STABILITY FOR DETERMINATION OF RIEMANNIAN METRICS BY SPECTRAL DATA AND DIRICHLET-TO-NEUMANN MAP LIMITED ON ARBITRARY SUBBOUNDARY

dedicated to the memory of Professor Yaroslav Kurylev

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Abstract. In this paper, we establish conditional stability estimates for two inverse problems of determining metrics in two dimensional Laplace-Beltrami operators. As data, in the first inverse problem we adopt spectral data on an arbitrarily fixed subboundary, while in the second, we choose the Dirichlet-to-Neumann map limited on an arbitrarily fixed subboundary. The conditional stability estimates for the two inverse problems are stated as follows. If the difference between spectral data or Dirichlet-to-Neumann maps related to two metrics \( g_1 \) and \( g_2 \) is small, then \( g_1 \) and \( g_2 \) are close in \( L^2(\Omega) \) modulo a suitable diffeomorphism within a priori bounds of \( g_1 \) and \( g_2 \). Both stability estimates are of the same double logarithmic rate.

1. Introduction and main results. Let \( \Omega \) be a simply connected bounded domain in \( \mathbb{R}^2 \) with boundary \( \partial \Omega \in C^\infty \), and \( \tilde{\Gamma} \) be an arbitrarily fixed relatively open subset of \( \partial \Omega \), \( \Gamma_0 = \partial \Omega \setminus \tilde{\Gamma} \). Let \( \Omega \) be equipped with the Riemannian metric

\[
\text{(1) } g = \{g_{jk}\} \in C^{2+\alpha}(\overline{\Omega}) \quad \text{for some } \alpha \in (0,1), \quad g_{kj} = g_{jk} \quad \forall j, k \in \{1,2\}
\]

satisfying

\[
\text{(2) } (g(x)\xi, \xi) > 0 \quad \forall \xi \in \mathbb{R}^2, \quad |\xi| = 1.
\]

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Throughout this paper, \( \{g_{jk}\} := \{g_{jk}\}_{1 \leq j, k \leq 2} \) means a \( 2 \times 2 \) matrix and \( \{g^{jk}(x)\} \) denotes the inverse to \( g(x) = \{g_{jk}(x)\} \). Moreover let \( \Delta_g \) denote the Laplace-Beltrami operator associated to the Riemannian metric \( g = \{g_{jk}\} \):

\[
\Delta_g = \frac{1}{\sqrt{\det g}} \sum_{j,k=1}^{2} \partial_{x_k}(\sqrt{\det g} g^{jk} \partial_{x_j}).
\]

Henceforth we write \( \Delta_g = \Delta \) if \( g \) is the \( 2 \times 2 \) identity matrix.

Let \( \lambda_k \) be the eigenvalues of the Laplace operator with the zero Dirichlet boundary condition, where we number \( \lambda_k \) repeatedly according to their multiplicities. Let \( e_k \) be the corresponding eigenfunctions:

\[
-\Delta_g e_k = \lambda_k e_k \quad \text{in} \quad \Omega, \quad e_k|_{\partial \Omega} = 0
\]
satisfying

\[
(e_k, e_j)_{L^2_{\rho_0}(\Omega)} = 0 \quad \text{if} \quad k \neq j, \quad (e_k, e_k)_{L^2_{\rho_0}(\Omega)} = 1 \quad \forall k, j \in \mathbb{N}, \quad \rho_0 = \det g.
\]

Here and henceforth let \( i = \sqrt{-1}, \mathbb{N} = \{1, 2, 3, \ldots\}, \mathbb{R}_+ = \{s; s \geq 0\}, B(x_0, r) = \{x \in \mathbb{R}^2; |x - x_0| < r\}, \mathbb{D} = \{z \in \mathbb{C}; |z| < 1\}, W^{1,1}_p(G) \) is the Sobolev space \( W^{1,1}_p(G) \) with the norm \( \|w\|_{W^{1,1}_p(G)} = \|w\|_{L^1(G)} + \|\nabla w\|_{L^1(G)} \), \( L^1_{\rho_0}(\Omega) = \{w; \int_\Omega \rho_0 |w|^2 dx < \infty\} \), \( (u, w)_{L^2_{\rho_0}(\Omega)} = \int_\Omega \rho_0 u w dx \) and let \( \nu = (\nu_1, \nu_2) \) be the outward unit normal vector to \( \Omega \). We set

\[
\frac{\partial}{\partial \nu g} = \sqrt{\det g} \sum_{j,k=1}^{2} g^{jk} \nu_k \partial_{x_j}.
\]

By \( [A, B] \) we denote the commutator for operators \( A \) and \( B \): \( [A, B] = AB - BA \). Henceforth we identify \( \mathbb{C} \) with \( \mathbb{R}^2 \).

In this paper, we consider two inverse problems and the first is stated as follows:

**Inverse spectral problem.**

*Suppose that the following spectral data are given:*

\[
\{ \lambda_k, \frac{\partial e_k}{\partial \nu g} |_{\partial \Omega} \}_{k=1}^{\infty}.
\]

*Then can we determine the metric \( g \)?*

The uniqueness results for this inverse problem were established for example in Belishev [3], Belishev and Kurylev [4] or in Katchalov, Kurylev and Lassas [18] (Theorem 3.3 on p.151): the metric \( g \) can be uniquely determined up to the gauge equivalence. That is, if \( g_1 \) and \( g_2 \) are Riemannian metrics with the same spectral data, then there exists a diffeomorphism

\[
F: \Omega \to \Omega, \quad F(x) = x \quad \forall x \in \overline{\Gamma}, \quad F \in C^{3+\alpha}(\overline{\Omega}) \quad \text{for some} \quad \alpha \in (0, 1)
\]
such that

\[
F_* g_1 = g_2.
\]

Throughout this paper, we set

\[
F_* g = [(F')^{-1} g((F')^{-1})^T] \circ F^{-1}.
\]

Here and henceforth \( F' \) is the Jacobian matrix of the mapping \( F \) and \( \circ \) denotes the composition of mappings.

Let \( \lambda_k(j) \) and \( e_k(j) \) be the eigenvalues and the eigenvectors of the Laplace-Beltrami operator associated to the Riemannian metric \( g_j \) and satisfying (3) and (4).
Denote
\[
\Lambda(g_1, g_2) = \sum_{k=1}^{\infty} \left( \max\{\lambda_k^{-1}(1), \lambda_k^{-1}(2)\} |\lambda_k(1) - \lambda_k(2)| + \max\{\lambda_k^{-\frac{1}{2}}(1), \lambda_k^{-\frac{1}{2}}(2)\} \left\| \frac{\partial e_k(1)}{\partial \nu_{g_1}} - \frac{\partial e_k(2)}{\partial \nu_{g_2}} \right\|_{L^2(\Gamma)} \right).
\]
Therefore the function \(\Lambda(g_1, g_2)\) is defined by the spectral data (5) for the metrics \(g_1\) and \(g_2\), provided that the infinite series is convergent.

The first main result of this paper is the stability estimate in the determination of the metric \(g\) by the spectral data:

**Theorem 1.1.** Let \(\alpha \in (0, 1)\) and let \(M_1, M_2\) be positive constants. We assume that metrics \(g_1, g_2 \in C^{2+\alpha}(\Omega)\) satisfy (1), (2) and
\[
\|g_j\|_{C^{2+\alpha}(\Omega)} \leq M_1, \quad (g_j(x)\xi, \xi) \geq M_2 > 0 \quad \forall \xi \in \mathbb{R}^2, |\xi| = 1
\]
for any \(j \in \{1, 2\}\) and \(x \in \Omega\) and
\[
\Lambda(g_1, g_2) < \infty.
\]

Then there exist a diffeomorphism \(F : \Omega \to \Omega, F \in C^{3+\alpha}(\Omega)\) in general depending on \(g_1, g_2\), satisfying (6) and a constant \(C(M_1, M_2) > 0\) such that
\[
\|F_{*}g_1 - g_2\|_{L^2(\Omega)} \leq C(M_1, M_2)G(\Lambda(g_1, g_2)),
\]
where the function \(G(\eta)\) is defined by
\[
G(\eta) = \begin{cases} (1 + |\ln |\ln \eta||)^{-\frac{1}{2}} & \text{if } 0 < \eta \leq 1/2 \\ (1 + |\ln 2|)^{-\frac{1}{2}} & \text{if } \eta \geq 1/2. \end{cases}
\]

In Theorem 1.1, we notice that in general \(\Lambda(g_1, g_2) < \infty\) does not necessarily hold for \(g_1, g_2\) satisfying (8), and if \(\Lambda(g_1, g_2) = \infty\), then the conclusion of the theorem holds trivially. In other words, we cannot expect the reverse inequality for (9). We emphasize that the conclusion in Theorem 1.1 is a conditional stability estimate under condition (8), and we do not know whether a weaker distance than \(\Lambda(g_1, g_2)\) can yield the same stability as (9) or not. The situation is the same also for Theorem 1.2 stated below under condition (14).

For the inverse spectral problem for the Schrödinger equation, we refer to Belishev [3], Bellassoued, Choulli and Yamamoto [5], [6], Choulli and Stefanov [9], Isozaki [17], Kurylev, Lassas and Weder [19], Nachman, Sylvester and Uhlmann [23], Novikov [24], Päivärinta and Serov [27].

In particular, the paper [9] proves the following: in the case of the Schrödinger equation \(\Delta + q(x)\), if spectral data are asymptotically close, that is,
\[
|\lambda_k(1) - \lambda_k(2)| = O(k^{-\theta_1}), \quad \left\| \frac{\partial e_k(1)}{\partial \nu_{g_1}} - \frac{\partial e_k(2)}{\partial \nu_{g_2}} \right\|_{L^2(\partial \Omega)} = O(k^{-\theta_2}),
\]
where \(\theta_1 > 1\) and \(\theta_2 > \frac{3}{4}\), then the uniqueness holds in the inverse problem. This closeness condition for the spectral data for \(\Delta + q_k\) is a weaker assumption than \(\lambda_k(1) = \lambda_k(2)\) and \(\nu_{g_1} e_k(1) = \nu_{g_2} e_k(2)\) on \(\Gamma\) for the case of \(\Delta g_k\). We compare the condition \(\theta_1 > 1\) and \(\theta_2 > \frac{3}{4}\) in [9] with our condition \(\Lambda(g_1, g_2)\). By \(\lambda_k(j) \sim k\) as \(k \to \infty\), we see that if \(\theta_1 > 0\) and \(\theta_2 > \frac{1}{2}\), then \(\Lambda(g_1, g_2) < \infty\), and we can interpret that \(\Lambda(g_1, g_2) < \infty\) is a more generous condition than in [9].
In order to formulate our second inverse problem we introduce the operator
\[
\Delta_{\beta,g} = \frac{1}{\sqrt{\det g}} \sum_{j,k=1}^{2} \partial_{x_j}(\beta(x)\sqrt{\det g} \beta^{jk} \partial_{x_k}).
\]

The second inverse problem is

**Inverse boundary value problem.**

We define the following Dirichlet-to-Neumann map by
\[
\Lambda_{\beta,g} f = \beta \frac{\partial u}{\partial \nu g} |_{\Gamma},
\]
where
\[
\Delta_{\beta,g} u = 0 \quad \text{in} \; \Omega, \quad u|_{\partial \Omega} = f, \quad \text{supp} \; f \subset \overline{\Gamma}.
\]

Then determine \( \beta \) and \( g \) by \( \Lambda_{\beta,g} \).

We see that \( \Lambda_{\beta,g} \) is a bounded linear operator from \( L^2(\overline{\Gamma}) \) to \( H^{-1}(\overline{\Gamma}) \). By \( \|L\|_{L^2(X,Y)} \) we denote the norm of an operator \( L \) from a Banach space \( X \) to a Banach space \( Y \).

In [15], the metric \( g \) is uniquely determined by the Dirichlet-to-Neumann map up to diffeomorphism and a multiplication by a positive function. As for further uniqueness results, see e.g., Imanuvilov and Yamamoto [16]. However, to the best knowledge of the authors, no stability results have been published, and our second main result is a positive answer.

**Theorem 1.2.** Let \( \alpha \in (0, 1) \) and let positive constants \( M_1, M_2, M_3 \) be arbitrarily fixed. We assume that metrics \( g_1, g_2 \in C^{2+\alpha}(\overline{\Omega}) \) and functions \( \beta_1, \beta_2 \in C^{2+\alpha}(\overline{\Omega}) \) satisfy (1), (2) and
\[
\|g_j\|_{C^{2+\alpha}(\overline{\Omega})} + \|\beta_j\|_{C^{2+\alpha}(\overline{\Omega})} \leq M_1,
\]
for any \( j \in \{1, 2\} \) and all \( x \in \Omega \), and \( g_1 = g_2 \) and \( \beta_1 = \beta_2 \) on \( \overline{\Gamma} \). Then there exist a diffeomorphism \( F : \Omega \to \Omega \) satisfying (6) and \( F \in C^{3+\alpha}(\overline{\Omega}) \) which depends on \( g_1 \) and \( g_2 \), a strictly positive function \( \tilde{\beta} \in C^{2+\alpha}(\overline{\Omega}) \), constants \( C_1 = C_1(M_1, M_2, M_3) > 0 \), \( C_2 = C_2(M_1, M_2, M_3) > 0 \) and \( \gamma \in (0, 1) \) such that
\[
\|\tilde{\beta} F_* g_1 - g_2\|_{L^2(\Omega)} + \|F_* \beta_1 - \beta_2\|_{L^2(\Omega)} \leq C_1 \left( \ln \left( \frac{C_2 + G(\|A_{\beta_1,g_1} - A_{\beta_2,g_2}\|_{L^2(\overline{\Omega})};L^2(\overline{\Omega}))}{G(\|A_{\beta_1,g_1} - A_{\beta_2,g_2}\|_{L^2(\overline{\Omega})};L^2(\overline{\Omega}))} \right) \right)^{-\gamma}
\]
with the function \( G \) defined by (10).

**Remark 1.** It is well-known that the inverse problem (12) - (13) admits the following non-uniqueness: pairs \((\beta, g)\) and \((\beta, g)\) where \( \beta \) satisfies the conditions of Theorem 1.2, generate the same Dirichlet-to-Neumann map. Therefore one can not avoid the appearance of the function \( \tilde{\beta} \) on the left-hand side of inequality (15).

As for stability and related results, we further refer to Bellassoued and Dos Santos Ferreira [7], Blästen, Imanuvilov and Yamamoto [8], Mandache [21], Montalto [22], Novikov and Santacesaria [25], [26], Santacesaria [29].

Theorems 1.1 and 1.2 are first results for the respective related inverse problems, and for the technical feasibility, we assume the class in \( C^{2+\alpha}(\overline{\Omega}) \) as admissible set.
of Riemannian metrics under consideration, and we do not pursue admissible sets in less regular function spaces.

The paper is composed of five sections. The proof of Theorem 1.1 is divided into three steps according to the main ideas used for the proof: Sections 2-4 are devoted to these three steps respectively.

Section 2: the proofs of the existence of a diffeomorphism and an estimate of 

\[ F_j g_1 - g_2 \text{ on } \tilde{\Gamma} \] (Propositions 1 and 2).

Section 3: the establishment of \( L^p \)-Carleman estimates for the Laplacian (Propositions 6 and 7).

Section 4: the completion of the proof of Theorem 1.1: after the construction of a conformal diffeomorphism, we reduce the proof to the stability result in [8] for an inverse boundary value problem for Schrödinger equations.

In Section 5, we complete the proof of Theorem 1.2, which is based on the stability result in [8].

2. First Step of the proof of Theorem 1.1: boundary estimate. We define an elliptic operator by 

\[ L_{g_j}(x, D, s) = -\Delta_{g_j} + s, \quad j = 1, 2 \] with \( s \geq 0 \).

Let 

\[ u_j = u_j(x, s), \quad j = 1, 2, \] be the solutions to the boundary value problems

\[ L_{g_j}(x, D, s)u_j = 0 \text{ in } \Omega, \quad u_j|_{\partial\Omega} = f, \tag{16} \]

where \( f \in C_0^\infty(\tilde{\Gamma}) \).

In terms of the eigenfunctions \( e_k(j) \) and the eigenvalues \( \lambda_k(j) \), one can write the solution to equation (16) in the form of the infinite series which is convergent in \( L^2(\Omega) \) as follows:

\[ u_j(x, s) = \sum_{k=1}^{\infty} \frac{e_k(j)(x)}{\lambda_k(j) + s} \int_{\tilde{\Gamma}} \frac{\partial e_k(j)}{\partial \nu_{g_j}} f(y) d\sigma_y. \]

Then the normal derivative of the function \( u_j \) can be represented in the form of infinite series converging in \( W_2^{-\frac{3}{2}}(\partial\Omega) \):

\[ \frac{\partial u_j}{\partial \nu_{g_j}}(x, s) = \sum_{k=1}^{\infty} \frac{\partial e_k(j)(x)}{\lambda_k(j) + s} \int_{\tilde{\Gamma}} \frac{\partial e_k(j)}{\partial \nu_{g_j}} f(y) d\sigma_y. \tag{17} \]

Indeed, for \( N \in \mathbb{N} \), we set

\[ u_{j,N}(x, s) = \sum_{k=1}^{N} \frac{e_k(j)(x)}{\lambda_k(j) + s} \int_{\tilde{\Gamma}} \frac{\partial e_k(j)}{\partial \nu_{g_j}} f(y) d\sigma_y. \]

We need the estimate of the difference of the normal derivatives \( \frac{\partial u_j}{\partial \nu_{g_j}} \). The following is known (see e.g., [13]) that there exists a constant \( C_1 \) independent of \( k \) and \( j \) such that

\[ \left\| \frac{\partial e_k(j)}{\partial \nu_{g_j}} \right\|_{L^2(\partial\Omega)}^2 \leq C_1 (|\lambda_k(j)||e_k(j)|^2_{L^2(\Omega)} + \|
abla e_k(j)\|^2_{L^2(\Omega)}). \tag{18} \]

In order to prove (18), we write equation (3) as

\[ -\sum_{m, \ell=1}^{2} \partial_{x_{m\ell}}(\det g_j, g_j^m_\ell \partial_{x_m} e_k(j)) = \sqrt{\det g_j} \lambda_k(j) e_k(j) \text{ in } \Omega. \tag{19} \]
Let \( P(x, D) = \sum_{p=1}^{2} a_p(x) \partial_{x_p} \) be a first order differential operator with coefficients \( a_p \in C^{2+\alpha} (\Omega) \) such that \( a_p(x) = \nu_p(x) \) on \( \partial \Omega \). Taking the scalar product of \( P(x, D) e_k(j) \) and equation (19) in \( L^2(\Omega) \) after integration by parts, we have

\[
\frac{1}{2} \int_{\Omega} \lambda_k(j) e_k^2(j) P^*(x, D) \sqrt{\det g_{j}} \, dx \\
= \int_{\Omega} \sum_{m, \ell = 1}^{2} \sqrt{\det g_{j}} g_{j}^{m \ell} \partial_{x_m} e_k(j)(P(x, D) \partial_{x_j} e_k(j) + [\partial_{x_j}, P(x, D)] e_k(j)) \, dx \\
- \int_{\partial \Omega} \sum_{m, \ell = 1}^{2} \sqrt{\det g_{j}} g_{j}^{m \ell} \nu_m \frac{\partial e_k(j)}{\partial \nu} \nu_{\ell} \frac{\partial e_k(j)}{\partial \nu} \, d\sigma \\
= \int_{\Omega} \sum_{m, \ell = 1}^{2} (P^*(x, D) \sqrt{\det g_{j}} g_{j}^{m \ell}) \partial_{x_m} e_k(j) \partial_{x_j} e_k(j) \, dx \\
+ \int_{\Omega} \sum_{m, \ell = 1}^{2} \sqrt{\det g_{j}} g_{j}^{m \ell} \partial_{x_m} e_k(j) [\partial_{x_j}, P(x, D)] e_k(j) \, dx \\
- \frac{1}{2} \int_{\partial \Omega} \sum_{m, \ell = 1}^{2} \sqrt{\det g_{j}} g_{j}^{m \ell} \nu_m \frac{\partial e_k(j)}{\partial \nu} \nu_{\ell} \frac{\partial e_k(j)}{\partial \nu} \, d\sigma.
\]

Here we set \( g_{j}^{-1} := \{g_{j}^{m \ell}\} \) for \( j = 1, 2 \). This equality and (8) imply (18).

Let \( \epsilon, \epsilon_1 \in (0, 1) \) be small parameters. Consider a sequence of functions \( \rho_\epsilon \) such that \( \rho_\epsilon \in C^\infty_0 (\tilde{\Gamma}), \ 0 \leq \rho_\epsilon \leq 1 \) on \( \tilde{\Gamma}, \ \rho_\epsilon(x) = 1 \ \forall x \in \{y \in \tilde{\Gamma}; \dist (\partial \tilde{\Gamma}, y) \geq \epsilon\} \).

Using (17), we obtain:

\[
\left\| \frac{\partial u_1}{\partial u_1} - \frac{\partial u_2}{\partial u_2} \right\|_{L^2(\tilde{\Gamma})} = \lim_{\epsilon \to +0} \left\| \rho_\epsilon \left( \frac{\partial u_1}{\partial u_1} - \frac{\partial u_2}{\partial u_2} \right) \right\|_{L^2(\tilde{\Gamma})} \\
= \lim_{\epsilon \to +0} \left( \lim_{\epsilon_1 \to +0} \sup_{w \in W^{\frac{3}{2}}_{2}(\partial \Omega)} \left( \frac{\partial u_1}{\partial u_1} - \frac{\partial u_2}{\partial u_2}, \rho_\epsilon w \right) \right)_{L^2(\partial \Omega)} \\
= \lim_{\epsilon \to +0} \left( \lim_{\epsilon_1 \to +0} \sup_{w \in W^{\frac{3}{2}}_{2}(\partial \Omega)} \left( \frac{\partial u_1}{\partial u_1} - \frac{\partial u_2}{\partial u_2}, \rho_\epsilon w \right) \right)_{L^2(\partial \Omega)} \\
\leq \lim_{N \to +\infty} \left( \sum_{k=1}^{N} \frac{\partial e_k(1)}{\partial \nu_{\xi}} \frac{1}{\lambda_k(1)} + \frac{1}{s} \int_{\tilde{\Gamma}} \frac{\partial e_k(1)}{\partial \nu_{\xi}} f(y) \, d\sigma y \right) \\
- \sum_{k=1}^{N} \frac{\partial e_k(2)}{\partial \nu_{\xi}} \frac{1}{\lambda_k(2)} + \frac{1}{s} \int_{\tilde{\Gamma}} \frac{\partial e_k(2)}{\partial \nu_{\xi}} f(y) \, d\sigma y \right)_{L^2(\tilde{\Gamma})} \\
\leq \lim_{N \to +\infty} \left( \sum_{k=1}^{N} \left( \frac{\partial e_k(1)}{\partial \nu_{\xi}} - \frac{\partial e_k(2)}{\partial \nu_{\xi}} \right) \frac{1}{\lambda_k(1)} + \frac{1}{s} \int_{\tilde{\Gamma}} \frac{\partial e_k(1)}{\partial \nu_{\xi}} f(y) \, d\sigma y \right)_{L^2(\tilde{\Gamma})}
\]

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Combining estimates (21) - (23), from (20) we have

\[
+ \lim_{N \to +\infty} \left\| \sum_{k=1}^{N} \frac{\partial e_k(2)}{\partial \nu g_2} \int_{\tilde{\Gamma}} \left( \frac{1}{\lambda_k(1)} - \frac{\partial e_k(1)}{\partial \nu g_1} \right) f(y) d\sigma(y) \right\|_{L^2(\tilde{\Gamma})} \\
\leq \lim_{N \to +\infty} \left\| \sum_{k=1}^{N} \left( \frac{\partial e_k(1)}{\partial \nu g_1} - \frac{\partial e_k(2)}{\partial \nu g_2} \right) \frac{1}{\lambda_k(1)} \int_{\tilde{\Gamma}} \frac{\partial e_k(1)}{\partial \nu g_1} f(y) d\sigma(y) \right\|_{L^2(\tilde{\Gamma})} \\
+ \lim_{N \to +\infty} \left\| \sum_{k=1}^{N} \frac{\partial e_k(2)}{\partial \nu g_2} \int_{\tilde{\Gamma}} \left( \frac{1}{\lambda_k(1)} + s \right) \frac{\partial e_k(1)}{\partial \nu g_1} f(y) d\sigma(y) \right\|_{L^2(\tilde{\Gamma})} \\
+ \lim_{N \to +\infty} \left\| \sum_{k=1}^{N} \frac{\partial e_k(2)}{\partial \nu g_2} \int_{\tilde{\Gamma}} \left( \frac{1}{\lambda_k(2)} + s \right) \left( \frac{\partial e_k(1)}{\partial \nu g_1} - \frac{\partial e_k(2)}{\partial \nu g_2} \right) f(y) d\sigma(y) \right\|_{L^2(\tilde{\Gamma})} \\
=: I_1 + I_2 + I_3.
\]

We estimate the terms \( I_j \) separately. For the first term \( I_1 \), using (18) we have

\[
I_1 \leq \sum_{k=1}^{\infty} \lambda_k^{-1}(1) \left\| \frac{\partial e_k(1)}{\partial \nu g_1} - \frac{\partial e_k(2)}{\partial \nu g_2} \right\|_{L^2(\tilde{\Gamma})} \left\| \frac{\partial e_k(1)}{\partial \nu g_1} \right\|_{L^2(\tilde{\Gamma})} \left\| f \right\|_{L^2(\tilde{\Gamma})} =
\]

\[
\sum_{k=1}^{\infty} \lambda_k^{-\frac{1}{2}}(1) \left\| \frac{\partial e_k(1)}{\partial \nu g_1} - \frac{\partial e_k(2)}{\partial \nu g_2} \right\|_{L^2(\tilde{\Gamma})} \left\| \frac{\partial e_k(1)}{\partial \nu g_1} \right\|_{L^2(\tilde{\Gamma})} \lambda_k^{-\frac{1}{2}}(1) \left\| f \right\|_{L^2(\tilde{\Gamma})} \\
\leq C_2 \sum_{k=1}^{\infty} \lambda_k^{-\frac{1}{2}}(1) \left\| \frac{\partial e_k(1)}{\partial \nu g_1} - \frac{\partial e_k(2)}{\partial \nu g_2} \right\|_{L^2(\tilde{\Gamma})} \left\| f \right\|_{L^2(\tilde{\Gamma})}.
\]

For \( I_2 \) and \( I_3 \), using (18) we obtain

\[
I_2 \leq C_3 \sum_{k=1}^{\infty} \left\| \frac{\partial e_k(2)}{\partial \nu g_2} \right\|_{L^2(\tilde{\Gamma})} \left\| \frac{\partial e_k(1)}{\partial \nu g_1} \right\|_{L^2(\tilde{\Gamma})} \left\| \lambda_k(1) - \lambda_k(2) \right\|_{L^2(\tilde{\Gamma})} \left\| f \right\|_{L^2(\tilde{\Gamma})} \\
\leq C_4 \sum_{k=1}^{\infty} \left\| \lambda_k(1) - \lambda_k(2) \right\|_{L^2(\tilde{\Gamma})} \left\| f \right\|_{L^2(\tilde{\Gamma})}
\]

and

\[
I_3 \leq \sum_{k=1}^{\infty} \lambda_k^{-1}(2) \left\| \frac{\partial e_k(1)}{\partial \nu g_1} - \frac{\partial e_k(2)}{\partial \nu g_2} \right\|_{L^2(\tilde{\Gamma})} \left\| \frac{\partial e_k(1)}{\partial \nu g_1} \right\|_{L^2(\tilde{\Gamma})} \left\| f \right\|_{L^2(\tilde{\Gamma})} =
\]

\[
\sum_{k=1}^{\infty} \lambda_k^{-\frac{1}{2}}(2) \left\| \frac{\partial e_k(1)}{\partial \nu g_1} - \frac{\partial e_k(2)}{\partial \nu g_2} \right\|_{L^2(\tilde{\Gamma})} \left\| \frac{\partial e_k(2)}{\partial \nu g_2} \right\|_{L^2(\tilde{\Gamma})} \lambda_k^{-\frac{1}{2}}(2) \left\| f \right\|_{L^2(\tilde{\Gamma})} \\
\leq C_5 \sum_{k=1}^{\infty} \lambda_k^{-\frac{1}{2}}(1) \left\| \frac{\partial e_k(1)}{\partial \nu g_1} - \frac{\partial e_k(2)}{\partial \nu g_2} \right\|_{L^2(\tilde{\Gamma})} \left\| f \right\|_{L^2(\tilde{\Gamma})}.
\]

Combining estimates (21) - (23), from (20) we have

\[
\left\| \frac{\partial u_1}{\partial \nu g_1} - \frac{\partial u_2}{\partial \nu g_2} \right\|_{L^2(\tilde{\Gamma})} \leq C_6 A(g_1, g_2) \left\| f \right\|_{L^2(\tilde{\Gamma})}.
\]

Let \( \Omega_1 \) be a bounded domain in \( \mathbb{R}^2 \) and \( F \in C^{\alpha}(\overline{\Omega}_1, \overline{\Omega}) \) be a diffeomorphism of the domain \( \Omega_1 \) on \( \Omega \) such that

\[
\| F \|_{C^{3+\alpha}(\overline{\Omega}_1, \overline{\Omega})} + \| F^{-1} \|_{C^{3+\alpha}(\overline{\Omega}, \overline{\Omega}_1)} \leq C_7.
\]
Let \( u_j \) be some solution to problem (16). Then the function \( u_j = \tilde{u}_j \circ F^{-1} \) is a solution to the following boundary value problem

\[
L_{F, \mathbf{g}_j}(x, D, s)\tilde{u}_j = 0 \quad \text{in} \quad \Omega_1, \quad \tilde{u}_j|_{\partial \Omega_1} = \tilde{f},
\]

where \( \tilde{f} = f \circ F \). Moreover short computations imply

\[
\frac{\partial u_j}{\partial \nu_{\mathbf{g}_j}} = \left( \gamma \frac{\partial \tilde{u}_j}{\partial \nu_{F, \mathbf{g}_j}} \right) \circ F^{-1}, \quad \gamma = |\det F'|/|((F^{-1})'\nu)|,
\]

where \( \nu \) is the outward unit normal vector to \( \Omega \).

In order to prove (27) we observe

\[
\frac{\partial u}{\partial \nu} = \sqrt{\det F_{\mathbf{g}}} |\det F'| \sum_{j,k=1}^2 g_{jk} \nu_k \partial u \times u = \sqrt{\det F_{\mathbf{g}}} |\det F'| g_\nu, \nabla u.
\]

Let a function \( v(x) \) satisfy the following conditions

\( v(x) > 0 \) for \( x \in \Omega_1, \quad v|_{\partial \Omega_1} = 0, \quad \nabla v|_{\partial \Omega_1} \neq 0 \).

We set \( v^*(x) = v \circ F^{-1} \). Observe that

\[
\nabla u = ((F^{-1})'\nabla u) \circ F^{-1}, \quad \nabla v^* = ((F^{-1})'\nabla v) \circ F^{-1}.
\]

Since \( \nu = \frac{\nabla v^*}{|\nabla v^*|} \), short computations imply

\[
\frac{\partial u}{\partial \nu} = \sqrt{\det F_{\mathbf{g}}} |\det F'| ((F^{-1})'g((F^{-1})'T_{\nu} \nabla u) \circ F^{-1}).
\]

The function \( \gamma \) is independent of the metrics \( \mathbf{g}_j \) and, provided that diffeomorphism \( F \) satisfies (25), there exist positive constants \( C_8, C_9 \) such that

\[
\|\gamma\|_{C^{2+\alpha}(\Gamma)} \leq C_8, \quad \gamma(x) \geq C_9 > 0 \quad \forall x \in \Gamma.
\]

Since the eigenvalues of the operators \( \Delta_{\mathbf{g}} \) and \( \Delta_{F, \mathbf{g}} \) are exactly the same, from (28) there exist constants \( C_{10}, C_{11} > 0 \) such that

\[
C_{10} \Lambda(\mathbf{g}_1, \mathbf{g}_2) \leq \Lambda(F_{\mathbf{g}_1}, F_{\mathbf{g}_2}) \leq C_{11} \Lambda(\mathbf{g}_1, \mathbf{g}_2).
\]

Hence, since the domain \( \Omega \) is simply connected, after possible change of variables, we can assume that

\[
\Omega \subset \{x \in \mathbb{R}^2; \quad 0 < x_2 < 1\} \quad \text{and} \quad \Gamma = \{x \in \mathbb{R}^2; \quad x_2 = 1, \quad -1 \leq x_1 \leq 1\},
\]

\( \Omega \) is symmetric with respect to the \( x_2 \)-axis.

We have

**Proposition 1.** Let \( \Gamma_1 \) be an open subset of \( \Gamma \) such that \( \overline{\Gamma_1} \subset \Gamma \). For any metrics \( \mathbf{g}_1, \mathbf{g}_2 \) satisfying (8), there exist constants \( C_{12}, C_{13} \) independent of \( \mathbf{g}_1, \mathbf{g}_2 \) and a diffeomorphism

\[
F \in C^{3+\alpha}(\overline{\Omega}), \quad F|_{\Gamma} = I : \quad \text{the identity map}
\]

such that

\[
\|F_{\mathbf{g}_1} - \mathbf{g}_2\|_{C(\Gamma_1)} \leq C_{12} \Lambda^{\frac{1}{2}}(\mathbf{g}_1, \mathbf{g}_2)
\]

and

\[
\|F\|_{C^{3+\alpha}(\overline{\Omega})} + \|F^{-1}\|_{C^{3+\alpha}(\overline{\Omega})} \leq C_{13}.
\]
Proof. We observe that in order to prove the statement of the proposition, it suffices to show the following: For any metrics $g_1, g_2$ satisfying (8), there exist constants $C_{14}, C_{15}$ and a diffeomorphisms $F(j) \in C^{3+\alpha}(\overline{\Omega})$ such that $F(j)|_{\Gamma} = I$ and

\[ \|F^*(1)g_1 - F^*(2)g_2\|_{C^2(\Omega)} \leq C_{14}A^2(g_1, g_2) \]

and

\[ \sum_{j=1}^{2} (\|F(j)\|_{C^{3+\alpha}(\overline{\Omega})} + \|L^{-1}(j)\|_{C^{3+\alpha}(\overline{\Omega})}) \leq C_{15}. \]

Indeed, if (34) and (35) are both proved, then we set $F = F(1) \circ F^{-1}(2)$. This mapping is a diffeomorphism of the domain $\Omega$ onto $\Omega$ such that $F|_{\overline{\Gamma}} = (F(1) \circ F^{-1}(2))|_{\overline{\Gamma}} = (I \circ I)|_{\overline{\Gamma}} = I$. Estimate (32) follows from (34):

\[ \|F_\ast g_1 - g_2\|_{C^2(\Gamma)} = \|F'(2)(F'(1)_\ast g_1)(F'(2)^T) - g_2\|_{C^2(\Gamma)} = \|F'(2)(F'(1)_\ast g_1 - F'(2)_\ast g_2)(F'(2)^T)\|_{C^2(\Gamma)} \leq C_{15}^1 g_1^2 \]

(36) \leq \|F'(2)^2\|_{C^2(\Gamma)} \leq \|F'(1)_\ast g_1 - F'(2)_\ast g_2\|_{C^2(\Gamma)} \leq C_{15}^2 C_{14}^2 A^2(g_1, g_2).

First we prove the proposition for the case $\Gamma \subset \overline{\Gamma}_2 \subset \{x \in \mathbb{R}^2; x_1 = 1\}, \Omega \subset \mathbb{R} \times \mathbb{R}_+$ and

\[ g_{j}^{21} = g_{j}^{21} = 0 \quad \text{on} \quad \Gamma, \quad g_{j}^{22} = 1 \quad \text{on} \quad \Gamma, \quad j \in \{1, 2\}. \]

In this case, we have

\[ \frac{\partial}{\partial \nu_{g_j}} = \sqrt{\det g_j} \sum_{k=1}^{2} g_{j}^{k \ell} \nu_{g_j} \partial_{x_k} = \sqrt{\det g_j} \frac{1}{g_{j}^{11}} \partial_{x_2} = \frac{1}{g_{j}^{11}} \partial_{x_2} \quad \text{on} \quad \Gamma. \]

Let $w_j \in C^{3+\alpha}(\overline{\Omega})$ be the solution to the boundary value problem

(38) \[ L_{g_j}(x, D, s)w_j = 0 \quad \text{in} \quad \Omega, \quad w_j|_{\partial\Omega} = f, \quad \supp \ f \subset \Gamma. \]

Next we prove some regularity results for the function $w_j$. Let $\rho \in C^2(\mathbb{R} \times \mathbb{R}_+)$ be identically equal to one in some neighborhood of $\Gamma$ and identically equal to zero on the set $\{x; \text{dist} (x, \Gamma) \geq \epsilon_0\}$ with some small positive $\epsilon_0$. We set $\bar{w}_j = \rho w_j$. Then $L_{g_j}(x, D, s)\bar{w}_j = r_j$ on $\mathbb{R} \times \mathbb{R}_+$ and $\bar{w}_j|_{\partial\Omega} = f$. From the standard estimates in the Schauder spaces and the Sobolev spaces for elliptic equations, there exists a constant $C_{16} > 0$, independent of $s$, such that

\[ \|r_j\|_{C^{2+\alpha}(\overline{\Omega})} \leq C_{16} \|f\|_{L^2(\mathbb{R})}. \]

Indeed, by taking into consideration that $g_j \in C^{2+\alpha}(\overline{\Omega})$, the standard regularity property for the solution to the boundary value problem (e.g., Chapter 2 in [20]) yields

\[ \|w_j\|_{H^{2+\alpha}(\Omega)} \leq C_{17} \|f\|_{L^2(\overline{\Gamma})}. \]

Let $\rho_k \in C^2(\overline{\Omega}), \ k = 1, 2$ and $\rho_1 \equiv 0$ on $\{x; \text{dist} (x, \Gamma) \leq \frac{\epsilon_0}{3}\}, \rho_1 \equiv 1$ on $\{x; \text{dist} (x, \Gamma) \geq \frac{\epsilon_0}{3}\}$ and $\rho_2 \equiv 0$ on $\{x; \text{dist} (x, \Gamma) \leq \frac{\epsilon_0}{3}\}$ and $\rho_2 \equiv 1$ on $\{x; \text{dist} (x, \Gamma) \geq \frac{\epsilon_0}{3}\}$. Then $L_{g_j}(x, D, s)(\rho_k w_j) = [\rho_k, L_{g_j} w_j]$: the commutator, where

\[ \|[\rho_1, L_{g_j} w_j]\|_{H^{-\frac{1}{2}}(\Omega)} \leq C_{15} \|f\|_{L^2(\overline{\Gamma})}. \]
Therefore, applying the regularity property (e.g., [20]) again, we have
\[ \| \rho_1 w_j \|_{H^2(\Omega)} \leq C_{19} \| f \|_{L^2(\tilde{\Gamma})}. \]

Observe that
\[ \| \rho_2, Lg w_j \|_{L^2(\Omega)} \leq C_{20} \| f \|_{L^2(\tilde{\Gamma})}. \]

Therefore the regularity (e.g., [20]) and the Sobolev embedding theorem imply
\[ \| \rho_2 w_j \|_{C^{\beta}(\overline{\Omega})} \leq C_{21} \| \rho_2 w_j \|_{H^2(\Omega)} \leq C_{22} \| f \|_{L^2(\tilde{\Gamma})} \]
for any \( \beta \in (0, 1) \).

Then we apply the Schauder internal regularity estimates (e.g., Chapter 6 in [11])
to obtain the above estimate for \( r_j \).

Henceforth we understand the double signs correspond.

Next we consider a solution to boundary value problem (38) for highly oscillatory boundary data \( f \). Our goal is to construct an asymptotic behavior of the normal derivative of the solution \( w_j \) on \( \Gamma \). In order to do that, we employ the standard techniques based on the factorization of elliptic operator into a product of two pseudodifferential operators.

We introduce symbols \( \lambda_{\pm, 0}, \lambda_{\pm, -1}, \lambda_{\pm, 1} \) as follows:
\[ \lambda_{\pm, 0} : = \lambda_{\pm, 1}(x, \xi, s, j) = -ig_j^{12}(\xi_1 \pm \bar{\mu}(\xi_1)\xi_1)\sqrt{-g_j^{12}2 + \frac{\xi_1^2}{\bar{\mu}(\xi_1)}}, \]
where \( \bar{\mu} \in C^\infty(\mathbb{R}), \bar{\mu}[0, 1] = 0, \bar{\mu}(t) = 1 \) for all \( t \geq 2 \). Next, the symbols \( \lambda_{\pm, 0}(x, \xi, s, j) \) and \( \lambda_{\pm, -1}(x, \xi, s, j) \) are defined by
\[ \lambda_{+1, 0} \lambda_{-1} + \lambda_{+0, 0} \lambda_{-1} + \partial x_j \lambda_{-1} = (\partial \xi_1 \lambda_{+1}) \partial x_j \lambda_{-1} \]
\[ = \partial x_1(\sqrt{\det g_j g_j^{12}})i \xi_1 + \partial x_j(\sqrt{\det g_j g_j^{12}})i \xi_1, \]
\[ \lambda_{-0} + \lambda_{+0} = \partial x_1(\sqrt{\det g_j g_j^{12}}) + \partial x_j(\sqrt{\det g_j g_j^{12}}), \]
\[ \lambda_{+0} \lambda_{-0} + \lambda_{+1} \lambda_{-1} + \partial x_j \lambda_{-0} = (\partial \xi_1 \lambda_{+1}) \partial x_j \lambda_{-0} \]
\[ = \lambda_{-1} + \lambda_{+1} = 0. \]

Let us show that the system (39)-(42) is solvable. In order to determine the symbols \( \lambda_{\pm, 0} \), we consider the equations (39) and (40). Using the identity (40), we write
\[ \lambda_{+1}(\partial x_1, \sqrt{\det g_j g_j^{12}}) + \partial x_j(\sqrt{\det g_j g_j^{12}}) + \lambda_{+0}(\lambda_{-1} - \lambda_{+1}), \]
Using the above identity in the equality (39), for any \( |\xi_1| \geq 2 \) we obtain
\[ \lambda_{+0}(x, \xi, s, j) = (\lambda_{-1} - \lambda_{+1})^{-1} \left[ -\partial x_j \lambda_{-1} - (\partial \xi_1 \lambda_{+1}) \partial x_j \lambda_{-1} \right. \]
\[ + \partial x_1(\sqrt{\det g_j g_j^{12}}) + \partial x_j(\sqrt{\det g_j g_j^{12}}) \xi_1 \]
\[ \left. - \lambda_{+1}(\partial x_1, \sqrt{\det g_j g_j^{12}}) \right]. \]
From (40), we determine the formulae for the symbol $\lambda_{-\cdot}$. Finally, from (41) and (42) we determine $\lambda_{\pm,-1}$ for any $|\xi_1| \geq 2$ by the following formulae

$$\lambda_{+,\cdot} = -\lambda_{\cdot,-1},$$

$$\lambda_{+,1} \lambda_{-\cdot,-1} + \lambda_{+,1} \lambda_{-,1} = \frac{\lambda_{+\cdot,0} \lambda_{-\cdot,0} + \partial_{x_2} \lambda_{-\cdot,0} + \partial_{\xi_1} \lambda_{+\cdot,0} \partial_{x_2} \lambda_{-\cdot,1} + \partial_{\xi_1} \lambda_{+,1} \partial_{x_2} \lambda_{-\cdot,0}}{\lambda_{+,1} - \lambda_{-,1}}.$$

Now we introduce two symbols

$$\lambda_{\pm}(x, \xi_1, s, j) = \lambda_{\pm,1}(x, \xi_1, s, j) + \lambda_{\pm,0}(x, \xi_1, s, j) + \lambda_{\pm,-1}(x, \xi_1, s, j), \quad j \in \{1, 2\}.$$

Then we can write

$$(\partial_{x_2} - \lambda_{+}(x, D_1, s, j)) (\partial_{x_2} - \lambda_{-}(x, D_1, s, j)) \tilde{w}_j =: r_j + K_j(x, D, s) \tilde{w}_j,$$

where the operators $K_j(x, D, s)$ satisfy the estimate

$$\|K_j(x, D, s) \tilde{w}_j\|_{L^2(\mathbb{R} \times \mathbb{R}_+)} \leq C_{23} \|\tilde{w}_j\|_{L^2(\mathbb{R} \times \mathbb{R}_+)}. \tag{43}$$

Here the constant $C_{23}$ is independent of $s$ and $j \in \{1, 2\}$.

Denote $p_j(x_1, s) = (\partial_{x_2} - \lambda_{-}(x, D_1, s, j)) \tilde{w}_j|_{x_2=0}$. Applying Lemma A.7 of [14], we obtain

$$\|\partial_{x_2} - \lambda_{-}(x, D_1, s, j) \tilde{w}_j|_{x_2=0}\|_{L^2(\mathbb{R})} \leq C_{24} \|\tilde{w}_j\|_{L^2(\mathbb{R} \times \mathbb{R}_+)}, \tag{44}$$

where the constant $C_{24}$ is independent of $s$ and $j \in \{1, 2\}$. From (37), (44) and (43), we have

$$\|\frac{1}{\sqrt{g_1}} \lambda_{-}(x, D_1, s, 1) \tilde{w}_1|_{x_2=0} - \frac{1}{\sqrt{g_2}} \lambda_{-}(x, D_1, s, 2) \tilde{w}_2|_{x_2=0}\|_{L^2(\mathbb{R})} \leq \left\| \frac{1}{\sqrt{g_1}} \frac{\partial \tilde{w}_1}{\partial x_2} - \frac{1}{\sqrt{g_2}} \frac{\partial \tilde{w}_2}{\partial x_2} + p_2 - p_1 \right\|_{L^2(\mathbb{R})} \leq \left\| \frac{1}{\sqrt{g_1}} \frac{\partial \tilde{w}_1}{\partial x_2} - \frac{1}{\sqrt{g_2}} \frac{\partial \tilde{w}_2}{\partial x_2} \right\|_{L^2(\mathbb{R})} + \|p_2 - p_1\|_{L^2(\mathbb{R})} \leq \left| C_{25}(\lambda(g_1, g_2)) \right| f\|_{L^2(\mathbb{R})} + \|\tilde{w}_j\|_{L^2(\mathbb{R} \times \mathbb{R}_+)} + \|f\|_{L^2(\mathbb{R})} \right\|_{L^2(\mathbb{R})} \leq C_{26}(\lambda(g_1, g_2)) \|f\|_{L^2(\mathbb{R})} + \|f\|_{L^2(\mathbb{R})}.$$

Consider the function $f = e^{ix_1} g$ where $g \in C^\infty_0(\tilde{\Gamma})$ and $g = 1$ on $\tilde{\Gamma}_1$.

Denote by $q(x_1, \xi_1, s)$ the principal symbol of the pseudodifferential operator

$$\frac{1}{\sqrt{g_1^*(x_1,0)}} \lambda_{-}(x_1, 0, D_1, s, 1) - \frac{1}{\sqrt{g_2^*(x_1,0)}} \lambda_{-}(x_1, 0, D_1, s, 2).$$

We can represent

$$q(x_1, \xi_1, s) = \frac{s^2}{\sqrt{s_1^2 + s^2_1(x_1,0)}} - \frac{s^2}{\sqrt{s_1^2 + s^2_2(x_1,0)}}.$$
Then
\[ q(x_1, D_1, s)(e^{isx_1}) = \int_{\mathbb{R}} \int_{\mathbb{R}} q(x_1, \xi_1, s)e^{i(x_1 - y_1)\xi_1 + sy_1}) g(y_1) d\xi_1 dy_1 \]
(46)
\[ = \int_{\mathbb{R}} \int_{\mathbb{R}} sq(x_1, s\xi_1, s)e^{is(x_1 - y_1)\xi_1 + y_1}) g(y_1) d\xi_1 dy_1. \]

Applying the stationary phase argument (e.g., [10]), we obtain
\[ q(x_1, D_1, s)(e^{isx_1}) = \frac{2\pi \left( \frac{s}{g_1'(x_1, 0)} - \frac{s}{g_2'(x_1, 0)} \right) e^{isx_1} g(x_1)}{\sqrt{\frac{1}{g_1'(x_1, 0)} + \frac{1}{g_2'(x_1, 0)}}} + M(s, x_1), \]
where the term \( M \) satisfies
\[ \|M(s, \cdot)\|_{C(\bar{\Gamma})} \leq C_{27}. \]

Observe that Lemma A.2 of [14] yields
\[ \left\| \frac{1}{\sqrt{g_1}} \lambda_-(x, D_1, s, 1)\tilde{w}_1|_{x_2=0} - \frac{1}{\sqrt{g_2}} \lambda_-(x, D_1, s, 2)\tilde{w}_2|_{x_2=0} - q(x_1, D_1, s)(e^{isx_1} g) \right\|_{L^2(\mathbb{R})} \]
\[ \leq C_{28}\|e^{isx_1} g\|_{L^2(\bar{\Gamma})}. \]
(49)

Therefore by (45) and (49) we have
\[ \|q(x_1, D_1, s)(e^{isx_1} g)\|_{L^2(\bar{\Gamma})} \]
\[ \leq \left\| \frac{1}{\sqrt{g_1}} \lambda_-(x, D_1, s, 1)\tilde{w}_1|_{x_2=0} - \frac{1}{\sqrt{g_2}} \lambda_-(x, D_1, s, 2)\tilde{w}_2|_{x_2=0} - q(x_1, D_1, s)(e^{isx_1} g) \right\|_{L^2(\bar{\Gamma})} \]
\[ + \left\| \frac{1}{\sqrt{g_1}} \lambda_-(x, D_1, s, 1)\tilde{w}_1|_{x_2=0} - \frac{1}{\sqrt{g_2}} \lambda_-(x, D_1, s, 2)\tilde{w}_2|_{x_2=0} \right\|_{L^2(\bar{\Gamma})} \]
\[ \leq C_{29}(\Lambda(g_1, g_2))\|g\|_{L^2(\bar{\Gamma})} + \|g\|_{L^2(\bar{\Gamma})}. \]
(50)

Using (47) and (48), in order to estimate the left-hand side of (50) from below, we see that there exists a constant \( C_{30} > 0 \), independent of \( s \), such that
\[ s \left\| \frac{1}{g_1} - \frac{1}{g_2} \right\|_{L^2(\bar{\Gamma})} \leq C_{30}(\Lambda(g_1, g_2))\|g\|_{L^2(\bar{\Gamma})} + 1. \]

Taking \( s = 1/\Lambda^{\frac{1}{2}}(g_1, g_2) \) from the above inequality, we obtain (34) under assumptions (37).

Now we will remove the restriction (37).

Let \( K > 0 \) be some parameter which we choose later. First we construct a diffeomorphism \( R = (R_1, R_2) : \mathbb{R} \times \mathbb{R}^+ \rightarrow \mathbb{R} \times \mathbb{R}^+ \) such that \( R = I \) on \( \{(x_1, 0); |x_1| \leq K\} \subset \Sigma(\bar{\Gamma}) \) and \( (R_1, g_1)^{12} = 0 \) on \( \{(x_1, 0); |x_1| \leq K\} \).

For all \( |x_1| \leq K \), direct computations yield
\[ R'(x_1, 0) = \left( \begin{array}{cc} 1 & \frac{\partial x_2 R_1}{\partial x_1} \\ 0 & \frac{\partial x_2 R_2}{\partial x_1} \end{array} \right) (x_1, 0), \]
Let us show that the constructed mapping is injective. Indeed let

\[(R')^{-1}(x_1,0) = \frac{1}{\partial x_2 R_2(x_1,0)} \begin{pmatrix} \partial x_2 R_2 & -\partial x_2 R_1 \\ 0 & 1 \end{pmatrix} (x_1,0).\]

Here we recall that \(R'\) is the Jacobian matrix. The formula (7) implies

\[R_*g(x_1,0) = \frac{1}{(\partial x_2 R_2)^2} \begin{pmatrix} \partial x_2 R_2 & -\partial x_2 R_1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} g_1^{11} & g_1^{12} \\ g_1^{12} & g_1^{22} \end{pmatrix} \begin{pmatrix} \partial x_2 R_2 & 0 \\ -\partial x_2 R_1 & 1 \end{pmatrix} (x_1,0) = \frac{1}{(\partial x_2 R_2(x_1,0))^2} \begin{pmatrix} \partial x_2 R_2(x_1,0) & 0 \\ -\partial x_2 R_1(x_1,0) & 1 \end{pmatrix} \begin{pmatrix} \partial x_2 R_2(x_1,0) & -\partial x_2 R_1(x_1,0) \\ g_2^{11} - (\partial x_2 R_2(x_1,0))^2 g_2^{22} & (\partial x_2 R_2(x_1,0))^2 g_2^{22} \end{pmatrix} (x_1,0)
\]

for \(|x_1| \leq K\).

We extend the functions \(g_1^{12}(x_1,0), g_1^{22}(x_1,0)\) on \([-2K, 2K]\) as positive \(C^{2+\alpha}\) functions. Let \(\bar{P} \in C^{1+\alpha}([-2K, 2K] \times [0, K]) \cap C^{10+\alpha}([-2K + \epsilon, 2K - \epsilon] \times [\epsilon, K])\) satisfy

\[\bar{P}(x_1,0)|_{[-2K,2K]} = 0, \quad \partial x_2 \bar{P}(x_1,0)|_{[-2K,2K]} = \frac{g_1^{12}(x_1,0)}{g_2^{22}(x_1,0)}.
\]

Let \(\eta \in C_0^{\infty}(-2K, 2K)\) and \(\eta|_{[-K,K]} = 1\). We set \(P(x) = \eta(x_1) \bar{P}(x) + (1 - \eta(x_1))x_2\). As such an \(R\), we take

\[R(x) = \left(x_1 + \eta\left(\frac{x_2}{\epsilon}\right) P(x), x_2\right).
\]

Obviously \(R(x_1,0) = (x_1,0)\) and \(R(x) \in \mathbb{R} \times \mathbb{R}_+\). Moreover

\[\partial x_2 R_1(x_1,0) = \frac{(\partial x_2 R_2)(x_1,0) g_2^{12}(x_1,0)}{g_2^{22}(x_1,0)} \quad \text{on } [-K, K].
\]

This equality implies \((R_* g)^{12} = 0\) on \(\{(x_1,0); |x_1| \leq K\}\).

The Jacobi matrix of the mapping \(R\) is given by

\[R'(x) = \begin{pmatrix} 1 + \eta\left(\frac{x_2}{\epsilon}\right) \partial x_1 P, & \partial x_2 \eta\left(\frac{x_2}{\epsilon}\right) P \end{pmatrix}.
\]

We can choose \(\epsilon > 0\) such that

\[\eta\left(\frac{x_2}{\epsilon}\right) \partial x_1 P(x) \geq -\frac{1}{2} \quad \text{on } \mathbb{R} \times \mathbb{R}_+.
\]

Then (53) yields

\[\det R'(x) = 1 + \eta\left(\frac{x_2}{\epsilon}\right) \partial x_1 P(x) \geq \frac{1}{2}, \quad \forall x \in \mathbb{R} \times \mathbb{R}_+.
\]

From (52) and (54), there exists a constant \(C_{31} > 0\), which is independent of \(g\), such that

\[\sup_{x \in \mathbb{R} \times \mathbb{R}_+} \|(R')^{-1}(x)\| \leq C_{31}.
\]

Let us show that the constructed mapping is injective. Indeed let \(R(x) = R(y)\) for some \(x, y \in \mathbb{R} \times \mathbb{R}_+\). This immediately implies \(x_2 = y_2\). Consider the function \(r(t) = t + \eta\left(\frac{x_2}{\epsilon}\right) P(t, y_2)\). Since \(r'(t) > 0\) for \(t > 0\), we see that \(y_1 = x_1\).
Next we prove that the mapping $F$ is surjective. For $(\tilde{y}_1, \tilde{y}_2) \in \mathbb{R} \times \mathbb{R}_+$, we have to solve the equation $R(x) = \tilde{y}$. Obviously $x_2 = \tilde{y}_2$. Hence we need to solve the equation $R_1(x_1, \tilde{y}_2) = r(x_1) = \tilde{y}_1$. Thanks to (53) this equation has a unique solution, which means that $P$ is surjective.

In the next step, we construct a diffeomorphism $G = (G_1, G_2) : \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R} \times \mathbb{R}_+$ such that

\begin{equation}
(G_1 g)^{12} = 0 \quad \text{and} \quad (G_2 g)^{22} = 1 \quad \text{on} \quad \{|x_1| \leq K\}.
\end{equation}

Using the diffeomorphism constructed above and making the change of coordinates in $\mathbb{R} \times \mathbb{R}_+$, we can assume $g^{12}(x_1, 0) = 0$ on $\{|x_1| \leq K\}$. We take $G_1(x) = x_1$. Then

\[
G_1 g(x_1, 0) = \begin{pmatrix}
g^{11}(x_1, 0) & 0 \\
0 & \frac{g^{22}(x_1, 0)}{(\partial_x g^{22}(x_1, 0))^2}
\end{pmatrix}, \quad \forall x_1 \in \mathbb{R}.
\]

We extend the function $g^{22}(x_1, 0)$ on $[-2K, 2K]$ as positive function of the class $C^{2+\alpha}$. Let $P \in C^{3+\alpha}([-2K, 2K] \times [0, K]) \cap C^{10+\alpha}([-2K + \epsilon, 2K - \epsilon] \times \epsilon, K)$ satisfy

\begin{equation}
P(x_1, 0) = 0, \quad \partial_{x_2} P(x_1, 0) = \sqrt{g^{22}(x_1, 0)}. \quad (57)
\end{equation}

By (57), there exist constants $\delta_0 > 0$ and $\beta_1 > 0$ such that

\[
P(x) > 0 \quad \forall x \in [-2K, 2K] \times [0, \delta_0], \quad \partial_{x_2} \tilde{P}(x) > \beta_1 > 0 \quad \forall x \in [-2K, 2K] \times [0, \delta_0].
\]

Let $\eta \in C^\infty_0([-2K, 2K], \eta|_{-2K, 2K} = 1$ and $\eta_1$ be a monotone decreasing function such that $\eta_1|_{[0, \delta_0/4]} = 1$, $\eta_1(x_2) \geq 0$ for $x_2 \geq 0$, $\eta_1(x_2) = 0$ for $x_2 \geq \delta_0/2$. We set

\[
P(x) = \eta(x_1) \tilde{P}(x) + (1 - \eta(x_1))\kappa \beta_1 \quad \text{and} \quad \kappa = ||\tilde{P}||_{C^0([-2K, 2K] \times [0, \delta_0])}
\]

Finally we set

\[
G_2(x) = \kappa_1 x_2 (1 - \eta_1(x_2)) + \eta_1(x_2) P(x) \quad \text{and} \quad \kappa_1 = ||P||_{C^0(\mathbb{R} \times \frac{\delta_0}{2})}. \quad (58)
\]

By (57), we obtain

\[
\partial_{x_2} G_2(x_1, 0) = \sqrt{g^{22}(x_1, 0)} \quad \text{on} \quad [-K, K]
\]

and so (56). Moreover

\[
\partial_{x_2} P = \eta(x_1) \partial_{x_2} P(x) + (1 - \eta(x_1))\kappa \geq \eta(x_1)\beta_1 + (1 - \eta(x_1))\kappa \geq \min\{\beta_1, \kappa\}
\]

on $[-2K, 2K] \times [0, \delta_0]$. The Jacobi matrix $G'$ of the mapping $G$ is given by

\begin{equation}
G'(x) = \begin{pmatrix}
1 & 0 \\
\partial_{x_1} G_2(x) & \kappa_1 (1 - \eta_1(x_2)) + \eta_1(x_2) \partial_{x_2} P(x) + (P(x) - \kappa_1 x_2) \eta'_1(x_2)
\end{pmatrix}.
\end{equation}

Taking $\epsilon > 0$ sufficiently small, we see that there exists a constant $\beta > 0$, independent of $x \in \mathbb{R} \times \mathbb{R}_+$, such that

\begin{equation}
\text{det} G'(x) = \kappa_1 (1 - \eta_1(x_2)) + \eta_1(x_2) \partial_{x_2} P(x) + (P(x) - \kappa_1 x_2) \eta'_1(x_2) \geq \kappa_1 (1 - \eta_1(x_2)) + \eta_1(x_2) \beta_1 > \beta > 0.
\end{equation}

Let us show that the constructed mapping is injective. Indeed if $G(x) = G(y)$, then we immediately have $x_1 = y_1$. Consider the function $\tilde{r}(t) = G_2(x_1, t)$. The derivative of this function is positive for $t > 0$, so that $x_2 = y_2$.

Next we prove that the equation $G(x) = \tilde{y}$ has a solution for an arbitrary $(\tilde{y}_1, \tilde{y}_2) \in \mathbb{R} \times \mathbb{R}_+$. We set $x_1 = \tilde{y}_1$. By (57) we see $G(\tilde{y}_1, 0) = 0$ and $\lim_{x_2 \to +\infty} G_2(\tilde{y}_1, x_2) = +\infty$. Therefore, for fixed $\tilde{y}_1$, the function $x_2 \to G_2(\tilde{y}_1, x_2)$ takes all the values from $\mathbb{R}_+$. Outside of a ball with sufficiently large radius, the
mapping $G$ is linear. Then from (58) and (59), we obtain that there exists a constant $M > 0$, which is independent of $g$, such that

$$(60) \quad \sup_{x \in \mathbb{R} \times \mathbb{R}^+} \|(G')^{-1}(x)\| \leq C_{32}.$$  

Let $\Psi$ be a conformal diffeomorphism of $\Omega$ on $D$. We recall that $D = \{x \in \mathbb{R}^2; |x| < 1\} = \{z \in \mathbb{C}; |z| < 1\}$. Without loss of generality we can assume that $(0, -1) \in \Psi(\Gamma_0)$. Let $\Sigma(z) = \frac{(z+1)}{z+1}$ be a conformal diffeomorphism of $D$ on $\mathbb{R} \times \mathbb{R}_+$. Consider the mapping $\Psi^{-1} \circ \Sigma^{-1} \circ G \circ R \circ \Sigma \circ \Psi : \Omega \to \Omega$. Obviously this mapping is injective and surjective.

Outside of the intersection of any neighborhood of the point $(0, -1)$ with the unit disc $D$ centered at 0, the mapping belongs to the Hölder space $C^{3+\alpha}$. On the other hand, for some sufficiently large $r_0$, we see that $R = G = I$: the identity mapping in $\mathbb{R} \times \mathbb{R}^+ \setminus B(0, r_0)$. Hence the mapping $\Psi^{-1} \circ \Sigma^{-1} \circ G \circ R \circ \Sigma \circ \Psi : \Omega \to \Omega$ is a diffeomorphism. Taking $F(j)$ in the form $\Psi^{-1} \circ \Sigma^{-1} \circ G \circ R \circ \Sigma \circ \Psi$, where the mappings $G$ and $R$ are constructed for the corresponding metrics $g_j$, $j \in \{1, 2\}$, we obtain (37) and, as we have already established, we see that $F(j)_{\ast} g_j$ satisfies (34).

From (60) and (55), we obtain (35).

Using Proposition 1, we construct a diffeomorphism $F$ from the domain $\Omega$ into itself satisfying (31), (32) and (33). Observe that for the operators $\Delta_{\ast} g_j$ and $\Delta_{F_{\ast} g_j}$, the eigenvalues are the same. Moreover by $\tilde{e}_k(1) = e_k(1) \circ F^{-1}$ we denote the corresponding eigenvectors of the operator $\Delta_{F_{\ast} g_j}$. By $F = I$ on $\Gamma$, we have (see e.g., [18])

$$\frac{\partial e_k(1)}{\partial \nu_{g_j}|_{\tilde{\Gamma}}} = \frac{\partial e_k(1)}{\partial \nu_{F_{\ast} g_j}|_{\tilde{\Gamma}}}, \forall k \in \mathbb{N}.$$  

This implies that $\Lambda(g_1, g_2) = \Lambda(F_{\ast} g_1, g_2)$. Hence, without loss of generality, we can assume that there exists a constant $C_{33}$ independent of $g_1, g_2$ such that

$$(61) \quad \|g_1 - g_2\|_{C(\overline{\Gamma}_1)} \leq C_{33} \Lambda_{\tilde{\Gamma}}(g_1, g_2).$$  

Let $\overline{\Gamma}_1$ be fixed. We shrink the set $\Gamma$ up to $\overline{\Gamma}_1$ and later, in order to simplify the notations, we use $\tilde{\Gamma}$ instead of $\overline{\Gamma}_1$.

Next we construct a diffeomorphism $F$ of domain $\Omega$ on itself such that there exists a smooth positive function $\mu_2$ such that $F_{\ast} g_2 = \mu_2 I$, where $I$ is the $2 \times 2$ unit matrix and the mapping $F$ and the function $\mu$ depend on metric $g$ continuously. We follow the approach developed in [2] and [31]. More precisely we have the following proposition.

**Proposition 2.** For any metric $g$ satisfying (8), there exist a diffeomorphism $F = (F_1, F_2) : \Omega \to \Omega, F \in C^{3+\alpha}(\overline{\Omega})$ and $\mu \in C^{2+\alpha}(\overline{\Omega})$ continuously depending on $g$, constants $\beta, C_{35}, C_{36} > 0$, depending on the metric $g$, such that

$$(62) \quad F_{\ast} g = \mu I, \quad \|\mu\|_{C^{2+\alpha}(\overline{\Omega})} \leq C_{34}, \quad \mu(x) \geq \beta > 0, \quad \forall x \in \Omega,$$

where $\|\mu\|_{C^{2+\alpha}(\overline{\Omega})}$ and

$$\|F\|_{C^{3+\alpha}(\overline{\Omega})} + \|F^{-1}\|_{C^{3+\alpha}(\overline{\Omega})} \leq C_{35}.$$  

Moreover if

$$\|\mu\|_{C(\overline{\Gamma})} \leq \epsilon,$$

where $\|\mu\|_{C(\overline{\Gamma})} \leq M$ and

$$\mu(x) \geq \beta > 0, \quad \forall x \in \overline{\Gamma},$$

$$\epsilon,$$
then there exists a continuous function $C_{36}$ such that
\[(64)\quad \|\partial_z(F_1 + iF_2)\|_{C^0(\Gamma)} \leq C_{36}(M, \beta)\epsilon.\]

**Proof.** Without loss of generality, we can assume that $0 \in \Omega$. Using the conformal mapping if necessary, we can assume that $\Omega = \mathbb{D}$. Since the domain $\Omega$ is simply connected, there exists a conformal diffeomorphism $P$ of $\Omega$ in $\mathbb{D} := \{ z \in \mathbb{C}^1; |z| < 1 \}$ such that $P(0) = 0$. It is known that for any $\hat{z} \in \mathbb{C}$ there exists a conformal diffeomorphism $A$ of $\mathbb{D}$ into itself such that $A(0) = 0$ and $\partial_z A(0) = \hat{z}$. After an appropriate choice of $\hat{z}$, we find the above mentioned conformal diffeomorphism, which we denote as $A_g$, such that $(A_g^* P)_g(0) = \hat{C}$ for some positive constant $\hat{C}$. Therefore, without loss of generality, we can assume that $g(0) = \hat{C}$. We choose $C_{37}, C_{38} > 0$, independent of the metric $g$, such that $0 < C_{37} < \hat{C} C_{38}$.

Consider the Beltrami equation
\[(65)\quad \frac{\partial f}{\partial \bar{z}} = \mu_g \frac{\partial f}{\partial z}, \quad z \in \mathbb{D},\]

where
\[(66)\quad \mu_g = \frac{g_{11} - g_{22} + 2i g_{12}}{g_{11} + g_{22} + 2\sqrt{g_{11}g_{22} - g_{12}^2}}.\]

Since by (8) $\det g(x) > M_2 > 0$ for $x \in \overline{\mathbb{D}}$, there exists $\beta_3 \in (0, 1)$, independent of $g$, such that
\[(67)\quad \|\mu_g\|_{C(\overline{\mathbb{D}})} \leq C_{39} < 1.\]

Following ([2], p.264), we consider a solution $f_g$ to the Beltrami equation (65) in the form
\[(68)\quad f_g = z \exp(\Gamma_0 g),\]

\[(69)\quad \Gamma_0 g = \frac{1}{\pi} \int_{\mathbb{D}} \left( \frac{\omega_g(\tau)}{\bar{z} - \tau} - \frac{2\omega_g(z)}{1 - \bar{z} z} \right) d\tau - \frac{1}{\pi} \int_{\mathbb{D}} \left( \frac{\omega_g(\bar{z})}{1 - \bar{z} z} - \frac{\omega_g(z)}{1 - \bar{z} z} \right) d\bar{z}.\]

The function $\omega_g$ is uniquely determined by the integral equation
\[(70)\quad \omega_g = \frac{\mu_g}{z} + \mu_g S_0 \omega_g, \quad S_0 \omega = -\frac{1}{\pi} \int_{\mathbb{D}} \left( \frac{1}{(z - \bar{z})^2} - \frac{\bar{z}^2}{(1 - \bar{z} z)^2} \right) \omega(\bar{z}) d\bar{z}.\]

Since $\mu_g(0) = 0$ by $g(0) = \hat{C} I$, we see that $\frac{\mu_g}{z} \in L^\infty(\mathbb{D})$. By (67), there exist $s > 2$ and a constant $0 < \beta_1 < 1$ such that the operator norm of $\mu_g S_0$ satisfies
\[\|\mu_g S_0\|_{L^\infty(\mathbb{D})} \leq C_{40} < 1.\]

Hence, equation (70) has a solution $\mu_g$ which is uniformly bounded in $L^s(\mathbb{D})$. It is shown in [2] (pp. 264-265) that the mapping given by (68) and (69) is a homeomorphism of $\mathbb{D}$ onto itself. Thanks to condition (67), the operator $\frac{\partial}{\partial \bar{z}} - \mu_g \frac{\partial}{\partial z}$ is an elliptic operator. By the theory of elliptic operators in any domain $\Omega' \subset \subset \mathbb{D}$, there exists a constant $C_{41}(\Omega')$ such that
\[(71)\quad \|f_g\|_{C^{2+s}(\Omega')} \leq C_{41}(\Omega').\]
Next we obtain the estimates in Hölder spaces near the boundary. Let \( \Psi : \mathbb{D} \to \mathbb{R} \times \mathbb{R}_+ \) be a conformal homeomorphism. Set \( \bar{f}_g = \Psi \circ f_g \circ \Psi^{-1} \) and \( \bar{\mu}_g = \bar{\mu}_g \circ \Psi^{-1} \).

The equation (65) is transformed into

\[
\frac{\partial \bar{f}_g}{\partial z} = \bar{\mu}_g \frac{\partial \bar{f}_g}{\partial \bar{z}} \quad \text{on} \ \mathbb{R} \times \mathbb{R}_+.
\]

Denoting the real and the imaginary parts of the function \( \bar{f}_g \) by \( \bar{f}_1 \) and \( \bar{f}_2 \) respectively, we write the above equation as the system of two equations:

\[
K_1(x) \partial x_2 \begin{pmatrix} \bar{f}_1 \\ \bar{f}_2 \end{pmatrix} = K_2(x) \partial x_1 \begin{pmatrix} \bar{f}_1 \\ \bar{f}_2 \end{pmatrix},
\]

\[
K_1 = \begin{pmatrix} -\mu_2 & -(1 + \mu_1) \\ 1 + \mu_1 & -\mu_2 \end{pmatrix}, \quad K_2 = \begin{pmatrix} \mu_1 - 1 & -\mu_2 \\ \mu_2 & \mu_1 - 1 \end{pmatrix}.
\]

The matrix

\[
K_3 = K_1^{-1} K_2 = \frac{1}{\mu_2 + (1 + \mu_1)^2} \begin{pmatrix} 2\mu_2 & \mu_1^2 + \mu_2^2 - 1 \\ 2\mu_2 & -\mu_1^2 - \mu_2^2 \end{pmatrix}
\]

has two complex eigenvalues \( \lambda_{\pm} = \frac{1}{\mu_2 + (1 + \mu_1)^2} (2\mu_2 \pm i(1 - (\mu_1^2 + \mu_2^2))) \) and the corresponding eigenvectors \( \bar{e}_{\pm} = (1, \pm i) \) respectively. Finally we observe that \( f_2 = 0 \) on \( \{x_2 = 0\} \). Therefore the Lopatinsiki condition for the system (73) is satisfied and in any bounded domain \( G \subset \mathbb{R} \times \mathbb{R}_+ \), we have

\[
\| \bar{f}_g \|_{C^{4,0}(\mathbb{D})} \leq C_{42}(G).
\]

Combining the estimates (71) and (74), we obtain

\[
\| f_g \|_{C^{4,0}(\mathbb{D})} \leq C_{43}.
\]

Setting \( F(g)(\cdot) = (\text{Re} f_g, \text{Im} f_g) \), we obtain

\[
\| F(g)(\cdot) \|_{C^{3,0}(\mathbb{D})} \leq C_{44}.
\]

In order to obtain the corresponding estimate (62) for \( F^{-1}(g)(\cdot) \), we first claim that there exists a constant \( \beta_2 > 0 \) independent of \( g \) such that

\[
| \partial_x f_g(x) | \geq \beta_2 > 0 \quad \forall x \in \mathbb{D}.
\]

Assume that (77) fails. Then, for some metric \( g_0 \) and some point \( x_0 \in \mathbb{D} \), one should have \( \partial_x f_{g_0}(x_0) = 0 \). Let \( g_{g_0} \) be another solution to the Beltrami equation in \( \mathbb{D} \) such that \( \partial_x g_{g_0}(x_0) \neq 0 \). By the Stoilow factorization (see e.g., [2] p.179), there exists a holomorphic function \( \Phi \) in \( \mathbb{D} \) such that \( g_{g_0} = \Phi g_0 \circ f_{g_0} \). Let \( x_0 \in \mathbb{D} \). Since \( \partial_x f_{g_0}(x_0) = 0 \) from the Beltrami equation, we see that \( \nabla f_{g_0}(x_0) = 0 \). Therefore \( \nabla g_{g_0}(x_0) = (\nabla \Phi g_0 \circ f_{g_0})(x_0) = 0 \). Now we have to consider the case \( x_0 \in \partial \mathbb{D} \). The homeomorphism \( f_g \) can be extended to a homeomorphism: \( \overline{\mathbb{D}} \to \overline{\mathbb{D}} \). Hence \( \Phi_{g_0} \in C(\overline{\mathbb{D}}) \) and \( \Phi_{g_0} : \partial \mathbb{D} \to \partial \mathbb{D} \). Let \( g_{g_0} \) be a homeomorphism of \( \mathbb{D} \to \mathbb{D} \) and \( \Psi : \mathbb{D} \to \mathbb{R}_+^2 \) be a conformal homeomorphism. The function \( \Psi \circ \Phi_{g_0} \circ \Psi^{-1} \) is holomorphic in \( \mathbb{R}_+^2 \) and \( \text{Im} \Psi \circ \Phi_{g_0} \circ \Psi^{-1} \neq 0 \) on \( \{x_2 = 0\} \). Hence the function \( \Psi \circ \Phi_{g_0} \circ \Psi^{-1} \) is smooth in \( \mathbb{C} \). Then \( \Phi_{g_0} \in C^1(\overline{\mathbb{D}}) \) and \( \nabla g_{g_0}(x_0) = (\nabla \Phi_{g_0} \circ f_{g_0})(x_0) = 0 \). We reach a contradiction. Thus we verify (77).

The inequality (77) immediately implies

\[
| \det F'(g)(x) | \geq C_{45} > 0, \quad \forall x \in \mathbb{D}.
\]
Then the implicit function theorem yields
\begin{equation}
\|F^{-1}(\mathbf{g})(\cdot)\|_{C^{2+\alpha}(\overline{\Omega})} \leq C_{46}.
\end{equation}
Observe that by (77) and (8) there exist constants $C_{47} > 0$ and $C_{48} > 0$, which are independent of $\mathbf{g}$, such that
\[
\mu = (F^*_{\mathbf{g}}, \mathbf{e}, \mathbf{e}_1) = (\mathbf{g}((\mathbf{F}')^{-1})^T \mathbf{e}_1, ((\mathbf{F}')^{-1})^T \mathbf{e}_1) \circ F^{-1} \\
\geq C_{47}((\mathbf{F}')^{-1})^T \circ F^{-1} \mathbf{e}_1 \geq C_{48} > 0,
\]
where $\mathbf{e}_1 = (1, 0)$. From (8), (79) and the above formula, we obtain the inequality $\|\mu\|_{C^{2+\alpha}(\overline{\Omega})} \leq C_{49}$ with some constant $C_{49} > 0$.

Next we prove the inequality (64). By (63), we have
\begin{equation}
\|(Ag^P)^*_{\mathbf{g}} - (Ag^P)^*\|_{C(\overline{\Gamma})} \leq \epsilon\|(Ag^P)^*\|_{C(\overline{\Gamma})},
\end{equation}
Since $Ag^P$ is a conformal diffeomorphism of $\Omega$ onto $\mathbb{D}$, we have $(Ag^P)^*\|_{C(\overline{\Omega})} = \mu_* I$ with some function $\mu_* \in C^{2+\alpha}(\overline{\mathbb{D}})$. Hence we rewrite the inequality (80) as
\begin{equation}
\|(Ag^P)^*_{\mathbf{g}} - \mu_* I\|_{C(\overline{\mathbb{D}})} \leq \epsilon\|(Ag^P)^*\|_{C(\overline{\mathbb{D}})}.
\end{equation}
We set $F = (Ag^P)^{-1} \circ f \circ Ag^P$. The function $f = f_{\mathbf{g}} = z \exp(\Gamma \omega_{(Ag^P)^*}) \mathbf{g}$ satisfies
\begin{equation}
\frac{\partial f}{\partial z} = \mu(Ag^P)^*_{\mathbf{g}} \frac{\partial f}{\partial z}, \quad z \in \mathbb{D}.
\end{equation}
Let the function $\mu(Ag^P)^*_{\mathbf{g}}$ be defined by (66). By (81) and (82), we see
\begin{equation}
\|\mu(Ag^P)^*_{\mathbf{g}}\|_{C(\overline{\mathbb{D}})} \leq C_{50}(\beta)\epsilon.
\end{equation}
This inequality and (82) imply
\[
\|\partial_z f\|_{C(\overline{\mathbb{D}})} \leq C_{51}(\beta)\epsilon.
\]
Then
\[
\|\partial_z(F_1 + iF_2)\|_{C(\overline{\Gamma})} \leq \|\partial_z(Ag^P)^{-1}\|_{C(f \circ Ag^P(\overline{\Gamma}))} \|f\|_{C(\overline{\mathbb{D}})} \|\partial_z(Ag^P)^{-1}\|_{C(\overline{\Gamma})} \leq C_{52}(\beta)\epsilon.
\]
The inequality (64) is proved.

By Proposition 2 there exists a diffeomorphism $F_2$ of the domain $\Omega$ into itself such that $(F_2)_{\mathbf{g}} = \mu_2 I$ and
\begin{equation}
\|\mu_2\|_{C^{2+\alpha}(\overline{\Omega})} \leq M_3, \quad \mu_2(x) \geq \beta > 0, \quad \forall x \in \Omega,
\end{equation}
where the constants $M_3$ and $\beta$ are independent of $\mathbf{g}_1, \mathbf{g}_2$. Then we make the same change of variables in the operator $L_{\mathbf{g}_1}(x, D, s)$. Observe that inequality (29) holds true.

Then, by (61) we have
\begin{equation}
\|(F_2)^*_{\mathbf{g}_1} - (F_2)^*_{\mathbf{g}_2}\|_{C(\overline{\Gamma})} \leq C_{53}\Lambda^{\frac{3}{2}}(\mathbf{g}_1, \mathbf{g}_2).
\end{equation}
Hence, without loss of generality, we can assume that
\begin{equation}
\mathbf{g}_2(x) = \mu_2(x) I \quad \text{on} \quad \Omega, \quad \|\mathbf{g}_1 - \mathbf{g}_2\|_{C(\overline{\Gamma})} \leq C_{54}\Lambda^{\frac{3}{2}}(\mathbf{g}_1, \mathbf{g}_2).
\end{equation}
Henceforth we set $x_\pm = (\pm 1, 1)$. By Proposition 2, there exists a diffeomorphism $F_1 = (F_{1,1}, F_{1,2})$ of the domain $\Omega$ onto itself such that

$$(87) \quad F_1 = (F_{1,1}, F_{1,2}) \in C^{3+\alpha}(\bar{\Omega}), \quad \|F_1\|_{C^{3+\alpha}(\bar{\Omega})} + \|F_1^{-1}\|_{C^{3+\alpha}(\bar{\Omega})} \leq C_{55}$$

and

$$(88) \quad F_1(\Gamma_0) = \Gamma_0, \quad F_1(x_\pm) = x_\pm, \quad x_\pm := (\pm 1, 1)$$

and the operator $L_{g_j}(x, D, s)$ is transformed to

$$(89) \quad Q_{\mu_1}(y, D, s) = -\frac{1}{\mu_1(y)}\Delta + s, \quad y \in \Omega$$

with a smooth positive function $\mu_1 \in C^{2+\alpha}(\bar{\Omega})$ with some $\alpha \in (0, 1)$ satisfying

$$(90) \quad \|\mu_1\|_{C^{2+\alpha}(\bar{\Omega})} \leq C_{56}, \quad \mu_1(y) \geq \beta > 0, \quad \forall y \in \Omega.$$ 

Moreover, provided that $\Lambda(g_1, g_2)$ is sufficiently small, the inequalities (64) and (86) imply

$$(91) \quad \|\partial_\tau F_1\|_{C(\bar{\Gamma})} \leq C_{57}(M, \alpha)\Lambda^{1/2}(g_1, g_2).$$

We set $w_j = w_j(x, 1)$ and $L_{g_j}(x, D) = L_{g_j}(x, D, 1)$. Then

$$(92) \quad L_{g_j}(x, D)w_j = (-\Delta g_j + 1)w_j = 0 \quad \text{in} \quad \Omega, \quad w_j|_{\partial \Omega} = f, \quad \forall j \in \{1, 2\}.$$ 

Observe that the Dirichlet-to-Neumann maps are given as follows:

$$(93) \quad \tilde{\Lambda}_{g_j}(f) = \frac{1}{\sqrt{\det g_j}} \frac{\partial w_j}{\partial y_j},$$

$$(94) \quad L_{g_j}(x, D)w_j = 0 \quad \text{in} \quad \Omega, \quad w_j|_{\partial \Omega} = f, \quad \text{supp} \ f \subset \tilde{\Gamma}.$$ 

We construct a solution to the boundary value problem

$$Q_{\mu_1}(y, D, 1)w = 0 \quad \text{in} \quad \Omega, \quad w|_{\Gamma_0} = 0$$

of the form:

$$(95) \quad w = \text{Re} \left\{ \sum_{j=0}^{\infty} e^{P(z)}(-1)^j U_j \right\}, \quad z = y_1 + iy_2.$$ 

Here $P(z)$ is a smooth holomorphic function constructed in the following proposition and the convergence in (95) is justified after the proof of Proposition 3. Henceforth let $\Gamma_{X,Y}$ denote the arc between points $x$ and $y$ which is clockwise oriented under consideration.

**Proposition 3.** Let $x(t), \ t \in [0, 1]$ be a parameterization of the curve $\tilde{\Gamma}$ such that $x_- = x(0)$ and $x_+ = x(1)$. There exists a holomorphic function $P \in C^{3+\alpha}(\bar{\Omega})$ such that this function does not have critical points in $\Omega$, $\text{Im} \ P|_{\Gamma_0} = 0$, $\text{Im} \ P > 0$ on $\Gamma$. Moreover there exist two points $y_k \in \tilde{\Gamma}, k \in \{1, 2\}$ with $y_1 = x(t_1), y_2 = x(t_2), 0 < t_1 < t_2 < 1$ such that $\text{Re} \ P$ is strictly monotone decreasing on $\Gamma_{x_-, y_1}$ and $\Gamma_{y_2, x_+}$. On the arc $\Gamma_{y_1, y_2}$, the function $\text{Re} \ P$ is strictly monotone decreasing on the arc $\Gamma_{y_2, x_+}$. Finally $\text{Re} \ P(x_-) = -1, \text{Re} \ P(y_1) = -2, \text{Re} \ P(y_2) = 2$ and $\text{Re} \ P(x_1) = 1$. 

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Proof. Let $\Omega \subset \mathbb{R} \times \mathbb{R}_+$ be a bounded domain with smooth boundary such that $\partial \Omega \cap \{x \in \mathbb{R}^2; x_2 = 0\} = \{(-2, 0), (2, 0)\}$, the function $P_0 \in C^{3+\alpha}(\overline{\Omega})$ be a conformal diffeomorphism of the domain $\Omega$ on $\Omega$ and $R_{0} \in C^{3+\alpha}(\overline{\Omega}, \mathbb{D})$ be a conformal diffeomorphism of the domain $\Omega$ on $\mathbb{D}$ (see e.g., [28]). Moreover for $x_\pm = \partial \Gamma_0$, we denote $y_\pm = R_{0} \circ P_0(x_\pm)$ and $R_{0} \circ P_0(\Gamma_0)$ is an arc located between the points $y_-$ and $y_+$ when we are moving clockwise along the unite circle centered at origin. Let $y_\pm \in \partial \mathbb{D}$ satisfy $R_{0}^{-1}(y_\pm) = (\pm 1, 0)$. Then denote by $\mathcal{V}$ the part of $\partial \mathbb{D}$ when one moves from $y_-$ to $y_+$ clockwise. Then either $R_{0}^{-1}(\mathcal{V}) = \{(-1, 0), (0, 1)\}$ or $R_{0}^{-1}(\partial \mathbb{D} \setminus \mathcal{V}) = \{(-1, 0), (1, 0)\}$. Henceforth we set

$$\text{Aut} \mathbb{D} = \left\{ z \to e^{i\theta} \frac{z - a}{1 - a \bar{z}}; \ a \in \mathbb{C}, \ |a| < 1, \ \theta \in \mathbb{R} \right\}.$$ 

In the first case let $M \in \text{Aut} \mathbb{D}$ such that $M \circ R_{0} \circ P_0(\Gamma_0) = \mathcal{V}$ and in the second case let $M \in \text{Aut} \mathbb{D}$ such that $M \circ R_{0} \circ P_0(\Gamma_0) = \partial \mathbb{D} \setminus \mathcal{V}$. Finally we set $P(z) = R_{0}^{-1} \circ M \circ R_{0} \circ P_0$. The proof of the proposition is complete. \hfill \Box

Now we define the functions $U_j$. First we set

$$R_{\tau} g = \frac{1}{2} e^{\tau(P(z)-P(z))} \partial_{\bar{z}}^{-1} (ge^{\tau(P(z)-P(z))}), \quad \partial_{\bar{z}}^{-1} g = \frac{1}{\pi} \int_{\Omega} \frac{g(\xi, \xi)}{z - (\xi + it_1 \xi_2)} \mathrm{d} \xi_1 \mathrm{d} \xi_2.$$ 

Let

$(96) \quad U_0 = i.$

Then the functions $U_j$ for any $j \geq 1$ are determined by

$$U_j = \frac{1}{2} (R_{\tau} (\partial_{\bar{z}}^{-1} (\mu_1 U_{j-1}))) + e^{\tau(P(z)-P(z))} a_j(\tau, \pi),$$

where $a_j$ satisfy

$$\text{Re} a_j(\tau, \pi)|_{\partial \Omega} = -\text{Re} \{R_{\tau} (\partial_{\bar{z}}^{-1} (\mu_1 U_{j-1}))\}.$$

The norms of the functions $U_j$ can be estimated as

$(97) \quad \|U_j\|_{c^{2-\alpha}(\Omega)} \leq C_{58}(\alpha) \tau^{(2-\alpha)j}, \quad \forall j \in \mathbb{N}, \ \forall \alpha \in (0, 1),$ 

where the constant $C_{58} > 0$ is independent of $j$.

In order to prove estimate $(97)$ we observe

$$e^{\tau(P(z)-P(z))} \partial_{\bar{z}}^{-1} (e^{\tau(P(z)-P(z))} (\mu_1 U_{j-1}))$$

$$= e^{\tau(P(z)-P(z))} \partial_{\bar{z}}^{-1} \left( \left( \partial_{\bar{z}} e^{\tau(P(z)-P(z))} \frac{\partial_{\bar{z}}^{-1} (\mu_1 U_{j-1})}{\tau \partial_{\bar{z}} P} \right) \right)$$

$$= -\frac{\partial_{\bar{z}}^{-1} (\mu_1 U_{j-1})}{\tau \partial_{\bar{z}} P} - e^{\tau(P(z)-P(z))} \partial_{\bar{z}}^{-1} \left( e^{\tau(P(z)-P(z))} \frac{\partial_{\bar{z}}^{-1} (\mu_1 U_{j-1})}{\tau \partial_{\bar{z}} P} \right)$$

$$+ e^{\tau(P(z)-P(z))} \int_{\partial \Omega} (\nu_1 - i \nu_2) e^{\tau(P(z)-P(z))} \frac{\partial_{\bar{z}}^{-1} (\mu_1 U_{j-1})}{\tau \partial_{\bar{z}} P} \frac{1}{\partial_{\bar{z}} P(\zeta - \bar{\zeta})} \mathrm{d} \sigma$$

$$= -\frac{\partial_{\bar{z}}^{-1} (\mu_1 U_{j-1})}{\tau \partial_{\bar{z}} P} - \frac{1}{\tau \partial_{\bar{z}} P} \left( \partial_{\bar{z}} \left( \frac{\partial_{\bar{z}}^{-1} (\mu_1 U_{j-1})}{\tau \partial_{\bar{z}} P} \right) \right)$$

$$- e^{\tau(P(z)-P(z))} \partial_{\bar{z}}^{-1} \left( e^{\tau(P(z)-P(z))} \frac{1}{\tau \partial_{\bar{z}} P(\zeta - \bar{\zeta})} \frac{1}{\partial_{\bar{z}} P(\zeta - \bar{\zeta})} \right)$$

$$\int_{\partial \Omega} (\nu_1 - i \nu_2) e^{\tau(P(z)-P(z))} \frac{1}{\tau \partial_{\bar{z}} P(\zeta - \bar{\zeta})} \left( \partial_{\bar{z}}^{-1} (\mu_1 U_{j-1}) + \partial_{\bar{z}} \left( \frac{\partial_{\bar{z}}^{-1} (\mu_1 U_{j-1})}{\tau \partial_{\bar{z}} P(\zeta - \bar{\zeta})} \right) \right) \mathrm{d} \sigma.$$
Obviously
\[(98) \quad \|J_k(\tau, \cdot)\|_{C^{2-\alpha}([0,1])} \leq \frac{C_{69}(\alpha)}{\tau} \|U_{j-1}\|_{C^{2-\alpha}([0,1])} \quad \forall k \in \{1, 2\}, \quad \forall \alpha \in (0, 1)\]
and
\[(99) \quad \|J_k(\tau, \cdot)\|_{C^{2-\alpha}([0,1])} \leq \frac{C_{60}(\alpha)}{\tau} \|U_{j-1}\|_{C^{2-\alpha}([0,1])} \quad \forall \alpha \in (0, 1).\]

We set
\[
b(\tau, \bar{\tau}) = \int_{\partial \Omega} (\nu_1 - i \nu_2) e^{(P - \bar{P})} \frac{1}{\tau \partial_\zeta P(\zeta) - \bar{N}} \left\{ \partial_\zeta^{-1}(\mu_1 U_{j-1}) + \partial_\bar{\zeta} \left( \frac{\partial_\zeta^{-1}(\mu_1 U_{j-1})}{\tau \partial_\zeta P} \right) \right\} d\sigma_\zeta.
\]
Let an antiholomorphic function \(\bar{b}(\tau, \bar{\tau})\) satisfy the following boundary value problem
\[
\partial_\bar{\tau} \bar{b} = 0 \quad \text{in } \Omega, \quad \text{Re } \bar{b} |_{\partial \Omega} = -\sum_{k=1}^{3} J_k(\tau, x).
\]
By (98) and (99), one can choose a function \(\bar{b}\) such that
\[(100) \quad \|J_k(\tau, \cdot)\|_{C^{2-\alpha}([0,1])} \leq \frac{C_{61}(\alpha)}{\tau} \|U_{j-1}\|_{C^{2-\alpha}([0,1])} \quad \forall \alpha \in (0, 1).\]
(see e.g., [32]).

We set \(a(\tau, \bar{\tau}) = b(\tau, \bar{\tau}) + \bar{b}(\tau, \bar{\tau})\). Inequalities (98)-(100) imply (97). Hence the infinite series given by (95) is convergent in the space \(C^{2-\alpha}(\overline{\Omega})\) for all sufficiently large \(\tau\) and \(\alpha \in (0, 1)\).

Then the function
\[(101) \quad w_1(x) = w(F_1(x))\]
satisfies (94).

3. **Second step of the proof of Theorem 1.1: Carleman estimates.** In this section we prove two Carleman estimates (Propositions 6 and 7) for the Laplace operator. Our weight is harmonic in the domain \(\Omega\) but allows to have the isolated singularities at some points on boundary.

We start with the construction of the weight for the Carleman estimates. Let \(\varphi_*(x, g_1) = \varphi_*(x)\) be a harmonic function in \(\Omega\) such that
\[(102) \quad \frac{\partial \varphi_*}{\partial \nu}\big|_{\Gamma_0} = 0, \quad \varphi_* = \text{Re } \{P \circ F_1\} \quad \text{on } \overline{\Gamma}, \quad F_1 = (F_{1,1} + iF_{1,2}).\]

Let \(\psi_*\) be a conjugate function to \(\varphi_*\) and \(\Phi_* = \varphi_* + i\psi_*\). The function \(\varphi_*\) is of the class \(C^{3+\alpha}\) at any points of \(\overline{\Omega}\) possibly except \(x_*\) and in order to estimate the derivatives of the function \(\Phi_*\) near possible singularity points, we provide more details of the construction of this function.

Consider the holomorphic function \(f(z) = \sin^{-1}(\pi z)\) and the corresponding conformal mapping \(f(x_1, x_2) = (\text{Re } \sin^{-1}(\pi x_1), \text{Im } \sin^{-1}(\pi x_2))\).

It is known that this maps \(\mathbb{R} \times \mathbb{R}_+\) onto strip \(\{y \in \mathbb{R}; -1 \leq y_1 \leq 1; y_2 \geq 0\}\). Let \(\Omega_1 \subset \mathbb{R} \times \mathbb{R}_+\) be a bounded domain such that \(\partial \Omega_1 \cap \{x \in \mathbb{R}; x_2 = 0\} = \{x \in \mathbb{R}; x_2 = 0, x_1 \in [-2, 2]\}\). Let \(\Psi \in C^{10}(\overline{\Omega}, \overline{\Omega})\) be a conformal diffeomorphism of the domain \(\Omega\) on domain \(\Omega_1\), such that \(\Psi(x_\pm) = (\pm 1, 0), \Psi(\overline{\Gamma}) = \{x \in \mathbb{R} \times \mathbb{R}_+; x_2 = 0, x_1 \in [-1, 1]\}\) and \(\{x \in \mathbb{R} \times \mathbb{R}_+; x_2 = 0, x_1 \in [-2, 2]\}\). Let \(\overline{\varphi} \in C^3(\overline{\Omega})\) be a
harmonic function such that $\frac{\partial^3 g}{\partial r^3}(x) = 0$ and $\frac{\partial^k g}{\partial x_i} = \frac{\partial^k \varphi}{\partial x_i}(x_\pm)$, $\forall k \in \{0, \ldots, 3\}$. For example one can take
\[
\varphi(x) = \text{Re} \left( (z - 1)^{10} \sum_{p=0}^{6} \frac{d^p g}{dx_1^p} (-1)^p \frac{z^p}{p!} \right) + \text{Re} \left( (z + 1)^{10} \sum_{p=0}^{6} \frac{d^p g}{dx_1^p} (1)^p \frac{z^p}{p!} \right),
\]
where $r(x_1) = \text{Re} \{P \circ F\}(x_1, 1)$. Set $g = \varphi - \tilde{\varphi}$. Let
\[
(103) \quad \Delta \phi = 0 \quad \text{in} \quad \Pi \supset f(\Omega), \quad \partial \varphi(x_1, 0)|_{[-1, 1]} = 0, \quad \phi|_{f(\Omega)} = g \circ \Omega^{-1} \circ f^{-1}.
\]

Let $\psi_0$ be the conjugate function to the harmonic function $\phi$ and $\psi_2$ be the conjugate function to the harmonic function $\tilde{\varphi}$. We set
\[
(104) \quad \Phi_* = (\phi + i\psi_0) \circ f \circ \Psi + \tilde{\varphi} + i\psi_2.
\]

We have

**Proposition 4.** A solution to problem (102) is given by
\[
\varphi_* = \phi \circ f \circ \Psi + \tilde{\varphi} \in C^2(\Omega),
\]

where $\phi \in C^{3+\alpha}(\Pi)$ satisfies (103). Moreover there exists a constant $C_1$ independent of $g_1$ such that
\[
(106) \quad |\partial^\alpha \Phi_*(x)| \leq C_1 (|x - x_-|^{-|\alpha| + \frac{1}{2}} + |x - x_+|^{-|\alpha| + \frac{1}{2}}) \quad \forall x \in \Omega \quad \text{and} \quad |\alpha| \leq 2.
\]

**Proof.** The conformal mapping generated by the holomorphic function $f$ is a diffeomorphism of $\Psi(\Omega)$ on $\Pi$. The function $\phi \circ f \circ \Psi$ is harmonic in $\Omega$. Moreover, thanks to (103), the boundary conditions in (102) hold true. Hence the function $\varphi_* = \phi \circ f \circ \Psi$ is a possible solution to problem (102). Then the statement of the lemma follows from the uniqueness of solution to problem (102). In order to prove the regularity of the function $\varphi_*$, it suffices to establish the $C^{3+\alpha}(\Pi)$-regularity of the function $\phi$. Since the normal derivative of $\phi$ on $\{(y_1, 0); y_1 \in [-1, 1]\}$ is zero, we consider the odd extension of the function $\phi$ across the $y_1$-axis. For any $m$ from $\{1, 2\}$, we set
\[
g_m(y_2) = \begin{cases} g \circ \Psi^{-1} \circ f^{-1}((-1)^m, -y_2) & \text{for } y_2 > 0 \\ g \circ \Psi^{-1} \circ f^{-1}((-1)^m, y_2) & \text{for } y_2 < 0. \end{cases}
\]

Then for some small positive $\epsilon$, the function $\phi$ satisfies
\[
(107) \quad \begin{cases} \Delta \phi = 0 & \text{in } \{y \in \mathbb{R}^2; |y_2| \leq \epsilon, |y_1| \leq 1\}, \\ \phi((-1)^m, y_2) = g_m(y_2), \forall y_2 \in [-\epsilon, \epsilon]. \end{cases}
\]

Since $\frac{\partial^6 g}{\partial x_1^6}(x_\pm) = 0$ for $k \in \{0, \ldots, 6\}$, the functions $g_m$ belong to the space $C^{3+\alpha}[-\epsilon, \epsilon]$ and $f \in C^2(\Psi(\Omega))$, we see that $\phi \in C^{3+\alpha}([-1, 1] \times [-\epsilon, \epsilon])$. Consequently the composition of the two functions $\phi$ and $f$ belongs to $C^2(\Omega)$. Now we prove the estimate (106). We establish this estimate for the point $x_-$ since the proof of (106) for the point $x_+$ is the same. Let $\psi_1$ be the conjugate function to $\phi$ and $\Phi_1 = \phi + \psi_1$. In a neighborhood of zero, we have $f(z) = \sqrt{z}(1 + g(z))$ where $g(z)$ is a holomorphic function in a neighborhood of zero and $g(0) = 0$. Then
\[
\Phi_1 \circ f \circ \Psi(x) = 3 \sum_{\alpha=0}^{3} \frac{1}{\alpha!} \frac{\partial^\alpha \Phi_1(0)}{(\Psi(z))^2} (1 + g(\Psi(z)))^\alpha + o(|x - x_-|^2).
\]

This formula implies (106) immediately.
Denote by $I = \{ x \in \overline{\Omega}; \nabla \varphi(x) = 0 \}$ the set of the critical points of the function $\varphi_*$ on $\overline{\Pi}$. Next we investigate the set of critical points of the function $\varphi_*$ in the domain $\Omega$.

**Proposition 5.** Let a function $\varphi_*(x, g_j)$ satisfy (102) and metrics $g_1, g_2$ satisfy (2). Then there exist positive constants $\epsilon$ and $\beta$ independent of $g_1, g_2$ such that if $\Lambda(g_1, g_2) \leq \epsilon$, then

$$\text{card } I \leq 2, \quad I \subset \Gamma_0,$$

and if $\hat{x}_1, \hat{x}_2 \in I$, then there exists a constant $C_2$ such that

$$|\nabla \varphi_*(x)|^2 \geq C_2 \sum_{j=1}^2 |x - \hat{x}_j|^2$$

and

$$\inf_{x \in \Gamma \setminus \{x_\pm\}} |\partial_x \Phi_*(x)| \geq \beta > 0.$$

**Proof.** In order to prove (108) and (109) we consider three cases.

**Case A.** First let us consider the case when $\varphi_*(x, g_1) \in C^1(\overline{\Pi})$. We show that the function $\varphi_*(x, g_1)$ does not have any critical points on $\overline{\Pi}$. Our proof is by contradiction. Suppose that the inequality (110) fails. Then there exists a sequence $\{\hat{x}_j\} \subset \overline{\Pi}$ such that $\lim_{j \to +\infty} |\partial_{x_j} \varphi_*(\hat{x}_j)| = 0$. On the other hand $\partial_{x_j} F_{1,2}|_{\overline{\Pi}} = 0$, and so $\det F_{1,2}^1(\hat{x}_j) = \partial_{x_j} \varphi_*(\hat{x}_j) \partial_{x_j} F_{1,2}(\hat{x}_j) \to 0$ as $j \to +\infty$. Therefore we obtain a contradiction to (87).

Next, we claim that there are no critical points of the function $\varphi_*(x, g_1)$ in $\Omega$.

Let $\psi_*$ be the conjugate function to $\varphi_*$ and $\Phi_* = \varphi_* + i\psi_*$. Since $\partial_{x_j} \Phi_* = \partial_{x_j} \varphi_* - i \partial_{x_j} \psi_*$, the assumption of the regularity of $\varphi_*$, Proposition 3 and (102) imply $\partial_{x_j} \Phi_*(x_\pm) = a_\pm < 0$, where the points $x_\pm$ are defined by (88). Let $x_j$ be two points from $\overline{\Pi}$ such that $y_j = F_1(x_j)$ where $y_j$ are defined in Proposition 3. Observe that $\partial_{x_j} \varphi_*(x_j) = 0$ for any $j$ from $\{1, 2\}$ and $\partial_{x_j} \varphi_*(x_1) < 0$, because $x_1$ is a point of minimum of the function $\varphi_*$. Moreover, since $x_2$ is a point of maximum of the function $\varphi_*$, by the Hopf lemma we have $\partial_{x_2} \varphi_*(x_2) > 0$. Hence the vector $\partial_{x_j} \Phi_*$ is moving from $x_-$ to $x_+$ counter-clockwise and the increment of $\arg \partial_{x_j} \Phi_*$ is $-2\pi$. Now let us compute the increment of $\arg \partial_{x_j} \Phi_*$ when we are moving from $x_+$ to $x_-$ clockwise. First assume that the function $\Phi_*$ does not have critical points on $\partial \Omega \setminus [x_-, x_+]$. We set $y_{\min} = (y_{\min,1}, y_{\min,2})$ where $y_{\min,1} = \min_{(y_1, y_2) \in \partial \Omega} y_1$ and $y_{\max} = (y_{\max,1}, y_{\max,2})$ where $y_{\max,1} = \max_{(y_1, y_2) \in \partial \Omega} y_1$. The function $\partial_{x_j} \varphi_*$ is equal to zero on $\partial \Omega \setminus [x_-, x_+]$ only at the two points $y_{\min}$ and $y_{\max}$. Moreover $\partial_{x_2} \varphi_*(y_{\min}) > 0$ and $\partial_{x_2} \varphi_*(y_{\max}) < 0$. Finally observe that $\frac{\partial_{x_j} \Phi_*(y_{\min})}{|\partial_{x_j} \Phi_*(y_{\min})|} = -i$ and $\frac{\partial_{x_j} \Phi_*(y_{\max})}{|\partial_{x_j} \Phi_*(y_{\max})|} = i$. Hence the increment of $\arg \partial_{x_j} \Phi_*$ when we are moving from $x_+$ to $x_-$ clockwise, is equal to $2\pi$. Therefore $\arg \partial_{x_j} \Phi_* = 0$ along $\partial \Omega$, and by the argument principle, there are no critical points of the function $\partial_{x_j} \Phi_*$ in $\Omega$. Consider the holomorphic extension of the function $\Phi_*$ across the curve $\partial \Omega \setminus [x_-, x_+]$ which can be constructed in the following way. Let $\Pi_1 \subset \mathbb{R} \times \mathbb{R}_+$ be a bounded domain with smooth boundary and $\Upsilon \in C^{2+\alpha}(\overline{\Pi_1, \overline{\Pi}})$ be a conformal diffeomorphism of $\Pi_1$ on $\Omega$ such that $\Upsilon(\partial \Pi_1 \cap \{ x \in \mathbb{R}^2; x_2 = 0 \}) = \Gamma_0$. Let $\Pi_1^*$ be the reflection of $\Pi_1$ across the $x_1$-axis. We extend the functions $\Phi_* \circ \Upsilon, \Upsilon$ on $\Pi_1^*$ as $\Phi_* \circ \Upsilon(z) = \overline{\Phi_* \circ \Upsilon(\overline{z})}$ and $\Upsilon(z) = \overline{\Upsilon(\overline{z})}$. We set $\Omega^* = \Omega \cup \Gamma_0 \cup \Upsilon(\Pi_1^*)$. The increment of $\arg \partial_{x_j} \Phi_*$ when we
are moving from \( x_+ \) to \( x_- \) clockwise along \( \partial \Omega^* \), is equal to \( 2\pi \). By the argument principle in the domain \( \Omega^* \), the function \( \partial_2 \Phi_* \) has at most two critical points in \( \Omega^* \). These critical points should belong to \( \partial \Omega \setminus [x_-, x_+] \). Observe that these critical points are nondegenerate. Indeed, suppose that the function \( \varphi_* \) has a degenerate critical point \( \tilde{y} \) on \( \partial \Omega \setminus [x_-, x_+] \). The level set \( \{ x \in \Omega \colon \varphi_*(x) = \varphi_*(\tilde{y}) \} \) consists of at least three curves which intersect at \( \tilde{y} \). This set should intersect \( \partial \Omega \setminus [x_-, x_+] \) at least at three points. This is impossible by Proposition 3 and (102). The proof of the proposition for Case A is complete.

Let
\[
(111) \quad \tilde{\varphi}_* = \phi + \varphi \circ \Psi^{-1} \circ \Gamma^{-1}
\]
be a harmonic function in \( \Pi \), where the domain \( \Pi \) is given by (103). Denote by \( \tilde{\varphi}_* \) the even extension of the function \( \tilde{\varphi}_* \) on the domain \( \Pi_2 = \Pi \cup \{(x_1, x_2); \ -x_2 \in \Pi \} \).

Let \( \psi_* \) be the conjugate function to \( \tilde{\varphi}_* \) and \( \tilde{\Phi}_* = \tilde{\varphi}_* + i\psi_* \in C(\Pi_2) \). Then
\[
\partial_{x_1} \tilde{\Phi}_* = \partial_{x_1} \tilde{\varphi}_* - i \partial_{x_2} \tilde{\varphi}_*. \quad \text{Denote by } \tilde{\nu} = (\tilde{\nu}_1, \tilde{\nu}_2) \text{ the outward unit normal vector to } \Pi_2.
\]

**Case B.** Let \( \nabla \tilde{\varphi}_*(-1, 0) \neq 0 \) and \( \nabla \tilde{\varphi}_*(1, 0) \neq 0 \).

We compute the increment of \( \arg \partial_2 \Phi_* \) when we are moving from \((-1, 0)\) to \((1, 0)\). The function \( \tilde{\varphi}_* \) has one point of minimum \( y_m = f \circ \Psi(x_1) \) and one point of maximum \( y_M = f \circ \Psi(x_2) \). Computing \( \partial_2 \tilde{\Phi}_* \) at these two points, we obtain
\[
\partial_{x_1} \tilde{\Phi}_*(y_m) = \left( \frac{\partial_{x_1} \tilde{\varphi}_*}{\tilde{\nu}_1} (\tilde{\nu}_1 - i\tilde{\nu}_2) \right) (y_m) \text{ if } \tilde{\nu}_1(y_m) \neq 0,
\]
\[
\partial_{x_1} \tilde{\Phi}_*(y_M) = \left( \frac{\partial_{x_1} \tilde{\varphi}_*}{\tilde{\nu}_1} (\tilde{\nu}_1 - i\tilde{\nu}_2) \right) (y_M) \text{ if } \tilde{\nu}_1(y_M) \neq 0
\]
and
\[
\partial_{x_2} \tilde{\Phi}_*(y_m) = -i \partial_{x_2} \tilde{\varphi}_*(y_m) \text{ if } \tilde{\nu}_1(y_m) = 0, \quad \partial_{x_2} \tilde{\Phi}_*(y_M) = -i \partial_{x_2} \tilde{\varphi}_*(y_M) \text{ if } \tilde{\nu}_1(y_M) = 0.
\]

On the other hand, if \( \partial_2 \Phi_*(y_0) = a(\tilde{\nu}_1(y_0) - i\tilde{\nu}_2(y_0)) \) for some \( a \in \mathbb{R}^1 \setminus \{0\} \) and some \( y_0 \) from \( \partial \Pi_2 \), then we have
\[
(112) \quad \frac{\partial \tilde{\varphi}_*}{\partial \tilde{\nu}}(y_0) = 0.
\]
(Here \( \tilde{\nu} \) is a tangential vector to \( \Pi_2 \).) Hence, by Proposition 3 either \( y_0 = y_m \) or \( y_0 = y_M \). Without loss of generality we may assume that
\[
\partial_{x_1} \tilde{\Phi}_*(-1, 0) =: a_- \neq 0, \quad \partial_{x_1} \tilde{\Phi}_*(1, 0) =: a_+ \neq 0.
\]

Let \( y(t) \) be a parametrization of \( \partial \Omega \setminus \{ x \in \mathbb{R}^2; \ x_2 = 0 \} = \{ y(t); \ t \in [0, 1] \} \) such that \( y(0) = (-1, 0) \) and \( y(1) = (1, 0) \). Then for some \( 0 < t_1 < t_2 < 1 \) we have \( y(t_1) = y_m \) and \( y(t_2) = y_M \). We compute the increment of \( \arg \partial_2 \Phi_* \) along the curve \( y(t) \). The increment of \( \arg (\tilde{\nu}_1(y(t)) - i\tilde{\nu}_2(y(t))) \) along the curve \( y(t) \) is \( \pi \). By (112) there are only two points on \( \{ y(t); \ t \in [0, 1] \} \) such that the vector \( \partial_{x_2} \Phi_*(y(t)) \) is parallel to the vector \( \tilde{\nu}_1(y(t)) - i\tilde{\nu}_2(y(t)) \). Moreover
\[
\frac{\partial_{x_2} \Phi_*(y)}{|\partial_{x_2} \Phi_*(y)|} = \tilde{\nu}_1(y) - i\tilde{\nu}_2(y)
\]

at least at one point \( y \in \{ y_m, y_M \} \). Therefore the absolute value of the increment of \( \arg \partial_2 \Phi_* \) over curve \( \{ y(t); \ t \in [0, 1] \} \) is at most \( 2\pi \). Hence the increment of \( \arg \partial_2 \Phi_* \)
over $\partial \Pi_2$ is at most $4\pi$. From the argument principle, we will obtain (108) and (109).

**Case C.** Let either $\nabla \tilde{\varphi}_*(1, 0) = 0$ or $\nabla \tilde{\varphi}_*(-1, 0) = 0$. Assume for example that $\nabla \tilde{\varphi}_*(1, 0) = 0$. Since $\nabla \varphi_*(x_\epsilon) \neq 0$, the critical point $(1, 0)$ is nondegenerate for the function $\tilde{\varphi}_*$. Set $\Pi_{1, \epsilon} = \Pi \setminus B(1, 0, \epsilon)$, where $\epsilon$ is positive and sufficiently small. The Taylor expansion yields $\tilde{\varphi}_*(x) = c((x_1 - 1)^2 - x_2^2) + o(|x - (1, 0)|^2)$ for some constant $c \neq 0$. By Proposition 3 the constant $c$ is negative. Without loss of generality, we can assume that $\partial_2 \Phi_*(1, 0, \epsilon) = 0$, then $\varphi_\epsilon \in C^1(\Omega)$ and this case is already treated in Case A. Short computations imply $\frac{\partial_2 \Phi_*(y_m)}{\partial_2 \Phi_*(y_M)} = -\tilde{\nu}_1(y_m) - i\tilde{\nu}_2(y_m)$ and $\frac{\partial_2 \Phi_*(y_M)}{\partial_2 \Phi_*(y_M)} = (\tilde{\nu}_1(y_M) - i\tilde{\nu}_2(y_M))$ and $\lim_{\epsilon \to 0} \frac{\partial_2 \Phi_*(1, \epsilon)}{\partial_2 \Phi_*(1, \epsilon)} = i$ or $\lim_{\epsilon \to 0} \frac{\partial_2 \Phi_*(1, \epsilon)}{\partial_2 \Phi_*(1, \epsilon)} = -i$. When we are moving counterclockwise from $(1, \epsilon)$ to $(1, -\epsilon)$ over the sphere centered at $(1, 0)$ of radius $\epsilon$, the increment of arg $\partial_2 \Phi$ is $-\pi$. Consider two subcases.

**Subcase 1.** Let $\frac{\partial_2 \Phi_*(1, 0, \epsilon)}{\partial_2 \Phi_*(1, 0, \epsilon)} = -1$. Then the increment of $\arg \partial_2 \Phi$ as we are moving counterclockwise from point $(1, 0, \epsilon)$ to $(1, \epsilon)$ over $\partial \Pi_2$, converges to $\frac{3\pi}{2}$ as $\epsilon \to +0$. Finally, the increment of $\arg \partial_2 \Phi$ as we are moving counterclockwise from point $(1, -\epsilon)$ to $(-1, 0)$ over $\partial \Pi_2$, converges to $\frac{3\pi}{2}$ as $\epsilon \to +0$. Hence the increment of $\arg \partial_2 \Phi_*$ over $\partial(\Pi_{2, \epsilon} \cap S_{\epsilon}(1, 0, \epsilon))$ is equal to $2\pi$.

**Subcase 2.** Let $\frac{\partial_2 \Phi_*(1, 0, \epsilon)}{\partial_2 \Phi_*(1, 0, \epsilon)} = 1$. Then the increment of $\arg \partial_2 \Phi$ as we are moving counterclockwise from $(-1, 0, \epsilon)$ to $(1, \epsilon)$ over $\partial \Pi_2$, converges to $\frac{\pi}{2}$ as $\epsilon \to +0$. Finally, the increment of $\arg \partial_2 \Phi$ as we are moving counterclockwise from point $(1, -\epsilon)$ to $(1, 0, \epsilon)$ over $\partial \Pi_2$, converges to $\frac{3\pi}{2}$ as $\epsilon \to +0$. Hence the increment of $\arg \partial_2 \Phi_*$ over $\partial(\Pi_{2, \epsilon} \cap S_{\epsilon}(1, 0, \epsilon))$ is equal to $0$.

Now we prove the estimate (110) by contradiction. Suppose that (110) fails. Then there exist a sequence of metrics $g_j$ and a sequence of points $\{x_j\} \subset \Omega$ such that

$$\lim_{j \to +\infty} \nabla \varphi_*(g_j, x_j) = 0 \quad \text{and} \quad \lim_{j \to +\infty} x_j \in \{x_\pm\}, \lim_{j \to +\infty} g_j = g,$$

where the metric $g$ satisfies (1), (2) and (8).

We denote $\varphi_*(g, \cdot)$ where $\varphi_*(g, \cdot)$ is given by (102) and the mapping $F_1$ is given by Proposition 2. Let $\tilde{\varphi}_*(g_j, \cdot) = \Re \Phi_*(g_j, \cdot) \circ \Psi^{-1} \circ f^{-1}$. As we proved in Case A, the function $\varphi$ does not have critical points on $\Gamma$. By (111), the sequence $\{\varphi_*(g_j, \cdot)\}$ is bounded in $C^2(\Omega)$ and $\lim_{j \to +\infty} \nabla \tilde{\varphi}_*(g_j, y_j) = 0$ where $\{y_j\} = \{f \circ \Psi(x_j)\}$. By (113) $\lim_{j \to +\infty} y_j \in \{\pm 1, 0\}$. Therefore taking a subsequence of the sequence $\{\tilde{\varphi}_*(g_j, \cdot)\}$, we obtain that there exists a metric $g$ such that the corresponding function $\tilde{\varphi}_*$ given by (111) has a critical point from the set $\{\pm 1, 0\}$. Hence the function $\varphi_*(g, \cdot)$ has a critical point in a neighborhood of the point $x_-$ or $x_+$ and belongs to the space $C^1$ in some neighborhood of this critical point. In Case A and Case B, we proved that this is impossible. The proof of the proposition is complete.

Next we prove an $L^p$-Carleman estimate for the Schrödinger equation $\Delta + q$ where a harmonic function $\varphi_*$ is taken as the weight function. Typically the proof of $L^p$-Carleman estimate with $p \neq 2$ involves some harmonic analysis technique, but such a usual technique can not be directly applied to our case, due to possible singularities of the weight function on the boundary. In our case we developed...
Then there exist constants \( \tau \) defined by (102). Let

\[
\Delta u + qu = f \quad \text{in } \Omega, \quad u|_{\Gamma_0} = 0.
\]

Then there exist constants \( \tau_0 > 0 \) and \( C_3 > 0 \) which is independent of \( \tau \), such that

\[
\Delta u + qu = f \quad \text{in } \Omega, \quad u|_{\Gamma_0} = 0.
\]

More precisely, we have

**Proposition 6.** Let \( p \in (2, +\infty), \ q \in L^\infty(\Omega) \) and \( \varphi_* \) be the harmonic function defined by (102). Let \( u \in H^2(\Omega) \) satisfy

\[
\Delta u + qu = f \quad \text{in } \Omega, \quad u|_{\Gamma_0} = 0.
\]

Then there exist constants \( \tau_0 > 0 \) and \( C_3 > 0 \) which is independent of \( \tau \), such that

\[
|u|_{L^p(\Omega)} \leq C_3 \left( |u|_{L^p(\Omega)} + \|\varphi_* u\|_{L^p(\Omega)} \right) \quad \text{for all } \|\tau\| \geq \tau_0.
\]

**Proof.** We set \( w = (\partial_\tau u)e^{i\varphi_*} \) and \( f_\tau = (f - qu)e^{i\varphi_*} \), where the holomorphic function \( \Phi_* \) is given by (104). Proposition 4 yields \( w \in H^1(\Omega) \). Then

\[
4\partial_\tau w = f_\tau \quad \text{in } \Omega.
\]

Let \( W = \partial_z^{-1}(|w|^{p-2}w) - a(\tau, \varphi) \), where \( a \in H^1(\Omega) \) is an antiholomorphic function satisfying

\[
\text{Re} a|_{\Gamma_0} = 0 = \text{Re} \partial_z^{-1}(|w|^{p-2}w), \quad \|a\|_{W^1_{p-1}(\Omega)} \leq C_4 \|w\|_{L^p(\Omega)}^{p-1},
\]

By (114), we have

\[
\|W\|_{W^1_{p-1}(\Omega)} \leq C_5 \|w\|_{L^p(\Omega)}^{p-1}.
\]

Taking the scalar product in \( L^2(\Omega) \) of equation (116) with the function \( W \), we obtain

\[
4\|w\|_{L^p(\Omega)}^p + 2\text{Re} \int_{\partial \Omega_*} (\nu_1 - i\nu_2)wWd\sigma = \text{Re} (f_\tau, W)_{L^2(\Omega)}.
\]

By (114), we have

\[
\text{Re} \{(\nu_1 - i\nu_2)w\} = \text{Re} \left( \frac{\partial u}{\partial \nu} e^{i\varphi_*} W \right) = \text{Re} \frac{\partial u}{\partial \nu} e^{i\varphi_*} \text{Re} \{W\} = 0 \quad \text{on } \Gamma_0.
\]

Therefore (117) yields

\[
\text{Re} \int_{\partial \Omega} (\nu_1 - i\nu_2)wWd\sigma = \text{Re} \int_{\Gamma} (\nu_1 - i\nu_2)wWd\sigma \leq C_6 \|w\|_{L^p(\Omega)} \|W\|_{L^p(\Omega)^*} \leq C_7 \|w\|_{L^p(\Omega)} \|w\|_{L^p(\Omega)}^{p-1}.
\]

In terms of (119), (118) and (117), we have

\[
\|\partial_\nu w e^{i\varphi_*} \|_{L^p(\Omega)} \leq C_9 \left( \|f_\tau\|_{L^p(\Omega)} + \|\nabla u e^{i\varphi_*}\|_{L^p(\Gamma)} \right).
\]

Since \( u \) is a real-valued function, the above inequality implies

\[
\|\partial_\nu w e^{i\varphi_*} \|_{L^p(\Omega)} \leq C_9 \left( \|f_\tau\|_{L^p(\Omega)} + \|\nabla u e^{i\varphi_*}\|_{L^p(\Gamma)} \right).
\]
For small positive $\epsilon$ we set $\Omega_{\epsilon} = \Omega \setminus (B(x_-, \epsilon) \cup B(x_+, \epsilon))$. We set $v = u e^{\tau \varphi_{\ast}}$. Then

$$\partial_{x_k} v - \partial_{x_k} \varphi_{\ast} v = \partial_{x_k} u e^{\tau \varphi_{\ast}}.$$  

Taking the scalar product of this equation in $L^2(\Omega_{\epsilon})$ with $-\partial_{x_k} \varphi_{\ast} |v|^{p-2} v$, for any $k \in \{1, 2\}$ we obtain

$$- \int_{\Omega_{\epsilon}} \tau (\partial_{x_k} \varphi_{\ast})^2 |v|^p + \frac{1}{p} \partial^2_{x_k} \varphi_{\ast} |v|^p \, dx \geq \int_{\partial \Omega_{\epsilon} \setminus \Gamma_0} \nu_k \partial_{x_k} \varphi_{\ast} |v|^p \, d\sigma =$$

$$\int_{\Omega} \partial_{x_k} \varphi_{\ast} |v|^{p-2} v \partial_{x_k} u e^{\tau \varphi_{\ast}} \, dx.$$

Adding these two equations, we obtain

$$\int_{\Omega_{\epsilon}} \tau |\nabla \varphi_{\ast}|^2 |v|^p \, dx + \sum_{k=1}^{2} \int_{\partial \Omega_{\epsilon} \setminus \Gamma_0} \nu_k \partial_{x_k} \varphi_{\ast} |v|^p \, d\sigma =$$

$$-\int_{\Omega_{\epsilon}} \partial_{x_k} \varphi_{\ast} |v|^{p-2} v \partial_{x_k} u e^{\tau \varphi_{\ast}} \, dx.$$

Using (106) we pass to the limit in (122) as $\epsilon \to +0$:

$$\int_{\Omega} \tau^p |\nabla \varphi_{\ast}|^2 |v|^p \, dx + \tau^p \sum_{k=1}^{2} \int_{\Gamma} \nu_k \partial_{x_k} \varphi_{\ast} |v|^p \, d\sigma =$$

$$-\tau^p \int_{\Omega} \partial_{x_k} \varphi_{\ast} |v|^{p-2} v \partial_{x_k} u e^{\tau \varphi_{\ast}} \, dx.$$

Estimating the right-hand side of equality (123), we obtain

$$\left| \tau^p \int_{\Omega} \partial_{x_k} \varphi_{\ast} |v|^{p-2} v \partial_{x_k} u e^{\tau \varphi_{\ast}} \, dx \right| \leq \left| (\nabla u) e^{\tau \varphi_{\ast}} \right|_{L^p(\Omega)} \tau^{p-1} \left| \nabla \varphi_{\ast} \right|_{L^p(\Omega)}$$

$$\leq C_{10} \left| (\nabla u) e^{\tau \varphi_{\ast}} \right|_{L^p(\Omega)} + \frac{\tau^p}{2} \left| |\nabla \varphi_{\ast}| \right|_{L^p(\Omega)}.$$

By (121), (123) and (124), we have

$$\int_{\Omega} |\tau^p |\nabla \varphi_{\ast}|^2 |v|^p \, dx + \left| (\nabla u) e^{\tau \varphi_{\ast}} \right|_{L^p(\Omega)}$$

$$\leq C_{11} \left( |f|_{L^p(\Omega)} + \left| \nabla u e^{\tau \varphi_{\ast}} \right|_{L^p(\Gamma)} + |\tau|^{p-1} \int_{\partial \Omega \setminus \Gamma_0} |(\nu, \nabla \varphi_{\ast})| |v|^p \, d\sigma \right)$$

$$\leq C_{12} \left( |f|_{L^p(\Omega)} + \left| u e^{\tau \varphi_{\ast}} \right|_{L^p(\Omega)} + \left| \nabla u e^{\tau \varphi_{\ast}} \right|_{L^p(\Gamma)} + |\tau|^{p-1} \int_{\partial \Omega \setminus \Gamma_0} |(\nu, \nabla \varphi_{\ast})| |v|^p \, d\sigma \right)$$

for all $\tau \geq \tau_1$ with sufficiently large $\tau_1$.

By Proposition 5, the function $\varphi_{\ast}$ has at most two critical nondegenerate points. Let $\tilde{x}_1$ and $\tilde{x}_2$ be positive critical points of the function $\varphi_{\ast}$. Moreover by Proposition 5, inequality (109) holds true. Therefore there exist constants $C_{13}$ and $\epsilon_0 > 0$ independent of $g$ such that

$$\left| \nabla \varphi_{\ast}(x) \right| \geq C_{13} > 0 \quad \forall x \in \Omega \cap (B(x_-, \epsilon_0) \cup (B(x_+, \epsilon_0)).$$

This inequality and (125) imply that

$$\int_{\Omega} \tau^p |\nabla \varphi_{\ast}|^2 |v|^p \, dx + \left| (\nabla u) e^{\tau \varphi_{\ast}} \right|_{L^p(\Omega)}$$
From (126) and (129), we have (115). The proof of the proposition is complete.

(129) \[ \Delta \] defined by (102). Consider the boundary value problem:

(128) \[ \tilde{\gamma} > 0 \] for any \( \gamma \). There exist positive constants \( \alpha \) satisfying \( \alpha \) independent of \( \tau \) and \( \gamma \).

Let \( \gamma > 0 \) and \( \tilde{\gamma} > 3 \) satisfy \( \gamma \tilde{\gamma} = 4 \). Then \( \frac{\tilde{\gamma}}{q - 1} < 2 \). We set \( q_0 = \frac{p - 1}{p} (p \tilde{\gamma} - 1) \) and \( q = \frac{4}{q_0} \).

The Hölder inequality implies:

\[
\|v\|_{L^{p}(\Omega_0)} \leq C_{16} \|\tilde{g}v\|_{L^{p}(\Omega_0)} \leq C_{17} \|\tilde{g}^\gamma v\|_{L^{2}(\Omega_0)} \leq C_{18} \|\tilde{g}^{\frac{2}{p}} v\|_{L^{2}(\Omega_0)} \leq C_{19} \|\tilde{g}^{\frac{2}{p}} v\|_{L^{p}(\Omega_0)} + \|v\|_{H^{1}(\Omega_0)}.
\]

From (126) and (129), we have (115). The proof of the proposition is complete.

Let \( \tilde{v} = (\nu_1, \nu_2) \) be the outward unit normal vector to \( \partial \Omega \) and \( \tilde{\tau} = (-\nu_2, \nu_1) \) be a tangential vector on \( \partial \Omega \). We set:

\[
\int_{x}^{\tau} f = \int_{-1}^{x_1} f(t, 1) dt.
\]

Using Proposition 6, we prove:

**Proposition 7.** Let \( f \in L^{p}(\Omega) \) for some \( p > 2 \) and \( \varphi_{*} \) be the harmonic function defined by (102). Consider the boundary value problem:

(130) \[ \Delta u = f \quad \text{in} \quad \Omega, \quad u|_{\Gamma_0} = 0. \]

There exist positive constants \( \alpha_0(p) \in (0, 1), \kappa(p, \alpha_0) \) and \( C_{20} = C_{20}(\kappa, \alpha_0) \), independent of \( \tau \) and \( \tau_0 \), such that problem (130) possesses a solution \( u = u(\tau, \cdot) \) satisfying:

\[
\|\varphi_{*}^{\Delta}ue^{-\tau \varphi_{*}}\|_{L^{p}(\Omega)} + \|\varphi_{*}^{\Delta}ue^{-\tau \varphi_{*}}\|_{L^{p}(\Omega)} \leq C_{20} e^{-\tau \varphi_{*}} f \|_{L^{p}(\Omega)}
\]

for all \( \tau \geq \tau_0 \).

**Proof.** Let \( \psi_{*} \) be a conjugate function to \( \varphi_{*} \) and \( \Phi_{*} = \varphi_{*} + i\psi_{*} \). We set:

(132) \[ u = \frac{1}{4} \text{Re} \left\{ e^{\varphi_{*}} \partial_{z}^{-1}(e^{\varphi_{*}} - \Phi_{*}) (\partial_{\overline{z}}^{-1}(e^{-\varphi_{*}} f - M(z)) + e^{\varphi_{*}} a(\tau, z) \right\}, \]

where \( a \) is a holomorphic function such that:

(133) \[ \text{Re} a = -\text{Re} \left\{ \partial_{z}^{-1}(e^{\varphi_{*}} - \Phi_{*}) (\partial_{\overline{z}}^{-1}(e^{-\varphi_{*}} f - M(z)) \right\} < 0 \quad \text{on} \ \partial \Omega, \ \forall \tau \geq \tau_0. \]

and \( M(z) \) is an antiholomorphic function in \( \Omega \) such that \( M(x_\pm) = \partial_{z}^{-1}(e^{-\varphi_{*}} f)(x_\pm) \) and \( M(y) = \partial_{z}^{-1}(e^{-\varphi_{*}} f)(y) \) for all \( y \in \mathbb{I} \).
We claim that the function \( u \) given by (132) and (133) satisfies (130). Indeed, by Proposition 4 the function \( u \) satisfies the equation (130) pointwise in \( \Omega \). Moreover Proposition 4 implies that \( e^{-\tau \Phi} f \in L^p(\Omega) \). By the classical properties of the operator \( \partial_{z}^{-1} \) (see e.g., [32]), the function \( \partial_{z}^{-1}(e^{-\tau \Phi} f) \) belongs to \( W^1_p(\Omega) \) and therefore \( \partial_{z}^{-1}(e^{-\tau \Phi} \cdot \partial_{z}^{-1}(e^{-\tau \Phi} f)) \in W^1_p(\Omega) \). Hence one can choose a harmonic function \( a \) from the space \( W^1_p(\Omega) \). Then the function \( \partial_{z}^{-1}(e^{-\tau \Phi} \cdot \partial_{z}^{-1}(e^{-\tau \Phi} f)) + e^{\tau \Phi} a(\tau, z) \in C^\gamma(\Omega) \) for some \( \gamma \in (0, 1) \) and the trace of this function on \( \Gamma_0 \) is identically equal to zero. Using the estimate (106) we see that \( u \in H^1(\Omega) \) and \( u|_{\Gamma_0} = 0 \).

Now we start the proof of estimate (131). Integrating by parts, we obtain the identity

\[
\partial_{z}^{-1}(e^{\tau(\Phi, -\Phi)}(\partial_{z}^{-1}(e^{-\tau \Phi} f) - M(\tau)))
\]

\[
= \partial_{z}^{-1}\left(\left(\partial_{z}e^{\tau(\Phi, -\Phi)}\right) \frac{\partial_{z}^{-1}(e^{-\tau \Phi} f) - M(\tau)}{\tau \partial_{z} \Phi}\right)
\]

\[
= e^{\tau(\Phi, -\Phi)} \frac{\partial_{z}^{-1}(e^{-\tau \Phi} f) - M(\tau)}{\tau \partial_{z} \Phi} + \partial_{z}^{-1}\left(\left(\partial_{z}e^{\tau(\Phi, -\Phi)}\right) \frac{\partial_{z}^{-1}(e^{-\tau \Phi} f) - M(\tau)}{\tau \partial_{z} \Phi}\right)
\]

\[
- \int_{\partial \Omega} (\nu_1 - i \nu_2)e^{\tau(\Phi, -\Phi)} \frac{\partial_{z}^{-1}(e^{-\tau \Phi} f) - M(\tau)}{\tau \partial_{z} \Phi} \frac{d \sigma}{\zeta - \tau} = \sum_{j=1}^{3} I_j(x), \quad \zeta = \xi_1 + i \xi_2.
\]

Let \( \tilde{\Omega} = \Omega \setminus \bar{I}_\delta \), where \( I_\delta \) is a \( \delta \)-neighborhood of the set \( I \), and let \( \tilde{\mu} \in C^\infty_c(I_\delta) \) and \( \tilde{\mu}|_{I_\delta} = 1 \). By (110) there exists a constant \( \delta > 0 \) independent of \( g_\delta \) such that one can find a constant \( \delta_1 = \delta_1(\delta) > 0 \) satisfying

\[ \Gamma_{\delta_1} \cap \bar{\Omega} \subset \tilde{\Omega} \quad \text{where} \quad \Gamma_{\delta_1} = \{ x \in \partial \Omega; \text{dist} (x, \tilde{\Omega}) \leq \delta_1 \}. \]

By Proposition 5 we have

\[
\| I_1 \|_{W^{-\frac{1}{2}, \frac{1}{p}}(\tilde{\Omega})} \leq C_{21} \left\| e^{\tau(\Phi, -\Phi)} \frac{\partial_{z}^{-1}(e^{-\tau \Phi} f) - M(\tau)}{\tau \partial_{z} \Phi} \right\|_{W^1_p(\tilde{\Omega})} \leq C_{22}(p) \| e^{-\tau \Phi} f \|_{L^p(\Omega)}.
\]

By the classical properties of the operator \( \partial_{z}^{-1} \) (see [32]), for any \( p > 2 \) there exist constants \( C_{24}(p) \) and \( C_{25}(p) \), independent of \( \tau \), such that

\[
\| I_2 \|_{W^{-\frac{1}{2}, \frac{1}{p}}(\tilde{\Omega})} \leq C_{23}(p) \| \tilde{\mu} I_2 \|_{W^{1, \frac{1}{p}}(\tilde{\Omega})} \leq C_{24}(p) \| \tilde{\mu} e^{\tau(\Phi, -\Phi)} \partial_{z} \left(\frac{\partial_{z}^{-1}(e^{-\tau \Phi} f) - M(\tau)}{\tau \partial_{z} \Phi}\right) \|_{L^p(\Omega)} +
\]

\[
(1 - \tilde{\mu}) e^{\tau(\Phi, -\Phi)} \partial_{z} \left(\frac{\partial_{z}^{-1}(e^{-\tau \Phi} f) - M(\tau)}{\tau \partial_{z} \Phi}\right) \|_{L^1(\Omega)} \leq C_{25}(p) \| e^{-\tau \Phi} f \|_{L^p(\Omega)}.
\]

Here \( \tilde{\mu} \in C^1(\Omega) \) is a function satisfying \( \tilde{\mu}(x) = 1 \) in some neighborhood of \( \tilde{\Omega} \) and \( \text{supp} (1 - \tilde{\mu}) \cap \text{supp} \tilde{\mu} = \emptyset \). In order to estimate the trace of the harmonic function
Let $I_3$ on the boundary, we observe that
\[
\left\| (\nu_1 - i\nu_2)e^{\tau(\Phi_i - \Phi_\ast)} \frac{\partial z^{-1}(e^{-\tau\Phi_i}) - M}{\tau \partial_z \Phi_i} \right\|_{W_p^{-\frac{1}{2} \tau, \gamma}(\Gamma_\ast)} \leq C_{26}(p)\| e^{-\tau\Phi_i} f \|_{L^p(\Omega)}.
\]
This estimate implies
\[
\| I_3 \|_{W_p^{-\frac{1}{2} \tau, \gamma}(\Gamma)} \leq C_{27}(p)\| e^{-\tau\Phi_i} f \|_{L^p(\Omega)}.
\]
For any $p > 2$, we see that (133) and (135)-(137) yield
\[
\| a \|_{W_p^{1, \tau}(\Omega)} \leq C_{28}(p)\| \Re a \|_{W_p^{-1, \frac{1}{2} \tau, \gamma}(\partial \Omega)} \leq C_{29}(p)\| e^{-\tau\Phi_i} f \|_{L^p(\Omega)}.
\]
By (132), (135)-(138), and the trace theorem, for any $p$ from $(2, +\infty)$ we have
\[
\| e^{-\tau\Phi} \|_{C^{1, -\frac{1}{2}}(\partial \Omega)} \leq C_{30}(p)\| e^{-\tau\Phi} f \|_{L^p(\Omega)}.
\]
Hence for any $0 < \alpha_0 < 1 - \frac{2}{p}$, there exists a positive constant $c(\alpha_0)$ such that
\[
tau^{\kappa(\alpha_0)}\| e^{-\tau\Phi} \|_{C^{\alpha_0}(\partial \Omega)} \leq C_{31}\| e^{-\tau\Phi} f \|_{L^p(\Omega)}.
\]
Now we estimate $\int_{x_+}^x \frac{\partial}{\partial \tau}$. Denote $W = \frac{1}{4} \partial_z^{-1}(e^{\tau(\Phi_i - \Phi_\ast)})(\partial_z^{-1}(e^{-\tau\Phi_i}) - M(z)) + a(\tau, z)$. Obviously
\[
\partial_z(e^{\tau\Phi_i} W) = \frac{1}{4} e^{\tau\Phi_i} (\partial_z^{-1}(e^{-\tau\Phi_i}) - M(z)) \quad \text{in } \Omega
\]
for all $\tau$. Therefore
\[
\frac{\partial(e^{\tau\Phi_i} W)}{\partial \nu} = i\partial_{\nu} e^{\tau\Phi_i} W + i\frac{1}{4} e^{\tau\Phi_i} (\partial_z^{-1}(e^{-\tau\Phi_i}) - M(z)) \quad \text{on } \Gamma = [x_-, x_+].
\]
Integrating the above equality, we obtain
\[
\left( \int_{x_-}^x \partial_{\nu} e^{\tau\Phi_i} W \right) (x) = -e^{\tau\Phi_i(x)} W(x) + e^{\tau\Phi_i(x)} W(x_-).
\]
By (135)-(137), (138) and $\Re \Phi_\ast(x_-) = F_{1,1}(x_-) = \min_{x \in \Gamma} F_{1,1}(x)$, there exists a positive constant $c_1$ such that
\[
tau^{\kappa(\alpha_0)}\left\| e^{-\tau\Phi_i} \left( \int_{x_-}^x \partial_{\nu} W \right) (x) \right\|_{C^{\alpha_0}(\Gamma)} \leq C_{32} \tau^{\kappa_1}\| e^{-\tau\Phi} f \|_{L^p(\Omega)}.
\]
Let $b \in C^1(\overline{\Gamma})$. Consider the operator $Rq = e^{-\tau\Phi_i} \int_{x_-}^x (be^{\tau\Phi_i} q)$. This operator is continuous from $L^2(\Gamma) \to C(\Gamma)$ and from $H^1(\Gamma) \to C(\Gamma)$. Moreover we claim that
\[
\| R \|_{L^p(L^2(\Gamma); C(\Gamma))} \leq C_{33}, \quad \| R \|_{L^p(H^1(\Gamma); C(\Gamma))} \leq C_{34}/\sqrt{\tau},
\]
where the constants $C_{33}, C_{34}$ are independent of $\tau$. Indeed, the function $\varphi_\ast(x, 1)$ is monotone increasing on the segment $[-1, 1]$. Therefore
\[
\sup_{x \in \Gamma} |Rq(x)| \leq \sup_{x \in \Gamma} e^{-\tau\Phi_i} \int_{x_-}^x (be^{\tau\Phi_i} q) \leq \sup_{x \in \Gamma} e^{-\tau\Phi_i} \int_{x_-}^x e^{\tau\varphi_\ast} |bq| \leq \| b \|_{C(\Gamma)} \| q \|_{L^1(\Gamma)}.
\]
Therefore we obtain the first estimate in (141). Now we prove the second estimate in (141). Let \( \rho_\tau \in C^1(\partial \Omega) \) satisfy \( \text{supp} \rho_\tau \subseteq \{ x \in \partial \Omega : \text{dist}(x, x_\pm) \leq 2/\tau \} \) and \( \rho_\tau(x) = 1 \) for \( x \in \{ x \in \partial \Omega : \text{dist}(x, x_\pm) \leq 2/\tau \} \). Moreover
\[
\| \rho_\tau \|_{C^k(\partial \Omega)} \leq C_{35}(k)/\tau^{\frac{k}{2}}.
\]
Then, using (142), we have
\[
\sup_{x \in \Gamma} |Rq(x)| \leq \sup_{x \in \Gamma} |R(\rho_\tau q)(x)| + \sup_{x \in \Gamma} |R((1 - \rho_\tau)q)(x)|
\leq C_{39} \frac{\| q \|_{L^2(\tilde{\Gamma})}}{\sqrt{\tau}} + \sup_{x \in \Gamma} |R((1 - \rho_\tau)q)(x)|.
\]
Estimating the second term on the right-hand side of (143) and taking into account (142) and (110), we have
\[
\leq \sup_{x \in \Gamma} \left| e^{-i \text{Im} \Phi_\tau(x)} \frac{b(x)q(x)(1 - \rho_\tau(x))}{\tau \partial_\tau \tilde{\Phi}_\tau} - e^{-\tau \varphi^*} \int_{x_\tau}^{x} \left( \frac{\partial_x}{\tau \partial_\tau \tilde{\Phi}_\tau} e^{-\tau \varphi} \right) \right|
\leq \frac{\| b \|_{L^\infty(\tilde{\Gamma})} \| q \|_{L^2(\tilde{\Gamma})}}{\tau} \| 1 - \rho_\tau \|_{L^\infty(\tilde{\Gamma})} + \sup_{x \in \Gamma} \left| e^{-\tau \varphi^*} \int_{x_\tau}^{x} \left( \frac{\partial_x}{\tau \partial_\tau \tilde{\Phi}_\tau} e^{-\tau \varphi} \right) \right|
\leq C_{37} \frac{\| q \|_{H^1(\tilde{\Gamma})}}{\sqrt{\tau}}.
\]
From (143) and (144), we obtain the second estimate in (141). The estimate (141) implies
\[
\| R \|_{L(H^\frac{1}{2}(\tilde{\Gamma}); C(\tilde{\Gamma}))} \leq C_{38}/\tau^\frac{1}{2}.
\]
Next we estimate the norm of the operator \( R \) in the H"older spaces. Let \( x, y \) be some points from \( \Gamma \) such that \( x_1 > y_1 \). Short computations imply
\[
\sup_{x, y \in \Gamma} \frac{|Rq(x) - Rq(y)|}{|x - y|^\frac{1}{2}} \leq \sup_{x, y \in \Gamma} \left| \left( e^{-\tau \varphi^*(x)} - e^{-\tau \varphi^*(y)} \right) - (1)Rq(y) \right| + \sup_{x, y \in \Gamma} \left| \left( e^{-\tau \varphi^*(x)} \int_y^x (e^{\tau \varphi^* b}) \right) \right|
\leq C_{40}(\tau \| \varphi^* \|_{C^\frac{1}{2}(\tilde{\Gamma})}) \| q \|_{L^2(\tilde{\Gamma})} \| b \|_{L^\infty(\tilde{\Gamma})} + \| q \|_{L^2(\tilde{\Gamma})} \| b \|_{L^\infty(\tilde{\Gamma})}.
\]
The estimate (146) implies
\[
\| R \|_{L(L^2(\tilde{\Gamma}); C^\frac{1}{2}(\tilde{\Gamma}))} \leq C_{40}(\tau).
\]
Therefore by (141), there exists a constant \( C_{41}(\alpha) \) such that
\[
\| R \|_{L(L^2(\tilde{\Gamma}); C^\frac{1}{2}(\tilde{\Gamma}))} \leq C_{41}(\tau^\alpha)
\]
for all \( \alpha \in (0, 1) \). Using (145) and (147), we have
\[
\| R(\partial_z^{-1}(e^{-\tau \tilde{\Phi}_\tau} f) - M) \|_{C^\frac{1}{2}(\tilde{\Gamma})}
\]
From (148) and (140), we obtain
\[
\leq C_{42} \| R(\partial_z^{-1}(e^{-\tau \Phi} f) - M) \|_{C(H)}^{1/2} \| R(\partial_z^{-1}(e^{-\tau \Phi} f) - M) \|_{C(H)}^{1/2} \\
\leq \frac{C_{43}}{\tau^{1/4}} \| \partial_z^{-1}(e^{-\tau \Phi} f) - M \|_{H^{1/4}(\partial \Omega)} \leq \frac{C_{44} \| e^{-\tau \varphi_{\ast}} f \|_{L^2(\Omega)}}{\tau^{1/4}}.
\]
(148)

From (148) and (140), we obtain
\[
\tau^{1/2} \left\| e^{-\tau \varphi_{\ast}} \int_{x_-}^{x} \frac{\partial u}{\partial \nu} \right\|_{C(H)} \leq C_{45} \| e^{-\tau \varphi_{\ast}} f \|_{L^p(\Omega)}.
\]
Thus the proof of the estimate for boundary integrals in (131) is complete.

Next we estimate the $L^p$ norm of the function $ue^{-\tau \varphi_{\ast}}$. Let $e_j \in C_0^\infty(\Omega)$, $j = 1, 2$ and let $\text{supp} \ e_j$ be located in a small neighborhood of the point $\bar{x}_j$ ($\bar{x}_j$ are the critical points of the function $\varphi_{\ast}$), and $e_j |_{B(\bar{x}_j, \delta_j)} = 1$ on some ball $B(\bar{x}_j, \delta_j)$. The function $\partial_z^{-1}(e^{-\tau \Phi} f) - M(\tau)$ satisfies the estimate
\[
\| \partial_z^{-1}(e^{-\tau \Phi} f) - M(\tau) \|_{W^2_2(\Omega)} \leq C_{46} \| e^{-\tau \varphi_{\ast}} f \|_{L^p(\Omega)}.
\]
(149)

On the other hand, by (149), (110), (109) and Morrey’s inequality (see e.g., [10]), we obtain
\[
\begin{align*}
\left\| \nabla \varphi_{\ast} \left[ \partial_z^{-1}(e^{-\tau \Phi} f) - M(\tau) \right] \right\|_{L^p(\Omega)} &
\leq C_{47} \left\| \nabla \varphi_{\ast} \left\| \partial_z^{-1}(e^{-\tau \Phi} f) - M(\tau) \right\|_{L^p(\Omega)} \\
&\leq C_{48} \left\| \nabla \varphi_{\ast} \left\| \partial_z^{-1}(e^{-\tau \Phi} f) - M(\tau) \right\|_{L^p(\Omega)} \right.
\end{align*}
\]
(150)

The classical estimate implies
\[
\left\| \int_{\partial \Omega} (\nu_1 - i\nu_2) e^{\tau(\Phi_{\ast} - \Phi_{\ast})} \partial_z^{-1}(e^{-\tau \Phi} f) - M(\tau) \right\|_{L^p(\Omega)} \leq C_{50} \| e^{-\tau \varphi_{\ast}} f \|_{L^p(\Omega)}.
\]
(151)

Estimating the remaining term in (134), using Morrey’s inequality and Young’s convolution inequality, we have
\[
\begin{align*}
\left\| \nabla \varphi_{\ast} \left[ \partial_z^{-1}(e^{-\tau \Phi} f) - M(\tau) \right] \right\|_{L^p(\Omega)} &
\leq C_{51} \left\| \partial_z^{-1}(e^{\tau(\Phi_{\ast} - \Phi_{\ast})}) |\nabla \varphi_{\ast}| \frac{\partial_z^{-1}(e^{-\tau \Phi} f) - M(\tau)}{\tau \partial_z \Phi_{\ast}} \right\|_{L^p(\Omega)} \\
&+ \left\| \partial_z^{-1}(e^{\tau(\Phi_{\ast} - \Phi_{\ast})}) |\nabla \varphi_{\ast}| \frac{\partial_z^{-1}(e^{-\tau \Phi} f) - M(\tau)}{\tau \partial_z \Phi_{\ast}} \right\|_{L^p(\Omega)} \\
&\leq C_{52} \left\| \frac{1}{|z - \zeta|} \left[ \frac{1}{|k|} \right]_{L^p(\Omega)} + \frac{1}{|z - \zeta|^{1 - \frac{2}{p}} \left[ \frac{1}{|\zeta|^{1 + \frac{2}{p}}} \right]_{L^p(\Omega)}} \right\| e^{-\tau \varphi_{\ast}} f \|_{L^p(\Omega)}.
\end{align*}
\]
(152)
Estimates (150) - (152) and equality (134) imply the $L^p(\Omega)$-estimate for the function $ue^{-\tau \varphi_*}$ in (131). The proof of the proposition is complete.

4. Third step of the proof of Theorem 1.1: construction of conformal diffeomorphism. Thanks to (86) one can apply the Carleman estimate (115) to equation (92). Hence, there exist constants $\tau_0$ and $C_1$ independent of $\tau$ such that

\begin{equation}
\begin{aligned}
\|e^{-\tau \varphi_*} |\nabla \varphi_*|^\frac{1}{4} w_2\|_{L^4(\Omega)}^4
\leq C_1 \left( \frac{1}{\tau^4} \|e^{-\tau \varphi_*} \nabla w_2\|_{L^4(\tilde{\Gamma})}^4 + \frac{1}{|\tau|} \|e^{-\tau \varphi_*} |(\nu, \nabla \varphi_*)|^\frac{1}{4} w_1\|_{L^4(\tilde{\Gamma})}^4 \right)
\leq C_2 \left( \frac{1}{\tau^4} \|e^{-\tau \varphi_*} \nabla (w_1 - w_2)\|_{L^4(\tilde{\Gamma})}^4 + \frac{1}{|\tau|} \|e^{-\tau \varphi_*} \nabla w_1\|_{L^4(\tilde{\Gamma})}^4 \right.
\left. + \frac{1}{|\tau|} \|((\nu, \nabla \varphi_*))|^\frac{1}{4} e^{-\tau \varphi_*} w_1\|_{L^4(\tilde{\Gamma})}^4 \right)
\end{aligned}
\end{equation}

for all $\tau \geq \tau_0$. We set $g_1(x) = |x - x_-| + |x - x_+|$. Short computations imply

\begin{equation}
\begin{aligned}
\|((\nu, \nabla \varphi_*))|^\frac{1}{4} e^{-\tau \varphi_*} w_1\|_{L^4(\tilde{\Gamma})}^4 \leq C_3 \|g_1 - \frac{4}{\tau^4} \|e^{-\tau \varphi_*} w_1\|_{L^4(\tilde{\Gamma})}^4 \leq C_4 \|e^{-\tau \varphi_*} w_1\|_{L^4(\tilde{\Gamma})}^4.
\end{aligned}
\end{equation}

This inequality and (153) yield

\begin{equation}
\begin{aligned}
\|e^{-\tau \varphi_*} |\nabla \varphi_*|^\frac{1}{4} w_2\|_{L^4(\Omega)}^4
\leq C_5 \left( \frac{1}{\tau^4} \|e^{-\tau \varphi_*} \nabla (w_1 - w_2)\|_{L^4(\tilde{\Gamma})}^4 + \frac{1}{|\tau|} \|e^{-\tau \varphi_*} \nabla w_1\|_{L^4(\tilde{\Gamma})}^4 + \frac{1}{|\tau|} \|e^{-\tau \varphi_*} \nabla w_1\|_{L^4(\tilde{\Gamma})}^4 \right)
\leq C_6 \left( 1 + \frac{e^{\frac{\|\varphi_*\|_{C(\tilde{\Gamma})}}{\tau}}}{\tau} \|\tilde{\Lambda} g_1 - \tilde{\Lambda} g_2\|_{L(W^1_4(\tilde{\Gamma});L^4(\tilde{\Gamma}))} \|w_1\|_{W^1_4(\tilde{\Gamma})}^4 \right).
\end{aligned}
\end{equation}

Let $\tilde{w}$ be a solution to the initial value problem:

$\Delta \tilde{w} = \mu_2 w_2$ in $\Omega$, $\tilde{w}|_{\Gamma_0} = 0$.

Since $w_2$ is a real-valued function, by Proposition 7, one can find a real-valued solution $\tilde{w}$ satisfying: there exist constants $\kappa > 0, \alpha_0 > 0, \tau_1$, and $C_7$ which is independent of $\tau$, such that

\begin{equation}
\begin{aligned}
|\tau||\nabla \varphi_*|^\frac{1}{4} \tilde{w} e^{-\tau \varphi_*}\|_{L^4(\Omega)} + \tau^\kappa \left( \tilde{w}, \int_{x_-}^x \frac{\partial \tilde{w}}{\partial r} e^{-\tau \varphi_*}\right)_{C^{\alpha_0}(\tilde{\Gamma}) \times C^{\alpha_0}(\tilde{\Gamma})}
\leq C_7 \|w_2 e^{-\tau \varphi_*}\|_{L^4(\Omega)} \leq C_8 \|\tilde{g} - g\|_{L^4(\Omega)} \|\tilde{w}_2\|_{L^4(\tilde{\Gamma})} \|\tilde{w}_1\|_{L^4(\tilde{\Gamma})}^4
\leq C_9 \|\nabla \varphi_*|^\frac{1}{4} \tilde{w}_2 e^{-\tau \varphi_*}\|_{L^4(\Omega)}
\end{aligned}
\end{equation}

for all $\tau \geq \tau_1$ and function $\tilde{g}$ is given by (128). Here, in order to obtain the last inequality, we used (109).

Applying (154) to the right-hand side of inequality (155), we obtain

\begin{equation}
\begin{aligned}
|\tau||\nabla \varphi_*|^\frac{1}{4} \tilde{w} e^{-\tau \varphi_*}\|_{L^4(\Omega)} + \tau^\kappa \left( \tilde{w}, \int_{x_-}^x \frac{\partial \tilde{w}}{\partial r} e^{-\tau \varphi_*}\right)_{C^{\alpha_0}(\tilde{\Gamma}) \times C^{\alpha_0}(\tilde{\Gamma})}
\leq C_{10} \left( 1 + \frac{1}{\tau} e^{\frac{\|\varphi_*\|_{C(\tilde{\Gamma})}}{\tau}} \|\tilde{\Lambda} g_1 - \tilde{\Lambda} g_2\|_{L(W^1_4(\tilde{\Gamma});L^4(\tilde{\Gamma}))} \|w_1\|_{W^1_4(\tilde{\Gamma})} \right).
\end{aligned}
\end{equation}
The harmonic function $\mathbf{v} = \mathbf{w}_2 - \bar{\mathbf{w}}$ verifies

$$
\Delta \mathbf{v} = 0 \text{ in } \Omega, \quad \mathbf{v}|_{\Gamma_0} = 0, \quad \left( \mathbf{v}, \frac{\partial \mathbf{v}}{\partial \nu} \right) \big|_\Gamma = \left( \mathbf{w}_2, \frac{\partial \mathbf{w}_2}{\partial \nu} \right) - \left( \bar{\mathbf{w}}, \frac{\partial \bar{\mathbf{w}}}{\partial \nu} \right).
$$

Since the domain $\Omega$ is simply connected, we can construct a function which is conjugate to $\mathbf{v} \in C^\infty(\overline{\Omega})$. Denote such a function by $\psi_0(\tau, x)$. We set $\phi_0(x) = \mathbf{v}(x) - \mathbf{v}(0)$ and $\Psi = \phi_0 + i\psi_0$. Then, thanks to the Cauchy-Riemann equations, we have

$$
\partial_\tau \Psi = 0 \quad \text{in } \Omega, \quad \text{Im} \Psi|_{\Gamma_0} = 0,
$$

(157) \quad $\Psi(\tau, z)|_{\Gamma} = \mathbf{w}_2 - i \int_{X_-}^x \frac{\partial \mathbf{w}_2}{\partial \nu} - (\mathbf{w}_2 - \bar{\mathbf{w}})(x) - \bar{\mathbf{w}} + i \int_{X_-}^x \frac{\partial \bar{\mathbf{w}}}{\partial \nu}.$

Denote $I_0(\tau, \cdot) = \bar{\mathbf{w}} - i \int_{X_-}^x \frac{\partial \bar{\mathbf{w}}}{\partial \nu}$. By (156) we have

$$
\|e^{-\tau \varphi_i} I_0(\tau, \cdot)\|_{C^\alpha(\Gamma)} = \left\| e^{-\tau \varphi_i} \left( \bar{\mathbf{w}} - i \int_{X_-}^x \frac{\partial \bar{\mathbf{w}}}{\partial \nu} \right) \right\|_{C^\alpha(\Gamma)} \leq C_{11} \left\| \left( \bar{\mathbf{w}}, \int_{X_-}^x \frac{\partial \bar{\mathbf{w}}}{\partial \nu} \right) e^{-\tau \varphi_i} \right\|_{C^\alpha(\Gamma) \times C^\alpha(\Gamma)}
$$

(158) \quad $\leq C_{12} \frac{1 + \frac{1}{\tau} \|\psi_0\|_{C(\Gamma)} \|\tilde{\Lambda}_{g_1} - \tilde{\Lambda}_{g_2}\|_{L^1(\Gamma)} \|\mathbf{w}_1\|_{W^{1,1}(\Gamma)}}{\tau^\alpha}.
$

Next we compute $i \int \frac{\partial \mathbf{w}_2}{\partial \nu}$:

$$
\int_{X_-}^x \frac{\partial \mathbf{w}_2}{\partial \nu} = \int_{X_-}^x \frac{\partial \mathbf{w}_1}{\partial \nu} + \int_{X_-}^x \left( \frac{\partial \mathbf{w}_2}{\partial \nu} - \frac{\partial \mathbf{w}_1}{\partial \nu} \right) =: \int_{X_-}^x \frac{\partial \mathbf{w}_1}{\partial \nu} + I_1(\tau, \cdot) \text{ on } \overline{\Gamma}.
$$

Estimating the norm of the difference between the Dirichlet-to-Neumann maps defined by (93) and (94), by (86) and (8) we have

(159) \quad $\frac{1}{\sqrt{\det g_1}} \frac{\partial \mathbf{w}_1}{\partial \nu} - \frac{1}{\sqrt{\det g_2}} \frac{\partial \mathbf{w}_1}{\partial \nu} \left\| \tilde{\Lambda}_{g_1}(\mathbf{w}_1) - \tilde{\Lambda}_{g_2}(\mathbf{w}_1) \right\|_{L^1(\Gamma)} \leq C_{13} \left( \|\mathbf{g}_1 - \mathbf{g}_2\|_{C(\Gamma)} \|\mathbf{w}_1\|_{W^{1,1}(\Gamma)} + \Lambda(\mathbf{g}_1, \mathbf{g}_2) \|\mathbf{w}_1\|_{W^{1,1}(\Gamma)} \right)
$

\begin{align*}
&\leq C_{14} \lambda^\alpha \|\mathbf{g}_1 - \mathbf{g}_2\|_{L^1(\Gamma)} \|\mathbf{w}_1\|_{W^{1,1}(\Gamma)}.
\end{align*}

We set $\Upsilon(\mathbf{g}_1, \mathbf{g}_2) = \|\tilde{\Lambda}_{g_1} - \tilde{\Lambda}_{g_2}\|_{L^1(\Gamma)} + \|\mathbf{g}_1 - \mu_2 I\|_{C(\Gamma)}$.

Inequality (159) yields

(160) \quad $\frac{1}{\sqrt{\det g_1}} \frac{\partial \mathbf{w}_1}{\partial \nu} - \frac{1}{\sqrt{\det g_2}} \frac{\partial \mathbf{w}_1}{\partial \nu} \left\| \tilde{\Lambda}_{g_1}(\mathbf{w}_1) - \tilde{\Lambda}_{g_2}(\mathbf{w}_1) \right\|_{L^1(\Gamma)} \leq C_{15} \Upsilon(\mathbf{g}_1, \mathbf{g}_2) \|\mathbf{w}_1\|_{W^{1,1}(\Gamma)}.$

Then, by (95), (96) and (101), we have

$$
\int_{X_-}^x \frac{\partial \mathbf{w}_1}{\partial \nu} = \int_{X_-}^x \frac{\partial \mathbf{w}_2}{\partial \nu} \mathbf{w}_1 =: \tau \int_{X_-}^x \text{Re} \left\{ i\partial_{x_2}(P \circ \mathcal{F}_1)e^{\tau P \circ \mathcal{F}_1} \right\} + I_2(\tau, \cdot),
$$

where the function $\mathcal{F}_1$ is given by (102), with mapping $F_1$ is determined by (87)-(90), and there exist a constant $C_{16}$ independent of $\tau$ such that function $I_2(\tau, \cdot)$.
satisfies the estimate
\[
\frac{1}{\sqrt{\tau}} \| e^{-\tau \varphi} I_2(\tau, \cdot) \|_{C_{\overline{T}}(\Gamma)} + \| e^{-\tau \varphi} I_2(\tau, \cdot) \|_{C_{\overline{T}}(\Gamma)} \leq C_{16} \frac{1}{\tau}.
\]

We set
\[
I_3(\tau, x) = i \tau \int_{X^-} (\partial_x (P \circ F_1) - \partial_x (\tau P \circ F_1)) e^\tau P_{\circ F_1}.
\]

By (87) and (91) we have
\[
\| I_3(\tau, \cdot) \|_{C_{\overline{T}}(\Gamma)} \leq C_{17} \tau \Psi(g_1, g_2).
\]

Using this notation, we obtain
\[
\int_{X^-} \frac{\partial W_1}{\partial \nu} = \tau \int_{X^-} \text{Re} \{ (-i \partial_x (P \circ F_1) e^{\tau P_{\circ F_1}} + \sum_{k=1}^{3} I_k(\tau, \cdot)\}
\]
\[
= \int_{X^-} \text{Re} \{ \partial_x e^{\tau P_{\circ F_1}} \} + \sum_{k=1}^{3} I_k(\tau, x) = -\text{Re} \{ e^{\tau P_{\circ F_1}(x)} + e^{\tau P_{\circ F_1}(X^-)} \} + \sum_{k=1}^{3} I_k(\tau, \cdot).
\]

From the above equality, by (157), (95) and (101), we obtain
\[
\Psi(\tau, z) \|_{\overline{T}} = ie^{\tau P_{\circ F_1}} - i \sum_{k=0}^{3} I_k(\tau, x).
\]

Consider the function \( \Pi(\tau, z) = \frac{1}{2} \ln(\Psi(\tau, z) / i) \), where \( \ln(z) = \ln(r) + i\theta \) and \( \tau \) is a large positive parameter. Then we locally define a holomorphic function \( \Pi(\tau, z) \) as
\[
\Pi(\tau, z) =: P \circ F_1 + I_4(\tau, x) \quad \text{on} \quad \overline{\Gamma}, \quad \text{Im} \Pi(\tau, z) \|_{\Gamma_0} = 0,
\]
where the mapping \( F_1 = (F_{1,1}, F_{1,2}) \) is given by (87)-(90) and
\[
I_4(\tau, x) = \frac{1}{\tau} \ln \left( 1 - e^{-\tau P_{\circ F_1}} \sum_{k=0}^{3} I_k(\tau, x) \right) \quad \text{on} \quad \overline{\Gamma}.
\]

By (160)-(162) we have
\[
\| I_4(\tau, x) \| \leq \frac{1}{\tau} \left( \max_{k \in \{0, 1\}} \left| \ln \left( 1 + C_{18} \left( \frac{1}{\tau^r} + \frac{1}{\tau} + \tau \Psi(g_1, g_2) e^{2\|\varphi\|_{C(\overline{T})}} \right) \right) \right| \right.
\]
\[
+ \left| \arctan \left( \frac{b(\tau, x)}{a(\tau, x)} \right) \right|, \quad x \in \overline{\Gamma},
\]
provided that \( \tau \) is sufficiently large. Here \( a(\tau, x) \) and \( b(\tau, x) \) are the real and the imaginary parts of \( 1 - e^{-\tau P_{\circ F_1}} \sum_{k=0}^{3} I_k(\tau, x) \). Taking \( \hat{\tau} \) satisfying
\[
\hat{\tau} \Psi(g_1, g_2) \leq e^{2\|\varphi\|_{C(\overline{T})}} \leq 1,
\]
we obtain
\[
\Pi(\hat{\tau}, z) = P \circ F_1 + I_4(\hat{\tau}, x) \quad \text{on} \quad \overline{\Gamma}, \quad \text{Im} \Pi(\hat{\tau}, z) \|_{\Gamma_0} = 0,
\]
where
\[
\| I_4(\hat{\tau}, \cdot) \|_{C_{\overline{T}}(\Gamma)} \leq \frac{C_{19}}{\ln \Psi(g_1, g_2)}.
\]

We prove that the domain of the function \( \Pi(\hat{\tau}, \cdot) \), which is considered as a function of \( z \), coincides with \( \Omega \) and this function can be extended as holomorphic on \( \overline{\Omega} \) for all sufficiently large \( \hat{\tau} \). Indeed, let us show that the function \( \Psi(\hat{\tau}, z) \) does not have zeros on \( \overline{\Omega} \) for all sufficiently large \( \hat{\tau} \). Obviously, by (163), (158), (161) and (162),
for all sufficiently large \( \hat{\gamma} \), the function \( \Psi(\hat{\gamma}, z) \) does not have any zeros on \( \partial \Omega \). Hence the function \( \Pi(\hat{\gamma}, \cdot) \) can be determined locally as a holomorphic function.

Then, by the argument principle, the function \( \Psi(\hat{\gamma}, z) \) does not have zeros in \( \Omega \). Let \( \gamma(t) : [0, 1] \to \Omega \) be a Jordan closed curve. Let \( \gamma(0) = \gamma(1) = z_0 \in \Omega \). We extend the function \( \Psi \) along this curve, and the extended holomorphic function \( \Pi \) is a continuation of \( \Psi_0 \) along the Jordan curve \( \gamma \). Then \( \Pi(\hat{\gamma}, z) - \Pi(\hat{\gamma}_0, z) = 2\pi k, k \in \mathbb{Z} \).

On the other hand, by the argument principle, the function \( \Psi \) should have at least one zero in the area bounded by the curve \( \gamma \). We reach a contradiction.

Let \( \tilde{\psi}_2 \in C^{3+\alpha}(\overline{\Omega}) \) be a harmonic function in \( \Omega \) such that \( \tilde{\psi}_2|_{\partial \Omega} = \text{Im} \, P \circ F_1 \) and \( \tilde{\varphi}_2 \) be the complex conjugate. By the construction of the holomorphic function \( P \), we have

\[
\tilde{\psi}_2|_{\Gamma_0} = 0 \quad \text{and} \quad \tilde{\psi}_2(x) > 0 \quad \forall x \in \overline{\Gamma}.
\]

Denote

\[
\Phi_2(z, g_1) = \tilde{\varphi}_2(x, g_1) + i\tilde{\psi}_2(x, g_1).
\]

By (102), (166), (167) and the maximum principle for the Laplace operator, we see that

\[
\|\tilde{\psi}_2(\cdot, g_1) - \text{Im} \Pi(\hat{\gamma}, \cdot)\|_{C^{\alpha}(\partial \Omega)} \leq \frac{C_{20}}{|\ln T(g_1, g_2)|}.
\]

The Cauchy-Riemann equations and (170) yield

\[
\|\Phi_2(\cdot, g_1) - \Pi(\hat{\gamma}, \cdot)\|_{C^{\alpha}(\overline{\Omega})} \leq \frac{C_{21}}{|\ln T(g_1, g_2)|}.
\]

We claim that there exist \( \kappa > 0 \) and \( \beta_0 > 0 \) such that

\[
|\partial_2 \Phi_2(z, g_1)| > \beta_0 \quad \text{for each } z \in \overline{\Omega} \text{ and } g_1 \text{ satisfying (8)}
\]

and \( T(g_1, g_2) \leq \kappa \).

Suppose that the inequality (172) is false for every positive \( \beta_0 \) and \( \kappa \). Then there exist two sequences of metrics \( \{g_{1,k}\}, \{g_{2,k}\} \), (metrics \( g_{p,k} \) satisfy (1), (2) and (8)) and a sequence of points \( \{z_k\} \subset \overline{\Omega} \) such that

\[
\partial_2 \Phi_2(z_k, g_{1,k}) \to 0, \quad T(g_{1,k}, g_{2,k}) \to 0 \text{ as } k \to +\infty,
\]

\[
z_k \to \hat{z} \in \overline{\Omega} \quad \text{as } k \to +\infty.
\]

By (102), (8) and (166), some subsequences of the holomorphic functions \( \tilde{\Phi}_2(z, g_{1,k}) \) and the metrics \( \{g_{1,k}\} \) converge to a holomorphic function \( \Psi_0 \in C^{3+\alpha}(\overline{\Omega}) \) and a metric \( g \) respectively satisfying

\[
\Psi_0 = P \circ F_1 \quad \text{on } \overline{\Gamma}, \quad \text{Im} \Psi_0|_{\Gamma_0} = 0
\]

and

\[
g = \{g_{j,k}\} \in C^{2+\alpha}(\overline{\Omega}), \quad \text{for some } \alpha_1 \in (0, 1), \ \alpha_1 < \alpha, \ g_{k,j} = g_{j,k}, \ \forall j, k \in \{1, 2\},
\]

\[
(g(x)\xi, \xi) > 0, \quad \forall (x, \xi) \in \overline{\Omega} \times \mathbb{R}^2 \setminus \{0\}.
\]

Since the mapping \( F_1 : \overline{\Omega} \to \overline{\Omega} \) is a diffeomorphism, we have \( det F_1'(x) \neq 0 \) on \( \overline{\Omega} \). On the other hand \( F_{1,2}|_{\Gamma} \equiv 0 \) and so \( \partial_{x_1} F_{1,1}|_{\Gamma} \neq 0 \). The Cauchy-Riemann equations imply \( \partial_1 \Psi_0 = \partial_{x_2} \text{Im} \Psi_0 + i\partial_{x_1} \text{Im} \Psi_0 \). Hence the construction of the holomorphic function \( P \) on \( \overline{\Gamma} \) yields \( \partial_{x_1} \text{Im} P = 0 \) only at point \( \hat{y} \in \overline{\Gamma} \). Therefore \( \partial_{x_1} \text{Im} \Psi_0 = 0 \) only for \( \hat{x} \in \overline{\Gamma} \) such that \( \hat{y} = F_1(\hat{x}) \).
On the other hand, $\hat{y}$ is a point of absolute maximum of the function $\text{Im} P$ on $\Gamma$. Consequently $\partial_{x_2} \text{Im} P(\hat{y}) \neq 0$. Observe that $\text{Re} \partial_2 \Psi_0(\hat{x}) = \partial_2 \text{Re} P(\hat{y}) \partial_{x_1} F_{11}(\hat{x}) = \partial_2 \text{Im} P(\hat{y}) \partial_{x_1} F_{11}(\hat{x}) \neq 0$. Therefore the function $\Psi_0(z) \neq 0$ does not have critical points on $\Gamma$. Suppose that a critical point $\hat{z}$ of the function $\Psi_0$ belongs to $\Gamma_0$. Then this is a critical point of the harmonic function $\text{Im} \Psi_0$. Since $\hat{z}$ is a point of the global minimum of the function $\text{Im} \Psi_0$ on $\Omega$, the Hopf lemma yields that $\frac{\partial \text{Im} \Psi_0(\hat{z})}{\partial r} \neq 0$. Therefore we reach a contradiction.

Suppose that the function $\Psi_0(z)$ has a critical point $\hat{z} \in \Omega$. Since there are no critical points of the function $\Psi_0$ on the boundary, one can apply the argument principle. Using the Cauchy–Riemann equations, we compute $\frac{\partial \Psi_0(\hat{z})}{\partial x} = -1$. On the segment $[x_-, x_+]$ there exists a unique pair $(\hat{x}_+, \hat{x}_-)$ of points in $\Gamma_0$ such that $\frac{\partial \Psi_0(\hat{x}_+)}{\partial x} = \frac{\partial \Psi_0(\hat{x}_-)}{\partial x}$. On the arc between the points $x_+$ and $x_-$, there exists a point $\hat{x}_+$ such that $\frac{\partial \Psi_0(\hat{x}_+)}{\partial x} = 1$. The vector $\partial \Psi_0(\hat{z})$ makes the full circle around the origin, when the parameter $z$ is moving clockwise from $x_+$ to $x_-$. On the other hand, moving clockwise on segment $[x_-, x_+]$, the vector $\frac{\partial \Psi_0(\hat{z})}{\partial x}$ rotating in the opposite direction, and so $\arg \partial_2 \Psi_0 = 0$. We reach a contradiction.

We have

**Proposition 8.** Let $\Upsilon(g_1, g_2) \leq \epsilon$ with sufficiently small $\epsilon > 0$. Then for any $\alpha_1 \in (0, \alpha)$ there exist a positive number $\kappa = \kappa(\alpha_1)$ and a conformal diffeomorphism $\Xi: \Omega \rightarrow \Omega$ such that $\Xi \in C^{3+\alpha}(\Omega)$, $\Xi(\Gamma_0) = \Gamma_0$ and

\begin{equation}
\|\Xi \circ F_1^{-1} - I\|_{C^{3+\alpha}(\Gamma)} \leq \frac{C_{22}}{|\ln \Upsilon(g_1, g_2)|^\kappa}
\end{equation}

and

\begin{equation}
\|\Xi\|_{C^{3+\alpha}(\Omega)} + \|\Xi^{-1}\|_{C^{3+\alpha}(\Omega)} \leq C_{23}.
\end{equation}

Here the mapping $F_1$ is determined by (87)-(90).

**Proof.** Let $\Omega_1 \subset \mathbb{R} \times \mathbb{R}$ be a bounded domain with smooth boundary such that $\partial \Omega_1 \cap \{x \in \mathbb{R}^2; x_2 = 0\} = [-2, 2]$, and let $P \in C^{3+\alpha}(\overline{\Omega_1})$ be a conformal diffeomorphism of the domain $\Omega$ on $\Omega_1$ such that $P(\Gamma_0) = \{x; -1 \leq x_1 \leq 1, x_2 = 0\}$, $P(x_+) = \bar{x}_+, \bar{x}_- = (\pm 1, 0)$. Let $\Gamma_0 = P(\Gamma_0)$, $\Gamma_\ast = \overline{P(\Gamma)}$, and $\Phi_3 = P \circ P^{-1} \circ \Phi_2 \circ P^{-1} = \tilde{\varphi}_3(x, g_1) + i\tilde{\psi}_3(x, g_1)$, where the function $\tilde{\Phi}_2$ is given by (169), and the holomorphic function $P$ is determined by Proposition 3. Consider the mapping $P_{g_1}(x) = (\tilde{\varphi}_3(x, g_1), \tilde{\psi}_3(x, g_1))$. Since $\tilde{\varphi}_3(x, g_1), \tilde{\psi}_3(x, g_1) \in C^{3+\alpha}(\overline{\Omega_1})$ and $\partial \Phi_3 \neq 0$ on $\overline{\Omega_1}$, the domain $P_{g_1}(\Omega_1)$ is bounded and simply connected in $\mathbb{R}^2$ with $\partial P_{g_1}(\Omega_1) \in C^{3+\alpha}$.

By (166) and (167), (171) we have

\begin{equation}
|P_{g_1}(\bar{x}_+) - \bar{x}_+| \leq \frac{C_{24}}{|\ln \Upsilon(g_1, g_2)|}
\end{equation}

Let $\ell_1 = 2/(\tilde{\varphi}_3(\bar{x}_+, g_1) - \tilde{\varphi}_3(\bar{x}_-, g_1))$ and $\ell_2 = 1 - \ell_1 \tilde{\varphi}_3(\bar{x}_+, g_1)$. We set $G_{g_1} = \ell_1 P_{g_1} + (\ell_2, 0)$. Observe that

\begin{equation}
G_{g_1}(\Gamma_0) = \Gamma_0^\ast.
\end{equation}
Denote $F_3 = \mathcal{P} \circ F_1 \circ \mathcal{P}^{-1}$. By (175) and (171), we have

\begin{equation}
\|G_{g_1} - F_3\|_{C(\overline{\Gamma}_1)} \leq \frac{C_{25}}{\ln Y(g_1, g_2)}.
\end{equation}

Let $G_{g_1}(x)$ be a solution to

\begin{equation}
\Delta G_{g_1} = 0 \quad \text{in } G_{g_1}(\Omega_1), \quad G_{g_1}|_{\partial G_{g_1}(\Omega_1)} = -\ln |x|.
\end{equation}

Let $R_{g_1}(x)$ be the conjugate function to $G_{g_1}(x)$ in $G_{g_1}(\Omega_1)$.

Consider the holomorphic function $\Psi_{g_1}(x) = G_{g_1}(x) + iR_{g_1}(x) + \ln z$. Since the boundary $\partial G_{g_1}(\Omega_1)$ is smooth, it is known that the mapping $e^{\Psi_{g_1}}$ is a diffeomorphism of the domain $G_{g_1}(\Omega_1)$ in $\mathbb{D}$ (see e.g., [12], p. 251, problem 73).

Let $G_{g_2}$ solve the boundary value problem

\begin{equation}
\Delta G_{g_2} = 0 \quad \text{in } \Omega_1, \quad G_{g_2}|_{\partial \Omega_1} = -\ln |x|.
\end{equation}

Let $R_{g_2}(x)$ be the conjugate function to $G_{g_2}(x)$ in $\Omega_1$.

Consider the holomorphic function $\Psi_{g_2}(x) = G_{g_2}(x) + iR_{g_2}(x) + \ln z$, $z = x + ix_2$. Since the boundary $\partial \Omega_1$ is smooth, it is known that the mapping $e^{\Psi_{g_2}}$ is a diffeomorphism of the domain $\Omega_1$ onto $\mathbb{D}$ (see e.g., [12], p. 251, problem 73).

By (87) there exists a constant $C_{26}$, independent of $g_1, g_2$ such that

\begin{equation}
\|F_3\|_{C^{3+\alpha}(\overline{\Gamma}_1)} + \|F_3^{-1}\|_{C^{3+\alpha}(\overline{\Gamma}_1)} + \|e^{\Psi_{g_1}}\|_{C^{3+\alpha}(\overline{\Gamma}_1)} + \|\Psi_{g_1}\|_{C^{3+\alpha}(\overline{\Gamma}_1)} + \|\Psi_{g_2}\|_{C^{3+\alpha}(\overline{\Gamma}_1)} + \|\Psi_{g_2}^{-1}\|_{C^{3+\alpha}(\overline{\Gamma}_1)} \leq C_{26}.
\end{equation}

Let $0 < \alpha_1 < \alpha$. By (180) and (177), there exists $\kappa = \kappa(\alpha_1)$ such that

\begin{equation}
\|G_{g_1} - F_3\|_{C^{3+\alpha_1}(\overline{\Gamma}_1)} \leq \frac{C_{27}}{\ln Y(g_1, g_2)^{\kappa}}.
\end{equation}

We claim that the following estimate is true:

\begin{equation}
\|e^{\Psi_{g_1}} \circ G_{g_1} - e^{\Psi_{g_2}} \circ F_3\|_{C^{3+\alpha_1}(\overline{\Gamma}_1)} \leq \frac{C_{28}}{\ln Y(g_1, g_2)^{\kappa}}.
\end{equation}

In order to prove (182), we construct a diffeomorphism $J$ of the domain $G_{g_1}(\Omega_1)$ onto $\Omega_1$ which is close to the unit mapping.

Let $\Omega_s = \{x \in G_{g_1}(\Omega_1): \text{dist}(x, G_{g_1}(\Gamma_s)) < \frac{1}{s}\}$, where $s$ is a large positive parameter. Let $\rho_s(x) = 1$ for $x \in \Omega_s$ and $\rho_s(x) = 0$ for $x \in \Omega_1 \setminus \Omega_s$. We set

\begin{equation}
J = (1 - \rho_s)I + \rho_s(F_3 \circ G_{g_1}^{-1}).
\end{equation}

Then $J' = I + (\rho_s(F_3 \circ G_{g_1}^{-1} - I))'$. On the set $G_{g_1}(\Omega_1) \setminus \text{supp } \rho_s$, the mapping $J$ is the identity mapping. The short computations imply

\begin{equation}
\|\rho_s(F_3 \circ G_{g_1}^{-1} - I)\|_{C^{1+\alpha_1}(\text{supp } \rho_s)} \leq C_{29}(s^{2\alpha_1}\|F_3 \circ G_{g_1}^{-1} - I\|_{C^{1+\alpha_1}(\text{supp } \rho_s)} + s^{1+\alpha_1}\|F_3 \circ G_{g_1}^{-1} - I\|_{C^{1+\alpha_1}(\text{supp } \rho_s)}).
\end{equation}
We take $s = |\ln \Upsilon(g_1, g_2)|^{\frac{1}{2}}$. By (180), (181) and (184), we obtain
\[
\| (\rho_+(F_3 \circ G_{g_1}^{-1} - I))^{\nu} \|_{C^{1+\alpha}, \{g_1\}}^C \leq C_{30} \left( |\ln \Upsilon(g_1, g_2)|^{\frac{1}{4}} \left\| F_3 \circ G_{g_1}^{-1} - I \right\|_{C^{1+\alpha}, \{g_1\}}^C \right.
\]
\[
+ |\ln \Upsilon(g_1, g_2)|^{\frac{(1+\alpha)}{4}} \left\| (F_3 \circ G_{g_1}^{-1})^{\nu} \right\|_{C^{1+\alpha}, \{g_1\}}^C \leq C_{31} \left( |\ln \Upsilon(g_1, g_2)|^{\frac{1}{4}} \left\| F_3 \circ G_{g_1}^{-1} - I \right\|_{C^{2+\alpha}, \{g_1\}}^C \right.
\]
\[
+ |\ln \Upsilon(g_1, g_2)|^{\frac{(1+\alpha)}{4}} \left\| (F_3 \circ G_{g_1}^{-1})^{\nu} \right\|_{C^{2+\alpha}, \{g_1\}}^C \leq C_{33} \left( |\ln \Upsilon(g_1, g_2)|^{\frac{1}{4}} \left\| F_3 \circ G_{g_1}^{-1} - I \right\|_{C^{3+\alpha}, \{g_1\}}^C \right)
\]
Therefore there exists a positive constant $\epsilon$ such that if $\Upsilon(g_1, g_2) \in (0, \epsilon)$, then
\[ |\det J'(x)| > 0, \quad \forall x \in G_{g_1}(\Omega_1). \]

By Proposition 3, we have $\Gamma_0^* \subset \partial G_{g_1}(\Omega_1)$ and $F_3(\Gamma_0^*) = \Gamma_0^*$. Hence, if $x \in \Gamma_0^* \setminus \supp \rho|_{\ln \Upsilon(g_1, g_2)}^{\frac{1}{2}}$, then $J(x) = (1 - \rho|_{\ln \Upsilon(g_1, g_2)}^{\frac{1}{2}})I x = x \in \Gamma_0^*$ and $\rho|_{\ln \Upsilon(g_1, g_2)}^{\frac{1}{2}} F_3 \circ G_{g_1}^{-1}(x) = 0$ for $x \in \Gamma_0^*$. Moreover (175), (176) and (88) yield
\[ J(\tilde{x}_+) = \tilde{x}_+. \]

On the other hand, provided that $\epsilon > 0$ is sufficiently small, there exist points $\tilde{x}_x \in \tilde{x}_0 \setminus \supp \rho|_{\ln \Upsilon(g_1, g_2)}^{\frac{1}{2}}$ such that $J(\tilde{x}_x) = (\tilde{x}_x, 0)$. Consider the part of $\Gamma_0^*$ located between the points $\tilde{x}_-$ and $\tilde{x}_-$ when we are moving along $\Gamma_0^*$ from $\tilde{x}_-$ to $\tilde{x}_-$ clockwise. By (183) we have
\[ J(x) = (J_1(x), 0), \quad \forall x \in [\tilde{x}_-, \tilde{x}_-] \subset \Gamma_0^*. \]

Since the function $f(t) = J_1(t \tilde{x}_- + (1 - t) \tilde{x}_-)$ is continuous, it takes all the values between $f(0) = \tilde{x}_-$ and $f(1) = \tilde{x}_-$. Then for any $x = (x_1, 0) \in [\tilde{x}_-, \tilde{x}_-]$ there exists $t_1 \in [0, 1]$ such that $f(t_1) = (x_1, 0)$. Hence $J_1(t_1 \tilde{x}_- + (1 - t_1) \tilde{x}_-) = x_1$ and therefore for any $x = (x_1, 0) \in [\tilde{x}_-, \tilde{x}_-]$ one can find $t \in [0, 1]$ such that $J(t_1 \tilde{x}_- + (1 - t_1) \tilde{x}_-) = x_1$. Similarly $J([\tilde{x}_+, \tilde{x}_+]) = [\tilde{x}_+, \tilde{x}_+]$. Hence
\[ J(\Gamma_0^*) = \Gamma_0^*. \]

Now let $x \in \partial G_{g_1}(\Omega_1) \setminus \Gamma_0^* = G_{g_1}(\Gamma_*)$. Then $G_{g_1}^{-1}(x) \in \tilde{\Gamma}_*$ and $F_3(y) \in \tilde{\Gamma}_*$ for any $y \in \tilde{\Gamma}_*$. Hence $J(x) \in \tilde{\Gamma}_*$. On the other hand, $G_{g_1}^{-1}(\partial G_{g_1}(\Omega_1) \setminus \Gamma_0^*) = \tilde{\Gamma}_*$, and so
\[ J(\partial G_{g_1}(\Omega_1) \setminus \Gamma_0^*) = \tilde{\Gamma}_*. \]

By (188) and (189), we obtain
\[ J(\partial G_{g_1}(\Omega_1)) = \partial \Omega_1. \]

Since (186) implies $\det J'(x) \neq 0$ in $G_{g_1}(\Omega_1)$, the mapping $J$ is a diffeomorphism of $G_{g_1}(\Omega_1)$ onto $\Omega_1$.

Making the change of variables in (178), we obtain that the function $\tilde{G}_{g_1} = G_{g_1} \circ J^{-1}$ solves the boundary value problem
\[ \Delta_{J, t} \tilde{G}_{g_1} = 0 \quad \text{in } \Omega_1, \quad \tilde{G}_{g_1}|_{\partial \Omega_1} = -(\ln |x|) \circ J. \]
By (190) and (179), we have
\[
\Delta(\bar{G}_{g_1} - G_{g_2}) = (\Delta - \Delta_{J, l})\bar{G}_{g_1} \quad \text{in } \Omega_1,
\]
\[
(\bar{G}_{g_1} - G_{g_2})|_{\partial\Omega_1} = \ln |x| - \ln |x| \circ J.
\]
Observe that by (183) and (184)
\[
\|J_* I - I\|_{C^{1+\alpha_1}(\overline{\Omega}_1)} \leq \frac{C_{34}}{|\ln Y(g_1, g_2)|^\frac{3}{2}}.
\]
We apply to the boundary value problem (191) the standard a priori estimate (e.g., [11]) in the Hölder spaces. By (192) we obtain
\[
\|\bar{G}_{g_1} - G_{g_2}\|_{C^{2+\alpha_1}(\overline{\Omega}_1)} \leq C_{35}((\Delta - \Delta_{J, l})\bar{G}_{g_2}\|_{C^{\alpha_1}(\overline{\Omega}_1)} + \|\ln |x| \circ J - \ln |x|\|_{C^{2+\alpha_1}(\partial\Omega_1)})
\]
\[
\leq \frac{C_{36}}{|\ln Y(g_1, g_2)|^\frac{3}{2}}.
\]
Next we estimate the norm of the mapping \(e^{\psi g_1} \circ G_{g_1} - e^{\psi g_2} \circ F_3\) on \(\overline{\Gamma}_x\):
\[
J := \|e^{\psi g_1} \circ G_{g_1} - e^{\psi g_2} \circ F_3\|_{C^{2+\alpha_1}(\overline{\Gamma}_x)}
\]
\[
= \|e^{\psi g_1} \circ J^{-1} \circ J \circ G_{g_1} - e^{\psi g_2} \circ J \circ F_3\|_{C^{2+\alpha_1}(\overline{\Gamma}_x)}
\]
\[
\leq \|e^{\psi g_2} \circ J \circ G_{g_1} - e^{\psi g_2} \circ F_3\|_{C^{2+\alpha_1}(\overline{\Gamma}_x)}
\]
\[
+ \|e^{\psi g_1} \circ J^{-1} \circ J \circ G_{g_1} - e^{\psi g_2} \circ J \circ G_{g_1}\|_{C^{2+\alpha_1}(\overline{\Gamma}_x)}.
\]
By (183), we have \(J(x) = (F_3 \circ G_{g_1}^{-1})\) for any \(x \in G_{g_1}(\overline{\Gamma}_x)\). Consequently
\[
e^{\psi g_2} \circ J \circ G_{g_1} - e^{\psi g_2} \circ F_3 = 0 \quad \text{on } \overline{\Gamma}_x.
\]
From (181) and (193) there exists a positive \(\kappa_1\) such that
\[
\|e^{\psi g_1} \circ J^{-1} - e^{\psi g_2}\|_{C^{2+\alpha_1}(\overline{\Gamma}_x)} \leq \frac{C_{37}}{|\ln Y(g_1, g_2)|^\kappa_1}.
\]
By (195) and (177) it follows from (194) that
\[
J \leq \frac{C_{38}}{|\ln Y(g_1, g_2)|^\kappa_1}.
\]
Inequality (196) proves (182).

By (176), we see that \(G_{g_1}(\overline{x}_\pm) = \overline{x}_\pm\). Observe that points \(\overline{x}_\pm\) also belong to \(\partial\Omega_1\). Hence
\[
|e^{\psi g_1}(\overline{x}_\pm) - e^{\psi g_2}(\overline{x}_\pm)|
\]
\[
\leq |e^{\psi g_1}(\overline{x}_\pm) - e^{\psi g_1} \circ J^{-1}(\overline{x}_\pm)| + |e^{\psi g_1} \circ J^{-1}(\overline{x}_\pm) - e^{\psi g_2}(\overline{x}_\pm)|
\]
\[
\leq |e^{\psi g_1}(\overline{x}_\pm) - e^{\psi g_1} \circ J^{-1}(\overline{x}_\pm)| + \frac{C_{39}}{|\ln Y(g_1, g_2)|^\kappa_1}
\]
\[
\leq C_{40}|\overline{x}_\pm - J^{-1}(\overline{x}_\pm)| + \frac{C_{41}}{|\ln Y(g_1, g_2)|^\kappa_1}.
\]
By (183) and (185), from the above inequality we obtain
\[
|e^{\psi g_1}(\overline{x}_\pm) - e^{\psi g_2}(\overline{x}_\pm)| \leq \frac{C_{42}}{|\ln Y(g_1, g_2)|^\kappa_1}.
\]
By (197) there exists a mapping \( \Xi_0 \in \text{Aut} \mathbb{D} \) such that
\[
||I - \Xi_0||_{C^3[\mathbb{D}]} \leq \frac{C_{43}}{4 \ln \overline{\mathcal{T}(g_1, g_2)}} \quad \text{and} \quad \Xi_0(e^{\psi}g_1(\Gamma^*_0)) = e^{\psi}g_2(\Gamma^*_0).
\]
We set
\[
\Xi = \mathcal{P}^{-1} \circ (e^{\psi}g_2)^{-1} \circ \Xi_0 \circ e^{\psi}g_1 \circ \mathcal{G}_{g_1} \circ \mathcal{P}.
\]
By the construction of the mappings \( \mathcal{P}, e^{\psi}g_2, e^{\psi}g_1, \) and \( \mathcal{G}_{g_1} \) we see that the mapping \( \Xi \) is a conformal diffeomorphism of the domain \( \Omega \) on itself. Inequalities (180) and (181) imply (174). From (176) it follows that
\[
\mathcal{G}_{g_1}(\Gamma^*_0) \circ \mathcal{P}(\Gamma_0) = \Gamma^*_0.
\]
This equality and (198) imply
\[
(e^{\psi}g_2)^{-1} \circ \Xi_0 \circ e^{\psi}g_1 \circ \mathcal{G}_{g_1} \circ \mathcal{P}(\Gamma^*_0) = \Gamma^*_0.
\]
Therefore \( \Xi(\Gamma_0) = \Gamma_0 \). Finally
\[
\frac{\Xi \circ F_1^{-1} - I}{C^{3+\alpha_1}(\Gamma)} \leq C_{44} ||\Xi - F_1||_{C^{3+\alpha_1}(\Gamma)} \leq C_{45} \left( (e^{\psi}g_2)^{-1} \circ e^{\psi}g_1 \circ \mathcal{G}_{g_1} \circ F_1 \right)_{C^{3+\alpha_1}(\Gamma_0)} \leq \frac{C_{46}}{4 \ln \overline{\mathcal{T}(g_1, g_2)}}.
\]
The proof of estimate (173) is finished. This complete the proof of the proposition.

In the operator \( L_{g_1}(x, D) \), we make the change of variables by \( \mathbf{F} = \Xi \circ F_1 \). Consider the operator \( L_{\mathbf{F}, g_1}(x, D) \). Then \( \mathbf{F}_*g_1 = \bar{\mu}_1 I \) with \( \bar{\mu}_1 = \mu_1 \circ \Xi^{-1} \in C^{2+\alpha}(\overline{\Omega}) \) and there exists a constant \( \beta(M_1, M_2) > 0 \) such that
\[
\bar{\mu}_1(x) > \beta > 0, \quad \forall x \in \Omega.
\]
Moreover (159) and (173) yield
\[
||\mathbf{F}_{g_1} - \bar{\mathbf{F}}_{g_1}||_{L^{(2)}(\partial\Omega);W^{1,2}_{2}(\partial\Omega))} \leq \frac{C_{47}}{\sqrt{\ln \overline{\mathcal{T}(g_1, g_2)}}} \leq \frac{C_{48}}{\sqrt{\ln \overline{\mathcal{A}(g_1, g_2)}}}.
\]
Observe that \( g_2 = \mu_2 I \). We introduce the function \( \tilde{G} \) by formula
\[
\tilde{G}(\eta) = \left\{ \begin{array}{ll}
(1 + ||\ln \eta||)^{-\frac{1}{2}} & \text{if } 0 < \eta \leq 1/2, \\
1 & \text{if } \eta \geq 1/2.
\end{array} \right.
\]
Thanks to (199) one can apply stability result of [8] to the Schrödinger operators with potentials \( \bar{\mu}_1 \) and \( \mu_2 \).

Using (199) we obtain:
\[
||\bar{\mu}_1 - \mu_2||_{L^{2}(\Omega)} \leq C_{49} \tilde{G}(||\mathbf{F}_{g_1} - \bar{\mathbf{F}}_{g_1}||_{L^{(2)}(\partial\Omega);W^{1,2}_{2}(\partial\Omega))}) \leq C_{50} \mathcal{G}(\mathbf{A}(g_1, g_2)).
\]
Thus the proof of Theorem 1.1 is complete.
5. Proof of Theorem 1.2. Consider a diffeomorphism of $F$ to the domain $\Omega$ into itself such that $F_*g_2 = \mu_2 I$ and $F(\Gamma) = \tilde{\Gamma}, F(x_\pm) = x_\pm$. Since $g_1 = g_2$ on $\Gamma$, we have $\Lambda_{F,\beta_1,F_*g_2} = \Lambda_{F,\beta_2,F_*g_2} = \Lambda_{F,\beta_2,\mu_2 I}$. Let $F_1$ be a diffeomorphism of $\Omega$ into itself such that $(F_1 \circ F)_*g_1 = \mu_1 I$ and $F_1(\tilde{\Gamma}) = \Gamma$. Repeating the arguments of Proposition 8, we see the existence of a conformal diffeomorphism $\Xi : \Omega \to \Omega$ such that $\Xi \in C^{\alpha}(\Omega)$ for some $\alpha \in (0, 1), \Xi(\Gamma_0) = \Gamma_0, \Xi(x_\pm) = x_\pm$ and there exist constants $\kappa \in (0, 1)$ and $C_1 > 0$, independent of $g_1, g_2$ and $\beta_1, \beta_2$, such that

$$
\|\Xi \circ F_1 - I\|_{C^{3+\alpha}(\tilde{\Gamma})} \leq \frac{C_1}{\ln \|\Lambda_{\beta_1,g_1} - \Lambda_{\beta_2,g_2}\|_{L(W^1_2(\tilde{\Gamma});L^2(\tilde{\Gamma}))}^{\kappa}},
$$

where

$$
\|\Xi\|_{C^{3+\alpha}(\Omega)} + \|\Xi^{-1}\|_{C^{3+\alpha}(\tilde{\Gamma})} \leq C_2.
$$

We set

$$
\tilde{\beta}_2 = F_*\beta_2, \quad \tilde{\beta}_1 = (\Xi \circ F_1 \circ F)_*\beta_1.
$$

By (202) and assumption that $\beta_1 = \beta_2$ on $\Gamma$ we have

$$
\|\tilde{\beta}_1 - \tilde{\beta}_2\|_{C(\Gamma)} \leq C_3 |\ln \|\Lambda_{\beta_1,g_1} - \Lambda_{\beta_2,g_2}\|_{L(W^1_2(\tilde{\Gamma});L^2(\tilde{\Gamma}))}|^{-\kappa}.
$$

Let $\Gamma_1 \subset \subset \Gamma$ and $v_j$ solve

$$
\nabla \cdot (\tilde{\beta}_j \nabla v_j) = 0 \quad \text{in} \quad \Omega \quad v_j|_{\partial \Omega} = f, \quad \text{supp} f \subset \tilde{\Gamma}, \quad j \in \{1, 2\}.
$$

Consider the Dirichlet-to-Neumann maps

$$
\Lambda_{\tilde{\beta}_j} f = \frac{\partial v_j}{\partial \nu}, \quad j \in \{1, 2\}.
$$

By (204) and (203), for constants $\kappa_4 > 0$ and $C_4 > 0$ we have

$$
\|\Lambda_{\tilde{\beta}_1} - \Lambda_{\tilde{\beta}_2}\|_{L(W^1_2(\tilde{\Gamma});L^2(\tilde{\Gamma}))} \leq \frac{C_4}{|\ln \|\Lambda_{\beta_1,g_1} - \Lambda_{\beta_2,g_2}\|_{L(W^1_2(\tilde{\Gamma});L^2(\tilde{\Gamma}))}|^{\kappa_4}}.
$$

With the help of following proposition, we estimate the difference between the normal derivatives of $\beta_j$.

**Proposition 9.** There exist positive constants $\kappa_1$ and $C_5(\Gamma_1)$, independent of $g_1, g_2, \beta_1, \beta_2$, such that

$$
\left\|\frac{\partial \tilde{\beta}_1}{\partial \nu} - \frac{\partial \tilde{\beta}_2}{\partial \nu}\right\|_{C(\Gamma_1)} \leq C_5 |\ln \|\Lambda_{\beta_1,g_1} - \Lambda_{\beta_2,g_2}\|_{L(W^1_2(\tilde{\Gamma});L^2(\tilde{\Gamma}))}|^{-\kappa_1}.
$$

**Proof.** Let $\Phi_0 \in C^2(\Omega)$ be a holomorphic function such that

$$
\text{Re} \Phi_0|_{\Gamma_2} = 0, \quad \text{Re} \Phi_0(x) < 0 \quad \text{on} \partial \Omega \setminus \Gamma_2.
$$

Here $\Gamma_2$ is some subboundary satisfying $\Gamma_1 \subset \subset \Gamma_2 \subset \subset \tilde{\Gamma}$. By Hopf’s lemma, we see

$$
\partial_{x_2} \text{Re} \Phi_0 > 0 \quad \text{on} \quad \Gamma_2.
$$

Therefore there exists a constant $\delta_0 > 0$, so that

$$
\partial_{x_2} \text{Re} \Phi_0 > 0 \quad \text{on} \quad G_{\delta_0} = \{x \in \Omega : \text{dist}(x, \Gamma_2) < \delta_0\}.
$$

Hence there exists a constant $\delta_1 > 0$ such that

$$
\text{Re} \Phi_0(x) \leq -\delta_1 \quad \forall x \in \Omega \setminus G_{\delta_0}.
$$

For arbitrary $\tilde{x} = (\tilde{x}_1, 1) \in \Gamma_1$, we set $f = \tilde{g}(\tilde{x})e^{-\tau_i(\text{Im} \Phi_0(x_1 - \tilde{x}_1)^2)}$ where $\tilde{g} \in C^\infty_c(\Gamma_2)$ satisfies $\tilde{g}|_{\Gamma_1} = 1$. 


Noting that $W^{1,2}_2(\Omega) \subset L^4(\Omega)$ by the Sobolev embedding, by the standard energy estimates (see e.g., Chapter 2 in [20]) for the boundary value problem (205), we have

\begin{equation}
\|v_j\|_{L^4(\Omega)} + \left\| \frac{\partial v_j}{\partial \nu} \right\|_{W^{-1,2}_2(\partial \Omega)} \leq C_6 \|f\|_{L^2(\Gamma_2)}.
\end{equation}

We set $r_j = e^{\tau \Phi_0} \sqrt{\beta_j}$. Taking the scalar product of (205) and the function $r_j/\sqrt{\beta_j}$ in $L^2(\Omega)$, we have

\[ \int_{\Omega} v_j (\Delta r_j - \text{div} (\nabla \ln \beta_j r_j)) dx + \int_{\partial \Omega} \left( \frac{\partial v_j}{\partial \nu} r_j - \frac{\partial r_j}{\partial \nu} v_j + \frac{\partial \ln \beta_j}{\partial \nu} v_j r_j \right) d\sigma = 0. \]

From the above equations, we easily obtain

\[ \sum_{j=0}^{1} (-1)^j \left\{ \int_{\partial \Omega} (1 - \rho) \left( \frac{\partial v_j}{\partial \nu} r_j - \frac{\partial r_j}{\partial \nu} v_j + \frac{\partial \ln \beta_j}{\partial \nu} v_j r_j \right) d\sigma \right\} = 0. \]

Simple computations and estimate (211) imply

\begin{equation}
\sum_{j=0}^{1} (-1)^j \left\{ \int_{\Omega} v_j (\Delta r_j - \text{div} (\nabla \ln \beta_j r_j)) dx \right\} \leq C_7 \|v_j\|_{L^4(\Omega)} \|e^{\tau \Phi_0}\|_{L^\frac{3}{2}(\Omega)}.
\end{equation}

Observe (209), (210) and

\begin{equation}
\|e^{\tau \Phi_0}\|_{L^\frac{3}{2}(\Omega)} \leq \|e^{\tau \Phi_0}\|_{L^\frac{3}{2}(\tilde{\Gamma} \setminus \Gamma_1)} + \|e^{\tau \Phi_0}\|_{L^\frac{3}{2}(\Omega \setminus \tilde{\Gamma})} \leq C_8 (e^{-\delta_1} + \tau^{-\frac{3}{4}}).
\end{equation}

From (212) and (213) we have

\begin{equation}
\sum_{j=0}^{1} (-1)^j \int_{\Omega} v_j (\Delta r_j - \text{div} (\nabla \ln \beta_j r_j)) dx \leq C_9 \tau^{-\frac{3}{4}}.
\end{equation}

Let $\rho \in C^2_0(\tilde{\Gamma})$ satisfy $\rho|_{\Gamma_1} = 1$. By (208) there exists some positive constant $\delta$ such that

\[ \left\| \sum_{j=0}^{1} (-1)^j \int_{\Omega} \left(1 - \rho \right) \left( \frac{\partial v_j}{\partial \nu} r_j - \frac{\partial r_j}{\partial \nu} v_j + \frac{\partial \ln \beta_j}{\partial \nu} v_j r_j \right) d\sigma \right\| \leq C_{10} e^{-\delta \tau}. \]

Obviously we have

\begin{equation}
\sum_{j=0}^{1} (-1)^j \int_{\partial \Omega} \rho \frac{\partial v_j}{\partial \nu} r_j d\sigma \leq C_{11} (\|\beta_1 - \beta_2\|_{C^1(\tilde{\Gamma})} + \tau \|\Lambda \beta_1 - \Lambda \beta_2\|_{L^2(W^{1,2}_p(\tilde{\Gamma}), L^p(\tilde{\Gamma}))}).
\end{equation}
Direct computations imply
\[
\sum_{j=0}^{1} (-1)^j \int_{\partial\Omega} \left( - \frac{\partial r_j}{\partial \nu} v_j + \frac{\partial \ln \beta_j}{\partial \nu} v_j r_j \right) \, d\sigma
= \sum_{j=0}^{1} (-1)^j \int_{\partial\Omega} \left( - e^{\tau \phi_0} v_j \sqrt{\beta_j} + \frac{\partial \ln \beta_j}{\partial \nu} v_j r_j \right) \, d\sigma
\]

(216) \[+ \frac{1}{2} \int_{|x-x_i|} \rho \left( \frac{\partial \sqrt{\beta_1}}{\partial \nu} - \frac{\partial \sqrt{\beta_2}}{\partial \nu} \right) g(x)e^{-\tau i(x_1-x_1)} \, d\sigma =: K_1 + K_2.\]

Using (208) and (205), we have
\[
|K_1| \leq C_{12} \left( \|\beta_1 - \beta_2\|_{C^1(\bar{\Gamma})} + \tau \|\Lambda_{\beta_1} - \Lambda_{\beta_2}\|_{L^2(\bar{\Gamma})} \right).
\]

Using the stationary phase argument (see e.g., [30] p.334 and p.337), we obtain
\[
K_2 = \frac{\tilde{g}(x) \sqrt{\pi}}{\sqrt{\tau}} \left( \frac{\partial \sqrt{\beta_1}}{\partial \nu} - \frac{\partial \sqrt{\beta_2}}{\partial \nu} \right) (\bar{x}) + O(\frac{1}{\tau}) \quad \text{as } \tau \to +\infty.
\]

By (214)-(218) and (204), we see
\[
\left\| \frac{\partial \sqrt{\beta_1}}{\partial \nu} - \frac{\partial \sqrt{\beta_2}}{\partial \nu} \right\|_{C^0(\bar{\Gamma})} \leq C_{13} \left( \sqrt{7} \|\beta_1 - \beta_2\|_{C^1(\bar{\Gamma})} + \tau \|\Lambda_{\beta_1} - \Lambda_{\beta_2}\|_{L^2(\bar{\Gamma})} + \frac{1}{\tau^{\frac{2}{7}}} \right)
\]

(219) \[\leq C_{14} \left( \|\ln \|\Lambda_{\beta_1} - \Lambda_{\beta_2}\|_{L^2(\bar{\Gamma})}\|_{L^2(\bar{\Gamma})}) \|\beta_1 - \beta_2\|_{C^1(\bar{\Gamma})} + \frac{1}{\tau^{\frac{2}{7}}} \right).
\]

Setting \( \tau = \|\ln \|\Lambda_{\beta_1} - \Lambda_{\beta_2}\|_{L^2(\bar{\Gamma})}\|_{L^2(\bar{\Gamma})}) \|\beta_1 - \beta_2\|_{C^1(\bar{\Gamma})} \) in (219), we obtain (207). The proof of the proposition is complete. \( \square \)

Setting \( q_j = -\frac{\Delta \sqrt{\beta_j}}{\sqrt{\beta_j}} \) and \( w_j = \sqrt{\beta_j} v_j \), we have
\[
L_{q_j}(x,D)w_j = \Delta w_j + q_j w_j = 0 \quad \text{in } \Omega \quad w_j|_{\partial\Omega} = \sqrt{\beta_j} f.
\]

Then for the Dirichlet-to-Neumann map \( \Lambda(q) f = \frac{\partial w}{\partial \nu} \) where \( L_{q_j}(x,D)w_j = 0 \) on \( \Omega \) and \( w_j|_{\partial\Omega} = f \) with supp \( f \subseteq \bar{\Gamma} \), we observe that \( \frac{\partial w_j}{\partial \nu} = \sqrt{\beta_j} \Lambda_{\beta_j} f + \frac{\partial \sqrt{\beta_j}}{\partial \nu} f \). This observation combined with (207) and (206) implies the existence of a positive constant \( \kappa_3 \) such that
\[
\|\Lambda(q_1) - \Lambda(q_2)\|_{L^2(W^{\frac{1}{2}}(\partial\Omega);W^{\frac{1}{2}}(\partial\Omega))} \leq C_{15} \|\Lambda_{\beta_1} - \Lambda_{\beta_2}\|_{L^2(\bar{\Gamma})} \|\beta_1 - \beta_2\|_{C^1(\bar{\Gamma})} \kappa_3
\]

(220) \[\leq \|\ln \|\Lambda_{\beta_1} - \Lambda_{\beta_2}\|_{L^2(\bar{\Gamma})}\|_{L^2(\bar{\Gamma})}) \|\beta_1 - \beta_2\|_{C^1(\bar{\Gamma})} \kappa_3 \kappa_4.
\]
Applying the stability result of [8] to the Schrödinger operators with potentials
\[-\frac{\Delta \sqrt{\beta_1}}{\sqrt{\beta_1}} \text{ and } -\frac{\Delta \sqrt{\beta_2}}{\sqrt{\beta_2}}\] by (220) we obtain:
\[
\|q_1 - q_2\|_{L^2(\Omega)} \leq C_{17} G(\|\Lambda(q_1) - \Lambda(q_2)\|_{L^2(\partial\Omega); L^2(\partial\Omega)})
\]
\[
\leq C_{16} (1 + |\ln(\|\Lambda_{\beta_1, g_1} - \Lambda_{\beta_2, g_2}\|_{L^2(\partial\Omega); L^2(\partial\Omega)})|)^{-\frac{1}{4}}.
\]
We set \(w = \sqrt{\beta_1} w_1 - \sqrt{\beta_2} w_2, g = (q_1 - q_2) \sqrt{\beta_2} w_2\). By (204), (207) and (221) we have
\[
L_{q_1}(x, D)w = g \quad \text{in } \Omega,
\]
where
\[
\|g\|_{L^2(\Omega)} + \left\|(w, \frac{\partial w}{\partial \nu})\right\|_{C^2(\Gamma) \times C^2(\Gamma)} \leq C_{19} G(\|\Lambda_{\beta_1, g_1} - \Lambda_{\beta_2, g_2}\|_{L^2(\partial\Omega); L^2(\partial\Omega)}).
\]
Applying the estimate (1.39) established in [1] to problem (222), we can choose some \(\gamma \in (0, 1)\) independent of \(g_j\) such that
\[
\|w\|_{L^2(\Omega)} \leq \frac{C_{20}}{(\ln \left(\frac{\|w\|_{H^1(\Omega)} + \|g\|_{L^2(\Omega)} + \|w, \frac{\partial w}{\partial \nu}\|_{C^2(\Gamma) \times C^2(\Gamma)}}{\|g\|_{L^2(\Omega)} + \|w, \frac{\partial w}{\partial \nu}\|_{C^2(\Gamma) \times C^2(\Gamma)}}\right))^\gamma}.
\]
\[
\leq \frac{C_{21}}{(\ln \left(\frac{\|w\|_{H^1(\Omega)} + G(\|\Lambda_{\beta_1, g_1} - \Lambda_{\beta_2, g_2}\|_{L^2(\partial\Omega); L^2(\partial\Omega)})\|w, \frac{\partial w}{\partial \nu}\|_{C^2(\Gamma) \times C^2(\Gamma)}}{G(\|\Lambda_{\beta_1, g_1} - \Lambda_{\beta_2, g_2}\|_{L^2(\partial\Omega); L^2(\partial\Omega)})\|w, \frac{\partial w}{\partial \nu}\|_{C^2(\Gamma) \times C^2(\Gamma)}}\right))}^\gamma.
\]
The proof of Theorem 1.2 is complete.

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