Microfield distribution in plasmas

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Abstract

We calculate stationary distribution of force acting on charged particle in plasma environment with magnetic interaction between particle and moving charges taken into account. Obtained result is a generalization of Holtsmark distribution and coincides with it in a limit of zero temperature $(\langle u^2 \rangle = 0)$. We suppose, this result to be valuable for spectra broadening calculations and studies of Brownian motion in plasma.

Introduction

First works on stochastic diffusion in plasma date back to the first half of XX century. For example, Taylor [1] considers diffusion of plasma in presence of external magnetic field with emerging fluctuations of electric field taken into account. Similar problem of Brownian motion in magnetic field was presented in [2] by Kurşunoğlu. This work was an extension of [3] by Chandrasekhar, where probability distribution function of Brownian particle associated with the magnitude of the velocity was calculated. Recent interest to this problem reemerged with works [4–10]. They take into account electrostatic interaction between particles and between particle and external magnetic field; the latter two [9,10] generalize Chandrasekhar approach and provide calculation of correlation function of charge density.

In current contribution we aim to take magnetic component of the inter-particle interaction into account (e.g. consider them as moving charges). We will find distribution of force acting on a particle immersed in plasma environment. It will be shown that relatively simple correction to Holtsmark distribution suffices to describe this physical system. We suppose, this result can be valuable for spectra broadening calculations and studies of Brownian motion in plasma.
1 Distribution of force

We aim to calculate the expression of force acting on a charged particle $Q$ in an environment containing many other discrete charges $\pm q$. To begin with, let’s write a force acting between particle, positioned at coordinates’ origin, and $i$-th charge

$$\vec{F}_i = -\frac{Q q_i}{r_i^3} \vec{r}_i + \frac{Q q_i}{c^2 r_i^3} [\vec{u}_i \times [\vec{v} \times \vec{r}_i]],$$

(1)

where $\vec{r}_i$, $\vec{u}_i$, $q_i$ are position, velocity, and charge of surrounding particles respectively, while the particle of interest has velocity $\vec{v}$. The total force acting on the particle can be written as

$$\vec{F} = \sum_{i=1}^{N} \vec{F}_i,$$

(2)

that can be treated as some sort of random walk. Here $N$ designates the total number of particles.

Now we apply Markov’s method that relates probability density function $W$ of some force $\vec{F}$ acting on particle to its characteristic function $A_N$

$$W_N (\vec{F}) = \frac{1}{8\pi^3} \int_{-\infty}^{+\infty} \exp \left( i \vec{\rho} \cdot \vec{F} \right) A_N (\vec{\rho}) d\vec{\rho},$$

(3a)

$$A_N (\vec{\rho}) = \prod_{i=1}^{N} \int_{|\vec{u}_i|=0}^{c} \int_{|\vec{r}_i|=0}^{R} \exp \left( i \vec{\rho} \cdot \vec{F}_i \right) \tau_i (\vec{r}_i; \vec{u}_i) d\vec{r}_i d\vec{u}_i,$$

(3b)

where $\tau_i$ is the probability distribution function for $i$-th particle. As integration limits we chose $R$ — radius of sphere containing $N$ charged particles and $c$ — speed of light that no particle may exceed.

Now we suppose that particles are uniformly randomly distributed, i.e. the only fluctuations occurring are these compatible with a constant average density

$$\tau_i (\vec{r}_i; \vec{u}_i) = \frac{3}{4\pi R^3} \tau (\vec{u})$$

and simplify (3b) to get the following

$$A_N (\vec{\rho}) = \left[ \frac{3}{4\pi R^3} \int_{|\vec{u}_i|=0}^{c} \int_{|\vec{r}_i|=0}^{R} \exp \left( i \vec{\rho} \cdot \vec{\phi} \right) \tau (\vec{u}) d\vec{r}_i d\vec{u}_i \right]^N,$$

(4)

where

$$\vec{\phi} = -\frac{Q q}{r^3} \vec{r} + \frac{Q q}{c^2 r^3} [\vec{u} \times [\vec{v} \times \vec{r}]].$$

Here we used the fact that all charges $q_i$ are equal by magnitude and may differ in sign only. But signs $\pm$ can be dropped due to the symmetry (action of positive...
charge $q$ at $\vec{r}$ is equivalent to the action of $-q$ at $-\vec{r}$ and because we suppose charges to be evenly distributed (probability of finding $+q$ at some point is equal to the probability of finding there $-q$).

Now we use following property of probability distribution

$$
\frac{3}{4\pi R^3} \int_0^R \int_0^R \tau (\vec{u}) \, d\vec{r} \, d\vec{u} = 1.
$$

Equation (4) can be written as

$$
A_N (\vec{\rho}) = \left[ 1 - \frac{3}{4\pi R^3} \int_0^c \int_0^R \left[ 1 - \exp \left( i\vec{\rho} \cdot \vec{\phi} \right) \right] \tau (\vec{u}) \, d\vec{r} \, d\vec{u} \right]^N.
$$

We use the fact that $\lim_{|\vec{r}| \to \infty} \vec{\phi} = 0$ to conclude the inner integral converges for big $R$ and change its upper bound to infinity, but fix the particles density $n = 3N / (4\pi R^3)$ as $R \to \infty$ and $N \to \infty$. Then we substitute $N = 4n\pi R^3 / 3$ and use the definition of $e$ constant to get the final form of (5)

$$
W \left( \vec{F} \right) = \frac{1}{8\pi^3} \int_{-\infty}^{+\infty} \exp \left( i\vec{\rho} \cdot \vec{F} \right) A (\vec{\rho}) \, d\vec{\rho} \quad (5a)
$$

$$
A (\vec{\rho}) = \exp \left[ -n \int_0^c \int_0^\infty \left[ 1 - \exp \left( i\vec{\rho} \cdot \vec{\phi} \right) \right] \, d\vec{r} \, \tau (\vec{u}) \, d\vec{u} \right] \quad (5b)
$$

$$
\vec{\phi} = -\frac{Qq}{r^3} \vec{r} + \frac{Qq}{c^2 r^3} \left[ \vec{u} \times [\vec{u} \times \vec{r}] \right] \quad (5c)
$$

1.1 Approximation for small velocities

Integrations in (5) are quite complicated, but certain approximation may simplify it drastically. As the appropriate one we will assume velocity of the particle $\vec{v}$ to be small.

First, we use (5c) to express $\vec{r}$ in terms of $\vec{\phi}$ by rewriting vector product from (5c) in terms of two scalar products, $(\vec{u} \cdot \vec{r})$ and $(\vec{u} \cdot \vec{v})$, multiplying it by $\vec{u}$ to express $Qq (\vec{u} \cdot \vec{r}) / r^3 = - \left( \vec{\phi} \cdot \vec{u} \right)$, and substitute this back to (5c). As a result $\vec{r} / r^3$ can be easily obtained, squaring it we get $1 / r^4$ and with this information easily obtain

$$
\vec{r} = -\sqrt{Qq} \sqrt{c^2 + (\vec{u} \cdot \vec{u})} \frac{c^2 \vec{\phi} + (\vec{u} \cdot \vec{\phi}) \vec{v}}{c^2 \vec{\phi} + (\vec{u} \cdot \vec{\phi}) \vec{v}^3 / 2}.
$$
Jacobian is obtained by first designating \( \mathbf{s} = c^2 \mathbf{\phi} + (\mathbf{u} \cdot \mathbf{v}) \mathbf{v} \) and representing the total Jacobian as product \( \det J = \det J_F(\mathbf{s}) \cdot \det J_F(\mathbf{\phi}) \). Then by writing appropriate matrices and calculating their determinants we get \( d\mathbf{s} = c^4 \left( c^2 + (\mathbf{u} \cdot \mathbf{v}) \right) d\mathbf{\phi} \) and \( d\mathbf{r} = d\mathbf{s} (Qq)^{3/2} \left( c^2 + (\mathbf{u} \cdot \mathbf{v}) \right)^{3/2} / (2|\mathbf{s}|^{9/2}) \). As a result we get

\[
A(\tilde{r}) = \exp \left( -n \int_{|\mathbf{\phi}|=0}^{\infty} \left[ 1 - e^{i(\bar{\mathbf{r}} \cdot \mathbf{\phi})} \right] \right) d\tilde{r} \cdot \left[ \frac{(Qq)^{3/2} c^4 \left( c^2 + (\mathbf{u} \cdot \mathbf{v}) \right)^{5/2}}{2 |c^2 \mathbf{\phi} + (\mathbf{u} \cdot \mathbf{\phi}) \mathbf{v}|^{9/2}} \right]^{1/2} f(\tilde{r}) d\mathbf{\phi}.
\]

Integration order was changed since \( \tilde{\mathbf{r}} \) is now variable of integration and does not depend on anything — all dependences were taken into account by the Jacobian.

Now we perform two assumptions. First of all we suppose \( \mathbf{v} \) to be small and approximate \( f(\mathbf{\tilde{v}}) \) with its Tailor series

\[
f \left( \mathbf{\tilde{v}} \approx \mathbf{0} \right) \approx f \left( \mathbf{\tilde{0}} \right) + \left[ \mathbf{\tilde{v}} \cdot \nabla \mathbf{\xi} \right] f \left( \mathbf{\tilde{0}} \right)_{\mathbf{\xi}=\mathbf{0}} + \frac{1}{2} \left[ \mathbf{\tilde{v}} \cdot \nabla \mathbf{\xi} \right]^2 f(\mathbf{\tilde{0}})_{\mathbf{\xi}=\mathbf{0}}, \tag{6}
\]

where

\[
f \left( \mathbf{\tilde{0}} \right) = \frac{(Qq)^{3/2}}{2^{9/2} \phi^3}
\]

\[
\left[ \mathbf{\tilde{v}} \cdot \nabla \mathbf{\xi} \right] f \left( \mathbf{\tilde{0}} \right)_{\mathbf{\xi}=\mathbf{0}} = \frac{(Qq)^{3/2}}{4 \phi^3} \left[ 5 \left( \mathbf{\tilde{\phi}} \cdot \mathbf{\phi} \right) (\mathbf{\tilde{v}} \cdot \mathbf{\tilde{u}}) + 9 \left( \mathbf{\tilde{v}} \cdot \mathbf{\phi} \right) \left( \mathbf{\tilde{\phi}} \cdot \mathbf{\tilde{u}} \right) \right]
\]

\[
\frac{1}{2} \left[ \mathbf{\tilde{v}} \cdot \nabla \mathbf{\xi} \right]^2 f \left( \mathbf{\tilde{0}} \right)_{\mathbf{\xi}=\mathbf{0}} = \frac{3(Qq)^{3/2}}{8 \phi^3} \left[ 5 \phi^4 (\mathbf{\tilde{u}} \cdot \mathbf{\tilde{v}})^2 + 39 (\mathbf{\tilde{u}} \cdot \mathbf{\phi})^2 \left( \mathbf{\tilde{\phi}} \cdot \mathbf{\tilde{u}} \right) - 6 \phi^2 (\mathbf{\tilde{u}} \cdot \mathbf{\phi})^2 v^2 - 30 \phi^2 (\mathbf{\tilde{v}} \cdot \mathbf{\phi}) (\mathbf{\tilde{u}} \cdot \mathbf{\tilde{v}}) (\mathbf{\tilde{v}} \cdot \mathbf{\phi}) \right]
\]

The second assumption is that we expect \( \mathbf{u} \) to be uniformly distributed with respect to directions and thus integration over angles can be performed

\[
\int f \left( \mathbf{v}; \mathbf{u} \right) \tau(\mathbf{u}) d\mathbf{u} = \int f \left( \mathbf{v}; \mathbf{u} \right) \frac{\tau(\mathbf{u})}{4 \pi u^2} d\mathbf{u} = \int_0^c \int_0^{2\pi} d\mathbf{\phi} \cdot \frac{\tau(\mathbf{u})}{4\pi} \sin(\mathbf{u}_\phi)\, du_\phi\, du_\theta\, du.
\tag{7}
\]

Uniform distribution with respect to direction is taken into account by changing \( \tau(\mathbf{u}) \to \tau(u)/(4\pi u^2) \). The latter means, if length of \( \mathbf{u} \) is fixed it is equally probable to find it anywhere on the sphere radius \( u \) (has area \( 4\pi u^2 \)).
Now we can integrate everything with respect to angles. First term in Taylor series does not depend on \( \vec{u} \) and thus integration over angles is equivalent to multiplication by \( 4\pi \). Integration of the second term is rather easy as well. We integrate every summand separately choosing coordinate system so that \( Z \) axis coincides with \( \vec{u} \) and \( \vec{\phi} \) respectively. Obviously it leads to integration\( \cos(u_\phi) \sin(u_\theta) \) over \([0; \pi]\) and the whole result is equal to zero. The third term of Taylor series is integrated similarly, but instead of \( \cos(u_\phi) \sin(u_\theta) \) we get \( \cos^2(u_\phi) \sin(u_\theta) \) for the first three summands (is integrated to \( 2/3 \) and \( 2\pi \) comes from integration over \( u_\phi \)) and the last summand can be integrated in Cartesian coordinates (relatively easily if symmetries are used). As a result we get

\[
\int f(\vec{v}; \vec{u}) \tau(\vec{u}) d\vec{u} = \frac{(Qq)^{3/2}}{2\phi^{9/2}} + \frac{(Qq)^{3/2}}{8\phi^{13/2}} \left( 9 \left(\vec{v} \cdot \vec{\phi}\right)^2 - \vec{\phi}^2 \vec{u}^2\right) = g(\vec{v}).
\]

Mean value \( \left\langle |\vec{u}|^2 \right\rangle \) comes from the integration of \( u^2 \) with probability distribution function \( \tau(u) \). This is convenient result, since we can express this term through environment temperature.

Now we perform integration over \( \vec{\phi} \) in \( A \). First of all we use symmetry \( (g \text{ is invariant under } \vec{\phi} \rightarrow -\vec{\phi} \text{ substitution}) \) and change \( \exp(\vec{\rho} \cdot \vec{\phi}) \) to \( \cos(\vec{\rho} \cdot \vec{\phi}) \), since the other part of complex exponent \( -i\sin(\vec{\rho} \cdot \vec{\phi}) \) is antisymmetric and vanishes as integration over the whole space is performed

\[
A(\vec{\rho}) = \exp \left[ -n \int_0^{\infty} \int_0^{\infty} \int_0^{2\pi} \left( 1 - \cos(\vec{\rho} \cdot \vec{\phi}) \right) g(\vec{v}) \phi^2 \sin(\tau) d\omega d\phi d\tau \right]. \tag{8}
\]

Integration of (8) is a bit tedious. We describe a general pathway and some intermediate results without too much details. First of all we choose convenient coordinate system — \( Z \) axis should be directed along \( \vec{\rho} \) while \( X \) axis should be chosen in a such way that \( \vec{v} \) has no \( v_y \) component. In this system \( \vec{\phi} \) has coordinates \( \vec{\phi} = \{ \phi \sin(\tau) \cos(\omega); \phi \sin(\tau) \sin(\omega); \phi \cos(\tau) \} \). Since \( \tau \) is angle between \( \vec{\phi} \) and \( \vec{\rho} \) we can deduce \( \vec{\rho} \cdot \vec{\phi} = \rho \phi \cos(\tau) \). Coordinates for vector \( \vec{v} \) are in this system \( \vec{v} = \{ v \sin(\nu); 0; v \cos(\nu) \} \). We should notice that \( (\vec{v} \cdot \vec{\phi})^2 \) is the only part of integrand dependent on \( \omega \) — all the rest parts of integral are just multiplied by \( 2\pi \) during integration over \( \omega \). But in a chosen coordinate system \( (\vec{v} \cdot \vec{\phi})^2 \) is easily integrated as well \( \int_0^{2\pi} (\vec{v} \cdot \vec{\phi})^2 d\omega = \pi v^2 \phi^2 \left( 1 - \cos^2(\tau) - \cos^2(\nu) + 3\cos^2(\tau) \cos^2(\nu) \right) \). Now we designate \( t = \cos(\tau) \), mention that \( \sin(\tau) d\tau = dt \) and \( v^2 \cos^2(\nu) = (\vec{v} \cdot \vec{\rho})^2 / \rho^2 \). This leads to

\[
\bar{g}(\vec{v}) = \frac{\pi(Qq)^{3/2}}{\phi^{9/2}} + \frac{\pi(Qq)^{3/2}}{8\phi^{13/2}} \left( 7v^2 \rho^2 - 9(\vec{v} \cdot \vec{\rho}) + 9\nu^2 \left[ 9(\vec{v} \cdot \vec{\rho}) - v^2 \rho^2 \right] \right),
\]

\[
A(\vec{\rho}) = \exp \left[ -n \int_0^1 \int_{-1}^1 \left[ 1 - \cos(\rho \phi t) \right] \bar{g}(\vec{v}) dt \phi^2 d\phi \right].
\]
Integration over \( t \) is now easily performed (doing it by parts when \( t^2 \) is integrated). And as a last step everything is integrated by \( \phi \) and we finally get

\[
A(\vec{\rho}) = \exp \left[ -\frac{8\sqrt{2} n\pi^{3/2} \rho^{3/2} (Qq)^{3/2}}{15} \left( 1 + \frac{\langle u^2 \rangle}{4c^4 \rho^2} [v^2 \rho^2 + (\vec{v} \cdot \vec{\rho})^2] \right) \right].
\tag{9}
\]

Minor simplification can be achieved if we suppose that \( \langle u^2 \rangle \ll c^2 \) and \( v^2 \ll c^2 \). This means we can approximate \( 9 \) with two first terms of Taylor series

\[
A(\vec{\rho}) = e^{-a(\rho)} \left( 1 - \frac{a(\rho) \langle u^2 \rangle}{4c^4 \rho^2} [v^2 \rho^2 + (\vec{v} \cdot \vec{\rho})^2] \right),
\tag{10}
\]

where \( a(\rho) = 8\sqrt{2} n\pi^{3/2} \rho^{3/2} (Qq)^{3/2} / 15 \). Expression \( 11 \) can be directly substituted into \( 5a \), or we may want to find its mean over all directions of \( \vec{v} \) in case we are interested in microfield distribution dependent on particle’s kinetic energy only (not velocity vector). In this case we get

\[
W(F|v) = \frac{2}{\pi F} \int_0^{+\infty} x \sin(x) e^{-a(x/F)} \left( 1 - \frac{a(x/F) \langle u^2 \rangle v^2}{3c^4} \right) dx.
\tag{11}
\]

It is worth noting that we moved from vector-valued force \( \vec{F} \) to its absolute value \( F \) by taking the mean value \( W(F) = 4\pi F^2 W(\vec{F}) \).

Equation \( 11 \) coincides with Holtsmark distribution \( 3 \) if \( Q = q \) and \( \langle u^2 \rangle = 0 \). The latter means Holtsmark distribution is partial case of \( 11 \) when temperature is zero \( \langle m(u^2) \sim kT \rangle \).

**Conclusion**

In conclusion, we have obtained stationary distribution of force \( W(F) \) acting on charged particle in plasma environment with magnetic interaction taken into account. Presented result can be treated as a generalization of Holtsmark distribution. The latter is obtained from the given one in a limit of zero temperature, i.e. \( \langle u^2 \rangle = 0 \). We suppose, this result can be valuable for spectra broadening calculations and studies of Brownian motion in plasma.

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