SHEAVES ON GROTHENDIECK CONSTRUCTIONS

N. GOLUB

Abstract. In this paper we introduce a generalisation of a covariant Grothendieck construction to the setting of sites. We study the basic properties of defined site structures on Grothendieck constructions as well as we treat the cohomological aspects of corresponding toposes of sheaves. Despite the fact that the toposes of $G$-equivariant sheaves $\mathcal{S}h_G(X)$ have been introduced in literature, their cohomological aspects have not been treated properly in a desired fashion. So in the end of the paper we study some of the acyclic families, introduce new type of acyclic resolutions which we call the $G$-equivariant Godement resolutions, the degree of actions, and some other basic cohomological concepts arising in $\mathcal{S}h_G(X)$.

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1. Introduction

Given a (small) category $\mathcal{C}$ with a coverage on it $J$ (see for definition [102 C2.1, page 537]) a pair $(\mathcal{C}, J)$ is called a (small) site. If $c \in \text{ob}(\mathcal{C})$ then $S \in J(c)$ is called a covering family of morphisms on $c$. Following [102] a family of morphisms $S := \{f: \text{dom}(f) \to c\}$ is called a sieve on $c$ if it is a right ideal under the composition, meaning that for any $g$ composable with $f$ we have $fg \in S$ if $f \in S$.

If a coverage on $\mathcal{C}$ consists of sieves, then it is called a sifted coverage. We shall copy [102 C2.1, Definition 2.1.8] for a Grothendieck coverage here for the sake of convenience:

Definition 1.1. Let $\mathcal{C}$ be a category and $J$ which assigns to each object $c$ a collection of families (called covering) of morphisms $J(c)$ in $\mathcal{C}$ with $c$ being the codomain, and satisfies the following conditions.

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1
(1) (Stability under a pullback) Given $S \in J(c)$, $g : \text{dom}(g) \to c$ in $\mathcal{C}$, then $g^*(S) := \{\gamma : \text{dom}(\gamma) \to \text{dom}(g) : g\gamma \in S\} \in J(c)$.

(2) Maximal sieves are covering, meaning that for any $c \in \text{ob}(\mathcal{C})$ we have $\mathcal{C}(-, c) \in J(c)$.

(3) Let $S$ be a sieve on $c$, $R \in J(c)$ such that for any $g \in R$ we have that $g^*(S) \in J(\text{dom}(g))$, then $S \in J(c)$.

Then, $J$ is called a Grothendieck coverage on $\mathcal{C}$ and $(\mathcal{C}, J)$ is called a Grothendieck site.

Further given a sieve $S$ as in (3) we say that $S$ is submitted to $J$ coverage. Hence, (3) states in conjunction with (1) that for a sieve being submitted to $J$ is equivalent to being a member of $J$.

In topos theory for each site $(\mathcal{C}, J)$ one defines a topos of sheaves denoted as $\text{Sh}(\mathcal{C}, J)$ (see [102, A2.1]) as a full subcategory of the category of contravariant functors to $\text{Set}$ generated by sheaves on $(\mathcal{C}, J)$. $\mathcal{A} : \mathcal{C}^{\text{op}} \to \text{Set}$ is a sheaf on $(\mathcal{C}, J)$ provided the following is satisfied: given some $R \in J(U)$, where $U \in \text{ob}(\mathcal{C})$ and some $(s_f \in \mathcal{A}(\text{dom}(f)))_f$ compatible family for $\mathcal{A}$ and $R$, there is unique $s \in \mathcal{A}(U)$ such that $\mathcal{A}(f)(s) = s_f$ for any $f \in R$. Compatibility means that $\mathcal{A}(g)(s_f) = s_{fg}$ for any $f \in R$ and composable with it $g$.

A geometric morphism between toposes $\mathcal{E} \to \mathcal{G}$ is a pair of functors denoted usually as $(f_*, f^*)$, where $f_* : \mathcal{E} \to \mathcal{G}$ is a right adjoint to $f^*$, and $f^*$ preserves finite limits. Further we shall refer to the following definition, so we copy that from [102].

**Definition 1.2.** A category $\mathcal{C}$ is called cofiltered if the following conditions hold:

1. $\mathcal{C}$ is nonempty.
2. (Equalizing property) given $g, f : a \to b$ in $\mathcal{C}$, then there is $h : c \to a$ such that $gh = fh$.
3. (Downward directedness) given objects $a, b$ in $\mathcal{C}$, then there is $c$ with arrows:

$$\begin{array}{ccc}
 a & \rightarrow & c \\
 \downarrow & & \downarrow \\
 b & & \\
\end{array}$$

Given small sites $(\mathcal{C}, J)$ and $(\mathcal{D}, K)$ a functor $F : \mathcal{C} \to \mathcal{D}$ is said to be cover-reflecting if for any $S \in K(F(c))$ we have that $F^{-1}(S) := \{g : \text{dom}(g) \to c : F(g) \in S\}$ is covering. A functor $F : \mathcal{C} \to \mathcal{D}$ is said to be flat provided that for any object $d$ of $\mathcal{D}$, the coslice category is cofiltered $d \setminus F$ (see [102, C2.3, page 566]).

Given a cover-reflecting functor $F : \mathcal{C} \to \mathcal{D}$ as above, it induces the following geometric morphism $\left(\lim_{\mathcal{F}} \circ F, F^* : \text{Sh}(\mathcal{C}, J) \to \text{Sh}(\mathcal{D}, K)\right)$, where $F^*(\mathcal{A}) := \mathcal{A} \circ F$ is a precomposition functor, $a$ is a sheffification (see [102, C2.2, page 551]),

$$\text{Sh}(\mathcal{D}, K) \xrightarrow{\text{in}} \mathcal{PSh}(\mathcal{D}, K) \xrightarrow{F^*} \mathcal{PSh}(\mathcal{C}, J) \xrightarrow{a} \text{Sh}(\mathcal{C}, J)$$
and,

\[
\begin{array}{ccc}
\mathcal{C} & \xrightarrow{F} & \mathcal{D} \\
\downarrow & & \downarrow \\
\text{Set} & \xleftarrow{\lim_{\to} F} & \mathcal{A}
\end{array}
\]

is a right Kan extension (see for definition [M71, “Kan extensions”]).

Given a flat cover-preserving functor (flatness is a necessary and sufficient condition for the left Kan extension below to be cartesian) \( F : \mathcal{C} \to \mathcal{D} \) as above, then the induced geometric morphism \( (F^*, a \circ \lim_{\to} F) : \operatorname{Sh}(\mathcal{D}, K) \to \operatorname{Sh}(\mathcal{C}, J) \).

Now we remind the definition of the so called covariant Grothendieck construction \( \int_{\mathcal{C}} F \) of a functor \( F : \mathcal{C} \to \operatorname{Cat} \), where \( \operatorname{Cat} \) denotes a category of small categories and functors being the morphisms. \( \int_{\mathcal{C}} F \) is a category defined as follows: objects are pairs \((n, x)\) where \( n \) object of \( \mathcal{C} \), \( x \) object of \( \mathcal{F}(n) \); \( (g, f) : (n', x') \to (n, x) \) is a morphism if \( g : n' \to n \) in \( \mathcal{C} \), \( f : \mathcal{F}(g)(x') \to x \) in \( \mathcal{F}(n) \); composition is defined by the formula \( (g', f') \circ (g, f) := (g g', f' \circ \mathcal{F}(g')(f)) \).

In [V04], or [J02, B1.2, B1.3] one may find a basic theory of the indexed categories and fibrations (opfibrations), and the fact that the (covariant) Grothendieck construction provides a 2-equivalence of categories of fibrations (opfibrations) over \( \mathcal{C} \). The Grothendieck construction may be generalised to the case of enriched categories as it is shown in [BW19]. Explicitly, given \( F \) as above, the evident functor \( \int_{\mathcal{C}} F \to \mathcal{C} \) sending \((n, x)\) to \( n \) (morphisms respectively) is an opfibration.

In this section one of our problems is the natural one: generalise the Grothendieck construction to sites. Namely, let \((J, K)\) be a small site, consider a functor \( F : J \to \operatorname{Site} \) and define on the covariant Grothendieck construction (corresponding to \( F \) when we compose \( F \) with forgetful to \( \operatorname{Cat} \)) the Grothendieck topology so that the opfibration is a cover-reflecting, cover-preserving functor. We shall study the properties of the site structure on the generalised Grothendieck construction, and the corresponding toposes of sheaves on them.

1.1. Motivation. Let \( \operatorname{Top} \) denote the 2-category of toposes, geometric morphisms and natural transformations between them (see [J02, A4]).

Topos theory provides a tool for introducing cohomology theories on categories. The main advantage is that these cohomology theories are free from fixation to particular chain complexes. Explicitly, given a category \( \mathcal{C} \), then most of the cohomology theories are defined by \( \mathcal{F} : \mathcal{C}^{\text{op}} \to \mathcal{C}^+(R) \), and \( H^q(c; -) := H^q(\mathcal{F}(c)) \), where \( c \in \text{ob}(\mathcal{C}) \) and \((-)\) stands for some coefficients in a theory. For example, the de Rham cohomology is defined for \( \mathcal{C} = \text{SmoothMan} \) the category of smooth manifolds, and \( \mathcal{F} \) is a de Rham complex functor; \( \mathcal{C} = \text{Space} \) category of topological spaces, and \( \mathcal{F} \) is a complex of cocycles \( \text{hom}_2(\Delta(-), c, A) \), for \( A \) coefficient group. The topos theoretical cohomology is defined, on the other hand, by considering \( \mathcal{G} : \mathcal{C} \to \operatorname{Top} \), and \( H^q(c; A) := H^q(\mathcal{G}(c); A) \), where the last one is the cohomology of a topos \( \mathcal{G}(c) \) with coefficients in an abelian group object in \( \mathcal{G}(c) \) ([MM92, “Cohomology and homotopy”]). The fact that definition is given
via the derived functors rather than fixed chain complexes provides more freedom in computation.

Moreover, sheaf theoretic approach to cohomology provides unifications of other cohomology theories defined via functors to cochain complexes. The main idea is this: given a category $C$ with a $F_1, F_2: C^{op} \to C^+(\mathbb{Z})$ defining cohomology theories, one finds a functor from $C$ to $\text{Site}$, so that $F_1(c), F_2(c)$ extend to differential sheaves on an associated site with $c \in \text{ob}(C)$. Further, one may prove separately that these differential sheaves are in fact acyclic resolutions of a fixed sheaf. This, immediately implies that theories are isomorphic.

For example, considering a category of smooth manifolds and diffeomorphisms $\text{SmoothMan}$ as a site whose covering families are jointly surjective families of open embeddings, the de Rham complex functor $\Omega^*(-): \text{SmoothMan} \to C^+(\mathbb{R})$ now extends to a sheaf on $\text{SmoothMan}$, and, in particular, it defines a differential sheaf on each particular smooth manifold $M$; it turns out, this differential sheaf forms an acyclic resolution of a constant sheaf $\mathbb{R}M$ and hence, coincides with a sheaf cohomology of $M$ with coefficients in $\mathbb{R}M$. On the other hand, the singular cochain complex functor $\text{hom}_\mathbb{Z}(\Delta(-), c), A)$ extends analogously to a presheaf on $\text{SmoothMan}$ whose sheafification is an acyclic resolution of a constant sheaf $A_M$ (this holds for a rather general class of spaces, namely the so called homologically locally connected spaces [Br67, Example 1.2, page 35]). This immediately implies that the de Rham cohomology and the singular cohomology are isomorphic.

One of the methods to introduce a cohomology theory via toposes is to consider covariant, or contravariant functors from $C$ to $\text{Site}$ (the category of sites, with morphisms chosen to be either cover-reflecting functors or flat cover-preserving functors) and compose with $\text{Sh}: \text{Site} \to \text{Top}$. Now, considering the cohomology theories defined this way on $C$ we need to extend them over the functor category $C^\gamma$, where $\gamma$ is some site. We propose the following solution to this: given a functor $F: \gamma \to \text{Site}$ (now we assume morphisms in $\text{Site}$ to be cover-reflecting), we define a site $\int F$ whose underlying category is a covariant Grothendieck construction and further study the cohomology of a topos $\text{Sh}(\int F)$ interpreting it as a cohomology of $\gamma$-diagrams in $C$.

Construction of sheaf theoretic cohomology on categories of the type $C^\gamma$ is an important problem, since there is even a greater variety of cohomology theories defined on functor categories (diagrams of topological spaces, $G$-equivariant cohomology in particular) [H88, S94, GZ18] which need to be related to each other. This relations should be provided via the idea sketched above.

1.2. Grothendieck construction of an indexed site. In this section we give a definition of a Grothendieck construction of indexed sites and study it as a functor.

**Definition 1.3.** Let $\text{Site}$ be a category of small sites with morphisms being the cover-reflecting functors. Consider a site $(J, K)$, a functor $F: J \to \text{Site}$, then $((J, K), F)$ is called an *indexed site*. We form a category of indexed sites by defining morphisms as $(\alpha, \beta): ((J, K), F) \to ((J', K'), F')$, where $\beta: J \to J'$ is a cover-reflecting morphism, $\alpha: F \to F'\beta$ such that $\alpha(n)$ are
under the conditions of the lemma we may conclude that

by $K$

$F$

the proof.

$J,K$
The following implications hold: if

Condition 1 of Definition 1.1 holds: given $(g,f) : (n',x') \to (n,x) : g \in S, f \in \Psi$, where $S \in K(n), \Psi$ is a covering sieve on $x$ in $F(i)$ site.

Lemma 1.4. Given $(C,J)$, $(D,K)$ small sites, $F: C \to D$ functor. Then, the following implications hold: if $K$ satisfies [1] [2] of Definition 1.1, $(C,J)$ is a Grothendieck site, $F$ reflects covers of $K$, then $F$ is cover reflective with respect to the Grothendieck topology generated by $J$; $J$ satisfies [1], [2] of Definition 1.1, $(D,K)$ is a Grothendieck site, $F$ preserves covers of $K$, then $F$ is cover preserving with respect to the Grothendieck topology generated by $J$.

Proof. As the above remark indicates, it is sufficient for us to show that under the conditions of the lemma we may conclude that $F$ reflects (preserves respectively) the submitted sieve to a coverage $K$ (resp. $J$). For let $R$, $S$ be sieves on $F(d)$ where $R \in K(F(d))$, such that for all $g \in R$ one has $g^*S \in K(dom(g))$, i.e., $S$ is submitted to $K$. This implies that $F^{-1}((F(g_0))^*S) = \{ \Psi : F(g_0)F(\Psi) \in S \}$ is covering in $(C,J)$ for $g_0 \in F^{-1}R$. Note that $g_0^*F^{-1}(S) = \{ \Psi : F(g_0 \circ \Psi) \in S \}$ and thus is covering, which by [3] of Definition 1.1 implies $F^{-1}(S)$ is covering, whence the first part of the lemma. Given a sieve $T$ in $C$, a sieve generated by $\{ F(g) : g \in T \}$ we denote as $F(T)$. Note that

$F(g^*(S)) = \{ F(\Psi)g : g \Psi \in S \} \subset F(g)^*(F(S)) = \{ \xi : \exists \Psi, \gamma \text{ such that } F(g)\xi = F(\Psi)\gamma \}$

for any sieve. Hence, if $S$ is submitted to $J$ via the covering sieve $R$, then $F(S)$ is submitted to $K$, since for each $g \in R$ we have that $F(g)^*(F(S))$ contains a covering sieve (hence itself is covering as it follows from the axioms of Definition 1.1), and by [3] it implies $F(S)$ is covering, whence we conclude the proof. □
Proposition 1.5. Let $\Sigma_i: \mathcal{F}(i) \to \int_j \mathcal{F}$ be inclusion functor. Then, for each $(n_0, x_0) \in \text{ob}(\int_j \mathcal{F})$ we have that a slice decomposes

$$(n_0, x_0) \downarrow \Sigma_i = \bigsqcup_{g: n_0 \to i} \Gamma_g,$$

where $\Gamma_g$ are the full subcategories and cofiltered. Hence, $\Sigma_i$ is flat iff $i$ is final in $J$.

Proof. Indeed, if $(x, (g, f)): (n_0, x_0) \to (i, x)$, then given any morphism $\beta$ to $(x', (g', f')): (n_0, x_0) \to (i', x')$, i.e., $(\text{id}, \beta) \circ (g, f) = (g, \beta \circ f) = (g', f')$, and thus $g = g'$. We see that $(n_0, x_0) \downarrow \Sigma_i = \cup_{g: n_0 \to i} \Gamma_g$ decomposes onto $\Gamma_g$ full subcategory generated by objects with $(g, f)$ for some $f$ in definition. Now we need only to show that $\Gamma_g$ is cofiltered for each $g: n_0 \to i$. Given $g: n_0 \to i$, define $(\mathcal{F}(g)(x_0), (g, \text{id}_{\mathcal{F}(g)(x_0)}): (n_0, x_0) \to (i, \mathcal{F}(g)(x_0)))$ to prove that $\Gamma_g$ is nonempty. Equalizing condition (see Definition [1,2]):

$$f'_0: (\mathcal{F}(g'_0)(x_0), (g'_0, \text{id}_{\mathcal{F}(g'_0)(x_0)})(n_0, x_0)) \to (i, \mathcal{F}(g'_0)(x_0))) \to (x', (g'_0, f'_0): (n_0, x_0) \to (i', x'))$$

clearly equalizes any pair $g_1, g_2$:

$$(x', (g'_0, f'_0): (n_0, x_0) \to (i', x')) \to (x, (g_0, f_0): (n_0, x_0) \to (i, x)),$$

since we have a condition

$$(\text{id}, g_1) \circ (g'_0, f'_0) = (g'_0, g_1 \circ f'_0) = (g_0, f_0) = (g'_0, g_2 \circ f'_0).$$

Downward directedness follows from the above argument.

For flatness $\Sigma_i$ we need only existence of a unique $g: n_0 \to i$ for any $i$. \(\square\)

Remark 1.6. The opfibration $\int_j \mathcal{F} \to J$ turns out to have a section in some interesting cases:

1. If $x_i \in \mathcal{F}(i)$ we choose the final objects, then $\mu: J \to \int_j \mathcal{F}$ is defined by $\mu(i) := (i, x_i)$, and given $g: i \to k$, we put $\mu(g, !): \mathcal{F}(g)(x_i) \to x_k$.

2. $\mathcal{F}(i)$ are monoids (whose unique objects we shall denote $\bullet$) for $i \in \text{ob}(J)$, then $\mu(i) := (i, \bullet)$, and $\mu(g) := (g, e)$, where $e$ is identity element of $\mathcal{F}(k)$.

Proposition 1.7. In the case [1,2] of Remark 1.6 $\mu$ is flat iff $J$ is locally cofiltered. In the case [2] if $\mathcal{F}(g_1) \Delta \mathcal{F}(g_2): \mathcal{F}(\text{dom}(g_1)) \to \mathcal{F}(g_1) \times \mathcal{F}(g_2)$ are surjective homomorphisms of monoids for each pair $g_1, g_2$ of morphisms with a common domain and $J$ is locally cofiltered, then $\mu$ is flat.

Proof. For the case [1] we construct an isomorphism of categories $(n_0, x_0) \downarrow (\int_j \mathcal{F}) \to n_0 \downarrow J$ as follows: objects of $(n_0, x_0) \downarrow \mu$ are $(i, (g, !): (n_0, x_0) \to (i, x_i))$, and thus a functor sending this to $(i, g: n_0 \to i)$ is a desired isomorphism, whence the first part.

For the case [2] denote $(i, \bullet) =: i$ for objects of $\int_j \mathcal{F}$, and the objects of $i \downarrow \mu$ are of the form $(g, \Psi)$ where $g: i \to k, \Psi \in \mathcal{F}(k)$. It is nonempty and for the downward directedness we have: given $(g, \Psi): i \to k, (g', \Psi'): i \to k'$;
i ↓ J being downward directed we find \( k_0 \in \text{ob}(J) \), \( g_0, \beta, \beta' \) such that

\[
\begin{array}{c}
g_0 \\
\downarrow \\
I \\
\beta \\
\downarrow \\
k \\
\end{array}
\xrightarrow{k_0}
\begin{array}{c}
g' \\
\downarrow \\
k' \\
\beta' \\
\downarrow \\
k \\
\end{array}
\]

Then, \((\beta, e), (\beta', e) : ((g_0, \Psi^*): i \to k_0) \to ((g', \Psi'): i \to k')\) and \((g, \Psi): i \to k\) exist since there is some \( \Psi^* \in \mathcal{F}(k_0) \) such that \( \mathcal{F}(\beta)(\Psi^*) = \Psi, \mathcal{F}(\beta')(\Psi^*) = \Psi' \).

For equalizing property: consider \( g_0, g_1 : ((g, \Psi): i \to k) \to ((g', \Psi'): i \to k') \), then we have \( g_0 g = g_1 g \) and \( \mathcal{F}(g_0)(\Psi) = \mathcal{F}(g_1)(\Psi) = \Psi' \), which defines a parallel pair of arrows in \( i \downarrow J \), and yields some \( \gamma : i \to k \) with \( \xi : k_0 \to K \) such that \( g = \xi \gamma, g_0 \xi = g_1 \xi \), then for find \( \Psi_0 \) such that \( \mathcal{F}(\xi)(\Psi_0) = \Psi \) and that extends to an equalizing map in \( i \downarrow \int J \mathcal{F} \). □

**Example 1.8.** Consider a functor \( F : \mathcal{S} \to \mathcal{C} \), where \( \mathcal{S}, \mathcal{C} \) are sites and \( \mathcal{C} \) has pushout of pair of morphisms. Let \( \gamma \) be a pushout category, then on \( \gamma \) has the Grothendieck topology consisting of families of morphisms

\[
\begin{array}{c}
\gamma_1 \\
\downarrow \\
f_1 \\
\gamma_2 \\
\downarrow \\
f_2 \\
\gamma_3 \\
\downarrow \\
f_3 \\

g_1 \\
\end{array}
\]

where \( f_1 \in R \), and \( R \) is a covering family on \( \gamma_1 \).

Given \( f : I \to J \) in we define \( \mathcal{C}(f) : \mathcal{C} \setminus F(I) \to \mathcal{C} \setminus F(J) \) as a pushout:

\[
\begin{array}{c}
F(I) \\
\downarrow \\
c \\
\end{array}
\xrightarrow{F(J)}
\begin{array}{c}
\mathcal{C}(f) \\
\downarrow \\
c \\
\end{array}
\]

Clearly, if \( \mathcal{C} \to \mathcal{C} \) is cover-preserving (resp. cover-reflecting), then the defined functor \( \mathcal{C}(f) \) is cover-preserving (resp. cover-reflecting). We notice that the section of an opfibration \( \int_{\mathcal{S}} \mathcal{C} \setminus (\cdot) \to \mathcal{S} \) is cover-preserving, and via the Proposition 1.7 is flat iff \( \mathcal{S} \) is locally cofiltered.

**Proposition 1.9.** Given \( J \to J' \to \mathcal{F} \mathbf{Site} \), such that \( \beta \) is flat, then \( \hat{\beta} : \int_{J'}(\mathcal{F} \beta) \to \int_{J}(\mathcal{F} \beta) \) is flat.

**Proof.** Fix \( (n, y) \) in \( \int_{J}(\mathcal{F}) \). There is \( n \to \beta(i) \), so it defines \( (g, id) : (n, y) \to (\beta(i), \mathcal{F}(g)(y)) \), i.e., \( (n, y) \downarrow \hat{\beta} \) is nonempty. Consider \((i, x), (g, f) : (n, y) \to (\beta(i), x) \) and \((i', x'), (g', f') : (n, y) \to (\beta(i'), x') \), this defines a pair of morphisms in \( n \downarrow \beta \) for which there are \( i_0 \in \text{ob}(J), \gamma, \gamma' \) as follows:
So we define the following morphisms:

\[ ((i_0, \mathcal{F}(g_0)(y)), (g_0, id_{\mathcal{F}(g_0)(y)})) : (n, y) \rightarrow (i_0, \mathcal{F}(g_0)(y)) \rightarrow \]

\[ ((i, x), (g, f)) : (n, y) \rightarrow (i, x), \]

and

\[ ((i_0, \mathcal{F}(g_0)(y)), (g_0, id_{\mathcal{F}(g_0)(y)})) : (n, y) \rightarrow (i_0, \mathcal{F}(g_0)(y)) \rightarrow \]

\[ ((i, x), (g', f')) : (n, y) \rightarrow (i', x'), \]

and we have

\[ (\beta(\gamma), f) \circ (g_0, id_{\mathcal{F}(g_0)(y)}) = (\beta(\gamma)g_0, f \circ \mathcal{F}(\gamma)(id)) = (g, f), \]

as well as

\[ (\beta(\gamma'), f') \circ (g_0, id_{\mathcal{F}(g_0)(y)}) = (\beta(\gamma')g_0, f' \circ \mathcal{F}(\gamma')(id)) = (g', f'). \]

Now it suffices to verify the equalizing property. Given \((\bar{\alpha}, \bar{\gamma}), (\alpha, \gamma) : ((i, x), (g, f)) \rightarrow ((i', x'), (g', f'))\), we have

\[ (\beta(\alpha)g, \gamma \circ \mathcal{F}\beta(\alpha)(f)) = (g', f') = (\beta(\bar{\alpha})g, \bar{\gamma} \circ \mathcal{F}\beta(\bar{\alpha})(f)), \]

\(\alpha, \bar{\alpha} : (i, n \rightarrow i) \rightarrow (i', n \rightarrow i')\), there are \(i_0, g_0 : n_0 \rightarrow i_0, \alpha_0 : i_0 \rightarrow i\) such that \(\bar{\alpha}\alpha_0 = \alpha_0\alpha\) and \(\beta(\alpha_0)g_0 = g\). Then, \(((i_0, \mathcal{F}(g_0)(y)), (g_0, id_{\mathcal{F}(g_0)(y)}) \rightarrow ((i, x), (g, f)))\), is an equalizing map:

\[ (\beta(\alpha_0), f) \circ (g_0, id_{\mathcal{F}(g_0)(y)}) = (\beta(\alpha_0)g_0, f \circ \mathcal{F}\beta(\alpha_0)(id_{\mathcal{F}(g_0)(y)})) = (g, f); \]

\[ (\bar{\alpha}, \bar{\gamma}) \circ (\alpha_0, f) = (\bar{\alpha}\alpha_0, \bar{\gamma} \circ \mathcal{F}\beta(\bar{\alpha}))(f) = (\alpha\alpha_0, \gamma \circ \mathcal{F}\beta(\alpha)(f)) = (\alpha, \gamma) \circ (\alpha_0, f). \]

\[ \square \]

1.3. Dense subindexes. In [J02, C2.2, page 546] defines a notion of a dense subcategory of a site. Nevertheless, the “comparison lemma” (Theorem 2.2.3) turns out to be incorrect, since a nontrivial group and a trivial subcategory do satisfy the conditions (i) and (ii) of Definition 2.2.1 and by the Theorem 2.2.3 it would imply that the category of sets is equivalent to the category of right \(G\)-sets, which, by Cauchy completeness of groupoids, implies \(G\) must be trivial. This is because the condition (ii) is weaker than the fullness and is not really sufficient for the “comparison lemma” to hold. Nonetheless, as it is noted by Johnstone in page 546 usually the definition is given as follows:
Definition 1.10. Given a site \((\mathcal{C}, J)\), a full subcategory \(D\) of \(\mathcal{C}\) such that for any object \(c\) of \(\mathcal{C}\) there is a \(J\)-covering sieve on it generated by morphisms with domains in \(D\). Then, \(D\) is called a dense subcategory of a site \((\mathcal{C}, J)\).

One may readily verify that this version of the “comparison lemma” is correct indeed:

Theorem 1.11. Given a small site \((\mathcal{C}, J)\) and a dense subcategory \(D\), then the inclusion of sites \((D, J_D) \to (\mathcal{C}, J)\) yields the equivalence of toposes \(\mathcal{Sh}(D, J_D) \simeq \mathcal{Sh}(\mathcal{C}, J)\).

We are interested in finding a suitable generalisation of the concept of density to the context of indexed sites. Explicitly, we define for an indexed site \(((J, K), \mathcal{F})\) a dense subindex \(((\mathcal{D}, L_D), \mathcal{F}')\) such that the Grothendieck construction sends a subindex to the dense subcategory of \(\int_{(J,K)} \mathcal{F}\) (see 1.13).

Definition 1.12. Let \(((J, K), \mathcal{F})\) be an indexed site, \(\mathcal{F}' : D \to \text{Site}\), where \(D\) is a dense subcategory of a site \((\mathcal{C}, J)\) such that:

1. \(\mathcal{F}' \subseteq \mathcal{F}|D\) is a subfunctor.
2. For all \(i \in \text{ob}(J)\) one has
   \[
   \bigcap_{g_0 : i_0 \to i, i_0 \in \text{ob}(D)} \text{im}(\mathcal{F}(g_0)|\mathcal{F}'(i_0))
   \]
   being dense in \(\mathcal{F}(i)\).
3. For all \(g_0 : i_0 \to i, i_0 \in \text{ob}(D)\) one has \(\mathcal{F}(g_0)\) being full.

Then, \(((D, \mathcal{F}')\) is called a dense subindex of \(((J, K), \mathcal{F})\).

Theorem 1.13. Given a dense subindex as in Definition 1.12 then \(\int_D \mathcal{F}' \subset \int_{(J,K)} \mathcal{F}\) is a dense subcategory.

Proof. Fullness of \(\int_D \mathcal{F}'\) in \(\int_{(J,K)} \mathcal{F}\) is evident. Thus, it suffices to show that any object \((i, x)\) in \(\int_{(J,K)} \mathcal{F}\) has a covering sieve generated by morphisms with domains in \(\int_D \mathcal{F}'\). There is a \(K\)-covering sieve \(S\) on \(i\) generated by morphisms with domains in \(D\), and \(\Psi\) a covering sieve on \(x\) generated by morphisms in \(\bigcap_{g_0 : i_0 \to i, i_0 \in \text{ob}(D)} \text{im}(\mathcal{F}(g_0)|\mathcal{F}'(i_0))\). Indeed, given \((g, f) : (i', x') \to (i, x)\), find \(g', g_0\) and \(f'\) by condition (2) in Definition 1.12 and \(f_0\):

Then, clearly \((g, f) = (g_0, f_0) \circ (g', f')\), for \(f' : \mathcal{F}(g')(x') \to x_0\).

Example 1.14. Grothendieck constructions along monoids shall be of our interest considering in particular the toposes of \(G\)-equivariant sheaves on \(G\)-spaces. Let \(M := \text{End}_J(i_0)\), then \(M\) may be considered as a full subcategory of \(J\), and put \(\mathcal{F}' : M \to \text{Site}\) index, such that \(\mathcal{F}'(i_0) \subseteq \mathcal{F}(i_0)\) and \(\mathcal{F}(i_0)\) is preserved by \(\mathcal{F}(g)\) for any endomorphism \(g : i_0 \to i_0\). The density of this subindex in \(((J, K), \mathcal{F})\) is equivalent now to the following: for any \(g : i_0 \to i\)
we have that \( \mathcal{F}(g) \) must be full, and for any \( \bigcap_{g^{-1}} \text{im}(\mathcal{F}(g)\vert \mathcal{F}'(i_0)) \) must be dense in \( \mathcal{F}(i) \) for each \( i \). Hence, if \( i_0 \) is the last object, meaning that any morphism in \( J \) with domain \( i_0 \) must be endomorphic, we need only to verify that an intersection along \( M \) of all the image categories as above is dense in \( \mathcal{F}(i_0) \), which holds if \( M \) is a group.

For the sake of completeness observation above should be attached hereby as a claim:

**Corollary 1.15.** If \( i_0 \) is the last object of \( J \) and any endomorphism of it is invertible, then given any \( F : J \to \text{Site} \), \( F' : \text{Aut}(i_0) \to \text{Site} \) is dense subindex iff \( F'(i_0) \subset F(i_0) \) is stable under \( F(g) \) for any \( g \in \text{Aut}(i_0) \) and dense in \( F(i_0) \). In this case we have an equivalence of the toposes \( \mathcal{S}h_{\text{Aut}(i_0)}(\mathcal{F'}) \simeq \mathcal{S}h(\{J,K\}, \mathcal{F}) \).

### 1.4. Simplicial \( G \)-sites associated with actions.

Given a group \( G \) we consider \( \text{Site}^G \) category of \( G \)-sites and define a functor from a category of sites to the category of \( G \)-sites, such that [TODO].

Here we shall be concerned with the Grothendieck construction along the \( G \)-sites, such that \( \text{[TODO]} \).

Group actions on categories and the lax \( G \)-functors have been studied widely (see for example [Sh17], [HK13]). Nonetheless, we could not find though simple yet natural construction proposed in this section.

For the sake of terminological convenience (relation with left \( G \)-functors) we shall use an equivalent but different notation for the Grothendieck construction: given a \( G \)-category \( C \), we denote \( \mathcal{G}(C) \) category whose objects are precisely the objects of \( C \), and \( \mathcal{G}(C)(a,b) := \{(g,f) : a \to b : (f : a \to gb) \in C(a,gb)\} \). This category is isomorphic with \( \int_C C \), and the isomorphism is provided by \( (g,f) \mapsto (g^{-1}, g^{-1}(f)) \) change of morphisms. Further, we define a \( G \)-category structure on \( \mathcal{G}(C) \): given \( a \in \text{ob}(\mathcal{G}(C)) = \text{ob}(C), g(a) := ga \), where the left hand side indicates the action defined on \( C \); given \( (g_0,f) : a \to b \) in \( \mathcal{G}(C) \), then \( g(g_0,f) := (gg_0g^{-1}, g(f)) \) action on morphisms. This by induction defines \( \mathcal{G}^{n+1}(C) := \mathcal{G}(\mathcal{G}^n(C)) \), categories with the same classes of objects, and morphisms being (as seen by induction) are of the form \((g_0,\ldots,g_n,f) : a \to b, \) where \( f : a \to g_n \ldots gb \) in \( C \).

A left \( G \)-functor is a \( G \)-category \( C \) with a functor \( F : C \to D \) with a family of natural transformations \( \Phi_g : Fg \to F \) subject to the condition

\[
(\forall g_1, g_2 \in G, c \in \text{ob}(C))(\Phi_{g_1g_2}(c) = \Phi_{g_2}(c) \circ \Phi_{g_1}(g_2c))
\]

(see [SJ01]).

Left \( G \)-functor structures on \( F : C \to D \) correspond bijectively to extensions over \( \epsilon : \mathcal{G}(C) \) of \( F \), where the correspondence being given by the following: if \( \epsilon \) is a retraction, then \( \Phi_g := \epsilon((g,id_{g(-)})). \) In particular, retractions \( \mathcal{G}(C) \to C \) correspond bijectively to left \( G \)-functor structures on the identity functor on \( C \).
Given a functor between $G$-categories $F : \mathcal{C} \to \mathcal{D}$ is said to be $G$-equivariant if the following diagrams commute:

$$
\begin{align*}
G \times \text{ob}(\mathcal{C}) & \longrightarrow \text{ob}(\mathcal{C}) & G \times \text{Mor}(\mathcal{C}) & \longrightarrow \text{Mor}(\mathcal{C}) \\
\text{id}_G \times F & \downarrow & \downarrow & \downarrow F \\
G \times \text{ob}(\mathcal{D}) & \longrightarrow \text{ob}(\mathcal{D}) & G \times \text{Mor}(\mathcal{D}) & \longrightarrow \text{Mor}(\mathcal{D})
\end{align*}
$$

where the vertical arrows are actions. Hence, one may readily verify that a retraction $\epsilon$ is equivariant iff for the corresponding left $G$-functor structure on the identity functor satisfies:

$$(\forall g_1, g_2 \in G, c \in \text{ob}(\mathcal{C}))(\Phi_{g_1 g_2}(c) = \Phi_{g_1}(c) \circ g_1(\Phi_{g_2}(c))) \quad (2)$$

**Remark 1.16.** All the constructions may be proceeded in the more general lax approach. Nevertheless, we decided to work with the functorial case rather than with pseudofunctorial in order to avoid notational difficulties.

**Lemma 1.17.** $G^n(\mathcal{C})$ have the following composition formulas in our notation:

$$(\tilde{g}_1, \ldots, \tilde{g}_n, \tilde{f}) \circ (g_1, \ldots, g_n, f) = (g_1 \tilde{g}_1, \ldots, (g_n \cdots g_1 \tilde{g}_n g_1^{-1} \cdots g_{n-1}^{-1}), (g_n, \ldots, g_1)(\tilde{f}) \circ f).$$

**Proof.** The proof is proceeded inductively on $n$. Consider a composition in $G(\mathcal{C})$: $(\tilde{g}, \tilde{f}) \circ (g, f) : a \to b \to c$, i.e., $\tilde{f} : b \to \tilde{g} c$, and $f : a \to g b$. So the composition formula has the form $(g \tilde{g}, g(\tilde{f}) \circ f)$, which verfies the base of induction.

Assume the formula has already been checked for $n$ and

$$(\tilde{g}_1, \ldots, \tilde{g}_{n+1}, \tilde{f}) \circ (g_1, \ldots, g_{n+1}, f) = (g_1 \tilde{g}_1, g_1((\tilde{g}_2, \ldots, \tilde{g}_{n+1}, \tilde{f}) \circ (g_2, \ldots, g_{n+1}, f))) =
= (g_1 \tilde{g}_1, (g_1 \tilde{g}_2 g_1^{-1}, \ldots, g_1 \tilde{g}_{n+1} g_1^{-1}, g_1(\tilde{f})) \circ (g_2, \ldots, g_{n+1}, f)) =
= (g_1 \tilde{g}_1, g_2 \tilde{g}_1 g_2^{-1}, \ldots, g_n \tilde{g} g_1^{-1}, g_{n+1}^{-1}, (g_n \cdots g_2) g_1(\tilde{f}) \circ f),$$

which concludes the inductive step. \hfill \square

**Theorem 1.18.** Put $X_{-1} := \mathcal{C}$, $X_n := G^{n+1}(\mathcal{C}), n \geq 0$. If there is an equivariant retraction $\epsilon : G(\mathcal{C}) \to \mathcal{C}$, then $\{X_i\}_{i \geq -1}$ forms an augmented simplicial $G$-category with $\epsilon$ being the augmentation functor.

**Proof.** $d_i : X_n \to X_{n-1}, i = 0, \ldots, n$ coboundaries are defined identically on objects, and

$$d_i((g_0, \ldots, g_n, f)) := (g_0, \ldots, g_{i-1}, g_i g_{i+1}, \ldots, g_n, f)$$

for $i < n$, and

$$d_n((g_0, \ldots, g_n, f)) := (g_0, \ldots, g_{n-1}, \Phi_{g_n}((g_{n-1} \cdots g_0) \text{cod}((g_0, \ldots, g_n, f))) \circ f).$$

Degeneracies are defined as in the Bar resolution (see [W94, 8.7.1] for instance),

$$s_i((g_0, \ldots, g_n, f)) := (g_0, \ldots, g_{i-1}, e, g_i, \ldots, g_n, f)$$

for $i = 0, \ldots, n$. 

Augmentation $\epsilon : X_0 \to X_{-1}$ satisfies $\epsilon d_0((g_0, g_1, f)) = \epsilon((g_1, g_0, f))$, whilst $\epsilon d_1((g_0, g_1, f)) = \epsilon((g_0, \Phi_{g_1}((g_0, g_1, f)) \circ f))$. Thus, to check the augmentation identity we need to verify the commutativity of the following diagram:

$$
\begin{array}{ccc}
(g_1, g_0)(\text{cod}((g_0, g_1, f))) & \Phi_{g_1}((g_1, g_0)(\text{cod}((g_0, g_1, f)))) & \text{cod}((g_0, g_1, f)) \\
\downarrow & & \downarrow \\
g_0(\text{cod}((g_0, g_1, f))) & & \Phi_{g_0}(\text{cod}((g_0, g_1, f)))
\end{array}
$$

which holds from $[1]$. Further we verify functoriality of degeneracies and coboundaries. Let $0 \leq i < n$, then $d_i$ is functorial as the following indicates:

$$
d_i(((\tilde{g}_0, \ldots, \tilde{g}_n, \tilde{f}) \circ (g_0, \ldots, g_n, f)) =
\begin{cases}
\Phi_{g_0(\text{cod}((\tilde{g}_0, \ldots, \tilde{g}_n, \tilde{f}) \circ (g_0, \ldots, g_n, f)))}
\end{cases}
$$

Functoriality of $d_n$ is more subtle:

$$
d_n((g_0\tilde{g}_n, \ldots, g_n \cdot g_0\tilde{g}_n g_0^{-1} \ldots g_{n-1}, (g_n \cdot g_0)(\tilde{f}) \circ f)) = (g_0\tilde{g}_n, \ldots, g_n \cdot g_0 g_n^{-1} \ldots g_{n-1}, \ldots, (g_n \cdot g_0)(\tilde{f}) \circ f)
$$

On the other hand we have a composition: $(\tilde{g}_0, \ldots, \tilde{g}_{n-1}, \Phi_{g_n}(\text{cod}((\tilde{g}_0, \ldots, \tilde{g}_n, \tilde{f}))) \circ (g_0, \ldots, g_{n-1}, \Phi_{g_n}(\text{cod}((g_0, \ldots, g_n, f))) \circ f)).$ Hence, it suffices to verify that the following diagram commutes:

$$
\begin{array}{ccc}
g_{n-1} \cdot g_0 g_{n-1} \cdot g_0 a & \to & g_{n-1} \cdot g_0 g_{n-1} \cdot g_0 a \\
\downarrow & & \downarrow \\
(g_{n-1} g_0, f) & & (g_{n-1} g_0, f)
\end{array}
$$

where $a := \text{cod}((\tilde{g}_0, \ldots, \tilde{g}_n, \tilde{f}))$, $b := \text{cod}((g_0, \ldots, g_n, f))$. For that we shall prove the commutativity of the inner diagrams, where the dotted arrow is $\Phi_{g_n}(g_{n-1} \cdot g_0 g_{n-1} \cdot g_0 a)$.

The right hand triangle:

$$
(g_{n-1} g_0, \Phi_{g_n}(g_{n-1} \cdot g_0 a)) \circ \Phi_{g_n}(g_{n-1} \cdot g_0 g_n \cdot g_0 a) =
= (\Phi_{g_n}(g_{n-1} \cdot g_0 a))^{-1} \circ \Phi_{g_n}(g_{n-1} \cdot g_0 a)
$$
which is equivalent to
\((g_{n-1} \cdots g_0)(\Phi_{g_n}(g_{n-1} \cdots g_0a)) = (\Phi_{(g_{n-1} \cdots g_0)}(g_{n-1} \cdots g_0a))^{-1} \circ \Phi_{g_{n-1} \cdots g_0\tilde{g}_n}(\tilde{g}_{n-1} \cdots \tilde{g}_0a).\)

We used here 4 and get the following equality:
\[ \Phi_{g_{n-1} \cdots g_0\tilde{g}_n}(\tilde{g}_{n-1} \cdots \tilde{g}_0a) = \Phi_{g_{n-1} \cdots g_0\tilde{g}_n}(\tilde{g}_{n-1} \cdots \tilde{g}_0a) \circ \Phi_{g_n}(g_{n-1} \cdots \tilde{g}_n \cdots \tilde{g}_0a). \]

Denoting \(c := \tilde{g}_{n-1} \cdots \tilde{g}_0a, \beta := \tilde{g}_n,\) we finally obtain
\[ (\Phi_\alpha(c))^{-1} \circ \Phi_{\alpha\beta}(c) = \alpha(\Phi_\beta(c)), \]
i.e., 2 (equivariance of \(\epsilon).\)

Functionality of degeneracies:
\[ s_i((g_0\tilde{g}_0, \ldots, g_n \cdots \tilde{g}_0\tilde{g}_n g_0^{-1} \cdots g_{n-1}^{-1}, (g_n \cdots g_0)(\tilde{f} \circ f)) = (g_0\tilde{g}_0, \ldots, g_i-1 \cdots \tilde{g}_0 g_0^{-1} \cdots g_{i-2}^{-1}, e, g_i \cdots g_0\tilde{g}_i-1 g_0^{-1} \cdots g_{i-2}^{-1} \cdots g_{i-1}^{-1}, \ldots, f). \]

On the other hand, the composition
\[ (g_0, \ldots, \tilde{g}_{i-1}, e, \tilde{g}_i, \ldots, \tilde{g}_n, \tilde{f}) \circ (g_0, \ldots, g_{i-1}, e, g_i, \ldots, g_n, f) = (g_0\tilde{g}_0, \ldots, g_{i-1} \cdots \tilde{g}_0 g_0^{-1} \cdots g_{i-2}^{-1}, g_{i-1} \cdots g_0\tilde{g}_i-1 g_0^{-1} \cdots g_{i-2}^{-1} \cdots g_{i-1}^{-1}, \ldots, f). \]

Now we verify equivariance of defined functors: let \(0 \leq i < n, \gamma \in G,\) then
\[ d_i((g_0 \gamma^{-1}, \ldots, g_n \gamma^{-1}, \gamma(f))) = (g_0 \gamma^{-1}, \ldots, g_i \gamma^{-1}, \ldots, g_n \gamma^{-1}, \gamma(f)) = \gamma(g_0, \ldots, g_i^{-1} g_i, \ldots, g_n, f). \]

For \(i = n\) we have:
\[ d_n(\gamma((g_0, \ldots, g_n, f))) = (g_0 \gamma^{-1}, \ldots, g_{n-1} \gamma^{-1}, \Phi_{g_n} g_{n-1} \cdots g_0 \gamma^{-1} \mathrm{cod}((g_0 \gamma^{-1}, \ldots, g_n \gamma^{-1}, \gamma(f))) \circ \gamma(f))). \]

Whilst,
\[ \gamma d_n() = (g_0 \gamma^{-1}, \ldots, g_{n-1} \gamma^{-1}, \gamma(\Phi_{g_n}(g_{n-1} \cdots g_0 \mathrm{cod}((g_0, \ldots, g_n, f)))) \circ \gamma(f)). \]

Then, it suffices to check
\[ \gamma(\Phi_{g_n}(c)) = \Phi_{\gamma g_n}^{-1}(\gamma(c)), \]
for \(c := g_{n-1} \cdots g_0 \mathrm{cod}((g_0, \ldots, g_n, f)).\) By 4 we have
\[ \Phi_{(\gamma g_n)^{-1}}(\gamma(c)) = \Phi_{\gamma^{-1}}(\gamma(c)) \circ \Phi_{\gamma g_n}(g_n c), \]
and
\[ \Phi_{\gamma g_n}(g_n c) = \Phi_{\gamma}(g_n c) \circ \gamma(\Phi_{g_n}(g_n c)) \]
by 2 so it is equivalent to
\[ \Phi_{\gamma^{-1}}(\gamma(c)) \circ \Phi_{\gamma}(g_n c) = \Phi_{\epsilon}(g_n c) = \mathrm{id}_{\gamma g_n c}. \]

Equivariance of degeneracies is evident, and \(\epsilon\) is equivariant by an assumption.

Let us attach here the simplicial identities (see [M71 page 175]):
\[ d_id_{j+1} = d_j d_i, i \leq j; \quad (3) \]
\[ s_{j+1}s_i = s_is_j, i \leq j; \]
\[ d_i s_j = \begin{cases} 
  s_{j-1} d_i, & i < j; \\
  1, & i = j, j + 1; \\
  s_j d_{i-1}, & i > j + 1. 
\end{cases} \]

For \( i < n \) identities are as in the Bar construction, so the only nontriviality is with \( n' \)th terms. For \( 3 \) we need to check \( d_{n-1} d_n = d_{n-1} d_{n-1} \): let \( f := \Phi_{g_n}(g_{n-1} \ldots g_0 \text{cod}((g_0, \ldots, g_n, f))) \circ f \), then
\[
 d_{n-1} d_n ((g_0, \ldots, g_n, f)) = (g_0, \ldots, g_{n-2}, \Phi_{g_{n-1}}(g_{n-2} \ldots g_0 \text{cod}((g_0, \ldots, g_{n-1}, f)))).
\]

Let \( a := \text{cod}((g_0, \ldots, g_n, f)) = \text{cod}((g_0, \ldots, g_{n-1}, f)) \). Then
\[
 d_{n-1} d_{n-1} ((g_0, \ldots, g_{n-2}, f)) = (g_0, \ldots, g_{n-2}, \Phi_{g_{n-2}}(g_{n-3} \ldots g_0 a) \circ f).
\]

The identity \( d_i d_n = d_{n-1} d_i \) for \( i < n - 1 \) is evident. The identity \( d_n s_{j+1} = s_{j} d_{n-1} \) is evident also. Now we have only to check: \( d_n s_n (g_0, \ldots, g_{n-1}, f) = d_n (g_0, \ldots, e, f) = (g_0, \ldots, g_{n-1}, \Phi_e \circ f) \).

**Remark 1.19.** A left \( G \)-functor structure on the identity functor exists on \( G(C) \) considered as a \( G \)-category, so it satisfies the condition of the Theorem [1.18]. Indeed, \( \Phi_g := (g, \text{id}_{g(-)}) \), then naturality is seen from
\[
 \xymatrix{ 
 g x \ar[d]_{\Phi_g(x)} \ar[r]^{(g_{x} \gamma x^{-1} g(f))} & g y \ar[d]_{\Phi_g(y)} \\
 x \ar[r]^{(\gamma, f)} & y
} \]

and an equivariance condition [2]
\[
 \Phi_{g_1 g_2}(x) := (g_2, \text{id}_{g_2 x}) \circ (g_1, \text{id}_{g_1 g_2 x}) = \Phi_{g_2}(x) \circ \Phi_{g_1}(g_1 g_2 x).
\]

**Remark 1.20.** As indicated in [3] a geometric realization of a Grothendieck construction along a \( G \)-category \( C \) is of type \( EG \times_G BC \). Thus, the Theorem [1.18] induces the structure of a simplicial \( G \)-space of the form:
\[
 \cdots \rightarrow EG \times_G BC (\cdots (EG \times_G BC) \cdots) \rightarrow \cdots \rightarrow EG \times_G BC \rightarrow BC
\]
for \( BC \) being the classifying space of a category, and \( EG \) is universal cover of the classifying space \( BG \).

1.5. **Site properties of** \( \int(J,K)F \). Here we shall be concerned with the problems of finding the implications on the basic properties of the \( \int(J,K)F \) site induced by the properties of \((J,K)\) and \(F(i), i \in \text{ob}(J)\).

Recall that a sieve in \( C \) category is said to be effective-epimorphic provided that any representable presheaf satisfies the sheaf axiom for it (see [1.02, C2.1, page 542]).

**Definition 1.21.** Let \((J,K), F\) be an indexed site. It is said to be surjectively full if for any \((i,x)\) object of \( \int(J,K)F \), \( S \in K(i) \), \( \Psi \) covering sieve on \( x \) the projections (sending \((g,f) \rightarrow g, \Psi\) covering sieve on \( x \)) \( \alpha(S,\Psi) \rightarrow S, \Psi \) are surjective.

**Lemma 1.22.** Let \((i,x) \in \text{ob}(\int(J,K)F), S \in K(i)\). Then
\[
 (1) \: \alpha(S,\Psi) \rightarrow S \text{ is surjective iff for all } g \in S \text{ there exists } (a \rightarrow x) \in \Psi \text{ and } (F(g)(a') \rightarrow a)_{a' \in F(\text{dom}(g))};
\]
(2) \(\alpha(S, \Psi) \to \Psi\) is surjective iff \(\{F(g) : g \in S\}\) is jointly objectwise surjective, meaning that for each \(x \in \text{ob}(F(i))\) there is \(g \in S\) with \(x' \in \text{ob}(F(\text{dom}(g)))\) such that \(F(g)(x') = x\).

Proof. The only slight nontriviality is in the left hand implication of the first part. Here we use that \(S\) is a sieve and stable under the composition on the right (see Section 1), since then \(F(g)(a') \to a \to x\) will be in \(\Psi\). \(\square\)

Remark 1.23. Further in the claims we shall rather use the stronger condition: \(F(g)\) functors to be objectwise surjective, full. Nonetheless, for the future research one may require this particular definition.

Lemma 1.24. Assume that \(F(g)\) are surjective objectwise full and cover-preserving for \(((J, K), F)\) index. Then, we have the following:

1. any sieve on \(\int_J F\) is contained in one of the form \(\alpha(S, \Psi)\).
2. If \(R\) is covering sieve in \(\int_{(J,K)} F\), then there are \(S, \Psi\) covering sieves such that \(R \subset \alpha(S, \Psi)\).

Proof. Indeed, \(\epsilon\) is full evidently, and thus \(\epsilon(R)\) is a sieve on \(n\) if \(R\) is a sieve on \((n, x)\). Put

\[
R|_{\mathcal{F}(n)} := \{f : x' \to x : \exists g : n' \to n \text{ such that } (g, f) \in R\},
\]

is a sieve: given \(f' : x'' \to x'\), by objectwise surjectiveness and fullness of \(F(g)\) we have that this morphism may be represented as \(F(g)(\tilde{f}) : F(g)(y'') \to F(g)(y')\), and thus \((g, f \circ f') = (g, f) \circ (\text{id}_{n'}, \tilde{f}) \in R\) (\(R\) is a sieve). Clearly \(R \subset \alpha(\epsilon(R), R|_{\mathcal{F}(n)})\), which proves (1).

To prove (2) we again use the transfinite argument (see the beginning of the Section 2) by showing that if \(R\) is submitted sieve to a coverage whose sieves \(R'\) satisfy the following: \(\epsilon(R')\) and \(R'|_{\mathcal{F}(n)}\) are covering; then \(R\) satisfies this condition as well. The base of induction is satisfies, since we start with the coverage consisting of \(\alpha(S, \Psi)\) sieves, for which we obviously have \(\epsilon(\alpha(S, \Psi)) = S, \alpha(S, \Psi)|_{\mathcal{F}(n)} = \Psi\). Now, assume that \(R\) is submitted to a coverage satisfying the above condition, and \(R'\) is a sieve in this coverage such that \((\forall(g, f) \in R')(\exists (g, f)^*R)\) is in this coverage. Since \(\epsilon\) is cover-preserving, whence \(\epsilon(R)\) is covering. Further we show that \(R|_{\mathcal{F}(n)}\) is submitted to \(R'|_{\mathcal{F}(n)}\), which shall immediately imply that \(R|_{\mathcal{F}(n)}\) is covering by (3) of Definition 1.12. For consider \(f \in R'|_{\mathcal{F}(n)}\), there is \(g : n' \to n\) such that \((g, f) \in R', (g, f)^*R|_{\mathcal{F}(n)} \equiv \{\gamma : \exists \tilde{g} : (\tilde{g}, f \circ \gamma) \in R\}, (F(g)((g, f)^*R)|_{\mathcal{F}(\text{dom}(g))} \equiv \{F(g)(\Psi) : \exists \beta : (g\beta, f \circ F(g)(\Psi)) \in R\}\) is covering by cover-preservance of \(F(g)\) and that the pullback inside is in the coverage. \(F(((g, f)^*R)|_{\mathcal{F}(\text{dom}(g))}) \subset (f^*(R|_{\mathcal{F}(n)})\) and thus is covering since it contains a covering sieve. Hence, \(R|_{\mathcal{F}(n)}\) is covering, which conclude an inductive step. \(\square\)

Recall that \((J, K)\) is a rigid site provided the full subcategory \(J_0 \subset J\) generated by \(K\)-irreducible objects in \((J, K)\) is dense (see Definition 1.12), an object \(c\) in \(J\) is \(K\)-irreducible if admits no \(K\)-covering sieve except the maximal.

Now, let \(\mathcal{F}_0 : J_0 \to \text{Site} \) be defined as \(\mathcal{F}_0(i) := (\mathcal{F}(i))_0\) the full subcategory of \(\mathcal{F}(i)\) generated by irreducible objects. Notice that \((n, x)\) is irreducible in \(\int_{(J,K)} F\) if only \(n\) and \(x\) are irreducible.
Lemma 1.25. Assume that for any $\gamma \to F(g)(x)$ there are $\gamma, f$:

$$
\begin{array}{ccc}
\gamma & \xleftarrow{f'} & F(g)(x) \\
\downarrow & & \downarrow \\
F(g)(a) & \xrightarrow{g} & F(g)(f)
\end{array}
$$

Then, $F_0$ is a subfunctor, meaning that the following holds:

$$
\begin{array}{ccc}
F_0(i) & \xrightarrow{F_0(i)} & F_0(k) \\
\downarrow & & \downarrow \\
F(i) & \xrightarrow{F(g)} & F(k)
\end{array}
$$

Proof. Indeed, if $x \in \text{ob}(F_0(i))$, we need to show that $F(g)(x)$ is irreducible. For that, consider $S$ to be a covering sieve on $F(g)(x)$, then $F(g)^{-1}(S)$ is maximal since $x$ is irreducible. Now, given $f'$, find $u, f$ as in the diagram above, then $F(g)(f) \in S$, and thus $f' \in S$, which proves irreducibility of $F(g)(x)$.

\[\square\]

Definition 1.26. Given $((J, K), F)$ indexed site satisfying the above lemma, we say that it is rigidly compatible.

Proposition 1.27. Let $((J, K), F)$ be a rigidly compatible indexed site, then $(\int_{(J, K)} F)_0 = \int_{(J, K)_0} F_0$, where in the right hand side we have a full subcategory generated by irreducible objects of a site $((J, K), F)$.

The Propositions 1.27, 1.28 may be useful in the problem to detect whether a presheaf on $\int J F$ is actually a sheaf, or in defining the geometric morphisms to the topos $\text{Sh}(\int_{(J, K)} F)$.

Proposition 1.28. Assume that $(J, K)$ is rigid, $\bigcap_{i \in J_0, i \in J_0} \text{im}(F(g_0))$ dense in $F(i)$ for each $i \in \text{ob}(J)$ (e.g $F(g)$ are objectwise surjective and full). Then, the inclusion $\int_{(J, K)_0} F((J, K)_0) \to \int_{(J, K)} F$ yields the equivalence of toposes $\text{Sh}(\int_{(J, K)} F) \simeq \text{Sh}(\int_{(J, K)_0} F)$.

Proof. Indeed, the conditions immediately imply that $((J, K)_0, F((J, K)_0)$ is a dense subindex of $((J, K), F)$ and thus, by Proposition 1.13 we conclude that $\int_{(J, K)_0} F((J, K)_0$ is dense in $\int_{(J, K)} F$, so applying the Theorem 1.11 we obtain the desired equivalence.

\[\square\]

Another case in which it suffices to verify that a presheaf $A$ on $\int J F$ satisfies sheaf condition on $F(i)$ to prove that $A$ is a sheaf on $\int_{(J, K)} F$. Despite the fact that we do not have an example of its application we hereby attach the following:

Lemma 1.29. Let $(n, x) \in \text{ob}(\int_{(J, K)} F)$, $S$ a covering sieve on $n$ and $\Psi$ on $x$ respectively, and assume we have a section $g[-] : \Psi \to \alpha(S, \Psi)$ of a projection $\alpha(S, \Psi) \to \Psi$ satisfying the following conditions:

1. Given $(g, f) \in \alpha(S, \Psi)$, then $g[f] \circ g = g$.
2. $g[f] = g[f']$ for any $f \in \Psi$ and $f'$ composable with $f$
3. $F(g[f]) = \text{id}_{F(n)}$ for any $f \in \Psi$
Then, $A : (\int_f F)^{op} \to \text{Set}$ satisfies the sheaf axiom for $\alpha(S, \Psi)$ if $A|\mathcal{F}(n)$ satisfies the sheaf axiom for $\Psi$.

Proof. Consider a compatible family $(s_{(g,f)} \in A(\text{dom}(g,f)))$ for $\alpha(S, \Psi)$, meaning that for any $(g', f') : \text{dom}(g', f') \to \text{dom}(g, f)$, we have that $s_{(g,f)\circ(g', f')} = A(g', f')(s_{(g,f)})$. Define $\check{s}_f := s_{(g,f), f} \in A|\mathcal{F}(n)(\text{dom}(f))$. This family is compatible: given $f' : \text{dom}(f') \to \text{dom}(f)$, $\check{s}_f f' = s_{(g,f)'\circ(g', f')} = A(id_n, f')(s_{(g,f), f}) \equiv A|\mathcal{F}(n)(f')(\check{s}_f)$, by (2), (3). Then, there is unique $\check{s} \in A|\mathcal{F}(n)(x) \equiv A(n, x)$ such that $(\forall f \in \Psi)(A(id_n, f)(\check{s}) = s_{(g,f), f})$. Let $((g, f) : (n', x') \to (n, x)) \in \alpha(S, \Psi)$, $(g, f) = (id_n, f) \circ (g, id_{x'})$ and hence $A(g, f)(\check{s}) = A(g, id_{x'})A(id_n, f)(\check{s}) = A(g, id_{x'})(s_{(g,f), f}) = s_{(g,f)'\circ(g, id_{x'})} = s_{(g,f)\circ(g, f)} = s_{(g,f), f}$, where the last equality follows form (1). Uniqueness of $\check{s}$ is evident. \qed

Recall that a category $C$ satisfies the right Ore condition if for any $f, f'$ there is an extension to a diagram:

\[
\begin{array}{ccc}
C & \longrightarrow & A' \\
\downarrow & & \downarrow f' \\
A & \underset{f}{\longrightarrow} & B \\
\end{array}
\]

Lemma 1.30. Let $((J, K), \mathcal{F})$ an indexed site such that $\mathcal{F}(g)$ are object-wise surjective full and cover-preserving. If for each $i$ in $J$ we have that $\mathcal{F}(i)$ satisfy the right Ore condition (e.g have pullbacks), then the family of collections of sieves $R$ on $(n, x)$ in $\int_J \mathcal{F}$ such that $\epsilon(R), R|\mathcal{F}(n)$ are covering forms a coverage satisfying (1), (2) as in Definition 1.1.

Proof. Let us denote by $L$ the families $L(n, x)$ of sieves $R$ such that $\epsilon(R), R|\mathcal{F}(n)$ are covering. By Lemma 1.24 any covering sieve of $\int_J \mathcal{F}$ is in $L$. Hence, $L$ contain maximal sieves. Now we need only to verify that $L$ is stable under pullback ((1) in Definition 1.1).

Consider some $R \in L(n, x)$, $(g, f) : (n', x') \to (n, x)$. Then, $\epsilon((g, f)^\ast(R)) \equiv \{g_0 : \exists f_0 : (g, f) \circ (g_0, f_0) = (gg_0, f \circ \mathcal{F}(g)(f_0)) \in R\}$ and this is contained in $g^\ast(\epsilon(R)) \equiv \{g_0 : \exists \check{f} : (gg_0, \check{f}) \in R\}$ which is covering. We shall show the inverse inclusion, and henceforth $\epsilon((g, f)^\ast(R)$ is covering: let $\check{f}$ be such that $(gg_0, \check{f}) \in R$, then find $z$ by right Ore condition and $z' : \mathcal{F}(g)(z') = z$ with $f_0 : z' \to x'$ such that $\mathcal{F}(g)(f_0) = f_0$:

\[
\begin{array}{ccc}
z & \longrightarrow & \mathcal{F}(gg_0)(\check{x}) \\
\downarrow & & \downarrow f \\
\mathcal{F}(x') & \underset{f}{\longrightarrow} & x \\
\end{array}
\]

By commutativity of this diagram we have $\check{f} \circ \gamma = f \circ \mathcal{F}(g)(f_0)$, and $(gg_0, \check{f} \circ \gamma) = (g, \check{f}) \circ (g_0, f) \in R$, which proves that $f_0$ satisfies: $(gg_0, f \circ \mathcal{F}(g)(f_0)) \in R$.

Further we show that $\mathcal{F}(g)^{-1}(f^\ast(R)|\mathcal{F}(n')) = ((g, f)^\ast(R))|\mathcal{F}(n')$ and that the latter is a covering sieve. Indeed, $f_0$ is in the right hand side if there is $g_0 : (gg_0, f \circ \mathcal{F}(g)(f_0)) \in R$, where $f_0 \in \mathcal{F}(g)^{-1}(f^\ast(R)|\mathcal{F}(n'))$. Conversely, if
f_0 such that (\hat{g}, f \circ \mathcal{F}(g)(f_0)) \in R, we have \mathcal{F}(\hat{g})(\hat{x}) = \mathcal{F}(g)(dom(f_0)) \to \mathcal{F}(f_0) \to \mathcal{F}(g)(x') \to x for some \hat{x} \in ob(\mathcal{F}(dom(\hat{g}))); by the right Ore condition we find n_0, \gamma, \gamma:

\[
\begin{array}{ccc}
n_0 & \xrightarrow{\gamma} & n \\
\downarrow & & \downarrow \\
n' & \xrightarrow{\gamma} & n
\end{array}
\]

There are \hat{g} \in ob(\mathcal{F}(n_0)), \hat{y}' \in ob(\mathcal{F}(n_0)) such that \mathcal{F}(\hat{g})\gamma(\hat{y}') = \mathcal{F}(\hat{g}, \gamma)(\hat{x}) = \mathcal{F}(\hat{g})(dom(f_j)), and then (\hat{g}, f \circ \mathcal{F}(g)(f_0)) \circ (\gamma, id_{\mathcal{F}(\gamma)(g)}) = (g\gamma, o f \circ \mathcal{F}(g)(f_0)) in R, (so if we replace \gamma by g_0) we find the inverse inclusion, i.e., f_0 \in ((g, f)^\ast(R)|_{\mathcal{F}(n)}).

**Remark 1.31.** Assumption that \mathcal{F}(g) must be cover-preserving in the Lemma 1.30 is not used in the proof. We only needed to show that the maximal sieves are in L, which holds without an application of the Lemma 1.24 implying that a Grothendieck topology of \int_{(J, K)} \mathcal{F} is contained in L. Nonetheless, we intended to keep this assumption for the sake of terminological convenience.

**Definition 1.32.** The coverage L in Lemma 1.30 is called a surjective refinement of the topology induced by \{\alpha(S, \Psi) : S, \Psi\} coverage of the site \int_{(J, K)} \mathcal{F}. Provided an indexed site ((J, K), \mathcal{F}) satisfies the conditions of the Lemma 1.30, a site (\int_J \mathcal{F}, L) is denoted as (\int_{(J, K)} \mathcal{F})^*.

Recall that \int_J is a dense topology on a category C provided it consists precisely of the sieves whose pullbacks are nonempty, or explicitly S is in \int_J if (\forall g : dom(g) \to c)(\exists f : dom(f) \to dom(g) : g f \in S). Such sieves we shall further refer to as dense sieves. A Grothendieck topology we shall call subdense provided it is contained in \int_J.

**Lemma 1.33.** Given an indexed site ((J, K), \mathcal{F}), the following implications hold:

1. Assume that the topology on \int_{(J, K)} \mathcal{F} is surjectively full (see the Definition 1.21), then if \alpha(S, \Psi) is a dense sieve so are S, \Psi dense sieves.
2. Let \mathcal{F}(g) be objectwise surjective full and on \mathcal{F}(i) for any i \in ob(J), (J, K) the topologies are subdense. Then, \int_{(J, K)} \mathcal{F} is subdense.
3. If \mathcal{F}(g) are objectwise surjective full, then \alpha(S, \Psi) is a dense sieve iff S, \Psi are dense sieves.
4. Assume together with 1 that J, \mathcal{F}(i) satisfy the right Ore condition, R is a dense sieve on ((n, x) iff \epsilon(R), R|_{\mathcal{F}(n)} are dense sieves on n and x respectively.

**Proof.** Assume that S, \Psi are dense sieves on n and x respectively, (g, f) : (n', x') \to (n, x). Find g' : n'' \to n' : gg' \in S, as well as some

\[
\mathcal{F}(gg')(x') \xrightarrow{f(g')} \mathcal{F}(g)(x') \xrightarrow{f} x
\]

such that f \circ \mathcal{F}(g)(f') \in \Psi. Then, (g, f) \circ (g', f') \in \alpha(S, \Psi), whence (1).
To prove (2): \( \alpha(S, \Psi) \) are dense sieves by (3), and the Grothendieck topology on \( \mathcal{F}(i) \) is defined by a transfinite process (see the beginning of the section 2) of attaching the submitted sieves to a coverage of a previous inductive step; now, we shall show that given a sieve \( R \) submitted to a coverage satisfying (1) of Definition 1.1 consisting of subdense sieves, then so \( R \) is dense as a sieve. Indeed, let \( R \) be a sieve on \( i, \hat{R} \) be in this coverage on \( i \) such that \((\forall g \in \hat{R})(g^*R \) is in the coverage), consider some \( \gamma \) with a codomain being \( i \), find some \( \gamma' \) such that \( \gamma \gamma' \in \hat{R} \), and hence \( (\gamma')^*(\gamma^*(R)) \) is covering and by density is nonempty, which implies \( \gamma^*(R) \) is nonempty.

The necessary condition of (4) follows with no assumption of the right Ore condition: given \( R \) sieve on \( (n, x) \), \( g : n' \to n \in \epsilon(R) \). There is some \( f : \mathcal{F}(g)(x') \to x \) (by definition of \( \epsilon(R) \) sieve) such that \( (g, f) : (n', x') \to (n, x) \in R \). There is (by density of \( R \) sieve) some \( (g_0, f_0) : (n'', x'') \to (n', x') : (g_0 g, f \circ \mathcal{F}(g)(f_0)) \in R \). Then, \( g_0 g \in \epsilon(R) \), whence density of \( \epsilon(R) \) sieve. If \( (f : \text{dom}(f) \to x) \in R|_{\mathcal{F}(n)} \), find some \( x' \in \text{ob}(\mathcal{F}(n')) \) : \( \text{dom}(f) = \mathcal{F}(g)(x') \), where \( g : n' \to n \) is such that \( g, f \in R \). Then again, finding \( (g_0, f_0) : (g_0 g, f \circ \mathcal{F}(g)(f_0)) \in R \) we prove \( f \circ \mathcal{F}(g)(f_0) \in R|_{\mathcal{F}(n)} \).

 Sufficiency in (4): if \( \epsilon(R), R|_{\mathcal{F}(n)} \) are dense, then by (3) we have that \( \alpha(\epsilon(R), R|_{\mathcal{F}(n)}) \) is dense. Then, we note that: given \( (g, f) \in \alpha(\epsilon(R), R|_{\mathcal{F}(n)}) \), there is some \( \gamma \) such that \( (g, f) \circ (id_{\text{dom}(g)}, \gamma) \in R \). Indeed, find some \( \hat{f} \) such that \( (g, \hat{f}) \in R \), use the right Ore condition:

\[
\begin{align*}
\mathcal{F}(g)(\gamma) &\xrightarrow{f} \mathcal{F}(g)(\hat{f}) \\
\mathcal{F}(g)(\gamma) &\xrightarrow{\mathcal{F}(g)(\gamma)} \mathcal{F}(g)(x') \\
&\xrightarrow{f} x
\end{align*}
\]

so we have \( (g, \hat{f}) \circ (id_{\text{dom}(g)}, \gamma) = (g, f) \circ (id_{\text{dom}(g)}, \gamma) \in R \). With this in hand one may easily show that \( R \) is dense as a sieve: given \( \gamma \) find some \( \gamma' \) such that \( \gamma \gamma' \in \alpha(\epsilon(R), R|_{\mathcal{F}(n)}) \), and some \( \gamma'' \) such that \( (\gamma \gamma') \gamma'' = (\gamma'\gamma'') \in R \).

To prove (3) we have to verify only the inverse implication by (1). Let \( S, \Psi \) be dense sieves on \( n, x \in \text{ob}(\mathcal{F}(n)) \) respectively, consider \( (\gamma, \sigma) : (n', x') \to (n, x) \), find some \( g : n'' \to n' \) such that \( \gamma \circ g \in S \), and by objectwise surjectiveness with fullness \( f' : x'' \to x' \) : \( \sigma \circ \mathcal{F}(\gamma)(f') \in \Psi \), and thus \( (\gamma, \sigma) \circ (g, f) \in \alpha(S, \Psi) \), so that \( (\gamma, \sigma)^*(\alpha(S, \Psi)) \) is nonempty. \( \square \)

**Remark 1.34.** Notice that in the proof of the sufficiency of (4) in the above lemma we use the following property of \( R \subset \alpha(\epsilon(R), R|_{\mathcal{F}(n)}) \) that any morphism of \( \alpha(\epsilon(R), R|_{\mathcal{F}(n)}) \) admits a composable with it morphism such that the composition is in \( R \).

**Proposition 1.35.** Let \((J, K), \mathcal{F}\) be an indexed site satisfying the conditions of the Lemma 1.30. Then, if \( K \) is a dense topology and on \( \mathcal{F}(i) \) sitites the topologies are dense, we have that on \((\int_{(J, K)} \mathcal{F})^* \) the topology is dense.

**Proof.** By Lemma 1.33 (4) given \( R \in \mathcal{L}(n, x) \), then \( \epsilon(R), R|_{\mathcal{F}(n)} \) be the covering sieves must be dense, and hence \( \hat{R} \) must be dense. Conversely, given a dense sieve \( R \), by Lemma 1.33 (4) we infer that \( \epsilon(R), R|_{\mathcal{F}(n)} \) are dense, and thus covering, so by the definition of \( L \) coverage we have that \( \hat{R} \in \mathcal{L}(n, x) \).
This proves the density of $L$ topology (since the generated topology of a coverage is subdense if the coverage itself consists of the dense sieves).

Recall that an atomic topology on a category consists precisely of nonempty (or as in [102] says inhabited) sieves. A topology is subatomic provided that all of its covering sieves are nonempty.

**Corollary 1.36.** Let $((J, K), F)$ be an indexed site satisfying the conditions of the Lemma 1.30. Then, if $K$ is an atomic topology and on $F(i)$ sites the topologies are atomic, we have that on $\left( \int_{(J, K)} F \right)^+$ the topology is dense and hence, subatomic.

**Proof.** It is a trivial fact that in a category with the right Ore condition for a sieve being nonempty is equivalent to being dense. □

2. **Geometric morphisms and toposes of sheaves on $\int_{(J, K)} F$**

Recall (see for instance [MM92]) that a geometric morphism $f : E \to F$ of toposes is

(1) Surjective, if the inverse image functor $f^*$ is faithful

(2) Connected, if the inverse image functor $f^*$ is full and faithful

Let $(C, J)$ be a site. A presheaf $A$ on $C$ is said to be $J$-separated provided for any pair $s, s' \in A(U), U \in ob(C)$ such that $A(f)(s) = A(f)(s')$ for any $f \in R$, where $R \in J(U)$ we have that $s = s'$ (see [102]). The full subcategory of $\mathcal{PSh}(C)$ generated by $J$-separated presheaves shall be denoted by $\text{Sep}(C, J)$. From [102, pages 551–552], we remind the construction of a sheafification functor for general sites. This is a composition $\mathcal{PSh}(C) \to \text{Sep}(C, J) \to \mathcal{Sh}(C, J)$. Given a presheaf $A$ on $C$, then $A^+$ is a $J$-separated presheaf defined by $A^+(U) := \lim_{R \in J(U) \setminus A} (R)$, where $R$ are understood as sets of morphisms, and $J(U) \setminus A(R) \subset (\mathcal{PSh}(C) \setminus A)^{op}$ is a subcategory with morphisms $(R \to A) \to (R', A)$ being the inverse inclusions $R' \subset R$:

$$
\begin{array}{ccc}
R' & \longrightarrow & A \\
\downarrow & & \uparrow \\
R & \quad &
\end{array}
$$

Elements of $(A)^+(U)$ are the equivalence classes of pairs $(R, (s_f)_{f \in R})$, where $(s_f)_{f \in R}$ is a compatible family for $A$ and $R \in J(U); (R, (s_f)_{f \in R}) \sim (R', (s'_f)_{f \in R'})$ iff there is a covering sieve $R_0 \subset R' \cap R$ such that for any $f \in R_0$ we have $s_f = s'_f$.

**Definition 2.1.** Given $(C, J)$ site and a functor $F : C \to D$. We say that $F$ is a $J$-faithful functor provided that for any $J$-covering sieve $R$ there is $R' \subset R$ such that $F|_{R'}$ is injective.

**Lemma 2.2.** Let $F : (C, J) \to (D, K)$ be a cover-reflecting functor. Then, we have the following implications:

(1) $F$ is objectwise surjective cover-preserving, $A$ is a $K$-separated presheaf, then $A \circ F$ is $J$-separated.

(2) $F$ is $J$-faithful connected (hence objectwise surjective) full and cover-preserving, $A$ is a sheaf on $(D, K)$, then $A \circ F$ is a sheaf on $(C, J)$.
Proof. We firstly prove (1). Given \( s, s' \in \mathcal{A}F(c) \), such that for some \( \Psi \in J(c) \) we have \((\forall g \in \Psi)(\mathcal{A}F(g)(s) = \mathcal{A}F(g)(s'))\). Then, since \( F(\Psi) = \{ F(g)\gamma : g \in \Psi \} \) is a covering sieve \((F \text{ is cover-preserving}), \) and \( s, s' \) coincide for any morphism in \( F(\Psi) \), which implies \( s = s' \). A presheaf \( \mathcal{B} \) is \( J \)-separated, then the canonical map is monic \( \mathcal{B} \to \mathcal{B}^+ \).

Further we prove (2). (2) condition implies (1), so we have only left to prove that we may glue sections for \( \mathcal{A} \circ F \). For we shall show that the canonical map \( \mathcal{A} \circ F \to \mathcal{A} \circ F^+ \) is surjective. Now, consider some object \( U \) of \( C \), an element \((R,(s_f)_{f \in R}) \in (\mathcal{A} \circ F)^+(U) \). There is some \( R' \subset R \) such that \( F|_{R'} \) is injective. Then, \( F(R') = \{ g \in R' \} \) is a covering sieve, and by putting \( \delta f := s_g \), where \( F(g) = f \) is unique, we define a compatible family for \( \mathcal{A} \) and \( F(R') \). There is some \( \delta \in \mathcal{A}F(U) \), such that \( \mathcal{A}F(g)(\delta) = s_g \) for each \( g \in R' \). Hence, we have just established that \((R,(s_f)_{f \in R}) \sim (M_U, (\mathcal{A}F(g)(\delta))_{g \in M_U}) \), for \( M_U \) the maximal sieve on \( U \), i.e an element of an image \( \mathcal{A}F \to (\mathcal{A}F)^+ \).

\[ \square \]

Remark 2.3. For \( F : C \to D \) the following implications are known from the basic theory of geometric morphisms:

(1) If \( F \) is objectwise surjective, then \( (\varprojlim F^*) : [C,\mathcal{Set}] \to [D,\mathcal{Set}] \) is surjective

(2) If \( F \) is connected, then \( (\varprojlim F^*) : [C,\mathcal{Set}] \to [D,\mathcal{Set}] \) is connected

Proposition 2.4. Let \((\alpha,\beta) : ((J,K),\mathcal{F}) \to ((J',K'),\mathcal{F}') \) be a morphism of indexed sites such that \( \beta \) and \( \alpha(n) \) are cover-preserving. We have the following:

(1) \( \beta, \alpha(n) \) are objectwise surjective, then the induced geometric morphism \( \mathcal{Sh}(\int_{(J,K)}\mathcal{F}) \to \mathcal{Sh}(\int_{(J',K')}\mathcal{F}') \) is surjective.

(2) If \( \mathcal{F}(g) \) are objectwise surjective full cover-preserving, and \( \beta \) is \( K \)-faithful, \( \alpha(n) \) are \( \Sigma_n \)-faithful \( (\Sigma_n \text{ is a Grothendieck topology on } \mathcal{F}(n)) \) and full, connected , then the induced geometric morphism \( \mathcal{Sh}(\int_{(J,K)}\mathcal{F}) \to \mathcal{Sh}(\int_{(J',K')}\mathcal{F}') \) is connected.

Proof. We shall prove separately for \( \alpha \) and \( \beta \), since the composition of surjective (connected) geometric morphisms is surjective (connected), and a morphism \((\alpha,\beta)\) of indexed sites decomposes as \((id,\beta) \circ (\alpha,id)\).

For the first part, given such \( \alpha \), then \( \tilde{\alpha} \) (see Definition 1.3) is full objectwise surjective cover-preserving, so we apply the Lemma 2.2. Since the unit of adjunction \( \mathcal{A}\tilde{\alpha} \to (\mathcal{A}\tilde{\alpha})^+ \) is mono, the composition \((-)^+ \circ \tilde{\alpha}^* \) is faithful then. The case of \( \beta \) is analogous.

For the second part, let \( \alpha \) satisfy (2), then \( \tilde{\alpha} \) is connected, so we have only to show that it is \( \int_K \Sigma_n \)-faithful, where \( \int_K \Sigma_n \) is the topology on \( \int_{(J,K)}\mathcal{F} \). For consider a covering sieve \( R \) on an object \((n,x) \), use Lemma 1.24 and find some covering sieves \( S, \Psi \) on \( n \) and \( x \) respectively, such that \( R \subset \alpha(S,\Psi) \), then since \( R|\mathcal{F}(n) = \Psi \) is covering, there is \( \Psi' \subset \Psi \) such that \( \alpha(n)|\Psi' \) is injective. Then, \( R' = \alpha(S,\Psi') \cap R \) is covering (intersection of a pair of covering sieves is covering by [102, pp 541] and \( \alpha|R' \) is injective. The case of \( \beta \) is treated analogously. Further we apply the second part of Lemma 1.23.

And fullness of a precomposition functor implies the fullness of the inverse image functor, and whence connectedness. \( \square \)
3. Cohomology aspects and further research

3.1. Global sections on $\mathcal{Sh}(\int_{(J,K)} F)$ and some spectral sequences.

We have an inclusion of toposes $\mathcal{Sh}(\int_{(J,K)} F) \to \mathcal{Sh}(\int_{j} F)$, so the global section functor on $\Gamma \equiv \text{hom}(1,-) : \mathcal{Sh}(\int_{(J,K)} F) \to \text{Set}$, where 1 is a terminal object of a topos, can be described as a composition with inclusion and a global section functor on $\mathcal{Sh}(\int_{j} F)$. Further we shall be interested only in $\mathcal{Sh}(\int_{j} F)$ toposes.

As indicated in the previous section, the restriction of a sheaf $A \in \mathcal{Sh}(\int_{j} F)$ on $F(i)$ is a sheaf denoted as $A|F(i) \equiv A \circ \Sigma_i$, where $\Sigma : F(i) \to \int_{j} F$ is an inclusion. So we have a jointly surjective family of geometric morphisms $(\lim_{\Sigma_i}, \Sigma_i^*): \mathcal{Sh}(F(i)) \to \mathcal{Sh}(\int_{j} F)$. This implies that the diagonal of units $\mu_i : id_{\mathcal{Sh}(F(j))} \to \lim_{\Sigma_i} \circ \Sigma_i^*$ of adjunctions is monic $\triangle_i \mu_i : id_{\mathcal{Sh}(F(j))} \to \prod_{i} \lim_{\Sigma_i} \circ \Sigma_i^*$. Hence, applying the global section functor we obtain a mono of functors $\Gamma \to \prod_{i} \Gamma_i \circ \Sigma_i^*$, where $\Gamma_i$ is the global section functor on $\mathcal{Sh}(F(i))$.

**Definition 3.1.** Let $\mathcal{P}$ be the set of points of $\mathcal{Sh}(J,K)$ topos such that $\{p^* : \mathcal{Sh}(J,K) \to \text{Set} : p \in \mathcal{P}\}$ is a jointly conservative family of functors. We say that $\mathcal{A}$ is $\mathcal{P}$-locally acyclic sheaf of $k$-modules on $\mathcal{Sh}(\int_{(J,K)} F)$ provided for any $p \in \mathcal{P}$ we have $p^*(H^q(\mathcal{Sh}(F(-)); \mathcal{A}|F(-))) = 0$.

Given $g : i \to k$ it yields a geometric morphism $\mathcal{Sh}(F(i)) \to \mathcal{Sh}(F(k))$, and we denote the natural transformation induced by a unit of adjunction as $A_g : \Gamma_k \circ \Sigma_k \to \Gamma_i \circ \alpha \circ \mathcal{F}(g)$.

**Lemma 3.2.** The direct image functor of a geometric morphism $\lim_p : \mathcal{Sh}(\int_{(J,K)} F) \to \mathcal{Sh}(J,K)$ is described as follows: $\lim_p (A)(i) = \Gamma_i(A|F(i))$, and given a $g : i \to k$ in $J$, $\lim_p (g) = \Gamma_k(A|F(k)) \xrightarrow{\alpha_g(A)} \Gamma_i(A|F(k) \circ \mathcal{F}(g)) \xrightarrow{\Gamma_i(\mu^*_g(A))} \Gamma_i(A|F(i))$.

**Proof.** $\lim_p (A)(i) = \text{lim}(p \downarrow i)^{\text{op}} \to (\int_{j} F)^{\text{op}} \xrightarrow{\mathcal{A} \mathcal{B}}$, recovering the definition of $(p \downarrow i)^{\text{op}}$ category this limit becomes $\text{lim}(A(k,x)((k,x),\alpha:k\to i))$ where the morphisms are described as $A(g,f) : A(k',x') \to A(k,x)$

$$
\begin{array}{c}
(k,x) \xrightarrow{(g,f)} (k',x') \\
\downarrow \quad \alpha \downarrow \\
\quad \alpha'
\end{array}
$$

Clearly, given $((k,x),\alpha : k \to i)$ it defines $((k,x),\alpha : k \to i) \xrightarrow{((\alpha, id_{F(\alpha)(x)}))} ((i, F(\alpha)(x)), id : i \to i)$ which yields $A(i,F(\alpha)(x)) \to A(k,x)$. This proves that the family $\{A(i,y)((i,y),id:i\to i)\}$ is final, so the limit reduces to $\text{lim}_{F(\alpha)}(A|F(i)) \simeq \Gamma_i(A|F(i))$. \qed

**Theorem 3.3.** Let $\mathcal{P}$ be the set of points in $\mathcal{Sh}(J,K)$ as in Definition 3.1. Assume that any injective sheaf on $\int_{(J,K)} F$ is $\mathcal{P}$-locally acyclic, then $R^q(\lim_p(A)) \simeq \text{a}(H^q(\mathcal{Sh}(F(-)); \mathcal{A}|F(-)))$. 

\[\text{\square}\]
Proof. Indeed, if $\mathcal{J}$ is an injective sheaf on $\int_{(J,K)} \mathcal{F}$, it is $\mathcal{P}$-locally acyclic. Then, this means that the sheafification of $H^q(\mathcal{Sh}(\mathcal{F}(-)); \mathcal{A}|\mathcal{F}(-))$ presheaf is zero. This implies that $(a(H^q(\mathcal{Sh}(\mathcal{F}(-)); \mathcal{A}|\mathcal{F}(-))))_{q \geq 0}$ forms a universal cohomological $\delta$-functor (see [W94, page 32]), whose zeroth term is the direct image functor as proven in the Lemma 3.2, which by universality implies that $a(H^q(\mathcal{Sh}(\mathcal{F}(-)); \mathcal{A}|\mathcal{F}(-))) \simeq R^q(\lim_p)(\mathcal{A})$. □

**Corollary 3.4.** Under the conditions of the Theorem 3.3 there is a spectral sequence $E_*$ with $E_2^{p,q} = H^p(\mathcal{Sh}(\mathcal{J}, \mathcal{K}); a(H^q(\mathcal{Sh}(\mathcal{F}(-)); \mathcal{A}|\mathcal{F}(-)))) \Rightarrow H^{p+q}(\mathcal{Sh}(\int_{(J,K)} \mathcal{F}); \mathcal{A})$.

Proof. This is precisely the Leray spectral sequence applied to the geometric morphism $\mathcal{Sh}(\int_{(J,K)} \mathcal{F}) \to \mathcal{Sh}(\mathcal{J}, \mathcal{K})$. □

For the definition of a locally constant object of a topos we address to [M95, “Cohomology and homotopy”, page 17].

**Proposition 3.5.** Given a locally constant sheaf $\mathcal{A}$ on $\int_{\mathcal{J}} \mathcal{F}$, the cohomology $(\phi_i: H^k(\mathcal{Sh}(\mathcal{F}(i)); \mathcal{A}) \to H^k(\mathcal{Sh}(\mathcal{F}(\mathcal{J})); \mathcal{A}|\mathcal{F}(\mathcal{J}))_{i \in \mathcal{J}}$ forms a pseudocone.

Proof. We need only to show that given a map $g : i \to k$, we have that the following diagram commutes up to an isomorphism:

$$
\begin{array}{ccc}
H^k(\mathcal{Sh}(\mathcal{F}(k)); \mathcal{A}|\mathcal{F}(k)) & \xrightarrow{\mu} & H^k(\mathcal{Sh}(\mathcal{F}(i)); \mathcal{A}|\mathcal{F}(i)) \\
\downarrow & & \downarrow \\
H^k(\mathcal{Sh}(\int_{\mathcal{J}} \mathcal{F}); \mathcal{A})
\end{array}
$$

For, consider a transformation $\Sigma_i \to \Sigma_k \mathcal{F}(g)$:

$$
\begin{array}{ccc}
\mathcal{F}(i) & \xrightarrow{\Sigma_i} & \int_{\mathcal{J}} \mathcal{F} \\
\downarrow & & \downarrow \\
\mathcal{F}(g) & \xrightarrow{\Sigma_k} & \mathcal{F}(k)
\end{array}
$$

defined by $\mu_g(x) := (g, \text{id}_{\mathcal{F}(g)}): (i, x) \to (k, \mathcal{F}(g)(x)) = \Sigma_k(\mathcal{F}(g)(x))$.

This yields a Sierpinski homotopy of induced geometric morphisms:

$$
\begin{array}{ccc}
\mathcal{Sh}(\mathcal{F}(i)) & \xrightarrow{\lim_{\Sigma_k \circ \mathcal{F}(g), \text{a.o.}(\Sigma_k \circ \mathcal{F}(g))^*}} & \mathcal{Sh}(\int_{\mathcal{J}} \mathcal{F}) \\
\downarrow & & \downarrow \\
\mathcal{Sh}(\mathcal{F}(i)) & \xrightarrow{\lim_{\Sigma_i, \text{a.o.}\Sigma_i^*}} & \mathcal{Sh}(\int_{\mathcal{J}} \mathcal{F})
\end{array}
$$

This defines an actual homotopy between this geometric morphisms ([M95, page 19]). Then, we exploit the fact that for locally constant sheaves the induced homomorphisms on cohomology must coincide up to an isomorphism ([M95, page 19]): $(\Sigma_i)^* \simeq (\Sigma_k \circ \mathcal{F}(g))^*: H^k(\mathcal{Sh}(\int_{\mathcal{J}} \mathcal{F})) \to H^k(\mathcal{Sh}(\mathcal{F}(i)); \mathcal{A}|\mathcal{F}(i))$, which proves the point. □
3.2. Cohomology of $\mathcal{S}h_G(X)$ toposes. Recall from [J02, page 76] the definition of a topos of $G$-equivariant sheaves denoted as $\mathcal{S}h_G(X)$. Consider a continuous action of a topological group $G$ on a topological space $X$. This defines an indexed site and taking the Grothendieck construction one obtains the following site $\Theta_G(X)$. Its underlying category has the following description: objects are open sets of $X$, and morphisms $(g, U, V) : U \to V, gU \subset V$. The Grothendieck topology is generated by the coverage: $(g_i, U_i, U_j)$ is covering provided $\bigcup_i g_i U_i \subset U$. The topos of sheaves on $\Theta_G(X)$ is $\mathcal{S}h_G(X)$.

Recall that a topological category is a category $C$ whose object and morphism classes are endowed with topological space structures denoted $C_0$ and $C_1$ respectively, and the domain (source) $s : C_1 \to C_0$, codomain (target) $t : C_1 \to C_0$, $id(-) : C_0 \to C_0$ are continuous.

Another definition of $G$-equivariant sheaves may be found in (M95, “Classifying topos of a topological category”, page 28), and [St78], which we copy here: given a topological category $C$, a $C$-sheaf is a sheaf $p : S \to C_0$ with a continuous action $\alpha : S \times C_0 \to C_1$, where

$$
\begin{array}{ccc}
S \times C_0 & \longrightarrow & C_1 \\
\alpha \downarrow & & \downarrow \iota \\
S & \longrightarrow & C_0 \\
\end{array}
$$

and $\alpha$ satisfies the following conditions: $\alpha(x, fg) = \alpha(\alpha(x, f), g)$, $\alpha(x, 1_{p(x)}) = x$ for any $x \in S$ and $p(\alpha(x, f)) = s(f)$. If we consider a topological group $G$ as a topological category with one object which acts on a space $X$ one defines the so called translation category $X_G$ (which is a Grothendieck construction induced by an action of $G$ on $X$ considered as a discrete category). $X_G$ is topological category whose object space is $X$, morphism space is $X \times G$ and $(x, g)$ is a map of $gx \to x$. Then, a $G$-equivariant sheaf on $X$ is a sheaf on a topological category $X_G$.

$G$-equivariant sheaves provide a natural extension of sheaf cohomology of topological spaces to $G$-spaces. We may point out that the topos itself $\mathcal{S}h_G(X)$ loses information on topology of a group $G$. $\Theta_G(X)$ underlying category is enriched over the topological spaces, $\Theta_G(X)(U, V) \simeq \{g \in G : gU \subset V\}$ with induced topology from a topological group $G$. We propose the following problem: find a generalization of a functor $\mathcal{S}h$ to the category of toposes over the sites whose underlying category is enriched over the category of topological spaces, or other monoidal category. Here, we shall only be concerned with $\mathcal{S}h_G(X)$ which are effective to work with when considering actions of discrete groups.

Here we consider particular acyclic families of $G$-equivariant sheaves (soft and flabby), define a new type of acyclic resolutions, that we call the $G$-equivariant Godement resolutions. Despite the fact that $\mathcal{S}h_G(X)$ have been already constructed we could not find a proper cohomological treatment of them in a fashion of the [Br67, Section 2], except, possibly, for [St78] which lacks some proofs (for instance, the fact that under particular assumptions the family of soft sheaves is acyclic, so that one can build resolutions with them). Here, we prove a weaker version of these results, define a notion of
a degree of an action of a group on a space, apply the basic results to the invariant de Rham cohomology of smooth G-manifolds.

Recall the so called Godement resolutions [Br67] Section 2.2, page 36: given A ∈ Sh(X), ζ0(X; A) is a sheaf of all sections of A, i.e. ζ0(X; A)(U) := \prod_{x \in U} A_x where A_x ≡ \lim_{x \in U} \{A(V) : x \in V\} is a stalk of A at x, and the restriction homomorphisms are simply the projections. ζ0(X; A) are flabby sheaves (restriction homomorphisms are surjective) so they are acyclic. By induction one defines the desired acyclic resolution of A:

\[
0 \to A \to ζ^0(X; A) \to ζ^1(X; A) \to \ldots \to ζ^n(X; A) \to ζ^{n+1}(X; A)
\]

where the mono is defined by A → ζ0(X; A)(U), s → (s_x)_{x \in U}. ζ^n+1(X; A) := ζ^1(X; ζ^n(X; A)), and ζ^n(X; A) := ζ^0(X; ζ^n(X; A)), and obtain

\[
0 \to A \to ζ^0(X; A) \to \ldots \to ζ^n(X; A) \to ζ^{n+1}(X; A)
\]

By acyclic families we undermine the following:

**Definition 3.6.** Let F : A → B be an additive functor between abelian categories, where A has enough injectives. If S is some family of objects of A satisfies the following, we shall call it F-acyclic: it contains injective objects, and for any short exact sequence 0 → D′ → D → D″ → 0, where D′′ ∈ S:

1. D ∈ S iff D″ ∈ S
2. 0 → F(D′) → F(D) → F(D″) → 0.

From standard arguments of homological algebra it is seen that F-acyclic families constitute families of F-acyclic objects. Continuing to follow the notation of a book [Br67], ΓΦ denotes a functor of global sections sheaves with support in Φ. There are various types of ΓΦ-acyclic families in Sh(X, R), where R stands for a category of R-modules:

1. A class of flabby sheaves
2. A class of Φ-soft sheaves if Φ is paracompactifying (see [Br67] Section 2).

Given a topos of sheaves on a site Sh(C, J), the category of abelian objects on it we shall denote as Sh((C, J), Z) We have a cover-reflecting functor Θ(X) → Θ_G(X) that yields Sh(X) → Sh_G(X) which is surjective as indicated by us in the section 3. This implies that the inverse image being conservative, it will be sufficient for us to check exactness of sequences applying the inverse image functor (forgetful functor, that we shall denote as R : Sh_G(X) → Sh(X)). The direct image functor we denote as f : Sh(X) → Sh_G(X).

Further we describe f. Let B be a sheaf on X, then f(B) is the right Kan extension, so f(B)(U) = \lim_{x \in U} (i \downarrow U)^{op} \to Θ(X)^{op} \to \textbf{Set}, where i : Θ(X) → Θ_G(X) is inclusion. (i \downarrow U)^{op} as a category has the following description: (V, (g, V, U)) ≡ (V, g) are objects, where gV ⊂ U, and (V, g) → (V′, g′) morphism is V′ ⊂ V and g = g′. Thus, it immediately follows that f(B)(U) = \prod_{g \in G} B(gU), with the following structure of a G-equivariant sheaf: given (g_0, U, g_0U), then \prod_{g \in G} B(gU) → \prod_{g \in G} B(gg_0U) sends (s_g ∈
Lemma 3.13. we hereby attach a quite simple calculation: equivariant sheaf such that $\dim_\delta B$ of actually occurs in toposes of sheaves on Grothendieck constructions.

Definition 3.11. Let $\text{Sh}(\mathcal{X})$ be some $G$-equivariant sheaf $\mathcal{B}$ on $\mathcal{X}$. We define an invariant, whose proper treatment we could not find in literature. Basically, this would generalise the sheaf dimension theory of $G$-spaces.

Remark 3.10. In the Theorem we considered the case when for each $i \in \text{ob}(i)$ the sheafification of $H^k(\mathcal{F}(\mathcal{X}); \mathcal{A}|_{\mathcal{F}(\mathcal{X})})$ presheaf on $\mathcal{X}$ is zero. In the case of $\mathcal{S}h_G(\mathcal{X})$ we see that this is a particular example of when this actually occurs in toposes of sheaves on Grothendieck constructions.

Definition 3.11. Let $\mathcal{B}$ be some $G$-equivariant sheaf. Define $\xi^*_G(\mathcal{X}; \mathcal{B})$ as:

\[
\begin{align*}
0 \rightarrow \mathcal{B} \rightarrow \xi^0_G(\mathcal{X}; \mathcal{B}) \rightarrow \xi^1_G(\mathcal{X}; \mathcal{B}) \rightarrow 0
\end{align*}
\]

By induction $\xi^{n+1}_G(\mathcal{X}; \mathcal{B}) := \xi^1_G(\mathcal{X}; \xi^n_G(\mathcal{X}; \mathcal{B}))$, and $\xi^{n+1}_G(\mathcal{X}; \mathcal{B}) := \xi^0_G(\mathcal{X}; \xi^n_G(\mathcal{X}; \mathcal{B}))$. This defines an acyclic resolution of $\mathcal{B}$: $0 \rightarrow \mathcal{B} \rightarrow \xi_*^G(\mathcal{X}; \mathcal{B})$, which we call the $G$-equivariant Godement resolution.

Given a topos of sheaves $\mathcal{S}h(\mathcal{C}, J)$ we denote $\text{dim}_k(\mathcal{S}h(\mathcal{C}, J))$ for a minimal $k$ such that for any sheaf of $k$-modules $\mathcal{A}$ on $(\mathcal{C}, J)$ we have $H^k(\mathcal{S}h(\mathcal{C}, J); \mathcal{A}) = 0$. Further we define an invariant, whose proper treatment we could not find in literature. Basically, this would generalise the sheaf dimension theory of topological spaces to that of $G$-spaces.

Definition 3.12. Since the direct image functor $\mathcal{S}h(\mathcal{X}, k) \rightarrow \mathcal{S}h_G(\mathcal{X}, k)$ is exact (where $k$ stands for an arbitrary ring) we imply that $\text{dim}_k(\mathcal{S}h(\mathcal{X})) \equiv \text{dim}_k(\mathcal{S}h_\mathcal{G}(\mathcal{X}))$. We propose to call the number $\text{deg}_k(\xi) \equiv \text{dim}_k(\mathcal{S}h_G(\mathcal{X})) - \text{dim}_k(\mathcal{X})$ as a degree in $K$ of an action $\xi$ of $G$ on $\mathcal{X}$.

Even though here we are not concerned with studying the invariant $\text{deg}_k$ we hereby attach a quite simple calculation:

Lemma 3.13. Let $G$ be a finite group of order $\text{ord}(G) \equiv m$, $\mathcal{B}$ is a $G$-equivariant sheaf such that $H^q(\mathcal{X}; \mathcal{B}(\mathcal{X}))$ are $\mathbb{Z}G[\frac{1}{m}]$-modules for $q > 0$,
hom_2(H^1(G;B(X)), H^2(X;\mathcal{R}(B))^G) = 0 (e.g., B(X) is a trivial G-module and torsionless, or is \(\mathbb{Z}[\frac{1}{m}]\)-module). Then, there are short exact sequences:

\[ 0 \longrightarrow H^n(X;B(X)) \longrightarrow H^n_G(X;B) \longrightarrow H^n(X;\mathcal{R}(B))^G \longrightarrow 0 \]

Proof. Indeed, here we simply use the spectral sequence of Proposition 6.1.10 whose second page \(E_2^{0,q} \equiv H^p(G;H^q(X;\mathcal{R}(B))) \Rightarrow H^{p+q}_G(X;B)\). By Proposition 6.1.10 the only possible nontrivial differential is \(E_2^{0,2} \rightarrow E_2^{0,0}\), which by our assumption is zero. Hence, we have a filtration \(0 = F^{n+1}H^n_G(X;B) \subset F^nH^n_G(X;B) = F^nH^n_G(X;B) \subset H^n_G(X;B)\) and \(F^nH^n_G/F^{n+1}H^n \simeq E_2^{0,n-s}\), whence we obtain the desired short exact sequence:

\[ 0 \longrightarrow E_2^{0,0} \longrightarrow H^n_G(X;B) \longrightarrow E_2^{0,n} \longrightarrow 0 \]

\[ \square \]

Corollary 3.14. Let \(G\) be a finite group with an action \(\xi\) on some topological space, then \(\text{deg}_G(\xi) = 0\), and \(H^n_G(X;B) \simeq H^n(X;B)^G\) for each \(n\) and a \(G\)-equivariant sheaf of \(\mathbb{Q}\)-modules \(B\).

Lemma 3.15. Let there be an action of a group on a paracompact space \(X\) such that the factorization map is closed, and \(0 \rightarrow A' \rightarrow A \rightarrow A'' \rightarrow 0\) is exact sequence of \(G\)-equivariant sheaves. Then if for any \(x \in X\) and saturated neighbourhood \(W\), there is saturated \(x \in V \subset W\) such that the restriction map \(H^1(W;\mathcal{R}(A'\vert_W)) \rightarrow H^1(V;\mathcal{R}(A'\vert_V))\) is zero (e.g., \(\dim_k(\pi^{-1}\pi(x)) = 0\) for all \(x \in X\) ), we have the following exact sequence:

\[ 0 \longrightarrow A'(\pi^{-1}\pi(x)) \longrightarrow A(\pi^{-1}\pi(x)) \longrightarrow A''(\pi^{-1}\pi(x)) \longrightarrow 0 \]

Proof. We simply apply and closeness of \(\pi\). Explicitly, the closeness implies that for any open \(W\) such that \(x \in W\), there is some open saturated \(V\) such that \(x \in V \subset W\). Hence, a colimit along all open saturated neighbourhoods of \(x\) is isomorphic to the colimit along all open neighborhoods of \(x\), which is isomorphic by the [Br67, Theorem 10.6, page 73] to the group of sections over \(\pi^{-1}\pi(x)\). We have the following:

\[
\lim\frac{(A'(W))}{\Rightarrow} \xrightarrow{\Rightarrow} \lim\frac{(A(W))}{\Rightarrow} \xrightarrow{\Rightarrow} \lim\frac{(A''(W))}{\Rightarrow} \lim\frac{(H^1(X;\mathcal{R}(A'\vert_W)))}{\Rightarrow} \lim\frac{(A'(\pi^{-1}\pi(x)))}{\Rightarrow} \lim\frac{(A(\pi^{-1}\pi(x)))}{\Rightarrow} \lim\frac{(A''(\pi^{-1}\pi(x)))}{\Rightarrow}
\]

The limit of the \(H^1(W;\mathcal{R}(A\vert_W))\) is zero by an assumption. Here we have used the fact that \(\lim\Rightarrow\) is exact when taken along a filtered category ([W94, Theorem 2.6.15]).

Exactness of this sequence implies that for any \(x \in X\), \(s \in A''(X)\), there is saturated \(x \in W, t \in A(W)\) such that \(p(W)(t) = s\vert_W\) where \(p\) is an epimorphism \(A \rightarrow A''\). \(\square\)

Lemma 3.16. Let \(X\) be some space with an action of a finite group \(G\), \(X/G\) is hereditarily paracompact and \(A\) is an arbitrary saturated subset, \(\mathcal{F}\) is a \(G\)-equivariant sheaf of \(k\)-modules, where \(k\) is a \(\mathbb{Z}[\frac{1}{\text{ord}(G)}]\)-module. Then,
the canonical map \( \lim \{ \mathcal{F}(U)^G : U = \pi^{-1} \pi(U), A \subset U \} \rightarrow \mathcal{F}(A)^G \) is an isomorphism.

Proof. Clearly, this morphism is mono, since if sections coincide on some subset then they must coincide on some neighbourhood of it. Invariance of section implies they must coincide on a saturation of this open neighbourhood as well.

Now, given a section \( s \in \mathcal{A}(A)^G \), following the proof of the analogous result (see Theorem 9.5 of [Br67 Section 2]) we define an open cover of \( A \) by \( \{ x_\alpha \in U_\alpha \} \), \( x_\alpha \in U_\alpha \cap A \) with sections \( s_\alpha \in \mathcal{A}(U_\alpha) \) such that \( s_\alpha|_{U_\alpha \cap A} = s|_{U_\alpha \cap A} \) for each \( \alpha \), only that we require that \( \pi^{-1} \pi(x_\alpha) \subset U_\alpha \) for each \( \alpha \).

Then, there is open saturated neighbourhood \( V_\alpha \) of \( x_\alpha \) such that \( V_\alpha \subset U_\alpha \), and \( \frac{1}{\text{ord}(G)} \sum_{g \in G} A_{V_\alpha}^{g, V_\alpha}(s_\alpha|_{V_\alpha}) \equiv \tilde{s}_\alpha \). Hence, we have defined an open cover of \( A \) by saturated open sets \( V_\alpha \) with invariant sections \( \tilde{s}_\alpha \in \mathcal{A}(V_\alpha)^G \) such that for each \( \alpha \) we have \( \tilde{s}_\alpha|_{V_\alpha \cap A} = s|_{V_\alpha \cap A} \).

Further, we copy the argument in the Theorem 9.5 of [Br67 Section 2] in order to construct the desired extension of \( s \). Namely, since \( X/G \) is hereditarily paracompact there is locally finite refinement by open saturated sets \( W_\alpha \) such that \( cI(W_\alpha) \subset V_\alpha \) for each \( \alpha \). Consider \( W \equiv \{ x \in X : (\forall \alpha, \beta : x \in cI(V_\alpha) \cap cI(V_\beta))(\tilde{s}_\alpha(x) = \tilde{s}_\beta(x)) \} \), which is clearly saturated since \( \tilde{s}_\alpha \) are invariant sections, and \( A \subset W \). We shall prove that \( W \) is open, and immediately glue \( \tilde{s}_\alpha|_{V_\alpha \cap W} \) sections into \( \tilde{s} \in \mathcal{A}(W) \), which is invariant and restricts to \( u \) on \( A \). Given \( x \in W \), find an open neighbourhood \( N(x) \) such that for any \( y \in N(x) \) we have implication \( y \in cI(V_\alpha) \Rightarrow x \in cI(V_\alpha) \); then, since sections \( \tilde{s}_\alpha \) for \( \alpha : x \in cI(V_\alpha) \) coincide on some neighbourhood we imply openness of \( W \). □

The following definition of soft \( G \)-equivariant sheaves can be found in [St78].

**Definition 3.17.** A \( G \)-equivariant sheaf \( \mathcal{A} \) is said to be flasque provided for any \( U \) open saturated subset of \( X \) the restriction homomorphism \( \mathcal{A}(X)^G \rightarrow \mathcal{A}(U)^G \) is surjective.

A \( G \)-equivariant sheaf \( \mathcal{A} \) is said to be soft provided for any \( A \) closed saturated subset of \( X \) the restriction homomorphism \( \mathcal{A}(X)^G \rightarrow \mathcal{A}(A)^G \) is surjective.

In [St78] the results stronger than the following can be found, nevertheless, some of them (for instance, the proof that soft \( G \)-equivariant sheaves if \( X/G \) is paracompact must form an acyclic family Proposition 2.3 “Acyclic \( G \)-sheaves”) lack the proofs. We could not find a proof without averaging method applied in the next two theorems.

**Theorem 3.18.** Let \( G \) be the finite group of order \( \text{ord}(G) \) acting on a paracompact space \( X \) and \( k \) be a ring which is \( \mathbb{Z}[\frac{1}{\text{ord}(G)}] \)-module (e.g., \( \mathbb{Q} \subset K \)).

Then, the family of flasque \( G \)-equivariant sheaves must be acyclic.

Proof. For finite group actions the factorisation maps are closed. Thus, we may exploit the Lemma 3.15. One may immediately note that injective \( G \)-equivariant cohomology theories do belong to this family. Indeed, given an open saturated open set \( U \), an injective \( G \)-equivariant sheaf \( \mathcal{F} \) and a section
$s \in \mathcal{J}(U)^G$, consider a morphism $h : \mathcal{K}|_U \to \mathcal{J}|_U$, where $\mathcal{K}$ is a constant $G$-equivariant sheaf defined by the ring $k$, and $h(\rho)_x := \rho_x s_x$, where $\rho$ section of $k|_U$. Then, by injectivity we find some $g$

\[
\begin{array}{ccc}
k_U & \overset{h}{\longrightarrow} & \mathcal{J} \\
\downarrow & & \downarrow \\
k & \overset{g}{\longrightarrow} & \mathcal{J}
\end{array}
\]

where $k_U$ is an extended by zero sheaf on $k|_U$. Clearly, $g(1) \in \mathcal{J}(X)$ extends $s$ and is invariant since $1$ is a constant section (sends to each point a unit of $k$).

Further we verify that for any

\[
0 \longrightarrow \mathcal{A}' \overset{i}{\longrightarrow} \mathcal{A} \overset{p}{\longrightarrow} \mathcal{A}'' \longrightarrow 0
\]

where $\mathcal{A}'$ is flasque, then we have:

\[
0 \longrightarrow \mathcal{A}'(X)^G \overset{i(X)}{\longrightarrow} \mathcal{A}(X)^G \overset{p(X)}{\longrightarrow} \mathcal{A}''(X)^G \longrightarrow 0
\]

For that it suffices to show that $p(X)$ is surjective. Let $s \in \mathcal{A}''(X)^G$, define the following ordered set $\Lambda := \{(W,t) : W = \pi^{-1}(W), t \in \mathcal{A}(W)^G, p(W)(t) = s|_W\}$, with the order relation: $(W_1,t_1) \leq (W_2,t_2)$ if $W_1 \subset W_2, t_2|_{W_1} = t_1$.

Firstly, we notice that $s|_W \in \mathcal{A}''(W)^G$ for any saturated open $W$ by the commutativity of the following diagram:

\[
\begin{array}{ccc}
\mathcal{A}''(X) & \overset{A''_{\alpha,\tau}}{\longrightarrow} & \mathcal{A}'(X) \\
\downarrow & & \downarrow \\
\mathcal{A}''(W) & \overset{A''_{\alpha,\tau}}{\longrightarrow} & \mathcal{A}'(W)
\end{array}
\]

Further, by the lemma 3.15 for any $x \in X$ there is saturated neighbourhood $W$, such that there is $\tau \in A(W) : p(W)(t) = s|_W$. We shall show that $\Lambda$ is nonempty and satisfies the Zorn lemma: take $t \in A(W) : p(W)(t) = s|_W$, then $t' := \frac{1}{|G|} \sum_{g \in G} A''_{\alpha,\tau}(s|_W) = s|_W$ given a chain $(W_\alpha, t_\alpha)$ in $\Lambda$, then $t(x) := t_\alpha(x)$ if $x \in W_\alpha$ is invariant extension evidently.

Thus, there is a maximal element $(V_0, t_0) \in \Lambda$ and we need to show that $V_0 = X$. Assume that $x \in X \setminus V_0$, there is saturated neighbourhood $V_1$ of $x$ with $t_1 \in A(V_1)^G, p(V_1)(t_1) = s|_{V_1}$. Then, considering the difference $t_0|_{V_0 \cap V_1} - t_1|_{V_0 \cap V_1} \in ker(p(V_0 \cap V_1))$ there is $\tau \in A'(V_0 \cap V_1) : i(V_0 \cap V_1)(\tau) = t_0|_{V_0 \cap V_1} - t_1|_{V_0 \cap V_1}$, $\tau$ is invariant and thus, by flasqueness of $\mathcal{A}'$ it extends to some invariant element of $\mathcal{A}'(V_1)$ whose image under $i(V_1)$ we denote as $t_2$. Finally, we have $(t_2 + t_1)|_{V_0 \cap V_1} = t_0|_{V_0 \cap V_1}$, and hence we can glue this sections to some $t \in A(V_0 \cup V_1)^G$, and $p(V_0 \cup V_1)(t) = s|_{V_0 \cup V_1}$, this contradicts maximality of $(V_0, t_0)$. 
The following diagram proves the property (1).

\[
\begin{array}{cccccc}
0 & \rightarrow & \mathcal{A}'(X)^G & \rightarrow & \mathcal{A}(X)^G & \rightarrow & \mathcal{A}''(X)^G & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \rightarrow & \mathcal{A}'(U)^G & \rightarrow & \mathcal{A}(U)^G & \rightarrow & \mathcal{A}''(U)^G & \rightarrow & 0
\end{array}
\]

\[\square\]

**Theorem 3.19.** Let \( X \) be hereditarily paracompact space with \( \xi \) action of a finite group and \( k \) is a \( \mathbb{Z}[\frac{1}{\text{ord}(G)}] \)-module (e.g., \( \mathbb{Q} \subset K \)). Then, the class of soft \( G \)-equivariant sheaves is acyclic.

**Proof.** Since \( X \) is hereditarily paracompact, then so is the quotient. By lemma 3.16 and the theorem above we conclude that injective \( G \)-equivariant sheaves are soft (injective \( G \)-equivariant sheaves are flasque). Again we consider an exact sequence of \( G \)-equivariant sheaves as in previous theorem, we shall show that \( p(X) \) is epi on \( k \)-modules of invariants. Let \( s \in \mathcal{A}''(X)^G \) we shall construct its representative in \( \mathcal{A}(X)^G \). Analogously to the previous theorem, we apply the lemma 3.15 to construct an open saturated cover \( U_\alpha \), such that for each \( \alpha \) there is \( t_\alpha \in \mathcal{A}(U_\alpha)^G, p(U_\alpha)(t_\alpha) = s|_{U_\alpha}; \{\pi(U_\alpha)\}_\alpha \) is a cover of \( X/G \), by paracompactness we may find its locally finite refinement, and thus \( \{U_\alpha\}_\alpha \) may be chosen to be locally finite and there is \( \{V_\alpha\}_\alpha \) saturated refinement \( V_\alpha \subset U_\alpha \). The last implication follows from the fact that the factorisation mappings are open for group actions: find \( \{W_\alpha\}_\alpha \) be refinement of \( \{\pi(U_\alpha)\}_\alpha \), with \( W_\alpha \subset \pi(U_\alpha) \), and \( \pi^{-1}(W_\alpha) = \pi^{-1}(W_\alpha) \equiv V_\alpha \).

Let \( \alpha_0 \) be the limit ordinal, then since by local finiteness we can glue \( \{t_\beta\}_{\beta(\alpha_0)} \) and obtain the invariant section over \( F_{\alpha_0} \). If \( \alpha + 1 \) is considered, then \( \tilde{V}_{\alpha+1} \cap F_\alpha = \tilde{V}_{\alpha+1} \cap (\bigcup_{\beta \leq \alpha} V_\beta) = \tilde{V}_{\alpha+1} \cap (\tilde{V}_{\alpha_1} \cup \cdots \cup \tilde{V}_{\alpha_k}) \), where \( \beta_1 < \cdots < \beta_k \leq \alpha \). Then, this intersection equals \( \tilde{V}_{\alpha+1} \cap F_{\beta_k} \). Consider \( \sigma \in \mathcal{A}(\tilde{V}_{\alpha+1})^G : p(\tilde{V}_{\alpha+1})(\sigma) = s|_{\tilde{V}_{\alpha+1}} \). Then, \( t_{\beta_k}|_{\tilde{V}_{\alpha+1} \cap F_{\beta_k}} - \sigma|_{\tilde{V}_{\alpha+1} \cap F_{\beta_k}} \) is in the kernel of \( p(\tilde{V}_{\alpha+1} \cap F_{\beta_k}) \), and by exactness there is \( \psi \in \mathcal{A}''(\tilde{V}_{\alpha+1} \cap F_{\beta_k})^G : \ i(\tilde{V}_{\alpha+1} \cap F_{\beta_k})(\psi) = t_{\beta_k}|_{\tilde{V}_{\alpha+1} \cap F_{\beta_k}} - \sigma|_{\tilde{V}_{\alpha+1} \cap F_{\beta_k}} \). By softness we extend \( \psi \) over \( \tilde{V}_{\alpha+1} \) by an invariant section \( \Psi \), and thus by \( t_{\alpha+1} := t_\alpha \vee (i(\tilde{V}_{\alpha+1})(\Psi) + \sigma) \) we conclude the inductive step. \[\square\]
Lemma 3.20. Let $G$ be a finite group of order $m$ act on a space $X$. Assume $X/G$ is hereditarily paracompact, and we have a $G$-equivariant sheaf $\mathcal{Y}$ of $m$ divisible rings with a $G$-equivariant sheaf $\mathcal{B}$ which is a $\mathcal{Y}$-module. If $\mathcal{Y}$ is soft, then so is $\mathcal{B}$.

Proof. Indeed, given a closed saturated subset $A \subset X$, and a section $s \in \mathcal{B}(A)^G$ we use that $\mathcal{B}$ is a $G$-equivariant sheaf of $\mathbb{Z}[\frac{1}{m}]$-modules and apply Lemma 3.16 to extend this section to $\hat{s}$ over some open saturated neighbourhood $A \subset U$. Then, finding $\pi(A) \subset V \subset \text{cl}(V) \subset \pi(U)$ we restrict $\hat{s}$ to $\pi^{-1}(\text{cl}(V)) = \text{cl}(\pi^{-1}(V))$. Let $t$ be a section of $\mathcal{Y}$ over $B \cup U$, where $B$ is the boundary of $V' \equiv \pi^{-1}(V)$ defined by $t(x) := 1$ (unit of $\mathcal{Y}_x$ ring) for $x \in A$ and zero on $B$. This section is invariant and by softness extends over $\text{cl}(V')$. Then, consider an invariant section $\xi'$ of $\mathcal{B}$ over $\text{cl}(V')$. This clearly extends by zero over $X$. \qed

Example 3.21. Let $M$ be a smooth connected $n$-manifold, $G$ a Lie group acting on it by diffeomorphisms. One defines a subcomplex of the de Rham complex $\Omega^\ast(M)^G \subset \Omega^\ast(M)$ consisting of invariant differential forms under the action of $G$ (see [GZ18, Definition 2.1]). Here one immediately applies an approach sketched in Subsection 2.1., namely the de Rham complex extends to the $G$-equivariant sheaf on $M$ and $\Omega^\ast(M)^G$ is an application of the global section functor on $\text{Sh}(G)(X)$ to it. Given a $G$-equivariant sheaf $\mathcal{B}$ which is soft as a sheaf and assuming that we have an averaging operation (integration $\int_G g^\ast \omega$ of differential forms if $\mathcal{B} \equiv \Omega^\ast(M)$, or $\frac{1}{|\text{ord}(G)|} \sum_{g \in G} B^0_X(\omega)$ if $G$ is finite) then it is soft as a $G$-equivariant sheaf. In particular, if $G$ is a finite group the de Rham $G$-equivariant differential sheaf is soft and defines a soft resolution of a constant $G$-equivariant sheaf $\mathcal{R}$. Thus, in addition, applying the Lemma 3.13 we obtain $H^k(\Omega^\ast(M)^G) \simeq H^k_G(X; \mathbb{R}) \simeq H^k(X; \mathbb{R})^G$. For free actions it is known that $H^k(\Omega^\ast(M)^G) \simeq H^k(M/G; \mathbb{R})$, whence we obtain that the $G$-equivariant sheaf cohomology for free actions by diffeomorphisms on $M$ is isomorphic to the cohomology of the quotients.

Further, we find a left exact subfunctor of $\mathcal{R}f$:

Definition 3.22. For each action $\xi$ (section 4) we may define the following endofunctor $\xi^* : \text{Sh}(X) \rightarrow \text{Sh}(X) : \xi^*(\mathcal{F}) := \text{sheaf}(U \mapsto \mathcal{F}(\pi_\xi^{-1}\pi_\xi(U)))$, where $\pi_\xi : X \rightarrow X/G$ is a quotient map to the orbit space.

This endofunctor is clearly left exact. Clearly, it is a subfunctor of $\mathcal{R}f$. Indeed, consider a presheaf $\mathcal{F}_\xi \equiv \xi^*(\mathcal{F}) \equiv \xi^*(\mathcal{F}(\pi_\xi^{-1}\pi_\xi(U)))$, then $\mathcal{F}_\xi(U) \subset \prod_{g \in G} \mathcal{F}(gU)$, where $s \in \mathcal{F}_\xi(U)$ is identified with $(s|_{gU})_{g \in G}$ (here we use that $\mathcal{F}$ is a monopresheaf to define this embedding). Further applying a sheafification (which is exact) we obtain an embedding $0 \rightarrow \xi^* \rightarrow \mathcal{R}f$.

Theorem 3.23. Let $\mathcal{F}$ be some sheaf on $X$, $\xi$ is an action of $G$ on $X$. The right derived functors $R^k(\xi^*)$ are characterized as: $R^k(\xi^*)\mathcal{F} \simeq \text{sheaf}(U \mapsto H^k(U^\ast, \mathcal{F}(U^\ast)))$. If $X$ is paracompact and $\xi$ is a $G$-space structure such that the factorization map $\pi : X \rightarrow X/G$ is closed, then $\max\{n : \exists \mathcal{F} \in \text{Sh}(X, k) : R^n(\xi^*)\mathcal{F} \neq 0\} = \sup\{\text{dim}_k(\pi^{-1}\pi(x)) : x \in X\}$

Proof. $\xi^0$ is a composition

$$\text{Sh}(X, k) \xrightarrow{\text{incl}} \mathcal{P}\text{Sh}(X, k) \xrightarrow{\text{sheaf}} \text{Sh}(X, k)$$
where the first functor is an inclusion - left exact, with right derived functor: 
\( R^k \text{incl}(F) \simeq (U \mapsto H^k(U, F|_U)) \) sends a sheaf \( F \) to a presheaf assigning to open sets the sheaf cohomology with coefficients in the restriction of \( F \); the second functor is a sheafification, and known to be exact (see [Br67]). Hence, \( F \mapsto (U \mapsto H^k(U^*, F|_{U^*})) \) is a universal cohomological \( \delta \)-functor, and we may compose it with sheafification, which by universality implies existence of the following isomorphism of functors \( \xi^k(F) = R^k(\xi^0)(F) \simeq \text{sheaf}(U \mapsto H^k(U^*, F|_{U^*})) \) for each \( k \).

For the second part, consider some point \( x \in X \), \( F \) sheaf and stalk \( \xi^k(F)_x \). By definition of stalks \( \xi^k(F)_x = \varprojlim \{ H^k(U^*, F|_{U^*}) : x \in U \} = \varprojlim \{ H^k(U, F|_U) : U = U^*, x \in U \} \), which by closeness of the factorisation map is isomorphic to \( \varprojlim \{ H^k(U, F|_U) : \pi^{-1}\pi(x) \subset U \} \simeq H^k(\pi^{-1}\pi(x), F|_{\pi^{-1}\pi(x)}) \), where the last isomorphism follows from Theorem 9.5 in [Br67]. \( \square \)

**Theorem 3.24.** Assume \( X \) is paracompact and \( \xi \) is a \( G \)-space structure such that the factorization map \( \pi : X \to X/\xi \) is closed. Then, the etale space of \( \xi^*F \) may characterized as follows: \( \mathcal{E}(\xi^*F) = \bigsqcup_{x \in X} F(\pi^{-1}\pi(x)) \), with finest topology making the following sections \( f_{(W,t)}(x) := t|_{\pi^{-1}\pi(x)} \) continuous for each pair \( t \in W, W = \pi^{-1}\pi(W) \) where \( W \) is open.

**Proof.** By closeness of \( \pi \) we have shown that stalks of a sheaf \( \xi^*F \) are indeed isomorphic to \( F(\pi^{-1}\pi(x)) \) groups of sections over orbits. The topology now is an evident reformulation of the topology on etale space of \( \xi^*F \) in terms of the sections of over \( \pi^{-1}\pi(x) \). \( \square \)

**Corollary 3.25.** If \( X \) locally compact space hereditarily paracompact with transitive exact action \( \xi \) and closed factorization map \( X \to X/G \), then \( X \) is totally disconnected.

**Proof.** Indeed, transitivity would imply that \( X \) is an orbit and by the above characterization of \( R^k\xi \) we obtain that any sheaf on the space is acyclic. Using Corollary 16.21 from [Br67], page 116] we conclude that \( X \) is totally disconnected. \( \square \)

**Theorem 3.26.** If \( \xi \) such that a factorization map is closed and the space is hereditarily paracompact, then there is split mono of functors \( \Gamma \to \Gamma \circ \xi^* \).

**Proof.** Let \( \mathcal{E}(\xi^*F) \xrightarrow{\sim} \mathcal{E}(F) \) be a canonical map, it assigns to each section \( (t : \pi^{-1}\pi(x) \to \mathcal{E}(F)) \in \mathcal{E}(\xi^*F) \) its value at \( x \), \( t(x) \). Define a section of \( \alpha \eta(s)(x) := s|_{\pi^{-1}\pi(x)} \), \( \eta : \mathcal{E}(F) \to \mathcal{E}(\xi^*F) \): continuity of \( \eta(s) \) follows from: let \( V \subset \mathcal{E}(\xi^*F) \) be open \( \eta(s)^{-1}(V) = \{ x \in X : s|_{\pi^{-1}\pi(x)} \in V \} \), which is open by definition of topology of stale space \( \mathcal{E}(\xi^*F) \). Analogously continuity of \( \eta \) is readily verified.

\( \square \)

Recall that \( \xi \) is a discrete action of a group \( G \) on \( X \) provided that for any point \( x \in X \) there is some open neighbourhood of it \( U \) such that \( gU \cap U = \emptyset \) if \( g \neq e \), where \( e \) identity element of \( G \).

**Proposition 3.27.** \( \xi^* = Rf \) if an action \( \xi \) is discrete.
Proof. Indeed, if an action is discrete, then the canonical inclusion \( \xi^* R f \) stalkwise has the following form: \( \lim_{\longleftarrow} x \in U (A(\pi^{-1}(\pi(U)))) \to \prod_{g \in G} A_{gx}, \) such that sends an equivalence class of pairs \((U, s) \in A(\pi^{-1}(\pi(U)))\) to \(((\pi^{-1}(\pi(U)), s)_{gx})_{g \in G} \). If \( U \) is chosen to be so that for any \( g \in G \setminus \{e\} \) we have \( gU \cap U = \emptyset \), then \( g_1 U \cap g_2 U = \emptyset \) if \( g_1 \neq g_2 \), which implies \( A(\pi^{-1}(\pi(U))) = A(\prod_{g \in G} gU) = \prod_{g \in G} A(gU) \), which proves that the canonical injection is in fact surjective.

To conclude this section we may point out that the following invariants have not gained enough attention (we could not find a proper treatment of them in literature): \( \text{deg}_k(\xi) \) (see Definition 3.12), etale homotopy groups of toposes \( \text{Sh}_G(X) \) (see for the etale homotopy groups \([M95, \text{“Cohomology and homotopy”}]\)). More generally, we could not find a proper sheaf theoretic treatment and unification of cohomology theories of diagrams of spaces (even in \([S94]\)) as well as studying the etale homotopy groups of associated toposes.

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