SUMMARY The capacity (i.e., maximum flow) of a unicast network is known to be equal to the minimum s-t cut capacity due to the max-flow min-cut theorem. If the topology of a network (or link capacities) is dynamically changing or unknown, it is not so trivial to predict statistical properties on the maximum flow of the network. In this paper, we present a probabilistic analysis for evaluating the accumulate distribution of the minimum s-t cut capacity on random graphs. The graph ensemble treated in this paper consists of undirected graphs with arbitrary specified degree distribution. The main contribution of our work is a lower bound for the accumulate distribution of the minimum s-t cut capacity. The feature of our approach is to utilize the correspondence between the cut space of an undirected graph and a binary LDGM (low-density generator-matrix) code. From some computer experiments, it is observed that the lower bound derived here reflects the actual statistical behavior of the minimum s-t cut capacity of random graphs with specified degrees.

**key words:** minimum cut, random graphs, LDGM code

1. Introduction

Rapid growth of information flow over a network such as a backbone network for mobile terminals requires efficient utilization of full potential of the network. In a multicast communication scenario, it is well known that appropriate network coding achieves its multicast capacity. Emergence of the network coding have broad network design strategies for efficient use of wired and wireless networks [1].

The multicast capacity of a directed graph is closely related to the s-t maximum flow of the graph, which is equal to the minimum s-t cut capacity due to the max-flow min-cut theorem [2]. The symbols s and t represent two distinct nodes in the graph. On the other hand, on a unicast network, the minimum s-t cut capacity of the network determines the unicast capacity between the nodes s and t. Therefore, it is meaningful to study the minimum s-t cut capacity for designing an efficient network.

If the topology of a network is static, the corresponding s-t maximum flow of the network can be efficiently evaluated in polynomial time [2]. However, if the topology of a network and its link capacities are dynamically changing or have stochastic nature, it is not so trivial to predict statistical properties on the maximum flow. For example, in a case of wireless network, the link capacities may fluctuate because of the effect of time-varying fading. Another example is an ad-hoc network whose link connections are stochastically determined.

In order to obtain an insight for statistical properties of the minimum s-t cut capacity for such random networks, it is natural to investigate statistical properties of the minimum s-t cut capacity over a random graph ensemble. Such a result may unveil typical behaviors of the minimum s-t cut capacity (or maximum flow) for given parameters of a network such as the number of vertices, edges and degree distributions.

Several theoretical works on the maximum flow of random graphs (i.e., graph ensembles) have been made. In a context of randomized algorithms, Karger showed a sharp concentration result for maximum flow [3]. Ramamoorthy et al. presented another concentration result. The network coding capacities of weighted random graphs and weighted random geometric graphs concentrate around the expected number of nearest neighbors of the source and the sinks [4]. These concentration results indicate an asymptotic properties of the maximum flow of random networks. Wang et al. shows statistical properties of the maximum flow in an asymptotic setting as well. They discussed the random graphs with Bernoulli distributed weights [5].

In this paper, we will present a lower bound for the accumulate distribution of the minimum s-t cut capacity of random graphs with specified degree distribution. The approach presented here is totally different from those used in the conventional works [3]–[5]. The basis of the analysis is the correspondence between the cut space of an undirected graph and a binary LDGM (low-density generator-matrix) code [6].

Based on this correspondence, Yano and Wadayama [7] presented an ensemble analysis for the network reliability problem. Fujii and Wadayama [8] proposed a probabilistic analysis for the global minimum cut capacity over the weighted Erdős-Rényi random graphs [9]. In this paper, we extend the idea in [7] and [8] to random graphs with arbitrary specified degree distribution, which may be applicable to a realistic network with a degree distribution such as power-law degree distribution. The random graphs assumed in this paper essentially the same as the configuration model treated in [13]. Moreover, this paper deals with s-t cut capacity which is more informative on network capacities instead of the global cut capacity [8].

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2. Preliminaries

In this section, we first introduce several basic definitions and notation used throughout the paper. Then, an ensemble of undirected graphs treated in this paper is defined.

2.1 Notation and Definitions

A graph \( G \overset{\Delta}{=} (V, E) \) is a pair of a vertex set \( V \overset{\Delta}{=} \{v_1, v_2, \ldots, v_n\} \) and an edge set \( E \overset{\Delta}{=} \{e_1, e_2, \ldots, e_m\} \) where \( e_j = (u, v), u, v \in V \) is an edge. If \( e_j = (u, v) \) is not an ordered pair, i.e., \( (u, v) = (v, u) \), the graph \( G \) is called an undirected graph. Assume that an undirected graph \( G \overset{\Delta}{=} (V, E) \) is given. A non-overlapping bi-partition \( (X, V \setminus X) \) such that \( V = X \cup (V \setminus X) \) is called a cut where \( X \) is a non-empty proper subset of \( V (X \neq V) \). The set of edges bridging \( X \) and \( V \setminus X \) is referred to as the cut-set corresponding to the cut \( (X, V \setminus X) \), which is denoted by \( \partial(X) \) (or equivalently \( \partial(V \setminus X) \)). The cut weight (i.e., cut capacity) of \( X \) is defined as \( |\partial(X)| \) which represents the cardinality of the cut-set \( \partial(X) \). This means that each edge weight in a graph is assumed to be one (i.e., unit link capacity) in our setting.

If a cut \( (X, V \setminus X) \) separates two vertices \( s, t \in V (s \neq t) \), the cut \( (X, V \setminus X) \) is called an \( s-t \) cut and the corresponding cut-set is called an \( s-t \) cut-set. The minimum \( s-t \) cut is an \( s-t \) cut whose cut weight is the smallest among all the \( s-t \) cut-sets in a given graph.

2.2 Random Graphs with Specified Degree Distribution

In the following, we will define an ensemble of undirected graphs. The random graph ensemble presented here is essentially the same as the configuration model treated in [13].

Let \( n (n \geq 1) \) be the number of vertices and \( d_i \) be the fraction of vertices having degree \( i \) such that \( nd_i \) is an non-negative integer and \( \sum_{i=0}^{\infty} in_i \) is even. It should be remarked that multiple edges and self-loops are allowed to exist in this model. We define \( d(x) \overset{\Delta}{=} \sum_{i=0}^{\infty} d_i x^i \) to be the generating function representing a degree distribution. Due to these assumptions, the number of edges \( m \) is given by \( 1/2 \sum_{i=0}^{\infty} in_i \). The set \( R_{n,d} \) denotes the set of all the undirected graphs satisfying the above assumption. We here assign the probability

\[
P(G) \overset{\Delta}{=} \frac{1}{|R_{n,d}|}
\]

for each \( G \in R_{n,d} \). The pair \( (R_{n,d}, P) \) defines an ensemble of random graphs treated in this paper.

3. Cut Weight Distribution

In this section, we define an incidence graph corresponding to a undirected graph. Then, we will discuss the cut weight distribution.

3.1 Incidence Graph

In this paper, we use a bipartite graph, which is called an incidence graph, corresponding to a given undirected graph. The incidence graph clarifies the close relationship between the incidence vectors of cut and cut-sets. In the following, we will explain the definition of the incidence graph.

Suppose that an undirected graph \( G \) is given. In order to construct the incidence graph from \( G \), for each edge \( e = (x, y) \in E \), we insert a new vertex \( v_e \) between \( x \) and \( y \). The new vertex \( v_e \) is, thus, adjacent to \( x \) and \( y \). Formally, the triple \((V_1, V_2, E')\) for the incidence graph \( G' \) is defined by

\[
V_1 \overset{\Delta}{=} V, \quad V_2 \overset{\Delta}{=} \{v_e \mid e \in E\},
\]

\[
E' \overset{\Delta}{=} \{(x, v_e), (y, v_e) \mid e = (x, y) \in E\}.
\]

From this definition, it is clear that the degree of all vertices in \( V_2 \) is 2. Figure 1 illustrates the correspondence between the original graph (left) and the incidence graph (right).

3.2 Relationship Between Cut-Set Vector and Cut Vector

For a given undirected graph \( G \overset{\Delta}{=} (V, E) \), the cut vector \( \text{cut}(X) \overset{\Delta}{=} (a_1, \ldots, a_n) \) of a cut \((X, V \setminus X)\) is defined by \( a_i \overset{\Delta}{=} 1 [v_i \in X] \) for \( i \in [1, n] \). The notation \([a, b]\) denotes the set of consecutive integers from \( a \) to \( b \). The function \( 1 [\cdot] \) is the indicator function that takes value 1 if the condition is true; otherwise it takes value 0. Namely, the cut vector \( \text{cut}(X) \) is the incidence vector of the cut \((X, V \setminus X)\). In a similar manner, we will define the cut-set vector as follows. The cut-set vector \( \text{cutset}(X) \overset{\Delta}{=} (b_1, \ldots, b_m) \) corresponding to a cut \((X, V \setminus X)\) is defined by \( b_i \overset{\Delta}{=} 1 [e_i \in \partial(X)] \) for \( i \in [1, m] \).

The incidence graph naturally connects a cut vector \( \text{cut}(X) \) and the corresponding cut-set vector \( \text{cutset}(X) \) for any \( X \subset V (X \neq \emptyset) \) in the following way. Suppose that an undirected graph \( G = (V, E) \) and the corresponding incidence graph \( G' = (V_1, V_2, E') \) are given. The vertices in \( V_1 \) are called variable nodes which are depicted by circles in Fig.1. We assume that a binary value \((0 \text{ or } 1)\) can be assigned to a variable node. The vertices in \( V_2 \) are

\footnote{A cut vector can be considered as a factor graph. The details will be given in Sect. 3.2.}
called function nodes which are represented by squares in Fig. 1 (right). The function node also have a binary value which is determined by the bitwise exclusive-OR (sum over $\mathbb{F}_2$) of values in adjacent variable nodes. Let us assume that $x \triangleq (x_1, \ldots, x_n) \in \{0,1\}^n$ is assigned to the variable nodes (i.e., $x_i$ is the assigned value for $v_i$) and that $y \triangleq (y_1, \ldots, y_m) \in \{0,1\}^m$ is the resulting values (i.e., $y_i$ is the exclusive-OR value at $v_i$).

From this definition, the graph $G'$ defines a linear map from $x$ to $y$. We will denote the linear map from $x$ to $y$ by $y = F_G(x)$. The next lemma presents the linear relation between a cut vector and the corresponding cut-set vector.

**Lemma 1.** Assume that an undirected graph $G = (V, E)$ is given. For any $X \subset V \setminus \emptyset$, the following linear relation
\[
\text{cutset}(X) = F_G(\text{cut}(X))
\]
holds.

**Proof.** Let $(y_1, \ldots, y_m) \triangleq F_G(\text{cut}(X))$ be a vector at the function nodes and $G' \triangleq (V_1, V_2, E')$ be the incidence graph corresponding to $G$. Two variable nodes adjacent to $v_i$ are denoted by $a, b \in V_1$. If $a \in X, b \in V \setminus X$, then $y_i = 1$. Otherwise, $y_i = 0$. From the definition of the incidence graph, $y_i = 1$ is equivalent to $e_i \in \partial(X)$. This proves the relation $\text{cutset}(X) = F_G(\text{cut}(X))$. \hfill $\Box$

This lemma indicates that a linear row space spanned by the incidence matrix of $G$ coincides with the set of incidence vectors of cut-sets. It should be remarked that the linear relation in Lemma 1 has been long known in the field of graph theory; e.g., [6].

### 3.3 Global Cut Weight Distribution

For a given undirected graph $G \triangleq (V, E)$, we can enumerate the number of cut-sets with cut weight $w$. The global cut weight distribution is defined by
\[
B_G(w) \triangleq \sum_{E' \subset E} \mathbb{I}[E' \text{ is a cut-set, } |E'| = w]
\]
for non-negative integer $w$. Note that $E'$ can be the empty set. From this definition, if $G$ is disconnected, then $E' = \emptyset$ becomes a cut-set. This means that a global minimum cut weight of unconnected graph is zero. The fact will be used in Sect. 4.5.

We use the term global to clarify the distinction from the $s$-$t$ cut weight distribution defined later. The global cut weight distribution $B_G(w)$ represents the number of cut-sets with cut weight $w$. In this paper, we assume the situation where an unit weight is assigned to an edge in order to keep the following discussion simple. Note that a generalization to a random graph with stochastic edge weights is straightforward.

The following lemma plays an important role for evaluating the ensemble average of the cut weight distribution $B_G(w)$.

**Lemma 2.** The global cut weight distribution $B_G(w)$ can be upper bounded by
\[
B_G(w) \leq \frac{1}{2} \sum_{u=1}^{n-1} A_G(u, w),
\]
for $w \in [0, m]$. The quantity $A_G(u, w)$ is defined by
\[
A_G(u, w) \triangleq \sum_{a \in Z^{(u)}} \sum_{b \in Z^{(u+1)}} \mathbb{I}[F_G(a) = b],
\]
for $u \in [1, n-1], w \in [0, m]$. The set of the constant weight binary vectors $Z^{(u)}$ is defined as $Z^{(u)} \triangleq \{(z_1, \ldots, z_u) \in \{0,1\}^u \mid \sum_{i=1}^{u} z_i = w \}$.

**Proof.** For any undirected graph $G = (V, E)$, $B_G(w)$ can be upper bounded by
\[
B_G(w) \leq \frac{1}{2} \sum_{X \subset V \setminus \emptyset} \mathbb{I}[|\partial(X)| = w].
\]
The factor 1/2 is required for compensating the double counting for $X$ and $V \setminus X$. The equality is attained if and only if $G$ is connected. Namely, in such a case, $\partial(X) = \partial(V \setminus X)$ holds for any $X \subset V$. Suppose that $G$ is an unconnected graph with two-connected components. The set of vertices in the two-connected components are denoted by $A$ and $B$ where no edges exist between $A \subset V$ and $B = V \setminus A$. Assume that $A$ is divided into two disjoint sets $A_1$ and $A_2$ and that $B$ is divided into two disjoint sets $B_1$ and $B_2$. Due to the unconnectivity between $A$ and $B$, we have $\partial(A_1 \cup B_1) = \partial(A_1 \cup B_2) = \partial(A_2 \cup B_1) = \partial(A_2 \cup B_2)$.

This means that we count the same cut-set four-times in the enumeration. Similar multiple counts occur if we consider an unconnected graph with many-connected components. Although, in such a case, the factor 1/2 is too small to give correct compensation, the inequality (7) is still valid.

Due to Lemma 1, the right-hand side of (7) can be rewritten as
\[
\frac{1}{2} \sum_{X \subset V \setminus \emptyset} \mathbb{I}[|\partial(X)| = w] = \frac{1}{2} \sum_{u=1}^{n-1} A_G(u, w).
\]
Substituting (8) into (7), we obtain the claim. \hfill $\Box$

### 3.4 $s$-$t$ Cut Weight Distribution

In the previous subsection, we derived an upper bound on the global cut weight distribution $B_G(w)$. In a similar manner, we can enumerate the number of $s$-$t$ cut-sets with cut weight $w$. Assume that an undirected graph $G = (V, E)$ and two vertices $s, t \in V \setminus \{s \neq t\}$ are given. The $s$-$t$ cut weight distribution is defined by
for non-negative integer \( w \). The \( s \)-cut weight distribution \( B_G^{(s)}(w) \) represents the number of \( s \)-cut sets with cut weight \( w \).

The following lemma for the \( s \)-cut weight distribution can be proved with an argument similar to the proof of Lemma 2.

**Lemma 3.** The \( s \)-cut weight distribution \( B_G^{(s)}(w) \) can be upper bounded by

\[
B_G^{(s)}(w) \leq \frac{1}{2} \sum_{u=1}^{n-1} A_G^{(s)}(u, w),
\]

for \( w \in [0, m] \), where the quantity \( A_G^{(s)}(u, w) \) is defined by

\[
A_G^{(s)}(u, w) \triangleq \sum_{a \in Y^{(s)} \cap Z^{(u)}} \sum_{b \in Z^{(m-u)}} E[I_F(a) = b],
\]

for \( u \in [1, n-1] \) and \( w \in [0, m] \). The set \( Y^{(s)} \) denotes the set of all cut vectors representing \( s \)-cut, which is given by

\[
Y^{(s)} = \{ a \in [0, 1]^n \mid a_s \neq a_t \}.
\]

The number of possible ways that the Hamming weight of a cut vector is \( u \) and \( w \) given \( U \) is conditionally independent of \( U \) and \( W \). We thus have

\[
\Pr (W = w \mid U = u) = \sum_{h=0}^{2m} \Pr (W = w, H = h \mid U = u) = \sum_{h=0}^{2m} \Pr (W = w \mid H = h, U = u) \Pr (H = h \mid U = u).
\]

We first focus on the conditional probability \( \Pr (H = h \mid U = u) \), which is the probability that the number of active edges is \( h \) for a given cut vector of weight \( u \). There are \( \binom{n}{u} \) possible ways that the Hamming weight of a cut vector is \( u \). The number of possible ways that \( h \) active edges connect to \( u \) variable nodes having the value one can be enumerated as \( \text{coef} \left( \prod_{i=0}^{\infty} (1 + x^i y^{u-i}), x^h y^a \right) \). We thus have

\[
\Pr (H = h \mid U = u) = \frac{\text{coef} \left( \prod_{i=0}^{\infty} (1 + x^i y^{u-i}), x^h y^a \right)}{\binom{n}{u}}.
\]

We then consider the conditional probability \( \Pr (W = w \mid H = h, U = u) \). Since \( W \) is conditionally independent of \( U \) given \( H \), we have

\[
\Pr (W = w \mid H = h, U = u) = \Pr (W = w \mid H = h).
\]
There are \((2^n)_h\) possible ways that the number of active edges is \(h\). A function node with the value one is connected to only one active edge because the value of a function node is given by exclusive-OR of values of the adjacent variable nodes. Since the weight of the cut-set vector is \(w\), the number of such function nodes with the value one is \(w\) and remaining \(m - w\) function nodes have the value zero. Note that a function node with the value zero is connected to two active edges or to non-active edges. There are \(\binom{n}{a}\) possible ways that \(w\) function nodes have the one value. We have two possibility that only one active edge is connecting to the function node. Remaining \(h - w\) active edges must be connect to \((h - w)/2\) function nodes which accepts two active edges. Consequently, the number of possible ways satisfying \(W = w\) and \(U = u\) is given by \(2^{(n)}_w\binom{m-w}{(h-w)/2}\). Note that \(\binom{n}{a}\) is defined by \(0\) if \(a\) or \(b\) is not a non-negative integer. We finally have

\[
\Pr(W = w \mid H = h) = \frac{2^n \binom{m-w}{(h-w)/2}}{\binom{m}{h}}.
\]

Combining (16), (17), (18), (19) and (20), we obtain the lemma. \(\square\)

As a special case of Lemma 4, if \(d(x) = x^c\) (i.e., \(G\) is a \(c\)-regular graph), (13) can be simplified as

\[
\mathbb{E}[A_G(u, w)] = \frac{2^w \binom{m-w}{(h-w)/2}}{\binom{m}{h}}.
\]

The following theorem provides an upper bound on average cut weight distribution that is the basis of our analysis.

**Theorem 1.** The expectation of the global cut weight distribution \(B_G(w)\) over \((R_{n,d}, P)\) can be upper bounded by

\[
\mathbb{E}[B_G(w)] \leq 2^{n-1} \left( \frac{m}{w} \right)^{n-1} \sum_{h=0}^{2m-n} \binom{m-w}{(h-w)/2} \text{coef} \left( \prod_{i=0}^{n} (1 + x^i)^{h_i}, x^h \right) \left( \frac{2^n}{h} \right).
\]

Proof. Applying Lemma 4 to the inequality (12), we obtain the claim of this theorem. \(\square\)

4.2 Upper Bound on Average \(s\)-\(t\) Cut Weight Distribution

We can extend the Lemma 4 and Theorem 1 to \(s\)-\(t\) cut weight distribution \(B_G^{(s,t)}(w)\).

**Lemma 5.** For any pair of \(s, t \in V\) \((s \neq t)\), the expectation of \(A_G^{(s,t)}(u, w)\) over \((R_{n,d}, P)\) is given by

\[
\mathbb{E}[A_G^{(s,t)}(u, w)] = \frac{2^{n+1} u(n-u) m}{n(n-1)} B_G^{(s,t)}(w).
\]

Applying Lemma 5 to the inequality (29), we obtain the claim of this theorem. \(\square\)

4.3 Minimum Global Cut Weight

The knowledge on the ensemble average of \(B_G(w)\) can be utilized to estimate the statistical behavior of the global minimum cut weight. Let \(\lambda_G\) be the global minimum cut weight.
of the graph $G$ and
\[
C_G(\delta) = \sum_{w=0}^{\delta-1} B_G(w)
\] (30)
be the accumulate global cut weight of $G$, where $\delta$ is a positive integer. From this definition, it is clear that the graph $G$ does not contain a cut with weight smaller than $\delta$ if $C_G(\delta)$ is zero. This implies that $C_G(\delta) = 0$ is equivalent to $\lambda_G \geq \delta$ and that
\[
Pr(\lambda_G \geq \delta) = Pr(C_G(\delta) = 0) = 1 - Pr(C_G(\delta) \geq 1). \tag{31}
\]
The second equality is due to the non-negativity of $C_G(\delta)$. The following theorem gives a lower bound on the probability $Pr(\lambda_G \geq \delta)$, which is a direct consequence of the first moment method.

**Theorem 3.** The probability for minimum global cut weight $Pr(\lambda_G \geq \delta)$ is lower bounded by
\[
Pr(\lambda_G \geq \delta) \geq 1 - \sum_{w=0}^{\delta-1} \frac{m^w}{w!} \left[ \prod_{\ell=0}^{\infty} (1 + x^{\delta} y^{\ell m}, x^{\delta} y^{\ell}) \right]
\] (32)
for $\delta \in \mathbb{N}$ over the ensemble $(R_{n,d}, P)$. The set $\mathbb{N}$ represents the set of positive integers.

**Proof.** The Markov inequality provides an lower bound on $Pr(\lambda_G \geq \delta)$ as follows:
\[
Pr(\lambda_G \geq \delta) = 1 - Pr(C_G(\delta) \geq 1)
\geq 1 - E[C_G(\delta)]
\geq 1 - \sum_{w=0}^{\delta-1} E[B_G(w)]. \tag{33}
\]
Applying the lower bound (22) in Theorem 1 to the inequality (33), we obtain the claim of this theorem. \hfill \Box

### 4.4 Minimum $s$-$t$ Cut Weight

Let $\lambda_G^{(s,t)}$ be the minimum $s$-$t$ cut weight of the graph $G$. As in the previous subsection, we can derive a lower bound on $Pr(\lambda_G^{(s,t)} \geq \delta)$. The quantity
\[
C_G^{(s,t)}(\delta) = \sum_{w=0}^{\delta-1} B_G^{(s,t)}(w)
\] (34)
is the accumulate $s$-$t$ cut weight of $G$ where $\delta$ is a positive integer. The following lemma for the minimum $s$-$t$ cut weight is obtained due to an argument similar to the proof of Theorem 3. This theorem is the main contribution of this work.

**Theorem 4.** The probability for minimum $s$-$t$ cut weight $Pr(\lambda_G^{(s,t)} \geq \delta)$ can be lower bounded by
\[
Pr(\lambda_G^{(s,t)} \geq \delta) \geq 1 - \sum_{w=0}^{\delta-1} \frac{2^w (m)}{w!} \left[ \prod_{\ell=0}^{\infty} (1 + x^{\delta} y^{\ell m}, x^{\delta} y^{\ell}) \right]
\] (35)
for $\delta \in \mathbb{N}$ over the ensemble $(R_{n,d}, P)$.

The minimum $s$-$t$ cut weight is equal to the maximum flow due to the max-flow min-cut theorem [2]. Therefore, the probability $Pr(\lambda_G^{(s,t)} \geq \delta)$ can be considered as the accumulate probability distribution for the maximum flow (or unicast capacity).

### 4.5 Upper Bound on Unconnected Probability

The statistical property on the connectivity of random graphs is one of fundamental problems in random graph theory [9], [15]. The unconnected probability is the probability that randomly chosen graph from the ensemble is unconnected. We can obtain the upper bound on the unconnected probability over the ensemble $(R_{n,d}, P)$ due to the lower bound of the global minimum cut weight in Theorem 3. In this subsection, we will give an upper bound on the unconnected probability.

**Theorem 5.** The unconnected probability $P_U$ over $(R_{n,d}, P)$ can be upper bounded by
\[
P_U \leq \frac{1}{2} \sum_{w=0}^{m/2} \frac{2^w (m/2)}{w!} \left[ \prod_{\ell=0}^{\infty} (1 + x^{\delta} y^{\ell m}, x^{\delta} y^{\ell}) \right]
\] (36)

**Proof.** Since the global minimum cut weight of unconnected graphs must be 0, we have
\[
P_U = Pr(\lambda_G = 0) = 1 - Pr(\lambda_G \geq 1). \tag{37}
\]
Applying the lower bound (32) in Theorem 3 to the last equation, we obtain the claim of this theorem. \hfill \Box

This theorem implies that the ensemble average of the cut weight distributions also contains certain statistical information on the connectivity of graphs in the ensemble.

### 5. Numerical Result

In order to evaluate the tightness of the lower bound shown in Theorem 4, we made the following computer experiments. In the experiments, we generated $10^4$-instances of undirected graphs from the random graph ensemble defined in Sect. 2.2. The minimum $s$-$t$ cut weight for each instance was computed by using Ford-Fulkerson algorithm [2].

Figure 2 presents the accumulate distribution of the minimum $s$-$t$ cut weight $Pr(\lambda_G^{(s,t)} \geq \delta)$ of sparse and dense graph ensembles. In the sparse case, the number of vertices...
6. Conclusion

In this paper, a lower bound on the accumulate distribution of the minimum cut capacity for random graphs with specified degree distribution is presented. From the computer experiments, it is observed that the lower bound reflects actual statistical behavior of the minimum cut capacity.

The proof technique used in this paper has close relationship to the analysis for the average weight distributions of irregular LDGM codes [14]. An advantage of the proposed technique is its applicability for a graph ensemble with finite number of vertices and edges. Most related studies deal with asymptotic behaviors and cannot directly be applied to a finite size graph ensemble. The second advantage of the proposed technique is extensibility. In this paper, we discussed the minimum cut capacity of random graphs with specified degree distribution, but we can also apply the technique for some problems in graph theory (e.g., minimum vertex cover).

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References

[1] R. Ahlswede, S.Y. Li, and R. Yeung, “Network information flow,” IEEE Trans. Inf. Theory, vol.46, no.4, pp.1204–1216, July 2000.
[2] A. Schrijver, Combinatorial Optimization: Polyhedra and Efficiency, Springer-Verlag, Berlin, 2003.
[3] D.R. Karger, “Random sampling in cut, flow, and network design problems,” Math. Oper. Res., vol.24, no.2, pp.383–413, May 1999.
[4] A. Ramamoorthy, J. Shi, and R. Wesel, “On the capacity of network coding for random networks,” IEEE Trans. Inf. Theory, vol.51, no.8, pp.2878–2885, Aug. 2005.
[5] H. Wang, P. Fan, and K. Letaief, “Maximum flow and network capacity of network coding for ad-hoc networks,” IEEE Trans. Wireless Commun., vol.6, no.12, pp.4193–4198, Dec. 2007.
[6] S. Hakimi and H. Frank, “Cut-set matrices and linear codes,” IEEE Trans. Inf. Theory, vol.11, no.3, pp.457–458, July 1965.
[7] A. Yano and T. Wadayama, “Probabilistic analysis of the network reliability problem on a random graph ensemble,” 2012 International Symposium on Information Theory and its Applications, pp.327–331, Oct. 2012.
[8] Y. Fuji and T. Wadayama, “A coding theoretic approach for evaluating accumulate distribution on minimum cut capacity of weighted random graphs,” 2012 International Symposium on Information Theory and its Applications, pp.332–336, Oct. 2012.
[9] P. Erdős and A. Rényi, “On random graphs I,” Publicationes Mathematicae, vol.6, pp.290–297, 1959.
[10] A. Barabási and J. Frangos, Linked: The New Science of Networks, Perseus, 2002.
[11] R. Albert and A.L. Barabási, “Statistical mechanics of complex networks,” Reviews of Modern Physics, vol.74, no.1, p.47, 2002.
[12] A.L. Barabási, R. Albert, and H. Jeong, “Mean-field theory for scale-free random networks,” Physica A: Statistical Mechanics and its Applications, vol.272, no.1, pp.173–187, 1999.
[13] M.E. Newman, “The structure and function of complex networks,” SIAM Review, vol.45, no.2, pp.167–256, 2003.
[14] C.H. Hsu and A. Anastasopoulos, “Asymptotic weight distributions of irregular repeat-accumulate codes,” Glob. Telecomm. Conf., pp.1147–1151, 2005.
[15] B. Bollobás, Random Graphs, Cambridge University Press, 2001.
[16] Y. Fujii and T. Wadayama, “An analysis on minimum s-t cut capacity of random graphs with specified degree distribution,” 2013 IEEE International Symposium on Information Theory, pp.2895–2899, July 2013.

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