**Abstract.** Resistance of social graphs to active attacks is a very important feature which must be maintained in the modern networks. Recently introduced \(k\)-metric antidimension graph invariant is used to define a new measure for resistance of social graphs. In this paper we have found and proved the \(k\)-metric antidimension for generalized Petersen graphs \(GP(n, 1)\) and \(GP(n, 2)\). It is proven that \(GP(2m+1, 1)\) and \(GP(8, 2)\) are 2-metric antidimensional, while all other \(GP(n, 1)\) and \(GP(n, 2)\) graphs are 3-metric antidimensional.

1. Introduction

The notion of \((k, l)\)-anonymity was introduced by Trujillo-Rasua and Yero (2016) in [8]. As explained in that paper the motivation was to establish a new measure for evaluating the resistance of social graphs against active attacks. This measure uses a new graph invariant: \(k\)-metric antidimension.

Let \(G = (V, E)\) be a simple connected graph and \(d(u, v)\) is the length of the shortest path between the vertices \(u\) and \(v\). The metric representation \(r(v|S)\) of vertex \(v\) with respect to an ordered set of vertices \(S = \{u_1, ..., u_t\}\) is defined as \(r(v|S) = (d(v, u_1), ..., d(v, u_t))\). Values \(d(v, u_i)\) are considered as metric coordinates of \(v\) with respect to vertices \(u_i\).

**Definition 1.1.** ([8]) Let \(k\) be the largest positive integer with the property that for every vertex \(v \in V(G) \setminus S\) there exist at least \(k - 1\) different vertices \(v_1, ..., v_{k-1} \in V(G) \setminus S\) with \(r(v|S) = r(v_1|S) = ... = r(v_{k-1}|S)\). In other words, \(v\) and \(v_1, ..., v_{k-1}\) have the same metric representation with respect to \(S\). Then, set \(S\) is called a \(k\)-antiresolving set for \(G\).

**Definition 1.2.** ([8]) For fixed \(k\), the minimum cardinality amongst all \(k\)-antiresolving sets in \(G\) is called the \(k\)-metric antidimension of graph \(G\), and it is denoted by \(\text{adim}_k(G)\). A \(k\)-antiresolving set of that minimum cardinality \(\text{adim}_k(G)\) is called a \(k\)-antiresolving basis of \(G\).

**Definition 1.3.** ([8]) If \(k = \max\{|\text{adim}_t(G)\text{ exists}\}\) then graph \(G\) is called \(k\)-metric antidimensional.

**Observation 1.4.** ([8]) If \(G\) has maximum degree \(\Delta\) and \(G\) is \(k\)-metric antidimensional then \(1 \leq k \leq \Delta\) holds.
In the sequel we shall use the equivalence relation defined in [1, 2]. Let \( S \subseteq V(G) \) be a subset of vertices of a connected graph \( G \) and let \( \rho_S \) be equivalence relation on \( V(G) \setminus S \) defined by

\[
(\forall a, b \in V(G) \setminus S \ (a \rho_S b \iff r(a|S) = r(b|S))
\]

and let \( S_1, \ldots, S_m \) be the equivalence classes of \( \rho_S \). Then the following property can be proved.

**Proposition 1.5.** (1, 2) Let \( k \) be a fixed integer, \( k \geq 1 \). Then \( S \) is a \( k \)-antiresolving set in \( G \) if and only if

\[
\min_{1 \leq i \leq m} |S_i| = k.
\]

In [2, 10] it has been proved that the problem of determining the \( k \)-metric antidimension of a graph for a fixed \( k \) is NP-complete in general case.

For some graphs with special structures it would be interesting to investigate the privacy measure based on the \( k \)-metric antidimension. Such investigations are considered in the literature:

- In [9] are considered 1-metric antidiendimensional trees and unicyclic graphs;
- Privacy violation properties of eight real social networks and large number of synthetic networks generated by both the classical Erdös-Rényi model and the Barabási-Albert preferential-attachment model were analyzed in [4];
- First privacy-preserving graph transformation improving privacy is presented in [6]. Experiments on random graphs show that the proposed method effectively counteracts active attacks;
- \( k \)-metric antidiendimensions of wheels and grid graphs are given in [11].

In this paper we study the \( k \)-metric antidimension of generalized Petersen graphs introduced by Coxeter [3]. The generalized Petersen graph \( GP(n, k) \) \( (n \geq 3; 1 \leq k < n/2) \) has \( 2n \) vertices and \( 3n \) edges, where vertex set \( V \) and edge set \( E \) are defined as follows: \( V = \{u_i, v_i \mid 0 \leq i \leq n - 1\} \), \( E = \{[u_i, u_{i+1}], [u_i, v_i], [v_i, v_{i+k}] \mid 0 \leq i \leq n - 1\} \), with vertex indices taken modulo \( n \). In this notation the well-known Petersen graph presented on Figure 1 is \( GP(5, 2) \).

There are a lot of papers devoted to generalized Petersen graphs and their invariants. Some recent results include: metric dimension [7], strong metric dimension [5], and power domination [11].

**Example 1.6.** Consider the Petersen graph \( G \) given on Figure 1. By total enumeration it is easy to see that \( G \) is 3-antidimensional: 1-antiresolving basis is \( \{u_0, u_2\} \), 2-antiresolving basis is \( \{u_0, v_0\} \), while 3-antiresolving basis is \( \{v_0\} \). Therefore, \( \text{adim}_k(G) = \begin{cases} 2, & k = 1, 2 \\ 1, & k = 3 \end{cases} \).
It should be noted that, according to Definition 1.3, if a graph is \( k \)-metric antidimensional, it does not mean that there exists an \( l \)-antiresolving set for each \( l \in \{2, \ldots, k - 1\} \). For example, wheel graphs studied in [1] are \( n \)-metric antidimensional, but for \( 4 \leq l \leq n - 1 \) there are no \( l \)-antiresolving sets in wheel graphs. Therefore, as mentioned and presented in [2, 4, 11], it is an interesting problem to find families of graphs for which there exist \( l \)-antiresolving sets for all values of \( l \), such that \( 2 \leq l \leq k - 1 \). In the next two sections we show that \( GP(n, 1) \) and \( GP(n, 2) \) satisfy the previous property.

In Section 2 we prove that \( GP(2m, 1) \) is 3-metric antidimensional, while \( GP(2m + 1, 1) \) is 2-metric antidimensional. In Section 3 it is shown that \( GP(n, 2) \) is 3-metric antidimensional, except for \( n = 8 \), when it is 2-metric antidimensional.

2. \( k \)-metric antidimension of \( GP(n,1) \)

\[ \text{Figure 2: Graph } GP(6,1) \]

**Theorem 2.1.** Graph \( GP(2m, 1) \) is 3-metric antidimensional and

(i) \( \text{adim}_1(GP(2m, 1)) = 1 \)

(ii) \( \text{adim}_2(GP(2m, 1)) = 4 \)

(iii) \( \text{adim}_3(GP(2m, 1)) = 2 \)

**Proof.** (i) Let us consider set \( S = \{u_0\} \). The equivalence classes of \( \rho_S \) are given in Table 1. More precisely, the first column of Table 1 contains set \( S \), while in the second one the equivalence classes of relation \( \rho_S \) are given, and in the third column the metric representations with respect to \( S \) are shown for all their vertices. Since the minimal cardinality of equivalence classes is one, according to Property 1.5 it follows that \( S = \{u_0\} \) is 1-antiresolving set. Since \( |S| = 1 \), \( S = \{u_0\} \) is a 1-antiresolving basis of \( GP(2m, 1) \), so \( \text{adim}_1(GP(2m, 1)) = 1 \).

(ii) Due to symmetry of \( GP(2m, 1) \) and the fact that set \( \{u_0\} \) is 1-antiresolving, it follows that every set \( S \) consisting of only one vertex of \( GP(2m, 1) \) is 1-antiresolving. Let us consider sets \( S \) of cardinality two. From symmetry properties of \( GP(2m, 1) \), without loss of generality we can assume \( u_0 \in S \). We have two cases.

**Case 1.** \( v_m \notin S \). Then from Table 1 it follows that \( v_m \) is the only vertex with the metric coordinate with respect to vertex \( u_0 \) which is equal to \( m + 1 \) and, consequently, \( S \) is 1-antiresolving.

**Case 2.** If \( v_m \in S \) then \( S = \{u_0, v_m\} \) and the corresponding equivalence classes are given in Table 1. From Table 2 and Property 1.5 it follows that set \( \{u_0, v_m\} \) is 3-antiresolving.

Cases 1 and 2 demonstrate that there does not exist set \( S \) of cardinality 2 which is 2-antiresolving for \( GP(2m, 1) \).

Next we consider sets \( S \) with cardinality three. Again, we can suppose that \( u_0 \in S \). If we \( v_m \notin S \), as in Case 1, we can conclude that \( S \) is 1-antiresolving. Suppose that \( v_m \in S \) and consider cases \( v_0 \in S \) or \( u_m \in S \). If \( v_0 \in S \), i.e. \( S = \{u_0, v_0, u_m\} \), then equivalence class \( \{u_m, v_{m-1}, v_{m+1}\} \) from Table 1 is partitioned into 2 classes: \( \{u_m\} \) with metric representation equal to \((m, 1, m + 1)\) and \( \{v_{m-1}, v_{m+1}\} \) with metric representation equal to...
Similarly, if $u_m \in S$, i.e. $S = \{u_0, u_m, v_m\}$, then class $\{u_1, u_{m-1}, v_0\}$ from Table 1 is partitioned into $\{u_1, u_{m-1}\}$ with metric representation equal to $(1, m, m-1)$ and $\{v_0\}$ with metric representation equal to $(1, m, m+1)$. Hence, if $u_0, v_m \in S$ and $v_0 \in S$ or $u_m \in S$ set $S$ is 1-antiresolving. Finally, if $u_0, v_m \in S$ and $v_0 \not\in S$ and $u_m \not\in S$ we consider equivalence class $\{u_m, v_{m-1}, v_{m+1}\}$ from Table 1. Table 2 contains distances of $u_m, v_{m-1}, v_{m+1}$ from all possible third elements of $S$. From Table 2 it follows that in all cases equivalence class $\{u_m, v_{m-1}, v_{m+1}\}$ is partitioned with respect to the third coordinate into two classes, one of cardinality 2 and the other of cardinality 1. Consequently, set $S$ is again 1-antiresolving. Therefore, there does not exist set $S$ of cardinality 3 which is 2-antiresolving for $GP(2m, 1)$.

Consider now set $S = \{u_0, v_0, u_m, v_m\}$ of cardinality 4 and the corresponding classes in Table 1. Since all classes have cardinality 2, it follows that $S$ is 2-antiresolving for $GP(2m, 1)$. Since $adim_2(GP(2m, 1)) = 3$, we conclude $adim_2(GP(2m, 1)) = 4$.

(iii) Let $S = \{u_0, v_m\}$. As we have already concluded in (ii), from Table 1 it follows that $S$ is 3-antiresolving set for $GP(2m, 1)$ and consequently $adim_2(GP(2m, 1)) = 2$. Let us prove that there does not exist a 3-antiresolving set $S'$ of cardinality one. By symmetry, we can suppose that $S' = \{u_0\}$. As proved in (i), $S'$ is 1-antiresolving set.

Since $GP(2m, 1)$ is 3-regular, according to Observation 1.4, it follows that $GP(2m, 1)$ is $k$-metric antidimensional for some $k \leq 3$. From (i)-(iii) it follows that $GP(2m, 1)$ is 3-metric antidimensional.

**Theorem 2.2.** Graph $GP(2m+1, 1)$ is 2-metric antidimensional and

(i) $adim_1(GP(2m+1, 1)) = 2$

(ii) $adim_2(GP(2m+1, 1)) = 1$

**Proof.** (i) Let $S = \{u_0, v_1\}$. It is easy to see that vertex $v_2$ has unique metric representation with respect to $S$ equal to $(3, 1)$. According to Property 1.5, $S$ is 1-antiresolving set of $GP(2m+1, 1)$. Let us prove that $S$ is 1-antiresolving basis of $GP(2m+1, 1)$. Suppose contrary, that there exists 1-antiresolving
set \( S' \) of cardinality 1. Without loss of generality, due to the symmetry of \( GP(2m + 1, 1) \), we can assume that \( S' = \{u_0\} \). The equivalence classes of \( \rho_\varphi \) are given in Table 3. From Table 3 it follows that set \( S' \) is 2-antiresolving, which is a contradiction. Therefore, \( S = \{u_0, v_1\} \) is an 1-antiresolving basis of \( GP(2m + 1, 1) \), i.e. \( adim_1(GP(2m + 1, 1)) = 2 \).

(ii) Let \( S = \{u_0\} \). From Table 3 it is evident that set \( S = \{u_0\} \) is 2-antiresolving set of \( GP(2m + 1, 1) \). Since \( |S| = 1, S \) is a 2-antiresolving basis of \( GP(2m + 1, 1) \) and hence \( adim_2(GP(2m + 1, 1)) = 1 \).

From (i) and (ii) it follows that \( GP(2m + 1, 1) \) is \( k \)-metric antidimensional for \( k \geq 2 \). On the other side, according to Observation \( \square \) \( k \leq 3 \). Let us prove that \( GP(2m + 1, 1) \) is not 3-metric antidimensional, i.e. that in this graph there does not exist a 3-antiresolving set. Let \( S \) be a set of vertices from \( V \). Without loss of generality, we can assume \( u_0 \in S \). Consider the following two cases:

Case 1. \( v_m \notin S \) or \( v_{m+1} \notin S \). According to Table 3 the equivalence class with respect to \( S' = \{u_0\} \) with metric coordinate \( m + 1 \) is \([v_m, v_{m+1}]\). Therefore, the equivalence class with respect to \( S, S \supseteq S' \), whose members have distance from \( u_0 \) equal to \( m + 1 \) has cardinality less or equal to 2. It follows that \( S \) is not a 3-metric antidimensional set.

Case 2. Suppose that \( v_m \in S \) and \( v_{m+1} \in S \). Then each vertex \( u_i, i = 1, ..., n - 1, v_i, j = 0, ..., n - 1, j \neq m, m + 1 \) has unique metric representation with respect to \( \{u_0, v_m, v_{m+1}\} \subseteq S \) and therefore \( S \) is 1-antiresolving set. Cases 1 and 2 demonstrate that in \( GP(2m + 1, 1) \) there does not exist a 3-antiresolving set. Therefore, \( GP(2m + 1, 1) \) is 2-metric antidimensional.

Table 3: Equivalence classes of \( \rho_\varphi \) on \( GP(2m, 1) \)

| \( S' \) | Equivalence class | Metric representation |
|-------|-------------------|-----------------------|
| \( \{u_0\} \) | \([u_1, u_{n-1}, v_0]\) | (1) |
|          | \([u_j, u_{n-j}, v_{j-1}, v_{n-j+1}]\) | (i), \( 2 \leq i \leq m \) |
|          | \([v_m, v_{m+1}]\) | \((m + 1)\) |

3. \( k \)-metric antidimension of \( GP(n,2) \)

![Figure 3: Graph GP(9,2)](image)

**Theorem 3.1.** For \( m \neq 2 \) graph \( GP(4m, 2) \) is 3-metric antidimensional and

(i) \( adim_1(GP(4m, 2)) = 2 \)
(ii) \(\text{adim}_2(GP(4m, 2)) = 1\)

(iii) \(\text{adim}_3(GP(4m, 2)) = 1\)

Proof. (i) Let \(S = \{u_0, u_{2m}\}\). It is easy to see that \(v_0\) has unique metric representation \((1, m + 1)\) with respect to \(S\). Therefore, \(S\) is \(1\)-antiresolving set. Suppose that there exists \(1\)-antiresolving set \(S'\) of cardinality 1. Due to the symmetry of \(GP(4m, 2)\), we can assume that \(S' = \{u_0\}\) or \(S' = \{v_0\}\). From Table 5 it can be seen that the equivalence classes in both cases have cardinality at least 2, which is a contradiction. Hence, \(\text{adim}_1(GP(4m, 2)) = 2\).

(ii) Let \(S = \{v_0\}\). According to Table 4, \(S\) is a \(2\)-antiresolving basis of cardinality 1, so \(\text{adim}_2(GP(4m, 2)) = 1\).

(iii) Let \(S = \{u_0\}\). From Table 4 we conclude that \(S\) is a \(3\)-antiresolving basis of \(GP(4m, 2)\), i.e. \(\text{adim}_3(GP(4m, 2)) = 1\).

From (i)-(iii) it follows that \(GP(4m, 2)\) is \(k\)-metric antidimensional for \(k \geq 3\). Since \(GP(4m, 2)\) is \(3\)-regular, according to Observation 1.4, it follows that \(k = 3\), i.e. \(GP(4m, 2)\) is \(3\)-metric antidimensional.

| Table 4: Equivalence classes of \(\rho_5\) on \(GP(4m, 2)\) |
|-------------|-----------|----------------|----------------|
| \(S\) | Equivalence class | Metric representation |
| \(\{u_0\}\) | \(\{u_1, u_{4m-1}, v_0\}\) | \(\{u_0, v_2, v_{4m-2}\}\) | \(\{u_1, u_{2}, u_{4m-2}, u_{4m-1}, v_4, v_{4m-4}\}\) |
| \(\{v_0\}\) | \(\{u_2, u_{2m}, v_{4m+3}\}\) | \(\{u_3, u_{2m+2}, v_{4m+3}\}\) | \(\{u_2, u_{2m+2}, v_{4m+3}\}\) |

Theorem 3.2. Graph \(GP(4m + 1, 2)\) is \(3\)-metric antidimensional and

(i) \(\text{adim}_1(GP(4m + 1, 2)) = 2\)

(ii) \(\text{adim}_2(GP(4m + 1, 2)) = 2\)

(iii) \(\text{adim}_3(GP(4m + 1, 2)) = 1\)

Proof. (i) The proof is similar to the proof of (i) in Theorem 3.1. Let \(S = \{u_0, u_{2m}\}\). Then vertex \(v_0\) has unique metric representation \((1, m + 1)\), which implies that \(S\) is an \(1\)-antiresolving set. Using Table 5 and the same argument as in (i) of Theorem 3.1 we conclude that \(\{u_0\}\) and \(\{v_0\}\) are not \(1\)-antiresolving sets, and due to the symmetry of \(GP(4m + 1, 2)\) the same holds for all singleton subsets of \(V\). Therefore, \(\text{adim}_1(GP(4m + 1, 2)) = 2\).

(ii) Let \(S = \{u_0, v_0\}\). According to Table 5, \(S\) is a \(2\)-antiresolving set since all equivalence classes are of cardinality at least 2. Since by Table 5 equivalence classes for sets \(\{u_0\}\) and \(\{v_0\}\) are of cardinality at least 3, similarly as in (i) we conclude \(\text{adim}_3(GP(4m + 1, 2)) = 2\).

(iii) For \(S = \{v_0\}\), directly from Table 5 it follows that \(\text{adim}_3(GP(4m + 1, 2)) = 1\).

From (i)-(iii) it follows that \(GP(4m + 1, 2)\) is \(k\)-metric antidimensional for \(k \geq 3\). By Observation 1.4 it follows that \(k = 3\), i.e. \(GP(4m + 1, 2)\) is \(3\)-metric antidimensional.
Table 5: Equivalence classes of $\rho_5$ on $GP(4m + 1, 2)$

| S          | Equivalence class                                                                 | Metric representation |
|------------|-----------------------------------------------------------------------------------|-----------------------|
| $[u_0]$    | $\{u_1, u_{4m}, v_0\}$                                                            | (1)                  |
|            | $\{u_i, u_{4m-i+1}, v_{2i-3}, v_{2i-2}, v_{4m-2i+3}, v_{4m-2i+4}\}$              | (i), $i = 2, 3, 4$   |
|            | $\{u_{2i-5}, u_{2i-4}, u_{4m-2i+5}, u_{4m-2i+6}, v_{2i-3}, v_{2i-2}, v_{4m-2i+3}, v_{4m-2i+4}\}$ | (ii), $i = 5, ..., m + 1$ |
|            | $\{u_{2m-1}, u_{2m}, u_{2m+1}, u_{2m+2}\}$                                       | (m + 2)              |
| $[v_0]$    | $\{u_1, u_2, v_{2m-1}\}$                                                          | (1)                  |
|            | $\{u_1, u_2, u_{4m-1}, u_{4m}, v_4, v_{4m-3}\}$                                  | (2)                  |
|            | $\{u_{2i-3}, u_{2i-2}, u_{4m-2i+3}, u_{4m-2i+4}, v_{2i-5}, v_{2i-2}, v_{4m-2i+3}, v_{4m-2i+4}\}$ | (i), $i = 3, ..., m$ |
|            | $\{u_{2m-1}, u_{2m}, u_{2m+1}, u_{2m+2}, v_{2m-3}, v_{2m-1}, v_{2m+2}, v_{2m+4}\}$ | (m + 1)              |
| $[u_0, v_0]$| $\{u_1, u_{4m}\}$                                                                 | (1, 2)               |
|            | $\{v_{2i}, v_{4m-1}\}$                                                            | (2, 1)               |
|            | $\{u_{2i}, u_{4m-1}\}$                                                            | (2, 2)               |
|            | $\{v_{1i}, v_{4m}\}$                                                              | (2, 3)               |
|            | $\{v_{4i}, v_{4m-3}\}$                                                            | (3, 2)               |
|            | $\{u_{4i}, u_{4m-2}\}$                                                            | (3, 3)               |
|            | $\{v_{5i}, v_{4m-2}\}$                                                            | (3, 4)               |
|            | $\{u_{4i}, u_{4m-3}, v_0, v_{4m-5}\}$                                             | (4, 3)               |
|            | $\{u_{2i-5}, u_{2i-4}, u_{4m-2i+5}, u_{4m-2i+6}, v_{2i-2}, v_{4m-2i+3}\}$         | (i, $i - 1$), $i = 5, ..., m + 1$ |
|            | $\{v_{2i-3}, v_{4m-2i+4}\}$                                                       | (i, $i + 1$), $i = 4, ..., m$ |
|            | $\{u_{2m-1}, u_{2m+2}\}$                                                          | (m + 1, m + 1)       |
|            | $\{u_{2m-1}, u_{2m}, u_{2m+1}, u_{2m+2}\}$                                        | (m + 2, m + 1)       |

**Theorem 3.3.** For $m \geq 3$ graph $GP(4m + 2, 2)$ is 3-metric antidimensional and

(i) $adim_1(GP(4m + 2, 2)) = 1$

(ii) $adim_2(GP(4m + 2, 2)) = 2$

(iii) $adim_3(GP(4m + 2, 2)) = 2$

**Proof.** (i) Let $S = \{u_0\}$. Then vertex $u_{2m+1}$ has the unique metric representation $(m + 3)$ and therefore, $adim_1(GP(4m + 2, 2)) = 1$.

(ii) $S = \{u_0, u_{2m+1}\}$. From Table 6, $S$ is a 2-antiresolving set. If we consider singleton subsets of $V$, due to symmetry it is sufficient to analyze cases $\{u_0\}$ and $\{v_0\}$. By (i), $\{u_0\}$ is 1-antiresolving and since $v_{2m+1}$ has unique metric representation $(m + 3)$ with respect to $\{v_0\}$, set $\{v_0\}$ is also 1-antiresolving. It means that all singleton subsets of $V$ are not 2-antiresolving. This implies that $adim_2(GP(4m + 2, 2)) = 2$.

(iii) For $S = \{v_0, v_{2m+1}\}$ from Table 6 it follows that $S$ is a 3-antiresolving set. Since all singleton vertices are 1-antiresolving sets it follows that $adim_3(GP(4m + 2, 2)) = 2$.

From (i)-(iii) it follows that $GP(4m + 2, 2)$ is $k$-metric antidimensional for $k \geq 3$. According to Observation 1.4, it follows that $k = 3$, i.e. $GP(4m + 2, 2)$ is 3-metric antidimensional. □
Table 6: Equivalence classes of $\rho_5$ on $GP(4m + 2, 2)$

| $S$ | Equivalence class | Metric representation |
|-----|-------------------|------------------------|
| $[u_0, u_{2m+1}]$ | $[u_1, u_{4m+1}, v_0]$ | $(1, m + 2)$ |
| | $[u_1, u_{4m+i+1}]$ | $(i, m - i + 2), i = 2, 3, 4$ |
| | $[v_{2i-3}, v_{2i-2}, v_{4m-2i+5}, v_{4m-2i+6}]$ | $(i, m - i + 3), i = 2, ..., m$ |
| | $[u_{2i-5}, u_{2i-4}, u_{4m-2i+6}, u_{4m-2i+7}]$ | $(i, m - i + 5), i = 5, ..., m + 1$ |
| | $[u_{2m-2}, u_{2m+4}]$ | $(m + 1, 3)$ |
| | $[u_{2m-3}, u_{2m+5}]$ | $(m + 1, 4)$ |
| | $[u_{2m}, u_{2m+2}, v_{2m+1}]$ | $(m + 2, 1)$ |
| | $[u_{2m-1}, u_{2m+3}]$ | $(m + 2, 2)$ |
| $[v_0, v_{2m+1}]$ | $[u_0, v_2, v_{4m}]$ | $(1, m + 2)$ |
| | $[u_1, u_2, u_{4m}, u_{4m+1}, v_4, v_{4m-2}]$ | $(2, m + 1)$ |
| | $[u_{2i-3}, u_{2i-2}, u_{4m-2i+4}, u_{4m-2i+5}, v_{2i-5}, v_{4m-2i+2}, v_{4m-2i+7}]$ | $(i, m - i + 3), i = 3, ..., m$ |
| | $[u_{2m-1}, u_{2m}, u_{2m+1}, u_{2m+2}, v_{2m-3}, v_{2m+5}]$ | $(m + 1, 2)$ |
| | $[u_{2m-1}, v_{2m-1}, v_{2m+3}]$ | $(m + 2, 2)$ |

Theorem 3.4. For $m \geq 2$ graph $GP(4m + 3, 2)$ is 3-metric antidimensional and

(i) $adim_1(GP(4m + 3, 2)) = 2$

(ii) $adim_2(GP(4m + 3, 2)) = 1$

(iii) $adim_3(GP(4m + 3, 2)) = 1$

Proof. (i) Let $S = \{u_0, u_2\}$. Then vertex $u_1$ has unique metric representation $(1,1)$ and consequently, $S$ is 1-antiresolving set. Since by Table 7 sets $\{u_0\}$ and $\{v_0\}$ are 2-antiresolving and 3-antiresolving, respectively, then $adim_1(GP(4m + 3, 2)) = 2$.

(ii) and (iii) follow directly from Table 7.

Since $GP(4m + 3, 2)$ is 3-regular, according to Observation 1.4 it follows that $GP(4m + 3, 2)$ is $k$-metric antidimensional for some $k \leq 3$. From (i)-(iii) it follows that $GP(4m + 3, 2)$ is 3-metric antidimensional.

Table 7: Equivalence classes of $\rho_5$ on $GP(4m + 3, 2)$

| $S$ | Equivalence class | Metric representation |
|-----|-------------------|------------------------|
| $[u_0]$ | $[u_1, u_{4m+2}, v_0]$ | $(1)$ |
| | $[u_1, u_{4m+i+3}, v_{2i-3}, v_{2i-2}, v_{4m-2i+5}, v_{4m-2i+6}]$ | $(i), i = 2, 3, 4$ |
| | $[u_{2i-5}, u_{2i-4}, u_{4m-2i+7}, u_{4m-2i+8}, v_{2i-3}, v_{2i-2}, v_{4m-2i+5}, v_{4m-2i+6}]$ | $(i), i = 5, ..., m + 1$ |
| | $[u_{2m-1}, u_{2m}, u_{2m+3}, u_{2m+4}, v_{2m+1}, v_{2m+2}]$ | $(m + 2)$ |
| | $[u_{2m+1}, u_{2m+2}]$ | $(m + 3)$ |
| $[v_0]$ | $[u_0, v_0, v_{4m+1}]$ | $(1)$ |
| | $[u_1, u_2, u_{4m+1}, u_{4m+2}, v_4, v_{4m-1}]$ | $(2)$ |
| | $[u_{2i-3}, u_{2i-2}, u_{4m-2i+5}, u_{4m-2i+6}, v_{2i-5}, v_{4m-2i+3}, v_{4m-2i+8}]$ | $(i), i = 3, ..., m$ |
| | $[u_{2m-1}, u_{2m}, u_{2m+1}, u_{2m+4}, v_{2m-3}, v_{2m+1}, v_{2m+2}, v_{2m+6}]$ | $(m + 1)$ |
| | $[u_{2m+1}, u_{2m+2}, v_{2m-1}, v_{2m+4}]$ | $(m + 2)$ |

The values for the metric antidimension of the cases which are not covered by Theorems 3.1-3.4 are obtained by total enumeration and given in the next two observations.

Observation 3.5. Graph $GP(8, 2)$ is 2-metric antidimensional and $adim_1(GP(8, 2)) = 1$ and $adim_2(GP(8, 2)) = 1$. 

Observation 3.6. Graphs \( GP(6, 2) \), \( GP(7, 2) \) and \( GP(10, 2) \) are 3-metric antidimensional and

\[
adim_k(GP(6, 2)) = \begin{cases} 
1, & k = 1, 2 \\
2, & k = 3 
\end{cases}
\]

\[
adim_k(GP(7, 2)) = \begin{cases} 
2, & k = 1, 2 \\
1, & k = 3 
\end{cases}
\]

\[
adim_k(GP(10, 2)) = \begin{cases} 
4, & k = 2 \\
2, & k = 3 
\end{cases}
\]

4. Conclusions

In this article the recently introduced \( k \)-metric antidimension problem is considered. We have studied mathematical properties of the \( k \)-antiresolving sets and the \( k \)-metric antidimension of some generalized Petersen graphs. Exact formulas for the \( k \)-metric antidimension of \( GP(n, 1) \) and \( GP(n, 2) \) are obtained.

A possible direction of future research could be considering the \( k \)-metric antidimension of some other challenging classes of graphs.

References

[1] M. Ćangalović, V. Kovačević-Vujičić, J. Kratica, \( k \)-metric antidimension of wheels and grid graphs, In: XIII Balkan Conference on Operational Research Proceedings, pp. 17–24. Belgrade May 2018.

[2] T. Chatterjee, B. DasGupta, N. Mobasheri, V. Srinivasan, I.G. Yero, On the computational complexities of three problems related to a privacy measure for large networks under active attack, Theoretical Computer Science, 775 (2019) 53-67.

[3] H. Coxeter, Self-dual configurations and regular graphs, Bulletin of American Mathematical Society 56 (1950) 413–455.

[4] B. DasGupta, N. Mobasheri, I.G. Yero, On analyzing and evaluating privacy measures for social networks under active attack, Information Sciences, 473 (2019) 87-100.

[5] J. Kratica, V. Kovačević-Vujičić, M. Ćangalović, The strong metric dimension of some generalized Petersen graphs, Applicable Analysis and Discrete Mathematics 11 (2017) 1–10.

[6] S. Mauw, R. Trujillo-Rasua, B. Xuan, Counteracting active attacks in social network graphs, In: IFIP Annual Conference on Data and Applications Security and Privacy, Springer, pp. 233–248, Trento, Italy, July 2016.

[7] S. Naz, M. Salman, U. Ali, I. Javaid, S.A.H. Bokhary, On the constant metric dimension of generalized Petersen graphs \( P(n, 4) \), Acta Mathematica Sinica, English Series 30(7) (2014) 1145–1160.

[8] R. Trujillo-Rasua, I.G. Yero, \( k \)-metric antidimension: A privacy measure for social graphs, Information Sciences 328 (2016) 403–417.

[9] R. Trujillo-Rasua, I.G. Yero, Characterizing 1-metric antidimensional trees and unicyclic graphs, The Computer Journal, 59(8) (2016) 1264-1273.

[10] C. Zhang, Y. Gao, On the complexity of \( k \)-metric antidimension problem and the size of \( k \)-antiresolving sets in random graphs, In: International Computing and Combinatorics Conference - COCOON 2017, Springer, pp. 555–567, Hong Kong, China, August 2017.

[11] M. Zhao, E. Shan, L. Kang, Power domination in the generalized Petersen graphs, Discussiones Mathematicae Graph Theory (2018) doi:10.7151/dmgt.2137