CONFORMING FINITE ELEMENTS
FOR $H(\text{sym} \, \text{Curl})$ AND $H(\text{dev} \, \text{sym} \, \text{Curl})$

OLIVER SANDER

ABSTRACT. We construct conforming finite elements for the spaces $H(\text{sym} \, \text{Curl})$ and $H(\text{dev} \, \text{sym} \, \text{Curl})$. Those are spaces of matrix-valued functions with symmetric or deviatoric-symmetric Curl in a Lebesgue space, and they appear in various models of nonstandard solid mechanics. The finite elements are not $H(\text{Curl})$-conforming. We show the construction, prove conformity and unisolvence, and point out optimal approximation error bounds.

Keywords: sym Curl, dev sym Curl, conforming finite elements, incompatible linear elasticity

1. INTRODUCTION

In [5], Lewintan, Müller, and Neff introduced the spaces

$$W^{1,p}(\text{sym} \, \text{Curl}; \Omega; \mathbb{R}^{3 \times 3}) := \{ P \in L^p(\Omega; \mathbb{R}^{3 \times 3}) : \text{sym} \, \text{Curl} \, P \in L^p(\Omega; \mathbb{R}^{3 \times 3}) \}$$

and

$$W^{1,p}(\text{dev} \, \text{sym} \, \text{Curl}; \Omega; \mathbb{R}^{3 \times 3}) := \{ P \in L^p(\Omega; \mathbb{R}^{3 \times 3}) : \text{dev} \, \text{sym} \, \text{Curl} \, P \in L^p(\Omega; \mathbb{R}^{3 \times 3}) \},$$

where $L^p(\Omega; \mathbb{R}^{3 \times 3})$, $p \geq 1$, is the Lebesgue space of $\mathbb{R}^{3 \times 3}$-valued functions that are $p$-integrable on a domain $\Omega$. The operator Curl denotes the classical curl operator acting row-wise on a matrix; to distinguish the two we write the matrix form with an upper-case letter. The operators dev and sym produce the deviatoric and symmetric parts of a $3 \times 3$-matrix

$$\text{dev} \, A := A - \frac{1}{3} \text{trace} \, A \quad \text{and} \quad \text{sym} \, A := \frac{1}{2} (A + A^T),$$

respectively. For brevity, we will call the spaces above $H(\text{sym} \, \text{Curl})$ and $H(\text{dev} \, \text{sym} \, \text{Curl})$ in this manuscript.

Lewintan, Müller, and Neff presented several potential applications from the field of solid mechanics. For numerical simulations it is therefore of interest to construct conforming finite elements for these spaces. As

$$H(\text{Curl}) \subsetneq H(\text{sym} \, \text{Curl}) \quad \text{and} \quad H(\text{Curl}) \subsetneq H(\text{dev} \, \text{sym} \, \text{Curl}),$$

a possible candidate are finite elements that are row-wise $H(\text{curl})$-conforming (for example, the Nédélec elements [6, 7]). However, the subset relations (1) are strict, and more structure of the new spaces can be captured by finite element spaces that are larger, i.e., not necessarily subspaces of $H(\text{Curl})$.

The author would like to thank Adam Sky and Patrizio Neff for the interesting discussions.
The classical way to construct conforming finite elements for a particular Sobolev space is to combine piecewise polynomials on a grid $T$ by certain continuity conditions. Define

$$\text{Anti} : \mathbb{R}^3 \to \mathbb{R}^{3 \times 3}, \quad \text{Anti} a := \begin{pmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{pmatrix},$$

which implies

$$(\text{Anti} a) b = a \times b$$

for every $a, b \in \mathbb{R}^3$. On any element $T \in T$, we can integrate by parts:

$$\int_T Q : \text{sym Curl} U \, dx - \int_T \text{Curl} \text{sym} Q : U \, dx = -\int_{\partial T} \text{sym}(U \text{Anti} n) Q \, d\sigma$$

and

$$\int_T Q : \text{dev sym Curl} U \, dx - \int_T \text{Curl} \text{dev sym} Q : U \, dx = -\int_{\partial T} \text{dev sym}(U \text{Anti} n) Q \, d\sigma,$$

if the functions $U, Q : T \to \mathbb{R}^{3 \times 3}$ are sufficiently smooth (see [5, Chapter 3]). From these formulas, we get the following characterization result.

**Theorem 1.** Let $\Omega$ be bounded. A piecewise continuously differentiable function $U : \Omega \to \mathbb{R}^{3 \times 3}$ on a grid $T$ is in $H(\text{sym Curl})$ if and only if

$$\text{sym}(\lfloor U \rfloor \text{Anti} n) = 0$$

on every inner face of the grid, where $\lfloor U \rfloor$ is the jump of $U$ at the face, and $n$ is a face normal. The function is in $H(\text{dev sym Curl})$ if and only if

$$\text{dev sym}(\lfloor U \rfloor \text{Anti} n) = 0$$

on every inner face of the grid.

The set of matrices that fulfill (2), but not the corresponding condition $\lfloor U \rfloor \text{Anti} n = 0$ for $H(\text{Curl})$ is spanned by the identity matrix. Multiples of the identity therefore play a special role, and are treated separately in the finite element construction.

We have

$$H(\text{sym Curl}) \subset H(\text{dev sym Curl}),$$

and by Theorem 4.1 of [5] the two spaces are not equal when $\Omega$ is bounded. Curiously, however, Observation 2.3 in [5] shows that

$$\text{sym}(U \text{Anti} n) = 0 \iff \text{dev sym}(U \text{Anti} n) = 0$$

for any $U \in \mathbb{R}^{3 \times 3}$ and $n \in \mathbb{R}^3$, and therefore Theorem 4.1 leads to identical finite element spaces for $H(\text{sym Curl})$ and $H(\text{dev sym Curl})$. In the following we will therefore only consider elements that are $H(\text{sym Curl})$-conforming, which are then automatically $H(\text{dev sym Curl})$-conforming. Construction of finite element spaces that are $H(\text{dev sym Curl})$-conforming but not $H(\text{sym Curl})$-conforming will require nonstandard ideas.

The modern treatment of $H(\text{curl})$-conforming and related finite element spaces is a particularly beautiful part of numerical mathematics, because it fits into the framework of finite element exterior calculus [2]. This framework builds on the observation that the space $H(\text{curl})$ forms part of the de Rham complex for the classical vector calculus operators grad, curl, and div. A similar construction that involves the spaces $H(\text{sym Curl})$
CONFORMING FINITE ELEMENTS FOR $H(\text{sym Curl})$ AND $H(\text{dev sym Curl})$

or $H(\text{dev sym Curl})$ is currently under investigation. Closely related is the div Div complex of Pauly and Zulehner [8], which deals with

$$H(\text{sym Curl}) \cap \{U : \text{trace } U = 0\}$$

instead of $H(\text{sym Curl})$ itself. There is a recent construction of a finite element subcomplex of the div Div complex in [4], which reproduces the exact-sequence property of the original complex.

Besides dealing only with trace-free matrix functions, the finite element functions of [4] are remarkably complicated. In this paper we construct $H(\text{sym Curl})$-conforming finite elements of any approximation order that are much simpler. We state the construction for tetrahedra, but a generalization to hexahedra is straightforward. The elements use full polynomial spaces, and the degrees of freedom are certain directional point evaluations. We show well-posedness, conformity and $H(\text{Curl})$-nonconformity, and optimal interpolation error bounds. However, the elements do not fulfill any exact-sequence properties, and no inf–sup condition is shown either. This is because the applications envisioned in [5] are not saddle-point problems, and therefore these structural properties are of lesser importance.

The gain is a vastly simplified construction compared to [4], and finite elements with fewer degrees of freedom per element.

The continuity conditions (1) force our finite element functions to be (almost) continuous at the grid vertices. This is not surprising, as some related finite elements also require vertex continuity [1, 4]. If the grid is such that all faces meeting at a vertex have one of three normals, then we can get $H(\text{sym Curl})$-conformity with less vertex continuity. This alternative construction is described in Chapter 4.

2. Conformity-preserving degrees of freedom

The degrees of freedom of our element are certain point evaluations. As a preparatory step we therefore first consider conditions that ensure conformity at a single point.

2.1. Face degrees of freedom. We begin by conditions for points on a face of an element. To every face $F$ in the (three-dimensional) grid we associate a unit normal vector $\mathbf{n}$, whose orientation does not change throughout this manuscript. Although normality is not used by the following lemma, the vector $\mathbf{n}$ that appears there will later be that face normal.

**Lemma 2** (Conformity). Let $\mathbf{a}_1, \mathbf{a}_2, \mathbf{n}$ be a basis of $\mathbb{R}^3$, and let $U \in \mathbb{R}^{3 \times 3}$. If

$$\begin{align*}
\mathbf{a}_1^T U (\mathbf{n} \times \mathbf{a}_1) &= 0 \\
\mathbf{a}_2^T U (\mathbf{n} \times \mathbf{a}_2) &= 0 \\
\mathbf{a}_1^T U (\mathbf{n} \times \mathbf{a}_2) + \mathbf{a}_2^T U (\mathbf{n} \times \mathbf{a}_1) &= 0 \\
\mathbf{n}^T U (\mathbf{n} \times \mathbf{a}_1) &= 0 \\
\mathbf{n}^T U (\mathbf{n} \times \mathbf{a}_2) &= 0,
\end{align*}$$

then

$$\text{sym}(U \text{ Anti } \mathbf{n}) = 0. \quad (4)$$

**Proof.** Equation (4) is equivalent to

$$\text{sym}(U \text{ Anti } \mathbf{n}) : Q = 0 \quad \forall Q \in \mathbb{R}^{3 \times 3},$$

and this in turn is equivalent to

$$\begin{align*}
(U \text{ Anti } \mathbf{n}) : Q &= 0 \\
\forall Q &\in \mathbb{S}^3,
\end{align*}$$

(5)
where $S^3$ is the set of symmetric $3 \times 3$ matrices. By linearity, it is sufficient to test (5) only for six matrices $Q_1, \ldots, Q_6 \in \mathbb{R}^{3\times3}$ that form a basis of $S^3$. One such basis is
\[
Q_1 = a_1 \otimes a_1 \quad Q_2 = a_2 \otimes a_2 \quad Q_3 = \text{sym}(a_1 \otimes a_2)
\]
\[
Q_4 = \text{sym}(a_1 \otimes n) \quad Q_5 = \text{sym}(a_2 \otimes n) \quad Q_6 = n \otimes n.
\]
Indeed, these matrices are linearly independent. To see this, let $\alpha_1, \ldots, \alpha_6 \in \mathbb{R}$ be such that
\[
\sum_{i=1}^{6} \alpha_i Q_i = 0.
\]
Then, write $Q_j = AQ_j^e A^T$, where $A$ is the matrix with columns $a_1, a_2, n$, and $Q_1^e, \ldots, Q_6^e$ are the canonical basis vectors
\[
Q_1^e = e_1 \otimes e_1 \quad Q_2^e = e_2 \otimes e_2 \quad Q_3^e = \text{sym}(e_1 \otimes e_2)
\]
\[
Q_4^e = \text{sym}(e_1 \otimes e_3) \quad Q_5^e = \text{sym}(e_2 \otimes e_3) \quad Q_6^e = e_3 \otimes e_3.
\]
Equation (6) implies $A(\sum_{i=1}^{6} \alpha_i Q_i^e)A^T = 0$, and as $A$ is invertible and the $Q_i^e$ are trivially linearly independent, it follows that $\alpha_1 = \ldots = \alpha_6 = 0$.

As it turns out, not all six test matrices are required. Note that for any two vectors $a, b \in \mathbb{R}^3$ we get
\[
(U \text{ Anti } n) : (a \otimes b) = \sum_{i,j=1}^{3} (U \text{ Anti } n)_{ij} \cdot a_i b_j = a^T(U \text{ Anti } n)b,
\]
and therefore
\[
(U \text{ Anti } n) : Q_6 = n^T(U \text{ Anti } n)n = 0.
\]
As a consequence, the other five basis vectors are enough to ensure (4). Using (7) to compute
\[
(U \text{ Anti } n) : Q_1 = a_1^T(U \text{ Anti } n)a_1 = a_1^T U(n \times a_1)
\]
\[
(U \text{ Anti } n) : Q_2 = a_2^T(U \text{ Anti } n)a_2 = a_2^T U(n \times a_2)
\]
\[
(U \text{ Anti } n) : Q_3 = \frac{1}{2} [a_1^T(U \text{ Anti } n)a_2 + a_2^T(U \text{ Anti } n)a_1]
\]
\[
= \frac{1}{2} [a_1^T U(n \times a_2) + a_2^T U(n \times a_1)]
\]
\[
(U \text{ Anti } n) : Q_4 = \frac{1}{2} [a_1^T(U \text{ Anti } n)n + n^T(U \text{ Anti } n)a_1]
\]
\[
= \frac{1}{2} n^T U(n \times a_1)
\]
\[
(U \text{ Anti } n) : Q_5 = \frac{1}{2} [a_2^T(U \text{ Anti } n)n + n^T(U \text{ Anti } n)a_2]
\]
\[
= \frac{1}{2} n^T U(n \times a_2)
\]
we get the assertion. $\square$

Note how we would get the same set of equations when trying to satisfy the seemingly weaker condition $\text{dev sym}(U \text{ Anti } n) = 0$. In that case, provided that $a_1, a_2, n$ have equal length, one possible set of test matrices for the five-dimensional space $\{Q \in S^3 : \text{trace } Q = 0\}$ would be $Q_3, Q_4, Q_5$ from above, together with $a_1 \otimes a_1 - a_2 \otimes a_2$ and $a_2 \otimes a_2 - n \otimes n$. However, by (6), testing with the latter is equivalent to testing with $a_2 \otimes a_2 = Q_2$, which, together with $a_1 \otimes a_1 - a_2 \otimes a_2$ spans the same space as $Q_1$ and $Q_2$. In view of (5), this is no surprise.
Lemma 3 (Unisolvence). If the five conditions of Lemma 2 hold, and if additionally
\[ a_1^T U(n \times a_2) - a_2^T U(n \times a_1) = 0 \quad a_1^T U n = 0 \]
\[ a_2^T U n = 0 \quad n^T U n = 0, \]
then
\[ U = 0. \]

Proof. By adding and subtracting the third condition of Lemma 2 and the first condition of Lemma 3, we find that the two are equivalent to
\[ a_1^T U(n \times a_2) = 0 \quad \text{and} \quad a_2^T U(n \times a_1) = 0. \]

Let \( A \) be the matrix with columns \( a_1, a_2, n \), and \( B \) the matrix with columns \( a_1 \times n, a_2 \times n, n \). Then the nine conditions can be written in matrix form
\[ A^T U B = 0. \]

\( A \) is invertible because the columns \( a_1, a_2, n \) form a basis. To see that \( B \) is invertible, note first that both \( n \times a_1 \) and \( n \times a_2 \) are in the plane orthogonal to \( n \). Secondly, they are not collinear, because otherwise \( a_1, a_2, n \) would all be in a common plane. Multiplying (9) with \( A^{-T} \) from the left and with \( B^{-1} \) from the right then yields the assertion. \( \square \)

2.2. Edge degrees of freedom. We now consider values on an edge \( E \) of an element. Here, the conformity conditions for the two adjacent faces \( F_1 \) and \( F_2 \) interact. We equip every edge of the grid with a unit tangent vector \( t_E \), and two further vectors \( n_{E,1} \) and \( n_{E,2} \) that span the plane orthogonal to \( E \). For each pair \((F_i, E)\) of face \( F_i \), \( i = 1, 2 \) and adjacent edge \( E \) we define a unit conormal \( \partial_i := t_E \times n_i \). The conormal \( \partial_i \) is tangent to \( F_i \) and orthogonal to \( t_E \).

To state conformity-ensuring conditions we again use Lemma 2. For face \( F_1 \) with normal \( n_1 \) we set \( a_1 = t_E \) and \( a_2 = \partial_1 \). For face \( F_2 \) with normal \( n_2 \) we set \( a_1 = t_E \) and \( a_2 = \partial_2 \). The conditions for the two faces are:

| \( F_1 \) | \( F_2 \) |
|------------------|------------------|
| \( t_E^T U(n_1 \times t_E) = 0 \) | \( t_E^T U(n_2 \times t_E) = 0 \) |
| \( \partial_1^T U(n_1 \times \partial_1) = 0 \) | \( \partial_1^T U(n_2 \times \partial_2) = 0 \) |
| \( \partial_1^T U(n_1 \times t_E) = 0 \) | \( \partial_1^T U(n_2 \times t_E) = 0 \) |
| \( n_1^T U(n_1 \times \partial_1) = 0 \) | \( n_2^T U(n_2 \times \partial_2) = 0 \) |

Together, these are more conditions than there are variables, but as it turns out we can unify the two pairs of conditions marked by arrows. We first assign the others to the faces meeting at \( E \). We get

- For the face \( F_1 \):
  \begin{align*}
  (10a) & \quad t_E^T U(n_1 \times t_E) = 0 \\
  (10b) & \quad t_E^T U(n_1 \times \partial_1) + \partial_1^T U(n_1 \times t_E) = 0 \\
  (10c) & \quad n_1^T U(n_1 \times t_E) = 0
  \end{align*}
For the face $F_2$:

\begin{align*}
(11a) \quad & t_E^T U(n_2 \times t_E) = 0 \\
(11b) \quad & t_E^T U(n_2 \times \partial_2) + \partial_2^T U(n_2 \times t_E) = 0 \\
(11c) \quad & n_2^T U(n_2 \times t_E) = 0
\end{align*}

As $n_1 \times \partial_1 = n_2 \times \partial_2 = t_E$ by definition of the conormal, the remaining four conditions are equivalent to requiring that $U t_E \in \mathbb{R}^3$ is collinear to $t_E$. This can be stated without explicit reference to the faces $F_1$ and $F_2$ by replacing the four conditions with

\begin{align*}
\text{(12)} \quad & n_{E,1}^T U t_E = 0 \\
& n_{E,2}^T U t_E = 0,
\end{align*}

where $n_{E,1}$ and $n_{E,2}$ are the two vectors associated to the edge $E$ that span the normal space of $t_E$.

By construction, for either face, the face and edge conditions are enough to control conformity.

**Lemma 4 (Conformity).** Let $U \in \mathbb{R}^{3 \times 3}$, and let $F$ be a face with normal $n$, bordering the edge $E$. If the three face conditions

\begin{align*}
& t_E^T U(n \times t_E) = 0 \\
& t_E^T U(n \times \partial) + \partial^T U(n \times t_E) = 0 \\
& n^T U(n \times t_E) = 0
\end{align*}

and the two edge conditions \text{(12)} hold, then $\text{sym}(U \text{ Anti } n) = 0$.

**Proof.** The edge conditions \text{(12)} are equivalent to

\begin{align*}
& \partial^T U(n \times \partial) = 0 \\
& n^T U(n \times \partial) = 0.
\end{align*}

Together with the three face conditions they form a set of conditions as given by Lemma \text{2} for the three vectors $t_E$, $\partial$, $n$. The assertion then follows from Lemma \text{2}.

However, the eight conditions for $F_1$, $F_2$, and $E$ together are not enough to uniquely determine the value of $U \in \mathbb{R}^{3 \times 3}$. The joint kernel is spanned by the identity matrix. Indeed, let $U$ be the identity matrix. Then $t_E^T U(n_1 \times t_E) = t_E^T (n_1 \times t_E) = 0$, and likewise for the five other conditions consisting of only one addend. For the remaining condition for face $F_1$ we get

\[ t_E^T U(n_1 \times \partial_1) + \partial_1^T U(n_1 \times t_E) = t_E^T (n_1 \times \partial_1) + t_E^T (\partial_1 \times n_1) = 0, \]

by invariance under circular shift of the triple product, and likewise for face $F_2$. We therefore need one further condition to control multiples of the identity matrix. A suitable choice is

\[ t_E^T U t_E = 0. \]

This will later turn into a degree of freedom assigned to the element.

**Lemma 5 (Unisolvence).** Let $U \in \mathbb{R}^{3 \times 3}$ be such that the conditions \text{(10a)} - \text{(10c)}, \text{(11a)} - \text{(11c)}, the two conditions \text{(12)}, and the condition \text{(13)} holds. Then $U = 0$.

**Proof.** Suppose that all nine conditions hold. Subtracting \text{(13)} from \text{(10b)} and \text{(11b)} yields

\[ t_E^T U t_E = 0 \quad \partial_1^T U(n_1 \times t_E) = 0 \quad \partial_2^T U(n_2 \times t_E) = 0. \]
These nine conditions can then be written as three vector equations

\[
\begin{pmatrix}
\frac{t_E^T}{n_1^T} \\
\partial_1^T
\end{pmatrix} U(n_1 \times t_E) = 0 \in \mathbb{R}^3
\]

\[
\begin{pmatrix}
\frac{t_E^T}{n_2^T} \\
\partial_2^T
\end{pmatrix} U(n_2 \times t_E) = 0 \in \mathbb{R}^3
\]

\[
\begin{pmatrix}
\frac{t_E^T}{n_{2,1}^T} \\
\frac{t_E^T}{n_{2,2}^T}
\end{pmatrix} U t_E = 0 \in \mathbb{R}^3.
\]

Each of the matrices on the left is invertible, and we therefore conclude that

\[U(n_1 \times t_E) = U(n_2 \times t_E) = U t_E = 0 \in \mathbb{R}^3.\]

This can be rewritten as

\[U \left( n_1 \times t_E \left| n_2 \times t_E \left| t_E \right. \right. \right) = 0 \in \mathbb{R}^{3 \times 3},\]

and since the vectors \( n_1 \times t_E, n_2 \times t_E \) and \( t_E \) are linearly independent we obtain that \( U = 0 \).

2.3. **Vertex degrees of freedom.** Let \( V \) be a vertex of an element, and let \( F_1, F_2, F_3 \) be the faces that meet at \( V \). The three corresponding face normals \( n_1, n_2, n_3 \) form a basis, and we can therefore formulate the conformity conditions of Lemma 2.2 for each of the three faces in terms of \( n_1, n_2, n_3 \). As two conditions can be shared for each pair of adjacent faces, we obtain nine degrees of freedom in total, and a natural assignment to the edges and faces at the vertex \( V \):

\[
\begin{align*}
n_2^T U(n_1 \times n_1) + n_3^T U(n_1 \times n_2) &= 0 \\
n_1^T U(n_1 \times n_2) &= 0 \\
n_2^T U(n_1 \times n_2) &= 0 \\
n_3^T U(n_1 \times n_3) &= 0 \\
n_1^T U(n_1 \times n_3) &= 0 \\
n_2^T U(n_1 \times n_3) &= 0
\end{align*}
\]

\[
\begin{align*}
n_3^T U(n_2 \times n_1) + n_1^T U(n_2 \times n_2) &= 0 \\
n_2^T U(n_2 \times n_3) &= 0 \\
n_1^T U(n_2 \times n_3) &= 0 \\
n_3^T U(n_3 \times n_2) &= 0 \\
n_1^T U(n_3 \times n_2) &= 0 \\
n_2^T U(n_3 \times n_2) &= 0
\end{align*}
\]

Using the same trick as in Section 2.2, the edge conditions can be formulated in terms of the edge tangents \( t_{E,1} \) and edge normal vectors \( n_{E,1}, n_{E,2} \). However, the nine conditions do not form a unisolvent set, because they do not control the identity matrix. This is because the three face degrees of freedom are not independent—any two imply the third one.
A natural way out would be to take any two, and assign them to the vertex itself. However, for an unstructured grid it is unclear how to do this in a way that is independent of the geometry of $T$. We are therefore forced to abandon the approach and force full continuity at the vertex—with the exception of the subspace spanned by the identity matrix. To this end we set up the continuity conditions

$$\begin{align*}
U_{12} &= 0 & U_{13} &= 0 & U_{23} &= 0 \\
U_{21} &= 0 & U_{31} &= 0 & U_{32} &= 0
\end{align*}$$

for the off-diagonals, together with

$$\begin{align*}
U_{11} - U_{22} &= 0 & U_{22} - U_{33} &= 0
\end{align*}$$

for the diagonal entries, and

$$U_{11} + U_{22} + U_{33} = 0$$

for multiples of the identity.

The first two sets of conditions then directly imply conformity.

**Lemma 6** (Conformity). Let $U \in \mathbb{R}^{3 \times 3}$ be such that Conditions (14) and (15) hold. Then $\text{sym}(U \text{ Anti } n) = 0$ for any $n \in \mathbb{R}^3$.

Together with (16), which controls multiples of the identity matrix, we obtain a unisolvent set of degrees of freedom.

**Lemma 7** (Unisolvency). Let $U \in \mathbb{R}^{3 \times 3}$ be such that Conditions (14), (15), and (16) hold. Then $U = 0$.

### 3. A Tetrahedral Element Based on Complete Polynomials

We can now construct the finite element for tetrahedra. It is based on a full polynomial space, and the degrees of freedom are directional point evaluations of the types developed in the previous chapter. In what follows, $\Pi_k(T; \mathbb{R}^{3 \times 3})$ is the space of $\mathbb{R}^{3 \times 3}$-valued polynomials of order $k$ or less on a tetrahedron $T$. The definition makes use of a set of Lagrange points on $T$, which should have the usual layout. Note again that the same construction also works for hexahedral elements and the ansatz space $\Pi_k^0$ of $\mathbb{R}^{3 \times 3}$-valued functions that are $k$th-order polynomials in each local coordinate direction.

Remember that we define a unit normal $n$ for each face of the grid, and a basis of unit vectors $t_E, n_{E,1}, n_{E,2}$ for each edge $E$ such that $t_E$ is tangent to $E$, and $n_{E,1}, n_{E,2}$ span the orthogonal space of $E$. The orientation of each of these vectors is arbitrary but fixed for the entire grid.

**Definition 8.** Let $T$ be a tetrahedron in $\mathbb{R}^3$. The $k$th-order $H(\text{sym Curl})$ finite element on $T$ is the space of all functions $U \in \Pi_k(T; \mathbb{R}^{3 \times 3})$, with the following degrees of freedom:

1. For each vertex $V$ of $T$:
   
   a. The off-diagonal values of $U$ at $V$
   
   $$U \mapsto U(V)_{ij} \quad i, j = 1, 2, 3, \quad i \neq j,$$

   ($4 \times 6$ degrees of freedom),

   b. the differences between the diagonal entries
   
   $$U \mapsto U(V)_{11} - U(V)_{22} \quad \text{and} \quad U \mapsto U(V)_{22} - U(V)_{33},$$

   ($4 \times 2$ degrees of freedom),
Theorem 9. Let $U \in \Pi_k(T; \mathbb{R}^{3\times 3})$ be such that all degrees of freedom of Definition 8 are zero. Then $U \equiv 0$.

Proof. We simply show that $U$ is zero at all Lagrange points. This is shown in Lemmas 5 and 7 for face, edge, and vertex degrees of freedom, respectively. For interior degrees of freedom it follows directly from the definition.

At the same time, the degrees of freedom allow to control the required conformity.

(c) the trace of $U$ at $V$

$$U \mapsto U(V)_{11} + U(V)_{22} + U(V)_{33}$$

(4 degrees of freedom).

(2) If $k \geq 2$: For each edge $E$ of $T$, and each inner Lagrange point $L$ on $E$:

(a) The quantities

$$U \mapsto n_{E,1}^T U(L) t_E \quad \text{and} \quad U \mapsto n_{E,2}^T U(L) t_E,$$

(6 x $(k - 1) \times 2$ degrees of freedom),

(b) for each of the two adjacent faces $F_i$, $i = 1, 2$, the quantities

$$U \mapsto t_{E_i}^T U(L)(n_i \times t_E) \quad U \mapsto n_{E_i}^T U(L)(n_i \times t_E)$$

$$U \mapsto t_{E_i}^T U(L)(n_i \times \partial_i) + \partial_i^T U(L)(n_i \times t_E)$$

where $\partial_i$ is the conormal to $F_i$ at $L$, with orientation such that $t_E = n_i \times \partial_i$.

(6 x $(k - 1) \times 2 \times 3$ degrees of freedom),

(c) the quantity

$$U \mapsto t_{E_i}^T U(L) t_E$$

(6 x $(k - 1)$ degrees of freedom).

(3) If $k \geq 3$: For each face $F$ of $T$ with normal $n$ and two vectors $a_1$, $a_2$ (not necessarily tangent, but such that $a_1$, $a_2$, $n$ are linearly independent):

(a) For each inner Lagrange point $L$ of $F$ the quantities

$$U \mapsto a_1^T U(L)(n \times a_1) \quad \text{and} \quad U \mapsto a_2^T U(L)(n \times a_2)$$

$$U \mapsto a_1^T U(L)(n \times a_2) + a_2^T U(L)(n \times a_1)$$

$$U \mapsto n^T U(L)(n \times a_1) \quad \text{and} \quad U \mapsto n^T U(L)(n \times a_2)$$

(4 x $\binom{k-1}{2}$ x 5 degrees of freedom),

(b) the quantities

$$U \mapsto a_1^T U(n \times a_2) - a_2^T U(n \times a_1) \quad U \mapsto a_1^T Un$$

$$U \mapsto a_2^T Un \quad \text{and} \quad U \mapsto n^T Un$$

(4 x $\binom{k-1}{2}$ x 4 degrees of freedom).

(4) If $k \geq 4$: For each inner Lagrange point $L$ of $T$: The entries of $U(L)$ ($\binom{k-1}{3}$ x 9 degrees of freedom).

Note that the list contains $\binom{k+3}{3} \times 9$ degrees of freedom in total, which is precisely the dimension of the polynomial space $\Pi_k(T; \mathbb{R}^{3\times3})$. To prove unisolvence we therefore show that $U \in \Pi_k(T; \mathbb{R}^{3\times3})$ is zero if all degrees of freedom are zero.
Theorem 10 (Conformity). Let $F$ be a face of $T$ with normal $\mathbf{n}$. Let $U \in \Pi_k(T; \mathbb{R}^{3 \times 3})$ be such that the degrees of freedom of types (1\text{a}), (1\text{b}), (2\text{a}), (2\text{b}), and (3\text{a}) corresponding to $F$ are zero. Then
\[ \text{sym}(U \text{ Anti } \mathbf{n}) = 0 \]
on $F$.

Proof. The function $\text{sym}(U \text{ Anti } \mathbf{n})$ is a polynomial of degree $k$. Therefore its restriction to $F$ is determined uniquely by its values on the $\binom{k+2}{2}$ Lagrange nodes on $F$. The assertion holds for the face boundary Lagrange points by Lemmas 4 and 8 and for the inner points of $F$ by Lemma 2. \qed

From their description in Definition 8, the degrees of freedom have natural associations to faces, edges, etc. of the tetrahedron $T$. However, in the definition of the global finite element space we only identify those degrees of freedom that control conformity:

Definition 11. Let $\mathcal{T}$ be a conforming grid of $\Omega$. The finite element space $V_{h,k}^\text{sym Curl}$ is the set
\[ V_{h,k}^\text{sym Curl} := \left\{ U_h \in L^2(\Omega; \mathbb{R}^{3 \times 3}) : U_h \in \Pi_k(T; \mathbb{R}^{3 \times 3}) \ \forall T \in \mathcal{T} \right\}, \]
with the restriction that
- vertex degrees of freedom of types (1\text{a}) and (1\text{b}) coincide for elements that share the vertex,
- edge degrees of freedom of types (2\text{a}) and (2\text{b}) coincide for elements that share the edge,
- face degrees of freedom of type (3\text{a}) coincide for elements that share the face.

Combining Theorem 10 with the characterization result of Theorem 1, we directly get the following conformity relation.

Corollary 12. $V_{h,k}^\text{sym Curl} \subset H(\text{sym Curl}) \subset H(\text{dev sym Curl})$.

Observe that Definition 11 does not require type (3\text{b}) degrees of freedom or degrees of freedom related to identity matrices (types (1\text{c}) and (2\text{c})) to match for adjacent elements, even though they are presented as belonging to the element boundary in Definition 8. These are the degrees of freedom that allow to violate $H(\text{Curl})$-conformity.

Theorem 13. For any $k \geq 1$ and any connected grid $\mathcal{T}$ with more than one element, there is a function $U_h \in V_{h,k}^\text{sym Curl}$ such that $U_h \notin H(\text{Curl})$.

Proof. Let $U_h : \Omega \to \mathbb{R}^{3 \times 3}$ be a function that is zero everywhere except on one element $T$ of $\mathcal{T}$, where $U_h$ is the identity matrix. This is a finite element function in the sense of Definition 11. Indeed, on $T$, the only nonzero degrees of freedom are of types (1\text{a}), (2\text{a}, (3\text{b}), and (4), and hence the coupling restrictions of Definition 11 are fulfilled. The function $U_h$ is not in $H(\text{Curl})$, because on any face of $T$ with normal $\mathbf{n}$ we have
\[ [U_h] \text{ Anti } \mathbf{n} = \text{ Anti } \mathbf{n} \neq 0. \]

Interpolation error bounds for the space $V_{h,k}^\text{sym Curl}$ are standard, because the polynomial space is invariant under affine transformations, and the degrees of freedom are essentially point evaluations. Optimal bounds for the interpolation error therefore follow from the standard arguments [3].

Let $\mathcal{T}$ be a tetrahedral grid, and let $W_{\mathcal{T},m,p}$ be the space of all functions $U \in H(\text{sym Curl})$ such that for each $T \in \mathcal{T}$, the restriction $U|_T$ is in $W^{m,p}(T; \mathbb{R}^{3 \times 3})$. For these
functions, if \( m \geq 2 \) and \( p \geq 2 \), the degrees of freedom of Definition 8 imply a well-defined interpolation operator.

**Theorem 14.** Let \( I_h : W_{T,m,p} \to V^\text{sym Curl}_{h,k} \) be the interpolation operator associated to the degrees of freedom of Definition 8. For any \( U \in W_{T,m,p} \) and \( 0 \leq k \leq m \), there is a constant \( c > 0 \) independent from the grid resolution and quality, such that

\[
|U - I_h U|_{W^{k,p}(\Omega)} \leq c \max_{T \in T} h_T^{-m-p+1} |U|_{W^m,p(\Omega)},
\]

where \( h_T \) is the diameter of \( T \), and \( \rho_T \) is its incircle radius.

4. A larger space on hexahedral grids with few normals

The tight conformity conditions at the vertices can be loosened a bit if all faces meeting at a vertex are normal to one of three vectors \( n_{V,1}, n_{V,2}, n_{V,3} \) associated to that vertex. In that case, these three vectors can be used to define global vertex degrees of freedom that do not couple (almost) the entire matrix as in Chapter 2.3.

Grids with the required property necessarily consist of hexahedral elements only. By far the most important example are axis-aligned grids, but there are more, in particular if curvilinear hexahedra are used.

4.1. Vertex degrees of freedom revisited. Let again \( V \) be a vertex of an element. We now assume that the vertex \( V \) comes equipped with a basis of unit vectors \( n_{V,1}, n_{V,2}, n_{V,3} \), and that for any element \( H \) adjacent to \( V \), the three faces \( F_1, F_2, F_3 \) of \( H \) at \( V \) are orthogonal to \( n_{V,1}, n_{V,2}, n_{V,3} \), respectively. Under these new circumstances we retry the failed attempt of Chapter 2.3 and formulate the conformity conditions of Lemma 2 for each of the three faces in terms of \( n_{V,1}, n_{V,2}, n_{V,3} \). Omitting the subindex \( V \) in the following we obtain nine degrees of freedom in total, and a natural assignment to the edges and faces at the vertex \( V \):

\[
\begin{align*}
n_2^T U (n_1 \times n_3) + n_3^T U (n_1 \times n_2) &= 0 \\
n_{E_1,1}^T U t_{E_1} &= 0 \\
n_{E_2,2}^T U t_{E_2} &= 0 \\
n_{E_3,2}^T U t_{E_3} &= 0
\end{align*}
\]

In contrast to the illustration in Chapter 2.3, we have here formulated the edge conditions in terms of the edge geometries. For \( i = 1, 2, 3 \), the vector \( t_{E_i} \) denotes the unit tangent vector of the edge \( E_i \) opposite of \( F_i \), and \( n_{E_i,1}, n_{E_i,2} \) are the given vectors that span the orthogonal plane of \( E_i \). That way, the edge quantities can be used as global edge degrees of freedom.
The special geometric situation comes into play for the three face conditions. Remember that they are not independent. However, the fact that the normals \( \mathbf{n}_1, \mathbf{n}_2, \mathbf{n}_3 \) are now the same for all elements adjacent to \( V \) allows to turn the face conditions into vertex degrees of freedom. For this, we pick any two of the three face conditions, e.g.,

\[
\begin{align*}
(17) & \quad \mathbf{n}_2^T U (\mathbf{n}_1 \times \mathbf{n}_3) + \mathbf{n}_3^T U (\mathbf{n}_1 \times \mathbf{n}_2) = 0 \\
\text{and} & \quad \mathbf{n}_1^T U (\mathbf{n}_2 \times \mathbf{n}_1) + \mathbf{n}_2^T U (\mathbf{n}_2 \times \mathbf{n}_3) = 0,
\end{align*}
\]

and we associate them to \( V \). This means that these quantities will be the same for all elements at \( V \), which is possible because by assumption all elements that meet at \( V \) share the same three normal vectors.

The following conformity result then again follows from Lemma \( \text{2} \).

**Lemma 15** (Conformity). Let \( U \in \mathbb{R}^{3 \times 3} \), and let \( F_i, i = 1, 2, 3 \), be a face at a vertex \( V \), with normal \( \mathbf{n}_i \). If the four edge conditions hold for the two edges adjacent to \( F_i \) and \( V \), and if additionally the conditions \((17)\) and \((18)\) hold, then \( \text{sym}(U \text{ Anti } \mathbf{n}_i) = 0 \).

**Proof.** For simplicity we show the assertion for \( F_1 \) only. The four edge conditions

\[
\begin{align*}
\mathbf{n}_{E_2,1}^T U \mathbf{t}_{E_2} &= 0 & \mathbf{n}_{E_2,1}^T U \mathbf{t}_{E_2} &= 0 \\
\mathbf{n}_{E_3,1}^T U \mathbf{t}_{E_3} &= 0 & \mathbf{n}_{E_3,1}^T U \mathbf{t}_{E_3} &= 0,
\end{align*}
\]

signify that \( U \mathbf{t}_{E_2} \) is collinear to \( \mathbf{t}_{E_2} \), and that \( U \mathbf{t}_{E_3} \) is collinear to \( \mathbf{t}_{E_3} \). As \( \mathbf{n}_1 \) and \( \mathbf{n}_3 \) are both normal to \( \mathbf{t}_{E_2} \) (without being collinear to each other), we get the equivalent conditions

\[
\begin{align*}
\mathbf{n}_1^T U (\mathbf{n}_1 \times \mathbf{n}_3) &= 0 & \mathbf{n}_3^T U (\mathbf{n}_1 \times \mathbf{n}_1) &= 0,
\end{align*}
\]

and likewise

\[
\begin{align*}
\mathbf{n}_1^T U (\mathbf{n}_1 \times \mathbf{n}_2) &= 0 & \mathbf{n}_2^T U (\mathbf{n}_1 \times \mathbf{n}_2) &= 0.
\end{align*}
\]

Together with \((17)\), \( \text{sym}(U \text{ Anti } \mathbf{n}_1) = 0 \) then follows from Lemma \( \text{2} \). The proof for face \( F_2 \) is identical. For \( F_3 \), the required fifth condition is the sum of \((17)\) and \((18)\).

To complement the eight conditions to a unisolvent set we need to additionally control the identity matrix. One suitable condition for this is

\[
\begin{align*}
\mathbf{n}_1^T U (\mathbf{n}_2 \times \mathbf{n}_3) + \mathbf{n}_2^T U (\mathbf{n}_1 \times \mathbf{n}_3) + \mathbf{n}_3^T U (\mathbf{n}_1 \times \mathbf{n}_2) &= 0.
\end{align*}
\]

This condition will later be assigned to the grid element itself, and not shared across vertices.

**Lemma 16** (Unisolvency). Let \( U \in \mathbb{R}^{3 \times 3} \) be such that the six edge conditions of a vertex \( V \), as well as conditions \((17)\), \((18)\), and \((21)\) hold. Then \( U = 0 \).

**Proof.** The argument is similar to Lemma \( \text{5} \). Subtracting \((17)\) from \((21)\) leads to \( \mathbf{n}_2^T U (\mathbf{n}_2 \times \mathbf{n}_3) = 0 \), and inserting this into \((18)\) implies \( \mathbf{n}_2^T U (\mathbf{n}_1 \times \mathbf{n}_2) = 0 \). Inserting this back into \((17)\) yields \( \mathbf{n}_2^T U (\mathbf{n}_3 \times \mathbf{n}_1) = 0 \). Rewriting the edge conditions as in \((19)\) or \((20)\), the resulting set of nine conditions can be written as \( A^T U B = 0 \), where \( A \) is the matrix with columns \( \mathbf{n}_1, \mathbf{n}_2, \) and \( \mathbf{n}_3 \), and \( B \) is the matrix with columns \( \mathbf{n}_1 \times \mathbf{n}_2, \mathbf{n}_3 \times \mathbf{n}_1, \) and \( \mathbf{n}_2 \times \mathbf{n}_3 \). As both \( A \) and \( B \) are invertible, the assertion follows.
4.2. A hexahedral element with discontinuous vertices. We can use the new set of vertex conditions to construct a larger finite element space for hexahedral grids where all faces at a vertex $V$ have one of three normals $n_{V,1}$, $n_{V,2}$, $n_{V,3}$. Let $\Pi_k^\mathbb{R}^3(T;\mathbb{R}^{3\times3})$ be the space of $\mathbb{R}^{3\times3}$-valued polynomials on a tetrahedron $T$ that are of order $k$ or less in each of the three local coordinate directions of $H$.

Definition 17. Let $H$ be a hexahedron in $\mathbb{R}^3$. The $k$th-order $H$(sym Curl) finite element on $H$ is the space of all functions $U \in \Pi_k^\mathbb{R}(H;\mathbb{R}^{3\times3})$, with the following degrees of freedom:

1. For each vertex $V$ of $H$:
   a. The quantities
      \[
      U \mapsto n_{V,2}^T U(V)(n_{V,1} \times n_{V,3}) + n_{V,3}^T U(V)(n_{V,1} \times n_{V,2})
      \]
      \[
      U \mapsto n_{V,3}^T U(V)(n_{V,2} \times n_{V,1}) + n_{V,1}^T U(V)(n_{V,2} \times n_{V,3})
      \]
      where $n_{V,1}$, $n_{V,2}$, $n_{V,3}$ are the normal vectors of the three adjacent faces ($4 \times 2$ degrees of freedom),
   b. for each edge $E_i$, $i = 1, 2, 3$, at $V$, the two quantities
      \[
      U \mapsto n_{E_{i,1}}^T U(V)t_{E_i} \quad \text{and} \quad U \mapsto n_{E_{i,2}}^T U(V)t_{E_i},
      \]
      ($4 \times 3 \times 2 = 24$ degrees of freedom),
   c. the quantity
      \[
      U \mapsto n_{T,1}^T U(V)(n_{V,2} \times n_{V,3}) + n_{T,2}^T U(V)(n_{V,1} \times n_{V,3}) + n_{T,3}^T U(V)(n_{V,1} \times n_{V,2})
      \]
      ($4$ degrees of freedom).

Edge, face, and element degrees of freedom are as in Definition 8.

This makes $(k+1)^3 \times 9$ degrees of freedom in total, which equals the dimension of the polynomial space $\Pi_k^\mathbb{R}(H;\mathbb{R}^{3\times3})$. Unisolvency and conformity of this element are proved just as in Theorems 9 and 10.

In the definition of the global finite element space, local degrees of freedom are identified across common vertices, edges or faces.

Definition 18. Let $\mathcal{T}$ be a grid such that at each vertex $V$, all adjacent faces are orthogonal to one of three unit vectors $n_{V,1}$, $n_{V,2}$, or $n_{V,3}$. The finite element space $H_{h,k}^{\text{sym Curl}}$ is the set

$$H_{h,k}^{\text{sym Curl}} := \left\{ U_h \in L^2(\Omega;\mathbb{R}^{3\times3}) : U_h \in \Pi_k^\mathbb{R}(H;\mathbb{R}^{3\times3}) \quad \forall H \in \mathcal{T} \right\},$$

with the restriction that

- vertex degrees of freedom of types 1a) and 1b) coincide for elements that share the vertex,
- edge degrees of freedom of types 2a) and 2b) coincide for elements that share the edge,
- face degrees of freedom of type 3a) coincide for elements that share the face.

This space differs from the one of Definition 11 only at the vertices. To show that the new space is larger, simply count the degrees of freedom at an inner grid vertex $V$. By construction, eight elements meet at $V$. Then Definition 11 has 8 global vertex degrees of freedom at $V$, and 8 further ones for the identity matrix components at $V$ of the 8 adjacent elements. Definition 18, in contrast, has only two global vertex degrees of freedom, the same 8 identity degrees of freedom, and additionally two degrees of freedom for each of the six adjacent edges. This makes 22 in total, in contrast to only 16 for Definition 11.
Using the same argument as in Chapter 3, we get conformity in $H(\text{sym Curl})$ and $H(\text{Curl})$-nonconformity.

**Theorem 19.** We have the subset relation

$$H_{h,k}^{\text{sym Curl}} \subset H(\text{sym Curl}) \subset H(\text{dev sym Curl}).$$

On the other hand, for any $k \geq 1$ and any connected grid $T$ with more than one element, there is a function $U_h \in H_{h,k}^{\text{sym Curl}}$ such that $U_h \notin H(\text{Curl})$.

Optimal interpolation error bounds follow along the standard arguments as for the tetrahedral element in Chapter 3.

**References**

[1] D. N. Arnold, G. Awanou, and R. Winther. “Finite Elements for Symmetric Tensors in Three Dimensions”. In: *Mathematics of Computation* 77.263 (2008), pp. 1229–1251.

[2] D. N. Arnold, R. S. Falk, and R. Winther. “Finite element exterior calculus, homological techniques, and applications”. In: *Acta Numerica* 15 (2006), pp. 1–155. DOI: 10.1017/S0962492906210018.

[3] S. Bartels. *Numerical Approximation of Partial Differential Equations*. Springer, 2016.

[4] J. Hu, Y. Liang, and R. Ma. “Conforming finite element DIVDIV complexes and the application for the linearized Einstein–Bianchi system”. In: *arXiv e-prints* (2021). arXiv: 2103.00088.

[5] P. Lewintan, S. Müller, and P. Neff. “Korn inequalities for incompatible tensor fields in three space dimensions with conformally invariant dislocation energy”. In: *arXiv e-prints* (2020). arXiv: 2011.10573.

[6] J. Nédélec. “A New Family of Mixed Finite Elements in $\mathbb{R}^3$”. In: *Numerische Mathematik* 50 (1986), pp. 57–81. DOI: 10.1007/BF01388668.

[7] J. Nédélec. “Mixed finite elements in $\mathbb{R}^3$”. In: *Numerische Mathematik* 35 (1980), pp. 315–341. DOI: 10.1007/BF01396415.

[8] D. Pauly and W. Zulehner. “The divDiv-complex and applications to biharmonic equations”. In: *Applicable Analysis* 99.9 (2020), pp. 1579–1630. DOI: 10.1080/00036811.2018.1542685.

Oliver Sander, Technische Universität Dresden, Fakultät für Mathematik, Zellescher Weg 12–14, 01069 Dresden, Germany

Email address: oliver.sander@tu-dresden.de