WEYL GROUPS OF FINE GRADINGS
ON SIMPLE LIE ALGEBRAS OF TYPES A, B, C AND D

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Abstract. Given a grading $\Gamma : \mathcal{L} = \bigoplus_{g \in G} \mathcal{L}_g$ on a nonassociative algebra $\mathcal{L}$ by an abelian group $G$, we have two subgroups of $\text{Aut}(\mathcal{L})$: the automorphisms that stabilize each component $\mathcal{L}_g$ (as a subspace) and the automorphisms that permute the components. By the Weyl group of $\Gamma$ we mean the quotient of the latter subgroup by the former. In the case of a Cartan decomposition of a semisimple complex Lie algebra, this is the automorphism group of the root system, i.e., the so-called extended Weyl group. A grading is called fine if it cannot be refined. We compute the Weyl groups of all fine gradings on simple Lie algebras of types $A, B, C$ and $D$ (except $D_4$) over an algebraically closed field of characteristic different from 2.

1. Introduction

In [EKb], we computed the Weyl groups of all fine gradings on matrix algebras, the Cayley algebra $\mathcal{C}$ and the Albert algebra $\mathcal{A}$ over an algebraically closed field $F$ (char $F \neq 2$ in the case of the Albert algebra). It is well known that $\text{Der}(\mathcal{C})$ is a simple Lie algebra of type $G_2$ (char $F \neq 2, 3$) and $\text{Der}(\mathcal{A})$ is a simple Lie algebra of type $F_4$ (char $F \neq 2$). Since the automorphism group schemes of $\mathcal{C}$ and $\text{Der}(\mathcal{C})$, respectively $\mathcal{A}$ and $\text{Der}(\mathcal{A})$, are isomorphic, the classification of fine gradings on $\text{Der}(\mathcal{C})$, respectively $\text{Der}(\mathcal{A})$, is the same as that on $\mathcal{C}$, respectively $\mathcal{A}$ [EKa] and, moreover, the Weyl groups of the corresponding fine gradings are isomorphic. The situation with fine gradings on the simple Lie algebras belonging to series $A, B, C$ and $D$ is more complicated, because the fine gradings on matrix algebras yield only a part of the fine gradings on the simple Lie algebras of series $A$ (so-called Type I gradings). In order to obtain the fine gradings for series $B, C$ and $D$ and the remaining (Type II) fine gradings for series $A$, one has to consider fine $\varphi$-gradings on matrix algebras, which were introduced and classified in [Eld10].

The purpose of this paper is to compute the Weyl groups of all fine gradings on the simple Lie algebras of series $A, B, C$ and $D$, with the sole exception of type $D_4$ (which differs from the other types due to the triality phenomenon), over an algebraically closed field $F$ of characteristic different from 2. To achieve this, we first determine the automorphisms of each fine $\varphi$-grading on the matrix algebra $\mathcal{R} = M_n(F), n \geq 3$, and then use the transfer technique of [BK10] to obtain the Weyl group of the corresponding fine grading on the simple Lie algebra $\mathcal{L} = $
(\mathcal{R}, \mathcal{R})/(Z(\mathcal{R}) \cap [\mathcal{R}, \mathcal{R}]) \text{ or } \mathcal{K}(\mathcal{R}, \varphi) \text{ stands for the set of skew-symmetric elements with respect to } \varphi.

We adopt the terminology and notation of [EKb], which is recalled in Section 2 for convenience of the reader. In Section 3, we restate the classification of fine \( \varphi \)-gradings on matrix algebras [Eld10] in more explicit terms and determine the relevant automorphism groups of each fine \( \varphi \)-grading (Theorem 3.12). In Section 4, we deal with the simple Lie algebras of series \( A \) (Theorems 4.6 and 4.7) and, in Section 5, with those of series \( B, C \) and \( D \) (Theorems 5.6 and 5.7).

### 2. Generalities on gradings

Let \( A \) be an algebra (not necessarily associative) over a field \( \mathbb{F} \) and let \( G \) be a group (written multiplicatively).

**Definition 2.1.** A \( G \)-grading on \( A \) is a vector space decomposition

\[
\Gamma : A = \bigoplus_{g \in G} A_g
\]

such that

\[A_g A_h \subset A_{gh} \quad \text{for all } g, h \in G.\]

If such a decomposition is fixed, we will refer to \( A \) as a \( G \)-graded algebra. The nonzero elements \( a \in A_g \) are said to be homogeneous of degree \( g \); we will write \( \deg a = g \). The support of \( \Gamma \) is the set \( \text{Supp } \Gamma := \{ g \in G \mid A_g \neq 0 \} \).

There are two natural ways to define equivalence relation on graded algebras. We will use the term “isomorphism” for the case when the grading group is a part of definition and “equivalence” for the case when the grading group plays a secondary role. Let

\[
\Gamma : A = \bigoplus_{g \in G} A_g \quad \text{and} \quad \Gamma' : B = \bigoplus_{h \in H} B_h
\]

be two gradings on algebras, with supports \( S \) and \( T \), respectively.

**Definition 2.2.** We say that \( \Gamma \) and \( \Gamma' \) are equivalent if there exists an isomorphism of algebras \( \psi : A \rightarrow B \) and a bijection \( \alpha : S \rightarrow T \) such that \( \psi(A_s) = B_{\alpha(s)} \) for all \( s \in S \). Any such \( \psi \) will be called an equivalence of \( \Gamma \) and \( \Gamma' \) (or of \( A \) and \( B \) if the gradings are clear from the context).

The algebras graded by a fixed group \( G \) form a category where the morphisms are the homomorphisms of \( G \)-graded algebras, i.e., algebra homomorphisms \( \psi : A \rightarrow B \) such that \( \psi(A_g) \subset B_g \) for all \( g \in G \).

**Definition 2.3.** In the case \( G = H \), we say that \( \Gamma \) and \( \Gamma' \) are isomorphic if \( A \) and \( B \) are isomorphic as \( G \)-graded algebras, i.e., there exists an isomorphism of algebras \( \psi : A \rightarrow B \) such that \( \psi(A_g) = B_g \) for all \( g \in G \).

It is known that if \( \Gamma \) is a grading on a simple Lie algebra, then \( \text{Supp } \Gamma \) generates an abelian group (see e.g. [Koc09 Proposition 3.3]). From now on, we will assume that our grading groups are abelian. Given a group grading \( \Gamma \) on an algebra \( A \), there are many groups \( G \) such that \( \Gamma \) can be realized as a \( G \)-grading, but there is one distinguished group among them [PZ89].
Definition 2.4. Suppose that $\Gamma$ admits a realization as a $G_0$-grading for some group $G_0$. We will say that $G_0$ is a universal group of $\Gamma$ if, for any other realization of $\Gamma$ as a $G$-grading, there exists a unique homomorphism $G_0 \to G$ that restricts to identity on $\text{Supp} \, \Gamma$.

One shows that the universal group, which we denote by $U(\Gamma)$, exists and depends only on the equivalence class of $\Gamma$. Indeed, $U(\Gamma)$ is generated by $S = \text{Supp} \, \Gamma$ with defining relations $s_is_j = s_3$ whenever $0 \neq A_{s_1} A_{s_2} \subset A_{s_3}$ ($s_i \in S$).

As in [PZ99], we associate to $\Gamma$ three subgroups of the automorphism group $\text{Aut}(A)$ as follows.

**Definition 2.5.** The automorphism group of $\Gamma$, denoted $\text{Aut}(\Gamma)$, consists of all automorphisms of $A$ that permute the components of $\Gamma$. Each $\psi \in \text{Aut}(\Gamma)$ determines a self-bijection $\alpha = \alpha(\psi)$ of the support $S$ such that $\psi(A_s) = A_{\alpha(s)}$ for all $s \in S$. The stabilizer of $\Gamma$, denoted $\text{Stab}(\Gamma)$, is the kernel of the homomorphism $\text{Aut}(\Gamma) \to \text{Sym}(S)$ given by $\psi \mapsto \alpha(\psi)$. Finally, the diagonal group of $\Gamma$, denoted $\text{Diag}(\Gamma)$, is the subgroup of the stabilizer consisting of all automorphisms $\psi$ such that the restriction of $\psi$ to any homogeneous component of $\Gamma$ is the multiplication by a (nonzero) scalar.

Thus $\text{Aut}(\Gamma)$ is the group of self-equivalences of the graded algebra $A$ and $\text{Stab}(\Gamma)$ is the group of automorphisms of the graded algebra $\tilde{A}$. Also, $\text{Diag}(\Gamma)$ is isomorphic to the group of characters of $U(\Gamma)$ via the usual action of characters on $\tilde{A}$: if $\Gamma$ is a $G$-grading (in particular, we may take $G = U(\Gamma)$), then any character $\chi \in \tilde{G}$ acts as an automorphism of $\tilde{A}$ by setting $\chi \ast a = \chi(g)a$ for all $a \in A_g$ and $g \in G$.

If $\dim A < \infty$, then $\text{Diag}(\Gamma)$ is a diagonalizable algebraic group (quasitorus). If, in addition, $\mathbb{F}$ is algebraically closed and $\text{char} \mathbb{F} = 0$, then $\Gamma$ is the eigenspace decomposition of $A$ relative to $\text{Diag}(\Gamma)$ (see e.g. [Koc09]), the group $\text{Stab}(\Gamma)$ is the centralizer of $\text{Diag}(\Gamma)$, and $\text{Aut}(\Gamma)$ is its normalizer. If we want to work over an arbitrary field $\mathbb{F}$, we can define the subgroups $\text{Diag}(\Gamma)$ of the automorphism group scheme $\text{Aut}(A)$ as follows:

$$\text{Diag}(\Gamma)(\mathbb{F}) := \{ f \in \text{Aut}_{\mathbb{F}}(A \otimes \mathbb{F}) \mid f|_{A_g \otimes \mathbb{F}} \in \mathbb{F}^\times \idd_{A_g \otimes \mathbb{F}} \text{ for all } g \in G \}$$

for any unital commutative associative algebra $S$ over $\mathbb{F}$. Thus $\text{Diag}(\Gamma)$ is the group of $\mathbb{F}$-points of $\text{Diag}(\Gamma)$. One checks that $\text{Diag}(\Gamma) = U(\Gamma)^D$, the Cartier dual of $U(\Gamma)$, also $\text{Stab}(\Gamma)$ is the centralizer of $\text{Diag}(\Gamma)$ and $\text{Aut}(\Gamma)$ is its normalizer with respect to the action of $\text{Aut}(A)$ on $\text{Aut}(A)$ by conjugation (see e.g. [Eka99a] §2.2).

**Definition 2.6.** The quotient group $\text{Aut}(\Gamma)/\text{Stab}(\Gamma)$, which is a subgroup of $\text{Sym}(S)$, will be called the Weyl group of $\Gamma$ and denoted by $W(\Gamma)$.

It follows from the universal property of $U(\Gamma)$ that, for any $\psi \in \text{Aut}(\Gamma)$, the bijection $\alpha(\psi) : \text{Supp} \, \Gamma \to \text{Supp} \, \Gamma$ extends to a unique automorphism of $U(\Gamma)$. This gives an action of $\text{Aut}(\Gamma)$ by automorphisms of $U(\Gamma)$. Since the kernel of this action is $\text{Stab}(\Gamma)$, we may regard $W(\Gamma) = \text{Aut}(\Gamma)/\text{Stab}(\Gamma)$ as a subgroup of $\text{Aut}(U(\Gamma))$. Given a $G$-grading $\Gamma : A = \bigoplus_{g \in G} A_g$ and a group homomorphism $\alpha : G \to H$, we obtain the induced $H$-grading $^\alpha \Gamma : A = \bigoplus_{h \in H} A_h^\alpha$ by setting $A_h^\alpha = \bigoplus_{g \in \alpha^{-1}(h)} A_g$. Clearly, an automorphism $\alpha$ of $U(\Gamma)$ belongs to $W(\Gamma)$ if and only if the $U(\Gamma)$-gradings $^\alpha \Gamma$ and $\Gamma$ are isomorphic.

Given gradings $\Gamma : A = \bigoplus_{g \in G} A_g$ and $\Gamma' : A = \bigoplus_{h \in H} A_h'$, we say that $\Gamma'$ is a coarsening of $\Gamma$, or that $\Gamma$ is a refinement of $\Gamma'$, if for any $g \in G$ there exists
$h \in H$ such that $A_g \subset A_g'$. The coarsening (or refinement) is said to be proper if the inclusion is proper for some $g$. (In particular, $\alpha \Gamma$ is a coarsening of $\Gamma$, which is not necessarily proper.) A grading $\Gamma$ is said to be fine if it does not admit a proper refinement in the class of (abelian) group gradings. Any $G$-grading on a finite-dimensional algebra $A$ is induced from some fine grading $\Gamma$ by a homomorphism $\alpha: U(\Gamma) \to G$. The classification of fine gradings on $A$ up to equivalence is the same as the classification of maximal diagonalizable subgroupschemes of $\text{Aut}(A)$ up to conjugation by $\text{Aut}(A)$ (see e.g. [EKa, §2.2]). Fine gradings on simple Lie algebras belonging to the series $A$, $B$, $C$ and $D$ (including $D_4$) were classified in [Eld10] assuming $\mathbb{F}$ algebraically closed of characteristic 0. If we replace automorphism groups by automorphism group schemes, as was done in [BK10], then the arguments of [Eld10] for all cases except $D_4$ (which required a completely different method) work under the much weaker assumption — which we adopt from now on — that $\mathbb{F}$ is algebraically closed of characteristic different from 2.

3. Fine $\varphi$-gradings on matrix algebras

The goal of this section is to determine certain automorphism groups of fine $\varphi$-gradings on matrix algebras. These groups will be used in the next two sections to compute the Weyl groups of fine gradings on simple Lie algebras of series $A$, $B$, $C$ and $D$.

3.1. Classification of fine $\varphi$-gradings on matrix algebras. Here we present the results of [Eld10] §3 in a more explicit form. We also introduce certain objects that will appear throughout the paper.

**Definition 3.1.** Let $A$ be an algebra and let $\varphi$ be an anti-automorphism of $A$. A $G$-grading $\Gamma: A = \bigoplus_{g \in G} A_g$ is said to be a $\varphi$-grading if $\varphi(A_g) = A_{\varphi g}$ for all $g \in G$ (i.e., $\varphi$ is an anti-automorphism of the $G$-graded algebra $A$) and $\varphi^2 \in \text{Diag}(\Gamma)$. The universal group of a $\varphi$-grading is defined disregarding $\varphi$.

We have natural concepts of isomorphism and equivalence for $\varphi$-gradings. In addition, we will need another relation, which is weaker than equivalence.

**Definition 3.2.** If $\Gamma_1$ is a $\varphi_1$-grading on $A$ and $\Gamma_2$ is a $\varphi_2$-gradings on $B$, we will say that $(\Gamma_1, \varphi_1)$ is isomorphic (respectively, equivalent) to $(\Gamma_2, \varphi_2)$ if there exists an isomorphism (respectively, equivalence) of graded algebras $\psi: A \to B$ such that $\varphi_2 = \psi \varphi_1 \psi^{-1}$. In the special case $A = B$ and $\varphi_1 = \varphi_2$, we will simply say that $\Gamma_1$ is isomorphic (respectively, equivalent) to $\Gamma_2$. We will say that $(\Gamma_1, \varphi_1)$ is weakly equivalent to $(\Gamma_2, \varphi_2)$ if there exists an equivalence of graded algebras $\psi: A \to B$ such that $\xi \varphi_2 = \psi \varphi_1 \psi^{-1}$ for some $\xi \in \text{Diag}(\Gamma_2)$.

Note that if $\varphi$ is an involution, then the condition $\varphi^2 \in \text{Diag}(\Gamma)$ is satisfied for any $\Gamma$. Also, any $\varphi$-grading $\Gamma$ on $A$ restricts to the space of skew-symmetric elements $\mathcal{K}(A, \varphi)$.

Suppose $\mathcal{R}$ is a matrix algebra equipped with a $G$-grading $\Gamma$. Then $\mathcal{R}$ is isomorphic to $\text{End}_D(V)$ where $D$ is a matrix algebra with a division grading (i.e., a grading that makes it a graded division algebra) and $V$ is a graded right $D$-module (which is necessarily free of finite rank). Let $T \subset G$ be the support of $D$. Then $T$ is a group and $D$ can be identified with a twisted group algebra $\mathbb{F}^\sigma T$ for some 2-cocycle $\sigma: T \times T \to \mathbb{F}^\times$, i.e., $D$ has a basis $X_t$, $t \in T$, such that $X_u X_t = \sigma(u, v)X_{ut}$ for all
$u,v \in T$ (we may assume $X_e = I$, the identity element of $\mathcal{D}$). Let $\beta(u,v) = \frac{\sigma(u,v)}{\sigma(v,u)}$, so

$$X_uX_v = \beta(u,v)X_vX_u.$$ 

Then $\beta : T \times T \to \mathbb{F}^\times$ is a nondegenerate alternating bicharacter — see e.g. [BK10 §2]. A division grading on a matrix algebra with a given support $T$ and bicharacter $\beta$ can be constructed as follows. Since $\beta$ is nondegenerate and alternating, $T$ admits a “symplectic basis”, i.e., there exists a decomposition of $T$ into the direct product of cyclic subgroups:

$$T = (H'_1 \times H''_1) \times \cdots \times (H'_n \times H''_n)$$

such that $H'_i \times H''_i$ and $H'_j \times H''_j$ are $\beta$-orthogonal for $i \neq j$, and $H'_i$ and $H''_i$ are in duality by $\beta$. Denote by $\ell_i$ the order of $H'_i$ and $H''_i$. (We may assume without loss of generality that $\ell_i$ are prime powers.) If we pick generators $a_i$ and $b_i$ for $H'_i$ and $H''_i$, respectively, then $\varepsilon_i := \beta(a_i, b_i)$ is a primitive $\ell_i$-th root of unity, and all other values of $\beta$ on the elements $a_1, b_1, \ldots, a_r, b_r$ are 1. Define the following elements of the algebra $M_{\ell_1}(\mathbb{F}) \otimes \cdots \otimes M_{\ell_r}(\mathbb{F})$:

$$X_{a_i} = I \otimes \cdots I \otimes X_i \otimes I \otimes \cdots I \quad \text{and} \quad X_{b_i} = I \otimes \cdots I \otimes Y_i \otimes I \otimes \cdots I,$$

where

$$X_i = \begin{bmatrix} \varepsilon_i^{n-1} & 0 & 0 & \cdots & 0 & 0 \\ 0 & \varepsilon_i^{n-2} & 0 & \cdots & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots \\ 0 & 0 & 0 & \cdots & \varepsilon_i & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{bmatrix} \quad \text{and} \quad Y_i = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix}$$

are the generalized Pauli matrices in the $i$-th factor, $M_{\ell_i}(\mathbb{F})$. Finally, set

$$X_{(a_1^{i_1}, b_1^{j_1}, \ldots, a_r^{i_r}, b_r^{j_r})} = X_{a_1}^{i_1}X_{b_1}^{j_1} \cdots X_{a_r}^{i_r}X_{b_r}^{j_r}.$$ 

Identify $M_{\ell_1}(\mathbb{F}) \otimes \cdots \otimes M_{\ell_r}(\mathbb{F})$ with $M_{\ell}(\mathbb{F})$, $\ell = \ell_1 \cdots \ell_r = \sqrt{|T|}$, via Kronecker product. Then

$$M_{\ell}(\mathbb{F}) = \bigoplus_{t \in T} \mathbb{F} X_t$$

is a division grading with support $T$ and bicharacter $\beta$.

Let $\varphi$ be an anti-automorphism of $\mathcal{R}$ such that $\Gamma$ is a $\varphi$-grading. It is shown in [Edi10 §3] that there exists an involution $\varphi_0$ of the graded algebra $\mathcal{D}$ and a $\varphi_0$-sesquilinear form $B : V \times V \to \mathcal{D}$, which is nondegenerate, homogeneous and balanced, such that, for all $r \in \mathcal{R}$, $\varphi(r)$ is the adjoint of $r$ with respect to $B$, i.e., $B(x, \varphi(r)y) = B(rx, y)$ for all $x, y \in V$ and $r \in \mathcal{R}$. By $\varphi_0$-sesquilinear we mean that $B$ is $\mathbb{F}$-bilinear and, for all $x, y \in V$ and $d \in \mathcal{D}$, we have $B(xd, y) = \varphi_0(d)B(x, y)$ and $B(x, yd) = B(x, y)d$; by balanced we mean that, for all homogeneous $x, y \in V$, $B(x, y) = 0$ is equivalent to $B(y, x) = 0$. Moreover, the existence of $\varphi_0$ forces $T$ to be an elementary 2-group. The pair $(\varphi_0, B)$ is uniquely determined by $\varphi$ up to the following transformations: for any nonzero homogeneous $d \in \mathcal{D}$, we may simultaneously replace $\varphi_0$ by $\varphi_0' : a \mapsto d\varphi_0(a)d^{-1}$ and $B$ by $B' = dB$. Using Pauli matrices (of order 2) as above to construct a realization of $\mathcal{D}$, we see that matrix transpose $X \mapsto {}^t X$ preserves the grading: for any $u \in T$, the transpose of $X_u$ equals $\pm X_u$. It follows from [BK10 Proposition 2.3] that $(\varphi_0, B)$ can be adjusted so that $\varphi_0$ coincides with the matrix transpose. We will always assume that $(\varphi_0, B)$
Denote by \( \tilde{\beta} \) generated by \( T \) from the above relations that \( \tilde{\beta} : T \to \{ \pm 1 \} \) is a quadratic form whose polar form is the bicharacter \( \beta : T \times T \to \{ \pm 1 \} \).

We will say that a \( \varphi \)-grading is fine if it is not a proper coarsening of another \( \varphi \)-grading. The following construction of fine \( \varphi \)-gradings on matrix algebras was given in [Eld01] starting from \( D \). We start from \( T \), an elementary 2-group of even dimension, i.e., \( T = \mathbb{Z}_2^{\dim T} \), which we continue to write multiplicatively. Let \( \beta \) be a nondegenerate alternating bicharacter on \( T \). Fix a realization, \( D \), of the matrix algebra endowed with a division grading with support \( T \) and bicharacter \( \beta \), and let \( \varphi_0 \) be the matrix transpose on \( D \). Let \( q \geq 0 \) and \( s \geq 0 \) be two integers. Let

\[
\tau = (t_1, \ldots, t_q), \quad t_i \in T.
\]

Denote by \( \tilde{G} = \tilde{G}(T, q, s, \tau) \) the abelian group generated by \( T \) and the symbols \( \tilde{g}_1, \ldots, \tilde{g}_{q+2s} \) with defining relations

\[
\tilde{g}_1 t_1 = \ldots = \tilde{g}_q t_q = \tilde{g}_{q+1} g_{q+2} = \ldots = \tilde{g}_{q+2s-1} g_{q+2s}.
\]

**Definition 3.3.** Let \( M(D, q, s, \tau) \) be the \( \tilde{G} \)-graded algebra \( \text{End}_D(V) \) where \( V \) has a \( D \)-basis \( \{ v_1, \ldots, v_{q+2s} \} \) with \( \deg v_i = \tilde{g}_i \). Let \( n = (q+2s)2^{\dim T} \) and \( R = M_n(F) \).

The grading \( \Gamma \) on \( R \) obtained by identifying \( R \) with \( M(D, q, s, \tau) \) will be denoted by \( \Gamma_M(D, q, s, \tau) \). In other words, we define this grading by identifying \( R = M_{q+2s}(D) \) and setting \( \deg(E_{ij} \otimes X_t) := \tilde{g}_i t_1^{-1} \). By abuse of notation, we will also write \( \Gamma_M(T, q, s, \tau) \).

Let \( \tilde{G}^0 \) be the subgroup of \( \tilde{G} \) generated by \( \text{Supp } \Gamma \), which consists of the elements \( z_{i,j,t} := \tilde{g}_i t_t^{-1}, t \in T \) (so \( z_{i,t,t} = t \) for all \( t \in T \)). Set \( z_i := z_{i,1+1,e} \) for \( i = 1, \ldots, q \) (if \( q = 0 \)).

Let \( \tilde{G}^1 \) be the subgroup generated by \( T \) and \( z_1, \ldots, z_{q-1} \). Let \( \tilde{G}^2 \) be the subgroup generated by \( z_1, \ldots, z_q \) if \( q = 0 \) and by \( z_{q+1}, \ldots, z_{q+s-1} \) if \( q > 0 \). Then it is clear from the above relations that \( \tilde{G}^0 = \tilde{G}^1 \times \tilde{G}^2, \tilde{G}^2 \cong \mathbb{Z}_2^s \), while \( \tilde{G}^1 = T \) if \( q = 0 \) and \( \tilde{G}^1 \cong \mathbb{Z}_2^{2\dim T-2} \times \mathbb{Z}_2^{4\dim T_0} \times \mathbb{Z}_2^{4\dim T_0} \) if \( q > 0 \), where \( T_0 \) is the subgroup of \( T \) generated by the elements \( t_i t_{i+1}, i = 1, \ldots, q - 1 \). To summarize:

\[
\tilde{G}^0 \cong \mathbb{Z}_2^{2\dim T-2} \times \mathbb{Z}_2^{4\dim T_0} \times \mathbb{Z}_2^s.
\]

Note that relations (2) are also equivalent to the following:

\[
\begin{align*}
z_{i,j,t,t} & = z_{j,i,t}, \quad i,j \leq q, \quad t \in T; \\
z_{i,q+2j-1,t} & = z_{q+2j,i}, \quad z_{i,q+2j,t,t} = z_{q+2j-1,i}, \quad i \leq q, \quad j \leq s, \quad t \in T; \\
z_{q+2i-1,q+2j-1,t} & = z_{q+2j,q+2i}, \quad i,j \leq q, \quad t \in T; \\
z_{q+2i-1,q+2j,t} & = z_{q+2j-1,q+2i}, \quad i,j \leq q, \quad i \neq j, \quad t \in T.
\end{align*}
\]
One verifies that, apart from the above equalities and \( z_{i,i,t} = t \), the elements \( z_{i,j,t} \) are distinct, so the support of \( \Gamma = \Gamma_M(\tilde{G}, D, \kappa, \bar{\gamma}) \) is given by

\[
\text{Supp } \Gamma = \{ z_{i,j,t} \mid i < j \leq q, \ t \in T \} \cup \{ z_{i,q+j,t} \mid i \leq q, \ j \leq 2s, \ t \in T \}
\]

\[
\cup \{ z_{q+2i-1,q+2j-1,t} \mid i < j \leq s, \ t \in T \} \cup \{ z_{q+2i,q+2j,t} \mid i < j \leq s, \ t \in T \}
\]

\[
\cup \{ z_{q+2i-1,q+2j,t} \mid i, j \leq s, \ i \neq j, \ t \in T \}
\]

\[
\cup \{ z_{q+2i-1,q+2i,t} \mid i \leq s, \ t \in T \} \cup \{ z_{q+2i,q+2i-1,t} \mid i \leq s, \ t \in T \} \cup T,
\]

where the union is disjoint and all homogeneous components except those that appear in the last line have dimension 2, the components of degrees \( z_{q+2i-1,q+2i,t} \) and \( z_{q+2i,q+2i-1,t} \) have dimension 1, and the components of degree \( t \) have dimension \( q + 2s \).

**Proposition 3.4.** Let \( \Gamma = \Gamma_M(D, q, s, \tau) \). Then \( \tilde{G}^0 = \tilde{G}^0(T, q, s, \tau) \) is the universal group of \( \Gamma \), and \( \text{Diag}(\Gamma) \) consists of all automorphisms of the form \( X \mapsto DXD^{-1}, \ X \in \mathbb{R}, \) where

\[
D = \text{diag}(\lambda_1, \ldots, \lambda_{q+2s}) \otimes X_t, \ \lambda_i \in \mathbb{F}^\times, \ t \in T,
\]

satisfying the relation

\[
\lambda_0^2 \beta(t, t_1) = \ldots = \lambda_q^2 \beta(t, t_q) = \lambda_{q+1} = \ldots = \lambda_{q+2s-1} = \lambda_{q+2s}.
\]

Proof. The relations \( z_{i,t,u}z_{j,v} = z_{i,j,u,v}, \ u, v \in T \), can be rewritten in terms of the elements of \( \text{Supp } \Gamma \), producing a set of defining relations for \( \tilde{G}^0 \). It follows that \( \tilde{G}^0 \) is the universal group of \( \Gamma \).

Since \( \tilde{G}^0 \) is the universal group of \( \Gamma \), \( \text{Diag}(\Gamma) \) consists of all automorphisms of the form \( X \mapsto \chi \ast X \) where \( \chi \) is a character of \( \tilde{G}^0 \). Since \( \mathbb{F}^\times \) is a divisible group, we can assume that \( \chi \) is a character of \( \tilde{G} \). Let \( \lambda_i = \chi(g_i), \ i = 1, \ldots, q + 2s \). Let \( t \) be the element of \( T \) such that \( \chi(u) = \beta(t, u) \) for all \( u \in T \). Looking at relations (2), we see that (5) must hold. Conversely, any \( t \in T \) and a set of \( \lambda_i \in \mathbb{F}^\times \) satisfying (5) will determine a character \( \chi \) of \( G \). It remains to observe that the action of \( \chi \) on \( \mathbb{R} \) coincides with the conjugation by \( D \) as in (4). \( \square \)

The following is Proposition 3.3 from [Elk10].

**Theorem 3.5.** Consider the grading \( \Gamma = \Gamma_M(D, q, s, \tau) \) on \( \mathbb{R} = M_{q+2s}(\mathbb{D}) \) by \( \tilde{G}^0 = \tilde{G}^0(T, q, s, \tau) \) where \( \tau \) is given by (1). Let \( \mu = (\mu_1, \ldots, \mu_s) \) where \( \mu_i \) are scalars in \( \mathbb{F}^\times \). Let \( \varphi = \varphi_{\tau, \mu} \) be the anti-automorphism of \( \mathbb{R} \) defined by \( \varphi(X) = \Phi^{-1}(t)X\Phi, \ X \in \mathbb{R}, \) where \( \Phi \) is the block-diagonal matrix given by

\[
\Phi = \text{diag} \left( X_{t_1}, \ldots, X_{t_s}, \begin{bmatrix} 0 & I \\ \mu_1 I & 0 \end{bmatrix}, \ldots, \begin{bmatrix} 0 & I \\ \mu_s I & 0 \end{bmatrix} \right)
\]

and \( I \) is the identity element of \( \mathbb{D} \). Then \( \Gamma \) is a fine \( \varphi \)-grading unless \( q = 2, \ s = 0 \) and \( t_1 = t_2 \). In the latter case, \( \Gamma \) can be refined to a \( \varphi \)-grading that makes \( \mathbb{R} \) a graded division algebra. \( \square \)

This result and the discussion preceding Proposition 3.8 in [Elk10] yield

**Theorem 3.6.** Let \( \Gamma \) be a fine \( \varphi \)-grading on the matrix algebra \( \mathbb{R} = M_n(\mathbb{F}) \) over an algebraically closed field \( \mathbb{F} \), \( \text{char} \mathbb{F} \neq 2 \). Then \( (\Gamma, \varphi) \) is equivalent to some \( (\Gamma_M(T, q, s, \tau), \varphi_{\tau, \mu}) \) as in Theorem 3.5 where \( (q + 2s)2^{\frac{\dim T}{2}} = n \). \( \square \)
In [Eld10], in order to obtain the classification of fine gradings on simple Lie algebras of series A, one classifies, up to weak equivalence, all pairs \((\Gamma, \varphi)\) where \(\Gamma\) is a fine \(\varphi\)-grading on a matrix algebra. At the same time, for series \(B, C\) and \(D\), one classifies, up to equivalence, such pairs where \(\varphi\) is an involution of appropriate type: orthogonal for series \(B\) and \(D\) (we write \(\text{sgn}(\varphi) = 1\)) and symplectic for series \(C\) (we write \(\text{sgn}(\varphi) = -1\)). The classifications involve equivalences \(\mathcal{D} \to \mathcal{D}'\) satisfying certain conditions, where \(\mathcal{D}\) and \(\mathcal{D}'\) are matrix algebras with division gradings. If \(T\) is the support of \(\mathcal{D}\) and \(T'\) is the support of \(\mathcal{D}'\), then the graded algebras \(\mathcal{D}\) and \(\mathcal{D}'\) are equivalent if and only if the groups \(T\) and \(T'\) are isomorphic. Identifying \(T\) and \(T'\), we may assume that \(\mathcal{D} = \mathcal{D}'\) and look at self-equivalences of \(\mathcal{D}\), i.e., the elements of \(\text{Aut}(\mathcal{D})\) when \(\mathcal{D}\) is the grading on \(\mathcal{D}\). By [EKb, Proposition 2.7], the Weyl group \(W(\Gamma_0)\) is isomorphic to \(\text{Aut}(T, \beta)\), the group of automorphisms of \(T\) that preserve the bicharacter \(\beta\). Explicitly, if \(\psi_0 \in \text{Aut}(\Gamma_0)\), then \(\psi_0(X_t) \in F X_{\alpha(t)}\), for all \(t \in T\), where \(\alpha \in \text{Aut}(T, \beta)\), and the mapping \(\psi_0 \mapsto \alpha\) yields an isomorphism \(\text{Aut}(\Gamma_0) \to \text{Aut}(T, \beta)\). Hence the conditions in [Eld10] can be rewritten in terms of the group \(T\) rather than the graded division algebra \(\mathcal{D}\). Note that \(\text{Aut}(T, \beta)\) can be regarded as a sort of symplectic group; in particular, if \(T\) is an elementary 2-group, then \(\text{Aut}(T, \beta) \cong \text{Sp}_m(2)\) where \(m = \dim T\).

**Definition 3.7.** Given \(\tau\) as in (1), we will denote by \(\Sigma(\tau)\) the multiset in \(T\) determined by \(\tau\), i.e., the underlying set of \(\Sigma(\tau)\) consists of the elements that occur in \((t_1, \ldots, t_q)\), and the multiplicity of each element is the number of times it occurs there.

The group \(\text{Aut}(T, \beta)\) acts naturally on \(T\), so we can form the semidirect product \(T \rtimes \text{Aut}(T, \beta)\), which also acts on \(T\): a pair \((u, \alpha)\) sends \(t \in T\) to \(\alpha(t)u\). Clearly, if \(\dim T = 2r\), then \(T \rtimes \text{Aut}(T, \beta)\) is isomorphic to \(\text{ASp}_2(2r)\), the affine symplectic group of order \(2r\) over the field of two elements ("rigid motions" of the symplectic space of dimension \(2r\)).

Using this notation, Theorem 3.17 of [Eld10] can be recast as follows:

**Theorem 3.8.** Consider two pairs, \((\Gamma, \varphi)\) and \((\Gamma', \varphi')\), as in Theorem 3.5, namely, \(\Gamma = \Gamma_M(T, q, s, r), \varphi = \varphi_{r, \alpha}\) and \(\Gamma' = \Gamma_M(T', q', s', r'), \varphi' = \varphi_{r', \alpha'}\), where \(T = \mathbb{Z}_2^r\) and \(T' = \mathbb{Z}_2^{r'}\). Then \((\Gamma, \varphi)\) and \((\Gamma', \varphi')\) are weakly equivalent if and only if \(r = r', q = q', s = s'\), and \(\Sigma(\tau)\) is conjugate to \(\Sigma(\tau')\) by the natural action of \(T \rtimes \text{Aut}(T, \beta)\). \hfill \(\square\)

Let \(\psi_0: \mathcal{D} \to \mathcal{D}\) be an equivalence. Then the map \(\psi_0^{-1}\varphi_0\psi_0\) is an involution of the graded algebra \(\mathcal{D}\), which has the same type as \(\varphi_0\) (orthogonal). Hence there exists a nonzero homogeneous element \(d_0 \in \mathcal{D}\) such that

\[
d_0\varphi_0(d)d_0^{-1} = (\psi_0^{-1}\varphi_0\psi_0)(d) \text{ for all } d \in \mathcal{D}.
\]

Note that \(d_0\) is determined up to a scalar in \(F\). Moreover, \(d_0\) is symmetric with respect to \(\varphi_0\). By a similar argument, \(\psi_0(d_0)\) is also symmetric. Let \(\alpha\) be the element of \(\text{Aut}(T, \beta)\) corresponding to \(\psi_0\) and let \(t_0\) be the degree of \(d_0\). Then (7) is equivalent to the following:

\[
\beta(t_0, t)\beta(t) = \beta(\alpha(t)) \text{ for all } t \in T,
\]

so \(t_0\) depends only on \(\alpha\). Moreover, \(\beta(t_0) = \beta(\alpha(t_0)) = 1\).
**Definition 3.9.** For any $\alpha \in \text{Aut}(T, \beta)$, the map $t \mapsto \beta(\alpha^{-1}(t))\beta(t)$ is a character of $T$, so there exists a unique element $t_\alpha \in T$ such that $\beta(t_\alpha, t) = \beta(\alpha^{-1}(t))\beta(t)$ for all $t \in T$. We define a new action of the group $\text{Aut}(T, \beta)$ on $T$ by setting

$$\alpha \cdot t := \alpha(t)t_\alpha$$

for all $\alpha \in \text{Aut}(T, \beta)$ and $t \in T$.

In other words, $\text{Aut}(T, \beta)$ acts through the (injective) homomorphism to $T \rtimes \text{Aut}(T, \beta)$, $\alpha \mapsto (t_\alpha, \alpha)$, and the natural action of $T \rtimes \text{Aut}(T, \beta)$ on $T$.

Comparing this definition with equation (8), which defines the element $t_\alpha$, we see that $t_\alpha = \alpha(t_0)$. In particular, $\beta(t_\alpha) = 1$. This implies that $\beta(\alpha \cdot t) = \beta(t)$ for all $t \in T$, so the sets

$$T_+ := \{ t \in T \mid \beta(t) = 1 \} \quad \text{and} \quad T_- := \{ t \in T \mid \beta(t) = -1 \},$$

which correspond, respectively, to symmetric and skew-symmetric homogeneous components of $D$ (relative to $\varphi_0$), are invariant under the twisted action of $\text{Aut}(T, \beta)$.

Now Proposition 3.8(2) and Theorem 3.22 of [Eld10] can be recast as follows:

**Theorem 3.10.** Let $\varphi = \varphi_{\tau, \mu}$ be as in Theorem 3.9. Then $\varphi$ is an involution with $\text{sgn}(\varphi) = \delta$ if and only if

$$\delta = \beta(t_1) = \ldots = \beta(t_q) = \mu_1 = \ldots = \mu_s.$$

For gradings $\Gamma = \Gamma_M(T, q, s, \tau)$ with $T = \mathbb{Z}_2^{2r}$ and $\Gamma' = \Gamma_M(T', q', s', \tau')$ with $T' = \mathbb{Z}_2^{2r'}$ and for involutions $\varphi = \varphi_{\tau, \mu}$ and $\varphi' = \varphi_{\tau', \mu'}$, the pairs $(\Gamma, \varphi)$ and $(\Gamma', \varphi')$ are equivalent if and only if $r = r'$, $q = q'$, $s = s'$, $\text{sgn}(\varphi) = \text{sgn}(\varphi')$, and $\Sigma(\tau)$ is conjugate to $\Sigma(\tau')$ by the twisted action of $\text{Aut}(T, \beta) \cong \text{Sp}_{2r}(2)$ as in Definition 3.9.

3.2. **Automorphism groups of fine $\varphi$-gradings on matrix algebras.** We are now going to study automorphisms of the fine $\varphi$-gradings $\Gamma_M(T, q, s, \tau)$. We begin with some general observations. Let $D$ and $D'$ be graded division algebras, with the same grading group $G$. Let $V$ be a graded right $D$-module and $V'$ a graded right $D'$-module, both of nonzero finite rank. By an *isomorphism* from $(D, V)$ to $(D', V')$ we mean a pair $(\psi_0, \psi_1)$ where $\psi_0: D \rightarrow D'$ is an isomorphism of graded algebras, $\psi_1: V \rightarrow V'$ is an isomorphism of graded vector spaces over $F$, and $\psi_1(vd) = \psi_1(v)\psi_0(d)$ for all $v \in V$ and $d \in D$.

Let $R = \text{End}_D(V)$ and $R' = \text{End}_{D'}(V')$. If $\psi: R \rightarrow R'$ is an isomorphism of graded algebras, then there exist $g \in G$ and an isomorphism $(\psi_0, \psi_1)$ from $(D, V^{[g]})$ to $(D', V')$ such that $\psi_1(rv) = \psi(r)\psi_1(v)$ for all $r \in R$ and $v \in V$ (see e.g. [Eld10] Proposition 2.5). Here $V^{[g]}$ denotes a shift of grading: the $(R, D)$-bimodule structure of $V^{[g]}$ is the same as that of $V$, but we set $V^{[g]}_h = V_{hg^{-1}}$ for all $h \in G$. Conversely, given an isomorphism $(\psi_0, \psi_1)$ of the above pairs, there exists a unique isomorphism $\psi: R \rightarrow R'$ of graded algebras such that $\psi_1(rv) = \psi(r)\psi_1(v)$ for all $r \in R$ and $v \in V$. Two isomorphisms $(\psi_0, \psi_1)$ and $(\psi'_0, \psi'_1)$ determine the same isomorphism $R \rightarrow R'$ if and only if there exists a nonzero homogeneous $d \in D'$ such that $\psi'_1(x) = d^{-1}\psi_0(x)d$ and $\psi'_1(v) = \psi_1(v)d$ for all $x \in D$ and $v \in V$.

**Lemma 3.11.** Let $\psi: R \rightarrow R'$ be the isomorphism of graded algebras determined by an isomorphism $(\psi_0, \psi_1)$ from $(D, V^{[g]})$ to $(D', V')$. Suppose that the graded algebras $R$ and $R'$ admit anti-automorphisms $\varphi$ and $\varphi'$, respectively, determined by a $\varphi_0$-sesquilinear form $B: V \times V \rightarrow D$ and a $\varphi'_0$-sesquilinear form $B': V' \times V' \rightarrow D'$.
Then \( \varphi' = \psi \varphi \psi^{-1} \) if and only if there exists a nonzero homogeneous \( d_0 \in \mathcal{D} \) such that

\[
(9) \quad B'(\psi_1(v), \psi_1(w)) = \psi_0(d_0 B(v, w)) \quad \text{for all} \quad v, w \in V.
\]

Moreover, \( d_0 \psi_0(d) d_0^{-1} = (\psi_0^{-1} \varphi' \psi_0)(d) \) for all \( d \in \mathcal{D} \).

**Proof.** Set \( \varphi'' := \psi^{-1} \varphi' \psi \) and \( B''(v, w) := \psi_0^{-1} \{ B'(\psi_1(v), \psi_1(w)) \} \) for all \( v, w \in V \).

Then we compute:

\[
B''(v, wd) = \psi_0^{-1} \left( B'(\psi_1(v), \psi_1(w) \psi_0(d)) \right) = \psi_0^{-1} \left( B'(\psi_1(v), \psi_1(w)) \psi_0(d) \right) = B''(v, w)d;
\]

\[
B''(vd, w) = \psi_0^{-1} \left( B'(\psi_1(v), \psi_1(w)) \psi_0(d) \right) = \psi_0^{-1} \left( \psi_0(\psi_0(d) B'(\psi_1(v), \psi_1(w))) \right) = (\psi_0^{-1} \varphi' \psi_0)(d) B''(v, w);\]

\[
B''(v, \varphi''(r)w) = \psi_0^{-1} \left( B'(\psi_1(v), \psi(\varphi''(r)) \psi_1(w)) \right) = \psi_0^{-1} \left( B'(\psi_1(v), \varphi'(\psi(r)) \psi_1(w)) \right) = \psi_0^{-1} \left( B'(\psi(r) \psi_1(v), \psi_1(w)) \right) = B''(rv, w).
\]

We have shown that \( B'' \) is a \((\psi_0^{-1} \varphi' \psi_0)\)-sesquilinear form corresponding to \( \varphi'' \). Hence \( \varphi'' = \varphi \) if and only if there exists a nonzero homogeneous element \( d_0 \in \mathcal{D} \) such that \( B'' = d_0 B \), i.e., equation (9) holds. \( \square \)

Now consider \( \Gamma = \Gamma_M(T, q, s, \tau) \) and \( \varphi = \varphi_{\tau, \mu} \) as in Theorem 3.5. There are two kinds of automorphism groups that we will need. Namely, there is

\[
\text{Aut}^*(\Gamma, \varphi) := \{ \psi \in \text{Aut}(\Gamma) \mid \psi \varphi \psi^{-1} = \xi \varphi \quad \text{for some} \quad \xi \in \text{Diag}(\Gamma) \},
\]

which will be relevant to computing the Weyl group of the corresponding fine grading on the simple Lie algebra of type \( A \), and there is

\[
\text{Aut}(\Gamma, \varphi) := \{ \psi \in \text{Aut}(\Gamma) \mid \psi \varphi \psi^{-1} = \varphi \},
\]

which will be relevant to computing the Weyl groups of fine gradings on the simple Lie algebras of types \( B, C \) and \( D \). Hence, we are interested in \( \text{Aut}(\Gamma, \varphi) \) only if \( \varphi \) is an involution. Similarly, define

\[
\text{Stab}(\Gamma, \varphi) := \{ \psi \in \text{Stab}(\Gamma) \mid \psi \varphi \psi^{-1} = \varphi \}.
\]

(We could also define \( \text{Stab}^*(\Gamma, \varphi) \), but we will not need it.)

Recall that \( \Gamma \) is the grading on \( \mathcal{R} = \text{End}_D(V) \) where \( D \) is a matrix algebra equipped with a division grading with support \( T = \mathbb{Z}_{2s}^{2} \) and bicharacter \( \beta \), and \( V \) has a \( D \)-basis \( \{ v_1, \ldots, v_k \} \) with \( \text{deg} v_i = \beta_i \) and \( k = q + 2s \). We will use the universal group \( \tilde{G}^0 \) for the grading \( \Gamma \). If \( \psi: \mathcal{R} \to \mathcal{R} \) is an equivalence, then there exists an automorphism \( \alpha \) of the group \( \tilde{G}^0 \) such that \( \psi \) sends \( \alpha \Gamma \) to \( \Gamma \). In other words, \( \psi: \mathcal{R}' \to \mathcal{R} \) is an isomorphism of graded algebras where \( \mathcal{R}' = \mathcal{R} \) as an algebra, but equipped with the grading \( \alpha \Gamma \). Define \( \mathcal{D}' \) similarly to \( \mathcal{R} \), using the restriction of \( \alpha \) to \( T \subset \tilde{G}^0 \). The support of \( \mathcal{D}' \) is \( T' = \alpha(T) \). Since \( V(\beta^{-1}) \) is \( \tilde{G}^0 \)-graded, we can also define \( \mathcal{V}' \) so that \( \mathcal{V}' = \text{End}_D(V') \) as a graded algebra. Therefore, \( \psi \) is determined by \( (\psi_0, \psi_1) \) where \( \psi_0: \mathcal{D}' \to \mathcal{D} \) is an isomorphism of graded algebras and \( \psi_1: \mathcal{V}' \to \mathcal{V} \) is an isomorphism up to a shift of grading. Hence \( T' = T \) and \( \psi_0 \in \text{Aut}(\Gamma_0) \), so \( \psi_0(X_t) \in FX_\alpha(t) \), for all \( t \in T \), and the map \( \alpha: T \to T \) belongs
to $\text{Aut}(T, \beta) \cong \text{Sp}_{2n}(2)$. Also, if $\Psi$ is the matrix of $\psi_1$ relative to $\{v_1, \ldots, v_k\}$, we have
\[
\psi(X) = \Psi \psi_0(X) \Psi^{-1} \quad \text{for all} \quad X \in \mathcal{R}.
\]
Since all $\tilde{g}_i$ are distinct modulo $T$, matrix $\Psi$ necessarily has the form $\Psi = PD$ where $P$ is a permutation matrix and $D = \text{diag}(d_1, \ldots, d_k)$ where $d_i$ are nonzero homogeneous elements of $D$. Moreover, the permutation $\pi \in \text{Sym}(k)$ corresponding to $P$ and the coset of $\psi_0$ modulo $\text{Stab}(\Gamma_0)$ are uniquely determined by $\psi$. Hence, we have a well-defined homomorphism
\[
\text{Aut}(\Gamma) \to \text{Sym}(k) \times \text{Aut}(T, \beta)
\]
that sends $\psi$ to the corresponding $(\pi, \alpha)$.

Now we turn to the anti-automorphism $\varphi: \mathcal{R} \to \mathcal{R}$, which is given by the adjoint with respect to a $\varphi_0$-sesquilinear form $B$ on $V$ where $\varphi_0: D \to D$ is given by matrix transpose, $X_t \mapsto \beta(t)X_t$ for all $t \in T$. Recall that such $B$ is determined up to a scalar in $\mathbb{F}$. We can take for $B$ the $\varphi_0$-sesquilinear form whose matrix with respect to $\{v_1, \ldots, v_k\}$ is $\Phi$ displayed in Theorem 3.5. Pick $\xi \in \text{Diag}(\Gamma)$ and let $B'$ be a $\varphi_0$-sesquilinear form on $V$ corresponding to $\xi\varphi$. By Lemma 3.11, $\psi$ satisfies $\psi\varphi\psi^{-1} = \xi\varphi$ if and only if condition (10) holds for some nonzero homogeneous $d_0 \in D$. Clearly, (10) is equivalent to (11)
\[
\hat{\Phi} = \psi_0(d_0\Phi),
\]
where $\hat{\Phi}$ is the matrix of $B'$ relative to $\{v_1(1), \ldots, v_k(1)\}$. Recall that (11) is equivalent to condition (5) on $t_0 := \deg d_0$. To summarize, $\psi$ satisfies $\psi\varphi\psi^{-1} = \xi\varphi$ if and only if
\[
\hat{\Phi} = d_0\psi_0(\Phi)
\]
for some $d_0 \in D$ of degree $t_0$ as in Definition 3.9. (we have replaced $\psi_0(d_0)$ in (10) by $d_0$ to simplify notation).

The matrix of $B'$ relative to $\{v_1, \ldots, v_k\}$ is $\Phi(D')^{-1}$ where $\xi(X) = D'\text{X}(D')^{-1}$, for all $X \in \mathcal{R}$, with $D'$ of the form given by Proposition 3.3. $D' = \text{diag}(\nu_1X_1, \ldots, \nu_kX_k)$ for some $u \in T$ and $\nu_i \in \mathbb{F}^\times$ satisfying
\[
\nu_i^2 = \ldots = \nu^2 \beta(u,t_1) = \ldots = \nu^2 \beta(u,t_q) = \nu_{q+1}\nu_{q+2} = \ldots = \nu_{q+2s-1}\nu_{q+2s}.
\]
It follows at once that, for $\psi \in \text{Aut}^*(\Gamma, \varphi)$, the permutation $\pi$ must preserve the set $\{1, \ldots, q\}$ and the pairing of $q + 2i - 1$ with $q + 2i$, for $i = 1, \ldots, s$. It is convenient to introduce the group $W(s) := \mathbb{Z}_2 \rtimes \text{Sym}(s)$ (i.e., the wreath product of $\text{Sym}(s)$ and $\mathbb{Z}_2$), which will be regarded as the group of permutations on $\{q + 1, \ldots, q + 2s\}$ that respect the block decomposition $\{q + 1, q + 2\} \cup \ldots \cup \{q + 2s - 1, q + 2s\}$. The reason for the notation $W(s)$ is that $\mathbb{Z}_2 \rtimes \text{Sym}(s)$ is the classical Weyl group of type $B_s$ or $C_s$ (and also the extended Weyl group of type $D_s$ if $s > 4$). By the above discussion, we have a homomorphism:
\[
\text{Aut}^*(\Gamma, \varphi) \to \text{Sym}(q) \times W(s) \times \text{Aut}(T, \beta).
\]
We need some more notation to state the main result of this section. Let $\Sigma$ be a multiset of cardinality $q$ and let $m_1, \ldots, m_\ell$ be the multiplicities of the elements of $\Sigma$, written in some order. Thus, $m_i$ are positive integers whose sum is $q$. We will denote by $\text{Sym}(\Sigma)$ the subgroup $\text{Sym}(m_1) \times \cdots \times \text{Sym}(m_\ell)$ of $\text{Sym}(q)$, which may be thought of as “interior symmetries” of $\Sigma$. For a multiset $\Sigma$ in $T$, let $\text{Aut}^* \Sigma$ be the stabilizer of $\Sigma$ under the natural action of $T \times \text{Aut}(T, \beta)$ on $T$, i.e., $\text{Aut}^* \Sigma$
is the set of “rigid motions” of the symplectic space $T$ that permute the elements of $\Sigma$ preserving multiplicity. These are “exterior symmetries” of $\Sigma$. Note that each bijection $\theta : T \to T$ that stabilizes $\Sigma$ determines an element of $\text{Sym}(q)$ that permutes the blocks of sizes $m_1, \ldots, m_t$ in the same way $\theta$ permutes the elements of $\Sigma$ (thus, only blocks of equal size may be permuted) and preserves the order within each block; we will call this permutation the restriction of $\theta$ to $\Sigma$. Hence, we obtain a restriction homomorphism $\text{Aut}^* \Sigma \to \text{Sym}(q)$. In particular, $\text{Aut}^* \Sigma$ acts naturally on $\text{Sym}\Sigma$ by permuting factors (of equal order). Finally, let $\text{Aut} \Sigma$ be the stabilizer of $\Sigma$ under the twisted action of $\text{Aut}(T, \beta)$ on $T$ as in Definition 3.9. Note that $\text{Aut} \Sigma$ may be regarded as a subgroup of $\text{Aut}^* \Sigma$.

**Theorem 3.12.** Let $\Gamma = \Gamma_M(T, q, s, \tau)$ and let $\varphi$ be as in Theorem 3.5 such that $\Gamma$ is a fine $\varphi$-grading. Let $\Sigma = \Sigma(\tau)$, so $|\Sigma| = q$.

1) $\text{Stab}(\Gamma, \varphi) = \text{Diag}(\Gamma)$.

2) $\text{Aut}^*(\Gamma, \varphi)/\text{Stab}(\Gamma, \varphi)$ is isomorphic to an extension of the group $((T^{q+s-1} \times Z_2^s) \times (\text{Sym} \Sigma \times \text{Sym}(s))) \rtimes \text{Aut}^* \Sigma$ by $\mathbb{Z}_2^{q+s-1}$, with the following actions: $T^{q+s-1}$ is identified with $T^{q+s}/T$ and $\mathbb{Z}_2^{q+s-1}$ is identified with $\mathbb{Z}_2^{q+s}/\mathbb{Z}_2$, where $T$ and $\mathbb{Z}_2$ are imbedded diagonally, then

- $\text{Sym}(q)$ acts on $T^{q+s}/T$ and $\mathbb{Z}_2^{q+s}/\mathbb{Z}_2$ by permuting the first $q$ components and trivially on $\mathbb{Z}_2^s$;
- $\text{Sym}(s)$ acts on $T^{q+s}/T$ and $\mathbb{Z}_2^{q+s}/\mathbb{Z}_2$ by permuting the last $s$ components and naturally on $\mathbb{Z}_2^s$;
- $\text{Aut}^* \Sigma$ acts on $\text{Sym} \Sigma$ and $\mathbb{Z}_2^{q+s}/\mathbb{Z}_2$ through the restriction homomorphism $\text{Aut}^* \Sigma \to \text{Sym}(q)$, trivially on $\text{Sym}(s)$, and as follows on $(T^{q+s}/T) \times \mathbb{Z}_2^s$: an element $(u, \alpha) \in \text{Aut}^* \Sigma \subset T \times \text{Aut}(T, \beta)$ sends a pair $((u_1, \ldots, u_q, u_{q+1}, \ldots, u_{q+s}), \varphi) \in (T^{q+s}/T) \times \mathbb{Z}_2^s$ to $((\alpha(u_{q-1}), \ldots, \alpha(u_{q-1})), \alpha(u_{q+1})u^{x_1}, \ldots, \alpha(u_{q+s})u^{x_s})T, \varphi)$, where $\pi$ is the image of $(u, \alpha)$ under the restriction homomorphism;
- $T^{q+s-1} \times \mathbb{Z}_2^s$ acts trivially on $\mathbb{Z}_2^{q+s-1}$.

3) If $\varphi$ is an involution, then $\text{Aut}(\Gamma, \varphi)/\text{Stab}(\Gamma, \varphi)$ is isomorphic to $((T^{q+s-1} \times Z_2^s) \times (\text{Sym} \Sigma \times \text{Sym}(s))) \rtimes \text{Aut} \Sigma$, with the following actions: $T^{q+s-1}$ is identified with $T^{q+s}/T$, where $T$ is imbedded diagonally, then

- $\text{Sym}(q)$ acts on $T^{q+s}/T$ by permuting the first $q$ components and trivially on $\mathbb{Z}_2^s$;
- $\text{Sym}(s)$ acts on $T^{q+s}/T$ by permuting the last $s$ components and naturally on $\mathbb{Z}_2^s$;
- $\text{Aut} \Sigma$ acts on $\text{Sym} \Sigma$ as a subgroup of $\text{Aut}^* \Sigma$, i.e., through the twisted action on $T$ (Definition 3.9) and restriction to $\Sigma$, trivially on $\text{Sym}(s)$, and as follows on $(T^{q+s}/T) \times \mathbb{Z}_2^s$: an element $\alpha \in \text{Aut} \Sigma \subset \text{Aut}(T, \beta)$ sends a pair $((u_1, \ldots, u_q, u_{q+1}, \ldots, u_{q+s}), \varphi) \in (T^{q+s}/T) \times \mathbb{Z}_2^s$ to $((\alpha(u_{q-1}), \ldots, \alpha(u_{q-1})), \alpha(u_{q+1})t^{x_1}, \ldots, \alpha(u_{q+s})t^{x_s})T, \varphi)$, where $\pi$ is the image of $(t, \alpha)$ under the restriction to $\Sigma$.

**Proof.** 1) If $\psi \in \text{Stab}(\Gamma, \varphi)$, then $\Psi = PD$ where $P$ corresponds to $\pi \in \text{Sym}(q) \times W(s)$, and $\psi_0 \in \text{Stab}(\Gamma_0)$. Adjusting $D$ if necessary, we may assume $\psi_0 = \text{id}$. We claim that $\pi$ is the trivial permutation. Since $\psi$ does not permute the homogeneous components of $\Gamma$, $\pi$ must act trivially on $G^0/T$. So, we consider the action of $\text{Sym}(q) \times W(s)$ on $G^0/T$ in terms of the generators $z_i (i = 1, \ldots, q - 1$ if $s = 0$ and $i = 1, \ldots, q + s$ if $s > 0)$ that were introduced after Definition 3.3.
Sym(q) acts trivially on the subgroup \( \langle z_{q+1}, \ldots, z_{q+s} \rangle \) and via the action of the classical Weyl group of type \( A_{q-1} \), taken modulo 2, on the subgroup \( \langle z_1, \ldots, z_{q-1} \rangle \cong \mathbb{Z}_2^{q-1} \) where \( z_i \) is identified with the element \( \varepsilon_i - \varepsilon_{i+1} \), with \( \{\varepsilon_1, \ldots, \varepsilon_q\} \) being the standard basis of \( \mathbb{Z}_2^q \), on which Sym(q) acts naturally.

\( W(s) \) acts trivially on the subgroup \( \langle z_1, \ldots, z_{q-1} \rangle \) and via the action of the classical Weyl group of type \( B_s \) or \( C_s \) on the subgroup \( \langle z_{q+1}, \ldots, z_{q+s} \rangle \cong \mathbb{Z}^s \) where \( z_{q+i} \) is identified with the element \( \varepsilon_i - \varepsilon_{i+1} \) for \( i \neq s \) and \( z_{q+s} \) is identified with the element \( 2\varepsilon_1 \), with \( \{\varepsilon_1, \ldots, \varepsilon_s\} \) being the standard basis of \( \mathbb{Z}^s \). The easiest way to see this is to extend \( G \) by adding a new element \( \bar{g}_0 \) satisfying \( (\bar{g}_0)^{-2} = \bar{g}_1 \bar{g}_2 \) and set \( \hat{g}_i = \bar{g}_i \bar{g}_0 \). The elements of the subgroup \( \hat{G}_0 \) are not affected if we replace \( \bar{g}_i \) by \( \hat{g}_i \), but then we have \( \hat{g}_{q+2j} = \hat{g}_{q+2j-1} \) for \( j = 1, \ldots, s \), so we can map \( \hat{g}_{q+2j-1} \) to \( \varepsilon_j \) and \( \hat{g}_{q+2j} \) to \( -\varepsilon_j \).

Note that the action of \( W(s) \) on \( \langle z_{q+1}, \ldots, z_{q+s} \rangle \) is always faithful, while the action of Sym(q) on \( \langle z_1, \ldots, z_{q-1} \rangle \) is faithful unless \( q = 2 \). If \( q > 0 \) and \( s > 0 \), then we also have the generator \( z_\eta \), on which \( \pi \in \text{Sym}(q) \times W(s) \) acts in this way (note that \( \pi(q) \leq q \) and \( \pi(q+1) > q \)):

\[
 z_\eta \mapsto \begin{cases} \varepsilon_{\pi(q)} \cdots z_{\eta} z_{\eta+1} \cdots z_{\eta+j} & \text{if} \: \pi(q+1) = q + 2j + 1; \\ \varepsilon_{\pi(q)} \cdots z_{\eta} z_{\eta+1} \cdots z_{\eta+j} z_{\eta+s} & \text{if} \: \pi(q+1) = q + 2j + 2. \end{cases}
\]

If \( \pi \) acts trivially on \( \langle z_{q+1}, \ldots, z_{q+s} \rangle \), then \( \pi(q+1) = q + 1 \). Hence, if \( \pi \) also acts trivially on \( z_\eta \), then \( \pi(q) = q \). It follows that the action of Sym(q) \( \times W(s) \) on \( \hat{G}_0/T \) is faithful unless \( g = 2 \) and \( s = 0 \). In this remaining case, we have \( \tau = (t_1, t_2) \) (otherwise \( \Gamma \) is not a fine \( \varphi \)-grading). If \( \psi_1 \) yields \( \pi = (12) \), then \( \psi_1(v_1) = v_2 d_1 \) and \( \psi_1(v_2) = v_1 d_2 \) for some nonzero homogeneous \( d_1, d_2 \in D \), but then \( B(v_1(v_1), v_1(v_1)) \) has degree \( t_2 \), while \( B(v_1, v_1) \) has degree \( t_1 \). This contradicts (11), because here we have \( \psi_0 = \text{id}, d_0 \in \mathbb{F}^\times \) and \( B' = B \). The proof of the claim is complete.

Since \( P = I \), we have \( \Psi = \text{diag}(d_1, \ldots, d_k) \), where the \( d_i \) must necessarily have the same degree, say, \( t \), so \( \Psi = \text{diag}(\lambda_1, \ldots, \lambda_k) \otimes X_1 \), but then (11) implies that (3) must hold, hence \( \psi \in \text{Diag}(\Gamma) \). We have proved that \( \text{Stab}(\Gamma, \varphi) \subset \text{Diag}(\Gamma) \). The opposite inclusion is obvious.

2) We can extract more information about an element \( \psi \in \text{Aut}^*(\Gamma, \varphi) \) than given by its image under the homomorphism (14) if we look at the action of \( \Psi \) on \( \varphi \). Write \( \psi_\varphi \varphi^{-1} = \xi_\psi \varphi \varphi^{-1} \) where \( \xi_\psi \) is a uniquely determined element of Diag(\Gamma). Clearly, we will have \( \xi_{\psi_\varphi} = \xi_\psi (\psi_{\xi_\psi})^{-1} \). Since \( \xi_\psi \) is the conjugation by \( \text{diag}(\nu_1, \ldots, \nu_k) \otimes X_{u_\psi} \), for a uniquely determined \( u_\psi \in T \), we obtain \( u_{\psi_\varphi} = u_\psi \alpha_\psi (u_\psi) \) where \( \alpha_\psi \) is the element of \( \text{Aut}(T, \beta) \) corresponding to \( \psi \) under (13). Hence, we can construct a homomorphism

\[
\text{Aut}^*(\Gamma, \varphi) \rightarrow \text{Sym}(q) \times W(s) \times (T \times \text{Aut}(T, \beta)),
\]

where the first two components are as in (13) and the third is \( \psi \mapsto (u_\psi, \alpha_\psi) \).

Theorem 5.8 implies that we may assume without loss of generality that

\[
\Phi = \text{diag} \left( X_{t_1}, \ldots, X_{t_n}, \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}, \ldots, \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} \right).
\]

(In other words, the scalars \( \mu_i \) are all equal to 1.) Then, for \( \psi \) given by \( \Psi = PD \) and \( \psi_0 \in \text{Aut}(\Gamma_0) \), with \( P \) corresponding to \( \pi \in \text{Sym}(q) \times W(s) \), condition (11) is
equivalent to the following, with $u = u_\varphi$:
\begin{equation}
\varphi_0(d_t)X_{\pi(u)},\nu^{-1}_{\varphi(1)}X_u^{-1}d_t = d_0\psi_0(X_t), \quad i = 1, \ldots, q,
\end{equation}
and, for each $j = 1, \ldots, s$, one of the following depending on whether $\pi(q+2j-1) < \pi(q+2j)$ or $\pi(q+2j-1) > \pi(q+2j)$:
\begin{equation}
\varphi_0(d_{q+2j-1})\nu^{-1}_{\pi(q+2j)}X_u^{-1}d_{q+2j} = \varphi_0(d_{q+2j})\nu^{-1}_{\pi(q+2j-1)}X_u^{-1}d_{q+2j-1} = d_0
\end{equation}
in the first case, and
\begin{equation}
\varphi_0(d_{q+2j-1})\nu^{-1}_{\pi(q+2j-1)}X_u^{-1}d_{q+2j} = \varphi_0(d_{q+2j})\nu^{-1}_{\pi(q+2j-1)}X_u^{-1}d_{q+2j-1} = d_0
\end{equation}
in the second case.

If $\psi \in \text{Aut}^* \Sigma$, then, looking at the degrees in (15), we obtain
\begin{equation}
t_{\pi(i)} = \alpha_\psi(t_i)t_{\alpha_\psi}u_\psi, \quad i = 1, \ldots, q,
\end{equation}
which implies that $(t_{\alpha_\psi}u_\psi, \alpha_\psi)$ belongs to $\text{Aut}^* \Sigma$. Composing the third component of the homomorphism (14) with the automorphism $(u, \alpha) \mapsto (t_\alpha u, \alpha)$ of the group $T \rtimes \text{Aut}(T, \beta)$, we obtain a homomorphism
\begin{equation}
\text{Aut}^*(\Gamma, \varphi) \to \text{Sym}(q) \times W(s) \times \text{Aut}^* \Sigma.
\end{equation}
For any element $(t_\alpha u, \alpha) \in \text{Aut}^* \Sigma$, let $\pi_{u, \alpha} \in \text{Sym}(q)$ be its restriction to $\Sigma$. Then (15) implies that the permutation $\pi\nu^{-1}_{u, \alpha}$ does not move the elements of the underlying set of $\Sigma$, so it belongs to $\text{Sym} \Sigma$. It follows that (19) can be rearranged as follows:
\begin{equation}
f: \text{Aut}^*(\Gamma, \varphi) \to W(s) \times (\text{Sym} \Sigma \rtimes \text{Aut}^* \Sigma).
\end{equation}
We claim that $f$ is surjective. We will construct representatives in $\text{Aut}^*(\Gamma, \varphi)$ for the elements of each of the subgroups $W(s), \text{Sym} \Sigma$ and $\text{Aut}^* \Sigma$.

For any $\pi \in W(s)$, let $P$ be the corresponding permutation matrix and let $\psi_\pi$ be given by $\Psi = P$ and $\psi_0 = \text{id}$. Let $\alpha$ be the automorphism of $G$ that restricts to identity on $T$ and sends $\tilde{g}_i$ to $\tilde{g}_{\pi(i)}$ (in particular, $\tilde{g}_i$ are fixed for $i = 1, \ldots, q$). Then $\psi_\pi$ sends $^{\pi^T} \Gamma$ to $\Gamma$, so $\psi_\pi \in \text{Aut}(\Gamma)$. Also, conditions (15) through (17) are satisfied with $d_0 = I$, $u = e$ and $\nu_1 = 1$, so $\psi_\pi \in \text{Aut}(\Gamma, \varphi)$.

For any $\pi \in \text{Sym} \Sigma$, let $P$ be the corresponding permutation matrix and let $\psi_\pi$ be given by $\Psi = P$ and $\psi_0 = \text{id}$. Since we have $t_{\pi(i)} = t_i$ for all $i = 1, \ldots, q$, we can define the automorphism $\alpha$ of $G$ in the same way as above (this time, $\tilde{g}_i$ are fixed for $i = q+1, \ldots, q+2s$). Then $\psi_\pi$ sends $^\pi \Gamma$ to $\Gamma$, so $\psi_\pi \in \text{Aut}(\Gamma)$. Also, conditions (15) and (16) are satisfied with $d_0 = I$, $u = e$ and $\nu_1 = 1$, so $\psi_\pi \in \text{Aut}(\Gamma, \varphi)$.

Now, for any $(t_\alpha u, \alpha) \in \text{Aut}^* \Sigma$, let $\pi = \pi_{u, \alpha}$. Then $t_{\pi(i)} = \alpha(t_i)t_\alpha u$ for $i = 1, \ldots, q$ and hence we can extend $\alpha: T \to T$ to an automorphism of $G$ by setting $\alpha(\tilde{g}_i) = \tilde{g}_{\pi(i)}$ for $i = 1, \ldots, q$, $\alpha(\tilde{g}_{q+2j-1}) = \tilde{g}_{q+2j-1}$ and $\alpha(\tilde{g}_{q+2j}) = \tilde{g}_{q+2j}t_\alpha u$ for $j = 1, \ldots, s$. Choose $\nu_\pi \in \mathbb{F}^\times$ such that $\nu_i^2 = \beta(u, t_\pi)\beta(u)$, $i = 1, \ldots, q$, and set $\nu_{q+2j} = 1$ and $\nu_{q+2j-1} = \beta(u)$, $j = 1, \ldots, s$. Then (12) holds, so the conjugation by $\text{diag}(\nu_1 X_u, \ldots, \nu_s X_u)$ is an element $\xi \in \text{Diag}(\Gamma)$. Choose $\psi_0$ such that $\psi_0(X_t) \in \text{FX}_{\alpha(i)}$. Let $P$ be the permutation matrix corresponding to $\pi$ and let
\begin{align*}
D = \text{diag}(\lambda_1 I, \ldots, \lambda_q I, I, X_u X_{t_1} \ldots, I, X_u X_{t_s}),
\end{align*}
where $\lambda_i \in \mathbb{F}^\times$ are selected in such a way that condition (15) holds with $d_0 = X_{t_\pi}$ (the degrees of both sides match, so it is indeed possible to find such $\lambda_i$). Since $\beta(t_\alpha) = 1$, condition (16) also holds. Finally, let $\psi_{u, \alpha}$ be given by $\Psi = PD$ and $\psi_0$. 

Then $\psi_{u,\alpha}$ sends $\alpha \Gamma$ to $\Gamma$ and $\varphi$ to $\xi \varphi$, with $\alpha$ and $\xi$ indicated above. Therefore, $\psi_{u,\alpha}$ belongs to $\text{Aut}^*(\Gamma, \varphi)$.

We have proved that the homomorphism $f$ is surjective. Let $K$ be the kernel of $f$. It consists of the conjugations by matrices of the form $D = \text{diag}(d_1, \ldots, d_k)$ such that (15) and (16) are satisfied with $\pi = \text{id}$, $\psi_0 = \text{id}$, $d_0 \in \mathbb{F}^\times$ and $u = e$. Hence $\deg d_{q+2j-1} = \deg d_{q+2j}$ for all $j = 1, \ldots, s$. Conversely, given $(u_1, \ldots, u_k) \in T^k$ with $u_{q+2j-1} = u_{q+2j}$ for $j = 1, \ldots, s$, we can find elements $d_i$ with $\deg d_i = u_i$ such that the conjugation by $D$ belongs to $\text{Aut}^*(\Gamma, \varphi)$.

According to 1), the subgroup

$$N = \{ \psi \in K \mid \deg d_1 = \cdots = \deg d_k \}$$

contains $\text{Stab}(\Gamma, \varphi)$. Clearly, $N$ is normal in $\text{Aut}^*(\Gamma, \varphi)$. From the previous paragraph it follows that $K/N \cong T^{q+s}/T$ where $T$ is imbedded into $T^{q+s}$ diagonally. The representatives $\psi$ that we constructed above for $\pi \in W(s)$ and for $\pi \in \text{Sym}\Sigma$ form subgroups of $\text{Aut}(\Gamma, \varphi)$ that commute with one another. But observe also that the representatives $\psi_{u,\alpha}$ for $(t_{u,\alpha}) \in \text{Aut}^* \Sigma$ form a subgroup modulo $N$. Moreover, for $\pi \in \text{Sym}(s) \subset W(s)$ the elements $\psi_{u,\alpha}$ and $\psi_\pi$ commute modulo $N$, while for $\pi \in \text{Sym}\Sigma$ we have $\psi_{u,\alpha} \psi_\pi \psi_{u,\alpha}^{-1} \in \psi_{u,\alpha,\pi,\pi^{-1}} N$. Finally, for the transposition $\pi = (q+2j-1, q+2j)$, we have $\psi_\pi \psi_{u,\alpha} \psi_\pi \psi_{u,\alpha}^{-1} \in \psi N$ where $\psi$ is the conjugation by $d_1, \ldots, d_k$ with $d_{q+2j-1} = d_{q+2j} = X_{t_{u,\alpha}}$ and all other $d_i = I$. It follows that $\text{Aut}^*(\Gamma, \varphi)/N$ is isomorphic to $((T^{q+s-1} \times Z_2) \ltimes (\text{Sym}\Sigma \times \text{Sym}(s))) \ltimes \text{Aut}^* \Sigma$, with the stated actions.

It remains to compute the quotient $N/\text{Stab}(\Gamma, \varphi)$. Since any element $\psi \in N$ belongs to $\text{Stab}(\Gamma)$, the mapping $\psi \mapsto \xi_\psi$ is a homomorphism $N \to \text{Diag}(\Gamma)$ whose kernel is exactly $\text{Stab}(\Gamma, \varphi)$. Hence, it suffices to compute the image. Since here $u = e$ and $\deg d_{q+2j-1} = \deg d_{q+2j}$, condition (16) implies that $\nu_{q+2j-1} = \nu_{q+2j}$ for $j = 1, \ldots, s$. But then (12) implies that all $\nu_i^2$ are equal to each other. Since multiplying all $\nu_i$ by the same scalar in $\mathbb{F}^\times$ does not change $\xi$, we may assume that $\nu_i \in \{ \pm 1 \}$. In fact, for $D = \text{diag}(\lambda_1 I, \ldots, \lambda_k I)$, conditions (15) and (16) reduce to the following: up to a common scalar multiple, $\nu_i = \lambda_i^2$ for $i = 1, \ldots, q$, and $\nu_{q+2j-1} = \nu_{q+2j} = \lambda_{q+2j-1} \lambda_{q+2j}$ for $j = 1, \ldots, s$. Hence every $(\nu_1, \ldots, \nu_k)$ with $\nu_i \in \{ \pm 1 \}$ and $\nu_{q+2j-1} = \nu_{q+2j}$ indeed appears in $\xi_\psi$ for some $\psi \in N$. Therefore, the quotient $N/\text{Stab}(\Gamma, \varphi)$ is isomorphic to $Z_2^{q+s}/Z_2$ where $Z_2$ is imbedded into $Z_2^{q+s}$ diagonally.

3) The proof is similar to 2), so we will merely point out the differences. According to Theorem 3.10 here we have

$$\Phi = \text{diag} \left( X_{t_1}, \ldots, X_{t_q}, \left[ \begin{array}{ccc} 0 & I & \cdots & 0 \\ \delta I & 0 & \cdots & 0 \end{array} \right] \right),$$

where $\delta = \text{sgn}(\varphi)$ and $\beta(t_i) = \delta$ for $i = 1, \ldots, q$. Also, $B'$ equals $B$ and hence, for $\psi$ given by $\Psi = PD$ and $\psi_0 \in \text{Aut}(\Gamma_0)$, with $P$ corresponding to $\pi \in \text{Sym}(q) \times W(s)$, condition (11) is equivalent to the following:

$$\varphi_0(d_i)X_{t_{\pi(i)}}d_i = d_0\psi_0(X_{t_i}), \quad i = 1, \ldots, q,$$

and, for each $j = 1, \ldots, s$, one of the following depending on whether $\pi(q+2j-1) < \pi(q+2j)$ or $\pi(q+2j-1) > \pi(q+2j)$:

$$\varphi_0(d_{q+2j-1})d_{q+2j} = d_0$$
in the first case, and

\[(22) \quad \varphi_0(d_{q+2j-1})d_{q+2j} = \delta d_0\]

in the second case. Here we took into account that, since \(\varphi_0(d_0) = d_0\), either \((21)\) or \((22)\) implies \(\varphi_0(d_{q+2j-1})d_{q+2j} = \varphi_0(d_{q+2j})d_{q+2j+1}\).

If \(\psi \in \text{Aut}(\Gamma, \varphi)\), then, looking at the degrees in \((20)\), we obtain

\[(23) \quad t_{\pi(i)} = \alpha_\psi(t_i) t_{\alpha_0}, \quad i = 1, \ldots, q,\]

which implies that \((t_{\alpha_0}, \alpha_\psi)\) stabilizes \(\Sigma\), i.e., \(\alpha_\psi\) belongs to \(\text{Aut} \Sigma\). Hence we obtain a homomorphism

\[(24) \quad \text{Aut}(\Gamma, \varphi) \to \text{Sym}(q) \times W(s) \times \text{Aut} \Sigma.\]

For any element \(\alpha \in \text{Aut} \Sigma\), let \(\pi_\alpha \in \text{Sym}(q)\) be the restriction of its twisted action to \(\Sigma\). Then \((23)\) implies that the permutation \(\pi \pi_\alpha^{-1}\) does not move the elements of the underlying set of \(\Sigma\), so it belongs to \(\text{Sym} \Sigma\). It follows that \((24)\) can be rearranged as follows:

\[f : \text{Aut}(\Gamma, \varphi) \to W(s) \times (\text{Sym} \Sigma \rtimes \text{Aut} \Sigma).\]

To prove that \(f\) is surjective, we construct representatives in \(\text{Aut}(\Gamma, \varphi)\) for the elements of each of the subgroups \(W(s)\), \(\text{Sym} \Sigma\) and \(\text{Aut} \Sigma\).

For \(\pi\) in \(\text{Sym} \Sigma\) or in \(\text{Sym}(s) \subset W(s)\), we take the same representatives as in the proof of 2). For \(\pi = (q + 2j - 1, q + 2j) \in W(s)\), a slight modification is needed: we take \(\Psi = PD\) rather than just \(P\), where \(d_{q+2j} = \delta I\) and all other \(d_i = I\). For any \(\alpha \in \text{Aut} \Sigma\), let \(\pi = \pi_\alpha\). Then \(t_{\pi(i)} = \alpha(t_i) t_\alpha\) for \(i = 1, \ldots, q\) and hence we can extend \(\alpha : T \to T\) to an automorphism of \(\tilde{G}\) by setting \(\alpha(g_j) = \tilde{g}_{\pi(i)}\) for \(i = 1, \ldots, q\), \(\alpha(g_{q+2j-1}) = \tilde{g}_{q+2j-1}\) and \(\alpha(g_{q+2j}) = \tilde{g}_{q+2j} t_\alpha\) for \(j = 1, \ldots, s\). Choose \(\psi_0\) such that \(\psi_0(X_i) \in FX_\alpha(t_i)\). Let \(P\) be the permutation matrix corresponding to \(\pi\) and let

\[D = \text{diag}(\lambda_1 I, \ldots, \lambda_q I, I, X_{t_\alpha}, \ldots, I, X_{t_\alpha}),\]

where \(\lambda_i \in \mathbb{F}^\times\) are selected in such a way that condition \((21)\) holds with \(d_0 = X_{t_\alpha}\).

Clearly, condition \((21)\) also holds. Finally, let \(\psi_\alpha\) be given by \(\Psi = PD\) and \(\psi_0\). Then \(\psi_\alpha\) sends \(^\ast\Gamma\) to \(\Gamma\) and fixes \(\varphi\), so \(\psi_\alpha\) belongs to \(\text{Aut}(\Gamma, \varphi)\).

Let \(K\) be the kernel of \(f\) and let

\[N = \{ \psi \in K \mid \deg d_1 = \cdots = \deg d_k \}.\]

The same arguments as in 2) show that \(K/N \cong T^q/d/T\) and \(\text{Aut}(\Gamma, \varphi)/N\) is isomorphic to \(((T^q/d) \times \mathbb{Z}_2) \ltimes (\text{Sym} \Sigma \times \text{Sym}(s)) \times \text{Aut} \Sigma\), with the stated actions. But here we have \(N = \text{Stab}(\Gamma, \varphi)\), which completes the proof. \(\square\)

4. Series A

In this section we describe the Weyl groups of fine gradings on the simple Lie algebras of series A. Thus, we take \(\mathcal{R} = M_n(F)\), \(n \geq 2\), and \(\mathcal{L} = \text{psl}_n(F) = \text{[R, R]}/(\text{Z}(\mathcal{R}) \cap [\text{R, R}])\). First we review the classification of fine gradings on \(\mathcal{L}\) from \(\text{EkHo}\) (extended to positive characteristic using automorphism group schemes) and then derive the Weyl groups for \(\mathcal{L}\) from what we already know about automorphisms of fine gradings (\(\text{EkKb}\)) and fine \(\varphi\)-gradings (Section 3) on \(\mathcal{R}\).
4.1. Classification of fine gradings. The case \( n = 2 \) is easy, because the restriction from \( \mathcal{R} \) to \( \mathcal{L} \) yields an isomorphism \( \text{Aut}(\mathcal{R}) \to \text{Aut}(\mathcal{L}) \). It follows that the classification of fine gradings on \( \mathcal{L} \) is the same as that on \( \mathcal{R} \). Namely, there are two fine gradings on \( \mathfrak{sl}_2(\mathbb{F}) \), up to equivalence: the Cartan grading, whose universal group is \( \mathbb{Z} \), and the Pauli grading, whose universal group is \( \mathbb{Z}_2^2 \).

Now assume \( n \geq 3 \). Then the restriction and passing modulo the center yields a closed imbedding \( \text{Aut}(\mathcal{R}) \to \text{Aut}(\mathcal{L}) \), which is not an isomorphism. To rectify this, one introduces the affine group scheme \( \text{Aut}(\mathcal{R}) \) corresponding to the algebraic group of automorphisms and anti-automorphisms of \( \mathcal{R} \) (see [BK10, §3]). Unless \( n = \text{char}\, \mathbb{F} = 3 \), we obtain an isomorphism \( \text{Aut}(\mathcal{R}) \to \text{Aut}(\mathcal{L}) \). It is convenient to divide gradings on \( \mathcal{L} \) into two types: for Type I the corresponding diagonalizable subgroup scheme of \( \text{Aut}(\mathcal{L}) \) is contained in the image of the closed imbedding \( \text{Aut}(\mathcal{R}) \to \text{Aut}(\mathcal{L}) \), while for Type II it is not. In other words, a grading on \( \mathcal{L} \) is of Type I if and only if it is induced from a (unique) grading on \( \mathcal{R} \) by restriction and passing modulo the center.

In [BK10], the distinguished element of a Type II grading \( \Gamma \) is introduced. It can be characterized as the unique element \( h \) of order 2 in the grading group \( G \) such that the coarsening \( \Gamma' \) induced from \( \Gamma \) by the quotient map \( G \to \bar{G} := G/\langle h \rangle \) is a Type I grading. The original grading \( \Gamma \) can be recovered from \( \Gamma' \) if we know the action of some character \( \chi \) of \( G \) with \( \chi(h) = -1 \). Indeed, we just have to split each component of \( \Gamma' \) into eigenspaces with respect to the action of \( \chi \). We can transfer this procedure to \( \mathcal{R} \) in the following way. The action of \( \chi \) on \( \mathcal{L} \) is induced by \( -\varphi \) where \( \varphi \) is an anti-automorphism of \( \mathcal{R} \). The Type I grading \( \Gamma' \) on \( \mathcal{L} \) comes from a grading \( \Gamma'' \) on \( \mathcal{R} \). Since \( -\varphi \) is an automorphism of \( \mathcal{R}^{\sim} \) (the Lie algebra \( \mathcal{R} \) under commutator) and \( \varphi^2 \) acts as a scalar on each component of \( \Gamma'' \), we can refine the \( G \)-grading \( \Gamma' \) : \( \mathcal{R} = \bigoplus_{g \in G} \mathcal{R}_g \) to a \( G \)-grading \( \Gamma' : \mathcal{R}^{\sim} = \bigoplus_{g \in G} \mathcal{R}_g \) by splitting each component \( \mathcal{R}_g \) into eigenspaces of \( \varphi \). In detail, \( \varphi^2 \) acts on \( \mathcal{R}_g \) as multiplication by \( \chi^2(g) \) (where we regard \( \chi^2 \) as a character of \( \bar{G} \), since \( \chi^2(h) = 1 \)), so we set

\[
\mathcal{R}_g = \{ X \in \mathcal{R}_g \mid \varphi(X) = -\chi(g)X \} = \{ \varphi(X) - \chi(g)X \mid X \in \mathcal{R}_g \}.
\]

Then \( \Gamma' \) induces the original Type II grading \( \Gamma \) on \( \mathcal{L} \) by restriction and passing modulo the center.

Now we apply the above to fine gradings on \( \mathcal{L} \). The fine gradings of Type I come from the fine gradings on \( \mathcal{R} \) that do not admit an anti-automorphism \( \varphi \) making them \( \varphi \)-gradings. All fine gradings on \( \mathcal{R} \) are obtained as follows. We start from \( T \), a finite abelian group that admits a nondegenerate alternating bicharacter \( \beta \) (hence \( |T| \) is a square). Fix a realization, \( T \), of the matrix algebra endowed with a division grading with support \( T \) and bicharacter \( \beta \). Let \( k \geq 1 \) be an integer. Denote by \( \bar{G} = \bar{G}(T,k) \) the abelian group freely generated by \( T \) and the symbols \( \bar{g}_1, \ldots, \bar{g}_k \).

**Definition 4.1.** Let \( \mathcal{M}(T,k) \) be the \( \bar{G} \)-graded algebra \( \text{End}_\mathcal{D}(V) \) where \( V \) has a \( \mathcal{D} \)-basis \( \{v_1, \ldots, v_k\} \) with \( \text{deg} \, v_i = \bar{g}_i \). Let \( n = k\sqrt{|T|} \) and \( \mathcal{R} = \mathcal{M}_n(\mathcal{F}) \). The grading on \( \mathcal{R} \) obtained by identifying \( \mathcal{R} \) with \( \mathcal{M}(T,k) \) will be denoted by \( \Gamma_M(T,k) \). In other words, we define this grading by identifying \( \mathcal{R} = \mathcal{M}_k(\mathcal{D}) \) and setting \( \text{deg}(E_{ij} \otimes X_i) := \bar{g}_i \bar{g}_j^{-1} \). By abuse of notation, we will also write \( \Gamma_M(T,k) \).

The universal group of \( \Gamma_M(T,k) \) is the subgroup \( \bar{G}^0 = \bar{G}(T,k)^0 \) of \( \bar{G} \) generated by the support, i.e., by the elements \( z_{i,j,t} := \bar{g}_i \bar{g}_j^{-1}, t \in T \). Clearly, \( \bar{G}^0 \cong T \times \mathbb{Z}^{k-1} \). By [EHR10, Proposition 3.24], \( \Gamma_M(T,k) \) is a \( \varphi \)-grading for some \( \varphi \) if and only if \( T \)
is an elementary 2-group and \( k \leq 2 \). Two gradings, \( \Gamma_M(T, k) \) and \( \Gamma_M(T', k') \), are equivalent if and only if \( T \cong T' \) and \( k = k' \).

**Definition 4.2.** Consider the grading \( \Gamma_M(T, k) \) on \( \mathcal{R} \) by the group \( \tilde{G}(T, k)^0 \) where \( k \geq 3 \) if \( T \) is an elementary 2-group. The \( \tilde{G}(T, k)^0 \)-grading on \( \mathcal{L} \) obtained by restriction and passing modulo the center will be denoted by \( \Gamma_M^{(1)}(T, k) \).

The grading \( \Gamma_M^{(1)}(T, k) \) is fine, and \( \tilde{G}(T, k)^0 \) is its universal group. To deal with fine gradings of Type II, we will need the following general observation:

**Lemma 4.3.** Let \( \Gamma \) be a \( \varphi \)-grading on an algebra \( A \) and let \( \Gamma_G \) be its universal group. Then there exist an abelian group \( \chi \). Define a fine grading of Type II, we will need the following general observation:

**Proof.** For each \( \varphi \in \Gamma_G \), \( \varphi^2 \) acts on \( A \) as multiplication by some \( \lambda(\varphi) \in \mathbb{F}^\times \). Since \( \Gamma_G \) is the universal group of \( \Gamma \), \( \lambda: \Gamma_G \rightarrow \mathbb{F}^\times \) is a homomorphism. For each \( \varphi \in \Gamma_G \), we select \( \mu(\varphi) \in \mathbb{F}^\times \) such that \( \mu(\varphi)^2 = \lambda(\varphi) \) (there are two choices). It will be convenient to choose \( \mu(T) = 1 \). It follows that

\[
\mu(\varphi \varphi) = \varepsilon(\varphi, \varphi) \mu(\varphi) \mu(\varphi) \quad \text{for all} \quad \varphi, \varphi \in \Gamma \]

where \( \varepsilon(\varphi, \varphi) \in \{\pm 1\} \). One immediately verifies that \( \varepsilon \) is a symmetric 2-cocycle on \( \Gamma \) with \( \varepsilon(\varphi, \varphi) = 1 \) for all \( \varphi \in \Gamma \), and, moreover, the class of \( \varepsilon \) in \( H^2(\Gamma_G, \mathbb{Z}/2) \) (where we identified \( \{\pm 1\} \) with \( \mathbb{Z}/2 \)) does not depend on the choices of \( \mu(\varphi) \). Let \( G \) be the central extension of \( \Gamma_G \) by \( \mathbb{Z}/2 \) determined by \( \varepsilon \), i.e., \( G \) consists of the pairs \( (\varphi, \delta) \), \( \varphi \in \Gamma \), \( \delta \in \{\pm 1\} \), with multiplication given by

\[
(\varphi, \delta_1)(\varphi, \delta_2) = (\varphi \varphi, \varepsilon(\varphi, \varphi) \delta_1 \delta_2) \quad \text{for all} \quad \varphi, \varphi \in \Gamma \quad \text{and} \quad \delta_1, \delta_2 \in \{\pm 1\}.
\]

Define \( \chi: G \rightarrow \mathbb{F}^\times \) by \( \chi(\varphi, \delta) = \mu(\varphi) \delta \). Comparing (26) and (27), we see that \( \chi \) is a homomorphism. Set \( h = (\varphi, -1) \in G \). Then \( h \) has order 2 and \( \chi(h) = -1 \). By construction, the action of \( \chi^2 \) on \( A \) determined by \( \Gamma \) coincides with \( \varphi^2 \).

Let \( T \) be an elementary 2-group of even dimension. Recall the group \( \tilde{G}(T, q, s, \tau) \), which was introduced before Definition 3.3 and its subgroup \( \tilde{G}(T, q, s, \tau)^0 \).

**Definition 4.4.** Consider the grading \( \Gamma = \Gamma_M(T, q, s, \tau) \) on \( \mathcal{R} \) by the group \( \Gamma_G = \tilde{G}(T, q, s, \tau)^0 \) where \( t_1 \neq t_2 \) if \( q = 2 \) and \( s = 0 \). Let \( \Phi \) be the matrix given by

\[
\Phi = \text{diag} \left( X_{t_1}, \ldots, X_{t_q}; 0 \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}, \ldots, 0 \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix} \right).
\]

Define \( \varphi(X) = \Phi^{-1}(t^i)X\Phi \). Let \( G, h \) and \( \chi \) be as in Lemma 4.3 so we obtain a \( G \)-grading on \( \mathcal{R}^{(-)} \) defined by (25). The \( G \)-grading on \( \mathcal{L} \) obtained by restriction and passing modulo the center will be denoted by \( \Gamma_M^{(1)}(T, q, s, \tau) \).

The grading \( \Gamma_M^{(1)}(T, q, s, \tau) \) is fine, and \( G \) is its universal group. Note that \( \varphi^4 = \text{id} \). It can be shown (cf. Example 3.21) that the extension \( (h) \rightarrow G \rightarrow \Gamma \) is split if and only if there exists \( t \in T \) such that \( t_it \) are in \( T_+ \) for all \( i \) or in \( T_- \) for
all $i$. Taking into account (3), we see that $G$ is isomorphic to
\[
\begin{cases}
\mathbb{Z}_2^{\dim T - 2 \dim T_0 + \max(0,q-1) + 1} \times \mathbb{Z}_4^{\dim T_0} \times \mathbb{Z}^s & \text{if } \exists t \in T \quad \beta(t_1 t) = \ldots = \beta(t_q t);
\mathbb{Z}_2^{\dim T - 2 \dim T_0 + \max(0,q-1)} \times \mathbb{Z}_4^{\dim T_0 + 1} \times \mathbb{Z}^s & \text{otherwise},
\end{cases}
\]
where $T_0$ is the subgroup of $T$ generated by the elements $t_i t_{i+1}$, $i = 1, \ldots, q - 1$.

Now Theorem 4.2 of [Eld10] can be extended to positive characteristic and recast as follows:

**Theorem 4.5.** Let $\mathbb{F}$ be an algebraically closed field, $\text{char } \mathbb{F} \neq 2$. Let $n \geq 3$ if $\text{char } \mathbb{F} \neq 3$ and $n \geq 4$ if $\text{char } \mathbb{F} = 3$. Then any fine grading on $\mathfrak{psl}_n(\mathbb{F})$ is equivalent to one of the following:

- $\Gamma_{A}^{(I)}(T, k)$ as in Definition 4.2 with $k \sqrt{|T|} = n$,
- $\Gamma_{A}^{(II)}(T, q, s, r)$ as in Definition 4.3 with $(q + 2s) \sqrt{|T|} = n$.

Gradings belonging to different types listed above are not equivalent. Within each type, we have the following:

- $\Gamma_{A}^{(I)}(T_1, k_1)$ and $\Gamma_{A}^{(I)}(T_2, k_2)$ are equivalent if and only if $T_1 \cong T_2$ and $k_1 = k_2$;
- $\Gamma_{A}^{(II)}(T_1, q_1, s_1, r_1)$ and $\Gamma_{A}^{(II)}(T_2, q_2, s_2, r_2)$ are equivalent if and only if $T_1 \cong T_2$, $q_1 = q_2$, $s_1 = s_2$ and, identifying $T_1 = T_2 = \mathbb{Z}_2^{2r}$, $\Sigma(r_1)$ is conjugate to $\Sigma(r_2)$ by the natural action of $\text{ASp}_{2r}(2)$. $\square$

The missing case $n = \text{char } \mathbb{F} = 3$ can be treated using octonions, because in characteristic 3 the algebra of traceless octonions under commutator is a Lie algebra isomorphic to $\mathfrak{psl}_3(\mathbb{F})$ (cf. [BRK10 Remark 4.11]).

### 4.2. Weyl groups of fine gradings.

By [EK13 Theorem 2.8], the Weyl group of $\Gamma_M(T, k)$ is isomorphic to $T^{k-1} \ltimes (\text{Sym}(k) \times \text{Aut}(T, \beta))$, with $\text{Sym}(k)$ and $\text{Aut}(T, \beta)$ acting on $T^{k-1}$ through their natural action on $T^k$ and identification of $T^{k-1}$ with $T^k/\text{ker } T$ where $T$ is imbedded into $T^k$ diagonally. Thanks to the isomorphism $\text{Aut}(\mathfrak{h}_2(\mathbb{F})) \to \text{Aut}(\mathfrak{sl}_2(\mathbb{F}))$, it follows that the Weyl group of the Cartan grading on $\mathfrak{sl}_2(\mathbb{F})$ is $\text{Sym}(2)$ (the classical Weyl group of type $A_1$) and the Weyl group of the Pauli grading on $\mathfrak{sl}_2(\mathbb{F})$ is $\text{Sp}_2(2) = \text{GL}_2(2)$ (this is known in the case $\text{char } \mathbb{F} = 0$ — see [HPPT02]).

To state our result for $\mathfrak{psl}_n(\mathbb{F})$, $n \geq 3$, it is convenient to introduce the following notation:

\[\text{Aut}(T, \beta) := \text{Aut}(T, \beta) \times \langle \sigma \rangle,\]

where $\sigma$ is an element of order 2 acting as the automorphism of $T$ that sends $a_i$ to $a_i^{-1}$ and $b_i$ to $b_i$, where $a_i$ and $b_i$ are the generators of $T$ used for the chosen realization of $\mathfrak{d}$ (a “symplectic basis” of $T$ with respect to $\beta$). We observe that $\beta(\sigma \cdot u, \sigma \cdot v) = \beta(u, v)^{-1}$, for all $u, v \in T$, and hence we obtain an induced action of $\sigma$ on $\text{Aut}(T, \beta)$ by setting $(\sigma \cdot \alpha)(t) := \sigma \cdot \alpha(\sigma \cdot t)$ for all $\alpha \in \text{Aut}(T, \beta)$ and $t \in T$. The elements of $\text{Aut}(T, \beta)$ act as automorphisms of $T$ that send $\beta$ to $\beta \pm 1$. However, this action is not faithful if $T$ is an elementary 2-group.

**Theorem 4.6.** Let $\mathbb{F}$ be an algebraically closed field, $\text{char } \mathbb{F} \neq 2$. Let $n \geq 3$ if $\text{char } \mathbb{F} \neq 3$ and $n \geq 4$ if $\text{char } \mathbb{F} = 3$. Consider the fine grading $\Gamma = \Gamma_{A}^{(I)}(T, k)$ on $\mathfrak{psl}_n(\mathbb{F})$ as in Definition 4.2, $k \sqrt{|T|} = n$. Then

\[W(\Gamma) \cong T^{k-1} \ltimes (\text{Sym}(k) \times \text{Aut}(T, \beta)),\]
with Sym\( (k) \) and \( \text{Aut}(T, \beta) \) acting on \( T^{k-1} \) through their natural action on \( T^k \) and identification of \( T^{k-1} \) with \( T^k / T \) where \( T \) is imbedded into \( T^k \) diagonally.

**Proof.** The grading \( \Gamma \) on \( \mathcal{L} = \mathfrak{psl}_n(\mathbb{F}) \) is induced by the grading \( \Gamma' = \Gamma_{M}(T, k) \) on \( \mathcal{R} = M_n(\mathbb{F}) \). The universal group of both gradings is \( G = G(T, k) \). Since restriction is a bijection between gradings on \( \mathcal{R} \) and Type I gradings on \( \mathcal{L} \), an automorphism \( \psi' \) of \( \mathcal{R} \) sends \( \alpha \Gamma \) to \( \Gamma' \), for some automorphism \( \alpha \) of \( G \), if and only if the induced automorphism \( \psi \) of \( \mathcal{L} \) sends \( \alpha \Gamma \) to \( \Gamma \). The automorphism group of \( \mathcal{L} \) is the semidirect product of \( \text{Aut}(\mathcal{R}) \), in its induced action on \( \mathcal{L} \), and \( \langle \sigma \rangle \), where \( \sigma \) is given by the negative of matrix transpose. To compute the action of \( \sigma \), recall that \( (u_1, \ldots, u_k) T \in T^k / T \) can be represented by the automorphism \( X \mapsto DXD^{-1} \) where \( D = \text{diag}(X_{u_1}, \ldots, X_{u_k}), \pi \in \text{Sym}(k) \) can be represented by \( X \mapsto PXP^{-1} \) where \( P \) is the permutation matrix corresponding to \( \pi \), and \( \alpha \in \text{Aut}(T, \beta) \) can be represented by \( X \mapsto \psi_0(X) \) where \( \psi_0 \) is an automorphism of \( \mathcal{D} \) such that \( \psi_0(\alpha(X_i)) = F_X\alpha(i) \) for all \( t \in T \). The conjugation by \( \sigma \) sends the automorphism \( X \mapsto \Psi X \Psi^{-1} \) to the automorphism \( X \mapsto (\Psi^{-1})X(\Psi) \), i.e., replaces \( \Psi \) by \( \Psi^{-1} \). Hence, \( \sigma \) commutes with \( \text{Sym}(k) \), while the conjugation by \( \sigma \) sends \( (u_1, \ldots, u_k) T \) to \( (\sigma \cdot u_1, \ldots, \sigma \cdot u_k) T \), where the action of \( \sigma \) on \( T \) is as indicated above. Also, the action of \( \sigma \) on \( G \) sends \( z_{i,j,t} := g_{i}g_{j}^{-1} \) to \( z_{i,j,s \cdot t} \), so \( \sigma \) belongs to \( \text{Stab}(G) \), but not to \( \text{Stab}(\Gamma) \). Hence we obtain \( \text{Aut}(\Gamma) = \text{Aut}(\Gamma') \times \langle \sigma \rangle \) and \( \text{Stab}(\Gamma) = \text{Stab}(\Gamma') \). The result follows.

**Theorem 4.7.** Let \( \mathbb{F} \) be an algebraically closed field, \( \text{char} \mathbb{F} \neq 2 \). Let \( n \geq 3 \) if \( \text{char} \mathbb{F} \neq 3 \) and \( n \geq 4 \) if \( \text{char} \mathbb{F} = 3 \). Consider the fine grading \( \Gamma = \Gamma_{A}^{(1)}(T, q, s, \tau) \) on \( \mathfrak{psl}_n(\mathbb{F}) \) as in Definition 4.4 \( (q + 2s)\sqrt{|T|} = n \). Let \( \Sigma = \Sigma(\tau) \). Then \( W(\Gamma) \) contains a normal subgroup \( N \) isomorphic to \( Z_2^{q+s-1} \times Z_2^{q+s-1} \) such that

\[
W(\Gamma)/N \cong \left((T^{q+s-1} \times Z_2^{q+s-1}) \times (\text{Sym}\Sigma \times \text{Sym}(s)) \times \text{Aut}^{\ast} \Sigma, \right.
\]

where the actions are described naturally if we identify \( T^{q+s-1} \times Z_2^{q+s-1} \) with \( Z_2^{q+s-1} \) with \( Z_2^{q+s-1} \) (diagonal imbeddings). Moreover, \( W(\Gamma) \) contains a subgroup isomorphic to \( ((T^{q+s-1} \times Z_2^{q+s-1}) \times (\text{Sym}\Sigma \times \text{Sym}(s)) \times \text{Aut} \Sigma \) that is disjoint from \( N \).

**Proof.** The grading \( \Gamma = \Gamma_{A}^{(1)}(T, q, s, \tau) \) on \( \mathcal{L} = \mathfrak{psl}_n(\mathbb{F}) \) is induced by the grading \( \Gamma' \) on \( \mathcal{R} = M_n(\mathbb{F}) \), obtained from \( \Gamma = \Gamma_{M}(T, q, s, \tau) \) and \( \varphi \) as in Definition 4.4 \( \mathfrak{G} \). The universal group of \( \Gamma \) is \( \mathfrak{G} = G(T, q, s, \tau) \), while the universal group of \( \Gamma \) is the extension \( G \) of \( \mathfrak{G} \) as in Lemma 4.3. Similarly to Type I, an automorphism \( \psi' \) of \( \mathcal{R} \) sends \( \alpha \Gamma \) to \( \Gamma' \), for some automorphism \( \alpha \) of \( G \), if and only if the induced automorphism \( \psi \) of \( \mathcal{L} \) sends \( \alpha \Gamma \) to \( \Gamma \). Note that \( \alpha \) fixes the distinguished element \( h = (e, -1) \) and hence yields an automorphism \( \varphi \) of \( \mathcal{G} \). It follows that \( \psi' \) sends \( \varphi \) to \( \Gamma' \). For any \( g \in G \) and \( X \in \mathcal{R} \), we have \( \varphi(X) = -\chi(g)X \). Since \( (\psi')^{-1}(X) \in \mathcal{R}_{\alpha^{-1}(g)} \), we also have \( (\varphi(\psi')^{-1}(X) = -\chi(\alpha^{-1}(g))(\psi')^{-1}(X) \). It follows that \( \psi' \varphi(\psi')^{-1} \) is \( \xi \varphi \) where \( \xi \) is the action of the character \( \chi \alpha^{-1} \) on \( \mathcal{R} \) determined by the \( G \)-grading \( \Gamma' \). Since \( (\psi')^{-1}(X) \in \mathcal{R}_{\alpha^{-1}(g)} \), we have \( \varphi(X) = \nu \psi'(X) \) where \( \nu \in \mathbb{F}^\times \) depends only on \( \varphi \). It follows that \( \psi' \) permutes the components of \( \Gamma' \) and hence sends \( \alpha \Gamma \) to \( \Gamma' \) where \( \alpha \) is a lifting of \( \varphi \). We have proved that an automorphism \( \psi' \) of \( \mathcal{R} \) belongs
to $\text{Aut}^*(\Gamma, \varphi)$, respectively $\text{Stab}(\Gamma, \varphi)$, if and only if the induced automorphism $\psi$ of $\mathcal{L}$ belongs to $\text{Aut}(\Gamma)$, respectively $\text{Stab}(\Gamma)$. Finally, note that $-\varphi$ induces an automorphism of $\mathcal{L}$ that belongs to $\text{Stab}(\Gamma)$. It follows that the Weyl group of $\Gamma$ is isomorphic to $\text{Aut}^*(\Gamma, \varphi)/\text{Stab}(\Gamma, \varphi)$. The latter group was described in Theorem 3.12.

If $\text{char} \mathbb{F} = 3$, there are two fine gradings on $\mathfrak{psl}_3(\mathbb{F})$: the Cartan grading, whose universal group is $\mathbb{Z}^2$, and the grading induced by the Cayley–Dickson doubling process for octonions, whose universal group is $\mathbb{Z}_2^3$. The Weyl groups of these gradings are, respectively, the classical Weyl group of type $G_2$ \cite[Theorem 3.3]{EKb} and $\text{GL}_3(2)$ \cite[Theorem 3.5]{EKb}.

5. Series $B$, $C$ and $D$

In this section we describe the Weyl groups of fine gradings on the simple Lie algebras of series $B$, $C$ and $D$ with exception of type $D_4$. Thus, we take $\mathcal{R} = M_n(\mathbb{F})$, $n \geq 4$, and $\mathcal{L} = \mathcal{K}(\mathcal{R}, \varphi)$ where $\varphi$ is an involution on $\mathcal{R}$. If $\varphi$ is symplectic, then, of course, $n$ has to be even. If $\varphi$ is orthogonal, we assume $n > 5$ and $n \neq 8$. First we review the classification of fine gradings on $\mathcal{L}$ from \cite{EKb} (extended to positive characteristic using automorphism group schemes) and then derive the Weyl groups for $\mathcal{L}$ from what we already know about automorphisms of fine $\varphi$-gradings (Section 3) on $\mathcal{R}$.

5.1. Classification of fine gradings. Under the stated assumptions on $n$, the restriction from $\mathcal{R}$ to $\mathcal{L}$ yields an isomorphism $\text{Aut}(\mathcal{R}, \varphi) \to \text{Aut}(\mathcal{L})$ (see \cite[§3]{BK10}). It follows that the classification of fine gradings on $\mathcal{L}$ is the same as the classification of fine $\varphi$-gradings on $\mathcal{R}$ (here $\varphi$ is fixed).

The case of series $B$ is quite easy, because $n$ is odd and hence the elementary 2-group $T$ must be trivial. Let $G = \bar{G}((e), q, s, \tau)^0$ where $\tau = (e, \ldots, e)$, so $G \cong \mathbb{Z}_2^{q-1} \times \mathbb{Z}^s$.

**Definition 5.1.** Consider the grading $\Gamma = \Gamma_M((e), q, s, \tau)$ on $\mathcal{R}$ by $G$. Let $\Phi$ be the matrix given by

$$\Phi = \text{diag} \left(1, \ldots, 1, \frac{[0 \ 1]}{q}, \ldots, \frac{[0 \ 1]}{[1 \ 0]} \right).$$

Then $\Gamma$ is a fine $\varphi$-grading for $\varphi(X) = \Phi^{-1}(X)\Phi$ and hence its restriction is a fine grading on $\mathcal{L} \cong \mathfrak{so}_n(\mathbb{F})$. We will denote this grading by $\Gamma_B(q, s)$.

Now we turn to series $C$ and $D$, where $n$ is even and hence $T$ may be nontrivial. So, let $T$ be an elementary 2-group of even dimension. Choose $\tau$ as in (11) with all $t_i \in T_-$ in case of series $C$ and all $t_i \in T_+$ in case of series $D$. Let $G = \bar{G}(T, q, s, \tau)^0$, so $G \cong \mathbb{Z}_2^{\dim T - 2 \dim T_0} \times \mathbb{Z}_2^{\dim T_0} \times \mathbb{Z}^s$ where $T_0$ is the subgroup of $T$ generated by the elements $t_it_{i+1}$, $i = 1, \ldots, q - 1$.

**Definition 5.2.** Consider the grading $\Gamma = \Gamma_M(D, q, s, \tau)$ on $\mathcal{R}$ by $G$ where $t_1 \neq t_2$ if $q = 2$ and $s = 0$. Let $\Phi$ be the matrix given by

$$\Phi = \text{diag} \left(X_{t_1}, \ldots, X_{t_q}, \frac{[0 \ I]}{\delta I \ 0}, \ldots, \frac{[0 \ I]}{\delta I \ 0} \right),$$
Theorem 5.3. Let $\mathbb{F}$ be an algebraically closed field, $\text{char} \mathbb{F} \neq 2$. Let $n \geq 5$ be odd. Then any fine grading on $\mathfrak{so}_n(\mathbb{F})$ is equivalent to $\Gamma_B(q, s)$ where $q + 2s = n$. Also, $\Gamma_B(q_1, s_1)$ and $\Gamma_B(q_2, s_2)$ are equivalent if and only if $q_1 = q_2$ and $s_1 = s_2$. 

Theorem 5.4. Let $\mathbb{F}$ be an algebraically closed field, $\text{char} \mathbb{F} \neq 2$. Let $n \geq 4$ be even. Then any fine grading on $\mathfrak{sp}_n(\mathbb{F})$ is equivalent to $\Gamma_C(T, q, s, \tau)$ where $(q + 2s)\sqrt{|T|} = n$. Moreover, $\Gamma_C(T_1, q_1, s_1, \tau_1)$ and $\Gamma_C(T_2, q_2, s_2, \tau_2)$ are equivalent if and only if $T_1 \cong T_2$, $q_1 = q_2$, $s_1 = s_2$ and, identifying $T_1 = T_2 = \mathbb{Z}_2^r$, $\Sigma(\tau_1)$ is conjugate to $\Sigma(\tau_2)$ by the twisted action of $\text{Sp}_{2r}(2)$ as in Definition 3.9. 

Theorem 5.5. Let $\mathbb{F}$ be an algebraically closed field, $\text{char} \mathbb{F} \neq 2$. Let $n \geq 6$ be even. Assume $n \neq 8$. Then any fine grading on $\mathfrak{so}_n(\mathbb{F})$ is equivalent to $\Gamma_D(T, q, s, \tau)$ where $(q + 2s)\sqrt{|T|} = n$. Moreover, $\Gamma_D(T_1, q_1, s_1, \tau_1)$ and $\Gamma_D(T_2, q_2, s_2, \tau_2)$ are equivalent if and only if $T_1 \cong T_2$, $q_1 = q_2$, $s_1 = s_2$ and, identifying $T_1 = T_2 = \mathbb{Z}_2^r$, $\Sigma(\tau_1)$ is conjugate to $\Sigma(\tau_2)$ by the twisted action of $\text{Sp}_{2r}(2)$ as in Definition 3.9. 

5.2. Weyl groups of fine gradings. Let $\Gamma = \Gamma_B(q, s)$, $\Gamma_C(T, q, s, \tau)$ or $\Gamma_D(T, q, s, \tau)$, so $\Gamma$ is the restriction of the grading $\Gamma' = \Gamma_M(T, q, s, \tau)$ on $\mathfrak{g}$ to $\mathfrak{l} = \mathfrak{k}(\mathfrak{g}, \varphi)$. By arguments similar to the proof of Theorem 4.7, one shows that the Weyl group of $\Gamma$ is isomorphic to $\text{Aut}(\Gamma', \varphi)/\text{Stab}(\Gamma', \varphi)$, which was described in Theorem 3.12. 

For $\Gamma = \Gamma_B(q, s)$, $T$ is trivial and $\Sigma$ is a singleton of multiplicity $q$, so we obtain: 

Theorem 5.6. Let $\mathbb{F}$ be an algebraically closed field, $\text{char} \mathbb{F} \neq 2$. Let $n \geq 5$ be odd. Consider the fine grading $\Gamma = \Gamma_B(q, s)$ on $\mathfrak{so}_n(\mathbb{F})$ as in Definition 5.1, where $q + 2s = n$. Let $\Sigma = \Sigma(\tau)$. Then $W(\Gamma) \cong \text{Sym}(q) \times W(s)$ where $W(s) = \mathbb{Z}_2^r \rtimes \text{Sym}(s)$ (wreath product of $\text{Sym}(s)$) and $Z_2$. 

For $\Gamma_C(T, q, s, \tau)$ and $\Gamma_D(T, q, s, \tau)$, $T$ may be nontrivial, so the answer is more complicated: 

Theorem 5.7. Let $\mathbb{F}$ be an algebraically closed field, $\text{char} \mathbb{F} \neq 2$. Let $n \geq 4$ be even. Consider the fine grading $\Gamma = \Gamma_C(T, q, s, \tau)$ on $\mathfrak{sp}_n(\mathbb{F})$ or $\Gamma = \Gamma_D(T, q, s, \tau)$ on $\mathfrak{so}_n(\mathbb{F})$ as in Definition 5.2, where $(q + 2s)\sqrt{|T|} = n$ and $n \neq 4, 8$ in the case of $\mathfrak{so}_n(\mathbb{F})$. Let $\Sigma = \Sigma(\tau)$ in Definition 5.2. Then 

$$W(\Gamma) \cong ((\mathbb{T}^{q+s-1} \rtimes \mathbb{Z}_2^r) \rtimes (\text{Sym}\Sigma \times \text{Sym}(s)) \rtimes \text{Aut} \Sigma,$$

where the actions on $\mathbb{T}^{q+s-1}$ are via the identification with $T^{q+s}/T$ (diagonal embedding).

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