Plasma planar filaments instability and Alfvén waves

by

L.C. Garcia de Andrade

Departamento de Física Teórica – IF – Universidade do Estado do Rio de Janeiro-UERJ
Rua São Francisco Xavier, 524
Cep 20550-003, Maracanã, Rio de Janeiro, RJ, Brasil
Electronic mail address: garcia@dft.if.uerj.br

Abstract

Inhomogeneous plasmas filaments instabilities are investigated by using the techniques of classical differential geometry of curves where Frenet torsion and curvature describe completely the motion of curves. In our case the Frenet frame changes in time and also depends upon the other coordinates taking into account the inhomogeneity of the plasma. The exponential perturbation method so commonly used to describe cosmological perturbations is applied to magnetohydrodynamic (MHD) plasma equations to find longitudinal modes describing Alfvén waves propagation modes describing plasma waves in the medium. Stability is investigated in the imaginary axis of the spectra of complex frequencies \( \omega \) or \( Im(\omega) \neq 0 \). PACS numbers: 02.40.Hw-classical differential geometry
I Introduction

The topology and geometry of hydrodynamical and MHD instability have been called [1] one of the most important parts of plasma science. Arnold and Khesin [2] have investigated the role of topology and Riemannian geometry to investigate MHD dynamos so important for use in geophysics and solar physics [3]. Use of chaotic flows have been developed recently by Thiffeault and Boozer[4]. Twisted filamentary magnetic structures have been applied in solar physics [5] and in plasma filaments electric carrying-current loops [5]. In this paper we consider the generalized filamentary structures and its unstable profiles. One of the simplest methods to investigate instabilities is the so-called exponential instability which is characterized by the relation \( \text{Im}(\omega) > 0 \) where \( \text{Im} \) denotes the imaginary part of complex structure of the spectra of perturbations where any physical quantity in equilibrium \( Q_0 \) is perturbed by a quantity \( Q_1 = Q_1^0 \exp[-i(\omega t - (k_{||} s + k_{\perp} n))] \) where \( s \) is the coordinate along the filament and \( n \) is along the filament direction. The quantities \( k_{||} \) and \( k_{\perp} \) represent the respective wave numbers of propagation. Thus \( Q \) could represent any perturbed physical quantity such as magnetic fields or flow speed. Throughout the paper we use the notation of a previously paper on vortex filaments in MHD [6]. The paper is organized as follows: In section 2 we decompose MHD equations on a Frenet frame along the thin filament and perturb the MHD vector equations. In section 3 we compute the perturbations of the magnetic filament and analyse the stable and unstable modes and the transition to stable to unstable ones. In section 4 we present the conclusions.

II Scalar perturbations in MHD filamentary structures

Let us now start by considering the MHD field equations

\[
\nabla \cdot \vec{B} = 0 \quad (\text{II.1})
\]

\[
\nabla \times \vec{B} = \partial_t \vec{B} \quad (\text{II.2})
\]

\[
\nabla (\rho \vec{v}) + \partial_t \rho = 0 \quad (\text{II.3})
\]
\[ \nabla \times (\vec{v} \times \vec{B}) = \partial_t \vec{B} \]  
\hfill (II.4)

\[ \frac{d}{dt} \left[ \frac{p}{\rho} \gamma \right] = 0 \]  
\hfill (II.5)

\[ \rho \frac{d\vec{v}}{dt} = \vec{J} \times \vec{B} - \nabla p \]  
\hfill (II.6)

where the equilibrium quantities are

\[ \vec{B} = B_0 \vec{t} \]  
\hfill (II.7)

\[ \vec{J}_0 = 0 \]  
\hfill (II.8)

\[ \vec{v}_0 = 0 \]  
\hfill (II.9)

\[ p = p_0 + p_1 \]  
\hfill (II.10)

\[ \rho = \rho_0 + \rho_1 \]  
\hfill (II.11)

magnetic field \( \vec{B} \) along the filament is defined by the expression \( \vec{B} = B_{s,n,t} \vec{t} \) and \( B_{s,t} \) is the component along the arc length \( s \) of the filament depending upon time. Here we consider that \( B_0 \) does not depend on time and also does not depend on normal coordinates \( n \), so \( B_0(s) \).

The vectors \( \vec{t} \) and \( \vec{n} \) along with binormal vector \( \vec{b} \) together form the Frenet frame which obeys the Frenet-Serret equations

\[ \vec{t}' = \kappa \vec{n} \]  
\hfill (II.12)

\[ \vec{n}' = -\kappa \vec{t} + \tau \vec{b} \]  
\hfill (II.13)

\[ \vec{b}' = -\tau \vec{n} \]  
\hfill (II.14)

the dash represents the ordinary derivation with respect to coordinate \( s \), and \( \kappa(s,t) \) is the curvature of the curve where \( \kappa = R^{-1} \). Here \( \tau \) represents the Frenet torsion. We follow the assumption that the Frenet frame may depend on other degrees of freedom such as that the gradient operator becomes

\[ \nabla = \vec{t} \frac{\partial}{\partial s} + \vec{n} \frac{\partial}{\partial n} + \vec{b} \frac{\partial}{\partial b} \]  
\hfill (II.15)

The other equations for the other legs of the Frenet frame are

\[ \frac{\partial}{\partial n} \vec{t} = \theta_{ns} \vec{n} + [\Omega_b + \tau] \vec{b} \]  
\hfill (II.16)

\[ \frac{\partial}{\partial n} \vec{n} = -\theta_{ns} \vec{t} - (\text{div}\vec{b}) \vec{b} \]  
\hfill (II.17)
\[
\frac{\partial}{\partial n} \vec{b} = -[\Omega_b + \tau] \vec{t} - (\text{div} \vec{b}) \vec{n}
\] (II.18)

\[
\frac{\partial}{\partial b} \vec{t} = \theta_b \vec{b} - [\Omega_n + \tau] \vec{n}
\] (II.19)

\[
\frac{\partial}{\partial b} \vec{n} = [\Omega_n + \tau] \vec{t} - \kappa + (\text{div} \vec{n}) \vec{b}
\] (II.20)

\[
\frac{\partial}{\partial b} \vec{b} = -\theta_b \vec{t} - [\kappa + (\text{div} \vec{n})] \vec{n}
\] (II.21)

The equations [?] for the time evolution of the Frenet frame yields

\[
\dot{\vec{t}} = -\tau \kappa \vec{n} + \kappa' \vec{b}
\] (II.22)

\[
\dot{\vec{n}} = -\kappa \tau \vec{t}
\] (II.23)

\[
\dot{\vec{b}} = -\kappa' \vec{t}
\] (II.24)

where \( \kappa' = \frac{\partial}{\partial s} \kappa \).

### III Unstable solutions of MHD plasma filaments and Alfven waves

Substitution of the above equations into the LHS of magnetic equation reads

\[
\nabla \times \vec{B}_0 = \mu_0 \vec{J}_0 = 0
\] (III.25)

Expansion of this equation on the Frenet frame yields

\[
B_0 [\vec{n} \times \partial_n \vec{t} + \vec{b} \times \partial_b \vec{t}] = 0
\] (III.26)

where we have used that \( B_0 \) is constant. Substitution of the above dynamical relations for the Frenet frame yields the following geometrical constraints

\[
\Omega_n = -\tau = 0
\] (III.27)

for planar filaments condition implies that torsion \( \tau = 0 \). The remaining constraint is

\[
\kappa = \Omega_b
\] (III.28)
where the $\Omega_a$, where $(a = s, n, b)$ represent the abnormalities [?]. In particular if $\Omega_s = 0$ we say that the filament bundles are geodesic. Note also that we consider that the curvature and torsion are perturbed but we assume that the torsion remains zero after perturbation so the motion is constrained to be perturbed in the plane. In mathematical terms, $\tau = \tau_0 + \tau_1$ and $\kappa = \kappa_0 + \kappa_1$ where $\tau_0 = 0$ and $\tau_1 = 0$. Since the current density is written as $\vec{J} = \rho \vec{v}$ one obtain
\[ \nabla \times \vec{B}_1 = \mu_0 \vec{J}_1 \tag{III.29} \]
which yields
\[ \partial_s B_1 + [\theta_{ns} + \theta_{nb}] B_1 = 0 \tag{III.30} \]
which reduces to
\[ [ik_\perp - \kappa_0] B_1 = \mu_0 J_1 \tag{III.31} \]
By calling $\theta := [\omega t - (k_\| s + k_\perp n)]$, where here $\omega = Re\omega + Im\omega$ and using the Moivre law
\[ exp[-i\theta] = \cos\theta - i\sin\theta \]
into equation (III.31) yields
\[ ik_\perp B_0^0 [\cos\theta - i\sin\theta] = (\kappa_0 B_0^0 + \mu_0 J_0^0) [\cos\theta - i\sin\theta] \tag{III.32} \]
This complex equation yields two scalar real equations which solution is
\[ [k_\perp B_0^0]^2 = [\kappa_0 B_0^0 + \mu_0 J_0^0]^2 \tag{III.33} \]
Since $B_0(s)$ the other Maxwell equation $\nabla \cdot \vec{B} = 0$ becomes
\[ \partial_s B_0 + [\theta_{bs} + \text{div}\vec{b}] B_0 = 0 \tag{III.34} \]
which yields the solution
\[ B_0 = -c_0 e^{\int (\theta_{bs} + \theta_{ns}) ds} \tag{III.35} \]
where $c_0$ is an integration constant. Now the perturbed equation is
\[ \nabla \cdot \vec{B}_1 = 0 \tag{III.36} \]
which reduces to the expression
\[ \partial_s B_1 + [\theta_{bs} + \text{div}\vec{b}] B_1 = 0 \tag{III.37} \]
which in turn produces the following complex equation

\[ ik_\| B^0_1 [\cos \theta - i \sin \theta] = -B^0_1 [\cos \theta - i \sin \theta] \]  

(III.38)

which being analogous to equation (III.32) can be solved in the same way to yield

\[ k_\| = \pm [\theta_{ns} + \theta_{sb}] \]  

(III.39)

Now let us solve the conservation of mass solution as

\[ i \omega \rho_1 = v_1 \rho_0 \text{div} \vec{b} \]  

(III.40)

Expanding this complex relation we are able to find out

\[ -i [Re \omega + i Im \omega] \rho_1 = v_1 \rho_0 \text{div} \vec{b} \]  

(III.41)

which yields two real equations which together yields

\[ [Re \omega \rho_1^0]^2 = [Im \omega \rho_1^0 + v_1^0 \rho_0]^2 \]  

(III.42)

In the branch \( Re \omega = 0 \) a simple solution of this equation allows us to investigate the instability of the plasma filaments, which is

\[ Im \omega = \frac{v_1^0 \rho_0}{\rho_1^0} \]  

(III.43)

Note that in this branch exponential instability is possible since \( Im \omega > 0 \) implies

\[ \frac{v_1^0 \rho_0}{\rho_1^0} > 0 \]  

(III.44)

Since the mass densities of the fluid are always positive the instabilities imposes constraints on the velocity \( v_1^0 > 0 \), thus if the velocity is negative or attractive the plasma filament is stable and the amplification of the magnetic field as happens in dynamos is not possible and magnetic field is damped. Let us now to investigate the remaining Maxwell MHD equations. They are

\[ \omega_0 := Im \omega = \pm \frac{k_\| LB_0 (\kappa_0 + \text{div} \vec{d})}{B_1^0} \]  

(III.45)

where \( \int ds = L \) which is the length of the filament. In the case of solar loops for example \( L = \pi R \) by considering that the half of the solar filament is under the surface of the Sun.
Writhing the expression for the Alfvén wave frequency as \( \omega_0^2 = [k||V_a]^2 \) and comparing it with the expression (III.45) one obtains the Alfvén velocity as

\[
V_a^2 = \left[ \frac{LB_0(\kappa_0 + \text{div} \vec{n})}{B_0^0} \right]^2
\]  

(III.46)

To simplify our physical analysis we assume that the plasma filament bundle obeys the relation \( \text{div} \vec{n} = 0 \) which reduces expression (III.46) to

\[
V_a = \pm \left[ \frac{LB_0\kappa_0}{B_0^0} \right]
\]  

(III.47)

once we have taken the plus sign or example we show that Alfvén waves propagates along the ilamet with the same sign of velocity as the one of the Frenet curvature of the equilibrium. The very last equation yields a relation etween components of the pressure as

\[
\frac{\rho_1^0}{\rho_0}p_0 = p_1^0
\]  

(III.48)

The solution described here is also well suitable for plasma filaments tokamaks where the curvature is not perturbed since is fixed by the topology of the plasma device.

**IV Conclusions**

In conclusion, plasma MHD instability is investigated in the framework of the Frenet inhomogeneous frame. Alfvén waves are found, where the velocity is expressed in terms of the Frenet curvature of planar filaments. This effect plays an important role in the construction of tokamaks and other plasma devices. Amplification of the magnetic fields is possible in the case of unstable filaments. Future work in the field of perturbations would include the plasma metric perturbations.

**Acknowledgements**

Thanks are due to CNPq and UERJ for financial supports.
References

[1] J. Freidberg, Plasma Physics and Fusion energy (2007) Cambridge University Press.

[2] V. Arnold and B. Khesin, Topological Methods in Hydrodynamics, Applied Mathematics Sciences 125 (1991).

[3] R. Ricca, Solar Physics 172,241 (1997). P.K. Newton,The N-Vortex problem: Analytic techniques,(Springer)2001.

[4] J. Thiffault and A.H.Boozer, The Onset of Dissipation in the Kinematic Dynamo,Los Alamos arXiv:nlin.CD/0209042v1.

[5] L.C. Garcia de Andrade, Physics of Plasmas 13, 022309 (2006).

[6] L.C. Garcia de Andrade, Phys. Scripta 73 (2006).

[7] P.K. Newton,The N-Vortex problem: Analytic techniques,(Springer)2001.