A symmetry-breaking phase transition in a dynamical decision model

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Abstract. We consider a simple decision model in which a set of agents randomly choose one of two competing shops selling the same perishable products (typically food). The satisfaction of agents with respect to a given store is related to the freshness of the previously bought products. Agents select with a higher probability the store that they are most satisfied with. Studying the model from a statistical physics perspective, both through numerical simulations and mean-field analytical methods, we find a rich behaviour with continuous and discontinuous phase transitions between a symmetric phase where both stores maintain the same level of activity, and a phase with broken symmetry where one of the two shops attracts more customers than the other.

Keywords: critical phenomena of socio-economic systems, interacting agent models, stochastic processes
1. Introduction

Phase transitions and symmetry breaking are standard phenomena occurring in physical systems due to the presence of interactions between the microscopic entities involved (most often atoms or molecules) [1]. A prominent example is the ferromagnetic Ising model [2], in which microscopic magnetic moments, schematically represented as spin variables $s_i = \pm 1$, get aligned at low temperature due to the interaction energy that couples neighbouring spins. Outside physics, phase transitions are also often found in complex systems composed of many interacting ‘agents’ (virtual entities representing in a schematic way human beings, animals or bacteria for instance) [3]–[5]. In these cases, interaction rules do not derive from an energy function as in physical systems, but still correspond to direct interactions between agents.

There do however exist interesting situations in which agents have no direct interactions between them, but rather interact with an external system, having its own dynamics, and producing a feedback on the agents. A simple example of such a situation is the case of an assembly of customers who go for shopping on a regular time basis—say, every week—and need to choose between several equivalent stores. In the absence of bias between shops, that is if customers choose any of the shops with equal probability, all stores should maintain the same activity level. If, however, a fluctuation of activity occurs, one shop may become slightly less attractive because, for instance, the products sold get older (which may have significant consequences when selling, e.g., fresh food). One can intuitively expect that this feedback loop may, in some cases, result in an amplification mechanism leading to a symmetry breaking between shops.
Modelling the competition between stores is a difficult issue, which raised interest in the economic literature from different perspectives [6]–[10] as well as, marginally, in the physics literature [11]. Such models are often grounded in discrete choice theory [12,13]. Despite the central role played by prices in economic modelling, empirical studies show that pricing mechanisms for fresh products like fruits and vegetables are complex, and that other criteria related to the quality of the products also play an important role in customer decisions [14]. The freshness of products thus turns out to be an important ingredient to take into account when modelling shops selling perishable goods [7,8,15].

In the present paper, we investigate from a statistical physics viewpoint a simple model in which the ageing of products, and the corresponding decrease of utility, are the central ingredients. At variance with economic models, our approach neglects all financial aspects (like the price of products and the fixed costs that the shopkeeper has to pay), in order to single out a unique dynamical feedback mechanism (the freshness of products) between shops and customers, thus leading to a minimal dynamical model of shop competition and decision making. Focusing on the symmetry-breaking phenomenon, we find through both numerical simulations and analytical mean-field approaches that a phase transition takes place between a symmetric state in which the two stores maintain the same level of activity, and a symmetry-broken state where a majority of customers choose a given store, in which they can find fresher products.

2. The model of competing stores

2.1. Definitions

The model is defined as follows. A set of $N_a$ agents randomly visit, on a regularly time basis, one of two competing stores selling the same products. Each store has the same number $N_p$ of products, which are all identical except for their age. The age $\tau$ defines the freshness $h(\tau)$ of the corresponding product, where $h(\tau)$ is a decreasing function, taking values between 0 and 1, of the variable $\tau \geq 0$. A typical functional form for $h(\tau)$ is

$$h(\tau) = e^{-\tau/\tau_1}$$

where $\tau_1$ is the characteristic ageing time of products.

At discrete time steps $t_k = k\delta t$, with $\delta t = \tau_0/N_a$, an agent is chosen at random with uniform probability. This agent $i$ then randomly selects one of the two stores with a probability related to its satisfaction $S_{ij}$ with respect to each store $j = 1, 2$. To be more specific, the probability $p_i$ that agent $i$ chooses the first store is given by

$$p_i = \frac{1}{1 + e^{-\Delta S_i/T}}$$

where $\Delta S_i = S_{i1} - S_{i2}$ is the satisfaction difference of agent $i$ with respect to the two shops, and $T$ is a parameter playing the role of a temperature, taking into account the influence of other factors that are not described explicitly. The second store is chosen with probability $1 - p_i$. This dynamical rule preserves the symmetry between the two shops. Note that equation (2) is known as the logit rule in the socio-economic literature [12], and that it is also formally similar to the Glauber transition probability used in physics, provided $-S_i$ can be interpreted as an energy (which is not necessarily the case in social models; see e.g. [16]).

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2. Qualitative behaviour of the model

Simulating numerically the above model, we observe for low enough temperature a symmetry breaking between the two stores (see the left panel of figure 1): one of the stores reaches a high level of commercial activity, resulting in a high freshness of the sold products, while the second one only maintains a low activity and a low freshness level. Interestingly, symmetry can be restored by varying the parameter $\tau_1$, which controls the ageing rate of products, as seen in the right panel of figure 1; the two shops then retain the same level of freshness. Less surprisingly, symmetry is also restored at high temperature (not shown).

In the following sections, we quantitatively study this symmetry-breaking phase transition, through both analytical and numerical methods. Section 3 is devoted to a mean-field approach that allows us to obtain the phase diagram of the model. We find in particular a transition line in the plane defined by the temperature $T$ and a control parameter $R$ related to the characteristic ageing time $\tau_1$. Section 4 is then devoted to
the comparison between mean-field results and numerical simulations. Finally, section 5 briefly discusses an analogy with the Ising model, one of the most standard models exhibiting symmetry breaking.

3. A mean-field approach

3.1. The framework and basic dynamical equations

In this section, we present an analytical approach for the above model, in order to characterize quantitatively the phase transition between symmetric and non-symmetric states of the two stores. Our analytical framework relies on the simplifying assumption that the fluctuations of the agents’ satisfaction can be neglected, so all agents are considered to have a satisfaction difference \( \Delta S \) equal to the average value \( \bar{\Delta S} \). In socio-economic terms, one would say that the set of agents is replaced by a ‘representative agent’ [12]. To lighten the notation, we introduce

\[
\sigma \equiv \Delta S.
\]

From equation (3), we get, assuming that the time spent between two visits to a shop is exactly \( \tau_0 \) (while this is only true on average),

\[
\sigma(t + \tau_0) = \alpha \sigma(t) + (1 - \alpha)(\bar{h}_1(t) - \bar{h}_2(t)),
\]

where \( \bar{h}_j(t) \) is the average freshness of products in the store \( j \) at time \( t \). This equation can be reformulated as

\[
\frac{1}{\tau_0} (\sigma(t + \tau_0) - \sigma(t)) = \frac{1 - \alpha}{\tau_0} (-\sigma(t) + \bar{h}_1(t) - \bar{h}_2(t)).
\]

Hence \( \sigma(t) \) varies significantly over a timescale of the order of \( \tau_0/(1 - \alpha) \). In the limit of strong memory, \( 0 < 1 - \alpha \ll 1, \tau_0/(1 - \alpha) \gg \tau_0 \), so the variations of \( \sigma \) are very small over a time \( \tau_0 \). Hence \( \sigma(t) \) can be considered as a continuous function of the variable \( t \), and the lhs of equation (5) can be replaced by a time derivative. We thus end up with

\[
\frac{d\sigma}{dt} = \gamma(-\sigma(t) + \bar{h}_1(t) - \bar{h}_2(t)),
\]

with \( \gamma = (1 - \alpha)/\tau_0 \). It is convenient to express the dynamics in terms of the fraction \( p \) of customers choosing the first store, instead of keeping the variable \( \sigma \). Using equation (2) and replacing \( \Delta S_i \) by the average value \( \sigma \), we get the relation

\[
\sigma = -T \ln \left( \frac{1 - p}{p} \right),
\]

which can be plugged into equation (6), leading to

\[
\frac{dp}{dt} = \frac{\gamma}{T} p(1 - p) \left[ T \ln \left( \frac{1 - p}{p} \right) + \bar{h}_1(t) - \bar{h}_2(t) \right].
\]

We now need to compute the average freshness \( \bar{h}_j \) as a function of \( p \). To this end, we first determine the age distribution of products.
3.2. The age distribution of products

Let us denote as $\lambda_j(t) \, dt$ the average fraction of products sold in store $j$ in the time interval $[t, t + dt]$, and as $\phi_j(\tau, t)$ the probability distribution of the age $\tau$ of products in store $j$ at time $t$, normalized according to $\int_0^\infty \phi_j(\tau, t) \, d\tau = 1$. This distribution evolves in time according to the following relations:

$$\phi_j(\tau + dt, t + dt) = \phi_j(\tau, t) - \lambda_j(t) \, dt \, \phi_j(\tau, t) \quad \text{for } \tau > 0,$$

$$\phi_j(0, t) \, dt = \lambda_j(t) \, dt.$$

(9)

The second equation in (9) encodes the fact that sold products, selected at random, are immediately replaced by new ones with age $\tau = 0$. Expanding the first equation in (9) to linear order in $dt$, one finds

$$\frac{\partial \phi_j}{\partial \tau}(\tau, t) + \frac{\partial \phi_j}{\partial t}(\tau, t) = -\lambda_j(t) \, \phi_j(\tau, t).$$

(10)

Note that integrating this equation over $\tau > 0$ yields $\phi_j(0, t) \, dt = \lambda_j(t) \, dt$, consistently with the second equation in (9). The stationary solution of equation (10) is exponential, namely

$$\phi^{st}_j(\tau) = \lambda_j \, e^{-\lambda_j \tau}.$$

(11)

Using the exponential ageing function given in equation (1), the corresponding average freshness is thus

$$\bar{h}_j = \int_0^\infty d\tau \phi^{st}_j(\tau) \, e^{-\tau/\tau_1} = \left(1 + \frac{1}{\lambda_j \tau_1}\right)^{-1}.$$

(12)

In addition, the fraction $\lambda_j \, dt$ of products sold in the time interval $[t, t + dt]$ is obtained as the ratio of the number $N_a \, p_j \, dt/\tau_0$ of customers visiting shop $j$ in this interval, and of the number $N_p$ of products, yielding

$$\lambda_j = \frac{N_a \, p_j}{N_p \, \tau_0}.$$

(13)

with $p_1 = p$ and $p_2 = 1 - p$. We thus end up, as far as the average freshness is concerned, with

$$\bar{h}_j = \left(1 + \frac{R}{2p_j}\right)^{-1}$$

(14)

where we have defined the control parameter

$$R = \frac{2N_p \, \tau_0}{N_a \, \tau_1}.$$

(15)

The dimensionless ratio $R$ compares $N_p$, the total number of products available in a shop, to $(N_a/2\tau_0) \, \tau_1$, the typical number of agents visiting a shop during the characteristic ageing time of the products (assuming a symmetric state). A small value of $R$ thus corresponds to a situation where products are often renewed, while for large $R$ products are rarely selected and become old on average. In the following, we use as an approximation the above steady-state distribution of the age of products, assuming that the satisfaction of agents evolves on timescales much longer than the turnover time of products. This assumption has, however, no consequence for the determination of the fixed points of the dynamics.
3.3. Determination of the phase transition

Gathering results, we find for the time evolution of \( p \) the following equation:

\[
\frac{dp}{dt} = \frac{\gamma}{T} p(1-p) \left[ T \ln \left( \frac{1-p}{p} \right) + \left( 1 + \frac{R}{2p} \right)^{-1} - \left( 1 + \frac{R}{2(1-p)} \right)^{-1} \right].
\]  

(16)

Introducing an order parameter \( m \) through \( p = \frac{1 + m}{2} \), we can formally rewrite equation (16) as

\[
\frac{dm}{dt} = \frac{\gamma}{T} (1 - m^2) g_{T,R}(m),
\]

(17)

with \( g_{T,R}(m) \) given by

\[
g_{T,R}(m) = \frac{1}{2} \left[ T \ln \left( \frac{1+m}{1-m} \right) + \left( 1 + \frac{R}{1+m} \right)^{-1} - \left( 1 + \frac{R}{1-m} \right)^{-1} \right].
\]

(18)

The reason for excluding the factor \( \frac{\gamma(1 - m^2)}{T} \) from the definition of \( g_{T,R}(m) \) will be discussed in section 5. In order to study possible phase transitions, it is convenient to define an effective free energy (or potential function) \( f_{T,R}(m) \) through

\[
f_{T,R}(m) = -\int_0^m g_{T,R}(m') \, dm',
\]

(19)

so

\[
\frac{dm}{dt} = -\frac{df_{T,R}}{dm}(m).
\]

(20)

It is easily seen that \( m = 0 \), corresponding to the symmetric state, is a steady-state solution of equation (17). To test the local stability of this solution, we expand the free energy \( f_{T,R}(m) \) for small \( m \), to order \( m^2 \), leading to

\[
f_{T,R}(m) = \frac{1}{2} \left[ T - \frac{R}{(1+R)^2} \right] m^2 + O(m^4).
\]

(21)

The solution \( m = 0 \) is linearly stable if the bracket on the rhs is positive. The critical temperature \( T_c \) is reached when the bracket vanishes, yielding

\[
T_c = \frac{R}{(1+R)^2}.
\]

(22)

When this transition coincides with the onset of two symmetric minima for \( T < T_c \), the transition is continuous. For low values of \( R \), however, two symmetric secondary minima appear already for a value \( T_s > T_c \), when \( m = 0 \) is still the global minimum as illustrated in figure 2. This case corresponds to a discontinuous transition, which should be reached (in analogy with the equilibrium theory of phase transitions) at a temperature \( T_d \) such that all three minima have equal values \( f_{T,R}(m) = 0 \).

The resulting mean-field phase diagram is presented in figure 3. In region A, a single steady-state solution \( (m = 0) \) exists, while in region D, only the two symmetric solutions remain. In region B, between the lines \( T_s \) and \( T_d \), the symmetric solution \( m = 0 \) is the most stable, while in region C, between \( T_d \) and \( T_c \), the most stable solutions are the non-zero ones. The lines \( T_s, T_d \) and \( T_c \) meet at a value \( R^* = \sqrt{3} - 1 \approx 0.732 \).

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A phase transition in a dynamical decision model

Figure 2. Illustration of the behaviour of the free energy $f_{T,R}(m)$, for four different temperatures (from top to bottom: $T = 0.2$, 0.185, 0.16, and 0.12) and the same value $R = 0.2$, showing the onset of the two symmetric secondary minima.

The nature of the discontinuous transition however calls for a short comment. First of all, we note that the evolution equation (17) is deterministic, so the system should remain trapped forever once it has reached a locally stable stationary solution. However, in a more realistic description, the finite size of the system should result in a small effective noise, making the system explore the vicinity of the stationary state. In this case, the most stable stationary states correspond to the absolute minima of $g_{T,R}(m)$ (either $m = 0$, or two symmetric values $m_0$ and $-m_0$).

4. Comparison between mean-field and numerical results

4.1. The measure of the order parameter

We now turn to the results of numerical simulations of the agent-based model, and compare them with mean-field results. The numerical value for $m$ is obtained as a time average (over $10^4 \tau_0$) for a simulation usually involving $N_a = 500$ agents and as many products as agents ($N_p = N_a$). The memory parameter is set to $\alpha = 0.8$, a value sufficiently large for the continuous time description used in equation (6) to be approximately valid. We have however checked that setting this parameter in the range $0.5 \leq \alpha \leq 0.95$ only changes the numerical value of $m$ by less than a few per cent for most values of $R$ and $T$.

We first plot in figure 4 the steady-state value of the order parameter $m$ as a function of temperature $T$ and parameter $R$. An overall good agreement is observed with the mean-field predictions, that are obtained as follows. The fixed points of equation (17) are solutions of $g_{T,R}(m) = 0$. Besides the trivial solution $m = 0$, a solution $m_0 > 0$ exists below the transition line. An analytical expression for $m_0(T, R)$ is hard to obtain, but one can instead easily express $T$ as a function of $m$ and $R$:

$$T(m, R) = \frac{(1 + R/(1 + m))^{-1} - (1 + R/(1 - m))^{-1}}{\ln(1 + m) - \ln(1 - m)}.$$  

(23)
A phase transition in a dynamical decision model

Figure 3. Mean-field phase diagram, with four different regions. A: $m = 0$ is the only minimum of $f_{T,R}(m)$. B: $m = 0$ is the global minimum, but secondary minima have appeared. C: $m = 0$ is only a local minimum, and the two symmetric minima are the most stable. D: $m = 0$ is a local maximum, and the two symmetric minima are the only stable states. The transition lines between those regions are respectively defined by $T(R) = T_s$ between A and B, $T_d$ between B and C, and $T_c$ between C and D. $T_c$ also separates A and D for $R > R^* \approx 0.732$, when the transition is continuous. The lower panels show the typical shape of $f_{T,R}(m)$ for $m > 0$ ($f_{T,R}(m)$ is a symmetric function) in each region.

Fixing $R$, it is then straightforward to plot $T$ as a function of $m$. One can also invert $T(R, m)$ to get $R(T, m)$, as shown on figure 4.

4.2. Determination of the transition line

We also compare in figure 6 numerical results for the phase diagram with the mean-field predictions. To determine the transition line, we have distinguished two methods for the cases of continuous and discontinuous phase transitions. For the discontinuous case $(R < R^*)$, the transition point can be determined from the jump of the order parameter $m$. In the continuous case, the transition point is more difficult to determine precisely from the average value of the order parameter, due to finite size effects. We thus have used the standard technique of the Binder cumulant, which characterizes the deviation from Gaussianity of the probability distribution of the order parameter [17]. For a centred variable $m$, the Binder cumulant is defined as

$$B = 1 - \frac{\langle m^4 \rangle}{3\langle m^2 \rangle^2}. \quad (24)$$

The main interest of the Binder cumulant is that when plotted as a function of the control parameter, curves corresponding to different system sizes intersect at the critical point,
A phase transition in a dynamical decision model

Figure 4. Order parameter $m$, plotted either as a function of $R$ or as a function of temperature $T$. Lines correspond to the mean-field predictions, while dots with error bars are the results from numerical simulations. In all four figures, a phase transition, either continuous or discontinuous, is observed between a symmetric phase (the two stores have the same activity) and a phase with broken symmetry (one of the stores has a higher activity level).

with only small finite size effects. This method allows for a rather precise estimate of the transition point. An illustration of the behaviour of the Binder cumulant is given in figure 5.

Altogether, a rather good agreement is obtained between mean-field and numerical results, though some systematic deviations appear, as seen in figure 6. These deviations might be due to underevaluated finite size effects (the number of agents used in the simulation is $N_a \sim 500$), but it is more likely to result from the simplifying assumptions made in the derivation the mean-field equations. Note however that the ranges in which continuous and discontinuous transitions are observed numerically are rather well reproduced by the mean-field argument.

5. Discussion

In this section, we briefly discuss the relationship between the present model and one of the most standard models exhibiting a symmetry-breaking phase transition, namely the Ising model, before drawing some conclusions and giving perspectives.

5.1. Relation to the Ising model

In the present model, each agent has to choose between two options, namely going to the first or the second store. Such a binary choice can be encoded into a ‘spin’ variable $s_i$, equal
to 1 if agent \( i \) chooses the first shop, and to \(-1\) in the opposite case. Though somewhat artificial, this definition has the advantage that the order parameter \( m \) introduced in section 3.3 can be re-expressed as \( m = N^{-1}_a \sum_i s_i \). This naturally suggests that the present model may exhibit, to some extent, an analogy with the Ising model. This analogy can be made more precise, at the mean-field level, in the following way. Let us modify the model in such a way that the freshness of products (which drives the evolution of the satisfaction of agents) is replaced by the opposite of the energy \( E_{\text{Ising}} \) of the mean-field Ising model, with \( E_{\text{Ising}} = \frac{1}{2}N_a Jm^2 \).\(^3\) Then the mean satisfaction \( \sigma \) introduced in section 3 evolves according to

\[
\sigma(t + \tau_0) = \alpha \sigma(t) - (1 - \alpha) \Delta E,
\]

with

\[
\Delta E = -\frac{1}{2}N_a J \left[ \left( m + \frac{1}{N_a} \right)^2 - \left( m - \frac{1}{N_a} \right)^2 \right] = -2Jm.
\]

Note that the last equality has been obtained in the large \( N_a \) limit. We then get the evolution equation, in the large memory limit \( 1 - \alpha \ll 1 \):

\[
\frac{d\sigma}{dt} = \gamma [ -\sigma + 2Jm ],
\]

which is the analogue of equation (6), with here again \( \gamma = (1 - \alpha)/\tau_0 \). It is easy to check, using relation (7) to eliminate the variable \( \sigma \), that the time evolution equation for \( m \) is given by

\[
\frac{dm}{dt} = \frac{\gamma}{T} (1 - m^2) \left[ \frac{T}{2} \ln \left( \frac{1 - m}{1 + m} \right) + Jm \right].
\]

\(^3\) Strictly speaking, the magnetization \( m \) should be computed from the last choice of the \( N_a - 1 \) other agents.
Figure 6. Phase diagram ($T$–$R$), showing the transition line between the symmetric phase and the symmetry-broken phase. Lines are mean-field predictions, and dots correspond to the phase transition observed numerically, from the order parameter curve for $R < R^* \approx 0.732$ (discontinuous transition) and from the Binder cumulant for $R > R^*$ (continuous transition). The mean-field transition line (either continuous or discontinuous) is indicated by a full line; it corresponds to $T_c$ for $R > R^*$ and to $T_d$ (the set of points for which the three minima of the free energy have the same value) for $R < R^*$. In the region $R < R^*$, $T_c$ is indicated as a dashed line. The inset shows a zoom for the region where $T_c$ and $T_d$ intersect.

Like for equation (17), the term in brackets can be written as the opposite of the derivative of a free energy $f(m)$:

$$f(m) = \frac{T}{2} (1 + m) \ln (1 + m) + \frac{T}{2} (1 - m) \ln (1 - m) - \frac{1}{2} Jm^2$$

(29)

which is, up to an additive constant $T \ln 2$, nothing but the standard mean-field free energy of the Ising model.

However, beyond the mean-field case, it is clear that the relation to the Ising model is at most an analogy. Indeed, the fact that the interaction between agents is here mediated by the ageing of products in each store rather than by an internal energy function makes the present model quite different from standard equilibrium models.

5.2. Conclusion and perspectives

In this paper, we have studied a simple model of competing shops selling fresh products, retaining as main ingredients the ageing of products and the satisfaction of customers with respect to the freshness of products—considered to be the main factor influencing choices made by customers. Under these very simplifying assumptions, we found a surprisingly rich dynamics, with both continuous and discontinuous transitions between symmetric and symmetry-broken states. The overall agreement between numerical simulations and mean-field predictions was found to be satisfactory.

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A phase transition in a dynamical decision model

Several extensions could be interesting to study. For instance, other functional forms could be considered for the ageing function \(h(\tau)\) introduced in equation (1), like

\[
h(\tau) = \left(1 - \frac{\tau}{\tau_1}\right) \theta \left(1 - \frac{\tau}{\tau_1}\right)
\]

where \(\theta(x)\) is the Heaviside function. Such a form would be rather natural for products that can be sold only up to an expiry date. We have checked that the form (30) preserves the qualitative behaviour of the model. Another natural extension of this work would be to include more than two shops. Although we did not study this case in detail, a simple argument can already be given here. With a fixed overall number of agents \(N_a\), adding shops reduces the nominal number of customers for a given shop. As a first approximation, a model with more than two shops can be described as a two-shop model with an effective number of agents \(N_{a\text{eff}} < N_a\), resulting in an effective parameter \(R_{\text{eff}} > R\). If one starts from a low value of \(R\) corresponding to the symmetric region, increasing the number of shops would then eventually lead to a value \(R_{\text{eff}}\) in the symmetry-broken phase. It is thus likely that, for a given total number of agents, there is a maximal number of shops that can maintain a good level of activity, which seems to be consistent with conventional wisdom.

Further possible extensions may include taking into account explicitly the prices of products, as well as the costs associated with the shops (e.g., rents or salaries). In this context, one can imagine including, in a simplified way, a dynamics of a shopkeeper who could modify the prices, or diversify the products, as a function of the dynamical state of the store. One might guess that such a dynamics could contribute to stabilization of the symmetric states.

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