STABILIZATION OF THE WAVE EQUATION WITH INTERIOR INPUT DELAY AND MIXED NEUMANN-DIRICHLET BOUNDARY

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Abstract. This paper considers the stabilization of a wave equation with interior input delay: \( \mu_1 u(x, t) + \mu_2 u(x, t - \tau) \), where \( u(x, t) \) is the control input. A new dynamic feedback control law is obtained to stabilize the closed-loop system exponentially for any time delay \( \tau > 0 \) provided that \( |\mu_1| \neq |\mu_2| \). Moreover, some sufficient conditions are given for discriminating the asymptotic stability and instability of the closed-loop system.

1. Introduction. Time delay effects arise in many applications and practical problems, and an arbitrarily small delay may destabilize a system in many cases (see e.g. [4, 5, 15]). How to design proper controllers to stabilize such systems, which puts forward new challenges to the control problems related to the time delay. For a hyperbolic system defined on bounded domain, time delay can be divided into internal time delay and boundary time delay according to the location of the delay. In recent decades, a lot of work is devoted to investigate the control and stabilization problems of boundary delay systems, we refer to [13, 3, 18, 9, 7, 24, 19, 22] and the references therein for examples. But, few pay attention to research those systems with internal delay.

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First discussed the exponential stabilization of the following wave system with internal time delay

\[
\begin{aligned}
\frac{\partial^2 w}{\partial t^2}(x,t) - \frac{\partial^2 w}{\partial x^2}(x,t) + \alpha(x)[\mu_1 w(x,t) + \mu_2 w_t(x,t - \tau)] &= 0, & \text{in } \Omega \times (0,\infty), \\
w(x,t) &= 0, & \text{on } \Gamma_D \times (0,\infty), \\
\frac{\partial w}{\partial v}(x,t) &= 0, & \text{on } \Gamma_N \times (0,\infty), \\
w(x,0) &= w_0(x), & \frac{\partial w}{\partial t}(x,0) = w_1(x), & \text{in } \Omega.
\end{aligned}
\]

The exponential stability of this system was proved by using a suitable observability estimate for \(0 < \mu_2 < \mu_1\). For the opposite situation, a sequence \(\{\tau_k\}\) of delays with \(\tau_k \to 0\) was constructed for which the corresponding solutions \(w_k\) have an increasing energy. [1] discussed above model with \(\alpha(x) = 1\) and \(\mu_1 = 0, \mu_2 > 0\). Under some Lions geometric conditions and the dissipative boundary feedback control \(\frac{\partial w}{\partial x}(x,t) = -kw_1(x,t)\) \((k > 0)\), they obtained a uniform decay estimate for a suitable Lyapunov function based on the multiplier method. \([16]\) considered the model of system (1) by setting \(\tau = \tau(t)\), \(\alpha(x) \equiv 1\). The feedback gains \(\mu_1, \mu_2\) were generalized to \(\mu_2 \in \mathbb{R}, \mu_1 > 0\) and \(|\mu_2| < \sqrt{1-d}\mu_1\), where the time-varying delay \(\tau(t)\) satisfies \(0 < \tau_0 < \tau(t) < \tau_1, \tau'(t) \leq d < 1, \forall t > 0\). The exponential stability estimate also has been obtained under Dirichlet boundary by introducing suitable Lyapunov functions. In addition, for other models, [2] studied a Timoshenko system with internal constant delay feedbacks and established the well-posedness and asymptotic stability results of the system with feedback gains \(\mu_1, \mu_2\) satisfying \(0 \leq \mu_2 < \mu_1\). A natural problem is that can the requirements imposed on \(\mu_1, \mu_2\) be further weakened while the corresponding closed-loop system remains stable? However, to the author’s knowledge, there is no result for general \(\mu_1, \mu_2 \in \mathbb{R}\) about interior delay systems.

In this paper, we intend to discuss the following 1-d wave system with internal time delay and mixed Neumann-Dirichlet boundary

\[
\begin{aligned}
w_{tt}(x,t) - w_{xx}(x,t) + \alpha(x)[\mu_1 u(x,t) + \mu_2 u(x,t - \tau)] &= 0, & x \in (0,1), t > 0, \\
w(0,t) &= 0, & u_x(1,t) = 0, & t > 0, \\
y(t) &= u_t(1,t), & t > 0, \\
w(x,0) &= w_0(x), & w_t(x,0) = w_1(x), & x \in (0,1), \\
u(x,t - \tau) &= f_0(x,t - \tau), & x \in (0,1), & t \in (0,\tau),
\end{aligned}
\]

where \(u(x,t)\) is the control input; \(y(t)\) is the output; \(\alpha(x) \in L^2(0,1), \alpha(x) \geq a > 0, x \in (0,1); \tau > 0\) is the time delay constant; \(\mu_1, \mu_2 \in \mathbb{R}\), \(\mu_2 \neq 0\) are feedback gain constants and the initial data \((w_0, w_1, f_0)\) belongs to a suitable space.

We are interested in giving an exponential stability result for such a problem by seeking for a dynamic feedback control law. This idea follows from [18]. In [18], Shang and Xu proved that a dynamic feedback controller can exponentially stabilize the cantilever Euler-Bernoulli beam system for \(|\mu_2| \neq |\mu_1|, \mu_1, \mu_2 \in \mathbb{R}\). Afterwards, this method was applied to other beam models with boundary control delays (see [10, 11, 23]).
To design a suitable controller for (2), we suppose the state of the system is measurable for the sake of simplicity, and introduce the following auxiliary system

\[
\begin{aligned}
\dot{\bar{w}}_{ss}(x,s,t) - \bar{w}_{xx}(x,s,t) + \alpha(x)\mu_2 u(x, t - \tau + s) &= 0, \\
x \in (0,1), \ s \in (0, \tau), t > 0, \\
\bar{w}(0, s, t) = 0 = \bar{w}_x(1, s, t), \ s \in (0, \tau), t > 0, \\
\bar{w}(x, 0, t) = w(x, t), \ \bar{w}_x(x, 0, t) = \bar{w}_t(x, t), \ x \in (0,1), \ t > 0,
\end{aligned}
\]

which will help us to translate (2) into a system without time delay, so called \((p,q)\)-system. Accordingly, we will get the suitable control signal by \((p,q)\)-system.

We organize the rest as follows. In section 2, we design a suitable control signal for (2) and present the main results of this paper. In section 3, we prove Theorem 2.2. For this end, we show that the system is well-posed by employing semigroup theory at first. Then we discuss the asymptotic stability. The exponential stabilization follows from the exact observability of its coupled system. In section 4, we verify Theorem 2.3 by estimating the error between the solutions to the closed loop system of (2) and (16). Finally, in section 5, we summarize our main idea and give some concluding remarks.

2. Design of the controller and main results. In this section, we will construct the \((p,q)\)-system to obtain the design of the controller. The following lemma and Green functions are needed.

**Lemma 2.1.** (see [21, Proposition 3.5.2]) Define the differential operator in \(L^2[0,1]\) as follows

\[
\mathcal{L}(z) = z''
\]

with domain \(\mathcal{D}(\mathcal{L}) = \{z \in H^2(0,1) : z(0) = z'(1) = 0\}\).

Then \(\mathcal{L}\) is a self-adjoint operator with compact resolvent in \(L^2(0,1)\). The eigenvalues of \(\mathcal{L}\) are \(\lambda_n = -(n\pi - \frac{\pi}{2})^2, \ n = 1,2,\cdots\). The eigenfunctions \(\phi_n = \sqrt{2}\sin \sqrt{|\lambda_n|}\) corresponding to the eigenvalue \(\lambda_n (n = 1,2,\cdots)\) form a normalized orthogonal basis for \(L^2[0,1]\).

Now, define the Green functions \(G_1, G_2\) by

\[
G_1(x, x', t) = \sum_{n=1}^{\infty} \phi_n(x') \phi_n(x) \cos(\sqrt{|\lambda_n|}t),
\]

\[
G_2(x, x', t) = \sum_{n=1}^{\infty} \frac{1}{\sqrt{|\lambda_n|}} \phi_n(x') \phi_n(x) \sin(\sqrt{|\lambda_n|}t).
\]

Obviously,

\[
\frac{\partial G_2}{\partial t}(x, x', t) = G_1(x, x', t),
\]

\[
\frac{\partial G_1}{\partial t}(x, x', t) = \mathcal{L}G_2(x, x', t).
\]
Take
\[
  w(x,t)
\]
\[
  = \int_0^1 G_1(x,x',t)w_0(x')dx' + \int_0^1 G_2(x,x',t)w_1(x')dx' \\
  - \int_0^t \int_0^1 G_2(x,x',t-t')\alpha(x')[\mu_1u(x',t') + \mu_2u(x',t'-\tau)]dx'dt',
\]
\(\bar{w}(x,s,t)\)
\[
  = \int_0^1 G_1(x,x',s)w(x',t)dx' + \int_0^1 G_2(x,x',s)w_t(x',t)dx' \\
  - \int_s^\alpha \int_0^1 G_2(x,x',s-s')\alpha(x')\mu_2u(x',t+s'-\tau)dx'ds'.
\]
Then \((w(x,t), w_t(x,t)), (\bar{w}(x,s,t), \bar{w}_s(x,s,t))\) solve (2) and (3), respectively.

Denote by \(p(x,t) := \bar{w}(x,\tau,t), q(x,t) := \bar{w}_s(x,\tau,t)\). A direct but maybe a little complicated calculation (see Appendix) shows that \(p(x,t), q(x,t)\) satisfy the following relations
\[
  \begin{cases}
  p_t(x,t) = q(x,t) - \int_0^1 G_2(x,x',\tau)\alpha(x')\mu_1u(x',t)dx', \ x \in (0,1), \ t > 0, \\
  q_t(x,t) = p_{xx}(x,t) - \int_0^1 \alpha(x')u(x',t)[\mu_1G_1(x,x',\tau) + \mu_2G_1(x,x',0)]dx', \ x \in (0,1), \ t > 0, \\
  p(0,t) = p_x(1,t) = 0, \ t > 0, \\
  p(x,0) = \int_0^1 G_1(x,x',\tau)w_0(x')dx' + \int_0^1 G_2(x,x',\tau)w_1(x')dx' \\
  - \int_0^\tau \int_0^1 G_2(x,x',\tau-s')\alpha(x')\mu_2f_0(x',s'-\tau)dx'ds', \ x \in (0,1), \\
  q(x,0) = \int_0^1 G_{2x}(x,x',\tau)w_0(x')dx' + \int_0^1 G_1(x,x',\tau)w_1(x')dx' \\
  - \int_0^\tau \int_0^1 G_1(x,x',\tau-s)\alpha(x')\mu_2f_0(x',s-\tau)dx'ds, \ x \in (0,1).
\end{cases}
\]
Let \(E(t)\) be the energy function of (12), defined by
\[
  E(t) := \frac{1}{2} \int_0^1 \left[ |p_x(x,t)|^2 + |q(x,t)|^2 \right] dx.
\]
We calculate
\[
  \frac{dE(t)}{dt} = \int_0^1 [p_x(x,t)p_{xx}(x,t) + q(x,t)q_t(x,t)] dx \\
  = -\int_0^1 p_t(x,t)p_{xx}(x,t)dx + \int_0^1 q(x,t)q_t(x,t)dx \\
  = -\int_0^1 u(x',t)\alpha(x')dx' \int_0^1 p_x(x,t)G_{2x}(x,x',\tau)\mu_1 dx \\
  - \int_0^1 u(x',t)\alpha(x')dx' \int_0^1 q(x,t)[G_1(x,x',\tau)\mu_1 + G_1(x,x',0)\mu_2] dx.
\]
If we set the control function 
\[ u(x', t) \]
\[ = \alpha(x') \left[ \int_0^1 p_x(x, t) G_2(x, x', \tau) \mu_1 + q(x, t) [G_1(x, x', \tau) \mu_1 + G_1(x, x', 0) \mu_2] dx \right] \]
\[ =: U(p, q)(x', t), \tag{15} \]
then
\[ \frac{dE(t)}{dt} = - \int_0^1 |U(p, q)(x', t)|^2 dx' \leq 0. \]

Now, with \( u(x', t) \) defined by \((15)\), the closed-loop system associated with \((12)\) can be expressed as
\[
\begin{cases}
  p_t(x, t) = q(x, t) - \int_0^1 G_2(x, x', \tau) \alpha(x') \mu_1 U(p, q)(x', t) dx', \\
  q_t(x, t) = p_{xx}(x, t) - \int_0^1 \alpha(x') U(p, q)(x', t) \\
  p(0, t) = p_x(1, t) = 0, t > 0, \\
  p(x, 0) = \int_0^1 G_1(x, x', \tau) w_0(x') dx' + \int_0^1 G_2(x, x', \tau) w_1(x') dx' \quad \tag{16} \\
  - \int_0^T \int_0^1 G_2(x, x', \tau - s') \alpha(x') \mu_2 f_0(x', s' - \tau) dx' ds', \ x \in (0, 1), \\
  q(x, 0) = \int_0^1 G_{2xx}(x, x', \tau) w_0(x') dx' + \int_0^1 G_1(x, x', \tau) w_1(x') dx' \\
  - \int_0^T \int_0^1 G_1(x, x', \tau - s) \alpha(x') \mu_2 f_0(x', s - \tau) dx' ds, \ x \in (0, 1). 
\end{cases}
\]

So far, we can describe our main results as follows.

**Theorem 2.2.** (1) System \((16)\) is exponentially stable if \(|\mu_1| \neq |\mu_2|\).

(2) System \((16)\) is asymptotically stable if \(\tau\) is an irrational number, or if \(\tau = \frac{2^{k_0} \rho}{\varrho}\) where \(k_0\) is some constant, \(\rho\) is an odd number such that \(2^{k_0} \rho\), \(\varrho\) are relatively prime numbers, and one of the following two conditions holds (i) \(k_0 \geq 2\) and \(\mu_1 = \mu_2\); (ii) \(k_0 = 0\).

(3) System \((16)\) is unstable if \(\tau = \frac{2^{k_0} \rho}{\varrho}\) where \(k_0\) is some constant, \(\rho\) is an odd number such that \(2^{k_0} \rho\), \(\varrho\) are relatively prime numbers, and one of the following two conditions holds (i) \(k_0 \geq 2\) and \(\mu_1 = -\mu_2\); (ii) \(k_0 = 1\) and \(|\mu_1| = |\mu_2|\).

Moreover, let \((w(x, t), w_t(x, t))\) be the solution to system \((2)\) with \((x', t)\) defined by \((15)\). Let \((p(x, t), q(x, t))\) be the solution to system \((16)\). We compare the two systems by estimating the error between \((w(x, t), w_t(x, t))\) and \((p(x, t), q(x, t))\)
\[ \|p(\cdot, t) - w(\cdot, t + \tau)\|_{L^2(0, 1)} + \|q(\cdot, t) - w_t(\cdot, t + \tau)\|_{L^2(0, 1)}, \]
and give the following result.
Theorem 2.3. With the control input \( u(x, t) \) defined by (15), system (2) and system (12) have the same stability. That is, if system (12) is exponentially stable (asymptotically stable), the same is true for system (2).

3. The proof of Theorem 2.2. To prove Theorem 2.2, we put system (16) into a suitable Hilbert space and show the well-posedness of it at first.

3.1. The well-posedness of system (16). Let \( H^k(0, 1)(k = 1, 2) \) be the usual Sobolev space and \( L^2(0, 1) \) be the usual Hilbert space.

Set \( H^1_E(0, 1) := \{ g \in H^1(0, 1) : g(0) = 0 \}, \quad X := H^1_E(0, 1) \times L^2(0, 1) \). Define inner product in \( X \) by
\[
\langle (g, h), (w, v) \rangle := \int_0^1 [g'(x)p'(x) + h(x)p(x)]dx,
\]
\( \forall (g, h), (w, v) \in X \).

Define the operator \( A \) in \( X \) by
\[
A\left( \begin{array}{c} g \\ h \end{array} \right) = \left( \begin{array}{c} h - \int_0^1 G_2(\cdot, x', \tau)\alpha(x')\mu_1U(g, h)(x')dx' \\ g'' - \int_0^1 \alpha(x')U(g, h)(x')[G_1(\cdot, x', \tau)\mu_1 + G_1(\cdot, x', 0)\mu_2]dx' \end{array} \right),
\]
with domain
\[
D(A) = \left\{ (g, h) \in X \cap [H^2(0, 1) \times H^1_E(0, 1)] : g'(1) = 0; U(g, h)(x) = \alpha(x) \left[ \int_0^1 g'(x')G_2\alpha(x', x, \tau)\mu_1 \\
+ h(x')[G_1(x', x, \tau)\mu_1 + G_1(x', x, 0)\mu_2]dx' \right] \right\}.
\]

Then we rewrite system (16) as an evolutionary equation in \( X \)
\[
\begin{align*}
\frac{d\tilde{Y}(t)}{dt} &= A\tilde{Y}(t), \quad t > 0, \\
\tilde{Y}(0) &= \tilde{Y}_0,
\end{align*}
\]
where \( \tilde{Y}(t) = (p(\cdot, t), q(\cdot, t))^T, \quad \tilde{Y}(0) = (p(\cdot, 0), q(\cdot, 0))^T \in X \).

The well-posedness of system (19) can be achieved by semigroup theory.

Theorem 3.1. The operator \( A \) defined by (17)–(18) generates a \( C_0 \) semigroup of contractions on \( X \). It follows that system (19) is well-posed.

Proof. To begin with, we point out that \( A \) is a closed and densely definite linear operator in \( X \). The proof is a routine work, we omit it here.

Next, we will show that both the operators \( A \) and \( A^* \) are dissipative, where \( A^* \) is the adjoint operator of \( A \) defined by
\[
A^*\left( \begin{array}{c} w \\ v \end{array} \right) = \left( \begin{array}{c} v + \int_0^1 G_2(\cdot, x', \tau)\alpha(x')\mu_1U(w, v)(x')dx' \\ w'' + \int_0^1 \alpha(x')[G_1(\cdot, x', \tau)\mu_1 + G_1(\cdot, x', 0)\mu_2]U(w, v)(x')dx' \end{array} \right)
\]
(20)
with domain
\[
D(A^*) = \left\{ (w, v) \in \mathcal{X} \cap [H^2(0, 1) \times H^1_E(0, 1)] : w'(1) = 0, \right. \\
\left. U(w, v)(x) = \alpha(x) \int_0^1 [w'(x')G_{2x}(x, x', \tau)\mu_1 \right. \\
\left. + (G_1(x, x', \tau)\mu_1 + G_1(x, x', 0)\mu_2)v(x')]dx' \right\}.
\]

(21)

Now, for any \((g, h) \in D(A)\), we have
\[
\Re < A(g, h), (g, h) > \mathcal{X} = \Re < h - \int_0^1 G_2(\cdot, x', \tau)\alpha(x')\mu_1U(g, h)(x')dx', \\
g'' - \int_0^1 \alpha(x')[G_1(\cdot, x', \tau)\mu_1 + G_1(\cdot, x', 0)\mu_2]U(g, h)(x')dx', \Re < g, h > \mathcal{X}.
\]
\[
= -\Re \int_0^1 \int_0^1 G_{2x}(x, x', \tau)\alpha(x')\mu_1U(g, h)(x')dx'dx \\
- \Re \int_0^1 \int_0^1 \alpha(x')[G_1(x, x', \tau)\mu_1 + G_1(x, x', 0)\mu_2]U(g, h)(x')dx'dx \\
= -\Re \int_0^1 \alpha(x')U(g, h)(x')dx'\int_0^1 [G_1(x, x', \tau)\mu_1 + G_1(x, x', 0)\mu_2]U(g, h)(x')dx \\
- \Re \int_0^1 \alpha(x')U(g, h)(x')dx'\int_0^1 \mu_1G_{2x}(x, x', \tau)U(g, h)(x')dx \\
= -\int_0^1 |U(g, h)(x')|^2dx' \leq 0.
\]

(22)

Similarly, we establish that
\[
\Re < A^*(w, v), (w, v) > \mathcal{X} = -\int_0^1 |U(w, v)(x')|^2dx' \leq 0.
\]

(23)

So, the Lumer-Phillips theorem (see [17]) guarantees that \(A\) generates a \(C_0\) semigroup of contractions on \(\mathcal{X}\). As a consequence, system (19) is well-posed in \(\mathcal{X}\) ([17]).

The following description about the operator \(A\)’s spectrum will be needed in the proof of Theorem 2.2.

**Theorem 3.2.** Let \(A\) be defined by (17)-(18). Then \(0 \in \rho(A)\). Moreover, \(A^{-1}\) is compact so that \(\sigma(A)\), the spectrum of \(A\), consists of all isolated eigenvalues of finite algebraic multiplicity only.

**Proof.** Let \((g, h) \in D(A)\) such that \(A(g, h) = (0, 0)\). It is easy to find that \((g, h) \equiv (0, 0)\) from (17)-(18) and (10)-(11), which implies that \(A^{-1}\) exists. Now, we seek the solution \((g, h) (\in D(A))\) of equation \(A(g, h) = (w, v)\) for any \((w, v) \in \mathcal{X}\). Concretely,
and (iii) in Theorem 2.2.

Theorem 2.2.

3.2. The proof of Theorem 2.2. In this section, we divide Theorem 2.2 into two theorems and prove them, respectively. The following theorem is the results (ii) and (iii) in Theorem 2.2.
Theorem 3.3. (1) System (16) is asymptotically stable if \( \tau \) is an irrational number, or if \( \tau = \frac{2k_0}{q} \) where \( k_0 \) is some constant, \( q \) is an odd number such that \( 2k_0, q \) are relatively prime numbers, and one of the following two conditions holds: (i) \( k_0 \geq 2 \) and \( \mu_1 = \mu_2 \); (ii) \( k_0 = 0 \).

(2) System (16) is unstable if \( \tau = \frac{2k_0}{q} \) where \( k_0 \) is some constant, \( q \) is an odd number such that \( 2k_0, q \) are relatively prime numbers, and one of the following two conditions holds: (i) \( k_0 \geq 2 \) and \( \mu_1 = -\mu_2 \); (ii) \( k_0 = 1 \) and \( |\mu_1| = |\mu_2| \).

Proof. For (1), we will show that the spectrum of \( \mathcal{A} \) does not intersect the imaginary axis. Since \( \sigma(\mathcal{A}) = \sigma_p(\mathcal{A}) \) in view of Theorem 3.2, we come to consider the eigenvalue problem of \( \mathcal{A} \) for \( \lambda = ir, r \in \mathbb{R} \).

Let \((g, h) \in D(\mathcal{A})\) satisfy \( \mathcal{A}(g, h) = \lambda(g, h) \) for \( \lambda = ir, r \in \mathbb{R} \). Then

\[
0 = \mathbb{R} < \mathcal{A}(g, h), (g, h) > = \int_0^1 |U(g, h)(x')|^2 dx' \leq 0, \tag{33}
\]

which means \( U(g, h)(x) \equiv 0, \forall x \in (0, 1) \). Besides that, equation \( \mathcal{A}(g, h) = \lambda(g, h) \) gives

\[
\begin{align*}
  h(x) &= irg(x), \\
  g''(x) &= irh(x), \\
  g(0) &= g'(1) = 0, \\
  0 &= U(g, h)(x) = \alpha(x) \int_0^1 \{g'(x')G_{2x'}(x', x, \tau)\mu_1 \\
  & \quad + h(x')[G_1(x', x, \tau)\mu_1 + G_1(x', x, 0)\mu_2] \} dx'. \tag{34}
\end{align*}
\]

So (34) has nonzero solutions \((g_n, h_n)\) if and only if \( r = n\pi - \frac{\pi}{2} = \sqrt{-\lambda_n}, \; n = 1, 2, \ldots, \) and \( U(g_n, h_n) \equiv 0 \), where \( \lambda_n(n = 1, 2, \ldots) \) are the eigenvalues of \( \mathcal{L} \) (see Lemma 2.1). In this case, \( g_n(x) = c\sin((n\pi - \frac{\pi}{2})x) = c\phi_n, c \) is an arbitrary constant here. We thereby obtain

\[
0 = U(g_n, h_n)(x) \\
= \alpha(x) \left[ \int_0^1 \phi'_n(x')G_{2x'}(x', x, \tau)\mu_1 \\
  + i\sqrt{-\lambda_n}\phi_n(x')[G_1(x', x, \tau)\mu_1 + G_1(x', x, 0)\mu_2] \right] dx'. \tag{35}
\]

Obviously, \( \sin(-\sqrt{-\lambda_n}\tau) \neq 0 \) if \( \tau \) is an irrational number, or \( \tau = \frac{2k_0}{q} \) and \( k_0 = 0 \). If \( \tau = \frac{2k_0}{q}, k_0 \geq 2 \) and \( \mu_1 = \mu_2 \), we see

\[
\mu_1 e^{-i\sqrt{-\lambda_n}\tau} + \mu_2 = \left[ \cos \frac{2k_0-1}{q}(2n-1)\pi + 1 - i\sin \frac{2k_0-1}{q}(2n-1)\pi \right] \mu_1 \neq 0 \tag{36}
\]
since \( \cos \frac{2k_0 - 1}{2} \rho (2n-1) \pi = -1 \), \( \sin \frac{2k_0 - 1}{2} \rho (2n-1) \pi = 0 \) cannot be simultaneously true for any \( n = 1, 2, \cdots \). So it must be \( c = 0 \) by (35), which implies that only \((0, 0)\) solves (35). Thus, there is no eigenvalue of \( A \) on the imaginary axis. System (16) is asymptotically stable (12).

(2) Set \( \tau = \frac{2k_0}{n} \). If \( k_0 \geq 2 \) and \( \mu_1 = -\mu_2 \), or \( k_0 = 1 \) and \( \mu_1 = -\mu_2 \), we take \( n = \frac{2k_0 + 1}{2}, k = 1, 2, \cdots \). Then

\[
\mu_1 e^{-i\sqrt{-\lambda_n} \tau} + \mu_2 = \left[ \cos \frac{2k_0 - 1}{2} \rho (2n-1) \pi \rho \right] - 1 - i \sin \frac{2k_0 - 1}{2} \rho (2n-1) \pi \mu_1 \\
= \left[ \cos(\rho k_0 \pi) - 1 - i \sin(\rho k_0 \pi) \right] \mu_1 \\
= 0.
\]

(37)

If \( k_0 = 1 \) and \( \mu_1 = \mu_2 \), we take \( n = \frac{3k_0 + 1}{2}, k = 1, 2, \cdots \). Then

\[
\mu_1 e^{-i\sqrt{-\lambda_n} \tau} + \mu_2 = \left[ \cos(\rho \pi) + 1 + i \sin(\rho \pi) \right] \mu_1 = 0
\]

(38)
as \( \rho \) is an odd number. In view of (37)-(38), we have \( U(g_n, h_n)(x) = 0 \) with

\[
g_n(x) = \frac{\phi_n(x)}{\sqrt{\lambda_n}}, \quad h_n(x) = i\phi_n(x).
\]

So that \((g_n, h_n) \in D(A)\) solves \( A(g_n, h_n) = i(n\pi - \frac{\pi}{2})(g_n, h_n)\), which means that the spectrum of the operator \( A \) contains a countable set of eigenvalues \( i(n\pi - \frac{\pi}{2}) \).

Thus, system (16) is unstable. \( \square \)

In what follows, we will prove the exponential stabilization of system (16) or (19) by applying the following lemmas.

**Lemma 3.4.** (See [20, Corollary 3.1] and [8]) Let \( H \) be a Hilbert space. Consider the linear control system

\[
\dot{x}(t) = Ax(t) + Bu(t).
\]

(39)

If the following conditions are fulfilled:

(i) \( A \) is a skew-adjoint operator with compact resolvent, and hence \( A \) generates a \( C_0 \) group \( T(t) \),

(ii) the spectrum of \( A \), \( \sigma(A) = \{ \mu_n, n \in \mathbb{N} \} \), satisfies the spectral gap condition, i.e.

\[
\inf_{n \neq k} |\mu_n - \mu_k| > 0,
\]

(iii) \( B \) is a bounded linear operator, and satisfies that there exists a positive constant \( \delta \) such that for any eigenfunction \( \varphi \) with \( \|\varphi\| = 1 \), \( \|B^* \varphi\| > \delta \),

then system (39) is exactly controllable on some \([0, T]\), \( T > 0 \), and can be stabilized exponentially by \( u(t) = -B^* x(t) \).

By duality principle, we only need to show that the observation system corresponding to (39) is exactly observable on some \([0, T]\), \( T > 0 \). The following result supplies us a simple but practical method for verifying the exact observability of an observation system.

**Lemma 3.5.** ([18]) Let \( H \) be a Hilbert space and \( A_0 \) be a skew-adjoint operator with compact resolvent in \( H \), i.e. \( A_0^* = -A_0 \). Let \( \sigma(A_0) = \{ \lambda_k : k = \pm 1, \pm 2, \cdots \} \) and \( \{ \phi_k \} \) be the corresponding eigenvectors. Let \( Y \) be another Hilbert space and
\( C \in L(D(A_0), Y) \). If \( A_0 \) and \( C \) satisfy the following conditions:

1) Spectral gap condition:
\[
\inf_{k \neq m} |\lambda_k - \lambda_m| = \delta > 0,
\]
(40)

2) Bounded condition:
\[
0 < m = \inf_k \|C\phi_k\|_Y < \sup_k \|C\phi_k\|_Y = M < \infty,
\]
(41)

then \( C \) is an admissibly observable operator for \( A_0 \), and \((A_0, C)\) is exactly observable in finite time.

Now, according to Lemma 3.4 and Lemma 3.5, we give the proof of the first result of Theorem 2.2.

**Theorem 3.6.** If \( |\mu_1| \neq |\mu_2| \), system (16) is exponentially stable.

**Proof.** In order to apply Lemma 3.5, we consider the observation system corresponding to (16):

\[
\begin{cases}
g_t(x, t) = h(x, t), \ x \in (0, 1), \ t > 0, \\
h_t(x, t) = g_{xx}(x, t), \ x \in (0, 1), \ t > 0, \\
g(0, t) = g_x(1, t) = 0, \ t > 0, \\
g(x, 0) = g_0(x), \ h(x, 0) = h_0(x), \ x \in (0, 1), \\
y(x, t) = U(g, h)(x, t) = \alpha(x) \int_0^1 g_{x'}(x', t)G_2(x', x, \tau)\mu_1 dx' \\
+ \alpha(x) \int_0^1 h(x', t)[G_1(x', x, \tau)\mu_1 + G_1(x', x, 0)\mu_2]dx'.
\end{cases}
\]
(42)

Then \( A_0 \) is a skew-adjoint operator with compact resolvent. (42) can be rewritten as

\[
\begin{cases}
dt(g(x, t), h(x, t)) = A_0(g(x, t), h(x, t)), \ t > 0, \\
(g(x, 0), h(x, 0)) = (g_0(x), h_0(x)), \\
y(x, t) = U(g, h)(x, t).
\end{cases}
\]
(45)

A direct calculation shows that the eigenvalues of \( A_0 \) are

\[
\gamma_k = \begin{cases} 
 i\sqrt{-\lambda_k}, & k = 1, 2, \cdots, \\
 -i\sqrt{-\lambda_k}, & k = -1, -2, \cdots,
\end{cases}
\]
(46)
where \( \lambda_k = -(k\pi - \frac{x}{2})^2 \). The corresponding eigenvectors

\[
\Phi_k(x) := (g_k, h_k)(x) = \begin{cases} \left( \frac{-i}{\sqrt{-2\lambda_k}} \phi_k(x), \frac{1}{\sqrt{2}} \phi_k(x) \right), & k = 1, 2, \ldots, \\
\left( \frac{i}{\sqrt{-2\lambda_k}} \phi_{-k}(x), \frac{1}{\sqrt{2}} \phi_{-k}(x) \right), & k = -1, -2, \ldots. 
\end{cases} \tag{47}
\]

Clearly, \( \inf_{k \neq m} |\gamma_k - \gamma_m| = \pi > 0 \), and \( \{(g_k, h_k)\}_{k=-\infty}^{+\infty} \) forms a normalized orthogonal basis for \( X \).

To check out the bounded condition of Lemma 3.5, we see that \( C = U \in D(A_0, L^2(0,1)) \) and

\[
C\Phi_k(x) = U(g_k, h_k)(x) = \alpha(x) \int_0^1 h_k(x')|G_1(x', x, \tau)| \mu_1 + G_4(x', x, 0) \mu_2 \, dx' + \alpha(x) \int_0^1 g'_k(x')G_2(x', x, \tau) \mu_1 \, dx' \\
= \frac{\alpha(x)}{\sqrt{2}} \phi_k(x) \left[ \mu_1[\cos(\sqrt{-\lambda_k} \tau) - i \sin(\sqrt{-\lambda_k} \tau)] + \mu_2 \right] \\
= \frac{\alpha(x)}{\sqrt{2}} \phi_k(x) \left[ e^{-i\sqrt{-\lambda_k} \tau} \mu_1 + \mu_2 \right] \tag{48}
\]

for \( k = 1, 2, \ldots, \) and

\[
C\Phi_k(x) = \frac{\alpha(x)}{\sqrt{2}} \phi_k(x) \left[ e^{i\sqrt{-\lambda_k} \tau} \mu_1 + \mu_2 \right] \tag{49}
\]

for \( k = -1, -2, \ldots. \)

Denote by \( M = \sup_{x \in (0,1)} \{\alpha(x)\}, \ m = \inf_{x \in (0,1)} \{\alpha(x)\} \) and notice that

\[
||\mu_1| - |\mu_2|| \leq |e^{i\sqrt{-\lambda_k} \tau} \mu_1 + \mu_2| \leq |\mu_1| + |\mu_2|, \tag{50}
\]

\[
||C\Phi_k||_Y^2 = ||U(g_k, h_k)||_{L^2(0,1)}^2 = \int_0^1 \left| \frac{\alpha(x)}{\sqrt{2}} \phi_k(x)[e^{i\sqrt{-\lambda_k} \tau} \mu_1 + \mu_2] \right|^2 \, dx. \tag{51}
\]

We get

\[
m \frac{1}{\sqrt{2}} (|\mu_1| - |\mu_2|) \leq ||C\Phi_k||_Y \leq M \frac{1}{\sqrt{2}} (|\mu_1| + |\mu_2|). \tag{52}
\]

Hence, system (42) is exactly observable in finite time by Lemma 3.5.

Finally, notice that \( A = A_0 - U^*U, \ U \) defined in (18) is a bounded linear operator from \( X \) to \( L^2(0,1) \). The principle of duality and Lemma 3.4 indicate that system (16) is exponentially stable. \( \square \)

4. The proof of Theorem 2.3. Let \( H^1_k(0,1) \) be the Hilbert space defined in section 3.1. We introduce the following inner product:

\[
< f, g >_{H^1_k(0,1)} = \int_0^1 \left( \frac{df(x)}{dx} \frac{dg(x)}{dx} \right) \, dx, \forall f, g \in H^1_k(0,1).
\]
Proof of Theorem 2.2. Let \((p, q)\) be the solution to system (16), \((w, w_t)\) be the solution to system (2) with \(u(x, t)\) defined by (15). We calculate the following error
\[
\|p(\cdot, t) - w(\cdot, t + \tau)\|_{H^1_x(0,1)}^2 + \|q(\cdot, t) - w_t(\cdot, t + \tau)\|_{L^2_x(0,1)}^2
= \int_0^1 |p_x(x, t) - w_x(x, t + \tau)|^2 \, dx + \int_0^1 |q(x, t) - w_t(x, t + \tau)|^2 \, dx. \tag{53}
\]
In fact, we can employ (6) and the appendix to find
\[
p(x, t) - w(x, t + \tau)
= \int_0^t G_1(x, x'', t + \tau)w_0(x'') \, dx'' + \int_0^t G_2(x, x'', t + \tau)w_1(x'') \, dx''
- \int_0^t \int_0^1 G_2(x, x'', t + \tau - t')\alpha(x'')[\mu_1 u(x'', t') + \mu_2 u(x'', t' - \tau)] \, dx'' \, dt'
- \int_0^t \int_{t-\tau}^1 G_2(x, x', t - t')\alpha(x'[\mu_2 u(x', t') + \mu_2 u(x', t' - \tau)] \, dx' \, dt'
- \int_0^1 G_1(x, x', t + \tau)w_0(x') \, dx' - \int_0^1 G_2(x, x', t + \tau)w_1(x') \, dx'
+ \int_{t-\tau}^t \int_{t-\tau}^1 G_2(x, x', t + \tau - t')\alpha(x'[\mu_1 u(x', t') + \mu_2 u(x', t' - \tau)] \, dx' \, dt'
= \int_{t-\tau}^t \int_{t-\tau}^1 G_2(x, x', t - t')\alpha(x'[\mu_1 u(x', t' + \tau)] \, dx' \, dt'.
\]
Consequently,
\[
p_x(x, t) - w_x(x, t + \tau) = \int_{t-\tau}^t \int_{t-\tau}^1 G_2(x, x', t - t')\alpha(x'[\mu_1 u(x', t' + \tau)] \, dx' \, dt'. \tag{54}
\]
So
\[
\|p(\cdot, t) - w(\cdot, t + \tau)\|_{H^1_x(0,1)}^2 + \|q(\cdot, t) - w_t(\cdot, t + \tau)\|_{L^2_x(0,1)}^2
= \int_0^1 \left| \int_{t-\tau}^t \int_{t-\tau}^1 G_2(x, x', t - t')\alpha(x'[\mu_1 u(x', t' + \tau)] \, dx' \, dt' \right|^2 \, dx
+ \int_0^1 \left| \int_{t-\tau}^t \int_{t-\tau}^1 G_1(x, x', t - t')\alpha(x'[\mu_1 u(x', t' + \tau)] \, dx' \, dt' \right|^2 \, dx
= \sum_{n=1}^{\infty} \left| \int_0^1 \phi_n(x') \int_0^\tau \sin(\sqrt{|\lambda_n|}(\tau - t'))\alpha(x'[\mu_1 u(x', t' + \tau) \, dx' \, dt' \right|^2
+ \sum_{n=1}^{\infty} \left| \int_0^1 \phi_n(x') \int_0^\tau \cos(\sqrt{|\lambda_n|}(\tau - t'))\alpha(x'[\mu_1 u(x', t' + \tau) \, dx' \, dt' \right|^2. \tag{55}
\]
In view of that \(\{\phi_n(x) \sin(\sqrt{|\lambda_n|}t), \phi_n(x) \cos(\sqrt{|\lambda_n|}t)\}_{n=1}^{\infty} \subseteq L^2([0, 1] \times (0, \tau])\) is a Bessel sequence, there exists a constant \(B > 0\) such that
\[
\|p(\cdot, t) - w(\cdot, t + \tau)\|_{H^1_x(0,1)}^2 + \|q(\cdot, t) - w_t(\cdot, t + \tau)\|_{L^2_x(0,1)}^2
\]
\[
\begin{align*}
&\leq B \int_0^\tau \int_0^1 |\alpha(x')\mu_1 u(x',t'+t)|^2 dx'dt' \\
&\leq \mu_1^2 \sup_{x \in (0,1)} \{\alpha(x)\}|^2 B \int_0^\tau \int_0^1 |u(x',t'+t)|^2 dx'dt' \\
&= -\mu_1^2 M^2 B \int_0^\tau \frac{dE(t'+t)}{dt'} dt' = \mu_1^2 M^2 B[E(t) - E(t + \tau)],
\end{align*}
\]

where \(M = \sup_{x \in (0,1)} \{\alpha(x)\}, E(t) \) is defined by (13).

Now, if (16) is exponentially stable, there exist constants \(\tilde{M} > 0, \omega > 0\) such that \(E(t) \leq \tilde{M}e^{-\omega t}\). So
\[
\|p(\cdot,t) - w(\cdot,t+\tau)\|_{H^1_1(0,1)}^2 + \|q(\cdot,t) - w_1(\cdot,t+\tau)\|_{L^2(0,1)}^2 \\
\leq B\mu_1^2 M^2 [E(t) - E(t + \tau)] \\
\leq 2B\mu_1^2 M^2 \tilde{M}e^{-\omega t}.
\]

That is, the closed-loop system of system (2) is also exponentially stable.

If \(\lim_{t \to +\infty} E(t) = 0\), we also have
\[
\lim_{t \to +\infty} [\|p(\cdot,t) - w(\cdot,t+\tau)\|_{H^1_1(0,1)}^2 + \|q(\cdot,t) - w_1(\cdot,t+\tau)\|_{L^2(0,1)}^2] = 0
\]
by (56). The proof is then completed. \(\Box\)

5. **Conclusion.** In this paper, we construct a partial state predictor to obtain a feedback controller, which stabilizes the corresponding closed-loop wave system with interior input delay and mixed Neumann-Dirichlet boundary provided that the time delay \(\tau\) and the feedback gain constants \(\mu_1, \mu_2\) satisfy some conditions. These conditions simplify the judgement of the stabilization of the system compared with those in [18, 23]. In fact, we obtain that whenever the time delay \(\tau\) is an irrational number, the closed-loop system is asymptotically stable, and if there exists a spectrum point on the imaginary axis, it must be have countable spectrum points on it. As a result, the closed-loop system is unstable. As far as concerned the exponential stability of the system, only the condition that \(|\mu_1| \neq |\mu_2|\) is needed. Furthermore, we point out that the traditional Multiplier Method fails to prove the exponential stability of the closed-loop system (16), and Lemma 3.5 provides us a effective way, which also can be used to verify the exponential stability of other one-dimensional elastic systems except for the multi-dimensional ones because of the “spectral gap condition” used in this lemma.

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**Appendix A. Representation of system (12).** In this appendix, we will construct the \((p, q)\) system. In fact, from (10)–(11), we have
\[
\begin{align*}
p(x, t) &= \int_0^1 G_1(x, x', \tau) w(x', t) dx' + \int_0^1 G_2(x, x', \tau) w_1(x', t) dx' \\
&\quad - \int_0^\tau \int_0^1 G_2(x, x', \tau - s') \alpha(x') \mu_2 u(x', t + s' - \tau) dx' ds'
\end{align*}
\]
\[
\begin{align*}
&= \int_0^1 G_1(x, x', \tau)w(x', t)dx' + \int_0^1 G_2(x, x', \tau)w_i(x', t)dx' \\
&\quad - \int_{t-\tau}^t \int_0^1 G_2(x, x', t-t')\alpha(x')\mu_2 u(x', t')dx'dt' \\
&= \int_0^1 G_1(x, x'', t+\tau)w_0(x'')dx'' + \int_0^1 G_2(x, x'', t+\tau)w_1(x'')dx'' \\
&\quad - \int_0^t \int_0^1 G_1(x, x'', t+\tau-t')\alpha(x'')[\mu_1 u(x'', t') + \mu_2 u(x'', t'-\tau)]dx''dt' \\
&\quad - \int_{t-\tau}^t \int_0^1 G_1(x, x', t-t')\alpha(x')\mu_2 u(x', t')dx'dt',
\end{align*}
\]

and

\[
q(x, t) = \int_0^1 \Sigma G_2(x, x'', t+\tau)w_0(x'')dx'' + \int_0^1 G_1(x, x'', t+\tau)w_1(x'')dx'' \\
- \int_0^t \int_0^1 G_1(x, x'', t+\tau-t')\alpha(x'')[\mu_1 u(x'', t') + \mu_2 u(x'', t'-\tau)]dx''dt' \\
- \int_{t-\tau}^t \int_0^1 G_1(x, x', t-t')\alpha(x')\mu_2 u(x', t')dx'dt'.
\]

According to (58) and (59), we calculate

\[
\begin{align*}
p_i(x, t) &= \int_0^1 G_1(x, x', \tau)w_i(x', t)dx' + \int_0^1 G_2(x, x', \tau)w_i(x', t)dx' \\
&\quad + \int_0^1 G_2(x, x', \tau)\alpha(x')\mu_2 u(x', t-\tau)dx' \\
&\quad - \int_{t-\tau}^t \int_0^1 G_1(x, x', t-t')\alpha(x')\mu_2 u(x', t')dx'dt' \\
&= \int_0^1 G_1(x, x', \tau)w_i(x', t)dx' + \Sigma \int_0^1 G_2(x, x', \tau)w(x', t)dx' \\
&\quad - \int_0^1 G_2(x, x', \tau) \int_0^1 G_1(x, x'', 0)\alpha(x'')[\mu_1 u(x'', t) + \mu_2 u(x'', t-\tau)]dx''dx' \\
&\quad + \int_0^1 G_2(x, x', \tau)\alpha(x')\mu_2 u(x', t-\tau)dx' \\
&\quad - \int_{t-\tau}^t \int_0^1 G_1(x, x', t-t')\alpha(x')\mu_2 u(x', t')dx'dt' \\
&\quad = q(x, t) - \int_0^1 \mu_1 G_2(x, x', \tau)\alpha(x')u(x', t)dx'.
\end{align*}
\]

\[
q_i(x, t) = \Sigma p(x, t) - \int_0^1 \alpha(x')u(x', t)[\mu_1 G_1(x, x', \tau) + \mu_2 G_1(x, x', 0)]dx'.
\]
Now, (60)-(61) together with the boundary conditions of (60) gives the following 
(p, q) system (12).

\[
\begin{aligned}
    p_t(x, t) &= q(x, t) - \int_0^1 G_2(x, x', \tau)\alpha(x')\mu_1 u(x', t)dx', \\ q_t(x, t) &= p_{xx}(x, t) - \int_0^1 \alpha(x')u(x', t)[\mu_1 G_1(x, x', \tau) + \mu_2 G_1(x, x', 0)]dx', \\
    p(0, t) &= p_x(1, t) = 0, t > 0; \\
    p(x, 0) &= \int_0^1 G_1(x, x', \tau)w_0(x')dx' + \int_0^1 G_2(x, x', \tau)w_1(x')dx' \\
        &\quad - \int_0^\tau \int_0^1 G_2(x, x', \tau - s')\alpha(x')\mu_2 f_0(x', s' - \tau)dx'ds', \\ q(x, 0) &= \int_0^1 G_{2xx}(x, x', \tau)w_0(x')dx' + \int_0^1 G_1(x, x', \tau)w_1(x')dx' \\
        &\quad - \int_0^\tau \int_0^1 G_1(x, x', \tau - s)\alpha(x')\mu_2 f_0(x', s - \tau)dx'ds, 
\end{aligned}
\]  
(62)

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