Embedding and extension results in fractional Musielak–Sobolev spaces

Elhoussine Azroul, Abdelmoujib Benkirane, Mohammed Shimi and Mohammed Srati
Faculty of Sciences Dhar El Mahraz, Laboratory of Mathematical Analysis and Applications, Sidi Mohamed Ben Abdellah University, Fez, Morocco

ABSTRACT
In this paper, we are concerned with some qualitative properties of the new fractional Musielak–Sobolev spaces $W^{s,L}_{\Phi,x,y}$ such that the generalized Poincaré type inequality and some continuous and compact embedding results. Moreover, we prove that any function in $W^{s,L}_{\Phi,x,y}(\Omega)$ may be extended to a function in $W^{s,L}_{\Phi,x,y}(\mathbb{R}^N)$, with $\Omega \subset \mathbb{R}^N$ is a bounded domain of class $C^{0,1}$. In addition, we establish a result that relates to the complemented subspace in $W^{s,L}_{\Phi,x,y}(\mathbb{R}^N)$. As an application, using the mountain pass theorem and some variational methods, we investigate the existence of a nontrivial weak solution for a class of nonlocal fractional type problems with Dirichlet boundary data.

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1. Introduction and preliminaries results
In the last decades, the fractional Sobolev spaces have been a classical topic in functional and harmonic analysis all along, see, for instance, [1–3], which treat the topic in detail. On the other hand, fractional Sobolev spaces and the corresponding nonlocal problems are now experiencing impressive applications in different subjects, both for the pure mathematical research and in view of concrete applications, such as, phase transitions [4], crystal dislocation [5], conservation laws [6], mathematical finance [7], anomalous diffusion [8], stratified materials [9], etc.

As well known, when $s \in (0,1)$, $p \in [1,\infty)$ and $\Omega$ is an open subset of $\mathbb{R}^N$ with $C^{0,1}$-regularity, any function in fractional Sobolev space $W^{s,p}(\Omega)$ may be extended to a function in $W^{s,p}(\mathbb{R}^N)$ (see [10]). Extension results are quite important in applications and to improve certain embeddings theorems, in the integer classical case [11–15], as well as in the fractional case [10,16–18]. All these previous results are held under certain crucial regularity assumptions on the domain $\Omega$.

The problem of characterizing the class of sets that are extension domains for $W^{s,p}_{\Phi,x,y}$ is open, where $W^{s,p}_{\Phi,x,y}$ is the new fractional Musielak–Sobolev space introduced recently in [19]. In general, an arbitrary open set $\Omega$ is not an extension domain for $W^{s,p}_{\Phi,x,y}$. When $s$ is an integer, we cite [13] for a complete characterization in the special case when $s = 1$, $\Phi_{x,y}(t) = t^2$ and $N = 2$, and we refer the interested reader to the nice monograph of Leoni [20] in which this problem is very well discussed (see, in particular, Chapter 11 and Chapter 12).

The study of variational problems where the modular function satisfies nonpolynomial growth conditions instead of having the usual $p$-structure arouses much interest in the development of applications to electrorheological fluids as an important class of non-Newtonian fluids (sometimes referred...
to as smart fluids). The electro-rheological fluids are characterized by their ability to drastically change the mechanical properties under the influence of an external electromagnetic field. A mathematical model of electro-rheological fluids was proposed by Rajagopal and Ruzicka (we refer the reader to [21,22] for more details). Another important application is related to image processing [23] where this kind of diffusion operator is used to underline the borders of the distorted image and to eliminate the noise.

Recently, great attention has been devoted to the study of a new class of fractional Sobolev spaces and related nonlocal problems, in particular, in the fractional Orlicz–Sobolev spaces \( W^{s,p}(\Omega) \) (see [24–32]) and in the fractional Sobolev spaces with variable exponents \( W^{s,p(x,y)}(\Omega) \) (see [33–40]), in which the authors establish some basic properties of these modular spaces and the associated nonlocal operators, they also obtained certain existence results for nonlocal problems involving this type of integro-differential operators. Furthermore, in that context, the authors in [19] introduced a new functional framework which can be seen as a natural generalization of the above-mentioned functional spaces.

Our main purpose in this paper is to continue the study of the aforementioned new class of fractional Sobolev spaces and the related nonlocal operators, by introducing the new fractional Musielak–Sobolev spaces \( W^{s,p}(\Omega) \). In addition, we establish certain basic properties of these spaces and we prove some continuous and compact embedding theorems of these spaces into Musielak spaces. Moreover, we show that any function in fractional Musielak–Sobolev spaces \( W^{s,p}(\Omega) \) may be extended to a function in \( W^{s,p}(\mathbb{R}^N) \) with \( \Omega \subset \mathbb{R}^N \) is a bounded domain of class \( C^{0,1} \). As an application, we are concerned with the existence of weak solutions of the following nonlocal problem

\[
(P_d) \begin{cases}
(-\Delta)^{s_1}_{a(x,\cdot)} u + (-\Delta)^{s_2}_{a(x,\cdot)} u = f(x, u) & \text{in } \Omega, \\
u = 0 & \text{in } \mathbb{R}^N \setminus \Omega,
\end{cases}
\]

where \( \Omega \) is an open bounded subset in \( \mathbb{R}^N, N \geq 1 \), with Lipschitz boundary \( \partial \Omega, 0 < s_2 \leq s_1 < 1, f : \Omega \times \mathbb{R} \to \mathbb{R} \) is a Carathéodory function satisfies some suitable conditions. Moreover, for any \( i = 1, 2, (-\Delta)^{s_i}_{a(x,\cdot)} \) is the nonlocal integro-differential operator of elliptic type defined as follows

\[
(-\Delta)^{s_i}_{a(x,\cdot)} u(x) = 2 \lim_{\varepsilon \to 0} \int_{\mathbb{R}^N \setminus B_\varepsilon(x)} a_{(x,y)} \left( \frac{|u(x) - u(y)|}{|x - y|^{s_i}} \right) \frac{u(x) - u(y)}{|x - y|^{s_i}} \, dy.
\]

for all \( x \in \mathbb{R}^N \), where \( (x,y,t) \mapsto a_{(x,y)}(t) := a(x,y,t) : \overline{\Omega} \times \overline{\Omega} \times \mathbb{R} \to \mathbb{R} \) is such that: \( \varphi(\cdot,\cdot) : \overline{\Omega} \times \overline{\Omega} \times \mathbb{R} \to \mathbb{R} \) defined by

\[
\varphi_{x,y}(t) := \varphi(x,y,t) = \begin{cases} a(x,y,|t|) t & \text{for } t \neq 0, \\
0 & \text{for } t = 0,
\end{cases}
\]

is increasing homeomorphism from \( \mathbb{R} \) onto itself. Let

\[
\Phi_{x,y}(t) := \Phi(x,y,t) = \int_0^t \varphi_{x,y}(\tau) \, d\tau \quad \text{for all } (x,y) \in \overline{\Omega} \times \overline{\Omega}, \text{ and } t \geq 0.
\]

Then, \( \Phi_{x,y} \) is a Musielak function (see [41]), that is

- \( \Phi(x,y,\cdot) \) is a \( \Phi \)-function for every \( (x,y) \in \overline{\Omega} \times \overline{\Omega} \), i.e. is continuous, nondecreasing function with \( \Phi(x,y,0) = 0, \Phi(x,y,t) > 0 \text{ for } t > 0 \) and \( \Phi(x,y,t) \to \infty \text{ as } t \to \infty \).
- For every \( t \geq 0, \Phi(\cdot,\cdot,t) : \overline{\Omega} \times \overline{\Omega} \to \mathbb{R} \) is a measurable function.

We introduce the function \( \widetilde{a}(x,t) := a(x,t) \) for all \( (x,t) \in \overline{\Omega} \times \mathbb{R} \). Then, we define the function \( \tilde{\Phi}(\cdot,\cdot) : \overline{\Omega} \times \mathbb{R} \to \mathbb{R} \) by
\[ \hat{\varphi}_x(t) := \hat{\varphi}(x, t) = \begin{cases} \hat{\varphi}(x, |t|)t & \text{for } t \neq 0, \\ 0 & \text{for } t = 0, \end{cases} \]

is an increasing homeomorphism from \( \mathbb{R} \) onto itself. If we set

\[ \hat{\Phi}_x(t) := \hat{\Phi}(x, t) = \int_0^t \hat{\varphi}_x(\tau) \, d\tau \quad \text{for all } t \geq 0. \quad (1) \]

Then, \( \hat{\Phi}_x \) is also a Musielak function.

**Remark 1.1:**
- For all \((x, y) \in \Omega \times \Omega\), \( \Phi_{x,y} \) and \( \hat{\Phi}_x \) are two convex and increasing functions from \( \mathbb{R}^+ \) to \( \mathbb{R}^+ \).
- Note that \( (-\Delta)^\alpha_{a(x)} \) is a nonlocal integro-differential operator of elliptic type which can be seen as a generalization of the fractional \( p(x,.) \)-Laplacian operator \( (-\Delta p(x,))^{\frac{\alpha}{2}} \) (when \( a(x,y)(t) = |t|^{p(x,y)-2} \), see for instance [33,40]), and of the fractional \( p \)-Laplacian operator \( (-\Delta)^\alpha_p \) in the constant exponent case (when \( p(x,y) = p \) constant, i.e. \( a(x,y)(t) = |t|^{p-2} \)).
- The operator \( (-\Delta)^\alpha_{a(x)} \) reduces to the fractional \( a(.) \)-Laplacian if \( a(x,y)(t) = a(t) \), i.e. the function \( a \) is independent of variables \( x, y \) (see for example [24,31]). On the other hand, we remark that is the fractional version of the well-known \( a(.) \)-Laplacian operator \( div(a(x,|\nabla u(x)|)\nabla u(x)) \) which is associated with the Musielak–Sobolev spaces (see [42–45]).
- In contrast to the classical \( p(x) \)-Laplacian and \( a(.) \)-Laplacian, which are local operators, the integro-differential operator \( (-\Delta)^\alpha_{a(x)} \) is a paradigm of the vast family of nonlocal nonlinear operators, and this has immediate consequences in the formulation of basic questions such as the Dirichlet problem. For this, the Dirichlet datum is given in \( \mathbb{R}^N \setminus \Omega \) (which is different from the classical case of the \( p(x) \)-Laplacian and \( a(.) \)-Laplacian) and not simply on \( \partial \Omega \). Moreover, the value of \( (-\Delta)^\alpha_{a(x)} u(x) \) at any point \( x \in \Omega \) depends not only on the values of \( u \) on \( \Omega \), but actually on the entire space \( \mathbb{R}^N \), which implies that the first equation in \((\mathcal{P}_a)\) is no longer a pointwise equation, it is no longer a pointwise identity. Hence, it is often called nonlocal problem. This causes some mathematical difficulties which make the study of such a problem particularly interesting.

Next, For the function \( \hat{\Phi}_x \) given in (1), we introduce the Musielak class as follows

\[ K_{\hat{\Phi}_x}(\Omega) = \left\{ u : \Omega \rightarrow \mathbb{R} \text{ measurable} : \int_\Omega \hat{\Phi}_x(|u(x)|) \, dx < \infty \right\}, \]

and the Musielak space

\[ L_{\hat{\Phi}_x}(\Omega) = \left\{ u : \Omega \rightarrow \mathbb{R} \text{ measurable} : \int_\Omega \hat{\Phi}_x(\lambda,|u(x)|) \, dx < \infty \quad \text{for some } \lambda > 0 \right\}. \]

The space \( L_{\hat{\Phi}_x}(\Omega) \) is a Banach space endowed with the Luxemburg norm

\[ ||u||_{\hat{\Phi}_x} = \inf \left\{ \lambda > 0 : \int_\Omega \hat{\Phi}_x \left( \frac{|u(x)|}{\lambda} \right) \, dx \leq 1 \right\}. \]

The conjugate function of \( \Phi_{x,y} \) is defined by \( \overline{\Phi}_{x,y}(t) = \int_0^t \overline{\varphi}_{x,y}(\tau) \, d\tau \) for all \((x, y) \in \Omega \times \Omega\) and all \( t \geq 0 \), where \( \overline{\varphi}_{x,y} : \mathbb{R} \rightarrow \mathbb{R} \) is given by \( \overline{\varphi}_{x,y}(t) := \overline{\varphi}(x,y,t) = \sup\{s : \varphi(x,y,s) \leq t\} \). Furthermore, we have
the following Hölder type inequality
\[
\left| \int_{\Omega} uv \, dx \right| \leq 2\|u\|_{\Phi_x} \|v\|_{\Phi_x} \quad \text{for all } u \in L_{\widehat{\Phi}_x}(\Omega) \quad \text{and} \quad v \in L_{\widehat{\Phi}_x}(\Omega).
\] (2)

Throughout this paper, we assume that there exist two positive constants \( \varphi^+ \) and \( \varphi^- \) such that
\[
1 < \varphi^- \leq \frac{t \varphi_{x,y}(t)}{\Phi_{x,y}(t)} \leq \varphi^+ < +\infty \quad \text{for all } (x, y) \in \overline{\Omega} \times \overline{\Omega} \quad \text{and all } t \geq 0.
\] (\( \Phi_1 \))

This relation implies that
\[
1 < \varphi^- \leq \frac{t \varphi_x(t)}{\Phi_x(t)} \leq \varphi^+ < +\infty, \quad \text{for all } x \in \overline{\Omega} \quad \text{and all } t \geq 0.
\] (3)

It follows that \( \Phi_{x,y} \) and \( \widehat{\Phi}_x \) satisfy the global \( \Delta_2 \)-condition (see [45]), written \( \Phi_{x,y} \in \Delta_2 \) and \( \widehat{\Phi}_x \in \Delta_2 \), that is,
\[
\Phi_{x,y}(2t) \leq K_1 \Phi_{x,y}(t) \quad \text{for all } (x, y) \in \overline{\Omega} \times \overline{\Omega}, \quad \text{and all } t \geq 0,
\] (4)

and
\[
\widehat{\Phi}_x(2t) \leq K_2 \widehat{\Phi}_x(t) \quad \text{for any } x \in \overline{\Omega}, \quad \text{and all } t \geq 0,
\] (5)

where \( K_1 \) and \( K_2 \) are two positive constants. The inequality (5) implies that \( L_{\widehat{\Phi}_x}(\Omega) = K_{\widehat{\Phi}_x}(\Omega) \) (see [41]).

An important role in manipulating the Musielak spaces is played by the modular of the space \( L_{\widehat{\Phi}_x}(\Omega) \). It is worth noticing that the relation between the norm and the modular shows an equivalence between the topology defined by the norm and that defined by the modular.

**Proposition 1.1 ([45, Proposition 2.1]):** Assume that the condition (\( \Phi_1 \)) is satisfied. Then, for all \( u \in L_{\widehat{\Phi}_x}(\Omega) \), the following relations hold true

(i) \( \|u\|_{\Phi_x} > 1 \Rightarrow \|u\|_{\phi_x}^{-} \leq \int_{\Omega} \widehat{\Phi}_x(|u|) \, dx \leq \|u\|_{\Phi_x}^{+} \),

(ii) \( \|u\|_{\Phi_x} < 1 \Rightarrow \|u\|_{\phi_x}^{+} \leq \int_{\Omega} \widehat{\Phi}_x(|u|) \, dx \leq \|u\|_{\Phi_x}^{-} \).

**Definition 1.1:** Let \( A_x(t), B_x(t) : \mathbb{R}^+ \times \Omega \rightarrow \mathbb{R}^+ \) be two Musielak functions. \( A_x \) is stronger (resp. essentially stronger) than \( B_x \), \( A_x \gg B_x \) (resp. \( A_x \gg B_x \)) in symbols, if for almost every \( x \in \overline{\Omega} \)
\[
B(x, t) \leq A(x, at), \quad t \geq t_0 \geq 0,
\]
for some (resp. for each) \( a > 0 \) and \( t_0 \) depending on \( a \).

**Remark 1.2 ([46, Section 8.5]):** \( A_x \gg B_x \) is equivalent to the condition
\[
\lim_{t \to 0} \sup_{x \in \overline{\Omega}} \left( \frac{B(x, \lambda t)}{A(x, t)} \right) = 0,
\]
for all \( \lambda > 0 \).

**Theorem 1.1 ([47]):** Let \( \Omega \) be an open subset of \( \mathbb{R}^N \) which has a finite volume. Let \( A_x(t), B_x(t) : \mathbb{R}^+ \times \Omega \rightarrow \mathbb{R}^+ \) be two Musielak functions such that \( B_x \ll A_x \). Then any bounded subset \( S \) of \( L_{A_x}(\Omega) \) which is precompact in \( L^1(\Omega) \) is also precompact in \( L_{B_x}(\Omega) \).
Now, due to the nonlocality of the operator $(-\Delta)^\varepsilon_{\bar{a}(x,y)}$, we define the new fractional Musielak–Sobolev space as introduced in [19] as follows
\[
W^sL_{\Phi_{x,y}}(\Omega) = \left\{ u \in L_{\bar{\Phi}}(\Omega) : \int_\Omega \int_\Omega \Phi_{x,y} \left( \frac{\lambda|u(x) - u(y)|}{|x - y|^s} \right) \frac{dx \, dy}{|x - y|^N} < \infty \text{ for some } \lambda > 0 \right\}.
\]
This space can be equipped with the norm
\[
||u||_{s,\Phi_{x,y}} = ||u||_{\bar{\Phi}} + [u]_{s,\Phi_{x,y}}, \tag{6}
\]
where $[\cdot]_{s,\Phi_{x,y}}$ is the Gagliardo seminorm defined by
\[
[u]_{s,\Phi_{x,y}} = \inf \left\{ \lambda > 0 : \int_\Omega \int_\Omega \Phi_{x,y} \left( \frac{|u(x) - u(y)|}{\lambda|x - y|^s} \right) \frac{dx \, dy}{|x - y|^N} \leq 1 \right\}.
\]

**Theorem 1.2 ([19]):** Let $\Omega$ be an open subset of $\mathbb{R}^N$, and let $s \in (0, 1)$. The space $W^sL_{\Phi_{x,y}}(\Omega)$ is a Banach space with respect to the norm (6), and a separable (resp. reflexive) space if and only if $\Phi_{x,y} \in \Delta_2$ (resp. $\Phi_{x,y} \in \Delta_2$ and $\bar{\Phi}_{x,y} \in \Delta_2$). Furthermore, if $\Phi_{x,y} \in \Delta_2$ and $\Phi_{x,y}(\sqrt{t})$ is convex, then the space $W^sL_{\Phi_{x,y}}(\Omega)$ is an uniformly convex space.

**Definition 1.2 ([19]):** We say that $\Phi_{x,y}$ satisfies the fractional boundedness condition, written $\Phi_{x,y} \in \mathcal{B}_f$, if
\[
\sup_{(x,y) \in \Omega \times \Omega} \Phi_{x,y}(1) < \infty. \tag{\Phi_2}
\]

**Theorem 1.3 ([19]):** Let $\Omega$ be an open subset of $\mathbb{R}^N$, and $0 < s < 1$. Assume that $\Phi_{x,y} \in \mathcal{B}_f$. Then,
\[
C^2_0(\Omega) \subset W^sL_{\Phi_{x,y}}(\Omega).
\]

**Lemma 1.1 ([19]):** Assume that $(\Phi_1)$ is satisfied. Then
\[
\bar{\Phi}_{x,y}(\varphi_{x,y}(t)) \leq \varphi^+\Phi_{x,y}(t) \quad \text{for all } (x,y) \in \Omega \times \Omega \quad \text{and all } t \geq 0.
\]

**Lemma 1.2 ([19]):** Assume that $(\Phi_1)$ is satisfied. Then the following inequalities hold true:
\[
\begin{align*}
\Phi_{x,y}(\sigma t) & \geq \sigma^{\varphi^-} \Phi_{x,y}(t) \quad \text{for all } t > 0 \text{ and any } \sigma > 1, \tag{7} \\
\Phi_{x,y}(\sigma t) & \geq \sigma^{\varphi^+} \Phi_{x,y}(t) \quad \text{for all } t > 0 \text{ and any } \sigma \in (0,1), \tag{8} \\
\Phi_{x,y}(\sigma t) & \leq \sigma^{\varphi^-} \Phi_{x,y}(t) \quad \text{for all } t > 0 \text{ and any } \sigma > 1, \tag{9} \\
\Phi_{x,y}(t) & \leq \sigma^{\varphi^+} \Phi_{x,y} \left( \frac{t}{\sigma} \right) \quad \text{for all } t > 0 \text{ and any } \sigma \in (0,1). \tag{10}
\end{align*}
\]

For any $u \in W^sL_{\Phi_{x,y}}(\Omega)$, we define the modular function on $W^sL_{\Phi_{x,y}}(\Omega)$ as follows
\[
\Psi(u) = \int_\Omega \int_\Omega \Phi_{x,y} \left( \frac{|u(x) - u(y)|}{|x - y|^s} \right) \frac{dx \, dy}{|x - y|^N} + \int_\Omega \bar{\Phi}_x ||u(x)|| \, dx.
\]

**Proposition 1.2 ([19]):** Assume that $(\Phi_1)$ is satisfied. Then, for any $u \in W^sL_{\Phi_{x,y}}(\Omega)$, the following relations hold true:
\[
\begin{align*}
||u||_{s,\Phi_{x,y}} > 1 & \implies ||u||_{s,\Phi_{x,y}}^\varphi^- \Psi(u) \leq ||u||_{s,\Phi_{x,y}}^\varphi^+, \tag{11} \\
||u||_{s,\Phi_{x,y}} < 1 & \implies ||u||_{s,\Phi_{x,y}}^\varphi^+ \Psi(u) \leq ||u||_{s,\Phi_{x,y}}^\varphi^- . \tag{12}
\end{align*}
\]
Theorem 1.4: Let $X$ be a real Banach space and $J \in C^1(X, \mathbb{R})$ with $J(0) = 0$. Suppose that the following conditions hold:

\begin{align*}
\text{(G1)} & \text{ There exist } \rho, r > 0 \text{ such that } J(u) \geq r \text{ for } u \in X \text{ with } ||u|| = \rho. \\
\text{(G2)} & \text{ There exists } e \in X \text{ with } ||e|| > \rho \text{ such that } J(e) \leq 0.
\end{align*}

Let

\begin{align*}
c := \inf \max_{\gamma \in \Gamma, t \in [0,1]} J(\gamma(t)) \quad \text{with } \Gamma = \{ \gamma \in C([0,1], X); \gamma(0) = 0, \gamma(1) = e \}.
\end{align*}

Then there exists a sequence $\{u_n\}$ in $X$ such that

\begin{align*}
J(u_n) \to c \quad \text{and} \quad J'(u_n) \to 0.
\end{align*}

2. Embeddings and extension results

In this section, we will establish some embeddings and extension results of the new fractional Musielak–Sobolev spaces.

2.1. Some embedding results

This subsection is devoted to the embedding results of the new fractional Musielak–Sobolev spaces $W^{s_1}L_{\phi_{x,y}}(\Omega)$. To this end, we follow the same approach used to obtain the embedding results in the fractional Orlicz–Sobolev space $W^sL_{\Phi}(\Omega)$ established in [24].

At first, it is worth noticing that, as in the classical case with $s_1, s_2$ being an integers, the space $W^{s_1}L_{\phi_{x,y}}(\Omega)$ is continuously embedded in $W^{s_2}L_{\phi_{x,y}}(\Omega)$ when $s_2 \leq s_1$, as next result points out.

Proposition 2.1: Let $\Omega$ be a bounded open subset of $\mathbb{R}^N$ and let $0 < s_2 \leq s_1 < 1$. Assume that $(\Phi_1) - (\Phi_2)$ holds true. Then, $W^{s_1}L_{\phi_{x,y}}(\Omega)$ continuously embedded in $W^{s_2}L_{\phi_{x,y}}(\Omega)$.

Proof: Let $u \in W^{s_1}L_{\phi_{x,y}}(\Omega)$ and $\lambda > 0$. If we define

\begin{align*}
\Omega_1 = \left\{(x,y) \subset \Omega \times \Omega : \frac{|D^s_i u|}{\lambda} \leq 1 \right\} \quad \text{and} \quad \Omega_2 = \Omega \times \Omega \setminus \Omega_1,
\end{align*}

where $D^s_i u = \frac{u(x) - u(y)}{|x-y|^s}$, $i = 1, 2$. Then

\begin{align*}
\int_\Omega \int_\Omega \Phi_{x,y} \left( \frac{|D^{s_2} u|}{\lambda} \right) \frac{1}{|x-y|^{N}} \, dx \, dy &= \int_\Omega \int_\Omega \Phi_{x,y} \left( \frac{|D^{s_1} u|}{\lambda} \right) \frac{1}{|x-y|^{N}} \, dx \, dy \\
&\leq \int_\Omega \int_\Omega \Phi_{x,y} \left( \frac{|D^{s_1} u|}{\lambda} \right) \frac{1}{|x-y|^{N+p(s_2-s_1)}} \, dx \, dy.
\end{align*}
\[
\left( \int_{\Omega_1} \int_{\Omega_2} + \int_{\Omega_2} \int_{\Omega_1} \right) \chi_{\Omega_1} \left( \frac{|D^{s_1} u|}{\lambda} \right) \frac{dx \, dy}{|x - y|^{N + p(s_2 - s_1)}} = I_{\Omega_1} + I_{\Omega_2},
\]

where \( p = 1 \) if \( |x - y| \geq 1 \) or \( p = \varphi^+ \) if \( |x - y| < 1 \) for all \((x, y) \in \Omega \times \Omega\). Notice that

\[
I_{\Omega_1} = \int_{\Omega_1} \int_{\Omega_1} \chi_{\Omega_1} \left( \frac{|D^{s_1} u|}{\lambda} \right) \frac{dx \, dy}{|x - y|^{N + p(s_2 - s_1)}} \leq \sup_{(x, y) \in \overline{\Omega} \times \overline{\Omega}} \chi_{\Omega_1} \left( \frac{|D^{s_1} u|}{\lambda} \right) \int_{\Omega_1} \int_{\Omega_1} \frac{dx \, dy}{|x - y|^{N + p(s_2 - s_1)}} := c_1.
\]

Since \( N > N + p(s_2 - s_1) \), it follows that the above integral is finite. On the other hand, we have

\[
I_{\Omega_2} \leq \int_{\Omega_2} \int_{\Omega_2} \chi_{\Omega_2} \left( \frac{|D^{s_1} u|}{\lambda} \right) \frac{dx \, dy}{|x - y|^{N + p(s_2 - s_1)}} \leq d^{p(s_1 - s_2)} \int_{\Omega_2} \int_{\Omega_2} \chi_{\Omega_2} \left( \frac{|D^{s_1} u|}{\lambda} \right) \frac{dx \, dy}{|x - y|^N} \leq d^{p(s_1 - s_2)} \int_{\Omega_2} \int_{\Omega_2} \chi_{\Omega_2} \left( \frac{|D^{s_1} u|}{\lambda} \right) \frac{dx \, dy}{|x - y|^N} < \infty,
\]

where \( d = \sup_{(x, y) \in \overline{\Omega} \times \overline{\Omega}} |x - y| \). Hence,

\[
\int_{\Omega} \int_{\Omega} \chi_{\Omega} \left( \frac{|D^{s_2} u|}{|u|_{\partial \Omega, \chi_{\Omega}}} \right) \frac{dx \, dy}{|x - y|^N} \leq c_1 + d^{p(s_1 - s_2)}.
\]

This fact implies that

\[
[u]_{\partial \Omega, \chi_{\Omega}} \leq (c_1 + d^{p(s_1 - s_2)}) [u]_{\partial \Omega, \chi_{\Omega}},
\]

Therefore,

\[
||u||_{\partial \Omega, \chi_{\Omega}} \leq c ||u||_{\partial \Omega, \chi_{\Omega}},
\]

where \( c = (1 + c_1 + d^{p(s_1 - s_2)}) \).

Given \( s \in (0, 1) \) and let \( \hat{\Phi}_x \) as defined in (1). We denote by \( \hat{\Phi}_x^{-1} \) the inverse function of \( \hat{\Phi}_x \) which satisfies the following conditions:

\[
\int_0^1 \frac{\hat{\Phi}_x^{-1}(\tau)}{\tau^{N+2}} \frac{d\tau}{N} < \infty \quad \text{for all } x \in \overline{\Omega},
\]

\[
\int_1^\infty \frac{\hat{\Phi}_x^{-1}(\tau)}{\tau^{N+2}} \frac{d\tau}{N} = \infty \quad \text{for all } x \in \overline{\Omega}.
\]

Note that, if \( \varphi_{x,y}(t) = |t|^{p(x,y) - 1} \), then (16) holds precisely when \( sp(x,y) < N \) for all \((x, y) \in \overline{\Omega} \times \overline{\Omega}\).

If (17) is satisfied, we define the inverse Musielak conjugate function of \( \hat{\Phi}_x \) as follows

\[
(\hat{\Phi}_{x,s}^*)^{-1}(t) = \int_0^t \frac{\hat{\Phi}_x^{-1}(\tau)}{\tau^{N+2}} \frac{d\tau}{N}.
\]
Theorem 2.1: Let \( \Omega \) be a bounded open subset of \( \mathbb{R}^N \) with \( C^{0,1} \)-regularity and bounded boundary. If (16) and (17) hold, then
\[
W^s L_{\Phi x, y}(\Omega) \hookrightarrow L_{\hat{\Phi}_{x,s}^{*}}(\Omega).
\] (19)

Theorem 2.2: Let \( \Omega \) be a bounded open subset of \( \mathbb{R}^N \) and \( C^{0,1} \)-regularity with bounded boundary. If (16) and (17) hold, then the embedding
\[
W^s L_{\Phi x, y}(\Omega) \hookrightarrow L_{Bx}(\Omega)
\] (20)
is compact for all \( B_x \ll \hat{\Phi}_{x,s}^{*} \).

The proof will be carried out in a several lemmas. The first one establishes an estimate for the Musielak conjugate function \( \hat{\Phi}_{x,s}^{*} \) defined in (18).

Lemma 2.1: Let \( s \in (0, 1) \), we assume that (16) and (17) hold true and let \( \hat{\Phi}_{x,s}^{*} \) be as defined in (18). Then for all \( s' \in (0, s) \), the following conclusions may be drawn.

1. \[
\left[ \hat{\Phi}_{x,s}^{*}(t) \right]^\frac{N-s'}{N} \text{ is a Musielak function, in particular, } \hat{\Phi}_{x,s}^{*} \text{ is a Musielak function.}
\]
2. For every \( \varepsilon > 0 \), there exists a constant \( K_\varepsilon > 0 \) such that for every \( t \),
\[
\left[ \hat{\Phi}_{x,s}^{*}(t) \right]^\frac{N-s'}{N} \leq \frac{1}{2\varepsilon} \hat{\Phi}_{x,s}^{*}(t) + \frac{K_\varepsilon}{\varepsilon} t.
\] (21)

Proof: The proof is similar to [24, Lemma 3]. ■

Lemma 2.2: Let \( \Omega \) be an open subset of \( \mathbb{R}^N \), and \( 0 < s < 1 \). Let \( f \) satisfies the Lipschitz-condition on \( \mathbb{R} \) and \( f(0) = 0 \), then,

1. For every \( u \in W^{s,1}_{loc}(\Omega) \), if \( g(x) = f(|u(x)|) \), then \( g \in W^{s,1}_{loc}(\Omega) \).
2. For every \( u \in W^s L_{\Phi x, y}(\Omega) \), if \( g(x) = f(|u(x)|) \), then \( g \in W^s L_{\Phi x, y}(\Omega) \).

In particular, for every \( u \in W^{s,1}(\Omega) \), if \( g(x) = f(|u(x)|) \), then \( g \in W^{s,1}(\Omega) \).

Proof: To prove Lemma 2.2, we follow the same approach as in [24, Lemma 4]. ■

Lemma 2.3: Let \( \Omega \) be a bounded open subset of \( \mathbb{R}^N \) and let \( 0 < s' < s < 1 \). Assume that the condition \( (\Phi_1) \) and \( (\Phi_2) \) are satisfied, then the space \( W^s L_{\Phi x, y}(\Omega) \) is continuously embedded in \( W^{s',q}(\Omega) \) for all \( q \in [1, \varphi^-] \).

Proof: By (3) there exist \( c_1 > 0 \) such that
\[
|t|^{\varphi^-} \leq c_1 \hat{\Phi}_x(t) t > 1 \quad \text{for all } x \in \overline{\Omega}.
\] (22)

Indeed, form (3) we have
\[
\varphi^- \leq \frac{f(\hat{\Phi}_x(t))}{\hat{\Phi}_x(t)} \quad \forall \ t > 1,
\]
so,
\[
\varphi^- [\ln(t)]' \leq [\ln(\hat{\Phi}_x(t))]' \quad \forall \ t > 1,
\]
Hence, assume the value unity for some positive value decreases continuously from infinity to zero as

\[ \int_\Omega [u(x)]^{\varphi^-} \, dx \leq \int_\Omega \left[ \int_\Omega \left| \frac{u(x)}{\Phi_x(x)} \right| \, dx \right]^{\varphi^-} \, dx \leq \int_\Omega \left[ \frac{1}{\Phi_x(x)} \right] \, dx. \]

where \( c_1 = \frac{1}{\inf_{x \in \Omega} \Phi_x(x)} \), note that by the definition of \( \Phi_x \) and \((\Phi)\), \( 0 < c_1 < \infty \).

Then, for \( u \in W^{s,L}_{x,y} (\Omega) \), we have

\[ \int_\Omega [u(x)]^{\varphi^-} \, dx \leq \int_\Omega \left[ \int_\Omega \left| \frac{u(x)}{\Phi_x(x)} \right| \, dx \right]^{\varphi^-} \, dx \]

\[ \leq |\Omega| + c_1 \int_\Omega \Phi_x(|u(x)|) \, dx. \]

Hence,

\[ ||u||_{\varphi^-} \leq c_2 ||u||_{\Phi_x}, \]  

where \( c_2 = |\Omega| + c_1 \). On the other hand, similar to Proposition 2.1 we have

\[ [u]_{\varphi^-} \leq c_3 [u]_{5,\Phi_x,y}. \]

Then, combining (23) with (24), we obtain

\[ ||u||_{\varphi^-} \leq c_4 [u]_{5,\Phi_x,y}, \]

where \( c_4 = \max\{c_2, c_3\} \). This completes the proof.

**Proof of Theorem 2.1**: Let \( 0 < \sigma' < s < 1 \), \( \sigma (t) = [\Phi_{x,s}^\ast (t)]^{N-\sigma'} \) and \( u \in W^{s,L}_{x,y} (\Omega) \), we suppose for the moment that \( u \) is bounded on \( \Omega \) and not equal to zero in \( L_{x,s} (\Omega) \), then \( \int_\Omega \Phi_{x,s}^\ast \left( \frac{|u(x)|}{k} \right) \, dx \)

decreases continuously from infinity to zero as \( \lambda \) increases from zero to infinity and accordingly assumes the value unity for some positive value \( k \) of \( \lambda \), thus

\[ \int_\Omega \Phi_{x,s}^\ast \left( \frac{|u(x)|}{k} \right) \, dx = 1, \quad \text{and} \quad k = ||u||_{\Phi_{x,s}^\ast}. \]  

Let \( f(x) = \sigma \left( \frac{|u(x)|}{k} \right) \). Using Lemma 2.1, we have \( u \in W^{s',1} (\Omega) \) and \( \sigma \) is a Lipschitz function, so from Lemma 2.2, we have \( f \in W^{s',1} (\Omega) \). Moreover, since \( N > s' \), then by [16, Theroem 4.58], one has

\[ W^{s',1} (\Omega) \hookrightarrow L^{\frac{N}{N-s'}} (\Omega). \]

Hence

\[ ||f||_{L^{\frac{N}{N-s'}}} \leq k_1 \left( ||f||_{L^1} + [f]_{s',1} \right), \]

and by (25), we get

\[ 1 = \left( \int_\Omega \Phi_{x,s}^\ast \left( \frac{|u(x)|}{k} \right) \, dx \right)^{\frac{N-\sigma'}{N}} = ||f||_{L^{\frac{N}{N-s'}}}. \]

This implies that

\[ 1 \leq k_1 \left( ||f||_{L^1} + [f]_{s',1} \right) \]

\[ = k_1 \left( \int_\Omega \sigma \left( \frac{|u(x)|}{k} \right) \, dx + \int_\Omega \int_\Omega \frac{|f(x) - f(y)|}{|x-y|^{N+s'}} \, dx \, dy \right). \]
Next, by Lemma 2.1, we have

\[ k_1 I_1 = \frac{1}{2} \int_\Omega \sigma \left( \frac{|u(x)|}{k} \right) \, dx + \int_\Omega \int_\Omega \frac{\sigma \left( \frac{u(x)}{k} \right) - \sigma \left( \frac{u(y)}{k} \right)}{|x-y|^{N+s'}} \, dx \, dy \]

\[ = k_1 I_1 + k_1 I_2. \tag{26} \]

From (21) and since \( L_{\hat{\Phi}_s}(\Omega) \hookrightarrow L^1(\Omega) \), then for \( \varepsilon = k_1 \), we have

\[ k_1 I_1 \leq \frac{1}{2} \int_\Omega \hat{\Phi}_{s,s} \left( \frac{|u(x)|}{k} \right) \, dx + \frac{k_2}{k} \int_\Omega |u(x)| \, dx \leq \frac{1}{2} + \frac{k_2}{k} ||u|| \hat{\Phi}_s. \tag{27} \]

On the other hand, as \( \sigma \) is a Lipschitz function, then there exists \( c_1 > 0 \) such that

\[ k_1 I_2 \leq \frac{k_1 c_1}{k} \int_\Omega \int_\Omega \frac{|u(x) - u(y)|}{|x-y|^{N+s'}} \, dx \, dy. \]

Next, by Lemma 2.1, we have

\[ \int_\Omega \int_\Omega \frac{|u(x) - u(y)|}{|x-y|^{N+s'}} \, dx \, dy \leq c_2 [u]_{s,\Phi_{x,y}}, \tag{28} \]

thus,

\[ k_1 I_2 \leq \frac{k_1 c_1 c_2}{k} [u]_{s,\Phi_{x,y}}. \tag{29} \]

We pose \( k_3 = k_1 c_1 c_2 \). Combining (27)–(29), we obtain

\[ 1 \leq \frac{1}{2} + \frac{k_2}{k} ||u|| \hat{\Phi}_s + \frac{k_3}{k} [u]_{s,\Phi_{x,y}}, \]

this implies that,

\[ \frac{k}{2} \leq k_2' ||u|| \hat{\Phi}_s + k_3 [u]_{s,\Phi_{x,y}}. \]

Hence, we obtain,

\[ k = ||u|| \hat{\Phi}_{s,s} \leq k_4 ||u||_{s,\Phi_{x,y}}, \]

where \( k_4 = \max\{2k_2', 2k_3\} \).

Now, for \( u \in W^s L_{\Phi_{x,y}}(\Omega) \) arbitrary, we define

\[ u_n(x) = \begin{cases} u(x) & \text{if } |u(x)| \leq n, \\ n \text{ sgn } u(x) & \text{if } |u(x)| > n. \end{cases} \]

Then, \( \{u_n\} \) is bounded and by Lemma 2.2, \( u_n \in W^s L_{\Phi_{x,y}}(\Omega) \). Moreover

\[ ||u_n||_{\hat{\Phi}_{s,s}} \leq k_4 ||u_n||_{s,\Phi_{x,y}} \leq k_4 ||u||_{s,\Phi_{x,y}}. \]

Let \( \lim_{n \to \infty} ||u_n||_{\hat{\Phi}_{s,s}} = k \), then \( k \leq k_4 ||u||_{s,\Phi_{x,y}} \). Using the Fatou’s Lemma, we get

\[ \int_\Omega \hat{\Phi}_{s,s} \left( \frac{|u(x)|}{k} \right) \, dx \leq \lim_{n \to \infty} \int_\Omega \hat{\Phi}_{s,s} \left( \frac{|u_n(x)|}{||u_n||_{\hat{\Phi}_{s,s}}} \right) \, dx < 1. \]

Consequently, \( u \in L_{\hat{\Phi}_{x,y}}(\Omega) \) and \( ||u||_{\hat{\Phi}_{s,s}} \leq k \leq k_4 ||u||_{s,\Phi_{x,y}} \). \[ \blacksquare \]
Proof of Theorem 2.2: From Lemma 2.3, we get

$$W^{s}L_{\Phi_{s,y}}(\Omega) \hookrightarrow W^{s',1}(\Omega) \hookrightarrow L^{1}(\Omega).$$

By [16, Theroem 4.58], the latter embedding being compact. Since a bounded subset $S$ of $W^{s}L_{\Phi_{s,y}}(\Omega)$ is also a bounded subset of $L_{\Phi_{s,y}}(\Omega)$ and precompact in $L^{1}(\Omega)$. Hence, by Theorem 1.1 it is precompact in $L_{\mathcal{B}_{k}}(\Omega)$. $lacksquare$

Now, combining Lemma 2.3 and [16, Theroem 4.58], we obtain the following results.

Corollary 2.1: Let $\Omega$ be a bounded open subset of $\mathbb{R}^{N}$ with $C^{0,1}$-regularity and bounded boundary. Let $0 < s' < s < 1$ and let $\Phi_{s,y}$ be a Musielak function satisfies the condition $(\Phi_{1})$, we define

$$\varphi_{s}^{+} = \begin{cases} \frac{N\varphi}{N - s\varphi} & \text{if } N > s'\varphi \\ \infty & \text{if } N \leq s'\varphi. \end{cases}$$

- If $s'\varphi < N$, then $W^{s}L_{\Phi_{s,y}}(\Omega) \hookrightarrow L^{q}(\Omega)$, for all $q \in [1, \varphi_{s}^{+}]$ and the embedding $W^{s}L_{\Phi_{s,y}}(\Omega) \hookrightarrow L^{q}(\Omega)$ is compact for all $q \in [1, \varphi_{s}^{+})$.
- If $s'\varphi = N$, then $W^{s}L_{\Phi_{s,y}}(\Omega) \hookrightarrow L^{q}(\Omega)$, for all $q \in [1, \infty]$ and the embedding $W^{s}L_{\Phi_{s,y}}(\Omega) \hookrightarrow L^{q}(\Omega)$ is compact for all $q \in [1, \infty)$.
- If $s'\varphi > N$, then the embedding $W^{s}L_{\Phi_{s,y}}(\Omega) \hookrightarrow L^{\infty}(\Omega)$ is compact.

Next, we introduce a closed linear subspace of $W^{s}L_{\Phi_{s,y}}(\Omega)$ as follows

$$W^{s}_{0}\Phi_{s,y}(\Omega) = \left\{ u \in W^{s}L_{\Phi_{s,y}}(\mathbb{R}^{N}); u = 0 \text{ a.e in } \mathbb{R}^{N} \setminus \Omega \right\}.$$ 

Then, we have the following generalized Poincaré type inequality.

Theorem 2.3: Let $\Omega$ be a bounded open subset of $\mathbb{R}^{N}$ with $C^{0,1}$-regularity and bounded boundary, let $s \in (0, 1)$. Then there exists a positive constant $\gamma$ such that

$$||u||_{\Phi_{s}} \leq \gamma ||u||_{s,\Phi_{s,y}} \text{ for all } u \in W^{s}_{0}\Phi_{s,y}(\Omega).$$

Proof: Let $\tilde{\sigma}(t) = (\tilde{\Phi}_{s}(t))^{\frac{N-s'}{N}}$ where $0 < s < s' < 1$ and $u \in W^{s}_{0}\Phi_{s,y}(\Omega)$, then similarly to Lemma 2.2, for $f(x) = \tilde{\sigma}(\frac{u(x)}{k})$ with $k = ||u||_{\Phi_{s,y}}$, we have $f \in W^{s',1}_{0}(\Omega)$, and since $N > s'$, then by [16, Theroem 4.58], one has

$$W^{s',1}_{0}(\Omega) \hookrightarrow L^{\frac{N}{N-s'}}(\Omega).$$

It follows that

$$||f||_{L^{\frac{N}{N-s'}}(\Omega)} \leq k_{5}||f||_{s',1},$$

and

$$1 = \left( \int_{\Omega} (\tilde{\Phi}_{s}(\frac{|u(x)|}{k}))^{\frac{N-s'}{N}} \ dx \right)^{\frac{N}{N-s'}} = ||f||_{L^{\frac{N}{N-s'}}(\Omega)},$$

this implies that

$$1 \leq k_{5}||f||_{s',1}$$

$$= k_{5} \int_{\Omega} \int_{\Omega} \frac{|f(x) - f(y)|}{|x - y|^{N+s'}} \ dx \ dy.$$
From Theorem 2.3, we deduce that
\[
\int \int \frac{\sigma \left( \frac{u(x)}{k} \right) - \sigma \left( \frac{u(y)}{k} \right)}{|x-y|^{N+s'}} \, dx \, dy
\]
\[
\leq \frac{k \sigma}{k} \int \int \frac{|u(x) - u(y)|}{|x-y|^{N+s'}} \, dx \, dy
\]
\[
\leq \frac{k \sigma}{k} |u|_{s, \nu_{x,y}}.
\]
Thus
\[
||u||_{s, \nu_{x,y}} \leq \gamma |u|_{s, \nu_{x,y}}
\]
where \( \gamma = k \sigma. \)

**Remark 2.1:**

(a) As a trivial consequence of Theorem 2.3, for a bounded open subset \( \Omega \) of \( \mathbb{R}^N \), there exists a positive constant \( \lambda_1 \) such that,
\[
\int \int \phi(x) |u(x)| \, dx \leq \lambda_1 \int \int \phi_{s,x,y} \left( \frac{|u(x) - u(y)|}{|x-y|^s} \right) \, dx \, dy
\]
for all \( u \in W_0^s L_{\phi_{s,x,y}}(\Omega). \)

(b) From Theorem 2.3, we deduce that \([u]_{s, \phi_{s,x,y}}\) is a norm on \( W_0^s L_{\phi_{s,x,y}}(\Omega) \) which is equivalent to the norm \( \|u\|_{s, \phi_{s,x,y}}\).

### 2.2. Extending a \( W^s L_{\phi_{s,x,y}}(\Omega) \) function to the whole of \( \mathbb{R}^N \)

As usual, for any \( k \in \mathbb{N} \) and \( \alpha \in (0, 1] \), we say that \( \Omega \) is of class \( C^{k,\alpha} \) if there exists \( M > 0 \) such that for any \( x \in \partial \Omega \) there exists a ball \( B = B_r(x), r > 0 \), and an isomorphism \( T : Q \to B \) such that
\[
T \in C^{k,\alpha}(Q), \quad T^{-1} \in C^{k,\alpha}(B), \quad T(Q_+) = B \cap \Omega, \quad T(Q_0) = B \cap \partial \Omega
\]
and
\[
||T||_{C^{k,\alpha}(Q)} + ||T^{-1}||_{C^{k,\alpha}(B)} \leq M,
\]
where
\[
Q := \{ x = (x', x_N) \in \mathbb{R}^{N-1} \times \mathbb{R} : |x'| < 1 \text{ and } |x_N| < 1 \},
\]
\[
Q_+ := \{ x = (x', x_N) \in \mathbb{R}^{N-1} \times \mathbb{R} : |x'| < 1 \text{ and } 0 < x_N < 1 \},
\]
and
\[
Q_0 := \{ x \in Q : x_N = 0 \}.
\]

Given an open bounded domain \( \Omega \in \mathbb{R}^N \), for any \( s \in (0, 1) \) and any Musielak function \( \phi_{s,x,y} \), we say that an open set \( \Omega \) is an extension domains for \( W^s L_{\phi_{s,x,y}}(\Omega) \) if for every function \( u \in W^s L_{\phi_{s,x,y}}(\Omega) \) there exists \( \tilde{u} \in W^s L_{\phi_{s,x,y}}(\mathbb{R}^N) \) with \( \tilde{u}(x) = u(x) \) for all \( x \) in \( \Omega \).

Hence, our aim in this subsection is to show that any open bounded set \( \Omega \) of class \( C^{0,1} \) with bounded boundary is an extension domain for \( W^s L_{\phi_{s,x,y}} \). To this end, we follow the same approach as in [10,49].

In this subsection, we assume that \( \phi_{s,x,y} \in B_f \) and
\[
\lim_{t \to \infty} \frac{|t|^{s'}}{\phi_{s,x,y}(kt)} = 0 \quad \forall \, k > 0.
\]

We start with some preliminary lemmas, in which we will construct the extension to the whole of \( \mathbb{R}^N \) of a function \( u \) defined on \( \Omega \) in two separated cases: when the function \( u \) is identically zero in a neighborhood of the boundary \( \partial \Omega \) and when \( \Omega \) coincides with the half-space \( \mathbb{R}^N_+ \).
**Lemma 2.4:** Assume that \((\Phi_1)\)–(\(\Phi_3\)) holds. Then for all \(u \in W^sL_{\Phi_{x,y}}(\Omega)\)

\[
\int_{\Omega} \Phi_{x,y}(|u(x)|) \, dx < \infty \quad \forall \ y \in \overline{\Omega}.
\]

In particular, for every \(u \in W^sL_{\Phi_{x,y}}(\Omega)\), we have

\[
\int_{\Omega} \sup_{y \in \overline{\Omega}} \Phi_{x,y}(|u(x)|) \, dx < \infty.
\]

**Proof:** First, seminary to (22), we have

\[
\Phi_{x,y}(t) \leq |t|^\psi^+, \quad \forall \ (x, y) \in \overline{\Omega} \times \overline{\Omega}, \quad \forall \ t > 1.
\]

Then,

\[
\int_{\Omega} \Phi_{x,y}(|u(x)|) \, dx \leq \int_{\Omega} \Phi_{x,y}(1) \, dx + \int_{\Omega} |u(x)|^{\psi^+} \, dx
\]

\[
\leq \sup_{(x,y) \in \overline{\Omega} \times \overline{\Omega}} \Phi_{x,y}(1) |\Omega| + \|u\|^{\psi^+}_{\psi^+}.
\]

On the other hand, by \((\Phi_3)\) we can use Theorem 2.2 and we have

\[
\|u\|^{\psi^+} \leq c\|u\|_{s, \Phi_{x,y}}.
\]

So,

\[
\int_{\Omega} \Phi_{x,y}(|u(x)|) \, dx \leq \sup_{(x,y) \in \overline{\Omega} \times \overline{\Omega}} \Phi_{x,y}(1) |\Omega| + \|u\|^{\psi^+}_{\psi^+}
\]

\[
\leq \sup_{(x,y) \in \overline{\Omega} \times \overline{\Omega}} \Phi_{x,y}(1) |\Omega| + c\|u\|^{\psi^+}_{s, \Phi_{x,y}}
\]

\[
< \infty. \quad \blacksquare
\]

**Lemma 2.5:** Let \(\Omega\) be an open bounded subset of \(\mathbb{R}^N\), \(s \in (0,1)\), and \(u \in W^sL_{\Phi_{x,y}}(\Omega)\). If there exists a compact subset \(K \subset \Omega\) such that \(u \equiv 0\) in \(\Omega \setminus K\), then the extension function \(\tilde{u}\) defined as

\[
\tilde{u}(x) = \begin{cases} 
 u(x) & \text{if } x \in \Omega, \\
 0 & \text{if } x \in \mathbb{R}^N \setminus \Omega,
\end{cases}
\]

belongs to \(W^sL_{\Phi_{x,y}}(\mathbb{R}^N)\).

**Proof:** Let \(u \in W^sL_{\Phi_{x,y}}(\Omega)\). So, clearly \(\tilde{u} \in L_{\Phi_{x}}(\mathbb{R}^N)\). On the other hand, since \(\tilde{u}\) in \(\mathbb{R}^N \setminus \Omega\), then for some \(\lambda > 0\), we have

\[
\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \Phi_{x,y} \left( \frac{|u(x) - \tilde{u}(y)|}{\lambda |x-y|^s} \right) \frac{dx \, dy}{|x-y|^N} = \int_{\Omega} \int_{\Omega} \Phi_{x,y} \left( \frac{|u(x) - \tilde{u}(y)|}{\lambda |x-y|^s} \right) \frac{dx \, dy}{|x-y|^N}
\]

\[
+ 2 \int_{\mathbb{R}^N \setminus K} \int_{\Omega} \Phi_{x,y} \left( \frac{|u(x)|}{\lambda |x-y|^s} \right) \frac{dx \, dy}{|x-y|^N},
\]

where the first term on the right-hand side is finite. Moreover, for any \(y \in \mathbb{R}^N \setminus K\) we have

\[
\int_{\mathbb{R}^N \setminus K} \int_{\Omega} \Phi_{x,y} \left( \frac{|u(x)|}{\lambda |x-y|^s} \right) \frac{dx \, dy}{|x-y|^N}
\]
\[
\leq \int_{\mathbb{R}^N \setminus \Omega} \int_{\Omega} \Phi_{x,y} \left( \frac{|u(x)|}{\lambda} \right) \frac{dx \, dy}{|x-y|^{sp+N}}
\]
\[
\leq \int_{\mathbb{R}^N \setminus \Omega} \int_{\Omega} \Phi_{x,y} \left( \frac{\mathbb{I}_K|u(x)|}{\lambda} \right) \frac{dx \, dy}{\text{dis}(y, \partial K)^{sp+N}}
\]
\[
\leq \int_{\mathbb{R}^N \setminus \Omega} \text{sup}_{y \in \Omega} \Phi_{x,y} \left( \frac{|u(x)|}{\lambda} \right) \frac{dy}{\text{dis}(y, \partial K)^{sp+N}},
\]

where \( p = 1 \) if \(|x - y| \geq 1 \) or \( p = \varphi^+ \) if \(|x - y| < 1 \) for some \((x, y) \in \Omega \times \Omega\). Since \( \text{dis}(\partial \Omega, \partial K) > 0 \) and \( N + sp > N \). Then, by Lemma 2.4

\[
\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \Phi_{x,y} \left( \frac{|\tilde{u}(x) - \tilde{u}(y)|}{\lambda |x-y|^s} \right) \frac{dx \, dy}{|x-y|^N} < \infty.
\]

\[\square\]

**Lemma 2.6**: Let \( s \in (0, 1) \) and let \( \Omega \) be an open subset of \( \mathbb{R}^N \), symmetric with respect to the coordinate \( x_N \), and consider the sets \( \Omega_+ = \{x \in \Omega : x_N > 0\} \) and \( \Omega_- = \{x \in \Omega : x_N \leq 0\} \). Let \( u \) be a function in \( W^s L_{\Phi_{x,y}}(\Omega_+) \), we define

\[
\tilde{u}(x) = \begin{cases} 
    u(x', x_N) & \text{if } x_N \geq 0, \\
    u(x', -x_N) & \text{if } x_N < 0.
\end{cases}
\]

Then \( \tilde{u} \) belongs to \( W^s L_{\Phi_{x,y}}(\Omega) \).

**Proof**: We set \( \tilde{x} = (x', -x_N) \), then by splitting the integrals, for some \( \lambda > 0 \), we get

\[
\int_{\Omega} \hat{\Phi}_x \left( \frac{|\tilde{u}(x)|}{\lambda} \right) \frac{dx}{\lambda} = \int_{\Omega_+} \hat{\Phi}_x \left( \frac{|u(x', x_N)|}{\lambda} \right) \frac{dx}{\lambda} + \int_{\Omega_-} \hat{\Phi}_x \left( \frac{|u(x', -x_N)|}{\lambda} \right) \frac{dx}{\lambda}
\]
\[
= \int_{\Omega_+} \hat{\Phi}_x \left( \frac{|u(x)|}{\lambda} \right) \frac{dx}{\lambda} + \int_{\Omega_+} \hat{\Phi}_x \left( \frac{|u(x', \tilde{x}_N)|}{\lambda} \right) \frac{dx}{\lambda}
\]
\[
= 2 \int_{\Omega_+} \hat{\Phi}_x \left( \frac{|u(x)|}{\lambda} \right) \frac{dx}{\lambda}
\]
\[
< \infty.
\]

On the other hand, for all \( \lambda > 0 \), we have

\[
\int_{\Omega} \int_{\mathbb{R}^N \setminus \Omega_+} \Phi_{x,y} \left( \frac{|u(x) - u(y') - \tilde{u}_N|}{\lambda |x-y'|^s} \right) \frac{dy}{|x-y'|^N} = \int_{\mathbb{R}^N \setminus \Omega} \int_{\mathbb{R}^N \setminus \Omega_+} \Phi_{x,y} \left( \frac{|u(x', x_N) - u(y', -y_N)|}{\lambda |x-y'|^s} \right) \frac{dy}{|x-y'|^N}
\]
\[
+ 2 \int_{\Omega} \int_{\mathbb{R}^N \setminus \Omega_+} \Phi_{x,y} \left( \frac{|u(x) - u(y') - \tilde{u}_N|}{\lambda |x-y'|^s} \right) \frac{dy}{|x-y'|^N} + \int_{\Omega} \int_{\mathbb{R}^N \setminus \Omega_+} \Phi_{x,y} \left( \frac{|u(x) - u(y') - \tilde{u}_N|}{\lambda |x-y'|^s} \right) \frac{dy}{|x-y'|^N}.
\]

By changing variable \( \tilde{x} = (x', -x_N) \) and \( \tilde{y} = (y', -y_N) \), we get

\[
\int_{\Omega} \int_{\mathbb{R}^N \setminus \Omega_+} \Phi_{x,y} \left( \frac{|\tilde{u}(x) - \tilde{u}(y)|}{\lambda |x-y'|^s} \right) \frac{dx \, dy}{|x-y'|^N} = 4 \int_{\Omega_+} \int_{\mathbb{R}^N \setminus \Omega_+} \Phi_{x,y} \left( \frac{|u(x) - u(y)|}{\lambda |x-y'|^s} \right) \frac{dx \, dy}{|x-y'|^N}
\]
\[
< \infty.
\]

This concludes the proof. \[\square\]
**Lemma 2.7:** Let $\Omega$ be an open subset of $\mathbb{R}^N$ and $s \in (0, 1)$. Let $u \in W^s L_{\Phi_{x,y}}(\Omega)$ and $\psi \in C^{0,1}(\Omega)$, $0 \leq \psi \leq 1$. Then $\psi u \in W^s L_{\Phi_{x,y}}(\Omega)$.

**Proof:** Let $u \in W^s L_{\Phi_{x,y}}(\Omega)$. Then, it is clear that $\psi u \in L_{\Phi_{x,y}}(\Omega)$. In addition, since $|\psi| \leq 1$, it follows that $||\psi u||_{L_{\Phi_{x,y}}(\Omega)} \leq ||u||_{L_{\Phi_{x,y}}(\Omega)}$. On the other hand, adding and subtracting the factor $\psi(x)u(y)$, we have

\[
\int_{\Omega} \int_{\Omega} \Phi_{x,y} \left( \frac{|\psi(x)u(x) - \psi(y)u(y)|}{\lambda|y-x|^s} \right) \frac{dx}{|x-y|^N} \cdot \\
\leq \int_{\Omega} \int_{\Omega} \Phi_{x,y} \left( \frac{2|\psi(x)u(x) - \psi(y)u(y)|}{\lambda|y-x|^s} \right) \frac{dx}{|x-y|^N} \\
\leq \frac{2^\varphi^+ - 1}{2} \int_{\Omega} \int_{\Omega} \Phi_{x,y} \left( \frac{|u(x) - u(y)|}{\lambda|y-x|^s} \right) \frac{dx}{|x-y|^N} \\
+ \int_{\Omega} \int_{\Omega} \Phi_{x,y} \left( \frac{|u(y)(\psi(x) - \psi(y))|}{\lambda|y-x|^s} \right) \frac{dx}{|x-y|^N}.
\]

Since $\psi \in C^{0,1}(\Omega)$, we obtain

\[
\int_{\Omega} \int_{\Omega} \Phi_{x,y} \left( \frac{|u(y)(\psi(x) - \psi(y))|}{\lambda|y-x|^s} \right) \frac{dx}{|x-y|^N} \\
= \int_{\Omega} \int_{\Omega \cap \{|x-y| \geq 1\}} \Phi_{x,y} \left( \frac{|u(y)(\psi(x) - \psi(y))|}{\lambda|y-x|^s} \right) \frac{dx}{|x-y|^N} \\
+ \int_{\Omega} \int_{\Omega \cap \{|x-y| \leq 1\}} \Phi_{x,y} \left( \frac{|u(y)(\psi(x) - \psi(y))|}{\lambda|y-x|^s} \right) \frac{dx}{|x-y|^N} \\
\leq \int_{\Omega} \int_{\Omega \cap \{|x-y| \geq 1\}} \Phi_{x,y} \left( \frac{|u(y)|}{\lambda|y-x|^s} \right) \frac{dx}{|x-y|^N} \\
+ \int_{\Omega} \int_{\Omega \cap \{|x-y| \leq 1\}} \Phi_{x,y} \left( \frac{L|u(y)||x-y|}{\lambda|y-x|^s} \right) \frac{dx}{|x-y|^N} \\
\leq C \int_{\Omega} \Phi_{x,y} \left( \frac{|u(y)|}{\lambda} \right) dy,
\]

where $L$ denotes the Lipschitz constant of $\psi$ and

\[
C = \int_{\Omega \cap \{|z| \geq 1\}} \frac{1}{|z|^{s+N}} dz + \int_{\Omega \cap \{|z| \leq 1\}} \frac{L^\alpha}{|z|^{(s-1)+N}} dz
\]

with $\alpha = 1$ if $L < 1$ and $\alpha = \varphi^+$ if $L > 1$. Note that the two above integrals are finite. In fact, the kernel $|z|^{-s+N}$ is summable when $|z| \geq 1$ since $N + s > N$. Moreover, as $(s-1) + N < N$, then the kernel $|z|^{-(s-1)+N}$ is summable when $|z| \leq 1$. Therefore $\psi u \in W^s L_{\Phi_{x,y}}(\Omega)$.

Now, we are in position to state and prove the main theorem of this subsection.
**Theorem 2.4:** Let \( \Omega \) be an open subset of \( \mathbb{R}^N \) with \( C^{0,1} \)-regularity and bounded boundary. Then, \( W^sL_{\phi_{x,y}}(\Omega) \) is continuously embedded in \( W^sL_{\phi_{x,y}}(\mathbb{R}^N) \), namely for any \( u \in W^sL_{\phi_{x,y}}(\Omega) \) there exists \( \tilde{u} \in W^sL_{\phi_{x,y}}(\mathbb{R}^N) \).

**Proof:** Since \( \partial \Omega \) is compact, we can find a finite number of balls \( B_j \) such that \( \partial \Omega \subset \bigcup_{j=1}^k B_j \) and so we can write \( \mathbb{R}^N = \bigcup_{j=1}^k B_j \cup (\mathbb{R}^N \setminus \partial \Omega) \).

If we consider this covering, there exists a partition of unity related to it, that is, there exist \( k + 1 \) smooth functions \( \psi_0, \psi_1, \ldots, \psi_k \) such that \( \text{supp}(\psi_0) \subset \mathbb{R}^N \), \( \text{supp}(\psi_j) \subset B_j \) for all \( j \in \{1, \ldots, k\} \), \( 0 \leq \psi_j \leq 1 \) for all \( j \in \{0, \ldots, k\} \), and \( \sum_{j=0}^k \psi_j = 1 \). Clearly,

\[
\tilde{u} = \sum_{j=0}^k \psi_j u.
\]

From Lemma 2.7, we know that \( \psi_0 u \in W^sL_{\phi_{x,y}}(\Omega) \). Furthermore, since \( \psi_0 u = 0 \) in a neighborhood of \( \partial \Omega \), we can extend it to the whole of \( \mathbb{R}^N \) by setting

\[
\tilde{\psi}_0 u(x) = \psi_0 u(x) \quad \text{if } x \in \Omega,
\]

\[
0 \quad \text{if } x \in \mathbb{R}^N \setminus \Omega.
\]

and \( \tilde{\psi}_0 u \in W^sL_{\phi_{x,y}}(\Omega) \). For all \( j \in \{1, \ldots, k\} \), let us consider \( u|_{B_j \cap \Omega} \) and set

\[
v_j(y) := u(T_j(y)) \quad \text{for any } y \in Q_+,
\]

where \( T_j : Q \rightarrow B_j \) is the isomorphism of class \( C^{0,1}(\Omega) \) defined in (11)32. Note that such a \( T_j \) exists because \( \Omega \) is an open subset of class \( C^{0,1} \).

Next, we claim that \( v_j \in W^sL_{\phi_{x,y}}(Q_+) \). Indeed, using the standard changing variable formula by setting \( x = T_j(x) \), we obtain

\[
\int_{Q_+} \int_{Q_+} \Phi_{x,y} \left( \frac{|v_j(x) - v_j(y)|}{\lambda|x - y|^s} \right) \frac{d\tilde{x}d\tilde{y}}{|\tilde{x} - \tilde{y}|^N} = \int_{Q_+} \int_{Q_+} \Phi_{x,y} \left( \frac{|u(T_j(x)) - u(T_j(y))|}{\lambda|x - y|^s} \right) \frac{d\tilde{x}d\tilde{y}}{|\tilde{x} - \tilde{y}|^N}
\]

\[
= \int_{B_j \cap \Omega} \int_{B_j \cap \Omega} \Phi_{x,y} \left( \frac{|u(x) - u(y)|}{\lambda|T_j^{-1}(x) - T_j^{-1}(y)|^s} \right) \det(T_j^{-1}) \frac{dx dy}{|T_j^{-1}(x) - T_j^{-1}(y)|^N}
\]

\[
\leq \frac{\tilde{C}}{\int_{B_j \cap \Omega} \int_{B_j \cap \Omega} \Phi_{x,y} \left( \frac{|u(x) - u(y)|}{\lambda|x - y|^s} \right) \frac{d\tilde{x}d\tilde{y}}{|\tilde{x} - \tilde{y}|^N}.
\]

Hence \( v_j \in W^sL_{\phi_{x,y}}(Q_+) \). Moreover, using Lemma 2.6, we can extend \( v_j \) to all \( Q \), then the extension \( \tilde{v}_j \in W^sL_{\phi_{x,y}}(Q) \).

We set

\[
w_j(x) := \tilde{v}_j(T_j^{-1}(x)) \quad \text{for any } x \in B_j.
\]

By arguing as above, since \( T_j \) is bi-Lipschitz, it follows that \( w_j \in W^sL_{\phi_{x,y}}(B_j) \). Note that \( w_j \equiv u \) (and consequently \( \psi_j w_j \equiv \psi_j u \)) on \( B_j \cap \Omega \). By definition \( \psi_j w_j \) has compact support in \( B_j \) and therefore, as done for \( \psi_0 u \), we can consider the extension \( \tilde{\psi}_j w_j \) to all \( \mathbb{R}^N \) in such a way that \( \tilde{\psi}_j w_j \in W^sL_{\phi_{x,y}}(\mathbb{R}^N) \).
Finally, we set
\[ \tilde{u} = \tilde{\psi}_0 u + \sum_{j=1}^{k} \psi_j w_j \]
which is the extension of \( u \) defined on all \( \mathbb{R}^N \). By construction, it is clear that \( \tilde{u} |_{\Omega} = u \). \hfill \blacksquare

### 2.3. Complemented subspaces in \( W^sL_{\Phi_{x,y}}(\mathbb{R}^N) \)

Let \( (M, \| \cdot \|_M) \) be a Banach space of measurable functions on \( \mathbb{R}^N \) and \( A \) is a measurable subset of \( \mathbb{R}^N \) with positive Lebesgue measure. We define the trace space \( M|_A \) as follows
\[ M|_A := \{ u : A \rightarrow \mathbb{R} \text{ such that } \exists U \in M \text{ with } U|_A = u \text{ a.e.} \} . \]
This space is endowed with the norm
\[ \| u \|_{M|_A} = \inf \{ \| U \|_M : U \in M, U|_A = u \text{ a.e.} \} . \]
We denote by \( T U = U|_A \) the trace operator on \( A \). In particular, the aforementioned construction can be applied to the new fractional Musielak–Sobolev space \( W^sL_{\Phi_{x,y}}(\mathbb{R}^N) \).

Note that the definitions of \( \mathcal{L}^\Phi_\phi(\mathbb{R}^N) \) and \( W^sL_{\Phi_{x,y}}(\mathbb{R}^N) \) is analogous to \( L^\Phi_\phi(\Omega) \) and \( W^sL_{\Phi_{x,y}}(\Omega) \), one just replaces the occurrence of \( \Omega \) by \( \mathbb{R}^N \).

Next, let us consider the trace operator
\[ T : W^sL_{\Phi_{x,y}}(\mathbb{R}^N) \longrightarrow W^sL_{\Phi_{x,y}}(\Omega) \]
\[ \tilde{u} \longrightarrow T \tilde{u} = \tilde{u}|_{\Omega} = u . \]

**Remark 2.2:** In other words, an open domain \( \Omega \subset \mathbb{R}^N \) is a \( W^sL_{\Phi_{x,y}} \)-extension domain if there exists a continuous linear extension operator
\[ \mathcal{E} : W^sL_{\Phi_{x,y}}(\Omega) \longrightarrow W^sL_{\Phi_{x,y}}(\mathbb{R}^N) \]
\[ u \longrightarrow \mathcal{E} u = \tilde{u} . \]

**Definition 2.1:** A closed subspace \( Y \) of a Banach space \( X \) is complemented if there is another closed subspace \( Z \) of \( X \) such that \( X = Y \oplus Z \). That is, \( Y \cap Z = 0 \) and every element \( x \in X \) can be written as \( x = y + z \), with \( y \in Y \) and \( z \in Z \).

In this subsection, our result relates to the complemented subspace in \( W^sL_{\Phi_{x,y}}(\mathbb{R}^N) \). More precisely, we have the following theorem.

**Theorem 2.5:** Suppose that \( \Omega \subset \mathbb{R}^N \) is an open bounded set of class \( C^{0,1} \). Then
\[ W^sL_{\Phi_{x,y}}(\mathbb{R}^N) = \ker T \oplus \mathcal{E}(W^sL_{\Phi_{x,y}}(\Omega)) . \]

**Proof:** From Theorem 2.4 and Remark 2.2, we know that there exists a linear operator \( \mathcal{E} : W^sL_{\Phi_{x,y}}(\Omega) \longrightarrow W^sL_{\Phi_{x,y}}(\mathbb{R}^N) \) such that
1. \( \mathcal{E} u|_{\Omega} = u \),
2. \( \mathcal{E} u \in W^sL_{\Phi_{x,y}}(\mathbb{R}^N) \).
On the one hand, \( E(W^{s_0}L_{\Phi_{x,y}}(\Omega)) \) is a closed subspace of \( W^{s}L_{\Phi_{x,y}}(\mathbb{R}^N) \). On the other hand, for every element \( u \) in \( W^{s}L_{\Phi_{x,y}}(\mathbb{R}^N) \) can be written as \( u = u - E(T(u)) + E(T(u)) \), as \( u - E(T(u)) \in \ker T \) and \( E(T(u)) \in E(W^{s_0}L_{\Phi_{x,y}}(\Omega)) \). Moreover, \( \ker T \cap E(W^{s_0}L_{\Phi_{x,y}}(\Omega)) = 0 \), we deduce that
\[
W^{s}L_{\Phi_{x,y}}(\mathbb{R}^N) = \ker T \oplus E(W^{s_0}L_{\Phi_{x,y}}(\Omega)).
\]

### 3. Application to a nonlocal problem

In this section, we investigate the existence of weak solution for problem \((\mathcal{P}_a)\) in the new fractional Musielak–Sobolev space.

We start by considering the function \( \alpha = \mathbb{R} \rightarrow \mathbb{R} \) such that \( g : \mathbb{R} \rightarrow \mathbb{R} \) defined by
\[
g(t) = \begin{cases} 
\alpha(|t|)t & \text{for } t \neq 0, \\
0 & \text{for } t = 0,
\end{cases}
\]
which is an increasing homeomorphism from \( \mathbb{R} \) onto itself. We assume that \( f : \Omega \times \mathbb{R} \rightarrow \mathbb{R} \) is a Carathéodory function that satisfies the following condition
\[
|f(x,t)| \leq c_1 (1 + g(|t|)),
\]
where \( c_1 \) is a nonnegative constant. Furthermore, if we set
\[
G(t) = \int_0^t g(\tau) \, d\tau, \quad \overline{G}(t) = \int_0^t \overline{g}(\tau) \, d\tau
\]
with \( \overline{g}(t) = \sup\{s : g(s) \leq t\} \), then we obtain complementary Musielak function which define their corresponding Musielak spaces \( \overline{L}_G \) and \( \overline{L}_{\overline{G}} \).

Moreover, we assume that there exist two positive constants \( g^+ \) and \( g^- \) such that
\[
1 < g^- \leq \frac{tg(t)}{G(t)} \leq g^+ < +\infty.
\]
\[
\lim_{t \to -\infty} \frac{G(kt)}{\Phi^*_{s_0}(x,t)} = 0 \quad \text{for all } k > 0
\]
\[
\text{if } (x,t) \geq \theta F(x,t) \geq 0, \quad \text{for all } |t| \geq r \quad \text{and } \text{a.e. } x \in \Omega,
\]
where \( \theta > \varphi^+, r > 0 \) and \( F(x,t) := \int_0^t f(x,\tau) \, d\tau \).
\[
\limsup_{t \to 0} \frac{F(x,t)}{\Phi_{s_0}(t)} < \frac{1}{\lambda_1} \quad \text{uniformly for a.e. } x \in \Omega,
\]
where \( \lambda_1 \) is as in (31).

**Remark 3.1:** In Problem \((\mathcal{P}_a)\), since \( 0 < s_2 \leq s_1 < 1 \), then by Proposition 2.1, \( W^{s_0}_0L_{\Phi_{x,y}}(\Omega) \) is continuously embedded in \( W^{s_2}_0L_{\Phi_{x,y}}(\Omega) \). Thus a solution for a problem of type \((\mathcal{P}_a)\) will be sought in \( W^{s_1}_0L_{\Phi_{x,y}}(\Omega) \).

To simplify the notations, for any \( i = 1, 2 \), we set
\[
D^i u := \frac{u(x) - u(y)}{|x - y|^{s_i}}, \quad \Psi_{s_i}(u) := \int_\Omega \int_\Omega \Phi_{s_0}(\frac{|u(x) - u(y)|}{|x - y|^{s_i}}) \, dx \, dy.
\]
\[ d\mu = \frac{dx
dy}{|x - y|^N}, \quad \text{and} \quad ||u||_{s_i} := ||u||_{W_{0}^{s_i}L_{\Phi_{x,y}}(\Omega)}. \]

Next, we give the definition of a weak solution for problem \((P_a)\).

**Definition 3.1:** \( u \in W_{0}^{s_1}L_{\Phi_{x,y}}(\Omega) \) is called a weak solution of problem \((P_a)\), if
\[
\int_{\Omega} \int_{\Omega} a_{x,y}(|D^{s_1}u|)D^{s_1}u D^{s_1}v \, d\mu + \int_{\Omega} \int_{\Omega} a_{x,y}(|D^{s_2}u|)D^{s_2}u D^{s_2}v \, d\mu = \int_{\Omega} f(x,u)v \, dx, \tag{35}
\]

for all \( v \in W_{0}^{s_1}L_{\Phi_{x,y}}(\Omega) \).

Now, we are ready to state our existence result.

**Theorem 3.1:** Suppose that \((H_1)-(H_5)\) hold true. Then problem \((P_a)\) has a nontrivial weak solution \( u \in W_{0}^{s_1}L_{\Phi_{x,y}}(\Omega) \) which is a critical point of mountain pass type for the energy functional \( J: W_{0}^{s_1}L_{\Phi_{x,y}}(\Omega) \longrightarrow \mathbb{R} \) defined by
\[
J(u) = \Psi_{s_1}(u) + \Psi_{s_2}(u) - \int_{\Omega} F(x,u) \, dx. \tag{36}
\]

Let us denote by \( \Psi, I: W_{0}^{s_1}L_{\Phi}(\Omega) \longrightarrow \mathbb{R} \) the functionals
\[
\Psi(u) = \Psi_{s_1}(u) + \Psi_{s_2}(u) \quad \text{and} \quad I(u) = \int_{\Omega} F(x,u) \, dx.
\]

**Remark 3.2:** We note that the functional \( J: W_{0}^{s_1}L_{\Phi_{x,y}}(\Omega) \longrightarrow \mathbb{R} \) in (36) is well defined. Indeed, if \( u \in W_{0}^{s_1}L_{\Phi_{x,y}}(\Omega) \), then by Proposition 2.1, we have \( u \in W_{0}^{s_2}L_{\Phi_{x,y}}(\Omega) \), thus, \( \Psi(u) < \infty \). Moreover, by condition \((H_3)\), we have that \( u \in L_{G}(\Omega) \) and thus \( u \in L^1(\Omega) \). Hence, by the condition \((H_1)\)
\[
|F(x,u)| \leq \int_{0}^{u} |f(x,t)| \, dt = c_1(||u|| + G(||u||)).
\]

It follows that
\[
I(u) = \int_{\Omega} |F(x,u)| \, dx < \infty.
\]

By a standard argument (as in [24,26]) we have \( J \in C^1( W_{0}^{s_1}L_{\Phi_{x,y}}(\Omega), \mathbb{R} ) \) and its Gâteaux derivative is given by
\[
\langle J'(u), v \rangle = \int_{\Omega} \int_{\Omega} a_{x,y}(|D^{s_1}u|)D^{s_1}u D^{s_1}v \, d\mu + \int_{\Omega} \int_{\Omega} a_{x,y}(|D^{s_2}u|)D^{s_2}u D^{s_2}v \, d\mu
\]
\[
- \int_{\Omega} f(x,u)v \, dx,
\]

where \( \langle \,, \, \rangle \) denotes the usual duality between \( (W_{0}^{s_1}L_{\Phi_{x,y}}(\Omega), ||\,||_{s_1}) \) and its dual space \( ((W_{0}^{s_1}L_{\Phi_{x,y}}(\Omega))^*, ||\,||_{s_1,*}) \).

Next, we show an important lemma, namely that if the functional \( J \) defined in (36) satisfies the conclusion of Theorem 1.4, then it has a critical point.
Lemma 3.1: Let \((H_1)-(H_3)\) hold true. Let \(J\) be the functional defined in (36), and let \(\{u_n\}\) be a sequence in \(W_0^{2,1} L_{\phi_{x,y}}(\Omega)\) such that

\[
J(u_n) \to c_2 > 0, \quad \|J'(u_n)\|_{s,\omega} \to 0. \tag{37}
\]

Then there exists \(u \in W_0^{2,1} L_{\phi_{x,y}}(\Omega)\) such that

\[
J(u) = c_2, \quad J'(u) = 0.
\]

Proof: It follows from (37) that there exists \(c_3 > 0\) such that \(|J(u_n)| \leq c_3\) and \(\|J'(u_n), u_n\| \leq c_4\|u_n\|\).

By assumptions \((\Phi_1)\) and \((H_1)-(H_3)\), we have

\[
0 < t\phi_{x,y}(t) \leq \phi^+(\phi_{x,y})(t) \quad \text{for all } t > 0, \tag{38}
\]

\[
0 < t g(t) \leq g^+(G)(t) \quad \text{for all } t > 0, \tag{39}
\]

and

\[
\int_{\Omega \cap \{|u_n| \leq r\}} (F(x, u_n) - \theta^{-1} f(x, u_n) u_n) \, dx \leq c_1 [ (1 + \theta^{-1}) r + (1 + \theta^{-1}) g^+(G)(r) ] \leq c_5. \tag{40}
\]

Thus, by (38)–(40) and Proposition 1.3, we get

\[
c_3 + c_4\|u_n\| \geq J(u_n) - \frac{1}{\theta}\langle J'(u_n), u_n \rangle \\
\geq \Psi_{s_1}(u_n) + \Psi_{s_2}(u_n) - \frac{1}{\theta} \int_{\Omega} \int_{\Omega} \phi_{x,y} (D^{s_1} u_n) D^{s_1} u_n \, d\mu \\
- \frac{1}{\theta} \int_{\Omega} \int_{\Omega} \phi_{x,y} (D^{s_2} u_n) D^{s_2} u_n \, d\mu \\
- \int_{\Omega \cap \{|u_n| \leq r\}} (F(x, u_n) - \theta^{-1} f(x, u_n) u_n) \, dx \\
\geq \Psi_{s_1}(u_n) + \Psi_{s_2}(u_n) - \frac{\phi^+}{\theta} \Psi_{s_1}(u_n) - \frac{\phi^+}{\theta} \Psi_{s_2}(u_n) - c_5 \\
\geq \left(1 - \frac{\phi^+}{\theta}\right) (\Psi_{s_1}(u_n) + \Psi_{s_2}(u_n) - c_5 \\
\geq \left(1 - \frac{\phi^+}{\theta}\right) \Psi_{s_1}(u_n) - c_5 \\
\geq \left(1 - \frac{\phi^+}{\theta}\right) \|u_n\|_{\phi_{x,y}} - c_5. \tag{41}
\]

Hence, \(\{u_n\}\) is bounded in \(W_0^{2,1} L_{\phi_{x,y}}(\Omega)\). Since \(W_0^{2,1} L_{\phi_{x,y}}(\Omega)\) is a reflexive space, we may assume that \(u_n\) converges weakly to \(u\) in \(W_0^{2,1} L_{\phi}(\Omega)\). Further, since the embedding of \(W_0^{2,1} L_{\phi_{x,y}}(\Omega)\) into \(L_G(\Omega)\) is compact, we obtain that \(u_n \rightharpoonup u\) in \(L_G(\Omega)\).

Then, since \(I \in C^1(W_0^{2,1} L_{\phi_{x,y}}(\Omega), \mathbb{R})\), using \((H_1)-(H_3)\), we get \(\lim_{n \to \infty} I(u_n) = I(u)\) and \(\lim_{n \to \infty} J'(u_n) = J'(u)\) in \((W_0^{2,1} L_{\phi_{x,y}}(\Omega))^*\). Moreover, from (37), we have \(J'(u_n) \to 0\) in
\[(W_0^{1,2} L_{\phi_{x,y}}(\Omega))^*]. Hence,
\[
\Psi'(u_n) \longrightarrow I'(u) \quad \text{in} \quad (W_0^{1,2} L_{\phi_{x,y}}(\Omega))^*.
\] (42)

Now, since \(\Psi\) is convex, then we have
\[
\Psi(u_n) \leq \Psi(u) + \langle \Psi'(u_n), u_n - u \rangle.
\]

Therefore, using (42), we may deduce that
\[
\limsup_{n \to \infty} \Psi(u_n) \leq \Psi(u).
\]
It further follows from the convexity of \(\Psi\) that it is weakly lower semicontinuous and hence
\[
\liminf_{n \to \infty} \Psi(u_n) \geq \Psi(u),
\]
which implies that
\[
\lim_{n \to \infty} \Psi(u_n) = \Psi(u).
\]
Thus
\[
\lim_{n \to \infty} J(u_n) = J(u).
\]

Consequently, by the uniqueness of the limit, we deduce that, \(J'(u) = 0\). The convexity of \(\Psi\) implies that \(\Psi'\) is monotone and hence
\[
\langle \Psi'(u_n), u_n - v \rangle \geq \langle \Psi'(v), u_n - v \rangle \quad \text{for all} \quad v \in W_0^{1,2} L_{\phi_{x,y}}(\Omega).
\]

By (42), we have
\[
\langle I'(u) - \Psi'(v), u - v \rangle \geq 0 \quad \text{for all} \quad v \in W_0^{1,2} L_{\phi_{x,y}}(\Omega).
\]

Setting \(v = u - th\) with \(h \in W_0^{1,2} L_{\phi_{x,y}}(\Omega)\) and \(t \in \mathbb{R}^+\), then, we get
\[
\langle I'(u) - \Psi'(u - th), h \rangle \geq 0
\]
for all \(h \in W_0^{1,2} L_{\phi_{x,y}}(\Omega)\). Letting \(t \to 0\) and using the fact that \(h\) is arbitrary in \(W_0^{1,2} L_{\phi_{x,y}}(\Omega)\), we find that
\[
J'(u) = \Psi'(u) - I'(u) = 0.
\]

It follows that \(u\) is a critical point of \(J\).

### 3.1. On the geometry of the functional \(J\)

In this subsection, we will show that under the conditions which we have imposed on the functions \(a_{x,y}\) and \(f\), the geometric conditions \((G_1)\) and \((G_2)\) of Theorem 1.4 will hold.

**Lemma 3.2:** Under the assumptions of Theorem 3.1, the first geometric condition \((G_1)\) of the mountain pass Theorem 1.4 hold for the functional \(J\) defined in (36).

**Proof:** For all \(u \in W_0^{1,2} L_{\phi_{x,y}}(\Omega) \setminus \{0\}\), the functional \(J\) is satisfied:
\[
J(u) = \Psi(u) - \int_{\Omega} F(x, u) \, dx
\]
\[
\begin{align*}
= \Psi(u) \left[ 1 - \frac{\int_{\Omega} F(x, u) \, dx}{\Psi(u)} \right].
\end{align*}
\] (43)

Using the condition \((H_5)\), we have that there exist \(\varepsilon \in (0, 1)\) and \(t_0 > 0\) such that

\[
F(x, t) \leq \frac{1 - \varepsilon}{\lambda_1} \hat{\Phi}_x(t) \quad \text{for all } |t| \leq t_0 \quad \text{and all } x \in \overline{\Omega}.
\]

We pose \(\Omega_0 := \{x \in \Omega : |u(x)| \geq t_0\}\), then we have

\[
\int_{\Omega} F(x, u(x)) \, dx \leq \frac{1 - \varepsilon}{\lambda_1} \int_{\Omega \setminus \Omega_0} \hat{\Phi}_x(u(x)) \, dx + \int_{\Omega_0} F(x, u(x)) \, dx. \quad (44)
\]

By (31), we have

\[
(1 - \varepsilon) \int_{\Omega \setminus \Omega_0} \hat{\Phi}_x(|u(x)|) \, dx \leq (1 - \varepsilon) \int_{\Omega \setminus \Omega_0} \hat{\Phi}_x(|u(x)|) \, dx \leq 1 - \varepsilon. \quad (45)
\]

Next, from \((H_1)\), we have

\[
F(x, t) \leq c_1 (|t| + G(|t|)) \quad \text{for all } |t| \geq t_0 \quad \text{and for a.e. } x \in \Omega.
\]

By proposition 1.1, it follows that

\[
\int_{\Omega_0} F(x, u) \, dx \leq c_1 \left( ||u||_{L^1(\Omega)} + \int_{\Omega} G(|u|) \, dx \right) \leq c_1 \left( ||u||_{L^1(\Omega)} + ||u||_{G^-} + ||u||_{G^+} \right).
\]

Since \(W^{s_1}_0 L\Phi_x(\Omega) \hookrightarrow L_G(\Omega)\) and \(W^{s_1}_0 L\Phi(\Omega) \hookrightarrow L^1(\Omega)\), we obtain

\[
\int_{\Omega_0} F(x, u(x)) \, dx \leq c_6 c_1 (||u||_{s_1} + ||u||_{s_1}^{g^-} + ||u||_{s_1}^{g^+}). \quad (46)
\]

Then, for \(||u||_{s_1} \leq 1\), we find

\[
\int_{\Omega_0} F(x, u(x)) \, dx \leq 3c_6 c_1 ||u||_{s_1}, \quad (47)
\]

where \(c_6\) denote various positive constants. By Proposition 1.3, we have

\[
\int_{\Omega_0} \frac{F(x, u)}{\Psi_{s_1}(u)} \, dx \leq 3c_6 c_1 ||u||_{s_1}^{1 - \psi^+}. \quad (48)
\]

Now, using (43), (44), (45) and (48), we obtain that

\[
J(u) \geq \Psi(u) \left( \varepsilon - 3c_6 c_1 ||u||_{s_1}^{1 - \psi^+} \right) \geq \frac{\varepsilon}{2} \Psi(u),
\]
whenever

\[ \rho \leq \min \left\{ 1, \left( \frac{\varepsilon}{6c_6c_1} \right)^{\frac{1}{1-\varphi^+}} \right\}. \tag{49} \]

Finally, by Proposition 1.3, we get

\[ ||u||_{s_1} \to 0 \iff \Psi(u) \to 0. \]

Hence for \( \rho > 0 \) as given in (49), there exists a \( \alpha = \alpha(\rho) > 0 \) such that for all \( u \) with \( ||u||_{s_1} = \rho \), we have

\[ \Psi(u) \geq \alpha. \]

We therefore obtain

\[ J(u) \geq \frac{\alpha \varepsilon}{2}. \]

Thus, if we set \( r = \alpha \frac{\varepsilon}{2} \), we obtain that \((G_1)\) is satisfied. \( \blacksquare \)

**Lemma 3.3:** Under the assumptions of Theorem 3.1, the second geometric condition \((G_2)\) of the mountain pass Theorem 1.4 hold for the functional \( J \) defined in (36).

**Proof:** First, by \((H_4)\), it follows that

\[ F(x, \xi) \geq r^{-\theta} \min \{F(x, r), F(x, -r)\} ||\xi||^\theta \]

for all \( ||\xi|| > r \) and a.e. \( x \in \Omega \). Thus by (50) and \( F(x, \xi) \leq \max_{||\xi|| \leq r} F(x, \xi) \) for all \( ||\xi|| \leq r \), we obtain

\[ F(x, \xi) \geq r^{-\theta} \min \{F(x, r), F(x, -r)\} ||\xi||^\theta - \max_{||\xi|| \leq r} F(x, \xi) - \min_{||\xi|| \leq r} F(x, r, F(x, -r)) \]

(51)

for any \( \xi \in \mathbb{R} \) and a.e. \( x \in \Omega \).

From Theorem 1.3, we can fix \( u_0 \in C_0^\infty (\Omega) \) such that \( ||u_0||_{s_1} = 1 \) and let \( t \geq 1 \). By (51), we have

\[
J(tu_0) = \Psi_{s_1}(tu_0) + \Psi_{s_2}(tu_0) - \int_\Omega F(x, tu_0) \, dx \\
\leq ||tu_0||_{s_1}^{\varphi^+} + ||tu_0||_{s_2}^{\varphi^+} + ||tu_0||_{s_1}^{\varphi^-} - \int_\Omega F(x, tu_0) \, dx \\
\leq (1 + c_7^{\varphi^+})t^{\varphi^+} + c_7^{\varphi^+} ||tu_0||_{s_1}^{\varphi^-} + c_7^{\varphi^-} ||tu_0||_{s_1}^{\varphi^+} - \int_\Omega F(x, tu_0) \, dx \\
\leq (1 + c_7^{\varphi^+})t^{\varphi^+} + c_7^{\varphi^+} ||tu_0||_{s_1}^{\varphi^-} + c_7^{\varphi^-} ||tu_0||_{s_1}^{\varphi^+} - \int_\Omega \min \{F(x, r), F(x, -r)\} ||u_0(x)||^\theta \, dx \\
+ \int_\Omega \max F(x, \xi) + \min \{F(x, r), F(x, -r)\} ||u_0(x)||^\theta \, dx.
\]

From assumptions \((H_1)\) and \((H_5)\), we get \( 0 < F(x, \xi) \leq c_0(|r| + G(|r|)) \) for \( ||\xi|| \leq r \) a.e. \( x \in \Omega \). Thus, \( 0 < \min \{F(x, r), F(x, -r)\} < c_0(|r| + G(|r|)) \), a.e. \( x \in \Omega \). Since \( \theta > \varphi^+ \geq \varphi^- \) by assumption \((H_4)\), passing to the limit as \( t \to \infty \), we obtain that \( J(tu_0) \to -\infty \). Thus, the assertion \((G_2)\) follows by taking \( e = Tu_0 \) with \( T \) sufficiently large. \( \blacksquare \)

**3.2. Proof of Theorem 3.1.**

It follows from Lemma 3.2 and Lemma 3.3 that the hypotheses of Theorem 1.4 are satisfied. So Lemma 3.1 implies the existence of a nontrivial critical point of the functional \( J \) which is a weak solution problem \((P_d)\).
Disclosure statement

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