A Lower Bound of The First Eigenvalue of a Closed Manifold with Positive Ricci Curvature *

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Abstract

We give an estimate on the lower bound of the first non-zero eigenvalue of the Laplacian for a closed Riemannian manifold with positive Ricci curvature in terms of the in-diameter and the lower bound of the Ricci curvature.

1 Introduction

If $M$ is an n-dimensional closed Riemannian manifold whose Ricci curvature has a positive lower bound $(n-1)K$ for some constant $K > 0$, A. Lichnerowicz [6] gave the following lower bound of the first non-zero eigenvalue $\lambda$ of the Laplacian on $M$

$$\lambda \geq nK. \tag{1}$$

This estimate gives no information when the above constant $K$ vanishes. In such case, Li-Yau [5] and Zhong-Yang [12] provided another lower bound

$$\lambda \geq \frac{\pi^2}{d^2}.$$

It is an interesting problem to find a unified lower bound of the first non-zero eigenvalue $\lambda$ in terms of the lower bound $(n-1)K$ of the Ricci curvature and the diameter $d$, in-diameter $\hat{d}$ and other geometric quantities, which do not vanish as $K$ vanishes, of the manifold with positive Ricci curvature. D. Yang [11] showed a lower bound $(1/4)(n-1)K + \pi^2/d^2$. In this paper we give a new estimate on the lower bound of the first non-zero eigenvalue of a closed Riemannian manifold with positive lower bound of Ricci curvature in terms

*2000 Mathematics Subject Classification: Primary 58J50, 35P15; Secondary 53C21
of the in-diameter and the lower bound of Ricci curvature. Instead of using the Zhong-Yang’s canonical function or the "midrange" of the normalized eigenfunction of the first eigenvalue in the proof, we use a function $\xi$ that the author constructed in [8] for the construction of the suitable test function and use the structure of the nodal domains of the eigenfunction. That provides a new way to sharpen the bound. We have the following result.

**Theorem 1.** If $M$ is an $n$-dimensional closed Riemannian manifold. Suppose that Ricci curvature $\text{Ric}(M)$ of $M$ is bounded below by $(n-1)K$ for some positive constant $K$, i.e. $\text{Ric}(M)$ satisfies (2)

\[(2) \quad \text{Ric}(M) \geq (n-1)K,\]

then the first non-zero eigenvalue $\lambda$ of the Laplacian of $M$ has the following lower bound

\[(3) \quad \lambda \geq \frac{1}{2}(n-1)K + \frac{\pi^2}{\tilde{d}^2},\]

where $\tilde{d}$ is the diameter of the largest interior ball in the nodal domains of the first eigenfunction.

We derive some preliminary estimates and conditions for test functions in the next section and construct the needed test function and prove the main result in the last section.

## 2 Preliminary Estimates

The classic Lichnerowicz Theorem [6] states that if $M$ is an $n$-dimensional compact manifold without boundary whose Ricci curvature satisfies (2) then the first positive closed eigenvalue has a lower bound in (1). For the completeness and consistency, we use gradient estimate in [3]-[5] and [10] to derive the Lichnerowicz estimate.

**Lemma 1 (Lichnerowicz).** Under the conditions in Theorem 1, the estimate (1) holds.

**Proof.** Let $v$ be a normalized eigenfunction of the first closed eigenvalue such that

\[(4) \quad \sup_M v = 1, \quad \inf_M v = -k\]
with $0 < k \leq 1$. The function $v$ satisfies the following equation

(5) $\Delta v = -\lambda v$ in $M$,

where $\Delta$ is the Laplacian of $M$.

Take an orthonormal frame $\{e_1, \ldots, e_n\}$ of $M$ about $x_0 \in M$. At $x_0$ we have

$$\nabla e_j(|\nabla v|^2)(x_0) = \sum_{i=1}^n 2v_i v_{ij}$$

and

$$\Delta(|\nabla v|^2)(x_0) = 2 \sum_{i,j=1}^n v_{ij} v_{ij} + 2 \sum_{i,j=1}^n v_i v_{jj} + 2 \sum_{i,j=1}^n R_{ij} v_i v_j$$

$$\geq 2 \sum_{i=1}^n v_i^2 + 2 \nabla v \nabla (\Delta v) + 2 \nabla v \nabla (\Delta v) + 2(n - 1) K |\nabla v|^2$$

$$\geq \frac{2}{n} (\Delta v)^2 - 2 \lambda |\nabla v|^2 + 2(n - 1) K |\nabla v|^2.$$ 

Thus at all point $x \in M$,

(6) $\frac{1}{2} \Delta(|\nabla v|^2) \geq \frac{1}{n} \lambda^2 v^2 + [(n - 1) K - \lambda] |\nabla v|^2$.

On the other hand, after multiplying (5) by $v$ and integrating both sides over $M$, we have

$$\int_M \lambda v^2 \, dx = - \int_M v \Delta v \, dx = \int_M |\nabla v|^2 \, dx.$$ 

Integrating (6) over $M$ and using the above equality, we get

(7) $0 \geq \int_M (nK - \lambda) \frac{n-1}{n} \lambda v^2 \, dx.$

Therefore (1) holds.
Lemma 2. Let \( v \) be, as the above, the normalized eigenfunction for the first non-zero eigenvalue \( \lambda \). Then \( v \) satisfies the following

\[
|\nabla v|^2 \frac{b^2 - v^2}{b^2} \leq \lambda,
\]

where \( b > 1 \) is an arbitrary constant.

Proof. Consider the function

\[
P(x) = |\nabla v|^2 + Av^2,
\]

where \( A = \lambda(1 + \epsilon) \) for small \( \epsilon > 0 \). Function \( P \) must achieve its maximum at some point \( x_0 \in M \). We claim that \( \nabla v(x_0) = 0 \).

If on the contrary, \( \nabla v(x_0) \neq 0 \), then we can rotate the local orthonormal about \( x_0 \) such that \( v_1(x_0) \neq 0 \) and \( v_i(x_0) = 0 , \ i \geq 2 \).

Since \( P \) achieves its maximum at \( x_0 \), we have,

\[
\nabla P(x_0) = 0 \quad \text{and} \quad \Delta P(x_0) \leq 0.
\]

That is, at \( x_0 \) we have

\[
0 = \frac{1}{2} \nabla_i P = \sum_{j=1}^{n} v_j v_{ji} + Av_i,
\]

(10) \( v_{11} = -Av \) and \( v_{1i} = 0 \) \( i \geq 2 \),

and

\[
0 \geq \frac{1}{2} \Delta P(x_0) = \sum_{i,j=1}^{n} (v_{ji} v_{ji} + v_j v_{jii} + Av_i v_i + Av_{ii})
\]

\[
= \sum_{i,j=1}^{n} (v_{ji}^2 + v_j (v_{ii})_j + R_{ji} v_j v_i + Av_{ii}^2 + Av_{ii})
\]

\[
= \sum_{i,j=1}^{n} v_{ji}^2 + \nabla v \nabla (\Delta v) + \text{Ric}(\nabla v, \nabla v) + A|\nabla v|^2 + Av \Delta v
\]

\[
\geq v_{11}^2 + \nabla v \nabla (\Delta v) + (n - 1)K|\nabla v|^2 + A|\nabla v|^2 + Av \Delta v
\]

\[
= (-Av)^2 - \lambda |\nabla v|^2 + (n - 1)K|\nabla v|^2 + A|\nabla v|^2 - \lambda Av^2
\]

\[
= (A - \lambda + (n - 1)K)|\nabla v|^2 + Av^2(A - \lambda)
\]
where we have used (10) and (2). Therefore at \(x_0\),
\[
0 \geq (A - \lambda)|\nabla v|^2 + A(A - \lambda)v^2, \tag{11}
\]
that is,
\[
|\nabla v(x_0)|^2 + \lambda(1 + \epsilon)v(x_0)^2 \leq 0.
\]
Thus \(\nabla v(x_0) = 0\). This contradicts \(\nabla v(x_0) \neq 0\). So the above claim is right.

Therefore we have \(\nabla v(x_0) = 0\),
\[
P(x_0) = |\nabla v(x_0)|^2 + Av(x_0)^2 = Av(x_0)^2 \leq A,
\]
and at all \(x \in M\)
\[
|\nabla v(x)|^2 + Av(x)^2 = P(x) \leq P(x_0) \leq A.
\]
Letting \(\epsilon \to 0\) in the above inequality, the estimate (8) follows.

We want to improve the upper bound in (8) further and proceed in the following way.

Define a function \(Z\) on \([-\sin^{-1}(k/b), \sin^{-1}(1/b)]\) by
\[
Z(t) = \max_{x \in M, t = \sin^{-1}(v(x)/b)} \frac{|\nabla v|^2}{b^2 - v^2}/\lambda.
\]
From (8) we have
\[
Z(t) \leq 1 \quad \text{on } [-\sin^{-1}(k/b), \sin^{-1}(1/b)] \tag{12}
\]
For convenience, in this paper we let
\[
\alpha = \frac{1}{2}(n - 1)K \quad \text{and} \quad \delta = \alpha/\lambda. \tag{13}
\]
By (11) we have
\[
\delta \leq \frac{n - 1}{2n}. \tag{14}
\]
We have the following conditions on the test function \(Z\).

**Theorem 2.** If the function \(z : [-\sin^{-1}(k/b), \sin^{-1}(1/b)] \to \mathbb{R}^1\) satisfies the following

1. \(z(t) \geq Z(t) \quad t \in [-\sin^{-1}(k/b), \sin^{-1}(1/b)]\),
2. there exists some \( x_0 \in M \) such that at point \( t_0 = \sin^{-1}(v(x_0)/b) \)
\[ z(t_0) = Z(t_0), \]
3. \( z(t_0) > 0 \), and
4. \( z'(t_0) \sin t_0 \geq 0 \),
then we have the following
\[ 0 \leq \frac{1}{2} z''(t_0) \cos^2 t_0 - z'(t_0) \cos t_0 \sin t_0 - z(t_0) + 1 - 2\delta \cos^2 t_0. \]

Proof. Define
\[ J(x) = \left\{ \frac{|\nabla v|^2}{b^2 - v^2} - \lambda z \right\} \cos^2 t, \]
where \( t = \sin^{-1}(v(x)/b) \). Then
\[ J(x) \leq 0 \text{ for } x \in M \text{ and } J(x_0) = 0. \]
If \( \nabla v(x_0) = 0 \), then
\[ 0 = J(x_0) = -\lambda z \cos^2 t. \]
This contradicts Condition 3 in the theorem. Therefore
\[ \nabla v(x_0) \neq 0. \]

The Maximum Principle implies that
\[ \nabla J(x_0) = 0 \quad \text{and} \quad \Delta J(x_0) \leq 0. \]

\( J(x) \) can be rewritten as
\[ J(x) = \frac{1}{b^2} |\nabla v|^2 - \lambda z \cos^2 t. \]
Thus (16) is equivalent to
\[ \frac{2}{b^2} \sum_i v_i v_{ij} \bigg|_{x_0} = \lambda \cos t [z' \cos t - 2 \sin t] |t_j \bigg|_{x_0} \]
and
\[ 0 \geq \frac{2}{b^2} \sum_{i,j} v_{ij}^2 + \frac{2}{b^2} \sum_{i,j} v_i v_{ijj} - \lambda (z'' |\nabla t|^2 + z' \Delta t) \cos^2 t \]
\[ + 4 \lambda z' \cos t \sin t |\nabla t|^2 - \lambda z \Delta \cos^2 t \bigg|_{x_0}. \]
Rotate the frame so that \( v_1(x_0) \neq 0 \) and \( v_i(x_0) = 0 \) for \( i \geq 2 \). Then (17) implies

\[
\begin{align*}
\frac{\partial}{\partial x} = \frac{\lambda b}{2} (z' \cos t - 2z \sin t) \\
\end{align*}
\]

and \( v_{11} \big|_{x_0} = 0 \) for \( i \geq 2 \).

Now we have

\[
\begin{align*}
|\nabla v|_x = \lambda b^2 z \cos t, \\
|\nabla t|_x = \frac{|\nabla v|_x}{b^2 - v^2} = \lambda z, \\
\frac{\Delta v}{b}|_{x_0} = \Delta \sin t = \cos t \Delta t - \sin t |\nabla t|_x, \\
\frac{\Delta t}{b}|_{x_0} = \frac{1}{\cos t} (\sin t |\nabla t|_x^2 + \frac{\Delta v}{b}) \\
&= \frac{1}{\cos t} [\lambda z \sin t - \lambda \frac{b}{v} ]|_{x_0}, \quad \text{and} \\
\Delta \cos^2 t|_{x_0} = \Delta \left(1 - \frac{v^2}{b^2}\right) = -\frac{2}{b^2} |\nabla v|_x^2 - \frac{2}{b^2} v \Delta v \\
&= -2 \lambda z \cos^2 t + \frac{2}{b^2} \lambda v^2|_{x_0}.
\end{align*}
\]

Therefore,

\[
\begin{align*}
\frac{2}{b^2} \sum_{i,j} v_{ij}^2 &\geq \frac{2}{b^2} v_{11}^2 \\
&= \frac{\lambda^2}{2} (z')^2 \cos^2 t - 2 \lambda^2 z' \cos t \sin t + 2 \lambda^2 z^2 \sin^2 t|_{x_0}, \\
\frac{2}{b^2} \sum_{i,j} v_i v_{ij} &\geq \frac{2}{b^2} (\nabla v \nabla (\Delta v) + \text{Ric} (\nabla v, \nabla v)) \\
&\geq \frac{2}{b^2} (\nabla v \nabla (\Delta v) + (n - 1)K |\nabla v|_x^2) \\
&= -2 \lambda^2 z \cos^2 t + 4 \alpha \lambda \cos^2 t|_{x_0}
\end{align*}
\]
−λ(\(z''|\nabla t|^2 + z'|t\Delta t\) \cos^2 t \bigg|_{x_0})
= −\(\lambda^2 zz'' \cos^2 t - \lambda^2 zz' \cos t \sin t\) + \frac{1}{b}\lambda^2 z'v \cos t \bigg|_{x_0},

and

\(4\lambda z' \cos t|\nabla t|^2 - \lambda z \Delta \cos^2 t \bigg|_{x_0}\)
= \(4\lambda^2 z' \cos t \sin t + 2\lambda^2 z^2 \cos^2 t - \frac{2}{b}\lambda^2 zv \sin t \bigg|_{x_0}\).

Putting these results into (18) we get

\[
0 \geq -\frac{1}{2} z''(t_0) \cos^2 t_0 + \frac{1}{2} z'(t_0) \cos t_0 \left( \sin t_0 + \frac{\sin t_0}{z(t_0)} \right) + z(t_0) - 1 + 2\delta \cos^2 t_0 + \frac{1}{4z(t_0)}(z'(t_0))^2 \cos^2 t_0.
\]

(20)

where we used (19). Now

(21)

\(z(t_0) > 0\),

by Condition 3 in the theorem. Dividing two sides of (20) by \(2\lambda^2 z \bigg|_{x_0}\), we have

\[
0 \geq -\frac{1}{2} z''(t_0) \cos^2 t_0 + z'(t_0) \cos t_0 \sin t_0 + z(t_0) - 1 + 2\delta \cos^2 t_0 + \frac{1}{4z(t_0)}(z'(t_0))^2 \cos^2 t_0.
\]

Therefore,

\[
0 \geq -\frac{1}{2} z''(t_0) \cos^2 t_0 + z'(t_0) \cos t_0 \sin t_0 + z(t_0) - 1 + 2\delta \cos^2 t_0
\]

\[
+ \frac{1}{4z(t_0)}(z'(t_0))^2 \cos^2 t_0 + \frac{1}{2} z'(t_0) \sin t_0 \cos t_0 \left[ \frac{1}{z(t_0)} - 1 \right].
\]

(22)

Conditions 1, 2 and 4 in the theorem imply that \(0 < z(t_0) = Z(t_0) \leq 1\) and \(z'(t_0) \sin t_0 \geq 0\). Thus the last two terms in (22) are nonnegative and (15) follows.
3 Proof of the Main Result

Proof of Theorem 1. Let

\[ z(t) = 1 + \delta \xi(t), \]

where \( \xi \) is the functions defined by (31) in Lemma 3. We claim that

\[ Z(t) \leq z(t) \quad \text{on} \quad [-\sin^{-1}(k/b), \sin^{-1}(1/b)]. \]

Lemma 3 implies that for \( t \in [-\sin^{-1}(k/b), \sin^{-1}(1/b)] \) we have the following

\[ \frac{1}{2}z'' \cos^2 t - z' \cos t \sin t - z = -1 + 2\delta \cos^2 t, \]

\[ z'(t) \sin t \geq 0, \]

\[ 0 < 1 - \left( \frac{\pi^2}{4} - 1 \right) \frac{n-1}{2n} \leq 1 - \left( \frac{\pi^2}{4} - 1 \right) \delta = z(0) \leq z(t), \quad \text{and} \]

\[ z(t) \leq z\left( \frac{\pi}{2} \right) = 1. \]

Let \( P \in \mathbb{R}^1 \) and \( t_0 \in [-\sin^{-1}(k/b), \sin^{-1}(1/b)] \) such that

\[ P = \max_{t \in [-\sin^{-1}(k/b), \sin^{-1}(1/b)]} (Z(t) - z(t)) = Z(t_0) - z(t_0). \]

Thus

\[ Z(t) \leq z(t) + P \quad \text{for} \quad t \in [-\sin^{-1}(k/b), \sin^{-1}(1/b)] \quad \text{and} \quad Z(t_0) = z(t_0) + P. \]

Suppose that \( P > 0 \). Then \( z + P \) satisfies the conditions in Theorem 2 \( \text{(15)} \) implies

\[ z(t_0) + P = Z(t_0) \]

\[ \leq \frac{1}{2} (z + P)''(t_0) \cos^2 t_0 - (z + P)'(t_0) \cos t_0 \sin t_0 + 1 - 2\delta \cos^2 t_0 \]

\[ = \frac{1}{2} z''(t_0) \cos^2 t_0 - z'(t_0) \cos t_0 \sin t_0 + 1 - 2\delta \cos^2 t_0 \]

\[ = z(t_0). \]

This contradicts the assumption \( P > 0 \). Thus \( P \leq 0 \) and \( \text{(24)} \) holds. That means

\[ \sqrt{\lambda} \geq \frac{\| \nabla t \|}{\sqrt{z(t)}}. \]
Note that the eigenfunction $v$ of the first nonzero eigenvalue has exactly two nodal domains $D^+ = \{ x : v(x) > 0 \}$ and $D^- = \{ x : v(x) < 0 \}$ and the nodal set $v^{-1}(0)$ is compact (see [1] and [2]). Take $q_1$ on $M$ such that $v(q_1) = 1 = \sup_M v$ and and $q_2 \in v^{-1}(0)$ such that distance $d(q_1, q_2) = d(q_1, v^{-1}(0))$. Let $L$ be the minimum geodesic segment between $q_1$ and $q_2$. We integrate both sides of (29) along $L$ and change variable and let $b \to 1$. Let $d_+, d_-$ be the diameter of the largest interior ball in $D^+, D^-$ respectively,

$$d_+ = 2r_+ \quad \text{and} \quad r_+ = \max_{x \in D_+} \text{dist}(x, v^{-1}(0))$$

and

$$d_- = 2r_- \quad \text{and} \quad r_- = \max_{x \in D_-} \text{dist}(x, v^{-1}(0)).$$

Then $\tilde{d} = \max\{d_+, d_-\}$ and

$$\sqrt{\lambda} \frac{d_+}{2} \geq \int_L \frac{\sqrt{|t|}}{\sqrt{z(t)}} \, dt \geq \int_0^{\pi/2} \frac{1}{\sqrt{z(t)}} \, dt \geq \left( \int_0^{\pi/2} \frac{\pi/2}{z(t) \, dt} \right)^{\frac{3}{2}} \geq \left( \int_0^{\pi/2} \frac{\pi/2}{z(t) \, dt} \right)^{\frac{1}{2}} \geq \left( \int_0^{\pi/2} \frac{\pi/2}{z(t) \, dt} \right)^{\frac{1}{2}} \geq \pi^3 \int_0^{\pi/2} \frac{1}{z(t) \, dt}.$$ 

Square the two sides. Then

$$\lambda \geq \frac{\pi^3}{2(d_+)^2 \int_0^{\pi/2} z(t) \, dt}.$$ 

Now

$$\int_0^{\pi/2} z(t) \, dt = \int_0^{\pi/2} [1 + \delta \xi(t)] \, dt = \frac{\pi}{2} (1 - \delta),$$

by (34) in Lemma 3. That is,

$$\lambda \geq \frac{\pi^2}{(1 - \delta)(d_+)^2} \quad \text{and} \quad \lambda \geq \frac{1}{2}(n - 1)K + \frac{\pi^2}{(d_+)^2}.$$ 

Noticing that $\tilde{d} \geq d_+$ and $\tilde{d} \geq d_-$, we complete the proof. 

We now present a lemma that is used in the proof of Theorem 1.

**Lemma 3.** Let

$$\xi(t) = \frac{\cos^2 t + 2t \sin t \cos t + t^2 - \pi^2}{\cos^2 t} \quad \text{on} \quad \left[ -\frac{\pi}{2}, \frac{\pi}{2} \right].$$

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Then the function $\xi$ satisfies the following

\begin{align*}
(32) & \quad \frac{1}{2} \xi'' \cos^2 t - \xi' \cos t \sin t - \xi = 2 \cos^2 t \quad \text{in } (-\frac{\pi}{2}, \frac{\pi}{2}), \\
(33) & \quad \xi' \cos t - 2 \xi \sin t = 4t \cos t \\
(34) & \quad \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \xi(t) \, dt = -\frac{\pi}{2} \\
1 - \frac{\pi^2}{4} & = \xi(0) \leq \xi(t) \leq \xi(\pm \frac{\pi}{2}) = 0 \quad \text{on } [-\frac{\pi}{2}, \frac{\pi}{2}], \\
\xi' & \text{ is increasing on } [-\frac{\pi}{2}, \frac{\pi}{2}] \text{ and } \xi'(\pm \frac{\pi}{2}) = \pm \frac{2\pi}{3}, \\
\xi'' & < 0 \text{ on } (-\frac{\pi}{2}, 0) \text{ and } \xi'' > 0 \text{ on } (0, \frac{\pi}{2}), \\
\xi''' & = 2, \quad \xi''(\pm \frac{\pi}{2}) = \frac{8\pi}{15}, \quad \xi'''(\pm \frac{\pi}{2}) = \pm \frac{2\pi}{3}. \\
\end{align*}

\text{Proof.} \quad \text{For convenience, let } q(t) = \xi'(t), \text{ i.e.,}

\begin{equation}
q(t) = \xi'(t) = \frac{2(2t \cos t + t^2 \sin t + \cos^2 t \sin t - \frac{\pi^2}{4} \sin t)}{\cos^3 t}.
\end{equation}

Equation (32) and the values $\xi(\pm \frac{\pi}{2}) = 0$, $\xi(0) = 1 - \frac{\pi^2}{4}$ and $\xi'(\pm \frac{\pi}{2}) = \pm \frac{2\pi}{3}$ can be verified directly from (31) and (35). The values of $\xi''$ at 0 and $\pm \frac{\pi}{2}$ can be computed via (32). By (33), $(\xi(t) \cos^2 t)' = 4t \cos^2 t$. Therefore $\xi(t) \cos^2 t = \int_{\frac{\pi}{2}}^{t} 4s \cos^2 s \, ds$, and

\begin{align*}
\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \xi(t) \, dt & = 2 \int_{0}^{\frac{\pi}{2}} \xi(t) \, dt - 8 \int_{0}^{\frac{\pi}{2}} \left( \frac{1}{\cos^2(t)} \int_{t}^{\frac{\pi}{2}} s \cos^2 s \, ds \right) \, dt \\
& = -8 \int_{0}^{\frac{\pi}{2}} \left( \int_{0}^{s} \frac{1}{\cos^2(t)} \, dt \right) s \cos^2 s \, ds = -8 \int_{0}^{\frac{\pi}{2}} s \cos s \sin s \, ds = -\pi.
\end{align*}

It is easy to see that $q$ and $q'$ satisfy the following equations

\begin{equation}
\frac{1}{2} q'' \cos t - 2q' \sin t - 2q \cos t = -4 \sin t,
\end{equation}

\text{on } (-\frac{\pi}{2}, \frac{\pi}{2}).
The last equation implies \( q' = \xi'' \) cannot achieve its non-positive local minimum at a point in \( (-\frac{\pi}{2}, \frac{\pi}{2}) \). On the other hand, \( \xi''(\pm \frac{\pi}{2}) = 2 \), by equation (32), \( \xi'(\pm \frac{\pi}{2}) = \pm \frac{2\pi}{3} \). Therefore \( \xi''(t) > 0 \) on \( [-\frac{\pi}{2}, \frac{\pi}{2}] \) and \( \xi' \) is increasing. Since \( \xi'(t) = 0 \), we have \( \xi'(t) < 0 \) on \( (-\frac{\pi}{2}, 0) \) and \( \xi'(t) > 0 \) on \( (0, \frac{\pi}{2}) \). Similarly, from the equation

\[
\frac{\cos^2 t}{2(1+\cos^2 t)}(q'')'' - \frac{\cos t \sin t(3+2\cos^2 t)}{(1+\cos^2 t)^2} (q'')' - \frac{2(5\cos^2 t+\cos^4 t)}{(1+\cos^2 t)^2} (q'') = -\frac{8\cos t \sin t}{(1+\cos^2 t)}.
\]

we get the results in the last line of the lemma.

Set \( h(t) = \xi''(t)t - \xi'(t) \). Then \( h(0) = 0 \) and \( h'(t) = \xi'''(t)t > 0 \) in \( (0, \frac{\pi}{2}) \). Therefore \( \frac{\xi'(t)}{t} = \frac{h(t)}{t^2} > 0 \) in \( (0, \frac{\pi}{2}) \). Note that \( \frac{\xi'(t)}{t} \big|_{t=0} = \frac{\xi''(0)}{2} = 2(3 - \frac{\pi^2}{4}) \) and \( \frac{\xi'(t)}{t} \big|_{t=\pi/2} = \frac{4}{3} \). This completes the proof of the lemma.

\( \square \)

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