Network Synthetic Interventions: 
A Framework for Panel Data with 
Network Interference

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Abstract

We propose a generalization of the synthetic controls and synthetic interventions methodology to incorporate network interference. We consider the estimation of unit-specific treatment effects from panel data where there are spillover effects across units and in the presence of unobserved confounding. Key to our approach is a novel latent factor model that takes into account network interference and generalizes the factor models typically used in panel data settings. We propose an estimator, “network synthetic interventions”, and show that it consistently estimates the mean outcomes for a unit under an arbitrary sequence of treatments for itself and its neighborhood, given certain observation patterns hold in the data. We corroborate our theoretical findings with simulations.

1 Introduction

There is growing interest in the identification and estimation of causal effects in the context of networks, in which the outcomes of a unit (e.g., an individual, customer cohort, or region) are affected by the treatments (e.g., recommendations, discounts, or legislation) assigned to other units, known as the unit’s “neighbors”. For example, whether an individual (i.e., the unit) gets COVID-19 (i.e., the outcome) is a function of not only the individual’s vaccination status (i.e., the treatment), but also the vaccination status of that individual’s social network. That is, there is network interference, also known as spillover effects.

Works thus far have generally considered the setting of a single measurement or dataset, whether collected from a randomized experiment or an observational study. Under fully arbitrary interference, it has been shown that it is impossible to estimate any desired causal estimands as the model is not fully identifiable Manski (2013); Aronow et al. (2017); Basse and Airoldi (2018a); Karwa and Airoldi (2018). As a result, there have been many proposed models that impose assumptions on exposure functions Manski (2013); Aronow et al. (2017); Viviano (2020); Auerbach and Tabord-Meehan (2021); Li et al. (2021), interference neighborhoods Ugander et al. (2013); Bargagli-Stoffi et al. (2020); Sussman and Airoldi (2017a); Bhattacharya et al. (2020), parametric structure Toulis and Kao (2013); Basse and Airoldi (2018b); Cai et al. (2015); Gui et al. (2015); Eckles et al. (2017), two-sided platforms Johari et al. (2022); Bajari et al. (2021) or a combination of these, each leading to a different solution concept. In this work, we focus on network interference that is additive across the neighbors, referred to in the literature as the joint assumptions of neighborhood interference, additivity of main effects, or additivity of interference effects Sussman and Airoldi (2017a); Yu et al. (2022); Cortez et al. (2022a,b).

Distinct to our work is that we consider a panel data setting in which there are multiple measurements for each unit, as arises when units are observed across time. The potential outcomes function are thus also time dependent. Additionally, we allow for estimation of counterfactuals under multiple treat-
ments, whereas the existing literature has largely focused on binary treatments. Key to our approach is a novel latent factor model that takes into account network interference and is a generalization of the factor models typically used in panel data settings. Although adding time to our analysis might appear to introduce complexity, we show that being able to measure potential outcomes across time actually enables the inference of unit-specific mean outcomes under different intervention sequences, as long as the dataset is “sufficiently rich” (specifically, as long as there is sufficient diversity in the observed treatments). Estimating unit-specific causal effects is typically not feasible in the single measurement setup unless one imposes strong parametric model assumptions on the potential outcomes function. As a result, previous work has focused on causal estimands that capture population-wide effects, such as the average direct treatment effect (the average difference in outcomes if only one unit and none of its neighbours get treated Basse and Airoldi (2018b); Jagadeesan et al. (2020); Sävje et al. (2021); Sussman and Airoldi (2017a); Leung (2019); Ma and Tresp (2021)) and the average total treatment effect (the average difference in outcomes if all units get treated versus if they do not Ugander et al. (2013); Eckles et al. (2017); Chin (2019); Yu et al. (2022); Cortez et al. (2022a,b)). Alternately there has been some literature that focus on hypothesis testing for the presence of network interference Aronow (2012); Bowers et al. (2013); Athey et al. (2018); Pouget-Abadie et al. (2017); Saveski et al. (2017); these results do not immediately extend to estimation as they are based on randomization inference with a fixed network size and study testing sharp null hypotheses.

While a majority of the literature focuses on randomized experiment, there is growing interest as well to develop theory for accounting for network interference when analyzing observational studies. The existing literature generally assumes partial interference, where the network consists of many disconnected sub-communities Tchetgen and VanderWeele (2012); Perez-Heyrich et al. (2014); Liu et al. (2016); DiTraglia et al. (2020); Vazquez-Bare (2022). Without this strong clustering condition, other works impose strong parametric assumptions on the potential outcomes function, assuming that the potential outcomes only depends on a known statistic of the neighborhood treatment, e.g. the number or fraction of treated Verbitsky-Savitz and Raudenbush (2012); Chin (2019); Ogburn et al. (2017). This reduces estimation to a regression task under requirements of sufficient diversity in the treatments. Forastiere et al. (2021) considers a general exposure mapping model alongside an inverse propensity weighted estimator, but the estimator has high variance when the exposure mapping is complex. Further, building on recent works in panel data Agarwal et al. (2020), we allow for unobserved confounding in treatment assignment as long as there exist low-rank latent factors that capture the unobserved confounding signals.

2 Problem Statement

Setup. Consider a setting with \( N \geq 1 \) units, \( D \geq 1 \) treatments, and \( T \geq 1 \) measurements of interest. Unless otherwise stated, we index units with \( n \in [N] \), measurements with \( t \in [T] \), and treatments with \( a \in [D]_0 \).\(^1\) Let \( G = ([N], \mathcal{E}) \in \mathcal{G} \) denote a graph over the \( N \) units, where \( \mathcal{E} \subset [N] \times [N] \) denotes the edges of the graph. Throughout, we shall assume \( G \) to be fixed and observed. Let \( \mathcal{N}(n) \) denote the neighbors of unit \( n \in [N] \) with respect to \( G \) such that \( j \in \mathcal{N}(n) \iff (j, n) \in \mathcal{E} \).\(^2\) Under network interference, the potential outcome for a given unit \( n \) and measurement \( t \) is a real-valued random variable denoted by \( Y_{tn}^{(a)} \), where \( a \in [D]_0 \) denotes the treatments over all \( N \) units. We impose the following additional structure on the potential outcomes.

**Assumption 1** (Network SUTVA). The potential outcome of measurement \( t \in [T] \) for unit \( n \in [N] \) under treatments \( a \in [D]_0 \) is given by

\[
Y_{tn}^{(a)} = Y_{tn}^{(a_{\mathcal{N}(n)})},
\]

where \( a_{\mathcal{N}(n)} \in [D]_{0}^{[\mathcal{N}(n)]} \) denotes the treatments assigned to the units in \( n \)'s neighborhood \( \mathcal{N}(n) \) for measurement \( t \). That is, the potential outcome of unit \( n \) depends on its neighbors’ treatments but does not depend the treatment of any other unit \( j \in [N] \setminus \mathcal{N}(n) \).

Several prior works on network interference also assume Network SUTVA, e.g., as the Neighborhood Interference Assumption (NIA) (Sussman and Airoldi, 2017b).

\(^1\)Let \([X]_0 = \{0, 1, \ldots, X - 1\}\) and \([X] = \{1, \ldots, X\}\) for any positive integer \(X\).

\(^2\)For simplicity of notation, we include self-edges: \((n, n) \in \mathcal{N}(n)\) for all \(n \in [N]\).
We describe a simple estimator, which we term the **Network Synthetic Intervention (NSI)** estimator. It is a natural extension of the Synthetic Interventions estimator Agarwal et al. (2020) in the presence of network interference. Below, we describe the estimator formally, with a caricature example in Figure 1.

**NSI Estimator.** Consider the causal parameter $\theta_{tn}(\hat{A}^n_{pr})$ of interest, as given in (1). Let

$$A^n_u = [a_{N(n)}^1, a_{N(n)}^2, \ldots, a_{N(n)}^{T_{pe}}] \in [D]^{N(n)} \times T_u,$$

$$z_{u,n} = [Y_{nt} : t \in T_u] \in \mathbb{R}^{T_u}.$$
We now introduce the model that we use to develop our formal results. We note that this model, given \( \theta \) Then, estimation proceeds in a two-step procedure with a parameter \( \alpha \):

1. There exists a way \( \pi_j \) to permute \( N(j) \) such that: \( a^j_{\pi_j(N(j))} = c^j \) for all \( t \in T_n \cup T_{pr} \).

To estimate \( \theta_{tn}^A \), the relevant donor set is turns out to be \( \mathcal{I}(A^a_n, \tilde{A}^a_n, |N(n)|) \subset [N] \). For simplicity, we use the shorthand \( T^n := \mathcal{I}(A^a_n, \tilde{A}^a_n, |N(n)|) \). Let the donors’ training observations be given by

\[
Z_{tn} = [Y_{jt} : t \in T_n, j \in T^n] \in \mathbb{R}^{T_n \times |T^n|}.
\]  

Then, estimation proceeds in a two-step procedure with a parameter \( \kappa \).

**Step 1: Principal component regression.** Perform a singular value decomposition (SVD) of \( Z_{tn} \) to obtain \( \hat{\alpha} = \sum_{t=1}^{\kappa} \hat{s}_t \hat{\mu}_t \hat{\mu}_t^\top z_{tn} \in \mathbb{R}^{T^n} \).

**Step 2: Estimator.** Using \( \hat{\alpha} = [\hat{\alpha}_j : j \in T^n] \), construct the estimate

\[
\hat{\alpha}_j Y_{jt}, \quad \text{for all } t \in T_{pr},
\]

and, accordingly,

\[
\hat{\alpha}_j \tilde{\alpha}_j.
\]

**4 Formal Analysis: Model and Results**

In this section, we provide a formal analysis of the NSI estimator. We start by presenting a model for network interference. This is followed by formal results for identification and finite sample analysis.

**4.1 Model**

We now introduce the model that we use to develop our formal results. We note that this model, given in Assumption 2 below, satisfies Assumption 1.

**Assumption 2.** Let the potential outcome of measurement \( t \in [T] \) for unit \( n \in [N] \) under graph \( G \in \mathcal{G} \) if assigned treatments \( a \in [D]_0^N \) be given by:

\[
Y_{tn}(a_{N(n)}) = \sum_{k \in N(n)} \langle u_{kn}, w_{lk}, a_k \rangle + \epsilon_{tn}(a_{N(n)}),
\]

where \( u_{kn}, w_{lk} \in \mathbb{R} \) represent latent (unobserved) factors; \( \epsilon_{tn}(a_{N(n)}) \) is an additive, zero-mean, independent (or idiosyncratic) noise term; and \( r \) is the rank or model complexity.

Intuitively, the potential outcome of unit \( n \) with neighbors \( N(n) \) at time \( t \) is determined by two factors: (a) the effect of the treatment assigned to unit \( n \) and (b) the spillover effects from the treatments assigned to \( n \)’s neighbors. Since \( n \in N(n) \), both effects are captured in the summation in (5). Note that (5) can be written as

\[
Y_{tn}(a_{N(n)}) = [\hat{u}_{kn}(N(n)), \hat{w}_{lk}(a_{N(n)})] + \epsilon_{tn}(a_{N(n)}),
\]
where
\[
\begin{align*}
\tilde{\mathbf{u}}_{n,N(n)} &= [\mathbf{u}_{N_1(n),n}^\top, \mathbf{u}_{N_2(n),n}^\top, \ldots, \mathbf{u}_{N_N(n),n}^\top]^\top, \\
\tilde{\mathbf{w}}_{t,a,N(n)} &= [\mathbf{w}_{t,a,N_1(n)}^\top, \mathbf{w}_{t,a,N_2(n)}^\top, \ldots, \mathbf{w}_{t,a,N_N(n)}^\top]^\top.
\end{align*}
\]
Here, \(\tilde{\mathbf{u}}_{n,N(n)} \in \mathbb{R}^{r|N(n)|}\) and \(\tilde{\mathbf{w}}_{t,a,N(n)} \in \mathbb{R}^{r|N(n)|}\) are the network-adjusted latent factors, and \(r|N(n)| \in \mathbb{N}_{>0}\) denotes the network-adjusted “rank”.

### 4.2 Formal results

In this section, we present an identification result for (1) under (6), then establish finite-sample consistency of the NSI estimator. We restrict our attention to a specific unit \(n \in [N]\) and counterfactual treatments \(A_{n}^m \in [D]_{0}^{N(n)}\) of interest. The proofs are relegated to Appendix A.

We begin with some notation and assumptions. Let \(\mathcal{O}\) and \(LF\) be given by
\[
\mathcal{O} = \{ (j,t,a) : Y_{t,j}^{(a,N(j))} \text{ is observed} \} \subseteq [N] \times [T] \times [D]^{N},
\]
\[LF = \{ \mathbf{u}_{k,j}, \mathbf{w}_{t,a} : k, j \in [N], t \in [T], \text{ and } a \in [D]_0 \}.\]

**Assumption 3** (Conditional exogeneity). We assume that \(E[\epsilon_{t,j}^{(a,N(j))} \mid LF] = 0\) and \(\epsilon_{t,j}^{(a,N(j))} \perp O \mid LF\) for all \(j \in [N], t \in [T]\), and \(a \in [D]_0\).

**Assumption 4** (Linear span inclusion). Given a unit \(n \in [N]\) and sequence of counterfactual treatments \(A_{n}^m \in [D]_{0}^{N(n)}\) of interest, consider the donor set \(I^n\). We assume that \(I^n\) is non-empty and that there exists \(\alpha \in \mathbb{R}^{|I^n|}\) such that
\[
\tilde{\mathbf{u}}_{n,N(n)} = \sum_{j \in I^n} \alpha_j \tilde{\mathbf{u}}_{j,\pi_j(N(j))},
\]
where \(\pi_j\) is defined in Definition 1.

Together, Assumptions 2-3 imply that \(Y_{t,n}^{(a,N(n))} \perp O \mid LF\), which is analogous to requiring “selection on network-adjusted latent factors”: that, conditioning on all latent factors, the treatment assignments are independent of the potential outcome. This requirement is analogous to “selection on latent factors” in (Agarwal et al., 2020). While the treatment assignment is allowed to depend on the latent factors, Assumption 4 requires that the treatment assignment is “diverse” enough that the target unit’s latent factor lies in the linear span of the donor units. We now state the identification result.

**Theorem 1** (Identification). Consider a unit \(n \in [N]\) and sequence of counterfactual treatments \(A_{n}^m \in [D]_{0}^{N(n)}\) of interest. Suppose that Assumptions 1-4 hold. Let \(\alpha\) denote the coefficients from Assumption 4 for the donor set \(I^n\), where \(I^n\) is defined in Section 3. Then,
\[
E[\mathbf{Y}_{t,n}^{(a,N(n))} \mid LF] = \sum_{j \in I^n} \alpha_j E[Y_{t,j} \mid LF, O] \quad \text{and} \quad \theta_{t,n}^{(a,N(n))} = \frac{1}{T_{tr}} \sum_{t \in T_{tr}} \sum_{j \in I^n} \alpha_j E[Y_{t,j} \mid LF, O].
\]

Theorem 1 implies that estimating (1) comes down to acquiring good estimates of \(\alpha\). Estimating \(\alpha\) using observational data is precisely what the NSI estimator does. Next, we give conditions under which the NSI estimator achieves finite-sample consistency.

To that end, let \(M = |I^n|\) and \(Z_{post,I^n} = [Y_{t,j} : t \in T_{pr}, j \in I^n] \in \mathbb{R}^{T_{pr} \times M}\). Recall \(Z_{tr,I^n}\) from (2) and let \(r_{tr} \in [r|N(n)|]\) be the rank of \(E[Z_{tr,I^n} \mid LF, O]\), \(s_1 \geq \ldots \geq s_{r_{tr}} > 0\) denote its singular values, and \(R_{tr} \in \mathbb{R}^{M \times r_{tr}}\) denote its right singular vectors. Let \(\alpha_{L} = R_{tr}^\top \alpha\), where \(\alpha\) is defined in Assumption 4. Finally, let \(\|\cdot\|_2\) denote the Orlicz norm and \(O_p\) denote a probabilistic version of big-\(O\) notation.

**Assumption 5** (Sub-Gaussian noise). Assume that \(\|\epsilon_{t,j}^{(a,N(j))} \mid LF, O\|_2 \leq c\sigma\) for some constant \(c > 0\) and for all \(j \in [N], t \in [T]\), and \(a \in [D]_0\).

**Assumption 6** (Boundedness). \(E[Y_{t,j}^{(a,N(j))} \mid LF, O] \in [-1, 1]\) for all \(j \in [N], t \in [T]\), and \(a \in [D]_0\).
Assumption 7 (Well-balanced spectrum). For universal constants $c', c'' > 0$, assume $s_{r_\ell}/s_1 \geq c'$ and $\|E[Z_{r_\ell}I \mid LF, O]\|_p^p \geq c''T_r[I]^n$, where $I^n$ is defined in Definition 1.

Assumption 8 (Subspace inclusion). Assume that the row-space of $E[Z_{post,I} \mid LF, O]$ lies within the row-space of $E[Z_{r,I} \mid LF, O]$.

Theorem 2 (Finite-sample consistency). Let Assumptions 1-8 hold and $\kappa = r_{tr}$. Then,

$$
\left| \theta_0^{(\hat{A}^n_{tr})} - \theta_0^{(A^n_{tr})} \right| = O_P \left( \frac{\sqrt{T_r}}{T_{pr}} + \frac{\|\alpha_1\|_2}{\sqrt{T_{pr}}} + \frac{\|\alpha_1\|_1^{3/2} \sqrt{\log (T_{pr})}}{\min \left( \sqrt{T_r}, \sqrt{M} \right)} \right) \left( LF, O \right),
$$

where we assume $\|\alpha_1\|_2 \geq c'''$ for a universal constant $c''' > 0$.

4.3 Subspace Inclusion and Implications for Network-Aware Experiment Design

The key enabling condition for finite-sample consistency of the NSI estimator (Theorem 2) is Assumption 8, i.e., the subspace inclusion assumption (SIA). Below, we show that SIA implies that the training treatments $A^n_{tr}$ must be diverse enough with respect to the prediction treatments of interest $A^n_{pr}$. In terms of experiment design, Propositions 3-4 suggest that the treatments assigned during the training period must be carefully designed.

To this end, consider a scenario where the treatments are binary such that $D = 2$ and the training period is split into $L$ sub-periods, denoted by $T_{r,1}$ through $T_{r,L}$. During each sub-period, let the treatments assigned to each unit be constant, i.e., for all $\ell \in [L]$, $a^\ell = \tilde{a}^\ell$ for all $t \in T_{r,\ell}$. Let

$$
W_{r,\ell} = [w_{t,a}^\top : t \in T_{r,\ell}, a \in \{0, 1\}] \in \mathbb{R}^{|T_{r,\ell}| \times 2r},
$$

$$
B_{r} = [1 - \tilde{a}^L_{N(n)}, \tilde{a}^L_{N(n)}, \ldots, 1 - \tilde{a}^L_{N(n)}, \tilde{a}^L_{N(n)}] \top = \{0, 1\}^{2L \times |N(n)|}.
$$

Let $W_{r} \in \mathbb{R}^{T_r \times 2rL}$ be a block diagonal matrix, with $W_{r,1}$ through $W_{r,L}$ along the diagonal.

Proposition 3. SIA holds for any $A^n_{tr}$ if $W_{r}$ and $B_{r}$ have linearly independent columns.

Proposition 4. Suppose $W_{r}$ has linearly independent columns. Then, SIA holds for any $\{u_k,j : k, j \in [N]\}$ if $\tilde{a}^L_{N(n)}$ and $1 - \tilde{a}^L_{N(n)}$ are in the rowspace of $B_{tr}$ for all $t \in T_{pr}$.

Recall that the latent factors are, by definition, unobserved. As such, $W_{r}$ is also unobserved, and it is not possible to verify that $W_{r}$ has linearly independent columns, as required in Propositions 3-4. However, as an example that, suppose $w_{t,a}$ are sampled i.i.d. from a multivariate Gaussian and $|T_{r,\ell}| \geq 2r$ for all sub-periods $\ell \in [L]$. Then, with high probability, $W_{r}$ has linearly independent columns. As for $B_{r}$, consider the following illustrative examples.

Example 1. Suppose $L = 1$, i.e., $a^\ell = \tilde{a}^1$ for all $t \in T_{r}$. Then, $B_{r} = [1 - \tilde{a}^1, \tilde{a}^1] \top$. Suppose that the treatment assignment during the prediction period is not equal to $\tilde{a}^1$, but we are interested in estimating what would have happened if all units had remained under $\tilde{a}^1$, i.e., if $\tilde{a}^t = \tilde{a}^1$ for all $t \in T_{pr}$. Under this setup, $\tilde{a}^1, 1 - \tilde{a}^1 \in B_{r}$, as required by Proposition 4. This setup could be viewed as Synthetic Control for Panel Data (Abadie, 2021) under network interference.

Example 2. Suppose $L = 1$ and $\tilde{a}^1 = 0_N$. As such, $B_{r} = [0, 1] \top$. Then, unless all units in $N(n)$ receive the same treatment as one another under $\tilde{a}^1$, neither $\tilde{a}^t$ nor $1 - \tilde{a}^t$ are in the rowspace of $B_{tr}$.

Example 3. Suppose that $L = |N(n)|$. Suppose that during each sub-period $T_{r,\ell}$, a single distinct unit in $N(n)$ is assigned treatment 1 and all other are assigned 0. Suppose that which unit in $N(n)$ is assigned treatment 1 rotates at each subsequent sub-period. Then, $B_{r}$ has linearly independent columns, as required in Proposition 3.

5 Simulations

In this section, we present simulation results illustrating the behavior of the NSI estimator and compare it to two related estimators. Experimental details can be found in Appendix B.

In particular, we consider the following setting. Suppose $G$ is a regular graph with degree $d$, and the treatments are binary. For simplicity, suppose that $a^t = a^0$ for all $t \in T_{pr}$ and, similarly, $\tilde{a}^t = \tilde{a}^0$ for
under the training treatments are assigned as described in Section 4.3 with $L = d + 1$. More precisely, let the training period be divided into $d + 1$ sub-periods. During each of the sub-periods, 1 out of every $d + 1$ units receives treatment 1, and all others receive treatment 0. Each subsequent sub-period rotates which units are treated such that each unit is only treated during one of the sub-periods.

Under this setup, Fig. 2(a) shows an example of NSI estimates for the ring graph ($d = 2$) with $N = 400$. On top, it plots the spectrum $\{\hat{s}_\ell\}_{\ell=1}^{t_{\text{tr}}}$ produced in Step 1 of Section 3, where the vertical line marks $\kappa$. On bottom, it gives the NSI estimates, where the vertical line separates the training and prediction periods. The ground-truth values are given as lines, and the predictions are marked with *’s. As shown, the predictions closely match the ground-truth values. Under the same setup, Fig. 2(b) plots the histogram of NSI residuals (the difference between the estimated and ground-truth potential outcomes) of 200 simulations, verifying that the residuals are consistent. Fig. 2(c) gives the MSE across $d = 2, 4, 6$, and 8. The left (blue) bars are for $N = 100$ and $T_{\text{tr}} = T_{\text{pr}} = 100$; the middle (red) bars for $N = 100$ and $T_{\text{tr}} = T_{\text{pr}} = 50$; and the right (yellow) bars for $N = 500$ and $T_{\text{tr}} = T_{\text{pr}} = 50$. As expected, the MSE typically increases with degree, fewer nodes, and less training time.

We also compare the NSI estimator to two others: the SI estimator (which does not account for network interference) (Agarwal et al., 2020) and a baseline estimator. The baseline estimator finds donor units that satisfy Definition 1, then averages the donor units’ observed outcomes. We compare the estimators for a ring graph (details given in Appendix B). The MSEs and R-squared values for the NSI estimator, SI estimator, and baseline estimators are, respectively, $(0.08013, 0.9994), (53.10, 0.9101)$, and $(576.1, -1.389)$. Both the NSI and baseline estimators use donor sets that contained, on average, 16 units. The SI estimator used donor sets with, on average, 66 units.

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A Proofs

The notation $O_p$ is a probabilistic version of big-$O$ notation. Formally, for any sequence of random vectors $X_n$, $X_n = O_p(\chi_n)$ if, for any $\varepsilon > 0$, there exists constants $c_\varepsilon$ and $n_\varepsilon$ such that $P(\|X_n\|_2 > c_\varepsilon \chi_n) < \varepsilon$ for every $n \geq n_\varepsilon$. Equivalently, we say that $X_n/\chi_n$ is “uniformly tight” or “bounded in probability”.

A.1 Proof of Theorem 1

Proof. Below, the symbol $\overset{AX}{=} = \overset{DX}{=}$ imply that the equality follows from Assumption X and Definition X, respectively. Recall that $\mathcal{I}^n$ is shorthand for $\mathcal{I}(\hat{A}_n, \hat{A}_n, \{X(n)\})$. Then, for $t \in \mathcal{T}_P$,

$$
\mathbb{E}\left[ Y_{t\hat{n}}(a_{X(n)}^\varepsilon) \bigg| LF \right] \overset{A2}{=} \mathbb{E}\left[ \left\langle \tilde{u}_{n,n}(N(n)), \tilde{w}_{t,a_{X(n)}^\varepsilon} \right\rangle + \varepsilon_{t\hat{n}}(a_{X(n)}^\varepsilon) \bigg| LF \right]
$$

$$
\overset{A3}{=} \left\langle \tilde{u}_{n,n}(N(n)), \tilde{w}_{t,a_{X(n)}^\varepsilon} \right\rangle \bigg| LF
$$

$$
= \left\langle \tilde{u}_{n,n}(N(n)), \tilde{w}_{t,a_{X(n)}^\varepsilon} \right\rangle \bigg| \{LF, \mathcal{O}\}
$$

$$
\overset{A4}{=} \sum_{j \in \mathcal{I}^n} \alpha_j \left\langle \tilde{u}_{j,\pi_j(N(j))}, \tilde{w}_{t,a_{X(n)}^\varepsilon(\pi_j(N(j)))} \right\rangle \bigg| \{LF, \mathcal{O}\}
$$

$$
= \sum_{j \in \mathcal{I}^n} \alpha_j \left\langle \tilde{u}_{j,\pi_j(N(j))}, \tilde{w}_{t,a_{X(n)}^\varepsilon(\pi_j(N(j)))} \right\rangle \bigg| \{LF, \mathcal{O}\}
$$

$$
\overset{D1}{=} \sum_{j \in \mathcal{I}^n} \alpha_j \left\langle \tilde{u}_{j,\pi_j(N(j))}, \tilde{w}_{t,a_{X(n)}^\varepsilon(\pi_j(N(j)))} \right\rangle \bigg| \{LF, \mathcal{O}\}
$$

$$
= \sum_{j \in \mathcal{I}^n} \alpha_j \left\langle \tilde{u}_{j,\pi_j(N(j))}, \tilde{w}_{t,a_{X(n)}^\varepsilon(\pi_j(N(j)))} \right\rangle \bigg| \{LF, \mathcal{O}\}
$$

$$
\overset{A4}{=} \sum_{j \in \mathcal{I}^n} \alpha_j \left\langle \tilde{u}_{j,\pi_j(N(j))}, \tilde{w}_{t,a_{X(n)}^\varepsilon(\pi_j(N(j)))} \right\rangle \bigg| \{LF, \mathcal{O}\}
$$

$$
= \sum_{j \in \mathcal{I}^n} \alpha_j \left\langle \tilde{u}_{j,\pi_j(N(j))}, \tilde{w}_{t,a_{X(n)}^\varepsilon(\pi_j(N(j)))} \right\rangle \bigg| \{LF, \mathcal{O}\}
$$

where (7) follows from the fact that, conditioned on $LF$, the left-hand side is deterministic, which implies that event on which it is conditioned can be exchanged for $\{LF, \mathcal{O}\}$. Therefore,

$$
\phi_{t\hat{n}}^{(\hat{A}_n^\varepsilon)} = \frac{1}{|\mathcal{T}_P|} \sum_{t \in \mathcal{T}_P} \mathbb{E}\left[ Y_{t\hat{n}}(a_{X(n)}^\varepsilon) \bigg| LF \right]
$$

$$
= \frac{1}{|\mathcal{T}_P|} \sum_{t \in \mathcal{T}_P} \sum_{j \in \mathcal{I}^n} \alpha_j \mathbb{E}\left[ Y_{t_j} \bigg| LF, \mathcal{O} \right],
$$

where the first equality follows from the definition of $\phi_{t\hat{n}}^{(\hat{A}_n^\varepsilon)}$ and the second equality follows from (9). Note that Assumption 1 immediately holds from Assumption 2.

A.2 Proof of Theorem 2

As indicated in the main text, Theorem 2 is adapted from Theorem 4.2 of (Agarwal et al., 2020). Below, we explain how to adapt Theorem 4.2 for this work.

Model. The model in (Agarwal et al., 2020) is given by (in their notation)

$$
Y_{t\hat{n}}^{(d)} = \left\langle \tilde{u}_{t}^{(d)}, v_{n} \right\rangle + \varepsilon_{t\hat{n}}^{(d)},
$$

(12)
where $u^{(d)}_n, v_n \in \mathbb{R}^r$ are latent factors; $\varepsilon^{(d)}_{tn}$ is a zero-mean, independent noise term; and $Y^{(d)}_{tn}$ is the potential outcome of interest.

Recall from (6) that our model is given by (in our notation)
\[
Y^{(a_{N(n)})}_{tn} = \langle \tilde{u}_{n,A_{N(n)}}, \tilde{w}_{t,a_{N(n)}} \rangle + \epsilon^{(a_{N(n)})}_{tn},
\]
where $\tilde{u}_{n,A_{N(n)}}, \tilde{w}_{t,a_{N(n)}} \in \mathbb{R}^{n_{N(n)}}$ are latent factors; $\epsilon^{(a_{N(n)})}_{tn}$ is a zero-mean, independent noise term; and $Y^{(a_{N(n)})}_{tn}$ is the potential outcome of interest.

As such, our setup model is analogous to the model used by Agarwal et al. (2020), with a change of notation. Specifically, $\tilde{u}_{n,A_{N(n)}}$ in this work corresponds to $u^{(d)}_n$ in (Agarwal et al., 2020), $\tilde{w}_{t,a_{N(n)}}$ to $v_n$, and $\epsilon^{(a_{N(n)})}_{tn}$ to $\varepsilon^{(d)}_{tn}$.

**Assumptions of Theorem 4.2 in Agarwal et al. (2020).** Given that our model (6) can be mapped to the model in Agarwal et al. (2020), it remains to check whether the assumptions in Theorem 4.2 of (Agarwal et al., 2020) are satisfied by those in Theorem 2.

In particular, one of the main differences between our work and Agarwal et al. (2020) is the observation pattern. In this work, the observation pattern is more general, allowing for any sequence of treatments during the training and prediction periods (referred to as the “pre-intervention” and “post-intervention” periods in Agarwal et al. (2020)). In Agarwal et al. (2020), the treatment must be constant across each period, and it is assumed that all units are under treatment 0 during the pre-intervention (i.e., training) period. This difference only affects Theorem 2 via the donor set. In other words, once we adjust the choice of donor set (see Definition 1) to suit the network interference setting, Theorem 4.2 can be mapped directly to Theorem 2.

We now go through the assumptions one-by-one. As we saw above, Assumption 2 is equivalent to Assumption 2 in (Agarwal et al., 2020), with a change of notation. Furthermore, as discussed in Section 4, Assumption 1 is automatically satisfied when Assumption 2 holds. Assumptions 3-6 map one-to-one to Assumptions 3-6 of (Agarwal et al., 2020) under the change of notation. Lastly, Assumptions 7-8 also map one-to-one to Assumptions 7-8 under the new definition of a donor set, as given by Definition 1.

### A.3 Proof of Proposition 3

**Proof.** Recall that for unit $n \in [N]$, measurement $t \in [T]$, and treatments $a \in [A]_0^N$,
\[
Y^{(a)}_{tn} = \sum_{k \in N(n)} \langle u_{k,n}, w_{t,k} \rangle + \epsilon^{(a_{N(n)})}_{tn},
\]
where $n \in N(n)$. Because $D = 2$, $a_k \in \{0, 1\}$ for all $k \in [N]$. As such,
\[
Y^{(a)}_{tn} - \epsilon^{(a_{N(n)})}_{tn} = \sum_{k \in N(n)} 1(a_k = 0)u_{k,n}^T w_{t,0} + \sum_{k \in N(n)} 1(a_k = 1)u_{k,n}^T w_{t,1} + \sum_{k \in N(n)} 1(a_k = 1)u_{k,n}^T w_{t,1}
\]
\[
= \sum_{k \in N(n)} 1(a_k = 0)u_{k,n}^T w_{t,0} + \sum_{k \in N(n)} 1(a_k = 1)u_{k,n}^T w_{t,1} + \sum_{k \in N(n)} a_k u_{k,n}^T w_{t,1}
\]

Given a unit $n \in [N]$ and sequence of counterfactual, prediction treatments of interest $A^n_{tn} \in \{0, 1\}^{N(n)} \times T_p$. Recall that we use $Z^{n}$ as a shorthand for $Z(A^n_{tn}, A^n_{tn}, N(n))$. Further, we let $T^n_j$ refer to the $j$-th donor in the donor set $Z^n$.

Recall that:
\[
[Z^{n} x^n] = \begin{bmatrix}
Y_{n, T^n_1} & Y_{1, T^n_2} & \cdots & Y_{1, T^n_{|Z^n|}} \\
Y_{2, T^n_1} & Y_{2, T^n_2} & \cdots & Y_{2, T^n_{|Z^n|}} \\
\vdots & \vdots & \ddots & \vdots \\
Y_{T^n_1, T^n_1} & Y_{T^n_2, T^n_2} & \cdots & Y_{T^n_{|Z^n|}, T^n_{|Z^n|}}
\end{bmatrix} \in \mathbb{R}^{T_p \times |Z^n|},
\]

11
denotes the observations across all training periods, and
\[
Z_{pr,T^n} = \begin{bmatrix}
Y_{T,T-n+1,T^n_1} & Y_{T,T-n+1,T^n_2} & \cdots & Y_{T,T-n+1,T^n_{|T^n|}} \\
Y_{T,T-n+2,T^n_1} & Y_{T,T-n+2,T^n_2} & \cdots & Y_{T,T-n+2,T^n_{|T^n|}} \\
\vdots & \vdots & \ddots & \vdots \\
Y_{T,T^n_1} & Y_{T,T^n_2} & \cdots & Y_{T,T^n_{|T^n|}} 
\end{bmatrix} \in \mathbb{R}^{T_n \times |T^n|},
\]
(19)
denotes the observations during the prediction period.
Without loss of generality, we assume that the first of the \( L \) sub-periods occupies the first \( T_{r,1} \) time steps of \( T_r \), the second sub-period occupies the next \( T_{r,2} \) time steps of \( T_r \), and so on.
The **subspace inclusion assumption** (SIA) requires that \( \text{rowspace}(Z_{pr,T^n}) \subset \text{rowspace}(Z_{r,T^n}) \).
Let \( \tilde{N}(j) \) denote the \( \pi_j(\tilde{N}(j)) \), where \( \pi_j \) is specified in Definition 1, i.e., \( \tilde{N}(j) \) corresponds to the already-permuted neighborhood of donor \( j \), where the permutation is fixed under Definition 1.

\[
U_{T^n} = \begin{bmatrix}
\mathbf{u}_{\tilde{N}_1(T^n_1),T^n_1} & \cdots & \mathbf{u}_{\tilde{N}_1(T^n_{|T^n|}),T^n_{|T^n|}} \\
\vdots & \ddots & \vdots \\
\mathbf{u}_{\pi_1(\tilde{N}_{|\tilde{N}(n)|}(T^n_1)),T^n_1} & \cdots & \mathbf{u}_{\pi_1(\tilde{N}_{|\tilde{N}(n)|}(T^n_{|T^n|}),T^n_{|T^n|})}
\end{bmatrix} \in \mathbb{R}^{r \times |\tilde{N}(n)| \times |T^n|}
\]

We now define \( W_r \) and \( B_r \). For ease of exposition, we express them for \( L = 2 \). Let \( \mathbb{I}_r \) denote the \( r \times r \) identity matrix. Using (16),
\[
\mathbb{E}[Z_{r,T^n}] = \begin{bmatrix}
\mathbf{w}^T_{1,0} & \mathbf{w}^T_{1,1} & 0 & 0 \\
\mathbf{w}^T_{2,0} & \mathbf{w}^T_{2,1} & 0 & 0 \\
0 & 0 & \mathbf{w}^T_{T,T_n+1,0} & \mathbf{w}^T_{T,T_n+1,1} \\
0 & 0 & \mathbf{w}^T_{T,T_n+2,0} & \mathbf{w}^T_{T,T_n+2,1}
\end{bmatrix} \begin{bmatrix}
1 - \bar{a}_{N_1(n)}^1 & 1 - \bar{a}_{N_2(n)}^1 & \cdots \\
\bar{a}_{N_1(n)}^1 & \bar{a}_{N_2(n)}^1 & \cdots \\
1 - \bar{a}_{N_1(n)}^2 & 1 - \bar{a}_{N_2(n)}^2 & \cdots \\
\bar{a}_{N_1(n)}^2 & \bar{a}_{N_2(n)}^2 & \cdots
\end{bmatrix} \otimes \mathbb{I}_r \begin{bmatrix}
\mathbf{u}_{N_1(1),T^n_1} & \cdots & \mathbf{u}_{N_2(1),T^n_1} \\
\vdots & \ddots & \vdots \\
\mathbf{u}_{N_1(1),T^n_{|T^n|}} & \cdots & \mathbf{u}_{N_2(1),T^n_{|T^n|}}
\end{bmatrix}
\]
(20)
and, analogously,
\[
\mathbb{E}[Z_{pr,T^n}] = \text{diag}_{T_r} \left( \left( \mathbf{W}^T_{1,0}, \mathbf{W}^T_{1,1} \right) \right) \begin{bmatrix}
1 - T_{n,1} & 1 - T_{n,2} & \cdots \\
\bar{a}_{N_1(n)} & \bar{a}_{N_2(n)} & \cdots \\
1 - T_{n,1} & 1 - T_{n,2} & \cdots \\
\bar{a}_{N_1(n)} & \bar{a}_{N_2(n)} & \cdots
\end{bmatrix} \otimes \mathbb{I}_{pr} \begin{bmatrix}
\mathbf{u}_{N_1(1),T^n_1} & \cdots & \mathbf{u}_{N_2(1),T^n_1} \\
\vdots & \ddots & \vdots \\
\mathbf{u}_{N_1(1),T^n_{|T^n|}} & \cdots & \mathbf{u}_{N_2(1),T^n_{|T^n|}}
\end{bmatrix}
\]
(21)
Let \( K_{r} = W_r (B_r \otimes \mathbb{I}_r) \) and \( K_{pr} = W_{pr} (B_{pr} \otimes \mathbb{I}_r) \).
Note that any matrix that has linearly independent columns has full row space. Hence to complete the proof, is suffices to show that \( K_{r} \) has linearly independent columns. Now if \( B_{nr} \) and \( W_{nr} \) have linearly independent columns, then it immediately implies that \( K_{r} \) has linearly independent columns.  

\section{A.4 Proof of Proposition 4}

\textbf{Proof}. Below, we use the same notation as in the proof of Proposition 3.
Subspace inclusion effectively requires that, for every $i \in [T_{pr}]$ there exists some $\phi \in \mathbb{R}^{T_n}$ such that
\[ e_i^T W_{pr} (B_{pr} \otimes I_r) W_{tr} = \phi^T W_{tr} (B_{tr} \otimes I_r) U_{T_n}. \]
Therefore, subspace inclusion holds for any $U_{T_n}$ if there exists some $\phi \in \mathbb{R}^{T_n}$ such that
\[ e_i^T W_{pr} (B_{pr} \otimes I_r) = e_i^T K_{pr} = \phi^T K_{tr} = \phi^T W_{tr} (B_{tr} \otimes I_r). \] (22)

Therefore, by the second equality, subspace inclusion requires that rowspace $(K_{pr}) \subset$ rowspace $(K_{tr})$. Note. Given that (i) rowspace $(K_{pr}) \subset$ rowspace $(B_{pr} \otimes I_r)$ and (ii) rowspace $(K_{tr}) = \text{rowspace} (B_{tr} \otimes I_r)$ since $W_{tr}$ has linearly independent columns, it suffices to show that rowspace $(B_{pr} \otimes I_r) \subset$ rowspace $(B_{pr} \otimes I_r)$. This is equivalent to showing that rowspace $(B_{pr}) \subset$ rowspace $(B_{tr})$.

Since the rows of $B_{pr}$ are $\tilde{a}_{N(n)}^t$ and $1 - \tilde{a}_{N(n)}^t$ for all $t \in T_{pr}$, rowspace $(B_{pr}) \subset$ rowspace $(B_{tr})$ holds when $\tilde{a}_{N(n)}^t, 1 - \tilde{a}_{N(n)}^t \in$ rowspace $(B_{tr})$ for all $t \in T_{pr}$.

\section{Simulations}

Below, we re-present the results given in Section 5, providing additional simulation results.

Figure 3: Simulation results for NSI estimator under $e_{tn}^{(a_{N(n)})} \sim N(0, 0.1)$. (a) Consider a ring graph with $N = 400$, $T_{tr, \ell} = T_{pr} = 50$, $r = 2$, and $L = 3$. The top graph plots the spectrum $\{s_\ell\}_{\ell=1}^N$, and the vertical line marks $\kappa$ from Section 3. The NSI estimates are plotted below, where the vertical line separates the training and prediction periods. The ground-truth values are given as lines, and the predictions are marked with ‘*’s. (b) gives the histogram of residuals (the difference between the estimated and ground-truth potential outcomes) of 200 simulations, averaged over 50 units and all possible counterfactual treatments. The experimental parameters match those (a). (c) plots the MSE of the NSI estimator across regular graphs of different degrees. The left (blue) bars are for $N = 1000$ and $T_{tr, \ell} = T_{pr} = 100$. The middle (red) bars are for $N = 1000$, $T_{tr, \ell} = T_{pr} = 50$. The right (yellow) bars are for $N = 500$, $T_{tr, \ell} = T_{pr} = 50$. All other parameters match those for (a).

All results are given for binary treatments, i.e., $D = 2$. The latent factors $u_{k,n}$ and $w_{0,a}$ are drawn from a standard random normal distribution, and $w_{0,a}$ is a Gaussian random walk. Our experiments use a simple donor-finding algorithm. In particular, instead of searching for donors over all possible permutations $\pi_j$, as defined in Definition 1, we fix an ordering of units (as described in Section 1) and restrict ourselves to the identity permutation $\pi_j(i) = i$.

Figure 2 shows results for the NSI estimator over a ring graph, such that the size of each neighborhood set is 3. For Fig. 2(a)-(b), we adopt the setup described in Section 4.3, where $T_n$ is divided into $L = 3$ sub-periods, each of length $T_{tr, \ell} = 50$ and $\bar{a}^1 = (1, 0, 0, 1, 0, 0, \ldots)$, $\bar{a}^2 = (0, 1, 0, 0, 1, 0, 0, \ldots)$, and $\bar{a}^3 = (0, 0, 1, 0, 0, 1, \ldots)$. For Fig. 2(c), we study the Network Synthetic Control setting described in Example 1, where $L = 1$, $\bar{a}^1 = \bar{a}^0 = 0_N$, $\bar{a}^{T_{pr}+1} = 1$ is drawn uniformly at random, and $\bar{a}^t$ is constant across $T_{pr}$.

We also compare the NSI estimator to two others: the SI estimator (which does not account for network interference) (Agarwal et al., 2020) and a baseline estimator. The baseline estimator finds donor units that satisfy Definition 1, then averages the donor units’ observed outcomes. We compare the estimators for a ring graph under the same hyper-parameters as those used in Fig. 2(a), averaging across 200 simulations, 50 units, and all possible counterfactual treatments. The MSEs and R-squared values for the NSI estimator, SI estimator, and baseline estimators are, respectively, (0.08013, 0.08013, 0.08013).
Both the NSI and baseline estimators used donor sets that contained, on average, 16 units. The SI estimator used donor sets with, on average, 66 units. These results as well as those for Fig. 2(a)-(b) are given for $\kappa \geq 3r$, and the results for Fig. 2(c) are given for $\kappa \geq r$.

The simulations were simple. They were run on a local machine with a 2.3 GHz processor. The simulations were completed in under two hours.