Discrete spectrum of a pair of nonsymmetric waveguides coupled by a window

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Abstract

In the paper we study the discrete spectrum of a pair of quantum two-dimensional waveguides having common boundary in which a window of finite length is cut out. We study the phenomenon of new eigenvalues emerging from the threshold of the essential spectrum when the length of window passes through critical values. We construct the asymptotics expansions for the emerging eigenvalues with respect to small parameter which is the difference between current length of the window and the nearest critical value. We also study the behaviour of the spectrum when the length of the window increases unboundedly and construct asymptotics expansions with respect to great parameter which is a length of the window.

Introduction

In last years much attention was paid to the study of spectral properties of the elliptic operators in unbounded domains with various perturbations. First of all this is due various applications of such problem in quantum mechanics and acoustics. Moreover, these problems possess various features interesting from mathematical point of view. One of such examples is a problem on bound states of two quantum waveguides coupled by a window. Mathematically this corresponds to an eigenvalue problem for the Dirichlet Laplacian in a domain formed by two parallel strips having common boundary in which a window of finite length is cut out (cf. figure). Such model was suggested in the paper [1]; physical aspects of this problem were discussed there as well (see also [2]). Besides, in [1] the authors obtained two-sided estimates for the eigenvalues and proved that the presence of the window leads to a non-empty discrete spectrum, the number of isolated eigenvalues increases when the length of the window does, eigenvalues appear when the length of the window passes through some critical values. A number of numerical results was obtained as well. The existence of at least one isolated eigenvalue in the case of the same widths of the strips was proved independently in [3]. For a sufficiently small window this system has exactly one isolated eigenvalue. In the case of symmetric strips a two-sided estimate was obtained for this eigenvalue in [4]. In [5] similar

The work is partially supported by RFBR and the program of supporting leading scientific schools. The author is also supported by Marie Curie International Fellowship of 6th European Community Framework (MIF1-CT-2005-006254)
result was established for several windows and non-symmetric strips as well as for two parallel layers coupled by a window. The case of small window was also considered in [6], where the asymptotics expansion for the aforementioned eigenvalue was formally constructed. The rigorous proof of the asymptotics expansions in the case of small window was adduced recently in [7]. In [8] the case of the strips of the same width and finite window was treated. The phenomenon of new eigenvalues emerging was studied. For the emerging eigenvalues the asymptotics expansions were obtained as the lengths of the window close to critical ones. The behaviour of the associated eigenfunction was described as well. Scattering for the system of two waveguides was considered in [1], [9]. The case in which the Neumann condition is imposed on the boundary instead of the Dirichlet one, was studied in [10]. The existence of at least one isolated eigenvalue was proven. In the paper [11] the system of two symmetric waveguides put in a magnetic field was considered. It was shown that a magnetic field can eliminate the influence of the window presence, namely, for sufficiently small window the system has no bound states. At the same time, the system has a bound state if the window is large enough.

In the present paper we consider a pair of nonsymmetric waveguides coupled by a finite window. The first part of the work is devoted to the studying of the eigenvalue appearing under the length of the window increasing. As it was mentioned, the eigenvalues emerge when the length of the window passes through some critical values. In the paper we give the criterion of the "criticality" for a given value of the length. We also obtain the asymptotics expansions for the emerging eigenvalues and describe the behaviour of the associated eigenfunctions. Moreover, we improve the two-sided estimates obtained in [1].

In the second part of the work we study the behaviour of the discrete spectrum as the window widens. We obtain the asymptotics expansions for the eigenvalues in this case. Under the window widening the shift of the essential spectrum occurs.
in the limits. We describe how this happens.

1 Statement of the problem and formulation of the results

Let \( x = (x_1, x_2) \) be Cartesian coordinates, \( \Pi^+ := \{ x : 0 < x_2 < \pi \}, \Pi^- := \{ x : -d < x_2 < 0 \} \). The width \( d \) of the strip \( \Pi^- \) is assumed to be not exceeding the width of the strip \( \Pi^+ \). In the axis \( x_2 = 0 \) we select an interval \( \gamma_l \) of length \( 2l \) centered at zero which will be called window in what follows. The union of the strips \( \Pi^- \) and \( \Pi^+ \) and the interval \( \gamma_l \) is denoted by \( \Pi \), i.e., the set \( \Pi \) are strips \( \Pi^+ \) and \( \Pi^- \) coupled by the window \( \gamma_l \). The boundary of the domain \( \Pi \) is indicated as \( \Gamma_l \) (cf. figure).

The main object of our study is the spectrum of the operator \( H_l := -\Delta_l^{(D)} \) in \( L_2(\Pi) \), where \( \Delta_l^{(D)} \) is the Friedrich’s extension of the Laplace operator from the set \( C_0^\infty(\Pi) \). The essential spectrum of the operator \( H_l \) coincides with the real semi-axis \([1, +\infty)\). For \( l = 0 \) (i.e., in the case \( \gamma_l = \emptyset, \Pi = \Pi^+ \cup \Pi^- \)) it is obvious, while the essential spectrum of the operators \( H_0 \) and \( H_l, l > 0 \), are same. The proof of this fact reproduces word for word the proof of Theorem 2.1 in \([11]\) and based on the ideas of the work \([12]\). One just needs to take into account that the domain \( \Pi \) possesses the cone property (see definition in \([13, Item 4.3, Ch. IV]\)), thus by Rellich-Kondrashov theorem \([13, Theorem 6.2, Ch. VI]\) the embedding \( W_1^2(\Pi) \to L_2(\Pi) \) is compact for any bounded subdomain \( Q \subset \Pi \) with smooth boundary.

As it has been mentioned in Introduction, the presence of the window \((l > 0)\) gives rise to a non-empty discrete part of the spectrum of the operator \( H_l \), i.e., to the existence of the isolated eigenvalues \( \lambda_m(l), m \geq 1 \). We take these eigenvalues in ascending order with the multiplicity taken into account.

In \([1]\) the following statement was proved.

Lemma 1.1. For any \( l > 0 \) the operator \( H_l \) has a non-empty discrete spectrum consisting of finitely many eigenvalues. There exists an infinite set of critical values \( 0 = l_1 < l_2 < \ldots < l_n < \ldots \) of length of the window \( \gamma_l \), such that as \( l \in (l_n, l_{n+1}] \) the operator \( H_l \) has exactly \( n \) isolated eigenvalues. These eigenvalues are non-increasing functions on \( l \) and satisfy two-sided estimates:

\[
\Lambda_{m-1}(l) \leq \lambda_m(l) \leq \Lambda_m(l), \quad m \geq 1, \quad l > l_m,
\]

where

\[
\Lambda_m(l) := \frac{\pi^2}{(\pi + d)^2} + \frac{\pi^2 m^2}{4l^2}.
\]

The number of eigenvalues \( \lambda_m(l) \) meets the inequalities

\[
\left[ \frac{2l}{\pi} \sqrt{1 - \frac{\pi^2}{(\pi + d)^2}} \right] \leq \text{card} (\sigma_{\text{disc}}(H_l)) \leq \left[ \frac{2l}{\pi} \sqrt{1 - \frac{\pi^2}{(\pi + d)^2}} \right] + 1,
\]
where \( [\cdot] \) indicates an integer part.

Throughout the work by \( W^2_1(\Omega, \gamma) \) we indicate the completion in the norm of \( W^2_1(\Omega) \) of the set of functions from \( C^\infty(\overline{\Omega}) \) having compact support and vanishing in a neighbourhood of the set \( \gamma \). We also set \( \Pi_a := \{ x : |x_1| < a \} \cap \Pi, \Gamma^a_t := \Gamma_t \cap \partial \Pi_a \). By \( \Xi \) we denote the set of all bounded subdomains \( Q \subset \Pi \) with smooth boundary separated from the edges of the window \( \gamma_t \) by a positive distance. The case \( \partial Q \cap \partial \Pi \neq \emptyset \) is not excluded.

Let us formulate the main results of the present work.

**Theorem 1.1.** The statements are valid:

1. The eigenvalues \( \lambda_m(l) \) of the operator \( H_l \) are continuous on \( l \), simple and satisfy the estimates
   \[
   \Lambda_{m-1}(l) < \lambda_m(l) < \Lambda_m(l), \quad m \geq 1, \quad l > l_m. \tag{1.2}
   \]
   The associated eigenfunctions are even on \( x_1 \) for odd \( m \), and odd on \( x_1 \) for even \( m \).

2. The length \( l = l_n \) is critical, if and only if a boundary value problem
   \[
   -\Delta \phi_n = \phi_n, \quad x \in \Pi, \quad \phi_n = 0, \quad x \in \Gamma_t, \tag{1.3}
   \]
   has a bounded solution belonging to \( W^1_2(\Pi, \Gamma^0_t) \) for any \( a > 0 \) and being even on \( x_2 \) in the case \( d = \pi \), and obeying an asymptotics representation
   \[
   \phi_n(x) = \sqrt{\frac{2}{\pi}} \sin x_2 + O(e^{-\sqrt{3}x_1}), \quad x_1 \to +\infty, \quad x_2 \in (0, \pi). \tag{1.4}
   \]
   In the case such solution exists, it is unique and even on \( x_1 \) for odd \( n \) and odd on \( x_1 \) for even \( n \).

3. The asymptotics expansion of the eigenvalue \( \lambda_n(l) \), \( n \geq 2 \), as \( l \to l_n + 0 \) is as follows:
   \[
   \lambda_n(l) = 1 - \mu_n^2 (l - l_n)^2 + O((l - l_n)^3), \tag{1.5}
   \]
   \[
   \mu_n = \frac{1}{l_n} \int_\Pi \left| \frac{\partial \phi_n}{\partial x_1} \right|^2 \, dx \quad \text{as} \quad d < \pi, \tag{1.6}
   \]
   \[
   \mu_n = \frac{1}{2l_n} \int_\Pi \left| \frac{\partial \phi_n}{\partial x_1} \right|^2 \, dx \quad \text{as} \quad d = \pi. \tag{1.7}
   \]
   The associated eigenfunction can be chosen such that it meets the asymptotics representation
   \[
   \psi_n(x) = \sqrt{\frac{2}{\pi}} e^{-\sqrt{1 - \lambda_n(l)|x_1|}} \sin x_2 + O(e^{-\sqrt{3 - \lambda_n(l)}x_1}), \quad x_1 \to +\infty, \quad x_2 \in (0, \pi). \tag{1.8}
   \]
At the same time for any $R > 0$ the equality

$$
\psi_n(x) = \phi_n(x) + O((l - l_n)^{1/2}) \quad \text{in norm } W^1_2(\Pi_R),
$$

(1.9)

holds true.

Remark 1.1. In Item 2 of Theorem 1.1 a solution to the boundary value problem 1.3 is regarded in a generalized sense. Namely, a solution is a function belonging to the space $W^1_2(\Pi_a, \Gamma^a_l)$ for each $a > 0$, and solving an integral equation:

$$
(\nabla_x \phi_n, \nabla_x \zeta)_{L^2(\Pi)} = (f, \zeta)_{L^2(\Pi)}
$$

(1.10)

for each function $\zeta \in C_0^\infty(\Pi)$. In accordance with the theorems on improving smoothness of solutions to elliptic problems [14, Ch. 4, § 2], the function $\phi_n$ belongs to $C^\infty(Q)$ for each $Q \in \Xi$. This is why the asymptotics (1.4) should be understood in the usual sense. In what follows all the boundary value problems are treated in the sense of an integral equation similar to (1.10). Moreover, due to the theorems on improving smoothness solutions to all boundary value problems posed in unbounded domains are infinitely differentiable functions as the absolute value of $x_1$ is large enough. This allows us to understand all the statements on behaviour of these solutions at infinity in the usual sense.

Remark 1.2. The function $\phi_n$ in Item 2 of Theorem 1.1 is supposed to be even on $x_2$ as $d = \pi$. Such a restriction is needed to exclude from consideration the function $\sqrt{2/\pi} \sin x_2$ which is a bounded solution to the problem (1.3) and satisfy the asymptotics representation (1.4) for all $l \geq 0$ in the case $d = \pi$. In the case $d < \pi$ a solution similar to $\sqrt{2/\pi} \sin x_2$ is absent and the requirement of being even on $x_2$ is not introduced.

Remark 1.3. It should be noted that Item 3 of Theorem 1.1 was proved in [8] for the case of symmetric strips ($d = \pi$).

Theorem 1.2. The following statements are valid:

1. The eigenvalues $\lambda_m(l)$ have the following asymptotics expansion as $l \to +\infty$:

$$
\lambda_m(l) = \frac{\pi^2}{(\pi + d)^2} + \frac{\pi^2 m^2}{4l^2} + O(l^{-3}).
$$

(1.11)

2. Each point of semi-interval $\left[\frac{\pi^2}{(\pi + d)^2}, 1\right]$ is the accumulation point for the eigenvalues $\lambda_m(l)$ as $l \to +\infty$, namely, for each point $\xi \in \left[\frac{\pi^2}{(\pi + d)^2}, 1\right]$ there exists a sequence of indexes $m = m(l, \xi)$ tending to infinity as $l \to +\infty$, such that the convergence

$$
\lambda_m(l, \xi) \to \xi \quad \text{as } l \to +\infty
$$

holds true.
Let us discuss the results of the work. Theorem 1.1 is devoted mostly to phenomenon of new eigenvalues of the operator $H_l$ emerging as the window $\gamma_l$ widens. The first item of the theorem improves the estimate (1.1), the second one provides the criterion determining the critical value of the window $\gamma_l$. As it follows from the third item of Theorem 1.1, new eigenvalues emerge from the threshold of the essential spectrum of the operator $H_l$ and have the asymptotics expansion (1.5)–(1.7). The leading term of this expansion is nonzero. This fact follows easily from the formula for $\mu_n$ and the boundary value problem for $\phi_n$. Formulas (1.6) and (1.7) imply that the coefficient $\mu_n$ is discontinuous as $d \to \pi$. Earlier similar phenomenon for the eigenvalue $\lambda_1(l)$ as $l$ is small enough was found formally in [6].

The second part of the results given in Theorem 1.2 describe the behaviour of the spectrum of the operator $H_l$ as the length of the window increases. As it follows from the first item of Theorem 1.2, all the eigenvalues $\lambda_m(l)$ tend to the threshold of the essential spectrum of the "limiting" operator, coinciding up to a quantity of order $O(l^{-3})$ with the right end-points of the intervals from Item 1 of Theorem 1.1. We stress that the estimate for the error term in (1.11) is not uniform on $m$. We also note that the leading term in the asymptotics expansion (1.11) is independent on $d$ in contrast to the formula (1.6) where this parameter plays a crucial role.

As the length of the window increases, it is appropriate to compare the spectra of the original operator $H_l$ and a "limiting" operator $H_* := -\Delta^{(D)}$, where $\Delta^{(D)}$ is the Friedrich's extension of the Laplace operator from a set $C^\infty_0(\Pi^*)$, $\Pi^* := \{x : -d < x_2 < \pi\} \setminus \{x : x_1 \geq 0, x_2 = 0\}$. This "limiting" operator appears if in the original problem one makes a shift $x_1 \mapsto x_1 - l$ and pass formally to the limit as $l \to +\infty$. The spectrum of the operator $H_*$ consists of its essential part only and coincides with the semi-axis $\left[\pi^2, +\infty\right)$. In order to prove this fact one just needs to estimate the threshold of the essential spectrum of the operator $H_*$ both from above and below by bracketing [17, Ch. 13, §15], introducing in the domain $\Pi^*$ an additional boundary $\{x : x_1 = 0, -d < x_2 < \pi\}$ and imposing Dirichlet or Neumann condition on it.

The second item of Theorem 1.2 describes how the shift of the essential spectrum occurs as $l \to +\infty$: each point of the semi-interval which is the shift of the essential spectrum in the limit is an accumulation point as $l \to +\infty$ for the eigenvalues $\lambda_m(l)$ whose indexes increases unboundedly together with $l$.

Let us describe briefly the structure of the present work. In the next section we prove Item 1 of Theorem 1.1 as well as the convergence of the eigenvalues to the threshold of the essential spectrum as the length of the window tends to a critical size. The third section is devoted to the studying behaviour of the resolvent as the spectral parameter tends to the threshold of the essential spectrum. Basing on the results of the third section, in the fourth one we prove Items 2 and 3 of Theorem 1.1. The proof of Theorem 1.2 is adduced in the last section.
2 Estimates, continuity and convergence of eigenvalues

The present section is devoted to the proof of Item I of Theorem 1. We will also prove the convergence of the eigenvalues $\lambda_n(l)$ to the threshold of the essential spectrum as $l \to l_n + 0$.

**Lemma 2.1.** The eigenvalues $\lambda_m(l)$ are continuous on $l$. As $l \to l_n + 0$ the convergence $\lambda_n \to 1 - 0$ holds true.

**Proof.** According to Lemma 1.1, the operator $H_l$ is lower semibounded and its lower bound is $\frac{\pi^2}{(\pi + a)^2}$. Therefore, for each value of $l$ there exists a bounded inverse operator $H_l^{-1} : L_2(\Pi) \to L_2(\Pi)$. The functions $\lambda_m^{-1}(l)$ are isolated eigenvalues of the operator $H_l^{-1}$. Let us prove that they are continuous on $l$. Let $l_*$ be a given length of the window $\gamma_l$ and $\lambda_m(l_*)$ is an isolated eigenvalue of the operator $H_{l_*}$. The eigenvalue $\lambda_m(l)$ is obviously to be an eigenvalue of the boundary value problem

$$-\Delta \psi = \lambda \psi, \quad x \in \Pi, \quad \psi = 0, \quad x \in \Gamma_l.$$

We remind that a solution to this boundary value problem is regarded in the generalized sense (see Remark 1.1). Due to Theorem 4.6.8 from [15] it guarantees the belonging of a generalized solution to the domain of the operator $H_l$, if its solution is an element of $L_2(\Pi)$.

Let $\chi_1(x_1)$ be an infinitely differentiable cut-off odd function which equals minus one as $x_1 \in [-l_* - \varepsilon_0, -l_* + \varepsilon_0]$, is one as $x_1 \in [l_* - \varepsilon_0, l_* + \varepsilon_0]$, and vanishes as $x_1 \in (-\infty, -l_* - 2\varepsilon_0] \cup [-l_* + 2\varepsilon_0, l_* - 2\varepsilon_0] \cup [l_* + 2\varepsilon_0, +\infty)$, where $\varepsilon_0$ is a small fixed number. In the problem (2.1) we make a change of variables

$$y_1 = x_1 - \varepsilon \chi_1(x_1), \quad y_2 = x_2, \quad \varepsilon = l - l_*, \quad \varepsilon \in [-\varepsilon_0, \varepsilon_0].$$

Such change, as it can be checked easily, leads us to a new boundary value problem:

$$-(\Delta_y + \varepsilon L_\varepsilon) \psi = \lambda \psi, \quad y \in \Pi, \quad \psi = 0, \quad y \in \Gamma_{l_*},$$

$$L_\varepsilon = A_{11}(y_1, \varepsilon) \frac{\partial^2}{\partial y_1^2} + A_1(y_1, \varepsilon) \frac{\partial}{\partial y_1},$$

$$A_{11}(y_1, \varepsilon) = -2\chi_1'(x_1(y_1, \varepsilon)) + \varepsilon \left(\chi_1'(x_1(y_1, \varepsilon))\right)^2,$$

$$A_1(y_1, \varepsilon) = -\chi_1'(x_1(y_1, \varepsilon)).$$

Therefore, the function $\lambda_m^{-1}(l)$ is an eigenvalue of the operator $(H_{l_*} + \varepsilon L_\varepsilon)^{-1} : L_2(\Pi) \to L_2(\Pi)$. This operator is well-defined and bounded. Indeed, the operator $H_l^{-1}$ is a bounded operator from $L_2(\Pi)$ into $W_2^1(\Pi)$ and $W_2^2(\Pi)$ for each $Q \in \Xi$. The boundedness of the operator $H_{l_*}^{-1} : L_2(\Pi) \to W_2^1(\Pi)$ is obvious while the boundedness of the operator $H_{l_*}^{-1} : L_2(\Pi) \to W_2^2(\Pi)$ follows from theorems on improving smoothness of solutions to elliptic boundary value problems [14, Ch. 4, §2].
Taking into account these facts as well as boundedness and compactness of supports of the coefficients of the operator $L$, we conclude that the operator $H^{-1}_\varepsilon L$ is bounded uniformly on $\varepsilon$ as an operator from $L^2(\Pi)$ into $L^2(\Pi)$. Thus, the operator $(H + \varepsilon L)^{-1} : L^2(\Pi) \to L^2(\Pi)$ is well-defined for sufficiently small $\varepsilon$. It is easy to check that it is determined by the formula $(H + \varepsilon L)^{-1} = (I + \varepsilon H^{-1}_\varepsilon L)^{-1} H^{-1}_\varepsilon$. The last representation proves also the convergence of the operator $(H + \varepsilon L)^{-1}$ to $H^{-1}$ in the operator norm as $\varepsilon \to 0$. From [16, Ch. 4, §2.6, Theorem 2.23] it follows that the operator $(H + \varepsilon L)^{-1}$ converge to $H^{-1}$ in a generalize sense as well. In its turn, due to [16, Ch. 4, §3.5] it implies the convergence $\lambda_m^{-1}(l) \to \lambda_m^{-1}(l_*)$ as $l \to l_*$, what proves the needed continuity of the eigenvalues of the operator $H_l$.

Let us prove the convergence of the eigenvalues $\lambda_n(l)$ to the threshold of the essential spectrum as $l \to l_n - 0$. The convergence $\lambda_1(l) \to 1$ follows from [3] Theorem 2.1]. The eigenvalues $\lambda_n(l)$ are monotonically nondecreasing functions on $l$ bounded from above by one. This yields the existence of the limits $c_n = \lim_{l \to l_n + 0} \lambda_n(l)$. Suppose that one of these limits is strictly less than one. Then the number $c_n$ is an eigenvalue of the operator $H_l$ as $l = l_n$ (see the proof of the continuity of the eigenvalues on $l$ adduced above). Hence, as $l = l_n$ the operator $H_l$ has $n$ isolated eigenvalues what contradicts to Lemma 1.1.

\section*{Lemma 2.2.} The Item 1 of Theorem 1.1 is valid.

\textbf{Proof.} The continuity of the eigenvalues was proved in the previous lemma. The simplicity of the eigenvalues $\lambda_m$ is surely to be an implication of the estimates 1.2. Let us prove the latter. According to minimax principle the eigenvalues of the operator $H_l$ are given by the formulas

$$\lambda_m(l) := \inf_{u \in W^1_2(\Pi), u \neq 0, \langle u, \psi_j \rangle_{L^2(\Pi)} = 0, j = 1, \ldots, m-1} \frac{\|\nabla u\|_{L^2(\Pi)}^2}{\|u\|_{L^2(\Pi)}^2},$$

(2.4)

where, we remind, $\psi_j$ are the eigenfunctions associated with $\lambda_j(l)$. We introduce the functions

$$u_j(x) = \begin{cases} \frac{\sqrt{2}}{\sqrt{\pi + d}} \sin \frac{\pi}{\pi + d} (x_2 - \pi) \sin \frac{\pi j}{2l} (x_1 + l), & x \in \Pi_l, \\ 0, & x \not\in \Pi_l. \end{cases}$$

Clear, the functions $u_j$ belong to the space $W^1_2(\Pi, \partial \Pi)$. Let us prove the right-hand side of the estimates 1.2. Suppose the opposite, namely, let for some $l$ and $m$ the equality $\lambda_m(l) = \lambda_m(l)$ is true. The functions $u_j$ are linear independent, this is why in the linear space spanned on the functions $u_j$, $j = 1, \ldots, m$, there exists a nonzero function $u = \sum_{j=1}^m \alpha_j u_j$ being orthogonal in $L^2(\Pi)$ to each function $\psi_i$, $i = 1, \ldots, m - 1$. By (2.4) we have

$$\lambda_m(l) \leq \frac{\|\nabla u\|_{L^2(\Pi)}^2}{\|u\|_{L^2(\Pi)}^2} = \frac{\sum_{j=1}^m \alpha_j^2 \lambda_j}{\sum_{j=1}^m \alpha_j^2}. \quad (2.5)$$

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The fraction in the right-hand side of this relation does not exceed $\Lambda_m(l)$. The equality

$$\frac{\sum_{j=1}^{m} \alpha_j^2 \Lambda_j}{\sum_{j=1}^{m} \alpha_j^2} = \Lambda_m(l)$$

is possible only in the case $\alpha_m \neq 0, \alpha_j = 0, j = 1, \ldots, m - 1$. In this case the function $u_m$ is an eigenfunction of the operator $H_l$ associated with the eigenvalue $\lambda_m(l)$. This contradicts to the fact that all the eigenfunctions of the operators $H_l$ belong to $C^\infty(\Pi)$. Thus, at least one of numbers $\alpha_j, j = 1, \ldots, m - 1$, is nonzero, what by (2.5) yields the estimate for $\lambda_m(l)$:

$$\lambda_m(l) \leq \frac{\|\nabla u\|^2_{L^2(\Pi)}}{\|u\|^2_{L^2(\Pi)}} = \frac{\sum_{j=1}^{m} \alpha_j^2 \Lambda_j}{\sum_{j=1}^{m} \alpha_j^2} < \Lambda_m(l).$$

This contradicts to the original assumption that $\lambda_m(l) = \Lambda_m(l)$.

We proceed to the proof of the left-hand side of the estimates (1.2). Let the operator $H_l$ has $n$ eigenvalues, what due to Lemma 2.1 implies that $\Lambda_j(l) < 1, j = 0, \ldots, n - 1$. Let $\delta > 0$ be some small number. Through the points $(-l + \delta, 0)$ and $(l - \delta, 0)$ we pass the segments being parallel to the axis $x_1 = 0$ and dissecting $\Pi$ into three disjoint parts. Isolated eigenvalues of the Laplacian in $\Pi$ subject to Dirichlet condition on $\Gamma_l$ and Neumann condition on the segments introduced estimate the eigenvalues $\lambda_m(l)$ from below. The essential spectrum of such operator coincides with real semi-axis $[1, +\infty)$, what can be established in same way as the equality $\sigma_{ess}(H_l) = [1, +\infty)$. The discrete spectrum of this operator is a union $\sigma_1 \cup \sigma_2$, where $\sigma_1$ is a set of the eigenvalues of the operator $S_1$ that are less than one. Here the operator $S_1$ is the Laplacian in an rectangle $\{x : |x_1| < l - \delta, -d < x_2 < \pi\}$ subject to Neumann condition on the lateral sides and to Dirichlet one on the upper and lower sides. The set $\sigma_2$ is the discrete spectrum of the Laplacian in the semi-strip $\Pi \cap \{x : x_1 > l - \delta\}$ subject to Neumann condition on $\{x : x_1 = l - \delta, -d < x_2 < \pi\}$ and to Dirichlet condition on the remaining part of the boundary. We denote this operator by $S_2$. For sufficiently small $\delta$ the eigenvalues forming $\sigma_1$ are the functions $\Lambda_j(l - \delta), j = 0, \ldots, n - 1$. Each eigenfunction of the operator $S_2$ can be continued through the boundary $\{x : x_1 = l - \delta, -d < x_2 < \pi\}$ in the odd way on $x_1$. The function obtained in this way is the eigenfunction of the operator $H_\delta$ (up to the change $x_1 \to x_1 - l + \delta$). Therefore, $\sigma_2 \subseteq \sigma_{disc}(H_\delta)$. In accordance with Lemma (17) for sufficiently small $\delta$ the discrete spectrum of the operator $H_\delta$ consists of the only eigenvalue converging to one as $\delta \to 0$. We choose $\delta > 0$ such that this eigenvalue is greater than each of the functions $\Lambda_j(l - \delta), j = 0, \ldots, n - 1$. Therefore, due to bracketing (17) Ch. 13, §15] we can write $\Lambda_{j-1}(l) < \Lambda_{j-1}(l - \delta) \leq \lambda_j(l), j = 1, \ldots, n$, what completes the proof of the needed estimates.

In conclusion let us prove the parity of the eigenfunctions of the operator $H_l$. The set $\Pi$ being symmetric on $x_1$, all the eigenfunctions of the operator $H_l$ can be chosen as being odd or even on $x_1$. The simplicity of the eigenvalues
\( \lambda_m(l) \) means that the eigenfunction of a certain parity is associated with each of these eigenvalues. The even eigenfunctions satisfy the Neumann condition as \( x_1 = 0 \), while the odd ones meet the Dirichlet condition. Moreover, the operator \( H_l \) is an orthogonal sum of the operators \( H_l^+ \) and \( H_l^- \) those are, respectively, restrictions of \( H_l \) on even and odd on \( x_1 \) functions from the domain of the operator \( H_l \). Completely by analogy with how in \( [1, \S 2] \) the estimates \( (1.1) \) were obtained, one can easily show that the isolated eigenvalues of the operator \( H_l^+ \) satisfy the estimates \( (1.1) \) for odd \( m \), while the ones of the operator \( H_l^- \) meet the estimates \( (1.1) \) with even \( m \). This proves the needed parity of the eigenfunctions of the operator \( H_l \), if one takes into account that \( \sigma_{\text{disc}}(H_l) = \sigma_{\text{disc}}(H_l^+) \cup \sigma_{\text{disc}}(H_l^-) \). □

3  The behaviour of the resolvent of the operator \( H_l \) in a vicinity of the threshold of the essential spectrum

This section is devoted to the studying the behaviour of the operator \((H_l - \lambda)^{-1}\) as \( \lambda \) close to one. The results of this section is the basis for the proof of Items 2, 3 of Theorem 1.1.

In studying the operator \((H_l - \lambda)^{-1}\) we employ the same approach as that used in \([8, 18]\) for the case of symmetric strips \( d = \pi \). We study the dependence on \( k \) of a solution to the boundary value problem

\[
-\Delta u = (1 - k^2)u + f, \quad x \in \Pi, \quad u = 0, \quad x \in \partial \Pi, \quad (3.1)
\]

which behaves as follows

\[
\begin{align*}
  u(x, k) &= c_\pm(k)e^{-k|x_1|}\sin x_2 + \mathcal{O}\left(e^{-\sqrt{1-k^2}|x_1|}\right), \quad x_2 \in (0, \pi), \\
  u(x, k) &= \tilde{c}_\pm(k)e^{-\sqrt{1-\frac{\pi^2}{d^2}+k^2}|x_1|}\sin x_2 + \mathcal{O}\left(e^{-\sqrt{1-\frac{\pi^2}{d^2}+k^2}|x_1|}\right), \quad x_2 \in (-d, 0),
\end{align*}
\]

(3.2)
as \( x_1 \to \pm\infty \). Here the function \( f \) is an element of \( L_2(\Pi) \) whose support lies inside \( \Pi_a, a > l \), \( c_\pm(k), \tilde{c}_\pm(k) \) are some constants. In the case \( d = \pi \) in the latter of the asymptotics representations \( (3.2) \) we set \( \sqrt{1-\frac{\pi^2}{d^2}+k^2} = k \). The parameter \( k \) is supposed to belong to a small neighbourhood of the zero in the complex plane. We denote this neighbourhood by \( B \). We note that a solution to the boundary value problem \( (3.1), (3.2) \) decays exponentially as \( \text{Re} k > 0 \), and, therefore, is an element of \( L_2(\Pi) \) in this case. In view of Remark \([14]\) and \([15, \text{Theorem 4.6.8}]\) it implies the belonging of this solution to the domain of the operator \( H_l \), i.e., the function \( u \) coincide with \((H_l - 1 + k^2)^{-1}f\) (of course, if the operator \((H_l - 1 + k^2)\) is invertible). This is why the linear mapping \( f \mapsto u \) defined by the boundary value problem \( (3.1), (3.2) \) can be regarded as an extension of the operator \((H_l - 1 + k^2)^{-1}\) on \( k \) in the domain \( \text{Re} k \leq 0 \). Such extension is surely to widen the range of the
operator $\left(H_t - 1 + k^2\right)^{-1}$ and the range of the extension is not a subset of the space $L_2(\Pi)$. At the same time we will show that in a certain sense this extension is analytic on $k$ and the operator $\left(H_t - 1 + k^2\right)^{-1}$ after extension is happened to be meromorphic on $k$.

Let us introduce the notations. If $X$ and $Y$ are Banach spaces, the symbol $\mathcal{L}(X, Y)$ indicate the set of all linear bounded operator from $X$ into $Y$. The set of all holomorphic (meromorphic) on $k \in B$ function whose values are elements of $X$ is denoted by $\mathcal{H}(X)$ ($\mathcal{M}(X)$). We also set $\mathcal{H}(X, Y) := \mathcal{H}(\mathcal{L}(X, Y))$, $\mathcal{M}(X, Y) := \mathcal{M}(\mathcal{L}(X, Y))$.

In order to study the boundary value problem (3.1) we employ the scheme borrowed from [19] Ch. 16, §4. Let $g$ be some function from $L_2(\Pi_a)$ continued by zero in $\Pi \setminus \Pi_a$. We consider the boundary value problems:

$$-\Delta v_i = (1 - k^2)v_i + g, \quad x \in \Omega_i, \quad v_i = 0, \quad x \in \partial\Omega_i, \quad i = 1, \ldots, 4,$$  \hspace{1cm} (3.3)

where $\Omega_1 := \Pi^+ \cap \{x : x_1 > 0\}$, $\Omega_2 := \Pi^+ \cap \{x : x_1 > 0\}$, $\Omega_3 := \Pi^+ \cap \{x : x_1 < 0\}$, $\Omega_4 := \Pi^- \cap \{x : x_1 < 0\}$. The problems (3.3) are easily solved by separation of variables:

$$v_i(x, k) = \sum_{j=1}^{\infty} \int_{\Omega_i} G_j^i(x, t, k)g(t) \, dt,$$  \hspace{1cm} (3.4)

$$G_j^1(x, t, k) := \frac{1}{\pi s_j^+} \left( e^{-s_j^+|x_1 - t_1|} - e^{-s_j^+(x_1 + t_1)} \right) \sin jx_2 \sin jt_2,$$

$$G_j^2(x, t, k) := \frac{1}{s_j^+d} \left( e^{-s_j^+|x_1 - t_1|} - e^{-s_j^+(x_1 + t_1)} \right) \sin \frac{\pi j}{d} x_2 \sin \frac{\pi j}{d} t_2,$$

$$G_j^3(x, t, k) := \frac{1}{\pi s_j^-} \left( e^{-s_j^-|x_1 - t_1|} - e^{s_j^-(x_1 + t_1)} \right) \sin jx_2 \sin jt_2,$$

$$G_j^4(x, t, k) := \frac{1}{s_j^-d} \left( e^{-s_j^-|x_1 - t_1|} - e^{s_j^-(x_1 + t_1)} \right) \sin \frac{\pi j}{d} x_2 \sin \frac{\pi j}{d} t_2,$$

where $s_1^+ = k$, $s_j^+ = \sqrt{d^2 - 1 + k^2}$, $j \geq 2$, $s_1^- = \sqrt{\frac{\pi^2}{d^2} - 1 + k^2}$ as $d < \pi$, $s_1^- = k$ as $d = \pi$, $s_j^- = \sqrt{\frac{\pi^2}{d^2} - 1 + k^2}$, $j \geq 2$. The functions $G_1^1, G_3^1$ at $k = 0$ are defined by continuity:

$$G_1^1(x, t, 0) := \frac{1}{\pi} (x_1 + t_1 - |x_1 - t_1|) \sin x_2 \sin t_2,$$

$$G_3^1(x, t, 0) := - \frac{1}{\pi} (x_1 + t_1 + |x_1 - t_1|) \sin x_2 \sin t_2,$$

In the case $d = \pi$ the functions $G_1^1(x, t, 0)$ and $G_3^1(x, t, 0)$ are defined in the same way. We denote $\Omega_b^i := \Omega_i \cap \Pi_b$.

**Lemma 3.1.** Let $b > 0$. The series (3.4) converge in the norm of $W_2^2(\Omega_b^i)$. The functions $v_i(x)$ meet the asymptotics formulas (3.2). Linear operators $T_i(k)$ defined
by a rule \( T_i(k)g := v_i \) are elements of \( \mathcal{L}(L_2(\Pi_a), W^2_2(\Omega^b_i)) \). The belonging \( T_i(\cdot) \in \mathcal{H}(L_2(\Pi_a), W^2_2(\Omega^b_i)) \) takes place.

In proof of this lemma we will employ an auxiliary statement.

**Lemma 3.2.** In the norm of \( L_2(\Omega^a_i) \) the equality

\[
 g(x) := \sum_{j=1}^{\infty} g_j(x_1) \sin jx_2, \quad g_j(x_1) := \frac{2}{\pi} \int_0^\pi g(x) \sin jx_2 \, dx_2, \quad i = 1, 3,
\]

\[
 g(x) := \sum_{j=1}^{\infty} g_j(x_1) \sin jx_2, \quad g_j(x_1) := \frac{2}{d} \int_{-d}^d g(x) \sin \frac{\pi j}{d} x_2 \, dx_2, \quad i = 2, 4,
\]

holds true.

**Proof.** We will give the proof for \( \Omega^a_1 \) only, in the other cases the arguments are same. Since \( g \in L_2(\Omega^a_1) \), by Fubini theorem for almost each \( x_1 \in (0, a) \) we have \( g(x_1, \cdot) \in L_2(0, \pi) \). Therefore, the functions \( g_j(x_1) \) are well-defined for almost each \( x_1 \in (0, a) \) and belong to \( L^2(0, a) \) due to an estimate:

\[
 \|g_j\|_{L^2(0, a)} \leq \|g\|_{L^2(\Pi)}.
\]

We introduce the functions

\[
 \mathcal{E}_N(x_1) = \int_0^\pi \left| g(x) - \sum_{j=1}^N g_j(x_1) \sin jx_2 \right|^2 \, dx_2.
\]

The functions \( \{\sin jx_2\}_{j=0}^\infty \) form basis in \( L_2(0, \pi) \), this is why the convergence \( \mathcal{E}_N(x_1) \overset{N \to \infty}{\longrightarrow} 0 \) is valid for almost each \( x_1 \in (0, a) \). Using the definition of the functions \( \mathcal{E}_N \), one can check easily that

\[
 0 \leq \mathcal{E}_N(x_1) = \int_0^\pi |g(x)|^2 \, dx_1 - \frac{\pi}{2} \sum_{j=1}^N |g_j(x_1)|^2 \leq \int_0^\pi |g(x)|^2 \, dx_1.
\]

Therefore, nonnegative functions \( \mathcal{E}_N \) are bounded from above by an integrable over \([0, a]\) function uniformly on \( N \). Bearing in mind this fact as well as the convergence of the functions \( \mathcal{E}_N \) to zero almost everywhere, in view of Lebesgue bounded convergence theorem we conclude that

\[
 \left\| g(x) - \sum_{j=1}^N g_j(x_1) \sin jx_2 \right\|_{L^2(\Pi_a)}^2 = \int_0^a \mathcal{E}_N(x_1) \, dx_1 \overset{N \to \infty}{\longrightarrow} 0.
\]

This completes the proof.
Proof of Lemma 3.1. We will give the proof for $\Omega^b_1$ only, the other cases are proved in the same way. We define the functions $g_j$ in accordance with Lemma 3.2. We indicate the terms of the series (3.4) as $V_j(x, k)$. By the definition of these functions the estimates

$$\left\| \sum_{j=N}^M V_j \right\|_{W^1_2(\Omega^b_1)}^2 \leq C \sum_{j=N}^M \|g_j\|_{L^2(0, b)}^2$$

hold true for all $b > 0$ with constant $C$ independent on $g$, $M$ and $N$. The right-hand side in this inequality tends to zero as $M, N \to \infty$ due to Lemma 3.2. Thus, for all $b > 0$ the series (3.4) converges in the norm of $W^1_2(\Omega^b_1)$ to some function $v_1(x, k)$, which meets the estimate

$$\|v_1\|_{W^1_2(\Omega^b_1)} \leq C\|g\|_{L^2(\Pi_a)}, \quad (3.5)$$

where the constant $C$ is independent on $g$. The function $v_1$, as one can check easily, is a generalized solution to the boundary value problem (3.3). Therefore, by theorems on improving smoothness and the estimate (3.5) the function $v_1$ is an element of $W^2_2(\Omega^b_1)$ and an estimate

$$\|v_1\|_{W^2_2(\Omega^b_1)} \leq C\|g\|_{L^2(\Pi_a)},$$

is valid, where the constant $C$ is independent on $g$. It follows the belonging $T_1(k) \in \mathcal{L}(L^2(\Pi_a), W^2_2(\Omega^b_1))$ for all $b > 0$ and each $k \in B$. By analogy with how the latter estimate for $v_1$ has been obtained, one can deduce that

$$\|V^N\|_{W^2_2(\Omega^b_1)} \leq C\|g^N\|_{L^2(\Pi_a)}, \quad (3.6)$$

where the constant is independent on $g^N$ and $N$,

$$V^N(x, k) := v_1(x, k) - \sum_{j=1}^N V_j(x, k), \quad g^N(x) := g(x) - \sum_{j=1}^N g_j(x) \sin jx_2.$$ 

The estimate (3.6) and Lemma 3.2 yield the convergence of the series (3.4) in $W^2_2(\Omega^b_1)$.

For $x_1 > a$ the functions $V_j$ are of the form

$$V_j(x, k) = -\sum_{j=1}^{\infty} \frac{2}{s^+_j \pi} e^{-s^+_j x_1} \sin jx_2 \int_{\Omega^b_1} g(x) \sinh s^+_j x_1 \sin jx_2 \, dx.$$ 

Thus, for $x_1 > a$ the estimate

$$|V_j(x, k)| \leq C e^{-s^+_j (x_1 - a)} \|g\|_{L^2(\Pi_a)}$$

hold true, where $C$ is a some constant independent on $j$ and $x_1$. In view of this inequality as $x_1 \geq 2a$ the function $(v_1 - V_1)$ can be estimated as follows:

$$|v_1(x, k) - V_1(x, k)| \leq C\|g\|_{L^2(\Pi_a)} \sum_{j=2}^{\infty} e^{-s^+_j (x_1 - a)} \leq$$

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\[ \leq C \|g\|_{L_2(\Pi_a)} e^{-s_2^+(x_1-a)} \sum_{j=2}^{\infty} e^{-(s_j^+-s_j^+)a} (x_1-a) \leq \]
\[ \leq C \|g\|_{L_2(\Pi_a)} e^{-s_2^+(x_1-a)} \sum_{j=2}^{\infty} e^{-(s_j^+-s_j^+)a} \leq \widetilde{C} \|g\|_{L_2(\Pi_a)} e^{-s_2^+x_1}, \]

where the constant \( \widetilde{C} \) is independent on \( x_1 \). The estimate obtained yields that as \( x_1 \geq 2a, \ x_2 \in [0, \pi] \) the series (3.4) is a continuous on \( x \) function and the asymptotics formula (3.2) takes place for the function \( v_1 \) as \( x_1 \to +\infty, \ x_2 \in (0, \pi) \).

Clear, for each function \( g \) we have \( V_j(x, \cdot) \in \mathcal{H}(W_2^j(\Omega_b^1)) \). Since the series (3.4) converges in \( W_2^j(\Omega_b^1) \), by Weierstrass theorem the sum of the series is holomorphic on \( k \) in the norm of \( W_2^j(\Omega_b^1) \), i.e., for each function \( g \in L_2(\Pi_a) \) and any \( b > 0 \) we have \( T_1(\cdot)g \in \mathcal{H}(W_2^j(\Omega_b^1)) \). Since the notions of being holomorphic for bounded operator-valued functions in the sense of weak, strong and uniform convergences are same (see, for instance, \([10, \text{Ch. 7, \S1.1}]\)), we conclude that \( T_1(\cdot) \in \mathcal{H}(L_2(\Pi_a), W_2^j(\Omega_b^1)) \) for all \( b > 0 \).

Remark 3.1. The functions \( v_i \) being elements of the spaces \( W_2^j(\Omega_b^1) \), the equations (3.3) take place not only in the sense of the corresponding integral equality (see remark 1.1), but also as the equality of two functions from \( L_2(\Omega_b^1) \).

We denote \( v(x, k) := v_i(x, k), \ x \in \Omega_i \). Let us consider one more boundary value problem:

\[ \Delta w = \Delta v, \quad x \in \Pi_a, \quad w = v, \quad x \in \partial \Pi_a. \quad (3.7) \]

The first derivatives of the function \( v \) have discontinuities on the boundaries of the sets \( \Omega_i \), this is we should explain what we mean by \( \Delta v \). This function is defined by the equality \( \Delta v := \Delta v_i, \ x \in \Omega_i \). It is obvious that the function \( \Delta v \) defined in such way is an element of \( L_2(\Omega_b^1) \). The set \( \Pi_a \) possesses a cone property, this is why an embedding \( W_2^1(\Pi_a) \subset L_2(\Pi_a) \) is compact due to Rellich-Kondrashov theorem \([13, \text{Theorem 6.2, Ch. VI}]\). The right hand side in the boundary condition (3.7) is a trace of a function belonging to \( W_2^1(\Pi_a) \). This function can be chosen as \( v(x)\chi_2(x_1) \), where \( \chi_2(x_1) \) is an infinitely differentiable cut-off function being equal to one as \( |x_1| > (2a + l)/3 \) and vanishing as \( |x_1| < (a + 2l)/3 \). Due to Lemma 3.1 the functions \( v_i \) being elements of the spaces \( W_2^j(\Omega_b^1) \), it yields that \( v\chi_3 \in W_2^j(\Pi_a, \Gamma_i^\alpha) \). The equality \( v(x)\chi_3(x_1) = v(x), \ x \in \partial \Pi_a \) follows from the definition of the function \( \chi_3 \) and the relation \( v(x) = 0, \ x \in \Gamma_i \). Employing the aforementioned facts, and following the idea of the proof of Theorem 10 in \([14, \text{Ch. IV, \S1.8}]\), one can check easily that the boundary value problem (3.7) is uniquely solvable in the space \( W_2^1(\Pi_a, \Gamma_i^\alpha) \).

The problem (3.7) is uniquely solvable in the space \( W_2^1(\Pi_a, \Gamma_i^\alpha) \) (see, for instance, \([14]\)). Moreover, \( w \) is an element of the space \( W_2^1(Q) \) for each \( Q \in \Xi \) due to theorems on improving the smoothness of solutions to elliptic boundary value problems. In particular, it means that in addition to the integral equality
corresponding to the problem (3.7) (see Remark 1.1) the equation in (3.7) holds also as the equality of two functions from $L_2(Q)$ for each $Q \in \Xi$. Therefore, $\Delta w \in L_2(\Pi_a)$ and the equation in (3.7) holds also as the equality of two functions from $L_2(\Pi_a)$. The function $w$ can be also considered as a value of linear bounded operator $T_5 : \bigoplus_{i=1}^4 W_2^2(\Omega_i^a, \partial\Omega_i^a) \rightarrow W_2^1(\Pi_a, \Gamma_i^\ell)$, $T_5 v := w$. It is clear that $T_5$ is also a linear bounded operator from $\bigoplus_{i=1}^4 W_2^2(\Omega_i^a, \partial\Omega_i^a \cap \partial\Omega_i)$ into $W_2^2(Q)$ for each $Q \in \Xi$.

Let $\chi_3(x_1)$ be an infinitely differentiable cut-off even function which equals minus one as $|x_1| < (a+2l)/3$ and vanishes as $|x_1| > (2a+l)/3$. We construct the function $u$ by the rule:

$$u(x, k) := w(x, k) \chi_3(x_1) + v(x, k)(1 - \chi_3(x_1)).$$

(3.8)

The function $u$ can also be regarded as $u = T_6(k)g$ where $T_6(k)$ is a linear bounded operator from $L_2(\Pi_a)$ into $W_2^1(\Pi_a, \Gamma_i^\ell)$ and $W_2^2(Q)$ for any $b > 0$ and each $Q \in \Xi$. Moreover, $T_6(\cdot) \in \mathcal{H}(L_2(\Pi_a), W_2^1(\Pi_a, \Gamma_i^\ell))$ and $T_6(\cdot) \in \mathcal{H}(L_2(\Pi_a), W_2^2(Q))$.

Let us apply the operator $-(\Delta+1-k^2)$ to $u$ and take into account the equations for $v$ and $w$ (see (3.3), (3.7)). As a result we get:

$$-(\Delta+1-k^2)u = g + (v-w)(\Delta+1-k^2)\chi_3 + 2(\nabla \chi_3, \nabla(v - w))_{\mathbb{R}^2} = g + T_7(k)g.$$  

(3.9)

The function $u$ defined by (3.8) satisfies the homogeneous Dirichlet condition on $\partial\Pi$ and asymptotics formulas (3.2). Therefore, this function is a solution to the boundary value problem (3.1), (3.2) if and only if it meets the equation from (3.9). Due to (3.9) this leads us to the equation for the function $g$:

$$g + T_7(k)g = f.$$  

(3.10)

Completely by analogy with Propositions 3.1 and 3.2 from [8] one can prove the following lemma.

**Lemma 3.3.** The operator $T_7(k)$ is a linear compact operator from $L_2(\Pi_a)$ into $L_2(\Pi_a)$ for each $k \in B$ and $T_7(\cdot) \in \mathcal{H}(L_2(\Pi_a), L_2(\Pi_a))$. For each $k \in B$ the equation (3.10) is equivalent to the boundary value problem (3.1), (3.2). Namely, for each solution $g$ of the equation (3.10) there exists a solution to the boundary value problem (3.1), (3.2) given by the formula $u = T_6(k)g$. For each solution $u$ to the boundary value problem (3.1), (3.2) there exists a unique solution of the equation (3.10) associated with $u$ by the equality $u = T_6(k)g$.

The operator $T_7$ being compact, Fredholm alternatives can be applied to the equation (3.10). Due to Lemma 3.3 this solves the solvability questions for the boundary value problem (3.1), (3.2). It should be also noted that in the case of unique solvability of the equation (3.10) the solution $u$ to the problem (3.1), (3.2) generated by the rule $u = T_6(k)(I + T_7(k))^{-1}f$ from the solution of the equation
The function \((H_l - 1 + k^2)^{-1} f\) as \(\text{Re} \ k > 0\) (see the asymptotics formulas (3.2)). This is why the operator \(T_6(k)(I + T_7(k))^{-1}\) can be interpreted as an analytic continuation of the operator \((H_l - 1 + k^2)^{-1} f\). At the same time it should be stressed that as \(\text{Re} \ k \leq 0\) the function \(u = T_6(k)(I + T_7(k))^{-1} f\), generally speaking, is not an element of the space \(L_2(\Pi)\).

Lemma 3.4. There exists a point \(k_* \in B\) such that the operator \((I + T_7(k_*))\) has a bounded inverse.

Proof. It is clear that it is sufficient to find the point \(k_* \in B\) for which the equation (3.10) is uniquely solvable. The unique solvability of the latter is equivalent to that of the boundary value problem (3.1), (3.2). We choose a point \(k_*\) as \(k_* = \delta(1 + i), \delta > 0\). For such \(k_*\) the problem (3.1) with \(f = 0\) has no nontrivial solution meeting the asymptotics formulas (3.2), since otherwise this function would be an element \(L_2(\Pi)\), and \(\lambda_* = 1 - k_*^2\) would be a complex-valued eigenvalue of the operator \(H_l\). This contradicts to the reality of the spectrum of the operator \(H_l\).

The proven lemma, compactness and holomorphy of the operator \(T_7(k)\) allow us to apply Theorem 7.1 from [19, Ch. 15, §7] to the operator \((I + T_7(k))^{-1}\), what leads us to the following statement.

Lemma 3.5. The belonging \((I + T_7(\cdot))^{-1} \in \mathcal{M}(L_2(\Pi_\alpha), L_2(\Pi_\alpha))\) takes place.

Due to this lemma the only possible singularities of the operator \((I + T_7(k))^{-1}\) are isolated poles. We are interesting only on presence and absence of the pole at the point \(k = 0\). This is why we suppose that the neighbourhood \(B\) of zero contains no poles except possible pole at zero. The presence of pole at zero implies the existence of a nontrivial solution of the equation (3.10) with \(k = 0, f = 0\), what is equivalent to the existence of the bounded nontrivial solution of the problem (3.1) (see asymptotics (3.2)) with \(k = 0, f = 0\). The next lemma describes possible options of such solutions to exist.

Lemma 3.6. Let \(k = 0, f = 0\). Then

1. The boundary value problem (3.1) has at most one nontrivial solution meeting the asymptotics formulas (3.2) and being even on \(x_2\) in the case \(d = \pi\). This solution has a definite parity on \(x_1\).

2. If \(d = \pi\), then the boundary value problem (3.1) has a unique nontrivial solution which is odd on \(x_2\) and meets the asymptotics formulas (3.2), where \(c_+(0) = 1\). This solution is \(\sin x_2\).

Proof. As \(k = 0\) the boundary value problem (3.1) being equivalent to the equation (3.10), owing to compactness of the operator \(T_7(0)\) the problem (3.1) can have only finitely many bounded linear independent solutions. Boundedness in this case is an implication of the asymptotics (3.2). We denote these solutions by \(u_j, j = 1, \ldots, q\). The change of variables \(x_1 \mapsto -x_1\) maps a solution to the problem (3.1) into a
solution, this is why without loss of generality we can assume that each of solutions $u_j$ has a definite parity on $x_1$. In the case $d = \pi$ we also assume that each of these solutions is even on $x_2$. Moreover, all the functions $u_j$ can be supposed to be real. We also observe that due to theorems on improving smoothness we have $u_j \in C^{\infty}(Q)$ for each $Q \in \Xi$.

First we prove that the coefficients $c_{\pm}(0)$ are always non-zero. Suppose the opposite, namely, let there exists a nontrivial solution $u = u_j$ whose coefficients $c_{\pm}(0)$ are zero. In view of asymptotics (3.2) it means that the function $u$ decays exponentially as $|x_1| \to \infty$, $x_2 \in (-d, \pi)$. Let the function $u$ be even on $x_1$. We introduce the function

$$U(x) := x_1 \int_0^{x_1} u(t, x_2) \, dt. \quad (3.11)$$

The function $U$ is surely to be infinitely differentiable at all interior points of $\Pi$. Moreover, it is an element of the space $W^{1,2}_1(\Pi, \Gamma_a^a)$ for any $a > 0$. Since the function $u$ is even on $x_1$, it follows that its derivative on $x_1$ vanishes as $x_1 = 0$. Taking into account this fact and the equation for $u$, it is not difficult to check that the function $U$ is a solution to the equation

$$(\Delta + 1)U = 2u, \quad x \in \Pi.$$

Moreover, the function $U$ satisfies the homogeneous Dirichlet condition on the lines $x_2 = -d$ and $x_2 = \pi$. We are going to prove that it vanishes on $\Gamma_l$ as well. In order to do it, due to evenness of $u$ on $x_1$, it is sufficient to establish the equality:

$$\int_{\gamma_l} u \, dx_1 = 0.$$

This equality can be proved easily by integration by parts:

$$0 = \int_{\Pi^+} \sin x_2 (\Delta + 1) u \, dx = \int_{\gamma_l} u \, dx_1. \quad (3.12)$$

Here we have also used the boundary condition for the function $u$ and its exponential decaying at infinity. The function $U$ behaves like $O(x_1)$ as $x_1 \to \pm \infty$, what follows from the exponential decaying of $u$ at infinity.

Bearing in mind the properties of the functions $u$ and $U$, we can integrate by parts:

$$0 = \int_{\Pi} U(\Delta + 1) u \, dx = 2 \int_{\Pi} |u|^2 \, dx,$$

what implies $u = 0$. The same equality can be also proved in the case the function $u$ being odd on $x_1$. Here the function $U$ should be defined as

$$U(x) := \int_0^{x_1} tu(t, x_2) \, dt.$$
This function possesses the same properties as the function \( U \) in (3.11). The only difference in the proof of these properties is a modification of (3.12), in this case the integral \( 0 = \int x_1 \sin x_2 (\Delta + 1) u \, dx \) should be taken as a source integral for integration by parts. It should be also noted that the function \( U \) in this case is bounded as \( x_1 \to +\infty \).

Thus, each of the functions \( u_j \) has nonzero coefficients \( c_{\pm}(0) \) in the asymptotic formulas (3.2). It means that the number of the functions \( u_j \) does not exceed two. Indeed, otherwise it would be possible to change a linear combination of the functions \( u_j \) whose coefficients \( c_{\pm}(0) \) would be zero. It would mean that this combination is identically zero, and, as a result, that the functions \( u_j \) are linear dependent. It also obvious that in the case two functions \( u_j \) are present, they have different parity on \( x_1 \). Let us stress that for \( d = \pi \) the assumed parity of \( u_j \) on \( x_2 \) is essential in these arguments otherwise the possible number of the functions \( u_j \) increases up to four.

Let the number of the functions \( u_j \) be two and let \( u_1 \) be even on \( x_1 \) and \( u_2 \) be odd. Without loss of generality we assume that the coefficients \( c_{\pm}(0) \) of the functions \( u_1 \) and \( u_2 \) are respectively of the form \( c_{\pm}(0) = \sqrt{2/\pi}, c_{\pm}(0) = \pm \sqrt{2/\pi} \). We set

\[
U_1(x) := \int_0^{x_1} u_1(t, x_2) \, dt.
\]

By analogy with how the properties of the function \( U \) in (3.11) have been found, one can show easily that \( U_1 \) is a solution to the boundary value problem (3.1) meets the asymptotics representation

\[
U_1(x) = \frac{\sqrt{2}}{\pi} (x_1 \pm c) \sin x_2 + \mathcal{O}(e^{-\sqrt{3}|x_1|}), \quad x_1 \to \pm \infty, \quad x_2 \in (0, \pi),
\]

where \( c \) is some constant. In the case \( d = \pi \) the function \( U_1 \) has exactly the same asymptotics as \( x_2 \in (-\pi, 0) \). If \( d < \pi \), then the function \( U_1 \) decays exponentially as \( x_1 \to \pm \infty, \, x_2 \in (-d, 0) \), what follows from the boundary value problem for \( U_1 \) and boundedness of \( U_1 \) as \( x_1 \to \pm \infty, \, x_2 \in (-d, 0) \). Taking into account the properties of the functions \( U_1 \) and \( u_2 \), integrating by parts in an integral \( \int_{\Pi} U_1(\Delta + 1) u_2 \, dx \) and passing after that to limit as \( R \to +\infty \), we get:

\[
0 = \int_{\Pi} U_1(\Delta + 1) u_2 \, dx = \begin{cases} -2, & d < \pi, \\ -4, & d = \pi, \end{cases}
\]

a contradiction. Thus, the number of the functions \( u_j \) is at most one and if exists, this function is unique and has a definite parity on \( x_1 \). Item 1 is proven.

Statement of Item 2 is obvious if one takes into account that odd on \( x_2 \) solution vanishes as \( x_2 = 0 \).
Let us introduce auxiliary notations. In the case the nontrivial solution to the problem (3.1) described in Item 1 of Lemma 3.6 exists, we denote this solution by $\phi(x)$. The associated solution of the equation (3.10) is indicated as $\Phi(x)$, $\phi(x) = (T_6(0)\Phi)(x)$. If such solution does not exists, we set $\phi = 0$, $\Phi = 0$. In the case $d = \pi$ the solution of the equation (3.10) associated with $\sin x_2$ is denoted by $\tilde{\Phi}(x)$, $\sin x_2 = (T_6(0)\tilde{\Phi})(x)$.

The next lemma describes the structure of the operator $(I + T_7(k))^{-1}$ for small $k$.

**Lemma 3.7.** The operator $(I + T_7(k))^{-1}$ can be represented as:

\[
(I + T_7(k))^{-1} = \frac{1}{k} T_8 + T_9(k),
\]

\[
T_8 f := \frac{1}{2} \Phi \int_{\Pi} f(x) \phi(x) \, dx, \quad \text{as } d < \pi,
\]

\[
T_8 f := \frac{1}{4} \Phi \int_{\Pi} f(x) \phi(x) \, dx + \frac{1}{2\pi} \tilde{\Phi} \int_{\Pi} f(x) \sin x_2 \, dx, \quad \text{as } d = \pi,
\]

where $T_9(\cdot) \in \mathcal{H}(L_2(\Pi_a), L_2(\Pi_a))$.

The proof of this lemma is analogous to that of Theorem 3.4 in [8].

## 4 Asymptotics expansions of emerging eigenvalues

In the present section we will prove Items 2, 3 of Theorem 1.1 finishing by this the proof of this theorem. For calculating the asymptotics expansions we will employ the scheme which is analogous to that employed in [8] in the case $d = \pi$. The main ideas of this scheme are borrowed from the works [20], [21]. Let $l_*$ be some value of the length of the window $\gamma_l$. We give an increment $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$ to this length. Here $\varepsilon_0$ is from (2.2). As it was shown in the proof of Lemma 2.1, the eigenvalues of the operator $H_{l_* + \varepsilon}$ are those of the boundary value problem (2.3).

We denote $\lambda = 1 - k^2$, then in accordance with the results of the previous section the boundary value problem (2.3) is equivalent to an operator equation in $L_2(\Pi_a)$:

\[
(I + T_7(k) - \varepsilon L_\varepsilon T_6(k))g = 0. \quad (4.1)
\]

Here the parameter $a$ should be chosen great enough and independent on $\varepsilon$ so that the supports of the coefficients of the operator $L_\varepsilon$ lie inside $\Pi_a$ for all $\varepsilon$ small enough.

Since $T_6(\cdot) \in \mathcal{H}(L_2(\Pi_a), W^2_2(Q))$ for each $Q \in \Xi$, in view of the form of the coefficients of the operator $L_\varepsilon$ (see (2.3)) we conclude that $L_\varepsilon T_6(\cdot) \in \mathcal{H}(L_2(\Pi_a), L_2(\Pi_a))$. Moreover, the operator $L_\varepsilon T_6(k)$ is bounded uniformly on $\varepsilon$. 

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The question on existence of the eigenvalues of the operator $H_l$ emerging from the threshold of the essential spectrum is surely to be equivalent to the question on existence of the function $k = k_\varepsilon \to 0$ so that the equation (4.1) have a nontrivial solution $g_\varepsilon$ such that $T_6(k_\varepsilon)g_\varepsilon \in L_2(\Pi)$. This is why it is sufficient to study the question on existence of such function $k_\varepsilon$.

**Lemma 4.1.** Let for $l = l_*$ there exist no solution $\phi$ described in Item 2 of Theorem 1.1. Then there exist $\varepsilon_0 > 0$ and $\delta > 0$ such that as $|l - l_*| < \varepsilon_0$ the operator $H_l$ has no eigenvalues in an interval $(1 - \delta, 1 + \delta)$.

**Proof.** We begin with the case $d < \pi$. In accordance with this assumption and Lemma 3.7 the operator $(I + T_7(k))$ is invertible for each $k \in B$. Since the operator $L_\varepsilon T_6(k)$ is bounded uniformly on $\varepsilon$ and $k \in B$, for sufficiently small $\varepsilon$ the operator in (4.1) is also invertible for each $k \in B$. Therefore, the equation (4.1) has no nontrivial solutions.

In the case $d = \pi$ the proof is analogous. The set $\Pi$ being symmetric w.r.t. the axis $x_2 = 0$, all the eigenfunctions of the operator $H_l$ are even on $x_2$, since odd eigenfunctions would satisfy Dirichlet condition on $x_2 = 0$ and would be the eigenfunctions of the operator $H_0$. At the same time, the discrete spectrum of the latter is empty. Taking into account the parity of the eigenfunctions on $x_2$, it is sufficient to consider the equation (4.1) on even on $x_2$ functions $g$ only (clear, the operator $T_6(k)$ preserves the parity on $x_2$). We denote by $V$ the subspace of $L_2(\Pi_a)$ consisting of even on $x_2$ functions. Then the operator $(I + T_7(k))^{-1}L_\varepsilon T_6(k) : V \to L_2(\Pi_a)$ is bounded uniformly on $\varepsilon$ (see Lemma 3.7). Using this fact, one can easily deduce the absence of nontrivial even on $x_2$ solution of the equation (4.1).

Now we are going to prove that the existence of nontrivial solution $\phi$ from Lemma 3.7 for $l = l_*$ implies the existence of the function $k = k_\varepsilon \to 0$ so that the equation (4.1) have a nontrivial solution. We are also going to show that the function $k_\varepsilon$ meets the equality

$$k_\varepsilon = \varepsilon \mu + O(\varepsilon^2),$$

(4.2)

where $\mu$ is defined by the formulas (1.6), (1.7) with $l_n$ and $\phi_n$ replaced by $l_*$, $\phi$, respectively. We adduce the proof in the case $d < \pi$ only; the case $d = \pi$ was proven in [8].

In the equation (4.1) we invert the operator $(I + T_7(k))$ taking into account Lemma 3.7

$$g - \frac{\varepsilon}{2k} \int_{\Pi} \phi L_\varepsilon T_6(k)g dy + \varepsilon T_9(k)L_\varepsilon T_6(k)g = 0.$$

The operator $T_9(k)L_\varepsilon T_6(k)$ is bounded uniformly on $\varepsilon$, this is why for sufficiently small $\varepsilon$ there exists a bounded inverse $T_{10}(k, \varepsilon) := (I + \varepsilon T_9(k)L_\varepsilon T_6(k))^{-1}$. Applying
this operator to the latter equation, we obtain:
\[
g - \frac{\varepsilon}{2k} \left( \int_{\Pi} \phi L_\varepsilon T_6(k) g \, dy \right) T_{10}(k, \varepsilon) \Phi = 0.
\] (4.3)

If the integral in the right-hand side is zero, it immediately leads us to the trivial solution \(g = 0\). Therefore, on a nontrivial solution this integral is nonzero. Bearing in mind this fact, we apply the operator \(L_\varepsilon T_6(k)\) to the equation (4.3), multiply then by \(2k\Phi\) and integrate over \(\Pi\). This results in the following equation:
\[
2k - \varepsilon \int_{\Pi} \phi L_\varepsilon T_6(k) T_{10}(k, \varepsilon) \Phi \, dy = 0.
\] (4.4)

In fact, this is an equation for the function \(k = k_\varepsilon\). Due to (4.3) the corresponding nontrivial solution of the equation (4.1) is given by the formula:
\[
g_\varepsilon = CT_{10}(k_\varepsilon, \varepsilon) \Phi,
\] (4.5)

where \(C\) is an arbitrary constant. The function
\[
(k, \varepsilon) \mapsto \varepsilon \int_{\Pi} \phi L_\varepsilon T_6(k) T_{11}(k, \varepsilon) \Phi \, dy
\]
is holomorphic on \(k\) and tends to zero as \(\varepsilon \to 0\) uniformly on \(k\). Therefore, on the boundary of the domain \(B\) it will be less by absolute value than \(2|k|\) if \(\varepsilon\) is small enough. By Rouche theorem it follows that for sufficiently small \(\varepsilon\) the equation (4.1) has the same number of roots in \(B\) as the number of zeros for the function \(k \mapsto 2k\), i.e., the unique root. We denote this root by \(k_\varepsilon\). Clear, the convergence \(T_{10}(k, \varepsilon) \xrightarrow{\varepsilon \to 0} I\) holds true in the operator norm uniformly on \(k \in B\). In view of the equality \(\phi = T_6(0)\Phi\) and (2.3) it allows to the rewrite the equation (4.4) as
\[
k_\varepsilon = \frac{\varepsilon}{2} \int_{\Pi} \phi L_0 \phi \, dy + \mathcal{O}(\varepsilon|k_\varepsilon| + \varepsilon^2),
\]

\[
L_0 := -2\chi_1'(y_1) \frac{\partial^2}{\partial y_1^2} - \chi_1''(y_1) \frac{\partial}{\partial y_1}.
\]

Here we have also taken into account the form of the coefficients of the operator \(L_\varepsilon\). Since \(k_\varepsilon \to 0\), the equalities obtained imply that \(k_\varepsilon = \mathcal{O}(\varepsilon)\). Hence,
\[
k_\varepsilon = \frac{\varepsilon}{2} \int_{\Pi} \phi L_0 \phi \, dy + \mathcal{O}(\varepsilon^2).
\] (4.6)
Lemma 4.2. In a vicinity of the right edge of the window $\gamma_1$ the function $\phi(x)$ behaves as:

$$\phi(y) = \alpha r^{1/2} \sin \frac{\theta}{2} + O(r), \quad \partial_{y_i} \phi(y) = \alpha \partial_{y_i} r^{1/2} \sin \frac{\theta}{2} + O(1), \quad r \to 0, \quad (4.7)$$

where $(r, \theta)$ are polar coordinates centered at the right edge of the window $\gamma_1$, $\alpha$ is some constant.

Proof. Let $\chi_4 = \chi_4(r)$ be an infinitely differentiable cut-off function which equals one as $r \leq \delta$ and vanishes as $r \geq 2\delta$. We denote $\Theta := \{ y : r < 2\delta, 0 < \theta < 2\pi \}$. We choose the number $\delta$ so that the circle $\Theta$ not to intersect the left edge of the window $\gamma_1$ and lie inside $\Pi$. As it was mentioned in Remark [14] $\phi \in C^\infty(\Pi)$. Taking into account this fact, one can easily check that the function $\tilde{\phi}(y) = \chi_4(r)\phi(y) \in W^1_2(\Theta, \partial\Theta)$ is a solution to the boundary value problem:

$$-\Delta \tilde{\phi} = \lambda \tilde{\phi} + \tilde{f}, \quad y \in \Theta, \quad \tilde{\phi} = 0, \quad y \in \partial\Theta,$$

where $\tilde{f} \in C^\infty(\Theta)$, $\tilde{f} \equiv 0$ as $r \leq \delta$. Changing variables in this problem $\tilde{r} = r^{1/2}$, $\tilde{\theta} = \theta/2$, we arrive at the following boundary value problem:

$$-\Delta_{\tilde{r}} \tilde{\phi} = 4\lambda \tilde{r}^2 \tilde{\phi} + 4\tilde{r}^2 \tilde{f}, \quad \tilde{y} \in \tilde{\Theta}, \quad \tilde{\phi} = 0, \quad \tilde{y} \in \partial\tilde{\Theta},$$

where $\tilde{y}$ are Cartesian coordinates associated with $(\tilde{r}, \tilde{\theta})$, $\tilde{\Theta} := \{ \tilde{y} : \tilde{r} < \sqrt{2\delta}, \tilde{\theta} \in (0, \pi) \}$. Since $\tilde{\phi} \in W^1_2(\tilde{\Theta})$,

$$\int_{\tilde{\Theta}} |\nabla_{\tilde{y}} \tilde{\phi}|^2 \, d\tilde{y} = \int_{\tilde{\Theta}} |\nabla_{\tilde{y}} \tilde{\phi}|^2 \, d\tilde{y} < \infty$$

and the function $\tilde{\phi}$ vanishes at the boundary of the domain $\tilde{\Theta}$, the belonging $\tilde{\phi} \in W^1_2(\tilde{\Theta}, \partial\tilde{\Theta})$ holds true. We continue the functions $\tilde{f}$ and $\tilde{\phi}$ into the domain $\{ y : \tilde{r} < 2\delta, \pi < \tilde{\theta} < 2\pi \}$ as follows: $\tilde{\phi}(\tilde{y}_1, \tilde{y}_2) = -\tilde{\phi}(-\tilde{y}_1, \tilde{y}_2)$, $\tilde{f}(\tilde{y}_1, \tilde{y}_2) = -\tilde{f}(-\tilde{y}_1, \tilde{y}_2)$, $\tilde{y}_2 < 0$. We preserve former notations $\phi$ and $f$ for the functions continued. We denote $\tilde{\Omega} := \{ \tilde{y} : \tilde{r} < \sqrt{2\delta} \}$. It is clear that $\tilde{f} \in L^2(\tilde{\Omega})$, $\tilde{f} \equiv 0$ as $r \leq \sqrt{\delta}$, and the function $\tilde{\phi} \in W^1_2(\tilde{\Omega}, \partial\tilde{\Omega})$ is a solution to the boundary value problem:

$$-\Delta \tilde{\phi} = 4r^2 \tilde{\phi} + \tilde{f}, \quad \tilde{y} \in \tilde{\Omega}, \quad \tilde{\phi} = 0, \quad \tilde{y} \in \partial\tilde{\Omega}.$$ 

Due to the theorems on improving smoothness of solutions to elliptic boundary value problems (see [14] Ch. 4, §2] the function $\tilde{\phi}$ is infinitely differentiable at zero. Moreover, in view of the boundary value problem for $\tilde{\phi}$ we have:

$$\tilde{\phi}(\tilde{y}) = \alpha \tilde{y}_2 + O(\tilde{r}^2), \quad \partial_{\tilde{y}_i} \tilde{\phi}(\tilde{y}) = \alpha \partial_{\tilde{y}_2} + O(\tilde{r}), \quad \tilde{r} \to 0.$$ 

Returning now to the variables $y$ and taking into account the definition of the function $\chi_4$, we arrive at the statement of the lemma. \qed
Remark 4.1. The idea of the proof of Lemma 4.2 is borrowed from the proof of Lemma 3.1 in [22, Ch. III, §2].

The function $\phi$ has a definite parity on $x_1$, this is why the formulas similar to (4.7) are also valid for the left edge of the window $\gamma_l$. Taking into account the behaviour of the function $\phi(x)$ in the vicinities of the edges of the window $\gamma_l$ and the boundary value problem for $\phi$, we can evaluate the integral in (4.6) by integrating by parts twice:

$$\int_{\Pi} \phi L_0 \varphi dy = - \int_{\Pi} \phi (\Delta + 1) \left( \chi(y_1) \frac{\partial \phi}{\partial y_1} \right) dy = \pi \alpha^2. \tag{4.8}$$

In the same way we check that

$$0 = \int_{\Pi} y_1 \frac{\partial \phi}{\partial y_1} (\Delta + 1) \phi dy = \pi l_* \alpha^2 + 2 \int_{\Pi} \phi \frac{\partial^2 \phi}{\partial y_1} dy = \pi l_* \alpha^2 - 2 \int_{\Pi} \left| \frac{\partial \phi}{\partial y_1} \right|^2 dy, \tag{4.9}$$

what together with (4.8) imply the equality:

$$\int_{\Pi} \phi L_0 \varphi dy = \frac{2}{l_*} \int_{\Pi} \phi \frac{\partial^2 \phi}{\partial y_1} dy = 2\mu, \tag{4.10}$$

where $\mu$ is defined by the formula (1.6) with $l_n$ and $\phi_n$ replaced by $l_*$ and $\phi$. Substituting the relations (4.10) into (4.6), we arrive at the asymptotics (4.2).

We put $C = 1$ in (4.5), then in view of the form of the operator $T_{10}(k, \varepsilon)$ the obtained solution of the equation (4.1) satisfies an asymptotic formula:

$$g_{\varepsilon} = \Phi + O(\varepsilon) \quad \text{in the norm of } L_2(\Pi_a). \tag{4.11}$$

Due to Lemma 3.3 and relation $\Phi \neq 0$ this equality implies that the function $\psi^\varepsilon(y) = (T_0(k_\varepsilon)g_{\varepsilon})(y)$ is not identically zero. Therefore, it is an eigenfunction of the boundary value problem (2.3). Due to the asymptotics representations (4.2), (4.2) and (4.10), the inequality $\text{Re} k_\varepsilon > 0$ holds true only as $\varepsilon > 0$. Hence, the function $\psi^\varepsilon$ is an element of $L_2(\Pi)$ only as $\varepsilon > 0$. Passing to the variables $x$ (see (2.2)), we conclude that a quantity $\lambda^\varepsilon := 1 - k_\varepsilon^2$ is an eigenvalue of the operator $H_{l^* + \varepsilon}$ only as $\varepsilon > 0$, and $\psi^\varepsilon(y(x, \varepsilon))$ is the associated eigenfunction in this case. As $\varepsilon \leq 0$ the operator $H_{l^* + \varepsilon}$ has no eigenvalues close to the threshold of the essential spectrum, i.e., the eigenvalue $\lambda^\varepsilon$ disappears as $\varepsilon \leq 0$. Therefore, $l_*$ is a critical value of the length of the window $\gamma_l$, and the corresponding eigenvalue emerging as $l > l_*$ has the asymptotics (1.5), (1.6), what follows from (4.2) and (4.10). We assume that $l_* = l_n$, then $\lambda_n = \lambda^\varepsilon$, $\psi_n(x) = c_\varepsilon \psi^\varepsilon(y(x, \varepsilon))$, where $c_\varepsilon$ is a some constant.

To finish the proof we need just to establish the relationships (1.8), (1.9). The functions $\psi_n$ and $\phi_n$ having the same parity on $x_1$ follows from (1.9) and Item 4 of Theorem 1.1.
The function $\psi_n(x)$ introduced above meets the asymptotics formulas \ref{2.2}, where $c_+(k_\varepsilon) = c_\varepsilon \left(\frac{2}{\sqrt{\pi}} + O(\varepsilon)\right)$. This equality is implied by \ref{1.11}, the definition of the operator $T_0(k)$ and the functions $\phi$ and $\Phi$, the asymptotics for $k_\varepsilon$ established above, and the equality $y_1(x_1, \varepsilon) \equiv x_1$ as $x_1$ large enough. Thus, the constant $c_\varepsilon$ can be chosen such that the function $\psi_n$ to satisfy the asymptotics expansion \ref{1.8}. Moreover, in this case we have $c_\varepsilon = 1 + O(\varepsilon)$. This equality and \ref{1.11} yield:

$$
\psi_n(x) = \phi_n(y(x, \varepsilon)) + O(\varepsilon)
$$

in the norm of $W^1_2(\Pi_R)$ for each $R > 0$ (the norm here is treated in the sense of the variables $x$). Thus, in order to prove the assertions \ref{1.8} it is sufficient to check that

$$
\|\phi_n(x) - \phi_n(y(x, \varepsilon))\|_{W^1_2(\Pi_R)} = O(\varepsilon^{1/2})
$$

for each $R > 0$. Clear, it is sufficient to check this equality only as $R > l$. In the domain $\Pi_R$ we select two rectangles $P_\pm := \{x : \pm x_1 \in (l - 2\varepsilon_0, l + 2\varepsilon_0), -d < x_2 < \pi\}$, where $\varepsilon_0$ is from \ref{2.2}. We choose $\varepsilon_0$ small enough so that the minimal eigenvalue of Dirichlet Laplacian in the rectangles be greater than one. We denote this eigenvalue by $\tau$. We set $\varphi(x) := \phi_n(x) - \phi_n(y(x, \varepsilon))$. The function $\varphi$ is an element of $W^1_2(P_\pm, \partial P_\pm)$, what implies the estimate:

$$
\tau \|\varphi\|^2_{L_2(P_\pm)} \leq \|\nabla \varphi\|^2_{L_2(P_\pm)}.
$$

Integrating by parts in the equality

$$
\int_{P_+} \varphi(x) \left((\Delta_x + 1)\varphi(x) + \varepsilon L_\varepsilon \phi_n(y)\right) \, dx = 0,
$$

in view of properties of $\phi_n$ we obtain

$$
\|\nabla \varphi\|^2_{L_2(P_+)} = \|\varphi\|^2_{L_2(P_+)} + \varepsilon \int_{P_+} \varphi L_\varepsilon \phi_n(y) \, dx +
\int_{l_+ - \varepsilon}^{l_+} \phi_n(x_1,0) \left(\frac{\partial}{\partial x_2} \phi_n(y_1(x_1,\varepsilon), -0) - \frac{\partial}{\partial x_2} \phi_n(y_1(x_1,\varepsilon), +0)\right) \, dx_1.
$$

The latter term in the right-hand side of this equality can be estimated taking into account \ref{1.7}:

$$
\left|\int_{l_+ - \varepsilon}^{l_+} \phi_n(x_1,0) \left(\frac{\partial}{\partial x_2} \phi_n(y_1(x_1,\varepsilon), -0) - \frac{\partial}{\partial x_2} \phi_n(y_1(x_1,\varepsilon), +0)\right) \, dx_1\right| \leq C_1 \varepsilon,
$$

where $C_1$ is a some constant independent on $\varepsilon$. The second term in the right-hand side of \ref{4.14} is estimated as follows:

$$
\left|\int_{P_+} \varphi L_\varepsilon \psi_n(y) \, dx\right| \leq C_2.
$$
where $C_2$ is a some constant independent on $\varepsilon$. In order to establish the latter estimate one just needs to take into account the smoothness of the function $\phi_n(y(x, \varepsilon))$ as well as the coefficients of the operator $L_\varepsilon$ being separated from the window $\gamma_l$, by a positive distance uniformly on $\varepsilon$. Substituting two last estimates into (4.14), we get
\[
\|\nabla \varphi\|^2_{L^2(P_\pm)} \leq C_1 \varepsilon + C_2 \varepsilon \|\varphi\|_{L^2(P_\pm)} + \|\varphi\|_{L^2(P_\pm)}^2.
\] (4.15)
Similar estimate is valid for $P_-$ as well.

Bearing in mind the obtained estimates for $\|\nabla \varphi\|_{L^2(P_\pm)}$, by (4.13) we deduce:
\[
(\tau - 1) \|\varphi\|_{L^2(P_\pm)}^2 - C_2 \varepsilon \|\varphi\|_{L^2(P_\pm)} - C_1 \varepsilon \leq 0.
\]
Solving this square inequality with the relations $\tau > 1$ and (4.15) taken into account, we get:
\[
\|\varphi\|_{L^2(P_\pm)} = O(\varepsilon^{1/2}), \quad \|\nabla \varphi\|_{L^2(P_\pm)} = O(\varepsilon^{1/2}).
\]
To finish the proof of (4.12) it is sufficient now to note that the estimate
\[
\|\varphi\|_{W^1_2(\Pi_R \setminus (P_+ \cup P_-))} = O(\varepsilon)
\]
holds true due to the definition of the function $\varphi$ and the function $\phi_n$ being infinitely differentiable on the set $\Pi_R \setminus (P_+ \cup P_-)$. The proof of Theorem 1.1 is complete.

5 Asymptotics expansions of the eigenvalues as $l \to +\infty$

This section is devoted to the proof of Theorem 1.2. We begin with the proof of Item 1.

In accordance with Theorem 1.1 all the eigenfunction of the operator $H_l$ have a certain parity on $x_1$. Thus, we may bisect the set $\Pi$ by a segment $\{0\} \times [-d, \pi]$ and impose on it Dirichlet or Neumann condition subject to the parity of an eigenfunction studied. Hence, we just to need to deal with the eigenvalue problem for the Laplacian in the right half of $\Pi$. In this problem we make the change of the variables by the rule $x_1 \mapsto x_1 - l$ what leads us to the problem on the spectrum of the Laplacian in a domain $\Pi^{*l} := \Pi \cap \{x_1 > -l\}$ subject to appropriate boundary conditions. Below it will be shown that as $l \to +\infty$ such problem can be treated as a problem on perturbation of the operator $H_\ast$ defined in the first section. It will allow us to get the needed asymptotics expansions (1.11).

First we study the behaviour of the operator $(H_\ast - \lambda)^{-1}$ as $\lambda$ close to $\kappa := \frac{\pi}{\pi + \delta}$. In order to do it we will employ the same approach as that used in the third section.
We set \( \Omega^*_b := \Pi^* \cap \{ x : |x_1| < a \}, \lambda = \kappa^2 + k^2 \). For small complex \( k \in B \) we consider the boundary value problem

\[
- \Delta u = (\kappa^2 + k^2)u + f, \quad x \in \Pi^*, \quad u = 0, \quad x \in \partial \Pi^*, \quad (5.1)
\]

\[
u(x, k) = c(k) e^{i k x_1 \sin \kappa (x_2 - \pi)} + O(-e^{-\sqrt{3} \kappa^2 - k^2 x_1}), \quad x_1 \to -\infty,
\]

\[
u(x, k) = O(e^{-\sqrt{\frac{\pi^2}{\kappa^2} - \kappa^2 - k^2 x_1}}), \quad x_1 \to +\infty, \quad x_2 \in (0, \pi), \quad (5.2)
\]

\[
u(x, k) = O(e^{-\sqrt{\frac{\pi^2}{\kappa^2} - k^2 x_1}}), \quad x_1 \to +\infty, \quad x_2 \in (-d, 0).
\]

Here \( f \in L_2(\Pi^*) \) is a function whose supports lies in \( \Pi^*_a \), \( a > 0 \) is a some fixed number, \( c(k) \) is a some constant. Let \( g \) be a function from \( L_2(\Pi^*_a) \) continued by zero in \( \Pi^* \setminus \Pi^*_a \). We denote \( \Omega_0 := \Pi^* \cap \{ x : x_1 < 0 \} \). The boundary value problems

\[
- \Delta v_i = (\kappa^2 + k^2)v_i + g, \quad x \in \Omega_i, \quad v_i = 0, \quad x \in \partial \Omega_i, \quad i = 0, 1, 2, \quad (5.3)
\]

are solved by separation of variables:

\[
v_i(x, k) = \sum_{j=1}^{\infty} \int_{\Omega_i} G_j^i(x, t, k) g(t) dt, \quad (5.4)
\]

\[
G_j^0(x, k) := \frac{1}{(\pi + d)s_j} \left( e^{-s_j|x_1-|t_1|} - e^{s_j|x_1+|t_1|} \right) \sin j \kappa (x_2 - \pi) \sin j \kappa (t_2 - \pi),
\]

\[
G_j^1(x, k) := \frac{1}{\pi s_j} \left( e^{-s_j|x_1-|t_1|} - e^{s_j|x_1+|t_1|} \right) \sin j x_2 \sin j t_2,
\]

\[
G_j^2(x, k) := \frac{1}{s_j d} \left( e^{-s_j|x_1-|t_1|} - e^{s_j|x_1+|t_1|} \right) \sin \frac{\pi j}{d} x_2 \sin \frac{\pi j}{d} t_2,
\]

where \( s_j^0 = ik, \ s_j^0 = \sqrt{\kappa^2 j^2 - \kappa^2 - k^2}, \ j \geq 2, \ s_j^1 = \sqrt{j^2 - \kappa^2 - k^2}, \ s_j^2 = \sqrt{\frac{\pi^2}{\kappa^2} - \kappa^2 - k^2} \). As \( k = 0 \) the function \( G_j^0 \) is defined by continuity:

\[
G_j^0(x, k) := -\frac{1}{(\pi + d)s_j} (|x_1-|t_1| + x_1 + t_1) \sin \kappa (x_2 - \pi) \sin \kappa (t_2 - \pi).
\]

We set \( \Omega_b^0 := \Omega_0 \cap \Pi_b \). An analogue of Lemma 3.1 holds true.

**Lemma 5.1.** Let \( b > 0 \). The series (5.4) converge in the norm of \( W_2^2(\Omega_b^0) \). The functions \( v_i(x) \) meet the asymptotics formulas (5.2). The mapping \( g \mapsto v_i \) are linear bounded operators from \( L_2(\Pi^*_a) \) into \( W_2^2(\Omega_b^0) \) as functions on \( k \) belonging to \( \mathcal{H}(L_2(\Pi^*_a), W_2^2(\Omega_b^0)) \).

Let \( v(x, k) := v_i(x, k), \ x \in \Omega_i \). We introduce the function \( w(x, k) \) as a solution to a boundary value problem

\[
\Delta w = \Delta v, \quad x \in \Pi^*_a, \quad w = v, \quad x \in \partial \Pi^*_a, \quad (5.5)
\]
Here $\Delta v$ is treated in the same sense as in (3.7). We denote $\Gamma_a^* := \partial \Pi_a^* \cap \{ x : |x| < a \}$. The function $w$ can be regarded as $w = T_{11}v$, where $T_{11} : H^1(\Omega^*_a, \partial \Omega^*_a) \to H^1(\Pi^*_a, \Gamma^*_a)$ is a linear bounded operator. Moreover, the operator $T_{11}$ is bounded as an operator from $H^1(\Omega^*_a, \partial \Omega^*_a \cap \partial \Omega_a)$ into $W^2(Q)$ for each $Q \in \Xi^*$, where $\Xi^*$ is a subset of all bounded subdomains of $\Pi_*$ having smooth boundary and separated from zero by a positive distance. Let $\chi_5(x_1)$ be an infinitely differentiable cut-off function which equals one as $|x_1| < a/3$ and vanishes as $|x_1| > 2a/3$. We define the function $u(x,k)$ by a rule:

$$u(x,k) := w(x,k)\chi_5(x_1) + v(x,k)(1 - \chi_5(x_1)).$$  \hfill (5.6)

The function $u$ is treated as a value of a linear operator $T_{12}(k)g$ defined by a rule $T_{12}(k)g := u$. The operator $T_{12} : L_2(\Pi^*_a) \to W^2(\Pi^*_a, \Gamma^*_a)$, $T_{12} : L_2(\Pi^*_a) \to W^2(Q)$ is bounded for all $b > 0$ and each $Q \in \Xi^*$. Moreover, $T_{12} \in \mathcal{H}(L_2(\Pi^*_a), W^2(\Pi^*_a, \Gamma^*_a))$ and $T_{12} \in \mathcal{H}(L_2(\Pi^*_a), W^2(Q))$ for all $b > 0$ and each $Q \in \Xi^*$.

By analogy with the deriving the equation (3.10) it can be shown that the function $u$ from (5.6) is a solution to the boundary value problem (5.1), (5.2), if $u$ is a solution to the equation

$$g + T_{13}(k)g = f,$$  \hfill (5.7)

where

$$T_{13}(k)g := (v - w) \left( \Delta + x^2 + k^2 \right) \chi_5(x_1) + 2(\nabla_x \chi_5, \nabla_x (v - w))_{\mathbb{R}^2}.$$

By analogy with Lemmas 3.3, 3.5 one can establish the following statement.

**Lemma 5.2.** The operator $T_{13}(k)$ is a linear compact operator from $L_2(\Pi^*_a)$ into $L_2(\Pi^*_a)$ for all $k \in B$ and $T_{13}(\cdot) \in \mathcal{H}(L_2(\Pi^*_a), L_2(\Pi^*_a))$. For each $k \in B$ the equation (5.7) is equivalent to the boundary value problem (5.1), (5.2). Namely, for each solution of the equation (5.7) the function $u = T_{12}(k)g$ is a solution to the boundary value problem (5.1), (5.2), and for each solution $u$ to the boundary value problem (5.7), (5.2) there exists the unique solution $g$ of the equation (5.7) related with $u$ by the equality $u = T_{12}(k)g$. The belonging $(I + T_{13}(\cdot))^{-1} \in \mathcal{M}(L_2(\Pi^*_a), L_2(\Pi^*_a))$ holds true.

As in the third section, we are interesting in the behaviour of the operator $(I + T_{13}(k))^{-1}$ for small $k$, namely, we are interested in the presence of the pole at the point $k = 0$. As the next statement shows, in distinction to Lemma 3.7, here the answer is always negative.

**Lemma 5.3.** If the vicinity $B$ of the zero is small enough, then $(I + T_{13}(\cdot))^{-1} \in \mathcal{H}(L_2(\Pi^*_a), L_2(\Pi^*_a))$. 

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Proof. Clear, it is sufficient to prove the absence of the pole of the operator \((I + T_{13}(k))^{-1}\). The presence of pole is equivalent to the existence of a nontrivial solution of the equation (5.7) as \(k = 0, f = 0\). The latter is equivalent to the presence of nontrivial solution to the boundary value problem (5.1) meeting the asymptotics formulas (5.2). Suppose that there exists such solution to the boundary value problem (5.1) and denote it by \(U(x)\). The function \(U\) can be chosen being real-valued. Moreover, at the point \(x = 0\) the function \(U\) possess the following asymptotic behaviour

\[
U(x) = \alpha r^{1/2} \sin \frac{\theta}{2} + O(r), \quad \frac{\partial}{\partial x_i} U(x) = \alpha \frac{\partial}{\partial x_i} r^{1/2} \sin \frac{\theta}{2} + O(r), \quad r \to 0,
\]

where \((r, \theta)\) are polar coordinates associated with \(x\). These asymptotics representations can be proven in analogy with Lemma 1.2. Integrating by parts and taking into account these asymptotics and (5.2), we obtain

\[
0 = \int_{\Pi^*} x_1 U(\Delta + x^2) \frac{\partial U}{\partial x_1} \, dx = 2 \int_{\Pi^*} \left| \frac{\partial U}{\partial x_1} \right|^2 \, dx.
\]

This implies that the function \(U\) is independent on \(x_1\), what in view of the asymptotics representations (5.2) taken as \(x_1 \to +\infty\) leads us to the equality \(U = 0\). The proof is complete.

As it was mentioned in the beginning of the section, the eigenvalues of the operator \(H_l\) coincides with those of a pair of boundary value problems

\[
-\Delta \Psi = \lambda \Psi, \quad x \in \Pi^*; \quad \Psi = 0, \quad x \in \partial \Pi^* \setminus K_l, \quad pu = 0, \quad x \in K_l, (5.8)
\]

where \(K_l := \{ x : x_1 = -l, x_2 \in (-d, 0) \}\), \(p\) is a boundary operator which is \(pu = u\) or \(pu = \frac{\partial u}{\partial x_1}\). As \(x_1 \to \pm \infty\) a function \(\Psi\) is assumed to meet the asymptotics representations (5.2) with \(k = \sqrt{\lambda - \kappa^2}\). The eigenfunctions of the operator \(H_l\) are related with those of the boundary value problem (5.8) by the equalities \(\Psi_m(x) = \psi_m(x_1 + l, x_2), x_1 > -l\), this is why the boundary operator \(p\) gives the Dirichlet condition in the case of odd on \(x_1\) functions \(\psi_m(x)\) and Neumann condition in the case of even on \(x_1\) functions \(\psi_m(x)\).

Our main aim at this stage is to reduce the boundary value problem (5.8) to an operator equation similar to (5.7). In order to do it we again employ the approach which allowed us to get the equation (5.7). We start with the case \(pu = u\). Suppose that \(l > a\). We define the function \(v_0^l\) as a solution to the boundary value problem

\[
-\Delta v_0^l = (x^2 + k^2)v_0^l, \quad x \in \Omega_l^l, \\
v_0^l = 0, \quad x \in \partial \Omega_l^l \setminus K_l, \quad v_0^l = -v_0, \quad x \in K_l.
\]

A solution of such problem in view of the formula (5.4) is of the form

\[
v_0^l(x, k) = \sum_{j=1}^{\infty} \frac{e^{-s_j^l t}}{\sinh s_j^l} \beta_j(k)[g] \sinh s_j^l x_1 \sin x_2 - \pi), \quad (5.9)
\]

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\[
\beta_j(k)[g] = -\frac{2}{(\pi + d)s_j^0} \int_{\Omega_0} g(x) \sinh s_j^0 x_1 \sin j\xi(x_2 - \pi) \, dx.
\]

The first term of this series contains the function \(\sinh s_j^0 l = i \sin kl\) in the denominator. This function vanishes as \(kl = \pi q, \, q \in \mathbb{Z}\). At the same time, in accordance with Item 1 of Theorem 1.1 the values \(k\) corresponding to the eigenvalues of the operator \(H_l\) lie strictly inside the intervals \((\frac{\pi(m-1)}{2l}, \frac{\pi m}{2l})\). This is why the values \(k = \frac{\pi m}{2l}\) are excluded from the consideration what allows us to avoid indefiniteness in (5.9).

By analogy with Lemma 3.1 one can prove that the series (5.9) converge in the norm of \(W_2^0(\Omega_0^l)\). We set \(v^l(x, k) := v_0^l(x, k), \, -l < x_1 < 0, \, v^l(x, k) := 0, x_1 > 0\). We define the function \(w^l\) as the solution to the boundary value problem (5.8) with the function \(v^l\) in the right-hand side, i.e., \(w^l = T_{11} v^l\). A solution to the boundary value problem (5.8) is sought as

\[
\Psi(x) := (w(x) + w^l(x)) \chi_5(x_1) + (1 - \chi_5(x_1)) (v(x) + v^l(x)), \quad (5.10)
\]

where \(v(x), w(x)\) are from (5.3), (5.5). The function \(\Psi(x)\) satisfies the boundary conditions (5.8), meets the asymptotics formulas (5.2) as \(x_1 \to +\infty\) and is a solution of the equation in (5.8), if the operator equation

\[
g + T_{13}(k)g + T_{14}(k, l)g = 0, \quad (5.11)
\]

holds true, where the operator \(T_{14}(k, l)\) is defined by a rule:

\[
T_{14}(k, l)g := (v^l - w^l) (\Delta + \xi^2 + k^2) \chi_5(x_1) + 2 (\nabla \chi_5, \nabla (v^l - w^l))_{\mathbb{R}^2}.
\]

The equation (5.11) is equivalent to the boundary value problem (5.8), what can be proved by analogy with Lemma 3.1.

Let \(k = k(l)\) correspond to an eigenvalue \(\lambda_m(l)\) of the operator \(H_l\) by the rule \(\lambda_m(l) = \xi^2 + k^2(l)\), and a corresponding solution \(g\) of the equation (5.11) generates an eigenfunction \(\Psi_m\) in accordance with (5.10). It follows from Lemma 1.1 that each eigenvalue of the operator \(H_l\) tends to \(\xi\) as \(l \to +\infty\), i.e., \(k(l) \to +\infty\) as \(l \to +\infty\).

Therefore, choosing \(l\) great enough, we can always make \(k(l)\) to belong \(B\) for \(l\) great enough. In what follows the value \(l\) is assumed to be chosen in such a way.

We denote \(\hat{v}(x, k) := \sin kx_1 \sin \xi(x_2 - \pi), \, x_1 < 0, \, \hat{v}(x, k) := 0, x_1 > 0, \hat{w} := T_{11} \hat{v}, \hat{F} := (\hat{v} - \hat{w}) (\Delta + \xi^2 + k^2) \chi_5 + 2 (\nabla \chi_5, \nabla (\hat{v} - \hat{w}))_{\mathbb{R}^2}\).

**Lemma 5.4.** As \(k\) small enough the operator \(T_{14}\) can be represented as:

\[
T_{14}(k, l) = T_{15}(k, l) + T_{16}(k, l), \quad T_{15}(k, l)g = \frac{e^{-ikl}}{\sin kl} \beta_0(k)[g] \hat{F},
\]

where the operator \(T_{16}(k, l) \in \mathcal{L}(L_2(\Omega_0^k), L_2(\Omega_0^k))\) obeys an estimate:

\[
\|T_{16}\| \leq C e^{-2\sqrt{3}\xi - \delta} l.
\]

Here \(C, \delta\) are some constants independent on \(l\), \(0 < \delta < 2\sqrt{3}\xi\).
The statement of this lemma follows easily from the definition of the function \( v_l \) (see (5.9)) and the definition of the operator \( T_{14} \).

In view of Lemma 5.4 the equation (5.11) can be rewritten as:

\[
g + T_{13}(k)g + T_{15}(k, l)g + T_{16}(k, l)g = 0.
\] (5.12)

The operator \((I + T_{13}(k))\) has the bounded inverse due to Lemma 5.3 and the operator \( T_{16}(k, l) \) is exponentially small as \( l \to +\infty \) in view of Lemma 5.4. Therefore, the operator \((I + T_{13}(k) + T_{16}(k, l))^{-1}\) has also the bounded inverse for \( l \) large enough. We apply this operator to (5.12), what results in:

\[
g + \frac{e^{-ikl}}{\sin kl} \beta_0(k)[g](I + T_{13}(k) + T_{16}(k, l))^{-1} \hat{F} = 0.
\] (5.13)

It is clear that \( \beta_0(k)[g] \neq 0 \), since otherwise it would follow from the equation obtained that \( g = 0 \), while \( g \) corresponds to the eigenfunction \( \Psi_m \). Applying now the functional \( \beta_0(k)[g] \) to (5.13), we arrive at the equation

\[
1 + \frac{e^{-ikl}}{\sin kl} \beta_0(k) \left[ (I + T_{13}(k) + T_{16}(k, l))^{-1} \hat{F} \right] = 0.
\] (5.14)

Directly from the definition of the function \( \hat{F} \) and Lemmas 5.3, 5.4 it follows the equality

\[
\beta_0(k) \left[ (I + T_{13}(k) + T_{16}(k, l))^{-1} \hat{F} \right] = ck + O \left( k^2 + e^{-(2\sqrt{3}c-\delta)l} \right),
\] (5.15)

where \( c \) is a some constant, \( \delta \) is the same as in Lemma 5.4. Since \( k = k(l) \) corresponds to the eigenvalue \( \lambda_m \), from Lemma 1.1 it follows that \( k(l) = \mathcal{O}(l^{-1}) \) as \( l \to +\infty \). Taking into account this equality and the realness of \( k \), we substitute (5.15) into (5.14):

\[
\sin kl = \mathcal{O}(l^{-1}),
\] (5.16)

what implies

\[
kl = \pi q + \mathcal{O}(l^{-1}), \quad q \in \mathbb{Z}.
\] (5.17)

In view of Item 4 of Theorem 1.1 the index \( m \) of the eigenvalue \( \lambda_m \) must be even and a two-sided estimate

\[
\frac{\pi(m-1)}{2} \leq \pi q \leq \frac{\pi m}{2}
\]

should take place. The index \( m \) being even, it follows that \( q = m/2 \), what by (5.17) and the equality \( \lambda_m(l) = \kappa^2 + k^2(l) \) proves the asymptotics expansions (1.11).

The case of even on \( x_1 \) function \( \psi_m \) can be proved in the same way. The function \( v_l' \) should be chosen as a solution to the boundary value problem

\[
-\Delta v_0' = (\kappa^2 + k^2)v_0', \quad x \in \Omega_0,
\]

\[
v_0' = 0, \quad x \in \partial \Omega_0' \setminus K_l, \quad \frac{\partial v_0'}{\partial x_1} = -\frac{\partial v_0}{\partial x_1}, \quad x \in \partial \Omega_0'.
\]
which is solved by separation of variables
\[ v_0^l(x, k) = -\sum_{j=1}^{\infty} \frac{e^{-s_0 j}}{\cosh s_0 j} \beta_j(k)[g] \sinh s_0 x_1 \sin \varphi_j(x_2 - \pi), \]
where \( \beta_j(k) \) are same as in (5.9). The other arguments are valid till Lemma 5.4 if by \( v_0^l \) we mean the function just defined. The statement of Lemma 5.4 is valid as well, if by \( T_{15} \) we mean the operator
\[ T_{15}(k,l)g = -\frac{e^{-ikl}}{\cos kl} \beta_0(k)[g] \hat{F}. \]
The deduction of the analogue of equation (5.14) needs no changes. In this case it is of the form:
\[ 1 - \frac{e^{-ikl}}{\cos kl} \beta_0(k) \left( (I + T_{13}(k) + T_{16}(k, l))^{-1} \hat{F} \right) = 0. \]
Using this equation, one can easily obtain an analogue of the equation (5.16):
\[ \cos kl = O(l^{-1}), \]
what gives the equality
\[ kl = \frac{\pi}{2} + \pi q + O(l^{-1}), \quad q \in \mathbb{Z}. \]
Again due to Item 1 of Theorem 1.1 the index \( m \) of the eigenvalue \( \lambda_m \) should be odd and inequalities
\[ \frac{\pi(m-1)}{2} \leq \frac{\pi}{2} + \pi q \leq \frac{\pi m}{2} \]
should take place. This implies that \( q = (m - 1)/2 \). It proves the asymptotics expansions in the case of even on \( x_1 \) eigenfunction \( \psi_m \). The proof of Item 1 of Theorem 1.2 is complete.

We proceed to the proof of Item 2 of Theorem 1.2. Let \( \xi \) be an arbitrary point of the segment \([\varphi, 1)\). For each value \( l \) we choose the number \( m = m(l, \xi) \) so that the belonging \( \xi \in [\Lambda_{m-1}, \Lambda_m) \) be valid. Then by Item 1 of Theorem 1.1 the estimate
\[ |\lambda_m(l, \xi)(l) - \xi| \leq \Lambda_m - \Lambda_{m-1} = \frac{\pi^2(2m-1)}{4l^2} \]
takes place. Since \( \Lambda_{m-1} \leq \xi < 1 \), it follows that
\[ m \leq 1 + \frac{2l\sqrt{1 - \xi^2}}{\pi}. \]
Two last estimates yield:
\[ |\lambda_m(l, \xi)(l) - \xi| \leq \frac{\pi^2}{4l^2} \left( 1 + \frac{4l\sqrt{1 - \xi^2}}{\pi} \right), \]
what implies that \( \lambda_m(l, \xi) \to \xi \) as \( l \to +\infty \). The proof of Theorem 1.2 is complete.

The author thanks R. Gadyl’shin, P. Exner and T. Weidl for discussion of the work and useful remarks.
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