THE MINIMAL OUTPUT ENTROPY OF EPOSIC CHANNELS
AND THEIR E.B.T PROPERTY

Muneerah Al Nuwairan
Department of Mathematics
University of Ottawa, Ottawa, ON.
E-mail: malnu009@uottawa

Abstract. In this paper we show that the EPOSIC channel \( \Phi_{m,n,h} \), introduced in [1], has zero minimal output entropy if and only if \( h = 0 \). We compute the minimal output entropy for special cases, and find a bound of the minimal output entropy of the tensor product of two \( SU(2) \)-irreducibly covariant channels. We also examine the E.B.T property of EPOSIC channels.

INTRODUCTION

The carrier of the states (information) from one part to another in quantum systems is known as quantum channel. A perfect channel carries a state form one system to another without losing information. However, the existence of noise in all information processing systems affects the channel’s performance in any transmission of such information. A well-known measure of channel’s performance is the minimal output entropy (MOE) of the channel.

In a previous paper [1], we introduced the EPOSIC channels, a class of quantum channels that forms the extreme points of all \( SU(2) \)-irreducibly covariant channels. In the present paper, we study the minimal output entropy of them and examine their E.B.T property. The first section contains definitions of EPOSIC channels and some preliminaries results. In Section 2, we show that the MOE of the EPOSIC channel \( \Phi_{m,n,h} \) is zero if and only if \( h = 0 \), while in the third section, we compute the MOE for special cases of EPOSIC channels and get a bound for the minimal output entropy.
entropy of some $SU(2)$-irreducibly covariant channels. In Section 4, we obtain a bound of MOE of the tensor product of two $SU(2)$-irreducibly covariant channels. Section 5 contains the basic definitions and result related to E.B.T (Entanglement Breaking Trace preserving) channels. In Section 6, we obtain some partial results on the fulfillment of the E.B.T property of EPOSIC channels. We assume that all vector spaces are finite dimensional.

This work was done under the supervision of professors B. Collins and T. Giordano as a part of a PhD thesis of the author. I greatly appreciate their patient advice and help.

1. Basic Definitions and Results

Recall that mathematically, a quantum system is represented by a Hilbert space $H$ that is described by its state, a positive operator in $End(H)$ that has trace one. The set of states of $H$ is denoted by $D(H)$. A rank one state of $H$ is called a pure state, such a state can be written in the form $ww^*$ with $w$ a unit vector in $H$. The set $\mathcal{P}(H)$ denotes the set of pure states on $H$. If $K$ is another Hilbert space then a quantum channel $\Phi : End(H) \to End(K)$, is a completely positive trace preserving map; such a map carries the states of $H$ into states of $K$ [5, ch.5]. For any quantum channel $\Phi : End(H) \to End(K)$, there exist a Kraus representation [18, p.54-p.56] which is a set of operators $\{T_j \in End(H, K) : 1 \leq j \leq n\}$ satisfying $\sum_{j=1}^{n} T_j^* T_j = I_H$ such that $\Phi(A) = \sum_{j=1}^{n} T_j A T_j^*$. If $\Phi : End(H) \to End(K)$ is a quantum channel that has Kraus operators $\{T_j : 1 \leq j \leq n\}$, then the image of a pure state $ww^*$ under $\Phi$ can be written in the form $\Phi(ww^*) = \sum_{j=1}^{n} u_j u_j^*$ where $u_j = T_j w \in K$.

Notation: Let $H, K$ be Hilbert spaces, and $\Phi : End(H) \to End(K)$ is a quantum channel with Kraus operators $\{T_j : 1 \leq j \leq n\}$. For a pure state $\varrho = ww^*$ of $H$, let $U_{\Phi, \varrho}$ denote the set $\{u_j = T_j w : 1 \leq j \leq n\}$.
Remark 1.1. The set $U_{\Phi, \varrho}$ contains a nonzero vector, for if $ww^*$ is a state then $\Phi(ww^*) = \sum_{j=1}^n u_j u_j^*$ must be a state.

Definition 1.2. [14, Ch.11] Let $H, K$ be Hilbert spaces and $\Phi : End(H) \rightarrow End(K)$ be a quantum channel. The minimal output entropy of $\Phi$, denoted by $S_{\text{min}}(\Phi)$, is defined by

$$S_{\text{min}}(\Phi) = \min_{\varrho \in \mathcal{P}(H)} S(\Phi(\varrho))$$

where $\mathcal{P}(H)$ is the set of all pure states on $H$, and where $S(\varrho) = -tr(\varrho \log \varrho)$ is the von Neumann entropy of the state $\varrho$.

Proposition 1.3. Let $H$ and $K$ be Hilbert spaces, and $\Phi : End(H) \rightarrow End(K)$ be a quantum channel. Then

1. $S_{\text{min}}(\Phi) = 0$ if and only if there exist a pure state $\varrho$ of $H$ such that $\Phi(\varrho)$ is a pure state.

2. If for each pure state $\varrho$ of $H$, the set $U_{\Phi, \varrho}$ contains at least two linearly independent vectors, then $S_{\text{min}}(\Phi) \neq 0$.

Proof.

1. By the continuity of von Neumann entropy, and the compactness of the set of states, the minimum entropy is achieved, i.e. there is a state $\varrho$ so that $S(\Phi(\varrho)) = 0$. But, it is easily seen that $S(\sigma) = 0$ if and only if $\sigma$ is pure state.

2. Since $U_{\Phi, \varrho}$ has two linearly independent vectors then $\Phi(\varrho) = \sum_{u_j \in U_{\varrho}} u_j u_j^*$ has rank at least two. Thus $\Phi(\varrho)$ is not pure. The result now follows from this and (1).

\[\square\]
Definition 1.4. Let $G$ be a group, and $\pi_H, \pi_K$ be two irreducible representations of $G$ on the Hilbert spaces $H, K$. Then a quantum channel $\Phi : \text{End}(H) \longrightarrow \text{End}(K)$ is a $G$-irreducibly covariant channel, if

$$\Phi(\pi_H(g) A \pi_H^*(g)) = \pi_K(g) \Phi(A) \pi_K^*(g)$$

for each $A \in \text{End}(H)$ and $g \in G$.

For $m \in \mathbb{N}$, let $P_m$ denote the space of homogeneous polynomials of degree $m$ in the two variables $x_1, x_2$. It is a complex vector space of dimension $m + 1$ whose

$$\{x_1^i x_2^{m-i} : 0 \leq i \leq m\}$$

is a basis, the space $P_{-1}$ will denote the zero vector space. For any $m \in \mathbb{N}$, the compact group $SU(2) = \left\{ \begin{bmatrix} a & b \\ -\bar{b} & \bar{a} \end{bmatrix} : a, b \in \mathbb{C}, |a|^2 + |b|^2 = 1 \right\}$ has a representation $\rho_m$ on $P_m$ given by

$$\rho_m(g)f(x_1, x_2) = f((x_1, x_2)g) = f(ax_1 - \bar{b}x_2, bx_1 + \bar{a}x_2)$$

for $f \in P_m$ and $g \in SU(2)$. For $m \in \mathbb{N}$, $\rho_m$ is a unitary representation with respect to the inner product on $P_m$ given by

$$\langle x_1^l x_2^{m-l}, x_1^k x_2^{m-k} \rangle_{P_m} = l! (m-l)! \delta_{lk}$$

The set $\{\rho_m : m \in \mathbb{N}\}$ constitutes the full list of the irreducible representations of $SU(2)$, see [17, p.276-p.279].

To facilitate the computations, we choose the orthonormal basis of $P_m$ given by

$$\{f_{l,1}^m = a_{l,1}^m x_1^l x_2^{m-l} : 0 \leq l \leq m\}, \text{ where } a_{l,1}^m = \frac{1}{\sqrt{l!(m-l)!}}.$$ We will call this basis, the standard basis of the $SU(2)$-irreducible space $P_m$. The corresponding standard basis of $\text{End}(P_m)$ will be $\{E_{l,k} = f_{l,1}^m f_{k,1}^m^* : 1 \leq l, k \leq m + 1\}$.

Recall that EPOSIC channels forms the extreme points of $SU(2)$-irreducibly covariant channels [1]. In the following, we give a definition of EPOSIC channels and some of their properties. For another equivalent definition see [1].
Definition 1.5. [1] For \( m, n, h \in \mathbb{N} \) with \( 0 \leq h \leq \min\{m, n\} \), let \( r = m + n - 2h \).

Then the EPOSIC channel \( \Phi_{m,n,h} : \text{End}(P_r) \to \text{End}(P_m) \), is the quantum channel given by the Kraus operators

\[
T_j = \sum_{i=\max\{0,j-h\}}^{\min\{r,m-h+j\}} \varepsilon_i^j f_i f_i^* , \quad 0 \leq j \leq n
\]

where \( l_{ij} = i - j + h \),

\[
\varepsilon_i^j (m,n,h) = \sum_{s=\max\{0,j-i+j-h-n\}}^{\min\{h,j+j-m-i\}} \beta_{i,s,j}^{m,n,h}, \quad \beta_{i,s,j}^{m,n,h} = \frac{(-1)^s \binom{h}{j-s} \binom{n-h}{i-j+s} \binom{m-h}{s}}{\sqrt{\binom{i}{j-s} \binom{m}{i-j+s} \binom{n}{s} \sum_{k=0}^{h} \binom{h}{k}^2}},
\]

and \( \{f_i^r : 0 \leq i \leq r\}, \{f_i^m : 0 \leq l \leq m\} \) are the standard bases of \( P_r, P_m \) respectively.

The Kraus operators given in the definition above are called the EPOSIC Kraus operators of the channel \( \Phi_{m,n,h} \).

Lemma 1.6. [1] Let \( m, n, h \in \mathbb{N} \) with \( 0 \leq h \leq \min\{m, n\} \), and \( r = m + n - 2h \).

1. The EPOSIC channel \( \Phi_{m,n,h} : \text{End}(P_{m+n-2h}) \to \text{End}(P_m) \) is an SU(2)-irreducibly covariant channel.

2. \( \Phi_{m,n,h}(f_i^r f_i^r^*) = \sum_{j=\max\{h,-m+i\}}^{\min\{i+h,n\}} (\varepsilon_i^j (m,n,h))^2 f_{l_{ij}} f_{l_{ij}}^* \) where \( l_{ij} = i - j + h \).

2. THE MINIMAL OUTPUT ENTROPY OF \( \Phi_{m,n,h} \)

In this section, we determine the EPOSIC channels with zero minimal output entropy. Namely, we show that the minimal output entropy is zero if and only if the index \( h \) in \( \Phi_{m,n,h} \) is zero. Note that both the channel \( \Phi_{m,0,0} \), which is the identity channel [1] Remark 2.23] on \( \text{End}(P_m) \), and the channel \( \Phi_{0,n,0} : \text{End}(P_n) \to \text{End}(P_0) \)
THE MINIMAL OUTPUT ENTROPY OF EPSIC CHANNELS AND THEIR E.B.T PROPERTY

6

given by \( \Phi_{0,n,0}(A) = tr(A) \), both have zero minimal output entropy. More generally, we have the following proposition:

**Proposition 2.1.** For \( m, n \in \mathbb{N} \), the channel \( \Phi_{m,n,0} \) has zero minimal output entropy.

**Proof.**

For \( k \in \mathbb{N} \), let \( \{ f^k_i : 0 \leq i \leq k \} \) denote the standard basis for \( P_k \). By Lemma [1.6] we have \( \Phi_{m,n,0}(f^r_0 f^*_0) = \sum_{j=0}^{0} (\varepsilon_{(m,n,0)}^{j})^2 f^m_0 f^n_0 = f^m_0 f^n_0 \). Thus by Proposition [1.3] the minimal output entropy of \( \Phi_{m,n,0} \) is zero. \( \square \)

Next, we show that for \( h > 0 \) the minimal output entropy of \( \Phi_{m,n,h} \) is not zero. The following lemma and proposition are needed. The lemma can be proved by direct computation using the formula \( \varepsilon_i^j(m,n,h) = \sum_{s=\max(0,i-j-h-n)}^{\min(h,j+m-i-h)} \beta_{i,s,j}^{m,n,h} \).

**Lemma 2.2.** For \( m, n, h \in \mathbb{N} \) with \( 0 \leq h \leq \min\{m, n\} \), let \( \varepsilon_i^j : \varepsilon_i^j(m,n,h) \) and \( r = m + n - 2h \), then :

1. \( \varepsilon_i^0 \neq 0 \), for \( 0 \leq i \leq m - h \).
2. \( \varepsilon_i^{m-h} \neq 0 \), for \( m - h \leq i \leq r \).
3. \( \varepsilon_i^h \neq 0 \), for \( 0 \leq i \leq n - h \).
4. \( \varepsilon_i^n \neq 0 \), for \( n - h \leq i \leq r \).

**Lemma 2.3.** For \( m, n, h \in \mathbb{N} \) with \( 0 < h \leq \min\{m, n\} \), let \( \varepsilon_i^j : \varepsilon_i^j(m,n,h) \) and \( r = m + n - 2h \). Then for \( 0 \leq i \leq r \) there exist \( j_1 < j_2 \) such that \( \varepsilon_i^{j_1} \neq 0 \) and \( \varepsilon_i^{j_2} \neq 0 \).
By the Lemma 2.2, both $\varepsilon_{j_1}^i$ and $\varepsilon_{j_2}^i$ are non zeros. If $j_1 = 0$, then $j_1 < h \leq j_2$. If $j_1 = i - m + h$, then $j_1 = i - (m - h) \leq i < i + h$, and $j_1 \leq r - m + h = n - h < n$, thus $j_1 < \min\{i + h, n\} = j_2$.

Recall that for any pure state $\varrho = ww^*$, and channel $\Phi$ the set $U_{\Phi, \varrho}$ denote the set $\{u_j = T_jw : 1 \leq j \leq n\}$, where $\{T_j : 1 \leq j \leq n\}$ a set of Kraus operators of $\Phi$.

Proposition 2.4. Let $m, n, h \in \mathbb{N}$ with $0 < h \leq \min\{m, n\}$, and $\Phi_{m,n,h}$ be the associated EPOSIC channel. Then for any pure state $\varrho$ the set $U_{\Phi_{m,n,h}, \varrho}$ contains at least two linearly independent vectors.

Proof.

Let $r = m + n - 2h$ and $\varrho = ww^*$ be any pure state in $\text{End}(P_r)$ where $w = \sum_{i=0}^r w_i f_i^r \in P_r$ and $\sum_{i=0}^r |w_i|^2 = 1$.

For $0 \leq j \leq n$, the vector $u_j = T_jw = \sum_{i=0}^r w_i T_j(f_i^r) = \sum_{i = \max\{0, j-h\}}^{\min\{r, m-h+j\}} w_i \varepsilon_{j}^{i} f_{i-j+h}^m$.

Assume that $U_{\Phi, \varrho} = \{u_j : 0 \leq j \leq n\}$ does not contain two linearly independent vectors. Pick $i_1$ minimal so that $w_{i_1} \neq 0$. By Lemma 2.3, there exist $j_1 < j_2$ such that $\varepsilon_{j_1}^{i_1} \neq 0$ and $\varepsilon_{j_2}^{i_1} \neq 0$. Thus $u_{j_1} = \sum_{i=\max\{0,j_1-h\}}^{\min\{r,m-h+j_1\}} w_i \varepsilon_{j_1}^{i} f_{i-j_1+h}^m \neq 0$ and $u_{j_2} = \sum_{i=\max\{0,j_2-h\}}^{\min\{r,m-h+j_2\}} w_i \varepsilon_{j_2}^{i} f_{i-j_2+h}^m \neq 0$, so there exist $\alpha \neq 0$ such that $u_{j_2} = \alpha u_{j_1}$.

In particular, comparing the coefficients of $f_{i_1-j_2+h}^m$, we obtain $0 \neq w_{i_1} \varepsilon_{j_2}^{i_1} = \alpha w_{i_2} \varepsilon_{j_1}^{i_2}$, where $i_1 - j_2 + h = i_2 - j_1 + h$ i.e $i_2 = i_1 - (j_2 - j_1) < i_1$ and $w_{i_2} \neq 0$, contradicting the minimality of $i_1$. 

The following corollary follows directly from the last proposition and Lemma 1.6.
Corollary 2.5. For strictly positive integers $m, n$ and $0 < h \leq \min\{m, n\}$, we have $S_{\min}(\Phi_{m,n,h})$ is not zero.

In conclusion, the results in the present section can be summarized in the following theorem

Theorem 2.6. Let $m, n, h \in \mathbb{N}$ with $0 \leq h \leq \min\{m, n\}$, and $\Phi_{m,n,h}$ be the associated EPOSIC channel. Then $S_{\min}(\Phi_{m,n,h}) = 0$ if and only if $h = 0$.

3. Computing the minimal output entropy for special cases

Although the last theorem gives that $S_{\min}(\Phi_{m,n,h})$ is not zero when $h > 0$, it doesn’t provide a way to compute its value. In the following, we compute the minimal output entropy for $\Phi_{m,n,h}$ for special cases such as $S_{\min}(\Phi_{m,1,1})$, $S_{\min}(\Phi_{m,m+1,m})$, and $S_{\min}(\Phi_{m,m-1,m-1})$ for $m \in \mathbb{N} \setminus \{0\}$. We also obtain a bound for $S_{\min}(\Phi)$ where $\Phi : \text{End}(P_1) \rightarrow \text{End}(P_m)$, $m \in \mathbb{N}$ is an $SU(2)$-irreducibly covariant channel.

3.1. The Minimal Output Entropy of $\Phi_{m,1,1}$

The main result for this section is Corollary 3.4 which computes $S_{\min}(\Phi_{m,1,1})$ for any $m \in \mathbb{N} \setminus \{0\}$. We use the definition of minimal output entropy given in Definition 1.2 to get our result. We start by computing the eigenvalues of $\Phi_{m,1,1}(\rho)$ for any pure state $\rho$, then minimize $\Phi_{m,1,1}(\rho)$ over such states.

To compute the eigenvalues of $\Phi_{m,1,1}(\rho)$, note that the channel $\Phi_{m,1,1} : \text{End}(P_{m-1}) \rightarrow \text{End}(P_m)$ has only two Kraus operators $[1]$, so for any pure state $\rho = ww^* \in \text{End}(P_{m-1})$, we have $\Phi_{m,1,1}(ww^*) = \sum_{j=0}^{1} u_j u_j^*$. By Proposition 2.4 the vectors $u_0, u_1$ are linearly independent in $P_m$. Complete $u_0, u_1$ to a basis $\{u_0, u_1, u_2, u_3, ..., u_m\}$ for $P_m$, where $\{u_2, u_3, ..., u_m\}$ is an orthonormal basis for $\{u_0, u_1\}^\perp$. Writing the matrix $\Phi_{m,1,1}(ww^*)$ in the basis $\{u_0, u_1, ..., u_m\}$ we get the $(m + 1) \times (m + 1)$ matrix given by
\[ \Lambda_{\Phi_{m,1,1}} := \begin{pmatrix} \langle u_0 | u_0 \rangle & \langle u_0 | u_1 \rangle & 0 & \cdots & 0 \\ \langle u_1 | u_0 \rangle & \langle u_1 | u_1 \rangle & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix} \]

The nonzero eigenvalues of \( \Lambda_{\Phi_{m,1,1}} \) are eigenvalues of \( \begin{pmatrix} \langle u_0 | u_0 \rangle & \langle u_0 | u_1 \rangle \\ \langle u_1 | u_0 \rangle & \langle u_1 | u_1 \rangle \end{pmatrix} \), thus can be found by solving the equation

\[ \det \left( \begin{pmatrix} \langle u_0 | u_0 \rangle & \langle u_0 | u_1 \rangle \\ \langle u_1 | u_0 \rangle & \langle u_1 | u_1 \rangle \end{pmatrix} - \lambda I_2 \right) = 0 \]

for \( \lambda \).

The following lemma follows by direct computations. The item (3) follows from the fact that \( \Phi_{m,1,1} \) is trace preserving. Note that the vectors \( u_0, u_1 \) for \( \Phi_{m,1,1} \) are given by \( u_0 = \sum_{l=1}^{m} \varepsilon_{l-1}^{0} w_{l-1} f_{l}^{m} \), and \( u_1 = \sum_{l=0}^{m-1} \varepsilon_{l}^{1} w_{l} f_{l}^{m} \).

**Lemma 3.1.** Let \( m \in \mathbb{N} \setminus \{0\} \) then

1. For \( 0 \leq l \leq m - 1 \), we have \( \varepsilon_{l}^{0} = \sqrt{\frac{l+1}{m+1}} \), \( \varepsilon_{l}^{1} = -\sqrt{\frac{m-l}{m+1}} \), and \( (\varepsilon_{l}^{0})^2 + (\varepsilon_{l}^{1})^2 = 1 \).
2. For \( 1 \leq l \leq m - 1 \), \( (\varepsilon_{l}^{0})^2 = (\varepsilon_{l-1}^{0})^2 + \frac{1}{m+1} \), and \( (\varepsilon_{l-1}^{1})^2 = (\varepsilon_{l}^{1})^2 + \frac{1}{m+1} \).
3. \( \|u_0\|^2 + \|u_1\|^2 = 1 \).

**Lemma 3.2.** Let \( m \in \mathbb{N} \setminus \{0\} \). For a pure state \( \varrho \in \text{End}(P_{m-1}) \), let \( u_0, u_1 \) be the elements in \( U_{\varrho} \) associated with \( \Phi_{m,1,1} \). Let \( R = \|u_0\|^2 \|u_1\|^2 - |\langle u_0 | u_1 \rangle|^2 \), then the minimal value of \( R \) is \( \frac{m}{(m+1)^2} \).

**Proof.**

Since \( u_0 = \sum_{l=1}^{m} \varepsilon_{l-1}^{0} w_{l-1} f_{l}^{m} \) and \( u_1 = \sum_{l=0}^{m-1} \varepsilon_{l}^{1} w_{l} f_{l}^{m} \), we have
\begin{align*}
\langle u_0 \mid u_1 \rangle & = \sum_{l=1}^{m-1} \epsilon^0_{i-1} w_{i-1} \epsilon^1_i w_i = \sum_{l=1}^{m-1} \epsilon^0_{i-1} w_i \epsilon^1_i w_{i-1} = \langle v_0 \mid v_1 \rangle \text{ where } v_0 = \sum_{l=1}^{m-1} \epsilon^0_{i-1} w_i f^m_i, \\
\text{and } v_1 & = \sum_{l=1}^{m-1} \epsilon^1_i w_{i-1} f^m_i.
\end{align*}

Using Lemma 3.1 we obtain
\begin{align*}
\|v_0\|^2 & = \sum_{l=1}^{m-1} (\epsilon^0_{i-1})^2 |w_i|^2 = \sum_{l=1}^{m-1} (\epsilon^0_{i-1})^2 |w_i|^2 + \|w\|^2 - \|w\|^2 \\
& = \frac{1}{m+1} |w_0|^2 + \sum_{l=1}^{m-1} \left( (\epsilon^0_{i-1})^2 + \frac{1}{m+1} \right) |w_i|^2 - \frac{1}{m+1} \\
& = (\epsilon^0_0)^2 |w_0|^2 + \sum_{l=1}^{m-1} (\epsilon^0_i)^2 |w_i|^2 - \frac{1}{m+1}
\end{align*}
(Note that \(\|w\|^2 = \sum_{l=0}^{m-1} |w_l|^2 = 1\).)

Thus
\begin{align*}
\|v_0\|^2 & = \sum_{l=0}^{m-1} (\epsilon^0_l)^2 |w_l|^2 - \frac{1}{m+1} = \sum_{l=1}^{m} (\epsilon^0_{i-1})^2 |w_{i-1}|^2 - \frac{1}{m+1} = \|u_0\|^2 - \frac{1}{m+1}
\end{align*}

Similarly \(\|v_1\|^2 = \|u_1\|^2 - \frac{1}{m+1}\). So
\begin{align*}
|\langle u_0 \mid u_1 \rangle|^2 & = |\langle v_0 \mid v_1 \rangle|^2 \leq \|v_0\|^2 \|v_1\|^2 = \left( \|u_0\|^2 - \frac{1}{m+1} \right) \left( \|u_1\|^2 - \frac{1}{m+1} \right) \\
& = \|u_0\|^2 \|u_1\|^2 - \frac{1}{m+1} \left( \|u_0\|^2 + \|u_1\|^2 \right) + \frac{1}{(m+1)^2} = \|u_0\|^2 \|u_1\|^2 - \frac{m}{(m+1)^2}
\end{align*}

Thus \(R = \|u_0\|^2 \|u_1\|^2 - |\langle u_0 \mid u_1 \rangle|^2 \geq \frac{m}{(m+1)^2}\), and \(\frac{m}{(m+1)^2}\) is a lower bound for \(R\).

To verify that \(\frac{m}{(m+1)^2}\) is the minimal value of \(R\), compute \(R\) at \(w_0 = (1, 0, \ldots, 0)^t\) to get \(R_{w_0} = \frac{m}{(m+1)^2}\).

\[\square\]

By solving the identity \(\text{det} \left( \begin{pmatrix} \langle u_0 \mid u_0 \rangle & \langle u_0 \mid u_1 \rangle \\ \langle u_1 \mid u_0 \rangle & \langle u_1 \mid u_1 \rangle \end{pmatrix} - \lambda I_2 \right) = 0\), we get the following lemma.
Lemma 3.3. Let $ww^*$ be a pure state in $\text{End}(P_{m-1})$. Then the non zero eigenvalues of $\Phi_{m,1,1}(ww^*)$ are given by

$$\lambda_{1,2} = \frac{1 \pm \sqrt{1 - 4R}}{2}$$

where $R = \|u_0\|^2 \|u_1\|^2 - |\langle u_0 | u_1 \rangle|^2$.

Note that using Cauchy Schwartz inequality and Lemma 3.1, we have $0 \leq 1 - 4R \leq 1$ holds for any $m \geq 1$ which affirms that $\lambda_{1,2}$ are non-negative numbers.

By concavity of von Neumann entropy [14, ch.11], we have that the von Neumann entropy of $\Phi_{m,1,1}(ww^*)$ achieves its minimum when the difference between $\lambda_1$ and $\lambda_2$ is maximal, this is when $R = \frac{m}{(m+1)^2}$. Consequently, we get:

Corollary 3.4. Let $m \in \mathbb{N} \setminus \{0\}$. Then

$$S_{\text{min}}(\Phi_{m,1,1}) = -\left[\frac{1}{m+1} \log_2 \frac{1}{m+1} + \frac{m}{m+1} \log_2 \frac{m}{m+1}\right]$$

where the state $\sigma = f_{m-1} f_{m-1}^*$ is one of the minimizer states for the entropy.

3.2. Lower bound of the minimal output entropy for an element in $QC_{SU(2)}(\rho_1, \rho_m)$.

Recall that the set $QC_{SU(2)}(\rho_1, \rho_m)$ consists of all $SU(2)$-irreducibly covariant channels $\Phi : \text{End}(P_1) \to \text{End}(P_m)$. To find the minimal output entropy for these channels, we begin with the following lemma:

Lemma 3.5. [1] Sec.6] Let $\Phi : \text{End}(P_1) \to \text{End}(P_m)$ be an $SU(2)$-equivariant map, and $w$ be a unit vector in $P_1$. Then $\Phi(ww^*)$ and $\Phi(E_{11})$ have the same eigenvalues.

The following corollary is straightforward application for the definition of $S_{\text{min}}(\Phi)$ and the above lemma.

Corollary 3.6. Let $\Phi : \text{End}(P_1) \to \text{End}(P_m)$ be an $SU(2)$-covariant channel then

$S_{\text{min}}(\Phi) = S(\Phi(E_{11}))$. 
THE MINIMAL OUTPUT ENTROPY OF EPOSIC CHANNELS AND THEIR E.B.T PROPERTY

The corollary above reduces the problem of finding $S_{\min}(\Phi)$ for $\Phi \in QC(\rho_1, \rho_m)$ to finding $S(\Phi(E_{11}))$. Recall that any element in $QC(\rho_1, \rho_m)$ is spanned by the set of EPOSIC channels $EC(1, m)$ [1, Cor 3.17] which has only two elements for any $m \geq 1$, namely $\Phi_{m,m+1,m}$ and $\Phi_{m,m-1,m-1}$. Thus, the problem descends to finding the number $p$ that spans $\Phi$. To explain the idea in detail, we begin with the following lemma:

**Lemma 3.7.** [1, Sec.6] For $m \in \mathbb{N} \setminus \{0\}$, we have

1. $\Phi_{m,m+1,m}(E_{11}) = \sum_{j=0}^{m} \frac{2(m-j+1)}{(m+1)(m+2)} E_{m-j+1,m-j+1}$.
2. $\Phi_{m,m-1,m-1}(E_{11}) = \sum_{j=0}^{m-1} \frac{2(j+1)}{m(m+1)} E_{m-j,m-j}$.

The channel $\Phi \in QC(\rho_1, \rho_m)$ is a convex combination of the channels $\Phi_{m,m+1,m}$ and $\Phi_{m,m-1,m-1}$ [1, sec4]. Thus there exist $0 \leq p \leq 1$ such that $\Phi = p\Phi_{m,m+1,m} + (1-p)\Phi_{m,m-1,m-1}$. Using Lemma 3.7 we are able to compute $\Phi(E_{11})$. Note the by convention $0 \log 0$ is 0.

**Proposition 3.8.** Let $m \in \mathbb{N}$, and $\Phi : End(P_1) \to End(P_m)$ be an $SU(2)$-covariant channel. Then there exist $0 \leq p \leq 1$ such that $S_{\min}(\Phi) = -\sum_{j=0}^{m} \lambda_j \log_2 \lambda_j$ where

$$\lambda_j = \frac{2(m-j+1)}{(m+1)(m+2)}p + \frac{2j}{m(m+1)}(1-p).$$

**Proof.**

By Corollary 3.6, it suffices to compute $S(\Phi(E_{11}))$. Since $\Phi \in QC(\rho_1, \rho_m)$, there exist $0 \leq p \leq 1$ such that

$$\Phi = p\Phi_{m,m+1,m} + (1-p)\Phi_{m,m-1,m-1}$$

i.e

$$\Phi(E_{11}) = p\Phi_{m,m+1,m}(E_{11}) + (1-p)\Phi_{m,m-1,m-1}(E_{11})$$

$$= \frac{2p}{m+2} E_{m+1,m+1} + \sum_{j=1}^{m} \left[ \frac{2p(m-j+1)}{(m+1)(m+2)} + \frac{2(1-p)j}{m(m+1)} \right] E_{m-j+1,m-j+1}$$
The minimal output entropy of Eposic channels and their E.B.T property

\[
E_m - j + 1, m - j + 1.
\]

The special cases where \( \Phi = \Phi_{m,m+1} \) and \( \Phi = \Phi_{m,m-1} \) are given by

1. \( S_{\min}(\Phi_{m,m+1}) = -\sum_{j=1}^{m+1} \frac{2j}{(m+1)(m+2)} \log_2 \frac{2j}{(m+1)(m+2)} \).
2. \( S_{\min}(\Phi_{m,m-1}) = -\sum_{j=1}^{m} \frac{2j}{m(m+1)} \log_2 \frac{2j}{m(m+1)} \).

Next we use the last proposition to find a lower bound for the minimal output entropy of a channel \( \Phi \in QC \) \( SU(2) \)(\( \rho_1, \rho_m \)).

**Lemma 3.9.**

1. The map \( -x \ln x \) has non negatives values on \([0, 1]\), and it is increasing on \([0, \frac{1}{e}]\). By convention \( 0 \ln 0 = 0 \).
2. \( \int -x \ln x \, dx = \frac{x^2}{4} - \frac{x^2 \ln x}{2} \).
3. The map \( \frac{x^2}{4} - \frac{x^2 \ln x}{2} \) is increasing map on \([0, 1]\).

**Proof.**

1. It is clear that the \( -x \ln x \) has non negatives values on \([0, 1]\). By taking the derivative \( (-1 - \ln x) \geq 0 \) if \( \ln x < -1 = \ln e \). that is \(-x \ln x \) is increasing on \([0, \frac{1}{e}]\).
2. \( \frac{d}{dx} \left( \frac{x^2}{4} - \frac{x^2 \ln x}{2} \right) = -x \ln x \, dx \).
3. Since \( \frac{d}{dx} \left( \frac{x^2}{4} - \frac{x^2 \ln x}{2} \right) = -x \ln x \, dx \) takes a non negative values on \([0, 1]\). The result follows.

**Theorem 3.10.** Let \( m \in \mathbb{N} \), and \( m \geq 5 \). Let \( \Phi : \text{End}(P_1) \rightarrow \text{End}(P_m) \) be an \( SU(2) \)-covariant channel. Then \( S_{\min}(\Phi) \geq \frac{1}{ln2} \frac{(m-2)^2}{16m(m+1)^2} \).

**Proof.**

By Proposition 3.8.
\[ S_{\text{min}}(\Phi) = \frac{1}{m} \sum_{j=0}^{m} - \lambda_j \ln \lambda_j \text{ where } \lambda_j = \frac{2(m-j+1)}{(m+1)(m+2)}p + \frac{2j}{m(m+1)}(1-p). \]

Let \( f(x) = -x \ln x \), and \( g(x) = \int f(x) \, dx = \frac{x^2}{4} - \frac{2x \ln x}{2} \). Let \( x_j = \frac{2(m-j+1)}{(m+1)(m+2)} \), and \( y_j = \frac{2j}{m(m+1)} \). As \( x_j \leq \frac{2(m+1)}{(m+1)(m+2)} = \frac{2}{m+2} \), and \( y_j \leq \frac{2m}{m(m+1)} = \frac{2}{m+1} \), then

\[
0 \leq \min\{x_j, y_j\} \leq px_j + (1-p)y_j \leq \max\{x_j, y_j\} \leq \frac{2}{m+1} \leq \frac{1}{3} < \frac{1}{e}
\]

Thus by Lemma 3.9, \( f(\min\{x_j, y_j\}) \leq f(px_j + (1-p)y_j) \).

As \( \min\{x_j, y_j\} = y_j \) if and only if \( j \leq \frac{m}{2} \), and \( f(px_j + (1-p)y_j) \geq 0 \) for any \( j \),

\[
\ln 2 \cdot S_{\text{min}}(\Phi) = \sum_{j=0}^{m} f(px_j + (1-p)y_j) \geq \sum_{j=0}^{\left\lfloor \frac{m}{2} \right\rfloor} f(px_j + (1-p)y_j)
\]

\[
\geq \sum_{j=0}^{\left\lfloor \frac{m}{2} \right\rfloor} f(\min\{x_j, y_j\}) = \sum_{j=0}^{\left\lfloor \frac{m}{2} \right\rfloor} f\left(\frac{2j}{m(m+1)}\right) = \sum_{j=1}^{\left\lfloor \frac{m}{2} \right\rfloor} f\left(\frac{2j}{m(m+1)}\right)
\]

But for any \( x \in [0, \left\lfloor \frac{m}{2} \right\rfloor] \), we have \( 0 \leq \frac{2x}{m(m+1)} \leq \frac{1}{e} \), thus for \( j \in [0, \left\lfloor \frac{m}{2} \right\rfloor] \), we have

\[
f\left(\frac{2j}{m(m+1)}\right) = \int_{j-1}^{j} f\left(\frac{2j}{m(m+1)}\right) \, dx \geq \int_{j-1}^{j} f\left(\frac{2x}{m(m+1)}\right) \, dx
\]

Consequently,

\[
\ln 2 \cdot S_{\text{min}}(\Phi) \geq \sum_{j=1}^{\left\lfloor \frac{m}{2} \right\rfloor} \int_{j-1}^{j} f\left(\frac{2x}{m(m+1)}\right) \, dx = \int_{0}^{\left\lfloor \frac{m}{2} \right\rfloor} f\left(\frac{2x}{m(m+1)}\right) \, dx
\]

Let \( u = \frac{2x}{m(m+1)} \), then

\[
\int_{0}^{\left\lfloor \frac{m}{2} \right\rfloor} f\left(\frac{2x}{m(m+1)}\right) \, dx = \frac{m(m+1)}{2} \int_{0}^{\left\lfloor \frac{m}{2} \right\rfloor} (-u \ln u) \, du = g\left(\frac{2\left\lfloor \frac{m}{2} \right\rfloor}{m(m+1)}\right) - g(0)
\]

\[
g\left(\frac{2\left\lfloor \frac{m}{2} \right\rfloor}{m(m+1)}\right) \geq g\left(\frac{2(m-1)}{m(m+1)}\right) = g\left(\frac{m-2}{2m(m-1)}\right) = \left(\frac{m-2}{2m(m-1)}\right)^2 \left[\frac{1}{4} - \frac{\ln\left(\frac{m-2}{2m(m-1)}\right)}{2}\right]
\]
Since \( \frac{m-2}{2m(m+1)} < 1 \) then \( g\left(\frac{m-2}{2m(m+1)}\right) > \frac{1}{4} \left(\frac{m-2}{2m(m+1)}\right)^2 \). Thus

\[
S_{\text{min}}(\Phi) \geq \frac{1}{4 \ln 2} \left(\frac{m-2}{2m(m+1)}\right)^2.
\]

\[\square\]

4. Bound for the minimal output entropy of the tensor product of two \( SU(2) \)-irreducibly covariant channels.

There are very few tools that used to understand the minimal output entropy of the tensor product of two channels in general. In this section, we restrict ourselves to the set of \( SU(2) \)-irreducibly covariant channels and we obtain a bound on the minimal output entropy for the tensor product of two of them. This bound is given by the minimal output entropy of a convex combination of a number of EPOSIC channels. Recall that according to Clebsch-Gordan decomposition [1, 3], we have: for \( m_1, m_2 \in \mathbb{N} \) the \( SU(2) \)-space \( P_{m_1} \otimes P_{m_2} \) decomposes into irreducible invariant subspaces. Namely \( P_{m_1} \otimes P_{m_2} = \bigoplus_{l=0}^{\min\{m_1, m_2\}} W_l \) where \( W_l \cong P_{m_1 + m_2 - 2l} \). The orthogonal projections \( q_{m_1, m_2, l}, 0 \leq l \leq \min\{m_1, m_2\} \) on the irreducible subspaces \( W_l \), \( 0 \leq l \leq \min\{m_1, m_2\} \) are \( SU(2) \)-equivariant maps that satisfy \( \sum_{l=0}^{\min\{m_1, m_2\}} q_{m_1, m_2, l} = I_{P_{m_1 \otimes P_{m_2}}} \) [1].

**Lemma 4.1.** Let \( m_1, m_2 \in \mathbb{N} \) and \( \mathfrak{P} = \{ q_l := q_{m_1, m_2, l} : 0 \leq l \leq \min\{m_1, m_2\} \} \) be the set of \( SU(2) \)-equivariant projections of \( P_{m_1} \otimes P_{m_2} \) onto the \( SU(2) \)-irreducible invariant subspaces \( W_l \). Then the map

\[
E_{\mathfrak{P}} : \text{End}(P_{m_1} \otimes P_{m_2}) \longrightarrow \bigoplus_{l=0}^{\min\{m_1, m_2\}} \text{End}(W_l)
\]

defined as

\[
E_{\mathfrak{P}}(A) = \sum_{l=0}^{\min\{m_1, m_2\}} q_l A q_l
\]

is a unital \( SU(2) \)-covariant channel.
Proof.

The map $\text{Ad}_{q_l} : \text{End}(P_{m_1} \otimes P_{m_2}) \rightarrow \text{End}(W_l)$, defined by taking $A \rightarrow q_l A q_l$ is a completely positive map. It is an $SU(2)$-equivariant since $q_l$ is. Consequently the map $E_{\mathfrak{P}}$ is a completely positive $SU(2)$-equivariant map. Since \[
\sum_{l=0}^{\min(m_1,m_2)} q_l = I_{P_{m_1} \otimes P_{m_2}}
\] then $E_{\mathfrak{P}}$ is a trace preserving unital map. $\square$

The following lemma is straightforward, the second statement uses the fact that the set of pure state of a subspace can be considered as subset of the pure states of the original space.

**Lemma 4.2.** Let $H, K$ be Hilbert spaces and $W$ a subspace of $H$. Let $\Phi : \text{End}(H) \rightarrow \text{End}(K)$ a quantum channel. Then

1. The restriction of $\Phi$ on $\text{End}(W)$ is quantum channel.
2. $S_{\text{min}}(\Phi) \leq S_{\text{min}}(\Phi|_{\text{End}(W)})$

**Corollary 4.3.** For $r_1, r_2, m_1, m_2 \in \mathbb{N}$, let $\Phi_1 : \text{End}(P_{r_1}) \rightarrow \text{End}(P_{m_1})$, $\Phi_2 : \text{End}(P_{r_2}) \rightarrow \text{End}(P_{m_2})$ be $SU(2)$ covariant channels, and $\Phi_1 \otimes \Phi_2 : \text{End}(P_{r_1} \otimes P_{r_2}) \rightarrow \text{End}(P_{m_1} \otimes P_{m_2})$ their tensor product. Let $P_{r_1} \otimes P_{r_2} = \bigoplus_{k=0}^{\min(r_1,r_2)} V_k$ be the decomposition of $P_{r_1} \otimes P_{r_2}$ into a direct sum of $SU(2)$-irreducible invariant subspaces, and $P_{m_1} \otimes P_{m_2} = \bigoplus_{l=0}^{\min(m_1,m_2)} W_l$ be the decomposition of $P_{m_1} \otimes P_{m_2}$ into a direct sum of $S(2)$-irreducible invariant subspaces. Then,

1. For $0 \leq k \leq \min\{r_1, r_2\}$, the restriction of $\Phi_1 \otimes \Phi_2$ to $\text{End}(V_k)$ is an $SU(2)$-covariant channel.
2. If $\mathfrak{P} = \{q_l : 0 \leq l \leq \min\{m_1, m_2\}\}$ is the set of projections of $P_{m_1} \otimes P_{m_2}$ into the $SU(2)$-irreducible invariant subspaces $W_l$, then the map

$$
E_{\mathfrak{P}} \left( \Phi_1 \otimes \Phi_2 \right)_{\text{End}(V_k)}^{\min(m_1,m_2)} : \text{End}(V_k) \rightarrow \bigoplus_{l=0}^{\min(m_1,m_2)} \text{End}(W_l)
$$

is an $SU(2)$-covariant channel.
Proposition 4.4. Let \( E_{\Phi} \left( \Phi_1 \otimes \Phi_2 \bigg|_{\text{End}(V_k)} \right) \) as in Corollary 4.3. Then for each 
\[ 0 \leq k \leq \min\{r_1, r_2\}, \] 
we have 
\[ E_{\Phi} \left( \Phi_1 \otimes \Phi_2 \bigg|_{\text{End}(V_k)} \right) = \sum_{l=0}^{\min\{m_1, m_2\}} \lambda_{k,l} \psi_{k,l} \]
for some \( \{\psi_{k,l} : \text{End}(V_k) \rightarrow \text{End}(W_l) : 0 \leq l \leq \min\{m_1, m_2\} \} \) a set of \( SU(2) \)-irreducibly covariant channels, and non-negative numbers \( \lambda_{k,l} \) such that \( \sum_{l=0}^{\min\{m_1, m_2\}} \lambda_{k,l} = 1 \).

Proof.

The map \( q_l \left( \Phi_1 \otimes \Phi_2 \bigg|_{\text{End}(V_k)} \right) q_l : \text{End}(V_k) \rightarrow \text{End}(W_l) \) is a completely positive \( SU(2) \)-irreducibly equivariant map (note that this mapping is not necessarily trace preserving). Thus \( q_l \left( \Phi_1 \otimes \Phi_2 \bigg|_{\text{End}(V_k)} \right) q_l \) is a multiple of \( SU(2) \)-irreducibly covariant channel \( [\Pi, \text{Sec.4}] \). i.e. there exist \( SU(2) \)-covariant channel \( \psi_{k,l} : \text{End}(V_k) \rightarrow \text{End}(W_l) \), and a non-negative number \( \lambda_{k,l} \) such that \( q_l \left( \Phi_1 \otimes \Phi_2 \bigg|_{\text{End}(V_k)} \right) q_l = \lambda_{k,l} \psi_{k,l} \).

Since \( E_{\Phi}(A) = \sum_{l=0}^{\min\{m_1, m_2\}} q_l A q_l \) for any \( A \in \text{End}(P_{m_1} \otimes P_{m_2}) \) then 
\[ E_{\Phi} \left( \Phi_1 \otimes \Phi_2 \bigg|_{\text{End}(V_k)} \right) = \sum_{l=0}^{\min\{m_1, m_2\}} \lambda_{k,l} \psi_{k,l}. \]
Finally, taking the trace of both side for any state \( \varrho \) we get that \( \sum_{l=0}^{\min\{m_1, m_2\}} \lambda_{k,l} = 1 \). \( \square \)
Lemma 4.5. [5, p.226] Let $H, K$ and $L$ be Hilbert spaces, $\Phi : \text{End}(H) \rightarrow \text{End}(K)$ be a quantum channel and $\Psi : \text{End}(K) \rightarrow \text{End}(L)$ be a unital quantum channel. Then $S(\Phi(\varrho)) \leq S(\Psi(\Phi(\varrho)))$ for $\varrho \in D(H)$.

The last lemma implies that $S_{\text{min}}(\Phi) \leq S_{\text{min}}(\Psi \circ \Phi)$ which we combine with Lemma 4.2 to get:

Corollary 4.6. Let $\Phi_1 : \text{End}(P_{r_1}) \rightarrow \text{End}(P_{m_1})$, $\Phi_2 : \text{End}(P_{r_2}) \rightarrow \text{End}(P_{m_2})$ be $SU(2)$-covariant channels. Then for any $0 \leq k \leq \min\{r_1, r_2\}$ there exist $\psi_{k,l} : \text{End}(V_k) \rightarrow \text{End}(W_l) : 0 \leq l \leq \min\{m_1, m_2\}$ a set of $SU(2)$-irreducibly covariant channels and $\{\lambda_{k,l} : 0 \leq l \leq \min\{m_1, m_2\}\}$ a probability distribution such that

$$S_{\text{min}}(\Phi_1 \otimes \Phi_2) \leq S_{\text{min}}\left(\sum_{l=0}^{\min\{m_1, m_2\}} \lambda_{k,l} \psi_{k,l}\right)$$

Proof.

Pick $k$ such that $0 \leq k \leq \min\{r_1, r_2\}$, let $E_\varphi$ and $V_k \cong P_{r_1+r_2-2k}$ be as in Proposition 4.4 then $S_{\text{min}}(\Phi_1 \otimes \Phi_2) \leq S_{\text{min}}(\Phi_1 \otimes \Phi_2 |_{\text{End}(V_k)}) \leq S_{\text{min}}(E_\varphi(\Phi_1 \otimes \Phi_2 |_{\text{End}(V_k)})) = S_{\text{min}}\left(\sum_{l=0}^{\min\{m_1, m_2\}} \lambda_{k,l} \psi_{k,l}\right)$ (Proposition 4.4) \hfill \Box

5. ENTANGLED BREAKING CHANNELS: BACKGROUND DEFINITIONS AND RESULTS

A property of quantum channels that has been studied and used to classify the quantum channel, is the elimination of entanglement between the input states of a composite systems. Such channels are called the Entanglement Breaking Channels denoted by E.B.T. Here is a description by P. Shor [16] for the E.B.T channels

“Entanglement breaking channels are channels which destroy entanglement with other quantum systems. That is, when the input state is entangled between the input space $H_{in}$ and another quantum system $H_{ref}$, the output of the channel is no longer entangled with the system $H_{ref}$.”
THE MINIMAL OUTPUT ENTROPY OF EPOSIC CHANNELS AND THEIR E.B.T PROPERTY

Definition 5.1. Let $H_1$ and $H_2$ be Hilbert spaces. A state $\rho \in D(H_1 \otimes H_2)$ is said to be a separable if it can be written as a probabilistic mixture of states of the form $\sigma \otimes \tau$ where $\sigma \in D(H_1)$, $\tau \in D(H_2)$. A non-separable state is called an entangled state.

Let $H, K$ be Hilbert spaces and $\Phi : \text{End}(H) \to \text{End}(K)$ be a quantum channel. For $n \in \mathbb{N}$, let $\Phi \otimes I_n$ be the map $\Phi \otimes I_n : \text{End}(H \otimes \mathbb{C}^n) \to \text{End}(K \otimes \mathbb{C}^n)$ taking $A \otimes B$ to $\Phi(A) \otimes B$ and extends by linearity. Then, the map $\Phi \otimes I_n$ is also a quantum channel, and we have the following definition:

Definition 5.2. [10] Let $H, K$ be Hilbert spaces. A quantum channel $\Phi : \text{End}(H) \to \text{End}(K)$ is said to be entanglement breaking if $\Phi \otimes I_n(\rho)$ is separable for any $\rho \in D(H \otimes \mathbb{C}^n)$ and $n \in \mathbb{N}$.

Let $H, K$ be Hilbert spaces, let $\Phi^*$ denote the dual map of the quantum channel $\Phi : \text{End}(H) \to \text{End}(K)$. It is evident that if $\{T_j : 1 \leq j \leq k\}$ is Kraus operators for $\Phi$ then $\{T_j^* : 1 \leq j \leq k\}$ will be Kraus operators for $\Phi^*$. Since we have $T_j = uv^* \iff T_j^* = vu^*$ then by Proposition 5.4(3),

Lemma 5.3. Let $\Phi$ be a quantum channel then $\Phi$ is an E.B.T map if and only if its dual $\Phi^*$ is an E.B.T map.

Recall that in a finite dimensional setting, a characterization of a quantum channel $\Phi$ is given by its Choi matrix [18], a matrix that is given by $C(\Phi) = \sum_{i,j=1}^{dim} \Phi(E_{ij}) \otimes E_{ij}$ where $E_{ij}$ is the standard basis for $\text{End}(H)$.

Proposition 5.4. [10] Th.4] Let $H, K$ be Hilbert spaces and $\Phi : \text{End}(H) \to \text{End}(K)$ is a quantum channel then the following statements are equivalent

(1) $\Phi$ is an E.B.T channel.

(2) The Choi matrix of $\Phi$ is separable.
THE MINIMAL OUTPUT ENTROPY OF EPOSIC CHANNELS AND THEIR E.B.T PROPERTY

(3) $\Phi$ can be written in operator sum form using only Kraus operators of rank one.

The following proposition follows directly by [11] Thm 1 and Proposition 5.4. The corollary to it, is just a generalization of [10] Theorem 6.

**Proposition 5.5.** Let $H, K$ be Hilbert spaces of dimension $d_H, d_K$, and $\Phi : \text{End}(H) \to \text{End}(K)$ be a quantum channel. If $\text{rank}(C(\Phi)) < \max\{d_H, \text{rank}(\text{Tr}_H(C(\Phi)))\}$ then $\Phi$ is not E.B.T.

**Corollary 5.6.** Let $H, K$ be Hilbert spaces of dimension $d_H, d_K$ respectively such that $d_H \geq d_K$. Let $\Phi : \text{End}(H) \to \text{End}(K)$ be a quantum channel. Then if $\Phi$ can be written in Kraus operator fewer than $d_n$ then $\Phi$ is not E.B.T.

6. THE E.B.T PROPERTY OF EPOSIC CHANNEL.

In this section, we classify EPOSIC channels $\{\Phi_{m,n,h} : m, n, h \in \mathbb{N}, 0 \leq h \leq \min\{m, n\}\}$ according to their E.B.T property. Unfortunately we didn’t obtain a full classification. We state below the partial results that we obtained.

**Theorem 6.1.** For $m \in \mathbb{N}$, the channel $\Phi_{m,m,m}$ and $\Phi_{0,m,0}$ are E.B.T channels.

**Proof.**

Let $\{T_j, 0 \leq j \leq n\}$ EPOSIC Kraus operators for $\Phi_{m,n,h}$. Then By Definition 1.5, we have $T_j = \sum_{i=\max\{0,j-h\}}^{\min\{r,m-r+h\}} \epsilon_i^j f_{ij}^{m} f_{i}^{*}$ for $0 \leq j \leq n$ where $r = m + n - 2h$. Thus for each $0 \leq j \leq n$, $\text{rank}(T_j) \leq \min\{r, m, m - h + j, r + h - j\} + 1$. In case of $\Phi_{m,m,m}$ we have $\text{rank}(T_j) \leq 1$ for any Kraus operator $T_j$ ($r = 0$). Thus by Proposition 5.4(3), $\Phi_{m,m,m}$ is E.B.T. Since $\Phi_{m,m,m}^* = \frac{1}{m+1} \Phi_{0,m,0}$ [II Sec.5] then $\Phi_{0,m,0}$ is E.B.T by Lemma 5.3. \qed

**Proposition 6.2.** Let $m, n, h \in \mathbb{N}$ with $0 \leq h \leq \min\{m, n\}$

1. If $n \geq 2h$ then $\Phi_{m,n,h}$ is not E.B.T for any $m > 2h$. 
THE MINIMAL OUTPUT ENTROPY OF EPOSIC CHANNELS AND THEIR E.B.T PROPERTIES

(2) If \( n \leq 2h \) then \( \Phi_{m,n,h} \) is not E.B.T for any \( m > n \).

Proof.

Let \( r = m + n - 2h \). Then \( \Phi_{m,n,h} : \text{End}(P_r) \to \text{End}(P_m) \) is written using \( n + 1 \) Kraus operators.

1. If \( n \geq 2h \) then \( \text{dim}(P_r) \geq \text{dim}(P_m) \), thus by Corollary 5.6 we get that \( \Phi_{m,n,h} \) is not E.B.T whenever \( m > 2h \).

2. If \( n \leq 2h \) then \( r \leq m \) and by (1) the channel \( \Phi_{r,n,n-h} : \text{End}(P_m) \to \text{End}(P_r) \) is not E.B.T whenever \( r > 2(n-h) \) i.e whenever \( m > n \). Thus \( \Phi_{m,n,h} = \Phi_{r,n,n-h}^* \) is not E.B.T whenever \( m > n \).

\qed

Corollary 6.3. Let \( m,n,h \in \mathbb{N} \) with \( 0 \leq h \leq \min\{m,n\} \). Then \( \Phi_{m,n,h} \) is not E.B.T whenever \( m > n \). In particular, \( \Phi_{m,h,h} \) is not E.B.T for any \( 0 \leq h < m \).

References

[1] M. Al Nuwairan, SU(2)-irreducibly covariant and EPOSIC channels, arXiv:1306.5321v1

[2] C. Bennett and P. Shor, Quantum Information Theory, IEEE Trans. Info. Theory Vol 44 Issue 6, 2724–2748 (1998).

[3] T. Brocker and T. Dieck, Representations of Compact Lie Groups, Springer-Verlag New York Inc 1985.

[4] M. Fannes, B. Haegeman, M. Mosonyi and D. Vanpeteghem, Additivity of minimal entropy output for a class of covariant channels, arXiv: quant-ph/040195v1.

[5] M. Hayashi, Quantum Information An Introduction, Springer-Verlag Berlin Heidelberg, 2006.

[6] A. Holevo, Additivity conjecture and covariant channel, Int. J. Quant.Inform. 3 (1) (2005) 41-48.

[7] A. Holevo, A note on covariant dynamical semigroup, Rep. Math. Phys. 32 (1993) 211-216.

[8] A. Holevo and V. Giovannetti, Quantum channels and their entropic characteristics, Rep. Prog. Phys. 75 (2012) 046001 (30pp).

[9] A. Holevo, Complementary channel and the additivity problem, Theory Probab. Appl, Vol 51, No 1, pp.92-100.
THE MINIMAL OUTPUT ENTROPY OF EPOSIC CHANNELS AND THEIR E.B.T PROPERTY

[10] M. Horodecki, P. Shor, and M. Ruskai, General Entanglement Breaking Channels, Rev.Math.Phys 15, 629-641 (2003)

[11] P. Horodecki, J.Smolin, B.Terhal, and A.Thapliya, Rank two bipartite bound entangled states do not exist, J. Theor. Computer Science 292, 589-596(2003). ArXiv.org preprint quant-ph/9910122.

[12] M. Horodecki, P.Horodecki, Reduction criterion of separability and limits for a class of distillation protocols, Phys. Rev A 59, 4206-4216(1999), LANL preprint quant-ph/9708015.

[13] P. Massey, Non-commutative Schur-Horn theorems and extended majorization for hermitian matrices, arXiv:0712.2246v1

[14] M. Nielsen, I. Chuang, Quantum Computation and Quantum Information, Cambridge university press 2000.

[15] J.-P.Serre, Linear Representations of Finite Groups, Springer-Verlag New York Inc, 1977.

[16] Peter W. Shor, Additivity of the Classical Capacity of Entanglement-Breaking Quantum channels, J.Math.Phys.Vol.43, (9) 4334-4340 (2002).

[17] N.Ja. Vilenkin, A.U. Klimyk, Representation of Lie Groups and Special Functions, Volume 1:Simplest Lie Groups, Special Functions and Integral Transforms. Kluwer academic publishers. Dordrecht/Boston/London (1991).

[18] J. Watrous, CS 766/QIC Theory of Quantum Information, Institute for Quantum Computing, University of Waterloo, Fall 2011.