Horseshoe shrinkage methods for Bayesian fusion estimation

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Abstract

We consider the problem of estimation and structure learning in a high dimensional normal sequence model, where the underlying parameter vector is piecewise constant, or has a block structure. We develop a Bayesian fusion estimation method by using the Horseshoe prior to induce a strong shrinkage effect on successive differences in the mean parameters, simultaneously imposing sufficient prior concentration for non-zero values of the same. The proposed method thus facilitates consistent estimation and structure recovery of the signal pieces. We provide theoretical justifications of our approach by deriving posterior convergence rates and establishing selection consistency under suitable assumptions. We demonstrate the superior performance of the Horseshoe based Bayesian fusion estimation method through extensive simulations and two real-life examples.

Keywords: Bayesian shrinkage; Fusion estimation; Horseshoe prior; posterior convergence rate; piecewise constant function.

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1 Introduction

With modern technological advances and massive data storage capabilities, large datasets are becoming increasingly common with applications in finance, econometrics, bioinformatics, engineering, signal-processing, among others. Flexible modeling of such datasets often require parameters whose dimension exceed the available sample size. Plausible inference is possible by finding a lower dimensional embedding of the high dimensional parameter.

Sparsity plays an important role in statistical learning in a high-dimensional scenario, where the underlying parameters are ‘nearly black’ (Donoho et al., 1992), implying that only a small subset of the same is non-zero. Several regularization based methods have been proposed in this regard in the literature, that induce sparsity in the models. From a frequentist perspective, such methods include penalization based methods like ridge regression (Hoerl and Kennard, 1970), lasso (Tibshirani, 1996), elastic net (Zou and Hastie, 2005), SCAD (Fan and Li, 2001), fused lasso (Tibshirani et al., 2005; Rinaldo, 2009), graphical lasso (Friedman et al., 2008), among others; see Bühlmann and Van De Geer (2011) for frequentist methods in high dimensions. Bayesian methods in high dimensions have been developed more recently, where sparsity is induced via suitable prior distributions on the parameters. These include spike-and-slab priors (Mitchell and Beauchamp, 1988; Ishwaran and Rao, 2005), Bayesian lasso (Park and Casella, 2008), Bayesian graphical lasso (Wang, 2012), and other shrinkage priors like spike-and-slab lasso (Roˇckov´a and George, 2018), Normal Exponential Gamma (Griffin and Brown, 2010), Horseshoe (Carvalho et al., 2009, 2010) and other variants like Horseshoe+ (Bhadra et al., 2017) and Horseshoe-like (Bhadra et al., 2019), Dirichlet-Laplace (Bhattacharya et al., 2015), R2-D2 (Zhang et al., 2016), among others. Theoretical guarantees of such Bayesian procedures have been developed recently as well. We refer the readers to Banerjee et al. (2021) for a comprehensive overview of Bayesian methods in a high-dimensional framework.

In this paper, we consider the normal sequence model

\[ y_i = \theta_i + \epsilon_i, \quad i = 1, \ldots, n, \]  

(1)

where \( \epsilon_i \overset{iid}{\sim} N(0, \sigma^2) \), \( \sigma^2 \) being the error variance, and \( \theta = (\theta_1, \ldots, \theta_n)^T \in \mathbb{R}^n \) is the vector of mean parameters. We further assume that the true parameter \( \theta_0 = (\theta_{0,1}, \ldots, \theta_{0,n})^T \in \mathbb{R}^n \) is
piecewise constant, or has an underlying block structure, in the sense that the transformed parameters \( \eta_{0,j} = \theta_{0,j} - \theta_{0,j-1}, 2 \leq j \leq n \) belong to a nearly-black class \( l_0[s] = \{ \eta \in \mathbb{R}^{n-1} : \#\{j : \eta_j \neq 0, 2 \leq j \leq n\} \leq s, 0 \leq s = s(n) \leq n \} \). Here \# denotes the cardinality of a finite set. The number of true blocks in the parameter vector is \( s_0 := \#\{j : \eta_{0,j} \neq 0, 2 \leq j \leq n\} \).

Our fundamental goals are two-fold – (i) estimating the mean parameter \( \theta \), and (ii) recovering the underlying piecewise constant structure. Piecewise constant signal estimation and structure recovery is an important and widely studied problem. Such signals occur in many applications, including bioinformatics, remote sensing, digital image processing, finance, and geophysics. We refer the readers to Little and Jones (2011) for a review on generalized methods for noise removal in piecewise constant signals, along with potential applications.

Frequentist methods in addressing the above problem include penalization methods like the fused lasso method (Tibshirani et al., 2005) and the \( L_1 \)-fusion method (Rinaldo, 2009). The Bayesian equivalent of the fused lasso penalty is using a Laplace shrinkage prior on the successive differences \( \eta \) (Kyung et al., 2010). However, the Laplace prior leads to posterior inconsistency (Castillo et al., 2015). To overcome this problem, Song and Liang (2017) used a heavy-tailed shrinkage prior for the coefficients in a linear regression framework. Motivated by this, Song and Cheng (2020) proposed to use a \( t \)-shrinkage prior for Bayesian fusion estimation. Shimamura et al. (2019) use a Normal Exponential Gamma (NEG) prior in this context, and make inference based on the posterior mode.

Motivated by strong theoretical guarantees regarding estimation and structure learning induced by the Horseshoe prior in a regression model (Datta and Ghosh, 2013; Van Der Pas et al., 2014; van der Pas et al., 2017), we propose to investigate the performance of the same in our fusion estimation framework. The Horseshoe prior induces sparsity via an infinite spike at zero, and also possess a heavy tail to ensure consistent selection of the underlying pieces or blocks. Furthermore, the global scale parameter that controls the level of sparsity in the model automatically adapts to the actual level of sparsity in the true model, as opposed to choosing the scale parameter in the \( t \)-shrinkage prior on the basis of the underlying dimension \( n \). The conjugate structure of the posterior distribution arising out of modeling via a Horseshoe prior allows fast computation along with working with the full posterior.
distribution, so that the samples obtained via MCMC can further be used for uncertainty quantification.

The paper is organized as follows. In the next section, we specify the Bayesian model along with the prior specifications, followed by posterior computations in Section 3. In Section 4, we provide theoretical guarantees of our proposed method via determining posterior convergence rates and establishing posterior selection consistency of the signal pieces. We demonstrate the numerical performance of our method along with other competing methods via simulations in Section 5, followed by real-life applications in two different areas – DNA copy number analysis and Solar X-Ray flux data analysis. We conclude our paper with a brief discussion of our proposed fusion estimation method along with future directions for research. Additional lemmas, proofs of main results, and a practical solution for block structure recovery along with numerical results for the same are provided in the Appendix.

The notations used in the paper are as follows. For real-valued sequences \( \{a_n\} \) and \( \{b_n\} \), \( a_n = O(b_n) \) implies that \( a_n/b_n \) is bounded, and \( a_n = o(b_n) \) implies that \( a_n/b_n \to 0 \) as \( n \to \infty \). By \( a_n \lesssim b_n \), we mean that \( a_n = O(b_n) \), while \( a_n \asymp b_n \) means that both \( a_n \lesssim b_n \) and \( b_n \lesssim a_n \) hold. \( a_n \prec b_n \) means \( a_n = o(b_n) \). For a real vector \( x = (x_1, \ldots, x_n)^T \), the \( L_r \)-norm of \( x \) for \( r > 0 \) is defined as \( \|x\|_r = \left( \sum_{i=1}^{n} |x_i|^r \right)^{1/r} \). We denote the cardinality of a finite set \( S \) as \( \#S \). The indicator function is denoted by \( 1_l \).

2 Bayesian modeling and prior specification

We consider the normal sequence model and assume that the successive differences of the means belong to a nearly-black class \( l_0[s] \). As discussed earlier, frequentist procedures induce sparsity in the model via suitable regularization based procedures like penalization of the underlying parameters, whereas Bayesian methods usually induce sparsity via imposing suitable prior distributions on the same. For example, for learning a high-dimensional parameter \( \theta \in \mathbb{R}^n \), a general version of a penalized optimization procedure can be written as \( \arg\min_{\theta \in \mathbb{R}^n} \{l(\theta; y) + \pi(\theta)\} \), where \( l(\theta; y) \) is the empirical risk and \( \pi(\theta) \) is the penalty function. If \( l(\theta; y) \) is defined as the negative log-likelihood (upto a constant) of the observations \( y \), the above optimization problem becomes equivalent to finding the mode of the posterior
distribution $p(\theta \mid y)$, corresponding to the prior density $p(\theta) \propto \exp(-\pi(\theta))$.

In our context, Tibshirani et al. (2005) proposed the fused lasso estimator $\hat{\theta}^{FL}$ that induces sparsity on both $\theta$ and $\eta$, defined as

$$
\hat{\theta}^{FL} = \arg \min_{\theta \in \mathbb{R}^n} \left\{ \frac{1}{2} \| y - \theta \|^2_2 + \lambda_1 \| \theta \|_1 + \lambda_2 \| \eta \|_1 \right\},
$$

(2)

for suitable penalty parameters $\lambda_1$ and $\lambda_2$. Rinaldo (2009) considered the $L_1$-fusion estimator $\hat{\theta}^F$ with penalization of the successive differences only, given by,

$$
\hat{\theta}^F = \arg \min_{\theta \in \mathbb{R}^n} \left\{ \frac{1}{2} \| y - \theta \|^2_2 + \lambda \| \eta \|_1 \right\},
$$

(3)

where $\lambda$ is the corresponding penalty (tuning) parameter. A Bayesian equivalent of the fused lasso estimator (3) can be obtained by putting a Laplace($\lambda/\sigma$) prior on the successive differences $\eta$ and finding the corresponding posterior mode. As in a normal regression model with Bayesian lasso (Park and Casella, 2008), the Bayesian fused lasso estimator given by the posterior mode will converge to the true $\eta_0$ at a nearly optimal rate. However, the induced posterior distribution has sub-optimal contraction rate (Castillo et al., 2015), owing to insufficient prior concentration near zero.

The posterior inconsistency of the Bayesian fused lasso method motivates us to explore other approaches that would mitigate the problems leading to the undesirable behavior of the posterior distribution. Shrinkage priors qualify as naturally good choices for our problem, as they can address the dual issue of shrinking true zero parameters to zero, and retrieving the ‘boundaries’ of the blocks effectively by placing sufficient mass on the non-zero values of successive differences of the normal means. In the normal sequence model, optimal posterior concentration has been achieved via using shrinkage prior distributions with polynomially decaying tails (Song and Liang, 2017). This is in contrast to the Laplace prior, that has exponentially light tails.

The Horseshoe prior is a widely acclaimed choice as a shrinkage prior owing to its infinite spike at 0 and simultaneously possessing a thick tail. The tails decay like a second-order polynomial, and hence the corresponding penalty function behaves like a logarithmic penalty, and is non-convex (see Carvalho et al. (2010); Bhadra et al. (2019) for more details). The Horseshoe prior can be expressed as a scale mixture of normals with half-Cauchy prior, thus
acting as a global-local shrinkage prior. We put a Horseshoe prior on the pairwise differences in the parameters \( \eta_i = \theta_i - \theta_{i-1}, \; i = 2, \ldots, n \). We also need to put suitable priors on the mean parameter \( \theta_1 \) and the error variance \( \sigma^2 \). The full prior specification is given by,

\[
\theta_1 \mid \lambda_1^2, \sigma^2 \sim N(0, \lambda_1^2 \sigma^2), \; \eta_i \mid \lambda_i^2, \tau^2, \sigma^2 \overset{ind}{\sim} N(0, \lambda_i^2 \tau^2 \sigma^2), \; 2 \leq i \leq n,
\]

\[
\lambda_i \overset{iid}{\sim} C^+(0, 1), \; 2 \leq i \leq n, \; \tau \overset{iid}{\sim} C^+(0, 1), \; \sigma^2 \overset{iid}{\sim} IG(a_\sigma, b_\sigma). \tag{4}
\]

Here \( C^+(\cdot, \cdot) \) and \( IG(\cdot, \cdot) \) respectively denote the half Cauchy density and Inverse Gamma density. The level of sparsity induced in the model is controlled by the global scale parameter \( \tau \), and choosing the same is a non-trivial problem. Using a plug-in estimate for \( \tau \) based on empirical Bayes method suffers from a potential danger of resulting in a degenerate solution resulting in a heavily sparse model. There are several works (Carvalho et al., 2010; Piironen and Vehtari, 2017b,a) that suggest effective methods regarding the choice of \( \tau \). In this paper, we have proposed to take a fully Bayesian approach as suggested in Carvalho et al. (2010); Piironen and Vehtari (2017b) and use a half-Cauchy prior.

The half-Cauchy distribution can further be written as a scale-mixture of Inverse-Gamma distributions. For a random variable \( X \sim C^+(0, \psi) \), we can write,

\[
X^2 \mid \phi \sim IG(1/2, 1/\phi), \; \phi \sim IG(1/2, 1/\psi^2).
\]

Thus, the full hierarchical prior specification for our model is given by,

\[
\theta_1 \mid \lambda_1^2, \sigma^2 \sim N(0, \lambda_1^2 \sigma^2), \; \eta_i \mid \lambda_i^2, \tau^2, \sigma^2 \overset{ind}{\sim} N(0, \lambda_i^2 \tau^2 \sigma^2), \; 2 \leq i \leq n,
\]

\[
\lambda_i^2 \mid \nu_i \overset{iid}{\sim} IG(1/2, 1/\nu_i), \; 2 \leq i \leq n, \; \tau^2 \mid \xi \sim IG(1/2, 1/\xi), \; \nu_2, \ldots, \nu_n, \xi \overset{iid}{\sim} IG(1/2, 1), \; \sigma^2 \sim IG(a_\sigma, b_\sigma) \tag{5}
\]

The hyperparameters \( a_\sigma \) and \( b_\sigma \) may be chosen in such a way that the corresponding prior becomes non-informative. The local scale parameter \( \lambda_1 \) is considered to be fixed as well.

### 3 Posterior computation

The conditional posterior distributions of the underlying parameters can be explicitly derived via exploring the conjugate structure. Hence, the posterior computations can be ac-
accomplished easily via Gibbs sampling. We present the conditional posterior distributions of the parameters below.

The normal means have the conditional posterior distribution

$$\theta_i | \cdots \sim N(\mu_i, \zeta_i), \ 1 \leq i \leq n, \ (6)$$

where $\mu_i$ and $\zeta_i$ are given by,

$$\zeta_1^{-1} = \frac{1}{\sigma^2} \left( 1 + \frac{1}{\lambda_2^2 \tau^2} + \frac{1}{\lambda_1^2} \right), \ \mu_1 = \frac{\zeta_1}{\sigma^2} \left( y_1 + \frac{\theta_2}{\lambda_2^2 \tau^2} \right),$$

$$\zeta_i^{-1} = \frac{1}{\sigma^2} \left( 1 + \frac{1}{\lambda_{i+1}^2 \tau^2} + \frac{1}{\lambda_i^2 \tau^2} \right), \ \mu_i = \frac{\zeta_i}{\sigma^2} \left( y_i + \frac{\theta_{i+1}}{\lambda_{i+1}^2 \tau^2} + \frac{\theta_{i-1}}{\lambda_i^2 \tau^2} \right), \ 2 \leq i \leq n.$$ 

Here $\lambda_{n+1}$ is considered to be infinity. The conditional posteriors for the rest of the parameters are given by,

$$\lambda_i^2 | \cdots \sim IG \left( 1, \frac{1}{\nu_i} + \frac{(\theta_i - \theta_{i-1})^2}{2 \tau^2 \sigma^2} \right), \ 2 \leq i \leq n,$$

$$\sigma^2 | \cdots \sim IG \left( n + a_\sigma, b_\sigma + \frac{1}{2} \left[ \sum_{i=1}^{n} (y_i - \theta_i)^2 + \frac{1}{\tau^2} \sum_{i=2}^{n} \frac{(\theta_i - \theta_{i-1})^2}{\lambda_i^2} + \frac{\theta_1^2}{\lambda_1^2} \right] \right),$$

$$\tau^2 | \cdots \sim IG \left( \frac{n}{2 \xi}, \frac{1}{2 \sigma^2} \sum_{i=2}^{n} \frac{(\theta_i - \theta_{i-1})^2}{\lambda_i^2} \right),$$

$$\nu_i | \cdots \sim IG \left( 1, 1 + \frac{1}{\lambda_i^2} \right), \ 2 \leq i \leq n,$$

$$\xi | \cdots \sim IG \left( 1, 1 + \frac{1}{\tau^2} \right). \ (7)$$

4 Theoretical results

In this section, we present the theoretical validations of using a Horseshoe prior for fusion estimation. We first discuss the result involving posterior convergence rate of the mean parameter $\theta$ under certain assumptions and then proceed to discuss the result on posterior selection of the underlying true block structure of the mean parameter.

Assumption 1. The number of true blocks in the model satisfies $s_0 < n/ \log n$. 
Assumption 2. The true mean parameter vector $\theta_0 = (\theta_{0,1}, \ldots, \theta_{0,n})^T$ and the true error variance $\sigma_0^2$ satisfy the following conditions:

(i) Define $\eta_{0,j} = \theta_{0,j} - \theta_{0,j-1}, \ 2 \leq j \leq n$. Then, $\max_j |\eta_{0,j}|/\sigma_0 < L$, where $\log L = O(\log n)$.

(ii) $\theta_{0,1}/(\lambda_1 \sigma_0^2) + \log \lambda_1 = O(\log n)$, where $\lambda_1$ is the prior hyperparameter appearing in the prior for $\theta_1$ in (4).

Assumption 3. The global scale parameter $\tau$ in the prior specification (4) satisfies $\tau < n^{-(3+u)}$ for some constant $u > 0$, and $-\log \tau = O(\log n)$.

The first assumption above involves the true block size, that is ubiquitous in structure recovery problems in high-dimensions. In Assumption 2 we have considered $L$ as a bound on the maximum value of $|\eta_{0,j}|/\sigma_0$. Such a restriction is necessary for block recovery at a desired contraction rate. We shall see (in Lemma 7) that the aforesaid condition on $L$, along with one of the conditions on the global scale parameter $\tau$ as in Assumption 3 would guarantee that the tail of the prior is not too sharp. This would ensure a minimum prior concentration for large non-zero values of the successive difference in the means. An equivalent prior condition cannot hold uniformly over an unbounded parameter space, thus prompting a restriction on the successive differences, and also on $\theta_{0,1}$. Similar restrictions in the context of linear and generalized regression models have been considered in the literature; for example, see [Wei and Ghosal (2020); Song (2020)]. Assumption 3 involves an upper bound on the global scale parameter $\tau$, that would be necessary for ensuring that the prior puts sufficient mass around zero for the successive differences in means, thus leading to effective fusion estimation via imposing sparsity. However, $\tau$ should not be too small, otherwise it would lead to degeneracy in inference by picking up too sparse a model. The lower bound on $\tau$, along with the condition on $L$ in Assumption 2 guarantees that the Horseshoe prior is ‘thick’ at non-zero parameter values, so that it is not too sharp. We now present our main result on posterior contraction rate.

Theorem 4. Consider the Gaussian means model (4) with prior specification as in (4), and suppose that assumptions 1, 2 and 3 hold. Then the posterior distribution of $\theta$, given by
\( \Pi^n(\cdot \mid y) \), satisfies
\[
\Pi^n(\|\theta - \theta_0\|_2 / \sqrt{n} \geq M\sigma_0\epsilon_n \mid y) \rightarrow 0, \text{ as } n \rightarrow \infty,
\] (8)
in probability or in \( L_1 \) wrt the probability measure of \( y \), for \( \epsilon_n \approx \sqrt{s_0 \log n}/n \) and a constant \( M > 0 \).

The above result implies that the posterior convergence rate for \( \|\theta - \theta_0\|_2 / \sqrt{n} \) is of the order \( \sigma_0\sqrt{s_0 \log n}/n \). When the exact piecewise constant structure is known, the proposed Bayesian fusion estimation method achieves the optimal convergence rate \( O(\sigma_0\sqrt{s}/n) \) up to a logarithmic term in \( n \). The posterior convergence rate also adapts to the (unknown) size of the pieces. The rate directly compares with the Bayesian fusion estimation method as proposed in Song and Cheng (2020).

The Horseshoe prior (and other global-local shrinkage priors) are continuous shrinkage priors, and hence exact block structure recovery is not possible. However, we can consider discretization of the posterior samples via the posterior projection of the samples \((\theta, \sigma)\) to a discrete set \( S(\theta, \sigma) = \{ 2 \leq j \leq n : |\theta_j - \theta_{j-1}|/\sigma < \epsilon_n/n \} \). The number of false-positives resulting from such a discretization can be expressed as the cardinality of the set \( A(\theta, \sigma) = S^c(\theta, \sigma) = \{ 2 \leq j \leq n : \theta_{0,j} - \theta_{0,j-1} \neq 0 \} \). The induced posterior distribution of \( S(\theta, \sigma) \) (and hence that of \( A(\theta, \sigma) \)) can be shown to be ‘selection consistent’, in the sense that the number of false-positives as defined above is bounded in probability. We formally present this result below.

**Theorem 5.** Under the assumptions of Theorem 4, the posterior distribution of \( A(\theta, \sigma) \) satisfies
\[
\Pi^n(\#A(\theta, \sigma) > Ks_0 \mid y) \rightarrow 0,
\] (9)
in probability or in \( L_1 \) wrt the measure of \( y \) for some fixed constant \( K > 0 \).

Note that the thresholding rule depends on the posterior contraction rate, that involves the number \( (s_0) \) of true blocks. However, in practical scenarios, knowledge of \( s_0 \) may not be readily available. To tackle such situations, we propose to use a threshold analogous to the concept of shrinkage weights as in the sparse normal sequence model. We outline the details in the appendix.
5 Simulation Study

To evaluate the performance of our method and compare with other competing approaches, we carry out simulation studies for varying signal and noise levels. Three different types of piecewise constant functions of length $n = 100$ are considered – (i) 10 evenly spaced signal pieces, with each of the pieces having length 10, (ii) 10 unevenly spaced signal pieces, with the shorter pieces each having length 5, and (iii) 10 very unevenly spaced signal pieces, with the shorter pieces each having length 2. The response variable $y$ is generated from an $n$-dimensional Gaussian distribution with mean $\theta_0$ and variance $\sigma^2$, where $\theta_0$ is the true signal vector, and $\sigma \in \{0.1, 0.3, 0.5\}$.

We estimate the true signal using our Horseshoe prior based fusion estimation approach, and compare with three other approaches – (a) Bayesian $t$-fusion method, as proposed in Song and Cheng (2020), (b) Bayesian fusion approach based on a Laplace prior, and (c) the $L_1$-fusion method. For the Bayesian methods, we consider 5000 MCMC samples, with initial 500 samples as burn-in. The MCMC-details including Gibbs sampler updates for the $t$-fusion and Laplace fusion approaches, and the choice of the corresponding scale parameters for the above priors are taken as suggested in Song and Cheng (2020). The hyperparameter values for the prior on error variance are taken as $a_\sigma = b_\sigma = 0.5$ across all the Bayesian methods, with the local scale parameter $\lambda_1 = 5$. For the frequentist fused lasso method based on $L_1$-fusion penalty, we use the genlasso package in R, and choose the fusion penalty parameter using a 5-fold cross-validation approach. To evaluate the performance of the estimators, we use the Mean Squared Error (MSE) and the adjusted MSE, respectively defined as,

$$
\text{MSE} = \|\hat{\theta} - \theta_0\|^2_2/n,
$$

$$
\text{adj.MSE} = \|\hat{\theta} - \theta_0\|^2_2/\|\theta_0\|^2_2,
$$

where $\hat{\theta}$ is the estimated signal, given by the posterior mean in case of the Bayesian methods and the fused lasso estimate for the frequentist method. All the computations were performed in R on a laptop having an Intel(R) Core(TM) i7-10750H CPU @ 2.60GHz with 16GB RAM and a 64-bit OS, x64-based processor.

The summary measures for the performance of the four methods using MSE, adjusted MSE and their associated standard errors (in brackets) based on 100 Monte-Carlo replica-
tions are presented in Table 1. We find that our proposed Horseshoe based fusion estimation method has excellent signal estimation performance across all the different types of signal pieces and noise levels. Within a Bayesian framework, the performance of the Laplace based fusion estimation method is not at all promising. Though the estimation performances for the Horseshoe-fusion, $t$-fusion and fused lasso are comparable in case of low noise levels, our proposed method performs much better in the higher noise scenario. Additionally, the Bayesian methods provide credible bands as well that can be utilized further for uncertainty quantification. In that regard, we observe from Figures 1, 2 and 3 that the Horseshoe based estimates have comparatively narrower credible bands as compared to other Bayesian competing methods considered here. Overall, we could successfully demonstrate the superiority of using a heavy-tailed prior for successive differences in the signals in contrast to an exponentially lighter one. In addition to that, using a prior with a comparatively sharper spike at zero results in better signal and structure recovery, especially in situations where the noise level is higher.

6 Real-data examples

We check the performance of our proposed method on two real-world data examples where the observed data can be modeled using piecewise constant functions. The first example involves DNA copy-number analysis of Glioblastoma multiforme (GBM) data, and the second one involves analysis of Solar X-Ray flux data for the period of October-November 2003 that encompasses the Halloween 2003 Solar storm events.

6.1 DNA copy-number analysis of Glioblastoma multiforme (GBM)

Array Comparative Genomic Hybridization (aCGH) is a high-throughput technique used to identify chromosomal abnormalities in the genomic DNA. Identification and characterization of chromosomal aberrations are extremely important for pathogenesis of various diseases, including cancer. In cancerous cells, genes in a particular chromosome may get amplified or deleted owing to mutations, thus resulting in gains or losses in DNA copies of the genes. Array CGH data measure the $\log_2$-ratio between the DNA copy number of genes in tumor cells
Figure 1: Fusion estimation performance with differently spaced signals and error sd $\sigma = 0.1$. Observations are represented in red dots, true signals in blue dots, point estimates in black dots, and 95% credible bands of the Bayesian procedures in green.
Figure 2: Fusion estimation performance with differently spaced signals and error sd $\sigma = 0.3$. Observations are represented in red dots, true signals in blue dots, point estimates in black dots, and 95% credible bands of the Bayesian procedures in green.
Figure 3: Fusion estimation performance with differently spaced signals and error sd $\sigma = 0.5$. Observations are represented in red dots, true signals in blue dots, point estimates in black dots, and 95% credible bands of the Bayesian procedures in green.
and those in control cells. The aCGH data thus may be considered as a piecewise constant signal with certain non-zero blocks corresponding to aberrations owing to additions/deletions at the DNA level.

Glioblastoma multiforme (GBM) is the most common malignant brain tumors in adults (Holland, 2000). Lai et al. (2005) analyzed array CGH data in GBM using glioma data from Bredel et al. (2005) along with control samples. Tibshirani and Wang (2008) used the data from Lai et al. (2005), and created aCGH data corresponding to a pseudo-chromosome so that the resulting genome sequence have shorter regions of high amplitude gains (copy number additions) as well as a broad region of low amplitude loss (copy number deletions). To be precise, the combined array consists of aCGH data from chromosome 7 in GBM29 and chromosome 13 in GBM31. The aCGH data is presented in Figure 4 and is available from the archived version of the cghFlasso package in R.

We apply the Horseshoe prior based Bayesian fusion estimation method and also compare the performance with the fused lasso method. We post-process the Bayesian estimates by discretization via the thresholding method as outlined in the appendix. The results are displayed in Figure 5. We find that our proposed Bayesian fusion estimation method has successfully detected the high amplitude regions of copy number additions, as well as the broad low amplitude region of copy number deletions. The results are very much comparable with the fused lasso estimates, and the findings are similar to that in Lai et al. (2005). The Bayesian credible intervals obtained using our proposed method may further be utilized in addressing future research problems, like controlling false discovery rates.

6.2 Solar X-Ray flux analysis: Halloween Solar Storms 2003

The Earth’s ionosphere is responsible for blocking high frequency radio waves on the side facing the Sun. Large solar flares affect the ionosphere, thus interrupting satellite communications, and posing threats of radiation hazards to astronauts and spacecrafts. Large solar flares are also associated with Coronal Mass Ejections (CMEs) that have the potential to trigger geomagnetic storms. Geomagnetic storms have the ability to disrupt satellite and radio communications, cause massive black-outs via electric grid failures, and affect global navigation systems. The Geostationary Operational Environmental Satellite (GOES), op-
Figure 4: Figure showing array CGH data for Chromosome 7 in GBM29 and chromosome 13 in GBM31.

erated by the United States’ National Oceanic and Atmospheric Administration (NOAA), records data on solar flares. Solar X-ray flux is measured as Watts-per-square-meter ($W/m^2$) unit. They are primarily classified in five categories, namely, A, B, C, M, and X, depending on the approximate peak flux with wavelengths in the range of $1 - 8$ Angstroms (long wave) and $0.5 - 4$ Angstroms (short wave). Solar flares in category A have the least peak flux ($< 10^{-7} W/m^2$), and those in category X are the most powerful ones ($> 10^{-4} W/m^2$).

Around Halloween in 2003, a series of strong solar storms affected our planet, leading to disruptions in satellite communication, damages to spacecrafts, and massive power outages in Sweden. The increased levels of solar activity made it possible to observe the Northern lights (Aurora Borealis) even from far south latitudes in Texas, USA. We consider the short wave solar X-ray flux data for the two months of October and November 2003. The data have been accessed from NOAA’s website [https://www.ngdc.noaa.gov/stp/satellite/goes/dataaccess.html](https://www.ngdc.noaa.gov/stp/satellite/goes/dataaccess.html). We use the 5-minute average flux data for all the days in the two months, and extract the information for the flux level for shortwave X-rays.

The data contain some missing values, which we impute via a linear interpolation method. Also, we further use 1-hour averages of the X-ray flux data, so as to arrive at 24 data points for each day, thus leading to a total of 1464 data points. Figure 6 shows the plot of the
Figure 5: Figure showing estimation performances of (a) the proposed Bayesian Horseshoe prior based fusion method, and (b) the $L_1$ fused lasso method. The blue lines represent the estimated signals, and the red dots represent the $\log_2$ ratios between the DNA copy numbers of genes in tumor cells and those in control cells.
Figure 6: Figure showing scatter plot of logarithm of solar X-ray flare flux values for the months of October and November 2003, recorded by GOES.

logarithm of the flux levels for the two months. As mentioned in [Little and Jones (2011)], we can approximate this data as a piecewise constant signal, and use fusion-based estimation methods to identify days of extreme solar activities. We apply our Horseshoe prior based Bayesian fusion estimation method, and post-process the estimates by using the thresholding approach as mentioned in the appendix. We also apply the $L_1$-fusion estimation method. The fitted estimates are shown in Figure 7. We find that our Horseshoe prior based Bayesian approach has been able to detect the periods of increased solar activity around October 28 and also on November 04, 2003, besides giving piecewise constant estimates. Our findings match with related analyses; see [Pulkkinen et al. (2005)]. On the other hand, the fused lasso method suffers from serious over-fitting problems.

7 Discussion

In this paper, we have addressed the problem of fusion estimation in a normal sequence model where the underlying mean parameter vector is piecewise constant. Noise removal and block structure learning is an important and widely studied problem with applications in a plethora of areas. We developed a Horseshoe prior based fusion estimation method and
Figure 7: Figure showing estimation performances of (a) the proposed Bayesian Horseshoe prior based fusion method, and (b) the $L_1$ fused lasso method. The blue lines represent the estimated signals, and the red dots represent the logarithm of the shortwave X-ray solar flare flux values as recorded by GOES.
provided theoretical guarantees via establishing posterior convergence and selection consistency. Numerical studies demonstrated excellent performance of our method as compared to other competing Bayesian and frequentist approaches.

There are several directions for extending our work. In many applications, we observe noisy signals defined over an undirected graph. In a recent work, Banerjee and Shen (2020) proposed to use the $t$-shrinkage prior to address the graph signal denoising problem via reducing the original graph to a linear chain graph. The excellent performance of the Horseshoe prior based learning can be extended to the graph signal denoising problem as well. Another direction to work is the graph fused lasso method for estimation and structure learning in multiple Gaussian graphical models (Danaher et al., 2014). We propose to explore these ideas as future work.

Data Availability

Data for the DNA copynumber analysis are available for free from the archived version of the cghFlasso package in R. Data on solar X-ray flux are available for free from the National Oceanic and Atmospheric Administration (NOAA)’s website at https://www.ngdc.noaa.gov/stp/satellite/goes/dataaccess.html.

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Appendix

Additional lemmas and proofs of results

**Lemma 6.** Under the assumptions on the true parameter $\theta_0$ and the prior distribution on $\theta$ as outlined in Theorem 4, for some $u' > 0$, we have,

\[
\int_{-s_0 \log n/n^2}^{s_0 \log n/n^2} p_{HS}(\eta; \tau) \, d\eta \geq 1 - n^{-(1+u')},
\]

where $p_{HS}(\eta; \tau)$ denotes the Horseshoe prior density on $\eta$ with hyperparameter $\tau > 0$.

**Proof.** Define $a_n = s_0 \log n/n^2$. Note that, $a_n < \sqrt{n \epsilon_n}/n$, where $\epsilon_n$ is the posterior convergence rate $\sqrt{s_0 \log n/n}$. We will show that $\int_{-a_n}^{a_n} p_{HS}(\eta; \tau) \, d\eta \geq 1 - n^{-(1+u')}$ for some $u' > 0$ under the assumptions outlined in Theorem 4. Our technique of the proof closely follows that in Wei and Ghosal (2020), who explore posterior contraction results in a logistic regression framework.

We have,

\[
1 - \int_{-a_n}^{a_n} p_{HS}(\eta; \tau) \, d\eta = 2 \int_0^{\infty} \left\{1 - \Phi \left( \frac{a_n}{\lambda \tau} \right) \right\} f_{C^+}(\lambda) \, d\lambda \\
= 2 \int_0^{n^{1+u}} \left\{1 - \Phi \left( \frac{a_n}{\lambda \tau} \right) \right\} f_{C^+}(\lambda) \, d\lambda + 2 \int_{n^{1+u}}^{\infty} \left\{1 - \Phi \left( \frac{a_n}{\lambda \tau} \right) \right\} f_{C^+}(\lambda) \, d\lambda \\
\leq 2 \left\{1 - \Phi \left( \frac{a_n}{\lambda \tau} \right) \right\} + 2 \int_{n^{1+u}}^{\infty} \frac{1}{\pi} \frac{1}{\pi + \lambda^2} \, d\lambda \\
\leq \sqrt{2/\pi} \exp \left( -a_n^2 n^{-2(1+u)}/\tau^2 \right) + n^{-(1+u)} \\
\leq n^{-(1+u')},
\]

where $0 < u' < u$, for $\tau \leq a_n n^{-(1+u)} \asymp n^{-(3+u)}$, and

\[
\int_{n^{1+u}}^{\infty} \frac{1}{\pi} \frac{1}{1 + \lambda^2} \, d\lambda = \frac{2}{\pi} \left( \frac{\pi}{2} - \arctan (n^{1+u}) \right) \asymp n^{-(1+u)}.
\]

This completes the proof. $\square$

As mentioned earlier, the above result guarantees that the Horseshoe prior puts sufficient mass around zero for the successive differences $\eta_i$ so as to facilitate Bayesian shrinkage for the block-structured mean parameters. However, the prior should be able to retrieve the blocks
effectively as well, so that successive differences that are non-zero should not be shrunk too much. The following result below guarantees that the Horseshoe prior is ‘thick’ at non-zero parameter values, so that it is not too sharp.

**Lemma 7.** Consider the prior structure and the assumptions as mentioned in Theorem 4. Then, we have,

\[ -\log \left( \inf_{\eta/\sigma \in [-L,L]} p_{HS}(\eta; \tau) \right) = O(\log n). \]  

(11)

*Proof.* The Horseshoe prior does not have an explicit functional form. However, Carvalho et al. (2010) provide a lower bound for the Horseshoe prior density that we would utilise for proving the above result. We have,

\[
\inf_{\eta/\sigma \in [-L,L]} p_{HS}(\eta; \tau) = \int_0^\infty \frac{1}{\sqrt{2\pi \lambda \tau}} \exp \left( -\frac{L^2}{2\lambda^2 \tau^2} \right) f_{C+}(\lambda) d\lambda \\
\geq \frac{1}{\sqrt{2\pi^3}} \log \left( 1 + \frac{4\tau^2}{L} \right) \approx \frac{\tau^2}{L} \\
= O\left( n^{-(1+u)} \right),
\]

for \(-\log(\tau) = O(\log n)\) and \(\log L = O(\log n)\). Thus, we get, \(-\log \left( \inf_{\eta/\sigma \in [-L,L]} p_{HS}(\eta; \tau) \right) = O(\log n)\), hence completing the proof.

We now present the proofs of the main results in our paper.

**Proofs of Theorem 4 and Theorem 5.** The proof readily follows from Theorems 2.1 and 2.2 in Song and Cheng (2020), if we can verify the conditions (see display (2.5) in the aforementioned paper) for posterior convergence specified therein. Lemma 6 and Lemma 7 imply that the first two conditions are satisfied. The third condition is satisfied for a Normal prior distribution on \(\theta_1\) and for a fixed hyperparameter \(\lambda_1 > 0\). The fourth condition is trivially satisfied for fixed choices of (non-zero) hyperparameters \(a_\sigma\) and \(b_\sigma\), and for some fixed (unknown) true error variance \(\sigma_0^2\). 

\(\square\)
A practical solution to block structure recovery

The Horseshoe prior is a continuous shrinkage prior, and hence block structure recovery is not straightforward. In Bayesian fusion estimation with Laplace shrinkage prior or with $t$-shrinkage prior, one may discretize the samples obtained via MCMC using a suitable threshold. [Song and Cheng (2020)] recommended using the $1/2n$-th quantile of the corresponding prior for discretization of the scaled samples. In Section 4, we proposed a threshold based on the posterior contraction rate. However, that would also require an idea of the true block size, which may not be available always, or may be difficult to ascertain beforehand. We provide a practical approach for discretization of the samples. Note that in a sparse normal sequence problem, the posterior mean of the mean parameter $\theta_i$ is given by $\kappa_i y_i$, where $\kappa_i$ is the shrinkage weight. These shrinkage weights mimic the posterior selection probability of the means, and hence thresholding the weights to 0.5 provide excellent selection performance, which is justified numerically ([Carvalho et al., 2010]) and later theoretically ([Datta and Ghosh, 2013]). Motivated by the same, we propose to use the following thresholding rule for selection of the blocks for our proposed method: $\theta_{j_1}$ and $\theta_{j_2}$ are equal if $|\hat{\theta}_{j_1} - \hat{\theta}_{j_2}|/\hat{\sigma} < 0.5|y_{j_1} - y_{j_2}|$, where $\hat{\theta}_{j_1}, \hat{\theta}_{j_2}, \hat{\sigma}$ are the posterior means of $\theta_{j_1}, \theta_{j_2}$ and $\sigma$ respectively, for $1 \leq j_1 \neq j_2 \leq n$. We thus estimate the block structure indicator $s_{j_1,j_2} = 1\{\theta_{j_1} = \theta_{j_2}\}$ by $\hat{s}_{j_1,j_2} = 1\{|\hat{\theta}_{j_1} - \hat{\theta}_{j_2}|/\hat{\sigma} < 0.5|y_{j_1} - y_{j_2}|\}$. To evaluate the performance of the structure recovery method using the proposed thresholding approach, we define the metrics $W$ and $B$, where $W$ is the average within-block variation defined as $W := \text{mean}_{s_{j_1,j_2} \neq 0}|\hat{\theta}_{j_1} - \hat{\theta}_{j_2}|$, and $B$ is the between-block separation defined as $B := \min_{s_{j_1,j_2} = 0}|\hat{\theta}_{j_1} - \hat{\theta}_{j_2}|$.

This practical thresholding approach resulted in excellent block structure recovery performance with respect to the metrics $W$ and $B$. We compare our results with the other Bayesian fusion estimation methods and also with the frequentist fused lasso method. The fused lasso method results in exact structure recovery, and hence no thresholding is required. As mentioned earlier, the thresholding rules for the competing Bayesian methods are taken as suggested in [Song and Cheng (2020)]. We notice that for low error variances, the within-block average variation is lower in case of the Horseshoe fusion and $t$-fusion priors, with the
Horseshoe fusion method having larger between-block separation especially in case of higher noise, thus indicating superior structure learning capabilities.

Other possible directions for coming up with a threshold include using a multiple hypothesis testing approach for successive differences in the means, and also using simultaneous credible intervals. We leave the theoretical treatment of using (or improving, if possible) our practical thresholding approach and exploring other proposed methods as a future work.

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| Signal    | $\sigma$ | HS-fusion MSE | adj MSE | $t$-fusion MSE | adj MSE | Laplace fusion MSE | adj MSE | $L_1$ fusion MSE | adj MSE |
|----------|----------|---------------|--------|----------------|--------|-------------------|--------|-------------------|--------|
| Even     | 0.1      | 0.002 (0.000) | 0.000 (0.000) | 0.002 (0.000) | 0.001 (0.000) | 0.504 (0.002) | 0.114 (0.004) | 0.005 (0.003) | 0.001 (0.001) |
|          | 0.3      | 0.020 (0.001) | 0.004 (0.000) | 0.030 (0.002) | 0.007 (0.001) | 0.516 (0.002) | 0.116 (0.004) | 0.035 (0.019) | 0.008 (0.004) |
|          | 0.5      | 0.062 (0.002) | 0.014 (0.000) | 0.133 (0.009) | 0.030 (0.002) | 0.539 (0.003) | 0.121 (0.004) | 0.087 (0.039) | 0.019 (0.009) |
| Uneven   | 0.1      | 0.002 (0.000) | 0.001 (0.000) | 0.002 (0.000) | 0.001 (0.000) | 0.566 (0.002) | 0.249 (0.003) | 0.005 (0.003) | 0.002 (0.001) |
|          | 0.3      | 0.019 (0.001) | 0.008 (0.000) | 0.036 (0.002) | 0.016 (0.001) | 0.575 (0.002) | 0.253 (0.004) | 0.037 (0.019) | 0.016 (0.008) |
|          | 0.5      | 0.060 (0.002) | 0.026 (0.001) | 0.212 (0.012) | 0.094 (0.006) | 0.596 (0.003) | 0.263 (0.004) | 0.090 (0.044) | 0.039 (0.019) |
| V. Uneven| 0.1      | 0.002 (0.000) | 0.002 (0.000) | 0.005 (0.000) | 0.005 (0.000) | 0.479 (0.001) | 0.520 (0.001) | 0.009 (0.002) | 0.010 (0.002) |
|          | 0.3      | 0.020 (0.001) | 0.022 (0.001) | 0.114 (0.007) | 0.124 (0.007) | 0.489 (0.001) | 0.531 (0.001) | 0.066 (0.024) | 0.072 (0.026) |
|          | 0.5      | 0.064 (0.002) | 0.070 (0.002) | 0.581 (0.020) | 0.631 (0.021) | 0.507 (0.002) | 0.551 (0.002) | 0.143 (0.057) | 0.155 (0.062) |

Table 1: MSE and adjusted MSE values (with associated standard errors in parentheses) for Horseshoe-fusion, $t$-fusion, Laplace fusion, and fused lasso method, when the true signal is evenly spaced (“Even”), unevenly spaced (“Uneven”), and very unevenly spaced (“V. Uneven”).
| Signal   | $\sigma$ | HS-fusion       |   | t-fusion       |   | Laplace fusion |   | $L_1$ fusion |   |
|----------|----------|-----------------|---|-----------------|---|-----------------|---|-----------------|---|
|          |          | W               | B | W               | B | W               | B | W               | B |
| Even     | 0.1      | 0.039 (0.001)   | 0.846 (0.005) | 0.035 (0.001)   | 0.892 (0.010) | 0.678 (0.002)   | 0.003 (0.000) | 0.069 (0.003)   | 0.739 (0.011) |
|          | 0.3      | 0.140 (0.003)   | 0.425 (0.020) | 0.121 (0.005)   | 0.381 (0.035) | 0.687 (0.003)   | 0.003 (0.000) | 0.181 (0.007)   | 0.333 (0.023) |
|          | 0.5      | 0.250 (0.005)   | 0.139 (0.016) | 0.221 (0.007)   | 0.079 (0.017) | 0.704 (0.004)   | 0.003 (0.000) | 0.285 (0.009)   | 0.119 (0.016) |
| Uneven   | 0.1      | 0.037 (0.001)   | 0.846 (0.005) | 0.034 (0.001)   | 0.864 (0.010) | 0.537 (0.002)   | 0.006 (0.001) | 0.070 (0.003)   | 0.751 (0.010) |
|          | 0.3      | 0.134 (0.003)   | 0.457 (0.020) | 0.106 (0.004)   | 0.266 (0.031) | 0.548 (0.003)   | 0.005 (0.001) | 0.182 (0.006)   | 0.348 (0.021) |
|          | 0.5      | 0.238 (0.005)   | 0.146 (0.015) | 0.173 (0.006)   | 0.022 (0.007) | 0.570 (0.004)   | 0.004 (0.000) | 0.284 (0.009)   | 0.131 (0.015) |
| V. Uneven| 0.1      | 0.036 (0.001)   | 0.800 (0.007) | 0.034 (0.001)   | 0.670 (0.022) | 0.325 (0.002)   | 0.008 (0.001) | 0.106 (0.001)   | 0.710 (0.010) |
|          | 0.3      | 0.128 (0.002)   | 0.401 (0.025) | 0.092 (0.003)   | 0.013 (0.004) | 0.344 (0.003)   | 0.007 (0.001) | 0.276 (0.006)   | 0.269 (0.023) |
|          | 0.5      | 0.222 (0.004)   | 0.149 (0.019) | 0.124 (0.004)   | 0.001 (0.000) | 0.376 (0.004)   | 0.006 (0.001) | 0.391 (0.010)   | 0.105 (0.017) |

Table 2: Within (W) and between (B) block average variation (with associated standard errors in parentheses) for different fusion estimation methods. For good structure recovery, W values should be low and B values should be high.