Application of the dynamic programming method to obtain of the angular velocity control law of a spacecraft with a small geometric asymmetry in the atmosphere

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Abstract. The atmospheric descent of a spacecraft with a small geometric asymmetry is considered. A quasi-linear dynamic system describes the motion of a spacecraft relative to the center of mass. The aim of the paper is to obtain the approximate optimal angular velocity control law, taking into account the change of the geometric asymmetry. We apply the method of dynamic programming and the non-resonant scheme of the averaging method for the synthesis of the optimal control law. Numerical results confirm the effectiveness of the obtained approximate law for optimal control of the angular velocity with respect to the problem of descent of the spacecraft with a small geometric asymmetry in the Martian atmosphere.

1. Introduction
It is known that it is required to comply with certain limits on the magnitude of the angular velocity at descent of a spacecraft in the atmosphere. Indeed, failure to comply with the limits on the angular velocity can lead to an emergency situation when a brake parachute system is deployed. Note that at present there is a significant amount of scientific publications considering uncontrolled atmospheric descent of the spacecraft with a small asymmetry, for example [1-5]. In addition, there are also publications on the control of the magnitude of the spacecraft asymmetry in the implementation of secondary resonance effects at atmospheric descent [6-7].

At the same time, the authors do not know the publications containing the solution of the problem of controlling the angular velocity by changing the small geometric asymmetry of the descending spacecraft, based on the application of the dynamic programming method.

The aim of the paper is to obtain an approximate optimal law for angular velocity control realized by changing the small magnitude of the geometric asymmetry.

The small geometric asymmetry consists of a small mass asymmetry and a small inertial asymmetry. In solving this optimization problem, the method of dynamic programming (the Bellman method) is applied [8]. In accordance with the Bellman method, it is required to solve the problem of finding the optimal control law the angular velocity of a spacecraft at the atmospheric stage of its descent $[t_0, T]$. The optimal law $u(t)$ should provide necessary fast-acting at the each point of the time interval $[t_0, T]$. It should be noted that a non-resonant scheme of the averaging method is used to obtain the solution of the Bellman equation [9].
2. Mathematical model

The quasi-linear equations of the motion of a spacecraft relative to the center of mass [2] with a variable magnitude of the small geometric asymmetry \(\vec{m}_x^A = u\) and a constant magnitude of the angle of attack \(\alpha\) take the form:

\[
\dot{\omega}_x = \varepsilon m_\omega \omega_x + \varepsilon \alpha \vec{m}_x^A \omega^2 \sin(\theta + \theta_1),
\]

\[
\frac{d\theta}{dt} = \omega_x - \omega_t,
\]

\[
\frac{d\omega}{dt} = \frac{\varepsilon q}{2q} \dot{q}.
\]

Here \(\varepsilon\) is a small parameter that characterizes the small magnitude of the damping torque \(m_\omega \omega_x\), the small magnitude of the geometric asymmetry parameter \(\vec{m}_x^A\) and the small magnitude of the right-hand side of the third equation (1), \(\omega_t\) is the spacecraft angular velocity relative to the axis OX, \(\theta\) is the fast phase, \(\theta = \varphi - \pi/2\), \(\varphi\) is the aerodynamic roll angle, \(\vec{m}_x^A = m_x^A / \omega^2\) is the dimensionless parameter characterizing the magnitude of mass and inertial asymmetries, \(m_x^A = \sqrt{(m_{x1}^A)^2 + (m_{x2}^A)^2}\),

\(m_{x1}^A = -\frac{\omega^2}{m_z x} C_{y1} \Delta y - \bar{T}_{xy} \omega_t^2\), \(m_{x2}^A = -\frac{\omega^2}{m_z x} C_{y1} \Delta z - \bar{T}_{xz} \omega_t^2\); \(\theta_1\) is the angle that determines the relative position of mass and inertial asymmetry; \(\sin \theta_1 = -m_{x1}^A / m_x^A; \cos \theta_1 = m_{x2}^A / m_x^A; C_{y1}\) is a coefficient of aerodynamic lift, \(m_z x\) is coefficient of restoring aerodynamic torque, \(\bar{T}_{xy} = I_{xy} / I; \ I_{xy} = I_{yx}\), \(I_{xz} = I_{xz} / I; I_{xy}\), \(I_{xz}\) is centrifugal moments of a spacecraft; \(\Delta y = \Delta y / L\), \(\Delta z = \Delta z / L\) is a small displacement of the spacecraft centre of mass; \(\omega_t = \frac{\bar{T}_{x} \omega_t}{2} + \omega_x\); \(\omega\) is the precession frequency with velocity \(\omega_t = 0\);

\(\omega = \sqrt{-m_{z1}^\alpha q S l} / l\), \(m_{z1}^\alpha\) is the partial derivative of \(m_{z1}\) with respect to \(\alpha\), \(q\) is the dynamic pressure, \(S\) is the characteristic spacecraft area, \(l\) is the characteristic spacecraft dimension; \(\omega_x - \omega_t\) is the resonant ratio of frequencies. It is known [5] that the main resonance in system (1) is realized when the following equality is satisfied \(\omega_x - \omega_t = 0\).

Numerical results show that the action of the torque caused by the geometric asymmetry and the damping torque does not lead to the main resonance in the system (1). This fact can be proved by using the conditions of realization of a long resonance [2]. Thus, in this paper we consider only the nonresonance case, which is realized without a long-term realization of equality \(\omega_x - \omega_t = 0\).

3. Optimal control law

Let us assume \(\omega_0 = \omega(t_0)\) to be the initial value of the angular velocity. Need to find control \(u_0\), which takes the angular velocity \(\omega_0 = \omega(t_0)\) in a position \(\omega_1 = \omega(T) = 0\) in the minimum time. In this case, the optimality criterion takes the form [10]:

\[
I = \varepsilon \int_0^T (C_1 \omega_x^2 + C_2 u^2) dt,
\]

(2)
where $C_1, C_2$ are weight coefficients of the criterion (2).

Note that the Lyapunov function is $V = C_1\omega_x^2 + C_2u^2$.

We use the Bellman method to solve the optimization problem. Thus, for system (1), taking into account the criterion (2), we obtain the following equation:

$$\min_u \left\{ \varepsilon C_1\omega_x^2 + \varepsilon C_2u^2 + \frac{\partial V}{\partial \omega_x} \frac{d\omega_x}{dt} + \frac{\partial V}{\partial \theta} \frac{d\theta}{dt} \right\} = 0.$$ (3)

Here $v(\omega_x, \theta)$ is the generating function.

By substituting in (3) the derivatives $\frac{d\omega_x}{dt}, \frac{d\theta}{dt}$ from Eqs. (1):

$$\min_u (\varepsilon b\omega_x^2 + \varepsilon cu^2 + \frac{\partial V}{\partial \omega_x} - \varepsilon \frac{m_\omega \omega_x}{I_x} + \varepsilon \frac{u\omega^2}{I_x} \sin \theta + \theta_1 + \frac{\partial V}{\partial \theta} \Delta) = 0.$$ (4)

By separating in Eq.(4) the terms that depend on the control $u$, we obtain the following function:

$$F(u) = \varepsilon C_2u^2 - \varepsilon \frac{\partial V}{\partial \omega_x} \alpha u \omega^2 \sin \theta + \theta_1.$$ (5)

A necessary minimum condition of the function (5) takes the form:

$$\frac{\partial F}{\partial u} = 2\varepsilon C_2u - \varepsilon \frac{\partial V}{\partial \omega_x} \alpha u \omega^2 \sin \theta + \theta_1 = 0.$$ (6)

It should be noted that a sufficient condition for a minimum the function (5) is fulfilled. Indeed, we obtain

$$\frac{\partial^2 F}{\partial u^2} = 2\varepsilon C_2 > 0.$$ (7)

Thus, we find the optimal control $u^0$ from solution of the Eq.(6) as follows:

$$u^0 = \frac{2\omega^2 \sin(\theta + \theta_1)}{2I_1C_2} \frac{\partial V}{\partial \omega_x}.$$ (8)

Substituting optimal control (8) into the equation (4), we find the Bellman equation:

$$\varepsilon C_1\omega_x^2 - \varepsilon C_2 \left( \frac{\partial V}{\partial \omega_x} \right)^2 \frac{\omega^2 \sin^2 \theta + \theta_1}{4I_1^2C_2^2} + \varepsilon \left( \frac{\partial V}{\partial \omega_x} \right) \frac{m_\omega \omega_x}{I_x} + \varepsilon \left( \frac{\partial V}{\partial \omega_x} \right)^2 \frac{\alpha \omega^2 \sin^2 \theta + \theta_1}{4I_1^2C_2^2} + \frac{\partial V}{\partial \theta} \Delta = 0.$$ (9)

Further, we seek a solution of (9), taking into account (1) in the form of following series of the non-resonant scheme of the averaging method:
\[ \omega_x = \omega_x^0 + \varepsilon \omega_x^1, \theta^0 + \ldots, \]
\[ \theta = \theta^0 + \varepsilon \theta_1, \omega_x^0, \theta^0 + \ldots, \]
\[ \nu = \nu^0 + \varepsilon \nu_1, \omega_x^0, \theta^0 + \ldots. \] \hspace{1cm} (10)

Here \( \omega_x^0, \theta^0 \) are the averaged variables of system (1), \( u_x^0, \omega_x^0, \theta^0, \theta_1, \omega_x^0, \theta^0, \nu^0, \omega_x^0, \nu_1, \omega_x^0, \theta^0, \) \( j = 1, 2, \ldots \) are the function to be determined.

In accordance with this non-resonant scheme of the averaging method, we obtain:
\[ \left\{ \omega_x^1, \omega_x^0, \theta^0 \right\}_{\theta^0} = \left\{ \theta_1, \omega_x^0, \theta^0 \right\} = \left\{ \nu_1, \omega_x^0, \theta^0 \right\}_{\theta^0} = 0, \]
\[ \frac{\partial \nu}{\partial \theta^0} = \varepsilon \frac{\partial \nu_1}{\partial \theta^0} + \varepsilon^2 \ldots. \] \hspace{1cm} (11)

Substituting the asymptotic series (10) into the Bellman equation (9) and taking into account (11), we obtain up to terms of the second order of smallness:
\[ \varepsilon C_1 \omega_x^0 - \varepsilon C_2 \left( \frac{\partial \nu^0}{\partial \omega_x^0} \right)^2 \frac{\omega_x^0}{C_1} \sin \theta^0 + \theta_1 + \varepsilon \left( \frac{\partial \nu^0}{\partial \omega_x^0} \right) \frac{m_0 \omega_x^0}{I_x} + \varepsilon^2 \frac{\partial \nu_1}{\partial \theta^0} \Delta = 0. \] \hspace{1cm} (12)

Here the first approximation of the averaging method is:
\[ C_1 \omega_x^0 - \frac{\left( \frac{\partial \nu^0}{\partial \omega_x^0} \right)^2 \alpha_x^2 \omega_x^0}{C_2} + \frac{\partial \nu^0}{\partial \omega_x^0} \frac{m_0 \omega_x^0}{I_x} = 0. \] \hspace{1cm} (13)

We seek a generating solution of Eq. (13), applying the method of indeterminate coefficients [10]:
\[ \nu^0 = A \omega_x^0. \] \hspace{1cm} (14)

In order to obtain this result, we substitute the solution (14) in Eq. (13):
\[ \omega_x^0 \left[ C_1 - \frac{A^2 \omega_x^0}{2 \frac{1}{I_x} C_2} + \frac{2 A m_0}{I_x} \right] = 0. \] \hspace{1cm} (15)

Assuming that \( \alpha^0 \neq 0 \), we obtain from Eq. (15):
\[ b_1 A^2 + b_2 A + b_3 = 0. \] \hspace{1cm} (16)

Here \( b_1 = -\frac{\omega_x^0}{2 \frac{1}{I_x} C_2}, b_2 = \frac{2 m_0}{I_x}, b_3 = C_1 \).

In what follows we omit the subscript "o" for the averaged values of the variables, the control (8) and the generating solutions (14). We solve the equation (16). As a result, we find the two roots
Among the solutions (17), only positive ones should be chosen, since these solutions ensure the positive definiteness of the generating function, which is also the Lyapunov function $\nu = A\omega_\lambda^2$. We will consider the roots (17) in more detail:

$$A_{1,2} = \frac{-b_2 \pm \sqrt{b_2^2 - 4b_1b_3}}{2b_1}. \quad (17)$$

Here $m_\omega < 0$ is damping coefficient. Consequently, the condition $A_2 > |A_1|$ is fulfilled. Note, that $A_2 > 0$, $A_1 < 0$. Consequently, from two solutions (18) we choose the positive solution $A_2$.

In addition, the condition of non-negativity of the discriminant in solution (18) must also be fulfilled. We substitute the generating solution (14) into the control law (8):

$$u = \frac{\alpha \omega_\lambda^2 \sin \theta + \theta \lambda_1}{C_2} \frac{A\omega_\lambda}{I_x}. \quad (19)$$

Further, we substitute control law (20) into the first equation (1). Taking into account the terms of the first approximation, we obtain:

$$\frac{d\omega_\lambda}{dt} = \varepsilon \frac{\omega_\lambda m_\omega}{I_x} - \varepsilon \frac{A\alpha_\lambda^2 \omega_\lambda^4}{2I_x} \omega_\lambda. \quad (20)$$

It follows from equation (20) that in order to achieve the asymptotic stability of the point $\omega_\lambda = 0$, the following condition is fulfilled:

$$\left\{\frac{m_\omega - \alpha_\lambda^2 \omega_\lambda^4}{I_x} - \frac{2\alpha_\lambda^2 \omega_\lambda^4}{2I_x^2} \right\} < 0. \quad (21)$$

Consequently, the second term in the numerator of solution $A$ in expression (19) satisfies the condition:

$$-\frac{1}{2} \sqrt{\frac{4\omega_\lambda^2}{I_x} + \frac{2\alpha_\lambda^2 \omega_\lambda^4 C_1}{I_x^2 C_2}} < 0. \quad (22)$$
4. Numerical results
We conducted a numerical simulation of the motion of the spacecraft with a small mass and small inertial asymmetry in the problem of descent in the Martian atmosphere. When obtaining numerical results, a quasilinear system of equations of relative motion of the spacecraft (1) was applied. The system (1) was solved numerically together with the system of three equations of motion of the center of mass for the velocity of the center of mass, the altitude of flight and the angle of inclination of the trajectory [1]. It was assumed that the descent spacecraft had mass-inertial characteristics corresponding to «Mars Polar Lander» [11]: the largest radius of cone r = 1,25 m, the height of cone l = 2 m, the mass m=576 kg, the axial moments of inertia Ix = 270 kgm², Iy = 443 kgm², Iz = 443 kgm². In this case, the spacecraft descends in the atmosphere of Mars (the average radius of Mars is R₀ = 3390 km). The average acceleration of gravity on Mars is equal to g₀=3,86 ms⁻². The initial conditions at the entrance to the Martian atmosphere are equal: the velocity of the center of mass V(0) = 3400 ms⁻¹, the altitude of the flight H(0) = 120 km, the angle of inclination of the trajectory θ(0) = -0.017 rad.

Figure 1 shows the evolution in the angular velocity of the spacecraft reentry «Mars Polar Lander» when uncontrolled descent in the atmosphere of Mars considering the effect of damping torque and geometric asymmetry.

Figure 2 shows the evolution in the angular velocity of the descending spacecraft "Mars Polar Lander" in the Martian atmosphere with optimal control, performed in accordance with the approximate expression (19).

**Figure 1.** Evolution of the angular velocity, taking into account the aerodynamic damping without optimal control

**Figure 2.** Evolution of the angular velocity, taking into account the aerodynamic damping and optimal control
From comparing the results of numerical integration to Figures 1 and 2 it follows that the approximate optimum control angular velocity «Mars Polar Lander» spacecraft by expression (19) leads to a reduction of the angular velocity to a value less than 0.1 s\(^{-1}\) at four times faster than it occurs in the absence of control.

Consequently, the problem of approximate optimal control of the angular velocity of the descent vehicle with a small variable geometric asymmetry is completely solved.

5. Conclusion

Thus, the application of the Bellman method in combination with the non-resonant averaging scheme made it possible to synthesize an approximate fast-acting optimal control of the angular velocity of a spacecraft with a small variable geometric asymmetry during its descent in the atmosphere. Numerical results confirm that the found new approximate expression for the optimal control law provides an effective speed reduction of the angular velocity of the descent of the spacecraft to small quantities.

It should be noted that the expression for approximate control law assumes that the angle of attack of the spacecraft was unchanged, and the angular velocity was positive. By analogy, we can consider the case of the negative angular velocity of a spacecraft having a constant value of the angle of attack.

In addition, it is of practical interest to consider the case of controlling the orientation of a spacecraft with a small variable asymmetry at a varying angular velocity and a non-constant angle of attack. However, these studies are beyond the scope of this paper. However, they can be considered in further publications.

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