Potts model: Duality, Uniformization and the Seiberg–Witten modulus

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Abstract

The introduction of a modulus $z(K)$, analogous to $u = \langle tr\phi^2 \rangle$ in the $N = 2$ SUSY $SU(2)$ gauge theory solved by Seiberg and Witten, and whose defining property is the invariance under the symmetry and duality transformations of the effective coupling $K$, reveals an intriguing correspondence between the $D = 2$ Ising and Potts models on the square lattice. The moduli spaces of both models, the spaces of inequivalent effective temperatures $K$, correspond to a three-punctured sphere $\mathcal{M}_3 = \mathbb{P}^1(\mathbb{C})\{z = \pm 1, \infty\}$. Furthermore, in both models, the locus of Fisher zeroes is given by the segment joining $z_c = -1$ to $z_c = +1$.

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1 Introduction

Duality plays a crucial role in the latest developments of string theory and has proven to be a tremendously powerful tool in the solution of non trivial $D = 4$ supersymmetric field theories, namely the Seiberg–Witten models [1].

In [3] [4], Fendley and Saleur showed that various $(1+1)$–dimensional quantum impurity systems display an exact form of self–duality, analogous to the famous Kramers–Wannier duality relation in the $D = 2$ Ising model [6]. They achieved this by expressing the relevant quantities in terms of contour integrals over certain hyperelliptic curves.

Previously, in [2], Fendley had expressed the magnetization in the Kondo model and the current in the boundary sine–Gordon model in terms of such contour integrals, noticing the similarity with the Seiberg–Witten results.

Fendley and Saleur also posed the question whether the duality relation itself is enough to solve the specific model, without making use of the Bethe ansatz technique.

This problem motivates our analysis which will focus on the $D = 2$ Ising and Potts models on a square lattice. We will see that the Kramers–Wannier duality relation [3] and its generalization to the Potts model [3] naturally lead to identify a three–punctured Riemann sphere $\mathcal{M}_3 = \mathbb{P}^1(\mathbb{C}) \setminus \{z = \pm 1, \infty\}$ as the moduli space of the given model.

This is achieved by the introduction of a modulus or uniformizing coordinate $z$, with the property of being invariant under the symmetry and duality transformations. In particular

$$z = \frac{1 + \sinh^4(2K)}{2\sinh^2(2K)},$$

$$z(K) = z(K^*) = z(-K) = z(K + i\frac{\pi}{2}),$$

(1)

in the Ising case, and

$$z = \frac{(e^K - 1)^2 + q}{2\sqrt{q}(e^K - 1)},$$

$$z(K) = z(K^*) = z(K + 2\pi i),$$

(2)

in the Potts case, where $K$ is the effective coupling between neighbouring spins.

Furthermore, in terms of the appropriate modulus $z$, the locus of Fisher zeroes both in the Ising and in the $q$–state Potts model is given by the segment joining the complex temperature singularity $z = -1$ and the physical singularity $z = +1$. 

1
2 Ising model

The Ising model is defined as follows \[5\]. At each site of a \(D\)-dimensional lattice, there is a spin variable \(\sigma_i = \pm 1\) interacting with its nearest neighbours only through the energy

\[ H = -J \sum_{\langle i,j \rangle} \sigma_i \sigma_j, \tag{3} \]

where the symbol \(\langle i, j \rangle\) denotes the sum over nearest neighbours pairs. The partition function \(Z\) is given by

\[ Z = \sum_\{\sigma\} e^{-\beta H} = \sum_\{\sigma\} \exp \left( K \sum_{\langle i,j \rangle} \sigma_i \sigma_j \right), \tag{4} \]

where \(K = \beta J\) and the sum is over all possible spin configurations.

The free energy per spin \(f(K)\) is given by

\[ -\beta f(K) = \lim_{N \to \infty} \frac{\ln Z(K)}{N}, \tag{5} \]

Let us define

\[ F(K) \equiv -\beta f(K). \tag{6} \]

In the following, we will consider a two–dimensional square lattice with \(N\) sites.

At high temperature and correspondingly small \(K\)

\[ Z_{\text{high}}(K) = \sum_\{\sigma\} \prod_{\langle i,j \rangle} \cosh(K)(1 + \sigma_i \sigma_j \tanh(K)). \tag{7} \]

The sum over all \(\sigma_i = \pm 1\) selects only those terms with even powers of \(\sigma_i\), while the others cancel exactly. We can represent each term diagrammatically by a line connecting the sites \(i\) and \(j\) for each factor \(\tanh(K)\sigma_i \sigma_j\). Therefore, the non vanishing contributions come from all closed loops on the lattice

\[ Z_{\text{high}}(K) = 2^N (\cosh K)^{2N} \sum_{\text{loops}} (\tanh K)^l. \tag{8} \]

where \(l\) denotes the length of the loop, that is the number of bonds in the loop. Note that the loops can be disconnected. By \(\{8\}\), we see that the partition function is invariant under

\[ K \to -K, \tag{9} \]

due to the fact that \(l\) is even.
Likewise, at low temperature, large $K$, a spin configuration is identified by the boundaries of positive spin droplets in a negative spin background or vice versa. These boundaries form loops and the partition function can be expressed as

$$Z_{\text{low}}(K) = 2e^{2NK} \sum_{\text{loops}} e^{-2Kl}. \quad (10)$$

From this we easily see that under the shift

$$K \rightarrow K + \frac{i\pi}{2}, \quad (11)$$

the free energy transforms like

$$F_{\text{low}}(K + \frac{i\pi}{2}) = F_{\text{low}}(K) + \frac{i\pi}{2}. \quad (12)$$

Furthermore, by $(8)$ and $(10)$, we see that if we set

$$e^{-2K^\ast} = \tanh K, \quad (13)$$

the high-temperature and low-temperature expansions are mapped into each other

$$\frac{Z_{\text{low}}(K^\ast)}{2e^{2NK^\ast}} = \frac{Z_{\text{high}}(K)}{2^N \cosh(K)2^N}. \quad (14)$$

This is the Kramers–Wannier self-duality relation $(8)$. In terms of the free energy $F(K)$, $(14)$ becomes

$$F_{\text{low}}(K^\ast) - \frac{1}{2} \ln \sinh(2K^\ast) = F_{\text{high}}(K) - \frac{1}{2} \ln \sinh(2K), \quad (15)$$

where we used

$$e^{-2K^\ast} = \tanh K \leftrightarrow \sinh(2K^\ast) \sinh(2K) = 1. \quad (16)$$

Arguing that $F(K)$ could have only one singularity for $K > 0$, Kramers and Wannier concluded that the transition temperature had to coincide with the fixed point $K^\ast(K_c) = K_c$ under the duality mapping

$$\sinh^2(2K_c) = 1 \leftrightarrow K_c = \frac{1}{2} \ln (1 + \sqrt{2}). \quad (17)$$

Note that the duality mapping $(16)$ connects the low temperature region with the high temperature region and vice versa: in this sense it is a mapping between a strong coupling regime and a weak coupling one. Furthermore, it is involutive

$$K^\ast(K^\ast(K)) = K. \quad (18)$$
The modulus $z$

In [1], Seiberg and Witten managed to calculate the low–energy Wilsonian effective action of the $N = 2$ supersymmetric $SU(2)$ Yang–Mills theory. This effective action is given in terms of the so–called prepotential $\mathcal{F}(a)$, which is a function of the vacuum expectation value $a$ of the scalar field $\phi$ contained in the supersymmetric multiplet. Most importantly, $N = 2$ supersymmetry constrains $\mathcal{F}(a)$ to be a polymorphic function, that is a multivalued analytic function.

The crucial feature is that the theory has a moduli space $\mathcal{M}$, namely a manifold of physically inequivalent vacua, which is parametrized by the gauge–invariant coordinate

$$u = \langle tr\phi^2 \rangle.$$  \hspace{1cm} (19)

The low–energy description breaks down at certain points $u_c$, due to the appearance of extra massless particles. Furthermore, as $u$ loops around these singular points, $a(u)$ and its dual $a_D(u) = \frac{\partial \mathcal{F}(a)}{\partial a}$ undergo a monodromy, they transform into a linear combination of themselves

$$
\begin{pmatrix}
    a_D(\tilde{u}) \\
    a(\tilde{u})
\end{pmatrix} =
\begin{pmatrix}
    A & B \\
    C & D
\end{pmatrix}
\begin{pmatrix}
    a_D(u) \\
    a(u)
\end{pmatrix},
$$  \hspace{1cm} (20)

where $\tilde{u} = u_c + e^{2\pi i}(u - u_c)$ and $A, B, C, D$ are integers satisfying $AD - BC = 1$.

These monodromies correspond to duality transformations relating two theories with different effective coupling constants $\tau = \frac{\partial^2 \mathcal{F}(a)}{\partial a^2}$, $\text{Im}\tau > 0$. Eq. (20) implies that

$$
\tau = \frac{\partial a_D}{\partial a} \rightarrow \gamma \tau = \frac{\partial a_D}{\partial \tilde{a}} = \frac{A\tau + B}{C\tau + D}, \quad \text{Im}\gamma \tau > 0,
$$  \hspace{1cm} (21)

and

$$
u(\gamma \tau) = \nu(\tau).$$  \hspace{1cm} (22)

Arguing that the theory had only three singularities, which could be set at $\{u_c = \pm 1, \infty\}$, Seiberg and Witten were able to determine $a(u)$ and $a_D(u)$ exactly in terms of period integrals of a suitable meromorphic 1–form over the two canonical homology basis cycles of an elliptic curve parametrized by $u$.

These period integrals have the desired monodromies and are known to satisfy second–order linear differential equations called Picard–Fuchs equations. In particular

$$
\left( \frac{\partial^2}{\partial u^2} + \frac{1}{4(u^2 - 1)} \right) \begin{pmatrix}
    a_D(u) \\
    a(u)
\end{pmatrix} = 0.
$$  \hspace{1cm} (23)
Note that, when \( u \) loops around one of the punctures, the differential equation does not change because the potential is single-valued. Therefore, \( a_D(\tilde{u}) \) and \( a(u) \) must be a linear combination of \( a_D(u) \) and \( a(u) \) with constant coefficients, as in (20). In general, multivalued functions with non-trivial monodromies are naturally associated to linear differential equations with meromorphic potentials.

From Eq.(23), Matone derived a non-perturbative identity relating \( u \) and \( \mathcal{F}(a) \)
\[
  u = \pi i \left( \mathcal{F}(a) - \frac{1}{2}a a_D \right),
\]
which allowed to find recursion relations for the instanton contributions to the prepotential \([11]\).

The identity (24) was verified by instanton calculations \([12]\ [13]\), and it follows from the superconformal Ward identities as well \([14]\). In \([14]\), Bonelli, Matone and Tonin took it as a starting point for a rigorous derivation of the Seiberg–Witten result by reflection symmetry, without any assumption on the number of singularities \([15]\). They univocally identified the three–punctured sphere \( \mathcal{M}_3 = \mathbb{P}^1(\mathbb{C}) \backslash \{ u_c = \pm 1, \infty \} \) as the moduli space of the model.

The Kramers–Wannier self–duality relation and its generalization in the Potts model imply that the physical properties of the system at low and high temperatures, equivalently high and low \( K \), are related.

Furthermore, \( K \) plays the same role as the effective coupling constant \( \tau \) in the Seiberg–Witten model. Therefore, it is natural to look for a new variable \( z(K) \), analogous to \( u(\tau) \), that is left invariant under both the duality mapping \([16]\) and the symmetries \([9], [11]\)
\[
  z(K) = z(K^*) = z(K + \frac{i\pi}{2}) = z(-K).
\]
Consequently, we may regard the free energy \( F(K) \) as a multivalued function of \( z, \mathcal{F}(z) \), and the various transformation properties of \( F(K) \) under \([9], [11], \) and \([16]\) will be reflected in the polymorphicity of \( \mathcal{F}(z) \). In fact, performing one of these mappings is equivalent to the modulus \( z \) going around a non trivial closed path in the moduli space \( \mathcal{M} \). Correspondingly, the free energy \( \mathcal{F}(z) \) will undergo a monodromy.

This procedure will show us that the natural setting or moduli space for the Ising model in two dimensions is again a three–punctured sphere \( \mathcal{M}_3 = \mathbb{P}^1(\mathbb{C}) \backslash \{ z = \pm 1, \infty \} \). As we will discuss below, this is of course encoded in Onsager’s solution \([7]\). We will consider the Potts case later.

By \([14]\), it is natural to consider the effect of the various transformations on \( s \equiv \sinh(2K) \). In particular, \( s \to -s \) under both \([9]\) and \([11]\), and \( s \to s^{-1} \) under \([16]\). We see that the
maps
\[ f : s \to -s \quad g : s \to \frac{1}{s}, \quad (26) \]
are involutive and commute
\[ f \circ f = id = g \circ g, \quad f \circ g = g \circ f \to (f \circ g) \circ (f \circ g) = id. \quad (27) \]
Hence the polynomial in the variable \( x \)
\[ P(x, s) = (x - s)(x - f(s))(x - g(s))(x - f(g(s))), \quad (28) \]
will satisfy
\[ P(x, s) = P(x, f(s)) = P(x, g(s)) = P(x, f(g(s))), \quad (29) \]
and its coefficients will be invariant under (11), (12) and (16). We have
\[ P(x, s) = (x - s)(x + s)(x - \frac{1}{s})(x + \frac{1}{s}) = x^4 - \left(\frac{s^4 + 1}{s^2}\right) + 1, \quad (30) \]
which implies that \( \frac{s^4 + 1}{s^2} \) is the quantity we are looking for. We remark that there are no transformations other than (11), (12) and (16) which leave \( z \) unchanged. This is a minimal choice in a sense, since no extra spurious transformations enter the picture. For instance we can easily check that all the quantities:
\[ H(s, n) = 1 + s^{4n} \quad \frac{1}{s^{2n}}, \quad (31) \]
are invariant under (11), (12) and (16). But these are not the only symmetry transformations. For example, \( H(s, 2) \) would be invariant under \( s \to is \) as well: however, we do not know how the free energy transforms under this map.

Finally, we will set:
\[ z = \frac{s^4 + 1}{2s^2}, \quad (32) \]
so that the critical temperature corresponds to \( z_c(K) = 1 \).

We remark that there is no loss of generality in picking \( s \) instead of \( \tanh(K) \) or \( e^{-2K} \) as the building block. The invariants one finds using the latter are equivalent to \( z \).

We would like to view the free energy as a function of \( z \). To this end, we have to invert (32) and obtain \( s = s(z) \), which is of course a multivalued function of \( z \). In general, by construction, for a given value of \( z \) there are four different values of \( s \), obtained by solving the equation
\[ A(s, z) = s^4 - 2zs^2 + 1 = 0. \quad (33) \]
However, there are certain critical values $z_c$ such that the polynomial (33) has multiple roots. Equivalently, $z_c = z(s_i)$, where $s_i$ are ramification points of the map $z(s)$. As $z$ winds around one of these critical values, we move from one branch of $s(z)$ to another. This is analogous to $y^2 = x$: as $x$ winds around 0 or equivalently $\infty$, we move from $y = \sqrt{x}$ to $y = -\sqrt{x}$.

If $s_0$ is a root of $A(s, z)$, then the others will be given by $-s_0, 1/s_0, -1/s_0$. Therefore, we can distinguish three cases:

1. $s_0 = -s_0 \rightarrow s_0 = 0, \infty \rightarrow z_c = \infty$.
2. $s_0 = 1/s_0 \rightarrow s_0^2 = 1 \rightarrow z_c = 1$.
3. $s_0 = -1/s_0 \rightarrow s_0^2 = -1 \rightarrow z_c = -1$.

Hence, we have found six $s_i$, $\{\pm 1, \pm i, 0, \infty\}$, which correspond to three singular points, namely $z_c = \{\pm 1, \infty\}$. By (32), we find

$$s(z) = \pm \sqrt{z \pm \sqrt{z^2 - 1}}.$$  \hfill (34)

Hence, when $z$ loops around $z_c = 1$

$$z - 1 \rightarrow e^{2\pi i}(z - 1) \Rightarrow s(z) = \pm \sqrt{z \pm \sqrt{z^2 - 1}} \rightarrow \pm \sqrt{z \mp \sqrt{z^2 - 1}} = \frac{1}{s(z)},$$  \hfill (35)

which is equivalent to $K \rightarrow K^*$. On the other hand, we see that looping around $\infty$ may correspond to either $K \rightarrow -K$ or $K \rightarrow K + i\pi/2$, since

$$z \rightarrow e^{2\pi i}z \Rightarrow s(z) \sim \pm (2z)\pm^{1} \rightarrow \mp (2z)^{\pm 1} \sim -s(z).$$  \hfill (36)

Indeed, as $z \rightarrow \infty$

$$c(z) = \cosh 2K(z) = \pm \sqrt{1 \pm \sqrt{z^2 - 1}},$$  \hfill (37)

may have the following asymptotic behaviours

$$c(z) \sim \pm \sqrt{2z},$$  \hfill (38)

or

$$c(z) \sim \pm \sqrt{1 + \frac{1}{2z}}.$$  \hfill (39)

In the first case we see that

$$z \rightarrow e^{2\pi i}z \Rightarrow c(z) \rightarrow -c(z),$$  \hfill (40)
which is equivalent to $K \rightarrow K + \frac{i\pi}{2}$, whereas in the other case

$$z \rightarrow e^{2\pi i z} \Rightarrow c(z) \rightarrow c(z)$$

(41)

which is equivalent to $K \rightarrow -K$.

Let us set

$$\mathcal{F}(z) = F_{\text{high}}(K(z)),$$

(42)

and

$$\mathcal{F}_D(z) = F_{\text{low}}(K(z)).$$

(43)

By (14) and (15), we see that looping around $z_c = 1$

$$\mathcal{F}(z) \rightarrow \mathcal{F}(\tilde{z}) = F_{\text{high}}(K(\tilde{z})) = F_{\text{high}}(K^*(z)) = F_{\text{low}}(K(z)) - \ln \sinh 2K(z) = \mathcal{F}_D(z) - \ln s(z).$$

(44)

In a more symmetric fashion

$$\mathcal{F}(z) - \frac{1}{2} \ln s(z) \rightarrow \mathcal{F}(\tilde{z}) - \frac{1}{2} \ln s(\tilde{z}) = \mathcal{F}_D(z) - \frac{1}{2} \ln s(z),$$

(45)

where we used the fact that $s(\tilde{z}) = s(z)^{-1}$. Furthermore, by (10) and (12), with $s(z) = \sqrt{z + \sqrt{z^2 - 1}}$

$$z \rightarrow e^{2\pi i z} \Rightarrow \mathcal{F}_D(z) \rightarrow \mathcal{F}_D(z) + \frac{i\pi}{2}.$$  

(46)

4 The Uniformization Equation

We said above that this geometrical structure is already encoded in Onsager’s solution. Indeed, the derivative of the free energy, the internal energy per spin $U(K)$ can be expressed in terms of elliptic integrals (see for example [10])

$$U(K) = \frac{\partial}{\partial \beta} [\beta f(K)] = -J \coth 2K \left[ 1 + k_1' \frac{2}{\pi} \mathcal{K}(k_1) \right],$$

(47)

where $\mathcal{K}(k_1)$ is the complete elliptic integral of the first kind

$$\mathcal{K}(k_1) = \int_0^{\frac{\pi}{2}} \frac{d\phi}{\sqrt{1 - k_1^2 \sin^2 \phi}}$$

(48)

and

$$k_1 = \frac{2s}{s^2 + 1} \quad k_1' = \frac{s^2 - 1}{s^2 + 1} \quad k_1^2 + (k_1')^2 = 1.$$  

(49)
In terms of $F(K(z)) = F(z)$, using (47) and $K = \beta J$

$$U(K) = -\frac{\partial K}{\partial \beta} \partial_K F(K) = -J \frac{\partial z}{\partial K} \partial_z F(z), \quad (50)$$

which implies, after some algebra, that

$$\partial_z F(z) = \frac{\sigma}{4\sqrt{z^2 - 1}} + \frac{1}{\pi(z + 1)} K(k_1), \quad (51)$$

where $\sigma = \pm 1 \leftrightarrow s^2 = z + \sigma \sqrt{z^2 - 1}$.

Furthermore, note that $\sqrt{z - 1} K(k_1(z))$ is a solution of the uniformization equation for the three-punctured sphere $M_3 = \mathbb{P}^1(\mathbb{C}) \setminus \{z = \pm 1, \infty\}$

$$\left[ \frac{\partial^2}{\partial z^2} + \frac{3 + z^2}{4(z^2 - 1)^2} \right] \begin{pmatrix} \psi_D(z) \\ \psi(z) \end{pmatrix} = 0. \quad (52)$$

Therefore, by (51), $F(z)$ solves the following third-order equation

$$\partial_z^3 F(z) + \frac{3z - 1}{z^2 - 1} \partial_z^2 F(z) + \frac{2z + 1}{2(z + 1)^2(z - 1)} \partial_z F(z) - \frac{\sigma}{8} \frac{z + 1}{(z^2 - 1)^2} = 0. \quad (53)$$

Let us denote by $H$ the Poincaré upper half plane

$$H = \{ \tau \in \mathbb{C} \mid \text{Im}\tau > 0 \}. \quad (54)$$

The uniformization theorem tells us that the universal covering of the three-punctured sphere $M_3$ is $H$. Indeed, there exists a holomorphic covering map

$$J_H : H \rightarrow M_3$$

$$\tau \mapsto J_H(\tau) = z(\tau), \quad (55)$$

such that

$$z(\gamma\tau) = z(\tau) \iff \gamma \in \Gamma(2), \quad (56)$$

where $\Gamma(2)$ is a subgroup of $PSL(2, \mathbb{Z})$ defined by

$$\Gamma(2) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid ad - bc = 1, \ a, d \equiv 1 \text{ mod } 2, \ b, c \equiv 0 \text{ mod } 2 \right\},$$

and

$$\gamma \tau = \frac{a\tau + b}{c\tau + d} \quad (56)$$

Hence, $M_3 \cong H/\Gamma(2)$. 

9
The ratio of two linearly independent solutions of (52) gives the inverse of the uniformization map up to an overall $PSL(2, \mathbb{C})$ transformation

$$J_H^{-1} : \mathcal{M}_3 \to H$$

$$\tau(z) = J_H^{-1}(z) = \frac{\psi_D(z)}{\psi(z)}.$$  

(57)

The function $\tau(z)$ is polymorphic: when $z$ loops around one of the punctures, $\{\pm 1, \infty\}$, $\psi_D(z)$ and $\psi(z)$ undergo a monodromy

$$\begin{pmatrix}
\psi_D(\tilde{z}) \\
\psi(\tilde{z})
\end{pmatrix} = \begin{pmatrix}
a & b \\
c & d
\end{pmatrix} \begin{pmatrix}
\psi_D(z) \\
\psi(z)
\end{pmatrix},$$

(58)

and correspondingly

$$\tau(\tilde{z}) = \frac{a\tau(z) + b}{c\tau(z) + d}, \quad \begin{pmatrix}
a & b \\
c & d
\end{pmatrix} \in \Gamma(2),$$

(59)

in accordance with (55).

5 Potts model

The Potts model [9] generalizes the Ising model in the sense that each spin $s_i$ can have $q$ values, $(1, 2, \ldots, q)$, and the Hamiltonian reads

$$\mathcal{H} = -J \sum_{\langle i,j \rangle} \delta_{s_i s_j}.$$  

(60)

The partition function is given by

$$Z(K) = \sum_{\{\sigma\}} e^{-\beta \mathcal{H}} = \sum_{\{\sigma\}} \exp \left( K \sum_{\langle i,j \rangle} \delta_{s_i s_j} \right),$$

(61)

where as usual $K = \beta J$. Although the problem of finding the exact free energy for the square lattice $D = 2$ Potts model has not been solved, some exact results are known. In particular, there exists the analogue of the Kramers–Wannier duality relation [9], which determines the critical temperature

$$K_c = \ln(1 + \sqrt{q}).$$

(62)

For $q = 3, 4$ the phase transition is second order, while for $q > 4$ the transition is first order and the latent heat is known exactly [18]. Furthermore, for $q > 4$ the spontaneous
magnetization \[19\] and the correlation length at \(T_c\) \[20\] are also known. Finally, for \(q = 3, 4\) the critical exponents are given exactly \[21\]. For a comprehensive review see \[17\].

Let us define

\[
F(K) \equiv \lim_{N \to \infty} \frac{\ln Z(K)}{N}.
\]  
(63)

The high temperature expansion reads

\[
Z_{\text{high}}(K) = q^N C(K)^{2^N} \sum_{\text{loops}} T(K)^l,
\]  
(64)

where

\[
C(K) = \frac{e^K + q - 1}{q}, \quad T(K) = \frac{e^K - 1}{e^K + q - 1},
\]  
(65)

and \(l\) denotes the length of the loop. Conversely, for the low temperature expansion we have

\[
Z_{\text{low}}(K) = q e^{2KN} \sum_{\text{loops}} e^{-Kl}.
\]  
(66)

Note that, in contrast to the Ising model, the length \(l\) can be odd. For instance, consider \(q = 3\), say blue green and red spins. In the low temperature expansion there are excitations corresponding to one green and one red spin next to each other in a background of blue spins. In this case \(l\) is seven. Therefore, the analogue of (11) is

\[
K \to K + 2\pi i.
\]  
(67)

Under this shift, by (66)

\[
F_{\text{low}}(K + 2\pi i) = F_{\text{low}}(K) + 4\pi i.
\]  
(68)

The generalization of the Kramers–Wannier relation for the Potts model is \[8\]

\[
\frac{Z_{\text{high}}(K^*)}{q^N C(K^*)^{2^N}} = \frac{Z_{\text{low}}(K)}{qe^{2KN^*}},
\]  
(69)

implying that

\[
F_{\text{high}}(K^*) - \ln q - 2 \ln C(K^*) = F_{\text{low}}(K) - 2K,
\]  
(70)

where

\[
e^{-K^*} = T(K) = \frac{e^K - 1}{e^K + q - 1} \leftrightarrow (e^{K^*} - 1)(e^K - 1) = q.
\]  
(71)

Therefore, we see that the critical temperature is indeed given by (62). By virtue of (71), we can rewrite Eq.(70) in a more symmetrical way, namely

\[
F_{\text{high}}(K^*) - \ln(e^{K^*} - 1) = F_{\text{low}}(K) - \ln(e^K - 1).
\]  
(72)
As before, the mapping $K^*(K)$ is involutive.

The simplicity of transformation (74) makes it easier to find a variable invariant under this mapping and (71). We can basically choose either

$$w(K) = T(K)T(K^*) = e^{-K}e^{-K^*} = \frac{e^K - 1}{e^K(e^K + q - 1)},$$

or

$$v(K) = T(K) + T(K^*),$$

which are related by (71), in particular

$$v(K) = (1 - q)w(K) + 1.$$

Solving Eq.(73) we find

$$e^K = \frac{(1 - q)w + 1 \pm (1 - q)\sqrt{w - \frac{1}{(1 + \sqrt{q})^2}}(w - \frac{1}{(1 - \sqrt{q})^2})}{2w}.$$  

By construction, there are two values of $w$ where the solutions of (76) coincide. These critical values $w_c = w(K_c)$ are given by

$$w_1 = \frac{1}{(1 + \sqrt{q})^2}, \quad w_2 = \frac{1}{(1 - \sqrt{q})^2},$$

where $w_1$ corresponds to the physical singularity and $w_2$ to the complex temperature singularity. These are the fixed points of the duality transformation (71). However, note that $F(K)$ is singular as $K \to 0, \infty$

$$K \to \infty \Rightarrow F_{low}(K) \sim 2K.$$  

Hence, we argue that there is a further singularity at $w(K = 0) = w(K = \infty) = 0$.

In order to show the close relationship with the Ising model, we will perform a Möbius transformation on $w$ so that the critical points coincide with $\{\pm 1, \infty\}$.

The linear fractional transformation that maps $w_1$ to 1, $w_2$ to $-1$ and $w = 0$ to $\infty$ is given by

$$z = \frac{1 - (1 + q)w}{2\sqrt{q}w} \Leftrightarrow w = \frac{1}{2\sqrt{q}z + (1 + q)}.$$  

Hence

$$z = \frac{(e^K - 1)^2 + q}{2\sqrt{q}(e^K - 1)},$$

and

$$e^K = 1 + \sqrt{q}(z \pm \sqrt{z^2 - 1}).$$
By (81), we see that looping around 
\( z_c = 1 \), \( K \) goes to \( K^* \)

\[
K(\tilde{z}) = \ln(1 + \sqrt{q}(z \mp \sqrt{z^2 - 1})) \to (e^{K(\tilde{z})} - 1)(e^{K(z)} - 1) = q \to K(\tilde{z}) = K^*(z),
\]

(82)

where \( \tilde{z} - 1 = e^{2\pi i}(z - 1) \). Furthermore, choosing the plus sign in (81), as \( z \) loops around \( \infty \)

\[
z \to e^{2\pi i}z \Rightarrow K \sim \ln z \to \ln z + 2\pi i = K + 2\pi i,
\]

(83)

that is we retrieve the symmetry (67).

6 Fisher zeroes

In [16], Fisher emphasized the role of the zeroes of the partition function in the complex temperature plane in the study of phase transitions. In particular, using Kaufman’s expression of \( Z(K) \) for the two–dimensional Ising model [8], he showed that, in the thermodynamic limit, the zeroes are dense on two circles in the \( \text{tanh} K \) or equivalently \( e^{-K} \) plane given by

\[
\text{tanh} K = 1 + \sqrt{2}e^{i\theta}, \quad \text{tanh} K = -1 + \sqrt{2}e^{i\theta}.
\]

(84)

These are the antiferromagnetic and ferromagnetic circles respectively and they are related by the map \( K \to -K \).

In terms of \( z \), both circles correspond to the segment joining \( z_c = -1 \) to \( z_c = +1 \). By (84), we have

\[
s^2(\theta) = \left( \frac{2 \text{tanh} K(\theta)}{1 - \text{tanh}^2 K(\theta)} \right)^2 = \left( \frac{\pm 1 + \sqrt{2}e^{i\theta}}{e^{i\theta} \pm \sqrt{2}e^{i\theta}} \right)^2 = \\
\left( \frac{\sqrt{2} \pm e^{i\theta}}{\sqrt{2} \pm e^{i\theta}} \right)^2.
\]

Then

\[
\bar{s}^2(\theta) = \frac{1}{s^2(\theta)},
\]

which implies that

\[
z(\theta) = \frac{1}{2} \left( s^2(\theta) + \frac{1}{s^2(\theta)} \right) = \bar{z}(\theta).
\]

Therefore, the two circles are constrained on the real axis and

\[
z(\theta) = \frac{1}{2} \left[ \left( \frac{\sqrt{2} \pm \cos \theta \mp i \sin \theta}{\sqrt{2} \pm \cos \theta \pm i \sin \theta} \right)^2 + \text{c.c.} \right] = \frac{1}{2} \left[ \frac{(\sqrt{2} \pm \cos \theta \mp i \sin \theta)^2}{(\sqrt{2} \pm \cos \theta)^2 + \sin^2 \theta} + \frac{(\sqrt{2} \pm \cos \theta)^2 + \sin^2 \theta}{(\sqrt{2} \pm \cos \theta)^2 + \sin^2 \theta} \right] = \\
\frac{(\sqrt{2} \pm \cos \theta)^4 - 6(\sqrt{2} \pm \cos \theta)^2 \sin^2 \theta + \sin^4 \theta}{((\sqrt{2} \pm \cos \theta)^2 + \sin^2 \theta)^2},
\]

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implying that \(|z(\theta)| \leq 1\), as it can be easily checked. Hence, since both \(z_c = -1\) and \(z_c = 1\) are in the image, \(z(\theta)\) maps the interval \([0, 2\pi]\) into the interval \([-1, +1]\).

In [22] [23], Martin, Maillard and Rammal conjectured that the Fisher zeroes in the two-dimensional \(q\)-state Potts model lie on a circle in the complex \(e^{-K}\) plane given by

\[
e^{-K(\theta)} = -\frac{1}{q-1} + \frac{\sqrt{q}}{q-1} e^{i\theta}.
\]

Actually, they conjectured that the locus was given by the condition

\[
e^{-K^*} = e^{-\bar{K}},
\]

where \(K^*\) is the dual of \(K\) and \(\bar{K}\) denotes the complex conjugate of \(K\). This yields precisely (85). Note also that setting \(q = 2\) recovers the ferromagnetic circle of the Ising model.

Numerical investigations for small lattices at \(q = 3\) and \(q = 4\) [22] [23] provided evidence for (85). Further progress was made in [24], for \(3 \leq q \leq 8\). In [25], on the basis of numerical results on small lattices for \(q \leq 10\), it was conjectured that for finite lattices with self-dual boundary conditions, and for other boundary conditions in the thermodynamic limit, the zeroes in the ferromagnetic regime are on the above circle. This conjecture was actually proved for infinite \(q\) in [26].

In [28], using a general result concerning the partition function zeroes of models displaying first order phase transition obtained by Lee [27], Kenna proved that the locus for \(q > 4\) is indeed given by (85). His argument relies on the fact that the thermodynamic limit and the application of the duality transformation (71) to the Fisher zeroes should commute.

We will now show that, in terms of the appropriate modulus \(z\), the locus of the Fisher zeroes for the \(q\)-state Potts model (85) still corresponds to the segment joining \(z_c = -1\) to \(z_c = +1\).

First, note that the locus is equivalent to

\[
e^{K(\theta)} = 1 + \sqrt{q} e^{i\theta}.
\]

Indeed, \((e^K)^{-1} \to e^K\) is a Möbius transformation and thus it maps circles into circles. Furthermore, Eq. (85) implies that

\[
e^{K(\theta)} = \frac{q-1}{\sqrt{q} e^{i\theta} - 1} \Rightarrow e^{K(\theta)} - 1 = \frac{\sqrt{q}(\sqrt{q} - e^{i\theta})}{e^{i\theta}(\sqrt{q} - e^{-i\theta})} \Rightarrow
\]

\[|e^{K(\theta)} - 1|^2 = q.\]  

Finally, by (87), we immediately recognize that (87) corresponds to \(z(\theta) = \cos \theta\).
\[ z \pm \sqrt{z^2 - 1} = e^{i\theta} \Rightarrow z^2 - 2ze^{i\theta} + e^{2i\theta} = z^2 - 1 \Rightarrow z(\theta) = \cos \theta. \]  

(89)

Therefore, as in the case of the Ising model, the locus of Fisher zeroes for the Potts model corresponds to the segment joining the complex temperature singularity \( z_c = -1 \) to the physical singularity \( z_c = +1 \).

7 Conclusions

In summary, we have seen that the introduction of a proper modulus or uniformizing coordinate \( z \), which is given by

\[ z = \frac{1 + \sinh^4(2K)}{2 \sinh^2(2K)}, \]  

(90)
in the Ising case, and

\[ z = \frac{(e^K - 1)^2 + q}{2\sqrt{q}(e^K - 1)}, \]  

(91)
in the Potts case, unveils an intriguing correspondence between the \( D = 2 \) Ising and Potts models. Indeed, their moduli spaces are both equivalent to the three–punctured sphere

\[ \mathcal{M}_3 = \mathbb{P}^1(\mathbb{C})\{z = \pm 1, \infty\}. \]  

(92)
The moduli (90) and (91) are characterized by the invariance under the symmetry and duality transformations (9)(11)(13) and (67)(71) respectively.

This connection is further strengthened by the fact that the locus of Fisher zeroes in both models is given by the segment joining \( z_c = -1 \) to \( z_c = +1 \).

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