Data-based design of stabilizing switching signals for discrete-time switched linear systems

Atreyee Kundu

Abstract—This paper deals with stabilization of discrete-time switched linear systems when explicit knowledge of the state-space models of their subsystems are not available. Given the sets of indices of the stable and unstable subsystems, the set of admissible switches between the subsystems, the admissible dwell times on the subsystems and a simulation model from which finite traces of state trajectories of the switched system can be collected, we devise an algorithm that designs periodic switching signals which preserve stability of the resulting switched system. We combine two ingredients: (a) data-based stability analysis of discrete-time linear systems and (b) multiple Lyapunov-like functions and graph walks based design of stabilizing switching signals, for this purpose. A numerical example is presented to demonstrate the proposed algorithm.

I. INTRODUCTION

A switched system has two ingredients — a family of systems and a switching signal. The switching signal selects an active subsystem at every instant of time, i.e., the system from the family that is currently being followed [13, §1.1.2]. Switched systems find wide applications in power systems and power electronics, automotive control, aircraft and air traffic control, network and congestion control, etc. [3, p. 5].

A. Motivation

It is well-known that a switched system does not necessarily inherit qualitative properties of its constituent subsystems. For instance, divergent trajectories may be generated by switching appropriately among stable subsystems while a suitably constrained switching signal may ensure stability of a switched system even if all subsystems are unstable (see e.g., [13, p. 19] for examples with two subsystems). Due to such interesting features, the problem of designing stabilizing switching signals for switched systems has attracted considerable research attention in the recent past. This design problem typically requires the availability of mathematical models of the subsystems, see e.g., the recent works [11], [9], [7], [8] and the references therein. However, in many real-world scenarios, particularly for large-scale complex systems, accurate mathematical models of the subsystems, such as transfer functions, state-space models or kernel representations, are often difficult to infer. This interesting fact motivates the current paper. We devise an algorithm to design stabilizing switching signals for discrete-time switched linear systems when explicit knowledge of the state-space models of their subsystems are not available.

B. Literature survey

Stability analysis and control synthesis of switched systems without explicitly involving mathematical models of their subsystems, are dealt with recently in [6], [12], [1]. The work [6] addresses the problem of deciding stability of a discrete-time switched linear system under arbitrary switching from a set of finite traces of state trajectories. Probabilistic stability guarantees are provided as a function of the number of available state observations and a desired level of confidence. In [12] reinforcement learning techniques are employed for optimal control of switched linear systems. A Q-learning based algorithm is proposed to design a discrete switching signal and a continuous control signal such that a certain infinite-horizon cost function is minimized. The convergence guarantee of the proposed algorithm is, however, not available. A randomized polynomial-time algorithm for the design of switching signals under the availability of certain information about the multiple Lyapunov(-like) functions [13, §3.1] corresponding to the individual subsystems in an expected sense, is presented in [1]. The authors show that if it is allowed to switch from any subsystem to a certain number of stable subsystems, then a switching signal obtained from the proposed algorithm is stabilizing with overwhelming probability.

C. Our contributions

We assume that the knowledge of the sets of indices of the stable and unstable subsystems, the set of admissible switches between the subsystems, and the admissible dwell times on the subsystems are available. In addition, we have access to a simulation model from which finite traces of state-trajectories of the switched system can be collected. In modern industrial setups, simulation is of prior importance and it is often possible to simulate very complex systems and collect data. However, the simulation model may be provided by the system manufacturer to study the system behaviour with respect to various sets of inputs prior to their application to the actual system. As a result, the mathematical model underlying the simulator is not known to the user. We will utilize such a simulation model towards collecting useful data about the subsystems.

We combine two ingredients: (a) data-based techniques for stability analysis of discrete-time linear systems and (b) multiple Lyapunov-like functions and graph walks based techniques for the design of stabilizing switching signals for switched systems, towards arriving at an algorithm that designs stabilizing switching signals in the absence of state-
space models of the subsystems. Our algorithm involves the following steps:

- First, sets of data, satisfying certain conditions, corresponding to each subsystem are collected from the given simulation model.
- Second, multiple Lyapunov-like functions, one for each subsystem, and a set of corresponding scalars are computed from the above set of data.
- Third, a cycle on the underlying directed graph that satisfies certain conditions involving the above set of scalars and the admissible dwell times on the subsystems, is identified.

A stabilizing periodic switching signal is designed by activating the vertices that appear in the above cycle and by dwelling on them for appropriate durations. If no favourable cycle is found on the underlying directed graph of the switched system, then the algorithm reports a failure.

The design of stabilizing switching signals for switched systems by involving multiple Lyapunov-like functions and graph-theoretic tools is standard in the literature, see e.g., [10], [7], [8] and the references therein. However, the suitable Lyapunov-like functions employed in this task, are commonly designed under complete knowledge of the state-space models of the subsystems. We assume certain properties of the data collected from the given simulation model, and involve techniques of data-based stability analysis of linear systems proposed in [15] to design multiple Lyapunov-like functions and compute a set of relevant scalars. These scalars together with the admissible switches between the subsystems and admissible dwell times on the subsystems are then employed to design stabilizing switching signals. At this point, it is worth highlighting that we do not opt for the construction of state-space models of the subsystems from the given data, and hence the proposed technique does not involve system identification of the subsystems. To the best of our knowledge, this is the first instance in the literature where the design stabilizing switching signals for switched systems in the setting of multiple Lyapunov-like functions and graph-theoretic tools is addressed without explicit knowledge of the state-space models of the subsystems.

D. Paper organization

The remainder of this paper is organized as follows: In §II we formulate the problem under consideration. A set of preliminaries required for our result is presented in §III. Our result appears in §IV. We present a numerical example in §V and conclude in §VI with a brief discussion of future research directions.

E. Notation

$\mathbb{R}$ is the set of real numbers and $\mathbb{N}$ is the set of natural numbers, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. We denote by $[k_1 : k_2]$ the set \( \{n \in \mathbb{N}_0 : k_1 \leq n \leq k_2\} \). $I_d$ denotes the $d \times d$ identity matrix, $0_n$ and $\overline{0}_n$ denote the $d \times 1$ zero matrix and $d \times d$ zero matrix, respectively. For a matrix $B \in \mathbb{R}^{d \times d}$, $B > 0$ (resp., $B < 0$) denotes that $B$ is positive definite (resp., negative definite), and $\lambda_{\text{max}}(B)$ denotes the maximal eigenvalue of $B$.

II. Problem statement

We consider a family of discrete-time linear systems

$$x(t + 1) = A_i x(t), \quad x(0) = x_0, \quad i \in \mathcal{P}, \quad t \in \mathbb{N}_0,$$

where $x(t) \in \mathbb{R}^d$ is the vector of states at time $t$, $\mathcal{P} = \{1, 2, \ldots, N\}$ is an index set, and

$$A_i = \begin{pmatrix}
-a_{i,1} & \cdots & -a_{i,j} & \cdots & -a_{i,0} \\
1 & \cdots & 0 \\
\vdots & \ddots & \vdots \\
\cdots & & 1 & 0
\end{pmatrix} \in \mathbb{R}^{d \times d}, \quad i \in \mathcal{P}$$

are full-rank constant matrices. Let $\sigma : \mathbb{N}_0 \to \mathcal{P}$ be a switching signal. A discrete-time switched linear system generated by the family of systems and a switching signal $\sigma$ is described as

$$x(t + 1) = A_{\sigma(t)} x(t), \quad x(0) = x_0, \quad t \in \mathbb{N}_0,$$

where we have suppressed the dependence of $x$ on $\sigma$ for notational simplicity.

Let $\mathcal{P}_S$ and $\mathcal{P}_U$ denote the sets of indices of the Schur stable and unstable subsystems, respectively, $\mathcal{P} = \mathcal{P}_S \cup \mathcal{P}_U$. We let $E(\mathcal{P})$ be the set of ordered pairs $(i, j)$ such that a switch from subsystem $i$ to subsystem $j$ is admissible, $i, j \in \mathcal{P}$, $i \neq j$, and $\delta \in \mathbb{N}$ and $\Delta \in \mathbb{N}$ be the admissible minimum and maximum dwell times on all subsystems, respectively, $\delta \leq \Delta$. Let $0 =: \tau_0 < \tau_1 < \tau_2 < \cdots$ be the switching instants; these are the points in time where $\sigma$ jumps. A switching signal $\sigma$ is said to be admissible if it satisfies the following conditions: $(\sigma(\tau_k), \sigma(\tau_{k+1})) \in E(\mathcal{P})$ and $\tau_{k+1} - \tau_k \in [\delta : \Delta]$, $k = 0, 1, 2, \ldots$. Let $\mathcal{S}$ denote the set of all admissible $\sigma$. We are interested in stability of the switched system. Recall that

Definition 1: The switched system is globally asymptotically stable (GAS) for a given switching signal $\sigma$ if it is Lyapunov stable and globally asymptotically convergent, i.e., for all $x(0), \lim_{t \to \infty} \|x(t)| = 0$.

The scalars $a_{i,j}, j = 0, 1, \ldots, d - 1, i \in \mathcal{P}$, are unknown. The sets $\mathcal{P}_S$, $\mathcal{P}_U$, $E(\mathcal{P})$, and the scalars $\delta, \Delta$ are known. In addition, a simulation model, $\mathcal{M}$, is available. The following task can be performed with $\mathcal{M}$: input a pair of finite sequences $\left(\sigma(\tau_0), \sigma(\tau_1), \ldots, \sigma(\tau_M), (D_{\sigma(\tau_0)} D_{\sigma(\tau_1)} \ldots D_{\sigma(\tau_M)})^{M}ight)$, where $M \in \mathbb{N}$, $\sigma(\tau_k) \in \mathcal{P}$, $D_{\sigma(\tau_k)} \in [\delta : \Delta]$, $k = 0, 1, \ldots, M$, and obtain a trace of state trajectories, $x(0), x(1), \ldots, x(L)$, of the switched system, where $\tau_0 = 0$, $x(0) \in \mathbb{R}^d$ is chosen randomly by $\mathcal{M}$, $\tau_{k+1} = \tau_k + D_{\sigma(\tau_k)}, k = 0, 1, \ldots, M = 1$, $x(T + 1) = A_j x(T), T = \tau_1, \tau_2 + 1, \ldots, \tau_{k+1} - 1, j = \sigma(\tau_k)$.

A matrix $B \in \mathbb{R}^{d \times d}$ is Schur stable if all its eigenvalues are inside the open unit disk. We call $B$ unstable if it is not Schur stable.
we have

\[ V_i(\gamma_i(t + 1)) \leq \lambda_i V_i(\gamma_i(t)), \quad t \in \mathbb{N}_0, \] 

and \( \gamma_i(\cdot) \) solves the \( i \)-th recursion in (1). The functions \( V_i, i \in \mathcal{P} \) are Lyapunov-like functions corresponding to the subsystems \( i \in \mathcal{P} \). The scalar \( \lambda_i, i \in \mathcal{P} \) gives a quantitative measure of (in)stability of subsystem \( i \).

The Lyapunov-like functions corresponding to the individual subsystems are related as follows:

**Fact 2:** [10, Fact 2] There exists \( \mathbb{R} \ni \mu_{ij} > 0 \) such that

\[ V_i(\xi) \leq \mu_{ij} V_j(\xi) \quad \text{for all} \quad \xi \in \mathbb{R}^{d_i}, \quad (i, j) \in \mathcal{E}(\mathcal{P}). \]  

A tight estimate of \( \mu_{ij} \) is provided below.

**Proposition 1:** [10, Proposition 1] The scalars \( \mu_{ij}, (i, j) \in \mathcal{E}(\mathcal{P}) \) can be computed as follows:

\[ \mu_{ij} = \lambda_{\max}(P_i P_j^{-1}), \quad (i, j) \in \mathcal{E}(\mathcal{P}). \]

We associate a directed graph \( G(\mathcal{P}, \mathcal{E}(\mathcal{P})) \) with the family of systems (1) and the set of admissible switches between the subsystems, \( \mathcal{E}(\mathcal{P}) \), as follows: the set of vertices of \( G \) is the set of indices of the subsystems, \( \mathcal{P} \), and the set of edges of \( G \) is the set of admissible switches between the subsystems, \( \mathcal{E}(\mathcal{P}) \). A walk on \( G \) is a finite alternating sequence of vertices and edges, \( W = v_0, (v_0, v_1), v_1, \ldots, v_{\ell-1}, (v_{\ell-1}, v_\ell), \ell \leq \ell \) where \( v_k \in \mathcal{P}, (v_k, v_{k+1}) \in \mathcal{E}(\mathcal{P}), k = 0, 1, \ldots, \ell - 1 \). A walk is a cycle if \( v_\ell := v_0 \) and \( v_0, v_1, \ldots, v_{\ell-1} \) are distinct.

Let \( \bar{w} : \mathcal{P} \rightarrow \mathbb{R} \) and \( \omega : \mathcal{E}(\mathcal{P}) \rightarrow \mathbb{R} \) be defined as

\[
\bar{w}(i) = \begin{cases} 
-|\ln \lambda_i|, & \text{if } i \in \mathcal{P}_S, \\
|\ln \lambda_i|, & \text{if } i \in \mathcal{P}_U,
\end{cases}
\]

\[ \omega((i, j)) = \ln \mu_{ij}, \quad (i, j) \in \mathcal{E}(\mathcal{P}), \]

where the scalars \( \lambda_i, i \in \mathcal{P} \) and \( \mu_{ij}, (i, j) \in \mathcal{E}(\mathcal{P}) \) are as described in Facts 1 and 2 respectively.

**Definition 2:** A cycle \( W = v_0, (v_0, v_1), v_1, \ldots, v_{\ell-1}, (v_{\ell-1}, v_\ell), v_\ell \) on \( G \) is called contractive if there exist \( D_{v_k} \in [\delta : \Delta], k = 0, 1, \ldots, \ell - 1 \), such that the following condition holds:

\[ \sum_{k=0}^{\ell-1} \bar{w}(v_k) D_{v_k} + \sum_{k=0}^{\ell-1} \sum_{v_{k+1} \neq v_k} \omega((v_k, v_{k+1})) < 0. \]  

Given a contractive cycle \( W = v_0, (v_0, v_1), v_1, \ldots, v_{\ell-1}, (v_{\ell-1}, v_\ell), v_\ell \) on \( G \) and the corresponding \( D_{v_0}, D_{v_1}, \ldots, D_{v_{\ell-1}} \), the following algorithm [8, Algorithm 1] constructs a switching signal \( \sigma \).

**Algorithm 1 Construction of switching signals**

**Input:** A contractive cycle, \( W = v_0, (v_0, v_1), v_1, \ldots, v_{\ell-1}, (v_{\ell-1}, v_\ell), v_\ell \) on \( G \) and the corresponding \( D_{v_0}, D_{v_1}, \ldots, D_{v_{\ell-1}} \).

**Output:** A switching signal, \( \sigma \).

1: Set \( p = 0 \) and \( \tau_0 = 0 \).
2: for \( k = p\ell, p\ell + 1, \ldots, (p + 1)\ell - 1 \) do
3: \[ \text{Set } \sigma(\tau_k) = v_{k-p\ell} \text{ and } \tau_{k+1} = \tau_k + D_{v_{k-p\ell}}. \]
4: end for
5: Set \( p = p + 1 \) and go to 2.

By construction, \( \sigma \) is periodic with period \( \sum_{k=0}^{\ell-1} D_{v_k} \).

**Lemma 1:** [8, Theorem 1] Consider a switching signal \( \sigma \) obtained from Algorithm 1. The following are true:

i) \( \sigma \in \mathcal{S} \), and

ii) \( S \) is GAS under \( \sigma \).

**Algorithm 2 Detection of contractive cycle on**

**Input:** The underlying directed graph, \( G(\mathcal{P}, \mathcal{E}(\mathcal{P})) \), of the switched system (1), the scalars \( \lambda_i, i \in \mathcal{P} \) and \( \mu_{ij}, (i, j) \in \mathcal{E}(\mathcal{P}) \), the admissible minimum and maximum dwell times on the subsystems, \( \delta \) and \( \Delta \).

**Output:** A contractive cycle, \( W \) on \( G \), if exists.

1: Associate edge weights, \( \alpha : \mathcal{E}(\mathcal{P}) \rightarrow \mathbb{R} \), to \( G \), as follows:

\[ \alpha(i, j) = \begin{cases} 
|\ln \mu_{ij} - |\ln \lambda_i| \Delta, & \text{if } i \in \mathcal{P}_S, \\
|\ln \mu_{ij} + |\ln \lambda_i| \delta, & \text{if } i \in \mathcal{P}_U.
\end{cases} \]

2: Apply a negative cycle detection algorithm to \( G \) to obtain a cycle \( W = v_0, (v_0, v_1), v_1, \ldots, v_{\ell-1}, (v_{\ell-1}, v_\ell), v_\ell \) that satisfies

\[ \sum_{k=0}^{\ell-1} \alpha(v_k, v_{k+1}) < 0. \]
B. Data-based design of multiple Lyapunov-like functions

Notice that the existence of contractive cycles on the underlying directed graph of the switched system (3) depends on three factors: (i) the scalars $\lambda_i$, $i \in P$ and $\mu_{ij}$, $(i,j) \in E(P)$, (ii) the connectivity of $G$, and (iii) the scalars, $\delta$ and $\Delta$. While (ii) and (iii) are pre-specified, we require a mechanism to design (i) by collecting data from $M$. The remainder of this section caters to the above purpose.

Let $x_i(T)$ denote the $T$-th element of the trace $x(0), x(1), \ldots, x(L)$ obtained from $M$ corresponding to an input $\left((\sigma(\tau_0)), (D_{\sigma(\tau_0)})\right)$. where $M = 0$, $\sigma(\tau_0) = i$ and $[\delta : \Delta] \ni D_{\sigma(\tau_0)} = L + 1$, $T = 0, 1, \ldots, L$. We let $x_i^{(p)}(T)$ be the $p$-th element of $x_i(T)$, $p = 1, 2, \ldots, d$. We define

$$q_i(T) = \begin{pmatrix} x_i^{(1)}(T + 1) \\ x_i^{(1)}(T) \\ \vdots \\ x_i^{(d)}(T) \end{pmatrix}, T = 0, 1, \ldots, L - 1.$$ 

Let

$$\Psi_i = (q_i(T), q_i(T+1) \cdots q_i(T + d - 1)) \in \mathbb{R}^{(d + 1)d},$$

$T \in \{0, 1, \ldots, L - 1\}$ be such that the column vectors are linearly independent.

Fix $i \in P$. Notice that whether $\Psi_i$ is well-defined or not, depends on the initial value, $x_i(0)$, and the length, $L + 1$, of the available trace $(x_i(0), x_i(1), \ldots, x_i(L))$. Indeed, to design $\Psi_i$, we need $L \geq d$ and $x_i(0)$ is such that the vectors $q_i(T), q_i(T + 1), \ldots, q_i(T + d - 1)$ are defined and linearly independent for some $T \in \{0, 1, \ldots, L - 1\}$. From [14, Lemma 3] it follows that for every $A_i$ in the companion form described in (2), there exists $x_i(0) \in \mathbb{R}^d$ such that $\Psi_i$ is well-defined with $T = 0$ and $L = d$. We will, therefore, operate under Assumption $1: \Delta = d + 1$.

Clearly, a multiple (if required) traces of state trajectories of the subsystem $i$ can be collected from $M$ to arrive at a well-defined $\Psi_i$.

Lemma 2: For each subsystem $i \in P_S$, condition (5) is equivalent to the following: there exists a symmetric and positive definite matrix $P_i \in \mathbb{R}^{d \times d}$ and a scalar $0 < \lambda_i < 1$ such that

$$\Psi_i^T \begin{pmatrix} I_n & 0_n \\ 0_n & I_n \end{pmatrix}^T \begin{pmatrix} P_i & \overline{0}_n \\ \overline{0}_n & -\lambda_i I_n \end{pmatrix} \begin{pmatrix} I_n & 0_n \\ 0_n & I_n \end{pmatrix} \Psi_i < 0. \quad (9)$$

Proof: Follows under the set of arguments employed in [15, Theorem 2].

Lemma 3: For each subsystem $i \in P_U$, condition (5) is equivalent to the following: there exists a symmetric and positive definite matrix $P_i \in \mathbb{R}^{d \times d}$ and a scalar $\lambda_i > 1$ such that (9) holds.

Proof: Fix $i \in P_U$. There exists $0 < \eta_i < 1$ such that $\eta_i A_i$ is Schur stable.

Now, condition (5) with $\gamma_i(t + 1) = \eta_i A_i \gamma_i(t)$ and $\lambda_i = \overline{\lambda}_i$ can be rewritten as

$$\gamma_i(t)^T \eta_i A_i^T P_i \eta_i A_i \gamma_i(t) \leq \overline{\lambda}_i \gamma_i(t)^T P_i \gamma_i(t),$$
or equivalently,

$$\gamma_i(t)^T A_i^T P_i \gamma_i(t) \leq \frac{\overline{\lambda}_i}{\eta_i} \gamma_i(t)^T \gamma_i(t).$$

In view of Lemma 2, the satisfaction of the above inequality is equivalent to the existence of $P_i > 0$ and $\frac{\overline{\lambda}_i}{\eta_i} \in \mathbb{R}$ such that condition (5) holds with $\lambda_i = \frac{\overline{\lambda}_i}{\eta_i}$. Since $0 < \overline{\lambda}_i, \eta_i < 1$, it follows that $\lambda_i > 1$.

Lemmas 2 and 3 provide a mechanism to design pairs $(A_i, P_i)$, $i \in P$, defined in Fact 1 from the data collected from $M$. The scalars $\mu_{ij}$, $(i,j) \in E(P)$ can then be computed by employing Proposition 1.

We are now in a position to present our solution to Problem 1.

IV. Result

Given the indices of the stable and unstable subsystems, $P_S$ and $P_U$, the set of admissible switches between the subsystems, $E(P)$, the admissible minimum and maximum dwell times on the subsystems, $\delta$ and $\Delta$, and a simulation model, $M$, Algorithm 5 designs a stabilizing periodic switching signal $\sigma \in S$ by employing the following steps:

1. First, the underlying directed graph, $G(P, E(P))$, of the switched system (3) is constructed from the knowledge of $P_S$, $P_U$, and $E(P)$.

2. Second, the matrices $\Psi_i$, $i \in P$ are constructed by collecting data from $M$. The inputs $(i, \Delta)$, $i \in P$ are provided to $M$, and state trajectories $x_i(0), x_i(1), \ldots, x_i(\Delta - 1)$ are collected until well-defined $\Psi_i$, $i \in P$ are obtained.

3. Third, the matrices $\Psi_i$, $i \in P$ are used to compute the scalars $\lambda_i$, $i \in P$ and $\mu_{ij}$, $(i,j) \in E(P)$ by employing Lemmas 2-3 and Proposition 1 respectively. Notice that the choice of the pairs $(P_i, \lambda_i)$, $i \in P$ (and hence the corresponding scalars $\mu_{ij}$, $(i,j) \in E(P)$) is not unique. We store all pairs $(P_i, \lambda_i)$, $i \in P$ that satisfy condition (9) in $\Lambda_i$, $i \in P$, and their corresponding $\mu_{ij}$, $(i,j) \in E(P)$ in $\chi_{ij}$, $(i,j) \in E(P)$.

4. Fourth, corresponding to all elements in $\Lambda_i$, $i \in P$ and their corresponding elements in $\chi_{ij}$, $(i,j) \in E(P)$, Algorithm 2 is applied to $G$ until a contractive cycle $W$ is obtained. If $G$ does not admit any such cycle, then Algorithm 4 reports a failure and terminates. Otherwise, $W$ is employed to design a stabilizing periodic switching signal that activates the subsystems whose corresponding vertices appear in $W$ and dwells on each stable and unstable subsystems for $\Delta$ and $\delta$ units of time, respectively.

Observe that solving (9) with both $P_i$ and $\lambda_i$, $i \in P$, unknown is a numerically difficult task. To address this issue, in Algorithm 4 we employ a line search technique [2] as follows:

1. For $i \in P_S$, a finite set of values of $\lambda_i$ on the interval $[0, 1]$ is fixed, and corresponding to each element of this set, the feasibility problem (10) is solved for symmetric and positive definite matrices, $P_i$, $i \in P$.
Algorithm 3 Model-free computation of stabilizing periodic switching signals

Input: The sets of indices of the stable and unstable subsystems, $\mathcal{P}_S$ and $\mathcal{P}_U$, the set of admissible switches between the subsystems, $E(\mathcal{P})$, the admissible minimum and maximum dwell times on the subsystems, $\delta$ and $\Delta$, and a simulation model, $M$.

Output: A stabilizing periodic switching signal, $\sigma$, or a failure message.

1: **Step I:** Construct the underlying directed graph, $G(\mathcal{P}, E(\mathcal{P}))$, of the switched system (3).
2: **Step II:** Collect data from $M$ and construct $\Psi_i$, $i \in \mathcal{P}$.
3: for $i = 1, 2, \ldots, N$ do
4: Input $(i, \Delta)$ to $M$ and obtain $x_i(0), x_i(1), \ldots, x_i(\Delta - 1)$ until a well-defined $\Psi_i$ is obtained.
5: end for
6: **Step III:** Apply Algorithm 4 to compute $\lambda_i$, $i \in \mathcal{P}$ and $\mu_{ij}$, $(i, j) \in E(\mathcal{P})$ from $\Psi_i$, $i \in \mathcal{P}$.
7: **Step IV:** Detect a contractive cycle on $G$.
8: for each $(P_i, \lambda_i) \in \Lambda_i$ and the corresponding $\mu_{ij}$, $(i, j) \in E(\mathcal{P})$ do
9: Apply Algorithm 2.
10: if a contractive cycle $W = v_0, (v_0, v_1), v_1, \ldots, v_{s-1}, (v_{s-1}, v_0), v_0$ on $G$ is detected then
11: Go to Step V.
12: else
13: Output FAIL and terminate.
14: end if
15: end for
16: **Step V:** Design a stabilizing periodic switching signal.
17: Apply Algorithm 1 with $\mathcal{P}$, $\mathcal{P}_i$, $i \in \mathcal{P}$, and the matrices $\Psi_i$, $i \in \mathcal{P}$ obtained from Algorithm 3. Consequently, a failure message obtained from Algorithm 3 does not indicate the non-existence of a set of pairs $(P_i, \lambda_i)$, $i \in \mathcal{P}$, that $G$ admits a contractive cycle. In addition, Lemma 1 provides a sufficient condition for the stability of (3). As a result, a failure message obtained from Algorithm 3 also does not conclude the non-existence of stabilizing elements in $S$.

Algorithm 4 Computation of $\lambda_i$, $i \in \mathcal{P}$ and $\mu_{ij}$, $(i, j) \in E(\mathcal{P})$ from $\Psi_i$, $i \in \mathcal{P}$

Input: The matrices $\Psi_i$, $i \in \mathcal{P}$.

Output: Sets of scalars $\lambda_i$, $i \in \mathcal{P}$ and $\mu_{ij}$, $(i, j) \in E(\mathcal{P})$.

1: **Step I:** Compute $(P_i, \lambda_i)$ from $\Psi_i$, $i \in \mathcal{P}_S$.
2: Fix $h_u > 0$ (small enough) and compute $k_u \in \mathbb{N}$ such that $k_u$ is the largest integer satisfying $k_u h_u < 1$.
3: for all $i \in \mathcal{P}_S$ do
4: Set $\Lambda_i = \emptyset$.
5: for $\lambda_i = h_u, 2h_u, \ldots, k_u h_u$ do
6: Solve the following feasibility problem in $P_i$:
7: minimize $1$
8: subject to $\begin{cases} \text{condition (9)} \\ P_i = P_i > 0. \end{cases}$ (10)
9: if a solution to (10) exists, then
10: Set $\Lambda_i = \Lambda_i \cup \{(P_i, \lambda_i)\}$.
11: end if
12: end for
13: for all $i \in \mathcal{P}_U$ do
14: Set $\Gamma_i = \emptyset$ and $\Lambda_i = \emptyset$.
15: for $\eta_i = h_u, 2h_u, \ldots, k_u h_u$ do
16: if $\eta_i \Lambda_i$ is Schur stable then
17: Set $\Gamma_i = \Gamma_i \cup \{\frac{1}{\eta_i}\}$.
18: end if
19: end for
20: end for
21: for all $\lambda_i \in \Gamma_i$ do
22: Solve the feasibility problem (10) in $P_i$.
23: if a solution to (10) exists then
24: Set $\Lambda_i = \Lambda_i \cup \{(P_i, \lambda_i)\}$.
25: end if
26: end for
27: end for
28: **Step III:** Compute $\mu_{ij}$, $(i, j) \in E(\mathcal{P})$ from $P_i$, $i \in \mathcal{P}$.
29: Set $\chi_{ij} = \emptyset$.
30: for all $(i, j) \in E(\mathcal{P})$ do
31: for each pair $(P_i, \lambda_i), (P_j, \lambda_j) \in \Lambda_i$ do
32: Compute $\mu_{ij} = \lambda_{\max}(P_{ij} P_{ij}^{-1})$.
33: Set $\chi_{ij} = \chi_{ij} \cup \{\mu_{ij}\}$.
34: end for
35: end for
Consider a switched system \( \mathcal{P} = \{1, 2, 3, 4, 5\} \) with \( \mathcal{P}_S = \{4, 5\} \), \( \mathcal{P}_U = \{1, 2, 3\} \), \( E(\mathcal{P}) = \{(1, 2), (1, 4), (1, 5), (2, 3), (2, 4), (2, 5), (3, 4), (3, 5), (4, 5), (5, 1), (5, 4)\} \), \( \delta = 2 \) and \( \Delta = 6 \). The numerical values of \( A_i, i \in \mathcal{P} \), are given in Table I.

| \( i \) | \( A_i \) |
|------|--------|
| 1    | \( 0.68009919  -0.0450156  0.8335225  0.2474328  0.6490691 \) |
| 2    | \( -0.5271101  0.3125321  -0.6457325  0.8071877  0.6392888 \) |
| 3    | \( -0.442885  0.5338433  0.6488334  0.8689565  0.3665932 \) |
| 4    | \( 0.6482512  0.0275798  -0.4435161  0.1849962  0.053028 \) |
| 5    | \( 0.2486157  0.0089903  -0.0930776  0.3252463  -0.1238403 \) |

We first demonstrate that the switched system under consideration is not stable under all \( \sigma \in \mathcal{S} \). Towards this end, we fix \( \tau_0 = 0 \), pick \((\sigma(\tau_k), \sigma(\tau_k+1)) \in E(\mathcal{P}), \tau_{k+1} - \tau_k \in [\delta : \Delta]\), \( k = 0, 1, 2, \ldots \), uniformly at random, and plot \( \|x(t)\|_{t \in [0, 5]} \) corresponding to \( x(0) \in \mathbb{R}^5 \) chosen uniformly at random from the interval \([-1, 1]^5\) and the \( \sigma \) designed above. We observe instability of \( \mathcal{P} \), see Figure 1. It is, therefore, of interest to identify stabilizing elements of \( \sigma \).

The values of \( a_{i,j}, j = 0, 1, \ldots , d-1, i \in \mathcal{P} \) are not known, but the sets \( \mathcal{P}_S, \mathcal{P}_U, E(\mathcal{P}) \) and the scalars \( \delta, \Delta \) are known.

In addition, a simulation model, \( \mathcal{M} \), is available\(^3\). We will apply Algorithm \( \mathcal{M} \) to design a stabilizing periodic switching signal \( \sigma \in \mathcal{S} \). It involves the following steps:

Step I. The underlying directed graph, \( G(\mathcal{P}, E(\mathcal{P})) \), of the switched system \( \mathcal{P} \) is constructed with \( \mathcal{P} \) and \( E(\mathcal{P}) \) as described above.

Step II. The matrices, \( \Psi_i, i \in \mathcal{P} \), are constructed by collecting data from the simulation model, \( \mathcal{M} \). The numerical values of \( \Psi_i, i \in \mathcal{P} \) are given in Table I.

Step III. Algorithm \( \mathcal{M} \) is applied with \( h_s = h_a = 0.1 \) to compute the sets \( \Lambda_i, i \in \mathcal{P} \) and \( \chi_{ij}, (i, j) \in \mathcal{E}(\mathcal{P}) \).\(^2\)

Step IV. Algorithm \( \mathcal{M} \) is applied on \( G \). A contractive cycle \( W = (4, 5, 5, 4, 4) \) is obtained with \( \lambda_4 = \lambda_5 = 0.7 \).

We now pick 100 different \( x(0) \in \mathbb{R}^5 \) from the interval \([-1, 1]^5\) uniformly at random, and plot \( \|x(t)\|_{t \in [0, 5]} \) under the switching signal \( \sigma \) obtained from Algorithm \( \mathcal{M} \). See in Figure 2. GAS of the switched system \( \mathcal{P} \) is demonstrated.

VI. Conclusion

In this paper we presented an algorithm that designs stabilizing periodic switching signals for discrete-time switched linear systems under restricted switching. The proposed design technique can be extended to the setting of stabilizing non-periodic switching signals by employing a cycle \( W = v_0, v_1, v_1, \ldots , v_{\ell-1}, v_{\ell-1}, v_0 \) that is contractive on \( G \) with multiple choices of \( D_{V_k} \in [\delta : \Delta] \), \( k = 0, 1, \ldots , \ell - 1 \) and/or multiple contractive cycles \( W_1, W_2, \ldots , W_p \) on \( G \), see e.g., [7, Remarks 6.7,10] for discussions on multiple Lyapunov-like functions and graph walks based design of stabilizing non-periodic switching signals.

\begin{thebibliography}{9}
\bibitem{Balachandran19} N. Balachandran, A. Kundu, and D. Chatterjee, Randomized algorithms for stabilizing switching signals, Math. Control Related. Fields, 9 (2019), pp. 159–174.
\end{thebibliography}

\(^1\) For this experiment, we generate the elements of the matrices \( A_i, i \in \mathcal{P} \) with numbers from the interval \([-1, 1]\) chosen uniformly at random. We also build a simulation model, \( \mathcal{M} \), in Scilab 6.0.2, to generate the state trajectories of \( \mathcal{M} \).

\(^2\) The feasibility problem \( \mathcal{M} \) is solved using the Imisolver tool in Scilab 6.0.2.
Fig. 2: Plot of $\|x(t)\|_{F \in \mathcal{D}}$ under $\sigma$ obtained from Algorithm

\[ \begin{array}{cccc}
\sigma & 1 & 2 & 3 & 4 & 5 \\
\|x(t)\|_{F \in \mathcal{D}} & 1.1109433 & 0.1262875 & 1.1877076 & 1.0762463 & 1.9978929 \\
& 0.7862729 & 1.1190433 & 0.1262875 & 1.1877076 & 1.0762463 \\
& -0.3303575 & 0.7862729 & 1.1190433 & 0.1262875 & 1.1877076 \\
& 0.8915796 & -0.3303575 & 0.7862729 & 1.1190433 & 0.1262875 \\
& -0.8272248 & 0.8915796 & -0.3303575 & 0.7862729 & 1.1190433 \\
& 0.0349338 & -0.8272248 & 0.8915796 & -0.3303575 & 0.7862729 \\
\end{array} \]

TABLE II: Description of $\Psi_i$, $i \in \mathcal{P}$.