HARMONIC ANALYSIS ON LAGRANGIAN MANIFOLDS
OF INTEGRABLE HAMILTONIAN SYSTEMS

JULIA BERNATSKA AND PETRO HOLOD

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Abstract. For an integrable Hamiltonian system we construct a representation
of the phase space symmetry algebra over the space of functions on a Lagrangian
manifold. The representation is a result of the canonical quantization of the inte-
grable system in terms of separation variables. The variables are chosen in such
way that a half of them parameterizes the Lagrangian manifold, which coincides
with the Liouville torus of the integrable system. The obtained representation is
indecomposable and non-exponentiated.

1. Introduction

The problem of quantization on a Lagrangian manifold has arisen from the theory
of geometric quantization [4]. But the question how to choose a proper Lagrangian
manifold remains open. Dealing with a dynamical system we use its Liuoville torus
as a Lagrangian manifold. This choice guarantees that the representation space
consists of holomorphic functions - functions on the special Lagrangian manifold
whose complexification serves as a phase space of the system.

According to the orbit method one can construct an integrlable soliton hierarchy
(hierarchy of equations of soliton type) on orbits of a loop group [3]. Finite gap
phase spaces for the integrable hierarchy is appeared to consist of orbits of finite
quotient algebras corresponding to the loop group. On such phase space one can
introduce canonical variables of separation (Darbu coordinates), which represent
points of a spectral curve [2]. The curve is hyperelliptic for many interesting inter-
grable systems. A half of the variables of separation parametrizes the Lagrangian
manifold which is the Liouville torus for the integrable system in question, and the
complexified Lagrangian manifold serves as a generalized Jacobian of the spectral
curve.

Canonical quantization in terms of the variables of separation gives rise to a rep-
resentation for the symmetry group of the phase space. We construct such repre-
sentation in the space of holomorphic functions on the complexified Lagrangian
manifold, and perform a harmonic analysis of the representation for the system of isotropic Landau-Lifshits equation (for a finite gap phase space).

2. Preliminaries

We deal with systems on orbits of the loop algebra $\tilde{g} = \mathfrak{sl}(2, \mathbb{C}) \times \mathcal{P}(z, z^{-1})$. In particular, on these orbits one can construct the integrable hierarchies of modified Korteweg-de Vries equation, sinh-Gordon equation, nonlinear Schrödinger equation, and isotropic Landau-Lifshits equation, for more details see [2]. The systems obey the Lax equation
\[
\frac{dL(z)}{dt} = [A(z), L(z)], \quad \tilde{g}^* \ni L(z) = \begin{pmatrix} \alpha(z) & \beta(z) \\ \gamma(z) & -\alpha(z) \end{pmatrix}
\]
where $\alpha_N, \beta_N, \gamma_N$ are constant. The matrix $A \in \tilde{g}$ defines a hierarchy. For example, the hierarchy of Landau-Lifshits equation is obtained by means of
\[
A(z) = -\frac{1}{z} \begin{pmatrix} \alpha_1 & \beta_1 \\ \gamma_1 & -\alpha_1 \end{pmatrix} - \frac{1}{z^2} \begin{pmatrix} \alpha_0 & \beta_0 \\ \gamma_0 & -\alpha_0 \end{pmatrix}.
\]

2.1. Phase Space of the Integrable System

According to the Kostant-Adler scheme, the coadjoint action of finite quotient algebra $\mathfrak{g} \times \mathcal{P}(z, z^{-1})$ over the finite subspace $\mathcal{M}^N = \tilde{g}^* \times \mathcal{P}(1, z, \ldots, z^N)$ of $\tilde{g}^*$ produces a set of orbits $O^N \in \mathcal{M}^N$, which serves as an $N$-gap phase space of an integrable system. Choosing different $\nu$, one can construct different Hamiltonian systems generated by a series of Poisson structures.

The Lax equation guarantees that evolution of a system preserves the spectrum of matrix $L$. Thus the quantities $\text{Tr}L^k$ are automatically constants of motion, and one gets as many as the order of $L$. A half of these constants defines an orbit $O^N$, the rest forms a complete set of integrals of motion, which we call Hamiltonians.

All such systems are algebraic integrable, that is integrable in Kowalewska sense: every solution of the system admits a holomorphic continuation in time. So every solution is associated with a Riemannian surface $\mathcal{R}$. The constant spectrum provides existence of a spectral curve, which is usually defined by the equation
\[
\det(L(z) - w) = 0.
\]
The spectral curve serves as a Riemannian surface $\mathcal{R}$ from the definition of integrability in Kowalewska sense.

As mentioned above, the orbits $O^N$ form the phase space of an integrable system. On the other hand, the phase space is the Abelian torus arising as a complexification of the Liouville torus of the system. The complexified Liouville torus coincides with a generalized Jacobian of the mentioned Riemannian surface $\mathcal{R}$:

$$\widetilde{\text{Jac}}(\mathcal{R}) = \text{Symm}_N \mathcal{R} \times \mathcal{R} \times \cdots \times \mathcal{R}, \quad N > g,$$

where $g$ is the genus of $\mathcal{R}$. The necessity of generalization arises in hierarchies of soliton type equations because the number $N$ of gaps is usually greater than $g$, see [7] for finite gap systems of the nonlinear Schrödinger hierarchy.

2.2. Separation of Variables and Quantization

Original variables in the phase space are coefficients of the polynomials $\gamma$, $\beta$, $\alpha$ which are the entries of matrix $L$. The set of coefficients $\{\gamma_j; j = 0, \ldots, N\}$ are eliminated by means of orbit equations. So $\{\beta_j, \alpha_j; j = 0, \ldots, N - 1\}$ serve as independent variables, and normally they are not canonically conjugate.

In order to construct a Lagrangian manifold, it is suitable to find conjugate variables. We use the scheme from [2], its idea is the following. Let $\{z_k, w_k; k = 1, \ldots, N\}$ be a set of variables of separation. If one requires every conjugate pair $(z_k, w_k)$ be a point of the spectral curve, then $\{z_k\}$ should be the roots of polynomial $\beta$.

The proposed scheme enables to construct variables of separation. Then we define a Lagrangian manifold as the submanifold parameterized by $\{z_k; k = 1, \ldots, N\}$ (all $w_k$ are fixed), it coincides with the Liouville torus of the system in question.

Quantization in the Schrödinger picture

$$z_k \mapsto \hat{z}_k, \quad w_k \mapsto \hat{w}_k = -i \frac{\partial}{\partial z_k}, \quad \{z_k, w_l\} = \delta_{kl} \mapsto [\hat{z}_k, \hat{w}_l] = i\delta_{kl} \mathbb{I}$$

in a very natural way gives a representation of the algebra corresponding to the phase space symmetry group, which we call the phase space symmetry algebra. The obtained algebra representation is realized by differential operators of high order (higher than one), and so can not be exponentiated to a group. It happens because we restrict the domain of functions from the phase space to a Lagrangian manifold. This is the difference from the standard geometric quantization.
3. The Integrable System of Isotropic Landau-Lifshits Equation

Here we consider the 2-gap system from the hierarchy of isotropic Landau-Lifshits equation, also called the continuous Heisenberg magnetic chain:

$$\frac{\partial \mu}{\partial t} = \frac{1}{2c_0} \left[ [\mu, \frac{\partial^2 \mu}{\partial x^2}] + c_1 \frac{\partial \mu}{\partial x} \right],$$

(1)

where the vector $\mu$ describes a magnetization, $c_0, c_1$ are constants.

3.1. Phase Space, $\mathfrak{e}(3)$ Structure

The Lax matrix $L$ looks as follows

$$L(z) = \begin{pmatrix} i\mu_3(z) & \mu_1(z) - i\mu_2(z) \\ -\mu_1(z) - i\mu_2(z) & -i\mu_3(z) \end{pmatrix}$$

$$\mu_{1,2}(z) = \sum_{j=0}^{N-1} \mu_{1,2}^{(j)} z^j; \quad \mu_3(z) = \frac{1}{2} z^N + \sum_{j=0}^{N-1} \mu_3^{(j)} z^j.$$

The vector $(\mu_1^{(0)}, \mu_2^{(0)}, \mu_3^{(0)}) = \mu$ obeys the Landau-Lifshits equation (1). In the case of 2-gap system ($N = 2$) one has

$$\mu_1(z) = \mu_1^{(0)} + \mu_1^{(1)} z$$

$$\mu_2(z) = \mu_2^{(0)} + \mu_2^{(1)} z$$

$$\mu_3(z) = \mu_3^{(0)} + \mu_3^{(1)} z + z^2/2.$$

The coefficients $\{\mu_{1,2,3}^{(0)}, \mu_{1,2,3}^{(1)}\}$ serve as dynamic variables, they form a phase space, which we equip with the Poisson structure

$$\{\mu_k^{(0)}, \mu_l^{(1)}\} = 0, \quad \{\mu_k^{(0)}, \mu_l^{(0)}\} = \varepsilon_{klj} \mu_j^{(0)}, \quad \{\mu_k^{(1)}, \mu_l^{(1)}\} = \varepsilon_{klj} \mu_j^{(1)}.$$  (2)

This is $\mathfrak{e}(3)$ algebra structure, therefore the Euclidian group $\text{E}(3)$ serves as a phase space symmetry group of the system. We also call $\mathfrak{e}(3)$ the phase space symmetry algebra.

Invariance of the matrix $L$ spectrum provides constants of motion: $h_0, h_1, h_2, h_3$ obtained from the equation

$$\text{const} = -\text{Tr} L^2(z) = z^4/4 + h_3 z^3 + h_2 z^2 + h_1 z + h_0$$

$$h_0 = (\mu^{(0)}, \mu^{(0)})$$

$$h_1 = 2(\mu^{(1)}, \mu^{(0)})$$

$$h_2 = (\mu^{(1)}, \mu^{(1)}) + \mu_3^{(0)}$$

$$h_3 = \mu_3^{(1)}.$$
The functions $h_0, h_1$ annihilate the Lie-Poisson bracket, they define an orbit $O$:

$$h_0 = c_0, \quad h_1 = c_1,$$

where $c_0$ and $c_1$ are arbitrary constants. The functions $h_2, h_3$ serve as integrals of motion called Hamiltonians.

The spectral curve, which is the Riemannian surface $R$, is of genus 2:

$$z^4w^2 = z^4/4 + h_3z^3 + h_2z^2 + c_1z + c_0.$$ 

In what follows we change notations from $\mu^{(0)}$ and $\mu^{(1)}$ to $p$ and $L$ vectors:

$$\mu^{(0)} \equiv p, \quad \mu^{(1)} \equiv L.$$ 

Then the orbit equations get the form

$$p^2 = c_0 \quad (3a)$$

$$(p, L) = c_1/2. \quad (3b)$$

Evidently, the orbit is a bundle of the planes (3b) over the sphere (3a); the plane is attached to every point $p$ of the sphere. Using different values of $c_0$ and $c_1$ one obtains a set of orbits. All such orbits form the phase space of the system. There exists a degenerate orbit collapsed into the point $p = 0$, that corresponds to the case $c_0 = 0, c_1 = 0$.

In the new notations the Hamiltonians look as follows

$$h_2 = L^2 + p_3, \quad h_3 = L_3.$$ 

### 3.2. Canonical Quantization

In order to obtain a representation of the phase space symmetry algebra we use the canonical quantization (see Preliminaries). By separation of variables we prepare the system for the quantization, which gives a representation over the space of functions on the Lagrangian manifold formed by a half of conjugate variables.

Variables of separation are obtained in the following way, for more details see [2]. According to the scheme, the variables $z_1, z_2$ are roots of the polynomial $\beta$. But this is a polynomial of degree 1 in our case. The situation is improved by means of the similarity transformation

$$P^{-1}L(z)P = \begin{pmatrix}
    i\mu_2(z) & \mu_1(z) + i\mu_3(z) \\
    -\mu_1(z) + i\mu_3(z) & -i\mu_2(z)
\end{pmatrix}, \quad P = \frac{1}{\sqrt{2}} \begin{pmatrix}
    1 & -1 \\
    1 & 1
\end{pmatrix},$$
Now the polynomial $\mu_1(z) + i\mu_2(z)$ has two roots: $z_1, z_2$. The conjugate variables are calculated by the formula $w_k = i\mu_2(z_k)/z_k^2$. Explicit expressions for all dynamic variables are given below

\[
p_1 = i \left( \frac{z_1 z_2}{4} - \frac{c_0}{z_1 z_2} - \frac{z_1 z_2 (z_1 w_1 - z_2 w_2)^2}{(z_1 - z_2)^2} \right)
\]

\[
p_2 = i z_1 z_2 \frac{z_1 w_1 - z_2 w_2}{z_1 - z_2}
\]

\[
p_3 = \frac{z_1 z_2}{4} + \frac{c_0}{z_1 z_2} + \frac{z_1 z_2 (z_1 w_1 - z_2 w_2)^2}{(z_1 - z_2)^2}
\]

\[
L_1 = i \left( -\frac{z_1 + z_2}{4} - \frac{c_1}{z_1 z_2} - \frac{c_0 (z_1 + z_2)}{z_1^2 z_2^2} + \frac{z_1^2 w_1^2 - z_2^2 w_2^2}{z_1 - z_2} \right)
\]

\[
L_2 = -i \frac{z_1^2 w_1 - z_2^2 w_2}{z_1 - z_2}
\]

\[
L_3 = -\frac{z_1 + z_2}{4} + \frac{c_1}{z_1 z_2} + \frac{c_0 (z_1 + z_2)}{z_1^2 z_2^2} - \frac{z_1^2 w_1^2 - z_2^2 w_2^2}{z_1 - z_2}
\]

After the canonical quantization: $z_k \rightarrow \hat{z}_k, w_k \rightarrow \hat{w}_k = -i\partial/\partial z_k$ and checking commutation relations we come to a representation of $\epsilon(3)$. We write the algebra in the form

\[
\epsilon(3) = \{ \hat{L}_3, \hat{L}_\pm = \hat{L}_1 \pm i\hat{L}_2, \hat{p}_3, \hat{p}_\pm = \hat{p}_1 \pm i\hat{p}_2 \}
\]

\[
[\hat{L}_3, \hat{L}_\pm] = \pm \hat{L}_\pm, \quad [\hat{L}_+, \hat{L}_-] = 2\hat{L}_3, \quad [\hat{p}_3, \hat{p}_\pm] = 0, \quad [\hat{p}_+, \hat{p}_-] = 0
\]

\[
[\hat{L}_3, \hat{p}_\pm] = [\hat{p}_3, \hat{L}_\pm] = \pm \hat{p}_\pm, \quad [\hat{L}_+, \hat{p}_-] = [\hat{p}_+, \hat{L}_-] = 2\hat{p}_3.
\]

The representation of $\epsilon(3)$ is the following:

\[
\hat{L}_3 = \frac{z_1^2}{z_1 - z_2} \left( \frac{\partial^2}{\partial z_1^2} - \frac{1}{4} + \frac{c_1}{z_1} - \frac{c_0}{z_1^2} \right) - \frac{z_2^2}{z_1 - z_2} \left( \frac{\partial^2}{\partial z_2^2} - \frac{1}{4} - \frac{c_1}{z_2^2} + \frac{c_0}{z_2} \right)
\]

\[
\hat{L}_\pm = \frac{iz_1^2}{z_1 - z_2} \left( -\frac{\partial^2}{\partial z_1^2} - \frac{1}{4} + \frac{c_1}{z_1} + \frac{c_0}{z_1^2} + \frac{\partial}{\partial z_1} \right) - \frac{iz_2^2}{z_1 - z_2} \left( \frac{\partial^2}{\partial z_2^2} - \frac{1}{4} + \frac{c_1}{z_2^2} + \frac{c_0}{z_2} - \frac{\partial}{\partial z_2} \right)
\]

\[
\hat{p}_3 = -\frac{z_1 z_2}{(z_1 - z_2)^2} \left( \frac{\partial^2}{\partial z_1^2} + \frac{\partial^2}{\partial z_2^2} - 2z_1 z_2 \frac{\partial^2}{\partial z_1 \partial z_2} \right) + \frac{z_1 z_2}{4} + \frac{c_0}{z_1 z_2} + \frac{2z_1^2 z_2^2}{(z_1 - z_2)^3} \left( \frac{\partial}{\partial z_1} - \frac{\partial}{\partial z_2} \right)
\]

\[
\hat{p}_\pm = i \left[ \frac{z_1 z_2}{(z_1 - z_2)^2} \left( \frac{\partial^2}{\partial z_1^2} + \frac{\partial^2}{\partial z_2^2} - 2z_1 z_2 \frac{\partial^2}{\partial z_1 \partial z_2} \right) + \frac{z_1 z_2}{4} - \frac{c_0}{z_1 z_2} \right]
\]
\[-\frac{2z_1^2z_2^2}{(z_1 - z_2)^3} \left( \frac{\partial}{\partial z_1} - \frac{\partial}{\partial z_2} \right) \pm \frac{z_1z_2}{z_1 - z_2} \left( \frac{\partial}{\partial z_1} - \frac{z_2}{\partial z_2} \right) \].

One can easily see that \(L_\pm\) and \(L_3\) admit separation of variables, but \(p_\pm\) and \(p_3\) have not a good structure for separation.

Then we calculate the Hamiltonians, which also fit separation of variables:

\[
\hat{h}_2 = -\frac{z_1^2z_2^2}{z_1 - z_2} \left( \frac{\partial^2}{\partial z_1^2} - \frac{1}{4} - \frac{c_0}{z_1^3} - \frac{c_1}{z_1^3} \right) + \frac{z_1z_2^2}{z_1 - z_2} \left( \frac{\partial^2}{\partial z_2^2} - \frac{1}{4} - \frac{c_0}{z_2^3} - \frac{c_1}{z_2^3} \right)
\]

\[
\hat{h}_3 = -\frac{z_1^2}{z_1 - z_2} \left( \frac{\partial^2}{\partial z_1^2} - \frac{1}{4} - \frac{c_0}{z_1^3} - \frac{c_1}{z_1^3} \right) - \frac{z_2^2}{z_1 - z_2} \left( \frac{\partial^2}{\partial z_2^2} - \frac{1}{4} - \frac{c_0}{z_2^3} - \frac{c_1}{z_2^3} \right).
\]

The obtained representation of \(e(3)\) is realized by differential operators of the second order, therefore it can not be exponentiated to a group. This is a representation over the space of smooth symmetric functions on the Lagrangian manifold.

4. Representation and Harmonic Analysis

Now we come to a harmonic analysis, which we develop with respect to the subalgebra \(sl(2) \subset e(3)\). Firstly we consider the simplest case of degenerate orbit, collapsed into a point:

\[
p^2 = 0, \quad (p, L) = 0.
\]

Its spectral curve \(R\) is reduced to genus 1: \(z^2w^2 = z^2/4 + h_3z + h_2\). As a result the operators \(\hat{L}_3, \hat{L}_\pm\) decompose to the corresponding one-particle operators. Then we investigate the case of a generic orbit

\[
p^2 = c_0, \quad (p, L) = c_1/2.
\]

We construct a representation space and obtain conditions of quantization.

4.1. Degenerate Orbit: Representation Space

We start from an action of \(sl(2) = \{\hat{L}_+, \hat{L}_-, \hat{L}_3\}\). Thus, we solve the equation

\[
\hat{L}_3f(z_1, z_2) = mf(z_1, z_2)
\]

by the method of separation of variables: \(f(z_1, z_2) = W_1(z_1)W_2(z_2)\). Both functions \(W_1, W_2\) obey the same equation

\[
W'' + \left( -\frac{1}{4} - \frac{m}{z} - \frac{C}{z^2} \right) W = 0,
\]
which is the Whittaker equation with \( C = \mu^2 - 1/4 \) and solutions \( W_{-m,\mu} \). If \( C = m(m+1) \), the function \( f \) also serves as an eigenfunction of \( \hat{L}^2 \). We fix a value of \( m \) and denote it by \( J \), then \( \mu = \pm (J + 1/2) \). At \( \mu = -(J + 1/2) \) the Whittaker function has a very simple form: \( W_{-J,-J-1/2}(z) = z^{-J}e^{-z/2} \). This brings to the function

\[
\hat{f}_{J,J}(z_1, z_2) = (z_1 z_2)^{-J}e^{-(z_1+z_2)/2}
\]

(5)

which is annihilated by \( \hat{L}_+ \). We obtain the highest weight vector of the \( \mathfrak{sl}(2) \) Verma module \( \mathcal{M}^J \) produced by the action of \( \hat{L}_- \):

\[
f_{Jm}(z_1, z_2) = i^{J-m}(J-m)!(z_1 z_2)^{-J}e^{-(z_1+z_2)/2}\mathcal{L}_{J-m}^{2J-1}(z_1 + z_2)
\]

\[m = J, J-1, \ldots\]

where \( \mathcal{L}_n^\alpha \) denotes an associated Laguerre polynomial. Using the known formula

\[
\mathcal{L}_n^\alpha(z_1)\mathcal{L}_n^\alpha(z_2) = \sum_{k=0}^{n} (\alpha + k + 1) \cdots (\alpha + n) \frac{(z_1 z_2)^k}{k!} \mathcal{L}_{n-k}^{\alpha+2k}(z_1 + z_2)
\]

(6)

one can expand every function \( f_{Jm} \) into a sum of products \( W_{-m,\mu}(z_1)W_{-m,\mu}(z_2) \) over \( \mu \) from \(-(J + 1/2)\) to \(-(m + 1/2)\), that accords with the variable separation method.

The algebra \( \{\hat{L}_+, \hat{L}_-, \hat{L}_3\} \) acts in the following way:

\[
\hat{L}_3 f_{Jm} = m f_{Jm}, \quad \hat{L}_- f_{Jm} = f_{J,m-1}, \quad \hat{L}_+ f_{Jm} = (J - m)(J + m + 1) f_{J,m+1}.
\]

The obtained Verma module has the invariant subspace \( \mathcal{M}^{-J-1} \) with the highest weight vector \( f_{J,-J-1} \). Thus, a representation over the quotient \( \mathcal{V} = \mathcal{M}^J \backslash \mathcal{M}^{-J-1} \) is irreducible.

### 4.2. Degenerate Orbit: ‘Unitarization’ of \( \mathfrak{sl}(2) \) Representation

The obtained representation is not canonical. Reduction to a canonical representation we call ‘unitarization’, because normally this procedure brings to a unitary group. On account of inability to exponentiate the proposed representation we use quotation marks.

A canonical representation can be constructed by means of the intertwining operator \( \hat{A} \) defined as follows:

\[
\hat{f}_{Jm} \equiv \hat{A} f_{Jm} = \sqrt{\frac{\Gamma(J + m + 1)}{\Gamma(J - m + 1)}} f_{Jm} = i^{J-m} \sqrt{\frac{\Gamma(J + m + 1)}{\Gamma(J - m + 1)}} \frac{\Gamma(J - m + 1)}{(z_1 z_2)^{-J}e^{-(z_1+z_2)/2}\mathcal{L}_{J-m}^{2J-1}(z_1 + z_2)}.
\]
Indeed, one easily checks that \( \mathfrak{sl}(2) \) algebra has the canonical action:

\[
\hat{L}_\pm \tilde{f}_{Jm} = \sqrt{(J \mp m)(J \pm m + 1)} \tilde{f}_{J,m \pm 1}, \quad \hat{L}_3 \tilde{f}_{Jm} = m \tilde{f}_{Jm}.
\]

Also we make the basis \( \{ \tilde{f}_{Jm} ; -J \leq m \leq J, J = 0, 1, \ldots \} \) orthonormal by introducing the inner product

\[
\langle \tilde{f}_{Jm}, \tilde{f}_{Jn} \rangle = \int_0^\infty \int_0^\infty \tilde{f}_{Jm}^*(z_1, z_2) \tilde{f}_{Jn}(z_1, z_2) \times \frac{\Gamma(J - m + 1) \Gamma(J + m + 1)}{z_1^{-J+1} z_2^{-J+1} \sum_{i=0}^J \frac{\Gamma(-J+i) \Gamma(-n-i)}{i! (J-n-i)!}} = \delta_{nm}.
\]

Here we use the summation theorem and the orthogonal relation from [1]. One can observe that ‘unitarization’ by means of the intertwining operator is equivalent to the Shapovalov formula [8].

### 4.3. Degenerate Orbit: Action of \( \hat{p}_3, \hat{p}_\pm \)

With respect to the canonical representation one gets the following action of the operators \( \hat{p}_3, \hat{p}_\pm \):

\[
\begin{align*}
\hat{p}_+ \tilde{f}_{Jm} &= -i \sqrt{(J - m)(J - m - 1)} \tilde{f}_{J-1,m+1} \\
\hat{p}_3 \tilde{f}_{Jm} &= -i \sqrt{(J - m)(J + m)} \tilde{f}_{J-1,m} \\
\hat{p}_- \tilde{f}_{Jm} &= i \sqrt{(J + m)(J + m - 1)} \tilde{f}_{J-1,m-1},
\end{align*}
\]

which matches with the abstract action formulas for \( \epsilon(3) \).

### 4.4. Generic Orbit: Representation Space

In the similar way we deal with a generic orbit.

Again we start with the equation (4), and come to a more complicated equation for the functions \( W_1, W_2 \):

\[
W'' + \left( -\frac{1}{4} - \frac{m}{z} - \frac{C}{z^2} - \frac{c_1}{z^3} - \frac{c_0}{z^4} \right) W = 0.
\]

(7)

Requiring \( \hat{L}_+ W(z_1) W(z_2) = 0 \), we find the following solution of (7):

\[
W(z) = z^{-m} e^{-z/2 + a/z}
\]
with an arbitrary $a$. In order to make this function an eigenfunction of $\hat{L}^2$ we should assign $C = J(J+1) + a$, $c_0 = a^2$, $c_1 = 2a(J+1)$, we again use $J$ for the highest value of $m$. Then the highest weight vector has the form

$$f_{J,J}(z_1, z_2) = (z_1 z_2)^{-J} e^{-(z_1 + z_2)/2 + a/z_1 + a/z_2}.$$  

By the action of $\hat{L}_-$ we produce the $\mathfrak{sl}(2)$ Verma module $M^J$

$$f_{Jm}(z_1, z_2) = i^{J-m}(J - m)!(z_1 z_2)^{-J} e^{-(z_1 + z_2)/2 + a/z_1 + a/z_2} L_{J-m}^{-2J-1}(z_1 + z_2)$$

$m = J, J - 1, \ldots$

Being applied to the function $f_{Jm}$ the formula (6) does not lead to a separation variable expansion, because the function $z^{-J} e^{-z/2 + a/z} L_{J-m}^{-2J-1}$ with $m < J$ does not obey (7).

Nevertheless, we obtain a proper representation of the algebra $\mathfrak{sl}(2)$. Indeed, one can easily check:

$$\hat{L}_3 f_{Jm} = m f_{Jm}, \quad \hat{L}_- f_{Jm} = f_{J,m-1}, \quad \hat{L}_+ f_{Jm} = (J - m)(J + m + 1) f_{J,m+1},$$

that coincides with the action formulas in the case of degenerate orbit ($a = 0$).

4.5. Quantization of a Generic Orbit

As shown above, one can quantize only certain orbits: with an arbitrary value $c_0 = a^2$ one should take the fixed value $c_1 = 2a(J+1)$. The latter means that a projection of $L$ along $p$ quantizes:

$$\text{pr}_p L = J + 1.$$

This result agrees with [5], where it is proven that a phase space admits quantization if its symplectic form is integer:

$$\frac{1}{4\pi} \int_{S^2} \omega \in \mathbb{Z}.$$

Indeed, after restriction to the orbit [3] the Poisson bracket (2) becomes nonsingular, and the restricting 2-form $\omega$ is symplectic. Moreover, it is shown in [6] that

$$\frac{1}{4\pi} \int_{S^2} \omega = \frac{c_1}{2\sqrt{c_0}} = J + 1$$

for the same Poisson structure on the same orbit as we consider.
4.6. Generic Orbit: ‘Unitarization’ of sl(2) Representation

Again we need to reduce the obtained representation to the canonical form, for this purpose we use the same intertwining operator $\hat{A}$:

$$\tilde{f}_{jm} \equiv \hat{A}f_{jm} = \sqrt{\frac{\Gamma(J + m + 1)}{\Gamma(J - m + 1)}} f_{jm} = i^{J-m} \sqrt{\Gamma(J + m + 1)} \times$$

$$\times \sqrt{\Gamma(J - m + 1)} (z_1 z_2)^{-J} e^{-(z_1 + z_2)/2 + a/z_1 + a/z_2} L_{J-m}^{2J-1}(z_1 + z_2).$$

The representation space becomes Hilbert after introducing the inner product

$$\langle \tilde{f}_{jm}, \tilde{f}_{jn} \rangle = \int_0^\infty \int_0^\infty \frac{f^*_m(z_1, z_2) f_{jn}(z_1, z_2)}{\Gamma(J - m + 1) \Gamma(J + m + 1)} \times$$

$$\times e^{-2a/z_1 - 2a/z_2} \frac{dz_1 dz_2}{z_1^{J+1} z_2^{J+1}} \sum_{i=0}^{J-n} \frac{\Gamma(-J+i) \Gamma(n-i)}{n! (J-n)!} = \delta_{nm}.$$  

4.7. Generic Orbit: Action of $\hat{p}_3, \hat{p}_\pm$

With respect to the canonical representation one obtains the action of $\hat{p}_3, \hat{p}_\pm$:

$$\hat{p}_+ \tilde{f}_{jm} = -i \left(1 + \frac{a(z_1 + z_2)}{J z_1 z_2}\right) \sqrt{(J - m)(J - m - 1)} \tilde{f}_{j-1,m+1} +$$

$$+ \frac{a}{J} \sqrt{(J - m)(J + m + 1)} \tilde{f}_{j,m+1}$$

$$\hat{p}_3 \tilde{f}_{jm} = -i \left(1 + \frac{a(z_1 + z_2)}{J z_1 z_2}\right) \sqrt{(J - m)(J + m)} \tilde{f}_{j-1,m} + \frac{a}{J} m \tilde{f}_{j,m}$$

$$\hat{p}_- \tilde{f}_{jm} = i \left(1 + \frac{a(z_1 + z_2)}{J z_1 z_2}\right) \sqrt{(J + m)(J + m - 1)} \tilde{f}_{j-1,m-1} +$$

$$+ \frac{a}{J} \sqrt{(J + m)(J - m + 1)} \tilde{f}_{j,m-1}.$$  

which does not match with the abstract action formulas for $e(3)$. This situation is probably caused by the mentioned absence of a separation variable expansion.

5. Conclusion and discussion

A combination of algebraic geometry methods applied to integrable Hamiltonian systems with methods of representation theory for Lie algebras gives a new approach to harmonic analysis on a Lagrangian manifold. Dealing with an integrable
system we have a definite rule how to chose a Lagrangian manifold - it should coincides with the Liouville torus of the system. This provides holomorphic functions as a representation space. Restriction of the function domain to the Lagrangian manifold entails that the phase space symmetry algebra is represented by differential operators of high order, and so can not be exponentiated to a group. Nevertheless, there are a lot of integrable systems, among them Gaudin’s model [9], where the proposed scheme gives a good basis in the phase space.

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References

[1] Abramowitz M., Stegun I. *Handbook of Mathematical Functions*, Dover Publications, New York 1972.

[2] Bernatska J., Holod P., *On Separation of Variables for Integrable Equations of Soliton Type*, Jornal of Nonlinear Mathematical Physics 14 (2007) 353–374.

[3] Holod P., *Hamiltonian Systems on the Orbits of Affine Lie Groups and Finite-Band Integration of Nonlinear Equations*, Nonlinear and Turbulebt Processes in Physics, Vol. 3, R. Sagdeev (Ed), Harwood Academic Publishers, New York 1984, pp 1361–1367.

[4] Karasev M., *Quantization by Membranes and Integral Representations of Wave Functions*, In: Geometric Methods in Physics : Quantization and Infinite-Dimensional Systems, S. Ali, J.-P. Antoine, W. Lisiecki, I. Mladenov, A. Odzijewicz (Eds.), Plenum, New York 1994, pp.9-19.

[5] Kostant B., *Quantization and Unitary Representation*, Lecture Notes in Math. 170 (1970) 87–208.

[6] Novikov S., Shmel’tser I. *Periodic Solutions of Kirchhoff’s Equations for the Free Motion of a Rigid Body in a Fluid and the Extended Theory of Lyusternik-Shnirel’man - Morse (LSM). I*, Funct. Anal. Appl. 15 (1981) 197–207.

[7] Previato E., *Hyperelliptic Quasi-Periodic and Soliton Solution of the Nonlinear Schrodinger Equation*, Duke Math. J. 52 (1985) 323–332.

[8] Shapovalov N., *On a Bilinear Form on the Universal Enveloping Algebra of a Complex Semisimple Lie Algebra*, Funct. Anal. Appl. 6 (1972) 307–312.
[9] Sklyanin E., *Separation of Variables in the Gaudin Model*, J. Sov. Math. **47** (1989) 2473–2488.

[10] Szego G., *Orthogonal Polynomials*, Amer. Math. Soc., Rhode Island 1939

Julia Bernatska  
Department for Mathematical & Physical Sciences  
National University of Kyiv-Mohyla Academy  
2, Skovorody str.  
Kyiv 04655, Ukraine  
E-mail address: BernatskaJM@ukma.kiev.ua

Petro Holod  
Department for Mathematical & Physical Sciences  
National University of Kyiv-Mohyla Academy  
2, Skovorody str.  
Kyiv 04655, Ukraine  
E-mail address: Holod@ukma.kiev.ua