A lower bound on the size of an absorbing set in an arc-coloured tournament

L. Beaudou\textsuperscript{a,}\textsuperscript{*}, L. Devroye\textsuperscript{b}, G. Hahn\textsuperscript{c}

\textsuperscript{a}Université Clermont Auvergne, Clermont-Ferrand, France  
\textsuperscript{b}McGill University, Montréal, Québec, Canada  
\textsuperscript{c}Université de Montréal, Montréal, Québec, Canada

Abstract

Bousquet, Lochet and Thomassé recently gave an elegant proof that for any integer \( n \), there is a least integer \( f(n) \) such that any tournament whose arcs are coloured with \( n \) colours contains a subset of vertices \( S \) of size \( f(n) \) with the property that any vertex not in \( S \) admits a monochromatic path to some vertex of \( S \). In this note we provide a lower bound on the value \( f(n) \).

A directed graph or digraph \( D \) is a pair \((V,A)\) where \( V \) is a set called the \textit{vertex set} of \( D \) and \( A \) is a subset of \( V^2 \) called the \textit{arc set} of \( D \). When \((u,v)\) is in \( A \), we say there is an arc from \( u \) to \( v \). A \textit{tournament} is a digraph for which there is exactly one arc between each pair of vertices (in only one direction).

Given a colouring of the arcs of a digraph, we say that a vertex \( x \) is \textit{absorbed} by a vertex \( y \) if there is a monochromatic path from \( x \) to \( y \). A subset \( S \) of \( V \) is called an \textit{absorbing set} if any vertex in \( V \setminus S \) is absorbed by some vertex in \( S \).

In the early eighties, Sands, Sauer and Woodrow \cite{SandsSauerWoodrow1984} suggested the following problem (also attributed to Erdős in the same paper):

\textbf{Problem} (Sands, Sauer and Woodrow \cite{SandsSauerWoodrow1984}). For each \( n \), is there a (least) positive integer \( f(n) \) so that every finite tournament whose arcs are coloured with \( n \) colours contains an absorbing set \( S \) of size \( f(n) \)?

This problem has been investigated in weaker settings by forbidding some structures (see \cite{HahnIlleWoodrow2007,DevroyeGyarfas2010}), or by making stronger claims on the nature of the tournament (see \cite{PalvolgyiGyarfas2015}). Hahn, Ille and Woodrow \cite{HahnIlleWoodrow2007} also approached the infinite case.

Recently, Pálvölgyi and Gyárfás \cite{PalvolgyiGyarfas2015} have shown that a positive answer to this problem would imply a new proof of a former result from Bárány and Lehel \cite{BaranyLehel1995} stating that any set \( X \) of points in \( \mathbb{R}^d \) can be covered by \( f(d) \) \( X \)-boxes (each

\textsuperscript{*}Corresponding author

\textit{Email addresses}: laurent.beaudou@uca.fr (L. Beaudou), lucdevroye@gmail.com (L. Devroye), hahn@iro.umontreal.ca (G. Hahn)

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box is defined by two points in $X$). In 2017, Bousquet Lochet and Thomassé [2] gave a positive answer to the problem of Sands, Sauer and Woodrow.

**Theorem 1** (Bousquet, Lochet and Thomassé [2]). Function $f$ is well defined and $f(n) = O(ln(n) \cdot n^{n+2})$.

In this note, we provide a lower bound on the value of $f(n)$.

**Theorem 2.** For any integer $n$, let

$$p = \left( \frac{n - 1}{\sqrt{2}} \right).$$

There is a tournament arc-coloured with $n$ colours, such that no set of size less than $p$ absorbs the rest of the tournament, and thus $f(n) \geq p$.

**Proof.** Let $n$ be an integer and let $\mathcal{P}$ be the family of all subsets of $\{1, \ldots, n-1\}$ of size exactly $\lfloor \frac{n-1}{2} \rfloor$. Note that $\mathcal{P}$ has size $p$.

For any integer $m$, let $V(m)$ be the set $\{1, \ldots, m\} \times \mathcal{P}$ and let $\mathcal{T}(m)$ be the probability space consisting of arc-coloured tournaments on $V(m)$ where the orientation of each arc is fairly determined (probability $1/2$ for each orientation), and the colour of an arc from $(i, P)$ to $(j, P')$ is $n$ if $P = P'$, and picked randomly in $P' \setminus P$ otherwise. Note that if $P$ and $P'$ are distinct, then $P' \setminus P$ can not be empty since both sets have same size.

For any set $P$ in $\mathcal{P}$, we shall denote by $B(P)$ the set of vertices having $P$ as second coordinate. We may call this set, the bag of $P$. A key observation is that the arcs coming in a bag, the arcs leaving a bag and the arcs contained in a bag share no common colour. As a consequence, a monochromatic path is either contained in a bag or of length 1.

Let us find an upper bound on the probability that such a tournament is absorbed by a set of size strictly less than $p$. Let $S$ be a subset of $V(m)$ of size $p - 1$. There must exist a set $P$ in $\mathcal{P}$ such that $S$ does not hit $B(P)$. For $S$ to be absorbing, each vertex in $B(P)$ must have an outgoing arc to some vertex of $S$. Let $x$ be a vertex in $B(P)$,

$$Pr(x \text{ is absorbed by } S) = 1 - \left( \frac{1}{2} \right)^{p-1}.$$  

The events of being absorbed by $S$ are pairwise independent for elements of $B(P)$. Then,

$$Pr(B(P) \text{ is absorbed by } S) = \left( 1 - \left( \frac{1}{2} \right)^{p-1} \right)^m.$$  

This is an upper bound on the probability for $S$ to absorb the whole tournament. We may sum this for every possible choice of $S$.

$$Pr(\text{some } S \text{ is absorbing}) \leq \binom{mp}{p - 1} \left( 1 - \left( \frac{1}{2} \right)^{p-1} \right)^m.$$
Finally, by using the classic inequalities \( \binom{n}{k} \leq \left( \frac{e}{n} \right)^k \) and \((1 + x)^n \leq e^{nx}\), we obtain

\[
Pr(\text{some } S \text{ is absorbing}) \leq \left( \frac{emp}{p-1} \right)^{p-1} e^{-m(\frac{1}{2})^{p-1}}.
\]

When \( m \) tends to infinity, this last quantity tends to 0. So that, for \( m \) large enough, there is a tournament which is not absorbed by \( p - 1 \) vertices. \( \square \)

**Remark.** By Stirling’s approximation we derive that the bound obtained in Theorem 2 is of order \( \frac{2^n}{\sqrt{n}} \).

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