Quantum optimal control via polynomial optimization:
A globally convergent approach

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The problems of optimal quantum control, Hamiltonian identification, and minimal-time control are reformulated as polynomial optimization tasks, where the objective function and constraints are polynomials. The proposed formulations have the unique properties that (i) they have globally convergent solvers and (ii) finding the optimum does not require solving the Schrödinger equation, hence does not depend on the dimension of quantum system. The polynomial formulations are applicable as long as both the Magnus expansion for a quantum propagator and the Chebyshev expansion for the exponential are valid, which are used in the derivation. The proposed formulations offer a new approach to quantum information science and technology.

I. INTRODUCTION

Quantum optimal control [1–6] plays a pivotal role in the development of quantum technologies such as quantum computing, quantum simulations and quantum sensing [7]. A fundamentally important problem specifically in quantum computing [8] is the implementation of quantum operations with high precision and speed while satisfying various constraints [9–11]. The most common quantum control approaches [12–16] are based on numerical optimization with respect to a suitably defined objective function, such as fidelity [17] for example. This is particularly relevant in the NISQ era [18], where the fidelity of quantum computing operations directly translates into quantum advantage. This task, however, is far from trivial and draws on a number of advances in physics, engineering, and applied mathematics. For example, higher fidelity of quantum gates can be achieved by removing sources of decoherence by improved shielding or material engineering for the fabrication of the quantum information processors. Gate fidelity can also be improved by identifying and compensating for various errors [19–21]. This allows for a more accurate execution of quantum algorithms. An important motivation for this work is an observation that optimal quantum control problems, including those used for the purpose of high-fidelity implementation of quantum computing, would benefit from advances in solving non-commutative optimization problems (NCOPs) involved in shaping the pulses that implement the gates.

Here, we reformulate several challenges, including optimal quantum control, time-optimal quantum control, and robust extensions, as polynomial optimization problems, which provably allow for solvers with arbitrarily small error. Our formulation relies on the use of Magnus expansion [22, 23], in combination with Chebyshev approximation of matrix exponentiation [24], and commutative polynomial optimization exploiting certain sparse structure of the problem. Crucially, the minimization of the proposed polynomial formulations does not require discretising Schrödinger equation. Thus, the complexity does not depend on the number of time steps in the discretisation of time. We have implemented this approach in Julia, which we use in our numerical illustrations on transmon qubits.

II. MAIN RESULT: FORMULATION OF QUANTUM OPTIMAL CONTROL AS A POLYNOMIAL OPTIMIZATION PROBLEM

Consider an N-level quantum system described by a Hilbert space $\mathcal{H}$. A unitary evolution operator acting on $\mathcal{H}$ satisfies the Schrödinger equation

$$\partial_t U(t) = A(t)U(t),$$

where the anti-Hermitian operator $A(t)$ is defined as

$$A(t) = -i(H_0 + E(t)V),$$

where $H_0$ and $V$ are the drift and control Hamiltonians, respectively, and $E(t) : [0, T] \to \mathbb{R}$ is a control field. A generalization to the case of multiple controls is straightforward. Furthermore we set $\hbar = 1$ and required the initial condition $U(0) = 1$.

The task of terminal coherent quantum control is to find control $E(t)$ such that for a given terminal time $T$
we approach the target unitary $U^*$, i.e.,

$$\text{minimize } E(t) \quad \|U(T) - U^*\|^2_F,$$

subject to $\partial_t U(t) = A(t)U(t)$,
\quad $U(0) = \mathbb{1}$. \hfill (3)

The choice of the Frobenius norm is dictated by convenience as will become evident later \[see the discussion after Eq. (15)].

To reformulate the coherent control as a polynomial optimization problem is to recall the solution to Eq. (1) can be written via the celebrated Magnus expansion [22]

$$U(T) = \exp \Omega^{(\infty)}, \quad \Omega^{(n)} = \sum_{k=1}^{n} \Omega_k,$$ \hfill (4)

whose first three terms read

$$\begin{align*}
\Omega_1 &= \int_0^T dt_1 A(t_1), \\
\Omega_2 &= \frac{1}{2} \int_0^T dt_1 \int_0^{t_1} dt_2 [A(t_1), A(t_2)], \\
\Omega_3 &= \frac{1}{6} \int_0^T dt_1 \int_0^{t_1} dt_2 \int_0^{t_2} dt_3 \left[ [A(t_1), [A(t_2), A(t_3)]]ight. \\
&\quad \left. + [[A(t_1), A(t_2)], A(t_3)] \right).
\end{align*}$$ \hfill (5-7)

The brackets $[\cdot, \cdot]$ denote the commutator of the underlying Lie algebra of the one-parameter family of operators $\{A(t)\}$, $i = 1, 2, \ldots$.

The Magnus series converges if

$$\int_0^T \|A(t)\|_2 dt < \pi.$$ \hfill (8)

It is noteworthy that by construction $\exp \Omega^{(n)}$ is unitary even for the truncated Magnus series $\Omega^{(n)}$, $n < \infty$.

Since the Frobenius norm is unitary invariant, the objective function in Eq. (3) can be written as

$$\|U(T) - U^*\|^2_F = \left\| \exp \left( \frac{\Omega^{(n)}}{2} - \frac{\Omega^{(n)}}{2} U^* \right) \right\|^2_F.$$ \hfill (9)

Computationally, it is very efficient to represent the matrix exponent via the expansion into the Chebyshev polynomials

$$\begin{align*}
\exp \left( \frac{\Omega^{(n)}}{2} \right) &= \exp_{\infty} \left( \frac{\Omega^{(n)}}{2} \right), \\
\exp_p \left( \frac{\Omega^{(n)}}{2} \right) &= J_0(1/2) \mathbb{1} + 2 \sum_{k=1}^{p} J_k(1/2) T_k,
\end{align*}$$ \hfill (10-11)

where $J_k(x)$ is the Bessel function and $T_k$ is the matrix Chebyshev polynomial defined via the recurrent relation

$$T_0 = \mathbb{1}, \quad T_1 = \Omega^{(n)}, \quad T_{k+1} = 2 \Omega^{(n)} T_k + T_{k-1}.$$ \hfill (12)

The series (10) converges if the spectral radius of $\Omega^{(n)}$ does not exceed one. See also Sec. V A in the Supplementary Material.

Using the triangle inequality, the truncation of the Chebyshev expansion for $\exp$ in Eq. (9) leads to an upper bound on the objective function,

$$\|U(T) - U^*\|^2_F \leq \left\| \exp_p \left( \frac{\Omega^{(n)}}{2} - \frac{\Omega^{(n)}}{2} U^* \right) \right\|^2_F.$$ \hfill (13)

The larger the $p$, the tighter the bound. In Sec. IV below, we use $p = 5$ yielding a nearly machine accuracy for $\exp$.

Hence, the solution to

$$\begin{align*}
\text{minimize } &\left\| \exp_p \left( \frac{\Omega^{(n)}}{2} - \frac{\Omega^{(n)}}{2} U^* \right) \right\|^2_F \\
\text{subject to } &\partial_t U(t) = A(t)U(t), \\
&U(0) = \mathbb{1}.
\end{align*}$$ \hfill (14)

gives an upper bound to the terminal coherent control problem (3).

If we restrict to the polynomial form of control

$$E(t) = \sum_{k=1}^{m} x_k t^{k-1}, \quad x = (x_1, \ldots, x_m) \in \mathbb{R}^m,$$ \hfill (15)

the truncated Magnus expansion $\Omega^{(n)}$ as directly seen from Eqs. (5)-(7) becomes a polynomial of $x$. Also recall that the square of the Frobenius norm of a matrix is a polynomial function of its matrix elements, $\|A\|^2_F = \text{Tr}(A^T A) = \sum_{j,k=1}^{N} A_{jk}^2$.

Hence, the optimal control problem (3) can be approximated by the unconstrained polynomial optimization problem as

$$\begin{align*}
\text{minimize } &\left\| \exp_p \left( \frac{\Omega^{(n)}}{2} - \frac{\Omega^{(n)}}{2} U^* \right) \right\|^2_F. \\
\text{subject to } &\partial_t U(t) = A(t)U(t), \\
&U(0) = \mathbb{1}.
\end{align*}$$ \hfill (16)

The higher the $n$, the better the approximation. In Sec. IV below, we utilize $n = 3$. Once the optimal solution is found, it can be used to verify the convergence condition (8) ensuring the validity of formulation (14).

Instead of the problem of synthesizing the target unitary (3), it is often sufficient to consider a computationally less challenging problem of finding the control $E(t)$ to reach a desired state $|\psi^*(T)\rangle$ at a terminal time $T$ starting from an initial state $|\psi(0)\rangle$,

$$\begin{align*}
\text{minimize } &\|U(T) |\psi(0)\rangle - |\psi^*(T)\rangle\|^2 \\
\text{subject to } &\partial_t U(t) = A(t)U(t), \\
&U(0) = \mathbb{1}.
\end{align*}$$ \hfill (17)

Using the unitary invariance of the vector norm $\| |\psi\rangle \| = \sqrt{\langle\psi|\psi\rangle}$, this task is also reduced to the following polynomial optimization problem

$$\text{minimize } \left\| \exp_p \left( \frac{\Omega^{(n)}}{2} |\psi(0)\rangle - \exp_p \left( \frac{\Omega^{(n)}}{2} |\psi^*(T)\rangle \right) \right. \right.$$ \hfill (18)
Unlike Eq. (17), this formulation does not involve solving the Schrödinger equation explicitly, but only via the Magnus expansion.

Another important class of problems for quantum technology is the time-optimal quantum control \([25,26]\). Its objective is to find the time-optimal evolution and the optimal Hamiltonian for a quantum system and a given pair of initial and final states. In our context, this translates into finding the control \(E(t)\) such that the target unitary \(U^\star\) is reached within the shortest possible time \(T\) with the desired accuracy \(\varepsilon\).

\[
\begin{align*}
\text{minimize} & \quad T \\
\text{subject to} & \quad T \geq 0, \\
& \quad \|U(T) - U^\star\|^2_F \leq \varepsilon^2, \\
& \quad \partial_t U(t) = A(t)U(t), \\
& \quad U(0) = 1.
\end{align*}
\]  

Using the method presented above, this problem can be readily reformulated as the following constrained polynomial optimization problem:

\[
\begin{align*}
\text{minimize} & \quad T \\
\text{subject to} & \quad T \geq 0, \\
& \quad \left\|\exp\left(\frac{\Omega(n)}{2} t\right) - \exp\left(-\frac{\Omega(n)}{2} U^\star\right)\right\|_F^2 \leq \varepsilon^2.
\end{align*}
\]

The quantum brachistochrone problem of minimizing time \(T\) for changing state \(|\psi(0)\rangle\) into \(|\psi^\star\rangle\) also reduces to a constrained polynomial optimization problem:

\[
\begin{align*}
\text{minimize} & \quad T \\
\text{subject to} & \quad T \geq 0, \\
& \quad \left\|\exp\left(\frac{\Omega(n)}{2} \sigma_0\right) - \exp\left(-\frac{\Omega(n)}{2} \sigma^\star\right)\right\|_F^2 \leq \varepsilon^2.
\end{align*}
\]

The latter two polynomial optimization problems do not involve the Schrödinger equation explicitly.

III. POLYNOMIAL OPTIMIZATION

Positivstellensätze and Convexifications

Non-convex optimization problems can have multiple points satisfying first-order optimality conditions, some of which are local optima, whose subset, in turn, are global optima. So-called first-order algorithms provide points satisfying first-order optimality conditions, but little to no global information. In contrast, recently proposed approaches to polynomial optimization utilize Positivstellensätze to form convex relaxations with (asymptotic) guarantees of global optimality.

In real algebraic geometry, Positivstellensätze characterize polynomials that are positive on a semialgebraic set, which can be thought of as real analogues of Hilbert’s Nullstellensatz. Using Putinar’s Positivstellensatz \([27]\), Lasserre and Parrilo \([28]\) obtained their moment/Sum-Of-Squares (SOS) hierarchy, while others have used it to obtain the second-order cone programming hierarchies \([29]\). Using Krivine-Stengle Positivstellensatz \([30]\) has yielded a linear programing hierarchy of historic importance and the more recent bounded-degree SOS hierarchy \([31]\). Generally, each Positivstellensatz can yield multiple hierarchies of relaxations of various properties.

For example, let us consider the moment/SOS hierarchy \([28,32]\) (see \([33]\) for a survey) and the following polynomial optimization problem:

\[
P : \quad f^\star = \min \{ f(x) \mid g_j(x) \geq 0, \quad j = 1, \ldots, m\}
\]

where \(f, g_j \in \mathbb{R}[x]\) are real valued polynomials in \(x \in \mathbb{R}^n\).

By defining the basic semi-algebraic set

\[
K = \{ x \in \mathbb{R}^n \mid g_j(x) \geq 0\},
\]

\(P\) can be reformulated as

\[
P : \quad f^\star = \min \{ f(x) \mid x \in K\}.
\]

\[
P_\lambda : \quad f^\star = \sup \{ \lambda \mid f(x) - \lambda \geq 0, \quad \forall x \in K\}
\]

To show that Prob. \([26]\) is in NP, one requires succinct certificates of positivity of \(f(x) - \lambda\) on \(K\). Lasserre \([34]\) and Parrilo \([35]\) provided a systematic algorithmic procedure to construct these by combining tools from real algebraic geometry and functional analysis. The certificates of positivity on \(K\) can also be implemented to obtain the solution of Prob. \([25]\) in practice.

Specifically, assuming a compact set \(K\), defined as in Eq. \([23]\), Parrilo used Putinar’s Positivstellensatz which states the following: If a polynomial \(f \in \mathbb{R}[x]\) is strictly positive on \(K\) then it can be written as

\[
f(x) = \sigma_0(x) + \sigma_1(x)g_1(x) + \ldots + \sigma_m(x)g_m(x),
\]

for any \(x \in \mathbb{R}^n\) where \(\sigma_j \in \mathbb{R}[x]\) are SOS polynomials. Testing whether \(f(x)\) can be written as in Eq. \([26]\), signifies the desired result \(f(x) > 0\) for any \(x \in K\), amounts to solving an SDP.

The dual of this SDP, which features prominently in the approach of Lasserre \([34]\), is the following problem: given a real sequence \(y = \{y_\alpha\}\) with \(\alpha \in \mathbb{N}^n\), if there exists a probability measure \(\mu\) on \(K\) such that

\[
y_\alpha = \int_K x_\alpha(d\mu),
\]

for any \(\alpha \in \mathbb{N}^n\), then \(y\) represents a measure supported on \(K\). Then, one introduces a linear map \(L_y : \mathbb{R}[x] \to \mathbb{R}\) defined as

\[
f = \sum_{\alpha \in \mathbb{N}^n} f_\alpha x^\alpha \mapsto L_y(f) = \sum_{\alpha \in \mathbb{N}^n} f_\alpha y^\alpha.
\]
$L_y$ is called the Riesz functional. The sequence $y$ above has a representing measure on $K$ if and only if for every polynomial $h \in \mathbb{R}[x]$ it holds that

$$L_y(h^2) \geq 0, \quad L_y(h^2 g_j) \geq 0,$$

for $j = 1, \ldots, m$. If Eq. (29) holds for all $h \in \mathbb{R}[x]$, polynomials in $\mathbb{R}[x]_d$ with maximum degree $d$, certain $m+1$ moment and localizing matrices, with entries linear in $\{y_n\}$, are positive semidefinite:

$$M_d(y) \succeq 0, \quad M_d(g_j y) \succeq 0,$$

for $j = 1, \ldots, m$. The latter problem, Prob. (30), can be modelled as a so-called “generalized moment problem”:

$$\inf_{\mu_i} \left\{ \sum_{i=1}^{\ell} f_i d\mu_i \bigg| \sum_{i=1}^{\ell} h_{i,j} d\mu_i \geq b_j \right\},$$

where $K_i \subset \mathbb{R}^{n_i}$ is defined analogously to Eq. (28).

In either view of the moment/SOS approach, using Putinar’s certificate amounts to solving a hierarchy of semidefinite programs of increasing size. The idea is to replace Prob. (25) with

$$f_d^* = \sup_{\lambda, \sigma_j} \left\{ \lambda \left| \sigma_0 + \sum_{j=1}^{m} \sigma_j g_j \right|, \text{with } \deg(\sigma_j g_j) \leq 2d \right\}.$$

The sequence $\{f_d^*\}$, where $d \in \mathbb{N}$, is monotone and non-decreasing. When $K$ is compact, one obtains the global optimum of Prob. (25) as the limit

$$f^* = \lim_{d \to \infty} f_d^*.$$

where finite convergence is generic. As a result, it has found applications across various fields, including quantum information theory.

**Exploiting sparsity**

Polynomial optimization over a basic semialgebraic set is NP-hard [33]. While the moment/SOS hierarchy of relaxations, which was discussed above, establishes a framework for tackling such problems, they remain hard and despite the popularity and power of the framework, certain limitations to the scalability of the moment/SOS hierarchy exist. Specifically, for the number $n$ of variables of the POP at hand, the size of the matrices appearing in the $d$-th step of the moment/SOS relaxation is of the order of $\binom{n+e}{n}$.

To address this challenge, several proposals have been put forth. Putinar [27] suggested removing monomials not appearing in the moment/SOS decomposition. Lasserre [36] suggested exploiting possible sparsity patterns observed in the polynomial optimization problem.

In some sense, this amounts to re-indexing the SDP matrices involved in the moment/SOS relaxation as follows: consider subsets $I_1, \ldots, I_p \subseteq \{1, \ldots, n\}$ of the initial input SDP variables, with cardinalities $\{|I_1|, \ldots, |I_p|\}$ respectively. The sparse moment/SOS hierarchies [35,33], consider (quasi) block-diagonal SDP matrices where the size of block $j$ is given by $|I_j|$, $1 \leq j \leq p$. With this reformulation of the original SDP, if the cardinalities above are small compared to the initial variables, that is if $|I_j| \ll n$ for all $j$, applying the moment/SOS relaxations achieves significant computational savings, which in turn enables scalability. Under mild assumption, theoretical guarantees on the global convergence of the sparse moment/SOS relaxations exist.

**IV. ILLUSTRATIONS**

**A. Coherent control**

In this section, we provide numerical illustrations for the polynomial formulation (16) of the terminal quantum control problem of synthesizing a desired unitary gate [37,38]. The code used in the analysis is available as a Julia Jupyter notebook [39]. In particular, we utilize the drift and control Hamiltonians of the form

$$H_0 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0.515916 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad V = \begin{pmatrix} 0 & 0.707107 & 0 & 0 \\ 0.707107 & 0 & 0 & 1 \end{pmatrix},$$

which are scaled versions of the Hamiltonians for IBM Q 2-qubit devices. The analysis is started by randomly drawing components of the vector $x^*$ ($m = 3$) from a uniform distribution on the interval $[-1, +1]$. In total, 1000 samples are generated. Then, for each sample, we perform the following steps: Setting $x = x^*$ in Eq. (15), we obtain the control field used to get the target unitary $U^*$ by solving the Schrödinger equation (1). The terminal time is chosen at $T = 0.5$. Hence, we know that $x^*$ is an exact solution to the terminal quantum control problem [3]. Figure 1(A) shows that the inequality (5) is satisfied, therefore justifying the usage of the truncated Magnus series to approximate the evolution operator. The polynomial formulation (16) with $n = 3$ and $p = 5$ is then solved using the TSSOS Julia package [37,38]. This library extracts the minimizer $\hat{x}$, i.e., the value of $x$ yielding the global minimum estimate for (16). It is noteworthy that the obtained global minimum estimate via TSSOS is larger than the value of the objective function (16) at $x^*$ (see Fig. 2) as expected from the monotone convergence property of the hierarchy of SDP relaxations (see Sec. III). Hence, the polynomial optimization yields $\hat{x} \neq x^*$. Notice also the well-known fact that quantum control problems typically have non-unique solutions. Figure 1(B) shows that the Magnus expansion for $\hat{x}$ converges significantly faster than for $x^*$. The control field [Eq. (15)] obtained from $\hat{x}$ generates the evolution operator $U$ that is supposed to be very
FIG. 1: Histogram plots for Magnus series convergence tests [see Eq. (8)] (A) when exact solutions $x^*$ for quantum control problem (3) are used; (B) when global minimizers $\hat{x}$ for (16), obtained by the TSSOS Julia package, are employed.

FIG. 2: Comparing the global minima for (16), obtained by the TSSOS Julia package, with the value of the polynomial objective function (16) when an exact solution $x^*$ for quantum control problem (3) is used.

FIG. 3: A relation between the quantum control problem (3) and its approximate polynomial formulation (16). $\hat{U}$ is the evolution operator obtained via polynomial optimization (16).

close to the target $U^*$. Figure 3 indeed confirms that.

We conclude that the approximate polynomial formulation (16) provides a viable alternative for the original quantum control problem (3).

B. Robust Extensions and Hamiltonian Identification

In practice, we often do not have the perfect specification of the quantum system available. Then, the mis-specified problem could be considered within the robust optimization [40] or robust control [41].

Let us now consider an easy starting point in this direction. For convenience, assume that we know the drift Hamiltonian $H_0$ (34), but we do not know the values of nonzero coupling constants in the interaction Hamiltonian $V$ (34). Hence, $V$ is assumed to be of the form

$$V = \begin{pmatrix} 0 & z_1 & 0 \\ z_1 & 0 & z_2 \\ 0 & z_2 & 0 \end{pmatrix},$$

and our goal is to find the unknown coupling constants $z = (z_1, z_2)$. We note that this a physically important formulation of the problem of Hamiltonian parameter identification. The diagonal elements correspond to eigenenergies of the non-driven quantum system, which are easy to measure, and thence can be assumed to be known accurately. The positions of zeros elements in the interaction Hamiltonian $V$ (34) follows from the symmetry of physical systems, e.g., via the Wigner–Eckart theorem.

To determine the unknown coupling constants $z$, we drive the quantum system with a known control signal $x$, repeatedly, and perform the tomography at the end of the evolution to determine the synthesized unitary $U^*$. 
FIG. 4: An illustration of the effectiveness of polynomial optimization formulation (36) for the problem of Hamiltonian identification. Distribution of the error between the exact $z$ and recovered $\hat{z}$ non-zero element of the interaction Hamiltonian (35).

Experimentally, we show that it is often sufficient to know only a single pair $x \rightarrow U^*$, in order to very accurately recover the unknown couplings $z$ by solving the following unconstrained polynomial optimization problem:

$$\min_z \| \exp_p \frac{Q^{(n)}}{2} - \exp_p \frac{-Q^{(n)}}{2} U^* \|_F^2.$$

(36)

The formulation (36) is a minor modification of the polynomial optimization of quantum control (16). The difference lies in the fact that in the latter, the control signal $x$ is to be found, using a known Hamiltonian (i.e., known $z$). In the former (36), we wish to find parameters of the interaction Hamiltonians, using the known control signal.

To numerically illustrate the effectiveness of formulation (36), we perform an analysis similar to that of Sec. IV A. We start by randomly drawing components of the vector $x$ ($m = 3$) from a uniform distribution on the interval [$-1, +1$]. In total, 1000 samples are generated. Then, for each sample, we perform the following steps: We obtain the terminal unitary $U^*$ at $T = 0.5$ by solving the Schrödinger equation (1) using the exact Hamiltonian (34). The polynomial minimization (36) with $n = 3$ and $p = 5$ is solved via the TSSOS Julia package [37, 38] yielding the estimate $\hat{z}$ of the global minimizer for $z$. The obtained estimate is then refined by performing a local minimization. The code used for this analysis can be found in the accompanying Julia Jupyter notebook [42].

The results are shown in Fig. 4 where the distribution of the errors between the exact couplings $z$ and recovered couplings $\hat{z}$ are shown for all samples. As can be readily seen, the Hamiltonian is recovered with a very high accuracy even from a single pair $x \rightarrow U^*$.

V. CONCLUSIONS

We have shown how to reformulate important problems of quantum engineering in terms of commutative polynomial optimization. This enabled us to utilize global optimization techniques. We have illustrated the effectiveness of the formulations in empirical testing. This could lead to important advances in the problem of designing high-fidelity gates and within the problem of state preparation [43], both of which could contribute towards practical quantum advantage, ultimately.

A natural question arises, as to the applicability of non-commutative optimization (NCOP). Solvers for NCOP have received a considerable interest, recently, following the introduction of the convergent hierarchy of semidefinite programs by Navascués, Pironio and Acín (NPA, [44]). The NPA hierarchy has been developed in the context of characterizing the correlations of observables when studying local measurements on separate quantum systems. Further use-cases arise in quantum state tomography [45] and quantum process tomography [46], when one considers the read-out pulse as an input, and does not assume the number of levels of the quantum systems is known. Application of the NPA hierarchy has also been considered in the context of quantum optimal control [23]. Unfortunately, the scaling of the NPA hierarchy with the problem size seems rather limiting, so far.

This also closes certain questions in the theory of computing. In Ref. [47], quantum control was proven to be uncomputable on Turing machines. The proof was based on the fact that digitized quantum control could solve Diophantine equations, i.e., polynomial equations in unknowns restricted to unbounded integers. The latter problem is well known to be Turing undecidable. The current work implies that switching the computational paradigm from the Turing machine, which models digital computations with a finite precision, to the Blum-Shub-Smale (BBS) machine, which allows for unit-time arithmetic operations with an infinite precision, makes quantum control computable, since there are algorithms for solving polynomial equations on BSS machines. This work shows that such an approach is not only theoretically possible, but also computationally feasible.
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[36] A. Chebysev Polynomials

We summarize basic features of the Chebyshev polynomial expansion. Let $f : \mathbb{R} \to \mathbb{R}$ be a smooth function. For $x \in [-1, 1]$, The Chebyshev polynomials are defined by the relations

$$T_n(x) = 1,$$

$$T_1(x) = x,$$

$$T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x),$$

where $\{T_n\}$ form an orthogonal basis. As a result, $f$ can be expanded as

$$f(x) = \sum_{n=0}^{\infty} b_n T_n(x) \approx \sum_{n=0}^{N} b_n T_n(x),$$

where $b_n$ are the Bessel coefficients corresponding to the $n$-th term of the expansion. We choose to use the expansion via the Chebyshev polynomials due to the special property that when the order of the polynomial becomes larger than its argument, the function decays exponentially fast, favoring convergence.

SUPPLEMENTARY MATERIAL

A. Chebysev Polynomials