Regular biorthogonal pairs and Pseudo-bosonic operators

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Abstract

The first purpose of this paper is to show a method of constructing a regular biorthogonal pair based on the commutation rule: $ab - ba = I$ for a pair of operators $a$ and $b$ acting on a Hilbert space $\mathcal{H}$ with inner product $(\cdot | \cdot)$. Here, sequences $\{\phi_n\}$ and $\{\psi_n\}$ in a Hilbert space $\mathcal{H}$ are biorthogonal if $(\phi_n | \psi_m) = \delta_{nm}$, $n, m = 0, 1, \cdots$, and they are regular if both $D_\phi \equiv \text{Span}\{\phi_n\}$ and $D_\psi \equiv \text{Span}\{\psi_n\}$ are dense in $\mathcal{H}$. Indeed, the assumption to construct the regular biorthogonal pair coincide with the definition of pseudo-bosons as originally given in Ref [8]. Moreover, we study the connections between the pseudo-bosonic operators $a$, $b$, $a^\dagger$, $b^\dagger$ and the pseudo-bosonic operators defined by a regular biorthogonal pair $\{\phi_n\}$, $\{\psi_n\}$ and an ONB $e$ of $\mathcal{H}$ in appeared Ref [2]. The second purpose is to define and study the notion of $\mathcal{D}$-pseudo bosons in Ref [6, 5] and give a method of constructing $\mathcal{D}$-pseudo bosons on some steps. Then it is shown that for any ONB $e = \{e_n\}$ in $\mathcal{H}$ and any operators $T$ and $T^{-1}$ in $\mathcal{L}(\mathcal{D})$, we may construct operators $A$ and $B$ satisfying $\mathcal{D}$-pseudo bosons, where $\mathcal{D}$ is a dense subspace in a Hilbert space $\mathcal{H}$ and $\mathcal{L}(\mathcal{D})$ the set of all linear operators $T$ from $\mathcal{D}$ to $\mathcal{D}$ such that $T^* \mathcal{D} \subset \mathcal{D}$, where $T^*$ is the adjoint of $T$. Finally, we give some physical examples of $\mathcal{D}$-pseudo bosons based on standard bosons by the method of constructing $\mathcal{D}$-pseudo bosons stated above.

1 Introduction

In this paper, we shall show a method of constructing a regular biorthogonal pair based on the following commutation rule under some assumptions. Here, the commutation rule is that a pair of operators $a$ and $b$ acting on
a Hilbert space $H$ with inner product $(\cdot|\cdot)$ satisfies $ab - ba = I$. Indeed, the assumptions to construct the regular biorthogonal pair coincide with the definition of pseudo-bosons as originally given in Ref [8], where in the recent literature many researchers have investigated in Ref [6, 8, 7, 9, 11]. Furthermore, in Ref [2] the author have studied the general theory of operators $A_e, B_e, A_e^\dagger, B_e^\dagger, N_e, N_e^\dagger$, defined by a regular biorthogonal pair $(\{\phi_n\}, \{\psi_n\})$ and an ONB $e$ in $H$. Though we will describe the formulas of these operators $A_e, B_e, A_e^\dagger, B_e^\dagger, N_e, N_e^\dagger$ in detail in Section 2, the operators connect with quasi-hermitian quantum mechanics and its relatives. Many researchers have investigated such operators mathematically in Ref [1, 4, 2]. Therefore, we study the connections between the operators $a, b, a^\dagger, b^\dagger$ and the operators $A_e, B_e, A_e^\dagger, B_e^\dagger$. By Proposition 2.1, (3) and Proposition 2.2, (3), we show that $\{a,b\}$ and $\{a^\dagger,b^\dagger\}$ have an algebraic structure, respectively, that is, algebras generated by $\{1,a\left[D_e,b\right]\left[D_e\right], b\left[D_e\right]\}$ and by $\{1,a^\dagger\left[D_e,b\right]\left[D_e\right], b^\dagger\left[D_e\right]\}$ are defined, respectively. However, $\{1,a,b,a^\dagger,b^\dagger\}$ do not have a $\ast$-algebraic structure, in general. From this reason, the second purpose is to define and study the notion of $D$-pseudo bosons. Here, $D$ is a dense subspace in a Hilbert space $H$. We denote by $L^\dagger(D)$ the set of all linear operators $T$ from $D$ to $D$ such that $T^*D \subset D$, where $T^*$ is the adjoint of $T$. $L^\dagger(D)$ is a $\ast$-algebra under the usual operators $aT, S + T, ST$ and an involution $T^\dagger \equiv T^*\left[D\right]$, and it is called a maximal $O^\ast$-algebra on $D$. A pair of operators $a$ and $b$ is said $D$-pseudo bosons if $a, b \in L^\dagger(D)$ satisfy following conditions (i), (ii) and (iii):

(i) $ab - ba = I$.

(ii) There exists a non-zero element $\phi_0 \in D$ such that $a\phi_0 = 0$.

(iii) There exists a non-zero element $\psi_0 \in D$ such that $b^\dagger\psi_0 = 0$.

Furthermore, we show a method of constructing $D$-pseudo bosons on some steps. Then it is shown that for any ONB $e = \{e_n\}$ in $H$ and any operators $T$ and $T^{-1}$ in $L^\dagger(D)$, we may construct operators $A$ and $B$ satisfying $D$-pseudo bosons. Finally, we give some physical examples of $D$-pseudo bosons based on a method of constructing $D$-pseudo bosons for some steps.

This article is organized as follows. In Section 2, we introduce a method of constructing a regular biorthogonal pair based on the commutation rule under some assumption. And we investigate the connections between the operators $a, b, a^\dagger, b^\dagger$ and the operators $A_e, B_e, A_e^\dagger, B_e^\dagger$. In Section 3, we define and study the notion of $D$-pseudo bosons and a method of constructing $D$-pseudo bosons for some steps. In Section 4, we give some physical examples of $D$-pseudo bosons based on the method of constructing $D$-pseudo bosons in Section 3.
2 Regular biorthogonal pairs and Pseudo-bosons

Let $\mathcal{H}$ be a Hilbert space with inner product $(\cdot|\cdot)$. We introduce a pair of operators $a$ and $b$ acting on $\mathcal{H}$ satisfying the following commutation rules

$$ab - ba = I.$$ 

In particular, this collapses to the canonical commutation rule (CCR) if $b = a^\dagger$. Furthermore, we introduce the notions of biorthogonal sequences and the regularity as follows: Sequences $\{\phi_n\}$ and $\{\psi_n\}$ in a Hilbert space $\mathcal{H}$ are biorthogonal if $(\phi_n|\psi_m) = \delta_{nm}$, $n, m = 0, 1, \cdots$, and they are regular if both $\text{Span}\{\phi_n\}$ and $\text{Span}\{\psi_n\}$ are dense in $\mathcal{H}$. Then $(\{\phi_n\}, \{\psi_n\})$ is said to be a regular biorthogonal pair. We construct biorthogonal pairs based on the above commutation rule under some assumptions. At first, we assume the following statement.

**Assumption 1.** There exists a non-zero element $\phi_0$ of $\mathcal{H}$ such that

1. $a\phi_0 = 0$,
2. $\phi_0 \in D^\infty(b) \equiv \cap_{k=0}^\infty D(b^k)$,
3. $b^n\phi_0 \in D(a)$, $n = 0, 1, \cdots$.

Then, we may define a sequence $\{\phi_n\}$ in $\mathcal{H}$ by

$$\phi_n \equiv \frac{1}{\sqrt{n!}} b^n \phi_0, \quad n = 0, 1, \cdots$$

$$= \frac{1}{\sqrt{n}} b^n \phi_{n-1}, \quad n = 1, 2, \cdots.$$ 

If we only define the above sequence $\{\phi_n\}$ in $\mathcal{H}$, we do not need Assumption 1, (iii). However, we need Assumption 1, (iii) to define the operator $N \equiv ba$. Then we have the following

**Proposition 2.1.** The following statements hold.

1. $b^n\phi_0 \in D(a^m)$ and

$$a^m b^n \phi_0 = \begin{cases} n P_m b^{n-m} \phi_0, & m \leq n, \\ 0, & m > n. \end{cases}$$

2. $\phi_n \in D(N^m)$ and $N^m \phi_n = n^m \phi_n$, $n, m = 0, 1, \cdots$. In particular, $N\phi_n = n\phi_n$, $n = 0, 1, \cdots$. 

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Proof. (1) We prove this statement based on mathematical induction. We prove this statement when \( m = 1 \):

\[
ab^n \phi_0 = \begin{cases} 
  nb^{n-1} \phi_0 & , n = 1, 2, \cdots, \\
  0 & , n = 0.
\end{cases}
\]  

Since \( a\phi_0 = 0 \) by Assumption 1, (i), (2.1) holds when \( n = 0 \). Let \( n = 1 \). Since \( \phi_0 \in D(ab) \cap D(ba) \) by Assumption 1, (i), (iii), we have

\[
ab\phi_0 = (ba + 1)\phi_0 = \phi_0.
\]  

Hence (2.1) holds when \( n = 1 \). Assume that (2.1) holds when \( n = k \geq 2 \), that is,

\[
b^k \phi_0 \in D(a) \quad \text{and} \quad ab^k \phi_0 = kb^{k-1} \phi_0.
\]  

Let \( n = k + 1 \). Then we have \( b^k \phi_0 \in D(ab) \cap D(ba) \) by Assumption 1, (iii) and (2.3), which implies by making use of the commutation relation \( ab - ba = I \) that

\[
ab^{k+1} \phi_0 = ab(b^k \phi_0) = (ba + 1)b^k \phi_0 = bab^k \phi_0 + b^k \phi_0 = (k + 1)b^k \phi_0.
\]

Thus the statement (2.1) holds. Assume that when \( m = k \) the statement (2.1) holds: \( b^n \phi_0 \in D(a^k) \) and

\[
a^k b^n \phi_0 = \begin{cases} 
  nP_kb^{n-k} \phi_0 & , k \leq n, \\
  0 & , k > n.
\end{cases}
\]  

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We show the statement (2.1) when \( m = k + 1 \). Let \( n \geq k + 1 \). By (2.4), we have \( b^n \phi_0 \in D(a^{k+1}) \) and furthermore, by (2.4) and (2.1)

\[
a^{k+1} b^n \phi_0 = a(a^k b^n \phi_0) = a(n P_k b^{n-k} \phi_0) = n P_k (n-k) b^{n-k-1} \phi_0 = n P_{k+1} b^{n-(k+1)} \phi_0.
\]

Let \( n = k \). By (2.4) we have \( b^k \phi_0 \in D(a^{k+1}) \) and

\[
a^{k+1} b^k \phi_0 = a(a^k b^k \phi_0) = k! a \phi_0 = 0.
\]

Let \( n < k \). By (2.4) we have

\[
a^{k+1} b^n \phi_0 = a(a^k b^n \phi_0) = 0.
\]

Thus the statement (2.1) holds when \( m = k + 1 \). This completes the proof of (1).

(2) We prove this statement in case of \( m = 1 \). It follows from (1) that \( \phi_n \in D(N) \) and

\[
N \phi_n = b a \left( \frac{1}{\sqrt{n!}} b^n \phi_0 \right) = b \left( \frac{1}{\sqrt{n!}} n b^{n-1} \phi_0 \right) = n \phi_n.
\]

Thus by the above statement and (1), it is easily shown that \( \phi_n \in D(N^m) \) and \( N^m \phi_n = n^m \phi_n, n, m = 0, 1, \ldots \).

(3) It follows from (1) and (2). This completes the proof.

The statement of Proposition 2.1, (2) means that \( N \) is the number operator for \( \{ \phi_n \} \). Next we assume the following statement.
Assumption 2. There exists a non-zero element $\psi_0$ of $\mathcal{H}$ such that

(i) $b^\dagger \psi_0 = 0$,
(ii) $\psi_0 \in D^\infty(a^\dagger) \equiv \bigcap_{k=0}^\infty D((a^\dagger)^k)$,
(iii) $(a^\dagger)^n \psi_0 \in D(b^\dagger)$, $n = 0, 1, \ldots$.

Then, we may define a sequence $\{\psi_n\}$ in $\mathcal{H}$ by

$$\psi_n \equiv \frac{1}{\sqrt{n!}} (a^\dagger)^n \psi_0, \quad n = 0, 1, \ldots$$

$$= \frac{1}{\sqrt{n}} a^\dagger \psi_{n-1}, \quad n = 1, 2, \ldots.$$ 

And we put an operator $N^\dagger \equiv a^\dagger b^\dagger$. Then we have the following

**Proposition 2.2.** The following statements hold.

(1) $(a^\dagger)^n \psi_0 \in D((b^\dagger)^m)$ and

$$(b^\dagger)^m (a^\dagger)^n \psi_0 = \begin{cases} n P_m (a^\dagger)^{n-m} \psi_0 & , m \leq n, \\ 0 & , m > n. \end{cases}$$

(2) $\psi_n \in D((N^\dagger)^m)$ and $(N^\dagger)^m \psi_n = n^m \psi_n$, $n, m = 0, 1, \ldots$. In particular, $N^\dagger \psi_n = n \psi_n, n = 0, 1, \ldots$.

(3) $a^\dagger \psi_n = \sqrt{n+1} \psi_{n+1}$, $n = 0, 1, \ldots$,

$\quad b^\dagger \psi_n = \begin{cases} 0 & , n = 0, \\ \sqrt{n} \psi_{n-1} & , n = 1, 2, \ldots. \end{cases}$

**Proof.** This is shown similarly to Proposition 2.1.

Here, we consider about the sequences $\{\phi_n\}$ and $\{\psi_n\}$. It is easily shown that $(\phi_n | \psi_m) = (\phi_0 | \psi_0) \delta_{nm}$. From now on, we may assume that $(\phi_0 | \psi_0) = 1$ without loss of generality. Then $\{\phi_n\}$ and $\{\psi_n\}$ are biorthogonal. Furthermore, we assume the following statements with respect to $\{\phi_n\}$ and $\{\psi_n\}$. 


Assumption 3.
(i) $\text{Span}\{b^n \phi_0\}$ is dense in $\mathcal{H}$.
(ii) $\text{Span}\{(a^\dagger)^n \psi_0\}$ is dense in $\mathcal{H}$.

Then $D_\phi \equiv \text{Span}\{\phi_n\} = \text{Span}\{b^n \phi_0\}$ and $D_\psi \equiv \text{Span}\{\psi_n\} = \text{Span}\{(a^\dagger)^n \psi_0\}$ are dense in $\mathcal{H}$. If a pair of operator $a$ and $b$ acting on $\mathcal{H}$ satisfy Assumption 1-3, $(\{\phi_n\}, \{\psi_n\})$ becomes a regular biorthogonal pair. The assumption to construct the regular biorthogonal pair coincide with the definition of pseudo-bosons as originally given in Ref [8].

In Ref [2], the author have studied the general theory of regular biorthogonal pairs. In particular, it has been shown that if $(\{\phi_n\}, \{\psi_n\})$ is a regular biorthogonal pair in a Hilbert space $\mathcal{H}$, then for any ONB $e = \{e_n\}$ in $\mathcal{H}$, there exists a densely defined closed operator $T$ in $\mathcal{H}$ with densely defined inverse such that $\{e_n\} \subset D(T) \cap D((T^{-1})^*)$, $\phi_n = Te_n$ and $\psi_n = (T^{-1})^* e_n$, $n = 0, 1, \cdots$ and the minimum in such operators $T$ exists and denoted by $T_e$, and furthermore there exists a unique ONB $f = \{f_n\}$ in $\mathcal{H}$ such that $T_f$ is a non-singular positive self-adjoint operator in $\mathcal{H}$.

Furthermore, the author have investigated the following operators defined by a regular biorthogonal pair $(\{\phi_n\}, \{\psi_n\})$ as follows: for any ONB $e = \{e_n\}$,

$$A_e = \sum_{k=0}^{\infty} \sqrt{k+1} \bar{e}_k \otimes e_{k+1} + 1 \cdot T_e^{-1}$$

$$B_e = \sum_{k=0}^{\infty} \sqrt{k} \bar{e}_{k+1} \otimes e_k + 1 \cdot T_e^{-1}$$

$$A_e^\dagger = (T_e^{-1})^* \left( \sum_{k=0}^{\infty} \sqrt{k} \bar{e}_{k+1} \otimes e_k + 1 \right) T_e^*$$

$$B_e^\dagger = (T_e^{-1})^* \left( \sum_{k=0}^{\infty} \sqrt{k} \bar{e}_k \otimes e_{k+1} + 1 \right) T_e^*$$

$$N_e = \sum_{k=0}^{\infty} \sqrt{k+1} \bar{e}_{k+1} \otimes e_{k+1} + 1 \cdot T_e^{-1}$$

$$N_e^\dagger = (T_e^{-1})^* \left( \sum_{k=0}^{\infty} \sqrt{k} \bar{e}_k \otimes e_{k+1} + 1 \right) T_e^*$$
where the tensor $x \otimes \bar{y}$ of elements $x, y$ of $\mathcal{H}$ is defined by

$$(x \otimes \bar{y})\xi = (\xi | y)x, \quad \xi \in \mathcal{H}.$$ 

Indeed, these operators defined by ONB do not depend on methods of taking ONB. Furthermore, $A_e$ and $B_e$ are lowering and raising operators for $\{\phi_n\}$, respectively, and $A_e^\dagger$ and $B_e^\dagger$ are raising and lowering operators for $\{\psi_n\}$, respectively, and $N_e$ and $N_e^\dagger$ are number operators for $\{\phi_n\}$ and $\{\psi_n\}$, respectively, and

$$A_e B_e - B_e A_e \subset I \quad \text{and} \quad B_e^\dagger A_e^\dagger - A_e^\dagger B_e^\dagger \subset I.$$ 

Moreover, in Ref [2] Proposition 3.4, the following statements hold with respect to the operators $A_e, B_e, A_e^\dagger$ and $B_e^\dagger$.

(i)

$$\phi_n = \frac{1}{\sqrt{n!}}B_e^n \phi_0, \quad n = 0, 1, \ldots,$$

$$\psi_n = \frac{1}{\sqrt{n!}}(A_e^\dagger)^n \psi_0, \quad n = 0, 1, \ldots.$$ 

(ii)

$$A_e D_\phi = D_\phi, \quad B_e D_\phi = D_\phi,$$

and

$$A_e^\dagger D_\psi = D_\psi, \quad B_e^\dagger D_\psi = D_\psi.$$ 

Here we apply the above results to the regular biorthogonal pairs $\{\phi_n\}, \{\psi_n\}$ obtained by a pseudo-bosonic operators $\{a, b, a^\dagger, b^\dagger\}$ satisfying Assumption 1-3, and may construct a new pseudo-bosonic operators $\{A_e, B_e, A_e^\dagger, B_e^\dagger\}$. Therefore, we investigate the connections between the operators $a, b, a^\dagger, b^\dagger$ and the operators $A_e, B_e, A_e^\dagger, B_e^\dagger$.

We put

$$a_\phi = a \left[ D_\phi \right] \quad \text{and} \quad b_\phi = b \left[ D_\phi \right],$$

$$(a^\dagger)_\psi = a^\dagger \left[ D_\psi \right] \quad \text{and} \quad (b^\dagger)_\psi = b^\dagger \left[ D_\psi \right].$$ 

For relationships between $a_\phi, b_\phi$ and $A_e, B_e$, we have the following...
Lemma 2.3. The following statements hold.
(1) $a_\phi = A_e|D_\phi$ and $b_\phi = B_e|D_\phi$.
(2) $\bar{a}_\phi \subset \bar{A}_e$ and $\bar{b}_\phi \subset \bar{B}_e$.
(3) If $T_e^{-1}$ is bounded, then $A_e$ and $B_e$ are closed, that is, $\bar{a}_\phi \subset A_e$ and $\bar{b}_\phi \subset B_e$.
(4) If $T_e$ is bounded, then $\bar{a}_\phi = \bar{A}_e$ and $\bar{b}_\phi = \bar{B}_e$.
(5) If $\{\{\phi_n\}, \{\psi_n\}\}$ is a pair of Riesz base, that is, $T_e$ and $T_e^{-1}$ are bounded, then $\bar{a}_\phi = A_e$ and $\bar{b}_\phi = B_e$.

Proof. (1) This follows from Proposition 2.1, (3).
(2) Take an arbitrary $x \in D(\bar{a}_\phi)$. Then there exists a sequence $\{x_n\}$ in $D_\phi$ such that $\lim_{n \to \infty} x_n = x$ and $a_\phi x_n = \lim_{n \to \infty} a_\phi x_n = \lim_{n \to \infty} A_e x_n$. Thus, we have $x \in D(\bar{A}_e)$ and $\bar{A}_e x = \bar{a}_\phi x$. Therefore, we have $\bar{a}_\phi \subset \bar{A}_e$.
(3) Since $A_e$ and $B_e$ are closed by Ref. [1], it follows from (1) that $\bar{a}_\phi \subset A_e$ and $\bar{b}_\phi \subset B_e$.
(4) Since $D_\phi$ is a core for $\bar{A}_e$ and $\bar{B}_e$ by Ref. [2], it follows from (1) that $\bar{A}_e = A_e|D_\phi = \bar{a}_\phi$ and $\bar{B}_e = B_e|D_\phi = \bar{b}_\phi$.
(5) This follows from (2) and (3).

For relationships between $a^\dagger_\psi$, $b^\dagger_\psi$ and $A^\dagger_\psi$, $B^\dagger_\psi$, we have following

Lemma 2.4. The following statements hold.
(1) $(a^\dagger)_\psi = A^\dagger_e|D_\psi$ and $(b^\dagger)_\psi = B^\dagger_e|D_\psi$.
(2) $\bar{a}^\dagger_\psi \subset \bar{A}^\dagger_e$ and $\bar{b}^\dagger_\psi \subset \bar{B}^\dagger_e$.
(3) If $T_e$ is bounded, then $A^\dagger_e$ and $B^\dagger_e$ are closed, that is, $\bar{a}^\dagger_\psi \subset A^\dagger_e$ and $\bar{b}^\dagger_\psi \subset B^\dagger_e$.
(4) If $T_e^{-1}$ is bounded, then $\bar{a}^\dagger_\psi = \bar{A}^\dagger_e$ and $\bar{b}^\dagger_\psi = \bar{B}^\dagger_e$.
(5) If $\{\{\phi_n\}, \{\psi_n\}\}$ is a pair of Riesz base, that is, $T_e$ and $T_e^{-1}$ are bounded, then $\bar{a}^\dagger_\psi = A^\dagger_e$ and $\bar{b}^\dagger_\psi = B^\dagger_e$.

Proof. The statements are proved similarly to Lemma 2.3.

Proposition 2.5. The following statements hold.
(1) Suppose $D_\phi$ is a core for $\bar{a}$ and $\bar{b}$, then $\bar{a} \subset \bar{A}_e$ and $\bar{b} \subset \bar{B}_e$. In particular, if $T_e^{-1}$ is bounded, then $\bar{a} \subset A_e$ and $\bar{b} \subset B_e$, and if $T_e$ is bounded, then $\bar{a} = \bar{A}_e$ and $\bar{b} = \bar{B}_e$.
(2) Suppose $D_\psi$ is a core for $\bar{a}^\dagger$ and $\bar{b}^\dagger$, then $\bar{a}^\dagger \subset \bar{A}^\dagger_e$ and $\bar{b}^\dagger \subset \bar{B}^\dagger_e$. In par-
ticular, if $T_e$ is bounded, then $\bar{a}^\dagger \subset A_e^\dagger$ and $\bar{b}^\dagger \subset B_e^\dagger$, and if $T_e^{-1}$ is bounded, then $\bar{a}^\dagger = A_e^\dagger$ and $\bar{b}^\dagger = B_e^\dagger$.

3 A construction of $\mathcal{D}$-pseudo bosons

In Section 2, we investigate the relationships between regular biorthogonal pairs and pseudo-bosons, and the connections between the pseudo-bosonic operators $a, b, a^\dagger, b^\dagger$ and the pseudo-bosonic operators $A_e, B_e, A_e^\dagger, B_e^\dagger$. By Proposition 2.1, (3) and Proposition 2.2, (3), $\{a, b\}$ and $\{a^\dagger, b^\dagger\}$ have an algebraic structure, respectively, that is, algebras generated by $\{1, a \lceil D_\phi, b \rceil D_\phi\}$ and by $\{1, a^\dagger \lceil D_\psi, b^\dagger \lceil D_\psi\}$ are defined, respectively. However, $\{1, a, b, a^\dagger, b^\dagger\}$ do not have a $*$-algebraic structure, in general. From this reason, we define the notion of $\mathcal{D}$-pseudo bosons in Ref [6, 5] Let $\mathcal{D}$ be a dense subspace in a Hilbert space $\mathcal{H}$. We denote by $\mathcal{L}(\mathcal{D})$ the set of all linear operators $T$ from $\mathcal{D}$ to $\mathcal{D}$ such that $T^* \mathcal{D} \subset \mathcal{D}$. Here $T^*$ is the adjoint of $T$. $\mathcal{L}(\mathcal{D})$ is a $*$-algebra under the usual operators $\alpha T, S + T, ST$ and an involution $T^\dagger = T^* \lceil \mathcal{D}$, and a $*$-subalgebra of $\mathcal{L}(\mathcal{D})$ is called an $O$-$*$-algebra on $\mathcal{D}$ in Ref [3].

**Definition 3.1.** A pair of operator $a$ and $b$ is $\mathcal{D}$-pseudo bosons if $a, b \in \mathcal{L}(\mathcal{D})$ satisfy the following conditions (i), (ii) and (iii):

(i) $ab - ba = I$.

(ii) There exists a non-zero element $\phi_0 \in \mathcal{D}$ such that $a\phi_0 = 0$.

(iii) There exists a non-zero element $\psi_0 \in \mathcal{D}$ such that $b^\dagger \psi_0 = 0$.

Let $\mathcal{H}$ be any separable Hilbert space. We construct $\mathcal{D}$-pseudo bosons on the following steps:

**Step 1.** Take an arbitrary ONB $e = \{e_n\}$ in $\mathcal{H}$. We put

$$\mathcal{D} \equiv \left\{ x \in \mathcal{H}: \sum_{k=0}^{\infty} (k+1)^{n}|(x|e_n)|^2 < \infty, \ n = 0, 1, \cdots \right\}.$$ 

Then $\mathcal{D}$ is a dense subspace in $\mathcal{H}$ such that

$$\sum_{k=0}^{\infty} \sqrt{k+1}(e_k \otimes \bar{e}_{k+1})\mathcal{D} \subset \mathcal{D}$$
and

\[ \sum_{k=0}^{\infty} \sqrt{k+1}(e_{k+1} \otimes \bar{e}_k) \mathcal{D} \subset \mathcal{D}. \]

**Step 2.** Take an arbitrary operator \( T \) and the inverse \( T^{-1} \) in \( \mathcal{L}^\dagger(\mathcal{D}) \).

**Step 3.** We define operators \( A \) and \( B \) on \( \mathcal{D} \) by

\[ A = T \left( \sum_{k=0}^{\infty} \sqrt{k+1} e_k \otimes \bar{e}_{k+1} \right) T^{-1}, \]
\[ B = T \left( \sum_{k=0}^{\infty} \sqrt{k+1} e_{k+1} \otimes \bar{e}_k \right) T^{-1}. \]

Then,

\[ A^\dagger = (T^{-1})^\dagger \left( \sum_{k=0}^{\infty} \sqrt{k+1} e_{k+1} \otimes \bar{e}_k \right) T^\dagger, \]
\[ B^\dagger = (T^{-1})^\dagger \left( \sum_{k=0}^{\infty} \sqrt{k+1} e_k \otimes \bar{e}_{k+1} \right) T^\dagger, \]

and

\[ AB - BA = I \quad \text{on } \mathcal{D}. \]

Furthermore, we put

\[ \phi_n = T e_n, \quad \psi_n = (T^{-1})^\dagger e_n, \quad n = 0, 1, \ldots. \]

Then \( \{ \phi_n \}, \{ \psi_n \} \) is a regular biorthogonal pair, and \( A \) and \( B \) are lowering and raising operators for \( \{ \phi_n \} \), respectively, and \( A^\dagger \) and \( B^\dagger \) are raising and lowering operators for \( \{ \psi_n \} \), respectively. Thus we have the following

**Theorem 3.2.** For any ONB \( e = \{ e_n \} \) in \( \mathcal{H} \), we may construct operators \( A \) and \( B \) satisfying \( \mathcal{D} \)-pseudo bosons.

By a method of taking an ONB \( e = \{ e_n \} \) and a operator \( T \) satisfying Step 2, we can construct many different \( \mathcal{D} \)-pseudo bosons.
4 Examples of $\mathcal{D}$-pseudo bosons

In this section, we give some physical examples of $\mathcal{D}$-pseudo bosons constructed by Step 1-3 in Section 3. We may construct various models by a method of taking the ONB $f = \{f_n\}$ in the Hilbert space $L^2(\mathbb{R})$ consisting of

$$f_n(x) = \frac{1}{\sqrt{2^n n!}} H_n(x) e^{-\frac{1}{2}x^2},$$

where $H_n(x)$ is the $n$th Hermite polynomial, and an operator $T$ in $L^2(\mathbb{R})$ satisfying Step 2.

Let $S(\mathbb{R})$ be the Schwartz space of all infinitely differentiable rapidly decreasing functions on $\mathbb{R}$. The ONB $f = \{f_n\}$ is contained in $S(\mathbb{R})$. We define the momentum operator $p$ and the position operator $q$ by

$$D(p) : \text{the set of all differentiable functions } f \text{ on } \mathbb{R} \text{ such that } \frac{df}{dx} \in L^2(\mathbb{R}),$$

$$(pf)(x) = -i \frac{df}{dx}, \quad f \in D(p),$$

$$D(q) = \left\{ f \in L^2(\mathbb{R}); \int_{-\infty}^{\infty} |xf(x)|^2 dx < \infty \right\},$$

$$(qf)(x) = xf(x), \quad f \in D(q).$$

These operators $p$ and $q$ are self-adjoint operators in $L^2(\mathbb{R})$ and $S(\mathbb{R})$ is a core for $p$ and $q$, and furthermore $pS(\mathbb{R}) \subset S(\mathbb{R})$ and $qS(\mathbb{R}) \subset S(\mathbb{R})$. We introduce the standard bosonic operators $a = \frac{1}{\sqrt{2}} (q + ip)$ and $a^\dagger = \frac{1}{\sqrt{2}} (q - ip)$, which obey $[a, a^\dagger] = I$. Here we consider some examples

**Example 1.** (The extended quantum harmonic oscillator)

Let $\beta > 0$. We put

$$T = e^{-\frac{1}{\beta^2} e^{-\frac{x^2}{\beta^2}}} e^{-\frac{1}{\beta^2} \frac{x^2}{\beta^2}} q.$$

Then $T$ and $T^{-1}$ are self-adjoint operators in $L^2(\mathbb{R})$ such that $TS(\mathbb{R}) \subset S(\mathbb{R})$ and $T^{-1}S(\mathbb{R}) \subset S(\mathbb{R})$. We denote the restriction $T$ to $S(\mathbb{R})$ by the same $T$. Thus the ONB $f = \{f_n\}$ in $L^2(\mathbb{R})$ and $T$ satisfy the conditions of Step
1-3 in Section 3 for \( \mathcal{D} \equiv S(\mathbb{R}) \). Hence we may construct \( \mathcal{D} \)-pseudo bosonic operators

\[
A = T \left( \sum_{k=0}^{\infty} \sqrt{k+1} f_k \otimes \bar{f}_{k+1} \right) T^{-1},
\]

\[
B = T \left( \sum_{k=0}^{\infty} \sqrt{k+1} f_{k+1} \otimes \bar{f}_k \right) T^{-1}.
\]

Then it is shown that \( A = a - \frac{1}{\beta} \) and \( B = a^\dagger + \frac{1}{\beta} \), and the non self-adjoint hamiltonian \( H_\beta = \frac{\beta}{2}(p^2 + q^2) + \sqrt{2} ip \), introduced in Ref [5] and Ref [6], can be written by

\[
H_\beta = \beta \left( BA + \frac{2 + \beta^2}{2\beta^2} I \right).
\]

**Example 2.** (The Swanson model)

Let \( \theta \) be a real parameter value in \((-\frac{\pi}{4}, \frac{\pi}{4}) \setminus \{0\}\). We put

\[
T = e^{i\theta (a^2 - a^\dagger 2)} = e^{-\frac{i}{2}(qp + pq)}.
\]

Then \( T \) and \( T^{-1} \) are self-adjoint operators in \( L^2(\mathbb{R}) \) such that \( TS(\mathbb{R}) \subset S(\mathbb{R}) \) and \( T^{-1}S(\mathbb{R}) \subset S(\mathbb{R}) \). We denote the restriction \( T \) to \( S(\mathbb{R}) \) by the same \( T \). Thus the ONB \( \mathbf{f} = \{f_n\} \) in \( L^2(\mathbb{R}) \) and \( T \) satisfy the conditions of Step 1-3 in Section 3 for \( \mathcal{D} \equiv S(\mathbb{R}) \). Hence we may construct \( \mathcal{D} \)-pseudo bosonic operators

\[
A = T \left( \sum_{k=0}^{\infty} \sqrt{k+1} f_k \otimes \bar{f}_{k+1} \right) T^{-1},
\]

\[
B = T \left( \sum_{k=0}^{\infty} \sqrt{k+1} f_{k+1} \otimes \bar{f}_k \right) T^{-1}.
\]

Then it is shown that \( A = \cos(\theta) \ a + i \sin(\theta) \ a^\dagger \) and \( B = \cos(\theta) \ a^\dagger + i \sin(\theta) \ a \), and the non self-adjoint hamiltonian \( H_\theta = \frac{1}{2}(p^2 + q^2) - \frac{i}{2} \tan(2\theta)(p^2 - q^2) \), introduced in Ref [6] and Ref [10], can be written by

\[
H_\theta = \frac{1}{\cos(2\theta)} \left( BA + \frac{1}{2} I \right).
\]
In next Example 3 and 4 we shall take the operators $T$ that are different to those in Example 1 and 2, respectively to construct various models.

**Example 3.** Let $\beta > 0$. We put

$$T = e^{\frac{1}{\beta^2} e^{i\beta a - a^\dagger}} = e^{-\frac{1}{\beta^2} \beta^2 \frac{\sqrt{2}}{\beta} p}.$$ 

Then $T$ and $T^{-1}$ are self-adjoint operators in $L^2(\mathbb{R})$ such that $TS(\mathbb{R}) \subset S(\mathbb{R})$ and $T^{-1}S(\mathbb{R}) \subset S(\mathbb{R})$. We denote the restriction $T$ to $S(\mathbb{R})$ by the same $T$. Thus the ONB $f = \{f_n\}$ in $L^2(\mathbb{R})$ and $T$ satisfy the conditions of Step 1-3 in Section 3 for $D \equiv S(\mathbb{R})$. Hence we may construct $D$-pseudo bosonic operators

$$A = T \left( \sum_{k=0}^{\infty} \sqrt{k+1} f_k \otimes \bar{f}_{k+1} \right) T^{-1},$$

$$B = T \left( \sum_{k=0}^{\infty} \sqrt{k+1} f_{k+1} \otimes \bar{f}_k \right) T^{-1}. $$

Then it is shown that $A = a + \frac{i}{\beta}$ and $B = a^\dagger + \frac{i}{\beta}$.

**Example 4.** Let $\theta$ be a real parameter value in $(-\pi/4, \pi/4) \setminus \{0\}$. We put

$$T = e^{\frac{\theta}{2} (a^2 - a^{12})} = e^{\frac{\theta}{2} (qp + pq)}.$$ 

Then $T$ and $T^{-1}$ are non self-adjoint operators in $L^2(\mathbb{R})$ such that $TS(\mathbb{R}) \subset S(\mathbb{R})$ and $T^{-1}S(\mathbb{R}) \subset S(\mathbb{R})$. We denote the restriction $T$ to $S(\mathbb{R})$ by the same $T$. Thus the ONB $f = \{f_n\}$ in $L^2(\mathbb{R})$ and $T$ satisfy the conditions of Step 1-3 in Section 3 for $D \equiv S(\mathbb{R})$. Hence we may construct $D$-pseudo bosonic operators

$$A = T \left( \sum_{k=0}^{\infty} \sqrt{k+1} f_k \otimes \bar{f}_{k+1} \right) T^{-1},$$

$$B = T \left( \sum_{k=0}^{\infty} \sqrt{k+1} f_{k+1} \otimes \bar{f}_k \right) T^{-1}. $$

Then it is shown that $A = \cosh(\theta) a + \sinh(\theta) a^\dagger$ and $B = \cosh(\theta) a^\dagger + \sinh(\theta) a$. 

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