Quantizing Gravitational Collapse
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1 Introduction

It is generally believed that the following is true: the classical gravitational field of a sufficiently massive star will overcome its neutron degeneracy pressure and, barring quantum gravitational effects in the final stages of collapse, the star will form a singularity of space-time. The initial conditions determine the precise nature of the singularity of the collapse: either the singularity will be covered, in which case a black hole is formed, or it will be naked, i.e., not covered by a horizon. Naked singularities lead to violations of causality, which encouraged Penrose to propose the Cosmic Censorship Hypothesis (CCH)\cite{1}. The CCH simply states that physically reasonable initial data cannot produce naked singularities. Nevertheless, to date there is little evidence that it is true in general relativity, even though most of our current emphasis on black hole physics is based on the validity of this hypothesis.

The Lemaître-Tolman-Bondi (LTB) collapse of spherical, inhomogeneous dust\cite{2} is (arguably) the simplest model that allows for the formation both of black holes and of naked singularities. It is defined by specifying two functions, \( F(\rho) \) and \( f(\rho) \), where \( \rho \) is a shell label coordinate. The former represents the weighted mass contained within the matter shell labeled by \( \rho \), and the latter is related to the velocity profile within the collapsing cloud at the initial time. The configuration space for all LTB models consists of the dust proper time, \( \tau \), and the area radius, \( R \). The classical collapse may end in a black hole or in a naked singularity depending on the behavior of \( F(\rho) \) and \( f(\rho) \) near the center\cite{3}. When it ends in a black hole, a significant fraction of the star is expected to evaporate via Hawking radiation\cite{4} during the semi-classical phase. However, the back-reaction of the space-time prevents us from making definitive predictions at the final stages. A different result is obtained when the collapse is toward a naked singularity\cite{5}. It is quantum mechanically unstable (the radiated power behaves as \( P \sim (U_0 - U)^{-\alpha} \) for some – model dependent – value of \( \alpha > 0 \)) but, during the validity of the semi-classical approximation (curvatures should be less than Planck scale), the collapsing cloud emits only about one Planck unit of energy\cite{6}. Because the back-reaction does not become important so long as gravity can be treated classically, the future evolution of the star is governed exclusively by quantum gravity and it is impossible to say, from the semi-classical approximation, whether the star radiates away its energy on a short time scale or settles down into a black hole state. Quantum gravitational effects are therefore expected to modify the very
nature of the classical singularities that result from the collapse of a relativistic star, serving even as the Cosmic Censor. However, the true end state of the collapse can only be fully understood by an application of a quantum theory of gravity.

2 Quantization

The canonical dynamics of the collapsing cloud is described by embedding the spherically symmetric ADM 4-metric in the LTB space-time, and by casting the action for the Einstein-dust system in canonical form. Performing a version of the canonical transformation developed by Kuchar, one may use the chart consisting of \((\tau, R, F, P_\tau, P_R, P_F)\) whence the time evolution of the wave-functional, \(\Psi[\tau, R, F]\), is determined by the functional Schroedinger equation,

\[
\frac{1}{\hbar} \frac{\delta \Psi}{\delta \tau} = \frac{\sqrt{\pm c^2 \frac{\delta^2}{\delta R^2} \pm m_\phi^2 c^4 \frac{F'^2}{4|F|}}}{\Psi[\tau, R, F]},
\]

the upper sign within the square-root referring to the region outside the horizon and the lower sign to the region inside. Invariance under spatial diffeomorphisms is implemented by the momentum constraint,

\[
\left[ \tau' \frac{\delta}{\delta \tau} + R' \frac{\delta}{\delta R_*} + F' \frac{\delta}{\delta F} \right] \Psi[\tau, R, F] = 0
\]

and an inner product on the Hilbert space of wave-functionals may be defined in a natural way, by exploiting the fact that the DeWitt super-metric is manifestly flat in the configuration space \((\tau, R_*)\), as the functional integral

\[
\langle \Psi_1 | \Psi_2 \rangle = \int_{R_*(0)}^{\infty} D R_* \Psi_1^\dagger \Psi_2.
\]

This inner product is defined on \(\tau = \text{constant}\) hypersurfaces and the set-up in equations (1)–(3) implies a specific choice, albeit a natural one, of operator ordering. For a solution to the quantum constraints, we note that momentum constraint is obeyed by any functional that is a spatial scalar. In particular the functional

\[
\Psi = A \exp \left[ -\frac{i c}{2\hbar} \int_0^\infty dr F' [\tau + \mathcal{U}(R, F)] \right],
\]

where \(A = A(F)\), is a spatial scalar if \(\mathcal{U}(R, F)\) has no explicit dependence on \(r\) and describes a stationary state of proper energy \(F'/2\). Throughout we will use this as a solution ansatz.

3 The “eternal” black hole: Mass Spectrum and Entropy

A mass function describing an eternal black hole of mass \(M\) may be taken to be that of a spherical shell \(F(r) = 2M \theta(r)\), where \(\theta(x)\) is the Heaviside func-
tion. Equation (4) then tells us that the problem of a single shell is essentially quantum mechanical,

\[ \tilde{\Psi}[\tau, R, F] = \exp[MW_0(\tau, R, M)], \quad (5) \]

where \( \tau = \tau(0), R = R(0) \) and \( F(0) = 2M \) represent, respectively, the proper time, the radial coordinate and the total mass of the single shell. In the WKB approximation, the stationary states of the black hole are easily described: in the interior they are a superposition of ingoing and outgoing waves,[11]

\[ \tilde{\Psi}[\tau, R^\ast] = A_{\pm}e^{-iM(\tau \pm R^\ast)}, \quad -\pi M < R^\ast < \pi M \quad (6) \]

and in the exterior, they are exponentially decaying

\[ \tilde{\Psi}[\tau, R^\ast] = Be^{-iM(\tau - iR^\ast)}, \quad R^\ast > \pi M \]
\[ = Ce^{-iM(\tau + iR^\ast)}, \quad R^\ast < -\pi M, \quad (7) \]

which is in keeping with the fact that there are no wave solutions in vacuum spherical gravity. Matching the wave function and its derivative at the horizon, one finds that the energy (mass) squared of the black hole is quantized in half integer units,

\[ M^2 = \left(n + \frac{1}{2}\right)M_p^2, \quad \forall \ n \in \mathbb{N} \cup \{0\} \quad (8) \]

where \( M_p \) is the Planck mass. This is just the mass spectrum originally proposed by Bekenstein[12].

The above considerations may be extended to the case of many shells. A simple generalization of the mass function above, which describes many shells, is \( F(r) = 2\sum_{j=1}^{N} \mu_j \theta(r - r_j) \) and a straightforward application of the boundary conditions appropriate to each shell gives the following quantization condition for the states of shell \( k \),[13]

\[ \mu_k M_k = \left(n_k + \frac{1}{2}\right)M_p^2, \quad (9) \]

where \( M_k \) represents the total mass contained within shell \( k \). These conditions, if applied recursively, show that the mass of shell \( k \) is determined by \( k \) quantum numbers. Thus the total mass depends on \( N \) quantum numbers for a black hole formed out of \( N \) quantum shells. The appearance of \( N \) quantum numbers means a quantum black hole is not simply described by its total mass: such a description ignores the manner in which the mass is distributed among the shells. The entropy counts the number of distributions for a given total mass. For an eternal black hole, \( M_k \) in (9) should be replaced by \( M \), the mass of the hole. The total mass (squared) of the hole continues quantized as before and the problem of counting the number of distributions is precisely the problem of asking for the number of ways in which \( N \) integers may be added to give another integer[14]. This result depends on the number of shells that have collapsed to form the black hole, which we do not know but which can be independently
determined by maximizing the entropy with respect to $N$. When both $N$ and $M/M_p$ are large, one readily finds, to leading order,

$$S \approx 0.962 \times \frac{A}{4}$$

in units of Planck area, which agrees well with the Bekenstein-Hawking value.

4 Hawking Radiation

When applied to a matter distribution that is appropriate to a massive black hole surrounded by dust whose total energy is small compared with the mass of the black hole, the WKB approximation describes Hawking radiation. For this purpose, let us assume that the mass function $F(r)$ is of the form

$$F(r) = 2M\theta(r) + f(r),$$

where $\theta(r)$ is the Heaviside step-function, and $f(r)$ is differentiable, representing a dust distribution with $f(r)/2M \ll 1$. It can be interpreted as the presence of a Schwarzschild black hole of mass $M$ at the origin, and $f(r)$ is a dust matter perturbation on the black hole, which can be related to Hawking radiation. In a sense, it plays the role of the quantum field used in standard derivations of Hawking radiation, except that now it is quantized along with the gravitational field.

Inserting this mass function in the wave-functional we have $\Psi = \Psi_{\text{bh}} \times \Psi_f$ where $\Psi_{\text{bh}}$ represents the black hole and

$$\Psi_f \sim \exp \left[ \frac{1}{2} \int_0^\infty dr f' W_f^f (\tau(r), R(r), M) \right].$$

The solution for $W_f^f$ outside the apparent horizon and written in terms of Schwarzschild time is given by

$$W_{\text{out}}^f = -i \left[ T + 4\sqrt{2}M \left( \frac{\sqrt{2}M}{2} \ln \left[ \frac{\sqrt{R} + \sqrt{2}M}{\sqrt{R} - \sqrt{2}M} \right] \right) \right].$$

The projection of our solution on the negative frequency modes of an outgoing basis on $I^+$ represents the negative frequency modes present in the solution. So we compute $|\langle \Psi_{\text{out}}^f | \Psi_{\text{hor}}^f \rangle |^2$ because these are the analogs of the Bogoliubov coefficients near the apparent horizon, $|\beta(f, \omega)|^2$. We find

$$|\beta(f, \omega)|^2 = \prod_r 2\pi M \Delta f \left[ \frac{1}{e^{8\pi M \Delta f} - 1} \right].$$

This is interpreted as the eternal black hole being in equilibrium with a thermal bath at the Hawking temperature $(8\pi M)^{-1}$. Our derivation provides a functional Schrödinger picture for dust Hawking radiation, consistent with the WKB wave-functional which solves the Wheeler-DeWitt equation.
5 Beyond the WKB approximation

Because of Hawking radiation, we do not expect to have a sharp boundary between the collapsing matter and an external vacuum (as is generally considered in the classical studies) owing to the evaporation process. Thus $F(r)$ will increase with increasing $r \in [0, \infty)$ and we could equivalently label shells by $F(r)$ instead of $r$. The Hamiltonian constraint is ill-defined and requires regularization, which we perform on a lattice (for details on the lattice construction, see[15]). If $j$ represent lattice points, the wavefunctional can be expressed as the product

$$\Psi[\tau, R] = \prod_j \Psi_j(\tau_j, R_j) = \prod_j \exp \left[-\omega_j \left[\tau_j + U_j(R_j, F_j)\right]\right].$$ (14)

where $\epsilon_j = \hbar \omega_j$ is the energy of shell $j$ and the Wheeler-DeWitt equation is to be solved independently at each label. Each lattice point physically represents a shell of matter. Thus we get an infinite set of formally identical equations, one for each shell. After carrying out the time derivative, defining the dimensionless variables $z_j = R_j/F_j$ and $\gamma_j = F_j \omega_j/c$, denoting $y_j = \tilde{\Psi}_j$ (the time independent shell wave function, and suppressing the subscript $j$ throughout, the equation for each shell takes the form

$$z(z-1)^2 \frac{d^2 y}{dz^2} + \frac{(z-1)}{2} \frac{dy}{dz} + \gamma^2 z^2 y = 0$$ (15)

both in the exterior ($z > 1$) and in the interior ($z < 1$). Every shell’s evolution is therefore completely determined by the total mass contained within it.

Solutions to [16] can be obtained as a power series about the non-essential singularities at the origin ($z = 0$) and the horizon ($z = 1$). It is possible to show[15] that the Hawking spectrum itself arises from retaining only the lowest order terms and dropping all terms of $O(\hbar)$. When we include the first order correction (in $O(\hbar)$) we find,

$$|\beta(\omega, \omega')|^2 \approx 2\pi^2 F^2 \sqrt{\frac{2c}{F\omega'}} \frac{kT_H}{\epsilon} \frac{1}{e^{\pi kT_H} - 1} \left[1 - \frac{1}{2} \ln \left(\frac{\pi kT_H}{\epsilon}\right)\right],$$ (16)

which completes the first correction to the WKB approximation. The correction term cannot be accounted for by modifying the Hawking temperature and one must conclude that it renders the Hawking radiation non-thermal.

6 Conclusions

There are two points worth noting: (a) equation [16] does not represent a correction to the Hawking temperature (equivalently, the Bekenstein-Hawking entropy). It shows instead that the spectrum is non-thermal, and (b) this non-thermal spectrum is associated with the horizon because we have considered only the near horizon wave-functional in our calculation. It is not, therefore, related
to the non-thermal spectrum observed at large distances from the horizon, e.g., at spatial infinity, due to grey body factors arising from the scattering of the radiation against the geometry.

Within the context and limitations of the canonical theory, our result indicates that unitarity is preserved in quantum gravitational evolution[16]. Many open questions, such as what meaning is to be assigned to the (logarithm of) the number of states, or the statistical entropy, of the black hole in the absence of a thermal spectrum and what is the final fate of black holes and naked singularities, remain to be examined.

7 Acknowledgements

Much of the work reported here was done in collaboration with Louis Witten and T.P. Singh. I am very grateful for our close collaboration and many discussions. I also acknowledge the financial support from the Fundação para a Ciência e a Tecnologia (FCT) under contract number POCTI/32694/FIS/2000 (III Quadro Comunitário de Apoio).

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