Multiple-Scales Approach for Addressing The Averaging Problem in Cosmology

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The Universe is homogeneous and isotropic on large scales, so on those scales it is usually modelled as a Friedmann-Lemaître-Robertson-Walker (FLRW) space-time. The non-linearity of the Einstein field equations raises concern over averaging over small-scale deviations from homogeneity and isotropy, with possible implications on the applicability of the FLRW metric to the Universe, even on large scales. Here I present a technique, based on the multiple-scales method of singular perturbation theory, to handle the small-scale inhomogeneities consistently. I obtain a leading order effective Einstein equation for the large-scale space-time metric, which contains a back-reaction term. The derivation of this equation is done in harmonic gauge, and conversion to other gauges is discussed. I estimate the magnitude of the back-reaction term relative to the critical density of the Universe in an example, and find it to be of the order of a few percent.
I. INTRODUCTION

In Newtonian mechanics, Newton’s third law ensures, that the internal forces acting inside a material body are immaterial to understanding how it moves – *that* is determined only by the forces exerted on it by other objects. A collection of particles may be viewed as a single object when “zooming out”, and treated as a point particle on scales that are large enough (provided it does not break up). If there are no spatial boundaries (their existence impacts on integration by parts), there is no back-reaction in Newtonian gravity, as it is a linear theory [1].

Einstein’s equations are non-linear though, and therefore concern has risen that when one studies the large-scale structure of space-time, the homogeneity of matter on large scales does not imply that inhomogeneities on small scale do not influence the large-scale metric [2, 3]. Indeed, the non-linearity of the Einstein equations

\[ R_{ab} - \Lambda g_{ab} = 8\pi G \rho_{ab}, \]

(1)

where

\[ \rho_{ab} = T_{ab} - \frac{1}{2} T g_{ab}, \]

(2)

(\( T^{ab} \) is the energy-momentum tensor) implies that averaging over spatial scales cannot be done easily – it does not commute with metric inversion, the connection, *et cetera*. Considered as an initial value problem, the evolution of the averaged spatial metric due to the exact equations is not, in general, the same as the evolution of the metric generated by the averaged equation; this affects the extent to which the Friedmann-Lemaître-Robertson-Walker (FLRW) solution is valid as a description of the Universe on large scales [4]. The difference arises from small-scale inhomogeneities, that react back on large scales through the non-linearity of the Einstein equations. This problem has been studied a lot in recent years (see e.g. [4–15]), but the magnitude of this so-called ‘back-reaction’ and of its influence on the large-scale gravitational dynamics of the Universe is still subject to some debate [16–18]. Some numerical relativistic simulations were conducted to investigate the averaging problem [13, 19, 20], leading to the conclusion that the over-all effect is probably small, and depends on the space-time slicing.

It is clear that standard cosmological perturbation theory does not suffice to handle the averaging problem [21], and a novel technique is needed – be it an averaging technique [4, 9],
or a special asymptotic expansion \cite{10,14,15}. Here, I wish to propose such an approach, which utilises the multiple-scales method of singular perturbation theory. The multiple-scales method \cite{22–25} has wide-ranging applications throughout physics; for instance, the Chapman-Enskog expansion, used in deriving the Navier-Stokes equations from the Boltzmann equation relies on it (see, e.g., \cite{26}), as well as any homogenisation technique used to study diffusion or transport processes in inhomogeneous media \cite{25}. It differs from other frameworks (e.g. \cite{10}) by treating the small scale differently from the large scale.

The paper is structured as follows: I start by putting forward my assumptions on the energy-momentum tensor in \S II; then, in \S III, I present the multiple-scales expansion of equation (1), assuming that the over-densities are no more than of the same order as the background density, so as to allow the reader to focus on the perturbative expansion, and in \S IV I remove this restriction, which allows me to show which terms in the averaged equations yield a possible back-reaction effect. I finish with a discussion in \S V. I always neglect, however, the presence of highly-relativistic objects, such as black holes or neutron stars.

I use harmonic gauge throughout the paper, and, of course, some of the results may be gauge-dependent. In harmonic co-ordinates equations (1) are quasi-linear hyperbolic equations, with the Ricci tensor given by \cite{27}

\[ R_{ab} \equiv R_{ab}^{(h)} = -\frac{1}{2} g^{cd} \partial^2_{cd} g_{ab} + P_{ab}^{cdfg}(g) \partial_c g_{ef} \partial_d g_{gh}, \]  

(3)

where

\[ P_{ab}^{cdfg}(g) \partial_c g_{ef} \partial_d g_{gh} = -\frac{1}{2} \left( \partial_b g^{cd} \partial_d g_{ad} + \partial_a g^{cd} \partial_d g_{bd} \right) - \Gamma^c_{ad} \Gamma^d_{bc}. \]  

(4)

When it does not cause confusion, I shorten \( P_{ab}^{cdfg}(g) \partial_c g_{ef} \partial_d g_{gh} \) to \( P_{ab}(g) \partial g \partial g \). I use the following order notation: \( g(\varepsilon) = O(f(\varepsilon)) \) if \( \lim_{\varepsilon \to 0} |g(\varepsilon)/f(\varepsilon)| \) is bounded, \( g(\varepsilon) = o(f(\varepsilon)) \) if \( g(\varepsilon)/f(\varepsilon) \to 0 \) as \( \varepsilon \to 0 \), and \( g(\varepsilon) = \text{ord } (f(\varepsilon)) \) if \( g(\varepsilon)/f(\varepsilon) \to \text{ const } \neq 0 \) as \( \varepsilon \to 0 \).

**II. ASSUMPTIONS CONCERNING THE ENERGY-MOMENTUM TENSOR**

The fundamental theoretical assumption that is made in this paper, is that matter in the Universe has a distribution that looks differently on different scales. These are comprised of stellar scales, galaxy-size scales, large-scale-structure (LSS) scales, and the largest, cosmological, scales \cite{2}. This implies that the energy-momentum tensor \( \rho_{ab} \) describing the
matter distribution in the exact solution of equation (1) must naturally depend on all of those scales. If one measures distances in cosmological scales, in units of 100 Mpc, then all the other scales, $\varepsilon_{\text{stars}} \ll \varepsilon_{\text{gal}} \ll \varepsilon_{\text{LSS}}$, are much smaller than unity; they are also well-separated. Indeed, $\rho_{ab}$, as a function of position, must depend on it in all of these scales; its most general form is

$$\rho_{ab} = \rho_{ab}(x, \frac{x}{\varepsilon_{\text{stars}}}, \frac{x}{\varepsilon_{\text{gal}}}, \frac{x}{\varepsilon_{\text{LSS}}}).$$  \tag{5}$$

One can always separate the dependence on $x$ into different scales in this way, at least formally: in Fourier space, one simply groups all modes whose wavelengths are smaller than $2\pi \varepsilon_{\text{stars}} \times 100$ Mpc into one group, then all the modes with wavelengths between $2\pi \varepsilon_{\text{stars}} \times 100$ Mpc and $2\pi \varepsilon_{\text{gal}} \times 100$ Mpc into another, et cetera, while all the long modes are grouped into the cosmological-scales dependence of $\rho_{ab}$.

In a wider context, a separation of scales in the matter distribution is common in the theory of large-scale structure, in the context of a peak-background split (see, e.g., [28]), where it is used to describe the clustering of over-dense regions. It is also used in the study of gravitational waves, as a basis for a WKBJ expansion, aimed at deriving the geometric optical description of progressive gravitational waves (e.g. [29, 30]).

For simplicity, however, I will focus on matter on just two scales: galactic scales and cosmological scales. This will allow me to keep the introduction of the application of the multiple-scales method in this paper – its main aim – as simple as possible, while retaining the essential physics; thus, set $\varepsilon = \varepsilon_{\text{gal}}$, whence equation (5) therefore becomes

$$\rho_{ab} = \rho_{ab}(x, X),$$  \tag{6}$$

where $X = x/\varepsilon$. All modes with wavelengths smaller than $2\pi \varepsilon_{\text{gal}}$ fall into the galactic-scales dependence – on $X$ – while all other modes are understood to contribute to the $x$ dependence. In general, the metric would depend on $x$ in this way, too.

Equation (6) is not begging the question – writing $\rho_{ab}$ as a function of two variables, the large scale and the small scale, does not yield that there is no back-reaction. Rather, it is an observational statement on the cosmological principle: the matter distribution in the universe depends on many scales, and its the energy-momentum tensor should do, too. Indeed, there are space-times with and without sizeable back-reaction whose energy-momentum tensor satisfies equation (6) (e.g. [31–34]).
In what follows, I treat $\rho_{ab}$ as a known function of $(x, X)$, a source for the Einstein equations, rather than as a variable which both determines the gravitational field and is determined by it. This is merely a conceptual simplification that allows me to derive a leading order (in $\varepsilon$) equation for the metric. Treating $\rho_{ab}$ as a source term in the Einstein equations means that the formalism developed here may be used to study the averaging problem as in a ‘post-processing’ manner; that is, given an exact solution of the Einstein equations coupled to matter fields, one may ask if there is any back-reaction of small-scale variations in the matter distribution on the large-scale behaviour of the metric. I emphasise, though, that this is a conceptual change, rather than a restriction, because one may use it to determine if there is back-reaction in any specified solution.

Let me emphasise that $\rho_{ab}$ is only due to matter. However, when one makes cosmological observations to measure $\rho_{ab}$, the result is always dependent on the underlying metric. Observationally, it is impossible to disentangle the effect of the metric from a measurement of the matter content of the universe, without some other assumption, which might constitute some form of back-reaction; this is true, at least, for the large-scale behaviour of $\rho_{ab}$. The inferred matter density may be, for example, in part due to back-reaction of curvature terms, and have nothing to do with the number of particles in the Universe. However, if it were found, somehow, that the back-reaction term in the “averaged” Einstein equations are small, given the true $\rho_{ab}$ (which is only due to matter), then it would follow a posteriori that a measurement attempting to discover $\rho_{ab}$ would yield a tensor that is close to it.

If $x$ is measured in units where space starts to be homogeneous, 100 Mpc, and as $\varepsilon$ describes galactic scales of 1 kpc, then $\varepsilon = 10^{-5}$ (and then $X$ also has units of 100 Mpc). Any ord (1) change in the value of $\varepsilon$ does not affect the results below significantly. The way to define $x$ and $X$ is therefore observational: one determines what the typical size of a galaxy is, and then one splits, mathematically, the functional dependence of the metric and the energy-momentum tensor into a dependence on $x$ on larger scales, and a dependence on the same co-ordinate on smaller scales. Then one calls the latter $X$.

Such a splitting into a small scale and a large one is part of the mathematical model that is used in this paper to address the averaging problem; form the point of view of the model, it is an assumption. From a physical point of view, on the other hand, the justification of this assumption rests on the existence of observational evidence for a scale-separation in the matter distribution in the Universe.
In the next section I start by ignoring over-densities in $\rho_{ab}$ that are larger than $\text{ord}(1)$, as it is simpler to explain the method in this case; in §IV I include over-densities due to galaxies. In doing so I assume that these small-scale over-densities in $\rho_{ab}$ are known to be only due to matter, and are observable directly. I still do not assume anything about the large-scale structure of the $O(1)$ component of $\rho_{ab}$ relative to the one inferred from observations, which is where back-reaction terms are important.

III. SMALL-SCALE VARIATIONS

I use the method of multiple scales (see, e.g., §6.3 of [23] or §4.4 of [24]), treating $x$ and $X$ as independent variables. This procedure is valid as long as one can identify, observationally, different behaviours of the matter distribution on different physical scales. This amounts to requiring that only the short modes (with wave-length no more than $2\pi \varepsilon \times 100 \text{ Mpc}$) have fluctuations of amplitude $\gg \text{ord}(1)$. The smaller $\varepsilon$ can be chosen such that it still satisfies this condition, the more accurate the asymptotic expansion derived in this section is. Besides, if the intermediate-scale modes of $\rho_{ab}$ have a small, $o(1)$, amplitude relative to both cosmological and galactic scales, this sharpens the distinction between $x$ and $X$, and thus improves the accuracy of the approximation.

The dimension of the manifold is then increased to 8, and the partial derivative becomes

$$\frac{\partial}{\partial x^a} \mapsto \frac{\partial}{\partial x^a} + \frac{1}{\varepsilon} \frac{\partial}{\partial X^a}. \quad (7)$$

Treating $X$ and $x$ independently is akin to taking a function $f(u, v)$, with $v = \alpha u$, and then identifying the total derivative $\frac{df}{du}$ with

$$\frac{\partial f}{\partial u} + \alpha \frac{\partial f}{\partial v}, \quad (8)$$

but going in the opposite direction. In this view, $x$ specifies the position of a galaxy, whereas $X$ describes motion inside the galaxy (see figure 1). From the point of view of the $x$-space-time galaxies are point particles, and $X$ zooms in to each individual galaxy. The effect of anything that happens inside a given galaxy on the large scale emerges consistently from the coupling between $x$ and $X$ in the Einstein equations.
FIG. 1. An illustration of the splitting of $x$ and $X$. The former describes the position of a galaxy in space-time, and the latter describes ‘zoomed-in’ motion inside it.

A. Geometrical Interpretation

A geometrical interpretation of this construction may be provided in terms of fibre bundles. Let $M$ be space-time, and let $\mathcal{B} = (M, F)$ be the bundle whose base is $M$, and whose fibres $F = \{F_x\}_{x \in M}$ are defined by the $X$-space-time. Thus, one inserts a new manifold, $F_x$, at every point $x \in M$. This might not be the trivial (product) bundle, as harmonic co-ordinates generally exist only locally. At present I do not specify the metric on $F$, but only require that it depend smoothly on $M$ (it will be shown that $F_x$ may be treated as a flat space-time in §III C to zeroth order, in the scenarios I consider here, but this is not strictly necessary). $F_x$ is a bounded manifold, with boundaries corresponding roughly to galaxy sizes. Local triviality follows from the equivalence principle.

The tensors I consider here are those appearing in equation (1), but they are (at present) defined only on the tangent space of $M$. However, if $T\mathcal{B}$ is the tangent space of $\mathcal{B}$, then it is locally spanned by

$$\left\{ \frac{\partial}{\partial x^a}, \frac{\partial}{\partial X^b} \right\}_{a,b=0,\ldots,3}.$$

So, any tensor field $W^{ab}(x, x/\varepsilon)$ on $M$ may be identified with a tensor field of the same rank...
on $\mathcal{B}$, via
\begin{align}
W &= W^{ab}(x, \frac{x}{\varepsilon}) \frac{\partial}{\partial x^a} \otimes \frac{\partial}{\partial x^b} \\
&\approx W^{ab}(x, X) \frac{\partial}{\partial x^a} \otimes \frac{\partial}{\partial x^b} + 0^{ab} \frac{\partial}{\partial X^a} \otimes \frac{\partial}{\partial X^b} + 0^{ab} \frac{\partial}{\partial x^a} \otimes \frac{\partial}{\partial X^b} + 0^{ab} \frac{\partial}{\partial X^b} \otimes \frac{\partial}{\partial x^b},
\end{align}

where $0^{ab}$ vanishes for each $a, b = 0, \ldots, 3$. This means that $W$ is viewed as a tensor field on $\mathcal{B}$, whose components depend smoothly both on $M$ and on $F$, but which really lies in the vector subspace of $T^{\otimes 2} \mathcal{B}$ corresponding to $T^{\otimes 2} M$. Contrary to equation (7), the basis vectors do not split as the partial derivatives there – that equation is relevant only for computing the components of $W^{ab}$.

B. Asymptotic Expansion

I take the Universe to be well-described by standard cosmological perturbation theory up to recombination, which I set as the initial data for the multiple-scales calculation. The initial potential fluctuation (at recombination, i.e. deep in matter-domination) has a power spectrum
\begin{equation}
P(k) = \frac{18\pi^2}{25} A_s \frac{k^{n_s-4}}{k_p^{n_s-1}} D^2(a_{rec}) T^2(k),
\end{equation}

where $D(a)$ is the linear growth factor, $n_s \approx 1$, $A_s = 2.1 \times 10^{-9}$ [35], the pivot scale is $k_p = 0.05$ Mpc$^{-1}$, and the transfer function is $T(k) \approx \frac{12k_p^2}{k^2} \frac{k^2}{k_p^2} \ln \left( \frac{k}{k_p} \right)$ [36], where $k_{eq} \approx 0.01$ Mpc$^{-1}$ [35]. (This power spectrum is strictly valid for small scales only, which are precisely the scales I need it for.) This shape of the power-spectrum implies that the mean squared amplitude of a small-scale metric perturbation is
\begin{equation}
\sigma^2 \propto \int_{k_0/\varepsilon}^{\infty} dk \frac{k^{n_s+2} \ln^2(k)}{k^8} \approx \int_{k_0/\varepsilon}^{\infty} dk \frac{\ln^2(k)}{k^5} = \frac{8 \ln^2(k_0/\varepsilon) + 4 \ln(k_0/\varepsilon) + 1}{32(k_0/\varepsilon)^4},
\end{equation}

where $k_0$ is some order-unity wave-vector whose exact value is immaterial. For $\varepsilon = 10^{-5}$, $\ln^2 \varepsilon = \text{ord } (1)$, whence $\sigma^2 \varepsilon = \text{ord } (\varepsilon^4)$, and the initial small scale (i.e. large $k$) metric perturbations have a root-mean-square amplitude proportional to $\varepsilon^2$. Thus, it is reasonable to expand in integer powers of $\varepsilon$ in the asymptotic expansion of the metric and the energy-
momentum tensor. Explicitly,

\[ g_{ab}(x, X) \sim g^0_{ab}(x, X) + \varepsilon g^1_{ab}(x, X) + \varepsilon^2 g^2_{ab}(x, X) + \text{h.o.t.} \]  \hspace{1cm} (14)

\[ g^{ab}(x, X) \sim g^0_{0}^{ab}(x, X) + \varepsilon g^1_{1}^{ab}(x, X) + \varepsilon^2 g^2_{2}^{ab}(x, X) + \text{h.o.t.} \]  \hspace{1cm} (15)

\[ \rho_{ab}(x, X) \sim \rho^0_{ab}(x, X) + \text{h.o.t.} \]  \hspace{1cm} (16)

For consistency, the various terms in this expansion would have to remain bounded, so that the hierarchy of orders is preserved throughout the system’s evolution. I do not use any higher order terms in the expansion of \( \rho_{ab} \) in this paper, so in fact, it is possible to expand it in different powers of \( \varepsilon \) from those in the series expansion of \( g_{ab} \). Note, that the asymptotic series for \( g^{ab} \) is determined in terms of that of \( g_{ab} \) completely, to ensure that \( g^{ab} \) is indeed the inverse of \( g_{ab} \), at each order; for example \( g^{1}_{ab} = -g^{ac}_{0} g^{bd}_{0} g^{1}_{cd} \). The reader should bear in mind that so far \( g^0_{ab} \) is a completely general tensor-valued function of both \( x \) and \( X \), and may differ from an FLRW metric considerably.

In this expansion, the second-order differential operator in equation (3) becomes

\[
- \frac{1}{2\varepsilon^2} \left[ \left( g^{cd}_{0}(x, X) + \varepsilon g^{cd}_{1}(x, X) + \varepsilon^2 g^{cd}_{2}(x, X) \right) \times \\
\left( \varepsilon^2 \partial^2_{x^c x^d} + 2\varepsilon \partial^2_{X^c X^d} + \partial^2_{X^c X^d} \right) \times \\
\left( g^0_{ab}(x, X) + \varepsilon g^1_{ab}(x, X) + \varepsilon^2 g^2_{ab}(x, X) \right) \right] + \ldots,
\]  \hspace{1cm} (17)

while the first-order differential term reads

\[
\frac{1}{\varepsilon} \left\{ P_{ab}(g_0 + \varepsilon g_1 + \varepsilon^2 g_2) \times \\
(\varepsilon \partial_x + \partial_X) \left( g_0 + \varepsilon g_1 + \varepsilon^2 g_2 \right) \times \\
(\varepsilon \partial_x + \partial_X) \left( g_0 + \varepsilon g_1 + \varepsilon^2 g_2 \right) \right\} + \ldots.
\]  \hspace{1cm} (18)

Let me remind the reader that in this section, I assume that \( 8\pi G \rho_{ab} \) is \( O(1) \), at most (in \( \text{§IV} \) I relax this assumption), to be able to describe the method more easily, without the complications arising from a large energy-momentum tensor. To progress, I multiply

\footnote{The proportionality coefficient \( \sqrt{\frac{2592\pi^2}{25}} A_s k_p^{3-n_s} D^2(a_{rec}) \) is about \( 10^{-3} \), which might lead one to add an additional power of \( \sqrt{\varepsilon} \) to \( \sigma_\varepsilon \), making it ord \( (\varepsilon^{5/2}) \). Doing this does not make any difference to what follows, so, for the sake of generality, I still include \( \sigma_\varepsilon \) in the ord \( (\varepsilon^2) \) equations below, as a worst-case possibility. This also simplifies the expansion, relieving one of the need to expand in powers of \( \sqrt{\varepsilon} \). Naturally, different initial power spectra might, in general, require different expansions in \( \varepsilon \).}
the Einstein equations by $\varepsilon^2$, whence the $\Lambda g_{ab}$ and $8\pi G \rho_{ab}$ terms contribute only at second order.\(^2\) The zeroth order equation is

$$-\frac{1}{2} g^c_d \partial^2_{X^c X^d} g_{ab}^0 + P_{ab}(g_0) \partial_X g_0 \partial_X g_0 = 0. \quad (19)$$

This is a vacuum Einstein equation with no cosmological constant, in the $X$ part of the manifold. Now, if the initial conditions are such that there are no order unity small-scale contributions to the metric, $g_0(\cdot, X)$ satisfies a vacuum Einstein equation, with constant initial conditions (i.e. flat space), whence by uniqueness, $g_0(\cdot, X)$ is independent of $X$. This is not a *petitio principii*, for the only initial conditions used are at (say) recombination. Then, there are no zeroth-order small-scale perturbations to the metric, whence, at any $x$ such that $t = t_{\text{rec}}$, $\partial_X g_0(x, X) = 0$, even when $X^0$ reaches its maximum value. Hence, at a slightly later time $t = t_{\text{rec}} + \delta t$, $\partial_X g_0 = 0$ (in effect, one has a matching condition to ensure that the small scale behaviour does correspond to $X = x/\varepsilon$). At this new (slow) time, I also solve equation (19), giving the same result. The final consequence of this analysis is, that if there are $X$-independent initial conditions for a function $f(x, X)$ at recombination, and if the differential equation satisfied by $f$ (differential with respect to $X$ – the large-scale coordinate $x$ is treated as a parameter) is such that $\partial_X f$ remains zero as a function of $X$, then $\partial_X f(x, X) = 0$, even for later times $t$.

**C. Low-Order Equations**

The order $\varepsilon$ equation is

$$-\frac{1}{2} g^c_d \partial^2_{X^c X^d} g_{ab}^1 = 0, \quad (20)$$

which is a wave-equation in the $X$ co-ordinates, endowed with a constant metric.

Suppose that the initial conditions for the metric (i.e. the initial tensor perturbations) are given by a the power-spectrum in equation (12); then initially, $\partial_X g_1 = 0$, whence $\partial_X g_1 = 0$ always.

\(^2\) Even in generalised harmonic co-ordinates [27], the additional contribution to the Ricci tensor is $O(\varepsilon^2)$. 
D. Second-Order Equations

The second-order terms in equations (17) and (18) yield (again, using $\partial X g_0 = \partial X g_1 = 0$)

$$-\frac{1}{2} g_0^{cd} \partial_x^2 g_0^{ab} - \frac{1}{2} g_0^{cd} \partial_{X^c X^d} g_0^2 + P_{ab}(g_0) \partial_x g_0 \partial_x g_0 - \Lambda g_0^{ab} = 8\pi G \rho_0^{ab}. \quad (21)$$

Rearranging gives:

$$\left[ -\frac{1}{2} g_0^{cd} \partial_x^2 g_0^{ab} + P_{ab}(g_0) \partial_x g_0 \partial_x g_0 - \Lambda g_0^{ab} \right] - \frac{1}{2} \left[ g_0^{cd} \partial_{X^c X^d} g_0^2 \right] = 8\pi G \rho_0^{ab}. \quad (22)$$

The first line is nothing but the Einstein tensor (and the $\Lambda$-term) for large scales. The other – an oscillating part (with non-trivial initial conditions), that would vanish upon averaging, which includes the term $g_0^{cd} \partial_{X^c X^d} g_0^2$, that dictates the evolution of the second-order perturbation of the metric. I consider this term in §III F.

E. The Averaged Part

Indeed, equation (22) may be broken into two parts: an averaged part (integrated, so to speak, over $X$), and an oscillating part. An advantage of the multiple-scales method is that averaging is only performed in flat space-time, as opposed to other approaches to the averaging problem [3]. There are co-ordinates $\tilde{X}$ in which $g_{ab}^0$ is the Minkowski metric (these co-ordinates depend on $x$, of course, but this is innocuous; see also appendix A). In this co-ordinate system one may introduce a Fourier transform, which is carried out solely in a flat space-time, and is therefore unambiguous; then the average, $\langle f \rangle$, is simply the $\tilde{k} = 0$ component of the $\tilde{X}$-Fourier transform of $f$ (divided by the 4-volume). The Jacobian, $\sqrt{\text{det} g_0}$ is a constant, which is removed upon division by the 4-volume. The oscillating part of $f$ is then $\{f\}_{\text{osc}} = f - \langle f \rangle$.

I shall show below that the oscillatory part of equation (22) may be solved consistently, leaving

$$-\frac{1}{2} g_0^{cd} \partial_{x^c x^d} g_0^0 + P_{ab}(g_0) \partial_x g_0 \partial_x g_0 - \Lambda g_0^{ab} = 8\pi G \langle \rho_0^{ab} \rangle + \frac{1}{2} g_0^{cd} \langle \partial_{X^c X^d} g_0^2 \rangle. \quad (23)$$

The last term on the right vanishes by integration by parts (cf. §III F). Then one is left with an Einstein equation in the $x$ co-ordinates for $g_0(x)$ – a large-scale equation, sourced only by the averaged part of the energy-momentum tensor. This implies that, if $\rho_{ab} = O(1)$, then
the small scales do not react back on the large-scale metric, to leading order. The exact functional form of $\langle \rho^0_{ab} \rangle$ may be guessed from symmetries – from the cosmological principle – to yield that $g_0(x)$ is an FLRW metric.

**F. The Oscillatory Part**

The oscillating part of equation (22) reads

$$-\frac{1}{2} g_0^{cd} \partial^2_{X^c X^d} g^{2}_{ab} = 8\pi G \{\rho^0_{ab}\}_{osc}. \tag{24}$$

This equation is a partial differential equation for $g^2$ – a wave equation with a source. By the existence and uniqueness theorem for the wave equation in flat space-time, this has a solution for any $\{\rho^0_{ab}\}_{osc}$; but my concern is to show that this solution does not break the asymptotic series, i.e. that the $g_2$ thus obtained does not become too big ($O(\varepsilon^{-2})$). Let me perform a Fourier transform in $\tilde{X}$. The problem arises only from the resonant part of the energy-momentum tensor – from its components that satisfy $\tilde{k} \cdot \tilde{k} = 0$, i.e. from relativistic motion on small scales. The other Fourier components of $\{\rho^0_{ab}\}_{osc}$ are chiefly non-relativistic matter particles, such as dark matter or stars, for whom $\tilde{k}^0 v \gg \|\tilde{k}\|^2$. There is negligible contribution to the overall energy density from small-scale relativistic particles, but let us consider it anyway. Indeed, by linearity one may write $g^2 = g^2_{non-rel} + g^2_{rel} + g^2_{init}$, and let each one of the first two summands be the solution to the wave equation, sourced by the non-relativistic and the relativistic parts of $\{\rho^0_{ab}\}_{osc}$ respectively, with zero initial conditions; the third satisfies a homogeneous wave-equation, with initial conditions that are prescribed by the power-spectrum, as explained above. The solution for $g^2_{init}$ is

$$g^2_{init,ab}(x, X) = \frac{1}{(2\pi)^4} \int d^4 \tilde{k} e^{-i \tilde{k} \cdot \tilde{X}} g^2_{init,ab}(x, \tilde{k}) \delta(\tilde{k} \cdot \tilde{k}), \tag{25}$$

where $a \cdot b = g_0^{cd} a^c b^d$. This solution does not increase its amplitude, and therefore $\varepsilon^2 g^2_{init}$ remains $O(\varepsilon^2)$, thus maintaining consistency. By the same argument, $g^2_{non-rel}$ maintains an amplitude that remains $O(1)$ throughout its evolution, which leaves only $g^2_{rel}$ as a potential problem.

Suppose that $\rho_{ab}$ contained a plane wave term $\exp(i \tilde{k} \cdot \tilde{X})$, where $\tilde{k} \cdot \tilde{k} = 0$. This would resonate with the wave-equation differential operator, producing a growing amplitude. If it becomes too large, there is a possibility that $\varepsilon^2 g_2$ would grow larger than $\varepsilon g_1$, thus ruining
the asymptotic expansion. Plane waves due to the small-scale modes of cosmic microwave background are negligible due to diffusion damping [37]. $8\pi G\rho_{ab}$ due to galaxy-scale electromagnetic fields and neutrinos is assumed to be so small, that the resonant behaviour of the amplitude of $g_{ab}^2$ induced by it does not violate the asymptotic expansion (recall from §III A that the $X$-space-time is bounded).

If $\varepsilon$ pertains to galactic scales of $\sim 1$ kpc, then the scales of coherent, relativistic motion of other particles tend to be much smaller, so that there are numerous such, spatio-temporally confined resonant sources for $g_{\text{rel}}^2$. The associated Fourier components would, in general, have different phases, so, in effect, this contribution to $g_{\text{rel}}^2$ is the sum of waves emanating for point-like sources, with random phases. To find what $g_{\text{rel}}^2$ at each point $X$ is, one needs to superpose all the waves, each weighted by its source’s distance from $X$. This problem has been considered in the past by [38–40], and the upshot is that, if the number of sources is finite, then the probability of $g_{\text{rel}}^2$ being higher than $h$ is $\sim h^{-3}$, for large $h$. Thus,

$$P(g_{\text{rel}}^2 \gtrsim \varepsilon^{-p}) \propto N_{\text{tot}}\varepsilon^{3p}. \quad (26)$$

Therefore, $g_2$ is small in all probability (more rigorously, the asymptotic series may only converge in probability, but this is not a problem). Needless to say, the mean number of relativistic sources inside a galaxy is finite.

IV. NEWTONIAN OBJECTS

So far, I have explained why the leading order metric is unaffected by back-reaction caused by the small-scale oscillations of the energy-momentum tensor, as long as their amplitude is up to order unity. In fact, it turned out to be completely independent of the small scale.

But usually, when one considers the averaging problem, one has the putative effect of over-densities $\delta\rho/\rho \gg 1$ in mind, which are typically present inside galaxies, primarily in stars. This would lead to an asymptotic series for $\rho_{ab}$ that includes terms of order, say, $\varepsilon^{-2}$, which would change the low-order equations, leading to non-zero $X$ derivatives in the low-order terms in the asymptotic series of the metric $g$. Such terms could, conceivably, affect the $O(\varepsilon^2)$ equation, which, as was shown earlier, dictates the large-scale behaviour of $g_0$.

My primary concern in this paper is to show that the technique I present here can be
used to address this issue, and, given specific initial data (as well as a $\rho_{ab}$), to determine
the extent to which the small-scale physics reacts back on the large-scale metric. To do so I
endeavour to find which terms in the $O(\varepsilon^2)$ equation arise due to small-scale effects, below.

To make things less cumbersome, I include in $g_0$ any possible terms that are larger than
ord (1). The equation for $g_0(\cdot, X)$ now reads

$$-rac{1}{2}g_0^{cd}\partial^2_{X^c X^d}g_0^{ab} + P_{ab}(g_0)\partial_X g_0 \partial_X g_0 = 8\pi G \rho_{ab}^2. \quad (27)$$

This equation is an Einstein equation with zero cosmological constant, whose sources are
basically Newtonian point particles; the initial conditions are independent of $X$.

I assume that $\rho_{ab}^2$ is due solely to Newtonian objects, which, as is well-known, generate, by
themselves, a metric which is a perturbation relative to flat space-time. This implies that in
solving equation (27) one obtains two terms: $g_0(x)$, which describes the large-scale variation,
and a perturbation, $h_{ab}(x, X)$, due to the stars. Its magnitude is of order $GM/R$ for a star
which is about $\lambda \equiv 10^{-6} = 0.1\varepsilon$; even for a galactic potential with circular rotation velocity
of a few hundred kilometres per second, the magnitude of $h$ does not exceed this amount.
Therefore, $h_{ab} = O(\varepsilon)$ (in fact, $O(\lambda) \leq O(\varepsilon)$), and may be safely absorbed into $g_1$, at the
cost of its $X$-dependent parts increasing to $O(\lambda/\varepsilon)$. See appendix A for a more detailed
calculation.

The second-order equation acquires two additional source terms, and reads

$$\left[\left[ -\frac{1}{2}g_0^{cd}\partial^2_{X^c X^d}g_0^{ab} + P_{ab}(g_0)\partial_X g_0 \partial_X g_0 - \Lambda g_{ab}^0 \right] \right. \\
\left. + \left[ -g_0^{cd}\partial^2_{X^c X^d}g_1^{ab} - \frac{1}{2}g_0^{cd}\partial^2_{X^c X^d}g_2^{ab} + P_{ab}(g_0) \left( \partial_X g_1 \partial_X g_0 + \partial_X g_0 \partial_X g_1 \right) \right] \right. \\
\left. + \left\{ -\frac{1}{2}g_1^{cd}\partial^2_{X^c X^d}g_1^{ab} + P_{ab}(g_0)\partial_X g_1 \partial_X g_1 \right\}_{osc} \right] = 8\pi G \rho_{ab}^0. \quad (28)$$

Let us consider the two middle rows first: they constitute the updated equation (24), with
$g_1$ having an additional $O(\lambda/\varepsilon)$ component that is due to Newtonian sources. The latter
would not give rise to resonances precisely because it arises from non-relativistic objects,
and the former contributes only at higher order. In any case, together with $\{\rho_{ab}^0\}_{osc}$, these
average out to zero on large scales.

\footnote{Strictly speaking, this is correct in Newtonian gauge, relative a Minkowski background in the $\tilde{X}$ co-
ordinates for $X$-space-time, defined as in \S III F.}
What one is left with is
\[ -\frac{1}{2}g_0^{cd}\partial_x^2g_{ab}^0 + P_{ab}(g_0)\partial_xg_0\partial_xg_0 - \Lambda g_{ab}^0 - B_{ab} = 8\pi G\langle \rho_0^0 \rangle, \] (29)
where
\[ B_{ab} = \left\langle \frac{1}{2}g_1^{cd}\partial_X^2\tilde{g}_{ab}^1 - P_{ab}(g_0)\partial_X\tilde{g}_1\partial_X\tilde{g}_1 \right\rangle. \] (30)
The tensor (on the $x$-space-time) $B_{ab}$ may constitute a possible back-reaction of the Newtonian sources, propagated through the non-linearity of the Einstein equations, on the large-scale properties of $g_0$ – finding these was the goal of this section. Equations (29) and (30) constitute something akin to a homogenised equation for cosmological back-reaction – they describe the dynamics of the large-scale (leading-order) metric, taking small-scale inhomogeneities into account in a consistent manner. Recall that the averaging $\langle \cdot \rangle$ is carried out only in the (flat) fibre $F_x$.

Appendix A implies that the $O(\lambda/\varepsilon)$ term in $g_1^{ab}$, that corresponds to $h_{ab}$, is given, in the frame of reference of a freely-falling observer on $M$, by $\tilde{\zeta}_{ab}$, which is defined there. This frame is associated with an orthonormal tetrad $e_a^b(x)$, which is used to convert from abstract indices to concrete ones (and vice versa). In this frame, the components $B_{ab}$ of $B_{ab}$ are given by the expression in equation (30), with the derivatives in the $X$ system, and $g_1$ set to $\tilde{\zeta}$, viz.
\[ \varepsilon^2 B_{ab} = \left\langle \frac{1}{2}\tilde{\zeta}^{cd}\partial_X^2\tilde{g}_{ab}^1 - P_{ab}(g_0)\partial_X\tilde{g}^1\partial_X\tilde{g}^1 \right\rangle. \] (31)
The reason is, that even though $g_1$ and $g_0$ in equation (30) are tensors on $M$ (and thus scalars on $F_x$), one actually performs two co-ordinate transformations here: one, on $M$, from harmonic co-ordinates to those of a freely-falling observer in $g_0$, at $x$, and then, an additional transformation on $F_x$ that takes $X$ to $\tilde{X}$. The average (as the zero mode of a Fourier transform) is invariant under the latter, which implies that the cumulative effect of both transformations justifies equation (31).

Let me try to estimate its magnitude: Consider, for instance, a galaxy with a constant-in-time Navarro-Frenk-White [41, 42] profile
\[ \Phi(r) = -4\pi G\rho_0r_0^2\frac{\ln(1 + r/r_0)}{r/r_0} = -4\pi G\rho_0r_0^2f(y), \] (32)
with $r_{200} = 15r_0$, $M = 10^{12} M_\odot = 200\rho_{\text{crit}}\frac{4}{3}\pi r_{200}^3$, $y = r/r_0$, where $\rho_{\text{crit}} = 27.75 \times 10^{10} h^2 M_\odot \text{ Mpc}^{-3}$, and $H_0 = 67.4 \text{ km s}^{-1} \text{ Mpc}^{-1} = 100 h \text{ km s}^{-1} \text{ Mpc}^{-1}$ [35]. Consider, for
example, \( a = b = i \) for some Cartesian index. Then, by equation (4)

\[
B_{ii} = \left\langle \frac{\partial_i g^{1j} \partial_j g_{1i}}{2} + \text{Christoffel product + boundary term} \right\rangle, \tag{33}
\]

where I have integrated the second derivative by parts once (the boundary term turns out to be minuscule). Hence (the integral \( dX^0 \) cancels out with the \( X^0 \) dimension of the 4-volume) each of the terms consists of a product of two first derivatives of \( \Phi \), each of which is about

\[
\langle \partial_X g_1 \partial_X g_1 \rangle \sim \epsilon^{-2} \times \frac{G^2 \rho_0^2 a^4}{c^4} \frac{3(4\pi)^3}{4\pi(r_{200}/r_0)^3} \int_{r_{200}/r_0} f'(y)^2 y^2 dy \approx 6.6 \times 10^{-5} \ (100 \text{ Mpc})^{-2}
\]

(34)

Of course, the units of 100 Mpc are those in which \( x \) is expressed. This is small (a few percent) compared with \( \frac{8\pi G\rho_{\text{crit}}}{c^2} \approx 0.0015 \ (100 \text{ Mpc})^{-2} \), whence it emerges that the back-reaction due to averaging exists, but is small relative to the background. Appendix B explains how to calculate the components of \( B_{ab} \) in a given gauge (which is not necessarily harmonic), provided that equation (29) is taken as an effective Einstein-like equation for \( M \), and that one does not perform co-ordinate transformations whose derivative matrix is not \( \text{ord} \ (1) \).

Taking equation (29) as an Einstein equation, with \( R_{ab} \) replacing \( R_{ab}^{(h)} \), and with \( B_{ab} \) now calculable in any gauge, one may perform a 3 + 1 splitting and derive, \textit{inter alia}, a Raychaudhuri equation; the simplest way to do so is to move \( B_{ab} \) to the matter side of the Einstein equation, and consider it as a correction to the energy-momentum tensor.

\section{V. DISCUSSION}

In this paper I presented an approach to study the averaging problem in cosmology using the method of multiple scales. The small and the large scales were treated as independent variables in harmonic gauge, and the Einstein field equations were expanded in the small scale. This yielded perturbative equations for both the small and the large scales, which were solved iteratively, until I reached second order in \( \epsilon \); at this order one obtains an effective equation for the large-scale dependence of the metric, which also includes a back-reaction term. I showed that this term vanishes completely if the energy-momentum tensor is always of the same order as the averaged one (at most), but it does not in general. If the \( O(\epsilon^{-2}) \) density variations are due to Newtonian objects, then the back-reaction terms are small.
However, a detailed model of the small-scale system is needed to study back-reaction in a realistic system \cite{12}.

Throughout the paper I assumed that the effect of black holes and neutron stars may be neglected. While this might be justified by Birkhoff’s theorem for isolated bodies (if the distance between them and the next over-density is $\gg GM$), it cannot be used to treat the fully relativistic case. I have also not proved that the asymptotic expansion in \S III does not break down at higher orders, but I did show that it does not up to second order, and I have not obtained estimates on how well it approximates the exact solution. However, one can generalise the approach I presented here to account for these issues. On the other hand, this approach has the advantage that it does not require any averaging over curved manifolds, and is effective in revealing the terms in the Einstein equations that lead to possible back-reaction, and how to gauge their magnitude.

Due to the well-separation of stellar, galactic and LLS scales, one can extend the formalism presented in this paper to account for back-reaction due to inhomogeneities on all these scales, by introducing $X_{\text{stars}}, X_{\text{gal}}, X_{\text{LSS}}$, in addition to the cosmological-scale $x$, and treating all four variables as independent.

The approach I proposed here and its relation to the averaging problem, are quite analogous to Hamiltonian perturbation theory, when faced with a resonance (say, in the context of celestial mechanics). Usually, the equations of motions are obtained there by averaging over the fast variables – the mean anomalies of the individual bodies (analogous to small scales) \cite{13} – thereby generating averaged equations of motion which govern the evolution of the slow variables (such as the energies and angular momenta). But na"ive averaging cannot be done when a resonance is present, which is simply another way of saying that the fast variables react back on the slow ones. Instead, resonant perturbation theory is required, which also draws on the method on multiple scales.

**ACKNOWLEDGMENTS**

I wish to thank Vincent Desjacques, Robert Reischke, Robert Lilow, Stefano Anselmi and Dennis Stock for helpful discussions and comments. I acknowledge funding from the Israel
Science Foundation (grant no. 1395/16 and 255/18).

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Appendix A: Explicit Solution With Newtonian Objects

The purpose of this appendix is the calculation of $h_{ab}$ is Iv. Equation (27) consists of a metric with constant (in $X$) boundary conditions, with an energy-momentum tensor that is comprised of Newtonian masses. This equation, qua a partial differential equation, is an Einstein equation (written in harmonic co-ordinates), which describes 10 components of what one would like to identify with a metric on $F_x$. Making this identification is akin to studying the tensor $g_F \in T^{*} \otimes^2 \mathcal{B}$, given by

$$g_F = g_{ab}(x, X) \frac{\partial}{\partial X^a} \otimes \frac{\partial}{\partial X^b}.$$ (A1)

(As before this is actually in a sub-space corresponding to $T^{*} \otimes^2 F_x$.) This tensor is not to be confused with the metric $g^0_M = g_{ab}^0$ on $M$, although they have the same components.

As the energy-momentum tensor is small (it is generated by Newtonian sources), one can solve this equation perturbatively, writing $g_F = g_F^0 + g_F^1 = g_{ab}^0 + \zeta_{ab}$, where $g_{ab}^0$ is a function of $x$ only (i.e. a constant on $F_x$), and $\zeta_{ab} = \text{ord} (\lambda)$. As in III.F one transforms to a co-ordinate system $\tilde{X}$ on $F_x$ where $\tilde{g}_F^0 = \eta$ is the Minkowski metric; the transformation is $\tilde{X}^b = P_a^b X^a$ (it exists due to Sylvester’s law of inertia). The transformation matrix $P^b_a = P^b_a (x)$ has the same components as the matrix that transforms $g_M$ to the co-ordinates of a freely-falling observer on $M$ with metric $g_0$ (although, as before, the former lies in the tangent space of $F_x$ whereas the latter – in the tangent space of $M$), i.e. $\eta_{ab} P^a_c P^b_d = g_{cd}^0$. This transformation leaves $\zeta_{ab}$ in harmonic gauge. However, Newtonian gauge is harmonic for particles whose (peculiar) velocities are much lower than the speed of light, so in this

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4 I assume that this matrix is ord (1) in $\varepsilon$. 
approximation I take $\zeta_{ab}$ to be in Newtonian gauge, whence it emerges that

$$\tilde{\zeta}_{00} = -2\Phi$$  \hspace{1cm} (A2)
$$\tilde{\zeta}_{ij} = -2\Phi\delta_{ij}$$  \hspace{1cm} (A3)
$$\tilde{\zeta}_{0i} = 0,$$  \hspace{1cm} (A4)

where $\Phi$ is the Newtonian gravitational potential.

**Appendix B: Gauge Transformations**

Having computed $B_{ab}$ in a frame attached to a freely-falling observer on $M$ in $g_0$, and having seen that it is small relative to $\rho_{\text{crit}}$, one may wish to calculate $B_{ab}$ in a specific co-ordinate system; for instance, in conformal Newtonian gauge. To see how this is done, I ignore any large-scale perturbations to the density field (and to the velocity field). Then $g_0$ is the FLRW metric, and a freely-falling observer there is co-moving.

The observer’s tetrad may be taken as $e^a_0 = u^a$, and $e^b_i = \delta^b_i / a(\eta) \sqrt{\gamma_{ii}}$ (no sum is implied), where $a(\eta)$ is the scale-factor and the spatial part of the metric is $g^0_{ij} = a^2 \gamma_{ij}$. Then

$$B^{ab} = B^{cd}e^a_c e^b_d,$$ \hspace{1cm} (B1)

explicitly (no sum implied),

$$B^{00} = \frac{B^{00}}{a^2}$$
$$B^{0i} = \frac{B^{0i}}{a^2 \sqrt{\gamma_{ii}}}$$ \hspace{1cm} (B2)
$$B^{ij} = \frac{B^{ij}}{a^2 \sqrt{\gamma_{ii}}} \gamma_{jj}.$$

If there exist large-scale perturbations, then one has to perform an additional asymptotic expansion in both the magnitude of these perturbations, and $\varepsilon$. Re-summing the former would imply that all the large-scale perturbations are present in the $g_0$ of this paper; the procedure for obtaining $B_{ab}$ in this case is the same as was outlined above, *mutatis mutandis*. To first order in this re-summation in Newtonian gauge, the metric is a perturbed FLRW metric, given by

$$ds_0^2 = a^2 \left[ -(1 + 2\Phi) d\eta^2 + (1 - 2\Psi) \delta_{ij} dx^i dx^j \right],$$ \hspace{1cm} (B3)

which means that $u^a = a^{-1} (1 - \Phi, \mathbf{v}_{\text{pec}})$, and $e^a_1$ are chosen to make the tetrad orthonormal. Equation (B1) still holds, of course. Besides, a perturbed FLRW metric is necessary in
general, because even if there are no large-scale perturbations, $B_{ab}$ itself would produce them. Equations (B2) are already at first order in the large-scale perturbation, so they receive contributions from the change of $u^a$ only at second order.