Upper bounds on $f_D$ and $f_{D_s}$ from two-point correlation function in QCD

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Abstract

The correlation function of two pseudoscalar charmed quark currents with a positive hadronic spectral density is employed to obtain upper bounds on the decay constants of $D$ and $D_s$ mesons. Including all known terms of the operator-product-expansion of this correlation function in QCD and taking into account the estimated uncertainties, we obtain $f_D < 230$ MeV and $f_{D_s} < 270$ MeV. Comparison with the decay constants determined from $D \to l\nu$ and $D_s \to l\nu$ measurements, reveals a tension between the bound and current experimental value of $f_{D_s}$. 
1 Introduction

The decay constants of charmed $D^+$ and $D_s$ mesons, defined via the hadronic matrix elements:

$$(0| \bar{d} \gamma_\mu \gamma_5 c| D^+(p)) = i f_D p_\mu, \quad (0| \bar{s} \gamma_\mu \gamma_5 c| D_s(p)) = i f_{D_s} p_\mu,$$

are extracted from the branching fractions of purely leptonic decays $D^+ \to l^+ \nu_l$ and $D_s \to l^+ \nu_l$ ($l = \mu, \tau$), respectively. Recent CLEO measurements of these decays yield $f_D$ and $f_{D_s}$ are consistent with the above intervals (see [2] for a review). Note that the observed $SU(3)_H$-violation in $D_s$ and $D$ decay constants turns out to be larger than in the light pseudoscalar mesons where $f_K = 155.5$ MeV and $f_\pi = 130.4$ MeV.

The decay constants of heavy mesons are accessible in lattice QCD. The recent results: $f_D = 207 \pm 4$ MeV and $f_{D_s} = 241 \pm 3$ MeV [5], obtained with the number of sea-quark flavours $N_f = 3$ have quite small errors. The lattice value of $f_D$ is in a good agreement with [2], whereas $f_{D_s}$ is smaller than [3]. This puzzling situation caused discussions of a possible non-standard physics (see e.g., [6] [7]). The previous $N_f = 3$ lattice QCD result [3]: $f_D = 201 \pm 3 \pm 17$ MeV, $f_{D_s} = 249 \pm 3 \pm 16$ MeV, as well as the two recent $N_f = 2$ calculations: $f_D = 205 \pm 7 \pm 7$ MeV, $f_{D_s} = 248 \pm 3 \pm 8$ MeV [9] and $f_{D_s} = 257 \pm 3 \pm 5$ MeV (preliminary) [10] quote larger errors, still their central values for $f_D$ are smaller than in [3]. Recent results for $f_{D_{(s)}}$ in quenched lattice QCD can be found in [11] [12] and a discussion of the accuracy of lattice determinations in [12].

An alternative way to calculate $f_{D_{(s)}}$ is provided by QCD sum rules [13]. The method is based on the operator-product expansion (OPE) of the two-point correlation function evaluated in deep spacelike region, in terms of perturbatively calculable Wilson coefficients and QCD vacuum condensates. The result of OPE is related with the hadronic matrix element via dispersion relation and quark-hadron duality. The QCD sum rule calculation of the heavy-meson ($D_{(s)}$) and $B_{(s)}$) decay constants has a long history, starting from [14] [15] [16]. More recent calculations using QCD sum rules are in [17] [18] [19] and finite-energy sum rules in [20]. A review of earlier results can be found in [21]. Not going into further details, let us only mention that QCD sum rules yield $f_{D_{(s)}} > f_D$, and the predicted intervals for both decay constants are in the ballpark of the lattice QCD results, albeit with larger uncertainties. In particular, a “systematic” uncertainty of QCD sum rules which is difficult to quantify in a model-independent way, is caused by the quark-hadron duality approximation. Having in mind a possible confrontation with experiment, an update of the QCD sum rule predictions for $f_{D_{(s)}}$ is a timely task which is however beyond our scope here.

The aim of this paper is to remind that the same correlation function which is used to obtain the QCD sum rule, provides upper bounds on $f_D$ and $f_{D_s}$. These bounds simply follow from the positivity of the hadronic spectral density of the correlation function and are independent of the quark-hadron duality.
approximation. In what follows, we will calculate the upper bounds for $D$ and $D_s$ meson decay constants and find that the bound on $f_{D_s}$ is on the verge of disagreement with the experimental value.

2 Derivation of the bounds

We start from the correlation function of two charmed pseudoscalar quark currents $j_5 = (m_c + m_d)\bar{d}\gamma_5 c$, the divergence of the axial-vector current in \(1\):

$$\Pi(q^2) = i \int d^4xe^{iqx} \langle 0 | T\{j_5(x)j_5^+(0)\} | 0 \rangle = \sum_{h=D,D^*,\pi,...} \frac{\langle 0 | j_5 | h \rangle \langle h | j_5^+ | 0 \rangle}{m_h^2 - q^2}. \quad (4)$$

For definiteness, the channel of the charged $D$ meson is considered, the corresponding correlation function $\Pi_s(q^2)$ for $D_s$-channel is obtained by a simple replacement of the light-quark flavour $d \rightarrow s$. On the r.h.s. of (4), the correlation function, via unitarity condition, is written as a sum over all hadronic states with $D$ quantum numbers, a very schematic representation of the dispersion integral over hadronic spectral density. The ground $D$-state contribution to (4) contains the square of the decay constant:

$$\langle 0 | j_5 | D \rangle = f_D m_D^2. \quad (5)$$

At large virtualities, $q^2 \ll m_c^2$, the correlation function (4) is dominated by short distances. In that region we approximate $\Pi(q^2)$ by OPE, including the contributions of the perturbative quark-loop diagrams (with gluon radiative corrections) and the terms with vacuum condensates. The latter are suppressed by inverse powers of $m_c^2 - q^2$ and it is sufficient to include condensates up to dimension $d \leq 6$. For virtual $c$ and light quarks propagating in the correlation function, it is convenient to adopt the \(\overline{MS}\) scheme. The most complete expressions for OPE in this scheme are presented in [17] where they were used for the QCD sum rule calculation of $f_B$ and $f_{B_s}$. The corresponding expressions for $D_{(s)}$-meson case are simply obtained by replacing the heavy quark mass $m_b \rightarrow m_c$, hence, there is no need to represent here the explicit formulae. We only write down a schematic form of the OPE result after Borel transformation:

$$\Pi(M^2) = \sum_{n=0,1,2} \int_{(m_c + m_d)^2}^{\infty} ds \left( \frac{\alpha_s}{\pi} \right)^n \rho^{(n)}(s)e^{-s/M^2}$$

$$+ \sum_{n=0,1} \left( \frac{\alpha_s}{\pi} \right)^n \Pi_{(qq)}^{(n)}(M^2) + \sum_{d=4,5,6} \Pi_d(M^2). \quad (5)$$

The Borel parameter $M^2$ replaces $q^2$ and represents the virtuality scale at which the correlation function is calculated. In the above, the perturbative contributions of quark-loop diagrams are represented in a convenient form of dispersion integrals over spectral densities, $\rho(s) = \frac{1}{\pi} \text{Im} \Pi(s)$. The $\alpha_s$-expansion of $\rho(s)$ includes the two-loop [16] and three-loop [22] gluon radiative corrections. For the latter the program ‘Rvs.m’, kindly provided by the authors of [22] was used. The second line in (5) includes the contributions of the quark condensate with the $O(\alpha_s)$ correction calculated in [17], as well as the standard gluon-, quark-gluon- and 4-quark-condensate terms of OPE denoted by the respective condensate dimension $d = 4, 5, 6$.  

\[ \text{2} \]
Borel-transforming \[ \Pi_{(s)}(M^2) \] we use the positivity of the hadronic spectral density. Hence, \[ \Pi_{(s)}(M^2) \] provides an upper limit for the ground-state \[ D_{(s)} \]-meson contribution to the hadronic sum in \[ (\text{4}) \]. The resulting upper bound for the \[ D_{(s)} \] decay constant is:

\[
f_{D_{(s)}} < \sqrt{\frac{\Pi_{(s)}(M^2)}{m^2_{D_{(s)}}}} \sqrt{\frac{m^2_{D_{(s)}}}{M^2}}.
\]  

where \[ \Pi_{(s)}(M^2) \] is calculated from \[ (\text{4}) \]. The bound is valid at any \[ M^2 \], at which one can trust the OPE \[ (\text{5}) \]. Naturally, one has to find the most restrictive value of the r.h.s. in \[ (\text{6}) \].

### 3 Numerical analysis

Numerical results for the bounds are obtained adopting the \( c \)-quark mass interval \( \tilde{m}_c = 1.29 \pm 0.03 \text{ GeV} \) which covers the recent determinations \[ (\text{23}) \] from charmonium sum rules with \( O(\alpha^4_s) \) accuracy, and, conservatively, we double the uncertainty. Note a good agreement of this determination with the recent lattice QCD result \[ (\text{23}) \] for \( \tilde{m}_c \). For the strange quark mass we adopt \( m_s(2\text{ GeV}) = 98 \pm 16 \text{ MeV} \), the interval of QCD sum rule determinations with \( O(\alpha^4_s) \) accuracy \[ (\text{23}) \]. Fixing \( m_s \) and using the ChPT relations \[ (\text{26}) \] we obtain \( m_u(2\text{ GeV}) = 2.8 \pm 0.6 \text{ MeV} \) and \( m_d(2\text{ GeV}) = 5.2 \pm 0.9 \text{ MeV} \). The quark-gluon coupling and quark masses are taken with 4-loop running, employing the program provided in \[ (\text{27}) \], with \( \alpha_s(m_Z) = 0.1176 \pm 0.002 \) \[ (\text{4}) \]. We assume equal renormalization scales for the quark masses and \( \alpha_s \). The quark condensate density is obtained using Gell-Mann-Oakes-Renner relation: \( (0|\bar{q}q|0)/(1\text{ GeV}) = -(250+16\text{ MeV})^2 \), (with \( f_\pi = (130.4 \pm 0.04 \pm 0.2) \text{ MeV} \) and \( m_\pi = 139.57 \text{ MeV} \) \[ (\text{4}) \]).

The ratio of the strange and nonstrange quark condensate densities \( (0|\bar{s}s|0)/(0|\bar{q}q|0) \) is \( (0.8 \pm 0.3)/(0.5 \pm 0.2) \), as well as the intervals for the quark-gluon, gluon and four-quark condensate densities not quoted here for brevity, are taken as in \[ (\text{29}) \], where a different correlation function using these universal parameters was calculated (see also \[ (\text{30}) \] for a review of vacuum condensates). The suppression of \( d = 4, 5, 6 \) terms in OPE makes uncertainties of the respective condensate densities inessential for our numerical results.

The upper bounds on \( f_D \) and \( f_{D_s} \) calculated from \[ (\text{6}) \] are plotted in Fig.1 as a function of \( M^2 \) at central values of all input parameters. The default renormalization scale is \( \mu = 1.5 \text{ GeV} \), close to \( \mu \sim \sqrt{m_D^2 - m_E^2} \) used in sum rule calculations. In \( \overline{\text{MS}} \) scheme the perturbative expansion works reasonably well: \( O(\alpha_s) \) and \( O(\alpha^2_s) \) terms in \[ (\text{4}) \] are, respectively, \( \leq 30\% \) and \( \leq 10\% \) of the total perturbative contribution.

As expected, the bound for \( f_{D_s} \) is larger than the one for \( f_D \). Both bounds grow and become less restrictive at larger \( M^2 \), implicitly due to increase of the relative weight of the continuum and excited states in the Borel-transformed hadronic sum. The most restrictive upper bounds

\[
f_D < 220 \text{ MeV, } f_{D_s} < 250 \text{ MeV},
\]  

\footnote{Positivity bounds of various form for this correlation function can be found in the literature starting from \[ (\text{28}) \].}

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are reached around $M^2 = 1.2$ GeV$^2$. At these values of the Borel scale, the sum of power suppressed $d = 4,5,6$ condensate contributions is less than 4% of the total $\Pi(M^2)$, hence the condensate expansion can be trusted. The scale dependence turns out to be mild, e.g., increasing the scale from $\mu = 1.5$ GeV to $\mu = 3$ GeV, one obtains the most restrictive bounds at $M^2 \simeq 1.4$ GeV$^2$, and they are only slightly (by about $+5$ MeV) shifted (see Fig. 1). Decreasing the scale up to $\mu = 1.0$ GeV, produces a more pronounced shift, by about -$15$ MeV, so that the bounds are even more restrictive. However, the NNLO correction at this scale reaches $\simeq 20\%$ of the total perturbative contribution, signalling that the perturbative expansion is less convergent numerically. Hence, to be on a conservative side, we use the bounds obtained at the default scale, including their uncertainty (at fixed $M^2$), caused by the variation $1 \text{ GeV} < \mu < 3 \text{ GeV}$, in the error budget discussed below.

The bounds have uncertainties caused by the limited accuracy of QCD parameters (quark masses, $\alpha_s$ and condensate densities) and by the scale-dependence. The individual uncertainties are estimated by varying all inputs one-by-one within the adopted intervals. The results are collected in the table:

| Variation of input | $f_D$-bound uncertainty | $f_{D_s}$-bound uncertainty |
|--------------------|--------------------------|-----------------------------|
| $m_c$              | $\pm 2.0\%$              | $\pm 2.0\%$                |
| $m_s$              | $-$                       | $\pm 1.4\%$                |
| $\alpha_s$         | $\pm 0.7\%$              | $\pm 0.7\%$                |
| $\langle 0|\bar{q}q|0\rangle$ | $-3.5\%$                  | $+2.7\%$                    |
| $\langle 0|\bar{s}s|0\rangle/\langle 0|\bar{q}q|0\rangle$ | $-$                       | $\pm 5\%$                   |
| $d=4,5,6$ condensates | $\pm 1.0\%$              | $\pm 1.0\%$                |
| scale $\mu$        | $+3.0\%$                  | $+3.0\%$                    |
| total in quadr.    | $\pm 4.8\%$              | $\pm 6.9\%$                |

Summing up separate uncertainties in quadratures, we obtain $\simeq \pm 10$ MeV ($\pm 20$ MeV) for the $f_D$ ($f_{D_s}$) bound in (7). The uncertainty of the $f_{D_s}$-bound is naturally larger, due to the variation of $m_s$ and the spread in the ratio of strange and nonstrange condensates. Conservatively, we shift the bounds (7) up by their respective total uncertainties, yielding our final estimate:

$$f_D < 230 \text{ MeV}, \quad f_{D_s} < 270 \text{ MeV}.$$  \hspace{1cm} (8)
4 Discussion

The lattice results \[5, 8, 9, 10, 11, 12\] for \(f_D\) and \(f_{D_s}\) are consistent with the upper bounds \[8\]. The decay constant of \(D\)-meson \[2\] extracted from experiment also obeys its bound. Formally, the experimental value \[3\] of \(f_{D_s}\) does not contradict \[8\] as well, especially if one takes into account the experimental errors. However, it seems quite unnatural for this bound to be almost completely saturated by the \(D_s\) contribution. In the correlation function \(\Pi_s(q^2)\) at timelike \(q^2\) there are intermediate hadronic states located above \(D_s\), starting from \(D^*K\) \((D^*_s\pi\) is forbidden by isospin symmetry), and including the radial excitations of \(D_s\) and continuum hadronic states with \(D_s\) quantum numbers. It is difficult to evaluate in a model-independent way the size of individual contributions of these states to the hadronic spectral density. Their Borel-transformed sum, a positive quantity, can be estimated, subtracting the \(D_s\)-contribution from the correlation function \(\Pi_s(M^2)\) calculated from OPE:

\[
\sum_{h=D^*K,...} \langle 0 | j_5^h | h \rangle \langle h | j_5^h \rangle | 0 \rangle e^{-m_h^2/M^2} = \Pi_s(M^2) - f_{D_s}^2 m_{D_s}^4 e^{-m_{D_s}^2/M^2},
\]

and substituting the experimentally determined \(f_{D_s}\) from \[3\]. For normalization we divide both parts of this equation by the \(D_s\) contribution and calculate the ratio:

\[
R_s(M^2) = \frac{\sum_{h=D^*K,...} \langle 0 | j_5^h | h \rangle \langle h | j_5^h \rangle | 0 \rangle e^{-m_h^2/M^2}}{f_{D_s}^2 m_{D_s}^4 e^{-m_{D_s}^2/M^2}}
\]

from \[9\], as well as the analogous ratio \(R(M^2)\) for the \(D\)-meson channel where, correspondingly, the experimental result for \(f_D\) is used. Note that in the \(SU(3)_{fl}\) symmetry limit the correlation functions for \(D_s\) and \(D\) channels and their hadronic components are equal: \(\Pi_s(M^2) = \Pi(M^2)\), \(f_{D_s} = f_D\), \(m_{D_s} = m_D\), \(R_s(M^2) = R(M^2)\), etc. Our calculation of \(\Pi_s(M^2)\) and \(\Pi(M^2)\) explicitly takes the \(SU(3)_{fl}\)-violation effects into account, via differences of \(s\)- and \(u, d\)- quark masses and \(\bar{q}q\) \((q = u, d)\) and \(s\bar{s}\) condensates. Since here we also use quite different values of \(f_D\) and \(f_D\), the results for the hadronic quantities \(R_s(M^2)\) and \(R(M^2)\), are drastically different, as seen from Fig. 2. At \(1 < M^2 < 2\) GeV\(^2\) the hadronic sum for \(D_s\) is even negative, violating unitarity. At larger \(M^2\), the share of higher states in the \(D_s\)-channel correlation function is strongly suppressed with respect to the same characteristics for \(D\)-channel. Maximizing the uncertainties in OPE (as we did it in obtaining the conservative bounds \[8\]), one can shift both curves upwards, making \(R_s\) marginally consistent with positivity, however the large \(SU(3)_{fl}\)-difference remains. A natural question is then: why is \(SU(3)_{fl}\)-symmetry so strongly violated in this correlation function?

In general, the reliability of the bounds \[3\] depends on the convergence of OPE. Including the \(O(\alpha_s^2)\) perturbative loops and condensate contributions up to \(d = 6\) we obtain a reasonable numerical convergence. One, however, cannot completely exclude, that some nonperturbative, e.g., instanton-like effects contribute to \(\Pi(M^2)\) beyond OPE, and modify the bounds. However, even if such effects influence the convergence of OPE, it is hardly possible that they simultaneously produce a drastic \(SU(3)_{fl}\)-violation. Moreover, a similar OPE in terms of perturbative loops and power-suppressed local condensates for the light-quark (pseudo)scalar currents was successfully used to obtain the lower bounds for the...
light-quark masses up to $O(\alpha_s^4)$ accuracy. If there were noticeable effects beyond OPE in the correlation function of charmed pseudoscalar currents, their role would have become more pronounced for the light-quark mass bounds, obtained with a light quark replacing the virtual $c$ quark.

5 Conclusion

Concluding, we find that the current experimental result for $f_{D_s}$ only marginally obeys the upper bound for this decay constant obtained from the OPE for the two-point correlation function. This is in contrast to the $D$-meson case where the analogous bound is consistent with $f_D$ inferred from experiment. A more precise knowledge of $c$- and $s$-quark masses and of the ratio of strange and non-strange quark condensates will make the bounds more accurate. Furthermore, the contributions of higher states to the correlation function in $D_s$ channel estimated using the experimental value of $f_{D_s}$, are strongly suppressed as compared with the corresponding contributions for the $D$ meson channel.

If the noticed difference between strange and nonstrange charmed meson channels remains in future, from QCD point of view, that could indicate a presence of unaccounted nonperturbative effects in the pseudoscalar correlation function of charmed quarks which violate not only OPE, but also $SU(3)_f$ symmetry. However such effects seem not to manifest themselves in the lattice calculations.

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