KÄHLER STRUCTURES ON QUANTUM IRREDUCIBLE FLAG MANIFOLDS

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ABSTRACT. We prove that all quantum irreducible flag manifolds admit Kähler structures, as defined by Ó Buachalla. In order to show this result, we also prove that the differential calculi defined by Heckenberger and Kolb are differential $\ast$-calculi in a natural way.

INTRODUCTION

Within the realm of non-commutative geometry, the study of structures coming from complex geometry is a relatively new trend, see for instance the papers [FGR99, BeSm13, ÓBu16]. Here we are interested in Kähler structures, which were defined recently in [ÓBu17]. Recall that the existence of a Kähler structure on a complex manifold has many far-reaching consequences, see [Huy05] for an overview. As shown in [ÓBu17], many of these consequences also hold in the non-commutative setting, provided they are reformulated accordingly.

The main problem then becomes to prove the existence of such Kähler structures. In the paper [ÓBu17] it was shown that they do exist for the class of quantum projective spaces. More generally, it was conjectured that they should exist for all quantum irreducible flag manifolds. The aim of this paper is to answer this conjecture in the affirmative.

Recall that a (generalized) flag manifold is a homogeneous space of the form $G/P$, where $P$ is a parabolic subgroup of $G$. These spaces admit natural Kähler structures and moreover they exhaust all compact homogeneous Kähler manifolds [Wan54]. The condition of being irreducible is equivalent to $G/P$ being a symmetric space. Hence the class of irreducible flag manifolds coincides with that of irreducible compact Hermitian symmetric spaces.

Quantum flag manifolds can be defined straightforwardly in terms of quantum subgroups of the quantum groups $\mathbb{C}_q[G]$, see [StDi99]. The class of quantum irreducible flag manifolds is singled out by a series of important results of Heckenberger and Kolb [HeKo04, HeKo06]. They show that these quantum spaces admit a canonical $q$-analogue of the de Rham complex, with the homogenous components having the same dimensions as in the classical case. We stress that this is definitely not the case for general quantum spaces.

Since the definition of a Kähler structure on a quantum space requires the existence of a differential calculus, quantum irreducible flag manifolds clearly provide the best avenue for testing this concept. However there is an obstacle that needs to be overcome: to study the existence of Kähler structures we actually need to have a differential $\ast$-calculus, a structure which has not been introduced yet for the Heckenberger-Kolb calculi.

For this reason the paper contains two main results. The first result is Theorem 4.2, which shows that the Heckenberger-Kolb calculus $\Omega^\ast$ over $\mathbb{C}_q[G/P_S]$ becomes a differential $\ast$-calculus in a natural way. The second result is Theorem 5.10, which shows the existence of a Kähler structure on $\Omega^\ast$, thus proving the conjecture formulated in [ÓBu17, Conjecture 4.25].

These results provide some further steps in the general understanding of complex geometry within the quantum setting. Of course, many more questions still remain to be answered.
As an example, the question of positive-definiteness of the quantum metric coming from the Kähler structure, as defined in [OBu17], certainly deserves further study.

The organization of the paper is as follows. In Section 1 we discuss various preliminaries related to quantized enveloping algebras and quantum coordinate rings. In Section 2 we recall various basic definitions regarding differential calculi, as well as the notions of Hermitian and Kähler structures. In Section 3 we review the description of quantum flag manifolds in terms of generators and relations. In Section 4 we present the Heckenberger-Kolb calculi and we prove our first main result, namely that they are naturally differential $*$-calculi. In Section 5 we prove our second main result, namely the existence of Kähler structures for these differential $*$-calculi. Finally in Appendix A we prove some identities for the components of the braiding, which are used in some of the proofs in the main text.

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1. Notations and preliminaries

In this section we recall some basic facts concerning quantized enveloping algebras and quantum coordinate rings, as well as fixing some notations. More details and missing explanations can be found in textbooks such as [KlSC97, NeTu13].

1.1. Quantized enveloping algebras. Let $\mathfrak{g}$ be a complex simple Lie algebra, with Cartan subalgebra $\mathfrak{h}$, and denote by $(\cdot, \cdot)$ the non-degenerate symmetric bilinear form on $\mathfrak{h}^*$ induced by the Killing form. We denote by $U_q(\mathfrak{g})$ the quantized enveloping algebra of $\mathfrak{g}$, the Hopf algebra with generators $\{K_i, E_i, F_i\}_{i=1}^r$ and relations as in [HeKo06]. In particular we have

\[
\Delta(K_i) = K_i \otimes K_i, \quad \Delta(E_i) = E_i \otimes K_i + 1 \otimes E_i, \quad \Delta(F_i) = F_i \otimes 1 + K_i^{-1} \otimes F_i,
\]

\[
S(K_i) = K_i^{-1}, \quad S(E_i) = -E_i K_i^{-1}, \quad S(F_i) = -K_i F_i.
\]

With these conventions we have the identity $S^2(X) = K_{2\rho}X K_{2\rho}^{-1}$, where $\rho$ is the half-sum of the positive roots of $\mathfrak{g}$. We will consider $0 < q < 1$, so that we have a $*$-structure corresponding to the compact real form of $\mathfrak{g}$. For instance we can take

\[
K_i^* = K_i, \quad E_i^* = F_i K_i, \quad F_i^* = K_i^{-1} E_i.
\]

We remark that the precise form of the $*$-structure will not matter in the following, hence we are free to replace it with any other equivalent one.

For $0 < q < 1$ the representation theory of $U_q(\mathfrak{g})$ is essentially the same as for $U(\mathfrak{g})$. Hence for any dominant weight $\lambda$ we have a $U_q(\mathfrak{g})$-module $V(\lambda)$. Recall that the dual space $V^*$ becomes a $U_q(\mathfrak{g})$-module by $(X f)(v) := f(S(X) v)$, where $v \in V$ and $f \in V^*$.

Next we will consider the braiding on the category of $U_q(\mathfrak{g})$-modules, namely a collection of $U_q(\mathfrak{g})$-module isomorphisms $\hat{R}_{V,W} : V \otimes W \to W \otimes V$, where $V$ and $W$ are $U_q(\mathfrak{g})$-modules. We will follow the choice of [HeKo06]: the braiding is uniquely determined by the requirement that $\hat{R}_{V,W}$ is a $U_q(\mathfrak{g})$-module isomorphism and by the condition

\[
\hat{R}_{V,W}(v_{hw} \otimes w_{lw}) := q^{(\text{wt} v_{hw}, \text{wt} w_{lw})} w_{lw} \otimes v_{hw},
\]

where $v_{hw}$ is a highest weight vector of $V$ and $w_{lw}$ is a lowest weight vector of $W$. Choosing a basis $\{v_i\}_i$ of $V$ and a basis $\{w_j\}_j$ of $W$, we will write

\[
\hat{R}_{V,W}(v_i \otimes w_j) = \sum_{k,l} (\hat{R}_{V,W})_{ij}^{kl} w_k \otimes v_l.
\]
1.2. **Quantum coordinate rings.** Next we recall the quantum coordinate rings $\mathbb{C}_q[G]$, which are essentially the Hopf $\ast$-algebras dual to $U_q(\mathfrak{g})$. They are obtained from the matrix coefficients of the finite-dimensional (type 1) representations of $U_q(\mathfrak{g})$. Recall that, given a $U_q(\mathfrak{g})$-module $V$, the matrix coefficients are given by

$$c^V_{f,v}(X) := f(Xv), \quad f \in V^*, \ v \in V.$$ 

By a $U_q(\mathfrak{g})$-invariant inner product on $V$ we mean an inner product $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{C}$, conjugate-linear in the first variable, such that

$$\langle Xv, w \rangle = \langle v, X^*w \rangle, \quad \forall v, w \in V, \ \forall X \in U_q(\mathfrak{g}).$$

It is well-known that if $V$ is a simple module then this inner product is unique, up to a constant. Now fix $\langle \cdot, \cdot \rangle$ on $V$ and take an orthonormal basis $\{v_i\}$. Then the elements of the dual basis $\{f_i\}$ of $V^*$ can be identified with $f_i = (v_i, \cdot)$. In this case we write

$$u^V_{ij} := c^V_{f_i,v_j}.$$ 

If $V = V(\lambda)$ is a simple $U_q(\mathfrak{g})$-module of highest weight $\lambda$, we will also write

$$c^\lambda_{f,v} := c^V_{f,v}(\lambda), \quad u^\lambda_{ij} := u^V_{ij}(\lambda).$$

It is easy to see that the elements $u^V_{ij}$ satisfy the relations

$$\Delta(u^V_{ij}) = \sum_k u^V_{ik} \otimes u^V_{kj}, \quad (u^V_{ij})^* = S(u^V_{ji}).$$

Moreover, since $S^2(X) = K_{2p} \cdot X K_{2p}^{-1}$ we have

$$S^2(u^V_{ij}) = q^{(2p, \mathrm{wt} v_i - \mathrm{wt} v_j)} u^V_{ij}.$$ 

2. **Differential calculi**

In this section we recall various notions related to the description of differential calculi on quantum spaces. In particular we consider the notions of Hermitian and Kähler structures, as defined in [ÖBu17]. Moreover we recall some aspects of Takeuchi’s categorical equivalence, which is a quite useful tool when dealing with differential calculi.

2.1. **First order differential calculus.** A first order differential calculus (FODC) over an algebra $\mathcal{A}$ is an $\mathcal{A}$-bimodule $\Gamma$ together with a linear map $d : \mathcal{A} \to \Gamma$, such that $\Gamma = \text{span}\{\text{ad}(b)c : a, b, c \in \mathcal{A}\}$ and $d$ satisfies the Leibnitz rule

$$d(ab) = d(a)b + ad(b).$$

If $\mathcal{A}$ is a $\ast$-algebra, then $\Gamma$ is a $\ast$-FODC if the $\ast$-structure of $\mathcal{A}$ extends to a $\ast$-structure of $\Gamma$ in such a way that $d(a)\ast = d(a\ast)$.

Suppose in addition that $\mathcal{H}$ is a Hopf algebra and $\Delta_{\mathcal{A}} : \mathcal{A} \to \mathcal{H} \otimes \mathcal{A}$ is a left $\mathcal{H}$-comodule algebra structure on $\mathcal{A}$. Then $\Gamma$ is called left-covariant if there exists a left $\mathcal{H}$-comodule structure $\Delta_{\Gamma} : \Gamma \to \mathcal{H} \otimes \Gamma$ on $\Gamma$ such that

$$\Delta_{\Gamma}(ad(b)c) = \Delta_{\mathcal{A}}(a)(\text{id} \otimes d)(\Delta_{\mathcal{A}}(b))\Delta_{\mathcal{A}}(c).$$

Given a family of FODCs $\{(\Gamma_i, d_i)\}_{i=1}^n$, their direct sum is the FODC $(\Gamma, d)$ with $\Gamma = \bigoplus_i d_i$, and $\Gamma = \mathcal{A} \cdot d(\mathcal{A}) \subset \bigoplus_i \Gamma_i$. If the calculi are left covariant then so is their direct sum.

Finally, suppose that $\mathcal{B} \subset \mathcal{A}$ is a subalgebra and $(\Gamma, d)$ is a FODC over $\mathcal{A}$. Then there is a FODC $(\Gamma|_{\mathcal{B}}, d|_{\mathcal{B}})$ over $\mathcal{B}$ defined by

$$\Gamma|_{\mathcal{B}} := \{ad(b) : a, b \in \mathcal{B}\}, \quad d|_{\mathcal{B}}(a) := d(a), \ \forall a \in \mathcal{B}.$$
This FODC is called the \textit{FODC over $\mathcal{B}$ induced by $\Gamma$}.

2.2. \textbf{Higher order differential calculus.} A \textit{differential calculus} over $\mathcal{A}$ is a differential graded algebra $(\Gamma^\wedge = \bigoplus_{k \in \mathbb{N}} \Gamma^{\wedge k}, d)$ such that $\Gamma^{\wedge 0} = \mathcal{A}$ and $\Gamma^\wedge$ is generated by $\mathcal{A}$ and $d\mathcal{A}$. If $\mathcal{A}$ is a $*$-algebra, then $\Gamma^\wedge$ is a \textit{differential $*$-calculus} if the $*$-structure of $\mathcal{A}$ extends to an involution of $\Gamma$ such that $d(\omega)^* = d(\omega^*)$ for any $\omega \in \Gamma$ and moreover,

$$(\omega \wedge \chi)^* = (-1)^{pq} \chi^* \wedge \omega^*, \quad \omega, \chi \in \Gamma^{\wedge q}.$$ 

An element $\omega \in \Gamma^\wedge$ is called \textit{real} if $\omega^* = \omega$.

Given a FODC $\Gamma$ over $\mathcal{A}$, there exists a \textit{universal differential calculus} $(\Gamma^\wedge_0, (d_u))$, uniquely determined by the following property: if $\Gamma^\wedge$ is any differential calculus over $\mathcal{A}$ such that $\Gamma^{\wedge 1} = \Gamma$, then $\Gamma$ is isomorphic to a quotient of $\Gamma^\wedge_0$.

The $*$-structure naturally lifts to the universal differential calculus, meaning that if $\Gamma$ is a $*$-FODC then $\Gamma^\wedge_0$ is a differential $*$-calculus, see [KISC97, Chapter 12, Proposition 4].

2.3. \textbf{Hermitian and Kähler structures.} Many structures from complex geometry can be adapted to the quantum setting, as discussed in [ÖBu17]. We will now recall the notions of Hermitian and Kähler structures, as defined in the cited paper. In this subsection $(\Omega^\bullet, d)$ will denote a differential $*$-calculus of dimension $2n$.

\textbf{Definition 2.1.} An \textit{almost symplectic form} is a central real 2-form $\sigma \in \Omega^\bullet$ satisfying the following property: denoting by $L_\sigma : \Omega^\bullet \rightarrow \Omega^\bullet$ the Lefschetz map given by $L_\sigma(\omega) := \sigma \wedge \omega$, the map $L_\sigma^{n-k} : \Omega^k \rightarrow \Omega^{2n-k}$ is an isomorphism for all $k = 0, \cdots, n - 1$.

We will omit the subscript $\sigma$ in the following, as the dependence will be clear.

\textbf{Definition 2.2.} A \textit{Hermitian structure} for $\Omega^\bullet$ is a pair $(\Omega^{(\bullet \bullet)}, \sigma)$, where $\Omega^{(\bullet \bullet)}$ is a complex structure and $\sigma$ is an almost symplectic form, called the \textit{Hermitian form}, such that $\sigma \in \Omega^{(1,1)}$.

For the definition of a complex structure we refer to [ÖBu17].

\textbf{Definition 2.3.} A \textit{Kähler structure} for $\Omega^\bullet$ is a Hermitian structure $(\Omega^{(\bullet \bullet)}, \kappa)$, such that the Hermitian form $\kappa$ is $d$-closed.

The existence of such structures on a differential calculus $\Omega^\bullet$ has various consequences, as in the classical case. We refer to [ÖBu17] for these results and more background material.

2.4. \textbf{Takeuchi’s categorical equivalence.} We will now briefly review some aspects of Takeuchi’s categorical equivalence [Tak79], following [HeKo06, Section 2.2.8].

Let $\mathcal{B} \subset \mathcal{A}$ be a left coideal subalgebra of a Hopf algebra $\mathcal{A}$ with bijective antipode. Then $\mathcal{C} := \mathcal{A}/\mathcal{B}^+\mathcal{A}$ is a right $\mathcal{A}$-module coalgebra, where $\mathcal{B}^+ := \{ b \in \mathcal{B} : \varepsilon(b) = 0 \}$. Let $\Box_\mathcal{C} \mathcal{M}$ denote the category of left $\mathcal{A}$-covariant left $\mathcal{B}$-modules and let $\Box_\mathcal{C} \mathcal{M}$ denote the category of left $\mathcal{C}$-comodules. Then there exist functors

$$\Phi : \Box_\mathcal{B} \mathcal{M} \rightarrow \Box_\mathcal{C} \mathcal{M}, \quad \Psi(\Gamma) := \mathcal{A}/\mathcal{B}^+\Gamma,$$

$$\Psi : \Box_\mathcal{C} \mathcal{M} \rightarrow \Box_\mathcal{B} \mathcal{M}, \quad \Psi(V) := \mathcal{A} \Box_\mathcal{C} V.$$

Here $\Box_\mathcal{C}$ denotes the cotensor product over $\mathcal{C}$.

\textbf{Theorem 2.4 ([Tak79, Theorem 1])}. Suppose that $\mathcal{A}$ is a faithfully flat right $\mathcal{B}$-module. Then $\Phi$ and $\Psi$ give rise to an equivalence of categories between $\Box_\mathcal{B} \mathcal{M}$ and $\Box_\mathcal{C} \mathcal{M}$.

For all the algebras considered in this paper the condition of being faithfully flat will be satisfied, hence we will be able to use Takeuchi’s categorical equivalence.
3. Quantum flag manifolds

In this section we define the quantum flag manifolds $\mathbb{C}_q[G/P_S]$ and recall their presentation by generators are relations, as given by Heckenberger and Kolb. Moreover we will discuss the $*$-structure on $\mathbb{C}_q[G/P_S]$ and its action on the generators.

3.1. Generators. Let $S \subseteq \Pi$ be a subset of the simple roots of $\mathfrak{g}$. Corresponding to any such choice we have the Levi factor $I_S$, which is a subalgebra of the standard parabolic subalgebra $p_S$. In the quantum setting we define the quantized Levi factor by

$$U_q(I_S) := \langle K_\lambda, E_i, F_i : i \in S \rangle \subseteq U_q(\mathfrak{g}).$$

Here $\langle \cdot \rangle$ denotes the algebra generated by these elements in $U_q(\mathfrak{g})$. Note that this is a Hopf $*$-subalgebra. Then we define the quantum flag manifold corresponding to $G/P_S$ by

$$\mathbb{C}_q[G/P_S] := \mathbb{C}_q[G]^{U_q(I_S)} = \{a \in \mathbb{C}_q[G] : Xa = \varepsilon(X)a, \forall X \in U_q(I_S)\}.$$

In the classical case the following realization of $G/P_S$ is well-known. Consider the dominant weight $\lambda := \sum_{i \in S} \omega_i$ and write $N := \dim V(\lambda)$. Then $G/P_S$ is isomorphic to the $G$-orbit of the highest weight vector $v_\lambda \in V(\lambda)$ in the projective space $\mathbb{P}(V(\lambda))$.

We will fix a weight basis $\{v_i\}_{i=1}^N$ of $V(\lambda)$, with the convention that $v_N$ is a highest weight vector. Denote by $\{f_i\}_{i=1}^N$ the dual basis of $V(\lambda)^*$, $\varepsilon(\cdot) = \varepsilon(\cdot)$. Then we define

$$z_{ij} := c^\lambda_{f_i v_\lambda} c^{-v_\lambda}_{u_j f_N} \in \mathbb{C}_q[G], \quad i, j = 1, \ldots, N.$$

Here in writing $c^{-v_\lambda}_{u_j f_N}$, we consider the element $v_j \in V(\lambda)$ as an element of $V(\lambda)^*$.

Let us explain this point in more detail, as it can give rise to some confusion. Given a finite-dimensional $U_q(\mathfrak{g})$-module $V$, we have a map $V \rightarrow V^*$ which assign to $v \in V$ the linear functional $\tilde{v} \in V^*$ given by $\tilde{v}(f) := f(v)$, where $f \in V^*$. The map $V \rightarrow V^*$ is a vector space isomorphism, but not an isomorphism of $U_q(\mathfrak{g})$-modules, since we have the relation

$$X \tilde{v}(f) = f(S^2(X)v) = S^2(X)f(v).$$

The upshot is that we can identify $V^*$ with $V$ in terms of the action $X \cdot v = S^2(X)v$.

Proposition 3.1 ([HeKo06, Proposition 3.2]). The elements $\{z_{ij}\}_{i,j=1}^N$ generate $\mathbb{C}_q[G/P_S]$.

Observe that $\mathbb{C}_q[G/P_S]$ has a natural factorization in terms of the algebras

$$\mathbb{C}_q[G/P_S] = \{c^\lambda_{f_i v_\lambda} : f \in V(\lambda)^*\}, \quad S_q[G/P_S^\text{op}] := \{c^{-v_\lambda}_{v_j f_N} : v \in V(\lambda)^*\}.$$

These are the quantum analogues of the homogeneous coordinate rings of $G/P_S$ and $G/P_S^\text{op}$.

3.2. Relations. In order to present the relations for the quantum flag manifold $\mathbb{C}_q[G/P_S]$ it is convenient to introduce some auxiliary algebras. First we define two algebras $A_+$ and $A_-$ as follows. The algebra $A_+$ has generators $f_1, \ldots, f_N$ and relations

$$f_i f_j = q^{-(\lambda, \lambda)} \sum_{k,l=1}^N (\tilde{R}_{V,V})_{ij}^{kl} f_k f_l,$$

while the algebra $A_-$ has generators $v_1, \ldots, v_N$ and relations

$$v_i v_j = q^{-(\lambda, \lambda)} \sum_{k,l=1}^N (\tilde{R}_{V^*,V^*})_{ij}^{kl} v_k v_l.$$
They become $U_q(\mathfrak{g})$-module algebras by identifying $\{f_i\}_{i=1}^N$ with the basis of $V(\lambda)^*$ and $\{v_i\}_{i=1}^N$ with the basis of $V(\lambda)^{**}$. In this way we obtain $U_q(\mathfrak{g})$-module algebra isomorphisms between $S_q[G/P_S]$ and $A_+$ and between $S_q[G/P_S^{00}]$ and $A_-$, given by

$$f_i \mapsto c^{\lambda}_{f_i,v_{iN}}, \quad v_i \mapsto c_{v_i,f_{iN}}^{-\mu_0\lambda}.$$  

Next we define the algebra $A_C := A_+ \otimes A_-$ with the exchange relations

$$v_i f_j := q^{(\lambda,\lambda)} \sum_{k,l=1}^N (\hat{R}_{V,V'})_{ij}^{kl} f_k v_l.$$  

The algebra $A_C$ admits a central invariant element defined by

$$c := \sum_{i=1}^N v_i f_i.$$  

Hence we can define $A := A_C/(c-1)$. It becomes a $\mathbb{Z}$-graded algebra upon setting $\deg f_i = -1$ and $\deg v_i = 1$. Finally we write $B := A^0$ for the degree zero part of $A$.

**Proposition 3.2** ([HeKo06, Proposition 3.2]). We have an isomorphism of $U_q(\mathfrak{g})$-module algebras $B \cong \mathbb{C}_q[G/P_S]$ given by $f_i v_j \mapsto z_{ij}$.

3.3. $*$-structure. The quantum flag manifolds $\mathbb{C}_q[G/P_S]$ are naturally $*$-algebras, since they are defined by invariance with respect to the Hopf $*$-algebras $U_q(\mathfrak{g})$. This $*$-structure can be transported to the algebras $A$ and $B$. However we will introduce a $*$-structure on $A$ from scratch, as a warm-up to the case of the $*$-calculi to be discussed in the next section.

To make our life easier, we will assume from now on that the basis $\{v_i\}_{i=1}^N$ of $V(\lambda)$ is orthonormal with respect to a $U_q(\mathfrak{g})$-invariant inner product.

**Proposition 3.3.** The algebras $A$ and $B$ become $*$-algebras upon setting $f_i^* = v_i$.

**Proof.** First we check that we obtain a $*$-structure on $A_C$ in this way. It suffices to check that the relations of $A_C$ are preserved under $*$. We compute

$$(f_j f_i)^* = v_i v_j = q^{-(\lambda,\lambda)} \sum_{k,l=1}^N (\hat{R}_{V,V'})_{ij}^{kl} v_k v_l$$

$$= q^{-(\lambda,\lambda)} \sum_{k,l=1}^N (\hat{R}_{V,V'})_{ij}^{kl} v_k v_l = \left( q^{-(\lambda,\lambda)} \sum_{k,l=1}^N (\hat{R}_{V,V'})_{ij}^{kl} f_k f_l \right)^*,$$

where we have used the first identity of Proposition A.2, which we note requires the use of the orthonormal basis $\{v_i\}_{i=1}^N$. Similarly for the relation between the $v_i$'s.

Next we look at the cross-relations. We compute

$$(v_j f_i)^* = v_i f_j = q^{(\lambda,\lambda)} \sum_{k,l=1}^N (\hat{R}_{V,V'})_{ij}^{kl} f_k v_l$$

$$= q^{(\lambda,\lambda)} \sum_{k,l=1}^N (\hat{R}_{V,V'})_{ij}^{kl} f_k v_l = \left( q^{(\lambda,\lambda)} \sum_{k,l=1}^N (\hat{R}_{V,V'})_{ij}^{kl} f_l f_k \right)^*,$$

where we have used the second identity of Proposition A.2. Finally this $*$-structure descends to the quotient $A = A_C/(c-1)$, since we have $c^* = c$. \qed

Now we show that this $*$-structure agrees with the one on $\mathbb{C}_q[G/P_S]$. 

Corollary 3.4. The map \( f_i v_j \mapsto z_{ij} \) is an isomorphism of \( U_q(\mathfrak{g}) \)-module \(*\)-algebras.

Proof. We have \((f_i v_j)^* = f_j v_i\), hence it suffices to show that \( z_{ij}^* = z_{ji} \).

First we claim that \( S(e_{f,v}^\Gamma) = e_{\overline{v} \circ f}^\Gamma \), where \( \overline{v} = f(v) \). Indeed we have
\[
S(e_{f,v}^\Gamma(X)) = e_{\overline{v} \circ f}^\Gamma(S(X)) = f(S(X)v) = (Xf)(v) = \overline{v}(Xf) = e_{\overline{v} \circ f}^\Gamma(X).
\]

Then \( z_{ij} = e_{f,v}^\Gamma S(e_{N,v}^\Gamma) \). Moreover, since \( \{v_i\}_{i=1}^N \) is an orthonormal basis, we can write \( u_i^\lambda = e_{f,v}^\Gamma \) and use \( (u_i^\lambda)^* = S(u_i^\lambda) \). Therefore \( z_{ij} = u_i^\lambda u_j^\lambda \), which shows \( z_{ij}^* = z_{ji} \).

4. Heckenberger-Kolb calculi and \(*\)-structures

In this section we will consider the Heckenberger-Kolb calculus over \( B = \mathbb{C}[G/P_S] \), where \( G/P_S \) is an irreducible flag manifold. We will show that this calculus is naturally a \(*\)-calculus, where the \(*\)-structure on \( B \subset A \) is the one introduced in the previous section. After giving a brief presentation of the Heckenberger-Kolb calculi, we give the necessary definitions for the two FODCs \( \Gamma_{\theta} \) and \( \Gamma_{\overline{\theta}} \), whose direct sum gives the FODC \( \Gamma_d \). To show that \( \Gamma_d \) is a \(*\)-calculus we will check that the relations are compatible with the \(*\)-structure of \( B \).

4.1. The calculi in brief. From this point on we will restrict to the case of irreducible flag manifolds \( G/P_S \). These spaces can be characterized by the following condition: we have \( S = \Pi \{ \alpha_s \} \) and \( \alpha_s \) has multiplicity 1 in the highest root of \( \mathfrak{g} \). In the following the index \( s \) will always be associated to the simple root \( \alpha_s \) removed from the set \( S \).

In the paper [HeKo04], Heckenberger and Kolb show that there exist exactly two non-isomorphic covariant FODCs over \( \mathbb{C}[G/P_S] \). We denote them by \( \Gamma_{\theta} \) and \( \Gamma_{\overline{\theta}} \), as they classically correspond to the holomorphic and anti-holomorphic calculi on the complex manifold \( G/P_S \), and write \( \Gamma_d = \Gamma_{\theta} \oplus \Gamma_{\overline{\theta}} \) for their direct sum. In the follow-up paper [HeKo06], they investigate the universal differential calculi \( \Gamma_{\theta}^{\wedge}, \Gamma_{\overline{\theta}}^{\wedge} \), built from \( \Gamma_d \). They show that these calcui have classical dimensions and have many of the features of the classical calculi over \( G/P_S \).

Before giving the details, let us first outline the main steps of this construction. First we define a FODC \( \Gamma_+ \) over \( A_+ \). Then using \( \Gamma_+ \) we construct a FODC \( \Gamma_+;C \) over \( A_C \). By taking an appropriate quotient, we obtain a FODC \( \Gamma_+;c/\Lambda_+ \) over \( A \). Finally the calculus \( \Gamma_{\theta} \) over \( B \) is simply the calculus induced by \( \Gamma_+;c/\Lambda_+ \) over \( A \). A similar construction gives \( \Gamma_{\overline{\theta}} \) starting from \( A_- \). Hence we obtain the FODC \( \Gamma_d \) over \( B \) as the direct sum of \( \Gamma_{\theta} \) and \( \Gamma_{\overline{\theta}} \).

Therefore we get the universal differential calculi \( \Gamma_{\theta}^{\wedge,\Lambda} \), which we will also denote by \( \Omega^\wedge \). Hence \( d = \partial + \overline{\partial} \) is a differential and we have the relation \( \partial \circ \overline{\partial} = -\overline{\partial} \circ \partial \).

4.2. The FODC \( \Gamma_{\theta} \). First we present the construction of the FODC \( \Gamma_{\theta} \) over \( B \). We start from the left \( A_+ \)-module \( \Gamma_+ \) generated by the elements \( \{df_i\}_{i=1}^N \) and the relations
\[
\sum_{\iota,j=1}^N (\hat{P}_V \hat{Q}_V)_{ij} f_i d f_j = 0. \tag{4.1}
\]
Here we make use of the notations
\[
\hat{P}_V := \hat{R}_{V,V} - q^{(\alpha_s,\alpha_s)} \mathrm{id}, \quad \hat{Q}_V := \hat{R}_{V,V} + q^{(\alpha_s,\alpha_s)} \mathrm{id}.
\]
We can make \( \Gamma_+ \) into an \( A_+ \)-bimodule by setting
\[
(d_f)_{ij} = q^{(\alpha_s,\alpha_s)} \sum_{k,l=1}^N \hat{R}_{V,V}^{ij}_{kl} f_k d f_l. \tag{4.2}
\]
Next we consider the left $\mathcal{A}_\mathbb{C}$-module $\Gamma_{+,\mathbb{C}}$ defined by

$$\Gamma_{+,\mathbb{C}} := \mathcal{A}_\mathbb{C} \otimes_{\mathcal{A}_+} \Gamma_+.$$ 

It becomes an $\mathcal{A}_\mathbb{C}$-bimodule by setting

$$(df_i)v_j = q^{-(\omega_s,\omega_s)} \sum_{k,l=1}^N (\hat{R}_{V,V^*})^{-1}_{kl} v_k df_i.$$  

(4.3)

We have a differential $\partial : \mathcal{A}_\mathbb{C} \to \Gamma_{+,\mathbb{C}}$ given by

$$\partial(f_i) = df_i, \quad \partial(v_i) = 0.$$ 

In the following we will always write $df_i = \partial(f_i)$.

Let $\Lambda_+ \subset \Gamma_{+,\mathbb{C}}$ be the sub-bimodule generated by $\partial(c)$, $(c-1)\Gamma_{+,\mathbb{C}}$ and $\Gamma_{+,\mathbb{C}}(c-1)$. Then the quotient $\Gamma_{+,\mathbb{C}}/\Lambda_+$ is a FODC over $\mathcal{A}$. Finally $\Gamma_0$ is the FODC over $\mathcal{B}$ induced by $\Gamma_{+,\mathbb{C}}/\Lambda_+$.

4.3. The FODC $\Gamma_{\overline{\mathbb{F}}}$. Next we introduce the second covariant FODC $(\Gamma_{\overline{\mathbb{F}}}, \overline{\partial})$ over $\mathcal{A}$. Consider the left $\mathcal{A}_-$-module $\Gamma_-$ generated by the elements $\{dv_i\}_{i=1}^N$ and the relations

$$\sum_{i,j=1}^N (\hat{P}_{V^*} \hat{Q}_{V^*})_{ij} v_i dv_j = 0.$$  

(4.4)

Here we are using the notations

$$\hat{P}_{V^*} := \hat{R}_{V^* V^*} - q^{(\omega_s,\omega_s)} \text{id}, \quad \hat{Q}_{V^*} := \hat{R}_{V^* V^*} + q^{(\omega_s,\omega_s) - (\alpha_s,\alpha_s)} \text{id}.$$ 

We turn the left $\mathcal{A}_-$-module $\Gamma_-$ into an $\mathcal{A}_-$-bimodule by

$$(dv_i)v_j = q^{(\omega_s,\omega_s) - (\alpha_s,\alpha_s)} \sum_{k,l=1}^N (\hat{R}_{V^* V^*})^{-1}_{kl} v_k dv_i.$$  

(4.5)

Next we consider the left $\mathcal{A}_\mathbb{C}$-module $\Gamma_{-,\mathbb{C}}$ defined by

$$\Gamma_{-,\mathbb{C}} := \mathcal{A}_\mathbb{C} \otimes_{\mathcal{A}_-} \Gamma_-.$$ 

It becomes an $\mathcal{A}_\mathbb{C}$-bimodule by setting

$$(dv_i)f_j = q^{(\omega_s,\omega_s)} \sum_{k,l=1}^N (\hat{R}_{V^* V^*})^{-1}_{kl} f_k dv_i.$$  

(4.6)

We have a differential $\overline{\partial} : \mathcal{A}_\mathbb{C} \to \Gamma_{-,\mathbb{C}}$ given by

$$\overline{\partial}(f_i) = 0, \quad \overline{\partial}(v_i) = dv_i.$$ 

In the following we will always write $dv_i = \overline{\partial}(v_i)$.

Let $\Lambda_- \subset \Gamma_{-,\mathbb{C}}$ be the sub-bimodule generated by $\partial(c)$, $(c-1)\Gamma_{-,\mathbb{C}}$ and $\Gamma_{-,\mathbb{C}}(c-1)$. Then the quotient $\Gamma_{-,\mathbb{C}}/\Lambda_-$ is a FODC over $\mathcal{A}$. Finally $\Gamma_{\overline{\mathbb{F}}}$ is the FODC over $\mathcal{B}$ induced by $\Gamma_{-,\mathbb{C}}/\Lambda_-$. 

4.4. Differential \(*\)-calculus. We will now investigate whether the FODC \(\Gamma_d\) over the \(*\)-algebra \(\mathcal{B}\) can be made into a differential \(*\)-calculus. We start with a simple lemma.

**Lemma 4.1.** The relations (4.1), (4.2) and (4.3) in \(\Gamma_{+\mathbb{C}}\) are equivalent to

\[
\sum_{i,j=1}^{N} (\hat{P}_V \hat{Q}_V)_{ij}^{kl} \partial(f_i) f_j = 0,
\]

\[
f_i \partial(f_j) = q^{(\omega, \omega)} - (\alpha, \alpha) \sum_{k,l=1}^{N} (\hat{R}_{V,V}^{-1})_{ij}^{kl} \partial(f_k) f_l;
\]

\[
v_i \partial(f_j) = q^{(\omega, \omega)} \sum_{k,l=1}^{N} (\hat{R}_{V,V}^{-1})_{ij}^{kl} \partial(f_k) v_l.
\]

Similarly the relations (4.4), (4.5) and (4.6) in \(\Gamma_{-\mathbb{C}}\) are equivalent to

\[
\sum_{i,j=1}^{N} (\hat{P}_V \hat{Q}_V)_{ij}^{kl} \overline{\partial}(v_i) v_j = 0,
\]

\[
v_i \overline{\partial}(v_j) = q^{(\omega, \omega)} - (\alpha, \alpha) \sum_{k,l=1}^{N} (\hat{R}_{V,V}^{-1})_{ij}^{kl} \overline{\partial}(v_k) v_l,
\]

\[
f_i \overline{\partial}(v_j) = q^{-(\omega, \omega)} \sum_{k,l=1}^{N} (\hat{R}_{V,V}^{-1})_{ij}^{kl} \overline{\partial}(v_k) f_l.
\]

**Proof.** Most of the identities follow straightforwardly by using the inverse of the appropriate braiding. The only non-trivial identities are those following from (4.1) and (4.4). Let us consider the first one. Plugging the identity for \(f_i \partial(f_j)\) into (4.1) we obtain

\[
q^{(\omega, \omega)} - (\alpha, \alpha) \sum_{i,j=1}^{N} (\hat{P}_V \hat{Q}_V \hat{R}_{V,V}^{-1})_{ij}^{kl} \partial(f_i) f_j = 0.
\]

Then using the relation \(\hat{P}_V \hat{Q}_V \hat{R}_{V,V}^{-1} = \hat{R}_{V,V}^{-1} \hat{P}_V \hat{Q}_V\) and multiplying on the left by \(\hat{R}_{V,V}\) we obtain the result. The second identity is obtained similarly. \(\square\)

We are now ready to prove the main result of this section.

**Theorem 4.2.** The differential calculus \((\Gamma_{d,\mathbb{C}}\mathcal{d})\) over \(\mathcal{B}\) is a differential \(*\)-calculus.

**Proof.** We have already mentioned that it suffices to show that the FODC \((\Gamma_{d,\mathbb{C}})\) is a \(*\)-FODC. Consider the requirement \(\mathcal{d}(a^*) = \mathcal{d}(a^*)\). Using \(f_i^* = v_i\), together with the relations \(\mathcal{d}(f_i) = \partial(f_i)\) and \(\mathcal{d}(v_i) = \overline{\partial}(v_i)\), we immediately find \(\partial(f_i^*) = \overline{\partial}(v_i)\). Now we have to check that the relations in \(\Gamma\) are preserved by this candidate \(*\)-structure.

We start with the relations (4.1) and (4.4). Observe that \((\hat{P}_V)_{ij}^{kl} = (\hat{P}_V^*)_{ji}^{lk}\) and \((\hat{Q}_V)_{ij}^{kl} = (\hat{Q}_V^*)_{ji}^{lk}\), which follows from their definitions and Proposition A.2. Then we have

\[
\left( \sum_{i,j=1}^{N} (\hat{P}_V \hat{Q}_V)_{ij}^{kl} f_i \partial(f_j) \right)^* = \sum_{i,j=1}^{N} (\hat{P}_V \hat{Q}_V^*)_{ji}^{lk} \overline{\partial}(v_j) v_i = 0,
\]

where the last identity follows from Lemma 4.1. Similarly for the relation (4.4).
Next consider the relations (4.2) and (4.5). Using Lemma 4.1 we compute
\[
(\partial(f_i)f_j)^* = v_j\overline{\partial}(v_i) = q^{(\alpha_s,\alpha_s) - (\omega_s,\omega_s)} \sum_{k,l=1}^{N} (\hat{R}_{V^*,V})^{ij}_{kl} f_l \overline{\partial}(v_k) v_l
\]
\[
= \left( q^{(\alpha_s,\alpha_s) - (\omega_s,\omega_s)} \sum_{k,l=1}^{N} (\hat{R}_{V^*}^{-1})^{ij}_{lk} f_l \partial(f_k) \right)^*,
\]
where we have used \((\hat{R}_{V^*}^{-1})^{kl}_{ij} = (\hat{R}_{V^*}^{-1})^{ik}_{jl}\) from Proposition A.2.

Next let us consider (4.3) and (4.6). Using Lemma 4.1 we compute
\[
(\partial(f_i)v_j)^* = f_j\overline{\partial}(v_i) = q^{-(\omega_s,\omega_s)} \sum_{k,l=1}^{N} (\hat{R}_{V^*}^{-1})^{ij}_{lk} v_l \overline{\partial}(f_k) v_l
\]
\[
= \left( q^{-(\omega_s,\omega_s)} \sum_{k,l=1}^{N} (\hat{R}_{V^*}^{-1})^{ij}_{lk} v_l \partial(f_k) \right)^*,
\]
where we have used \((\hat{R}_{V^*}^{-1})^{kl}_{ij} = (\hat{R}_{V^*}^{-1})^{ik}_{jl}\) from Proposition A.2.

Finally we have to check that the sub-bimodules \(\Lambda^+\) and \(\Lambda^-\) are preserved under \(*\). But this is clear, since we have \(c^* = c\).

5. HERMITIAN AND KÄHLER STRUCTURES

In this section we will show the existence of Hermitian and Kähler structures on the Heckenberger-Kolb calculi \(\Omega^*\) over quantum irreducible flag manifolds \(\mathbb{C}_q[G/P_S]\). For the most technical part of the proof, namely the fact that the Lefschetz map gives appropriate isomorphisms, we will use a filtration introduced by Heckenberger and Kolb.

5.1. Some identities. Recall the \(U_q(\mathfrak{g})\)-module algebra isomorphism \(B \cong \mathbb{C}_q[G/P_S]\) from Proposition 3.2. By a small abuse of notation, we will also write \(z_{ij} = f_i v_j\) for the generators of the algebra \(B\). We will now obtain some identities for these elements.

**Lemma 5.1.** We have \(\sum_{k=1}^{N} z_{ik} z_{kj} = z_{ij}\) and \(z_{ij}^* = z_{ji}\).

**Proof.** The projection relation follows from \(\sum_{i=1}^{N} v_i f_i = 1\) and
\[
\sum_{k=1}^{N} z_{ik} z_{kj} = \sum_{k=1}^{N} f_i v_k f_k v_j = f_i v_j = z_{ij}.
\]
The \(*\)-relation follows immediately from the definition of \(z_{ij}\). \(\square\)

Next we consider some identities involving the differentials.

**Lemma 5.2.** We have the identities
\[
\sum_{k=1}^{N} z_{ik} \partial(z_{kj}) = 0, \quad \sum_{k=1}^{N} \partial(z_{ik}) z_{kj} = \partial(z_{ij}),
\]
\[
\sum_{k=1}^{N} z_{ik} \overline{\partial}(z_{kj}) = \overline{\partial}(z_{ij}), \quad \sum_{k=1}^{N} \overline{\partial}(z_{ik}) z_{kj} = 0.
\]
Proof. We start with the differential $\partial$. We compute

$$
\sum_{k=1}^{N} \partial(z_{ik})z_{kj} = \sum_{k=1}^{N} \partial(f_k v_k) f_k v_j = \sum_{k=1}^{N} \partial(f_k) v_k f_k v_j
$$

$$
= \partial(f_i) v_j = \partial(f_i v_j) = \partial(z_{ij}).
$$

On the other hand, using the fact that $z$ is a projection, we have

$$
\partial(z_{ij}) = \sum_{k=1}^{N} \partial(z_{ik} z_{kj}) = \sum_{k=1}^{N} \partial(z_{ik}) z_{kj} + \sum_{k=1}^{N} z_{ik} \partial(z_{kj}).
$$

Using the previous identity, this implies the vanishing of last term.

The identities for the differential $\overline{\partial}$ are similar. We compute

$$
\sum_{k=1}^{N} z_{ik} \overline{\partial}(z_{kj}) = \sum_{k=1}^{N} f_k v_k \overline{\partial}(f_k v_j) = \sum_{k=1}^{N} f_k v_k \overline{\partial}(v_j)
$$

$$
= f_i \overline{\partial}(v_j) = \overline{\partial}(f_i v_j) = \overline{\partial}(z_{ij}).
$$

On the other hand using the Leibnitz rule we get

$$
\overline{\partial}(z_{ij}) = \sum_{k=1}^{N} \overline{\partial}(z_{ik} z_{kj}) = \sum_{k=1}^{N} \overline{\partial}(z_{ik}) z_{kj} + \sum_{k=1}^{N} z_{ik} \overline{\partial}(z_{kj}).
$$

This immediately implies the fourth identity. \qed

5.2. The Kähler form. We will now introduce a 2-form $\kappa \in \Omega^*$, which we will later show to satisfy the conditions defining a Kähler form. It is given by

$$
\kappa := i \sum_{i,j=1}^{N} q^{(2p, wt_{tv})} d(z_{ij}) \wedge d(z_{ji}).
$$

Here $i := \sqrt{-1}$ denotes the imaginary unit.

We begin by showing that $\kappa$ is left $\mathbb{C}_q[G]$-coinvariant. In this proof we will consider $z_{ij}$ as elements of $\mathbb{C}_q[G/P_S]$, so that $\Delta$ denotes the coproduct of $\mathbb{C}_q[G]$.

Lemma 5.3. The element $\kappa$ is left $\mathbb{C}_q[G]$-coinvariant.

Proof. We have $z_{ij} = u_{N_j}^\lambda S(u_{N_j})$, as seen in the proof of Corollary 3.4. We will omit the superscript $\lambda$ for notational convenience. Then we easily compute

$$
\Delta(z_{ij}) = \sum_{a,b=1}^{N} u_{ia} S(u_{bj}) \otimes z_{ab},
$$

in accordance with the fact that $\mathbb{C}_q[G/P_S]$ is a left coideal. Next recall that in a left-covariant FODC $\Gamma$ we have $\Delta_{\Gamma}(adb) = \Delta(a)(\text{id} \otimes d)(\Delta(b))$. Then we obtain

$$
\Delta_{\Gamma}(\kappa) = i \sum_{i,j=1}^{N} \sum_{a,b,c,d=1} q^{(2p, wt_{tv})} u_{ia} S(u_{bj}) u_{jc} S(u_{di}) \otimes d(z_{ab}) \wedge d(z_{cd}).
$$

Next we use the identities following from the Hopf algebra structure

$$
\sum_{k=1}^{N} S(u_{ik}) u_{kj} = \delta_{ij}, \quad \sum_{k=1}^{N} q^{(2p, wt_{tv} - wt_{tv})} u_{kj} S(u_{ik}) = \delta_{ij},
$$

where $\delta_{ij}$ is the Kronecker delta.
where the second one follows from the fact that $S^2(u_{ij}) = q^{(2p, wt_v_i - wt_v_j)}u_{ij}$. Finally

$$\Delta_T(\kappa) = i \sum_{a,b,d,s = 1}^N q^{(2p, wt_v_i)}u_{ia}S(u_{db}) \otimes d(z_{ab}) \wedge d(z_{bd})$$

$$= i \sum_{a,b = 1}^N q^{(2p, wt_v_i)}1 \otimes d(z_{ab}) \wedge d(z_{ba}) = 1 \otimes \kappa. \quad \Box$$

We can now easily show that $\kappa$ satisfies most of the requirements of a Kähler form.

**Proposition 5.4.** The 2-form $\kappa$ satisfies the following properties.

1. It is closed, central and real.
2. It belongs to $\Omega^{(1,1)}$, in particular $\kappa = i \sum_{i,j=1}^N q^{(2p, wt_v_i)}\partial(z_{ij}) \wedge \overline{\partial}(z_{ji})$.

**Proof.** (1) It is clear that $d(\kappa) = 0$, since $\kappa = i \sum_{i,j=1}^N q^{(2p, wt_v_i)}d(z_{ij}) \wedge d(z_{ji})$. Next it follows from Lemma 5.3 that $\kappa$ is central, since every left $C_q[G]$-coinvariant $d$-closed form is central, see [ÓBu17, Corollary 4.6]. To show that $\kappa$ is real we use $z_{ij}^* = z_{ji}$ and compute

$$\kappa^* = i \sum_{i,j=1}^N d(z_{ij})^* \wedge d(z_{ij})^* = i \sum_{i,j=1}^N d(z_{ij}) \wedge d(z_{ji}) = \kappa.$$

(2) From Lemma 5.2 we easily derive the identities

$$\sum_{k=1}^N \partial(z_{ik}) \wedge \partial(z_{kj}) = 0, \quad \sum_{k=1}^N \overline{\partial}(z_{ik}) \wedge \overline{\partial}(z_{kj}) = 0.$$

Hence $\kappa$ can be rewritten in the form

$$\kappa = i \sum_{i,j=1}^N q^{(2p, wt_v_i)}\partial(z_{ij}) \wedge \overline{\partial}(z_{ji}) + i \sum_{i,j=1}^N q^{(2p, wt_v_i)}\overline{\partial}(z_{ij}) \wedge \partial(z_{ji}).$$

To prove that $\kappa \in \Omega^{(1,1)}$ it suffices to show that the second term vanishes. Consider

$$\sum_{j=1}^N \overline{\partial}(z_{ij}) \wedge \partial(z_{ji}) = \sum_{j=1}^N \partial(v_j) \wedge \partial(f_j)v_j.$$

Then, using $\sum_{j=1}^N v_j f_j = 1$ and $\partial \circ \overline{\partial} = -\overline{\partial} \circ \partial$, we compute

$$\sum_{j=1}^N \overline{\partial}(v_j) \wedge \partial(f_j) = \sum_{j=1}^N \overline{\partial}(v_j f_j) = 0. \quad \Box$$

To prove that $\kappa$ is a Kähler form we need the condition on the Lefschetz map $L : \Omega^* \to \Omega^*$, as given in Definition 2.1. To show this we will Takeuchi’s categorical equivalence.

5.3. **Completing the proof.** Recall that for an irreducible flag manifold we have $S = \Pi \setminus \{\alpha_s\}$ for some $s \in \{1, \cdots, r\}$. Write $I := \{1, \cdots, N\}$ and define the index set

$I' := \{i \in I : (\omega_s, \omega_s - \alpha_s - wt_v_i) = 0\}.$

This set is denoted by $I_{(1)}$ in [HeKo06, Section 3.2.1]. It it known that $\#I' = \dim_C(g/p_S)$. We will use the shorter notation $M := \dim_C(g/p_S)$.
Recall the functor $\Phi$ which features in Takeuchi’s categorical equivalence, which on the algebra $\Omega^*$ gives $\Phi(\Omega^*) = \Omega^*/B^+\Omega^*$. We have that $\Phi(\Omega^*)$ is an algebra, since $B^+\Omega^* = \Omega^*B^+$ as explained in [HeKo06, Section 3.3.4]. We introduce the notation

$$x_i := \Phi(\partial z_{N_i}), \quad y_i := \Phi(\partial \bar{z}_{N_i}), \quad i \in I'.$$

**Lemma 5.5.** We have $\Phi(\partial z_{ij}) = 0$ if $i \notin I'$ or $j \neq N$ and $\Phi(\partial \bar{z}_{ij}) = 0$ if $i \neq N$ or $j \notin I'$. The algebra $\Phi(\Omega^*) = \Omega^*/B^+\Omega^*$ can be endowed with a certain filtration $\mathcal{H}$, for details see [HeKo06, Sections 3.3.1 and 3.3.4]. The reason for defining this filtration is that in the associated graded algebra $Gr_{\mathcal{H}}\Omega^*/B^+\Omega^*$ the relations are very similar to the classical case.

**Proposition 5.6.** In the associated graded algebra $Gr_{\mathcal{H}}\Omega^*/B^+\Omega^*$ we have the relations

$$x_i \wedge x_j = -b_{ij}x_j \wedge x_i, \quad y_i \wedge y_j = -b'_{ij}y_j \wedge y_i, \quad y_i \wedge x_j = -c_{ij}x_j \wedge y_i.$$

Here the coefficients satisfy the following properties: we have $b_{ii} = 0$ and $b_{ij} > 0$ for $i \neq j$ and similarly for $b'_{ij}$; on the other hand we have $c_{ij} > 0$ for all $i$ and $j$.

**Proof.** This is a reformulation of [HeKo06, Proposition 3.11(iii)].

Let us write $V^k := \Omega^k/B^+\Omega^k$. We introduce the set of multi-indices

$$\mathcal{I}_n := \{J = (j_1, \cdots, j_n) : j_i \in I', \, j_1 < \cdots < j_n\}.$$ Given $J \in \mathcal{I}_n$ we will write $x_J := x_{j_1} \wedge \cdots \wedge x_{j_n}$ and similarly for $y_J$. Then we have a vector space basis of $V^k$ given by the elements

$$\{x_J \wedge y_K : J \in \mathcal{I}_a, \, K \in \mathcal{I}_b, \, a + b = k\}.$$ **Lemma 5.7.** We have $\kappa^k \neq 0$ for $k = 0, \cdots, M$.

**Proof.** By Takeuchi’s equivalence, it suffices to show that $\Phi(\kappa^k) \neq 0$. We have

$$\kappa^k = i^k \sum_{i_1, j_1 \in I} \cdots \sum_{i_k, j_k \in I} q^{(2p, \lambda_1, \cdots, \lambda_k)} \partial z_{j_1 i_1} \wedge \partial \overline{z}_{j_1 i_1} \wedge \cdots \wedge \partial z_{j_k i_k} \wedge \overline{\partial} z_{j_k i_k}.$$ Applying the map $\Phi$ and using **Lemma 5.5** we obtain

$$\Phi(\kappa^m) = i^k \sum_{i_1, \cdots, i_k \in I'} q^{(2p, \lambda_1, \cdots, \lambda_k)} x_{i_1} \wedge y_{i_1} \wedge \cdots \wedge x_{i_k} \wedge y_{i_k}.$$ Next, using the commutation relations from **Proposition 5.6**, it is straightforward to see that the indices $i_1, \cdots, i_k$ have to be pairwise distinct. Finally, using the various commutation relations, it is possible to rewrite $\Phi(\kappa^k)$ in the form

$$\Phi(\kappa^k) = \sum_{J \in \mathcal{I}_k} a_J x_J \wedge y_J, \quad a_J \neq 0.$$ Hence this element is non-zero, as the $x_J \wedge y_J$ give a vector space basis of of $V^{2k}$. We observe the following simple corollary.

**Corollary 5.8.** We have that $\kappa^m$ is a volume form of $\Omega^*$. We are now ready to show that $\kappa$ is an almost symplectic form, as in **Definition 2.1**.

**Proposition 5.9.** The map $L^{M-k} : \Omega^k \to \Omega^{2M-k}$ is an isomorphism for $k = 0, \cdots, M - 1$. 

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Proposition 5.6. We have \( \Phi(L^{M-k}) : V^k \to V^{2M-k} \) is an isomorphism. We have \( \Phi(L^{M-k})v = \Phi(\kappa^{M-k}) \land v \) for \( v \in V^k \). We have seen in the previous lemma that
\[
\Phi(\kappa^{M-k}) = \sum_{J \in I_{M-k}} a_J x_J \land y_J, \quad a_J \neq 0.
\]
Consider the action of \( x_J \land y_J \) with \( J \in I_{M-k} \) on a basis element \( x_K \land y_L \in V^k \). By the commutation relations of Proposition 5.6, we have that \( x_J \land y_J \land x_K \land y_L = 0 \) if \( J \cap K \neq \emptyset \) or \( J \cap L \neq \emptyset \). Now consider the case \( J \cap K = J \cap L = \emptyset \). Note that there exists at least one multi-index \( J \in I_{M-k} \) with this property, since \( K \cup L \) has at most \( k \) distinct elements and \( \#I' = M \). Then using the commutation relations we can rewrite
\[
x_J \land y_J \land x_K \land y_L = C_{J,K,L}x_{J\cup K} \land y_{J\cup L},
\]
where \( C_{J,K,L} \neq 0 \) and the multi-indices \( J \cup K \) and \( J \cup L \) are ordered. Moreover observe that different choices of \( J \) give different basis elements, hence linearly independent. From this we conclude that \( \Phi(\kappa^{M-k}) \land x_K \land y_L \neq 0 \), and hence \( \ker \Phi(L^{M-k}) = 0 \). \( \square \)

Using this fact, it is immediate to prove the main result of this section.

Theorem 5.10. The pair \((\Omega^{\bullet \bullet}, \kappa)\) is a Kähler structure for \( \Omega^\bullet \).

Proof. This follows by combining Proposition 5.4 and Proposition 5.9. We have that \( \kappa \) is a central, real 2-form such that \( L^{M-k} : \Omega^k \to \Omega^{2M-k} \) is an isomorphism for \( k = 0, \cdots, M-1 \), hence is an almost symplectic form according to Definition 2.1. Next \( \Omega^\bullet \) has a natural complex structure and \( \kappa \in \Omega^{(1,1)} \), hence it is a Hermitian form according to Definition 2.2. Finally \( \kappa \) is d-closed and hence a Kähler form according to Definition 2.3. \( \square \)

This proves the conjecture formulated in [ÔBu17, Conjecture 4.25].

Appendix A. Some identities for the braiding

In this appendix we will derive some identities for the components of the braiding, which are related to its behaviour under the operations of duality and adjoint. These are used in the main text to prove the compatibility of the differential calculi with the \( \ast \)-structure.

A.1. Basic facts. Let \( V \) and \( W \) be finite-dimensional \( U_q(\mathfrak{g}) \)-modules. Fix bases \( \{v_i\}_i \subset V \) and \( \{w_i\}_i \subset W \). Then for the braiding \( \hat{R}_{V,W} : V \otimes W \to W \otimes V \) we write
\[
\hat{R}_{V,W}(v_i \otimes w_j) = \sum_{k,l} (\hat{R}_{V,W})_{ij}^{kl} w_k \otimes v_l.
\]

Consider the double-dual module \( V^{\ast \ast} \). To any vector \( v \in V \) we associate the linear functional \( \hat{v} \in V^{\ast \ast} \) on \( V^\ast \) defined by \( \hat{v} f := f(v) \). The map \( v \mapsto \hat{v} \) is an isomorphism of vector spaces but not of \( U_q(\mathfrak{g}) \)-modules. On the other hand, it is simple to check that the map \( \eta_v : V \to V^{\ast \ast} \) given by \( \eta_v(v) := \hat{K}_{2p} v \) is an isomorphism of \( U_q(\mathfrak{g}) \)-modules. This is because in our conventions we have \( S^2(X) = K_{2p} X K_{2p}^{-1} \) for all \( X \in U_q(\mathfrak{g}) \).

We will also need \( U_q(\mathfrak{g}) \)-invariant inner products. Fixing \( (\cdot, \cdot)_V \) on \( V \), we will write \( j_V : V \to V^\ast \) for the conjugate-linear map given by \( j_V(v)(w) := (v, w)_V \). Then
\[
(j_V(v), j_V(w))_{V^\ast} := (K_{2p} w, v)_V
\]
is an invariant inner product on \( V^\ast \). To check this claim one uses the fact that \( (\cdot, \cdot)_V \) is invariant and the identities \( S^2(X) = K_{2p} X K_{2p}^{-1} \) and \( S(X)^* = S^{-1}(X^*) \).
A.2. Duality and adjoint. We will now investigate the effect of the operations of duality and adjoint on the braiding $\hat{R}_{V,W}$. Recall that, given a morphism $T : V \rightarrow W$, the transpose $T^\tau : W^* \rightarrow V^*$ is defined by $T^\tau(g) := g \circ T$ for $g \in W^*$. Hence we have

$$\hat{R}^\tau_{V,W} : (W \otimes V)^* \rightarrow (V \otimes W)^*.$$ 

We have a natural isomorphism $\iota_{V,W} : V^* \otimes W^* \rightarrow (W \otimes V)^*$ given by $\iota_{V,W}(f \otimes g) := g \otimes f$. Using this isomorphism, we obtain the map $\hat{R}^\vee_{V,W} : V^* \otimes W^* \rightarrow W^* \otimes V^*$ given by

$$\hat{R}^\vee_{V,W} := \iota_{W,V}^{-1} \circ \hat{R}^\tau_{V,W} \circ \iota_{V,W}.$$ 

Then we have the identity $\hat{R}^\vee_{V,W} = \hat{R}^\vee_{W,V^*}$, see for instance [EGNO16, Exercise 8.9.2]. This is actually a general result, valid for any braided monoidal category with duals.

Next we consider the adjoint of the braiding $\hat{R}_{V,W}$, which is defined using the natural invariant inner products on $V \otimes W$ and $W \otimes V$. In this way we obtain a map $\hat{R}^*_{V,W} : W \otimes V \rightarrow V \otimes W$. Then we have the identity $\hat{R}^*_{V,W} = \hat{R}_{W,V}$, see for instance [NeTu13, Example 2.6.4].

Now we look at the effect of these identities on the components of $\hat{R}_{V,W}$.

**Lemma A.1.** We have the following identities.

1. Suppose $\{f_i\} \subset V^*$ and $\{g_i\} \subset W^*$ are dual bases. Then $\hat{R}_{V^*,W^*}^{\tau j} = (\hat{R}_{V,W})^{ji\tau}$. 
2. Suppose $\{v_i\} \subset V$ and $\{w_i\} \subset W$ are orthonormal bases. Then $\hat{R}_{V,V}^{\tau j} = (\hat{R}_{W,V})^{ji\tau}$. 

**Proof.** 1) A simple computation shows that $\hat{R}^\tau_{V,W}(g_j \otimes f_i) = \sum_{kl}(\hat{R}_{V,W})^{ji\tau}_{kl}g_k \otimes f_l$, from which it easily follows that $\hat{R}^\vee_{V,W}(f_i \otimes g_j) = \sum_{kl}(\hat{R}_{V,W})^{ji\tau}_{kl}g_k \otimes f_l$. Then using the identity $\hat{R}^\vee_{V,W} = \hat{R}_{V^*,W^*}$ and comparing the coefficients we obtain the result.

2) This easily follows from $\hat{R}^*_{V,W} = \hat{R}_{W,V}$ and the fact that we use orthonormal bases. 

Using these it is possible to obtain more identities for the components of the braiding. Here we will derive two of them, which are needed to prove some results in the main text.

**Proposition A.2.** Let $\{v_i\} \subset V$ be an orthonormal basis and $\{f_i\} \subset V^*$ the dual basis.

1. We have $\hat{R}^{\tau}_{V,W} = (\hat{R}_{V,V^*})^{ji\tau}$. 
2. We have $\hat{R}^{\tau}_{V,V} = (\hat{R}_{V,V^*})^{ji\tau}$.

**Proof.** 1) Using the identities of **Lemma A.1** we easily get

$$\hat{R}_{V,V^*}^{\tau ji} = (\hat{R}_{V,V^*})^{ji\tau} = (\hat{R}_{V^*,V})^{ji\tau}.$$ 

2) First we observe that the dual basis $\{f_i\}$ is not orthonormal. Indeed, using the fact that $f_i = j_V(v_i)$ and the definition of the inner product on $V^*$, we compute

$$(f_i, f_j)_{V^*} = (K_{2p}v_j, v_i)_V = \delta_{ij}q^{(2p,w_{2p,v_i})}.$$ 

Hence we obtain an orthonormal basis by setting $f'_i = q^{-\langle p, w_{2p,v_i} \rangle} f_i$. We write the formulae for the braiding with respect to the orthonormal bases $\{v_i\}$ and $\{f'_i\}$ as follows

$$\hat{R}_{V,V^*}(v_i \otimes f'_j) = \sum_{k,l} b_{ij}^{kl} f'_k \otimes v_l, \quad \hat{R}_{V^*,V}(f'_i \otimes v_j) = \sum_{k,l} a_{ij}^{kl} v_k \otimes f'_l.$$ 

From these we immediately get the relations

$$\hat{R}_{V,V^*}^{\tau ji} = q^{\langle p, w_{2p,v_j} - w_{2p,v_i} \rangle} a_{ij}^{ji\tau}, \quad (\hat{R}_{V^*,V})^{ji\tau} = q^{\langle p, w_{2p,v_i} - w_{2p,v_j} \rangle} b_{ij}^{ji\tau}.$$
Then using the second identity of Lemma A.1 we get

\[(\hat{R}_{V,V^*})_{kl}^{ij} = q^{\langle \rho, wt v_j - wt v_k \rangle} a_{kl}^{ij} = q^{\langle \eta_V, wt v_j - wt v_k \rangle} (\hat{R}_{V^*,V})_{kl}^{ij}.\]

Next we have \((\hat{R}_{V^*,V^*})_{kl}^{ij} = (\hat{R}_{V^*,V^*})_{ji}^{kl}\), by the first identity of Lemma A.1. To proceed we use the \(U_q(\mathfrak{g})\)-module isomorphism \(\eta_V : V \to V^{**}\) given by \(\eta_V(v) = \tilde{K}_{2\rho} v\). It follows from the naturality of the braiding that we have the relation

\[\hat{R}_{V^*,V^*} = (\text{id} \otimes \eta_V) \circ \hat{R}_{V,V^*} \circ (\eta_V^{-1} \otimes \text{id}).\]

Using this identity we compute \(\hat{R}_{V^*,V^*}(\tilde{u}_i \otimes f_j) = \sum_k q^{-\langle 2\rho, wt v_i - wt v_k \rangle} (\hat{R}_{V^*,V^*})_{kl}^{ij} f_k \otimes \tilde{u}_l\), that is \((\hat{R}_{V^*,V^*})_{ij}^{kl} = q^{-\langle 2\rho, wt v_i - wt v_k \rangle} (\hat{R}_{V,V^*})_{ij}^{kl}\). Putting everything together we have

\[(\hat{R}_{V,V^*})_{kl}^{ij} = q^{\langle 2\rho, wt f_j - wt f_k \rangle} (\hat{R}_{V^*,V^*})_{ij}^{lk} = q^{\langle 2\rho, wt v_i - wt v_k \rangle} q^{-\langle 2\rho, wt v_j - wt v_k \rangle} (\hat{R}_{V,V^*})_{ki}^{lj} = (\hat{R}_{V,V^*})_{ji}^{lk}.\]

\[\square\]

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