4D DOUBLETON CONFORMAL THEORIES, CPT AND II B STRING ON $AdS_5 \times S^5$

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Abstract

We study the unitary supermultiplets of the $\mathcal{N} = 8$ $d = 5$ anti-de Sitter (AdS) superalgebra $SU(2,2|4)$ which is the symmetry algebra of the IIB string theory on $AdS_5 \times S^5$. We give a complete classification of the doubleton supermultiplets of $SU(2,2|4)$ which do not have a Poincare limit and correspond to $d = 4$ conformal field theories (CFT) living on the boundary of $AdS_5$. The CPT self-conjugate irreducible doubleton supermultiplet corresponds to $d = 4$ $\mathcal{N} = 4$ super Yang-Mills theory. The other irreducible doubleton supermultiplets come in CPT conjugate pairs. The maximum spin range of the general doubleton supermultiplets is 2. In particular, there exists a CPT conjugate pair of doubleton supermultiplets corresponding to the fields of $\mathcal{N} = 4$ conformal supergravity in $d = 4$ which can be coupled to $\mathcal{N} = 4$ super Yang-Mills theory in $d = 4$. We also study the "massless" supermultiplets of $SU(2,2|4)$ which can be obtained by tensoring two doubleton supermultiplets. The CPT self-conjugate "massless" supermultiplet is the $\mathcal{N} = 8$ graviton supermultiplet in $AdS_5$. The other "massless" supermultiplets generally come in conjugate pairs and can have maximum spin range of 4. We discuss the implications of our results for the conjectured CFT/AdS dualities.
1 Introduction

Since the original conjecture of Maldacena relating the large N limits of certain conformal field theories (CFT) in d-dimensions to M-theory/string theory compactified to d+1-dimensional AdS spacetimes, a lot of work has been done on CFT/AdS duality. Maldacena’s conjecture was originally based on certain properties of the physics of N Dp-branes in the near horizon limit and the old knowledge about 10-d IIB supergravity compactified on $AdS_5 \times S^5$, and 11-d supergravity compactified on $AdS_7 \times S^4$ and $AdS_4 \times S^7$ [3, 4, 5, 6, 7, 8, 9]. In particular, Maldacena’s conjecture was made more precise in [10, 11].

The relation between Maldacena’s conjecture and the dynamics of the singleton and doubleton fields that live on the boundary of AdS spacetimes was reviewed in [12] and [13].

In this paper we want to focus on the prime example of this CFT/AdS duality, namely the duality between the large N limit of $\mathcal{N}=4$ SU($N$) super Yang-Mills theory in $d=4$ and the IIB string theory over $AdS_5 \times S^5$.

The $\mathcal{N}=4$ super Yang-Mills multiplet corresponds to the CPT self-conjugate irreducible doubleton supermultiplet of the $\mathcal{N}=8$ AdS superalgebra $SU(2,2\mid 4)$ in $d=5$. There exist other doubleton supermultiplets of $SU(2,2\mid 4)$ which are not CPT self-conjugate. One of the goals of this paper is to emphasize this point and give a complete list of all doubleton supermultiplets of $SU(2,2\mid 4)$. In particular, we find doubleton supermultiplets corresponding to $\mathcal{N}=4$ conformal supergravity living on the boundary of $AdS_5$, which can be identified with the $d=4$ Minkowski space.

The $d=4$, $\mathcal{N}=4$ super Yang-Mills matter can be coupled to $\mathcal{N}=4$ conformal supergravity. Since the resulting $d=4$ theory is conformally invariant we expect that Maldacena’s conjecture can be generalized so as to include the degrees of freedom of the conformal supergravity sector.

We also study "massless" supermultiplets of $\mathcal{N}=8$ AdS$_5$ superalgebra $SU(2,2\mid 4)$. The long "massless" supermultiplets have spin range 4. However, there exist other "massless" supermultiplets whose spin range is less than 4.

The paper is organized as follows: In section 2. we present a short review of the oscillator method. Section 3. gives the general construction of the positive energy representations of $SU(2,2)$. In section 4. we give a complete list of doubleton representations of $SU(2,2)$. Likewise, in sections 5. and 6. we give complete lists of "massless" and "massive" positive energy representations of $SU(2,2)$. In section 7. the oscillator method for the general
supergroup $SU(m, n|p + q)$ is reviewed. In section 8, we give a complete list of doubleton supermultiplets of $SU(2, 2|4)$ and in section 9, we study the ”massless” irreducible supermultiplets of $SU(2, 2|4)$. We conclude the article with a discussions of the implications our results have for CFT/AdS duality.

2 Short Review of the Oscillator Method

In [14] a general oscillator method was developed for constructing the unitary irreducible representations (UIR) of the lowest (or highest) weight type of non-compact groups. The oscillator method yields the UIR’s of lowest weight type of a noncompact group over the Fock space of a set of bosonic oscillators. To achieve this one realizes the generators of the noncompact group as bilinears of sets of bosonic oscillators transforming in a finite dimensional representation of its maximal compact subgroup. The minimal realization of these generators requires either one or two sets of bosonic annihilation and creation operators transforming irreducibly under its maximal compact subgroup. These minimal representations are fundamental in that all the other ones can be obtained from the minimal representations by a simple tensoring procedure. These fundamental representations are nothing but a generalization of the celebrated remarkable representations of the $AdS_4$ group $SO(3, 2)$ discovered by Dirac [15] long time ago, which were later named singletons [17] (indicating the fact that the remarkable representations of Dirac corresponding to the fields living on the boundary of $AdS_4$ are singular when the Poincare limit is taken). In the language of the oscillator method, these singleton representations require a single set of oscillators transforming in the fundamental representation of the maximal compact sugroup of the covering group $Sp(4, R)$ of $SO(3, 2)$ [7, 18, 19] (a fact that meshes nicely with the name singleton). In some cases (as with the $AdS_5$ group $SU(2, 2)$) the fundamental representations require two sets of oscillators, and they were called doubletons in [8, 4]. The general oscillator construction of the lowest (or highest) weight representations of non-compact supergroups (i.e. the case when the even subgroup is non-compact) was given in [17]. The oscillator method was further developed and applied to the spectra of Kaluza-Klein supergravity theories in references [4, 7, 8].

A non-compact group $G$ that admits unitary representations of the lowest weight type has a maximal compact subgroup $G^0$ of the form $G^0 = H \times U(1)$ with respect to whose Lie algebra $g^0$ one has a three grading of the Lie
algebra $g$ of $G$,

$$g = g^{-1} \oplus g^{0} \oplus g^{1}$$  \hspace{1cm} (2 - 1)

which simply means that the commutators of elements of grade $k$ and $l$ satisfy

$$[g^{k}, g^{l}] \subseteq g^{k+l}.$$  \hspace{1cm} (2 - 2)

Here $g^{k+l} = 0$ for $|k + l| > 1$.

For example, for $SU(1,1)$ this corresponds to the standard decomposition

$$g = L_{+} \oplus L_{0} \oplus L_{-}$$  \hspace{1cm} (2 - 3)

where

$$[L_{0}, L_{\pm}] = \pm L_{\pm}$$
$$[L_{\pm}, L_{-}] = 2L_{0}$$  \hspace{1cm} (2 - 4)

The three grading is determined by the generator $E$ of the $U(1)$ factor of the maximal compact subgroup

$$[E, g^{+1}] = g^{+1}$$
$$[E, g^{-1}] = -g^{-1}$$
$$[E, g^{0}] = 0$$  \hspace{1cm} (2 - 5)

In most physical applications $E$ turns out to be the energy operator. In such cases the unitary lowest weight representations correspond to positive energy representations.

The bosonic annihilation and creation operators in terms of which one realizes the generators of $G$ transform typically in the fundamental and its conjugate representation of $H$. In the Fock space $\mathcal{H}$ of all the oscillators one chooses a set of states $|\Omega\rangle$ which transform irreducibly under $H \times U(1)$ and are annihilated by all the generators in $g^{-1}$. Then by acting on $|\Omega\rangle$ with generators in $g^{+1}$ one obtains an infinite set of states

$$|\Omega\rangle, \quad g^{+1}|\Omega\rangle, \quad g^{+1}g^{+1}|\Omega\rangle, \ldots$$  \hspace{1cm} (2 - 6)

which form an UIR of the lowest weight (positive energy) type of $G$. The infinite set of states thus obtained corresponds to the decomposition of the UIR of $G$ with respect to its maximal compact subgroup.

As we have already emphasized, whenever we can realize the generators of $G$ in terms of a single set of bosonic creation (and annihilation) operators transforming in an irreducible representation (and its conjugate) of the
compact subgroup $H$ then the corresponding UIRs will be called singleton representations and there exist two such representations for a given group $G$. For the AdS group in $d = 4$ the singleton representations correspond to scalar and spinor fields. In certain cases we need a minimum of two sets of bosonic creation and annihilation operators transforming irreducibly under $H$ to realize the generators of $G$. In such cases the corresponding UIRs are called doubleton representations and there exist infinitely many doubleton representations of $G$ corresponding to fields of different "spins". For example, the non-compact group $Sp(2N, R)$ with the maximal compact subgroup $U(N)$ admits singleton representations \cite{7, 14, 20}. On the other hand, the non-compact groups $SO^*(2N)$ \cite{3, 4} and $SU(N, M)$ \cite{14, 17} with maximal compact subgroups $U(N)$ and $S(U(M) \times U(N))$ respectively, admit doubleton representations.

The noncompact supergroups also admit either singleton or doubleton supermultiplets corresponding to some minimal fundamental unitary irreducible representations, in terms of which one can contract all the other UIR’s of the lowest weight type by a simple tensoring procedure. For example, the non-compact supergroup $OSp(2N/2M, R)$ with the even subgroup $SO(2N) \times Sp(2M, R)$ admits singleton supermultiplets, while $OSp(2N^*|2M)$ and $SU(N, M|P)$ with even subgroups $SO^*(2N) \times USp(2M)$ and $SU(N, M) \times SU(P) \times U(1)$ admit doubleton supermultiplets.

Even though the Poincare limit of the singleton (or doubleton) representations is singular, the tensor product of two singleton (or doubleton) representations decomposes into an infinite set of "massless" irreducible representations which do have a smooth Poincare limit \cite{7, 16, 18}. Based on this fact the following definition of "massless" representations in AdS space-time was proposed in \cite{19}:

A representation (or a supermultiplet) of an AdS group (or supergroup) is "massless" if it occurs in the decomposition of the tensor product of two singleton or two doubleton representations (or supermultiplets).

This should be taken as a working definition which agrees with some other definitions of "masslessness" in $d \leq 7$. Note also that recent work on CFT/AdS duality gives support to the above definition from a dynamical point of view \cite{11, 23}. Furthermore, tensoring more than two singletons or doubletons representations leads to "massive" representations of AdS groups and supergroups \cite{19}.
3 Oscillator Construction of the Positive Energy Representations of $SU(2,2)$

Unitary representations of the covering group $SU(2,2)$ of the conformal group $SO(4,2)$ in $d = 4$ have been studied extensively \[21\]. The group $SO(4,2)$ is also the AdS group of $d = 5$ spacetime with Lorentzian signature. Positive energy or equivalently the lowest weight representations of $SU(2,2)$ can be constructed very simply by the oscillator method outlined in the previous section \[4, 17\].

Let us denote the two $SU(2)$ subgroups of $SU(2,2)$ as $SU(2)_L$ and $SU(2)_R$ respectively. The generator of the Abelian factor in the maximal compact subgroup of $SU(2,2)$ is the AdS energy operator in $d = 5$ (or the conformal Hamiltonian in $d = 4$ whose eigenvalues give the conformal dimensions) and will be denoted as $E$. To construct the positive $AdS_5$ energy (or $d=4$ conformal) representations we realize the generators of $SU(2,2)$ as bilinears of pairs of bosonic oscillators transforming in the fundamental representation of the two $SU(2)$ subgroups. The oscillators satisfy the canonical commutation relations

\[
[a_i(\xi),a^\dagger_j(\eta)] = \delta^j_i \delta_{\xi\eta} \quad i, j = 1, 2, \quad (3 - 1)
\]

\[
[b_r(\xi),b^\dagger_s(\eta)] = \delta^s_r \delta_{\xi\eta} \quad r, s = 1, 2 \quad (3 - 2)
\]

Here $\xi, \eta = 1, ..., P$ label different generations of oscillators and

\[
[a_i(\xi),b_r(\eta)] = [a_i(\xi),b^\dagger_r(\eta)] = [a_i(\xi),a_j(\eta)] = [b_r(\xi),b_s(\eta)] = 0 \quad (3 - 3)
\]

The bosonic oscillators with an upper index ($i$ or $r$) are creation operators while those with lower indices are annihilation operators. The vacuum vector is defined as

\[
a_i(\xi)|0\rangle = 0 = b_r(\xi)|0\rangle \quad (3 - 4)
\]

for all values of $i, r, \xi$. The non-compact generators of $SU(2,2)$ are realized by the following bilinears

\[
L_{ir} = \vec{a}_i \cdot \vec{b}_r \quad L^{ir} = \vec{a}^\dagger_i \cdot \vec{b}^\dagger_r \quad (3 - 5)
\]

where $\vec{a}_i \cdot \vec{b}_r = \sum_{\xi=1}^P a_i(\xi)b_r(\xi)$ etc. They close into the generators of the compact subgroup $SU(2)_L \times SU(2)_R \times U(1)$

\[
[L_{ir},L^{js}] = \delta^s_r L^{ij}_i + \delta^j_r R^{is}_r + \delta^j_r \delta^s_i E
\]
\[ [L_{ir}, L_{js}] = [L^{ir}, L^{js}] = 0 \quad (3 - 6) \]

where
\[
L^i_j = \bar{a}^j \cdot \bar{a}_i - \frac{1}{2} \delta^j_i \bar{a}^i \cdot \bar{a}_i
\]
\[
R^r_s = \bar{b}^r \cdot \bar{b}_s - \frac{1}{2} \delta^r_s \bar{b}^s \cdot \bar{b}_t
\]
\[
E = \frac{1}{2} (\bar{a}_i \cdot \bar{a}^i + \bar{b}^r \cdot \bar{b}_r) \quad (3 - 7)
\]

Here \( L^i_j \) and \( R^r_s \) are the generators of \( SU(2)_L \) and \( SU(2)_R \), respectively.

Defining the number operators
\[
N_a = \bar{a}^i \cdot \bar{a}_i = \sum_{i=1}^{P} \sum_{\xi=1}^{P} a^i(\xi) a_i(\xi)
\]
\[
N_b = \bar{b}^r \cdot \bar{b}_r
\]
\[
N = N_a + N_b \quad (3 - 8)
\]

we can write the AdS energy operator \( E \) as
\[
E = \frac{1}{2} (N_a + N_b + 2P) = \frac{1}{2} N + P \quad (3 - 9)
\]

The quadratic Casimir operator \( C_2 \) of \( SU(2, 2) \) is uniquely defined up to an overall multiplicative constant. We choose this constant such that
\[
C_2 = -\frac{1}{2} (L_{ir} L^{ir} + L^{ir} L_{ir}) + \frac{1}{2} (L^i_j L^j_i + R^r_s R^s_r + E^2) \quad (3 - 10)
\]

This expression can be rewritten in terms of number operators \( N_a, N_b \) and \( N \) for the case of \( P = 1 \)
\[
C_2 = (\frac{N}{2} + 1)(\frac{N}{2} - 3) + N_a(\frac{N_a}{2} + 1) + N_b(\frac{N_b}{2} + 1) \quad (3 - 11)
\]

The positive energy irreducible unitary representations of \( SU(2, 2) \) are uniquely defined by a lowest weight vector \(|\Omega\rangle\) transforming irreducibly under the maximal compact subgroup \( S(U(2) \times U(2)) \) and that is annihilated by \( L_{ir} \)
\[
L_{ir} |\Omega\rangle = 0 \quad (3 - 12)
\]

\[^4\text{From this expression it is clear that the eigenvalues of the energy operator in AdS}_5\text{ (or conformal dimensions in d=4) take on integer or half-integer values. To get the generic, real, values of the conformal dimension (which includes the anomalous dimension) one needs to take the infinite covering of the conformal group }SU(2, 2)\text{ and consider its lowest weight UIRs [21].}\]
Then by acting on $|\Omega\rangle$ repeatedly with the generators $L^{ir}$ one generates an infinite set of states

$$|\Omega\rangle, \quad L^{ir}|\Omega\rangle, \quad L^{ir}L^{j^s}|\Omega\rangle,...$$

(3 - 13)

that form the basis of a unitary irreducible representation of $SU(2, 2)$ \[4, 17\]. This infinite set of states corresponds to the decomposition of a positive energy UIR of $SU(2, 2)$ with respect to the maximal compact subgroup. They can be identified with the Fourier modes of a field in $AdS_5$ that is uniquely defined by the lowest weight vector. If the lowest weight vector transforms in the $(j_L, j_R)$ representation of $SU(2)_L \times SU(2)_R$ and has AdS energy $E$ the corresponding field in $AdS_5$ will be denoted as

$$\Xi_{(j_L,j_R)}(E)$$

(3 - 14)

Note that the local fields that transform covariantly under the Lorentz group in $d = 5$ correspond in general to direct sums of such fields and their conjugates.

The eigenvalues of the quadratic Casimir operator $C_2$ of $SU(2, 2)$ can be, in general (i.e. for arbitrary $P$), expressed in the following form

$$C_2 = E(E - 4) + 2j_L(j_L + 1) + 2j_R(j_R + 1)$$

(3 - 15)

where $E$ denotes the AdS energy and $j_L, j_R$ the two $SU(2)$ quantum numbers. For $P = 1$, $j_L, j_R$ are determined by the number operators $N_a, N_b$ as follows

$$j_L = \frac{N_a}{2}, \quad j_R = \frac{N_b}{2}$$

(3 - 16)

Similarly, the eigenvalues of the cubic, $C_3$, and quartic Casimir operators, $C_4$, can be expressed in terms of $E, j_L$ and $j_R$ as

$$C_3 = -(E - 2)[j_L(j_L + 1) - j_R(j_R + 1)]$$

(3 - 17)

$$C_4 = \frac{1}{4}(E - 2)^4 - (E - 2)^3[j_L(j_L + 1) + j_R(j_R + 1)]$$

$$+ 4j_L(j_L + 1)j_R(j_R + 1)$$

(3 - 18)

4 Doubleton Representations of the $AdS_5$ Group

$SU(2, 2)$

The minimal oscillator realization of $SU(2, 2)$ requires a single pair of oscillators i.e. $P = 1$, corresponding to doubleton representations \[4\] that have
no Poincare limit. Possible lowest weight vectors in this case are of the form

\[ a^{i_1}...a^{i_{n_L}} |0\rangle \]  \hspace{1cm} (4 - 1)

and

\[ b^{r_1}...b^{r_{n_R}} |0\rangle \]  \hspace{1cm} (4 - 2)

where \( n_L \) and \( n_R \) are some non-negative integers (including zero).

The \( AdS_5 \) fields corresponding to the UIR’s defined by these lowest weight vectors are

\[ a^{i_1}...a^{i_{n_L}} |0\rangle \leftrightarrow \Xi(n_L,0)\left(\frac{n_L}{2} + 1\right) \]  \hspace{1cm} (4 - 3)

\[ b^{r_1}...b^{r_{n_R}} |0\rangle \leftrightarrow \Xi(0,n_R)\left(\frac{n_R}{2} + 1\right) \]  \hspace{1cm} (4 - 4)

The eigenvalues of the quadratic Casimir operator \( C_2 \) on these \( AdS_5 \) fields are given by

\[ C_2 \Xi(n_L,0)\left(\frac{n_L}{2} + 1\right) = 3\left(\frac{n_L^2}{4} - 1\right) \Xi(n_L,0)\left(\frac{n_L}{2} + 1\right) \]  \hspace{1cm} (4 - 5)

\[ C_2 \Xi(0,n_R)\left(\frac{n_R}{2} + 1\right) = 3\left(\frac{n_R^2}{4} - 1\right) \Xi(0,n_R)\left(\frac{n_R}{2} + 1\right) \]  \hspace{1cm} (4 - 6)

We should note that since the Poincare limit of the doubleton representations of the \( AdS_5 \) group is singular they are to be interpreted as fields living on the boundary of \( AdS_5 \) space which can be identified with the four dimensional Minkowski space-time with some points added. Then the group \( SU(2,2) \) acts as the conformal group on the boundary.

5 "Massless" Representations of the \( AdS_5 \) Group \( SU(2,2) \)

"Massless" representations of the \( AdS_5 \) group \( SU(2,2) \) are obtained when we take two pairs \( (P = 2) \) of oscillators in the oscillator construction. In this case we have lowest weight vectors with the same \( SU(2) \) transformation properties as in the case of doubletons, but with \( AdS \) energies one unit higher:

\[ a^{(i_1}...a^{i_{n_L})} |0\rangle \leftrightarrow \Xi(n_L,0)\left(\frac{n_L}{2} + 2\right) \]  \hspace{1cm} (5 - 1)

\[ b^{(r_1}...b^{r_{n_R})} |0\rangle \leftrightarrow \Xi(0,n_R)\left(\frac{n_R}{2} + 2\right) \]  \hspace{1cm} (5 - 2)
where the round brackets indicate symmetrization of indices.

We also have a new type of lowest weight vectors with both $j_L$ and $j_R$ nonvanishing, which have no analogs in the case of doubletons:

$$a^{i_1(1)...a^{i_nL}(1)}b^{r_1(2)...b^{r_nR}(2)}|0\rangle \Leftrightarrow \Xi(\frac{n_L}{2}, \frac{n_R}{2})(\frac{n_L + n_R}{2} + 2) \quad (5 - 3)$$

The eigenvalues of the quadratic Casimir operator on the UIR’s defined by the above lowest weight vectors are

$$C_2 \Xi(\frac{n_L}{2}, 0)(\frac{n_L}{2} + 2) = \left(\frac{3}{4}n_L^2 + n_L - 4\right)\Xi(\frac{n_L}{2}, 0)(\frac{n_L}{2} + 2) \quad (5 - 4)$$

$$C_2 \Xi(0, \frac{n_R}{2})(\frac{n_R}{2} + 2) = \left(\frac{3}{4}n_R^2 + n_R - 4\right)\Xi(0, \frac{n_R}{2})(\frac{n_R}{2} + 2) \quad (5 - 5)$$

$$C_2 \Xi(\frac{n_L}{2}, \frac{n_R}{2})(\frac{n_L + n_R}{2} + 2) = \left(\frac{3}{4}(n_L^2 + n_R^2) + n_R + n_L + \frac{1}{2}n_Ln_R - 4\right)\Xi(\frac{n_L}{2}, \frac{n_R}{2})(\frac{n_R + n_L}{2} + 2) \quad (5 - 6)$$

In addition to the above lowest weight vectors for $P > 2$ we have the following possible lowest weight vectors and the corresponding fields:

$$a^{[j_1a^{k_1]}...a^{[j_La^{k_L}]}a^{(i_1...a^{i_nL})}|0\rangle \Leftrightarrow \Xi(\frac{n_L}{2}, 0)(\frac{n_L}{2} + L + 2) \quad (5 - 7)$$

$$b^{[s_1b^{t_1]}...b^{[s_Rb^{t_R}]}b^{(r_1...b^{r_nR})}|0\rangle \Leftrightarrow \Xi(0, \frac{n_R}{2})(\frac{n_R}{2} + R + 2) \quad (5 - 8)$$

where the square brackets indicate antisymmetrization of indices.

6 "Massive" Representations of the $AdS_5$ Group $SU(2, 2)$

"Massive" representations of the $AdS_5$ group $SU(2, 2)$ can be obtained by the oscillator method by taking $P > 2$. For example, the lowest weight vectors (5.1), (5.2) and (5.3) correspond to "massive" fields for $P > 2$:

$$a^{i_1...a^{i_nL}}|0\rangle \Leftrightarrow \Xi(\frac{n_L}{2}, 0)(\frac{n_L}{2} + P) \quad (6 - 1)$$

$$b^{r_1...b^{r_nR}}|0\rangle \Leftrightarrow \Xi(0, \frac{n_R}{2})(\frac{n_R}{2} + P) \quad (6 - 2)$$

$$a^{i_1(1)...a^{i_nL}(1)}b^{r_1(2)...b^{r_nR}(2)}|0\rangle \Leftrightarrow \Xi(\frac{n_L}{2}, \frac{n_R}{2})(\frac{n_L + n_R}{2} + P) \quad (6 - 3)$$
Some of these "massive" fields with "spin" less than or equal to two appear in the spectrum of the $S^5$ compactification of IIB theory \[4, 5\]. We should note, however, that the mass operator is not an invariant operator of the $AdS_5$ group $SU(2,2)$. To illustrate the problems associated with defining the concept of mass in $AdS$ space, consider the UIR associated with the vacuum $|0\rangle$ chosen as the lowest weight vector (lwv) for $P = L + 2$

$$|0\rangle \leftrightarrow \Xi_{(0,0)}(L+2) \quad (6 - 4)$$

On the other hand, the lowest weight vector

$$a^{[i_1 a^{j_1}]...a^{[i_L a^{j_L}]}|0\rangle \leftrightarrow \Xi_{(0,0)}(L+2) \quad (6 - 5)$$

with $P = 2$ yields the same UIR. The analysis based on the wave equations in $AdS$ spacetimes suggests that the UIR defined by $|0\rangle$ for $P = L + 2$ ($L > 0$) is "massive" \[4\]. On the other hand, the state $a^{[i_1 a^{j_1}]...a^{[i_L a^{j_L}]}|0\rangle$ for $P = 2$ occurs in the tensor product of two doubletons and must be "massless" \[4, 16\].

The resolution of the puzzle is that the standard analysis using wave equations assumes that the field in question is an elementary field \[5\]. An elementary scalar $\Phi_{(0,0)}(L+2)$ field corresponding to the lowest weight vector $|0\rangle$ with $P = L + 2$ has the same $AdS$ energy as the composite scalar built out of $2L$ massless "spin 1/2" fields $\Psi$ of the form $(\bar{\Psi}\Psi)$.\[14\].

Much more useful and unambiguous concept than mass is that of $AdS$ energy or conformal dimension (conformal energy) when $SU(2,2)$ is interpreted as the 4-d conformal group \[11\]. When we consider $SU(2,2)$ as the conformal group then the conformal dimension of a composite operator is simply given by the sum of the conformal dimensions of its elementary constituents and we do not have any interpretational problems.

7 $SU(m, n|p + q)$ via the Oscillator Method

The symmetry group of the compactification of type IIB superstring over the five sphere is the supergroup $SU(2,2|4)$ with the even subgroup $SU(2,2) \times SU(4) \times U(1)$ where $SU(4)$ is the isometry group of the five sphere \[4\]. The Abelian $U(1)$ factor comes directly from the ten dimensional theory itself and is the subgroup of the $SU(1,1)$ symmetry of the IIB supergravity. This $U(1)$ generator commutes with all the other generators and acts like a central charge. Therefore, $SU(2,2|4)$ is not a simple Lie superalgebra. By factoring out this Abelian ideal one obtains a simple Lie superalgebra, denoted
as $PSU(2,2|4)$, whose even subalgebra is simply $SU(2,2) \times SU(4)$ \[1\]. By orbifolding the five sphere with some discrete subgroup one can obtain consistent backgrounds for the compactification of IIB superstring \[22\]. These backgrounds have fewer supersymmetries corresponding to supergroups of the form $SU(2,2|k)$ with $k < 4$. In the following sections we apply the general oscillator method outlined in the introduction to construct positive energy unitary irreducible representations of $SU(2,2|4)$. Before specializing to the supergroup $SU(2,2|4)$ relevant for the IIB superstring we shall discuss the oscillator realization of general supergroups of the form $SU(m,n|p+q)$ following \[17, 4\].

The superalgebra $SU(m,n|p+q)$ has a three graded decomposition with respect to its compact subsuperalgebra $SU(m|p) \times SU(n|q) \times U(1)$

$$g = L^+ \oplus L^0 \oplus L^-$$

where

$$[L^0, L^\pm] = L^\pm$$
$$[L^+, L^-] = L^0$$
$$[L^+, L^+] = 0 = [L^-, L^-]$$

Here $L^0$ represents the generators of $SU(m|p) \times SU(n|q) \times U(1)$.

The Lie superalgebra $SU(m,n|p+q)$ can be realized in terms of bilinear combinations of bosonic and fermionic annihilation and creation operators $\xi_A (\xi^A = \xi_A^\dagger)$ and $\eta_M (\eta^M = \eta_M^\dagger)$ which transform covariantly and contravariantly under the $SU(m|p)$ and $SU(n|q)$ subsupergroups of $SU(m,n|p+q)$

$$\xi_A = \begin{pmatrix} a_i \\ \alpha_\mu \end{pmatrix}, \quad \xi^A = \begin{pmatrix} a^i \\ \alpha^\mu \end{pmatrix}$$

(7 - 3)

and

$$\eta_M = \begin{pmatrix} b_r \\ \beta_x \end{pmatrix}, \quad \eta^M = \begin{pmatrix} b^r \\ \beta^x \end{pmatrix}$$

(7 - 4)

with $i, j = 1, 2, \ldots, m; \mu, \nu = 1, 2, \ldots, p; r, s = 1, 2, \ldots, n; x, y = 1, 2, \ldots, q$ and

$$[a_i, a^j] = \delta_i^j, \quad \{\alpha_\mu, \alpha^{\nu}\} = \delta^\nu_\mu$$

(7 - 5)

$$[b_r, b^s] = \delta_r^s, \quad \{\beta_x, \beta^y\} = \delta_x^y$$

(7 - 6)

---

5 In \[4\] the symmetry supergroup of the $S^5$ compactification of IIB theory was denoted as $U(2,2|4)$ to stress the fact that it contains an Abelian ideal.
Again we denote the annihilation and creation operators with lower and upper indices, respectively. The generators of $SU(m, n|p + q)$ are given in terms of the above superoscillators as

\[
\begin{align*}
L^- &= \xi^A \cdot \eta^M \\
L^0 &= \xi^A \cdot \xi^B \oplus \eta^M \cdot \eta^N \\
L^+ &= \xi^A \cdot \eta^M
\end{align*}
\]

(7 - 7)

where the arrows over $\xi$ and $\eta$ indicate that we are taking an arbitrary number $P$ of superoscillators and the dot represents the summation over the internal index $k = 1, ..., P$, i.e $\xi^A \cdot \eta^M \equiv \sum_{k=1}^{P} \xi_A(k) \eta_M(k)$.

The $SU(p+q)$ generators, written in terms of fermionic oscillators $\alpha$ and $\beta$, read as follows

\[
\begin{align*}
A^\nu_{\mu} &= \alpha^\nu \cdot \alpha_\mu - \frac{1}{P} \delta^\nu_{\mu} N_\alpha \\
B^y_x &= \beta^y \cdot \beta_x - \frac{1}{q} \delta^y_x N_\beta \\
C &= \frac{1}{P} N_\alpha - \frac{1}{q} N_\beta + P \\
L_{\mu x} &= \alpha_{\mu} \cdot \beta_x, \quad L^{x \mu} = \beta^x \cdot \alpha^\mu
\end{align*}
\]

(7 - 8)

where $N_\alpha = \alpha^\nu \cdot \alpha_\nu$ and $N_\beta = \beta^x \cdot \beta_x$ are the fermionic number operators.

Similarly, the $SU(m, n)$ generators, written in terms of bosonic oscillators $a$ and $b$, read

\[
\begin{align*}
L_{ir} &= \tilde{a}_i \cdot \tilde{b}_r, \quad L^{ir} = \tilde{a}^i \cdot \tilde{b}^r \\
L^k_i &= \tilde{a}_k \cdot \tilde{a}_i - \frac{1}{m} \delta^k_i N_a \\
R^r_s &= \tilde{b}_r \cdot \tilde{b}_s - \frac{1}{n} \delta^r_s N_b \\
E &= \frac{1}{m} N_a + \frac{1}{n} N_b + P
\end{align*}
\]

(7 - 9)

where $N_a \equiv \tilde{a}^i \cdot \tilde{a}_i, N_b \equiv \tilde{b}^r \cdot \tilde{b}_r$ are the bosonic number operators.

The following closure relations are valid

\[
\begin{align*}
[\tilde{a}_i \cdot \tilde{b}_r, \tilde{b}^s \cdot \tilde{a}^k] &= \delta^s_i L^k_r + \delta^k_i R^s_r + \delta^s_k \delta^i_r E \\
[\tilde{a}_\mu \cdot \tilde{b}_x, \tilde{b}^z \cdot \tilde{a}^\nu] &= -\delta^z_x A^\mu_{\nu} - \delta^x_{\nu} B^z_{\mu} + \delta^z_{\mu} \delta^x_{\nu} C \\
\{\tilde{a}_i \cdot \tilde{b}_x, \tilde{b}^z \cdot \tilde{a}^k\} &= \delta^z_x L^k_i - \delta^k_i B^z_x + \frac{1}{2} \delta^z_\xi \delta^k_i (E + C + D - F) \\
\{\tilde{a}_\mu \cdot \tilde{b}_r, \tilde{b}^\nu \cdot \tilde{a}^\rho\} &= -\delta^\nu_{\rho} A^\mu_r + \delta^\rho_{\nu} R^\mu_r + \frac{1}{2} \delta^\nu_{\rho} \delta^\mu_r (E + C - D + F)
\end{align*}
\]
\[ \{ \bar{\alpha} \cdot \bar{a}_\mu, \bar{a}^\rho \cdot \bar{a}_k \} = \delta_\mu^\rho L_k^\rho + \delta_k^\mu A_\rho^\mu + \frac{1}{2} \delta_\mu^\rho \delta_k^\rho (E - C + D - F) \]
\[ \{ \bar{\beta} \cdot \bar{\beta}_x, \bar{\beta}_z \cdot \bar{b}_i \} = \delta_x^x R_i^x + \delta_i^x B_z^x + \frac{1}{2} \delta_x^x \delta_i^x (E - C - D + F) \quad (7 - 10) \]

where \( D \) and \( F \) are defined as
\[
D = \frac{1}{m} N_\alpha - \frac{1}{q} N_\beta + P
\]
\[
F = \frac{1}{n} N_b - \frac{1}{p} N_\alpha + P \quad (7 - 11)
\]

Note that only three of the four \( U(1) \) charges \( E, C, D, F \) are linearly independent. In what follows, we will choose \( E, C \) and \( (D - F) \) as the linearly independent \( U(1) \) generators.

The quadratic Casimir operator must be of the form
\[
C_2 = \lambda_1 \bar{a}_i \cdot \bar{b}_i + \lambda_2 \bar{a}_i \cdot \bar{\beta}_x \cdot \bar{\beta}_z \cdot \bar{b}_i + \lambda_3 \bar{a}_i \cdot \bar{\beta}_x \cdot \bar{\beta}_z \cdot \bar{a}_i + \lambda_4 \bar{a}_i \cdot \bar{\beta}_x \cdot \bar{\beta}_z \cdot \bar{a}_i + \lambda_5 \bar{a}_i \cdot \bar{\beta}_x \cdot \bar{\beta}_z \cdot \bar{b}_i + \lambda_6 \bar{\beta}_x \cdot \bar{\beta}_z \cdot \bar{a}_i \cdot \bar{b}_i + \lambda_7 \bar{\beta}_x \cdot \bar{\beta}_z \cdot \bar{\beta}_x \cdot \bar{\beta}_z + \lambda_8 \bar{\beta}_x \cdot \bar{\beta}_z \cdot \bar{\beta}_x \cdot \bar{\beta}_z
\]
\[
+ \mu_1 L_i^1 L_i^1 + \mu_2 R_i^x R_i^x + \mu_3 A_\rho^\mu A_\rho^\mu + \mu_4 B_z^x B_z^x
\]
\[
+ \rho_1 \bar{a}_i \cdot \bar{a}_i \cdot \bar{a}_i + \rho_2 \bar{a}_i \cdot \bar{a}_i \cdot \bar{a}_i + \rho_3 \bar{\beta}_x \cdot \bar{\beta}_x \cdot \bar{b}_i + \rho_4 \bar{\beta}_x \cdot \bar{\beta}_x \cdot \bar{b}_i + \rho_5 \bar{\beta}_x \cdot \bar{\beta}_x \cdot \bar{b}_i + \rho_6 \bar{\beta}_x \cdot \bar{\beta}_x \cdot \bar{b}_i
\]
\[
+ \sigma_1 E^2 + \sigma_2 C^2 + \sigma_3 (D - F)^2 + \sigma_4 E C + \sigma_5 E (D - F) + \sigma_6 C (D - F) + \sigma_7 E (D - F) + \sigma_8 C (D - F) \quad (7 - 12)
\]

The requirement that \( C_2 \) commutes with the generators fixes the coefficients up to an overall constant. It turns out that the following commutators
\[
[C_2, \bar{\beta}_x \cdot \bar{a}_i] = [C_2, \bar{\beta}_x \cdot \bar{a}^\rho] = [C_2, \bar{\beta}_z \cdot \bar{a}_i] = [C_2, \bar{\beta}_z \cdot \bar{a}^\rho] = 0 \quad (7 - 13)
\]
determine all the coefficients up to an overall multiplicative constant (for which \( \sigma_3 \) turns out to be a convenient choice)
\[
\lambda_1 = -\lambda_2 = -\lambda_3 = \lambda_4 = \lambda_5 = \lambda_6 = \lambda_7 = \lambda_8
\]
\[
= -\mu_1 = -\mu_2 = \mu_3 = \mu_4 = \rho_1 = -\rho_2 = \rho_3 = -\rho_4 = \frac{4(m + n - p - q)}{(m + n)(q + p)} \sigma_3
\]
\[
\sigma_1 = \frac{-4mn + (m + n)(p + q)}{(m + n)(q + p)} \sigma_3
\]
\[
\sigma_2 = \frac{-4pq + (m + n)(p + q)}{(m + n)(q + p)} \sigma_3
\]
\[
\sigma_4 = \sigma_4 = \frac{(n - m)(q - p)}{(m + n)(q + p)} \sigma_3
\]

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The UIRs of $SU(i, j, \ldots)$ listed in the introduction. To this end we let the indices 8 Unitary Supermultiplets of $SU$ transforming in the fundamental representation of $SU$ and can not be realized in terms of bilinears of superoscillators $q_m$ for $P SU$ commutes with all the generators of $\mathcal{N}$. The smallest non-trivial representation of $P SU$ is simply the generator that commutes with all the generators of $PSU(m,n|p+q)$. We should note that for $m+n = p+q$ the operator $C_2$ is not a Casimir operator of $PSU(m,n|p+q)$. The smallest non-trivial representation of $PSU(m,n|p+q)$ is the adjoint representation and can not be realized in terms of bilinears of superoscillators transforming in the fundamental representation of $SU(m,n|p+q)$.

b) For the case of $P = 1$ the most general form (valid for arbitrary $m,n,p,q$) of $C_2$ reduces to

$$C_2 = (m+n-p-q-1)[(n-q-m+p)(N_a - N_b + N_a - N_\beta) - (N_a - N_b + N_a - N_\beta)^2 + (m-p)(n-q)]$$  \hspace{1cm} (7 - 17)

8 Unitary Supermultiplets of $SU(2,2|4)$

The UIRs of $SU(2,2|4)$ are constructed following the general procedure outlined in the introduction. To this end we let the indices $i, j, \ldots; r, s, \ldots; \mu, \nu, \ldots; x, y, \ldots$
run from 1 to 2. Starting from the ground state $|\Omega\rangle$ in the Fock space transforming irreducibly under $SU(2|2) \times SU(2|2) \times U(1)$ and is annihilated by $L^-$ (given in terms of oscillators $\xi$ and $\eta$ as in eq. (7.7)), one can generate the UIRs of $SU(2,2|4)$ by repeated application of $L^+$

$$|\Omega\rangle, \quad L^+|\Omega\rangle, \quad L^+L^+|\Omega\rangle, \ldots \quad (8\cdot1)$$

The irreducibility of UIRs of $SU(2,2|4)$ follows from the irreducibility of $|\Omega\rangle$ under $SU(2|2) \times SU(2|2) \times U(1)$.

When restricted to the subspace involving purely bosonic oscillators we get the subalgebra $SU(2,2)$ and the above construction yields its positive energy UIR's. Similarly, when restricted to the subspace involving purely fermionic oscillators we get the compact internal symmetry group $SU(4)$. Then the above construction yields the representations of $SU(4)$ in its $SU(2) \times SU(2) \times U(1)$ basis.

The positive energy UIR’s of $SU(2,2|4)$ decompose into a direct sum of finitely many positive energy UIR’s of $SU(2,2)$ transforming in certain representations of the internal symmetry group $SU(4)$. Thus each positive energy UIR of $SU(2,2|4)$ corresponds to a supermultiplet of fields living in $AdS_5$ or its boundary. The bosonic and fermionic fields in $AdS_5$ or its boundary will be denoted as $\Phi_{(j_L,j_R)}^{(E)}$ and $\Psi_{(j_L,j_R)}^{(E)}$, respectively.

We shall assume that the vacuum vector $|0\rangle$ is CPT invariant and we shall call the supermultiplets defined by taking $|0\rangle$ as the lowest weight vector the CPT self-conjugate supermultiplets. Furthermore, we shall refer to the supermultiplets obtained by the interchange of $\xi$ and $\eta$ type superoscillators as conjugate supermultiplets.

### 8.1 Doubleton Supermultiplets of $SU(2,2|4)$

More explicitly, consider $|0\rangle$ as the lowest weight vector of $SU(2,2|4)$. $|0\rangle$ is automatically a lowest weight vector of $SU(2,2) \times U(4)$. By acting on $|0\rangle$ with the supersymmetry generators $\vec{a}^i \cdot \vec{\beta}^2$ and $\vec{b}^r \cdot \vec{a}^\mu$, one generates additional lowest weight vectors of $SU(2,2) \times U(4)$. The action of $L^{ir}$ on these lowest weight vectors generates the higher Fourier modes of the corresponding fields and the action of $L^{i\mu}$ corresponds to moving within the respective $SU(4)$ representation. We find that the AdS scalar fields corresponding to $|0\rangle$ transform as $\mathbf{6}$ of $SU(4)$, the spinor fields corresponding to $L^{i\mu}|0\rangle$ and $L^{j\mu}|0\rangle$ transform respectively as $\bar{\mathbf{4}}$ and $\mathbf{4}$ of $SU(4)$ and finally, the self-dual and anti-self-dual tensor fields corresponding to $L^{i\mu}L^{j\nu}|0\rangle$ and $L^{i\mu}L^{j\nu}|0\rangle$ transform as two singlets of $SU(4)$.
This supermultiplet is the CPT self-conjugate doubleton supermultiplet (we have used a pair of oscillators in its construction) and it is the supermultiplet of $\mathcal{N} = 4$ supersymmetric Yang-Mills theory in 4-d \[4\]. The contents of this particular doubleton supermultiplet are given in Table 1. (We will continue to use this form of representing our results in what follows.)

\[
\begin{array}{|c|c|c|c|c|}
\hline
SU(2, 2) \times SU(4) \text{ lwv} & E & (j_L, j_R) & SU(4) & Y \\
\hline
|0\rangle & 1 & (0,0) & 6 & 0 \Phi_{0,0} \\
\hline
a^i \beta^r |0\rangle & 3/2 & (1/2,0) & 4 & -1 \Psi_{1/2,0} \\
\hline
b^r \alpha^b |0\rangle & 3/2 & (0,1/2) & 4 & 1 \Psi_{0,1/2} \\
\hline
a^i a^j \beta^r \beta^p |0\rangle & 2 & (1,0) & 1 & -2 \Phi_{1,0} \\
\hline
b^r b^s \alpha^b \alpha^c |0\rangle & 2 & (0,1) & 1 & 2 \Phi_{0,1} \\
\hline
\end{array}
\]

Table 1. The doubleton supermultiplet corresponding to the lwv $|\Omega\rangle = |0\rangle$. The first column indicates the lowest weight vectors (lwv) of $SU(2, 2) \times SU(4)$. Also, $Y = N_\alpha - N_\beta; E = (N_a + N_b)/2 + P \equiv N/2 + 1$. $\Phi$ and $\Psi$ denote bosonic and fermionic fields respectively.

Note, that by insisting on CPT self-conjugacy of the irreducible supermultiplet (which means, in this case, that $\Omega = |0\rangle$) we get only the above $\mathcal{N} = 4$ Yang-Mills supermultiplet. But there are other irreducible doubleton supermultiplets which are not CPT self-conjugate, and which are different from the 4-d $\mathcal{N} = 4$ supersymmetric Yang-Mills supermultiplet.

If we take
\[
|\Omega\rangle = \xi^A |0\rangle \equiv a^i |0\rangle \oplus \alpha^\mu |0\rangle = | \quad ,1 \rangle \quad \quad (8 - 1)
\]
we get the supermultiplet represented in Table 2. (See the appendix for a quick review of the supertableaux notation \[24\].)

\[
\begin{array}{|c|c|c|c|c|}
\hline
SU(2, 2) \times SU(4) \text{ lwv} & E & (j_L, j_R) & SU(4) & Y \\
\hline
a^i |0\rangle & 3/2 & (1/2,0) & 6 & 0 \Psi_{1/2,0} \\
\hline
a^i a^j \beta^r |0\rangle & 2 & (1,0) & 4 & -1 \Phi_{1,0} \\
\hline
a^i a^j a^k \beta^r \beta^p |0\rangle & 5/2 & (3/2,0) & 1 & -2 \Psi_{3/2,0} \\
\hline
\alpha^\mu |0\rangle & 1 & (0,0) & 4 & 1 \Phi_{0,0} \\
\hline
b^r \alpha^b \alpha^c |0\rangle & 3/2 & (0,1/2) & 1 & 2 \Psi_{0,1/2} \\
\hline
\end{array}
\]

Table 2. The doubleton supermultiplet corresponding to the lwv $|\Omega\rangle = \xi^A |0\rangle = | \quad ,1 \rangle$. 

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The conjugate supermultiplet to the one above can be obtained if we take

\[ |\Omega\rangle = \eta^A|0\rangle \equiv b^r|0\rangle \oplus \beta^x|0\rangle = |1, \square\rangle \]  \hspace{1cm} (8 - 2)

Then we get the supermultiplet represented in Table 3.

| \( SU(2,2) \times SU(4) \) lwv | E | \((j_L, j_R)\) | \( SU(4) \) | \( Y \) | Field |
|---|---|---|---|---|---|
| \( b^r|0\rangle \) | 3/2 | (0,1/2) | 6 | 0 | \( \Psi_{0,1/2} \) |
| \( b^r b^c \alpha^\mu|0\rangle \) | 2 | (0,1) | 4 | 1 | \( \Phi_{0,1} \) |
| \( b^r b^c \alpha^\mu \beta^\nu|0\rangle \) | 5/2 | (0,3/2) | 1 | 2 | \( \Psi_{0,3/2} \) |
| \( \beta^\mu|0\rangle \) | 1 | (0,0) | 4 | -1 | \( \Phi_{0,0} \) |
| \( a^i \beta^\mu \beta^\nu|0\rangle \) | 3/2 | (1/2,0) | 1 | -2 | \( \Psi_{1/2,0} \) |

Table 3. The doubleton supermultiplet corresponding to the lwv \( |\Omega\rangle = \eta^A|0\rangle = |1, \square\rangle \).

The conjugate irreducible supermultiplet is obtained by taking

\[ |\Omega\rangle = \xi^A \xi^B|0\rangle \equiv a^i a^j|0\rangle \oplus a^i \alpha^\mu|0\rangle \oplus a^\mu \alpha^\nu|0\rangle = |1, \square\rangle \]  \hspace{1cm} (8 - 3)

we get the supermultiplet represented in Table 4.

| \( SU(2,2) \times SU(4) \) lwv | E | \((j_L, j_R)\) | \( SU(4) \) | \( Y \) | Field |
|---|---|---|---|---|---|
| \( a^i a^j|0\rangle \) | 2 | (1,0) | 6 | 0 | \( \Phi_{1,0} \) |
| \( a^i a^j a^k \beta^\mu|0\rangle \) | 5/2 | (3/2,0) | 4 | -1 | \( \Psi_{3/2,0} \) |
| \( a^i a^j a^k \beta^\mu \beta^\nu|0\rangle \) | 3 | (2,0) | 1 | -2 | \( \Phi_{2,0} \) |
| \( a^i \alpha^\mu|0\rangle \) | 3/2 | (1/2,0) | 4 | 1 | \( \Psi_{1/2,0} \) |
| \( \alpha^\mu \alpha^\nu|0\rangle \) | 1 | (0,0) | 1 | 2 | \( \Phi_{0,0} \) |

Table 4. The doubleton supermultiplet corresponding to the lwv \( |\Omega\rangle = \xi^A \xi^B|0\rangle = |1, \square\rangle \).

The conjugate irreducible supermultiplet is obtained by taking

\[ |\Omega\rangle = \eta^A \eta^B|0\rangle \equiv b^r b^i|0\rangle \oplus b^r \beta^x|0\rangle \oplus \beta^\mu \beta^\nu|0\rangle = |1, \square\rangle \]  \hspace{1cm} (8 - 4)

The result is given in Table 5.
\[
SU(2,2) \times SU(4) \text{ lwv E (}j_L,j_R\text{) SU(4) Y Field}
\]
\begin{center}
\begin{tabular}{|c|c|c|c|c|}
\hline
\(b^*b^0[0]\) & 2 & (0,1) & 6 & 0 \(\Phi_{0,1}\) \\
\(b^*b^\alpha[0]\) & 5/2 & (0,3/2) & 4 & 1 \(\Psi_{0,3/2}\) \\
\(b^*b^\alpha b^\beta[0]\) & 3 & (0,2) & 1 & 2 \(\Phi_{0,2}\) \\
\(b^*\beta^2[0]\) & 3/2 & (0,1/2) & 4 & -1 \(\Psi_{0,1/2}\) \\
\(\beta^2\beta^\gamma[0]\) & 1 & (0,0) & 1 & -2 \(\Phi_{0,0}\) \\
\hline
\end{tabular}
\end{center}

Table 5. The doubleton supermultiplet corresponding to the lwv \(|\Omega\rangle = \eta^A \eta^B |0\rangle = |1, \square\rangle\).

The direct sum of the supermultiplets defined by the lowest weight vectors \(\xi^A \xi^B |0\rangle\) and \(\eta^A \eta^B |0\rangle\) is parity invariant and corresponds to the \(N = 4\) conformal supergravity multiplet in \(d = 4\).

In general, we could take
\[
|\Omega\rangle = \xi^A_1 \xi^A_2 \cdots \xi^A_{2J} |0\rangle \\
|\Omega\rangle = \eta^A_1 \eta^A_2 \cdots \eta^A_{2J} |0\rangle
\]  
(8 - 5)

as our lowest weight vectors.

For \(j \geq 1\), the general doubleton supermultiplets, obtained by taking
\[
|\Omega\rangle = \xi^A_1 \xi^A_2 \cdots \xi^A_{2J} |0\rangle = |\square \cdots \square, 1\rangle
\]  
(8 - 6)

are represented in Table 6.

\[
\begin{center}
\begin{tabular}{|c|c|c|c|c|}
\hline
E & \((j_L,j_R)\) & SU(4) & \(U(1)_Y\) & Field \\
\hline
j+1 & (j,0) & 6 & 0 & \(\Phi_{j,0}\) \\
\hline
j+3/2 & (j+1/2,0) & 4 & -1 & \(\Psi_{j+1/2,0}\) \\
\hline
j+1/2 & (j-1/2,0) & 4 & 1 & \(\Psi_{j-1/2,0}\) \\
\hline
j+ & (j+1,0) & 1 & -2 & \(\Phi_{j+1,0}\) \\
\hline
j & (j-1,0) & 1 & 2 & \(\Phi_{j-1,0}\) \\
\hline
\end{tabular}
\end{center}
\]

Table 6. The doubleton supermultiplet corresponding to the lwv \(|\Omega\rangle = \xi^A_1 \xi^A_2 \cdots \xi^A_{2J} |0\rangle = |\square \cdots \square, 1\rangle\).

Here we assume that \(j\) takes integer values. For \(j\) half-integer the roles of \(\Phi\) and \(\Psi\) are reversed.
The conjugate supermultiplets to the ones above are obtained by taking

$$|\Omega\rangle = \eta^1 \eta^2 \ldots \eta^{2j} |0\rangle = |1, \underbrace{\ldots |} \rangle_{2j}$$  \hspace{0.5cm} (8-7)$$

and have the form represented in Table 7.

| E  | (j_L, j_R) | SU(4) | U(1)_γ | Field            |
|-----|------------|-------|--------|------------------|
| j+1 | (0, j)    | 6     | 0      | Φ_{0,j}          |
| j+3/2 | (0, j+1/2) | 4     | 1      | Ψ_{0,j+1/2}     |
| j+1/2 | (0, j-1/2) | 4     | -1     | Ψ_{0,j-1/2}     |
| j+2 | (0, j+1)  | 1     | 2      | Φ_{0,j+1}       |
| j   | (0, j-1)  | 1     | -2     | Φ_{0,j-1}       |

Table 7. The doubleton supermultiplet corresponding to the lwv $|\Omega\rangle = \eta^1 \eta^2 \ldots \eta^{2j} |0\rangle = |1, \underbrace{\ldots |} \rangle_{2j}$.

9 "Massless" Irreducible Representations of SU(2, 2|4)

The "massless" supermultiplets of SU(2, 2|4) are obtained by tensoring two doubleton supermultiplets, i.e. by taking $P = 2$. In this case the vacuum $|0\rangle$ taken as the lowest weight vector of SU(2, 2|4) leads to the "massless" $\mathcal{N} = 8$ graviton supermultiplet in AdS$_5$. It is a short supermultiplet having spin range 2. However, generic "massless" supermultiplets have spin range 4. We also have "massless" supermultiplets that are short in addition to the graviton supermultiplet.

For example, consider the "massless" supermultiplet $(P \equiv 2)$ defined by the lowest weight vector

$$|\Omega\rangle = \xi^A(2)|0\rangle \equiv a^i(2)|0\rangle \oplus a^\mu(2)|0\rangle = | \underbrace{\ldots |} \rangle_{1}$$  \hspace{0.5cm} (9-1)$$

The application of the susy generators $\vec{a} \cdot \vec{\beta}$ and $\vec{b} \cdot \vec{a}^\mu$ leads to the "massless" supermultiplet represented in Table 8. The possible SU(4) representations and the corresponding lowest weight vectors for $P = 2$ are given in the appendix.
Table 8. The "massless" supermultiplet corresponding to the lwv $|\Omega\rangle = \xi^A(2)|0\rangle \equiv a^i(2)|0\rangle \oplus \alpha^\mu(2)|0\rangle = |1, 1\rangle$.

Similarly, consider the "massless" supermultiplet defined by

$$|\Omega\rangle = \eta^A(2)|0\rangle \equiv b^r(2)|0\rangle \oplus \beta^x(2)|0\rangle = |1, 1\rangle \quad (9 - 2)$$

Again, the application of the susy generators $\vec{a} \cdot \vec{\beta}$ and $\vec{b} \cdot \vec{\alpha}$ leads to the "massless" supermultiplet represented in Table 9.

| E     | (j_L, j_R) | SU(4) Dynkin | $U(1)_Y$ | Field       |
|-------|------------|--------------|----------|-------------|
| 2     | (0,0)      | (1,1,0)      | 1        | $\Phi_{0,0}$|
| 5/2   | (1/2,0)    | (0,2,0)      | 0        | $\Psi_{1/2,0}$|
| 5/2   | (1/2,0)    | (1,0,1)      | 0        | $\Psi_{1/2,0}$|
| 5/2   | (0,1/2)    | (0,1,0)      | 2        | $\Psi_{0,1/2}$|
| 5/2   | (0,1/2)    | (2,0,0)      | 2        | $\Psi_{0,1/2}$|
| 3     | (1,0)      | (1,0,0)      | -1       | $\Phi_{1,0}$|
| 3     | (0,1)      | (1,0,0)      | 3        | $\Phi_{0,1}$|
| 3     | (0,0)      | (1,0,0)      | 3        | $\Phi_{0,0}$|
| 3     | (0,0)      | (0,1,1)      | -1       | $\Phi_{0,0}$|
| 3     | (1,0)      | (0,1,1)      | -1       | $\Phi_{1,0}$|
| 3     | (1/2,1/2)  | (1,1,0)      | 1        | $\Phi_{1/2,1/2}$|
| 3     | (1/2,1/2)  | (0,0,1)      | 1        | $\Phi_{1/2,1/2}$|
| 7/2   | (3/2,0)    | (0,1,0)      | -2       | $\Psi_{3/2,0}$|
| 7/2   | (1/2,2)    | (0,0,2)      | -2       | $\Psi_{1/2,0}$|
| 7/2   | (1/2,0)    | (0,1,0)      | -2       | $\Psi_{1/2,0}$|
| 7/2   | (1/2,1)    | (0,1,0)      | 2        | $\Psi_{1/2,1}$|
| 7/2   | (1,1/2)    | (1,0,1)      | 0        | $\Psi_{1,1/2}$|
| 7/2   | (0,1/2)    | (0,0,0)      | 4        | $\Psi_{0,1/2}$|
| 7/2   | (1,1/2)    | (0,0,0)      | 0        | $\Psi_{1,1/2}$|
| 4     | (1,0)      | (0,0,1)      | -3       | $\Phi_{1,0}$|
| 4     | (0,0)      | (0,0,1)      | -3       | $\Phi_{0,0}$|
| 4     | (3/2,1/2)  | (1,0,0)      | -1       | $\Phi_{3/2,1/2}$|
| 4     | (1,1)      | (0,0,1)      | 1        | $\Phi_{1,1}$|
| 9/2   | (1/2,0)    | (0,0,0)      | -4       | $\Psi_{1/2,0}$|
| 9/2   | (3/2,1)    | (0,0,0)      | 0        | $\Psi_{3/2,1}$|
**Table 9.** The "massless" supermultiplet corresponding to the lwv $|\Omega\rangle = \eta^A(2)|0\rangle \equiv b^\tau(2)|0\rangle \oplus \beta^x(2)|0\rangle = |1, \square\rangle$.

The above supermultiplets have spin range $5/2$.

The general form of the "massless" supermultiplet that is obtained from

$$|\Omega\rangle = \xi^{A_1}(1)\xi^{A_2}(1)\ldots\xi^{A_{2j}}(1)|0\rangle = |\underbrace{\begin{array}{c} \square \hspace{0.5cm} \square \hspace{0.5cm} \cdots \hspace{0.5cm} \square \end{array}}_{2j}, 1\rangle \quad (9 - 3)$$

is represented in Table 10 (we take $j > 3/2$).
| E | \((j_L, j_R)\) | SU(4) Dynkin | \(U(1)_Y\) | Field          |
|---|----------------|-------------|---------|---------------|
| j+1 | (j-1,0)        | (0,1,0)    | 2       | \(\Phi_{j-1,0}\) |
| j+3/2 | (j-3/2,0)     | (0,0,1)    | 1       | \(\Psi_{j-3/2,0}\) |
| j+3/2 | (j-1/2,0)     | (0,0,1)    | 1       | \(\Psi_{j-1/2,0}\) |
| j+3/2 | (j-1,1/2)     | (1,0,0)    | 3       | \(\Psi_{j-1,1/2}\) |
| j+3/2 | (j-1/2,0)     | (1,1,0)    | 1       | \(\Psi_{j-1/2,0}\) |
| j+2  | (j-1,0)       | (0,0,0)    | 0       | \(\Phi_{j-1,0}\)  |
| j+2  | (j-1,1/2)     | (0,1,0)    | 2       | \(\Phi_{j-1/2,1/2}\) |
| j+2  | (j,0)         | (1,0,1)    | 0       | \(\Phi_{j,0}\)     |
| j+2  | (j-1,0)       | (1,0,1)    | 0       | \(\Phi_{j-1,0}\)  |
| j+2  | (j-1,2,1/2)   | (2,0,0)    | 2       | \(\Phi_{j-1,2,1/2}\) |
| j+2  | (j,0)         | (0,2,0)    | 0       | \(\Phi_{j,0}\)     |
| j+2  | (j,0)         | (0,0,0)    | 0       | \(\Phi_{j,0}\)     |
| j+2  | (j-2,0)       | (0,0,0)    | 0       | \(\Phi_{j-2,0}\)   |
| j+2  | (j-1,1)       | (0,0,0)    | 4       | \(\Phi_{j-1,1}\)   |
| j+5/2 | (j+1/2,0)    | (0,1,1)    | -1      | \(\Psi_{j+1/2,0}\) |
| j+5/2 | (j-1/2,0)    | (0,1,1)    | -1      | \(\Psi_{j-1/2,0}\) |
| j+5/2 | (j,1/2)       | (1,1,0)    | 1       | \(\Psi_{j,1/2}\)   |
| j+5/2 | (j-1/2,1)    | (1,0,0)    | 3       | \(\Psi_{j-1/2,1}\) |
| j+5/2 | (j+1/2,0)    | (1,0,0)    | -1      | \(\Psi_{j+1/2,0}\) |
| j+5/2 | (j-3/2,0)    | (1,0,0)    | -1      | \(\Psi_{j-3/2,0}\) |
| j+5/2 | (j-1/2,0)    | (1,0,0)    | -1      | \(\Psi_{j-1/2,0}\) |
| j+3  | (j+1,0)       | (0,1,0)    | -2      | \(\Phi_{j+1,0}\)  |
| j+3  | (j,0)         | (0,0,2)    | -2      | \(\Phi_{j,0}\)     |
| j+3  | (j,0)         | (0,1,0)    | -2      | \(\Phi_{j,0}\)     |
| j+3  | (j-1,0)       | (0,1,0)    | -2      | \(\Phi_{j-1,0}\)   |
| j+3  | (j,1)         | (0,1,0)    | 2       | \(\Phi_{j,1}\)     |
| j+3  | (j+1/2,1/2)  | (1,0,1)    | 0       | \(\Phi_{j+1/2,1/2}\) |
| j+7/2 | (j+1,1/2)    | (1,0,0)    | -1      | \(\Psi_{j+1,1/2}\) |
| j+7/2 | (j+1/2,1)    | (0,0,1)    | 1       | \(\Psi_{j+1/2,1}\) |
| j+7/2 | (j+1/2,0)    | (0,0,1)    | -3      | \(\Psi_{j+1/2,0}\) |
| j+7/2 | (j+1/2,0)    | (0,0,1)    | -3      | \(\Psi_{j+1/2,0}\) |
| j+4  | (j,0)         | (0,0,0)    | -4      | \(\Phi_{j,0}\)     |
| j+4  | (j+1,1)       | (0,0,0)    | 0       | \(\Phi_{j+1,1}\)  |

Table 10. The "massless" supermultiplet corresponding to the lwv \(|\Omega\rangle = \xi^{A_1}(1)\xi^{A_2}(1)\cdots\xi^{A_{2j}}(1)|0\rangle = |\underbrace{\cdots}_{2j}, 1\rangle\).
Note that $j$ is again assumed to take only integer values in Table 10. For $j$ half-integer, $\Phi$ and $\Psi$ should be interchanged.

On the other hand, the general form of the "massless" supermultiplet that is obtained from

$$|\Omega\rangle = \eta^A_1(1)\eta^A_2(1)\cdots\eta^A_{2j}(1)|0\rangle = |1, \underbrace{\ldots}_2 \rangle \quad (9 - 4)$$

is represented in Table 11. ($j > 3/2$)
Table 11. The "massless" supermultiplet corresponding to the lwv $|\Omega\rangle = \eta^A_1(1)\eta^A_2(1)\cdots \eta^A_{2j}(1)|0\rangle = |1, \underbrace{\cdots}_{2j}\rangle$. 

| E   | $(j_L, j_R)$ | SU(4) Dynkin | $U(1)_Y$ | Field |
|-----|--------------|--------------|----------|-------|
| j+1 | (0,j-1)      | (0,1,0)      | -2       | $\Phi_{0,j-1}$ |
| j+3/2 | (0,j-3/2)   | (1,0,0)      | -1       | $\Psi_{0,j-3/2}$ |
| j+3/2 | (0,j-1/2)   | (1,0,0)      | -1       | $\Psi_{0,j-1/2}$ |
| j+3/2 | (1/2,j-1)   | (0,0,1)      | -3       | $\Psi_{1/2,j-1}$ |
| j+3/2 | (0,j-1/2)   | (0,1,1)      | -1       | $\Psi_{0,j-1/2}$ |
| j+2  | (0,j-1)     | (0,0,0)      | 0        | $\Phi_{0,j-1}$ |
| j+2  | (1/2,j-1/2) | (0,1,0)      | -2       | $\Phi_{1/2,j-1/2}$ |
| j+2  | (j)         | (1,0,1)      | 0        | $\Phi_{0,j}$ |
| j+2  | (0,j)       | (0,2,0)      | 0        | $\Phi_{0,j}$ |
| j+2  | (0,j-2)     | (0,0,0)      | 0        | $\Phi_{0,j-2}$ |
| j+2  | (1,j-1)     | (0,0,0)      | -4       | $\Phi_{1,j-1}$ |
| j+5/2 | (0,j+1/2)   | (1,1,0)      | 1        | $\Psi_{0,j+1/2}$ |
| j+5/2 | (0,j-1/2)   | (1,1,0)      | 1        | $\Psi_{0,j-1/2}$ |
| j+5/2 | (1/2,j)     | (0,1,1)      | -1       | $\Psi_{1/2,j}$ |
| j+5/2 | (1,j-1/2)   | (0,0,1)      | -3       | $\Psi_{1,j-1/2}$ |
| j+5/2 | (0,j+1/2)   | (0,0,1)      | 1        | $\Psi_{0,j+1/2}$ |
| j+5/2 | (0,j-3/2)   | (0,0,1)      | 1        | $\Psi_{0,j-3/2}$ |
| j+5/2 | (0,j-1/2)   | (0,0,1)      | 1        | $\Psi_{0,j-1/2}$ |
| j+3  | (0,j+1)     | (0,1,0)      | 2        | $\Phi_{0,j+1}$ |
| j+3  | (0,j)       | (2,0,0)      | 2        | $\Phi_{0,j}$ |
| j+3  | (0,j)       | (0,1,0)      | 2        | $\Phi_{0,j}$ |
| j+3  | (0,j-1)     | (0,1,0)      | 2        | $\Phi_{0,j-1}$ |
| j+3  | (1,j)       | (0,1,0)      | -2       | $\Phi_{1,j}$ |
| j+3  | (1/2,j+1/2) | (1,0,1)      | 0        | $\Phi_{1/2,j+1/2}$ |
| j+7/2 | (1/2,j+1)   | (0,0,1)      | 1        | $\Psi_{1/2,j+1}$ |
| j+7/2 | (1,j+1/2)   | (1,0,0)      | -1       | $\Psi_{1,j+1/2}$ |
| j+7/2 | (0,j+1/2)   | (1,0,0)      | 3        | $\Psi_{0,j+1/2}$ |
| j+7/2 | (0,j-1/2)   | (1,0,0)      | 3        | $\Psi_{0,j-1/2}$ |
| j+4  | (0,j)       | (0,0,0)      | 4        | $\Phi_{0,j}$ |
| j+4  | (1,j+1)     | (0,0,0)      | 0        | $\Phi_{1,j+1}$ |

\[24\]
As before, $j$ is assumed to take only integer values in Table 11.

Finally we list another allowed irreducible "massless" supermultiplet which can be obtained from the following lowest weight vector

$$|\Omega\rangle = \xi^{A_1}(1)\xi^{A_2}(1)\ldots\xi^{A_{2j_L}}(1)\eta^{B_1}(2)\eta^{B_2}(2)\ldots\eta^{B_{2j_R}}(2)|0\rangle = \left[ \begin{array}{c} \cdots \xi^{A_1(1)} \xi^{A_2(2)} \eta^{B_1(2)} \eta^{B_2(2)} \cdots \eta^{B_{2j_R}}(2) \end{array} \right]$$

(9 - 5)
| E         | (j_L, j_R) | SU(4) Dynkin | U(1)_{Y} | Field                  |
|-----------|------------|--------------|---------|------------------------|
| j_L + j_R | (j_L - 1, j_R - 1) | (0,0)       | 0       | Φ_{j_L-1,j_R-1}         |
| j_L + j_R + 1/2 | (j_L - 1/2, j_R - 1) | (1,0)       | -1      | Ψ_{j_L-1/2,j_R-1}       |
| j_L + j_R + 1/2 | (j_L - 1, j_R - 1/2) | (0,0,1)     | 1       | Ψ_{j_L-1,j_R-1/2}       |
| j_L + j_R + 1 | (j_L - 1/2, j_R - 1/2) | (0,0,0)     | 0       | Φ_{j_L-1/2,j_R-1/2}     |
| j_L + j_R + 1 | (j_L - 1, j_R) | (0,1,0)     | 2       | Φ_{j_L-1,j_R}           |
| j_L + j_R + 1 | (j_L, j_R - 1) | (0,1,0)     | -2      | Φ_{j_L,j_R-1}           |
| j_L + j_R + 1 | (j_L - 1/2, j_R - 1/2) | (1,0,1)     | 0       | Φ_{j_L-1/2,j_R-1/2}     |
| j_L + j_R + 3/2 | (j_L - 1/2, j_R) | (1,0,0)     | -1      | Ψ_{j_L,j_R-1/2}         |
| j_L + j_R + 3/2 | (j_L + 1/2, j_R - 1) | (0,0,1)     | -3      | Ψ_{j_L+1/2,j_R-1}       |
| j_L + j_R + 3/2 | (j_L - 1, j_R + 1/2) | (1,0,0)     | 3       | Ψ_{j_L-1,j_R+1/2}       |
| j_L + j_R + 3/2 | (j_L, j_R - 1/2) | (0,1,1)     | -1      | Ψ_{j_L,j_R-1}           |
| j_L + j_R + 3/2 | (j_L - 1/2, j_R) | (1,1,0)     | 1       | Ψ_{j_L-1/2,j_R}         |
| j_L + j_R + 2 | (j_L, j_R) | (1,0,0)     | 0       | Φ_{j_L,j_R}             |
| j_L + j_R + 2 | (j_L, j_R) | (0,0,0)     | 0       | Φ_{j_L,j_R}             |
| j_L + j_R + 2 | (j_L + 1, j_R) | (0,0,0)     | 0       | Φ_{j_L+1,j_R}           |
| j_L + j_R + 2 | (j_L + 1/2, j_R - 1/2) | (0,0,2)     | -2      | Φ_{j_L+1/2,j_R-1/2}     |
| j_L + j_R + 2 | (j_L + 1/2, j_R - 1/2) | (0,1,0)     | -2      | Φ_{j_L+1/2,j_R-1/2}     |
| j_L + j_R + 2 | (j_L + 1, j_R - 1) | (0,0,0)     | -4      | Φ_{j_L+1,j_R-1}         |
| j_L + j_R + 2 | (j_L + 1/2, j_R + 1/2) | (2,0,0)     | 2       | Φ_{j_L+1/2,j_R+1/2}     |
| j_L + j_R + 2 | (j_L - 1/2, j_R + 1/2) | (0,1,0)     | 2       | Φ_{j_L-1/2,j_R+1/2}     |
| j_L + j_R + 2 | (j_L - 1, j_R + 1) | (0,0,0)     | 4       | Φ_{j_L-1,j_R+1}         |
| j_L + j_R + 5/2 | (j_L + 1/2, j_R) | (1,0,0)     | -1      | Ψ_{j_L+1/2,j_R}         |
| j_L + j_R + 5/2 | (j_L, j_R + 1/2) | (0,0,1)     | 1       | Ψ_{j_L,j_R+1/2}         |
| j_L + j_R + 5/2 | (j_L + 1/2, j_R) | (0,1,1)     | -1      | Ψ_{j_L+1/2,j_R}         |
| j_L + j_R + 5/2 | (j_L, j_R + 1/2) | (1,1,0)     | 1       | Ψ_{j_L,j_R+1/2}         |
| j_L + j_R + 5/2 | (j_L + 1, j_R - 1/2) | (0,0,1)     | -3      | Ψ_{j_L+1,j_R-1/2}       |
| j_L + j_R + 5/2 | (j_L - 1/2, j_R + 1) | (1,0,0)     | 3       | Ψ_{j_L-1/2,j_R+1}       |
| j_L + j_R + 5/2 | (j_L + 1/2, j_R + 1) | (0,0,0)     | 0       | Φ_{j_L+1/2,j_R+1/2}     |
| j_L + j_R + 5/2 | (j_L + 1, j_R) | (0,1,0)     | -2      | Φ_{j_L+1,j_R}           |
| j_L + j_R + 5/2 | (j_L, j_R + 1) | (0,1,0)     | 2       | Φ_{j_L,j_R+1}           |
| j_L + j_R + 5/2 | (j_L + 1, j_R + 1) | (1,0,1)     | 0       | Φ_{j_L+1/2,j_R+1/2}     |
| j_L + j_R + 7/2 | (j_L + 1, j_R) | (1,0,0)     | -1      | Ψ_{j_L+1,j_R+1/2}       |
| j_L + j_R + 7/2 | (j_L + 1/2, j_R + 1) | (0,0,1)     | 1       | Ψ_{j_L+1/2,j_R+1}       |
| j_L + j_R + 4 | (j_L + 1, j_R + 1) | (0,0,0)     | 0       | Φ_{j_L+1,j_R+1}         |

Table 12. The "massless" supermultiplet corresponding to the lwv $|\Omega\rangle =$
We assume that both $j_L, j_R$ are either integers or half-integers, and that $j_L, j_R \geq 1$. Otherwise $\Phi$ and $\Psi$ must be interchanged. Note that for $j_L, j_R \geq 1$ this supermultiplet can also be obtained by tensoring the doubleton supermultiplet from Table 6 with the doubleton supermultiplet from Table 7. For $j_L = j_R = 1$ the above supermultiplet can also be interpreted as the $N = 4$, spin 4 conformal supermultiplet in four dimensions.[25]

10 Implications for CFT/AdS duality

We saw in section 8, that there exist doubleton supermultiplets of ever increasing spin. The CPT self-conjugate irreducible doublet on supermultiplet is that of $N = 4$ super Yang-Mills multiplet in $d = 4$. One may wonder what the physical meaning of other doubletons supermultiplets is in light of CFT/AdS duality.

In particular, there exist doubleton supermultiplets whose spin range is $3/2$. It is also known that there exist 1/4 BPS states in $N = 4$ super Yang-Mills theory in $d = 4$ that correspond to medium long supermultiplets with spin range $3/2$. It would be interesting to find out if these supermultiplets of 1/4 BPS states correspond to the medium long doubleton supermultiplets we found [24].

We also found doubleton supermultiplets corresponding to $N = 4$ conformal supergravity in $d = 4$. Since $N = 4$ super Yang-Mills theory can be coupled to $N = 4$ conformal supergravity in $d = 4$, one might wonder if this coupled conformal theory might describe the dynamics of some higher dimensional theory. Now, $N = 4$ conformal supergravity has two scalars parametrizing the coset space $SU(1,1)/U(1)$. The $d = 10$ IIB supergravity also has two scalars parametrizing the coset space $SU(1,1)/U(1)$[28]. But the original Maldacena’s conjecture applies to the $SL(2, Z)$ invariant sector. This suggests that the full $SL(2, Z)$ covariant dynamics of IIB supergravity theory over $S^5$ may be described in terms of $N = 4$ super Yang-Mills theory in $d = 4$ coupled to $N = 4$ conformal supergravity in $d = 4$, which might point toward a connection between F-theory[30] and CFT/AdS duality.

We should note that the spectrum of scalar fields of the $S^5$ compact-
ification of $d = 10$ IIB supergravity corresponds to symmetric tensors of $SO(6)$ ($20, 50, ...$). In terms of $\mathcal{N} = 4$ superfields, transforming in the adjoint of $SU(N)$, which represent the $SU(N)$ gauged doubleton supermultiplet [23], the entire spectrum corresponds to gauge invariant symmetric tensors. We should note that the tensor product of the doubleton with itself includes the $\mathcal{N} = 4$ conformal supergravity multiplet in addition to the $\mathcal{N} = 8$ AdS$_5$ graviton supermultiplet [27]. The conformal supergravity multiplet corresponds to the trace part which is not taken into account by the original conjecture on CFT/AdS duality. The fact that the gauge invariant trace component of the product of two doubletons includes the conformal supergravity supermultiplet suggests again that the original conjecture of Maldacena ought to be generalized so as to include the coupling of $\mathcal{N} = 4$ super Yang-Mills in $d = 4$ to the $\mathcal{N} = 4$ conformal supergravity in $d = 4$ as already mentioned above. The fuller discussion of the implications of our results to the CFT/AdS duality, some of which we have already mentioned, will be given elsewhere [27].

CFT/AdS duality has been studied in the literature for fewer supersymmetries corresponding to the orbifolding of $S^5$ [22]. Such theories have fewer supersymmetries described by superalgebras $SU(2, 2|k)$ ($k = 1, 2, 3$). Our methods can be extended to these cases in a straightforward manner.

Finally, one may also wonder how is it possible that the super Yang-Mills theory, which comes from the open-string sector in $d = 10$, captures the dynamics of the closed IIB string theory over $S^5$ which contains gravity.

Now, the oscillator construction of the spectrum of the closed IIB string over $S^5$ in terms of the doubleton supermultiplet gives an algebraic realization of this dynamical relationship between the maximally supersymmetric Yang-Mills theory and supergravity. Note also that the oscillator construction works very much in parallel with the construction of closed string states in terms of open string states in perturbative string theory.

But the oscillator method of constructing the doubleton supermultiplets (coming from the open string sector) and the graviton supermultiplets (coming from the closed string sector), which is obtained by tensoring two doubleton supermultiplets, should be true non-perturbatively if the CFT/AdS duality conjecture is indeed correct. In particular, it should have a deeper dynamical meaning.

It would be important to extend the use of the oscillator method beyond the calculation of the spectrum to the calculation of the correlation functions in $d = 4$ conformal field theory as was done in [11, 31], in order to uncover the method’s true dynamical meaning. We hope to return to this
question in the future.

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11 Appendix

The allowed lowest weight vectors (lwv) of $SU(4)$ are given in the following table

| lwv | SU(4) Dynkin (dim) | $Y$ |
|------|-------------------|-----|
| $|0\rangle$ | $(0,2,0)$ (20') | 0 |
| $\beta^x(1)|0\rangle$ | $(0,1,1)$ (20) | -1 |
| $\alpha^\mu(1)|0\rangle$ | $(1,1,0)$ (20) | 1 |
| $\beta^x(1)\beta^y(1)|0\rangle$ | $(0,1,0)$ (6) | -2 |
| $\alpha^\mu(1)\alpha^\nu(1)|0\rangle$ | $(0,1,0)$ (6) | 2 |
| $\beta^x\beta^y\beta^z\beta^w|0\rangle$ | $(0,0,0)$ (1) | -4 |
| $\alpha^\mu\alpha^\nu\alpha^\rho\alpha^\lambda|0\rangle$ | $(0,0,0)$ (1) | 4 |
| $\beta^x(1)\beta^y(2)|0\rangle$ | $(0,0,2)$ (10) | -2 |
| $\alpha^\mu(1)\alpha^\nu(2)|0\rangle$ | $(2,0,0)$ (10) | 2 |
| $\beta^x\beta^y\beta^z|0\rangle$ | $(0,0,1)$ (4) | -3 |
| $\alpha^\mu\alpha^\nu\alpha^\rho|0\rangle$ | $(1,0,0)$ (4) | 3 |
| $\alpha^\mu(1)\beta^x(2)|0\rangle$ | $(1,0,1)$ (15) | 0 |
| $\alpha^\mu(1)\beta^x(2)\beta^y(2)|0\rangle$ | $(1,0,0)$ (4) | -1 |
| $\alpha^\mu(1)\alpha^\rho(1)\beta^x(2)|0\rangle$ | $(0,0,1)$ (4) | 1 |
| $\alpha^\mu(1)\alpha^\rho(1)\beta^x(2)\beta^y(2)|0\rangle$ | $(0,0,0)$ (1) | 0 |

For the decomposition of the supertableaux of $U(m/n)$ in terms of the tableaux of its even subgroup $U(m) \times U(n)$ we refer to \[24\]. Here we give a few examples.
\[ U(m/n) \supset U(m) \times U(n) \]

\[
\begin{align*}
\bullet & = (\bullet, 1) + (1, \bullet) \\
\bullet & = (\bullet, 1) + (\bullet, \bullet) + (1, \bullet) \\
\bullet & = (\bullet, 1) + (\bullet, \bullet) + (1, \bullet) \\
\bullet & = (\bullet, 1) + (\bullet, \bullet) + (\bullet, \bullet) \\
+ (1, \bullet) & + (\bullet, \bullet) + (\bullet, \bullet)
\end{align*}
\]

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Erratum

to

" 4D Doubleton Conformal Theories, CPT and IIB Strings on $AdS_5 \times S^5 $"

by M. Günaydin, D. Minic and M. Zagermann, Nucl. Phys. B534 (1998) 96-120.

In the first paragraph of section 7 the central charge-like $U(1)$ is incorrectly identified as the automorphism group $U(1)_Y$. Therefore, the first part of this paragraph should be replaced by the following:

The centrally extended symmetry supergroup of the compactification of type IIB superstring over the five sphere is the supergroup $SU(2,2|4)$ with the even subgroup $SU(2,2) \times SU(4) \times U(1)$, where $SU(4)$ is the isometry group of the five sphere [4]. The generator of the Abelian $U(1)$ factor in the even subgroup of $SU(2,2|4)$ commutes with all the generators and acts like a central charge. Therefore, $SU(2,2|4)$ is not a simple Lie superalgebra. By factoring out this Abelian ideal one obtains a simple Lie superalgebra, denoted as $PSU(2,2|4)$, whose even subalgebra is simply $SU(2,2) \times SU(4)$ [6]. Both $SU(2,2|4)$ and $PSU(2,2|4)$ have an outer automorphism group $U(1)_Y$ that can be identified with a $U(1)$ subgroup of the $SU(1,1)_{\text{global}} \times U(1)_{\text{local}}$ symmetry of IIB supergravity in ten dimensions [4]. By orbifolding ....

\footnote{We would like to thank Kenneth Intriligator for informing us of this.}

\footnote{In [4] the symmetry supergroup of the $S^5$ compactification of IIB theory was denoted as $U(2,2|4)$.}