Supersymmetric manipulation of quasienergy states. Application to the Berry phase

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Time-dependent supersymmetry allows one to delete quasienergy levels for time-periodic Hamiltonians and to create new ones. We illustrate this by examining an exactly solvable model related to the simple harmonic oscillator with a time-varying frequency. For an interesting nonharmonic example we present the change of the Berry phase due to a supersymmetry transformation.

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One may say without exaggeration that non-relativistic quantum mechanics is essential to understanding the basic laws of nature. Thus, the Schrödinger equation is one of the most fundamental constructs of modern natural science and over the past century, as evidenced by a vast literature, great progress has been made in developing various consequences of the time-independent Schrödinger equation describing stationary processes. Key recent advances concern the manipulation of bound states \[\text{[1]}.\] In the real world, however, nearly everything changes with time, and stationary processes are exceptional (perhaps even impossible), but there is ample evidence for believing that there will soon be similar advances in the study of non-stationary processes and the consequences of the time-dependent Schrödinger equation. The end of the last century saw some progress in this area: an extensive investigation into the properties of time periodic Hamiltonians, their dynamical invariants \[\text{[2]}\] and symmetries \[\text{[3]}\], dynamical pulsed coherent states \[\text{[4]}\], Floquet quanta \[\text{[5]}\] and application in optics \[\text{[6]}\]. An area that is now attracting interest is calculating the Berry, or geometric, phase for time dependent Hamiltonians, although so far the only system treated in detail is the, exactly solvable, harmonic oscillator with time dependent mass and/or frequency \[\text{[7]}\].

It has long been known how to construct new exactly solvable time-independent Hamiltonians from an initial one by means of the Darboux, or supersymmetry, transformation (see e.g. \[\text{[8]}\]), and the procedure has recently been generalized to include time dependence \[\text{[9], [10]}\]. Although stationary state energies do not exist for a time-dependent Hamiltonian, if the time dependence is periodic, quasienergies play a similar role \[\text{[11], [12]}\]. In this letter, based on the simple harmonic oscillator model, having a time-periodic frequency, we show that this analogy is much deeper than previously recognized and that it can be extended to the level of supersymmetry. In particular, we show that processes such as the creation and annihilation of quasienergies may be effected easily by time-dependent supersymmetry (or Darboux) transformations. This also opens the door for performing more complex operations on quasienergy states by applying integral transformations which are essentially inverses to the differential supersymmetry transformations \[\text{[9], [11, 12]}\].

Since exact eigenstates are known for these new time-dependent Hamiltonians their remarkable properties, such as the the Berry phase, can be investigated in detail. We demonstrate here that the Berry phase, for systems generated from our model harmonic oscillator Hamiltonian, simply acquires an additive correction under a Darboux transformation.

We begin with the Schrödinger equation for the Hamiltonian

\[ h_0 = -\partial_x^2 + \omega^2(t)x^2, \quad \omega(t + T) = \omega(t) \]  

(1)

where \[\partial_x = \partial/\partial_x\] and appropriate units have been chosen. A complete set of solutions normalized to unity on the real line is \[\text{[13], [14]}\].

\[ \psi_n(x,t) = (\pi/\varepsilon)^{n/2+1/4} \exp(i\gamma x^2) f_n(x,t) \]  

(2)

where

\[ f_n(x,t) = N_n \gamma^{-1/4} H_n(z) e^{-z^2/2}, \quad z(x,t) = x/\sqrt{8\gamma(t)}, \]  

(3)

Here \[N_n = (2^{n+1}n!\sqrt{2\pi})^{-1/2}, \quad \gamma = \varepsilon \varphi, \quad H_n(z) \] is a Hermite polynomial and \[\varepsilon(t), \varphi(t)\] are two linearly independent solutions of the classical equation of motion for the harmonic oscillator

\[ \ddot{\varepsilon}(t) + 4\omega^2(t)\varepsilon(t) = 0. \]  

(4)

A dot over a symbol represents the derivative with respect to time and we choose these solutions so that their Wronskian \[\varepsilon(t)\dot{\varepsilon}(t) - \varepsilon(t)\ddot{\varepsilon}(t) = 1\]. In general, the solutions of \[\text{[15]}\] are of three types: (i) both \[\varepsilon\] and \[\varphi\] are stable, (ii) they are both unstable, (iii) one is stable the other is unstable. It is known \[\text{[12]}\] that only in the first case the functions \[\text{[16]}\] have the property

\[ \psi_n(x,t + T) = e^{-iE_nT} \psi_n(x,t), \]  

(5)
where $E_n = (n + \frac{1}{2})\delta$ is called the quasienergy \[1\] and $\delta$ is defined by the equation

$$\varepsilon(t + T) = \varepsilon(t)e^{i\delta T}. \quad (6)$$

According to Floquet’s theorem a (complex) solution $\varepsilon(t)$ to equation \[2\] satisfying \[3\] can always be found in case (i). We choose the complex conjugate of $\varepsilon$ to be the second, linearly independent, solution to \[4\]. Here and below a bar over a quantity denotes its complex conjugate and we note that $\gamma(t)$ is real. It is clear from \[11\] that quasienergies $E_n$ are defined modulo $2\pi$ if they are commensurable with $\frac{2\pi}{T}$, the functions $\psi_n$ are periodic.

A further useful property of the functions \[2\] is that there exist ladder operators

$$a = \varepsilon \partial_x - i\varepsilon x/2, \quad a^+ = -\bar{\varepsilon} \partial_x + i\bar{\varepsilon} x/2 \quad (7)$$

such that

$$a\psi_n = \sqrt{n} \psi_{n-1}, \quad a^+\psi_n = \sqrt{n+1} \psi_{n+1}. \quad (8)$$

These functions are eigenfunctions of the symmetry operator $g = \frac{(aa^++a^+a)}{2}$, $g\psi_n = g_n\psi_n$ with $g_n = \frac{1}{2}(2n+1)$.

Next we shall apply the approach presented in \[11\] for constructing new exactly solvable time-dependent Hamiltonians having the same quasienergy spectrum, with the possible exception of a few levels, as a given one. The procedure is simply the time-dependent generalization of the usual supersymmetry construction \[8\]. A new hamiltonian $h_1$ is defined by means of a nodeless solution $u(x,t)$, called the transformation function, to the initial Schrödinger equation, $i\partial_t u(x,t) = h_0 u(x,t)$, subject to the additional condition $\partial_x^2 (\ln u/\gamma) = 0$, which guarantees that $h_1$ has the form

$$h_1 = h_0 - \partial_x^2 \ln |u(x,t)|^2. \quad (9)$$

Solutions of the corresponding Schrödinger equation are obtained by applying the differential operator

$$L = L_1(t)[\partial_x + \partial \ln u(x,t)/\partial x] \quad (10)$$

to $\psi_n$:

$$\varphi_n(x,t) = M_n^{-1/2}L\psi_n(x,t) \quad (11)$$

where the time-independent factor $M_n^{-1/2}$ guarantees that the functions \[11\] are normalized to unity. The operator $L$ is defined in terms of the same function $u(x,t)$ as in (9) and

$$L_1(t) = \exp \left[2 \int \text{Im} (\partial_x^2 \ln u) \, dt \right]. \quad (12)$$

The operator $L^+ = L_1(t)[\partial_x + \partial \ln \bar{u}(x,t)/\partial x]$, adjoint to $L$, realizes the transformation in the opposite direction, from solutions of the Schrödinger equation with Hamiltonian $h_1$ to those of the Hamiltonian $h_0$. Thus, the superposition $L^+L$ is a symmetry operator for the initial Schrödinger equation and the function $u(x,t)$ is an eigenfunction of this operator: $L^+Lu(x,t) = g_u u(x,t)$. The normalization factor in \[11\] is equal to its mean value $M_n = \langle \psi_n | L^+L | \psi_n \rangle$.

Among the functions \[2\] only $\psi_0$ is nodeless and suitable for using as a transformation function; it produces only a shift of the Hamiltonian by an $x$-independent value. This is simply a manifestation of the well-known shape-invariance property \[8\] in the time-dependent case. Any other function $u_k(x,t) = \psi_k(x,t)$, $k > 0$ taken as the transformation function will produce a potential with $k$ poles corresponding to the zeros of $\psi_k(x,t)$ which clearly has no physical meaning if the variable $x$ runs over the whole real line. The transformed Hamiltonian $h_1$ can be taken as the initial one for the next transformation step and if this is realized with the transformation function $\bar{u}_{k+1}(x,t) = Lu_{k+1}(x,t)$ all the poles are removed and the resulting Hamiltonian $h_2(k) = h_0 - A_2(k)(x,t)$ is physically admissible. For the potential difference one gets

$$A_2(k)(x,t) = \frac{1}{4\gamma} \left[ \frac{J'_k(\zeta)}{J_k(\zeta)} - \left( \frac{J'_j(\zeta)}{J_j(\zeta)} \right)^2 - 2 \right]. \quad (13)$$

Here

$$J_k(z) = \sum_{j=0}^{k} \frac{k!}{j!(k-j)!} H_j^2(z)$$

and for the first three of these functions one has the simple expressions:

$$J_0(z) = 1, \quad J_1(z) = 2z + 1, \quad J_2(z) = 4z^2 + 3.$$
functions. They produce the new Hamiltonians $h^{(2l)}_1 = -\partial_x^2 + V^{(2l)}_1(x,t)$, $l = 0, 1, \ldots$ with potentials

$$V^{(2l)}_1(x,t) = \omega^2(t)x^2 - A^{(2l)}_1(x,t), \quad L_1(t) = \sqrt{\gamma(t)},$$

where

$$A^{(2l)}_1 = \frac{1}{4\gamma} \left[ 1 + 4l(2l-1) \frac{q_{2l-2}(z)}{q_{2l}(z)} - 8l^2 \left( \frac{q_{2l-1}(z)}{q_{2l}(z)} \right)^2 \right],$$

and $q_k(z) = (-i)^k 2^{-k/2} H_k(iz)$. Using the recursion relation for Hermite polynomials one finds

$$q_0(z) = 1, \quad q_1(z) = \sqrt{2}z, \quad q_{k+1}(z) = \sqrt{2}z q_k(z) + k q_{k-1}(z).$$

The fact that a new quasienergy level is created by this process follows from the property that the function $v = 1/(L_1 \psi)$ is a square integrable solution of the transformed Schrödinger equation. It is easy to see that it corresponds to the quasienergy $E = -\delta(k + \frac{1}{2})$, which, in general, is different from all the other quasienergies $E_n = \delta(n + \frac{1}{2})$, $n = 0, 1, \ldots$. The case $k = 0$ reproduces the harmonic oscillator Hamiltonian shifted by an $x$-independent quantity. The first nontrivial case corresponds to $k = 2$. We display a typical potential, $V^{(2)}_1(x,t)$, in Fig. 2.

After applying the operator (10) to the functions (2) one gets solutions of the transformed equation corresponding to the quasienergies $E_n$. For the case $k = 2$ they can easily be expressed in terms of $\psi_n(x,t)$:

$$\varphi_n(x,t) = \frac{1}{\sqrt{\gamma}} \left[ \sqrt{n+1} \mathcal{I}^{1/2} \psi_{n+1}(x,t) \right]$$

The normalization factor here is calculated by noting that the symmetry operator $L^+L$ is simply the shifted $g$ operator, $L^+L = g + 5/8$.

We now turn to the transformation of the Berry phase and consider for simplicity the first nontrivial case $k = 2$. In these calculations we are using the standard approach (see e.g. [15]). The phase change of a solution of the Schrödinger equation $\psi(x,t)$ during time $T$ is given in terms of the mean energy value

$$\chi = -\int_0^T \langle \psi | i \dot{\psi} \rangle dt$$

and the Berry phase $\beta$ is obtained from this overall phase change, to be extracted from (2) for the harmonic oscillator potential and from (17) for transformed potentials, by subtracting the dynamical component.

For the functions in (2) one gets

$$\langle \psi_n | i \dot{\psi}_n \rangle = \frac{i}{4}(2n + 1) \frac{d}{dt} \ln(\gamma) - \frac{1}{8} \langle \psi_n | x^2 | \psi_n \rangle \frac{d^2}{dt^2} \ln \gamma$$

where the mean value of the square coordinate is

$$\langle \psi_n | x^2 | \psi_n \rangle = 4\gamma(2n + 1).$$

The structure of the functions $\varphi_n(x,t)$ is similar to those in (2) and therefore the average energy is given by (19) with the replacement $\psi_n \rightarrow \varphi_n$. To calculate the mean value of the square coordinate for the functions (17) we use the property $L^+L = g + 5/8$ and the fact that the functions (2) are eigenfunctions for $g$ to get

$$\langle \varphi_n | x^2 | \varphi_n \rangle = \langle \psi_n | x^2 | \psi_n \rangle + 4\frac{\gamma}{\sqrt{n+3}} \langle \psi_n | x | \varphi_n \rangle.$$}

The first integral here is standard and the second has the expression

$$\langle \psi_n | x | \varphi_n \rangle = \sqrt{\gamma n + 3 - I_n(\frac{1}{2})},$$

where

$$I_n(a) = \int_{-\infty}^{\infty} \frac{H_n^2(x)}{x^2 + a} e^{-x^2} dx$$

which obeys the recursion relation

$$I_n(a) = -2I_{n-1}(a) + (4n + 1)^2 I_{n-2}(a) - 4a I_{n-1}(a),$$

which follows directly from that for the Hermite polynomials. The initial values are $I_0(a) = \frac{2}{\sqrt{\pi} a} e^{-a^2}$ and $I_1(a) = 4\sqrt{\pi} - 4a I_0(a)$. Therefore, the mean energy in the states (17) is given by

$$\langle \varphi_n | i \dot{\varphi}_n \rangle = \langle \psi_n | i \dot{\psi}_n \rangle - \left( 1 - \frac{I_n(\frac{1}{2})}{n+3} \right) \gamma \frac{d^2}{dt^2} \ln \gamma.$$ (20)

After being integrated over a period $T$ the first term in (19) gives us the overall phase change both for the initial
states and for the transformed ones (see [20]); the Berry phase is determined only by the second term. Thus, the Berry phases \( \beta_n^\alpha \) for all states \( \psi_n(x, t) \) are determined by the Berry phase for the ground state: 
\[
\beta_n^0 = \frac{1}{2} \int_0^T \gamma \frac{d^2}{dt^2} \ln \gamma dt = -\frac{1}{2} \int_0^T \frac{\dot{\gamma}^2}{\gamma} dt.
\]
For the transformed Berry phase from [20] one gets \( \beta_n^1 = \beta_n^0 + 2(1 - \frac{k(t/2)}{g(t)})\beta_n^0 \). Hence, once the quantity \( \beta_n^0 \) is known, we can easily calculate the Berry phase both for the harmonic oscillator and for the Hamiltonian \( h_1^{(2)} \).

As a numerical illustration we have chosen a model for which analytic solutions of equation (4) are available: 
\[
\omega(t) = \sqrt{\omega_0^2 - \frac{1}{4} \sigma^2 (t + \omega_i)}.
\]
Here \( \omega_0 \) is a parameter of the model along with the real \( (\omega_r = T) \) and imaginary \( (\omega_i) \) periods of the Weierstrass \( \wp \) function. If we eliminate the parameter \( \omega_0 \) in favor of \( d \) given by \( \sigma(d) = -4\omega_i^2 \), the solutions of the equation (4) have the form [16]
\[
\varepsilon(t) = \frac{\sigma(t + \omega_i + d)}{\sigma(t + \omega_i)} e^{-\zeta(t)}.
\]
Here \( \sigma \) and \( \zeta \) are (non-elliptic) Weierstrass functions. Figures 1 and 2 are plotted with \( \omega_r = -i\omega_i = 2 \) and \( \omega_0 \equiv 0.5978 \). We have found also the value \( \beta_n^0 = -0.0149 \).

We make one further comment concerning the time-dependent supersymmetry underlying our approach. The transformation operators are related to eigenfunctions (not necessarily “physical”) of the symmetry operator 
\[
g = \frac{1}{2}(a^\dagger a + aa^\dagger), \quad g_{uk} = g_{uk}, \quad g_u = 5/8,
\]
which accounts for the factorization \( L^+L = g - g_u \). When the order of transformation operators is interchanged, one gets a symmetry operator for the transformed Schrödinger equation \( \tilde{g} \), \( LL^+ = \tilde{g} - g_u \). The operators \( g \) and \( \tilde{g} \) are supersymmetric partners from which a matrix operator \( G = \text{diag}(g, \tilde{g}) \) can be constructed. It acts in the space spanned by the basis vectors \( \Psi_n(1) = (\psi_n, 0)^t \) and \( \Psi_n(2) = (0, \varphi_n)^t \) where the superscript “\( t \)” stands for transposition and the functions \( \psi_n \) and \( \varphi_n \) are given in [20] and [17] respectively. So, just as in the usual supersymmetric approach, one has a two-fold degenerate spectrum except for the ground state level which is non-degenerate. This means that we have constructed here a model with unbroken supersymmetry. Finally we note that supercharge operators, closing superalgebra, can be constructed as usual with the help of the transformation operators \( L \) and \( L^+ \).

In summary, in this note we have constructed, in principle, an infinite number of time-periodic Hamiltonians having almost unlimited complexity, for which the Berry phase can be determined explicitly and have illustrated the procedure for a Hamiltonian whose time dependence is given by the square root of an elliptic function. This vastly extends the set of previously known cases, all based on the simple Harmonic oscillator. We feel that this opens the way for a systematic investigation of the Berry phase for one dimensional quantum systems.

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