Locality of the heat kernel on metric measure spaces

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Abstract. We will discuss what it means for a general heat kernel on a metric measure space to be local. We show that the Wiener measure associated to Brownian motion is local. Next we show that locality of the Wiener measure plus a suitable decay bound of the heat kernel implies locality of the heat kernel. We define a class of metric spaces we call manifold-like that satisfy the prerequisites for these theorems. This class includes Riemannian manifolds, metric graphs, products and some quotients of these as well as a number of more singular spaces. There exists a natural Dirichlet form based on the Laplacian on manifold-like spaces and we show that the associated Wiener measure and heat kernel are both local. These results unify and generalise facts known for manifolds and metric graphs. They provide a useful tool for computing heat kernel asymptotics for a large class of metric spaces. As an application we compute the heat kernel asymptotics for two identical particles living on a metric graph.

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1. Introduction and motivation

This article is about the well-known statement: ‘The heat kernel is local’. We will discuss what this means exactly and when it holds. This statement first appeared around 1950, for example in the works of [MP49]. They studied the short time expansion of the heat kernel of the Laplace-Beltrami operator on Riemannian manifolds. In order to go from boundaryless manifolds to manifolds with boundary one uses the following trick. A small neighbourhood of a point on the boundary looks like a half space where the heat kernel is known explicitly, a small neighbourhood of a point away from the boundary looks like a neighbourhood of a point in a boundaryless manifold, where the expansion of the heat kernel is also known. One can get an expansion of the heat kernel of a manifold with boundary by combining these two situations. The fact that this strategy works is the essence of what is meant by saying the heat kernel is local. For that reason, the same fact is also referred to as ‘the heat kernel does not feel the boundary’, see [Kac51].

A first naive understanding of ‘the heat kernel is local’ would be the following. Given a space $M$ with some operator $D$ on it and some nice set $U \subset M$, one could think that locality means the heat kernel on $U$ is determined by knowing $U$ and $D|_U$. Unfortunately this is not true. Consider a simple example, the unit interval with the standard Laplace operator, once with Neumann boundary conditions at both ends $\Delta_{\text{Neu}}$ and once with Dirichlet boundary conditions $\Delta_{\text{Dir}}$. Let $U = (a, 1-a)$ for some $a > 0$. One can compute both heat kernels $p_{\Delta_{\text{Neu}}}$ and $p_{\Delta_{\text{Dir}}}$ explicitly and $p_{\Delta_{\text{Neu}}}|_U \neq p_{\Delta_{\text{Dir}}}|_U$. For large times one should not expect this to be true anyway, the Neumann Laplace operator preserves energy, whereas in the Dirichlet case all heat eventually leaks out of the system.

What is true however, is that the difference between the two heat kernels is quite small for small times $t$, namely it can be bounded by $e^{-\varepsilon t}/t$ for some explicit constant $\varepsilon > 0$ depending on $a$. In particular, such a bound implies that the asymptotic expansion for $t \to 0$ of these two heat kernels is equal on $U$. Hence, getting a bound of this form in a quite general setting is our primary goal.

Locality in this sense holds for Riemannian manifolds with boundary and the Laplace-Beltrami operator. It still holds if we add lower order terms to the operator. It is also known for quantum graphs with the Laplacian and some suitable boundary conditions at the vertices. Some works on orbifolds also make indirect use of it.

A slightly weaker notion of locality is proved in [Hsu95] for Riemannian manifolds and in [Gün17] for general metric spaces. Their definition of the principle of not feeling the boundary only makes a statement about the behaviour in the $t \to 0^+$ limit. Hence our locality definition implies theirs but not vice versa.
While in the original works from the 50s the locality statement is first rigorously proven before it is used, in some of the more modern works it seems to be the other way round. Some asymptotic expansion of the heat kernel is derived, the derivation clearly makes use of the same version of the locality of the heat kernel. Sometimes this locality is just referred to as a well known fact, sometimes not even that but little consideration is given to the question whether it actually holds in the given setting and how this might be proven. Here we are going to show that locality does indeed hold in a very general setting.

Our strategy is based on the Wiener measure. One can represent the heat kernel at time \( t \) between two points \( x \) and \( y \) as the Wiener measure of the set of paths from \( x \) to \( y \) in time \( t \). In the unit interval example the Neumann and the Dirichlet Laplacian give rise to Wiener measures \( W^{\text{Neu}} \) and \( W^{\text{Dir}} \). These Wiener measures satisfy locality in the literal sense of the word, that is \( W^{\text{Neu}}|_U = W^{\text{Dir}}|_U \), where the restriction means the restriction to paths that stay inside of \( U \) during the entire time interval \([0,t]\). Our first Main Theorem 3.13 states that locality of the Wiener measure holds for fairly general metric measure spaces.

Our next Main Theorem 3.22 states that if the heat kernels satisfy a suitable decay bound (see Definition 3.15), then locality of the Wiener measure implies locality of the heat kernel. Decay bounds for general heat kernels on metric spaces are an area of active research, see for example [Stu95, GT12, GK17].

To show that this heat kernel decay and hence locality is satisfied on a wide class of examples, we define a class of metric spaces we call manifold-like. They are an extension of spaces that satisfy the measure contraction property introduced by Sturm in [Stu98] and [Stu06b]. This class includes Riemannian manifolds with our without boundary, quantum graphs, products and certain quotients of these and a number of other spaces. On these spaces there exists a natural Dirichlet form, and the associated operator corresponds to the Laplace operator in the case of Riemannian manifolds. The associated heat kernel satisfies a bound of the form \( p_t(x,y) \leq e^{-d(x,y)^2/(c-t)} \), that is the heat kernel decays exponentially fast with distance between the points. Our third Main Theorem 5.23 shows that locality of the heat kernel holds for manifold-like spaces.

This paper is organised as follows. Section 2 defines heat kernels and the Wiener measure in a metric space setting and collects some facts on how the operator, the Dirichlet form, the semigroup, the heat kernel and the Wiener measure all relate to each other and imply each others existence. Section 3 contains our Main Theorems 3.13 and 3.22, locality of the Wiener measure and locality of the heat kernel on metric measure spaces. In Section 4 we define our class of manifold-like metric spaces and in Section 5 we define Laplace-type operators through Dirichlet forms on them. We then show that these operators satisfy the conditions for locality of the Wiener measure and the heat kernel in Main Theorem 5.23. Finally, Section 6 contains an example computation of the heat kernel asymptotics of a two particle system on a metric graph through decomposing the state space into simple pieces.

2. General heat kernels

A heat kernel can arise in a variety of ways, one can define a heat kernel directly, it can arise as the fundamental solution of a suitable operator, which itself can be defined through a Dirichlet form, it can be seen as the transition function of a Markov process
or as the integral kernel of a semigroup. In full generality not all these interpretations might exist but we are interested in situations where they are. We will first collect some well-known facts that hold in great generality.

To introduce a heat kernel and a Markov process we need an underlying space \( M \) with Hilbert space \( L_2(M) \). Throughout the paper we let \((M,d)\) be a compact metric space (\(M\) is then automatically complete and separable). The metric is a map \( d: M \times M \rightarrow [0, \infty] \), in particular we do not assume that \( M \) is connected. Let \( \mu \) be a Radon measure on \( \mathcal{B}(M) \), the Borel sets on \( M \). We assume \( \mu \) has full support on \( M \). The triple \((M,d,\mu)\) is called a metric measure space.

A semigroup \( P_t \) is called Markovian if for any \( f \) satisfying \( 0 \leq f \leq 1 \) \( \mu \)-almost everywhere we also have \( 0 \leq P_t f \leq 1 \) \( \mu \)-almost everywhere. It is contracting if \( \|P_t\| \leq 1 \).

A closed, densely defined linear operator \( D \) is called a Dirichlet operator if for all \( f \in \text{dom} \, D \) we have \( \langle Df, \max\{f-1,0\} \rangle \geq 0 \).

As usual, we use the same symbol for a symmetric form \( \mathcal{E} \) and the associated quadratic form given by \( \mathcal{E}(f) := \mathcal{E}(f,f) \). We assume that \( \mathcal{E} \) is non-negative, i.e., \( \mathcal{E}(f) \geq 0 \) for all \( f \in \text{dom} \, \mathcal{E} \). Such a form is closed if \( \text{dom} \, \mathcal{E} \) equipped with the norm given by \( \|f\|_\mathcal{E}^2 := \|f\|_{L_2(M)}^2 + \mathcal{E}(f,f) \) is complete, i.e., itself a Hilbert space. A closed symmetric form \( \mathcal{E} \) is a Dirichlet form if the unit contraction operates on it, i.e., if for \( f \in \text{dom} \, \mathcal{E} \), then \( f^\# \in \text{dom} \, \mathcal{E} \) and \( \mathcal{E}(f) \geq \mathcal{E}(f^\#) \), where \( f^\# := \min(\max(f,1),0) \) denotes the unit contraction operator.

A Dirichlet form is called regular if \( C(M) \cap \text{dom} \, \mathcal{E} \) is dense in \( C(M) \) (with respect to the supremum norm) and also dense in \( \text{dom} \, \mathcal{E} \) with respect to \( \|\cdot\|_\mathcal{E} \). A Dirichlet form is called local if \( \mathcal{E}(f,g) = 0 \) whenever \( f, g \in \text{dom} \, \mathcal{E} \) have disjoint support. If \( \mathcal{E}(f,g) = 0 \) whenever \( f \) is constant on a neighbourhood of the support \( \text{supp} \, g \) of \( g \), then \( \mathcal{E} \) is called strongly local.

The following is well known, see for example [Fuk80], [Kat80], [Dav80], [FOT11], [MR92] or [BH91].

2.1. Theorem. Let \( H \) be a Hilbert space. Then the existence of any of the following implies the existence of the others.

- a linear non-negative self-adjoint and densely defined operator \( D \)
- a closed symmetric form \( \mathcal{E} \)
- a strongly symmetric continuous contraction semigroup \( \{P_t\}_{t \geq 0} \) of self-adjoint operators

If \( H = L_2(M) \) we can additionally say the following. The semigroup satisfies the Markov property if and only if the operator is a Dirichlet operator if and only if the symmetric form is a Dirichlet form.

Proof. The equivalence of operator and form is given by \( \langle Df, g \rangle = \mathcal{E}(f,g) \) and \( \text{dom} \, \mathcal{E} = \text{dom} \, \sqrt{D} \). The operator \( D \) is the generator of the semigroup \( P_t := e^{tD} \).

2.2. Definition. A family \( \{p_t\}_{t > 0} \) of functions \( p_t: M \times M \rightarrow \mathbb{R} \) is called a heat kernel if it satisfies the following conditions for all \( t > 0 \).

(i) Measurability: \( p_t(\cdot, \cdot) \) is \( \mu \times \mu \) measurable
(ii) Markov property: \( p_t(x,y) \geq 0 \) for \( \mu \)-almost all \( x, y \) and \( \int_M p_t(x,y) \, d\mu(x) = 1 \) for \( \mu \)-almost all \( y \)
(iii) Symmetry: \( p_t(x,y) = p_t(y,x) \) for \( \mu \)-almost all \( x, y \)
(iv) Semigroup property: for any $s, t > 0$ and $\mu$-almost all $x, y$ we have

$$p_{s+t}(x, y) = \int_M p_s(x, z)p_t(z, y) \, d\mu(z)$$

(v) Approximation of identity: for any $f \in L^2(M)$ we have

$$\int_M p_t(x, y)f(y) \, d\mu(y) \to_{t \to 0^+} f(x)$$

2.3. Remark. One can consider more general heat kernels that satisfy only the sub-Markov property $\int_M p_t(x, y) \, d\mu(x) \leq 1$. In this case one usually expands the space $M$ by a cemetery point $\Delta$ and then extends the heat kernel so that it satisfies the Markov property on this larger space. In a similar way one can generalise the definition of a Markov process below. This construction is well known and explained for example in [FOT11]. However it makes various formulations a lot more chunky and gets somewhat technical. In the interest of clear exposition we will restrict to the strict Markov property here but remark that the generalization to sub-Markov is straightforward.

2.4. Remark. Heat kernels are only defined up to behaviour on a set of $\mu$-measure zero. We will thus identify heat kernels that agree $\mu$-almost everywhere.

If the heat kernel is continuous, this distinguishes this particular heat kernel, so that all the $\mu$-almost everywhere statements can be replaced by everywhere statements.

2.5. Definition. A Markov process on the set of continuous paths $\mathcal{P}(M)$ consists of a family of probability measures $\{P_x\}_{x \in M}$ on $\mathcal{P}(M)$ such that $P_x(\omega(0) = x) = 1$ and a stochastic process $X_t(\omega) := \omega(t)$ on $\mathcal{P}(M)$ with values in $M$. It additionally satisfies the Markov property. See the appendix for more details.

If it satisfies the strong Markov property, that is

$$P_x(X_{\zeta+s} \in U | \mathcal{F}_\zeta) = P_{X_\zeta}(X_s \in U)$$

holds for any time stopping function $\zeta$, then it is called a continuous Hunt process or a diffusion.

By convention one refers to $X_t$ as the Hunt process, the probability measure is implicit.

2.6. Theorem ([Gri03]). A heat kernel $p_t$ defines a semigroup via

$$P_t f(x) := \int_M p_t(x, y)f(y) \, d\mu(y)$$

This semigroup is strongly continuous, self-adjoint, contracting and Markov.

The Markov property is not mentioned in [Gri03], but it follows trivially from the definition.

2.7. Theorem ([BG68]). A continuous Hunt process $X_t$ on $M$ defines a heat kernel via

$$\int_U p_t(x, y) d\mu(y) = P_x(X_t \in U)$$

The heat kernel is the density function of the transition function $p_t(x, U) := P_x(X_t \in U)$. See the appendix for details.

Two stochastic processes are called equivalent if their transitions functions agree outside of a properly exceptional set. Note that all properly exceptional sets have $\mu$-measure zero.

2.8. Theorem ([FOT11, Thm 7.2.1 and Thm 4.2.8]). Given a regular local Dirichlet form $\mathcal{E}$ on $L^2(M)$, there exists a continuous Hunt process $X_t$ on $M$ whose Dirichlet form is the given one $\mathcal{E}$. This Hunt process is unique up to equivalence.
2.9. Remark. This theorem is by far the most difficult part in the equivalence. The full proof goes over several dozen pages. The basic construction is as follows. Given a family of probability measures $p_t(x,U)$, for $x$ fixed and $t \in [0,T]$, the Kolmogorov extension theorem guarantees the existence of a stochastic process $X_t$ and a probability measure $\mathbb{P}_x$ such that

$$p_t(x,U) = \mathbb{P}_x(X_t \in U)$$

but proving that this process has the claimed regularity is very hard.

3. Locality of the Wiener measure and the heat kernel

In this section we will show that the Wiener measure is local and then that the heat kernel is local provided it satisfies a suitable decay bound. We will first introduce the notion of martingales and then quote a uniqueness and existence theorem for the Wiener measure. Next, if two spaces are identical on some subset, we can define a new measure on the set of paths of one of the spaces by using one of the measures inside the subset and the other one outside. This is called splicing and will be explained in further detail. One can then show that this spliced Wiener measure is also compatible with the operator, by uniqueness this implies that the spliced measure is identical to the original measure. In other words, on the subset where the spaces and operators agree, so do the Wiener measures. Combined with a decay bound (see Definition 3.15) this implies that the heat kernel is local as well.

3.1. Local isometries

Let $(M,d,\mu)$ and $(M',d',\mu')$ be two metric measure spaces with energy forms $\mathcal{E}$ and $\mathcal{E}'$ and associated operators $D$ and $D'$, respectively.

Assume that $U \subset M$ and $U' \subset M'$ are open and that there exists a local isometry $\psi: U \rightarrow U' = \psi(U)$. For a function $f: M \rightarrow \mathbb{R}$ with $\text{supp} f \subset U$ we denote by $\psi_* f$ the function $f|_U \circ \psi^{-1}$ extended by 0 onto $M'$. Note that $\psi_*: L_2(U) \rightarrow L_2(U')$ is unitary.

3.1. Definition.

(i) We say that $\mathcal{E}$ and $\mathcal{E}'$ agree on $U$ and $U'$ if there is a measure preserving isometry $\psi: U \rightarrow U' = \psi(U) \subset M'$ such that for any $f \in \text{dom} \mathcal{E}$ with $\text{supp} f \subset U$ we have $\psi_* f \in \text{dom} \mathcal{E}'$ and $\mathcal{E}'(\psi_* f) = \mathcal{E}'(\psi_* f)$.

(ii) Similarly, we say that $D$ and $D'$ agree on $U$ and $U'$ if there is a measure preserving isometry $\psi: U \rightarrow U' = \psi(U) \subset M'$ such that for any $f \in \text{dom} D$ with $\text{supp} f \subset U$ we have $\psi_* f \in \text{dom} D'$ and $\psi_*(Df) = D'(\psi_* f)$.

3.2. Lemma. Assume that $\mathcal{E}$ and $\mathcal{E}'$ are local Dirichlet forms, then $\mathcal{E}$ and $\mathcal{E}'$ agree on $U$ and $U'$ if and only if $D$ and $D'$ agree on $U$ and $U'$.

Proof. Note first that $\mathcal{E}$ is local (i.e., $\mathcal{E}(f,g) = 0$ for all $f,g \in \text{dom} \mathcal{E}$ with $\text{supp} f \cap \text{supp} g = \emptyset$) if and only if $D$ is local (i.e., $\text{supp} Df \subset \text{supp} f$ for all $f \in \text{dom} D$).

Let $f \in \text{dom} D$ with $\text{supp} f \subset U$. Then there is an open set $V$ such that $\text{supp} f \subset V \subset \overline{V} \subset U$. If $g \in \text{dom} \mathcal{E}$ with $\text{supp} g \subset U$, then

$$\langle \psi_*(Df), \psi_* g \rangle_{L_2(M')} = \langle Df, g \rangle_{L_2(M)} = \mathcal{E}(f,g) = \mathcal{E}'(\psi_* f, \psi_* g)$$

as $\psi_*$ is an isometry for functions with support in $U$ and $U'$ (we used the locality of $D$ here) and as $\mathcal{E}$, $\mathcal{E}'$ agree on $U$ and $U'$. 

For \( g' \in \text{dom} \mathcal{E}' \) with \( \text{supp} g' \cap V = \emptyset \), we have \( \langle \psi_s(Df), g' \rangle_{L_2(M')} = 0 \) (again by locality of \( D \)) and \( \mathcal{E}'(\psi_s f, g') = 0 \). Since all \( \psi_s g \) with \( g \in \text{dom} \mathcal{E} \) and \( \text{supp} g \subseteq U \) and \( g' \in \text{dom} \mathcal{E}' \cap V = \emptyset \) span \( \text{dom} \mathcal{E}' \), we have shown that \( \psi_s f \in \text{dom} D' \) and \( D'(\psi_s f) = \psi_s(Df) \).

The opposite implication can be seen similarly. \( \square \)

### 3.2. Martingales

Recall that \( \mathcal{P}(M) \) denotes the set of continuous paths on a metric measure space \( (M, d, \mu) \).

#### 3.3. Definition.

A stochastic process \( Y : [0, \infty) \times \mathcal{P}(M) \to \mathbb{R} \) is called a martingale with respect to the family of probability measures \( \mathbb{P} = \{ \mathbb{P}_x \}_{x \in M} \) and the increasing sequence of \( \sigma \)-algebras \( \mathcal{F}_t \) if the following conditions are fulfilled:

- (i) Measurability: \( Y(t, \cdot) \) is \( \mathcal{F}_t \) measurable.
- (ii) Right continuity: for every \( \omega \in \mathcal{P}(M) \) the map \( t \mapsto Y(t, \omega) \) is right continuous.
- (iii) Conditional constancy: for \( 0 \leq s < t \) we have
  \[
  Y(s, \cdot) = \mathbb{E}^{\mathbb{P}_s}[Y(t, \cdot) | \mathcal{F}_s]
  \]
  holds \( \mathbb{P}_s \)-almost surely.

In most of our applications the probability measures and the \( \sigma \)-algebras will come from a Markov process \( X_t \), \( \mathcal{F}_t = \sigma(X_s | s \leq t) \) and we will just write \( Y \) is a martingale with respect to \( X_t \).

#### 3.4. Definition.

Let \( X_t \) be a Markov process on \( \mathcal{P}(M) \) and let \( D \) be a non-negative self-adjoint operator on \( M \). For each \( f \in \text{dom} D \), we define a stochastic process \( M_f : [0, \infty) \times \mathcal{P}(M) \to \mathbb{R} \) by setting

\[
M_f(t, X) := f(X_t) - \int_0^t (Df)(X_s) \, ds.
\]

We say that the Markov process \( X_t \) solves the martingale problem for \((M, D)\) if \( M_f \) is a martingale with respect to \( X_t \) for each \( f \in \text{dom} D \).

#### 3.5. Theorem ([EK86]).

Let \((M, d, \mu)\) be a compact metric measure space and \( D \) a non-negative self-adjoint operator on it. Then the continuous Hunt process \( X_t \) associated to \( D \) is the unique solution of the martingale problem for \((M, D)\).

### 3.3. Splicing measures

We will follow the construction of [Str05] for splicing measures on \( \mathbb{R}^n \) and extend it to the more general setting of metric measure spaces.

#### 3.6. Definition.

A measurable function \( \zeta : \mathcal{P}(M) \to [0, \infty] \) such that for all \( t \geq 0 \), \( \{ \zeta \leq t \} \in \mathcal{F}_t^0 \), is called a stopping time function.

Let \( \omega \in \mathcal{P}(M) \) and \( U \subseteq M \) be open, let

\[
\zeta_U(\omega) := \inf \{ t \geq 0 | \omega(t) \notin U \}
\]

be the first exit time from \( U \) of the path \( \omega \). The function \( \zeta_U \) is an example of a stopping time function. All stopping time functions we are going to use are of this form.

#### 3.7. Definition (Splicing measures on the same space).

Let \( \mathbb{P}' = \{ \mathbb{P}'_x \}_{x \in M} \) be a family of Borel probability measure on \( \mathcal{P}(M) \) with \( \mathbb{P}'_x(\omega(0) = x) = 1 \). Let \( U \subseteq M \) be open
and let $\chi_U$ denote the characteristic function of $U$. Define a family of Borel probability measures $\delta_\omega \otimes t \mathbb{P}$ on $\mathcal{P}(M)$ indexed by $t \in [0, \infty)$ and $\hat{\omega} \in \mathcal{P}(M)$ by setting

$$(\delta_\omega \otimes t \mathbb{P})(\omega(s) \in U) := \begin{cases} \chi_U(\hat{\omega}(s)) & s < t \\ \mathbb{P}^\omega_{\xi(t)}(\omega(s - t) \in U) & s \geq t \end{cases}$$

Next, given another family of Borel probability measures $\mathbb{P}$ on $\mathcal{P}(M)$ and a stopping time function $\zeta: \mathcal{P}(M) \rightarrow [0, \infty]$ define a new family of spliced measures $\mathbb{P} \otimes \zeta \mathbb{P}'$ by setting

$$(\mathbb{P} \otimes \zeta \mathbb{P}')(\omega(t) \in U) := \int_{\{\hat{\omega} \in \mathcal{P}(M) | \zeta(\hat{\omega}) < \infty\}} \delta_\omega \otimes \zeta(\hat{\omega}) \mathbb{P}'(\omega(t) \in U) \, d\mathbb{P}_x(\hat{\omega}) + \mathbb{P}_x(\omega(t) \in U \{\zeta(\omega) = \infty\}).$$

This can be interpreted as follows. Each path $\omega$ is measured with $\mathbb{P}$ until time $\zeta(\omega)$. After time $\zeta(\omega)$ it is measured by $\mathbb{P}'$ shifted back in time by $\zeta(\omega)$.

3.8. Remark. This spliced measure $\mathbb{P} \otimes \zeta \mathbb{P}'$ is also completely determined by stating that $\mathbb{P} \otimes \zeta \mathbb{P}' | _{\mathcal{F}_t} = \mathbb{P} | _{\mathcal{F}_t}$ and that the conditional distribution of shifted paths $\omega(\tau + \zeta(\omega))$ under $\mathbb{P} \otimes \zeta \mathbb{P}'$ with $\mathcal{F}_\zeta$ given, is just $\mathbb{P}^{\omega(\zeta)}$.

3.9. Remark. Note that if the stopping time function is of the form $\zeta_U$ defined above, paths that leave $U$ but reenter it at a later point would be measured with $\mathbb{P}'$ upon reentering. Hence the spliced measure is not just using one measure inside the set $U$ and the other one outside of it.

3.10. Definition (Splicing measures on different spaces). Let $(M, d, \mu)$ and $(M', d', \mu')$ be two metric measure spaces. Assume there exists an open set $U \subset M$ and a measure preserving isometry $\psi: U \rightarrow \psi(U) \subset M'$. Let $\mathbb{P}$ and $\mathbb{P}'$ be two families of Borel probability measures on $\mathcal{P}(M)$ and $\mathcal{P}(M')$.

For $A \subset U$ and $x \in U$ we let

$$\mathbb{P}^U_x(\omega(t) \in A) := \mathbb{P}'_{\psi(x)}(\psi(\omega(t)) \in \psi(A))$$

This defines a family of Borel measures on $\mathcal{P}(U)$.

We can now define the spliced measure $\mathbb{P}^U \otimes \zeta_{\psi} \mathbb{P}$ which is a family of Borel probability measures on $\mathcal{P}(M)$ as in Definition 3.7.

3.4. Locality of the Wiener measure

For $a, b \in \mathbb{R}$ let $a \wedge b := \min(a, b)$.

3.11. Theorem (Doob’s time stopping theorem [Str11]). If $Y(t, \omega)$ is a martingale with respect to a Markov process $X_t$, then for any time stopping function $\zeta$, $Y(t \wedge \zeta(\omega), \omega)$ is also a martingale with respect to the Markov process $X_t$.

3.12. Lemma ([SV79]). Let $\zeta$ be a stopping time and $X_t$ a Markov process. Recall that $\mathcal{F}_t = \sigma(X_s | s \leq t)$ and let $\{Q_\omega\}_{\omega \in \mathcal{P}(M)}$ be the conditional probability distribution of $\mathbb{P}$ with respect to $\mathcal{F}_t$ (see Definition A.5). Let $M: [0, \infty) \times \mathcal{P}(M) \rightarrow \mathbb{R}$ be a stochastic process. Assume $M$ is $\mathbb{P}$-integrable, $M(t, \cdot)$ is $\mathcal{F}_t^0$-measurable and $M(\cdot, \omega)$ is continuous. Then the following two statements are equivalent:

(i) $M(t, \omega)$ is a martingale with respect to $X_t$.

(ii) $M(t \wedge \zeta(\omega), \omega)$ is a martingale with respect to $X_t$ and $M(t, \omega) - M(t \wedge \zeta(\omega), \omega)$ is a martingale with respect to the measures $Q_\omega$ and the $\sigma$-algebra $\mathcal{F}_t^0$ for all $\omega \in \mathcal{P}(M)$ outside of a $\mathbb{P}$-null-set.
Recall that two (local) operators $D$ and $D'$ agree on some subsets if there is a measure preserving local isometry intertwining $D$ and $D'$ (see Definition 3.1 (ii)).

We now formulate our first main theorem:

**3.13. Main Theorem (Locality of the Wiener measure).** Let $(M, d, \mu)$ and $(M', d', \mu')$ be two metric measure spaces with non-negative self-adjoint operators $D$ and $D'$. Let $P$ and $P'$ be the associated Wiener measures. Assume that $D$ and $D'$ agree on some open subsets $U \subset M$ and $U' \subset M'$, then

$$P = P'' \otimes_{\zeta_U} P'.$$

In words, the spliced measure that uses the Wiener measure from $U$ from the original Wiener measure on $M$.

**3.14. Corollary.** Under the assumptions of the previous theorem we have

$$P|_{\mathcal{F}(U)} = P'|_{\mathcal{F}(U')},$$

i.e., when restricted to paths that stay inside $U$, the two Wiener measures are identical.

**Proof of Main Theorem 3.13.** This proof is a generalization of a proof in [Str11] where Stroock shows the above theorem for $\mathbb{R}^n$ instead of metric spaces.

We are going to show that the Markov process $X_t(\omega) := \omega(t)$ with the family of measures $P'' \otimes_{\zeta_U} P$ solves the martingale problem for $D$. Then uniqueness of the solution (Theorem 3.5) shows the equality of the measures. Thus we need to check that for all $f \in \text{dom} D$ the map

$$M_f(t, \omega) = f(\omega(t)) - \int_0^t (Df)(\omega(s))ds$$

is a martingale with respect to $P'' \otimes_{\zeta_U} P$.

We let $f' = f \circ \psi^{-1}$ and define $M'_{f'}$ analogously to $M_f$, hence $M'_{f'}(t, \omega')$ is a martingale with respect to $X'_t$. Through the isometry $\psi$ we get $M_f(t \land \zeta_U(\omega), \omega) = M'_{f'}(t \land \zeta_U(\omega), \omega')$ which is a martingale with respect to $X'_t$ by Doob's time stopping Theorem 3.11.

Note that up to time $\zeta_U(\omega)$ the measures $P'$ and $P'' \otimes_{\zeta_U} P$ are identical, so this implies that $M_f(t \land \zeta_U(\omega), \omega)$ is a martingale with respect to $P'' \otimes_{\zeta_U} P$ as well.

$M_f(t, \omega) - M_f(t \land \zeta_U(\omega), \omega)$ is just the function $M_f(t, \omega)$ starting at time $\zeta_U(\omega)$. Hence $M_f(t, \omega) - M_f(t \land \zeta_U(\omega), \omega)$ is a martingale for a shifted version of some $P$ for $t \geq 0$ if and only if $M_f(t, \omega)$ is a martingale for $P$ for $t \geq \zeta_U(\omega)$. Here being a martingale for $t \geq \zeta_U(\omega)$ means that the conditionally constant property only holds for these $t$ and not for $t \geq 0$ as in the original definition. For $t \geq \zeta_U(\omega)$, the measure $\delta_\omega \otimes_{\zeta_U(\omega)} P$ is the time shifted version of $P$. Thus $M_f(t, \omega) - M_f(t \land \zeta_U(\omega), \omega)$ is a martingale with respect to $\delta_\omega \otimes_{\zeta_U(\omega)} P$ if and only if $M_f(t, \omega)$ is a martingale with respect to $P$. The latter is true by assumption.

Next we have that $\delta_\omega \otimes_{\zeta_U(\omega)} P$ is the conditional probability distribution of $P'' \otimes_{\zeta_U} P$ with respect to $\mathcal{F}_{\zeta_U}(\omega)$, by which definition means

$$(P'' \otimes_{\zeta_U} P)(\omega(s) \in A \cap B) = \int_A (\delta_\omega \otimes_{\zeta_U(\omega)} P)(\omega(s) \in B) dP(\omega)$$

for $A \in \mathcal{F}_t$ and $B \in \mathcal{F}$. This is just the definition of the spliced measure Definition 3.7.
Hence we can apply Lemma 3.12 and conclude that $M_f$ is a martingale with respect to the measures $P^U \otimes_{\mathcal{C}_U} P$. □

3.5. Locality of the heat kernel

Here we are going to prove that exponential decay of the heat kernel together with the locality of the Wiener measure imply locality of the heat kernel.

3.15. Definition. Let $(M, d, \mu)$ be a metric measure space. Let $\mathcal{D}'$ be a strongly local regular Dirichlet form on $M$. Then we say the heat kernel satisfies an exponential decay bound if there exist constants $C, c > 0$ and $n \geq 0$ such that

$$p_t(x, y) \leq C t^{-n/2} \exp\left(-\frac{d^2(x, y)}{ct}\right)$$

holds for all $x, y \in M$ and all $t$ as long as $0 < t < T$ for some $T$.

In most applications $n$ is the dimension of $M$.

3.16. Theorem (Existence of conditional Wiener measure [BP11]). Let $(M, d, \mu)$ be a metric measure space. Assume the heat kernel $p$ satisfies an exponential decay bound as in Definition 3.15. Then there exists a unique point-to-point Wiener measure $P^y_x$ on the set $C^y_x([0, T], M)$, that is the set of continuous paths with start point $x$ and end point $y$ at time $T$. Moreover, it satisfies

$$P^y_x(\omega(t) \in U) = \int_U p_t(z, y) p_{T-t}(x, z) \, d\mu(z)$$

for $t \in (0, T)$ and $U \in \mathcal{B}(M)$. It is compatible with the Wiener measure $P^y_x$ in the sense that

$$\int_{C^y_x([0, t], M)} f(\omega) \, dP^y_x(\omega) = \int_M \int_{C^y_x([0, t], M)} f(\omega) \, dP^y_x(\omega) \, d\mu(y)$$

for any function $f : C^y_x([0, t], M) \rightarrow \mathbb{R}$ that is integrable with respect to $P^y_x$.

3.17. Remark. Note that the definition of $P^y_x$ immediately implies

$$P^y_x(C^y_x([0, t], M)) = p_t(x, y).$$

Theorem 3.16 cited above from [BP11] requires a decay bound of the heat kernel that is much weaker than the one we use here.

3.18. Definition. For $U \subseteq M$ be open and let $p_t$ be a heat kernel on $M$, let

$$p^U_t(x, y) := P^y_x(C^y_x([0, t], U)).$$

This means $p^U_t$ kills off all paths that leave the set $U$ and corresponds to the heat kernel on $U$ with Dirichlet boundary conditions. In particular $p^U_t(x, y) \leq p_t(x, y)$.

3.19. Proposition. Let $(M, d, \mu)$ and $(M', d', \mu')$ be two metric measure spaces with non-negative self-adjoint operators $D$ and $D'$ and let $p_t$ and $p'_t$ be the associated heat kernels. Assume $D$ and $D'$ agree on some open subsets $U$ and $U'$ via a measure preserving isometry $\psi : U \rightarrow U'$. Then

$$p^U_t(x, y) = p^{\psi(U)}(\psi(x), \psi(y)).$$

Proof. Using Theorem 3.16 this is exactly the statement of Corollary 3.14. □

The following lemma is based on an argument of [Hsu95]. The authors are indebted to Batu Güneysu for providing us with this reference and useful comments on the proof.
Locality of the heat kernel

3.20. Lemma. [Hsu95] Let $\zeta := \zeta_U$ be the time stopping function for the first exit time from the open set $U \subseteq M$. Let $x, y \in U$. Then the following decomposition of the heat kernel

$$p_t(x, y) = p^U_t(x, y) + \int_{\{\omega \in \mathcal{P}_x(M) | \zeta(\omega) \leq t\}} p_{t-\zeta(\omega)}(\omega(\zeta(\omega)), y) \, d\mathbb{P}_x(\omega)$$

holds for $\mu$-almost all $x, y \in U$.

This can be interpreted as follows. The set of all paths from $x$ to $y$ in time $t$ is decomposed into the set of paths that stay inside $U$ and those that do not. The paths that leave the set $U$ can be represented as an integral using the time $\zeta(\omega)$ and place $\omega(\zeta(\omega))$ where they leave the set $U$ for the first time.

If the heat kernel is continuous as a function of $x$ and $y$ one can replace the $\mu$-almost all $x, y \in U$ by all $x, y \in U$.

Proof. Let $f \in C(M)$ be such that $\text{supp } f \subset U$. Then we have

$$\int_U f(y)p_t(x, y) \, d\mu(y) = \int_{\mathcal{P}_x(M)} f(\omega(t)) \, d\mathbb{P}_x(\omega) = \int_{\zeta > t} f(\omega(t)) \, d\mathbb{P}_x(\omega) + \int_{\zeta \leq t} f(\omega(t)) \, d\mathbb{P}_x(\omega) = \int_U f(y)p^U_t(x, y) \, d\mu(y) + \mathbb{E}^x[f(\omega(t)) \chi_{\zeta \leq t}(\omega)]$$

by the definition of $p^U_t$. Here $\chi_{\zeta \leq t}$ denotes the characteristic function of the set $\{\zeta \leq t\}$ in $\mathcal{P}_x(M)$.

Using the substitution $\tau := t - \zeta$ we can write

$$\mathbb{E}^x[f(\omega(t)) \chi_{\zeta \leq t}(\omega)] = \mathbb{E}^x[f(\omega(\tau + \zeta)) \chi_{\tau > 0}(\omega)] = \mathbb{E}^x[f(\omega(\tau + \zeta))] = \mathbb{E}^x[f(\omega(\tau))] = \int_{\mathcal{P}_x(M)} \chi_{\tau > 0}(\omega) \mathbb{E}^\zeta[f(\omega(\tau))] \, d\mathbb{P}_x(\omega)$$

where we first used the fact that the condition expectation $\mathbb{E}^\zeta$ is just the identity projection in this case and then applied the strong Markov property (see Definition A.9).
We have
\[
\int_{\mathcal{P}^\infty(M)} \mathbb{E}^\xi [f(\omega(\tau))] \chi_{\tau \geq 0}(\omega) \, d\mathbb{P}^\xi(\omega)
\]
\[= \int_{\mathcal{P}^\infty(M)} \mathbb{E}^\xi [f(\omega(t - \zeta))] \chi_{\zeta < t}(\omega) \, d\mathbb{P}^\xi(\omega)
\]
\[= \int_{\mathcal{P}^\infty(M)} \int_{\mathcal{P}(M)} f(\hat{\omega}(t - \zeta)) \, d\mathbb{P}_{\omega(\zeta)}(\hat{\omega}) \chi_{\zeta < t}(\omega) \, d\mathbb{P}^\xi(\omega)
\]
\[= \int_{\mathcal{P}(M)} \int_{M} p_t - \zeta(\omega(\zeta), y) f(y) \, d\mu(y) \chi_{\zeta < t}(\omega) \, d\mathbb{P}^\xi(\omega)
\]
\[= \int_{U} \int_{\{\omega \in \mathcal{P}(M) \mid \zeta(\omega) < t\}} p_t - \zeta(\omega(\zeta), y) \, d\mathbb{P}^\xi(\omega) f(y) \, d\mu(y)
\]
where we used Fubini’s theorem in the last step.

This holds for all \( f \in C(M) \) with \( \text{supp} \, f \subset U \), hence we proved the lemma for \( \mu \)-almost all \( y \). \( \square \)

### 3.21. Lemma

Let \((M, d, \mu)\) be a metric measure space. Let \( p_t \) be a heat kernel on it. Assume the heat kernel satisfies a decay bound as in Definition 3.15. Let \( U \subset M \) be open. Then for \( \mu \)-almost all \( x, y \in U \) we have
\[
\mathbb{P}^u_x \left( C^u_x([0, t], M) \setminus C^u_x([0, t], U) \right) < Ct^{-\frac{n}{2}} e^{-\varrho^2/(ct)}
\]
where
\[
\varrho := \inf d(\{x, y\}, \partial U)
\]
is the infimum of the distance of \( x \) and \( y \) from the boundary and \( C, c > 0 \) are some constants that are independent of \( x, y \) for all \( x, y \) such that \( \varrho \) is bounded away from zero.

This is a bound on the set of paths from \( x \) to \( y \) in time \( t \) that leave the set \( U \). If the set \( U \) is geodesically convex, these paths are longer than the distance realizing paths.

**Proof.** We have
\[
\mathbb{P}^u_x \left( C^u_x([0, t], M) \setminus C^u_x([0, t], U) \right)
\]
\[= p_t(x, y) - p^u_t(x, y)
\]
\[= \int_{\{\omega \in \mathcal{P}(M) \mid \zeta(\omega) < t\}} p_t - \zeta(\omega(\zeta(\omega)), y) \, d\mathbb{P}^\xi(\omega)
\]
\[\leq C \int_{\{\omega \in \mathcal{P}(M) \mid \zeta(\omega) < t\}} (t - \zeta(\omega))^{-\frac{n}{2}} e^{-d(\omega(\zeta(\omega)), y)^2/(c(t - \zeta(\omega)))} \, d\mathbb{P}^\xi(\omega)
\]
\[\leq C \int_{\{\omega \in \mathcal{P}(M) \mid \zeta(\omega) < t\}} (t - \zeta(\omega))^{-\frac{n}{2}} e^{-d(\partial U, y)^2/(c(t - \zeta(\omega)))} \, d\mathbb{P}^\xi(\omega)
\]
where we used the heat kernel decomposition from Lemma 3.20 and then the heat kernel decay bound.

Let \( f(t) := t^{-n/2} e^{-\alpha/t} \) with \( \alpha > 0 \) and let \( T > 0 \) be fixed. Then for any \( 0 < s < t < T \) we have
\[
f(s) < \frac{f_{\text{max}}}{f(T)} f(t)
\]
where \( f_{\text{max}} \) denotes the unique maximum of the function \( f \).
Plugging in this estimate with \( s = t - \zeta(\omega) \) we get
\[
\mathbb{P}_x^y(C^y_x ([0, t], M) \setminus C^y_x ([0, t], U)) \\
\leq C \int_t^{\text{max}} f(T) t^{-\frac{2}{\omega}} e^{-d(\partial U, y)^2/(ct)} \mathbb{P}_x(\zeta \leq t) \\
\leq C' t^{-\frac{2}{\omega}} e^{-d(\partial U, y)^2/(ct)}
\]
The constant \( f_{\text{max}}/f(T) \) depends on \( y \) but if one restricts to values of \( y \) such that \( d(\partial U, y) \) is bounded away from zero one can pick a universal constant \( C' \) that works for all such \( y \).

By symmetry we can get the same estimate with \( d(\partial U, x) \). \( \square \)

Recall again that two (local) operators \( D \) and \( D' \) agree on some subsets if there is a measure preserving local isometry intertwining \( D \) and \( D' \) (see Definition 3.1 (ii)).

We now state our second main result:

**3.22. Main Theorem (Locality of the heat kernel).** Let \( (M, d, \mu) \) and \( (M', d', \mu') \) be two metric measure spaces with non-negative self-adjoint operators \( D \) and \( D' \). Assume \( D \) and \( D' \) agree on some open subsets \( U \) and \( U' \) via a measure preserving isometry \( \psi: U \rightarrow U' \). Assume in addition that the associated heat kernels \( p_t \) and \( p'_t \) each satisfy an exponential decay bound as stated in Definition 3.15. Let \( V \) be open with \( V \subset U \) and let \( x, y \in V \). Then

\[
|p_t(x, y) - p'_t(\psi(x), \psi(y))| \leq C e^{-\varepsilon/t}
\]

for \( \mu \)-almost all \( x, y \in V \) and all \( t \in (0, T] \), where the constants \( C \) and \( \varepsilon \) depend only on \( U, V \) and \( T \), but not on \( x, y \) or \( t \).

**Proof.** We can write the heat kernel \( p_t(x, y) \) with the help of the Wiener measure and separate the set of paths from \( x \) to \( y \) into the local part that stays in \( U \) and the part that leaves \( U \) as in Lemma 3.20. As \( \psi \) is an isometry, the set of local paths in \( U \) is the same as the local paths in \( U' \). By Main Theorem 3.13 the Wiener measures on these sets are also identical. Hence the heat kernels \( p_t \) and \( p'_t \) differ only by the Wiener measures of the non-local paths. Let \( \tilde{\varrho} := \inf(d(\partial V, \partial U)) \). Then \( \tilde{\varrho} > 0 \) because we assumed that the closure of \( V \) is contained in \( U \) and \( \tilde{\varrho} \) is the infimum over the \( (x, y) \)-dependent \( \varrho \) in Lemma 3.21 taken over all \( x, y \in V \). Hence if we apply Lemma 3.21 we get the estimate

\[
|p_t(x, y) - p'_t(\psi(x), \psi(y))| \leq 2C t^{-\frac{2}{\omega}} e^{-\tilde{\varrho}^2/ct}
\]

One can remove the \( t^{-n} \) term by using the following elementary estimate. For all \( \alpha > 0 \) and all \( 0 < b < a \) there exists a \( \tilde{C} > 0 \) such that

\[
t^{-\alpha} e^{-a/t} < \tilde{C} e^{-b/t}
\]

holds for all \( t > 0 \). \( \square \)

**3.23. Corollary.** Under the assumptions of Main Theorem 3.22, the asymptotic expansions for \( p_t \) and \( p'_t \) are identical over \( V \), i.e.,

\[
\left| \int_V p_t(x, x) \, d\mu(x) - \int_{\psi(V)} p'_t(x', x') \, d\mu'(x') \right| \rightarrow_{t \rightarrow 0^+} \tilde{C} e^{-\varepsilon/t}.
\]
3.24. Remark. We believe that Main Theorem 3.22 is sharp in the sense that some exponential decay bound on the heat kernel is needed for locality of the heat kernel to hold. Grigoryan [Gri03] considers heat kernel estimates for very general metric spaces. He shows that some fractals satisfy heat kernel estimates of the form

\[ p_t(x, y) \leq Ct^{-c_1} \exp \left( -\frac{d(x, y)^2}{t^{c_2}} \right) \]

for some suitable constants. One can probably extend Lemma 3.21 and Main Theorem 3.22 to this setting.

However, he also shows that the heat kernel for the operator \((-\partial^2_x - \partial^2_y)^{\frac{1}{2}}\) on subsets of \(\mathbb{R}^2\) with reasonably nice boundary satisfies a non-exponential decay bound of the form \(1/C(t^2 + td(x, y)) \leq p_t(x, y) \leq C/(t^2 + td(x, y))\) and one can easily show that these heat kernels do not satisfy locality in the sense of Main Theorem 3.22.

4. Manifold-like spaces

In this section, we will define manifold-like spaces. They provide a rich class of examples where the conditions for locality can be explicitly checked and proven. We start with metric measure spaces which satisfy the measure contraction property (MCP), a concept first introduced in [Stu98]. Then we define a manifold-like space as a quotient of an MCP space with only a finite number of points in each equivalence class being identified (and some other conditions).

4.1. The measure contraction property

We need a few more notions from metric geometry. For details we refer to the book [BBI01]. Let \((M, d)\) be a metric space and \(\gamma\) a path in \(M\), i.e., a continuous map \(\gamma: [a, b] \to M\) with \(a < b\). For a finite number of points \(T := \{t_0, \ldots, t_N\}\) with \(t_0 = a < t_1 < \cdots < t_N = b\) let

\[ L_d(\gamma, T) := \sum_{j=1}^{N} d(\gamma(t_{j-1}), \gamma(t_j)). \]

The length \(L_d(\gamma)\) of \(\gamma\) is defined as the supremum of \(L(\gamma, T)\) over all partitions \(T\) of \([a, b]\). The path \(\gamma\) is called rectifiable if \(L_d(\gamma)\) is finite.

For a subset \(M_0 \subseteq M\) we define the intrinsic metric \(d_{M_0}(x, y)\) of \(M_0\) in \(M\) as the infimum of \(L_d(\gamma)\) over all rectifiable paths \(\gamma\) from \(x\) to \(y\) which stay entirely in \(M_0\).

We say that \(M_0\) is geodesically complete if for all points \(x, y \in M_0\), the intrinsic metric \(d_{M_0}(x, y)\) is achieved by a shortest path \(\gamma\) joining \(x\) and \(y\) in \(M_0\), i.e., if \(d_{M_0}(x, y) = L_d(\gamma)\). We say that \(M_0\) is (geodesically) convex in \(M\) if \(M_0\) is geodesically complete and if \(d_{M_0} = d|_{M_0 \times M_0}\), i.e., all pairs of points \((x, y) \in M_0 \times M_0\) are joined by a geodesic \(\gamma\) in \(M_0\) with length given by the original metric \(d\), i.e., with \(L_d(\gamma) = d(x, y)\). If \(M\) is geodesically convex in itself, we say \(M\) is a geodesic space. We say that \(M_0\) is (geodesically) strictly convex (in \(M\)) if the geodesic joining any pair of points is unique.

Let \(B_r(x) := \{ y \in M \mid d(x, y) \leq r \} \subseteq M\) denote the (closed) ball of radius \(r\) around \(x\) and let \(B_r^*(x)\) denote the ball without the point \(x\). Let \(C_{\text{Lip}}(M)\) denote the Lipschitz continuous functions on \(M\). For \(t \in (0, 1)\), a point \(z\) is \(t\)-intermediate between \(x\) and \(y\) if \(d(x, z) = td(x, y)\) and \(d(y, z) = (1-t)d(x, y)\). If \(M\) is geodesically strictly convex,
the $t$-intermediate point between $x$ and $y$ is unique but in general there can be multiple $t$-intermediate points between $x$ and $y$.

For $N = 1$ set $\zeta_{K,1}^{(t)}(\theta) = t$. For $N > 1$ and $K < 0$ define

$$
\zeta_{K,N}^{(t)}(\theta) := t \left( \frac{\sinh(t\sqrt{-K/((N-1))})}{\sinh(\theta\sqrt{-K/((N-1))})} \right)^{N-1}
$$

This function defines a reference constant which represents the ratio of volumes of the radius $t\theta$ ball to the radius $\theta$ ball in the constant curvature $K$ space of dimension $N$. One can make suitable adjustments for $K = 0$ or $K > 0$.

### 4.1. Definition

A Markov kernel from $(\Omega_1, \mathcal{F}_1)$ to $(\Omega_2, \mathcal{F}_2)$ with $\Omega_i$ being measurable spaces and $\mathcal{F}_i$ the $\sigma$-algebras of measurable sets, is a map $P$ that associates to each $x \in \Omega_1$ a probability measure $P(x, \cdot)$ on $\Omega_2$ such that for any $B \in \mathcal{F}_2$ the map $x \mapsto P(x, B)$ is $\mathcal{F}_1$-measurable.

### 4.2. Definition ([Stu06b])

Let $N \geq 1$ and $K \in \mathbb{R}$. A metric measure space $(M, d, \mu)$ satisfies the $(K, N)$ measure contraction property or $(K, N)$-MCP for short if for every $t \in (0, 1)$ there exists a Markov kernel $P_t$ from $M \times M$ to $M$ such that

(i) $P_t(x, y; dz) = \delta_{\gamma_t(x, y)}(dz)$ with $\gamma_t(x, y)$ a $t$-intermediate point between $x$ and $y$ holds for $\mu^2$-almost all $(x, y) \in M \times M$.

(ii) for $\mu$-almost every $x \in M$ and every measurable $B \subseteq M$ we have

$$
\int_M \zeta_{K,N}^{(t)}(d(x, y)) P_t(x, y; B) \, d\mu(y) \leq \mu(B)
$$

$$
\int_M \zeta_{K,N}^{(1-t)}(d(x, y)) P_t(x, y; B) \, d\mu(y) \leq \mu(B)
$$

As written, this definition implies that $M$ is connected. By a slight abuse of notation we will also include disconnected spaces provided they have at most finitely many connected components and satisfy the measure contraction property on each component.

### 4.3. Remark

This definition can be interpreted as a way to generalize the notion of a lower Ricci curvature bound and an upper dimensional bound on a Riemannian manifold. A Riemannian manifold with Ricci curvature at least $K$ and dimension $N$ satisfies the $(K, N)$-MCP.

For a list of classes of spaces that satisfy this property and a few more explicit examples see Subsection 4.3 below.

### 4.4. Lemma

If $(M, d, \mu)$ satisfies the $(K, N)$-MCP then (each connected component of) $M$ is a geodesic space.

**Proof.** The definition of the $(K, N)$-MCP implies that for $\mu \otimes \mu$-almost all points $(x, y) \in M \times M$ and all $t \in (0, 1)$ a $t$-intermediate point exists. As we have assumed that $M$ is complete, we can replace ‘$\mu \otimes \mu$-almost all $(x, y)$’ by ‘all $(x, y)$’. Existence of $t$-intermediate points for all $t$ and all $x, y$ is equivalent to being a geodesic space by [Stu06a].

### 4.5. Theorem (see [Stu06b])

Assume the metric measure space $(M, d, \mu)$ satisfies the $(K, N)$-MCP for some $K \in \mathbb{R}$ and some $N \geq 1$. Then

(i) $(M, d, \mu)$ also satisfies the $(K', N')$-MCP for any $K' \leq K$ and any $N' \geq N$.

(ii) If $M' \subseteq M$ is convex, then $(M', d|_{M' \times M'}, \mu|_{M'})$ also satisfies the $(K, N)$-MCP.

(iii) $(M, d, \mu)$ has Hausdorff dimension at most $N$. 

(iv) For every $x \in M$ the function $r \mapsto \mu(B_r(x))/r^N$ is bounded away from zero for $r \in (0,1]$.

(v) $M$ satisfies the volume doubling property, that is there exists a constant $v_M$ such that for all $r > 0$ and all $x \in M$ we have

$$\mu(B_{2r}(x)) \leq v_M \mu(B_r(x))$$

Note that property (v) follows from property (iv).

4.6. Assumption. We assume from now on the following:

(i) The number $N$ is the exact Hausdorff dimension of $M$, that is the $N$ in the $(K,N)$-MCP is sharp.

(ii) The space $M$ is $N$-Ahlfors-regular, i.e., there exists a constant $c > 0$ such that

$$\frac{1}{c} r^N \leq \mu(B_r(x)) \leq cr^N$$

for all $x \in M$ and all $r \leq 1$.

4.7. Lemma. Assume the metric measure space $(M,d,\mu)$ satisfies the $(K,N)$-MCP for some $K \in \mathbb{R}$ and some $N \geq 1$. Then the limit

$$\tau(x) := \lim_{r \to 0} \frac{\mu(B_r(x))}{r^N}$$

exists for all $x \in M$.

Note that the limit function $\tau : M \to (0, \infty)$ is in general not continuous, but globally bounded.

Proof. MCP spaces satisfy the Bishop-Gromov inequality by [Stu06b], so $\frac{\mu(B_r(x))}{r^N}$ is increasing as $r \to 0$, as it’s bounded this implies the limit exists. \(\square\)

4.8. Remark. These assumptions restrict the class of examples compared to [Stu06b] but we feel we mostly excluded some pathological cases. We will call a space that satisfies the $(K,N)$-MCP and these assumptions an MCP space.

4.2. Glueing and manifold-like spaces

The class of MCP spaces is already fairly large but it does not contain some of the examples we want to study. We will extend this class by introducing a glueing operation.

4.9. Definition. Let $(M,d,\mu)$ be a metric measure space. We say that $\tilde{M}$ is obtained from $M$ by glueing

- if there are closed subsets $F_1$ and $F_2$ of $M$ such that there is a measure preserving isometry $\varphi : F_1 \to F_2$ and
- if $\tilde{M} := M/\sim$, where $\sim$ is the equivalence relation defined by $x \sim \varphi(x)$;
- we assume that there exists a $k \in \mathbb{N}$ such that each equivalence class contains at most $k$ elements.

Denote the natural projection map by $\pi : M \to \tilde{M}$. This projection defines a metric measure space $(\tilde{M},\tilde{d},\tilde{\mu})$ as follows: The induced distance is given by $\tilde{d}(\tilde{x},\tilde{y}) := \min \{ d(x,y) \mid \pi(x) = \tilde{x}, \pi(y) = \tilde{y} \}$, and the induced measure is the push forward measure $\tilde{\mu} := \pi^* \mu$ (i.e., $\tilde{\mu}(\tilde{B}) := \mu(\pi^{-1}(\tilde{B}))$).

Note that this construction includes both the possibility of glueing a metric space to itself as well as the possibility of glueing together two components of a disconnected metric space.
4.10. Definition. A manifold-like space is a connected metric measure space \((\tilde{M}, \tilde{d}, \tilde{\mu})\) that is obtained from a (possibly not connected) MCP space \((M, d, \mu)\) through a finite number of glueings.

Note that \(\tilde{M} = M/\sim\) where \(x \sim y\) if and only if there is a finite sequence of isometries \(\varphi_1, \ldots, \varphi_r\) defining the gluing such that \(x = x_0, x_1 = \varphi_1(x_0), \ldots, y = \varphi_r(x_{r-1})\). We still write \(\pi : M \to \tilde{M}\) for a manifold-like space.

4.11. Remark. Note that glueing does not preserve the \((K, N)\)-MCP property, as we will see in the example Subsection 4.3.

4.12. Theorem. Let \((\tilde{M}, \tilde{d}, \tilde{\mu})\) be a manifold-like space obtained from the \((K, N)\)-MCP space \((M, d, \mu)\). Then \(\tilde{M}\) inherits the following properties.

(i) \((\tilde{M}, \tilde{d}, \tilde{\mu})\) has Hausdorff dimension \(N\) and is \(N\)-Alfors-regular, see (4.1).

(ii) \((\tilde{M}, \tilde{d}, \tilde{\mu})\) satisfies the volume doubling property. There exists a constant \(v_{\tilde{M}}\) such that for all \(r > 0\) and all \(\tilde{x} \in \tilde{M}\) we have

\[
\tilde{\mu}(B_{2r}(\tilde{x})) \leq v_{\tilde{M}} \tilde{\mu}(B_r(\tilde{x}))
\]

(iii) The limit

\[
\tilde{\tau}(\tilde{x}) := \lim_{r \to 0} \frac{\tilde{\mu}(B_r(\tilde{x}))}{r^N}
\]

exists for every \(\tilde{x} \in \tilde{M}\). The limit function \(\tilde{x} \mapsto \tilde{\tau}(\tilde{x})\) is globally bounded on \(\tilde{M}\).

Proof. It is clearly sufficient to prove this for one glueing. By definition each point in \(\tilde{M}\) has only finitely many preimages in \(M\) under the projection map \(\pi\). This shows that \(\tilde{M}\) still has Hausdorff dimension \(N\) and is \(N\)-Alfors-regular.

As \(\tilde{\mu}\) is just the push forward metric of \(\mu\), properties (ii) and (iii) are directly inherited from \(M\). \(\square\)

4.3. Examples

In this section we will show that various classes of spaces are MCP spaces or manifold-like. We will also exhibit a few concrete examples and counter examples.

4.13. Lemma ([Stu06b]). If \((M, d, \mu)\) is a metric measure space with Hausdorff dimension \(N\) and with Alexandrov curvature bounded from below by \(\kappa\), then \(M\) satisfies the \(((N - 1)\kappa, N)\)-MCP.

4.14. Corollary. Compact Riemannian manifolds without boundary or with smooth boundary are MCP spaces.

Proof. Compact \(N\)-dimensional manifolds have Alexandrov curvature bounded from below by the Cartan-Alexandrov-Toponogov triangle comparison theorem and have Hausdorff dimension \(N\). \(\square\)

A closed subset \(D \subset \mathbb{R}^n\) is called a special Lipschitz domain if there is a Lipschitz-continuous function \(\psi : \mathbb{R}^{n-1} \to \mathbb{R}\) such that

\[
D = \{ (x', x_n) \in \mathbb{R}^n \mid \psi(x') \leq x_n \forall x' \in \mathbb{R}^{n-1} \}.
\]

4.15. Definition (cf. [MT99]). A pair of a compact metric measure space \((M, d, \mu)\) and a smooth Riemannian manifold without boundary \((\tilde{M}, \tilde{g})\) is a smooth manifold with Lipschitz boundary if the following holds.

- \(M \subseteq \tilde{M}\);
4.16. Corollary. If \((M, \tilde{M})\) is a manifold with Lipschitz boundary and \(M\) is convex in \(\tilde{M}\), then \(M\) is an MCP space.

**Proof.** The \((K, N)\)-MCP property is inherited on convex subsets by Theorem 4.5 (ii). The dimension Assumption 4.6 is inherited for subsets with Lipschitz boundary. Note that Lipschitz continuity is crucial here. If there are cusps, this assumption may fail, see Example 4.21 (v) below. □

4.17. Lemma. Compact metric graphs are manifold-like spaces.

**Proof.** A compact metric graph can be obtained from a finite number of finite intervals, that is manifolds with boundary, through glueing of the end points. □

Note that any vertex of degree at least 3 has Alexandrov curvature \(-\infty\).

4.18. Example. A compact good orbifold is a manifold-like space. See [DGGW08] for the exact definition and a general introduction to orbifolds. A good orbifold is the orbit space of an isometric action by a discrete group on a manifold. In other words it can be obtained through glueing from a manifold.

4.19. Lemma ([Oht07]). If \((M_1, d_1, \mu_1)\) and \((M_2, d_2, \mu_2)\) satisfy the \((K_1, N_1)\)-MCP and \((K_2, N_2)\)-MCP respectively, then \((M_1 \times M_2, d_1 + d_2, \mu_1 \times \mu_2)\) satisfies the \((\min(K_1, K_2), N_1 + N_2)\)-MCP. In other words, the MCP property is preserved under products.

4.20. Example. Let \((M, d, \mu)\) be a \((K, N)\)-MCP space. Then \(M \times M\) is a \((K, 2N)\)-MCP space. This can be seen as a physical model of the state space of two distinguishable particles.

Let \(\varphi\colon M \times M \to M \times M\) be the isometry given by \(\varphi(x, y) = (y, x)\). Then \((M \times M)/\sim\) with \((x, y) \sim \varphi(x, y)\) is a manifold-like space. This corresponds to the state space of two indistinguishable particles. The same construction applies to multi-particle systems.

4.21. Examples. Some concrete examples and counter-examples:

(i) If \(M\) is a flat cone (i.e. a wedge like segment of the unit disk in \(\mathbb{R}^2\) with boundaries identified) it satisfies the \((0, 2)\)-MCP and is an MCP space. The Alexandrov curvature is \(+\infty\) at the cone point and zero elsewhere.

(ii) Let \(M\) be constructed as follows. Cut open the unit disk in \(\mathbb{R}^2\) along the negative \(x\)-axis and glue in another quarter of the unit disk. \(M\) is a pseudo cone with angle \(5\pi/2\). It has Alexandrov curvature \(-\infty\) at the cone point and does not satisfy the \((K, N)\)-MCP for any \(K, N\) (see [Stu06b]) but \(M\) is a manifold-like space (glued out of 3 pieces to make the Lipschitz domains convex).

(iii) Let \(M\) consist of two copies of the unit disk in \(\mathbb{R}^2\) glued together at the origin. This is a manifold-like space but does not satisfy the \((K, N)\)-MCP.

(iv) Let \(M\) be the set of points in \(\mathbb{R}^3\) that is the union of \(\{(x, y, z) \mid x^2 + y^2 \leq 1, z = 0\}\) and \(\{(x, y, z) \mid x^2 + z^2 \leq 1, x, z \geq 0, y = 0\}\). This is a manifold-like space.
There exists a natural Dirichlet form on MCP spaces and it induces a Dirichlet form on manifold-like spaces.

5. The natural Dirichlet forms

5.1. Definition of the natural Dirichlet form

There exists a natural Dirichlet form on MCP spaces and it induces a Dirichlet form on manifold-like spaces.

5.1. Definition ([Stu06b]). Let $(M, d, \mu)$ be a metric measure space. Let

$$\mathcal{E}_r(f) := \int_M \frac{N}{r^n} \int_{B_r(x)} \left( \frac{f(y) - f(x)}{d(y, x)} \right)^2 d\mu(y) d\mu(x)$$

for all $f \in C_{Lip}(M)$.

5.2. Theorem (see [Stu06b]). Assume $(M, d, \mu)$ is an MCP space. Then the limit

$$\mathcal{E}(f) := \lim_{r \to 0} \mathcal{E}_r(f)$$

exists for all $f \in C_{Lip}(M)$. Furthermore the closure of $\mathcal{E}$ is a regular strongly local Dirichlet form on $(M, d, \mu)$ with core $C_{Lip}(M)$.

5.3. Remark. This and other theorems quoted from [Stu06b] also hold for non-compact MCP spaces. In this case one needs to replace the function spaces with the compactly supported versions.

We now define a Dirichlet form $\tilde{\mathcal{E}}$ on the quotient $\tilde{M}$ from our Dirichlet form $\mathcal{E}$ on the original space $M$ via $\pi: M \to \tilde{M} = M/\sim$. We can see $\tilde{\mathcal{E}}$ as a restriction of the form $\mathcal{E}$ (see remark below):

5.4. Proposition. Let $(\tilde{M}, d, \tilde{\mu})$ be a manifold-like space obtained from the MCP space $(M, d, \mu)$. Then the Dirichlet form $\mathcal{E}$ on $M$ induces a Dirichlet form $\tilde{\mathcal{E}}$ on $\tilde{M}$ as a pull back. This form $\tilde{\mathcal{E}}$ is also regular, strongly local and has core $C_{Lip}(\tilde{M})$.

Proof. As $\tilde{\mu}$ is the push forward measure of $\mu$, the map $\pi^*: L_2(\tilde{M}) \to L_2(M)$, $\pi^* f := \tilde{f} \circ \pi$, is an isometry onto its image. The image of $C_{Lip}(M) \subset L_2(\tilde{M})$ under $\pi^*$ is given by

$$\pi^*(C_{Lip}(M)) = \{ f \in C_{Lip}(M) \mid f(x) = f(y) \text{ whenever } x \sim y \} \subset \text{dom } \tilde{\mathcal{E}}$$

Hence for $\tilde{f} \in C_{Lip}(\tilde{M})$ we define $\tilde{\mathcal{E}}(\tilde{f}) := \mathcal{E}(\pi^* \tilde{f})$. We then define $\tilde{\mathcal{E}}$ to be the closure of this form with respect to the norm given by $\| \cdot \|_{\tilde{\mathcal{E}}}^2 = \| \cdot \|_{L_2(\tilde{M})}^2 + \tilde{\mathcal{E}}(\cdot)$.

The unit contraction property is inherited from $\mathcal{E}$, i.e. $\tilde{f} \in \text{dom } \tilde{\mathcal{E}}$ implies that $\tilde{f}^# \in \text{dom } \tilde{\mathcal{E}}$ and $\tilde{\mathcal{E}}(\tilde{f}^#) \leq \tilde{\mathcal{E}}(\tilde{f})$. Similarly, locality is inherited from $\mathcal{E}$.

For the regularity of $\tilde{\mathcal{E}}$, we note first that $C_{Lip}(\tilde{M}) \subset C(\tilde{M}) \cap \text{dom } \tilde{\mathcal{E}}$ by definition. Hence $C(\tilde{M}) \cap \text{dom } \tilde{\mathcal{E}}$ is dense in $\text{dom } \tilde{\mathcal{E}}$. By Stone-Weierstrass, $C_{Lip}(\tilde{M})$ is dense in $C(\tilde{M})$.
in the supremum norm. Thus \( C(\tilde{M}) \cap \text{dom } \tilde{\mathcal{E}} \) in also dense in \( C(\tilde{M}) \) in the supremum norm.

5.5. Remark. By definition, \( \pi^* \colon \text{dom } \tilde{\mathcal{E}} \to \text{dom } \mathcal{E} \) (endowed with the natural norms \( \| \cdot \|_{\tilde{\mathcal{E}}} \) and \( \| \cdot \|_{\mathcal{E}} \)) is also an isometry onto its image (as
\[
\| \pi^* \tilde{f} \|^2 = \| \pi^* \tilde{f} \|_{L^2(\tilde{M})}^2 + \mathcal{E}(\pi^* \tilde{f}) = \| \tilde{f} \|_{L^2(M)}^2 + \tilde{\mathcal{E}}(\tilde{f})
\]
for \( \tilde{f} \) in the core \( C_{\text{Lip}}(\tilde{M}) \)). Hence, we can also work with the corresponding image form \( \tilde{\mathcal{E}} \) on \( L_2(M) \), which is the restriction \( \hat{\mathcal{E}} := \mathcal{E}|_{\text{dom } \tilde{\mathcal{E}}} \) of \( \mathcal{E} \) with domain given by
\[
\text{dom } \hat{\mathcal{E}} = \{ f \in C_{\text{Lip}}(M) \mid f(x) = f(y) \text{ whenever } x \sim y \} \subset \text{dom } \mathcal{E}.
\]

Note that it can happen that \( \text{dom } \hat{\mathcal{E}} = \text{dom } \mathcal{E} \) although \( \pi^* C_{\text{Lip}}(\tilde{M}) \subset C_{\text{Lip}}(M) \). This happens because the \( \| \cdot \|_x \)-norm cannot see subsets of codimension at least two (see e.g. [CF78]). This effect can be seen in Example 4.21 (iii)). The Dirichlet form of two copies of the unit disk identified at a point is the same as the Dirichlet form on two disjoint copies.

5.2. Local isometries on manifold-like spaces

5.6. Proposition. (i) Let \( (M, d, \mu) \) and \( (M', d', \mu') \) be two MCP spaces with associated Dirichlet forms \( \mathcal{E} \) and \( \mathcal{E}' \) as constructed in Theorem 5.2. If there is a measure preserving isometry \( \psi : U \to U' = \tilde{\psi}(U) \) for open subsets \( U \subset M \) and \( U' \subset M' \), then \( \mathcal{E} \) and \( \mathcal{E}' \) agree on \( U \) and \( U' \).

(ii) Let \( (M, d, \mu) \) and \( (\tilde{M}, d, \tilde{\mu}) \) be two manifold-like spaces with associated Dirichlet forms \( \tilde{\mathcal{E}} \) and \( \tilde{\mathcal{E}}' \) as constructed in Proposition 5.4. Assume that \( \pi : M \to \tilde{M} \) and \( \pi' : M' \to \tilde{M}' \) are the corresponding projections from MCP spaces \( M \) and \( M' \), respectively (see Definition 4.10).

If there is a measure preserving isometry \( \tilde{\psi} : \tilde{U} \to \tilde{U}' = \tilde{\psi}(\tilde{U}) \) for open subsets \( \tilde{U} \subset \tilde{M} \) and \( \tilde{U}' \subset \tilde{M}' \) that lifts to a measure preserving isometry \( \psi : U \to U' \) with \( U = \pi^{-1}(\tilde{U}) \) and \( U' = (\pi')^{-1}(\tilde{U}') \) (i.e., \( \tilde{\psi} \circ \pi = \pi' \circ \psi \)), then \( \tilde{\mathcal{E}} \) and \( \tilde{\mathcal{E}}' \) agree on \( U \) and \( U' \).

Proof. (i) The Dirichlet form on the MCP space \( (M, d, \mu) \) is defined by \( \mathcal{E}(f) = \lim_{r \to 0} \mathcal{E}_r(f) \) and similarly for \( (M', d', \mu') \). As \( \mathcal{E}_r \) and \( \mathcal{E}'_r \) are expressed entirely in terms of the metric \( d \) and the measure \( \mu \), we have \( \mathcal{E}_r(f) = \mathcal{E}'_r(\psi_* f) \) for \( f \in C_{\text{Lip}}(M) \) with \( \text{supp } f \subset U \) and \( 0 < r < d(\text{supp } f, M \setminus U) \). Passing to the limit \( r \to 0 \) yields the first result.

(ii) By part (i), the lifted forms \( \tilde{\mathcal{E}} \) and \( \tilde{\mathcal{E}}' \) agree on \( U \) and \( U' \), i.e., \( \tilde{\mathcal{E}}(f) = \tilde{\mathcal{E}}'(\psi_* f) \). Moreover,
\[
\tilde{\mathcal{E}}(\tilde{f}) = \tilde{\mathcal{E}}(\pi^* \tilde{f}) = \tilde{\mathcal{E}}'(\psi_* \pi^* \tilde{f}) = \tilde{\mathcal{E}}'(\pi^* \psi_* \tilde{f}) = \tilde{\mathcal{E}}'(\psi_* \tilde{f})
\]
using the lift property of \( \tilde{\psi} \) and \( \psi \).

5.7. Corollary. Under the assumptions of the previous proposition, the associated operators on MCP resp. manifold-like spaces also agree.

Proof. This follows directly from Lemma 3.2.
5.3. The boundary conditions for the operator

We defined a Dirichlet form and the associated operator in a quite general setting. In this section we are going to show that for nice spaces the operator and the Dirichlet form are very natural and familiar.

The main motivation for the definition of the Dirichlet form Definition 5.1 in [Stu06b] is the fact that if $M$ is a Riemannian manifold, the corresponding form is $\mathcal{E}(f) = \int_M |\nabla f|^2 \, d\mu$. Additionally, his definition makes sense in a much broader metric space setting. This statement is also true in a local version:

5.8. Proposition. Assume that $(M, d, \mu)$ is a manifold-like space, and $M'$ a boundaryless Riemannian manifold with its natural metric $d'$ and Riemannian measure $\mu'$. If there is a measure preserving isometry $\psi: U \rightarrow U'$ with $U \subset M$ and $U' \subset M'$ open, then on $U$, the form $\mathcal{E}$ just acts as $\int_U |\nabla f|^2 \, d\mu'$ and the operator $D$ acts as the Laplace Beltrami operator on $U'$.

Proof. This follows directly from Proposition 5.6.

5.9. Definition. Assume that $(\bar{M}, \bar{d}, \bar{\mu})$ is a manifold-like space with MCP lift $(M, d, \mu)$ and projection $\pi: M \rightarrow \bar{M}$. We say that $\bar{U} \subset \bar{M}$ is an $r$-fold smooth fibration glued at a closed subset $\bar{F} \subset M$ if the following holds:

(i) $\bar{U}$ is open and connected and $\pi^{-1}(\bar{U}) = U = \bigcup_{j=1}^r U_j$, where each $U_j$ is connected and the closure of each $U_j$ is isometric to a subset of a Riemannian manifold with smooth boundary. The sets $U_j$ are called leaves.

(ii) $\bar{F}$ is connected and $\pi^{-1}(\bar{F}) = \bigcup_{j=1}^r F_j$ with $F_j$ connected and $F_j \subset \partial U_j$, hence $F_j$ is isometric to a subset of the boundary of the Riemannian manifold.

To simplify notation, we assume that $U_j$ and $F_j$ are already subsets of a Riemannian manifold (the former open in the interior, the latter a closed subset of the boundary). If $\bar{f}: \bar{M} \rightarrow \mathbb{R}$ denote by $f: M \rightarrow \mathbb{R}$ the lift of $\bar{f}$ onto $M$, i.e., $f \circ \pi = \bar{f}$. If $f$ is smooth enough on each $U_j$, we define $\partial_n f_j$ as the normal (outward) derivative of $f$ on $\bar{U}_j := F_j \cup U_j$, and we pull back all functions $\partial_n f_j |_{F_j}$ formally defined on $F_j \subset M$ onto $\bar{F}$ via the isometries and denote them by $\partial_n \bar{f}_j: \bar{F} \rightarrow \mathbb{R}$.

Note that $U \setminus \bigcup_{j=1}^r F_j$ is naturally the same as $\bar{U} \setminus \bar{F}$, as $\pi$ does not identify any points here. Moreover, these two sets also have the same measure, and integrals over them agree. Therefore, we consider these sets as the same.

5.10. Proposition. Let $(\bar{M}, \bar{d}, \bar{\mu})$ be a manifold-like space obtained from the MCP space $(M, d, \mu)$. Let $\bar{U} \subset \bar{M}$ be an $r$-fold smooth fibration with leaves $U_j$ glued at $F_j$.

If $\bar{f}$ is in the domain of the associated operator $\bar{D}$ on $\bar{M}$ with $\text{supp} \, \bar{f} \subset \bar{U}$ then $\bar{f}$ acts on $\bar{U}$ as the usual Laplacian ($\bar{D} \bar{f} = -\Delta \bar{f}$).

Moreover, the normal derivatives on the leaves satisfy the so-called Kirchhoff condition on the glued part $\bar{F}$. This means that

$$\sum_{j=1}^r \partial_n \bar{f}_j = 0 \quad \text{on } \bar{F}$$

and that $\bar{f}$ is continuous on $\bar{F}$. 

Locality of the heat kernel
Note that the derivatives are only weak derivatives. This theorem does not make any statements on the regularity of $\text{dom } \tilde{D}$. If we are only on parts which are $r$-fold smooth fibrations, then the solutions are in $H^2$.

**Proof.** Let $\tilde{g} \in \text{dom } \tilde{E}$ with supp $\tilde{g} \subset \tilde{U}$ and $g$ its lift. Now after our notes made above, we have

$$
\int_{\tilde{U}} \tilde{D} \tilde{f} \cdot \tilde{g} \, d\tilde{\mu} = \tilde{E}(\tilde{f}, \tilde{g}) = \sum_{j=1}^{r} \int_{U_j} (-\Delta f) \cdot g \, d\mu + \int_{F_j} \partial_n f_j \cdot g \, d\sigma
$$

using Green’s formula on the Riemannian manifold (third equality). Here, $\sigma$ denotes the canonical measure on the boundary of the Riemannian manifold and $\tilde{\sigma}$ the push forward measure on $\tilde{F}$ (counting each measure from the leaves boundary, hence $\tilde{\sigma}(\tilde{F}) = r\sigma(F_j)$). We first see that $\tilde{D} \tilde{f} = -\Delta \tilde{f}$ (choose $\tilde{g}$ with support away from $\tilde{F}$). Then we let $\tilde{g} \in \text{dom } \tilde{E}$ with supp $\tilde{g} \subset \tilde{U}$; as $\tilde{g}|_{\tilde{F}}$ runs through a dense subspace of $L^2(\tilde{F}, \tilde{\sigma})$, the result follows.

If $r = 1$, this reduces to the manifold case with Neumann boundary conditions.

**5.11. Corollary.** With the same notation as above and the additional assumption that $r = 1$, functions $\tilde{f} \in \text{dom } \tilde{D}$ with supp $\tilde{f} \subset \tilde{U}$ satisfy Neumann boundary conditions $\partial_n \tilde{f} = 0$ on $\tilde{F}$.

**5.12. Examples.** The simplest example of the situation in Proposition 5.10 is a metric graph. The MCP space consists of a collection of disjoint intervals, one for each edge of the metric graph. The glueing then identifies the end points of the intervals that correspond to adjacent edges in the metric graph.

Example 4.21 (iv) is a higher dimensional version.

**5.4. Heat kernel estimates**

**5.13. Theorem ([CKS87]).** Let $(M, d, \mu)$ be a compact metric measure space and $\mathcal{E}$ a regular Dirichlet form on it. Then there exists a measure $\Gamma(f)$ such that

$$
\mathcal{E}(f) = \int_{M} d\Gamma(f)(x)
$$

for any $f \in C(M) \cap \text{dom } \mathcal{E}$.

**5.14. Lemma (Subpartitioning lemma).** Let $(M, d, \mu)$ be a $(K, N)$-MCP space and let $U \subset M$ be open and convex. Then

$$
\int_{U} \frac{N}{r^N} \int_{B_r(x) \cap U} \left( \frac{f(y) - f(x)}{d(y, x)} \right)^2 \, d\mu(y) \, d\mu(x) \leq \int_{U} d\Gamma(f)(x)
$$
**Proof.** For any MCP space and \(0 = t_0 < t_1 < \ldots < t_{n-1} < t_n = 1\) a partition of the unit interval we have the estimate

\[
\mathcal{E}_r(f) \leq \sum_{i=1}^{n} (t_i - t_{i-1}) \mathcal{E}_{(t_i-t_{i-1})r}(f)
\]

by [Stu06b]. We will apply this directly to \(U\), which is also a \((K,N)\)-MCP space by Theorem 4.5 (ii). Let \(r_n := 2^{-nr}\) then \(\mathcal{E}(f) = \lim_{n \to \infty} \mathcal{E}_{r_n}(f)\). Using the partition \(0, \frac{1}{2}, 1\) this is an increasing sequence. Hence \(\mathcal{E}_r(f) \leq \mathcal{E}(f)\). \(\Box\)

**5.15. Definition.** Let \((M, d, \mu)\) be a metric measure space and \(\mathcal{E}\) a regular Dirichlet form on it. Then we define the **energy metric** \(\varrho\) on \(M\) as follows

\[
\varrho(x,y) := \sup \left\{ f(x) - f(y) \mid f \in \mathcal{C}(M) \cap \text{dom } \mathcal{E}, \frac{d\Gamma(f)}{d\mu} \leq 1 \text{ on } M \right\}
\]

where \(d\Gamma(f)/d\mu\) represents the Radon-Nikodým derivative. Note that this includes the implicit assumption that \(d\Gamma(f)\) is absolutely continuous with respect to \(d\mu\).

This metric is often called the intrinsic metric, especially when \(M\) is only a (sufficiently nice) topological space. To avoid confusion with the distance induced via the length of paths we use the term energy metric. A priori, the energy metric need not be a proper metric, it can be degenerate.

**5.16. Remark.** Let \((M, d, \mu)\) be an MCP space and let \(\mathcal{E}\) be the associated Dirichlet form. Then \(d\Gamma(f)\) is absolutely continuous with respect to \(d\mu\) for all \(f \in \mathcal{C}(M) \cap \text{dom } \mathcal{E}\) by [Stu06b, Cor. 6.6 (iii)].

**5.17. Lemma.** Let \((M, d, \mu)\) be a \((K,N)\)-MCP space. Let \(\mathcal{E}\) be the associated Dirichlet form from Theorem 5.2. Then the energy metric is equivalent to the metric \(d\). In particular they induce the same topology on \(M\).

**Proof.** This is proven in [Stu98] for a different version of the MCP. This proof is an adaptation of his proof.

We have \(\mathcal{E}(f) = \int_M d\Gamma(f) = \int_M \frac{d\Gamma(f)}{d\mu}(x) \, d\mu(x)\) by Theorem 5.13 and Remark 5.16.

For \(z \in M\) and \(C > 0\) fixed, let \(f(x) := Cd(x,z)\). Then we have

\[
\mathcal{E}_r(f) = \int_M \int_{B_r(z)} C^2 \left( \frac{d(x,z) - d(y,z)}{d(x,y)} \right)^2 \, d\mu(y) \, d\mu(x)
\]

\[
\leq \int_M \int_{B_r(z)} C^2 \, d\mu(y) \, d\mu(x).
\]

We assumed in (4.1) that \(\mu(B_r(x))/r^N\) is globally bounded by some constant \(c\). Hence we can apply the dominated convergence theorem and get

\[
\frac{d\Gamma(f)}{d\mu}(x) = \lim_{r \to 0} \frac{1}{r^N} \int_{B_r(z)} C^2 \left( \frac{d(x,z) - d(y,z)}{d(x,y)} \right)^2 \, d\mu(y).
\]

This shows \(d\Gamma(f)/d\mu \leq 1\) for \(C \leq (cN)^{-1/2}\).

Plugging \(f\) into the definition of the energy metric, we obtain \(\varrho(x,y) \geq C(d(x,z) - d(y,z))\) valid for any \(C \leq (cN)^{-1/2}\) and any \(z \in M\). In particular, for \(z := y\) we get the lower bound \(\varrho(x,y) \geq C d(x,y)\).

Let \(f \in \mathcal{C}\text{lip}(M)\) with \(d\Gamma(f)/d\mu \leq 1\) and let \(L_f\) denote the sharp Lipschitz constant of \(f\). Then \(f(x) - f(y) \leq L_f d(x,y)\). Hence if we show that there exists a global Lipschitz
constant \( L \) for all functions \( f \in C_{\text{Lip}}(M) \) that satisfy \( d\Gamma(f)/d\mu \leq 1 \) we get the estimate \( g(x, y) \leq Ld(x, y) \).

Let \( x_0, y_0 \in M \) be such that \( f(x_0) - f(y_0) \geq \frac{L_f}{6}d(x_0, y_0) \). We can assume without loss of generality that \( d_0 := d(x_0, y_0) \) is arbitrarily small by repeatedly taking midpoints. Hence \( d_0 \) can be bounded from above independent of \( f \). Let \( x \in B_{d_0/6}(x_0) \) and \( y \in B_{d_0/6}(y_0) \). Then

\[
|f(x) - f(y)| \geq |f(x) - f(y_0)| - |f(x) - f(x_0)| - |f(y) - f(y_0)| \\
\geq \left( \frac{L_f}{2} - \frac{L_f}{6} - \frac{L_f}{6} \right) d_0 = \frac{L_f d_0}{6} \geq \frac{L_f}{12} d(x, y)
\]

Let \( U := B_{2d_0}(x_0) \) and \( r = 2d_0 \) in Lemma 5.14, then

\[
\mu(B_{2d_0}(x_0)) \geq \int_{B_{2d_0}(x_0)} d\Gamma(f) \\
\geq \int_{B_{2d_0}(x_0)} \frac{N}{(2d_0)^N} \int_{B_{d_0}(x) \cap B_{2d_0}(x)} \left( \frac{f(y) - f(x)}{d(y, x)} \right)^2 d\mu(y) d\mu(x) \\
\geq \int_{B_{d_0/6}(x_0)} \frac{N}{(2d_0)^N} \int_{B_{d_0/6}(y_0)} \left( \frac{f(y) - f(x)}{d(y, x)} \right)^2 d\mu(y) d\mu(x) \\
\geq \frac{L_f^2}{144} \frac{N}{(2d_0)^N} \mu(B_{d_0/6}(x_0)) \mu(B_{d_0/6}(y_0))
\]

By (4.1) we have uniform global bounds for the volumes of balls. All the \( d_0 \) cancel out, so this proves an upper bound for \( L_f \) that is independent of \( f \) completing the proof. \( \Box \)

5.18. Lemma. Let \((\tilde{M}, \tilde{d}, \tilde{\mu})\) be a manifold-like space with induced Dirichlet form \( \tilde{\mathcal{E}} \). Then the energy metric is equivalent to the \( \tilde{d} \) metric.

Proof. It is sufficient to prove this for one glueing with glueing map \( \varphi \) and projection \( \pi \). We can write \( \tilde{d}(\tilde{x}, \tilde{y}) = \min \{ d(x, y) \mid \pi(x) = \tilde{x}, \pi(y) = \tilde{y} \} \) and similarly for the energy metric. Hence the equivalence of metrics is inherited through glueing. \( \Box \)

5.19. Definition. Let \((M, d, \mu)\) be a metric measure space that is \( N \)-Alfohrs regular and let \( \mathcal{E} \) be a regular Dirichlet form on \( M \). Let \( N^* := \max \{ 3, N \} \). Then we say \( \mathcal{E} \) satisfies the Sobolev inequality if there exists a \( C > 0 \) such that for all \( f \in \text{dom} \mathcal{E} \cap C_c(B_r(x)) \) we have

\[
\left( \int_{B_r(x)} |f|^{(2N^*)/(N^* - 2)} d\mu \right)^{(N^* - 2)/N^*} \leq C \frac{r^2}{\mu(B_r(x))^{2/N^*}} \left( \int_{B_r(x)} d\Gamma(f) + \frac{1}{r^2} \int_{B_r(x)} |f|^2 d\mu \right)
\]

for all \( r > 0 \).

5.20. Theorem ([Stu95]). Let \((M, d, \mu)\) be a metric measure space and \( \mathcal{E} \) a strongly local regular Dirichlet form on it.

Assume \( M \) satisfies the volume doubling property, the Sobolev inequality and the topology induced by \( g \) is the same as the one induced by \( d \). Then for any \( T > 0 \) and any \( \varepsilon > 0 \) there exists a \( C > 0 \) such that the heat kernel estimate

\[
p_t(x, y) \leq C \mu(B_{\sqrt{t}}(x))^{-1/2} \mu(B_{\sqrt{t}}(y))^{-1/2} \exp \left( -\frac{\varphi^2(x, y)}{4 + \varepsilon} t \right)
\]

is valid for all \( x, y \in M \) and all \( 0 < t < T \).
5.21. Corollary. Let \((M, d, \mu)\) be a \((K, N)\)-MCP space. Let \(E\) be the strongly local regular Dirichlet form from Theorem 5.2. Then the heat kernel satisfies an exponential decay bound as in Definition 3.15. That is

\[ p_t(x, y) \leq Ct^{-N/2} \exp\left(-\frac{d^2(x, y)}{ct}\right) \]

holds for some \(C, c > 0\) independent of \(x, y \in M\) and of \(t\) as long as \(0 < t < T\) for some \(T\).

Proof. \((M, d, \mu)\) satisfies a parabolic Harnack inequality by \([Stu06b]\). This also implies that \(E\) satisfies the Sobolev inequality \([Stu06b]\). Lemma 5.17 gives the equivalence of the energy metric and \(d\). Hence the assumptions of Theorem 5.20 are satisfied.

As the volume of radius \(\sqrt{t}\)-balls is bounded and the metrics \(d\) and \(\rho\) are equivalent, we can reformulate the bound from \([Stu95]\) as written. □

5.22. Corollary. Let \((\tilde{M}, \tilde{d}, \tilde{\mu})\) be a manifold-like space with induced Dirichlet form \(\tilde{E}\). Then the heat kernel satisfies the exponential decay bound in Definition 3.15.

Proof. We are going to use Theorem 5.20 again. The only assumption that is missing is the Sobolev inequality.

Let \((M, d, \mu)\) be the \((K, N)\)-MCP space that \(\tilde{M}\) was obtained from. The Sobolev inequality holds on \(M\) by \([Stu06b]\). As \(\tilde{E}\) is defined as a restriction of \(E\) and the measure \(\tilde{\mu}\) on \(\tilde{M}\) is just the push forward measure, the Sobolev inequality also holds on \(\tilde{M}\). □

The manifold-like spaces we defined in Section 4 provide a large class of examples where locality holds:

5.23. Main Theorem. Let \((M, d, \mu)\) and \((M', d', \mu')\) be two manifold-like spaces. Let \(E\) and \(E'\) be the natural Dirichlet forms on \(M\) and \(M'\) from Proposition 5.6. Let \(p_t\) and \(p'_t\) be the associated heat kernels and let \(P\) and \(P'\) the associated Wiener measures.

Let \(U \subset M\) be open and assume there exists a measure preserving isometry \(\psi: U \to U' \subset M'\).

Then the Wiener measures \(P\) and \(P'\) are identical on \(U\).

Let \(V\) be open with \(\overline{V} \subset U\) and let \(x, y \in V\). Then the difference of the heat kernels is exponentially small, that is

\[ |p_t(x, y) - p'_t(\psi(x), \psi(y))| \leq Ce^{-\varepsilon/t} \]

for \(\mu\)-almost all \(x, y \in V\) and all \(t \in (0, T]\). The asymptotic expansions of \(p_t\) and \(p'_t\) agree on \(V\).

Proof. If \(\psi\) is a measure preserving isometry, then the Dirichlet forms and the operators are equivalent by Proposition 5.6. Hence we get equivalence of the Wiener measures by Main Theorem 3.13. The heat kernels associated to these Dirichlet forms satisfy the heat kernel decay bound by corollary Corollary 5.21. Thus we can apply Main Theorem 3.22 and get locality of the heat kernel. □
6. Example application: a two particle system on a metric graph

Let $G = (V,E)$ be a compact metric graph with vertex set $V$ and edge set $E$. A metric graph is a combinatorial graph together with an assignment of edge lengths. The operator is the Laplacian, that is the second derivative on the edges seen as intervals. We impose the standard Kirchhoff boundary conditions at all vertices. This means functions are continuous and the sum of the first derivatives on all edges adjacent to a vertex is zero (the derivatives are oriented away from the vertex). A metric graph together with the operator is called a quantum graph. See for example [BK13] for an introduction and a survey of quantum graphs.

The manifold-like space we will look at is $M := G \times G/\sim$ with $(x,y) \sim (y,x)$. This is a model from physics, it corresponds to two particles moving freely on a metric graph. The particles do not interact and they are indistinguishable, hence we factor out by the symmetry.

This is a 2-dimensional space which is neither a manifold nor an orbifold. To the best of our knowledge, the results in this paper are the first to explicitly show that these kind of spaces do have ‘well-behaved’ heat kernels.

In order to compute the heat asymptotics of the heat kernel for this system, we will decompose the state space of the two particles into various pieces. This is were the locality of the heat kernel comes in. For each piece we will explicitly compute the heat kernel of a different space which is locally isometric to the piece of $M$ but globally a much simpler space. For these much simpler spaces one can write down an explicit expression of the heat kernel and use it to compute the asymptotics.

Pick a universal $\varepsilon > 0$ much smaller than any edge length. We say that a particle is in the neighbourhood of a vertex if it less than $\varepsilon$ away from it. We decompose $M$ into the following types of pieces.

(A) both particles are away from vertices and on distinct edges
(B) both particles are away from vertices and on the same edge
(C) one particle is in a neighbourhood of a vertex, the other one is away from the vertices on an edge
(D) the particles are in neighbourhoods of two distinct vertices
(E) both particles are in the neighbourhood of a vertex

Note that the cutoffs between the pieces need to be made in a way that intersects the singular pieces of $M$ orthogonally otherwise these cutoffs will produce additional terms in the asymptotics.

For the pieces of type A, C and D the particles cannot run into each other. So the heat kernel is just the product of the heat kernels of the two pieces. For pieces of type B and E the heat kernel is the product of the individual heat kernels modded out by the symmetry.

For a single particle away from vertices we can just use the real line as a comparison space, the heat kernel is $p_R(t,x,y) = \frac{1}{\sqrt{4\pi t}} e^{-(x-y)^2/4t}$. For a particle in the neighbourhood of a vertex we use the star shaped metric graph consisting of $k$ half-infinite edges all meeting in a single central vertex as a comparison space. Its heat kernel can be written down explicitly. For $\alpha, \beta \in [1, \ldots, \deg(v)]$, we write $x^\alpha$ when $x$ is on the edge $\alpha$. The
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heat kernel is then given by

\[ p_S(t, x^\alpha, y^\beta) = \frac{1}{\sqrt{4\pi t}} \left( \delta_{\alpha\beta} e^{-(x^\alpha - y^\beta)^2/4t} + \sigma_{\alpha\beta} e^{-(x^\alpha + y^\beta)^2/4t} \right) \]

(see e.g. [Rot84] or [KPS07] for a more general version), where \( \delta_{\alpha\beta} \) is the Kronecker-\( \delta \) and \( \sigma_{\alpha\beta} \) is the matrix of the boundary conditions at the vertex. For Kirchhoff conditions we have \( \sigma_{\alpha\beta} = -\delta_{\alpha\beta} + \frac{2}{\deg(v)} \).

We will just carry out the computations for pieces of type \( C \) and \( E \). The other types work in exactly the same way.

Let \( N(v) \) denote the \( \varepsilon \)-neighbourhood of the vertex \( v \) and \( l_\gamma \) the edge where the other particle is located. The particles do not interact, hence we just multiply a heat kernel of the star graph with a heat kernel on the real line.

We can now apply Main Theorem 5.23 which says that the heat kernel on \( M \) in the region \( C \) and the heat kernel on the comparison space differ only by an exponentially small error term. Hence, on the diagonal the heat kernel of the region \( C \) is given by

\[ p_C(t, (x_1^\alpha, x_2), (x_1^\alpha, x_2)) = \frac{1}{4\pi t} \left( 1 + \sigma_{\alpha\alpha} e^{-(x_1^\alpha)^2/4t} \right) + O(t^\infty) \]

Here and in further computations we use the notation \( O(t^\infty) \) to mean an error term that can be bounded as \( O(t^k) \) for any \( k \). Such an error will make no contribution to the asymptotics we are interested in. Thus the contribution to the asymptotics is

\[
\int_C p_C(t, (x_1, x_2), (x_1, x_2)) \, dx_1 \, dx_2 \\
= \frac{1}{4\pi t} (l_\gamma - 2\varepsilon) \varepsilon \deg(v) + \frac{1}{4\pi t} (l_\gamma - 2\varepsilon) \sum_{\alpha\sim v} \sigma_{\alpha\alpha} \int_0^\varepsilon e^{-(x_1^\alpha)^2/4t} \, dx_1^\alpha \\
= \frac{1}{4\pi t} \text{vol}(C) + \frac{1}{8\sqrt{\pi t}} (l_\gamma - 2\varepsilon) \sum_{\alpha\sim v} \sigma_{\alpha\alpha} + O(t^\infty)
\]

To get an expression for the heat kernel on the diagonal of the region \( E \), we also need to know it in a neighbourhood of the diagonal. In a neighbourhood of the diagonal we can just assume that both points on \( M \) are in the region \( E \).

For two distinguishable particles, the heat kernel on the domain \( N(v) \times N(v) \) is just the product, that is

\[
p_{N(v) \times N(v)}(t, (x_1^\alpha, x_2^\beta), (y_1^\gamma, y_2^\delta)) \\
= \frac{1}{4\pi t} \left( \delta_{\alpha\gamma} e^{-(x_1^\alpha - y_1^\gamma)^2/4t} + \sigma_{\alpha\gamma} e^{-(x_1^\alpha + y_1^\gamma)^2/4t} \right) \cdot \left( \delta_{\beta\delta} e^{-(x_2^\beta - y_2^\delta)^2/4t} + \sigma_{\beta\delta} e^{-(x_2^\beta + y_2^\delta)^2/4t} \right)
\]
the one on \( E' \) up to an exponentially small error term. Hence

\[
p_E(t, (x_1^\alpha, x_2^\beta), (y_1^\alpha, y_2^\beta)) = \frac{1}{4\pi t} \left( \delta_{\alpha\gamma} e^{-(x_1^\gamma - y_1^\gamma)^2/4t} + \sigma_{\alpha\gamma} e^{-(x_1^\gamma + y_1^\gamma)^2/4t} \right) \cdot \left( \delta_{\beta\delta} e^{-(x_2^\delta - y_2^\delta)^2/4t} + \sigma_{\beta\delta} e^{-(x_2^\delta + y_2^\delta)^2/4t} \right) + O(t^\infty)
\]

On the diagonal this gives

\[
p_E(t, (x_1^\alpha, x_2^\beta), (x_1^\alpha, x_2^\beta)) = \frac{1}{4\pi t} \left( 1 + \sigma_{\alpha\alpha} e^{-(x_1^\gamma)^2/4t} \right) \cdot \left( 1 + \sigma_{\beta\beta} e^{-(x_2^\gamma)^2/4t} \right) + O(t^\infty)
\]

We will separate the integration into two parts, first the region where \( \alpha < \beta \) and second the region where \( \alpha = \beta \) and \( x_1^\alpha \leq x_2^\beta \). In the first region we can integrate over the rectangle \([0, \epsilon] \times [0, \epsilon]\) for each pair of edges.

\[
\sum_{\alpha < \beta} \int_0^\epsilon \int_0^\epsilon \sigma_{\alpha\beta} e^{-(x_1^\gamma)^2/4t} \alpha \int_0^\epsilon \sigma_{\beta\gamma} e^{-(x_2^\gamma)^2/4t} \beta dx_\alpha dx_\beta
\]

\[
= \frac{1}{4\pi t} \sum_{\alpha < \beta} \left( \epsilon + \sigma_{\alpha\beta} \frac{\sqrt{\pi}}{2} t^{1/2} \right) \cdot \left( \epsilon + \sigma_{\beta\alpha} \frac{\sqrt{\pi}}{2} t^{1/2} \right) + \frac{1}{4\pi t} \sum_{\alpha < \beta} (\sigma_{\alpha\beta})^2 t + O(t^\infty)
\]

For the second part, where \( \alpha = \beta \) we will drop the superscript and write \( x_1 \) and \( x_2 \) for \( x_1^\alpha \) and \( x_2^\beta \) to simplify notation.

\[
e_E(t, (x_1, x_2), (x_1, x_2)) = \frac{1}{4\pi t} \left( 1 + \sigma_{\alpha\alpha} e^{-(x_1^\gamma)^2/4t} \right) \cdot \left( 1 + \sigma_{\alpha\alpha} e^{-(x_1^\gamma)^2/4t} + \sigma_{\beta\alpha} e^{-(x_1^\gamma + x_2^\gamma)^2/4t} \right) + O(t^\infty)
\]

\[
= \frac{1}{4\pi t} + \frac{1}{4\pi t} \sigma_{\alpha\alpha} e^{-(x_1^\gamma)^2/4t} + \frac{1}{4\pi t} \sigma_{\alpha\alpha} e^{-(x_2^\gamma)^2/4t} + \frac{1}{4\pi t} (\sigma_{\alpha\alpha})^2 e^{-(x_1^\gamma + x_2^\gamma)^2/4t}
\]

\[
+ \frac{1}{4\pi t} e^{-(x_1^\gamma)^2/2t} + \frac{1}{2\pi t} \sigma_{\alpha\alpha} e^{-(x_1^\gamma + x_2^\gamma)^2/2t} + \frac{1}{4\pi t} (\sigma_{\alpha\alpha})^2 e^{-(x_1^\gamma + x_2^\gamma)^2/2t} + O(t^\infty)
\]

In the second region we will integrate over a difference of two triangles to ensure that the region meets the boundary orthogonally. The first triangle is described by
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\[ 0 \leq x_2 \leq x_1 \leq \varepsilon \]

with area \( \frac{1}{2} \varepsilon^2 \). The second triangle is the region where \( 2^{-1/2} \varepsilon \leq x_1 \leq \varepsilon \) and \( 2^{1/2} \varepsilon - x_1 \leq x_2 \leq x_1 \) with area \( \frac{1}{2} (1 - 2^{-1/2}) \varepsilon^2 = \left( \frac{3}{8} - 2^{-1/2} \right) \varepsilon^2 \). Therefore

\[
4\pi t \int_0^\varepsilon \int_0^{x_1} e_E(t, (x_1, x_2), (x_1, x_2)) \, dx_2 \, dx_1 \\
- 4\pi t \int_{2^{-1/2} \varepsilon}^\varepsilon \int_{2^{1/2} \varepsilon - x_1}^{x_1} e_E(t, (x_1, x_2), (x_1, x_2)) \, dx_2 \, dx_1
\]

\[
= \frac{1}{2} \varepsilon^2 + \sigma_{\alpha \alpha}^{-1} \int_0^\varepsilon \int_0^{x_1} x_1 e^{-x_1^2/t} \, dx_1 + \sigma_{\alpha \alpha}^{-1} \int_0^\varepsilon \int_0^{x_1} e^{-x_2^2/t} \, dx_2 \, dx_1
\]

\[
+ (\sigma_{\alpha \alpha}^{-1})^2 \int_0^\varepsilon \int_0^{x_1} e^{-(x_1^2 + x_2^2)/t} \, dx_2 \, dx_1 + \int_0^\varepsilon \int_0^{x_1} e^{-(x_1 - x_2)^2/2t} \, dx_2 \, dx_1
\]

\[
+ 2\sigma_{\alpha \alpha}^{-1} \int_0^\varepsilon \int_0^{x_1} e^{-(x_1^2 + x_2^2)/2t} \, dx_2 \, dx_1 - (\sigma_{\alpha \alpha}^{-1})^2 \int_0^\varepsilon \int_0^{x_1} e^{-(x_1 + x_2)^2/2t} \, dx_2 \, dx_1
\]

\[
- (\sigma_{\alpha \alpha}^{-1})^2 \int_0^\varepsilon \int_0^{x_1} e^{-(x_1 + x_2)^2/2t} \, dx_2 \, dx_1 + O(t^\infty)
\]

\[
= \frac{1}{2} \varepsilon^2 + (\sigma_{\alpha \alpha}^{-1})^2 \left( \frac{1}{2} t + \sqrt{\pi} \varepsilon t^{1/2} \right) + (\sigma_{\alpha \alpha}^{-1})^2 \frac{1}{8} \pi t
\]

\[
- t + 2^{-1/2} \sqrt{\pi} \varepsilon t^{1/2} + 2\sigma_{\alpha \alpha}^{-1} \frac{1}{4} \pi t + (\sigma_{\alpha \alpha}^{-1})^2 \frac{1}{2} t
\]

\[
- (\sigma_{\alpha \alpha}^{-1})^2 \frac{1}{8} \pi t - t + \sigma_{\alpha \alpha}^{-1} \frac{1}{2} \pi t + (\sigma_{\alpha \alpha}^{-1})^2 \frac{1}{2} t + \frac{t}{2} + O(t^\infty)
\]

The various terms that give zero contribution can all be bounded by noticing that the integrated function can be bounded as \( e^{-C/t} \) for some constant \( C > 0 \) over the entire domain of integration.

Let \( \Omega \) denote the region we integrated over, then this gives the following contribution

\[
\int_\Omega e_E(t, (x_1, x_2), (x_1, x_2)) \, dvol
\]

\[
= \frac{1}{4\pi t} \text{vol}(\Omega) + \frac{1}{8\sqrt{\pi} t} \left( \sigma_{\alpha \alpha}^{-1} + 1 \right) \varepsilon
\]

\[
- \frac{1}{8\pi} + \frac{1}{8} \sigma_{\alpha \alpha}^{-1} + \left( \frac{1}{32} + \frac{1}{8\pi} \right) (\sigma_{\alpha \alpha}^{-1})^2 + O(t^\infty)
\]
so for the entire region $E$ we get

$$\int_{E} e_{E}(t,(x_{1}, x_{2}),(x_{1}, x_{2})) \operatorname{dvol} = \frac{1}{4\pi t} \operatorname{vol}(E) + \frac{1}{8\sqrt{\pi t}} \sum_{\alpha} \left(\sigma_{\alpha\alpha}^{v} + 1\right) \varepsilon + \frac{1}{8\sqrt{\pi t}} \sum_{\alpha<\beta} \left(\sigma_{\alpha\alpha}^{v} + \sigma_{\beta\beta}^{v}\right) \varepsilon$$

$$- \frac{\deg(v)}{8\pi} + \frac{1}{8} \sum_{\alpha} \sigma_{\alpha\alpha}^{v} + \left(\frac{3}{32} + \frac{1}{8\pi}\right) \sum_{\alpha} \left(\sigma_{\alpha\alpha}^{v}\right)^{2}$$

$$+ \frac{1}{16} \sum_{\alpha<\beta} \sigma_{\alpha\alpha}^{v} \sigma_{\beta\beta}^{v} + \frac{1}{4\pi} \sum_{\alpha<\beta} \left(\sigma_{\alpha\beta}^{v}\right)^{2} + O(t^{\infty})$$

After doing a similar computation for the pieces of type $A$, $B$ and $D$ we get the following heat asymptotics.

**6.1. Theorem.** Let $M := G \times G/((x, y) \sim (y, x))$ where $G = (V, E)$ is a metric graph with the standard Kirchhoff boundary conditions at all vertices. Then the heat kernel asymptotics of $M$ are

$$\int_{M} p(t,(x_{1}, x_{2}),(x_{1}, x_{2})) \operatorname{d}x_{1} \operatorname{d}x_{2}$$

$$\rightarrow_{t \to 0^{+}} \frac{1}{4\pi t} \operatorname{vol}(M) + \frac{1}{8\sqrt{\pi t}} \left(2L(G)(|V| - |E|) + \sqrt{2}L(G)\right)$$

$$+ \frac{1}{16} \sum_{v \neq v'} (2 - \deg(v))(2 - \deg(v')) + \sum_{v \in V} \left(\frac{3}{8} - \frac{\deg(v)}{4} + \frac{\deg(v)^{2}}{32}\right) + O(t^{\infty})$$

The first two terms of the asymptotic expansion are the volume and the lengths of the boundary (the boundary here consists of the terms from the product and the symmetrization). The constant term describes the corners and again has contributions from the product and the symmetrization.

A vertex of degree 2 in a metric graph with Kirchhoff boundary condition imposes no conditions on the functions. Hence this vertex should be invisible to the heat kernel and one can easily check that degree two vertices give no contribution to the asymptotics above. A vertex of degree one will produce a wedge with opening angle $\frac{\pi}{4}$ in $M$. This can be compared to the known heat asymptotics of planar polygons [vdBS88]. The constant term contribution matches the expected $\frac{5}{32}$.

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Appendix A. Measure theoretic background

A.1. Definition. A function \((t, x, U) \mapsto p_t(x, U)\) with \(t \in (0, \infty)\), \(x \in M\) and \(U \in \mathcal{B}(M)\) is called a \(\mu\)-symmetric Markovian transition function if it satisfies the following

(i) Measurability: for fixed \(t\) and \(x\), the function \(p_t(x, \cdot)\) is a positive measure and for fixed \(t\) and \(U\), the function \(p_t(\cdot, U)\) is \(\mathcal{B}(M)\)-measurable.

(ii) Semigroup property: \(p_{t+s}f = p_t p_s f\) for all \(f \in \mathcal{B}_b(M)\)

(iii) Markov property: \(p_t(x, M) = 1\)

(iv) \(\mu\)-symmetry: \(\int_M f(x)p_t g(x)\, \mu(dx) = \int_M (p_t f)(x)g(x)\, d\mu(x)\) for any \(f, g\) non-negative measurable functions

(v) Unit mass: \(\lim_{t \to 0} p_t f(x) = f(x)\)

Here \((p_t f)\) is defined as \((p_t f)(x) := \int_M f(y)p_t(x, dy)\) where \(p_t(x, dy)\) means we are integrating with respect to the measure \(p_t(x, \cdot)\).

The following proposition says that heat kernels and transition functions are equivalent in our setting. The absolute continuity condition is satisfied by Remark 5.16.

A.2. Proposition. For \(x\) and \(t > 0\) fixed, the probability measure \(p_t(x, U)\) admits a probability density function \(p_t(x, y)\) such that

\[ p_t(x, U) = \int_U p_t(x, y)\, d\mu(y) \]

for all measurable \(U \in \mathcal{B}(M)\) if and only if the measure \(p_t(x, \cdot)\) is absolutely continuous with respect to \(\mu(\cdot)\), that is \(p_t(x, U) = 0\) for any set \(U\) with \(\mu(U) = 0\).

Proof. As \(M\) is a compact metric space, \(\mu\) is \(\sigma\)-finite. Hence, this is the exact statement of the Radon-Nikodým theorem, the density function is the Radon-Nikodým derivative. \(\square\)

A.3. Definition. Let \((\Omega, \mathcal{F}, \mathbb{P}_{\Omega})\) be a probability space \((\Omega\) a set, \(\mathcal{F}\) a \(\sigma\)-algebra and \(\mathbb{P}_{\Omega}\) a probability measure on \(\mathcal{F}\)). Let \((M, \mathcal{B}, \mu)\) be a measure space and let \(I\) be a subset of \([0, \infty)\). A family \(\{X_t\}_{t \in I}\) of measurable maps \(X_t: (\Omega, \mathcal{F}) \to (M, \mathcal{B})\) is called a stochastic process on \(\Omega\) with values in \(M\) and index set \(I\).

A.4. Definition. For a stochastic process \(X_t\), let

\[ \mathcal{F}^0_t := \sigma(X_s, s \in [0, t]) \],
Localities of the heat kernel

where $\sigma(\cdot)$ denotes the smallest $\sigma$-algebra containing all sets in the brackets and we use the convention that $X_s$ understood as a $\sigma$-algebra means the $\sigma$-algebra $X_{s^{-1}}(\mathcal{B})$.

Note that $\mathcal{F}_s^0 \subseteq \mathcal{F}_t^0 \subseteq \mathcal{F}$ for any $s < t$ by definition. Hence this is an increasing sequence of $\sigma$-algebras.

A.5. Definition ([SV79]). Let $\mathcal{G} \subset \mathcal{F}$ by a sub-$\sigma$-algebra, then the conditional probability distribution $Q_\omega$ of $\mathbb{P}_\Omega$ given $\mathcal{G}$ is a family of probability measures on $(\Omega, \mathcal{F})$ indexed by $\omega \in \Omega$ such that

- for each $B \in \mathcal{F}$, the function $Q_\omega(B)$ is $\mathcal{G}$-measurable as a function of $\omega$
- for $A \in \mathcal{G}$ and $B \in \mathcal{F}$ we have

$$\mathbb{P}_\Omega(A \cap B) = \int_A Q_\omega(B) \, d\mathbb{P}_\Omega(\omega).$$

A conditional probability distribution is unique in the sense that two choices agree $\mathbb{P}_\Omega$-almost surely. If $\Omega$ is a Polish space, a conditional probability distribution always exists.

A.6. Remark. If $\mathcal{G} = \mathcal{F}$, then $Q_\omega(B) = \chi_B(\omega)$ that is $Q_\omega(B)$ is the indicator function of the set $B$. When $\mathcal{G}$ is a proper sub-$\sigma$-algebra, one can picture the $Q_\omega$ as $\mathcal{G}$-measurable approximations of the indicator function.

A.7. Remark. One writes $\mathbb{P}_\Omega(X_t \in U|\mathcal{F}_0^0)$ with $U \in \mathcal{B}$ as a shorthand for the map from $\Omega$ to $\mathbb{R}$ that maps $\omega \mapsto Q_\omega(X_t \in U)$ where $Q_\omega$ is the conditional probability distribution of $\mathbb{P}_\Omega$ given $\mathcal{F}_0^0$.

A.8. Definition. Let $(M, d, \mu)$ be a metric measure space with Borel $\sigma$-algebra $\mathcal{B}(M)$. A continuous normal Markov process on the set of continuous paths consists of four elements

(i) $\Omega := \mathcal{P}(M)$ is the set of continuous paths $\omega : [0, \infty) \to M$;
(ii) $\mathcal{B}(\mathcal{P}(M))$ is the $\sigma$-algebra of Borel sets on it;
(iii) $\{\mathbb{P}_x\}_{x \in M}$ are probability measures on $\mathcal{P}(M)$ that satisfy $\mathbb{P}_x(\omega(0) = x) = 1$;
(iv) $X_t(\omega) := \omega(t)$ is a stochastic process on $\mathcal{P}(M)$ with values in $M$

satisfying the Markov property

$$\mathbb{P}_x(X_{t+s} \in U|\mathcal{F}_t^0) = \mathbb{P}_{X_t}(X_s \in U)$$

$\mathbb{P}_x$-almost surely for all $U \in \mathcal{B}(M)$ and all $s, t \geq 0$.

By convention one often refers to $X_t$ as the Markov process. The existence of the other objects is then implicitly assumed. We will also write $\mathbb{P}$ to denote the family of Borel probability measures $\mathbb{P}_x$. We will refer to the measures $\mathbb{P}_x$ as Wiener measures.

We have

$$\mathbb{E}_x[f(X_t)] = \int_{\mathcal{P}(M)} f(\omega(t)) \, d\mathbb{P}_x(\omega) = \int_M f(y)p_t(x, y) \, d\mu(y)$$

for a continuous function $f$ on $M$.

A.9. Definition. A stochastic process satisfies the strong Markov property if

$$\mathbb{P}_x(X_{\zeta+s} \in U|\mathcal{F}_\zeta) = \mathbb{P}_{X_\zeta}(X_s \in U)$$

holds for all $s \geq 0$, $U \in \mathcal{B}(M)$ and all stopping times $\zeta$ (see Definition 3.6). In this case the Markov process is called a continuous Hunt process or a diffusion.

Note that the strong Markov property also implies

$$\mathbb{E}^{\mathcal{F}_\zeta}[f(X_{s+\zeta})] = \mathbb{E}^{\zeta}[f(X_s)].$$