On the distinctive dispersive signature of 2D magnetically-interacting particle systems

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We report on the unconventional dynamic characteristics exhibited by special classes of 2D interacting particle systems, in which the particles are in equilibrium at rest under the action of self-balancing, repulsive static forces exchanged with their nearest neighbors. Through a theoretical model informed by current-configuration kinematics, we document the emergence of large intrinsic shift in the natural frequencies of these systems - compared to their conventional spring-mass counterparts - which can be fully ascribed to the static repulsive forces. In the context of wave propagation, the effect manifests as a massive mode-selective correction of the dispersive characteristics, which is here elucidated using the benchmark example of a triangular lattice of magnetized particles. To corroborate these findings, we perform laser vibrometry experiments carried out on a metamaterial prototype consisting of a lattice of magnetized particles supported by an elastic foundation of thin pillars. These tests confirm unequivocally the existence of the dispersion shifting phenomenon predicted by the model.

I. INTRODUCTION

Lattices of interacting particles have been used as a versatile building block to model many physics and mechanics problems. For example, a gas of electrons interacting with long-range Coulomb forces can crystallize into one-dimensional or two-dimensional lattices, known as Wigner crystals [1-3]. At a higher scale, charged microparticle systems, including colloidal dispersions and dusty plasmas, can be organized, through a Yukawa or a screened-Coulomb interaction, into ordered spatial structures referred to as Yukawa lattices [4-8]. Crystalline particle ensembles can also be formed at the macroscopic level. A classical example is offered by constrained granular systems composed of spherical beads interacting through Hertzian contacts [9-13], possibly under the confining action of compressive loads. Recently, conceptually similar implementations have been obtained using arrays of repulsive magnets [14-17]. Although the above-mentioned examples are drawn from different physical domains, their dynamical properties are controlled by analogous laws and therefore captured by similar analytical models. One important feature shared across all these problems is that, in these systems, each particle is in equilibrium, even at rest, under the action of self-balancing static forces exchanged with its neighbors.

This article explores the notion that the existence of these self-balancing static forces is responsible for unique effects on the dynamical properties that are germane to this class of system, and therefore is overlooked in the relevant literature relying on classical treatments. Specifically, we revisit from this perspective the problem of vibration and wave dynamics of 2D interacting particle systems. Through a rigorous treatment based on a complete geometric description of the particle kinematics, we demonstrate that the self-balancing static forces are responsible for intrinsic (and possible large) frequency shifts in the oscillatory characteristics of these systems when we compare their responses against that of their conventional spring-mass counterparts. These shifts, in turn, determine macroscopic corrections in the dispersive characteristics (i.e., band diagrams) that are heavily wavevector-dependent and mode-sensitive. Our theoretical findings are supported by explicit time-domain numerical simulations and by laser vibrometry experiments carried out on a prototype lattice of magnetized particles supported by an elastic foundation of thin pillars. Moreover, we show that the dispersion shifting effects can be exploited to achieve tunability of the lattice spatial directivity, thus adding a dimension to the existing landscape of tuning strategies based on magneto-mechanical coupling [18-20].

In Sec. II, to properly set the background for the investigation, we briefly revisit the procedure for derivation and linearization of general equations of motion for particle systems. In Sec. III, we establish an analytical framework for the vibration analysis of a magnetic particle resonator, revealing the existence of intrinsic natural frequency shifts and we validate the results via numerical simulations. In Sec. IV, the approach is adapted to capture the distinctive signature of the inter-particle repulsive forces on the dispersive characteristics of a 2D magnetic particle lattice, and we corroborate the results with numerical simulations and experiments. The significance and potential impacts of this work are summarized in Sec. V.

II. THEORETICAL BACKGROUND

In the interest of generality, we consider a 2D particle system subjected to an arbitrary inter-particle interaction law. The general form of the governing equations
can be written as

\[ \ddot{\mathbf{M}} \ddot{\mathbf{u}}(t) + \mathbf{F}_{\text{int}}(\mathbf{r}, t) = \mathbf{F}_{\text{ext}}(\mathbf{r}, t) \quad (1) \]

where \( \mathbf{M} \) is the mass matrix, \( \mathbf{u} \) is the displacement vector, and \( \mathbf{r} \) is the current position vector. \( \mathbf{F}_{\text{int}} \) is a vector of internal forces due to the inter-particle interactions, and \( \mathbf{F}_{\text{ext}} \) is a vector of external forces. The relation between \( \mathbf{r} \) and \( \mathbf{u} \) reads \( \mathbf{r} = \mathbf{r}_0 + \mathbf{u} \), with \( \mathbf{r}_0 \) being the reference position vector at \( t = 0 \). Accordingly, we can rewrite Eq. (1) as

\[ \ddot{\mathbf{M}} \ddot{\mathbf{u}}(t) + \mathbf{F}_{\text{int}}(\mathbf{r}_0 + \mathbf{u}, t) = \mathbf{F}_{\text{ext}}(\mathbf{r}_0 + \mathbf{u}, t) \quad (2) \]

When the displacement \( \mathbf{u} \) is sufficiently small compared to \( \mathbf{r}_0 \), Eq. (2) can be linearized to

\[ \ddot{\mathbf{M}} \ddot{\mathbf{u}} + \nabla_u \mathbf{F}_{\text{int}}|_{\mathbf{u}=0} \cdot \mathbf{u} = \mathbf{F}_{\text{ext}}|_{\mathbf{u}=0} + \nabla_u \mathbf{F}_{\text{ext}}|_{\mathbf{u}=0} \cdot \mathbf{u} \quad (3) \]

If the system is initially in equilibrium, the second term on the LHS vanishes. Moreover, in the absence of external forces the two terms on the RHS also vanish, leading to the general equation

\[ \ddot{\mathbf{M}} \ddot{\mathbf{u}} + \nabla_u \mathbf{F}_{\text{int}}|_{\mathbf{u}=0} \cdot \mathbf{u} = 0 \quad (4) \]

where \( \nabla_u \mathbf{F}_{\text{int}}|_{\mathbf{u}=0} \) yields the stiffness matrix \( \mathbf{K} \). In the contexts where a potential function \( \phi(r) \) is given, the above equation can be recast in terms of the potential through the relation \( \mathbf{F} = -\nabla \phi \). In the remainder of the article, we will illustrate how, under the special circumstances of internal forces that are self-balancing at rest, a second stiffness component arises with strong implications on the dynamic behavior of 2D interacting particle systems.

### III. FREQUENCY SHIFTS IN 2D MAGNETIC RESONATORS

#### A. Analytical model

Consider the system of magnetized particles shown in Fig. 1. Assume that the four particles are identical with mass \( \mathbf{M} \) and subjected to mutually repulsive forces. At the initial equilibrium conditions, a constant vertical force \( \mathbf{f} \) is applied on the free particle 1 (red dot) to balance the repulsive forces exerted on 1 by the fixed particles 2, 3, and 4 (black dots). The force between pairs of adjacent particles can be written in the form

\[ \mathbf{F} = f(r) \mathbf{n} \quad (5) \]

where \( \mathbf{n} \) is the unit vector in the direction connecting the particles, and \( f(r) \), in general, is taken to obey an inverse power law, i.e., \( f(r) = \beta r^{-\alpha} \).

The governing equation for particle 1 is

\[ \ddot{\mathbf{M}} \ddot{\mathbf{u}} - \sum_{i=1}^{3} \mathbf{F}_i = \mathbf{f} \quad (6) \]

where \( \mathbf{M} = \begin{bmatrix} M & 0 \\ 0 & M \end{bmatrix} \), \( \mathbf{u} = \begin{bmatrix} u \\ v \end{bmatrix} \), \( \mathbf{F}_i = f(r_i) \mathbf{n}_i \), \( \mathbf{f} = -\sum_{i=1}^{3} f(L_0) \mathbf{n}^i(\mathbf{u} = 0) \), and

\[ r_i = \left\| L_0 \mathbf{e}_0^i + \mathbf{u} \right\| \]

\[ \mathbf{n}_i = \frac{L_0 \mathbf{e}_0^i + \mathbf{u}}{\left\| L_0 \mathbf{e}_0^i + \mathbf{u} \right\|} \quad (7) \]

where \( \mathbf{e}_0^i (i = 1, 2, 3) \) can be expressed as \( \mathbf{e}_0^1 = \left\{ \cos \theta_0, \sin \theta_0 \right\} \), \( \mathbf{e}_0^2 = \left\{ 0, 1 \right\} \), and \( \mathbf{e}_0^3 = \left\{ -\cos \theta_0, \sin \theta_0 \right\} \).

In the small perturbation limit, the linear stiffness matrix of the system, according to Eq. 3 can be obtained as (note that in Eq. 7 both \( r_i \) and \( \mathbf{n}_i \) are functions of \( \mathbf{u} \))

\[ \mathbf{K} = -\left( \nabla_u \sum_{i=1}^{3} \mathbf{F}_i \right)|_{\mathbf{u}=0} = \mathbf{K}_0 + \mathbf{K}_1 \]

\[ = -\sum_{i=1}^{3} f(L_0) \mathbf{e}_0^i \otimes \mathbf{e}_0^i - \sum_{i=1}^{3} \frac{f(L_0)}{L_0} \left( \mathbf{I} - \mathbf{e}_0^i \otimes \mathbf{e}_0^i \right) \quad (8) \]

where \( \mathbf{I} \) is the identity matrix and \( \otimes \) denotes the dyadic product. Details of this calculation are given in the SI. Eq. 8 shows that the stiffness matrix has two contributions denoted as \( \mathbf{K}_0 \) and \( \mathbf{K}_1 \). \( \mathbf{K}_0 \) depends on the derivative of the interaction law \( f(r) \) evaluated at the initial separation distance \( L_0 \), which follows the standard procedure for deriving the linearized stiffness matrix. Interestingly, \( \mathbf{K}_1 \) is a secondary source of stiffness proportional to the initial static magnetic force \( f(L_0) \). Note that this stiffness contribution is directly linked to the existence of an initial static force and to the 2D nature of the system, and can only be revealed through an exact deformation description involving current-configuration kinematics.

The natural frequency \( \omega_0 \) of the system is the only admissible root of the characteristic equation obtained by solving the eigenvalue problem

\[ \left( -\omega^2 \mathbf{M} + \mathbf{K}_0 + \mathbf{K}_1 \right) \mathbf{u} = 0 \quad (9) \]
To quantify the separate contributions of the two stiffness terms, we consider the reference natural frequency \( \bar{\omega}_0 \) obtained by solving the reference eigenvalue problem

\[
(-\omega^2M + K_0) \mathbf{u} = 0
\]

(10)
corresponding to a system featuring the same elastic response of the links to deformation \( f'(L_0) \), but without the initial static force component, i.e., \( f(L_0) = 0 \). In the analytical model, each particle is treated as an ideal magnetic dipole, and the repulsive force can be expressed as \[15\] \[21\]

\[
F = \frac{3\mu_0}{4\pi} \frac{m^2}{r^4} \mathbf{n}
\]

(11)
where \( \mu_0 \) is the permeability of the medium and \( m \) is the magnetic moment. Combining Eq. 8 and Eq. 11 yields

\[
K = \sum_{i=1}^{3} \frac{4\gamma}{L_0} \mathbf{e}_0^i \otimes \mathbf{e}_0^i - \sum_{i=1}^{3} \frac{\gamma}{L_0} \left( \mathbf{I} - \mathbf{e}_0^i \otimes \mathbf{e}_0^i \right)
\]

(12)
where \( \gamma = \frac{3\mu_0 m^2}{4\pi} \). For an arbitrary choice of parameters \( (M = 1, L_0 = 1, \gamma = 10^4) \) with standard SI units used throughout the paper except where specified), the natural frequencies are calculated from Eq. 9 and Eq. 10 and the values will be compared with simulation results provided in the next section.

B. Time-domain numerical simulations

We perform time-domain numerical simulations assuming a harmonic excitation applied vertically on particle 1 (note that, in the simulation, the horizontal motion of the particle may need to be constrained in some cases, as explained in the SI). The governing equation (Eq. 6) is integrated in time using Verlet algorithm \[22\], and the magnitude of the harmonic response is recorded after steady-state conditions are reached. To effectively establish steady-state conditions, we add viscous damping to the system and we consider sufficiently long excitation times to fully dissipate the signature of the transient response. We also keep the amplitude of excitation sufficiently low to neglect the effects of nonlinearity (naturally embedded in the constitutive model of Eq. 11), which are not relevant for this treatment.

In Fig. 2 we plot the numerically-obtained frequency response function (FRF) for the complete system (red curve with circular markers) and for the reference system (blue curve with plus markers) for four different orientations of the links: \( \theta_0 \in [0 \, \pi/6 \, \pi/4 \, \pi/2] \). We also superimpose the natural frequency bar \( \omega_0 \) (red line) and the reference frequency bar \( \bar{\omega}_0 \) (blue dashed line) predicted from Eq. 9 and Eq. 10, respectively. We observe that the computed natural frequencies match the peaks of the FRF curves. Since the numerical simulations are not subjected to any restrictive assumptions, as

![Fig. 2. Frequency response function (FRF) for the complete (red) and reference (blue) systems for four configurations, sketched in the insets: (a) \( \theta_0 = 0 \), (b) \( \theta_0 = \pi/6 \), (c) \( \theta_0 = \pi/4 \), and (d) \( \theta_0 = \pi/2 \). The natural frequencies computed from the analytical model are also reported as vertical lines.](image)

they explicitly involve updating the most general form of the interaction law at each integration step, this result confirms the validity of the linearized model in Eq. 8. Moreover, the difference between \( \omega_0 \) and \( \bar{\omega}_0 \) decreases as \( \theta_0 \) increases: Moreover, in the limit case of \( \theta_0 = \pi/2 \), the corresponding system is reduced to a 1D problem and the two frequencies become identical. This result indicates that the frequency shift (i.e., \( \Delta \omega_0 = \omega_0 - \bar{\omega}_0 \)) due to the existence of \( K_1 \) is germane to 2D configurations. This marks a fundamental difference with other stiffness correction effects available in particle systems - e.g. those merely due to changes in precompression in granular systems - which in general do not vanish for 1D configurations and fully captured by the stiffness term \( K_0 \). Another interesting feature is that, since the magnetic force is repulsive in our framework, the frequency shift \( \Delta \omega_0 \) is necessarily negative (softening effect). Finally, we note that, for certain parameter choices, a special condition may occur when \( |\Re(\Delta \omega_0)| \) is no longer less than \( \omega_0 \), resulting in dynamic instabilities (a preliminary stability analysis of the magnetic system based on Eq. 12 is reported in the SI).

IV. DISPERSION SHIFTS IN 2D TRIANGULAR MAGNETIC LATTICES

A. Analytical model

We proceed now to explore the frequency shifting effects on the dispersion relation of a triangular lattice consisting of repulsive particles, as shown in Fig. 3. Under
where \( \Delta \) is a constant, and \( \phi = \{ \phi_u \phi_v \} \) is a modal vector. According to Floquet-Bloch theorem, the relations between displacements at neighboring sites can be expressed as
\[
\mathbf{u}_{i\pm 1,j\pm 1} = \mathbf{u}_{i,j} e^{i(\pm \mathbf{k} \cdot \mathbf{r}_0 + \mathbf{k} \cdot \mathbf{r}_0^2)}
\]
(17)

Substituting Eq. (16) and Eq. (17) into Eq. (15) yields the wavevector-dependent eigenvalue problem
\[
[-\omega^2 \mathbf{M} + \mathbf{K}(k)] \phi = 0
\]
(18)
where
\[
\mathbf{K}(k) = 2 \sum_{l=1}^{3} \left\{ \begin{array}{c} f'(L_0) \mathbf{e}_0^l \otimes \mathbf{e}_0^l \left[ \cos(\mathbf{k} \cdot \mathbf{r}_0) - 1 \right] \\ + 2 \sum_{l=1}^{3} \left[ \frac{f(L_0)}{L_0} \left( \mathbf{I} - \mathbf{e}_0^l \otimes \mathbf{e}_0^l \right) \left[ \cos(\mathbf{k} \cdot \mathbf{r}_0) - 1 \right] \right] \right\}
\]
(19)
is a wavevector-dependent stiffness matrix. Again, we observe the appearance of two terms in the stiffness matrix. Canonically, the linear dispersion relation of the magnetic system is obtained by solving the eigenvalue problem for wavevectors along the contour of the irreducible Brillouin zone. Similarly to what we did for the first example, we also define a reference system, whose dispersion relation is obtained from the following eigenvalue problem
\[
[-\omega^2 \mathbf{M} + \mathbf{K}_0(k)] \phi = 0
\]
(20)
where \( \mathbf{K}_0(k) \) is the first term of \( \mathbf{K}(k) \) in Eq. (19).

In Fig. 4, we plot the band diagram (red curves) of the 2D triangular magnetic lattice with repulsive interaction law (according to Eq. (11)), and we superimpose the reference one (blue dashed curves) obtained solving Eq. (20) Clearly, macroscopic softening dispersion shifts towards low frequencies are observed, especially for the first mode.
B. Full-scale simulations

To validate our analytical model, we perform a suite of full-scale simulations. To this end, we simulate the wave response of a finite lattice (shown in Fig. 5) and we compare it against that of a corresponding reference system. The particles located on the boundary are fixed in order to establish initial equilibrium conditions.

Our first task is to numerically reconstruct the band diagram and demonstrate the dispersion shifting effect. To this end, we focus on wavevectors \( k \) sampled along the \( \Gamma-M \) direction, which correspond to wave propagation in the vertical direction. Nearly plane-wave conditions are established by considering an array of excitation points collocated at the particles denoted as green dots in Fig. 5. The force excitation is prescribed as a five-cycle tone burst with carrier frequency \( \Omega_0 \) chosen to fall within the frequency range of first (shear) and of the second (longitudinal) modes, respectively. Accordingly, the force is applied in the horizontal and in the vertical direction to optimally excite the respective modes. The spatio-temporal displacement response is sampled at nodes located along lattice vector \( r_2 \) (black dots shown in Fig. 5) and transformed via 2D discrete Fourier transform (2D-DFT). For an excitation at \( \Omega_0 = 120 \text{ rad/s} \) (in the shear mode range), the normalized spectral amplitude is plotted in Fig. 6 (a) and (b) for the magnetic and reference lattice, respectively. For excitation at \( \Omega_0 = 400 \text{ rad/s} \) (in the longitudinal mode range), the results are plotted in Fig. 6 (c) and (d). From a visual inspection, we observe a large dispersion shift for the shear mode and a minor correction of the longitudinal mode, both matching the dispersion relation corrections obtained from the analytical model.

The second task is to explore how the frequency shifts affect the spatial directivity of magnetic lattices. The isofrequency contours of the two dispersion surfaces are given in Fig. 7 (a) and (b) for the reference system, and in Fig. 7 (d) and (e) for the magnetic system, respectively. Comparing Fig. 7 (a) and (d) (especially tracking the red lines evaluated at \( \Omega_0 = 250 \text{ rad/s} \)), one can observe a dramatic change in directivity for the shear mode. In contrast, the minor differences between Fig. 7 (b) and (e) confirm the fact that the shifting effect on the longitudinal mode is small. Full-scale simulations are performed by exciting the lattices at the mid point of the bottom edge (one layer insider the fixed boundary) with a tone burst (\( \Omega_0 = 250 \text{ rad/s} \)). The wavefield obtained for the reference system is plotted in Fig. 7 (c) and two distinct wave signatures are identified. The one with larger phase velocity and circular wave front corresponds to the quasi nondispersive longitudinal mode, while the slower wave propagating predominantly along the lattice vectors captures the highly dispersive shear mode. Fig. 7 (f) shows the manifestation of the dispersion shift in the wavefield of the magnetic lattice. In this case, the shear wave is significantly more dispersive, and therefore interferes more conspicuously with the longitudinal wave, while featuring a less pronounced spatial directivity.

C. Experiments

To corroborate the theoretical predictions, we perform a series of dynamical experiments on a lattice prototype, shown in Fig. 8 involving finite-size magnets supported by simple structural elements, which represents a practical implementation of the idealized system considered in our model. The role of the particles is played by small ring magnets (with outer diameter 1/4 inch \( \times \) inter diameter 1/16 inch \( \times \) thickness 1/8 inch, Grade N42) interacting repulsively in their own plane and thus naturally occupying the nodal locations of a triangular lattice at equilibrium. To enforce planarity of the lattice, the magnets
FIG. 7. Spatial directivity of wave propagation in the reference and magnetic lattices. (a and d) Isofrequency contours of the shear mode. (b and e) Isofrequency contours of the longitudinal mode. (c and f) Snapshots of wavefields for excitation at $\Omega_0 = 250$ rad/s, displaying distinct spatial patterns. (a, b, and c) are obtained for the reference system, while (d, e, and f) are obtained for the magnetic system.

are supported by Aluminum beams inserted in the magnets at one tip and clamped to an Acrylic base through a lattice of drilled holes. The interior magnets are supported by slender beams (with cross sectional diameter 1/16 inch) to allow minimally impeded displacement of the tip magnets, while the exterior magnets, located along the perimeter of a half hexagon, are supported by thick beams (with cross sectional diameter 1/4 inch) featuring large bending stiffness so that fixed boundary conditions are established (Fig. 8 (b-c)). To capture the effects of the supporting beams, which effectively act as an elastic foundation, we modify our model by endowing each particle with an additional in-plane spring with elastic constant proportional to the equivalent bending stiffness of a cantilever beam. Moreover, precise values of the repulsive forces between magnet pairs are obtained from a static characterization of the magnet-magnet interaction (see details in the SI).

The experimental setup is shown in Fig. 8 (a). A 3D scanning laser Doppler vibrometer (SLDV, Polytec PSV-400-3D) is used to scan the magnets and measure their in-plane response. The excitation is prescribed in the vertical direction at the magnet located at the center of the bottom edge (one layer insider the fixed boundary) through a Bruel & Kjaer Type 4809 shaker (powered by a Bruel & Kjaer Type 2718 amplifier), as shown in Fig. 8 (b). In Fig. 9, we plot the spectral response obtained via 2D-DFT of the experimental spatio-temporal data sampled along a lattice vector for tone-burst excitations with carrier frequencies centered at 30 Hz, 40 Hz, 50 Hz, and 60 Hz, respectively. For comparison, we superimpose the dispersion relations predicted using our modified analytical model (involving a double stiffness contribution as per Eq. 19). In contrast with the previous cases, the band diagram is here fully gapped at low frequencies, which is a typical feature of systems with elastic foundations. Notwithstanding small deviations at higher frequencies (which can be easily attributed to non-idealities and unavoidable minor differences between particle model and physical specimen), the experimental results show remarkable agreement with the analytical dispersion branches endowed with the dispersion shifting associated with the stiffness augmentation. This result provides unequivocal experimental evidence confirming the existence of macroscopic dispersion shifting effects in 2D repulsive magnet lattices. Moreover, the experimental spectra confirm that the shifts in the shear mode are indeed much stronger that those observed for the longitudinal mode, which is another peculiar result emerging from the analytical model.
FIG. 9. Experimental response spectra for tone-burst excitations at (a) 30 Hz, (b) 40 Hz, (c) 50 Hz, and (d) 60 Hz. The amplitude spectra conform to the corrected band diagram with shifted branches predicted by the theoretical model.

V. CONCLUSIONS

In this study, we have predicted theoretically and demonstrated experimentally the existence of frequency shifting effects in the dynamics of 2D interacting particle systems subjected to self-balancing repulsive forces, such as lattices of magnetized particles. First, we have elucidated the requirements necessary for establishing the shifting effect through the simple example of a single magnetic resonating particle. We have further shown that this effect is intrinsically tied to the 2D nature of the particle arrangements and disappears when the system reduces to a 1D configuration. Then, we have discussed the implications of this effect on the wave propagation characteristics of triangular lattices, and we have highlighted the opportunities for mode-selective dispersion correction and tuning of the spatial directivity. Finally, we have experimentally demonstrated all these findings in a triangular lattice prototype assembled using magnets supported by a foundation of beam elements. The framework of this study is not restricted to systems featuring magnetic interactions, but can be conceptually extended to seek new insights in a variety of physical systems with repulsive interactions, ranging from plasma crystals to granular media.

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SUPPLEMENTARY INFORMATION (SI)

Derivation of the complete stiffness matrix $K$

The complete stiffness matrix $K$ for the 2D magnetic resonator (i.e., Eq. 8) is derived as follows

$$
K = -\nabla u \sum_{i=1}^{3} F_i \bigg|_{u=0} 
= -\sum_{i=1}^{3} [n_i \otimes \nabla_u f(r_i) + f(r_i) \nabla_u n_i]_{u=0}
= -3 \sum_{i=1}^{3} \left[ f'(r_i) n_i \otimes (L_0 e_i^0 + u) \right]_{u=0} - 3 \sum_{i=1}^{3} f(r_i) \left[ \frac{I}{r_i} - \frac{(L_0 e_i^0 + u) \otimes (L_0 e_i^0 + u)}{r_i^3} \right]_{u=0}
= -3 \sum_{i=1}^{3} f'(L_0) e_i^0 \otimes e_i^0 - 3 \sum_{i=1}^{3} f(L_0) \left( I - e_i^0 \otimes e_i^0 \right)
\equiv K_0 + K_1
$$

(S1)

where $K_0$ is referred to as the reference stiffness matrix in our framework, and $K_1$ is the new stiffness component responsible for the frequency shifting effects.

Preliminary analysis of the magnetic resonator for the case of $\theta = \pi/2$

As discussed in the example of the magnetic resonator, instabilities may occur for certain configurations. For instance, in the case of $\theta = \pi/2$ shown in FIG.2 (d), the system approaches the one-dimensional limit, and we will show that its dynamic behavior becomes unstable in the horizontal direction. For the choice of parameters used in the main article ($M = 1$, $L_0 = 1$, $\gamma = 10^4$), the complete stiffness matrix $K$ is calculated as

$$
K = K_0 + K_1
= \begin{bmatrix}
0 & 0 \\
0 & 120000 \\
0 & 0
\end{bmatrix} + \begin{bmatrix}
-30000 & 0 \\
0 & 0 \\
0 & 0
\end{bmatrix}
= \begin{bmatrix}
-30000 & 0 \\
0 & 120000
\end{bmatrix}
$$

(S2)

where we notice that the first diagonal component is negative. From basic structural dynamics, a negative stiffness component is often related to dynamic instabilities. In this case, the instabilities occur when the resonator is perturbed in the horizontal direction, which is in response to the fact that there is no horizontal resistance provided in the system. Also, this is the reason why, for this configuration, the horizontal motion of the resonator reported in the main manuscript is constrained in the simulation. This result provides additional evidence that our analytical model is capable of capturing the complete dynamic behavior of the 2D magnetic system.

A modified analytical model for the lattice specimen

To provide guidelines for the experiment, we propose a modified analytical model to approximately capture the dynamics of the lattice specimen, as shown in Fig. S1. The magnets are modeled as point masses connected by springs featuring repulsive interaction law $f(r)$. The supporting beams are treated as an elastic foundation with in-plane equivalent spring constant

$$
k_{eq} = \frac{3EI}{H^3}
$$

(S3)

typical of cantilever beams, where $E$ is the Young's modulus, $I = \pi R^4/4$ is the second moment of area of the beam's circular cross-section, and $R$ and $H$ are the radius and height of the beam. With the elastic foundation incorporated
to the analytical model introduced in the section IV. A, the governing equation for this modified system becomes

$$\mathbf{M}\ddot{\mathbf{u}}_{i,j} + \sum_{l=1}^{3} \left\{ f'(L_0)\mathbf{e}_0 \otimes \mathbf{e}_0 + \frac{f(L_0)}{L_0} \left( \mathbf{I} - \mathbf{e}_0 \otimes \mathbf{e}_0 \right) \right\} \Delta \mathbf{u}_l \right\} + \mathbf{K}_f \mathbf{u}_{i,j} = 0 \quad (S4)$$

where the additional stiffness term $$\mathbf{K}_f = \begin{bmatrix} k_{eq} & 0 \\ 0 & k_{eq} \end{bmatrix}$$ accounts for the resistance effect of the elastic foundation, and the rest quantities are defined in the main text.

Considering Bloch conditions (Eq. 17), the wavenumber-dependent stiffness matrix for the modified analytical model can be written as

$$\mathbf{K}' = \mathbf{K} + \mathbf{K}_f \quad (S5)$$

where $$\mathbf{K} = \mathbf{K}_0 + \mathbf{K}_1$$ has the same expression as Eq. 19. The dispersion relation can be obtained by solving the corresponding eigenvalue problem with the modified stiffness matrix $$\mathbf{K}'$$, which requires determining $$f(L_0)$$ and $$f'(L_0)$$ from the repulsive interaction law between pairs of magnets. To this end, we conduct a static test on two magnets. The experimental setup is shown in Fig. S2 (a). The two magnets are mounted on appropriately machined holder such that their sides can progressively moved against each other. A micrometer is used to control by small amounts the distance between the two magnets, and a highly sensitive load cell (shown in Fig. S2 (b)) is attached to the top magnet grip to measure the repulsive force. The measured force values are plotted as circles in Fig. S3, and fitted by an inverse power law $$f(r) = \beta r^{-\alpha}$$ using “lsqcurvefit” in Matlab (based on the method of nonlinear least squares). We determine $$\alpha = 4.5824$$ and $$\beta = 1.6209 \times 10^{-10}$$ (the dashed red line in Fig. S3).

The other parameters necessary for the calculation of the eigenvalue problem are listed below: the mass of the magnet $$M = 7.07 \times 10^{-4}$$ kg, the Young’s modulus of Aluminum $$E = 71$$ Gpa, the diameter and height of the beam $$R = 7.9375 \times 10^{-4}$$ m and $$H = 0.15$$ m, and the distance between two magnets (at rest) in the lattice $$L_0 = 0.01$$ m. Finally, we determine from the modified analytical model the dispersion relation for wavevectors along the $$\Gamma$$-M direction, which is plotted in Fig. S4. For comparison, the reference dispersion relation, obtained for the system with stiffness matrix $$\mathbf{K}_0 + \mathbf{K}_f$$, is superimposed as dashed blue lines.

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**FIG. S1.** Equivalent spring-mass model of the lattice specimen, featuring an elastic foundation with in-plane spring constant.
FIG. S2. Experimental characterization of the repulsive force between magnets. (a) Testing apparatus. (b) Detail of the magnets position.

FIG. S3. Experimental data and fitted force-displacement curve for two magnets with repulsive interaction.
FIG. S4. Comparison between dispersion relation of the modified analytical model and the reference case, showing large mode-selective dispersion shifts.