Row-column factorial designs with strength at least 2

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Abstract

The $q^k$ (full) factorial design with replication $\lambda$ is the multi-set consisting of $\lambda$ occurrences of each element of each $q$-ary vector of length $k$; we denote this by $\lambda \times [q]^k$. An $m \times n$ row-column factorial design $q^k$ of strength $t$ is an arrangement of the elements of $\lambda \times [q]^k$ into an $m \times n$ array (which we say is of type $I_k(m,n,q,t)$) such that for each row (column), the set of vectors therein are the rows of an orthogonal array of degree $k$, size $n$ (respectively, $m$), $q$ levels and strength $t$. Such arrays are used in experimental design. In this context, for a row-column factorial design of strength $t$, all subsets of interactions of size at most $t$ can be estimated without confounding by the row and column blocking factors.

In this manuscript we study row-column factorial designs with strength $t \geq 2$. Our results for strength $t = 2$ are as follows. For any prime power $q$ and assuming $2 \leq M \leq N$, we show that there exists an array of type $I_k(q^M,q^N,q,2)$ if and only if $k \leq M + N$, $k \leq (q^M - 1)/(q - 1)$ and $(k,M,q) \neq (3,2,2)$. We find necessary and sufficient conditions for the existence of $I_k(4m,n,2,2)$ for small parameters. We also show that $I_{k+\alpha}(2^\alpha b,2^k,2,2)$ exists whenever $\alpha \geq 2$ and $2^\alpha + \alpha + 1 \leq k < 2^\alpha b - \alpha$, assuming there exists a Hadamard matrix of order $4b$.

For $t = 3$ we focus on the binary case. Assuming $M \leq N$, there exists an array of type $I_k(2^M,2^N,2,3)$ if and only if $M \geq 5$, $k \leq M + N$ and $k \leq 2^{M-1}$. Most of our constructions use linear algebra, often in application to existing orthogonal arrays and Hadamard matrices.

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1 Introduction

For any integer \( s \), let \([s] = \{0, 1, \ldots, s-1\}\). An orthogonal array of size \( N \), degree \( k \), \( q \) levels and strength \( t \), denoted \( \text{OA}(N, k, q, t) \) is an \( N \times k \) array with entries from \([q] \) such that in every \( N \times t \) submatrix, every \( 1 \times t \) row vector appears \( N/q^t \) times. The \( q^k \) (full) factorial design with replication \( \lambda \) is the multi-set consisting of \( \lambda \) occurrences of each element of \([q]^k \); we denote this by \( \lambda \times [q]^k \).

An \( m \times n \) row-column factorial design \( q^k \) is any arrangement of the elements of \( \lambda \times [q]^k \) into an \( m \times n \) array. We say that such an array has strength \( t \) if for each row (column), the set of vectors therein are the rows of an orthogonal array of size \( k \), degree \( n \) (respectively, \( m \)), \( q \) levels and strength \( t \). That is, if we consider any subset of \( t \) positions within the vectors in a fixed row (or column), we obtain a \([q]^t \) full-factorial design with replication \( n/q^t \) (respectively, \( m/q^t \)). We denote such a row-column factorial design by \( I_k(m, n, q, t) \), where the replication number or index \( \lambda \) of the design is given by \( \lambda = mn/q^k \).

For example, in Table 1 the elements of \([3]^4 \) are arranged into a \( 9 \times 9 \) array such that the vectors in each row and column are the rows of an \( \text{OA}(9, 4, 3, 2) \). Thus this is an array of type \( I_4(9, 9, 3, 2) \).

```
  0000 1011 2022 0112 1120 2101 0221 1202 2210
  0111 1122 2100 0220 1201 2212 0002 1010 2021
  0222 1200 2211 0001 1012 2020 0110 1121 2102
  1021 2002 0010 1100 2111 0122 1212 2220 0201
  1102 2110 0121 1211 2222 0200 1020 2001 0012
  1210 2221 0202 1022 2000 0011 1101 2112 0120
  2012 0020 1001 2121 0102 1110 2200 0211 1222
  2120 0101 1112 2202 0210 1221 2011 0022 1000
  2201 0212 1220 2010 0021 1002 2122 0100 1111
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Table 1: A row-column factorial design \( I_4(9, 9, 3, 2) \).

Within experimental design, a row-column design can refer to a variety of combinatorial designs, all with the property of being arranged in a rectangular array, where regularity conditions may be imposed in order to estimate effects without confounding. Table 1 for example, could be used to study the effects of 4 drugs on cows, each at 3 dosage levels while controlling for the effects of 9 breeds (the rows) and 9 age groups (the columns). Here the vector \((2, 0, 2, 2)\) in the first row and third column indicates that the first breed and third age group are given the highest dosage of the first, third and fourth drug and the lowest dosage of the second drug. The property of being an array of type \( I_4(9, 9, 3, 2) \) eliminates, for example, confounding between breed or age group and the interaction between any pair of drugs. We refer the reader to [7, 13] for a literature review on the application of row-column factorial designs to statistical experimental design.

In this paper two arrays are equivalent under any: (a) reordering of rows; (b) reordering of columns; (c) reordering of levels (applied globally); and (d) reordering of the entries in each vector (with the same reordering applied globally). It is also convenient to use the terms “array” and “matrix” interchangeably. When linear algebra is applied we often work over the field of order \( q \), with an understanding that when \( q \) is a prime power, the levels
are relabelled with \([q]\) as a final step.
By definition, if there exists an array of type \(I_k(m, n, q, t)\) then \(q^k|mn\) and there exists an \(OA(m, k, q, t)\) and there exists an \(OA(n, k, q, t)\). If \((k, m, n, q, t)\) is a 5-tuple satisfying these three necessary conditions we say that \((k, m, n, q, t)\) is admissible.

Necessary conditions for the existence of orthogonal arrays of strength \(t\) imply further necessary conditions for the existence of row-column factorial designs of strength \(t\). It is impractical to list all known necessary conditions (in particular as \(t\) grows large); we refer the reader to surveys in III.6 and III.7 of the Handbook of Combinatorial Designs [4]. Elementary conditions imply that \(t \leq k\) and \(q^t|n\). In summary:

**Lemma 1.** If \((k, m, n, q, t)\) is admissible then \(q^t|m\), \(q^t|n\), \(q^k|mn\) and \(t \leq k\).

Necessary and sufficient conditions for a row-column factorial design of strength 1 are given in [13], generalizing [7] and [16].

**Theorem 1.** ([13]) Let \(m \leq n\). There exists \(I_k(m, n, q, 1)\) (that is, an \(m \times n\) row-column factorial design \(q^k\) of strength 1) if and only if:

i. \(q|m\) and \(q|n\);

ii. if \(k = q = m = 2\) then 4 divides \(n\); and

iii. \((k, m, n, q) \neq (2, 6, 6, 6)\).

Note that an array \(I_k(n, n, q, t)\) implies the existence of a set of \(k\) mutually orthogonal frequency squares (MOFS) of size \(n\) based on a set of size \(q\). Thus, the existence of row-column factorial designs also relates to the existence of frequency squares and Latin squares. For example, the exceptions in the previous theorem include pairs of orthogonal Latin squares of orders 2 and 6, which are well-known not to exist. Some results, including a table of lower bounds, related to the existence of MOFS can be found in [2, 10, 11].

In this manuscript we focus on row-column factorial designs of strength 2 and higher. Binary row-column factorial designs of strength 1 which come as close as possible to strength 2 are studied in [7], in the case when the dimensions of the array are powers of 2. The motivation in [7] is to be able to estimate as many two-factor interactions as possible without confounding, given fixed parameters.

In the binary strength 2 case we will frequently make use of Hadamard matrices. A Hadamard matrix \(H(n)\) is a square matrix of order \(n\), having entries from the set \(\{1, -1\}\) such that any two rows are orthogonal, i.e., it satisfies the equation: \(H(n)H(n)^T = nI_n\). If a Hadamard matrix \(H(n)\) exists, then either \(n = 2\) or \(n\) is divisible by 4. However, the converse is an open problem known as the Hadamard conjecture; the smallest value for which it is not known whether a Hadamard matrix exists or not is 668 [6]. The following lemma gives a relationship between binary orthogonal arrays of strength 2 and 3 and the Hadamard matrices of order \(4m\).

**Lemma 2.** [8, p. 148] Let \(m \geq 4\). Orthogonal arrays \(OA(4m, 4m-1, 2, 2)\) and \(OA(8m, 4m, 2, 3)\) exist if and only if there exists a Hadamard matrix order \(4m\).

It is worth mentioning how orthogonal arrays can be constructed from Hadamard matrices as per the previous lemma, as this idea is frequently applied in Sections 5 and 6, where we focus on binary arrays. Let \(H\) be a Hadamard matrix of order \(4m\). Assume that \(H\) is in normalized form; that is, we assume the first row and column of \(H\) only contain the
entry 1. Now delete the first column and replace each \(-1\) with \(0\). The resultant array is an OA\((4m, 4m - 1, 2, 2)\). Next, consider the array \([H - H]^T\) and again replace each \(-1\) with \(0\); the resultant array is an OA\((8m, 4m, 2, 3)\). The rows of such an array are the codewords of a code known as a Hadamard code \[9\].

For a binary vector \(v\), we often say that its weight \(\omega(v)\) is equal to the number of 1’s in \(v\). It is also often convenient to say that two binary vectors \(v\) and \(w\) of length \(4k\) are orthogonal if each has weight \(2k\) and \(v \cdot w = k\). So in a binary orthogonal array of strength 2, each pair of columns is necessarily orthogonal.

The previous lemma, together with the Bose-Bush bound for orthogonal arrays \([4]; [12]\) originally) implies the following necessary conditions for strength 2 row-column factorial designs.

**Lemma 3.** Let \(m \leq n\). If there exists an array of type \(I_k(m, n, q, 2)\), then \(k \leq (m-1)/(q-1)\). If there exists an array of type \(I_{m-1}(m, n, 2, 2)\), then there is a Hadamard matrix of order \(m\).

Since the Hadamard conjecture is a well-studied but unsolved open problem, it is likely that generalizing Theorem \[1\] to the strength 2 case, that is finding necessary and sufficient conditions for the existence of a row-column factorial design of strength 2, is untenable even in the binary case.

The following result on strength 3 binary orthogonal arrays is well-known \([1]; [14]\).

**Lemma 4.** If an OA\((m, n, 2, 3)\) exists, then \(m \leq 2^{n-1}\). Moreover an OA\((2^{n-1}, n, 2, 3)\) exists, the rows of which are all the binary vectors of length \(n\) and odd weight.

Let \(C\) and \(R\) be orthogonal arrays each of degree \(k\) with \(q\) levels with the zero vector in the first row. We define \(C \boxplus R\) to be the array such that row \(i\) and column \(j\) contain the vector sum, calculated in \(\mathbb{F}_q\), of the \(i\)th row of \(C\) and the \(j\)th row of \(R\). In turn, we call an array \(L\) of type \(I_k(m, n, q, t)\) abelian if and only if there exists \(C\) and \(R\) such that \(L = C \boxplus R\), where \(C\) is an OA\((m, k, q, t)\) and \(R\) is an OA\((n, k, q, t)\). (Here we use \(C\) and \(R\) to remind the reader that the first Column and first Row of \(L\) are, respectively, the orthogonal arrays \(C\) and \(R\).) If the replication is 1, then such an array is abelian if and only if it is the subarray of the addition table for \(\mathbb{F}_q^k\). Most constructions in this paper are abelian, however Section 5 contains some non-abelian constructions.

In Section 2, we give some general recursive constructions that apply to all row-column factorial designs. In Section 3 we focus on the abelian case, using linear algebra to show that row-column factorial designs can be constructed from orthogonal arrays and matrices with certain independence properties. These are applied in Section 4 where we consider the strength 2 case with an arbitrary number of levels. We solve this case completely when the number of levels \(q\) is a prime power and the dimensions of the array are each a power of \(q\); see Theorem \[8\]. This generalizes the binary case solved in \[7\].

In Section 5 we find necessary and sufficient conditions for the existence of \(I_k(4m, n, 2, 2)\) whenever \(m \leq 5\); or \(m\) is odd assuming the truth of Conjecture \[4\]. We also show that \(I_{k+\alpha}(2^\alpha b, 2^k, 2, 2)\) exists whenever \(\alpha \geq 2\) and \(2^\alpha + \alpha + 1 \leq k < 2^\alpha b - \alpha\), assuming there exists a Hadamard matrix of order \(4b\) (Theorem \[7\]). Finally in Section 6 we consider the strength 3 binary case, solving this whenever the dimensions are powers of 2 (Theorem \[10\]).
2 General results

In this section we list some general observations and results that can be applied to row-column factorial designs of any strength.

We start with some straightforward lemmas.

Lemma 5. If $D$ is an array of type $I_k(m, n, q, t)$ then:

- $D$ is also an array of type $I_k(m, n, q, t')$ for each $t'$ such that $1 \leq t' \leq t$;
- there exists an array of type $I_k'(m, n, q, t')$ for each $k'$ such that $1 \leq k' \leq k$.

Lemma 6. If there exist arrays of type $I_k(m, n, q, t)$ and $I_k(m', n, q, t)$ there exists an array of type $I_k(m+m', n, q, t)$. If there exist arrays of type $I_k(m, n, q, t)$ and $I_k(m, n', q, t)$ there exists an array of type $I_k(m, n+n', q, t)$.

The proof of the following lemma is a Kronecker product construction based on a similar construction for orthogonal arrays (Theorem III.7.20 from [4], originally [3]).

Lemma 7. If there exist arrays of type $I_k(m, n, q, t)$ and $I_k(m', n', q', t')$ then there exists an array of type $I_k(mm', nn', qq', tt')$.

Proof. Let $D$ and $D'$ be arrays of type $I_k(m, n, q, t)$ and $I_k(m', n', q', t')$, respectively. We construct an $mn' \times mn$ array $D \otimes D'$ as follows. For each $(i, j) \in [mn'] \times [mn']$, write $i = xm' + x'$ and $j = yn' + y'$ where $x \in [m], x' \in [m'], y \in [n], y' \in [n']$, noting that the choices of $x, x', y$ and $y'$ are unique and depend on $i$ and $j$. In cell $(i, j)$ we place the vector $q' D(x, y) + D'(x', y')$, where $D(x, y)$ and $D'(x', y')$ are the vectors in cells $(x, y)$ and $(x', y')$ of $D$ and $D'$, respectively.

We next verify that $D \otimes D'$ is an array of type $I_k(mm', nn', qq', tt')$. Fix a set $T$ of $t$ coordinates in column $j$ of $D \otimes D'$ and let $(v_1, v_2, \ldots, v_t) \in [qq']^t$. As above, write $j = yn' + y'$ for unique $y \in [n], y' \in [n']$. For each $\alpha \in [t]$, let $x_\alpha \in [q]$ and $x'_\alpha \in [q']$ be unique solutions to $v_\alpha = x_\alpha q' + x'_\alpha$.

Since $D$ is of strength $t$, the vector $(x_1, x_2, \ldots, x_t)$ appears $n/q^t$ times in column $y$ of $D$ in the set of positions $T$. Similarly, the vector $(x'_1, x'_2, \ldots, x'_t)$ appears $n'/q'^t$ times in column $y'$ of $D'$ in the same set of positions $T$. Thus $(v_1, v_2, \ldots, v_t)$ appears precisely $mn'/qq'^t$ times in column $j$ of $D \otimes D'$. By the same argument in transpose, each vector in $[qq']^t$ appears $mm'/qq'^t$ in each row of $D \otimes D'$.

It remains to show that each vector in $[qq']^k$ appears the same number of times in the array $D \otimes D'$. The idea is similar to above. Let $(v_1, v_2, \ldots, v_k) \in [qq']^k$. For each $\alpha \in [k]$, let $x_\alpha \in [q]$ and $x'_\alpha \in [q']$ be unique solutions to $v_\alpha = x_\alpha q' + x'_\alpha$. By the parameters of $D$ and $D'$, the vectors $(x_1, x_2, \ldots, x_k)$ and $(x'_1, x'_2, \ldots, x'_k)$ appear $mn/q^k$ and $mn'/q'^k$ times, respectively, in the arrays $D$ and $D'$. Thus $(v_1, v_2, \ldots, v_k)$ appears precisely $mm'nn'/qq'^k$ times in the array $D \otimes D'$. 

The $m \times n$ matrix of 0 vectors of dimension $k$ is trivially an array $I_k(m, n, 1, t)$ for any $1 \leq k \leq t$. The following corollary is then immediate.

Corollary 1. If there exists an array $I_k(m, n, q, t)$, then there exists an array $I_k(mm', nn', q, t)$ for any integers $m', n' \geq 1$. 

5
3 Abelian row-column factorial designs

In this section we find necessary and sufficient conditions on orthogonal arrays $C$ and $R$ such that $C \boxplus R$ is a row-column factorial design of strength $t$, where at least one of $C$ or $R$ is a vector space.

Since every row (column) in $C \boxplus R$ is equivalent to the first row (respectively, column), we have the following observation.

**Lemma 8.** Let $C$ be an OA$(m, k, q, t)$ and let $R$ be an OA$(n, k, q, t)$. Then $L = C \boxplus R$ is an array of type $I_k(m, n, q, t)$ if and only if $L$ is a row-column factorial design, that is, the set of entries of the cells of $L$ is the $\lambda \times [q]^k$ factorial design.

We next consider the extreme case when every row is a factorial design.

**Lemma 9.** If there exists an orthogonal array OA$(m, k, q, t)$ then there exists an array of type $I_k(m, q^k, q, t)$.

**Proof.** Let $R$ be an OA$(q^k, k, q, t)$ where $t \leq k$. That is, the row vectors of $R$ are the factorial design $[q]^k$. Let $C$ be an OA$(m, k, q, t)$. Since the entries in each row of $C \boxplus R$ are trivially $[q]^k$, by the previous lemma $C \boxplus R$ is an array of type $I_k(m, q^k, q, t)$.

In the following, given a matrix $A$ over the field $\mathbb{F}_q$, let $< A >$ be a matrix whose row vectors are the rowspace of $A$; that is the vector space generated by the row vectors of $A$.

**Corollary 2.** Let $K$ be a binary $m \times (n - m)$ matrix. Then $< [I | K] >$ is an OA$(2^m, n, 2, 2)$ if and only if the columns of $K$ are distinct and each column of $K$ contains at least 2 non-zero elements.
Example 1. From the previous corollary, \( < A > \) is an OA\((2^6, 8, 2, 2)\):

\[
A = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\
\end{pmatrix}
\]

Theorem 2. Let \( G \) be an \( m \times k \) matrix and let \( A \) be an \( N \times k \) matrix of full rank, each over the field \( \mathbb{F}_q \) where \( N \leq k \). Let \( A^\perp \) be a \( k \times (k - N) \) matrix the columns of which are a basis for the nullspace of \( A \). Then \( G \boxplus < A > \) is an \( m \times q^N \) row-column factorial design if and only if \( GA^\perp \) is an OA\((m, k - N, q, k - N)\).

Proof. Let \( M = G \boxplus < A > \) and let \( v_i \in \mathbb{F}_q^k \) denote the \( i^{th} \) row vector of \( G \). Then the elements of the \( i^{th} \) row of \( M \) form the coset \( v_i + < A > \) of \( < A > \) in \( \mathbb{F}_q^k \). Note that \( < A > \) has exactly \( q^{k-N} \) distinct cosets in \( \mathbb{F}_q^k \). We show that each of these cosets appears the same number of times as sets of entries in a row of \( M \). To this end, observe that for \( 1 \leq i, j \leq m \):

\[ v_i + < A > = v_j + < A > \iff v_i - v_j \in < A > \iff v_i A^\perp = v_j A^\perp. \]

Thus the set of entries in two rows of \( M \) are identical if and only if the corresponding rows in \( GA^\perp \) are identical. Thus \( M \) is a row-column factorial design (that is, the set of entries of \( M \) form a factorial design) if and only if each element of \( (\mathbb{F}_q^k)^{k-N} \) occurs the same number of times as a row of \( GA^\perp \). In turn, this is true if and only if \( GA^\perp \) is an OA\((m, k - N, q, k - N)\). \( \square \)

Theorem 3. Let \( G \) be an OA\((m, k, q, t)\) and let \( < A > \) be an OA\((q^N, k, q, t)\) where \( A \) is an \( N \times k \) matrix of full rank and \( N \leq k \). Let \( A^\perp \) be a \( k \times (k - N) \) matrix whose columns generate the nullspace of \( A \). Suppose that \( GA^\perp \) is an OA\((m, k - N, q, k - N)\). Then \( G \boxplus < A > \) is an array of type \( I_k(m, q^N, q, t) \).

Proof. Let \( M = G \boxplus < A > \). The result follows from Theorem 2 and Lemma 8. \( \square \)

In the next example (and in Section 5) we make use of the result (well-known to coding theorists) that over any field, the nullspace of the matrix \( \begin{bmatrix} I & K \end{bmatrix} \) is equal to the columnspace of the matrix \( \begin{bmatrix} -K^T & I \end{bmatrix}^T \) (Remark 1.5, [4, p. 677]).

Example 2. We continue with Example 1 to show that \( G \boxplus < A > \) is an \( I_8 \)\((12, 2^6, 2, 2)\), using Theorem 3 the following \( G = \text{OA}(12, 8, 2, 2) \).

\[
A^\perp = \begin{pmatrix}
1 & 0 \\
0 & 1 \\
1 & 1 \\
1 & 0 \\
0 & 1 \\
-1 & 0 \\
0 & 1 \\
\end{pmatrix} \quad G = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 1 & 0 & 1 \\
1 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\
1 & 0 & 1 & 0 & 0 & 0 & 1 & 1 \\
1 & 1 & 0 & 1 & 0 & 0 & 1 & 1 \\
0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\
1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 \\
1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 \\
0 & 1 & 1 & 0 & 1 & 0 & 1 & 1 \\
0 & 0 & 1 & 1 & 1 & 0 & 1 & 1 \\
\end{pmatrix} \quad GA^\perp = \begin{pmatrix}
0 & 0 \\
0 & 1 \\
1 & 0 \\
0 & 1 \\
1 & 0 \\
0 & 0 \\
1 & 1 \\
1 & 1 \\
0 & 0 \\
0 & 0 \\
0 & 1 \\
1 & 1 \\
\end{pmatrix}
\]
The following is a generalization of Theorem 5 in [13].

**Theorem 4.** Let $q$ be a prime power. Let $M, N \geq 1$ and $k \leq M + N$. Suppose there exists a $k \times M$ matrix $A$ and a $k \times N$ matrix $B$ such that (a) the $k \times (M + N)$ matrix $[A|B]$ has rank $k$; (b) the rows of $A$ are a $t$-independent set of vectors; and (c) the rows of $B$ are a $t$-independent set of vectors. Then there exists an abelian array of type $I_k(q^M, q^N, q, t)$.

**Proof.** Assuming the conditions of the theorem, the rows of $[A|B]$ are a linearly independent set of $k$ vectors:

$$
(a_r,0 + a_r,1 + \cdots + a_r,M+N-1); \ r \in [k]
$$

in the vector space $\mathbb{F}_q^{M+N}$ such that:

(i) The set of vectors $\{(a_r,0, \ldots, a_r,M-1) : r \in [k]\}$ is $t$-independent; and

(ii) the set of vectors $\{(a_r,M, \ldots, a_r,M+N-1) : r \in [k]\}$ is $t$-independent.

Corresponding to each vector in $[k]$ we construct a $q^M \times q^N$ array $A_r$ by using a polynomial $f_r$, where

$$
f_r(x_0, \ldots, x_{M+N-1}) = a_{r,0}x_0 + \cdots + a_{r,M+N-1}x_{M+N-1}.
$$

Label the rows and columns of $A_r$ by using the set of all $M$-tuples and $N$-tuples, respectively, over the field $\mathbb{F}_q$. We place the element $f(b_0, \ldots, b_{M-1}, c_0, \ldots, c_{N-1})$ in the intersection of row $(b_0, \ldots, b_{M-1})$ and column $(c_0, \ldots, c_{N-1})$ of the array $A_r$.

We next form an array $D$ by overlapping the arrays $A_r$, $r \in [k]$. That is, cell $(i, j)$ of $D$ contains a vector of dimension $k$ the $r$th coordinate of which is the entry of cell $(i, j)$ of $A_r$. We claim that the array $D$ is an $I_k(q^M, q^N, q, t)$.

We first show that $D$ is an array of strength $t$. Let $T$ be a subset of $[k]$ of size $t$ and consider a sequence $(\alpha_1, \ldots, \alpha_t)$ in $[q]^t$. For a fixed column in $D$, the system of equations

$$
f_r(x_0, \ldots, x_{M+N-1}) = \alpha_r; \quad r \in T
$$

reduces to:

$$
ar_{r,0}x_0 + a_{r,1}x_1 + \cdots + a_{r,M-1}x_{M-1} = \alpha_r + K_r; \quad r \in T
$$

where $K_r$ is a constant in $\mathbb{F}_q$ for each $r \in T$. By condition (i), the above system with $M$ variables has rank $t$. Therefore it has exactly $q^{M-t}$ solutions in $\mathbb{F}_q$. Thus each column of $D$ is an orthogonal array of type OA($q^M, k, q, t$). Similarly, we can show that each row of $D$ also forms an orthogonal array of type OA($q^N, k, q, t$). Hence the strength $t$ conditions is satisfied.

Now to show that $D$ is a $q^k$-full factorial design. Consider a sequence $(\alpha_0, \alpha_1, \ldots, \alpha_{k-1})$ in $\mathbb{F}_q^k$. Since the system of equations:

$$
ar_{r,0}x_0 + a_{r,1}x_1 + \cdots + a_{r,M+N-1}x_{M+N-1} = \alpha_r, \quad r \in [k],
$$

has $M+N$ variables and rank $k$, it has exactly $q^{M+N-k}$ solutions in $\mathbb{F}_q$. Thus each sequence in $\mathbb{F}_q^k$ appears exactly $q^{M+N-k}$ times in $D$. \hfill \Box
4 Strength 2 with arbitrary number of levels

In this section we consider row-column factorial designs of the form \( I_k(q^M, q^N, q, 2) \). The aim of this section is to prove the following theorem.

**Theorem 5.** Let \( 2 \leq M \leq N \), let \( q \) be a prime power and let \( k \geq 2 \). Then there exists an array of type \( I_k(q^M, q^N, q, 2) \) if and only if \( k \leq M + N \), \( k \leq (q^M - 1)/(q - 1) \) and \((k, M, q) \neq (3, 2, 2)\).

Lemma 9 and the previous theorem imply the following corollary.

**Corollary 3.** Let \( q_1 \leq q_2 \leq \ldots \leq q_\alpha \) be powers of distinct primes and \( q_1 \neq 4 \). Let \( q = q_1q_2\ldots q_\alpha \), \( k \leq (q^M - 1)/(q - 1) \), \( 2 \leq M \leq N \) and \( 2 \leq k \leq M + N \). Then there exists an array of type \( I_k(q^M, q^N, q, 2) \).

The elements of \([2]^2\) form the rows of an OA\((4, 2, 2, 2)\). In turn, Lemma 9 implies the existence of \( I_2(4, 4, 2, 2) \). This observation, together with the following two lemmas and Theorem 4, imply the above theorem.

**Lemma 11.** Let \( N \geq M \geq 2 \) be integers and \( q \geq 2 \) be a prime power, with \( M + N \leq (q^M - 1)/(q - 1) \) and \((M, q) \neq (2, 2)\). Then there exists an \((M + N) \times M\) matrix \( A \) and an \((M + N) \times N\) matrix \( B \) such that (a) \([A|B]\) has full rank; (b) no two rows of \( A \) are parallel; and (c) no two rows of \( B \) are parallel.

**Proof.** We split the proof in different cases. In each case we describe a square matrix \( L = [A|B] \) with the required properties.

**Case I:** When \( M = 3 \) and \( q = 2 \).

In this case, \( 3 \leq N \leq 4 \). For \( N = 4 \) we define the matrix \( L \) to be,

\[
L = \begin{pmatrix}
1 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 \\
0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\
1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\
1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\
1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\
1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\
\end{pmatrix}
\]

(2)

The above matrix has full rank over \( \mathbb{F}_2 \) and also satisfies the conditions (b) and (c). Now for the case \( N = 3 \), we can take the \( 6 \times 6 \) sub-matrix (as shown in (2)) of the above matrix obtained by deleting the final row and column. Observe that this matrix also has full rank and satisfies the conditions in the lemma. This completes Case I.

For all other cases, we take \( L \) to be of the form:

\[
L = \begin{pmatrix}
I_M & & & I_M & & & O \\
& C_M - I_M & & & C_M & & & O \\
& & S & & & O & & & I_{N-M} \\
\end{pmatrix}
\]

(3)

Where \( I_M \) is an identity matrix of order \( M \) and \( O \) is a matrix of zeroes of appropriate size. Matrices \( C_M \) and \( S \) are to be defined later.
Label the columns of the matrix $L$ in $[3]$ by $c_i$, $i \in [M + N]$. Note that for any choices of matrices $C_M$ and $S$, the column operations;

$$c_{M+i} - c_i \rightarrow c_{M+i}, \text{ for each } i \in [M],$$

transfers the matrix $L$ into a lower triangular matrix with entry 1 on the main diagonal. Thus condition (a) is satisfied for any choice of $C_m$ and $S$.

**Case II:** When $M \geq 4$.

In this case let $C_M$ be a $M \times M$ matrix with exactly one 0 in each row and column, 1’s on the main diagonal and 1’s in every other cell. Observe that condition (c) is satisfied. Let $D$ be the set of all rows in $I_M$ and $C_M - I_M$. It is easy to see that no two vectors in $D$ are parallel. Let $W$ be the largest set of non-parallel vectors in $\mathbb{F}_q^M$ containing $D$. Then $|W| = (q^M - 1)/(q - 1)$. We define $S$ to be the matrix for which each row is a distinct element in $W \setminus D$. Thus condition (b) is satisfied. This completes Case II.

In the remaining cases the matrix $S$ can be obtained in the similar manner from the corresponding $C_M$, again satisfying condition (b).

**Case III:** When $2 \leq M \leq 3$ and $q$ is odd.

In this case we define the matrices $C_M, M \in \{2, 3\}$ to be:

\[
C_2 = \begin{pmatrix} 2 & 2 \\ 1 & 2 \end{pmatrix} \quad \text{and} \quad C_3 = \begin{pmatrix} 2 & 0 & 1 \\ 1 & 2 & 0 \\ 0 & 1 & 2 \end{pmatrix}.
\]

**Case IV:** When $M = 3$ and $q = 2^l$, $l \geq 2$.

In this case we define $C_3$ as follows.

\[
C_3 = \begin{pmatrix} \alpha + 1 & 1 & 1 \\ 1 & \alpha + 1 & 1 \\ 1 & 1 & \alpha + 1 \end{pmatrix},
\]

where $\alpha$ is a primitive element of the field $\mathbb{F}_q$. \hfill \Box

**Lemma 12.** If $b$ is odd, there does not exist an array of type $I_3(4, 4b, 2, 2)$.

**Proof.** Suppose that an array $D$ exists of type $I_3(4, 4b, 2, 2)$. Then each column of $D$ is an OA($4, 3, 2, 2$). By inspection, the vectors in any column of $D$ are either all the vectors of even weight or all the vectors of odd weight; we refer to these columns as type A or B, respectively.

\[
\begin{array}{c|c}
A & B \\
000 & 001 \\
101 & 100 \\
011 & 010 \\
110 & 111 \\
\end{array}
\]

Since the vectors in $A$ and $B$ form a partition of $\mathbb{F}_2^3$ and the entries of $D$ form a factorial design, $D$ must contain exactly $2b$ columns of each type.

Now consider a row $R$ in $D$; without any loss of generality we may assume that the first two coordinates of $R$ have the following form:
where each $B_i$ has size $b$. Let $x$ be the number of zeros at the third coordinate in $B_1$, then without loss of generality $x \geq (b+1)/2$. By strength two property, the number of zeros in the third coordinate in $B_2$, $B_3$ and $B_4$ is $b-x$, $b-x$ and $x$ respectively. This implies that there are $x$ vectors of type $A$ in each $B_i$. Consequently, $R$ contains $4x \geq 2b+2$ vectors of type $A$. This is a contradiction since $D$ contains exactly $2b$ columns of each type. \hfill $\Box$

### 5 Binary row-column factorial designs of strength 2

In this section we restrict ourselves to the binary case. We exploit the theory developed in Section 3 to give existence results for arrays of the form $I_k(4m,n,2,2)$. We focus on the case where $m$ is odd, however the next theorem is also true when $m$ is even. The main results in this section are given in Theorems 7, 8 and 9.

**Theorem 6.** Let $k \geq 5$. Let $m \geq 3$ be odd and suppose there exists an OA$(4m,k,2,2)$ with two subsets of column vectors $V$ and $W$ such that:

- $|V|,|W| \geq 3$;
- there exists $v \in V \setminus W$ and $w \in W \setminus V$ such that $V \setminus \{v\} \neq W \setminus \{w\}$;
- $(\sum_{x \in V} x)$ is orthogonal to $(\sum_{y \in W} y)$.

Then there exists an abelian $I_k(4m,2^{k-2},2,2)$.

**Proof.** Let $G = OA(4m,k,2,2)$ be an orthogonal array satisfying the conditions of the theorem. Let $G = [v_1|v_2|\ldots|v_k]$. Without loss of generality, assume that $v_{k-1} \in V \setminus W$ and $v_k \in W \setminus V$. Define a $(k-2) \times 2$ matrix $K$ over $\mathbb{F}_2$ such that the first column of $K$ contains a 1 in the $j$th row if and only if $v_j \in V$. Similarly, the second column of $K$ contains a 1 in the $j$th row if and only if $v_j \in W$. Observe furthermore that $[K^T|I]^T$ has the same property.

Next, let $A$ be the $(k-2) \times k$ matrix defined by $A = [I|K]$. If the columns of $K$ are identical, then $V \setminus \{v\} = W \setminus \{w\}$, a contradiction. Moreover, since $|V|,|W| \geq 3$, the columns of $K$ each have at least two 1’s. Thus, by Corollary 2 $<A>$ is an OA$(2^{k-2},k,2,2)$.

Define $A^\perp = [K^T|I]^T$. Observe that $GA^\perp$ is a $4m \times 2$ matrix with columns given by $\sum_{x \in V} x$ and $\sum_{y \in W} y$. By definition, $GA^\perp$ is an OA$(4m,2,2,2)$. Thus, by Theorem 3 $G \boxplus <A>$ is an $I_k(4m,2^{k-2},2,2)$. \hfill $\Box$

Now, observe that the matrix $G$ from Example 2 is an $OA(12,8,2,2)$ with the property that $v_1 + v_3 + v_4 + v_6 + v_7$ is orthogonal to $v_2 + v_3 + v_5 + v_6 + v_8$. Moreover, $G$ embeds in the Hadamard matrix $H(12)$ of order 12 (had.12, [13]). Thus we have the following corollary.

**Corollary 4.** There exists an abelian $I_k(12,2^{k-2},2,2)$ where $8 \leq k \leq 11$.

**Corollary 5.** There exists an abelian $I_k(20,2^{k-2},2,2)$ where $8 \leq k \leq 19$. 

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Proof. The following is a transpose of an OA(20, 8, 2, 2) which has the property that $v_1 + v_3 + v_4 + v_6 + v_7$ is orthogonal to $v_2 + v_3 + v_5 + v_6 + v_8$. The result follows by Theorem 6.

\[
\begin{bmatrix}
1 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 \\
1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 1 \\
1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 \\
1 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\
1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 1 \\
\end{bmatrix}
\]

The above array consists of columns 2 to 9 of the Hadamard matrix (had.20.toncheviv, [15]) with the permutation $(2, 4)(5, 7)(3, 8, 6, 9)$ applied to its columns. □

Corollary 6. For any odd $m \geq 3$, there exists an abelian $I_k(4m, 2^{k-2}, 2, 2)$ where $8 \leq k \leq 11$.

Proof. By Theorem 5 there exists an $I_k(16, 2^{k-2}, 2, 2)$ where $8 \leq k \leq 15$. Thus the result follows by previous two corollaries and Lemma 6. □

Via a counting argument, Theorem 6 cannot work for $m$ odd if $k \leq 6$. We outline this argument in the conclusion in Lemma 23. Moreover, computational results show that $k = 7$ does not work in the cases $m \in \{3, 5\}$.

The above and Theorem 6 thus motivate the following conjecture, which is stronger than the Hadamard conjecture.

Conjecture 1. For each odd $m$, there exists a Hadamard matrix $4m$ which yields an orthogonal array OA($4m, 8, 2, 2$) satisfying the conditions of Theorem 6.

If the above conjecture is true, then by Theorem 6 there exists an $I_k(4m, 2^{k-2}, 2, 2)$ for any $8 \leq k \leq 4m - 1$.

We next focus on a strategy for the case $k \leq 7$. Our constructions are typically non-abelian.

In the following, $\oplus$ is a binary operation that gives the concatenation of two vectors. That is,

$$(a_1, a_2, \ldots, a_r) \oplus (b_1, b_2, \ldots, b_s) = (a_1, a_2, \ldots, a_r, b_1, b_2, \ldots, b_s).$$

The following lemma is implied by the definition of an orthogonal array. Note that 1 is the vector containing only 1’s.

Lemma 13. Consider a set $S$ of $2^k$ binary vectors of dimension $k + 2$ with the following properties:

- For each $v \in 2^k$, the vector $v \oplus (i, j) \in S$, for some $i, j \in 2$;
- For each $(i, j) \in 2^2$, there are precisely $2^{k-2}$ vectors in $S$ of the form $v \oplus (i, j)$ for some $v$;
- The vector $v \oplus (i, j) \in S$ if and only if $(1 + v) \oplus (i, j) \in S$.

Then the vectors of $S$ are the rows of an OA($|S|, k + 2, 2, 2$).
Lemma 14. Let \( \{w, x, y, z, 1\} \) be a set of linearly independent binary vectors of dimension \( k \geq 5 \). Consider the \( 4 \times 2^k \) array of vectors of dimension \( k + 2 \) given by \( C \oplus R \), where \( R \) is the set of \( 2^k \) vectors with 0 in the final two positions and

\[
C = (w \oplus (0, 0), x \oplus (0, 1), y \oplus (1, 0), z \oplus (1, 1))^T.
\]

Then the elements in each column of \( C \oplus R \) can be rearranged so that each row is an OA\((2^k, k + 2, 2, 2)\).

Proof. Observe that \( H \) is a \( 4 \times 8 \) subarray of \( C \oplus R \):

\[
H = \begin{bmatrix}
w & x & y & z & t & u & s & v \\
x & w & t & u & y & z & v & s \\
y & t & w & s & x & v & z & u \\
z & u & s & w & v & x & y & t
\end{bmatrix}
\]

where \( s = w + y + z \), \( t = w + x + y \), \( u = w + x + z \), \( v = x + y + z \) and the vectors in the first, second, third and fourth rows are concatenated with \((0,0), (0,1), (1,0)\) and \((1,1)\), respectively.

Next, let \( H' \) be the \( 4 \times 8 \) array formed by replacing each vector \( a \) in \( H \) with the vector \( a + (1 \oplus (0,0)) \). We next arrange the entries in each column of \([H|H']\). We mark the elements of \( H \) as follows:

\[
H = \begin{bmatrix}
w & x^o & y^* & z^* & t^* & u^* & s^* & v^* \\
x^* & w & t & u^o & y^o & z^o & v^o & s^o \\
y & t^* & w^o & s & x & v^o & z^* & u \\
z^o & u & s & w^* & v^* & x & y & t^o
\end{bmatrix}
\]

Next, rearrange the elements in each column of \( H \) so that elements with the same mark are in the same row, with a corresponding permutation applied to each column of \( H' \). Let the resultant \( 4 \times 16 \) matrix be \( J \).

Now, replace each vector of the form \( a \) in \( J \) with the vector \((v \oplus (0,0)) + a\) to obtain a \( 4 \times 16 \) matrix \( J' \). Observe that for each row of \( K = [J|J'] \) and for each \( g \in G = \langle w, x, y, z, 1 \rangle \), there exists \( i \) and \( j \) such that \( g \oplus (i, j) \) is in that row. Moreover, each column from \( K \) is a column from \( C \oplus R \) with elements permuted.

Let \( G, z_0 + G, \ldots, z_{\alpha - 1} + G \) be the cosets of \( G \) in \((\mathbb{F}_2)^k\), where \( \alpha = 2^{k-5} \). For each \( i \in [\alpha] \), let \( K_i \) be formed from \( K \) by replacing each entry \( a \) of \( K \) with \((z_i \oplus (0,0)) + a\).

Then, observe that \([K_0|K_1|\ldots|K_\alpha]\) can be formed from \( C \oplus R \) by permuting the elements in each column. Moreover, the resultant rows each now satisfy the conditions of Lemma 13.

Example 3. Let \( w = 10000, x = 10111, y = 01101, \) and \( z = 01011 \). Then \( H \) and \( H' \) in the proof of above lemma are as follows:

\[
H = \begin{bmatrix}
100000 & 1011000^* & 0110100^* & 0101000 & 0101000 & 0101000^* & 1011000^* & 1000100^* \\
101101^* & 1000001 & 0101001 & 0110001^* & 0110101^* & 0110101 & 1001001 & 1010101^* \\
0110110 & 0101010^* & 1000001^* & 1011010 & 1011110 & 1000110^* & 0101110^* & 0110010 \\
0101111^* & 0110011 & 1011111 & 1000011^* & 1000111^* & 1011111 & 0110111 & 0101011^*
\end{bmatrix}
\]
Corollary 7. Let $G = OA(4m, k + 2, 2, 2)$ be an orthogonal array such that the rows partition into sets of 4 vectors of the form

$$\{w \oplus (0, 0), x \oplus (0, 1), y \oplus (1, 0), z \oplus (1, 1)\}$$

where $\{w, x, y, z, 1\}$ is a linearly independent set. Let $R$ be the set of $2^k$ vectors with 0 in the final two positions. Then the elements in each column of $G \boxplus R$ can be rearranged to create an $I_{k+2}(4m, 2^k, 2, 2)$.

Proof. From the previous lemma it suffices to check that $G \boxplus R$ is a factorial design. Let $A$ be a $k \times (k + 2)$ matrix of the form $[I \mid 0]$. Observe that $R = <A>$. The nullspace of $A$ is generated by the columns of $A^\perp = [0 \mid I]^T$. Thus the columns of $GA^\perp$ are the last two columns of $G$ which are by definition orthogonal. The result then follows from Theorem 2.

Corollary 8. There exists $I_7(12, 32, 2, 2)$.

Proof. We present an orthogonal array of type $OA(12, 7, 2, 2)$ in Table 2, that satisfies the conditions of Corollary 7. The dashed lines partition the rows into three sets of the form $\{w \oplus (0, 0), x \oplus (0, 1), y \oplus (1, 0), z \oplus (1, 1)\}$ such that in each case $\{w, x, y, z, 1\}$ is linearly independent.

Table 2: An orthogonal array of type $OA(12, 7, 2, 2)$.

We next generalize the above ideas to the case where linear independence is not assumed.
Lemma 15. Let \( m = 2^\alpha \) where \( \alpha \geq 2 \) and \((\mathbb{F}_2)^\alpha = \{ \mathbf{e}_i \mid i \in [m] \}\). That is, label the binary vectors of dimension \( \alpha \) with \( \mathbf{e}_0, \mathbf{e}_1, \ldots, \mathbf{e}_{m-1} \). Let \( \mathbf{v}_i, i \in [m] \) be any vectors (possibly non-distinct) of dimension \( k \geq \alpha + m + 1 \). Consider the \( m \times 2^k \) array of vectors of dimension \( k + \alpha \) given by \( C \oplus R \), where the rows of \( R \) are the set of \( 2^k \) vectors with \( 0 \) in the final \( \alpha \) positions and the \( i \)th row of \( C \) is given by \( \mathbf{v}_i \oplus \mathbf{e}_i \), where \( i \in [m] \). Then the elements in each column of \( C \oplus R \) can be rearranged so that the vectors in each row form an OA\((2^k, k + \alpha, 2, 2)\).

Proof. Let \( \mathbf{e}_0 = \mathbf{0} \) be the first row of \( R \). Let \( G \) be the subgroup generated by the set of vectors \( \{ \mathbf{v}_i : i \in [m] \} \cup \{ \mathbf{1} \} \). Then \( |G| = 2^\ell \) for some \( 1 \leq \ell \leq m+1 \). Let \( G = \{ \mathbf{g}_i \mid i \in [2^\ell] \} \) where \( \mathbf{g}_0 = \mathbf{0} \). Define a \( 2^\ell \times (k + \alpha) \) array \( R_0 \) so that row \( i \) of \( R_0 \) is \( \mathbf{g}_i \oplus \mathbf{0} \), \( i \in [2^\ell] \). Let \( K_0 \) be the \( m \times 2^\ell \) array of vectors of dimension \( k + \alpha \) given by \( C \oplus R_0 \).

Next, let \( \mathbf{z}_0 + G, \mathbf{z}_1 + G, \ldots, \mathbf{z}_{2^\ell - 1} + G \) be the cosets of \( G \) in \((\mathbb{F}_2)^k\), where \( \beta = 2^{k-\ell} \). For each \( j \in [\beta] \), let \( K_j \) be formed from \( K_0 \) by adding \( \mathbf{z}_j \oplus \mathbf{0} \) to each vector in \( K \). Next, cyclically permute the elements in each column of \( K_j \) by \( j \) places (modulo \( m \)) to create \( K_j' \).

Note that \( L = [K_0'K_1' \ldots K_{\beta-1}'] \) is equal to \( C \oplus R \) after a permutation of the elements in each column. Moreover, let \( S \) be the set of vectors of dimension \( k + \alpha \) that occur in a given row of \( L \). Then we claim the following:

(a) For each \( \mathbf{w} \in [2]^k \), the vector \( \mathbf{w} \oplus \mathbf{e}_i \in S \), for some \( \mathbf{e}_i \in (\mathbb{F}_2)^\alpha \);
(b) For each \( \mathbf{e}_i \in (\mathbb{F}_2)^\alpha \), there are precisely \( 2^{k-\alpha} \) vectors in \( S \) of the form \( \mathbf{w} \oplus \mathbf{e}_i \) for some \( \mathbf{w} \);
(c) For each \( \mathbf{e}_i \in (\mathbb{F}_2)^\alpha \), the vector \( \mathbf{w} \oplus \mathbf{e}_i \in S \) if and only if \( (\mathbf{w} + \mathbf{1}) \oplus \mathbf{e}_i \in S \).

Similarly to Lemma 13 if this claim is true, it follows that the set of vectors in \( S \) form the rows of an orthogonal array of strength 2. So it suffices to show that the above claim is true.

To see (a), let \( j \in [\beta] \). Observe that in every row of \( K_j \), and for every element \( \mathbf{w} \in \mathbf{z}_j + G \), the vector \( \mathbf{w} \oplus \mathbf{e}_i \) occurs in that row. The same property holds for \( K_j' \). Next, from the conditions of the lemma, \( \beta \geq m \); indeed \( m \) divides \( \beta \). Thus (b) is true. Finally (c) is true because \( 1 \in G \).

Corollary 9. Let \( \alpha \geq 2 \) and \( 2^\alpha + \alpha + 1 \leq k \). Suppose there exists an OA\((2^\alpha b, k + \alpha, 2, 2)\) such that the last \( \alpha \) columns are an OA\((2^\beta b, 2, 2)\). Then \( I_{k+\alpha}(2^k, 2^\alpha b, 2, 2) \) exists.

Proof. We shall construct the transpose design \( I_{k+\alpha}(2^\alpha b, k + \alpha, 2, 2) \) such that the last \( \alpha \) columns are an OA\((2^\beta b, 2, 2)\). Note that \( C \) can be partitioned into subarrays \( C_0, C_1, \ldots, C_{b-1} \), each of dimension \( 2^\alpha \times (k + \alpha) \), such that for each \( i \in [b] \), the last \( \alpha \) columns of \( C_i \) contain each element of \((\mathbb{F}_2)^\alpha \) exactly once.

Next, let \( A \) be the \( k \times (k + \alpha) \) matrix of the form \([I\mathbf{0}]\) and let \( R \) be an orthogonal array whose rows are the elements of \(< A >\). Observe that the rows of \( R \) are the set of \( 2^k \) vectors with \( 0 \) in the final \( \alpha \) positions. Moreover, the nullspace of \( A \) is generated by the columns of \( A^\perp = [\mathbf{0}\mathbf{I}]^T \). Thus the columns of \( CA^\perp \) are the last \( \alpha \) columns of \( C \).

Thus, by Theorem 2, \( C \oplus R \) is a row-column factorial design. Moreover, the columns of \( C \oplus R \) are each of strength 2 since \( C \) is of strength 2. Finally, apply the previous lemma with \( m = 2^\alpha \) to rearrange the elements in each column of subarray \( C_i \oplus R \) so that \( C \oplus R \) becomes an \( I_{k+\alpha}(2^\alpha b, 2^k, 2, 2) \).

The following lemma uses a standard doubling technique.
Lemma 16. Suppose there exists a Hadamard matrix of order $4b$ for some integer $b$. Let $\alpha \geq 2$. Then there exists an OA($2^\alpha b, k + \alpha, 2, 2$) such that the final $\alpha$ columns form an OA($2^\alpha b, \alpha, 2, 2$), for any $\alpha \leq k + \alpha < 2^\alpha b$.

Proof. We proceed by induction on $\alpha$. Suppose $\alpha = 2$. The existence of a Hadamard matrix of order $4b$ implies the existence of an OA($4b, k + 2, 2, 2$) for any $k + 2 \leq 4b - 1$ by Lemmas 2 and 5. By the definition of the strength of an orthogonal array, the final two columns must contain each ordered pair $b$ times, so the final two columns form an OA($4b, 2, 2, 2$).

Next assume that the lemma is true for a fixed value of $\alpha \geq 2$. Then there exists an L = OA($2^\alpha b, 2^\alpha b - 1, 2, \alpha$) with the specified properties. Observe that the following matrix $L'$ is an OA($2^\alpha + 1b, 2^\alpha + 1b - 1, 2, 2$):

$$L' = \begin{bmatrix} L & L \end{bmatrix},$$

where $\overline{L}$ is formed from $L$ by replacing each 0 with 1. Moreover, the final $\alpha + 1$ columns of $L'$ contain each binary sequence of dimension $\alpha + 1$ exactly once. Thus the final $\alpha + 1$ columns form an OA($2^\alpha + 1b, \alpha + 1, 2, \alpha + 1$). Hence an OA($2^\alpha + 1b, k + \alpha + 1, 2, \alpha + 1$) can be obtained for any $k$ such that $k + \alpha + 1 < 2^\alpha b$ by deletion of columns. This completes the induction and the proof.

From the previous lemma and Corollary 9, we have the following.

Theorem 7. If there exists a Hadamard matrix $H(4b)$, then there exists $I_{k+\alpha}(2^\alpha b, 2^k, 2, 2)$ for any $2 \leq \alpha; 2^\alpha + \alpha + 1 \leq k < 2^\alpha b - \alpha$.

Before we completely deal with the case when $m$ is odd and $k$ is small, we need some constructions for specific parameters.

Lemma 17. There exists $I_5(12, 8, 2, 2)$, $I_6(12, 16, 2, 2)$ and $I_4(12, 12, 2, 2)$.

Proof. In Table 3 we present an abelian array of the form $C \boxplus R$, where $R$ is the rowspace of the $3 \times 5$ matrix $[I_3|0]$ and $C$ is an OA($12, 5, 2, 2$) (constructed from 5 columns of a Hadamard matrix of order 12). Now, the columns of the matrix $C[I_3|0]^T$ are in turn distinct columns of $C$; thus $C[I_3|0]^T$ is an OA($12, 3, 2, 2$). Hence, by Theorem 2 $C \boxplus R$ is a row-column factorial design. Moreover, each column is an orthogonal array of strength 2. We can then rearrange the elements in each column to create an $I_5(12, 8, 2, 2)$; the rearrangement is indicated by the use of superscripts. That is, we permute entries within each column so that vectors with the same superscript belong to the same row.
Thus Table 5.

In Table 4, first consider the matrix $A$ formed by the first 4 columns. This matrix is abelian of the form $C \oplus R$, where $C$ is a row-column factorial design. Thus $[A^2 A]$, as shown in Table 4, is a row-column factorial design with each column an orthogonal array of strength 2. It thus remains to rearrange the elements within each column so that the rows are each of strength 2. The superscripts $A, B, C$ and $D$ indicate 4 rows of strength 2. The remaining rows are formed by cyclic shifts of each of these by 4 rows and then 8 rows; also indicated by superscripts. This results in the array given in Table 6.

| $0000^A$ | $0001^B$ | $1111^C$ | $1110^D$ | $0000^E$ | $0001^F$ | $1111^G$ | $1110^H$ | $0000^I$ | $0001^J$ | $1111^K$ | $1110^L$ |
|---------|---------|---------|---------|---------|---------|---------|---------|---------|---------|---------|---------|
| $0110^J$ | $0111^I$ | $1001^A$ | $1000^B$ | $0110^B$ | $0111^A$ | $1001^E$ | $1000^F$ | $1011^B$ | $1011^I$ | $1001^J$ | $1000^K$ |
| $1011^B$ | $1010^A$ | $1000^D$ | $0101^C$ | $1101^C$ | $0010^E$ | $1001^G$ | $1100^H$ | $1100^I$ | $1100^J$ | $1100^K$ | $1100^L$ |
| $1101^D$ | $1100^G$ | $0010^B$ | $0011^A$ | $0000^D$ | $0001^C$ | $1111^G$ | $1110^H$ | $0000^I$ | $0001^J$ | $1111^K$ | $1110^L$ |
| $0000^L$ | $0001^K$ | $1111^K$ | $1110^L$ | $1001^B$ | $1000^E$ | $0111^A$ | $1100^E$ | $1010^A$ | $1110^B$ | $1110^K$ | $1110^L$ |
| $0111^H$ | $0110^G$ | $1000^D$ | $0101^H$ | $1101^E$ | $1000^F$ | $0111^I$ | $1100^I$ | $1100^J$ | $1100^K$ | $1100^L$ | $1100^I$ |
| $0011^C$ | $0001^D$ | $1100^J$ | $1101^I$ | $0010^A$ | $0110^B$ | $1010^B$ | $1011^A$ | $0101^B$ | $0101^F$ | $1101^E$ | $1100^I$ |
| $0101^I$ | $0100^J$ | $1010^I$ | $1011^I$ | $1101^A$ | $1100^B$ | $1000^I$ | $0110^C$ | $1100^D$ | $1100^E$ | $1100^F$ | $1100^G$ |
| $1001^K$ | $1000^L$ | $0101^E$ | $0111^F$ | $1001^C$ | $1000^D$ | $0110^I$ | $1111^A$ | $1110^B$ | $1110^C$ | $1110^D$ | $1110^E$ |
| $1110^E$ | $1111^F$ | $0010^D$ | $0000^E$ | $1101^J$ | $1110^J$ | $1110^I$ | $1110^I$ | $1110^J$ | $1110^K$ | $1110^L$ | $1110^I$ |
| $1010^G$ | $1011^H$ | $0101^L$ | $0100^K$ | $1010^K$ | $1011^L$ | $1010^D$ | $0100^C$ | $0011^C$ | $0010^B$ | $0000^A$ | $0000^C$ |
| $1100^F$ | $1101^E$ | $0011^B$ | $0010^G$ | $1100^J$ | $1101^J$ | $1011^I$ | $0011^L$ | $0010^K$ | $0011^B$ | $0001^A$ | $0001^D$ |

Table 3: An array of type $I_5(12, 8, 2, 2)$, with rows indicated by superscripts.

Table 4: A factorial row-column design with each row strength 2.
Since Proof and only if We can now give necessary and sufficient conditions for the case when the number of rows
Finally, I exists for odd OA I exists by Corollary 8. Also, I exists for each I by Lemma 6,
Table 5: An array of type $I_4(12, 12, 2, 2)$.

Finally, $I_6(12, 16, 2, 2)$ is given in the Appendix. Similarly to above, this is presented first as an abelian row-column factorial design where each column is of strength 2. The superscripts indicate how to permute the entries within each column.

We can now give necessary and sufficient conditions for the case when the number of rows is congruent to 4 (mod 8), assuming the truth Conjecture 1.

**Theorem 8.** Let $m$ and $b$ be odd. If Conjecture 1 is true, then $I_k(4m, 2^ab, 2, 2)$ exists if and only if $(k, 4m, 2^ab, 2, 2)$ is admissible and

$$(k, 4m, 2^ab, 2, 2) \not\in \{(3, 4m, 4, 2, 2), (3, 4, 4m, 2, 2) \mid m \text{ is odd}\}.$$

**Proof.** Since $(k, 4m, 2^a, 2, 2)$ is admissible, from Lemmas 1 and 2 $a \geq 2$, $k \leq a + 2$, $k \leq 4m - 1$ and $k \leq 2^ab - 1$.

**Case 1:** $a = 2$ and $b = 1$. Then $k \leq 3$. Suppose $k = 2$. Now, $[00, 01, 10, 11]^T$ is an OA$(4, 2, 2, 2)$, so by Lemma 2 there exists $I_2(4, 4, 2, 2)$. Thus by Corollary 1, $I_2(4m, 4b, 2, 2)$ exists for any integers $m$ and $b$. Otherwise $k = 3$. By Lemma 12, $I_3(4m, 4, 2, 2)$ does not exist for odd $m$.

**Case 2:** $a = 2$ and $b \geq 3$. If $m = 1$, this is the transpose of Case 1, so we may assume $m \geq 3$. Thus $k \leq 4$ implies admissibility. From Lemma 17 there exist $I_4(12, 12, 2, 2)$ and $I_5(12, 8, 2, 2)$. From Theorem 5, $I_6(8, 8, 2, 2)$ exists. In turn, by Lemma 5, $I_4(12, 8, 2, 2)$ and $I_4(8, 8, 2, 2)$ exist. By adjoining copies of $I_4(12, 12, 2, 2)$, $I_4(12, 8, 2, 2)$, $I_4(8, 12, 2, 2)$ and $I_5(8, 8, 2, 2)$ as needed using Lemma 6, there exists $I_4(4m, 4b, 2, 2)$ for any $m, b \geq 3$.

**Case 3:** $m = 1$ and $a \geq 3$. Since $m = 1$, $k \leq 3$. Then there exists $I_3(4, 8, 2, 2)$ by Theorem 5. Thus there exists $I_3(4, 2^ab, 2, 2)$ for any $a \geq 3$ by Corollary 1.

**Case 4:** $m \geq 3$ and $a \in \{3, 4\}$. Here $k \leq a + 2$ implies admissibility. Now, $I_{a+2}(12, 2^a, 2, 2)$ exists for each $a \in \{3, 4\}$ from Lemma 17. Next, $I_8(8, 8, 2, 2)$ exists by Theorem 5. Thus by Lemma 6, $I_{a+2}(4m, 2^a, 2, 2)$ exists for any odd integer $m$. In turn, $I_{a+2}(4m, 2^ab, 2, 2)$ exists by Corollary 1.

**Case 5:** $m \geq 3$ and $a = 5$. Then $k \leq 7$ implies admissibility. Now, $I_7(12, 2^5, 2, 2)$ exists by Corollary 8. Also, $I_7(8, 2^5, 2, 2)$ exists by Theorem 5. Thus by Lemma 6 and Corollary 1, $I_7(4m, 2^ab, 2, 2)$ exists for all odd $m \geq 3$ and odd $b$. 18
Case 6: $m \geq 3$ and $a \geq 6$. From Corollary 6 and assuming the truth of Conjecture 1, $I_{a+2}(4m, 2^a, 2, 2)$ exists for all $6 \leq a \leq 4m - 3$. Thus $I_k(4m, 2^a b, 2, 2)$ exists for all $k \leq a + 2$.

Theorem 9. Let $m \leq 5$ and $b$ odd. Then $I_k(4m, 2^a b, 2, 2)$ exists for all admissible

$$(k, 4m, 2^a b, 2, 2) \notin \{(3, 4m, 4, 2, 2), (3, 4, 4m, 2, 2) \mid m \text{ is odd}\}.$$ \hfill \Box

Proof. From the previous theorem and the fact that Conjecture 1 is true for $m \in \{3, 5\}$, we can assume $m \in \{2,4\}$.

Let $m = 2$. By Theorem 5 there exists $I_3(8, 4, 2, 2)$ and $I_7(8, 8, 2, 2)$. Also there exists $I_5(4b, 8, 2, 2)$ (and thus $I_5(8, 4b, 2, 2)$) by the previous theorem, where $b \geq 3$ is odd. The result then follows by Lemma 6 and Corollary 1.

Otherwise $m = 4$. Then by Theorem 5 there exists $I_3(16, 4, 2, 2)$ and $I_{a+4}(16, 2^a, 2, 2)$ for any $3 \leq a \leq 11$. Also there exists $I_k(16, 4b, 2, 2)$, where $b \geq 3$ is odd, by the previous theorem. The result then follows by Lemma 6 and Corollary 1. \hfill \Box

6 Binary row-column factorial designs with strength $t = 3$

In this section we restrict ourselves to binary row-column factorial designs of strength 3. We completely classify these when the dimensions of the arrays are powers of 2. The aim of this section is to prove the following theorem.

Theorem 10. Let $M \leq N$. Then an array of type $I_k(2^M, 2^N, 2, 3)$ exists if and only if $3 \leq k \leq M + N$, $3 \leq M$, $k \leq 2M - 1$ and $(k, M, N) \notin \{(4, 3, 3), (8, 4, 4)\}$.

Lemma 18. Let $M \leq N$. Then $(k, 2^M, 2^N, 2, 3)$ is admissible if and only if $3 \leq k \leq M + N$, $3 \leq M$ and $k \leq 2^{M-1}$.

Proof. By Lemma 4, $3 \leq k \leq M + N$ and $3 \leq M$. The bound $k \leq 2^{M-1}$ (and sufficiency) follows by Lemma 4. \hfill \Box

To establish the two exceptions in Theorem 10, we first need the following result on orthogonal arrays. This result is a fairly standard observation for researchers in Hadamard codes but we include a proof for thoroughness.

Lemma 19. Let $M \geq 3$. In any OA$(2^M, 2^{M-1}, 2, 3)$, the weight of any two rows has the same parity.

Proof. Let $K$ be an OA$(2^M, 2^{M-1}, 2, 3)$. Without loss of generality assume that, restricting ourselves to the first two columns of $K$, the first $2^{M-2}$ rows contain the ordered pairs $(1,1)$, the next $2^{M-2}$ rows contain the ordered pairs $(1,0)$, the next $2^{M-2}$ rows contain the ordered pairs $(0,1)$ and the final $2^{M-2}$ rows contained the ordered pairs $(0,0)$.

For the rest of the proof, we assume that in a Hadamard matrix each $-1$ has been replaced by 0. It follows, from the strength 3 property of the orthogonal array, that: (a) the first $2^{M-1}$ rows form a Hadamard matrix; (b) the last $2^{M-1}$ rows form a Hadamard matrix; and (c) the first $2^{M-2}$ rows together with the third set of $2^{M-2}$ rows forms a Hadamard matrix.

Now, in a normalized Hadamard matrix of order at least 4, the weight of any row or column is even. Equivalent Hadamard matrices are formed by rearranging rows or columns, taking
a transpose or swapping 0 with 1 in any row or column. All of these equivalences preserve
the property that the weight of each pair of rows shares the same parity. The result
follows.

**Corollary 10.** There exists neither an array of type \( I_4(8,8,2,3) \) nor an array of type
\( I_5(16,16,2,3) \).

*Proof.* If an array of type \( I_4(8,8,2,3) \) exists, then the vectors in any row or column,
by definition, form an OA\((8,4,2,3)\). Thus, from the previous lemma, the weight of every vector
in the array has the same parity. Hence the vectors in all the cells of the array do not
form a factorial design. Similarly, there does not exist an array of type \( I_5(16,16,2,3) \). □

We now focus on proving Theorem 10 in the case where \( M \geq 5 \). We will use Theorem 4
for this case. We first need some preliminary lemmas.

We remind the reader that a set \( S \) of vectors is \( t \)-independent if and only if each subset of
\( S \) of size \( t \) is independent.

**Lemma 20.** Let \( C \) be a set consisting of \( M \) cyclic permutations of the vector
\((1,1,1,0,0,...,0)\) over \( \mathbb{F}_2^M \), where \( M \geq 5 \) and \( M \neq 6 \). Then the vectors in \( C \) are
3-independence.

*Proof.* Note that all the vectors in \( C \) have weight 4. Also, any three vectors \( t,u,v \) in
\((\mathbb{F}_2)^M \) are linearly dependent if and only if \( u + v = t \). Now for any two vectors \( u \) and \( v \)
in \( C \) we have the following possibilities:

- **Case I:** There is at most one \( i \) such that \( u_i = v_i = 1 \). In this case \( \omega(u + v) = 6 \) and
  therefore \( u + v \notin C \).

- **Case II:** There are exactly two values of \( i \) for which \( u_i = v_i = 1 \). In this case \( \omega(u + v) = 4 \)
  and \( M \geq 7 \) (since \( M \neq 6 \)). However, notice that the vector \( u + v \) contains the values
  \( 1,1,0,0,1,1,0 \) at seven consecutive positions (modulo \( n \)) and therefore does not belong to
  \( C \).

- **Case III:** There are exactly three values of \( i \) for which \( u_i = v_i = 1 \). In this case \( \omega(u + v) = 2 \).

**Corollary 11.** For \( M \geq 5 \) and \( M \neq 6 \), let \( B = \{e_1,...,e_M\} \) be the standard basis for
\( \mathbb{F}_2^M \) and \( C \) be the set defined in Lemma 20. Then the set \( W = B \cup C \) is 3-independent.

**Lemma 21.** For \( N \geq M \geq 5 \) there exists an \( I_k(2^M,2^N,2,3) \) if and only if \( (k,2^M,2^N,2,3) \)
is admissible.

*Proof.* From Lemma 18 and Corollary 1 it suffices to assume \( k = \min\{2^{M-1}, M + N\} \).

We split the proof into different cases. In each case we define a matrix \( L \) satisfying the
required conditions of Theorem 4.

- **Case I:** \( M \neq 6 \). Let \( C_M \) be an \( M \times M \) matrix such that the rows are the elements of the set \( C \) defined in Lemma 20 with the main diagonal of \( C_M \) containing only entry 1. Let \( D \) be the set of all the rows in \( I_M \) and \( C_M - I_M \). Let \( W \) be the set of all vectors of odd
  weight in \( \mathbb{F}_2^M \). By Lemma 3 \( W \) is a 3-independent set of vectors. Let \( S \) be a \( (k-2M) \times M \)
  matrix such that each row is a distinct element in \( W \setminus D \). Then the matrix \( L \) is as follows.

\[
L = \begin{pmatrix}
I_M & I_M & 0 \\
C_M - I_M & C_M & 0 \\
S & 0 & I_{k-2M}
\end{pmatrix}
\]  \hspace{1cm} (5)

\[20\]
**Case II**: When $M = 6$. In this case we can take the above matrix $L$ using the following $C_M$:

$$C_M = \begin{pmatrix}
1 & 1 & 1 & 0 & 0 \\
0 & 1 & 1 & 1 & 0 \\
0 & 0 & 1 & 1 & 1 \\
1 & 0 & 0 & 1 & 1 \\
1 & 0 & 1 & 0 & 1 \\
1 & 0 & 1 & 1 & 0 \\
\end{pmatrix}.$$

\[\square\]

**Lemma 22.** There exists $I_3(8,8,2,3), I_4(8,16,2,3), I_7(16,16,2,3)$ and $I_8(16,32,2,3)$.

**Proof.** The eight binary vectors of dimension 3 give the rows of an OA$(8,3,2,3)$. Thus by Lemma 9 there exists an array of type $I_3(8,8,2,3)$. Next, the 8 binary vectors of dimension 4 and even weight give the rows of an OA$(8,4,2,3)$. Using Lemma 9 again, there exists an array of type $I_4(8,16,2,3)$.

By Theorem 4 and the following array $L$, there exists $I_8(16,32,2,3)$. Moreover, we get $I_7(16,16,2,3)$ for free by deleting the last row and column, as indicated by dotted lines.

$$L = \begin{pmatrix}
1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\
1 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 \\
1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 \\
1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 0 \\
0 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 1 \\
\end{pmatrix}.$$

\[\square\]

We now have all the tools, base cases and exceptions we need to prove Theorem 10. From Lemma 21, we may restrict ourselves to the case $M \in \{3,4\}$ and $N \geq M$. If $M = 3$ then by Lemma 18, $3 \leq k \leq 2^{M-1} = 4$. An $I_3(8,8,2,3)$ exists by the previous lemma. Thus by Corollary 1 there exists $I_3(2^M,2^N,2,3)$ whenever $M,N \geq 3$. Next, $I_4(8,8,2,3)$ does not exist by Corollary 10. However $I_4(8,16,2,3)$ exists by the previous lemma. Thus by Corollary 1 there exists an array of type $I_4(2^M,2^N,2,3)$ whenever $M \geq 3$ and $N \geq 4$.

Finally suppose that $N \geq M = 4$. By Lemma 18, $3 \leq k \leq 8$. Now, $I_8(16,16,2,3)$ does not exist by Corollary 10 but $I_7(16,16,2,3)$ exists by the previous lemma and $I_8(16,32,2,3)$. The result then follows by Corollary 1.
Conclusion

We first discuss some limitations to the approach given in Section 5. Firstly, the idea in Theorem 6 cannot work for theoretical reasons when $k \leq 6$, and for computational reasons (inspection of possible cases) when $k = 7$ and $m \in \{3, 5\}$. The reason that $k \geq 7$ is necessary for the approach is as follows. The counting argument in the following lemma shows that if $m$ is odd, then in any $\text{OA}(4m, n, 2, 2)$, if the sum of $\ell$ columns has weight $2m$ (that is, contains $2m$ occurrences of 1), then $\ell \equiv 1$ or 2 (mod 4). In turn, from Corollary 2 the columns of $K$ must be distinct and have weight at least 2. This precludes a suitable $K$ for $k \leq 6$.

**Lemma 23.** Let $H$ be an $\text{OA}(4m, \ell, 2, 2)$, where $m$ is an odd integer. Let $v$ be the sum of columns of $H$ over $\mathbb{F}_2$.

- If $\ell \equiv 0$ or 3 (mod 4) then $\omega(v) \equiv 0$ (mod 4).
- If $\ell \equiv 1$ or 2 (mod 4) then $\omega(v) \equiv 2$ (mod 4).

**Proof.** Let $x_i$ be the weight of the $i$th row of $H$. The total number of $(1,1)$ pairs such that both of them lie in the same row is given by $\sum_{i=1}^{4m} (x_i/2)$. Also, since each pair of columns contain exactly $m$ $(1,1)$ pairs, therefore this number can also be given by $m(\ell/2)$. Thus we have,

$$\sum_{i=1}^{4m} \left(\frac{x_i}{2}\right) = m\left(\frac{\ell}{2}\right)$$

which simplifies to:

$$\sum_{i=1}^{4m} x_i^2 = m\ell(\ell + 1).$$

(6)

Also notice that, $x_i^2 \equiv 1$ (mod 4) if $x_i$ is odd and $x_i^2 \equiv 0$ (mod 4) otherwise. Thus we have:

$$\omega(v) = \sum_{i=1}^{4m} (x_i \pmod{2}) = \sum_{i=1}^{4m} (x_i^2 \pmod{2}) = \sum_{i=1}^{4m} (x_i^2 \pmod{4}).$$

The result is now follows from (6) and the fact that $m$ is odd. \qed

We have intentionally structured our paper so that abelian and non-abelian constructions are distinguished. By inspection, we have determined that there does not exist an abelian $I_5(12, 8, 2, 2)$; a non-abelian example is given in Lemma 17. However we do not know at this stage whether there are infinitely many parameters for which their exists only a non-abelian binary strength 2 row-column factorial design.

As observed in the introduction, finding necessary and sufficient conditions for the existence of a strength 2 binary row-column factorial design depends on the Hadamard conjecture. However, the following may be more within reach.

**Conjecture 2.** If there exists a Hadamard matrix of order $4m$, then there exists an $I_k(4m, 4n, 2, 2)$ for any $n \geq m$, $2^k|16mn$ and $k \leq 4m - 1$, with the exception $m = 1$ and $n$ is odd.

From Theorem 6 the above conjecture is true for $m \leq 5$. For $m = 6$, the unknown cases with minimal parameters are: $I_{k+3}(24, 2^k, 2, 2); 5 \leq k \leq 11$. From Theorem 7 and the existence of a Hadamard matrix of order 12, $I_{k+3}(24, 2^k, 2, 2)$ exists for $12 \leq k \leq 20$. 

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Table 6: An array of type $I_6(12, 16, 2, 2)$. 

|      | 000000 | 100000 | 010000 | 001000 | 000100 | 001100 | 010000 | 011000 | 100000 | 101000 | 110000 | 111000 |
|------|--------|--------|--------|--------|--------|--------|--------|--------|--------|--------|--------|--------|
| 110101 | 010101 | 100101 | 111101 | 111001 | 100001 | 101101 | 011001 | 001001 | 000101 | 010001 | 011001 | 100001 |
| 110010 | 010010 | 100010 | 111010 | 111100 | 100100 | 101100 | 011000 | 001000 | 000100 | 010000 | 011000 | 100000 |
| 000011 | 100011 | 010011 | 001111 | 001110 | 001110 | 001110 | 001110 | 001110 | 001110 | 001110 | 001110 | 001110 |

| | 001100 | 101100 | 011100 | 001000 | 001000 | 001000 | 111000 | 111000 | 111000 | 111000 | 111000 | 111000 |
| | 010010 | 110101 | 000100 | 001000 | 001000 | 001000 | 111000 | 111000 | 111000 | 111000 | 111000 | 111000 |
| | 011110 | 111010 | 011000 | 011000 | 011000 | 011000 | 111000 | 111000 | 111000 | 111000 | 111000 | 111000 |
| | 011010 | 110110 | 001100 | 001100 | 001100 | 001100 | 111000 | 111000 | 111000 | 111000 | 111000 | 111000 |

| | 111000 | 010000 | 100000 | 101000 | 110000 | 111000 | 111000 | 111000 | 111000 | 111000 | 111000 | 111000 |
| | 101001 | 000101 | 111001 | 100001 | 101001 | 111001 | 111001 | 111001 | 111001 | 111001 | 111001 | 111001 |
| | 100110 | 000010 | 110100 | 100010 | 101010 | 110100 | 111000 | 111000 | 111000 | 111000 | 111000 | 111000 |
| | 101110 | 001110 | 111110 | 101110 | 100110 | 111010 | 111100 | 111100 | 111100 | 111100 | 111100 | 111100 |
References

[1] R. Bose. On some connections between the design of experiments and information theory. *Bull. Inst. Inter. Statist.*, 38:257–271, 1961.

[2] T. Britz, N. Cavenagh, A. Mammoliti, and I. Wanless. Mutually orthogonal binary frequency squares. *The Electronic Journal of Combinatorics*, 27:P3.7, 2020.

[3] K. Bush. A generalization of a theorem due to macneish. *Ann. Math. Statist.*, 23:293–295, 1953.

[4] C. J. Colbourn and J. H. Dinitz. *Handbook of Combinatorial Designs, Second Edition (Discrete Mathematics and Its Applications)*. Chapman & Hall/CRC, 2006.

[5] S. Damelin, G. Michalski, and G. L. Mullen. The cardinality of sets of k-independent vectors over finite fields. *Monatshefte für Mathematik*, 150:289–295, 2007.

[6] D. Z. Dokovic. Hadamard matrices of order 764 exist. *Combinatorica*, 28:487–489, 2008.

[7] J. Godolphin. Construction of row–column factorial designs. *Journal of the Royal Statistical Society: Series B (Statistical Methodology)*, 81:335–360, 2019.

[8] S. Hedayat, Sloane. *Orthogonal Arrays*. Springer, New York, NY, 1999.

[9] K. J. Horadam. *Hadamard matrices and their applications*. Princeton university press, 2012.

[10] C. F. Laywine and G. L. Mullen. A table of lower bounds for the number of mutually orthogonal frequency squares. *Ars Combinatoria*, 59:85–96, 2001.

[11] M. Li, Y. Zhang, and B. Du. Some new results on mutually orthogonal frequency squares. *Discrete Mathematics*, 331:175–187, 2014.

[12] R. Plackett and J. Burman. The design of optimum multifactorial experiments. *Biometrika*, pages 305–325, 1946.

[13] F. Rahim and N. J. Cavenagh. Row-column factorial designs with multiple levels. *Journal of Combinatorial Designs*, 29:750–764, 2021.

[14] C. R. Rao. Factorial experiments derivable from combinatorial arrangements of arrays. *Supplement to the Journal of the Royal Statistical Society*, 9:128–139, 1947.

[15] N. J. Sloane. A library of hadamard matrices. Accessed Apr 2022. [http://neilsloane.com/hadamard/](http://neilsloane.com/hadamard/)

[16] P. Wang. Orthogonal main-effect plans in row–column designs for two-level factorial experiments. *Communications in Statistics-Theory and Methods*, 46:10685–10691, 2017.