The Phase Space Structure of Dark Matter Halos

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ABSTRACT

The phase space structure of dark matter halos can be used to measure the mass of the halo, infer mass accretion rates, and probe the effects of modified gravity. Previous studies showed that the splashback radius can be measured in position space using the slope of the density profile. Using N-body simulations, we show that the phase space structure of the dark matter halo does not end at this splashback radius. Instead, there exists a region where infalling, splashback, and virialized halos are mixed spatially. We model the distribution of the three kinematically distinct populations and show that there exists an "edge radius" beyond which a dark matter halo has no orbiting substructures. This radius is a fixed multiple of the splashback radius as defined in previous works, and can be interpreted as a radius which contains a fixed fraction of the apocenters of dark matter particles. Our results provide a firm theoretical foundation to the satellite galaxy model adopted in the companion paper by Tomooka et al., where we analyzed the phase space distribution of SDSS redMaPPer clusters.

1 INTRODUCTION

Over the past several decades, numerical simulations have provided significant insights into our understanding of the structure and formation of dark matter halos in the concordance $\Lambda$CDM model (Frenk & White 2012, for review). The density profile of a halo in N-body simulations is typically characterized by a self-similar profile (e.g. NFW Navarro et al. 1996), with a shallow slope inside the cluster that gets progressively steeper with increasing radius. The velocity dispersion profile is governed by the density profile of the halo through Jeans equilibrium, and has been found to be increasing when the density slope is shallow, and decreasing outward when the density slope is steep (Cole & Lacey 1996; Taylor & Navarro 2001). Studies have shown that the density profile, and thus the velocity dispersion profile, reflect the initial density peaks and assembly history of the halo (Dalal et al. 2010; Ludlow et al. 2014). Recent simulations showed that halos have a sharp drop in the slope of the density profile at large radii, where the precise location of this feature is dependent on the mass accretion rate of the halo (Diemer & Kravtsov 2014).

The simple spherical collapse model predicts that there exists a physical caustic in the phase space structure of the halo (Bertschinger 1985). This so-called "splashback radius" is defined by the apocenters of the recently accreted spherical shells of particles that are at their second turnaround, and manifests as a sharp jump in the slope of the spherically symmetric density profile (Adhikari et al. 2014). Even without perfect spherical symmetry, such a density drop can be detected in the spherically-averaged density profile in N-body simulations, and has been regarded as a physical boundary of the halo that encompasses most of the bounded particles (Diemer & Kravtsov 2014; More et al. 2015). The detailed analysis of individual particle trajectories in N-body simulations revealed a broad distribution in the apocenters of the splashback particle population (Diemer et al. 2017). In practice, the radius measured based on the density slope in Mansfield et al. (2017) encloses 87\% of the apocenters of the particle trajectories, regardless of the mass accretion rate and mass of the halos. Analysis of hydrodynamics simulations also reveals that some galaxies outside the splashback radius of a halo have been inside the halo before (Haggar et al. 2020). Consequently, halos appear to extend at least somewhat past the splashback radius as it is traditionally defined.

Motivated by analysis in the companion paper by Tomooka et al, in this work we set out to determine whether a detailed study of the phase space structure of dark matter halos can shed light on what the true extent of a halo is. The phase space structure of a dark matter halo can be used to constrain cosmology through cluster mass measurements (Evrard et al. 2008; Munari et al. 2013; Bocquet et al. 2015; Hamabata et al. 2019), to constrain modified gravity models (Schmidt 2010; Lam et al. 2012; Zu et al. 2014; Mitchell et al. 2018) and to understand astrophysical processes such as assembly bias (Hearin 2015; Xu & Zheng 2018). The velocity structure inside the halo is virialized, with an average radial velocity of 0 and a dispersion directly related to the mass of the halo (Evrard et al. 2008). In the outskirts of the halo, the virialized structure disappears and most of the matter corresponds to radially infalling particles (Cole & Lacey 1996; Eke et al. 1998). Detailed characterization of the phase space structure of dark matter halos, however, reveals that near the splashback radius, the tracers of the potential well cannot be cleanly separated into virialized and infall matter using a simple radial cut. Instead, the spatial distribution usually exhibits a mix of these two types of tracers. Indeed, the infalling stream may pen-
etrate all the way into the halo center (Zu & Weinberg 2013, hereafter ZW13). For these reason, halo models that split the density distribution into a one-halo term at small scales and a two-halo term at large scales usually break down near the edge of the halo, with differences in velocity dispersion as large as 20% (Lam et al. 2013). This difference is comparable to the changes in phase space which arise from assembly bias, and is much larger than the effects from modified gravity. Thus, proper understanding of the phase space structure of dark matter halos is needed to make reliable testable predictions for cosmology and astrophysics using galaxy surveys.

In this paper, we analyze the phase space structure of dark matter halos with the goal of understanding the transition from the virialized to infalling region better. In particular, we identify the “edge radius” beyond which one does not find any additional virialized structures. Specifically, we (1) characterize the phase space structure of dark matter halos in and around the edge radius, (2) show how this radius differs from the “splashback radius” defined by the steep feature of the slope of the density profile, and (3) relate this radius to the splashback radius, and interpret it as enclosing a certain percentile of splashback particles. To analyze the phase space structure of the dark matter halos, we use the dark matter halos and subhalos from the MDPL2 (Multi-Dark Planck) N-body simulation as tracers. §2 describes the simulations and mock catalog. We present our results in §3. We summarize our findings in §4.

2 METHODOLOGY

In this work, we analyze the MDPL2 dark matter-only N-body simulation performed with the Adaptive-Refinement Tree (ART) code. The code uses the adaptive mesh refinement (AMR) technique to achieve high spatial refinement (AMR) technique to achieve high spatial resolution (Kravtsov et al. 1997; Kravtsov 1999). The simulation has a box size of 1 Gpc/h, with a force resolution of 5−13 kpc/h. The mass resolution for dark matter particle is 1.51 × 10^8 M☉/h, corresponding to 3840^3 particles. It assumes the Planck 2013 cosmology with Ω_m = 0.307, ΩΛ = 0.693, α_s = 0.823, and H_0 = 68 km (s Mpc)^{-1}. More details of the simulation can be found in Klypin et al. (2016). The halos and subhalos are identified using the Rockstar 6D phase space halo finder (Behroozi et al. 2013a), and the merger tree is built using the Consistent-Tree algorithm (Behroozi et al. 2013b). The subhalos and halos around the main halos are selected with a peak mass cut M_p > 3 × 10^{11} M☉/h, which corresponds to at least 200 particles before falling onto the halos. The main halos are selected using a mass cut M_{200m} > 10^{14} M☉/h. This ensures that the every halo has at least 20 dark matter subhalos with the above mass cut. All analyses are performed using the stacked profiles of the main halos.

3 RESULTS

3.1 The Virialized & Infalling Components of Dark Matter Structures in Phase Space

To understand the phase space around halos, we study the radial and tangential velocities using dark matter subhalos and halos as tracers. The velocity of the tracer with respect to the central halo is given by \( \vec{v} = \vec{v}_{\text{tracer}} - \vec{v}_{\text{cen}} \). The radial velocity is \( v_r = \vec{v} \cdot \hat{r} \) and the tangential velocity is \( v_\theta = \vec{v} \times \hat{r} \). Thus, the radial velocity is directional, positive for outgoing, and negative for infalling, while the tangential component is only a magnitude.

Fig. 1 shows the phase space structure of dark matter halos, illustrated as the 2D histograms of radial and tangential velocities in 4 representative radial bins. Note that the velocities are normalized by the circular velocity at \( r_{200m} \) of the halo, \( v_{\text{c}} = \sqrt{GM_{200m}/r_{200m}} \). All other radial bins are qualitatively similar to one of the four bins shown below.

The top-left panel shows the distributions of halos and subhalos in the \( v_r - v_\theta \) plane for the radial bin, \( r/r_{200m} = [0.5 - 0.55] \). The phase space structure at this radius is typical of a virialized halo, with approximately zero mean radial velocity. However, we can see a faint split between low and high total velocities for negative radial velocity component. The valley between low and high total velocity (or kinetic energy) components is roughly approximated by the blue dashed line, which is determined as the local minimum in the distribution \( P(v|v_r < 0) \). The average infall time of the high energy subhalos is less than a dynamical time, indicating that these are subhalos that have recently fallen into the central halo. Turning to the distribution of subhalos with positive radial velocities, we can see a large population of subhalos with large kinetic energy, similar to those in the infall stream. As we move radially outward in the top-right panel to \( r/r_{200m} = [0.8 - 0.85] \), the splitting between the low and high velocity components for the negative radial velocity becomes more apparent. However, the “arc” of outgoing material with large velocities becomes less distinct and appears to merge with the virialized component. Here again, subhalos with very high kinetic energies (i.e. those outside the dashed-blue line) and negative radial velocity fell into the halo less than one dynamical time ago.

The bottom-left panel shows the result at \( r/r_{200m} = [1.1 - 1.15] \) and exhibits features that are quite similar to those found in the previous radial bin at \( r/r_{200m} = [0.8 - 0.85] \), despite the fact that this bin is past the \( r_{200m} \) radius of the central halo. In both cases, there are two kinematically distinct populations. The first one has a slightly positive average radial velocity, indicating structures similar to the virialized and splashback populations within the central halo. The second population has a negative radial velocity on average, corresponding to infalling halos. In addition, there is also a small population with the total velocity larger than the blue dashed line and positive radial velocity, associated with the splashback stream. Traditionally, the structures with zero radial velocity found at \( r = [1.1 - 1.15]r_{200m} \) are not considered substructures of the central halo, because they lie outside most halo radius definitions (such as \( r_{200m} \) or \( r_{2200m} \)). However, it is clear that these halos are kinematically distinct from the infalling population, and are better thought of as substructures associated with the central halo.

We can see in these 3 panels that in general the infall streams have the largest total velocity, followed by the splashback stream, and then virialized substructures. As shown in the top-left panel, the infall and splashback streams almost mirror each other in the inner part of the halo, but the difference between these two becomes more pronounced at larger radii. The difference in kinetic en-

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¹ The dynamical time is the timescale for halos at \( r_{200m} \) to fall into the center of halo given a typical circular velocity, \( t_{\text{dyn}} = r_{200m}/\sqrt{GM_{200m}/r_{200m}} \).
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Figure 1. The 2D histograms of the radial and tangential velocity distribution at 4 representative radii. The top-left panel shows the inner region \((r/r_{200m} = [0.5 - 0.55])\) where bulk of subhalos seem "virialized" with infalling and splashback components concentrating at large velocity. The top-right panel also shows a mix of virialized halos as well as infalling and splashback stream at \(r/r_{200m} = [0.8 - 0.85]\). The blue dashed line separates virialized population from the two streams with higher total velocity. The bottom-left panel shows similar structure outside halo, but with virialized and splashback stream less prominent at \(r/r_{200m} = [1.1 - 1.15]\). The bottom-right panel at \(r/r_{200m} = [1.95 - 2]\) shows an infalling region. A small panel inside each histogram shows the radial position of the histogram along with average radial velocity. Note that the color for the histogram is normalized such that the sum of the distribution in each panel is normalized to unity.

Energy of the two populations stems from the different times at which they fell into and became part of the central halo. Subhalos which fell into the central halo earlier see smaller gravitational potential since the halo was less massive then. By the same token, virialized subhalos, which fell in even earlier, have even lower kinetic energy than the splashback subhalos. Moreover, virialized structures have lost energy to the central halo as part of the virialization process. At small radii, the difference in the infall time between the infalling and splashback populations becomes small, leading to the "mirroring" between these two populations as shown in the top-left panel of Figure 1.

Finally, the bottom-right panel of fig. 1 shows that the virialized and splashback populations have disappeared by \(r/r_{200m} = [1.95 - 2]\), leaving behind only the infalling component. As we move further away from the central halo, the average velocity of the infalling component becomes less negative, being eventually overtaken by the Hubble flow at the turnaround radius \(r_{ta}\). Beyond this radius, the distance between halos increases due to the expansion of the Universe.

Fig. 2 illustrates the halos from the bottom-left panel of fig. 1 separated into two categories: (1) top panel: halos that have been in the central halo at least once in the last 2 Gyr (approximately 1 dynamical time at \(z = 0\), or 1.5 dynamical time at \(z = 0.36\); (2) bottom panel: halos that
3.2 The Edge of Dark Matter Halos

Fig. 1 presents a simple yet compelling way of describing the phase space structure of orbiting dark matter structures around a central halo. At small radii, dark matter substructures in a halo belong to one of three categories: (1) substructures in approximate virial equilibrium with the halo; (2) an infalling stream of substructures; and (3) an outgoing population of splashback substructures. As we move towards larger radii, the virialized and splashback populations disappear, eventually leaving only a stream of infalling structures. In this work, we want to identify the radius $r_{\text{edge}}$ which defines the transition from a mix of infall and virialized/splashback populations to an infall only region based on the kinematics of subhalos.

In a previous study of the phase space structure of dark matter halos, ZW13 defined the virial extent of a halo by modeling the distribution of galaxies near a halo as a mixture of virialized and infalling galaxies. The infall stream was modeled using a skewed $t$-distribution, whereas the virialized structures were modeled as a Gaussian distribution with mean of 0. However, the model fit produced a decreasing virialized fraction in the inner part of the halo. This is in contrast to the phase space structure in the fig. 1, which shows the virialized population increases toward the halo center as expected. The degree of freedom of the $t$-distribution also hits the upper bound, turning the $t$-distribution into a Gaussian. In fitting the ZW13 model to our data, we find that these peculiarities arise because the $t$-distribution shifts so that it ends up describing the wide-peaked virialized/splashback population, rather than the infalling stream in the interior of the halo.

We find that modeling the virialized and infalling streams as a double Gaussian distribution suffices to describe all the populations adequately, thereby removing the peculiarities seen in ZW13. We also find that the unskewed $t$-distribution for infalling stream produces a good fit for the region of interest (upto $r \lesssim 2.5 r_{200m}$), with the skewed distribution only needed when we move further away from the halo. Our final model for the radial velocity distribution of substructures is

$$
\text{PDF}(v_r) = f_{\text{orb}} \left[ f \frac{G(v_r, \mu, \sigma)}{G(v_r, \mu + \mu_x, \sigma)} + (1 - f) \frac{G(v_r, \mu + \mu_x, \sigma)}{G(v_r, \mu + \mu_x, \sigma)} \right]
+ (1 - f_{\text{orb}}) t \left( \frac{v_r - \mu_{\text{int}}}{\sigma_{\text{int}}}, v \right),
$$

where $G(x, \mu, \sigma)$ is a normalized Gaussian distribution with mean $\mu$ and standard deviation $\sigma$, and $t(x, v)$ is a normalized standard $t$-distribution with $v$ degree-of-freedom. $f_{\text{orb}}$ is the fraction of orbiting substructures (i.e. virialized+splashback, any substructure that has had one pericenteric passage), whereas $f$ controls the relative weight of the two Gaussians. We also set a prior to ensure that the mean radial velocity of infalling population $\mu_{\text{int}}$ decreases monotonically, and that the orbiting fraction increases toward the inner region of the halo. This ensures that our model utilizes the $t$-distribution for capturing the behavior of the infalling stream at all radii.

Fig. 3 shows the fraction of orbiting substructures as a function of radius recovered by our model. We can see that the fraction starts out at 0 at large radii, and constantly rises after $r \lesssim 1.7 - 1.8 r_{200m}$. It then asymptotically approaches toward but not equal to unity as we move towards smaller radii. $f_{\text{orb}}(r)$ is well fit by a slight modification to the original function used in ZW13, namely $f_{\text{orb}}(r) = a \exp(-(r/r_0)^7)$, where $a = 0.986$ is the asympt-
3.3 Relation Between the Edge Radius and the Splashback Radius

We now compare the edge radius we have identified based on the subhalo kinematics to the splashback radius defined using the \textit{SPARTA} algorithm, calculated using the fitting formula in Diemer et al. (2017) for the median mass (and thus peak height) and mass accretion rate of the halos in each bin using COLOSSUS (Diemer 2018). \textit{SPARTA} identifies the splashback radius of individual particles by tracking their trajectories. The splashback radius of a particle is defined as the apocenter of the orbit at the second turnaround. The splashback radius of the halo is defined as the radius within which a specified percentile of the particle apocenters lie. The splashback radius identified using the slope of the spherically-averaged dark matter density profile corresponds to 75 to 87-percentile of particle apocenters (Xhalaj et al. 2019), while the splashback radius defined by line-of-sight density slopes corresponds to 87-percentile (Mansfield et al. 2017). In other words, at least 13% of the particles in a halo lie outside the splashback radius identified using density profile.

Fig. 4 illustrates the mass and redshift dependence of the ratio of \( r_{\text{edge}} \) and \( r_{\text{sp,87\%}} \). We see that this ratio \( (r_{\text{edge}}/r_{\text{sp,87\%}}) \) is approximately constant throughout the entire mass and redshift range we sampled. Since the peak-height is a function of mass and redshift, the ratio also stays constant as a function of the peak-height as well. Thus, we interpret the edge radius as a splashback radius containing specific percentiles of subhalos. Specifically, we can see that the edge radius \( (r_{\text{edge}}) \) extends further out than the radius encompassing 87-percentile of the dark matter particles. Beyond the 87-percentile, the splashback radius defined using particle apocenters diverges quickly Diemer et al. (2017). We conclude that \( r_{\text{edge}} = 1.6r_{\text{sp,87\%}} \) provides a better definition of the boundary of halo as we can infer from our fitting function that roughly 40% of subhalos at \( r_{\text{sp,87\%}} \) are still orbiting subhalos.

Fig. 5 also shows the ratio \( (r_{\text{edge}}/r_{\text{sp,87\%}}) \) as a function of the mass accretion rate \( (\Gamma) \), where the mass accretion rate is defined as \( \Gamma = d \log M / d \log a \) evaluated in the \( a = [0.600 - 0.733] \) range which spans one dynamical time. This figure further demonstrates the constancy of the ratio \( (r_{\text{edge}}/r_{\text{sp,87\%}}) \). It has the same mass accretion rate dependence as the splashback radius, and is again roughly a fixed multiple of \( r_{\text{sp,87\%}} \).

Analysis of the relative change of \( r_{\text{edge}} \) using different subhalo mass cuts also shows splashback-like behavior as seen in fig. 5. \( r_{\text{edge}} \) serves as the furthest splashback radius for all matter orbiting around the halo. When working with substructures, this radius is expected to be sensitive to the effects of dynamical friction. Dynamical friction tends to increase with the mass squared, so the higher the subhalo mass, the more kinetic energy the subhalo will lose and the smaller the splashback radius will be (Adhikari et al. 2016). Thus, \( r_{\text{edge}} \) decreases for a subhalo sample of larger \( M_p \).

Our findings demonstrate that \( r_{\text{edge}} \) is approximately related to the 87\% splashback radius defined using \textit{SPARTA} by a constant factor. As such, the steep slope of the spherically average density profile at the splashback radius, for example, occurs at a constant radius when normalized using \( r_{\text{edge}} \). Notably, the virialized region of a halo extends significantly beyond the traditionally defined splashback radius, and must be taken into account when modeling the virialized structures of dark matter halos.

4 CONCLUSIONS

In this work, we analyzed the phase space structure of dark matter halos using dark matter subhalos and nearby halos as tracers. Our main findings are summarized as follows:

- The phase space structure inside dark matter halos can be modeled as a mixture of subhalos on their first infall, a splashback stream of subhalos that are on their way to their first apocentric passage, and virialized subhalos which have orbited the main halo at least once. We refer to the splashback and virialized subhalo population as...
Figure 5. The ratio of $r_{\text{edge}}$ and $r_{\text{ap}, 87\%}$ as a function of mass accretion rate. The two have similar mass accretion rate dependence and the ratio remains roughly constant except at very low accretion regime. The three dashed lines indicate $r_{\text{edge}}$ computed using subhalos within different $M_p$ bins. Higher mass subhalos have smaller $r_{\text{edge}}$ due to dynamical friction similar to splashback radius.

“orbiting”, in that they are in an orbit around the central halo, bounded or unbounded.

- The edge of the halo can be defined by the radius ($r_{\text{edge}}$), beyond which little (<1%) virialized and splashback populations exist. Inside the edge radius ($r < r_{\text{edge}}$), virialized and infalling structures are mixed in physical space, but they are distinct in velocity space. Outside $r_{\text{edge}}$ and up to the turnaround radius $r_{\text{ta}}$, the halos are infalling to the central halo. Outside $r_{\text{ta}}$, the halos are receding away from the central halo due to the Hubble flow.

- The edge radius ($r_{\text{edge}}$) coincides with a fixed multiple of the splashback radius defined either using the slope of the density profile or the splashback radius containing 87-percentage of apocenters of dark matter particles. We reinterpret the edge radius $r_{\text{edge}}$, which has been previously found as part of the phase space analysis in ZW13, as the radius within which all apocenters of splashback tracers lie. This is supported by the fact that it has similar mass, redshift and mass accretion rate dependence as the splashback radii.

Our results suggest a new way of defining the halo boundaries based on the phase space structure of subhalos around dark matter halos. The edge radius ($r_{\text{edge}}$) is larger than the traditional splashback radius defined based on the slope of the dark matter density profile. Our finding is consistent with previous studies showing that the splashback radius defined based on the density slope does not encompass all the splashback particles. We show, however, that the edge radius $r_{\text{edge}}$ is clearly defined in phase space, and encompasses more than 99% of all virialized and splashback structures. That is, the edge radius ($r_{\text{edge}}$) defines a real kinematic boundary for a dark matter halo. In addition, we improved upon the previous characterization of a phase space model by ZW13, by enforcing that the $t$-distribution used in the model corresponds to the same physical population of structures at all radii (namely infalling structures).

The improved modeling and phase space and new definition of the halo boundary will allow us to use phase space measurements of cluster galaxies for cosmology and astrophysics. In the companion paper by Tomooka et al. 2020, we present the first detection of the outer edge of galaxy clusters based on spectroscopic measurements of SDSS cluster galaxy kinematics. Our study presents the physical interpretation of the edge radius defined based on the subhalo kinematics and its connection to the splashback radius and its properties. In future work, we plan to investigate observational and systematic uncertainties in extracting the 3D phase space information from line-of-sight velocity measurements and test the robustness of the method used by Tomooka et al. to infer the cluster edge radius. Such work is particularly important for measuring the phase space structures of dark matter halos accurately and precisely with the next generation spectroscopic galaxy surveys, e.g. DESI (DESI Collaboration et al. 2016) and Subaru PFS (Takada et al. 2014).

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REFERENCES

Adhikari S., Dalal N., Chamberlain R. T., 2014, J. Cosmology Astropart. Phys., 11, 19
Adhikari S., Dalal N., Clampitt J., 2016, J. Cosmology Astropart. Phys., 2016, 022
Behroozi P. S., Wechsler R. H., Wu H.-Y., 2013a, ApJ, 762, 109
Behroozi P. S., Wechsler R. H., Wu H.-Y., Busha M. T., Klypin A. A., Primack J. R., 2013b, ApJ, 763, 18
Bertschinger E., 1985, ApJS, 58, 39
Bocquet S., et al., 2015, ApJ, 799, 214
Cole S., Lacey C., 1996, MNRAS, 281, 716
DESI Collaboration et al., 2016, preprint, (arXiv:1611.00036)
Dalal N., Lithwick Y., Kuhlen M., 2010, arXiv e-prints, p. arXiv:1010.2539
Diemer B., 2018, ApJS, 239, 35
Diemer B., Kravtsov A. V., 2014, ApJ, 789, 1
Diemer B., Mansfield P., Kravtsov A. V., More S., 2017, ApJ, 843, 140
Eke V. R., Navarro J. F., Frenk C. S., 1998, ApJ, 503, 569
Evrard A. E., et al., 2008, ApJ, 672, 122
Frenk C. S., White S. D. M., 2012, Annalen der Physik, 524, 507
Haggar R., Gray M. E., Pearce F. R., Knebe A., Cui W., Mostoghiu R., Yepes G., 2020, MNRAS, p. 277
Hamabata A., Oguri M., Nishimichi T., 2019, MNRAS, 489, 1344
Hearin A. P., 2015, MNRAS, 451, L45
Klypin A., Yepes G., Gottl¨ober S., Prada F., Heß S., 2016, MNRAS, 457, 4340
Kravtsov A. V., 1999, PhD thesis, NEW MEXICO STATE UNIVERSITY
Kravtsov A. V., Klypin A. A., Khokhlov A. M., 1997, ApJS, 111, 73

MNRAS 000, 000–000 (0000)
Lam T. Y., Nishimichi T., Schmidt F., Takada M., 2012, Phys. Rev. Lett., 109, 051301
Lam T. Y., Schmidt F., Nishimichi T., Takada M., 2013, Physical Review D, 88, 023012
Ludlow A. D., Navarro J. F., Angulo R. E., Boylan-Kolchin M., Springel V., Frenk C., White S. D. M., 2014, MNRAS, 441, 378
Mansfield P., Kravtsov A. V., Diemer B., 2017, ApJ, 841, 34
Mitchell M. A., He J.-h., Arnold C., Li B., 2018, MNRAS, 477, 1133
More S., Diemer B., Kravtsov A. V., 2015, ApJ, 810, 36
Munari E., Biviano A., Borgani S., Murante G., Fabjan D., 2013, MNRAS, 430, 2638
Navarro J. F., Frenk C. S., White S. D. M., 1996, ApJ, 462, 563
Schmidt F., 2010, Phys. Rev. D, 81, 103002
Takada M., et al., 2014, PASJ, 66, R1
Taylor J. E., Navarro J. F., 2001, ApJ, 563, 483
Xhakaj E., Diemer B., Leauthaud A., Wasserman A., Huang S., Luo Y., Adhikari S., Singh S., 2019, arXiv e-prints, p. arXiv:1911.09295
Xu X., Zheng Z., 2018, MNRAS, 479, 1579
Zu Y., Weinberg D. H., 2013, MNRAS, 431, 3319
Zu Y., Weinberg D. H., Jennings E., Li B., Wyman M., 2014, MNRAS, 445, 1885