Dust reference frame in quantum cosmology

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Abstract
We give a formulation of quantum cosmology with pressureless dust and arbitrary additional matter fields. The system has the property that its Hamiltonian constraint is linear in the dust momentum. This feature provides a natural time gauge, leading to a physical Hamiltonian that is not a square root. Quantization leads to a Schrödinger equation for which unitary evolution is directly linked to geodesic completeness. Our approach simplifies the analysis of both Wheeler–deWitt and loop quantum cosmology (LQC) models and significantly broadens the applicability of the latter. This is demonstrated for arbitrary scalar field potential and cosmological constant in LQC.

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1. Introduction

In attempts to formulate a non-perturbative theory of quantum gravity, one of the obstacles is the so-called problem of time [1]. Its manifestation in the canonical formulation of general relativity is that the Hamiltonian vanishes, which is a direct consequence of the time reparametrization invariance of the theory. For matter with the usual kinetic energy, the Hamiltonian constraint is quadratic in all momenta, including the pure gravity term.

There appears to be an obvious way to address this problem: impose an explicit time gauge fixing to obtain a true Hamiltonian that describes the evolution of physical degrees of freedom. This explicit deparametrization brings with it some difficulties: the true Hamiltonian is generically a square root, which is difficult to define as an operator (see e.g. [2]). A second problem is that different time gauge fixings give radically different true Hamiltonians, so in the final analysis the quantum theory ends up being time-gauge dependent.

In a recent paper [3], we presented a complete non-perturbative quantization of gravity that solves both problems. This approach uses a single timelike dust field (motivated by the Brown and Kuchař (BK) form) [4] coupled to general relativity and standard model matter. Like the BK action, the canonical theory has the feature that the Hamiltonian constraint is linear in the dust momentum. However, unlike the BK action, it also has the remarkable feature
that in a natural time gauge fixing, the true Hamiltonian is not a square root. Furthermore, this Hamiltonian is identical in form to what would be the Hamiltonian constraint without the dust field. It is this feature, combined with the well-developed kinematical framework for loop quantum gravity, that leads to a complete quantum theory.

In this paper, we apply this development to quantum cosmology in both the Wheeler–DeWitt and LQG quantization. We show that (i) the true Hamiltonian is first order, and in the loop quantum cosmology (LQC) case it poses no self-adjoint extension problem; (ii) the non-zero cosmological constant is analytically solvable for both signs; and (iii) any type of matter is included.

Our approach differs significantly from existing LQC models, all of which use the scalar field as a clock; this relies on the scalar field momentum being a constant of motion and leads to a Hamiltonian constraint operator that is second order in time (and so is not a time-dependent Schrödinger equation). For these reasons, it is difficult to implement scalar field time for arbitrary scalar potential, including polynomial or slow roll cases used in inflationary models.

2. General relativity with dust

The theory is given by the Einstein–Hilbert action with timelike dust field $T$ and an arbitrary matter Lagrangian $L_m$ (assumed to be at most second order in time):

$$S = \frac{1}{4G} \int d^4x \sqrt{-g} R - \int d^4x \sqrt{-g} L_m - \frac{1}{2} \int d^4x \sqrt{-g} M (g^{ab} \partial_a T \partial_b T + 1),$$

where $M$ is a Lagrange multiplier. The last term resembles the BK dust action [4] but contains only one scalar.

The canonical degrees of freedom arising from this action are the ADM variables $(q_{ab}, \pi^{ab})$, the dust variables $(T, p_T)$ and other matter variables in $L_m$, subject to the (first-class) Hamiltonian and spatial diffeomorphism constraints. The Hamiltonian constraint is linear in the dust momentum $p_T$ conjugate to $T$ [3], which makes the theory resemble a parametrized system and suggests the gauge $T = t$. In this gauge, the true Hamiltonian is $-p_T$, given by

$$H \equiv -p_T = H_G + H_m,$$

where $H_G$ and $H_m$ are the gravitational and matter parts of the Hamiltonian constraint coming from the first two terms in the action. We note that preservation of this gauge condition under evolution leads to lapse function $N = 1$ and arbitrary shift vector field $N^a$ [3].

In this natural gauge, quantization gives the time-dependent Schrödinger equation

$$i \frac{\partial \Psi}{\partial t} = (\hat{H}_G + \hat{H}_m) \Psi.$$
3. Quantum cosmology with dust

As a first example, we consider FRW cosmology with only dust, i.e. $H_m = 0$ in (3), and give a quantization of this system using both the Wheeler–DeWitt (WDW) and the background-independent LQG method. These methods differ in the choice of Hilbert space used. This will demonstrate a first advantage over scalar field time used in all previous LQC work: classically the singularity is reached in finite dust time, so it allows a direct comparison of the mechanism of singularity resolution in background-independent quantization, and the lack of it in WDW quantization.

The phase space variables are the canonical pair $\{a, P_a\} = 1$ where $a$ is the FRW scale factor. Classically $a$ is positive; therefore, to have phase space topology $\mathbb{R}^2$, one needs to equip $a$ with an orientation. To do this, we can either work with the original variables or use a canonical transformation to the oriented volume $v (= a^{-1} a^3)$ and its conjugate momentum $b$, which satisfy the Poisson bracket $\{v, b\} = 2$. In these variables, $H_G = (3\pi G/2a) b^2 |v|$. 

3.1. Wheeler–DeWitt quantization

In WDW quantization, one uses the standard Schrödinger representation. The canonical variables $(v, b)$ are promoted to operators defined on the Schwartz space $S(\mathbb{R}) \in \mathcal{H}_G = L^2_2(\mathbb{R}, dv)$, where $s$ denotes restriction to the functions symmetric in $v$. Since the classical Hamiltonian is linear in $v$, a naive realization of the Hamiltonian operator in momentum space is

$$\hat{H}_G := 3\pi\ell_\hbar^2 \alpha^{-1} \hat{b}^2 \hat{b}_v. \quad (4)$$

It acts on a dense domain $\mathcal{H}_G$ of functions spanned by the eigenfunctions $\theta(\pm x) \exp(i k x)$, where $k \in \mathbb{R}$ and $x = -1/b$. This operator however breaks the positive definiteness of the classical Hamiltonian since its spectrum $\text{Sp}(\hat{H}_G) = \mathbb{R}$. An obvious way to fix this problem is just to restrict attention to the positive part of (4). However, there is a rigorous way to arrive at this conclusion in the following steps. (i) Begin with the natural factor ordering $[5, 6]$

$$\hat{H}_G = \frac{3\pi G}{2\alpha} \sqrt{|v|} \hat{b}^2 \sqrt{|v|}. \quad (5)$$

This is a symmetric positive definite operator on $S(\mathbb{R})$. (ii) Analysis of its deficiency subspaces shows that it admits a family of extensions labeled by $e^{i\beta} \in U(1)$, and that the spectrum of each extension $\hat{H}_\beta$ with domain $\mathcal{H}_\beta$ is nondegenerate. (iii) Due to this fact, for each extension $\hat{H}_\beta$, there is an invertible map $P_\beta : \mathcal{H}_\beta \to \mathcal{H}_G$ to a certain auxiliary space such that $\hat{H}_G$ is mapped onto the positive part $[\mathcal{H}_G]^{\beta+}$ of $\hat{H}_G$. (The auxiliary spaces $\tilde{\mathcal{H}}_\beta$ are determined by the self-adjoint extensions $\tilde{\hat{H}}_\beta$ of $\hat{H}_G$, and construction of the map uses the fact that the symmetric state is encoded in one orientation $\theta(v)\Psi(v)$.)

In each extension, the physical states are

$$[P_\beta \Psi](x) = \int_{\mathbb{R}^+} dk \hat{\Psi}(k) [\theta(x) e^{ikx} + \theta(-x) e^{ikx}], \quad (6)$$

where $\hat{\Psi} \in L^2(\mathbb{R}, dk)$, with the inner product

$$\langle P_\beta \Psi | P_\beta \Phi \rangle = \int_{\mathbb{R}} dx [\hat{P}_{\beta} \hat{\Psi}](x) [\hat{P}_\beta \hat{\Phi}](x). \quad (7)$$

1 This gives a simple transformation between the configuration and momentum representations.

2 $a \approx 1.35 \ell^3_\text{Pl}$. This choice synchronizes the variable $v$ with what is used in LQC. $a$ is expressed in terms of the Barbero–Immirzi parameter and the LQC area gap $[5]$. 

3
Time evolution is given by
\[ \hat{\Psi}(k) = e^{i\omega(k)(t-t_0)}\hat{\Psi}_{t_0}(k), \quad \omega(k) = 3\pi \ell_P^2 \alpha^{-1} k. \] (8)
This is a plane wave packet moving with the constant speed \( v := 3\pi \ell_P^2 \alpha^{-1} \) till it hits \( x = 0 \), where it gets rotated by an extension-dependent phase \( e^{i\theta} \).

The dynamics of the system is described by the observable \( \hat{V} = |\tilde{a}|^2 \). Under the mapping \( \tilde{V} = \hat{P}_\phi \hat{V} \hat{P}_\phi^{-1} \), it takes a simple form involving only \( \hat{a} \) and \( x^2 \). Its time evolution and dispersion are analytically determined:
\[ \langle \hat{V}(t) \rangle = V(t) = 2\alpha^{-1} \langle \hat{k} \rangle [3\pi \ell_P^2 (t-t_i)]^2 + 2\alpha \sigma_x^2, \quad \hat{k} = k\hat{1}, \] (9)
where \( t_i \) corresponds to the (always existing) point where \( \langle \hat{a} \hat{\delta}_0 \hat{a} \rangle = 0 \) (the colons denote symmetrization) and \( \sigma_x^2 \) can be related to the dispersion of \( x \) at this moment. In the same way, one can determine the time dependence of the Hubble parameter \( H \). Physically these trajectories follow the classical ones and describe a contracting universe until the big crunch singularity at the time \( t_c \). At that moment, \( V \) becomes comparable to its dispersion. In this sense, the singularity \( V = 0 \) is reached dynamically.

To determine the evolution past the singular point, one needs to know the self-adjoint extension. This is equivalent to imposing additional boundary conditions at the singularity \( x = 0 \); it cannot be determined by an analysis of the state for \( t < t_c \). Given a choice of extension, it is possible to continue the evolution. The resulting trajectory for \( t > t_c \) describes a universe that expands out of the singularity and subsequently follows a classical trajectory. Thus, we see that in WDW quantization the big bang singularity is dynamically reached, and evolution beyond it is not uniquely determined without additional boundary data. In this sense, the WDW quantization does not resolve the singularity.

### 3.2. Loop quantum cosmology

We now consider the same problem in LQC. To quantize the system, we follow the improved dynamics prescription of [5]. As in the WDW case, we can apply directly the results of kinematical quantization [5, 7]. The canonical variables are again \((v, b)\), but the physical Hilbert space is now \( \mathcal{H} = L^2(\mathbb{R}, d\mu_B) \), where \( \mathbb{R} \) is the Bohr compactification of the reals and \( d\mu_B \) is the Haar measure on it. The (physical) gravitational Hamiltonian \( H_G \) is a second-order difference operator in \( v \). With the same factor ordering as in (5), it takes the form
\[ \hat{H}_G = -\frac{3\pi G}{8\epsilon} \sqrt{|v|} (\hat{N}^2 - \hat{N}^{-2}) \sqrt{|v|}, \quad \hat{N} |v\rangle = |v + 1\rangle. \] (10)
This operator is non-negative definite and, unlike in WDW quantization, it is essentially self-adjoint [8]. Its action distinguishes superselection sectors of functions supported on \( \mathbb{C} = \epsilon + 4\mathbb{Z} \). As the physical predictions of LQC systems do not depend qualitatively on \( \epsilon \) [9], we select \( \epsilon = 0 \). Then \( H_G \) does not couple the sets \( v < 0, v = 0 \) and \( v > 0 \) which permits with construction of the mapping \( P : \mathcal{H}_G \rightarrow \mathcal{H}_G \) into an auxiliary space analogous to that for WDW quantization. Due to the self-adjointness of \( \hat{H}_G \), its construction here is even simpler. The auxiliary Hamiltonian now takes the form
\[ \hat{H}_G := P\hat{H}_G P^{-1} = \left[ 3\pi \ell_P^2 \alpha^{-1} \sin^2(b) \partial_b \right]^+, \] (11)
where due to discreteness in \( v \), the momentum \( b \) is periodic with period \( \pi \). The auxiliary Hilbert space \( \mathcal{H}_G \) consists of the states
\[ |P\Psi \rangle(x) = \int_{\mathbb{R}^+} dk \tilde{\Psi}(k) e^{-ikx}, \quad \tilde{\Psi} \in L^2(\mathbb{R}^+, k dk). \] (12)

\(^3\) The explicit dependence of the first term on \( \alpha \) is compensated by analogous dependence in \( \langle \hat{k} \rangle \).
where now \( x = -\cot(b) \). The inner product (in new coordinate) is given by the same formula (7). At this point, it is worth noting that \( \tilde{H}_G \) is a first-order differential operator so its spectrum is nondegenerate. This fact is crucial in identifying the images of the mapping of the original Hilbert space basis to the auxiliary one. In the models with a scalar field, an analogous construction would lead to the second-order operator with the degenerate spectrum. This makes the identification of the bases much more involved and required an alternative approach [7] (which cannot be extended to more complicated systems.)

The physical evolution is again given by (8), but due to self-adjointness the evolution across \( x = 0 \) is now unique. In particular, the expectation value of the volume is now

\[
V(t) = 2\alpha^{-1}(\hat{k}) \left[ 3\pi \ell_{Pl}^2 (t - t_s) \right]^2 + 2\alpha \tilde{\sigma}^2 + 2\alpha \langle \hat{k} \rangle,
\]

where the meaning of \( t_s \) and \( \tilde{\sigma}^2 \) is the same as in (9). We note here the presence of a nontrivial minimum volume \( V_m = 2\alpha \langle \hat{k} \rangle \), which implies dynamical resolution of the singularity through the bounce, as in the models with scalar field time [5, 7].

In summary, we have seen that the dust reference frame gives exactly solvable models in WDW and LQC, a feature present in earlier models [7]. We now turn to more complicated models, where the real advantage of dust time becomes apparent.

3.3. LQC with a cosmological constant

FRW cosmologies with the non-vanishing cosmological constant and scalar field time have been investigated in detail in LQC [5, 10, 11]. In these works, the appearance of the square root in the evolution operator rendered the models analytically unsolvable. Furthermore, for the case \( \Lambda > 0 \), the Hamiltonian was not essentially self-adjoint; the multitude of extensions complicated the analysis even further, in particular, raising an issue of the dependence of global evolution on the choice of time [6]. As we will see, the situation simplifies considerably with dust time.

With the cosmological constant, the gravitational Hamiltonian is

\[
H_\Lambda = H_G + \frac{3\rho_c}{16\alpha} \Lambda |v|,
\]

(14)

(where \( \rho_c \approx 0.81 \rho_{Pl} \) is the critical energy density of LQC [5].) The polymer quantization of this operator is straightforward [5, 10]: it is positive definite for \( \Lambda < 0 \) and essentially self-adjoint for all values of \( \Lambda \). Remarkably, the cosmological constant term does not disrupt the properties required for the mapping \( P_\Lambda : H_G \rightarrow \tilde{H}_\Lambda \) between the physical and auxiliary Hilbert spaces. The auxiliary Hamiltonian is

\[
\tilde{H}_\Lambda := P_\Lambda \hat{H}_\Lambda P_\Lambda^{-1} = \tilde{H}_G - \frac{3i\rho_c}{8\alpha} \Lambda \partial_b.
\]

(15)

This is simple enough to determine its eigenfunctions analytically. For all values of \( \Lambda \), these can be expressed in the form \( e^{ikx(b)} \), where \( x(b) \) depends analytically on \( \Lambda \).

\( \Lambda < 0 \). For this case, we have

\[
x(b) = \arctan \left[ \sqrt{\left( 3\rho_c |\Lambda| + 24\pi \ell_{Pl}^2 \right) / \left( 3\rho_c |\Lambda| \right) \tan(b)} \right].
\]

(16)

This inherits the periodicity properties of \( b \). Therefore, the spectrum of the auxiliary Hamiltonian \( \{ \tilde{H}_\Lambda \}^+ \) is discrete. Nondegeneracy of its spectrum allows again to relate the original Hilbert space basis elements to the auxiliary counterparts in a straightforward way. The (auxiliary) physical states are

\[
\tilde{H}_\Lambda \ni [P_\Lambda \Psi](x) = \sum_n \tilde{\Psi}_n e^{2i\alpha x},
\]

(17)
and the time evolution is given by

\[ \hat{\Psi}_n(t) = \hat{\Psi}_n(t_0) e^{i\omega_n (t-t_0)}, \quad \omega_n = \frac{n}{4\alpha} \sqrt{3\rho_c |\Lambda| (3\rho_c |\Lambda| + 24\pi \ell_P^2)}. \]  

(18)

The standard observables used for \( \Lambda = 0 \) also map in a straightforward way onto an auxiliary system. In particular, \( \hat{V} := P_\Lambda \hat{V} P_\Lambda^{-1} \) of \( \hat{V} \) acts directly on the spectral profile \( \hat{\Psi}_n \), in terms of the shift and multiplication operators.

The calculation of expectation values reproduces the dynamics qualitatively similar to the one of the scalar field system: the universe goes through an infinite chain of quantum bounces and classical recollapses. Now however, unlike in [10] due to exact uniformity of \( \text{Sp}(\hat{H}_\Lambda) \), the evolution is exactly periodic; thus, the wave packets do not spread out between the cycles of the evolution.

\( \Lambda > 0 \). As in the model with scalar field, we observe qualitative changes at the critical value \( \Lambda_c = 8\pi G \rho_c \). The most interesting case is for \( 0 < \Lambda < \Lambda_c \). Let us write \( \Lambda \) as \( \Lambda = \Lambda_c \sin^2(\beta) \), with \( \beta \in [0, \pi/2] \). Then the domain of \( b \) ([0, \pi]) admits two uncoupled deSitter sectors. Furthermore, the spectrum of auxiliary Hamiltonian (15) is degenerate, which makes the identification of the basis vectors of \( \hat{H}_\Lambda \) considerably more involved. Fortunately a careful analysis of the asymptotic properties of the eigenfunctions of \( \hat{H}_\Lambda \) allows them to be uniquely identified. This gives that \( \text{Sp}(\hat{H}_\Lambda) = \mathbb{R} \) with the eigenfunctions

\[ e_{+|k|}(b) = \theta(\sin(b) - \sin(\beta)) e^{i\xi_+(b)}, \]

\[ e_{-|k|}(b) = \theta(\sin(\beta) - \sin(b)) e^{i\xi_-(b)}, \]

(19)

where the coordinates \( x_\pm \) are related to \( b \) by

\[ \tan(\beta) [\tanh(x_\pm)]^{\pm 1} = \tan(b), \quad x_\pm \in \mathbb{R}. \]

(20)

We have thus two sectors, one of positive and the other of negative energy, supported respectively on \( \sin(b) > \sin(\beta) \) and \( \sin(b) < \sin(\beta) \). In each sector, physical states are of a form analogous to (12), with \( k \in \mathbb{R}, a \) a basis defined via (19) and \( \hat{\Psi} \in L^2(\mathbb{R}, |k| \, dk) \). They evolve as free plane wave packets in the respective coordinates \( x_\pm \). Time evolution is again given by equation (8), now with the frequencies \( \omega(k) = 3\pi \ell_P^2 a^{-1} \cos^2(\beta) k \). The volume operator is in this case involves \( \partial_{x_\pm} \) and multiplications by hyperbolic functions of \( x_\pm \). One can thus apply the methods of [7] to calculate its expectation value and dispersion analytically.

The positive energy sector gives analytical results similar to those found with scalar field time in LQC, i.e. a contracting deSitter-like universe bouncing once into an expanding one. In addition, we obtain negative energy solutions which give DeSitter universes with ‘phantom’ dust and classical bounce, i.e. due to classical negative energy. The last feature has interesting implications for the case \( \Lambda \geq \Lambda_c \): there is a large solution space of phantom dust solutions with classical bounce (\( \Lambda = \Lambda_c \)) or cyclic evolutions consisting of an infinite chain of classical bounces and quantum recollapses (\( \Lambda > \Lambda_c \)). These cases however are just mathematical curiosity and are not of physical interest.

4. Other applications

As we have seen, the use of dust time considerably simplifies the analysis of systems with cosmological constant. Its usefulness is not however restricted to only these: the physical Hamiltonian is self-adjoint for any non-exotic matter, including Yang–Mills gauge fields and scalar fields with any potential term. This follows from the fact that \( H_G \) is self-adjoint for any

4 Each sector is represented by \( k > 0 \) and \( k < 0 \), respectively.
value of the cosmological constant [12], and that the only gravitational variables the matter terms contain are factors of the volume operator and its inverse. This holds also for other spatial topologies.

Inflaton potential. One of potentially important applications is the analysis of the cosmologies with massive scalar field, including those with inflaton potential. In this situation, the phase space is coordinatized by \((v, b, \phi, p_\phi)\), where \(\phi\) is the scalar field value and \(p_\phi\) its momentum.

The physical Hilbert space is simply a product

\[ H = H_G \otimes H_\phi, \quad H_\phi = L^2(\mathbb{R}, d\phi), \]

and the Hamiltonian is \(\tilde{H}_G + \tilde{H}_\phi\), where \(\tilde{H}_\phi\) is a polymer-quantized scalar field Hamiltonian [13]. The physical evolution is given by the Schrödinger equation (3) and physical observables can be taken as the kinematical ones of [5], say \(\tilde{V}, \tilde{\phi}\). Furthermore, the mappings into auxiliary systems found for \(\tilde{H}/\Lambda_1\) can be extended to this situation, thus, casting the physical system into one that is at least numerically manageable.

Effective Friedman equation. Another application is a derivation of the effective Friedmann equation. Here we have the advantage that LQC kinematical operators become physical ones. Given a physical state, the derivation requires computation of the expectation values of the square of the Hubble operator and of the physical energy density operator of the dust. The latter is given by the operator \(\hat{\rho}_D := - : V^{-1}\hat{H}_G :\) and the inverse volume operator has a suitable definition that descends from LQG. In a chosen factor ordering, the Hubble operator \(\hat{H}\) and matter density \(\hat{\rho}\) have the simple forms

\[
\hat{H} = \pi \ell^2_{Pl} \alpha^{-1} \sin(2\tilde{b}), \quad \hat{\rho} = \rho_c \sin^2(\tilde{b}).
\]

This allows us to express \(\hat{H}^2\) in terms of \(\hat{\rho}\). Taking the expectation values, one arrives at the modified Friedmann equation

\[
\langle \hat{H}^2 \rangle = \frac{8\pi G}{3} \langle \hat{\rho}_D \rangle \left(1 - \frac{\langle \hat{\rho}_D \rangle}{\rho_c}\right) - \left[\frac{8\pi G}{3} \frac{\sigma_H^2}{\rho_c} + \frac{\sigma_{\rho}^2}{H^2}\right], \tag{23}
\]

where \(\sigma_H\) and \(\sigma_{\rho}\) are the dispersions of \(\hat{H}\) and \(\hat{\rho}\), respectively.

This equation is exact for all elements of \(H_G\) and generalizes to any matter type. For if the term \(H_\phi\) in (3) is nonzero, the above equation remains valid: the formula for energy density is unchanged, but the density is now to be interpreted as that of all non-gravitational matter: \(\rho_{\text{tot}} := - : V^{-1}\hat{H}_G :\). The form of (23) implies a bounce for all matter for which energy density grows unboundedly with \(V^{-1}\).

Classical effective dynamics. The dust frame formalism can also be applied to the ‘effective’ approach developed in [14], where it suggests a useful improvement. There one begins with a pair of canonical variables \((x, p)\) (or set of pairs) subject to constraints, and defines quantum evolution via a set of equations of motion for the quantities \(\langle \hat{x} \rangle, \langle \hat{p} \rangle\) and \(G^{m,n} = \langle (\hat{x} - \langle \hat{x} \rangle)^m(\hat{p} - \langle \hat{p} \rangle)^n \rangle\). Provided that the constrained system is under sufficient control, the equations for expectation values can describe to a good precision the dynamics for sufficiently large classes of physical states. However, implementation of this approach requires a procedure for converting kinematical observables to the physical ones. Explicit deparametrization produces a square root problem which requires careful approximation, and this has not yet been satisfactorily achieved. In the dust frame presented here, this problem vanishes allowing this approach to be unambiguously implemented.

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5. Summary

We have presented applications to cosmology of a formulation of quantum gravity with dust time in which the gravitational part of the Hamiltonian constraint becomes the true physical Hamiltonian. This has several consequences which follow directly from the fact that the kinematical Hilbert space and observables become physical. Foremost among them is that the physical Hamiltonian is not a square root and so allows an analytical treatment in cases where it was not possible before, and permits a clear numerical approach where necessary.

We studied several examples in the WDW and LQC with cosmological constant, all of which were shown to be analytically solvable. We also outlined possible further developments, including extensions to any matter type and potential, which are not possible with scalar field time.

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