Bootstrap Percolation on Complex Networks with Community Structure

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(Dated: January 23, 2014)

Abstract

Real complex networks usually involve community structure. How innovation and new products spread on social networks which have internal structure is a practically interesting and fundamental question. In this paper we study the bootstrap percolation on a single network with community structure, in which we initiate the bootstrap process by activating different fraction of nodes in each community. A previously inactive node transfers to active one if it detects at least $k$ active neighbors. The fraction of active nodes in community $i$ in the final state $S_i$ and its giant component size $S_{gci}$ are theoretically obtained as functions of the initial fractions of active nodes $f_i$. We show that such functions undergo multiple discontinuous transitions; The discontinuous jump of $S_i$ or $S_{gci}$ in one community may trigger a simultaneous jump of that in the other, which leads to multiple discontinuous transitions for the total fraction of active nodes $S$ and its associated giant component size $S_{gc}$ in the entire network. We have further obtained the phase diagram of the total number of jumps with respect to the inner-degrees of the two communities on Erdős-Rényi networks. If their inner-degrees are comparable or one of which is small, the system exhibits at most one discontinuous jump; otherwise it undergoes two discontinuous transitions. The number of discontinuous transitions reveals the internal structure of the network.
I. INTRODUCTION

In the recent decade, small world navigation and epidemics are widely studied to understand the structure of complex networks and how virus or information propagates on social networks [1–5]. In general the spreading of information on a network is via connectivity links. In epidemic model, the infection happens through the links by certain probabilities. However there exists a different type of spreading process. For instance in neuronal networks, a neuron becomes excited if the number of its excited neighbors reaches a threshold; In the Ising model, the physical state of a site is determined by the physical states of its neighboring sites via interactions [6, 7]: below the Curie point, sites favor the same state of their neighbors; And in social networks, individuals may be influenced or convinced by their neighbors [8]: If the number of one’s friends (linked neighbors) who have acquired certain innovation or bought certain new product reaches some threshold value, one is prone to the same behavior. Such spreading process, namely bootstrap percolation, is iteratively enhanced by the increase of influenced nodes in the network, which eventually reaches a final state that no more nodes can be further affected.

Bootstrap percolation has been studied on two and three dimensional lattices [9–12], d-dimensional lattices [10, 11], infinite trees [13], random regular graph [14, 15] and even finite random graph [16]. Recently it is successfully applied to unstructured networks [17] and multiplex networks [18, 19]. In graph theory, the maximal subgraph for which each node has at least \( k \) neighbors is called the \( k \)-core of the graph. K-core percolation is an extensively studied problem [20, 21] and it is closely related to, yet different from bootstrap percolation [22]. Another variation of bootstrap percolation is proposed by Watts to model the spreading of opinions in social networks in which the thresholds of activation of each node is a certain fraction of the neighbors, rather than an absolute number [8]. The model also has wide applications in analyzing phenomena such as neuronal activity [23–25], jamming transition [26–28] and magnetic systems [29–31].

It is worth noting that by various reasons community structure commonly exists in social, biological, transportation and energy supply networks [32–34]. It is of importance as it reveals the internal structure of the networks and contributes to the percolation robustness of them [35–37].

In this paper we consider under the framework of bootstrap percolation how information spreads on a single undirected network with community structure. Bi-community structure is simple and representative, as multi-community system can reduce to bi-community one in the “one vs all” manner. We assign each community a different probability \( f_i (i = 1, 2) \) by which the nodes
are randomly chosen to be initially active, otherwise inactive in the beginning. Next we exam all inactive nodes. If any of them has at least \( k \) active neighbors, it becomes activated \((k \geq 2)\) — This activation, by increasing the number of active nodes, may potentially trigger more inactive nodes to become activated. Therefore we repeat this process until no more nodes can be activated, namely we reach a final state.

We analytically express the fractions of active nodes(and also the giant component size) in each community in the final state as functions of the initial fractions of active nodes. We find that such functions undergo multiple discontinuous transitions. A discontinuous jump of a function for one community may trigger a simultaneous jump of the function for the other community. We have further obtained the phase diagram of the total number of jumps in terms of the inner-degrees of the two communities for Erdős-Rényi networks. We also discuss the emergence of such discontinuous transitions.

II. ANALYTICAL MODEL

We consider a single complex network with community structure. Bi-community structure is representative as it can be easily generalized to multi-community in a “one vs all” manner. Therefore we discuss an undirected network with two communities, namely subnetwork 1 and 2. Let \( P^1(i, j) \) and \( P^2(i, j) \) be the probability distribution of inner-degree and outer-degree of a node in subnetwork 1 and 2 respectively, with \( i \) the number of edges of a node connected to subnetwork 1, and \( j \) the number of edges of a node connected to subnetwork 2. In \( P^1(i, j) \), \( i \) is the inner-degree and \( j \) is the outer-degree, whereas in \( P^2(i, j) \), \( i \) is the outer-degree and \( j \) is the inner-degree.

Let \( f_i \) be the probability of a node in subnetwork \( i (i = 1, 2) \) being initially active. An initially inactive node becomes active if it has at least \( k \) active downstream neighbors in either subnetwork 1 or 2. These active neighbors must satisfy the same criteria, that they are either active in the very beginning or they have \( k \) further downstream neighbors of their own that are previously active. A final (equilibrium) state is reached when no more nodes can be activated by the above criteria.

At equilibrium of the bootstrap percolation process, a fraction \( S_i \) (with \( i = 1, 2 \)) of nodes in subnetwork \( i \) are active. \( S_i \) can also be interpreted as the probability that an arbitrarily chosen node in subnetwork \( i \) is active in the final state. In the following we are also interested in calculating the probability of an arbitrarily chosen node in each subnetwork belonging to the giant active component of the whole network, denoted by \( S_{gc1} \) and \( S_{gc2} \) respectively.
To calculate $S_1$ and $S_2$, we randomly choose an edge from subnetwork $i$ to subnetwork $j$ ($i, j = 1, 2$), and let $Z_{i,j}$ denote the probability that the node arrived at is active in the final state. In other words, the node is either active in the beginning, or has at least $k$ active downstream neighbors reached by edges except the one we arrive from.

Randomly choose an edge in subnetwork 1, there are two possibilities for the arriving node to be active: one is that the node is active in the beginning, which has a probability $f_1$; the other is that the node is inactive initially, but it has $k$ active downstream neighbors. These neighbors must also be in one of the two possibilities to be active. So if the node we arrived at by the chosen edge has $i$ other edges in subnetwork 1 except the one we arrived from and $j$ neighbors in subnetwork 2, then the probability is $\frac{(i+1)p_1(i+1,j)}{\sum_{i, j \geq 0} i p_1(i,j)}$. If among the $i$ edges in subnetwork 1, there are $l$ edges connected to active nodes in subnetwork 1, and among the $j$ edges linked with subnetwork 2, there are $m$ edges connected to active nodes, then $l$ and $m$ should satisfy $l + m \geq k$. So we can construct the following equation for $Z_{11}$:

$$Z_{11} = f_1 + (1 - f_1) \sum_{i+j \geq k} \frac{(i+1)p_1(i+1,j)}{\sum_{i, j \geq 0} i p_1(i,j)} \sum_{l+m=k}^{l \leq i, m \leq j} \binom{i}{l} Z_{11}^{i-l} \binom{j}{m} Z_{12}^{m-j} (1 - Z_{11})^{i-l} (1 - Z_{12})^{j-m}. \quad (1)$$

Equations for $Z_{12}, Z_{21}, Z_{22}$ are shown in appendix A.

A randomly chosen node in subnetwork $i$ is active at equilibrium if either it is active at first, or it has at least $k$ edges connected with active downstream neighbors in either subnetwork 1 or 2. So by writing down mathematical expressions for the probabilities, we can construct the following equation for $S_1$:

$$S_1 = f_1 + (1 - f_1) \sum_{i+j \geq k} p_1(i, j) \sum_{l+m=k}^{l \leq i, m \leq j} \binom{i}{l} Z_{11}^{i-l} \binom{j}{m} Z_{12}^{m-j} (1 - Z_{11})^{i-l} (1 - Z_{12})^{j-m}. \quad (2)$$

The equation for $S_2$ is in appendix A. These equations can be solved numerically for a given network degree distribution.

### III. RESULTS

#### 3.1 Erdős-Rényi Networks

The Erdős-Rényi (ER) network (which has a Poisson degree distribution in the infinite size limit) is a representative random graph, so we discuss ER network in particular. In ER network,
the presence or absence of an edge between two nodes is independent of the presence or absence
of any other edge, so that each edge may be considered to be present with independent probability
\( p \). The degree of any particular node (that is the number of edges of the node) has a probability
distribution \( p_k \) given by
\[
p_k = \binom{N}{k} p^k (1 - p)^{N-k} \approx \frac{k^{z^2}}{z!},
\]
where \( N \) is the number of nodes in the network, \( z \) is the average degree of any node, and the second equality becomes exact in the limit of large \( N \).

In the ER network consisting of two subnetworks, suppose any edge from a node in subnetwork
\( i \) to a node in subnetwork \( j \) is present with probability \( p_{ij} \) \((i,j=1,2)\). So we have
\[
P^i_{ij} = \binom{N_i}{i} p^i_{11} (1 - p_{11})^{N_i - i} \binom{N_j}{j} p^j_{12} (1 - p_{12})^{N_j - j}
\approx \frac{k^i_{11} e^{-k_{11}}}{i!} \cdot \frac{k^j_{12} e^{-k_{12}}}{j!},
\]
where \( k_{ij} \) is the average edges of any node in subnetwork \( i \) connecting with nodes in subnetwork \( j \). In ER network, we get that
\[
Z_{11} = Z_{21} = S_1, \tag{3}
\]
\[
Z_{22} = Z_{12} = S_2, \tag{4}
\]
and
\[
S_1 = f_1 + (1 - f_1) \sum_{r \geq k} \frac{(S_1 k_{11} + S_2 k_{12})^r}{r!} e^{-(S_1 k_{11} + S_2 k_{12})}, \tag{5}
\]
\[
S_2 = f_2 + (1 - f_2) \sum_{r \geq k} \frac{(S_1 k_{21} + S_2 k_{22})^r}{r!} e^{-(S_1 k_{21} + S_2 k_{22})}, \tag{6}
\]
where \( k_{11} = N_1 \cdot p_{11}, k_{12} = N_2 \cdot p_{12}, k_{21} = N_1 \cdot p_{21}, k_{22} = N_2 \cdot p_{22} \), and \( p_{12} = p_{21} \). The details can be seen in appendix B.

Suppose subnetworks 1 and 2 are of identical size, and their initial fractions of active nodes
are identical. Fig. 1 shows the active fraction \( S = (S_1 + S_2) / 2 \) in the whole network as a function
of the initial active fraction \( f_1 = f_2 = f \) for three groups of values of \( k_{ij} \) in an ER network, both
theoretically and by simulations. We find that there are three subcases. There may be one or two
discontinuous jumps, or no jumps in curve of the size of the active component as a function of \( f \).
Fig. 2 shows the diagram of the phase transitions, with respect to different values of \( k_{12} \) and \( k_{21} \).

Given small values of \( k_{ij} \), for example \( k_{11} = 6.5 \) and \( k_{22} = 6.5 \), the jump phenomenon does not
appear. Once there is a jump in \( S_1 \) or \( S_2 \), then the jump appears in \( S \). When \( k_{11} = 10, k_{22} = 7, \)
there are two jumps in \( S \), as \( S_1 \) and \( S_2 \) jump at two different values of \( f \).
FIG. 1. (Color online) Comparison of theoretical results and simulations in terms of the fraction $S$ of the active nodes in the entire network. Symbols are simulation data. Solid lines are theoretical solutions. We assume $S = \frac{S_1 + S_2}{2}$, and $f_1 = f_2 = f$. Here we set $k_{12} = 0.5$, $k_{21} = 0.5$, and $k = 5$. When $k_{11} = 12$, $k_{22} = 12$, there is one jump (the red line). When $k_{11} = 10$, $k_{22} = 7$, there are two jumps (the blue line). When $k_{11} = 6.5$, $k_{22} = 6.5$, there is no jump (the green line).

In the network with community structure, we find a phenomenon which is different from single network without community structure. As functions of $f$, $S_1$ and $S_2$ interact with each other. The interactions depend on the interconnection of the two subnetworks. If they are closely interacted, that is, when the value of $k_{12}$ or $k_{21}$ is larger, the interactions between $S_1$ and $S_2$ are stronger. Fig. 3 shows the interaction between $S_1$ and $S_2$. This is the case that $S$ has two jumps. We can see that when a jump appears in $S_i$, there is also a jump in $S_j (i \neq j)$. So there are two discontinuous jumps in both $S_1$ and $S_2$.

The equations above can be solved numerically. If multiple solutions exist, the physical solution is always the smallest value. To find the location of the discontinuous jump of $S_i$, we can observe that the jump appears at the disappearance of the smallest solution of Eq. 5 and Eq. 6.

Fig. 4 shows the process of the disappearance of the smallest solution, which explains the jump in $S_i(f)$. To find the location of the discontinuous transition theoretically, we argue in the following way. Given the values of $k_{ij}$, $f_1 = f_2 = f$ and $k$, Eq. 5 determines $S_1$ as a function of $S_2$, i.e. $S_1(S_2)$, and Eq. 6 determines $S_2$ as a function of $S_1$, i.e. $S_2(S_1)$. The intersection point of the curves of the two functions is the solution. We notice that the functions $S_1(S_2)$ and $S_2(S_1)$ may be
The diagram of the phase transitions. Given values of $k_{12}$, $k_{21}$ and $k$, the function $S(f)$ (also $S_i(f)$) depends on the values of $k_{11}$ and $k_{22}$. Here $k_{12} = k_{21} = 0.5$, $k = 5$. Given the values of $k_{11}$ and $k_{22}$ in different areas in the above graph, there may be one or two discontinuous jumps, or no jumps in the size of the active component of the graph as a function of initial active fraction $f$.

The interaction between $S_1$ and $S_2$. There are two discontinuous jumps in both $S_1$ and $S_2$. When a jump appears in $S_i$, there is also a jump in $S_j (i \neq j)$. Here $k_{11} = 8$, $k_{22} = 7$, $k_{12} = 2.4$, $k_{21} = 0.5$, $k = 5$. 
The disappearance of the smallest solution. Here $f = 0.12, k_{11} = 10, k_{22} = 7, k_{12} = 0.5, k_{21} = 0.5, k = 5$. (b) The curves intersecting at the smallest solution, which is also the jump point of function $S_1(S_2)$ (see figure 4). Here $f = f_{c1} = 0.151, k_{11} = 10, k_{22} = 7, k_{12} = 0.5, k_{21} = 0.5, k = 5$. (c) The disappearance of the smallest solution. Here $f = 0.18, k_{11} = 10, k_{22} = 7, k_{12} = 0.5, k_{21} = 0.5, k = 5$.

discontinuous, as shown in Fig. 4. Given $k_{11} = 10, k_{22} = 7, k_{12} = 0.5, k_{21} = 0.5, k = 5$, as shown in Fig. 1 there are two jumps in the curve of function $S(f)$, also in the curves of functions $S_i(f)$. The jump points are $f = f_{c1} = 0.151$ and $f = f_{c2} = 0.215$. When $f = f_{c1} = 0.151$, the function $S(f)$ also $S_i(f)$ jumps in Fig. 1. Meanwhile, in Fig. 4(b), the physical solution point is the jump point of the curve of function $S_1(S_2)$. Therefore, we argue that at the jump point $f_c$ of function $S(f)$ (also $S_i(f)$), the smallest solution of Eq. 5 and Eq. 6 is related to the jump point of curve of one of the two functions $S_1(S_2)$ and $S_2(S_1)$.

The jump point of $S_1(S_2)$ can be found in the following way. Given values of $k_{ij}$ and $k$, let $f = f_{c1}$, and let $S_2 = S_2$, which is the value of $S_2$ of the physical solution. Let $F(S_1)$ denote the right side of Eq. 5 as a function of $S_1$, and let $G(S_1) = S_1$. Then the value of $S_1$ at the intersection point of curves of the two functions $F(S_1)$ and $G(S_1)$ is the value of $S_1$ of the physical solution, denoted by $S_1$ as before. Fig. 5 shows the appearance of the jump point $f = f_{c1} = 0.151$ of function $S_1(S_2)$, which is also the process of the disappearance of the smallest solution of $F(S_1) = G(S_1)$.

To be precise, we arrive at the jump point $f_{c1} = 0.151$ of $S_1(S_2)$

$$\frac{dF(S_1)}{dS_1} = 1.$$  

(7)

Similarly, we can analyze the appearance of the jump point $f = f_{c2} = 0.215$ of $S_2(S_1)$. The jump point $f_{c2} = 0.215$ of $S_2(S_1)$ can be found by

$$\frac{dH(S_2)}{dS_2} = 1.$$  

(8)
FIG. 5. (Color online) The appearance of the jump point of function $S_1(S_2)$, which is also the process of the disappearance of the smallest solution of $F(S_1) = G(S_1)$. (a) The curves of the two functions $G(S_1)$ and $F(S_1)$ intersecting at the physical solution. Here $f = 0.12$, $k_{11} = 10$, $k_{22} = 7$, $k_{12} = 0.5$, $k_{21} = 0.5$, $k = 5$. (b) The curves of the two functions $G(S_1)$ and $F(S_1)$ being tangential to each other at the smallest intersection. Here $f = f_c = 0.151$, $k_{11} = 10$, $k_{22} = 7$, $k_{12} = 0.5$, $k_{21} = 0.5$, $k = 5$. (c) The disappearance of the smallest solution of $F(S_1) = G(S_1)$. Here $f = 0.18$, $k_{11} = 10$, $k_{22} = 7$, $k_{12} = 0.5$, $k_{21} = 0.5$, $k = 5$.

where $H(S_2)$ denotes the right side of Eq. 6 with given values of $k_{ij}$ and $k$, and $S_1 = \tilde{S}_1$, $f = f_c$. Finally, the jump points $f_c$ of functions $S_i(f)$ can be found from any solution of one of the two groups of equations: Eq. 5, Eq. 6 and Eq. 7, or Eq. 5, Eq. 6 and Eq. 8. If there are two different solutions, then there are two jumps in function $S(f)$.

3.2 The analysis of the fraction of the giant active component $S_{gci}$

Now we consider the probability $S_{gci}(i = 1, 2)$ that an arbitrarily chosen node in subnetwork $i$ belongs to the giant active component. In the infinite size limit, the giant active component is an active subtree of infinite extent. Define $X_{ij}$ to be the probability that the node arrived at by following an arbitrarily chosen edge from subnetwork $i$ to subnetwork $j$ satisfying the conditions for $Z_{ij}$ and has at least one edge leading to an active subtree of infinite extent except the arbitrarily chosen edge. There are two possibilities: one is that the node arrived at is active at first and has at least one edge leading to an active subtree of infinite extent; the other one is that the node arrived at is inactive initially, but it has more than $k$ edges, except the arbitrarily chosen edge, leading to active nodes at equilibrium in subnetwork 1 or 2, and at least one of those edges leads to an active subtree of infinite extent in subnetwork 1 or 2.

Writing down the mathematical expressions of the probabilities for the above two possibilities,
we get

\[ X_{11} = f_1 \sum_{i+j \geq 1} \frac{(i+1)P^1(i+1,j)}{\sum_{i \geq 0, j \geq 0} iP^1(i,j)} \left[ \sum_{0 \leq m \leq i, \ 0 \leq n \leq j, \ m+n \geq 1} \binom{i}{m} X_{11}^m (1-X_{11})^{i-m} \binom{j}{n} X_{12}^n (1-X_{12})^{j-n} \right] \]

\[ + (1 - f_1) \sum_{i+j \geq k} \frac{(i+1)P^1(i+1,j)}{\sum_{i \geq 0, j \geq 0} iP^1(i,j)} \left\{ \sum_{0 \leq m \leq i, \ 0 \leq n \leq j} \binom{i}{m} \binom{j}{n} \right\} \left[ \sum_{1 \leq s+t \leq m+n \atop 0 \leq s \leq m \atop 0 \leq t \leq n} \binom{m}{s} \binom{n}{t} X_{11}^s (Z_{11} - X_{11})^{m-s} X_{12}^t (Z_{12} - X_{12})^{n-t} (1 - Z_{11})^{i-m} (1 - Z_{12})^{j-n} \right] \]  

(9)

The equations of \( X_{12} \), \( X_{21} \) and \( X_{22} \) are similar. And we get the probability \( S_{gc1} \)

\[ S_{gc1} = f_1 \sum_{i+j \geq 1} P^1(i,j) \left[ \sum_{0 \leq m \leq i \atop 0 \leq n \leq j \atop m+n \geq 1} \binom{i}{m} X_{11}^m (1-X_{11})^{i-m} \binom{j}{n} X_{12}^n (1-X_{12})^{j-n} \right] \]

\[ + (1 - f_1) \sum_{i+j \geq k} P^1(i,j) \left\{ \sum_{0 \leq m \leq i \atop 0 \leq n \leq j} \binom{i}{m} \binom{j}{n} \right\} \left[ \sum_{1 \leq s+t \leq m+n \atop 0 \leq s \leq m \atop 0 \leq t \leq n} \binom{m}{s} \binom{n}{t} X_{11}^s (Z_{11} - X_{11})^{m-s} X_{12}^t (Z_{12} - X_{12})^{n-t} (1 - Z_{11})^{i-m} (1 - Z_{12})^{j-n} \right] \]  

(10)

The equation for \( S_{gc2} \) can be seen in appendix C.

As we have shown in appendix B, in ER network,

\[ \frac{(i+1)P^1(i+1,j)}{\sum_{i \geq 0, j \geq 0} iP^1(i,j)} = P^1(i,j), \]

so

\[ X_{11} = X_{21} = S_{gc1}, \]  

(11)

\[ X_{12} = X_{22} = S_{gc2}. \]  

(12)
FIG. 6. (Color online) The fraction $S_{gc}$ of the giant active component in the whole network, for both the theory and simulations. Symbols are simulation data. Solid lines are theoretical solutions. We suppose $S_{gc} = \frac{S_{gc1} + S_{gc2}}{2}$, and $f_1 = f_2 = f$. Here we set $k_{12} = 0.5$, $k_{21} = 0.5$, and $k = 5$. When $k_{11} = 12$, $k_{22} = 12$, there is one jump (the red line). When $k_{11} = 10$, $k_{22} = 7$, there are two jumps (the blue line). When $k_{11} = 6.5$, $k_{22} = 6.5$, there is no jump (the green line).

Finally, we have

$$S_{gc1} = f_1 \left\{ \sum_{i+j \geq 1} P^1(i, j) \left[ \sum_{\substack{0 \leq m \leq i \\ 0 \leq n \leq j \\ m+n \geq 1}} \binom{i}{m} S_{gc1}^m (1 - S_{gc1})^{i-m} \binom{j}{n} S_{gc2}^n (1 - S_{gc2})^{j-n} \right] \right\}$$

$$+ (1 - f_1) \sum_{i+j \geq k} P^1(i, j) \left\{ \sum_{\substack{k \leq m+n \leq i+j \\ 0 \leq m \leq i \\ 0 \leq n \leq j}} \binom{i}{m} \binom{j}{n} \right\}$$

$$\left[ \sum_{\substack{1 \leq s+t \leq m+n \\ 0 \leq s \leq m \\ 0 \leq t \leq n}} \binom{m}{s} \binom{n}{t} S_{gc1}^s (S_1 - S_{gc1})^{m-s} S_{gc2}^t (S_2 - S_{gc2})^{n-t} (1 - S_1)^{i-m} (1 - S_2)^{j-n} \right].$$

The equation of $S_{gc2}$ in ER network can be seen in appendix D.

Fig. 6 shows the giant active component fraction $S_{gc} = (S_{gc1} + S_{gc2})/2$ in the whole network as a function of the initial active fraction $f_1 = f_2 = f$ for three groups of values of $k_{ij}$ in an ER network, both theoretically and by simulations. There are also three possibilities, which is similar to the fraction of active nodes. Fig. 7 shows the interaction between $S_{gc1}$ and $S_{gc2}$. In the case with two jumps, when a jump appears in $S_{gc1}$, there is also a jump appearing in $S_{gc2}(i \neq j)$. So there are two discontinuous jumps in both $S_{gc1}$ and $S_{gc2}$. 
FIG. 7. (Color online) The interaction between $S_{gc1}$ and $S_{gc2}$. There are two discontinuous jumps in both $S_{gc1}$ and $S_{gc2}$. When a jump appears in one of $S_{gc i}(i = 1, 2)$, a jump appears in the other one too. Here $k_{11} = 8$, $k_{22} = 7$, $k_{12} = 2.4$, $k_{21} = 0.5$, $k = 5$.

IV. CONCLUSION

We have studied the bootstrap percolation on complex network with bi-community structure, in which we observe either continuous appearance of the giant active component, or discontinuous hybrid transition of that. In contrast to the unstructured network, community structured networks may exhibit multiple discontinuous transitions for the fraction of the active nodes as well as the size of the giant active component. We find that the discontinuous transition in one community may trigger a simultaneous discontinuous jump in the other. The number of discontinuous transitions depends on the degree distributions of the two communities and their correlations. In Erdős-Rényi networks we observe that if the inner-degrees are comparable or one of which is small, the system shows at most one jump; otherwise it undergoes two discontinuous transitions. Our results exhibit important properties of information spreading dynamics on real social networks. The number of discontinuous jumps of bootstrap percolation elucidates the network’s internal structure. One may harness the phase diagram to study the community structure of network of interest.

V. ACKNOWLEDGEMENT

This work is supported by the NSFC Grants No. 61203156.
VI. APPENDIX

A

Randomly choose an edge from subnetwork $i$ to subnetwork $j(i, j = 1, 2)$. $Z_{i,j}$ denotes the probability that the node arrived at is active in the steady state. Similar to $Z_{11}$,

$$Z_{22} = f_2 + (1 - f_2) \sum_{i+j \geq k} \frac{(i+1)P^2(i, j)}{\sum_{i \geq 0, j \geq 0} iP^2(i, j)} \sum_{i+m=k} \binom{i}{l} Z^l_{21} (1 - Z_{21})^{i-l} \left( \frac{j}{m} \right) Z^m_{22} (1 - Z_{22})^{j-m}$$ (14)

$$Z_{12} = f_2 + (1 - f_2) \sum_{i+j \geq k} \frac{(i+1)P^2(i+1, j)}{\sum_{i \geq 0, j \geq 0} iP^2(i, j)} \sum_{i+m=k} \binom{i}{l} Z^l_{21} (1 - Z_{21})^{i-l} \left( \frac{j}{m} \right) Z^m_{22} (1 - Z_{22})^{j-m}$$ (15)

$$Z_{21} = f_1 + (1 - f_1) \sum_{i+j \geq k} \frac{(j+1)P^1(i, j+1)}{\sum_{i \geq 0, j \geq 0} jP^1(i, j)} \sum_{i+m=k} \binom{i}{l} Z^l_{11} (1 - Z_{11})^{i-l} \left( \frac{j}{m} \right) Z^m_{12} (1 - Z_{12})^{j-m}$$ (16)

The equation for the possibility $S_2$ of a node in subnetwork 2 to be active at the steady state is similar to the equation of $S_1$:

$$S_2 = f_2 + (1 - f_2) \sum_{i+j \geq k} P^2(i, j) \sum_{i+m=k} \binom{i}{l} Z^l_{21} (1 - Z_{21})^{i-l} \left( \frac{j}{m} \right) Z^m_{22} (1 - Z_{22})^{j-m}$$ (17)

B

We show that in ER network $Z_{11} = Z_{21} = S_1$. Notice that $\sum_{l \geq 0} \frac{k^{l_{i}} e^{-k_{11}}}{l!} = 1$, we have

$$\frac{(i+1)P^1(i+1, j)}{\sum_{i \geq 0, j \geq 0} iP^1(i, j)} = \frac{(i+1) \cdot \frac{k^{i+1} e^{-k_{11}}}{(i+1)!} \cdot \frac{k_{21} e^{-k_{12}}}{j!}}{\sum_{i \geq 0, j \geq 0} i \cdot \frac{k^{i} e^{-k_{11}}}{i!} \cdot \frac{k_{21} e^{-k_{12}}}{j!}}$$

$$= \frac{k^{i+1} e^{-k_{11}} k_{21} e^{-k_{12}}}{\sum_{j \geq 0} (\sum_{i \geq 1} \frac{k^{i} e^{-k_{11}}}{(i-1)!} k_{11} k_{21} e^{-k_{12}})}$$

$$= P^1(i, j).$$ (18)

So $Z_{11} = S_1$. Similarly $Z_{21} = S_1$ and $Z_{22} = Z_{12} = S_2.$
Suppose a randomly selected node in subnetwork 1 has \( r \) active neighbors. Let \( P(r) \) be the corresponding probability. Suppose \( l \) active neighbors are in subnetwork 1. Let \( N_i (i = 1, 2) \) be the number of the nodes in each subnetwork respectively. Then the number of all neighbors of the node must be larger than \( r \), and less than \( N_1 + N_2 \). Let \( m \) be the number of all neighbors and \( i \) be the number of neighbors in subnetwork 1, then \( l \leq i \leq m \) and \( r \leq m \leq N_1 + N_2 \).

\[
P(r) = \sum_{l=0}^{r} \sum_{m-r}^{m} \sum_{i=l}^{m} P^1(i,m-i) \left( \frac{i}{l} \right) Z_{11}^l (1-Z_{11})^{i-l} \left( \frac{m-i}{r-l} \right) Z_{12}^{r-l} (1-Z_{12})^{m-i-(r-l)},
\]

and

\[
S_1 = f_1 + (1 - f_1) \sum_{r \geq k} P(r).
\]

\[
P(r) = \sum_{l=0}^{r} \sum_{m-r}^{m} \sum_{i=l}^{m} \left( \begin{array}{c} N_1 \\ i \end{array} \right) (1-p_{11})^{N_1-i} \left( \begin{array}{c} N_2 \\ m-i \end{array} \right) P^1(m-i) (1-p_{12})^{N_2-(m-i)}
\]

\[
\left( \frac{i}{l} \right) Z_{11}^l (1-Z_{11})^{i-l} \left( \frac{m-i}{r-l} \right) Z_{12}^{r-l} (1-Z_{12})^{m-i-(r-l)}.
\]

Here let \( i = l + l', m = r + m', m - i = r - l + m' - l' \). Notice that when \( i = m, r = l \), we have \( m' + r - l = m' \), then

\[
P(r) = \sum_{l=0}^{r} \sum_{m' \geq 0} \sum_{i' = 0}^{m'} \left( \begin{array}{c} N_1 \\ l \end{array} \right) (1-p_{11})^{N_1-l} \left( \begin{array}{c} N_2 \\ m-i \end{array} \right) Z_{11}^l (1-Z_{11})^{l} \left( \begin{array}{c} N_1 \\ i' \end{array} \right) Z_{12}^{r-l} (1-Z_{12})^{m'-l'}
\]

\[
= \sum_{l=0}^{r} Z_{11}^l (1-Z_{11})^{l} \left( \begin{array}{c} N_1 \\ l \end{array} \right) Z_{12}^{r-l} (1-Z_{12})^{m'-l'} \sum_{m' \geq 0} \sum_{i' = 0}^{m'} \left( \begin{array}{c} N_1 \\ i' \end{array} \right) (1-p_{11})^{N_1-i'} Z_{12}^{r-l} (1-Z_{12})^{m'-l'}
\]

On the other hand,

\[
P_{11}^{k} (1-p_{11})^{N_1-l-k} \left( \begin{array}{c} N_1 \\ k \end{array} \right) \approx \frac{k_{11}^k e^{-k_{11}}}{k!}.
\]

Therefore,

\[
P(r) = \sum_{l=0}^{r} \sum_{m' \geq 0} \sum_{i' = 0}^{m'} \left( \frac{[k_{11}(1-Z_{11})]^{i'} e^{-k_{11}}}{i'!} \right) \left( \frac{[k_{12}(1-Z_{12})]^{m'-i'} e^{-k_{12}}}{(m'-i')!} \right)
\]

\[
= \sum_{l=0}^{r} \sum_{m' \geq 0} \sum_{i' = 0}^{m'} \left( \begin{array}{c} N_1 \\ l \end{array} \right) Z_{11}^l (1-Z_{11})^{l} \left( \begin{array}{c} N_2 \\ m-i \end{array} \right) Z_{12}^{r-l} (1-Z_{12})^{m'-l'}
\]

\[
\sum_{m' = 0}^{N_1+N_2-r} \left( \frac{[k_{11}(1-Z_{11})]^{i'} e^{-k_{11}}}{i'!} \right) \left( \frac{[k_{12}(1-Z_{12})]^{m'-i'} e^{-k_{12}}}{(m'-i')!} \right) e^{-(k_{11}Z_{11}+k_{12}Z_{12})}.
\]
Notice that the last sum is 1 in the infinite size limit, so

$$P(r) = \sum_{l=0}^{r} Z_{11}^l p_{11}^l \left( \frac{N_1}{l} \right) Z_{12}^{r-l} p_{12}^{r-l} \left( \frac{N_2}{r-l} \right) e^{-(k_1 Z_{11} + k_2 Z_{12})}.$$

(19)

When $N$ is large enough, $l$ is relatively small, $\frac{N_l}{N(N-l)!} \approx 1$, so

$$P(r) = \sum_{l=0}^{r} Z_{11}^l p_{11}^l \left( \frac{N_1}{l} \right) Z_{12}^{r-l} p_{12}^{r-l} \left( \frac{N_2}{r-l} \right) e^{-(k_1 Z_{11} + k_2 Z_{12})}$$

$$= \sum_{l=0}^{r} \frac{r! (Z_{11} k_{11})! (Z_{12} k_{12})!}{l! (r-l)!} e^{-(k_1 Z_{11} + k_2 Z_{12})}$$

$$= \frac{(Z_{11} k_{11} + Z_{12} k_{12})^r}{r!} e^{-(k_1 Z_{11} + k_2 Z_{12})},$$

(20)

So we can get the equation for $S_1$.

C

The equation for the possibility $S_{gc2}$ that an arbitrarily chosen node in subnetwork 2 belongs to
the giant active component is similar to the equation for $S_{gc1}$:

$$S_{gc2} = f_2 \sum_{i+j \geq 1} P^2(i, j) \left\{ \sum_{0 \leq m \leq i \atop 0 \leq n \leq j} \left( \frac{i}{m} \right) X_{21}^m (1 - X_{21})^{i-m} \left( \frac{j}{n} \right) X_{22}^n (1 - X_{22})^{j-n} \right\}$$

$$+ (1 - f_2) \sum_{i+j \geq k} P^2(i, j) \left\{ \sum_{k \leq m+n \leq i+j \atop 0 \leq m \leq i \atop 0 \leq n \leq j} \left( \frac{i}{m} \right) \left( \frac{j}{n} \right) \left\{ \sum_{1 \leq s+t \leq m+n \atop 0 \leq s \leq m \atop 0 \leq t \leq n} \left( \begin{array}{c} m \\ s \end{array} \right) \left( \begin{array}{c} n \\ t \end{array} \right) X_{21}^s (Z_{21} - X_{21})^{m-s} X_{22}^t (Z_{22} - X_{22})^{n-t} (1 - Z_{21})^{i-m} (1 - Z_{22})^{j-n} \right\} \right\}.$$
In ER network, the equation for the possibility $S_{gc2}$ that an arbitrarily chosen node in subnetwork 2 belongs to the giant active component is similar to the equation for $S_{gc1}$:

$$S_{gc2} = f_2 \left\{ \sum_{i+j \geq 1} p^2(i, j) \left[ \sum_{0 \leq m \leq i} \sum_{0 \leq n \leq j} \binom{i}{m} S_{gc1}^m (1 - S_{gc1})^{i-m} \binom{j}{n} S_{gc2}^n (1 - S_{gc2})^{j-n} \right] \right\}$$

$$+ (1 - f_2) \left\{ \sum_{i+j \geq k} p^2(i, j) \left[ \sum_{0 \leq m \leq i} \sum_{0 \leq n \leq j} \binom{i}{m} \binom{j}{n} \right]$$

$$\left[ \sum_{1 \leq s+t \leq m+n} \binom{m}{s} \binom{n}{t} S_{gc1}^s (S_1 - S_{gc1})^{m-s} S_{gc2}^t (S_2 - S_{gc2})^{n-t} (1 - S_1)^{i-m} (1 - S_2)^{j-n} \right] \right\}.$$  \hspace{1cm} (21)
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