Universal Thermodynamics with state finder parameters: A general prescription

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Abstract

The work deals with universal thermodynamics for flat FRW model, considering interacting $n$−fluid system. By introducing the state finder parameters, the feasible regions in the $\{r, s\}$−plane has been examined for the validity of the Generalized Second Law of Thermodynamics. A general prescription for the evolution of the horizon has been derived for general non-static spherically symmetric space time and then it is applied to FRW universe. A general restriction on the matter has been found for the validity of GSLT.

Keywords: Holographic Dark energy, State finder parameter, Ricci’s length scale.

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1 Introduction

In 1974, Hawking \cite{1, 2} discovered that a black hole (BH) behaves like a black body, emitting thermal radiations with a temperature proportional to its surface gravity at the horizon and the related entropy is proportional to its horizon area \cite{3, 4}. This entropy, temperature and the mass of the BH are related by the first law of BH thermodynamics ($dM = TdS$) \cite{4, 5}. Due to the above thermodynamics-geometry relationship i.e.

$$(\text{temperature, Entropy}) \approx (\text{Surface gravity, Area}),$$

There had been speculations in the literature that thermodynamic laws are in some how related to the Einstein equations. The real picture came in 1995 when Jacobson \cite{6} was able to formulate Einstein equations from the thermodynamical laws. Subsequently, Padmanabhan \cite{7, 8} showed the equivalence in the otherway. Most discussions on BH thermodynamics have been related to static BH. Subsequently, Hayward \cite{9, 10} have initiated the study of non-static BH (dynamic BH) by introducing the concept of trapping horizon. For spherically symmetric space-times he was able to write down the Einstein field equations in a form, termed as 'Unified first law' and first law of thermodynamics for a dynamical BH can be obtained by projecting this unified first law along the trapping horizon. Subsequently, a lot of works has been going on related to universal thermodynamics. These studies \cite{11, 12, 13, 14, 15, 16, 17, 18, 19, 20} are mostly concentrated to homogeneous and isotropic FRW model of the universe bounded by apparent or event horizon and conditions for the validity of generalized second law of thermodynamics(GSLT) have been evaluated.

In the present work, we formulate a general prescription of universal thermodynamics for FRW model starting from an interacting $n$−fluid system. by introducing, the state finder parameters, we derive the feasible region in the $r, s$−plane for the validity of GSLT. also formulating the evolution of the horizon, a general restriction on the matter has been derived and it is applied to the FRW model.

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2 Universal Thermodynamics: A General Prescription

We consider a homogeneous, isotropic, flat, FRW model of the universe. Suppose the universe is filled up with interaction n-fluids having energy densities and thermodynamic pressures are \((\rho_i, p_i), i = 1, 2, \ldots n\). So the conservation equations are

\[ \dot{\rho}_i + 3H(\rho_i + p_i) = Q_i, \quad i = 1, 2, \ldots n \]  

where \(Q_i\) stands for the interaction term related to \(i\)-th fluid it may be positive or negative so that \(\sum_{i=1}^n Q_i = 0\). Now combining the \(n\)-equations in (1) we have

\[ \dot{\rho} + 3H(\rho + p) = 0 \]  

where \(\rho = \sum_{i=1}^n \rho_i\), is the total energy density of the \(n\)-fluids and \(p = \sum_{i=1}^n p_i\), is the total thermodynamic pressure of the \(n\)-fluid. Thus equation (2) can be considered as the energy conservation equation of the resulting single fluid having energy density \(\rho\) and thermodynamic pressure \(p\).

The Einstein field equations for the above interacting \(n\)-fluid system are given by

\[ 3H^2 = \rho \]  

and

\[ 2\dot{H} = -(\rho + p) \]  

To study the thermodynamics of the universe as a thermodynamical system we start with the amount of energy crossing the horizon \((R_h)\) during an infinitesimal interval of the time \(dt\) as

\[ -dE = 4\pi R_h^2 T_{\mu \nu} l^\mu l^\nu dt = 4\pi R_h^3 (\rho + p) dt \]  

Then from the clausius relation

\[ -dE = dQ = T_h dS_h \]  

We have,

\[ \frac{dS_h}{dt} = 4\pi R_h^3 \frac{H}{T_h} (\rho + p) \]  

where \(dQ\) is the heat flux crossing the horizon during the time interval \(dt\), \(T_h\) and \(S_h\) are respectively the temperature and entropy at the horizon.

Now the entropy \((S_I)\) of the matter of the universe inside the horizon can be related to its energy and pressure in the horizon by Gibb’s equation

\[ T dS_I = dE_I + pdV \]  

where \(S_I = \sum_{i=1}^n S_i\) is the sum of the entropies of different matter components, \(V = \frac{4}{3}\pi R_h^3\) is the volume of the universe enclosed by the horizon and \(E_I = \rho V\) is the total energy of the matter bounded by the horizon. Then using the conservation equation (2) and after a little bit of calculations we obtain the variation of the inside matter entropy as

\[ \frac{dS_I}{dt} = \frac{1}{T_h} \left[-4\pi R_h^3 H(\rho + p) + 4\pi R_h^3 (\rho + p) \dot{R}_h\right] \]  

where for equilibrium thermodynamics we chose the temperature of the inside matter same as that of the horizon. Thus combining (7) and (9) the time variation of total entropy (matter and horizon) is given by

\[ \frac{dS_{tot}}{dt} = 4\pi R_h^3 \frac{T_h}{T_h} \dot{R}_h(\rho + p) \]  

Thus generalized second law of thermodynamics (GSLT) (i.e. \(\frac{dS_{tot}}{dt} \geq 0\)) will always be true for any one of the following two possibilities:
(a) The equivalent single fluid for the interacting $n$–fluid system is in quintessence era and horizon is non-decreasing with the evolution.

(b) The equivalent single fluid system is in phantom era and horizon is contracting with the evolution.

It should be noted that the validity of the GSLT does not depend on any individual fluid nor on the interaction among the fluids.

Now we shall express the universal thermodynamics in terms of state finder parameters \( \{r, s\} \), proposed by Sahni et al [21, 22]. According to them, statefinder parameters are defined as

\[
\begin{align*}
  r &= \frac{1}{aH^3} \frac{d^3a}{dt^3} \quad \text{and} \quad s = \frac{(r - 1)}{3(q - \frac{1}{2})} \\
\end{align*}
\]

where \( q = -\frac{\ddot{a}}{a \dot{a}} \) is the deceleration parameter.

The parameters ‘\( r' \) and ‘\( s' \) are dimensionless and ‘\( r' \) forms the next step in the hierarchy of geometrical cosmological parameters after \( H \) and \( q \) [21]. It is speculated that these parameters together with coming SNAP observations may discriminate between different dark energy models. Further, trajectories in the \( \{r, s\} \) plane corresponding to different cosmological models demonstrate qualitatively different behaviour.

Now using these parameters \( q, r, \) and \( s \), the rate of change of the Hubble parameter can be written as

\[
\dot{H} = -(1 + q)H^2 = -\frac{3H^2}{2} \left[ 1 + \frac{2(r - 1)}{9s} \right]
\]

So from the Einstein field equation (4) we have

\[
\rho + p = 3H^2 \left( 1 + \frac{2(r - 1)}{9s} \right)
\]

Thus variation of total entropy can be expressed in terms of \( \{r, s\} \) parameters as

\[
\frac{dS_{\text{tot}}}{dt} = \frac{12\pi R_h^2 \left( 1 + \frac{2(r - 1)}{9s} \right)}{T_h} \dot{R}_h
\]

If we now draw the straight line \( 2(r - 1) + 9s = 0 \) in the \( \{r, s\} \)–plane then region I (see Fig1) corresponds to the valid region for GSLT to be satisfied provided horizon radius grows with the evolution of the universe otherwise region II will be the possible region for validity of GSLT. Therefore, we can classify regions in the \( \{r, s\} \)–plane where universe can be considered as a thermodynamical system.
Further, for the commonly used horizon in the literature, namely the apparent horizon given by

$$R_A = \frac{1}{H}$$  \hspace{1cm} (15)

we have

$$\dot{R}_A = -\frac{1}{H^2} \dot{H} = \frac{3}{2} \left\{ 1 + \frac{2(r - 1)}{9s} \right\}$$  \hspace{1cm} (16)

So

$$\frac{dS_{tot}}{dt} = \frac{18\pi R_L^2 H^2}{T_H} \left\{ 1 + \frac{2(r - 1)}{9s} \right\}^2 \geq 0$$  \hspace{1cm} (17)

Thus we have the known result that GSLT is always satisfied at the apparent horizon.

On the other hand, the event horizon (which is defined for accelerating model) is defined as

$$R_E = a \int_t^\infty \frac{dt}{a}$$  \hspace{1cm} (18)

and we have

$$\dot{R}_E = H(R_E - R_A)$$

and it cannot be expressed in terms of \( \{r, s\} \) parameters. So in this case \( R_E > R_A \) and region I or region II is the possible solution for the validity of GSLT. However, if we use the Ricci’s length scale namely

$$R_L = 2H^2 + \dot{H}^{-\frac{1}{2}}$$  \hspace{1cm} (19)

as the boundary of the universe, then

$$\dot{R}_L = \frac{1}{2} R_L^3 H^3 (q + 2 - r) = \frac{1}{2} R_L^3 H^3 \left\{ \frac{3}{2} - (r - 1)(1 - \frac{1}{3s}) \right\}$$  \hspace{1cm} (20)

The argument behind \( R_L \) as the IR cut off in the holographic bound \cite{23, 24} is that it corresponds to the size of the maximal perturbation, leading to the formation of a black hole. Thus in this case the possible part of the \( \{r, s\} \)-plane for the validity of GSLT is classified as
Region A: \[ \frac{3}{2} - (r - 1) \left(1 - \frac{1}{3s}\right) > 0 \text{ and } 1 + \frac{2(r - 1)}{9s} > 0 \]

or

Region B: \[ \frac{3}{2} - (r - 1) \left(1 - \frac{1}{3s}\right) < 0 \text{ and } 1 + \frac{2(r - 1)}{9s} < 0 \]

In figure 2 we have identified regions A and B which are the valid region for the universal thermodynamics.

3 Evolution of the horizon in general spherically symmetric black hole

In this section, we study the growth or contraction of the horizon of a general non-static spherically symmetric black hole due to exchange of matter with its environment and then we extend this idea to the universal thermodynamics. Any spherically symmetric gravitational field can be written as

\[ dS^2 = -f(r, t)dt^2 + \frac{1}{g(r, t)}dr^2 + r^2d\Omega_2^2 \]  \hspace{1cm} (21)

The horizon is located at \( r = r_h \) such that \( f(r_h, t) = 0 = g(r_h, t) \). To have a smooth system across the horizon we make the following Painleve-type coordinate transformation:

\[ dt \rightarrow dt - \sqrt{\frac{1-g}{fg}} \]  \hspace{1cm} (22)

so that metric (21) becomes

\[ dS^2 = -f dt^2 + 2\sqrt{\frac{f(1-g)}{g}}dtdr + dr^2 + r^2d\Omega_2^2 \]  \hspace{1cm} (23)

The incoming and outgoing radial null geodesics are defined by \( dS^2 = 0 = d\theta = d\phi \) and so from equation (23)

\[ \frac{dr}{dt} = -\sqrt{\frac{f}{g}} \left[ \sqrt{1-g} \pm 1 \right] \]  \hspace{1cm} (24)

Now we define the evolving horizon by the condition

\[ \sqrt{1-g} = 1 \]  \hspace{1cm} (25)

Thus if \( \sqrt{1-g} > 1 \) then out going light rays are dragged backwards towards origin. As a result we can define a function \( r_h(t) \) such that \( g(r_h(t), t) = 0 \) and is termed as evolving horizon. We can identify the mass function as

\[ 2m(r, t) = r[1 - g(r, t)] \]  \hspace{1cm} (26)

and hence the horizon

\[ 2m(r_h(t), t) = r_h(t) \]  \hspace{1cm} (27)

Further due to spherical symmetry, the Hawking-Isreal quasi local mass function \[ (25) \] namely,

\[ m_{HI}(r, t) = \frac{r}{2} [1 - g^{\mu\nu}\nabla_\mu r \nabla_\nu r] \]  \hspace{1cm} (28)

is identified with \( m(r, t) \) defined in \[ (26) \]. So we can identify the function \( 'm' \) as the 'mass inside radius \( r \) at time \( t \)'.

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Now taking total time derivative of both sides of equation \((27)\) we have

\[
\dot{m}(r_h(t), t) = \frac{1 - 2m'(r_h(t), t)}{16\pi r_h(t)} dA_h(t) \frac{dt}{dt}
\]  

(29)

where \(A_h = 4\pi r_h^2\) is the surface area of the horizon and here dot and dash stands for partial derivatives with respect to \(t'\) and \(r'\) respectively.

Now comparing with the first law of black hole mechanics namely,

\[
dm = \frac{k}{8\pi} dA,
\]  

(30)

the surface gravity on the horizon has the expression

\[
k_h = \frac{1 - 2m'(r_h(t), t)}{2r_h(t)}
\]  

(31)

It is to be noted that the surface gravity \(k\) can also be obtained from the evolution equation of outward radial null vector

\[
l^\mu \nabla_\nu l^\nu = kl^\nu
\]  

(32)

where

\[
l^\mu = \left\{ \sqrt{g}, \sqrt{1 - g}, 0, 0 \right\}
\]  

(33)

is the outgoing radial null vector.

As the metric coefficient \(g(r, t)\) is positive definite for \(r > r_h(t)\) so from \((26)\) \(2m(r, t) < r\) for \(r > r_h(t)\) or equivalently

\[1 - 2m'(r_h(t), t) \geq 0 \text{ i.e. } r_h g'(r_h, t) \geq 0
\]  

(34)

Hence the surface gravity on the horizon is positive definite. Further, using Maple (or similar software) one can calculate (for details see ref \([2]\)) for any \(r\)

\[
G_{\mu\nu} l^\mu l^\nu = \frac{2}{r^2} \sqrt{\frac{g}{f}} \frac{\dot{m}(r, t)}{2} \frac{\partial}{\partial r} \left\{ \sqrt{f} \right\} \left( 1 - \sqrt{1 - g} \right)^2
\]  

(35)

So on the horizon,

\[
\dot{m}(r_h(t), t) = \frac{r_h^2}{2} \sqrt{f'(r_h, t)} \frac{g'(r_h, t)}{g'(r_h, t)} (8\pi T_{\mu\nu} l^\mu l^\nu),
\]  

(36)

where in deriving the last equation we have used Einstein equations \(G_{\mu\nu} = 8\pi T_{\mu\nu}\). So from equation \((26)\) we obtain

\[
\frac{dr_h}{dt} = 8\pi r_h T_{\mu\nu} l^\mu l^\nu \sqrt{f'(r_h(t), t)} g'(r_h(t), t)^3
\]  

(37)

Further using \((27)\) the total time derivative of the mass function at the horizon gives

\[
\frac{dm(r_h(t), t)}{dt} = 4\pi r_h T_{\mu\nu} l^\mu l^\nu \sqrt{f'} g'^3
\]  

(38)

This gives the total mass change i.e. contributions both from mass flux across the horizon and from the motion of the horizon itself. Thus due to inequality \((34)\) we may conclude that as long as null energy condition (NEC) is satisfied both the horizon as well as the mass bounded by the horizon can never decrease.

As a particular case if we consider the flat FRW model of the universe we can write the metric similar to \((23)\) as
\[ dS^2 = -(1 - H^2 R^2)dt^2 - 2HRdRdt + dR^2 + R^2 d\Omega_2^2 \] (39)

So the horizon is located at \( R = R_h = \frac{1}{H} = R_A \), the apparent horizon. The mass term is defined as

\[ m(R, t) = \frac{1}{2} H^2 R^3 \] (40)

and hence the mass bounded by the horizon is

\[ m(R_h(t), t) = \frac{R_h(t)}{2}, \] (41)

which is the usual Mishner-Sharp mass at the apparent horizon of the FRW model. Hence from (37) the evolution of the horizon is given by

\[ \frac{dR_h(t)}{dt} = 4\pi R_h^2 T_{\mu \nu}l^\mu l^\nu \] (42)

Hence from (14) using (42) we can say that validity of GSLT does not depend on the type of fluid we are considering- the only criteria is whether fluid satisfies the null energy condition or not.

4 Summary and Conclusion:

In this work we have presented a general formulation of universal thermodynamics for flat, homogeneous and isotropic FRW model of the universe. We have started with an interacting \( n^-\) fluid system and have shown that the thermodynamics of the universe does not depend on any individual fluid nor on the interactions, it depends on the total matter density and thermodynamic pressure of the fluids i.e. for thermodynamical study we can take the interacting \( n^-\) fluid system as a single fluid satisfying the energy conservation relation \( \Box \). Then we have introduced the state finder parameters and expressed the restrictions for the validity of GSLT in terms of these parameters. Also in the \( \{r, s\}^-\) plane we have shown the feasible regions graphically. In section (III), we have derived the evolution of the horizon for a general non-static spherically symmetric gravitating system and we have found (see equation (38)) that horizon radius increases or decreases depending on whether null energy condition is satisfied or not. Then we have calculated the time derivative of the horizon radius (see equation (42)) for the FRW model. Therefore we can conclude that for a general prescription of universal thermodynamics in FRW model, we need to know the feasible region in \( (r, s)^-\) parameter plane and have to examine whether the resulting single fluid satisfies the NEC or not. For future work, we shall try, whether similar general features for universal thermodynamics can be obtained for a general space-time model of the universe.

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