The evolution fractional p-Laplacian equation in $\mathbb{R}^N$. Fundamental solution and asymptotic behaviour

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Abstract

We consider the natural time-dependent fractional $p$-Laplacian equation posed in the whole Euclidean space, with parameters $p > 2$ and $s \in (0, 1)$ (fractional exponent). We show that the Cauchy Problem for data in the Lebesgue $L^q$ spaces is well posed, and show that the solutions form a family of non-expansive semigroups with regularity and other interesting properties. As main results, we construct the self-similar fundamental solution for every mass value $M$, and prove that general finite-mass solutions converge towards that fundamental solution having the same mass in all $L^q$ spaces.

1 Introduction. The problem

We consider the initial-value problem for the evolution equation

\begin{equation}
\partial_t u + \mathcal{L}_{s,p} u = 0
\end{equation}

posed in the Euclidean space $x \in \mathbb{R}^N, N \geq 1$. The nonlinear operator $\mathcal{L}_{s,p}$ (technically called $s$-fractional $p$-Laplacian operator) is defined by the formula

\begin{equation}
\mathcal{L}_{s,p}(u) := \text{P.V.} \int_{\mathbb{R}^N} \frac{\Phi(u(x,t) - u(y,t))}{|x - y|^{N+sp}} \, dy,
\end{equation}

where $\Phi(z) = |z|^{p-2}z$. It is the Euler-Lagrange operator corresponding to the power-like functional with nonlocal kernel of the $s$-Laplacian type:

\begin{equation}
\mathcal{J}_{p,s}(u) = \frac{1}{p} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^p}{|x - y|^{N+sp}} \, dxdy.
\end{equation}

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We will take fractional exponent $0 < s < 1$ and also $p > 2$, though in principle we could have taken the larger range $1 < p < \infty$. We supplement the equation with an initial datum

$$\lim_{t \to 0} u(x, t) = u_0(x),$$

where $u_0(x)$ belongs to some class of locally integrable functions in $\mathbb{R}^N$. The cases $u_0 \in L^q(\mathbb{R}^N)$ with $1 \leq q < \infty$ are considered, and in particular $q = 1$ which leads to the class of finite-mass data and solutions. We will be specially interested in taking a Dirac delta as initial datum. In that case the solution is called a fundamental solution, and also a source-type solution (mainly in the Russian literature).

Before presenting our results, let us comment on motivations and related equations. The study of nonlinear equations like (1.1) is motivated by an increasing interest in studying the combined effect of nonlinear and nonlocal terms in the formulation of the basic models of nonlinear diffusion in view of a large number of applications. These arise in fields like continuum mechanics, stochastic processes, image processing, finance, population dynamics, and so on. It also has a theoretical interest for PDEs, Nonlinear Functional Analysis and Potential Theory. Some of these nonlinear nonlocal diffusion models are presented in the survey paper [41], where the nonlinearities are mainly of porous medium type, see also [38, 40].

The simplest equation in the fractional family is found in the limit case where $p = 2$

$$u_t + (-\Delta)^s u = 0,$$

i.e., the heat equation associated to the fractional Laplacian $(-\Delta)^s$, a nonlocal generalization of the Laplace operator studied in classical monographs like [27, 34]. The $s$-Laplacian is a linear operator that coincides with $\mathcal{L}_{s,2}$ up to a constant. Equation (1.5) inherits many of the well-known properties of the classical heat equation (case $s = 1$) except for rates of space propagation, reflected in the fact the solutions with compactly supported data develop, for all positive times, spatial profiles with tails at infinity that decay like a power of distance, $u(x, t) \sim c(t)|x|^{-(N+2s)}$. The equation has been amply discussed in the literature, see recent results in [5, 11] on the existence theory for optimal classes of data.

On the other hand, it is proved that in the limit $s \to 1$ with $p \neq 2$, and after inserting a normalizing constant, we get the well-known evolution $p$-Laplacian equation $\partial_t u = \Delta_p(u) = \nabla \cdot (|\nabla u|^{p-2} \nabla u)$, that has also been widely investigated since the early 1970’s because of a number of applications (cf. for instance [36], Section 11) and for its remarkable mathematical properties. The semigroup method proved to be an effective method to treat the equation, see early works by Bénilan and Véron [4, 43]. Regularity theory is contained in the monograph by DiBenedetto [18]. The recent literature is very large and has many novel features.

The nonlinear fractional operator we are dealing with in this paper was mentioned in the paper [24] by Ishii and Nakamura, see also [16] by Chambolle, Lindgren and
Monneau. There are a number of works that cover the evolution equation (1.1) in the case where the space domain is a bounded subdomain $\Omega \subset \mathbb{R}^N$, see [31, 41] and references. References to the equation posed in the whole space are more recent, like [13, 35].

1.1 Outline of the paper

We focus on Problem (1.1)-(1.4), posed in $\mathbb{R}^N$. It is not difficult to prove that this Cauchy problem is well-posed in all $L^q(\mathbb{R}^N)$ spaces, $1 \leq q < \infty$. This parallels what is known in the case of bounded domains. In view of such works, we first review the main facts of the theory in Section 2. In particular, we define the class of continuous strong solutions that correspond to $L^2$ and $L^1$ initial data and derive its main properties in detail.

We want to stress the differences brought about but the consideration of the whole space. In this respect, a most interesting question is that of finding the fundamental solution, i.e., the solution such that

$$
\lim_{t \to 0} \int_{\mathbb{R}^N} u(x, t) \varphi(x) \, dx = M \varphi(0),
$$

for every smooth and compactly supported test function $\varphi \geq 0$, and some $M > 0$.

**Theorem 1.1.** For every given mass $M > 0$ there exists a unique self-similar solution of Problem (1.1)-(1.4). It has the form

$$
U(x, t; M) = M^{sp \beta} t^{-\alpha} F(M^{-(p-2)\beta} x t^{-\beta}),
$$

with self-similarity exponents

$$
\alpha = \frac{N}{N(p-2) + sp}, \quad \beta = \frac{1}{N(p-2) + sp}.
$$

The profile $F(r)$ is a continuous, positive, radially symmetric ($r = |x| t^{-\beta}$), and decreasing function such that $F(r) = O(r^{-\left(N + sp\right)})$ as $r \to \infty$.

We see that all fundamental solutions with $M > 0$ are obtained from the one with unit mass, $M = 1$ by a simple rescaling. Fundamental solutions with $M < 0$ are obtained by just reversing the sign of the solution. For $M = 0$ the fundamental solution becomes the null function. The theorem is proved in Section 6. Important preliminaries take up Sections 3, 4, and 5.

The fundamental solution is the key to the study of the long-time behaviour of our problem with general initial data, since it represents, in Barenblatt’s words, the intermediate asymptotics, cf. [3]. This is the asymptotic result in our case.
Theorem 1.2. Let \( u \) be a solution of Problem (1.1)-(1.4) with initial data \( u_0 \in L^1(\mathbb{R}^N) \) of integral \( M \), and let \( U_M \) be the fundamental solution with that mass. Then,

\[
\lim_{t \to \infty} \| u(t) - U_M(t) \|_1 = 0.
\]

We also have the \( L^\infty \)-estimate

\[
\lim_{t \to \infty} t^\alpha \| u(t) - U_M(t) \|_\infty = 0.
\]

The theorem is proved in Section 7. There is no restriction on the sign of the solution. By interpolation rates in all \( L^q \) spaces are easily obtained, \( 1 < q < \infty \), see for instance examples in [42]. Of course, for \( M = 0 \) we just say that \( \| u(t) \|_1 \) goes to zero.

Finally, as added information, the existence of the source-type solution in a bounded domain is also shown in Section 8. It is not relevant for the long-time behaviour.

Comments. The importance of the Gaussian fundamental solution in the classical heat equation is well-known in the mathematics literature and needs no reminder, [22, 44]. In the linear fractional case \( p = 2 \) with \( 0 < s < 1 \), the fundamental solution of the fractional heat equation is also known thanks to Blumenthal and Getoor [8] who studied it in 1960. In such a case the fundamental solution also allows to construct the class of all nonnegative solutions of the Cauchy problem in the whole space by using the representation formula, see the theory of [11] where an optimal class of data is considered and well-posedness shown.

In the case of nonlinear problems, the importance of fundamental solutions has been proved in numerous examples, even if, contrary to what happens for linear equations, representation formulas for general solutions in terms of such a special solution are not available. Their interest lies mainly in the description of the asymptotic behaviour as \( t \to \infty \) of general solutions. The fundamental solution is well-known in the standard \( p \)-Laplacian case, \( p > 2, s = 1 \). Its existence comes from [2], hence the name Barenblatt solution, and its uniqueness was established in [25], see also [28]. For the standard porous medium equation the situation is well-known, see the historical comment in the monograph [37]. A recent example for nonlinear fractional equations is given by the fundamental solution of the fractional porous medium equation constructed by the author in [39]. For the so-called porous medium equation with fractional potential pressure the fundamental solution was first constructed in [7] and [15], and the asymptotic behaviour was established in the latter reference. In all cases the application to the asymptotic behaviour as \( t \to \infty \) is carried through, and convergence of a general class of finite-mass solutions to the corresponding fundamental solution is proved.

On the other hand, for the problem posed in a bounded domain the special solution that is relevant concerning the asymptotic behaviour as \( t \to \infty \) is the separate-variables solution called the friendly giant, that was constructed for our equation by the author in [41].
Notations. We sometimes write a function \( u(x,t) \) as \( u(t) \) or \( u \) when one some of the variables can be safely understood. We use the notation \( u_+ = \max\{u,0\} \). The letters \( \alpha \) and \( \beta \) will be fixed at the values given in the self-similar formula (1.8). We also use the symbol \( \| u \|_q \) as shortened notation for the norm of \( u \) in the \( L^q \) space over the corresponding domain when no confusion is to be feared. We denote the duality product in \( L^q \times L^{q'} \), with \( q \) and \( q' \) dual exponents, by \( \langle \cdot, \cdot \rangle \). For a function \( u(x) \geq 0 \) we call mass or total mass the integral \( \int_{\mathbb{R}^N} u(x) \, dx \), either finite or infinite. For signed functions that integral does not coincide with the \( L^1 \) norm, so the use of the term is only justified by analogy and usually refers to the \( L^1 \) norm.

2 Basic theory

We establish well-posedness of Problem (1.1)-(1.4) in different functional spaces, starting by the consideration of the equation as a gradient flow in \( L^2(\mathbb{R}^N) \). We obtain unique strong solutions that are \( C^\delta \)-Hölder continuous and space and time, and decay as expected by dimensional considerations. We give a detailed account of the main qualitative and quantitative properties, some of them correspond to known work done in bounded domains, but some are particular to the whole-space setting. Some of the results of the section are new.

2.1 Existence and uniqueness

We can solve the evolution problem for equation (1.1) with initial data \( u_0 \in L^2(\mathbb{R}^N) \) by using the fact that the equation is the gradient flow of a maximal monotone operator associated to the convex functional (1.3), see for instance [31, 32, 41]. The domain of that operator is

\[
D_2(\mathcal{L}_{s,p}) = \{ \phi \in L^2(\mathbb{R}^N) : J_{s,p}(u) < \infty, \mathcal{L}_{s,p}u \in L^2(\mathbb{R}^N) \}.
\]

Well known theory implies that for every initial \( u_0 \in L^2(\mathbb{R}^N) \) there is a unique strong solution \( u_t \) and \( \mathcal{L}_{s,p}u \in L^2(\mathbb{R}^N) \) for every \( t > 0 \), and the equation is satisfied a.e in \( x \) for every \( t > 0 \). The semigroup is denoted as \( S_t(u_0) = u(t) \), where \( u(t) \) is the solution emanating from \( u_0 \) at time 0. Typical a priori estimates for gradient flows follow, cf. [14]. The next results are part of the standard theory:

\[
\frac{1}{2} \frac{d}{dt} \| u(t) \|_2^2 = -\langle \mathcal{L}_{s,p}u(t), u(t) \rangle = -p \mathcal{J}(u(t)),
\]

where \( \mathcal{J} = \mathcal{J}_{p,s} \) as in the introduction, and also

\[
\frac{d}{dt} \mathcal{J}(u(t)) = \langle \mathcal{L}_{s,p}u(t), u_t(t) \rangle = -\| u_t(t) \|_2^2,
\]
where integrals and norms are taken in $\mathbb{R}^N$. It follows that both $\|u(t)\|_2$ and $J(u(t))$ are decreasing in time, and we get the easy estimate $J(u(t)) \leq \|u_0\|_2^2/t$ for every $t > 0$. See other properties below.

Moreover, for given $p > 1$ (the index of the operator) and every $1 \leq q \leq \infty$, the $L^q$ norm of the solution is non-increasing in time. We can extend the set of solutions to form a continuous semigroup of contractions in $L^q(\mathbb{R}^N)$ for every $1 \leq q < \infty$: for every $u_0 \in L^q(\mathbb{R}^N)$ there is a unique strong solution such that $u \in C([0, \infty) : L^q(\mathbb{R}^N))$. The class of solutions can be called the $L^q$ semigroup for equation (1.1) posed in $\mathbb{R}^N$. These $q$-semigroups coincide on their common domain. The Maximum Principle applies, and more precisely $T$-contractivity holds in the sense that for two solutions $u_1, u_2$ and any $q \geq 1$ we have

$$\|(u_1(t) - u_2(t))_+\|_q \leq \|(u_1(0) - u_2(0))_+\|_q.$$  

This implies that we have an ordered semigroup for every $q$ and $p$. An operator with these properties in all $L^q$ spaces is called completely accretive. We can also obtain the solutions by Implicit Time Discretization, cf. the classical references [17, 21]. The word mild solutions is used in that context, but mild and strong solutions coincide by uniqueness. The operator is also accretive in $L^\infty$ and this allows to generate a semigroup of contractions in $C_0(\mathbb{R}^N)$ the set of continuous functions that go to zero at infinity.

This part of the theory can be done for solutions with two signs, but we will often reduce ourselves in the sequel to nonnegative data and solutions. Splitting the data into positive and negative parts most of the estimates apply to signed solutions. To be precise, for a signed initial function $u_0$ we may consider its positive part, $u_{0,+}$ and its negative part $u_{0,-} = -u_0 + u_{0,+} = -\max\{-u_0, 0\}$. Then, both $u_{0,+}$ and $u_{0,-}$ are nonnegative and $-u_{0,-} \leq u_0 \leq u_{0,+}$. It follows from the comparison property of the $L^q$ semigroups that

$$-S_t(u_{0,-}) \leq S_t(u_0) \leq S_t(u_{0,+}).$$

Therefore, we may reduce many of the estimates to the case of nonnegative solutions.

• An alternative construction approach is to prove that the solutions in $\mathbb{R}^N$ are obtained as limits of the solutions of the Dirichlet problem posed in expanding balls $\Omega_R = B_R(0)$, as constructed in [31, 41]. For nonnegative solutions with a common initial datum this limit is monotone in $R$. The proof that the two ways of construction give the same solutions is easy in the nonnegative case and will be omitted. In this way the $L^q$ semigroups are obtained as limit as the ones on bounded domains and the many properties, like $L^q$ boundedness, contractivity or comparison are inherited.

### 2.2 Scaling

In our study we will use the fact that the equation admits a one-parameter scaling group that conserves the mass of the solutions. Thus, if $u$ is a weak or strong solution
of the equation, then we obtain a family of solutions of the same type, \( u_k = \mathcal{T}_k u \), given by
\[
(2.4) \quad \mathcal{T}_k u(x, t) = k^N u(kx, k^{N(p-2)+sp}t)
\]
for every \( k > 0 \). This scaling transformation can be combined with a second one that keeps invariant the space variable
\[
(2.5) \quad \hat{\mathcal{T}}_M u(x, t) = Mu(x, M^{p-2}t)
\]
for every \( M > 0 \). This one can be used to reduce the calculations to solutions with unit mass, \( M = 1 \). Together, these transformations form the two-parameter scaling group under which the equation is invariant.

Let us point out that the set of solutions of the equation is invariant under a number of isometric transformations, like: change of sign: \( u(x, t) \) into \( -u(x, t) \), rotations and translations in the space variable, and translations in time. They will also be used below.

### 2.3 A priori bounds

- Our operator is homogeneous of degree \( d = p - 1 > 1 \) in the sense that \( \mathcal{L}_{s,p}(\lambda u) = \lambda^{p-1} \mathcal{L}_{s,p} u \). Using the general results by Bénilan-Crandall [6] for homogeneous operators in Banach spaces, we can prove the a priori bound
\[
(2.6) \quad (p-2)tu_t > -u,
\]
which holds for all nonnegative solutions, in principle in the sense of distributions. This a priori bound is quite universal, independent of the solutions. It is based on the scaling properties and comparison. Therefore, we have almost monotonicity in time if \( u \geq 0 \). In particular, if a strong solution is positive at a certain point \( x_0 \) at \( t = t_0 \), then for all later times \( u(x_0, t) > 0 \). This is called conservation of positivity (for nonnegative solutions).

Combined with the decay of the space integral in time, we conclude another interesting result for nonnegative solutions:
\[
(2.7) \quad \|u(\cdot, t)\|_{L^1(\mathbb{R}^N)} \leq C_p \|u(\cdot, t)\|_{L^1(\mathbb{R}^N)} t^{-1}.
\]
- On the other hand, paper [4] also implies the estimate for all \( p > 2 \) and \( q > 1 \) we have
\[
(2.8) \quad \|u_t\|_q \leq \frac{2}{(p-2)t} \|u_0\|_q
\]
for every \( 1 < q \leq \infty \). For \( q = 1 \) it is formulated as different quotients. Formula (2.8) is valid for all signed solutions.
2.4 Energy estimates

- As we have seen before, for solutions with data in $L^2(\mathbb{R}^N)$ and times $0 \leq t_1 < t_2$ we have the identity

$$\int_{\mathbb{R}^N} u^2(x, t_1) dx - \int_{\mathbb{R}^N} u^2(x, t_2) dx = 2 \int_{t_1}^{t_2} \int_{\mathbb{R}^N} |u(x, t) - u(y, t)|^p d\mu(x, y) dt,$$

where $d\mu(x, y) = p|x - y|^{-(N+sp)} dx dy$. In the sequel we omit the domain of integration of most space integrals when it is $\mathbb{R}^N$ and the time interval when it can easily understood from the context.

We point out that this estimate shows that solutions with $L^2 \cap L^p$ data belong automatically to the space $L^p(0, \infty : W^{s,p}(\mathbb{R}^N))$.

- Arguing in the same way, for solutions with data in $L^q(\mathbb{R}^N)$ with $q > 1$ and $0 \leq t_1 < t_2$ we have for nonnegative solutions

$$\int u^q(x, t_1) dx - \int u^q(x, t_2) dx = q \int \int \int |u(x, t) - u(y, t)|^{p-2}((u(x) - u(y)), (u^{q-1}(x, t) - u^{q-1}(y, t))) d\mu(x, y) dt,$$

with integration in the same sets as before. We use the inequality

$$(a - b)^{p-1}(a^{q-1} - b^{q-1}) \geq C(p, q) |a^{(p+q-2)/p} - b^{(p+q-2)/p}|^p$$

which is valid for all $a > b > 0$ and $p, q > 1$. This inequality is also true when $b \geq a > 0$ by symmetry and when $a$ and $b$ have different signs in an elementary way. We get the new inequality

$$C(p, q) \int \int \int |u(x, t)^{(p+q-2)/p} - u(y, t)^{(p+q-2)/p}|^p d\mu(x, y) dt \leq \int u^q(x, t_1) dx - \int u^q(x, t_2) dx,$$

which applies the solutions of the $L^q$ semigroup, $q > 1$. This gives a precise estimate of the dissipation of the $L^q$ norm along the flow.

Case of signed solutions. The above results hold on the condition that we use the notation $a^{p-1}$ to mean $|a|^{p-2}a$ and so on (this is a usual convention). The equality to prove is

$$\int |u|^q(x, t_1) dx - \int |u|^q(x, t_2) dx = q \int \int \{(u(x) - u(y))^{p-1}, (u^{q-1}(x, t) - u^{q-1}(y, t))\} d\mu(x, y) dt,$$

and the dissipation estimate is also true in this case.

Note that these estimates can be obtained as limit of the ones already obtained for the problem posed in a bounded domain.
2.5 Difference estimates

It is well known that the semigroup is contractive in all $L^q$ norms, $1 \leq q \leq \infty$. At some moments we would like to know how the norms of the difference of two solutions decrease in time. Such decrease is called dissipation. We present here the easiest case, decrease in $L^2$ norm.

$L^2$ dissipation. For solutions with data in $L^2(\mathbb{R}^N)$ and times $0 \leq t_1 < t_2$ we have the identity for the difference of two solutions $u = u_1 - u_2$

$$
(2.14) \quad \int_{t_1}^{t_2} \int_{\mathbb{R}^N} u_1^2(x, t) \, dx - \int_{\mathbb{R}^N} u_2^2(x, t) \, dx = 2 \int_{t_1}^{t_2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} (|u_1(x, t) - u_1(y, t)|^{p-1}(u_1(x, t) - u_1(y, t))

- |u_2(x, t) - u_2(y, t)|^{p-1}(u_2(x, t) - u_2(y, t)))

(u_1(x, t) - u_2(x, t) - u_1(y, t) + u_2(y, t)) \, d\mu(x, y) \, dt,
$$

where $d\mu(x, y) = |x - y|^{-(N+sp)} \, dx \, dy$ as before. Putting $a = u_1(x, t) - u_1(y, t)$ and $b = u_2(x, t) - u_2(y, t)$ and using the numerical inequality as before we bound below the last integral by

$$
C(p) \iint (|u_1(x, t) - u_1(y, t)|^{p/2} - (u_2(x, t) - u_2(y, t))^{p/2})^2 \, d\mu(x, y) \, dt.
$$

This is an estimate of the $L^2$ dissipation of the difference $u = u_1 - u_2$.

Later on, we will need the expression of the $L^1$ dissipation in the study of the asymptotic behaviour, but we will postpone it until conservation of mass is proved.

2.6 Boundedness for positive times. Continuity

• An important result valid for many nonlinear diffusion problems with homogeneous operators is the so-called $L^1$-$L^\infty$ smoothing effect. In the present case we have

Theorem 2.1. For every solution with initial data $u_0 \in L^1(\mathbb{R}^N)$ we have

$$
(2.15) \quad |u(x, t)| \leq C(N, p, s)\|u_0\|_1 \, t^{-\alpha},
$$

with exponents

$$
\alpha = N/(N(p - 2) + sp), \quad \gamma = sp\alpha,
$$

given essentially by the scaling rules.

The result has been recently proved by Bonforte and Salort [9], Theorem 5.3, where an explicit value for the constant $C(N, p, s)$ is given. It can also be derived as a consequence of the results of Strömqvist [35]. Note that this formula has to be invariant under the scaling transformations of Subsection 2.2. For reference to the
similar result in a number of similar nonlinear diffusion theories, including linear and fractional heat equation, porous medium and its fractional versions, $p$-Laplacian, and so on, cf. for instance [12].

• Once we know that solutions are bounded, we can prove further regularity. We can rely on Theorem 1.2 of [13] by Brasco-Lindgren-Strömqvist that we state in short form as follows:

**Theorem 2.2.** Let $\Omega \subset \mathbb{R}^N$ be a bounded and open set, let $I = (t_0, t_1]$, $p \geq 2$ and $0 < s < 1$. Suppose that $u$ is a local weak solution of (1.1) in the cylinder $Q = \Omega \times I$ such that it is bounded in the sense that $u \in L^\infty_{\text{loc}}(I; L^\infty(\mathbb{R}^N))$. Then, there exist positive constants $\Theta(s, p)$ and $\Gamma(s, p)$ such that

$$u \in C^\delta_{x, \text{loc}}(Q) \cap C^\gamma_{t, \text{loc}}(Q)$$

for every $0 < \delta < \Theta$ and $0 < \gamma < \Gamma$. Moreover, the Hölder bounds in both space and time are uniform in any cylinder $Q' = B_R(x_0) \times I'$ strictly included in $Q$, and they depend only on $N, s, p$, the distance of $Q'$ to the parabolic boundary of $Q$ and on the norm of $u$ in $L^\infty(\mathbb{R}^N \times I')$.

Explicit values for $\Theta$ and $\Gamma$ are given in [13]. We can check that the conditions of this theorem apply to our setting whenever $t_0 > 0$, hence we have

**Corollary 2.1.** The solutions of our evolution problem (1.1) - (1.4) are uniformly Hölder continuous in space with exponent $\delta < \Theta$ and in time with exponent $\gamma < \Gamma$, always for $t \geq t_0 > 0$.

For completeness we recall a number of previous papers on the elliptic equation $L_{s,p}u(t) = f$ that proved different results on continuity of solutions of the elliptic version under assumptions on $f$. Let us quote Kuusi-Mingione-Sire [26] who first proved continuity for $sp < N$, Lindgren [29] who proved Hölder continuity for continuous $f$, Iannizzotto et al. [23] who proved Hölder regularity for bounded $f$ with $u = 0$ outside of $\Omega$ and finally Brasco-Lindgren-Schikorra [12] who proved Hölder regularity for $f \in L^q_{\text{loc}}$ with $q > N/sp$, $q \geq 1$. This last result was the basis of an alternative but more complicated former proof we had for our corollary.

### 2.7 Positivity of nonnegative solutions

Nonnegative strong solutions of equation (1.1) enjoy the property of strict positivity at least in the almost everywhere sense. Indeed, at every point $(x_0, t_0)$ where a solution reaches the minimum value $u = 0$ and $(L_{s,p}u)(x_0, t_0)$ exists, then it must be strictly negative according to the formula for the operator. On the other hand, if $u_t$ exists it must to zero. From this contradiction we conclude that a.e. $u(x, t)$ must be positive. By the already proved conservation of positivity, for any $t > t_0$ we have $u(x, t_0) > 0$ for a.e. $x \in \mathbb{R}^N$.

Since we know that the nonnegative solution is continuous, then $u$ is positive everywhere unless if it is zero everywhere.
2.8 On the fundamental solutions

The existence and properties of the fundamental solution of Problem (1.1)-(1.4) are a main concern of this paper. We expect it to be unique, positive and self-similar for any given mass $M > 0$. Self-similar solutions have the form

$$U(x, t; M) = t^{-\alpha} F(x t^{-\beta}; M).$$

Substituting this formula into equation (1.1), we see that time is eliminated as a factor in the resulting formula on the condition that:

$$\alpha + 1 = \frac{(p-1)\alpha + \beta sp}{N(p-2) + sp}.$$

We also want integrable solutions that will enjoy the mass conservation property, which implies $\alpha = N\beta$. Imposing both conditions, we get

$$\alpha = \frac{N}{N(p-2) + sp}, \quad \beta = \frac{\alpha}{N} = \frac{1}{N(p-2) + sp},$$

as announced in the Introduction. The profile function $F(y; M)$ must satisfy the nonlinear stationary fractional equation

$$(2.16) \quad \mathcal{L}_{s,p} F = \beta \nabla \cdot (y F).$$

Cf. a similar computation for the Porous Medium Equation in [37], page 63. Using rescaling $\hat{T}_M$, we can reduce the calculation of the profile to mass 1 by the formula

$$F(y; M) = M^{sp\beta} F(M^{-(p-2)\beta} y; 1).$$

In view of past experience with $p = 2$, we will look for $F$ to be radially symmetric, monotone in $r = |y|$, and positive everywhere with a certain behaviour as $|y| \to \infty$.

We have proved that all solutions with $L^1$ data at one time will be uniformly bounded and continuous later on. Thus, $F$ must be bounded and continuous. Moreover, bounded solutions have a bounded $u_t$ for all later times. In the case of the fundamental solution this means that $(rF(r))'$ is bounded, hence $F$ must be $C^1$ regular for all $r > 0$.

The self-similar fundamental solution takes a Dirac mass as initial data, at least in the sense of initial trace, $u(x, t) \to M\delta(x)$ as $t \to 0$ in a weak sense. It will be invariant under the scaling group $\mathcal{T}_k$ of Subsection 2.2. All of this will be proved in this paper. The detailed statement is contained in Theorems 6.1 and 6.3 and whole proofs follow there.

2.9 Self-similar variables

In several instances in the sequel it will be convenient to pass to self-similar variables, by zooming the original solution according to the self-similar exponents (2.8). More precisely, the change is done by the formulas

$$(2.17) \quad u(x, t) = (t + a)^{-N\beta} v(y, \tau) \quad y = x (t + a)^{-\beta}, \quad \tau = \log(t + a),$$
with \( \beta = (N(p - 2) + sp)^{-1} \), and any \( a > 0 \). It implies that \( v(y, \tau) \) is a solution of the corresponding PDE:

\[
\partial_\tau v + L_{s,p} v - \beta \nabla \cdot (y v) = 0.
\]

This transformation is usually called continuous in time rescaling to mark the difference with the transformation with fixed parameter \( 24 \).

Note that the rescaled equation does not change with the time-shift \( a \) but the initial value in the new time does, \( \tau_0 = \log(a) \). If \( a = 0 \) then \( \tau_0 = -\infty \) and the \( v \) equation is defined for \( \tau \in \mathbb{R} \). The mass of the \( v \) solution at new time \( \tau \geq \tau_0 \) equals that of the \( u \) at the corresponding time \( t \geq 0 \).

Sometimes \( \tau \) is defined as \( \tau = \log((t + a)/a) \) without change in the equation. It is just a displacement in the new time, but it is important to take it into account in the computations.

Denomination: for convenience we sometimes refer in the sequel to the solutions of the rescaled equation \( (2.18) \) as \( v \)-solutions, while the original ones are \( u \)-solutions.

### 2.10 Comparison via symmetries. Almost radiality

The Aleksandrov symmetry principle \( 1 \) has found wide application in elliptic and parabolic linear and nonlinear problems. An explanation of its use for the Porous Medium Equation is given in \( 37 \), pages 209–211. In the parabolic case it says that whenever an initial datum can be compared with its reflection with respect to a space hyperplane, say \( \Pi \), so that they are ordered, and the equation is invariant under symmetries, then the same space comparison applies to the solution at any positive time \( t > 0 \).

The result has been applied to elliptic and parabolic equations of Porous Medium Type involving the fractional Laplacian in \( 39 \), section 15. The argument of that reference can be applied in the present setting. We leave the verification to the reader-

The standard consequence we want to derive is the following

**Proposition 2.1.** Solutions of our Cauchy Problem having compactly supported data in a ball \( B_R(0) \) are radially decreasing in space for all \( |x| \geq 2R \). Moreover, whenever \( |x| > 2R \) and \( |x'| < |x| - 2R \), then we have \( u(x, t) \leq u(x', t) \) for all \( t > 0 \).

### 3 Barrier construction and tail behaviour

Here we want to construct an upper barrier \( \hat{u} \) for the solutions of the Cauchy problem with suitable data. The barrier will be needed in the construction of the fundamental solution as limit of approximations with the same mass as the initial Dirac delta.
We will only need to consider nonnegative data and solutions. It will be enough to do it for bounded radial functions with compact support as initial data, and then use some comparison argument to eliminate the restrictions of radial symmetry and compact support. The barrier will be radially symmetric, decreasing in $|x|$ and will have behaviour $\tilde{u}(x,t) = O(|x|^{-N-\gamma})$ for very large $|x|$.

**Construction.** We will use the rescaled solution and the equation (2.18) introduced in Subsection 2.9. Translating previous a priori bounds for the original equation into the present rescaled version, we see that all the rescaled solutions are bounded $v(r,t) \leq A((t + a)/t)^{N\beta}$. We also get a bound of the form $v(r,t) \leq Br^{-N}$, as a consequence of finite mass, radially symmetry and monotonicity in $|x|$, and the decay at infinity is uniform in time. $A$ and $B$ depend on the mass of the solution.

Therefore, we only need to refine the latter estimate for large $r$ so that we get an integrable barrier in a region $r \geq R_1 \gg 0$. We use the notation $r = |y| > 0$ in this section where we work with self-similar variables.

The upper barrier we consider in self-similar variables will be stationary in time, $\tilde{v}(r)$. The barrier will have the form of an inverse power in the far field region. We need to compare $v(r,t)$ with $\tilde{v}(r)$ in an outer domain, and make a correction of the solution concept into a Dirichlet problem in a time interval.

To be precise, the barrier will be defined by different expressions in three regions: we select two radii $1 < R < R_1$. For $r > R_1$ it has the form

$$\tilde{v}(y) = C_1 r^{-(N+\gamma)}, \quad r = |y|,$$

with a $\gamma > 0$, we will later make the choice $\gamma = sp$. For $r \leq R$ it is smooth and proportional to $A$. Finally, in the intermediate region $R < r < R_1$

$$\tilde{v}(y,t) \sim C_2 r^{-N}.$$ 

We have to glue these regions: $A \sim C_2 R^{-N}$, $C_2 R_1^{-N} \sim C_1 R_1^{-N-\gamma}$. We can do it in a smooth way, the details are not important.

**Comparison.** We choose as comparison domain the exterior of a big ball for some time, $D = \{r > 2R_1\} \times (0,T)$. We will prove that given a solutions $v$ with small initial data, then $\tilde{v} \geq v$ by in $D$ by using the equation in rescaled form plus the exterior and initial conditions. Let us first remark that by choosing $A$ large and $C_2,C_3$ large we will have $\tilde{v} \geq v$ for all $r < 2R_1$ and all $t$ (the exterior conditions). The initial data will also be satisfied since we are assuming smallness and compact support. As a last step, we need to prove that

$$\mathcal{L}_{s,p} \tilde{v} - \beta r^{1-N}(r^{N}\tilde{v})_r \geq 0$$

in $D$, i.e., for all $r \geq 2R_1$. We see that

$$-\beta r^{1-N}(r^{N}\tilde{v})_r = \beta \gamma C_1 r^{-N-\gamma} > 0.$$
On the other hand, $\mathcal{L}_{s,p}\hat{v}$ may be negative. We have to estimate the contribution of the different regions against the previous bound. The first term to come from the influence of the inner core $\{r < R\}$ where we have $\hat{v} \approx A$. We get for the contribution from this region to the integral $\mathcal{L}_{s,p}\hat{v}$ the quantity

$$\mathcal{L}_{s,p}\hat{v}\big|_1 \sim -A^{p-1}R^N r^{-N-sp}$$

For the moment we need $A^{p-1}R^N r^{-N-sp} \leq C_1 r^{-N-\gamma}$, that holds if $\gamma = sp$ and $A^{p-1}R^N \leq C_1$. We fix $\gamma = sp$ in the sequel. We need $A^{p-1}R^N \leq \varepsilon_1 C_1$.

We still need to calculate the contribution of the remaining regions. Let us fix the point $r = r_0 > 2R_1$. The contribution of the region $\{r > r_0\}$ does not count since it is positive, see the formula. So we have to calculate it in the annulus $\{R < r < r_0\}$. In the smaller annulus $D_2 = \{r_0/2 < r < r_0\}$ we have

$$\hat{v}'(r) = c C_1 r^{-(N+sp)+1}$$

hence putting $\rho = r - r_0$ we get a second contribution

$$-\mathcal{L}_{s,p}\hat{v}(r_0)\big|_2 \leq \int_0^{r_0/2} c C_1^{p-1} r_0^{-(N+sp+1)(p-1)} \rho^{p-1} \rho^{-(N+sp)} \rho^{N-1} d\rho \sim c C_1^{p-1} r_0^{-\gamma'}$$

with

$$\gamma' = (N + sp + 1)(p-1) - p + 1 + sp = (N + sp)(p-1) + sp > N + sp,$$

which fits. We then need $C_1^{p-2} \leq \varepsilon_2 R_1^{\gamma'}$. For $D_3 = \{R_1 < r < r_0/2\}$ we get the contribution:

$$-\mathcal{L}_{s,p}\hat{v}(r_0)\big|_3 \leq \int_{R_1}^{r_0/2} c C_1^{p-1} r^{-(N+2s)(p-1)} r_0^{-(N+sp)} r^{N-1} dr \sim c C_1^{p-1} R_1^{-\gamma_1} r_0^{-(N+sp)},$$

with

$$\gamma_1 = ((N + 2s)(p-1) - N = N(p-2) + 2s(p-1) > 0.$$ 

We need $C_1^{p-2} \leq \varepsilon_2 R_1^{\gamma_1}$. Finally, for $D_4 = \{R < r < R_1\}$

$$-\mathcal{L}_{s,p}\hat{v}(r_0)\big|_4 \leq \int_R^{R_1} c C_2^{p-1} r^{-(N(p-1)-N+sp)} r^{N-1} dr \sim c C_2^{p-1} R^{-N(p-2)} r_0^{-(N+sp)},$$

so that we need $C_2^{p-1} \leq \varepsilon_3 C_1 R^{N(p-2)}$.

- List of inequalities

$$A^{p-1}R^N \leq \varepsilon_1 C_1, \quad C_2^{p-1} \leq \varepsilon_3 C_1 R^{N(p-2)}, \quad C_1^{p-2} \leq \varepsilon_2 R_1^{\gamma_1}, \quad C_1^{p-2} \leq \varepsilon_2 R_1^{\gamma_1}.$$ 

After choosing $A$ and $C_2$ we fix $R$ with $A = C_2 R^{-N}$. Then we need a large $C_1$ large to satisfy the first inequalities and then a large $R_1$ with respect to $C_1^{p-2}$. The construction is done and becomes a supersolution with respect to small initial data.
with compact support. If the data are not small apply transformation $\tilde{T}$ to reduce the data.

**Decay of solutions.** The barrier can be used to find a rate of decay in space of the solutions which is uniform for bounded mass and bounded support. The main result is the following

**Theorem 3.1.** Let $u$ be a solution with nonnegative and bounded and compactly supported data $u_0$. Then, for every $x \in \mathbb{R}^N$, $t > 0$ we have

$$u(x, t) \leq U(x, t) := (t + 1)^{-\alpha}G(|x| (t + 1)^{-\beta}),$$

where $G$ is a positive and bounded function such that $G(r) \leq Cr^{-(N+sp)}$. $C$ depends only on $s, p, N$ and the bounds on the data.

It follows that for $|x| \geq c(t + 1)^{\beta}$ we have

$$u(x, t) \leq C|x|^{-(N+sp)}(t + 1)^{sp\beta}.$$  

**Remark.** We have taken $a = 1$ for convenience since then $\tau_0 = 0$, $x = y$ and $v(y, 0) = u_0(x)$. The same formula holds with $(t + a)$ instead of $(t + 1)$ but then $C$ changes.

## 4 Mass conservation

We now proceed with the mass analysis. The main result is the conservation of the total mass for the Cauchy problem posed in the whole space with nonnegative data.

**Theorem 4.1.** Let $u(x, t)$ be the semigroup solution of Problem (1.1), (1.4), with $u_0 \in L^1(\mathbb{R}^N)$, $u_0 \geq 0$. Then for every $t > 0$ we have

$$\int_{\mathbb{R}^N} u(x, t) \, dx = \int_{\mathbb{R}^N} u_0(x) \, dx.$$  

Before we proceed with the proof we make two reductions: i) We may always assume that $u_0 \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ and compactly supported. If mass conservation is proved under these assumptions then it follows for all data $u_0 \in L^1(\mathbb{R}^N)$ by the contraction semigroup.

We recall that the $L^1$ mass is not conserved in the case of the Cauchy-Dirichlet problem posed in a bounded domain since mass flows out at the boundary. Indeed, the mass decays in time according to a power rule. On the other hand, mass conservation holds for the most typical linear and nonlinear diffusion problems posed in $\mathbb{R}^N$, like the Heat Equation, the Porous Medium Equation or the evolution $p$-Laplacian equation. It also holds for Neumann Problems with zero boundary data posed on bounded domains.

The proof of the theorem is divided into several cases in order to graduate the difficulties.
4.1 First case: $N < sp$.

Here the mass calculation is quite straightforward. We do a direct calculation for the tested mass. Taking a smooth and compactly supported test function $\varphi(x) \geq 0$, we have for $t_2 > t_1 > 0$

\begin{align}
\left| \int u(t_1) \varphi \, dx - u(t_2) \varphi \, dx \right| &\leq \iiint \frac{\Phi(u(y, t) - u(x, t))(\varphi(y) - \varphi(x))}{|x - y|^{N+sp}} \, dydxdt \\
&\leq \left( \iiint |u(y, t) - u(x, t)|^p \, d\mu(x, y) \, dt \right)^{\frac{1}{p}} \left( \iiint |\varphi(y) - \varphi(x)|^p \, d\mu(x, y) \, dt \right)^{\frac{1}{p}},
\end{align}

with space integrals over $\mathbb{R}^N$. Use now the sequence of test functions $\varphi_n(x) = \varphi(x/n)$ where $\varphi(x)$ is a cutoff function. Then,

$$\int \int |\varphi_n(y) - \varphi_n(x)|^p \, d\mu(x, y) = n^{N-sp} \int \int |\varphi(y) - \varphi(x)|^p \, d\mu(x, y)$$

and this tends to zero as $n \to \infty$. Using (2.9) we conclude that the triple integral involving $u$ is also bounded in terms of $\|u(\cdot, t_1)\|_2^2$, which is bounded independently of $t_1$. Therefore, taking the limit as $n \to \infty$ so that $\varphi_n(x) \to 1$ everywhere, we get

$$\int u(x, t_1) \, dx = \int u(x, t_2) \, dx,$$

hence the mass is conserved for all positive times for data in $L^2 \cap L^1$. The statement of the theorem needs to let $t_1 \to 0$, but this can be done thanks to the continuity of solution of the $L^1$ semigroup as a curve in $L^1(\mathbb{R}^N)$.

The limit case $N = sp$ also works by revising the integrals, but we get no rate.

4.2 Case $N \geq sp$.

In order to obtain the mass conservation in this case we need to use a uniform estimate of the decrease of the solutions in space so that they help in estimating the convergence of the integral. This will be done by using the barrier estimate that we have proved.

- The proof relies on some calculations with the double integrals. We also have to consider different regions. We first deal with exterior region $A_n = \{(x, y): |x|, |y| \geq n\}$, where recalling (4.2) we have

$$I(A_n) := \int_{t_1}^{t_2} \iiint_{A_n} \frac{\Phi(u(y, t) - u(x, t))|\varphi_n(y) - \varphi_n(x)|}{|x - y|^{N+sp}} \, dydxdt$$

$$\leq \left( \iiint |u(y, t) - u(x, t)|^p \, d\mu(x, y) \, dt \right)^{\frac{1}{p}} \left( \iiint |\varphi_n(y) - \varphi_n(x)|^p \, d\mu(x, y) \, dt \right)^{\frac{1}{p}},$$

which we write as $I = I_1, I_2$. In the rest of the calculation we omit the reference to the limits that is understood.
We already know that \( I_2 \leq C_p n^{(N-sp)/p(t_2 - t_1)} \). On the other hand, we want to compare \( I_1 \) with the dissipation \( D_\varepsilon \) of the \( L^r \) norm, for \( r = 1 + \varepsilon \). We recall that
\[
D_\varepsilon = \iint \int |(u(y,t) - u(x,t))|^{p-1} |u^\varepsilon(y,t) - u^\varepsilon(x,t)| \, d\mu dt \leq C(\varepsilon, p) \int |u|^{1+\varepsilon}(x, t_1) \, dx \leq C(\varepsilon, p, u_0).
\]
Next, we use the elementary equivalence: for all \( \varepsilon \in (0, 1) \) and all \( a, b > 0 \) we have
\[
|a^\varepsilon - b^\varepsilon| \geq C(\varepsilon)(a - b)(a + b)^{\varepsilon-1}.
\]
It follows that
\[
D_\varepsilon \geq C\varepsilon \iint \int |(u(y,t) - u(x,t))|^{p-1} (|u|^{\varepsilon}(y,t) + |u|^{\varepsilon}(x,t))^{\varepsilon-1} \, d\mu dt.
\]
After comparing the formulas, we conclude that
\[
I_1^{p/(p-1)} \leq D_\varepsilon \|2u\|^{1-\varepsilon}_\infty,
\]
In view of the value of \( u \) in the region \( A_n, u \approx n^{N+sp} \), we have \( I(A_n) \leq Cn^{-\sigma} \) with
\[
\sigma = \frac{1}{p}((N + sp)(p - 1)(1 - \varepsilon) - (N - sp)).
\]
Since \( p\sigma = N(p - 2) + sp^2 - \varepsilon(N + sp)(p - 1) > 0 \) for \( \varepsilon \) small, this gives the vanishing in the limit \( n \to \infty \) of this term that contributes to the conservation of mass. Note that the argument holds for all \( p \geq 2 \) and \( 0 < s < 1 \).

* We still have to make the analysis in the other regions. In the inner region \( B_n = \{(x, y) : |x|, |y| \leq 2n\} \) we get \( \varphi_n(x) - \varphi_n(y) = 0 \), hence the contribution to the integral \([4.2] \) is zero. It remains to consider the cross regions \( C_n = \{(x, y) : |x| \geq 2n, |y| \leq n\} \) and \( D_n = \{(x, y) : |x| \leq n, |y| \geq 2n\} \). Both are similar so we will look only at \( C_n \).

The idea is that we have an extra estimate: \( |x - y| > n \) so that
\[
I(C_n) \leq n^{-(N+sp)} \int_{t_1}^{t_2} \int_{C_n} |\Phi(u(y,t) - u(x,t))| |\varphi_n(y) - \varphi_n(x)| \, dydx dt
\]
\[
\leq Cn^{-(N+sp)}(t_1 - t_2) \int dy \int |u(x,t)|^{p-1} dy \leq Cn^{-(N+sp)}(t_1 - t_2)n^N \|u_0\|_{p-1}^{-1},
\]
which tends to zero as \( n \to \infty \) with rate \( O(n^{-sp}) \). Same for \( I(D_n) \). This concludes the proof.

**Signed data.** Theorem [4.1] holds also for signed data and solutions. However, the denomination mass for the integral over \( \mathbb{R}^N \) is physically justified only when \( u \geq 0 \). For signed solutions the theorem talks about conservation of the whole space integral. The above proof has been reviewed. Subsection [4.1] needs no change. As for Subsection [4.2] the elementary equivalence has to be written for all \( a, b \in \mathbb{R} \)
\[
|a^\varepsilon - b^\varepsilon| \geq C(\varepsilon)|a - b|(|a| + |b|)^{\varepsilon-1}.
\]
\section{L\textsuperscript{1} dissipation for differences}

In subsequent sections we will need the very interesting case of the dissipation of the difference \( u = u_1 - u_2 \) in the framework of the \( L^1 \) semigroup. We multiply the equation by \( \phi = s_+(u_1 - u_2) \), where \( s_+ \) denotes the sign-plus or Heaviside function, and then integrate in space and time. We get in the usual way, with \( u = u_1 - u_2 \), \( u_+ = \max\{u, 0\} \),

\[
\int u_+(x, t_1) \, dx - \int u_+(x, t_2) \, dx = \int_{t_1}^{t_2} \int s_+(u) \, dt \, dx
\]

\[= \int_{t_1}^{t_2} \int (\mathcal{L}_{s,p} u_1 - \mathcal{L}_{s,p} u_2) \, s_+(u_1 - u_2) \, dx \]

\[
\int_{t_1}^{t_2} \int \left( |u_1(x, t) - u_1(y, t)|^{p-1}(u_1(x, t) - u_1(y, t)) - |u_2(x, t) - u_2(y, t)|^{p-1}(u_2(x, t) - u_2(y, t)) \right) (s_+(u(x, t)) - s_+(u(y, t))) \, d\mu(x, y).
\]

We recall that \( s_+(u(x, t)) = 1 \) only when \( u_1(x, t) > u_2(x, t) \), and \( s_+(u(y, t)) = 0 \) only when \( u_1(y, t) < u_2(y, t) \). If we call the last factor in the above display

\[I = s_+(u_1(x, t) - u_2(x, t)) - s_+(u_1(y, t) + u_2(y, t)),\]

we see that \( I = 1 \) if \( u_1(x, t) > u_2(x, t) \) and \( u_1(y, t) \leq u_2(y, t) \). Therefore, on that set

\[u_1(x, t) - u_1(y, t) > u_2(x, t) - u_2(y, t).\]

In that case we examine the other factor,

\[F = |u_1(x, t) - u_1(y, t)|^{p-1}(u_1(x, t) - u_1(y, t)) - |u_2(x, t) - u_2(y, t)|^{p-1}(u_2(x, t) - u_2(y, t))\]

and conclude that it is positive. The whole right-hand integrand is positive.

In the same way, \( I = -1 \) if \( s_+(u(x, t)) = 0 \) and \( s_+(u(y, t)) = 1 \) i.e., only when \( u_1(x, t) \leq u_2(x, t) \) and \( u_1(y, t) > u_2(y, t) \). Then, \( u_1(x, t) - u_1(y, t) < u_2(x, t) - u_2(y, t) \) and \( F < 0 \). The whole right-hand integrand is again positive. We conclude that

\textbf{Proposition 5.1.} \textit{In the above situation we have the following dissipation estimate:}

\[
\int (u_1 - u_2)_+(x, t_1) \, dx - \int (u_1 - u_2)_+(x, t_2) \, dx
\]

\[
\geq \iint_{D} \left| |u_1(x : y, t)|^{p-1}u_1(x : y, t) - |u_2(x : y, t)|^{p-1}u_2(x : y, t) \right| \, d\mu \, dt.
\]

where \( u_1(x : y, t) = u_1(x, t) - u_1(y, t) \), \( u_2(x : y, t) = u_2(x, t) - u_2(y, t) \), and \( D \subset \mathbb{R}^2 \) is the domain where

\[
\{u(x, t) > 0, \, u(y, t) \leq 0\} \cup \{u(x, t) \leq 0, \, u(y, t) > 0\},
\]

that includes the whole domain where \( u(x, t) u(y, t) < 0 \). There is no dissipation on the set where \( u(x, t) u(y, t) > 0 \).
6 Existence of a fundamental solution

This section deals only with nonnegative solutions unless mention to the contrary. This the first main result.

Theorem 6.1. For any value of the mass $M > 0$ there exists a fundamental solution of Problem (1.1) - (1.4) having the following properties: (i) it is a nonnegative strong solution of the equation in all $L^q$ spaces, $q \geq 1$, for $t \geq t_0 > 0$. (ii) It is radially symmetric and decreasing in the space variable. (ii) It decays in space as predicted by the barrier, $u(t) = O(|x|^{-N+sp})$. (iii) It decays in time $O(t^{-\alpha})$ uniformly in $x$.

Proof. We will use the rescaling method to construct the fundamental solution as a consequence of some asymptotic behaviour as $t \to \infty$. This method has been used in typical nonlinear diffusion problems like the Porous Medium Equation, see [37], and relies on suitable a priori estimates, that are available after the previous sections. The version of the method we use here is the continuous rescaling, that can be of independent interest for the reader.

- We take an initial datum $\phi(x) \geq 0$ that is bounded, radially symmetric and supported in the ball of radius 1 and has total mass $M = 1$. We consider the strong solution $u_1(r,t)$ with such initial datum and then perform the transformation

\[ u_k(x,t) := T_k u(x,t) = k^N u_1(kx, k^{N(p-2)+sp}t) \]

for every $k > 1$. We want to let $k \to \infty$ in the end. We will apply the continuous rescaling transformation and study the rescaled flow (2.17) (with $a = 0$). First, a lemma.

Lemma 6.2. If $v_1$ is the rescaled function form $u_1$ and $v_k$ from $u_k$, then

\[ v_k(y, \tau) = v_1(y, \tau + h), \quad h = \log(k). \]

This means that the transformation $T_k$ on the original semigroup becomes a forward time shift in the rescaled semigroup

\[ S_h v(t) = v(t + h), \quad h = \log(k). \]

Proof. We have

\[ v_k(y, \tau) = (t + 1)^\alpha u_k(y(t + 1)^\beta, t) = k^N (t + 1)^\alpha u(ky(t + 1)^\beta, k^{1/\beta}t), \]

\[ v_k(y, \tau) = e^{\tau^\alpha} u_k(ye^{\tau^\beta}, e^\tau) = k^N e^{\tau^\alpha} u_1(kye^{\beta\tau}, k^{1/\beta}e^\tau), \]

where $t = e^\tau$, $\tau > -\infty$. Put $k = e^{\beta h}$ so that $ke^{\beta\tau} = e^{\beta(\tau + h)}$. Then

\[ v_k(y, \tau) = e^{(\tau + h)^\alpha} u_1(ye^{\beta(\tau + h)}, e^{\tau + h}), \]
Putting $τ = U$ has a Dirac delta as initial data. The self-similar solution is constructed.

\[ v_k(y, τ) = e^{(τ + h)α} u_1(y e^{β(τ + h)}, e^{(τ + h)}) = e^{(τ + h - τ')α} v_1(y' e^{β(τ + h - τ')}, τ'(e^h t)) \]

Putting $τ' = τ + h$, we get $v_k(y, τ) = v_1(y, τ + h)$.  

1. We may pass to the limit in the original family $\{u_k(x, t)\}_k$ or in the rescaled family $\{v_k(y, τ)\}_h$. The latter is more convenient since it is just the orbit $v_1(τ)$ and its forward translations. We will work in finite time intervals $0 < t_1 \leq t \leq t_2$, that means $-∞ < τ_1 \leq τ \leq τ_2$. From the boundedness estimates we know that both families are bounded and more precisely, the $v$-sequence has a uniform bound that does not depend on $h$. The family is also uniformly bounded in $L^1(\mathbb{R}^N)$. We also have uniform estimates on $v_t$ in $L^∞_t(L^2_x)$ and $v$ in $L^∞_t(W^{s,p}_x)$ (Hint: transform the ones for $u_k$). Using the Aubin-Lions compactness results as presented in Simon’s [33], the orbit forms a relatively compact subset of $L^1(\mathbb{R}^N) \cap L^2(\mathbb{R}^N)$. Therefore, we can pass to the limit $h → ∞$ and get a limit $V$ with strong convergence in $\mathbb{R}^N \times [t_1, t_2]$.

The limit $V(y, τ)$ is a nonnegative solution of the rescaled equation (2.18) for $τ ≥ τ_i$ with some initial value at $τ_i$. It satisfies the same bounds as before so it is strong solution in all $L^q$ spaces for $τ > t_1 = C$. The function is radially decreasing and symmetric in space for all times. The mass is conserved thanks to the uniform tail decay.

2. Going back to the original variables by inverting transformation (2.17), we get

\[ U(x, t) = t^{-α} V(x t^{-β}, \log t). \]

This a strong solution of the original equation (1.1) that has all the aforementioned properties. Let check the initial trace. Using the barrier for $u = u_1$ and its decay (3,5) there is a $C > 0$ such that

\[ u_1(x, t) ≤ C |x|^{-(N + sp)} (t + 1)^{-spβ}. \]

for all and $t > 0$ and $x ≥ C(t + 1)β$. It follows that

\[ u_k(x, t) ≤ C k^N |x|^{-(N + sp)} (k^{1/β} t + 1)^{-spβ} = C |x|^{-(N + sp)} (t + k^{-1/β})^{-spβ}. \]

for all $x ≥ C(t + k^{-1/β})β$. In the limit this means that $U(x, t) ≤ C |x|^{-(N + sp)} t^{-spβ}$, thus $U$ has a Dirac delta as initial data. The self-similar solution is constructed.  

Remarks. 1) It is easy to see that set of self-similar solutions $\{U_M\}$ is invariant under the mass preserving scaling $T_k$. In other terms, the corresponding set of $v$-solutions $\{V_M\}$ is invariant under the time translations $S_h$. This has an important consequence; if we prove uniqueness of the general fundamental solution as constructed in this section, then it would imply self-similarity because it would imply that such $V$ is stationary in time, hence $U$ is self-similar. We will not pursue that path in this paper.
2) Whenever the given total mass is negative, $M < 0$, the fundamental solution is obtained by just putting $U_M(x, t) = -U_{-M}(x, t)$.

3) Any fundamental solution must be radial and decreasing. Use approximation of $\delta$ by $u(\cdot, t)$, with $t$ very small and cut to small support and bounded data and use Proposition 2.1

6.1 The fundamental self-similar solution

Since we did not address the question of uniqueness in the previous section, we study next the issue of existence of such a self-similar solution. It will be obtained by a method that in a first step proves existence of periodic $v$ solutions.

Theorem 6.3. There is a fundamental solution of Problem (1.1)-(1.4) with the properties of Theorem 6.1 that is also self-similar. Moreover, the self-similar fundamental solution is unique. The profile $F$ is a nonnegative and radial $C^1$ function that is non-increasing along the radius, is positive everywhere and goes to zero at spatial infinity like $O(r^{-\left(N+\epsilon_1\right)})$.

Proof of existence. Let $X = L^1(\mathbb{R}^N)$. We consider a subset $K \subset X$ consisting of nonnegative radial functions $\phi$, decreasing along the radial variable, bounded and such that $\|\phi\|_1 = 1$. This is a convex, closed and bounded subset with respect to the norm of the Banach space $X$.

Next, we consider the solution of the $v$-equation (2.18) starting at $\tau = 0$ with data $v(y, 0) = \phi(y)$ and consider the semigroup map $S_h : X \to X$ defined by $S_h(\phi) = v(\cdot, h)$. According to our analysis. The set of images $S_h(K)$ satisfies $S_h(K) \subset K$. Moreover, it is relatively compact in $X$. It follows from the Schauder Fixed Point Theorem that there exists at least fixed point $\phi_h \in K$, i.e. $S_h(\phi_h) = \phi_h$. Iterating the equality we get periodicity for the orbit $v_h(y, \tau)$ starting at $\tau = 0$:

$$v_h(y, \tau + kh)) = v_h(y, \tau) \quad \forall \tau > 0,$$

valid for all integers $k \geq 1$. We now consider the set of such data $\phi_h$ producing periodic orbits $v_h$ of period $h > 0$ and contained in $K$. We may pass to the limit along a subsequence of the dyadic sequence $h_n = 2^{-n}$ as $n \to \infty$ and thus find a limit solution $\hat{v}$ defined for all $\tau \geq 0$ and starting in $K$ such that the equality

$$\hat{v}(y, \tau + k2^{-n}) = \hat{v}(y, \tau) \quad \forall \tau > 0$$

holds for infinitely many $n$'s and all integers $k \geq 1$. By continuity of the orbit in $X$, $\hat{v}$ must be stationary in time. Going back to the original variables, it means that the corresponding function $\hat{u}(x, t)$ is a self-similar solution of equation (1.1). The initial data are the unit Dirac mass.

The fixed point idea can be found in the literature. We mention Escobedo and Mischler [20] in the study of the equations of coagulation and fragmentation.
Proof of uniqueness. We know that any self-similar profile $F$ is bounded, radially symmetric and non-increasing. We know that $0 \leq F \leq C$, that $F \leq Cr^{-(N+sp)}$. We prove regularity for the profile by using the regularity of the equation. We have $U_t(x, 1) = -\nabla \cdot (xF)$ is bounded so that $F$ is a $C^1$ function for $r > 0$.

The main step is to use mass difference analysis, since this is a strict Lyapunov functional, so that we can have a contradiction when two self-similar profiles meet. This is an argument taken from my book [37]. It goes as follows.

We take two profiles $F_1$ and $F_2$ and assume the same mass $\int F_1 \, dx = \int F_2 \, dx = 1$. If $F_1$ is not $F_1$ they must intersect and then $\int (F_1 - F_2)_+ \, dx = C$ is not zero. By self-similarity it must be constant. But we have proved that whenever $C > 0$ at one time, it must be a decreasing quantity in time.

7 Asymptotic Behaviour

We establish here the asymptotic behaviour of finite mass solutions, reflected in Theorem [1.2]. We may assume that $M > 0$ and the case $M < 0$ can be reduced to positive mass by changing the sign of the solution. We comment on $M = 0$ below.

(i) We prove first the $L^1$ convergence. By scaling we may also assume that $M = 1$. The proof relies on the previous results plus the existence of a strict Lyapunov functional, that happens to be

$$(7.1) \quad J(u_1, u_2; t) := \int (u_1(x, t) - u_2(x, t))_+ \, dx$$

where $u_1$ and $u_2$ are two solutions with finite mass.

**Lemma 7.1.** Let $u_1$ and $u_2$ are two solutions with finite mass. Then, $J(u_1, u_2; t)$ is strictly decreasing in time unless the solutions are ordered.

**Proof.** By previous analysis, Section [5] we know that

$$\frac{d}{dt} J(u_1, u_2; t) = -\int_D \left| \left| u_1(x, y, t) \right|^{p-1} u_1(x, y, t) - \left| u_2(x, y, t) \right|^{p-1} u_2(x, y, t) \right| \, d\mu(x, y),$$

with notation as in (5.2). In particular, the set $D \subset \mathbb{R}^{2N}$ contains the points where

$$(u_1(x, t) - u_2(x, t)) (u_1(y, t) - u_2(y, t)) < 0.$$

Now, in order to $dJ/dt$ to vanish at a time $t_0 > 0$ we need $u_1(x, y, t) = u_2(x, y, t)$ on $D$, i.e., $u_1(x, t) - u_2(x, t) = u_1(y, t) - u_2(y, t)$. But this is incompatible with the definition $D$, so $D$ must be empty, hence $u_1$ and $u_2$ must be ordered at time $t$. This implies that they have the same property for $t > t_0$.

**Proof of (1.9) continued.** It is convenient to consider the $v$ version of both solutions, namely $v_1$ and $V_M$. We can show that $v(y, \tau + n_k)$ converges strongly in $L^1(\mathbb{R}^N)$,
along a subsequence \( n_k \to \infty \), towards a new solution \( w_1 \) of the \( \nu \)-equation. Under our assumptions \( w_1 \) is a fundamental solution. On the other hand, \( V_M \) is stationary.

We know from the Lemma that \( J(v_1, V_M; t) \) is strictly decreasing in time, unless \( v_1(t) = V_M \) for all large \( t \), in which case we are done. If this is not the case, we continue as follows. By monotonicity there is a limit

\[
\lim_{t \to \infty} J(u_1, U_M; t) = \lim_{\tau \to \infty} J(v_1, V_M; \tau) = C \geq 0.
\]

We want to prove that \( C = 0 \), which implies our result. If the limit is not zero, we consider the evolution of the new solution \( w_1 \) together with \( V_M \). We have

\[
J(w_1, V_M; t_0) = \lim_{\tau \to \infty} J(v_1, V_M; t_0 + \tau) = C,
\]

i.e., is constant for all \( t_0 > 0 \), which means that \( w_1 = V_M \) by equality of mass and the lemma. By uniqueness of the limit, we get convergence along the whole half line \( t > 0 \) instead of a sequence of times.

For general data \( u_0 \in L^1(\mathbb{R}^N), M > 0 \), we use approximation.

Finally, in the case \( M = 0 \) we just bound our solution from above and below by solutions of mess \( \varepsilon \) and \( -\varepsilon \) resp., apply the Theorem and pass to the limit \( \varepsilon \to 0 \).

(ii) Proof of convergence in uniform norm, formula (1.10). We return to the proof of the previous step and discover that the bounded sequence \( v(y, \tau + n_k) \) is locally relatively compact in the set of continuous functions in \( \mathbb{R}^N \times (\tau_1, \tau_2) \) thanks to the results on Hölder continuity of \([13]\) as commented in Subsection 2.8 once they are translated to the \( \nu \)-equation. Hence, it converges locally to the same limit as before, but now in uniform norm. In order to get global convergence we need to control the tails at infinity. We use the following argument: a sequence of space functions \( v(\cdot, \tau) \) that is uniformly bounded near infinity in \( L^1 \) (thanks to the convergence to \( V_M \)) and is also uniformly Hölder continuous must also be also uniformly small in \( L^\infty \). This implies that the previous uniform convergence was not only local but global in space. Using the correspondence (2.17), we get the convergence of the \( u(t) \) with factor \( t^{\alpha} \).

This part of the theorem is proved. \( \square \)

8 Source-type solution in a bounded domain

We can derive from the previous study the existence of source-type solutions for the problem posed in a bounded domain with zero Dirichlet outside conditions. They take a Dirac delta as initial data but we do not call them fundamental because they do not play such a key role in the theory.

Theorem 8.1. There exists a solution of the Dirichlet problem for equation (1.1) posed in a bounded domain \( \Omega \subset \mathbb{R}^N \) with initial data a Dirac delta located at an interior point, \( x_0 \in \Omega \), and zero Dirichlet data outside \( \Omega \). For \( t \geq \tau > 0 \), it is a bounded strong solution of the equation as described in [42].
Proof. (i) For convenience, we assume in the first step that $\Omega$ is the ball radius 1 centered at 0 and $x_0 = 0$. We may also assume that $M = 0$. Existence and uniqueness of solutions for the Cauchy-Dirichlet has been established in [41] and other references, and an ordered semigroup of contractions is generated in all $L^q$ spaces, $1 \leq q < \infty$. Further estimates and regularity are obtained, but beware of the long-time behaviour that is completely different. Here a question of small time behaviour is of concern, and luckily there is great similarity in that issue.

(ii) The existence of solutions of the approximate problems with data $u_{0n} \geq 0$ that converge to a Dirac delta does not offer any difficulty. Passing to the limit we easily obtain a solution $U(x,t)$ of the Cauchy-Dirichlet problem in $B_1$, using the a priori estimates and known compactness. The only important missing point is justifying that the initial data are taken. We recall that mass is not conserved in time for the Cauchy-Dirichlet problem in a bounded domain.

In order to solve the pending issue, it will be enough to show that the mass of the limit solution $U(\cdot,t)$ tends to 1 as $t \to 0$. We want to prove that for an approximating sequence of functions $u_{0n} \geq 0$, $\int_{B_1} u_{0n}(x) \, dx = 1$ and $u_{0n}(x) \to \delta(x)$ weakly, then for every $\epsilon > 0$ there is an $n_0$ and a $t_0$ such that

$$\int_{B_1} u_n(x, t) \, dx > 1 - \epsilon, \forall n \geq n_0, \, 0 < t < t_0. \tag{8.1}$$

We take the same initial data $u_{0n}$ as an approximating sequence for the problem in $\mathbb{R}^N$ and in this way we show that the corresponding solutions that we now call $u_{\infty}(x, t)$ converge to the self-similar fundamental solution that we call $U_{\infty}(x, t)$, and we have described in previous sections. By comparison we have

$$u_n(x, t) \leq u_{\infty}(x, t), \quad U(x, t) \leq U_{\infty}(x, t).$$

(iii) The novelty comes next. The following lemma provides a proof of the needed estimate $[8.1]$. We will also assume that the initial data $u_{0n}$ are a sequence of rescalings of an initial $u_{01}$ that is nonnegative, smooth, bounded and supported in a small ball $B_\delta(0)$.

**Lemma 8.2.** Under the previous assumptions, for every $\epsilon > 0$ there are $n_0$ and $\tau$ such that for $n \geq n_0$ the following inequality holds

$$u_{\infty}(x, t) - \epsilon \leq u_n(x, t) \quad \text{in} \quad B_1(0) \times (0, \tau).$$

Therefore, $U(x, t) \geq U_{\infty}(x, t) - \epsilon$ in $B_1(0) \times (0, \tau)$.

**Proof.** We first claim that $\tilde{u}(x, t) = u_{\infty}(x, t) - \epsilon$ is a solution of the same equation [(1.1)] posed in the context of the space $X_\epsilon$ obtained from $L^1(\mathbb{R}^N)$ by subjecting all functions to a downward shift. This is due to the fact the operator in invariant under vertical shifts. After the shift, the initial data are lower that before in $B_1$. In the
exterior of the ball, \(|x| \geq 1\), \(u_n(x, t)\) is extended but zero, while we can check that for large \(n\)
\[
(8.2) \quad u_n^\infty(x, t) - \varepsilon \leq 0 \quad \text{for all } |x| \geq 1, \quad \text{for } 0 < t < \tau,
\]
thanks to the a priori estimates on the decay of the solutions. Admitting this fact for the moment, we may now use comparison of the solutions in the ball to conclude that \(u_n^\infty(x, t) - \varepsilon \leq u_n(x, t)\) in \(B_1(0) \times (0, \tau)\) as desired, and this implies (8.1).

In order to prove (8.2) we use the a priori estimate for all the sequence \(u_n\) in terms of the barrier as stated at the end of Section 3
\[
u_n(x, t) \leq C|x|^{-(N+sp)}(t + a)^{sp\beta}.
\]
This constant depends on the initial data. We need \(u_n\) to be below the barrier at \(t = 0\) and for that need that for \(n\) large and putting \(|x| = \delta/n\) we have
\[
c_1 n^N \leq C(\delta/n)^{-(N+sp)}a^{sp\beta},
\]
i.e., \(C \geq c_1 \delta^{-(N+sp)}n^{-sp}a^{-sp\beta}\) near infinity. We conclude that we can fix a uniform \(C\) at for \(n \leq n_0\). We go back to the outer comparison. We need
\[
C|x|^{-(N+sp)}(t + a)^{sp\beta} \leq \varepsilon
\]
for \(|x| \geq 1\) and \(0 < t < \tau\). This holds if \(C(\tau + a)^{sp\beta} \leq \varepsilon\). \(\square\)

8.1 Other domains

(i) We consider first the case of balls \(B_R\) of radius \(R > 0\). Given some initial data \(u_0 \in L^1(B_1)\) we can solve the Cauchy-Dirichlet problem in \(B_1\) to obtain a function \(u(x, t) = S_t(u_0)\), where \(S_t\) is the semigroup generated by the equation in \(B_1\). Likewise, we denote the semigroup in \(B_R\) by \(S^R_t\), and the semigroup in \(\mathbb{R}^N\) by \(\overline{S}_t\).

It is easy to see that the scaling \(T_R u(x, t) = R^{-N} u(x, t/R^{1/\beta})\) generates a function \(u^R = T_R u\) that solves the same Cauchy-Dirichlet problem in \(B_R\). Moreover,
\[
u^R(x, 0) := T_R u_0(x) = R^{-N} u(x/R)
\]
is a rescaling of \(u_0\) that is defined for all \(x \in R\). Mass in conserved (at corresponding times). We have \(S^R_t(u^R(0)) = T_R S_t(u_0)\). The transformation can be inverted using \((T_R)^{-1} = T_{1/R}\). It is clear that \(T_R\) transforms a source-type solution in \(B_1\) into a source-type solution in \(B_R\). Besides, the Maximum Principle implies that for all \(u_0 \in L^1(\mathbb{R}^N)\), \(u_o \geq 0\) we have
\[
S_t(u_0) \leq S^R_t(u_0) \leq \overline{S}_t(u_0),
\]
A similar order applies to fundamental solutions.

(ii) For other domains \(\Omega \subset \mathbb{R}^N\) we use comparison with balls to make sure that the usual approximate solutions so not lose the initial trace when passing to the limit. More precisely, after translation we may assume that \(0 \in \Omega\) and that \(B_{R_1}(0) \subset \Omega \subset B_{R_2}(0)\). In this way the existence of a source-type solution in \(\Omega\) is proved. We leave the details to the reader.

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9 Comments and extensions

- We have proved uniqueness of the self-similar fundamental solution. The uniqueness of the general fundamental solution is a delicate issue that we did not settle here.

- The exact tail behaviour of the fundamental solution is to be established.

- Existence of solutions for measures as initial data should be investigated. This is related to the question of initial traces.

- The question of rates of convergence for the result (1.9) of Theorem 1.2 has not been considered. This issue has been addressed for many other models of nonlinear diffusion. It is solved for many of them, but well known cases remain open.

- We did not consider the case where $1 < p < 2$. For $p$ close to 1 there must exist a fundamental solution that explains the asymptotic behaviour, much as done here. This property is well known for the standard $p$-Laplacian equation with an explicit formula, cf. [36], formula (11.8). Likewise, there is a critical exponent for our equation when the self-similarity exponents blow up, i.e., for $p_c = 2N/(N+s)$. For $p < p_c$ such a fundamental solution does not exist. There is extinction in finite time, as proved in [9].

- In the existence theory we can consider wider classes of initial data, possibly growing at infinity. Optimal classes are known in the linear fractional equation (case $p = 2$), [11], and in the standard $p$-Laplacian equation (case $s = 1$), cf [19]. Of course, the asymptotic behaviour will not be the same. Also the presence of a right-hand side in the equation has been studied by a number of authors.

- We have considered a nonlinear equation of fractional type with nonlinearity $\Phi(u) = |u|^{p-2}u$, and we have used the fact that $\Phi$ is a power in a number of tools. We wonder how much of the theory holds for more general monotone nonlinearities $\Phi$.

Acknowledgments. Author partially funded by Projects MTM2014-52240-P and PGC2018-098440-B-I00 (Spain). Partially performed as an Honorary Professor at Univ. Complutense de Madrid. We thank M. Bonforte for information of ongoing parallel work with A. Salort [9], that covers a number of topics of this paper. Thus, it treats the case $p < 2$, but does not treat the self-similar solutions or asymptotic behaviour. We thank J. L. Rodrigo for pointing out reference [20] and E. Lindgren for his work [13].

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2020 Mathematics Subject Classification. 35K55, 35K65, 35R11, 35A08, 35B40.
Keywords: Nonlinear parabolic equations, $p$-Laplacian operator, fractional operators, fundamental solutions, asymptotic behaviour.