DYNAMICS OF SPIKE IN A KELLER-SEGEL'S MINIMAL CHEMOTAXIS MODEL

YAJING ZHANG∗

School of Mathematical Sciences, Shanxi University
Taiyuan, Shanxi 030006, China

XINFU CHEN

Department of Mathematics, University of Pittsburgh
Pittsburgh, PA 15260, USA

JIANGHAO HAO

School of Mathematical Sciences, Shanxi University
Taiyuan, Shanxi 030006, China

XIN LAI

Department of Mathematics, Harbin Institute of Technology
Harbin, Heilongjiang 150001, China

CONG QIN

Center for Financial Engineering, Soochow University
Suzhou, Jiangsu 215006, China

ABSTRACT. The dynamics are studied for the Keller-Segel’s minimal chemotaxis model
\[ \tau u_t = (u_{xx} - kuv_x)_x, \quad v_t = v_{xx} - v + u \]
on a bounded interval with homogeneous Neumann boundary conditions, where \( \tau \geq 0 \) and \( k \gg 1 \) are parameters and the total mass of \( u \) is scaled to be one. In general, the dynamics can be divided into three stages: the first stage is very short in which \( u \) quickly becomes a delta like function with mass concentrated near the point of global maximum of \( v \); in the second stage, the point of the global maximum of \( v \) drifts towards the boundary of the domain and reaches it at the end of the second stage; in the third stage, the profile of the solution evolves to a steady state profile. This paper considers a special case in which the relaxation parameter \( \tau \) is set to be zero, so the first stage takes no time. A free boundary problem describing the second stage is presented. Rigorous asymptotic behavior is proven for the third stage evolution.

2010 Mathematics Subject Classification. Primary: 35B40, 92C17; Secondary: 92D15.
Key words and phrases. Spike dynamics, chemotaxis, potential analysis, asymptotic behavior.
This work is partially supported by NSF DMS-1008905, NNSFC (No. 61374089), China Scholarship Council, NSF of Shanxi Province(No. 2014011005-2), Hundred Talent Program of Shanxi and International Cooperation Projects of Shanxi Province (No. 2014081026).
∗ Corresponding author: Yajing Zhang.
1. **Introduction.** Chemotaxis is movement of an organism in response to a chemical stimulus. Somatic cells, bacteria, and other single-cell or multicellular organisms direct their movements according to certain chemicals in their environment. This is important for bacteria to find food by swimming toward the highest concentration of food molecules, or to flee from poisons. Positive chemotaxis occurs if the movement is toward a higher concentration of the chemical in question; negative chemotaxis occurs if the movement is in the opposite direction.

Various PDE models for chemotaxis have been extensively studied; see the survey papers by Horstmann [23, 24], Hillen and Painter [21], and the references therein. One of the most important phenomenon about chemotaxis is cell aggregation, which typically is modeled by spiky steady states. The pioneering papers that prove the existence of such steady states are Lin, Ni and Takagi [28, 32]; see also [37] for a brief survey on spiky steady states.

One of the simplest and prominent model for chemotaxis is the following Keller and Segel’s [27] (see also Patlak [34]) minimal chemotaxis model, which in its dimensionless form, can be written as

\[
\begin{align*}
\tau u_t &= (u_x - kuv_x)_x \quad \text{in } \Omega \times (0, \infty), \\
v_t &= v_{xx} - v + u \quad \text{in } \Omega \times (0, \infty), \\
u_x &= v_x = 0 \quad \text{on } \partial \Omega \times (0, \infty), \\
\int_{\Omega} u(x, t) \, dx &= 1 \quad \text{for all } t \geq 0,
\end{align*}
\]

where \( \Omega = (0, \ell) \), \( t \) relates the time, \( x \) the space, \( u \) the cell density, and \( v \) the chemo-attractant density; here \( \tau \geq 0 \) and \( k > 0 \) are parameters; the total mass of the cell is scaled to be 1. First observed by Childress and Perkus [12], the minimal chemotaxis model possesses rich and interesting properties. The well-posedness of (1) is established by Nagai [29], Osaki and Yagi [33], and Hillen and Potapov [22]; see also Section 6 of this paper for a simple proof of key a priori estimates needed for global existence.

Note that a steady state of (1) is a solution of the non-local ode

\[
v'' - v + \frac{e^{kv} \int_{\Omega} e^{kv} \, dx}{\int_{\Omega} e^{kv} \, dx} = 0 \quad \text{in } \Omega, \quad v' = 0 \quad \text{on } \partial \Omega; \quad u = \frac{e^{kv} \int_{\Omega} e^{kv} \, dx}{\int_{\Omega} e^{kv} \, dx}.
\]

Since any steady state can be extended evenly and periodically, we can assume without loss of generality that \( \Omega \) is half-period and \( v_x > 0 \) in \( \Omega \). In [22], Hillen and Potapov constructed asymptotically, as \( k \to \infty \), a boundary spike steady state solution, i.e., a family of steady state solutions \( \{ (u(k, x), v(k, x)) \}_{k \geq 0} \) satisfying

\[
\lim_{k \to \infty} \int_{\Omega} u(k, x) \zeta(x) \, dx = \zeta(\ell) \quad \forall \zeta \in C(\bar{\Omega}).
\]

The first term of the expansion of the boundary spike solution of (2) is also derived in [26] by Kang, Kolokolnikov, and Ward. In [37], Wang and Xu proved rigorously the existence of a boundary spike solution of (1). Using a phase plane analysis, Chen, Hao, Wang, Wu, and Zhang established in [8] the existence, uniqueness, and local exponential stability of the spike steady state solution, together with rigorous arbitrary high order internal and boundary layer asymptotic expansions. To obtain more detailed information on the stability of the spike solution, we reinvestigated in [38] the associated eigenvalue problem in a general setting with a systematic method.
In this paper, we focus on the dynamical aspect of the minimal chemotaxis model \(^1\). According to numerical simulations (e.g. [22]), we divide the evolution into three stages:

1. The first stage generates the spike. It is very short in which \(u\) quickly becomes a delta like function with mass concentrated near the points of the global maximum of \(v(k, \cdot, t)\). After the generation, 
   \[
   u(k, \cdot, t) \approx \delta(\cdot - z(t))
   \]
   where \(z(t)\) is the point of global maximum of \(v(k, \cdot, t)\). In particular, if the relaxation parameter \(\tau\) is taken to be zero, then 
   \[
   u = e^{kv} / \int_{\Omega} e^{kv} dx,
   \]
   so this first stage takes no time.

2. Assume for simplicity that \(v(k, \cdot, t)\) admits a unique point of global maximum. Then in the second stage, the position, \(z(t)\), of the point of the global maximum of \(v(k, \cdot, t)\) drifts towards the boundary of the domain; see Figure 1. The second stage ends when \(z(t)\) reaches the boundary.

3. In the third stage, the profile of the solution evolves to the steady state profile, i.e., the solution of (2); see Figure 1.

---

\(^1\)We point out that our division of stages is different from that stated in [22, p1795].
derived that after a certain amount of time in the second stage, a quasi-steady state for \( v \) is reached:

\[ v(k, \cdot, t) \approx G(\cdot, z(t)), \]  

(3)

where \( G(x, y) \) is the Green’s function associated with the operator \(-\partial_{xx} + 1\) with homogeneous Neumann boundary condition; more precisely,

\[ G(x, y) = \frac{1}{\sinh \ell} \begin{cases} \cosh x \cosh(\ell - y) & \text{when } x < y, \\ \cosh y \cosh(\ell - x) & \text{when } x \geq y. \end{cases} \]  

(4)

Remarkably, they derived the law of the motion of the spike, i.e., equation (2.45) in [26], which, under the current notation, can be expressed as

\[ \frac{1}{2} \frac{dz(t)}{dt} \approx \frac{\sinh(2z(t) - \ell)}{\sinh \ell} \left. \frac{dG(x, x)}{dx} \right|_{x = z(t)} \quad (\text{if } k \gg 1, \ell \gg 1). \]  

(5)

When \( \ell \gg 1 \), for each \( \xi \in \Omega, G(\cdot, \xi) \) is a “quasi” steady state, so the spike moves slowly. For rigorous analysis of slow (meta-stable) dynamics, a successful example is for the Allen-Cahn equation by Fife and McLeod [14], Fife and Hsiao [13], Carr and Pego [6], Fusco [15], and Fusco and Hale [16]; see also Chen [7] for the evolution of all stages, i.e., generation, evolution, annihilation, and subsequent motion to equilibrium. Similar analysis can also be found for the (viscous) Cahn-Hilliard equation [1, 5, 2, 3, 36] and Gierer-Meinhardt system [9, 10].

We are interested in rigorous verification of the formal derivation of the dynamics. Since \( (1) \) is a rather complicated system, we shall consider only the simple case when the relaxation parameter \( \tau = 0 \), so the first stage takes no time. Hence, \( (1) \) with \( \tau = 0 \) and supplement of an initial condition can be written as, for \( v = v(k, x, t) \),

\[
\begin{cases}
    v_t = v_{xx} - v + \frac{k^v}{\int_\Omega e^{kv} dx} e^{kv} & \forall x \in \Omega, t > 0, \\
    v_x(k, 0, t) = 0, \quad v_x(k, \ell, t) = 0 & \forall t > 0, \\
    v(k, x, 0) = v_0(x) & \forall x \in \Omega.
\end{cases}
\]  

(6)

We shall assume that \( v_0 \) is a non-constant function that does not depend on \( k \). We study the asymptotic behavior of the solution as \( k \to \infty \).

Extending the formal analysis in [26] where \( \ell \gg 1 \), here we consider the case when \( \ell \) is not necessarily large. When \( \ell \) is not large, \( G(\cdot, z(t)) \) is no longer a quasi-steady state, so the approximation \( (3) \) does not hold. We can formally derive (with derivation omitted here) the following:

1. Assume that \( v_0 \) has exactly one local maximum, located at \( z_0 \in (0, \ell) \). Then

\[
\lim_{k \to \infty} v(k, x, t) = w(x, t) \quad \forall x \in [0, \ell], t \in [0, T)
\]

where \( w \), together with \( z \) and \( T \), form the solution of the following free boundary problem:
We propose the implementation of numerically reliable algorithm for (1), for large enough $k = 100$ and $\Delta x^2$. A numerical simulation confirming the convergence of the solution of (1) to the solution of (7) is shown in Figure 2.

We expect to prove rigorously that when $\ell \gg 1$, the function $w(x,t) := G(x,z(t))$ with $z(t)$ being the solution of (5) with initial condition $z(0) = z_0$ approximates the solution of (7) up to $T$ at which $z$ reaches the boundary.

2. Assume for simplicity that $T < \infty$ and that $w(\cdot,T^-)$ is increasing. Then

$$\lim_{k \to \infty} v(k,x,t) = w(x,t) \quad \forall x \in [0,\ell], t \in [T,\infty),$$

where $w$ is the solution of the initial boundary value problem

$$\begin{cases}
  w_t = w_{xx} - w, & \forall x \in \Omega, t > T, \\
  w_x(0,t) = 0, & \forall t \in (0,T), \\
  w_x(\ell,t) = 1, & \forall t > T, \\
  w(x,T) = w(x,T^-) & \forall x \in \Omega.
\end{cases}$$

A numerical simulation confirming the convergence of the solution of (1) to the solution of (7) is shown in Figure 2.

We shall prove the evolution limit system (7) in a subsequent paper. In this paper we prove only the limit system (8). For this, we assume for simplicity that $T = 0$ and $w(\cdot,T^-) = v_0$ is a non-constant increasing function. We shall prove that

$$\lim_{k \to \infty} \|v(k,\cdot) - v(\cdot)\|_{L^\infty(0,\ell)} = 0, \quad \lim_{t \to \infty} \|w(\cdot,t) - G(\cdot,t)\|_{H^1(\Omega)} = 0.$$

In our proof, we discover a new $L^\infty(0,\ell; W^{1,p}(\Omega))$ estimate. As an application, we shall use this estimate to provide a simple argument to establish a time independent a priori estimate for the solution of (1), improving those from [33, 22].

The rest of the paper is organized as follows. In Section 2 we state our main result about the convergence of $v$ to $w$ as $k \to \infty$, along with the idea of the proof. In Section 3, we prove the local in time convergence, using a potential analysis for an $L^\infty(0,\ell; W^{1,p}(\Omega))$ estimate. Observed that both (6) and (8) are gradient flows, we study in Section 4 the energy functionals and related estimates. We complete the proof of the convergence of the solution of (6) to the solution of (8) in Section 5. Finally in Section 6, we use our $L^\infty((0,\infty); W^{1,p}(\Omega))$ estimate to establish a time independent a priori estimate for the solution of (1), a key estimate needed for the well-posedness of (1).

\footnote{Due to computational limitation and non-sophistication of our algorithm, we cannot take large enough $k$ to make the solutions of (1) and (7)+(8) match perfectly. Indeed, since $k \Delta x$, with $k = 100$ and $\Delta x = \ell/400$ [and $\Delta t = \frac{1}{2}(\Delta x)^2$], is not small, our numerical scheme for (1) is not highly accurate. We propose the implementation of numerically reliable algorithm for (1), for large $k$, as an open problem.}
2. Main result and idea of the proof. To present our result, we introduce, for $p \geq 1$ and $k > 0$,

\[
\|\phi\|_{W^{1,p}(\Omega)} = \left( \int_{\Omega} \left[ |\phi|^p + |\phi_x|^p \right] dx \right)^{1/p}, \quad H^1(\Omega) = W^{1,2}(\Omega),
\]

\[
E_k[\phi] = \frac{1}{2} \int_{\Omega} \left[ \phi^2 + \phi_x^2 \right] dx - \frac{1}{k} \ln \left( \frac{1}{|\Omega|} \int_{\Omega} e^{k\phi} dx \right),
\]

\[
E[\phi] = \frac{1}{2} \int_{\Omega} \left[ \phi^2 + \phi_x^2 \right] dx - \max_{\Omega} \phi.
\]

Note that

\[
\frac{1}{2} \|\phi\|_{H^1(\Omega)}^2 - \max_{\Omega} \phi = E[\phi] \leq E_k[\phi] \leq \frac{1}{2} \|\phi\|_{H^1(\Omega)}^2 - \min_{\Omega} \phi.
\]

Our main result is the following:
Theorem 2.1. Let $v_0 \in H^1(\Omega)$ be a given non-constant increasing function and $w$ be the solution of
\[
\begin{cases}
  w_t = w_{xx} - w & \forall x \in \Omega, t > 0, \\
  w_x(0, t) = 0, \quad w_x(\ell, t) = 1 & \forall t > 0, \\
  w(x, 0) = v_0(x) & \forall x \in \Omega.
\end{cases}
\]
(11)

For each $k > 0$, let $v(k, \cdot, \cdot)$ be the solution of problem (4). Then
\[
\lim_{k \to \infty} \sup_{t \geq 0} \|v(k, \cdot, t) - w(\cdot, t)\|_{W^{1, p}(\Omega)} = 0 \quad \forall p \in [1, \infty),
\]
(12)
\[
\lim_{k \to \infty} \sup_{t \geq 0} |E_k[v(k, \cdot, t)] - E[w(\cdot, t)]| = 0,
\]
(13)
\[
\lim_{k \to \infty} \|v(k, \cdot, \cdot) - w_k(\cdot, \cdot)\|_{L^2(\Omega \times (0, \infty))} = 0.
\]
(14)

Remark 1. The estimate (12) is optimal since $w_x - v_x = 1$ at $x = \ell$.

Now we outline the proof of Theorem 2.1. We first establish the local in time convergence:
\[
\lim_{k \to \infty} \|v(k, \cdot, \cdot) - w(\cdot, \cdot)\|_{L^\infty(0, T; W^{1, p}(\Omega))} = 0 \quad \forall T > 0, p > 1.
\]
(15)

By a potential analysis, this suffices to show that
\[
u(k, \cdot, t) := \frac{e^{k v(k, \cdot, t)}}{\Omega e^{k v(k, \cdot, t)} dy}
\]
approaches a delta function concentrated at $\ell$. This is established by finding a positive lower bound of $v_x$, using maximum and comparison principles.

For global in time convergence, we first use the fact that (6) and (11) are gradient flows with energies $E_k$ and $E$ defined in (9) and (10), respectively. Indeed, we have the identities
\[
\frac{d}{dt} E_k[v(k, \cdot, t)] = -\int_{\Omega} v^2 dx, \quad \frac{d}{dt} E[w(\cdot, t)] = -\int_{\Omega} w^2 dx.
\]
(16)

Next by solving the minimization problems of the associated energy functionals of increasing functions, we find that
\[
E_k[\phi] - E[w^*] \geq E[\phi] - E[w^*] \geq \frac{1}{2} \|\phi - w^*\|_{H^1(\Omega)}^2
\]
(17)
where
\[
w^*(x) = G(x, \ell) = \frac{\cosh x}{\sinh \ell}
\]
is the global minimizer of $E$.

Finally, we notice that $w^*$ is the unique equilibrium of (11), so
\[
\lim_{t \to \infty} \|w(\cdot, t) - w^*(\cdot)\|_{H^1(\Omega)} = 0.
\]
(18)

Then from (15), (16), and (18) we find that
\[
\lim_{k \to \infty, t \to \infty} \left( E_k[v(k, \cdot, t)] - E[w^*] \right) \leq 0.
\]

It then follows from (17) that $v(k, \cdot, t)$ approaches $w^*$ in $H^1(\Omega)$ as $k$ and $t$ approach infinity. Using the boundedness of $v_x$ in $L^{p+1}$ and the local convergence, we then obtain (12).

The rest assertions of Theorem 2.1 can be proven by using energy identities and the convergence of energies.
3. Local in time convergence. In this section, we prove the local in time convergence, i.e., \([15]\). We shall first use Green’s formula to show that the convergence of \(v\) to \(w\) is equivalent to the convergence of \(u\) to the delta function concentrated on the boundary \(x = \ell\). Then we use comparison to establish a positive lower bound of \(v_x\) and show the smallness of \(u\) in \([0, \ell - \eta] \times [\eta^2, \eta^{-2}]\) for any fixed small positive \(\eta\). Finally, we establish \([15]\). In the meanwhile, we establish a \(W^{1,p}\) bound of \(v\), global in time.

3.1. The Green’s representation. Denote by \(\Gamma(x, y, t)\) the Green’s function associated with the linear differential operator \(\partial_t - \partial_{xx} + 1\) equipped with the Neumann boundary condition: for each \(y \in \Omega\),

\[
\begin{cases}
\Gamma_t - \Gamma_{xx} + \Gamma = 0 & \forall x \in \Omega, t > 0, \\
\Gamma_x(0, y, t) = 0, \quad \Gamma_x(\ell, y, t) = 0 & \forall t > 0, \\
\Gamma(x, y, 0) = \delta(x - y) & \forall x \in \Omega,
\end{cases}
\]

where \(\delta\) is the delta function. Then by Green’s formula, the solutions of \([11]\) and \([1]\) are given by

\[
v(k, x, t) = \int_{\Omega} \Gamma(x, y, t)v_0(y)dy + \int_0^t \int_{\Omega} \Gamma(x, y, t-s)u(k,y,s)dyds,
\]

\[
w(x,t) = \int_{\Omega} \Gamma(x, y, t)v_0(y)dy + \int_0^t \int_{\Omega} \Gamma(x, \ell, t-s)ds.
\]

Since \(\int_{\Omega} u(k, y, s)dy = 1\), we find that

\[
v(k, x, t) - w(x, t) = \int_0^t \int_{\Omega} \left[ \Gamma(x, y, t-s) - \Gamma(x, \ell, t-s) \right] u(k, y, s)dyds.
\]

Hence, to show the convergence \(v \to w\), we need only show that for each \(s > 0\), \(u(k, x, s)\) approaches a delta function with mass concentrated at \(x = \ell\), i.e., \(u(k, \cdot, s)\) is small in \([0, \ell - \eta]\) for any fixed small positive \(\eta\) and \(k \gg 1\).

3.2. Non-degeneracy of the maximum of \(v\) at \(x = \ell\).

**Lemma 3.1.** Assume that \(v_0 \in H^1(\Omega), \ v'_0 \geq 0\) and \(v'_0 \neq 0\). Then for each \(k \geq 0\),

\[v_x(k, x, t) > v'_0(x, t) > 0 \quad \forall x \in \Omega, t > 0,\]

where \(v^0(x, t) = \int_{\Omega} \Gamma(x, y, t)v_0(y)dy\) is the unique solution of the initial value problem

\[
\begin{cases}
(\partial_t - \partial_{xx} + 1)v^0 = 0 & \forall x \in \Omega, t > 0, \\
v^0_x(0, t) = 0, \ v^0_x(\ell, t) = 0 & \forall t > 0, \\
v^0(x, 0) = v_0(x) & \forall x \in \Omega.
\end{cases}
\]

**Proof.** Note that \(v_x\) solves the “linear” initial boundary value problem

\[
\begin{cases}
(\partial_t - \partial_{xx} + 1 - ku)v_x = 0 & \forall x \in \Omega, t > 0, \\
v_x(k, 0, t) = 0, \ v_x(k, \ell, t) = 0 & \forall t > 0, \\
v_x(k, x, 0) = v'_0(x) > 0 & \forall x \in \Omega.
\end{cases}
\]

It then follows by the maximum principle that

\[v_x(k, x, t) > 0 \quad \forall x \in \Omega, t > 0.\]
Now set \( \hat{v} = v_x - v_x^0 \). One finds that \( \hat{v}_t - \hat{v}_{xx} + \hat{v} = kvu_x > 0 \) in \( \Omega \times (0, \infty) \) and \( \hat{v} = 0 \) on the parabolic boundary of \( \Omega \times (0, \infty) \). Hence, by the maximum principle, \( \hat{v} > 0 \) in \( \Omega \times (0, \infty) \). Finally, applying the maximum principle to the equation for \( v_x^0 \), we find that \( v_x^0 > 0 \) in \( \Omega \times (0, \infty) \). The assertion of the lemma thus follows. \( \square \)

3.3. The function \( u \). Here we show that for each fixed \( t > 0 \), \( u(k, \cdot, t) \) approaches a delta function as \( k \to \infty \). Since \( u \geq 0 \), \( \int_\Omega u(k, y, t)dy = 1 \), and \( u_x = kvu_x \geq 0 \), we need only estimate the smallness of \( u(k, \ell - \eta, t) \) for any fixed small positive \( \eta \).

Let \( \eta \in (0, \ell) \) be an arbitrary constant. For \( t > 0 \), as \( v_x(k, \cdot, t) > 0 \) in \( \Omega \),

\[
\int_\Omega e^{kv(k, y, t)} dy > \int_{\ell - \eta/2}^\ell e^{kv(k, y, t)} dy > \frac{\eta}{2} e^{kv(k, \ell - \eta/2, t)}.
\]

In addition, as \( v_x(k, \cdot, t) > v_x^0(\cdot, t), v(k, \ell - \eta/2, t) - v(k, \ell - \eta, t) \geq v^0(\ell - \eta/2, t) - v^0(\ell - \eta, t) \). It then follows that, when \( x \in [0, \ell - \eta] \) and \( t > 0 \),

\[
0 < u(k, x, t) \leq \frac{e^{kv(k, \ell - \eta, t)}}{\int_\Omega e^{kv(k, x, t)} dy} \leq \frac{2e^{-k[v^0(\ell - \eta/2, t) - v^0(\ell - \eta, t)]}}{\eta}.
\]

This implies that, since \( v_x^0 > 0 \) in \( \Omega \times (0, \infty) \),

\[
\lim_{k \to \infty} \|u(k, \cdot, \cdot)\|_{L^\infty([0, \ell - \eta] \times [\eta^2, \eta^2 - \eta^2])} = 0 \quad \forall \eta \in (0, \min\{\ell, 1\}). \tag{21}
\]

3.4. An \( L^\infty(0, \infty; L^p(\Omega)) \) bound of \( v_x - w_x \). Fix \( t > 0 \). We estimate the \( W^{1,p}(\Omega) \) norm of \( \zeta(\cdot) = v(k, \cdot, t) - w(\cdot, t) \). Define \( u(k, \cdot, s) \equiv 0 \) for \( s < 0 \). Then from \( \text{[19]} \),

\[
\zeta(x) = \int_0^\infty \int_\Omega \left\{ \Gamma(x, y, s) - \Gamma(x, \ell, s) \right\} u(k, y, t-s) dyds,
\]

\[
\zeta_x(x) = \int_0^\infty \int_\Omega \left\{ \Gamma_x(x, y, s) - \Gamma_x(x, \ell, s) \right\} u(k, y, t-s) dyds.
\]

For \( p \in [1, \infty) \), we estimate the \( L^p(\Omega) \) norm of \( \zeta_x \). We have, by the Minkowski technique,

\[
\int_\Omega |\zeta_x(x)|^p dx \leq \int_0^\infty \int_\Omega u(k, y, t-s) \int_\Omega |\zeta_x(x)|^{p-1} |\Gamma_x(x, y, s) - \Gamma_x(x, \ell, s)| dx dy ds
\]

\[
\leq \|\zeta_x\|_{L^p(\Omega)}^{p-1} \int_0^\infty \int_\Omega u(k, y, t-s) \|\Gamma_x(\cdot, y, s) - \Gamma_x(\cdot, \ell, s)\|_{L^p(\Omega)}^p dy ds.
\]

Hence,

\[
\|\zeta_x\|_{L^p(\Omega)} \leq 2 \int_0^\infty \sup_{z \in \Omega} \|\Gamma_x(\cdot, z, s)\|_{L^p(\Omega)} \int_\Omega u(k, y, t-s) dy ds
\]

\[
\leq 2 \int_0^\infty \sup_{z \in \Omega} \|\Gamma_x(\cdot, z, s)\|_{L^p(\Omega)} ds.
\]

We now show that the last quantity is finite.

First we consider the case \( \Omega = \mathbb{R} \). Then \( \Gamma(x, y, t) = e^{-t} K(x - y, t) \) where

\[
K(z, s) = (4\pi s)^{-1/2} e^{-z^2/4s}.
\]
Lemma 3.2. For every

\[\int_0^\infty \sup_{z \in \Omega} \| \Gamma_x(\cdot, z, s) \|_{L^p(\Omega)} \, ds = \int_0^\infty \frac{e^{-s}}{4\sqrt{\pi} s^{3/2}} \left( \int_\mathbb{R} |z|^p e^{-pz^2/4} \, dz \right)^{1/p} \, ds = \left( \int_\mathbb{R} |x|^p e^{-px^2/4} \, dx \right)^{1/p} \int_0^\infty \frac{s^{3/2 - p} e^{-s}}{4\sqrt{\pi}} \, ds < \infty.\]

In the general case \( \Omega = (0, \ell) \), for \( x \in [0, \ell] \) and \( y \in [0, \ell] \), we can express \( \Gamma \) as

\[\Gamma(x, y, s) = e^{-s} \left\{ K(x - y, s) + K(x + y, s) + K(x + y - 2\ell, s) + H(x, y, s) \right\}, \tag{22}\]

where \( H \) is the solution of the heat equation \( H_t = H_{xx} \) in \( \Omega \times (0, \infty) \) subject to the initial and boundary conditions

\[H(x, y, 0) = 0, \quad H_y(0, y, t) = \frac{y - 2\ell}{4\sqrt{\pi t^3}} e^{-\frac{(y - 4\ell)^2}{4t}}, \quad H_y(\ell, y, t) = \frac{\ell + y}{4\sqrt{\pi t^3}} e^{-\frac{(\ell + y)^2}{4t}}.\]

It is easy to see that \( H \) is a smooth function with bounded derivatives of arbitrary order. Hence, for each \( p \in [1, \infty) \), we have

\[\sup_{z \in \Omega} \| \Gamma_x(\cdot, z, s) \|_{L^p(\Omega)} \leq C(p, \ell) \left[ s^{\frac{3}{2} - \frac{1}{p}} + 1 \right] e^{-s}. \tag{23}\]

Consequently,

\[\int_0^\infty \sup_{z \in \Omega} \| \Gamma_x(\cdot, z, s) \|_{L^p(\Omega)} \, ds < \infty \quad \forall p \in [1, \infty).\]

After a similar estimate for the \( L^p(\Omega) \) norm of \( \zeta \), we obtain the following:

**Lemma 3.2.** For every \( p \geq 1 \) there exists a constant \( C(p, \ell) \) such that

\[\sup_{k > 0, t \geq 0} \| v(k, \cdot, t) - w(\cdot, t) \|_{W^{1, p}(\Omega)} \leq C(p, \ell).\]

Later on we shall use, with \( q = 1 \), the estimate

\[\sup_{z \in \Omega} \| D^2 \Gamma(\cdot, z, s) \|_{L^q(\Omega)} \leq C(q, \ell) \left[ s^{-\frac{1}{2} + \frac{1}{4q}} + 1 \right] e^{-s}, \tag{24}\]

where \( D^2 \) represents all second order partial derivatives of \( \Gamma(x, y, s) \) with respect to \( x \) and \( y \). The proof is analogous to that for (23).

### 3.5. The local in time \( W^{1, p}(\Omega) \) convergence

Integrating the differential equation \( (v - w)_t - (v - w)_{xx} + (v - w) = u \) over \( \Omega \) and using the boundary condition \( w_x(\ell, t) = 1 = \int_\Omega u \, dx \), one can derive that

\[\int_\Omega [v(k, x, t) - w(x, t)] \, dx = 0 \quad \forall k > 0, t \geq 0.\]

Thus, to show the local in time convergence of \( v \) to \( w \) in the \( W^{1, p}(\Omega) \) norm, by Lemma 3.2, we need only show the \( L^1 \) convergence of \( v_x \) to \( w_x \). For this, we fix \( t \) and define \( \zeta(x) = v(k, x, t) - w(x, t) \) as in the previous subsection. For any fixed
small positive \( \eta \), we write \( \zeta_x = \zeta_1 + \zeta_2 + \zeta_3 \) where

\[
\zeta_1(x) = \left( \int_0^{\eta^2 \wedge (t-\eta^2)^2} \sup_{\eta^2 \wedge (t-\eta^2)^2} \left\| \Gamma_x(\cdot, z, s) \right\|_{L^1(\Omega)} ds \right)^\frac{1}{2}.
\]

\[
\zeta_2(x) = \int_{\eta^2 \wedge (t-\eta^2)^2}^t \int_{\eta^2 \wedge (t-\eta^2)^2}^{t-\eta} \left\{ \Gamma_x(x, y, s) - \Gamma_x(x, \ell, s) \right\} u(k, y, t-s) dy ds,
\]

\[
\zeta_3(x) = \int_{\eta^2 \wedge (t-\eta^2)^2}^t \int_{\eta^2 \wedge (t-\eta^2)^2}^{t-\eta} \left( \int_{t-\eta}^{t} \Gamma_{x,z}(x, z, s) dz \right) u(k, y, t-s) dy ds,
\]

where \( A \lor B := \max\{A, B\} \) and \( A \land B := \min\{A, B\} \). Denote by \( O(1) \) a generic constant depending only on \( \ell \). Then, by (23) and (24),

\[
\left\| \zeta_1 \right\|_{L^1(\Omega)} \leq 2 \left( \int_0^{\eta^2 \wedge (t-\eta^2)^2} \sup_{\eta^2 \wedge (t-\eta^2)^2} \left\| \Gamma_x(\cdot, z, s) \right\|_{L^1(\Omega)} ds \right)^\frac{1}{2} = O(1) \left( \int_0^{\eta^2 \wedge (t-\eta^2)^2} s^{-1/2} e^{-s} ds \right) = O(1) \eta,
\]

\[
\left\| \zeta_2 \right\|_{L^1(\Omega)} \leq 2 \ell \left\| u(k, \cdot, \cdot) \right\|_{L^\infty([0, \ell-\eta]) \times [\eta^2 \wedge (t-\eta^2)^2]} \int_0^\infty \sup_{\eta^2 \wedge (t-\eta^2)^2} \left\| \Gamma_x(\cdot, z, s) \right\|_{L^1(\Omega)} ds = O(1) \left\| u(k, \cdot, \cdot) \right\|_{L^\infty([0, \ell-\eta]) \times (0, \eta^2 \wedge (t-\eta^2)^2]} \eta^2 \wedge (t-\eta^2)^2 ds = O(1) \eta [1 + |\ln \eta|].
\]

This implies that there exists a positive constant \( C(\ell) \) such that when \( t \in [\eta, \eta^{-2}] \),

\[
\left\| \zeta_x \right\|_{L^1(\Omega)} \leq C(\ell) \left\{ \eta [1 + |\ln \eta|] + \left\| u(k, \cdot, \cdot) \right\|_{L^\infty([0, \ell-\eta]) \times (0, \eta^2 \wedge (t-\eta^2)^2]} \right\}.
\]

First sending \( k \to \infty \) then sending \( \eta \downarrow 0 \) we then obtain from (21) that

\[
\lim_{k \to \infty} \sup_{0 \leq t \leq T} \left\| \psi_x(k, \cdot, t) - w(\cdot, t) \right\|_{L^1(\Omega)} = 0 \quad \forall T > 0.
\]

From this \( L^1 \) convergence, the \( L^{p+1} \) estimate in Lemma 3.2 and the Hölder inequality

\[
\left\| \zeta_x \right\|_{L^p(\Omega)}^p \leq \left\| \zeta_x \right\|_{L^1(\Omega)} \left\| \zeta_x \right\|_{L^{p+1}(\Omega)}^{p-1},
\]

we then obtain the following:

**Lemma 3.3.** For each \( p \in [1, \infty) \) and \( T > 0 \),

\[
\lim_{k \to \infty} \left\| \psi(k, \cdot, t) - w(\cdot, t) \right\|_{L^\infty([0, T]; W^{1, p}(\Omega))} = 0.
\]

**3.6. The convergence of the energy.**

**Lemma 3.4.** For every \( T > 0 \)

\[
\lim_{k \to \infty} \sup_{t \in [0, T]} \left| E_k[v(k, \cdot, t) - E[w(\cdot, t)] = 0.
\]
Proof. By Lemma 3.3 and the definition of $E$, we have
\[
\lim_{k \to \infty} \sup_{t \in [0, T]} \|E[v(k, \cdot, t)] - E[w(\cdot, t)]\| = 0.
\]
To complete the proof, we need only estimate the difference of $E_k[v] - E[v]$. For this, using $\max_{\bar{\Omega}} v(k, \cdot, t) = v(k, \ell, t)$ we have
\[
0 > E[v] - E_k[v] = \frac{1}{k} \ln \left( \frac{1}{|\Omega|} \int_{\Omega} e^{k[v(k, x, t) - v(k, \ell, t)]} dx \right)
\geq \frac{1}{k} \ln \left( \frac{\eta}{|\Omega|} e^{k[v(k, \ell - \eta, t) - v(k, \ell, t)]} \right)
= v(k, \ell - \eta, t) - v(k, \ell, t) + \frac{1}{k} \ln \frac{\eta}{\ell}.
\]
It then follows from Lemma 3.3 that for each $T > 0$,
\[
\lim_{k \to \infty} \sup_{t \in [0, T]} \|E_k[v] - E[v]\| \leq \sup_{t \in [0, T]} |w(\ell - \eta, t) - w(\ell, t)|.
\]
Sending $\eta \to 0$, we then obtain the assertion of Lemma 3.4.

4. Energy estimates. In this section we first establish energy identities for solutions of (6) and (11). Then we find minimizers of the associated energy functionals. Finally, we estimate the $H^1(\Omega)$ distance of a generic function to the minimizer in terms of the distance of their energies.

4.1. Energy identities. Assume that $v$ is the solution of (6). Define $E_k[v]$ as (9). Then
\[
\frac{d}{dt} E_k[v(k, \cdot, t)] = \int_{\Omega} \left[ v_x v_x + v v_t \right] dx - \frac{1}{k} \int_{\Omega} e^{kv} v_t dx
\leq \int_{\Omega} v_t \left\{ - v_{xx} + v - \frac{e^{kv}}{\int_{\Omega} e^{kv} dy} \right\} dx = - \int_{\Omega} v_t^2 dx.
\]
Similarly, for solution $w$ of (11), one can show by maximum principle that $w_x > 0$ in $\Omega \times (0, \infty)$. Hence, $\max_{\Omega} w(\cdot, t) = w(\ell, t)$. Consequently, using $w_x = 1$ at $x = \ell$ we obtain
\[
\frac{d}{dt} E[w(\cdot, t)] = \int_{\Omega} \left[ w_x w_{xx} + w w_t \right] dx - w_t(\ell, t)
= \int_{\Omega} \left[ - w_{xx} + w \right] w_t dx = - \int_{\Omega} w_t^2 dx.
\]
Thus, we have the following

Lemma 4.1. For every $t \geq s \geq 0$,
\[
E_k[v(k, \cdot, t)] + \int_s^t \|v_t(\tau)\|_{L^2(\Omega)}^2 d\tau = E_k[v(k, \cdot, s)],
\]
\[
E[w(\cdot, t)] + \int_s^t \|w_t(\cdot, \tau)\|_{L^2(\Omega)}^2 d\tau = E[w(\cdot, s)].
\]
4.2. **Energy minimizers.** Here we find a lower bound of $E$. We introduce, for $\xi \in \Omega$ and $v \in H^1(\Omega)$,

$$E[\xi, \phi] = \frac{1}{2} \int_\Omega (\phi^2_x + \phi^2)dx - \phi(\xi) = \frac{1}{2} \|\phi\|^2_{H^1(\Omega)} - \phi(\xi).$$

(25)

One can show that

$$\inf_{\phi \in H^1(\Omega)} E[\phi] = \inf_{\xi \in \Omega} \inf_{\phi \in H^1(\Omega)} E[\xi, \phi].$$

Now for each fixed $\xi \in \bar{\Omega}$, we find minimizers of $E[\xi, \cdot]$ in $H^1(\Omega)$. For any $\phi \in H^2(\Omega)$ and $\varphi \in H^1(\Omega)$, we find the variation of $E[\xi, \cdot]$ at $\phi$ in the direction $\psi$ by the following:

$$\left< \frac{\delta E[\xi, \phi]}{\delta \phi}, \psi \right> = \int_\Omega (\phi_x \psi_x + \phi \psi)dx - \psi(\xi)$$

$$= \phi_x \psi|_0^\ell + \int_\Omega \left[ -\phi_{xx} + \phi - \delta(x - \xi) \right]dx,$$

where $\delta$ is the Dirac mass. Hence, $\left< \frac{\delta E[\xi, \phi]}{\delta \phi}, \psi \right> = 0$ for any $\psi \in H^1(\Omega)$, if $\phi$ is the solution of

$$\begin{cases} 
-\phi'' + \phi = \delta(\cdot - \xi) & \text{ in } \Omega, \\
\phi'(0) = 0, \quad \phi'(\ell) = 0 & \text{ on } \partial\Omega. 
\end{cases}$$

(26)

This problem admits a unique solution given by $\phi(x) = G(x, \xi)$ where $G$ is the Green’s function given in (4). In addition, with $\phi = G(\cdot, \xi)$,

$$E[\xi, \phi + \psi] - E[\xi, \phi] = \left< \frac{\delta E[\xi, \phi]}{\delta \phi}, \psi \right> + \frac{1}{2} \|\psi\|^2_{H^2(\Omega)} = \frac{1}{2} \|\psi\|^2_{H^2(\Omega)}.$$ 

Thus, $G(\cdot, \xi)$ is the only minimizer of $E[\xi, \cdot]$ in $H^1(\Omega)$ and

$$\min_{\phi \in H^1(\Omega)} E[\xi, \phi] = E[\xi, G(\cdot, \xi)] = -\frac{1}{2} G(\xi, \xi) = -\frac{\cosh(\ell - \xi) \cosh \xi}{2 \sinh \ell}.$$ 

Consequently,

$$\min_{\phi \in H^1(\Omega)} E[\phi] = \min_{\xi \in \Omega} \min_{\phi \in H^1(\Omega)} E[\xi, \phi] = \min_{\xi \in \Omega} \left\{ -\frac{1}{2} G(\xi, \xi) \right\}$$

$$= -\frac{1}{2} \coth \ell = E[G(\cdot, \ell)] = E[G(\cdot, 0)].$$

Thus, $E$ admits exactly two minimizers, $G(\cdot, 0)$ and $G(\cdot, \ell)$. We summarize our calculation by the following:

**Lemma 4.2.** The functions $G(\cdot, 0)$ and $G(\cdot, \ell)$ are the only minimizers of $E$ in $H^1(\Omega)$.

4.3. **The distance to the energy minimizer.**

**Lemma 4.3.** Let $w^*(x) = G(x, \ell)$. If $\phi$ is an increasing function then

$$\frac{1}{2} \|\phi - w^*\|^2_{H^1(\Omega)} = E[\phi] - E[w^*] \leq E[\phi] - E[w^*].$$

(27)
Proof. Since \( \phi \) and \( w^* \) are increasing functions, by Taylor expansion,

\[
E[\phi] - E[w^*] = E[\ell, \phi] - E[\ell, w^*] \\
= \left\langle \frac{\delta E[\ell, w^*]}{\delta \phi}, \phi - w^* \right\rangle + \frac{1}{2} \|\phi - w^*\|^2_{H^1(\Omega)} \\
= \frac{1}{2} \|\phi - w^*\|^2_{H^1(\Omega)},
\]

since \( w^* \) is a minimizer. As \( E_k[\phi] \geq E[\phi] \), the assertion of the lemma thus follows.

\[ \square \]

5. Global in time estimates and proof of Theorem 2.1. In this section, we first establish the longtime behavior of the solution \( w \) of \([11]\). Then we use local in time convergence and the energy dissipation to show that \( E[v(\cdot, k, T)] \) approaches its minimum as \( T \) and \( k \) approaches infinity. Then we use Lemma 4.3 to obtain the global in time convergence. Finally, we use energy identity and energy convergence to show the convergence of \( v_k \) to \( w \) in \( L^2 \).

5.1. Asymptotic behavior of \( w \) as \( t \to \infty \). Set \( \psi(x, t) = w(x, t) - w^*(x) \). Then

\[
\psi_t - \psi_{xx} + \psi = 0 \text{ in } \Omega \times (0, \infty), \quad \psi_x = 0 \text{ on } \partial \Omega \times (0, \infty).
\]

Integrating \( 2[\psi - \psi_{xx}] [\psi_t - \psi_{xx} + \psi] = 0 \) over \( \Omega \) we obtain

\[
\frac{d}{dt} \int_{\Omega} [\psi^2 + \psi_x^2] + 2 \int_{\Omega} [\psi^2 + 2 \psi_x^2 + \psi_{xx}^2] dx = 0.
\]

It then follows from the Gronwall’s inequality that

\[
\int_{\Omega} [\psi^2(x, t) + \psi_x^2(x, t)] dx \leq e^{-2t} \int_{\Omega} [\psi(x, 0)^2 + \psi_x^2(x, 0)] dx \quad \forall \ t > 0.
\]

Hence, we have the following:

**Lemma 5.1.** Let \( w \) be the solution of \([11]\). Then

\[
\lim_{t \to \infty} \{ \|\psi(\cdot, t) - w^*\|_{H^1(\Omega)} + |E[\psi(\cdot, t)] - E[w^*]| \} = 0.
\]

5.2. Global in time convergence. Let \( \varepsilon > 0 \) be an arbitrarily fixed constant. By Lemma 5.1 there exists \( T_\varepsilon > 0 \) such that

\[
\left| E[w(\cdot, T_\varepsilon)] - E[w^*] \right| \leq \varepsilon^2.
\]

From Lemmas 3.3 and 3.4 there exists \( K_\varepsilon > 0 \) such that when \( k \geq K_\varepsilon \),

\[
\sup_{t \in [0, T_\varepsilon]} \|v(k, \cdot, t) - w(\cdot, t)\|_{H^1(\Omega)} \leq \varepsilon, \quad \sup_{t \in [0, T_\varepsilon]} \left| E_k[v(k, \cdot, t)] - E[w(\cdot, t)] \right| \leq \varepsilon^2.
\]

It then follows that

\[
E_k[v(k, \cdot, T_\varepsilon)] - E[w^*] \leq 2\varepsilon^2.
\]

As the energy is decreasing in \( t \), we obtain from Lemmas 4.1 and 4.3 that for every \( t \geq T_\varepsilon \),

\[
\|v(k, \cdot, t) - w^*\|^2_{H^1(\Omega)} \leq 2 \left\{ E_k[v(k, \cdot, t)] - E[w^*] \right\} \leq 4\varepsilon^2,
\]

\[
\|w(\cdot, t) - w^*\|^2_{H^1(\Omega)} \leq 2 \left\{ E[w(\cdot, t)] - E[w^*] \right\} \leq 2\varepsilon^2,
\]

\[
\left| E_k[v(k, \cdot, t)] - E[w(\cdot, t)] \right| \leq 3\varepsilon^2.
\]
Thus,
\[ \sup_{t \geq T_0} \| v(k, \cdot, t) - w(\cdot, t) \|_{H^1(\Omega)} \leq 4 \varepsilon \quad \forall k \geq K_\varepsilon. \]
It then follows from (28) that
\[ \sup_{t \geq 0} \| v(k, \cdot, t) - w(\cdot, t) \|_{H^1(\Omega)} \leq 4 \varepsilon \quad \forall k \geq K_\varepsilon. \]

This implies that \( \sup_{t \geq 0} \| v(k, \cdot, t) - w(\cdot, t) \|_{H^1(\Omega)} \to 0 \) as \( k \to \infty \). Finally using Hölder inequality and the \( W^{1,p} \) estimate in Lemma 3.2 we then obtain the first assertion (12) of Theorem 2.1. The above proof also shows that the energy estimate (13) holds. To completes the proof of Theorem 2.1 it remains to show the convergence of \( v_t \) to \( w_t \).

5.3. Convergence of \( v_t \) to \( w_t \). Set \( \zeta(k, x, t) = v(k, x, t) - w(x, t) \). We have
\[ \zeta_t - \zeta_{xx} + \zeta = u \quad \text{in} \quad \Omega \times (0, \infty), \quad \zeta_x(k, 0, t) = 0 \quad \forall t > 0. \]

Fix any \( 0 < \eta < \min\{\ell/3, 1/3\} \). By local parabolic estimate, we have
\[ \| \zeta_t \|_{L^2(\{0, \ell - 2\eta \times [2\eta^2, \eta^{-2}]\})} \leq C(\eta) \left\{ \| u \|_{L^\infty(\{0, \ell - \eta \times [\eta^2, \eta^{-2}]\})} + \| \zeta \|_{L^\infty(\Omega \times (0, \infty))} \right\}. \]

Sending \( k \to \infty \) and using (21) and (12) we then obtain
\[ \lim_{k \to \infty} \| \zeta_t(k, \cdot) \|_{L^2(\{0, \ell - 2\eta \times [2\eta^2, \eta^{-2}]\})} = 0 \quad \forall \eta > 0. \quad (29) \]

Next fix \( T > 0 \). Using Lemma 4.1 and \( E_k[v] \geq E[v] \geq E[w^*] \), we obtain
\[
\int_T^\infty \int_\Omega (v_t - w_t)^2 dx dt \leq 2 \int_T^\infty \int_\Omega (v_t^2 + w_t^2) dx dt \\
\leq 2 \left( E_k[v(k, \cdot, T)] + E[w(\cdot, T)] - 2E[w^*] \right).
\]

Hence,
\[
\lim_{k \to \infty} \int_T^\infty \int_\Omega (v_t - w_t)^2 dx dt \leq 4 \left( E[w(\cdot, T)] - E[w^*] \right).
\]

Similarly,
\[
\lim_{k \to \infty} \int_0^T \int_\Omega (v_t - w_t)^2 dx dt \leq 4 \left( E[v_0] - E[w(\cdot, \eta)] \right).
\]

Using energy identity and convergence of energy, we also have
\[
\lim_{k \to \infty} \int_\Omega \int_\eta^T |v_t^2 - w_t^2| dx dt = \lim_{k \to \infty} \left( E_k[v(k, \cdot, t)] - E[w(\cdot, t)] \right) \bigg|_\eta^T = 0. \quad (30)
\]

Finally, using (29), (30), and
\[
\int_\eta^T \int_\Omega |v_t - w_t|^2 dx dt \leq 2 \int_\eta^T \int_\ell-\eta |v_t^2 + w_t^2| dx dt + \int_\eta^T \int_\ell-\eta |v_t - w_t|^2 dx dt \\
= 2 \int_\eta^T \int_\Omega |v_t^2 - w_t^2| dx dt + 2 \int_\eta^T \int_\ell-\eta |v_t^2 - w_t^2| dx dt \\
+ 4 \int_\eta^T \int_\ell-\eta w_t^2 dx dt + \int_\eta^T \int_\ell-\eta |v_t - w_t|^2 dx dt,
\]
In a similar manner one can establish the assertion (32) can be proved similarly. As an illustration, here we provide the following lemma. We denote by $C^\alpha,\alpha/(\Omega \times [0,\infty))$ the Hölder space.

**Lemma 6.1.** For any $p \in (1,\infty)$, there exists a constant $C(p,\ell)$ such that the solution of

$$\begin{align*}
V_t - V_{xx} + V &= f \quad \text{in } \Omega \times (0,\infty), \\
V_x &= 0 \text{ (or } V = 0) \quad \text{on } \partial \Omega \times [0,\infty), \\
V(\cdot, 0) &= 0 \quad \text{on } \Omega \times \{0\},
\end{align*}$$

satisfies

$$\begin{align*}
\|V\|_{L^\infty([0,\infty);W^{1,p}(\Omega))} &\leq C(p,\ell) \|f\|_{L^\infty([0,\infty);L^1(\Omega))}, \\
\|V\|_{C^{1-\frac{\ell}{p},1-\frac{\ell}{p}}(\Omega \times [0,\infty))} &\leq C(p,\ell) \|f\|_{L^\infty([0,\infty);L^p(\Omega))}.
\end{align*}$$

(31)

(32)

**Proof.** The assertion (31) follows from the potential analysis presented in Section 3.4. The assertion (32) can be proved similarly. As an illustration, here we provide an $L^\infty((0,\infty);C^{2-\frac{\ell}{p}}(\Omega))$ estimate. We write $V(x,t) = \int_0^\infty \int_\Omega \Gamma(x,y, s) f(y,t-s) dy ds$ where $f(y,s) = 0$ for $s < 0$. Then for $0 \leq x < x + h \leq \ell$, we have, for $\eta > 0$ and $q = p/(p-1)$,

$$\begin{align*}
|V_x(x,t) - V_x(x+h,t)| &\leq \int_0^\eta 2 \sup_{z \in \Omega} |\Gamma_z(z,\cdot, s)|_{L^p(\Omega)} \|f(\cdot,t-s)\|_{L^p(\Omega)} ds \\
&\quad + h \int_\eta^\infty \sup_{z \in \Omega} |\Gamma_{zz}(z,\cdot, s)|_{L^p(\Omega)} \|f(\cdot,t-s)\|_{L^p(\Omega)} ds \\
&\leq C(p,\ell) \left[ \eta^{\frac{1}{2p}} + h \eta^{-\frac{1}{2} + \frac{1}{p}} \right] \|f\|_{L^\infty([0,\infty);L^p(\Omega))}.
\end{align*}$$

Choosing $\eta = h^2$ we then obtain

$$\|V_x\|_{L^\infty([0,\infty);C^{1/(\Omega)})} \leq C(p,\ell) \|f\|_{L^\infty([0,\infty);L^p(\Omega))}.$$

In a similar manner one can establish the $C^{1-\frac{\ell}{p},1-\frac{\ell}{p}}(\Omega)$ and $C^{1-\frac{\ell}{p},1-\frac{\ell}{p}}([0,\infty);C^1(\Omega))$ estimates. This completes the proof.

We denote by $(u,v)$ the solution of (1). For simplicity of presentation, we assume that $v_0 := v(\cdot,0) \in W^{1,\infty}(\Omega)$, $u_0 := u(\cdot,0) \in L^\infty(\Omega)$, and $u_0 \geq 0$. Then by
maximum principle, $u > 0$ in $\bar{\Omega} \times (0, \infty)$. We denote by $v^0$ the solution of (20). Applying Lemma 6.1 to $V = v - v^0$ and $f = u$, we obtain

$$
\|v - v^0\|_{L^\infty(0, \infty; W^{1,p}(\Omega))} \leq C(p, \ell).
$$

Next, with $\tau > 0$, we write the equation $\tau u_x = [u_x - kuv_x]_x$ as

$$
Lu := \tau u_t - u_{xx} + u = u - [kuv_x]_x \quad \text{in} \quad \Omega \times (0, \infty).
$$

Then we have the decomposition

$$
u = u^0 + u_1 + u_2,
$$

where $u^0$, $u_1$, and $u_2$ are solutions of the following linear problems:

- $Lu^0 = 0$ in $\Omega \times (0, \infty)$, $u^0(\cdot, 0) = u_0$, $u^0_0 = 0$ on $\partial \Omega \times (0, \infty)$,
- $Lu_1 = u$ in $\Omega \times (0, \infty)$, $u_1(\cdot, 0) = 0$, $u_1x = 0$ on $\partial \Omega \times (0, \infty)$,
- $Lu_2 = -kuv_x$ in $\Omega \times (0, \infty)$, $u_2(\cdot, 0) = 0$, $u_2 = 0$ on $\partial \Omega \times (0, \infty)$.

Here we point out that on $\partial \Omega \times (0, \infty)$, $u_{2xx} = \tau u_{2x} + u_2 + kuv_x = 0$.

Then by Lemma 6.1 we have

$$
\|u_1\|_{L^\infty(0, \infty; W^{1,p}(\Omega))} \leq C(p, \ell) \|u\|_{L^\infty(0, \infty; L^p(\Omega))} = C(p, \ell),
$$

$$
\|u_2\|_{L^\infty(0, \infty; W^{1,p}(\Omega))} \leq kC(p, \ell) \|uv_x\|_{L^\infty(0, \infty; L^1(\Omega))}.
$$

Note that for $p > 1$, $r := \frac{2p}{p-1}$, and $t > 0$,

$$
\|u(t)v_x(t)\|_{L^1(\Omega)} \leq \|u\|_{L^1(\Omega)}^{1/2} \|u\|_{L^p(\Omega)}^{1/2} \|v_x\|_{L^r(\Omega)} = \|u\|_{L^1(\Omega)}^{1/2} \|v_x\|_{L^r(\Omega)}
$$

$$
\leq \frac{1}{2kC(p, \ell)} \|u\|_{L^p(\Omega)} + \frac{kC(p, \ell)}{2} \|v_x\|_{L^r(\Omega)}^2
$$

$$
\leq \frac{1}{2kC(p, \ell)} \|u\|_{L^\infty(0, \infty; L^p(\Omega))} + \frac{kC(p, \ell)}{2} \left( \|v^0\|_{L^\infty(0, \infty; W^{1,p}(\Omega))} + C(r, \ell) \right)^2.
$$

Hence, we obtain

$$
\|u\|_{L^\infty(0, \infty; L^p(\Omega))} \leq \|u^0\|_{L^\infty(0, \infty; L^p(\Omega))} + \|u_1\|_{L^\infty(0, \infty; L^p(\Omega))} + \|u_2\|_{L^\infty(0, \infty; W^{1,p}(\Omega))}
$$

$$
\leq \|u^0\|_{L^\infty(0, \infty; L^p(\Omega))} + C(p, \ell) + \frac{1}{2} \|u\|_{L^\infty(0, \infty; L^p(\Omega))}
$$

$$
+ \frac{kC(p, \ell)^2}{2} \left( \|v^0\|_{L^\infty(0, \infty; W^{1,p}(\Omega))} + C(r, \ell) \right)^2.
$$

In summary, we have the following:

**Theorem 6.2.** Assume that $u_0, v^0_0 \in L^\infty(\Omega)$, $u_0 \geq 0$ and $\int_\Omega u_0(x) dx = 1$. Let $(u, v)$ be the solution of (11) subject to $(u, v)|_{t=0} = (u_0, v_0)$. Then, for any $p > 1$,

$$
\|u\|_{L^\infty(0, \infty; W^{1,p}(\Omega))} \leq C(p, \ell) + \|u^0\|_{L^\infty(0, \infty; W^{1,p}(\Omega))},
$$

$$
\|u\|_{L^\infty(0, \infty; L^p(\Omega))} \leq 2 \|u^0\|_{L^\infty(0, \infty; L^p(\Omega))} + 2C(p, \ell)
$$

$$
+ \left[ kC(p, \ell)^2 \left( \|v^0\|_{L^\infty(0, \infty; W^{1,p}(\Omega))} + C\left( \frac{2p}{p-1}, \ell \right) \right)^2 \right],
$$

where $u^0$ and $v^0$ are solutions of the linear problem $\tau u_x^0 - u_{xx}^0 + u^0 = 0$, $v_x^0 - v_{xx}^0 + v^0 = 0$ in $\Omega \times (0, \infty)$ subject to the boundary conditions $u^0_x = 0$ and $v^0_x = 0$ on $\partial \Omega \times (0, \infty)$ and the initial conditions $v^0 = v_0$ and $u^0 = u_0$ at $t = 0$. 

DYNAMICS OF KELLER-SEGEL’S CHEMOTAXIS MODEL 1125
Remark 2. By the maximum principle, we have, for each \( t > 0 \),
\[
\|u_0(t)\|_{L^\infty(\Omega)} \leq \|u_0\|_{L^\infty(\Omega)} e^{-t/\tau},
\]
\[
\|v_0(t)\|_{L^\infty(\Omega)} \leq \|v_0\|_{L^\infty(\Omega)} e^{-t},
\]
\[
\|v_0'(t)\|_{L^\infty(\Omega)} \leq \|v_0'\|_{L^\infty(\Omega)} e^{-t}.
\]

Remark 3. First using (32) and then using standard local Hölder estimates for scalar parabolic equations and a bootstrap argument, one can show that under the conditions of Theorem 6.2, for any positive integer \( m \) and real number \( \varepsilon > 0 \),
\[
\|u\|_{C^m(\bar{\Omega} \times [\varepsilon, \infty))} + \|v\|_{C^m(\bar{\Omega} \times [\varepsilon, \infty))} < \infty.
\]
We omit the details of the proof.

Remark 4. In [33], Osaki and Yagi discovered the basic estimate (33) with \( p = 2 \). Together with energy estimates for high order derivatives, they established the existence of a global (in time) solution in certain Sobolev spaces, say \((u, v) \in L^\infty(0, \infty; H^2(\Omega) \times H^3(\Omega))\).

In [22], using a semi-group theory, Hillen and Potapov established the existence of a solution in
\[
\bigcap_{T > 0} L^\infty(0, T; L^\infty(\Omega) \times W^{1+s,p}(\Omega))
\]
where \( s \in (0, 1) \) and \( ps > 1 \). Their estimates for the bounds of \( u(t) \) and \( v(t) \) depend on \( t \).

REFERENCES

[1] N. Alikakos, P. W. Bates and G. Fusco, Slow motion for the Cahn-Hilliard equation in one space dimension, *J. Differ. Eqns.*, 90 (1991), 81–135.
[2] P. W. Bates and J. Xun, Metastable patterns for the Cahn-Hilliard equation, Part I, *J. Differ. Eqns.*, 111 (1994), 421–457.
[3] P. W. Bates and J. Xun, Metastable patterns for the Cahn-Hilliard equation, Part II, *J. Differ. Eqns.*, 117 (1995), 165–216.
[4] P. Biler, Local and global solvability of some parabolic systems modelling chemotaxis, *Adv. Math. Sci. Appl.*, 8 (1998), 715–743.
[5] L. Bronsard and D. Hilhorst, On the slow dynamics for the Cahn-Hilliard equation in one space dimension, *Proc. Roy. Soc. Lond.*, 439 (1992), 669–682.
[6] J. Carr and R. Pego, Metastable patterns in solutions of \( u_t = \varepsilon^2 u_{xx} - f(u) \), *Comm. Pure Appl. Math.*, 42 (1989), 523–576.
[7] X. Chen, Generation, propagation, and annihilation of metastable patterns, *J. Differ. Eqns.*, 206 (2004), 399–437.
[8] X. Chen, J. Hao, X. Wang, Y. Wu and Y. Zhang, Stability of spiky solution of the Keller-Segel’s minimal chemotaxis model, *J. Differ. Eqns.*, 257 (2014), 3102–3134.
[9] X. Chen and M. Kowalczyk, Dynamics of an interior spike in the Gierer-Meinhardt system, *Siam J. Math. Anal.*, 33 (2001), 172–193.
[10] X. Chen and M. Kowalczyk, Slow dynamics of interior spikes in the Shadow Gierer-Meinhardt system, *Adv. Differ. Eqns.*, 6 (2001), 847–872.
[11] S. Childress, Chemotactic collapse in two dimensions, in *Lecture Notes in Biomath.*, 55, Springer, (1984), 61–66.
[12] S. Childress and J. Perkus, Nonlinear aspects of chemotaxis, *Math. Bios.*, 56 (1981), 217–237.
[13] P. C. Fife and L. Hsiao, The generation and propagation of internal layers, *Nonlinear Anal.*, 12 (1988), 19–41.
[14] P. C. Fife and J. B. Mcleod, The approach of solutions of nonlinear diffusion equations to traveling front solutions, *Arch. Rational Mech. Anal.*, 65 (1977), 335–361.
[15] G. Fusco, A geometric approach to the dynamics of \( u_t = \varepsilon^2 u_{xx} + f(u) \) for small \( \varepsilon \), in *Problems Involving Change of Type*, Springer, 359 (1990), 53–73.
G. Fusco and J. K. Hale, Slow motion manifolds, dormant instability and singular perturbations, J. Dyn. Diff. Eqns., 1 (1989), 75–94.

H. Gajewski, K. Zacharias and Dr. Konrad Gröger, Global behavior of a reaction-diffusion system modelling chemotaxis, Math. Nachr., 195 (1998), 77–114.

M. Herrero and J. Velázquez, Singularity patterns in a chemotaxis model, Math. Ann., 306 (1996), 583–623.

M. Herrero and J. Velázquez, Chemotaxis collapse for the Keller-Segel model, J. Math. Biol., 35 (1996), 177–194.

T. Hillen and K. J. Painter, Global existence far a parabolic chemotaxis model with prevention of overcrowding, Adv. Appl. Math., 26 (2001), 280–301.

T. Hillen and K. J. Painter, A user’s guide to PDE models for chemotaxis, J. Math. Biol., 58 (2009), 183–217.

W. Jäger and S. Luckhaus, On explosions of solutions to a system of partial differential equations modelling chemotaxis, Trans. Amer. Math. Soc., 329 (1992), 819–824.

K. Kang, T. Kolokolnikov and M. J. Ward, The stability and dynamics of a spike in the one-dimensional Keller-Segel model, IMA J. Appl. Math., 72 (2007), 140–162.

E. Keller and L. Segel, Initiation of slime mold aggregation viewed as an instability, J. Theoret. Biol., 26 (1970), 399–415.

C.-S. Lin, W.-M. Ni and I. Takagi, Large amplitude stationary solutions to a chemotaxis system, J. Differ. Eqns., 72 (1988), 1–27.

T. Nagai, Blow-up of radially symmetric solutions to a chemotaxis system, Adv. Math. Sci. Appl., 5 (1995), 581–601.

T. Nagai, T. Senba and K. Yoshida, Application of the Trudinger-Moser inequality to a parabolic system of chemotaxis, Funkcial. Ekvac., 40 (1997), 411–433.

T. Nagai, Blowup of nonradial solutions to parabolic-elliptic systems modeling chemotaxis in two-dimensional domains, J. Inequal. Appl., 6 (2001), 37–55.

W.-M. Ni and I. Takagi, Locating the peaks of least-energy solutions to a semilinear Neumann problem, Duke Math. J., 70 (1993), 247–281.

K. Osaki and A. Yagi, Finite dimensional attractors for one dimensional Keller-Segel equations, Funkcial. Ekvac., 44 (2001), 441–469.

C. S. Patlak, Random walk with persistence and external bias, Bull. Math. Biophys., 15 (1953), 311–338.

T. Senba and T. Suzuki, Parabolic system of chemotaxis: Blowup in a finite and the infinite time, Methods Appl. Anal., 8 (2001), 349–367.

X. Sun and M. J. Ward, Dynamics and coarsening of interfaces for the viscous Cahn-Hilliard equation in one spatial dimension, Stud. Appl. Math., 105 (2000), 203–234.

X. Wang and Q. Xu, Spiky and transition layer steady states of chemotaxis systems via global bifurcation and Helly compactness theorem, J. Math. Biol., 66 (2013), 1221–1266.

Y. Zhang, X. Chen, J. Hao, X. Lai and C. Qin, An eigenvalue problem arising from spiky steady states of a minimal chemotaxis model, J. Math. Anal. Appl., 420 (2014), 684–704.

Received October 2014; revised March 2016.