Symplectic geometries on supermanifolds

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Extension of symplectic geometry on manifolds to the supersymmetric case is considered. In the even case it leads to the even symplectic geometry (or, equivalently, to the geometry on supermanifolds endowed with a non-degenerate Poisson bracket) or to the geometry on an even Fedosov supermanifolds. It is proven that in the odd case there are two different scalar symplectic structures (namely, an odd closed differential 2-form and the antibracket) which can be used for construction of symplectic geometries on supermanifolds.

1 Introduction

It is well-known that methods of symplectic geometry play very important role in the formulation of classical mechanics on manifolds (see, for example, [1]). Deformation quantization [2] is formulated in terms of symplectic manifolds with a symmetric connection compatible with the given symplectic structure (the so-called Fedosov manifolds [3]). Formulation of supersymmetric field theories, quantization of general gauge theories introduced a number of applications of differential geometry based on the notation of a supermanifold introduced and studied by Berezin [4]. In these cases, a supermanifold must be endowed with an appropriate symplectic structure or (and) a symmetric connection. Thus, investigating the geometrical contents of the well-known Batalin-Vilkovisky quantization [5] is based on using of the so-called antisymplectic supermanifolds which are supermanifolds equipped with the antibracket [6]. In several specific investigations in modern gauge field theory [7], flat even Fedosov supermanifolds (in the terminology adopted here) have been used.

Our aim of this work is to study extension of symplectic geometry on manifolds to supersymmetric case. In the even case it leads to the even symplectic geometry which is formulated on a supermanifold equipped with an even closed non-degenerate differential 2-form (a symplectic structure). It is equivalent to the geometry based on a supermanifold equipped with the non-degenerate Poisson bracket. If, in addition, a given symplectic supermanifold is endowed with a symmetric connection (covariant derivative) compatible with the symplectic structure, one has an even Fedosov supermanifolds which can be considered as generalization of Fedosov manifolds [3]. As for Fedosov manifolds the scalar curvature tensor for even Fedosov supermanifolds vanishes. It is proven that in the odd case there are two scalar symplectic structures which can be used for construction of symplectic geometries on supermanifolds. First, one can equip a supermanifold with an odd closed non-degenerate differential 2-forms to get an odd symplectic supermanifold. Second, one can equip a supermanifold with the antibracket (the antisymplectic structure) to get an antisymplectic supermanifolds. Moreover, if we equip an odd symplectic supermanifold with a symmetric connection compatible with the odd symplectic structure, we get the geometry in which the scalar curvature tensor is identically equal to zero. Situation is more interesting when an antisymplectic supermanifold is equipped with a symmetric connection compatible with a given antisymplectic structure. We prove that in this case the scalar curvature tensor is not, in general, equal to zero. However, the scalar curvature tensor squared is identically equal to zero.

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The paper is organised as follows. In Sect. 2, we study multiplication, contraction and symmetry properties of tensor fields on supermanifolds. In Sect. 3, we consider scalar structures which can be used for constructions of symplectic geometries on supermanifolds. In Sect. 4, we discuss symmetric affine connections and properties of their curvature tensors on supermanifolds. In Sect. 5, we present the notion of the even symplectic geometry. In Sect. 6, we introduce the notions of the odd symplectic geometry and the antisymplectic geometry and study their basic properties. In Sect. 7, we give a short summary.

We use the condensed notation suggested by DeWitt [8] and definitions and notations adopted in [9]. Derivatives with respect to the coordinates $x^i$ are understood as acting from the left and are standardly denoted by $\partial A/\partial x^i$. Right derivatives with respect to $x^i$ are labeled by the subscript "r" or the notation $A_{,i} = \partial_r A/\partial x^i$ is used. The Grassmann parity of any quantity $A$ is denoted by $\epsilon(A)$.

### 2 Tensor fields

In this section, we give the basic definitions and relations in tensor calculus on supermanifolds, to be used in calculations in what follows.

Let the variables $x^i, \epsilon(x^i) = \epsilon_i$ be local coordinates on a supermanifold $M, \text{dim} M = N$, in the vicinity of a point $P \in M$. Let the sets $\{e_i\}$ and $\{e^i\}$ be coordinate bases in the tangent space $T_PM$ and the cotangent space $T^*_PM$, respectively. If one goes over to another set $\bar{x}^i = \bar{x}^i(x)$ of local coordinates, the basis vectors in $T_PM$ and $T^*_PM$ transform as

$$\bar{e}_i = e_j \frac{\partial x^j}{\partial \bar{x}^i}, \quad \bar{e}^i = e^j \frac{\partial \bar{x}^i}{\partial x^j}. \quad (1)$$

For the transformation matrices the following relations hold:

$$\frac{\partial x^k}{\partial \bar{x}^j} \frac{\partial \bar{x}^i}{\partial x^k} = \delta^i_j, \quad \frac{\partial x^k}{\partial x^j} \frac{\partial x^i}{\partial \bar{x}^k} = \delta^i_j, \quad \frac{\partial \bar{x}^i}{\partial \bar{x}^k} \frac{\partial \bar{x}^j}{\partial x^k} = \delta^j_i, \quad \frac{\partial \bar{x}^i}{\partial x^j} \frac{\partial x^k}{\partial \bar{x}^i} = \delta^j_i. \quad (2)$$

A tensor field of type $(n, m)$ and rank $n + m$ is defined as a geometric object given by a set of functions with $n$ upper and $m$ lower indices in each local coordinate system $(x) = (x^1, \ldots, x^N)$ with certain transformation laws. We omit the general definition (see [9]) and restrict ourself to cases of vector fields $T^i$ and co-vector fields $\bar{T}_i$

$$T^i = T^n \frac{\partial x^i}{\partial x^n}, \quad \bar{T}_i = T^n \frac{\partial x^n}{\partial \bar{x}^i} \quad (3)$$

and of second-rank tensor fields of different types

$$\bar{T}^{ij} = T^{mn} \frac{\partial \bar{x}^j}{\partial x^m} \frac{\partial x^n}{\partial \bar{x}^i} (-1)^{\epsilon_j(\epsilon_i + \epsilon_m)}, \quad (4)$$

$$T_{ij} = T_{mn} \frac{\partial x^n}{\partial \bar{x}^j} \frac{\partial \bar{x}^m}{\partial x^i} (-1)^{\epsilon_j(\epsilon_i + \epsilon_m)}, \quad (5)$$

$$\bar{T}_{ij} = T_{mn} \frac{\partial x^n}{\partial x^i} \frac{\partial x^m}{\partial \bar{x}^j} (-1)^{\epsilon_j(\epsilon_i + \epsilon_m)}. \quad (6)$$

Note that the unit matrix $\delta^i_j$ is related to the unit tensor field $E^i_j$ transforming in accordance with (6) as

$$E^i_j = \delta^i_j. \quad (7)$$

From a tensor field of type $(n, m)$ and rank $n + m$, where $n \neq 0, m \neq 0$, one can construct a tensor field of type $(n - 1, m - 1)$ and rank $n + m - 2$ by the contraction of an upper and a lower index by the rules (for details, see [9]). In particular, for tensor fields of type $(1, 1)$, the contraction gives the supertrace,

$$T^{i}_{i} (-1)^{\epsilon_i}. \quad (8)$$
Using the multiplication operation, from two tensor fields of types \((n, 0)\) and \((0, m)\), one can construct new tensor fields of type \((n - 1, m - 1)\). In particular, vector \(U^i\) and covector \(V_i\) fields thus yield a scalar field

\[
(-1)^{\epsilon_i(\epsilon(V)+1)} U^i V_i = (-1)^{\epsilon(U)\epsilon(V)+\epsilon_i\epsilon(U)} V_i U^i,
\]

which is invariant with respect to the choice of local coordinates. Two second-rank tensor fields \(U_{ij}\) and \(V_{ij}\) yield the tensor fields

\[
(-1)^{\epsilon_i+\epsilon_k}\epsilon(V)+\epsilon_k U^{ik} V_{kj}
\]

and

\[
(-1)^{\epsilon_i+\epsilon_k}\epsilon(V)+\epsilon_k(\epsilon_i+\epsilon_j+1) U^{ki} V_{jk}
\]

transforming in accordance with (6). Further contracting indices yields the scalar

\[
(-1)^{\epsilon_i+\epsilon_k}\epsilon(V)+1 U^{ik} V_{ki} = (-1)^{\epsilon(U)\epsilon(V)+(\epsilon_i+\epsilon_k)\epsilon(U)} V_{ik} U^{ki}.
\]

Moreover, recalling (11) and (10), the inverse tensor field \(T_{ij}\) for a non-degenerate second-rank tensor field \(T^{ij}\) of type \((2, 0)\) should be defined via the relations

\[
(-1)^{\epsilon_i+\epsilon_k}\epsilon(T)+\epsilon_k T^{ik} T_{kj} = \delta^i_j,
\]

\[
(-1)^{\epsilon_i+\epsilon_k}\epsilon(T)+\epsilon_j T^{ik} T_{jk} = \delta^i_j,
\]

\[
\epsilon(T_{ij}) = \epsilon(T^{ij}) = \epsilon(T) + \epsilon_i + \epsilon_j,
\]

and similarly for tensor fields of type \((0,2)\).

It is well known that in constructing a tensor calculus on manifold, an important role is played by symmetric and antisymmetric tensor fields. In the supersymmetric case, supermatrices have more possible symmetry properties (eight types [10]), and a natural question is whether these properties are compatible with the tensor transformation laws. Among the eight types of supermatrices with possible symmetry properties there exist only two ones satisfying tensor transformation laws. In our definition of tensor fields on supermanifolds, only the supermatrices having the generalized symmetry or antisymmetry properties satisfy the tensor transformation laws. Indeed, let us consider a second-rank supermatrix of type \((2, 0)\) having the generalized symmetry (antisymmetry) property

\[
T^{ij} = (-1)^{\epsilon_i\epsilon_j} T^{ji} \quad (T^{ij} = (-1)^{\epsilon_i\epsilon_j} T^{ji}).
\]

This property is compatible with the transformation law [11],

\[
T^{ij} = T^{mn} \frac{\partial \bar{x}^j}{\partial x^m} \frac{\partial \bar{x}^i}{\partial x^n} (-1)^{\epsilon_j(\epsilon_i+\epsilon_m)} = T^{mn} \frac{\partial \bar{x}^i}{\partial x^m} \frac{\partial \bar{x}^j}{\partial x^n} (-1)^{\epsilon_i\epsilon_n} = (-1)^{\epsilon_i\epsilon_j} T^{ji}
\]

and similarly for antisymmetry property. Other possible symmetry types of supermatrices do not survive verification of the compatibility with adopted tensor transformation laws. We note that for non-degenerate symmetric and antisymmetric tensor fields, their inverse tensor fields also have the necessary symmetry properties. For example, we consider a second-rank tensor field \(T^{ij}\) with symmetry (antisymmetry) property. From definition (12) and (13), we can then find that the inverse tensor field

\[
T_{ij} = (-1)^{\epsilon_i\epsilon_j+(T)} T^{ji} \quad (T_{ij} = (-1)^{\epsilon_i\epsilon_j+(T)} T^{ji})
\]

also has the generalized symmetry (antisymmetry) property.

### 3 Scalar structures on supermanifolds

In this section, we discuss important scalar structures on supermanifolds which can be defined in terms of symmetric and antisymmetric tensor fields. Namely, we are going to consider definitions and basic properties of the super-Poisson bracket, the antibracket and the differential 2-form which are main objects in formulation of Quantum Field Theory.
The Poisson bracket is defined on a supermanifold $M$ with even dimension for any two scalar functions $A$ and $B$ as an even bilinear operation $\{A, B\}$, $\epsilon(\{A, B\}) = \epsilon(A) + \epsilon(B)$ having the generalized antisymmetry property

$$\{A, B\} = -(-1)^{\epsilon(A)\epsilon(B)}\{B, A\}$$  \hspace{1cm} (16)

and obeying the Jacobi identity

$$\{A, \{B, C\}\}(\epsilon(A) + 1) + \text{cycle}(A, B, C) \equiv 0.$$ \hspace{1cm} (17)

We can define the Poisson bracket by the relation

$$\{A, B\} = \partial_x A \omega^{ij} \frac{\partial B}{\partial x^j}, \quad \epsilon(\omega^{ij}) = \epsilon_i + \epsilon_j.$$  \hspace{1cm} (18)

If $\omega^{ij}$ is a second-rank tensor field of type $(2, 0)$ then this definition gives the invariance of the Poisson bracket under local coordinate transformations $x \rightarrow \bar{x}$, $\{\bar{A}, \bar{B}\} = \{A, B\}$. If $\omega^{ij}$ has the generalized antisymmetry property

$$\omega^{ij} = -(-1)^{\epsilon_i\epsilon_j}\omega^{ji}$$  \hspace{1cm} (19)

then the definition (18) reproduce the property (16). In terms of $\omega^{ij}$ the Jacobi identity means fulfilment of the following relations

$$\partial_x \omega^{jk} \omega^{ij}_n(-1)^{\epsilon_i\epsilon_k} + \text{cycle}(i, j, k) \equiv 0.$$  \hspace{1cm} (20)

Now, suppose that the tensor field $\omega^{ij}$ is non-degenerate. We can introduce the inverse tensor field $\omega_{ij}$ which has also the generalized antisymmetry property

$$\omega_{ij} = -(-1)^{\epsilon_i\epsilon_j}\omega^{ji}.$$  \hspace{1cm} (21)

In terms of $\omega_{ij}$, the Jacobi identity (20) can be rewritten in the form

$$\omega_{ijk}(-1)^{\epsilon_i\epsilon_k} + \text{cycle}(i, j, k) \equiv 0.$$  \hspace{1cm} (22)

The tensor field $\omega_{ij}$ defines the differential $2$-form on the supermanifold $M$

$$\omega = \omega_{ij} dx^j \wedge dx^i, \quad dx^j \wedge dx^i = -(-1)^{\epsilon_i\epsilon_j} dx^i \wedge dx^j, \quad \epsilon(\omega) = 0$$  \hspace{1cm} (23)

which is invariant under a change of the local coordinates, $\bar{\omega} = \omega$. The external derivative is given by

$$d\omega = \omega_{ij,k} dx^k \wedge dx^j \wedge dx^i.$$  \hspace{1cm} (24)

It is also invariant under a change of the local coordinates, $d\bar{\omega} = d\omega$. Moreover, due to the identities (22) the differential non-degenerate $2$-form $\omega$ (23) is closed

$$d\omega = 0.$$  \hspace{1cm} (25)

Therefore, any non-degenerate super-Poisson bracket on a supermanifold defines an even non-degenerate closed differential $2$-form and via verse.

The antibracket is defined for any two scalar functions $A$ and $B$ as an odd bilinear operation $(A, B)$, $\epsilon((A, B)) = \epsilon(A) + \epsilon(B) + 1$ having the generalized antisymmetry property

$$(A, B) = -(-1)^{\epsilon(A)+1}(\epsilon(B)+1)(B, A)$$  \hspace{1cm} (26)

and obeying the Jacobi identity

$$(A, (B, C))(-1)^{\epsilon(A)+1}(\epsilon(C)+1) + \text{cycle}(A, B, C) \equiv 0.$$  \hspace{1cm} (27)
We can define the antibracket by the relation
\[
(A, B) = \frac{\partial_x A}{\partial x^i} (-1)^{\epsilon_i \epsilon_j} \frac{\partial B}{\partial x^j}, \quad \epsilon(\Omega^{ij}) = \epsilon_i + \epsilon_j + 1.
\] (28)

If \(\Omega^{ij}\) is a second-rank tensor field of type \((2,0)\) then this definition leads to the invariance of the antibracket under local coordinate transformations \(x \to \tilde{x}\), \((\tilde{A}, \tilde{B}) = (A, B)\). If \(\Omega^{ij}\) has the generalized symmetry property
\[
\Omega^{ij} = (-1)^{\epsilon_i \epsilon_j} \Omega^{ji}
\] (29)
then the definition \((28)\) reproduce the property \((26)\). In terms of \(\Omega^{ij}\) the Jacobi identity means fulfilment of the following relations
\[
\Omega^{in} \frac{\partial \Omega^{jk}}{\partial x^n} (-1)^{\epsilon_i (\epsilon_k + 1)} + \text{cycle}(i,j,k) \equiv 0.
\] (30)

When the tensor field \(\Omega^{ij}\) \((29)\) is non-degenerate then the inverse tensor field \(\Omega_{ij}\) has the generalized antisymmetry property
\[
\Omega_{ij} = -(-1)^{\epsilon_i \epsilon_j} \Omega_{ji}, \quad \epsilon(\Omega_{ij}) = \epsilon_i + \epsilon_j + 1.
\] (31)

In terms of \(\Omega_{ij}\) the Jacobi identity \((30)\) can be rewritten in the form
\[
\Omega_{ij,k} (-1)^{\epsilon_k (\epsilon_i + 1)} + \text{cycle}(i,j,k) \equiv 0.
\] (32)

We can also introduce an odd closed non-degenerate differential 2-form on a supermanifold by the relation
\[
\omega = \omega_{ij} \, dx^j \wedge dx^i, \quad dx^j \wedge dx^i = -(-1)^{\epsilon_i \epsilon_j} \, dx^i \wedge dx^j, \quad \epsilon(\omega) = 1, \quad \omega_{ij} = -(-1)^{\epsilon_i \epsilon_j} \omega_{ji}
\] (33)
which has formally the same properties as in even case. In particular, in terms of tensor field \(\omega_{ij}\) the closure of \(\omega\) \((d\omega = 0)\) has the form \((22)\). We see that in the odd case any antibracket defines an odd closed differential 2-form and therefore an antisymplectic supermanifold should be considered as an odd symplectic supermanifold.

4 Covariant derivatives and curvature tensor

As in the case of tensor analysis on manifolds, on a supermanifold \(M\) one can introduce the covariant derivation (or affine connection) as a mapping \(\nabla\) (with components \(\nabla_i, \epsilon(\nabla_i) = \epsilon_i\)) from the set of tensor fields on \(M\) to itself by the requirement that it should be a tensor operation acting from the right and adding one more lower index and, when it is possible locally to introduce Cartesian coordinates on \(M\), that it should reduce to the usual (right–)differentiation. It allows to construct the action of covariant derivatives on tensor fields of different types. In particular, they are given as local operations acting on scalar, vector and co-vector fields by the rules
\[
T \nabla_i = T_{i,i},
\] (34)
\[
T^i \nabla_j = T^i_{j,j} + T^k \Gamma^i_{kj} (-1)^{\epsilon_k (\epsilon_i + 1)},
\] (35)
\[
T_i \nabla_j = T_{i,j} - T_k \Gamma^k_{ij},
\] (36)
and on second-rank tensor fields of type \((2,0)\), \((0,2)\) and \((1,1)\) by the rules
\[
T^{ij} \nabla_k = T^{ij}_{k,k} + T^{il} \Gamma^j_{lk} (-1)^{\epsilon_i (\epsilon_j + 1)} + T^{lj} \Gamma^i_{lk} (-1)^{\epsilon_i \epsilon_j + \epsilon_i (\epsilon_i + \epsilon_j + 1)},
\] (37)
\[
T_{ij} \nabla_k = T_{ij,k} - T_{il} \Gamma^j_{lk} - T_{lj} \Gamma^i_{lk} (-1)^{\epsilon_i \epsilon_j + \epsilon_i (\epsilon_j + 1)},
\] (38)
\[
T^i_j \nabla_k = T^i_{j,k} - T^i_j \Gamma^l_{jk} + T^l_j \Gamma^i_{lk} (-1)^{\epsilon_i \epsilon_j + \epsilon_i (\epsilon_i + \epsilon_j + 1)}.
\] (39)
Here, $\Gamma^i_{jk}$ are affine connection components. Similarly, the action of the covariant derivative on a tensor field of any rank and type is given in terms of their tensor components, their ordinary derivatives and the connection components.

In general, the connection components $\Gamma^i_{jk}$ do not have the property of (generalized) symmetry w.r.t. the lower indices. The deviation from this symmetry is the torsion, the connection components.

$$T^i_{jk} := \Gamma^i_{jk} - (-1)^{\epsilon_j \epsilon_k} \Gamma^i_{kj},$$

which transforms as a tensor field. If a supermanifold $M$ is torsionless, i.e., if a connection obey the relation

$$\Gamma^i_{jk} = (-1)^{\epsilon_j \epsilon_k} \Gamma^i_{kj},$$

then one says that a symmetric connection is defined on $M$. Here, we consider only symmetric connections.

The curvature tensor $R^i_{mjk}$ of a given symmetric connection is defined in a coordinate basis by the action of the commutator of covariant derivatives, $[\nabla_i, \nabla_j] = \nabla_i \nabla_j - (-1)^{\epsilon_i \epsilon_j} \nabla_j \nabla_i$, on a vector field $T^i$ as

$$T^i[\nabla_j, \nabla_k] = -(-1)^{\epsilon_m (\epsilon_i + 1)} T^m R^i_{mjk}.$$  

The choice of factor in r.h.s [42] is dictated by the requirement for product of tensor fields of types $(1,0)$ and $(1,3)$ to be a tensor field of type $(1,2)$. A straightforward calculation yields

$$R^i_{mjk} = -\Gamma^i_{mk,j} + \Gamma^i_{mk,j}(-1)^{\epsilon_j \epsilon_k} + \Gamma^i_{jn} \Gamma^m_{mk}(-1)^{\epsilon_j \epsilon_m} - \Gamma^i_{kn} \Gamma^n_{mj}(-1)^{\epsilon_k \epsilon_m + \epsilon_j}.$$  

The curvature tensor field possesses the following generalized antisymmetry property,

$$R^i_{mjk} = -(-1)^{\epsilon_j \epsilon_k} R^i_{mkj};$$

furthermore, it obeys the Jacobi identity,

$$(-1)^{\epsilon_m \epsilon_k} R^i_{mjk} + (-1)^{\epsilon_j \epsilon_m} R^i_{jkm} + (-1)^{\epsilon_k \epsilon_j} R^i_{kmj} \equiv 0.$$  

Using the Jacobi identity for the covariant derivatives,

$$[\nabla_i, [\nabla_j, \nabla_k]] (-1)^{\epsilon_i \epsilon_k} + [\nabla_k, [\nabla_i, \nabla_j]] (-1)^{\epsilon_k \epsilon_j} + [\nabla_j, [\nabla_k, \nabla_i]] (-1)^{\epsilon_j \epsilon_i} \equiv 0,$$

one obtains the Bianchi identity,

$$(-1)^{\epsilon_i \epsilon_j} R^n_{mjk;i} + (-1)^{\epsilon_i \epsilon_k} R^n_{mij;k} + (-1)^{\epsilon_k \epsilon_j} R^n_{mk;i,j} \equiv 0,$$

with the notation $R^n_{mjk;i} := R^n_{mjk} \nabla_i$.

### 5 Even symplectic geometry

Suppose now that we are given a supermanifold $M$ of an even dimension, $\dim M = 2n$. Let $\omega$ be an even non-degenerate differential 2-form [23] on $M$. Then, the pair $(M, \omega)$ is called an even almost symplectic supermanifold; it is called an even symplectic supermanifold if $\omega$ is closed, $d\omega = 0$. The inverse tensor field $\omega^{-1}$ defines the non-degenerate Poisson bracket. Supermanifolds equipped with this structure are called non-degenerate Poisson supermanifolds. From the above considerations it follows that, as in the case of ordinary symplectic geometry on manifolds, there exists one-to-one correspondence between an even symplectic supermanifold and the non-degenerate Poisson supermanifold.

Let $\nabla$ (or $\Gamma$) be a covariant derivative (a symmetric connection) on $M$ which preserves the 2-form $\omega$, $\omega \nabla = 0$. In a coordinate basis this requirement reads

$$\omega_{ij,k} - \omega_{im} \Gamma^n_{jk} + \omega_{jm} \Gamma^n_{ik} (-1)^{\epsilon_i \epsilon_j} = 0.$$  

(48)
If, in addition, $\Gamma$ is symmetric then we have an even symplectic connection (or symplectic covariant derivative) on $M$. Now, an even Fedosov supermanifold $(M, \omega, \Gamma)$ is defined as an even symplectic supermanifold with a given even symplectic connection.

Let us introduce the curvature tensor of an even symplectic connection, 

$$R_{ijkl} = \omega_{in} R^n_{jkl}, \quad \epsilon(R_{ijkl}) = \epsilon_i + \epsilon_j + \epsilon_k + \epsilon_l,$$  

where $R^n_{jkl}$ is given by (43). This leads to the following representation, 

$$R_{imjk} = -\omega_{in} \Gamma^n_{mj,k} + \omega_{in} \Gamma^n_{mk,j}(-1)^{\epsilon_j \epsilon_k} + \Gamma_{ijn} \Gamma^n_{mk}(-1)^{\epsilon_j \epsilon_m} - \Gamma_{ikn} \Gamma^n_{mj}(-1)^{\epsilon_k(\epsilon_m + \epsilon_j)},$$  

where we used the notation  

$$\Gamma_{ijk} = \omega_{in} \Gamma^n_{jk}, \quad \epsilon(\Gamma_{ijk}) = \epsilon_i + \epsilon_j + \epsilon_k.$$  

Using this, the relation (48) reads  

$$\omega_{ij,k} = \Gamma_{ijk} - \Gamma_{jik}(-1)^{\epsilon_i \epsilon_j}.$$  

Furthermore, from Eq. (43) it is obvious that  

$$R_{ijkl} = (-1)^{\epsilon_i \epsilon_j} R_{ijlk},$$  

and, using (43) and (45), one deduces the Jacobi identity for $R_{ijkl}$,  

$$(-1)^{\epsilon_i \epsilon_j} R_{ijkl} + (-1)^{\epsilon_i \epsilon_k} R_{iljk} + (-1)^{\epsilon_k \epsilon_l} R_{iklj} = 0.$$  

In addition, the curvature tensor $R_{ijkl}$ is generalized symmetric w.r.t. the first two indices,  

$$R_{ijkl} = (-1)^{\epsilon_i \epsilon_j} R_{jikl}.$$  

In order to prove this, let us consider the relations which follow from (48)  

$$\omega_{ij,k,l} = \Gamma_{ijkl} - \Gamma_{jikl}(-1)^{\epsilon_i \epsilon_j}.$$  

Then, using the relations  

$$\Gamma_{ijkl} = \omega_{in} \Gamma^n_{jkl} + \omega_{in} \Gamma^n_{jkl}(-1)^{\epsilon_i + \epsilon_j + \epsilon_k} \epsilon_l$$  

and the definitions (50) and (52), we get  

$$0 = \omega_{ij,k,l} - (-1)^{\epsilon_i \epsilon_j} \omega_{ij,kl} = \Gamma_{ijkl} - \Gamma_{jikl}(-1)^{\epsilon_i \epsilon_j} + \Gamma_{jikl}(-1)^{\epsilon_i \epsilon_j + \epsilon_i \epsilon_l} = -R_{ijkl} + (-1)^{\epsilon_i \epsilon_j} R_{jikl}.$$  

For any even symplectic connection there holds the identity  

$$R_{ijkl} + (-1)^{\epsilon_i(\epsilon_i + \epsilon_j) + \epsilon_j} R_{ijlk} + (-1)^{\epsilon_i(\epsilon_k + \epsilon_l)} R_{klij} + (-1)^{\epsilon_i(\epsilon_i + \epsilon_j + \epsilon_k + \epsilon_l)} R_{jikl} = 0.$$  

This is proved by using the Jacobi identity (75) together with a cyclic change of the indices (see [2]). The identity (59) involves components of the curvature tensor with cyclic permutation of all indices, but the sign factors depending on the Grassmann parities of the indices do not follow from a cyclic permutation, as is the case, for example, for the Jacobi identity, but are defined by the permutation of the indices that takes a given set into the original one. In the case of ordinary manifolds, i.e., when all the variables $x^i$ are even ($\epsilon_i = 0$), Eq. (59) obtains the symmetric form [3],  

$$R_{ijkl} + R_{iljk} + R_{klij} + R_{jikl} = 0.$$  

Having the curvature tensor, $R_{ijkl}$, and the inverse tensor field $\omega^i_j$, with allowance made for the symmetry properties of these tensors, [19], [53] and [54], one can construct the only tensor field of type $(0, 2)$,  

$$K_{ij} = \omega^{kn} R_{nij}(-1)^{\epsilon_i \epsilon_k + \epsilon_k + \epsilon_n} = R^k_{ik}(-1)^{\epsilon_k(\epsilon_i + 1)} \epsilon(K_{ij}) = \epsilon_i + \epsilon_j.$$  

(61)
This tensor has the generalized symmetry property
\[ K_{ij} = (-1)^{\epsilon_i \epsilon_j} K_{ji} \] (62)
and is called the Ricci tensor.

Now we can define the scalar curvature tensor \( K \) by the formula
\[ K = \omega^{ji} K_{ij} (-1)^{\epsilon_i + \epsilon_j}. \] (63)
From the symmetry properties of \( K_{ij} \) and \( \omega^{ij} \), it follows that
\[ K = 0. \] (64)
Therefore, as in the case of Fedosov manifolds [3], for any even symplectic connection the scalar curvature tensor necessarily vanishes.

6 Odd symplectic geometry

Consider now possible constructions of geometry on supermanifolds in odd supersymmetric extension of symplectic geometry on manifolds. We know that in the odd case there exist two independent structures constructed with the help of generalized symmetric (an antibracket) and antisymmetric (a 2-form) second-rank tensor fields.

Suppose that a supermanifold \( M \) of an even dimension (\( \dim M = 2n \)) is equipped both with an odd closed non-degenerate differential 2-form
\[ \omega = \omega_{ij} \, dx^j \wedge dx^i, \quad \omega_{ij} = -(-1)^{\epsilon_i \epsilon_j} \omega_{ji}, \quad \epsilon(\omega) = 1, \quad d\omega = 0 \] (65)
and a symmetric connection (covariant derivative) compatible with a given symplectic structure \( \omega \)
\[ \omega_{ij,k} - \omega_{im} \Gamma^m_{jk} + \omega_{jm} \Gamma^m_{ik} (-1)^{\epsilon_i \epsilon_j} = 0. \] (66)
Repeating all calculations of previous section and taking into account that
\[ \epsilon(R_{ijkl}) = \epsilon_i + \epsilon_j + \epsilon_k + \epsilon_l + 1, \quad \epsilon(\Gamma_{ijk}) = \epsilon_i + \epsilon_j + \epsilon_k + 1, \] (67)
we obtain that all relations and identities for the curvature tensor have the same forms as in the case of even symplectic supermanifolds. There are two essential differences only. The first one is connected with Ricci tensor which has no special symmetry properties. The second one is non-triviality of the scalar curvature tensor. Both these statements will be considered below within antisymplectic supermanifolds.

Consider the second possibility to construct an odd symplectic geometry. To do this let us equip a supermanifold \( M \) with an antibracket (28). In its turn, let tensor field \( \Omega^{ij} \) (an antisymplectic structure) be covariant constant
\[ \Omega^{ij} \nabla_k = 0. \] (68)
Then the inverse tensor field \( \Omega_{ij} \) will be covariant constant too
\[ \Omega_{ij} \nabla_k = 0, \quad \Omega_{ij,k} - \Omega_{il} \Delta^l_{jk} + \Omega_{jl} \Delta^l_{ik} (-1)^{\epsilon_i \epsilon_j} = 0 \] (69)
where \( \Delta^i_{jk} (\epsilon(\Delta^i_{jk}) = \epsilon_i + \epsilon_j + \epsilon_k) \) is a symmetric connection and the symmetry property of \( \Omega_{ij} \) was used.

Let us introduce the curvature tensor of an antisymplectic connection,
\[ \mathcal{R}_{ijkl} = \Omega_{im} \mathcal{R}^m_{jkl}, \quad \epsilon(\mathcal{R}_{ijkl}) = \epsilon_i + \epsilon_j + \epsilon_k + \epsilon_l + 1, \] (70)
where \( \mathcal{R}^n_{jkl} \) is given by (13) with natural replacement \( \Gamma^i_{jk} \) for \( \Delta^i_{jk} \). This leads to the following representation,
\[ \mathcal{R}_{nljk} = -\Delta_{nlj,k} + \Delta_{nl,k,j} (-1)^{\epsilon_j \epsilon_k} + \Delta_{nk} \Delta^l_{ij} (-1)^{\epsilon_i \epsilon_j} \epsilon_i + \epsilon_k + \epsilon_l + \epsilon_j \]
\[ -\Delta_{nj} \Delta^l_{ik} (-1)^{\epsilon_i \epsilon_j} \epsilon_i + \epsilon_j + \epsilon_l + \epsilon_j \] (71)
\[ \Delta_{ijk} = \Omega_{ln} \Delta^n_{jk}, \quad \epsilon(\Delta_{ijk}) = \epsilon_i + \epsilon_j + \epsilon_k + 1. \]  \hspace{1cm} (72)

Using Eq. (72), the relation (69) reads
\[ \Omega_{ij,k} = \Delta_{ijk} - \Delta_{jik}(-1)^{\epsilon_i \epsilon_j}. \]  \hspace{1cm} (73)

Furthermore, from Eq. (71) it follows that
\[ R_{ijkl} = (1)_{ijk} R_{jik,l} + (1)_{ikl} R_{ikj,l} + (1)_{jkl} R_{jik,l} = 0. \]  \hspace{1cm} (74)

In addition, the curvature tensor obeys the identities
\[ R_{ijkl} = (1)_{ikl} R_{jik,l}. \]  \hspace{1cm} (75)

In order to prove this, let us consider
\[ \Omega_{ij,kl} = \Delta_{jik,l} - \Delta_{jik,l}(-1)^{\epsilon_i \epsilon_j}. \]  \hspace{1cm} (76)

Then, using the relations
\[ 0 = \Omega_{ij,kl} - (1)_{ikl} \Omega_{ij,kl} \]
\[ = \Delta_{jik,l} - \Delta_{jik,l}(-1)^{\epsilon_i \epsilon_j} - \Delta_{jil,k}(-1)^{\epsilon_i \epsilon_l} + \Delta_{jil,k}(-1)^{\epsilon_i \epsilon_j + \epsilon_k \epsilon_l} \]
\[ = -R_{ijkl} + (1)_{ikl} R_{jik,l}. \]  \hspace{1cm} (77)

Moreover the curvature tensor obeys the identities

\[ R_{ijkl} = (1)_{ikl} R_{jik,l} + (1)_{ikl} R_{ikj,l} + (1)_{jkl} R_{jik,l} + (1)_{ikl} R_{jik,l} = 0. \]  \hspace{1cm} (78)

which have the same form as in the even case (59).

Ricci tensor can be defined by contracting two indices of curvature tensor
\[ R_{ij} = R^k_{ikj} (-1)^{\epsilon_k (\epsilon_i + \epsilon_k + \epsilon_j)} = \Omega^n_{mn} R_{nkj}(-1)^{\epsilon_i \epsilon_k + \epsilon_k + \epsilon_n}, \quad \epsilon(R_{ij}) = \epsilon_i + \epsilon_j. \]  \hspace{1cm} (79)

Notice that in contrast with the even case Ricci tensor (79) has no a special symmetry properties. The further contraction between antisymplectic tensor and Ricci tensor gives scalar curvature
\[ S = \Omega^n_{ij} R_{ij} (-1)^{\epsilon_i + \epsilon_j}, \quad \epsilon(S) = 1 \]  \hspace{1cm} (80)

which, in general, is not equal to zero. Notice that the scalar curvature tensor squared is identically equal to zero, \( R^2 = 0 \).

If we identify \( \Omega_{ij} \) with \( \omega_{ij} \) then we can find coincidence antisymplectic supermanifolds and odd symplectic supermanifolds.
7 Conclusion

We have considered possible generalizations of symplectic geometry on manifolds to the supersymmetric case. In the even case there are two scalar structures (an even closed non-degenerate differential 2-form and Poisson bracket) which can be used for clothing of a supermanifold. When the Poisson bracket is non-degenerate and is constructed with the help of tensor field inverse to a given symplectic structure, then an even symplectic supermanifold and the non-degenerate Poisson supermanifold coincide. If, in addition, an even symplectic supermanifolds is endowed with a symmetric connection (covariant derivative) compatible with a given symplectic structure, one has an even Fedosov supermanifolds which can be considered as generalization of Fedosov manifolds [3]. In particular, the scalar curvature tensor for even Fedosov supermanifolds vanishes.

In the odd case we have again two scalar structures (an odd closed non-degenerate differential 2-form and the antibracket) for clothing of a supermanifold. These structures lead to an odd symplectic supermanifold and an antisymplectic supermanifold having the similar geometry. The same statement is true if one equips these supermanifolds with a symmetric connection compatible with given structures. The more important deference in contrast with the even case is non-triviality of the scalar curvature tensor. Note that quite recently [12] the scalar curvature tensor non-triviality was used in generalization of the Batalin-Vilkovisky quantization scheme.

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