Universal glassy dynamics at noise-perturbed onset of chaos. A route to ergodicity breakdown.

A. Robledo
Instituto de Física,
Universidad Nacional Autónoma de México,
Apartado Postal 20-364, México 01000 D.F., Mexico.

The dynamics of iterates at the transition to chaos in one-dimensional unimodal maps is shown to exhibit the characteristic elements of the glass transition, e.g. two-step relaxation and aging. The properties of the bifurcation gap induced by external noise, including a relationship between relaxation time and entropy, are seen to be comparable to those of a supercooled liquid above a glass transition temperature. Universal time evolution obtained from the Feigenbaum RG transformation is expressed analytically via $q$-exponentials, and interpreted in terms of nonextensive statistics.

While at present the phenomenology of glass formation is in a well-documented advanced stage \cite{1,2}, the subject remains a prevailing and major theoretical challenge in condensed matter physics. In experiments and numerical simulations the transition of a liquid into a glass manifests itself as a dramatic dynamical slowing down where the characteristic structural relaxation time changes by many orders of magnitude in a relatively small space of temperatures. Associated to this process, atypical connections develop between dynamical and thermodynamic properties, such as the so-called Adam-Gibbs relationship between structural relaxation times and configurational entropy $\gamma$. These poorly understood connections pose very intriguing questions that suggest deep-lying, hence generic, physical circumstances which may manifest themselves in completely different classes of systems and therefore are capable of leading to novel but universal laws. Here we make a case for this premise by exhibiting that glassy behavior is in fact present in prototypical nonlinear maps close to the onset of chaos. There are clear indications that standard phase-space mixing is not entirely fulfilled during glass forming dynamics, i.e. upon cooling, caged molecules rearrange so slowly that they cannot sample configurations in the available time allowed by the process \cite{1,2}. Naturally, the question arises as to whether under conditions of ergodicity malfunction, and, as a final point, downright failure, the Boltzmann-Gibbs (BG) statistical mechanics is still capable of describing stationary states on the point of glass formation or those representing the glass itself. The aim of this letter is to make evident that the essential elements of glassy behavior are all actually present within the neighborhood of the chaos transition in simple nonlinear dissipative maps. By showing this hitherto unidentified association we put forward a minimal model for glass dynamics that endorses the idea of universality in this phenomenon, and at the same time provides the rare opportunity for detailed examination of properties, such as the connection between dynamics and statics and the departure from BG statistics.

A distinctive feature of supercooled liquids on approach to glass formation is the development of a two-step process of relaxation, as displayed by the time evolution of correlations e.g. the intermediate scattering function $F_k$. This consists of time $t$ power-law decays towards and away from a plateau, the duration $t_x$ of which diverges also as a power law of the difference $T - T_g$ as the temperature $T$ decreases to a critical value $T_g$ \cite{1,2}. This behavior is displayed by molecular dynamics simulations \cite{8} and successfully reproduced by mode coupling (MC) theory \cite{4}. Another important feature of the dynamic properties of glasses is the loss of time translation invariance. This is referred to as aging \cite{5}, and is due to the fact that properties of glasses depend on the procedure by which they are obtained. The slow time decline of relaxation functions and correlations display a scaling dependence on the ratio $t/t_w$ where $t_w$ is a waiting time. Interestingly, a recent important development is the finding that both two-step relaxation and aging are present in fixed-energy Hamiltonian systems of $N$ classical $XY$ spins with homogeneous but sufficiently long-ranged interactions. In these systems the length of the plateau diverges with infinite size $N \to \infty$ \cite{6} and aging, similar to that found for short-ranged interaction spin glasses, is observed in the long time behavior of the autocorrelation function of system trajectories \cite{6,8}. Remarkably, here we demonstrate that the same two features, two-step relaxation and aging, are displayed by simpler systems with only a few degrees of freedom, such as one-dimensional dissipative maps at the edge of chaos in the presence of stochastic noise, the amplitude $\sigma$ of which plays a role parallel to $T - T_g$ in the supercooled liquid or $1/N$ in the spin system. This adds to the idea of universality for the phenomenon of glass formation.

The experimentally observed relaxation behavior of supercooled liquids is effectively described, under simple heat capacity assumptions, by the above-mentioned Adam-Gibbs equation, $t_x = A \exp(B/T S_c)$, where the relaxation time $t_x$ can be identified with the viscosity, and the configurational entropy $S_c$ relates to the number of minima of the fluid’s potential energy surface (and $A$ and $B$ are constants) \cite{6}. Although at present a first principles derivation of this equation is lacking, it offers a sensible picture of progressive reduction in the number of configurations that the system is capable of sampling as $T - T_g \to 0$ as the origin of viscous slow-down in
supercooled liquids. As a parallel to the Adam-Gibbs formula, we show below that our one-dimensional dissipative map model for glassy dynamics exhibits a relationship between the plateau duration $t_x$, and the entropy $S_c$ for the state that comprises the largest number of (iterate positions) bands allowed by the bifurcation gap - the noise-induced cutoff in the period-doubling cascade. This entropy is obtained from the probability of chaotic band occupancy at position $x$.

So, our purpose here is to contrast the dynamics of a fluid near glass formation with that of iterates in simple nonlinear maps near the edge of chaos when subject to external noise. We chose to illustrate this by considering the known behavior of the logistic map under these conditions. The idea in mind is that the course of action that leads to glass formation is one in which the system is driven gradually into a nonergodic state by reducing its ability to pass through phase-space-filling configurational regions until it is only possible to go across a (multi)fractal subset of phase space. This situation is emulated in the logistic map with additive external noise, $x_{t+1} = f(x_t) = 1 - \mu x_t^2 + \xi_t$, $-1 \leq x_t \leq 1$, $0 \leq \mu \leq 2$, where $\xi_t$ is Gaussian-distributed with average $\langle \xi_t \xi_t \rangle = \delta_{t,t'}$, and $\sigma$ measures the noise intensity. As is well known, in the absence of noise $\sigma = 0$ the Feigenbaum attractor at $\mu_c(0) = 1.40115...$ is the accumulation point of both the period doubling and the chaotic band splitting sequences of transitions and it marks the threshold between periodic and chaotic orbits. The locations of period doublings, at $\mu = \mu_n < \mu_c(0)$, and band splittings, at $\mu = \tilde{\mu}_n > \mu_c(0)$, obey, for large $n$, the power laws $\mu_n - \mu_c(0) \sim \delta^{-n}$ and $\mu_c(0) - \tilde{\mu}_n \sim \delta^{-n}$, where $\delta = 0.46692...$ is one of the two Feigenbaum’s universal constants. The 2nd, $\alpha = 2.50290...$ measures the power-law period-doubling spreading of iterate positions. All the trajectories with $\mu_c(0)$ and initial condition $-1 \leq x_{in} \leq 1$ fall, after a (power-law) transient, into the attractor set of positions with fractal dimension $d_f = 0.5338...$. Therefore, these trajectories represent nonergodic states, as $t \rightarrow \infty$ only a Cantor set of positions is accessible within the entire phase space $-1 \leq x \leq 1$. For $\sigma > 0$ the noise fluctuations smear the sharp features of the periodic attractors as these broaden into bands similar to those in the chaotic attractors, but, and this is a sharp transition to chaos at $\mu_c(\sigma)$ where the Lyapunov exponent changes sign. The period-doubling of bands ends at a finite value $2^{N(\sigma)}$ as the edge of chaos transition is approached and then decreases in reverse fashion at the other side of the transition. The broadening of orbits with periods or bands of number smaller than $2^{N(\sigma)}$ and the removal of orbits of periods or bands of number larger than $2^{N(\sigma)}$ in the infinite cascades introduces a bifurcation gap with scaling features that we shall use below. When $\sigma > 0$ the trajectories visit sequentially a set of $2^n$ disjoint bands or segments leading to a cycle, but the behavior inside each band is completely chaotic. These trajectories represent ergodic states as the accessible positions have a fractal dimension equal to the dimension of phase space. Thus the elimination of fluctuations in the limit $\sigma \rightarrow 0$ leads to an ergodic to nonergodic transition in the map and we contrast its properties with those known for the molecular arrest occurring in a liquid as $T \rightarrow T_p$. Since the map clearly differs from a Hamiltonian model for a liquid in that it does not take into consideration its molecular nature we may gain information regarding universality in the processes of glass formation.

The main points in the following analysis are: 1) The quasi-stationary trajectories followed by iterates at $\mu_c(\sigma)$ are obtained via the fixed-point map solution $g(x)$ and the first noise perturbation eigenfunction $G_\lambda(x)$ of the RG doubling transformation consisting of functional composition and rescaling, $R f(x) = 1 - (q - 1)x^{1/1-q}$. Positions for time subsequences within these trajectories can be expressed analytically in terms of the $q$-exponential function $\exp_q(x) = [1 - (q-1)x]^{1/1-q}$. 2) The two-step relaxation occurring when $\sigma \rightarrow 0$ is determined in terms of the bifurcation gap properties, in particular, the plateau duration is given by the power law $t_x(\sigma) \sim \sigma^{-1}$ where $r \simeq 0.6332$ or $r - 1 \simeq -0.3668$. 3) The map equivalent of the Adam-Gibbs law is obtained as a power-law relation $t_x \sim S^{-c}, \lambda = (1 - r)/r \simeq 0.5792$, between $t_x(\sigma)$ and the entropy $S_c(\sigma)$ associated to the noise broadening of chaotic bands. 4) The trajectories at $\mu_c(\sigma \rightarrow 0)$ are shown to obey a scaling property, characteristic of aging in glassy dynamics, of the form $x_{t+t_w} = h(t_w)h( t/t_w)$ where $t_w$ is a waiting time. The dynamics of iterates for the logistic map at the onset of chaos $\mu_c(0)$ has recently been analyzed in detail. It was found that the trajectory with initial condition $x_{in} = 0$ (see Fig. 1) maps out the Feigenbaum attractor in such a way that (the absolute values) of succeeding (time-shifted $\tau = t + 1$) positions $x_t$ form subsequences with a common power-law decay of the form $\tau^{-1/1-q}$ with $q = 1 - \ln 2 / \ln \alpha \simeq 0.24497$. That is, the entire attractor can be decomposed into position subsequences generated by the time subsequences $\tau = (2k + 1)2^n$, each obtained by running over $n = 0, 1, 2, ...$ for a fixed value of $k = 0, 1, 2, ...$. Noticeably, the positions in these subsequences can be obtained from those belonging to the 'super-stable' periodic orbits of lengths $2^n$, i.e. the $2^n$-cycles that contain the point $x = 0$ at $\mu_c(0)$.

Specifically, the positions for the main subsequence $k = 0$, that constitutes the lower bound of the entire trajectory (see Fig. 1), were identified to be $x_{2n} = d_n = \alpha^{-n}$, where $d_n \equiv \left[f(2^{-n-1})\right](0)$ is the ‘$n$-th diameter’ defined at the $2^n$-supercycle. The main subsequence can be expressed as the $q$-exponential $x_t = \exp_{2^{-q}}(-\lambda t)$ with $\lambda_q = \ln \alpha / \ln 2$, and interestingly this analytical result for $x_t$ can be seen to satisfy the dynamical fixed-point relation, $h(t) = \alpha h(h(t)/\alpha)$ with $\alpha = 2^{1/1-q}$. Further, the sensitivity to initial conditions $\xi_t$ obeys the closely related form $\xi_t = \exp_{\lambda}(\lambda_q t)$.

These properties follow from the use of $x_{in} = 0$ in the scaling relation

$$x_{\tau} = \left[g^{(\tau)}(x_{in})\right] = \tau^{-1/1-q} \left[g(\tau^{1/1-q} x_{in})\right],$$

(1)
that in turn is obtained from the $n \to \infty$ convergence of the 2nd th map composition to $(-\alpha)^{-n}g(\alpha^n x)$ with $\alpha = 2^{1/(1-q)}$. When $x_{in} = 0$ one obtains in general

$$x_r = \left| g^{(2k+1)}(0)g^{(2n-1)}(0) \right| \alpha^{-n}. \quad (2)$$

When the noise is turned on (\(\sigma\) always small) the 2nd th map composition converges instead to $(-\alpha)^{-n}[g(\alpha^n x) + \xi \sigma n \sigma G_{c}(\alpha^n x)]$, where $\kappa$ a constant whose numerically determined [14], [15] value $\kappa \simeq 6.619$ is well approximated by $\nu = 2\sqrt{2\alpha}(1 + 1/\alpha^2)^{-1/2}$, the ratio of the intensity of successive subharmonics in the map power spectrum [15, 16]. The connection between $\kappa$ and the $\sigma$-independent $\nu$ stems from the necessary coincidence of two ratios, that of noise levels causing band-merging transitions for successive 2$^n$ and 2$^{n+1}$ periods and that of spectral peaks at the corresponding parameter values $\mu_n$ and $\mu_{n+1}$ [15, 16].

Following the same procedure as above we see that the orbits $x_r$ at $\mu_r(\sigma)$ satisfy, in place of Eq. (1), the relation

$$x_r = \tau^{-1/1-q} \left| g^{(\tau^{1/1-q} x)} + \xi \sigma^{r^{1/1-r}} G_{c}(\tau^{1/1-q} x) \right|, \quad (3)$$

where $r = 1 - \ln 2/\ln \kappa \simeq 0.6332$. So that use of $x_{in} = 0$ yields $x_r = \tau^{-1/1-q} \left| 1 + \xi \sigma^{r^{1/1-r}} \right| \text{ or}$

$$x_t = \exp_{-2q}-(-\lambda_q t) \left| 1 + \xi \sigma \exp_{c}(\lambda_q t) \right| \quad (4)$$

where $t = \tau - 1$ and $\lambda_q = \ln \kappa/\ln 2$.

At each noise level $\sigma$ there is a 'crossover' or 'relaxation' time $t_x = \tau_x - 1$ when the fluctuations start suppressing the fine structure imprinted by the attractor on the orbits with $x_{in} = 0$. This time is given by $\tau_x = \sigma^{r-1}$, the time when the fluctuation term in the perturbation expression for $x_r$ becomes $\sigma$-independent and so unrestrained, i.e. $x_{\tau_x} = \tau_x^{-1/1-q} \left| 1 + \xi \right|$. Thus, there are two regimes for time evolution at $\mu_r(\sigma)$. When $\tau < \tau_x$ the fluctuations are smaller than the distances between adjacent subsequence positions of the noiseless orbit at $\mu_r(0)$, and the iterate positions in the presence of noise fall within small non overlapping bands each around the $\sigma = 0$ position for that $\tau$. In this regime the dynamics follows in effect the same subsequence pattern as in the noiseless case. When $\tau \sim \tau_x$ the width of the fluctuation-generated band visited at time $\tau_x = 2^N$ matches the distance between two consecutive diameters, $d_N - d_{N+1}$ where $N \sim -\ln \sigma/\ln \kappa$, and this signals a cutoff in the advance through the position subsequences. At longer times $\tau > \tau_x$ the orbits are unable to stick to the fine period-doubling structure of the attractor. In this 2nd regime the iterate follows an increasingly chaotic trajectory as bands merge progressively. This is the dynamical image - observed along the time evolution for the orbits of a single state $\mu_r(\sigma)$ - of the static bifurcation gap first described in the map space of position $x$ and control parameter $\mu$ [15, 16].

In establishing parallels with glassy dynamics in supercooled liquids, it is helpful to define an 'energy landscape' for the map as being composed by an infinite number of 'wells' whose equal-valued minima coincide with the points of the attractor on the interval $[-1,1]$. The widths of the wells increase as an 'energy parameter' $U$ increases and the wells merge by pairs at values $U_N$ such that within the range $U_N < U \leq U_N$, the landscape is composed of a set of $2^N$ bands of widths $w_m(U)$, $m = 1, \ldots, 2^N$. This 'picture' of an energy landscape resembles the chaotic band-merging cascade in the well-known $(x, \mu)$ bifurcation diagram [10]. The landscape is sampled at noise level $\sigma$ by orbits that visit points within the set of $2^N$ bands of widths $w_m(U) \sim \sigma$, and, as we have seen, this takes place in time in the same way that period-doubling and band merging proceeds in the presence of a bifurcation gap when the control parameter is run through the interval $0 \leq \mu \leq 1$. That is, the trajectories starting at $x_{in} = 0$ duplicate the number of visited bands at times $\tau = 2^k, n = 1, \ldots, N$, the bifurcation gap is reached at $\tau_x = 2^N$, after which the orbits fall within bands that merge by pairs at times $\tau = 2^{N+n}, n = 1, \ldots, N$. The sensitivity to initial conditions grows as $\xi_t = \exp_{q}(\lambda_q t)$ ($q = 1 - \ln 2/\ln \alpha < 1$ as above) for $t < t_x$, but for $t > t_x$ the fluctuations dominate and $\xi_t$ grows exponentially as the trajectory has become chaotic and so one anticipates an exponential $\xi_t$ ($q = 1$). We interpret this behavior to be the dynamical system analog of the so-called $\alpha$ relaxation in supercooled fluids. The plateau duration $t_x \to \infty$ as $\sigma \to 0$. Additionally, trajectories with initial conditions $x_{in}$ not belonging to the attractor exhibit an initial relaxation stretch towards the plateau as the orbit falls into the attractor. This appears as the analog of the so-called $\beta$ relaxation in supercooled liquids.

Next, we proceed to evaluate the entropy of the orbits starting at $x_{in} = 0$ as they enter the bifurcation gap at $t_x(\sigma)$ when the maximum number $2^N$ of bands allowed by the fluctuations is reached. The entropy $S_c(\mu_r, t_x)$ associated to the state at $\mu_r(\sigma)$ at iteration time $t_x(\sigma)$ has the form $S_c(\mu_r, t_x) = 2^N \sigma$, since at $t_x(\sigma)$ each of the $2^N$ bands contributes with an entropy $\sigma$ where $s = -\int_0^\sigma \tilde{p}(\xi) \ln p(\xi) d\xi$ and where $\tilde{p}(\xi)$ is the distribution for the noise random variable. In terms of $t_x$ (since $2^N = 1 + t_x$ and $\sigma = (1 + t_x)^{-1/1-r}$) one has $S_c(\mu_r, t_x)/s = (1 + t_x)^{-r-1}$ or, conversely,

$$t_x = (s/S_c(1-r)/r. \quad (5)$$

Since $t_x \sim \sigma^{-r-1}, r \sim -0.3668$ and $(1-r)/r \sim 0.5792$ then $t_x \to \infty$ and $S_c \to 0$ as $\sigma \to 0$, i.e. the relaxation time diverges as the 'landscape' entropy vanishes. We interpret this relationship between $t_x$ and the entropy $S_c$ to be the dynamical system analog of the Adam-Gibbs formula for a supercooled liquid.

Finally, we draw attention to the aging scaling property of the trajectories $x_t$ at $\mu_r(\sigma)$. The case $\sigma = 0$ is more readily appraised because this property is, essentially, built into the same position subsequences $x_r = \left[ g^{(k+1)}(0) \right], \tau = (2k+1)2^n, k, n = 0, 1, \ldots$ that we have been using all along. These subsequences are relevant for
the description of trajectories that are at first held at a given attractor position for a waiting period of time \( t_w \) and then released to the normal iterative procedure. We chose the holding positions to be any of those along the top band shown in Fig. 1 for a waiting time \( t_w = 2k + 1 \), \( k = 0, 1, \ldots \). Notice that, as shown in Fig. 1, for the \( x_{in} = 0 \) orbit these positions are visited at odd iteration times. The lower-bound positions for these trajectories are given by those of the subsequences at times \((2k+1)2^n\) (see Fig. 1). Writing \( \tau = t_w + t \) we have that \( t/t_w = 2^n - 1 \) and \( x_{t+t_w} = g^{(t_w)}(0)g^{(t/t_w)}(0) \) or

\[
x_{t+t_w} = g^{(t_w)}(0)\exp(-\lambda t/t_w).
\] (6)

This property is gradually modified when noise is turned on. The presence of a bifurcation gap limits its range of validity to total times \( t_w + t < t_x(\sigma) \) and so progressively disappears as \( \sigma \) is increased.

In summary, we have shown that the dynamics of noise-perturbed logistic maps at the chaos threshold exhibit the peculiar features of glassy dynamics in supercooled liquids. These are: a two-step relaxation process, association between dynamic (relaxation time) and static (configurational entropy) properties, and an aging scaling relation for two-time functions. Our findings clearly on. The presence of a bifurcation gap limits its range of validity to total times \( t_w + t < t_x(\sigma) \) and so progressively disappears as \( \sigma \) is increased.

In summary, we have shown that the dynamics of noise-perturbed logistic maps at the chaos threshold exhibit the peculiar features of glassy dynamics in supercooled liquids. These are: a two-step relaxation process, association between dynamic (relaxation time) and static (configurational entropy) properties, and an aging scaling relation for two-time functions. Our findings clearly have a universal (in the RG sense) validity for the class of unimodal maps studied. The occurrence of this novel analogy may not be completely fortuitous since the limit of vanishing noise amplitude \( \sigma \to 0 \) (the counterpart of the limit \( T \to T_g \to 0 \) in the supercooled liquid) entails loss of ergodicity. The incidence of these properties in such simple dynamical systems, with only a few degrees of freedom and no reference to molecular interactions, suggests an all-encompassing mechanism underlying the dynamics of glass formation. As established [13], the dynamics of deterministic unimodal maps at the edge of chaos is a bona fide example of the applicability of nonextensive statistics. Here we have shown that this nonergodic state corresponds to the limiting state, \( \sigma \to 0 \), \( t_x \to \infty \), for a family of small \( \sigma \) noisy states with glassy properties, that are conspicuously described for \( t < t_x \) via the \( q \)-exponentials of the nonextensive formalism [13]. The fact that these features transform into the usual BG exponential behavior for \( t > t_x \) provides a significant opportunity for investigating the crossover from the ordinary BG to the nonextensive statistics in the physical circumstance of loss of mixing and ergodic properties.

Acknowledgments. I am grateful to Piero Tartaglia for guiding me into the concepts of glass formation and also for his kind hospitality at Dipartamento di Fisica, Università degli Studi di Roma “La Sapienza”. I thank Fulvio Baldwin for contributing the figure. Work partially supported by CONACyT grant P-40530-F.

[1] For a recent review see, P.G. De Benedetti and F.H. Stillinger, Nature 410, 267 (2001).
[2] P.G. De Benedetti, Metastable Liquids. Concepts and Principles (Princeton Univ. Press, Princeton, 1996).
[3] W. Kob and H.C. Andersen, Phys. Rev. E 51, 4626 (1995).
[4] W. Götze and L. Sjögren, Rep. Prog. Phys. 55, 241 (1992).
[5] See, for example, J.P. Bouchaud, L.F. Cugliandolo, J. Kurchan and M. Mezard, in Spin Glasses and Random Fields, A.P. Young, editor (World Scientific, Singapore, 1998).
[6] V. Latora, A. Rapisarda and C. Tsallis, Phys. Rev. E64, 056134 (2001).
[7] M.A. Montemurro, F. Tamarit and C. Anteneodo, Phys. Rev. E67, 031106 (2003).
[8] A. Pluchino, V. Latora and A. Rapisarda, cond-mat/0303081 cond-mat/0306374
[9] G. Adam and J.H. Gibbs, J. Chem. Phys. 43, 139 (1965).
[10] See, for example, H.G. Schuster, Deterministic Chaos. An Introduction, 2nd Revised Edition (VCH Publishers, Weinheim, 1988).
[11] J.P. Crutchfield, J.D. Farmer and B.A. Huberman, Phys. Rep. 92, 45 (1982).
[12] C. Beck and F. Schlogl, Thermodynamics of Chaotic Systems (Cambridge University Press, UK, 1993).
[13] F. Baldovin and A. Robledo, Phys. Rev. E 66, 045104(R) (2002), cond-mat/0304410.
[14] J. Crutchfield, M. Nauenberg and J. Rudnick, Phys. Rev. Lett. 46, 933 (1981).
[15] B. Shraiman, C.E. Wayne and P.C. Martin, Phys. Rev. Lett. 46, 935 (1981).
FIG. 1: Absolute values of positions in logarithmic scales of the first 1000 iterations \( \tau \) for a trajectory of the logistic map at the onset of chaos \( \mu_c(0) \) with initial condition \( x_{i0} = 0 \). The numbers correspond to iteration times. The power-law decay of the time subsequences described in the text can be clearly appreciated.