STRONG HAAGERUP INEQUALITIES FOR FREE $\mathcal{R}$-DIAGONAL ELEMENTS

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ABSTRACT. In this paper, we generalize Haagerup’s inequality [H] (on convolution norm in the free group) to a very general context of $\mathcal{R}$-diagonal elements in a tracial von Neumann algebra; moreover, we show that in this “holomorphic” setting, the inequality is greatly improved from its original form. We give an elementary combinatorial proof of a very special case of our main result, and then generalize these techniques. En route, we prove a number of moment and cumulant estimates for $\mathcal{R}$-diagonal elements that are of independent interest. Finally, we use our strong Haagerup inequality to prove a strong ultracontractivity theorem, generalizing and improving the one in [Bi2].

1. INTRODUCTION

There is an interesting phenomenon which often occurs in holomorphic spaces. A theorem in the context of a function space (for example a family of norm-estimates, such as the $L^p$-bound of the Riesz projection, [R]) takes on a stronger form when restricted to a holomorphic subspace. For example, $L^p$-bounds often shrink, and have meaningful extensions to the regime $p < 1$. For our purposes, the most relevant example is Janson’s strong hypercontractivity theorem [Ja], discussed below. In algebraic terms, this theorem states that a certain semigroup has better properties when acting on the algebra generated by i.i.d. complex Gaussians than on the algebra generated by i.i.d. real Gaussians. The latter is a $*$-algebra while the former is far from one; we will exploit this difference in what follows.

In this paper, we will primarily be concerned with one prominent non-commutative norm inequality: the Haagerup inequality. It first arose in [H], where it was the main estimate used to foster an example of a non-nuclear $C^*$-algebra with the metric approximation property. In the context of that paper, Haagerup’s inequality takes the following form:

**Theorem 1.1** ([H], Lemma 1.4). Let $\mathbb{F}_k$ be the free group on $k$ generators, and let $f \in \ell^2(\mathbb{F}_k)$ be a function supported on the subspace generated by words in $\mathbb{F}_k$ of length $n$. Then $f$ acts as a convolutor on $\ell^2(\mathbb{F}_k)$, and its convolution norm $\|f\|_* = \sup_{\|g\|_2 = 1} \|f \ast g\|_2$ satisfies

$$\|f\|_* \leq (n+1)\|f\|_2.$$

Note that the convolution product is just the usual product in the von Neumann algebra generated by the left-regular representation of $\mathbb{F}_k$ (known as the free group factor $L(\mathbb{F}_k)$), and so in the language of operator algebras, the statement is that the (non-commutative) $L^2$-norm controls the operator norm on subspaces of uniform finite word-length, where the bound grows linearly with word-length.

The Haagerup inequality, and its decendents, have played important roles in several different fields. In the context of geometric group theory, the Haagerup inequality (and other constructions presented in [H]) have evolved into $\alpha$-$T$-menability or property $T$ [Va2]; in the context of Lie theory, Haagerup’s inequality is related to property RD [Laf2]. It has proved useful for other operator algebraic applications: in [Laf1], Lafforgue uses the Haagerup inequality as a crucial tool in his proof of the Baum-Connes conjecture for cocompact lattices in $SL(3, \mathbb{R})$; in this context, the precise order of growth of the Haagerup constant is immaterial (so long as it is polynomial). On the other hand, the Haagerup inequality has proved useful in studying return probabilities and other
statistics of random walks on groups (see [CPS, Va1]), where the exact form of the Haagerup constant is important.

Our main theorem, Theorem 1.3 below, is a strong Haagerup inequality in a general "holomorphic" setting – i.e. a non-self-adjoint algebra. In the special case of the free group factor, this amounts to considering convolution operators which involve only generators of the group, not their inverses; the resulting Haagerup inequality (Corollary 1.4 below) then has growth of order $\sqrt{n}$, where $n$ is the word-length.

There are two main approaches to norm estimates in such a setting. A direct one (as used in the original approach of Haagerup) is to work directly in the concrete representation of the considered element as operator on a Hilbert space and try to estimate the operator norm by considering the action of the operator on vectors. A more indirect approach is by recovering the operator norm as the limit of the $L^p$-norms as $p \to \infty$, and therefore trying to get a combinatorial understanding of $L^p$-norms for $p = 2m$ even. It is the latter approach which we take. Thus, we need a good (at least asymptotic) understanding of the moments of the involved operators with respect to the underlying state. To our benefit, the moments of the generators of free groups possess a lot of structure: namely the generators are free in the sense of Voiculescu’s free probability theory.

Our strong Haagerup inequality is actually derived in a much more general setting: algebras generated by free $\mathcal{R}$-diagonal elements. We therefore handle not only the original framework of Haagerup (in the form of free Haar unitaries), but also free circular elements, and a wealth of other non-normal operators.

There have been some predecessor of our strong Haagerup inequality for the general $\mathcal{R}$-diagonal case. Namely, the one-dimensional case was mainly addressed in [HL] and, in particular, in [Lar]. Furthermore, [Lar] contains a very specialized multi-dimensional case, where the considered operator is a product of identically-distributed free $\mathcal{R}$-diagonal elements. All these results relied on analytic techniques, using the theory of $\mathcal{R}$- and $\mathcal{S}$-transforms for probability measures on $\mathbb{R}$. However, in the genuine non-commutative case of polynomials in several non-commuting $\mathcal{R}$-diagonal elements, as we treat it here, such analytical tools are unavailable to us, and so our analysis will rely on the combinatorial machinery of free cumulants, as powered by free probability theory.

Our main tool is the moment-cumulant formula (Equation 2.5, below), which expresses the moments of the considered elements in a very precise combinatorial way in terms of free cumulants. This allows us to reduce the multi-dimensional case essentially to the one-dimensional case. (Note that this reduction is usually the hardest part in such inequalities.) Whereas in some cases (as for circular elements) this reduction yields directly the desired result, in other cases – namely when the cumulants of the $\mathcal{R}$-diagonal element may be negative (as it happens for Haar unitaries, i.e., in the free group situation) – we need an additional step. Our strategy is to replace the original $\mathcal{R}$-diagonal element $a$ with a different $\mathcal{R}$-diagonal element $b$ whose cumulants are positive and dominate the absolute values of the cumulants of $a$; this has to be done in such a way that we have control over both the $L^2$-norm and the operator norm of $b$ in terms of the corresponding norms of $a$. The technique we develop will, we hope, have more general applicability.

Let us now give a precise definition of the arena for our Haagerup inequality. Section 2 contains brief introductions to all the terms used in what follows (and in the foregoing).

**Definition 1.2.** Let $I$ be any indexing set, and let $\{a_i : i \in I\}$ be $*$-free identically distributed $\mathcal{R}$-diagonal elements in a $C^*$-probability space with state $\varphi$; for convenience, let $a$ be a fixed $\mathcal{R}$-diagonal element with the same distribution. Define $\mathcal{H}(a, I)$ to be the norm-closed (non-$*$) algebra generated by the $a_i$. For each
\( n \geq 0, \text{define } \mathcal{H}^{(n)}(a, I) \text{ as the Hilbert subspace of } L^2(\mathcal{H}(a, I), \varphi) \text{ of all elements of the form} \\
T = \sum_{|i|=n} \lambda_i a_i, \\
\text{where } i = (i_1, \ldots, i_n) \in I^n, \lambda_i \in \mathbb{C}, \text{and } a_i = a_{i_1} \cdots a_{i_n}. \text{ We refer to } \mathcal{H}^{(n)}(a, I) \text{ as the } n\text{-particle space (relative to } a, I). \)

The motivation for considering the algebra \( \mathcal{H}(a, I) \) comes from the first author’s paper [Ke], and [Bi1]. If \( c \) is a circular element, then \( L^2(\mathcal{H}(c, I), \varphi) \) is a free analogue of the Segal-Bargmann space of [Ba] – i.e. the space \( \mathcal{H}L^2(\mathcal{H}, \gamma) \) of holomorphic functions on a Hilbert space \( \mathcal{H} \) of dimension \( |I| \), square-integrable with respect to a certain Gaussian measure \( \gamma \). The Segal-Bargmann space is the framework for the complex wave representation of quantum mechanics. It played an important role in the constructive quantum field theory program in the mid- to late-twentieth century.

There is a natural operator, the Ornstein-Uhlenbeck operator or number operator \( N \) on \( L^2(\mathcal{H}, \gamma) \), which is related to the energy operator in quantum field theory. In the classical (Gaussian) context, the Ornstein-Uhlenbeck semigroup \( e^{-tN} \) satisfies a regularity property called hypercontractivity: for \( 1 < p \leq r < \infty \) the semigroup \( e^{-tN} \) is a contraction from \( L^p(\mathcal{H}, \gamma) \) to \( L^r(\mathcal{H}, \gamma) \) for large enough time \( t \). When \( e^{-tN} \) is restricted to the Segal-Bargmann space and its holomorphic \( L^p \) generalizations, the time to contraction is shorter, as shown in [Ja] and generalized in [G]. This strong hypercontractivity demonstrates that contraction properties of the Ornstein-Uhlenbeck semigroup improve in the holomorphic category.

In [Bi2], Biane showed how to canonically generalize the Ornstein-Uhlenbeck operator to the setting of free group factor, and proved that the resulting semigroup \( e^{-tN_0} \) is hypercontractive. He further showed that the semigroup \( e^{-tN_0} \) satisfies an even stronger condition called ultracontractivity: it continuously maps \( L^2 \) into \( L^\infty \) for all \( t > 0 \), and for small time \( \|e^{-tN_0}\|_{2 \to \infty} \) is of order \( t^{-3/2} \). This result was proved using a version of the Haagerup inequality presented in [Bo1]. We should note that, although this result is for the free group factor, the \( n \)-particle spaces used in the proof are not the same as in Theorem 1.1, but are rather defined in terms of a generating family of semicircular elements defined in Section 2; nevertheless, the relevant Haagerup inequality can be proved from Theorem 1.1 using a central limit approach similar to the one in [VDN].

It is Biane’s free ultracontractivity theorem, along with our intuition that norm-inequalities improve in holomorphic categories, that motivated us to consider the same type of Haagerup inequality for \( \mathcal{R} \)-diagonal elements. In the special case of circular elements, the first author showed in [Ke] that, as in the Gaussian case, in the holomorphic category – in this case the spaces \( L^p(\mathcal{H}(c, I), \varphi) \) – Biane’s hypercontractivity result is trumped by Janson’s strong hypercontractivity. The first author further spelled out precisely the holomorphic structure inherent in \( \mathcal{H}(c, I) \). Our interpretation of \( \mathcal{R} \)-diagonal elements as “holomorphic” is more vague. Nevertheless, the algebra \( \mathcal{H}(a, I) \) is a triangular algebra much like the space of bounded Hardy functions \( H^\infty \) (as a Banach algebra acting on \( L^2(S^1) \)). More importantly, the kinds of norm estimates used in [Ke] have natural analogues for \( \mathcal{R} \)-diagonal elements.

The following theorem, which is our strong version of Haagerup’s inequality in the general \( \mathcal{R} \)-diagonal setting, is the main result of this paper.

**Theorem 1.3.** Let \( a \) be an \( \mathcal{R} \)-diagonal element in a \( C^* \)-probability space. There is a constant \( C_a < \infty \) such that for all \( T \in \mathcal{H}^{(n)}(a, I) \),

\[
\|T\| \leq C_a \sqrt{n} \|T\|_2. \tag{1.1}
\]

In general, \( C_a \) may be taken \( \leq 2^{10} \sqrt{e} \|a\|^2 / \|a\|_2^2 \); if \( a \) has non-negative free cumulants, \( C_a \) may be taken \( \leq \sqrt{e} \|a\| / \|a\|_2 \).
As a very special case (where the $a_i$ are free Haar unitaries), we deduce the following surprising strong version of the classical Haagerup inequality (Theorem 1.1).

**Corollary 1.4.** Let $k \geq 2$, let $\mathbb{F}_k$ be the free group on $k$ generators, and let $\mathbb{F}_k^+ \subset \mathbb{F}_k$ be the free semigroup (i.e. the set of all words in the generators, excluding their inverses). If $f \in \ell^2(\mathbb{F}_k^+) \subset \ell^2(\mathbb{F}_k)$ is supported on words of length $n$, then $f$ acts (via the left-regular representation on the full group $\mathbb{F}_k$) as a convolutor, with convolution norm

$$\|f\|_* \leq 2^{10} \sqrt{e} \sqrt{n} \|f\|_2.$$ 

This paper is organized as follows. In section 2, we give a brief introduction to free probability theory and $\mathcal{R}$-diagonal elements, in addition to setting the standard notation we will use throughout the paper. In Section 3, we provide a concrete bijection in order to calculate the moments of a circular element $c$; in it we derive, using more elementary techniques, a formula for $\|c^n\|$, confirming results in [O] and [Lar]. We then use this calculation, together with more involved combinatorial techniques, to estimate the norm of an element in the $n$-particle space $\mathcal{H}^{(n)}(c, I)$ for arbitrary indexing set $I$, and thus prove a special case of Theorem 1.3 in the circular context.

In Section 4, we show how to modify the techniques in Section 3 to prove Theorem 1.3 in general. In the process, we derive bounds on the growth of the free cumulants of $\mathcal{R}$-diagonal elements and, given an $\mathcal{R}$-diagonal $a$, show how to construct another $\mathcal{R}$-diagonal element $b$ with all positive cumulants dominating the cumulants of $a$. We also show that the Haagerup inequality affiliated to the space $\mathcal{H}L^2(\nu_a)$ of holomorphic functions square integrable with respect to the Brown measure $\nu_a$ of $a$ is consistent with Theorem 1.3, which shows that $\nu_a$ does carry some information about the mixed moments of $a$. Finally, in Section 5, we introduce a natural analogue of the Ornstein-Uhlenbeck semigroup affiliated with $\mathcal{H}(a, I)$, and prove a strong ultracontractivity theorem for it.

## 2. A Free Probability Primer

In this section we collect all the relevant results from free probability theory that will be used in what follows. Our descriptions will be brief, as this material is quite standard and is explained in depth in the book [NS3].

### 2.1. $C^*$-probability spaces

Let $\mathcal{A}$ be a unital $C^*$ algebra, and let $\varphi$ be a faithful state on $\mathcal{A}$ (i.e. for $a \in \mathcal{A}$, $\varphi(a^*a)$ only vanishes when $a = 0$). The pair $(\mathcal{A}, \varphi)$ is a $C^*$-probability space. Elements of $\mathcal{A}$ are non-commutative random variables (which we will often refer to simply as random variables). (Some authors prefer to reserve the term ‘random variable’ for self-adjoint elements; in our context, all elements of $\mathcal{A}$ are treated equally.) The motivating example is afforded by the commutative von Neumann algebra $L^\infty(\Omega, \mathcal{F}, P)$ of a probability space. It comes equipped with the faithful state $\varphi = \int_\Omega \cdot dP$; the random variables in this context are bounded random variables in the usual sense.

In classical probability theory, any random variable $X$ has a probability distribution $\nu_X$ – a measure on $\mathbb{C}$ which, among other things, determines the moments of $X$:

$$\int_\Omega X(\omega)^n \overline{X(\omega)}^m dP(\omega) = \int_{\mathbb{C}} z^n \overline{z}^m d\nu_X(z, \bar{z}).$$

In the case of a real random variable $X$, $\nu_X$ is supported in $\mathbb{R}$ and we have $\int X^n dP = \int_{\mathbb{R}} t^n d\nu_X(t)$. At least in the case of bounded random variables, these moment conditions uniquely determine the distribution, which is a compactly-supported probability measure. The same holds true for normal elements in a $C^*$-probability space – if $a$ is normal then there is a unique probability measure $\nu_a$ on $\mathbb{C}$ which satisfies

$$\varphi(a^n(a^*)^m) = \int_{\mathbb{C}} z^n \overline{z}^m d\nu_a(z, \bar{z}),$$

(2.1)
and the measure $\nu_a$ is compactly supported. Indeed, $\text{supp} \nu_a$ is the spectrum of $a$, and the measure can be constructed using the spectral theorem: $\nu_a = \varphi \circ E^a$ where $E^a$ is the spectral measure of $a$ in $\mathcal{A}$.

If $a$ is not a normal element, then there is no measure satisfying Equation 2.1; more generally, given two elements in $\mathcal{A}$ that do not commute, there is no measure which represents their joint probability distribution (this is one way to state the Heisenberg uncertainty principle). In the case where $(\mathcal{A}, \varphi)$ is a tracial $W^*$-probability space ($\mathcal{A}$ is a von Neumann algebra, $\varphi$ is a faithful normal tracial state) however, there is a best-approximation of a probability distribution called the Brown measure, introduced in [Br]. If $a$ is normal, then its Brown measure coincides with its spectral measure, and so the Brown measure is also denoted $\nu_a$. The Brown measure of $a$ always satisfies the moment condition $\varphi(a^n) = \int_{\mathbb{C}} z^n \, d\nu_a(z, \overline{z})$, however it does not respect mixed-moments.

2.2. The free group factors. Free probability was invented by Voiculescu in [Vo] in order to import tools from classical probability theory into the study of the free group factors (specifically to address the still-open question of whether different free group factors are isomorphic).

Let $k \geq 2$, and let $F_k$ denote the free group on $k$ generators $u_1, u_2, \ldots, u_k$. (We will also allow $k = \infty$ to denote the free group with countably-many generators.) The $k$th free group factor $L(F_k)$ is the von Neumann algebra generated by the left-regular representation of $F_k$ on $l^2(F_k)$. (Note: if $g \in F_k$, then the image of $g$ in $L(F_k)$ is an operator with $g^* = g^{-1}$.) There is a natural state $\varphi_k$ defined on $L(F_k)$ induced by the function $g \mapsto \delta_{e^g}$ on $F_k$ (here $e$ is the identity in the group). This state is faithful, normal, and tracial, making $(L(F_k), \varphi_k)$ into a $W^*$-probability space.

There is a canonical representation of the free group factor on the full Fock space. Let $\mathcal{H}$ be a real Hilbert space, and let $\mathcal{H}_{\mathbb{C}} = \mathbb{C} \otimes \mathcal{H}$ be its complexification. The full Fock space of $\mathcal{H}$ is $\mathcal{F}(\mathcal{H}) = \bigoplus_{j=0}^{\infty} (\mathcal{H}_{\mathbb{C}})^{\otimes j}$, where $\otimes$ and $\otimes$ are the Hilbert space direct sum and tensor product, and $(\mathcal{H}_{\mathbb{C}})^{\otimes 0}$ is defined to be the $\mathbb{C}$-span of an abstract vector $\Omega$ (not in $\mathcal{H}$) called the vacuum vector.

For each $h \in \mathcal{H}$, the creation operator $l(h)$ in $\mathcal{B}(\mathcal{F}(\mathcal{H}))$ is uniquely defined by its action $l(h)(h_1 \otimes \cdots \otimes h_j) = h \otimes h_1 \otimes \cdots \otimes h_j$ on $(\mathcal{H}_{\mathbb{C}})^{\otimes j}$ (and $l(h)\Omega = h$). The adjoint $l(h)^*$ is called the annihilation operator, and is given by $l(h)^*(h_1 \otimes h_2 \otimes \cdots \otimes h_j) = (h_1, h) h_2 \otimes \cdots \otimes h_j$ (and $l(h)^* \Omega = 0$). The operator $l(h)$ is not normal (if $h \neq 0$), but it is natural to consider the real part $X(h) = \frac{1}{2}(l(h)+l(h)^*)$. For any $k$-dimensional real Hilbert space $\mathcal{H}$, the von Neumann algebra generated by $\{X(h) : h \in \mathcal{H}\}$ is isomorphic to $L(F_k)$. What’s more, under this isomorphism, the state $\varphi_k$ conjugates to the vacuum expectation state $\tau(X) = \langle X \Omega, \Omega \rangle$.

Let $e_1, \ldots, e_k$ be an orthonormal basis for $\mathcal{H}$. The algebra $\mathcal{W}^*\{X(h) : h \in \mathcal{H}\} \cong L(F_k)$ is, of course, generated by the set $\{X(e_1), \ldots, X(e_k)\}$. It is important to note that the isomorphism does not carry the generators $u_1, \ldots, u_k$ in $F_k \subset L(F_k)$ to the generators $X(e_1), \ldots, X(e_k)$. Indeed, the two generating sets give two different, and important, families of non-commutative random variables: Haar unitary and semicircular elements, which we will discuss below. In both cases, the relationship between different generators is a model of a non-commutative version of independence called freeness.

2.3. Free cumulants and free independence. A normal random variable in a $C^*$-probability space is indistinguishable from a classical bounded complex random variable (indeed, one can construct a random variable with any given distribution $\nu$ as the identity function in the space $L^\infty(\nu)$.) The important classical notion of independence of random variables, however, has no direct analog for pairs of non-commuting random variables. The notion of free independence or freeness, introduced in [Vo] is a substitute, which is, in many ways, better.

Let $\pi = \{V_1, \ldots, V_r\}$ be a partition of the set $\{1, \ldots, n\}$. The partition is called crossing if for some $i \neq j$ there are numbers $p < q < p' < q'$ with $p, p' \in V_i$ and $q, q' \in V_j$. (Notation: we say
A partition $\pi$ of $\{1, \ldots, n\}$ is called non-crossing if $p \sim_\pi q$ for all $p, q$ in the same block of the partition $\pi$. Thus, $\pi$ is crossing iff there are $p < q < p' < q'$ with $p \sim_\pi p'$, $q \sim_\pi q'$, and $p' \sim_\pi q$. A non-crossing partition is one which is not crossing. We represent a partition by connecting numbers in the same block $V_i$ of the partition. The following figure gives four examples of non-crossing partitions of the set $\{1, \ldots, 6\}$.

![Non-crossing partitions](image)

**Figure 1.** Four elements of $NC(6)$, including the minimal and maximal elements $0_6$ and $1_6$.

The set of non-crossing partitions of $\{1, \ldots, n\}$, denoted $NC(n)$, is partially-ordered under reverse refinement. It is a lattice, in fact, with minimal element $0_n$ and maximal element $1_n$ as in Figure 1. The Möbius function $\mu_n$ of this lattice is well-known (see [Kr]). In particular, $\mu_n(0_n, 1_n) = (-1)^{n-1} C_{n-1}$, where $C_n$ are the Catalan numbers

$$C_n = \frac{1}{n} \binom{2n}{n-1}.$$  (2.2)

More generally, for any $\sigma \in NC(n)$,

$$|\mu_n(\sigma, 1_n)| \leq 4^{n-1}. \quad (2.3)$$

(The proof can be found contained in the proof of Proposition 13.15 in [NS3].) It is worth noting that $C_n \leq 4^n$ (and indeed $C_n \asymp 4^n$).

Let $(\mathcal{A}, \varphi)$ be a $C^*$-probability space. Let $n > 0$ and let $\pi$ be a partition in $NC(n)$. For each block $V = \{i_1, \ldots, i_k\}$ in $\pi$, define the function $\varphi_V : \mathcal{A}^n \to \mathbb{C}$ by $\varphi_V[a_1, \ldots, a_n] = \varphi(a_{i_1} \cdots a_{i_k})$. Then define $\varphi_\pi : \mathcal{A}^n \to \mathbb{C}$ by $\varphi_\pi[a_1, \ldots, a_n] = \prod_{V \in \pi} \varphi_V[a_1, \ldots, a_n]$. Finally, define the free cumulants of $(\mathcal{A}, \varphi)$ to be the functionals $\{\kappa_\pi : \pi \in NC(n) \text{ for some } n > 0\}$ by

$$\kappa_\pi[a_1, \ldots, a_n] = \sum_{\substack{\sigma \in NC(n) \\ \sigma \leq \pi}} \varphi_\sigma[a_1, \ldots, a_n] \mu_n(\sigma, \pi), \quad (2.4)$$

for each $\pi \in NC(n)$. An immediate consequence of this definition is that the moments can be recovered from the free cumulants,

$$\varphi_\pi[a_1, \ldots, a_n] = \sum_{\substack{\sigma \in NC(n) \\ \sigma \leq \pi}} \kappa_\sigma[a_1, \ldots, a_n].$$

(Indeed, this is the motivation for the inclusion of the coefficients $\mu_n(\sigma, \pi)$ in the definition of $\kappa_\pi$, for the Möbius function is the convolution-inverse of the Zeta-function for the lattice $NC(n)$.)
a special case, we have the formula

\[ \varphi(a_1a_2 \cdots a_n) = \sum_{s \in NC(n)} \kappa_{e(s)}[a_1, \ldots, a_n]. \quad (2.5) \]

Free cumulants allow a very easy statement of the definition of free independence, or freeness, of random variables. Let \( \kappa_{e(n)} \) denote the free cumulant \( \kappa_{e_1} \). (These cumulants in fact contain all information about the cumulants, since all others can be built up block-wise by multiplication.) Elements \( a_1, \ldots, a_n \) in \( \mathcal{A} \) are called free if, for \( j \geq 2 \) and \( 1 \leq i_1, \ldots, i_j \leq n \), \( \kappa_{e_j}[a_{i_1}, \ldots, a_{i_j}] = 0 \) whenever there is at least one pair \( 1 \leq \ell, m \leq j \) with \( i_\ell \neq i_m \). In other words, random variables are free if all their mixed free cumulants vanish.

One can calculate that the generators \( u_1, \ldots, u_n \) of \( \mathbb{F}_n \subset L(\mathbb{F}_n) \) are free, as are the generators \( X(e_1), \ldots, X(e_n) \) in the Fock-space representation of \( L(\mathbb{F}_n) \); hence, this notion generalizes freeness from the free group context. This approach mirrors the classical theory of cumulants in the probabilistic constructions work: given any countable list of probability measures \( \nu_j \), there is a \( C^* \) probability space in which there are free random variables with distributions \( \nu_j \) (one can construct the reduced free-product \( C^* \) algebra of the \( L^\infty(\nu_j) \), for example).

2.4. \( \mathcal{B} \)-diagonal elements. As commented above, the operators \( X(e_j) \) in the Fock-space representation of \( L(\mathbb{F}_n) \) are semicircular elements: \( s = X(e_j) \) has as distribution \( \nu_s \) with

\[ d\nu_s(t) = \frac{1}{2\pi} \sqrt{4 - t^2} \, dt. \]

Let \( s_1, s_2 \) be two free semicircular random variables. The operator \( c = (s_1 + is_2)/\sqrt{2} \) (where \( i = \sqrt{-1} \)) is called a circular element. It is non-normal, and so does not have a probability distribution. (Its Brown measure is known, however, to be the uniform measure on the closed unit disc in \( \mathbb{C} \).) The \( * \)-cumulants of a circular element (i.e. the free cumulants of tuples of operators all of the form \( c \) or \( c^* \)) have a particularly nice form. If \( \varepsilon_j \in \{1, *\} \) then \( \kappa_n[e^{\varepsilon_1}, \ldots, e^{\varepsilon_n}] = 0 \) for \( n \neq 2 \), and in fact only \( \kappa_2[c, c^*] = \kappa_2[c^*, c] = 1 \) are nonzero.

Consider also a generator \( u = u_j \) of \( \mathbb{F}_k \). Note that \( \varphi_k(u^n) = \delta_{n0} \), and the same holds true for \( u^* = u^{-1} \). The spectral measure of \( u \) is thus the Haar measure on the unit circle, and such random variables are called Haar unitary. The \( * \)-cumulants of a Haar unitary are not as restricted as those of a circular, but they follow a similar pattern. The only nonvanishing cumulants \( \kappa_n \) have \( n \) even, and must have alternating \( u \) and \( u^* \) arguments:

\[ \kappa_{2n}[u, u^*, \ldots, u, u^*] = \kappa_{2n}[u^*, u, \ldots, u^*, u] = (-1)^{n-1}C_{n-1}, \]

the same as the Möbius coefficients \( \mu_n(0_n, 1_n) \) of \( NC(n) \) (and this is no coincidence).

This connection between two widely known classes of non-selfadjoint random variables (circulars and Haar unitaries) motivated the second author, in [NS1], to introduce \( \mathcal{B} \)-diagonal elements. A random variable \( a \) in a \( C^* \)-probability space is \( \mathcal{B} \)-diagonal if its only nonvanishing cumulants are the alternating ones \( \kappa_{2n}[a, a^*, \ldots, a, a^*] \) and \( \kappa_{2n}[a^*, a, \ldots, a^*, a] \). (The notation \( \mathcal{B} \)-diagonal derives from a characterization of such elements in terms of the multivariate \( \mathcal{B} \)-transform, a combinatorial free version of the logarithmic Fourier transform in classical probability theory.)

Note that an \( \mathcal{B} \)-diagonal element’s odd cumulants vanish. (The term even element is used in this context, but it is usually formulated in terms of mixed moments, so we do not use it for \( \mathcal{B} \)-diagonal elements.) From Equations 2.4 and 2.5 we see vanishing of odd cumulants is equivalent to vanishing of odd moments. (A semicircular \( s \) is even: its mixed moments are just its moments.}
since it is self-adjoint, and, like a circular, only its second cumulant is nonzero: \( \kappa_n[s, \ldots, s] = \delta_{n2} \).

If \( a \) is \( \mathcal{R} \)-diagonal, its determining sequences are \( (\alpha_n[a])_{n=1}^{\infty} \) and \( (\beta_n[a])_{n=1}^{\infty} \) defined by

\[
\alpha_n[a] = \kappa_{2n}[a, a^*, \ldots, a, a^*], \\
\beta_n[a] = \kappa_{2n}[a^*, a, \ldots, a^*, a].
\]

(2.6)

If \( a \) is in a tracial probability space (better yet if \( \varphi \) restricted to the algebra generated by \( a \) and \( a^* \) is tracial), then \( \alpha_n[a] = \beta_n[a] \); in any case, these sequences contain all the information about the cumulants (and therefore mixed moments) of \( a \) and \( a^* \).

\( \mathcal{R} \)-diagonal elements form a large class of (mostly) non-normal elements about which a great deal is known. In a sense, they are non-normal analogues of rotationally invariant distributions in \( \mathbb{C} \); namely, the distribution of an \( \mathcal{R} \)-diagonal element is not changed if it is multiplied by a free Haar unitary. This results in a special polar decomposition and relations with maximization problems for free entropy [NS3, NSS, HP]. Our main theorem (1.3) supports the point of view that \( \mathcal{R} \)-diagonal elements can be considered as non-normal versions of holomorphic variables.

Finally, we comment that there is a precise description of the Brown measure of an \( \mathcal{R} \)-diagonal element in terms of its \( \mathcal{R} \)-transform (another formal power-series associated to the moments of \( a \)). The following theorem shows that \( \mathcal{R} \)-diagonal elements have rotationally-invariant Brown measures with nice densities. Let \( \times_p \) denote the polar Cartesian product (i.e. \( [x, y] \times_p [0, 2\pi] \) is the closed annulus with inner-radius \( x \) and outer-radius \( y \)).

**Theorem 2.1** (Corollary 4.5 in [HL]). If \( a \) is \( \mathcal{R} \)-diagonal (and is not a scalar multiple of a Haar unitary), then its Brown measure \( \nu_a \) is supported on \( (\|a^{-1}\|_2^{-1}, \|a\|_2) \times_p [0, 2\pi] \) if \( a \) is invertible, and on the disc \( [0, \|a\|_2] \times_p [0, 2\pi] \) if it is not. Moreover, \( \nu_a \) is rotationally-invariant with density

\[
dr a(r, \theta) = f(r) \, dr \, d\theta,
\]

where \( f \) is strictly positive on \( (\|a^{-1}\|_2^{-1}, \|a\|_2) \) or \( [0, \|a\|_2] \) and has an analytic continuation to a neighborhood of this interval in \( \mathbb{C} \).

### 3. Circul ar Elements

In this section, we prove Theorem 1.3 in the special case that \( a = c \) is circular. Our proof in Section 4 subsumes this one, but the techniques in this proof are new and interesting, and motivate the proof in what follows. In Section 3.1, we give a new combinatorial proof that the \( * \)-moments of the powers of a circular element are the Fuss-Catalan numbers, defined in Equation 3.5 below. (The main ideas of the construction in this section are due to Drew and Heather Armstrong, and we thank them for their contribution.) In Section 3.2, we use the asymptotics of the Fuss-Catalan numbers to demonstrate the strong Haagerup inequality for algebras generated by free circular elements.

#### 3.1. The powers of a circular element.

Let \( c \) be a (variance 1) circular element in a \( C^* \)-probability space \( (\mathscr{A}, \varphi) \). The moments of \( c^n \) were calculated first by Oravecz [O] and Larsen [Lar], each using a different approach to iterated free convolution of the \( \mathcal{R} \)-transform of \( c \). We will reproduce their results here, using more elementary combinatorial techniques.

From Equation 2.5, we have

\[
\varphi[(c^n (c^n)^*)^m] = \sum_{\pi \in NC(2nm)} \kappa_{\pi}[c_{n,m}],
\]

(3.1)

where \( c_{n,m} \) is the list

\[
c_{n,m} = \underbrace{c, \ldots, c}_{n}, \underbrace{c^*, \ldots, c^*}_{n}, \ldots, \underbrace{c, \ldots, c}_{n}, \underbrace{c^*, \ldots, c^*}_{n}
\]

(3.2)
Since \( c \) is circular, its only nonzero free cumulants are \( \kappa_2[c, c^*] = 1 \) and \( \kappa_2[c^*, c] = 1 \), hence the only nonzero terms in the above sum are those for which the partition \( \pi \) is a pair partition \( \pi \in NC_2(2mn) \) (each block is of size 2), and for which each \( c \) is paired to a \( c^* \) in \( c_{n,m} \). We call such pairings \(-\)-pairings, and denote the set of \(-\)-pairings in \( NC_2(2mn) \) by \( NC^*_2(n, m) \). Pictured below are two examples of elements in \( NC^*_2(3, 4) \).

![Figure 2. Two \(-\)-pairings in \( NC^*_2(3, 4) \).](image)

Since \( \kappa_\pi[c_{n,m}] = 1 \) whenever \( \pi \in NC^*_2(n, m) \) and is 0 otherwise, Equation 3.1 reduces to

\[
\|c^n\|_{2m}^{2m} = \sum_{\pi \in NC^*_2(n, m)} 1 = |NC^*_2(n, m)|.
\]  

(3.3)

A non-crossing partition can be represented linearly as in Figures 1 and 2, or equivalently on a circle, as seen below in Figure 3. As such, we can describe the problem of counting the elements in \( NC^*_2(n, m) \) in the following medieval terms:

**Knights and Ladies of the Round Table.** King Arthur’s Knights wish to bring their Ladies to a meeting of the Round Table. There are \( k = nm \) Knights (including Arthur himself) and each has one Lady. Arthur wishes to seat everyone so that men and women alternate in groups of \( n \), and in such a way that each Lady can converse with her Knight across the table without any conversations crossing. How many possible seating plans are there?

Letting \( c \) stand for “Knight” and \( c^* \) stand for “Lady,” the pictures in Figure 3 (which are the circular representations of the pairings from Figure 2) represent allowable seating plans.

A related counting problem asks for pairings of the pattern \( c_{n,m} \) where we relax the condition that each \( c \) must be paired to a \( c^* \), but still required that no two elements in a single \( n \)-block are paired together. Denote the set of all such non-crossing pairings as \( \mathcal{T}(n, m) \) (so \( NC^*_2(n, m) \subset \mathcal{T}(n, m) \)). As discussed in [BiS], this problem is the combinatorial counterpart to another moment problem, this time dealing with a semicircular element \( s \). Of course, since \( s \) is selfadjoint, \((s^n(s^*)^m)^m = s^{2nm}\), and calculating these moments is routine. Instead, the number of pairings in \( \mathcal{T}(n, m) \) equals the moment \( \varphi(T_n(s)^{2m}) \), where \( T_n \) are the Chebyshev polynomials. While we do not have a nice schema for calculating \( |\mathcal{T}(n, m)| \) explicitly (which we do for \( |NC^*_2(n, m)| \) below), functional calculus for selfadjoint operators immediately yields that \( \varphi(T_n(s)^{2m})^{1/2m} \to n + 1 \) as \( m \to \infty \) — the norm
$\|T_n(s)\|$ is linear in $n$, rather than in $\sqrt{n}$ as in Theorem 1.3 above. This difference in size precisely reflects the improvement of Haagerup’s inequality from $O(n)$ to $O(n^{1/2})$ behaviour for circular elements, and indeed for all $\mathcal{R}$-diagonal elements as discussed in Section 4.

As to the Knights and Ladies of the Round Table problem, let us introduce some notation which will be useful throughout what follows.

**Notation 3.1.** Label the entries in $c_{n,m}$ with decreasing indices $n$ through 1 in each block of $c$’s and increasing indices 1 through $n$ in each block of $c^*$’s.

\[ c_{n,m} = c_1, c_2, \ldots, c_{n-1}, c_n, c^*, c^*, \ldots, c^*, c_n, c_1, c_2, \ldots, c_{n-1}, c_n \]  \hfill (3.4)

We thus give each element of the list $c_{n,m}$ an address: $c(\ell, j)$ is the $c$ in the $\ell$th block of $c$’s, while $c^*(\ell, j)$ is the $c^*$ in the $\ell$th block of $c^*$’s.

**Lemma 3.2.** For $1 \leq j \leq n$, any $NC^*_2(n, m)$ must pair each $c$ to a $c^*$.

**Proof.** The number of $c$’s between $c(\ell, j)$ and $c^*(\ell', j')$ is $n|\ell - \ell'| + j$, while the number of $c^*$’s between them is $n|\ell - \ell'| + j'$. Let $\pi$ be a pairing which links $c(\ell, j)$ to (without loss of generality) $c(\ell', j')$ for some $j < j'$. Since the number of $c$’s between $c(\ell, j)$ and $c^*(\ell', j')$ is greater than the number of $c^*$’s between them, $\pi$ must match at least one $c(k, i)$ between $c(\ell, j)$ and $c^*(\ell', j')$ to $c^*(k', i')$ where $k' < \min\{\ell, \ell'\}$ or $k' > \max\{\ell, \ell'\}$. But then the blocks $\{c(\ell, j), c^*(\ell', j')\}$ and $\{c(k, i), c^*(k', i')\}$ in $\pi$ cross, and hence $\pi \not\in NC_2(2nm)$. Thus, $\pi \not\in NC^*_2(n, m)$. \hfill $\Box$

We may note further that any non-crossing pairing which respects the labels in Equation 3.4 is, in fact, a $*$-pairing, and so enumerating $NC^*_2(n, m)$ amounts to counting the non-crossing pairings which respect those labels. Using this observation, we proceed to define a bijection from $NC^*_2(n, m)$ to a set we can enumerate.

**Definition 3.3.** Let $\pi \in NC^*_2(n, m)$, and let $1 \leq j \leq n$. Say that $k, k' \in \{1, \ldots, m\}$ are $(\pi, j)$-**connected** if there are $1 \leq k_1, \ldots, k_r \leq m$ with $k_1 > k$ such that $c(k, j) \sim_\pi c^*(k_1, j)$, $c(k_1, j) \sim_\pi c^*(k_2, j)$, $\ldots$, and $c(k_r, j) \sim_\pi c^*(k', j)$. Similarly, say $k, k'$ are $(\pi^*, j)$-**connected** if there are $1 \leq k_1, \ldots, k_r \leq m$ with $k_1 < k$ such that $c^*(k, j) \sim_\pi c(k_1, j)$, $c^*(k_1, j) \sim_\pi c(k_2, j)$, $\ldots$, and $c^*(k_r, j) \sim_\pi c(k', j)$.
In other words, if we augment \( \pi \) by connecting each pair \( c(k, j), c^*(k, j) \), then \( k, k' \) are \((\pi, j)\)-connected if there is a (n initially increasing) path from \( c(k, j) \) to \( c^*(k', j) \) in the augmented pairing diagram; they are \((\pi^+, j)\)-connected if there is a (n initially decreasing) path from \( c^*(k, j) \) to \( c(k', j) \).

If we exclude the conditions \( k_1 > k \) in \( \pi \)-connectedness and \( k_1 < k \) in \( \pi^+ \)-connectedness, the two notions coincide (for example, 1 and 4 would be both \((\pi, 2)\)- and \((\pi^+, 2)\)-connected in Figure 4). We find it convenient to treat them separately, however.

**Lemma 3.4.** If \( k < k' \) are \((\pi, j)\)-connected, then the sequence \( k_1, \ldots, k_r, k' \) in definition 3.3 is decreasing. Likewise, if \( k < k' \) are \((\pi^+, j)\)-connected, then the sequence \( k_1, \ldots, k_r, k' \) in definition 3.3 is increasing.

**Proof.** If \( k < k' \) are \((\pi, j)\)-connected, we have \( k < k_1, c(k, j) \sim_\pi c^*(k_1, j) \) and \( c(k_1, j) \sim_\pi c^*(k_2, j) \). If \( k_2 > k_1 \), it follows that \( c(k, j) < c(k_1, j) < c^*(k_1, j) < c^*(k_2, j) \), and hence there is a crossing. The same argument applied at each pair \((k_\ell, k_{\ell+1})\) and at \((k_r, k')\) demonstrates the claim. The argument for \((\pi^+, j)\)-connectedness is similar. \( \square \)

**Definition 3.5.** Given \( \pi \in NC^*_2(n, m) \), define partitions \( \Phi^*_1, \ldots, \Phi^*_n \) of \( \{1, \ldots, m\} \) as follows: for \( k, k' \) in \( \{1, \ldots, m\} \), \( k \sim_{\Phi^*_j} k' \) iff \( k, k' \) are either \((\pi, j)\)-connected or \((\pi^+, j)\)-connected.

That is, \( \Phi^*_j \) is the image of \( \pi|_{\{c(k, j), c^*(j, k) : 1 \leq k \leq m\}} \) under the push-forward of the function \( f_j \) from \( \{c(j, k), c^*(j, k) : 1 \leq k \leq m\} \) to \( \{1, \ldots, m\} \) which maps \( c(j, k) \) and \( c^*(j, k) \) to \( k \). (Note that \( f_j \) is monotone.)

Figure 5 shows the partitions \( \Phi_j \) resulting from the \( * \)-pairings in 2; in it, we see that the \( \Phi_j \) are non-crossing, and moreover they are refinement-decreasing – in other words, they form a multichain (increasing sequence) in the lattice \( NC(4) \): \( \Phi_1 \leq \Phi_2 \leq \Phi_3 \). This holds generally for the \( \Phi^*_j \) corresponding to any \( \pi \in NC^*_2(n, m) \).

**Proposition 3.6.** Let \( \pi \in NC^*_2(n, m) \), and let \( \Phi^*_1, \ldots, \Phi^*_n \) be the partitions in Definition 3.5. Then the \( \Phi^*_j \) are in \( NC(m) \), and \( \Phi^*_1 \leq \cdots \leq \Phi^*_n \).

**Proof.** Since \( f_j \) is monotone increasing and \( \pi \) is non-crossing, \( \Phi^*_j = (f_j)_* \pi|_{\{c(k, j), c^*(j, k) : 1 \leq k \leq m\}} \) is non-crossing as well. Now, let \( 1 < j \leq n \), and suppose that \( k < k' \) are connected by \( \Phi^*_{j-1} \); thus, \( k \) and \( k' \) are either \((\pi, j-1)\)-connected or \((\pi^+, j-1)\)-connected.

Suppose \( k, k' \) are \((\pi, j-1)\)-connected, and let \( k_1, k_2, \ldots, k_r \) be a sequence connecting \( c(k, j-1) \) to \( c^*(k', j-1) \). By Lemma 3.4, \( k_1 > k_2 > \cdots > k_r > k' \). Note that \( c(k, j-1) < c^*(k, j) \), and so \( c^*(k, j) \)
must be paired to some \(c(\ell_1, j)\) with \(\ell_1 > k\) – otherwise \(c(\ell_1, j) < c(k, j-1) < c^*(k, j) < c^*(k_1, j-1)\) resulting in a crossing. If \(\ell_1 > k_1\) then there is a crossing at \(c(k, j-1) < c^*(k, j) < c^*(k_1, j-1) < c(\ell_1, j)\); hence \(\ell_1 \leq k_1\). Suppose that \(k' < \ell_1 < k\). Then there is a \(k_i\) with \(k_{i+1} \leq \ell_1 < k_i\), giving a crossing with \(c^*(k, j) < c^*(k_{i+1}, j-1) < c(\ell_1, j) < c(k_i, j-1)\). Hence, \(k < \ell_1 \leq k'\).

Inducting the previous argument, we find a chain \(k < \ell_1 < \ell_2 < \cdots\) with \(c^*(\ell_{i-1}, j) \sim c(\ell_i, j)\), and each \(\ell_i \leq k'\). Since there are only finitely many numbers between \(k\) and \(k'\), and since each \(c^*(\ell_i, j)\) must be paired to a \(c(\ell_{i+1}, j)\) with \(\ell_{i+1} > \ell_i\), it follows that \(\ell_i = k'\) for some \(i\). Thus, \(k, k'\) are \((\pi^*, j)\)-connected.

A similar argument shows that if \(k < k'\) are \((\pi^*, j-1)\)-connected then they are \((\pi, j)\)-connected. Hence, \(\Phi_{j-1}^\pi\) is a refinement of \(\Phi_j^\pi\), and so \(\Phi_{j-1}^\pi \leq \Phi_j^\pi\) in the lattice \(NC(m)\).

Denote by \(NC^{(n)}(m)\) the set of all multichains of length \(n\) in \(NC(m)\). Thus, Proposition 3.6 shows that the function \(\mathcal{P}: \pi \mapsto (\Phi_1^\pi, \ldots, \Phi_n^\pi)\) is a map \(NC^*_n(n, m) \to NC^{(n)}(m)\). In what follows, we will show that \(\mathcal{P}\) is a bijection. To do so, we exhibit its inverse.

To invert the above procedure for \(\Phi \in NC(m)\), the idea (heuristically) is to “fatten up” each connecting line on the right-hand side of Figure 5, and assign pairings by ignoring the top connections (which identify each \(c(k, j)\) with \(c^*(k, j)\)).

![Figure 6](image-url)
Proposition 3.8. Given an \( n \)-multichain \( \Phi_1 \leq \cdots \leq \Phi_n \) in \( \text{NC}^{(n)}(m) \), the pairing \( \pi_1^{\Phi_1} \sqcup \cdots \sqcup \pi_n^{\Phi_n} \) is in \( \text{NC}_2(n,m) \).

Note: the \( \sqcup \)'s above denote union of disjoint partial pairings.

Proof. First, note that \( \pi_j^{\Phi_j} \) is a refinement of the pull-back \( f_j^*\Phi_j \), and so, again since \( f_j \) is monotone and \( \Phi_j \) is non-crossing, \( \pi_j^{\Phi_j} \) is also non-crossing. Let \( k, k' \) be such that \( c(k, j) \sim c(k', j) \) in \( \pi_j^{\Phi_j} \), let \( j' > j \) be such that \( c(\ell, j') \sim c^*(\ell', j') \), and suppose there is a crossing between \( \{c(k, j), c^*(k', j)\} \) and \( \{c(\ell, j'), c^*(\ell', j')\} \). There are eight possible arrangements – we only treat the case \( k < \ell < k' < \ell' \), and note the others may be treated similarly. So, \( c(k, j) < c(\ell, j') < c^*(k', j) < c^*(\ell', j') \). Since \( j' > j \), we have also \( c(k, j) < c(\ell, j) < c^*(k', j) < c^*(\ell', j) \), and as \( \Phi_j \) is a refinement of \( \Phi_{j'} \), \( \ell \sim_{\Phi_j} \ell' \) as well. Thus there is a crossing in \( \Phi_j \), which is a contradiction. Hence, there are no crossings between \( \pi_j^{\Phi_j} \) and \( \pi_{j'}^{\Phi_{j'}} \) for any \( 1 \leq j < j' \leq n \), and it follows that \( \pi_1^{\Phi_1} \sqcup \cdots \sqcup \pi_n^{\Phi_n} \) is in \( \text{NC}_2(2n,m) \). By construction, it is a \( \sim \)-pairing, and so it is in \( \text{NC}_2(n,m) \).

Hence, the map \( \mathcal{D} : \text{NC}^{(n)}(m) \to \text{NC}_2^*(n,m) \) defined by \( \mathcal{D}(\Phi_1, \ldots, \Phi_n) = \pi_1^{\Phi_1} \sqcup \cdots \sqcup \pi_n^{\Phi_n} \) is a well-defined function. In fact, it is the inverse of \( \mathcal{P} \).

Proposition 3.9. The maps \( \mathcal{P} : \text{NC}_2^*(n,m) \to \text{NC}^{(n)}(m) \) and \( \mathcal{D} : \text{NC}^{(n)}(m) \to \text{NC}_2^*(n,m) \) are inverses of each other.

Proof. Let \( \pi \in \text{NC}_2^*(n,m) \), and suppose that \( c(k, j) \sim c^*(k', j) \). Then \( k, k' \) are in the same block of \( \Phi = \Phi_j^\pi \), and by Definition 3.7, \( c(k, j) \) and \( c^*(k', j) \) are connected in \( \pi_j^\Phi \). Hence, \( \pi \) is a refinement of \( \mathcal{D} \circ \mathcal{P}(\pi) \). On the other hand, suppose \( c(k, j) \) and \( c(k', j) \) are paired by \( \mathcal{D} \circ \mathcal{P}(\pi) \). Then \( \mathcal{P}(\pi) = (\Phi_1, \ldots, \Phi_n) \), where \( k, k' \) are in the same block \( V \) of \( \Phi_j \), and moreover \( k, k' \) are adjacent in the list \( V = \{k_1, \ldots, k_r\} \) since, by Definition 3.7, \( \mathcal{D} \) only creates pairings from adjacent elements of each block. So, by definition 3.5, \( k, k' \) are either \( (\pi, j) \)-connected or \( (\pi^*, j) \)-connected. In either case, if the path connecting them were of length greater than 1 then \( k, k' \) would not be adjacent in the block \( V \), since the sequence connecting them is monotone by Lemma 3.4. Hence, \( k, k' \) are, in fact, connected in \( \pi \). This demonstrates that \( \mathcal{D} \circ \mathcal{P}(\pi) \) is a refinement of \( \pi \), and so \( \mathcal{D} \circ \mathcal{P}(\pi) = \pi \).

Now, let \( (\Phi_1, \ldots, \Phi_n) \in \text{NC}^{(n)}(m) \), and let \( (\Psi_1, \ldots, \Psi_n) = \mathcal{P} \circ \mathcal{D}(\Phi_1, \ldots, \Phi_n) \). If \( k \sim_{\Phi_j} k' \) for \( k < k' \), then there is a block \( V \) of \( \Phi_j \) including \( k, k' \): \( V = \{k_1 < \cdots < k_r < k < k_{r+1} < \cdots < k_s < k_{s+1} < \cdots < k_t\} \). Then \( \pi_j^{\Phi_j} \) includes the pairings \( c(k, j) \sim c^*(k_{r+1}, j), \) \( c(k_{r+1}, j) \sim c(k_{r+2}, j), \ldots, c(k_s, j) \sim c^*(k', j) \); in particular, letting \( \pi = \mathcal{D}(\Phi_1, \ldots, \Phi_n) \), we have a path \( (\pi, j) \)-connecting \( k \) and \( k' \). Hence, by Definition 3.5, \( k \sim_{\Psi_j} k' \), and so \( \Phi_j \) is a refinement of \( \Psi_j \) for each \( j \). Conversely, if \( k \sim_{\Psi_j} k' \), then \( k, k' \) are either \( (\pi, j) \)-connected or \( (\pi^*, j) \)-connected. Hence, there is a path connecting \( k \) to \( k' \) in \( \pi_j^{\Phi_j} \), and so, by the action of \( \mathcal{D}, k \) and \( k' \) must lie in the same block of \( \Phi_j \) – i.e. \( k \sim_{\Phi_j} k' \). This shows that \( \Psi_j \) is a refinement of \( \Phi_j \) for each \( j \), and so we have shown that \( \Psi_j = \Phi_j - i.e. \mathcal{P} \circ \mathcal{D} = id_{\text{NC}^{(n)}(m)} \).

At this point, we have reproduced the results of Larsen using the above constructive approach. The set \( \text{NC}^{(n)}(m) \) is a well-studied combinatorial structure, and its enumeration was calculated by Edelman in [E]. The next result follows.

Corollary 3.10. For all positive integers \( n \) and \( m \), the number of \( \sim \)-pairings \( |\text{NC}_2^*(n,m)| \) is equal to \( |\text{NC}^{(n)}(m)| = C_m^{(n)} \), where \( C_m^{(n)} \) are the Fuss-Catalan numbers

\[
C_m^{(n)} = \frac{1}{m} \binom{m(n+1)}{m-1}.
\]
similar context in [BJ], where the central objects of study, the \textit{Fuss-Catalan algebras} (a generalization of the Temperley-Lieb algebras) are generated by diagrams like Figure 3, and hence the dimensions of the algebras (the number of essentially different such diagrams) are the numbers $C_m(n)$.

3.2. The Haagerup inequality in $\mathcal{H}(c, I)$. From Equation 3.3 and Corollary 3.10, we have calculated the $2m$-norms of the powers of a circular element,

$$\|c^n\|_{2m} = \left[ C_m(n) \right]^{1/2m} = \left[ \frac{1}{m} \left( \frac{m(n+1)}{m-1} \right) \right]^{1/2m}.$$  \hspace{1cm} (3.6)

In particular, the 2-norm is $\|c^n\|_2 = 1$. We can calculate the norm $\|c^n\|$ by taking the limit as $m \to \infty$, which may be computed using Stirling’s formula. The result is

$$\|c^n\|^2 = \lim_{m \to \infty} \left[ \frac{1}{m} \left( \frac{m(n+1)}{m-1} \right) \right]^{1/m} = \left( 1 + \frac{1}{n} \right)^n (n+1) \leq e(n+1).$$

Now, in line with Theorem 1.3, consider the algebra $\mathcal{H}(c) = \mathcal{H}(c, \{1\})$, the norm-closed algebra generated by $c$. In this case, the $n$-particle space $\mathcal{H}(n)(c)$ is spanned by $c^n$, and hence Equation 3.7 immediately yields the following strong Haagerup inequality.

**Proposition 3.11.** For $n \geq 0$ and $T \in \mathcal{H}(n)(c)$,

$$\|T\| \leq \sqrt{e} \sqrt{n+1} \|T\|_2.$$  

In fact, we can use similar techniques to achieve the same inequality for the algebra $\mathcal{H}(c, I)$ for any countable indexing set $I$. This jump, from 1 to many (even infinite) dimensions is usually the hardest part of such analyses; we will see below that the freeness does all the work for us. Note, the algebra $\mathcal{H}(c, I)$ is canonically isomorphic to the 0-holomorphic space $\mathcal{H}_0(\mathcal{H}_c)$ in [Ke] and the free Segal-Bargmann space $\mathcal{G}_{\text{hol}}(\mathcal{H})$ in [Bi], where $\mathcal{H}_c$ is a complex Hilbert space of dimension $|I|$.

Let $T \in \mathcal{H}(n)(c, I)$, so that $T = \sum_{|i| = n} \lambda_i c_i$ for some scalars $\lambda_i \in \mathbb{C}$ satisfying a summability condition guaranteeing that $\|T\|_2 < \infty$ (see Equation 3.10 below), where $c_i = c_{i_1} \cdots c_{i_n}$. By the definition of $\mathcal{H}(c, I)$, the generating elements $c_{i_j}$ are variance 1 and $c_{i_k}, c_{i_{k'}}$ are *-free whenever $i_k \neq i_{k'}$. Then we have the following multinomial expansion for the $2m$th moment of $|T|$:

$$\|T\|_{2m}^2 = \varphi[(TT^*)^m] = \sum_{|i(1)| = \cdots = |i(m)| = n} \lambda_{i(1)} \cdots \lambda_{i(m)} \overline{\lambda_{j(1)}} \cdots \overline{\lambda_{j(m)}} \varphi \left( c_{i(1)} c_{j(1)}^* \cdots c_{i(m)} c_{j(m)}^* \right).$$

In particular, setting $m = 1$,

$$\|T\|^2 = \sum_{|i| = |j| = n} \lambda_i \lambda_j \varphi \left( c_i c_j^* \right).$$

The expression $\varphi(c_i c_j^*)$ is a mixed moment of length $2n$, and can (by Equation 2.5) be expressed in terms of the cumulants of the $c_i$:

$$\varphi \left( c_i c_j^* \right) = \sum_{\pi \in NC(2n)} \kappa_{\pi} [c_{i_1}, \ldots, c_{i_n}, c_{j_1}^*, \ldots, c_{j_n}^*].$$

As the $c_i$ are circular (and so only the cumulants $\kappa_2[c, c^*] = \kappa_2[c^*, c] = 1$ are nonzero), only pair partitions $\pi$ which match $c$’s to $c^*$’s contribute to the sum. Any such partition is in $NC_2(n, 1)$, which contains only the partition $\varpi$.
\[ \mathcal{W} = \begin{array}{c|c|c|c|c|c|c|c} \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \hline \end{array} \]

(the fact that there is only one follows from the calculation in Section 3.1 that \( |NC_2^* (n, 1)| = C_1^{(n)} = 1 \). So, we have

\[ \|T\|_2^2 = \sum_{|i| = |j| = n} \lambda_i \lambda_j \kappa_{\mathcal{W}} [c_1, c_j^*]. \]  

(3.9)

A note on notation: in Equation 3.9, the \( c_1 \) and \( c_j^* \) stand for lists of length \( n \), not products of \( n \) elements; i.e. there are implied commas. We will use this convention whenever such expressions appear as arguments of cumulants in what follows. To be clear, for the pairing \( \mathcal{W} \) above, we have

\[ \kappa_{\mathcal{W}} [c_1, c_j] = \kappa_{\mathcal{W}} [c_{i_1}, \ldots, c_{i_n}, c_{j_1}^*, \ldots, c_{j_n}^*] = \kappa_2 [c_{i_1}, c_{j_1}^*] \cdot \kappa_2 [c_{i_2}, c_{j_2}^*] \cdots \kappa_2 [c_{i_n}, c_{j_n}^*]. \]

Now following Equation 3.9, since the \( c_{i_\ell} \) are \( * \)-free, \( \kappa_{\mathcal{W}} [c_1, c_j^*] = 0 \) unless each block of \( \mathcal{W} \) contains like-indexed elements – i.e. unless \( i = j \), in which case \( \kappa_{\mathcal{W}} = 1 \). Thus, we have the Pythagorean formula

\[ \|T\|_2^2 = \sum_{|i| = n} |\lambda_i|^2. \]  

(3.10)

Following suit, for general \( m > 1 \) we have

\[ \varphi \left( c_{i(1)} c_{j(1)}^* \cdots c_{i(m)} c_{j(m)}^* \right) = \sum_{\pi \in NC(2nm)} \kappa_\pi [c_{i(1)}, c_{j(1)}^*, \ldots, c_{i(m)}, c_{j(m)}^*]. \]

Once again, since the \( c_{i(k)\ell} \) are circular elements, the only partitions \( \pi \) which contribute to the sum are those which pair \( c \)'s with \( c^* \)'s – i.e. \( \pi \in NC_2^* (n, m) \). This, with Equation 3.8, yields

\[ \|T\|_{2m}^2 = \sum_{\pi \in NC_2^* (n, m)} \sum_{|i(1)| = \ldots = |i(m)| = n} \sum_{|j(1)| = \ldots = |j(m)| = n} \lambda_{i(1)} \cdots \lambda_{i(m)} \overline{\lambda_{j(1)}} \cdots \overline{\lambda_{j(m)}} \kappa_\pi [c_{i(1)}, c_{j(1)}^*, \ldots, c_{i(m)}, c_{j(m)}^*]. \]

Many of the above terms are in fact 0, since the \( c_{i(k)\ell} \) are \( * \)-free. Indeed, the mixed cumulant \( \kappa_\pi \) in the above sum is nonzero only when the indices of terms paired by \( \pi \) are all equal (and in this case it is 1). We record this with the function \( \delta(\pi, i(1), j(1), \ldots, i(m), j(m)) \) defined to equal 0 whenever \( \pi \) pairs any \( c_{i(k)\ell} \) with a \( c_{j(k')\ell}^* \), with \( i(k) \neq j(k') \), and 1 if \( \pi \) always pairs like-indexed \( c \)'s and \( c^* \)'s. Thus

\[ \|T\|_{2m}^2 = \sum_{\pi \in NC_{2m}^* (n, m)} \sum_{|i(1)| = \ldots = |i(m)| = n} \sum_{|j(1)| = \ldots = |j(m)| = n} \lambda_{i(1)} \cdots \lambda_{i(m)} \overline{\lambda_{j(1)}} \cdots \overline{\lambda_{j(m)}} \delta(\pi, i(1), j(1), \ldots, i(m), j(m)). \]

Now, let us re-index the above sum. Denote the indices \( \{i(1), \ldots, i(m)\} \) by \( p_1, \ldots, p_{nm} \), and let \( \lambda(p_1, \ldots, p_{nm}) = \lambda_{i(1)} \cdots \lambda_{i(m)} \). Note, in any nonzero term in the above sum, the indices appearing in the product \( \overline{\lambda_{j(1)}} \cdots \overline{\lambda_{j(m)}} \) are exactly those paired to \( p_1, \ldots, p_{nm} \) by \( \pi \); identifying the pairing \( \pi \) with its corresponding permutation, we then have

\[ \|T\|_{2m}^2 = \sum_{\pi \in NC_{2m}^* (n, m)} \sum_{p_1, \ldots, p_{nm}} \lambda(p_1, \ldots, p_{nm}) \overline{\lambda(p_{\pi(1)}, \ldots, p_{\pi(nm)})}. \]  

(3.11)
Applying the Cauchy-Schwarz inequality to the interior summation yields, for each $\pi$,

$$\sum_{p_1,\ldots,p_{nm}} \lambda(p_1,\ldots,p_{nm}) \overline{\lambda(p_{\pi(1)},\ldots,p_{\pi(nm)})} \leq \left[ \sum_{p_1,\ldots,p_{nm}} |\lambda(p_1,\ldots,p_{nm})|^2 \right]^{1/2} \cdot \left[ \sum_{p_1,\ldots,p_{nm}} |\lambda(p_{\pi(1)},\ldots,p_{\pi(nm)})|^2 \right]^{1/2}.$$

Since the sum is over all $nm$-tuples of indices and $\pi$ is a permutation, the second term may be reordered to cancel the apparent $\pi$-dependence, yielding the same summation in both factors; i.e. the interior sum in Equation 3.11 is just

$$\sum_{p_1,\ldots,p_{nm}} |\lambda(p_1,\ldots,p_{nm})|^2.$$

Returning to our original indexing scheme, this becomes

$$\sum_{p_1,\ldots,p_{nm}} |\lambda(p_1,\ldots,p_{nm})|^2 = \sum_{|i|=n} |\lambda_i|^2 m^{\sum_{|i|=n} |\lambda_i|^2} \leq \left[ \sum_{|i|=n} |\lambda_i|^2 \right]^{m} = \sum_{\pi \in NC^*_2(n,m)} \|T\|_{2m}^2 C_m^{(n)} \|T\|_{2m}^2.$$

Taking $m$th roots and letting $m \to \infty$, referring to the same limit calculated in Equation 3.7, we have thus proved the main theorem of this section:

**Theorem 3.12.** Let $c$ be a variance 1 circular, and let $T \in H^{(n)}(c,I)$ for some countable index set $I$. Then

$$\|T\| \leq \sqrt{e} \sqrt{n + 1} \|T\|_2.$$

We note that this inequality (with the $\sqrt{n + 1}$ factor) bears some resemblance to what Bożejko called Nelson’s inequality in [Bo1]. The context of his inequality is different, however (his estimate is for the creation and annihilation operators on the full Fock space separately), and our result cannot be derived from his.

### 4. $\mathcal{A}$-diagonal Elements

In this section, we extend the techniques developed in Section 3 to all $\mathcal{A}$-diagonal elements. A similar reduction of the multidimensional case to the one-dimensional case is possible, but there is an obstruction: the main argument goes through only when the mixed cumulants are non-negative. We address this problem by replacing an $\mathcal{A}$-diagonal element with negative cumulants with a different $\mathcal{A}$-diagonal whose cumulants are positive and dominate the original’s.

In Section 4.1, we calculate the 2-norm of an element $T$ in the $n$-particle space, and develop the main estimate (which generalizes the proof of Theorem 3.12) of higher moments of $\|T\|$ in terms of the absolute values of the cumulants. Then, in Section 4.2, we show how to replace a given $\mathcal{A}$-diagonal element with a different one who cumulants dominate the absolute values of the original’s, and use this substitution to prove Theorem 1.3.
4.1. Estimating moments for \( T \in \mathcal{H}(n)(a, I) \). Let \( a \) be an \( \mathcal{R} \)-diagonal element in a \( C^* \)-probability space, and let \( T \in \mathcal{H}(n)(a, I) \). So, \( T = \sum_{i=1}^{n} \lambda_i a_i \) for some scalars \( \lambda_i \in \mathbb{C} \), where \( \{ a_i : i \in I \} \) are \( * \)-free \( \mathcal{R} \)-diagonal elements each with the same \( * \)-distribution as \( a \). As in Equation 3.8 above, we have the following multinomial expansion for the \( 2m \)th moment of \( |T| \):

\[
\| T \|_{2m}^2 = \varphi[(TT^*)^m] = \sum_{|i(1)|=\ldots=|i(m)|=n} \lambda_i(1) \cdots \lambda_i(m) \varphi\left( a_i(1) a_j^*(1) \cdots a_i(m) a_j^*(m)\right). \tag{4.1}
\]

The term \( \varphi(a_i(1) a_j^*(1) \cdots a_i(m) a_j^*(m)) \) can be calculated, via Equation 2.5, as

\[
\varphi\left( a_i(1) a_j^*(1) \cdots a_i(m) a_j^*(m)\right) = \sum_{\pi \in NC(2nm)} \kappa_{\pi}[a_i(1), a_j^*(1), \ldots, a_i(m), a_j^*(m)].
\]

Since the \( a_i(k) \) are \( * \)-free, the above mixed cumulant is nonzero only when the indices of terms connected by \( \pi \) are all equal. We record this with the function \( \delta(\pi, i(1), j(1), \ldots, i(m), j(m)) \) defined above, which equals 0 whenever \( \pi \) connects two differently-indexed elements, and 1 if all connected elements have like-indices. It is, then, true that

\[
\varphi\left( a_i(1) a_j^*(1) \cdots a_i(m) a_j^*(m)\right) = \sum_{\pi \in NC(2nm)} \kappa_{\pi}[a_i(1), a_j^*(1), \ldots, a_i(m), a_j^*(m)] \delta(\pi, i(1), j(1), \ldots, i(m), j(m)).
\]

In the special case \( m = 1 \), this reduces to

\[
\varphi\left( a_i a_j^*\right) = \sum_{\pi \in NC(2n)} \kappa_{\pi}[a_i, a_j^*] \delta(\pi, i, j). \tag{4.2}
\]

Now, let \( \pi \) be a partition with \( \delta(\pi, i, j) = 1 \). Thus, each block of \( \pi \) connects only terms with a single index \( i \). Since \( a_i \) is \( \mathcal{R} \)-diagonal, its only nonzero \( * \)-cumulants are \( \kappa_{2n}[a_i, a_i^*, \ldots, a_i, a_i^*] \) and \( \kappa_{2n}[a_i^*, a_i, \ldots, a_i^*, a_i] \). Hence, \( \pi \) still contributes a zero in Equation 4.2 unless, in each block of \( \pi \), the \( a_i \)’s and \( a_i^* \)’s alternate. But in this case (\( m = 1 \)), all the \( a_i^* \)’s are to the right of all the \( a_i \)’s, and hence alternating sequences have length at most 2. So \( \pi \) contributes only if it is a pair partition. Since the cumulants \( \kappa_2[a_j, a_j] = \kappa_2[a_j^*, a_j^*] = 0 \) for each \( j \), such a \( \pi \) only pairs \( * \)’s to non-\( * \)’s, and so \( \pi \) is actually a \( * \)-pairing: \( \pi \in NC^*_2(n, 1) \). As shown in Section 3.2, the only element of \( NC^*_2(n, 1) \) is \( \varpi \). So the sum in Equation 4.2 reduces to at most a single term,

\[
\varphi\left( a_i a_j^*\right) = \kappa_{\varpi}[a_i, a_j^*] \delta(\varpi, i, j).
\]

Since \( a_i = a_i \cdots a_i \) and \( a_j^* = a_j^* \cdots a_j^* \), \( \delta(\varpi, i, j) = 1 \) if \( i = j \), and in this case, \( \kappa_{\varpi}[a_i, a_i^*] \) is equal to the product \( \kappa_2[a_i, a_i^*] \cdots \kappa_2[a_i, a_i^*] \) which (since the \( a_i \) are identically distributed) equals \( \kappa_2[a, a^*]^n \). So Equation 4.1 yields

\[
\| T \|^2_2 = \sum_{|i|=|j|=n} \lambda_i \lambda_j |a_i a_j^*| = \sum_{|i|=n} |\lambda_i|^2 \kappa_2[a, a^*]^n.
\]

Finally, we note that the second cumulant of a centred random variable is equal to its second moment (in general we may easily calculate that \( \kappa_2[a, a^*] = \text{Var}(a) \)), and since \( \mathcal{R} \)-diagonal elements have vanishing first moment, it follows that

\[
\| T \|^2_2 = \sum_{|i|=n} |\lambda_i|^2 \| a \|^2_2. \tag{4.3}
\]

Similar considerations are not enough to explicitly calculate higher moments, since alternating sequences can have greater length (e.g. in \( \| T \|^3_2 \), terms corresponding to partitions with blocks of sizes 2 and 4 may contribute), and calculations become unwieldy very quickly. Nevertheless, we
can estimate the higher norms using only pair partitions, to great effect. In general, from Equation 4.1 we have
\[
\|T\|_{2m}^2 = \sum_{\pi \in NC(2mn)} \sum_{\|i(1)\| = \cdots = \|i(m)\| = n} \lambda_i(1) \cdots \lambda_i(m) \overline{\lambda_j(1)} \cdots \overline{\lambda_j(m)} \cdot \vartheta[\pi, i(1), j(1), \ldots, i(m), j(m)],
\]
where
\[
\vartheta[\pi, i(1), j(1), \ldots, i(m), j(m)] = \kappa_\pi[a_i(1), a^*_i(1), \ldots, a_i(m), a^*_i(m)] \delta(\pi, i(1), j(1), \ldots, i(m), j(m)).
\]

Now, in any term where \(\delta(\pi, i(1), j(1), \ldots, i(m), j(m)) = 1\), each block of \(\pi\) connects only \(a_i\)'s and \(a^*_i\)'s for a single index \(i\). Since \(a_i\) is \(\mathbb{R}\)-diagonal, its only nonvanishing \(*\)-cumulants are alternating, and so the term is zero unless \(a\)'s and \(a^*\)'s alternate within each block of \(\pi\). This is an important set of non-crossing partitions; we call it \(NC^*(n, m)\) (so \(NC^*_2(n, m)\) is the subset of \(NC^*(n, m)\) consisting of only pair partitions). It is important to note that, as per our definition of alternating, the size of each block of a partition in \(NC^*(n, m)\) must be even. (The sequence \(a, a^*, \ldots, a, a^*, a\) is not alternating in our sense, since an \(\mathbb{R}\)-diagonal element still has vanishing cumulants for this list.)

Using this notation, the above summation becomes
\[
\|T\|_{2m}^2 = \sum_{\pi \in NC^*(n, m)} \sum_{\|i(1)\| = \cdots = \|i(m)\| = n} \lambda_i(1) \cdots \lambda_i(m) \overline{\lambda_j(1)} \cdots \overline{\lambda_j(m)} \cdot \vartheta[\pi, i(1), j(1), \ldots, i(m), j(m)].
\]

Fix \(i(1), j(1), \ldots, i(m), j(m)\), and let \(\pi \in NC^*(n, m)\) be such that \(\delta(\pi, i(1), j(1), \ldots, i(m), j(m)) = 1\). Let \(\{V_1, \ldots, V_k\}\) be the blocks of \(\pi\). Since all indices of elements in a single block \(V_j\) are equal (to, say, \(i\)), and since \(a_i\) has the same distribution as \(a\), we have that \(\varphi_{V_j}[a_i(1), a^*_i(1), \ldots, a_i(m), a^*_i(m)] = \varphi_{V_j}[a_{n,m}]\), where
\[
a_{n,m} = \underbrace{a, \ldots, a}_{n}, \underbrace{a^*, \ldots, a^*}_{n}, \underbrace{a, \ldots, a}_{n}, \underbrace{a^*, \ldots, a^*}_{n}
\]
is independent of the indices. Consequently, we have (for \(\pi\) with \(\delta(\pi, i(1), j(1), \ldots, i(m), j(m)) = 1\))
\[
\kappa_\pi[a_i(1), a^*_i(1), \ldots, a_i(m), a^*_i(m)] = \kappa_\pi[a_{n,m}]. \quad (4.4)
\]

Thus, for \(\pi \in NC^*(n, m)\), we have
\[
\vartheta[\pi, i(1), j(1), \ldots, i(m), j(m)] = \kappa_\pi[a_{n,m}] \delta(\pi, i(1), j(1), \ldots, i(m), j(m)),
\]
and so
\[
\|T\|_{2m}^2 = \sum_{\pi \in NC^*(n, m)} \kappa_\pi[a_{n,m}] \sum_{\|i(1)\| = \cdots = \|i(m)\| = n} \lambda_i(1) \cdots \lambda_i(m) \overline{\lambda_j(1)} \cdots \overline{\lambda_j(m)} \delta(\pi, i(1), j(1), \ldots, i(m), j(m)).
\]

We now estimate this sum by associating to each \(\pi \in NC^*(n, m)\) a refinement \(\pi_r \in NC^*_2(n, m)\) as follows: for each block \(V = \{k_1 < k_2 < \cdots < k_{2\ell}\}\) in \(\pi\), the pairings \(k_1 \sim k_2, k_3 \sim k_4, \ldots, k_{2\ell-1} \sim k_{2\ell}\) are in \(\pi_r\).

Since \(\pi_r\) is a refinement of \(\pi\), if \(\pi\) only connects like-indexed elements then \(\pi_r\) does as well, and so \(\delta(\pi, i(1), j(1), \ldots, i(m), j(m)) \leq \delta(\pi_r, i(1), j(1), \ldots, i(m), j(m))\). Hence, we may estimate (by taking absolute values)
\[
\|T\|_{2m}^2 \leq \sum_{\pi \in NC^*(n, m)} |\kappa_\pi[a_{n,m}]| \sum_{\|i(1)\| = \cdots = \|i(m)\| = n} |\lambda_i(1) \cdots \lambda_i(m) \cdot \overline{\lambda_j(1)} \cdots \overline{\lambda_j(m)}| \delta(\pi_r, i(1), j(1), \ldots, i(m), j(m)).
\]
We can now reindex the interior sum the same way we did in Section 3.2: denote the indices \(\{i(1), \ldots, i(m)\}\) by \(p_1, \ldots, p_{nm}\), and this time let \(\lambda(p_1, \ldots, p_{nm}) = |\lambda_{i(1)} \cdots \lambda_{i(m)}|\). Then allowing \(\pi_r\) to refer both to the pair-partition and the associated permutation, we have

\[
\sum_{|\pi|=n} |\lambda_{i(1)} \cdots \lambda_{i(m)}| \leq \left( \sum_{|\pi|=n} \lambda(p_1, \ldots, p_{nm}) \right)^{1/2} \left( \sum_{|\pi|=n} \lambda(p_{\pi_1(1)}, \ldots, p_{\pi_{nm}(m)}) \right)^{1/2},
\]

where we have applied the Cauchy-Schwarz inequality. Since the sum is over all indices \(p_1, \ldots, p_{nm}\) and since \(\pi_r\) is a permutation, the second term above can be reindexed to yield the first term, and hence the interior sum is

\[
\leq \sum_{p_1, \ldots, p_{nm}} \lambda(p_1, \ldots, p_{nm})^2 = \sum_{|i|=n} |\lambda_i|^{2m} \leq \left[ \sum_{|i|=n} |\lambda_i|^2 \right]^{m}.
\]

Combining this with Equation 4.3 yields the following estimate, which is the main lemma of this section.

**Lemma 4.1.** Let \(T \in H(n^{m}(a, I))\) for a \(\mathcal{R}\)-diagonal. Then for \(m \geq 1\),

\[
\|T\|_{2m} \leq \left[ \sum_{\pi \in NC^*(n, m)} |\kappa_{\pi}[a_{n,m}]| \right]^{1/2m} \frac{1}{\|a\|^{2}} \|T\|_{2}.
\]

If the cumulants of \(a\) are all non-negative, then \(\kappa_{\pi}[a_{n,m}] \geq 0\) as well, and the above summation reduces to a one-dimensional calculation.

**Corollary 4.2.** If the cumulants of \(a\) are non-negative, then \(\|T\| \leq \frac{\|a^n\|}{\|a\|^{2}} \|T\|_{2}\).

**Proof.** By Equation 2.5,

\[
\|a^n\|_{2m}^{2m} = \varphi(|a^n(a^*)|)^m = \sum_{\pi \in NC(2nm)} k_{\pi}[a_{n,m}].
\]
As explained above, since $a$ is $\mathcal{R}$-diagonal, $\kappa_\pi[a_{n,m}] = 0$ unless $\pi \in NC^*(n,m)$. Thus, from Lemma 4.1, we have

$$
\|T\|_{2m}^2 \leq \sum_{\pi \in NC^*(n,m)} \|\kappa_\pi[a_{n,m}]\| \left\| \kappa_\pi[a_{n,m}] \right\|_2 \|a\|_2^{2m} = \sum_{\pi \in NC(2nm)} \|\kappa_\pi[a_{n,m}]\| \left\| \kappa_\pi[a_{n,m}] \right\|_2 \|a\|_2^{2m} = \frac{1}{\|a\|_2^{2m}} \|T\|_2^{2m} = \frac{1}{\|a\|_2^{2nm}} \|T\|_2^{2m}.
$$

The result now follows by taking $2m$th roots, and letting $m$ tend to $\infty$. \hfill \square

Hence, in this case, the question of Haagerup’s inequality is reduced to determining the growth-rate of $\|a^n\|/\|a\|_2^n$, which was addressed in [Lar] (and will be discussed in the next section). However, if some cumulants of $a$ are negative, we must work harder to make such an estimate.

4.2. **Strong Haagerup inequalities.** To reduce the calculation in Section 4.1 to the one-dimensional case when $a$ can have negative cumulants, our strategy is to replace $a$ with a different $\mathcal{R}$-diagonal element $b$ whose cumulants are positive and dominate the absolute values of $a$’s cumulants. We will do this in a way that allows close control of both $\|b\|$ and $\|b\|_2$.

To begin, we bound the growth of the nonvanishing cumulants of $a$.

**Lemma 4.3.** Let $a$ be an $\mathcal{R}$-diagonal element in a $C^*$-probability space. Then the nonvanishing cumulants of $a$ satisfy

$$
|\alpha_n[a]|, |\beta_n[a]| \leq \frac{1}{2} (2^4 \|a\|)^{2n},
$$

where $\alpha_n[a]$ and $\beta_n[a]$ are the determining sequences of $a$ from Equation 2.6.

**Proof.** From Equation 2.4, we have

$$
\alpha_n[a] = \kappa_{2n}[a, a^*, \ldots, a, a^*] = \sum_{\sigma \in NC(2n)} \varphi_{\sigma}[a, a^*, \ldots, a, a^*] \mu(\sigma, 1_{2n}).
$$

(The sum is over all of $NC(2n)$ since all $\sigma$ are less than $1_{2n}$, the largest element.) Therefore, from Equation 2.3 we have

$$
|\alpha_n[a]| \leq \sum_{\sigma \in NC(2n)} |\varphi_{\sigma}[a, a^*, \ldots, a, a^*]| 4^{2n-1} = 4^{2n-1} \sum_{\sigma \in NC(2n)} \prod_{V \in \sigma} |\varphi_V[a, a^*, \ldots, a, a^*]|.
$$

Let $V_1, \ldots, V_r$ be the blocks of a given $\sigma \in NC(2n)$; so $|V_1| + \cdots + |V_r| = 2n$. Well, $\varphi_{V_j}[a, a^*, \ldots, a, a^*] = \varphi(a^{e_1} \cdots a^{e_{|V_j|}})$ where $e_i \in \{1, *\}$. Since $\varphi$ is a state on a $C^*$-algebra, this gives

$$
|\varphi_{V_j}[a, a^*, \ldots, a, a^*]| \leq \|a^{e_1} \cdots a^{e_{|V_j|}}\| \leq \|a\|^{2n},
$$

Hence, $|\alpha_n[a]| \leq 4^{2n-1} \sum_{\sigma \in NC(2n)} \|a\|^{2n} = 4^{2n-1} C_{2n} \|a\|^{2n}$.

The result for $\alpha_n[a]$ now follows from the fact that $C_{2n} \leq 4^{2n}$. The argument for $\beta_n[a]$ is identical. \hfill \square

Thus, we need only construct an $\mathcal{R}$-diagonal element whose determining sequences are positive and bounded below by $\frac{1}{2} (2^4 \|a\|)^{2n}$.

**Lemma 4.4.** Let $(\mathcal{A}, \varphi)$ be a $C^*$-probability space, and let $\gamma$ and $\lambda$ be positive constants. There exists an $\mathcal{R}$-diagonal element $b = b_{\gamma, \lambda} \in \mathcal{A}$ with $\alpha_n[b] = \beta_n[b] = \gamma \cdot \lambda^{2n}$. 

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Proof. As shown in [NS3] (and also in [S]), there is a free Poisson element \( p = p_{\gamma,\lambda} \) which is self-adjoint and satisfies \( \kappa_n[p, \ldots, p] = \frac{1}{2} \gamma \cdot \lambda^n \). Let \( p_1, p_2 \) be free copies of this Poisson element, and let \( q = p_1 - p_2 \). As \( \kappa_n \) is a linear combination of products of multilinear functionals \( \varphi_V \), and as \( p_1 \) and \( -p_2 \) are free (so their mixed cumulants vanish), we have

\[
\kappa_n[q, \ldots, q] = \kappa_n[p_1, \ldots, p_1] + \kappa_n[-p_2, \ldots, -p_2] = (1 + (-1)^n) \kappa_n[p, \ldots, p] = \begin{cases} \gamma \cdot \lambda^n, & n \text{ even}, \\ 0, & n \text{ odd}. \end{cases}
\]

Now, let \( u \) be a Haar unitary \( * \)-free from \( q \). By Theorem 4.2(2) in [NS2], \( b = qu \) is \( \mathcal{R} \)-diagonal. (The conditions of the theorem require the \( C^* \)-probability space to be tracial; however, we may simply restrict \( \varphi \) to the unital \( C^* \) algebra generated by the normal elements \( q \) and \( u \), where it is always a trace.) Since \( b \) is \( \mathcal{R} \)-diagonal, we can compute its determining sequences by

\[
\alpha_n[b] = \beta_n[b] = \sum_{\pi \in NC^*(n,1)} \varphi_\pi[b, b^*, \ldots, b, b^*] \mu(\pi, 12n).
\]

Well, since \( \pi \in NC^*(n,1) \), all blocks in \( \pi \) are of even size and alternately connect \( b \)'s and \( b^* \)'s. Hence, for each block \( V \) in \( \pi \),

\[
\varphi_V[b, b^*, \ldots, b, b^*] = \varphi[(bb^*)^{V/2}] = \varphi[(quu^*q)^{V/2}] = \varphi[q^{V}] = \varphi_V[q, \ldots, q],
\]

and thus \( \varphi_\pi[b, b^*, \ldots, b, b^*] = \varphi_\pi[q, \ldots, q] \) for \( \pi \in NC^*(n,1) \).

Now, suppose \( \sigma \) is a partition in \( NC(2n) \setminus NC^*(n,1) \) – i.e. \( \sigma \) contains a block \( V = \{k_1, \ldots, k_r\} \) with two successive elements \( k_\ell < k_{\ell+1} \) of the same parity. (Indeed, \( NC^*(n,1) \) consists of non-crossing partitions whose blocks always successively pair \( b \)'s and \( b^* \)'s in the pattern \( b, b^*, b, b^* \) – i.e. the blocks must alternately pair even and odd numbers in \( \{1, \ldots, 2n\} \).) But then there is an odd number of elements between \( k_\ell \) and \( k_{\ell+1} \), and so some block in \( \sigma \) must be of odd size. Since \( q \) is an even element, it follows that \( \varphi_\sigma[q, \ldots, q] = 0 \). Hence, we also have \( \kappa_{2n}[q, \ldots, q] = \sum_{\pi \in NC^*(n,1)} \varphi_\pi[q, \ldots, q] \mu_{2n}(\pi, 12n) \), and so from Equation 4.5,

\[
\alpha_n[b] = \sum_{\pi \in NC^*(n,1)} \varphi_\pi[b, b^*, \ldots, b, b^*] \mu_{2n}(\pi, 12n) = \sum_{\pi \in NC^*(n,1)} \varphi_\pi[q, \ldots, q] \mu_{2n}(\pi, 12n) = \kappa_{2n}[q, \ldots, q] = \gamma \cdot \lambda^{2n}.
\]

\( \blacksquare \)

Following the argument of Corollary 4.2, we see that if we choose an \( \mathcal{R} \)-diagonal element \( b \) which satisfies \( \alpha_n[b] \geq |\alpha_n[a]| \) for all \( n \) then letting \( b_{n,m} \) be the list corresponding to \( [b^n(b^*)^m] \), we have \( \kappa_\pi[b_{n,m}] \geq |\kappa_\pi[a_{n,m}]| \), and so

\[
\|b^n\|_{2m}^{2m} = \sum_{\pi \in NC^*(n,m)} \kappa_\pi[b_{n,m}] \geq \sum_{\pi \in NC^*(n,m)} |\kappa_\pi[a_{n,m}]|.
\]

Hence, from Lemma 4.1, we have

\[
\|T\|_{2m} \leq \frac{\|b^n\|_{2m} \|T\|_2}{\|a\|_2^{\frac{m}{2}}}. \tag{4.6}
\]

In order for this to yield useful information, we must choose \( b \) in such a way that its variance and norm are well-controlled by those of \( a \). In the following lemma, we choose \( b = b_{\gamma,\lambda} \) as in Lemma 4.4 to optimally bound the ratio \( \|b^n\|/\|a\|_2 \).
Lemma 4.5. Let \( a \) be \( \mathcal{R} \)-diagonal, and define \( \lambda = 2^8 \|a\|^2/\|a\|_2 \) and \( \gamma = \|a\|_2^3 \lambda^{-2} \). Set \( b = b_{\gamma, \lambda} \), as in Lemma 4.4. Then \( \|b\|_2 = \|a\|_2 \), and

\[
\frac{\|b^n\|}{\|a\|_2^n} \leq 2^{10} \sqrt{e} \sqrt{n} \|a\|^2 \|a\|_2.
\]

Proof. For \( \mathcal{R} \)-diagonal \( b \), Corollary 3.2 in [Lar] says that \( \|b^n\| \leq \sqrt{e} \sqrt{n} \|b\|_2^{n-1} \). Note that, since \( b \) is centred, \( \|b\|_2^n = \kappa_2[b, b^n] \) which, from Lemma 4.4, equals \( \gamma \cdot \lambda^2 = \|a\|_2^3 \). Hence,

\[
\|b^n\|/\|a\|_2^n = \|b^n\|/\|b\|_2^n \leq \sqrt{e} \sqrt{n} \|b\|_2/\|a\|_2^3 = \sqrt{e} \sqrt{n} \|b\|_2/\|a\|_2^3.
\]

(4.7)

For the norm \( \|b\| \), we have \( b = qu \) where \( u \) is unitary, and so \( \|b\| = \|q\| = \|p_1 - p_2\| \leq 2\|p_1\| \). The norm of a free Poisson was calculated in [VDN]; the result is \( \|p_1\| = \lambda(1 + \sqrt{\gamma/2})^2 \), so

\[
\sqrt{\gamma/2} = 2^{-1/2} \cdot \|a\|_2 \lambda^{-1} = 2^{-8.5} \|a\|_2^3 < 2^{-8.5},
\]

and so

\[
\|b\| \leq 2 \cdot 2^{8} \|a\|_2^3 \cdot (1 + 2^{-8.5})^2 \leq 2^{10} \|a\|_2^3,
\]

yielding the result. \( \square \)

We now stand ready to prove the main result of this paper.

Proof of Theorem 1.3. We will check that the element \( b = b_{\gamma, \lambda} \) with coefficients chosen as in Lemma 4.5 has all positive cumulants which dominate the absolute values of the cumulants of \( a \). First, we have (as used above) \( \alpha_1[b] = \|a_1[a]\| \). For higher cumulants, using Lemma 4.4,

\[
\alpha_n[b] = \gamma \cdot \lambda^{2n} = \|a\|_2^n \left( \frac{2^8 \|a\|^2}{\|a\|_2^2} \right)^{2n-2} = \frac{1}{2} \left( \frac{2^4 \|a\|_2}{\|a\|_2} \right)^{2n-4} \cdot \left( \frac{\|a\|_2}{\|a\|_2} \right)^{2n-15},
\]

and since \( n \geq 2 \) and \( \|a\|_2 \leq \|a\| \), this is \( \geq 2^{(1/2(2^4 \|a\|_2))^{2n}} \) which is, by Lemma 4.3, \( \geq \|a_n[a]\| \). Having shown that \( \alpha_n[b] \geq \|a_n[a]\| \) for all \( n \), we may now use Equation 4.6. We have (taking the limit as \( m \to \infty \))

\[
\|T\| \leq \frac{\|b^n\|}{\|a\|_2^n} \|T\|_2,
\]

and from Lemma 4.5 this yields the result:

\[
\|T\| \leq 2^{10} \sqrt{e} \frac{\|a\|^2}{\|a\|_2^2} \sqrt{n} \|T\|_2.
\]

If the cumulants of \( a \) are all non-negative, then Equation 4.6 holds with \( b = a \), and then Equation 4.7 yields the tighter estimate. \( \square \)

Corollary 1.4 follows directly from Theorem 1.3. To be precise: if \( u_1, \ldots, u_k \) are generators of \( \mathbb{F}_k \), then the inclusions of \( u_1, \ldots, u_k \) into \( L(\mathbb{F}_k) \) are free Haar units in the free group factor \( L(\mathbb{F}_k) \) (this is discussed in Section 2.2). The set of functions \( g \in L^2(\mathbb{F}_k) \) supported on words in the \( u_j \) (excluding their inverses) of length \( n \) is equal to the \( n \)-particle space \( H^{(n)}(u, I_k) \) \( (I_k = \{1, \ldots, k\}) \) in the \( W^* \)-probability space \( (L(\mathbb{F}_k), \varphi_k) \), and a short calculation verifies that the norm on \( L^2(\mathbb{F}_k) \) equals the norm in \( L^2(L(\mathbb{F}_k), \varphi_k) \). Finally, the convolution norm is defined by \( \|g\|_r = \sup_{f \neq 0} \|g * f\|_2/\|f\|_2 \), which is the definition of the norm in the von Neumann algebra \( L(\mathbb{F}_k) \). So, Corollary 1.4 is indeed a special case of Theorem 1.3.

Note, the proof of Lemma 4.5 actually produces a constant involving \( 2^9(1 + 2^{-8})^2 = 516.0078125 \), far less than the stated \( 2^{10} = 1024 \). However, since it is highly doubtful that this constant is
optimal, there is little point quibbling. That there is a constant at all – i.e. that the behaviour is $O(n^{1/2})$ rather than $O(n)$, is the important, and surprising, fact.

We also note that the sharp constant for $a$ with negative cumulants is greater than the sharp constant $\sqrt{e} \langle |a|/\|a\|_2 \rangle \sqrt{n}$ which holds when $\alpha_n[a] \geq 0$. For example, consider a Haar unitary $\nu$, and the corresponding algebra $\mathcal{H}(u, \mathbb{N})$. For $k > 1$ in $\mathbb{N}$, the element $T_k = u_1 + \cdots + u_k$ is in the 1-particle space, and satisfies $\|T_k\|_2 = \sqrt{k}$ (Equation 4.3) and $\|T_k\| = 2\sqrt{k-1}$ (as calculated in [HL]). Thus

$$\frac{\|T_k\|}{\|T_k\|_2} = 2 \cdot \sqrt{\frac{k-1}{k}}.$$ 

Thus, if the Haagerup inequality $\|T\| \leq C \|T\|_2$ (note $\|u\|_2 = 1$) holds for all $T \in \mathcal{H}^{(1)}(u, \mathbb{N})$, then $C \geq 2 > \sqrt{e}$. It may be that $2\sqrt{n}$ is the optimal constant for $\mathcal{H}(u, \mathbb{N})$, but we are as yet unable to calculate norms of elements in these $n$-particle spaces for $n > 1$.

We conclude this section with a discussion of Brown measure.

**Theorem 4.6.** Let $a$ be an $\mathcal{B}$-diagonal element which is not a scalar multiple of a Haar unitary, and let $\nu_a$ be its Brown measure. For $n \in \mathbb{N}$, there are constants $C(n) \propto \sqrt{n}$ such that

$$\|z^n\|_\infty = \sup_{\supp \nu_a} |z^n| \leq C(n) \left[ \int |z|^{2n} d\nu_a(z, \bar{z}) \right]^{1/2} = C(n) \|z^n\|_2.$$

**Proof.** First note from Theorem 2.1, there is a function $f : [0, \|a\|_2] \to \mathbb{R}_+$ which is continuous and satisfies $f(\|a\|_2) > 0$, such that $\nu_a = f(r) dr d\theta$ with supp $\nu_a$ equal to an annulus whose outer radius is $\|a\|_2$. Of course, this means that $\sup_{\supp \nu_a} |z^n| = \|a\|_2^n$. For the 2-norm, let $M$ be the supremum of $f$ on $[0, \|a\|_2]$; then

$$\int_{\supp \nu_a} |z^n|^2 d\nu_a(z, \bar{z}) = \int_0^{2\pi} \int_0^{\|a\|_2} r^{2n} f(r) dr d\theta \leq \frac{2\pi M}{2n+1} \|a\|_2^{2n+1} = 2\pi M \|a\|_2^2 \|z^n\|_2^2,$$

and this shows that $\|z^n\|_\infty/\|z^n\|_2 \geq \sqrt{n}$. For the reverse inequality, since $f$ is continuous and $f(\|a\|_2) > 0$, there are $\epsilon, m > 0$ such that $f(r) \geq m > 0$ for $r \in [\|a\|_2 - \epsilon, \|a\|_2]$, and so since $f \geq 0$ everywhere,

$$\int_0^{2\pi} \int_0^{\|a\|_2} r^{2n} f(r) dr d\theta \geq 2\pi m \int_0^{\|a\|_2-\epsilon} r^{2n} dr = \frac{2\pi m \|a\|_2}{2n+1} (1 - (\|a\|_2)^{2n+1}) \|a\|_2^{2n} \geq \frac{\|z^n\|_\infty^2}{n}.$$

□

As discussed in Section 2.1, the Brown measure of a non-normal element $a$ (as most $\mathcal{B}$-diagonal elements are) does not respect mixed moments; that is, $\varphi(a^*a) \neq \int |z|^2 d\nu_a(z, \bar{z})$ in general, and so forth. Nevertheless, as we see in Theorem 4.6, a Haagerup inequality with the same $O(n^{1/2})$-behaviour holds in the space $\mathcal{H}L^2(\nu_a)$ of holomorphic $L^2$ functions with respect to the Brown measure of any $\mathcal{B}$-diagonal element. $\mathcal{H}L^2(\nu_a)$ is, in some sense, the commutative model for our spaces $\mathcal{H}(a, I)$ (at least in the case where $|I| = 1$), and so we see that the Brown measure does retain some information about mixed moments.

## 5. Strong Ultracontractivity

In this final section, we apply our strong Haagerup inequality (Theorem 1.3) to give strong ultracontractive bounds for the Ornstein-Uhlenbeck semigroup on $\mathcal{H}(a, I)$. In Section 5.1 we define the O-U semigroup in this general context, and show that it is a natural generalization of the free O-U semigroup considered in [Bi2]. In Section 5.2, we prove optimal ultracontractive bounds, and discuss applications to free groups.
5.1. Ornstein-Uhlenbeck semigroups. Let a be $\mathcal{B}$-diagonal. Consider the operator $N_{\text{fin}}$, defined on the algebraic direct sum $\bigoplus_{n=0}^{\infty} \mathcal{H}^{(n)}(a, I)$ (which is, of course, dense in $L^2(\mathcal{H}(a, I), \varphi)$) as the linear extension of $N_{\text{fin}}(h_n) = n h_n$ for $h_n \in \mathcal{H}^{(n)}(a, I)$. Since $h_n \perp h_m$ for $n \neq m$ (this follows from the $*$-freeness of the $a_i$), the operator $N_{\text{fin}}$ is symmetric and lower-semi-bounded by 0. Thus, by the Friedrich’s extension theorem, $N_{\text{fin}}$ extends to a densely-defined (unbounded) self-adjoint operator $N$ on $L^2(\mathcal{H}(a, I), \varphi)$, and this operator is positive semidefinite. We will refer to $N$ as the number operator affiliated with $\mathcal{H}(a, I)$.

**Proposition 5.1.** The number operator $N$ affiliated with $\mathcal{H}(a, I)$ generates a $\mathcal{C}_0$ contraction semigroup $e^{-tN}$ on $L^2(\mathcal{H}(a, I), \varphi)$.

**Proof.** Since the spaces $\mathcal{H}^{(n)}(a, I)$ reduce $N$, we see easily that $e^{-tN}$ must act via

$$e^{-tN} \sum_{n=0}^{\infty} h_n = \sum_{n=0}^{\infty} e^{-nt} h_n.$$  

It is then immediately verified that $e^{-tN}$ is a contraction semigroup, since $e^{-nt} \leq 1$ for all $t \geq 0$. To prove that it is $\mathcal{C}_0$, it suffices to show that $\omega\lim_{t \downarrow 0} e^{-tN} h = h$ for each $h \in L^2(\mathcal{H}(a, I), \varphi)$. Let $h = \sum h_n$ and $g = \sum g_n$; since $h_n \perp g_m$ for $n \neq m$,

$$\langle e^{-tN} h, g \rangle = \left( \sum_{n=0}^{\infty} e^{-nt} h_n, \sum_{m=0}^{\infty} g_m \right) = \sum_{n=0}^{\infty} e^{-nt} \langle h_n, g_n \rangle.$$  

As both $h$ and $g$ are in $L^2$, the sequence $\langle h_n, g_n \rangle$ is in $\mathbb{L}^1$, and since $e^{-nt} \leq 1$, it follows from the dominated convergence theorem that

$$\lim_{t \downarrow 0} \sum_{n=0}^{\infty} e^{-nt} \langle h_n, g_n \rangle = \sum_{n=0}^{\infty} \langle h_n, g_n \rangle = \langle h, g \rangle. \quad \square$$

An important example of this number operator is given in the case of a circular element $a = c$. In this case, $\mathcal{H}(c, I)$ is naturally isomorphic to the holomorphic space $\mathcal{H}_0(\mathcal{K})$ over a Hilbert space $\mathcal{K}$ of dimension $|I|$, as defined in the first author’s paper [Ke], and the number operator $N$ above is just the free Ornstein-Uhlenbeck (number) operator $N_0$ considered in that paper. $N_0$ is the restriction to the holomorphic space $\mathcal{H}_0(\mathcal{K})$ of the free Ornstein-Uhlenbeck operator defined in [Bi2] on the free group factor $L(\mathcal{F}[I])$, which coincides with the 0-Gaussian factor $\Gamma_0(\mathcal{K})$ introduced in [Vo] and further developed in [BoS, BKLS]. There is a family of such spaces $\Gamma_q(\mathcal{K})$ for $-1 \leq q \leq 1$ (with $q = 1$ corresponding to the classical theory of Gaussian random variables, and $q = -1$ the hyperfinite $\Pi_1$-factor), and Biane introduced number operators $N_q$ affiliated to each of them. We should also note that, in [Bi1], Biane introduced a space isomorphic to $\mathcal{H}(c, I)$, but did not consider the action of a number operator on it.

The main theorem of [Ke] shows as a special case (the case $q = 0$) that the semigroup $e^{-tN}$ affiliated with $\mathcal{H}(c, I)$ is not only a contraction semigroup on $L^2(\mathcal{H}(c, I), \varphi)$ (for tracial $\varphi$), but is in fact strongly hypercontractive:

**Theorem 5.2** (Theorem 4 in [Ke]). Let $r > 2$ be an even integer, and let $t_f(2, r) = \frac{1}{2} \log \frac{2}{r}$. Then for $t \geq t_f(2, r)$, $e^{-tN}$ is a contraction from $L^2(\mathcal{H}(c, I), \varphi)$ to $L^r(\mathcal{H}(c, I), \varphi)$.

This strong hypercontractivity theorem is the precise analogue of the same theorem in the context of the spaces $\mathcal{H}L^r(C^n, \gamma)$ (where $\gamma$ is Gauss measure) proved by Janson in [Ja]. (We should note, however, that Janson’s theorem holds from $L^p \rightarrow L^r$ for $0 < p \leq r < \infty$, not just the discrete values in [Ke].) The time $t_f$ is shorter than the least time to contraction $t_N$ in the real spaces...
$L^r(\mathbb{R}^n, \gamma)$, where the hypercontractivity inequalities were first proved and studied by Nelson in [N]. The main theorem of [Bi2] is the generalization of Nelson’s hypercontractivity theorem to the $q$-Gaussian factors.

5.4 Ultracontractivity. In the classical holomorphic case studied by Janson, while the semigroup $e^{-tN}$ is a contractive map from $\mathcal{H}L^2$ to $\mathcal{H}L^r$ for any $r > 2$, once $t$ is large enough it is also unbounded for $t < t_f(2, r)$. As a result, the semigroup $e^{-tN}$ does not map $\mathcal{H}L^2$ into the algebra of bounded functions for any time. Of course, in the classical context, the algebra of bounded functions contains no holomorphic functions save constants; even in the full real spaces, the same effect holds. This is essentially due to the fact that the kernel of the semigroup $e^{-tN}$ in these cases, the Mehler kernel, is not a bounded function.

A semigroup is called ultracontractive if it maps $L^2$ into $L^\infty$ for all $t > 0$. The Ornstein-Uhlenbeck semigroups studied by Nelson and Janson (and many others) fail to be ultracontractive. Nevertheless, the non-commutative counterpart $e^{-tN_0}$ on the free group factor is ultracontractive, as shown in [Bi2] and essentially in [Bo1].

Proposition 5.3 (Corollary 3 in [Bi2]). The free Ornstein-Uhlenbeck semigroup $e^{-tN_0}$ is ultracontractive; there is $c > 0$ with

$$\|e^{-tN_0}\|_{L^2(\Gamma_0)\to L^\infty(\Gamma_0)} \leq c t^{-3/2}, \quad 0 < t < 1.$$  (In general the function $t \mapsto \|e^{-tN_0}X\|_r$ is decreasing for any $X$ and $r$, hence it is only small-time behaviour which is interesting.) Bożejko later generalized this theorem to all the $\Gamma_q$ factors with $-1 < q < 1$; see [Bo2].

The generators of the algebra $\Gamma_0$ (the free group factor) are $*$-free semicircular elements. Thus, the $*$-algebra generated by $\mathcal{H}(c, I)$ is contained in $\Gamma_0$, and the ultracontractive $O(t^{-3/2})$-bound of Proposition 5.3 also holds for the semigroup $e^{-tN}$ affiliated with $\mathcal{H}(c, I)$ defined above. Using our main theorem, Theorem 1.3, we may essentially follow Biane’s argument and prove a stronger form of Proposition 5.3 not only for the algebra $\mathcal{H}(c, I) \cong \mathcal{H}_0(\mathcal{H})$, but in fact for all $\mathcal{H}(a, I)$ with a $\mathcal{R}$-diagonal. Indeed, we find that the short-time behaviour in the $\mathcal{R}$-diagonal case is $O(t^{-1})$.

Theorem 5.4. Let $a$ be $\mathcal{R}$-diagonal, and let $N$ be the number operator affiliated with $\mathcal{H}(a, I)$. Then $e^{-tN}$ is ultracontractive; for each $h \in L^2(\mathcal{H}(a, I), \varphi)$, $e^{-tN}h \in \mathcal{H}(a, I)$ for $t > 0$, and moreover

$$\|e^{-tN}h\| \leq \frac{1}{2} C_a t^{-1} \|h\|_2 \quad t > 0.$$  (5.1)

(Here $C_a$ is the same constant as in Theorem 1.3.) We refer to Theorem 5.4 as strong ultracontractivity, as it is a stronger version of the inequality in Proposition 5.3 which holds when the semigroup is restricted to a holomorphic subspace. This is similar in spirit to the stronger form of hypercontractivity [Ja] which holds in the holomorphic version of Nelson’s setup in [N]. We emphasize, again, that ultracontractivity is a strictly non-commutative effect in this case, since the semigroup is unbounded from $L^2 \to L^\infty$ in the classical (real and holomorphic) contexts. Theorem 5.4 is thus an essentially non-commutative result which highlights the interesting phenomenon that many functional inequalities improve in the holomorphic category.

Proof. Let $h = \sum_{n=0}^{\infty} h_n$ with $h_n \in \mathcal{H}^{(n)}(a, I)$. We estimate

$$\|e^{-tN}h\| = \left\| \sum_{n=0}^{\infty} e^{-nt}h_n \right\| \leq \sum_{n=0}^{\infty} e^{-nt}\|h_n\|.$$  (5.1)

We now employ Theorem 1.3, which implies that $h_n \in \mathcal{H}(a, I)$ and $\|h_n\| \leq C_a \sqrt{n} \|h_n\|_2$. Thus,

$$\|e^{-tN}h\| \leq C_a \sum_{n=0}^{\infty} \sqrt{n} e^{-nt}\|h_n\|_2 \leq C_a \left[ \sum_{n=0}^{\infty} n e^{-2nt} \right]^{1/2} \left[ \sum_{n=0}^{\infty} \|h_n\|_2^2 \right]^{1/2},$$

25.
where we have used the Cauchy-Schwarz inequality. The second factor is just \( \| h \|_2^2 \). The first factor is the derivative of \(-\frac{1}{2} \sum_{n=0}^{\infty} e^{-2nt} = \frac{1}{2} \frac{1}{1-e^{-2t}} \). The reader may readily verify that we thus have
\[
\| e^{-tN} h \| \leq C_a \frac{e^{-t}}{1-e^{-2t}} \| h \|_2
\]
for all \( t > 0 \). This shows that \( e^{-tN} h \in \mathcal{H}(\alpha, I) \). Moreover, the function \( t \mapsto \frac{te^{-t}}{1-e^{-2t}} \) is decreasing on \( \mathbb{R}_+ \) and has limit 1/2 at \( t = 0 \). This proves Equation 5.1.

It is typical to prove, from a bound like Equation 5.1, a Sobolev inequality of the form \( \| h \|_p \leq c(Nh, h), \ h \in \mathcal{D}(N) \) for an appropriate \( p > 2 \); indeed, if \( e^{-tN} \) in Theorem 5.4 were a classical sub-Markovian semigroup defined on \( L^2 \) of a Radon measure, we could use the standard techniques in, for example, [CSV], to prove a strong Sobolev imbedding theorem (for any \( p < \infty \)) in this case. However, the techniques necessary to implement such a proof use the Marcinkiewicz interpolation theorem in a fundamental way. As pointed out in [Ke], holomorphic spaces like \( \mathcal{H}(\alpha, I) \) (in particular in the case \( \alpha = c \)) tend not to be complex interpolation scale (at least in the \( |I| = \infty \) case). Thus, we are unable to prove a Sobolev inequality for \( \mathcal{H}(\alpha, I) \) using known-techniques.

We finally remark that one interesting new application of this theorem is to the discrete O-U semigroup on the free semigroup \( \mathbb{F}_k \) (or rather its restriction to \( \mathbb{F}_k^+ \)). As noted above, the algebra \( \mathcal{H}(u, I_k) \) with \( u \) a Haar unitary and \( |I_k| = k \) is isomorphic to the convolution-norm closure of \( \mathbb{F}_k^+ \) in \( L^2(\mathbb{F}_k^+) \), and thus \( L^2(\mathcal{H}(u, I_k), \mathcal{D}(N)) \cong L^2(\mathbb{F}_k^+) \), where the number operator \( N \) acts by \( Nw = nw \) on a word \( w \) of length \( n \). The same semigroup \( e^{-tN} \) defined on all of \( \mathbb{F}_k \) was essentially introduced in [H], and has been studied in [JLX, JX] with a view towards \( L^p \)-contraction bounds; to the authors’ knowledge, Theorem 5.4 yields the first ultracontractive bound in that context.

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