ZERO TEMPERATURE LIMITS OF EQUILIBRIUM STATES FOR SUBADDITIVE POTENTIALS AND APPROXIMATION OF THE MAXIMAL LYAPUNOV EXPONENT

REZA MOHAMMADPOUR

Abstract. In this paper, we study ergodic optimization problems for subadditive sequences of function on topological dynamical system. We prove that for $t \to \infty$ any accumulation points of a family of equilibrium states is a maximizing measure. We show that the Lyapunov exponent and entropy of equilibrium states converges in the limit $t \to \infty$ to the maximum Lyapunov exponent and entropy of maximizing measures.

In the particular case of matrix cocycle we prove that the maximal Lyapunov exponent can be approximated by periodic point under certain assumptions.

1. Introduction and statement of the results

In every part of this paper $X$ is a compact metric space that is endowed with metric $d$. We call $(X,T)$ a topological dynamical system (TDS), if $T : X \to X$ is a continuous map on compact metric space $X$. We say that $\Phi := \{\log \phi_n\}_{n=1}^{\infty}$ is a subadditive potential if each $\phi_n$ is a continuous non-negative-valued function on $X$ such that

$$0 \leq \phi_{n+m}(x) \leq \phi_n(x)\phi_m(T^n(x)) \quad \forall x \in X, m, n \in \mathbb{N}.$$ 

Furthermore, $\Phi = \{\log \phi_n\}_{n=1}^{\infty}$ is said to be an almost additive potential if there exists a constant $C \geq 0$ such that for any $m, n \in \mathbb{N}$, $x \in X$, we have

$$C^{-1}\phi_n(x)\phi_m(T^n)(x) \leq \phi_{n+m}(x) \leq C\phi_n(x)\phi_m(T^n(x)).$$

For any $T$–invariant measure $\mu$ such that $\log \phi^+ \in L^1(\mu)$, the pointwise Lyapunov exponent

$$\chi(x, \Phi) := \lim_{n \to \infty} \frac{1}{n} \log \phi_n(x),$$

exists for a.e. points.

By Kingman’s subadditive theorem [30], the Lyapunov exponent of measure $\mu$

$$\chi(\mu, \Phi) := \lim_{n \to \infty} \frac{1}{n} \int \log \phi_n(x)d\mu(x)$$
exists. If $\mu$ is ergodic then $\chi(x, \Phi) = \chi(\mu, \Phi)$ for a.e. points. Even though the existence of pointwise Lyapunov exponent follows from Kingman’s subadditive theorem, Furstenberg and Kifer [15] show that the existence of pointwise Lyapunov exponent at first.

In this paper, we are interested in the **maximal Lyapunov exponent**, defined as

$$\beta(\Phi) := \limsup_{n \to \infty} \frac{1}{n} \log \sup_{x \in X} \log \phi_n(x).$$

We denote by $\mathcal{M}(X, T)$ the space of all $T$–invariant Borel probability measure on $X$. Morris [26] shows that one can define maximal Lyapunov exponent as supremum Lyapunov exponent of measure over invariant measures. That means,

$$\beta(\Phi) = \sup_{\mu \in \mathcal{M}(X, T)} \chi(\mu, \Phi).$$

Feng and Huang [13] give different proof for it.

Let us define the set of **maximizing measures** of $\Phi$ to be the set of measures on $X$ given by

$$\mathcal{M}_{\text{max}}(\Phi) := \{ \mu \in \mathcal{M}(X, T), \beta(\Phi) = \chi(\mu, \Phi) \}.$$

In this paper, we study behavior of the equilibrium measures $(\mu_t)$ for subadditive potential $t\Phi$ when $t \to \infty$. In the thermodynamic interpretation of the parameter $t$ is the **inverse temperature**, limit $t \to \infty$ is called **zero temperature limit**, and the accumulation points of the measure $(\mu_t)$ as $t \to \infty$ are called **ground states**.

The topic of **ergodic optimization** revolves around realizing invariant measures in maximizing the Lyapunov exponents. Zero temperature limits laws are also related to ergodic optimization, because for $t \to \infty$ any accumulation points of equilibrium measure $(\mu_t)$ will be a maximizing measure $\Phi$ (maximizing $\chi(\mu, \Phi)$). We refer the reader to [18] and [7].

The behavior of the equilibrium measure $(\mu_t)$ as $t \to \infty$ has also been analyzed. In particular, continuity of zero temperature limit $(\mu_t)_{t \to \infty}$ in the sense,

$$\chi(\mu, \Phi) = \lim_{t \to \infty} \chi(\mu_t, \Phi),$$

and

$$h_{\mu}(T) = \lim_{t \to \infty} h_{\mu_t}(T),$$

have been investigated by many authors [11], [17], [18], [19], [25], [31], [32].

In non-compact space setting, (1.2) and (1.3) were proved by Jenkinson, Mauldin and Urbański [19], and Morris [25] on additive potential $\psi : X \to \mathbb{R}$. In fact, the proof of Theorem 1.1 is based on ideas from those works. Moreover, this kind of result is known for almost subadditive potential by Zhao [32] under the specification property, upper semi-continuity of entropy and finite topological entropy assumption.

The goal of this paper extend above results for subadditive potential under the upper semi-continuity of entropy and finite topological entropy assumption.
Note that even though we know the existence of an accumulation point for the sequence \((\mu_t)\) (see Proposition 2.3), this does not imply that the \(\lim_{t \to \infty} \mu_t\) exists. In fact, Chazottes, and Hochman [10] constructed an example on compact subshifts of finite type and Hölder potentials, where there is no convergence. For more information about zero temperature see [18].

Our main results are Theorems 1.1 and 1.3 formulated as follows:

**Theorem 1.1.** Let \((X, T)\) be a TDS such that the entropy map \(\mu \mapsto h_\mu(T)\) is upper semi-continuous and topological entropy \(h_{top}(T) < \infty\). Suppose that \(\Phi = \{\log \phi_n\}_{n=1}^{\infty}\) is a subadditive potential on compact metric \(X\) which satisfies \(\beta(\Phi) > -\infty\). Then a family of equilibrium measure \((\mu_t)\) for potential \(t\Phi\), where \(t > 0\), has a weak* accumulation point as \(t \to \infty\). Any such accumulation point is a Lyapunov maximizing measure for \(\Phi\). Moreover,

\[
\begin{align*}
(\text{i}) & \quad \chi(\mu, \Phi) = \lim_{t \to \infty} \chi(\mu_t, \Phi), \\
(\text{ii}) & \quad h_\mu(T) = \lim_{t \to \infty} h_{\mu_t}(T) = \max\{h_\nu(T), \nu \in \mathcal{M}_{\text{max}}(\Phi)\}.
\end{align*}
\]

Furthermore, \(\beta(\Phi)\) can be approximated by Lyapunov exponents of equilibrium measures of a subadditive potential \(t\Phi\).

Let \(\varphi : X \to GL(d, \mathbb{R})\) be a measurable function. We can define a linear cocycle \(F : X \times \mathbb{R}^d \to X \times \mathbb{R}^d\) as

\[
F(x, v) = (T(x), \varphi(x)v).
\]

We say that \(F\) is generated by \(T\) and \(\varphi\), we will also denote by \((T, \varphi)\). Observe that \(F^n(x, v) = (T^n(x), \varphi_n(x)v)\) for each \(n \geq 1\), where

\[
\varphi_n(x) = \varphi(T^{-1}(x)) \varphi(T^{-2}(x)) \cdots \varphi(x).
\]

If \(T\) is invertiable then so is \(F\). Moreover, \(F^{-n}(x, v) = (T^{-n}(x), \varphi_{-n}(x)v)\) for each \(n \geq 1\), where

\[
\varphi_{-n}(x) = \varphi(T^{-n}(x))^{-1} \varphi(T^{-n+1}(x))^{-1} \cdots \varphi(T^{-1}(x))^{-1}.
\]

A very important class of linear cocycles are locally constant cocycles which are defined as follows.

**Example 1.2.** Let \(X = \{1, \ldots, k\}^\mathbb{Z}\) be a symbolic space. Let \(T : X \to X\) be a shift map, i.e. \(T(x)_i = (x_{i+1})_i\). Given a finite set of matrices \(A = \{A_1, \ldots, A_k\} \subset GL(d, \mathbb{R})\), we define the function \(A : X \to GL(d, \mathbb{R})\) by \(A(x)_i = A_{x_0}\).

We say that a homeomorphism \(T\) satisfies Anosov closing property if there exists \(C, \varepsilon, \delta > 0\) such that for any \(x \in X\) and \(k \in \mathbb{N}\) with \(d(x, T^k(x)) < \varepsilon\) there exists a point \(p \in X\) with \(T^k(p) = p\) such that the orbit \(O^+(f(x)) = \{f^n(x), n \in \mathbb{N}\}\), and \(O^+(f(p)) = \{f^n(p), n \in \mathbb{N}\}\) are exponentially close, i.e.

\[
d(f^i(x), f^i(p)) \leq C e^{-\delta \min\{i, n-i\}} d(f^n(x), x)
\]

for every \(i = 1, \ldots, n\).
Note that shifts of finite type, Axiom A diffeomorphism, and hyperbolic homeomorphism are particular systems satisfying the Anosov closing property. See for more information [22].

Kalinin and Sadovskaya [23] proved if homeomorphism $T$ satisfies Anosov closing property, and $\varphi : X \to GL(d, \mathbb{R})$ is a Hölder continuous Banach cocycle, then the maximal Lyapunov exponent can be approximated by Lyapunov exponents of measures supported on periodic orbits. For locally constant cocycle $(T, \varphi)$, where $\varphi : X \to GL(2, \mathbb{R})$, we show that the maximal Lyapunov exponent can be approximated by Lyapunov exponents of measures supported on periodic orbits.

We write $\phi_n := \|\varphi_n\|$, where $\|\|$ is operator norm.

**Theorem 1.3.** Let $(T, \varphi)$ be a locally constant cocycle. Satisfying the Anosov closing property. Then, maximal Lyapunov exponent $\beta(\Phi)$ can be approximated by Lyapunov exponents of measures supported on periodic orbits.

In general, Kalinin [20] shows that for a Hölder continuous map $\varphi : X \to GL(d, \mathbb{R})$, Lyapunov exponents can be approximated by Lyapunov exponents of measures supported on periodic orbits under a slightly stronger than the Anosov closing property.

This paper is organized as follows. In Section 2, we recall some preliminary material regarding convex function as well as some results in thermodynamic formalism for subadditive setting. In Section 3, we prove Theorem 1.1. In Section 4, we state a theorem about the continuity of Lyapunov exponents for locally constant cocycles, and we prove Theorem 1.3.

**Acknowledgements.** The author thanks M. Rams for his careful reading of an earlier version of this paper and many helpful suggestions. The author was partially supported by the National Science Center grant 2014/13/B/ST1/01033 (Poland).

2. Preliminaries

2.1. Convex functions. We first give some notation and basic facts in convex analysis. For details, one is referred to [16].

Let $x, y \in \mathbb{R}^n$, the line segment connecting $x$ and $y$ is the set $[x, y]$ formally given by

$$[x, y] = \{\beta x + (1 - \beta)y \mid \beta \in [0, 1]\}.$$  

We say that a set $X$ is convex when for any two points $x, y \in X$, the line segment $[x, y]$ also belongs to the set $X$, i.e., $\beta x + (1 - \beta)y \in X$ for any $x, y \in X$ and $\beta \in (0, 1)$. Let $C$ be a convex subset of $\mathbb{R}^n$. A point $x \in C$ is called an extreme point of $C$ if whenever $x = \beta y + (1 - \beta)z$ for some $y, z \in C$ and $0 < \beta < 1$, then $x = y = z$. We denote by $ext(C)$ the set of extreme points of $C$.

A function $f : \mathbb{R}^n \to \mathbb{R}$ is a convex if its domain $dom(f)$ is convex set and for all $x, y \in dom(f)$ and $\beta \in (0, 1)$, the following relation holds

$$f(\beta x + (1 - \beta)y) \leq \beta f(x) + (1 - \beta)f(y).$$
In other words, a function \( f : \mathbb{R}^n \to \mathbb{R} \) is convex when for every segment \([x_1, x_2]\), as the vector \( x_\beta = \beta x_1 + (1-\beta) x_2 \) varies within the line segment \([x_1, x_2]\), the points \((x_\beta, f(x_\beta))\) on the graph \(\{(x, f(x)) | x \in \mathbb{R}^n\}\) lie below the segment connecting \((x_1, f(x_1))\) and \((x_2, f(x_2))\), as illustrated in Figure 1.

![Figure 1. Convex line](image)

Let \( U \) be an open convex subset of \( \mathbb{R}^n \) and \( f \) be a real continuous convex function on \( U \). We say a vector \( a \in \mathbb{R}^n \) is a subgradient of \( f \) at \( x \) if for all \( z \in U \),

\[
    f(z) \geq f(x) + a^T(z-x),
\]

where the right hand side is the scalar product.

For each \( x \in \mathbb{R}^n \) set the subdifferential of \( f \) at point \( x \)

\[
    \partial f(x) := \{a : a \text{ is a subgradient for } f \text{ at } x\}.
\]

For \( x \in U \), the subdifferential \( \partial f(x) \) is a nonempty convex compact set. Define \( \partial^e f(x) := \text{ext}\{\partial f(x)\} \). In case \( n = 1 \), \( \partial^e f(x) = \{f^\prime(x_-), f^\prime(x_+)\} \), where \( f^\prime(x_-) \) (resp. \( f^\prime(x_+) \)) denotes the left (resp. right) derivative. We say that \( f \) is differentiable at \( x \) when \( \partial^e f(x) = \{a\} \).

We define

\[
    (2.1) \quad \partial f(U) = \bigcup_{x \in U} \partial f(x) \quad \text{and} \quad \partial^e f(U) = \bigcup_{x \in U} \partial^e f(x).
\]

In the case \( n = 1 \), since \( \partial^e f \) is monotone functions of one variable, then Lebesgue’s theorem for the differentiability of monotone functions said \( \partial^e f \) is differentiable almost everywhere. The case \( n = 2 \) was proven by H. Busemann and W. Feller [6]. The general case was settled by A. D. Alexandrov [11]. The following result is well known (cf. [29, Theorem 7.9]).
Theorem 2.1. Let $f$ be a continuous function on open interval that has a derivative at each point of $\mathbb{R}$ except on countable set, and $f' \leq 0$ a.e., then $f$ is a non-increasing function.

2.2. The thermodynamic formalism for subadditive potential. We require some elements from the subadditive thermodynamic formalism. Cao, Feng and Huang [9] extend the additive theory of thermodynamic formalism to the subadditive theory. Let $(X,T)$ be a TDS and let $\Phi = \{\log \phi_n\}_{n=1}^{\infty}$ be a subadditive potential on $(X,T)$.

We introduce the topological pressure of $\Phi$ as follows. The space $X$ is endowed with a metric $d$. For any $n \in \mathbb{N}$, one can define a new metric $d_n$ on $X$ by

$$d_n(x,y) = \max\{d(T^k(x),T^k(y)) : k = 0, ..., n\}.$$

For any $\varepsilon > 0$ a set $E \subset X$ is said to be a $(n,\varepsilon)$-separated subset of $X$ if $d_n(x,y) > \varepsilon$ for any two different points $x,y \in E$. We define for $\Phi$

$$P_n(T,\Phi,\varepsilon) = \sup\{\sum_{x \in E} \phi_n(x) : E \text{ is a } (n,\varepsilon) \text{-separated subset of } X\}.$$

Since $P_n(T,\Phi,\varepsilon)$ is a decreasing function of $\varepsilon$. We define

$$P(T,\Phi,\varepsilon) = \lim_{n \to \infty} \frac{1}{n} \log P_n(T,\Phi,\varepsilon),$$

and

$$P(T,\Phi) = \lim_{\varepsilon \to 0} P(T,\Phi,\varepsilon).$$

We call $P(T,\Phi)$ the topological pressure of $\Phi$. We define $h_{top}(T) := P(T,0)$.

Bowen [5] shows that for any Hölder continuous $\psi : X \to \mathbb{R}$ on transitive hyperbolic set $(X,T)$ there exists a unique equilibrium measure $\mu_t$ (which is also a Gibbs state) for the additive potential $t\log \psi$ for $t \in \mathbb{R}$.

Feng and Käenmäki [14] extend the Bowen’s result for subadditive potential $t\Phi$ of locally constant cocycle under the assumption that the matrices $\varphi_n$ do not preserve a common proper subspace of $\mathbb{R}^d$ (i.e. $(T,\varphi)$ is irreducible).

Recently, Park [28] shows that continuity of topological pressure, and uniqueness of equilibrium measure for general cocycles under generic assumptions. Author [24] announces that he proves the continuity of topological pressure under some assumption which is weaker than Park’s assumptions.

Let $(X,\tau,\mu)$ be a Borel probability space, and $T : X \to X$ be a measure preserving transformation.

A partition of $(X,\tau,\mu)$ is a subfamily of $\tau$ consisting of mutually disjoint elements whose union is $X$. We denote by $\alpha$ and $\beta$ the countable partition of $X$.

Let $\alpha = \{A_i, i \geq 1\}$, where $A_i \in \tau$. We define

$$H_\mu(\alpha) = -\sum_{A \in \alpha} \mu(A) \log \mu(A)$$

to be the entropy of $\alpha$ (with the convention $0 \log 0 = 0$).
We denote by $\alpha \lor \beta$ the joint partition $\{ A \cap B \mid A \in \alpha, B \in \beta \}$. Let $T^{-1}(\alpha) = \{ T^{-1}(A) \mid A \in \alpha \}$. We define
\[
  h(\mu, \alpha) = \lim_{n \to \infty} \frac{1}{n} H_\mu(\bigvee_{j=0}^{n-1} T^{-1}(\alpha))
\]
to be the entropy of $T$ relative to $\alpha$.

Then the metric entropy of $\mu$ is defined as
\[
  h_\mu(T) = \sup h(\mu, \alpha),
\]
where the supremum is taken over all countable partitions $\alpha$ with $H_\mu(\alpha) < \infty$.

We can define topological pressure by the following variational principle. It was proved by Cao, Feng and Huang [9].

**Theorem 2.2** ([9, Theorem 1.1]). Let $(X, T)$ be a TDS such that $h_{top}(T) < \infty$. Suppose that $\Phi = \{ \log \phi_n \}_{n=1}^{\infty}$ is a subadditive potential on compact metric space $X$ and $h_{top}(T) < \infty$. Then
\[
  P(t, \Phi) = \sup \{ h_\mu(T) + t \chi(\mu, \Phi) \mid \mu \in \mathcal{M}(X, T), \chi(\mu, \Phi) \neq \infty \}.
\]

Let us denote $P(t, \Phi) = P(t)$. The topological pressure is related to Lyapunov exponents in the following way.

**Theorem 2.3** ([13, Theorem 1.2]). Let $(X, T)$ be a TDS such that $h_{top}(T) < \infty$. Assume that $\Phi = \{ \log \phi_n \}_{n=1}^{\infty}$ is a subadditive potential on compact metric space $X$ which satisfies $\beta(\Phi) > -\infty$.

Then the pressure function $P(t)$ is a continuous real convex function on $(0, \infty)$. Furthermore, $P'(\infty) := \lim_{t \to \infty} \frac{P(t)}{t} = \beta(\Phi)$.

Let $t \in \mathbb{R}_+$, we denote by $\text{Eq}(t)$ the collection of invariant measure $\mu$ such that
\[
  h_\mu(T) + t \chi(\mu, \Phi) = P(t).
\]

If $\text{Eq}(t) \neq \emptyset$, then each element $\text{Eq}(t)$ is called an equilibrium state for $t\Phi$.

**Proposition 2.4** ([13, Theorem 3.3]). Let $(X, T)$ be a TDS such that the entropy map $\mu \mapsto h_\mu(T)$ is upper semi-continuous and $h_{top}(T) < \infty$. Suppose that $\Phi = \{ \log \phi_n \}_{n=1}^{\infty}$ is a subadditive potential on compact metric space $X$ which satisfies $\beta(\Phi) > -\infty$. Then,
\[
  \partial P(t) = \{ \chi(\mu_t, \Phi) : \mu_t \in \text{Eq}(t) \}.
\]
Moreover, $\text{Eq}(t)$ is a non-empty compact convex subset of $\mathcal{M}(X, T)$, for any $t \in \mathbb{R}_+$, and every extreme point of $\text{Eq}(q)$ is an ergodic measure.

\footnote{Limits exists by subadditivity.}
Theorem 2.5 ([18 Proposition 3.2]). Suppose that \( \Phi = \{ \log \phi_n \}_{n=1}^{\infty} \) is a subadditive potential on compact metric space \( X \). Assume that \( h_{\text{top}}(T) < \infty \) and \( \beta(\Phi) > -\infty \). Then
\[
\partial P(\mathbb{R}_+) \subseteq (-\infty, \beta(\Phi)],
\]
where \( \partial P(\mathbb{R}_+) \) defined in (2.7).

We denote by \( \mathcal{M}(X) \) the space of all Borel probability measure on \( X \) with weak* topology.

Theorem 2.6 ([9 Lemma 2.3]). Suppose \( \{ \nu_n \}_{n=1}^{\infty} \) is a sequence in \( \mathcal{M}(X) \) and \( \Phi = \{ \log \phi_n \}_{n=1}^{\infty} \) is a subadditive potential on compact metric space \( X \). We form the new sequence \( \{ \mu_n \}_{n=1}^{\infty} \) by
\[
\mu_n = \frac{1}{n} \sum_{i=0}^{n-1} \nu_i o T_i.
\]
Assume that \( \mu_{n_i} \) converges to \( \mu \) in \( \mathcal{M}(X) \) for some subsequence \( \{ n_i \} \) of natural numbers. Then \( \mu \in \mathcal{M}(X,T) \) and
\[
\limsup_{i \to \infty} \frac{1}{n_i} \int \log \phi_{n_i}(x) d\nu_i(x) \leq \chi(\mu, \Phi).
\]

3. Proof of the Theorem 1.1

We start the proof of Theorem 1.1 (i) with a key Proposition 2.4 which tells us that the subdifferentiable of topological pressure for subadditive potential is equal to the Lyapunov exponent of equilibrium measure. Let \( (X, T) \) be a TDS such that the entropy map \( \mu \mapsto h_{\mu}(T) \) is upper semi-continuous and \( h_{\text{top}}(T) < \infty \). Suppose that \( \Phi = \{ \log \phi_n \}_{n=1}^{\infty} \) is a subadditive potential on compact metric space \( X \) which satisfies \( \beta(\Phi) > -\infty \). We write Theorem 1.1 (i) as follows.

Theorem 3.1. For each \( t > 0 \), the family of equilibrium measures \( (\mu_t) \), has a weak* accumulation point as \( t \to \infty \). Any such accumulation point \( \mu \) is a Lyapunov maximizing measure for \( \Phi \). Moreover,
\[
\chi(\mu, \Phi) = \lim_{t \to \infty} \chi(\mu_t, \Phi).
\]

Proof. It is obvious that \( (\mu_t) \) has at least one accumulation point, let us call it \( \mu \).

By Theorem 2.3 \( P(t) \) is convex, then we have \( \partial P(t) = \{ \chi(\mu_t, \Phi) \} \) by Proposition 2.4. Moreover, since \( P(t) \) is convex for \( t > 0 \), \( t \mapsto \chi(\mu_t, \Phi) \) is non-decreasing and bounded above.\(^2\) It follows that
\[
\lim_{t \to \infty} \partial P(t) = \lim_{t \to \infty} \chi(\mu_t, \Phi) \text{ exists and is finite.}
\]
Since Lyapunov exponents are upper semi continuous
\[
\lim_{t \to \infty} \chi(\mu_t, \Phi) \leq \chi(\mu, \Phi).
\]
By the definition of \( Eq(t) \),
\[
\chi(\mu_t, \Phi) + \frac{h_{\mu_t}(T)}{t} \geq \chi(\mu, \Phi) + \frac{h_\mu(T)}{t}.
\]
Since the TDS \((X,T)\) has finite topological entropy, so when \(t \to \infty\), (3.1) implies

\[
\lim_{t \to \infty} \chi(\mu_t, \Phi) \geq \chi(\mu, \Phi).
\]

Now, we show that \(\mu\) is a Lyapunov maximizing measure.

By contradiction, let us assume that there exists \(\nu\) with \(\chi(\nu, \Phi) - \chi(\mu, \Phi) = \kappa > 0\). One can define the affine map \(T_\nu : \mathbb{R}_+ \to \mathbb{R}\) by \(T_\nu(t) = h_\nu(T) + t\chi(\nu, \Phi)\). We know that \(t \mapsto \chi(\mu_t, \Phi)\) is a function which increases to its limit \(\chi(\mu, \Phi)\), so

\[
\chi(\mu, \Phi) \geq \chi(\mu_{t_*}, \Phi) = \partial P(t_*), \text{ where } t_* = t_- \text{ or } t_+.
\]

and \(T_\nu(t) = \chi(\nu, \Phi) = \chi(\mu, \Phi) + \kappa \geq \partial P(t_*) + \kappa\).

Consequently, \(h_\nu(T) + t\chi(\nu, \Phi) > P(t)\) for all sufficiently large \(t > 0\), that contradicts by our assumption. So, \(\mu\) is Lyapunov maximizing measure.

Moreover, our proof implies that \(\beta(\Phi)\) can be approximated by Lyapunov exponents of equilibrium measures of a subadditive potential \(t\Phi\).

The Theorem 1.1(ii) is obtained by combining Lemmas 3.2 and 3.3 below.

**Lemma 3.2.** The maps \(t \mapsto h_{\mu_t}(T)\) and \(t \mapsto P(t\Phi - t\beta(\Phi))\) are non-increasing and bounded below on the interval \((0, \infty)\). Moreover, we have

\[
\lim_{t \to \infty} h_{\mu_t}(T) = \lim_{t \to \infty} P(t\Phi - t\beta(\Phi)) = \sup_{\nu \in M_{\text{max}}(\Phi)} h_\nu(T).
\]

*Proof.* The map \(t \mapsto P(t\Phi - t\beta(\Phi))\) is convex. By definition \(\beta(\Phi)\),

\[
\chi(\mu_t, \Phi) \leq \beta(\Phi) \text{ for all } \mu_t \in Eq(t).
\]

We assume that \(P(t) = P(t\Phi)\). By definition topological pressure, \(P(t\Phi - t\beta(\Phi)) = P(t\Phi) - t\beta(\Phi)\). Then,

\[
\partial P(t_* \Phi - t_* \beta(\Phi)) = \partial P(t_* \Phi) - \beta(\Phi) = \chi(\mu_{t_*}, \Phi) - \beta(\Phi) \leq 0,
\]

where \(t_* = t_-\) or \(t_+\). So, \(P(t\Phi - t\beta(\Phi))\) is non-increasing by Theorem 2.1. We are going to show that \(t \mapsto h_{\mu_t}(T)\) is non-increasing. Since \(\mu_t\) is an equilibrium measure,

\[
h_{\mu_t}(T) = P(t) - t\partial P(t_*)\]

For \(0 < x < y\) we have

\[
\partial P(x) \leq \frac{P(y) - P(x)}{y - x} \leq \partial P(y),
\]

so

\[
P(y) - P(x) \leq y\partial P(y) - x\partial P(y) \leq y\partial P(y) - x\partial P(x),
\]

and then

\[
P(y) - x\partial P(x) \leq P(x) - x\partial P(x).
\]

Since \(t \mapsto h_{\mu_t}(T)\) and \(t \mapsto P(t\Phi - t\beta(\Phi)) \geq 0\) are non-increasing and non-negative, we conclude that \(\lim_{t \to \infty} h_{\mu_t}(T)\) and \(\lim_{t \to \infty} P(t\Phi - t\beta(\Phi))\) both exist. It implies that the limit

\[
\lim_{t \to \infty} t\partial P(t) - t\beta(\Phi) = \lim_{t \to \infty} (P(t\Phi - t\beta(\Phi)) - h_{\mu_t}(T))
\]
exists. We know that \( \lim_{t \to \infty} \frac{P(t)}{t} = \beta(\Phi) \) by Theorem 2.3. Then,
\[
\lim_{t \to \infty} h_\mu(T) = \lim_{t \to \infty} P(t\Phi - t\beta(\Phi)).
\]
Last part follows from variational principle. □

**Lemma 3.3.** \( \mathcal{M}_{\max}(\Phi) \) is compact, convex and nonempty, and its extreme points are precisely its ergodic elements.

**Proof.** See [27, Appendix A]. □

**Theorem 3.4.** \( h_\mu(T) = \lim_{t \to \infty} h_\mu(T) = \max_{\nu \in \mathcal{M}_{\max}(\Phi)} \{ h_\nu(T) \} \).

**Proof.** By Theorem 3.1 and Lemmas 3.2 and 3.3,
\[
h_\mu(T) \leq \max_{\nu \in \mathcal{M}_{\max}(\Phi)} h_\nu(T) \leq \lim_{t \to \infty} h_\mu(T),
\]
other side follows from upper semi continuity of entropy. □

**Remark 1.** Let \((T, \varphi)\) be a locally constant cocycle. Then, one can prove Theorem 1.1 for Gibbs measures under the assumption that \((T, \varphi)\) is irreducible (see [14]). Moreover, if \( T : X \to X \) is a mixing subshift finite type and \( A : X \to GL(d, \mathbb{R}) \) is a Hölder continuous function, then one can prove Theorem 1.1 for Gibbs measures under the generic assumption on \((T, A)\) (see [28]).

**Remark 2.** Let \( \vec{q} = (q_1, \ldots, q_d) \in \mathbb{R}_+^d \), and \( \vec{\Phi} = (\Phi_1, \ldots, \Phi_d) = (\{ \log \phi_{n,1} \}_{n=1}^\infty, \ldots, \{ \log \phi_{n,d} \}_{n=1}^\infty) \). Assume that \( \vec{q} \cdot \vec{\Phi} = \sum_{i=1}^d q_i \Phi_i \) is a subadditive potential \( \{ q_i \log \phi_{n,i} \}_{n=1}^\infty \). We write topological pressure, and maximal Lyapunov exponent of \( \vec{\Phi} \), respectively
\[
P(\vec{q}) = P(T, \vec{q} \cdot \vec{\Phi}), \quad \beta(\vec{\Phi}) = \beta(\sum_{i=1}^d \Phi_i).
\]

Feng and Huang [13] proved the higher dimensional versions of Theorem 2.3, Proposition 2.4, and Theorem 2.5. So, one can obtain the higher dimensional versions of Theorem 1.1 by using [13].

4. **Proof of the Theorem 1.3**

In this section, we consider locally constant cocycles and we prove Theorem 1.3. Apart from the continuity of Lyapunov exponent Theorem 1.1 and the Anosov closing property, the main input of our argument will be Theorem 2.6. This theorem was proved by Cao, Feng, and Huang [12] for subadditive potentials.

Let \((T, \varphi)\) be the locally constant cocycle which is defined in Example 1.2, where \( \varphi : X \to GL(2, \mathbb{R}) \) and \( \phi_n = \| \varphi_n \| \).

Bocker, and Viana [3] proved continuity of Lyapunov exponents for 2D locally constant cocycles. In order to state the result of Bocker and Viana, we denote by \( \Delta_k \) the collection of strictly positive probability vectors in \( \mathbb{R}^k \) for \( k \geq 2 \). We denote by \( X \) the full shift space over \( k \) symbols. For \( p = (p_1, \ldots, p_k) \in \Delta_k \), let \( \mu \) be the Bernoulli product measure on \( X \).
Theorem 4.1. For every $\varepsilon > 0$ there exist $\delta > 0$ and a weak* neighborhood $V$ of $\mu$ in the space of probability measures on $GL(2,\mathbb{R})$ such that for every probability measure $\mu' \in V$ whose support is contained in the $\delta$-neighborhood of the support of $\mu$, we have

$$|\chi(\mu, \Phi) - \chi(\mu', \Phi)| < \varepsilon.$$ 

Now, we can prove the Theorem 1.3.

Theorem 4.2. Suppose that $T$ satisfies the Anosov closing property. Then, maximal Lyapunov exponent $\beta(\Phi)$ can be approximated by Lyapunov exponents of measures supported on periodic orbits.

Proof. Let $\alpha = \beta(\Phi)$. By (1.1),

$$\alpha = \max\{\chi(\mu, \Phi) : \mu \in \mathcal{M}(X, T)\}. \tag{4.1}$$

Let $\mu$ be an ergodic maximizing measure. That is, $\beta(\Phi) = \chi(\mu, \Phi)$.

Let $x \in \{x \in X, \chi(x, \Phi) = \beta(\Phi)\}$ be a generic point for $\mu$. Then, there exists $\mu_{n,x} := \frac{1}{n} \sum_{j=0}^{n-1} \delta_{T^j(x)}$, where $\delta_x$ is a Dirac measure at point $x$, so that $\mu_{x,n} \rightarrow \mu$. According to Theorem 2.6,

$$\lim_{i \rightarrow \infty} \frac{1}{n_i} \log \phi_{n_i}(x) = \chi(\mu, \Phi).$$

Let $p \in X$ be a periodic point associated to $\varepsilon, C, \delta$ and $\{x, T(x), ..., T^{n_i}(x)\}$ by the Anosov closing property. Denote by $\mu_p := \frac{1}{n_i} \sum_{j=0}^{n_i-1} \delta_{T^j(p)}$ the ergodic $T$–invariant measure supported on the orbit of $p$.

Lemma 4.3. $\mu_p \rightarrow \mu$ in weak* topology.

Proof. We will use the Anosov closing property.

Assume that $(f_m)$ is a sequence of continuous functions. The periodic orbit $p$ has length $n_i$ is close to the initial segment of the orbit of $x$. Since the $f_m$'s are continuous, the average of $f_m$ along the periodic orbit is very close to the average of $f_m$ along the first $n_i$ iterates of $x$, and that is very close to $\int f_m d\mu$ by the genericity condition. Then, for $i$ large enough, we get longer and longer the periodic orbits, approaching $x$ more closely, we obtain convergence of the measures to $\mu$. \qed

We now use Lemma 4.3 to finish the proof. By the Anosov closing property, periodic point $p$ is close to $x$, with iterates also close to the iterates of $x$. So, Theorem 4.1 implies for every $\varepsilon > 0$

$$\chi(p, \Phi) = \lim_{i \rightarrow \infty} \frac{1}{n_i} \log \phi_{n_i}(p) = \lim_{i \rightarrow \infty} \frac{1}{n_i} \log \phi_{n_i}(x) + \varepsilon. \tag{4.2}$$

Applying Lemma 4.3, Theorem 2.6 and (4.2), we obtain

$$\chi(p, \Phi) = \lim_{i \rightarrow \infty} \frac{1}{n_i} \log \phi_{n_i}(p) = \chi(\mu, \Phi) = \beta(\Phi).$$
Remark 3. Avila, Eskin and Viana [2] announced recently that the Theorem 4.1 remains true in arbitrary dimension. By their result, the proof given for Theorem 4.1 works for arbitrary dimension.

Remark 4. Morris [26] shows that the speed of convergence of Theorem 4.2 is always superpolynomial for locally constant cocycles. Moreover, Bochi and Garibaldi [7] show that it is true for general cocycles under certain additional assumption.

References

[1] A. D. Alexandrov, Almost everywhere existence of the second differential of a convex function and some properties of convex surfaces connected with it. Leningrad State Univ. Annals [Uchenye Zapiski] Math. Ser., 6 (1939), 3-35.
[2] A. Avila, A. Eskin and M. Viana, Continuity of Lyapunov exponents of random matrix-products, In preparation.
[3] R. Bowen, Markov partitions for axiom a diffeomorphisms. American Journal of Mathematics. 92 (1970), no. 3, 725-747.
[4] ————, Topological entropy for noncompact sets, American Journal of Mathematics, 184 (1973), 125-136.
[5] ————, Some systems with unique equilibrium states, Mathematical Systems Theory, 8 (1974), no. 3, 193-202.
[6] H. Busemann and W. Feller, Krümmungsindikatritizen konvexer Flächen, Acta Math. 66 (1936), 1-47.
[7] J. Bochi and E. Garibaldi, Extremal norms for fiber bunched cocycles, Journal de l’École polytechnique - Mathématiques, 6 (2019), 947-1004.
[8] C. Bocker-Neto and M. Viana, Continuity of Lyapunov exponents for random two-dimensional matrices, Ergod. Th. & Dynam. Sys. 5 (2017), 1413-1442.
[9] Y. Cao, D. Feng and W. Huang, The thermodynamic formalism for sub-additive potentials, Discrete Contin. Dyn. Syst. 20 (2008), no. 3, 639-657.
[10] J. Chazottes and M. Hochman, On the zero-temperature limit of Gibbs states, Comm. Math. Phys. 297 (2010), no. 1, 265-281.
[11] A. Davie, M. Urbanski and Zdunik, Maximizing measures of metrizable non-compact spaces, Proc. Edinb. Math. Soc. (2). 50 (2007), no. 1, 123-151.
[12] D. Feng, Lyapunov exponents for products of matrices and multifractal analysis. Part II. General matrices. J. Math. 170 (2009), 355-394.
[13] D. Feng and W. Huang, Lyapunov spectrum of asymptotically sub-additive potentials, Comm. Math. Phys. 297 (2010), no. 1, 1-43.
[14] D. Feng and A. Käenmäki, Equilibrium states of the pressure function for products of matrices, Discrete Contin. Dyn. Syst. 30 (2011), no. 3, 699-708.
[15] H. Furstenberg, and Yu. Kifer, Random matrix products and measures in projective spaces, Israel J. Math. 10 (1983), 12-32.
[16] J. Hiriart-Urruty and C. Lemaréchal, Fundamentals of convex analysis. Springer-Verlag, Berlin, 2001.
[17] G. Iommi and Y. Yayama, Zero temperature limits of Gibbs states for almost-additive potentials, J. Stat. Phys. 155 (2014), 23-46.
[18] O. Jenkinson, Ergodic optimization in dynamical systems, Ergod. Th. & Dynam. Sys. 39(2019), no. 10, 2593-2618.
[19] O. Jenkinson, R.D. Mauldin and M. Urbański, Zero temperature limits of Gibbs-equilibrium states for countable alphabet subshifts of finite type, *J. Stat. Phys.* **119** (2005), 765-776.
[20] B. Kalinin, Liššic theorem for matrix cocycles, *Ann. of Math.* **173** (2011), no. 2, 1025-1042.
[21] T. Kempton, Zero Temperature Limits of Gibbs Equilibrium States for Countable Markov Shifts, *J. Stat. Phys.* **143** (2011), no. 4, 795-806.
[22] A. Katok and B. Hasselblatt, *Introduction to the modern theory of dynamical systems*, Cambridge University Press, London-New York, 1995.
[23] B. Kalinin and V. Sadovskaya, Periodic approximation of Lyapunov exponents for Banach cocycles, *Ergod. Th. & Dynam. Sys.* **39** (2019), no. 3, 689-706.
[24] R. Mohammadpour, Continuity of topological pressure and Lyapunov spectrum, In preparation.
[25] I.D. Morris, Entropy for zero-temperature limits of Gibbs-equilibrium states for countable-alphabet subshifts of finite type, *J. Stat. Phys.* **126** (2007), no. 2, 315-324.
[26] I.D. Morris, A rapidly-converging lower bound for the joint spectral radius via multiplicative ergodic theory, *Adv. Math.* **225** (2010), no. 6, 3425-3445.
[27] I.D. Morris, Mather sets for sequences of matrices and applications to the study of joint spectral radii, *Proc. Lond. Math. Soc.(3).* **107** (2013), no. 1, 121-150.
[28] K. Park, Quasi-Multiplicativity of typical cocycles, [https://arxiv.org/abs/1903.03928](https://arxiv.org/abs/1903.03928)
[29] S. Saks, *Theory of the integral*, Instytut Matematyczny Polskiej Akademii Nauk, 1937.
[30] M. Viana, *Lectures on Lyapunov exponents*, Cambridge University Press, 2014.
[31] Q. Wang and Y. Zhao, Variational principle and zero temperature limits of asymptotically (sub)-additive projection pressure, *Front. Math. China.* **13** (2018), no. 5, 2018.
[32] Y. Zhao, Constrained ergodic optimization for asymptotically additive potentials, *J. Math. Anal. Appl.* **474** (2019), 612-639.

**Department of Dynamical System, Institute of Mathematics of the Polish Academy of Sciences, ul. Śniadeckich 8, 00-656 Warsaw, Poland**

**E-mail address:** rmohammadpour@impan.pl