Abstract

As one step in a working program initiated by Pudlák [Pud17] we construct an oracle relative to which \( P \neq \text{NP} \) and all non-empty sets in \( \text{NP} \cup \text{coNP} \) have \( P \)-optimal proof systems.

1 Introduction

The main motivation for the present paper is an article by Pudlák [Pud17] who lists several major conjectures in the field of proof complexity and discusses their relations. Among others, Pudlák conjectures the following assertions (note that within the present paper all reductions are polynomial-time-bounded):

- \( \text{CON} \) (resp., \( \text{SAT} \)): \( \text{coNP} \) (resp., \( \text{NP} \)) does not contain many-one complete sets that have \( P \)-optimal proof systems
- \( \text{CON}^N \): \( \text{coNP} \) does not contain many-one complete sets that have optimal proof systems, (note that \( \text{CON}^N \) is the non-uniform version of \( \text{CON} \))
- \( \text{DisjNP} \) (resp., \( \text{DisjCoNP} \)): The class of all disjoint \( \text{NP} \)-pairs (resp., \( \text{coNP} \)-pairs) does not have many-one complete elements,
- \( \text{TFNP} \): The class of all total polynomial search problems does not have complete elements,
- \( \text{NP} \cap \text{coNP} \) (resp., \( \text{UP} \)): \( \text{NP} \cap \text{coNP} \) (resp., \( \text{UP} \), the class of problems accepted by \( \text{NP} \) machines with at most one accepting path for each input) does not have many-one complete elements.

Pudlák asks for oracles separating corresponding relativized conjectures. Recently there has been made some progress in this working program [Kha19, DG19, Dos19a] which is documented by the following figure representing the current state of the art.

In the figure \( O \) denotes the oracle that we construct in the present paper. It shows that there is no relativizable proof for the implication \( P \neq \text{NP} \Rightarrow \text{CON} \lor \text{SAT} \), i.e. the conjectures \( P \neq \text{NP} \) and \( \text{CON} \lor \text{SAT} \) cannot be shown equivalent with relativizable proofs. More precisely, the relativization of \( \text{CON} \lor \text{SAT} \) (i.e., the statement “for all oracles \( D \) it holds (i) there is no \( A \in \text{NP}^D \) that has \( P^D \)-optimal proof systems or (ii) there is no \( A \in \text{coNP}^D \) that has \( P^D \)-optimal proof systems”) is strictly stronger than the relativization of \( P \neq \text{NP} \) (i.e., the statement “for all oracles \( D \) it holds \( P^D \neq \text{NP}^D \)”).
Figure 1: Solid arrows mean implications. All implications occurring in the graphic have relativizable proofs. A dashed arrow from one conjecture $A$ to another conjecture $B$ means that there is an oracle $X$ against the implication $A \Rightarrow B$, i.e., relative to $X$, it holds $A \land \neg B$. Pudlák [Pud17] also defines the conjecture $RFN_1$ and lists it between $\text{CON} \lor \text{SAT}$ and $P \neq \text{NP}$, i.e., $\text{CON} \lor \text{SAT} \Rightarrow RFN_1 \Rightarrow P \neq \text{NP}$. Khaniki [Kha19] even shows $\text{CON} \lor \text{SAT} \iff RFN_1$, which is why we omit $RFN_1$ in the figure. For a definition of $RFN_1$ we refer to [Pud17].

2 Preliminaries

Most parts of this section are copied from our previous papers [DG19] and [Dos19a].

Throughout this paper let $\Sigma$ be the alphabet $\{0, 1\}$. We denote the length of a word $w \in \Sigma^*$ by $|w|$. Let $\Sigma_{\leq n} = \{w \in \Sigma^* | |w| \leq n\}$. The empty word is denoted by $\varepsilon$ and the $i$-th letter of a word $w$ for $0 \leq i < |w|$ is denoted by $w(i)$, i.e., $w = w(0)w(1)\cdots w(|w| - 1)$. If $v$ is a prefix of $w$, i.e., $|v| \leq |w|$ and $v(i) = w(i)$ for all $0 \leq i < |v|$, then we write $v \subseteq w$. For each finite set $Y \subseteq \Sigma^*$, let $t(Y) \triangleq \sum_{w \in Y} |w|$.

The set of all integers is denoted by $\mathbb{Z}$. Moreover, $\mathbb{N}$ denotes the set of natural numbers and $\mathbb{N}^+$ denotes the set of positive natural numbers. The identity function $x \mapsto x$ is denoted by $\text{id}$.

We identify $\Sigma^*$ with $\mathbb{N}$ via the polynomial-time computable, polynomial-time invertible bijection $w \mapsto \sum_{i < |w|} (1 + w(i))2^{|w|-1-i}$, which is a variant of the dyadic encoding. Hence, notations, relations, and operations for $\Sigma^*$ are transferred to $\mathbb{N}$ and vice versa. In particular, $|n|$ denotes the length of $n \in \mathbb{N}$. We eliminate the ambiguity of the expressions $0^i$ and $1^i$ by always interpreting them over $\Sigma^*$.

Let $\langle \cdot \rangle : \bigcup_{i \geq 0} \mathbb{N}^i \to \mathbb{N}$ be an injective, polynomial-time computable, polynomial-time invertible pairing function such that $|\langle u_1, \ldots, u_n \rangle| = 2(|u_1| + \cdots + |u_n| + n)$.

Given two sets $A$ and $B$, $A - B$ denotes the set difference between $A$ and $B$. The complement of a set $A$ relative to the universe $U$ is denoted by $\overline{A} = U - A$. The universe will always be apparent from the context. The symmetric difference of sets $A$ and $B$ is denote by $A \Delta B = (A - B) \cup (B - A)$.

$\text{FP}$, $\text{P}$, and $\text{NP}$ denote standard complexity classes [Pap94]. Define $\text{coC} = \{A \subseteq \Sigma^* | \overline{A} \in \mathcal{C}\}$ for a class $\mathcal{C}$. We also consider all these complexity classes in the presence of an oracle $O$ and denote the corresponding classes by $\text{FP}^O$, $\text{P}^O$, $\text{NP}^O$, and so on.
Let $M$ be a Turing machine. $M^D(x)$ denotes the computation of $M$ on input $x$ with $D$ as an oracle. For an arbitrary oracle $D$ we let $L(M^D) = \{ x \mid M^D(x) \text{ accepts}\}$. A nondeterministic computation accepts if and only if it has an accepting path.

For a deterministic polynomial-time Turing transducer, depending on the context, $F^D(x)$ either denotes the computation of $F$ on input $x$ with $D$ as an oracle or the output of this computation.

**Definition 2.1** A sequence $(M_i)$ is called a standard enumeration of nondeterministic, polynomial-time oracle Turing machines, if it has the following properties:

1. All $M_i$ are nondeterministic, polynomial-time oracle Turing machines.
2. For all oracles $D$ and all inputs $x$ the computation $M_i^D(x)$ stops within $|x|^i + i$ steps.
3. For every nondeterministic, polynomial-time oracle Turing machine $M$ there exist infinitely many $i \in \mathbb{N}$ such that for all oracles $D$ it holds that $L(M^D) = L(M_i^D)$.
4. There exists a nondeterministic, polynomial-time oracle Turing machine $M$ such that for all oracles $D$ and all inputs $x$ it holds that $M^D((i, 0^{|x|^i} + i), x)$ nondeterministically simulates the computation $M_i^D(x)$.

Analogously we define standard enumerations of deterministic, polynomial-time oracle Turing transducers.

Throughout this paper, we fix some standard enumerations. Let $M_1, M_2, \ldots$ be a standard enumeration of nondeterministic polynomial-time oracle Turing machines. Then for every oracle $D$, the sequence $(M_i)$ represents an enumeration of languages in $\text{NP}^D$, i.e., $\text{NP}^D = \{ L(M_i^D) \mid i \in \mathbb{N}^+ \}$. Let $F_1, F_2, \ldots$ be a standard enumeration of polynomial time oracle Turing transducers.

By the properties of standard enumerations, for each oracle $D$ the problem

$$K^D = \{ (0^i, 0^i, x) \mid M_i^D(x) \text{ accepts within } t \text{ steps} \}$$

is $\text{NP}^D$-complete (in particular it is in $\text{NP}^D$).

**Definition 2.2** ([CR79]) A function $f \in \text{FP}$ is called a proof system for the set $\text{ran}(f)$. For $f, g \in \text{FP}$ we say that $f$ is simulated by $g$ (resp., $f$ is $\text{P}$-simulated by $g$) denoted by $f \leq g$ (resp., $f \leq^p g$), if there exists a function $\pi$ (resp., a function $\pi \in \text{FP}$) and a polynomial $p$ such that $|\pi(x)| \leq p(|x|)$ and $g(\pi(x)) = f(x)$ for all $x$. A function $g \in \text{FP}$ is optimal (resp., $\text{P}$-optimal), if $f \leq g$ (resp., $f \leq^p g$) for all $f \in \text{FP}$ with $\text{ran}(f) = \text{ran}(g)$. Corresponding relativized notions are obtained by using $\text{P}^O$, $\text{FP}^O$, and $\leq^{p,O}$ in the definitions above.

The following proposition states the relativized version of a result by Köbler, Messner, and Torán [KMT03], which they show with a relativizable proof.

**Proposition 2.3** ([KMT03]) For every oracle $O$, if $A$ has a $\text{PO}$-optimal (resp., optimal) proof system and $B \leq^{p,O}_m A$, then $B$ has a $\text{PO}$-optimal (resp., optimal) proof system.

**Corollary 2.4** For every oracle $O$,

1. if there exists a $\leq^{p,O}_m$-complete $A \in \text{NP}^O$ that has optimal (resp., $\text{P}$-optimal) proof systems, then all sets in $\text{NP}^O$ have optimal (resp., $\text{P}$-optimal) proof systems.
2. if there exists a $\leq^{p,O}_m$-complete $A \in \text{coNP}^O$ that has optimal (resp., $\text{P}$-optimal) proof systems, then all sets in $\text{coNP}^O$ have optimal (resp., $\text{P}$-optimal) proof systems.
We introduce some quite specific notations that are designed for the construction of oracles. The domain and range of a function $t$ are denoted by $\text{dom}(t)$ and $\text{ran}(t)$, respectively. If a partial function $t$ is not defined at point $x$, then $t \cup \{x \mapsto y\}$ denotes the continuation $t'$ of $t$ that at $x$ has value $y$ and satisfies $\text{dom}(t') = \text{dom}(t) \cup \{x\}$.

If $A$ is a set, then $A(x)$ denotes the characteristic function at point $x$, i.e., $A(x)$ is 1 if $x \in A$, and 0 otherwise. An oracle $D \subseteq \mathbb{N}$ is identified with its characteristic sequence $D(0)D(1) \cdots$, which is an $\omega$-word. (In this way, $D(i)$ denotes both, the characteristic function at point $i$ and the $i$-th letter of the characteristic sequence, which are the same.) A finite word $w$ describes an oracle that is partially defined, i.e., only defined for natural numbers $x < |w|$. We can use $w$ instead of the set $\{i \mid w(i) = 1\}$ and write for example $A = w \cup B$, where $A$ and $B$ are sets. For nondeterministic oracle Turing machines $M$ we use the following phrases: A computation $M^w(x)$ definitely accepts, if all paths accept and all queries are $< |w|$. A computation $M^w(x)$ definitely rejects, if it contains a path that rejects (within $t$ steps) and the queries on this path are $< |w|$. For deterministic oracle Turing machines $P$ we say: A computation $P^w(x)$ definitely accepts (resp., definitely rejects), if it accepts (resp., rejects) and the queries are $< |w|$.

For a deterministic or nondeterministic Turing machine $M$ we say that the computation $M^w(x)$ is defined, if it definitely accepts or definitely rejects. For a transducer $F$, the computation $F^w(x)$ is defined, if all queries are $< |w|$.

## 3 Oracle Construction

We now construct the announced oracle.

**Lemma 3.1 ([DG19])** For all $y \leq |w|$ and all $v \supseteq w$ it holds $K^v(y) = K^w(y)$.

**Proof** We may assume $y = (0^t, 0^t, x)$ for suitable $i, t, x$, since otherwise $K^w(y) = K^v(y) = 0$. For each $q$ that is queried within the first $t$ steps of $M^w_i(x)$ or $M^v_i(x)$ it holds that $|q| \leq t < |y|$ and thus, $q < y$. Hence, these queries are answered the same way relative to $w$ and $v$, showing that $M^w_i(x)$ accepts within $t$ steps if and only if $M^v_i(x)$ accepts within $t$ steps. \qed

**Theorem 3.2** There exists an oracle $O$ relative to which the following statements hold:

- $P^O \neq NP^O$
- $K^O$ has $P^O$-optimal proof systems.
- $\overline{K^O}$ has $P^O$-optimal proof systems.

The following corollary follows from Theorem 3.2 and Corollary 2.4.

**Corollary 3.3** There exists an oracle $O$ relative to which the following statements hold:

- $P^O \neq NP^O$
- Each set in $NP^O$ has $P^O$-optimal proof systems.
- Each set in $coNP^O$ has $P^O$-optimal proof systems.

**Proof of Theorem 3.2** We define $c(i, x, y) = (0^t, 0^{|x|^t + i}, x, y)$. Let $D$ be a (possibly partial) oracle and define

$$A^D = \{0^n \mid \exists y \in \Sigma^* 0y \in D\}.$$
We will construct the oracle such that \( A^O \in \text{NP}^O - \text{P}^O \) for the final oracle \( O \). Note that throughout this proof we sometimes omit the oracles in the superscript, e.g., we write NP or \( A \) instead of NP\(^D\) or \( A^D \). However, we do not do that in the “actual” proof but only when explaining ideas in a loose way in order to give the reader the intuition behind the occasionally very technical arguments.

Let us briefly sketch the idea of our construction.

**Preview of construction.** For each \( F_i \) we first try to ensure that \( F_i \) does not compute a proof system for \( K \) (resp., \( \overline{K} \)). If this is impossible, then \( F_i \) inherently computes a proof system for \( K \) (resp., \( \overline{K} \)). In that case we start to encode the values of \( F_i \) into the oracle so that \( F_i \) can be P-simulated by some proof system for \( K \) (resp., \( \overline{K} \)) that we will define later and finally show to be P-optimal.

Moreover, we diagonalize against all \( P_i \) such that \( A \) is not in \( \text{P} \) relative to the final oracle.

**Claim 3.4** ([DG19]) Let \( w \in \Sigma^* \) be an oracle, \( i \in \mathbb{N}^+ \), and \( x, y \in \mathbb{N} \) such that \( c(i, x, y) \leq |w| \). Then the following holds.

1. \( F_i^w(x) \) is defined and \( F_i^w(x) < |w| \).
2. For all \( v \supseteq w \), \( (F_i^w(x) \in K^w \iff F_i^w(x) \in K^v) \).

**Proof** As the running time of \( F_i^w(x) \) is bounded by \(|x|^i + i < |c(i, x, y)| < c(i, x, y) \leq |w| \), the computation \( F_i^w(x) \) is defined and its output is less than \( |w| \). Hence, 1 holds. Consider 2. It suffices to show that \( K^v(q) = K^w(q) \) for all \( q < |w| \) and all \( v \supseteq w \). This holds by Lemma 3.1. \( \square \)

During the construction we maintain a collection of requirements \( t : \{0, 1\} \times \mathbb{N}^+ \to \mathbb{N} \), where \( t \in \mathcal{T} \) for

\[
\mathcal{T} = \{ t : \{0, 1\} \times \mathbb{N}^+ \to \mathbb{N} \mid t \text{ has a finite domain} \}.
\]

A partial oracle \( w \) is called \( t \)-valid if it satisfies the following properties.

**V1** For all \( i \in \mathbb{N}^+ \),

1. if \( 10c(i, x, y) \in w \) for some \( x, y \in \mathbb{N} \), then \( F_i^w(x) = y \in K^w \).
2. if \( 11c(i, x, y) \in w \) for some \( x, y \in \mathbb{N} \), then \( F_i^w(x) = y \in \overline{K}^w \).

**V2** For all \( i \in \mathbb{N}^+ \), if \( t(0, i) = 0 \), then there exists \( x \) such that \( F_i^w(x) \) is defined and \( F_i^w(x) \notin K^v \) for all \( v \supseteq w \).

**V3** For all \( i \in \mathbb{N}^+ \), if \( t(0, i) > 0 \), then for all \( x \in \mathbb{N} \) with \( t(0, i) \leq 10c(i, x, F_i^w(x)) < |w| \), it holds \( 10c(i, x, F_i^w(x)) \in w \).

**V4** For all \( i \in \mathbb{N}^+ \), if \( t(1, i) = 0 \), then there exists \( x \) such that \( F_i^w(x) \) is defined and \( F_i^w(x) \notin \overline{K}^v \) for all \( v \supseteq w \).

**V5** For all \( i \in \mathbb{N}^+ \), if \( t(1, i) > 0 \), then for all \( x \in \mathbb{N} \) with \( t(1, i) \leq 11c(i, x, F_i^w(x)) < |w| \), it holds \( 11c(i, x, F_i^w(x)) \in w \).

The following claim follows directly from the definition of \( t \)-valid.

**Claim 3.5** Let \( t, t' \in \mathcal{T} \) such that \( t' \) is an extension of \( t \). If \( w \in \Sigma^* \) is \( t' \)-valid, then \( w \) is \( t \)-valid.

**Claim 3.6** Let \( t \in \mathcal{T} \) \( u, v, w \in \Sigma^* \) be oracles with \( u \subseteq v \subseteq w \). If \( u \) and \( w \) are \( t \)-valid, then \( v \) is \( t \)-valid.
Proof v satisfies V2 and V4 since u satisfies V2 and V4.

Let us argue for V1. Let \(10c(i, x, y) \in \nu\) for \(i \in \mathbb{N}^+\) and \(x, y \in \mathbb{N}\). Then \(10c(i, x, y) \in \nu\) and as \(\nu\) is \(t\)-valid, it holds by V1 that \(F_i^\nu(x) = y \in K^\nu\). By Claim 3.4, \(F_i^\nu(x) = F_i^\nu(x) = y\) and \(K^\nu(y) = K^\nu(y) = 1\). Analogously, \(11c(i, x, y) \in \nu\) for \(i \in \mathbb{N}^+\) and \(x, y \in \mathbb{N}\) implies \(F_i^\nu(x) = y \in K^\nu\).

Thus, \(v\) satisfies V1.

Consider V3/V5. Let \(i \in \mathbb{N}^+, x \in \mathbb{N},\) and \(b \in \{0, 1\}\) such that \(0 < t(b, i) \leq 1bc(i, x, F_i^\nu(x)) < |\nu|\). Then by Claim 3.4, \(F_i^\nu(x) = F_i^\nu(x)\). As \(\nu\) is \(t\)-valid, we obtain by V3/V5 that \(1bc(i, x, F_i^\nu(x)) \in \nu\).
Since \(1bc(i, x, F_i^\nu(x)) < |\nu|\) and \(v \subseteq \nu\), we have \(1bc(i, x, F_i^\nu(x)) = 1bc(i, x, F_i^\nu(x)) \in \nu\), which shows that \(v\) satisfies V3/V5.

Oracle construction. Let \(T : \mathbb{N} \rightarrow \{0, 1, 2\} \times \mathbb{N}^+\) be a bijection. Each value of \(T(s)\) for \(s \in \mathbb{N}\) stands for a task. We treat the tasks in the order specified by \(T\). We start with the nowhere defined function \(t_0\) and the \(t_0\)-valid oracle \(w_0 = \varepsilon\). Then we define functions \(t_1, t_2, \ldots\) in \(T\) such that \(t_{i+1}\) is an extension of \(t_i\) and partial oracles \(w_0 \sqsubseteq w_1 \sqsubseteq w_2 \sqsubseteq \ldots\) such that each \(w_i\) is \(t_i\)-valid. Finally, we choose \(O = \bigcup_{i=0}^\infty w_i\) (note that \(O\) is totally defined since in each step we strictly extend the oracle). We describe step \(s > 0\), which starts with a \(t_{s-1}\)-valid oracle \(w_{s-1}\) and extends it to a \(t_s\)-valid \(w_s \sqsupseteq w_{s-1}\) depending on the value of \(T(s)\). We will argue later that the construction is possible.

- task \((0, i)\) for \(i \in \mathbb{N}^+\): Let \(t' = t_{s-1} \cup \{(0, i) \mapsto 0\}\). If there exists a \(t'\)-valid \(v \sqsupseteq w_{s-1}\), then \(t_s = t'\) and define \(w_s = v\) for the least \(t'\)-valid \(v \sqsupseteq w_{s-1}\). Otherwise, let \(t_s = t_{s-1} \cup \{(0, i) \mapsto |w_{s-1}|\}\) and choose \(w_s = w_{s-1}b\) for \(b \in \{0, 1\}\) such that \(w_s\) is \(t_s\)-valid.

- task \((1, i)\) for \(i \in \mathbb{N}^+\): Let \(t' = t_{s-1} \cup \{(1, i) \mapsto 0\}\). If there exists a \(t'\)-valid \(v \sqsupseteq w_{s-1}\), then \(t_s = t'\) and define \(w_s = v\) for the least \(t'\)-valid \(v \sqsupseteq w_{s-1}\). Otherwise, let \(t_s = t_{s-1} \cup \{(1, i) \mapsto |w_{s-1}|\}\) and choose \(w_s = w_{s-1}b\) for \(b \in \{0, 1\}\) such that \(w_s\) is \(t_s\)-valid.

- task \((2, i)\) for \(i \in \mathbb{N}^+\): Let \(t_s = t_{s-1}\) and chose \(w_s \sqsupseteq w_{s-1}\) such that for some \(n \in \mathbb{N}\) the computation \(P_i^{w_s}(0^n)\) is defined, \(0^n \in A^v \Leftrightarrow 0^n \in A^{w_s}\) for all \(v \supseteq w_s\), and \(0^n \in A^{w_s} \Leftrightarrow P_i^{w_s}(0^n)\) rejects.

Claim 3.7 Let \(s \geq 0\) and \(w \supseteq w_s\) such that \(w\) is \(t_s\)-valid.

1. If \(z = 10c(i, x, F_i^\nu(x))\) for \(i \in \mathbb{N}^+\) and \(x \in \mathbb{N}\) with \(0 < t_s(0, i) \leq z\), then \(w_1\) is \(t_s\)-valid.
2. If \(z = 11c(i, x, F_i^\nu(x))\) for \(i \in \mathbb{N}^+\) and \(x \in \mathbb{N}\) with \(0 < t_s(1, i) \leq z\), then \(w_1\) is \(t_s\)-valid.
3. If \(z = 0y\) for \(y \in \Sigma^n\) and \(n \in \mathbb{N}\), then \(w_0\) and \(w_1\) are \(t_s\)-valid.

4. In all other cases (i.e., none of the assumptions in 1–3 holds) \(w_0\) is \(t_s\)-valid.

Proof First observe that V2 and V4 are not affected by extending the oracle. Moreover, by Claim 3.4, as \(w\) satisfies V1, V3, and V5, \(wb\) for \(b \in \{0, 1\}\) satisfies

(A) V1.1 unless \(b = 1\), \(z = 10c(i, x, y)\) for \(i, x, y \in \mathbb{N}\) with \(i > 0\) and \(\neg(F_i^\nu(x) = y \in K^\nu)\)

(B) V1.2 unless \(b = 1\), \(z = 11c(i, x, y)\) for \(i, x, y \in \mathbb{N}\) with \(i > 0\) and \(\neg(F_i^\nu(x) = y \in K^\nu)\)

(C) V3 unless \(b = 0\) and \(z = 10c(i, x, F_i^\nu(x))\) for \(i > 0\) and \(x \in \mathbb{N}\) with \(0 < t_s(0, i) \leq z\).

(D) V5 unless \(b = 0\) and \(z = 11c(i, x, F_i^\nu(x))\) for \(i > 0\) and \(x \in \mathbb{N}\) with \(0 < t_s(1, i) \leq z\).

This proves statement 3. Let us argue for statement 4. According to (A) and (B) \(w_0\) satisfies V1. If \(w_0\) does not satisfy V3 (resp., V5), then according to (C) (resp., (D)), \(z = 10c(i, x, F_i^\nu(x))\) (resp., \(z = 11c(i, x, F_i^\nu(x))\) for \(i \in \mathbb{N}^+\) and \(x \in \mathbb{N}\) as well as \(0 < t_s(0, i) \leq z\) (resp., \(0 < t_s(1, i) \leq z\)). However, this case is covered by statement 1 (resp., statement 2). This proves statement 4.
Let us consider statements 1 and 2 simultaneously. Due to the statements (C) and (D) it suffices to argue for V1.1 (resp., V1.2 when arguing for statement 2). Here it is sufficient to show $F^w_i(x) \in K^w$ (resp., $F^y_i(x) \in \overline{K^w}$). For a contradiction assume $F^w_i(x) \notin K^w$ (resp., $F^y_i(x) \notin \overline{K^w}$). Let $s' > 0$ be the step with $T(s') = (0, i)$ (resp., $T(s') = (1, i)$). Then $s' \leq s$. By Claim 3.5, the oracle $w$ is $t_{s'-1}$-valid and by Claim 3.4, $F^v_i(x)$ is defined and $F^v_i(x) \notin K^v$ (resp., $F^w_i(x) \notin \overline{K^v}$) for all $v \supseteq w$. Hence, $w$ is even $t$-valid for $t = t_{s'-1} \cup \{(0, i) \mapsto 0\}$ (resp., $t = t_{s'-1} \cup \{(1, i) \mapsto 0\}$). But then the construction would have chosen $t_{s'} = t$, in contradiction to $t_s(0, i) > 0$ (resp., $t_s(1, i) > 0$).

□

We now show that the described construction is possible: for a contradiction, assume that it is not. Hence, there exists a minimal $s > 0$ such that step $s$ fails. Then $w_{s-1}$ is $t_{s-1}$-valid.

Assume that in step $s$ some task $(a, i)$ for $a \in \{0, 1\}$ and $i \in \mathbb{N}^+$ is treated. Then $t_{s-1}(a, i)$ is not defined as this value is defined in the unique treatment of the task $(a, i)$. Thus, $t'$ is well defined. Moreover, if there exists a $t'$-valid oracle $v \supseteq w_{s-1}$, then step $s$ is clearly possible. Otherwise, by the (sufficiently large) choice of $t_{s}(a, i)$, the oracle $w_{s-1}$ is even $t_{s}$-valid and by Claim 3.7, there exists $b \in \{0, 1\}$ such that the oracle $w_{b} = w_{s-1}b$ is $t_{s}$-valid. Hence, if some task $(a, i)$ for $a \in \{0, 1\}$ is treated in step $s$, then we obtain a contradiction.

From now on we assume that step $s$ treats some task $(2, i)$ for $i > 0$. Thus, $t_s = t_{s-1}$ and we need to show that there exist some $t_{s}$-valid $w_s \supseteq w_{s-1}$ and some $n \in \mathbb{N}$ such that the computation $P^w_i(0^n)$ is defined, $0^n \in A^v \iff 0^n \in A^{w_s}$ for all $v \supseteq w_s$, and $(0^n \in A^v \iff P^w_i(0^n)$ rejects).

Choose $n$ large enough such that $2^n > 2(n^i + i)$. Let $u_0 \supseteq w_{s-1}$ be the minimal $t_s$-valid oracle that is defined for all words of length $n$. Such an oracle exists by Claim 3.7. Moreover, let $u \supseteq u_0$ be the minimal $t_s$-valid oracle that is defined for all words of length $2(n^i + i)$. Such an oracle exists by Claim 3.7 and by Claim 3.7.3, $u \cup 0^n = \emptyset$. If $P^w_i(0^n)$ accepts, then it definitely accepts by the choice of $u$ and since $0^n \in A^v$ for all $v \supseteq u$ (note that $u$ is defined for all words of length $n + 1$), we can choose $w_s = u$ and obtain a contradiction to the assumption that step $s$ is not possible.

From now on we assume that $P^w_i(0^n)$ rejects. Let $U$ be the set of oracle queries of $P^w_i(0^n)$ whose length is $\geq n + 1$. We define $Q_0(U) = U$ and for $m \in \mathbb{N}$

$$Q_{m+1}(U) := \bigcup_{j > 0, x, y \in \mathbb{N}} \{q \in \Sigma^2n+1 \mid q \text{ is queried by } F^w_j(x)\}.$$ 

Moreover, define $Q(U) = \bigcup_{m \in \mathbb{N}} Q_m(U)$.

Claim 3.8 $\ell(Q(U)) \leq 2(n^i + i)$.

Proof By definition of $Q_0(U)$ it holds $\ell(Q_0(U)) \leq n^i + i$. We show that for all $m \in \mathbb{N}$ it holds $\ell(Q_{m+1}(U)) \leq \ell(Q_m(U))/2$. Then for all $m \in \mathbb{N}$ it holds $\ell(Q_m(U)) \leq \ell(Q_0(U))/2^m$ and thus,

$$\ell(\bigcup_{k=0}^m Q_k(U)) \leq \ell(Q_0(U)) \cdot \sum_{k=0}^m 1/2^k \leq (n^i + i) \cdot \frac{1 - 1/2^{m+1}}{1/2} < 2(n^i + i),$$

which shows $\ell(Q(U)) \leq 2(n^i + i)$.

It remains to show that $\ell(Q_{m+1}(U)) \leq \ell(Q_m(U))/2$ for all $m \in \mathbb{N}$. Let $\alpha \in Q_m(U)$. If $\alpha$ is not of the form $10c(j, x, y)$ or $11c(j, x, y)$, then it generates no elements in $Q_{m+1}(U)$. Let $\alpha = 1bc(j, x, y)$ for $b \in \{0, 1\}$, $j \in \mathbb{N}^+$, and $x, y \in \mathbb{N}$. This affects that all queries of $F^w_j(x)$ are added into $Q_{m+1}(U)$. The computation time of $F^w_j(x)$ (and also the sum of the lengths of all queries asked by that computation) is bounded by $|x|^j + j \leq |c(j, x, y)/2$ (cf. the definition of $c(\cdot, \cdot, \cdot)$ and the definition of the pairing function).
Hence,
\[
\ell(Q_{m+1}(U)) = \ell\left( \bigcup_{j > 0, x, y \in \mathbb{N}} Q_{m+1}(U) \cap F_j(x) \right) \\
\leq \sum_{j > 0, x, y \in \mathbb{N}} \ell\left( \left\{ q \in \Sigma^{\geq n+1} \mid q \text{ is queried by } F_j(x) \right\} \right) \\
\leq \frac{1}{2} \cdot \sum_{j > 0, x, y \in \mathbb{N}} \ell\left( \left\{ q \in \Sigma^{\geq n+1} \mid q \text{ is queried by } F_j(x) \right\} \right) \\
\leq 1/2 \cdot \ell(Q_m(U))/2,
\]
which finishes the proof. \hfill \Box

As by the choice of \( n \), it holds \(|Q(U)| \leq \ell(Q(U)) \leq 2(n^i + i) < 2^n\), there exists \( \alpha \in 0\Sigma^n \) that is not in \( Q(U) \). Let \( u' \) be the minimal \( t_u \)-valid oracle \( \sqsubseteq u_0 \) that is defined for all words \( \leq 01^n \) and satisfies \( u' \cap 0\Sigma^n = \{ \alpha \} \). Such an oracle exists by Claim 3.7.3.

**Claim 3.9** There exists a \( t_u \)-valid oracle \( v \sqsubseteq u' \) that is defined for all words of length \( 2(n^i + i) \) and satisfies \( v(q) = u(q) \) for all \( q \in Q(U) \).

**Proof** As \( \alpha \notin Q(U) \) it holds \( u'(q) = u(q) \) for all \( q \in Q(U) \) that \( u' \) is defined for.

It suffices to show the following:

For each \( t_u \)-valid \( w \sqsubseteq u' \) with \( w(q) = u(q) \) for all \( q \in Q(U) \) that \( w \) is defined for, there exists \( b \in \{0, 1\} \) such that \( wb \) is \( t_u \)-valid and \( wb(q) = u(q) \) for all \( q \in Q(U) \) that \( wb \) is defined for.

Let some \( w \) with the properties of (1) be given. Moreover, let \( z = |w| \), i.e., \( z \) is the least word that \( w \) is not defined for. We study three cases.

1. Assume \( z = 1ac(j, x, F_j^w(x)) \) for \( a \in \{0, 1\} \), \( j \in \mathbb{N}^+ \), and \( x, y \in \mathbb{N} \) with \( 0 < t_u(a, j) \leq z \). Then choose \( b = 1 \). According to Claim 3.7.1 (resp., Claim 3.7.2 in case \( a = 1 \)) the oracle \( wb \) is \( t_u \)-valid.

   It remains to show that \( z \in Q(U) \Rightarrow z \in u \). For a contradiction assume \( z \in Q(U) \land z \notin u \). Then \( F_j^w(x) \neq F_j^u(x) \), since \( F_j^w(x) = F_j^u(x) \) and \( 0 < t_u(a, j) \leq z \) would imply by V3 (resp., V5 in case \( a = 1 \)), that \( 1ac(j, x, y) \notin u \). Hence, \( F_j^w(x) \neq F_j^u(x) \), which shows that there is some query \( q \in u \Delta w \) that is asked by both computations \( F_j^w(x) \) and \( F_j^u(x) \) (otherwise, the two queries would output the same value). In particular, \( q \in Q(U) \). As \( |q| \leq |x|^2 + j < |c(j, x, y)| < c(j, x, y) \), the oracle \( w \) is defined for \( q \) and by assumption \( w(q) = u(q) \), a contradiction.

2. If \( z = 0y \) for \( y \in \Sigma^m \) and \( m \in \mathbb{N} \), then we choose \( b = u(z) \) and hence, \( wb(q) = u(q) \) for all \( q \in Q(U) \) that \( wb \) is defined for. Moreover, by Claim 3.7.3, \( wb \) is \( t_u \)-valid.

3. For the remaining cases choose \( b = 0 \). Then Claim 3.7.4 states that \( wb \) is \( t_u \)-valid. It remains to show that \( z \in Q(U) \Rightarrow z \notin u \).

   For a contradiction assume \( z \in Q(U) \cap u \). Let \( u'' \) be the prefix of \( u \) that is defined for exactly the words \( < z \). As \( w_{s-1} \subseteq u'' \subseteq u \) and \( w_{s-1} \) as well as \( u \) are \( t_u \)-valid, \( u'' \) is \( t_u \)-valid as well by Claim 3.6.

   - Assume that Claim 3.7.1 or Claim 3.7.2 can be applied to \( u'' \). Then \( z = 1ac(j, x, F_j^{u''}(x)) \) for \( a \in \{0, 1\} \), \( j \in \mathbb{N}^+ \), and \( x \in \mathbb{N} \) with \( 0 < t_u(a, i) \leq z \). By Claim 3.4, \( F_j^w(x) = F_j^{u''}(x) \), which implies \( F_j^u(x) = F_j^{u''}(x) \) (otherwise, we were in a case that has already been treated). This shows that there is some query \( q \in u \Delta w \) that is asked by both computations \( F_j^u(x) \) and \( F_j^{u''}(x) \) (otherwise, the two computations would output the same value). In particular, \( q \in Q(U) \). As \( |q| \leq |x|^2 + j < |c(j, x, y)| < c(j, x, y) \), the oracle \( w \) is defined for \( q \) and by assumption \( w(q) = u(q) \), a contradiction.
• Now assume that Claim 3.7.3 or Claim 3.7.4 can be applied to \( u^n \) and yields that \( u^n0 \) is \( t_
abla \)-valid. By Claim 3.7, \( u^n0 \) can be extended to a \( t_
abla \)-valid oracle \( v' \) defined for exactly the words of length \( \leq 2(n^i + i) \). As \( u \) and \( v' \) agree on all words \( z \) and \( v'(z) < 0 \), it holds \( v' < u \), in contradiction to the choice of \( u \).

In both cases we obtain a contradiction. Hence, \( u(q) = w0(q) \) for all \( q \in Q(U) \) that \( w1 \) is defined for.

In all cases (1) holds. This completes the proof of Claim 3.9. \( \square \)

Recall that \( P_i^u(0^n) \) rejects. Let \( v \) be the oracle postulated by Claim 3.9. Since all queries of \( P_i^u(0^n) \) are in \( U \subseteq Q(U) \) and by Claim 3.9, \( u \) and \( v \) agree on all these queries, the computation \( P_i^u(0^n) \) rejects as well. Moreover, this computation is defined as \( v \) is defined for all words of length \( 2(n^i + i) \). However, as \( \alpha \in v \), we obtain \( 0^n \in A'^v \) for all \( v' \supseteq v \), which is a contradiction to the assumption that the construction fails in step \( s \) treating the task \((2, i)\).

We now have seen that the construction described above is possible. It remains to prove that

• \( NP^O \neq P^O \),

• \( K^O \) has \( P^O \)-optimal proof systems, and

• \( \overline{K^O} \) has \( P^O \)-optimal proof systems.

This is shown in the next three claims.

Claim 3.10 \( NP^O \neq P^O \).

**Proof** Assume \( NP^O = P^O \). Then there exists \( i > 0 \) such that \( L(P_i^O) = A^O \). Let \( s \) be the step with \( T(s) = (2, i) \). By construction, there exists some \( n \in \mathbb{N} \) such that the computation \( P_i^w_s(0^n) \) is defined, \( 0^n \in A^v \Leftrightarrow 0^n \in A^w_s \) for all \( v \supseteq w_s \), and \( (0^n \in A^w_s \Leftrightarrow P_i^w_s(0^n) \) rejects\). Hence, \( 0^n \in A^O \) if and only if \( P_i^O(0^n) \) definitely rejects. This contradicts \( L(P_i^O) = A^O \). \( \square \)

Claim 3.11 \( K^O \) has \( P^O \)-optimal proof systems.

**Proof** Let \( g \) be a proof system for \( K^O \) and \( a \in K^O \). Define

\[
 f(z) = \begin{cases} 
  y & \text{if } z = 010c(i, x, y) \text{ and } 10c(i, x, y) \in O \text{ for } i \in \mathbb{N}^+ \text{ and } x, y \in \mathbb{N} \\
  g(y) & \text{if } z = 1y \\
  a & \text{otherwise}
\end{cases}
\]

Then \( f \in FP^O \) and \( f(N) \supseteq K^O \) as \( g \) is a proof system for \( K^O \). We show \( f(N) \subseteq K^O \). As \( g \) is a proof system for \( K^O \) and \( a \in K^O \), it suffices to show \( f(z) \in K^O \) for \( z = 010c(i, x, y) \) with \( 10c(i, x, y) \in O \), \( i \in \mathbb{N}^+ \), and \( x, y \in \mathbb{N} \). Let \( s \) be large enough such that \( w_s \) is defined for \( 10c(i, x, y) \). Then by V1 and Claim 3.4, \( y \in K^v \) for all \( v \supseteq w_s \). It follows \( f(z) = y \in K^O \) and thus, \( f \) is a proof system for \( K^O \).

In order to show that \( f \) is \( P^O \)-optimal, let \( h \) be an arbitrary proof system for \( K^O \). Then there exists \( i \in \mathbb{N}^+ \) such that \( F_i^O \) computes \( h \). Let \( s \) be the step with \( T(0, i) = s \). It holds \( t_s(0, i) > 0 \) (otherwise, by V2 there exists \( x \) such that \( F_i^w(x) \) is defined and \( F_i^w(x) \notin K^v \) for all \( v \supseteq w \), which would imply that \( F_i^O \) is not a proof system for \( K^O \)). Define

\[
 \pi(x) = \begin{cases} 
  010c(i, x, F_i^O(x)) & \text{if } 10c(i, x, F_i^O(x)) \geq t_s(0, i) \\
  z & \text{if } 10c(i, x, F_i^O(x)) < t_s(0, i) \text{ and } z \text{ is minimal with } f(z) = F_i^O(x)
\end{cases}
\]
\(K\) is total as \(f\) and \(F_i^O\) are proof systems for \(K^O\) and thus, for each \(x\) there exists \(z\) with \(f(z) = F_i^O(x)\). Moreover, since \(t_s(0, i)\) is a constant, \(\pi\in FP^O\). It remains to show \(F_i^O(x) = f(\pi(x))\) for all \(x\). For all \(x\) with \(10c(i, x, F_i^O(x)) < t_s(0, i)\), this clearly holds. Assume \(10c(i, x, F_i^O(x)) \geq t_s(0, i)\). Choose \(s' \geq s\) large enough such that \(w_{s'}\) is defined for \(10c(i, x, F_i^O(x))\). By Claim 3.4, the computation \(F_i^{w_{s'}}(x)\) is defined and hence, \(F_i^{w_{s'}}(x) = F_i^O(x)\). Then by V3, \(10c(i, x, F_i^O(x)) = 10c(i, x, F_i^{w_{s'}}(x)) \in w_{s'} \subseteq O\) and thus, \(f(\pi(x)) = f(010c(i, x, F_i^O(x))) = F_i^O(x)\), which shows that \(F_i^O\) is \(P^O\)-simulated by \(f\). This completes the proof.

Claim 3.12 \(\overline{K^O}\) has \(P^O\)-optimal proof systems.

Proof Let \(g\) be a proof system for \(\bar{K}^O\) and \(a \in K^O\). Define

\[
f(z) = \begin{cases} y & \text{if } z = 011c(i, x, y) \text{ and } 11c(i, x, y) \in O \text{ for } i \in \mathbb{N}^+ \text{ and } x, y \in \mathbb{N} \\ g(y) & \text{if } z = 1y \\ a & \text{otherwise} \end{cases}
\]

Then \(f \in FP^O\) and \(f(\mathbb{N}) \supseteq \overline{K^O}\) as \(g\) is a proof system for \(\overline{K^O}\). We show \(f(\mathbb{N}) \subseteq \overline{K^O}\). As \(g\) is a proof system for \(\overline{K^O}\) and \(a \in K^O\), it suffices to show \(f(z) \in K^O\) for \(z = 011c(i, x, y)\) with \(11c(i, x, y) \in O\), \(i \in \mathbb{N}^+\), and \(x, y \in \mathbb{N}\). Let \(s\) be large enough such that \(w_s\) is defined for \(11c(i, x, y)\). Then by V1 and Claim 3.4, \(y \in \overline{K^O}\) for all \(v \supseteq w_s\). It follows \(f(z) = y \in \overline{K^O}\) and thus, \(f\) is a proof system for \(\overline{K^O}\).

In order to show that \(f\) is \(P^O\)-optimal, let \(h\) be an arbitrary proof system for \(\overline{K^O}\). Then there exists \(i \in \mathbb{N}^+\) such that \(F_i^O\) computes \(h\). Let \(s\) be the step with \(T(1, i) = s\). It holds \(t_s(1, i) > 0\) (otherwise, by V2 there exists \(x\) such that \(F_{w_s}^w(x)\) is defined and \(F_{w_s}^w(x) \in \overline{K^O}\) for all \(v \supseteq w_s\), which would imply that \(F_i^O\) is not a proof system for \(\overline{K^O}\)). Define

\[
\pi(x) = \begin{cases} 011c(i, x, F_i^O(x)) & \text{if } 11c(i, x, F_i^O(x)) \geq t_s(1, i) \\ z & \text{if } 11c(i, x, F_i^O(x)) < t_s(1, i) \text{ and } z \text{ is minimal with } f(z) = F_i^O(x) \end{cases}
\]

\(\pi\) is total as \(f\) and \(F_i^O\) are proof systems for \(\overline{K^O}\) and thus, for each \(x\) there exists \(z\) with \(f(z) = F_i^O(x)\). Moreover, since \(t_s(1, i)\) is a constant, \(\pi \in FP^O\). It remains to show \(F_i^O(x) = f(\pi(x))\) for all \(x\). For all \(x\) with \(11c(i, x, F_i^O(x)) < t_s(1, i)\), this clearly holds. Assume \(11c(i, x, F_i^O(x)) \geq t_s(1, i)\). Choose \(s' \geq s\) large enough such that \(w_{s'}\) is defined for \(11c(i, x, F_i^O(x))\). By Claim 3.4, the computation \(F_i^{w_{s'}}(x)\) is defined and hence, \(F_i^{w_{s'}}(x) = F_i^O(x)\). Then by V5, \(11c(i, x, F_i^O(x)) = 11c(i, x, F_i^{w_{s'}}(x)) \in w_{s'} \subseteq O\) and thus, \(f(\pi(x)) = f(011c(i, x, F_i^O(x))) = F_i^O(x)\), which shows that \(F_i^O\) is \(P^O\)-simulated by \(f\). This completes the proof.

This completes the proof of Theorem 3.2.

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