HOCHSCHILD COHOMOLOGY AND SUPPORT VARIETIES FOR TAME HECKE ALGEBRAS

SIBYLLE SCHROLL AND NICOLE SNASHALL

Abstract. We give a basis for the Hochschild cohomology ring of tame Hecke algebras. We then show that the Hochschild cohomology ring modulo nilpotence is a finitely generated algebra of Krull dimension 2, and describe the support varieties of modules for these algebras.

Introduction

Hecke algebras play an important role in representation theory. They arise as deformations of the group algebras of finite Coxeter groups and appear as endomorphism algebras of induced representations of finite or $p$-adic Chevalley groups. They also give rise to the Kazhdan-Lusztig polynomials which appear in the expression of the canonical basis in terms of the natural basis of the Hecke algebra. In particular, the Hecke algebras of type $A$ (which arise as a deformation of the group algebras of the symmetric group) have been well studied. A complete classification of the representation type of the blocks of the Hecke algebras of type $A$ was obtained in [EN]. For the tame Hecke algebras of type $A$, it was shown in [J] that there are precisely two Morita equivalence classes of blocks and they are represented by $H_q(S_4)$ and the principal block of $H_q(S_5)$ with $q = -1$. Moreover, they are in the same derived equivalence class. This follows from [M] and see also [BHS], since they are generalized Brauer tree algebras of the same type.

The algebra $H_q(S_5)$ with $q = -1$ is a Koszul symmetric special biserial algebra. Special biserial algebras occur in many aspects of representation theory and they are necessarily of finite or tame representation type. Symmetric special biserial algebras occur, for example, as Hopf algebras associated to infinitesimal groups whose principal block is tame [FS1, FS2], in the representation theory of $U_q(sl_2)$ [F, S1, X], of Drinfeld doubles of generalized Taft algebras [EGST] (and see also [ST1]) and as socle deformations of the latter [ST2]. For the symmetric special biserial algebras in [ST1] and [ST2], the Hochschild cohomology ring modulo nilpotence has been shown to be a finitely generated algebra of Krull dimension 2. In this paper we show that the same phenomenon holds for the tame Hecke algebras, so the question naturally arises as to whether or not the Hochschild cohomology ring modulo nilpotence of any symmetric special biserial algebra is a finitely generated algebra of Krull dimension at most 2.

In this paper we let $K$ be an algebraically closed field and study the Hochschild cohomology ring of the algebra $A$, which, in the case where the characteristic of $K$ is not 2, is precisely the principal block of $H_q(S_5)$ with $q = -1$. The algebra $A$ is a symmetric special biserial algebra of tame representation type. Furthermore, it is Koszul with radical cube zero, and so is one of the algebras recently classified in [B]; it corresponds to the algebra associated to the simply laced extended Dynkin diagram of type $\tilde{T}_2$ (we...
adopt the notation commonly used in articles in physics journals; note that in [B] and [ESo] this diagram is denoted by $\tilde{Z}_1$. It is well-known that the Hecke algebras of finite type arise from Brauer tree algebras, and thus [EH] shows that they are periodic algebras and describes their Hochschild cohomology ring. Hence the structure of the Hochschild cohomology ring for Hecke algebras of both tame and finite type is now known. Specifically, in Theorem 5.7 we prove that the Hochschild cohomology ring modulo nilpotence of a Hecke algebra has Krull dimension 1 if the algebra is of finite type and has Krull dimension 2 if the algebra is of tame type. For wild Hecke algebras, there has so far been very little progress in the study of Hochschild cohomology or support varieties.

The Hochschild cohomology ring modulo nilpotence was used in [SS] to define the concept of a support variety for any finitely generated module over a finite-dimensional algebra. It was shown in [EHSSST] that, under certain reasonable finiteness conditions on the algebra, these support varieties have many of the analogous properties to those satisfied by modules over a group algebra for a finite group or over a cocommutative Hopf algebra. In addition, when these conditions hold, the Hochschild cohomology ring is itself a finitely generated algebra, and so the finiteness conjecture of [SS] holds concerning the Hochschild cohomology ring modulo nilpotence. We show that these finiteness conditions hold for the tame Hecke algebras, which has also been independently shown by Erdmann and Solberg in [ESo]. We then consider the consequences for support varieties of modules over $A$.

The paper is structured as follows. In section 1, we give a minimal projective bimodule resolution of $A$, and in section 2 a basis for each of the Hochschild cohomology groups $\HH^n(A)$ for $n \geq 0$. This extends the results of [ESo], where the dimensions of the Hochschild cohomology groups were calculated for the Hecke algebra $H_q(S_4)$ with $q = -1$. The advantage of considering $A$ over the algebras in [ESo] is that $A$ is a Koszul algebra, and so has a linear projective resolution as an $A$-$A$-bimodule. In addition, in section 1.1 we give the minimal projective bimodule resolution of a more general family of algebras which contains the algebra $A$. In section 3 we recall the finiteness conditions [Fig2] and [EHSSST] and use them to give new results on the support varieties of finitely generated $A$-modules. Finally, in Theorem 5.7, we show that the Hochschild cohomology ring of $A$ modulo nilpotence is a finitely generated commutative algebra of Krull dimension 2.

The principal block of $H_q(S_5)$ for $q = -1$ was given by quiver and relations in [EN]. Let $K$ be an algebraically closed field (with no restrictions placed on char $K$). Let $Q$ be the quiver

\[
\begin{array}{c}
\varepsilon \quad 1 \quad \alpha \quad 2 \quad \varepsilon \\
\end{array}
\]

and let $I$ be the ideal of $KQ$ generated by \{$\alpha\varepsilon, \varepsilon\alpha, \bar{\alpha}\varepsilon, \varepsilon^2 - \alpha\bar{\alpha}, \varepsilon^2 - \bar{\alpha}\alpha$\}. Let $A = KQ/I$. Then $A$ is the principal block of $H_q(S_5)$ with $q = -1$ in the case where char $K$ is not 2. The Hochschild cohomology ring of $A$ is given by $\HH^*(A) = \Ext^*_A(A, A) = \oplus_{n=0}^\infty \Ext^n_A(A, A)$ with the Yoneda product, where $A^e = A^{op} \otimes_k A$ denotes the enveloping algebra of $A$.

We denote the trivial path at the vertex $i$ by $e_i$. We write paths from left to write. For any arrow $a$ in the quiver $Q$, we write $o(a)$ for the trivial path corresponding to the origin of $a$ and $t(a)$ for the trivial path corresponding to the terminus of $a$. Thus $o(\varepsilon) = e_1 = o(\alpha)$, $t(\varepsilon) = e_1$ and $t(\alpha) = e_2$ etc. There is an algebra isomorphism $b : A \to A$ induced by the involution given by $e_1 \mapsto e_2$, $e_2 \mapsto e_1$, $\alpha \mapsto \bar{\alpha}$, $\bar{\alpha} \mapsto \alpha$, $\varepsilon \mapsto \varepsilon$ and $\bar{\varepsilon} \mapsto \varepsilon$. 
1. A minimal projective bimodule resolution

The terms of the minimal projective bimodule resolution \( (R_n, \delta) \) of \( A \) can be calculated following [1]. We denote by \( P_{ij} \) the projective indecomposable \( A \)-\( A \)-bimodule \( Ae_i \otimes e_j A \), for \( i, j = 1, 2 \). The projectives \( R_n \) are periodic of period 4. We have \( R_0 = P_{11} \oplus P_{22} \) and, for \( 4k + i \geq 1 \),

\[
R_{4k+i} \cong \begin{cases} 
P_{11}^{2k+1} \oplus P_{12}^{2k+i} \oplus P_{21}^{2k+i} \oplus P_{22}^{2k+1} & \text{if } i = 0, 1 \\
P_{11}^{2k+i-1} \oplus P_{12}^{2k+2} \oplus P_{21}^{2k+2} \oplus P_{22}^{2k+i-1} & \text{if } i = 2, 3.
\end{cases}
\]

We follow the approach of [GHMS] in defining the minimal projective \( A \)-\( A \)-bimodule resolution of the Koszul algebra \( A \). As the starting point, we recall that [GSZ] explains how to recursively define sets - which we will denote by \( G^n \) - for each \( n \geq 0 \), in order to give an explicit construction of a minimal projective resolution \( (Q_n, \delta) \) of the right \( A \)-module \( A/J(A) \) where \( J(A) \) denotes the Jacobson radical of \( A \). These sets have the following properties.

(i) For each \( n \geq 0 \), \( Q_n = \oplus_{x \in G^n} t(x)A \).
(ii) For each \( x \in G^n \), there are unique elements \( r_j \in KQ \) such that \( x = \sum h_j^{n-1} r_j \) where the sum is over all elements \( h_j^{n-1} \in G^{n-1} \).
(iii) For each \( n \geq 1 \), using the decomposition of (ii), for \( x \in G^n \), the map \( d_n : Q_n \rightarrow Q_{n-1} \) is given by \( t(x) \lambda \mapsto \sum_j r_j t(x) \lambda \).

The minimal projective bimodule resolution of a Koszul algebra which was given in [GHMS] uses these same sets of [GSZ] in its construction.

Therefore, for our algebra \( A \), we start by defining the set \( G^0 \) to be the set of vertices of \( Q \), labelled so that \( G^0 = \{ g_1^0 = e_1, g_2^0 = e_2 \} \). Then, for \( n \geq 1 \), we will recursively define sets \( G^n = \{ g_i^n, \bar{g}_i^n, f_j^n, \bar{f}_j^n \}_{i,j} \) in \( KQ \) for appropriate indices \( i \) and \( j \) and such that \( |G^n| = 2(n+1) \) for all \( n \geq 0 \), so that these sets contain the requisite information to define the minimal projective \( A \)-\( A \)-bimodule resolution of \( A \). The elements of the sets \( G^n \) will be uniform elements of \( KQ \), that is, for each \( x \in G^n \) there are vertices \( v, w \) such that \( x = v w \). We set \( o(x) = v \) and \( t(x) = w \).

**Definition 1.1.** For \( n \geq 1 \), we define elements:

\[
\begin{align*}
g_1^n &= \begin{cases} 
g_1^{n-1} \varepsilon - f_1^{n-1} \alpha & \text{if } n = 4k \text{ or } n = 4k + 2 \\
g_1^n &\text{if } n = 1 \\
g_1^{n-1} \varepsilon - f_2^{n-1} \bar{\alpha} & \text{if } n = 4k + 1 \text{ with } k \geq 1, \text{ or } n = 4k + 3 
\end{cases} \\
g_2^n &= \begin{cases} 
g_2^{n-1} \varepsilon - f_2^{n-1} \bar{\alpha} & \text{if } n = 1 \\
g_2^{n-1} \varepsilon + f_2^{n-1} \bar{\alpha} & \text{if } 1 \leq l \leq k \text{ if } n = 4k \text{ or } 1 \leq l \leq k \text{ if } n = 4k + 2 \\
g_2^{n-1} \varepsilon + f_2^{n-1} \bar{\alpha} & \text{if } 1 \leq l \leq k \text{ if } n = 4k + 1 \text{ or } n = 4k + 3 
\end{cases} \\
g_2^{2l+1} &= \begin{cases} 
g_2^{n-1} \varepsilon + f_2^{n-1} \bar{\alpha} & \text{if } n = 1 \\
g_2^{n-1} \varepsilon + f_2^{n-1} \bar{\alpha} & \text{if } 1 \leq l \leq k \text{ if } n = 4k \text{ or } 1 \leq l \leq k \text{ if } n = 4k + 2 \\
g_2^{n-1} \varepsilon + f_2^{n-1} \bar{\alpha} & \text{if } 1 \leq l \leq k \text{ if } n = 4k + 1 \text{ or } 1 \leq l \leq k \text{ if } n = 4k + 3 
\end{cases} \\
g_2^{2k} &= g_2^{4k-1} \varepsilon \\
g_2^{4k+1} &= \begin{cases} 
f_2^{4k-1} \varepsilon & \text{if } n = 4k \\
g_2^{4k-1} \varepsilon & \text{if } n = 4k + 1 \text{ with } k \geq 1
\end{cases} \\
g_2^{4k+3} &= f_2^{4k+2} \varepsilon
\end{align*}
\]
Applying the bar involution we define \( \bar{g}_n = b(g^n) \) and \( \bar{f}_n = b(f^n) \) and we obtain in this way all the elements of \( G^n \).

**Remark.** (1) Note that \( g_1^1 = \varepsilon, f_1^1 = \alpha, \bar{g}_1^1 = \bar{\varepsilon} \) and \( \bar{f}_1^1 = \bar{\alpha} \) so that \( G^1 \) is the set of arrows of \( Q \). Also, \( \bar{g}_2^1 = \varepsilon^2 - \bar{\alpha} \varepsilon, f_2^1 = \alpha \varepsilon \) and \( \bar{f}_2^1 = \bar{\alpha} \varepsilon \). Thus \( G^2 \) is a minimal set of uniform elements which generate the ideal \( I \).

(2) It follows from above that

\[
G^n = \{ g_i^n, \bar{g}_i^n, f_j^n, \bar{f}_j^n \mid i = 1, \ldots, 2k + 1, j = 1, \ldots, n - 2k \} \quad \text{for } n = 4k \text{ or } n = 4k + 2
\]

\[
G^n = \{ g_i^n, \bar{g}_i^n, f_j^n, \bar{f}_j^n \mid i = 1, \ldots, \frac{n+1}{2}, j = 1, \ldots, \frac{n+1}{2} \} \quad \text{for } n = 4k + 1 \text{ or } n = 4k + 3
\]

where \( g_i^n \in e_1 K Q e_1, f_j^n \in e_1 K Q e_2, \bar{f}_j^n \in e_2 K Q e_1, \) and \( \bar{g}_i^n \in e_2 K Q e_2 \). (3) For each \( n \geq 1 \), the set \( G^n \) contains four monomials of length \( n \). These are the four subpaths of length \( n \) of the path \((\varepsilon \alpha \bar{\varepsilon} \bar{\alpha})^N\) for \( N > 0 \).

Thus

\[
R_\alpha = \bigoplus_{\varepsilon \in G^n} A o(x) \otimes t(x) A
\]

\[
= (\bigoplus_i A o(g_i^n) \otimes t(g_i^n) A) \oplus (\bigoplus_j A o(f_j^n) \otimes t(f_j^n) A) +
\]

\[
(\bigoplus_i A o(\bar{g}_i^n) \otimes t(\bar{g}_i^n) A) \oplus (\bigoplus_j A o(\bar{f}_j^n) \otimes t(\bar{f}_j^n) A)
\]

where, for each \( i \), the summand \( A o(g_i^n) \otimes t(g_i^n) A = P_{i1} \) and the summand \( A o(\bar{g}_i^n) \otimes t(\bar{g}_i^n) A = P_{22} \), and, for each \( j \), the summand \( A o(f_j^n) \otimes t(f_j^n) A = P_{12} \) and the summand \( A o(\bar{f}_j^n) \otimes t(\bar{f}_j^n) A = P_{21} \).

Following [GHMS], in order to define the differential \( \delta \) we need the following lemma so that we have two different ways of expressing the elements of the set \( G^n \) in terms of the elements of the set \( G^{n-1} \). Namely, we write the elements of \( G^n \) in the form \( \sum h_i^{n-1} p_i \) and in the form \( \sum q_i h_i^{n-1} \) for appropriate \( p_i, q_i \) in the ideal of \( K Q \) which is generated by all the arrows, and where the sums are over all \( h_i^{n-1} \in G^{n-1} \). The proof of Lemma 1.2 is straightforward and is omitted.
Lemma 1.2. For $n \geq 1$,

\[ g^n_1 = \begin{cases} \varepsilon g^0_1 - \alpha \bar{f}^0_1 & \text{if } n = 1 \\ \varepsilon g^{n-1}_1 - \alpha \bar{f}^{n-1}_1 & \text{if } n \geq 2 \end{cases} \]

\[ g^n_{2l} = \varepsilon g^{n-1}_{2l+1} + \alpha \bar{f}^{n-1}_{2l} \]

where $1 \leq l < k$ if $n = 4k$, or $1 \leq l \leq k$ if $n = 4k + 2$

\[ g^n_{2l+1} = \varepsilon g^{n-1}_{2l} - \alpha \bar{f}^{n-1}_{2l+1} \]

where $1 \leq l < k$ if $n = 4k$, or $1 \leq l \leq k$ if $n = 4k + 2$

\[ g^n_{4k} = \alpha \bar{f}^{n-1}_{4k} \]

\[ g^n_{4k+1} = \varepsilon g^{n-1}_{4k} \]

if $n = 4k$, or if $n = 4k + 1$ with $k \geq 1$

\[ g^n_{4k+2} = \alpha \bar{f}^{n-1}_{4k+2} \]

\[ f^n_1 = \begin{cases} \alpha \bar{f}^0_1 + \varepsilon \bar{g}^{n-1}_1 & \text{if } n = 1 \\ \varepsilon f^{n-1}_1 + \alpha \bar{g}^{n-1}_1 & \text{if } n = 4k + 1 \text{ with } k \geq 1, \text{ or } n = 4k + 3 \end{cases} \]

\[ f^n_{2l+1} = \varepsilon f^{n-1}_{2l+2} + \alpha \bar{g}^{n-1}_{2l+1} \]

where $0 \leq l < k$ if $n = 4k$, or if $n = 4k + 2$

\[ f^n_{2l+2} = \varepsilon f^{n-1}_{2l+1} - \alpha \bar{g}^{n-1}_{2l+2} \]

where $0 \leq l < k$ if $n = 4k$, or $n = 4k + 2$

\[ f^n_{4k+1} = \alpha \bar{g}^{n-1}_{4k+1} \]

if $n = 4k + 1$ with $k \geq 1$, or if $n = 4k + 2$

\[ f^n_{4k+2} = \varepsilon f^{n-1}_{4k+2} \]

if $n = 4k + 2$ or $n = 4k + 3$.

and the analogous set of equations holds if we apply $b$ to the equations above.

We now introduce some additional notation to enable us to distinguish the summands of $R_n$. First, let

\[ u := e_1 \otimes e_1 \in P_{11} = A e_1 \otimes e_1 A \]

\[ \bar{u} := e_2 \otimes e_2 \in P_{22} = A e_2 \otimes e_2 A \]

\[ p := e_1 \otimes e_2 \in P_{12} = A e_1 \otimes e_2 A \]

\[ \bar{p} := e_2 \otimes e_1 \in P_{21} = A e_2 \otimes e_1 A. \]

We add the superscript $n$ to indicate that each of these elements lies in the projective module $R_n$, and then add subscripts in order to distinguish, within $R_n$, the different summands of the form $A e_i \otimes e_j A$. Specifically, $u^n_i = e_1 \otimes e_1$ lies in the $i$th copy of $P_{11}$ as a summand of $R_n$, that is, in the copy of $P_{11}$ corresponding to the element $g^n_1$ of $G^n$. Similarly, $p^n_j = e_1 \otimes e_2$ lies in the $j$th copy of $P_{12}$ as a summand of $R_n$, that is, in the copy of $P_{12}$ corresponding to the element $f^n_j$ of $G^n$.

The algebra isomorphism $b : A \to A$ extends to an algebra isomorphism $b : A^e \to A^e$ where $b(a_1 \otimes a_2) = b(a_1) \otimes b(a_2)$ for $a_1 \in A^{op}, a_2 \in A$. Thus $b(u) = \bar{u}$ and $b(p) = \bar{p}$.

For better readability we omit in the following definition the superscripts on all expressions except the left most one with the understanding that the terms on the right hand side are in $R_{n-1}$.
Definition 1.3. We define an $A$-$A$-bimodule homomorphism $\delta_n : R_n \to R_{n-1}$, for all $n \geq 1$, by

\begin{align*}
\mu^n_1 &\mapsto \begin{cases} 
    u_1 \bar{e} - p_1 \alpha + \bar{u}_1 - \alpha \bar{p}_1 & \text{if } n = 4k \text{ or } n = 4k + 2 \\
    u_1 \bar{e} - \bar{u}_1 & \text{if } n = 1 \\
    u_1 \bar{e} - p_2 \alpha - (\bar{u}_1 - \alpha \bar{p}_1) & \text{if } n = 4k + 1 \text{ with } k \geq 1, \text{ or } n = 4k + 3
\end{cases} \\
\mu^n_2 &\mapsto \begin{cases} 
    u_{2l} \bar{e} - p_{2l+1} \alpha + \bar{u}_{2l+1} + \alpha \bar{p}_{2l} & 1 \leq l < k \text{ if } n = 4k \text{ or } 1 \leq l \leq k \text{ if } n = 4k + 2 \\
    u_{2l} \bar{e} + p_{2l-1} \alpha - (\bar{u}_{2l+1} + \alpha \bar{p}_{2l}) & 1 \leq l < k \text{ if } n = 4k + 1 \text{ or } n = 4k + 3
\end{cases} \\
\mu^n_{2l+1} &\mapsto \begin{cases} 
    u_{2l+1} \bar{e} + p_{2l+1} \alpha + \bar{u}_{2l+1} - \alpha \bar{p}_{2l+1} & 1 \leq l < k \text{ if } n = 4k \text{ or } 1 \leq l \leq k \text{ if } n = 4k + 2 \\
    u_{2l+1} \bar{e} - p_{2l+1} \alpha - (\bar{u}_{2l+1} - \alpha \bar{p}_{2l+1}) & 1 \leq l < k \text{ if } n = 4k + 1 \text{ or } 1 \leq l \leq k \text{ if } n = 4k + 3
\end{cases} \\
\mu^n_{2k} &\mapsto \begin{cases} 
    p_{2k} \alpha + \bar{u}_k & \text{if } n = 4k \\
    u_{2k+1} \bar{e} - \bar{u}_k & \text{if } n = 4k + 1 \text{ with } k \geq 1
\end{cases} \\
\mu^n_{2k+1} &\mapsto \begin{cases} 
    p_{2k+1} \alpha - \alpha \bar{u}_{2k+1} & \text{if } n = 4k + 1 \text{ with } k \geq 1, \text{ or } n = 4k + 3 \\
    p_{2k+1} \bar{e} + \bar{u}_{2k+1} - \alpha \bar{p}_{2k+1} & \text{if } n = 4k + 2 \\
    p_{2k+1} \bar{e} + p_{2k+2} \alpha + \bar{u}_{2k+2} + \alpha \bar{p}_{2k+1} & 0 \leq l < k \text{ if } n = 4k \text{ or if } n = 4k + 2 \\
    p_{2k+1} \bar{e} + u_{2l+1} \alpha - (\bar{u}_{2l+1} + \alpha \bar{u}_{2l+1}) & 1 \leq l < k \text{ if } n = 4k + 1 \text{ or } 1 \leq l \leq k \text{ if } n = 4k + 3
\end{cases} \\
\mu^n_{2k+2} &\mapsto \begin{cases} 
    p_{2k+2} \bar{e} + u_{2l+1} \alpha + \bar{u}_{2l+1} - \alpha \bar{p}_{2l+2} & 0 \leq l < k \text{ if } n = 4k \text{ or if } n = 4k + 2 \\
    p_{2k+2} \bar{e} - u_{2l+3} \alpha - (\bar{u}_{2l+1} + \alpha \bar{u}_{2l+2}) & 0 \leq l < k \text{ if } n = 4k + 1 \text{ or } n = 4k + 3
\end{cases} \\
\mu^n_{2k+1} &\mapsto \begin{cases} 
    u_{2k+1} \bar{e} + \bar{u}_{2k+1} & \text{if } n = 4k + 1 \text{ with } k \geq 1 \\
    p_{2k+1} \bar{e} + \bar{u}_{2k+1} - \alpha \bar{p}_{2k+1} & \text{if } n = 4k + 2 \\
    p_{2k+2} \bar{e} + \bar{u}_{2k+1} - \alpha \bar{p}_{2k+1} & \text{if } n = 4k + 3
\end{cases}
\end{align*}

together with the correspondences given by applying $b$, that is, for any of the above correspondences $\bar{x} \mapsto \delta(\bar{x})$ we also have $b(\bar{x}) \mapsto b(\delta(\bar{x}))$.

The next result is now immediate from [CHMS] Theorem 2.1.

Proposition 1.4. The complex $(R_{\bullet}, \delta)$ is a minimal projective $A$-$A$-bimodule resolution of $A$, where $\delta_0 : R_0 \to A$ is the multiplication map.

1.1. A more general class of special biserial algebras. The algebra $A$ belongs to the wider class of special biserial algebras whose quiver is $Q$ but where we replace the relations by $\alpha \bar{e} = \bar{e} \alpha = \bar{e} \alpha = \bar{e} \alpha = 0, \bar{e} \alpha - (\alpha \bar{e})^s = 0$ and $\bar{e} \alpha - (\alpha \bar{e})^s = 0$, where $r, s$ are integers with $r \geq 2, s \geq 1$. Let $I'$ be the ideal of $KQ$ generated by these relations and denote the resulting algebra by $\Lambda = KQ/I'$. We note that $\Lambda$ is Koszul if and only if $r = 2$ and $s = 1$, whence $\Lambda = A$. Although $\Lambda$ is, in general, not a Koszul algebra, we may still give an explicit minimal $\Lambda$-$A$-projective bimodule resolution of $\Lambda$.

The terms of the minimal projective bimodule resolution $(\tilde{R}_\bullet, \tilde{\delta})$ of $\Lambda$ follow the same pattern as those of the corresponding resolution of $A$ given in [1], that is, for all $r \geq 2$
and \( s \geq 1 \), we have \( \check{R}_0 = \check{P}_{11} \oplus \check{P}_{22} \) and, for \( 4k + i \geq 1 \),

\[
\check{R}_{4k+i} = \begin{cases}
\check{P}_{11}^{2k+1} \oplus \check{P}_{12}^{2k+1} \oplus \check{P}_{21}^{2k+i} \oplus \check{P}_{22}^{2k+1} & \text{if } i = 0, 1 \\
\check{P}_{11}^{2k+i-1} \oplus \check{P}_{12}^{2k+2} \oplus \check{P}_{21}^{2k+2} \oplus \check{P}_{22}^{2k+i-1} & \text{if } i = 2, 3
\end{cases}
\]

where \( \check{P}_{ij} = \Lambda e_i \otimes e_j \Lambda \) for \( i, j = 1, 2 \). By abuse of notation, we write \( u = e_1 \otimes e_1 \in \check{P}_{11}, \) \( \bar{u} = e_2 \otimes e_2 \in \check{P}_{22}, \) and denote by \( b \) the algebra homomorphism \( \Lambda^* \rightarrow \Lambda^* \) induced by the involution \( e_1 \leftrightarrow e_2, \bar{e} \leftrightarrow \bar{e}, \alpha \leftrightarrow \bar{\alpha} \) on \( \Lambda \). We keep the same conventions on the use of superscripts and subscripts as those preceding Definition 1.3 and make the following definition of \( \partial_n \); again we omit the superscripts on the right hand side with the understanding that all the terms are in \( \check{R}_{n-1} \).

**Definition 1.5.** We define a \( \Lambda-\Lambda \)-bimodule homomorphism \( \partial_n : \check{R}_n \rightarrow \check{R}_{n-1} \), for all \( n \geq 1 \), by

\[
\begin{align*}
\text{if } n & = 4k \text{ or } n = 4k + 2 \\
& \mapsto \begin{cases}
u_i^{e-1} - p_1 \bar{\alpha}(\alpha \bar{\alpha})^{s-1} + \varepsilon^{r-1} u_1 - (\alpha \bar{\alpha})^{r-1} \alpha \bar{p}_1 & \text{if } n = 1 \\
u_i^{e-1} - e^{r-1} u_1 - (\alpha \bar{\alpha})^{r-1} \alpha \bar{p}_1 & \text{if } n = 4k + 1 \text{ with } k \geq 1, \text{ or } n = 4k + 3 \\
\end{cases} \\
\text{if } 1 \leq l < k \text{ if } n = 4k \text{ or } 1 \leq l \leq k \text{ if } n = 4k + 2 \\
\text{if } 1 \leq l < k \text{ if } n = 4k + 1 \text{ or } n = 4k + 3 \\
\text{if } 1 \leq l < k \text{ if } n = 4k + 1 \text{ or } 1 \leq l \leq k \text{ if } n = 4k + 3 \\
\text{if } n = 4k + 1 \text{ with } k \geq 1 \\
\text{if } n = 4k \\
\text{if } n = 4k + 1 \text{ with } k + 1 \\
\text{if } n = 4k + 1 \text{ or } n = 4k + 3 \\
\text{if } 0 \leq l < k \text{ if } n = 4k \text{ or } n = 4k + 2 \\
\text{if } 1 \leq l < k \text{ if } n = 4k + 1 \text{ or } 1 \leq l \leq k \text{ if } n = 4k + 3 \\
\text{if } 0 \leq l < k \text{ if } n = 4k + 1 \text{ or } n = 4k + 3 \\
\text{if } n = 4k + 1 \text{ with } k \geq 1 \\
\text{if } n = 4k + 2 \\
\text{if } n = 4k + 2 \\
\text{if } n = 4k + 3 \\
\end{align*}
\]

together with the correspondences given by applying \( b \), that is, for any \( \bar{\chi} \rightarrow \partial(\bar{\chi}) \) given above, we also have \( b(\bar{\chi}) \rightarrow \partial(b(\bar{\chi})) \).

**Proposition 1.6.** The complex \( (\check{R}_n, \partial) \) is a minimal projective \( \Lambda-\Lambda \)-bimodule resolution of \( \Lambda \), where \( \partial : \check{R}_0 \rightarrow \Lambda \) is the multiplication map.

**Proof:** A straightforward verification shows that \( (\check{R}_n, \partial) \) is a complex. The fact that this complex is exact follows from an analogous argument to the one in [GS, Proposition 2.8].

\[\square\]
2. A BASIS OF THE HOCHSCHILD COHOMOLOGY RING

In this section we return to the algebra $A$ and give a basis of the Hochschild cohomology ring $HH^*(A)$ in the case where $\text{char } K \neq 2$. By [ESc], the dimensions of the Hochschild cohomology groups $HH^n(A)$ are as follows

$$\dim HH^{4k+i}(A) = \begin{cases} 2k + 3 & i = 0 \text{ and } k > 0 \text{ or } i = 1, 2 \text{ and } k \geq 0 \\ 2k + 4 & i = 3, k \geq 0 \end{cases}$$

and the set $\{e_1 + e_2, \varepsilon, \varepsilon^2, \varepsilon^3\}$ is a $K$-basis of $HH^0(A)$.

We describe the elements of a basis of each Hochschild cohomology group $HH^n(A)$, for $n \geq 1$, in terms of cocycles in $\text{Hom}(R_n, A)$. We write $\delta_n$ for both the map $\delta_n : R_n \to R_{n-1}$ and for the induced map $\text{Hom}(R_{n-1}, A) \to \text{Hom}(R_n, A)$. Thus our basis is given in terms of a set of elements in $\text{Ker} \delta_{n+1}$ as a subset of $\text{Hom}(R^n, A)$, such that the corresponding cosets in $\text{Ker} \delta_{n+1}/\text{Im} \delta_n$ form a basis of $HH^n(A) = \text{Ker} \delta_{n+1}/\text{Im} \delta_n$. We keep the notation of section [SS]. When describing a cocycle in $\text{Hom}(R_n, A)$, we simply write the images of the generators $u^n_i, u^n_{i-1} p^n_j$ or $p^n_j$ in $R_n$ where those images are non-zero.

It is easy to verify that the following elements are indeed cocycles in $\text{Hom}(R_n, A)$. They are used to define a basis of the Hochschild cohomology groups in Theorem 2.1.

(i) Suppose $i = 0$ and $k \geq 1$ or $i = 2$ and $k \geq 0$.

(a) Let $\phi_{4k+i} : \begin{cases} u_{4k+i}^{4k+i} \mapsto e_1, \\ u_{4k+i}^{4k+i} \mapsto e_2. \end{cases}$

(b) For $1 \leq l \leq k$, let $\theta^i_{4k+i} : \begin{cases} u_{4k+i}^{4k+i} \mapsto e_1, \\ u_{4k+i}^{4k+i} \mapsto e_2, \\ u_{4k+i}^{4k+i} \mapsto \varepsilon. \end{cases}$

(c) For $1 \leq l \leq k$, let $\psi^{4k+i} : u_{4k+i}^{4k+i} \mapsto \varepsilon_2$.

(ii) Suppose $i = 1$ or $i = 3$ and that $k \geq 0$.

For $1 \leq l \leq 2k + 1$ if $i = 1$ and $1 \leq l \leq 2k + 2$ if $i = 3$, let

$$\chi^i_{4k+i} : \begin{cases} u_{4k+i}^{4k+i} \mapsto \varepsilon, \\ p_{4k+i}^{4k+i} \mapsto (-1)^{i+1} \alpha, \\ u_{4k+i}^{4k+i} \mapsto \varepsilon. \end{cases}$$

**Theorem 2.1.** Suppose $\text{char } K \neq 2$, and let $k \geq 0$ and $0 \leq i \leq 3$ with $4k + i > 0$.

1. For $i = 0$ or $2$, $HH^{4k+i}(A)$ has basis $\{\phi_{4k+i}, \varepsilon \phi_{4k+i}, \varepsilon \phi_{4k+i}, \psi_{4k+i}, \theta_{4k+i} | 1 \leq l \leq k\}$.

2. For $i = 1$ or $3$, $HH^{4k+i}(A)$ has basis $\{\chi^i_{4k+i}, \varepsilon \chi^i_{4k+i}, \varepsilon \chi^i_{4k+i}, \chi^i_{4k+i} | 1 \leq l \leq m_i\}$, where $m_i = 2k + 1$ if $i = 1$ and $m_i = 2k + 2$ if $i = 3$.

We omit the proof since it is straightforward to check these maps represent linearly independent non-zero elements in $HH^{4k+i}(A)$.

3. **Fg1.** and **Fg2.**

Throughout this section, suppose again that $\text{char } K$ is arbitrary. In [SS], the Hochschild cohomology ring was used to define the support variety of a finitely generated module over any finite-dimensional algebra. It was shown in [EHSST] that, when certain (reasonable) finiteness conditions hold, then the support varieties of [SS] share many of the analogous
properties of support varieties for finite group rings or co-commutative Hopf algebras. We start by stating these finiteness conditions and showing that they hold in our setting.

Let $J(A)$ denote the Jacobson radical of $A$. Then the Yoneda algebra or Ext algebra of $A$ is given by $E(A) = \text{Ext}^n(A/J(A), A/J(A))$ with the Yoneda product. We use the notation $E(A)^n = \text{Ext}^n(A/J(A), A/J(A))$ for the $n$-th graded component of $E(A)$. The finiteness conditions of [EHSST] are:

**Fg1.** There is a graded subalgebra $H$ of $\text{HH}^*(A)$ such that $H$ is a commutative Noetherian ring and $H^0 = \text{HH}^0(A)$.

**Fg2.** $E(A)$ is a finitely generated $H$-module.

The algebra $A$ is Koszul, so we know from [BGSS] that the image of the natural ring homomorphism $\text{HH}^*(A) \to E(A)$ is precisely the graded centre $Z_{\text{gr}}(E(A))$ of $E(A)$, where $Z_{\text{gr}}(E(A))$ is the subring of $E(A)$ generated by all homogeneous elements $z$ in $E(A)^n$ ($n \geq 0$) such that $xy = (-1)^{nm}yx$ for all $y \in E(A)^m$. Moreover, $E(A)$ is given by quiver and relations as $E(A) = KQ/I^k$ where $I^k = \langle \epsilon^2 + \alpha \bar{\epsilon}, \bar{\epsilon}^2 + \bar{\alpha} \epsilon \rangle$ (from [GM, Theorem 2.2]). Thus $E(A)$ is the preprojective algebra associated to the simply laced extended Dynkin graph of type $T_2$.

Now, it is easy to check that the elements

$$x = \bar{\epsilon} \epsilon + \bar{\epsilon}^2$$

and

$$z = \epsilon \bar{\alpha} \bar{\epsilon} + \alpha \bar{\epsilon} \bar{\alpha} + \bar{\epsilon} \bar{\alpha} \epsilon + \bar{\alpha} \epsilon \epsilon$$

are both in $Z_{\text{gr}}(E(A))$.

**Theorem 3.1.** As a left $K[x, z]$-module, $E(A)$ is finitely generated with generating set

$$S = \{e_1, e_2, \epsilon, \alpha, \bar{\epsilon}, \bar{\alpha}, \bar{\epsilon} \bar{\alpha}, \epsilon \bar{\alpha}, \bar{\epsilon} \bar{\alpha} \epsilon, \epsilon \bar{\alpha} \epsilon, \bar{\epsilon} \bar{\alpha} \epsilon, \epsilon \bar{\alpha} \epsilon\}.$$ 

Before proving Theorem 3.1 we consider the conditions **Fg1.** and **Fg2.** The element $\phi^2 \in \text{HH}^2(A)$ is a pre-image of $x$ and the element $\theta^4_1 \in \text{HH}^4(A)$ is a pre-image of $z$. Let $H$ be the graded subalgebra of $\text{HH}^*(A)$ generated by $\text{HH}^0(A), \phi^2$ and $\theta^4_1$, so that $H$ is a pre-image of $K[x, z]$ in $\text{HH}^*(A)$. Then we have the following immediate consequence of Theorem 3.1.

**Corollary 3.2.** The conditions **Fg1.** and **Fg2.** hold for the algebra $A$ with respect to the subring $H$ of $\text{HH}^*(A)$.

This result was independently shown by Erdmann and Solberg in [ES0], where they consider all radical cube zero weakly symmetric algebras. We observe that Corollary 3.2 does not require the explicit structure of either $Z_{\text{gr}}(E(A))$ or $\text{HH}^*(A)$, and that it now follows from [ES0] that **Fg1.** and **Fg2.** also hold with respect to $\text{HH}^{\text{even}}(A) \supset H$.

In order to prove Theorem 3.1 we need the following lemma.

**Lemma 3.3.** For any $h \in S$ and any arrow $x$ in $Q$ we have $hx = \sum_{s \in S} z_s s$ where $z_s \in Z_{\text{gr}}(E(A))$.

**Proof:** We will show the result for $x = \alpha$ and $x = \epsilon$; the cases $x = \bar{\alpha}$ and $x = \bar{\epsilon}$ are analogous.

Clearly the result holds for $\alpha, \epsilon \alpha, \bar{\epsilon} \bar{\alpha} \epsilon, \epsilon \bar{\alpha} \epsilon$, $\bar{\epsilon}^2 \alpha = x \epsilon^2$, $\epsilon^2 \alpha = x \alpha$ and $\alpha \bar{\epsilon} \bar{\alpha} = -\epsilon^2 \alpha \bar{\epsilon} = -\epsilon x \epsilon$. Finally, $\epsilon \bar{\alpha} \epsilon \alpha = -\epsilon^3 \alpha \bar{\epsilon} = -x \epsilon \alpha \bar{\epsilon}$. Again
the result immediately holds for \( \varepsilon, \bar{a}\varepsilon \) and \( \bar{e}\varepsilon \). Furthermore, \( \varepsilon^2 = xe_1, \bar{a}\varepsilon^2 = x\bar{e}\bar{a}, \alpha\bar{e}\varepsilon = ze_1 - \varepsilon e\bar{a}\bar{e} \) and \( \varepsilon x\bar{e}\varepsilon = xz - x\alpha e\bar{a} \).

**Proof of Theorem 3.7.** A proof by induction will show that \( S \) is a generating set of \( E(A) \).

More precisely, we show by induction on \( n \) that each element of \( E(A)^n \) can be expressed as a linear combination of elements in \( S \) with coefficients in \( \mathbb{Z}_{\text{gr}}(E(A)) \).

The base case holds since \( e_1 \) and \( e_2 \) are in \( S \). Suppose that the result holds for \( E(A)^n \), that is, every \( x \in E(A)^n \) is of the form \( x = \sum_{s \in S} z_s s \) with \( z_s \in \mathbb{Z}_{\text{gr}}(E(A)) \). Since \( E(A) \) is Koszul, an element \( x \in E(A)^{n+1} \) is a linear combination of elements of the form \( y\gamma \) where \( y \in E(A)^n \) and \( \gamma \) is an arrow of \( Q \). By induction \( y = \sum_{s \in S} z_s s \) with \( z_s \in \mathbb{Z}_{\text{gr}}(E(A)) \) and thus \( y\gamma = \sum_{s \in S} z_s s\gamma \). By Lemma 3.3 we have \( y = \sum_{s \in S} z_s (\sum_{t \in S} z_t') t \) with \( z_t' \in \mathbb{Z}_{\text{gr}}(E(A)) \) and therefore \( y = \sum_{s \in S} \sum_{t \in S} (z_s z_t') t \). Thus \( x \) has the required form. \( \square \)

Following [SS], the variety of a finitely generated \( A \)-module \( M \) with respect to \( H \) is given by

\[
V_H(M) = \text{MaxSpec}(H/A_H(M, M))
\]

where \( A_H(M, M) \) is the annihilator of \( \text{Ext}^*_A(M, M) \) as an \( H \)-module. Moreover, \( V_H(A/J(A)) = \text{MaxSpec } H \). Corollary 3.2 and [EHSST, Theorems 2.5(b), 5.3] have the following direct consequences for our algebra \( A \).

**Proposition 3.4.**

1. \( HH^*(A) \) is a finitely generated \( K \)-algebra.
2. For \( M \in \text{mod } A \), the variety of \( M \) is trivial if and only if \( M \) is projective.
3. For \( M \in \text{mod } A \) and \( M \) indecomposable, the variety of \( M \) is a line if and only if \( M \) is periodic.

We now consider the simple \( A \)-modules \( S_1 \) and \( S_2 \) (corresponding to the vertices 1 and 2 respectively of the quiver \( Q \)). From [SS, Proposition 3.4], we have that \( V_H(A/J(A)) = V_H(S_1) \cup V_H(S_2) \) and \( V_H(S_1) = V_H(\text{rad}(e_1 A)) \). Using the exact sequence \( 0 \to S_1 \to \text{rad}(e_1 A) \to A/J(A) \to 0 \), we have \( V_H(A/J(A)) \subseteq V_H(S_1) \cup V_H(\text{rad}(e_1 A)) = V_H(S_1) \). Thus \( V_H(A/J(A)) = V_H(S_1) \). A similar argument for \( S_2 \) proves the following proposition.

**Proposition 3.5.** \( V_H(S_1) = V_H(S_2) = \text{MaxSpec } H \).

This is, in fact, a special case of the following result.

**Proposition 3.6.** Suppose that \( \hat{A} \) is a selfinjective algebra and that \( \hat{H} \) is a graded subalgebra of \( HH^*(\hat{A}) \) containing \( HH^0(\hat{A}) \). Let \( S \) be the simple module corresponding to the indecomposable projective \( P \), so that \( P/\text{soc } P \cong S \). If \( \text{soc } P \cong S \) and if \( A/J(A) \) is a summand of \( \text{rad } P/\text{soc } P \) then \( V_{\hat{H}}(S) = V_{\hat{H}}(\hat{A}/J(\hat{A})) = \text{MaxSpec } \hat{H} \).

Finally, we remark that it can be verified that \( \mathbb{Z}_{\text{gr}}(E(A)) \) and \( K[x, z] \) are isomorphic as algebras. Since \( A \) is Koszul, we know from [BGSS] that the Hochschild cohomology ring of \( A \) modulo nilpotence is isomorphic to \( \mathbb{Z}_{\text{gr}}(E(A)) \) modulo nilpotence. Hence \( HH^*(A)/\mathcal{N} \cong K[x, z] \), where \( \mathcal{N} \) denotes the ideal of \( HH^*(A) \) generated by all nilpotent elements. In contrast, for a Hecke algebra \( B \) of finite type, the results of [EH] show that the Hochschild cohomology ring modulo nilpotence of \( B \) is isomorphic to \( K[y] \), where \( y \) is in degree \( m > 1 \) and \( m \) is minimal such that \( \Omega_{E_0}^m(B) \cong B \) as \( B-B \)-bimodules. Thus we have the following description of the Hochschild cohomology ring modulo nilpotence for all Hecke algebras of tame and of finite type.

**Theorem 3.7.** The Hochschild cohomology ring modulo nilpotence of a Hecke algebra has Krull dimension 1 if the algebra is of finite type and has Krull dimension 2 if the algebra is of tame type.
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Department of Mathematics, University of Leicester, University Road, Leicester LE1 7RH, UK

E-mail address: schroll@mcs.le.ac.uk

Department of Mathematics, University of Leicester, University Road, Leicester LE1 7RH, UK

E-mail address: N.Snashall@mcs.le.ac.uk