The Koslowski–Sahlmann representation:  
gauge and diffeomorphism invariance

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Abstract
The discrete spatial geometry underlying loop quantum gravity (LQG) is degenerate almost everywhere. This is at apparent odds with the non-degeneracy of asymptotically flat metrics near spatial infinity. Koslowski generalized the LQG representation so as to describe states labeled by smooth non-degenerate triad fields. His representation was further studied by Sahlmann with a view to imposing gauge and spatial diffeomorphism invariance through group averaging methods. Motivated by the desire to model asymptotically flat quantum geometry by states with triad labels which are non-degenerate at infinity but not necessarily so in the interior, we initiate a generalization of Sahlmann’s considerations to triads of varying degeneracy. In doing so, we include delicate phase contributions to the averaging procedure which are crucial for the correct implementation of the gauge and diffeomorphism constraints, and whose existence can be traced to the background exponential functions recently constructed by one of us. Our treatment emphasizes the role of symmetries of quantum states in the averaging procedure. Semianalyticity, influential in the proofs of the beautiful uniqueness results for LQG, plays a key role in our considerations. As a by product, we re-derive the group averaging map for standard LQG, highlighting the role of state symmetries and explicitly exhibiting the essential uniqueness of its specification.

Keywords: loop quantum gravity, representations, group averaging

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1. Introduction

Loop quantum gravity (LQG) is an effort to construct a canonical quantization of a classical Hamiltonian description of the gravitational field. The phase space variables of this classical Hamiltonian description are an SU(2) connection and a conjugate electric (triad) field on a Cauchy slice $\Sigma$. Most work in LQG is in the context of compact (without boundary) Cauchy slices. In this context one of the key results of LQG is that its underlying representation endows the quantum spatial geometry described by the triad field with a fundamental discreteness [1].

It is of physical interest to generalize LQG to the context of asymptotically flat gravitational fields. Such a generalization faces two immediate issues. First, by virtue of the existence of asymptotia, the Cauchy slices must be non-compact. Second, by virtue of asymptotic flatness,
the spatial triad must asymptote to a smooth flat triad which is at odds with the discrete, non-smooth spatial geometry alluded to above.

We shall return to a discussion of the first issue in the concluding section of this paper. Here, let us focus on the second issue ignoring complications arising from non-compactness. It is expected that the effective smoothness of classical geometry arises through coarse graining of its LQG quantum counterpart. Thus one would expect that the asymptotic conditions translate to a requirement on suitably defined coarse grained properties of the quantum states. While a final understanding of the quantum states underlying asymptotically flat geometries would require such a treatment, as a first step, it is useful to enquire, already in the case of compact spatial topology, if there is some way in which the standard LQG representation can be modified so as to directly accommodate smooth spatial triads at the quantum level without explicit coarse graining. An affirmative answer to this query is provided by the representation constructed by Koslowski in his seminal contribution [2]. This representation assigns an extra label $\bar{E}_a^i$ to the standard kinematic LQG states. Here $\bar{E}_a^i$ is a smooth triad field. Triad dependent operators acquire an extra (smooth) contribution from $\bar{E}_a^i$ in addition to the standard discrete contributions [2, 3]. While Koslowski restricted attention to nondegenerate $\bar{E}_a^i$, this representation admits a straightforward and obvious generalization to triads of arbitrary (and, in general, spatially varying) degeneracy as well, and, indeed, the $\bar{E}_a^i = 0$ sector of this representation turns out be exactly the standard LQG one.

In a putative generalization of the representation to the asymptotically flat case it would perhaps be appropriate to retain standard LQG structures in the interior while capturing flatness at asymptotia. This would necessitate the consideration of states which are labeled by electric fields which are flat at asymptotia and vanish in the interior (more precisely, inside of compact sets). Hence, in anticipation of such a generalization of the representation to the asymptotically flat case, it is of relevance to study its properties in the compact case in the context of $\bar{E}_a^i$ of varying degrees of degeneracy.

As mentioned above such a study is trivial at the level of the kinematic representation. However, physically relevant configurations are those which are invariant under the action of the $SU(2)$ gauge group, as well as the action of spatial diffeomorphisms and the Hamiltonian constraint. Since the construction of the latter is an open issue even in standard LQG we are interested, as a first step, in imposing the diffeomorphism and $SU(2)$ gauge constraints in this representation in the context of ‘background’ electric fields $\bar{E}_a^i$ of spatially varying degeneracy. Sahlmann initiated the imposition of gauge and diffeomorphism invariance via an application of group averaging techniques to this representation in his pioneering work of [3] wherein he restricted attention to the case of non-degenerate triads. Accordingly, the first aim of this work is to initiate an investigation into the group averaging of triads of spatially varying degeneracy.

In [4], it was noted that this representation studied by Koslowski and Sahlmann (henceforth referred to as the Koslowski–Sahlmann or KS representation) supported, in addition to the action of the standard holonomy-electric flux operators, operator correspondents of certain connection dependent functions, called ‘background exponentials’. In this work we show that in order to implement the gauge transformation properties of these new functions in quantum theory, the action of quantum $SU(2)$ gauge transformations of [3] needs to be augmented with a phase factor. While this augmentation can be absorbed in a redefinition of states labeled by the nondegenerate triads of Sahlmann’s work\(^1\), in the degenerate case (of rank 1) this is no longer true. Since the $SU(2)$ Gauss Law and the spatial diffeomorphism constraints generate a gauge group $\mathcal{G} \rtimes \text{Diff}$ which is the semidirect product of the group of finite $SU(2)$ transformations $\mathcal{G}$ with the group of spatial diffeomorphisms Diff, these phases need to be understood when

\(^1\) There may exist exceptional geometries for which this is not true, see section 4.2.
both sets of constraints are imposed. Accordingly, the second aim of this work is to initiate an investigation into the structure and role of these phases in the group averaging procedure.

The necessity of extra phase factors can be easily seen in a $U(1)$ version of the KS representation\(^2\). Consider any $U(1)$ gauge invariant state in the standard LQG type representation for gauge group $U(1)$ and augment it with a background electric field label $\bar{E}^a$. For concreteness let the standard LQG type state be a $U(1)$ spin network. Then, $U(1)$ gauge transformations do not change the spin network labels because these are anyway gauge invariant and do not change the label $\bar{E}^a$ because electric fields are $U(1)$ gauge invariant. If we ignore the phase factors alluded to above, the state is left invariant by any $U(1)$ gauge transformation and hence should be annihilated by the Gauss Law constraint. But this is not true because the Gauss Law constraint being just the divergence of the electric field operator, yields the divergence of $\bar{E}^a$ when it acts on the state in question, and $\bar{E}^a$ may be chosen so that its divergence is non-vanishing! When the phase factors are included it turns out that the action of finite gauge transformations rephase the state with a phase proportional to the divergence of $\bar{E}^a$. The group averaging procedure averages over these phases and produces a vanishing result due to phase factor cancellation unless $\bar{E}^a$ is divergence free in which case the state is exactly invariant under gauge transformations as expected. Thus, to ensure correct results, it is crucial to keep account of the delicate phases alluded to above.

Let us return to the case of interest, namely that of gauge group $G \rtimes \text{Diff}$ with $G$ being the group of internal $SU(2)$ transformations. The KS Hilbert space is spanned by an orthonormal basis of ‘KS’ spin net states which are generalizations of the standard LQG spin net states. Each KS spin net is specified by an $SU(2)$ gauge invariant spin net label ‘$s$’ together with a background field label $\bar{E}^a$.\(^3\) The group averaging procedure applied to a KS spin net state seeks to construct a corresponding $G \rtimes \text{Diff}$ invariant state as an ‘average’ over $G \rtimes \text{Diff}$ related images of the spin net. Recall that in the standard LQG case, an important role is played by transformations which leave this spin net invariant. More in detail, given an $SU(2)$ invariant spin network $s$ these ‘symmetries’—corresponding to diffeomorphisms leaving the spin net label $s$ invariant—determine the superselection sector containing the spin net state $|s\rangle$ as well as the detailed group averaging map in each superselection sector. In particular (see [5, 6] as well as section 4 of this paper), the structure of these symmetries implies that LQG spinnets living on graphs which are not related by diffeomorphisms lie in distinct superselected sectors and that within each such sector the group averaging map is well defined and unambiguous. Therefore, we expect that in the KS case too the symmetries of the label set of the KS spin net being averaged play similar, key role in the averaging procedure. Accordingly, a large part of our analysis is focused on trying to understand the symmetries of these labels. Since one of these labels is a field with three dimensional support (in contrast to the LQG case wherein the graph label is a one dimensional set), these symmetries are (infinitely!) more complex than those encountered in LQG. As a result, explicit results of a general nature are hard to come by.

After these remarks, let us now summarize our main results. Our first set of results pertain to superselection criteria. Similar to the support of the spinnet graph being associated with superselection in LQG, here we show that appropriately defined support sets of the background field serve as (partial) superselection labels. Specifically, let the background field have rank 0, 1 or (2,3) in the ‘support’ sets $V_0, V_1$ and $V_2$. Then, if two KS states have support sets (up to sets of zero measure) which are unrelated by the action of any diffeomorphism, their group averages lie in distinct superselection sectors. We note here that semianalyticity of the

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\(^2\) This example is worked out in detail in section 4.1.

\(^3\) In section 9.2 we comment on the possibility of allowing for gauge variant spin networks.

\(^4\) States in a single superselection sector are mapped to each other by an appropriately defined set of gauge invariant observables whereas states in distinct superselection sectors cannot be mapped to each other by any such observable.
background field plays a key role in our analysis. More in detail, these support sets may be
written in terms of the zero sets of appropriate functions of the background field. Now, it is well
known that zero sets of smooth functions do not, in general, have any nice properties. Hence
it would be very hard, if not impossible, to proceed with our analysis for smooth (i.e. $C^\infty$)
background fields. However, standard LQG is most elegantly formulated for the semianalytic
category [7, 8]. In particular, the spatial diffeomorphisms considered in standard LQG are
semianalytic and preserve semianalyticity as opposed to smoothness. Hence it is necessary for
the KS background fields to also be semianalytic. It turns out that the zero sets of semianalytic
functions on compact manifolds have very nice properties, namely they are constructed as
finite unions of semianalytic submanifolds [7, 9]. It is this beautiful property which allows us
to perform a fairly detailed analysis and derive the above superselection result.

Our second set of results pertain to the role of phase contributions in the averaging
procedure. As in standard LQG, due to the absence of a well defined group invariant measure,
our starting point is the definition of a putative group averaging map as a formal sum over gauge
related states. Our results are as follows. First we show that if a KS state can be non-trivially
rephased by any gauge transformation, its image under the putative averaging map vanishes.
Next, we show that such phasings may manifest for non-degenerate triads with appropriate
symmetries and that such phases do manifest generically if the rank 1 support set is of non-zero
measure.

Our third set of results pertain to the case where the rank 1 support set is of zero measure
so that the triad is exclusively of rank 0 or (2,3) almost everywhere. We show that, modulo
one reasonable assumption (which needs to be proved), the group averaging map in this
superselection sector is well defined and unambiguous.

Besides these fairly general results, we also derive a variety of less general, ‘case by case’
results when the rank 1 set is of non-zero measure. Our hope is that the material in this paper
can serve as a starting point for further studies of the complex and structurally rich problem
of group averaging for this (i.e. rank 1 support set of non-zero measure) case.

The layout of this paper is as follows. Section 2 is devoted to a review of the
necessary background for our considerations. This includes: (a) the definition of classical
holonomy, flux and background exponential functions, (b) their transformations under the
gauge transformations generated by the classical $SU(2)$ Gauss Law and spatial diffeomorphism
constraints of gravity, (c) a demonstration [4] that the KS Hilbert space provides a
representation for the operator correspondents of the functions in (a) as well a unitary
representation of the transformations of (b). In particular, we highlight the additional phase
contributions alluded to above. In section 3 we outline our general strategy for the construction
of the group averaging map, emphasizing the importance of appropriate phase contributions
in its construction so as to anticipate the corresponding subtleties to be encountered in the
following sections. In particular we show that states which admit non-trivial rephasings average
to zero. In section 4 we derive the phasing related results alluded to above for nondegenerate
triads with symmetries and for generic triads with rank 1 sets of non-zero measure.

In section 5, we re-derive the standard LQG group averaging map from a slightly
different perspective to the standard one. Our treatment emphasizes the relative roles of
various symmetry structures in the group averaging process and serves as a preview for the
considerations which inform sections 6-8. In section 6 we derive our principal result on
superselection in the KS representation, namely its dependence on the support sets of rank 0,
1 and (2,3) of the background triad label. In section 7 we show (modulo certain reasonable
assumptions) the existence of a well defined and ambiguity free group averaging map for
superselection sectors labeled by rank 1 sets of zero measure. Section 8 is devoted to a
discussion of the various subtleties associated with the case when the rank 1 set is not of zero

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measure. Section 9 is devoted to a discussion of our results, open questions and to an account of work in progress. Assorted technical details are collected in the appendices.

In this work we use units such that \( c = 8 \pi \gamma G = \hbar = 1 \) where \( \gamma \) is the Barbero–Immirzi parameter. Further, all differential geometric structures of interest will be based on the semianalytic, \( C^k, k \gg 1 \) category (see appendix B for further details). Finally, the Cauchy slice \( \Sigma \) is assumed to be a compact without boundary semianalytic manifold.

2. Preliminaries

The present section introduces some background material. It serves to set up notation but most importantly it introduces the additional kinematical observables—referred to as ‘background exponentials’—on an equal footing as the traditional holonomies and fluxes. The key equation in this section is (2.24), which describes how gauge transformations act on KS states.

The standard LQG quantization (prior to imposition of \( SU(2) \) gauge and spatial diffeomorphism invariance) provides a representation of the algebra generated by holonomies (more precisely, their matrix components) and electric fluxes. As shown in [4], the KS quantization generalizes this standard LQG one so as to provide a representation of the enlargement of the holonomy-flux algebra to one which includes certain connection dependent functions referred to as ‘background exponential’ functions. As discussed in section 1, the gauge symmetry group of interest is the semidirect product of the group of local \( SU(2) \) rotations with that of spatial diffeomorphisms. Accordingly, in section 2.1 we review the classical phase space underlying LQG and the definition of the holonomy, flux and background exponential functions thereon. In section 2.2 we describe the action of the gauge symmetry group on the phase space and its induced action on the newly introduced background exponential functions. In section 2.3 we review the KS representation of the enlarged algebra of holonomies, fluxes and background exponentials and display a unitary representation of the gauge symmetry group in this representation.

We shall use the following nomenclature. Structures used prior to the imposition of the gauge symmetry group are referred to as kinematic. Local \( SU(2) \) gauge transformations are referred to as internal gauge transformations. The gauge symmetry group will be referred to as the bundle automorphism group [3] or, in short, as the automorphism group.

2.1. Classical phase space and functions thereon

The classical phase space is coordinatized by an \( SU(2) \) connection \( A_a \), and its conjugate (unit density weight) \( su(2) \)-valued electric field \( E^a \) on the Cauchy slice \( \Sigma \). In terms of their components in an \( su(2) \) basis \( \tau_i, i = 1, 2, 3 \) with \( [\tau_i, \tau_j] = \epsilon_{ijk} \tau_k, A_\alpha = A_\alpha^i \tau_i \) and \( E^a = E^a_i \tau_i \), their Poisson brackets read: \( \{A^\alpha_i(x), E^\beta_j(y)\} = \delta^\beta_i \delta^\alpha_j \delta(x, y) \). We define the following functions on phase space:

\[
h_e(A) := \mathcal{P} e^{iA},
\]

\[
F_{S,f}(E) := \int_S dS_a \text{Tr}[fE^a].
\]

\[
\beta_{E}(A) := e^{i\int_E \text{Tr}[E A]}.
\]

Here \( h_e(A) \) is the \( SU(2) \) matrix valued holonomy of the connection along the one dimensional oriented curve or ‘edge’ \( e \subset \Sigma \). The normalization of the trace \( \text{Tr} \) is taken so that \( \text{Tr}[\tau_i \tau_j] = \delta_{ij} \).

\( F_{S,f}(E) \) is the electric flux smeared with the \( su(2) \)-valued function \( f \) through the surface \( S \subset \Sigma \).
\( \beta_E(A) \) is a function which only depends on the connection and is obtained by exponentiating the integral of the connection smeared with an \( su(2) \) valued unit density weight vector field \( \tilde{E} \). We refer to the \( c \)-number smearing function \( \tilde{E} \) as a background electric field and to \( \beta_E(A) \) as a background exponential.

The Poisson brackets involving the background exponentials are\(^5\):

\[
\{ \beta_E, \beta_E \} = \{ \beta_E, h_a \} = 0, \quad \{ \beta_E, F_\phi \} = i f_{\phi, f} (\tilde{E}) \beta_E.
\]

### 2.2. The group of gauge symmetries

Let the group of internal \( SU(2) \) gauge transformations be \( \mathcal{G} \) and that of spatial diffeomorphisms be \( \text{Diff} \), both groups consisting of transformations connected to identity. Denote the gauge symmetry group, referred to by Sahlmann as the bundle automorphism group by \( \text{Aut} \) so that \( \text{Aut} := \mathcal{G} \rtimes \text{Diff} \). Elements of \( \text{Aut} \) will be denoted as \( a = (g, \phi) \in \text{Aut} \) with \( g \in \mathcal{G} \) and \( \phi \in \text{Diff} \). The product structure is given by (see appendix D for further details):

\[
(g, \phi) (g', \phi') = (g \phi_a g', \phi \circ \phi'),
\]

where \( \phi_a \) denotes the push-forward action so that \( (\phi_a g)(x) = g(\phi^{-1}(x)) \). Infinitesimal generators of \( \text{Aut} \) will be denoted by \( (\Lambda, \xi) \) with \( \Lambda \) an \( su(2) \) valued scalar and \( \xi \) a vector field. The infinitesimal version of (2.5) corresponds to the commutator\(^6\):

\[
[(\Lambda, \xi), (\Lambda', \xi')] = (-\mathcal{L}_\xi \Lambda' + \mathcal{L}_{\Lambda'} \Lambda + [\Lambda, \Lambda'], [\xi, \xi']),
\]

where \( [\Lambda, \Lambda'] \) is the \( 2 \times 2 \) matrix commutator and \( [\xi, \xi'] \) the vector field Lie bracket. The group \( \text{Aut} \) acts on phase space according to:

\[
(g, \phi) \cdot A_a := g \phi_a A_a g^{-1} - (\partial_a g) g^{-1},
\]

\[
(g, \phi) \cdot E^a := g \phi_a E^a g^{-1},
\]

with the corresponding infinitesimal version:

\[
(\Lambda, \xi) \cdot A_a := [\Lambda, A_a] - \mathcal{L}_\xi A_a - \partial_a \Lambda = [G[\Lambda] + D[\xi], A_a],
\]

\[
(\Lambda, \xi) \cdot E^a := [\Lambda, E^a] - \mathcal{L}_\xi E^a = [G[\Lambda] + D[\xi], E^a],
\]

generated by Poisson brackets with the Gauss Law and diffeomorphism constraints

\[
G[\Lambda] = -\int_\Sigma \text{Tr}[A (\partial_a E^a + [A_a, E^a])],
\]

\[
D[\xi] = \int_\Sigma \text{Tr}[E^a \mathcal{L}_\xi A_a].
\]

The action of the group \( \text{Aut} \) on the phase space functions (2.1)–(2.3) is given by:

\[
a \cdot h_c(A) := h_c(a^{-1} \cdot A) = g^{-1}(\phi(e_i)) h_{\phi(e_j)}(A) g(\phi(e_j)),
\]

\[
a \cdot F_{\phi, f} (E) := F_{\phi, f} (a^{-1} \cdot E) = F_{\phi(\Sigma), \phi, f}^{-1} (E),
\]

\(^5\) In the second equation of (2.4), \( F_{\phi, f} \) inside the brackets is a phase space function, whereas \( F_{\phi, f} (\tilde{E}) \) is just the number \( F_{\phi, f} (\tilde{E}) = \int d^4 x \text{Tr}[\tilde{F} \tilde{E}^a] \).

\(^6\) In the present setting of \( C^2 \) fields, the infinitesimal generators do not form a Lie algebra since the Lie bracket of two \( C^2 \) vector fields is in general \( C^{k-1} \). But they do generate well defined one parameter subgroups of \( \text{Aut} \).
where $e_i$ and $e_f$ denote the initial and final points of the edge $e$. The transformation law for the background exponentials is found to be

\[ a \cdot \bar{\beta}_E[A] := \bar{\beta}_E[a^{-1} \cdot A] = e^{\omega(a, E)\bar{\beta}_a E}[A], \tag{2.13} \]

where

\[ \omega(a, \bar{E}) := \int \Sigma \text{Tr} \left[ \phi_s(\bar{E}^s) g^{-1} \tilde{a}_g \right]. \tag{2.14} \]

By construction, (2.11)–(2.13) provide a representation of Aut on holonomies, fluxes and background exponentials. In particular, for the newly introduced functions we have:

\[ a \cdot (a' \cdot \bar{\beta}_E) = (aa') \cdot \bar{\beta}_E. \tag{2.15} \]

Since this fact may not be immediately obvious, an explicit verification of (2.15) is provided in appendix C.1.

We conclude the section by describing a key property of the phase factors. Given a background electric field $\bar{E}$, we define its symmetry group by:

\[ S_E := \{ a \in \text{Aut} : a \cdot \bar{E} = \bar{E} \} \subset \text{Aut}. \tag{2.16} \]

Consider the map $\pi_E : S_E \to U(1)$ defined by

\[ \pi_E(a) := \omega^a(a, \bar{E}), \quad a \in S_E. \tag{2.17} \]

Thus, $a \cdot \bar{\beta}_E = \pi_E(a)\bar{\beta}_E$ for $a \in S_E$. From the property (2.15) it follows that $\pi_E(a_1)\pi_E(a_2) = \pi_E(a_1a_2)$ for $a_1, a_2 \in S_E$, i.e. $\pi_E$ is a group homomorphism from $S_E$ into $U(1)$.

### 2.3. The KS representation

The kinematical Hilbert space of standard LQG is spanned by the orthonormal basis of spin network states $\{|s\rangle\}$. Let the dense domain of the finite linear span of spinnets be $\mathcal{D}$. Let $\hat{O}$ be an operator from $\mathcal{D}$ to $\mathcal{D}$ so that $\hat{O}|s\rangle$ is a finite linear combination of spinnets i.e. $\hat{O}|s\rangle = \sum_j O^{(i)}_j |s_j\rangle$ where $O^{(i)}_j$ are the complex coefficients in the sum over the spinnets $|s_j\rangle$.

It is useful to introduce the notation $|\hat{O}s\rangle$ to denote this linear combination of spinnets so that we have

\[ |\hat{O}s\rangle := \hat{O}|s\rangle = \sum_j O^{(i)}_j |s_j\rangle. \tag{2.18} \]

The KS Hilbert space is then spanned by states which have, in addition to their LQG spinnet label, an additional label $\bar{E}^a$ where $\bar{E}^a$ is a background electric field. We denote such a state by $|s, \bar{E}\rangle$. These states for all $s, \bar{E}$ provide an orthonormal basis for the KS kinematic Hilbert space so that the inner product between two such KS spinnets in this Hilbert space is

\[ \langle s', \bar{E}' | s, \bar{E} \rangle = |\langle s'|s\rangle\delta_{\bar{E}', \bar{E}}. \tag{2.19} \]

where $|\langle s'|s\rangle\rangle$ is just the standard LQG inner product and the second factor is the Kronecker delta which vanishes unless the two background fields agree in which case it equals unity.

The holonomy-flux operators act on the KS spinnets as:

\[ \hat{h}^A_{a_B}|s, \bar{E}\rangle := \hat{h}^{\text{LQG}A}_{a_B} |s, \bar{E}\rangle \tag{2.20}, \]

\[ \hat{F}_{s,f}|s, \bar{E}\rangle := \hat{F}^{\text{LQG}f}_{s} |s, \bar{E}\rangle + F_{s,f}(\bar{E})|s, \bar{E}\rangle. \tag{2.21} \]

Here $\hat{h}^A_{a_B}$ is the $A, B$ component of the holonomy operator. Since the holonomy of equation (2.1) is in the defining $j = 1/2$ representation, the indices $A, B$ take values in $\{1, 2\}$. Further, we
have used the notation of equation (2.18) so that \( \hat{\beta}_E^L_{QG} \) and \( \hat{F}_{S,f}^L_{QG} \) represent the standard LQG action on spin networks. The background exponential operators act through:

\[
\hat{\beta}_E|s, \bar{E}\rangle := |s, \bar{E} + \bar{E}\rangle. \tag{2.22}
\]

It then follows that (2.20) and (2.21) satisfy the standard holonomy-flux commutation relations. It is easy to verify that the only additional non-trivial commutator is given by

\[
[\hat{\beta}_E, \hat{F}_{S,f}] = -\hat{F}_{S,f}(\bar{E}) \hat{\beta}_E, \tag{2.23}
\]

in agreement with the Poisson bracket in (2.4).

The unitary action of the gauge group \( \text{Aut} \) on the KS Hilbert space is dictated by the transformation properties of the elementary phase space functions and reads:

\[
U(a)|s, \bar{E}\rangle := e^{i\alpha(a, \bar{E})[U^L_{QG}(a)s, a \cdot \bar{E}]- \frac{1}{2}\omega(a, \bar{E})[a \cdot \bar{E}, a \cdot \bar{E}]}, \tag{2.24}
\]

where \( U^L_{QG}(a)s \) denotes the usual action of \( \text{Aut} \) on spin networks, and \( \alpha(a, \bar{E}) \) and \( a \cdot \bar{E} \) are given in (2.14) and (2.7) for \( a = (g, \phi) \). It is immediate to verify that (2.24) preserves the inner product (2.19). Further, from the fact that \( U^L_{QG}(a_1a_2) = U^L_{QG}(a_1)U^L_{QG}(a_2) \) in conjunction with equation (2.15) it follows that \( U(a_1a_2) = U(a_1)U(a_2) \) so that equation (2.24) defines a unitary representation of \( \text{Aut} \). Finally, it is easy to verify that (2.24) reproduces the transformation rules (2.11)–(2.13) on the corresponding quantum operators. We refer to appendix C.2 for explicit verification of this fact.

We conclude by pointing out that the actual kinematical space to be used in the text is the subspace generated by states \( |s, \bar{E}\rangle \) such that \( s \) is \( SU(2) \) gauge invariant. This is the Hilbert space to be referred as ‘kinematical’ and denoted by \( \mathcal{H}_{\text{kin}} \). We comment on the reasons for this restriction in section 9.2.

### 3. Outline of our general strategy

The organization of the material is as follows. In section 3.1 we review the defining properties of any satisfactory group averaging map. Any such map yields gauge invariant states from appropriate kinematic ones. Following [3, 5], a candidate for such a map may be constructed as a formal sum over all distinct ‘bras’ which are gauge related to the kinematic one under consideration, the idea being that the action of any gauge transformation on this sum sends the sum into itself thus ensuring gauge invariance. In section 3.2 we organize this (formal) sum in a way which anticipates the context of the unitary representation of the group Aut described in section 2, with particular emphasis on the structure and effect of phase contributions of the type encountered in equation (2.24). This lead us to the key result, equation (3.9), that kinematical states admitting non-trivial rephasing are in the kernel of the group averaging map. The arguments we use to organize the sum are of a slightly formal nature and framed in a general context. Their purpose is to serve as heuristic motivation for the group averaging maps constructed in detail in the context of the KS representation. In section 3.3 we note that similar to the case of LQG [5], when applied to KS states, our strategy yields a putative averaging map with infinitely many unknown positive definite parameters. As in [5] we recall the relevance of the phenomenon of superselection in evaluating the import of these parameters and show how the ambiguities in their values can be reduced using properties of gauge invariant observables. Finally, in section 3.4 we use the material presented in sections 3.1–3.3 to formulate the strategy followed in sections 5–7.

### 3.1. Defining properties of a group averaging map

Let the gauge group \( \text{Aut} \) of interest be represented unitarily on the kinematic Hilbert space \( \mathcal{H}_{\text{kin}} \). A satisfactory Group Averaging map \( \eta \) is an anti linear map \( \eta : \mathcal{D} \to \mathcal{D}' \), from a dense
domain $\mathcal{D} \subset \mathcal{H}_{\text{kin}}$ that is preserved under the unitary action of Aut to the space $\mathcal{D}'$ of complex linear mappings on $\mathcal{D}$ ($\mathcal{D}'$ is called the algebraic dual of $\mathcal{D}$), satisfying the following three properties [5, 10]:

1. $\forall \psi_1 \in \mathcal{D}, \eta(\psi_1) \in \mathcal{D}'$ is Aut-invariant:
   \[ \eta(\psi_1)[U(a)\psi_2] = \eta(\psi_1)[\psi_2] \quad \forall a \in \text{Aut}, \psi_2 \in \mathcal{D} \]  
   (3.1)

2. $\eta$ is real and positive:
   \[ \eta(\psi_1)[\psi_2] = \eta(\psi_2)[\psi_1], \quad \eta(\psi_1)[\psi_1] \geq 0 \quad \forall \psi_1, \psi_2 \in \mathcal{D} \]  
   (3.2)

3. $\eta$ commutes with the observables:
   \[ \eta(\psi_1)[O\psi_2] = \eta(O^\dagger \psi_1)[\psi_2] \quad \forall \psi_1, \psi_2 \in \mathcal{D}, \forall O \in \mathcal{O}. \]  
   (3.3)

In the last condition ‘observables’ stand for ‘strong observables that preserve $\mathcal{D}'$, that is:

\[ \mathcal{O} := \{ O : \mathcal{D} \to \mathcal{D} \mid U(a)O = OU(a) \quad \forall a \in \text{Aut} \}. \]  
   (3.4)

Once one succeeds in finding such a map $\eta$, the Aut-invariant Hilbert space $\mathcal{H}_{\text{Aut}}$ is obtained as follows [10]: let $V_{\text{Aut}} \subset \mathcal{D}'$ be the span of dual vectors of the form $\eta(\psi)$. The sesquilinear form $(\eta(\psi_1), \eta(\psi_2))_{\text{Aut}} := \eta(\psi_2)[\psi_1]$ provides an inner product on $V_{\text{Aut}}/\sim$ where the quotient is over zero-norm states. Property 2 implies it is an inner product, and $\mathcal{H}_{\text{Aut}}$ is defined as the completion of $V_{\text{Aut}}/\sim$ under this inner product. Property 3 ensures that strong observables are well defined on $\mathcal{H}_{\text{Aut}}$, and satisfy the correct adjointness relations on $\mathcal{H}_{\text{Aut}}$ if they do so on $\mathcal{D}$ [6].

### 3.2. Group averaging as a sum over states

We write the Aut invariant image of the kinematic state $|\psi\rangle$ as the formal sum:

\[ \eta(|\psi\rangle) = \sum_{|\phi\rangle \in \text{Orb}(\psi)} |\phi\rangle^\dagger, \]  
   (3.5)

where $\text{Orb}(\psi)$ is the orbit of $|\psi\rangle$ under Aut i.e. the set of all distinct gauge related images of $|\psi\rangle$, and we have used the notation $|\phi\rangle^\dagger := \langle \phi |$. Every such element may be written as $|\phi\rangle = U(a)|\psi\rangle$ for some $a \in \text{Aut}$. Let us, in this manner, arbitrarily choose one such element $a$ for each element in $\text{Orb}(\psi)$ and call the resulting set $\text{Aut}_{\text{Orb}(\psi)}$ so that

\[ \eta(|\psi\rangle) = \sum_{a \in \text{Aut}_{\text{Orb}(\psi)}} (U(a)|\psi\rangle)^\dagger. \]  
   (3.6)

Elements of $\text{Aut}_{\text{Orb}(\psi)}$ can be characterized as follows. Let $\text{Sym}_\psi \subset \text{Aut}$ be the set of symmetries of $|\psi\rangle$ i.e. the set of automorphisms which leave $|\psi\rangle$ invariant so that $U(a)|\psi\rangle = U(b)|\psi\rangle$ iff $a = bs$ for some $s \in \text{Sym}_\psi$. This implies that elements of $\text{Aut}_{\text{Orb}(\psi)}$ are in correspondence with the cosets of Aut by $\text{Sym}_\psi$. Next, recall that elements of this coset space, $\text{Aut}/\text{Sym}_\psi$, are the equivalence classes $[a]$ where $b \in [a]$ iff there exists some $s \in \text{Sym}_\psi$ such that $a = bs$. Using the defining properties of $\text{Aut}_{\text{Orb}(\psi)}$, the coset space $\text{Aut}/\text{Sym}_\psi$ and the symmetry group $\text{Sym}_\psi$, it is easily verified that

1. for any $b \in \text{Aut}$ and any $a_1, a_2 \in \text{Aut}_{\text{Orb}(\psi)} a_1 \neq a_2$, it follows that $[ba_1] \neq [ba_2]$
2. given any $c \in \text{Aut}_{\text{Orb}(\psi)}$ and any $b \in \text{Aut}$, there exists a unique $a \in \text{Aut}_{\text{Orb}(\psi)}$ such that $[ba] = [c]$. 


It then follows from (i) and (ii) that for any $b \in \text{Aut}$:

\[
U^\dagger (b) \eta (|\psi\rangle) = \sum_{a \in \text{Aut}_{\Omega(\psi)}} \left( U(b) U(a) |\psi\rangle \right) = \sum_{a \in \text{Aut}_{\Omega(\psi)}} \left( U(ba) |\psi\rangle \right) = \sum_{c \in \text{Aut}_{\Omega(\psi)}} \left( U(c) |\psi\rangle \right),
\]

thus establishing the formal gauge invariance of the sum.

Note that the above argument requires the sum to range over all distinct images of $|\psi\rangle$. In particular, images of $|\psi\rangle$ which are distinct from each other but proportional to each other must be included in the sum. Thus, if $|\psi\rangle$ is in the sum and there exists $a \in \text{Aut}$ such that $U(a)|\psi\rangle = c|\psi\rangle$, then $c|\psi\rangle$ must also be in the sum. Note that $|c| = 1$ by unitarity of $U(a)$ so that $c = e^{i\theta}$, $\theta \in \mathbb{R}$ is a phase factor. It is convenient for future purposes to define the sum in equation (3.6) so as to sum over such phase related states first. This is done as follows.

Let $\text{Ph}_\psi \subset \text{Aut}$ be the set of gauge transformations which rephase $|\psi\rangle$ so that $U(a)|\psi\rangle$ and $U(b)|\psi\rangle$ are proportional iff $a = bp$ for some $p \in \text{Ph}_\psi$. Thus, the set of ‘phase unrelated states’ are in correspondence with the coset space $\text{Aut}/\text{Ph}_\psi$ each coset consisting of elements of an equivalence class $[a]_{\text{Ph}_\psi}$ where $a \equiv b$ iff $a = bp$ for some $p \in \text{Ph}_\psi$. Let us choose, arbitrarily, one element from each such equivalence class and call the resulting set $\text{Aut}_{\Omega(\psi)}$. It follows that every element of the orbit of $|\psi\rangle$ can be obtained by an appropriate rephasing of $U(a)|\psi\rangle$ for some $a \in \text{Aut}_{\Omega(\psi)}$. Next, note that distinct rephasings of $|\psi\rangle$ are in correspondence with cosets of $\text{Ph}_\psi$ by the symmetry group $\text{Sym}_\psi$. It is easily verified that $\text{Sym}_\psi$ is a normal subgroup of $\text{Ph}_\psi$ so that the coset space $\text{Ph}_\psi / \text{Sym}_\psi$ is just the quotient group obtained by quotienting $\text{Ph}_\psi$ by $\text{Sym}_\psi$. It is then straightforward to see that $\text{Ph}_\psi / \text{Sym}_\psi$ must be homomorphic to $U(1)$ or a subgroup of $U(1)$. In terms of this homomorphism we write the sum as:

\[
\eta(|\psi\rangle) = \left( \sum_{a \in \text{Aut}_{\Omega(\psi)}} U(a) \sum_{c \in \text{Ph}_\psi / \text{Sym}_\psi} e^{i\theta} |\psi\rangle \right)^\dagger.
\]

If $\text{Ph}_\psi / \text{Sym}_\psi$ is a non-trivial proper subgroup of $U(1)$ standard group theoretic results imply that this subgroup is finite and that the sum over phases vanishes i.e. $\sum_{c \in \text{Ph}_\psi / \text{Sym}_\psi} e^{i\theta} = 0$. If $\text{Ph}_\psi / \text{Sym}_\psi = U(1)$, we have an infinite sum of phases which we can plausibly define to vanish by virtue of the fact that for every element in the sum there is also its negative. Thus, in both these cases we have that $\eta(|\psi\rangle) = 0$. Thus, the only case for which the group averaging map could be non-trivial is when $\text{Ph}_\psi = \text{Sym}_\psi$. The group averaging sum (3.6) then takes the form:

\[
\eta(|\psi\rangle) = \begin{cases} 
0 & \text{if } \text{Sym}_\psi \subset \text{Ph}_\psi \\
\sum_{a \in \text{Aut}_{\Omega(\psi)}} \left( U(a) |\psi\rangle \right)^\dagger & \text{if } \text{Sym}_\psi = \text{Ph}_\psi.
\end{cases}
\]

In closing, we note that the first equation of (3.9) must be satisfied by any well defined group averaging map. To see this, consider the case where there exists $a \in \text{Ph}_\psi$ that rephases $|\psi\rangle$ by $e^{i\theta} \neq 1$. Then from the gauge invariance condition (3.1) we have that

\[
\eta(|\psi\rangle) = \eta(U(a)|\psi\rangle) = \eta(e^{i\theta}|\psi\rangle) = e^{-i\theta} \eta(|\psi\rangle),
\]

which implies the vanishing of $\eta(|\psi\rangle)$. From this point of view, the considerations in the main part of this section correspond to a particular mechanism for the vanishing of such $\eta(|\psi\rangle)$.

### 3.3. Ambiguities in the group averaging map

Let $|\psi\rangle$ be a KS spinnet and choose $D$ (see section 3.1) to be the finite linear span of KS spinnets. From equation (2.24) it follows that every state which is gauge related to $|\psi\rangle$ is
The Hilbert space $H$ is also a spinnet. To reduce notational clutter we shall call this set, referred to as Orb($\psi$) in section 3.2, as $[\psi]$. It follows that if $\sum_{\eta \in \text{Aut}} (U(\eta)\langle \psi \rangle)^* \sum_{\eta \in \text{Aut}} (U(\eta)\langle \psi \rangle)^\dagger$ is a gauge invariant state, so is $\eta_{[\psi]} \sum_{\eta \in \text{Aut}} (U(\eta)\langle \psi \rangle)^* \sum_{\eta \in \text{Aut}} (U(\eta)\langle \psi \rangle)^\dagger$ for any constant $\eta_{[\psi]}$ (from property 2 (3.2) this constant must be real and positive).

Recall that the dense set of gauge invariant states $D'$ is obtained as the image of the set $D$ (see section 3.1). It then follows that every choice of the set of coefficients $\eta_{[\psi]}$, one for each gauge equivalence class of KS spinnets $[\psi]$, yields a putative group averaging map. While some of the ambiguity in these choices can be absorbed into a rescaling of the gauge invariant Hilbert space inner product (see section 3.1, property 2), a vast ambiguity still remains. As in the case of LQG [5] we adopt the view that such ambiguities are only of physical relevance within a single superselection sector of the gauge invariant Hilbert space. Recall that, roughly speaking, if no observable maps a set of states to its complement, we say that the set of states is superselected. In particular we need only address the ambiguities in the superselection sector at a time. In particular we need only address the ambiguities in the superselection sector at a time. In more detail, using the notation of section 3.1, the notion of superselection sector is as follows.

Given two states $[\psi_1], [\psi_2] \in H_{\text{kin}}$ we say their corresponding Aut-invariant states $\eta([\psi_1]) \in H_{\text{Aut}}$ are superselected if

$$\langle \eta([\psi_1]), O\eta([\psi_2]) \rangle_{\text{Aut}} = 0 \quad \forall O \in \mathcal{O},$$

or equivalently if,

$$\langle \psi_1 | O \psi_2 \rangle_{\text{kin}} = 0 \quad \forall O \in \mathcal{O}, \forall a \in \text{Aut}. \quad (3.12)$$

The Hilbert space $H_{\text{Aut}}$ decomposes then into superselected subspaces, each of which is left invariant under the action of all observables $O \in \mathcal{O}$. The viewpoint of [5] is that each superselection sector is one possible realization of nature and it suffices to focus on one superselection sector at a time. In particular we need only address the ambiguities in the choice of group averaging map within the context of a single superselected sector. In practice, this reduces the choice of complex coefficients $[\psi]$ drastically because overall scaling of the group averaging map in each superselection sector can be reabsorbed into the Hilbert space inner product for that superselection sector.

Note that within a single superselection sector, observables can (and do) map different gauge orbits $[\psi]$ to each other. It then follows that the adjointness requirement 3 of section 3.1 yields consistency conditions between the ambiguity coefficients $\eta_{[\psi]}$ i.e. $\eta_{[\psi_1]}, \eta_{[\psi_2]}$ must be chosen so as to satisfy requirement 3 of section 3.1 whenever the action of an observable on a state in $[\psi_1]$ results in a state with overlap with state(s) in $[\psi_2]$. In practice, these consistency requirements often serve to remove the ambiguities in $\eta_{[\psi]}$ in a given superselection sector [5, 11].

Finally, we show that states admitting rephasings are superselected from those that do not. Let $[\psi]$ be a state admitting rephasing so that there exists $a \in \text{Aut}$ such that $U(\eta_{[\psi]}) = e^{i\theta} [\psi]$ with $e^{i\theta} \neq 1$. From the discussion of the previous section $\eta([\psi]) = 0$. Now, since our group averaging map satisfies property 1, we can use the same reasoning as in equation (3.10) to conclude that $\eta(O|\psi))$ vanishes for any observable $O$:

$$\eta(O|\psi)) = \eta(U(\eta_{[\psi]})) = \eta(OU(a|\psi)) = e^{-i\theta} \eta(O|\psi)). \quad (3.13)$$

This implies that property 3 of the group averaging trivializes to $0 = 0$ whenever one of the states admits rephasings. In particular the superselection condition (3.11) will follow whenever one of the states admits rephasings.
3.4. Our strategy

Based on sections 3.1–3.3, our strategy for the construction of a group averaging map for the KS representation is as follows. We choose \( D \) to be the finite span of KS spinnets. We write the group averaging map applied to any KS spinnet \( |\psi\rangle \) in the form of the sum (3.9) augmented with an ambiguity coefficient \( \eta \) i.e.

\[
\eta(|\psi\rangle) = \begin{cases} 
0 & \text{if } \text{Sym}_\psi \not\subseteq \text{Ph}_\psi \\
\sum_{a \in \text{AutOrb}(\psi)} (U(a)|\psi\rangle)^\dagger & \text{if } \text{Sym}_\psi = \text{Ph}_\psi.
\end{cases}
\]  

(3.14)

We verify that this sum defines an element of \( D' \). We then isolate a large superselection sector and reduce the ambiguity in the group averaging map when restricted to such a sector by imposing requirement 3 of section 3.1.

4. Importance of phases

In this section we illustrate the importance and non-trivial consequences of the presence of the phase term \( e^{i \theta(\alpha, \bar{E})} \) (2.24) in the unitary action of Aut on the KS space.

We first analyze \( U(1) \) gauge theory to highlight the crucial role of the non-trivial rephasings of section 3.2 in obtaining the correct gauge invariant state space. We then move to the \( SU(2) \) theory, where we show a close similarity of rank 1 triads with Abelian electric fields. We finally point out how phases may also have non-trivial effects in the case of rank 2 or 3 triads.

4.1. \( U(1) \) Abelian example

Consider \( U(1) \) gauge theory in a KS representation. Since our purpose in this section is to provide the reader with a setting wherein the importance of phases manifests in a direct and transparent manner, we restrict attention below to a \( G \) (rather than \( G \rtimes \text{Diff} \)) averaging of ‘pure background’ states of the form \( |\bar{E}\rangle := |0, \bar{E}\rangle \), where \( \bar{E} \) is now a single densitized vector field and ‘0’ refers to the trivial graph spin network.

The gauge group \( G \) is now given by local \( U(1) \) rotations \( g = e^{i \theta} \) with \( \theta : \Sigma \to \mathbb{R} \). From equation (2.14), the phase factor associated to \( g \) evaluates to

\[
\alpha(e^{i \theta}, \bar{E}) = \int_\Sigma \bar{E}^a \partial_a \theta = -\int_\Sigma \theta \partial_a \bar{E}^a, \tag{4.1}
\]

and the unitary action (2.24) becomes

\[
U(e^{i \theta})|\bar{E}\rangle = e^{-i \int_\Sigma \theta \partial_a \bar{E}^a}|\bar{E}\rangle, \tag{4.2}
\]

where we have used the fact that \( g\bar{E}g^{-1} = \bar{E} \).

Now, for a moment, let us see what happens if we ignore this phase factor so as to set \( U(e^{i \theta})|\bar{E}\rangle = |\bar{E}\rangle \) which implies that \( |\bar{E}\rangle \) is gauge invariant. Thus if we did this, all pure background states would be invariant under the group averaging procedure. This should be equivalent to the statement that the Gauss Law is satisfied. As the reader may verify, in \( U(1) \) theory, we are in the fortunate situation that the electric field operator is itself well defined and diagonalized by all KS spinnets (including, therefore, pure background ones). One obtains \( \partial_a \bar{E}^a|\bar{E}\rangle = \partial_a \bar{E}^a|\bar{E}\rangle \) which vanishes as expected only for divergence free electric fields. This is in contrast to invariance imposed by the group averaging procedure without phasing which leads to the physically erroneous conclusion that all pure background states, whether labeled by divergence free electric fields or not, are gauge invariant!
Let us now use the correct quantum implementation of elements of $\mathcal{G}$ with the phasing of equation (4.2). From equation (4.1), the phase is non-trivial only for $\tilde{E}^a$ which is not divergence free. Then, identical to the discussion just after equation (3.8), we have that the sum over such non-trivial phases vanishes in the group averaging procedure. This implies that, as expected on physical grounds, the only non-trivial gauge invariant states are the ones with divergence free $\tilde{E}^a$! This underlines the crucial role played by the phases (4.1) in the group averaging procedure.

Finally we rephrase our result above in the language developed in section 3.2. Equation (4.2) implies that $\text{Ph}_\theta = \mathcal{G}$. $\text{Sym}_\theta$ is given by those gauge transformations satisfying $e^{-i\int_\Sigma \xi a \tilde{E}^a} = 1$. It then follows that $\text{Ph}_\theta = \text{Sym}_\theta$ iff $\partial_\phi \tilde{E}^a = 0$, and the general group averaging sum (3.9) becomes

$$
\eta^{\mathcal{G}}(\tilde{E}) = \begin{cases} 
0 & \text{if } \partial_\phi \tilde{E}^a \neq 0 \\
\langle \tilde{E} \rangle & \text{if } \partial_\phi \tilde{E}^a = 0.
\end{cases}
$$

(4.3)

4.2. Phases in the SU(2) theory

We return to $SU(2)$ theory in the KS representation. As mentioned at the end of section 2, the states of interest are of the form $|s, \tilde{E}\rangle$ where $s$ is $SU(2)$ gauge invariant. The action of $a = (g, \phi) \in \text{Aut}$ on such states (2.24) takes the form

$$
U(a)|s, \tilde{E}\rangle = e^{\omega(a, \tilde{E})}|\phi(s), a \cdot \tilde{E}\rangle, \quad a = (g, \phi),
$$

(4.4)

so that the spin network label is insensitive to $SU(2)$ local rotations. States admitting non-trivial rephasings are those for which there exists $a \in \text{Aut}$ leaving the KS labels invariant with $e^{\omega(a, \tilde{E})} \neq 1$. From the discussion of section 3.2, such states are annihilated by the group averaging map.

The nonAbelian theory exhibits a general class of triads admitting rephasings in a way that closely resembles the previous Abelian example. Consider a state $|s, \tilde{E}\rangle$ with $\tilde{E}^a$ of rank 1, so that it can be written in the form:

$$
\tilde{E}^a = \hat{n} X^a,
$$

(4.5)

where $\hat{n} : \Sigma \rightarrow su(2)$ is an $su(2)$ valued scalar with unit norm so that $\text{Tr}[\hat{n} \hat{n}] = 1$, and $X^a$ is a unit density weight vector field. It is clear that local $SU(2)$ rotations of the form $g = e^{\hat{n} \hat{\theta}}$ with an arbitrary function $\theta : \Sigma \rightarrow \mathbb{R}$ leave (4.5) invariant, as well as the gauge invariant spin network $s$. The corresponding phases (2.14) are given by:

$$
\alpha(e^{\hat{n} \hat{\theta}}, \tilde{E}) = \int_\Sigma \text{Tr}[\tilde{E}^a e^{-\hat{n} \hat{\theta}} \partial_\phi e^{\hat{n} \hat{\theta}}] = \int_\Sigma X^a \text{Tr}[\tilde{E}^a e^{-\hat{n} \hat{\theta}} \partial_\phi e^{\hat{n} \hat{\theta}}]
$$

(4.6)

$$
= \int_\Sigma X^a \partial_\phi \theta = -\int_\Sigma \partial_\phi X^a.
$$

(4.7)

where in the third equality we used $\text{Tr}[\tilde{E}^a e^{-\hat{n} \hat{\theta}} \partial_\phi e^{\hat{n} \hat{\theta}}] = \partial_\phi \theta$, as can be verified for instance from the expression $e^{\hat{n} \hat{\theta}} = \cos \frac{\theta}{2} \mathbf{1} + 2 \sin \frac{\theta}{2} \hat{n}$.\footnote{Equation (4.7) also follows from the general formula (8.1) discussed in appendix A.2.} The situation is then as in the Abelian theory, with $e^{\hat{n} \hat{\theta}}$ playing the role of local $U(1)$ rotation and $X^a$ playing the role of Abelian electric field. Thus, if $\partial_\phi X^a \neq 0$, there exist non-trivial rephasings and the corresponding state is annihilated by the group averaging map.

Let us finally consider the case of a ‘pure background’ state $|0, \tilde{E}\rangle$ with rank($\tilde{E}$) $\geq 2$. It is easy to verify that there are no internal rotations leaving $\tilde{E}^a$ fixed. There could however
be symmetries associated to combinations of diffeomorphisms and local rotations. For this to happen, the tensor
\[
\tilde{q}^{ab} := \text{Tr}[\tilde{E}^a \tilde{E}^b],
\]
should admit symmetries. At an infinitesimal level, this corresponds to the existence of vector fields \(\xi^a\) satisfying:
\[
\mathcal{L}_\xi \tilde{q}^{ab} = 0.
\]
In the rank 3 case, condition (4.9) is not generically satisfied, but only in the special case when the metric admits Killing symmetries. One can similarly show that in the rank 2 case there are no generic solutions to (4.9), see appendix C.3 for an argument.

If the rank \(\geq 2\) triad does admit symmetries, either infinitesimal as in (4.9) or discrete ones, one would then need to determine the corresponding phases. Recall from the last paragraph of section 2.2 that such phases give rise to a homomorphism \(S_{\tilde{E}} \to U(1)\). Thus, a necessary condition for the existence of non-trivial phases is that the group \(S_{\tilde{E}}\) admits non-trivial homomorphism to \(U(1)\).

The possibility of rank 3 triads admitting symmetries with non-trivial phases is an intriguing one, as it would imply the corresponding states are annihilated by the group averaging map. This would be in striking contrast with the analogous quantization in metric variables [12], where all metrics yield non-trivial Diff-invariant states. It is however unknown to us whether there actually exist rank 3 triads admitting non-trivial rephasings. In appendix A.3 we discuss further situations where the phase can be shown to be vanishing.

5. Group averaging: the example of standard LQG

In this section we construct the Diff group averaging map of \(SU(2)\) gauge invariant spin networks [5, 13] through an application of the strategy described in section 3.8 Our considerations in the well understood context of LQG in this section serve as a preview of similar considerations in the context of the KS representation in the following sections.

As mentioned in section 1, standard LQG is the \(\tilde{E} = 0\) sector of the KS representation so that the kinematic states \(|s, \tilde{E} = 0\rangle = |s\rangle\) are labeled by \(SU(2)\) invariant spin networks. The automorphism group \(\text{Aut}\) then reduces to the spatial diffeomorphism group Diff. The sum over states form of the (putative) group averaging map (3.14) applied to the spin net state \(|s\rangle\) is
\[
\eta(|s\rangle) = \eta_{[s]} \sum_{a \in \text{Aut}_\text{Orb}(s)} (U(a)|s\rangle)^\dagger.
\]
We move to a less cumbersome notation tuned to this standard LQG context as follows. As indicated above we have that \(\text{Aut} = \text{Diff}\) so that \(\text{Sym}_s := \{\phi \in \text{Diff} : \phi(s) = s\}\) is its symmetry group. Let \(\text{Diff}/\text{Sym}_s\) be the set of right cosets of Diff by \(\text{Sym}_s\). This set is just the set of distinct \(\text{Sym}_s\)-orbits \(\phi \text{Sym}_s \subset \text{Diff}\) for all \(\phi \in \text{Diff}\). The group averaging map (5.1) may then be written as
\[
\eta(|s\rangle) := \eta_{[s]} \sum_{c \in \text{Diff}/\text{Sym}_s} (U(\phi_c)|s\rangle)^\dagger.
\]
Here \(\phi_c\) is a choice of representative diffeomorphism on each orbit \(c\) and we remind the reader that \(\eta_{[s]}\) is a yet to be determined positive number obeying \(\eta_{[\phi(s)]} = \eta_{[s]} \forall \phi \in \text{Diff}\).

\[\text{Whereas the end result of this section is well known, the derivation we present is slightly different, and in our opinion more transparent, than the standard derivation.}\]
The next step in our strategy is to identify the superselection sector containing $|s\rangle$. This is done as follows [5, 13]. Let $\mathcal{D}$ of 3.1 be the finite span of spin net states and so $O : \mathcal{D} \to \mathcal{D}$ of equation (3.4) is a diffeomorphism invariant operator. Let the coarsest graph underlying a spin net $s$ be denoted by $\gamma(s)$. Let $|s_1\rangle, |s_2\rangle$ be a pair of spin net states. Consider the matrix element $\langle s_1|O|s_2\rangle$. Suppose that there are infinitely many diffeomorphisms $\phi_i$ each of which leave $\gamma(s_1)$ invariant but yield a distinct image when applied to $\gamma(s_2)$. From the commutativity property of observables with diffeomorphisms one has,

$$
\langle s_2|O|s_1\rangle = \langle s_2|OU^\dagger (\phi)|s_1\rangle \quad (5.3)
$$

$$
= \langle s_2|U^\dagger (\phi)O|s_1\rangle. \quad (5.4)
$$

Thus, the state $O|s_1\rangle$ has the same component along spin networks of the form $|\phi(s_2)\rangle$. Since there are infinitely many of them, it follows from $O : \mathcal{D} \to \mathcal{D}$ that $\langle s_2|O|s_1\rangle = 0$. This implies that $|s_1\rangle, |s_2\rangle$ are superselected at the kinematic level.

We show now that if $\gamma(s_1) \neq \gamma(s_2)$, there are infinitely many elements of Diff which move one of them, say $\gamma(s_2)$, and keep the other invariant. Accordingly, let $\gamma(s_1) \neq \gamma(s_2)$. Consider the coarsest graph $\gamma(s_1, s_2)$ which underlies both $s_1$ and $s_2$. It follows that there exists an edge $e_{1,2} \in \gamma(s_1, s_2)$ such that $e_{1,2}$ is contained in some edge $e_2 \in \gamma(s_2)$ and such that $\text{Int}(e_{1,2}) \cap \gamma(s_1) = \emptyset$. Since $\gamma(s_1, s_2)$ is the finite union of closed semianalytic edges it follows that there exists a small enough open neighbourhood $U_p$ of a point $p \in \text{Int}(e_{1,2})$ such that $U_p \cap \gamma(s_1) = \emptyset$ and such that $U_p \cap \gamma(s_2)$ is a connected subset of $e_2$. Next, consider any semianalytic vector field $v_2$ on $\Sigma$ which is transverse to $e_2$ in $U_p \cap \gamma(s_2)$. Let $f$ be a semianalytic function compactly supported within $U_p$. It follows that there are infinitely many diffeomorphisms generated by the vector field $f v_2$ which yield distinct images of $\gamma(s_2)$ but leave $\gamma(s_1)$ invariant.

Thus, we have that $|s_1\rangle, |s_2\rangle$ are in different kinematical superselection sectors unless $\gamma(s_1)$ coincides with $\gamma(s_2)$. Equivalently from equation (3.12) it follows that $\eta(|s_1\rangle), \eta(|s_2\rangle)$ lie in the same superselection sector only if there exists a diffeomorphism $\phi$ such that $\phi(\gamma(s_2)) = \gamma(s_1)$. Thus the diffeomorphism invariant superselection sector containing $|s\rangle$ is made up of all spinnets $|s'\rangle$ such that $\gamma(s') = \phi(\gamma(s))$ for some diffeomorphism $\phi$.

Having determined the superselection sectors, the final step in our strategy is to determine the ambiguity coefficients in such sector. A useful heuristic idea underlying this determination is that the coefficients $\eta_{\text{Sym}}$ in (5.2) should be proportional to the ‘size’ of the symmetry group $\text{Sym}$ in order to compensate for the quotient space being summed over: $\sum_{\text{Diff}} = |\text{Sym}| \sum_{\text{Diff/Sym}}$. Now, even though there is no sense in ‘$|\text{Sym}|$’, one could attempt to make sense of relative sizes of symmetry groups of spinnets that belong to the same superselection sector. One way in which this could be done is to identify a suitable ‘reference subgroup’ $\text{Sym}_{\text{ref}}$ of $\text{Sym}$, which is the same for all $s$ in the same kinematic level superselection sector and such that $|\text{Sym}/\text{Sym}_{\text{ref}}| < \infty$.

To implement this idea we proceed as follows. Let

$$
\text{Sym}_{\text{ref}} := \{ \phi \in \text{Diff} : \phi(e) = e \quad \forall e \in \gamma(s) \}\), \quad (5.5)
$$

be the ‘trivial action group’ of diffeomorphisms preserving the oriented edges of the graph $\gamma(s)$ (denoted $\text{TDiff}_{\gamma(s)}$ in [13]). It is easy to verify that $\text{Sym}_{\text{ref}}$ is a normal subgroup of $\text{Sym}$. The corresponding quotient group

$$
D_s := \text{Sym}/\text{Sym}_{\text{ref}}^0, \quad (5.6)
$$

9 By kinematic level, we mean kinematic states which are mapped to each other exclusively by gauge invariant observables, as opposed to being mapped on to each by gauge invariant observables or diffeomorphisms.
is the group of discrete symmetries of allowed edge permutation of the spin network $s$. It is a finite group and we denote by $|D_s|$ its number of elements or 'size'. From the discussion around equation (5.4) and from the finiteness of $D_s$ it follows that $\text{Sym}_0^0$ can play the role of the desired reference group $\text{Sym}_0^{\text{ref}}$. Let us now see in detail that this is indeed what happens.

Let $s_1$ and $s_2$ be two spin networks based on diffeomorphic graphs (for otherwise property 3 trivializes). We want to impose the condition $\eta(O(s_1))|O(s_2)\rangle = \eta((s_1))|O^\dagger(s_2)\rangle$, \hspace{1cm} (5.7)

for all $O \in \mathcal{O}$. Since $O : \mathcal{D} \to \mathcal{D}$, the vector $O(s_1)$ admits an expansion of the form,

$$O(s_1) = \sum_{i=1}^{n} \lambda_i U(\phi_i)|s_2\rangle + |\chi\rangle \quad \text{with} \quad \langle s_2|U(\phi)|\chi\rangle = 0 \quad \forall \phi \in \text{Diff}. \hspace{1cm} (5.8)$$

The vectors $\lambda_i U(\phi_i)|s_2\rangle$ represent the components of $O(s_1)$ along the orbit of $|s_2\rangle$ and are taken to be orthogonal; $|\chi\rangle$ encodes the remaining vectors orthogonal to the span of the orbit of $|s_2\rangle$.

We now use (5.8) to evaluate both sides of (5.7). The left-hand side becomes

$$\eta(O(s_1))|s_2\rangle = \sum_{i} \bar{\lambda}_i \eta(U(\phi_i)|s_2\rangle)|s_2\rangle = \eta_{[s_1]} \sum_{i} \bar{\lambda}_i. \hspace{1cm} (5.9)$$

To evaluate the right-hand side, we first rewrite $\eta(|s_1\rangle)$ as a sum over $\text{Sym}_0^{\text{ref}}$ cosets as follows. Consider the auxiliary map defined by

$$\eta^0(|s\rangle) := \eta_{[s]} \sum_{c \in \text{Diff}/\text{Sym}_0^{\text{ref}}} (U(\phi_c)|s\rangle)^\dagger. \hspace{1cm} (5.10)$$

It then follows that:

$$\eta^0(|s\rangle) = |D_s|\eta(|s\rangle). \hspace{1cm} (5.11)$$

Using (5.10) and (5.11), the right-hand side of (5.7) becomes:

$$\eta(|s_1\rangle)|O^\dagger(s_2)\rangle = |D_{s_1}|^{-1} \eta_{[s_1]} \sum_{c \in \text{Diff}/\text{Sym}_0^{s_1}} \langle s_2|OU(\phi_c)|s_1\rangle$$

$$= |D_{s_1}|^{-1} \eta_{[s_1]} \sum_{c \in \text{Diff}/\text{Sym}_0^{s_1}} \langle s_2|U(\phi_c)O|s_1\rangle$$

$$= |D_{s_1}|^{-1} \eta_{[s_1]} \sum_{c \in \text{Diff}/\text{Sym}_0^{s_1}} \sum_{i} \bar{\lambda}_i \langle s_2|U(\phi_c)U(\phi_i)|s_2\rangle$$

$$= |D_{s_1}|^{-1} \eta_{[s_1]} \sum_{i} x_i \sum_{c \in \text{Diff}/\text{Sym}_0^{s_1}} \langle s_2|U(\phi_c)U(\phi_i)|s_2\rangle$$

$$=: \eta_{[s_1]} \sum_{i} x_i \lambda_i \hspace{1cm} (5.12)$$

where

$$x_i := |D_{s_1}|^{-1} \sum_{c \in \text{Diff}/\text{Sym}_0^{s_1}} \langle \phi_c U(\phi_c)|s^{(i)}_2\rangle. \hspace{1cm} (5.13)$$

$$s^{(i)}_2 := \phi(s_2). \hspace{1cm} (5.14)$$
Notice that the interchange in the sums order leading to (5.12) is valid since only a finite number of terms in the sum over Diff/Sym\textsubscript{0} is non-zero. We now focus on evaluating (5.13). We first notice that since ∣s\textsubscript{2}⟩\langle O|s\textsubscript{1}⟩ = λ\textsubscript{i} ≠ 0 it follows that γ\textsuperscript{0}(s\textsubscript{2}⟩\langle O|s\textsubscript{1}) = γ\textsuperscript{0}(s\textsubscript{1}⟩\langle O|s\textsubscript{1}) in particular Sym\textsubscript{0} = Sym\textsubscript{0} and the sum is independent of the orbit representative choices c → φ\textsubscript{c}. Let c\textsubscript{i} := φ\textsuperscript{-1}\textsubscript{c}Sym\textsubscript{0} be the Sym\textsubscript{0} orbit through φ\textsuperscript{-1}. Such term gives a contribution of 1 to the sum in (5.13). All other Sym\textsubscript{0} orbits can be obtained as φ\textsuperscript{-1}\textsubscript{c}Sym\textsubscript{0} for appropriate φ ∈ Diff. Non-zero contributions will then come from elements φ ∈ Sym\textsubscript{0}. It then follows that there are |D\textsubscript{2}⟩ of such terms and so we obtain:

\[ x\textsubscript{i} = |D\textsubscript{1}|^{-1}|D\textsubscript{2}|. \tag{5.15} \]

The key property of Diff invariance of the relative sizes implies the numbers above are independent of i:

\[ |D\textsubscript{2}| = |D\textsubscript{1}|. \tag{5.16} \]

Assuming O is such that \[ \sum \lambda\textsubscript{i} ≠ 0 \] (for if no such observable exist then the states would be superselected) we conclude that:

\[ \eta(s\textsubscript{1})/\eta(s\textsubscript{2}) = |D\textsubscript{1}|/|D\textsubscript{2}|. \tag{5.17} \]

Thus, in order to satisfy (5.7), within each sector the ambiguity coefficient must be set as

\[ \eta(s) = C|D\textsubscript{1}| \tag{5.18} \]

for some constant C > 0.

Since the above presentation slightly differs from the traditional one, let us explicitly see how the two coincide. In the following we take γ = γ(s). In [5, 13], the group of graph symmetries:

\[ \text{Sym}_{γ} := \{ φ ∈ \text{Diff} : φ(γ) = γ \} \tag{5.19} \]

(denoted Diff\textsubscript{γ} in [13]), is used instead of Sym\textsubscript{γ}, together with the corresponding discrete group

\[ D_{γ} := \text{Sym}_{γ}/\text{Sym}_{0}, \tag{5.20} \]

(denoted GS\textsubscript{γ} in [13], where Sym\textsubscript{0} ≡ Sym\textsubscript{0} is given by (5.5). We now use these groups to rewrite the group averaging map (5.2) (we set C = 1 in (5.18)):

\[ \eta(s) = |D\textsubscript{1}| \sum_{c ∈ \text{Diff}/\text{Sym}} (U(φ\textsubscript{c})(s))\dagger = |D\textsubscript{1}| \sum_{c ∈ \text{Diff}/\text{Sym}, d ∈ \text{Sym}/\text{Sym}} (U(φ\textsubscript{c, d})(s))\dagger = |D\textsubscript{1}| \sum_{c ∈ \text{Diff}/\text{Sym}} \sum_{d ∈ \text{Sym}/\text{Sym}} (U(φ\textsubscript{c, d})(s))\dagger = \sum_{c ∈ \text{Diff}/\text{Sym}} \sum_{d ∈ \text{Sym}} (U(φ\textsubscript{c, d})(s))\dagger \tag{5.21} \]

where φ\textsubscript{c} and φ\textsubscript{d} are representatives of the cosets c′ ∈ Diff/\textsuperscript{Sym} and d ∈ Sym\textsubscript{γ}/Sym\textsubscript{γ} ≡ D\textsubscript{γ}/D\textsubscript{c}. Expression (5.21) takes precisely the form of the group averaging as given in [5, 13].

We conclude with a few remarks regarding the role of the groups Sym\textsubscript{0} and Sym\textsubscript{0} (or Sym\textsubscript{0}). We first point out that for both the characterization of superselection sectors and the proof of well definedness of the group averaging, one does not require an explicit characterization of these groups. Indeed, we are not aware of such characterization and they...
may well be complicated groups with infinitely many connected components. Superselection sectors can be identified by a ‘large enough’ subgroup of $\text{Sym}_0$ of the type described at the beginning of the section. A perusal of our argumentation in this section indicates that this ‘large enough’ subgroup is as follows. Note that every edge $e$ with endpoints removed admits an open cover $\{U^e_\alpha\}$ in $\Sigma$ which does not intersect any other edge. Consider semianalytic vector fields which compactly supported in each open set $U^e_\alpha$ and which are parallel to the edge tangent wherever their support intersects $e$. Such vector fields generate the ‘large enough’ subgroup of $\text{Sym}_0^0$ which suffices to identify the kinematic and, from equation (3.12), the diffeomorphism invariant, superselection sector containing $|s\rangle$. As shown above the underlying graphs of spin nets in a single such kinematic sector are constrained to coincide. Thus all but a finite set of data (namely the edge colorings and the vertice intertwiners) are completely constrained. As a result the maps on the remaining finite ‘free data’ are easily understood: they are just the finite edge permutation groups referred to above.

To summarize: the identification of a sufficiently large (infinite dimensional) subset of $\text{Sym}_0^0$ suffices to (1) isolate generic superselection sectors and (2) constrain an ‘infinite dimensional’ part of the label set of states in each such sector.

The left over ‘free data’ in the label set is then sufficiently restricted so as to be finite. In the LQG case, this finiteness translates to that of the Diff invariant numbers $|D_s|$, which in turn allows a well defined, consistent evaluation of ambiguity coefficients.

Finally, we note that while the finiteness of $|D_s|$ allows a proof of existence of a consistent group averaging map, the explicit identification of $D_s$ and it cardinality is still a non-trivial exercise, the degree of non-triviality increasing with complexity of the graph $\gamma(s)$ underlying $s$.

6. Superselection sectors in the KS representation

A key ingredient in the construction of the group averaging map is the identification of superselection sectors. As reviewed in the previous section, in the case of standard LQG superselection sectors are labeled by (diffeomorphism equivalence class of) graphs. In the present section we describe superselection sectors in the KS representation. The labeling will now include certain sets of the manifold—referred to as rank sets—associated to the background triad label of the KS states. The characterization obtained can be summarized in equations (6.10)–(6.12) below\(^\text{10}\).

Given a triad $\vec{E}^a$, we define the following rank sets:

\[
V_0 := \{x \in \Sigma : \text{rank}(\vec{E}) = 0\},
\]

(6.1)

\[
V_1 := \{x \in \Sigma : \text{rank}(\vec{E}) = 1\},
\]

(6.2)

\[
V_2 := \{x \in \Sigma : \text{rank}(\vec{E}) \geq 2\}.
\]

(6.3)

As mentioned in the introduction, semianalyticity of the triad ensures these are ‘nice’ sets in the sense that they can be expressed as a finite union of semianalytic submanifolds. This follows from properties of the zero level sets, and their complements, of semianalytic functions (see [7, 9] and appendix B for definitions and results on semianalytic category). More in detail, the rank sets can be described by semianalytic functions as follows. Fix an auxiliary semianalytic metric $q_{ab}^0$ on $\Sigma$ and let

\[
u_i^a := \epsilon_{ijk}q^{\frac{1}{2}}n_{abc}\vec{E}^b_j\vec{E}^c_k.
\]

(6.4)

\(^\text{10}\)The labeling described here is not necessarily an exhaustive one (for instance additional labels are needed in the presence of rank 1 electric fields, see section 8.2). It however suffices to fully characterize the sectors studied in section 7.
where $\hat{q} = \det(q_{ab})$. It is then easy to verify that:

\[ V_0 = \{ h(x) = 0 \}, \]

\[ V_1 = \{ h(x) \neq 0 \} \cap \{ g(x) = 0 \}, \]

\[ V_2 = \{ g(x) \neq 0 \}. \]

As discussed in appendix B, sets that are defined by semianalytic functions such as (6.7)–(6.9) can always be expressed as finite union of semianalytic submanifolds. In particular, there is a notion of dimension of such sets, given by the maximum dimension of its submanifold components.

To appreciate the non-triviality of this property, let us compare with a case where we had smooth, rather than semianalytic, fields. Then $V_0$ would be given by the zero level set of a smooth function $h$. By continuity we know $V_0$ is a closed set. But it turns out that any closed set of the manifold can be realized as the zero level set of some smooth function (see for instance proposition 3.3.6 of [14]). Thus, we cannot a priori guarantee any property on $V_0$ (other than it being closed). For instance $V_0$ could have fractal-like structure and thus be very far from a ‘finite union of submanifolds’ as in the semianalytic case.

We now use the rank sets to characterize superselection sectors. As in the standard LQG case it is convenient to first characterize ‘kinematical superselection sectors’, i.e. sectors generated by states admitting non-zero matrix elements on observables. As reviewed in section 5, in the standard LQG case the kinematical sectors correspond to spin networks based on the same graph, while the superselection sectors of the group averaging map are given by spin networks based on diffeomorphic graphs.

We now show the following kinematical superselection conditions for KS states $|s, \bar{E}\rangle$ and $|s', \bar{E}'\rangle$:

\[ \langle s, \bar{E}|O|s', \bar{E}'\rangle \neq 0 \Rightarrow (i) \ V_0 = \bar{V}_0, \quad \bar{V}_2 = \bar{V}_2 \]

\[ (ii) \ dim(V_n \cap (\cup_{n \neq 0} V'_n)) < 3, \quad n = 0, 1, 2 \]

\[ (iii) \ \gamma(s) \cap \bar{V}_0 = \gamma(s') \cap \bar{V}_0, \]

where $V_n, V'_n$ are the rank sets of $\bar{E}, \bar{E}'$ and $\gamma(s), \gamma(s')$ are the graphs of the spin networks. Notice that the set $V_n \cap (\cup_{n \neq 0} V'_n)$ featuring in (ii) is again determined by zeros (and their complements) of semianalytic functions. It is thus composed of finite union of semianalytic submanifolds, and ‘dim’ in (ii) refers to the maximum dimension of its submanifold components.

The logic in showing (6.10)–(6.12) is the same as in the spin network case: the existence of infinite gauge transformations leaving one of the states fixed and producing infinitely many distinct images on the other implies their matrix elements on observables vanishes, since observables map a KS state into finite linear combination of KS states and strongly commute with gauge transformations. Thus in the following we show how if any of the conditions in (6.10)–(6.12) fails to be satisfied, there exist infinitely many of such gauge transformations.

We first focus on (6.10) and (6.11) in a ‘pure background’ case with no non-trivial spin networks. We start with (6.10). It will be convenient to consider the following unions of rank sets:

\[ V_{01} := V_0 \cup V_1 = \{ g(x) = 0 \}, \]

\[ V_{02} := V_0 \cup V_2 = \{ g(x) \neq 0 \}. \]
\[ V_{12} := V_1 \cup V_2 = \{ h(x) \neq 0 \}, \]

and similarly for the primed sets associated to \( E' \). To show the first equality in (6.10), assume by contradiction that \( V_0 \neq \tilde{V}_0 \). Then there exist a point \( x \) such that \( x \in \tilde{V}_0 \) and \( x \notin V_0 \). The latter means that there is no open neighborhood of \( x \) that is contained in \( V_0 \). This in turn is equivalent to the statement that for every open neighborhood \( U \) of \( x \), \( U \cap V_{12} \neq \emptyset \). Taking \( U \) such that \( U \subseteq V'_{0} \), and noting that \( V_{12} \) is open, we conclude that \( U \cap V_{12} \neq \emptyset \) is an open set. Diffeomorphisms with support inside such set will generate symmetries of \( \tilde{E}' \) but not of \( E \). The second equality in (6.10) is shown similarly, by noting that \( \tilde{V}_2 = \Sigma \setminus V_0 \). If by contradiction there exist a point \( x \) such that \( x \in \tilde{V}_{01} \) and \( x \notin \tilde{V}_{01} \) we can find an open set \( U \subseteq V_{01} \) such that \( U \cap V_2 \neq \emptyset \). Since \( V_2 \) is open we can again construct infinitely many symmetries of \( E' \) that are not symmetries of \( \tilde{E} \).

Let us now show condition (6.11). Assume by contradiction that \( \dim(V_n \cap (\bigcup_{r \neq \phi} V'_r)) = 3 \). A case by case analysis shows this leads to a contradiction. We use the fact that dimension 3 sets contain open sets.

- \( n = 0 \): there is an open set \( U \subseteq V_0 \cap (V'_1 \cup V'_2) \). Diffeos generated by vector fields with support in \( U \) generate infinite symmetries of \( \tilde{E} \) but not of \( E' \).
- \( n = 1 \): either \( \dim(V_1 \cap V'_0) = 3 \) or \( \dim(V_1 \cap V'_2) = 3 \) (or both). In the first case there are diffeos with support inside an open set of \( V_1 \setminus V'_0 \) that are symmetries of \( \tilde{E}' \) but not of \( E \). In the second case one can find local internal rotations with support inside \( U \subseteq V_1 \cap V'_2 \). These are symmetries of \( \tilde{E} \) but not of \( E' \).
- \( n = 2 \): either \( \dim(V_2 \cap V'_0) = 3 \) or \( \dim(V_2 \cap V'_1) = 3 \) (or both). In the first case one finds diffeos that are symmetries of \( \tilde{E}' \) but not of \( E \). In the second case one finds local internal rotations that are symmetries of \( E' \) but not of \( \tilde{E} \).

We thus conclude (6.11). In the case where spin networks are present, the argument above are still valid, provided one works with small enough open sets so that they do not intersect the spin network graphs.

Finally, condition (6.12) follows from a similar argument as the one given in section 5 for the standard LQG case. A detailed proof is given in appendix C.5.

The above kinematical superselection rules translate into the following conditions for superselection sectors of the group averaging map: a necessary condition for two KS states \( |s, E \rangle \) and \( |\tilde{E}', s' \rangle \) to lie on the same superselection sector is the existence of a diffeomorphism \( \phi \) such that conditions (6.10)–(6.12) hold with \( \phi(V'_n) \) in place of \( V'_n \) and \( \phi(s') \) in place of \( s' \).

### 7. Group averaging in the absence of rank 1 backgrounds

In this section we describe the Aut group averaging map along the lines of standard LQG as described in section 5. For technical reasons we do not consider the most general KS states but focus on certain (superselected) states as detailed below. In particular, rank 1 background triads are excluded in the present discussion (see section 8).

The superselection sectors we restrict attention to are given by KS states \( |s, \tilde{E} \rangle \) such that:

(a) The rank 1 set of the background label \( \tilde{E} \) is of zero measure.
(b) There are no infinitesimal symmetries on the rank 2 or 3 regions, that is, no vector fields exist such that equation (4.9) holds on the open set \( V_2 \).
(c) \( s \) is \( SU(2) \) gauge invariant.

It is easy to verify that a state that does not satisfy (a) and (b) will necessarily lie in a different superselection sector: for a state not satisfying (a) this follows from the discussion
of section 6; for a state not satisfying (b) this follows from the fact that such a state admits a one parameter family of symmetries that are not symmetries of states obeying (b) and hence they are superselected. We comment on restriction (c) in section 9.2.

The considerations of section 6 specialized to the present setting simplify to:

\[ \langle \tilde{E}, s \rangle | \tilde{E}', s' \rangle \neq 0 \Rightarrow (i) \tilde{V}_0 = \tilde{V}_0', V_2 = V_2' \quad (V_1 = \tilde{V}_1 = \emptyset) \tag{7.1} \]

\[ (ii) \gamma(s) \cap \tilde{V}_0 = \gamma(s') \cap \tilde{V}_0, \tag{7.2} \]

where \( V_n, V'_n \) are the rank sets of \( \tilde{E}, \tilde{E}' \) and \( \gamma(s), \gamma(s') \) are the graphs of the spin networks. Conditions (i) and (ii) are the analogue of the kinematical superselection condition of coincident graphs in the standard LQG case. It is easy to verify that in the present setting \( \tilde{V}_0 = V_0 \) and so \( \Sigma = V_0 \cup V_2. \)

Thus the sets in (7.1) cover all the manifold.

We will denote KS labels as: \( \psi = (s, \tilde{E}), \psi' = (s', \tilde{E}') \), etc; and the action of Aut on these labels as: \( a \cdot \psi \equiv (g, \phi) \cdot (s, \tilde{E}) = (\phi(s), a \cdot \tilde{E}) \). The group averaging sum takes the form discussed in section 3:

\[ \eta(|\psi \rangle) = \begin{cases} \eta_{|\psi \rangle} \sum_{e \in \text{Aut}/\text{Sym}_\psi} (U(a_e)|\psi \rangle)^\dagger & \text{if } \text{Sym}_\psi \subseteq \text{Ph}_\psi, \\ 0 & \text{if } \text{Sym}_\psi = \text{Ph}_\psi, \end{cases} \tag{7.3} \]

where \( \text{Sym}_\psi = \text{Sym}_{(s, \tilde{E})} = \{ a \in \text{Aut} : U(a)|s, \tilde{E} \rangle = |s, \tilde{E} \rangle \} = \text{Sym}_s \cap \text{Sym}_{\tilde{E}} \) and \( \phi_s = \text{Sym}_s \cap \text{Ph}_s = \text{Sym}_s \cap S_{\tilde{E}}. \) Since Aut transformations either rephase a KS state or map it into an orthogonal one, it follows that all states appearing in the sum (7.3) are orthogonal so that \( \eta(|\psi \rangle) \) is a well defined element of \( D. \) Then the presence of rephasings is an Aut invariant notion (see appendix A.1), it follows that (7.3) satisfies the first requirement of group averaging. The satisfaction of the second requirement is also easily verified. It thus remains to study the third requirement and the corresponding determination of ambiguity coefficients \( \eta_{|\psi \rangle}. \) The requirement reads:

\[ \eta(|\psi_1 \rangle) [|\psi_2 \rangle] = \eta(|\psi_1 \rangle)|O| \eta(|\psi_2 \rangle) \quad \forall O, |\psi_1 \rangle, |\psi_2 \rangle. \tag{7.4} \]

We focus on the case where \( |\psi_1 \rangle \) and \( |\psi_2 \rangle \) are in the same superselection sector and such that they do not admit non-trivial rephasings, for otherwise (7.4) trivializes to \( 0 = 0 \) (see section 3.3).

Similar to section 5 we expect that the ratio \( \eta_{|\psi_1 \rangle}/\eta_{|\psi_2 \rangle} \) of ambiguity parameters will be given by the ‘relative size’ of the groups \( \text{Sym}_{\psi_1} \) and \( \text{Sym}_{\psi_2}. \) In order to make sense of such ‘relative size’ in a way that is compatible with (7.4) we need, as in the spin network case, a common subgroup of (conjugated versions of) \( \text{Sym}_{\psi_1} \) and \( \text{Sym}_{\psi_2} \) such that (a) the respective quotient spaces are finite and (b) the size of these discrete finite spaces is Aut invariant. The natural analogue of the ‘reference group’ used in the spin network case (5.5) is now given by:

\[ \text{Sym}_\psi^0 = \text{Sym}_{(s, \tilde{E})}^0 = \{ a \in \text{Aut} : a|_{\tilde{E}} = \text{Id}, a(e) = e \quad \forall e \in \gamma(s) \}, \tag{7.5} \]

where \( e \) denotes the edges of \( \gamma(s) \), and \( a(e) = (g, \phi)(e) = \phi(e) \). It is easy to verify that \( \text{Sym}_\psi^0 \) is a normal subgroup of \( \text{Sym}_\psi \). We assume the corresponding quotient group

\[ D_\psi := \text{Sym}_\psi / \text{Sym}_\psi^0 \tag{7.6} \]

is finite, this finiteness being expected from that of the group of allowed edge permutations of \( \gamma(s) \) and of the discrete symmetries of the background triad in \( V_2. \) Finally, definition (7.5) is

\[ \text{since } V_1 \cap \tilde{V}_0 = V_1 \cap \tilde{V}_0, \text{ it follows that } V_1 \cap \tilde{V}_0 \text{ is an open set. Since } V_1 \text{ is of zero measure it cannot contain non-trivial open sets and we conclude that } V_1 \cap \tilde{V}_0 = \emptyset. \text{ This implies } V_1 \subset \tilde{V}_2 \text{ from which is easy to verify that } \tilde{V}_2 = V_1 \text{ and hence } \tilde{V}_0 = \emptyset. \]

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Aut covariant in the sense that it satisfies: \( \text{Sym} \cdot a \| \psi \rangle = a \| \text{Sym} \| \psi \rangle \). This in turn implies the Aut invariance property on the size of the discrete groups: \( |D_\psi| = |D_{\| \psi \rangle} | \).

We can now repeat the argument of section 5 by making the replacements \( s \to \psi, \phi \to a, \text{Diff} \to \text{Aut} \) as follows.

The fact that \( O \) preserves \( \mathcal{D} \) allows us to write \( O\| \psi_1 \rangle \) as a finite linear combination of states:
\[
O\| \psi_1 \rangle = \sum_{i=1}^{n} \lambda_i U(a_i)\| \psi_2 \rangle + \| \chi \rangle \quad \text{with} \quad \langle \psi_2 | U(a) | \chi \rangle = 0 \quad \forall \ a \in \text{Aut},
\]
where \( \lambda_i \) are non-zero complex numbers, the vectors \( U(a_i)\| \psi_2 \rangle \) are orthonormal and \( | \chi \rangle \) is in the orthogonal complement of the orbit of \( | \psi_2 \rangle \). The left-hand side of (7.4) becomes
\[
\eta(O(\| \psi_1 \rangle))|\| \psi_2 \rangle \rangle = \eta(\| \psi_1 \rangle) \sum_i \lambda_i.
\]

To evaluate the right-hand side, we write \( \eta(\| \psi_1 \rangle) \) as a sum over \( \text{Sym} \| \psi_1 \rangle \) orbits:
\[
\eta(\| \psi_1 \rangle) = |D_{\| \psi_1 \rangle} |^{-1} \sum_{c \in \text{Aut}/\text{Sym} \| \psi_1 \rangle} (U(a_i)|\| \psi_1 \rangle)^{\dagger}.
\]

After a few steps as in (5.12) we obtain:
\[
\eta(\| \psi_1 \rangle)\| O| \| \psi_2 \rangle \rangle = |D_{\| \psi_1 \rangle} |^{-1} \sum_i \lambda_i e^{-i\omega_i} \sum_{c \in \text{Aut}/\text{Sym} \| \psi_1 \rangle} \langle \psi_2 | U(a_i) | \| \psi_2 \rangle^{\dagger},
\]
where \( \psi_2^{(i)} \equiv a_i \cdot \psi_2 \) and \( a_i = \alpha(a_i, \vec{E}_2) \) so that \( U(a_i)|\| \psi_2 \rangle = e^{i\omega_i}|\| \psi_2 \rangle^{(i)} \). Since \( \langle \psi_2^{(i)} | O | \psi_1 \rangle = e^{i\omega_i} \lambda_i \neq 0 \) we conclude \( \text{Sym} \| \psi_2 \rangle = \text{Sym} \| \psi_1 \rangle \). Let \( c_i := a_i^{-1} \text{Sym} \| \psi_1 \rangle \in \text{Aut}/\text{Sym} \| \psi_1 \rangle \) be the \( \text{Sym} \| \psi_1 \rangle \) orbit along \( a_i^{-1} \). Such a term gives a contribution of \( e^{i\omega_i} \) in the second sum of (7.10). All other orbits can be obtained as \( a_i^{-1} \text{Sym} \| \psi_1 \rangle \) for appropriate \( a \in \text{Aut} \). Non-zero contributions come from elements \( a \in \text{Sym} \| \psi_1 \rangle \) and so there are \( |D_{\| \psi_1 \rangle}| \) terms contributing to the sum. Finally, the Aut invariance of the sizes of the discrete groups imply \( |D_{\| \psi_1 \rangle}| = |D_{\| \psi_2 \rangle}| \) i.e. 1, 2, ..., n.

Equating the result with (7.8) we obtain: \( \eta(\| \psi_1 \rangle) = |D_{\| \psi_1 \rangle}|^{-1} \eta(\| \psi_1 \rangle) \). Thus, in order to satisfy (7.4) for KS states \( \psi \) in the superselection sector of interest we must set:
\[
\eta(\| \psi \rangle) = C|D_{\| \psi \rangle}|
\]
for some constant \( C > 0 \).

8. Rank 1 background triads

In this section we summarize our (partial) understanding of group averaging related subtleties in the case of background triad labels with rank 1 support sets of non-zero measure.

The organization of the material is as follows. In section 8.1 we discuss symmetries of triads with rank 1 regions and their associated phases. In section 8.2 we describe superselection conditions arising from these symmetries and comment on difficulties when spin networks are present. In section 8.3 we illustrate subtleties of implementing the group averaging of section 7 in the presence of rank 1 backgrounds.

8.1. Symmetries of rank 1 triads and their phases

We focus on infinitesimal symmetries, from which the corresponding phase can be obtained by the following formula that we prove in appendix A.1:
\[
\alpha(e^{i(A,\vec{E})}, \vec{E}) = \int_\Sigma \text{Tr}[\vec{E}\partial_\xi A], \quad \text{for} \quad (\Lambda, \xi) : [\Lambda, \vec{E}^\mu] - \partial_\xi \vec{E}^\mu = 0.
\]
8.1.1. Constant rank case. Recall from section 4.2 that rank 1 triads:

\[ \hat{E}^a = \hat{n} X^a \]  
(8.2)

average to zero if \( \partial_a X^a \neq 0 \). Thus we restrict attention to ‘divergence free’ rank 1 triads, i.e. \( \partial_a X^a = 0 \). In this first part we focus in the case where (8.2) holds everywhere on \( \Sigma \) with \( X^a \neq 0 \); in the next subsection we discuss the case where (8.2) holds on some region of \( \Sigma \) of non-zero measure.

Symmetries of (8.2) are given by \( a = (g, \phi) \in \text{Aut} \) satisfying

\[ (g, \phi) \cdot (\hat{n} X^a) = g(\phi_n \hat{n}) g^{-1} \phi_a X^a = \hat{n} X^a. \]

Since ‘internal’ and ‘external’ indices are factorized, the symmetry condition (8.3) translates into the two equations: \( \phi_a X^a = X^a \) and \( g \phi_n \hat{n} g^{-1} = \hat{n} \). At the infinitesimal level, they correspond to the following conditions:

\[ 0 = \mathcal{L}_\xi X^a, \]
(8.4)
\[ 0 = [\Lambda, n] - \mathcal{L}_\xi \hat{n}. \]

The most general solution of (8.5) is given by,

\[ \Lambda = -[\mathcal{L}_\xi \hat{n}, \hat{n}] + \theta \hat{n}, \]

for any function \( \theta : \Sigma \to \mathbb{R} \). When \( \xi^a = 0 \), (8.6) reduces to the local rotations about \( \hat{n} \) discussed in section 4.2. Non-zero vector fields \( \xi^a \) satisfying (8.4) give rise to additional symmetries we now describe.

Let us first rewrite equation (8.4) in a way that facilitates the discussion of its solutions and corresponding phases. Denote the two-form dual to \( X \) by \( \omega \) so that \( \omega_{ab} := \eta_{abc} X^c \). The divergence-free property of \( X \) translates into the condition \( d\omega = 0 \). From the Lie derivative formula: \( \mathcal{L}_\xi \omega = (\partial_i + \xi_i) d\omega \), it follows that (8.4) is equivalent to \( d\mathcal{L}_\xi \omega = 0 \). Let us now restrict attention to the case of simply connected \( \Sigma \) so that we can rewrite this condition as \( \partial_i \omega = df \) for some function \( f \). To summarize: for divergence free vector fields \( X^a \), we have that

\[ \mathcal{L}_\xi X^a = 0 \iff \partial_i \omega = df \quad (\text{case of simply connected } \Sigma) \]
\[ \iff X^{[a} \xi^{b]} = -\frac{1}{2} \eta^{abc} \partial_c f \quad \text{for some } f : \Sigma \to \mathbb{R}, \]

where the third equation reformulates the second equation in terms of the original vector field \( X^a \). The simplest class of solutions to (8.7) are given by taking \( f = \text{constant} \) and \( \xi^a \) of the form \( \xi^a = g X^a \) for any density weight \( -1 \) scalar \( g \). There may also be ‘transversal’ symmetries with \( \partial_a f \neq 0 \). The issue of fully characterizing these additional transversal symmetries remains an open problem to us which we comment on below. Fortunately it is still possible to show the phases vanish without explicit knowledge of the solutions to (8.7): using (8.1) with \( (\Lambda, \xi) \) satisfying (8.6) and (8.7) we find,

\[ \alpha(e^{(\Lambda, \xi)}, \hat{n} X) = \int_\Sigma X^a \text{Tr}[\hat{n} \partial_a \Lambda] = -\int_\Sigma X^a \text{Tr}[\partial_a \hat{n} \Lambda] \]
\[ = \int_\Sigma X^a \xi^{[abc]} \text{Tr}[\partial_a \hat{n} \partial_b \hat{n}\partial_c \hat{n}] = \int_\Sigma X^a \xi^{[abc]} \text{Tr}[\partial_a \hat{n} \partial_b \hat{n}\partial_c \hat{n}] \]
\[ = -\int_\Sigma \eta^{abc} \partial_c f \text{Tr}[\partial_a \hat{n} \partial_b \hat{n}\partial_c \hat{n}] = \int_\Sigma \eta^{abc} \partial_c f \text{Tr}[\partial_a \hat{n} \partial_b \hat{n}\partial_c \hat{n}] = 0. \]

(8.8)

The vanishing follows from the fact that since \( n_1^2 + n_2^2 + n_3^2 = 1 \), the differentials \( \partial_i n_i \), \( i = 1, 2, 3 \) are not linearly independent, and thus their antisymmetric product vanishes.
We conclude the section by summarizing our understanding on the solutions of (8.7) and hence on the continuous symmetries of rank 1 triads. Some additional details are given in appendix C.4.

We start by describing a class of configurations for which we do have a complete description of \( \xi^a \) and \( f \) satisfying (8.7). Since \( \omega \) is closed and nowhere vanishing, it can be thought of as a presymplectic form on \( \Sigma \). Proceeding in analogy with the description of phase spaces with gauge symmetries (see for instance appendix B of [15]), consider the ‘reduced phase space’ \( \tilde{\Sigma} := \Sigma / \sim \) where \( x \sim y \) iff they lie along the same orbit of \( X^a \). Let us now restrict to the case where \( X^a \) is such that \( \tilde{\Sigma} \) is a (two dimensional) manifold. In this case, \( \omega_{ab} \) can be realized as the pull-back under the projection map \( \pi : \Sigma \to \tilde{\Sigma} \) of a closed two form \( \tilde{\omega}_{ab} \) on \( \tilde{\Sigma} \). Solutions to (8.7) can then be described as follows: take any function \( \tilde{f} \) on \( \tilde{\Sigma} \). Let \( \tilde{\xi}^a = \tilde{\alpha}^{ab} \tilde{\partial}_b \tilde{f} \) be the corresponding ‘Hamiltonian’ vector field on \( \tilde{\Sigma} \). Such vector field defines a (non-unique) vector field \( \xi^a \) on \( \Sigma \) such that \( \pi_* \xi^a = \xi^a \). The ambiguity in the definition of \( \xi^a \) is given by vector fields parallel to \( X^a \). It is then easy to verify that (i) the vector field \( \xi^a \) obtained in this way solves (8.7) with \( f = \tilde{f} \circ \pi \) and (ii) any solution to (8.7) can be seen as arising from this construction. In this manner we obtain an explicit description of the solution to (8.7).

In a general case, where \( \tilde{\Sigma} \) may not be a manifold we have less control over the ‘transversal’ symmetries. Even though there exists a simple local characterization of solutions to (8.7) (see appendix C.4), we do not know if these local solutions extend to global solutions of (8.7).

### 8.1.2. Varying rank case.

We now focus in the case of generic triads, by which we mean triads of variable rank that do not have infinitesimal symmetries associated to the rank \( \geq 2 \) regions. In this case the phases associated to infinitesimal symmetries of the triad are determined by the rank 1 region:

\[
\alpha \left( e^{(X,\xi)} , \tilde{E} \right) = \int_{\Sigma} \tilde{E}^a \partial_a \Lambda = \sum_{n=0}^{2} \int_{V_n} \tilde{E}^a \partial_a \Lambda = \int_{V_1} \tilde{E}^a \partial_a \Lambda.
\]  

(8.9)

Since \( \tilde{E}^a \vert_{V_0} = 0 \) and \( \Lambda \vert_{V_1} = 0 \). From the analysis of the constant rank 1 case, it is immediate to conclude that

\[
[L, \tilde{E}^a] - L_{\xi} \tilde{E}^a = 0 \Rightarrow \Lambda \vert_{V_1} = -[L_{\xi} \hat{n}, \hat{n}] + \theta \hat{n}, \quad L_{\xi} X^a \vert_{V_1} = 0,
\]  

(8.10)

for some function \( \theta \) on \( V_1 \). Let us now restrict attention to the case where \( \hat{V}_{01} \) is simply connected so that the second condition can be written as in (8.7):

\[
\partial_a \tilde{f} = -\eta_{abc} X^b \xi^c \quad \text{on} \, \hat{V}_{01}
\]  

(8.11)

for some function \( f \). In (8.11) we have extended \( X^a \) to \( \hat{V}_{01} \) by setting \( X^a \vert_{\hat{V}_0} := 0 \). In order to extend the computation (8.8) to the present case, we need to take into account boundary terms\(^{12}\). There are two integration by parts in (8.8), occurring in the second and sixth equalities. These yield the following contributions:

\[
\int_{\hat{V}_1} \tilde{E}^a \partial_a \Lambda = \int_{\hat{V}_1} dS_{\hat{n}} X^a \text{Tr}[\hat{n} \Lambda] - \int_{\partial \hat{V}_1} dS_{\hat{n}} \eta^{abc} f \text{Tr}[\partial_b \hat{n} \partial_c \hat{n} \hat{n}],
\]  

(8.12)

for \( \Lambda \) as in (8.10) and \( f \) as in (8.11). The first integral is given by

\[
\int_{\partial \hat{V}_1} dS_{\hat{n}} X^a \text{Tr}[\hat{n} \Lambda] = \int_{\partial \hat{V}_1} dS_{\hat{n}} X^a \theta.
\]  

(8.13)

\(^{12}\) We assume the set \( V_1 \) is such that we can apply Stokes theorem. It is likely that the rank sets are nice enough so that Stokes theorem applies, but we have not studied this issue in detail.
Since $X^a|_{\partial V_1\cap\partial V_2} = 0$ and $\theta|_{\partial V_1\cap\partial V_2} = 0$ (by continuity of $\bar{E}^a$ and $\Lambda$ respectively), it follows that (8.13) vanishes. The second integral requires lengthier discussion. To keep notation simple we now assume that $V_1$ is connected, the generalization to multiple components being straightforward. To begin with we notice that since $X^a|_{\partial V_1\cap\partial V_2} = 0$ and $\xi^a|_{\partial V_1\cap\partial V_2} = 0$, equation (8.11) implies $\partial_a f|_{\partial V_1} = 0$. In particular, $f$ takes a constant value on each connected component of the boundary of $V_1$. Denoting by $\partial V_1^{(k)}$, $k = 1, \ldots, N$ the connected components of $\partial V_1$ and by $f_k = f|_{\partial V_1^{(k)}}$ the constant value taken by $f$ on each component, the integral can be written as:

$$
\int_{\partial V_1} d\Sigma_0 \hat{n}^a f \text{Tr}[\partial_a \hat{n}\hat{n}, \hat{n}\hat{n}] = \frac{1}{2} \sum_{k=1}^N f_k \int_{\partial V_1^{(k)}} \text{Tr}[\hat{n}\hat{n} \wedge \hat{n}\hat{n}]. \tag{8.14}
$$

We notice that the two-form being integrated on the right-hand side is the pull-back of the area form on the two-sphere under the map $\hat{n} : \partial V_1^{(k)} \to S^2 \subset \text{su}(2)$. If $m_k \in \mathbb{Z}$ is the degree of such map$^{14}$, the integral is given by:

$$
\int_{\partial V_1^{(k)}} \text{Tr}[\hat{n}\hat{n} \wedge \hat{n}\hat{n}] = 4\pi m_k, \quad \sum_{k=1}^N m_k = 0. \tag{8.15}
$$

The condition that the degrees add to zero follows from Stokes theorem and the fact that $\text{Tr}[\hat{n}\hat{n} \wedge \hat{n}\hat{n}]$ is closed (see paragraph following (8.8)). In particular, if $\partial V_1$ is connected we can ensure the phases vanish.

To determine the fate of the phases in more general cases one needs to study the possible values $f_k$ can take. This in turn is related to the problem of understanding the ‘transversal’ symmetries of $X^a$. As mentioned at the end of 8.1.1, this remains an open problem to us and thus this is the case of $\tilde{V}_1$ with multiple boundaries remains unsettled.

To summarize, the phase associated to a infinitesimal symmetries $(\Lambda, \xi)$ takes the form (case of connected $\tilde{V}_1$ and simply connected $V_{01}$):

$$
\alpha(e^{i(\Lambda, \xi)}, \bar{E}) = -2\pi \sum_{k=1}^N f_k m_k \quad \text{with} \quad \sum_{k=1}^N m_k = 0, \tag{8.16}
$$

where the sum is over all connected components of $\partial V_1$, with $f_k$ the value of $f$ at each component and $m_k$ the degree of $\tilde{n}$ on each component.

### 8.2. Superselection conditions in the presence of rank 1 triads

The symmetries discussed in the previous section imply new superselection conditions in addition to the ones described in section 6. Let us first focus on the case of ‘pure background’ states. Accordingly consider two states $|\bar{E}\rangle$ and $|\bar{E}'\rangle$ with non-zero overlap on some observable. The condition $\tilde{V}_2 = \bar{V}_2$ (6.10) implies $\tilde{V}_{01} = \bar{V}_{01}$ since $\tilde{V}_{01} = \Sigma \setminus \bar{V}_2$. Diffeomorphisms generated by vector fields parallel to $X^a$ and $X^b$ can be used to conclude (see appendix C.6):

$$
|\bar{E}(O)|\bar{E}'\rangle \neq 0 \Rightarrow X^a X^b = 0 \quad \text{on} \quad \tilde{V}_{01} = \bar{V}_{01}, \tag{8.17}
$$

(recall the densitized vector field $X^a$ on $\tilde{V}_{01}$ is the result of extending $X^a$ to $V_0$ by setting $X^a|_{V_0} = 0$). The above condition is very general but at the same time not the strongest possible: in all configurations we can envisage, there are always enough ‘transversal’ symmetries of the vector fields $X^a$ that allows one to conclude the stronger condition:

$^{13}$ $f$ is defined up to an additive constant, and so are the $f_k$’s. The total integral (8.14) however, is independent of this constant, as it should be. This is explicitly seen in expression (8.16).

$^{14}$ The numbers $m_k$ are topological and in particular Aut-invariant. The invariance under gauge transformations can be seen explicitly from the identity $\text{Tr}[\hat{n}\hat{n} \wedge \hat{n}\hat{n}] = \text{Tr}[\hat{n}\hat{n} \wedge \hat{n}\hat{n}] = d(\text{Tr}[g^{-1}dg\hat{n}])$ where $\hat{n}_g = \hat{n}g\hat{g}^{-1}$. 




\[ \langle \bar{E}|O|\bar{E}' \rangle \neq 0 \Rightarrow X^a = cX'^a \text{ on } \hat{V}_0 \] for some constant $c$. However, as in the constant rank case, we have not been able to construct a proof (or counterexample) of this stronger condition.

Let us now consider the inclusion of spin networks. Thus we now have $|\psi⟩ = |s, \bar{E}\rangle$ and $|\psi'⟩ = |s', \bar{E}'\rangle$ with non-zero overlap on an observable. What can we say about the portions of the spin network graphs lying in $\hat{V}_0 \cup \tilde{V}'_0$? From the standard LQG argument we know that edges of the spin networks $s$ and $s'$ contained on $\hat{V}_0 = V_0'$ must agree.

The first observation is that edges $e$ that are entirely contained in $V_1 \cap \tilde{V}'_1$ must also agree, since otherwise one can use diffeomorphisms generated by vector fields parallel to $X^a$ to produce infinitely many symmetries of one state that change the other. Since edges on $\hat{V}_0$ must also agree, it remains to describe the situation for edges intersecting ‘transition’ regions between rank 1 and 0.

There are many different realizations of such a situation. While case by case studies of various examples seem tractable, in the absence of exhaustive results, we conclude our discussion of superselection here.

8.3. Subtleties in the group averaging

A key ingredient in showing well definedness of the group averaging of section 7 is the existence of a ‘reference group’ $\text{Sym}_0$ (7.5) with respect to which we could find the relative size of the symmetry groups $\text{Sym}_a$. In the presence of rank 1 regions however, we do not know of a general prescription to define the appropriate reference group.

To illustrate the difficulties let us focus on pure background states $|\bar{E}\rangle = |0, \bar{E}\rangle$ with $\bar{E} \neq 0$. We would like to find a general definition of ‘reference group’ $\text{Sym}_0^0 \subset \text{Sym}_E$ such that: (a) the quotient $\text{Sym}_E/\text{Sym}_0^0$ is always finite; (b) $\text{Sym}_0^0 = \text{Sym}_0$ if $|\bar{E}\rangle$ and $|\bar{E}'\rangle$ are in the same kinematical superselection sector; and (c) $a\text{Sym}_0^0 a^{-1} = \text{Sym}_0^0 \bar{E}$.

Since we want $\text{Sym}_E/\text{Sym}_0^0$ to be composed of discrete transformations, a natural candidate for $\text{Sym}_0^0$ is given by the continuous group generated by infinitesimal symmetries:

\[
\text{Sym}_0^0 = \{ e^{i(L_{\Lambda_1}+\ldots+L_{\Lambda_n})} \in \text{Aut} : [\Lambda_i, \bar{E}^a] - \mathcal{L}_\Lambda \bar{E}^a = 0, \int \text{Tr}[\bar{E}^a \partial_\alpha \Lambda_i] = 0 \}. \tag{8.18}
\]

It is easy to verify that $\text{Sym}_0^0$ is a normal subgroup of $\text{Sym}_E$. Unfortunately, as we show below, it is not necessary that $\text{Sym}_E/\text{Sym}_0^0$ is finite. Thus, although it is conceivable that (8.18) may serve as a reference group for certain superselection sectors, it cannot be used as a general prescription.

We now give an example where $\text{Sym}_E/\text{Sym}_0^0$ is of infinite size. The possibility of ‘infinite discrete symmetries’ arises from submanifolds of $\Sigma$ whose mapping class group $[16]$ is of infinite size\(^{15}\). We first give a two dimensional example and then show its three dimensional counterpart. Let $\Sigma = \mathbb{R}^2$ with polar coordinates $(r, \theta)$ and consider an $su(2)$ electric field $\bar{E}^a$ such that: (a) $V_1 = [1 \leq r \leq 2]$, $\bar{E}|_{V_1} = r_3 \xi F'$; (b) $V_2 = \Sigma \setminus V_1$ and $\bar{E}|_{V_2}$ does not admit infinitesimal symmetries.

Consider a vector field $\xi^a = f(r) \frac{\partial}{\partial \theta}$ where $f(r)$ is semianalytic and satisfies $f(r \geq 2) = 0$ and $f(r \leq 1) = 2\pi$. Let $\phi_t = e^{\frac{t}{2} \xi}$ be the one parameter group of diffeomorphisms generated by $\xi$. Then $\phi_t$ generates symmetries of $\bar{E}|_{V_t}$ for all $t$. These however do not represent one parameter group of symmetries of $\bar{E}$ since by assumption $\bar{E}|_{V_2}$ does not admit infinitesimal symmetries. However, for the values $t = n \in \mathbb{Z}$ they do yield symmetries of $\bar{E}$ since

\(^{15}\) Similar ‘infinite discrete symmetries’ could be encountered in the spin network case of section 5 if one (wrongly) uses as a reference group the subgroup of (5.5) given by diffeomorphisms of the type $\phi = e^{t \xi_1} \ldots e^{t \xi_n}$ with $e^{t \xi_e}(e) = e$ for all edges $e$ of the spin network. The difficulty in the present case is that we do not know what the analogue of $\text{Sym}_0^0$ (5.5) for general ‘pure background’ states is, let alone for general KS states.

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\( \phi_{\text{inv}}|_{V_1} = \text{Id} \). Now, the diffeomorphisms \( \phi_n|_{V_1} \) provide representatives of the so-called mapping class group of the annulus, \( \text{MCG}(V_1) = \mathbb{Z} \) [16], and so they yield different elements of the (diffeomorphism part) of the quotient group \( \text{Sym}_{\mathbb{E}}/\text{Sym}^0_{\mathbb{E}} \), which implies it is of infinite size.

A three dimensional version of the example above exhibiting an infinite quotient group can be constructed as follows. Let \( \Sigma = \mathbb{R}^3 \) with cylindrical coordinates \((z, \rho, \varphi)\) and take \( \bar{E} \) such that \( V_1 = \{ 1 \leq (\rho - a)^2 + z^2 \leq 4 \} \) with \( a \) large enough so that \( (\rho - a)^2 + z^2 = 4 \) is a well defined torus. Let \( \bar{E}|_{V_1} = \tau_{i\rho}(\rho - a)\partial_\rho - z\partial_z \) and \( \bar{E}|_{\Sigma|V_1} \) of rank \( \geq 2 \) admitting no infinitesimal symmetries. Thus a cross section \( \varphi = \text{constant} \) corresponds to the type of configuration of the two dimensional example. The \( t = n \) diffeomorphisms generated by \( \xi^a = f(\sqrt{(\rho - a)^2 + z^2})(\rho - a)\partial_\rho - z\partial_z \) with \( f(r) \) as before correspond to different elements in \( \text{Sym}_{\mathbb{E}}/\text{Sym}^0_{\mathbb{E}} \) and so the quotient group is of infinite size.

We emphasize that in the example above it is possible to identify the superselection sector as well as an appropriate reference group: by arguments similar of the type discussed in sections 6 and 8.2 one can show the kinematical superselection sector is given by states \( \langle \bar{E}' \rangle \) such that: \( \bar{V}_2 = \bar{V}_2, \bar{V}_1 = V_1, \bar{E}'|_{V_1} \) does not admit infinitesimal symmetries and \( \bar{E}'|_{V_1} = c\bar{E}|_{V_1} \) for some constant \( c \neq 0 \). The reference group can then be taken to be \( \text{Sym}_{\mathbb{E}}^\text{ref} = \{ a \in \text{Sym}_{\mathbb{E}} : a|_{V_1} = \text{Id} \} \). The quotients \( \text{Sym}_{\mathbb{E}}/\text{Sym}_{\mathbb{E}}^\text{ref} \) are then finite, corresponding to discrete symmetries of \( \bar{E}'|_{V_1} \). It is however not clear that this prescription will work in other sectors.

9. Discussion

9.1. Summary and discussion of results

The Koslowski–Sahlmann (KS) representation is based on a Hamiltonian formulation of gravity in terms of \( SU(2) \) connections and triads. The group of gauge transformations generated by the \( SU(2) \) Gauss Law and spatial diffeomorphism constraints is referred to in this work as the automorphism group \( \text{Aut} \). As shown here, the correct unitary action of \( \text{Aut} \) in the KS representation involves hitherto unnoticed phase contributions. In this paper we incorporated these phase contributions into an analysis of the imposition of invariance under the group \( \text{Aut} \) in the KS representation via group averaging techniques.

Since we do not know (and are not aware) of any well defined group invariant measure on \( \text{Aut} \), our starting point was to define the action of the putative group averaging map on a state to be a sum over gauge related states. The strategy was then to check if this sum over states leads to a well defined map which satisfies the properties of a group averaging map (see section 3.1). We showed that if there exist automorphisms which non-trivially rephase a state then such a state is in the kernel of this sum over states map. Since any group averaging map must have this property (see section 3.2), the putative sum-over-states group averaging map passes this test. We also showed, as required for the consistency of the group averaging map, that states in this ’rephasing’ related kernel of the sum-over-states map lie in superselection sectors distinct from the ones which contain states which do not admit such rephasings.

Next, we showed that non-trivial rephasings arise generically for KS states labeled by background triads whose rank 1 support sets are of non-zero measure. We also showed that such rephasings can occur, in principle, for states labeled by non-degenerate background triads with appropriate symmetries.

Next, we derived the main result of this paper, namely the (partial) labeling of superselection sectors by the (diffeomorphism equivalence classes of) rank support sets (up to sets of zero measure) of the background triad label of the KS spin net being averaged.
Finally, we showed, modulo the assumptions of section 9.3 that the group averaging map is well defined for the superselection sectors in which the rank 1 support set of the background triad label is of zero measure. We remind the reader that for generic states in this sector, there are no rephasing subtleties.

The open questions, therefore, all pertain to the case of rank 1 support sets of non-zero measure. Due to the rich structure of the symmetries of such triads, we were unable to derive results of wide generality. However, we did derive a number of case by case results which we presented in section 8. Notwithstanding our expectation of the complicated structure of the symmetry group, we do expect that it is most likely that further progress does not entail a detailed understanding of the symmetry group of the labels of the KS state being averaged. Rather, we expect that the identification of a sufficiently large (infinite dimensional) subset of this group suffices to (1) isolate generic superselection sectors and (2) constrain an ‘infinite dimensional’ part of the label set of states in each such sector. If the left over ‘free data’ in the label set is sufficiently restricted so as to be finite, it should be possible to demonstrate a consistent evaluation of ambiguity coefficients. Indeed, as we emphasized in section 5, this is exactly what happens in loop quantum gravity (LQG) wherein a sufficiently large subset of the symmetry group constrains the graphs of superselected states to coincide (up to diffeomorphisms) and the left over ‘free data’ of a state |s⟩ can be mapped to that of another by elements of the finite graph symmetry group Dγ. From this perspective (see section 5 for a detailed discussion) the two ingredients needed to complete the group averaging procedure are (i) the identification of a sufficiently large set of symmetries which constrain the values of all but a finite set of state labels for states within a superselection sector and (ii) a sufficiently large ‘reference’ subgroup of symmetries underlying all states in a superselection sector where ‘large enough’ implies that group of all symmetries of the label set for any state in this sector modulo the reference symmetries is finite.

9.2. On the restriction to gauge invariant spin nets

In this section we comment on our restriction to gauge invariant spin net labels and discuss ways to generalize our group averaging procedure to the gauge variant case. We provide only rough arguments which we hope will serve as guidelines for future attempts at an exhaustive treatment.

As in Sahlmann’s work [3], we have restricted attention to KS states with gauge invariant spin net labels. The reason to do so stems from the desire to avoid the following complication, related to the averaging over the ‘internal gauge transformation’ part of Aut, which arises when this restriction is relaxed. Consider a KS spin net |s, E⟩ with s being an SU(2) gauge variant spin net and consider its group average over SU(2) transformations defined through a sum of the type (3.5) over its SU(2) gauge related images. Suppose there are infinitely many internal SU(2) rotations which leave the label E invariant but change the label s. The action of these transformations on the state |s, E⟩ yields infinitely many states which are not orthogonal because of the non-orthogonality of such SU(2) gauge related states in standard LQG. As a result, it looks as if the action of the ‘sum over bras’-group averaged state on, say, the state |s, E⟩ is ill defined. In other words, it seems as if this sum does not reside in D′ so that the putative averaging map violates (3.1). If, on the other hand, the action of these transformations is integrated, as in standard LQG, over an appropriate measure deriving from the Haar measure on SU(2), we expect the resulting group averaged state to reside in D′. However, such a procedure would require the choice of group averaging ‘measure’ to be further tailored to the nature of the orbit of the state being averaged. A simple way to avoid this complication is to restrict the spin net labels to be gauge invariant.
The consequence of this restriction is that one is then obliged to restrict attention to gauge invariant observables which preserve the finite span of KS states with gauge invariant spin net labels. This is what we have done in this paper. However, as far as we can see, there is no reason for gauge invariant observables to map such states exclusively into themselves i.e. it is conceivable that there exist gauge invariant observables which map KS states with SU(2) gauge invariant spin net labels to ones with SU(2) gauge variant spin net labels. Therefore this restriction seems to entail a loss of generality and one would like to remove it. It is pertinent to note that in the case of generic triads of rank 2,3 there are no internal rotations which preserve the triad so that the argument of the previous paragraph cannot be applied to this case\(^{16}\). However, in the case of rank 0 and rank 1 triads, clearly there are infinitely many internal rotations which preserve the background triad. Let us focus on the rank 0 case first. Accordingly, consider the averaging of \(\langle s, \bar{E}\rangle\) with \(\bar{E}\) such that \(V_0\) is of non-zero measure. Further, let \(s\) be a gauge variant spin net which is an element of the basis of extended spin networks constructed in [20]. Recall that each such basis element has, in addition to the edge labels and invariant intertwiners at vertices, an assignation of vectors, \(v_v\) in appropriate representations, one for each vertex \(v\). For the subspace of gauge invariant spin nets, these vectors are all in the trivial \(j = 0\) representation. Let us refer to vertices \(v\) labeled by \(j_v \neq 0\) vectors as gauge variant and the remaining ones as gauge invariant vertices. Given any \(g(x) \in G\), such basis states transform by a rotation of the vector labels \(v_v \rightarrow g_{j_v}(v)v_v\) where \(g_{j_v}(v)\) is the matrix representative of \(g(x = v)\) in the spin \(j_v\) representation. Let \(V_0\) contain a gauge variant vertex \(v_0\). Clearly there are an \(SU(2)\) worth of gauge transformations which change \(v_{v_0}\) but otherwise leave \(\bar{E}\), \(s\) invariant. It then follows from [20] that the averaging over this set of \(SU(2)\) transformations with respect to the Haar measure yields a vanishing result. Further, it is easy to see that among these transformations, there are gauge transformations which rephase \(v_{v_0}\) by any desired amount. From [20], such transformations rephase the state \(\langle s, \bar{E}\rangle\) by any desired amount so that, our general arguments (3.10), once again indicate the vanishing of its group average. Moreover our general argument (3.13) shows that such states lie in a distinct superselection sector from states which do not admit rephasings. Our argumentation then suggests that \(V_0\) cannot contain any gauge invariant vertices if the resulting state is to be non-trivially averaged. Note that our argumentation concerning rephasing transformations applies equally well to a sum over gauge related images (as opposed to a Haar measure type integration) if we agree, as in section 3, that an infinite sum over phases vanishes.

Next, consider the rank 1 case so that \(V_1\) is of non-zero measure. Similar to section 3.2 let \(\bar{E}' = \hat{n}X^a\) and consider its symmetries \(e^{\theta\hat{a}}\) with \(\theta\) supported in \(V_1\). Consider a gauge variant vertex \(v\) of \(s, v \in V_1\). It is then easy to check that the integral over \(\theta\) of the action of these gauge transformations on \(v_v\) vanishes only if \(j_v\) is half integer. If \(j_v\) is integer valued the integration projects \(v_v\) into the zero eigenvalue subspace of \(\hat{n}\). Further work is needed to construct the group averaging map to accommodate this result. Note that in the case of integer \(j_v\), Sahlmann’s arguments of ill definedness of a ‘sum’ hold and one is obliged to use an integral over \(\theta\). In summary, once again, the rank 1 case exhibits subtleties not encountered in the rank 0, 2, 3 cases.

Let us then restrict attention to the case studied in section 7 i.e. the case of \(V_1\) of measure zero. In this case we take our rough arguments above as indicative that KS states with gauge variant vertices in \(V_0\) consistently average to zero. Thus all gauge variant vertices must be confined to \(V_2\). Let us then restrict attention to the (putative) superselection sector of states with generic \(\bar{E}\) in \(V_2\) and \(s\) such that its gauge variant vertices lie only in \(V_2\). Since for generic \(\bar{E}\), the only elements of \(\text{Aut}\) which preserve \(\bar{E}\) are identity on \(V_2\), there seems to be no obstruction

\(^{16}\) This was not realized in the pioneering work of [3].
to using a sum over states form of the averaging map applied to \( |s, \bar{E}\rangle \) and we anticipate that with this minor generalization, all the considerations of 7 can be repeated without obstruction.

9.3. Summary of technical assumptions

In the main body of the paper, we have made certain technical assumptions. We list them and discuss their status below:

1. We assume that semianalytic \( C^k \) vector fields on \( \Sigma \) generate semianalytic \( C^{k+1} \) diffeomorphisms. As far as we can tell this is implicitly assumed by researchers in LQG but we have never seen a proof of the statement. Proofs are available for the smooth \([17]\) and analytic categories (see for example \([18]\)) and we think it is a reasonable assumption to make for the semianalytic category as well.

2. We have assumed that the quotient groups, \( D_\psi \) of section 7, are discrete and finite. We think that this a plausible assumption modulo point (3) below. Nevertheless, even if the assumption is false, it does not necessarily mean that the group averaging map does not exist. Instead, we anticipate that if these groups are either continuous or discrete but infinite, there will be further superselection. More work would then be needed to see if the ambiguity parameters in these superselection sectors can be consistently determined. Note that we have been careful to define the ‘reference’ group \( \text{Sym}_0^\psi \) of equation (7.5) in terms of automorphisms which are identity on \( \bar{V}_2 \) irrespective of their extension into \( V_0 \). This is to avoid potential ‘infinite discreteness from topology’ of the type encountered in section 8.3 in the rank 1 support set context (see footnote 15).

3. We have provided a ‘counting’ argument for the genericity in the rank 2 case in appendix C.3. We feel the argument is a bit formal but that with further work, it should be possible to be converted into a rigorous definition of genericity.

4. We have assumed the existence of observables which map the finite span of spinnets into itself. This assumption also underlies the LQG analysis (see for example \([5]\) for the real analytic category). While such operators certainly do exist at the quantum level as linear maps from the finite span of spinnets into itself, it is not clear to us if classical correspondents to these observables exist. Indeed in the case of LQG, apart from the operator corresponding to the volume of the entire spatial slice, we do not know of any such operators which arise from the quantization of classical observables. A set of classical observables which are gauge and diffeomorphism invariant correspond to integrals of density weight one integrands constructed exclusively from phase space variables. Such integrands can be constructed from the triad and curvature and, in general, are expected to be connection dependent and may not even admit quantum analogues on the diffeomorphism (or automorphism) invariant Hilbert space which arises from group averaging. On the other hand, it may be the case, that the natural observables relevant to the interpretation of quantum geometry are not of this type. In any case, at the very least, we feel that the group averaged Hilbert spaces constructed here and in LQG (or their ‘habitat’ deformation \([19]\)) serve as useful arenas to explore properties of the Hamiltonian constraint \([19, 22, 23]\).

9.4. Avenues for further work

For the case of rank 1 support set of non-zero measure, much more work is needed to establish the existence of superselection sectors in which the group averaging map is well defined and unambiguous. As a first step, it would be very useful to characterize the conditions under
which background fields with rank 1 sets of non-zero measure admit symmetries generated by
vector fields transverse to the direction of the field (see end of section 8.1 and appendix C.4).
We anticipate that this would go a long way to supplying ingredient (i) of section 9.1 above.

Let us restrict attention to the well understood (modulo assumption (3) of section 9.3)
superselection sector of general KS states with backgrounds whose rank 1 set is of zero
measure. Then, the most direct application of our work here is, as mentioned in section 1, in
the context of asymptotically flat space times. While our work here is in the context of compact
manifolds, the spatial slice in the asymptotically flat context are non-compact. However, the
topology of such slices is fixed outside a compact region to be that of the complement of a
compact region in $\mathbb{R}^3$ (see for example [15]). In this $\mathbb{R}^3$ region, the triad boundary conditions
imply, in particular, that the triad is non-degenerate and hence of fixed rank 3. Moreover,
the automorphism group consists of those automorphisms which approach identity at spatial
infinity. We expect that these facts should allow us to identify an Aut invariant superselection
sector which is obtained by the group averaging of KS spinnets for which (a) the spinnet
graph is confined to a compact set where the background vanishes, (b) the background triad
label asymptotes to the fixed flat triad at spatial infinity and (c) the triad label is exclusively of
rank 0 or 2,3 almost everywhere. Work on this is in progress [24]. Of course, the Hamiltonian
constraint may map states out of the Aut invariant superselection sector under consideration.
Moreover, as mentioned in section 1, the use of the KS representation is only a first step; we
expect the final picture to involve only the LQG sector, suitably generalized to admit states
for which the asymptotically flat boundary conditions are satisfied in a suitably coarse grained
sense. However, since very little is known about asymptotically flat kinematics, let alone the
situation at the Aut invariant level, we believe that it is still a useful exercise to enquire as to
how asymptotically flat boundary conditions together with Aut invariance can be imposed in
the KS representation in the limited manner envisaged above.

As sketched above the KS representation can be used to satisfy the asymptotically flat
boundary conditions on the triad. However, there are also the conditions on the connection. We
believe that the incorporation of these conditions on the connection is a more subtle matter and
should be built into the very construction of the representation. More in detail, work in progress
seems to show, similar to the LQG case [25], that the KS representation in the compact case can
be understood as an $L^2$ representation on a suitable completion of the classical configuration
space of connections. We are hopeful that a generalization of this (putative) result exists for
the asymptotically flat case and that we can interpret the existence of this generalization as the
incorporation of the boundary conditions on the connection.

Let us turn briefly from kinematics to dynamics. One of the key open issues in LQG
is that of a satisfactory definition of the Hamiltonian constraint. Recently Laddha proposed
a model with internal gauge group $U(1)^3$ as good toy model to formulate such definitions.
This model is of interest in its own right as it corresponds to Smolin’s novel weak coupling
limit of Euclidean gravity [26]. Smolin’s idea was that this model could offer a background
independent setting about which one could define Euclidean gravity as a perturbation theory.
While there have been attempts to define the quantum theory of this model in standard
LQG like representations [22, 23], it is certainly of interest to explore alternate background
independent quantizations. Note that the ‘pure background’ sector of the KS representation
provides a quantum representation of the algebra of fluxes and background exponentials. Since
the fluxes and background exponentials do separate points on phase space, it follows that this
pure background sector does provide an alternative background independent quantization. It
is then of interest to enquire if the pure background representation for internal gauge group
$U(1)^3$ together with its Aut averaging provides an alternate setting in which to explore the
treatment of the Hamiltonian constraint in the model. Finally, another model of interest is that
of parametrized field theory (PFT) [11]. Recently Sengupta analyzed the quantization of PFT in a KS-like representation [27, 28]. This work provides valuable hints for future studies of LQG dynamics in the KS representation.

Returning from dynamics to kinematics, we touch on a final issue. The pure background KS representation and its group averaging are defined for semianalytic background fields. As mentioned above, this representation can be thought of as arising from a quantization of the classical algebra of electric fluxes and background exponentials. The background exponentials are exponentials of the connection integrated against the three dimensional, semianalytic background electric field. What if we replace these three dimensional smearings with one dimensional ones concentrated around loops in the spatial manifold? Well, one would then obtain an algebra of electric fluxes and effectively \( U(1)^3 \) holonomies! One could then enquire as to what would happen if we imposed SU(2) invariance in the resulting KS representation.

A heuristic treatment is outlined in appendix E, the main result being a hint that what one obtains is the standard LQG SU(2) representation in terms of \( U(1)^3 \) structures. It would be of interest to see if this treatment can be made less heuristic and if such a less-heuristic treatment leads to useful insights or applications in the standard LQG context.

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Appendix A. Phase factors

A.1. Gauge invariance of phase factors associated to symmetries

Let \( \bar{E} \) be any triad, and \( b \in S_{\bar{E}} \) an element of its symmetry group. Let \( a \) be any element in Aut and define \( \bar{E}' := a \cdot \bar{E} \) and \( b' := aba^{-1} \) so that \( b' \in S_{\bar{E}'} \). We show that the phase is gauge invariant in the sense that \( e^{ia(a;b)} = e^{ia(b)} \). First notice that the phase associated to the symmetry \( b \) of the triad \( \bar{E} \) can be written as

\[
e^{ia(b)} = \beta_{\bar{E}}^* (b \cdot \beta_{\bar{E}}), \quad \text{for} \quad b \in S_{\bar{E}}, \tag{A.1}
\]

where \( \beta_{\bar{E}}^* \) is the complex conjugated of \( \beta_{\bar{E}} \) so that \( \beta_{\bar{E}}^* \beta_{\bar{E}} = 1 \). Thus

\[
e^{ia(a;b')} = \beta_{\bar{E}}^* (b' \cdot \beta_{\bar{E}}) \tag{A.2}
\]

\[
= (a \cdot \beta_{\bar{E}}^*)^* (aba^{-1}) \cdot (a \cdot \beta_{\bar{E}}) \tag{A.3}
\]

\[
= e^{ia(a;\bar{E})}. \tag{A.4}
\]

A consequence of this result is that the existence of non-trivial rephasings is a gauge invariant property in the sense that \( \text{Sym}_{\bar{E}} \subseteq S_{\bar{E}} \Rightarrow \text{Sym}_{a \bar{E}} \subseteq S_{a \bar{E}} \).

A.2. Phase factor for one parameter subgroup of symmetries

Given a triad \( \bar{E} \), an infinitesimal Aut transformation \((\Lambda, \xi)\) satisfying

\[
(\Lambda, \xi) \cdot \bar{E}' \equiv [\Lambda, \bar{E}'] - \mathcal{L}_\xi \bar{E}' = 0, \tag{A.5}
\]

generates the one parameter subgroup of symmetries of \( \bar{E} \):

\[
a_t := e^{i(\Lambda, \xi)} =: (g_t, \phi_t). \tag{A.6}
\]
The condition \( \dot{\alpha} = (\Lambda, \xi) \dot{\alpha} \) translates into
\[
\dot{\phi}_t = \mathcal{L}_{\xi} \phi_t \quad (A.7)
\]
\[
\dot{g}_t = \mathcal{L}_{\xi} g_t \quad (A.8)
\]
Thus, whereas \( \phi_t \) is simply given by \( \phi_t = e^{t\mathcal{L}_\xi} \), \( g_t \) is not in general a 1 parameter subgroup of \( \mathcal{G} \). Let
\[
\dot{\alpha}(t) := \alpha(\phi_t, \hat{\mathcal{E}}) \quad (A.9)
\]
\[
= \int_{\Sigma} \text{Tr} [ (\phi_t)_*(\hat{\mathcal{E}}^a) g_t^{-1} \partial_\mathcal{G} g_t ] . \quad (A.10)
\]
Using (A.7), (A.8), and the fact that \( g_t(\phi_t)_*(\hat{\mathcal{E}}^a) g_t^{-1} = \hat{\mathcal{E}}^a \), a straightforward computation leads to:
\[
\dot{\alpha}(t) = \int_{\Sigma} \text{Tr}[\hat{\mathcal{E}}^a \partial_\mathcal{G} \Lambda] . \quad (A.11)
\]
Thus \( \dot{\alpha}(t) \) is independent of \( t \). Since \( \alpha(0) = 0 \) we conclude
\[
\alpha(t) = t \int_{\Sigma} \text{Tr}[\hat{\mathcal{E}}^a \partial_\mathcal{G} \Lambda] . \quad (A.12)
\]
Finally, setting \( t = 1 \) we obtain equation (8.1).

### A.3. The case of rank 3 triads

It was noted in section 4.2 that phases associated to symmetric rank 3 configurations are necessarily zero if the symmetry group does not admit non-trivial homomorphisms to \( U(1) \). Here we identify additional conditions in which we can guarantee the phase is vanishing.

Consider a case where the metric associated to the rank 3 triad \( \hat{\mathcal{E}}^a \) admits a Killing vector field \( \xi^a \), i.e.:
\[
\text{Tr}[\hat{\mathcal{E}}^a \mathcal{L}_\xi \hat{\mathcal{E}}^{b}] = 0 . \quad (A.13)
\]
Using the \( su(2) \) matrices identity \( [a, b, c] = \text{Tr}[ac]b - \text{Tr}[bc]a \) one can verify that \( (\Lambda, \xi) \) is a symmetry of \( \hat{\mathcal{E}}^a \) if \( \Lambda \) is chosen as:
\[
\Lambda = \frac{1}{2} [\hat{\mathcal{E}}_a, \mathcal{L}_\xi \hat{\mathcal{E}}^a] . \quad (A.14)
\]
Let \( \hat{\mathcal{E}}^a = \hat{\mathcal{E}}^a_\tau \). If there exist a gauge choice such that
\[
\hat{\mathcal{E}}^a_\tau \propto \xi^a , \quad (A.15)
\]
then the action of \( \xi^a \) is locally a rotation in the \( (\hat{\mathcal{E}}^a_\tau, \hat{\mathcal{E}}^b_\tau) \) plane, and thus \( \Lambda \) is proportional to \( \tau_3 \). From (8.1) we conclude that \( \alpha(e^{(\Lambda, \xi)}, \hat{\mathcal{E}}) = \int \partial_\mathcal{E} \hat{\mathcal{E}}^a_\tau \Lambda_3 \). But since \( \hat{\mathcal{E}}^a_\tau \) is parallel to \( \xi^a \), and \( \xi^a \) is divergence free, it follows that \( \partial_\mathcal{G} \hat{\mathcal{E}}^a_\tau = 0 \). Thus if the gauge choice (A.15) is available, we are guaranteed the phases associated to \( \xi^a \) are zero.

### Appendix B. Results on semianalytic category

We first recall some definitions and results from appendix A of [7]. In order to define semianalytic manifolds and functions, one first defines semianalytic functions on open sets of \( \mathbb{R}^n \). That requires the notion of semianalytic partition of an open set \( U \subset \mathbb{R}^n \). This is given by a decomposition of the form:
\[
U = \cup_{\sigma} V_\sigma , \quad V_\sigma \cap V_{\sigma'} = \emptyset \quad \text{if} \quad \sigma \neq \sigma' , \quad (B.1)
\]
where \( V_\sigma \) are described by analytic functions \( h_i , i = 1, \ldots, n , \) on \( U \) according to:
\[
V_\sigma \equiv V_{\sigma_1, \ldots, \sigma_n} := \cap_{i=1}^n [h_i(x)\sigma_i,0] , \quad \sigma_i = \{0, >, <\} . \quad (B.2)
\]
A key property of semianalytic partitions (see proposition A.9 of [7]) is that every \( x \in U \) has a neighborhood \( U_x \) such that \( V_x \cap U_x \subset U_x \) is a finite union of analytic submanifolds of \( U_x \). This result will allow us to conclude the sets (6.1)–(6.3) are finite unions of submanifolds.

Given an open set \( U \subset \mathbb{R}^n \), a function \( f : U \rightarrow \mathbb{R} \) is said to be semianalytic if for every \( x \in U \) there exists an open neighborhood \( U_x \) with a semianalytic partition as above such that

\[
f|_{U_x} = f_\sigma|_{U_x},
\]

where \( f_\sigma \) are analytic functions on \( U_x \).

Semianalytic manifolds and submanifolds are then defined as in the differentiable case, with the requirement that transition functions between local charts are semianalytic [7]. Likewise semianalytic functions on such manifolds are defined by the requirement that they are semianalytic on the local charts.

The main property of semianalytic functions that we will need is the following: given semianalytic functions \( f_i : \Sigma \rightarrow \mathbb{R}, i = 1, \ldots, n \) and a choice \( \sigma_i = \prec \cdot > \cdot \) for each \( i \), the set \( X = \bigcap_i f_i(\sigma_i) \subset \Sigma \) can be decomposed into a finite union of submanifolds.

For the sake of clarity, we show this result for the case of a single function, \( X := \{ f(x) = 0 \}. \) From the structure of the proof it will become evident that the general case follows. Let \( \{ x_i, U_i \} \) be a semianalytic atlas of \( \Sigma \) compatible with \( f \). That is, there is a semianalytic partition on each local chart:

\[
U_i' := x_i(U_i) = \cup_a V_i'^a,
\]

as in (B.1), (B.2) such that,

\[
f \circ x_i^{-1}|_{V_i'^a} = f_\sigma|_{V_i'^a}
\]

where \( f_\sigma \) are analytic functions on \( U_i' \subset \mathbb{R}^3 \). On each local chart the set of interest takes the form:

\[
\chi_i(X \cap U_i) = \cup_a \{ f_\sigma(x) = 0 \} \cap \{ h_i^1(x) \sigma_i 0 \} \cap \{ h_i^2(x) \sigma_i 0 \} \cap \{ h_i^3(x) \sigma_i 0 \} \cap \cdots \cap \{ h_i^n(x) \sigma_i 0 \}.
\]

The sets featuring in the union (B.6) can be realized as sets of a new partition of \( U_i' \) defined in terms of the functions \( [h_i^j] \cup [f_\sigma^j] \). By the proposition of [7] mentioned after equation (B.3), it follows that every \( x \in U_i \) has an open neighborhood \( U_i' \) such that

\[
\chi_i(X \cap U_i') = \text{finite union of analytic submanifolds.}
\]

This in turn implies that \( X \cap U_i' \) is a finite union of semianalytic submanifolds of \( \Sigma \). Since \( \Sigma \) is compact, there exist a finite subcover of the (uncountable) open cover \( \{ U_i' \} \). This allows to express \( X \) as a finite union of submanifolds.

**Appendix C. Additional proofs**

**C.1. Equation (2.15)**

The left- and right-hand side of (2.15) are given by:

\[
a \cdot (a' \cdot \beta_r) = \omega^{a(a', E)} \omega^{a'(a', \tilde{E})} \beta_{a(a', \tilde{E})}
\]

\[
(a a') \cdot \beta_r = \omega^{a(a', \tilde{E})} \beta_{(a a') \cdot \tilde{E}}.
\]

Since \( a \cdot (a' \tilde{E}) = (a a') \cdot \tilde{E} \), all we need to check is that the phases in (C.1) and (C.2) agree. Starting from the phase in (C.2) for \( a = (g, \phi), a' = (g', \phi') \) and using (2.5) and (2.14), we
find:
\[ \alpha(a', \bar{E}) = \int_{\Sigma} \text{Tr} \left[ (\phi \circ \phi')_a (\bar{E}' \bar{g}^{-1} \bar{g}^{-1}) \partial_a (g \phi \bar{s} g') \right] \]
\[ = \int_{\Sigma} \text{Tr} \left[ \phi_s (\phi'_s \bar{E}^a \bar{g}^{-1}) \bar{g}^{-1} \partial_a (g \phi \bar{s} g') \right] + \int_{\Sigma} \text{Tr} \left[ \phi_s (\phi'_s \bar{E}^a) \phi_s \bar{g}^{-1} \partial_a \bar{g}' \right] \]
\[ = \int_{\Sigma} \text{Tr} \left[ \phi_s (g \phi'_s \bar{E}^a \bar{g}^{-1}) \bar{g}^{-1} \partial_a \bar{g}' \right] + \int_{\Sigma} \text{Tr} \left[ \phi'_s \bar{E}^a \bar{g}^{-1} \partial_a \bar{g}' \right] \]
\[ = \alpha(a, a' \cdot \bar{E}) + \alpha(a', \bar{E}). \]  
(C.3)

Thus, the phases in (C.1) and (C.2) agree and (2.15) is indeed satisfied.

C.2. Action of gauge transformations on quantum kinematical observables

That (2.24) reproduces (2.11) for the holonomy operators (2.20) follows from the transformation rule being satisfied in the standard LQG space. For the flux operators (2.21) we have for \( a = (g, \phi) \):
\[ U(a) \hat{F}_s f U(a^{-1})|s, \bar{E}\rangle = e^{ia(a^{-1}, \bar{E})} U(a) \hat{F}_s f U(1)^{-1}(a^{-1}) s, a^{-1} \cdot \bar{E} \]
\[ = e^{ia(a^{-1}, \bar{E})} U(a) \left( \hat{F}_s f U(1)^{-1}(a^{-1}) s, a^{-1} \cdot \bar{E} \right) \]
\[ + F_s f (a^{-1}, \bar{E}) U(1)^{-1}(a^{-1}, a^{-1} \cdot \bar{E}) \]
\[ = \hat{F}^{\text{LQG}}_{\phi(s), \bar{g}_s f} |s, \bar{E}\rangle + F_{\phi(s), \bar{g}_s f} (\bar{E}) |s, \bar{E}\rangle \]
\[ = \hat{F}_{\phi(s), \bar{g}_s f} |s, \bar{E}\rangle , \]  
(C.4)

where in going from the second to third line we used the transformation rule for the standard LQG flux, as well as the identity \( e^{ia(a^{-1}, \bar{E})} e^{ia(a^{-1}, \bar{E})} = 1 \) that follows from (C.3) by setting \( a' = a^{-1} \). For the background exponential operators (2.3) we have:
\[ U(a) \hat{\phi}_{E} U(a^{-1})|s, \bar{E}\rangle = e^{ia(a^{-1}, \bar{E})} U(a) \hat{\phi}_{E} U(1)^{-1}(a^{-1}) s, a^{-1} \cdot \bar{E} \]
\[ = e^{ia(a^{-1}, \bar{E})} U(a) \hat{\phi}_{E} U(1)^{-1}(a^{-1}) s, \bar{E}' + a^{-1} \cdot \bar{E} \]
\[ = e^{ia(a^{-1}, \bar{E})} e^{-ia(a, \bar{E} + a^{-1} \bar{E})} |s, a \cdot \bar{E}' + \bar{E}\rangle \]
\[ = e^{i(a(a, \bar{E}))} |s, a \cdot \bar{E}' + \bar{E}\rangle \]
\[ = e^{ia(a, \bar{E})} \hat{\phi}_{a \bar{E}} |s, \bar{E}\rangle \]  
(C.5)

where in going from the third to fourth line we again used the identity \( e^{ia(a^{-1}, \bar{E})} e^{ia(a^{-1}, \bar{E})} = 1 \).

It follows from equations (C.4) and (C.5) that the unitary representation (2.24) reproduces the transformation rules (2.12), (2.13) for the flux and background exponential operators.

C.3. Generic rank 2 \( \hat{q}^{ab} \) do not admit symmetries

Rank 2 symmetric tensors \( \hat{q}^{ab} \) are characterized by the existence of a degenerate direction:
\[ \exists \lambda_a : \hat{q}^{ab} \lambda_b = 0. \]  
(C.6)

The number of independent components that are needed to specify such tensors can be counted as follows. A general symmetric tensor has 6 independent components. We then need to specify the degenerate direction \( \lambda_a \), which requires 2 independent components. Finally condition (C.6) gives 3 constraints on the components of the tensor. Thus the number of independent functions to define such tensor is given by \( 6 + 2 - 3 = 5 \). This in turn implies that the equation
\[ \mathcal{L} \hat{q}^{ab} = 0 \]  
(C.7)
will generically give five independent conditions for the vector field $\xi^a$ (which represents three unknowns to be solved for). The system is thus generically overdetermined and hence there are no generic solutions to (C.7).

C.4. Local description of rank 1 configurations and their symmetries

Consider a triad $\bar{E}^a$ with a non-trivial rank 1 region in the sense that there exist an open set $U \subset \Sigma$ where $\text{rank}(\bar{E})|_U = 1$. On $U$ we have the splitting $\bar{E}^a = \hat{n}X^a$, and we restrict attention to the case where $X^a$ is divergence free, since otherwise the corresponding state vanishes upon averaging. An infinitesimal symmetry $(\Lambda, \xi)$ of $\bar{E}$ will then have to satisfy equations (8.6) and (8.7) on $U$ (for simplicity we take $U$ to be simply connected). We now describe the local form of the vector fields $\xi^a$ satisfying (8.7).

As discussed in section 8.1, the divergence free property of $X^a$ translates in the closeness of the two form $\omega_{ab} = \eta_{abc}X^c$. This means that $\omega_{ab}$ is a presymplectic form on $U$. Darboux theorem for presymplectic forms (see for instance theorem 5.1.3 of [29]) tell us that every point in $U$ has an open neighborhood with local coordinates $x, y, z$ such that:

\begin{align}
\omega_{ab} &= dy \wedge dz \\
X^a &= \frac{\partial}{\partial x}.
\end{align}

(C.8)  
(C.9)

It is easy to see that the most general solution to $\mathcal{L}_\xi X^a = 0$ in the local coordinates takes the form:

\begin{align}
\xi^a &= g(x, y, z) \frac{\partial}{\partial x} + \partial_y f(y, z) \frac{\partial}{\partial y} - \partial_z f(y, z) \frac{\partial}{\partial z}
\end{align}

(C.10)

for arbitrary functions $f(y, z)$ and $g(x, y, z)$. In the language of section 8.1, the first term correspond to vector fields parallel to $X^a$ and the remaining part corresponds to the ‘transversal’ vector fields. Notice that this transversal part takes the form of a ‘Hamiltonian vector field’, with ‘Hamiltonian’ $f$ and symplectic form (C.8). The function $f$ in (C.10) corresponds to the function $f$ in (8.7), and the independence on the $x$ variable corresponds to the condition $X^a \partial_0 f = 0$ that follows from (8.7). This condition forbids a choice of $f$ with a compact support inside the Darboux chart. Thus, to obtain a well defined vector field $\xi^a$ associated to a non-trivial $f$ one has to appropriately patch together the solutions (C.10) for the different Darboux charts. Furthermore, in the varying rank case, one would need to take into account the regions where the rank changes from 1 to $\neq 1$. At such points Darboux charts are no longer available and one would need new local descriptions. Whereas it is clear how one should proceed for a given particular configuration, we are not able to give further characterization of these ‘transversal’ symmetries in a generic setting. Such characterization is also needed to determine the phases of the varying rank configurations in the case where $\partial \hat{V}_1$ is not connected, see equation (8.16).

C.5. Equation (6.12)

Let $\hat{e}$ denote the open edge that results from removing the endpoints of a closed edge $e$. Similarly, given a graph $\gamma = \bigcup_{n=1}^N e_n$ composed of $N$ closed edges, we define $\hat{\gamma} := \bigcup_{n=1}^N \hat{e}_n$. The following lemma will be used in obtaining (6.12).

Given a semianalytic graph $\gamma$, and a point $p \in \hat{e} \subset \hat{\gamma}$, there exists an open neighborhood $U_p$ such that $U_p \cap \hat{\gamma} = U_p \cap \hat{e}$.

To show this assertion, suppose by contradiction there exists a point $p \in \hat{e}$ that does not admit any such open neighborhood. Then $(\hat{\gamma} \setminus \hat{e}) \cap U_p \neq \emptyset$ for every open neighborhood $U_p$ of
p. It follows that there exists a sequence of points \( p_n \in (\tilde{\gamma} \setminus \hat{e}) \) that have \( p \) as an accumulation point. Since \( p, p_n \in \gamma \) and \( \gamma \) is compact, there exists a subsequence of \( p_n \) that converges to \( p \). This can only happen if \( p \) is a vertex of \( \gamma \), which then contradicts the condition \( p \in \hat{e} \).

We now show equation (6.12). Suppose by contradiction that \( s \) and \( s' \) satisfy

\[
\gamma(s) \cap V_0 \neq \gamma(s') \cap V_0.
\]  

(C.11)

Let \( \gamma(s, s') \) be the coarsest graph underlying \( s \) and \( s' \) and consider the graph \( \tilde{\gamma} := \gamma(s, s') \setminus (\gamma(s) \cap \gamma(s')) \). Semianalyticity of the original graphs ensures \( \tilde{\gamma} \) is itself semianalytic and condition (C.11) is equivalent to \( \tilde{\gamma} \cap V_0 \neq \emptyset \). Since the intersection of a graph with an open set cannot consist of isolated points (otherwise we would be able to find an open set \( U \) whose intersection with the graph consists of a single point), it follows that there exists an open interval \( I \subset \tilde{\gamma} \cap V_0 \). \( I \) is either contained in \( \gamma(s) \) or in \( \gamma(s') \). Consider the case where \( I \subset \gamma(s) \). From the lemma above, every point \( p \in I \) has an open neighborhood \( U_p \) such that \( U_p \cap \tilde{\gamma} = U_p \cap I \). Furthermore, we can take \( U_p \) such that \( U_p \subset V_0 \). We can then construct vector fields with support in \( U_p \) of the type used in the argument following equation (5.4) in order to produce infinitely many diffeomorphisms that change \( I \) (and hence \( s' \)) but leave \( s \) unchanged. This implies \( \langle s, \tilde{E}|O|s' \rangle, \tilde{E} \rangle \) vanishes, and hence the contradiction.

C.6. Equation (8.17)

Diffeomorphisms generated by vector fields of the form \( \xi^a = gX^a \) are symmetries of \( X^a \) for any density weight minus one scalar \( g \). These in turn generate continuous symmetries of \( \tilde{E} \). If \( \langle \tilde{E}|O|\tilde{E} \rangle \neq 0 \) they must also be symmetries of \( \tilde{E} \):

\[
\langle \tilde{E}|O|\tilde{E} \rangle \neq 0 \Rightarrow L_{\xi^a}X^a|_{\tilde{V}_0} = 0 \quad \forall \text{ density weight } -1 \text{ in } \tilde{V}_0
\]  

(C.12)

(recall that \( \tilde{V}_0 = \tilde{V}_0^\prime \)). Let us now use an auxiliary metric \( \tilde{q}_{ab} \) on \( \Sigma \) to express (C.12) in terms of the associated covariant derivative \( D, D_{\tilde{q}_{ab}} = 0 \):

\[
L_{\xi^a}X^a = g(X^bD_bX^a - X^bD_bX^a) + 2D_{\tilde{q}_{ab}}X^a = 0 \quad \forall g.
\]  

(C.13)

Taking \( g = \tilde{q}^{-1/2} \), the second term vanishes and we conclude that \( (X^bD_bX^a - X^bD_bX^a) = 0 \). Equation (C.13) then becomes:

\[
D_bgX^bX^a = 0 \quad \forall g \quad \Rightarrow X^bX^a = 0,
\]  

(C.14)

and we recover condition (6.12).

Appendix D. Semidirect product structure of Aut

Let us first recall the notion of semi direct product (we follow presentation from [30]). Let \( D \) and \( G \) be two Lie groups and let \( \sigma : D \rightarrow \text{Aut}(G) \) be an homomorphism, that is, for each \( \phi \in D, \sigma_\phi : G \rightarrow G \) is an invertible map satisfying:

\[
\sigma_\phi(gg') = \sigma_\phi(g)\sigma_\phi(g'), \quad \sigma_{\phi\phi'}(g) = \sigma_\phi(\sigma_{\phi'}(g)) \quad \forall g, g' \in G, \phi, \phi' \in D.
\]  

(D.1)

The semidirect product \( G \rtimes D \) is a new Lie group whose elements are pairs \( (g, \phi) \in G \times D \) satisfying the following product:

\[
(g, \phi)(g', \phi') := (g\sigma_\phi(g'), \phi\phi').
\]  

(D.2)

Properties (D.1) guarantee all the group product rules are satisfied. The Lie algebra structure is as follows. As a vector space is given by \( \text{Lie}(G \rtimes D) = \text{Lie}(G) \oplus \text{Lie}(D) \). The Lie bracket can be obtained from (D.2) and is given by:

\[
[[\Lambda, \xi], (\Lambda', \xi')] = (\tau_{\xi}(-\Lambda') - \tau_{\xi'}(\Lambda) + [\Lambda, \Lambda'], [\xi, \xi'])
\]  

(D.3)
where $\tau : \text{Lie}(D) \to \text{Lie}(G)$ is a Lie algebra homomorphism induced by $\sigma$ and defined as follows. Let $g(s)$ and $\phi(t)$ be one-parameter family of group elements such that $g(0) = \text{Id}_G$, $g(0) = \Lambda \in \text{Lie}(G)$ and $\phi(0) = \text{Id}_D$, $\phi(0) = \xi \in \text{Lie}(D)$. Then
\[
\tau_\xi(\Lambda) := \frac{d}{ds} \sigma_{\phi(t)}(g(s)))|_{t=0}.
\] (D.4)

In our case of interest we have: $D = \text{Diff}$, $G = \mathcal{G}$, $\sigma_\rho(g) = \phi_\rho g$ and $\tau_\xi(\Lambda) = -\mathcal{L}_\xi \Lambda$.

### Appendix E. Heuristic relation between spin networks and distributional background triads

In an Abelian $U(1)$ gauge theory, holonomies can be realized as background exponentials functions associated to distributional electric fields: given a closed curve $\gamma : [0, 1] \to \Sigma$ and $j \in \mathbb{Z}$, define the distributional electric field
\[
\tilde{E}^a_{(\gamma,j)}(x) := j \int d\gamma^a(t)\delta(x, \gamma(t))
\] (E.1)
so that
\[
\beta_{\tilde{E}^\gamma_{(\gamma,j)}}[A] = e^{i\int_{\gamma} A^a_{\gamma}(t) d\gamma^a} = e^{i\int_{\gamma} d\gamma^a A^a} = h^a_\gamma[A],
\] (E.2)
where $h^a_\gamma[A]$ is the holonomy along the closed curve $\gamma$ in the $j$ representation of $U(1)$. Notice that $\tilde{E}^\gamma_{(\gamma,j)}$ defined by (E.1) satisfies the divergence free condition $\delta_{\gamma} \tilde{E}^\gamma_{(\gamma,j)} = 0$.

The analogue of (E.1) in the $SU(2)$ theory is given by a rank 1 triad $\tilde{E}^a = \hat{n}X^a$ with $X^a$ a distributional vector field with support on $\gamma$, and $\hat{n}$ a normalized internal direction at each point of $\gamma$:
\[
\tilde{E}^a_{(\gamma,j,\hat{n})}(x) := j \int d\gamma^a(t)\delta(x, \gamma(t))\hat{n}(t), \quad j \in \mathbb{Z}/2,
\] (E.3)
so that
\[
\beta_{\tilde{E}^\gamma_{(\gamma,j,\hat{n})}}[A] = e^{i\int_{\gamma} d\gamma^a \text{Tr}[A_{\gamma}\hat{n}]} = e^{i\int_{\gamma} d\gamma^a \text{Tr}[A_{\gamma}\hat{n}]},
\] (E.4)
where the last equation represents the integral of the one form $\text{Tr}[A\hat{n}]$ along the one dimensional curve defined by $\gamma(t)$ (which we are assuming has no self-intersections; later we comment on the general graph case). The reason for taking $j \in \mathbb{Z}/2$ will become clear below. Let us now study the behavior of the distributional background exponential (E.4) under local $SU(2)$ rotations. Given $g \in \mathcal{G}$, the phase factor associated to (E.3) can be expressed as a difference of phases associated to $\hat{n}$ and $g\hat{n}g^{-1}$:
\[
\alpha(g, \tilde{E}_{(\gamma,j,\hat{n})}) = j \int_{\gamma} \text{Tr}[g^{-1} d\hat{n}]
\] (E.5)
\[
= j(\Phi[g\hat{n}g^{-1}] - \Phi[\hat{n}]),
\] (E.6)
where
\[
\Phi[\hat{n}] = \int_D \text{Tr}[\hat{n}\hat{n} \wedge \hat{n}]
\] (E.7)
is given by a two dimensional integral over a surface $D$ such that $\gamma = \partial D$. Geometrically it corresponds to the area enclosed by $\hat{n}(t) \subset S^2$. For $j \in \mathbb{Z}/2$, the phase $e^{i\phi[G]}$ is independent of whether one takes the ‘inside’ or ‘outside’ area of this curve.

17 If $\gamma$ cannot be realized as the boundary of a two dimensional surface in $\Sigma$, it is still possible to define $\Phi$ as a two dimensional integral on the parameter space where $t$ belongs.
Thus, if we denote by 

$$w_{\gamma,h}^j[A] \equiv e^{i\Phi [g]} P_{E_{\gamma,h}}[A],$$

(E.8)

be the rephased background exponential. It then follows that $w_{\gamma,h}^j$ transforms covariantly under the action of $g \in G$:

$$g \cdot w_{\gamma,h}^j[A] \equiv w_{\gamma,g\cdot h}^j[g^{-1} \cdot A]$$

(E.10)

$$= w_{\gamma,h}^j[A].$$

(E.11)

Thus, if we denote by $|\tilde{n} \gamma j\rangle$ the state associated to $w_{\gamma,h}^j$, the $G$-group averaging formula would take the simple form:

$$\eta(|\tilde{n} \gamma j\rangle)[A] = \sum_{\tilde{n}(t)} w_{\gamma,h}^j[A].$$

(E.12)

with the sum ranging over all possible internal directions $\tilde{n}(t)$ along the curve $\gamma(t)$ and with no additional phases present. At a formal level, this expression defines the following ‘wave function’ of the connection $A$:

$$\eta(|\tilde{n} \gamma j\rangle)[A] = \sum_{\tilde{n}(t)} w_{\gamma,h}^j[A].$$

(E.13)

It turns out that the formal expression on the right-hand side of (E.13) can be identified with a Wilson loop function as follows [31]. Let $h_j^i[A]$ be the holonomy in the $j$ representation along $\gamma$ and $W_j^i[A] = \text{trace}(h_j^i[A])$. $h_j^i[A]$ can formally be thought of as an ‘evolution operator’ (in the spin $j$ vector space) associated with the ‘time dependent Hamiltonian’ $H(t) := A_j^i(\gamma(t))\gamma^\alpha(t)\pi^\beta(\tau),$ where $\pi^\beta(\tau)$ are the $su(2)$ generators in the $j$ representation. One can then express this ‘evolution operator’ in a path integral form in the spin $j$ coherent state basis $|\tilde{j}\rangle$, $\tilde{n} \in S^2$. Doing so from $t = 0$ to $t = 1$ where $\gamma(0) = \gamma(1)$, and taking the trace, one arrives at [31]:

$$W_j^i[A] = \frac{1}{N!} \int \mathcal{D}\tilde{n}(t)[D\tilde{n}] w_{\gamma,h}^j[A],$$

(E.14)

where $\int \mathcal{D}\tilde{n}(t) = \lim_{N \to \infty} \frac{1}{2 \pi N} \int d\tilde{n}_1 \ldots \int d\tilde{n}_N$, with $\int d\tilde{n}_a$ the integral on the unit sphere corresponding to the discretized ‘time’ $\tau_a = n/N$, and $w_{\gamma,h}^j[A]$ given by (E.8). Expression (E.14) has structurally the same form as the formal ‘group averaged wave function’ (E.13).

The above formal procedure can be extended to general graphs to obtain ‘spin network wave functions’. In this case, there is additional gauge invariant information encoded in the relative directions of the internal vectors $\tilde{n}_e$ at the vertices where the edges $e$ meet. Upon averaging one obtains a spin network function in a coherent intertwiner representation [32], with intertwiners determined by the aforementioned gauge invariant information.

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