Interference-based universal decoupling and swapping for multimode bosonic systems

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A key requirement for bosonic quantum information processing is the ability to control interactions between desired modes of the system. In practical devices, however, this is often difficult to realize due to the presence of undesired coupling to additional modes. In this work, we develop interference-based protocols for decoupling and swapping selected modes of a multimode bosonic system. Specifically, for a generic coupler characterized by Gaussian unitary process, we show how to decouple a single mode or swap any pair of modes with a constant depth sequence of operations, while maintaining the coupling for the remaining system. These protocols require only multiple uses of the same coupler interleaved with single-mode Gaussian unitary operations, and thus enable efficient construction of operations crucial to quantum information science, such as high-fidelity quantum transduction. Our results are directly derived from fundamental physical properties of bosonic systems and are therefore broadly applicable to various existing platforms.

Introduction. With the increasing ubiquity of bosonic mode-based systems in quantum information processing, an emerging challenge is the undesired coupling with the environment leading to loss and decoherence errors. Recently, significant advances have been made in bosonic quantum error correction [1–3], which can actively correct excitation loss errors. Bosonic dynamical decoupling schemes have also been proposed to suppress bilinear coupling between the system and environment [4]. Both quantum error correction and dynamical decoupling are powerful techniques to suppress undesired coupling with the environment, when we can only control the system without any access to the surrounding modes. In practice, however, we may be able to control the surrounding modes, which can lead to much more efficient schemes to achieve some non-trivial quantum information processing tasks while completely suppressing the undesired coupling.

Consider the task of quantum transduction: a process by which quantum information is transferred from one bosonic mode to another. Recent advances include the conversion between microwave and optical frequency modes [5–12], between microwave and mechanical/spin-wave modes [13–15], and between processor and memory modes for quantum information storage [16–20]. In many such cases, direct coupling between the desired modes is infeasible, and protocols resort to using ancillary modes to mediate the coupling (e.g. mechanical modes in optical-to-microwave transduction [7, 21, 22]). Even in cases where direct coupling is possible, we might not have perfect conversion due to the inevitable presence of unwanted sideband modes [10, 23]. Hence, practical physical transducers are inherently multimode devices, though in practice we often do have access to these surrounding ancillary and sideband modes.

So far, most theoretical investigations [24–27] implicitly assume that the surrounding modes are not accessible; they consequently require minimizing the undesired coupling with the surrounding modes and resort to quantum error correction or dynamical decoupling to correct the errors. However, with the capability of controlling the surrounding modes, we might have much more efficient schemes to completely decouple or perfectly transfer quantum information between modes. To our knowledge, a general treatment of multimode bosonic processes does not exist in literature.

In this work, we solve the problem above by designing a scheme to decouple unwanted modes in a generic bosonic process, i.e. removing their interactions with the rest of the system while in general maintaining the desired coupling over the modes we want to control. Motivated by a similar observation made in Ref. [27], we find that the coupling between an arbitrary pair of selected quadratures can be removed via interference. This is done by implementing the given bosonic interaction twice, interspersed with local Gaussian unitary operations (i.e. phase-shifting and/or single-mode finite squeezing). By constructing an inductive multi-pass sequence of the form above, we can then successively remove all unwanted coupling terms quadrature-by-quadrature. Moreover, with minor modifications to this protocol, we can also efficiently construct perfect quantum transducers, by implementing a swap operation between any two modes of the multimode system using a fixed number of operations. Our protocols are generally applicable to Gaussian processes and can be implemented on various existing bosonic platforms. We will start with examples in two-mode systems:

Two-mode decoupling. Gaussian unitary physical processes involving linearly-coupled bosonic modes are completely determined by the change of the expectation values of the quadrature operators before and after the interaction. Specifically, we can organize such a transformation into a $2N \times 2N$ real symplectic matrix $S$ map-
(a) Sequence for two-mode decoupling

![Diagram showing sequence for two-mode decoupling](image)

(b) Sequence for two-mode perfect transduction

![Diagram showing sequence for two-mode perfect transduction](image)

FIG. 1. (a) To decouple two interacting modes, we construct a sequence involving four copies of the interaction $S$, interspersed by local operations. The operations $L_{(p)}^{(q)}$ are sandwiched between two copies of $S$ to construct an effective interaction $S'$ with the $\hat{q}_1$-quadrature decoupled from mode 2. We then repeat this step recursively using $S'$ and modified local operations $L_{(p)}^{(q)}$ to yield a net process with both $\hat{q}_1$ and $\hat{p}_1$ quadratures decoupled. (b) To construct a swap operation, we instead use two different interactions $S'$ and $S''$, where $S'$ is constructed so that $\hat{q}_1^{\text{out}} = \hat{p}_1^p$ and $\hat{q}_1^{\text{in}}$ is only present in $\hat{p}_2^{\text{out}}$. Using interference, we can then cancel the contamination of $\hat{p}_2^{\text{out}}$ by the other input quadratures besides $\hat{q}_1^{\text{in}}$ to yield (up to local operations) a perfect two-mode swap.

ping $S \rightarrow SS$ where $N$ is the number of modes involved and $X := (\hat{q}_1, \hat{p}_1, \ldots, \hat{q}_N, \hat{p}_N)^T$ is a collection of their respective quadrature operators [28] (see Supplementary Material for definitions and conventions [29]). Without loss of generality, we can work entirely in terms of these symplectic matrices.

Let us investigate a generic bosonic interaction process involving only two linearly-coupled modes. This can be described by a $4 \times 4$ real symplectic matrix

$$S = \begin{pmatrix}
S_{11} & S_{12} & S_{13} & S_{14} \\
S_{21} & S_{22} & S_{23} & S_{24} \\
S_{31} & S_{32} & S_{33} & S_{34} \\
S_{41} & S_{42} & S_{43} & S_{44}
\end{pmatrix}. \quad (1)
$$

The protocol that we develop is based on running this interaction multiple times to form a multi-pass sequence as shown in Fig. 1(a). After each pass, we can apply local (i.e. single-mode) operations to tune the interference between quadratures. It turns out that by carefully choosing these local operations, we can end up with a net process where the two modes are decoupled from one another. To see this, let us choose:

$$L_{(p)}^{(q)} = \begin{pmatrix}
0 & 1 & 0 & 0 \\
-1 & S_{22} & S_{23} & S_{24} \\
0 & S_{32} & S_{33} & S_{34} \\
0 & S_{42} & S_{43} & S_{44}
\end{pmatrix}, \quad (3)
$$

is of the form above, which corresponds to an effective interaction with no coupling between the $\hat{q}_1$ quadrature and quadratures $\hat{q}_2, \hat{p}_2$ for mode 2. (Note that the construction of the local operations throughout the text is not unique, and one can even modify them to relax the requirement of squeezing. See Supplementary Material [29] for more details). It now remains only to decouple the $\hat{p}_1$ quadrature, which is achieved via

$$L_{(p)}^{(q)} = \begin{pmatrix}
0 & 1 & 0 & 0 \\
-1 & S_{22} & S_{23} & S_{24} \\
0 & S_{32} & S_{33} & S_{34} \\
0 & S_{42} & S_{43} & S_{44}
\end{pmatrix}, \quad (4)
$$

and $L_{(p)}^{(q)} = -\vec{\omega} = \begin{pmatrix} 0 & 1 \\
1 & 0 \end{pmatrix}$, with $i, j \in \{1, 2\}$ [30]. If we then explicitly calculate

$$S_{\text{dc}} = S' \begin{pmatrix}
0 & 1 & 0 & 0 \\
-1 & S_{22} & S_{23} & S_{24} \\
0 & S_{32} & S_{33} & S_{34} \\
0 & S_{42} & S_{43} & S_{44}
\end{pmatrix}, \quad (5)
$$

we see that $S_{\text{dc}}$ has a block diagonal structure, reflecting the lack of coupling between the two modes in the net process. Note that, for now, we have assumed that the original matrix $S$ is generic, i.e. meaning that the denominator of Eq. (2) is always non-vanishing. The non-generic cases are treated in Supplementary Material [29].

**Two-mode quantum transduction.** With minor changes to the intermediate local operations, our protocol can be directly applied to construct perfect quantum transducers. Let us define the local operations $M_{k}^{(q)}$ via:

$$M_{k}^{(q)} = \begin{pmatrix}
S_{11} & S_{12} & S_{13} & S_{14} \\
S_{21} & S_{22} & S_{23} & S_{24} \\
S_{31} & S_{32} & S_{33} & S_{34} \\
S_{41} & S_{42} & S_{43} & S_{44}
\end{pmatrix}. \quad (6)
$$

which are chosen to ensure that $\hat{q}_2^{\text{out}} = \hat{p}_1^{\text{in}}$. Once again, $i, j, k \in \{1, 2\}$ and we have $i = 2k - 1$, $j = 2k$ if $i = 1$, and $i = 2k$, $j = 2k - 1$ if $i = 2$. Then, we can check by direct calculation that

$$S' = S \begin{pmatrix}
L_{1}^{(q)} & 0 \\
0 & L_{2}^{(q)}
\end{pmatrix} = \begin{pmatrix}
0 & 1 & 0 & 0 \\
-1 & S_{22} & S_{23} & S_{24} \\
0 & S_{32} & S_{33} & S_{34} \\
0 & S_{42} & S_{43} & S_{44}
\end{pmatrix}, \quad (7)
$$

which indeed has the desired form. Observe from the structure of this matrix that $\hat{q}_1^{\text{in}}$ is fully transferred to $\hat{p}_2^{\text{out}}$, though additional contamination from the other quadratures is still present. Our goal is then to cancel these
contributions by sandwiching local operations between $S^\ast$ and $S'$, as shown in Fig. 1(b). Here, we choose the local operations

$$(M_2^{(p)})_{ij} = \frac{(-1)^{j+1} S^*_i S_j'}{(S^*_i)^2 + (S^*_j)^2} - \frac{(-1)^{j+1} S'_i S_j}{(S_i)^2 + (S_j)^2}$$

and $M_1^{(p)} = -\omega = (\frac{0}{1 \ 0} \ 1)$. By explicit calculation, the matrix $S^{td}$ describing the net process will be of the form

$$S^{td} = S^\ast \left( \begin{array}{cc} M_1^{(p)} & 0 \\ 0 & M_2^{(p)} \end{array} \right) S' = \left( \begin{array}{ccc} 0 & 0 & S_{13}^{td} \\ 0 & 0 & S_{23}^{td} \\ 1 & 0 & 0 \end{array} \right)$$

which, up to local operations, is equivalent to swapping the two modes (referred to as perfect transduction).

**Multimode decoupling.** It turns out that the decoupling and transduction protocols described above are not accidental, but rather stem from fundamental physical properties of bosonic systems – they thus readily generalize to the multimode case. To see why, it will be helpful to first introduce a geometric interpretation of the symplectic matrices, whose rows (or columns) form an orthonormal symplectic basis [31]. Specifically, for an arbitrary $2N \times 2N$ symplectic matrix $S$, we can denote its rows via $S = (u_1, v_1, \ldots, u_N, v_N)^T$ and its columns via $S = (x_1, y_1, \ldots, x_N, y_N)$, where $u_i, v_i, x_i$ and $y_i$ are $2N$-dimensional vectors, with $1 \leq i \leq N$.

Since any unitary physical process described by a symplectic matrix $S$ must preserve the canonical commutation relations, the matrix $S$ must satisfy the defining condition $S\Omega S^T = \Omega$, where $\Omega = \bigoplus_{i=1}^N \omega \dagger = \text{diag}(\omega, \ldots, \omega)$, and where $\omega = (\begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array})$ is called the symplectic form. Therefore, the matrix $\Omega$ can be considered a ‘local’ operation as defined: it simply corresponds to a $\pi/2$ phase-shift on each mode. This condition gives us an explicit set of orthogonality relations between the rows and columns: $u_i^T \Omega u_i = v_j^T \Omega v_j = x_i^T \Omega x_j = y_j^T \Omega y_j = 0$, and $u_i^T \Omega v_j = x_i^T \Omega y_j = \delta_{ij}$, where $i, j \in \{1, 2, \ldots, N\}$. Comparing these relations to the similar properties of orthogonal matrices, one can thus think of symplectic matrices as geometric transformations on the spaces spanned the row (or column) vectors.

Let us consider the general decoupling protocol for multiple modes. As shown in Fig. 2, we can decouple an individual mode in two steps (i.e. first decoupling the $\hat{q}$-quadrature and then the $\hat{p}$-quadrature). The motivation of each step is to build up a certain destructive interference between the quadratures using the geometric relations above. For concreteness, we will demonstrate how to decouple the first mode $\hat{a}_1$ from the others. Nevertheless, with different choice of local operations, the same steps can be used to decouple any mode.

In the first recursive step, our goal is to construct an interaction $S' = SL_1^{(q)}S$ where the first quadrature of the first mode is decoupled from the other modes (i.e. this means $S'_{1j} = S'_{j1} = 0$ for $j > 2$). Explicitly, we have

$$S = \begin{pmatrix} -u_1^T & -v_1^T & \cdots & -u_N^T \\ -v_1^T & -u_1^T & \cdots & -v_N^T \end{pmatrix} L^{(q)} \begin{pmatrix} x_1 & \cdots & x_N & y_N \end{pmatrix}$$

using the row/column notation introduced before, where $L^{(q)} = \text{diag}(L_1^{(q)}, \ldots, L_N^{(q)})$ is a series of local operations. We claim that if there exists an $L^{(q)}$ such that $L^{(q)} x_1 = -\Omega u_1$, then this particular “sandwich operation” will decouple the $\hat{q}_1$ quadrature as desired. The reason for this is as follows: by the properties above, $x_1$ is naturally orthogonal to each of the other columns except $y_1$, and thus $L^{(q)} x_1 = -\Omega u_1$ by assumption, it will also be orthogonal to each of the rows except for $v_1$. Thus $SL^{(q)}S$ will be of the expected form:

$$S' = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ -1 & S_{22}^{'} & S_{23}^{'} & \cdots & S_{2,2N}^{'} \\ 0 & S_{32}^{'} & S_{33}^{'} & \cdots & S_{3,2N}^{'} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & S_{2N,2}^{'} & S_{2N,3}^{'} & \cdots & S_{2N,2N}^{'} \end{pmatrix} (11)$$

FIG. 2. (a) We define a “sandwich operation” to denote two copies of the interaction $S$ interspersed with local operations $L_1, L_2, \ldots, L_N$. (b) A decoupled mode has no effective interaction with the remaining modes of the system: here we show $\hat{a}_1^{out}$ decoupled so that it is equivalent to $\hat{a}_1^{out} \to a_1^{in}$ up to a local operation. (c) Our protocol demonstrates that a sequence of two sandwich operations interspersed by carefully chosen local operations can yield a net interaction with one mode decoupled. This process can then be repeated inductively to remove any number of coupling terms to unwanted modes.
With the $\hat{q}_1$-quadrature decoupled from the remaining modes, we now proceed to the second recursive step of our protocol to decouple $\hat{p}_1$. This involves repeating the technique above using $S'$ instead, with some minor modifications to the local operations. Once again, let us denote the rows of this matrix by $S' = (\alpha_1, \beta_1, \ldots, \alpha_N, \beta_N)^T$ and the columns by $S' = (\chi_1, \gamma_1, \ldots, \chi_N, \gamma_N)$. We want to build up a “sandwich operation” of the form $S^{dc} = S' L(p) S'$ such that the local operation $L(p)$ transforms the second column vector $\gamma_1 \rightarrow L(p) \gamma_1 = -2(S'^*_2) \Omega \alpha_1 + \Omega \beta_1$ while also transforming the first column vector $\chi_1 \rightarrow L(p) \chi_1 = -\Omega \alpha_1$.

Since $L(p) \chi_1$ and $L(p) \gamma_1$ are linearly independent, the two-dimensional plane spanned by the pair of vectors $\Omega \alpha_1, \Omega \beta_1$ is identical to that spanned by the vectors $L(p) \chi_1, L(p) \gamma_1$. Consequently, this plane is orthogonal to every other vector $\Omega \alpha_j, \Omega \beta_j, L(p) \chi_j, L(p) \gamma_j$ for $j \geq 2$, as guaranteed by the geometrical relations above. Thus, the “sandwich operation” $S' L(p) S'$ will be:

$$S^{dc} = S' L(p) S' = \begin{pmatrix}
0 & 0 & \ldots & 0 \\
-1 & S'^*_2 & 0 & \ldots & 0 \\
0 & 0 & S'^*_3 & \ldots & S'^*_3 \\
\vdots & \vdots & \ddots & \ddots & \ddots \\
0 & 0 & S'^*_4 & \ldots & S'^*_4 \\
\end{pmatrix}.$$

Observe that $S^{dc}$ is exactly of the form shown in Fig. 2(b): the first mode is decoupled from all the others. At this point, we can proceed inductively and apply the same protocol to the $N-1$ mode subblock of $S^{dc}$. Doing so, we can in principle decouple any number of modes from the system.

It remains to be shown that the appropriate local operations $L^{(q)}$ and $L^{(p)}$ can always be constructed. By definition, each of them is a direct sum of single-mode operations: e.g. $L^{(q)} = \text{diag}(L^{(q)}_1, \ldots, L^{(q)}_N)$, where $L^{(q)}_i$ are $2 \times 2$ matrices. Therefore, it suffices to show that we can transform any generic two-dimensional vector to another using just a single local operation. As demonstrated in Fig. 3, this is always generically satisfied: any required local operation can be realized using a sequence of three elementary local operations: (a) rotation to the $q$-axis, (b) dilation, and (c) rotation to the final direction. In the language of linear optics, rotation and dilation correspond to phase-shifting and finite squeezing, respectively.

The existence of $L^{(q)}$ and $L^{(p)}$ is thus always guaranteed unless either the initial or the final vector is the zero vector. We refer to generic scattering matrices as those for which the local operations can be constructed (i.e. they contain no zero subblock in any quadrature vector). The procedure for handling the non-generic cases is discussed in Supplementary Material [29].

**Multimode quantum transduction.** With minor modifications, we can also generalize the quantum transduction protocol to multimode systems. Similar to the two-mode case, we can use four copies of a generic $S$ to construct an effective ‘transducer-type’ interaction of the form:

$$S^{td} = \begin{pmatrix}
0 & 0 & S^{td}_{1,3} & \ldots & S^{td}_{1,2N} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & S^{td}_{N-1,3} & \ldots & S^{td}_{N-1,2N} \\
0 & 1 & 0 & \ldots & 0 \\
-1 & S^{td}_{2N,2} & 0 & \ldots & 0 \\
\end{pmatrix}.$$

Here, we have constructed an ‘asymmetric swap’ that transfers information from $\delta^q_{in}$ to $\delta^q_{out}$ but not vice versa. By combining this ‘transducer’-type interaction with the ‘decoupling’-type interaction in Eq. (12), however, it is possible to convert the asymmetric swap into a symmetric one. As shown in Fig. 4, we can combine $S^{td}$ with three copies of $S^{dc}$ in order to successively remove coupling terms in the upper-right subblock of $S^{td}$. Using the decoupling result, we can find three local operations $L^{(q)}$, $M^{(p)}$, and $M^{(q)}$ such that the net process $\hat{S}$ has the form:

$$\hat{S} = (S^{td} M^{(q)} S^{dc}) M^{(p)} (S^{dc} L^{(q)} S^{dc}) = \left(\begin{array}{ccc}
0 & 0 & 0 & \ldots & 0 & \hdotsfor{5}
\end{array}\right).$$

which corresponds to a perfect swap (up to local operations) between the first and the last modes, with no coupling to the remaining $N-2$ modes. We can even repeat this process recursively to obtain swap operations between any arbitrary pairs of modes involved in the original interaction process $S$. We note that $S^{dc}$ and $S^{td}$ can each be generated using four copies of $S$, provided this interaction is generic (i.e. we can construct the local operations as needed). Thus, generically, we require a fixed overhead of just sixteen copies of $S$ to realize this generalized two-mode swap. More details are given in Supplementary Material [29].
any protocols can be implemented without requiring mechanical-microwave transducer of Ref. [7, 21, 22]), our measurements that for certain practical systems (e.g. the optical-Material [29]).

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SUPPLEMENTARY MATERIAL

Definitions

The conventions used in this work closely follow the standard definitions for continuous-variable quantum information [28]. For completeness, we will review the salient details below.

We consider multimode systems comprised of N bosonic modes, which correspond to N pairs of bosonic field operators \((\hat{a}_1, \hat{a}^\dagger_1), \ldots, (\hat{a}_N, \hat{a}^\dagger_N)\) \(\equiv \hat{a}\), where \([\hat{a}_j, \hat{a}^\dagger_k] = \delta_{jk}\). We can equivalently describe the system using quadrature operators \(\hat{q}_k \equiv (\hat{a}_k + \hat{a}^\dagger_k) / \sqrt{2}\) and \(\hat{p}_k \equiv i(\hat{a}_k - \hat{a}^\dagger_k) / \sqrt{2}\), which satisfy the canonical commutation relations. We also define the quadrature vector \(\hat{x} \equiv (\hat{q}_1, \hat{p}_1, \ldots, \hat{q}_N, \hat{p}_N)^T\).

We consider Gaussian unitary operations of the form \(U = \exp(-i\hat{H}/2)\) where \(\hat{H}\) is bilinear in the field operators. Then, in the Heisenberg picture, such operations transform \(\hat{a} \rightarrow U^\dagger \hat{a} U\), or equivalently, transform the quadrature operators via \(\hat{x} \rightarrow S \hat{x}\). In order to respect the canonical commutation relations, this real \(2N \times 2N\) matrix \(S\) must be symplectic: \(S^T \Omega S^T = \Omega\), where \(\Omega\) is block diagonal:

\[
\Omega = \bigoplus_{i=1}^N \omega = \text{diag}(\omega, \ldots, \omega) \quad \text{with} \quad \omega = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}
\]

In the main text, we refer to single-mode transformations as ‘local’; these correspond to \(2 \times 2\) symplectic matrices. We also use the label ‘local’ to mean the direct sum of \(N\) single-mode operations.

The local transformation corresponding to phase-space rotation (i.e. phase shifting) is given by \(\hat{R}(\theta) = \exp[-i\theta \hat{a}^\dagger \hat{a}]\), represented in the quadrature basis by

\[
\hat{R}(\theta) = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix}.
\]

We also make use of single-mode squeezing \(\hat{Z}(r) = \exp[r(\hat{a}^2 - \hat{a}^\dagger^2)/2]\), given in the quadrature basis by

\[
\hat{Z}(r) = \begin{pmatrix} e^{-r} & 0 \\ 0 & e^{r} \end{pmatrix}.
\]

Construction of local operations

In this section, we will elaborate on the existence of local operations in our protocols and the various constraints these operations must satisfy. Particularly, we
will see why we can implement our protocol without squeezing each of the involved modes. For concreteness and without loss of generality, let us consider the example of two-mode decoupling, starting from the generic interaction

\[ S = \begin{pmatrix} S_{11} & S_{12} & S_{13} & S_{14} \\ S_{21} & S_{22} & S_{23} & S_{24} \\ S_{31} & S_{32} & S_{33} & S_{34} \\ S_{41} & S_{42} & S_{43} & S_{44} \end{pmatrix}. \] (18)

We also refer to the rows and columns of this matrix as \( S = (u_1, v_1, u_2, v_2)^T \) and \( S = (x_1, y_1, x_2, y_2) \), respectively.

The function of each local operation in the protocol is essentially to align two originally nonparallel 4-dimensional vectors (in the two-mode case). For instance, the first recursive step of the decoupling scheme involves choosing a local operation \( L^{(q)} \) such that \( L^{(q)} x_1 = -\Omega u_1 \). Note that each two-dimensional subvector can be transformed independently to the target subvector using a single-mode Gaussian operation, e.g. \( L^{(q)} (S_{11}, S_{21})^T = \omega(S_{11}, S_{21})^T = (S_{12}, -S_{11})^T \), which is constructed via

\[
L^{(q)}_1 = R(-\theta)Z(r)R(\varphi) = \begin{pmatrix} -\sin \theta & -\cos \theta & e^{-r} & 0 \\ \cos \theta & -\sin \theta & 0 & e^r \\ 0 & 0 & \cos \varphi & \sin \varphi \\ 0 & 0 & -\sin \varphi & \cos \varphi \end{pmatrix},
\] (19)

where \( \theta = \arctan(-S_{11}/S_{12}) \), \( r = \ln\left(\frac{\sqrt{S_{11}^2 + S_{12}^2}}{\sqrt{S_{11}^2 + S_{12}^2}}\right) \) and \( \varphi = \arctan(S_{21}/S_{11}) \). Similarly, we would transform \( L^{(q)}_2 (S_{31}, S_{41})^T = \omega(S_{13}, S_{43})^T \) using a similar combination of phase-shifting and local squeezing.

Clearly the choice of local operations above is not unique. In fact, it turns out that we can relax the squeezing requirement on one of the two modes. To see this, we note that the first recursive step of our protocol only requires use of the orthogonality relations. That is, it suffices to transform \( L^{(q)} x_1 = -c\Omega u_1 \), up to a rescaling factor \( c \), since this factor does not change the orthogonality between rows/columns. In the choice of \( L^{(q)} \) above, we have taken \( c = 1 \). However, we can also, for instance, relax the squeezing requirement on mode 1, i.e. set \( r \to 1 \) in Eq. (19) above. To compensate, we then correspondingly need additional local squeezing on mode 2, so that overall \( c = \|S_{11}, S_{12}\| \). In general, any local operation must be chosen so that the ratios of the norms of the two-dimensional subvectors are preserved, i.e. so that

\[
\left\|(-S_{12}, S_{11})^T\right\| / \left\|L^{(q)}_1 (S_{11}, S_{21})^T\right\| = \left\|(-S_{14}, S_{13})^T\right\| / \left\|L^{(q)}_2 (S_{31}, S_{41})^T\right\|.
\] (20)

For the case of \( c = 1 \), the left-hand side and right-hand side of this equality both have value 1. This is not a necessary condition, and so the two sides can take any (finite nonzero) value so long as they are equal.

Thus, it is possible to redefine the local operations in the first recursive step up to a rescaling factor such that \( L^{(q)}_1 \) is completely squeeze free (though \( L^{(q)}_2 \) is generally not). As a result, for the second recursive step, we also need to choose \( L^{(q)} \gamma_1 = -2(S_{22}/c)\Omega u_1 + \Omega \beta_1 \), where \( S' = (\alpha_1, \beta_1, \alpha_2, \beta_2)^T = (x_1, \gamma_1, x_2, \gamma_2)^T \) as given in the main text. This new transformation can still be realized by modifying \( L^{(q)}_2 \) while keeping \( L^{(q)}_1 = -\omega = R(-\pi/2) \). In this case, then, we see that both \( L^{(q)}_1 \) and \( L^{(q)}_2 \) require no squeezing and so, in effect, we have eliminated the requirement of local squeezing for mode 1 entirely.

The same relaxation holds for the two-mode transduction example as well. Here, in the first recursive step, we choose an \( M^{(q)} \) such that \( M^{(q)} x_1 = -\Omega u_2 \), as is shown in the main text. However, this only needs to satisfy the orthogonality relations, and so holds up to a constant. Thus, we can generally make \( M^{(q)}_1 \) squeezing free, and as shown in the text, we always pick \( M^{(q)}_1 = -\omega \). Thus, our two-mode transduction protocol likewise requires squeezing only one of the modes.

In the multimode case, a similar argument holds. Here, for \( N \) mode, we must transform one \( 2N \)-dimensional vector to another. We can thus relax squeezing on one of the modes, and correspondingly modify the intermediate local operations for the remaining \( N-1 \) modes, so as to maintain the ratios of each of the subvectors. Thus, in general, for an \( N \)-mode interaction, our protocol requires local squeezing on only \( N-1 \) of the modes.

**Strategy for the edge cases**

To decouple or swap modes, we assumed that the symplectic matrix \( S \) is generic such that the required local operations always exist. However, there are situations where this is not possible — specifically, when either the initial or final quadrature vectors is a zero vector. In this section, we explain a strategy for converting such an edge case into the generic case for which our protocols can be applied.

Before proceeding, we have to point out a subtle distinction between the decoupling protocol and the transduction protocol. Although the strategy introduced in this section can be applied to decoupling and swapping modes, the applicability of the transduction protocol requires some extra discussion. Firstly, it is easy to see that if a symplectic matrix is a permutation of modes, up to local operations, then whether it can be used to apply the transduction protocol entirely depends on the structure of the permutation. For the chosen pair of modes, one can implement the transduction protocol using the permutation-like symplectic matrix if and only if the corresponding permutation contains an order-2 cycle swapping these two modes. A quick counterexample is that a local operation, which can be viewed as a trivial permut-
tation, obviously cannot be used to implement the transduction protocol. This observation may be further generalized to more complicated symplectic matrices, which will be discussed in more detail in our future follow-up works.

Two-mode situation—We start with the two-mode decoupling case. An edge case happens when the local operation cannot be constructed using the geometric picture. For example, as shown in the main text, the local operation $L^{(q)}_1$ exists only if $(S_{11}, S_{21})$ and $(S_{12}, S_{22})$ are either both zero (in which case apply the identity) or both nonzero (in which case apply $L^{(q)}_1$ as prescribed). A non-generic interaction is where only one out of a pair of subvectors is nonzero, so that the denominator of the corresponding local operation is vanishing. Such an interaction must contain a two-by-two block

\[
\begin{pmatrix}
S_{2k-1,2l-1} & S_{2k-1,2l}
S_{2k,2l-1} & S_{2k,2l}
\end{pmatrix}
\]  

for some $k,l \in \{1,2\}$, with a certain zero row or column. Since we can always apply arbitrary single-mode operation before and after $S$ to modify the matrix elements, we must then further assume that the above two-by-two block is actually a zero matrix. Evidently, in order to make it an edge case, we will require

\[
\begin{pmatrix}
S_{2l-1,2k-1} & S_{2l-1,2k}
S_{2k-1,2l-1} & S_{2k-1,2l}
\end{pmatrix}
\]  

(21)
to be non-zero. Then for the two-mode edge case, without loss of generality, we consider the interaction

\[
S = \begin{pmatrix}
S_{11} & S_{12} & 0 & 0 \\
S_{21} & S_{22} & 0 & 0 \\
S_{31} & S_{32} & S_{33} & S_{34} \\
S_{41} & S_{42} & S_{43} & S_{44}
\end{pmatrix}
\]  

(22)

However, by the definition of symplectic matrix, $S = S^T \Omega$ (and $S^T \Omega S = \Omega$ equivalently), one can easily verify that the submatrix $\begin{pmatrix} S_{11} & S_{12} \\ S_{41} & S_{42} \end{pmatrix}$ must also be zero, meaning that $S$ is already a decoupled. Therefore, the two-mode edge cases will not increase the number of copies of $S$ required to decouple a bosonic mode.

Multimode situation—Likewise for the multimode decoupling case, our strategy for constructing the local operations works when each single-mode projection of the rows and columns to be either both zero or both nonzero, because we cannot use local operations to transform a nonzero/zero vector to a zero/nonzero vector. For the same reason explained in the two-mode case, without loss of generality, an $N$-mode edge-case interaction $S$ should at least contain a zero submatrix, for example $\begin{pmatrix} S_{1,3} & S_{1,4} \\ S_{4,1} & S_{4,2} \end{pmatrix}$ (meanwhile with $\begin{pmatrix} S_{3,1} & S_{4,1} \\ S_{3,2} & S_{4,2} \end{pmatrix} \neq 0$). We will also randomize each non-zero two-by-two block of $S$ by a pre-local operation $L^{(R)}$ and a post-local operation $M^{(R)}$ to make it full-ranked, i.e. $S^{\text{new}} = L^{(R)} S M^{(R)}$, with the local operations $L^{(R)}$, $M^{(R)}$ randomly chosen. Then it is easy to check that the number of the vanishing two-by-two blocks in the product $(L^{(R)} S M^{(R)})^\circ$ will only decrease before the resulting net interaction is generic, if $S$ is not a permutation of the bosonic interaction. Since the number of the vanishing two-by-two blocks should not exceed $N(N-1)$ (because the a scattering matrix should always be full-ranked) and the $N$-th product of any permutation is the identity operation, we conclude there always exists an integer $K \leq N(N-1)$ such that the net interaction is generic even though the initial $S$ is non-generic.

Local operations for multimode transduction

We first relabel the $N$ modes so that the original $N$-th mode is now the first mode, and the original first mode is now the $N$-th mode. Then the the local operation $L^{(q)}$ can be calculated using the same equation for $L^{(q)}$ only with $S$ replaced by $S^{de}$. The local operation $M^{(p)}$ can be calculated similarly using the formula of $M^{(p)}$ if we replace $S^{*}$ with $S^{\text{ed}} M^{(q)} S^{de}$ and $S^{*}$ with $S^{de} L^{(q)} S^{de}$. The remaining local operation $M^{(q)}$ can be calculated by the following equations:

\[
(M^{(q)}_k)_{ij} = \frac{(-1)^{i+1} S_{3i} S_{j,1}}{(S_{11})^2 + (S_{12})^2} - \frac{(-1)^{i+1} S_{3i} S_{j,1}}{(S_{31})^2 + (S_{32})^2} 
\]  

(24)

for $k < N$, and

\[
M^{(q)}_N = \text{Id.} 
\]  

(25)

Note that these constructions are not unique and our previous discussions on the local operations and edge cases also apply in this situation.

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