Global Hochschild (co-)homology of singular spaces

Ragnar-Olaf Buchweitz $^a$, Hubert Flenner $^b, *$

$^a$ Department of Computer and Mathematical Sciences, University of Toronto at Scarborough, Toronto, Ont. M1C 1A4, Canada

$^b$ Fakultät für Mathematik der Ruhr-Universität, Universitätsstr. 150, Geb. NA 2/72, 44780 Bochum, Germany

Received 30 June 2006; accepted 28 June 2007
Available online 31 August 2007
Communicated by Michael J. Hopkins

Abstract

We introduce Hochschild (co-)homology of morphisms of schemes or analytic spaces and study its fundamental properties. In analogy with the cotangent complex we introduce the so-called (derived) Hochschild complex of a morphism; the Hochschild cohomology and homology groups are then the Ext and Tor groups of that complex. We prove that these objects are well defined, extend the known cases, and have the expected functorial and homological properties such as graded commutativity of Hochschild cohomology and existence of the characteristic homomorphism from Hochschild cohomology to the (graded) centre of the derived category.

MSC: 14F43; 13D03; 32C35; 16E40; 18E30

Keywords: Hochschild cohomology; Hochschild homology; Derived category; Complex spaces; Resolvent

Contents

0. Introduction .................................................................................................................. 206
1. The Hochschild complex for complex spaces ......................................................... 208
   1.1. Resolvents of complex spaces ........................................................................ 208

© 2007 Elsevier Inc. All rights reserved.

E-mail addresses: ragnar@math.toronto.edu (R.-O. Buchweitz), Hubert.Flenner@rub.de (H. Flenner).
0. Introduction

The aim of this note is to define Hochschild (co-)homology in the global setting, for morphisms of schemes or analytic spaces.

Hochschild homology for a flat morphism of any type of spaces $X \to Y$ should coincide with $\text{Tor}^{X \times Y}_\bullet (\mathcal{O}_X, \mathcal{O}_X)$, and if $Y$ is a simple point, Hochschild cohomology should agree with $\text{Ext}^\bullet_{X \times Y} (\mathcal{O}_X, \mathcal{O}_X)$, where $X$ is considered as a subspace of $X \times_Y X$ via the diagonal embedding. In the algebraic, case, when $X$ is a quasi-projective scheme over some field $K$, Swan [30] showed that this requirement holds for any globalization of the concept of Hochschild (co-)homology that had been proposed earlier, e.g. in [17]. He also proved that for $X$ smooth the corresponding Hodge spectral sequences agree, thus the Hochschild–Kostant–Rosenberg (HKR) decomposition that results from the degeneration of that spectral sequence for $X$ smooth over a field $K$ of characteristic zero is the same in those globalizations.

To go beyond the flat or absolute situation, with the aim to include both non-flat morphisms of schemes and the complex analytic case, let us first review the pertinent issues in the affine case. For homomorphisms $A \to B$ of commutative rings, even more generally, for any associative $A$-algebra $B$, the bar complex $B$ provides a canonical resolution of $B$ as a (right) module over the enveloping algebra $B_e := B \otimes_A B$, and classically Hochschild (co-)homology is the (co-)homology of (the dual of) that complex, that is,

$$\text{HH}_{B/A}^\bullet (M) := H^\bullet (B \otimes_B M)$$

and

$$\text{HH}_{B/A}^\bullet (M) := H^\bullet (\text{Hom}_{B_e} (B_e, M)),$$

for any $B$-bimodule $M$. If $P$ is a projective resolution of $B$ as a $B^e$-module, then there exists a comparison map of complexes $P \to B$ over the identity on $B$, and that map is unique up to homotopy. Consequently, there are natural comparison maps

$$\alpha_M : \text{Tor}_{\bullet}^{B^e} (B, M) \to \text{HH}_{B/A}^\bullet (M) \quad \text{and} \quad \alpha^M : \text{HH}_{B/A}^\bullet (M) \to \text{Ext}_{B^e}^\bullet (B, M).$$

The maps $\alpha_M$ are isomorphisms whenever $B$ is flat over $A$, while the maps $\alpha^M$ are isomorphisms as soon as $B$ is projective as an $A$-module.
For a morphism of analytic algebras, one can, in principle, mimic this approach, replacing the standard tensor product by the analytic one, to obtain an analytic bar complex and analytic Hochschild (co-)homology. However, already in simple situations, for instance for a free power series ring extension, this construction will not return the expected cohomology groups. The main reason for this is that even for a free analytic extension $A \to B$ of analytic algebras, $B$ is not a projective module over $A$ unless $A$ is artinian; see, for example, [6] and [32, Satz 9].

The same caveat applies to the obvious extension of this definition to spaces: in any category of ringed spaces that admits fibre products, one can form, for a morphism $f : X \to Y$ and each integer $n \geq 1$, the $n$-fold fibre product $X \times^{y^n} := X \times_Y \cdots \times_Y X$, and then restrict the structure sheaf topologically to the diagonal $\Delta_n : X \subseteq X \times^y$. With the usual definition of the differential, one obtains the sheafified (analytic) bar complex on $X$, a complex of $\Delta_2^{-1}\mathcal{O}_{X \times^y X}$-modules, that agrees with the complex of the same name considered by Swan in the algebraic setting. Instead we follow the approach introduced by Quillen in [22] for morphisms of algebras. To deal with the case when $B$ is not $A$-flat, one replaces $B^e$ by the derived tensor product $B^{\text{op}} \otimes_A B$ that can be defined as a real world algebra using free simplicial resolutions of $B$, and then the same definition as before applies. There is a natural homomorphism $B^{\text{op}} \otimes_A B \to B^{\text{op}} \otimes_A B = B^e$ over the identity of $B$, and this implies the existence of natural comparison maps

$$\beta_M : \text{Tor}^B_{\bullet} \otimes_A (B, M) \to \text{Tor}^B_{\bullet} (B, M) \quad \text{and} \quad \beta^M : \text{Ext}^B_{\bullet} (B, M) \to \text{Ext}^B_{\bullet} \otimes_A (B, M).$$

The maps $\beta_M$ are again isomorphisms for $B$ flat over $A$, while the maps $\beta^M$ are isomorphisms as long as $\text{Tor}^i_A (B, B) = 0$ for $i \neq 0$, in particular for $B$ flat over $A$.

One advantage of this setup, as shown in [22], is that the crucial Hochschild–Kostant–Rosenberg (HKR) decomposition theorem for Hochschild (co-)homology generalizes to arbitrary morphisms between commutative rings of characteristic zero, with the module of Kähler differential forms replaced by the cotangent complex of $A \to B$. Moreover, the elegant treatment of the Eckmann–Hilton argument by Suarez-Alvarez in [29] yields essentially automatically that the natural ring structure on $\text{Ext}^B_{\bullet} \otimes_A (B, B)$ is graded commutative, thus providing the counterpart to Gerstenhaber’s fundamental result in [15] for classical Hochschild cohomology.

The second advantage is that, following Quillen’s guidance, one may extend the technique to not necessarily flat morphisms $X \to Y$ between schemes or analytic spaces by replacing $X \times^y X$ with the derived fibre product. We describe the more complicated case of complex analytic algebras, asking the reader to make the simplifications that occur in the case of schemes. Locally, one proceeds as follows. Given a morphism of analytic algebras $A \to B$, resolve $B$ by a free DG $A$-algebra $R$, so that $R$ is a DG algebra with an $A$-linear derivation $\partial$ as differential\(^1\) and comes equipped with a quasiisomorphism $R \to B$ over $A$. Here “free” means that $R$ is concentrated in degrees $\leq 0$, that $R^0$ is a free analytic power series ring over $A$, and that $R$ is free as a graded $R^0$-algebra in the usual sense. The derived analytic tensor product $B \otimes_A B$ is then represented by the analytic tensor product $S := R \otimes_A R := R \otimes_{R^0} (R^0 \otimes_A R^0) \otimes_{R^0} R$ that inherits naturally the structure of a free DG algebra over $A$. The multiplication map $S \to R$ followed by the quasiisomorphism $R \to B$ endows $B$ with a DG-module structure over $S$. Accordingly we will define the Hochschild complex $\mathbb{H}_{B/A}$ as the derived tensor product $B \otimes_S B$, a complex in

\(^1\) We use throughout the convention that differentials increase degrees by 1.
the derived category, with the Hochschild (co-)homology functors given by $\Tor^B_\bullet(\mathbb{H}_{B/A}, -)$ and $\Ext^\bullet_B(\mathbb{H}_{B/A}, -)$. Clearly, this definition works locally on any analytic space.

We should point out here that contrary to the theory of the cotangent complex that is quite sensitive to the characteristic and requires simplicial instead of DG algebra resolutions in case of positive or mixed characteristic, the construction of the Hochschild complex is oblivious to the characteristics of the rings involved, whence one can use DG algebras throughout.

To globalize, we use *resolvents* as developed in [5,12,21]. This allows to define first a Hochschild complex on the simplicial space over $Y$ that is associated to a locally finite covering of the given space $X$ by Stein compact sets, using the local construction and propagating it along the nerf of the simplicial structure, and then to descend to $X$ via a Čech construction to obtain a Hochschild complex $\mathbb{H}_{X/Y}$ in the derived category $D^{-}(X)$. We verify that this procedure indeed leads to a notion of Hochschild (co-)homology with the expected properties, such as graded commutativity of the natural product on Hochschild cohomology and existence of a characteristic ring homomorphism from the Hochschild cohomology to the graded centre of the derived category. For a flat morphism, this Hochschild complex reduces to the derived tensor product $\mathcal{O}_X \otimes_{\mathcal{O}_X \otimes \mathcal{O}_Y} \mathcal{O}_Y \mathcal{O}_X \otimes \mathcal{O}_X = (L\Delta^*)\Delta_* \mathcal{O}_X \in D^{-}(X)$, where $\Delta : X \to X \times_Y X$ is the diagonal embedding, and so the requirements laid out at the beginning are satisfied.

In a subsequent paper we will show that the HKR-decomposition theorem holds as well, thus globalizing Quillen’s theorem. We note that in the meantime F. Schuhmacher [24], familiar with but independent of an early version of this work, has used the machinery of [3] to define Hochschild cohomology for morphisms of complex spaces and to give another proof of the HKR-decomposition theorem in that situation.

One inconvenience of the approach here is the variety of choices involved, from the open covering by Stein compact sets, to the local models of the free resolutions, to the glueing data. Although one can track independence of choices step by step, this is indeed cumbersome. Here we use a different approach: to show that our construction is independent of the choices, and thus leads to a well defined object in the derived category, we employ *categories of models*, in a context as suitable for our purposes. These are related to but less sophisticated than Quillen’s model categories. This technique was already used implicitly in [12] to deduce that the cotangent complex of a morphism of complex spaces is well defined in the derived category. Here we formalize the treatment and give complete proofs along with the application that shows the Hochschild complex to be well defined and to behave functorially with respect to morphisms of complex spaces.

We use throughout the notations and techniques as set up in Section 2 of our previous paper [5], including here only quick reviews of some relevant facts. Moreover, as already mentioned, we treat explicitly just the case of morphisms of complex spaces, leaving the simplifications that occur in the case of morphisms of schemes to the reader.

1. The Hochschild complex for complex spaces

1.1. Resolvents of complex spaces

We begin with a short review of the notion of a resolvent of a morphism of complex spaces, see [12, p. 33], [5, 2.34], or [21].
Definition 1.1.1. A resolvent for a morphism \( f : X \to Y \) of complex spaces\(^2\), or simply for \( X \) over \( Y \), consists of a triple \( X = (X_*, W_*, \mathcal{R}_*) \) given as follows:

1. \( X_* = (X_\alpha)_{\alpha \in A} \) is the simplicial space associated to some locally finite covering \((X_i)_{i \in I} \) of \( X \) by Stein compact subsets; see [5, 2.2(a)]. In particular, \( A \) is the simplicial set of all subsets \( \alpha \) of \( I \) with \( X_\alpha := \bigcap_{i \in \alpha} X_i \neq \emptyset \).

2. \( W_* \) is a smoothing of \( f \), which means that there is given a factorization

\[
\begin{array}{ccc}
X_* & \xrightarrow{i} & W_* \\
\downarrow{f} & & \downarrow{\tilde{f}} \\
Y & & 
\end{array}
\]

where \( W_* = (W_\alpha)_{\alpha \in A} \) is a simplicial system of Stein compact sets\(^3\) on the same simplicial set as \( X_* \), the morphism \( i \) is a closed embedding, and the morphism \( \tilde{f} \) is smooth.\(^4\)

3. \( \mathcal{R}_* \) is a locally free DG algebra\(^5\) over \( \mathcal{O}_{W_*} \), with each graded component of \( \mathcal{R}_\alpha \) a coherent \( \mathcal{O}_{W_\alpha} \)-module for each simplex \( \alpha \), and there is given a quasiisomorphism \( \mathcal{R}_* \to \mathcal{O}_{X_*} \) of \( \mathcal{O}_{W_*} \)-algebras.

The smoothing \( W_* \) is called free if

(a) \( W_i \subseteq \mathbb{C}^{n_i} \times Y \) for some \( n_i \), and \( \mathcal{O}_{W_i} \cong \mathcal{O}_{\mathbb{C}^{n_i} \times Y}|_{W_i} \);

(b) \( W_\alpha \subseteq \prod_{i \in \alpha} W_i \), where \( \prod^Y \) denotes the fibre product over \( Y \), and \( \mathcal{O}_{W_\alpha} \) is the topological restriction of the structure sheaf on \( \prod_{i \in \alpha} W_i \) to \( W_\alpha \);

(c) the transition maps \( W_\beta \to W_\alpha, \alpha \subseteq \beta \), are induced by the corresponding projections

\[
\prod_{i \in \beta} W_i \to \prod_{i \in \alpha} W_i.
\]

The resolvent \( X \) itself is said to be free if \( W_* \) is a free smoothing and \( \mathcal{R}_* \) is furthermore a free \( \mathcal{O}_{W_*} \)-algebra (see [5, 2.31]).

Each morphism between complex spaces admits such resolvents, even free ones, with respect to any covering of \( X \) by Stein compact sets; see, for example, [12, 2.11(a)] or [5, 2.35]. Recall that these resolvents are constructed inductively along the nerve \( A \) of the given covering. The initial step uses the fact that the ring of global sections over the Stein compact set \( X_i \) is noetherian by Frisch’s theorem [13], whence \( \mathcal{O}_{X_i} \) can be obtained for some \( n_i \geq 0 \) as a quotient of \( \mathcal{O}_{\mathbb{C}^{n_i} \times Y} \) restricted to a suitable Stein compact set \( W_i \) that is smooth over \( Y \). The \( \mathcal{O}_{W_i} \)-algebra resolution \( \mathcal{R}_i \) of \( \mathcal{O}_{X_i} \) can then be constructed à la Tate [31]; see [12, 2.7] and [5, 2.32] for the relevant case of DG algebras over simplicial systems of Stein compact sets. As this result will be frequently used in the following, we state it along with the lifting property for free algebra resolutions [12, 2.8].

---

\(^2\) A complex space is always viewed as a ringed space \( X = (|X|, \mathcal{O}_X) \), with \(|X|\) as underlying topological space and \( \mathcal{O}_X \) as structure sheaf.

\(^3\) Stein compact sets are always assumed to be semianalytic.

\(^4\) That is, for every point \( x \in W_\alpha \), the analytic algebra \( \mathcal{O}_{W_\alpha, x} \) is smooth over \( \mathcal{O}_{Y, f(x)} \).

\(^5\) Our DG algebras are always assumed to be concentrated in non-positive degrees.
By convention, all DG algebras over simplicial systems of Stein compact sets considered in this paper are assumed to have coherent homogeneous components concentrated in only non-positive degrees. (Recall that the differential increases degree by one, thus, such a DG algebra is bounded, as a complex, in the direction of the differential.)

**Proposition 1.1.2.** (1) Given a morphism of DG algebras $A_* \to B_*$ on a simplicial system of Stein compact sets there is a free DG $A_*$-algebra resolution of $B_*$, that is, a free DG $A_*$-algebra $R_*$ together with a quasiisomorphism of DG $A_*$-algebras $R_* \to B_*$. 

(2) Given a commutative diagram in solid arrows

$$
\begin{array}{ccc}
A_* & \to & A'_* \\
\downarrow \phi & & \downarrow \pi \\
B_* & \to & B'_*
\end{array}
$$

of DG algebra morphisms on a simplicial scheme of Stein compact sets with $B_*$ free over $A_*$ and $\pi$ a surjective quasiisomorphism, there exists a morphism $\tilde{\phi} : B_* \to A'_*$ of DG $A_*$-algebras such that $\pi \circ \tilde{\phi} = \phi$. Moreover, the lifting $\tilde{\phi}$ of $\phi$ through $\pi$ is unique in the derived category.

**Proof.** For the proof of (1) we refer the reader to [5, 2.32]. To deduce (2), by induction along the Postnikov tower of the free algebra $A_* \to B_*$, see [5, discussion after 2.32], and in view of the structure of free graded modules over a simplicial DG algebra, [5, 2.13], one may reduce to the case that $B_*$ is obtained from $A_*$ by adjunction of a graded free $A_*$-module of the form $p_\alpha^* F_\alpha$ with $F_\alpha$ generated in a single degree $k \leq 0$ over $A_\alpha$. If $e$ is a generator of $F_\alpha$, then $\tilde{\phi}(\partial e)$ is already defined as a section of $A'_\alpha$ and $\phi(e)$ is a section in $B'_\alpha$. As $\pi$ is surjective, there exists, by Theorem B, a section $e'$ in $A'_\alpha$ with $\pi(e') = \phi(e)$. Now $b = \tilde{\phi}(\partial e) - \partial e'$ is a cycle that maps to zero under $\pi$. As $\pi$ is a quasiisomorphism, $b = \partial e''$ for some $e''$, and $\tilde{\phi}(e) = e' + e''$ yields the desired lifting of $\phi(e)$.

The last assertion follows as $\tilde{\phi} = \pi^{-1} \phi$ in the derived category. \qed

1.1.3. Given a resolvent $(X_*, W_*, R_*)$ of a morphism of complex spaces $f : X \to Y$, we can form the simplicial spaces $X_\alpha \times_Y X_* := (X_\alpha \times_Y X_\alpha)_{\alpha \in A}$ and $W_\alpha \times_Y W_* := (W_\alpha \times_Y W_\alpha)_{\alpha \in A}$ of Stein compact sets over $Y$. Note that $X_\alpha \times_Y X_\alpha$ is the simplicial space associated to the covering $(X_i \times_Y X_i)_{i \in I}$ of a neighbourhood of the diagonal subspace $X \subseteq X \times_Y X$. The simplicial space $W_\alpha \times_Y W_\alpha$ constitutes a smoothing of $X_\alpha \times_Y X_\alpha$ over $Y$. As in [5, 2.37] we set$^6$

$$S_* := R_* \otimes_{O_Y} R_* := R_* \otimes_{O_{W_*}} (O_{W_* \times_Y W_*}) \otimes_{O_{W_*}} R_* ,$$

and note that $S_*$ is naturally a DG algebra over the simplicial space $W_* \times_Y W_*$. The reader should keep in mind the following facts.

---

$^6$ Another plausible notation, modeled on the one sometimes used in algebraic geometry, is

$$S_* = R_* \otimes_{O_Y} R_* := p_1^* R_* \otimes_{O_{W_* \times_Y W_*}} p_2^* R_* ,$$

where $p_1, p_2 : W_* \times_Y W_* \to W_*$ denote the canonical projections.
Lemma 1.1.4. For a point \((w, w') \in W_\alpha \times_Y W_\beta\), the complex \(S_{\alpha,(w,w')}\) is exact if \((w, w') \notin X_\alpha \times_Y X_\alpha\), whereas it represents the derived analytic tensor product \(O_{X,w} \hat{\otimes}_{O_Y} O_{X,w'}\), if \((w, w') \in X_\alpha \times_Y X_\alpha\) with \(y = f(w) = f(w')\). In particular,

1. for any simplices \(\alpha \subseteq \beta\) and each \(z \in W_\beta \times_Y W_\beta\), the transition map \(p_{\alpha\beta} : W_\beta \times_Y W_\beta \to W_\alpha \times_Y W_\alpha\) induces a quasiisomorphism \(S_{\alpha,p_{\alpha\beta}(z)} \to S_{\beta,z}\);
2. if \(X\) is flat over \(Y\), then the natural morphism \(S_* \to O_{X_*} \times_Y X_*\) is a quasiisomorphism.

Proof. The given quasiisomorphism \(R_* \to O_{X_*}\) is a \(Y\)-flat resolution, whence at each point the DG algebra \(S_*\) represents the derived analytic tensor product \(O_{X_*} \hat{\otimes}_{O_Y} O_{X_*}\). This implies the first assertion as well as (1). If \(X \to Y\) is flat, the derived tensor product is represented by the non-derived one, and the quasiisomorphism \(R_* \to O_{X_*}\) of \(Y\)-flat resolutions of \(O_{X_*}\) yields the quasiisomorphism

\[
S_* = R_* \hat{\otimes}_{O_Y} R_* \to O_{X_*} \hat{\otimes}_{O_Y} O_{X_*} \cong O_{X_*} \times_Y X_*.
\]

\(\square\)

1.2. The Čech construction

We keep the notation from 1.1.3. To construct the Hochschild complex, we use the Čech functor as the basic tool to pass from modules on \(X_*\) to modules on \(X\). We remind the reader in brief of the relevant definitions and properties, see also [5, 2.27–2.30].

1.2.1. Restricting a given \(O_X\)-module \(M\) to the Stein compact sets of the given covering defines the \(O_{X_*}\)-module \(M_* = j^*M\) with \(M_\alpha := M|_{X_\alpha}\). This functor is exact and so induces directly a functor \(j^* : D(X) \to D(X_*)\) between the respective derived categories.

To describe a right adjoint, denote by \(j_\alpha : X_\alpha \hookrightarrow X\) the inclusion and order the vertices of the covering to associate to any module \(M_*\) on \(X_*\) the Čech complex \(C^*(M_*)\) with terms

\[
C^p(M_*) := \prod_{|\alpha| = p} j_{\alpha*}(M_\alpha),
\]

where the product is taken over all ordered simplices, and the differential is defined in the usual way by means of the transition morphisms for \(M_*\) and the given ordering on the vertices. The functor \(j_*(M_\alpha) := H^0(C^*(M_\alpha))\) from \(O_{X_*}\)-modules to \(O_X\)-modules is right adjoint to \(j^*\), and the canonical homomorphism of \(O_X\)-modules \(M \to j_*j^*(M)\) is an isomorphism. The Čech functor extends in the usual way to complexes \(M_*\) by taking the total complex of the double complex \(C^p(M^\alpha_*)\). We note the following important properties of the Čech functor:

1. \(C^*\) is exact, thus can be viewed directly as a functor \(C^* : D(X_*) \to D(X)\).
2. \(C^*\) represents \(Rj_*\), the right derived functor of \(j_*\).
3. The adjunction morphism \(M \to Rj_*j^*M \cong C^*(M_*)\) is just the Čech complex of sheaves associated to the given covering. It is always a quasiisomorphism, that is, the functor \(j^* : D(X) \to D(X_*)\) is fully faithful. The other adjunction morphism, \(j^*C^*(M_*) \to M_*\) on \(D(X_*)\), is a quasiisomorphism if and only if the transition morphisms \(p_{\alpha\beta}^*M_\alpha \to M_\beta\) are quasiisomorphisms for all pairs of simplices \(\alpha \subseteq \beta\).
The terms of the complex $C^\bullet(M_\alpha)$ are flat $O_X$-modules whenever $M_\alpha$ is flat over $O_{X_\alpha}$ for each simplex $\alpha$.

**Remark 1.2.2.** We end the general review of the setup recalling the following convention. If $\mathcal{M}, \mathcal{N}$ are objects of a triangulated category $D$, then

$$\text{Ext}^n_D(\mathcal{M}, \mathcal{N}) := \text{Hom}_D(\mathcal{M}, T^n\mathcal{N}), \quad n \in \mathbb{Z},$$

with $T$ the translation functor on $D$. This convention thus ignores whether $\text{Ext}^\bullet$ can be realized as the cohomology of a “concrete” complex, obtained from resolutions of some kind. This additional feature is however present in most, if not in all the (unbounded) derived categories we work with; see, for example, [28]. Indeed, these triangulated categories arise from abelian categories with enough projectives and/or injectives. In particular, the sheaves of the form $\mathcal{E}xt^\bullet_X(\mathcal{M}, \mathcal{N})$ or $\mathcal{T}or^{O_X}_\bullet(\mathcal{M}, \mathcal{N})$ on some space $X$, with $\mathcal{M}, \mathcal{N} \in D(X)$, exist and are well defined, regardless of the size of the complexes involved.

### 1.3. The Hochschild complex

The multiplication map $\mu : S_* = R_* \otimes_{O_Y} R_* \to R_*$ turns $R_*$ into an $S_*$-algebra, and there is a locally free, and by 1.1.2(1) even a free, DG $S_*$-algebra resolution $B_*$ of $R_*$ over $S_*$ that fits into a commutative diagram

$$
\begin{array}{ccc}
S_* & \xrightarrow{\mu} & R_* \\
\downarrow{\nu} & & \downarrow{\text{id}} \\
B_* & & R_*
\end{array}
$$

where $\nu$ is a quasiisomorphism of DG algebras.

We keep track of these additional data through the following notation.

**Definition 1.3.1.** The quadruple $\mathcal{X}^{(e)} := (X_*, W_*, R_*, B_*)$ is called an extended resolvent of $f : X \to Y$ if $\mathcal{X} := (X_*, W_*, R_*)$ is a resolvent and $B_*$ is a locally free algebra resolution of $R_*$ over $S_*$ that fits into a commutative diagram.

By the preceding discussion such an extended free resolvent always exists. To define Hochschild complexes, free extended resolvents are sufficient; however to deduce uniqueness we need the flexibility offered by arbitrary resolvents.

The tensor product of the algebra resolution $B_*$ over $S_*$ with the composition $S_* \xrightarrow{\mu} R_* \to O_{X_*}$, when restricted topologically to the diagonal $\Delta : X_* \hookrightarrow X_* \times_Y X_* \hookrightarrow W_* \times_Y W_*$, leads first to a Hochschild complex of $X_*$ over $Y$, and then through the Čech functor to a Hochschild complex of $X$ over $Y$.

**Definition 1.3.2.** We call the DG $O_{X_*}$-algebra

$$\mathcal{H}^{X_*/Y} := \Delta^*(B_*) = (B_* \otimes_{S_*} O_{X_*})|_{X_*}$$


a Hochschild complex of \( X_s \) over \( Y \). The associated Čech complex

\[
\mathbb{H}_{X/Y} := C^\bullet(\mathbb{H}_{X_s/Y})
\]

of \( \mathcal{O}_X \)-modules will be called a Hochschild complex of \( X \) over \( Y \).

Unlike the classical Hochschild complex for algebras, the complexes here are of course not canonically defined as they depend on the choice of the extended resolvent. However we have the following result the proof of which will be postponed to Section 2.

**Theorem 1.3.3.** The Hochschild complex is well defined as an object of the derived category \( D(X) \). Moreover, for any commutative diagram of morphisms of complex spaces

\[
\begin{array}{ccc}
X & \xrightarrow{g} & X' \\
\downarrow & & \downarrow \\
Y & \xrightarrow{} & Y'
\end{array}
\]

there is a functorial morphism

\[
Lg^*(\mathbb{H}_{X'/Y'}) \rightarrow \mathbb{H}_{X/Y}
\]

in \( D(X) \).

Let us note the following simple properties of Hochschild complexes.

**Proposition 1.3.4.** Let \( \mathbb{H}_{X_s/Y} \) and \( \mathbb{H}_{X/Y} \) be Hochschild complexes as just constructed.

1. The complex \( \mathbb{H}_{X_s/Y} \) is a graded free \( \mathcal{O}_{X_s} \)-algebra with coherent and locally free components concentrated in non-positive degrees.
2. The transition maps \( \mathbb{H}_{X_\alpha/Y} |_{X_\beta} \rightarrow \mathbb{H}_{X_\beta/Y} \) are quasiisomorphisms for all simplices \( \alpha \subseteq \beta \).
3. The adjunction morphism

\[
j^*\mathbb{H}_{X/Y} \rightarrow \mathbb{H}_{X_s/Y}
\]

is a quasiisomorphism.
4. The cohomology sheaves \( \mathcal{H}^p(\mathbb{H}_{X_s/Y}) \) vanish for \( p > 0 \) and \( \mathcal{H}^0(\mathbb{H}_{X_s/Y}) \cong \mathcal{O}_{X_s} \).
5. The complex \( \mathbb{H}_{X/Y} \) is locally bounded above with coherent cohomology, and its stalks \( \mathbb{H}_{X/Y,x} \) are free, so in particular flat \( \mathcal{O}_{X,x} \)-modules. The cohomology sheaves \( \mathcal{H}^p(\mathbb{H}_{X/Y}) \) vanish for \( p > 0 \), and \( \mathcal{H}^0(\mathbb{H}_{X/Y}) \cong \mathcal{O}_X \).

**Proof.** The assertions in (1) are immediate from the construction. In order to deduce (2), it suffices to argue locally. As \( \mathcal{B}_s \) is a free algebra over \( \mathcal{S}_s \), for each \( z \in W_\alpha \times_Y W_\alpha \), the DG \( \mathcal{S}_{\alpha,z} \)-module \( \mathcal{B}_{\alpha,z} \) is projective, resolving \( \mathcal{O}_{X,z} \) via the quasiisomorphisms \( \mathcal{B}_{\alpha,z} \rightarrow \mathcal{R}_{\alpha,z} \rightarrow \mathcal{O}_{X_{\alpha,z}} \cong \mathcal{O}_{X,z} \). By 1.1.3, for simplices \( \alpha \subseteq \beta \) and every point \( z \in W_\beta \times_Y W_\beta \), the corresponding algebra
homomorphism \( S_{\alpha,p\alpha}(z) \rightarrow S_{\beta,z} \) is a quasiisomorphism, where \( p_{\alpha\beta}: W_{\beta} \times_Y W_{\beta} \rightarrow W_{\alpha} \times_Y W_{\alpha} \) is as before the transition map. Applying now [4, X.66, §4 no.3], the morphism

\[
B_{\alpha,p\alpha}(z) \rightarrow B_{\alpha,p\alpha}(z) \otimes S_{\alpha,p\alpha}(z) S_{\beta,z}
\]

is seen to be a quasiisomorphism of \( S \) modules. Hence the module on the right constitutes along with \( B_{\beta,z} \) a projective resolution of \( O_{X,z} \) over \( S_{\beta,z} \) and the transition map \( B_{\alpha,p\alpha}(z) \otimes S_{\alpha,p\alpha}(z) S_{\beta,z} \rightarrow B_{\beta,z} \) is consequently a quasiisomorphism of projective resolutions of \( O_{X,z} \), thus

\[
\mathbb{H}_{X_a/Y} X_{\beta} = B_{\alpha} \otimes_{S_{\alpha}} O_{X_a} | X_{\beta} \cong p_{\alpha\beta}^*(B_{\alpha}) \otimes_{S_{\beta}} O_{X_{\beta}} | X_{\beta} \rightarrow B_{\beta} \otimes_{S_{\beta}} O_{X_{\beta}} | X_{\beta} = \mathbb{H}_{X_{\beta}/Y}
\]

is a quasiisomorphism as claimed. The last assertion of (2) follows now from 1.2.1(3), and these arguments prove as well (3).

It remains to establish (4). As the given covering of \( X \) is locally finite, \( (\mathbb{H}_{X/Y})_x = O^*(\mathbb{H}_{X_a/Y})_x \) for every \( x \in X \), and the localized complex is bounded above, whence \( \mathbb{H}_{X/Y} \) is locally bounded above. As the stalks \( (\mathbb{H}_{X_a/Y})_x \) are free \( O_{X_a} \)-modules by (1), the same holds for \( (\mathbb{H}_{X/Y})_x \). The quasiisomorphism \( j^* \mathbb{H}_{X/Y} \rightarrow \mathbb{H}_{X_a/Y} \) established in (2) yields the claims on the cohomology sheaves of \( \mathbb{H}_{X/Y} \).

1.4. The algebra structure on the Hochschild complex

Note that \( \mathbb{H}_{X/Y} \) will no longer necessarily be an \( O_X \)-algebra since the Čech functor is not compatible with tensor products. However, we will show in this part that the Hochschild complex is at least an \( O_X \)-algebra object in the derived category.

1.4.1. Let \( A \) be a complex of flat \( O_X \)-modules. Assume given a morphism

\[
m : A \otimes_{O_X} A = A \otimes_{O_X} A \rightarrow A,
\]

in \( D(X) \), called the multiplication. As usual, such a multiplication is said to be associative if the morphisms \( m \circ (m \otimes \text{id}_A) \) and \( m \circ (\text{id}_A \otimes m) \) from \( A \otimes_{O_X} A \otimes_{O_X} A \) to \( A \) are equal. With \( \sigma : A \otimes_{O_X} A \rightarrow A \otimes_{O_X} A \) the Koszul morphism that interchanges the two factors, the multiplication is (graded) commutative if \( m = m \circ \sigma \). Finally, a morphism \( \epsilon : O_X \rightarrow A \) is said to be a (left) unit if the morphism \( m \circ (\epsilon \otimes \text{id}_A) \) from \( O_X \otimes_{O_X} A \) to \( A \) is equal to the canonical isomorphism \( O_X \otimes_{O_X} A \cong A \).

In the following, an object \( A \) of \( D(X) \) consisting of flat \( O_X \)-modules and equipped with a commutative and associative multiplication \( m \) and with a unit \( \epsilon \) as above will be called in brief a commutative \( O_X \)-algebra in \( D(X) \). Morphisms of such commutative algebras are introduced in a straightforward manner. Moreover, if \( f : X' \rightarrow X \) is a morphism, then \( Lf^* = f^* \) on flat \( O_X \)-modules, thus it transforms commutative \( O_X \)-algebras in \( D(X) \) naturally into commutative \( O_{X'} \)-algebras in \( D(X') \).

In the same way we can introduce commutative \( O_{X_a} \)-algebras in \( D(X_a) \). For instance, the Hochschild complex \( \mathbb{H}_{X_a/Y} \) is already equipped with a commutative, associative and unitary \( O_{X_a} \)-bilinear multiplication that is a morphism of complexes, thus it represents trivially a commutative \( O_{X_a} \)-algebra in \( D(X_a) \).
We now show that the Čech functor $\mathcal{C}^\bullet$ transforms (commutative) $\mathcal{O}_{X_\ast}$-algebras in $D(X_\ast)$ into $\mathcal{O}_X$-algebras in $D(X)$. The reader will notice that all we require is indeed the fact that the left adjoint $j^*$ to $\mathcal{C}^\bullet$ commutes with (derived) tensor products.

**Lemma 1.4.2.** Let $\mathcal{M}_\ast, \mathcal{N}_\ast$ be complexes of $\mathcal{O}_{X_\ast}$-modules and assume that $\mathcal{M}_\ast$ is flat over $\mathcal{O}_{X_\ast}$ and locally bounded above. Then the complex of $\mathcal{O}_X$-modules $\mathcal{C}^\bullet(\mathcal{M}_\ast)$ is flat, and there is a natural morphism

$$\mathcal{C}^\bullet(\mathcal{M}_\ast) \otimes_{\mathcal{O}_X} \mathcal{C}^\bullet(\mathcal{N}_\ast) \to \mathcal{C}^\bullet(\mathcal{M}_\ast \otimes_{\mathcal{O}_{X_\ast}} \mathcal{N}_\ast)$$

in $D(X)$. If, moreover, for all simplices $\alpha \subseteq \beta$, the transition maps $p_{\alpha \beta}^* \mathcal{M}_\alpha \to \mathcal{M}_\beta$ and $p_{\alpha \beta}^* \mathcal{N}_\alpha \to \mathcal{N}_\beta$ are quasiisomorphisms, then the natural morphism is an isomorphism in $D(X)$.

**Proof.** Clearly $\mathcal{C}^\bullet(\mathcal{M}_\ast)$ is flat over $\mathcal{O}_X$. As $j^*$ commutes with tensor products of complexes, one obtains a natural morphism

$$j^*(\mathcal{C}^\bullet(\mathcal{M}_\ast) \otimes_{\mathcal{O}_X} \mathcal{C}^\bullet(\mathcal{N}_\ast)) \cong j^*\mathcal{C}^\bullet(\mathcal{M}_\ast) \otimes_{\mathcal{O}_{X_\ast}} j^*\mathcal{C}^\bullet(\mathcal{N}_\ast) \to \mathcal{M}_\ast \otimes_{\mathcal{O}_{X_\ast}} \mathcal{N}_\ast$$

from the adjunction morphisms $j^*\mathcal{C}^\bullet(\mathcal{M}_\ast) \to \mathcal{M}_\ast$ and $j^*\mathcal{C}^\bullet(\mathcal{N}_\ast) \to \mathcal{N}_\ast$. Adjunction yields the desired natural morphism.

If the transition morphisms for $\mathcal{M}_\ast$ and $\mathcal{N}_\ast$ are quasiisomorphisms, then the adjunction morphisms $j^*\mathcal{C}^\bullet(\mathcal{M}_\ast) \to \mathcal{M}_\ast$ and $j^*\mathcal{C}^\bullet(\mathcal{N}_\ast) \to \mathcal{N}_\ast$ are quasiisomorphisms by 1.2.1(3), whence the displayed morphism is a quasiisomorphism. As $j^*$ is fully faithful on $D(X)$ (see again 1.2.1(3)) the last assertion follows. □

**Lemma 1.4.3.** For each (commutative) flat $\mathcal{O}_{X_\ast}$-algebra $A_\ast$ in $D^-(X_\ast)$ the associated Čech complex $\mathcal{C}^\bullet(A_\ast)$ carries a natural (commutative) $\mathcal{O}_X$-algebra structure in $D^-(X)$. If, moreover, the transition morphisms on $A_\ast$ are quasiisomorphisms, then the adjunction morphism $j^*\mathcal{C}^\bullet(A_\ast) \to A_\ast$ is an isomorphism of algebra objects in $D^-(X_\ast)$.

**Proof.** In view of 1.4.2, we have a natural morphism

$$\mathcal{C}^\bullet(A_\ast) \otimes_{\mathcal{O}_X} \mathcal{C}^\bullet(A_\ast) \to \mathcal{C}^\bullet(A_\ast \otimes_{\mathcal{O}_{X_\ast}} A_\ast)$$

in $D^-(X)$. Following this morphism with $\mathcal{C}^\bullet(\mu)$, where $\mu: A_\ast \otimes_{\mathcal{O}_{X_\ast}} A_\ast \to A_\ast$ is the given multiplication on $A_\ast$, defines the multiplication on $\mathcal{C}^\bullet(A_\ast)$ in $D^-(X)$. That $\mathcal{C}^\bullet(A_\ast)$ inherits as well the unit from $A_\ast$ and that these data turn $\mathcal{C}^\bullet(A_\ast)$ into a (commutative) $\mathcal{O}_X$-algebra in $D^-(X)$ is clear from the naturality of the construction, and so is the last assertion. □

Applying the result just established to Hochschild complexes yields immediately the following.

**Proposition 1.4.4.** The complex $\mathbb{H}_{X/Y}$ admits a commutative $\mathcal{O}_X$-algebra structure in the derived category $D(X)$ that does not depend on the choice of $B_\ast$. The $\mathcal{O}_{X_\ast}$-algebra $j^*\mathbb{H}_{X/Y}$ is canonically isomorphic to the $\mathcal{O}_{X_\ast}$-algebra $\mathbb{H}_{X_\ast/Y}$ in the derived category $D(X_\ast)$. 

Remark 1.4.5. One can find an explicit morphism of complexes

\[ H_{X/Y} \times H_{X/Y} \to H_{X/Y} \]

that represents the multiplication. Indeed, the classical Alexander Whitney map for simplicial complexes; see, for example, [20, VIII 8.5]; yields an explicit associative unital product

\[ C^p(H_{X/Y}) \times C^q(H_{X/Y}) \to C^{p+q}(H_{X/Y} \otimes O_{X/Y}) \].

In general, this product is only graded commutative in the homotopy category of complexes; see [20, VIII 8.7]. Moreover, as we are considering the Čech complexes of alternating (or, equivalently, ordered) chains, this construction depends on an ordering of the vertex set of the simplices and is not functorial with respect to mappings. Note, however, that the Alexander–Whitney map becomes functorial as soon as we replace the alternating Čech complex by the larger complex of all Čech chains as in [18].

An algebra structure in \( D(X) \) induces similar structures in cohomology and on various Tor or Ext groups or sheaves. More precisely, we have the following standard application.

Lemma 1.4.6. An \( O_X \)-algebra structure on a complex \( A \in D^-(X) \) of flat \( O_X \)-modules induces a natural commutative graded \( H^0(X, O_X) \)-algebra structure on \( H^*(X, A) \). Moreover, for every complex \( M \in D(X) \), the groups

\[ \text{Tor}^X_*(M, A) \quad \text{and} \quad \text{Ext}^X_*(A, M) \]

carry natural graded module structures over \( H^*(X, A) \). Similarly, the sheaf \( \mathcal{H}^*(A) \) is a graded commutative \( O_X \)-algebra, and the sheaves

\[ T\text{or}^X_*(M, A) \quad \text{and} \quad T\text{ext}^X_*(A, M) \]

are modules over it.

Proof. Note that \( H^*(X, A) = \text{Ext}^X_*(O_X, A) \) by definition. If \( f, g : O_X \to A \) are morphisms in \( D^-(X) \) representing elements of this graded group of degree \( |f|, |g| \) respectively, then the composition

\[ O_X \xrightarrow{\sim} O_X \otimes O_X \xrightarrow{f \otimes g} A \otimes O_X \xrightarrow{\mu} A \]

defines the product \( fg \in H^{\max(|f|, |g|)}(X, A) \). It is easy to see that this multiplication is associative, with unit the image of 1 under \( H^*(X, O_X) \xrightarrow{\sim} H^*(X, A) \). To show that it is graded commutative, note that, with \( T \) the translation funtor on \( D(X) \), the diagram

\[
\begin{array}{ccc}
T^n A \otimes O_X & \xrightarrow{\sim} & T^{n+m} (A \otimes O_X \otimes X) \\
\sigma \downarrow & & \downarrow (-1)^{mn} \sigma \\
T^m A \otimes O_X & \xrightarrow{\sim} & T^{n+m} (A \otimes O_X \otimes X)
\end{array}
\]
commutes up to the sign \((-1)^{nm}\), where \(\sigma\) denotes as before the Koszul map interchanging the two factors. Taking cohomology, it follows easily that \(H^\bullet(X, A)\) is graded commutative. The remaining assertions are left to the reader as an exercise. \(\square\)

2. Hochschild (co-)homology of complex spaces

2.1. Definition and basic properties

With Hochschild complexes at our disposal, it is now immediate how to define Hochschild (co-)homology.

Definition 2.1.1. Let \(\mathcal{M} \in D(X)\) be a complex of \(\mathcal{O}_X\)-modules. The groups

\[
\text{HH}^\bullet_{X/Y}(\mathcal{M}) := \text{Tor}^\bullet_X(\mathcal{H}^\bullet_{X/Y}, \mathcal{M}) := H^{-\bullet}(X, \mathcal{H}^\bullet_{X/Y} \otimes_{\mathcal{O}_X} \mathcal{M}) \quad \text{and}
\]

\[
\text{HH}^\bullet_X(\mathcal{M}) := \text{Ext}^\bullet_X(\mathcal{H}^\bullet_{X/Y}, \mathcal{M})
\]

are called the Hochschild homology, respectively Hochschild cohomology of \(\mathcal{M}\). Similarly we introduce Hochschild (co-)homology sheaves

\[
\mathcal{H}\text{H}^\bullet_{X/Y}(\mathcal{M}) := \text{Tor}^\bullet_X(\mathcal{H}^\bullet_{X/Y}, \mathcal{M}) \quad \text{and} \quad \mathcal{H}\text{H}^\bullet_X(\mathcal{M}) := \text{Ext}^\bullet_X(\mathcal{H}^\bullet_{X/Y}, \mathcal{M}).
\]

If \(Y\) is just a point, we write simply \(\text{HH}^\bullet_X(\mathcal{M}), \text{HH}^\bullet_X(\mathcal{M})\), and so forth, and call these groups or sheaves absolute Hochschild (co-)homology of \(X\).

The following properties are immediate from the definition and the previous results.

Proposition 2.1.2. With Hochschild (co-)homology as just introduced, one has

1. \(\text{HH}^\bullet_{X/Y}\) is a cohomological functor, that is, every short exact sequence\(^7\) of complexes of \(\mathcal{O}_X\)-modules \(0 \to \mathcal{M}' \to \mathcal{M} \to \mathcal{M}'' \to 0\) induces a long exact cohomology sequence

\[
\cdots \to \text{HH}^{p-1}_{X/Y}(\mathcal{M}'') \to \text{HH}^p_{X/Y}(\mathcal{M}') \to \text{HH}^p_{X/Y}(\mathcal{M}) \to \text{HH}^p_{X/Y}(\mathcal{M}'') \to \cdots,
\]

and these long exact sequences depend functorially on the given short exact sequences. Analogously, \(\mathcal{H}\text{H}^\bullet_{X/Y}(-)\) is a cohomological functor, and similarly \(\text{HH}^\bullet_X(-)\) and \(\mathcal{H}\text{H}^\bullet_X(-)\) are homological functors on \(D(X)\).

2. If \(\mathcal{M} \in D^\cdot_{\text{coh}}(X)\), then the sheaves \(\mathcal{H}\text{H}^\bullet_{X/Y}(\mathcal{M})\) are coherent for each \(p \in \mathbb{Z}\). Similarly, the sheaf \(\mathcal{H}\text{H}^\bullet_X(\mathcal{M})\) is coherent for \(\mathcal{M} \in D^+_{\text{coh}}(X)\) and any \(p\).

3. For every \(\mathcal{O}_X\)-module \(\mathcal{M}\) and any \(p < 0\), the objects

\[
\text{HH}^p_{X/Y}(\mathcal{M}), \quad \mathcal{H}\text{H}^p_{X/Y}(\mathcal{M}), \quad \text{and} \quad \mathcal{H}\text{H}^\bullet_{X/Y}(\mathcal{M}),
\]

\(^7\) The reader may, of course, substitute “distinguished triangle” for “short exact sequence,” if needed.
vanish. Moreover, \( \text{HH}}_{X/Y}^0(\mathcal{M}) \cong \text{Hom}_X(\mathcal{O}_X, \mathcal{M}) = H^0(X, \mathcal{M}), \) and
\[
\mathcal{H}^0(X/Y) (\mathcal{M}) \cong \mathcal{M} \cong \mathcal{H}^0_{X/Y}(\mathcal{M}).
\]

(4) The Hochschild homology \( \text{HH}}_{X/Y}^0(\mathcal{O}_X) \) carries a natural graded commutative \( H^0(X, \mathcal{O}_X) \)-algebra structure, and for every complex \( \mathcal{M} \in D(X) \) the groups \( \text{HH}}_{X/Y}^0(\mathcal{M}) \) and \( \text{HH}}_{X/Y}^0(\mathcal{M}) \) are graded modules over this algebra. Similarly, \( \mathcal{H}^0_{X/Y}(\mathcal{O}_X) \) is a graded commutative sheaf of \( \mathcal{O}_X \)-algebras, and \( \mathcal{H}^0_{X/Y}(\mathcal{M}), \mathcal{H}^0_{X/Y}(\mathcal{M}) \) are graded modules over it.

**Proof.** Taking into account 1.3.4 and 1.4.4, (1) through (3) are standard properties of Ext- and Tor-groups or sheaves. Finally, 1.4.4 and 1.4.6 imply (4).

**Remark 2.1.3.** Note that the Hochschild homology groups \( \text{HH}}_{X/Y}^0(\mathcal{O}_X) \) will generally not be zero for \( p < 0 \), whence the graded algebra structure may involve both positive and negative degrees. If \( \mathcal{M} \) is an \( \mathcal{O}_X \)-module with proper support of dimension \( d \), then \( \text{HH}}_{X/Y}^0(\mathcal{M}) = 0 \) for \( p < -d \), while \( \text{HH}}_{X/Y}^0(\mathcal{M}) \) is isomorphic to \( H^d(X, \mathcal{M}) \). This follows from the hyper(co-)homology spectral sequence
\[
E^2_{p,q} = H^{-p}(X, \mathcal{H}^q_{X/Y}(\mathcal{M})) \Rightarrow \text{HH}}_{X/Y}^0(\mathcal{M})
\]
and 1.3.4(3).

### 2.2. Commutativity of Hochschild (co-)homology

We will now show that the Hochschild cohomology \( \text{HH}}_{X/Y}^*(\mathcal{O}_X) \) also admits a natural graded commutative algebra structure, induced from the Yoneda or composition product. For arbitrary (affine) algebras \( A \rightarrow B \), this is Gerstenhaber’s famous theorem [15]. The proof here applies the inspired treatment of the Eckmann–Hilton argument in [29]. The multiplicative structure in question is based on the following isomorphisms.

**Lemma 2.2.1.** (1) There are canonical isomorphisms
\[
\text{HH}}_{X/Y}^*(\mathcal{O}_X) \cong \text{Ext}^*_S(\mathcal{R}_\ast, \mathcal{R}_\ast) \cong \text{Ext}^*_S(\mathcal{B}_\ast, \mathcal{B}_\ast)
\]
and the Yoneda product on the last two terms endows thus \( \text{HH}}_{X/Y}^*(\mathcal{O}_X) \) with the (same) structure of a graded algebra.

(2) For every complex of \( \mathcal{O}_X \)-modules \( \mathcal{M} \), there are natural isomorphisms
\[
\text{HH}}_{X/Y}^*(\mathcal{M}) \cong \text{Ext}^*_S(\mathcal{R}_\ast, \mathcal{M}_\ast) \cong \text{Ext}^*_S(\mathcal{B}_\ast, \mathcal{M}_\ast)
\]
and the action of the corresponding Yoneda Ext-algebra through the contravariant argument of \( \text{Ext}^*_S(?, \mathcal{M}_\ast) \) realizes \( \text{HH}}_{X/Y}^*(\mathcal{M}) \) as a graded right module over \( \text{HH}}_{X/Y}^*(\mathcal{O}_X) \).

(3) With \( \mathcal{M} \) as in (2), there are natural isomorphisms
\[
\text{HH}}_{X/Y}^*(\mathcal{M}) \cong \text{Ext}^*_S(\mathcal{O}_X \otimes \mathcal{M}_\ast) \cong \text{Ext}^*_S(\mathcal{R}_\ast, \mathcal{R}_\ast \otimes \mathcal{M}_\ast).
\]
and the pairing
\[ \text{Ext}^*_{\mathcal{S}_*}(\mathcal{R}_*, \mathcal{R}_*) \times \text{Ext}^*_{\mathcal{S}_*}(\mathcal{R}_*, \mathcal{R}_* \otimes_{\mathcal{S}_*} \mathcal{M}_*) \to \text{Ext}^*_{\mathcal{R}_*}(\mathcal{R}_*, \mathcal{R}_* \otimes_{\mathcal{S}_*} \mathcal{M}_*) \]

that sends \( (\varphi, \psi) \) to \( (\varphi \otimes \text{id}_{\mathcal{M}_*}) \circ \psi \) endows \( \text{HH}^{x/y}_{-\bullet}(\mathcal{M}) \) with the structure of a graded left \( \text{HH}^{+\bullet}_{-\bullet}(\mathcal{O}_X) \)-module.

**Proof.** To establish the isomorphism in (2), note first that for all simplices \( \alpha \subseteq \beta \) the transition maps \( \mathbb{H}_{\alpha/y} \mid_{\mathbb{H}_{\beta/y}} \) are quasiisomorphisms by 1.3.4(2) and so, applying [5, 2.30(2)], there is an isomorphism
\[ \text{HH}^{x/y}_{-\bullet}(\mathcal{M}) = \text{Ext}^*_{x/y}(\mathcal{M}) \cong \text{Ext}^*_{x/y}(\mathcal{M}). \]

According to [5, 2.25(2)], the term on the right is isomorphic to \( \text{Ext}^*_{\mathcal{S}_*}(\mathcal{B}_*, \mathcal{M}_*) \). As \( \mathcal{B}_* \to \mathcal{R}_* \) is a quasiisomorphism, the latter group is as well isomorphic to \( \text{Ext}^*_{\mathcal{S}_*}(\mathcal{R}_*, \mathcal{M}_*) \). The isomorphism in (1) is just the special case \( \mathcal{M} = \mathcal{O}_X \), and in terms of the Ext-groups, the assertions on the multiplicative and linear structures hold for any triangulated category.

Finally, (3) is deduced in the same fashion, and we leave the details to the reader. \( \square \)

**Remark 2.2.2.** We remind the reader that a DG algebra is usually **not projective** as a DG module over itself or a subalgebra; see [5, Ex. 2.20]. However, the algebras in question here admit projective approximations according to (loc.cit). So, if \( P_* \to B_* \) is a projective approximation of the DG \( \mathcal{S}_* \)-module \( B_* \) and \( Q_* \to R_* \) one of \( R_* \) as a DG module over itself, then we may realize the groups in the preceding lemma as (co-)homology groups of complexes:

\[ \text{HH}^{x/y}_{-\bullet}(\mathcal{O}_X) \cong H^*(\text{Hom}_{\mathcal{S}_*}(P_*, P_*)), \]
\[ \text{HH}^{x/y}_{-\bullet}(\mathcal{M}) \cong H^*(\text{Hom}_{\mathcal{S}_*}(P_*, \tilde{M}_*)), \]
\[ \text{HH}^{x/y}_{-\bullet}(\mathcal{M}) \cong H^{-\bullet}(\text{Hom}_{\mathcal{R}_*}(Q_*, P_* \otimes_{\mathcal{S}_*} \tilde{M}_*)) \]

where \( \tilde{M}_* \) is a \( W_* \)-acyclic resolution of \( M_* \); see [5, 2.23].

In particular, the pairing in 2.2.1(3) can be expressed in terms of complexes as the pairing

\[ H^*(\text{Hom}_{\mathcal{S}_*}(P_*, P_*)) \times H^*(\text{Hom}_{\mathcal{R}_*}(Q_*, P_* \otimes_{\mathcal{S}_*} \tilde{M}_*)) \]

that sends \( (\varphi, \psi) \) to \( (\varphi \otimes \text{id}_{\tilde{M}_*}) \circ \psi \). In this explicit description, it becomes clear, for example, that \( \text{HH}^{x/y}_{-\bullet}(\mathcal{O}_X) \cong \text{Ext}^*_{\mathcal{S}_*}(\mathcal{R}_*, \mathcal{R}_*) \) acts naturally on \( \text{HH}^{x/y}_{-\bullet}(\mathcal{M}) \cong \text{Ext}^*_{\mathcal{R}_*}(\mathcal{R}_*, \mathcal{R}_* \otimes_{\mathcal{S}_*} \mathcal{M}) \) only through the covariant argument.

**Corollary 2.2.3.** For any morphism \( f : X \to Y \) and any complex \( \mathcal{M} \) of \( \mathcal{O}_X \)-modules, there are natural maps
\[ \text{HH}^\bullet_{X/Y}(M) \to \text{Tor}^\bullet_{O^X \times Y^X}(O_X, M), \]
\[ \text{Ext}^\bullet_{X \times Y^X}(O_X, M) \to \text{HH}^\bullet_{X/Y}(M). \]

For \( M = O_X \), the second map becomes a homomorphism of graded algebras, and the given maps are homomorphisms of graded modules over it.

**Proof.** We show the existence of the second map, leaving the analogous argument for the first one to the reader. From 2.2.1(2) we have first \( \text{HH}^\bullet_{X/Y}(M) \cong \text{Ext}^\bullet_{S^*}(B^*_s, M^*_s) \), and then \( \text{Ext}^\bullet_{S^*}(B^*_s, M^*_s) \cong \text{Ext}^\bullet_{S^*}(O^*_X, M^*_s) \) as \( B^*_s \to O^*_X \) is a quasiisomorphism. The algebra morphism \( S^* \to O^*_X \tilde{\otimes} O^*_Y O^*_X \) provides a “forgetful” algebra homomorphism

\[ \text{Ext}^\bullet_{O^*_X \tilde{\otimes} O^*_Y O^*_X} (O^*_X, M^*_s) \to \text{Ext}^\bullet_{S^*}(O^*_X, M^*_s), \]

and finally \( \text{Ext}^\bullet_{O^*_X \tilde{\otimes} O^*_Y O^*_X} (O^*_X, M^*_s) \cong \text{Ext}^\bullet_{O^*_{X \times Y^X}} (O_X, M) \), in view of 1.2.1(3).

Theorem 2.2.4. The Hochschild cohomology \( \text{HH}^\bullet_{X/Y}(O_X) \) is a graded commutative \( \Gamma(X, O_X) \)-algebra with respect to the Yoneda product.

**Proof.** To show commutativity of the Yoneda product, we apply [29, Theorem 1.7] to the derived category \( C = D^-(S^*_s) \) of DG \( S^*_s \)-modules. To this end, we show first that \( D^-(S^*_s) \) is a suspended monoidal category in the sense of Definition 1.4 in [29]. In fact, \( S^*_s \) has a \( R^*_s \)-algebra structure from the left and from the right, so the analytic tensor product \( S^*_s \tilde{\otimes} R^*_s S^*_s \) is an \( S^*_s \)-bimodule.

Given \( S^*_s \)-modules \( M^*_s \) and \( N^*_s \), their analytic tensor product

\[ M^*_s \tilde{\otimes} N^*_s := M^*_s \otimes_{S^*_s} (S^*_s \tilde{\otimes} R^*_s S^*_s) \otimes_{S^*_s} N^*_s \]

carries a natural \( R^*_s \)-structure from the right and from the left and thus it admits again an \( S^*_s \)-structure. As \( S^*_s \tilde{\otimes} R^*_s S^*_s \) is a flat \( S^*_s \)-module from the right we have, for \( N^*_s = R^*_s \),

\[ (S^*_s \tilde{\otimes} R^*_s S^*_s) \tilde{\otimes} R^*_s R^*_s = (S^*_s \tilde{\otimes} R^*_s S^*_s) \otimes_{S^*_s} R^*_s \cong S^*_s \tilde{\otimes} R^*_s R^*_s \cong S^*_s. \]

As a consequence, there are isomorphisms

\[ \varrho_{M^*_s} : M^*_s \tilde{\otimes} R^*_s \to M^*_s \quad \text{and, similarly,} \quad \lambda_{M^*_s} : R^*_s \tilde{\otimes} M^*_s \to M^*_s \]

that are functorial in \( M^*_s \in D^-(S^*_s) \). Hence \( R^*_s \) is a “unit” in the category \( D^-(S^*_s) \) with respect to \( \tilde{\otimes} \). As \( \tilde{\otimes} \) satisfies further the usual associativity and commutativity properties of tensor products, it follows that \( D^-(S^*_s) \) is indeed a suspended monoidal category. Applying the main result of [29], the commutativity of

\[ \text{Ext}^\bullet_{S^*_s}(R^*_s, R^*_s) = \bigoplus_{i \in \mathbb{Z}} \text{Hom}_{D^-(S^*_s)}(R^*_s, T^i R^*_s) \]

follows; here \( T \) denotes again the translation functor on the derived category. For the convenience of the reader we give a short outline of the argument: it is immediately seen from the construction
that $\varrho_{R_*} = \lambda_{R_*}$. Given morphisms $f : R_* \to R_*$, of degree $|f|$, and $g : R_* \to R_*$, of degree $|g|$, in the derived category, functoriality of $\varrho_{M_*}$ and $\lambda_{N_*}$ shows that the diagram

$$
\begin{array}{cccc}
R_* \otimes R_* & \xrightarrow{1 \otimes f} & R_* \otimes R_* & \xrightarrow{g \otimes 1} & R_* \otimes R_* \\
\downarrow \varrho_{R_*} & & \downarrow \varrho_{R_*} & & \downarrow \varrho_{R_*} \\
R_* & \xrightarrow{\lambda_{R_*}} & R_* & \xrightarrow{g} & R_* \\
\end{array}
$$

commutes in $D^-(S_*)$. In other words, identifying $R_*$ with $R_* \otimes R_*$ via $\varrho_{R_*} = \lambda_{R_*}$, the morphisms $g \circ f$ and $g \otimes f = (g \otimes 1) \circ (1 \otimes f)$ of degree $|f| + |g|$ are equal in $D^-(X)$. Similarly, $f \circ g$ is equal to $(1 \otimes f) \circ (g \otimes 1) = (-1)^{|f||g|} g \otimes f$, whence graded commutativity follows. □

Given a composable pair of morphisms $f : X \to Y$ and $g : Y \to Z$, of complex spaces, there is a natural morphism $\mathbb{H}^*_X \to \mathbb{H}^*_X Y$ in the derived category of $X$; see 1.3.3. It induces a natural map $\text{HH}^*_X/Y(\mathcal{M}) \to \text{HH}^*_X/Z(\mathcal{M})$ for every complex $\mathcal{M}$ of $\mathcal{O}_X$-modules.

We postpone the proof of the following result to 4.2.6.

**Proposition 2.2.5.** The natural map $\text{HH}^*_X/Y(\mathcal{O}_X) \to \text{HH}^*_X/Z(\mathcal{O}_X)$ is a homomorphism of graded commutative algebras, and the induced map $\text{HH}^*_X/Y(\mathcal{M}) \to \text{HH}^*_X/Z(\mathcal{M})$ is a homomorphism of modules over this algebra homomorphism.

In particular, there is a homomorphism of graded commutative algebras from the relative Hochschild cohomology of $X$ over $Y$ to the absolute Hochschild cohomology, $\text{HH}^*_X/Y(\mathcal{O}_X) \to \text{HH}^*_X(\mathcal{O}_X)$.

**Remark 2.2.6.** We end this section raising a subtle point that already appears in the affine case and provides further evidence that the derived version of Hochschild (co-)homology as originally proposed by Quillen in [22] for affine algebras and employed here for complex spaces is indeed appropriate.

For an associative, not necessarily commutative, algebra $A \to B$ over the commutative ring $A$, let $B^e := B^{op} \otimes_A B$ be the enveloping algebra and $B_* := B_*(B/A)$ the bar resolution of $B$ over $A$, a resolution of $B$ as a (right) module over $B^e$ via the multiplication map. The cohomology $H^\bullet(\text{Hom}_{B^e}(B_*, B)) \cong H^\bullet(\text{Hom}_{B^e}(B_*, B_*))$ constitutes then the classical, non-derived Hochschild cohomology of $B$ over $A$ and the composition of endomorphisms of $B_*$ induces the Yoneda product on it. Remarkably, this product on classical Hochschild cohomology is always graded commutative, a fact first observed by Gerstenhaber [15], who also supplied a simple direct proof in [16].

If $\mathbb{P}_* \to B$ is a projective resolution of $B$ as a $B^e$-module, then there exists a comparison map $\mathbb{P}_* \to B_*$ of complexes of $B^e$-modules over the identity on $B$. This comparison map is, of course, unique in the homotopy category and induces thus a natural homomorphism of graded $A$-algebras

$$
\alpha : H^\bullet(\text{Hom}_{B^e}(B_*, B)) \to \text{Ext}^\bullet_{B^e}(B, B)
$$

that is an isomorphism as soon as $B$ is projective over $A$, as then $B_*$ itself constitutes already a projective $B^e$-module resolution of $B$. 
On the other hand, consider Quillen’s [22] derived version of Hochschild cohomology, constructed by choosing a free (associative) DG $A$-algebra resolution $R$ of $B$ over $A$, and then taking cohomology,

$$\text{Ext}^\bullet_{B \otimes_A B}(B, B) := H^\bullet \left( \text{Hom}_{R^{op} \otimes_A R}(R, B) \right).$$

There is as well a natural comparison map

$$\beta : \text{Ext}^\bullet_{B^e}(B, B) \to \text{Ext}^\bullet_{B \otimes_A B}(B, B)$$

that is an $A$-algebra homomorphism with respect to the Yoneda product on either side. This homomorphism is an isomorphism, by [22], as soon as the “transversality condition”

$$\text{Tor}^A_i(B, B) = 0 \quad \text{for } i > 0$$

is satisfied, in particular, when $B$ is flat over $A$. The Eckmann–Hilton argument, as formulated by Suarez-Alvarez [29], that we employed above yields graded commutativity of $\text{Ext}^\bullet_{B \otimes_A B}(B, B)$ for any algebra $B$ over $A$.

However, the argument does not establish commutativity of $\text{Ext}^\bullet_{B^e}(B, B)$ in general! The rather delicate point is that, without transversality as in (†), the object $B$ is not necessarily a unit for the derived bifunctor

$$? \otimes_B ? : D(\text{Mod}B^e) \times D(\text{Mod}B^e) \to D(\text{Mod}B^e).$$

Indeed, the underlying bifunctor $? \otimes_B ?$ on the module category commutes with all colimits, that is, it is right exact and respects all direct sums in either argument. The left derived tensor product $B \otimes_B B$ is consequently always represented by the total complex of $P \otimes_B P$, where $P$ is a projective $B^e$-module resolution of $B$ as above. While this object comes equipped with the natural augmentation

$$B \otimes_B B = P \otimes_B P \to H_0(P \otimes_B P) \cong B,$$

which equals $\rho_B = \lambda_B$ in the notation adapted from the proof of 2.2.4 or [29], this augmentation need not be a quasiisomorphism, as, say, the case of the homomorphism of commutative rings $A = K[y] \to B = K[x, y]/(x^2, xy)$, $K$ some field, already demonstrates. For a different view of the importance of some kind of transversality conditions such as (†) in this context, see also [25].

### 2.3. The case of a flat morphism

Next we will show that for a flat morphism Hochschild (co-)homology is nothing but the usual (Ext-) Tor-algebra of the diagonal in $X \times_Y X$. The reader may wish to compare this to the corresponding result for quasi-projective schemes over a field, as described in [30].

**Proposition 2.3.1.** For any flat morphism $X \to Y$ of complex spaces, there is a canonical isomorphism of $O_X$-algebras in $D(X)$

$$H_{X/Y} \cong O_X \otimes_{O_{X \times_Y X}} O_X \cong (L \Delta^*) \Delta_* O_X,$$  \hspace{1cm} (1)
where we consider $\mathcal{O}_X$ as a module on $X \times Y X$ via the diagonal embedding $\Delta : X \hookrightarrow X \times Y X$. Accordingly, for every complex of $\mathcal{O}_X$-modules $\mathcal{M}$, the comparison maps from 2.2.3 become isomorphisms,

$$
\text{HH}_{X/Y}^\bullet (\mathcal{M}) \cong \text{Ext}_{X \times Y X}^\bullet (\mathcal{O}_X, \mathcal{M}),
$$

$$
\text{HH}_{X/Y}^X (\mathcal{M}) \cong \text{Tor}_{X \times Y X}^\bullet (\mathcal{O}_X, \mathcal{M}).
$$

For $\mathcal{M} = \mathcal{O}_X$, these isomorphisms are compatible with the algebra structures on either side.

**Proof.** First note that by definition

$$
\mathbb{H}_{X_s/Y} = B_s \otimes_{S_s} \mathcal{O}_{X_s} \cong (B_s \otimes_{S_s} \mathcal{O}_{X_s \times Y_s}) \otimes_{\mathcal{O}_{X_s \times Y_s}} \mathcal{O}_{X_s}.
$$

As $X \to Y$ is flat, the canonical map $S_s \to \mathcal{O}_{X_s \times Y_s}$ is a quasiisomorphism, by 1.1.4(2). Tensoring with $B_s$ from the left we get that

$$
B_s \to \tilde{B}_s := B_s \otimes_{S_s} \mathcal{O}_{X_s \times Y_s}
$$

is as well a quasiisomorphism (cf. [5, 2.25(3)]). This shows that $\tilde{B}_s$ is a flat resolution of $\mathcal{O}_{X_s}$ over $\mathcal{O}_{X_s \times Y_s}$ and so the right-hand side of (*) represents $\mathcal{O}_{X_s} \otimes_{\mathcal{O}_{X_s \times Y_s}} \mathcal{O}_{X_s}$. Taking the Čech complex, (1) follows.

The remaining isomorphisms (2) and the remark concerning algebra structures are immediate consequences of (1), the definitions and 2.2.3.

**Remark 2.3.2.** Observe that 2.3.1 applies in particular to the absolute case, when $Y$ is a simple point or $X$ a scheme over a field. Thus, if one were interested only in the absolute case, one could define Hochschild (co-)homology through these Tor- or Ext-groups, the point of view taken, say, in [30].

However, the definition given here allows to define (absolute) Hochschild (co-)homology as well for arbitrary schemes (over Spec $\mathbb{Z}$): while the structure morphism of such a scheme to Spec $\mathbb{Z}$ generally need not be flat, the construction via resolvents still applies mutatis mutandis, replacing Stein compacts by affine schemes. The comparison maps as in 2.2.3 still exist, but they will not be isomorphisms in general.

### 3. Hochschild cohomology and the centre of the derived category

The aim of this section is to show that there exists a characteristic homomorphism of graded commutative algebras from Hochschild cohomology of $f : X \to Y$ to the graded centre of the derived category of $X$. We begin with a review of some terms.
3.1. Centre of a category

Let \( C \) be any category. Recall that a natural transformation \( f : \text{id}_C \to \text{id}_C \), or endomorphism of the identity functor, consists of morphisms \( f_M : M \to M \), one for each \( M \) in \( C \), such that for every morphism \( \alpha : M \to N \) in \( C \) the diagram

\[
\begin{array}{ccc}
M & \xrightarrow{\alpha} & N \\
f_M \downarrow & & \downarrow f_N \\
M & \xrightarrow{\alpha} & N
\end{array}
\]

commutes. If \( C \) is small, these endomorphisms of \( \text{id}_C \) form a set, the so-called centre \( Z(C) \) of \( C \),

\[
Z(C) := \text{Hom}(\text{id}_C, \text{id}_C),
\]

see e.g. [20]. Composition of morphisms of functors provides a product on \( Z(C) \) with the identity transformation as unit. If \( f, g \) are elements of \( Z(C) \) then applying \((\ast)\) to the case \( M = N \) and \( \alpha = g_M \) yields that the centre is always commutative with respect to this product.

If \( C \) is furthermore a \( K \)-linear\(^8\) category, then endomorphisms of the identity functor form themselves a \( K \)-module and the centre \( Z(C) \) comes equipped with the structure of a commutative \( K \)-algebra.

3.2. Graded centre of a triangulated category

Now let \( C = (C, T, \Delta) \) be a triangulated category with translation functor \( T \) and collection of distinguished triangles \( \Delta \). The category \( C \) is in particular graded by \( T \), and we can consider more generally the abelian groups \( Z^n_{gr}(C) := \text{Hom}^T_C(\text{id}_C, T^n) \) of all natural transformations, or morphisms of functors, \( f : \text{id}_C \to T^n \) that anticommute with the shift functor, so that for every object \( M \) of \( C \) we have

\[
f_{TPM} = (-1)^{pn} T^p (f_M) : T^p M \to T^{p+n} M,
\]

or, in brief, \( f T^p = (1)^{pn} T^p f \). The direct sum

\[
Z^*_{gr}(C) = \bigoplus_{n \in \mathbb{Z}} \text{Hom}^T_C(\text{id}_C, T^n),
\]

is the graded centre of \( C \). It is a graded commutative ring: the product of two elements \( f \in \text{Hom}^T_C(\text{id}_C, T^n) \) and \( g \in \text{Hom}^T_C(\text{id}_C, T^m) \) is given by \( T^n(f) \circ g \in \text{Hom}^T_C(\text{id}_C, T^{n+m}) \).

What precisely is an element of \( Z^n_{gr}(C) \)? This question is answered by the following explicit description.

\(^8\) That is, \( K \) is a commutative ring, each Hom-set is a \( K \)-module, and composition is \( K \)-bilinear.
Lemma 3.2.1. An element $f \in \mathcal{Z}^n_{gr}(\mathcal{C})$ is represented by a collection of morphisms $f_M : \mathcal{M} \to T^n \mathcal{M}$, equivalently, elements $f_M \in \text{Ext}^n_{\mathcal{C}}(\mathcal{M}, \mathcal{M})$, one for each object $\mathcal{M}$ in $\mathcal{C}$, such that for each distinguished triangle

$$\mathcal{M}' \xrightarrow{u} \mathcal{M} \xrightarrow{v} \mathcal{M}'' \xrightarrow{w} T \mathcal{M},$$

thus, $(u, v, w)$ in $\Delta$, the following diagram commutes:

$$
\begin{array}{cccccccccccc}
M' & \xrightarrow{u} & M & \xrightarrow{v} & M'' & \xrightarrow{w} & TM' \\
\downarrow f_{M'} & & \downarrow f_M & & \downarrow f_{M''} & & \downarrow T(f_{M'}) \\
T^n M' & \xrightarrow{T^n u} & T^n M & \xrightarrow{T^n v} & T^n M'' & \xrightarrow{T^n w} & T^n + 1 M'.
\end{array}
$$

Proof. Any morphism $u : \mathcal{M}' \to \mathcal{M}$ in $\mathcal{C}$ occurs as the first component in a distinguished triangle by one of the axioms of a triangulated category, whence commutation of the leftmost square subsumes that $f$ is a natural transformation from $\text{id}_{\mathcal{C}}$ to $T^n$.

The bottom row in the diagram (**) is again a distinguished triangle in $\mathcal{C}$ by the translation axiom for a triangulated category, and anticommutation of $f$ with the translation functor just means that the rightmost square commutes as well. Thus, the elements of $\mathcal{Z}^n_{gr}(\mathcal{C})$ are precisely those morphisms from $\text{id}_{\mathcal{C}}$ to $T^n$ that are natural with respect to distinguished triangles. \qed

Remark 3.2.2. Note that applying $\text{Hom}_{\mathcal{C}}(N, ?)$ or $\text{Hom}_{\mathcal{C}}(?, N)$ to the diagram (**) returns the standard long exact sequences for the Ext-groups. Invoking Yoneda’s Lemma, commutativity of (**) now simply means that the family $(f_M)_{\mathcal{M}}$ is functorial with respect to long exact sequences of Ext-groups.

3.2.3. Given an object $\mathcal{M}$ in $\mathcal{C}$, the evaluation map,

$$ev_\mathcal{M} : \mathcal{Z}^\bullet_{gr}(\mathcal{C}) \to \text{Ext}^\bullet_{\mathcal{C}}(\mathcal{M}, \mathcal{M}), \quad f \mapsto f_M,$$

is a homomorphism of graded rings with image in the graded centre of $\text{Ext}^\bullet_{\mathcal{C}}(\mathcal{M}, \mathcal{M})$. Thus, $ev_\mathcal{M}$ endows $\text{Ext}^\bullet_{\mathcal{C}}(\mathcal{M}, \mathcal{M})$ with the structure of a graded $\mathcal{Z}^\bullet_{gr}(\mathcal{C})$-algebra.

Moreover, for each pair of objects $\mathcal{M}, \mathcal{N}$ in $\mathcal{C}$, the group $\text{Ext}^\bullet_{\mathcal{C}}(\mathcal{M}, \mathcal{N})$ is a graded bimodule, as a graded right module over $\text{Ext}^\bullet_{\mathcal{C}}(\mathcal{M}, \mathcal{M})$ and a graded left module over $\text{Ext}^\bullet_{\mathcal{C}}(\mathcal{N}, \mathcal{N})$. It becomes therefore a graded bimodule over $\mathcal{Z}^\bullet_{gr}(\mathcal{C})$ via $ev_\mathcal{M}$ on the right and $ev_\mathcal{N}$ on the left. These structures anticommute, in that

$$ev_\mathcal{N}(f) \cdot \varphi = T^m(f_{\mathcal{N}}) \circ \varphi = (-1)^{mn} f_{T^m \mathcal{N}} \circ \varphi = (-1)^{nm} \varphi \circ f_M$$

for $f \in \mathcal{Z}^n_{gr}(\mathcal{C})$, $(\varphi : \mathcal{M} \to T^n \mathcal{N}) \in \text{Ext}^n_{\mathcal{C}}(\mathcal{M}, \mathcal{N})$. Thus, $\text{Ext}^\bullet_{\mathcal{C}}(\mathcal{M}, \mathcal{N})$ is naturally a graded symmetric bimodule over $\mathcal{Z}^\bullet_{gr}(\mathcal{C})$ and the Yoneda pairing on the Ext-groups is bilinear over the graded centre of $\mathcal{C}$. 
3.3. The characteristic homomorphism

We first recall the setup of Fourier–Mukai transformations; see, for example, [7].

**Definition 3.3.1.** Let \( X \to Y \), \( X' \to Y \) be morphisms of complex spaces, with \( p : X \times_Y X' \to X \), \( p' : X \times_Y X' \to X' \) the projections from the fiber product. Let \( \mathcal{F} \in D(X \times_Y X') \) be a complex whose cohomology is supported on \( Z \subseteq X \times_Y X' \) with \( p'|Z : Z \to X' \) finite. These data define the Fourier–Mukai transformation, with kernel \( \mathcal{F} \),

\[
\Phi_{\mathcal{F}} = p'_*(\llcorner p^*(?) \otimes_{O_{X \times_Y X'}} \mathcal{F}) : D(X) \to D(X'),
\]

an exact functor between the indicated triangulated categories.

A morphism \( (a : \mathcal{F} \to T^m G) \in \text{Ext}^m_{X \times_Y X'}(\mathcal{F}, \mathcal{G}) \), with the supports of \( \mathcal{F}, \mathcal{G} \) finite over \( X' \), defines a morphism between the corresponding Fourier–Mukai transformations,

\[
\Phi_a := p'_*(\llcorner p^*(?) \otimes_{O_{X \times_Y X'}} a) : \Phi_{\mathcal{F}} \to \Phi_{T^m G} \cong T^m \Phi_{\mathcal{G}}.
\]

Remark that there is no need here to derive the functor \( p'_* \), as it is already exact on the subcategory of those complexes whose cohomology has finite support over \( X' \).

For \( \mathcal{F} = O_X \), the structure sheaf of the diagonal in \( X \times_Y X \), the associated Fourier–Mukai transformation is the identity on \( D(X) \), so \( \Phi_{O_X} = \text{id}_{D(X)} \). A morphism \( (g : O_X \to T^m O_X) \in \text{Ext}^m_{X \times_Y X}(O_X, O_X) \) yields consequently a morphism \( \Phi_g : \text{id}_{D(X)} \to T^m \) of endofunctors on \( D(X) \), and the commutation rules for \( \otimes \) and \( T \) imply that indeed \( \Phi_g \in \mathcal{Z}^m_{\text{gr}}(D(X)) \). This establishes the following result.

**Proposition 3.3.2.** For any morphism \( f : X \to Y \) of complex spaces, there exists a natural homomorphism of graded algebras

\[
\eta_{X/Y} \cong \Phi_f : \text{Ext}^\bullet_{X \times_Y X}(O_X, O_X) \to \mathcal{Z}^\bullet_{\text{gr}}(D(X)).
\]

If \( g : Y \to Z \) is a morphism of complex spaces, then \( \eta_{X/Y} \) factors as

\[
\eta_{X/Y} : \text{Ext}^\bullet_{X \times_Y X}(O_X, O_X) \xrightarrow{\rho_{Y/Z}} \text{Ext}^\bullet_{X \times_Z X}(O_X, O_X) \xrightarrow{\eta_{X/Z}} \mathcal{Z}^\bullet_{\text{gr}}(D(X))
\]

with \( \rho_{Y/Z} \) induced from the natural morphism \( X \times_Y X \to X \times_Z X \) defined by \( f \) and \( g \). It would thus be enough to define \( \eta_{X/Z} \) for \( Z \) a point and then to compose with \( \rho_{Y/Z} \) for the general case.

**Remark 3.3.3.** If \( g : Y \to Z \) is a morphism of complex spaces such that \( gf : X \to Z \) is flat, for example, taking \( Z \) to be a point, then there is the commutative diagram of homomorphisms of graded algebras

\[
\begin{array}{ccc}
\text{Ext}^\bullet_{X \times_Y X}(O_X, O_X) & \xrightarrow{\rho_{Y/Z}} & \text{Ext}^\bullet_{X \times_Z X}(O_X, O_X) \\
\downarrow & & \downarrow \eta_{X/Z} \\
\text{HH}^\bullet_{X/Y}(O_X) & \xrightarrow{\cong} & \text{HH}^\bullet_{X/Z}(O_X)
\end{array}
\]
with the vertical comparison maps coming from 2.2.3 and the algebra homomorphism at the bottom from 2.2.5. According to 2.3.1, the vertical morphism on the right is an isomorphism, whence we obtain, in a roundabout way, homomorphisms of graded commutative algebras

$$\text{HH}_{X/Y}^\bullet (\mathcal{O}_X) \to \mathcal{Z}_{gr}^\bullet (D(X)).$$

We now show directly that, without any flatness assumptions, the morphism $\eta_{X/Y}$ above factors through the comparison homomorphism $\text{Ext}_{X \times Y X}^\bullet (\mathcal{O}_X, \mathcal{O}_X) \to \text{HH}_{X/Y}^\bullet (\mathcal{O}_X)$, thus, that Hochschild cohomology is closer to the graded centre of the derived category than the self-extensions of the diagonal in $X \times_Y X$. The key, as always, is to replace the fibre product $X \times_Y X$ by its derived version by means of extended resolvents. This argument applies as well to arbitrary morphisms of schemes, thus, covers, for example, schemes over the integers, when there is no recourse to a flat situation.

**Theorem 3.3.4.** For a morphism $f : X \to Y$ of complex spaces, there exists a natural homomorphism of graded commutative algebras

$$\chi_{X/Y} : \text{HH}_{X/Y}^\bullet (\mathcal{O}_X) \to \mathcal{Z}_{gr}^\bullet (D(X))$$

that factors the map $\eta_{X/Y}$ in 3.3.2 through the comparison map from 2.2.3.

The proof of Theorem 3.3.4, including an analysis of the structure of this characteristic homomorphism, will occupy the remainder of this section.

We work with an extended resolvent $\mathfrak{X} = (X_*, W_*, R_*, B_*)$ of $X/Y$ as considered in 1.3.1. Recall from 1.3 that $B_*$ represents a locally free (or even free) DG $S_*$-algebra resolution of $R_*$ over $S_* = R_* \otimes \mathcal{O}_Y R_*$. The algebra homomorphisms

$$j_1 : R_* \xrightarrow{\approx} R_* \otimes 1 \subseteq S_*$$
$$j_2 : R_* \xrightarrow{\approx} 1 \otimes R_* \subseteq S_*$$

(3) (4)

define, by restriction of scalars, two flat $R_*$-algebra structures on $B_*$, and the corresponding structure maps, again denoted $j_{1,2} : R_* \to B_*$, are quasiisomorphisms of algebras. It follows that

$$\bar{B}_* := B_* \otimes 1 \otimes R_*, \mathcal{O}_{X_*}$$

carries two algebra structures as well: an $R_* = R_* \otimes 1$-structure from the action on the first factor, and the, again flat, $\mathcal{O}_{X_*}$-structure from the second factor. Moreover, the given (quasiiso-)morphism of algebras $\nu : B_* \to R_* \to \mathcal{O}_{X_*}$ induces a morphism

$$\bar{\nu} := \nu \otimes 1 \otimes R_* : \bar{B}_* \to \mathcal{O}_{X_*} \otimes R_* \mathcal{O}_{X_*} \cong \mathcal{O}_{X_*}.$$

The central tool is now the following lemma.

**Lemma 3.3.5.** With notation as just introduced, the following hold.

1. For any DG $R_*$-module $M_*$, the natural morphism $M_* \to M_* \otimes R_* \otimes 1 \bar{B}_*$ is a quasiisomorphism; in particular, the functor $^9 M_* \mapsto M_* \otimes R_* \otimes 1 \bar{B}_*$, where the target is considered

---

9 Quillen, in [23], denotes this functor $M_* \mapsto M_* \otimes^1_{\bar{R}_*} \bar{B}_*$ in the affine case.
a \( R_\alpha \)-module through the action of \( 1 \otimes R_\alpha \) on the right, is exact on the category of DG \( R_\alpha \)-modules.

(2) For any DG \( O_{X_\alpha} \)-module \( M_\alpha \), the natural morphism

\[
M_\alpha \otimes R_{\alpha} \otimes 1 \tilde{B}_\alpha \xrightarrow{id \otimes \tilde{v}} M_\alpha \otimes R_{\alpha} O_{X_\alpha} \cong M_\alpha
\]

is a quasiisomorphism in each of the following two cases:

(i) \( O_X \) is flat over \( Y \), or

(ii) \( M_\alpha \) is \( K \)-flat in the sense of [28], that is, \( M_\alpha \otimes f^{-1} O_Y f^{-1}(?) \) is exact on \( D(Y) \).

Furthermore, the (right) DG \( O_{X_\alpha} \)-module \( M_\alpha \otimes R_{\alpha} \otimes 1 \tilde{B}_\alpha \) represents each of these cases \( M_\alpha \otimes R_{\alpha} \otimes 1 \tilde{B}_\alpha \) as well as \( M_\alpha \otimes R_{\alpha} \otimes 1 \tilde{B}_\alpha \) over \( O_{X_\alpha} \).

(3) If \( O_X \) is flat over \( Y \) then the functor \( M_\alpha \mapsto M_\alpha \otimes R_{\alpha} \otimes 1 \tilde{B}_\alpha \) is exact on DG \( O_{X_\alpha} \)-modules.

In general, the functor \( ? \otimes R_{\alpha} \otimes 1 \tilde{B}_\alpha \) defines an exact auto-equivalence on \( D(O_{X_\alpha}) \) that is isomorphic to the identity functor via \( \text{id} \otimes \tilde{v} \).

**Proof.** It is sufficient to show the corresponding statements for every stalk. Thus, for a simplex \( \alpha \) and \( x \in X_\alpha \) consider the stalks

\[
M := M_{\alpha,x}, \quad R := R_{\alpha,x}, \quad S := S_{\alpha,x}, \quad A := O_{X_{\alpha,x}}, \quad B := B_{\alpha,x}, \quad \tilde{B} := \tilde{B}_{\alpha,x}.
\]

To deduce (1) consider the DG algebra \( R' := R \otimes R^0 B^0 \cong (R \otimes 1) \otimes R^0 \otimes 1 B^0 \). As \( R^0 \otimes 1 \to B^0 \) is flat, the functor \( M \mapsto M \otimes R R' \cong M \otimes R^0 B^0 \) is exact. Moreover, since \( \tilde{B} \) is a free DG algebra over \( R' \), by [5, 2.17(1)] the functor \( - \otimes \tilde{R} \tilde{B} \) is exact, whence \( M \mapsto M \otimes \tilde{R} \tilde{B} \cong M \otimes \tilde{R} R' \otimes \tilde{R} B \) is exact as well. This proves exactness of the functor \( M_\alpha \mapsto M_\alpha \otimes R_{\alpha} \otimes 1 \tilde{B}_\alpha \).

It remains to show for (1) that the natural map \( M \to M \otimes R B = M \otimes R_{\alpha} \otimes 1 \tilde{B}_\alpha \) is a quasiisomorphism. As tensor products are compatible with direct limits, we may suppose that \( M \) is a finitely generated \( R \)-module. As the functor \( - \otimes R_{\alpha} \otimes 1 \tilde{B} \) is exact, after replacing \( M \) by a quasiisomorphic complex of free \( R \)-modules, we may suppose that \( M \) is free over \( R \). By [5, 2.13(2)] the submodules \( M^{(i)} \) generated by all elements of degree \( > i \) form a filtration of \( M \) by free DG submodules, and the quotients \( M^{(i-1)} / M^{(i)} \) are as well free. Using a simple spectral sequence argument, we are thus reduced to the case that \( M \) is freely generated in one degree so that, up to a shift, \( M \) is isomorphic as a DG module to a direct sum of copies of \( R \). In this case, the map in question is a direct sum of copies of the quasiisomorphism \( j_1 : R \cong R \otimes 1 \to B \), and the assertion follows.

To prove the first claim in (2), it suffices, in view of (1), to show that \( M \otimes R B \to M \otimes \tilde{R} \tilde{B} \) is a quasiisomorphism, as the composition of the sequence of morphisms

\[
M \otimes R B \to M \otimes \tilde{R} \tilde{B} \xrightarrow{\text{id}_M \otimes \tilde{v}} M \otimes R A \otimes R A \cong M
\]

is the identity on \( M \), and the leftmost morphism is a quasiisomorphism by part (1). We will show this substitute claim more generally, replacing \( B \) by any complex of free \( S \)-modules, say, \( F \) that is bounded above; we have to replace then, of course, \( \tilde{B} \) by \( \tilde{F} := F \otimes R_{\alpha} \circ A \), and the assertion becomes that in this situation

\[
M \otimes R F \to M \otimes \tilde{R} \tilde{F} \quad (*)
\]

is a quasiisomorphism.
Now assume first that (ii) holds. As before, using again [5, 2.13(2)], the \( R \)-submodules \( F^{(i)} \) of \( F \) that are generated by all elements of degree \( > i \) form a filtration of \( F \) by free DG submodules with free successive quotients \( F^{(i-1)}/F^{(i)} \). Again, with a simple spectral sequence argument we reduce to the case that up to a shift \( F \) is isomorphic to a direct sum of copies of \( S \) as a DG module over \( S \). Clearly we may suppose that \( F \) is of rank 1, that is, \( F \cong S \). In this case, \( M \otimes_R F \cong M \otimes_R S \) is isomorphic to the analytic tensor product \( M \tilde{\otimes}_A R \), where \( \Lambda := \mathcal{O}_{Y,f(x)} \).

As \( M \) is \( \Lambda \)-flat over \( \Lambda \), the functor \( M \tilde{\otimes}_A - \) is exact, whence \( M \tilde{\otimes}_A R \) is quasiisomorphic to \( M \tilde{\otimes}_A \Lambda \cong M \otimes_{R \otimes_1 S} S \otimes_{1 \otimes_R} A \), as required.

Next assume that (i) holds in (2). \( M \) can be written as a direct limit of subcomplexes that are bounded above. As tensor products are compatible with direct limits, we may suppose that \( M \) itself is bounded above. Now we can proceed as in the previous case and reduce the assertion to the case that \( F = S \) so that as before \( M \otimes_R S \cong M \tilde{\otimes}_A R \). As \( A \) is flat over \( \Lambda \), the mapping cone over \( R \to A \) is an exact complex of \( \Lambda \)-flat finite \( R^0 \)-modules. Thus it remains exact after applying \( M^p \tilde{\otimes}_A - \) for any \( p \), and so \( M^p \tilde{\otimes}_A R \) is quasiisomorphic to \( M^p \tilde{\otimes}_A A \). Taking total complexes proves the assertion that the morphism in (5) is a quasiisomorphism also in this case.

Concerning the final claim in (2), note that \( \mathcal{B}_* \otimes_{1 \otimes_R \mathcal{R}_*} \mathcal{O}_{X_s} \) always represents \( \mathcal{B}_* \otimes_{1 \otimes \mathcal{R}_s} \mathcal{O}_{X_s} \), as \( \mathcal{B}_s \) is flat over \( R \) via \( j_2 : R \cong 1 \otimes R \to B \). In case (i), consider the following commutative diagram

\[
\begin{array}{ccc}
B & \xrightarrow{j_1} & \tilde{B} = B \otimes_{1 \otimes_R} A \\
\downarrow{\text{free}} & & \downarrow{\text{free}} \\
R \otimes 1 & \xrightarrow{j_2} & S \\
\downarrow & & \downarrow \\
1 \otimes R & \xrightarrow{j} & A \\
\end{array}
\]

in which the two squares, as well as the rectangle they form, represent tensor products of algebras. As \( S \to B \) is flat (even free), and \( \tilde{B} \cong B \otimes_S (S \otimes_{1 \otimes_R} A) \), we obtain that \( \tilde{B} \) is flat (even free) over \( S \otimes_{1 \otimes R} A \). On the other hand, \( A \) is flat over \( \Lambda \) by hypothesis (i), and so \( R \cong R \otimes 1 \to R \tilde{\otimes}_A A \) is a flat homomorphism of algebras. In summary, this exhibits \( j_1 : R \cong R \otimes 1 \to \tilde{B} \) as the composition of the two flat maps \( R \cong R \otimes 1 \to R \tilde{\otimes}_A A \) and \( R \tilde{\otimes}_A A \to \tilde{B} \). The second factor being flat and bounded above, \( M \otimes_{R \otimes 1} \tilde{B} \) represents \( M \otimes_{R \otimes 1} \tilde{B} \). In case (ii), the first factor \( M \) is \( \Lambda \)-flat by assumption, whence \( M \otimes_{R \otimes 1} \tilde{B} \) again represents \( M \otimes_{R \otimes 1} \tilde{B} \).

Finally, (3) is an immediate consequence of (2) and the fact that any DG module admits a \( K \)-flat resolution. \( \square \)

**Remark 3.3.6.** Note that the preceding result is already non-trivial and interesting in the affine case. For instance, let \( A = \Lambda/I \) be a quotient of a regular local ring \( \Lambda \) containing \( \mathbb{Q} \), and let \( R \to A \) be a resolution of \( A \) by a free DG \( A \)-algebra; of course we may assume that \( R^0 = \Lambda \). Choose an algebra resolution \( B \) of \( S := R \otimes_A R \to R \) so that \( B \to R \) is a quasiisomorphism and \( B \) is free as \( S \)-algebra.

Given an \( A \)-module \( M \) we can construct a free resolution of \( M \) over \( A \) as follows:

Choose a free resolution \( F \) of \( M \) over \( A \) that admits a DG module structure over \( R \). By 3.3.5(1) above, the complex \( F \otimes_{R \otimes 1} B \otimes_{1 \otimes R} A \) constitutes then an \( A \)-free resolution of \( M \).
For instance, if \( A = \Lambda / f \Lambda \) is a hypersurface, then we can take \( R \) to be the Koszul complex over \( f \), i.e. the free \( \Lambda \)-algebra \( \Lambda[\varepsilon] \) with a generator \( \varepsilon \) of degree \(-1\) and differential \( \partial(\varepsilon) = f \). Thus \( S = \Lambda[\varepsilon_1, \varepsilon_2] \) with \( \varepsilon_1 := 1 \otimes \varepsilon \) and \( \varepsilon_2 := \varepsilon \otimes 1 \). The free \( S \)-algebra \( B := S[\eta] \) with a generator \( \eta \) of degree \(-2\) and differential \( \partial(\eta) = \varepsilon_1 - \varepsilon_2 \) constitutes an algebra resolution\(^{10}\) of \( R \) via the map \( \eta \mapsto 0 \) and \( \varepsilon_i \mapsto \varepsilon_i \). If now \( M \) is a maximal Cohen–Macaulay module over \( A \) then its minimal resolution

\[
0 \to F_1 \to F_0 \to M \to 0
\]

over \( A \) is of length 1. Since \( M \) is annihilated by \( f \), we have \( fF_0 \subseteq F_1 \), whence multiplication by \( f \) yields a map \( \varepsilon : F_0 \to F_1 \). Via this map the complex \( F = (F_1 \to F_0) \) has a natural DG \( R \)-module structure. The resolution above is in this case just the periodic resolution constructed by Eisenbud and Shamash [10,26]; the periodicity is given by multiplication by \( \eta \). Applying this in a similar way to complete intersections \( A = \Lambda/(f_1, \ldots, f_r) \) leads to [1, 2.4].

3.3.7. We now turn to the explicit description of the algebra homomorphism \( \chi \). To this end, we exhibit the right action of \( \text{HH}^\bullet_X(\mathcal{O}_X) \) via \( ev_M \circ \chi_X \) on \( \text{Ext}^\bullet_X(M, N) \). We know from [5, 2.28], see 1.2.1(3), and 3.3.5(2(i)) that

\[
\text{Ext}^\bullet_X(M, N) \cong \text{Ext}^\bullet_S(B^*, B^*)
\]

where we consider \( M^* \otimes_{\mathcal{R}^* \otimes 1} \tilde{B}^* \) and \( N^* \otimes_{\mathcal{R}^* \otimes 1} \tilde{B}^* \) as \( \mathcal{O}_{X^*} \)-modules from the right. We also have from 2.2.1(1) that

\[
\text{HH}^\bullet_X(\mathcal{O}_X) \cong \text{Ext}^\bullet_S(B^*, B^*).
\]

The required module structure is now induced by the pairing

\[
\text{Ext}^\bullet_X(M^*, N^*) \times \text{Ext}^\bullet_S(B^*, B^*) \to \text{Ext}^\bullet_X(M^* \otimes_{\mathcal{R}^* \otimes 1} \tilde{B}^*, N^* \otimes_{\mathcal{R}^* \otimes 1} \tilde{B}^*)
\]

that sends \((f, g)\) to \( f \otimes_{\mathcal{R}^* \otimes 1} g \otimes_{\mathcal{R}^* \otimes 1} \mathcal{O}_{X^*} \). We thus obtain \( \chi(g)_{M^*} \), for a homogeneous element \( g \in \text{Ext}^{|g|}_{\mathcal{S}^*}(B^*, B^*) \), interpreted as a morphism \( g : B^* \to T^{|g|}B^* \) in \( D(S) \), by applying the Čech functor to the morphism on top of the commutative diagram

\[
\begin{array}{ccc}
\mathcal{O}^* & \xrightarrow{\chi(g)_{M^*}} & T^{|g|} \mathcal{O}^* \\
\cong & & \cong \\
\mathcal{O}^* \otimes_{\mathcal{R}^* \otimes 1} \tilde{B}^* & \xrightarrow{\text{id}_{\mathcal{O}^*} \otimes \tilde{g}} & \mathcal{O}^* \otimes_{\mathcal{R}^* \otimes 1} T^{|g|} \tilde{B}^*
\end{array}
\]

\(^{10}\) This is no longer true in positive characteristic, say \( p > 0 \), as then \( \partial(\eta^M) = 0 \). One either has to resolve further, or, more economically, use a divided power algebra. This is precisely the place, where the (DG algebras arising from) simplicial algebras become advantageous.
where $\tilde{g} = g \otimes \text{id}_{\mathcal{O}_X}$, and the vertical arrow on the left represents the quasiisomorphism from 3.3.5(2), while the vertical arrow on the right is the corresponding quasiisomorphism for $T^{[g]}\mathcal{M}$, composed with the (quasi)isomorphism

$$\mathcal{M}_s \otimes_{\mathcal{R}_e} T^{[g]} \mathcal{B}_s \xrightarrow{\sim} (T^{[g]}\mathcal{M}_s) \otimes_{\mathcal{R}_e} \mathcal{B}_s.$$ 

One sees now immediately that for every $g \in \text{Ext}^m_{\mathcal{S}}(\mathcal{B}_s, \mathcal{B}_s) \cong \mathbb{H}^m_X(\mathcal{O}_X)$, the family

$$\chi(g)_{\mathcal{M}} = \text{ev}_{\mathcal{M}}(\chi(g)) := (\gamma \otimes \tilde{g}): \mathcal{M} \to T^m \mathcal{M} \quad \text{for } \mathcal{M} \in D(X),$$

defines an element of $Z^m_{gr}(D(X))$, and that $\chi$ is a homomorphism of graded algebras.

This completes the proof of Theorem 3.3.4.

Remarks 3.3.8. (1) This characteristic homomorphism to the graded centre of the derived category has made its appearance in several forms for the affine situation of algebra homomorphisms, for example in [1, Sect. 3], [2], [27] or [11].

(2) For compact manifolds, Căldăraru, in [7,8], investigates higher order structure on the Hochschild cohomology of compact complex manifolds and notes the existence of the characteristic homomorphism in [7, 4.10]. He also points out that for the example of an elliptic curve $X$, the homomorphism $\chi_X$ is not injective: $\mathbb{H}^2_X(\mathcal{O}_X) \cong H^1(X, \mathcal{O}_X)$ is a one-dimensional vector space, but $Z^2_{gr}(D(X)) = 0$ as the category of quasicoherent sheaves on $X$ is hereditary, that is, of global dimension one.

4. Functoriality of the Hochschild complex

In this section we will show that the Hochschild complex is a well defined object of the derived category and that it behaves functorially with respect to morphisms of complex spaces. The idea of the proof is the same as the one for the cotangent complex given in [12]; however we will formalize the treatment as it seems useful as well in other situations, such as the one considered here. Moreover we give the full details of the (non trivial) proof that was left to the reader in [12].

4.1. Categories of models

Assume that we are given a commutative diagram of categories$^{11}$ and functors as indicated by the solid arrows:

$$\begin{array}{ccc}
\mathcal{M} & \xrightarrow{G} & \mathcal{F} \\
F \downarrow & & \downarrow \ U \\
\mathcal{C} & \longleftarrow \tilde{G} & \longleftarrow \mathcal{H} \\
& \phantom{F} & \phantom{\downarrow} \\
& \mathcal{C}. & \end{array}$$

In this subsection we will give a simple criterion as to when the given functor $G: \mathcal{M} \to \mathcal{F}$ can be factored through $F$ by a functor $\tilde{G}$, represented by the dotted arrow.

$^{11}$ We will henceforth only consider small categories.
To explain what we have in mind consider the following example.

Example 4.1.1. Let $\mathbf{C} = \mathbf{Mor}$ be the category of holomorphic mappings of complex spaces, where we write an object $X \to Y$ of $\mathbf{Mor}$ in brief as $X/Y$. Recall that the morphisms $f = (f_X, f_Y) : X'/Y' \to X/Y$ in $\mathbf{Mor}$ are given by commutative diagrams

$$
\begin{array}{ccc}
X' & \xrightarrow{f_X} & X \\
\downarrow & & \downarrow \\
Y' & \xrightarrow{f_Y} & Y.
\end{array}
$$

Furthermore, let $H : \mathbf{F} \to \mathbf{Mor}$ be the fibration in derived categories of complexes of modules that are bounded above so that the fibre of $\mathbf{F} \to \mathbf{Mor}$ over an object $X/Y$ is just the derived category $\mathbf{D}^{-}(X)$ and the fibre functor over a morphism $f = (f_X, f_Y) : X'/Y' \to X/Y$ is given by $Lf^* : \mathbf{D}(X) \to \mathbf{D}(X')$.

For the category of models $\mathbf{M}$, choose the category of extended resolvents of morphisms of complex spaces; we will define the morphisms in that category in 4.2.1. Let $F : \mathbf{M} \to \mathbf{C} = \mathbf{Mor}$ be the functor that assigns to the extended resolvent the map it resolves. Finally, for every extended resolvent $\mathcal{X} = (X_*, W_*, R_*, B_*)$ of $X/Y$ we have defined in 1.3 the Hochschild complex with respect to this extended resolvent $G(\mathcal{X}) = \mathbb{H}_X := \mathbb{C}^*(B_* \otimes_{S_*} \mathcal{O}_{X_*}|X_*)$,

and we need to know that this complex does not depend on the choice of the extended resolvent, equivalently, that there is a functor $G : \mathbf{Mor} \to \mathbf{F}$ that assigns to every morphism $X \to Y$ just one Hochschild complex $\mathbb{H}_{X/Y}$.

In the following, the objects of the category $\mathbf{M}$ over a given object $X$ of $\mathbf{C}$ will be called models of $X$. With this notation we introduce the following definition.

Definition 4.1.2. The category $\mathbf{M}$; or, more precisely, the functor $F : \mathbf{M} \to \mathbf{C}$; is called a category of models for $\mathbf{C}$ if the following conditions are satisfied.

1. Every object $X$ in $\mathbf{C}$ admits a model $\mathcal{X} \in \mathbf{M}$, that is, $F$ is surjective on objects.

2. Let $f : X \to Y$ be a morphism in $\mathbf{C}$. Given models $\mathcal{X}$ of $X$ and $\mathcal{Y}$ of $Y$, there is a model $\mathcal{X}'$ of $X$ together with a morphism $\mathcal{X}' \to \mathcal{X}$ over the identity of $X$ and a morphism $\mathcal{X}' \to \mathcal{Y}$ over $f$.

3. Let $f : X \to Y$ be a morphism in $\mathbf{C}$. If we are given two morphisms $f_1, f_2 : \mathcal{X} \to \mathcal{Y}$ over $f$ then we can find a model $\mathcal{X}'$ of $X$ together with commutative diagrams of models for $i = 1, 2$.

$$
\begin{array}{ccc}
\mathcal{X} & \xleftarrow{\text{id}} & \mathcal{X} \\
\downarrow & & \downarrow \\
\mathcal{X}' & \xleftarrow{f_1} & \mathcal{Y} \\
\downarrow & & \downarrow \\
\mathcal{X}' & \xrightarrow{f} & \mathcal{X}. \\
\end{array}
$$
4.1.3. Consider the set of morphisms \( \Sigma \) in \( M \) that lie over an isomorphism of \( C \), so that a morphism \( f \) of \( M \) belongs to \( \Sigma \) if and only if the underlying morphism \( f = F(f) \) in \( C \) is an isomorphism. With respect to this set, one can form the localized category \( M[\Sigma^{-1}] \) as in [14, 1.1]. Recall that the objects of \( M[\Sigma^{-1}] \) are just the objects of \( M \), whereas a morphism \( \mathcal{X} \to \mathcal{Y} \) in \( M[\Sigma^{-1}] \) consists of an equivalence class of a zigzag

\[
\begin{array}{ccccccc}
\mathcal{Y}_1 & \xrightarrow{\beta_1} & \mathcal{X}_1 & \xrightarrow{\alpha_1} & \mathcal{Y}_2 & \cdots & \xrightarrow{\beta_n} & \mathcal{X}_n \equiv \mathcal{Y}_n \\
\mathcal{X} = \mathcal{X}_0 & & & & & & & \mathcal{X}_n = \mathcal{Y} \\
\end{array}
\]

where \( F(\beta_i) \) is an isomorphism for \( i = 1, \ldots, n \). The construction comes with the obvious localization functor \( p : M \to M[\Sigma^{-1}] \) that is the identity on objects and considers each morphism \( f : \mathcal{X} \to \mathcal{Y} \) from \( M \) as the zigzag \( \mathcal{X} \xleftarrow{\text{id}} \mathcal{X} \xrightarrow{f} \mathcal{Y} \) in \( M[\Sigma^{-1}] \).

By the universal property of such localized categories; see [14, 1.1(ii)]; there is a canonical factorization of \( F \) through the localization, thus, a commutative diagram

\[
\begin{array}{ccc}
M & \xrightarrow{p} & M[\Sigma^{-1}] \\
\downarrow F & & \downarrow \tilde{F} \\
C & & \\
\end{array}
\]

where \( \tilde{F} : M[\Sigma^{-1}] \to C \) is the induced functor.

The main insight here is now contained in the following result.

**Theorem 4.1.4.** If \( M \) is a category of models for \( C \) then the functor \( \tilde{F} \) is an equivalence of categories.

For the proof we need the following lemma.

**Lemma 4.1.5.** (1) Assume given a model \( \mathcal{X} \) of \( X \), a model \( \mathcal{Y} \) of \( Y \), and a morphism \( f : X \to Y \) in \( C \). If \( f_1, f_2 : \mathcal{X} \to \mathcal{Y} \) are two morphisms over \( f \) then the localized morphisms \( p(f_1), p(f_2) : p(\mathcal{X}) \to p(\mathcal{Y}) \) in \( M[\Sigma^{-1}] \) are equal.

(2) Let \( h : p(\mathcal{X}) \to p(\mathcal{Y}) \) be a morphism in \( M[\Sigma^{-1}] \) over \( f : X \to Y \) in \( C \). There is then a model \( \mathcal{X}' \) of \( X \) and morphisms \( g : \mathcal{X}' \to \mathcal{X} \) over \( \text{id}_X \) and \( f : \mathcal{X}' \to \mathcal{Y} \) over \( f \) such that \( h \circ p(g) = p(f) \).

**Proof.** (1) If we choose a model \( \mathcal{X}' \) of \( X \) as in 4.1.2(3) then \( p(\pi) \) is an isomorphism in \( M[\Sigma^{-1}] \) whence \( p(g_1) = p(\pi)^{-1} = p(g_2) \) and then also \( p(f_1) = p(f_2) \).

For (2), it suffices to show by 4.1.2(2) that every morphism in \( M[\Sigma^{-1}] \) is represented by a zigzag (6) of length \( n = 1 \). Assume that the morphism \( \tilde{h} \) is represented by a zigzag as in (6), of minimal length \( n \geq 1 \). If \( n > 1 \) then we can find a model \( \mathcal{Y}'_{n-1} \) of \( Y_{n-1} := F(\mathcal{Y})_{n-1} \) together with
two morphisms $g_{n-1} : \mathcal{Y}_{n-1}' \to \mathcal{Y}_{n-1}$ over $\text{id}_{\mathcal{Y}_{n-1}}$ and $g_n : \mathcal{Y}_{n-1}' \to \mathcal{Y}_n$ over $F(\beta_n)^{-1} \circ F(\alpha_{n-1})$, that is, the diagram

$$
\begin{array}{ccc}
F(\mathcal{Y}_n) & \xrightarrow{F(g)} & F(\mathcal{Y}_n) \\
\downarrow & & \downarrow \\
F(\mathcal{Y}_{n-1}) & \xrightarrow{F(\alpha_{n-1})} & F(\mathcal{X}_{n-1})
\end{array}
$$

commutes. Using part (1), the diagram

$$
\begin{array}{ccc}
p(\mathcal{Y}_n) & \xrightarrow{p(g)} & p(\mathcal{Y}_n) \\
\downarrow & & \downarrow \\
p(\mathcal{Y}_{n-1}) & \xrightarrow{p(\alpha_{n-1})} & p(\mathcal{X}_{n-1})
\end{array}
$$

already commutes in $M[\Sigma^{-1}]$, whence $\bar{h}$ is represented by the shorter zigzag

$$
\begin{array}{ccccccc}
\beta_1 & \mathcal{Y}_1 & \alpha_1 & \mathcal{Y}_2 & \cdots & \mathcal{Y}_{n-2} & \beta'_{n-1} & \mathcal{Y}_{n-1}' \\
\downarrow & \xrightarrow{\ell_1} & \downarrow & \xrightarrow{\ell_2} & \cdots & \downarrow & \xrightarrow{\ell_{n-2}} & \downarrow & \xrightarrow{\ell_{n-1}} \\
\mathcal{X} = \mathcal{X}_0 & \xrightarrow{\mathcal{X}_1} & \cdots & \xrightarrow{\mathcal{X}_{n-2}} & \mathcal{X}_{n-2} & \xrightarrow{\mathcal{X}_n = \mathcal{Y}_n}
\end{array}
$$

where $\beta'_{n-1} := \beta_{n-1} \circ g_{n-1}$ and $\alpha'_{n-1} := \alpha_n \circ g_n$. □

**Proof of Theorem 4.1.4.** As $F$ is surjective on objects, so is $\bar{F}$. In order to show that $\bar{F}$ is fully faithful, let $\mathcal{X}, \mathcal{Y}$ be objects of $M$ over $X$, respectively $Y$, and consider the maps

$$
\text{Mor}_C(\mathcal{X}, \mathcal{Y}) \xrightarrow{p} \text{Mor}_{M[\Sigma^{-1}]}(p(\mathcal{X}), p(\mathcal{Y})) \xrightarrow{\bar{F}} \text{Mor}_C(X, Y).
$$

We need to show that the second map, labeled $\bar{F}$ is bijective. We first show that it is surjective. In view of condition 4.1.2(2), for a given morphism $f : X \to Y$, we can find a model $\bar{X}'$ of $X$ together with morphisms $\gamma : \bar{X}' \to \bar{X}$ over $\text{id}_X$ and $\bar{f} : \bar{X}' \to \mathcal{Y}$ over $f$, whence the morphism $p(\gamma) \circ p(\bar{f})^{-1} \in \text{Mor}_{M[\Sigma^{-1}]}(p(\bar{X}), p(\mathcal{Y}))$ maps to $f$ under $\bar{F}$.

To prove the injectivity of the second map in (*), let $\bar{f}_1, \bar{f}_2 : p(\mathcal{X}) \to p(\mathcal{Y})$ be morphisms over the same morphism $f : X \to Y$. By 4.1.5(2), $\bar{f}_1$ and $\bar{f}_2$ can be represented by zigzags of length 1,
where \( f_i \), for \( i = 1, 2 \), is a morphism over \( f \) and \( g_i \) is a morphism over \( \text{id}_X \) (in these diagrams we use the convention that the solid arrows are morphisms in \( \mathcal{M} \), whereas the dotted ones are only morphisms in \( \mathcal{M}[\Sigma^{-1}] \)). By condition 4.1.2(2), we can find a model \( \tilde{X} \) and morphisms \( \tilde{h}_i : \tilde{X} \to X_i \), \( i = 1, 2 \), over \( \text{id}_X \). By construction, we have then

\[
F(g_1 \circ \tilde{h}_1) = F(g_2 \circ \tilde{h}_2) \quad \text{and} \quad F(f_1 \circ \tilde{h}_1) = F(f_2 \circ \tilde{h}_2).
\]

In view of 4.1.5(1), this implies

\[
p(g_1 \circ \tilde{h}_1) = p(g_2 \circ \tilde{h}_2) \quad \text{and} \quad p(f_1 \circ \tilde{h}_1) = p(f_2 \circ \tilde{h}_2)
\]

in \( \mathcal{M}[\Sigma^{-1}] \). It follows now from

\[
\tilde{f}_i = p(f_i) \circ p(g_i)^{-1} = p(f_i \circ \tilde{h}_i) \circ p(g_i \circ \tilde{h}_i)^{-1}, \quad i = 1, 2,
\]

that \( \tilde{f}_1 = \tilde{f}_2 \) as we had to show. \( \square \)

Returning to the problem discussed at the beginning of this section we get the following application.

**Corollary 4.1.6.** Let \( F : \mathcal{M} \to \mathcal{C} \) be a category of models for \( \mathcal{C} \) and assume that \( G : \mathcal{M} \to \mathcal{F} \) is a functor such that for every morphism \( f : X \to Y \) over an isomorphism \( f : X \to Y \) in \( \mathcal{C} \), the morphism \( G(f) : G(X) \to G(Y) \) is also an isomorphism. Then there is a unique functor \( \tilde{G} : \mathcal{C} \to \mathcal{F} \) such that \( G = \tilde{G} \circ F \).

The proof follows immediately from the main result 4.1.4 above and the universal property of localizations; see [14, 1.1(b)].

### 4.2. Resolvents as a category of models

In a first step we introduce morphisms of resolvents.

**Definition 4.2.1.** Assume we are given a morphism \( f = (f_X, f_Y) : X'/Y' \to X/Y \) in \( \text{Mor} \) and extended resolvents

\[
\mathcal{X}^{(e)} = (X'_*, W'_*, R'_*, B'_*) \text{ of } X'/Y' \quad \text{and} \quad \mathcal{X}^{(e)} = (X_*, W_*, R_*, B_*) \text{ of } X/Y.
\]

A morphism \( f : \mathcal{X}^{(e)} \to \mathcal{X}^{(e)} \) over \( f \) consists of the following data.

1. A map \( \sigma : I' \to I \) with \( f_X(X'_i) \subseteq X_{\sigma(i)} \); this induces a map, denoted by the same symbol, \( \sigma : A' \to A \) of simplicial sets, and a morphism \( f_{X_*} : X'_* \to X_* \) of simplicial spaces.
2. A (simplicial) morphism \( f_{W_*} : W'_* \to W_* \) restricting to \( f_{X_*} \) on \( X_* \) such that the diagram

\[
\begin{array}{ccc}
W'_* & \xrightarrow{f_{W_*}} & W_* \\
\downarrow & & \downarrow \\
Y' & \xrightarrow{f_{X_*}} & Y
\end{array}
\]

commutes.
(3) A morphism of $\mathcal{O}_{W^e}$-algebras
\[ f^*(R^e) := f^*_{W^e}(R^e) \rightarrow R'^e \]
compatible with the projections to $\mathcal{O}_{X^e}$.

(4) With $S^e_*$ and $S^e_*$ as in 1.1.3 and denoting the pullback of an $\mathcal{O}_{W^e \times Y_{W^e}}$-algebra $D_*$ under $f_{W^e} \times f_{W^e} : W^e \times Y W^e \rightarrow W^e \times Y W^e$ by $f^*(D_*)$, there is given a morphism $f^*(B^e_*) \rightarrow B'^e_*$ such that the diagram
\[
\begin{array}{ccc}
\{f^*(S^e_*) & \rightarrow & f^*(B^e_*), \\
\downarrow & & \downarrow \\
S^e_* & \rightarrow & B'^e_* \end{array}
\]
commutes, where the first vertical arrow is the tensor product of the morphism $f^*(R^e_*) \rightarrow R'^e_*$ in (3) with itself.

One can compose morphisms of extended resolvents in an obvious way, and the identity is a morphism of extended resolvents. Thus the extended resolvents form a category over the category $\text{Mor}$ of morphisms of complex spaces.

In a similar way one can also form the category of resolvents; morphisms of resolvents will be given by the data (1)–(3) in 4.2.1.

**Proposition 4.2.2.** The category of (extended) resolvents constitutes a model category over $\text{Mor}$ in the sense of 4.1.2.

For the proof we need a few preparations. Let $\mathfrak{X}$ be a resolvent of $X/Y$ and $S_*$ a DG $\mathcal{O}_{W_0}$-algebra that, according to our conventions, is assumed to be concentrated in degrees $\leq 0$ and to have coherent homogeneous components. By Proposition 2.19 in [5], the category of those $S_*$-modules that are bounded above with coherent cohomology has enough projectives. More precisely, for a simplex $\alpha$ and an $S_\alpha$-module $P_\alpha$, one can form the module $p^*_{\alpha}(P_\alpha)$ defined by $p^*_{\alpha}(P_\alpha)_{\beta} := 0$, if $\alpha \nsubseteq \beta$, and $p^*_{\alpha}(P_\alpha)_{\beta} := p^*_{\alpha\beta}(P_\alpha) \otimes p^*_{\alpha\beta}(S_\beta)$ otherwise, where $p_{\alpha\beta} : W_\beta \rightarrow W_\alpha$ are the transition maps. This is a simplicial module with respect to the obvious transition maps, and it is projective if $P_\alpha$ is projective. Moreover, as is shown in [5, 2.13(1)], every projective module over $S_*$ is a direct sum of such modules. By definition, the graded free modules over $S_*$ are those that admit a direct sum decomposition $P_\alpha = \bigoplus_{\alpha} p^*_{\alpha}(P_\alpha)$, where each $P_\alpha$ is graded free over $S_\alpha$. We need the following observation.

**Lemma 4.2.3.** Let $\mathfrak{Y}' \rightarrow \mathfrak{X}$ be a morphism of resolvents and assume that $S_*$ is a DG $\mathcal{O}_{W_0}$-algebra. If, with the notation of 4.2.1, the map $\sigma : I' \rightarrow I$ is injective then the following hold.

1. For every projective (graded free) $S_*$-module $P_\alpha$, its pull back $f^*(P_\alpha)$ is again projective (graded free) over $f^*(S_*)$.
2. For every graded free $S_*$-algebra $T_\alpha$, its pull back $f^*(T_\alpha)$ is again a graded free algebra over $f^*(S_*)$. 
Proof. (2) is an immediate consequence of (1). For the proof of (1), we may assume, in view of the structure theorem for projective modules mentioned above, that \( P_e \cong p^*_e(P) \) for some projective (respectively graded free) module over \( S_e \). If now \( \alpha \not\subseteq \sigma (I') \), then \( f^*(P_e) = 0 \) and the assertion holds trivially true. Otherwise \( \alpha = \sigma (\alpha') \) for a unique simplex \( \alpha' \) of \( A' \), and then \( f^*(P_e) \cong p^*_e(f^*(P)) \), whence this module is projective, respectively graded free over \( f^*(S_e) \).

\[ \square \]

We remark that the lemma is in general no longer true if the map \( \sigma \) fails to be injective.

Proof of 4.2.2. We have to verify that the conditions (1)–(3) in 4.1.2 are satisfied. The existence of resolvents follows from [5, 2.34], and the existence of extended ones from the discussion in 1.3. In order to deduce (2), let \( f = (f_X, f_Y) : X'/Y' \to X/Y \) be a morphism in \( \text{Mor} \) and let \( X^{(e)} = (X'_*, W'_*, R'_*, B'_*) \) and \( X^{(e)} = (X_*, W_*, R_*, B_*) \) be extended resolvents of \( X'/Y' \), respectively \( X/Y \). The sets

\[ \tilde{X}'_k := X'_j \cap f^{-1}_X(X_i), \quad k := (i, j) \in \tilde{I}' := I \times I', \]

form again a covering of \( X' \) by Stein compact sets with associated nerve, say \( \tilde{X}' := (\tilde{X}'_\alpha)_{\alpha \in \tilde{A}'} \). The projections \( p : \tilde{I}' \to I' \) and \( q : \tilde{I}' \to I \) provide maps of simplicial schemes, again denoted \( p : \tilde{A}' \to A' \) and \( q : \tilde{A}' \to A \), and \( \tilde{X}'_\alpha \) can be embedded diagonally into the simplicial space \( \tilde{W}'_\alpha \) with

\[ \tilde{W}'_\alpha := W'_p(\alpha) \times_Y W'_q(\alpha), \quad \alpha \in \tilde{A}', \]

that is smooth over \( Y' \). The projections \( \pi : \tilde{W}'_\alpha \to W'_* \) and \( \pi' : \tilde{W}'_\alpha \to W'_* \) are then morphisms of simplicial spaces that restrict to the morphisms \( f_X \), respectively id, on \( \tilde{X}'_\alpha \) for every simplex \( \alpha \in \tilde{A}' \). We can now choose a free algebra resolution \( \tilde{R}'_* \) of the induced algebra homomorphism

\[ \pi'^*(R'_*) \otimes_{O_{W'_*}} \pi^*(R_*) \to O_{\tilde{X}'_*}. \]

The construction so far provides a resolvent \( \tilde{\chi}' := (\tilde{X}'_*, \tilde{W}'_*, \tilde{R}'_*) \) of \( X'/Y' \) together with morphisms of resolvents

\[ \tilde{\chi}' : \tilde{\chi}' := (\tilde{X}'_*, \tilde{W}'_*, \tilde{R}'_*) \to X' = (X'_*, W'_*, R'_*) \]

lying over id and \( f \), respectively. To construct as well a morphism of extended resolvents we note that \( \tilde{\chi}' \) and \( \tilde{\chi} \) induce algebra homomorphisms

\[ \tilde{\chi}'^*(S'_e) \to \tilde{S}'_e \quad \text{and} \quad \tilde{\chi}^*(S_e) \to \tilde{S}'_e \]

on \( \tilde{W}'_* \times_Y \tilde{W}'_* \) compatible with the projections onto \( \tilde{R}'_* \), where \( \tilde{S}'_*, S'_e, S_e \) are as explained in 1.1.3. Hence, if \( \tilde{B}'_* \) is a free algebra resolution of

\[ D_* := \tilde{\chi}'^*(B'_*) \otimes_{\tilde{\chi}'^*(S'_e)} \tilde{S}'_e \otimes_{\tilde{\chi}^*(S_e)} \tilde{S}'_e, \]

we have

\[ \tilde{\chi}'^*(D'_*) \to \tilde{R}'_* \]

as a distinguished triangle in \( \text{DGC} \).
on $\tilde{W}'_*$, then there are induced maps $\tilde{f}'(B'_*) \to \tilde{B}'_*$ and $f^*(B_*) \to \tilde{B}'_*$ as required in 4.1.2(4), and so $\tilde{f}'$ and $f$ extend to morphisms of extended resolvents $\tilde{\mathcal{X}}^{(e)} := (\tilde{X}'_*, \tilde{W}'_*, \tilde{R}'_*, \tilde{B}'_*) \to \mathcal{X}^{(e)}$ and $\mathcal{X}^{(e)} \to \mathcal{X}^{(e)}$, proving (2).

In order to show that 4.1.2(3) is satisfied, let $f_i : \mathcal{X}^{(e)} \to \tilde{\mathcal{X}}^{(e)}$, $i = 1, 2$, be two morphisms over $f$ and let $f_{W_*} : W_* \to W_*$ be the associated morphisms of smoothings. With $\tilde{\mathcal{X}}^{(e)}$ as before, the maps $f_i$ give rise to morphisms of simplicial spaces

$$g_{ni} := (id, f_{W_*}) : W_* \to \tilde{W}'_* = W'_* \times Y W_*$$

and to morphisms of simplicial algebras

$$g_{ni}^* \left( \pi'^*(R'_*) \otimes \mathcal{O}_{\tilde{W}'_*} \otimes \pi^*(R_*) \right) \cong \mathcal{R}'_* \otimes \mathcal{O}_{W_*} f_i^*(R_*) \to \mathcal{R}'_*. \quad (*)$$

The underlying map of simplicial schemes $A' \to \tilde{A}'$ is the graph of a map and thus is injective. Applying 4.2.3, $g_{ni}^* (\tilde{R}'_*)$ is a free algebra over the left-hand side of $(*)$ and so the morphism in $(*)$ extends to a morphism $g_{ni}^*(\tilde{R}'_*) \to \mathcal{R}'_*$. Thus we have constructed morphisms $g_i : \mathcal{X}' \to \tilde{\mathcal{X}}'$, $i = 1, 2$, of resolvents as required in 4.1.2(3).

To construct a well morphism of extended resolvents, again denoted $g_i : \mathcal{X}^{(e)} \to \tilde{\mathcal{X}}^{(e)}$, for $i = 1, 2$, we note that the data so far constructed provide for $i = 1, 2$ a diagram in solid arrows

$$\begin{array}{ccc}
B'_* \otimes \mathcal{S}'_* & \overset{g_i^*(\tilde{S}'_*) \otimes f_i^*(\mathcal{S}_*)}{\longrightarrow} & \mathcal{B}'_* \\
\downarrow & & \downarrow \\
g_i^*(\tilde{B}'_*) & \longrightarrow & \mathcal{R}'_*.
\end{array}$$

Applying the same argument as before, $g_i^*(\tilde{B}'_*)$ is a free algebra over $g_i^*(\mathcal{D}_*)$, whence there is a morphism of DG algebras $g_i^*(\tilde{B}'_*) \to B'_*$ as indicated by the dotted arrow, so that the resulting diagram is still commutative. This gives the required morphisms of extended resolvents. □

We are now able to deduce that Hochschild cohomology is well defined and functorial as claimed earlier in 1.3.3.

**Theorem 4.2.4.** (1) Assigning to an extended resolvent $f : \mathcal{X}^{(e)} = (X_*, W_*, \mathcal{R}_*, B_*)$ the Hochschild complex $\mathbb{H}_{\mathcal{X}^{(e)}} := \mathcal{C}^*(B_* \otimes \mathcal{S}_* \mathcal{O}_{X_*}) \in D(X)$ defines a functor on the category of extended resolvents. If $\mathcal{X}^{(e)} \to \mathcal{X}^{(e)}$ is a morphism of extended resolvents over an isomorphism $f = (f_X, f_Y) : X'/Y' \to X/Y$ then the induced morphism $f^*(\mathbb{H}_{\mathcal{X}^{(e)}}) \to \mathbb{H}_{\mathcal{X}^{(e)}}$ is a quasiisomorphism.

(2) The Hochschild complex $\mathbb{H}_{X/Y}$ is well defined and functorial in $X/Y$, that is, every diagram of complex spaces as displayed in 4.1.1 induces a well defined and functorial morphism of algebra objects $L^f^* (\mathbb{H}_{X/Y}) \to \mathbb{H}_{X'/Y'}$.

**Proof.** (2) is a consequence of (1) and 4.1.6 applied to the category $\mathcal{M}$ of extended resolvents. To show (1), let $f^*(B_*) \to B'_*$ be the morphism of DG algebras belonging to $f$, see 4.2.1(4). This
are topologically reduced. Let \( \sigma \) on \( X \) yields first a morphism of complexes \( f^*_X(B_\ast \otimes S_\ast \mathcal{O}_{X_\ast}) \to B'_\ast \otimes S'_\ast \mathcal{O}_{X'_\ast} \) on \( X'_\ast \), and then, after applying the \( \check{\text{C}} \)ech functor, a morphism of algebra objects

\[
f^*(\mathbb{H}_X) = f^*(\mathcal{C}^*(B_\ast \otimes S_\ast \mathcal{O}_{X_\ast})) \to \mathcal{C}^*(f^*_X(B_\ast \otimes S_\ast \mathcal{O}_{X_\ast})) \to \mathcal{C}^*(B'_\ast \otimes S'_\ast \mathcal{O}_{X'_\ast}) = \mathbb{H}_{X'}
\]
on \( X' \). As \( \mathbb{H}_X \) is a complex of flat \( \mathcal{O}_X \)-modules, the term on the left represents \( Lf^*(\mathbb{H}_X) \). This construction is compatible with compositions and transforms the identity into the identity, thus proving functoriality of \( \mathcal{X}(e) \mapsto \mathbb{H}_{X'(e)} \).

The second part of (1) can be deduced with the same reasoning as in 1.3.4(2); we leave the details to the reader. \( \square \)

**Corollary 4.2.5.** For every \( \mathcal{O}_X \)-module \( \mathcal{M} \) there are natural maps

\[
\mathbb{H}\mathbb{H}_{X/Y}^X(M) \to \mathbb{H}\mathbb{H}_{X'/Y'}^{X'}(Lf^*\mathcal{M}).
\]

The map \( \mathbb{H}\mathbb{H}_{X/Y}^X(\mathcal{O}_X) \to \mathbb{H}\mathbb{H}_{X'/Y'}^{X'}(\mathcal{O}_{X'}) \) is compatible with the algebra structures on both sides.

The following fact was already used in Section 2, see 2.2.5.

**Proposition 4.2.6.** The graded algebra structure on \( \mathbb{H}\mathbb{H}_{X_Y}^X(\mathcal{O}_X) \) is independent of the choice of the extended resolvent, and, for every morphism \( f = (\text{id}_X, f_Y) : X/Y' \to X/Y \), the induced map \( \mathbb{H}\mathbb{H}_{X/Y}^X(\mathcal{O}_X) \to \mathbb{H}\mathbb{H}_{X'/Y'}^{X'}(\mathcal{O}_{X'}) \) is a homomorphism of algebras.

Before giving the proof we make following preparation.

4.2.7. Let \( \mathcal{X}(e) = (X_\ast, W_\ast, R_\ast, B_\ast) \) be an extended resolvent of a morphism of complex spaces \( X/Y \). Restricting the sheaf \( \mathcal{O}_{W_\ast} \) to \( X_\ast \) topologically gives a new smoothing \( W_\ast^\tau = (|X_\ast|, \mathcal{O}_{W_r}|X_\ast) \), and the restrictions \( R^\tau_\ast := R_\ast|X_\ast \) and \( B^\tau_\ast := B_\ast|X_\ast \) resolve \( \mathcal{O}_{X_\ast} \) again as locally free DG algebras over \( \mathcal{O}_{W^\tau_\ast} \), respectively \( S^\tau_\ast = S_\ast|X_\ast \). In the following we will call \( \mathcal{X}(e)^{\tau} = (X_\ast, \mathcal{O}_{W^\tau_\ast}, R^\tau_\ast, B^\tau_\ast) \) in brief the topological reduction of \( \mathcal{X} \), and we will call \( \mathcal{X}(e)^{\tau} \) topologically reduced if \( \mathcal{X}(e) = \mathcal{X}(e)^{\tau} \). Obviously, forming the topological reduction is compatible with morphisms of resolvents. The natural morphism

\[
\operatorname{Ext}^\ast_{\mathcal{S}_\ast}(\mathcal{O}_{X_\ast}, \mathcal{O}_{X'_\ast}) \to \operatorname{Ext}^\ast_{\mathcal{S}_\ast^\tau}(\mathcal{O}_{X_\ast}, \mathcal{O}_{X'_\ast})
\]
is compatible with the algebra structures on both sides and, furthermore, it is an isomorphism as both sides represent the Hochschild cohomology \( \mathbb{H}\mathbb{H}_{X/Y}^X(\mathcal{O}_X) \) in view of 2.2.1(1).

**Proof of 4.2.6.** Let \( \check{f} : \mathcal{X}'(e) = (X'_\ast, W'_\ast, R'_\ast, B'_\ast) \to \mathcal{X}(e) = (X_\ast, W_\ast, R_\ast, B_\ast) \) be a morphism of resolvents over \( X/Y' \to X/Y \). By the preceding remark, we may assume that \( \mathcal{X}'(e) \) and \( \mathcal{X}(e) \) are topologically reduced. Let \( \sigma : A' \to A \), as in 4.2.1, be the map of simplices associated to the simplicial morphism \( X'_\ast = (X'_\alpha)_{\alpha \in A'} \to X_\ast = (X_\beta)_{\beta \in A} \). Forming the topological restrictions

\[
\mathcal{O}_{W_\alpha} := \mathcal{O}_{W_{\sigma(\alpha)}}|X'_\alpha, \quad \check{R}_\alpha := \mathcal{R}_{\sigma(\alpha)}|X'_\alpha, \quad \check{S}_\alpha := \mathcal{S}_{\sigma(\alpha)}|X'_\alpha \quad \text{and} \quad \check{B}_\alpha := \mathcal{B}_{\alpha}|X'_\alpha
\]
gives a new extended resolvent $\tilde{\mathfrak{X}}^{(e)} := (X'_s, \mathcal{O}_{W_s}, \tilde{R}_s, \tilde{B}_s)$ of $X/Y$. Using the fact that $\mathfrak{X}^{(e)}$ is topologically reduced, there is a factorization

$$\mathfrak{X}^{(e)} \rightarrow \tilde{\mathfrak{X}}^{(e)} \rightarrow \mathfrak{X}^{(e)}.$$  

Restricting scalars to $\tilde{S}_s$ gives a morphism of Ext-algebras

$$\text{HH}_X^{\bullet}(\mathcal{O}_X) \cong \text{Ext}_{\tilde{S}_s}^\bullet(\mathcal{O}_{X'_s}, \mathcal{O}_{X'_s}) \rightarrow \text{Ext}_{\tilde{S}_s}^\bullet(\mathcal{O}_{X'_s}, \mathcal{O}_{X'_s}) \cong \text{HH}_X^{\bullet}(\mathcal{O}_X)$$

as desired. $\square$

**Remark 4.2.8.** (1) Let $X/Y$ be a morphism of complex spaces and $\mathfrak{X}$ as before a resolvent. With the same arguments as in 4.2.4 it follows that the cotangent complex

$$\mathbb{L}_{X/Y} := C^*\left(\Omega^1_{R_s/Y} \otimes_{R_s} \mathcal{O}_{X_s}\right)$$

is a well defined object of the derived category $D(X)$ and that it is functorial with respect to morphisms.

(2) Categories of models can be used to derive (additive or non-additive) functors. Let $\mathbf{A}$ be an abelian category with enough projectives and let $\mathbf{M}$ be the category of all pairs $(P, M)$, where $P$ is a projective resolution of $M \in \mathbf{A}$, that is, there is an exact sequence $\cdots \rightarrow P^i \rightarrow P^{i+1} \rightarrow \cdots \rightarrow P^0 \rightarrow M \rightarrow 0$ with each $P^i$ projective. Morphisms of such resolutions are defined in the usual way. The reader may easily verify that the functor of opposite categories $F : \mathbf{M}^c \rightarrow \mathbf{A}^c$ with $(P, M) \mapsto M$ is a model category in the sense of 4.1.2. Let now $T : \mathbf{M} \rightarrow \mathbf{F}$ be a (not necessarily additive!) functor into an arbitrary category $\mathbf{F}$ such that for $f : (P, M) \rightarrow (Q, N)$ the morphism $T(f)$ is an isomorphism whenever the morphism $M \rightarrow N$ induced by $f$ is an isomorphism. Applying 4.1.6, such a functor admits a factorization $\tilde{T} : \mathbf{A} \rightarrow \mathbf{F}$.

For instance, if $T(P, M) := H^i(\text{Hom}(P, N))$ with a fixed object $N \in \mathbf{A}$ then $\tilde{T}(M)$ is just the group $\text{Ext}^i(M, N)$. Similarly, if there is an internal tensor product $\otimes$ on $\mathbf{A}$ with the usual properties and $T(P, M) := H^i(P \otimes N)$ then we recover the functors $\text{Tor}^i(M, N)$ from this construction.

(3) Let us consider the category $\mathbf{M}$ whose objects are all quadruples $\mathfrak{X} = (X_s, \mathcal{P}_s, X, \mathcal{M})$, where $X$ is a complex space, $X_s$ is the simplicial space associated to a locally finite covering of $X$ by Stein compact sets, $\mathcal{M}$ is a complex of $\mathcal{O}_X$-modules with coherent cohomology bounded above, and $\mathcal{P}_s$ is a $\mathcal{O}_{X_s}$-projective approximation, see [5, 2.19] of the complex of $\mathcal{O}_{X_s}$-modules $\mathcal{M}_s$ associated to $\mathcal{M}$. With the morphisms in this category defined analogous to the case of resolvents, the reader may easily verify that this is indeed a category of models for the category $\mathbf{C}$ of pairs $(X, \mathcal{M})$, where $X$ is a complex spaces and $\mathcal{M}$ is a complex of $\mathcal{O}_X$-modules with coherent cohomology that is bounded above.

Using this category of models, it is then possible to define derived symmetric and alternating powers, or even more generally arbitrary derived Schur functors, of complexes $\mathcal{M}$ as above on a complex space $X$, and to establish the functoriality of this construction; see also [9] and [19, I.4]. Let us sketch the construction in case of symmetric powers. Given a quadruple $\mathfrak{X}$ as above, set $S^p_{\mathfrak{X}} := C^*(S^p(\mathcal{P}_s))$, where $S^p(\mathcal{P}_s)$ denotes the $p$th symmetric power of the DG $\mathcal{O}_X$-module $\mathcal{P}_s$. This construction is clearly functorial in $\mathfrak{X}$, and any morphism $\mathfrak{X}' \rightarrow \mathfrak{X}$ induces a quasiisomorphism $f^* : S^p_{\mathfrak{X}} \rightarrow S^p_{\mathfrak{X}'}$, whenever the underlying map, say $(f, \psi)$, of pairs $(X', \mathcal{M}') \rightarrow (X, \mathcal{M})$ is an isomorphism. Applying 4.1.6, it follows that there are well defined derived symmetric powers
\( S^p(M) \in D^-(X) \) for any \( M \in D^-(X) \), functorial in \( M \), and even functorial with respect to morphisms of complex spaces.

References

[1] L.L. Avramov, R.-O. Buchweitz, Homological algebra modulo a regular sequence with special attention to codimension two, J. Algebra 230 (2000) 24–67.
[2] L.L. Avramov, Li-Chuan Sun, Cohomology operators defined by a deformation, J. Algebra 204 (2) (1998) 684–710.
[3] J. Bingener, Lokale Modulräume in der analytischen Geometrie, vols. 1 and 2, Aspects Math., Friedr. Vieweg & Sohn, Braunschweig, 1987, with the cooperation of S. Kosarew.
[4] N. Bourbaki, Algèbre, Chapitre 10, Masson, Paris, 1980.
[5] R.-O. Buchweitz, H. Flenner, A semiregularity map for modules and applications to deformations, Compos. Math. 137 (2003) 135–210.
[6] R.-O. Buchweitz, H. Flenner, Power series rings and projectivity, Manuscripta Math. 11 (2006) 107–114.
[7] A. Căldăraru, The Mukai pairing, I: The Hochschild structure, preprint, 32 pp., available at: http://arxiv.org/abs/math.AG/0308079, 2003.
[8] A. Căldăraru, The Mukai pairing, II: The Hochschild–Kostant–Rosenberg isomorphism, Adv. Math. 194 (1) (2005) 34–66.
[9] A. Dold, D. Puppe, Homologie nicht-additiver Funktoren. Anwendungen, Ann. Inst. Fourier 11 (1961) 201–312.
[10] D. Eisenbud, Homological algebra on a complete intersection, with an application to group representations, Trans. Amer. Math. Soc. 260 (1980) 35–64.
[11] K. Erdmann, M. Holloway, N. Snashall, R. Solberg, Ø. Taillefer, Support varieties for selfinjective algebras, K-Theory 33 (1) (2004) 67–87.
[12] H. Flenner, Deformationen holomorpher Abbildungen, Habilitationsschrift Osnabrück, 1978.
[13] J. Frisch, Points de platitude d’un morphisme d’espaces analytiques complexes, Invent. Math. 4 (1967) 118–138.
[14] P. Gabriel, M. Zisman, Calculus of Fractions and Homotopy Theory, Ergeb. Math. Grenzgeb. (neue Serie), vol. 35, Springer-Verlag, Berlin–Heidelberg–New York, 1967.
[15] M. Gerstenhaber, The cohomology structure of an associative ring, Ann. of Math. (2) 78 (1963) 267–288.
[16] M. Gerstenhaber, On the deformation of rings and algebras. III, Ann. of Math. (2) 88 (1968) 1–34.
[17] M. Gerstenhaber, S.D. Schack, Algebraic cohomology and deformation theory, in: M. Hazewinkel, M. Gerstenhaber (Eds.), Deformation Theory of Algebras and Structures and Applications, Kluwer, Dordrecht, 1988, pp. 11–264.
[18] R. Godement, Théorie des faisceaux, Hermann, Paris, 1964.
[19] L. Illusie, Complexe cotangent et déformations I, Lecture Notes in Math., vol. 239, Springer-Verlag, Berlin–Heidelberg–New York, 1971.
[20] S. MacLane, Homology, Grundlehren Math., vol. 114, Springer-Verlag, Berlin–Heidelberg–New York, 1963.
[21] V.P. Palamodov, Deformations of complex spaces, Russian Math. Surveys 31 (3) (1976) 129–197.
[22] D. Quillen, On the (co-)homology of commutative rings, in: Applications of Categorical Algebra, New York, 1968, in: Proc. Sympos. Pure Math., vol. XVII, Amer. Math. Soc., Providence, RI, 1970, pp. 65–87.
[23] D. Quillen, Cyclic cohomology and algebra extensions, K-Theory 3 (1989) 205–246.
[24] F. Schuhmacher, Hochschild cohomology of complex spaces and Noetherian schemes, Homology Homotopy Appl. 6 (1) (2004) 299–340.
[25] S. Schwede, An exact sequence interpretation of the Lie bracket in Hochschild cohomology, J. Reine Angew. Math. 498 (1998) 153–172.
[26] J. Shamash, The Poincaré series of a local ring, J. Algebra 12 (1969) 453–470.
[27] N. Snashall, Ø. Solberg, Support varieties and Hochschild cohomology rings, Proc. London Math. Soc. (3) 88 (3) (2004) 705–732.
[28] N. Spaltenstein, Resolutions of unbounded complexes, Compos. Math. 65 (2) (1988) 121–154.
[29] M. Suarez-Alvarez, The Hilton–Heckmann argument for the anticommutativity of cup products, Proc. Amer. Math. Soc. 132 (2004) 2241–2246 (electronic).
[30] R.G. Swan, Hochschild cohomology of quasiprojective schemes, J. Pure Appl. Algebra 110 (1996) 57–80.
[31] J. Tate, Homology of Noetherian rings and local rings, Illinois J. Math. 1 (1957) 14–27.
[32] K. Wolffhardt, Zur lokalen Hochschild-Homologie eines komplexen Raumes, Manuscripta Math. 37 (1982) 27–47.
Further reading

[33] M. Gerstenhaber, S.D. Schack, A Hodge type decomposition for commutative algebra cohomology, J. Pure Appl. Algebra 48 (1987) 229–247.
[34] J.-L. Loday, Cyclic Homology, Grundlehren Math. Wiss., vol. 301, Springer-Verlag, Berlin–Heidelberg–New York, 1992.
[35] S. MacLane, Categories for the Working Mathematician, Grad. Texts in Math., vol. 5, Springer-Verlag, Berlin–Heidelberg–New York, 1971.
[36] D. Quillen, Homotopical Algebra, Lecture Notes in Math., vol. 43, Springer-Verlag, Berlin–Heidelberg–New York, 1967.
[37] D. Quillen, Rational homotopy theory, Ann. of Math. 90 (1969) 205–295.
[38] J.L. Verdier, Des catégories dérivées des catégories abéliennes, Astérisque 239 (1997).