Gauging Yang-Mills Symmetries
In 1+1-Dimensional Spacetime

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ABSTRACT
We present a systematic and ‘from the ground up’ analysis of the ‘minimal coupling’ type of gauging of Yang-Mills symmetries in (2,2)-supersymmetric 1+1-dimensional spacetime. Unlike in the familiar 3+1-dimensional $N = 1$ supersymmetric case, we find several distinct types of minimal coupling symmetry gauging, and so several distinct types of gauge (super)fields, some of which entirely novel. Also, we find that certain (quartoid) constrained superfields can couple to no gauge superfield at all, others (haploid ones) can couple only very selectively, while still others (non-minimal, i.e., linear ones) couple universally to all gauge superfields.
1. Introduction

Ref. [1] presented an intrinsically 1+1-dimensional analysis, ‘from scratch’, of the basic building blocks in (2,2)-supersymmetric theories: the superconstrained superfields, most often used to represent ‘matter’. This analysis re-established some earlier results [2,3,4,5], generalized them in several different aspects, and uncovered some hitherto unknown phenomena and mechanisms leading to several open questions and new research topics.

While comprehensive in its study of the variously constrained ‘matter’ superfields, Ref. [1] postponed the study of gauge (super)fields and their couplings to ‘matter’. This important issue is analyzed herein, again from an intrinsically 1+1-dimensional approach, ‘from scratch’, and loosely following the methodology of §4.2.b of Ref. [6] (p. 169). We find two new types of symmetry gauging, besides the one that descends from 3+1-dimensional, \( N=1 \) supersymmetric theories by dimensional reduction and its mirror-twisted version [3,4].

This article is organized as follows: The remaining part of this section presents the basics of 1+1-dimensional (2,2)-superspace and sets up the notation; further details are found in Appendix A. The gauge-covariant (super)derivatives are defined and the field strength superfields calculated in Section 2, with the details of Jacobi identity calculations deferred to the Appendix B. Section 3 analyzes the most general minimal coupling of gauge superfields with ‘matter’. Some special types of symmetry gauging are discussed in Section 4. Section 5 discusses the component field content within the gauge superfields, with details deferred to appendix C. The choice of the Lagrangian density for ‘matter’ superfields with gauged symmetries is discussed in Section 6, generalizing the already immense palette presented in Ref. [1]. Finally, section 7 summarizes the presented results and discusses some further topics.

We emphasize that our purpose here is to enlist all logical possibilities, and explore their consistency and major features, but leave open the details and issues of application.

1.1. \((1,1|2,2)\) superspacetime

(Super)field theory in 1+1-dimensional spacetime is crucially distinct from that in higher dimensions because of its Lorentz group, \( SO(1,1) \): being Abelian, all of its irreducible representations are 1-dimensional. For example, the coordinate 2-vector \((\sigma^0, \sigma^1)\) decomposes into the (light-cone) characteristic coordinates \( \sigma^\pm \) such that \( \sigma^\pm \equiv \frac{1}{2} (\sigma^0 - \sigma^1) \) and \( \sigma^\mp \equiv \frac{1}{2} (\sigma^0 + \sigma^1) \). These are eigenfunctions of the Lorentz (boost) generator \( \mathbf{B} \) and the group elements of \( SO(1,1) \):

\[
\begin{align*}
B(\sigma^-, \sigma^+) &= (-\sigma^-, +\sigma^+) , & B &\equiv (\sigma^1 \partial_0 + \sigma^0 \partial_1) , \\
U_\alpha(\sigma^-, \sigma^+) &= (e^{-i\alpha} \sigma^-, e^{+i\alpha} \sigma^+) , & U_\alpha &\equiv e^{i\alpha B} .
\end{align*}
\]

\(^1\) The eigenvalue of the Lorentz boost operator, extended to include total angular momentum in the usual way, equals the spin projection, and will therefore be denoted by \( j_3 \) although it does not stem from its 3+1-dimensional namesake. Moreover, all representations of the 1+1-dimensional Lorentz group being 1-dimensional, ‘spin’ and ‘spin projection’ are one and the same, and we will call \( j_3 \) simply ‘spin’.
All other tensors (spinors) decompose similarly into 1-component objects. Upon the frequently practiced analytic continuation \( \sigma^0 \to i\sigma^0 \), \((\sigma^\pm, \sigma^\mp) \to (z, \bar{z})\): the light-cone structure becomes a complex structure and \( U_\alpha \) becomes the winding number (holomorphic homogeneity) operator. The sub- and superscripts “±” then simply denote the winding number (spin in real time) in units of \( \frac{1}{2}(h) \). Functions depending on only \( \sigma^\pm \) or only \( \sigma^\mp \) are called left- and right-movers, respectively, and become holomorphic (complex-analytic) and anti-holomorphic (complex-antianalytic) functions upon analytic continuation to imaginary time.

1.2. \((2,2)\)-superbasics

The \((2,2)\)-supersymmetry algebra involves the supersymmetry charges \( Q_\pm \) and \( \bar{Q}_\pm \), which satisfy (adapting from Refs. [7,2]; comparison with Refs. [5,6] is provided in appendix A):

\[
\{ Q_-, \bar{Q}_- \} = -2i\partial_- , \quad \{ Q_+, \bar{Q}_+ \} = -2i\partial_+ .
\]

Here \( \partial_+=\frac{\partial}{\partial\sigma^+}=(\partial_0+\partial_1) \) and \( \partial_-=\frac{\partial}{\partial\sigma^-}=(\partial_0-\partial_1) \). \( p= -i\partial_1 = -\frac{i}{2}(\partial_++\partial_-) \) is the linear momentum operator and \( H= -i\partial_0 = -\frac{i}{2}(\partial_+^2+\partial_-^2) \) is the Hamiltonian (energy) operator.

Equipping the world-sheet with anticommuting fermionic coordinates, \( \zeta^\pm, \bar{\zeta}^\pm \), the supercharges are realized as differential operators on the (super) world-sheet:

\[
Q_- \overset{\text{def}}{=} \partial_- + i\zeta^- \partial_- , \quad \bar{Q}_- \overset{\text{def}}{=} -\bar{\partial}_- - i\zeta^- \bar{\partial}_- ,
\]

\[
Q_+ \overset{\text{def}}{=} \partial_+ + i\zeta^+ \partial_+ , \quad \bar{Q}_+ \overset{\text{def}}{=} -\bar{\partial}_+ - i\zeta^+ \bar{\partial}_+ ,
\]

The spinorial derivatives

\[
\begin{align*}
D_- &\overset{\text{def}}{=} \partial_- - i\zeta^- \partial_- , & \bar{D}_- &\overset{\text{def}}{=} -\bar{\partial}_- + i\zeta^- \bar{\partial}_- , \\
D_+ &\overset{\text{def}}{=} \partial_+ - i\zeta^+ \partial_+ , & \bar{D}_+ &\overset{\text{def}}{=} -\bar{\partial}_+ + i\zeta^+ \bar{\partial}_+ ,
\end{align*}
\]

are covariant with respect to supersymmetry transformations: \( \{ Q_\pm, D_\pm \} = 0 = \{ \bar{Q}_\pm, D_\pm \} \), whereupon \([ U_{e,\bar{e}}, D_\pm \] = 0 = \([ U_{e,\bar{e}}, \bar{D}_\pm \] \), with \( U_{e,\bar{e}} \overset{\text{def}}{=} \exp \{ i(\zeta^\pm Q_\pm + \bar{\zeta}^\pm \bar{Q}_\pm) \} \).

Finally the \( D, \bar{D} \)'s close virtually the same algebra (1.2), as do the \( Q, \bar{Q} \)'s:

\[
\{ D_-, \bar{D}_- \} = 2i\partial_-, \quad \{ D_+, \bar{D}_+ \} = 2i\partial_+ .
\]

All anticommutators among the \( Q_\pm, \bar{Q}_\pm, D_\pm, \bar{D}_\pm \) other than (1.2) and (1.3), vanish.

Berezin superintegrals are by definition equivalent to partial superderivatives, and up to total (world-sheet) spacetime derivatives \(^2\) (which we ignore, assuming world-sheets without boundaries) equivalent to covariant superderivatives \([7,1,4,8,9]\). Following Ref. [4], we use (see appendix A for further definitions and conventions):

\[
\int d^4\zeta \ (\ldots) \overset{\text{def}}{=} \frac{1}{8} \{ \{ D_-, \bar{D}_- \}, \{ D_+, \bar{D}_+ \} \} (\ldots) \overset{\text{def}}{=} (D^4 \ldots) ,
\]

\(^2\) Hereafter, ‘total derivative’ will stand for ‘total (world-sheet) spacetime derivative’.
where “|” means setting $\zeta^\pm = 0 = \bar{\zeta}^\pm$. Integration over a fermionic subspace is formally achieved by inserting a fermionic Dirac delta-function (see Appendix A), so that:

$$\int d^2\zeta \ldots \equiv \frac{1}{2} [D_-, D_+] \ldots , \quad \int d^2\bar{\zeta} \ldots \equiv \frac{1}{2} [D_+, D_-] \ldots , \quad (1.7a, b)$$

$$\int d^2\bar{\zeta} \ldots \equiv \frac{1}{2} [D_-, \bar{D}_+] \ldots , \quad \int d^2\zeta \ldots \equiv \frac{1}{2} [D_+, \bar{D}_-] \ldots , \quad (1.7c, d)$$

$$\int d^2\zeta_R \ldots \equiv \frac{1}{2} [D_-, \bar{D}_-] \ldots , \quad \int d^2\zeta_L \ldots \equiv \frac{1}{2} [D_+, \bar{D}_+] \ldots , \quad (1.7e, f)$$

are integrals over the various halves of the fermionic coordinates.

2. Gauge-Covariant Superderivatives

In the presence of a gauge symmetry, all derivatives, including those in the Berezin integrals (1.6) and (1.7), need to be ‘covariantized’. This modifies the supersymmetry algebra (1.5), and this modification is used here as the starting point for the analysis. Somewhat formally, the Berezin integrals are easily turned into gauge-covariant ones, by replacing $D$’s $\rightarrow \nabla$’s; see appendix A for details.

2.1. Definitions

Much of the subsequent analysis follows the procedures used in Ref. [6], although the notation will be adjusted to conform with Refs. [2, 7]. We start by defining the covariant (super)derivatives

$$\nabla_\pm = D_\pm - i\Gamma_\pm , \quad \bar{\nabla}_\pm = D_\pm - i\bar{\Gamma}_\pm , \quad (2.1a)$$

$$\nabla_\mp = \partial_\mp - i\Gamma_\mp , \quad \bar{\nabla}_\mp = \partial_\mp - i\bar{\Gamma}_\mp . \quad (2.1b)$$

The $\Gamma$’s are Lie algebra valued gauge superfields and are, in general, linear combinations of gauge superfields for the direct summands in the possibly non-simple Lie algebra.

For brevity and convenience, the gauge coupling constants were absorbed in the definition of the gauge superfields $\Gamma$; these can be reinserted later by replacing $\Gamma \rightarrow g\Gamma$. Also, recall that the superderivatives (1.4) already include a connection ‘$\frac{1}{2}$-form coefficient’. For example, $D_-$ and $\bar{D}_-$ include $\bar{\zeta}^-\partial_\pm$ and $\zeta^-\partial_\pm$ as the (derivative-valued!) ‘$\frac{1}{2}$-form coefficients’: spacetime derivatives here play the rôle of the generators of the spacetime translation group, and the (fermionic) ‘gauge superfields’ here are simply the supercoordinates $\bar{\zeta}^-$ and $\zeta^-$. The (super)derivatives (2.1) are all covariant with respect to the general gauge transformation:

$$\nabla' = g \nabla g^{-1} , \quad X' = g X , \quad (2.2)$$

where $X$ represents any (homogeneously transforming) ‘matter’ (super)field, and $g$ is the operator implementing the gauge transformation. Typically, we assume the gauge
transformation operator, \( G \), to be unitary, so that the transformation of the ‘matter’ superfield (2.2) implies

\[
X' = X G^\dagger = X G^{-1}, \tag{2.3}
\]

whence preserving the norm (squared),

\[
\|X\|^2 \overset{\text{def}}{=} \text{Tr} \left( X, X^i \right) = \text{Tr} \left( X^i X_i \right), \tag{2.4}
\]

where \( i \) simply counts the superfields \( X^i \), each of which is a collection of superfields forming a given representation of the gauge group. Because of the transposition involved in (2.3), the action of gauge-covariant derivatives on \( X \) is awkward: the derivative part of \( \nabla \) should act from the left as usual, but the gauge superfield part should act from the right. We therefore calculate (implicitly) using a double transposition:

\[
(O X) \overset{\text{def}}{=} (O^\alpha X^\beta)^{\gamma} = (\bar{O} X)^{\dagger}, \tag{2.5}
\]

where \( O \) denotes any gauge-covariant operator, \( O^* \) and \( O^T \) its complex conjugate and transpose, respectively; over-bar and dagger interchangeably denote Hermitian conjugation: \( \bar{O} \equiv O^\dagger \). In practical calculation, and in cases when above ‘matrix’ notation would be ambiguous or confusing, we resort to the explicit gauge group index notation. Assuming that the matter fields form a representation of the gauge group the elements of which are indexed by \( \alpha, \beta, \ldots \), Eqs. (2.2), (2.3) and (2.5) become:

\[
\nabla'_{\alpha \beta} = G^\gamma_{\alpha \delta} D^\delta_{\beta \gamma}, \quad X'^{\alpha} = G^{\alpha \beta} X^\beta, \tag{2.6}
\]

\[
X'_{\alpha} = X_{\beta} G^{\beta}_{\alpha} = X_{\beta} G^{-1}_{\beta \alpha}, \tag{2.7}
\]

and

\[
(O X)_{\alpha} \overset{\text{def}}{=} (O^\beta X^\alpha)^{\beta}_{\gamma} = (\bar{O}^\beta X^\alpha)^{\dagger}, \tag{2.8}
\]

respectively. This disentangles ordering issues and the ‘matrix’ action of the gauge fields on the matter fields: re-ordering now solely depends on the spin/statistics of the involved superfields and operators. For most of this article, however, we suppress explicit gauge group indices, hoping that the Reader will always be able to discern the implied meaning of the more compact implicit ‘matrix’ notation.

The gauge superfields, \( \Gamma \), of course transform inhomogeneously:

\[
\Gamma' = G \Gamma G^{-1} - i G^{-1} (D G). \tag{2.9}
\]

Field strength superfields and torsion superfields, \( F \) and \( T \), are defined by (anti)commutation of the covariant derivatives (2.1), according the general formula \(^{3}\)

\[
[\nabla, \nabla] = T \cdot \nabla - i F, \tag{2.10}
\]

\(^{3}\) Here "\( [\ , \] \)" denotes the (anti)commutator, as appropriate for the (anti)commuted quantities; see appendix A for precise definition and use.
which determines \( F, T \) in terms of the gauge superfields \( \Gamma \) upon using Eqs. (2.1). Because of their definition (2.10), field strength superfields and torsion superfields are also covariant: with respect to the gauge transformation \( \mathcal{G} \), they transform just like the \( \nabla \)'s do (2.2).

Next, we generalize the standard supersymmetry algebra by inserting the so far unrestricted field strength superfields (choosing numerical coefficients for later convenience):

\[
\{\nabla_-, \nabla_+\} \overset{\text{def}}{=} \mathcal{A}, \quad \{\nabla_-, \nabla_+\} \overset{\text{def}}{=} \mathcal{B}, \quad \{\nabla_-, \nabla_+\} \overset{\text{def}}{=} \mathcal{C}, \quad \{\nabla_-, \nabla_+\} \overset{\text{def}}{=} \mathcal{W} ,
\]

where \( \Box \) is the \textit{gauge-covariant} d’Alembertian (wave operator). Notice that the field strengths \( \mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{W} \) are all complex \(^4\), the \( \mathcal{C}' \)'s being defined by the conjugates of Eqs. (2.11a), and no new field strength has been introduced in Eqs. (2.11a), as it would merely redefine the gauge superfields \( \Gamma_-, \Gamma_\pm \). Also, all torsion vanishes, except for \( T^-_- = 2i T^\pm_\pm \) as it appears already in Eqs. (2.11a). This is because the only derivative-valued connection 1-forms are within the superderivatives, \( D_\pm, \bar{D}_\pm \), and so the torsion content is unchanged by the covariantization (2.1).

The Reader familiar with the 3+1-dimensional supersymmetry algebra modified by a gauge symmetry and its dimensional reduction to 1+1-dimensional spacetime will realize that \( \mathcal{A} \) in (2.11a) contains the two gauge (super)field components that are transversal to the 1+1-dimensional subspace. Naturally, the anticommutators in (2.11a) yield no torsion- and-spacetime derivative term, \( T \cdot \partial \), on the right hand side, as these 3+1-dimensional spacetime derivatives are transversal to the 1+1-dimensional spacetime of our interest and by the assumption of dimensional reduction, everything is a constant in these transversal directions.

Finally, note that the Berezin integration formulae (1.6) and (1.7) also need to be ‘covariantized’; see Appendix A.

\(^4\) Being first order bosonic derivatives, the \( \nabla_\pm \) are \textit{antihermitian}; the above definitions of the \( \mathcal{W} \)'s then ensure them to be the hermitian conjugates of the \( \mathcal{W} \)'s.
2.2. Potential superfields

The covariant (super)derivatives \((\nabla, \mathcal{Y})\) must satisfy the graded Jacobi identities. After straightforward algebra (see Appendix B), we obtain the following relationships:

Eight of the Jacobi identities involving the covariant superderivatives \((\nabla, \mathcal{Y})\) completely determine the fermionic field strength superfields, \(\mathcal{W}\) and \(\bar{\mathcal{W}}\), in terms of the (gauge-covariant superderivatives\(^5\)) of \(A, \bar{A}, B, \bar{B}, C, \bar{C}\):

\[
\mathcal{W}_- = -\frac{i}{2}(\mathcal{A}_-, \mathcal{A}) + [\mathcal{A}_-, \mathcal{B}] , \quad \mathcal{W}_+ = -\frac{i}{2}[\mathcal{A}_-, \mathcal{C}_-] , \quad (2.12a)
\]

\[
\bar{\mathcal{W}}_- = -\frac{i}{2}(\mathcal{A}_-, \bar{A}) + [\mathcal{A}_-, \bar{B}] , \quad \bar{\mathcal{W}}_+ = -\frac{i}{2}[\mathcal{A}_-, \bar{C}_-] ; \quad (2.12b)
\]

\[
\mathcal{W}_+ = \frac{i}{2}(\mathcal{A}_+, \mathcal{A}) + [\mathcal{A}_+, \mathcal{B}] , \quad \mathcal{W}_\parallel = \frac{i}{2}[\mathcal{A}_+, \mathcal{C}_+] , \quad (2.12c)
\]

\[
\bar{\mathcal{W}}_+ = \frac{i}{2}(\mathcal{A}_+, \bar{A}) + [\mathcal{A}_+, \bar{B}] , \quad \bar{\mathcal{W}}_\parallel = \frac{i}{2}[\mathcal{A}_+, \bar{C}_+] . \quad (2.12d)
\]

We therefore call \(A, \bar{A}, B, \bar{B}, C, \bar{C}, C_\parallel, \bar{C}_\parallel\) the gauge potential superfields.

The four Jacobi identities involving three equal covariant superderivatives impose the constraints

\[
[\mathcal{A}_-, \mathcal{C}_-] = 0 \quad [\mathcal{A}_-, \bar{C}_-] = 0 , \quad (2.13a, b)
\]

\[
[\mathcal{A}_+, \mathcal{C}_+] = 0 \quad [\mathcal{A}_+, \bar{C}_+] = 0 . \quad (2.13c, d)
\]

Finally, the remaining eight Jacobi identities among the spinorial superderivatives \((\nabla, \mathcal{Y})\) imply the following curious relationships:

\[
[\mathcal{A}_+, \mathcal{A}] = -[\mathcal{A}_-, \mathcal{C}_+] , \quad [\mathcal{A}_-, \mathcal{A}] = -[\mathcal{A}_+, \mathcal{C}_-] , \quad (2.14a)
\]

\[
[\mathcal{A}_+, \bar{A}] = -[\mathcal{A}_-, \bar{C}_-] , \quad [\mathcal{A}_-, \bar{A}] = -[\mathcal{A}_+, \bar{C}_+] . \quad (2.14b)
\]

and:

\[
[\mathcal{A}_+, \mathcal{B}] = -[\mathcal{A}_-, \mathcal{C}_-] , \quad [\mathcal{A}_-, \mathcal{B}] = -[\mathcal{A}_+, \mathcal{C}_+] , \quad (2.15a)
\]

\[
[\mathcal{A}_+, \bar{B}] = -[\mathcal{A}_-, \bar{C}_-] , \quad [\mathcal{A}_-, \bar{B}] = -[\mathcal{A}_+, \bar{C}_+] . \quad (2.15b)
\]

Most of the Jacobi identities among two spinorial and one vectorial covariant derivative are identically satisfied upon using the previous identities\(^6\). Two of them, however, express the bosonic field strength superfield \(F\) in terms of the spinorial field strengths, and in two different ways:

\[
F = +\frac{i}{2}(\{\mathcal{A}_+, \mathcal{W}_-\} + \{\mathcal{A}_-, \bar{W}_-\}) , \quad (2.16a)
\]

\[
F = +\frac{i}{2}(\{\mathcal{A}_-, \mathcal{W}_+\} + \{\mathcal{A}_+, \bar{W}_+\}) . \quad (2.16b)
\]

\(^5\) We follow the practice of writing \([\nabla, \mathcal{Y}]\) for a gauge-covariant (super)derivative a Lie algebra-valued superfield \(\mathcal{Y}\), and \((\nabla \mathcal{X})\) for a gauge-covariant (super)derivative of a superfield \(\mathcal{X}\) in any representation of the Lie group other than the adjoint.

\(^6\) They do provide (super)differential relations between field strength superfields, such as \([\mathcal{A}, \mathcal{A}] = \{\mathcal{A}_-, \mathcal{W}_-\} + \{\mathcal{A}_+, \bar{W}_-\}\), \([\mathcal{A}, \mathcal{B}] = \{\mathcal{A}_-, \mathcal{W}_+\} + \{\mathcal{A}_+, \bar{W}_+\}\), \([\mathcal{A}, \mathcal{C}] = \{\mathcal{A}_-, \bar{W}_+\}\) and so on; for a complete listing, see appendix B.
The half-sum and the difference of these produce the standard results \[3\]:
\[
\begin{align*}
F &= \frac{1}{4} \left( \{\nabla_{-}, W_{+}\} + \{\nabla_{-}, \overline{W}_{+}\} \right), \\
0 &= \{\nabla_{-}, W_{+}\} + \{\nabla_{-}, \overline{W}_{+}\},
\end{align*}
\]
where grouping of indices in parentheses indicates their symmetrization, while bracketing indicates antisymmetrization. Eq. (2.16d), also known as the ‘bisection formula’ (Ref. \[3\], p. 158), is easy to rewrite in the more familiar form, $\nabla^\alpha W_\alpha = \nabla_\alpha \overline{W}^\alpha$ (Eq. (6.12) in Ref. \[7\]).

As another consequence of the Jacobi identities, the spin $\pm \frac{3}{2}$ field strengths $W_{\mp}, W_{\mp}$ are also related to their conjugates through ‘bisection’ formulae
\[
\begin{align*}
\{\nabla_{-}, W_{\mp}\} + \{\nabla_{-}, \overline{W}_{\mp}\} &= 0, \\
\{\nabla_{+}, W_{\mp}\} + \{\nabla_{+}, \overline{W}_{\mp}\} &= 0,
\end{align*}
\]
Note that the summands in the bisection formula (2.16d) have (total) spin 0, while those in (2.16c) have (total) spin $\pm 2$.

After using Eqs. (2.12), (2.16c) becomes:
\[
\begin{align*}
F &= \frac{1}{2} (f + \overline{f}) = \Re(f), \\
f &= \frac{i}{4} \left( \{\nabla_{-}, \nabla_{+}, A\} - \{\nabla_{+}, \nabla_{-}, A\} + \{\nabla_{-}, [\nabla_{+}, B]\} \right), \\
&\quad \equiv \frac{i}{4} \left( ([\nabla_{-}, \nabla_{+}]A) + ([\nabla_{-}, \nabla_{+}]B) \right), \\
\overline{f} &= \frac{i}{4} \left( ([\nabla_{-}, \nabla_{+}]\overline{A}) + ([\nabla_{-}, \nabla_{+}]\overline{B}) \right).
\end{align*}
\]
Note that $([\nabla_{-}, \nabla_{+}]A)$ is merely an abbreviation for the antisymmetrized second gauge-covariant superderivative of the Lie algebra-valued superfield $A$, written out in ‘long-hand’ as $\{\nabla_{-}, [\nabla_{+}, A]\} - \{\nabla_{+}, [\nabla_{-}, A]\}$.

We see that this field strength too is thus expressed entirely in terms of the gauge potential superfields $A, \overline{A}, B, \overline{B}$. Notice, however, that $F$ is not related to the $C, \overline{C}$’s, not even through the ‘curious relations’ (2.14) and (2.15)! In fact, the ‘usual’ bosonic field strengths for the $C, \overline{C}$’s, the spin-$\pm 2$ field strength superfields, $F_{\pm,\pm}$ and $F_{\mp,\mp}$, vanish identically; see appendix B. From the definition of $F$ in Eq. (2.11d), it is clear that its lowest component field is the standard Yang-Mills type field strength (a.k.a. curvature). The independence of $F$ from the $C, \overline{C}$’s then means that the type of extension of the supersymmetry algebra (2.11d) does not modify the standard Yang-Mills field strength.

The Jacobi identities among one spinorial and two vectorial covariant derivatives are all identically satisfied upon using the earlier identities. Finally, the Jacobi identities among three vectorial covariant derivatives are identically satisfied as there are only two such derivatives.
2.3. Gauge potential superfields

Having started by introducing fermionic and bosonic gauge superfields \((2.1)\), we now determine the relationship between these and the gauge potentials \(A, B, C\) and their conjugates by expanding the covariant derivatives in \((2.11)\). We find:

\[
A = -i \left[ \{ D_+, \Gamma_- \} + \{ D_-, \Gamma_+ \} - i \{ \Gamma_-, \Gamma_+ \} \right], \tag{2.18a}
\]

\[
\bar{A} = -i \left[ \{ \bar{D}_-, \bar{\Gamma}_+ \} + \{ \bar{D}_+, \bar{\Gamma}_- \} - i \{ \bar{\Gamma}_-, \bar{\Gamma}_+ \} \right], \tag{2.18b}
\]

\[
B = -i \left[ \{ D_+, \Gamma_+ \} + \{ D_-, \Gamma_- \} - i \{ \Gamma_-, \Gamma_+ \} \right], \tag{2.18c}
\]

\[
\bar{B} = -i \left[ \{ \bar{D}_-, \bar{\Gamma}_+ \} + \{ \bar{D}_+, \bar{\Gamma}_- \} - i \{ \bar{\Gamma}_-, \bar{\Gamma}_+ \} \right], \tag{2.18d}
\]

\[
C = -i \left[ \{ D_-, \bar{\Gamma}_- \} - \frac{i}{2} \{ \Gamma_-, \Gamma_- \} \right], \tag{2.18e}
\]

\[
C_\pm = -i \left[ \{ D_+, \bar{\Gamma}_+ \} - \frac{i}{2} \{ \Gamma_+, \Gamma_+ \} \right], \tag{2.18f}
\]

\[
\bar{C} = -i \left[ \{ \bar{D}_-, \bar{\Gamma}_- \} - \frac{i}{2} \{ \bar{\Gamma}_-, \bar{\Gamma}_- \} \right], \tag{2.18g}
\]

\[
\bar{C}_\pm = -i \left[ \{ \bar{D}_+, \bar{\Gamma}_+ \} - \frac{i}{2} \{ \bar{\Gamma}_+, \bar{\Gamma}_+ \} \right]. \tag{2.18h}
\]

Finally, Eqs. \((2.11a)\) imply that

\[
\Gamma_- = -\frac{i}{2} \left[ \{ D_-, \Gamma_- \} + \{ D_-, \Gamma_- \} - i \{ \Gamma_-, \Gamma_- \} \right], \tag{2.18i}
\]

\[
\Gamma_+ = -\frac{i}{2} \left[ \{ D_+, \Gamma_+ \} + \{ D_+, \Gamma_+ \} - i \{ \Gamma_+, \Gamma_+ \} \right]. \tag{2.18j}
\]

Thus, both the vectorial gauge superfields \(\Gamma_-, \Gamma_+\) from Eqs. \((2.11)\), and all the gauge potential superfields \(A, B, C\) and their conjugates from Eqs. \((2.11a\)–\(c)\) are completely defined in terms of the fermionic gauge superfields \(\Gamma_\pm, \bar{\Gamma}_\pm\) from Eqs. \((2.1a)\). Note that the lowest components of \(\Gamma_\pm, \bar{\Gamma}_\pm\) appear in the gauge potential and vectorial gauge superfields only through the non-linear terms produced if the gauge Lie group is nonabelian. Also, the lowest components of \(A, B, C\), their conjugates and of \(\Gamma_-, \Gamma_+\) are all related only through these non-linear terms.

Since all gauge superfields are in the adjoint representation of the gauge group, we can expand them over the generators, \(T_i\), which satisfy

\[
[T_j, T_k] = if_{jk}^l T_l, \tag{2.19}
\]

where \(f_{jk}^l\) are the structure constants. That is, \(\Gamma = \Gamma^i T_i\), where the \(\Gamma^i\) are now simply (anti)commutative gauge fields, as appropriate in \((2.1)\). The expressions \((2.18)\) now become:

\[
A = -i \left[ \{ D_-, \Gamma_+^l \} + \{ D_+, \Gamma_-^l \} + \Gamma_-^l \Gamma_+^k f_{jk}^l \right] T_l, \tag{2.18a'}
\]

\[
\bar{A} = -i \left[ \{ \bar{D}_-, \bar{\Gamma}_+^l \} + \{ \bar{D}_+, \bar{\Gamma}_-^l \} + \bar{\Gamma}_-^l \bar{\Gamma}_+^k f_{jk}^l \right] T_k, \tag{2.18b'}
\]

and so on.

Furthermore, it should be clear that any field strength superfield added to the r.h.s of Eqs. \((2.11a)\) would again appear on the r.h.s of Eqs. \((2.18a, b)\), so that these additional field strength superfields \textit{and} the bosonic gauge superfields become two independent degrees of
freedom—for each gauge transformation. Standard wisdom (see Ref. [6], p. 170) has it that this redundant duplication of gauge fields must be avoided—as we did do by introducing no field strength superfield on the r.h.s of Eqs. (2.11).

In retrospect, the modifications of the standard supersymmetry algebra shown in Eqs. (2.11) and parametrized by the superfields $A, B, C_-, C_+$, their conjugates and $\Gamma_-, \Gamma_+$ represent the most general ‘minimal coupling’ covariantization of (2,2)-supersymmetric theories in 1+1-dimensional spacetime.

3. Coupling to Constrained Matter

The above definitions provide gauge superfields, each of which contains component fields which are gauge fields for some symmetry. One of the primary interests in gauge (super)fields is of course their coupling to ‘matter’, typically represented by some sort of constrained (super)fields. A systematic list of such ‘matter’ superfields was given in Ref. [1], defined to satisfy a (system of) superdifferential equation(s).

Their coupling to the gauge superfields is induced by the ‘minimal coupling’ modification (2.1). Lagrangian densities (for a comprehensive listing, see Ref. [1]) are defined as Berezin integrals, which in turn are equivalent to (multiple) superderivatives. The ‘covariantization’ (2.1) then induces a ‘covariantization’ of the Lagrangian densities. Upon expansion into component fields, these will then exhibit explicit coupling terms between ‘matter’ and gauge component fields.

However, in the presence of gauged symmetries, the defining (super)constraints themselves must be modified, and we first explore the consequences of this modification.

3.1. Minimal gauge-covariantly haploid superfields

Adapting from Ref. [1], we recall the definition of the minimal (first-order constrained) gauge-covariantly haploid superfields:

1. Chiral:
   \[ (\nabla_+ \Phi) = 0 = (\nabla_- \Phi) \]  
   \[ (3.1a) \]

2. Antichiral:
   \[ (\nabla_+ \Phi^\dagger) = 0 = (\nabla_- \Phi^\dagger) \]  
   \[ (3.1b) \]

3. Twisted-chiral:
   \[ (\nabla_+ \Xi) = 0 = (\nabla_- \Xi) \]  
   \[ (3.1c) \]

4. Twisted-antichiral:
   \[ (\nabla_+ \Xi^\dagger) = 0 = (\nabla_- \Xi^\dagger) \]  
   \[ (3.1d) \]

5. Lefton:
   \[ (\nabla_- \Lambda) = 0 = (\nabla_- \Lambda) \]  
   \[ (3.1e) \]

6. Righton:
   \[ (\nabla_+ \Upsilon) = 0 = (\nabla_+ \Upsilon^\dagger) \]  
   \[ (3.1f) \]

It is easy to show that the equations (3.1) are gauge-covariant:

\[ (\nabla_+ \Phi) \rightarrow (\nabla_+ \Phi^\prime) = (G \nabla_+ G^{-1} G \Phi) = G (\nabla_+ \Phi), \]  
\[ (3.2) \]

so that

\[ (\nabla_\pm \Phi) \rightarrow G (\nabla_\pm \Phi) = 0 \]  
\[ (3.3) \]
Note that the above constraints imply that gauge-covariantly haploid superfields commute with the covariant derivatives which annihilate them; for example:

\[(\nabla_+ \Phi) = 0 \quad \Rightarrow \quad [\nabla_+, \Phi] = 0 \, . \quad (3.4)\]

This ‘commutator form’ of the constraints will be necessary when reconsidering the above superfields as multiplicative operators in quantum theory, in which case also the second of Eqs. (2.2) needs to read \(X' = G X G^{-1}\), where \(X\) stands for any multiplicative superfield operator. Of course, when \(X\) is in the adjoint representation, it makes perfect sense to regard it as an operator even as a classical superfield, and expand it over the gauge group generators, \(T_i\).

A simple argument [4] shows that a gauge-covariantly chiral superfield, \(\Phi\), cannot be charged with respect to that part of the gauged Lie algebra in which the \(B\)'s take value:

\[
\begin{align*}
(\nabla - \Phi) &= 0 \\
(\nabla + \Phi) &= 0
\end{align*}
\]

\[
\Rightarrow \quad (\{\nabla -, \nabla +\} \Phi) = 0 , \quad (3.5a)
\]

thus, using Eq. (2.11b) : \((B \Phi) = 0 \, . \quad (3.5b)\)

That is, the Lie group generators in which the \(B\)'s take values annihilate gauge-covariantly (anti)chiral superfields. But, as the generators of a Lie group are Hermitian, the \(B\)'s are valued in the same generators as are the \(B\)'s, and we obtain:

\[
(B \Phi) = 0 = (B \Phi) , \quad \text{and} \quad (B \Phi) = 0 = (B \Phi) . \quad (3.5c)
\]

Similarly, gauge-covariantly twisted-(anti)chiral superfields are annihilated by those Lie group generators in which the \(A, \tilde{A}\)'s take values:

\[
\begin{align*}
(\nabla - \Xi) &= 0 \\
(\nabla + \Xi) &= 0
\end{align*}
\]

\[
\Rightarrow \quad (\{\nabla -, \nabla +\} \Xi) = 0 , \quad (3.6a)
\]

thus, using Eq. (2.11a) : \((A \Xi) = 0 \, . \quad (3.6b)\)

Again, as for Eqs. (3.5c), it follows that

\[
(A \Xi) = 0 = (A \Xi) , \quad \text{and} \quad (A \Xi) = 0 = (A \Xi) . \quad (3.6c)
\]

By a similar calculation, covariant leftons are annihilated by the \(C_-, \tilde{C}_-\)'s:

\[
(\nabla - \Lambda) = 0 , \quad \Rightarrow \quad (\nabla^2 - \Lambda) = (\tilde{C}_- \Lambda) = 0 , \quad (3.7a)
\]

\[
(\nabla - \Lambda) = 0 , \quad \Rightarrow \quad (\nabla^2 - \Lambda) = (C_- \Lambda) = 0 , \quad (3.7b)
\]

and covariant rightons by the \(C_+, \tilde{C}_+\)'s:

\[
(\nabla + \Upsilon) = 0 , \quad \Rightarrow \quad (\nabla^2 + \Upsilon) = (\tilde{C}_+ \Upsilon) = 0 , \quad (3.8a)
\]

\[
(\nabla + \Upsilon) = 0 , \quad \Rightarrow \quad (\nabla^2 + \Upsilon) = (C_+ \Upsilon) = 0 . \quad (3.8b)
\]
In fact, the \(C=, \overline{C}=\)'s annihilate all but the covariant rightons:

\[
\begin{align*}
\left(\nabla_\perp \Phi\right) = 0 & \Rightarrow \left(C= \Phi\right) = 0 = \left(\overline{C}= \Phi\right), \\
\left(\nabla_\perp \overline{\Phi}\right) = 0 & \Rightarrow \left(C= \overline{\Phi}\right) = 0 = \left(\overline{C}= \overline{\Phi}\right), \\
\left(\nabla_\perp \Xi\right) = 0 & \Rightarrow \left(C= \Xi\right) = 0 = \left(\overline{C}= \Xi\right), \\
\left(\nabla_\perp \overline{\Xi}\right) = 0 & \Rightarrow \left(C= \overline{\Xi}\right) = 0 = \left(\overline{C}= \overline{\Xi}\right).
\end{align*}
\]

In all of these, the middle (second) equality is obtained by re-applying the superderivative from the left (first) equality, and the right (third) equality follows from the middle (second) one since the \(C\)'s and the \(C\)'s take values in the same, Hermitian, Lie group generators. Similarly, the \(C=, \overline{C}=\)'s annihilate all but the covariant leftons.

The last (third, left) equality in Eqs. (3.9) would seem to imply additional constraints on the ‘matter’ superfields, such as \(\nabla^2 \Phi = 0\). This, however, does not in the least restrict the gauge-covariantly haploid superfields. In the vanishing gauge coupling constant limit, \(\nabla_\perp^2 \rightarrow D_\perp^2 \equiv 0\), and the (would-be) additional constraints are vacuous. For any non-zero gauge coupling constant, the superfields \(\Phi\) are simply \(C=\)-chargeless. In particular, the first order superderivatives \(\nabla_\pm \Phi, \nabla_\perp \Xi, \nabla_+ \Xi\) and their conjugates remain unrestricted.

We have therefore proven that the minimal coupling type of interaction between the gauge superfields (2.1) and gauge-covariantly haploid ‘matter’ superfields (3.1) is highly selective, as summarized in Table 1.

| \(A, \overline{A}\) | \(B, \overline{B}\) | \(C=, \overline{C}=\) | \(C=, \overline{C}=\) |
|-----------------|-----------------|-----------------|-----------------|
| \(\Phi, \overline{\Phi}\)   | \(\sqrt{\checkmark}\) | \(-\) | \(-\) | \(-\) |
| \(\Xi, \overline{\Xi}\)   | \(-\) | \(\checkmark\) | \(-\) | \(-\) |
| \(\Lambda, \overline{\Lambda}\) | \(\checkmark\) | \(\checkmark\) | \(\checkmark\) | \(-\) |
| \(\Upsilon, \overline{\Upsilon}\) | \(\checkmark\) | \(\checkmark\) | \(-\) | \(\checkmark\) |

**Table 1**: The minimal coupling of gauge (super)fields (2.1) to gauge-covariantly haploid ‘matter’ superfields (3.1) is highly selective: the entry ‘\(\checkmark\)’ indicates that the minimal coupling type interaction is possible, and ‘\(-\)’ that it is impossible.

The results summarized in Table 1 may be ‘translated’ into the gauge group index notation of Eqs. (2.3)–(2.5) as follows. The gauge-covariantly chiral superfields, \(\Phi^\mu\) with \(\mu = 1, \ldots, N_c\), must form a representation \(^7\) of \(G_A\), the (factor of the) gauge group gauged by the \(A\)'s. The gauge-covariantly twisted chiral superfields, \(\Xi^\alpha\) with \(\alpha = 1, \ldots, N_t\), must form a representation of \(G_B\), the (factor of the) gauge group gauged by the \(B\)'s. The gauge-covariant leftons, \(\Lambda^a\) with \(a = 1, \ldots, N_L\), must form a representation of \(G_A\), of

\(^7\) We assume that the collections of the various gauge-covariantly haploid superfields all form *irreducible* representations. Reducible representations are then obtained simply by including formal sums of the irreducible ones.
Thus, for example, \((A \Phi)^\rho = A^\mu_\rho \Phi^\mu\), but \((A \Lambda)^a = A^a_b \Lambda^b\), and \((A \Upsilon)^i = A^i_j \Lambda^j\). Also, \((A \bar{\Phi})^{\bar{\rho}} = (\delta^{\mu \bar{\rho}} A^\mu_{\rho \bar{\sigma}}) \Phi^\sigma = \delta^{\mu \bar{\rho}} (A^\mu_{\rho \bar{\sigma}} \delta_{\rho \bar{\sigma}} \Phi^\sigma)\). Note that this is in perfect agreement with (and in fact clarifies) the ‘matrix’ notation (2.5). Raising and lowering indices with \(\delta^{\mu \bar{\rho}}\) and \(\delta_{\rho \bar{\sigma}}\), respectively, precisely corresponds to transposition:

\[
\Phi^T \leftrightarrow \Phi^\rho = \delta_{\rho \bar{\sigma}} \Phi^\sigma, \quad \text{and} \quad (\ldots)^T \leftrightarrow \delta^{\mu \bar{\rho}} (\ldots)_{\mu}.
\]  

(3.10)

As the matrix multiplication in \(A^\mu_\rho \bar{\Phi}^\rho\) is the transpose of that in \(A^\rho_\mu \Phi^\rho\),

\[
(A \bar{\Phi})^{\bar{\rho}} = \delta^{\mu \bar{\rho}} (A^\mu_{\rho \bar{\sigma}} \Phi^\sigma) \leftrightarrow (A \bar{\Phi}) \equiv (A^T \bar{\Phi}^T)^T,
\]  

(3.11)

re-derives Eq. (2.8).

3.2. Covariantly quartoid superfields

Besides the minimal (first-order constrained) gauge-covariantly haploid superfields defined in Eqs. (3.1), Ref. [1] also lists superfields which are constrained by three rather than two first-order superderivatives. Adapting for gauge covariance, we then define gauge-covariantly quartoid superfields:

1. Chiral Lefton: \((\nabla_+ \Phi_R) = (\nabla_- \Phi_R) = (\nabla_- \Phi_R) = 0\),  
(3.12a)

2. Antichiral Lefton: \((\nabla_+ \bar{\Phi}_R) = (\nabla_- \bar{\Phi}_R) = (\nabla_- \bar{\Phi}_R) = 0\),  
(3.12b)

3. Chiral Righton: \((\nabla_- \Phi_L) = (\nabla_+ \Phi_L) = (\nabla_+ \Phi_L) = 0\),  
(3.12c)

4. Antichiral Righton: \((\nabla_- \bar{\Phi}_L) = (\nabla_+ \bar{\Phi}_L) = (\nabla_+ \bar{\Phi}_L) = 0\).  
(3.12d)

Notice that all of these may be regarded as gauge-covariantly chiral superfields (or their hermitian conjugates) which so happen to satisfy an additional gauge-covariant superconstraint.

All of these being (anti)chiral, they must have no charges with respect to the gauge groups generated by the \(B, C\)‘s and their conjugates. However, each gauge-covariantly quartoid superfield obeys an additional constraint, which then implies that the \(A\)-charges must be zero too. For example, a chiral lefton satisfies both the constraints of a chiral superfield, and those of a twisted-antichiral superfield. The former preclude coupling to all but the type-A gauge superfields, while the latter precludes even that.

3.3. Non-minimal gauge-covariantly haploid superfields

Unlike the highly selective coupling of the minimal gauge-covariantly haploid superfields to the various gauge (super)fields as displayed in Table 1, non-minimal gauge-covariantly haploid superfields couple universally [1].
Adapting from Ref. [1], we recall the definitions of the non-minimal (second-order constrained) gauge-covariantly haploid superfields:

1. **NM-Chiral**: \(([\nabla_+, \nabla_-] \Theta) = 0\), (3.13a)

2. **NM-Antichiral**: \((([\nabla_-, \nabla_+] \bar{\Theta}) = 0\), (3.13b)

3. **NM-Twisted-chiral**: \((([\nabla_+, \nabla_-] \Pi) = 0\), (3.13c)

4. **NM-Twisted-antichiral**: \((([\nabla_-, \nabla_+] \Pi)) = 0\), (3.13d)

5. **NM-(Almost)-Lefton**: \((([\nabla_-, \nabla_-] A) = 0\), (3.13e)

6. **NM-(Almost)-Righton**: \((([\nabla_+, \nabla_+] U) = 0\), (3.13f)

It is absolutely essential to realize that the quadratic superderivatives in these defining constraints must be commutators, as this—and not the anticommutator—has the correct vanishing coupling limit: when \(\nabla, \nabla \to D, \bar{D}\), the symmetric products of the superderivatives vanish identically, except for Eqs. (1.5).

It is indeed a trivial observation—antisymmetric and symmetric products being linearly independent—that Eqs. (3.13) imply no exclusion on the minimal coupling type interaction between any of the \(A, B, C\) and their conjugates to any of the \(\Theta, \Pi, A, U\) and their conjugates. So, although in the massless free-field limit the physical component fields of non-minimal superfields occur in a 1–1 correspondence with those of the minimal ones (see p.200 of Ref. [6]), their differing couplings to the minimal coupling gauge (super)fields as defined in Eqs. (2.1) prove the physical inequivalence of the ‘minimal’ haploid superfields (3.1) from their non-minimal brethren (3.13) [10].

Similarly, the peculiar inhomogeneous (gauge) transformation which enables the minimal/non-minimal 1–1 identification even without a choice of the Lagrangian density [1] is manifestly broken by the couplings to the gauge (super)fields defined in Eqs. (2.1). After all, what really matters in nontrivial physics models are the types of interactions the various involved (super)fields can have.

---

Unlike the \(N=1\) supersymmetric 3+1-dimensional spacetime, \((2, 2)\)-supersymmetric 1+1-dimensional spacetime admits different types of symmetry gauging, and gauge-covariantly constrained superfields which represent:

1. ‘matter’ with no minimal-coupling gauge interaction: gauge-covariantly quartoid superfields (3.12);
2. ‘matter’ with highly selective minimal-coupling gauge interactions: minimal gauge-covariantly haploid superfields (3.1) — see Table 1;
3. ‘matter’ with universal minimal-coupling gauge interactions: non-minimal gauge-covariantly quartoid superfields (3.13).

---

8) The definitions of the non-minimal *gauge-covariantly* haploid superfields (3.13) are invariant under this peculiar inhomogeneous (gauge) transformation only in the limit of no gauge coupling.
Note that the above discussion focuses on the so-called ‘minimal coupling’, wherein the gauge (super)fields enter as connection superfields, through the modification of the (super)derivative, i.e., (super)momentum operator (2.1). Additional types of symmetry gauging are possible through introduction of ‘Pauli-terms’ or other higher-derivative terms; their study is left to the interested Reader.

4. Special Symmetry Gauging Types

With the consistency of the modifications (2.11) established, and the resulting constraints (2.14), (2.13), and relations (2.12), (2.16) and (2.17) in hand, we now seek interesting special cases.

The selectivity of couplings between gauge-covariantly haploid superfields and various gauge potentials shown in Table 1 suggests the following simplification. Assume that the gauge group is of the form \( G = G_A \times G_B \times G_c \times G_\pm \), and let the gauge potentials \( A, B, C, C_\pm \) each take values in the generators of the accordingly named factor. It is then possible to project on any one direct product factor, effectively setting the generators (charges) of all other factors to zero. This annihilates every (super)field valued only in the generators of the complementary factor of the gauge group. Therefore, projecting on a factor is equivalent to setting all (super)fields valued in the complementary factor to zero, and this is how we proceed.

4.1. Type-A gauging

Let us first project on the \( G_A \) factor in the gauge group, i.e., set \( B = 0 = C \). The peculiar relations (2.13) now imply that

\[
[\nabla_-, A] = 0 = [\nabla_+, A], \quad \text{and} \quad [\nabla_-, \bar{A}] = 0 = [\nabla_+, \bar{A}].
\]

That is, \( A \) is gauge-covariantly twisted-chiral and \( \bar{A} \) gauge-covariantly twisted-antichiral. Reassuringly, the conclusion of Eqs. (3.6) does not apply here. That is, Eqs. (3.6) notwithstanding, Eqs. (4.1) do not imply that \( A \) must itself be chargeless with respect to gauge group it generates which would preclude type-A gauging of nonabelian groups. Repeating the calculation of Eqs. (3.6), however with \( \Xi \rightarrow A \), we now obtain

\[
0 \equiv [A, A] = iA_j A^k f_{jk}^l T_l,
\]

which vanishes on account of the antisymmetry \( f_{jk}^l = -f_{kj}^l \) and the commutivity of the \( A_j \)'s. This allows \( A \) to have nonzero \( G_A \)-charges, as is indeed necessary when the gauge group \( G_A \) is nonabelian.

Superfield strengths

The formulae (2.12) and (2.17) simplify:

\[
W_- = -\frac{i}{2} [\nabla_-, \bar{A}] , \quad W_+ = \frac{i}{2} [\nabla_+, A] ,
\]

\[
\bar{W}_- = -\frac{i}{2} [\nabla_-, A] , \quad \bar{W}_+ = \frac{i}{2} [\nabla_+, \bar{A}] ,
\]
and \( F = \Re(f) \), with

\[
f = \frac{i}{4} ([\nabla_-, \nabla_+] A),
\]

(4.3c)

and the \( W_\equiv, W_\mp \) and their conjugates vanish. The Jacobi identities involving two spinorial and one vectorial \( \nabla \) then produce the familiar relations:

\[
\{ \nabla_-, W_\mp \} = 0, \quad \{ \nabla_+, W_\pm \} = 0, \quad (4.4a)
\]

\[
\{ \nabla_-, \bar{W}_\pm \} = 0, \quad \{ \nabla_+, \bar{W}_\pm \} = 0, \quad (4.4b)
\]

proving that \( W_\pm \) are gauge-covariantly chiral and \( \bar{W}_\pm \) are gauge-covariantly antichiral, related by the bisection formula (2.16d)—just as in \( N=1 \) supersymmetric 3+1-dimensional spacetime theories.

We will refer to \( \{ A, \bar{A}; W_\pm, \bar{W}_\pm; F \} \), subject to the superdifferential relations (4.3), as the type-A, or twisted gauge multiplet, since the gauge potential superfields \( A \) are gauge-covariantly twisted-chiral. Notice however that the spinorial gauge field strength superfields \( W_\pm \) are gauge-covariantly chiral. Recall also that these gauge fields do not couple to gauge-covariantly twisted-chiral ‘matter’ superfields and their conjugates (3.1c,d).

**Dimensional reduction**

The type-A gauge supermultiplet, \( \{ A, \bar{A}; W_\pm, \bar{W}_\pm; F \} \), was called VM-I in Ref. [4] and was identified there with the gauge supermultiplet obtained by dimensional reduction from \( N=1 \) supersymmetric 3+1-dimensional spacetime. Recall that all field strength superfields with two fermionic indices are conventionally set to zero to prevent the duplication of gauge field degrees of freedom per gauge symmetry (see p.171–172 of Ref. [6]). Appearances to the contrary, this is not violated by the inclusion of \( A, \bar{A} \) on the r.h.s. of Eqs. (2.11a).

To see this, consider the 1+1-dimensional spacetime (world-sheet) embedded in a 3+1-dimensional one. Let \( \nabla_-, \nabla_+ \) be locally tangent to the world-sheet, and \( \nabla_j = \partial_j - i\Gamma_j \), with \( j = 2, 3 \), be transversal to it. Dimensional reduction then implies that all (super)fields are set to be annihilated by \( \partial_j \), whereupon \( \nabla_j \to -i\Gamma_j \). Finally, since

\[
\{D\pm, D\mp\} = 2i(\nabla_2 \pm i\nabla_3), \quad (4.5)
\]

a comparison with Eqs. (2.11a) sets \( A = 2(\Gamma_2+i\Gamma_3) \). So, the superfields \( A \), which are scalars in the 1+1-dimensional sense, are in fact a linear combination of the ‘transversal’ components of the gauge 3+1-vector potential superfields. Also, although \( A, \bar{A} \) are gauge-covariant (being defined as the anticommutator of gauge-covariant superderivatives), they are identified as gauge potential superfields, and not as (super)field strengths.

Furthermore, although \( A, \bar{A}, W_\pm, \bar{W}_\pm, F \) are separate (2,2)-superfields related through the superdifferential equations (4.3), it is convenient to regard them as jointly forming a gauge superfield multiplet. Finally, the \( A, \bar{A} \) are thus seen to be rightly regarded as gauge potential superfields, which determine the field strength superfields through Eqs. (4.3).

**Abelian case**

A further simplification is easy to demonstrate when \( G_A \) is abelian, so that \( A \) are \( G_A \)-chargeless, being Lie-algebra valued, i.e., in the adjoint representation of \( G_A \). Thus—while
acting on $A, \bar{A}$—the $\mathcal{G}_A$-covariant (super)derivatives in Eqs. (4.1) act as ordinary (super)derivatives (1.4), $A$ and $\bar{A}$ become ‘plain’ twisted-chiral and twisted-antichiral superfields. These are then easily expressed in terms of an unconstrained, prepotential superfield:

$$A = ([D_+, \bar{D}_-] V^{(A)}), \quad {\text{and}} \quad \bar{A} = ([D_-, \bar{D}_+] V^{(A)}).$$

(4.6)

This produces

$$F = i\frac{1}{8} ([D_-, \bar{D}_+] [D_+, \bar{D}_-] V^{(A)} + [D_-, D_+] [D_-, \bar{D}_+] V^{(A)}) ,$$

$$= i\frac{1}{2} (D_- \bar{D}_- D_+ V^{(A)} - D_+ D_+ \bar{D}_+ \bar{V}^{(A)}) ,$$

(4.7)

and

$$W_ - = -i (D_+ \bar{D}_- D_+ \bar{V}^{(A)}) , \quad W_+ = -i (D_+ \bar{D}_- D_+ V^{(A)}) ,$$

$$\bar{W}_ - = -i (D_- D_+ \bar{D}_- V^{(A)}) , \quad \bar{W}_+ = -i (D_- D_+ \bar{D}_+ \bar{V}^{(A)}) ,$$

(4.8a)

(4.8b)

which are consistent with

$$\Gamma_- = -i (D_- \bar{V}^{(A)}) , \quad \bar{\Gamma}_- = i (D_- V^{(A)}) ,$$

$$\Gamma_+ = -i (D_+ V^{(A)}) , \quad \bar{\Gamma}_+ = i (D_+ \bar{V}^{(A)}) .$$

(4.9)

Finally, we must ensure that $B, \bar{B}$ do vanish as assumed. To this end, substituting Eqs. (1.3) into (2.18a, b), we see that $\bar{V}^{(A)} = V^{(A)}$ must be ensured, and no further restrictions transpire from the remaining equations (2.18). This reproduces the well-known reality of the gauge vector prepotential superfield, $\bar{V}$. After a little $D$-algebra, we also obtain

$$F = 2 (\partial_\mp V^{(A)} - \partial_\mp \bar{V}^{(A)}) ,$$

(4.10)

where

$$V_+ = \frac{1}{4} ([D_-, D_-] V^{(A)}) , \quad V_- = \frac{1}{4} ([D_+, D_+] V^{(A)}) .$$

(4.10)

so that the lowest component of $F$ in (4.10)’ is the usual (abelian) Yang-Mills field strength, in terms of the 2-vector potential obtained as the lowest components of $V_+^{(A)}, V_-^{(A)}$.

4.2. Type-B gauging

Let us next project on the $\mathcal{G}_B$ factor in the gauge group, i.e., set $A = 0 = C$. The peculiar relations (2.14) now imply that

$$[\bar{\nabla}_\pm, B] = 0 , \quad {\text{and}} \quad [\nabla_\pm, \bar{B}] = 0 .$$

(4.11)

That is, $B$ is gauge-covariantly chiral and $\bar{B}$ gauge-covariantly antichiral. Again, the conclusion of Eqs. (3.5) does not apply here. Repeating the calculation of Eqs. (3.4), with $\Phi \to B$, we obtain

$$0 \equiv [B, B] = iB^j B^k f_{jk}^l T_l ,$$

(4.12)

9) Unlike $F$ and the $W$’s, the $\Gamma$’s transform inhomogeneously under gauge transformations, and so remain defined only up to additive terms stemming from this.
which again vanishes on account of the antisymmetry of the $f_{jk}^l$’s and the commutivity of the $B^j$’s, allowing $B$ to have nonzero $\mathcal{G}_B$-charges.

Superfield strengths

The formulae (2.12) and (2.17) again simplify:

$$W_\mp = \mp \frac{i}{2} [\nabla_\mp, B], \quad \bar{W}_\mp = \mp \frac{i}{2} [\nabla_\mp, B] \quad (4.13a)$$

$$F = \Re (f), \quad f = \frac{i}{4} ([\nabla_-, \nabla_+] B) \quad (4.13b)$$

and $W_\pm, \bar{W}_\pm$ and their conjugates vanish. The Jacobi identities involving two spinorial and one vectorial $\nabla$ then produce the ‘mirror’ of the familiar relations (4.4):

$$\{\nabla_-, W_+\} = 0, \quad \{\nabla_+, W_+\} = 0 \quad (4.14a)$$

$$\{\nabla_-, \bar{W}_-\} = 0, \quad \{\nabla_+, \bar{W}_-\} = 0 \quad (4.14b)$$

and

$$\{\nabla_-, W_-\} = 0, \quad \{\nabla_+, W_-\} = 0 \quad (4.14c)$$

$$\{\nabla_-, \bar{W}_+\} = 0, \quad \{\nabla_+, \bar{W}_+\} = 0 \quad (4.14d)$$

proving that now $W_+, \bar{W}_-$ are gauge-covariantly twisted-chiral and $W_-, \bar{W}_+$ are gauge-covariantly twisted-antichiral, again related by the bisection formula (2.16d). The abelian version of this type-B gauge multiplet was labeled VM-II in Ref. [4] and was identified as the ‘mirror’ of the usual vector multiplet. Notice that now the spinorial field strength superfields $W_+, \bar{W}_-$ are gauge-covariantly twisted-chiral, while the gauge potentials $B$ are gauge-covariantly chiral.

We will refer to $\{B, B; W_\pm, \bar{W}_\pm; F\}$, subject to the superdifferential relations (4.13), as the type-B or chiral gauge multiplet. Recall that these gauge fields do not couple to gauge-covariantly chiral superfields and their conjugates (3.1a,b).

Dimensional reduction

The type-B gauge supermultiplet, $\{B, B; W_\pm, \bar{W}_\pm; F\}$, was called VM-II and identified with the ‘mirror’-twisted cousin of the type-A gauge superfield multiplet [4]. This gauge multiplet has no counterpart in the rather more familiar 3+1-dimensional models, where the anticommutator of any two (un)conjugate gauge-covariant superderivatives vanishes.

However, one may regard the superfields $B$ as the gauge potential superfield introduced to covariantize the (spin-0) central charges—were such to have been introduced in further generalizing the gauge-covariant supersymmetry algebra (2.11). That the present case (2.11) contains no such central charges may be understood merely as the statement that all involved superfields are chargeless with respect to these central charges.

Abelian case

Again, a further simplification is easy to demonstrate when $\mathcal{G}_B$ is abelian, so that $B$ are $\mathcal{G}_B$-chargeless. Again—while acting on the $B, \bar{B}$—the $\mathcal{G}_B$-covariant (super)derivatives in
Eqs. (4.11) act as ordinary (super)derivatives (1.4), \( B \) and \( \bar{B} \) become ‘plain’ chiral and antichiral superfields, and so can be expressed in terms of an unconstrained prepotential superfield:

\[
B = ([\bar{D}_+, D_-] \nabla^{(B)}), \quad \text{and} \quad \bar{B} = ([D_-, D_+] \nabla^{(B)}).
\]  

(4.15)

Easily then,

\[
F = \frac{i}{8} \left( [D_-, D_+] [\bar{D}_+, D_-] \nabla^{(B)} \right) + \left[ D_- \bar{D}_+ [D_-, D_+] \nabla^{(B)} \right],
\]

(4.16)

and

\[
W_\mp = \mp i (D_\mp \bar{D}_+ D_- \nabla^{(B)}), \quad \bar{W}_\mp = \mp i (\bar{D}_- D_- D_+ \nabla^{(B)}),
\]

(4.17)

which are consistent with

\[
\Gamma_\pm = \pm i (D_\pm \nabla^{(B)}), \quad \text{and} \quad \bar{\Gamma}_\pm = \mp i (\bar{D}_\pm \nabla^{(B)}).
\]

(4.18)

Again, to ensure that \( A, \bar{A} \) do vanish as assumed, substitute Eqs. (1.18) into (2.18c,d), to show that \( \nabla^{(B)} = \nabla^{(B)} \); no further restrictions transpire from the remaining equations (2.18). So, the mirror of the gauge vector prepotential superfield, \( \nabla \), must also be real. After a little \( D \)-algebra, this produces

\[
F = 2 (\partial_+ \nabla_-^{(B)} - \partial_- \nabla_+^{(B)}),
\]

(1.16)′

where

\[
\nabla_-^{(B)} \overset{\text{def}}{=} \frac{1}{4} ([D_-, \bar{D}_-] \nabla^{(B)}), \quad \nabla_+^{(B)} \overset{\text{def}}{=} \frac{1}{4} ([D_+, \bar{D}_+] \nabla^{(B)}),
\]

(4.19)

so that the lowest component of \( F \) in (1.16)′ is the usual (abelian) Yang-Mills field strength, in terms of the 2-vector potential obtained as the lowest components of \( \nabla_-^{(B)}, \nabla_+^{(B)} \).

4.3. Type-\( C_\pm \) gauging

Let us project on the \( G_\pm \) factor in the gauge group, i.e., set \( B = A = 0 = C_\pm \). The peculiar relations (2.14) and (2.15) now imply that

\[
[\nabla_+, C_\pm] = 0, \quad [\nabla_+, \bar{C}_\pm] = 0,
\]

(4.20)

while Eqs. (2.13) further constrain

\[
[\nabla_-, C_\pm] = 0, \quad [\nabla_-, \bar{C}_\pm] = 0.
\]

(4.21)

That is, \( C_\pm \)'s are gauge-covariantly chiral rightons (3.12c), while \( \bar{C}_\pm \)'s are gauge-covariantly antichiral rightons (3.12d); in fact, they are gauge-covariantly quartoid superfields. That is, they (effectively) depend on only one quarter of the four supercoordinates, \( \varsigma^\pm, \bar{\varsigma}^\pm \).
As with the \(B\)'s and the \(A\)'s, the conclusion of \(\S 3.1\) on the high selectivity of gauge couplings does not apply to the \(C\)'s themselves. In fact, we find that the \(C\) do couple to the \(G\)-gauge superfields, since

\[
0 \overset{!}{=} [C_\equiv, C_\equiv] = iC_\equiv^i C_\equiv^j f_{ij}^k T_k ,
\]

vanishes on account of the antisymmetry of the \(f_{ij}^k\)'s and the commutivity of the \(C\)'s. This allows \(C_\equiv\) and its conjugate to have nonzero \(G\)-charges, as is necessary for nonabelian \(G\).

### Superfield strengths

Again, the formulae (2.12) and (2.17) simplify:

\[
W_\equiv = -\frac{i}{2}[\nabla_-, C_\equiv] , \quad \bar{W}_\equiv = -\frac{i}{2}[\nabla_-, \bar{C}_\equiv] ,
\]

and \(W_\pm, \bar{W}_\pm\), their conjugates and \(F\) all vanish. Recall that the would-be ‘usual’ bosonic field strength, \(F_{\equiv,=}\), vanishes identically; see appendix B. The Jacobi identities involving two spinorial and one vectorial \(\nabla\) then produce the analogue of the familiar relations (4.4):

\[
\{\nabla_+, W_\equiv\} = 0 , \quad \{\nabla_+, \bar{W}_\equiv\} = 0 ,
\]

\[
\{\nabla_+, \bar{W}_\equiv\} = 0 , \quad \{\nabla_+, W_\equiv\} = 0 ,
\]

proving that \(W_\equiv, \bar{W}_\equiv\) are covariant rightons, related by the ‘bisection formula’ (2.10c). In fact, since

\[
-2i\{\nabla_-, W_\equiv\} = \{\nabla_-, [\nabla_-, C_\equiv]\} = \frac{1}{2} [C_\equiv, \{\nabla_-, \nabla_-\}] = [C_\equiv, \bar{C}_\equiv] ,
\]

the \(W_\equiv\) are gauge-covariantly \textit{antichiral} rightons if \(G\) is abelian. Of course, the \(\bar{W}_\equiv\) then are gauge-covariantly \textit{chiral} rightons.

We will refer to \(\{C_\equiv, C_\equiv; W_\equiv, \bar{W}_\equiv\}\), subject to the superdifferential relations (1.23), as the type-\(C\)\textsubscript{\equiv}, or chiral righton gauge multiplet. These gauge fields \textit{only couple} to covariant rightons (3.1f), and to all non-minimal haploid superfields (3.13).

### Dimensional reduction

The chiral righton gauge multiplet \(\{C_\equiv, C_\equiv; W_\equiv, \bar{W}_\equiv\}\) also has no counterpart in the rather more familiar 3+1-dimensional models. Somewhat in analogy with the \(B, \bar{B}\) potential superfields, the spin-1 gauge potential superfields \(C_\equiv, \bar{C}_\equiv\) also may be regarded as the gauge potentials introduced to covariantize the (now spin-1) central charges—were such to have been introduced on the r.h.s. of first of Eqs. (2.11c). Again, their absence from the present case (2.11b) may be understood merely as the statement that all involved superfield are chargeless with respect to these central charges.

### Abelian case

Again, \(C_\equiv, \bar{C}_\equiv\) are \(G\)-chargeless, and the \(G\)-gauge covariant (super)derivatives act on the \(C_\equiv, \bar{C}_\equiv\) as ordinary (super)derivatives do. Now, as noted in Ref. [4], it is not possible to express chiral rightons, and so also the \(C_\equiv, \bar{C}_\equiv\)'s, in terms of a superderivative of
an ambidextrous superfield. It is possible, however, to write \( C_- \equiv (D_- C_-) \), thereby specifying the \( G_- \)-gauge potential in terms of a spin-\( \frac{1}{2} \) righton prepotential superfield, \( C_- \).

Since now Eqs. (2.18) imply that \( C_- = -i(D_- \Gamma_-) \), it follows that the half of \( \Gamma_- \) which is not annihilated by \( D_- \) must be equal to the analogous half of \( iC_- \). In the same fashion, the half of \( \overline{\Gamma}_- \) which is not annihilated by \( D_- \) must be equal to the analogous half of \( i\overline{C}_- \). The complementary halves of these respective superfields remain unrelated. Of course, now \( \Gamma_+ = 0 = \overline{\Gamma}_+ \). Thus, unlike in the type-A and type-B gauging, the type-C \( G_- \)-gauge superfields \( \Gamma_-, \overline{\Gamma}_- \) are not completely determined by the prepotential superfields \( C-, \overline{C}_- \), and their introduction is less useful.

4.4. Type-C\( \pm \) gauging

Finally, let us project on the \( G_\pm \) factor in the gauge group, \textit{i.e.}, set \( B = A = 0 = C_- \). The peculiar relations (2.14) and (2.15) now imply that

\[
[\nabla_-, C_{\pm}] = 0, \quad [\nabla_-, \overline{C}_{\pm}] = 0, \quad [\nabla_+, C_{\pm}] = 0, \quad [\nabla_+, \overline{C}_{\pm}] = 0, \tag{4.26}
\]

while Eqs. (2.13) further constrain

\[
[\nabla_+, C_{\pm}] = 0, \quad \text{and} \quad [\nabla_+, \overline{C}_{\pm}] = 0. \tag{4.27}
\]

That is, \( C_{\pm}'s \) are gauge-covariantly chiral leftons (3.12a), while \( \overline{C}_- \)'s are gauge-covariantly antichiral leftons (3.12b).

As with the \( C_- \)'s, the \( C_{\pm}'s \) do couple to the \( G_\pm \)-gauge superfields, since

\[
0 \equiv [C_{\pm}, C_{\pm}] = iC_{\pm}^i C_{\pm}^j f_{ijk} T_k, \tag{4.28}
\]

vanishes on account of the antisymmetry of the \( f_{ijk} \)'s and the commutivity of the \( C \)'s. This allows \( C_{\pm} \) and its conjugate to have nonzero \( G_\pm \)-charges.

Superfield strengths

Again, the formulae (2.12) and (2.17) simplify:

\[
W_{\pm} = \frac{i}{2} [\nabla_+, C_{\pm}], \quad \overline{W}_{\pm} = \frac{i}{2} [\overline{\nabla}_+, \overline{C}_{\pm}], \tag{4.29}
\]

and \( W_{\pm}, \overline{W}_{\pm}, \) their conjugates and \( F \) all vanish. Again, recall that the would-be ‘usual’ bosonic field strength, \( F_{\pm, \pm} \), vanishes identically; see appendix B. The Jacobi identities involving two spinorial and one vectorial \( \nabla \) then produce the analogue of the familiar relations (4.4):

\[
\begin{align*}
\{ \nabla_-, W_{\pm} \} &= 0, \quad \{ \nabla_-, \overline{W}_{\pm} \} = 0, \quad \tag{4.30b} \\
\{ \nabla_-, \overline{W}_{\pm} \} &= 0, \quad \{ \nabla_-, W_{\pm} \} = 0. \quad \tag{4.30a}
\end{align*}
\]

proving that \( W_{\pm}, \overline{W}_{\pm} \) are covariant leftons, related by the ‘bisection formula’ (2.16c). Again, since

\[
2i\{ \nabla_+ , W_{\pm} \} = \{ \nabla_+ , [\nabla_+ , C_{\pm}] \} = \frac{1}{2} [C_{\pm}, \{ \nabla_+ , \nabla_+ \}] = [C_{\pm}, \overline{C}_{\pm}], \tag{4.31}
\]
the $W_\pm$ are gauge-covariantly \textit{antichiral} leftons if $G_\pm$ is abelian. Of course, the $\overline{W}_\pm$ then are gauge-covariantly \textit{chiral} leftons.

We will refer to $\{C_+, \overline{C}_+; W_+, \overline{W}_+\}$, subject to the superdifferential relations (4.29), as the type-C$_\pm$, or chiral lefton gauge multiplet. These gauge fields \textit{only couple} to covariant leftons (3.17), and to all non-minimal haploid superfields (3.13).

\textbf{Dimensional reduction}

The chiral lefton gauge multiplet $\{C_+, \overline{C}_+; W_+, \overline{W}_+\}$ also has no counterpart in the rather more familiar 3+1-dimensional models. Somewhat in analogy with the $B, \overline{B}$ and $C_-, \overline{C}_-$ potential superfields, the spin-$(-1)$ gauge potential superfields $C_+, \overline{C}_+$ also may be regarded as the gauge potentials introduced to covariantize the spin-$(-1)$ central charges—were such to have been introduced on the r.h.s. of second of Eqs. (2.11a). Again, their absence from the present case (2.11) may be understood merely as the statement that all involved superfield are chargeless with respect to these central charges.

\textbf{Abelian case}

Again, $C_+, \overline{C}_+$ are $G_+$-chargeless, and the $G_+$-gauge covariant (super)derivatives act on the $C_+, \overline{C}_+$ as ordinary (super)derivatives do. Now, as noted in Ref. [1], it is not possible to express chiral leftons, and so also the $C_+, \overline{C}_+$’s, in terms of a superderivative of an am-bidextrous superfield. It is possible, however, to write $C_+ \overset{\text{def}}{=} (D_+ C_+)$, thereby specifying the $G_+$-gauge potential in terms of a spin-$(-\frac{1}{2})$ lefton prepotential superfield, $C_+$.

Since now Eqs. (2.187) imply that $C_+ = -i(D_+ \Gamma_+)$, it follows that the half of $\Gamma_+$ which is not annihilated by $D_+$ must be equal to the analogous half of $iC_+$. In the same fashion, the half of $\overline{\Gamma}_+$ which is not annihilated by $D_-$ must be equal to the analogous half of $\overline{iC}_+$. The complementary halves of these respective superfields remain unrelated. Of course, this time $\Gamma_- = 0 = \overline{\Gamma}_-$. Again, unlike in the type-A and type-B gauging and just like in the type-C= gauging, the type-C$_\pm$ gauge superfields $\Gamma_+, \overline{\Gamma}_+$ are \textit{not} completely determined by the prepotential superfields $C_+, \overline{C}_+$.

Thus, for type-A and type-B gauging, both the spinorial field strength superfields, $W, \overline{W}$, and the potential superfields, $A, \overline{A}, B, \overline{B}$, are gauge-covariantly \textit{haploid} superfields, and of relatively twisted chirality: if the gauge potentials are (anti)chiral, the spinorial field strengths are twisted (anti)chiral, and \textit{vice versa}. For both type-C gaugings, however, the spinorial field strengths, $W, \overline{W}$, are gauge-covariantly unidexterous \textit{haploid} superfields, whereas the gauge potentials, $C, \overline{C}$, are gauge-covariantly \textit{quartoid} superfields.

Similarly, there are marked differences in the abelian case. While the type-A and type-B gauging of an abelian symmetry both allow a straightforward reduction of all gauge superfields to a real gauge prepotential superfield, this is not the case in the two type-C gauging. Here, only a pair of spinorial gauge prepotential superfields, $C_-, \overline{C}_-$ and $C_+, \overline{C}_+$, can be introduced as easily, but they \textit{do not} determine completely the gauge superfields $\Gamma, \overline{\Gamma}$, and so are of limited use. Also, while the spinorial field strengths for type-A and
type-B gauging remain gauge-covariantly \textit{haploid} superfields, those for the two type-C gaugings become gauge-covariantly \textit{quartoid} superfields.

Finally, unlike the type-A and type-B gauging, neither of the type-C gaugings contributes to the usual Yang-Mills field strength superfield, $\mathbf{F}$.

4.5. Mixed type gauging

The previous simple cases, when all but one (and its conjugate) of the $A, B, C\equiv, C_\perp$ gauge superfields is zero, represent the most restricted type of symmetry gauging.

It is also possible to have only some \textit{two}, or only some \textit{three}, or indeed \textit{all four} of the $A, B, C\equiv, C_\perp$'s valued in the generators of the \textit{same} irreducible factor of the total gauge group. Thus, the most general type of gauge symmetry is, in principle, of the form of a direct product of 15 factors:

$$
\mathcal{G}_{\text{Gen}} = \left( \bigotimes_{\mathcal{I}} \mathcal{G}_{\mathcal{I}} \right),
$$

(4.32)

where $\mathcal{I}$ is a multi-index, taking values in 1-, 2-, 3- and 4-element subsets of the label-set $\{A, B, \equiv, \perp\}$. $T_j$, the generators of $\mathcal{G}_{\text{Gen}}$ then have a block-diagonal matrix representation, such that the matrix-generators of the $n^{th}$ factor in (4.32) have only the $n^{th}$ diagonal block non-zero. Normalizing these generators so that

$$
\text{Tr} \{ T_j, T_k \} = \delta_{jk},
$$

(4.33)

we easily define projectors on any one of the 15 factors in (4.32):

$$
P_{\mathcal{I}}(X) \overset{\text{def}}{=} \sum_{j \in \mathcal{I}} T_j \text{Tr} \{ T_j, X \},
$$

(4.34)

where $X$ is an arbitrary superfield, assumed only to be expandable over the generators of $\mathcal{G}_{\text{Gen}}$: $X = T_k X^k$. Clearly then,

$$
P_{\mathcal{I}}(X) = \sum_{j \in \mathcal{I}} T_j \text{Tr} \{ T_j, T_k \} X^k = \begin{cases} X & \text{if } k \in \mathcal{I}, \\ 0 & \text{if } k \notin \mathcal{I}. \end{cases}
$$

(4.35)

The $A, \bar{A}$ gauge potential superfields are expanded over the generators of

$$
\mathcal{G}_{\text{Gen}}|_{\exists A} = \bigotimes_{\mathcal{I} \ni A} P_{\mathcal{I}}(\mathcal{G}_{\text{Gen}}),
$$

(4.36)

$$
= \mathcal{G}_A \otimes \mathcal{G}_{AB} \otimes \mathcal{G}_{A\equiv} \otimes \mathcal{G}_{A\perp} \otimes \mathcal{G}_{AB\equiv} \otimes \mathcal{G}_{AB\perp},
$$

and so on. The properties of the ‘mixed’ gauging types can be deduced from the above analysis of the ‘pure’ gauging types, and shortly we turn to a few sample cases.

Before that, however, a general remark is in order: all the ‘mixed’ types of gauging are beset with a \textit{common} property. As now more than one of the gauge superfields $A, B, C\equiv, C_\perp$ is non-zero, there will occur a \textit{duplication} of degrees of freedom per gauge transformation. This is against standard wisdom (see Ref. [4], p.170), but need not be
deleterious in 1+1-dimensional spacetime. We will comment on this below, but defer a detailed study for a later time.

**Type-AB gauging**

An example of this ‘mixed’ kind is provided by the gauge group which is denoted by $G_{AB}$ in (4.32). Projecting the (total) gauge group to this factor, we obtain $C = 0$, but $A, B \neq 0$. In this case, the spin-$\pm \frac{1}{2}$ superfield strengths are as given in Eqs. (2.12) and they obey the ‘bisection formula’ (2.16a), but the spin-$\pm \frac{3}{2}$ superfield strengths given in Eqs. (2.12) vanish. The bosonic superfield strength, $F$, is as given in Eq. (2.16d). Since the $C$’s vanish, so do the right hand sides of the ‘peculiar relations’ (2.14) and (2.15), so that the gauge potential superfields $A, B$ are twisted-chiral and chiral, respectively, just as in the simple cases of §4.1 and 4.2.

Reviewing the selectivity of coupling to matter (Table 1), it should be clear that $G_{AB}$ couples neither to chiral nor to twisted-chiral superfields (or their conjugates), but it can couple to leftons (3.1e) and rightons (3.1f), and of course to all non-minimal haploid superfields (3.13).

The two non-zero gauge superfields, $A, B$, now provide two independent complex gauge scalars, $a, b$, and four independent gauginos, $\alpha_-, \alpha_+, \beta_+; \beta_-$; see below. Doubling the physical degrees of freedom assigned to each gauge transformation, this will definitely lead to incorrect (redundant) interaction with matter. The similar duplication of the independent contributions into the bosonic field strength, $F$, is less alarming, as it is an auxiliary field in 1+1-dimensions.

The gauge symmetry in this context may be identified as the ‘diagonal’ subgroup $G_{AB} \subset (G_A \times G_B)$, where of course $G_A \approx G_B$. A properly non-redundant gauging would then imply an identification of the type-A gauge fields with the type-B ones, and in a supersymmetric manner. To this end, one has to impose additional (super)constraints. This approach however often has unexpected and undesired consequences [1]: $[\nabla_+, A] = [\nabla_+, B]$, for example, induces both $A$ and $B$ to become gauge-covariantly right-moving, i.e., $\nabla_+ A = 0 = \nabla_+ B$! In the nonabelian case this would also produce non-linear relationships between the component fields of $A, B$. Whether or not this redundancy of physically relevant gauge degrees of freedom can be eliminated in a manifestly supersymmetric fashion then remains an open question for now.

**Type-AC= gauging**

Another example of this ‘mixed’ kind is provided by the gauge group $G_{A=}$: here $B, C_+ = 0$, but $A, C_- \neq 0$. This time, the spin-$\pm \frac{1}{2}$ superfield strengths are as given in Eqs. (4.3a, 4.4) and they obey the ‘bisection formula’ (2.16d); the nonzero spin-$\pm \frac{3}{2}$ superfield strengths are as given in Eqs. (4.23). The bosonic superfield strength, $F$, is as given in Eq. (1.3e). The gauge potential superfields $A$ are no longer twisted-chiral, but instead satisfy only the single constraint, $[\nabla_+, A] = 0$, and are related to $C_-$ through the ‘curious relation’ (2.14d): $[\nabla_-, A] = -[\nabla_+, C_-]$. A combination of the remaining ‘curious relations’ (2.14), (2.15) and the constraints (2.13) imply that $C_-$ now is chiral.
Reviewing again the selectivity of coupling to matter (Table 1), it should be clear that \( G_{A=} \) again couples neither to chiral nor to twisted-chiral superfields (or their conjugates), but it can couple to leftons \((3.1c)\), rightons \((3.1f)\), and of course to all non-minimal haploid superfields \((3.13)\).

As compared to the type-A gauging, the gauge superfields \( A, C= \) now contribute additional spin-\((1, \frac{3}{2})\) degrees of freedom, none of which are physical. They do however complicate the auxiliary field structure of any model where this type of gauging is employed. In the quantum theory, these additional, higher-spin component fields will also induce the appearance of additional ghost degrees of freedom as compared to the type-A gauging. It is possible that this leads to essentially and usefully different dynamics, much as non-minimal haploid and minimal haploid superfields are physically inequivalent [10]. A definite answer to this question will however require a study of its own.

**Universal, \( i.e., \) type-ABC gauging**

Our final example of this ‘mixed’ kind is provided by the ‘universal’ gauge group \( G_{AB=\pm} \): here all of \( A, B, C \)'s are nonzero, and valued in the same irreducible Lie algebra. This is in fact the general case considered in §2, and no modification is needed.

The selectivity of coupling to matter (Table 1) now implies that \( G_{AB=\pm} \) couples to none of the gauge-covariantly haploid superfields, but of course can couple to all of the non-minimal haploid superfields \((3.13)\). This yet again reinforces the distinction between ‘minimal’ gauge-covariantly haploid superfields \((3.1)\), and their ‘non-minimal’ brethren \((3.13)\).

Clearly, the issue of duplication of physical degrees of freedom becomes most complicated here, and it combines the kind found in the type-AB and the type-AC= gaugings. Whether or not a suitable non-redundant gauging of this type is possible remains an open question for now.

**4.6. Gauge transformation**

The gauge transformation operator, \( G \), which appears in Eqs. \((2.2)\), has so far remained unspecified. In general, of course, \( G = \exp(i\mathcal{E}) \), where \( \mathcal{E} \) is a \((4.32)\)-Lie algebra valued superfield, \( i.e., \) a superfield expanded over the generators of \( G_{Gen} \) in \((4.32)\). Notice that the definitions of the constrained superfields \((3.1), (3.12)\) and \((3.13)\) are all covariant with respect to the gauge transformation \((2.2)\) — regardless of the superfield type of \( \mathcal{E} \)! For example, \( \mathcal{E} \) may be chosen to be an unconstrained superfield, yet the transformed gauge-covariantly chiral superfield \((3.1a)\), \( \Phi' \equiv G\Phi \) satisfies the transformed covariant chirality condition,

\[
(\nabla_{\mp}\Phi) = 0 \quad \Rightarrow \quad (\nabla_{\mp}\Phi') \equiv (G\nabla_{\mp}G^{-1}\Phi) = G(\nabla_{\mp}\Phi) = 0 .
\]  (4.37)

The same applies to all other superconstraints \((3.1), (3.12)\) and \((3.13)\). Note that as the gauge transformation operator, \( G \), is required to be unitary, the gauge transformation generator, \( \mathcal{E} \), must be Hermitian. Other that that, however, \( \mathcal{E} \) remains an unconstrained general superfield.
Depending on the choice of superfield type for \( E \), there exist several distinct ‘representations’ \(^6\), and we now discuss some of them in turn.

**Chiral representation**

The infinitesimal version of \( \Phi' \equiv G\Phi \) reads

\[
\delta \Phi = i\mathcal{E}\Phi ,
\]

and the infinitesimal version of (2.9) becomes

\[
\delta \Gamma_\pm = (D_\mp \mathcal{E}) - i[\Gamma_\pm ,\mathcal{E}] = \nabla_\mp \mathcal{E} ,
\]

\[
\delta \mathcal{G}_\pm = (D_\mp \mathcal{E}) - i[\mathcal{G}_\pm ,\mathcal{E}] = \nabla_\mp \mathcal{E} .
\]

The infinitesimal transformation of, say, the chirality condition (3.1a) becomes

\[
\delta (\nabla_\pm \Phi) = ((\delta \nabla_\pm )\Phi) + (\nabla_\pm (\delta \Phi)) .
\]

Now, since \( \delta \nabla_\pm = -i\delta \mathcal{G}_\pm \) and \( \delta \Phi = i\mathcal{E}\Phi \), we have that for a gauge-covariantly chiral superfield to remain so, it must be that

\[
- i\delta \mathcal{G}_\pm \Phi + \nabla_\pm i\mathcal{E}\Phi = 0 ,
\]

whereupon

\[
\delta \mathcal{G}_\pm = \nabla_\pm \mathcal{E} , \quad \text{and} \quad \delta \Gamma_\pm = \nabla_\mp \mathcal{E} ,
\]

in agreement with (4.39).

Frequently, however, one encounters a weaker notion of preserving the type of constrained superfields. In comparison to the preceding argument, let us regard the variation of the conjugate gauge superfields, \( \mathcal{G} \), as being of higher order. That is, the gauge parameter \( \mathcal{E} \) is now chosen to be a ‘slowly varying’ superfield, so that \( \nabla \mathcal{E} \ll \mathcal{E} \), and \( \delta \mathcal{G} \approx 0 \). In this limit, the condition (4.41) for a gauge-covariantly chiral superfield to remain so may be stated as

\[
\nabla_\pm \tilde{\mathcal{E}} = 0 ,
\]

*i.e.*, that \( \tilde{\mathcal{E}} \) is a gauge-covariantly chiral superfield itself, and with respect to the *untransformed* covariant derivative. This necessarily contradicts the originally required hermiticity of \( \mathcal{E} \), and so the unitarity of \( G \). For, if \( \tilde{\mathcal{E}} \) was also Hermitian, then the Hermitian conjugate of (4.43) would imply

\[
\nabla_\pm \tilde{\mathcal{E}}^\dagger = \nabla_\mp \tilde{\mathcal{E}} = 0 ,
\]

whereupon \( \nabla_\mp \tilde{\mathcal{E}} = 0 = \nabla_\pm \tilde{\mathcal{E}} \), and \( \tilde{\mathcal{E}} \) would have to be gauge-covariantly constant. Note that this *does not* necessarily imply absolute constancy; nevertheless, this is often considered too restrictive a choice. One hence leaves \( \tilde{\mathcal{E}} \) to be complex and chiral, and so the gauge transformation operator \( \tilde{G} \equiv \exp\{i\tilde{\mathcal{E}}\} \) is no longer unitary.

This complicates things, since now \( \tilde{G}^{-1} \neq \tilde{G}^\dagger \), and expressions such as \( \Tr(\Phi\bar{\Phi}) \), \( \Tr(\nabla^4 \Phi\bar{\Phi}) \), etc. are no longer gauge-invariant. The remedy (see, *e.g.*, Ref. \[7\], §3.6.4)
involves the introduction of two distinct covariant derivatives, $\nabla^{(\pm)}$, such that $(\nabla^{(+)}\Phi)$ and $(\nabla^{(-)}\widetilde{\Phi})$ transform covariantly:

$$
(\nabla^{(+)}\Phi)' = \tilde{G}(\nabla^{(+)}\Phi), \quad \text{and} \quad (\nabla^{(-)}\widetilde{\Phi})' = (\nabla^{(-)}\Phi)(\tilde{G}^{-1})^\dagger.
$$

(4.45)

We will return to this issue below.

This is the ‘simplest’ choice, in that Eq. (4.38) maintains chirality in this weaker, but more familiar sense: gauge-covariantly chiral superfield form a ring, and are in particular closed under multiplication—a product of two gauge-covariantly chiral superfields is again gauge-covariantly chiral. Note that this implies that, in this (chiral) representation,

$$
\delta\Gamma^+ = (\nabla^+\tilde{\epsilon}), \quad \text{and} \quad \delta\Gamma^- = (\nabla^-\tilde{\epsilon}) = 0,
$$

(4.42)

which is rather asymmetric [6]. In the “antichiral” representation, $\tilde{\epsilon}$ of course obeys (3.1b), and the asymmetry of (4.42)’ is reversed.

Other representations

It is of course equally easy to choose a ‘twisted-chiral representation’, in which $\tilde{\epsilon}$ obeys (3.1c), and is suitable for transforming twisted-chiral superfields. Similarly, in the ‘lefton representation’, $\tilde{\epsilon}$ is chosen to obey (3.1e) and is suitable for transforming leftons, and in the ‘righton representation’, $\tilde{\epsilon}$ is required to obey (3.1f) and is suitable for transforming rightons. In each of these case, one uses the closure under multiplication of the ‘minimal’ haploid superfields (3.1):

For any two superfields which both satisfy any one of the pairs of simple, first order superdifferential constraints (3.1), so does any analytic function thereof, and in particular, so does their product.

By contrast, ‘non-minimal’ haploid superfields (3.13) are not closed under multiplication:

For any two superfields which both satisfy any one of the simple, second order superdifferential constraints (3.13), only their linear combinations satisfy the same superconstraint.

However, since all ‘minimal’ gauge-covariantly haploid superfields (3.1) also satisfy the ‘non-minimal’ gauge-covariantly haploid superconstraints (3.13), there does exist a choice of $\tilde{\epsilon}$ which preserves the type of the ‘non-minimal’ gauge-covariantly haploid superfields (3.13). If the gauge parameter superfield $\tilde{\epsilon}$ is chosen to be a ‘minimal’ gauge-covariantly haploid superfield, its product with the corresponding ‘non-minimal’ gauge-covariantly haploid remains a ‘non-minimal’ gauge-covariantly haploid.

For example, if $\tilde{\epsilon}$ obeys (3.1a) while $\Theta$ obeys (3.13a), then $e^{i\tilde{\epsilon}}\Theta$ also obeys (3.13a), and so does $\delta\Theta = i\tilde{\epsilon}\Theta$. Thus, any one of the ‘minimal’ haploid representations of the gauge transformation may also be used for the corresponding ‘non-minimal’ haploid superfields, and their superconstraint will remain preserved also in the weaker sense (4.43).
4.7. Non-unitary (de)covariantization and gauge prepotentials

On physical grounds, the covariantization $D \rightarrow \nabla$, as done in Eqs. (2.1), relates the (super)derivatives with their gauge-covariant counterparts in a continuous fashion. Indeed, the (suppressed) coupling constant may be continuously turned off to recover $\nabla \rightarrow D$. This operation resembles the gauge transformation process, and it is indeed possible to find operators, $\mathcal{H}$, such that

$$\nabla_\mp = \mathcal{H}^{-1}D_\mp \mathcal{H}, \quad \text{and} \quad \nabla_\mp = \nabla_\mp^\dagger = \mathcal{H}D_\mp \mathcal{H}^{-1}. \quad (4.46)$$

It should be clear that $\mathcal{H}$ must not be unitary: if it were, there would always have to exist a gauge in which $\nabla \rightarrow D$, and where all the gauge fields $\Gamma$ would vanish—contradicting the fact that (at least some) gauge fields are physically relevant. As it is, $\mathcal{H}$ may be written as the exponential with a complex exponent, the imaginary part of which can always be annihilated by a suitable (unitary) gauge transformation, $G = e^{iE}$, where $E^\dagger = E$. In this gauge then, $\mathcal{H}$ is Hermitian, being an exponential with a Hermitian exponent: $\mathcal{H} = e^{-V}$, where $V^\dagger = V$ so also $H^\dagger = H = H$.

Note that the non-unitary transformation (4.46) induces a corresponding transformation on superfields. For example, the gauge-covariantly chiral superfields satisfy Eq. (3.14), which now becomes

$$0 = \nabla_\mp \Phi = \mathcal{H}D_\mp \mathcal{H}^{-1}\Phi. \quad (4.47)$$

This prompts the definitions

$$\Phi = \mathcal{H}\Phi^{(0)}, \quad \text{and so} \quad \Phi = \Phi^{(0)}\mathcal{H}, \quad (4.48)$$

where now $\Phi^{(0)}$ $(\Phi^{(0)})$ is simply (anti)chiral:

$$D_\mp \Phi^{(0)} = 0, \quad \text{and} \quad D_\mp \Phi^{(0)} = 0. \quad (4.49)$$

Using the non-unitary transformation (4.46) and (4.48), it is then always possible to rewrite a model involving gauge-covariantly (anti)chiral superfields, $\Phi, \Phi^\dagger$, coupled to gauge (super)fields in terms of the simply (anti)chiral superfields, $\Phi^{(0)}, \Phi^{(0)}$, coupled to the same gauge (super)fields. For example,

$$\text{Tr} \left( \Phi\Phi^\dagger \right) = \text{Tr} \left( \mathcal{H}\Phi^{(0)}\Phi^{(0)}\mathcal{H} \right) = \text{Tr} \left( \Phi^{(0)}\mathcal{H}\mathcal{H}\Phi^{(0)} \right). \quad (4.50)$$

Furthermore, upon the gauge transformation described above in which $\mathcal{H} = e^{-V}$ is Hermitian, we have

$$\text{Tr} \left( \Phi\Phi^\dagger \right) = \text{Tr} \left( \Phi^{(0)}e^{-2V}\Phi^{(0)} \right), \quad (4.51)$$

which recovers (upon 4-fermionic integration) the standard expressions for the gauge-covariant kinetic term in the Lagrangian for (anti)chiral superfields [3.3.4]. Moreover, this shows the origin of the gauge (pre)potential superfield, $V$, as the generator of the coset $G^c/G$, where $G^c$ denotes the complexification of the gauge group $G$. That is, having relaxed unitarity, $\mathcal{H}$ is an element of $G^c$; but, when taken modulo gauge transformations (which live in $G$), $\mathcal{H}$ belongs to the coset $G^c/G$. Indeed, this $V$ becomes the gauge prepotential $V^{(A)}$, as found in Eqs. (4.6) for the abelian case of type-A gauging, and $V^{(B)}$,

\[10\] ‘Non-gauge-covariantly . . .’ being such a clumsy mouthful, we write ‘simply . . .’ instead.
as found in Eqs. (4.15) for the abelian case of type-B gauging. See §3.6 of Ref. [[9]] for the derivation of the analogous statement from the standard (but opposite to our) approach, where one employs the simple (super)derivatives and simply constrained superfields but inserts explicitly gauge-covariantizing factors such as $e^{-2V}$.

In the present approach, gauge-covariance is made manifest through the use of gauge-covariant derivatives (2.1) and gauge-covariantly constrained superfields (3.1), (3.12) and (3.13). With the present level of generality, this seems preferable to determining the precise form of all the gauge-covariantizing insertions separately. Instead, the analogues of $e^{-2V}$ for other simply haploid superfields may be obtained by applying to the above results: (1) either of the (mirror map) discrete transformations, $C^\pm$, of Ref. [[1]] for twisted-(anti)chiral superfields, and (2) either of $q, \bar{q}$ of Ref. [[1]] to map (anti)chiral superfields into leftons (rightons). We leave this to the interested Reader.

We do note, however, that the appearance of the prepotential superfield, $V$, in expressions like (4.51) proves its existence and uncovers its geometric origins, for all types of symmetry gauging and all (compact) symmetry groups—not just the abelian cases shown explicitly in §4. A more complete discussion of such decovariantization(s) is presented elsewhere [[11]].

5. Component field content

The definition of component fields by means of projecting on the various terms in the Taylor expansion [[6,9,1]] is easy to adapt for the presence of gauge symmetries. One simply uses covariant superderivatives (2.1) in place of ordinary ones (1.4), and commutators for second order superderivatives. As usual, ‘$|$’ will denote setting $\varsigma^- = 0 = \bar{\varsigma}^-$. 

5.1. Matter fields and general convention

To begin, consider the ‘minimal’ gauge-covariantly haploid superfields (3.1).

\[
\begin{align*}
\phi & \equiv \Phi^1, & \psi^\pm & \equiv \frac{1}{\sqrt{2}} \nabla^\pm \Phi^1, & F & \equiv \frac{1}{4} [\nabla_-, \nabla_+ ] \Phi^1; \\
\bar{\phi} & \equiv \bar{\Phi}^1, & \bar{\psi}^\pm & \equiv \frac{1}{\sqrt{2}} \nabla^\pm \bar{\Phi}^1, & \bar{F} & \equiv \frac{1}{4} [\nabla_-, \nabla_+ ] \bar{\Phi}^1;
\end{align*}
\]

(5.1a)

are the component fields of the gauge-covariantly chiral and the gauge-covariantly antichiral spin-0 superfield (3.1a,b). We write $\Phi = (\phi; \psi_\pm; F)$.

Next,

\[
\begin{align*}
x & \equiv \Xi^1, & \xi^- & \equiv \frac{1}{\sqrt{2}} \nabla^- \Xi^1, & \chi^+ & \equiv \frac{1}{\sqrt{2}} \nabla^+ \Xi^1, & X & \equiv \frac{1}{4} [\nabla_-, \nabla_+ ] \Xi^1; \\
\bar{x} & \equiv \bar{\Xi}^1, & \bar{\xi}^- & \equiv \frac{1}{\sqrt{2}} \nabla^- \bar{\Xi}^1, & \bar{\chi}^+ & \equiv \frac{1}{\sqrt{2}} \nabla^+ \bar{\Xi}^1, & \bar{X} & \equiv \frac{1}{4} [\nabla_-, \nabla_+ ] \bar{\Xi}^1;
\end{align*}
\]

(5.1c)

are the component fields of the gauge-covariantly twisted-chiral and the gauge-covariantly twisted-antichiral spin-0 superfield (3.1c,d). We write $\Xi = (x; \xi_-, \chi^+; X)$. 

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Finally, we define
\[ \ell \overset{\text{def}}{=} \Lambda |, \quad \lambda_+ \overset{\text{def}}{=} \frac{1}{\sqrt{2}} \nabla_+ \Lambda |, \quad \lambda_- \overset{\text{def}}{=} \frac{1}{\sqrt{2}} \nabla_- \Lambda |, \quad L_\pm \overset{\text{def}}{=} \frac{1}{4} [\nabla_+, \nabla_-] \Lambda |; \quad (5.1e) \]
and
\[ r \overset{\text{def}}{=} \Upsilon |, \quad \rho_- \overset{\text{def}}{=} \frac{1}{\sqrt{2}} \nabla_- \Upsilon |, \quad \varrho_+ \overset{\text{def}}{=} \frac{1}{\sqrt{2}} \nabla_+ \Upsilon |, \quad R_- \overset{\text{def}}{=} \frac{1}{4} [\nabla_-, \nabla_-] \Upsilon |; \quad (5.1f) \]
are the component fields in the gauge-covariantly lefton and the gauge-covariantly righton superfields. We write \( \Lambda = (\ell; \lambda_+, \lambda_-; L_\pm) \) and \( \Upsilon = (r; \rho_-, \varrho_-; R_-) \).

Since gauge-covariantly quartoid superfields cannot couple to any of the minimally coupled gauge (super)fields, the definitions of their component fields remain as given in Ref. [1].

The component field content of the 'non-minimal' haploid superfields is just as straightforward to obtain, only this time there are more component fields than in Eqs. (5.1). For example,
\[
\begin{align*}
t & \overset{\text{def}}{=} \Theta |, \quad \theta_\pm \overset{\text{def}}{=} \frac{1}{\sqrt{2}} \nabla_\pm \Theta |, \quad \vartheta_\pm \overset{\text{def}}{=} \frac{1}{\sqrt{2}} \nabla_\pm \Theta |, \\
T & \overset{\text{def}}{=} \frac{1}{4} [\nabla_-, \nabla_+] \Theta |, \quad T_\mp \overset{\text{def}}{=} \frac{1}{4} [\nabla_-, \nabla_-] \Theta |, \quad T_\pm \overset{\text{def}}{=} \frac{1}{4} [\nabla_+, \nabla_+] \Theta |, \\
T_\mp & \overset{\text{def}}{=} \frac{1}{4} [\nabla_-, \nabla_-] \Theta |, \quad T_\pm \overset{\text{def}}{=} \frac{1}{4} [\nabla_+, \nabla_+] \Theta |
\end{align*}
\]
(5.2a)

Note: the ± and ± symbols on \( T_\pm, T_\mp \) indicate single components, whereas the ± on the fermions \( \theta_\pm, \vartheta_\pm, \tau_\pm \) indicates a choice of spin \(+\frac{1}{2}\) and \(-\frac{1}{2}\). We also write \( \Theta = (t; \theta_\pm, \vartheta_\pm; T_\mp, T_\pm, T_\mp, T_\pm; \tau_\pm) \). Similarly,
\[
\begin{align*}
p & \overset{\text{def}}{=} \Pi |, \quad \pi_\pm \overset{\text{def}}{=} \frac{1}{\sqrt{2}} \nabla_\pm \Pi |, \quad \varpi_\pm \overset{\text{def}}{=} \frac{1}{\sqrt{2}} \nabla_\pm \Pi |, \\
P & \overset{\text{def}}{=} \frac{1}{4} [\nabla_-, \nabla_+] \Pi |, \quad P_\mp \overset{\text{def}}{=} \frac{1}{4} [\nabla_-, \nabla_-] \Pi |, \quad P_\pm \overset{\text{def}}{=} \frac{1}{4} [\nabla_+, \nabla_+] \Pi |, \\
P_\mp & \overset{\text{def}}{=} \frac{1}{4} [\nabla_-, \nabla_-] \Pi |, \quad P_\pm \overset{\text{def}}{=} \frac{1}{4} [\nabla_+, \nabla_+] \Pi |
\end{align*}
\]
(5.2b)

The component fields of \( \bar{\Theta} \) and \( \bar{\Pi} \) are then obtained by hermitian conjugation. We write \( \Pi = (p; \pi_\mp, \varpi_\mp; P_\mp, P_\pm, P_\pm; \bar{\varphi}_-, \bar{\varphi}_+) \).

The components of the non-minimal gauge-covariantly (almost) lepton (3.13a) and (almost) righton (3.13f) can be defined as easily. However, to avoid confusion with the gauge potential superfield \( \Lambda \) and since we will not need these (almost) unidexeterous superfields, we leave the definition of the components of the non-minimal gauge-covariantly (almost) lepton (3.13a) and righton (3.13f) to the diligent Reader.
Figure 1. The sequence of multiple covariant superderivatives used in defining component fields. Component fields of conjugate superfields are found using the hermitian conjugates of the operators shown here. The operator $\nabla^4$ is obtained by replacing all $D$’s in Eq. (1.6) with $\nabla$’s.

5.2. Component diagrammatics

The Reader may find it convenient to use the diagram in Fig. 1 to find the component field content of any superfield. All components are obtained by acting on the superfield with one of the covariant operators from the diagram, and then setting $\varsigma^\pm=0=\bar{\varsigma}^\pm$. Finally, throughout this article, we set the numerical coefficients as in (5.1a): $\frac{1}{\sqrt{2}}$ for every superderivative, and $\frac{1}{2}$ for every commutator. The examples in (5.1) and (5.2) should suffice in clarifying this.

Applying additional (super)derivatives merely produces total derivatives of the already defined component fields. In this sense, the component fields, onto which the operators in the diagram in Fig. 1 project (upon setting $\varsigma^\pm=0=\bar{\varsigma}^\pm$), are considered independent.

5.3. Gauge fields

As with any of the ‘matter’ superfields in the preceding sections, the component fields of the gauge superfields are also defined by applying the operators from the sequence in Fig. 1, and setting $\varsigma=0=\bar{\varsigma}$’s. Bending somewhat the typographical conventions of Ref. [1], we list the component fields of the gauge superfields $\Gamma_\mp$ as follows

$$
\Gamma_- = \begin{pmatrix}
\Gamma = & \gamma^-_- & G = \\
\Gamma = & \gamma^-_\pm & G = \\
\Gamma = & \gamma^-_\mp & G = \\
\Gamma = & \gamma^-_+ & G = \\
g_- & & g_-
\end{pmatrix}, \quad (5.3a)
$$

$$
\Gamma_+ = \begin{pmatrix}
\Gamma = & \gamma^+_+ & G = \\
\Gamma = & \gamma^+_+ & G = \\
\Gamma = & \gamma^+_+ & G = \\
\Gamma = & \gamma^+_+ & G = \\
g_+ & & g_+
\end{pmatrix}, \quad (5.3b)
$$
and similarly for their conjugates. Many of these component fields either do not even show up in the Lagrangian densities of interest, or they show up as auxiliary fields, \textit{i.e.}, such that their equations of motion are algebraic, which allows for immediate elimination. Until however a Lagrangian density is chosen for such an elimination to take place, one must deal with all the component fields (5.3).

It is fairly standard in theories in 1+1-dimensional spacetime, to assign the coupling constant, \(g\), the canonical dimensions of \(\partial_\pm\), \(\partial_{\pm}\), \textit{i.e.}, of a mass parameter. The lowest components of the superfields \(\Gamma_\pm\), which through using Eqs. (2.18a–h) become

\[
\Gamma_\pm \mid = - \frac{i}{\sqrt{2}} [\Gamma^l_\pm + \Gamma^l_{\pm} + \frac{1}{\sqrt{2}} \bar{g}^j \bar{g}^k \phi_{jk}^l] T_l ,
\]

(5.4)
then acquire the proper canonical dimension, 0, only upon the rescaling \(\Gamma \rightarrow g\Gamma\). This makes it possible for the canonically rescaled superfields to appear in the ‘standard, flat’ kinetic term

\[
- \frac{1}{4} \int d^2 \sigma \, Tr \left\| (\partial_\pm \Gamma_{\pm}) \right\|^2 + \ldots
\]

\textit{Prior} to the \(\Gamma \rightarrow g\Gamma\) rescaling, such action terms ought to be premultiplied by \(g^{-2}\). Through the \(\Gamma \rightarrow g\Gamma\) rescaling, \(g_\pm\), the lowest components of \(\Gamma_\pm, \Gamma_{\pm}\), acquire the canonical dimension \(-\frac{1}{2}\); the (potentially) physical fields there being the gauge bosons \(\Gamma, \bar{\Gamma}\) and the gauginos \(\gamma, \bar{\gamma}\); the \(G, \bar{G}\)’s and the \(g_\pm\) stand for bosonic and fermionic ‘auxiliary’ components.

Essentially the same fate befalls the gauge potential superfields \(A, B, C\)’s and their conjugates (2.18a–h). Their lowest components—the gauge bosons—are linear combinations of the \(\Gamma, \bar{\Gamma}\)’s, and in the nonabelian case, of \(g-\Gamma\) anticommutators. Their next-to-lowest components—the gauginos—are linear combinations of the \(\gamma, \bar{\gamma}\)’s and derivatives of \(g\)’s, and in the nonabelian case, of \(g-\Gamma\) commutators, and so on.

5.4. Field strengths

Since the gauge superfields \(\Gamma\) transform inhomogeneously under the gauge transformations (2.9)—in the infinitesimal form (4.42), so do their components. This means that the gauge transformation parameter \(E\) may be \textit{chosen} so as to gauge away some of the degrees of component fields in \(\Gamma\). The Wess-Zumino gauge \([6,7,8,9]\) is the best known such choice.

On the other hand, the field strength superfields defined in Eqs. (2.11) all transform homogeneously, so that there exists no choice of the gauge parameter superfield, \(E\), that would ‘gauge away’ a component field of \(A, B, C\). This then guarantees that all the component fields of these superfields are unambiguously defined and independent of any gauge choice.

Furthermore, \(F\) is defined as a linear combination of gauge covariant superderivatives of the \(W\)’s, which in turn are all defined as gauge covariant superderivatives of the \(A, B, C\)’s. This guarantees that the lowest component field of \(F\) is a linear combination of the next-to-lowest component fields of the \(W\)’s. And in turn, the lowest component fields of the \(W\)’s are linear combinations of the next-to-lowest component fields of the \(A, B, C\)’s.
For this reason, all physically relevant (and unambiguously defined) component fields in the gauge multiplets must be expressible in terms of the component fields of the $A$, $B$, $C$ and their conjugates. Consequently, we leave to the interested Reader the tracing of the physical degrees of freedom in gauge multiplets all the way to the component fields of the $\Gamma$’s and specification of the Wess-Zumino-like gauge(s) in which the remaining component fields are annihilated. Herein, we proceed instead to use the component fields of the $A$, $B$, $C$’s.

Type-A gauge fields
In the simple type-A, -B and -C gauging, these latter superfields satisfy important superconstraints $(4.1)$, $(4.11)$, $(1.20)$ and $(4.21)$, $(4.26)$ and $(4.27)$. Thus, the independent component fields of the type-A gauge multiplet can be specified equivalently either as the component fields of the gauge-covariantly twisted-chiral superfield $\Gamma$, or as the lowest component fields of the type-A gauge multiplet $\{A; W_\pm; F\}$ given in §4.11:

$$a \overset{\text{def}}{=} A|$$

$$\alpha_- \overset{\text{def}}{=} \frac{1}{\sqrt{2}}[\nabla_-, A]| = \sqrt{2}iW_-|$$

$$\bar{\alpha}_- \overset{\text{def}}{=} -\frac{1}{\sqrt{2}}[\nabla_-, \bar{A}]| = -\sqrt{2}i\bar{W}_-|$$

$$a_+ \overset{\text{def}}{=} \frac{1}{\sqrt{2}}[\nabla_+, A]| = -\sqrt{2}iW_+|$$

$$\bar{a}_+ \overset{\text{def}}{=} -\frac{1}{\sqrt{2}}[\nabla_+, \bar{A}]| = \sqrt{2}i\bar{W}_+|$$

$$A \overset{\text{def}}{=} \frac{1}{4}[\nabla_-, \nabla_+]A| = -if|$$

$$\bar{A} \overset{\text{def}}{=} \frac{1}{4}[\nabla_+, \nabla_-]A| = i\bar{f}| .$$

Note the relative signs in the definition of the conjugate fermionic components; they stem from the antisymmetry of the commutator. These are the spin-$\left(0;\pm\frac{1}{2};0\right)$ component fields of the scalar gauge-covariantly twisted-chiral superfield $A = (a; \alpha_-, a_+; A)$ and its Hermitian conjugate, $\bar{A} = (\bar{a}; \bar{\alpha}_-, \bar{a}_+; \bar{A})$. The bosonic field strength, $F|_{A}$ then equals (the negative of) the imaginary part of $A$:

$$F_{A} \overset{\text{def}}{=} F| = \Re(f)| = -\Im(A) .$$

The real part of $A$ is easily shown to be the lowest component of the superfield

$$D_{A} \overset{\text{def}}{=} \frac{1}{2i}[\nabla_{\pm}, W_\pm]_{A} = \Im(f) ,$$

$$D_{A} \overset{\text{def}}{=} D_{A}| = \Im(m|) = \Re(A) .$$

where the subscript indicates projection on type-A gauge fields. Clearly both $F$ and $D_{A}$ are real. The lowest component of $D_{A}$ is indeed, up to some conventional overall factor, the familiar ‘auxiliary D field’, descending by dimensional reduction from $N=1$ supersymmetric Yang-Mills theory in 3+1-dimensions. This vindicates our use of the complex superfield $f$ as compared to using only its imaginary part, $\bar{f}$.

In the present incarnation, the lowest components of $F$ and $D_{A}$ appear as the ‘auxiliary’ components of the gauge-covariantly twisted-chiral superfield $A$. In particular, note

---

11) Recall: $([\nabla_-, \nabla_+]A) \text{ merely abbreviates } \{\nabla_-, [\nabla_+, A]\} - \{\nabla_+, [\nabla_-, A]\}; \text{ see appendix A.}$

---
that the canonical dimension of $F$ and $D_A$ is one more than that of $A$. Therefore, if a term in the Lagrangian density is *chosen* so as to provide a kinetic term for $a$—something like $h |\partial_0 a|^2 + \ldots$ with $h$ an arbitrary analytic function of the available fields, then the same term will also feature $F$ and $D_A$ with no derivatives on them.

**Type-B gauge fields**

The independent component fields of the type-B gauge multiplet can similarly be specified equivalently either as the component fields of the gauge-covariantly chiral superfield $B$, or as the lowest component fields of the multiplet $\{ B; W_\pm; F \}$ given in §4.2:

$$
\begin{align*}
  b \& \equiv |B|, & \quad \bar{b} \& \equiv \bar{|B|}, & \quad (5.8a) \\
  \beta_\pm \& \equiv \frac{1}{\sqrt{2}} |\nabla_+ B| = \pm \sqrt{2} |W_\pm|, & \quad \bar{\beta}_\mp \& \equiv -\frac{1}{\sqrt{2}} |\nabla_- B| = \mp \sqrt{2} |\bar{W}_\mp|, & \quad (5.8b) \\
  B \& \equiv \frac{1}{2} [\nabla_- + \nabla_+ B] = \mp i |f|, & \quad \bar{B} \& \equiv \frac{1}{2} [\nabla_+ - \nabla_- |B| = \mp i |\bar{f}|. & \quad (5.8c)
\end{align*}
$$

These are the spin-$(0; \pm \frac{1}{2}; 0)$ component fields of the scalar gauge-covariantly chiral superfield $B = (b; \beta_\pm; B)$ and its Hermitian conjugate, $\bar{B} = (\bar{b}; \bar{\beta}_\mp; \bar{B})$. The bosonic field strength, $F_B$ now equals (the negative of) the imaginary part of $B$:

$$
F_B \equiv F = \Re(f)| = -\Im(B). \quad (5.9)
$$

The real part of $B$ is easily shown now to be the lowest component of the superfield

$$
D_B \equiv D_B| = \Im(f) = \Re(B). \quad (5.10)
$$

where the subscript indicates projection on type-B gauge fields. Again, both $F$ and $D_A$ are real.

**Type-C= gauge fields**

The type-C= gauge multiplet is equivalently described either as the component fields of the gauge-covariantly chiral lefton $C=\pm$, or as the lowest component fields of the gauge multiplet $\{ C=\pm; W=\pm \}$ of §4.3:

$$
\begin{align*}
  C_= \& \equiv C_-|, & \quad \bar{C}_- \& \equiv \bar{C}-|, & \quad (5.11a) \\
  \gamma_-^\pm \& \equiv \frac{1}{\sqrt{2}} |\nabla_- C_+| = \sqrt{2} |W_-|, & \quad (5.11b) \\
  \bar{\gamma}_-^\pm \& \equiv -\frac{1}{\sqrt{2}} |\nabla_- \bar{C}_-| = -\sqrt{2} |\bar{W}_-|. & \quad (5.11c)
\end{align*}
$$

These are the spin-$(1; \frac{3}{2})$ component fields of the spin-(+1) gauge-covariantly chiral lefton superfield $C_= = (C_-=; \gamma_-^\pm)$ and its Hermitian conjugate, $\bar{C}_- = (\bar{C}_-; \bar{\gamma}^\pm_-)$. There are no $F$ and no $D$ field strength superfields in type-C= gauging.
Type-\(C_\pm\) gauge fields

The component fields of the type-\(C_\pm\) gauge multiplet either as the component fields of the gauge-covariantly chiral righton \(C_\pm\), or as the lowest component fields of the gauge multiplet \(\{C_\pm; W_\pm\}\) of §4.4:

\[
C_\pm \overset{\text{def}}{=} C_\pm|, \quad \overline{C}_\pm \overset{\text{def}}{=} \overline{C}_\pm|,
\]

\[\gamma_+ \overset{\text{def}}{=} \frac{1}{\sqrt{2}} [\nabla_+, C_\pm]| = \sqrt{2i} W_\pm|, \quad (5.11b)\]

\[\bar{\gamma}_+ \overset{\text{def}}{=} -\frac{1}{\sqrt{2}} [\nabla_+, \overline{C}_\pm]| = -\sqrt{2i} \overline{W}_\pm|, \quad (5.11c)\]

These are the spin-(\(\pm 1; \mp \frac{3}{2}\)) component fields of the spin-(\(-1\)) gauge-covariantly chiral lepton superfield \(C_\pm = (C_\pm; \gamma_\pm^+)\) and its Hermitian conjugate, \(\overline{C}_\pm = (\overline{C}_\pm; \overline{\gamma}_\pm^-)\). There are no \(F\) and no \(D\) field strength superfields in type-\(C_\pm\) gauging.

Based on the last remark of the previous subsection, the canonical dimension of \(a, b, C_\pm\) and \(C_\pm\) from (5.3)–(5.12) is the same as that of the \(\Gamma\)'s in (5.3); these are the gauge bosons. The \(\alpha_-, a_+, \beta_\pm, \gamma_\pm^+\) and \(\gamma_\pm^-\) from (5.3)–(5.12) have the canonical dimensions of the \(\gamma\)'s in (5.3); these are the gauginos. Notice that the type-C spin-(\(\pm \frac{3}{2}\)) gauginos, \(\gamma_-, \gamma_+^\pm\), accompany the spin-(\(\pm 1\)) gauge bosons \(C_\pm, \overline{C}_\pm\), and should not be confused with gravitini which are superpartners of the spin-(\(\pm 2\)) gravitons.

The definition of \(\gamma_\pm^+\) in (5.11b) and \(\gamma_\pm^-\) in (5.12b) differs from that in (the conjugates of) (5.3) by a numerical multiplicative factor, and the addition of two terms depending on the lowest components of \(\Gamma_\pm\). The latter discrepancy vanishes in the appropriate Wess-Zumino-like gauge, and the numerical multipliers are inessential. The precise relation of the component fields (5.5), (5.8), (5.11) and (5.12) to those in (5.3) is left to the diligent Reader; we proceed using the gauge covariant component fields (5.5), (5.8), (5.11) and (5.12).

6. Lagrangian Densities

All the numerous Lagrangian densities listed in Ref. [1] admit a straightforward extension to include minimal coupling type gauge interactions: the fermionic integration must be redefined so as to use the covariant derivatives (2.1) instead the ordinary ones. Section 1 and Appendix A contain all the necessary conventions and notation.

6.1. Kinetic terms for gauge fields

It is straightforward to write down (flat) kinetic terms for the pure type-A and type-B gauge fields [1]:

\[-\frac{1}{8} \int d^4 \varsigma \| A \|^2 , \quad \text{and} \quad +\frac{1}{8} \int d^4 \varsigma \| B \|^2 , \quad (6.1)\]

where the relative sign ensures the correct sign for the bosonic terms.

It is immediately obvious that no such (flat) kinetic terms exist for the type-C gauge superfields! This may sound alarming, as it is sharply counterintuitive when compared
with the familiar situation in 3+1-dimensional spacetime. However, as we have just seen in Eqs. (5.11) and (5.12), the type-C gauge multiplets contain spin-(±1, ±3/2) fields, neither of which is 

supposed to be a propagating degree of freedom in 1+1-dimensional spacetime. A quantum treatment would have to introduce appropriate (and propagating) spin-(0, ±1/2) ghost degrees of freedom, which then would carry the dynamics of this type of symmetry gauging, and the study of which we defer to a later effort.

For illustrative purposes, consider however the pure type-A gauging of an abelian symmetry, say U(1), with respect to which the A, A themselves are chargeless, and the integration measure in the first of the two integrals in (6.1) reduces to that one in Eq. (1.6).

After straightforward D-algebra, we obtain for a single type-A U(1) gauge multiplet:

$$-\frac{i}{8} \int d^4 \varsigma \| A \|^2 = \frac{1}{4} [ (\partial_+ a)(\partial_+ a) + (\partial_+ \bar{a})(\partial_+ \bar{a}) ] + \frac{1}{2} D_A^2 + \frac{1}{2} F_A^2$$

$$+ \frac{i}{4} \left[ (\bar{a}_- \bar{\partial}_+ a_-) + (\bar{a}_+ \bar{\partial}_- a_+) \right] ,$$

Identifying $F_A = \frac{1}{2} \epsilon^{ab} F_{ab}^{(A)}$ and using that $\epsilon^{ac} \epsilon^{bd} = (\delta^{ac} - \delta^{ac})$ since $\epsilon^{01} = 1 = \epsilon_{10}$, the last bosonic term becomes $-\frac{1}{4} F_{ab}^{(A)} F_{ab}^{(A)}$, which is the standard Lagrangian density for a Yang-Mills gauge boson, given in terms of its field strength. Furthermore, one can write

$$F_{ab}^{(A)} = (\partial_a A_b - \partial_b A_a) ,$$

but the Reader should be cautioned that the gauge vector potential $A_a$ carries no physical degree of freedom in 1+1 dimensions. Its only use is for identification with the components of the 3+1-dimensional gauge vector potential 'along' the 1+1-dimensional sub-spacetime; the 'transversal' components form the lowest components of the complex superfield, A, a scalar from the 1+1-dimensional point of view. In fact, it is not an accident that

$$-\frac{1}{4} F_{ab}^{(A)} F_{ab}^{(A)}$$

emerges, in (6.1), as a half of the norm-squared of the 'auxiliary' component field of A, with $\frac{1}{2} D_A^2$ being the other half.

Up to a few numerical differences in convention, this is in complete agreement with Ref. [4], and the dimensional reduction results obtained from Refs. [6,7,8,9].

The second of the two Lagrangians in (6.1) is equally easily evaluated for a single type-B U(1) gauge multiplet:

$$\frac{1}{8} \int d^4 \varsigma \| B \|^2 = \frac{1}{4} [ (\partial_+ \bar{b})(\partial_+ b) + (\partial_+ \bar{b})(\partial_+ \bar{b}) ] + \frac{1}{2} D_B^2 + \frac{1}{2} F_B^2$$

$$+ \frac{i}{4} \left[ (\bar{b}_- \bar{\partial}_+ b_-) + (\bar{b}_+ \bar{\partial}_- b_+) \right] ,$$

Up to some numerical differences of convention, this is again in agreement with the corresponding result of Ref. [4].

Again, writing somewhat formally $F_B = \frac{1}{2} \epsilon^{ab} F_{ab}^{(B)}$, the last bosonic term becomes the standard Yang-Mills Lagrangian density, $-\frac{1}{4} F_{ab}^{(B)} F_{ab}^{(B)}$. This, however, is not the result of dimensional reduction of any 3+1-dimensional Yang-mills gauge multiplet and the introduction of a gauge vector potential, $B_a$ through writing $F_{ab}^{(B)} = (\partial_a B_b - \partial_b B_a)$ is
merely for analogy. As mentioned in §4.2, the definition (2.11) permits us to identify, in a 3+1-dimensional framework prior to dimensional reduction to 1+1-dimensions, the lowest components of the superfield $B, \bar{B}$ as gauge fields covariantizing central charges, were such to have been introduced.

For nonabelian gauge symmetry, the component field expansions of the simple Lagrangian density terms (6.1) become rather more involved than their abelian versions (6.2) and (5.4). In addition, even for pure type-A and type-B gauge multiplets, one can also write down a ‘superpotential’ term:

$$\int d^2 \xi \text{Tr} \Sigma(A) + h.c., \quad \text{and} \quad \int d^2 \varsigma \text{Tr} W(B) + h.c., \quad (6.5)$$

owing to the fact that in pure type-A gauging $A$ is gauge-covariantly twisted-chiral, and in pure type-B gauging $B$ is gauge-covariantly chiral. The superpotentials, $\Sigma$ and $W$, are arbitrary analytic functions of their respective arguments, and ‘Tr’ makes the Lagrangian density terms gauge-invariant. Note that this annihilates all linear terms for all nonabelian factors in $G_A \otimes G_B$. For the abelian, $U(1)$, factors a linear (Fayet-Illiopoulos) term is permitted by gauge invariance, and has been given a pivotal role in the gauged linear $\sigma$-model of Refs. [2] and subsequent work.

Also, the ‘kinetic’ terms (6.1) may well be generalized into

$$\int d^4 \varsigma \text{Tr} K(A, \bar{A}; B, \bar{B}), \quad (6.6)$$

where $K$ is a general, real, function of its arguments, just as first given in Ref. [12]; gauge-invariance is ensured by taking its trace. This of course does not in general produce the ‘standard’ Yang-Mills terms of the type $-\frac{1}{4} F_{ab} F^{ab}$, i.e., $\frac{1}{4} [3m(A)^2 + 3m(B)^2]$, which may be regarded as ‘flat’. Instead, Eq. (5.6) yields an $A, B$-dependent set of terms the nonlinearity of which is governed by the choice of the function $K$ (6.6). Of course, the choice of the Lagrangian, and the adherence to the ‘standard’ one, is governed only by the intended application. From the intrinsically 1+1-dimensional point of view, there is no compelling reason in general not to consider the non-linear and rather general Lagrangian density term (6.6). Moreover, the D-term (6.6) may well be made dependent on all available fields:

$$\int d^4 \varsigma \text{Tr} K(A, \bar{A}; B, \bar{B}; C; \bar{C}; \ldots), \quad (6.7)$$

where the ellipses stand for any other available superfield, including those used to represent ‘matter’. This of course makes the kinetic terms for all involved component fields depend, in general, on all of the fields involved. Most applications will only need a special case of this general form, but we leave that to the decision of the interested Reader, depending on the intended application.

### 6.2. General terms for gauge fields

In ‘unpure’ gauging types, such as the type-AC=, discussed briefly in §4.5, the gauge superfield $A$ is no longer gauge-covariantly twisted-chiral, and neither is then the integrand
of the first Berezin integral in (6.5). Instead, the superfield and so also \( \text{Tr} (\Sigma(A)) \) are annihilated only by \( \nabla_+ \). The \textit{triple} Berezin integral

\[
\frac{1}{2} \int d\bar{\varsigma}^- d^2 \varsigma \text{Tr} (\Sigma(A))
\]

would be supersymmetric and gauge-invariant, but its result has spin \( +\frac{1}{2} \), and so is not suitable for adding to the Lagrange density, which has spin 0. Of course, if the considered model also includes a spin-(-\( \frac{1}{2} \)) superfield which is also annihilated by \( \nabla_+ \)—say, an antichiral \( \Phi_+ \), then

\[
\frac{1}{2} \int d\bar{\varsigma}^- d^2 \varsigma \text{Tr} (\Sigma(A)\Phi_+) = \frac{1}{4} \text{Tr} (\nabla_- [\nabla_-, \nabla_+] \Sigma(A)\Phi_+) \quad (6.9)
\]

is a candidate term for the Lagrangian density. Since \( C_\pm \equiv 0 \) in type-AC\(_-\) gauging, \( \nabla_+^2 = 0 \), and with a suitable twisted-antichiral superfield, \( \Xi' \), we may replace \( \Phi_+ \) in the above expressions freely with a linear combination of \( \nabla_+\Xi' \) and \( \Phi_+ \). Since also \( [\nabla_+, A]=0 \), the \( \nabla_+ \) may be passed to the left and this addition, upon including also the Hermitian conjugate, becomes proportional to the D-term

\[
\frac{1}{4} \text{Tr} (\nabla^4 \Sigma(A)\Xi') \quad (6.10)
\]

\textit{Typically}, models are \textit{chosen} to include the most general D-term (on the premise that quantum corrections will typically generate such terms anyway), so that this addition is already accounted for as a special case of a (D-)term in the total Lagrangian. However, under special circumstances (when quantum corrections will either not be considered or are sufficiently restricted by (additional) symmetry), it is perfectly possible to include no general D-term in the Lagrangian, whence the term (6.10) is indeed a new addition.

6.3. Interaction with matter

Ref. [1] has uncovered a great many candidate terms—each (2,2)-supersymmetric all by itself—for the Lagrangian density involving ‘matter’ represented by the constrained superfields (3.1), (3.12) and (3.13).

It is straightforward to turn all these Lagrangian density terms into their gauge-invariant counterparts, provided a trace over the gauge group action is taken in addition to fermionic integration, as illustrated in Eq. (6.7). This trace automatically projects out gauge-noninvariant choices of the Lagrangian densities. Consider, for example, a collection of \( n \) chiral superfields, \( \Phi_i \), coupled to a type-A gauged \( SU(n) \) symmetry with respect to which the \( \Phi_i \) transform as the fundamental representation. Then, the analytic function \( W(\Phi) \) appearing in the gauge-invariant \( F \)-term

\[
\frac{1}{2} \int d^2 \varsigma \text{Tr} W(\Phi) = \frac{1}{4} \text{Tr} (|\nabla_-, \nabla_+| W(\Phi)) \quad (6.11)
\]

\(^{12}\) Recall that \( \int d\varsigma^- \simeq \nabla_- \), so it has spin +\( \frac{1}{2} \), not -\( \frac{1}{2} \) as the naïve reading of the integration measure in Eq. (6.8) would suggest.
is restricted by the trace operation in front to

$$\Tr W(\Phi) = \sum_{k=0}^{\infty} w_k (\Phi^{[n]})^k, \quad \Phi^{[n]} \overset{\text{def}}{=} \Tr (\epsilon_{i_1 \ldots i_n} \Phi^{i_1} \ldots \Phi^{i_n}). \quad (6.12)$$

This follows on realizing that powers of $\Phi^{[n]}$ are the only holomorphic (chiral) $SU(n)$-invariants one can make from $\Phi^i$. Were the gauge group chosen to be the $SO(n) \subset SU(n)$ subgroup, $W(\Phi)$ would become

$$\Tr W(\Phi) = \sum_{k,l=0}^{\infty} w_{k,l} (\Phi^{[n]})^k (\Phi^{(2)})^l, \quad \Phi^{(2)} \overset{\text{def}}{=} \Tr (\Phi^i \delta_{ij} \Phi^j), \quad (6.13)$$

where the Kronecker $\delta_{ij}$ is the $SO(n)$-invariant metric. Note that in both cases the initial (constant) term in the series is irrelevant, as it is annihilated in the Berezin integration $\langle 6.11 \rangle$.

Keeping such gauge group dependent restrictions in mind, we conclude that all the candidate Lagrangian density terms listed in $[1]$ can be used, and that all couplings of the matter (super)fields to the gauge (super)fields stem from the covariantization of the Berezin integration process, i.e., from replacing the superderivatives in the superdifferential equivalents of the various Berezin integrals with the gauge-covariant superderivatives. Of course, one should also add the gauge-kinetic terms $\langle 6.7 \rangle$. Moreover, the gauge superfields may also be included in the construction of all Lagrangian density terms, e.g., the superpotential function in $\langle 6.11 \rangle$, may also be allowed to depend on $\mathbf{B}$, these being chiral. Since the $\Phi$ are $\mathbf{B}$-chargeless, $[\mathbf{B}, \Phi]=0$, and the superpotential factorizes into a product of the (type-A gauge invariant) $W(\Phi)$ and a (type-B gauge invariant) function of $\mathbf{B}$.

With that in mind, we now turn to a sample calculation.

The general gauge-invariant D-term

Consider a model built from the following collection of gauge-covariantly constrained superfields:

\begin{align}
\Phi^\mu, \quad &\mu = 1, \ldots, N_c, \quad \text{satisfying Eqs. (3.1a)}; \quad (6.14a) \\
\Xi^\alpha, \quad &\alpha = 1, \ldots, N_t, \quad \text{satisfying Eqs. (3.1c)}; \quad (6.14b) \\
\Lambda^a, \quad &a = 1, \ldots, N_L, \quad \text{satisfying Eqs. (3.1e)}; \quad (6.14c) \\
\Upsilon^i, \quad &i = 1, \ldots, N_R, \quad \text{satisfying Eqs. (3.1f)}. \quad (6.14d)
\end{align}

This information suffices to work out the projections by all of the superderivative operators given in Fig. 1 in $\S \ref{sec:superderivatives}$, which in turn suffices to determine the component field Lagrangian densities in all cases. The Reader may wish to use the intermediate results collected in appendix C.

As an example, we present here the general D-term, for which we use the gauge-covariant form of $\langle 1.0 \rangle$:

$$\frac{1}{4} \int d^4 \varsigma \ K = \frac{1}{32} \Tr \left[ \{\nabla_-, \nabla_-\}, \{\nabla_+ , \nabla_+\} \right] K \right]. \quad (6.15)$$
The so defined Berezin integration is Hermitian; if $K = K^\dagger$ is a Hermitian function of its arguments, so is the resulting term in the Lagrangian density. Furthermore, this Berezin $d^4\zeta$-integral is even under parity, and odd under the ‘mirror map’, $C_+$ of Ref. [1]—as it should [12]. This latter property we find the crucial reason for adopting (6.15) over the choice of Ref. [1], which transcribed into our notation becomes

$$
\int d^4\zeta \rightarrow \frac{1}{32} \text{Tr} \left[ \{[\nabla_-, \nabla_+], [\nabla_+, \nabla_-]\} K \right].
$$

(6.16)

Under $C_+$, this D-term transforms into $\frac{1}{32} \text{Tr} \left[ \{[\nabla_-, \nabla_+], [\nabla_+, \nabla_-]\} K \right]$, which would have had to have been subtracted for the required antisymmetry with respect to mirror symmetry.

Terms for the Lagrangian density such as (6.15) (and all the many other found in Ref. [1]) may be expanded ‘for the general case’ as follows. Let $\Omega$ denote a string of all the superfields on which $K$ depends; we assume nothing about (anti)commutivity of $\Omega$. For example, we may set $\Omega = (\Phi^\mu, \Phi^{\bar{\mu}}, \Xi_a, \Xi^{\bar{a}}, \Lambda^a, \Lambda^{\bar{a}}, \Upsilon_i, \Upsilon^{\bar{i}}, \Upsilon^{a}, \Upsilon^{\bar{a}}, \Upsilon_i, \Upsilon^{\bar{i}}, \ldots)$, including all the ‘species’ of superfields on which $K$ may depend in addition to what was specified above. Then, expressions like $(O_1\Omega^1)(O_2\Omega^2)K_{12}$ simply abbreviate the sum over all ‘species’

$$
(O_1\Omega^1)(O_2\Omega^2)K_{12} \overset{\text{def}}{=} (O_1\Phi^\mu)(O_2\Phi^{\nu})K_{\mu\nu} + \cdots + \left(\begin{array}{c}
O_1\Phi^\mu \\
O_2\Phi^{\nu}
\end{array}\right)K_{\mu\nu} + \cdots
$$

(6.17)

The numerical index on $\Omega$ is a multi-index, taking values in the array if indices, $\mu_1, \nu_1, \alpha_1, \beta_1, \ldots$, each of which in turn takes on its appropriate range of values, counting the corresponding superfields. The indices on the (multiple) superderivative operators, $O_1, O_2$, simply distinguish one from another. The extreme compactness of this notation ought to be obvious.

Without any assumption regarding (anti)commutivity of $\Omega$, we expand the integrand of (6.15) as follows:

$$
\left(\{[\nabla_-, \nabla_-], [\nabla_+, \nabla_+]\} K \right)
= \left(\{[\nabla_-, \nabla_-], [\nabla_+, \nabla_+]\} \Omega^1\right) K_1
+ \left([([\nabla_+, \nabla_+] \Omega^1)([\nabla_-, \nabla_-] \Omega^2) + ([\nabla_-, \nabla_-] \Omega^1)([\nabla_+, \nabla_+] \Omega^2]\right) K_{12}
- 2 \left([\nabla_+ \nabla_- \Omega^1)(\nabla_+ \nabla_- \Omega^2) + (\nabla_- \nabla_+ \Omega^1)(\nabla_- \nabla_+ \Omega^2)\right)
+ \left([\nabla_+ \nabla_- \Omega^1)(\nabla_+ \nabla_- \Omega^2) + (\nabla_- \nabla_+ \Omega^1)(\nabla_- \nabla_+ \Omega^2)\right] K_{12}
+ 2 \left([\nabla_+ \nabla_- \Omega^1)(\nabla_+ \nabla_- \Omega^2) + (\nabla_- \nabla_+ \Omega^1)(\nabla_- \nabla_+ \Omega^2)\right)
$$

(6.18a)

(6.18b)

(6.18c)
\[\begin{align*}
+ (\nabla_+ \nabla_- \Omega^1)(\nabla_+ \nabla_- \Omega^2) + (\nabla_- \nabla_+ \Omega^1)(\nabla_- \nabla_+ \Omega^2) & \quad K_{12} \\
+ (-)^{\pi_1} [2(\nabla_+ [\nabla_-, \nabla_-] \Omega^1)(\nabla_+ \Omega^2) + 2(\nabla_- [\nabla_+, \nabla_+] \Omega^1)(\nabla_- \Omega^2)] & \quad K_{12} \\
+ (\nabla_- - \nabla_+) \nabla_- \Omega^1) \nabla_+ \Omega^2) + ([\nabla_+, \nabla_+] \nabla_- \Omega^1)(\nabla_- \Omega^2) & \quad K_{12} \\
- (\nabla_+ \Omega^1) (\nabla_- - \nabla_-) \nabla_+ \Omega^2) - (\nabla_- \Omega^1) (\nabla_+ - \nabla_-) \nabla_- \Omega^2) & \quad K_{12} \\
- (-)^{\pi_1} [2(\nabla_+ + \nabla_- \Omega^1)(\nabla_+ \Omega^2) + 2(\nabla_- + \nabla_+ \Omega^1)(\nabla_- \Omega^2)] & \quad K_{12} \\
+ (\nabla_- - \nabla_+) \nabla_- \Omega^1) \nabla_+ \Omega^2) + ([\nabla_+, \nabla_+] \nabla_- \Omega^1)(\nabla_- \Omega^2) & \quad K_{12} \\
- (\nabla_+ \Omega^1) (\nabla_- - \nabla_-) \nabla_+ \Omega^2) - (\nabla_- \Omega^1) (\nabla_+ - \nabla_-) \nabla_- \Omega^2) & \quad K_{12}
\end{align*}\]

\[\begin{align*}
+ (-)^{\pi_2} [((\nabla_+ \Omega^1)(\nabla_+ \Omega^2) + (-)^{\pi_1} (\nabla_+ \Omega^1)(\nabla_+ \Omega^2)] & \quad K_{123} \\
+ [\nabla_+ \nabla_- \Omega^1)(\nabla_- \Omega^2) + (-)^{\pi_1} (\nabla_+ \Omega^1)(\nabla_+ \Omega^2)] & \quad K_{123} \\
+ [\nabla_+ \nabla_- \Omega^1)(\nabla_- \Omega^2) + (-)^{\pi_1} (\nabla_+ \Omega^1)(\nabla_+ \Omega^2)] & \quad K_{123} \\
- (-)^{\pi_2} [((\nabla_- \Omega^1)(\nabla_+ \Omega^2) + (-)^{\pi_1} (\nabla_- \Omega^1)(\nabla_- \Omega^2)] & \quad K_{123} \\
+ [\nabla_- \nabla_+ \Omega^1)(\nabla_- \Omega^2) + (-)^{\pi_1} (\nabla_- \Omega^1)(\nabla_- \Omega^2)] & \quad K_{123} \\
+ [\nabla_- \nabla_+ \Omega^1)(\nabla_- \Omega^2) + (-)^{\pi_1} (\nabla_- \Omega^1)(\nabla_- \Omega^2)] & \quad K_{123} \\
+ (-)^{\pi_2 \cdot \pi_3} [((\nabla_- \Omega^1)(\nabla_- \Omega^2) - (\nabla_- \Omega^1)(\nabla_- \Omega^2)] & \quad K_{1234} \\
- (-)^{\pi_1 \cdot \pi_3} [((\nabla_- \Omega^1)(\nabla_- \Omega^2) - (\nabla_- \Omega^1)(\nabla_- \Omega^2)] & \quad K_{1234}
\end{align*}\]

where \((-)^{\pi_1} \equiv (-)^{\pi(\Omega^1)} = \pm 1\) when \(\Omega_1\) is (anti)commuting; see appendix A.

Most frequently, the \(\Omega\)'s simply stand for a string of bosonic (commuting) superfields, \((-)^{\pi_1} = \cdots = (-1)^{\pi_4} = +1\), and the multiple derivatives of \(K(\Omega)\) satisfy the usual symmetrization: \(K_{12} = K_{21}, K_{123} = K_{(123)}\) and \(K_{1234} = K_{(1234)}\), and the (anti)commutivity of (multiple) derivatives \((\mathcal{O}_1 \Omega)\) stems solely from the (anti)commutivity of the (multiple) derivative \(\mathcal{O}\). This permits the combination of many of the terms in (6.18), and results in:

\[\begin{align*}
\left\{[\nabla_-, \nabla_-], [\nabla_+, \nabla_+]\right\} K \\
= \left\{[\nabla_-, \nabla_-], [\nabla_+, \nabla_+]\right\} \Omega^1 K_1
\end{align*}\]

(6.19a)
\[ + 2[(\nabla_+, \nabla_+)^1(\nabla_-, \nabla_-)^2]K_{12} \]
\[ - 4[(\nabla_+, \nabla_-)^1(\nabla_+, \nabla_-^2) + (\nabla_-, \nabla_+^1)(\nabla_-, \nabla_+^2)]K_{12} \]
\[ + 4[(\nabla_+, \nabla_-^1)(\nabla_+, \nabla_-^2) + (\nabla_-, \nabla_+^2)(\nabla_-, \nabla_+^2)]K_{12} \]
\[ + 2[((\nabla_+, \nabla_-^1)(\nabla_+^1) + (\nabla_-, [\nabla_+, \nabla_-^1])(\nabla_-^2)]K_{12} \]
\[ - 2[((\nabla_+, \nabla_-^1)(\nabla_+^1) + (\nabla_-, [\nabla_+, \nabla_-^1])(\nabla_-^2)]K_{12} \]
\[ + 4[[\nabla_-^1)(\nabla_-^2)((\nabla_+, \nabla_-)^2) + ([\nabla_-, \nabla_-^1)(\nabla_+^2)]K_{123} \]
\[ + 4[[\nabla_-^1)(\nabla_-^2)((\nabla_+, \nabla_-)^2) + ([\nabla_-, \nabla_-^1)(\nabla_+^2)]K_{123} \]
\[ - 4[[\nabla_-^1)(\nabla_-^2)((\nabla_+, \nabla_-)^2) + ([\nabla_-, \nabla_-^1)(\nabla_+^2)]K_{123} \]
\[ + 8(\nabla_-^1)(\nabla_-^2)(\nabla_+^3)(\nabla_+^4)K_{1234} \]

All Lagrangian densities obtained in this way involve a final \( \varsigma, \zeta = 0 \) projection, which can be performed on the expansions (6.18) and (6.19) term by term and factor by factor, using the projections collected in appendix C. The final insertion of these projections and the collection of (very many) terms for the general D-term (6.15), and any other of the many possible terms \( \Pi \), we leave to the diligent Reader.

7. Summary, Outlook and Conclusions

The above analysis shows how to couple matter fields, represented by constrained superfields\(^{13}\) (3.1) and (3.13), to ‘minimally coupled’ gauge (super)fields.

Considering the most general gauge-covariant extension of the supersymmetry algebra (2.1) and the consistency requirements (appendix B), we find that 1+1-dimensional theories admit four distinct types of symmetry gauging: §4.1–4.4. Furthermore, allowing for a duplication among gauge superfield components, there also exist additional, ‘mixed’ types of symmetry gauging; see §4.5.

Given the wealth of candidate Lagrangian density terms listed in Ref. \( \Pi \), this uncovers a vast arena in which to generalize the results of Refs. \( \Pi, \Pi, \Pi, \Pi \). The methods of Refs. \( \Pi, \Pi \) then are well suited to explore the geometry of the target spaces as related to the quantum dynamics in these models.

The ‘projection method’ described above makes it possible to straightforwardly gauge-covariantize all of the candidate Lagrangian density terms listed in Ref. \( \Pi \). The process reduces to the following five steps:

1. Select a candidate Lagrangian density term from Ref. \( \Pi \). Above, we chose \( \int \mathrm{d}^4 \varsigma K \).
2. Gauge-covariantize the chosen term by substituting gauge-covariant superderivatives and applying the overall trace operation. Above, this produces the D-term (6.15).

\(^{13}\) Recall that the quartoid superfields (3.12) cannot couple to any of the gauge fields by means of ‘minimal coupling’.
3. Expand the (multiple) superderivatives; to this end, the identities in appendix A are useful. Above, this resulted in the expansions (6.18) and (6.19).

4. Perform the $\zeta, \bar{\zeta} = 0$ projections; to this end, the intermediate results in appendix C are useful. This final (and voluminous) collection of the results for the general D-term (6.15), and any other of the many possible terms [1], we leave to the interested Reader.

5. The previous two steps in the calculation are more easily performed using the gauge group index notation (2.6)–(2.8). The results can then be recast into the perhaps neater ‘matrix’ notation, tidying the results further by using the cyclicity of the overall trace operation.

For each of these candidate Lagrangian density terms, supersymmetry is most easily proven upon the non-unitary decovariantization described in § 4.7. On the other hand, the gauge-covariant formalism used throughout makes gauge invariance manifest.

The combination of the above results and techniques, and the large number of candidate Lagrangian density terms [1] guarantees an unsuspected wealth of various (2,2)-supersymmetric gauged $\sigma$-models in 1+1-dimensional spacetime. This provides a vast number of generalizations to the models of Refs. [2,3,4,5], and a stable framework for exploring these generalizations.
Appendix A. Absolutely Basic Conventions (ABC)

We follow Ref. [1]’s adaptation of Wess and Bagger’s definitions [7]. The $-$, $+$ (spinorial) indices actually indicate spin: $\psi^- = \psi_+$ has spin $-\frac{1}{2}$, whereas $\psi^+ = -\psi_-$ has spin $+\frac{1}{2}$. Since $\psi \chi \overset{\text{def}}{=} \psi^\alpha \chi_\alpha$, but $\bar{\psi} \bar{\chi} \overset{\text{def}}{=} \bar{\psi}_\dot{\alpha} \bar{\chi}^\dot{\alpha}$, it follows that

$$\psi^2 = [\psi^+, \psi^-] = 2\psi^+ \psi^-,$$

but

$$\bar{\psi}^2 = [\bar{\psi}^-, \bar{\psi}^+] = 2\bar{\psi}^- \bar{\psi}^+,$$

and so on. Similarly, $\delta^-(\varsigma^1) \equiv \varsigma^+$. Multiple delta-functions are of course products of simple ones, and we merely have to choose the order and the sign. Fixing $\delta^4(\varsigma) \overset{\text{def}}{=} \varsigma^- \varsigma^+ \varsigma^+ \varsigma^-$ so that $\int d^4 \varsigma \delta^4(\varsigma) = 1$, we set

$$\delta^2(\varsigma) = \varsigma^+ \varsigma^-, \quad \delta^2(\varsigma) = \varsigma^+ \varsigma^-,$$

so that Eqs. (1.7) follow from (A.6).

Switching to gauge-covariant (super)derivatives, the following operational identity may be of help. Let $\nabla_i$ range over the gauge-covariant superderivatives $\bar{2}\bar{1}$. Then:

$$\nabla_1 \nabla_2 = \frac{1}{2} \{\nabla_1, \nabla_2\} + \frac{1}{2} \{\nabla_1, \nabla_2\},$$

so that

$$[\nabla_1, \nabla_2] = -2 \nabla_2 \nabla_1.$$

Then, for example

$$[\nabla_-, \nabla_-] \equiv 2(\nabla_- \nabla_- - i \nabla_-) \equiv -2(\nabla_- \nabla_- - i \nabla_-),$$

and so on. Similarly,

$$[\nabla_-, \nabla_-] \equiv 2i \nabla_+ \nabla_- - 2\nabla_- \bar{\nabla}_+, \quad (A.8a)$$

$$[\nabla_-, \nabla_-] \equiv 2\nabla_- \bar{\nabla}_+ - 2i \nabla_+ \nabla_- \quad (A.8b)$$

$$[\nabla_+, \nabla_+] \equiv 2i \nabla_+ \nabla_- - 2\nabla_+ \bar{\nabla}_+, \quad (A.8c)$$

$$[\nabla_+, \nabla_+] \equiv 2\nabla_+ \bar{\nabla}_+ - 2i \nabla_+ \nabla_+, \quad (A.8d)$$
When using gauge-covariant derivatives to calculate gauge-covariant Berezin integrals—or, indeed, simply apply to superfields, the (anti)commutator notation is more precise, albeit also more cumbersome. Let

\[ \pi(X) \overset{\text{def}}{=} \begin{cases} 0 & \text{if } X \text{ is commutative,} \\ 1 & \text{anticommutative,} \end{cases} \]  

so that

\[ \{X, Y\} \overset{\text{def}}{=} XY - (-)^{\pi(X)\pi(Y)}YX \]  

is the graded commutator: a commutator unless both arguments are anticommutative in which case it is an anticommutator.

Then,

\[ \int d\varsigma X \overset{\text{def}}{=} D_1X \to [\nabla_1, X] , \]  

\[ \int d\varsigma d\varsigma^2 X \overset{\text{def}}{=} \frac{1}{2}[D_1, D_2]X \to \frac{1}{2}[[\nabla_1, [\nabla_2, X]]] , \]  

and so on, iterating these two expressions. Note that the index on the \(\nabla\)’s is meant to encode also the conjugation bar. Before switching to the nested (anti)commutator notation, the expression (1.6) merely needed covariantization:

\[ \int d^4\varsigma (\ldots) \overset{\text{def}}{=} \frac{1}{8}\{[\nabla_-, \nabla_-], [\nabla_+, \nabla_+]\}(\ldots) \overset{\text{def}}{=} (\nabla^4 \ldots) \]  

Finally, to obtain the gauge-covariant D-term projector corresponding to the Berezin \(d^4\varsigma\)-integration, the substitution (A.11) is iterated twice.

The full \(d^4\varsigma\) integration measure actually corresponds to the full, totally antisymmetrized product of the four superderivatives. However, it is not necessary to calculate with all the 4! permutations of \(\nabla_-\nabla_-\nabla_+\nabla_+\), as many are equivalent up to total derivatives. This leads to various choices of calculationally convenient subsets of the 4! quartic superderivatives for the assignment \(\int d^4\varsigma[\ldots] \to (\nabla^4 \ldots)\). Several such choices are readily found in the literature, typically obtained through dimensional reduction from 3+1-dimensional \(N=1\) supersymmetry. However, our discussion of 1+1-dimensional \((2, 2)\)-supersymmetric gauge theories follows the intrinsic approach of Ref. [1], disregarding as much as possible dimensional reduction. We then require the assignment (6.15) to satisfy the usual requirement of Hermiticity, and evenness under parity. In addition, we require that \(\nabla^4\) should be odd with respect to the ‘mirror map’, \(C_+\) [1]. The latter requirement would necessitate a doubling of terms for the Ref. [1]’s choice, but leaves (6.15) as it is. This choice provides for greater symmetry between type-A and type-B gauged models with chiral and twisted-chiral superfields, although it also causes a few minor differences when comparing with the framework set-up (but not the results presented) in Ref. [4].
The work with nested (anti)commutators is simplified a good deal by using the ‘derivative’ (operatorial) identities:

\[ [XY, Z] \equiv X[Y, Z] + (-)^{\pi(Y) \cdot \pi(Z)} [X, Z] Y , \quad (A.12a) \]
and

\[ [X, YZ] \equiv [X, Y] Z + (-)^{\pi(X) \cdot \pi(Y)} X [Y, Z] . \quad (A.12b) \]

In particular, then

\[ [\nabla, XY] \equiv [\nabla, X] Y + (-)^{\pi(X)} X [\nabla, Y] . \quad (A.13) \]

Iterating this for superderivatives \( \nabla_1, \nabla_2 \), we obtain

\[ \left[ \nabla_1, \left[ \nabla_2, XY \right] \right] \]
\[ \equiv \left[ \nabla_1, \left[ \nabla_2, X \right] \right] Y + (-)^{\pi(X)} \left[ \nabla_1, \left[ \nabla_2, X \right] \right] [\nabla_2, Y] + X \left[ \nabla_1, \left[ \nabla_2, Y \right] \right] . \quad (A.14) \]

where bracketing indices indicates their antisymmetrization, as in (2.16); it stems from the anticommutivity of the \( \nabla \)'s. When \( X \) and \( Y \) are simple (multiplicative) commuting superfields and \( \nabla_1, \nabla_2 \) superderivatives, this reduces to

\[ (\nabla_1 \nabla_2 XY) = (\nabla_1 \nabla_2 X) Y + (-)^{\pi(X)} 2(\nabla_1 X)(\nabla_2 Y) + X (\nabla_1 \nabla_2 Y) , \quad (A.15) \]
in agreement with Eq. (A.11) of Ref. [1].

The (anti)commutator analogue of the ‘chain rule’ is

\[ [\nabla, f(X)] \equiv [\nabla, X] f'(X) . \quad (A.16) \]

Iterating this, we obtain

\[ \left[ \nabla_1, \left[ \nabla_2, f(X) \right] \right] \equiv \left[ \nabla_1, \left[ \nabla_2, X \right] \right] f'(X) - \left[ \nabla_2, \left[ \nabla_1, X \right] \right] f''(X) , \quad (A.17) \]

which, after antisymmetrizing of the superderivatives, shows how to apply the expressions (A.11) for a two-fold gauge-covariant Berezin integration. When \( X^i \) are simple (multiplicative) commuting superfields and \( \nabla_1, \nabla_2 \) superderivatives, this reduces to

\[ (\nabla_1 \nabla_2 f(X)) \equiv (\nabla_1 \nabla_2 X^i) f_{,i}(X) - (\nabla_2 X^i)(\nabla_1 X^j) f_{,ij}(X) . \quad (A.18) \]

Note that the order of superderivatives becomes reversed in the second term; \((\nabla_1 X^j)\) is not trivial to ‘pass back to the left’, as the (superderivatives of) \( X^i \) may not (anti)commute.

Finally, the Jacobi identities (see appendix B) imply the following two identities:

\[ \{ \nabla_1, [\nabla_2, X] \} = - \{ \nabla_2, [\nabla_1, X] \} + \{ \nabla_1, \nabla_2 \}, \quad (A.19a) \]
\[ = \frac{1}{2} \{ \nabla_1, [\nabla_2, X] \} + \frac{1}{2} \{ \nabla_1, \nabla_2 \}, \quad (A.19b) \]

the latter of which is also seen as an application of the operatorial identity (A.3).

A combined iteration of the above identities typically simplifies most of the calculations significantly. Even so, however, the considerable technical tedium makes one wish for a mechanization of these calculations. Short of this, we present the details of the calculations involving the (graded) Jacobi identities and component field projections in the next two
appendices, and several sample results are presented within the body of the article; we hope this will suffice for the interested Reader to master the technique.

A final remark is in order, relating to Eq. (2.5). The original definitions in § 2.1 imply a ‘matrix’ representation of Lie algebra-valued objects, which is especially important in case of nonabelian symmetries. That is, the gauge-covariant derivatives $\nabla$, and all gauge (super)fields should be regarded as square matrices. On ‘column-vectors’ $X$, these act from the left, but on ‘row-vectors’ $\bar{X}$, they should act from the right. To avoid a meticulous indication of this leftward action, we adopt a ‘mixed’ notation in which various derivatives act always from the left, whereas Lie algebra valued objects act as appropriate. Thus (suppressing spacetime, i.e., spin indices):

$$\nabla X = DX - i\Gamma X, \quad \text{but} \quad \nabla \bar{X} = D\bar{X} - i\bar{X} \Gamma,$$  \hspace{1cm} (A.20)

which also follows from Eq. (2.5). For the record, in the explicit gauge group index notation, this is

$$\nabla^\alpha X^\beta = DX^\alpha - i\Gamma^\alpha X^\beta, \quad \text{but} \quad \nabla^\beta X^\alpha = D\bar{X}^\beta - i\bar{X}^\beta \Gamma^\alpha,$$  \hspace{1cm} (A.21)

where now the ordering of the superfields, both being bosonic, no longer matters.

As Lagrangian density terms must be gauge-invariant, the Fermionic integration is also meant to implicitly include a trace over the gauge group action. But then, owing to the cyclicity of the trace operator and using Eq. (2.5), the product rule yields, e.g.,

$$\text{Tr} [\nabla X \bar{X}] = \text{Tr} [\bar{X} \nabla X] + \text{Tr} [\bar{X} \nabla \bar{X}], \quad \text{Tr} [\bar{X}(\nabla X)] + \text{Tr} [(\nabla^\dagger X^\dagger)^\dagger X],$$  \hspace{1cm} (A.22)

where both terms are separately gauge-invariant, as they transform into

$$\text{Tr} [\bar{X} g^{-1} g \nabla g^{-1} g X] + \text{Tr} [\bar{X} g^{-1} g \nabla g^{-1} g X].$$  \hspace{1cm} (A.23)

Higher order superderivative analogues of this are not difficult to obtain iterating the chain rule (A.22). This proves their gauge-invariance and also allows explicit calculations in terms of the projections listed in appendix C.

The only alternative to this procedure is the use of the (de)covariantizing non-unitary transformation (1.46). This lets the Berezin integration to be calculated as a projection using gauge-noncovariant superderivatives $D_\mp, \bar{D}_\mp$, which act on all superfields standardly, from the left. The Berezin integrand, in turn, becomes covariantized through the explicit insertion of factors such as $\bar{\Gamma} \mathcal{H} = e^{-2V}$ in (1.51), in case of the ‘flat’ kinetic term for a chiral superfield. Such covariantizing factors would have to be determined separately for every other candidate term for the Lagrangian density. To the best of our knowledge, the use of this gauge non-covariant notation and framework is prevalent in the existing literature. In spite of the slight notational awkwardness described above, we choose instead the gauge covariant notation and framework for its universality.
Appendix B. Graded Jacobi Identities

The general, graded Jacobi (cyclic) identity \([X_{<i}, [X_j, X_k>]\} \equiv 0\), is a property of the binary operation \((A.10)\), and does not depend on the (derivative, multiplicative or otherwise) nature of \(X_i; \ldots\) here indicates a summation over cyclic permutations of the enclosed indices. It is therefore used to verify the consistency of the algebra \((2.11)\).

The Jacobi identities involving three spinorial \(\nabla\)'s are:

\[
0 \doteq 3[\nabla_-, \{\nabla_-, \nabla_\} ] = 6[\nabla_-, C_] ;
\]  
\[
0 \doteq [\nabla_-, \{\nabla_-, \nabla_\} ] + [\nabla_-, \{\nabla_-, \nabla_\} ] + [\nabla_-, \{\nabla_-, \nabla_\} ] ,
= 2[\nabla_-, C_+] - 4iW_\equiv ;
\]  
\[
0 \doteq [\nabla_+, \{\nabla_-, \nabla_\} ] + [\nabla_-, \{\nabla_-, \nabla_\} ] + [\nabla_-, \{\nabla_+, \nabla_\} ] ,
= 2[\nabla_+, C_+] + 2[\nabla_-, B] ;
\]  
\[
0 \doteq [\nabla_-, \{\nabla_-, \nabla_\} ] + [\nabla_-, \{\nabla_-, \nabla_\} ] + [\nabla_-, \{\nabla_-, \nabla_\} ] ,
= -4iW_\equiv + 2[\nabla_-, C_+] ;
\]  
\[
0 \doteq [\nabla_+, \{\nabla_-, \nabla_\} ] + [\nabla_-, \{\nabla_-, \nabla_\} ] + [\nabla_-, \{\nabla_+, \nabla_\} ] ,
= -2iW_- + [\nabla_-, A] + [\nabla_-, B] ;
\]  
\[
0 \doteq [\nabla_+, \{\nabla_-, \nabla_\} ] + [\nabla_-, \{\nabla_-, \nabla_\} ] + [\nabla_-, \{\nabla_+, \nabla_\} ] ,
= -2iW_- + [\nabla_-, B] + [\nabla_-, \bar{A}] ;
\]  
\[
0 \doteq [\nabla_+, \{\nabla_-, \nabla_\} ] + [\nabla_-, \{\nabla_-, \nabla_\} ] + [\nabla_+, \{\nabla_+, \nabla_\} ] ,
= 2[\nabla_+, \bar{A}] + 2[\nabla_-, C_+] ;
\]  
\[
0 \doteq 3[\nabla_-, \{\nabla_-, \nabla_\} ] = 6[\nabla_-, C_+] ;
\]  
\[
0 \doteq [\nabla_+, \{\nabla_-, \nabla_\} ] + [\nabla_-, \{\nabla_-, \nabla_\} ] + [\nabla_-, \{\nabla_+, \nabla_\} ] ,
= 2[\nabla_+, C_+] + 2[\nabla_-, B] ;
\]

and the ten identities \((B.1k-t)\), omitted here, but obtained from \((B.1a-j)\) upon a ‘\(\nabla_+\nabla_-\)’ swap in the subscripts (note that this changes the signs of the \(W\)'s, swaps \(A \leftrightarrow \bar{A}\) but leaves \(B, \bar{B}\) intact).

The Jacobi identities involving one vectorial and two spinorial \(\nabla\)'s are:

\[
0 \doteq [\nabla_-, \{\nabla_-, \nabla_\} ] + \{\nabla_-, [\nabla_-, \nabla_\} ] + \{\nabla_-, [\nabla_-, \nabla_\} ] ,
= 2[\nabla_-, C_+] - 2\{\nabla_-, W_\equiv} ,
\]  

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\[
0 \overset{!}{=} [\nabla_-, \{\nabla_-, \nabla_-\}] + \{\nabla_-, [\nabla_-, \nabla_-]\} + \{\nabla_-, [\nabla_-, \nabla_-]\} \\
= -\{\nabla_-, \mathbf{W}_\equiv\} - \{\nabla_-, \mathbf{W}_\equiv\}, \tag{B.2b}
\]
\[
0 \overset{!}{=} [\nabla_-, \{\nabla_-, \nabla_+\}] + \{\nabla_-, [\nabla_-, \nabla_+]\} + \{\nabla_+, [\nabla_-, \nabla_-]\} \\
= [\nabla_-, \mathbf{B}] - \{\nabla_-, \mathbf{W}_-\} - \{\nabla_+, \mathbf{W}_\equiv\}, \tag{B.2c}
\]
\[
0 \overset{!}{=} [\nabla_-, \{\nabla_-, \nabla_+\}] + \{\nabla_-, [\nabla_-, \nabla_+\}] + \{\nabla_+, [\nabla_-, \nabla_-\}] \\
= [\nabla_-, \mathbf{A}] - \{\nabla_-, \mathbf{W}_-\} - \{\nabla_+, \mathbf{W}_\equiv\}, \tag{B.2d}
\]
\[
0 \overset{!}{=} [\nabla_-, \{\nabla_-, \nabla_+\}] + \{\nabla_-, [\nabla_-, \nabla_+\}] + \{\nabla_-, [\nabla_-, \nabla_-\}] \\
= 2[\nabla_-, \mathbf{C}_-] - 2\{\nabla_-, \mathbf{W}_\equiv\}, \tag{B.2e}
\]
\[
0 \overset{!}{=} [\nabla_-, \{\nabla_-, \nabla_+\}] + \{\nabla_-, [\nabla_-, \nabla_+\}] + \{\nabla_+, [\nabla_-, \nabla_-\}] \\
= [\nabla_-, \mathbf{A}] - \{\nabla_-, \mathbf{W}_-\} - \{\nabla_+, \mathbf{W}_\equiv\}, \tag{B.2f}
\]
\[
0 \overset{!}{=} [\nabla_-, \{\nabla_-, \nabla_+\}] + \{\nabla_-, [\nabla_-, \nabla_+\}] + \{\nabla_+, [\nabla_-, \nabla_-\}] \\
= [\nabla_-, \mathbf{B}] - \{\nabla_-, \mathbf{W}_-\} - \{\nabla_+, \mathbf{W}_\equiv\}, \tag{B.2g}
\]
\[
0 \overset{!}{=} [\nabla_-, \{\nabla_+, \nabla_+\}] + \{\nabla_+, [\nabla_+, \nabla_-\}] + \{\nabla_+, [\nabla_+, \nabla_-\}] \\
= 2[\nabla_-, \mathbf{C}_+] - 2\{\nabla_+, \mathbf{W}_-\}, \tag{B.2h}
\]
\[
0 \overset{!}{=} [\nabla_-, \{\nabla_+, \nabla_+\}] + \{\nabla_+, [\nabla_+, \nabla_-\}] + \{\nabla_+, [\nabla_+, \nabla_-\}] \\
= 2\mathbf{F} - \{\nabla_+, \mathbf{W}_-\} - \{\nabla_+, \mathbf{W}_-\}, \tag{B.2i}
\]
\[
0 \overset{!}{=} [\nabla_-, \{\nabla_+, \nabla_+\}] + \{\nabla_+, [\nabla_+, \nabla_-\}] + \{\nabla_+, [\nabla_+, \nabla_-\}] \\
= 2[\nabla_-, \mathbf{C}_+] - 2\{\nabla_+, \mathbf{W}_-\}, \tag{B.2j}
\]

and the ten identities (B.2k–t), omitted here, but easily obtained from (B.2a-j) upon a ‘\(-\leftrightarrow+\)’ swap in the subscripts (recall that this changes the signs of the \(\mathbf{W}\)’s and of \(\mathbf{F}\), swaps \(\mathbf{A} \leftrightarrow \mathbf{\overline{A}}\) but leaves \(\mathbf{B}, \mathbf{\overline{B}}\) intact). Notice that in Eq. (B.2l),

\[
[\nabla_-, \{\nabla_-, \nabla_-\}] = 2i[\nabla_-, \nabla_-] \equiv 0, \tag{B.3}
\]

since the \(\Gamma^j_\equiv\) in \(\nabla_- = \partial_- - i\Gamma^j_\equiv T_j\) are commutative. This sets the ‘usual’ bosonic \(\mathcal{G}_\equiv\)-field strength, \(\mathbf{F}_\equiv\), to zero. Similarly, \(\mathbf{F}_{\iota\iota} \equiv 0\) for \(\mathcal{G}_\iota\).

There are four Jacobi identities involving two bosonic and one fermionic \(\nabla\):

\[
0 \overset{!}{=} [\nabla_-,[\nabla_+,\nabla_-]] + [\nabla_+, [\nabla_-, \nabla_-]] + [\nabla_-, [\nabla_-, \nabla_+]] \\
= -i[\nabla_-, \mathbf{W}_+] - i[\nabla_+, \mathbf{W}_\equiv] + i[\nabla_-, \mathbf{F}], \tag{B.4a}
\]
\[
0 \overset{!}{=} [\nabla_-,[\nabla_+,\nabla_-]] + [\nabla_+, [\nabla_-, \nabla_-]] + [\nabla_-, [\nabla_-, \nabla_+]] \\
= -i[\nabla_-, \mathbf{W}_+] - i[\nabla_+, \mathbf{W}_\equiv] + i[\nabla_-, \mathbf{F}], \tag{B.4b}
\]

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\[ 0 \overset{!}{=} [\nabla_-, [\nabla_+, \nabla_+]] + [\nabla_+, [\nabla_+, \nabla_-]] + [\nabla_+, [\nabla_-, \nabla_+]] \]
\[ = -i[\nabla_-, \bar{\nabla}_+^\perp] - i[\nabla_+, \bar{\nabla}_-^\perp] + i[\nabla_+, F], \]  \quad (B.4c)

\[ 0 \overset{!}{=} [\nabla_-, [\nabla_+, \nabla_+]] + [\nabla_+, [\nabla_+, \nabla_-]] + [\nabla_+, [\nabla_-, \nabla_+]] \]
\[ = -i[\nabla_-, W^{-\perp}] - i[\nabla_+, W_-^\perp] + i[\nabla_+, F]. \]  \quad (B.4d)

All these are identically satisfied upon substituting one of Eqs. (2.16) and some of the Eqs. (2.12). Finally the Jacobi identities involving only bosonic \( \nabla \)'s are trivially satisfied as there is only two bosonic \( \nabla \)'s.

**Appendix C. Component Projections**

The following is a collection of identities and component field projections which the diligent Reader may find useful.

Superderivatives of the fermionic field strength superfields

Below are listed a half of the non-zero superderivatives of the spinorial field strength superfields; the other half is implied by Hermitian conjugation.

Type-A: \( \{ \nabla_-, W_- \} = +[\nabla_-, A], \{ \nabla_+, W_- \} = f + \frac{i}{\bar{2}} [A, A], \)
\[ \{ \nabla_+, W_+ \} = -[\nabla_+, A], \{ \nabla_-, W_+ \} = f + \frac{i}{\bar{2}} [A, A]. \]

Type-B: \( \{ \nabla_-, W_- \} = -[\nabla_-, B], \{ \nabla_+, W_- \} = f + \frac{i}{\bar{2}} [B, B], \)
\[ \{ \nabla_+, W_+ \} = +[\nabla_+, B], \{ \nabla_-, W_+ \} = f - \frac{i}{\bar{2}} [B, B]. \]

Type-C: \( \{ \nabla_-, W^{-\perp} \} = +\frac{i}{\bar{2}} [C_-, C_-], \{ \nabla_+, W^{-\perp} \} = -[\nabla_+, C_-]. \)

Type-C: \( \{ \nabla_+, W_+^\perp \} = -\frac{i}{\bar{2}} [C_+, C_+], \{ \nabla_+, W_+^\perp \} = -[\nabla_+, C_+]. \)

Upon the \( \zeta, \xi = 0 \) projection, this provides (differential) relations among the component fields of the gauge superfields. However, the above (super)differential relations among the gauge superfields also hold without any projection.

Component projections of matter superfields

In sections 3 and 1, we need the non-zero gauge-covariant projections of a gauge-covariantly chiral superfield (3.1a), including for completeness those listed in (3.1a):
\[ \Phi = \phi; \quad \nabla^\pm \Phi = \sqrt{2} \psi_\pm^\dagger; \quad [\nabla_-, \nabla_+] \Phi = 4F, \quad \nabla_- \nabla_+ \Phi = a \phi, \quad \nabla^\pm \nabla^\mp \Phi = \bar{a} \phi, \]
\[ [\nabla_-, \nabla_-] \Phi = -2i(\nabla_- \phi), \quad [\nabla_+, \nabla_+] \Phi = -2i(\nabla_+ \phi); \]
\[ [\nabla_-, \nabla_-] \nabla_+ \Phi = 2\sqrt{2}(a \psi_- + a_- \phi) - 2\sqrt{2}i(\nabla_- \psi_+), \quad \nabla_+ [\nabla_-, \nabla_-] \Phi = -\sqrt{2} \bar{a} \phi, \]
\[ [\nabla_+, \nabla_+] \nabla_+ \Phi = 2\sqrt{2}(\bar{a} \psi_- - \bar{a}_- \phi) - 2\sqrt{2}i(\nabla_+ \psi_-), \quad \nabla_- [\nabla_+, \nabla_+] \Phi = +\sqrt{2}a \phi, \]
\[ \nabla_- [\nabla_-, \nabla_-] \Phi = \sqrt{2} \bar{a} \phi - 2i(\nabla_- \psi_+), \quad \nabla_- [\nabla_+, \nabla_+] \Phi = -\sqrt{2} \bar{a}_- \phi + 2i(\nabla_+ \psi_-); \]
and finally, \( \{[\nabla_-, \nabla_-], [\nabla_+, \nabla_+] \} \Phi = -8(\nabla \phi) + 8D_A \phi + 4(\bar{a}_- \psi_+ - a_+ \psi_-). \)

Note: since \( \{ \nabla_-, \nabla_+ \} \Phi = B \Phi = 0 \), then also \( \nabla_- \nabla_+ \Phi = 2F \) and \( \nabla_+ \nabla_- \Phi = -2F \).

The corresponding projections of a gauge-covariantly antichiral superfield are obtained by Hermitian conjugation, using Eq. (2.4):
The needed projections of a gauge-covariantly twisted-antichiral superfield are:

\[ \Xi = x; \quad \nabla_- \Xi = \sqrt{2} \xi^+, \quad \nabla_+ \Xi = \sqrt{2} \chi^+; \quad \nabla_- \Xi = b \tilde{x}, \quad \nabla_+ \Xi = b \tilde{x}, \]
\[ ([\nabla_-, \nabla_-] \Xi)^t = 4X, \quad ([\nabla_-, \nabla_-] \Xi)^t = -2i(\nabla = x), \quad ([\nabla_+, \nabla_+] \Xi)^t = 2i(\nabla = x); \]
\[ ([\nabla_-, \nabla_-] \Xi)^t = -2\sqrt{2}i(\nabla = x^+) + 2\sqrt{2}(b \xi_- + \beta_- x), \quad \nabla_+ \nabla_- \Xi = -\sqrt{2} \beta_- x, \]
\[ \nabla_+ \nabla_- \Xi = 2\sqrt{2}(\nabla = x^+) - 2\sqrt{2}(b \chi_- + \beta_- x), \quad \nabla_+ \nabla_+ \Xi = -\sqrt{2} \beta_- x, \]
\[ \nabla_+ \nabla_- \Xi = \sqrt{2}(\nabla = x^+) - 2i(\nabla = x^+ x^+ x^+), \quad \nabla_- \nabla_+ \Xi = \sqrt{2}(\nabla = x^+) + 2i(\nabla = x^+ x^+ x^+); \]
\[ \nabla_- \nabla_- \Xi = 4(\nabla = x^+) - 8D_B x + 4(\beta_- \xi^+ - \beta_- \chi^+); \]
\[ \nabla_- \nabla_- \Xi = 2X \quad \text{and} \quad \nabla_- \nabla_- \Xi = -2X. \]

Again, we also have \( \nabla_+ \nabla_- \Xi = 2X \quad \text{and} \quad \nabla_- \nabla_- \Xi = -2X. \)

The needed projections of a gauge-covariant leftton are:

\[ \Lambda = \delta; \quad \nabla_+ \Lambda = \sqrt{2} \lambda^+, \quad \nabla_- \Lambda = \sqrt{2} \lambda^-; \quad [\nabla_+, \nabla_+] \Lambda = 4L_\Lambda, \quad \nabla_- \nabla_+ \Lambda = \hat{a} \ell, \quad \nabla_- \nabla_+ \Lambda = \hat{a} \ell, \quad \nabla_- \nabla_- \Lambda = b \ell; \]
\[ \nabla_- \nabla_- \Lambda = 2(\alpha_- + \beta_-) \ell, \quad \nabla_- \nabla_+ \Lambda = 2 \sqrt{2}(b \lambda_+ - a \lambda^-) + 2 \sqrt{2}(b \lambda_- + a \lambda^+) \ell, \]
\[ \nabla_+ \nabla_- \Lambda = \sqrt{2}(\alpha_- + \beta_-) \ell, \quad \nabla_+ \nabla_+ \Lambda = 2 \sqrt{2}(a \lambda_+ - b \lambda^-) + 2 \sqrt{2}(a \lambda_- + b \lambda^+) \ell, \]
\[ \{\nabla_- \nabla_-, \nabla_+ \nabla_+ \} \Lambda = \{(a, a) - (b, b)\} \ell + 2[(\alpha_- + \beta_-) \lambda^+ - (\alpha_- + \beta_-) \lambda^-]. \]

The needed projections of a gauge-covariant rightton are:

\[ \Upsilon = r; \quad \nabla_- \Upsilon = \sqrt{2} \rho_-, \quad \nabla_- \Upsilon = \sqrt{2} \rho_-; \quad [\nabla_+, \nabla_-] \Upsilon = 4R_\Upsilon, \quad \nabla_- \nabla_- \Upsilon = \hat{a} \rho, \quad \nabla_- \nabla_- \Upsilon = \hat{a} \rho, \quad \nabla_- \nabla_- \Upsilon = b \rho; \]
\[ \nabla_- \nabla_- \Upsilon = 2(\beta_- + \tilde{a}^-) r, \quad \nabla_- \nabla_+ \nabla_- \Upsilon = 2 \sqrt{2}(b \rho_- - a \rho_-) - 2 \sqrt{2}(\alpha_- + \beta_-) r, \]
\[ \nabla_+ \nabla_- \nabla_- \Upsilon = 2(\beta_- + \alpha^-) r, \quad \nabla_+ \nabla_- \nabla_- \Upsilon = 2 \sqrt{2}(a \rho_- - b \rho_-) - 2 \sqrt{2}(\alpha_- + \beta_-) r, \]
\[ \{\nabla_- \nabla_- \nabla_- \nabla_+ \} \Upsilon = \{(a, \tilde{a}^-) - (b, b)\} r + 2[(\alpha_- + \beta_-) \rho^+ - (\alpha_- + \beta_-) \rho^-]. \]

When evaluating candidate Lagrangian density terms other than \((6.13)\), additional
projections may be necessary. Given the above list, the derivation of such additional
projections should pose no significant problem for the interested Reader.

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