Nonunital Operator Systems and Noncommutative Convexity

Nicholas Manor

University of Waterloo

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Unital Commutative C*-algebras

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*Unital commutative C*-algebras are dual to compact Hausdorff spaces.*
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**Theorem (Gelfand-Naimark)**

Unital commutative $C^*$-algebras are dual to compact Hausdorff spaces. The mutually inverse functors are given by

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Notice that the zero map is omitted as a representation, otherwise this duality would not hold.
Nonunital Commutative C*-algebras

If we include the zero map as an irreducible representation, Theorem (Nonunital Gelfand-Naimark) Commutative C*-algebras are dual to pointed compact Hausdorff spaces \((X, z)\). The mutually inverse functors are given by

\[ A \mapsto \text{Spec}(A) \cup \{0\}, \quad (X, z) \mapsto C_0(X \setminus \{z\}). \]

We may identify \(C_0(X \setminus \{z\})\) with the C*-algebra \(C(X, z) := \{f \in C(X) | f(z) = 0\}\).

Remark \(C(X, z)\) is unital if and only if \(z\) is an isolated point.
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**Theorem (Nonunital Gelfand-Naimark)**

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**Remark**

*\(C(X, z)\) is unital if and only if \(z\) is an isolated point.*
Function Systems

Definition
A closed subspace $E \subseteq C^0(X)$ is a function system if $E^* = E$.

Example
The space $A([-1,1])$ of continuous affine function on $[-1,1]$ has elements all of the form $f(x) = ax + b$ for $a, b \in \mathbb{C}$.

The space $A([-1,1], 0) := \{f \in A([-1,1]) | f(0) = 0\}$ of continuous affine functions vanishing at zero has elements all of the form $f(x) = ax$ for $a \in \mathbb{C}$. 
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Unital Function Systems

Theorem (Kadison's duality)

Unital function systems are dual to compact convex sets.

The mutually inverse functors are given by

\[ E \mapsto \rightarrow S(E) \]

\[ K \mapsto \rightarrow A(K) \].

Example

The C*-algebra \( C^*_2 \) has state space \( S(C^*_2) \sim [\mathbb{R}, 1] \). Therefore,

\[ A([\mathbb{R}, 1]) \sim = C^*_2. \]
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### Example

The C*-algebra \( \mathbb{C}^2 \) has state space \( S(\mathbb{C}^2) \cong [-1, 1] \). Therefore, \( A([-1, 1]) \cong \mathbb{C}^2 \).
Nonunital Function Systems

Definition

Let $E \subseteq C_0(X)$ be a function system. A positive linear functional $\phi: E \to \mathbb{C}$ is called a quasistate if $\|\phi\| \leq 1$.

We denote the quasistate space of $E$ by $Q(E)$.

Theorem (Nonunital Kadison’s duality)

Function systems are dual to “pointed” compact convex sets. The mutually inverse functors are given by $E \mapsto (Q(E), 0)$ $(K, z) \mapsto A(K, z)$. 
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If $E$ is unital, $Q(E)$ can be abstractly obtained from $S(E)$ by adding an “affinely independent” point.
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Figure: Compact convex set
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Noncommutative Setting

Definition

A closed subspace $E \subseteq B(H)$ is an operator system if $E^* = E$.

Definition

A compact noncommutative (nc) convex set over a dual operator space $Y$ is a graded subset $K = \bigoplus_{n \geq 1} K_n$ such that

1. all $K_n$ are compact in the dual space topology on $M_n(Y)$ and
2. $K$ is closed under nc convex combinations: if $\alpha_i \in M_n$, $n_i$ and $x_i \in K_{n_i}$ for all $i$ such that $\sum_i \alpha_i \alpha_i^* = 1$, then $\sum_i \alpha_i x_i \alpha_i^* \in K$.
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   \sum_i \alpha_i x_i \alpha_i^* \in K.
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Example

Let $E$ be a unital operator system. Let $S_n(E)$ denote the set of ucp maps from $E$ into $M_n$. The **nc state space** of $E$ is $S(E) := \bigsqcup_n S_n(E)$. It is a compact nc convex set over $E^d$. 
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If $E$ is not necessarily unital, then we instead take $Q_n(E)$ to be the space of cpcc maps from $E$ into $M_n$. The **nc quasistate space** of $E$ is $Q(E) := \bigsqcup_n Q_n(E)$.
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Example

The set $\bigcup_n \{ \alpha \in (M_n)_{sa} : -1 \leq \alpha \leq 1 \}$ is a compact nc convex set over $\mathbb{C}$. This is the nc state space of $\mathbb{C}^2$. 
Noncommutative Setting

**Definition**

Let $K$ be a compact nc convex set. Let $\mathcal{M} := \bigsqcup_n M_n$. We say a graded function $f := \bigsqcup_n f_n : K \to \mathcal{M}$ is an **affine nc function** if $f$ preserves nc convex combinations:

$$f(\sum_i \alpha_i x_i \alpha_i^*) = \sum_i \alpha_i f(x_i) \alpha_i^*. $$
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$$f\left(\sum_i \alpha_i x_i \alpha_i^*\right) = \sum_i \alpha_i f(x_i) \alpha_i^*.$$ 

Let $A(K)$ denote the unital operator system of continuous affine nc functions on $K$. 
Theorem (Webster-Winkler, Davidson-Kennedy)

Compact nc convex sets are dual to unital operator systems.

The mutually inverse functors are given by

\[ E \mapsto \mathcal{S}(E) \]
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Nonunital Operator Systems

Example
Let $K$ be a compact nc convex set and fix a point $z \in K$. The space $A(K, z) := \{ f \in A(K) : f(z) = 0 \} \subseteq A(K)$ is an operator system.

Example (Communications with C.K. Ng)
There is a large class of (nonunital) operator systems arising as duals of "well understood" operator systems, said to have the bounded decomposition property. These include the duals of all $C^*$-algebras and all unital operator systems.
Example

Let $K$ be a compact nc convex set and fix a point $z \in K_1$. The space $A(K, z) := \{ f \in A(K) : f(z) = 0 \} \subseteq A(K)$ is an operator system.
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The obstruction: \(E \mapsto (Q(E), 0)\) and \((K, z) \mapsto A(K, z)\) are not mutually inverse functors.
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We have \(A([-1/2, 1], 0) \cong A([-1, 1], 0)\).

In fact, \(Q(A([-1/2, 1], 0)) = Q(A([-1, 1], 0)) =([-1, 1], 0)\).
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Consequences of the Duality

Let \((K, z)\) be a pointed compact nc convex set. We denote by \(\partial K\) the set of extreme points of \(K\).

**Theorem (Kennedy-Kim-M)**

The space \((K, z)\) corresponds to a unital operator system iff there is some \(e \in A(K, z)\) such that \(e(x) = 1\) for all \(x \in \partial K \setminus \{z\}\).

**Theorem (Kennedy-Kim-M)**

The operator system \(A(K, z)\) generates a simple C*-algebra iff for every nonzero \(x \in \partial K\), \(K\) is the smallest pointed nc convex set containing \(x\).
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A Dynamical Consequence

Theorem (Kennedy-Kim-M)

Let $G$ be a second countable group. The following are equivalent:

1. $G$ has property $(T)$.
2. For every unital $G$-C*-algebra $A$, $S^1(A)^G$ is the state space of some C*-algebra.
3. For every $G$-C*-algebra $A$, $(Q^1(A)^G, 0)$ is the pointed quasistate space of some C*-algebra.

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