Group velocity in noncommutative spacetime

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**ABSTRACT**

The realization that forthcoming experimental studies, such as the ones planned for the GLAST space telescope, will be sensitive to Planck-scale deviations from Lorentz symmetry has increased interest in noncommutative spacetimes in which this type of effects is expected. We focus here on \(\kappa\)-Minkowski spacetime, a much-studied example of Lie-algebra noncommutative spacetime, but our analysis appears to be applicable to a more general class of noncommutative spacetimes. A technical controversy which has significant implications for experimental testability is the one concerning the \(\kappa\)-Minkowski relation between group velocity and momentum. A large majority of studies adopted the relation \(v = dE(p)/dp\), where \(E(p)\) is the \(\kappa\)-Minkowski dispersion relation, but recently some authors advocated alternative formulas. While in these previous studies the relation between group velocity and momentum was introduced through *ad hoc* formulas, we rely on a direct analysis of wave propagation in \(\kappa\)-Minkowski. Our results lead conclusively to the relation \(v = dE(p)/dp\). We also show that the previous proposals of alternative velocity/momentum relations implicitly relied on an inconsistent implementation of functional calculus on \(\kappa\)-Minkowski and/or on an inconsistent description of spacetime translations.
1 Introduction

Over the last few years there has been a sharp increase in the interest in experimental investigations of Planck-scale effects (see, e.g., Refs. [1, 2, 3, 4, 5, 6, 7]). In particular, studies such as the ones planned for the GLAST space telescope [8] would be sensitive to small, Planck-scale suppressed, deviations from the special-relativistic relation between group velocity and momentum. Within the framework of Planck-scale spacetime noncommutativity such modifications of the relation between group velocity and momentum are often encountered. A noncommutative spacetime which has been considered extensively in this respect is the $\kappa$-Minkowski spacetime [9, 10, 11]. Group velocity in $\kappa$-Minkowski has been discussed in several studies (see, e.g., Refs. [10, 11] and references therein), under the working assumption that the relation $v = dE(p)/dp$, which holds in Galilei spacetime and Minkowski spacetime, would also hold in $\kappa$-Minkowski. This leads to interesting predictions as a result of the fact that, upon identification of the noncommutativity scale $1/\kappa$ with the Planck length $L_p$, the dispersion relation $E(p)$ that holds in $\kappa$-Minkowski is characterized by Planck-length-suppressed deviations from the corresponding relation that holds in Minkowski spacetime. These deviations are very small, because of the Planck-length suppression, and they are in agreement with all available data [1, 2, 6], but the mentioned forthcoming experimental studies would be able [8] to test them.

Recently the validity of $v = dE(p)/dp$ in $\kappa$-Minkowski has been questioned in the studies reported in Refs. [12, 13]; moreover in the study reported in Ref. [14] the relation $v = dE(p)/dp$ was considered on the same footing as some alternative relations. Especially in light of the mentioned plans for experimental studies, this technical issue appears to be rather significant. We here report progress on an approach to the study of $\kappa$-Minkowski which was initiated in Ref. [11]. We argue that key ingredients for the correct derivation of the relation between group velocity and momentum are: (i) a fully developed $\kappa$-Minkowski differential calculus, and (ii) a proper description of energy-momentum in terms of generators of translations. Our analysis provides support for the adoption of the formula $v = dE(p)/dp$, already assumed in most of the $\kappa$-Minkowski literature. We discuss the ad hoc assumptions which led to alternatives to $v = dE(p)/dp$ in Refs. [12, 13], and we find that analysis in Ref. [13] was based on erroneous implementation of the $\kappa$-Minkowski differential calculus, while the analysis in Ref. [12] was interpreted as momenta some quantities which cannot be properly described in terms of translation generators.

2 Preliminaries on $\kappa$-Minkowski noncommutative spacetime

In this section we briefly review some results on $\kappa$-Minkowski spacetime, with emphasis on the aspects that are relevant for the analysis we present in the following sections. The $\kappa$-Minkowski spacetime

$$[x_0, x_j] = i\frac{1}{\kappa} x_j, \quad [x_j, x_k] = 0, \quad \text{(1)}$$

is an example of Lie-algebra noncommutative spacetime (spacetimes with commutation relations of the type $[x_\mu, x_\nu] = iC_{\mu\nu}^{\sigma} x_\sigma$). It came to the attention of the

\footnote{We set $\hbar = c = 1$.}
community primarily because of its role as dual algebra of the momentum sector of the popular "bicrossproduct" $\kappa$-Poincaré Hopf algebra:

\[
\begin{align*}
[M_j, M_k] &= i\epsilon_{jkl}M_l, \\
[M_j, N_k] &= i\epsilon_{jkl}N_l, \\
[N_j, N_k] &= -i\epsilon_{jkl}M_l
\end{align*}
\]

\[
[p_\mu, p_\nu] = 0, \\
[M_j, p_0] = 0, \\
[M_j, p_k] &= i\epsilon_{jkl}p_l
\]

\[
[N_j, p_0] = ip_j, \\
[N_j, p_k] &= i\delta_{jk} \left( \frac{1 - e^{-2\lambda p_0}}{2\lambda} + \frac{\lambda}{2} p^2 \right) - i\lambda p_j p_k,
\]

where $p_\mu = (p_0, p_j)$ are the translation generators, $M_j$ are ordinary rotation generators, $N_j$ are the $\kappa$-deformed boost generators, and we introduced the convenient notations $\lambda \equiv \kappa^{-1}$, $p \equiv |\vec{p}|$.

The algebraic relations (2) are accompanied by coalgebraic structures: the coproducts

\[
\begin{align*}
\Delta(p_0) &= p_0 \otimes 1 + 1 \otimes p_0 \\
\Delta(p_j) &= p_j \otimes 1 + e^{-\lambda p_0} \otimes p_j \\
\Delta(M_j) &= M_j \otimes 1 + 1 \otimes M_j \\
\Delta(N_j) &= N_j \otimes 1 + e^{-\lambda p_0} \otimes N_j + \lambda e_{jkl}p_k \otimes M_l
\end{align*}
\]

and the antipodes

\[
\begin{align*}
S(N_j) &= -e^{\lambda p_0} N_j + \lambda e^{\lambda p_0} \epsilon_{jkl} p_k M_l \\
S(p_j) &= -e^{\lambda p_0} p_j \\
S(p_0) &= -p_0.
\end{align*}
\]

The "mass Casimir" (dispersion relation) of this "bicrossproduct" $\kappa$-Poincaré Hopf algebra is

\[
C_\kappa(p_0, \vec{p}) = \left( \frac{2}{\lambda} \sinh \frac{\lambda p_0}{2} \right)^2 - p^2 e^{\lambda p_0}
\]

It is sometimes quickly stated that the bicrossproduct $\kappa$-Poincaré Hopf algebra (3),(2),(4) describes the symmetries of the $\kappa$-Minkowski noncommutative spacetime (1). Actually, the commutation relations (1) are not sufficient to fully characterize a noncommutative geometry: one must consider the enveloping algebra generated by (1) and introduce a differential calculus on this enveloping algebra. The natural differential calculus on the enveloping algebra of $\kappa$-Minkowski is

\[
\partial_j : f(x) := \frac{\partial f(x)}{\partial x_j} : \\
\partial_0 : f(x) := \frac{e^{\lambda \frac{\partial}{\partial t}} - 1}{i\lambda} f(x) := \frac{f(\vec{x}, t + i\lambda)}{i\lambda} - f(\vec{x}, t)
\]

Other examples of $\kappa$-Poincaré Hopf algebras are discussed in Refs. [10, 15]. Besides its relevance for the description of $\kappa$-Minkowski spacetime (endowed with the differential calculus here described), the bicrossproduct $\kappa$-Poincaré Hopf algebra is a particularly interesting example also because it has been shown to be suitable for the construction of a symmetry group of deformed finite Lorentz transformations obtained exponentiating the generators of its Lorentz sector.

The enveloping algebra is the algebra that contains all (Taylor-expandable) functions of the noncommutative coordinates.
The notation : \( f(x) \), conventional in the \( \kappa \)-Minkowski literature, is reserved for time-to-the-right-ordered functions of the noncommutative coordinates. The standard symbolism adopted in Eqs. (3)-(7) describes noncommutative differentials in terms of familiar actions on commutative functions. The symbols “\( \partial_j \)” and “\( \partial_0 \)” refer to elements of the differential calculus on \( \kappa \)-Minkowski, while the symbols “\( \partial / \partial x_j \)” and “\( \partial / \partial t \)” act as ordinary derivatives on a time-to-the-right-ordered function of the \( \kappa \)-Minkowski coordinates. For example, Eq. (3) states that in \( \kappa \)-Minkowski \( \partial_x(xt) = t \) and \( \partial_x[xt^2 + 2i\lambda xt − \lambda^2 x + x^2 t] = t^2 + 2i\lambda t − \lambda^2 + 2xt \), i.e. \( \partial_x \) acts as a familiar \( x \)-derivative on time-to-the-right-ordered functions. Of course, the \( \kappa \)-Minkowski commutation relations impose that, if derivatives are standard on time-to-the-right-ordered functions, derivatives must be accordingly modified for functions which are not time-to-the-right ordered. For example, since \( \partial_x(xt) = t \) and \( \partial_x(x) = 1 \) (the functions \( xt \) and \( x \) are time-to-the-right ordered), also taking into account the \( \kappa \)-Minkowski commutation relation \( xt = tx − i\lambda x \), one can obtain the \( x \)-derivative of the function \( tx \), which must be given by \( \partial_x(tx) = t + i\lambda \). Similarly, one finds that \( \partial_x[t^2 x + x^2 t] = t^2 + 2i\lambda t − \lambda^2 + 2xt \) (in fact, using the \( \kappa \)-Minkowski commutation relations one finds that \( t^2 x + x^2 t = xt^2 + 2i\lambda xt − \lambda^2 x + x^2 t \)).

The time derivative described by Eq. (6) has analogous structure, with the only difference that the special role of the time coordinate in the structure of \( \kappa \)-Minkowski spacetime forces \( [\mathfrak{l}] \) one to introduce an element of discretization in the time direction: the time derivative of time-to-the-right-ordered functions is indeed standard (just like the \( x \)-derivative of time-to-the-right-ordered functions is standard), but it is a standard \( \lambda \)-discretized derivative (whereas the \( x \)-derivative of time-to-the-right-ordered functions is a standard continuous derivative).

It is only once (the enveloping algebra of) \( \kappa \)-Minkowski is equipped with this differential calculus that one can address physically-meaningful questions, such as the ones concerning a description of the symmetries of \( \kappa \)-Minkowski. And it is at this level that the duality between \( \kappa \)-Minkowski and the bicrossproduct \( \kappa \)-Poincaré Hopf algebra \( (3),(2),(4) \) is established. In fact, from the coalgebraic structure of \( \kappa \)-Poincaré one can actually reconstruct the \( \kappa \)-Minkowski spacetime through the inner product

\[
\left\langle \frac{(i\omega)^{n_0}}{n_0!} \frac{(-ik_1)^{n_1}}{n_1!} \frac{(-ik_2)^{n_2}}{n_2!} \frac{(-ik_3)^{n_3}}{n_3!}, x_1^{m_1} x_2^{m_2} x_3^{m_3} t^{m_0} \right\rangle = \delta_{n_0m_0} \delta_{n_1m_1} \delta_{n_2m_2} \delta_{n_3m_3} \tag{8}
\]

which pairs the two algebras.

This duality between \( \kappa \)-Minkowski and bicrossproduct \( \kappa \)-Poincaré is also describable as a covariance (in Hopf-algebra sense) of \( \kappa \)-Minkowski under bicrossproduct \( \kappa \)-Poincaré actions. Let \( w \) be an element of \( U(\mathfrak{so}_{1,3}) \) (a function of boosts and rotations) and \( \chi \) a function of the momenta, then the (left) adjoint action is defined by\(^4\)

\[ w \triangleright \chi = w^{(1)} \chi S(w^{(2)}) \]

If \( w \in \mathfrak{so}_{1,3} \) is a linear combination of boost and rotation generators, from \( (3) \) and \( (4) \) it follows that the action on a function of the momenta simply reduces to the

\(^{4}\)In \( \kappa \)-Minkowski spacetime (with its commuting space coordinates and nontrivial commutation relations only when the time coordinate is involved), it is easy to see that the natural functional calculus should be introduced in terms of time-to-the-right-ordered functions or (the equivalent alternative of) intuitive rules for time-to-the-left-ordered functions. In other noncommutative spacetimes the choice of ordering may not be so obvious.

\(^{5}\)We introduce \( w^{(1)} \) and \( w^{(2)} \) in the sense of the “Sweedler notation” for the coproduct: \( \Delta(w) = \sum_i w_i^{(1)} \otimes w_i^{(2)} \equiv w^{(1)} \otimes w^{(2)} \) (we omit index and summation symbol).
commutator:

\[ M_i \triangleright \chi = [M_i, \chi] \quad N_i \triangleright \chi = [N_i, \chi] \quad (9) \]

and, by construction, satisfies the condition of covariance (in the Hopf-algebra sense)

\[ w \triangleright (\chi \chi') = (w(1) \triangleright \chi)(w(2) \triangleright \chi'), \quad (10) \]

which, for the action of a generator of \( U(so_{1,3}) \), is the familiar Leibniz rule.

The action of an element of the Lorentz sector on the coordinates is implicitly defined through the relation:

\[ \langle f(p), w \triangleright :g(x)\rangle = \langle S(w) \triangleright f(p), :g(x)\rangle \quad (11) \]

which amounts to stating that a finite transformation described by a grouplike element \( g \) be self-adjoint with respect to the inner product \( \langle , \rangle \). This guarantees the covariance of \( \kappa \)-Minkowski in the sense of the Hopf algebras.

By using (9) and (11) one obtains

\[ M_j \triangleright :f(x)\rangle = -i \epsilon_{jkl} x_k \partial_l :f(x)\rangle \quad (12) \]
\[ N_j \triangleright :f(x)\rangle = \left[ ix_0 \partial_j + ix_j \partial_0 - \frac{\lambda}{2} x_j \partial_\mu \partial^\mu \right] :f(x)\rangle \quad (13) \]

which for the generators of \( \kappa \)-Minkowski means

\[ M_j \triangleright x_0 = 0 \quad M_j \triangleright x_k = i \epsilon_{jkl} x_l \quad N_j \triangleright x_0 = ix_j \quad N_j \triangleright x_k = i \delta_{jk} x_0 \quad (14) \]

The canonical (left) action of \( p_\mu \) on the coordinate space is

\[ p_\mu \triangleright :f(x)\rangle = -i \partial_{x^\mu} :f(x)\rangle \quad (15) \]

so that the finite transformation is a simple translation:

\[ e^{iap_\mu} \triangleright :f(x)\rangle = :f(x + a)\rangle \quad (16) \]

A central role in the \( \kappa \)-Minkowski functional calculus is played by the ordered exponentials:

\[ e^{-iq_0 t} e^{iq_0 t}, \quad (17) \]

where \( \{q_j, q_0\} \) are four real numbers and \( \{x_j, t\} \) are \( \kappa \)-Minkowski coordinates. These ordered exponentials enjoy a simple property with respect to the generators \( p_\mu \) of translations of the \( \kappa \)-Minkowski coordinates:

\[ \langle p_\mu, e^{-iq_0 t} e^{iq_0 t} \rangle = q_\mu. \quad (18) \]

We also note that, using the \( \kappa \)-Minkowski commutation relations, one finds the relation

\[ e^{-iq_0 t} e^{iq_0 t} = \exp \left( iq_0 t - i q_0 t \frac{\lambda q_0}{1 - e^{-\lambda q_0}} \right) \quad (19) \]

\[ \text{If } g \text{ is a grouplike element, i.e. } \Delta(g) = g \otimes g, \text{ then } S(g) = g^{-1} \text{ so that } \langle g \triangleright f(p), g \triangleright :g(x)\rangle = \langle f(p), (g^{-1}) \triangleright :g(x)\rangle = \langle f(p), :g(x)\rangle. \]
which turns out to be useful in certain applications.

The ordered exponentials $e^{-i\vec{q}\cdot \vec{x}} e^{i\omega t}$ also play the role of plane waves in $\kappa$-Minkowski since on the mass-shell (i.e. $\mathcal{C}_\kappa(q_0, \vec{q}) = M^2$) they are solutions [13] of the relevant wave (deformed Klein-Gordon) equation:

\[
(\Box - M^2) [e^{-i\vec{q}\cdot \vec{x}} e^{i\omega t}] = 0
\]

(20)

where $\Box = \partial_\mu \partial^\mu L^{-1}$ is the $\kappa$-deformed D’Alembert operator, properly defined [17] in terms of the so-called “$\kappa$-Minkowski shift operator” $L$

$$L : f(\vec{x}, t) := e^{-\lambda p_0 \triangleright} : f(\vec{x}, t) : = : f(\vec{x}, t + i\lambda) :$$

The ordered exponentials are also the basic ingredient of the Fourier theory on $\kappa$-Minkowski. This Fourier theory [17] is constructed in terms of the canonical element

$$\sum_i e_i \otimes f^i,$$

where $\{e_i\}$ and $\{f^j\}$ are dual bases, which satisfy the relation $\langle e_i, f^j \rangle = \delta^j_i$. On the basis of (8) one finds that the canonical element is

\[
\psi_{(q_0, \vec{q})}(t, \vec{x}) = \sum_{n_0, n_1, n_2, n_3} \frac{(-i q_1 x_1)^{n_1}}{n_1!} \frac{(-i q_2 x_2)^{n_2}}{n_2!} \frac{(-i q_3 x_3)^{n_3}}{n_3!} \frac{(i q_0 t)^{n_0}}{n_0!} = e^{-i\vec{k}\cdot \vec{x}} e^{i\omega t}
\]

(21)

The canonical element (21) retains the notable feature that, if we define the transform $\tilde{f}(q)$ of an ordered function $f(x)$ through

$$f(x) = \int \tilde{f}(q) e^{-i\vec{q}\cdot \vec{x}} e^{i\omega t} \frac{e^{3\lambda q_0} d^4 q}{(2\pi)^4},$$

the choice of the integration measure $e^{3\lambda q_0}$ and the definition [14] of the actions of boosts/rotations on the coordinates guarantee that

$$w \triangleright : f(x) : = \int \left( S(w) \triangleright \tilde{f}(q) \right) e^{-i\vec{q}\cdot \vec{x}} e^{i\omega t} \frac{e^{3\lambda q_0} d^4 q}{(2\pi)^4}$$

for each $w \in U(so_{1,3})$. This is a relevant property because it implies that under a finite transformation both $f$ and $\tilde{f}$ change, but they remain connected by the Fourier-transform relations. The action of a transformation on the $x$ is equivalent to the inverse transformation on the $q$. This is exactly what happens in the classical-Minkowski case ($\lambda = 0$), through the simple relation

$$f(x) \mapsto f_\Lambda(x) = \int \tilde{f}(\Lambda^{-1} q) e^{i q x} \frac{d^4 q}{(2\pi)^4} = f(\Lambda x).$$

In $\kappa$-Minkowski the action of boosts does not allow description in terms of a matrix $\Lambda^{\nu}_\mu$, but it is still true that the action of a transformation on the $x$ is equivalent to the “inverse transformation” on the $q$ (where, of course, here the “inverse transformation” is described through the antipode).
3 Group velocity in $\kappa$-Minkowski noncommutative spacetime

In this section we first briefly discuss the familiar relation between group velocity and dispersion relation in classical commutative spacetime, then we use some of the tools reviewed in the previous section to derive the corresponding relation that holds in $\kappa$-Minkowski noncommutative spacetime.

3.1 Group velocity in commutative spacetimes

Both in theories in Galilei spacetime and in theories in Minkowski spacetime the relation between the physical velocity of signals (the group velocity of a wave packet) and the dispersion relation is governed by the formula
\[
v = \frac{dE}{dp},
\]
(22)
in components
\[
v_j = \frac{dx_j}{dt} = \frac{\partial E}{\partial p_j} = \frac{p_j \partial E}{p \partial p}.
\]
(23)
This is basically a result of the fact that our theories in Galilei and Minkowski spacetime admit Hamiltonian formulation. In classical mechanics this leads directly to
\[
\frac{dx_j}{dt} = \frac{\partial H(p)}{\partial p_j}.
\]
(24)
In ordinary quantum mechanics $\vec{x}$ and $\vec{p}$ are described in terms of operators that satisfy the commutation relations $[p_j, x_k] = i\delta_{jk}$, and in the Heisenberg picture the time evolution for the position operator is given by
\[
\frac{dx_j(t)}{dt} = i[x_j(t), H]
\]
Since $x_j \to \partial/\partial p_j$ and, again, $H \to E(p)$, also in ordinary quantum mechanics one finds $v = dE/dp$ (but in quantum mechanics $v_j$ is the operator $dx_j/dt$ and the group-velocity relation strictly holds only for expectation values).

Given a spacetime, the concept of group velocity can be most naturally investigated in the study of the propagation of waves. It is useful to review that discussion briefly. For simplicity we consider a classical 1+1-dimensional Minkowski spacetime. We denote by $\omega$ the frequency of the wave and by $k(\omega)$ the wave number of the wave. [Of course, $k(\omega)$ is governed by the dispersion relation, by the mass Casimir of the classical Poincaré algebra.] A plane wave is described by the exponential $e^{i\omega t - ikx}$. A wave packet is the Fourier transform of a function $a(\omega)$ which is nonvanishing in a limited region of the spectrum ($\omega_0 - \Delta, \omega_0 + \Delta$):
\[
\Psi_{(\omega_0, k_0)}(t, x) = \int_{\omega_0 - \Delta}^{\omega_0 + \Delta} a(\omega) e^{i\omega t - ikx} d\omega.
\]

The information/energy carried by the wave will travel at a sharply-specified velocity, the group velocity, only if $\Delta \ll \omega_0$. It is convenient to write the wave-packet
as $\Psi_{(\omega_0,k_0)}(t, x) = A(t, x)e^{i\omega_0 t - i k_0 x}$, from which the definition of the wave amplitude $A(t, x)$ follows:

$$A(t, x) = \int_{\omega_0-\Delta}^{\omega_0+\Delta} a(\omega)e^{i(\omega-\omega_0)(t-[\frac{d\omega}{dk}]_0 x)}d\omega$$  \hspace{1cm} (25)$$

The wave packet is therefore the product of the plane-wave factor $e^{i\omega_0 t - i k_0 x}$ and the wave amplitude $A(t, x)$. One can introduce a “phase velocity”, $v_{ph} = \omega_0 / k_0$, associated with the plane-wave factor $e^{i\omega_0 t - i k_0 x}$, but there is no information/energy that actually travels at this velocity (this “velocity” is a characteristic of a pure phase, with modulus 1 everywhere). It is the wave amplitude $A(t, x)$ that describes the time evolution of the energy/information actually carried by the wave packet. From (25) we see that the wave amplitude stiffly translates at velocity $v_g = d\omega_0 / dk_0$, the group velocity. In terms of the group velocity and the phase velocity the wave packet can be written as

$$\Psi_{(\omega_0,k_0)}(t, x) = e^{ik_0(v_{ph} t - x)} \int_{\omega_0-\Delta}^{\omega_0+\Delta} a(\omega)e^{i(\omega-\omega_0)(t-x/v_g)}d\omega .$$

In ordinary Minkowski spacetime the group velocity and the phase velocity both are 1 (in our units) for photons (light waves) travelling in vacuum. For massive particles or massless particles travelling in a medium $v_{ph} \neq v_g$. The causality structure of Minkowski spacetime guarantees that $v_g \leq 1$, whereas, since no information actually travels with the phase velocity, it provides no obstruction for $v_{ph} > 1$.

### 3.2 Group velocity in $\kappa$-Minkowski

The elements of $\kappa$-Minkowski functional analysis we reviewed in Section 2 allow us to implement a consistent deformation of the analysis that applies in commutative Minkowski spacetime, here reviewed in the preceding subsection. In order to present specific formulas we adopt the $\kappa$-Minkowski functional analysis based on time-to-the-right-ordered noncommutative functions, but the careful reader can easily verify that the same result for the group velocity is obtained adopting the time-to-the-left ordering prescription.

We are little concerned with the concept of phase velocity (which is not a physical velocity). In this respect we just observe that the phase velocity should be a property of the $\kappa$-Minkowski plane wave

$$\psi_{(\omega,k)} = e^{-ik\vec{x}}e^{i\omega t} ,$$  \hspace{1cm} (26)$$

and, since the $\kappa$-Minkowski calculus is structured in such a way that the properties of time-to-right-ordered functions are just the ones of the corresponding commutative function, this suggests that the relation

$$v_{ph} = \frac{\omega}{k}$$  \hspace{1cm} (27)$$

should be valid.

But let us focus on the more significant (physically meaningful) analysis of group velocity. Our starting point is the wave packet

$$\Psi_{(\omega_0,k_0)} = \int e^{-i\vec{k} \cdot \vec{x}}e^{i\omega t}d\mu .$$
In this equation (3.2) for simplicity we denote with $d\mu$ an integration measure which includes the spectrum of the packet. In fact, the precise structure of the wave packet is irrelevant for the analysis of the group velocity: it suffices to adopt a packet which is centered at some $(\omega_0, \vec{k}_0)$ (with $(\omega_0$ and $\vec{k}_0$) related through Eq. (5), the dispersion relation, the mass Casimir, of the classical Poincaré algebra) and has support only on a relatively small neighborhood of $(\omega_0, \vec{k}_0)$, i.e. $\omega_0 - \Delta \omega \leq \omega \leq \omega_0 + \Delta \omega$ and $\vec{k}_0 - \Delta \vec{k} \leq \vec{k} \leq \vec{k}_0 + \Delta \vec{k}$.

Next, in order to proceed just following the same steps of the familiar commutative-spacetime case, we should factor out of the integral a “pure phase” with frequency and wavelength fixed by the wave-packet center: $(\omega_0, \vec{k}_0)$. Consistently with the nature of the time-to-the-right-ordered functional calculus the phase $e^{i\vec{k}_0 \cdot \vec{x}}$ will be factored out to the left and the phase $e^{-i\omega_0 t}$ will be factored out to the right:

$$
\Psi_{(\omega_0, \vec{k}_0)} = e^{-i\vec{k}_0 \cdot \vec{x}} \left[ \int e^{-i\Delta \vec{k} \cdot \vec{x}} e^{i\Delta \omega t} d\mu \right] e^{i\omega_0 t} \tag{28}
$$

This way to extract the phase factor preserves the time-to-the-right-ordered structure of the wave $\Psi_{(\omega_0, \vec{k}_0)}$, and therefore, also taking into account the role that time-to-the-right-ordered functions have in the $\kappa$-Minkowski calculus, should allow an intuitive analysis of its properties.

From (28) one recognizes the $\kappa$-Minkowski group velocity as

$$
v_g = \lim_{\Delta \omega \to 0} \frac{\Delta \omega}{\Delta \vec{k}} = \frac{d\omega}{dk}, \tag{29}
$$

just as in Galilei and Minkowski spacetime. Just as one does in commutative Minkowski spacetime, the integral can be seen as the amplitude of the wave, the group velocity $v_g$ is the velocity of translation of this wave amplitude, which be meaningfully introduced only in the limit of narrow packet (small $\Delta \omega$ and $\Delta \vec{k}$).

Notice that

$$
e^{-i\Delta \vec{k} \cdot \vec{x}} e^{i\Delta \omega t} = \exp \left( i\Delta \omega t - i\Delta \vec{k} \cdot \vec{x} \frac{\lambda \Delta \omega}{1 - e^{-\lambda \Delta \omega}} \right), \tag{30}
$$

and

$$
\left[ \exp \left( i\Delta \omega t - i\Delta \vec{k} \cdot \vec{x} \frac{\lambda \Delta \omega}{1 - e^{-\lambda \Delta \omega}} \right) \right]_{\Delta \omega \to 0} = \exp \left( i\Delta \omega t - i\Delta \vec{k} \cdot \vec{x} \right), \tag{31}
$$

and therefore the evaluation of the velocity of translation of this wave amplitude turns out to be independent of the way in which the exponentials are arranged (but this is an accident due to the fact that for small $\Delta \omega$ and $\Delta \vec{k}$ one finds that $[e^{-i\Delta \vec{k} \cdot \vec{x}}, e^{i\Delta \omega t}] = 0$.

4 Comparison with previous analyses

Because of the mentioned interest in the phenomenological implications [1, 2, 6, 8], the introduction of group velocity in $\kappa$-Minkowski has been discussed in several studies. In the large majority of these studies the concept of group velocity was not introduced constructively (it was not a result obtained in a full theoretical scheme: it was just introduced through an ad hoc relation). This appeared to be harmless since the ad-hoc
assumption relied on the validity of the relation \( v_g = dE/dp \), which holds in Galilei spacetime and Minkowski spacetime (and for which the structure of \( \kappa \)-Minkowski appears to pose no obstacle).

Taking as starting point the approach to \( \kappa \)-Minkowski proposed in Ref. [11], we have here shown through a dedicated analysis that the validity of \( v_g = dE/dp \) indeed follows automatically from the structure of \( \kappa \)-Minkowski and of the associated functional calculus.

At this point it is necessary for us to clarify which erroneous assumptions led to the claims reported in Refs. [12, 13], which questioned the validity of \( v_g = dE(p)/dp \) in \( \kappa \)-Minkowski.

4.1 Tamaki-Harada-Miyamoto-Torii analysis

It is rather easy to compare our analysis with the study reported by Tamaki, Harada, Miyamoto and Torii in Ref. [13]. In fact, Ref. [13] explicitly adopted the same approach to \( \kappa \)-Minkowski calculus that we adopted here, with Fourier transform and functional calculus that make direct reference to time-to-the-right-ordered functions. Also the scheme of analysis is analogous to ours, in that it attempts to derive the group velocity from the analysis of the time evolution of a superposition of plane waves. However, the \( \kappa \)-Minkowski functional calculus was applied inconsistently in Ref. [13]: at the stage of the analysis were one should factor out the phases \( e^{-i\vec{k}_0 \cdot \vec{x}} \) and \( e^{i\omega_0 t} \) from the wave amplitude (as we did in Eq. (28)) Ref. [13] does not proceed consistently with the time-to-the-right-ordered functional calculus. Of course, as done here, in order to maintain the time-to-the-right-ordered form of the wave packet it is necessary to factor out the phases \( e^{-i\vec{k}_0 \cdot \vec{x}} \) and \( e^{i\omega_0 t} \) respectively to the left and to the right, as we did here. Instead in Ref. [13] both phases are factored out to the left leading to a form of the wave packet which is not time-to-the-right ordered. In turn this leads to the erroneous conclusion that \( v_g(k) \neq d\omega(k)/dk \), i.e. \( v_g(p) \neq dE(p)/dp \).

This inconsistency with the ordering conventions is the key factor that affected Ref. [13] failure to reproduce \( v_g(p) \neq dE(p)/dp \), but for completeness we note here also that Ref. [13] leads readers to the erroneous impression that in order to introduce the group velocity in \( \kappa \)-Minkowski one should adopt the approximation

\[
e^{-i\vec{k} \cdot \vec{x}} e^{i\omega t} \sim e^{-i\vec{k} \cdot \vec{x} + i\omega t},
\]

for generic values of \( \omega \) and \( \vec{k} \). Actually, unless \( \omega \) and \( \vec{k} \) are very small, this approximation is very poor: it only holds in zeroth order in the noncommutativity scale \( \lambda \) and therefore it does not describe reliably the structure of \( \kappa \)-Minkowski (since it fails already in leading order in \( \lambda \), it does not even reliably characterize the main differences between classical Minkowski and \( \kappa \)-Minkowski). As we showed here there is no need for the approximation (32) in the analysis of the group velocity of a wave packet in \( \kappa \)-Minkowski.

4.2 \( \kappa \)-Deformed phase space

As discussed in the preceding Subsection, it is very easy to compare our study with the study reported in Ref. [13], since both studies adopted the same approach. We must now provide some guidance for the comparison with the study reported by Kowalski-Glikman in Ref. [12]. Also this comparison is significant for us since Ref. [12], like
Ref. [13], questioned the validity of the relation \( v_g = dE(p)/dp \), which instead emerged from our analysis.

Our approach to \( \kappa \)-Minkowski, which originates from techniques developed in Refs. [9, 11], is profoundly different from the one adopted in Ref. [12]. In fact, the differences start off already at the level of the action of \( \kappa \)-Poincaré generators on \( \kappa \)-Minkowski coordinates. The actions we adopted are described in Section 2. They take a simple form on time-to-the-right ordered functions, but they do not allow description as a "commutator action" on generic ordering of functions in \( \kappa \)-Minkowski. Instead in Ref. [12] the action of the \( \kappa \)-Poincaré generators on \( \kappa \)-Minkowski coordinates was introduced in fully general terms as a commutator action. This would allow to introduce a "phase-space extension" of \( \kappa \)-Minkowski.

\[
[x_0, x_j] = i\lambda x_j \quad [p_0, x_0] = -i \quad [p_k, x_j] = i\delta_{jk} e^{-\lambda p_0} \quad [p_j, x_0] = [p_0, x_j] = 0
\]

Taking this phase space \( \{x_0, x_j\} \) as starting point, Kowalski-Glikman then found, after a rather lengthy analysis, that "massless particles move in spacetime with universal speed of light" \( c \), in conflict with the relation \( v_g = dE(p)/dp \) and the structure of the mass casimir \( \{E(p, m)\} \). Kowalski-Glikman argued that this puzzling conflict with the structure of the mass casimir might be due to a missmatch between the mass-casimir relation, \( E(p, m) \), and the dispersion relation, \( \omega(k, m) \): the puzzle could be explained \( [12] \) if the usual identifications \( k \sim p \) and \( \omega \sim E \) were to be replaced by \( k \sim pe^{\lambda E} \) and \( \omega \sim sinh(\lambda E)/\lambda + \lambda p^2 e^{\lambda E} / 2 \).

We observe that the correct explanation of the puzzling result obtained by Kowalski-Glikman is actually much simpler: the commutator action \( \{x_0, x_j\} \) adopted in Ref. [12], in spite of the choice of symbols \( p_j, x_k \), cannot describe the action of "momenta" \( p_j \) on coordinates \( x_k \). Momenta should generate translations of the coordinates, which requires that they may be represented as derivatives of functions of the coordinates, but the commutator action \( [p_k, x_j] = i\delta_{jk} e^{-\lambda p_0} \) clearly does not allow to represent \( p_k \) as a derivative with respect to the \( x_k \) coordinate, because of the spurious factor \( e^{-\lambda p_0} \). Similarly, those commutation relations do not allow to represent the \( x_k \) coordinate as a derivative with respect to \( p_k \), and therefore in a Hamiltonian theory, with Hamiltonian \( H \), one would find

\[
\dot{x}_j \sim [x_j, H] \neq \frac{dH}{dp_j} , \tag{34}
\]

and this is basically the reason for the puzzling result \( v_g(p) \neq dE(p)/dp \) obtained in Ref. [12]. Kowalski-Glikman finds a function \( v_g(p) \) but this function cannot be seen as describing the relation between velocity and momentum, since the "\( p \)" symbol introduced in \( \{x_0, x_j\} \) does not generate translations of coordinates, and therefore "\( p \)" is not a momentum.

5 Conclusions

Plans for experimental searches of a possible dependence of the group velocity on the Planck scale are already at an advanced stage [8]. The key motivation for these studies comes from the idea that Planck-scale (quantum) structure of spacetime might affect the group-velocity/wavelength relation. This type of Planck-scale effects is plausible (and in some cases inevitable) in most quantum-gravity approaches, including phenomenological models of spacetime foam [4], loop quantum gravity (see,
e.g., Ref. [18]), superstring theory (see, e.g., Ref. [19]), and noncommutative geometry. However, a detailed careful description of wave propagation is beyond the reach of our present technical understanding of most quantum-gravity scenarios. One noticeable exception is $\kappa$-Minkowski noncommutative spacetime, which is being considered as a possible flat-space remnant of quantum properties of spacetime induced by Planck-scale physics. The structure of $\kappa$-Minkowski spacetime is simple enough that, as shown here, a rigorous analysis of wave propagation is possible. In turn this allows us to provide experimentalists a definite quantum-spacetime scenario to use as reference for their sensitivity estimates.

We showed here that the formula $v = dE(p)/dp$, where $E(p)$ is fixed by the $\kappa$-Poincaré dispersion relation, holds in $\kappa$-Minkowski spacetime, just like $v = dE(p)/dp$ holds in classical Galileo and Minkowski spacetimes. The validity of $v = dE(p)/dp$ in $\kappa$-Minkowski had been largely expected in the literature, even before our direct analysis, but such a direct analysis had become more urgent after the appearance of some recent articles [12, 13] which had argued in favour of alternatives to $v = dE(p)/dp$ for $\kappa$-Minkowski. We have shown that these recent claims were incorrect: the analysis reported in Ref. [13] was based on an erroneous implementation of the $\kappa$-Minkowski differential calculus, while the analysis in Ref. [12] interpreted as momenta some quantities which cannot be properly described in terms of translation generators.

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