STABILIZATION OF PORT-HAMILTONIAN SYSTEMS WITH DISCONTINUOUS ENERGY DENSITIES

Jochen Schmid
Fraunhofer Institute for Industrial Mathematics (ITWM)
Fraunhofer-Platz 1
67663 Kaiserslautern, Germany

(Communicated by Luciano Pandolfi)

ABSTRACT. We establish an exponential stabilization result for linear port-Hamiltonian systems of first order with quite general, not necessarily continuous, energy densities. In fact, we have only to require the energy density of the system to be of bounded variation. In particular, and in contrast to the previously known stabilization results, our result applies to vibrating strings or beams with jumps in their mass density and their modulus of elasticity.

1. Introduction. In this paper, we are concerned with the stabilization of linear first-order port-Hamiltonian systems with discontinuous energy densities on a bounded interval \([a,b]\). Such a system evolves according to a partial differential equation of the form

\[
\frac{\partial}{\partial t} x(t,\zeta) = P_1 \partial_\zeta \left( \mathcal{H}(\zeta) x(t,\zeta) \right) + P_0 \mathcal{H}(\zeta) x(t,\zeta)
\]

for the states \(x(t,\cdot) : [a,b] \to \mathbb{K}^m\), and the energy of such a system at time \(t\) is given by an integral of the form

\[
E(x(t,\cdot)) = \frac{1}{2} \int_a^b x(t,\zeta)^* \mathcal{H}(\zeta) x(t,\zeta) \, d\zeta.
\]

In these equations, \(\mathcal{H}\) is the energy density of the system, that is, a suitable measurable function from \([a,b]\) to \(\mathbb{K}^{m \times m}\), and \(P_0, P_1\) are matrices in \(\mathbb{K}^{m \times m}\) with suitable symmetry properties. We want to stabilize such systems by linear boundary control and therefore we complement (1.1) by the linear boundary condition

\[
0 = W_{B,1} \begin{pmatrix} \mathcal{H}(b)x(t,b) \\ \mathcal{H}(a)x(t,a) \end{pmatrix}
\]

and the linear boundary in- and output relations

\[
\begin{align*}
 u(t) &= W_{B,2} \begin{pmatrix} \mathcal{H}(b)x(t,b) \\ \mathcal{H}(a)x(t,a) \end{pmatrix} & y(t) &= W_{C} \begin{pmatrix} \mathcal{H}(b)x(t,b) \\ \mathcal{H}(a)x(t,a) \end{pmatrix},
\end{align*}
\]

where \(W_{B,1} \in \mathbb{K}^{(m-k) \times 2m}\) and \(W_{B,2}, W_{C} \in \mathbb{K}^{k \times 2m}\) and \(k \in \{1,\ldots,m\}\). So, in abstract terms, we consider a linear evolution equation

\[
\dot{x} = Ax = P_1 \partial_\zeta (\mathcal{H}x) + P_0 \mathcal{H}x
\]
in the state space $X := L^2([a, b], \mathbb{K}^m)$ with additional linear boundary input and output conditions

$$u(t) = Bx(t) \quad \text{and} \quad y(t) =Cx(t),$$

(1.5)

where the linear differential operator $A : D(A) \subset X \to X$ is defined by the right-hand side of (1.4) with domain

$$D(A) := \left\{ x \in X : \mathcal{H}x \in W^{1,2}([a, b], \mathbb{K}^m) \text{ and } W_{B,1} \left( \begin{array}{c} \mathcal{H}x(b) \\ \mathcal{H}x(a) \end{array} \right) = 0 \right\}$$

and where the linear boundary in- and output operators $B, C : D(A) \subset X \to \mathbb{K}^k$ are defined by

$$Bx := W_{B,2} \left( \begin{array}{c} \mathcal{H}x(b) \\ \mathcal{H}x(a) \end{array} \right) \quad \text{and} \quad Cx := W_{C} \left( \begin{array}{c} \mathcal{H}x(b) \\ \mathcal{H}x(a) \end{array} \right).$$

What we show in this paper is that the input-output system (1.4)-(1.5) can be exponentially stabilized by means of the negative output-feedback law

$$u(t) = -\mu y(t)$$

(1.6)

with an arbitrary $\mu > 0$, that is, the system (1.4)-(1.5) with the additional feedback condition (1.6) is an exponentially stable linear system. We achieve this exponential stability of the closed-loop system (1.4)-(1.6) under the assumption that the energy density $\zeta \mapsto \mathcal{H}(\zeta)$ is of bounded variation and that the open-loop system (1.4)-(1.5) satisfies two additional natural conditions, namely (i) impedance-passivity and (ii) domination of the state value at one of the boundary points $(a$ or $b)$ by the input and the output. We apply this stability result to vibrating strings and to beams (modelled according to Timoshenko).

Since the energy density in our result is only required to be of bounded variation, we can treat strings and beams with jumps in their material characteristics like mass density or modulus of elasticity. With the previously known stabilization results, by contrast, such situations with jumps in the mass density and the modulus of elasticity cannot be dealt with. Indeed, the stability results from [18], [10], [4], [3] are restricted to port-Hamiltonian systems with continuously differentiable or Lipschitz continuous energy densities, and the stability result from [5] is restricted to vibrating strings with constant modulus of elasticity (while allowing bounded variation regularity for the mass density).

In the entire paper, we will use the following notations. As usual, $\mathbb{K}$ stands for the field $\mathbb{R}$ of real or the field $\mathbb{C}$ of complex numbers, $\mathbb{R}_+^\infty := [0, \infty)$ denotes the set of non-negative reals, and $|\cdot|$ denotes the standard norm on $\mathbb{K}^m$ for any $m \in \mathbb{N}$. Also, $L^p(S, \mathbb{K}^m)$ and

$$W^{k,p}([a, b], \mathbb{K}^m) := W^{k,p}((a, b), \mathbb{K}^m)$$

for $p \in [1, \infty) \cup \{ \infty \}$ (integrability index) and $k \in \mathbb{N}$ (differentiability index) are the usual Lebesgue and Sobolev spaces, respectively, and $\|\cdot\|_p$ and $\langle \cdot, \cdot \rangle_2$ will denote the standard norm and scalar product of $L^p(S, \mathbb{K}^m)$ and $L^2(S, \mathbb{K}^m)$, respectively. $AC([a, b], \mathbb{K}^m)$ denotes the set of absolutely continuous functions from $[a, b]$ to $\mathbb{K}^m$. And finally, for $J = [a, b]$ or $J = \mathbb{R}$,

$$BV(J, \mathbb{K}^m) := \{ \text{functions } f : J \to \mathbb{K}^m \text{ with } \text{Var}(f) < \infty \}$$
denotes the set of functions of bounded variation from \( J \) to \( \mathbb{K}^m \), where the variation \( \text{Var}(f) \) of a function \( f : J \to \mathbb{K}^m \) is defined as follows. In the case where the domain of \( f \) is \( J = [a, b] \),

\[
\text{Var}(f) := \sup \left\{ \sum_{i=1}^{L} |f(t_i) - f(t_{i-1})| : (t_i)_{i \in \{a, \ldots, L\}} \text{ a partition of } [a, b] \right\}
\]

(1.7)

and in the case where the domain of \( f \) is \( J = \mathbb{R} \),

\[
\text{Var}(f) := \sup \{ \text{Var}(f|_{[a,b]} : a < b) \}.
\]

(1.8)

2. Some technical preliminaries. In this section, we record some purely technical preliminaries needed later in the proof of our exponential stability result. At first reading, this section can be safely skipped. What we study here is when and in which sense a relation of the kind

\[
\left( \int_J f(s) \, ds \right)(\zeta) = \int_J f(s)(\zeta) \, ds
\]

holds true for almost every \( \zeta \in Z \), where \( s \mapsto f(s) \) is an integrable \( L^p(Z, \mathbb{K}^m) \)-valued function defined on some interval \( J \). In essence, the following lemma can be found in [8] (Section 3.4, paragraph about spaces of class \( L \)), but the importance of choosing the right representatives – which is demonstrated by our example below – is ignored there. We recall from [2] (Chapter X.1) that a function \( f : S \to X \) between a measurable set \( S \in \mathcal{L}_{\mathbb{R}^d} \) (Lebesgue-\( \sigma \)-algebra on \( \mathbb{R}^d \)) and a Banach space \( X \) is called \( \lambda \)-measurable if there is a sequence of integrable simple functions \( f_n : S \to X \) converging to \( f \) \( \lambda \)-almost everywhere, where \( \lambda \) is the Lebesgue measure on \( \mathcal{L}_{\mathbb{R}^d} \). In case \( X \) is separable, \( \lambda \)-measurability coincides with (plain) \( \mathcal{L}_{\mathbb{R}^d} \)-\( B_X \)-measurability by Pettis’ theorem (\( B_X \) being the Borel \( \sigma \)-algebra of \( X \)). A function \( f : S \to X \) as above will be called \( p \)-integrable for a \( p \in [1, \infty) \) if it is \( \lambda \)-measurable and

\[
\int_S \| f(s) \|^p_X \, ds := \int_S \| f(s) \|^p_X \, d\lambda(s) < \infty.
\]

Lemma 2.1. Suppose \( f : J \to X \) is a \( p \)-integrable function from a bounded interval \( J \subset \mathbb{R} \) to the space \( X := L^p(Z, \mathbb{K}^m) \), where \( p \in [1, \infty) \) and \( Z \in \mathcal{L}_{\mathbb{R}} \). Then

\begin{enumerate}[(i)]
\item for every \( s \in J \) there is a representative \( f(s) : Z \to \mathbb{K}^m \) of \( f(s) \) such that the function

\[
J \times Z \ni (s, \zeta) \mapsto f(s)(\zeta) \in \mathbb{K}^m
\]

is measurable and

\[
(2.1)
\]

(ii) for every choice of representatives \( f(s) \) as in (i) the function \( J \ni s \mapsto f(s)(\zeta) \) is integrable for a.e. \( \zeta \in Z \) and

\[
\zeta \mapsto \int_J f(s)(\zeta) \, ds
\]

is a representative of the element \( \int_J f(s) \, ds \in X \).
\end{enumerate}

Proof. We strictly distinguish between functions and equivalence classes of functions in this proof and, as usual, we use square brackets to denote equivalence classes.

(i) Since \( f : J \to X \) is \( p \)-integrable, there exist integrable simple functions \( f_n : J \to X \) such that \( f_n(s) \to f(s) \) for a.e. \( s \in J \) and we can also assume that

\[
\| f_n(s) \|_X \leq 2 \| f(s) \|_X
\]

Proof.
for all \( s \in J \) and \( n \in \mathbb{N} \) (if this bound does not hold for the initial choice of simple functions \( f^0_n \), just multiply them by the characteristic function of the measurable set \( \{ s \in J : \| f^0_n(s) \|_X \leq 2 \| f(s) \|_X \} \)). So, by the theorem of dominated convergence,

\[
\int_J \| f_n(s) - f(s) \|^p_X \, ds \longrightarrow 0 \quad (n \to \infty). \tag{2.3}
\]

Since the \( f_n \) are \( \lambda \)-measurable simple functions, they are of the form

\[
f_n(s) = \sum_{k=1}^{m_n} \alpha_{nk} \chi_{E_{nk}}(s) \quad (s \in J) \tag{2.4}
\]

for certain \( \alpha_{nk} \in X = L^p(Z, \mathbb{K}^m) \) and \( E_{nk} \in \mathcal{L}_R \). Choosing representatives \( \alpha_{nk} : Z \to \mathbb{K}^m \) of \( \alpha_{nk} \) and defining \( \varphi_n : J \times Z \to \mathbb{K}^m \) by

\[
\varphi_n(s, \zeta) := \sum_{k=1}^{m_n} \alpha_{nk}(\zeta) \chi_{E_{nk}}(s) \quad ((s, \zeta) \in J \times Z),
\]

we see that \( \varphi_n \) is measurable for every \( n \in \mathbb{N} \) so that by Tonelli’s theorem (Theorem X.6.7 of [2]) and (2.3) we have

\[
\int_{J \times Z} |\varphi_n(s, \zeta) - \varphi_m(s, \zeta)|^p \, d(s, \zeta) = \int_J \| f_n(s) - f_m(s) \|^p_X \, ds \longrightarrow 0
\]

as \( m, n \to \infty \). So, by the completeness of \( L^p(J \times Z, \mathbb{K}^m) \), there is a \( p \)-integrable function \( \varphi : J \times Z \to \mathbb{K}^m \) such that

\[
\int_{J \times Z} |\varphi_n(s, \zeta) - \varphi(s, \zeta)|^p \, d(s, \zeta) \longrightarrow 0 \quad (n \to \infty). \tag{2.5}
\]

We have by Tonelli’s theorem that \([\varphi(s, \cdot)], [\varphi_n(s, \cdot) - \varphi(s, \cdot)]\) belong to \( L^p(Z, \mathbb{K}^m) \) for a.e. \( s \in J \) (with exceptional sets \( N_0 \) and \( N_n \) respectively) and that

\[
J \setminus N' \ni s \mapsto \| [\varphi(s, \cdot)] \|_X^p, \| [\varphi_n(s, \cdot) - \varphi(s, \cdot)] \|_X^p
\]

are measurable for all \( n \in \mathbb{N} \), where \( N' := \bigcup_{n=0}^\infty N_n \). In view of (2.5) it now follows that

\[
\int_J \| f_n(s) - [\varphi(s, \cdot)] \|^p_X \, ds = \int_J \int_Z |\varphi_n(s, \zeta) - \varphi(s, \zeta)|^p \, d\zeta \, ds \longrightarrow 0 \tag{2.6}
\]

as \( n \to \infty \). Combining (2.3) and (2.6) we see that

\[
f(s) = [\varphi(s, \cdot)] \tag{2.7}
\]

for a.e. \( s \in J \) (with an exceptional set denoted by \( N'' \)). We now define \( f(s) : Z \to \mathbb{K}^m \) by

\[
f(s)(\zeta) := \begin{cases} 
\varphi(s, \zeta), & (s, \zeta) \in (J \setminus N'') \times Z \\
\varphi_n(s, \zeta), & (s, \zeta) \in N'' \times Z
\end{cases}
\]

where \( f(s) \) for \( s \in N'' \) is an arbitrary representative of \( f(s) \). It then follows by (2.7) that \( f(s) \) is a representative of \( f(s) \) for every \( s \in J \) and by the measurability of \( \varphi \) and \( \lambda(N'' \times Z) = 0 \) it follows that \( J \times Z \ni (s, \zeta) \mapsto f(s)(\zeta) \) is measurable, as desired.

(ii) Choose and fix for every \( s \in J \) a representative \( f(s) \) of \( f(s) \) such that \( (s, \zeta) \mapsto f(s)(\zeta) \) is measurable (which is possible by part (i)). \( \overline{f} \) follows by Tonelli’s theorem.
that \( s \mapsto f(s)(\zeta) \) is measurable for a.e. \( \zeta \in Z \) and that \( \zeta \mapsto \int_J |f(s)(\zeta)| \, ds \in [0, \infty) \cup \{ \infty \} \) is measurable as well. Also,
\[
\int_Z \left( \int_J |f(s)(\zeta)| \, ds \right)^p \, d\zeta \leq \int_Z \lambda(J)^{p/q} \int_J |f(s)(\zeta)|^p \, ds \, d\zeta = \lambda(J)^{p/q} \int_J \| f(s) \|_X^p \, ds < \infty
\]
by the boundedness of \( J \) and the \( p \)-integrability of \( f \) (where \( q \in (1, \infty] \) is the dual exponent of \( p \in [1, \infty) \), of course). Consequently,
\[
\int_J |f(s)(\zeta)| \, ds < \infty
\]
for a.e. \( \zeta \in Z \) and thus the function \( J \ni s \mapsto f(s)(\zeta) \) is integrable for a.e. \( \zeta \in Z \).

What we have to show now is that
\[
\zeta \mapsto \int_J f(s)(\zeta) \, ds
\]
(2.8)
is a representative of \( F := \int_J f(s) \, ds \in X \) (where the existence of this integral in \( X \) follows by means of Hölder’s inequality from the \( p \)-integrability of \( f \) and the boundedness of \( J \)). In order to do so, we show that for any given representative \( F \) of \( F \) one has
\[
\int_J f(s)(\zeta) \, ds = F(\zeta)
\]
(2.9)
for a.e. \( \zeta \in Z \). Choose integrable simple functions \( f_n : J \to X \) such that
\[
\int_J \| f_n(s) - f(s) \|_X^p \, ds \to 0 \quad (n \to \infty)
\]
(2.10)
(see the beginning of the proof of part (i)) and write \( F_n := \int_J f_n(s) \, ds \). Also, for every \( s \in J \) and \( n \in \mathbb{N} \) choose a representative \( f_n(s) \) of \( f_n(s) \) and \( F_n \) of \( F_n \) by choosing representatives of the values \( a_{nk} \in X \) of \( f_n \), see (2.4). Clearly,
\[
\int_J f_n(s)(\zeta) \, d\zeta = F_n(\zeta)
\]
(2.11)
for a.e. \( \zeta \in Z \) and every \( n \in \mathbb{N} \). In view of (2.10) it further follows that
\[
\int_Z |E_n(\zeta) - F(\zeta)|^p \, d\zeta = \| F_n - F \|_X^p \leq \lambda(J)^{p/q} \int_J \| f_n(s) - f(s) \|_X^p \, ds \to 0
\]
(2.12)
as \( n \to \infty \) and that
\[
\int_Z \left| \int_J f_n(s)(\zeta) \, ds - \int_J f(s)(\zeta) \, ds \right|^p \, d\zeta \leq \int_Z \lambda(J)^{p/q} \int_J \| f_n(s)(\zeta) - f(s)(\zeta) \|^p \, ds \, d\zeta
\]
\[
= \lambda(J)^{p/q} \int_J \| f_n(s) - f(s) \|_X^p \, ds \to 0
\]
(2.13)
as \( n \to \infty \). So by (2.12) and (2.13) there is a subsequence \( (n_k) \) such that
\[
E_{n_k}(\zeta) \to E(\zeta) \quad (k \to \infty)
\]
(2.14)
for a.e. \( \zeta \in Z \) and such that
\[
\int_J f_{n_k}(s)(\zeta) \, ds \to \int_J f(s)(\zeta) \, ds \quad (k \to \infty)
\]
(2.15)
for a.e. \( \zeta \in Z \). Combining now (2.11) with (2.14) and (2.15), we obtain the desired equality (2.9) for almost every \( \zeta \in Z \).
Corollary 2.2. Suppose \( f : J \to X \) is a continuous function from a compact interval \( J \subset \mathbb{R} \) to the space \( X := \mathbb{L}^p(Z, \mathbb{K}^m) \), where \( p \in [1, \infty) \) and \( Z \in \mathcal{L}_\mathbb{R} \). Then the conclusions of the previous lemma hold true.

Proof. Since \( f \) is continuous, it is \( \mathcal{L}_\mathbb{R} \)-\( \mathcal{B}_X \)-measurable and separably valued. So, \( f \) is \( \lambda \)-measurable by Pettis' theorem (Theorem X.1.4 of [2]). Since moreover \( J \) is compact, \( f \) is \( p \)-integrable and thus the assertion follows by the previous lemma.

In view of the previous lemma, the question arises whether (i) for every choice of representatives \( \bar{f}(s) \) of \( f(s) \), the function (2.1) is measurable and whether (ii) for every choice of representatives \( \bar{f}(s) \) of \( f(s) \) such that \( J \ni s \mapsto \int f(s)(\zeta) \) is integrable for a.e. \( \zeta \in Z \), the function (2.2) is a representative of the element \( \int_J f(s) \, ds \in X \). As the following example shows, the answers to both questions are negative.

Example 2.3. (i) Set \( J, Z := [0, 1] \) and choose a subset \( E \) of \( J \times Z \) such that \( E \) is not Lebesgue-measurable and such that each line in \( \mathbb{R}^2 \) intersects \( E \) in at most 2 points. Such a set was shown to exist by Sierpiński in [13] using the axiom of choice. Also, let \( f : J \to X := \mathbb{L}^2(Z, \mathbb{R}) \) and \( f(s) : J \to \mathbb{R} \) be defined by

\[
\begin{align*}
\int f(s) \, ds &= 0 \quad (s \in J) \quad \text{and} \quad \bar{f}(s)(\zeta) := \chi_E(s, \zeta) \quad ((s, \zeta) \in J \times Z). \quad (2.16)
\end{align*}
\]

Since the section \( E_s := \{ \zeta \in Z : (s, \zeta) \in E \} \) has at most 2 elements for every \( s \in J \), the function \( f(s) \) is a representative of \( f(s) \) for every \( s \in J \) but the function (2.1) is not measurable because \( E \not\in \mathcal{L}_{\mathbb{R}^2} \).

(ii) Set \( J, Z := [0, 1] \) and choose a subset \( E \) of \( J \times Z \) such that the section \( E_s := \{ \zeta \in Z : (s, \zeta) \in E \} \) is countable for every \( s \in J \) and such that the section \( E^c_s := \{ s \in J : (s, \zeta) \in E \} \) is co-countable for every \( \zeta \in Z \). Such a set \( E \) was shown to exist by Sierpiński in [14] assuming that the continuum hypothesis is true (which is not needed for [13]). (See also Example 8.9 (c) in [11] and Exercise 2.47 and Section 2.8 of [7].) Also, let \( f : J \to X := \mathbb{L}^2(Z, \mathbb{R}) \) and \( f(s) : J \to \mathbb{R} \) be defined by

\[
\begin{align*}
\int f(s) \, ds &= 0 \quad (s \in J) \quad \text{and} \quad \bar{f}(s)(\zeta) := \chi_E(s, \zeta) \quad ((s, \zeta) \in J \times Z). \quad (2.17)
\end{align*}
\]

Since the section \( E_s \) is countable for every \( s \in J \), the function \( f(s) = \chi_{E_s} \) is a representative of \( f(s) \) for every \( s \in J \), and since \( E_s \) is co-countable, \( J \ni s \mapsto \int \bar{f}(s)(\zeta) = \chi_{E^c(s)} \) is integrable for every \( \zeta \in Z \) but, as

\[
\int_J \bar{f}(s)(\zeta) \, ds = 1 \neq 0 \quad (\zeta \in Z), \quad (2.18)
\]

the function (2.2) is not a representative of \( 0 = \int_J f(s) \, ds \in X \).

3. Stability results. In this section, we establish the main stability results of this paper and to do so we need some preparations. In the entire section, \( X := \mathbb{L}^2([a, b], \mathbb{K}^m) \) with some fixed real numbers \( a < b \) and some fixed \( m \in \mathbb{N} \). We will call a matrix-valued function \([a, b] \ni \zeta \mapsto \mathcal{H}(\zeta) \in \mathbb{K}^{m \times m}\) on some compact interval \([a, b]\) an energy density if it is measurable, \( \mathcal{H}(\zeta) \) is self-adjoint for every \( \zeta \in [a, b] \), and there are constants \( \underline{m}, \overline{m} \in (0, \infty) \) such that

\[
m \leq \mathcal{H}(\zeta) \leq \overline{m} \quad (3.1)
\]

for every \( \zeta \in [a, b] \). Also, for a given energy density \( \mathcal{H} \), a linear operator \( A : D(A) \subset X \to X \) is called a first-order port-Hamiltonian operator with energy density \( \mathcal{H} \) iff the domain

\[
D(A) \subset \{ x \in X : \mathcal{H}x \in W^{1,2}([a, b], \mathbb{K}^m) \}.
\]
is a dense subspace of $X = L^2([a, b], \mathbb{K}^m)$ and if $A$ is of the form
\begin{equation}
Ax = P_1 \partial_t (\mathcal{H}x) + P_0 \mathcal{H}x \quad (x \in D(A))
\end{equation}
for some invertible self-adjoint matrix $P_1 = P_1^* \in \mathbb{K}^{m \times m}$ and some skew-adjoint matrix $P_0 = -P_0^* \in \mathbb{K}^{m \times m}$. An evolution equation $\dot{x} = Ax$ with $A$ being a first-order port-Hamiltonian operator is called a \textit{first-order port-Hamiltonian system}. Additionally, the scalar product $\langle \cdot, \cdot \rangle_X$ defined by
\begin{equation}
\langle x, y \rangle_X := \frac{1}{2} \int_a^b x(\zeta)^* \mathcal{H}(\zeta)y(\zeta) \, d\zeta
\end{equation}
is called the \textit{$\mathcal{H}$-energy scalar product} and the corresponding norm $\|\cdot\|_X$ is called the \textit{$\mathcal{H}$-energy norm}. In view of (3.1) it is clear that the $\mathcal{H}$-energy norm is equivalent to the standard norm of $L^2([a, b], \mathbb{K}^m)$. In view of the continuous embedding of $W^{1,2}([a, b], \mathbb{K}^m)$ in $C([a, b], \mathbb{K}^m)$ it is also clear that for $x \in D(A)$ the vector
\begin{equation}
(\mathcal{H}x)|_\partial := \left( \begin{array}{c}
(\mathcal{H}x)(b) \\
(\mathcal{H}x)(a)
\end{array} \right) \in \mathbb{K}^{2m}
\end{equation}
of stacked boundary values is well-defined. As usual, we do not distinguish here and in the following between $\mathcal{H}$ and the multiplication operator $M_\mathcal{H}$ associated with $\mathcal{H}$, that is, we will always write $\mathcal{H}x$ for $M_\mathcal{H}x$. Similarly, $\mathcal{H}^{-1}$ will stand for $\zeta \mapsto \mathcal{H}(\zeta)^{-1}$ as well as for the corresponding multiplication operator.

As a first preparatory lemma, we recall from [9] (Theorem 1.1) the following characterization of when a port-Hamiltonian operator generates a contraction semigroup.

\textbf{Lemma 3.1.} Suppose $A : D(A) \subset X \to X$ is a first-order port-Hamiltonian operator with energy density $\mathcal{H} : [a, b] \to \mathbb{K}^{m \times m}$, where $X := L^2([a, b], \mathbb{K}^m)$ is endowed with the $\mathcal{H}$-energy norm $\|\cdot\|_X$. Suppose further that the domain of $A$ incorporates linear boundary conditions, that is, it is of the form
\begin{equation}
D(A) = \{ x \in X : \mathcal{H}x \in W^{1,2}([a, b], \mathbb{K}^m) \text{ and } W(\mathcal{H}x)|_\partial = 0 \}
\end{equation}
for some matrix $W \in \mathbb{K}^{m \times 2m}$. Then $A$ generates a contraction semigroup on $X$ if and only if $A$ is dissipative in $X$, that is,
\[\text{Re } \langle x, Ax \rangle_X \leq 0 \quad (x \in D(A)).\]
In that case, the boundary matrix $W$ automatically has full rank $m$.

As a second preparatory lemma, we show the following differentiability result for certain sideways energy functions [5] along classical solutions of port-Hamiltonian systems with absolutely continuous energy densities. (As usual, a classical solution of such a system (3.5) is a continuously differentiable map $x : J \to X$ on some interval $J \subset \mathbb{R}^+_0$ such that for all $t \in J$ one has $x(t) \in D(A)$ and $\dot{x}(t) = Ax(t)$.) In [10], [3] such a differentiability result – for the special case of continuously differentiable or Lipschitz continuous energy densities – is used implicitly as well, but no proofs are given there. As we will see, the proof requires quite some work and care. In fact, some of the (formal) computations from [10], [3] will in general become false for careless choices of representatives. See the example below.

\textbf{Lemma 3.2.} Suppose $A : D(A) \subset X \to X$ is a first-order port-Hamiltonian operator on $X := L^2([a, b], \mathbb{K}^m)$ with energy density $\mathcal{H} \in AC([a, b], \mathbb{K}^{m \times m})$. Suppose further that $x : \mathbb{R}^+_0 \to X$ is a classical solution of the differential equation
\begin{equation}
\dot{x} = Ax
\end{equation}
and let \( F : [a, b] \to \mathbb{K} \) be the sideways energy defined by
\[
F(\zeta) := \int_{r(\zeta)}^{t(\zeta)} \overline{x}(s)(\zeta)^* \mathcal{H}(\zeta) x(s)(\zeta) \, ds,
\] 
(3.6)
where \( r, t \in C^1([a, b], \mathbb{R}_+^m) \) are given functions and where \( x(s) \) for every \( s \in \mathbb{R}_+^m \) is the continuous representative of \( x(s) \). It then follows that \( F \) is absolutely continuous and, in particular, integrable. In this equation, \( x \in \mathcal{H} \) is a solution of (3.5), we have that \( x \) is continuous and, in particular, integrable. In this equation, \( x \) is continuous as well. Since moreover
\[
F'(\zeta) = \overline{x}(s)(\zeta)^* (t'(\zeta) \mathcal{H}(\zeta) + P_1^{-1}) x(s)(\zeta) \bigg|_{s=r(\zeta)}
- \overline{x}(s)(\zeta)^* (r'(\zeta) \mathcal{H}(\zeta) + P_1^{-1}) x(s)(\zeta) \bigg|_{s=r(\zeta)}
- \int_{r(\zeta)}^{t(\zeta)} x(s)(\zeta)^* ((P_1^{-1} P_2 \mathcal{H}(\zeta))^* + \mathcal{H}'(\zeta) + P_1^{-1} P_2 \mathcal{H}(\zeta)) x(s)(\zeta) \, ds
\] 
(3.7)
for almost every \( \zeta \in [a, b] \). Additionally, for \( \mathcal{H} \in C^1([a, b], \mathbb{K}^{m \times m}) \) the sideways energy \( F \) defined above is even continuously differentiable.

**Proof.** We divide the proof into two parts. In part (i) we prove in five steps the assertion for \( \mathcal{H} \in AC([a, b], \mathbb{K}^{m \times m}) \) and in part (ii) we prove the strengthened assertion for \( \mathcal{H} \in C^1([a, b], \mathbb{K}^{m \times m}) \). In the entire proof, we abbreviate \( Z := [a, b] \).

(i) As a first step, we observe that \( x(s) \in W^{1,1}(Z, \mathbb{K}^m) \) for every \( s \in \mathbb{R}_+^m \) and that \( s \mapsto x(s) \in W^{1,1}(Z, \mathbb{K}^m) \) is continuous. (In particular, by the embedding of \( W^{1,2}(Z, \mathbb{K}^m) \) in \( C(Z, \mathbb{K}^m) \), the element \( x(s) \) has a unique continuous representative \( x(s) \) and, hence, (3.6) is well-defined in the first place.) Indeed, since \( x \) is a classical solution of (3.5), we have that
\[
\mathcal{H} x(s) \in W^{1,2}(Z, \mathbb{K}^m) \quad (s \in \mathbb{R}_+^m)
\] 
(3.8)
and that \( s \mapsto \mathcal{H} x(s) \in X = L^2(Z, \mathbb{K}^m) \) as well as
\[
s \mapsto \partial_\zeta (\mathcal{H} x(s)) = P_1^{-1} x(s) - P_1^{-1} P_2 \mathcal{H} x(s) \in X = L^2(Z, \mathbb{K}^m)
\] 
(3.9)
are continuous. So, \( s \mapsto \mathcal{H} x(s) \) is continuous as a function with values in \( W^{1,2}(Z, \mathbb{K}^m) \). Since \( W^{1,2}(Z, \mathbb{K}^m) \) is continuously embedded in \( W^{1,1}(Z, \mathbb{K}^m) \),
\[
s \mapsto \mathcal{H} x(s) \in W^{1,1}(Z, \mathbb{K}^m)
\] 
(3.10)
is continuous as well. Since moreover \( \mathcal{H} \) belongs to \( W^{1,1}(Z, \mathbb{K}^{m \times m}) \) by assumption, we also have that
\[
\mathcal{H}^{-1} \in W^{1,1}(Z, \mathbb{K}^{m \times m}) \quad \text{with} \quad \partial_\zeta \mathcal{H}(\zeta)^{-1} = - \mathcal{H}(\zeta)^{-1} \mathcal{H}'(\zeta) \mathcal{H}(\zeta)^{-1}.
\] 
(3.11)
Combining now the continuity of (3.10) with (3.11) and with the continuity of multiplication
\[
W^{1,1}(Z, \mathbb{K}) \times W^{1,1}(Z, \mathbb{K}) \ni (f, g) \mapsto fg \in W^{1,1}(Z, \mathbb{K})
\] 
(3.12)
in \( W^{1,1}(Z, \mathbb{K}) \) (Theorem 4.39 in [1]), we obtain the assertion of the first step.

As a second step, we observe that for every \( r, t \in \mathbb{R}_0^+ \) the map \( \Phi_{r,t} : Z \to \mathbb{K} \) defined by
\[
\Phi_{r,t}(\zeta) := \int_{r}^{t} x(s)(\zeta)^* \mathcal{H}(\zeta) x(s)(\zeta) \, ds
\] 
(3.13)
is continuous and, in particular, integrable. In this equation, \( x(s) \) for every \( s \in \mathbb{R}_0^+ \) is the continuous representative of \( x(s) \in W^{1,1}(Z, \mathbb{K}^m) \) (first step). Since \( s \mapsto \)
\( x(s) \in W^{1,1}(Z, \mathbb{K}^m) \) is continuous by the first step, it follows by the continuous embedding of \( W^{1,1}(Z, \mathbb{K}^m) \) in \( C(Z, \mathbb{K}^m) \) that

\[
(s, \zeta) \mapsto \mathcal{g}(s)(\zeta)
\]

(3.14) is continuous. And therefore, \( \Phi_{r,t} \) is continuous as well.

As a third step, we show that for every \( r, t \in \mathbb{R}_0^+ \) the map \( \Phi_{r,t} \) is weakly differentiable with integrable weak derivative given by

\[
\partial_s \Phi_{r,t}(\zeta) = \mathcal{g}(s)(\zeta)^* P_1^{-1} \mathcal{g}(s)(\zeta) \bigg|_{s=t} - \int_r^t \mathcal{g}(s)(\zeta)^* \big( (P_1^{-1} P_0 \mathcal{H}(\zeta))^* + \mathcal{H}'(\zeta) + P_1^{-1} P_0 \mathcal{H}(\zeta) \big) \mathcal{g}(s)(\zeta) \, ds
\]

(3.15)

for almost all \( \zeta \in Z \). So let \( r, t \in \mathbb{R}_0^+ \) be fixed with \( r \leq t \) and set \( J := [r, t] \).

Combining the continuity of (3.10) with (3.11) and with the continuity of (3.12), we see that

\[
s \mapsto \psi(s) := x(s)^* \mathcal{H} x(s) = (\mathcal{H} x(s))^* \cdot \mathcal{H}^{-1} \cdot \mathcal{H} x(s) \in W^{1,1}(Z, \mathbb{K})
\]

(3.16)
is continuous. With (3.9) and (3.11) it further follows that

\[
\partial_s \psi(s) = \dot{x}(s)^* P_1^{-1} x(s) + x(s)^* P_1^{-1} \dot{x}(s) - x(s)^* \big( (P_1^{-1} P_0 \mathcal{H}(\zeta))^* + \mathcal{H}'(\zeta) + P_1^{-1} P_0 \mathcal{H}(\zeta) \big) x(s)
\]

for all \( s \in \mathbb{R}_0^+ \).

Choose now for every \( s \in \mathbb{R}_0^+ \) a representative \( \mathcal{v}(s) \) of \( v(s) := \dot{x}(s) \) such that \( (s, \zeta) \mapsto \mathcal{g}(s)(\zeta) \) is measurable (Corollary 2.2) and define \( \psi(s), \omega(s) : Z \to \mathbb{K} \) by

\[
\psi(s)(\zeta) := \mathcal{g}(s)(\zeta)^* \mathcal{H}(\zeta) \mathcal{g}(s)(\zeta)
\]

\[
\omega(s)(\zeta) := \mathcal{g}(s)(\zeta)^* P_1^{-1} \mathcal{g}(s)(\zeta) + \mathcal{g}(s)(\zeta)^* P_1^{-1} \mathcal{g}(s)(\zeta)
\]

\[
- \mathcal{g}(s)(\zeta)^* \big( (P_1^{-1} P_0 \mathcal{H}(\zeta))^* + \mathcal{H}'(\zeta) + P_1^{-1} P_0 \mathcal{H}(\zeta) \big) \mathcal{g}(s)(\zeta)
\]

(3.17)

(3.18)

for all \( (s, \zeta) \in \mathbb{R}_0^+ \times Z \). Then \( \psi(s), \omega(s) \) are representatives of \( \psi(s), \partial_s \psi(s) \) for every \( s \in \mathbb{R}_0^+ \) and

\[
(s, \zeta) \mapsto \psi(s)(\zeta) \quad \text{and} \quad (s, \zeta) \mapsto \omega(s)(\zeta)
\]

are continuous or measurable, respectively. So, by Tonelli’s theorem and by the continuity of (3.16), it follows that

\[
\int_{J \times Z} |\psi(s)(\zeta)| \, ds, \zeta) = \int_J \|\psi(s)(\zeta)\| \, ds \leq \lambda(J) \sup_{s \in J} \|\psi(s)(\zeta)\|_1 < \infty
\]

(3.19)

\[
\int_{J \times Z} |\omega(s)(\zeta)| \, ds, \zeta) = \int_J \|\partial_s \psi(s)(\zeta)\|_1 \, ds \leq \lambda(J) \sup_{s \in J} \|\partial_s \psi(s)(\zeta)\|_1 < \infty
\]

(3.20)

We can thus apply Fubini’s theorem to see that for every \( \varphi \in C^\infty_c((a, b), K) \)

\[
\int_Z \varphi'(\zeta) \Phi_{r,t}(\zeta) \, d\zeta = \int_J \int_Z \varphi'(\zeta) \psi(s)(\zeta) \, d\zeta \, ds
\]

\[
= - \int_J \int_Z \varphi(\zeta) \omega(s)(\zeta) \, d\zeta \, ds = - \int_Z \varphi(\zeta) \int_J \omega(s)(\zeta) \, ds \, d\zeta.
\]

(3.21)

So, by (3.20) and (3.21) the map \( \Phi_{r,t} \) is weakly differentiable with integrable weak derivative given by

\[
\partial_s \Phi_{r,t}(\zeta) = \int_J \omega(s)(\zeta) \, ds = \int_r^t \omega(s)(\zeta) \, ds
\]

(3.22)
for almost every $\zeta \in Z$. Since
\[
\left. \int_r^t \dot{x}(s)^* P_1^{-1} x(s) + x(s)^* P_1^{-1} \dot{x}(s) \, ds \right|_{s=r}^{s=t} = x(s)^* P_1^{-1} x(s)
\]
we have (by virtue of Corollary 2.2)
\[
\left. \int_r^t \psi(s)(\zeta)^* P_1^{-1} \psi(s)(\zeta) + \psi(s)(\zeta)^* P_1^{-1} \psi(s)(\zeta) \, ds \right|_{s=r}^{s=t}
\]
for almost every $\zeta \in Z$. Combining now (3.22) with (3.18) and (3.23), we obtain the desired formula (3.15).

As a fourth step, we show that $F : Z \to \mathbb{K}$ is absolutely continuous. We immediately see from the second and third step, that for every $r, t \in \mathbb{R}_0^+$ the map $\Phi_{r,t}$ is absolutely continuous with
\[
\Phi_{r,t}(\zeta) = \Phi_{r,t}(\zeta_0) + \int_{\zeta_0}^\zeta \partial_{\zeta} \Phi_{r,t}(\eta) \, d\eta
\]
for every $\zeta, \zeta_0 \in Z$, where
\[
\Psi_{r,t}(\eta) := \left. \psi(s)(\eta)^* P_1^{-1} \psi(s)(\eta) \right|_{s=r}^{s=t}
\]
\[
= \Phi_{r,t}(\zeta_0) + \int_{\zeta_0}^\zeta \Psi_{r,t}(\eta) \, d\eta + \int_{\zeta_0}^\zeta \Psi_{r,t}(\eta) \, d\eta
\]
and
\[
\Psi_{r,t}(\eta) := \int_r^t \psi(s)(\eta)^* \mathcal{H}(\eta)^* \psi(s)(\eta) \, ds.
\]
We also have
\[
F(\zeta) = \Phi_{r(\zeta),t(\zeta)}(\zeta) = \int_{r(\zeta)}^{t(\zeta)} \psi(s)(\zeta) \, ds
\]
\[
= \Phi_{r(\zeta_0),t(\zeta_0)}(\zeta) + \int_{t(\zeta_0)}^{t(\zeta)} \psi(s)(\zeta) \, ds - \int_{r(\zeta_0)}^{r(\zeta)} \psi(s)(\zeta) \, ds
\]
(3.25) for every $\zeta, \zeta_0 \in Z$. So, by (3.24) and (3.25) we see that
\[
F(\zeta) - F(\zeta_0) = \int_{\zeta_0}^{\zeta} \Psi_{r(\zeta_0),t(\zeta_0)}(\eta) \, d\eta + \int_{\zeta_0}^{\zeta} \Psi_{r(\zeta_0),t(\zeta_0)}(\eta) \, d\eta
\]
\[
+ \int_{t(\zeta_0)}^{t(\zeta)} \psi(s)(\zeta) \, ds - \int_{r(\zeta_0)}^{r(\zeta)} \psi(s)(\zeta) \, ds
\]
(3.26) for every $\zeta, \zeta_0 \in Z$. Choose now a compact interval $J$ such that
\[
r(\zeta), t(\zeta) \in J \quad (\zeta \in Z)
\]
(3.27) ($r, t$ are continuously differentiable on the compact interval $Z$ by assumption). With the help of (3.27) it then follows by the definition of $\Psi_{r,t}$, $\Psi_{r,t}$, $\psi$ that
\[
\left| \int_{\zeta_0}^{\zeta} \Psi_{r(\zeta_0),t(\zeta_0)}(\eta) \, d\eta \right| \leq 2 \left( \| P_1^{-1} \| + \| P_1^{-1} P_0 \| \right) \| \psi \|_{J \times Z, \infty}^2 \cdot \left( 1 + \sup_{\eta \in Z} \| \mathcal{H}(\eta) \| \lambda(J) \right) |\zeta - \zeta_0|
\]
(3.28)
\[
\left| \int_{\zeta_0}^{\zeta} \tilde{\Psi}_{r,\zeta_0}, t, (\zeta_0) (\eta) \, d\eta \right| \leq \| Z \|^2_{J \times Z, \infty} \lambda(J) \left| \int_{\zeta_0}^{\zeta} \| H'(\eta) \| \, d\eta \right| \quad (3.29)
\]
\[
\left| \int_{t(\zeta_0)}^{(t(\zeta_0))} (\zeta) \, ds \right|, \left| \int_{r(\zeta_0)}^{(r(\zeta_0))} (\zeta) \, ds \right| \leq \| Z \|^2_{J \times Z, \infty} \sup_{\eta \in Z} \| H(\eta) \| \cdot \max\{\| r' \|_{\infty}, \| t' \|_{\infty} \} |\zeta - \zeta_0| \quad (3.30)
\]

for all \( \zeta, \zeta_0 \in Z \). Since the function \( H' \) is integrable and since, moreover, \( \| Z \|_{J \times Z, \infty} := \sup_{(s, \zeta) \in J \times Z} |z(s)(\zeta)| < \infty \) and \( \| t' \|_{\infty}, \| t'' \|_{\infty} < \infty \) by the continuity of (3.14) and by assumption respectively, it follows from (3.26) with the help of (3.28), (3.29), (3.30) that \( F \) is absolutely continuous, as desired.

As a fifth step, we show that the derivative of \( F \) – which by the fourth step exists almost everywhere – is given by the asserted formula (3.7) for almost every \( \zeta \). Since \( \tilde{\Psi}_{r, \zeta} \) and \( (s, \zeta) \mapsto \tilde{\psi}(s)(\zeta) \) are continuous, it follows that
\[
\frac{1}{\zeta - \zeta_0} \int_{\zeta_0}^{\zeta} \tilde{\Psi}_{r, \zeta_0}, t, (\zeta_0) (\eta) \, d\eta \rightarrow \tilde{\Psi}_{r, \zeta_0}, t, (\zeta_0) (\zeta_0) \quad (3.31)
\]
for every \( \zeta_0 \in Z \) and that
\[
\frac{1}{\zeta - \zeta_0} \int_{t(\zeta_0)}^{(t(\zeta_0))} \tilde{\psi}(s)(\zeta) \, ds - \frac{1}{\zeta - \zeta_0} \int_{r(\zeta_0)}^{(r(\zeta_0))} \tilde{\psi}(s)(\zeta) \, ds \rightarrow t'(\zeta_0) \tilde{\psi}(s)(\zeta_0) \big|_{s=t(\zeta_0)} - r'(\zeta_0) \tilde{\psi}(s)(\zeta_0) \big|_{s=r(\zeta_0)} \quad (3.32)
\]
for every \( \zeta_0 \in Z \). Choose now a null set \( N \) such that
\[
\frac{1}{\zeta - \zeta_0} \int_{\zeta_0}^{\zeta} \| H'(\eta) - H'(\zeta_0) \| \, d\eta \rightarrow 0 \quad (\zeta \rightarrow \zeta_0)
\]
for all \( \zeta_0 \in Z \setminus N \), which is possible by the integrability of \( H' \) and Lebesgue’s differentiation theorem. Since by the definition of \( \tilde{\Psi}_{r, \zeta} \)
\[
\left| \frac{1}{\zeta - \zeta_0} \int_{\zeta_0}^{\zeta} \tilde{\Psi}_{r, \zeta_0}, t, (\zeta_0) (\eta) \, d\eta \right| - \tilde{\Psi}_{r, \zeta_0}, t, (\zeta_0) (\zeta_0) \leq \| Z \|^2_{J \times Z, \infty} \lambda(J) \left| \int_{\zeta_0}^{\zeta} \| H'(\eta) - H'(\zeta_0) \| \, d\eta \right| \quad (3.33)
\]
for every \( \zeta, \zeta_0 \in Z \) (with \( J \) as in (3.27)), it follows that
\[
\frac{1}{\zeta - \zeta_0} \int_{\zeta_0}^{\zeta} \tilde{\Psi}_{r, \zeta_0}, t, (\zeta_0) (\eta) \, d\eta \rightarrow \tilde{\Psi}_{r, \zeta_0}, t, (\zeta_0) (\zeta_0) \quad (3.33)
\]
for every \( \zeta_0 \in Z \setminus N \). Combining now (3.31), (3.32), (3.33) with (3.26), we conclude that \( F \) is differentiable at every \( \zeta_0 \in Z \setminus N \) with derivative
\[
F'(\zeta_0) = \Psi_{r, \zeta_0}, t(\zeta_0) (\zeta_0) + \tilde{\Psi}_{r, \zeta_0}, t, (\zeta_0) (\zeta_0)
+ t'(\zeta_0) \tilde{\psi}(s)(\zeta_0) \big|_{s=t(\zeta_0)} - r'(\zeta_0) \tilde{\psi}(s)(\zeta_0) \big|_{s=r(\zeta_0)}
= \partial_{\zeta} \Phi_{r, \zeta_0}, (\zeta_0) (\zeta_0) + t'(\zeta_0) \tilde{\psi}(s)(\zeta_0) \big|_{s=t(\zeta_0)} - r'(\zeta_0) \tilde{\psi}(s)(\zeta_0) \big|_{s=r(\zeta_0)} \quad (3.34)
\]
for every \( \zeta_0 \in Z \setminus N \). In view of (3.15) from the third step, this is precisely the asserted formula (3.7) for the derivative (with \( \zeta \) replaced by \( \zeta_0 \)).
(ii) We finally show -- by some slight modifications of the arguments above -- that F is even continuously differentiable under the strengthened assumption that
\[ \mathcal{H} \in C^1(Z, \mathbb{K}^{m \times m}) \].

So, let \( \mathcal{H} \in C^1(Z, \mathbb{K}^{m \times m}) \). We can then argue until (3.26) in exactly the same way as above. And this equation (3.26), under our strengthened assumption, almost immediately yields the desired conclusion. Indeed, for \( \mathcal{H} \in C^1(Z, \mathbb{K}^{m \times m}) \) not only \( \Psi_{r,t} \) and \( (s, \zeta) \mapsto \psi(s)(\zeta) \) but also \( \Psi_{r,t} \) is continuous and therefore not only (3.31), (3.32) but also (3.33) holds true for every \( \zeta_0 \in Z \). So, from (3.26) we see that F is differentiable at every \( \zeta_0 \in Z \) with derivative given by (3.34). And this expression, in turn, is continuous in \( \zeta_0 \) under our strengthened assumption.

\[ \square \]

**Example 3.3.** Choose \( A \) to be the port-Hamiltonian operator on \( X := L^2(Z, \mathbb{R}) \) corresponding to the transport equation on \( Z := [0, 1] \), that is,
\[ A = \partial_\zeta \quad \text{with} \quad D(A) = W^{1, 2}(Z, \mathbb{R}) \] (3.35)
and thus \( \mathcal{H}(\zeta) = 1 \in \mathbb{R} \) for all \( \zeta \in Z \) and \( P_1 = 1, P_0 = 0 \in \mathbb{R} \). In particular, \( s \mapsto x(s) := 1 \) is a classical solution of \( \dot{x} = Ax \). Choose now \( g(s), \tilde{g}(s) : Z \to \mathbb{R} \) for \( s \in J := [0, 1] \) in the following way:
\[ g(s)(\zeta) := 1 \quad \text{and} \quad \tilde{g}(s)(\zeta) := \chi_E(s, \zeta) \] (3.36)
for every \( (s, \zeta) \in J \times Z \), where \( E \) is chosen as in Example 2.3 (ii). We then have that \( g(s) \) for every \( s \in J \) is the continuous representative of \( x(s) \) and that \( \tilde{g}(s) \) for every \( s \in J \) is a representative of \( 0 = \dot{x}(s) \), but with this specific choice of representatives the formula (3.23) -- and hence the formula for the first integral from the last equation on page 113 of [10] -- becomes false. Indeed,
\[ \int_0^1 g(s)(\zeta) P_1^{-1} g(s)(\zeta) + g(s)(\zeta) P_1^{-1} g(s)(\zeta) \, ds = 2 \int_0^1 \chi_E(s, \zeta) \, ds = 2 \neq 0 = g(s)(\zeta) P_1^{-1} g(s)(\zeta) \bigg|_{s=1}^{s=0} \] (3.37)
for every \( \zeta \in Z \).

As a third preparatory lemma, we show the following approximation result for an energy density \( \mathcal{H} \) of bounded variation by absolutely continuous energy densities \( \mathcal{H}_n \).

**Lemma 3.4.** Suppose \( \mathcal{H} \in BV([a, b], \mathbb{K}^{m \times m}) \) is an energy density with lower and upper bounds denoted by \( m, \overline{m} \). Then there exists a sequence of energy densities \( \mathcal{H}_n \in AC([a, b], \mathbb{K}^{m \times m}) \) such that
\begin{itemize}
  \item[(i)] \( \mathcal{H}_n(\zeta) \to \mathcal{H}(\zeta) \) as \( n \to \infty \) for almost every \( \zeta \in [a, b] \)
  \item[(ii)] \( m \leq \mathcal{H}_n(\zeta) \leq \overline{m} \) for every \( \zeta \in [a, b] \) and every \( n \in \mathbb{N} \)
  \item[(iii)] \( \int_a^b \| \mathcal{H}_n'(\zeta) \| \, d\zeta \leq \| \mathcal{H}(a) \| + \text{Var}(\mathcal{H}) + \| \mathcal{H}(b) \| \) for all \( n \in \mathbb{N} \).
\end{itemize}

**Proof.** We argue by mollification. So, let \( j \in C_c^\infty(\mathbb{R}) \) be such that
\[ j(r) \geq 0 \quad (r \in \mathbb{R}), \quad \int_\mathbb{R} j(r) \, dr = 1, \quad \text{supp } j \subset [-1, 1] \] (3.38)
and let \( j_\varepsilon(r) := 1/\varepsilon \cdot j(r/\varepsilon) \) for \( r \in \mathbb{R} \) and \( \varepsilon > 0 \). Also, let \( \hat{\mathcal{H}} : \mathbb{R} \to \mathbb{K}^{m \times m} \) be the extension of \( \mathcal{H} \) defined by
\[
\hat{\mathcal{H}}|_{(a-1,a)} = \mathcal{H}(a), \quad \hat{\mathcal{H}}|_{[a,b]} = \mathcal{H}, \quad \hat{\mathcal{H}}|_{(b,b+1]} = \mathcal{H}(b) \\
\hat{\mathcal{H}}|_{(-\infty,a-1)} = 0, \quad \hat{\mathcal{H}}|_{(b+1,\infty)} = 0.
\]
Since \( \mathcal{H}(\zeta) \) is self-adjoint with (3.1) for every \( \zeta \in [a,b] \), it follows that
\[
\overline{m} \leq \hat{\mathcal{H}}(\zeta) \leq \underline{m} \quad (\zeta \in [a-1,b+1]). \quad (3.39)
\]
Since \( \mathcal{H} \in BV([a,b], \mathbb{K}^{m \times m}) \), it further follows that
\[
\hat{\mathcal{H}} \in BV(\mathbb{R}, \mathbb{K}^{m \times m}) \quad \text{with} \quad \text{Var}(\mathcal{H}) = \|\mathcal{H}(a)\| + \text{Var}(\mathcal{H}) + \|\mathcal{H}(b)\|. \quad (3.40)
\]
In particular, \( \hat{\mathcal{H}} \in L^1(\mathbb{R}, \mathbb{K}^{m \times m}) \) and therefore \( j_\varepsilon \ast \hat{\mathcal{H}} \in C^\infty(\mathbb{R}, \mathbb{K}^{m \times m}) \) and \( j_\varepsilon \ast \hat{\mathcal{H}} \rightarrow \hat{\mathcal{H}} \) in \( L^1(\mathbb{R}, \mathbb{K}^{m \times m}) \) as \( \varepsilon \searrow 0 \). Consequently, there exists a sequence \((\varepsilon_n)\) in \((0,1]\) such that \( \varepsilon_n \searrow 0 \) and
\[
(j_\varepsilon \ast \hat{\mathcal{H}})(\zeta) \rightarrow \hat{\mathcal{H}}(\zeta) \quad (n \rightarrow \infty) \quad (3.41)
\]
for almost every \( \zeta \in \mathbb{R} \). Setting now
\[
\mathcal{H}_n := (j_\varepsilon \ast \hat{\mathcal{H}})|_{[a,b]},
\]
we obtain \( \mathcal{H}_n \in C^\infty([a,b], \mathbb{K}^{m \times m}) \subset AC([a,b], \mathbb{K}^{m \times m}) \) for all \( n \in \mathbb{N} \). Assertion (i) immediately follows from (3.41). Assertion (ii), in turn, follows from (3.39) using (3.38) and \( \varepsilon_n \leq 1 \) for \( n \in \mathbb{N} \). It thus remains to prove assertion (iii). Since \( \mathcal{H}_n \in C^1([a,b], \mathbb{K}^{m \times m}) \), it follows by a well-known formula for curve lengths (Theorem VIII.1.3 of [2]) that
\[
\int_a^b \|\mathcal{H}_n'(\zeta)\| \, d\zeta = \text{Var}(\mathcal{H}_n) \quad (3.42)
\]
for every \( n \in \mathbb{N} \). Also, for every partition \((t_l)_{l=1}^L \) of \([a,b] \), we see using (3.38) and (1.7), (1.8), (3.40) that
\[
\sum_{l=1}^L \|\mathcal{H}_n(t_l) - \mathcal{H}_n(t_{l-1})\| = \sum_{l=1}^L \left\| \int_{t_{l-1}}^{t_l} j_\varepsilon(r)(\hat{\mathcal{H}}(t_l - r) - \hat{\mathcal{H}}(t_{l-1} - r)) \, dr \right\|
\leq \int_{\mathbb{R}} j_\varepsilon(r) \sum_{l=1}^L \left\| \hat{\mathcal{H}}(t_l - r) - \hat{\mathcal{H}}(t_{l-1} - r) \right\| \, dr \leq \text{Var}(\hat{\mathcal{H}})
= \|\mathcal{H}(a)\| + \text{Var}(\mathcal{H}) + \|\mathcal{H}(b)\|
\]
and therefore
\[
\text{Var}(\mathcal{H}_n) \leq \|\mathcal{H}(a)\| + \text{Var}(\mathcal{H}) + \|\mathcal{H}(b)\| \quad (3.43)
\]
for every \( n \in \mathbb{N} \). Combining now (3.42) and (3.43), we obtain the desired conclusion (iii) and we are done.

With the above lemmas at hand, we can now establish the following exponential stability result for port-Hamiltonian operators with energy densities of bounded variation. It is a generalization of the respective stability results from [10] (Theorem 9.1.3) and [4] (Proposition 2.12), [3] (Theorem 4.1.5) where the energy densities are required to be continuously differentiable or Lipschitz continuous, respectively.
We point out that, under the assumptions of our exponential stability result, the mere strong stability
\[ e^{At}x \to 0 \quad (t \to \infty) \quad (x \in X) \quad (3.44) \]
could be established with a more direct proof than our exponential stability proof below. We also point out, however, that this more direct proof of strong stability probably does not extend to a more direct exponential stability proof. See the remarks after the proof below for details.

**Theorem 3.5.** Suppose \( A : D(A) \subset X \to X \) is a first-order port-Hamiltonian operator with energy density \( \mathcal{H} \in BV([a, b], \mathbb{K}^{m \times m}) \), where \( X := L^2([a, b], \mathbb{K}^{m}) \) is endowed with the \( \mathcal{H} \)-energy norm \( \| \cdot \|_X \). Suppose further that the domain of \( A \) is of the form
\[ D(A) = \{ x \in X : \mathcal{H}x \in W^{1,2}([a, b], \mathbb{K}^m) \text{ and } W(\mathcal{H}x)|_{\partial} = 0 \} \quad (3.45) \]
for some matrix \( W \in \mathbb{K}^{m \times 2m} \) and that there exists \( \kappa \in (0, \infty) \) such that for \( c = a \) or \( c = b \) one has
\[ \text{Re} \langle x, Ax \rangle_X \leq -\kappa |(\mathcal{H}x)(c)|^2 \quad (x \in D(A)). \quad (3.46) \]
Then \( A \) generates an exponentially stable contraction semigroup on \( X \).

**Proof.** It immediately follows from the assumption (3.46) by Lemma 3.1 that \( A \) generates a contraction semigroup on \( X \) and so we have only to show that \( e^{At} \) is exponentially stable. We do so in various steps by means of a suitable approximation argument. We write
\[ Jf := P_t \partial_x f + P_0 f \quad (f \in D(J)), \]
where \( P_t, P_0 \) are the matrices defining \( A \) and where
\[ D(J) := \{ f \in W^{1,2}([a, b], \mathbb{K}^m) : Wf|_{\partial} = 0 \}. \]
In particular, \( A = JH \). We also choose energy densities \( \mathcal{H}_n \in AC([a, b], \mathbb{K}^{m \times m}) \) as in Lemma 3.4, define
\[ A_n := J\mathcal{H}_n, \]
and endow \( X_n := L^2([a, b], \mathbb{K}^m) \) with the \( \mathcal{H}_n \)-energy norm \( \| \cdot \|_{X_n} \). In particular,
\[ D(A_n) = \{ x \in X_n : \mathcal{H}_n x \in W^{1,2}([a, b], \mathbb{K}^m) \text{ and } W(\mathcal{H}_n x)|_{\partial} = 0 \}. \]

As a first step, we show that \( A_n \) is a contraction semigroup generator on \( X_n \) for every \( n \in \mathbb{N} \). Indeed, \( A_n \) is a port-Hamiltonian operator with energy density \( \mathcal{H}_n \) which has a domain of the form (3.4) and is dissipative in \( X_n \). In order to see the dissipativity, note that for every \( x_n \in D(A_n) = D(J\mathcal{H}_n) \) one has \( f_n := \mathcal{H}_n x_n \in D(J) \) and therefore
\[ \mathcal{H}_n x_n = f_n = \mathcal{H} y_n \]
for \( y_n := \mathcal{H}^{-1} f_n \in D(J\mathcal{H}) = D(A) \). So, by the assumption (3.46), we have for \( c = a \) or \( c = b \) that
\[ \text{Re} \langle x_n, A_n x_n \rangle_{X_n} = \text{Re} \langle \mathcal{H}_n x_n, J\mathcal{H}_n x_n \rangle_2 = \text{Re} \langle \mathcal{H} y_n, J\mathcal{H} y_n \rangle_2 = \text{Re} \langle y_n, A y_n \rangle_X \]
\[ \leq -\kappa |(\mathcal{H} y_n)(c)|^2 = -\kappa |(\mathcal{H}_n x_n)(c)|^2 \quad (3.47) \]
for every \( x_n \in D(A_n) \), which implies the claimed dissipativity of \( A_n \) in \( X_n \). In view of Lemma 3.1 this concludes the proof of the first step.
As a second step, we show that there exist constants $\gamma_0, \kappa_0 \in (0, \infty)$ such that for every $n \in \mathbb{N}$ and $x_{n0} \in D(A_n)$ one has the following sideways energy estimates:

$$F^+_{n\tau}(\zeta) \leq F^+_{n\tau}(b) e^{\kappa_0(b-a)} \quad \text{and} \quad F^-_{n\tau}(\zeta) \leq F^-_{n\tau}(a) e^{\kappa_0(b-a)} \quad (3.48)$$

for every $\zeta \in [a,b]$ and every $\tau > 2\gamma_0(b-a)$, where

$$F^+_{n\tau}(\zeta) := \int_{\gamma_0(b-\zeta)}^{\tau-\gamma_0(b-\zeta)} \mathcal{E}_n(s)(\zeta^*) \mathcal{H}_n(\zeta) \mathcal{E}_n(s)(\zeta) \, ds$$

and

$$F^-_{n\tau}(\zeta) := \int_{\gamma_0(\zeta-a)}^{\tau-\gamma_0(\zeta-a)} \mathcal{E}_n(s)(\zeta^*) \mathcal{H}_n(\zeta) \mathcal{E}_n(s)(\zeta) \, ds$$

and where $\mathcal{E}_n(s)$ denotes the continuous representative of $x_n(s) := e^{A_n s} x_{n0}$ (first step of the proof of Lemma 3.2). We can argue similarly to [10], [3], the essential difference being that in contrast to [10], [3] the derivative $\mathcal{H}_n'$ here need not be in $L^\infty$ but is only in $L^1$. Set

$$\gamma_0 := \| P^{-1}_1 \| m / m \quad \text{and} \quad \kappa_0 := (2 \| P^{-1}_1 P_0 \| m + m') / m \quad (3.49)$$

where $m$, $m'$ are as in Lemma 3.4 and $m' := (\| \mathcal{H}(a) \| + \text{Var}(\mathcal{H}) + \| \mathcal{H}(b) \| ) / (b - a)$. Also, choose and fix $n \in \mathbb{N}$ and $x_{n0} \in D(A_n)$ and write $x_n := e^{A_n} x_{n0}$. Since $A_n$ is a port-Hamiltonian operator with energy density $\mathcal{H}_n \in AC([a,b], \mathbb{R}^{m \times m})$ and since $x_n = e^{A_n} x_{n0}$ is a classical solution of $\dot{x} = A_n x,$

it follows by Lemma 3.2 that $F^\pm_{n\tau} : [a,b] \rightarrow \mathbb{K}$ for every $\tau > 2\gamma_0(b-a)$ is absolutely continuous and hence differentiable almost everywhere with derivative given by

$$(F^\pm_{n\tau})' (\zeta) = \{ \mathcal{E}_n(s)(\zeta^*) (\pm \gamma_0 \mathcal{H}_n(\zeta) + P^{-1}_1 \mathcal{E}_n(s)(\zeta)) \}_{s=t^\pm(\zeta)} + \mathcal{E}_n(s)(\zeta^*) (\pm \gamma_0 \mathcal{H}_n(\zeta) - P^{-1}_1 \mathcal{E}_n(s)(\zeta)) \}_{s=t^\pm(\zeta)} - \int_{t^\pm(\zeta)}^{r^\pm(\zeta)} \mathcal{E}_n(s)(\zeta^*) \left( (P^{-1}_1 P_0 \mathcal{H}_n(\zeta)) + \mathcal{H}_n'(\zeta) + P^{-1}_1 P_0 \mathcal{H}_n(\zeta) \right) \mathcal{E}_n(s)(\zeta) \, ds \quad (3.50)$$

for a.e. $\zeta \in [a,b]$, where $r^+(\zeta) := \gamma_0(b - \zeta)$, $t^+(\zeta) := \tau - \gamma_0(b - \zeta)$ and $r^-(\zeta) := \gamma_0(\zeta - a)$, $t^-(\zeta) := \tau - \gamma_0(\zeta - a)$. In view of Lemma 3.4 (ii) and of (3.49) it follows from (3.50) that

$$(F^+_{n\tau})'(\zeta) \geq -\kappa_0(\zeta) \int_{r^+(\zeta)}^{t^+(\zeta)} m | \mathcal{E}_n(s)(\zeta)|^2 \, ds \geq -\kappa_0(\zeta) F^+_{n\tau}(\zeta) \quad (3.51)$$

and

$$(F^-_{n\tau})'(\zeta) \leq \kappa_0(\zeta) \int_{r^-(\zeta)}^{t^-(\zeta)} m | \mathcal{E}_n(s)(\zeta)|^2 \, ds \leq \kappa_0(\zeta) F^-_{n\tau}(\zeta) \quad (3.52)$$

for all $\tau > 2\gamma_0(b-a)$ and a.e. $\zeta \in [a,b]$, where

$$\kappa_0(\zeta) := \left( 2 \| P^{-1}_1 P_0 \| m + \| \mathcal{H}_n'(\zeta) \| / m \right) / m.$$

Since $F^\pm_{n\tau}$ is absolutely continuous, the differential inequalities (3.51) and (3.52) imply that $\zeta \mapsto F^+_{n\tau}(\zeta) \exp(-\int_\zeta^b \kappa_0(\eta) \, d\eta)$ and $\zeta \mapsto F^-_{n\tau}(\zeta) \exp(-\int_0^\zeta \kappa_0(\eta) \, d\eta)$ are monotonically increasing or decreasing, respectively. Consequently,

$$F^+_{n\tau}(\zeta) \leq (F^+_{n\tau})(b) e^{\kappa_0(b-a)} \quad (3.48)$$

and

$$F^-_{n\tau}(\zeta) \leq (F^-_{n\tau})(a) e^{\kappa_0(b-a)} \quad (3.48)$$
as desired, where for the second inequalities Lemma 3.4 (iii) has been used.

As a third step, we show that there exist constants \( C_0, t_0 \in (0, \infty) \) such that for every \( n \in \mathbb{N} \) and \( x_{n0} \in D(A_n) \) one has the following estimate:

\[
\|x_n(\tau)\|_{X_n}^2 \leq C_0 \int_0^\tau |(\mathcal{H}_n x_n(s))(c)|^2 \, ds \tag{3.53}
\]

for every \( \tau \geq t_0 \) and for \( c = a \) and \( c = b \), where \( x_n := e^{A_n} x_{n0} \). We can argue as in \([10], [3]\) building on our first and second step. Set

\[
t_0 := 2\gamma_0(b - a) + 1 \quad \text{and} \quad C_0 := \frac{e^{\kappa_0(b-a)}}{2\kappa} (b - a). \tag{3.54}
\]

Also, choose and fix \( n \in \mathbb{N} \) and \( x_{n0} \in D(A_n) \) and write \( x_n := e^{A_n} x_{n0} \). Since \( A_n \) generates a contraction semigroup on \( X_n \) by the first step, we see for every \( \tau \geq t_0 \) that

\[
\|x_n(\tau)\|_{X_n}^2 \leq (\tau - 2\gamma_0(b - a)) \|x_n(\tau)\|_{X_n}^2 = \int_{\gamma_0(b-a)}^{\tau-\gamma_0(b-a)} \|x_n(s)\|_{X_n}^2 \, ds
\]

\[
\leq \int_{\gamma_0(b-a)}^{\tau-\gamma_0(b-a)} \|x_n(s)\|_{X_n}^2 \, ds
\]

\[
= \frac{1}{2} \int_a^b \int_{\gamma_0(b-a)}^{\tau-\gamma_0(b-a)} \mathcal{L}_n(s)(\zeta)^* \mathcal{H}_n(\zeta) \mathcal{L}_n(s)(\zeta) \, ds \, d\zeta, \tag{3.55}
\]

where interchanging the integrals in the last equality is justified due to the continuity of \( (s, \zeta) \mapsto \mathcal{L}_n(s)(\zeta) \), see (3.14). Increasing the inner integration interval in (3.55) to \( [\gamma_0(b - \zeta), \tau - \gamma_0(b - \zeta)] \) or \( [\gamma_0(\zeta - a), \tau - \gamma_0(\zeta - a)] \) respectively and using the sideways energy estimates (3.48) from the second step, we conclude that

\[
\|x_n(\tau)\|_{X_n}^2 \leq \frac{1}{2} \int_a^b F_n^+(\zeta) \, d\zeta \leq \frac{1}{2} \min\{F_n^+(b), F_n^-(a)\} e^{\kappa_0(b-a)}(b-a) \tag{3.56}
\]

for every \( \tau \geq t_0 \). And from this, in turn, the desired estimate (3.53) immediately follows (using the definition of \( F_n^\pm \)) both for \( c = b \) and for \( c = a \).

As a fourth step, we show that there exist constants \( M_0 \in [1, \infty) \) and \( \omega_0 \in (-\infty, 0) \) such that

\[
\|e^{A_n t}\|_{X_n, X_n} \leq M_0 e^{\omega_0 t} \tag{3.57}
\]

for all \( t \in \mathbb{R}_0^+ \) and \( n \in \mathbb{N} \), where \( \|\|_{X_n, X_n} \) is the operator norm induced by \( \|\|_{X_n} \).

Indeed, from the third step and (3.47) it follows that for every \( n \in \mathbb{N} \) and \( x_{n0} \in D(A_n) \)

\[
\|e^{A_n t_0} x_{n0}\|_{X_n}^2 \leq C_0 \int_0^{t_0} |(\mathcal{H}_n x_n(s))(c)|^2 \, ds \leq -C_0/\kappa \int_0^{t_0} \Re \langle x_n(s), A_n x_n(s) \rangle \, ds
\]

\[
= C_0/(2\kappa) \|x_{n0}\|_{X_n}^2 - \|e^{A_n t_0} x_{n0}\|_{X_n}^2,
\]

where as usual \( x_n := e^{A_n} x_{n0} \). So, by the density of \( D(A_n) \) in \( X_n \), we obtain

\[
\|e^{A_n t_0}\|_{X_n, X_n} \leq \left( \frac{C_0/(2\kappa)}{1 + C_0/(2\kappa)} \right)^{1/2} =: \mu_0
\]
for every $n \in \mathbb{N}$. And from this, in turn, we conclude by the semigroup and the contraction semigroup property of $e^{A_n}$ that for arbitrary $t \in \mathbb{R}_0^+$ one has
\[
\|e^{A_n t}\|_{X_n, X_n} = \|(e^{A_n t_0})^l e^{A_n (t-t_0)}\|_{X_n, X_n} \leq \mu_0^l = \frac{1}{\mu_0} \mu_0^{l+1} \leq \frac{1}{\mu_0} \mu_0^{t/t_0},
\]
where we used the abbreviation $l := \lfloor t/t_0 \rfloor$ for the integer part of $t/t_0$ and the fact that $\mu_0 < 1$. Setting
\[
M_0 := \frac{1}{\mu_0} \in [1, \infty) \quad \text{and} \quad \omega_0 := (\log \mu_0)/t_0 \in (-\infty, 0),
\]
we recognize (3.58) as the desired estimate (3.57).

As a fifth and last step, we finally show that $e^{A_t}$ is exponentially stable. Indeed, since $\|\cdot\|_X$ is equivalent to $\|\cdot\|_X$ with equivalence constants independent of $n$ (Lemma 3.4 (ii)), it follows from the fourth step that there exists a constant $M \in [1, \infty]$ such that
\[
\|e^{A_n t}\|_{X, X} \leq M e^{\omega_0 t}
\]
for all $t \in \mathbb{R}_0^+$ and $n \in \mathbb{N}$, where $\|\cdot\|_{X, X}$ is the operator norm induced by $\|\cdot\|_X$. Also, for every $x \in D(A)$ there exists a sequence $(x_n)$ with $x_n \in D(A_n)$ and
\[
x_n \xrightarrow{X} x \quad \text{and} \quad A_n x_n \xrightarrow{X} Ax
\]
as $n \to \infty$. (Indeed, for $x \in D(A) = D(JH)$ one has $f := Hx \in D(J)$ and $x_n := H_n^{-1} f \in D(JH_n) = D(A_n)$ and $A_n x_n = Jf = Ax \to Ax$ as well as $x_n = H_n^{-1} f \to H^{-1} f = x$, where for the last convergence we used dominated convergence along with Lemma 3.4 (i) and (ii).) Combining now (3.60) and (3.61), we see by the theorem of Trotter and Kato (Theorem III.4.8 of [6]) that $e^{A_n t} \to e^{At}$ in the strong operator topology of $X$ as $n \to \infty$ for every $t \in \mathbb{R}_0^+$. So, by (3.60),
\[
\|e^{At}\|_{X, X} \leq M e^{\omega_0 t}
\]
for every $t \in \mathbb{R}_0^+$. And since $\omega_0 < 0$, this finally proves the asserted exponential stability.

As has been pointed out above, under the assumptions of our exponential stability result, the mere strong stability (3.44) could be proved in a more direct way than how we established exponential stability above. Indeed, from the dissipation assumption (3.46) one can show, using the uniqueness theorem for maximal Carathéodory solutions (Theorem 54 in Section C.2 of [15]), that
\[
\sigma_p(A) \cap i\mathbb{R} = \emptyset.
\]
Also, one can easily show that the contraction semigroup generator $A$ has compact resolvent, using Lemma 3.2.3 of [3] and the compact embedding of $W^{1,2}([a, b], \mathbb{R}^m)$ into $X$. And from these two facts, in turn, the strong stability (3.44) follows by virtue of the stability theorem of Arendt, Batty, Lyubich and Průša (Corollary V.2.24 in conjunction with Corollary V.2.15 of [6]). In order to extend this strong stability proof to a proof of exponential stability, however, an extra boundedness relation for the resolvent would have to be established. In fact, by the stability theorem of Gearhart and Průša (Theorem V.1.11 in conjunction with Theorem V.1.18), it would be necessary and sufficient to additionally prove
\[
\sup_{\omega \in \mathbb{R}} \|(i\omega - A)^{-1}\|_{X, X} < \infty.
\]
Yet, to prove this in a direct way is difficult (or even elusive). At least, the proof of (3.64) for $\mathcal{H} \in W^{1,\infty}([a,b],\mathbb{K}^{m \times m})$ from [4] (Proposition 2.12) cannot be extended to the case of $\mathcal{H} \in BV([a,b],\mathbb{K}^{m \times m})$, which we treat in our stability theorem above. (Indeed, the $W^{1,\infty}$-assumption from [4] is used in an essential way in the exponential stability proof of [4], namely first for partial integration and second for establishing a constant positive lower bound of a stability proof of [4], which we treat in our stability theorem above.

With the above theorem at hand, we can now easily prove the following stabilization result. See the remarks after the proof for a control-theoretic interpretation of this result and its assumptions.

**Corollary 3.6.** Suppose $\mathcal{A} : D(\mathcal{A}) \subset X \to X$ is a first-order port-Hamiltonian operator with energy density $\mathcal{H} \in BV([a,b],\mathbb{K}^{m \times m})$, where $X := L^2([a,b],\mathbb{K}^m)$ is endowed with the $\mathcal{H}$-energy norm $\| \cdot \|_X$ and where

$$D(\mathcal{A}) = \{ x \in X : \mathcal{H}x \in W^{1,2}([a,b],\mathbb{K}^m) \text{ and } W_{B,1}(\mathcal{H}x)\partial = 0 \}$$

for some matrix $W_{B,1} \in \mathbb{K}^{(m-k) \times 2m}$ and some $k \in \{1, \ldots, m\}$. Suppose further that $B, C : D(\mathcal{A}) \subset X \to \mathbb{K}^k$ are linear boundary operators given by

$$Bx := W_{B,2}(\mathcal{H}x)\partial \quad \text{and} \quadCx := W_C(\mathcal{H}x)\partial$$

with matrices $W_{B,2}, W_C \in \mathbb{K}^{k \times 2m}$ such that the following two conditions are satisfied:

(i) $\Re \langle x, Ax \rangle_X \leq \langle Bx \rangle^* Cx$ for all $x \in D(\mathcal{A})$

(ii) there exists a constant $\lambda \in (0, \infty)$ such that for $c = a$ or $c = b$ one has

$$|Bx|^2 + |Cx|^2 \geq \lambda (\mathcal{H}x)(c)^2 \quad (x \in D(\mathcal{A})).$$

Then for every $\mu \in (0, \infty)$ the operator $A := \mathcal{A}|_{D(\mathcal{A})}$, which arises by restricting the operator $\mathcal{A}$ above to the domain

$$D(A) := \{ x \in D(\mathcal{A}) : Bx = -\mu Cx \},$$

generates an exponentially stable contraction semigroup on $X$.

**Proof.** Choose and fix $\mu \in (0, \infty)$ and define the matrix

$$W := \begin{pmatrix} W_{B,1} \\ W_{B,2} + \mu W_C \end{pmatrix} \in \mathbb{K}^{m \times 2m}.$$ 

Then the domain of $A$ is of the form (3.45) and there exists a constant $\kappa \in (0, \infty)$ such that for $c = a$ or $c = b$ the estimate (3.46) holds true. Indeed, setting

$$\kappa := \lambda \min \left\{ \frac{1}{2\mu}, \frac{\mu}{2} \right\},$$

we conclude from our assumptions (i) and (ii) that for every $x \in D(\mathcal{A})$

$$\Re \langle x, Ax \rangle_X \leq \frac{1}{2} \langle Bx \rangle^* Cx + \frac{1}{2} \langle Bx \rangle^* Cx = -\frac{1}{2\mu} |Bx|^2 - \frac{\mu}{2} |Cx|^2 \leq -\kappa |(\mathcal{H}x)(c)|^2.$$

So, the assertion of the corollary follows from the previous theorem. \qed
In control-theoretic terms, the above corollary says that the linear input-output system
\[ \dot{x} = Ax \]  
\[ u(t) = B x(t) \quad \text{and} \quad y(t) = C x(t) \]
with control input \( u \) and observation output \( y \) is exponentially stabilized by means of the negative output-feedback law
\[ u(t) = -\mu y(t) \]
with an arbitrary amplification factor \( \mu > 0 \). Condition (i) of the above corollary means that the input-output system \((3.68)-(3.69)\) is impedance-passive in the sense of [16], [3]. Condition (ii), in turn, means that the control input and observation output dominate the value of the state at one of the boundary points \((a \text{ or } b)\). Also, if one sharpens the assumptions of the above corollary – namely by additionally requiring that \( H \in AC([a,b],\mathbb{R}^{m \times m}) \) and that \( \text{Re} \{x, Ax\}_X = (B x)^* C x \) for all \( x \in D(A) \) (impedance-energy-preservation) – then the system \((3.68)-(3.69)\) is classically approximately observable in infinite time in the sense of [12] (Condition 4.9). In fact, this can be proven in exactly the same way as Lemma 4.16 of [12].

4. Some applications. In this section, we apply our stabilization result to a vibrating string and a Timoshenko beam, respectively.

Example 4.1. Consider a vibrating string [17], [10], [3], that is, the transverse displacement \( w(t, \zeta) \) of the string at position \( \zeta \in [a,b] \) evolves according to the partial differential equation
\[ \rho(\zeta) \partial_t^2 w(t, \zeta) = \partial_t \left(T(\zeta) \partial_t w(t, \zeta)\right) \quad (t \in [0, \infty), \zeta \in [a,b]) \]  
(vibrating string equation) and the energy \( E_w(t) \) of the string at time \( t \) is given by
\[ E_w(t) = \frac{1}{2} \int_a^b \rho(\zeta) \left(\partial_t w(t, \zeta)\right)^2 + T(\zeta) \left(\partial_t w(t, \zeta)\right)^2 \, d\zeta. \]
In these equations, \( \rho, T \) are the mass density and the Young modulus of elasticity of the string and they are assumed to belong to \( BV([a,b], \mathbb{R}) \) and to be bounded below and above by positive finite constants. Also, assume that the string is clamped at its left end, that is,
\[ \partial_t w(t, a) = 0 \quad (t \in [0, \infty)) \]  
(4.2)
and that the control input \( u(t) \) and observation output \( y(t) \) are given respectively by the force and by the velocity at the right end of the string, that is,
\[ u(t) = T(b) \partial_t w(t, b) \quad \text{and} \quad y(t) = \partial_t w(t, b) \]  
(4.3)
for all \( t \in [0, \infty) \). With the choices
\[ x(t)(\zeta) := \begin{pmatrix} \rho(\zeta) \partial_t w(t, \zeta) \\ \partial_t w(t, \zeta) \end{pmatrix}, \quad H(\zeta) := \begin{pmatrix} 1/\rho(\zeta) & 0 \\ 0 & T(\zeta) \end{pmatrix}, \quad P_3 := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \]
and \( P_0 := 0 \in \mathbb{R}^{2 \times 2} \), the pde \((4.1)\) with the boundary condition \((4.2)\) takes the form \((3.68)\) of a first-order port-Hamiltonian system with \((3.65)\) and with \( W_{B,1} \in \mathbb{R}^{1 \times 4} \) and, moreover, the in- and output conditions \((4.3)\) take the desired form \((3.69)\) with \((3.66)\) and with matrices \( W_{B,2}, W_C \in \mathbb{R}^{1 \times 4} \). It is straightforward to verify that \( H \) is an energy density with \( H \in BV([a,b], \mathbb{R}^{2 \times 2}) \) and that condition (i) (even impedance-energy-preservation) and condition (ii) of Corollary 3.6 are satisfied. So,
Example 4.2. Consider a beam modelled according to Timoshenko [17], [10], [3], that is, the transverse displacement $w(t, \zeta)$ and the rotation angle $\varphi(t, \zeta)$ of the beam at position $\zeta \in [a, b]$ evolve according to the partial differential equations

$$\rho(\zeta)\partial_{\zeta}^2 w(t, \zeta) = \partial_{\zeta}\left( K(\zeta)(\partial_{\zeta} w(t, \zeta) - \varphi(t, \zeta)) \right)$$

$$I_{r}(\zeta)\partial_{\zeta}^2 \varphi(t, \zeta) = \partial_{\zeta}\left( EI(\zeta)\partial_{\zeta} \varphi(t, \zeta) \right) + K(\zeta)(\partial_{\zeta} w(t, \zeta) - \varphi(t, \zeta))$$

for $t \in [0, \infty), \zeta \in [a, b]$ (Timoshenko beam equations) and the energy $E_{w,\varphi}(t)$ of the beam at time $t$ is given by

$$E_{w,\varphi}(t) = \frac{1}{2} \int_{a}^{b} \rho(\zeta)(\partial_{\zeta} w(t, \zeta))^2 + K(\zeta)(\partial_{\zeta} w(t, \zeta) - \varphi(t, \zeta))^2$$

$$+ I_{r}(\zeta)(\partial_{\zeta} \varphi(t, \zeta))^2 + EI(\zeta)(\partial_{\zeta} \varphi(t, \zeta))^2 \, d\zeta.$$ 

In these equations, $\rho$, $E$, $I$, $I_{r}$, $K$ are respectively the mass density, the Young modulus, the moment of inertia, the rotatory moment of inertia, and the shear modulus of the beam and they are assumed to belong to $BV([a, b], \mathbb{R})$ and to be bounded below and above by positive finite constants. Also, assume that the beam is clamped at its left end, that is,

$$\partial_{\zeta} w(t, a) = 0 \quad \text{and} \quad \partial_{\zeta} \varphi(t, a) = 0 \quad (t \in [0, \infty)) \quad (4.6)$$

(velocity and angular velocity at the left endpoint are zero), and that the control input $u(t)$ is given by the force and the torsional moment at the right end of the beam and the observation output $y(t)$ is given by the velocity and angular velocity at the right end of the beam, that is,

$$u(t) = \begin{pmatrix} K(b)(\partial_{\zeta} w(t, b) - \varphi(t, b)) \\ EI(b)\partial_{\zeta} \varphi(t, b) \end{pmatrix}, \quad y(t) = \begin{pmatrix} \partial_{\zeta} w(t, b) \\ \partial_{\zeta} \varphi(t, b) \end{pmatrix} \quad (4.7)$$

for all $t \in [0, \infty)$. With the choices

$$x(t)(\zeta) := \begin{pmatrix} \partial_{\zeta} w(t, \zeta) - \varphi(t, \zeta) \\ \rho(\zeta)\partial_{\zeta} w(t, \zeta) \\ \partial_{\zeta} \varphi(t, \zeta) \\ I_{r}(\zeta)\partial_{\zeta} \varphi(t, \zeta) \end{pmatrix}, \quad \mathcal{H}(\zeta) := \begin{pmatrix} K(\zeta) & 0 & 0 & 0 \\ 0 & 1/\rho(\zeta) & 0 & 0 \\ 0 & 0 & EI(\zeta) & 0 \\ 0 & 0 & 0 & 1/I_{r}(\zeta) \end{pmatrix},$$

and an appropriate choice of $P_{1}, P_{0} \in \mathbb{R}^{4 \times 4}$, the pde (4.4), (4.5) with the boundary conditions (4.6) take the form (3.68) of a first-order port-Hamiltonian system with (3.65) and with $W_{B,1} \in \mathbb{R}^{2 \times 8}$ and, moreover, the in- and output conditions (4.7) take the desired form (3.69) with (3.66) and with matrices $W_{B,2}, W_{C} \in \mathbb{R}^{2 \times 8}$. It is straightforward to verify that $\mathcal{H}$ is an energy density with

$$\mathcal{H} \in BV([a, b], \mathbb{R}^{4 \times 4})$$

and that condition (i) (even impedance-energy-preservation) and condition (ii) of Corollary 3.6 are satisfied. So, by that corollary, the input-output system (4.4)-(4.7) is exponentially stabilized by means of the negative output-feedback law (3.70).
Acknowledgments. I would like to thank the German Research Foundation (DFG) for financial support through the grant “Input-to-state stability and stabilization of distributed-parameter systems” (DA 767/7-1). Additionally, I would like to thank the anonymous reviewers for their valuable comments.

REFERENCES

[1] R. A. Adams and J. J. F. Fournier, Sobolev Spaces, 2nd edition. Elsevier, 2003.
[2] H. Amann and J. Escher, Analysis I, II, III, Birkhäuser, 2005, 2008, 2009.
[3] B. Augner, Stabilisation of Infinite-Dimensional Port-Hamiltonian Systems via Dissipative Boundary Feedback, PhD thesis. Available at http://elpub.bib.uni-wuppertal.de/edocs/dokumente/fbc/mathematik/diss2016/augner/dc1613.pdf.
[4] B. Augner and B. Jacob, Stability and stabilization of infinite-dimensional linear port-Hamiltonian systems, Evol. Equ. Contr. Th., 3 (2014), 207–229.
[5] S. Cox and E. Zuazua, The rate at which energy decays in a string damped at one end, Indiana Univ. Math. J., 44 (1995), 545–573.
[6] K.-J. Engel and R. Nagel, One-Parameter Semigroups for Linear Evolution Equations, Springer, 2000.
[7] G. B. Folland, Real Analysis, 2nd edition, Wiley, 1999.
[8] E. Hille and R. S. Phillips, Functional Analysis and Semi-Groups, American Mathematical Society Colloquium Publications, 1957.
[9] B. Jacob, K. Morris and H. Zwart, C0-semigroups for hyperbolic partial differential equations on a one-dimensional spatial domain, J. Evol. Equ., 15 (2015), 493–502.
[10] B. Jacob and H. J. Zwart, Linear Port-Hamiltonian Systems on Infinite-Dimensional Spaces, Birkhäuser, 2012.
[11] W. Rudin, Real and Complex Analysis, 3rd edition. McGraw-Hill, 1987.
[12] J. Schmid and H. Zwart, Stabilization of port-Hamiltonian systems by nonlinear boundary control in the presence of disturbances, ESAIM Contr. Optim. Calc. Var., 27 (2021), Paper No. 53, 37 pp.
[13] W. Sierpiński, Sur un problème concernant les ensembles mesurables superficiellement, Fund. Math., 1 (1920), 112–115.
[14] W. Sierpiński, Sur les rapports entre l’existence des intégrales \( \int_0^1 f(x,y)dx \), \( \int_0^1 f(x,y)dy \) et \( \int_0^1 dx \int_0^1 f(x,y)dy \), Fund. Math., 1 (1920), 142–147.
[15] E. D. Sontag, Mathematical Control Theory. Deterministic Finite-Dimensional Systems, 2nd edition, Springer, 1998.
[16] M. Tucsnak and G. Weiss, Well-posed systems – the LTI case and beyond, Automatica, 50 (2014), 1757–1779.
[17] J. Villegas, A Port-Hamiltonian Approach to Distributed-Parameter Systems, Ph.D. thesis, Universität Twente, 2007.
[18] J. A. Villegas, H. Zwart, Y. Le Gorrec and B. Maschke, Exponential stability of a class of boundary control systems, IEEE Trans. Autom. Contr., 54 (2009), 142–147.

Received October 2018; revised October 2021; early access January 2022.

E-mail address: jochen.schmid@itwm.fraunhofer.de