A FAMILY OF MARTINGALES GENERATED BY
A PROCESS WITH INDEPENDENT INCREMENTS

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ABSTRACT. An explicit procedure to construct a family of martingales generated by a process with independent increments is presented. The main tools are the polynomials that give the relationship between the moments and cumulants, and a set of martingales related to the jumps of the process called Teugels martingales.

1. Introduction

In this work we present an explicit procedure to generate a family of martingales from a process \( X = \{X_t, t \geq 0\} \) with independent increments and continuous in probability. We extend our results exposed in [8], where we dealt with Lévy processes (independent and stationary increments); in that case, the martingales obtained were of the form \( M_t = P(X_t, t) \), where \( P(x, t) \) is a polynomial in \( x \) and \( t \), and then they are time–space harmonic polynomials relative to \( X \). Here, the martingales constructed are polynomials on \( X_t \) but, in general, not in \( t \). Part of the paper is devoted to define the Teugels martingales of a process with independent increments; such martingales, introduced by Nualart and Schoutens [5] for Lévy processes, are a building block of the stochastic calculus with that type of processes.

2. Independent increment processes and their Teugels martingales

Let \( X = \{X_t, t \geq 0\} \) be a process with independent increments, \( X_0 = 0 \), continuous in probability and càdlàg; such processes are also called additive processes, and we will indistinctly use both names. Moreover, assume that \( X_t \) is centered, and has moment of all orders. It is well known that the law of \( X_t \) is infinitely divisible for all \( t \geq 0 \). Let \( \sigma_t^2 \) the variance of the gaussian part of \( X_t \), and \( \nu_t \) its Lévy measure; for all these notions we refer to Sato [6] or Skorohod [7].

Denote by \( \tilde{\nu} \) the (unique) measure on \( B((0, \infty) \times \mathbb{R}_0) \) defined by

\[
\tilde{\nu}((0, t] \times B) = \nu_t(B), \quad B \in B(\mathbb{R}_0),
\]

where \( \mathbb{R}_0 = \mathbb{R} \setminus \{0\} \). By an standard approximation argument, we have that for a measurable function \( f : \mathbb{R}_0 \to \mathbb{R} \), and for every \( t > 0 \),

\[
\int \int_{(0,t] \times \mathbb{R}_0} |f(x)| \tilde{\nu}(ds, dx) < \infty \iff \int_{\mathbb{R}_0} |f(x)| \nu_t(dx) < \infty,
\]

and in this case,

\[
\int \int_{(0,t] \times \mathbb{R}_0} f(x) \tilde{\nu}(ds, dx) = \int_{\mathbb{R}_0} f(x) \nu_t(dx).
\]
Note that since for every \( t \geq 0 \), \( \nu_t \) is a Lévy measure, then \( \tilde{\nu} \) is \( \sigma \)-finite. To prove this, observe that 
\[ \nu_t([|x| > 1]) < \infty, \text{ and } \nu_t((1/(n + 1), 1/n)) < \infty \text{ and } \nu_t([-1/n, -1/(n + 1))] < \infty, \ n \geq 1. \]
So there is a numerable partition of \( \mathbb{R}_0 \) with sets of finite \( \nu_t \) measure, \( \forall t > 0 \). Then, we can construct a numerable partition of \((0, \infty) \times \mathbb{R}_0\), each set with finite \( \nu \)-measure.

Write
\[ N(C) = \#\{t : (t, \Delta X_t) \in C\}, \quad C \in \mathcal{B}((0, \infty) \times \mathbb{R}_0), \]
the jump measure of the process, where \( \Delta X_t = X_t - X_{t-} \). It is a Poisson random measure on \((0, \infty) \times \mathbb{R}_0\) with intensity measure \( \tilde{\nu} \) (Sato [6, Theorem 19.2]). Define the compensated jump measure
\[ d\tilde{N}(t, x) = dN(t, x) - d\tilde{\nu}(t, x). \]

The process admits the Lévy–Itô representation
\[ X_t = G_t + \int_{(0,t] \times \mathbb{R}_0} x \, d\tilde{N}(t, x), \tag{2} \]
where \( \{G_t, \ t \geq 0\} \) is a centered continuous Gaussian process with independent increments and variance \( \mathbb{E}[G_t^2] = \sigma_t^2 \).

It is also well known the relationship between the moments of an infinitely divisible law and the moments of its Lévy measure (see Sato [6, Theorem 25.4]). In our case, as the process has moments of all orders, for all \( t \geq 0 \),
\[ \int_{\{|x| > 1\}} |x| \nu_t(dx) < \infty \quad \text{and} \quad \int_{\mathbb{R}_0} |x|^n \nu_t(dx) < \infty, \ \forall n \geq 2. \]
Write
\[ F_2(t) = \sigma_t^2 + \int_{\mathbb{R}_0} x^2 \nu_t(dx) \quad \text{and} \quad F_n(t) = \int_{\mathbb{R}_0} x^n \nu_t(dx), \quad n \geq 3. \tag{1} \]
Since \( \int_{\{|x| > 1\}} |x| \nu_t(dx) < \infty \) and \( \mathbb{E}[X_t] = 0 \), the characteristic function of \( X_t \) can be written as
\[ \phi_t(u) = \exp \left\{ -\frac{1}{2} \sigma_t^2 u^2 + \int_{\mathbb{R}_0} (e^{iux} - 1 - iux) \nu_t(dx) \right\}. \]
It is deduced that for \( n \geq 2 \), \( F_n(t) \) is the cumulant of order \( n \) of \( X_t \) (for \( n = 2 \), \( \mathbb{E}[X_t^2] = F_2(t) \)). Also \( \sigma_t^2 \) is continuous and increasing (Sato [6, Theorem 9.8]).

**Proposition 1.** The functions \( F_n(t), \ n \geq 2 \), are continuous and have finite variation on finite intervals, and for \( n \) even, they are increasing.

**Proof.**
Consider \( 0 < u < t < v \), and write \( U = [u, v] \). From the continuity in probability of \( X \),
\[ \lim_{s \to t, s \in U} X_s^n = X_t^n, \quad \text{in probability.} \]
Moreover, \( \forall s \in U, \ |X_s| \leq \sup_{r \in U} |X_r|, \) and since \( X \) is a martingale, by Doob’s inequality,
\[ \mathbb{E} \left[ \sup_{r \in U} |X_r|^n \right] \leq C \sup_{r \in U} \mathbb{E}[|X_r|^m] \leq C \mathbb{E}[|X_v|^m] < \infty. \]
So by dominated convergence it follows that the function $t \mapsto \mathbb{E}[X^n_t]$ is continuous. Since the cumulants are polynomials of the moments, it is deduced the continuity of all functions $F_n(t)$.

To prove that $F_n(t)$ has finite variation on finite intervals, consider a partition of $[0,t]: 0 < t_0 < \cdots < t_k = t$. Then

$$
\sum_{j=1}^k |F_n(t_j) - F_n(t_{j-1})| = \sum_{j=1}^k \left| \int_{(t_{j-1}, t_j] \times \mathbb{R}_0} x^n \nu(ds, dx) \right|
\leq \sum_{j=1}^k \int_{(t_{j-1}, t_j] \times \mathbb{R}_0} |x|^n \nu(ds, dx) = \int_{[0,t] \times \mathbb{R}_0} |x|^n \nu(ds, dx) < \infty.
$$

Consider the variations of the process $X$ (see Meyer [4]):

$$
X^{(1)}_t = X_t,
X^{(2)}_t = [X, X]_t = \sigma_t^2 + \sum_{0<s\leq t} (\Delta X_s)^2
X^{(n)}_t = \sum_{0<s\leq t} (\Delta X_s)^n, \ n \geq 3.
$$

By Kyprianou [2, Theorem 2.7], for $n \geq 3$ (the case $n = 2$ is similar), the characteristic function of $X^{(n)}$ is

$$
\exp \left\{ \int_{[0,t] \times \mathbb{R}_0} (e^{ixx^n} - 1) \nu(ds, dx) \right\} = \exp \left\{ \int_{\mathbb{R}_0} (e^{ixx} - 1) \nu^{(n)}_t(dx) \right\},
$$

where $\nu^{(n)}_t$ is the measure image of $\nu_t$ by the function $x \mapsto x^n$, which is a Lévy measure. So $X^{(n)}$ has independent increments. Also by Kyprianou [2, Theorem 2.7], for $n \geq 2$,

$$
\mathbb{E}[X^{(n)}_t] = F_n(t) \quad \text{and} \quad \mathbb{E}[(X^{(n)}_t)^2] = F_{2n}(t) + (F_n(t))^2.
$$

Therefore, combining the independence of the increments and the continuity of $F_n(t)$, it is deduced that $X^{(n)}$ is continuous in probability.

By Proposition 1, $F_n(t)$ has finite variation on finite intervals. Hence, the process

$$
X^{(n)}_t = F_n(t) + (X^{(n)}_t - F_n(t))
$$

is a semimartingale.

The Teugels martingales introduced by Nualart and Schoutens [5] for Lévy processes can be extended to additive processes. In the same way as in [5], these martingales are obtained centering the processes $X^{(n)}$:

$$
Y^{(1)}_t = X_t,
Y^{(n)}_t = X^{(n)} - F_n(t), \ n \geq 2,
$$

They are square integrable martingales with optional quadratic covariation

$$
[Y^{(n)}, Y^{(m)}]_t = X^{(n+m)}
$$

and, since $F_{2n}(t)$ is increasing, the predictable quadratic variation of $Y^{(n)}$ is

$$
(Y^{(n)})_t = F_{2n}(t).
$$
3. The polynomials of cumulants

The formal expression

\[
\exp \left\{ \sum_{n=1}^{\infty} \frac{\kappa_n u^n}{n!} \right\} = \sum_{n=0}^{\infty} \frac{\mu_n u^n}{n!}.
\]  

(3)

relates the sequences of numbers \( \{\kappa_n, \ n \geq 1\} \) and \( \{\mu_n, \ n \geq 0\} \). When we consider a random variable \( Z \) with moment generating function in some open interval containing 0, then both series converge in a neighborhood of 0, and (3) is the relationship between the moment generating function, \( \psi(u) = E[e^{uZ}] \), and the cumulant generating function, \( \log \psi(u) \). Moreover, \( \mu_n \) (respectively, \( \kappa_n \)) is the moment (respectively, the cumulant) of order \( n \) of \( Z \), and the well known relationship between moments and cumulants can be deduced from (3). The first three are

\[
\begin{align*}
\mu_1 &= \kappa_1, \\
\mu_2 &= \kappa_2^2 + \kappa_2, \\
\mu_3 &= \kappa_3^3 + 3\kappa_1\kappa_2 + \kappa_3, \ldots
\end{align*}
\]

If the random variable \( Z \) has only finite moments up to order \( n \), the corresponding relationship is true up to this order.

There is a general explicit expression of the moments in terms of cumulants in Kendall and Stuart [1], or formulas involving the partitions of a set, see McCullagh [3]. In general, \( \mu_n \) is a polynomial of \( \kappa_1, \ldots, \kappa_n \), called Kendall polynomial. Denote by \( \Gamma_n(x_1, \ldots, x_n), \ n \geq 1 \), this polynomial, that is, we have

\[
\mu_n = \Gamma_n(\kappa_1, \ldots, \kappa_n).
\]

Also write \( \Gamma_0 = 1 \). These polynomials enjoy very interesting properties, as the recurrence formula that follows from Stanley [9, Proposition 5.1.7]:

\[
\Gamma_{n+1}(x_1, \ldots, x_{n+1}) = \sum_{j=0}^{n} \binom{n}{j} \Gamma_j(x_1, \ldots, x_j) x_{n+1-j}.
\]  

(4)

We also have

\[
\frac{\partial \Gamma_n(x_1, \ldots, x_n)}{\partial x_j} = \binom{n}{j} \Gamma_{n-j}(x_1, \ldots, x_{n-j}), \ j = 1, \ldots, n.
\]  

(5)

Computing the Taylor expansion of \( \Gamma_n(x_1 + y, x_2, \ldots, x_n) \) at \( y = 0 \), we get the following expression that we will need later:

\[
\Gamma_n(x_1 + y, x_2, \ldots, x_n) = \sum_{j=0}^{n} \binom{n}{j} \Gamma_{n-j}(x_1, \ldots, x_{n-j}) y^j.
\]  

(6)

Interchanging the role of \( x_1 \) and \( y \), and evaluating the function at 0 we obtain

\[
\Gamma_n(x_1, x_2, \ldots, x_n) = \sum_{j=0}^{n} \binom{n}{j} \Gamma_{n-j}(0, x_2, \ldots, x_{n-j}) x_1^j.
\]  

(7)
4. A FAMILY OF MARTINGALES RELATIVE TO THE ADDITIVE PROCESS

The main result of the paper is the following Theorem:

**Theorem 1.** Let $X$ be a centered additive process with finite moments of all orders. Then the process

$$M_t^{(n)} = \Gamma_n \left( X_t, -F_2(t), \ldots, -F_n(t) \right)$$

is a martingale.

**Proof.**

Let $n \geq 2$. We apply the multidimensional Itô formula to the semimartingales $X_t$, $F_2(t), \ldots, F_n(t)$. By Proposition 1, the functions $F_2(t), \ldots, F_n(t)$ and $\sigma_t^2$ are continuous and of finite variation. From (5) and the fact that $[X, X]_t^F = \sigma_t^2$ and $[F_j, F_j]_t^F = 0$, we have

$$M_t^{(n)} = n \int_0^t M_{s-}^{(n-1)} dX_s - \sum_{j=2}^n \left( \begin{array}{c} n \\ j \end{array} \right) \int_0^t M_{s-}^{(n-j)} dF_j(s) + \frac{1}{2} n(n-1) \int_0^t M_{s-}^{(n-2)} d(\sigma_s^2)$$

$$+ \sum_{0 < s \leq t} \left( \Gamma_n (X_{s-} + \Delta X_s, -F_2(s), \ldots, -F_n(s)) - \Gamma_n (X_{s-}, -F_2(s), \ldots, -F_n(s)) \right)$$

$$- n\Delta X_s \Gamma_n (X_{s-}, -F_2(s), \ldots, -F_n(s)).$$

Applying (6),

$$\Gamma_n (X_{s-} + \Delta X_s, -F_2(s), \ldots, -F_n(s)) = \sum_{j=0}^n \left( \begin{array}{c} n \\ j \end{array} \right) M_{s-}^{(n-j)} (\Delta X_s)^j.$$

Then, the jumps part given in the expression of $M_t^{(n)}$ is

$$\sum_{0 < s \leq t} \sum_{j=2}^n \left( \begin{array}{c} n \\ j \end{array} \right) M_{s-}^{(n-j)} (\Delta X_s)^j = \sum_{j=2}^n \left( \begin{array}{c} n \\ j \end{array} \right) \int_0^t M_{s-}^{(n-j)} dX_s^{(j)} - \left( \begin{array}{c} n \\ 2 \end{array} \right) \int_0^t M_{s-}^{(n-2)} d(\sigma_s^2)$$

$$= \sum_{j=2}^n \left( \begin{array}{c} n \\ j \end{array} \right) \int_0^t M_{s-}^{(n-j)} d(Y_s^{(j)} + F_j(s)) - \left( \begin{array}{c} n \\ 2 \end{array} \right) \int_0^t M_{s-}^{(n-2)} d(\sigma_s^2).$$

Therefore,

$$M_t^{(n)} = \sum_{j=1}^n \left( \begin{array}{c} n \\ j \end{array} \right) \int_0^t M_{s-}^{(n-j)} dY_s^{(j)}. \quad (8)$$

Moreover, $(M_t^{(k)})^2$ is a polynomial in $X_t$, $F_2(t), \ldots, F_k(t)$. Taking expectations and using the relationship between moments and cumulants, and that the cumulants of $X_t$ are $F_n(t)$, $n \geq 2$, we obtain that

$$E \left[ (M_t^{(k)})^2 \right] = P(F_2(t), \ldots, F_{2k}(t)),$$

for a suitable polynomial $P$. Then, for every $t \geq 0$, we have

$$E \left[ \int_0^t (M_{s-}^{(k)})^2 d(Y_s^{(j)})_s \right] = \int_0^t E \left[ (M_{s-}^{(k)})^2 \right] dF_{2j}(s) = \int_0^t P(F_2(s), \ldots, F_{2k}(s)) dF_{2j}(s) < \infty,$$

So all the stochastic integrals on the right hand side of (8) are martingales. □
Remark 1. It is worth to note that it follows from the preceding Theorem that the function

\[ g_n(x, t) = \Gamma_n(x, -F_2(t), \ldots, -F_n(t)) \]

is a time–space harmonic function with respect to \( X_t \). By (7)

\[ g_n(x, t) = \sum_{j=0}^{n} \Gamma_{n-j}(0, -F_2(t), \ldots, F_n(t)) x^j. \]

In general, \( g_n(x, t) \) is a polynomial in \( x \). If \( F_n(t), \ n \geq 2, \) are polynomials in \( t \), then \( g_n(x, t) \) is a time–space harmonic polynomial; this happens for all Lévy processes with moments of all orders, and for some additive process; see the example below.

Example. Let \( \Lambda(t) : \mathbb{R}_+ \rightarrow \mathbb{R}_+ \) be a continuous increasing function, and \( J \) be a Poisson random measure on \( \mathbb{R}_+ \) with intensity measure \( \mu(A) = \int_A \Lambda(dt), \ A \in B(\mathbb{R}_+) \). Then the process \( X = \{X_t, t \geq 0\} \) defined pathwise,

\[ X_t(\omega) = \int_0^t J(ds, \omega) - \Lambda(t) \]

is an additive process; it is a Cox process with deterministic hazard function \( \Lambda(t) \). From the characteristic function of \( X_t \) we deduce that the Lévy measure is

\[ \nu(dx) = \Lambda(t) \delta_1(dx), \]

where \( \delta_1 \) is a Dirac delta measure concentrated in the point 1. Hence,

\[ F_n(t) = \Lambda(t), \ n \geq 2. \]

Note that the conditions that we have assumed on \( \Lambda \) are necessary to obtain an additive process, but it is not necessary (though not very restrictive) to assume that \( \Lambda \) is absolutely continuous with respect to the Lebesgue measure.

The function defined in Remark 1 is

\[ g_n(x, t) = \Gamma_n(x, -\Lambda(t), \ldots, -\Lambda(t)). \]

Hence, when \( \Lambda(t) \) is a polynomial, \( g_n(x, t) \) is a time–space harmonic polynomial.

Denote by \( \mathcal{C}_n(x, t) \) the Charlier polynomial with leading coefficient equal to 1, then (see [8])

\[ g_n(x, t) = \sum_{j=1}^{n} \lambda_j^{(n)} \mathcal{C}_j(x, \Lambda(t)), \]

where \( \lambda_j^{(n)} = 1 \) and

\[ \lambda_k^{(n)} = \sum_{j=k}^{n-1} \binom{n-1}{j} \lambda_j^{(j)}, \ k = 1, \ldots, n - 1. \]
REFERENCES

1. M. Kendall and A. Stuart, *The Advanced Theory of Statistics, Vol. 1, 4th edition*, MacMillan Publishing CO., Inc., New York, 1977.

2. A. E. Kyprianou, *Introductory Lectures on Fluctuations of Lévy Processes with Applications*, Springer, Berlin, 2006.

3. P. McCullagh, *Tensor Methods in Statistics*, Chapman and Hall, London, 1987.

4. P. A. Meyer, *Un cours sur les intégrales stochastiques*, Séminaire de Probabilités X, Springer, New York, 1976, pp. 245–400. (French)

5. D. Nualart and W. Schoutens, *Chaotic and predictable representation for Lévy processes*, Stochastic Process. Appl. 90 (2000), 109–122.

6. K. Sato, *Lévy Processes and Infinitely Divisible Distributions*, Cambridge University Press, Cambridge, 1999.

7. A. V. Skorohod, *Random Processes with Independent Increments*, Kluwer Academic Publ., Dordrecht, Boston, London, 1986.

8. J. L. Solé and F. Utzet, *Time-space harmonic polynomials relative to a Lévy process*, Bernoulli (2007).

9. R. P. Stanley, *Enumerative Combinatorics, Vol. 2*, Cambridge University Press, Cambridge, 1999.

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