Adaptive optimal estimation of irregular mean and covariance functions

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Abstract

We propose straightforward nonparametric estimators for the mean and the covariance functions of functional data. Our setup covers a wide range of practical situations. The random trajectories are, not necessarily differentiable, have unknown regularity, and are measured with error at discrete design points. The measurement error could be heteroscedastic. The design points could be either randomly drawn or common for all curves. The definition of our nonparametric estimators depends on the local regularity of the stochastic process generating the functional data. We first propose a simple estimator of this local regularity which takes strength from the replication and regularization features of functional data. Next, we use the “smoothing first, then estimate” approach for the mean and the covariance functions. The new nonparametric estimators achieve optimal rates of convergence. They can be applied with both sparsely or densely sampled curves, are easy to calculate and to update, and perform well in simulations. Simulations built upon a real data example on household power consumption illustrate the effectiveness of the new approach.

Key words: Functional data analysis; Hölder exponent; Local polynomials; Minimax optimality

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1 Introduction

Motivated by a large number of applications, there is a great interest in models for observation entities in the form of a sequence of measurements recorded intermittently at several discrete points in time. Functional data analysis (FDA) considers such data as being values on the trajectories of a stochastic process, recorded with some error, at discrete random times. The mean and the covariances functions play a critical role in FDA.

To formalize the framework, let $\mathcal{T} \subset \mathbb{R}$ be a non-degenerate bounded, open interval, typically $(0, 1)$. Data consist of random realizations of sample paths from a second-order stochastic process $X = (X_t : t \in \mathcal{T})$ with continuous trajectories. The mean and covariance functions are $\mu(t) = \mathbb{E}(X_t)$ and

$$
\Gamma(s, t) = \mathbb{E} \{[X_s - \mu(s)][X_t - \mu(t)]\} = \mathbb{E}(X_sX_t) - \mu(s)\mu(t), \quad s, t \in \mathcal{T},
$$

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respectively. If the independent realizations \( X^{(1)}, \ldots, X^{(i)}, \ldots, X^{(N)} \) of \( X \) were observed, the ideal estimators would be

\[
\tilde{\mu}_N(t) = \frac{1}{N} \sum_{i=1}^{N} X_t^{(i)}, \quad t \in \mathcal{T},
\]

and

\[
\tilde{\Gamma}_N(s,t) = \frac{1}{N-1} \sum_{i=1}^{N} \{X_s^{(i)} - \tilde{\mu}_N(s)\} \{X_t^{(i)} - \tilde{\mu}_N(t)\}, \quad s, t \in \mathcal{T}.
\]

In real applications, the curves are rarely observed without error and never at each value \( t \in \mathcal{T} \). This is why we consider the following common and more realistic setup. For each \( X \), and given a positive integer \( M_i \), let \( T_m^{(i)} \in \mathcal{T} \), \( 1 \leq m \leq M_i \), be the observation times for the curve \( X^{(i)} \). The observations associated with a curve, or trajectory, \( X^{(i)} \) consist of the pairs \( (Y_m^{(i)}, T_m^{(i)}) \in \mathbb{R} \times \mathcal{T} \) where \( Y_m^{(i)} \) is defined as

\[
Y_m^{(i)} = X^{(i)}(T_m^{(i)}) + \varepsilon_m^{(i)}, \quad 1 \leq m \leq M_i, \quad 1 \leq i \leq N,
\]

and \( \varepsilon_m^{(i)} \) is an independent (centered) error variable. Here and in the following we use the notation \( X_t^{(i)} \) for the value at a generic point \( t \in \mathcal{T} \) of the realization \( X^{(i)} \) of \( X \), while \( X^{(i)}(T_m^{(i)}) \) denotes the measurement at \( T_m^{(i)} \) of this realization.

A commonly used idea is to build feasible versions of \( \tilde{\mu}_N(\cdot) \) and \( \tilde{\Gamma}_N(\cdot, \cdot) \) using nonparametric estimates of \( X_t^{(i)} \) and \( X_t^{(i)}X_s^{(i)} \), such as obtained by smoothing splines or local polynomials. This approach, usually called ‘smoothing first, then estimate’ or ‘two-stage procedure’, have been considered, amongst others, by Hall et al. (2006) and Zhang and Chen (2007). In general, the sample trajectories are required to admit at least second-order derivatives over the whole domain \( \mathcal{T} \). Li and Hsing (2010), Zhang and Wang (2016) and Zhang and Wang (2018) propose an alternative local linear smoothing approach where the estimators are determined by suitably weighting schemes which involve the whole sample of curves. This idea exploits the so-called replication and regularization features of functional data (see Ramsay and Silverman, 2005, ch. 22). In this alternative approach, the regularity assumptions are imposed on the mean and covariance functions, which are required to admit second, or higher, order derivatives over the whole domain. Since, in general, the mean and covariance functions are more regular than the sample trajectories, the approach based on weighting schemes using all the sample curves might be preferable. However, in some cases, for instance in energy, chemistry and physics, astronomy and medical applications, the mean and covariance functions could be quite irregular, of unknown irregularity.

Cai and Yuan (2011) and Cai and Yuan (2010) derive the optimal rates of convergence, in the minimax sense, for the mean and covariance functions, respectively, and propose optimal estimators. The optimal estimator of the mean function proposed by Cai and Yuan (2011) is a smoothing spline estimator which could be built only if the regularity of the sample paths is known. For the covariance function, Cai and Yuan (2010) used the representation of the covariance function in a tensor product reproducing kernel Hilbert space (RKHS) space. Next, they derived estimators for \( \Gamma(s,t) \) using a low dimension version of this representation obtained by a regularization procedure, provided the values \( M_i \) are not very different. This procedure does not require a given regularity, but involve some numerical optimization. See also Wong and Zhang (2019). The minimax optimal rates for the mean and covariance functions are defined
by the sum of two types of terms. One corresponds to the rate of convergence of the \( \tilde{\mu}_N(\cdot) \) and 
\( \tilde{\Gamma}_N(\cdot, \cdot) \), which is the standard rate of convergence for empirical means and covariances. The
other contribution to the optimal rates is given by the differences between \( \tilde{\mu}_N(\cdot) \) and \( \tilde{\Gamma}_N(\cdot, \cdot) \) and
their feasible versions. The optimal rates of the differences depend on the sample trajectories regularity. The reason is that the minimax lower bounds should also take into account, the case where the trajectories have the same regularity as the function to be estimated.

The estimation of the mean and covariance functions presents another specific feature. The minimax optimal rates of convergence depend on the nature of the measurement times \( T_m^{(i)} \). For now, two situations were investigated in the literature. On the one hand, the so-called independent design case where, given the \( M_i \)’s, the \( T_m^{(i)} \) are obtained as a random sample of size \( M_1 + \cdots + M_N \) from the same continuous distribution. On the other hand, the so-called common design case where the \( M_i \) are all equal to some integer value \( m \), and the \( T_m^{(i)} \), \( 1 \leq m \leq M \), are the same across the curves \( X^{(i)} \). In both cases, the best rates for the nonparametric estimators depend on the regularity of the sample trajectories. These rates also depend on the number of different observation times \( T_m^{(i)} \), that is equal to \( M_1 + \cdots + M_N \) with independent design, and equal to \( m \) with common design. In other words, the replication feature of functional data is less impactful with common design. See Cai and Yuan (2011) for the case of the mean function, and Cai and Yuan (2010) and Cai and Yuan (2016) for the covariance function case.

In this paper, we propose simple “smoothing first, then estimate” type methods, based on 1–dimensional smoothing, which achieve minimax optimal rates of convergence. The process is allowed to have a piecewise constant, unknown regularity. Our method does not require complex numerical optimization. It applies in the same way with common and independent design situations, and allows for general heteroscedastic measurement errors \( \varepsilon_m^{(i)} \). Moreover, our approach is suitable with both sparsely or densely sampled curves. The definition of sparse and dense regimes is recalled in Section 2.

Let \( \tilde{X}^{(i)} \) be a suitable nonparametric estimator of the curve \( X^{(i)} \) applied to the \( M_i \) pairs 
\( (Y_m^{(i)} , T_m^{(i)}) \), for instance a local polynomial estimator. What will make this estimator suitable is that it takes into account, the regularity of the process \( X \) and the final estimation purpose, that is the mean or the covariance function. These features can be achieved in an easy, data-driven way, as will be explained below. For now, with at hand the \( \tilde{X}^{(i)} \)'s tuned for the mean function estimation, let us define

\[
\tilde{\mu}_N(t) = \frac{1}{N} \sum_{i=1}^{N} \tilde{X}^{(i)}_t, \quad t \in T.
\] (2)

For the covariance function, we distinguish the diagonal from the non-diagonal points. For now, with at hand the \( \tilde{X}^{(i)} \)'s tuned for the covariance function estimation, and for some \( \delta \geq 0 \) determined using the data, let us define

\[
\hat{\Gamma}_N(s, t) = \frac{1}{N} \sum_{i=1}^{N} \tilde{X}^{(i)}_s \tilde{X}^{(i)}_t - \tilde{\mu}_N(s)\tilde{\mu}_N(t), \quad s, t \in T, \ |s - t| > \delta.
\] (3)

Moreover, for \( 0 \leq t - s \leq \delta \),

\[
\hat{\Gamma}_N(s, t) = \hat{\Gamma}_N(u - \delta/2, u + \delta/2), \quad \text{with} \quad u = (s + t)/2,
\] (4)

and \( \hat{\Gamma}_N(t, s) = \hat{\Gamma}_N(s, t) \). It is well known that the variance function \( \Gamma(s, s) \) induces a singularity when estimating the covariance function \( \Gamma(\cdot, \cdot) \). See, for instance, Zhang and Wang (2016),
Remark 4. This singularity causes little problem when studying pointwise rates of convergence and confidence intervals. In that cases, one could set $d = 0$ and use a modified covariance function estimator only for the diagonal $\hat{\Gamma}_N(s,s)$. However, the diagonal singularity could deteriorate the rate of convergence of $\hat{\Gamma}_N$ in the integrated squared norm. In general, this rate deterioration is propagated and affects the rates of convergence for the estimators of the eigenvalues and eigenfunctions of the covariance operator. The idea underlying (4) is natural and the diagonal tuning parameter $d$ decreases to zero according to a data-driven rule that we provide in the following.

Although the methodology we propose is general and can be used with different types of smoothers, we focus on the case where the $\hat{X}_t^{(i)}$ are obtained by local polynomials with a compactly supported kernel. In this case tuning the $\hat{X}_t^{(i)}$‘s means to suitably determine the rate of decrease and the constant defining the bandwidth. In our case, this is done completely data-driven by a one variable minimization of a new, suitable risk function.

To the best of our knowledge, there is no contribution following the ‘smoothing first, then estimate’ idea, which considers estimators of the curves $X^{(i)}$ adapted to their regularity and to the purpose of estimating mean or covariance functions. It is clear that trajectory-by-trajectory adaptive optimal smoothing, for instance using the Goldenshluger and Lepski (2011) method, in general yields sub-optimal rates of convergence for $\hat{\mu}_N(t)$ and $\hat{\Gamma}_N(s,t)$. The reason is that trajectory-by-trajectory smoothing ignores the information contained in the other $N-1$ curves in the sample generated according to the same stochastic process $X$. See Cai and Yuan (2011) for a discussion on the differences with the usual nonparametric rates. One can also use cross-validation for choosing the bandwidth with the suitably weighting schemes, such as proposed by Li and Hsing (2010) or Zhang and Wang (2016). However, this would require significant computational effort, and, to the best of our knowledge, the idea has not yet received a theoretical justification. Using the replication and regularization features of functional data, we propose a new effective estimator for the local regularity of the process $X$, a probabilistic concept which determines the analytic regularity of the trajectories of $X$. The replication feature of the functional data makes the concept of local regularity of the process a more meaningful parameter than the usual curve regularity, which is an analytic concept designed for a single function. Our local regularity estimator, inspired by, but different from, the estimator introduced by Golovkine et al. (2020), combines information both across and within curves. Moreover, it allows for general heteroscedastic measurement errors, does not involve any optimization and is obtained after a fast, possibly parallel, computation. With at hand the local regularity estimator, we derive the suitable estimators $\hat{X}_t^{(i)}$, and finally our optimal mean and covariance functions estimators. The smoothing parameter used to build the $\hat{X}_t^{(i)}$ depends on $M_i$ and $N$, but can be easily computed given the estimate of the local regularity of $X$. It is worth noting that our estimators have good theoretical and finite sample properties without using a bivariate nonparametric estimator of $X^{(i)}_s X^{(i)}_t$. We explain this somehow unexpected result by the particular separable structure of the bivariate functions $(s,t) \mapsto X^{(i)}_s X^{(i)}_t$ which can be estimated at the faster rate for univariate functions.

In Section 2 we provide insight on why the local regularity of the process $X$ is a natural feature to be considered. Moreover, we explain why the ‘smoothing first, then estimate’ approach could achieve optimal rates when the regularity of $X$ is known. In Section 3, we formally define the local regularity of the process $X$ and discuss the relationship between this regularity and the decrease rate for the eigenvalues of the covariance operator. Moreover, we introduce a new...
estimator for this regularity and present exponential bounds for the concentration under mild conditions. In particular, both independent and common designs are allowed, and the process regularity is allowed to be piecewise constant. Section 3 ends with a discussion on the relationship between the process regularity and the trajectories’ analytical regularity. In Section 4, we use the regularity estimate to build sharp bounds of the pointwise quadratic risk function between our estimators and the unfeasible estimators \( \hat{\mu}_N \) and \( \tilde{\Gamma}_N \), respectively. The bounds depend on quantities which could be estimated by sample averages. Minimizing the risk bounds with respect to the bandwidth, we derive the optimal bandwidth for the local polynomial estimates of the trajectories. These estimates are further used to optimally estimate the mean and covariance functions. The finite sample performance of the new estimators is illustrated in Section 5 using simulated samples generated according to the setup of a real data set on the power consumption of households. Some conclusions and discussions are gathered in Section 6. Few proofs are relegated to the Appendix. A Supplementary Material contains more proofs, technical arguments and simulation results.

2 From unfeasible to feasible optimal estimators

Let us first provide insight into the reason why the local regularity of the process generating the curves is a meaningful concept, and why our approach can achieve good performance. For this purpose, we analyze the difference \( \hat{\mu}_N(t) - \tilde{\mu}_N(t), s, t \in \mathcal{T} \), but similar apply to the covariance function estimation.

The data \((Y_m^{(i)}, T_m^{(i)}) \in \mathbb{R} \times \mathcal{T}\) are generated according to model (1) with

\[
\varepsilon_m^{(i)} = \sigma(T_m^{(i)}, X^{(i)}(T_m^{(i)})) e_m^{(i)}, \quad 1 \leq m \leq M_i, \; 1 \leq i \leq N, \tag{5}
\]

where the \(X^{(i)}\) are independent trajectories of \(X\), \(e_m^{(i)}\) are independent copies of a centered variable \(e\) with unit variance, and \(\sigma(t, x)\) is some unknown bounded function which account for possibly heteroscedastic measurement errors. The integers \(M_1, \ldots, M_N\) represent an independent sample of an integer-valued random variable \(M\) with expectation \(m\) which increases with \(N\). Thus, \(M_1, \ldots, M_N\) is the \(N\)th line of a triangular array of integer numbers. In the independent design case, for each \(1 \leq i \leq N\), the observation times \(T_m^{(i)}\) are random realizations of a variable \(T \in \mathcal{T}\).

We assume that the realizations of \(X\), \(M\) and \(T\) are mutually independent. Let \(\mathcal{T}_\text{obs}^{(i)}\) denote the set of observation times \(T_m^{(i)}, 1 \leq i \leq M_i\), over the trajectory \(X^{(i)}\). In the common design case, \(M \equiv m\), and the \(\mathcal{T}_\text{obs}^{(i)}\) are the same for all \(i\). Thus, if not stated differently, the issues discussed in this section apply to both independent design and common design cases.

Let

\[
\mathbb{E}_i(\cdot) = \mathbb{E}(\cdot \mid M_i, \mathcal{T}_\text{obs}^{(i)}, X^{(i)}) \quad \text{and} \quad \mathbb{E}_{M,T}(\cdot) = \mathbb{E}(\cdot \mid M_i, \mathcal{T}_\text{obs}^{(i)}, 1 \leq i \leq N).
\]

For any \(t \in \mathcal{T}\), we consider a generic linear nonparametric estimator

\[
\hat{X}_t^{(i)} = \sum_{m=1}^{M_i} Y_m^{(i)} W_m^{(i)}(t), \quad 1 \leq i \leq N. \tag{6}
\]

The weights \(W_m^{(i)}(t)\) are defined as functions of the elements in \(\mathcal{T}_\text{obs}^{(i)}\). The example we keep in mind is that of local polynomials which we investigate in detail in Section 4. Let

\[
\hat{X}_t^{(i)} - X_t^{(i)} = B_t^{(i)} + V_t^{(i)}, \quad t \in \mathcal{T}, \tag{7}
\]
where
\[ B_t^{(i)} := \mathbb{E}_i \left[ \hat{X}_t^{(i)} \right] - X_t^{(i)} \quad \text{and} \quad V_t^{(i)} := \hat{X}_t^{(i)} - \mathbb{E}_i \left[ \hat{X}_t^{(i)} \right] = \sum_{m=1}^{M_i} \varepsilon_m^{(i)} W_m^{(i)}(t). \]

The pairs of random variables \((B_t^{(i)}, V_t^{(i)})\), \(1 \leq i \leq N\), are independent and we could reasonably assume that they are squared integrable for all \(t\).

For illustration purposes, let us suppose that \(\hat{X}_t^{(i)}\) in (6) are well defined for all \(1 \leq i \leq N\). The general case is presented in Section 4. Then, for the mean, we can write
\[
\hat{\mu}_N(t) - \bar{\mu}_N(t) = \frac{1}{N} \sum_{i=1}^{N} B_t^{(i)} + \frac{1}{N} \sum_{i=1}^{N} V_t^{(i)}.
\]

All the variables \(\varepsilon_m^{(i)}\) are centered and conditionally independent, with bounded conditional variance, given all \(M_t\), \(T_t^{(i)}\) and \(X_t^{(i)}\). Thus, given \(M_t, \ldots, M_N\) and \(T_{\text{obs}}^{(i)}, \ldots, T_{\text{obs}}^{(N)}\),
\[
\mathbb{E}_{M,T} \left[ \left\{ N^{-1} \sum_{i=1}^{N} V_t^{(i)} \right\}^2 \right] = N^{-1} \mathbb{E}_{M,T} \left[ N^{-1} \sum_{i=1}^{N} \left\{ V_t^{(i)} \right\}^2 \right] \leq N^{-1} \sup_x \sigma^2(t,x) \times N^{-1} \sum_{i=1}^{N} \left\{ \max_m W_m^{(i)}(t) \times \sum_{m=1}^{M_i} W_m^{(i)}(t) \right\}.
\]

For local polynomials with bandwidth \(h\), under some mild conditions, the rate of decrease of the right-hand side in the last display, given the design, is \(O_P \left( (Nmh)^{-1} \right) \).

On the bias part, let us suppose for the moment that the trajectories are not differentiable. By Cauchy-Schwarz inequality, we have
\[
\mathbb{E}_{M,T} \left[ \left\{ N^{-1} \sum_{i=1}^{N} B_t^{(i)} \right\}^2 \right] \leq N^{-1} \sum_{i=1}^{N} \mathbb{E}_{M,T} \left[ \left( B_t^{(i)} \right)^2 \right] \leq N^{-1} \sum_{i=1}^{N} \sum_{m=1}^{M_i} \mathbb{E}_{M,T} \left[ \left\{ X_t^{(i)}(T_m^{(i)}) - X_t^{(i)} \right\}^2 \left| T_{\text{obs}}^{(i)} \right| W_m^{(i)}(t) \right].
\]

It now becomes clear that the rate of the square of the bias term in the difference \(\hat{\mu}_N(t) - \bar{\mu}_N(t)\) is determined by the second-order moment of the increments \(X_s^{(i)} - X_t^{(i)}\). If, for \(s\) and \(t\) which are close,
\[
\mathbb{E} \left( \left\{ X_s^{(i)} - X_t^{(i)} \right\}^2 \right) \approx L_0^2 |t - s|^{2H_0},
\]
with some constants \(0 < H_0 \leq 1\) and \(L_0\), then the rate of the right-hand side quantity in (8) can be bounded by
\[
N^{-1} \sum_{i=1}^{N} \left\{ \sum_{m=1}^{M_i} W_m^{(i)}(t) \times \sum_{m=1}^{M_i} L_0^2 \left| T_m^{(i)} - t \right| W_m^{(i)}(t) \right\}.
\]
For local polynomials with bandwidth \( h \), this leads to \( \mathcal{O}_P(h^{2H_0}) \). We call \( H_0 \) the Hölder exponent of \( X(i) \), whenever the trajectories are not differentiable.

With \( \delta \)-times differentiable trajectories, \( \delta \geq 1 \) integer, the condition (9) should be considered with the difference of the values of the \( \delta \)-th derivative of \( X(i) \) instead of the \( X(i) \) itself. Then, following the lines of Proposition 1.13 of Tsybakov (2009), we use the Taylor expansion and the property \( \sum_{m=1}^{M}(T_m^{(i)} - t)^dW_m^{(i)}(t) = 0 \) which is satisfied by any integers \( 0 \leq d \leq \delta \) and \( 1 \leq i \leq N \), in the case of local polynomials of order \( \delta \). The rate in (8) then becomes \( \mathcal{O}_P(h^{2(\delta + H_3)}) \), where \( H_3 \in (0,1] \) denotes the Hölder exponent of the \( \delta \)-th derivatives of \( X(i) \).

Gathering facts, we deduce that, in the case of \( \delta \)-times differentiable trajectories, \( \delta \geq 0 \), for local polynomial of order \( \delta \) and \( h \) of order \( (Nm)^{-1/(1+2(\delta + H_3))} \), as long as this bandwidth order does not yield degenerate estimates, we obtain

\[
\mathbb{E}_{M,T} \left[ (\hat{\mu}_N(t) - \tilde{\mu}_N(t))^2 \right] = \mathcal{O}_P( (Nm)^{-\frac{2(\delta + H_3)}{1+2(\delta + H_3)}} ) .
\]

Thus, given the local regularity \( \delta + H_3 \), the estimator \( \hat{\mu}_N(t) \) can achieve the minimax optimal rate for the estimation of the mean function \( \mu(t) \). See Cai and Yuan (2011).

Let us note that in some cases, in particular with local polynomials, the estimator defined in (6) could be degenerate, i.e., the weights \( W_m^{(i)}(t) \) are not well defined because \( h \) is too small. The trajectories for which this happens could change with \( t \). Then, \( \hat{\mu}_N(t) \) is defined as an average over the trajectories for which the estimator (6) is not degenerate. This can more likely happen in the so-called sparse regime, where \( m^{2(\delta + H_3)} \ll N \). A similar phenomenon occurs with estimators determined by suitably weighting schemes, see for instance equation (2.1) in Li and Hsing (2010), or equation (2.3) in Zhang and Wang (2016). However, in the independent case, one could benefit from the replication feature of functional data, because only a fraction of trajectories will yield non degenerate estimators \( \hat{X}_t^{(i)} \). The size of this fraction plays a central role in the sparse regime. This crucial aspect is taken into account in Sections 4.1 and 4.2, where we choose the bandwidths while penalizing the number of trajectories which yield degenerate estimators.

The case of common design requires some special attention. For simplicity, let us assume the common design points are equidistant and consider the local polynomials are built with the kernel supported on \([-1,1] \). In this case, the bandwidth cannot have a rate smaller than \( m^{-1} \), otherwise all the weights \( W_m^{(i)}(t) \) could all be equal to zero. This means that with a common design, the optimal bandwidth is given by the minimization of \( h^{2(\delta + H_3)} + (Nmh)^{-1} \) under the constraint that \( mh \) stays away from zero. Without loss of generality, we could set \( h = k/m \) with \( k \) a positive integer and search \( k \) which minimizes \( h^{2(\delta + H_3)} + (Nmh)^{-1} \). Balancing the two terms, one expects the optimal \( k/m \) to have the rate \( (Nm)^{-1/(1+2(\delta + H_3))} \). If \( m^{2(\delta + H_3)} \) is larger than \( N \), i.e. in the so-called dense regime, the optimal \( k \) is well defined and \( k \sim (m^{2(\delta + H_3)}/N)^{1/(1+2(\delta + H_3))} \) and, with this optimal choice, \( \mathbb{E}_{M,T} \left[ (\hat{\mu}_N(t) - \tilde{\mu}_N(t))^2 \right] = \mathcal{O}_P(N^{-1}) \) if \( m^{2(\delta + H_3)} \ll N \), then the constraint that \( k \geq 1 \) becomes binding, and it is no longer possible to balance the squared bias term and the variance term. The rate of \( h^{2(\delta + H_3)} \) dominates the rate \( (Nmh)^{-1} \). The minimal rate for \( \mathbb{E}_{M,T} \left[ (\hat{\mu}_N(t) - \tilde{\mu}_N(t))^2 \right] \) then corresponds to \( k = 1 \), and is \( \mathcal{O}_P\left(m^{-2(\delta + H_3)}\right) \). We obtain a similar conclusion with differentiable trajectories. Gathering facts, we recover the optimal rate for mean estimation with common design, that is \( \mathcal{O}_P\left(m^{-2(\delta + H_3)} + N^{-1}\right) \), as found by Cai and Yuan (2011). Finally, let us recall the somehow surprising message from Cai and Yuan (2011), p. 2332: the interpolation is rate optimal when \( m^{2(\delta + H_3)} \gg N \) in the case of common
design; smoothing does not improve converges rates. Our contribution on this aspect is a data-driven rule for the practitioner which completes this theoretical fact about the interpolation.

The adaptive bandwidth rule proposed in Section 4 automatically chooses between smoothing and interpolation with finite sample sizes.

We learn from the above that the ‘smoothing first, then estimate’ approach can lead to optimal rates of convergence for estimating the mean function with independent and common design, as derived by Cai and Yuan (2011), provided the regularity of the process \( X \) is known. We will show that achieving optimal rates using the local regularity is also possible for the covariance function. This probabilistic concept of regularity, summarized by (9), will be formally introduced in the next section. Next, we introduce a simple estimator of the local regularity of \( X \). It will be shown that, under some mild conditions, this estimator concentrates around the true local regularity faster than a suitable negative power of \( \log(m) \). This suffices to guarantee that our mean and covariance functions estimators achieve the same rates as when the local regularity is known.

Let us end this section with a discussion of the differences with the weighting schemes approach, as for instance considered by Li and Hsing (2010) and Zhang and Wang (2016). If the regularity of \( \mu(\cdot) \) is known, then one could define \( B_t^{(i)} \) and \( V_t^{(i)} \) in (7) centering by the mean function instead of the trajectory \( X_t^{(i)} \). Then, one could derive the rate of \( \mathbb{E} \left[ (\hat{\mu}_N(t) - \mu(t))^2 \right] \) and find the bandwidth which minimizes this rate, exactly as done in Li and Hsing (2010) and Zhang and Wang (2016) where \( \mu(\cdot) \) is assumed to be twice differentiable. However, the estimation of the regularity of \( \mu(\cdot) \) remains an open problem.

3 Local regularity estimator

We will allow the sample paths of the real-valued random process \( X = (X_t : t \in \mathcal{T}) \) to have piecewise constant regularity. That means that there exists a finite, but unknown, partition of \( \mathcal{T} \) in non-degenerate intervals, such that, in the interior of each interval of the partition, the regularity of the process is the same. To allow this generality, we first need to formally introduce the notion of local regularity of the process \( X \). Given this type of regularity, the Kolmogorov Extension Theorem allows us to determine the analytic regularity of the trajectories of \( X \). The details are provided in Section 3.4.

The most convenient way to obtain the theoretical properties of our ‘smoothing first, then estimate’ approach for the mean and covariance functions, will be to consider that the estimator of the local regularity of \( X \) is built from a separate, independent sample of \( N_0 \) trajectories. We will call the data obtained from this separate random sample of trajectories, generated by the same stochastic process \( X \), the learning sample. We will show that, in theory, the size \( N_0 \) of the learning sample can be as small as any positive power of \( m \), regardless of the type of design, independent or common. Our simulation experiences reveal that in practice one can confidently use the same sample for learning the local regularity of \( X \) and for estimating the mean and covariance functions.

3.1 Local regularity in quadratic mean

Let \( \mathcal{O}_* \) be an open subinterval of \( \mathcal{T} \), with length \( \Delta_* = \text{diam}(\mathcal{O}_*) \). Whenever it exists, let \( \nabla^d X_t \) denote the \( d \)-th derivative, \( d \geq 1 \), of the generic curve \( X_t \) at \( t \in \mathcal{T} \). By definition \( \nabla^0 X_t \equiv X_t \).
The following assumptions ensure that, locally on \( \mathcal{O}_* \), the derivatives exist up to the integer order \( \delta \geq 0 \) and that the \( \delta \)-th derivative is regular in quadratic mean.

**Assumptions.** Let \( \delta \in \mathbb{N} \) and \( 0 < H_\delta \leq 1 \).

(H1) With probability 1, for any \( d \in \{0, \ldots, \delta\} \) the \( d \)-th order derivative \( \nabla^d X_t \) of \( X_t \) exists for all \( t \in \mathcal{O}_* \), and satisfies:

\[
0 < a_d = \inf_{u \in \mathcal{O}_*} \mathbb{E} \left[ (\nabla^d X_u)^2 \right] \leq \sup_{u \in \mathcal{O}_*} \mathbb{E} \left[ (\nabla^d X_u)^2 \right] = \sigma_d < \infty.
\]

(H2) Two positive constants \( S_\delta \) and \( \beta_\delta \) exist such that:

\[
\mathbb{E} \left[ (\nabla^\delta X_t - \nabla^\delta X_s)^2 \right] - L_\delta^2 |t - s|^{2H_\delta} \leq S_\delta^2 |t - s|^{2H_\delta} \Delta_*^{2\beta_\delta}, \quad s, t \in \mathcal{O}_*.
\]

(H3) \( a > 0 \) and \( \mathfrak{A} > 0 \) exist such that, for any \( d \in \{0, \ldots, \delta\} \) and any \( p \geq 1 \):

\[
\mathbb{E} \left[ |\nabla^d X_t - \nabla^d X_s|^{2p} \right] \leq \frac{p!}{2} a \mathfrak{A}^{p-2}, \quad s, t \in \mathcal{O}_*.
\]

**Definition 1.** For any \( \delta \in \mathbb{N} \), \( 0 < H_\delta \leq 1 \) and \( L_\delta > 0 \), the class \( \mathcal{X}(\delta + H_\delta, L_\delta; \mathcal{O}_*) \) is the set of stochastic processes indexed by \( t \in \mathcal{O}_* \) for which the conditions (H1) to (H3) hold true. The quantity \( \alpha = \delta + H_\delta \) is the local regularity of the process on \( \mathcal{O}_* \), while \( L_\delta \) is the Hölder constant of the \( \delta \)-th derivative of the trajectories.

See, e.g., Blanke and Vial (2014) for some examples and references on processes satisfying the mild condition in (H2), which we impose on the \( \delta \)-th derivative of the trajectories. Examples include stationary or stationary increment processes. For some common processes with the ordered eigenvalues of the covariance operator such that, for some \( \nu > 1 \), \( \lambda_j \sim j^{-\nu}, j \geq 1 \), one has \( \delta + H_\delta = (\nu - 1)/2 \). Details are provided in the Supplementary Material. The condition in (H3) serves to derive the exponential bound for the concentration of the local regularity estimator. By definition, when the local regularity \( \alpha \) is an integer, \( \delta = \alpha - 1 \) and \( H_\delta = 1 \). In Lemma 1 in the Supplement we describe the embedding structure of the spaces \( \mathcal{X}(\cdot; \cdot; \mathcal{O}_*) \). More precisely, we show that whenever \( \delta \geq 1 \), for any \( 1 \leq d \leq \delta - 1 \), the process \( X \) restricted to \( \mathcal{O}_* \) belongs to \( \mathcal{X}(d + 1, L_d; \mathcal{O}_*) \) for some \( L_d > 0 \).

### 3.2 The local regularity estimation method

Let us assume that \( X \) restricted to \( \mathcal{O}_* \) belongs to \( \mathcal{X}(\delta + H_\delta, L_\delta; \mathcal{O}_*) \) for some \( \delta \in \mathbb{N} \), \( 0 < H_\delta \leq 1 \) and \( L_\delta > 0 \). Our first goal is to construct an estimator of \( \alpha = \delta + H_\delta \). To do so, we first have to estimate \( H_d \) for any \( d = 0, \ldots, \delta \). For simplicity, let us denote

\[
\theta_d(s, t) = \mathbb{E} \left[ (\nabla^d X_t - \nabla^d X_s)^2 \right], \quad s, t \in \mathcal{O}_*.
\]

Since the restriction of the process \( X \) belongs to \( \mathcal{X}(d + H_d, L_d; \mathcal{O}_*) \), with \( H_d = 1_{\{d \neq \delta\}} + H_\delta 1_{\{d = \delta\}} \), we have

\[
\theta_d(s, t) \approx L_d^2 |t - s|^{2H_d} \quad \text{if } \Delta_* \text{ is small. \quad (10)}
\]
Now, let \( t_1 \) and \( t_3 \) be such that \([t_1,t_3] \subset O_*\) and \( t_3 - t_1 = \Delta_* / 2 \). Denote by \( t_2 \) the middle point of \([t_1,t_3]\). It is easily seen that

\[
H_d \approx \tilde{H}_d = \frac{\log(\theta_d(t_1,t_3)) - \log(\theta_d(t_1,t_2))}{2\log(2)} \quad \text{if } \Delta_* \text{ is small.}
\]

Consider a learning sample of \( N_0 \) curves, generated according to (1), which yield

\[
(Y^{(n)}_m, T^{(n)}_m) \in \mathbb{R} \times T, \quad 1 \leq m \leq M_n, 1 \leq n \leq N_0.
\]

Then, given a nonparametric estimator \( \tilde{\nabla}_d X_t \) of \( \nabla^d X_t \), for \( t \in O_* \), we define a natural estimator of \( \tilde{H}_d \), and thus of \( H_d \), as

\[
\hat{H}_d = \frac{\log(\hat{\theta}_d(t_1,t_3)) - \log(\hat{\theta}_d(t_1,t_2))}{2\log(2)}, \quad (11)
\]

where

\[
\hat{\theta}_d(s,t) = \frac{1}{N_0} \sum_{n=1}^{N_0} \left( \tilde{\nabla}_d X^{(n)}_t - \tilde{\nabla}_d X^{(n)}_s \right)^2.
\]

By the definition of the local regularity, \( \delta = \min\{d \in \mathbb{N} : H_d < 1\} \), which suggests defining

\[
\hat{\delta} = \min\{d \in \mathbb{N} : \hat{H}_d < 1 - \varphi(m)\},
\]

for some decreasing function \( \varphi(\cdot) \) which is defined later. Typically it could decrease as fast as a negative power of \( \log(m) \). The estimator of the local regularity is then

\[
\hat{\alpha} = \hat{\delta} + \hat{H}_\hat{\delta}.
\]

Since (11) can be easily modified when \( N_0 \) increases, \( \hat{\alpha} \) can be updated with little effort. Our local regularity estimator is related to the estimator introduced by Golovkine et al. (2020). However, here we propose smoothing the trajectories to calculate \( \tilde{H}_d \) for \( d = 0 \), while Golovkine et al. (2020) directly uses the noisy measurements of the trajectories. To derive the properties of the local regularity estimator, for simplicity, we suppose that \( m \) is given. In applications, it could be estimated by the average of the realizations of \( M \).

### 3.3 Concentration properties of the local regularity estimator

The quality of the estimators \( \hat{\delta} \) and \( \hat{H}_d \) depends on the quality of the generic nonparametric estimators \( \tilde{\nabla}_d X \) of \( \nabla^d X \). To quantify their behavior, we consider the local \( L^p \)-risk

\[
R_p(d) = R_p(d; O_*) = \sup_{t \in O_*} \mathbb{E}(|\xi_d(t)|^p), \quad \text{where} \quad \xi_d(t) = \tilde{\nabla}_d X_t - \nabla^d X_t.
\]

Our method applies with any type of nonparametric estimator \( \tilde{\nabla}_d X \) (local polynomials, splines,...) as soon as, for any \( p \in \mathbb{N} \), its \( L^p \)-risk is suitably bounded. The following mild condition is satisfied by common estimators, see for instance Theorem 1 in Gaïffas (2007) for the case of local polynomials.

**Assumptions.**
(LP1) There exist two positive constants $c$ and $C$ such that

$$R_{2p}(d) \leq \frac{p!}{2} c e^{p-2}, \quad \forall p \geq 1, \ d \in \{0,\ldots,\delta\}. $$

We can now derive an exponential bound for the concentration of all the estimators $\hat{H}_d$, $d \in \{0,\ldots,\delta\}$. To make this exponential bound useful for deriving optimal rates for our estimators of the mean and covariance functions, we will require the largest risk among $R_2(0),\ldots,R_2(\delta)$ to tend to zero as $m$ increases to infinity.

**Theorem 1.** Assume that $X$ restricted to $O_*$ belongs to $\mathcal{X}(\delta+H_\delta,L_\delta;O_*)$, for some integer $\delta \geq 0$ and $0 < H_\delta \leq 1$, and that (LP1) holds. Assume also that there exists $\tau > 0$ and $B > 0$ such that:

$$\rho_* = \max_{d \in \{0,\ldots,\delta\}} R_2(d) \leq B m^{-\tau}. $$

Let $0 < \gamma < 1$ and $\Gamma > 0$, and consider

$$\Delta_* = 2 \exp(-\log^{\gamma}(m)) \quad \text{and} \quad \varphi(m) = \log^{-\Gamma}(m). $$

Then, for any $m$ larger than some constant $m_0$ depending on $B$, $\tau$, $\gamma$, $\Gamma$, $H_\delta$, $\beta_\delta$ and for some constant $\hat{f}$, we have

$$\mathbb{P}(|\hat{a} - a| > \varphi(m)) \leq 8(1 + \delta) \exp\left(-\hat{f} N_0 \varphi^2(m) \Delta_*^{4H_\delta}\right). $$

The proof of Theorem 1 follows the lines of the results of Golovkine et al. (2020) and is thus omitted. The three quantities $\rho_*$, $\Delta_*$ and $\varphi(m)$ are required to decrease to zero, as $m$ tends to infinity, in such a way that $\rho_*/\Delta_* + \Delta_*/\varphi(m) \to 0$. We propose $\Gamma = 2$ and $\gamma = 1/2$. The choices of the rates for $\rho_*$, $\Delta_*$ and $\varphi(m)$ satisfy some additional requirements. First, it will be shown below that, in order to achieve optimal rates of convergence for the mean and covariance estimators, the local regularity has to be estimated with a concentration rate $\varphi(m)$ faster than $\log^{-1}(m)$. This is a consequence of the identity $m^{1/\log(m)} = e$ for any $m > 1$ and of a mild condition on the rates of increase of $N$ and $m$; see (21) below. Second, we want to allow for reasonable rates of increase for $N_0$, the size of the learning set. In Theorem 1, $N_0$ can increase as fast as an arbitrary positive power of $m$. Third, since $\tau > 0$ could be arbitrarily small, the rate imposed on the nonparametric estimators $\widehat{\nabla^d X}$ of $\nabla^d X$ is a very mild consistency requirement which is achieved by the common estimators, with random or fixed design, under mild conditions, in particular on the distribution of the $M_i$. See, for instance, Tsybakov (2009) and Belloni et al. (2015). In particular, the required rate for the $\widehat{\nabla^d X}$ can be obtained under general forms of heteroscedasticity.

### 3.4 From the regularity of the process to the regularity of the trajectories

Let us now connect the probabilistic concept of local regularity with the regularity of the sample paths considered as functions. Let $O_*$ be some open subinterval of $\mathcal{T}$. We say that a function is $\beta$ times differentiable on $O_*$ if the function has an up to $[\beta]-$order derivative and the $[\beta]-$th derivative is Hölder continuous with exponent $\beta - [\beta]$. (For a real number $a$, let $[a]$ denotes the largest integer not exceeding $a$.) Let us call a Hölder space of local regularity $\beta$, the set of functions which are $\beta$ times differentiable on $O_*$. 

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By Assumption (H1), for almost all realizations of the process $X$, the derivatives of the sample path exist up to order $\delta$. Under a suitable moment condition on the increments the derivative process $\nabla^\delta X$, and using the refined version of Kolmogorov’s criterion stated in Revuz and Yor (2013), it can be proven that, for any $\delta < \beta < \delta + H_\delta = \alpha$, the sample path of the process $X$ restricted to $\mathcal{O}_s$ belong to the Hölder space of exponent $\beta$ over $\mathcal{O}_s$. Technical details are provided in the Supplementary Material. As an example, the Brownian motion has a local regularity equal to $1/2$ on any open interval. Almost surely, the sample paths of the Brownian motion belong to any Hölder space of local regularity $\beta < 1/2$, but cannot have a Hölder continuity of order $\alpha$, $\alpha \geq 1/2$. Hence, the probability theory indicates that imposing assumptions on the regularity of the sample paths could be a delicate issue. Indeed, even for some widely used examples, this regularity is not well defined in the sense required by the nonparametric statistics theory. Since the sample paths have a regularity which can be arbitrarily close to the (local) regularity of the process $X$ as defined above, the probabilistic concept of local regularity seems more appropriate for establishing the rates of convergence for the mean and covariance estimators.

4 Optimal mean and covariance function estimators

We now explain how to suitably select the bandwidths for the local polynomial smoothing of the trajectories, and next build mean and covariance function estimates. Our data-driven adaptive bandwidth rules lead to optimal rate estimates whenever the estimator of the local regularity concentrates to the true value faster than $\log^{-1}(m)$. Theorem 1 then guarantees that the suitable rate of the bandwidth can be achieved with high probability.

Motivated by the applications, we allow the process $X$ to have a piecewise constant regularity.

Assumptions.

(E1) There exist $K \geq 1$ and $\min T = t_0 < t_1 < \ldots t_{K-1} < t_K = \max T$ such that, for each $T_k^\circ = (t_{k-1}, t_k), X$ restricted to $T_k^\circ$ belongs to $\mathcal{X}(\delta_k + H_{\delta_k}, L_{\delta_k}; T_k^\circ),$ for some integer $\delta_k \geq 0, 0 < H_{\delta_k} \leq 1$ and $L_{\delta_k} > 0$. For any $t \in T$, let $\alpha_t$ be the local regularity from Definition 1 corresponding to a small neighborhood of $t$, and let $L_\delta$, with $\delta = |\alpha_t|$, denote the corresponding Hölder constant. By our assumptions, $\alpha_t$ is well defined everywhere except a finite number of points in $T$. For a small neighborhood of points like $t_k$, the values of $\alpha_t$ are different for $t$ to the left and to the right of $t_k$. However, for practical purposes, any convention would work, such as for instance, setting $\alpha_{t_k}$ equal to the average of the neighboring values. Hereafter therefore we assume that $\alpha_t$ is well-defined everywhere on $T$.

Hereafter, $\hat{\alpha}_t$ will be the estimator of $\alpha_t$ defined in (12) and (11), built with an independent learning sample, and with $\phi(m)$ replaced by $\log^{-2}(\tilde{m})$, where $\tilde{m} = N^{-1} \sum_{i=1}^N M_i$. If $t_1 < \ldots < t_{K-1}$ are known, then for each $t \in T_k^\circ$, $\hat{\alpha}_t$ can be defined as in (12) and (11) with $t_2$ the middle point in $T_k^\circ$. In practice the $t_1, \ldots, t_{K-1}$ are likely not given. One could then estimate the local regularity on a grid on points in $T$, and let each point on the grid play the role of $t_2$ in (11). Such implementation is used in Section 5.

For each $1 \leq i \leq N$, we use a local polynomials (LP) approach to build suitable nonparametric estimators $\hat{X}^{(i)}$. For $t \in T$, if the local regularity $\alpha_t$ is available, using the measurements

$\hat{X}^{(i)}(t) = \sum_{j=1}^N \hat{\alpha}_j \frac{N^{\frac{2}{\alpha_j}}}{\sqrt{N}} \hat{\phi}(\hat{\mathbf{x}}^{(i)}(t) - \mathbf{x}_j) \mathbf{1}_{(t \in \Delta_j)} \mathbf{1}_{(\alpha_j \leq \frac{1}{2})} + e^{\frac{1}{2}} \mathbf{1}_{(\alpha_j > \frac{1}{2})},$
of a generic trajectory $X^{(i)}$, we consider the $LP(\lfloor \alpha t \rfloor)$ estimator defined by
\[ \hat{X}^{(i)}_t = \hat{X}^{(i)}_t(h_t) = \sum_{m=1}^{M_i} Y^{(i)}_m W^{(i)}_m(t), \quad 1 \leq i \leq N, \] (13)
with a suitable bandwidth $h_t$ which depends on $\alpha_t$, and
\[ W^{(i)}_m(t) = W^{(i)}_m(t; h) = \frac{1}{M_i h} U^\top(0) A^{(i)}_{M_i}(t, h)^{-1} U \left( \frac{T^{(i)}_m - t}{h} \right) K \left( \frac{T^{(i)}_m - t}{h} \right), \quad 1 \leq m \leq M_i, \]
with
\[ A^{(i)}_{M_i}(t, h) = \frac{1}{M_i h} \sum_{m=1}^{M_i} U \left( \frac{T^{(i)}_m - t}{h} \right) U^\top \left( \frac{T^{(i)}_m - t}{h} \right) K \left( \frac{T^{(i)}_m - t}{h} \right). \]
Here, for any $z \in \mathbb{R}$, $U(z) = U(z; \alpha_t) = (1, z, \ldots, z^{[\alpha_t]}/[\alpha_t]!)^\top$ and $K : \mathbb{R} \to \mathbb{R}_+$ is a continuous kernel with the support in $[-1, 1]$.

For a given $t \in T$ and each $1 \leq i \leq N$, the weights $W^{(i)}_m(t)$, $1 \leq m \leq M_i$ are well defined as soon as the matrix $A^{(i)}_{M_i}(t, h)$ is invertible. When $A^{(i)}_{M_i}(t, h)$ is not invertible, we consider the smoothing of the curve $i$ at point $t$ as degenerate and the curve $i$ should not be considered in the construction of the estimator of $\mu(t)$. A similar reasoning applies to the covariance estimator. In the case of common design, for each $t$, the number of degenerate estimates $\hat{X}^{(i)}_t$ is either equal to $N$ or to zero. In the independent design case, this number could be any integer between 0 and $N$. A suitable bandwidth rule should be penalizing for the number of curves which are not considered for the estimation. In the following sections we propose a natural way to penalize which adapts to the sparse and dense regimes. Moreover, the two types of designs are automatically handled.

### 4.1 Adaptive mean estimation

Let $k_0$ be some integer, and let $1\{ \cdot \}$ denote the indicator function. For any $t \in T$, define
\[ w_i(t; h) = 1 \text{ if } \sum_{m=1}^{M_i} 1\{|T^{(i)}_m - t| \leq h\} \geq k_0, \quad \text{and } w_i(t; h) = 0 \text{ otherwise,} \] (14)
and let
\[ W_N(t; h) = \sum_{i=1}^{N} w_i(t; h). \]
Our adaptive mean function estimator is
\[ \hat{\mu}^*_N(t) = \hat{\mu}_N(t; h^*_\mu) \quad \text{with} \quad \hat{\mu}_N(t; h) = \frac{1}{W_N(t; h)} \sum_{i=1}^{N} w_i(t; h) \hat{X}^{(i)}_t. \] (15)
Here, $\hat{X}^{(i)}_t$ is the local polynomial estimator $LP(\lfloor \alpha_t \rfloor)$ defined in (13) with some suitable bandwidth $h^*_\mu$ which is defined below. The mean estimator $\hat{\mu}_N(t; h)$ is a practical version of that defined in (2) which takes into account that some trajectories could have less than $k_0$ observation times between $t - h^*_\mu$ and $t + h^*_\mu$. The threshold defined by $k_0$ avoids considering degenerate
For this purpose, it has to be greater than or equal to $\lfloor \alpha t \rfloor + 1$; more details are provided in the Supplementary Material. The normalization of the mean estimator by $W_N(t; h)$ is also implicitly used in the definition of the estimators proposed by Li and Hsing (2010) and Zhang and Wang (2016).

To introduce our bandwidth rule, for any $h > 0$, $\alpha > 0$, let

$$c_i(t; h) = \sum_{m=1}^{M_i} \left| W_m^{(i)}(t; h) \right| \quad \text{and} \quad c_i(t; h, \alpha) = \sum_{m=1}^{M_i} \left| (T_m^{(i)} - t) / h \right|^{\alpha} \left| W_m^{(i)}(t; h) \right|,$$

(16)

and

$$\bar{C}_1(t; h, \alpha) = \frac{1}{W_N(t; h)} \sum_{i=1}^{N} w_i(t; h) c_i(t; h) c_i(t; h, \alpha).$$

With the Nadaraya-Watson (NW) estimator, all the $c_i(t; h)$ are equal to 1. Moreover,

$$\bar{C}_1(t; h, \alpha) \approx \int |u|^\alpha K(u) du.$$

(17)

Using the equivalent kernels idea, see section 3.2.2 in Fan and Gijbels (1996), the same approximation could be used in the case of local linear estimators. The accuracy of the approximation (17) could be high since it involves the $T_m^{(i)}$ close to $t$ for all the curves with $w_i(t; h) = 1$. Next, using the rule $0/0 = 0$, let

$$N_i(t; h) = \frac{w_i(t; h)}{\max_{1 \leq m \leq M_i} W_m^{(i)}(t; h)} \quad \text{and} \quad N_{i,\mu}(t; h) = \left[ \frac{1}{W_N^2(t; h)} \sum_{i=1}^{N} w_i(t; h) c_i(t; h) N_i(t; h) \right]^{-1}.$$

(18)

With the NW estimator, $N_{i,\mu}(t; h)$ is equal to $W_N(t; h)$ times the harmonic mean of $N_i(t; h)$, over the curves with $w_i(t; h) = 1$. Moreover, under the mild condition (22) below, $N_i(t; h) = O_P(mh)$.

Quantities like $c_i(t; h)$, $c_i(t; h, \alpha)$ and $\max_m |W_m^{(i)}(t; h)|$ are commonly used to bound the risk of LP estimators, see also Section 2 above. For deriving theoretical results, $c_i(t; h)$ and $c_i(t; h, \alpha)$ are usually bounded by a constant. Meanwhile, the maximum of the absolute values of the LP weights could be bounded by a suitable constant divided by the number of observation times between $t-h_{\mu}^*$ and $t+h_{\mu}^*$. See Guillou and Klutchnikoff (2011) or Tsybakov (2009). In the context of functional data, we directly use the $c_i(t; h)$, $c_i(t; h, \alpha)$ and $N_i(t; h)$ and thus entirely exploit the information contained in all the curves. The computational complexity remains low and the reward is a sharper risk bound, a better bandwidth choice and an improved mean estimator in finite samples.

We define the bandwidth for computing $\hat{\mu}_{N}(t)$ such that it minimizes the mean squared difference between $\hat{\mu}_{N}(t; h)$ and $\mu(t)$. This leads us to define the optimal bandwidth

$$h_{\mu}^* = h_{\mu}^*(t) = \arg \min_{h > 0} \mathcal{R}_\mu(t; h),$$

(19)

with

$$\mathcal{R}_\mu(t; h) = q_1^2 h^{2\hat{\alpha}_t} + q_2^2 \frac{1}{N_{i,\mu}(t; h)} + q_3^2 \left[ \frac{1}{W_N(t; h)} - \frac{1}{N} \right],$$

(20)

and

$$q_1^2 = \bar{C}_1(t; h, 2\hat{\alpha}_t) \tilde{l}_\delta^2, \quad q_2^2 = \sigma_{\text{max}}^2, \quad q_3^2 = \text{Var}(X_t),$$

where $\tilde{l}_\delta = \sqrt{\frac{\hat{\mu}}{\hat{\alpha}_t}!^2 \tilde{l}_\delta^2}$.
where \( \sigma_{\text{max}} \) is a bound for the function \( \sigma(t, x) \) in (5) and \( \hat{L}_\delta \) is an estimate of the Hölder constant \( L_\delta \) from Assumption (E1). In Section 5, we propose a simple procedure to build \( \hat{L}_\delta \) based on the preliminary nonparametric estimates of the sample paths used for \( \hat{\alpha}_t \). We show in the Appendix that \( 2R_\mu(t; h) \) is a sharp bound for \( \mathbb{E}_{M,T} \left[ \{ \hat{\mu}_N(t; h) - \bar{\mu}_N(t) \}^2 \right] \). The maximization of \( R_\mu(t; h) \) can be easily performed on a grid of \( h \) values. With some additional effort, \( R_\mu(t; h) \) can also be minimized with respect to \( k_0 \) in (14) over a small set of integers greater than \( \lfloor \alpha_t \rfloor \).

The bandwidth rule (19) could be used with both independent and common design. With common design, the \( T^{(i)}_m \equiv T_m \) and \( W^{(i)}_m(t; h) = W_m(t; h) \) no longer depend on \( i \) and the solution \( h^*_\mu \) will always be a value in the set of \( h \) such that \( W_N(t; h) = N \). Moreover, whenever \( W_N(t; h) = N \),

\[
\mathcal{C}_1(t; h, 2\hat{\alpha}_t) = \sum_{m=1}^{m} |W_m(t; h)| \sum_{i=1}^{m} \frac{T_m-t}{h} |W_m(t; h)| \quad \text{and} \quad N_\mu(t; h) = \frac{N \sum_{m=1}^{m} |W_m(t; h)|}{\max_{1 \leq n \leq m} |W_m(t; h)|}.
\]

In a data-driven way, \( h^*_\mu \) automatically chooses between interpolation and smoothing.

The following result states that our estimator \( \hat{\mu}^*_N(t) \) defined by (15) and (19) achieves the optimal rate. For simplicity, we assume that

\[
\limsup_{N,m \to \infty} \{ \log(N)/\log(m) \} < \infty, \tag{21}
\]

a technical condition which matches general situations found in applications. We also impose the following mild technical condition in the independent design case:

\[
\exists c_L, C_U > 0 \text{ such that } c_L \leq M_i m^{-1} \leq C_U, \quad \text{for all } N \text{ and } 1 \leq i \leq N. \tag{22}
\]

Moreover, in the common design case, where \( M_i \equiv m \) and the \( T^{(i)}_m \) are not changing with \( i \), we suppose that:

\[
\exists C_U \geq 1 \text{ such that } \max_{1 \leq m \leq m-1} \{ T^{(i)}_m - T^{(i)}_{m+1} \} \leq C_U \min_{1 \leq m \leq m-1} \{ T^{(i)}_{m+1} - T^{(i)}_m \}. \tag{23}
\]

**Theorem 2.** Assume that \( T^{(i)}_m \) are either independently drawn, have a continuous density which is bounded away from zero and (22) holds true, or \( T^{(i)}_m \) represent the points of a common design satisfying (23). Moreover, (21) and Assumption (E1) hold true. For \( t \in T \), let \( \hat{\alpha}_t \) be an estimator of \( \alpha_t < 1 \) computed on a separate sample such that \( \hat{\alpha}_t - \alpha_t = O_P(\log^{-1}(m)) \). Then, the estimator \( \hat{\mu}^*_N(t) = \hat{\mu}_N(t; h^*_\mu) \) defined by (15) and (19) satisfies

\[
\hat{\mu}^*_N(t) - \bar{\mu}_N(t) = O_P \left( (N m)^{-\frac{\alpha_t}{1+2\alpha_t}} \right) \quad \text{and} \quad \hat{\mu}^*_N(t) - \mu(t) = O_P \left( (N m)^{-\frac{\alpha_t}{1+2\alpha_t}} + N^{-1/2} \right),
\]

in the independent design case. Meanwhile, with the common design,

\[
\hat{\mu}^*_N(t) - \bar{\mu}_N(t) = O_P \left( \max \left\{ (N m)^{-\frac{\alpha_t}{1+2\alpha_t}}, m^{-\alpha_t} \right\} \right) = O_P \left( m^{-\alpha_t} \right),
\]

and

\[
\hat{\mu}^*_N(t) - \mu(t) = O_P \left( \max \left\{ (N m)^{-\frac{\alpha_t}{1+2\alpha_t}}, m^{-\alpha_t} \right\} + N^{-1/2} \right) = O_P \left( m^{-\alpha_t} \right).
\]
The rates achieved by $\hat{\mu}^*_N(t)$ are the best one could expect in view of the results of Cai and Yuan (2011). To avoid additional technical arguments, we only prove our result for a local regularity less than 1 and Nadaraya-Watson estimators $\hat{X}_i \gamma^{(i)}$. We conjecture that it also true for $\alpha_t \geq 1$, but we leave formal justification for future work. The difference between the common and independent designs comes from the fact that, in order to avoid degenerate mean estimator, the bandwidth cannot decrease faster than $m^{-1}$.

4.2 Adaptive covariance function estimates

For any $s, t \in \mathcal{T}$, $s \neq t$, define

$$w_i(s, t; h) = w_i(s; h) w_i(t; h) \quad \text{and} \quad W_N(s, t; h) = \sum_{i=1}^N w_i(s, t; h),$$

with $w_i(s; h)$ and $w_i(t; h)$ as in (14). Our adaptive covariance function estimator is

$$\hat{\Gamma}^*_N(s, t) = \hat{\Gamma}_N(s, t; h^*_N) \quad \text{with} \quad \hat{\Gamma}_N(s, t; h) = \hat{\gamma}_N(s, t; h) - \hat{\mu}^*_N(s) \hat{\mu}^*_N(t),$$

(24)

where $\hat{\mu}^*_N(s)$, $\hat{\mu}^*_N(t)$ are defined according to (15) with the corresponding bandwidths, and

$$\hat{\gamma}_N(s, t; h) = \frac{N}{W_N(s, t; h)} \sum_{i=1}^N w_i(s, t; h) \hat{X}_s^{(i)} \hat{X}_t^{(i)}.$$ (25)

Here, $\hat{X}_s^{(i)}$ and $\hat{X}_t^{(i)}$ are the local polynomial estimators $LP([\hat{\alpha}_s])$ and $LP([\hat{\alpha}_t])$ built, respectively, with some suitable bandwidth $h^*_N$, which is defined below. This covariance function estimator is a practical version of that defined in (3). The normalization of the mean estimator by $W_N(s, t; h)$ is also implicitly used in the definition of the estimators proposed by Li and Hsing (2010) and Zhang and Wang (2016).

We define the bandwidth for computing $\hat{\gamma}_N(s, t; h)$, and eventually $\hat{\Gamma}^*_N(s, t)$, such that it minimizes the mean squared difference between $\hat{\gamma}_N(s, t; h)$ and the unfeasible estimator $\bar{\gamma}_N(s, t) = N^{-1} \sum_{i=1}^N X_s^{(i)} X_t^{(i)}$ of $\mathbb{E}(X_s^{(i)} X_t^{(i)})$. To this aim, we define modified versions of $\mathcal{N}_i(t; s, h)$ and $\mathcal{N}_{\hat{\mu}}(t; s, h)$, see (18), taking into account only the curves with $w_i(s, t; h) = 1$:

$$\mathcal{N}_i(t|s; h) = \frac{w_i(s, t; h)}{\max_{1 \leq m \leq M_i} |W^{(i)}_m(t, h)|},$$

and

$$\mathcal{N}_\Gamma(t|s; h) = \left[ \frac{1}{W^2_N(s, t; h)} \sum_{i=1}^N w_i(s, t; h) \frac{c_i(t; h)}{\mathcal{N}_i(t|s; h)} \right]^{-1},$$

where the $c_i(t; h)$ are defined as in (16). This idea leads us to define the optimal bandwidth as

$$h^*_\Gamma = h^*_\Gamma(s, t) = \arg\min_{h>0} \{ \mathcal{R}_\Gamma(s|t; h) + \mathcal{R}_\Gamma(t|s; h) \},$$

(26)

with

$$\mathcal{R}_\Gamma(t|s; h) = q_1 h^{2\alpha_t} + \frac{q_2}{N_\Gamma(t|s; h)} + q_3 \left[ \frac{1}{W_N(s, t; h)} - \frac{1}{N} \right].$$ (27)
In the last equation, \( q_\ell, 1 \leq \ell \leq 3 \), are defined by:
\[
q_1^2 = 2E(X_s^2) \frac{\overline{C}_1(t; h, 2\hat{\alpha}t)}{[\hat{\alpha}t]!^2} \hat{L}_2^2
\]
\[
q_2^2 = \sigma^2_{\text{max}} E(X_s^2),
\]
\[
q_3^2 = \frac{\text{Var}(X_s X_t)}{2},
\]
where
\[
\overline{C}_1(t|s; h, \alpha) = \sum_{i=1}^{N} \frac{w_i(s, t; h) c_i(t; h) c_i(t; \alpha)}{W_N(s, t; h)}.
\]

With NW or local linear estimators,
\[
\overline{C}_1(t|s; h, \alpha) \approx \int |u|^\alpha K(u) du.
\]

The details for (28) are provided in the Supplementary material. The definition of \( R_T(s|t; h) \) is the same as in (27) with the occurrences of \( s \) and \( t \) switched. We show in the Supplementary Material that the function of \( h \) minimized in (26) is a sharp bound for \( E_{M,T} \left[ \{ \tilde{\gamma}_N(s, t; h) - \gamma_N(s, t) \}^2 \right] / 2 \).

The sum of the first two terms in the expressions of \( R_T(s|t; h) \) and \( R_T(t|s; h) \) represents the quadratic risk of our estimator of \( E(X_s X_t) \) compared to the unfeasible one based on the true values \( X_s^{(i)} X_t^{(i)} \) from the curves yielding non-degenerate estimates \( \tilde{X}_s^{(i)} \tilde{X}_t^{(i)} \). Like for the mean function, the third term in (27) penalizes for the number of curves which are dropped when calculating our estimator. The minimization of \( R_T(s|t; h) + R_T(t|s; h) \) can be performed on a grid of values \( h \). The minimization could be also done over a set of values \( k_0 \) used in (14).

Like for the mean function, with obvious modifications, the definition (26) could be used with both independent and common design. Indeed, with common design, the solution \( h_T^* \) will always be a value in the set of \( h \) such that \( W_N(s, t; h) = N \). In a completely data-driven way, \( h_T^* \) will automatically choose between interpolation and smoothing.

**Theorem 3.** Assume that the conditions of Theorem 2 are met for \( s, t \in T \) such that \( s \neq t \). Moreover, \( \sup_{t \in T} E(X_t^4) < \infty \). Let \( \alpha(s, t) = \min \{ \alpha_s, \alpha_t \} \) and assume \( \alpha(s, t) < 1 \). Then the estimator \( \hat{\Gamma}_N^*(s, t) = \hat{\Gamma}_N^*(s, t; h_T^* \) defined by (24) and (26) satisfies
\[
\hat{\Gamma}_N^*(s, t) - \Gamma(s, t) = O_P \left( (Nm)^{-\alpha(s, t)} \right)
\]
and
\[
\hat{\Gamma}_N^*(s, t) - \Gamma(s, t) = O_P \left( (Nm)^{-\alpha(s, t)} + N^{-1/2} \right),
\]
in the independent design case. Meanwhile with the common design,
\[
\hat{\Gamma}_N^*(s, t) - \Gamma(s, t) = O_P \left( \max \left\{ (Nm)^{-\alpha(s, t)} + m^{-\alpha(s, t)} \right\} \right) = O_P \left( m^{-\alpha(s, t)} \right),
\]
and
\[
\hat{\Gamma}_N^*(s, t) - \Gamma(s, t) = O_P \left( \max \left\{ (Nm)^{-\alpha(s, t)} + m^{-\alpha(s, t)} \right\} + N^{-1/2} \right) = O_P \left( m^{-\alpha(s, t)} \right).
\]

The rates achieved by \( \hat{\Gamma}_N^*(s, t) \) are the best one could expect in view of the results of Cai and Yuan (2010). For now, we only prove our result for the case \( \min \{ \alpha_s, \alpha_t \} < 1 \) and Nadaraya-Watson estimators \( \tilde{X}_t^{(i)} \). We conjecture that it also true in the general case.
4.3 The estimator on the diagonal band of the covariance function

As mentioned in (3) and (4), we propose to use the estimator (25) only outside the diagonal set \( \{(s, t) : |s - t| \leq \delta\} \), for some suitable \( \delta > 0 \). It remains to give a data-driven rule for choosing \( \delta \) decreasing to zero, and propose an estimator for \( \mathbb{E}(X_s X_t) \) when \( s \) and \( t \) are in the diagonal set. Let \( T^o_k, 1 \leq k \leq K \), as defined in Assumption (E1). With a piecewise constant regularity, the value of \( \delta \) depends on \( k \).

To understand how to build a covariance estimator which is optimal in integrated squared norm, for \( 1 \leq k \leq K \) and \( s, t \in T^o_k \) such that \( s \leq t \), let \( u = (s + t)/2 \) and

\[
\tilde{D}(\delta) := \int_{t-\delta \leq s \leq t} \left\{ \Gamma_N(u - \delta/2, u + \delta/2) - \Gamma_N(s, t) \right\}^2 ds dt.
\]

Let \( \alpha \) denote the local regularity corresponding to \( u = (s + t)/2 \). Under mild assumptions on the moments of \( X_t \) and \( X_t - X_s \), we show in the Appendix that

\[
\tilde{D}(\delta) = O_P \left( \delta^{2\alpha + 1} \right). \tag{29}
\]

Thus, for \( (s, t) \) inside the diagonal band, it suffices to use \( \hat{\Gamma}_N(u - \delta/2, u + \delta/2) \) defined according to (24) when \( s \leq t \) and use the symmetry of the covariance function when \( s > t \). That is, in order to estimate the covariance function at a point from the diagonal set, we simply apply the covariance function estimator, designed for outside the diagonal set, for the closest point on the boundary of the diagonal band. The rate (29) indicates that in order to make this a suitable estimator, \( \delta \) should have a rate of decrease slower than the power \( (2\alpha + 1)^{-1} \) of the optimal rate achievable by a nonparametric estimator of the covariance. If \( \hat{\alpha} \) is the estimate of \( \alpha \), we can take

\[
\delta = \left\{ N^{-2} \sum_{i=1}^{N} \left( 1/M_i \right) \right\}^{c} \quad \text{for some} \quad \frac{2\hat{\alpha}}{\{2\hat{\alpha} + 1\}^2} < c < \frac{1}{2\hat{\alpha} + 1},
\]

for instance \( c = \{2\hat{\alpha} + 1/2\}/\{2\hat{\alpha} + 1\}^2 \).

5 Empirical study

5.1 Implementation aspects

The risks \( \mathcal{R}_\mu \) and \( \mathcal{R}_\Gamma \), defined in (20) and (27), respectively, depend on \( L^2_\delta \) and the conditional variance bound \( \sigma^2_{\max} \). In view of (10), if \( t_2 \) is the midpoint of \([t_1, t_3] \),

\[
L^2_\delta \approx \frac{1}{2} \left( \frac{\theta_\delta(t_2, t_3)}{|t_3 - t_2|^{2(\alpha - \delta)}} + \frac{\theta_\delta(t_1, t_2)}{|t_2 - t_1|^{2(\alpha - \delta)}} \right),
\]

provided \( t_3 - t_1 = \Delta_s/2 \) is small. Given the estimate \( \hat{\alpha}_t \) and the estimators \( \hat{\theta}_\delta(t_2, t_3) \) and \( \hat{\theta}_\delta(t_1, t_2) \) as in (11), with \( \delta = [\hat{\alpha}_t] \), we then define a natural estimator of \( L^2_\delta \) as

\[
\widetilde{L}^2_\delta \approx \frac{1}{2} \left( \frac{\hat{\theta}_{[\hat{\alpha}_t]}(t_2, t_3)}{|t_3 - t_2|^{2(\hat{\alpha}_t - [\hat{\alpha}_t])}} + \frac{\hat{\theta}_{[\hat{\alpha}_t]}(t_1, t_2)}{|t_2 - t_1|^{2(\hat{\alpha}_t - [\hat{\alpha}_t])}} \right).
\]
To estimate the conditional variance bound, let us first consider the case where \( \sigma^2(t, x) \) does not depend on \( x \). In this case, one can compute

\[
\tilde{\sigma}^2(t) = \frac{1}{N} \sum_{i=1}^{N} \frac{1}{2|S_i|} \sum_{m \in S_i} \left[ Y_{m}^{(i)} - Y_{m-1}^{(i)} \right]^2,
\]

where \( S_i \) is a subset of indices \( m \) for the \( i \)-th trajectory. When the variance of the errors is considered constant, \( S_i \) can be the set \( \{2, 3, \ldots, M_i\} \). When the variance depends on \( t \), one could define \( S_i \) as the set of indices corresponding to the \( K_0 \) values \( T_m^{(i)} \) closest to \( t \). The theory allows a choice such as \( K_0 = [\hat{m} \exp (- (\log \log \hat{m})^2)] \), with \( \hat{m} = N^{-1} \sum_{i=1}^{N} M_i \). Then \( \sigma_{\max}^2 \) could be \( \max_{t \in T} \tilde{\sigma}^2(t) \), and this choice was used in our empirical investigation. When the variance of the errors also depends on the realizations \( X_u \), in general it is no longer possible to consistently estimate \( \sigma^2(u, X_u) \). However, the simple inspection of the squared differences between the observations \( Y_{m}^{(i)} \) and the presmoothed values allows to build a bound \( \sigma_{\max}^2 \). In our simulation experiments, \( \sigma_{\max}^2 \) obtained from the estimate (30), with \( S_i \) given by \( K_0 \), works well.

5.2 Simulation design

Our simulation study is based on the Household Active Power Consumption dataset which was sourced from the UC Irvine Machine Learning Repository (https://archive.ics.uci.edu/ml/datasets/Individual+household+electric+power+consumption). This dataset contains diverse energy related features gathered in a house located near Paris, every minute between December 2006 and November 2010. In total, its represents around 1440 points. In Figure 1 in the Supplementary Material, we plot the logarithms of the ordered eigenvalues of a smoothed version of the empirical covariance matrix. Moreover, we present the empirical mean curve and three randomly selected curves from the sample. The time is normalized such that \( T = [0, 1] \). We also present in that figure the estimates of the local regularity obtained with the 708 voltage curves. Most of the estimates are smaller than 1.

The 708 voltage curves are used to build a mean function \( \mu(\cdot) \), and a covariance function \( \Gamma(\cdot, \cdot) \). We also derive a conditional variance function \( \sigma^2(t, x) \) for the noise which we plot in Figure 1a. Next, we generate samples of independent trajectories from the Gaussian process characterized by these \( \mu(\cdot) \) and \( \Gamma(\cdot, \cdot) \). Their local regularity is approximately equal to 0.7. Finally, we add the heteroscedastic noise. A random sample of curves generated according to our simulation setup are plotted in Figure 1b. The details on the construction of our simulation setup are provided in the Supplementary Material.

We consider four experiments, each of them replicated 500 times. In Experiment 1, \( N \in \{40, 100, 200\} \), \( m \in \{40, 100, 200\} \), and \( (1 - p)m \leq M_i \leq (1 + p)m \) with \( p = 0.2 \). Experiment 2 is the same as Experiment 1 except that \( p = 0.5 \). In Experiment 3 we modify Experiment 1 by multiplying the conditional variance of the error terms by 2. Finally, in Experiment 4 we consider the setup from Experiment 1 but with a higher regularity for the mean function. In the Supplementary Material we present plots of the mean functions and trajectories from Experiment 4.

For the presmoothing, we use the `locpoly` function in R. Given that most of the local regularity estimates are between 0 and 1, our mean and covariance functions estimators are built...
with the NW smoother. Moreover, the value $k_0$ in (14) is set equal to 2, and we use the biweight kernel $K(t) = (15/16) (1 - t^2)^2 1_{[-1,1]}(t)$. The estimates $\hat{\alpha}_t$ and the estimates of the mean and covariance functions are obtained using the same data. That means we did not use a learning sample for $\hat{\alpha}_t$. An implementation of the method used in the four experiments is available as a R package on Github at the URL address: https://github.com/StevenGolovkine/funestim. More details on the implementation are also available in the Supplementary Material.

5.3 Mean estimation

Concerning the estimation of the mean, the estimates $\hat{\alpha}_t$ are computed using (11), on a uniform grid of 50 points $t_2$ between 0.05 and 0.95, with $t_3 - t_1 = \Delta_s/2 = \exp(-\log^{1/2}(\hat{m}))$. For each value of the 50 estimates $\hat{\alpha}_t$, we compute the optimal bandwidths $h_{\hat{\alpha}}^* (t)$ by minimization with respect to $h$ over a logarithmic grid of 151 points.

Our mean estimator is compared to that of Cai and Yuan (2011), denoted $\hat{\mu}_{\text{CY}}$, and Zhang and Wang (2016), denoted $\hat{\mu}_{\text{ZW}}$. To compute $\hat{\mu}_{\text{CY}}$, we use smooth.splines in R, with the $M_1 + \ldots + M_N$ data points $(Y^{(i)}_m, T^{(i)}_m)$. To obtain $\hat{\mu}_{\text{ZW}}$, we use the R package fdapace, see Carroll et al. (2021). To compare the accuracy of the estimators, we use the ISE risk with respect to the target. For any $f$ and $g$ real-valued functions defined on $[0, 1]$, the ISE is defined as

$$\text{ISE}(f, g) = \|f - g\|^2 = \int_{[0,1]} (f(t) - g(t))^2 dt.$$ 

We approximate the integral using the mean estimates on a uniform grid of 101 points and the trapezoidal rule. For each configuration $N, m, p$, and each of the 500 samples, we compute the ISEs with respect to the infeasible $\tilde{\mu}$, and the ISEs with respect to the mean function $\mu$ used for generating the samples. The 101 bandwidth values used for our estimator are obtained from the 50 optimal bandwidths $h_{\hat{\alpha}}^*$ by linear interpolation.

The results obtained in Experiment 1 are plotted in the Figure 2, on a logarithmic scale. The results for the other three experiments are relegated to the Supplementary Material. Our mean function estimator reveals good performance. It provides a much more accurate estimate of the infeasible empirical mean function $\tilde{\mu}$. When compared to the true mean function, except the cases $N \in \{100, 200\}$ and $m = 40$, our estimator outperforms the competitors. In that cases, our estimator, $\hat{\mu}_{\text{CY}}$ and $\hat{\mu}_{\text{ZW}}$ have similar performance. The fact that the advantage of our estimator wanes in these cases is explained by the poorer performance of the infeasible empirical mean function. Similar conclusions could be drawn from the Experiments 2 to 4. In the setup with a more regular mean function, the advantage of our estimator diminishes for all pairs $N, m$.

5.4 Covariance estimation

Concerning the estimation of the covariance, the estimates $\hat{\alpha}_t$ are computed using (11), on a uniform grid of 10 points $t_2$ between 0.05 and 0.95, with $t_3 - t_1 = \Delta_s/2 = \exp(-\log^{1/2}(\hat{m}))$. For each values $s, t$ in the grid, we compute the optimal bandwidths $h_{\hat{\alpha}}^*(s, t)$ by minimization over a logarithmic grid of 41 points between 0.01 and 0.1. Our covariance estimator is compared to the ones from Cai and Yuan (2010), denoted $\hat{\Gamma}_{\text{CY}}$, and from Zhang and Wang (2016), denoted $\hat{\Gamma}_{\text{ZW}}$. We compute $\hat{\Gamma}_{\text{CY}}$ using the R package ssfcov, see Cai and Yuan (2010). For $\hat{\Gamma}_{\text{ZW}}$, we use
the R package \texttt{fdapace}, see Carroll et al. (2021). The risk we consider is defined as

\[
ISE(f, g) = \|f - g\|^2 = \int_{[0,1]} \int_{[0,1]} (f(s, t) - g(s, t))^2 dsdt,
\]

approximated by the trapezoidal rule applied with a uniform grid of 101 × 101 points. The 101 × 101 bandwidth values used for our estimator are obtained from the 10 × 10 optimal bandwidths $h_{T}^*$ by linear interpolation. For each configuration $N$, $m$, $p$, and each of the 500 samples, we compute the ISEs with respect to the infeasible $\tilde{\Gamma}$, and the ISEs with respect to the covariance function $\Gamma$ used for generating the samples.

The results obtained in \textit{Experiment 1} are plotted in the Figure 3, on a logarithmic scale. The results for the other three experiments are relegated to the Supplementary Material. We were not able to calculate $\hat{\Gamma}_{CY}$ when $M = 200$ and $N = 200$, and therefore some cases are not reported. Our estimator provides the most accurate approximation of $\tilde{\Gamma}$. Our estimator and $\hat{\Gamma}_{ZW}$ show better accuracy for estimating $\Gamma$ in all cases considered. Meanwhile, our estimator performs similarly or slightly better than $\hat{\Gamma}_{ZW}$. The advantage of our approach increases with $N$. Let us point out the clear advantage for our estimator in terms of computation times. For instance, with $N = 100$ and $M = 100$, our covariance estimator is computed about three times faster than that of Zhang and Wang (2016) (with fixed bandwidth) and about thirty times faster than that of Cai and Yuan (2010). More details on the computation time are provided in the Supplementary Material.
Figure 2: Results from Experiment 1 on the log-scale.
(a) ISE with respect to the empirical covariance \( \hat{\Gamma} \) in each simulated sample

(b) ISE with respect to the true covariance function \( \Gamma \)

Figure 3: Results from Experiment 1 on the log-scale
6 Discussion and conclusions

We propose new nonparametric estimators for the mean and covariance function. They are built using a novel ‘smoothing first, then estimate’ strategy based on univariate local polynomials. The main novelty comes from the fact that the optimal bandwidths for the local polynomials are selected by minimization of suitable penalized quadratic risks. The penalized risks for the mean and the covariance functions are quite similar, could be easily built from data and be optimized on a grid of bandwidths. What distinguishes them from the usual sum between the squared bias and the variance, is a penalty for the fact that not all the curves have enough observation points to be included in the final estimator. Removing curves from the nonparametric estimators of the mean and covariance functions is an aspect which characterizes practically all smoothing-based approaches. Indeed, to entirely benefit from the replication feature of functional data, one has to determine the amount of smoothing for the mean and covariance estimation using all the curves. In this case some curves could present too few observation points and thus will be dropped. This is more likely to happen in the so-called sparse regime. To our best knowledge, our bandwidth choice procedure is the first attempt to explicitly account for this aspect. We thus build estimators which achieve the optimal rates of convergence in a completely adaptive, data-driven way. The theoretical results are derived under very mild conditions. In particular, the curves could be observed with heteroscedastic errors at discrete observations points. These points could be common to all curves or they could change randomly from one curve to another. In the case of the common observation points, our procedure automatically chooses between smoothing and interpolation, the latter being known to be rate optimal, but is not necessarily the best solution with finite samples.

Our nonparametric estimation approach relies on a probabilistic concept of local regularity for the sample paths of the process generating the curves. In some common examples, this local regularity is related to the polynomial decrease rate of the eigenvalues of the covariance operator, a characteristic of the data generating process widely used in the literature and usually supposed to be known. The local regularity also determines the regularity of the trajectories, the usual concept used in nonparametric regression. It is well-known that the optimal rates, in the minimax sense, for estimating the mean and covariance functions depend on the regularity of the trajectories. Moreover, the so-called sparse and dense regimes, commonly invoked in functional data analysis, are defined using the regularity of the trajectories, which usually is supposed to be known. We therefore propose a novel simple estimator of the local regularity of the process and we use it to build our penalized quadratic risk. Applied to real data, the local regularity estimator reveals that the regularity of the trajectories could be quite far from what is usually assumed in the existing theoretical contributions.

Our method performs quite well in simulations and outperforms the main competitors when the mean and covariance functions have a regularity close to that of the trajectories. The approach is still satisfactory when the mean or the covariance functions are more regular than the trajectories. The reason is that, in some sense, our nonparametric estimators are as close as possible to the empirical mean and covariance, respectively, which are the ideal estimators if the trajectories were observed at any point without error. In the case where the mean and covariance function are smoother than the trajectories, our penalized quadratic risk should be built using the mean or covariance functions’ regularity instead of the trajectories’ regularity. However, for now, the estimation of the regularity of the mean or covariance function remains an open problem.
A Details on the definition (20)

To explain our empirical risk bound \( R_{\mu}(t; h) \) defined in (20), let

\[
\tilde{\mu}_W(t; h) = \frac{1}{W_N(t; h)} \sum_{i=1}^{N} w_i(t; h) X_t^{(i)},
\]

be the unfeasible estimator of \( \mu(\cdot) \) using only the curves for which \( \hat{X}_t^{(i)} \) is well-defined. Using the definitions, see (7), we then have,

\[
\mathbb{E}_{M,T} \left[ \{ \tilde{\mu}_N(t) - \tilde{\mu}_N(t; h) \}^2 \right] = \mathbb{E}_{M,T} \left[ \left\{ \tilde{\mu}_N(t) - \tilde{\mu}_W(t; h) - \frac{1}{W_N(t; h)} \sum_{i=1}^{N} w_i(t; h) \left( B_t^{(i)} + V_t^{(i)} \right) \right\}^2 \right]
\]

\[
\leq 2\mathbb{E}_{M,T} \left[ \{ \tilde{\mu}_N(t) - \tilde{\mu}_W(t; h) \}^2 \right] + 2\mathbb{E}_{M,T} \left[ \left\{ \frac{1}{W_N(t; h)} \sum_{i=1}^{N} w_i(t; h) \left( B_t^{(i)} + V_t^{(i)} \right) \right\}^2 \right]
\]

\[
= 2E_1 + 2E_2.
\]

In the following, for simplicity, we write \( w_i \) and \( W_N \) instead of \( w_i(t; h) \) and \( W_N(t; h) \), respectively. Since

\[
\tilde{\mu}_W(t; h) - \tilde{\mu}_N(t) = \frac{1}{W_N} \sum_{i=1}^{N} \left\{ X_t^{(i)} - \mu(t) \right\} \left\{ w_i - \frac{W_N}{N} \right\},
\]

and the trajectories are drawn independently, we have

\[
E_1 = \frac{\text{Var}(X_t)}{W_N^2} \sum_{i=1}^{N} \left\{ w_i - \frac{W_N}{N} \right\}^2 = \text{Var}(X_t) \left\{ \frac{1}{W_N} - \frac{1}{N} \right\}.
\]

For \( E_2 \), let us first look at the bias part. By the arguments used in the proof of Proposition 1.13 in Tsybakov (2009) and the Cauchy-Schwarz inequality, we have

\[
\mathbb{E}_{M,T} \left[ \left\{ \frac{1}{W_N} \sum_{i=1}^{N} w_i B_t^{(i)} \right\}^2 \right] \leq \frac{1}{[\hat{\alpha}_t]^2} \times \frac{1}{W_N} \sum_{i=1}^{N} w_i \mathbb{E}_{M,T} \left[ \left\{ B_t^{(i)} \right\}^2 \right]
\]

\[
\leq \frac{1}{[\hat{\alpha}_t]^2} \times \frac{1}{W_N} \sum_{i=1}^{N} w_i \sum_{m=1}^{M_t} W_{m}^{(i)}(t) \sum_{m=1}^{M_t} \mathbb{E} \left( \left\{ \nabla [\hat{\alpha}_t] X^{(i)}(T_m^{(i)}) - \nabla [\hat{\alpha}_t] X_t^{(i)} \right\}^2 \right| T_{\text{obs}}^{(i)} \mid W_{m}^{(i)}(t) \right)
\]

\[
= L_3^2 h^{2\hat{\alpha}_t} \{ 1 + o_P(1) \} \times \frac{1}{W_N} \sum_{i=1}^{N} w_i \sum_{m=1}^{M_t} W_{m}^{(i)}(t) \times \sum_{m=1}^{M_t} \left| \frac{T_{m}^{(i)} - t}{h} \right|^{2\hat{\alpha}_t} \left| W_{m}^{(i)}(t) \right|
\]

\[
= L_3^2 h^{2\hat{\alpha}_t} \left\{ 1 + o_P(1) \right\} \times \mathcal{C}_1(t; h, 2\hat{\alpha}_t) \times \{ 1 + o_P(1) \}.
\]
For the first equality we use the condition \((H2)\) on the event \(\{\hat{\alpha}_t = \delta\}\). By Theorem 1, this event has an exponentially small probability. When the kernel function is supported on \([-1,1]\), we could use the bound
\[
\mathbb{E}_{M,T}\left[\left\{\frac{1}{W_N}\sum_{i=1}^{N} w_i B_t^{(i)}\right\}^2\right] \leq \frac{L_2^2 h^{2\hat{\alpha}_t}\{1 + o_P(1)\}}{[\hat{\alpha}_t]^2} \times \frac{1}{W_N}\sum_{i=1}^{N} w_i \left\{\sum_{m=1}^{M_i} \left| W_m^{(i)}(t) \right| \right\}^2.
\]
In the case \(|\hat{\alpha}_t| = 0\), with the Nadaraya-Watson estimator, we could use the more refined bound
\[
\mathbb{E}_{M,T}\left[\left\{\frac{1}{W_N}\sum_{i=1}^{N} w_i B_t^{(i)}\right\}^2\right] \leq \frac{L_2^2 h^{2\hat{\alpha}_t}\{1 + o_P(1)\}}{[\hat{\alpha}_t]^2} \times \frac{1}{W_N}\sum_{i=1}^{N} w_i \left\{\sum_{m=1}^{M_i} \left| T_m^{(i)} - t \right| h \right\}^{2\hat{\alpha}_t} W_m^{(i)}(t)
\approx \frac{L_2^2 h^{2\hat{\alpha}_t}\{1 + o_P(1)\}}{[\hat{\alpha}_t]^2} \times \int |u^{2\hat{\alpha}_t} K(u) du.
\]
Using the equivalent kernels idea, see section 3.2.2 in Fan and Gijbels (1996), the bound on the last line of the last display could be used in the case of local linear estimators. To complete the bound for \(E_2\), note that by construction, \(\mathbb{E}_{M,T}\left\{V_t^{(i)} B_t^{(j)}\right\} = 0\) and
\[
\mathbb{E}_{M,T}\left\{V_t^{(i)} V_t^{(j)}\right\} = 0, \quad \forall 1 \leq i \neq j \leq N.
\]
Up to negligible terms, we can then write
\[
E_2 \leq \mathbb{E}_{M,T}\left[\left\{\frac{1}{W_N}\sum_{i=1}^{N} w_i B_t^{(i)}\right\}^2\right] + \frac{1}{W_N^2}\sum_{i=1}^{N} w_i \mathbb{E}_{M,T}\left[\left\{V_t^{(i)}\right\}^2\right]
\leq h^{2\hat{\alpha}_t} \frac{L_2^2}{[\hat{\alpha}_t]^2} \overline{C}_1(t;h,2\hat{\alpha}_t) + \frac{\sigma_{\text{max}}}{W_N^2}\sum_{i=1}^{N} w_i \left\{\max_{m} \left| W_m^{(i)}(t;h) \right| \right\} \times \sum_{m=1}^{M_i} \left| W_m^{(i)}(t;h) \right|
\]
\[
= h^{2\hat{\alpha}_t} \frac{L_2^2}{[\hat{\alpha}_t]^2} \overline{C}_1(t;h,2\hat{\alpha}_t) + \frac{\sigma_{\text{max}}}{W_N^2}\sum_{i=1}^{N} w_i \frac{c(t;h)}{N_i(t;h)},
\]
where \(\sigma_{\text{max}}\) is a bound for the function \(\sigma(t,x)\) in (5).

**B  Proofs**

*Proof of Theorem 2.* For simplicity, \(\hat{\mu}_N^*\) is built with \(k_0 = 1\) and the uniform kernel. The lines below adapt to other choices of \(k_0\) and \(K(\cdot)\) at the price of more involved technical arguments. First, let us prove that
\[
\frac{1}{W_N(t;h)} - \frac{1}{N} \leq \max\left\{h^{2\hat{\alpha}_t}, \overline{N}^{-1}(t;h) \right\} O_P(1), \quad \text{(A.1)}
\]
provided $Nmh \to \infty$ and $h \to 0$. Note that if $W_N(t; h) = 0$, then necessarily $N_\mu(t; h) = 0$. The
property (A.1) is implied by the following ones: there exist two constants $c_1, c_2 > 0$ such that
\[
\frac{1}{N_\mu(t; h)} \geq \frac{c_1 \{1 + o_P(1)\}}{Nmh},
\]  
(A.2)

and
\[
\frac{1}{W_N(t; h) + 1} \leq \max \left\{ \frac{1}{N + 1}, \frac{c_2 \{1 + o_P(1)\}}{Nmh} \right\}.
\]  
(A.3)

Indeed, the latter two properties imply
\[
\frac{1}{W_N(t; h) + 1} - \frac{1}{N + 1} = \max \left\{ 0, \frac{1}{N} \left( \frac{1}{mh} - 1 \right) \right\} O_P(1)
\]
\[
\leq \max \left\{ h^{2s_\alpha}, \frac{1}{N_\mu(t; h)} \right\} O_P(1),
\]

and from this we obtain (A.1) because
\[
\{W_N(t; h) + 1\}^{-1} - \{N + 1\}^{-1} = \{W_N^{-1}(t; h) - N^{-1}\} \frac{N\{N + 1\}}{W_N(t; h)\{W_N(t; h) + 1\}}.
\]

To justify (A.2) and (A.3), we omit the arguments $t$ and $h$. For instance, we write $W_N$ and $N_\mu$ instead of $W_N(t; h)$ and $N_\mu(t; h)$, respectively.

Using the fact that the harmonic mean is less than or equal to the mean we obtain
\[
\frac{1}{N_\mu(t; h)} \geq \frac{c_i}{\sum_{i=1}^N w_i N_i},
\]

with $c_i = c_i(t; h)$ and $N_i = N_i(t; h)$ defined in (16) and (18), respectively. In the case of $\tilde{\alpha}_i < 1$, for all $i$ we have $c_i \equiv 1$. To justify (A.2) it suffices to prove that there exists a sequence of variables $C_N$ such that almost surely $0 < \liminf_N C_N < \limsup_N C_N < \infty$, and
\[
C_N \{1 + o_P(1)\} = \frac{\sum_{i=1}^N w_i N_i}{Nmh}.
\]  
(A.4)

For this purpose, let us notice that in the case of an NW estimator with a uniform kernel, when $k_0 = 1$,
\[
\sum_{i=1}^N w_i N_i = \sum_{i=1}^N \sum_{m=1}^{M_i} \mathbb{1}\{|T_{m(i)} - t| \leq h\}.
\]

Thus $\sum_{i=1}^N w_i N_i$ is a binomial random variable with $M_1 + \ldots + M_N$ trials and success probability
\[
p(h) = p(h; t) = \int_{t-h}^{t+h} f_T(s)ds \approx 2f_T(t)h,
\]

provided $f_T$, the density of the $T_{m(i)}$, is continuous at $t$. Since $Nmh \to \infty$, by Bernstein’s
inequality the property (A.4) holds true with
\[
C_N = 2f_T(t)\frac{M_1 + \ldots + M_N}{Nm}.
\]
Condition (22) guarantees that almost surely $C_N$ stays away from zero and infinity. The property (A.2) follows.

For justifying (A.3), let us first recall that by definition, $W_N(t; h) \leq N$. Thus it remains to show that

$$\frac{1}{W_N(t; h) + 1} \leq c_2 \{1 + o_{\mathcal{P}}(1)\} \frac{1}{Nmh}.$$  

For this purpose, it suffices to show that there exists a positive constant $c_W > 0$ such that

$$\min\{N, c_W Nmh \{1 + o_{\mathcal{P}}(1)\}\} \leq W_N(t; h). \quad (A.5)$$

Let $\mathbb{P}_M(\cdot) = \mathbb{P}(\cdot \mid M_i, 1 \leq i \leq N)$. We now note that

$$p_i = p_i(t) := \mathbb{P}_M(w_i = 1) = 1 - [1 - p(h)]^{M_i}.$$  

Next, note that the variable $W_N(t; h)$ is a sum of $N$ independent Bernoulli variables with probabilities $p_i$. In the case where $mh \geq \zeta > 0$ for some constant $\zeta$, since $\{1 - p(h)\}^{-1/p(h)} > e$ for any $h < 1/2$, we can write

$$p_i = 1 - [1 - p(h)]^{M_i} \geq 1 - [1 - p(h)]^{c_L m} \geq 1 - \exp(-c_L mp(h))$$

$$\approx 1 - \exp(-2f_T(t) c_L mh) \geq 1 - \exp(-2f_T(t) c_L \zeta),$$

and the approximations are valid as soon as $h \to 0$. Thus, in the case where $mh \geq \zeta > 0$, $\liminf p_i > 0$. Then, Bernstein’s inequality yields the property (A.5). More precisely, $W_N(t; h)$ will increase at a rate at least as fast as $[1 - \exp(-2f_T(t) c_L \zeta)] N$. In the case $mh \to 0$ we have

$$p_i = \mathbb{P}_M(w_i = 1) \approx 1 - \exp(-M_i p(h)) \approx M_i p(h) \geq 2c_L f_T(t) mh.$$  

Again, Bernstein’s inequality guarantees that

$$\frac{W_N(t; h)}{p_1 + \ldots + p_N} = 1 + o_{\mathcal{P}}(1).$$

Since $p_1 + \ldots + p_N \geq 2c_L f_T(t) + Nmh$, condition (A.5) follows with $c_W = 2c_L f_T(t)$.

Finally, to complete the proof in the independent design case, it suffices first to notice that from above, we have

$$\max \{h^{2\alpha_t}, N^{-1}_\mu(t; h)\} = O_{\mathcal{P}} \left( h^{2\alpha_t} + (Nmh)^{-1} \right),$$

which is minimized by $h$ with the rate $(Nm)^{-2\alpha_t/(2\alpha_t + 1)}$. Next, condition (21) guarantees that $\log(Nm)/\log(m)$ is bounded, and thus $h^{2\alpha_t} = h^{2\alpha_t} \{1 + o_{\mathcal{P}}(1)\}$ whenever $\alpha_t - \alpha_t = o_{\mathcal{P}}(\log^{-1}(m))$. For the rate of $\tilde{\mu}_N(t) - \mu(t)$ we simply add the rate of $\tilde{\mu}_N(t) - \mu(t)$.

With a common design $W_N(t; h)$ can only take the values 0 or $N^{-1}$. Thus the penalty introduced by $W_N(t; h)^{-1} - N^{-1}$ plays a different role. It constrains the bandwidth to be greater than or equal to the lengths of the intervals $[T^{(i)}_m, T^{(i)}_{m+1}]$ including $t$. By condition (23), this means that the rate of convergence of $\tilde{\mu}_N(t) - \mu(t)$ could not be faster than $O_{\mathcal{P}} (m^{-2\alpha_t})$. This aspect is automatically included in the definition of $R_\mu(t; h)$ because, under the constraint $mh \geq c_L/Cu$,

$$O_{\mathcal{P}} \left( \max \{h^{2\alpha_t}, N^{-1}_\mu(t; h)\} \right) = O_{\mathcal{P}} \left( \max \{h^{2\alpha_t}, (Nmh)^{-1}, N^{-1}\} \right) = O_{\mathcal{P}} (m^{-2\alpha_t}).$$

Finally, $\alpha_t$ can be replaced by $\alpha_t$ using the arguments from the independent design case.  \qed
The proof of Theorem 3 follows the lines of that of the proof of Theorem 2 and is left to the Supplementary Material.

Proof of equation (29). To prove the rate of \( \widetilde{D}(\vartheta) \), we use the assumptions: \( \sup_{t \in \mathcal{T}} \mathbb{E}(X_t^4) < \infty \) and there exists a constant \( c \) such that

\[
\mathbb{E}(|X_s - X_t|^4) \leq c \mathbb{E}^2(|X_s - X_t|^2), \quad \forall s, t \in \mathcal{T}.
\]

Omitting the integration domain, we can now write

\[
\mathbb{E}[\widetilde{D}(\vartheta)] \leq 2 \int \int \mathbb{E} \left[ \left( \frac{1}{N} \sum_{i=1}^{N} \left\{ X_s^{(i)} X_t^{(i)} - X_{\frac{s+t-\vartheta}{2}}^{(i)} X_{\frac{s+t+\vartheta}{2}}^{(i)} \right\} \right)^2 \right] dsdt
\]

\[
+ 2 \int \int \mathbb{E} \left[ \left( \frac{1}{N} \sum_{i=1}^{N} \left\{ X_s^{(i)} \right\} \left\{ X_t^{(i)} \right\} - \left\{ \frac{1}{N} \sum_{i=1}^{N} X_{\frac{s+t-\vartheta}{2}}^{(i)} \right\} \left\{ \frac{1}{N} \sum_{i=1}^{N} X_{\frac{s+t+\vartheta}{2}}^{(i)} \right\} \right)^2 \right] dsdt
\]

\[
= 2\widetilde{D}_1 + 2\widetilde{D}_2.
\]

By the Cauchy-Schwarz inequality,

\[
\mathbb{E} \left[ \left( X_s^{(i)} X_t^{(i)} - X_{\frac{s+t-\vartheta}{2}}^{(i)} X_{\frac{s+t+\vartheta}{2}}^{(i)} \right)^2 \right] = O(\vartheta^{2\alpha}),
\]

where \( \alpha \) is the local regularity on \( \mathcal{T}^\circ_k \) to which \( s, t, (s + t + \vartheta)/2 \) and \( (s + t - \vartheta)/2 \) eventually belong. We deduce that \( \widetilde{D}_1 = O(\vartheta^{2\alpha+1}) \). On the other hand, by repeatedly application of the Cauchy-Schwarz inequality and the moment conditions imposed above,

\[
\widetilde{D}_2 \leq 2 \int \int \mathbb{E}^{1/2} \left[ \left( \frac{1}{N} \sum_{i=1}^{N} \left\{ X_s^{(i)} - X_{\frac{s+t-\vartheta}{2}}^{(i)} \right\} \right)^4 \right] \mathbb{E}^{1/2} \left[ \left( \frac{1}{N} \sum_{i=1}^{N} X_t^{(i)} \right)^4 \right] dsdt
\]

\[
+ 2 \int \int \mathbb{E}^{1/2} \left[ \left( \frac{1}{N} \sum_{i=1}^{N} X_{\frac{s+t-\vartheta}{2}}^{(i)} \right)^4 \right] \mathbb{E}^{1/2} \left[ \left( \frac{1}{N} \sum_{i=1}^{N} \left\{ X_s^{(i)} - X_{\frac{s+t+\vartheta}{2}}^{(i)} \right\} \right)^4 \right] dsdt
\]

\[
= O(\vartheta^{2\alpha+1}).
\]

Gathering facts, we deduce that \( \widetilde{D}(\vartheta) = O_F(\vartheta^{2\alpha+1}) \).

\[ \square \]

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Supplementary Material

In the Supplementary Material, we provide some additional technical arguments, proofs and simulation results. In Section A, we provide details on the relation between local regularity and the decreasing of the eigenvalues of the covariance operator, while in Section B, we detail the connection between local regularity as a probabilistic concept and the regularity of the sample paths. In Section C, we describe the embedding structure of the spaces $\mathcal{X}(\cdot,\cdot; O_x)$. In Section D, we provide details on some quantities and equations from the main text. In Section E, we prove Theorem 3. Details on our simulation experiments and additional results are gathered in Section F.

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Supplementary material for “Adaptive optimal estimation of irregular mean and covariance functions”

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In this Supplementary Material, we provide some additional technical arguments, proofs and simulation results. In Section A below, we provide details on the relation between local regularity and the decreasing of the eigenvalues of the covariance operator, while in Section B, we detail the connection between local regularity as a probabilistic concept and the regularity of the sample paths. In Section C, we describe the embedding structure of the spaces $X(\cdot, \cdot; \mathcal{O}_s)$. In Section D, we provide details on some quantities and equations from the main text. In Section E, we prove Theorem 3. Details on our simulation experiments and additional results are gathered in Section F.

A Local regularity and covariance operator eigenvalues decrease

Several examples of processes satisfy our condition (H2). See, for instance, Berman (1974), Blanke and Vial (2011), Blanke and Vial (2014), and the references therein. Let us here mention few of them. Before that, let us note that condition (H2) requires

$$\text{Var}(\nabla^\delta X_t) + \text{Var}(\nabla^\delta X_s) - 2\text{Cov}(\nabla^\delta X_s, \nabla^\delta X_t) = L_0^2 |t - s|^{2H_0} \{1 + o(1)\},$$  \hspace{1cm} (SM.1)

whenever $s,t \in \mathcal{O}_s$ and the length of $\mathcal{O}_s$ tends to zero.

Let us first focus on the case $\delta = 0$ and stationary processes. To guarantee (SM.1) in the case of a stationary process, it suffices to have the covariance function $\Gamma(s,t) = \eta(s-t)$, for some symmetric, positive definite function $\eta(\cdot)$ such that, for some constant $c$,

$$\eta(u) = \eta(0) + cu^{2H_0} \{1 + o(1)\} \hspace{1cm} \text{as} \hspace{0.5cm} |u| \to 0.$$  \hspace{1cm} (SM.2)

If $\eta(\cdot)$ is such that $\eta(u) = \int_{\mathbb{R}} e^{ixu} h(x) dx$ for some symmetric, non negative function $h(\cdot)$ that is squared integrable, and, for some $c' > 0$ and $\nu > 1$,

$$h(x) \approx c' x^{-\nu} \hspace{1cm} \text{as} \hspace{0.5cm} x \to \infty,$$  \hspace{1cm} (SM.3)

then it easy to see that (SM.2) holds true with $2H_0 = \nu - 1$. (Here, the symbol $h_1(x) \approx h_2(x)$ means $h_1(x)/h_2(x) \to 1$ as $x \to \infty$.) The function $h(\cdot)$ is the so-called spectral density and (SM.3)
is a common high-frequency property. Theorem 3 of Rosenblatt (1963) shows that whenever the spectral density has these properties, the asymptotic behavior of the eigenvalues of the covariance operator is
\[ \lambda_j = C_j - \nu \{1 + o(1)\} \] as \( j \to \infty \), where \( C > 0 \) is a constant depending on \( c' \) and \( \nu \). As an example, we mention the stationary fractional Ornstein-Uhlenbeck process with index \( \rho \in (0, 2) \). In that case,
\[ \Gamma(s, t) = \exp(-a|s - t|^\rho), \text{ for some } a > 0, \]
and
\[ h(x) \approx c' x^{-(1+\rho)} \text{ as } x \to \infty, \]
with
\[ c' = \frac{a \Gamma(1 + \rho) \sin(\pi \rho/2)}{\pi}, \]
and \( \Gamma(\cdot) \) the gamma function. See, for instance, Luschgy and Pagès (2004), page 1584. In this example, \( \rho \) is equal to twice our local regularity \( H_0 \).

Among the nonstationary processes satisfying our condition (H2) with \( \delta = 0 \), we can mention the fractional Brownian motion with Hurst exponent \( H_0 \in (0, 1) \), for which \( \lambda_j = C' j^{-(1+2H_0)} \{1 + o(1)\} \) as \( j \to \infty \), with
\[ C' = \frac{a \Gamma(1 + 2H_0) \sin(\pi H_0)}{2\pi}. \]
Examples with \( \delta > 0 \) could be obtained by \( \delta \)-times integration of the processes with \( \delta = 0 \). See for instance the so-called \( \delta \)-integrated Brownian motion, with \( \delta \) a positive integer. For this process \( H_\delta = 1/2 \) and \( \lambda_j = (\pi j)^{-\nu} \{1 + o(1)\} \) as \( j \to \infty \), with \( \nu = 2\delta + 2 = (2\delta + 2H_\delta) + 1 \). See Luschgy and Pagès (2004), page 1585.

## B Details on the regularity of the trajectories

Let us detail the connection between the probabilistic concept of local regularity with the regularity of the sample paths considered as functions. Let \( O_* \) be some open subinterval of \( T \). We say that a function is \( \beta \) times differentiable on \( O_* \) if the function has an up to \( \lfloor \beta \rfloor - \)order derivative and the \( \lfloor \beta \rfloor - \)th derivative is Hölder continuous with exponent \( \beta - \lfloor \beta \rfloor \). (For a real number \( a \), let \( \lfloor a \rfloor \) denotes the largest integer not exceeding \( a \).) Let us call a Hölder space of local regularity \( \beta \), the set of functions which are \( \beta \) times differentiable on \( O_* \).

Let us consider the following reinforcement of (H3).

**Assumptions.**

(H1) Two positive constants \( b \) and \( B \) exist such that, for any integer \( p \geq 1 \):
\[
\mathbb{E} \left[ \left| \nabla^\delta X_t - \nabla^\delta X_s \right|^{2p} \right] \leq \frac{p!}{2} b 2p-2 |t - s|^{2pH_\delta}, \quad s, t \in O_*. 
\]

Recall that, by Assumption (H1), for almost all realizations of the process \( X \), the derivatives of the sample path exist up to order \( \delta \). Assumption (H1) implies that the derivative process \( \nabla^\delta X \) is regular in mean of order \( p \). Using the refined version of Kolmogorov’s criterion stated
in Revuz and Yor (2013, Chapter 1, section §2), it can be proven that, for any $0 < \eta < H_\delta$ the random Hölder constant

$$\Lambda_\eta = \sup_{u \neq v \in O_\ast} \frac{\left| \nabla^\delta X_u - \nabla^\delta X_v \right|}{|u - v|^\eta},$$

admits $p$-order moments, $\forall p \geq 1$. In particular, for any $\delta < \beta < \delta + H_\delta = \alpha$, the sample path of the process $X$ restricted to $O_\ast$ belong to the Hölder space of exponent $\beta$ over $O_\ast$.

As an example, the Brownian motion has a local regularity equal to 1/2 on any open interval. Almost surely, the sample paths of the Brownian motion belong to any Hölder space of local regularity $\beta < 1/2$, but cannot have a Hölder continuity of order $\alpha$, $\alpha \geq 1/2$. See Revuz and Yor (2013). Hence, the probability theory indicates that imposing assumptions on the regularity of the sample paths could be a delicate issue. Indeed, even for some widely used examples, this regularity is not well defined in the sense required by the nonparametric statistics theory.

C Technical lemmas

**Lemma 1.** Let $O_\ast \subset \mathcal{T}$ and assume that $\Delta_\ast \leq 1$. Let $\delta \in \mathbb{N}^*$ and assume that $X$ restricted to $O_\ast$ belongs to $\mathcal{X}(\delta + H_\delta, L_\delta; O_\ast)$ for some $0 < H_\delta \leq 1$ and $L_\delta > 0$. Then, for any $d \in \{0, \ldots, \delta - 1\}$, two positive real numbers $L_d$ and $S_d$ exist such that

$$\left| E \left[ (\nabla^d X_t - \nabla^d X_s)^2 \right] - L^2_d |t - s|^2 \right| \leq S^2_d |t - s|^2 \Delta_\ast^{H_{d+1}}, \quad s, t \in O_\ast,$$

with $H_{d+1} = 1_{\{d \neq \delta - 1\}} + H_\delta 1_{\{d = \delta - 1\}}$.

**Proof of Lemma 1.** Using Taylor’s formula, there exists $\xi \in (s \wedge t, s \vee t)$ such that:

$$E \left[ \left| \nabla^d X_t - \nabla^d X_s \right|^2 \right] = (t - s)^2 E \left[ \left( \nabla^{d+1} X_\xi \right)^2 \right] = (t - s)^2 \left\{ L^2_d + 2E_1(d) + E_2(d) \right\},$$

where

$$L^2_d = E \left[ \left( \nabla^{d+1} X_{t_1} \right)^2 \right],$$

$$E_1(d) = E \left[ \nabla^{d+1} X_{t_1} \left( \nabla^{d+1} X_\xi - \nabla^{d+1} X_{t_1} \right) \right],$$

$$E_2(d) = E \left[ \left( \nabla^{d+1} X_\xi - \nabla^{d+1} X_{t_1} \right)^2 \right].$$

Remark that (H1) implies that $\underline{a}_{d+1} < L^2_d < \bar{a}_{d+1}$. Using the Cauchy-Schwartz inequality,

$$|E \left[ (\nabla^d X_t - \nabla^d X_s)^2 \right] - L^2_d |t - s|^2| \leq |2E_1(d) + E_2(d)| |t - s|^2 \leq \left( 2L_d \sqrt{E_2(d)} + E_2(d) \right) (t - s)^2.$$

Thus, it remains to bound $E_2(d)$. First, consider $d = \delta - 1$. Then using (H2) combined with the fact that $|\xi - t_1| \leq \Delta_\ast \leq 1$, we have:

$$E_2(\delta) = E \left[ \left( \nabla^\delta X_\xi - \nabla^\delta X_{t_1} \right)^2 \right] \leq (L^2_\delta + S^2_\delta) \Delta_\ast^{2H_\delta}.$$
This implies that

\[ |2E_1(\delta - 1) + E_2(\delta - 1)| \leq S_{\delta - 1} \Delta H_{\delta} \] with \( S_{\delta - 1} = 2L_d \sqrt{\frac{L_d^2}{S_d^2} + \frac{L_d^2}{S_d^2}} \).

Next, consider the case of \( d < \delta - 1 \). Using Taylor’s formula and (H1), we have

\[ E_2(d) = \mathbb{E} \left[ \left( \nabla^{d+1} \xi - \nabla^{d+1} X_{t_1} \right)^2 \right] \leq \bar{a}_{d+2}(\xi - t_1)^2 \leq \bar{a}_{d+2} \Delta_x^2, \]

which implies \( |2E_1(d) + E_2(d)| \leq S_d \Delta_x \) with \( S_d = 2L_d \sqrt{\bar{a}_{d+2} + \bar{a}_{d+2}} \). Lemma 1 is now proved. □

D Details on diverse quantities

D.1 Details on the role of \( k_0 \) in (14)

Following the lines of Lemma 1.5 of Tsybakov (2009), assume that \( K(\cdot) \geq 0 \) and there exists \( K_{\min} \geq 0 \) and \( 0 < \Delta \leq 1 \) such that \( K(u) \geq K_{\min} \{ |u| \leq \Delta \}, \forall u \in \mathbb{R} \). Then, \( \forall 1 \leq i \leq N \) and for any \( v \in \mathbb{R}^{[\hat{\alpha}_i] + 1} \),

\[ v^T A_M^{(i)}(t, h)v \geq \frac{K_{\min}}{M_i h} \sum_{i=1}^{M_i} \{ v^T U((T_m^{(i)} - t)/h) \}^2 \{ \{ |T_m^{(i)} - t|/h \leq \Delta \}. \]

The right-hand side of the last display is strictly positive for any \( \|v\| = 1 \), and thus the LP estimator for the \( i \)th curve is well-defined, as soon as the possible values of \( U((T_m^{(i)} - t)/h) \in \mathbb{R}^{[\hat{\alpha}_i] + 1} \) with \( |T_m^{(i)} - t|/h \leq \Delta, 1 \leq m \leq M_i \), are not contained in the hyperplane. In the independent case, this happens with probability 1 given that

\[ \sum_{m=1}^{M_i} \{ |T_m^{(i)} - t| \leq h \} \geq k_0 \quad \text{with} \quad k_0 \geq [\hat{\alpha}_i] + 1. \]

This also happens with usual common designs, as for instance the equidistant one.

D.2 Details on the approximations (17) and (28)

Recall that

\[ c_i(t; h, \alpha) = \sum_{m=1}^{M_i} \left| (T_m^{(i)} - t)/h \right|^\alpha \left| W_m^{(i)}(t; h) \right|, \]

and

\[ \bar{C}_1(t; h, \alpha) = \frac{1}{W_N(t; h)} \sum_{i=1}^{N} w_i(t; h)c_i(t; h, \alpha). \]

When using the Nadaraya-Watson (NW) estimator, for each \( 1 \leq i \leq N \),

\[ c_i(t; h, \alpha) = \frac{1}{\hat{f}_T^{(i)}(t)} \frac{1}{M_i h} \sum_{m=1}^{M_i} \left| (T_m^{(i)} - t)/h \right|^\alpha K \left( (T_m^{(i)} - t)/h \right), \quad (\text{SM.4}) \]
with
\[ \hat{f}_T^{(i)}(t) = \frac{1}{M_i h} \sum_{m=1}^{M_i} K \left( \frac{(T_m^{(i)} - t)/h}{\alpha} \right) \approx f_T(t). \]

Here, \( f_T \) denotes the density of the \( T_m^{(i)} \). By a standard change of variables,
\[ \mathbb{E}[c_i(t; h, \alpha) \hat{f}_T^{(i)}(t)] \approx f_T(t) \int |u|^\alpha K(u) du. \]

and this explains our proposal
\[ C_1(t; h, \alpha) \approx \int |u|^\alpha K(u) du, \quad \text{(SM.5)} \]

for the NW estimator. The same arguments apply for (28). Note that the approximation in the last display would be more accurate if, for each \( i \), \( \hat{f}_T^{(i)}(t) \) in (SM.4) would be replaced by the density kernel estimator of \( f_T \) built using the \( T_m^{(i)} \) from all the curves. In the case of a local linear estimator, it suffices to use the equivalent kernels for local polynomial smoothing. Approximation (SM.5) could remain the same in the local linear case, but has to be changed for higher-order polynomials. See section 3.2.2 in Fan and Gijbels (1996).

D.3 Details on the definition (27)

Recall that \( \tilde{\gamma}_N(s, t) = N^{-1} \sum_{i=1}^{N} X_s^{(i)} X_t^{(i)} \). Here, \( W_N \) and \( w_i \) are short notations for \( W_N(s, t; h) \) and \( w_i(s, t; h) \), respectively. Moreover, \( \tilde{X}_t^{(i)} - X_t^{(i)} = B_t^{(i)} + V_t^{(i)} \), where \( B_t^{(i)} := \mathbb{E}_i \left[ \tilde{X}_t^{(i)} \right] - X_t^{(i)} \) and \( V_t^{(i)} := \tilde{X}_t^{(i)} - \mathbb{E}_i \left[ \tilde{X}_t^{(i)} \right] \). Let us define
\[ \tilde{\gamma}_W(s, t; h) = \frac{1}{W_N} \sum_{i=1}^{N} w_i X_s^{(i)} X_t^{(i)}. \]

To explain our empirical risk bound \( R_T(s|t; h) \) defined in (27), let us write
\[ \tilde{\gamma}_N(s, t; h) - \tilde{\gamma}_W(s, t; h) = \frac{1}{W_N} \sum_{i=1}^{N} w_i \{ \tilde{X}_s^{(i)} - X_s^{(i)} \} X_t^{(i)} + \frac{1}{W_N} \sum_{i=1}^{N} w_i X_s^{(i)} \{ \tilde{X}_t^{(i)} - X_t^{(i)} \} \]
\[ + \frac{1}{W_N} \sum_{i=1}^{N} w_i \{ \tilde{X}_s^{(i)} - X_s^{(i)} \} \{ \tilde{X}_t^{(i)} - X_t^{(i)} \} \]
\[ = \frac{1}{W_N} \sum_{i=1}^{N} w_i \left\{ B_t^{(i)} X_t^{(i)} + X_s^{(i)} B_t^{(i)} \right\} \]
\[ + \frac{1}{W_N} \sum_{i=1}^{N} w_i \left\{ V_s^{(i)} X_t^{(i)} + X_s^{(i)} V_t^{(i)} \right\} \]
\[ + \frac{1}{W_N} \sum_{i=1}^{N} w_i \left\{ B_s^{(i)} B_t^{(i)} + V_s^{(i)} V_t^{(i)} \right\} \]
\[ + \frac{1}{W_N} \sum_{i=1}^{N} w_i \left\{ B_s^{(i)} V_t^{(i)} + V_s^{(i)} B_t^{(i)} \right\}. \]
By construction,
\[ EM,T \left\{ V_s^{(i)} B_t^{(j)} \right\} = EM,T \left\{ B_s^{(i)} V_t^{(j)} \right\} = 0, \quad \forall 1 \leq i, j \leq N. \]
Moreover, whenever \( h < |s - t| \), we have
\[ EM,T \left\{ V_s^{(i)} V_t^{(j)} \right\} = 0, \quad \forall 1 \leq i, j \leq N. \]
Using these properties and repeatedly applying the Cauchy-Schwarz inequality, we deduce
\[
EM,T \left[ \{ \gamma_N(s, t; h) - \gamma_W(s, t; h) \} \right]^2 = EM,T \left[ \left\{ \frac{1}{W_N} \sum_{i=1}^{N} w_i \left( B_s^{(i)} X_t^{(i)} + X_s^{(i)} B_t^{(i)} \right) \right\}^2 \right]
+ EM,T \left[ \left\{ \frac{1}{W_N} \sum_{i=1}^{N} w_i \left\{ V_s^{(i)} X_t^{(i)} + X_s^{(i)} V_t^{(i)} \right\} \right\}^2 \right] + \text{negligible terms}
\leq \left[ \frac{1}{W_N} \sum_{i=1}^{N} w_i EM,T \left\{ B_s^{(i)} X_t^{(i)} + X_s^{(i)} B_t^{(i)} \right\} \right]^2
+ \frac{1}{W_N^2} \sum_{i=1}^{N} w_i EM,T \left[ \left\{ V_s^{(i)} X_t^{(i)} \right\}^2 + \left\{ X_s^{(i)} V_t^{(i)} \right\}^2 \right] + \text{negligible terms}
\leq G_1 + G_2 + \text{negligible terms.}
\]
We can now write
\[
EM,T \left[ \left\{ X_s^{(i)} V_t^{(i)} \right\}^2 \right] = EM,T \left[ \left\{ X_s^{(i)} \right\}^2 \left\{ \sum_{m=1}^{M_i} e_m^{(i)} W_m^{(i)}(t; h) \right\}^2 \right]
= EM,T \left[ \left\{ X_s^{(i)} \right\}^2 \sum_{m=1}^{M_i} \mathbb{E}_t \left\{ \left| e_m^{(i)} \right|^2 \right\} \left| W_m^{(i)}(t; h) \right|^2 \right]
\leq \sigma_{\text{max}}^2 m_2(s) \left\{ \max_m \left| W_m^{(i)}(t; h) \right| \times \sum_{m=1}^{M_i} \left| W_m^{(i)}(t; h) \right| \right\},
\]
where \( m_2(s) = \mathbb{E} \left\{ X_s^{(i)} \right\}^2 \) and \( \mathbb{E}_t(\cdot) = \mathbb{E}(\cdot \mid M_i, T_{\text{obs}}^{(i)}, X^{(i)}) \). Let
\[
\mathcal{N}_i(t; s; h) = \frac{w_i(s, t; h)}{\max_{1 \leq m \leq M_i} \left| W_m^{(i)}(t; h) \right|} \quad \text{and} \quad \mathcal{N}_i(t; s; h) = \frac{w_i(s, t; h)}{\max_{1 \leq m \leq M_i} \left| W_m^{(i)}(s; h) \right|}, \quad (\text{SM.6})
\]
We deduce
\[
G_2 \leq \frac{\sigma_{\text{max}}^2}{W_N^2} \sum_{i=1}^{N} w_i \left[ m_2(t) c_i(s; h) / \mathcal{N}_i(t; s; h) + m_2(s) c_i(t; h) / \mathcal{N}_i(t; s; h) \right],
\]
where the \( c_i(t; h) \) are defined by (16) and the \( \mathcal{N}_i(t; s; h) \) and \( \mathcal{N}_i(t; s; h) \) are defined using (SM.6).
To bound the terms related to the bias of \( \hat{X}_t^{(i)} \), let us note that
\[
EM,T \left[ \left\{ X_s^{(i)} \right\}^2 \right] = \mathbb{E} \left[ \left\{ X_s^{(i)} \right\}^2 \right] = m_2(s)
\]
and

$$C_{1}(t|s; h, 2\hat{\alpha}_t) = \sum_{i=1}^{N} w_{i}(s, t; h) c_{i}(t; h) c_{i}(t; h, 2\hat{\alpha}_t) \overline{W}_{N}(s, t; h). \quad (SM.7)$$

By repeated application of Cauchy-Schwarz inequality, we can write

$$G_{1} \leq 2 \left[ \frac{1}{\overline{W}_{N}} \sum_{i=1}^{N} w_{i} E_{M, T} \left\{ \left| B_{s}^{(i)} X_{t}^{(i)} \right| \right\} \right]^{2} + 2 \left[ \frac{1}{\overline{W}_{N}} \sum_{i=1}^{N} w_{i} E_{M, T} \left\{ \left| B_{t}^{(i)} X_{s}^{(i)} \right| \right\} \right]^{2} \leq 2 \sum_{i=1}^{N} w_{i} E_{M, T} \left\{ \left| B_{s}^{(i)} \right| \right\}^{2} \sum_{i=1}^{N} w_{i} E_{M, T} \left\{ \left| X_{t}^{(i)} \right| \right\}^{2} \leq 2 \sum_{i=1}^{N} w_{i} E_{M, T} \left\{ \left| B_{s}^{(i)} \right| \right\}^{2} m_{2}(t) + 2 \sum_{i=1}^{N} w_{i} E_{M, T} \left\{ \left| B_{t}^{(i)} \right| \right\}^{2} m_{2}(s) \leq 2 m_{2}(t) \overline{C}_{1}(s; h, 2\hat{\alpha}_s) \hat{\sigma}_{s}^{2} \hat{L}_{\delta}^{2} + 2 m_{2}(s) \overline{C}_{1}(t; h, 2\hat{\alpha}_t) \hat{\alpha}_{t}^{2} \hat{L}_{\delta}^{2},$$

where $\overline{C}_{1}(s|t; h, 2\hat{\alpha}_s)$ and $\overline{C}_{1}(t|s; h, 2\hat{\alpha}_t)$ are defined according to (SM.7).

Gathering facts, we deduce that

$$E_{M, T} \left\{ \left( \tilde{\gamma}_{N}(s, t; h) - \tilde{\gamma}_{W}(s, t; h) \right) \right\}^{2} \leq 2 E^{2}(t) \overline{C}_{1}(s; h, 2\hat{\alpha}_s) \hat{\sigma}_{s}^{2} \hat{L}_{\delta}^{2} + 2 E^{2}(s) \overline{C}_{1}(t; h, 2\hat{\alpha}_t) \hat{\alpha}_{t}^{2} \hat{L}_{\delta}^{2} + \frac{\hat{\sigma}_{max}}{\overline{W}_{N}^{2}} \sum_{i=1}^{N} w_{i} \left[ m_{2}(t) c_{i}(s; h) c_{i}(s; h) + m_{2}(s) c_{i}(t; h) \right] + \text{negligible terms.}$$

On the other hand, we have

$$E_{M, T} \left\{ \left( \tilde{\gamma}_{N}(s, t; h) - \tilde{\gamma}_{N}(s, t; h) \right) \right\}^{2} = \frac{\text{Var}(X_{s} X_{t})}{\overline{W}_{N}^{2}} \sum_{i=1}^{N} \left\{ w_{i} - \frac{\overline{W}_{N}}{N} \right\}^{2} = \text{Var}(X_{s} X_{t}) \left\{ \frac{1}{\overline{W}_{N}} - \frac{1}{N} \right\}.$$

It remains to note that

$$E_{M, T} \left\{ \left( \tilde{\gamma}_{N}(s, t; h) - \tilde{\gamma}_{N}(s, t; h) \right) \right\}^{2} \leq 2 E_{M, T} \left\{ \left( \tilde{\gamma}_{N}(s, t; h) - \tilde{\gamma}_{W}(s, t; h) \right) \right\}^{2} + 2 E_{M, T} \left\{ \left( \tilde{\gamma}_{W}(s, t; h) - \tilde{\gamma}_{N}(s, t) \right) \right\}^{2}. $$
E Proof of Theorem 3

Proof of Theorem 3. For simplicity, \( \hat{\Gamma}_N \) is built with \( k_0 = 1 \) and the uniform kernel. Recall that \( s \neq t \) are fixed and without loss of generality we can consider \( h < |s - t|/2 \). First, we prove that

\[
\frac{1}{W_N(s, t; h)} - \frac{1}{N} \leq \min \left\{ \max \left\{ h^{2\alpha_s}, \left( \frac{1}{N} \right)^{-1}(s|t; h) \right\}, \max \left\{ h^{2\alpha_t}, \left( \frac{1}{N} \right)^{-1}(t|s; h) \right\} \right\} O_P(1).
\]

For this purpose we start by showing that there exists a constant \( c_1 > 0 \) such that

\[
\frac{1}{\mathcal{N}_T(t|s; h)} \geq \frac{c_1(1 + o_P(1))}{N\alpha h}, \quad \text{and} \quad \frac{1}{\mathcal{N}_T(s|t; h)} \geq \frac{c_1(1 + o_P(1))}{N\alpha h}.
\]

Using the fact that the harmonic mean is less than or equal to the mean, we obtain

\[
\frac{1}{\mathcal{N}_T(t|s; h)} \geq \frac{c_i}{\sum_{i=1}^N w_i(s; h)w_i(t; h)\mathcal{N}_i(t|s; h)},
\]

with \( w_i(t; h), c_i = c_i(t; h) \) and \( \mathcal{N}_i(t|s; h) \) defined in (14), (16) and (SM.6), respectively. In the case we consider, for all \( i \), we have \( c_i \equiv 1 \). To justify (SM.9), it suffices to prove that there exists a sequence of positive constants \( c_N \) such that \( \limsup_N c_N < \infty \)

\[
\sum_{i=1}^N w_i(s; h)w_i(t; h)\mathcal{N}_i(t|s; h) \leq c_N(1 + o_P(1)). \tag{SM.10}
\]

Let us notice that in the case of a NW estimator with a uniform kernel, when \( k_0 = 1 \),

\[
\sum_{i=1}^N w_i(s; h)w_i(t; h)\mathcal{N}_i(t|s; h) = \sum_{i=1}^N w_i(s; h) \sum_{m=1}^{M_i} 1\{|T_m^{(i)} - t| \leq h\}.
\]

Let

\[
m^{(i)}(s; h) = \sum_{m=1}^{M_i} 1\{|T_m^{(i)} - s| \leq h\},
\]

and

\[
S_N = \sum_{i=1}^N \sum_{m=1}^{M_i} 1\{|T_m^{(i)} - t| \leq h\}.
\]

The conditional distribution of \( S_N \) given \( m^{(i)}(s; h), 1 \leq i \leq N \), is that of a sum of \( \sum_{i=1}^N (M_i - m^{(i)}(s; h)) \) independent Bernoulli variables with success probabilities

\[
p(t; h) = \int_{t-h}^{t+h} f_T(u)du \approx 2f_T(t)h.
\]

By Bernstein’s inequality,

\[
\frac{\sum_{i=1}^N m^{(i)}(s; h)}{M_1 + \ldots + M_N} = 2f_T(s)h\{1 + o_P(1)\}.
\]
Thus, again using Bernstein’s inequality,
\[
SN = \{M_1 + \ldots + M_N\} \{1 - 2f_T(s)h\} 2f_T(t)h \{1 + o_P(1)\}^2 \\
= 2f_T(t)\frac{M_1 + \ldots + M_N}{Nm} \times Nh \times \{1 + o_P(1)\} \\
\leq 2f_T(t)C_U \times Nh \times \{1 + o_P(1)\},
\]
where for the last inequality we use (22). Since by construction
\[
\sum_{i=1}^{N} w_i(s;h)w_i(t;h)\mathcal{N}_i(t|s;h) \leq SN,
\]
we deduce (SM.10) with \(c_N = 2f_T(t)C_U\), and thus (SM.9). The arguments remain the same when switching the roles of \(s\) and \(t\). Thus, we proved (SM.8) with \(c_1 = 2f_T(t)C_U\).

The next step is to prove that
\[
\frac{1}{WN(s,t;h) + 1} \leq \max\left\{ \frac{1}{N+1}, \frac{c_2 \{1 + o_P(1)\}}{Nh} \right\},
\]
for some constant \(c_2\). For this purpose, it suffices to show that there exists a positive constant \(c_W > 0\) such that
\[
\min\{N, c_W Nh \{1 + o_P(1)\}\} \leq WN(s,t;h). \tag{SM.11}
\]
Let \(\mathbb{P}_M(\cdot) = \mathbb{P}(\cdot | M_i, 1 \leq i \leq N)\). We now note that given the \(m^{(i)}(s;h)\), \(1 \leq i \leq N\),
\[
\mathbb{P}_M(w_i(s;h)w_i(t;h) = 1 | m^{(j)}(s;h), 1 \leq j \leq N) \\
= \mathbb{P}_M(w_i(s;h)w_i(t;h) = 1 | m^{(i)}(s;h)) \\
= \mathbb{P}_M(w_i(t;h) | w_i(s;h) = 1, m^{(i)}(s;h)) \times \mathbb{P}_M(w_i(s;h) = 1 | m^{(i)}(s;h)) \\
= \left\{1 - [1 - p(t;h)]^{M_i - m^{(i)}(s;h)}\right\}1\{m^{(i)}(s;h) \geq 1\} =: p_i.
\]

Next, note that given the \(m^{(i)}(s;h)\), \(1 \leq i \leq N\), the variable \(WN(s,t;h)\) is a sum of \(N\) independent Bernoulli variables with probabilities \(p_i\). Let us define the event
\[
\mathcal{E}_i = \{m^{(i)}(s;h) \leq c_Lm/2\}.
\]

Note that Bernstein’s inequality and condition (21) guarantee that \(\mathbb{P}\left(\cap_{i=1}^{N} \mathcal{E}_i\right) \to 1\), provided that \(h \log(N) \to 0\). In the case where \(mh \geq c > 0\), since \(\{1 - p(t;h)\}^{-1/p(t;h)} > e\) for any \(h < 1/2\), on the event \(\mathcal{E}_i\) and whenever \(m^{(i)}(s;h) \geq 1\), we can write
\[
p_i \geq 1 - [1 - p(t;h)]^{c_Lm - m^{(i)}(s;h)} \\
\geq 1 - \exp\{- (c_Lm - m^{(i)}(s;h))p(t;h)\} \\
\approx 1 - \exp\{- 2f_T(t)\{c_Lm - m^{(i)}(s;h)\}h\} \\
\geq 1 - \exp(- f_T(t)c_Lc).
\]

Then, Bernstein’s inequality yields the property (SM.11).
In the case $mh \to 0$ we proceed as follows. First, let
\[
\mathcal{M}^{(i)} = m^{(i)}(s; h) + m^{(i)}(t; h).
\]
We can write
\[
\mathbb{P}_M(w_i(s, t; h) = 1) = \sum_{l \geq 2} \mathbb{P}_M(w_i(s, t; h) = 1 \mid \mathcal{M}^{(i)} = l) \times \mathbb{P}_M(\mathcal{M}^{(i)} = l).
\]

On the other hand, since $M_i h$ is small,
\[
\mathbb{P}_M(\mathcal{M}^{(i)} \geq 2) = 1 - [1 - \{p(t; h) + p(s; h)\}]^{M_i} \\
- M_i p(t; h)\{1 - p(s; h)\}^{M_i-1} - M_i p(s; h)\{1 - p(t; h)\}^{M_i-1} \\
\approx 1 - \exp(-M_i\{p(t; h) + p(s; h)\}) \\
- M_i p(t; h)\{1 - \exp(-M_i - 1)p(s; h)\} \\
\approx M_i\{p(t; h) + p(s; h)\} - M_i\{M_i - 1\}p(t; h)p(s; h) \\
\approx M_i\{p(t; h) + p(s; h)\}. 
\]

On the other hand, the conditional distribution of $w_i(s, t; h)$ given $\mathcal{M}^{(i)}$ is a Bernoulli random variable. The parameter of this binary variable depends on $\mathcal{M}^{(i)}$. When $\mathcal{M}^{(i)} = 2$,
\[
\mathbb{P}_M(w_i(s, t; h) = 1 \mid \mathcal{M}^{(i)} = 2) = 2 \frac{p(t; h)p(s; h)}{(p(t; h) + p(s; h))^2} \geq \frac{1}{4} \left[ \min\{f_T(s), f_T(t)\} \right]^2,
\]
provided $h$ is sufficiently small, while for any $l > 2$,
\[
\mathbb{P}_M(w_i(s, t; h) = 1 \mid \mathcal{M}^{(i)} = l) = 1 - (p(t; h))^l - (p(s; h))^l \leq C_1 \left[ \frac{\min\{f_T(s), f_T(t)\}}{\max\{f_T(s), f_T(t)\}} \right]^2,
\]
for some suitable constant $C_1$ depending only on $f_T(s)$ and $f_T(t)$. Gathering facts, we deduce that there exists a constant $C_2$, depending only on $f_T(s)$ and $f_T(t)$, such that
\[
\mathbb{P}_M(w_i(s, t; h) = 1) \geq C_2 \mathbb{P}_M(\mathcal{M}^{(i)} \geq 2) \\
\approx C_2 M_i\{p(t; h) + p(s; h)\} \\
\approx 2C_2 M_i\{f_T(s) + f_T(t)\}h \\
\geq 2C_2 c_L\{f_T(s) + f_T(t)\} mh
\]
Again, Bernstein’s inequality guarantees that
\[
\frac{W_N(t; h)}{\sum_{i=1}^N \mathbb{P}_M(w_i(s, t; h) = 1)} = 1 + o_p(1).
\]
Condition (SM.11) follows with $c_W = 2C_2 c_L\{f_T(s) + f_T(t)\}$. Finally, the result follows by the same arguments as used for Theorem 2.
F Additional simulation results

In this section we recall and add details on the construction of the simulation setup. Moreover, we provide more details on the implementation of our estimators and present results obtained from additional experiments.

F.1 Description of the real data set used to build the simulations

For the simulation experiments reported in this article we build mean and covariance functions, as well as conditional variance function, using a real data set. More precisely, our simulation study is based on the Household Active Power Consumption dataset which was sourced from the UC Irvine Machine Learning Repository (https://archive.ics.uci.edu/ml/datasets/Individual+household+electric+power+consumption). An implementation of the simulation methods is available as a R package on Github at the URL address https://github.com/StevenGolovkine/simulater.

F.2 Construction of the simulation design

For building the mean function, we consider the following model

$$\mu(t) = \beta_0t + \sqrt{2} \sum_{k=1}^{50} \{ \beta_{1,k} \cos(2k\pi t) + \beta_{2,k} \sin(2k\pi t) \}, \quad t \in [0,1].$$

The coefficients $\beta$ are estimated by LASSO regression with the outcomes given by the 1440 values of the empirical mean of the 708 curves and the covariates obtained with the uniform grid of 1440 points in $[0,1]$. The penalized regression was done using the function glmnet from the R package glmnet. The regularity of the mean function is controlled using the penalty parameter $\lambda$. The mean function obtained with $\lambda = \exp(-5.5)$, which was used in Experiments 1 to 3, is plotted with plain line in Figure 1b.

To build the covariance function for the simulation design, we first smooth each curve using the function smooth.splines in R software, and we compute the empirical covariance of the
sample of smoothed curves on the lattice grid with $1440 \times 1440$ points in $[0,1]^2$. We next fit a two dimensional local linear kernel smoother to this empirical covariance with a bandwidth 0.01 using the function \texttt{Lwls2D} from the R package \texttt{fdapace}. We use this model to estimate a smoothed covariance on a $2880 \times 2880$ equispaced grid. We refined the grid on points in $[0,1]^2$ to obtained a higher numerical precision for the target quantities. Then, after computing the eigenvalues of this smoothed covariance, we fit a linear model onto the logarithm of the 6th to the 105th eigenvalues onto the logarithm of their rank (the $R^2$ is equal to 0.9373). The slope is -2.4118 (standard error equal to 0.063), which corresponds to a local regularity equal to $0.7059 = (2.4118-1)/2$. We use this model to predict the eigenvalues from 6 to 2880. The covariance matrix we use for simulations is then that rebuilt using this new set of 2880 eigenvalues and the eigenfunctions of the smoothed covariance. The eigenvalues of the smoothed covariance are plotted in Figure 1. The eigenvalues used to fit the linear model correspond to the first almost linear part, after removing the first five eigenvalues, which resulted in an improved fit.

To estimate the conditional variance of the noise $\sigma^2(t, x)$, for each $1 \leq i \leq 708$, we first fit a smoothing spline curve to the subset of every second voltage measurement, that is 720 outcome values. Next we use the fitted splines to predict the other 720 voltage measurements on each curve. We compute the squares of the residuals from all the curves, that is 720 times 708 squared residuals. We finally define $\sigma^2(t, x)$ as the fitted function obtained from a generalized additive model (GAM) applied with all the squared residuals and using a full tensor product smooth on $[0,1] \times [\min_i X(i); \max_i X(i)]$. See Wood (2006). The model allows us to estimate the conditional variance of the noise at every point in $[0,1] \times [\min_i X(i); \max_i X(i)]$, as it will be needed with an independent design setup. The plot of the fitted function $\sigma^2(t, x)$ for $(t, x) \in [0,1] \times [230,245]$, is given on Figure 1a.

To simulate our data, we first fix $N$ and $m$ and $p \in (0,1)$. Next, for each $1 \leq i \leq N$, we generate $M_i$ according to a uniform distribution on the interval $[(1-p)m,(1+p)m]$. Given $M_i$, we draw the observations times $T_m^{(i)} \in [0,1]$, $1 \leq m \leq M_i$ according to a uniform distribution on $[0,1]$. Next, we compute the mean function at the times $T_m^{(i)}$ using the model (SM.12). We subset the $2880 \times 2880$ covariance matrix to build the $M_i \times M_i$–covariance matrix corresponding to the $T_m^{(i)}$. If the considered point $(s,t)$ does not exists in the large matrix, we use the closest one. We use the function \texttt{mvrnorm} from the R package \texttt{MASS} to generate the $M_i$ measurements $X(i)(T_m^{(i)})$. Finally, we generate the error term $\varepsilon_m^{(i)} = \sigma(T_m^{(i)},X(i)(T_m^{(i)}))\varepsilon_m^{(i)}$, with $\varepsilon_m^{(i)}$ a standard Gaussian variable and the function $\sigma(\cdot,\cdot)$ fitted using to the GAM, and define $Y_m^{(i)} = X(i)(T_m^{(i)}) + \varepsilon_m^{(i)}$. A random sample of three curves generated according to this type of simulation setup, obtained with $m = 200$ and $p = 0.2$, are plotted in Figure 1b.

F.3 Additional implementation details and simulation results

For presmoothing, we use NW smoothing with the Epanechnikov kernel and the bandwidth equal to $(\Delta_s / 2 \bar{m})^{1/3}$, where $\Delta_s = 2 \exp(-\log^{1/2}(\bar{m}))$ and $\bar{m} = N^{-1} \sum_{i=1}^N M_i$. This choice is designed for a sample size $2 \bar{m}/\Delta_s$, which is the rate of the average number of points in $[t_2, t_3]$, and the case of a regularity $\delta + H_\delta < 1$ where it yields oversmoothing. For the mean and covariance functions estimators we use the biweight kernel $K(t) = (15/16) \left(1 - t^2\right)^2_1[t_{[-1,1]}(t)$ for which $\int |u|^\alpha K(u)du = 15\{(\alpha + 1)^{-1} - 2(\alpha + 3)^{-1} + (\alpha + 5)^{-1}\}/8$. 

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F.3.1 Computation time

The average cumulated computation time for our estimators of the mean and covariance functions is about 30 seconds on a 2.6 GHz Intel-i5 laptop with 8Gb RAM. Our estimator of the mean is slower than the competitors, but our covariance function estimator requires far less computation time than \(\hat{\Gamma}_{CY}\) and \(\hat{\Gamma}_{ZW}\) and the difference increases with \(N\) and \(M\). Figure 2 presents the ratios of computation times for covariance estimation in Experiment 1 with \(N = 100\) and \(M = 100\).

F.3.2 Additional results

In addition to Experiment 1 reported in the main manuscript, we investigated three variations of the simulation setup to investigate the effects of three aspects: the variance of the \(M_i\), the variability of the conditional variance \(\sigma^2(t,x)\) and the regularity of the mean function. The details of the three additional experiments are provided below.

Experiment 2. The simulation design is similar to Experiment 1, but the \(M_i\) are generated with a uniform distribution \(\mathcal{U}(0.5m, 1.5m)\). We replicate the simulation 500 times. The results are plotted on the Figure 3 for the mean estimation and on the Figure 4 for the covariance estimation.

Experiment 3. The simulation design is similar to Experiment 1, but the conditional variance of the noise twice larger than Experiment 1. We replicate the simulation 500 times. The results are plotted on the Figure 5 for the mean estimation and on the Figure 6 for the covariance estimation.

Experiment 4. The simulation design is similar to Experiment 1, but the mean function has a higher regularity, that is is obtained by LASSO regression with penalty \(\lambda = \exp(-3.5)\). The mean function and three sample paths, with and without noise, are plotted in Figure 7. We replicate the simulation 500 times. The results are plotted on the Figure 8 for the mean estimation and on the Figure 9 for the covariance estimation.

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Figure 3: Results for Experiment 2 on the log scale
0.3 1.0 3.0 10.0 0.3 1.0 3.0 10.0 0.3 1.0 3.0 10.0
ISE (log scale)

(a) ISE with respect to the empirical covariance \( \hat{\Gamma} \) in each simulated sample

(b) ISE with respect to \( \Gamma \)

Figure 4: Results for Experiment 2 on the log scale
Figure 5: Results for Experiment 3 on the log scale

(a) ISE with respect to the empirical mean $\bar{\mu}$ in each simulated sample

(b) ISE with respect to $\mu$
(a) ISE with respect to the empirical covariance \( \tilde{\Gamma} \) in each simulated sample

(b) ISE with respect to \( \Gamma \)

Figure 6: Results for **Experiment 3** on the log scale
(a) Mean function $\mu(t)$ (bold line) and three randomly generated sample paths

(b) The mean function $\mu(t)$ (bold line) and three random noisy trajectories

Figure 7: Mean function and sample paths, without (left panel) and with (right panel) noise, in Experiment 4

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(a) ISE with respect to the empirical mean $\tilde{\mu}$ in each simulated sample

(b) ISE with respect to $\mu$

Figure 8: Results for Experiment 4 on the log-scale
Figure 9: Results for Experiment 4 on the log-scale

(a) ISE with respect to the empirical covariance $\hat{\Gamma}$ in each simulated sample

(b) ISE with respect to $\Gamma$