cdot) \text{ is an } \mathcal{F}_t\text{-adapted, } \mathcal{H}\text{-valued measurable process on } [a, b] \text{ and } \|\varphi(\cdot)\|_F < +\infty \right. \right\}$$

with the norm

$$\|\varphi(\cdot)\|_F = \left( E \left[ \int_a^b \|\varphi(t, \omega)\|^2_\mathcal{H} \, dt \right] \right)^{\frac{1}{2}}.$$

### 2.2 Problem Formulation

Consider an arbitrage-free financial market where $m + 1$ assets are traded continuously on a finite horizon $[0, T]$. One asset is a bond, whose price $S_0(t)$, $t \geq 0$, evolves according to the differential equation

$$\begin{cases}
    dS_0(t) = r(t)S_0(t) \, dt, & t \in [0, T], \\
    S_0(0) = s_0 > 0,
\end{cases}$$

where $r(t)$ is the interest rate of the bond at time $t$. The remaining $m$ assets are stocks, and their prices are modeled by the system of stochastic differential equations

$$\begin{cases}
    dS_i(t) = S_i(t)\{b_i(t) \, dt + \sum_{j=1}^m \sigma_{ij}(t) \, dW^j(t)\}, & t \in [0, T], \\
    S_i(0) = s_i > 0,
\end{cases}$$

where $b_i(t)$ is the appreciation rate of the $i$-th stock and $\sigma_{ij}(t)$ is the volatility coefficient at time $t$. Denote $b(t) := (b_1(t), \cdots, b_m(t))^\prime$ and $\sigma(t) := (\sigma_{ij}(t))$. We assume throughout that $r(t)$, $b(t)$ and $\sigma(t)$ are given deterministic, measurable, and uniformly bounded functions on $[0, T]$. In addition, we assume that the non-degeneracy condition on $\sigma(\cdot)$, that is,

$$y'\sigma(t)\sigma(t)'y \geq \delta y'y, \quad \forall \ (t, y) \in [0, T] \times \mathbb{R}^m, \quad (1)$$

is satisfied for some scalar $\delta > 0$. Also, we define the excess return vector

$$B(t) = b(t) - r(t)1,$$

where $1 = (1, 1, \cdots, 1)^\prime$ is an $m$-dimensional vector.

Suppose an agent has an initial wealth $x_0 > 0$ and the total wealth of his position at time $t \geq 0$ is $X(t)$. Denote by $\pi_i(t)$, $i = 1, \cdots, m$, the total market value of the agent’s
wealth in the \( i \)th stock at time \( t \). We call \( \pi(\cdot) := (\pi_1(\cdot), \ldots, \pi_m(\cdot))' \in L^2_T(0,T; \mathbb{R}^m) \) a portfolio. We will consider self-financing portfolios in this paper. Then it is well-known that \( X(t), t \geq 0 \), follows (see, e.g., [22])

\[
\begin{aligned}
    dX(t) &= [r(t)X(t) + \pi(t)'B(t)] \, dt + \pi(t)'\sigma(t) \, dW(t), \\
    X(0) &= x_0.
\end{aligned}
\] (2)

An important restriction considered in this paper is the convex cone portfolio constraints, that is \( \pi(\cdot) \in C \), where

\[
C = \{ \pi(\cdot) : \pi(\cdot) \in L^2_T(0,T; \mathbb{R}^m), \ C(t)'\pi(t) \in \mathbb{R}_+^k, \forall \ t \in [0,T] \},
\] (3)

and \( C(\cdot) : [0,T] \mapsto \mathbb{R}^{m \times k} \) is a given deterministic and measurable function. Another important yet relevant restriction considered in this paper is the prohibition of bankruptcy, i.e., we required that

\[
X(t) \geq 0, \ \forall \ t \in [0,T].
\] (4)

On the other hand, borrowing from the money market (at the interest rate \( r(\cdot) \)) is still allowed; that is, the money invested in the bond \( \pi_0(\cdot) = X(\cdot) - \sum_{i=1}^m \pi_m(\cdot) \) has no constraint.

**Definition 1** A portfolio \( \pi(\cdot) \) is called an admissible control (or portfolio) if \( \pi(\cdot) \in C \) and the corresponding wealth process \( X(\cdot) \) defined in (2) satisfies (4). In this case, the process \( X(\cdot) \) is called an admissible wealth process, and \((X(\cdot), \pi(\cdot))\) is called an admissible pair.

**Remark 1** In view of the boundedness of \( \sigma(\cdot) \) and the non-degeneracy condition [1], we have that \( \pi(\cdot) \in L^2_T(a,b; \mathbb{R}^m) \) if and only if \( \sigma(\cdot)'\pi(\cdot) \in L^2_T(a,b; \mathbb{R}^m) \). The latter is often used to define the admissible process in the literature, for instance, Bielecki, Jin, Pliska and Zhou [1].

**Remark 2** It is easy to show that both the set of all admissible controls and the set of all admissible wealth processes are convex. In particular, the set of expected terminal wealths

\[
\{ \mathbb{E}[X(T)] : X(\cdot) \text{ is an admissible process} \}
\]

is an interval.

Mean-variance portfolio selection refers to the problem of, given a favorable mean level \( d \), finding an allowable investment policy, (i.e., a dynamic portfolio satisfying all the
constraints), such that the expected terminal wealth \( \mathbb{E}[X(T)] \) is \( d \) while the risk measured by the variance of the terminal wealth

\[
\text{Var}(X(T)) = \mathbb{E}[X(T) - \mathbb{E}[X(T)]]^2 = \mathbb{E}[X(T) - d]^2
\]
is minimized.

We impose throughout this paper the following assumption:

**Assumption 1** The value of the expected terminal wealth \( d \) satisfies \( d \geq x_0 e^{\int_0^T r(s) \, ds} \).

**Remark 3** Assumption 1 states that the investor’s expected terminal wealth \( d \) should be no less than \( x_0 e^{\int_0^T r(s) \, ds} \) which coincides with the amount that he/she would earn if all of the initial wealth is invested in the bond for the entire investment period. Clearly, this is a reasonable assumption, for the solution of the problem under \( d < x_0 e^{\int_0^T r(s) \, ds} \) is foolish for rational investors.

**Definition 2** The mean-variance portfolio selection problem is formulated as the following optimization problem parameterized by \( d \):

\[
\min \quad \text{Var}(X(T)) = \mathbb{E}[X(T) - d]^2,
\]

subject to

\[
\begin{align*}
\mathbb{E}[X(T)] &= d, \\
\pi(\cdot) &\in C \quad \text{and} \quad X(\cdot) \geq 0, \\
(X(\cdot), \pi(\cdot)) &\text{satisfies equation (2)}. 
\end{align*}
\]

(5)

An optimal control satisfying (5) is called an efficient strategy, and \((\text{Var}(X(T)), d)\), where \( \text{Var}(X(T)) \) is the optimal value of (5) corresponding to \( d \), is called an efficient point. The set of all efficient points, when the parameter \( d \) runs over all possible values, is called the efficient frontier.

In the current setting, the admissible controls belong to a convex cone, so the expectation of the final outcome may not be arbitrary. Denote by \( V(d) \) the optimal value of problem (5). Denote

\[
\hat{d} = \sup \{ \mathbb{E}[X(T)] : X(\cdot) \text{ is an admissible process} \}.
\]

(6)

Taking \( \pi(\cdot) \equiv 0 \), we see that \( X(t) = x_0 e^{\int_0^t r(s) \, ds} \) is an admissible process, so \( \hat{d} \geq \mathbb{E}[X(T)] = x_0 e^{\int_0^T r(s) \, ds} \). The following nontrivial example shows that it is possible that \( \hat{d} = x_0 e^{\int_0^T r(s) \, ds} \).
Example 1  Let $B(t) = -C(t)\chi$, where $\chi$ is any positive vector of appropriate dimension. Then for any admissible control $\pi(\cdot) \in \mathcal{C}$, we have $\pi(\cdot)'B(\cdot) = -\pi(\cdot)'C(\cdot)\chi \leq 0$. Therefore, by [2],

$$d(E[X(t)]) = (r(t)E[X(t)] + E[\pi(t)'B(t)])\,dt \leq r(t)E[X(t)]\,dt,$$

which implies $E[X(T)] \leq x_0 e^{\int_0^T r(s)\,ds}$. Hence $\hat{d} = x_0 e^{\int_0^T r(s)\,ds}$.

Theorem 1  Assume that $\hat{d} = x_0 e^{\int_0^T r(s)\,ds}$. Then the optimal value of problem (5) is 0.

Proof. From Assumption 1 and with $\hat{d} = x_0 e^{\int_0^T r(s)\,ds}$, we obtain that the only possible value of $d$ is $x_0 e^{\int_0^T r(s)\,ds}$. Note that $(X(t),\pi(t)) \equiv (x_0 e^{\int_0^t r(s)\,ds}, 0)$ is an admissible pair satisfying the constraint of problem (5), so $V(d) \leq E[X(T) - d]^2 = E[x_0 e^{\int_0^T r(s)\,ds} - d]^2 = 0$. The proof is complete.

From now on we assume $\hat{d} > x_0 e^{\int_0^T r(s)\,ds}$. Denote $\mathfrak{D} = (0, \hat{d})$ and $\mathfrak{D}^+ = [x_0 e^{\int_0^T r(s)\,ds}, \hat{d}]$. In this case both $\mathfrak{D}$ and $\mathfrak{D}^+$ are nonempty intervals.

Lemma 1  The value function $V(\cdot)$ is convex on $\mathfrak{D}$ and strictly increasing on $\mathfrak{D}^+$.

Proof. Let $(\bar{X}(\cdot),\bar{\pi}(\cdot))$ and $(\tilde{X}(\cdot),\tilde{\pi}(\cdot))$ be any two admissible pairs such that $d_1 = E[\bar{X}(T)]$ and $d_2 = E[\tilde{X}(T)]$ are different and both in $\mathfrak{D}$. For any $0 < \alpha < 1$, define $(\hat{X}(\cdot),\hat{\pi}(\cdot)) = (\alpha \bar{X}(\cdot) + (1 - \alpha) \tilde{X}(\cdot), \alpha \bar{\pi}(\cdot) + (1 - \alpha) \tilde{\pi}(\cdot))$. Then $(\hat{X}(\cdot),\hat{\pi}(\cdot))$ satisfies [2] such that $\hat{\pi}(\cdot) \in \mathcal{C}$, $\hat{X}(\cdot) \geq 0$ and $E[\hat{X}(T)] = \alpha d_1 + (1 - \alpha) d_2 \in \mathfrak{D}$, that is, $(\hat{X}(\cdot),\hat{\pi}(\cdot))$ is an admissible pair. Therefore,

$$V(\alpha d_1 + (1 - \alpha) d_2) \leq \text{Var}(\hat{X}(T)) = \text{Var}(\alpha \bar{X}(\cdot) + (1 - \alpha) \tilde{X}(\cdot))$$

$$\leq \alpha \text{Var}(\bar{X}(T)) + (1 - \alpha) \text{Var}(\tilde{X}(T)),$$

where we used the convexity of square function to get the last inequality. Because $(\bar{X}(\cdot),\bar{\pi}(\cdot))$ and $(\tilde{X}(\cdot),\tilde{\pi}(\cdot))$ are arbitrary chosen, we conclude that

$$V(\alpha d_1 + (1 - \alpha) d_2) \leq \alpha V(d_1) + (1 - \alpha) V(d_2).$$

This establishes the convexity of $V(\cdot)$.

Taking $\pi(\cdot) \equiv 0$, we see that $X(T) = x_0 e^{\int_0^T r(s)\,ds}$. This clearly implies that

$$V\left(x_0 e^{\int_0^T r(s)\,ds}\right) = 0.$$

It is known that if there are no portfolio constraints (i.e. $C(t) = 0$ for all $t \in [0,T]$), then the optimal value is positive on $\mathfrak{D}^+$ (see [1]), so $V(\cdot)$ must be positive on $\mathfrak{D}^+$. Using the convexity, we conclude that $V(\cdot)$ is strictly increasing on $\mathfrak{D}^+$. 

\qed


Corollary 1 The value function $V(\cdot)$ is finite and continuous on $\mathcal{D}$.

Since problem (5) is a convex optimization problem, the mean constraint $\mathbf{E}[X(T)] = d$ can be dealt with by introducing a Lagrange multiplier. As well-known, then mean-variance portfolio selection problem (5) is meaningful only when $d \in \mathcal{D}^+$. We will focus on this case.

Because $V(\cdot)$ is convex on $\mathcal{D}$ and strictly increasing at any $d \in (x_0 e^{\int_t^T r(s) ds}, \hat{d})$, there is a constant $\lambda > 0$ such that $V(x) - 2\lambda x \geq V(d) - 2\lambda d$ for all $x \in \mathcal{D}$, where the factor 2 in front of the multiplier $\lambda$ is introduced just for convenience. In this way the portfolio selection problem (5) is equivalent to the following problem

$$\min \mathbf{E}[X(T) - d]^2 - 2\lambda (\mathbf{E}[X(T)] - d),$$

subject to

$$\pi(\cdot) \in \mathcal{C} \text{ and } X(\cdot) \geq 0,$$

$$(X(\cdot), \pi(\cdot)) \text{ satisfies equation } (2),$$

or equivalently,

$$\min \mathbf{E}[X(T) - (d + \lambda)]^2,$$

subject to

$$\pi(\cdot) \in \mathcal{C} \text{ and } X(\cdot) \geq 0,$$

$$(X(\cdot), \pi(\cdot)) \text{ satisfies equation } (2)$$

in the sense that these problems have exactly the same optimal pair if one of them admits one.

We plan to use dynamic programming to study the aforementioned problems, so we denote by $\hat{V}(t, x)$ the optimal value of problem

$$\min \mathbf{E}[X(T) - (d + \lambda)] | \mathcal{F}_t, X(t) = x]^2,$$

subject to

$$\pi(\cdot) \in \mathcal{C} \text{ and } X(\cdot) \geq 0,$$

$$(X(\cdot), \pi(\cdot)) \text{ satisfies equation } (2)$$

Lemma 2 The function $\hat{V}(t, \cdot)$ is strictly decreasing and convex on $(0, (d + \lambda)e^{-\int_t^T r(s) ds}]$ for each fixed $t \in [0, T]$.

PROOF. The proof is similar to that of Lemma 1. We leave the proof for the interested readers. □

Remark 4 If the initial wealth $X(t) = x$ is too big, then as well-known the mean-variance portfolio selection problem (8) is not meaningful. This make us focus on the small initials in $(0, (d + \lambda)e^{-\int_t^T r(s) ds}]$. 

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Lemma 3 If $X(\cdot)$ is a feasible wealth process with $X(t) = 0$ for some $t \in [0, T]$, then $X(s) = 0$ for all $s \in [t, T]$.

Proof. Since $X(\cdot)$ is a feasible wealth process, we have $X(s) \geq 0$, for all $s \in [t, T]$. If $P(X(s) > 0)$ is positive for some $s \in [t, T]$, then this would lead to an arbitrage opportunity. □

Lemma 4 We have that $\hat{V}(t, 0) = (d + \lambda)^2$ and $\hat{V}(t, (d + \lambda)e^{-\int_t^T r(s) ds}) = 0$ for all $t \in [0, T]$.

Proof. If $X(t) = 0$, then $X(T) = 0$ by Lemma 3. Hence, $\hat{V}(t, 0) = (d + \lambda)^2$. Suppose $X(t) = (d + \lambda)e^{-\int_t^T r(s) ds}$. Then taking $\pi(\cdot) \equiv 0$, we obtain that $X(T) = d + \lambda$, so $\hat{V}(t, (d + \lambda)e^{-\int_t^T r(s) ds}) \leq E[X(T) - (d + \lambda)^2] = 0$. The proof is complete. □

3 An Equivalent Stochastic Problem

Since the Riccati equation approach to solve problem (8) is not applicable in this case, we consider the corresponding Hamilton-Jacobi-Bellman (HJB) equation. This is the following partial differential equation:

$$\begin{cases}
L v = 0, & (t, x) \in \mathcal{S}, \\
v(t, (d + \lambda)e^{-\int_t^T r(s) ds}) = 0, & 0 \leq t \leq T, \\
v(T, x) = (x - (d + \lambda))^2, & 0 < x < d + \lambda,
\end{cases}$$

(9)

where

$$L v = v_t(t, x) + \inf_{\pi \in \mathcal{C}_t} \left\{ v_x(t, x) [r(t)x + \pi' B(t)] + \frac{1}{2} v_{xx}(t, x) \pi' \sigma(t) \sigma(t)' \pi \right\},$$

$$\mathcal{S} = \{(t, x) : 0 \leq t < T, 0 < x < (d + \lambda)e^{-\int_t^T r(s) ds}\},$$

and

$$\mathcal{C}_t = \{z \in \mathbb{R}^m : C(t)'z \in \mathbb{R}^k\}.$$

We need the following technical result.

Lemma 5 Suppose problem (9) admits a solution $v \in C^{1,2}(\mathcal{S})$ which is convex in the second argument. Then $v \leq (d + \lambda)^2$ on $\mathcal{S}$. 

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Proof. By the convexity of $v$ in the second argument, we have, for each $(t, x) \in S$,

$$v(t, x) \leq \max \left\{ v(t, 0), v \left( t, (d + \lambda)e^{-\int_t^T r(s) \, ds} \right) \right\} = (d + \lambda)^2.$$ 

The proof is complete. \( \square \)

Now we are ready to establish the following result:

**Theorem 2** Suppose problem \((9)\) admits a solution $v \in C^{1,2}(S)$ which is convex in the second argument. Then $\hat{V} = v$ on $S$.

Proof. Without loss of generality, we shall show $\hat{V}(0, x_0) = v(0, x_0)$.

Let $(X(\cdot), \pi(\cdot))$ be the corresponding optimal pair. Define

$$\tau = \inf \left\{ t \in [0, T] : X(t) = 0 \text{ or } X(t) = (d + \lambda)e^{-\int_t^T r(s) \, ds} \right\} \land T, \quad (10)$$

$$\tau_N = \sup \left\{ t \in [0, T] : \int_0^t \|v_x(s, X(s))\pi(s)\sigma(s)\|^2 \, ds \leq N \right\} \land T. \quad (11)$$

Applying Itô’s Lemma to $v(t, X_t)$ yields

$$v(\tau \land \tau_N, X(\tau \land \tau_N)) = \int_0^{\tau \land \tau_N} \left( v_t(t, X(t)) + v_x(t, X(t))r(t)X(t) + \pi(t)'B(t) \right) + \frac{1}{2} v_{xx}(t, X(t))\pi(t)\pi(t)' + \frac{1}{2} v_{xx}(t, X(t))\pi(t)\pi(t)' \, dt$$

$$+ \int_0^{\tau \land \tau_N} v_x(t, X(t))s(t) \, dW(t) + v(0, x_0)$$

$$\geq \int_0^{\tau \land \tau_N} \mathcal{L}v(t, X(t)) \, dt + \int_0^{\tau \land \tau_N} v_x(t, X(t))\pi(t)\sigma(t) \, dW(t) + v(0, x_0)$$

$$= \int_0^{\tau \land \tau_N} v_x(t, X(t))\pi(t)\sigma(t) \, dW(t) + v(0, x_0).$$

Taking expectation of both sides, we obtain

$$E[v(\tau \land \tau_N, X(\tau \land \tau_N))] \geq E \left[ \int_0^{\tau \land \tau_N} v_x(t, X(t))\pi(t)\sigma(t) \, dW(t) + v(0, x_0) \right] = v(0, x_0).$$

Because $v$ is continuous, and $\tau \land \tau_N$ and $X(\tau \land \tau_N)$ are both uniformly bounded, letting $N \to \infty$ and applying the dominated convergence theorem, we obtain

$$E[v(\tau, X(\tau))] \geq v(0, x_0). \quad (12)$$

If $X(\tau) = 0$, then by Lemma \[3\] we have $X(T) = 0$. Using Lemma \[5\] yields

$$E[v(T, X(T))|\mathcal{F}_\tau]1_{\{X(\tau)=0\}} = (d + \lambda)^21_{\{X(\tau)=0\}} \geq v(\tau, X(\tau))1_{\{X(\tau)=0\}}. \quad (13)$$
If \( X(\tau) = (d + \lambda)e^{-\int_0^{\tau} r(s) \, ds} \), then \( v(\tau, X(\tau)) = 0 \). This trivially leads to

\[
\mathbb{E}[v(T, X(T)) | \mathcal{F}_\tau] \mathbf{1}_{\{X(\tau) = (d + \lambda)e^{-\int_0^{\tau} r(s) \, ds}\}} \geq v(\tau, X(\tau)) \mathbf{1}_{\{X(\tau) = (d + \lambda)e^{-\int_0^{\tau} r(s) \, ds}\}}.
\] (14)

If \( 0 < X(\tau) < (d + \lambda)e^{-\int_0^{\tau} r(s) \, ds} \), then \( \tau = T \) by the definition. Hence,

\[
\mathbb{E}[v(T, X(T)) | \mathcal{F}_\tau] \mathbf{1}_{\{0 < X(\tau) < (d + \lambda)e^{-\int_0^{\tau} r(s) \, ds}\}} = \mathbb{E}[v(\tau, X(\tau)) | \mathcal{F}_\tau] \mathbf{1}_{\{0 < X(\tau) < (d + \lambda)e^{-\int_0^{\tau} r(s) \, ds}\}} = v(\tau, X(\tau)) \mathbf{1}_{\{0 < X(\tau) < (d + \lambda)e^{-\int_0^{\tau} r(s) \, ds}\}}.
\] (15)

From (13), (14) and (15), we obtain

\[
\mathbb{E}[v(T, X(T)) | \mathcal{F}_\tau] \geq v(\tau, X(\tau)),
\]

\[
\mathbb{E}[v(T, X(T))] \geq \mathbb{E}[v(\tau, X(\tau))] \geq v(0, x_0),
\]

where the last inequality holds due to (12). Note \( v(T, X(T)) = (X(T) - (d + \lambda))^2 \), so \( \tilde{V}(0, x_0) \geq v(0, x_0) \).

On the other hand, define a feasible portfolio in the feedback form,

\[
\pi(t) = \begin{cases} 
0, & \text{if } X(t) = 0; \\
0, & \text{if } X(t) = (d + \lambda)e^{-\int_0^t r(s) \, ds}; \\
\arg\min_{\pi \in C} \left\{ v_x(t, X(t))\pi' B(t) + \frac{1}{2} v_{xx}(t, X(t))\pi' \sigma(t) \sigma(t)' \pi \right\}, & \text{otherwise.}
\end{cases}
\]

It is not hard to see \((X(\cdot), \pi(\cdot))\) is an admissible pair. Then we see that (12), (13), (14) and (15) become identities, so

\[
\mathbb{E}[X(T) - (d + \lambda)^2] = \mathbb{E}[v(T, X(T))] = \mathbb{E}[v(\tau, X(\tau))] = v(0, x_0).
\]

This implies that \( \tilde{V}(0, x_0) \leq v(0, x_0) \). The proof is complete. \( \Box \)

Before we go further, we need the following key result.

**Lemma 6** Suppose \( A \in \mathbb{R}^{m \times k} \), \( B \in \mathbb{R}^m \), \( D \in \mathbb{R}^{m \times m} \) and \( DD' \) is positive definite, and \( C = \{ z \in \mathbb{R}^m : A'z \in \mathbb{R}^k_+ \} \). Then, for each fixed scalar \( \alpha > 0 \), the following two convex optimization problems

\[
\min_{z \in C} \frac{1}{2} z' DD' z - \alpha B' z
\] (16)

and

\[
\min_{z \in \mathbb{R}^m} \frac{1}{2} z' DD' z - \alpha z' DD' z
\] (17)

have the same optimal solution \( \alpha \tilde{z} \) and the same optimal value \( -\frac{1}{2} \alpha^2 \tilde{z}'DD' \tilde{z} \), where

\[
\tilde{z} = \arg\min_{z \in C} \| DD' z - D^{-1} B \|.
\] (18)
Proof. Because $C$ is a cone, it is sufficient to study the case $\alpha = 1$. From the definition, $\bar{z}$ solves

$$\min_{z \in C} \frac{1}{2} z' DD' z - B' z.$$  \hfill (19)

By the definition of $C$, the above problem is equivalent to

$$\min_{z \in \mathbb{R}^m, A' z \in \mathbb{R}^k_+} \frac{1}{2} z' DD' z - B' z.$$ \hfill (20)

Introducing a Lagrange multiplier yields the following unconstrained problem

$$\min_{z \in \mathbb{R}^m} \frac{1}{2} z' DD' z - B' z - \nu' A' z.$$ \hfill (21)

Problems (19) and (20) should have the same unique solution $\bar{z}$ and optimal value, for some $\nu \in \mathbb{R}^k$. Since $(DD')^{-1}(B + A\nu)$ is the unique solution to problem (20), we have $\bar{z} = (DD')^{-1}(B + A\nu)$ and $DD' \bar{z} = B + A\nu$. Therefore, the optimal solution and the optimal value of the above problem (20) are the same as those of the following unconstrained problem

$$\min_{z \in \mathbb{R}^m} \frac{1}{2} z' DD' z - \bar{z}' DD' z.$$ \hfill (22)

Note that $\alpha = 1$, so the above problem is actually the same as problem (17). The proof is complete. \hfill \Box

Remark 5. By Lemma 2, we know that the solution $\hat{V}(t, \cdot)$ to problem (9) is strictly decreasing and convex on $(0, (d + \lambda)e^{-\int_t^T r(s) ds})$ for $t \in [0, T]$. Therefore, $v_x(t, x) < 0$ and $v_{xx}(t, x) > 0$ for $t \in [0, T]$.

We now return to the dynamic problem (9). Let

$$\bar{z}(t) := \arg\min_{z \in C_t} \|\sigma'(t) z - \sigma(t)^{-1} B(t)\|.$$ \hfill (23)

By Lemma 6 with $\alpha = -\frac{v_x(t, x)}{v_{xx}(t, x)} > 0$, the infimum in the HJB equation (9) is attained by

$$\pi = -\frac{v_x(t, x)}{v_{xx}(t, x)} \bar{z}(t) \in C_t.$$ \hfill (24)

Moreover, the HJB equation (9) is equivalent to

$$\begin{cases}
 v_t(t, x) + \inf_{\pi \in \mathbb{R}^m} \left\{ v_x(t, x)[r(t)x + \pi' \tilde{B}(t)] + \frac{1}{2} v_{xx}(t, x)\pi' \sigma(t)\sigma(t)' \pi \right\} = 0, \\
 v(t, (d + \lambda)e^{-\int_t^T r(s) ds}) = 0, \quad v(t, 0) = (d + \lambda)^2, \\
 v(T, x) = (x - (d + \lambda))^2,
\end{cases} \quad 0 \leq t \leq T, \quad 0 < x < d + \lambda,$$ \hfill (25)
where
\[ \hat{B}(t) := \sigma(t)\sigma(t)'\hat{z}(t). \]

In fact, the solution to the above HJB equation also is the value function associated with the following problem
\[
\min \quad E[X(T) - (d + \lambda)|F_t, X(t) = x]^2,
\]
subject to
\[
\begin{cases}
\pi(\cdot) \in \hat{C} \text{ and } X(\cdot) \geq 0, \\
(X(\cdot), \pi(\cdot)) \text{ satisfies the following equation (27)},
\end{cases}
\]
where
\[ \hat{C} = \{ \pi(\cdot) : \pi(\cdot) \in L^2_x(0, T; \mathbb{R}^m) \}, \]
and
\[
\begin{cases}
dX(t) = [r(t)X(t) + \pi(t)'\hat{B}(t)] \, dt + \pi(t)'\sigma(t) \, dW(t), \\
X(0) = x_0.
\end{cases}
\]

Removing the Lagrange multiplier, the above problem has the same optimal control as the following mean-variance problem without constraints on the portfolio:
\[
\min \quad \text{Var}(X(T)) = E[X(T) - \tilde{d}]^2,
\]
subject to
\[
\begin{cases}
E[X(T)] = \tilde{d}, \\
\pi(\cdot) \in \hat{C} \text{ and } X(\cdot) \geq 0, \\
(X(\cdot), \pi(\cdot)) \text{ satisfies equation (27)},
\end{cases}
\]
for some \( \tilde{d} \). Because the optimal solution to the above problem (28) is also optimal to problem (5), the mean of the optimal terminal wealth should be the same, that is to say \( \tilde{d} = d \). Therefore, we conclude that problem (5) and problem
\[
\min \quad \text{Var}(X(T)) = E[X(T) - d]^2,
\]
subject to
\[
\begin{cases}
E[X(T)] = d, \\
\pi(\cdot) \in \hat{C} \text{ and } X(\cdot) \geq 0, \\
(X(\cdot), \pi(\cdot)) \text{ satisfies equation (27)},
\end{cases}
\]
have the same optimal solution.

The above mean-variance with bankruptcy prohibition problem was fully solved in [1]. Consequently, so is our problem (5). Moreover, these two problems have the same efficient frontier.

It is interesting to note that the market price of risk \( \hat{\theta}(\cdot) = \sigma(\cdot)^{-1}\hat{B}(\cdot) \) does not depend on the wealth process \( X(\cdot) \). This important feature allows us to give a linear feedback policy in \( X(\cdot) \) at or before the terminal time.
4 Optimal Portfolio

The result of the martingale pricing theory states that the set of random terminal payoffs that can be generated by feasible trading strategies corresponds to the set of nonnegative $\mathcal{F}_T$-measurable random payoffs $X(T)$ which satisfy the budget constraint $E[\phi(T)X(T)] \leq x_0$. Therefore, the dynamic problem (29), of choosing an optimal trading strategy $\pi(\cdot)$, is equivalent to the static problem of choosing an optimal payoff $X(T)$:

$$\min \quad \text{Var}(X(T)) = E[X(T) - d]^2,$$

subject to

$$\begin{cases} E[X(T)] = d, \\ E[\phi(T)X(T)] = x_0, \\ X(T) \geq 0, \end{cases}$$

where $\phi(\cdot)$ is the state price density, or stochastic discount factor, defined by

$$\begin{cases} d\phi(t) = \phi(t)\{-r(t) \, dt - \hat{\theta}(t)'dW(t)\}, \\ \phi(0) = 1, \end{cases}$$

and

$$\hat{\theta}(t) = \sigma(t)^{-1}\hat{B}(t) = \sigma(t)'\hat{z}(t).$$

The above static optimization problem (30) was solved in [1]. The optimal random terminal payoff is

$$X^*(T) = (\mu - \gamma\phi(T))^+, \tag{32}$$

where $x^+ = \max\{x, 0\}$, $\mu > 0$, $\gamma > 0$ and $(\mu, \gamma) \in \mathbb{R}^2$ solves the system of equations

$$\begin{cases} E[(\mu - \gamma\phi(T))^+] = d, \\ E[\phi(T)(\mu - \gamma\phi(T))^+] = x_0. \end{cases} \tag{33}$$

That is,

$$\begin{cases} \mu N\left(\frac{\ln \left( \frac{\mu}{\gamma} \right) + \int_0^T [r(s) + \frac{\hat{\theta}(s)^2}{2}] \, ds}{\sqrt{\int_0^T [\hat{\theta}(s)^2] \, ds}}\right) - \gamma e^{-\int_0^T r(s) \, ds} N\left(\frac{\ln \left( \frac{\mu}{\gamma} \right) + \int_0^T [r(s) - \hat{\theta}(s)^2] \, ds}{\sqrt{\int_0^T [\hat{\theta}(s)^2] \, ds}}\right) = d, \\ \mu N\left(\frac{\ln \left( \frac{\mu}{\gamma} \right) + \int_0^T [r(s) - \frac{3}{2}\hat{\theta}(s)^2] \, ds}{\sqrt{\int_0^T [\hat{\theta}(s)^2] \, ds}}\right) - \gamma e^{-\int_0^T r(s) \, ds} N\left(\frac{\ln \left( \frac{\mu}{\gamma} \right) + \int_0^T [r(s) - \frac{\hat{\theta}(s)^2}{2}] \, ds}{\sqrt{\int_0^T [\hat{\theta}(s)^2] \, ds}}\right) = x_0 e^{\int_0^T r(s) \, ds}, \end{cases} \tag{34}$$

where $N(y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^y e^{-\frac{t^2}{2}} \, dt$ is the cumulative distribution function of the standard normal distribution.
The investor’s optimal wealth is given by the stochastic process

\[ X^*(t) = \mathbb{E} \left[ \frac{\phi(T)}{\phi(t)} X^*(T) \bigg| \mathcal{F}_t \right] = f(t, \phi(t)), \quad (35) \]

where

\[ f(t, y) = \mu N \left( -d_2(t, y) \right) e^{-\int_t^T r(s) \, ds} - \gamma N \left( -d_1(t, y) \right) y e^{-\int_t^T [2r(s) - \bar{\theta}(s)]^2 \, ds}, \tag{36} \]

and

\[ d_1(t, y) := \ln \left( \frac{\gamma \mu y}{e^{\int_t^T [2r(s) - \bar{\theta}(s)]^2 \, ds}} \right) + \int_t^T [-r(s) + \frac{3}{2} \bar{\theta}(s)]^2 \, ds, \quad d_2(t, y) := d_1(t, y) - \sqrt{\int_t^T |\bar{\theta}(s)|^2 \, ds}. \]

Applying Itô’s lemma to \( f(\cdot, \phi(\cdot)) \) yields

\[ dX^*(t) = df(t, \phi(t)) = \{\cdots\} \, dt + \gamma \hat{\theta}(t) N \left( -d_1(t, \phi(t)) \right) \phi(t) e^{-\int_t^T [2r(s) - \bar{\theta}(s)]^2 \, ds} \, dW(t). \]

Comparing this to the wealth evolution equation (27), we obtain the efficient portfolio

\[ \pi^*(t) = \gamma (\sigma(t) \sigma(t)')^{-1} \hat{B}(t) N \left( -d_1(t, \phi(t)) \right) \phi(t) e^{-\int_t^T [2r(s) - \bar{\theta}(s)]^2 \, ds} \]

\[ = -\left( \sigma(t) \sigma(t)'' \right)^{-1} \hat{B}(t) \left[ X^*(t) - \mu N \left( -d_2(t, \phi(t)) \right) e^{-\int_t^T r(s) \, ds} \right]. \tag{37} \]

**Remark 6** The above results for the efficient portfolio and the associated wealth process were first derived in Bielecki et al. [1] and give a complete solution to the mean-variance portfolio selection problem with bankruptcy prohibition [24].

Based on the above analysis, we have the following main theorem.

**Theorem 3** Assume that \( \int_0^T |\hat{\theta}(s)|^2 \, ds > 0 \). Then there exists a unique efficient portfolio for (5) corresponding to any given \( d \geq x_0 e^{\int_0^T r(s) \, ds} \). Moreover, the efficient portfolio is given by (37) and the associated wealth process is expressed by (35).

### 5 Special Models

The main result of the constrained mean-variance portfolio selection model with bankruptcy prohibition derived in Section 4 is quite surprising but important. The mean-variance portfolio selection model, like many other stochastic optimization models, is based on averaging over all the possible random scenarios. We now discuss the model in terms of how its different constraints could guide real investment in practice.
5.1 Bankruptcy Prohibition with Unconstrained Portfolio

The mean-variance unconstrained portfolio problem with bankruptcy prohibition is an interesting but practically relevant model. In this case, \( k = m \) and \( \pi(\cdot) \in L^2_F(0, T; \mathbb{R}^m) \). It follows from (22) that

\[
\bar{z}(t) = \arg\min_{z \in \mathbb{R}^m} \| \sigma(t)'z - \sigma(t)^{-1}B(t)' \| = (\sigma(t)\sigma(t)')^{-1}B(t)'.
\]

Therefore,

\[
\hat{B}(t) = \sigma(t)\sigma(t)'\bar{z}(t) = B(t).
\]

**Proposition 1** Assume that \( \int_0^T |\hat{\theta}(s)|^2 \, ds > 0 \). Then there exists a unique efficient portfolio for this mean-variance model corresponding to any given \( d \geq x_0 e^{\int_0^T r(s) \, ds} \). Moreover, the efficient portfolio is given by (37) and the associated wealth process is expressed by (35), where \( \hat{B}(t) = B(t) \) and \( \hat{\theta}(t) = \sigma(t)^{-1}B(t) \).

The proof of Proposition 1 can be found in Bielecki et al. [1].

5.2 Bankruptcy Prohibition with No-shorting Constraint

The mean-variance portfolio problem with mixed bankruptcy and no-shorting constraints is another interesting and challenging model. In this case, \( k = m \) and \( \pi(\cdot) \in L^2_F(0, T; \mathbb{R}^m_+) \). It follows from (22) that

\[
\bar{z}(t) = \arg\min_{z \in \mathbb{R}^m_+} \| \sigma(t)'z - \sigma(t)^{-1}B(t)' \| = (\sigma(t)\sigma(t)')^{-1}(B(t) + \lambda(t))',
\]

where

\[
\lambda(t) := \arg\min_{y \in \mathbb{R}^m_+} \| \sigma(t)^{-1}y + \sigma(t)^{-1}B(t)' \|.
\]

Therefore,

\[
\hat{B}(t) = \sigma(t)\sigma(t)'\bar{z}(t) = B(t) + \lambda(t).
\]

**Proposition 2** Assume that \( \int_0^T |\hat{\theta}(s)|^2 \, ds > 0 \). Then there exists a unique efficient portfolio for this mean-variance model corresponding to any given \( d \geq x_0 e^{\int_0^T r(s) \, ds} \). Moreover, the efficient portfolio is given by (37) and the associated wealth process is expressed by (35), where \( \hat{B}(t) = B(t) + \lambda(t) \) and \( \hat{\theta}(t) = \sigma(t)^{-1}(B(t) + \lambda(t)) \).
5.3 No-shorting Constraint without Bankruptcy Prohibition

The mean-variance portfolio problem with no-shorting constraints is also an important model in financial investment. In this case, $k = m$ and $\pi(\cdot) \in L^2_T(0, T; \mathbb{R}^m_+)$. We again have

$$\hat{B}(t) = B(t) + \lambda(t),$$

where $\lambda(t)$ is determined by (38).

In particular, $d_1(t, \phi(t)) = -\infty$ and $d_2(t, \phi(t)) = -\infty$, that is, $N(-d_1(t, \phi(t))) = N(-d_2(t, \phi(t))) = 1$. The investor’s optimal wealth is the stochastic process

$$X^*(t) = \mu e^{-\int^T_t r(s) \, ds} - \gamma \phi(t) e^{-\int^T_t [2r(s) - |\hat{\theta}(s)|^2] \, ds}$$

and

$$\pi^*(t) = -(\sigma(t)\sigma(t)')^{-1} \hat{B}(t) \left[ X^*(t) - \mu e^{-\int^T_t r(s) \, ds} \right],$$

where

$$\mu = \frac{\mathbb{E}[\phi(T)^2]d - x_0 \mathbb{E}[\phi(T)]}{\text{Var}(\phi(T))} = \frac{d - x_0 e^{\int^T_0 [r(s) - |\hat{\theta}(s)|^2] \, ds}}{1 - e^{-\int^T_0 |\hat{\theta}(s)|^2 \, ds}},$$

$$\gamma = \frac{\mathbb{E}[\phi(T)]d - x_0}{\text{Var}(\phi(T))} = \frac{(d - x_0 e^{\int^T_0 r(s) \, ds}) e^{\int^T_0 [r(s) - |\hat{\theta}(s)|^2] \, ds}}{1 - e^{-\int^T_0 |\hat{\theta}(s)|^2 \, ds}}.$$  

where $\hat{\theta}(t) = \sigma(t)^{-1}(B(t) + \lambda(t))$.

**Proposition 3** Assume that $\int^T_0 |\hat{\theta}(s)|^2 \, ds > 0$. Then there exists a unique efficient portfolio for this mean-variance model corresponding to any given $d \geq x_0 e^{\int^T_0 r(s) \, ds}$. Moreover, the efficient portfolio is given by (39) and the associated wealth process is expressed by (41).

Another version of the proof of Proposition 3 can be found in Li, Zhou and Lim [16].

6 Example

In this section, a numerical example with constant coefficients is presented to demonstrate the results in the previous section. Let $m = 3$. The interest rate of the bond and the
appreciation rate of the $m$ stocks are $r = 0.03$ and $(b_1, b_2, b_3)' = (0.12, 0.15, 0.18)'$, respectively, and the volatility matrix is

$$
\sigma = \begin{bmatrix}
0.2500 & 0 & 0 \\
0.1500 & 0.2598 & 0 \\
-0.2500 & 0.2887 & 0.3227
\end{bmatrix}.
$$

Then we have

$$
\sigma^{-1} = \begin{bmatrix}
4.0000 & 0 & 0 \\
-2.3094 & 3.8490 & 0 \\
5.1640 & -3.4427 & 3.0984
\end{bmatrix},
$$

and $B = (b_1 - r, b_2 - r, b_3 - r)' = (0.09, 0.12, 0.15)'$. Hence,

$$
\theta := \sigma^{-1}B = (0.3600, 0.2540, 0.5164)'.
$$

In addition, we suppose that the initial prices of the stocks are $(S_1(0), S_2(0), S_3(0)) = (1, 1, 1)'$ and the initial wealth is $X(0) = 1$.

### 6.1 Bankruptcy Prohibition with Unconstrained Portfolio

In this subsection, we determine the optimal portfolio and the corresponding wealth process in Subsection 5.1 for the above market data. According to (34), we obtain the numerical results $\mu = 1.5046$ and $\gamma = 0.3154$. Hence, the wealth process (35) can be expressed by

$$
X^*(t) = \mu N\left(-d_2(t, \phi(t))\right)e^{-r(T-t)} - \gamma N\left(-d_1(t, \phi(t))\right)\phi(t)e^{-[2r-|\hat{\theta}|^2](T-t)},
$$

where

$$
d_1(t, \phi(t)) = \frac{\ln\left(\frac{\phi(t)}{\phi(T)}\right) - [r - \frac{3}{2} \hat{\theta}^2(T-t)]}{\sqrt{\hat{\theta}^2(T-t)}}, \\
d_2(t, \phi(t)) = d_1(t, \phi(t)) - \sqrt{\hat{\theta}^2(T-t)}, \\
\phi(t) = e^{-[r + \frac{1}{2} \hat{\theta}^2(T-t) - \hat{\theta}(W(T) - W(t))]}, \\
\hat{\theta} = \theta = \sigma^{-1}B = (0.3600, 0.2540, 0.5164)'.
$$

The efficient portfolio is given by

$$
\pi^*(t) = -(\sigma\sigma')^{-1}\hat{B}\left[X^*(t) - \mu N\left(-d_2(t, \phi(t))\right)e^{-r(T-t)}\right],
$$

where

$$
(\sigma\sigma')^{-1}\hat{B} = (\sigma\sigma')^{-1}B = (3.5200, -0.8000, 1.6000)'.
$$

In particular, the policy of investing in the second stock $\pi^*_2(t)$ is negative.
6.2 Bankruptcy Prohibition with No-shorting Constraint

From Subsection 6.1, we see that there exists a shorting case in policy (44). Using (38), we obtain the following $\lambda$ to re-construct the no-shorting policy

$$\lambda := \arg\min_{y \in \mathbb{R}_+^m} \|\sigma^{-1}y + \sigma^{-1}B'\| = (0, 0.03, 0)' .$$

Hence,

$$\begin{cases}
\hat{\theta} := \sigma^{-1}\hat{B} = \sigma^{-1}(B + \lambda) = (0.3600, 0.3695, 0.4131)', \\
(\sigma\sigma')^{-1}\hat{B} = (\sigma\sigma')^{-1}(B + \lambda) = (2.72, 0, 1.28)'.
\end{cases}$$

According to (34), we obtain the numerical results $\mu = 1.5253$ and $\gamma = 0.3368$. Hence, the wealth process (35) can be expressed by

$$X^*(t) = \mu N(- d_2(t, \phi(t))) e^{-r(T-t)} - \gamma N(- d_1(t, \phi(t))) \phi(t) e^{-[2r-|\hat{\theta}|^2](T-t)},$$

where

$$d_1(t, \phi(t)) = \frac{\ln\left(\frac{2}{\sigma^2} \phi(t) + [-r + \frac{1}{2}|\hat{\theta}|^2](T-t)\right)}{\sqrt{|\hat{\theta}|^2(T-t)}}, \quad d_2(t, \phi(t)) = d_1(t, \phi(t)) - \sqrt{|\hat{\theta}|^2(T-t)}, \quad \phi(t) = e^{-[r+\frac{1}{2}|\hat{\theta}|^2(T-t)-\hat{\theta}(W(T)-W(t))].}
$$

The

$$\pi^*(t) = - (\sigma\sigma')^{-1}\hat{B}\left[X^*(t) - \mu N(- d_2(t, \phi(t))) e^{-r(T-t)}\right].$$

Note that the policy in (48) is non-negative, that is, this is a no-shorting policy.

6.3 No-shorting Constraint without Bankruptcy Prohibition

In this subsection, we present the optimal no-shorting policy without the bankruptcy prohibition of Subsection 5.3 and its corresponding wealth process. According to (41), we find the numerical results $\mu = 1.5095$ and $\gamma = 0.3190$. Hence, the wealth process (35) can be expressed by

$$X^*(t) = \mu e^{-r(T-t)} - \gamma \phi(t) e^{-[2r-|\hat{\theta}|^2](T-t)}$$

and

$$\pi^*(t) = - (\sigma\sigma')^{-1}\hat{B}\left[X^*(t) - \mu e^{-r(T-t)}\right].$$
where \( \hat{\theta} \) and \((\sigma \sigma')^{-1} \hat{B}\) are given by (46), and

\[
\phi(t) = e^{-[r + \frac{1}{2}\|\hat{\theta}\|^2](T-t) - \hat{\theta}(W(T)-W(t))}.
\]

Note that the policy in (50) is non-negative, i.e., this is a no-shorting policy. However, its corresponding wealth (49) is possibly negative. We shall further discuss this point by simulation results in Subsection 6.4.

### 6.4 Simulation

In this subsection, we further analyze using simulation how the properties of the optimal portfolio strategies (44), (48) and (50) change according to the given target wealth \( d = 1.2X(0) \), and compare their wealth processes (43), (47) and (49).
Policy of Bankruptcy Prohibition with No-shorting Constraint

Policy of No-shorting Constraint without Bankruptcy Prohibition

Different Wealth Processes
7 Conclusion

We have studied the continuous-time mean-variance portfolio selection with mixed restrictions of bankruptcy prohibition (constrained state) and convex cone portfolio constraints (constrained controls). The main contribution of the paper is that we developed semi-analytical expressions for the pre-committed efficient mean-variance policy without the viscosity solution technique. A natural extension of our result to continuous-time linear-quadratic cone constrained controls with constrained states is straightforward, at least conceptually. On the other hand, if the rates of all market coefficients are random, the problem becomes more complicated. An important challenge appears when one considers general markets with convex portfolio constraints, so the constraints may not be cone.

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